GENERALIZED INTRANSITIVE DICE: MIMICKING AN ARBITRARY TOURNAMENT

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Abstract. A generalized $N$-sided die is a random variable $D$ on a sample space of $N$ equally likely outcomes taking values in the set $\{1, \ldots, N\}$. We say of independent $N$-sided dice $D_i, D_j$ that $D_i$ beats $D_j$, written $D_i \rightarrow D_j$, if $\text{Prob}(D_i > D_j) > \text{Prob}(D_j > D_i)$. Examples are known of intransitive 6-sided dice, i.e. $D_1 \rightarrow D_2 \rightarrow D_3$ but $D_3 \rightarrow D_1$. A tournament of size $n$ is a choice of direction $i \rightarrow j$ for each edge of the complete graph on $n$ vertices. We show that if $R$ is tournament on the set $\{1, \ldots, n\}$, then for sufficiently large $N$ there exist sets of independent $N$-sided dice $\{D_1, \ldots, D_n\}$ such that $D_i \rightarrow D_j$ if and only if $i \rightarrow j$ in $R$.

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Date: January, 2019.

Key words and phrases. intransitive dice, nontransitive dice, digraph, tournament, universal tournament, mimicking a tournament, modeling a tournament, homeomorphism groups.

2010 Mathematical Subject Classification 05C20, 05C25, 05C62, 05C70.
1. Intransitive Dice

A generalized die is a cube with each face labeled among the numbers $1, 2, \ldots, 6$. On the standard die each number occurs once. Of two dice $A$ and $B$ we say that $A$ beats $B$ (written $A \rightarrow B$) if, when they are rolled, the probability that $D_A > D_B$ is greater than the probability that $D_B > D_A$ where $D_A$ and $D_B$ are the independent random variables of the values displayed by the dice $A$ and $B$, respectively. There exist examples of nontransitive dice, or intransitive dice, three dice $A, B, C$ such that $A \rightarrow B, B \rightarrow C,$ and $C \rightarrow A$. The Wikipedia page on Nontransitive Dice contains a lovely exposition with a number of different examples constructed by Efron, Grime and others.

We will be interested in the stronger version of winning, writing $A \rightarrow B$ when $P(D_A > D_B) > \frac{1}{2}$. The notion given above corresponds to $P(D_A > D_B) > \frac{1}{2} \cdot (1 - P(D_A = D_B))$.

On a sample space of $N$ equally likely outcomes, which we will call the faces, an $N$-sided die is a random variable taking values in the set $\{1, \ldots, N\}$. Such a die is called proper when the sum of the values is $N(N + 1)/2$, or, equivalently, when the expected value of a roll is $(N+1)/2$. That is, the sum is the same as that of the standard $N$-sided die with each value occurring once. These have been considered by Gowers in his blog and by Corey et al [2]. Considering large numbers of such dice leads us to the theory of tournaments.

A digraph on a set $S$ is a subset $R \subset S \times S$ such that $R \cap R^{-1} = \emptyset$ with $R^{-1} = \{(j, i) : (i, j) \in R\}$. In particular, $R$ is disjoint from the diagonal $\Delta = \{(i, i) : i \in S\}$. We write $i \rightarrow j$ for $(i, j) \in R$. For $i \in S$, $R(i) = \{j : (i, j) \in R\}$ and so $R^{-1}(i) = \{j : (j, i) \in R\}$. If $S_0 \subset S$, then the restriction of $R$ to $S_0$ is $R|_{S_0} = R \cap (S_0 \times S_0)$.

A digraph $R$ is called a tournament when $R \cup R^{-1} = (S \times S) \setminus \Delta$. Thus, $R$ is a tournament on $S$ when for each pair of distinct elements $i, j \in S$ either $(i, j)$ or $(j, i)$ lies in $R$ but not both. Harary and Moser provide a nice exposition of tournaments in [4].

Our purpose here is to show that any tournament can be mimicked by a suitable choice of $N$ sided dice. That is, with $[n] = \{1, 2, \ldots, n\}$ we prove the following.

**Theorem 1.1.** If $R$ is a tournament on $[n]$, then there is a positive integer $M$ such that for every $N \geq M$, there exists a set $D_1, \ldots, D_n$ of independent, proper $N$-sided dice such that for $i, j \in [n]$, $D_i \rightarrow D_j$ if and only if $i \rightarrow j$ in $R$. That is, for $i, j \in [n]$,

$$P(D_i > D_j) > \frac{1}{2} \iff (i, j) \in R,$$
Since the number of tournaments on \([n]\) is finite \((= 2^n(n-1)/2)\), we may choose \(M\) large enough that for every \(N \geq M\), every tournament on \([n]\) can be mimicked by proper \(N\)-sided dice.

In Corey et al \([2]\) the authors’ numerical work led them to the much stronger conjecture that, with \(n\) fixed, and letting \(N\) tend to infinity, every tournament on \([n]\) becomes equally likely.

We will call \(X\) a \textit{continuous random variable on the unit interval} when the distribution function \(F\) of \(X\) is strictly increasing and continuous on the unit interval \([0, 1]\) with \(F(0) = 0\) and \(F(1) = 1\). Equivalently, the associated measure is nonatomic with support equal to \([0, 1]\).

To prove Theorem 1.1 we replace the discrete random variables given by the dice with continuous random variables on the unit interval.

For an independent pair of such random variables, say that \(X\) beats \(Y\), written \(X \rightarrow Y\), if \(P(X > Y) > \frac{1}{2}\). Notice that for independent continuous random variables the probability that \(X = Y\) is zero.

We will call \(X\), a continuous random variable on the unit interval, \textit{proper}, when the expected value, \(E(X)\), is equal to \(\frac{1}{2}\). This is the analogue of the proper condition for dice.

In the following sections we will prove:

**Theorem 1.2.** If \(R\) is a tournament on \([n]\), then exists a set \(X_1, \ldots, X_n\) of independent, proper, continuous random variables on the unit interval such that for \(i, j \in [n]\), \(X_i \rightarrow X_j\) if and only if \(i \rightarrow j\) in \(R\). That is, for \(i, j \in [n]\),

\[
P(X_i > X_j) > \frac{1}{2} \iff (i, j) \in R,
\]

We complete this section by showing how Theorem 1.1 follows from Theorem 1.2. That is, we obtain the discrete result from the continuous one.

**Lemma 1.3.** For a continuous \([0, 1]\) valued random variable \(X\) and \(\epsilon > 0\), there exists a random variable \(Y\) with finitely many values, all rational in \([0, 1]\), with rational probabilities for each value and such that such that \(P(|X - Y| > \epsilon) < \epsilon\). In addition, if \(E(X)\) is rational, \(Y\) can be chosen so that \(E(Y) = E(X)\).

**Proof.** For \(x \in \mathbb{R}\) let \([x]\) be the largest integer less than or equal to \(x\). We may assume \(\epsilon < 1\). Choose a positive integer \(M\) with \(M\epsilon > 2\), and let \(Y_1 = \frac{[M \cdot X]}{M}\) for \(X \neq 1\) and \(Y_1 = \frac{M-1}{M}\) if \(X = 1\). Hence, \(Y_1 \leq \min(X, \frac{M-1}{M})\). In addition, \(X - Y_1 \leq \frac{1}{M}\). Furthermore, the inequality is strict unless \(X = 1\) and since \(X\) is continuous, \(P(X = 1) = 0\). It
follows that
\[ E(X) - \frac{1}{M} < E(Y_1) \leq \min(E(X), \frac{M-1}{M}). \]

For each \( k = 1, \ldots, M-1 \) we can move some weight from \( k/M \) to 0 to obtain \( Y_2 \leq Y_1 \) with rational probabilities for each value. Technically, \( Y_2 = I \cdot Y_1 \) where \( I \) is a suitably chosen Bernoulli random variable. We can make the total weight change, i.e. \( P(I = 0) \), arbitrarily small so that \( P(\{|Y_1 - Y_2| > \epsilon\}) < \delta \) with \( \delta < \epsilon \) and \( \delta < E(Y_1) - E(X) + \frac{1}{M} \). Hence, we have
\[ E(X) - \frac{1}{M} < E(Y_2) \leq \min(E(X), \frac{M-1}{M}). \]

If \( E(X) \) is irrational, let \( Y = Y_2 \). If \( E(X) \) is rational, then let \( Y = Y_2 + E(X) - E(Y_2) \). Since \( 0 \leq E(X) - E(Y_2) < \frac{1}{M} \) it follows that \( Y \) has values in \([0, 1)\). Finally,
\[ P(|X - Y| > \frac{2}{M}) < P(\{Y_1 - Y_2 > 0\}) < \epsilon. \]

Now we show that there exists a set of dice with tournament \( R \).

Proof. We obtain a set of dice which mimics tournament \( R \) by approximating the sequence \( \{X_1, \ldots, X_n\} \) from Theorem 1.2.

There exists a positive \( \gamma \) such that \( P(X_i - X_j > \gamma) > \frac{1}{2} + \gamma \) for all \( (i, j) \in R \).

Now with \( \epsilon = \gamma/3 \) use Lemma 1.3 to choose finite, \([0, 1)\)-valued, rational \( Y_j \) so that \( P(\{|X_j - Y_j| > \epsilon\}) < \epsilon \) for \( j \in [n] \) and such that \( E(Y_j) = \frac{1}{2} \).

Let \( N \) be a common denominator for all of the values and probabilities. Thus, for \( j \in [n] \) and \( k = 0, \ldots, N-1 \) there exists a non-negative integer \( P_{jk} \) such that
\[ P(Y_j = \frac{k}{N}) = \frac{P_{jk}}{N}. \]

Since the probabilities sum to one, and the expected value is \( 1/2 \) we have for each \( j \in [n] \)
\[ \sum_{k=0}^{N-1} P_{jk} = N, \]
\[ \sum_{k=0}^{N-1} kp_{jk} = \frac{N^2}{2}. \]
In particular, we see that $N$ must be even.

Define an $N$-sided die $D_j$ so that on $P_{jk}$ of the faces, the value $k + 1$ is displayed. The outcome $\bar{Z}_j$ of a roll of the die $\bar{D}_j$ has the distribution of $NY_j + 1$. It follows that for $(i,j) \in R$

$$P(\bar{Z}_i - \bar{Z}_j > 0) = P(Y_i - Y_j > 0) =$$

$$P(X_i - X_j + (Y_i - X_i) + (X_j - Y_j) > 0) \geq$$

$$P(X_i - X_j > \gamma) - P(|Y_i - X_i| > \gamma/3) - P(|X_j - Y_j| > \gamma/3) > \frac{1}{2} + \gamma/3.$$

Thus, $\bar{D}_i \to \bar{D}_j$ if and only if $(i,j) \in R$.

The sum of the face values is:

$$\sum_{k=0}^{N-1} (k + 1)P_{jk} = \frac{N(N + 2)}{2}.$$

So these are not quite proper dice.

In order to obtain proper dice, we need some additional work. First, we can choose the common denominator arbitrarily large. We will require that

$$\left(\frac{1}{2} + \frac{\gamma}{3}\right) \cdot \left(\frac{N}{N + 1}\right)^2 > \frac{1}{2}.$$

The die $D_j$ will be $N + 1$ sided. On $P_{jk}$ of the faces, the value $k + 1$ is displayed as before. In addition there is one new face with the value $\frac{N}{2} + 1 = \frac{N + 2}{2}$ (Recall that $N$ is even). It follows that the sum of the values is

$$\frac{N(N + 2)}{2} + \frac{N + 2}{2} = \frac{(N + 1)(N + 2)}{2},$$

and so these are proper dice.

For each $j \in [n]$ let $I_j$ be the indicator of the event that when $D_j$ is rolled, one of the old faces turns up. Thus, $I_j$ is a Bernoulli random variable with $P(I_j = 1) = \frac{N}{N + 1}$. If $Z_j$ is the outcome of a roll of $D_j$ then

$$Z_j = I_j \cdot (\bar{Z}_j) + (1 - I_j) \cdot \left(\frac{N}{2} + 1\right).$$

Conditioned on the assumptions that $I_i = 1$ and $I_j = 1$, $Z_i > Z_j$ if and only if $\bar{Z}_i > \bar{Z}_j$. Hence, for $(i,j) \in R$

$$P(Z_i - Z_j > 0) \geq P(\bar{Z}_i - \bar{Z}_j > 0) \cdot P(I_i = 1, I_j = 1) =$$

$$P(\bar{Z}_i - \bar{Z}_j > 0) \cdot \left(\frac{N}{N + 1}\right)^2 > \left(\frac{1}{2} + \gamma/3\right) \cdot \left(\frac{N}{N + 1}\right)^2 > \frac{1}{2}.$$
Thus, \( \{D_1, \ldots, D_n\} \) is the required list of proper \( N + 1 \)-sided dice.

To complete the proof of Theorem 1.1 we show that the tournament can be mimicked by \( N \) sided dice for sufficiently large \( N \).

Assume that \( D \) is a proper \( N \)-sided die such that there are \( P_k \) faces with value \( k \) for \( k = 1, \ldots, N \). For \( M \) a positive integer and \( S \) an integer with \( 0 \leq S < N \), we define the \( MN+S \) sided extension \( \hat{D} \) with

\[
\text{For } Q = 1, \ldots, M, \text{ there are } P_k \text{ faces with value } (Q-1)N + k \text{ for } k = 1, \ldots, N, \\
\text{For } i = 1, \ldots, S, \text{ there is one face with value } MN + i.
\]

It is clear that with \( S = 0 \) the face sum is \( M \cdot \frac{N(N+1)}{2} + N^2 \cdot \frac{M(M-1)}{2} \) and this equals \( \frac{MN(MN+1)}{2} \). As the additional faces are added, the dice remain proper. To see this, note that if the \( MN \) die had been standard, it would have remained standard as the additional faces are added.

**Lemma 1.4.** Assume that \( R \) is a tournament on \([n]\) and that \( D_1, \ldots, D_n \) is a list of proper \( N \)-sided dice such that for some \( \epsilon > 0 \)

\[
(i,j) \in R \implies P(D_i > D_j) > \frac{1}{2} + \epsilon.
\]

If \( M \) is large enough that \( 2MN\epsilon > 1 \) then for all \( S = 0, \ldots, N-1 \) the \( MN+S \) extensions \( \hat{D}_1, \ldots, \hat{D}_n \) are proper dice satisfying

\[
(i,j) \in R \implies P(\hat{D}_i > \hat{D}_j) > \frac{1}{2}.
\]

**Proof.** For \( Q = 1, \ldots, M \), think of the values between \( (Q-1)N + 1 \) and \( QN \) as the \( Q^{th} \) full block of values and the values \( MN+1, \ldots MN+S \) as the partial block. With \( (i,j) \in R \) we condition on the following cases:

- Assuming \( \hat{D}_i \) and \( \hat{D}_j \) occur in different blocks, then \( P(\hat{D}_i > \hat{D}_j) = \frac{1}{2} \), because ties cannot occur and it is equally likely that \( \hat{D}_i \) or \( \hat{D}_j \) occurs in a higher block.
- Assuming \( \hat{D}_i \) and \( \hat{D}_j \) occur in the same full block, then \( P(\hat{D}_i > \hat{D}_j) > \frac{1}{2} + \epsilon \). The probability that they occur in the same full block is \( \frac{MN^2}{(MN+S)^2} \).
- Assuming that \( \hat{D}_i \) and \( \hat{D}_j \) both occur in the partial block, \( P(\hat{D}_i = \hat{D}_j) = \frac{1}{S} \) and so \( P(\hat{D}_i > \hat{D}_j) = \frac{1}{2} (1 - \frac{1}{S}) \). The probability that they both occur in the partial block is \( \frac{S^2}{(MN+S)^2} \).
It follows that for \((i, j) \in R\), \(P(\hat{D}_i > \hat{D}_j)\) is at least \(\frac{1}{2}\) plus the deviation

\[
\epsilon \cdot \frac{MN^2}{(MN + S)^2} - \frac{1}{2S} \cdot \frac{S^2}{(MN + S)^2}.
\]

Since \(S < N\), this deviation is positive when \(2MN\epsilon > 1\).

\[\square\]

**Remark:** Observe that for any positive integer \(M\), if \(S = 0\), then for \((i, j) \in R\), \(P(\hat{D}_i > \hat{D}_j)\) is at least \(\frac{1}{2}\) plus the deviation \(\epsilon/M\). With \(M = 2\) and \(S = 0\) we call the \(2N\) extension \(\hat{D}\) of an \(N\)-sided die \(D\) the double of \(D\).

### 2. Homeomorphism Groups

Let \(\mathcal{H}\) denote the group of orientation preserving homeomorphisms on \([0, 1]\). Thus, \(F \in \mathcal{H}\) when it is a strictly increasing, continuous real-valued function on \([0, 1]\) with \(F(0) = 0\) and \(F(1) = 1\). Let \(I\) be the identity element so that \(I(x) = x\) for \(x \in [0, 1]\). For \(F \in \mathcal{H}\) let \(\rho_F\) be the right translation map on \(\mathcal{H}\) given by \(\rho_F(G) = G \circ F\).

\(\mathcal{H}\) is a subset of the Banach space \(C([0, 1])\) of continuous real-valued functions on \([0, 1]\) and, when equipped with the metric induced by the sup norm \(\| \cdot \|\), \(\mathcal{H}\) is a topological group via composition.

We call \(X\) a continuous random variable on \([0, 1]\) exactly when its distribution function \(F_X\) is an element of the group \(\mathcal{H}\). Technically, the distribution function is defined on \(\mathbb{R}\) and is zero below 0 and one above 1, but we will restrict to \([0, 1]\).

Let \(U\) be a uniform random variable on \([0, 1]\), i.e. \(U \sim Unif(0, 1)\), so that \(F_U = I\). If \(F \in \mathcal{H}\), then the random variable \(X = F^{-1}(U)\) has distribution function \(F\). That is, for \(x \in [0, 1]\) \(P(X < x) = P(U < F(x)) = F(x)\), see, e.g. [1] Chapter 5 on the Universality of the Uniform. Notice that for the expected value of such a random variable, we can integrate by parts to get

\[
E(X) = \int_0^1 x \, dF(x) =
\]

\[
xF(x)|_0^1 - \int_0^1 F(x) \, dx = 1 - \int_0^1 F(x) \, dx.
\]

Now suppose that for \(i \in [n]\), \(X_i = F_i^{-1}(U_i)\) where the \(U_i\)'s are independent \(Unif(0, 1)\) random variables and \(F_i \in \mathcal{H}\).
\[ P(X_i > X_j) = P(F_i^{-1}(U_i) > F_j^{-1}(U_j)) \]
\[ = P(F_j(F_i^{-1}(U_i)) > U_j). \]

Conditioning on the assumption \( U_i = x \), this is
\[ P(F_j(F_i^{-1}(x)) > U_j) = F_j(F_i^{-1}(x)). \]

Since \( U_i \) is uniform, it follows that
\[ P(X_i > X_j) = \int_0^1 F_j(F_i^{-1}(x)) \, dx. \]

Each \( F \in \mathcal{H} \) is a strictly increasing function from \([0, 1]\) onto \([0, 1]\). Note that the integral \( \int_0^1 F^{-1}(x) \, dx \) is the area in the square \([0, 1] \times [0, 1]\) between the y-axis and the set
\[ \{(F^{-1}(y), y) : 0 \leq y \leq 1\} = \{(x, F(x)) : 0 \leq x \leq 1\} \]
which is the complement in the square of the region under the graph of \( F \). Thus,
\[ \int_0^1 F(x) + F^{-1}(x) \, dx = 1 \]
for all \( F \in \mathcal{H} \).

Define a partition of \( \mathcal{H} \) by
\[ \mathcal{H}_+ = \{F \in \mathcal{H} : \int_0^1 F(x) \, dx > \frac{1}{2}\} \]
\[ \mathcal{H}_- = \{F \in \mathcal{H} : \int_0^1 F(x) \, dx < \frac{1}{2}\} \]
\[ \mathcal{H}_0 = \{F \in \mathcal{H} : \int_0^1 F(x) \, dx = \frac{1}{2}\} \]

The sets \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) are open and with union dense in \( \mathcal{H} \), because if \( F \in \mathcal{H}_0 \) and \( G \in \mathcal{H} \) with \( G \geq F \) and \( G \neq F \), then \( G \in \mathcal{H}_+ \). Similarly \( F \) can be perturbed to an element of \( \mathcal{H}_- \). From \( (2.4) \) we see that \( F \in \mathcal{H}_+ \) if and only if \( F^{-1} \in \mathcal{H}_- \).

If \( X = F^{-1}(U) \) with \( U \sim \text{Unif}(0, 1) \) and \( F \in \mathcal{H} \), it follows from \( (2.1) \) that
\[ E(X) = \frac{1}{2} \iff F \in \mathcal{H}_0. \]

Define on \( \mathcal{H} \) the digraph \( \Gamma_{\mathcal{H}} \) by
\[ (F, G) \in \Gamma_{\mathcal{H}} \iff F \rightarrow G \iff G \circ F^{-1} \in \mathcal{H}_+. \]
Thus, \( I \rightarrow G \) if and only if \( G \in \mathcal{H}_+ \) and \( G \rightarrow I \) if and only if \( G \in \mathcal{H}_- \). The digraph is invariant with respect to right translation. Consequently, for every \( F \in \mathcal{H} \), the union
\[
(2.8) \quad \Gamma_\mathcal{H}(F) \cup \Gamma_\mathcal{H}^{-1}(F) = \rho_F(\mathcal{H}_+) \cup \rho_F(\mathcal{H}_-)
\]
is open and dense in \( \mathcal{H} \).

From (2.8) we see that \((F_i, F_j) \in \Gamma_\mathcal{H} \) if and only if \( X_i \to X_j \) when \( X_i \) and \( X_j \) are independent random variables with distribution functions \( F_i \) and \( F_j \), respectively. Thus, Theorem 1.2 is equivalent to the statement that every finite tournament can be embedded in the restriction to \( \mathcal{H}_0 \) of the \( \Gamma_\mathcal{H} \) digraph on \( \mathcal{H} \).

To prove this, it is convenient to shift the interval from \([-1, 1]\) to \([0, 1]\). Let \( \mathcal{G} \) denote the group of orientation preserving homeomorphisms on \([-1, 1]\) and let \( i \) be the identity element so that \( i(t) = t \) for \( t \in [-1, 1] \). Let \( \mathcal{C}([-1, 1]) \) denote the separable Banach space of continuous, real-valued functions on \([-1, 1]\) so that the subset \( \mathcal{G} \) is a topological group. On \( \mathcal{G} \) the distance \( d(f, g) = \max(||f - g||, ||f^{-1} - g^{-1}||) \) defines a complete metric, topologically equivalent to one induced by the sup norm.

Define \( q : [-1, 1] \to [0, 1] \) by \( x = q(t) = \frac{t+1}{2} \) so that \( t = 2x-1 \). The conjugation map \( A_q \) given by \( f = A_q(F) = q^{-1} \circ F \circ q \) is a topological group isomorphism from \( \mathcal{H} \) to \( \mathcal{G} \), with \( F(x) = \frac{f(2x-1)+1}{2} \).

It follows that
\[
(2.9) \quad \int_0^1 F(x) \, dx = \frac{1}{2} + \frac{1}{4} \int_{-1}^1 f(t) \, dt.
\]
In particular, from (2.8) we obtain
\[
(2.10) \quad \int_{-1}^1 f(t) + f^{-1}(t) \, dt = 0.
\]

Define the partition of \( \mathcal{G} \) by
\[
\mathcal{G}_+ = \{ f \in \mathcal{G} : \int_{-1}^1 f(t) \, dt > 0 \},
\]
\[
(2.11) \quad \mathcal{G}_- = \{ f \in \mathcal{G} : \int_{-1}^1 f(t) \, dt < 0 \},
\]
\[
\mathcal{G}_0 = \{ f \in \mathcal{G} : \int_{-1}^1 f(t) \, dt = 0 \}.
\]

We see that \( A_q \) maps \( \mathcal{H}_+, \mathcal{H}_- \) and \( \mathcal{H}_0 \) to \( \mathcal{G}_+, \mathcal{G}_- \) and \( \mathcal{G}_0 \), respectively. Since it is a group isomorphism, \( A_q \) provides an isomorphism between
the digraphs $\Gamma_3$ and $\Gamma_g$ with $\Gamma_g = \{(f, g) \in \mathcal{G} \times \mathcal{G} : g \circ f^{-1} \in \mathcal{G}_+\}$. That is,

\begin{equation}
(f, g) \in \Gamma_g \iff f \to g \iff \int_{-1}^{1} g(f^{-1}(t)) \, dt > 0.
\end{equation}

Thus, Theorem 1.2 is equivalent to the following result which we will prove in the next section.

**Theorem 2.1.** Every finite tournament can be embedded in the restriction of the $\Gamma_g$ digraph to $\mathcal{G}_0$.

We let $\bar{\mathcal{G}}$ denote the set of continuous, non-decreasing maps from $[-1, 1]$ onto itself. So $f \in \bar{\mathcal{G}}$ when $f \in C([-1, 1])$, $f(\pm 1) = \pm 1$, and for all $t_1, t_2 \in [-1, 1]$, $t_1 < t_2$ implies $f(t_1) \leq f(t_2)$. Since composition is jointly continuous, $\bar{\mathcal{G}}$ is a topological semigroup with $\mathcal{G}$ the group of invertible elements. Let

$$\bar{\mathcal{G}}_0 = \{f \in \bar{\mathcal{G}} : \int_{-1}^{1} f(t) \, dt = 0\}.$$

For $f \in \bar{\mathcal{G}}$ we let $\rho_f$ denote the right translation map on $\bar{\mathcal{G}}$ so that $\rho_f(g) = g \circ f$. If $f \in \mathcal{G}$ then $\rho_f$ is a homeomorphism with inverse $\rho_{f^{-1}}$ and it maps $\mathcal{G}$ to itself. As before, the digraph $\Gamma_g$ is $\rho_f$ invariant for all $f \in \mathcal{G}$.

Define $z : [-1, 1] \to [-1, 1]$ by $z(t) = -t$. The conjugation map $A_z$ given by $A_z(f) = z \circ f \circ z$ is a topological semigroup isomorphism from $\bar{\mathcal{G}}$ to itself which preserves $\mathcal{G}$. For $f \in \bar{\mathcal{G}}$ we write $f^*$ for $A_z(f)$ so that $f^*(t) = -f(-t)$. Using the substitution $s = -t$ we see that

\begin{equation}
\int_{-1}^{1} f^*(t) \, dt = -\int_{-1}^{1} f(s) \, ds.
\end{equation}

Clearly, $f^{**} = f$.

Since $A_z$ is a topological semigroup isomorphism and is a group isomorphism on $\mathcal{G}$ it is clear that

\begin{equation}
(f \circ g)^* = f^* \circ g^* \quad \text{for } f, g \in \bar{\mathcal{G}},
\end{equation}

\begin{equation}
(f^*)^{-1} = (f^{-1})^* \quad \text{for } f \in \mathcal{G}.
\end{equation}

Let $\bar{\mathcal{G}}_{00} = \{f \in \bar{\mathcal{G}} : f = f^*\}$ and let $\mathcal{G}_{00} = \mathcal{G} \cap \bar{\mathcal{G}}_{00}$. Thus, $\bar{\mathcal{G}}_{00}$ and $\mathcal{G}_{00}$ consist of the odd functions in $\bar{\mathcal{G}}$ and $\mathcal{G}$, respectively. From (2.14) it is clear that $\bar{\mathcal{G}}_{00}$ is a closed subsemigroup of $\bar{\mathcal{G}}$ that $\mathcal{G}_{00}$ is a subgroup of $\mathcal{G}$.

Clearly,

$$\mathcal{G}_{00} \subset \bar{\mathcal{G}}_0 \quad \text{and} \quad \mathcal{G}_{00} \subset \mathcal{G}_0.$$
We collect some elementary results.

**Proposition 2.2.** (i) \( \bar{\mathcal{G}}, \mathcal{G}_0, \bar{\mathcal{G}}_0, \mathcal{G}, \mathcal{G}_0 \) and \( \mathcal{G}_{00} \) are convex subsets of \( \mathcal{C}([−1, 1]) \). Moreover, \( ((1−x)f_1 + xf_2)^* = (1−x)f_1^* + xf_2^* \) for all \( f_1, f_2 \in \bar{\mathcal{G}} \) and \( x \in [0, 1] \).

(ii) \( \bar{\mathcal{G}}, \mathcal{G}_0 \) and \( \mathcal{G}_{00} \) are the closures in \( \mathcal{C}([−1, 1]) \) of \( \mathcal{G}, \mathcal{G}_0 \) and \( \mathcal{G}_{00} \), respectively. Moreover, \( \mathcal{G} \) is a dense \( G_δ \) subset of \( \bar{\mathcal{G}} \), and \( \bar{\mathcal{G}} \setminus \mathcal{G}_0 \) is a dense, relatively open subset of \( \bar{\mathcal{G}} \).

(iii) Each of the sets \( \bar{\mathcal{G}}, \mathcal{G}_0, \bar{\mathcal{G}}_0, \mathcal{G}, \mathcal{G}_0 \) and \( \mathcal{G}_{00} \) is preserved by \( A_z \) and the map \( f \mapsto \frac{1}{2}(f + f^*) \) is a retraction of \( \bar{\mathcal{G}} \) onto \( \mathcal{G}_{00} \), taking \( \mathcal{G} \) onto \( \mathcal{G}_{00} \).

(iv) If \( f_1, \ldots, f_n \in \mathcal{G} \), then \( f_M, f_m \in \mathcal{G} \) with

\[
f_M(t) = \max_i f_i(t) \quad \text{and} \quad f_m(t) = \min_i f_i(t).
\]

**Proof.** (i): Convexity of the various subsets is obvious, and it is clear that \( A_z \) commutes with convex combination.

(ii): The sets \( \bar{\mathcal{G}}, \mathcal{G}_0 \) and \( \mathcal{G}_{00} \) are clearly closed in \( \mathcal{C}([−1, 1]) \). Furthermore, if \( f_1 \in \bar{\mathcal{G}} \) and \( f_2 \in \mathcal{G} \), then \( (1−x)f_1 + xf_2 \in \mathcal{G} \) for all \( x \in (0, 1) \) and so \( \mathcal{G} \) is dense in \( \bar{\mathcal{G}} \). Similarly, if \( f_1 \in \mathcal{G}_0 \) and \( f_2 \in \bar{\mathcal{G}} \setminus \mathcal{G}_0 \), then \( (1−x)f_1 + xf_2 \in \bar{\mathcal{G}} \setminus \mathcal{G}_0 \) for all \( x \in (0, 1) \). Since \( \bar{\mathcal{G}} \setminus \mathcal{G}_0 \) is clearly nonempty, it is dense in \( \bar{\mathcal{G}} \).

For a fixed \( t_1 < t_2 \) the condition \( f(t_1) < f(t_2) \) is an open condition on \( f \). Intersecting on all such pairs with \( t_1 \) and \( t_2 \) rational, we obtain \( \mathcal{G} \) as a \( G_δ \) subset of \( \bar{\mathcal{G}} \).

(iii): \( \frac{f(t)+f^*(t)}{2} = \frac{f(t)−f^*(t)}{2} \) and so \( \frac{1}{2}(f + f^*) \) is just the odd part of \( f \).

(iv): If \( t_1 < t_2 \) and \( f_M(t_1) = f_i(t_1) \), then \( f_M(t_2) \geq f_i(t_2) > f_i(t_1) \) because \( f_i \in \mathcal{G} \). Similarly, \( f_m \in \mathcal{G} \).

\( \Box \)

**Remark:** The set \( \mathcal{G} \) is not open in \( \bar{\mathcal{G}} \). In fact, it is easy to check that \( \mathcal{G} \setminus \mathcal{G} \) is dense in \( \bar{\mathcal{G}} \).

Let \( \bar{q} : [−1, 1] \to [−1, 0] \) by \( \bar{q}(t) = \frac{t−1}{2} \).

We define the map \( \odot : \bar{\mathcal{G}} \times \bar{\mathcal{G}} \to \mathcal{G} \) so that the restriction \( f_1 \circ f_2([-1, 0]) \) equals \( \bar{q} \circ f_1 \circ (\bar{q})^{-1} \) and \( f_1 \circ f_2([0, 1]) \) equals \( q \circ f_2 \circ (q)^{-1} \). That is,

\[
\begin{align*}
2t−1 &\quad \text{for } t \in [−1, 0], \\
2t−1 &\quad \text{for } t \in [0, 1].
\end{align*}
\]

By using the substitutions, \( s = 2t + 1 \) on \([−1, 0] \) and \( s = 2t − 1 \) on \([0, 1] \) we obtain

\[
\int_{−1}^{1} f_1 \circ f_2(t) \, dt = \frac{1}{4}\int_{−1}^{1} f_1(s) \, ds + \int_{−1}^{1} f_2(s) \, ds.
\]
Proposition 2.3. Let \( f_1, f_2, g_1, g_2 \in \mathcal{G} \).

(i) \((f_1 \circ f_2) \circ (g_1 \circ g_2) = (f_1 \circ g_1) \circ (f_2 \circ g_2)\). Moreover, if \( f_1, f_2 \in \mathcal{G} \), then \( f_1 \circ f_2 \in \mathcal{G} \) with \((f_1 \circ f_2)^{-1} = (f_1)^{-1} \circ (f_2)^{-1}\).

(ii) For \( x \in [0, 1] \),

\[
(xf_1 + (1-x)g_1) \circ f_2 = xf_1 \circ f_2 + (1-x)(g_1 \circ f_2), \\
(f_1 \circ (xf_2 + (1-x)g_2)) = xf_1 \circ f_2 + (1-x)(f_1 \circ g_2).
\]

(iii) \((f_1 \circ f_2)^* = f_2^* \circ f_1^*\). In particular, for \( f \in \mathcal{G} \), \( f \circ f^* \in \mathcal{G}_{00}\).

(iv) For \( f \in \mathcal{G} \), there exist \( f_1, f_2 \in \mathcal{G} \) such that \( f = f_1 \circ f_2 \) if and only if \( f(0) = 0 \).

(v) For \( f \in \mathcal{G} \), \( f \in \mathcal{G}_{00} \) if and only if there exists \( f_1 \in \mathcal{G} \) such that \( f = f_1 \circ f_1^* \).

Proof. (i): The map \( A_q \) and the analogue for \( \tilde{q} \) are homomorphisms.

(ii): The maps \( q \) and \( \tilde{q} \) are affine.

(iii) For \( t \in [-1, 0], -t \in [0, 1] \) and so

\[-(f_1 \circ f_2)(-t) = -f_2(-2t - 1) + 1 = f_2^*(2t + 1) - 1 = (f_2^* \circ f_1^*)(t).\]

Similarly, for \( t \in [0, 1] \).

(iv): If \( f(0) = 0 \), then

\[
f_1 = \tilde{q}^{-1} \circ (f|[-1,0]) \circ \tilde{q}, \\
f_2 = q^{-1} \circ (f|[0,1]) \circ q.
\]

(v): If \( f \in \mathcal{G}_{00} \), then, because \( f \) is odd, \( f(0) = 0 \). By (iv), \( f = f_1 \circ f_2 \) for some \( f_1, f_2 \in \mathcal{G} \). Since \( f^* = f \), (iii) implies \( f_1 \circ f_2 = f_2^* \circ f_1^* \) and so \( f_2 = f_1^* \). The converse is in (iii).

\[\square\]

Corollary 2.4. The relatively open set \( \mathcal{G}_0 \setminus \mathcal{G}_{00} \) is dense in \( \mathcal{G}_0 \).

Proof. If \( g_1, g_2 \in \mathcal{G}_{00} \) are distinct, then \( g_1 \circ g_2 \in \mathcal{G}_0 \setminus \mathcal{G}_{00} \). In general, if \( g_1 \in \mathcal{G}_0 \setminus \mathcal{G}_{00} \) and \( g_2 \in \mathcal{G}_{00} \), then for \( x > 0 \), \( xg_1 + (1-x)g_2 \in \mathcal{G}_0 \setminus \mathcal{G}_{00} \) and these approach \( g_2 \) as \( x \) tends to 0.

\[\square\]

Define

\[
Q(f, g) = \int_{-1}^{1} f(g^{-1}(t)) \, dt \quad \text{for } f, g \in \mathcal{G},
\]

so that \( g \to f \iff Q(f, g) > 0 \).
From (2.10) and (2.13) we see that for \( f, g \in G \).
(2.20) \[ Q(g, f) = -Q(f, g) = Q(f^*, g^*). \]

So \( Q(f, g) = 0 \) if \( f = g \) or if \( f, g \in G_0 \).

Recall that \( \rho_h \) is right translation by \( h \in \bar{G} \): \( \rho_h(f) = f \circ h \).
(2.21) \[ Q(\rho_h(f), \rho_h(g)) = Q(f, g) \text{ for } f, g, h \in \bar{G}. \]

From Proposition 2.3 and (2.16) we see that for \( f_1, f_2, g_1, g_2 \in \bar{G} \).
(2.22) \[ Q( f_1 \circ f_2, g_1 \circ g_2 ) = \frac{1}{4} [Q(f_1, g_1) + Q(f_2, g_2)]. \]

Finally, \( Q \) is affine in each variable separately. That is for \( f_1, f_2, g_1, g_2 \in \bar{G} \) and \( x \in [0,1] \)
(2.23) \[
\begin{align*}
Q(x f_1 + (1-x) f_2, g) &= xQ(f_1, g) + (1-x)Q(f_2, g), \\
Q(f, x g_1 + (1-x) g_2) &= xQ(f, g_1) + (1-x)Q(f, g_2).
\end{align*}
\]
This is obvious for the first variable and so, from (2.20), it follows for the second.

3. Tournaments of Generic \( n \)-tuples

For \( f \in \bar{G} \) we define \( f^e = \frac{1}{2}(f - f^*) \) so that \( f^e(t) = \frac{f(t) + f(-t)}{2} \). Thus, \( f^e \) is the even part of \( f \). Observe that \( f^e(\pm 1) = 0 \) and so while \( f^e \) is a continuous function from \([-1,1]\) to itself, it is never in \( \bar{G} \).

Since \( f^e \) is even, (2.13) implies that
(3.1) \[ 2 \int_0^1 f^e(t) \, dt = \int_{-1}^1 f^e(t) \, dt = \int_{-1}^1 f(t) \, dt. \]

In particular,
(3.2) \[ \int_0^1 f^e(t) \, dt = 0 \iff f \in \bar{G}_0. \]

Clearly
(3.3) \[ f^e = 0 \iff f \in \bar{G}_{00}. \]

For any \( x \in [0,1] \)
(3.4) \[ (xf_1 + (1-x)f_2)^e = xf_1^e + (1-x)f_2^e. \]

The proofs below use the following step-functions.
Definition 3.1. With \( m \) a positive integer, an \( m \)-sequence pair on \([-1, 1]\)
\([-1 = x_0 < x_1 < \cdots < x_m = 1; -1 = y_0 < y_1 < \cdots < y_{m+1} = 1]\)
has associated step function \( h : [-1, 1] \to [-1, 1] \) defined by

\[
(3.5) \quad h(t) = \begin{cases} 
  y_i & \text{for } x_{i-1} < t < x_i, \ i = 1, \ldots, m, \\
  y_0 & \text{for } t = x_0, \\
  \frac{1}{2}(y_i + y_{i+1}) & \text{for } t = x_i, \ i = 1, \ldots, m - 1, \\
  y_{m+1} & \text{for } t = x_m.
\end{cases}
\]

For the \( m \)-sequence pair on \([0, 1]\)
\([0 < x_1 < \cdots < x_m = 1; 0 < y_1 < \cdots < y_m < 1]\)
the associated odd step function \( h : [-1, 1] \to [-1, 1] \) is the step function associated to the \( 2m \)-sequence pair
\([-1 = -x_m < \cdots < -x_1 < 0 < x_1 < \cdots < x_m = 1; \\
-1 < -y_m < \cdots < -y_1 < y_1 < \ldots y_m < 1]\).

For a sequence pair \([x_0, \ldots, x_m; y_0, \ldots, y_{m+1}]\) on \([-1, 1]\) we define
\[
(3.6) \quad \ell_i = x_i - x_{i-1} \text{ for } 1 = 1, \ldots, m
\]
so that \( x_i = x_0 + \sum_{k=1}^{i} \ell_i \) for \( 1 = 1, \ldots, m \).

Clearly,

\[
(3.7) \quad \int_{-1}^{1} h(t) \ dt = \sum_{i=1}^{m} \ell_i y_i.
\]

Lemma 3.2. For an \( m \)-sequence pair \([x_0, \ldots, x_m; y_0, \ldots, y_{m+1}]\) on \([-1, 1]\),
assume that \( \epsilon > 0 \) satisfies \( 2\epsilon, 2\bar{\epsilon} < \min_{i} \ell_i \) where \( \bar{\epsilon} = \epsilon \frac{1+y_1}{1-y_m} \). There is
a piecewise linear \( h_{\epsilon} \in \mathcal{G} \) with \( \int_{-1}^{1} h_{\epsilon}(t) \ dt = \int_{-1}^{1} h(t) \ dt \) and as \( \epsilon \) tends
to 0 (written \( \epsilon \to 0 \)), \( h_{\epsilon} \) converges pointwise to \( h \). So if \( g : [-1, 1] \to \mathbb{R} \)
is continuous, then

\[
(3.8) \quad \int_{-1}^{1} g(h_{\epsilon}(t)) \ dt \to \int_{-1}^{1} g(h(t)) \ dt = \sum_{i=1}^{m} \ell_i g(y_i).
\]

If \( h \) is the associated odd step function to the sequence pair
\([0, x_1, \ldots, x_m = 1; 0, y_1, \ldots, y_m, y_{m+1} = 1]\)
on \([0, 1]\), then each \( h_{\epsilon} \) is an odd function and so lies in \( \mathcal{G}_{00} \).
Proof. For \( i = 1, \ldots, m \) replace the jump at \( x_i \) by the line segment connecting the point \((x_i - \epsilon, y_i)\) to \((x_i + \epsilon, y_{i+1})\). Note that at \( x_i \) the line passes through the mid-point \((x_i, \frac{1}{2}(y_i + y_{i+1}))\). In the area under the graphs a rectangle with base \( \epsilon \) and height \( y_{i+1} - y_i \) is replaced by a triangle with base \( 2\epsilon \) and height \( y_{i+1} - y_i \) and so the integral remains unchanged.

At \( x_0 = -1 \) replace the jump by the line segment connecting \((-1, -1)\) to \((-1+\epsilon, y_1)\) and at \( x_m = 1 \) replace the jump by the line connecting \((1 - \bar{\epsilon}, y_m)\) to \((1, 1)\). In the area under the graphs a triangle on the left with base \( \epsilon \) and height \( y_1 - y_0 = 1 + y_1 \) is removed and a triangle on the right with base \( \bar{\epsilon} \) and height \( y_{m+1} - y_m = 1 - y_m \) is added. The definition of \( \bar{\epsilon} \) implies that the two triangles have the same area. It follows that \( \int_{-1}^1 h_{\epsilon}(t) \, dt = \int_{-1}^1 h(t) \, dt \).

When \( h \) is an associated odd step function, it is clear that each \( h_{\epsilon} \) is odd. In that case, note that \( \bar{\epsilon} = \epsilon \).

For any \( t \in [-1, 1] \) it is clear that for \( \epsilon \) sufficiently small \( h_{\epsilon}(t) = h(t) \) and so pointwise convergence is obvious. Since \( g \circ h_{\epsilon} \) then converges pointwise to \( g \circ h \), the integral results of (3.8) follow.

\[ \square \]

The following example illustrates why we will focus upon the proper elements of \( \mathcal{G} \), i.e. the elements of \( \mathcal{G}_0 \).

**Example 3.3.** Not every edge of the digraph \( \Gamma_\mathcal{G} \) is contained in a 3-cycle.

Proof. Recall that \( i \in \mathcal{G}_{00} \) is the identity with \( i(t) = t \). Assume that \( g \in \mathcal{G} \) with \( g \geq i \) and \( g \neq i \). That is, \( g(t) \geq t \) for all \( t \in [-1, 1] \) and the inequality is strict for some \( t \). It follows that \( \int_{-1}^1 g(t) \, dt > 0 \), i.e. \( g \circ i^{-1} \in \mathcal{G}_+ \). Thus, \( i \to g \) in \( \Gamma_\mathcal{G} \). If \( g \to f \), then

\[ (3.9) \quad f \circ i^{-1} = f = (f \circ g^{-1}) \circ g \geq f \circ g^{-1}, \]

and the latter inequality is not an equation. It follows that

\[ (3.10) \quad \int_{-1}^1 f(t) \, dt > \int_{-1}^1 f(g^{-1}(t)) \, dt > 0. \]

Hence, \( i \to f \) and \( f \not\in \mathcal{G}_0 \).

Thus, \((i, g)\) is not contained in a 3-cycle. Furthermore, there does not exists \( f \in \mathcal{G}_0 \) such that \( g \to f \).

\[ \square \]
In contrast with the example is the following result which is essentially the continuous time version of Theorem 2 of [3].

**Theorem 3.4.** If \( f \in \mathcal{S}_0 \setminus \{i\} \), then there exist \( g_1, g_2 \in \mathcal{S}_0 \) such that

\[
\int_{-1}^{1} f(g^{-1}_1(t)) \, dt > 0 > \int_{-1}^{1} f(g^{-1}_2(t)) \, dt.
\]

That is, \( Q(f, g_1) > 0 > Q(f, g_2) \) and so \( g_1 \to f \to g_2 \) in \( \Gamma_5 \).

If \( f \in \mathcal{S}_0 \setminus \mathcal{S}_{00} \), then \( g_1 \) and \( g_2 \) can be chosen in \( \mathcal{S}_{00} \).

**Proof.** We will generalize this result in Theorem 3.5 below. We will here only consider the case with \( f \in \mathcal{S}_0 \setminus \mathcal{S}_{00} \) as the simple proof motivates the later argument.

Because \( f \notin \mathcal{S}_{00} \), \( f^e \) is even and is not identically zero. So there exists \( a \in [0, 1] \) such that \( f^e(a) \neq 0 \). Because \( f \in \mathcal{S}_0 \), \( \int_0^1 f^e(t) dt = 0 \) and so \( f^e \) must change sign in the interval \([0, 1]\). We can therefore choose \( a_1, a_2 \in [0, 1] \) such that \( f^e(a_1) > 0 > f^e(a_2) \). That is, \( f(a_1) + f(-a_1) > 0 > f(a_2) + f(-a_2) \).

Consider the sequence pair \([0, 1; 0, a_1, 1]\) in \([0, 1]\) and let \( h \) be the associated odd step function. That is,

\[
(3.12) \quad h(t) = \begin{cases} 
-a_1 & \text{for } -1 < t < 0, \\
a_1 & \text{for } 0 < t < 1, \\
\pm 1 & \text{for } t = \pm 1, \\
0 & \text{for } t = 0.
\end{cases}
\]

Observe that

\[
(3.13) \quad \int_{-1}^{1} f(h(t)) \, dt = f(a_1) + f(-a_1) > 0.
\]

Let \( h_\epsilon \in \bar{\mathcal{S}}_{00} \) be be the functions obtained from Lemma 3.2 which converge pointwise to \( h \). For \( x \in (0, 1) \) \( xi + (1 - x)h_\epsilon \in \bar{\mathcal{S}}_{00} \). If \( g_1^{-1} = xi + (1 - x)h_\epsilon \) for \( x \) and \( \epsilon \) sufficiently small, then \( \int_{-1}^{1} f(g_1^{-1}(t)) \, dt > 0 \).

We define \( g_2 \), proceeding the same way with \( a_2 \) replacing \( a_1 \) in (3.12).

**Remark:** Note that \( Q(g, i) = 0 \) for all \( g \in \mathcal{S}_0 \) and so there does not exist \( g \in \mathcal{S}_0 \) such that \( g \to i \) or \( i \to g \) in \( \Gamma_5 \).

Our major result is an extension of Theorem 3.4

**Theorem 3.5.** Let \( f_1, \ldots, f_n \in \mathcal{S}_0 \) and let \( J \subset [n] \).
If the set of $n + 1$ functions $\{i, f_1, \ldots, f_n\}$ is linearly independent in $C([-1, 1])$, then there exists $g \in G_0$ such that

$$
Q(f_j, g) \begin{cases} > 0 & \text{for } j \in J, \\ < 0 & \text{for } j \in [n] \setminus J. 
\end{cases}
$$

(3.14)

In terms of the digraph $\Gamma_g$, $g \rightarrow f_j$ for $j \in J$ and $f_j \rightarrow g$ for $j \in [n] \setminus J$.

If the set of even functions $\{f_1^e, \ldots, f_n^e\}$ is linearly independent in $C([-1, 1])$, then $g$ can be chosen in $G_{00}$.

**Proof.** First, we assume that $\{f_1^e, \ldots, f_n^e\}$ is linearly independent and produce $g \in G_{00}$. Since no set containing 0 is linearly independent, it follows that no $f_i^e = 0$ and so $f_i \in G_0$ for all $i \in [n]$.

Define $E_j = f_j^e$ for $j \in J$ and $- f_j^e$ for $j \in [n] \setminus J$. The set $\{E_1, \ldots, E_n\}$ is linearly independent and from (3.2) it follows that $\int_0^1 E_j(t) \, dt = 0$ for all $j \in [n]$.

**Step 1:** There exist $a_1, \ldots, a_m \in (0, 1)$ and $p_1, \ldots, p_m > 0$ such that

$$
\sum_{k=1}^m p_k E_j(a_k) > 0, \quad \text{for all } j \in [n].
$$

(3.15)

Assuming the contrary we let $C$ be the cone in $\mathbb{R}^n$ generated by the vectors $\{(E_1(t), \ldots, E_n(t)) : t \in (0, 1)\}$. By assumption the cone $C$ is disjoint from positive orthant $\mathbb{R}^n_+ = \{(y_1, \ldots, y_n) : y_j > 0 \text{ for all } j \in [n]\}$. We apply the Theorem of the Separating Hyperplane (see, e.g. [5] p. 38). It implies that there exist $z \in \mathbb{R}^n_+ \setminus \{0\}$, $b \in \mathbb{R}$, such that $\sum_j z_j y_j \geq b$ for all $(y_1, \ldots, y_n) \in \mathbb{R}^n_+$ and $\sum_j z_j c_j \leq b$ for all $c \in C$.

Because 0 lies in the closure of both $C$ and $\mathbb{R}^n_+$ it follows that $b = 0$ and so from the definition of $C$ it follows that $\sum_j z_j E_j(t) \leq 0$ for all $t \in (0, 1)$ and so, by continuity, for $t = 0$ and $t = 1$ as well.

On the other hand, $\int_0^1 \sum_j z_j E_j(t) \, dt = 0$. Consequently, $\sum_j z_j E_j(t) = 0$ for all $t \in [0, 1]$. Since the vector $z$ is nonzero, it follows that the set of functions $\{E_1, \ldots, E_n\}$ is linearly dependent, contradicting our assumption of linear independence.

By renumbering and multiplying by a positive constant, we may assume that the points have been chosen with $0 < a_1 < \cdots < a_m < 1$ and $\sum_j p_j = 1$.

**Step 2:** Let $b_k = \sum_{s=1}^k p_s$ for $k = 1, \ldots, m$. Let $h$ be the associated odd step-function for the sequence pair $[0, b_1, \ldots, b_m = 1; 0, a_1, \ldots, a_m, 1]$ on $[0, 1]$. 

For \( j \in [n] \)

\[
(3.16) \quad \int_{-1}^{1} f_j(h(t)) \, dt = \sum_{k=1}^{m} p_k (f_j(a_k) + f_j(-a_k)) = 2 \sum_{k=1}^{m} p_k f_{j}^{0}(a_k). 
\]

By (3.15) and the definition of \( E_j \)'s, this is positive for \( j \in J \) and negative for \( j \in [n] \setminus J \).

**Step 3:** As in the proof of Theorem 3.4, replace \( h \) by \( g^{-1} = x + (1 - x) h \), where the functions \( h \) are the Lemma 3.2 approximations to \( h \). For sufficiently small \( x > 0 \) and \( \epsilon > 0 \) the required inequalities still hold with \( h \) replaced by \( g^{-1} \).

Now we assume merely that \( \{i, f_1, \ldots, f_n\} \) is linearly independent. Notice that if some linear combination \( \sum_k z_k f_k \in S_{00} \) then \( \sum_k z_k f_k^0 = 0 \). It follows that linear independence of \( \{f_1^0, \ldots, f_n^0\} \) implies linear independence of \( \{i, f_1, \ldots, f_n\} \).

Define \( E_0 = i \), \( E_j = f_j \) for \( j \in J \) and \( E_j = -f_j \) for \( j \in [n] \setminus J \). Thus, \( \{E_0, E_1, \ldots, E_n\} \) is linearly independent and \( \int_{-1}^{1} E_j(t) \, dt = 0 \) for \( j = 0, 1, \ldots, n \).

**Step 4:** There exist \( a_1, \ldots, a_m \in (-1, 1) \) and \( p_1, \ldots, p_m > 0 \) such that

\[
(3.17) \quad \sum_{k=1}^{m} p_k E_0(a_k) = 0 \quad \text{and} \quad \sum_{k=1}^{m} p_k E_j(a_k) > 0, \quad \text{for all} \quad j \in [n].
\]

Assuming the contrary we let \( C \) be the cone in \( \mathbb{R}^{n+1} \) generated by the vectors \( \{(E_0(t), E_1(t), \ldots, E_n(t)) : t \in (-1, 1)\} \). By assumption the cone \( C \) is disjoint from the cone \( \{0\} \times \mathbb{R}_+^n = \{(0, y_1, \ldots, y_n) : y_j > 0 \text{ for all } j \in [n]\} \). We again apply the Theorem of the Separating Hyperplane and so there exist \( z \in \mathbb{R}^{n+1} \setminus \{0\}, b \in \mathbb{R} \) such that \( \sum_{j=1}^{n} z_j y_j > b \) for all \( (y_1, \ldots, y_n) \in \mathbb{R}_+^n \) and \( \sum_{j=0}^{n} z_j c_j \leq b \) for all \( c \in C \).

Because 0 lies in the closure of both \( C \) and \( \{0\} \times \mathbb{R}_+^n \), it follows that \( b = 0 \) and so from the definition of \( C \) it follows that \( \sum_{j=0}^{n} z_j E_j(t) \leq 0 \) for all \( t \in (-1, 1) \) and so, by continuity, for \( t = -1 \) and \( t = 1 \) as well.

As in Step 1, \( \int_{-1}^{1} \sum_{j=0}^{n} z_j E_j(t) \, dt = 0 \) implies \( \sum_{j=0}^{n} z_j E_j(t) = 0 \) for all \( t \in [-1, 1] \) and so the set of functions \( \{E_0, E_1, \ldots, E_n\} \) is linearly dependent, contradicting our assumption of linear independence.
By renumbering and multiplying by a positive constant, we may assume that the points have been chosen with $-1 < a_1 < \cdots < a_m < 1$ and $\sum_j p_j = 2$.

**Step 5:** Let $b_0 = -1$ and $b_k = -1 + \sum_{s=1}^{k} p_s$ for $k = 1, \ldots, m$.

Let $a_0 = -1$ and $a_{m+1} = 1$.

Let $h$ be the associated step-function for the sequence pair $[-1 = b_0, b_1, \ldots, b_m = 1; -1 = a_0, a_1, \ldots, a_m, a_{m+1} = 1]$ on $[-1, 1]$.

\begin{equation}
(3.18) \quad \int_{-1}^{1} h(t) \, dt = \sum_{k=1}^{m} p_k a_k = \sum_{k=1}^{m} p_k E_0(a_k) = 0.
\end{equation}

So the approximating functions $h_\epsilon$ of Lemma 3.2 lie in $\bar{G}$.

For $j \in [n]$ \begin{equation}
(3.19) \quad \int_{-1}^{1} f_j(h(t)) \, dt = \sum_{k=1}^{m} p_k (f_j(a_k))
\end{equation}

By (3.17) and the definition of $E_j$’s, this is positive for $j \in J$ and negative for $j \in [n] \setminus J$.

**Step 6:** As before replace $h$ by $g^{-1} = xi + (1 - x)h_\epsilon$ which this time lies in $\bar{G}$. For sufficiently small $x > 0$ and $\epsilon > 0$ the required inequalities still hold with $h$ replaced by $g^{-1}$.

\[\square\]

**Remark:** Note that if $f \in \bar{G} \setminus \{i\}$, then $\{i, f\}$ is linearly independent and so from Theorem 3.5 it follows that there exist $g_1, g_2 \in \bar{G}$ such that $g_1 \to f \to g_2$. This completes the proof of Theorem 3.4.

In contrast with Example 3.3 we have the following.

**Corollary 3.6.** If $f_1, f_2 \in \bar{G}$ with $f_1 \to f_2$ in $\Gamma_\bar{G}$, then there exists $f_3 \in \bar{G}$ such that $f_2 \to f_3 \to f_1$ in $\Gamma_\bar{G}$. That is, every edge of the restriction $\Gamma_\bar{G}|\bar{G}_0$ is contained in a 3-cycle in $\Gamma_\bar{G}|\bar{G}_0$.

**Proof.** If $f_1 = x_1 i + x_2 f_2$, then $Q(f_1, f_2) = x_1 Q(i, f_2) + x_2 Q(f_2, f_2) = 0$. So $f_1 \to f_2$ implies that $\{i, f_1, f_2\}$ is linearly independent and so the existence of the required $f_3$ follows from Theorem 3.5.

\[\square\]

Recall from Proposition 2.2 (ii) that $\bar{G}$ is a dense, $G_\delta$ subset of $\bar{G}$ which is closed in the complete, separable metric space $C([-1,1])$. It
follows that the relatively closed subsets $G_0 = G \cap \bar{G}_0$ and $G_{00} = G \cap \bar{G}_{00}$ are $G_\delta$ subsets of $C([-1, 1])$ as well.

**Corollary 3.7.** For every positive integer $n$ the set of $n$-tuples

$$\mathcal{T}_n = \{ f = (f_1, \ldots, f_n) \in (\mathcal{G})^n : Q(f_i, f_j) \neq 0 \text{ for all } i, j \in [n] \text{ with } i \neq j \}$$

is a relatively open, dense subset of $(\mathcal{G})^n$ and the intersection with $(G_0)^n$ is dense in $(G_0)^n$.

**Proof.** Since the intersection of finitely many open, dense subsets is open and dense, it suffices to prove this for the case $n = 2$.

Because the group operations are continuous and $Q(f_1, f_2) = f_1^1 f_1(f_2^{-1}(t)) dt$, the condition $Q(f_1, f_2) \neq 0$ is an open condition on the pair $(f_1, f_2) \in (\mathcal{G})^2$.

Density in $(\mathcal{G})^2$ is easy. If $Q(f_1, f_2) = 0$, choose $h \in \mathcal{G}$ with $h \geq f_1$, but $h \neq f_1$. By replacing $h$ by $xh + (1 - x)f_1$ with small $x > 0$ we may choose $h$ arbitrarily close to $f_1$. Since $h \circ f_2^{-1} \geq f_1 \circ f_2^{-1}$ and is distinct from it, $Q(h, f_2) > 0$.

For a pair $(f_1, f_2) \in (G_0)^2$ we may first perturb to get $f_1 \in G_0 \setminus \{ i \}$. Now if $Q(f_1, f_2) = 0$, then by Theorem 3.4 there exists $g \in G_0$ such that $Q(f_1, g) \neq 0$. Let $f_2^x = xg + (1 - x)f_2$ so that by (2.23) $Q(f_1, f_2^x) = xQ(f_1, g)$ and this is not equal to zero for $x > 0$.

□

An $n$-tuple $f = (f_1, \ldots, f_n) \in (\mathcal{G})^n$ lies in $\mathcal{T}_n$ exactly when the restriction of $\Gamma_5$ to the set $\{ f_1, \ldots, f_n \}$ is a tournament. For such an $f = (f_1, \ldots, f_n) \in \mathcal{T}_n$ we denote by $R[f]$ the tournament on $[n]$ defined by

$$\forall (i, j) \in R[f] \iff Q(f_j, f_i) > 0 \iff (f_i, f_j) \in \Gamma_5.$$

The tournament $R[f]$ is called the tournament associated to $f \in \mathcal{T}_n$.

**Example 3.8.** For $n \geq 3$ there exist linearly dependent $n$-tuples in $\mathcal{T}_n$. Nonetheless, the general lifting result of Theorem 3.7 requires that $\{ i, f_1, \ldots, f_n \}$ be linearly independent.

**Proof.** Assume $f_1 \to f_0$ with $f_0, f_1 \in \mathcal{G}$. For $x \in [0, 1]$ let $f_x = (1 - x)f_0 + xf_1$. Because $-Q(f_1, f_0) = Q(f_0, f_1) > 0$ we have, for $a < b$,

$$Q(f_a, f_b) = b(1 - a)Q(f_0, f_1) + a(1 - b)Q(f_1, f_0) = (b - a)Q(f_0, f_1) > 0.$$ 

That is, $f_b \to f_a$ when $b > a$. If $0 < a_1 < \cdots < a_{n-2} < 1$, then $(f_0, f_{a_1}, \ldots, f_{a_{n-2}}, f_1) \in \mathcal{T}_n$. 


Suppose that \( f = (f_1, \ldots, f_n) \in \text{TOUR}_n \cap (G_0)^n \). Assume that for \( J_1 \subset [n-1] \) and \( J_2 = [n-1] \setminus J_1 \), \( f_n = \sum_{j \in J_1} a_j f_j - \sum_{j \in J_2} a_j f_j + a_0 t \) with \( a_j \geq 0 \) for \( j = 1, \ldots, n \). Since \( Q(f, i) = 0 \) for all \( f \in G_0 \), \( f_n \neq i \) and so \( a_j > 0 \) for at least one \( j \in [n] \). Now if \( g \in G_0 \) with \( Q(f_j, g) > 0 \) for all \( j \in J_1 \) and \( Q(f_j, g) < 0 \) for all \( j \in J_2 \), then \( Q(f, g) = \sum_{j \in J_1} a_j Q(f_j, g) - \sum_{j \in J_2} a_j Q(f_j, g) > 0 \). That is, \( g \to f_j \) for all \( j \in J_1 \) and \( f_j \to g \) for all \( j \in J_2 \) implies \( g \to f_n \). So we cannot lift with \( J = J_1 \subset [n] \) and \( g \in G_0 \).

\[ \square \]

**Lemma 3.9.** There is an infinite sequence \( g_1, g_2, \ldots \) in \( G_0 \) with \( g_i(0) = 0 \) for all \( i \) and such that \( \{g_1, g_2, \ldots\} \) is linearly independent.

**Proof.** If \( h_1, h_2 \in G_0 \) with \( h_1^* \neq h_2 \), then \( h_1 \circ h_2 \in G_0 \setminus G_{00} \). It is in \( G_0 \) by (2.10). For \( t \in [0, 1] \),

\[
(h_1 \circ h_2)^{(t)} = \frac{1}{2}[(h_1 \circ h_2)(t) + (h_1 \circ h_2)(-t)] = \frac{1}{4}[(h_2(2t-1) + 1) + (h_1(-2t + 1) - 1)] = \frac{1}{4}[h_2(2t - 1) - h_1^*(2t - 1)].
\]

In particular, this applies when \( h_1 \) and \( h_2 \) are distinct elements of \( G_{00} \).

Now choose a linearly independent sequence \( h_1, h_2, \ldots \in G_{00} \), for example, we may choose \( h_k(t) = t^{2k-1} \). If we let \( g_k = h_{2k-1} \circ h_{2k} \), then (3.22) implies that \( \{g_1, g_2, \ldots\} \) is linearly independent. From the definition of the \( \circ \) operator, \( g_i(0) = 0 \) for all \( i \).

\[ \square \]

We will need a bit of linear algebra folklore.

**Lemma 3.10.** Let \( V \) be a normed linear space.

(a) If \( V_0 \) is a finite dimensional subspace of \( V \), then \( V_0 \) is closed in \( V \).

(b) For any positive integer \( n \), the set \( \{v_1, \ldots, v_n\} \in V^n : \{v_1, \ldots, v_n\} \) is linearly independent \} is a \( G_5 \) subset of \( V^n \).

**Proof.** (a) Let \( v_1, \ldots, v_n \) be a basis for \( V_0 \) and let \( M = \max_i \|v_i\| \). For \( (x_1, \ldots, x_n) \in \mathbb{R}^n \) let \( J(x_1, \ldots, x_n) = \sum_i x_i v_i \). The linear isomorphism \( J \) is continuous, since the linear operations are continuous in \( V \). Define \( ||(x_1, \ldots, x_n)||_1 = ||J(x_1, \ldots, x_n)|| \leq nM \cdot ||(x_1, \ldots, x_n)||_0 \) where \( || \cdot ||_0 \) is the Euclidean norm on \( \mathbb{R}^n \). Since \( || \cdot ||_1 \) is continuous, and the unit sphere in \( \mathbb{R}^n \) is compact, there exists \( m > 0 \) such that \( || \cdot ||_1 \geq m \cdot || \cdot ||_0 \)
on $\mathbb{R}^n$. It follows that with respect to the norm $|| \cdot ||_1$, $\mathbb{R}^n$ is complete and so $V_0$ is a complete subspace of $V$. A subset of a metric space is closed if it is complete.

(b) For $K, k$ positive integers, let $A^k_K = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : K^{-1} \leq |x_i| \leq K$ for $i \in [k]\}$. The set $(v_1, \ldots, v_k) \in V^k$ such that $\sum_i x_i v_i = 0$ for some $(x_1, \ldots, x_k) \in A^k_K$ is closed because $A^k_K$ is compact. Take the union over the positive integers $K$, we obtain an $F_\sigma$ subset $W_k$ of $V^k$. An $n$-tuple $(v_1, \ldots, v_n) \in V^n$ is linearly dependent if and only if it projects to $W_k$ for some $k \leq n$, where the projection omits $n - k$ vectors and renumbers the remaining $k$. It follows that the set of linearly dependent $n$-tuples is an $F_\sigma$.

\begin{lemma}
For any positive integer $n$, the sets

\[(f_1, \ldots, f_n) \in (\mathcal{G}_0)^n : \{f^e_1, \ldots, f^e_n\} \text{ is linearly independent} \]

\{(f_1, \ldots, f_n) \in (\mathcal{G}_0)^n : \{i, f_1, \ldots, f_n\} \text{ is linearly independent} \}

are dense, $G_\delta$ subsets of $(\mathcal{G}_0)^n$.
\end{lemma}

\begin{proof}
The inclusion was shown during the proof of Theorem 3.5. By continuity of the linear map $f \rightarrow f^e$, it follows from Lemma 3.10(b) that the sets are $G_\delta$.

From the inclusion it suffices to show that the first set is dense.

By induction on $k$ we show we can perturb to get linear independence when $\{f^e_1, \ldots, f^e_{n-k}\}$ is linearly independent. No perturbation is needed when $k = 0$. Now assuming the result for $k - 1$, if $\{f^e_1, \ldots, f^e_{n-k}, f^e_{n-(k-1)}\}$ is linearly independent then we can apply the induction hypothesis to perturb to linear independence. If it is not, then $f^e_{n-(k-1)}$ is a linear combination of the linearly independent set $\{f^e_1, \ldots, f^e_{n-k}\}$. For a sequence $\{g_1, g_2, \ldots\}$ as constructed in Lemma 3.9, at least one among $g^e_1, \ldots, g^e_{n-(k-1)}$ is not a linear combination of $\{f^e_1, \ldots, f^e_{n-k}\}$. Suppose $g^e_i$ is not. Replace $f_{n-(k-1)}$ by $f^x_{n-(k-1)} = (1 - x)f_{n-(k-1)} + x g_i$. For any $x > 0$ $\{f^e_1, \ldots, f^e_{n-k}, (f^x_{n-(k-1)})^e\}$ is linearly independent. Choose $x > 0$ small and then apply the induction hypothesis as before.

\end{proof}

\begin{definition}
Call $f = (f_1, \ldots, f_n)$ a generic $n$-tuple when $f \in (\mathcal{G}_0)^n \cap \mathcal{TOUR}_n$ and the set $\{i, f_1, \ldots, f_n\}$ is linearly independent. Call it strongly generic if $f \in (\mathcal{G}_0)^n \cap \mathcal{TOUR}_n$ and the set $\{f^e_1, \ldots, f^e_n\}$ is linearly independent.
\end{definition}
We denote by $\mathcal{GEN}_n$ and $\mathcal{GEN}_n^+$ the set of generic $n$-tuples and the set of strongly generic $n$-tuples.

**Proposition 3.13.** The sets $\mathcal{GEN}_n$ and $\mathcal{GEN}_n^+$ of generic $n$-tuples are dense $G_\delta$ subsets of $(\mathcal{S}_0)^n$.

**Proof.** This is immediate from Corollary 3.7 and Lemma 3.11. □

**Theorem 3.14.** Assume that $f = (f_1, \ldots, f_n) \in \mathcal{GEN}_n$ with $R[f]$ the associated tournament on $[n]$ and assume that $R$ is a tournament on $[n+1]$ whose restriction to $[n]$ is $R[f]$. For $f_{n+1} \in \mathcal{S}_0$ write $f'$ for the $n+1$-tuple $(f_1, \ldots, f_n, f_{n+1})$.

(a) The set $\{f_{n+1} \in \mathcal{S}_0 : f' \in \mathcal{GEN}_{n+1}\}$ is open and dense in $\mathcal{S}_0$. If $f \in \mathcal{GEN}_n^+$, then $\{f_{n+1} \in \mathcal{S}_0 : f' \in \mathcal{GEN}_{n+1}^+\}$ is open and dense in $\mathcal{S}_0$.

(b) The set $\{f_{n+1} \in \mathcal{S}_0 : f' \in \mathcal{GEN}_{n+1}, \text{ and } R[f'] = R\}$ is open and nonempty in $\mathcal{S}_0$.

(c) If $f \in \mathcal{GEN}_n^+$, then the set $\{f_{n+1} \in \mathcal{S}_0 : f' \in \mathcal{GEN}_{n+1}, \text{ and } R[f'] = R\}$ is open in $\mathcal{S}_0$. There exists $f_{n+1} \in \mathcal{S}_0$ with $f_{n+1}(0) = 0$ such that $f' \in \mathcal{GEN}_{n+1}^+$, and $R[f'] = R$.

In particular, every possible tournament extension of $R[f]$ occurs as the associated tournament of some extension $f'$ of $f$ to a generic $n+1$-tuple.

**Proof.** By Lemma 3.10(a) the set of $f_{n+1}$ such that $f_{n+1}$ lies in the space spanned by $\{i, f_1, \ldots, f_n\}$ is closed in $\mathcal{S}_0$. Similarly, the set of $f_{n+1}$ such that $f_{n+1}^e$ lies in the space spanned by $\{f_1^e, \ldots, f_n^e\}$ is closed in $\mathcal{S}_0$. So the conditions on $f_{n+1}$ that $\{i, f_1, \ldots, f_n, f_{n+1}\}$ be linearly independent (or that $\{f_1^e, \ldots, f_n^e\}$ be linearly independent) is an open condition when $f$ is generic (resp. when $f$ is strongly generic). The condition $Q(f_{n+1}, f_i) > 0$ or $< 0$ for any $i \in [n]$ is an open condition. It follows that the $f_{n+1}$'s such that $f'$ is generic or strongly generic form an open set as do those with $R[f'] = R$.

By Theorem 3.5 there exists $g \in \mathcal{S}_0$ such that $g \rightarrow f_i$ for all $i \in R(n+1) \subset [n]$, i.e. $n+1 \rightarrow i$ in $R$, and $f_i \rightarrow g$ for all $i$ in $R^{-1}(n+1)$, the complementary subset in $[n]$. If $f$ is strongly generic then $g$ can be chosen in $\mathcal{S}_{00}$. 

Using the sequence \( \{g_1, g_2, \ldots \} \) from Lemma 3.9 we obtain \( g_i \in \mathcal{G}_0 \) such that \( g_i(0) = 0 \) and \( \{i, f_1, \ldots, f_n, g_i\} \) is linearly independent or \( \{f_1^e, \ldots, f_n^e, g_i^e\} \) is linearly independent in the strongly generic case.

Let \( f_{n+1}^x = xg_i + (1 - x)g \) with \( x \in [0, 1] \).

In the generic case, if \( \{i, f_1, \ldots, f_n, g\} \) is linearly independent, then we let \( f_{n+1} = g \). Otherwise, \( \{i, f_1, \ldots, f_n, f_{n+1}^x\} \) is linearly independent for \( x > 0 \).

In the strongly generic case then, since \( g \in \mathcal{G}_0 \), (3.4) implies that \( (f_{n+1}^x)^e = xg_i^e \). Hence, for \( x > 0 \), \( \{f_1^e, \ldots, f_n^e, (f_{n+1}^x)^e\} \) is linearly independent. Also, in this case, \( f_{n+1}^x(0) = 0 \).

For \( x \) small enough, \( f_{n+1}^x \rightarrow f_i \) for all \( i \in R(n+1) \) and and \( f_i \rightarrow f_{n+1}^x \) for all \( i \in R^{-1}(n + 1) \). Fix \( x > 0 \) small enough that these conditions hold and let \( f_{n+1} = f_{n+1}^x \).

Thus, \( f' = (f_1, \ldots, f_n, f_{n+1}) \) is a generic or strongly generic \( n + 1 \)-tuple with \( R[f'] = R \).

For arbitrary \( h \in \mathcal{G}_0 \) the \( n + 1 \)-tuple \( (f_1, \ldots, f_n, (1 - x)h + xf_{n+1}) \) is generic for \( x > 0 \) small enough (or strongly generic if \( f \) is). For example, if the original \( n + 1 \)-tuple with \( x = 0 \) has linearly independent even parts, then it still does for small \( x > 0 \). If it had linearly dependent even parts then they becomes linearly independent for \( x \in (0, 1] \). Similarly for the tournament inequalities. This completes the proof of density in (a).

\[ \square \]

**Corollary 3.15.** For every tournament \( R \) on \([n]\) and every \( f_1 \in \mathcal{G}_0 \setminus \{i\} \), there exists a generic \( n \)-tuple \( f \) which begins with \( f_1 \) and which has associated tournament \( R[f] \) equal to \( R \). If \( f_1 \in \mathcal{G}_0 \setminus \mathcal{G}_{00} \), we can choose \( f \) strongly generic and so that \( f_i(0) = 0 \) for all \( i > 1 \).

**Proof.** Use induction on \( n \), beginning with \( f_1 \) for \( n = 1 \) and then applying Theorem 3.14. \[ \square \]

**Remark:** The condition \( f_i(0) = 0 \) is of interest because of Proposition 2.3 (iv).

In particular, this completes the proof of Theorem 2.11 and so of Theorem 1.2.

We conclude with some constructions.
For $f = (f_1, \ldots, f_n), g = (g_1, \ldots, g_n) \in (\mathcal{G})^n$, $h \in \mathcal{G}$ and $\pi$ a permutation on $[n]$ we define

$$f^* = (f_1^*, \ldots, f_n^*),$$

$$g \circ f = (g_1 \circ f_1, \ldots, g_n \circ f_n),$$

$$\rho_h f = (\rho_h(f_1), \ldots, \rho_h(f_n)),$$

$$\pi f = (f_{\pi^{-1}1}, \ldots, f_{\pi^{-1}n}).$$

(3.23)

We call $f \circ f$ the double of $f$.

Notice that the operations $\cdot^*, \circ$ and $\pi$ leave $(\mathcal{G}_0)^n$ invariant while $\rho_h$ need not.

For a tournament $R$ we define $\pi R = \{(\pi i, \pi j) : (i, j) \in R\}$.

**Proposition 3.16.** Let $f = (f_1, \ldots, f_n) \in \mathcal{TOURE}_n \cap (\mathcal{G}_0)^n$.

(a) The $[n]$-tuple $f^* \in \mathcal{TOURE}_n$ with $R[f^*]$ the reversed tournament $R[f]^{-1}$. Furthermore, $f^*$ is generic (or strongly generic) if $f$ is generic (resp. strongly generic).

(b) The double $f \circ f \in \mathcal{TOURE}_n$ with $R[f \circ f] = R[f]$, and $f \circ f$ is generic (or strongly generic) if $f$ is generic (resp. strongly generic).

(c) If $g \in (\mathcal{G}_00)^n$, then $g \circ f \in \mathcal{TOURE}_n$ with $R[g \circ f] = R[f]$, and $g \circ f$ is generic (or strongly generic) if $f$ is generic (resp. strongly generic).

(d) If $h \in \mathcal{G}$, then $\rho_h (f) \in \mathcal{TOURE}_n$ with $R[\rho_h (f)] = R[f]$.

(e) If $\pi$ is a permutation on $[n]$, then $\pi f \in \mathcal{TOURE}_n$ with $R[\pi f] = \pi R[f]$ and $\pi f$ is generic (or strongly generic) if $f$ is generic (resp. strongly generic).

**Proof.** (a) Since $(f_i^*)^e = -f_i^e$ it follows that $\{(f_i^*)^e, \ldots, (f_n^*)^e\}$ is linearly independent when $\{(f_1)^e, \ldots, (f_n)^e\}$ is. It is clear that if $i, f_1, \ldots, f_n$ is linearly independent, then $\{i, f_1^*, \ldots, f_n^*\}$ is, since $i^* = i$.

Since $f_2^* \circ (f_2^*)^{-1} = (f_k \circ f_j^{-1})^*$ the reversal of the signs of the integrals follows from (2.13).

(b) and (c) If $z_0 i^* + \sum_{j=1}^n z_j g_j \circ f_j = 0$ then for $t \in [0, 1]$, $z_0 (2t - 1) + \sum_{j=1}^n z_j f_j (2t - 1)$ is a constant (Note that $i \circ i = i$). Since $i, f_1, \ldots, f_n \in \mathcal{G}_0$ the constant is zero. Hence, $z_0 i^* + \sum_{j=1}^n z_j f_j = 0$ on $[-1, 1]$. That is, for arbitrary $g \{i, g_1 \circ f_1, \ldots, g_n \circ f_n\}$ is linearly independent if $\{i, f_1, \ldots, f_n\}$ is.

(b) By (3.22)

$$(f_j \circ f_j)^e(t) = \frac{1}{4} [f_j^*(2t - 1) - f_j^*(2t - 1)] = \frac{1}{2} f_j^e(2t - 1)$$

for $t \in [0, 1]$. Consequently, $\{(f_1 \circ f_1)^e, \ldots, (f_n \circ f_n)^e\}$ is linearly independent when $\{(f_1)^e, \ldots, (f_n)^e\}$ is.
It follows from Proposition 2.3(i) and (2.16) that

\begin{equation}
Q((f_k \odot f_k), (f_j \odot f_j)) = \frac{1}{2} Q(f_k, f_j).
\end{equation}

So \( R[f \odot f] = R[f] \) and \( f \odot f \) is strongly generic if \( f \) is.

(c) By (3.22) \( (g_j \odot f_j)^e(t) = \frac{1}{4}[f_j(2t - 1) - g_j(2t - 1)] \) for \( t \in [0, 1] \).

If \( \sum_j x_j f_j \in \mathcal{G}_{00} \) then \( \sum_j x_j f_j^e = 0 \). If \( f \) is strongly generic and \( g \in (\mathcal{G}_{00})^n \) it follows that \( \{(g_1 \odot f_1)^e, \ldots, (g_n \odot f_n)^e\} \) is linearly independent. Moreover, since \( \mathcal{G}_{00} \) is a subgroup contained in \( \mathcal{G}_0 \), it follows from Proposition 2.3(i) and (2.16) that

\begin{equation}
Q((g_k \odot f_k), (g_j \odot f_j)) = \frac{1}{4} Q(f_k, f_j).
\end{equation}

So \( R[g \odot f] = R[f] \) and \( g \odot f \) is strongly generic if \( f \) is.

(d) Apply (2.21).

(e) It is clear that \( \pi f \) is generic (or strongly generic) if \( f \) is. Observe that \( (\pi f)_i \rightarrow (\pi f)_j \) means \( f_{\pi^{-1}i} \rightarrow f_{\pi^{-1}j} \) and so \( (\pi^{-1}i, \pi^{-1}j) \in R[f] \). This is equivalent to \((i, j) \in \pi R[f] \).

**Proposition 3.17.** Let \( f = (f_1, \ldots, f_n) \) be a strongly generic \( n \)-tuple, with \( R[f] = R \) the associated tournament on \([n]\). Assume that \( g = (g_1, \ldots, g_n) \in (\mathcal{G}_{00})^n \) is such that for all \((i, j) \in R \), \( g_i \rightarrow f_j \) and \( f_i \rightarrow g_j \). If for \( (x_1, \ldots, x_n) \in (0,1)^n \), we let \( h_i = x_i f_i + (1 - x_i) g_i \), then \( h = (h_1, \ldots, h_n) \) is a strongly generic \( n \)-tuple, with \( R[h] = R \).

**Proof.** Notice that \( g_j \in \mathcal{G}_{00} \) has been chosen so that for all \( i \in [n] \) with \( i \neq j \), \( Q(f_i, g_j) > 0 \) when \( Q(f_i, f_j) > 0 \) and \( Q(f_i, g_j) < 0 \) when \( Q(f_i, f_j) < 0 \). By Theorem 3.3 such \( g_j \) always exist.

Because \( h_j^e = x_j f_j^e \) it follows that \( \{h_1^e, \ldots, h_n^e\} \) is linearly independent. Furthermore, because \( Q \) is affine in each variable \( Q(h_i, h_j) \) is equal to

\begin{equation}
x_i x_j Q(f_i, f_j) + x_i (1 - x_j) Q(f_i, g_j) + (1 - x_i) x_j Q(g_i, f_j).
\end{equation}

All of these terms have the same sign as \( Q(f_i, f_j) \). So \( h \) is strongly generic with \( R[h] = R \).

\( \square \)
In this section, we will consider infinite as well as finite tournaments.  
We will write \((S, R)\) for a tournament \(R\) on a set \(S\) or just use \(R\) when \(S\) is understood.  
For a set \(S\) we write \(|S|\) for the cardinality of \(S\).

Let \((S_1, R_1)\) and \((S_2, R_2)\) be tournaments.  A tournament morphism  
\(\phi : R_1 \to R_2\) is a mapping \(\phi : S_1 \to S_2\) such that \((\phi \times \phi)^{-1}(R_2) \subset R_1\).  
That is, \(\phi(i) \to \phi(j)\) in \(R_2\) implies \(i \to j\) in \(R_1\).  
Because \(R_1\) and \(R_2\) are tournaments, \(i \to j\) in \(R_1\) implies \(\phi(i) \to \phi(j)\) in \(R_2\) unless \(\phi(i) = \phi(j)\).  

An injective morphism is called an embedding in which case \(i \to j\) in \(R_1\) if and only if \(\phi(i) \to \phi(j)\) in \(R_2\).  
A bijective morphism is called an isomorphism, in which case, the inverse map \(\phi^{-1} : S_2 \to S_1\) defines  
the inverse isomorphism \(\phi^{-1} : R_2 \to R_1\).  
An isomorphism from \(R\) to itself is called an automorphism of \(R\).

Recall that if \((S, R)\) is a tournament and \(S_1 \subset S\), then the tournament  
on \(S_1, R|_{S_1} = R \cap (S_1 \times S_1)\) is called the restriction of \(R\) to \(S_1\).  
The inclusion map \(\text{inc} : S_1 \to S\) defines an embedding of \(R|_{S_1}\) into \(R\).  
On the other hand, if \(\phi : R_1 \to R_2\) is an embedding and  
\(S_3 = \phi(S_1) \subset S_2\), then \(\phi : S_1 \to S_3\) defines an isomorphism of \(R_1\) onto the restriction \(R_2|_{S_3}\).

For tournaments \((S, R)\) and \((T, U)\) if \(S_0 \subset S\) and \(\phi : R|_{S_0} \to U\) is an embedding, we say that \(\phi\) extends to \(R\) if there exists an embedding  
\(\psi : R \to U\) such that \(\psi = \phi\) on \(S_0\).

**Definition 4.1.** If \((T, U)\) is a tournament and \(T_0 \subset T\), then we say that \(T_0\) satisfies the simple extension property in \(U\) if for every subset  
\(J \subset T_0\) there exists \(v_J \in T\) such that in \(U\)  
\[(4.1) \quad v_J \to j \quad \text{for all } j \in J, \quad \text{and} \quad j \to v_J \quad \text{for all } j \in T_0 \setminus J.\]

We will describe this by saying \(v_J\) chooses \(J \subset T_0\) for \(U\)

Since a tournament contains no diagonal pairs, it follows that the set \(\{v_J : J \subset T_0\}\) consists of \(2^{|T_0|}\) vertices disjoint from \(T_0\).  
Hence,  
\[|T| \geq |T_0| + 2^{|T_0|}.\]

**Lemma 4.2.** (a) Assume \((S, R)\) is a tournament, and \(S_0 \subset S\) with  
\(|S \setminus S_0| = 1\), i.e. \(S\) contains a single additional vertex.  
If \(\phi : R|_{S_0} \to U\) is a tournament embedding and \(\phi(S_0)\) satisfies the simple extension property in \(U\), then \(\phi\) extends to \(R\).

(b) If \((S_0, R_0)\) is a tournament, then there exists a tournament \(R\) on a set \(S\) with \(S_0 \subset S\) and \(R_0 = R|_{S_0}\) such that \(S_0\) satisfies the simple extension property in \(R\).  
Furthermore,  
\(|S| = |S_0| + 2^{|S_0|}\).
Proof. (a) If \(S = \{v\} \cup S_0\), then we let \(J = \phi(R(v)) \subset \phi(S_0)\) and we obtain the extension by mapping \(v\) to \(v_J\).

(b) If \(*\) is a point not in \(S_0\) and \(P(S_0)\) is the power set of \(S_0\), then we let \(S = S_0 \cup (\{*\} \times P(S_0))\). Let \(R\) be a tournament on \(S\) which contains \(R_0\) and such that
\[
\{((*, J), j) : j \in J\} \cup \{(j, (*, J)) : j \in S_0 \setminus J\} \subset R.
\]

Clearly, \(v_J = (*, J)\) chooses \(J \subset S_0\) for \(R\) and so \(S_0\) satisfies the simple extension property.

□

**Definition 4.3.** A tournament \((T, U)\) is called universal when it satisfies the following.

**Extension Property** If \(R\) is a tournament on a countable set \(S\), \(S_0\) is a finite subset of \(S\) and \(\phi : R|S_0 \rightarrow U\) is an embedding, then \(\phi\) extends to \(R\).

**Proposition 4.4.** In order that a tournament \((T, U)\) be universal, it is necessary and sufficient that every finite subset \(T_0\) of \(T\) satisfies the simple extension property in \(U\).

**Proof.** If \(U\) is universal and \(T_0\) is a finite subset of \(T\), then by Lemma 4.2 (b) there exists a finite set \(T_1 \supset T_0\) and a tournament \(R_1 \supset U|T_0\) such that \(T_0\) satisfies the simple extension property in \(R_1\). Let \(\psi : R_1 \rightarrow U\) be an extension of the inclusion of \(U|S_0\) into \(U\). If \(J \subset T_0\) and \(u_J \in T_1\) chooses \(J \subset T_0\) for \(R_1\), then \(v_J = \psi(u_J)\) chooses \(J \subset T_0\) for \(U\).

Now assume that every finite subset of \(T\) satisfies the simple extension property in \(U\).

Count the finite or countably infinite set of vertices \(v_1, v_2, \ldots\) of \(S \setminus S_0\). Let \(S_k = S_0 \cup \{v_1, \ldots, v_k\}\). Inductively, with \(\psi_0 = \phi\), Lemma 4.2(a) implies that we can define an embedding \(\psi_k : R|S_k \rightarrow U\) which extends \(\psi_{k-1}\) for \(k \geq 1\). If \(S\) is finite with \(|S \setminus S_0| = N\) then \(\psi = \psi_N\) is the required extension. If \(S\) is countably infinite then \(\psi = \bigcup_k \psi_k\), with \(\psi(v_i) = \psi_k(v_i)\) for all \(k \geq i\) is the required extension.

□

**Proposition 4.5.** Assume that \(\{(T_k, U_k) : k \in \mathbb{N}\}\) is an increasing sequence of tournaments, i.e. \(T_k \subset T_{k+1}\) and \(U_k = U_{k+1}|T_k\) for and \(k \in \mathbb{N}\). If \(T_k\) satisfies the simple extension property in \(U_{k+1}\) for all \(k \in \mathbb{N}\) then \(U = \bigcup_k U_k\) is a universal tournament on \(T = \bigcup_k T_k\).
Proof. It is clear that the union $U$ is a tournament on $T$. Also, $|T_{k+1}| \geq |T_k| + 2^{|T_k|}$ and so $T$ is infinite.

If $S_0$ is a finite subset of $T$ and $J \subset S$, then there exists $k \in \mathbb{N}$ such that $S_0 \subset T_k$ and so $J \subset T_k$. If $v_J \in T_{k+1}$ chooses $J \subset T_k$ for $U_{k+1}$, then it chooses $J \subset S_0$ for $U$. Thus, $S_0$ has the simple extension property in $U$. Hence, $U$ is universal by Proposition 4.4.

\[\Box\]

**Theorem 4.6.** Assume that $(T_1,U_1)$ and $(T_2,U_2)$ are countable, universal tournaments. If $S$ is a finite subset of $T_1$ and $\phi : U_1|S \longrightarrow U_2$ is an embedding, then $\phi$ extends to an isomorphism $\psi : U_1 \longrightarrow U_2$.

Proof. This is a standard back and forth argument. First note that a universal tournament contains copies of every finite tournament and so must be infinite. Let $u_1,u_2,\ldots$ be a counting of the vertices of $T_1 \setminus S_0$ and $v_1,v_2,\ldots$ be a counting of the vertices of $T_2 \setminus S_0$ with $S_0 = S$ and $\bar{S}_0 = \phi(S)$. Let $\psi_0 : U_1|S_0 \longrightarrow U_2|\bar{S}_0$ be the isomorphism obtained by restricting $\phi$.

Inductively, we construct for $k \geq 1$

- $S_k \supseteq S_{k-1} \cup \{u_k\}$,
- $\bar{S}_k \supseteq \bar{S}_{k-1} \cup \{v_k\}$,
- $\psi_k : U_1|S_k \longrightarrow U_2|\bar{S}_k$ an isomorphism which extends $\psi_{k-1}$.

Define $S_{k+5}$ to be $S_k$ together with the first vertex of $T_1 \setminus S_0$ which is not in $S_k$ and extend $\psi_k$ to define an embedding $\psi_{k+5}$ on $S_{k+5}$. Let $\bar{S}_{k+5} = \psi_{k+5}(S_{k+5})$ so that $\psi_{k+5}^{-1} : U_2|\bar{S}_{k+5} \longrightarrow U_1$ is an embedding. Define $\bar{S}_{k+1}$ to be $\bar{S}_{k+5}$ together with the first vertex of $T_2 \setminus \bar{S}_0$ which is not in it. Extend to define the embedding $\psi_{k+1}^{-1} : U_2|\bar{S}_{k+1} \longrightarrow U_1$ and let $S_{k+1} = \psi_{k+1}^{-1}(\bar{S}_{k+1})$.

The union $\psi = \bigcup_k \psi_k$ is the required isomorphism.

\[\Box\]

**Corollary 4.7.** There exist countable universal tournaments, unique up to isomorphism. In fact, if $(T_1,U_1)$ and $(T_2,U_2)$ are countable, universal tournaments with $i_1 \in T_1, i_2 \in T_2$ then there exists an isomorphism $\psi : U_1 \longrightarrow U_2$ with $\psi(i_1) = i_2$.

Any countable tournament can be embedded in any universal tournament.

Proof. Beginning with an arbitrary finite tournament we can use Lemma 4.2 (b) to construct inductively a sequence of finite tournaments to
which Proposition 4.3 applies, thus obtaining a countable universal tournament.

Since the restriction \( U_1|\{i_1\} \) is empty, the map \( i_1 \mapsto i_2 \) gives an embedding of \( U_1|\{i_1\} \) into \( U_2 \). It extends to an isomorphism by Theorem 4.6.

If \((S, R)\) is a countable tournament with \( i_1 \in S \), then, as above, the map taking \( i_1 \) to any point of \( i_2 \in T_2 \) is an embedding of \( R|\{i_1\} \) which extends to an embedding of \( R \) into \( U_2 \).

\[ \square \]

**Corollary 4.8.** Let \((T, U)\) be a universal tournament and \( S \) be a finite subset of \( T \).

(a) If \( i_1, i_2 \in T \), then there exists an automorphism \( \psi \) of \( U \) with \( \psi(x_1) = x_2 \).

(b) If \( \phi \) is an automorphism of \( U|S \), then there exists an automorphism \( \psi \) of \( U \) which restricts to \( \phi \) on \( S \).

**Proof.** (a) This follows from Corollary 4.7.

(b) Since the composition of \( \phi \) with the inclusion of \( S \) is an embedding, (b) follows from Theorem 4.6.

\[ \square \]

Let \((T, U)\) be a tournament and \( S \) be a nonempty, finite subset of \( T \). For \( J \subset S \), let

\[ T_J = \{ i \in T : (i, j) \in U \text{ for all } j \in J \text{ and } (j, i) \in U \text{ for all } j \in S \setminus J \}. \]

That is, \( T_J \) is the set of \( i \in T \) which choose \( J \subset S \) for \( U \).

Clearly, \( \{S\} \cup \{T_J : J \subset S\} \) is a partition of \( T \) into \( 1 + 2^{|S|} \) subsets.

**Proposition 4.9.** If \((T, U)\) is a universal tournament, \( S \) is a finite subset of \( T \) and \( J \) is a subset of \( S \), then the restriction \((T_J, U|T_J)\) is a universal tournament.

**Proof.** Assume that \( S_1 \) is a finite subset of \( T_J \) and \( K \subset S_1 \). Let \( S_1^+ = S_1 \cup S \) and \( K^+ = K \cup J \). Because \( K^+ \) satisfies the simple extension property in \( U \), there exists \( v_{K^+} \in T \) such that \( v_{K^+} \) chooses \( K^+ \subset S^+ \) for \( U \). It follows first that \( v_{K^+} \) chooses \( J \subset S \) for \( U \) and so \( v_{K^+} \in T_J \).

It then follows that \( v_{K^+} \) chooses \( K \subset S_1 \) for \( U|T_J \). Thus, \( S_1 \) satisfies the simple extension property in \( U|T_J \). As \( S_1 \) was arbitrary, it follows from Proposition 4.3 that \( U|T_J \) is universal.

\[ \square \]
Example 4.10. (a) For \((T, U)\) a countable, universal tournament, there exists \(T_0\) a proper infinite subset of \(T\) and an embedding of \(U|T_0\) into \(U\) which cannot be extended to an embedding of \(U\) into itself.

(b) There exists a tournament which is not universal but into which every countable tournament can be embedded.

Proof. Let \(i \in T, J = S = \{i\}, T_0 = T_J\) and \(T_1 = S \cup T_J\). Since \(U|T_0\) is universal by Proposition 4.9 Corollary 4.7 implies that there exists an isomorphism \(\phi : U|T_0 \rightarrow U\). Since \(\phi\) is surjective, it cannot be extended to an embedding even of \(U|T_1\) into \(U\).

Since \((T_0, U|T_0)\) is universal and \(T_0 \subset T_1\), it follows that every countable tournament can be embedded into \(U|T_1\). Let \(i_1 \in T_0\) so that \(i \rightarrow i_1\) in \(U\). The inclusion of \(i\) into \(T_1\) cannot be extended to an embedding of \(U|\{i, i_1\}\) into \(U_1\) since \(j \rightarrow i\) for every \(j \neq i\) in \(T_1\). So \(U|T_1\) is not universal.

\(\Box\)

We apply all this to the digraph \(\Gamma_3\) from the previous sections.

Theorem 4.11. Assume that \(f = (f_1, \ldots, f_n)\) is a strongly generic \(n\)-tuple in \((\mathcal{G}_0)^n\). The finite sequence \(\{f_1, \ldots, f_n\}\) can be extended to an infinite sequence \(\{f_1, f_2, \ldots\}\) in \(\mathcal{G}_0\) with \(f_i(0) = 0\) for all \(i > n\), such that the sequence of even functions \(\{f_1^e, f_2^e, \ldots\}\) is linearly independent and the restriction of \(\Gamma_3\) to the set \(\{f_1, f_2, \ldots\}\) is a universal tournament.

Proof. Let \(U\) be a universal tournament on \(\mathbb{N}\). The tournament \(R[f]\) on \([n]\) can be embedded in \(U\). By permuting \(\mathbb{N}\), we may assume that the embedding is given by the inclusion of \([n]\). That is, so that \(R[f] = U|[n]\).

Inductively apply Theorem 3.14(c) to construct for \(k > n\) the sequence \(\{f_1, \ldots, f_k\}\) with \(f_i(0) = 0\) for \(n < i \leq k\) such that \(f_k = (f_1, \ldots, f_k) \in (\mathcal{G}_0)^k\) is strongly generic and such that \(R[f_k] = U|[k]\).

Since every finite subset is linear independent, \(\{f_1^e, f_2^e, \ldots\}\) is linearly independent. Finally, \(k \mapsto f_k\) induces an isomorphism from \(U\) to the restriction \(\Gamma_3|\{f_1, f_2, \ldots\}\).

\(\Box\)

For a function \(f : \mathbb{N} \rightarrow \mathcal{G}_0\), i.e. an element of \((\mathcal{G}_0)^\mathbb{N}\), we define the associated digraph \(U[f]\) on \(\mathbb{N}\) to be the pullback of \(\Gamma_3\). That is,

\[(4.4) \quad i \rightarrow j\text{ in } U[f] \iff f_i \rightarrow f_j \text{ in } \Gamma_3.\]
Theorem 4.12. The sets
\[(4.5)\]
\[
\mathcal{UTOUR} = \{ f \in (\mathcal{G}_0)^N : U[f] \text{ is a universal tournament} \} \\
\mathcal{USEN} = \{ f \in \mathcal{UTOUR} : \{i, f_1, f_2, \ldots\} \text{ is linearly independent} \} \\
\mathcal{USEN}^+ = \{ f \in \mathcal{UTOUR} : \{f^e_1, f^e_2, \ldots\} \text{ is linearly independent} \}
\]
are dense $G_\delta$ subsets of $(\mathcal{G}_0)^N$.

Proof. The projection map from $(\mathcal{G}_0)^N$ to $(\mathcal{G}_0)^n$ is open map for every $n \in \mathbb{N}$. It follows that the preimage of a dense $G_\delta$ set is a dense $G_\delta$ set. Thus, the condition on $f \in (\mathcal{G}_0)^n$ that $(f_1, \ldots, f_n) \in \mathcal{TOUR}_n$ for every $n$, is a dense $G_\delta$ condition by Corollary 3.7 and the Baire Category Theorem. Similarly, by Proposition 3.13 the conditions $(f_1, \ldots, f_n) \in \mathcal{SEN}_n$ for every $n$ and $(f_1, \ldots, f_n) \in \mathcal{SEN}^+_n$ for every $n$, are dense $G_\delta$ conditions. The first condition says that the digraph $U[f]$ is a tournament. The latter conditions say, in addition, that $\{i, f_1, f_2, \ldots\}$ or $\{f^e_1, f^e_2, \ldots\}$ is linearly independent.

Given two disjoint finite subsets $J_1, J_2$ of $\mathbb{N}$, let
\[(4.6) \]
\[
W(J_1, J_2) = \{ f \in (\mathcal{G}_0)^N : \text{there exists } k \in \mathbb{N} \text{ such that } (k, j_1), (j_2, k) \in U[f] \text{ for all } j_1 \in J_1, j_2 \in J_2 \}.
\]
Because $f_k \to f_j$ is an open condition, it follows that $W(J_1, J_2)$ is an open subset. Intersecting over all disjoint pairs $J_1, J_2$ we obtain the $G_\delta$ set $W$. From Lemma 4.4 it follows that $f \in \mathcal{UTOUR}$ if and only if $f \in W$ and $(f_1, \ldots, f_n) \in \mathcal{TOUR}_n$ for all $n$. Furthermore, $f \in \mathcal{USEN}^+$ if and only if $f \in W$ and $(f_1, \ldots, f_n) \in \mathcal{SEN}^+_n$ for all $n$. Similarly, for $\mathcal{USEN}$. So these are all $G_\delta$ conditions.

For density, it suffices to show that $\mathcal{USEN}^+$ is dense. To begin with we may perturb $f$ to get the dense condition that $(f_1, \ldots, f_n) \in \mathcal{SEN}^+_n$ for all $n$. Let $N$ be arbitrarily large. We can apply Theorem 4.11 to obtain $f' \in \mathcal{USEN}^+$ with $f'_k = f_k$ for $k = 1, \ldots, N$. By definition of the product topology, choosing $N$ large enough we obtain $f'$ arbitrarily close to $f$.

\[
\square
\]

Theorem 4.13. If $(\mathbb{N}, U)$ is a universal tournament, then there exists a sequence $\{X_1, X_2, \ldots\}$ of independent, continuous random variables on $[0, 1]$ with $E(X_i) = \frac{1}{2}$ for all $i$, such that $P(X_i > X_j) > \frac{1}{2}$ if and only if $i \to j$ in $U$.

Proof. By Theorem 4.11 and uniqueness of the universal tournament, we may choose $f \in \mathcal{UTOUR}$ such that $U[f] = U$. For each $i \in \mathbb{N}$, let
\[ F_i \in \mathcal{H}_0 \text{ equal } A_t^{-1}(f_i) = q \circ f_i \circ q^{-1} \text{ with } q(t) = \frac{t+1}{2}. \] Let \( \{Z_1, Z_2, \ldots\} \) be a sequence of independent \( \text{Unif}(0,1) \) random variables and let \( X_i = F_i^{-1}(Z_i) \). From (2.3) and (2.9) it follows that \( P(X_i > X_j) > \frac{1}{2} \) if and only if \( \int_0^1 f_j(f_i^{-1}(t)) \, dt > 0 \) and so if and only if \( i \to j \) in \( U \). 

We conclude this section with a sketch of the measure version of the density result Theorem 4.12.

Let \( \mathcal{M} \) denote the space of Borel measures on \([0,1]\). With the weak* topology induced from the dual space of \( \mathcal{C}([0,1]) \), \( \mathcal{M} \) becomes a compact metrizable space. A continuous map \( G : [0,1] \to [0,1] \) induces the continuous map \( G^* : \mathcal{M} \to \mathcal{M} \) by \( G^* \mu(A) = \mu(G^{-1}(A)) \) for \( A \) a measurable subset of \([0,1]\). We call \( \mu \in \mathcal{M} \) a proper measure when the mean value \( \int_0^1 x \mu(dx) = \frac{1}{2} \). We let \( \mathcal{M}_0 \) denote the subset of proper measures. For example, the Lebesgue measure \( \lambda \) on \([0,1]\) is proper.

We call \( \mu \) a continuous measure when it is full and nonatomic, i.e. every point has measure zero and every nonempty open subset of \([0,1]\) has positive measure. We let \( \mathcal{M}^c \) denote the set of continuous measures with \( \mathcal{M}_0^c = \mathcal{M}^c \cap \mathcal{M}_0 \).

**Proposition 4.14.** The set \( \mathcal{M}^c \) is a dense \( G_\delta \) subset of \( \mathcal{M} \) and \( \mathcal{M}_0^c \) is a dense \( G_\delta \) subset of \( \mathcal{M}_0 \).

**Proof.** By Fubini’s Theorem, the measure \( \mu \) is nonatomic if and only if the diagonal \( \Delta \subset [0,1] \times [0,1] \) has \( \mu \times \mu \) measure zero. This holds if and only if for every \( \epsilon > 0 \) there exists a non-negative function \( h \) on \([0,1] \times [0,1] \) which is 1 on \( \Delta \) but whose \( \mu \times \mu \) integral is less than \( \epsilon \). Thus, \( (\mu \times \mu)(\Delta) = 0 \) is a \( G_\delta \) condition.

For a continuous non-negative function \( h \in \mathcal{C}([0,1]) \), the condition \( \int_0^1 h(x)\mu(dx) > 0 \) is an open condition. Intersecting with a suitable countable collection of functions \( h \) we see that having full support is a \( G_\delta \) condition as well.

If \( \mu \) has an atom at 0 or 1, we replace \( \mu \) by \( G^* \mu \) with \( G(x) = \epsilon + (1 - 2\epsilon)x \). If \( \mu \) has mean \( \frac{1}{2} \), then

\[ \int_0^1 xG^*\mu(dx) = \int_0^1 G(x)\mu(dx) = \epsilon + (1 - 2\epsilon) \cdot \frac{1}{2} = \frac{1}{2}. \]

With small \( \epsilon > 0 \) the new measure is close to \( \mu \) and has no atom at 0 or 1.

We may replace an atom at \( a \in (0,1) \) by a distribution with the same weight, uniform on \((a-\epsilon, a+\epsilon)\). With small \( \epsilon > 0 \) the new measure is arbitrarily close to \( \mu \) and has the same mean.
Finally, with small \( \epsilon > 0 \) the measure \( \epsilon \lambda + (1-\epsilon)\mu \) is full and has mean \( \frac{1}{2} \) if \( \mu \) does.

For \( \mu \in \mathcal{M} \) the distribution function \( F_\mu \) on \([0,1]\) is defined by \( F_\mu(x) = \mu([0,x)) \) for \( x \in [0,1] \). It is clear that \( \mu \) is a continuous measure if and only if \( F_\mu \in \mathcal{H} \). If \( F \in \mathcal{H} \), then \( \mu = (F^{-1})_\mu \lambda \) is the continuous measure with \( F_\mu = F \) because \( \mu([0,x)) = \lambda((F^{-1})^{-1}[0,x)) = \lambda([0,F(x))) = F(x) \). The map \( L : \mathcal{H} \to \mathcal{M}^c \) given by \( L(F) = (F^{-1})_\mu \lambda \) is a continuous bijection which maps \( \mathcal{H}_0 \) onto \( \mathcal{M}_0^c \).

We can identify \( \mu \in (\mathcal{M})^N \) with the product measure \( \mu_1 \times \mu_2 \times \ldots \) on the product space \([0,1]^N\). Given \( \mu \in (\mathcal{M})^N \) we define the associated digraph \( U[\mu] \) on \( \mathbb{N} \) by

\[
(i,j) \in U[\mu] \iff \mu(\{x \in [0,1]^N : x_i > x_j\}) > \frac{1}{2}.
\]

Define \( \mathcal{U} \mathcal{T} \mathcal{O} \mathcal{U} \mathcal{R} \mathcal{M} \) to be the set of \( \mu \in (\mathcal{M}_0^c)^N \) such that \( U[\mu] \) is a universal tournament.

**Theorem 4.15.** The set \( \mathcal{U} \mathcal{T} \mathcal{O} \mathcal{U} \mathcal{R} \mathcal{M} \) is a dense \( G_\delta \) subset of \( \mathcal{M}_0^N \).

**Proof.** The condition \( (i,j) \) or \( (j,i) \) \( \in U[\mu] \) is an open condition on \( \mu \) and so the condition that \( U[\mu] \) be a tournament is a \( G_\delta \) condition. Defining \( W(J_1,J_2) \) as in (4.6) and proceeding as in the proof of Theorem 4.12 we see that \( \mathcal{U} \mathcal{T} \mathcal{O} \mathcal{U} \mathcal{R} \mathcal{M} \) is a \( G_\delta \) set.

For \( f \in (\mathcal{G})^N \) we define \( F = A_q^{-1}(f) \in (\mathcal{H})^N \) letting \( F_i = A_q^{-1}(f_i) \) as in the proof of Theorem 4.13. It follows that \( \tilde{L} = L \circ A_q^{-1} \) is a continuous bijection from \( (\mathcal{G})^N \) to \( (\mathcal{M}^c)^N \) which maps \( \mathcal{G}_0^N \) onto \( (\mathcal{M}_0^c)^N \), dense in \( \mathcal{M}_0^N \).

The map \( \tilde{L} \) maps \( \mathcal{U} \mathcal{T} \mathcal{O} \mathcal{U} \mathcal{R} \) onto \( \mathcal{U} \mathcal{T} \mathcal{O} \mathcal{U} \mathcal{R} \mathcal{M} \). By Theorem 4.12 \( \mathcal{U} \mathcal{T} \mathcal{O} \mathcal{U} \mathcal{R} \mathcal{M} \) is dense in \( (\mathcal{G}_0)^N \). It follows that \( \mathcal{U} \mathcal{T} \mathcal{O} \mathcal{U} \mathcal{R} \mathcal{M} \) is dense in \( (\mathcal{M}_0^c)^N \) and so in \( \mathcal{M}_0^N \).

\[\square\]

5. Partition Tournaments

An \( n \)-partition \( \mathcal{A} = \{A_1,\ldots,A_n\} \) of \([Nn] = 1,\ldots,Nn\) consists of \( n \) disjoint subsets with union \([Nn]\). We call it a \( regular \ n \)-partition when the cardinality \( |A_i| = N \) for \( i = 1,\ldots,n \). There are \((nN)!/(N!)^n\) regular \( n \)-partitions of \([Nn]\).

We define for a regular \( n \)-partition on \([Nn]\) the digraph

\[
(5.1) \quad R[\mathcal{A}] = \{(i,j) \in [n] \times [n] : |\{(a,b) \in A_i \times A_j : a > b\}| > N^2/2\}.
\]
That is, \((i, j) \in R[\mathcal{A}]\) or \(A_i \rightarrow A_j\) if it is more likely that a randomly chosen element of \(A_i\) is greater than a randomly chosen element of \(A_j\) than the reverse.

If \(N\) is odd, then \(R[\mathcal{A}]\) is a tournament on \([n]\). That is, for every pair \(i, j \in [n]\) with \(i \neq j\) either \(A_i \rightarrow A_j\) or \(A_j \rightarrow A_i\) and not both. Note that for \(i = j\), \(|\{(a, b) \in A_i \times A_i : a > b\}| = N(N - 1)/2\).

We can think of the partition as the values on the faces on \(n\) different \(N\)-sided dice, but now with values selected from \([Nn]\), and with the \(Nn\) different faces all having different values. If \(D_i\) is the random variable associated with the die having faces with values from \(A_i\), then \(A_i \rightarrow A_j\) exactly when \(D_i \rightarrow D_j\) in the previous sense.

For \(N\) large enough we can obtain any tournament on \([n]\) by using a regular \(n\)-partition on \([Nn]\).

**Theorem 5.1.** If \(R\) is a tournament on \([n]\), then there is a positive integer \(M\) such that for every integer \(N \geq M\), there exists a regular \(n\)-partition of \([Nn]\) \(\mathcal{A} = \{A_1, \ldots, A_n\}\) such that for \(i, j \in [n]\), \(A_i \rightarrow A_j\) if and only if \(i \rightarrow j\) in \(R\). That is, \(R = R[\mathcal{A}]\).

*Proof.* From Theorem [12] we can choose \(X_1, \ldots, X_n\) independent, continuous random variables on \([0, 1]\) so that for some \(\epsilon > 0\), and all \((i, j) \in R\), \(P(X_i > X_j) > \frac{1}{2} + \epsilon\). Let \(N\) be an integer greater than 1.

Now for \(i \in [n]\) and \(\alpha \in [N]\) let \(\{X_i^\alpha\}\) be independent random variables with each \(X_i^\alpha\) distributed like \(X_i\). We think of \(\{X_i^\alpha : \alpha \in [N]\}\) as \(N\) points, independently chosen in \([0, 1]\) according to the distribution of \(X_i\). Since the random variables are continuous, the probability that \(X_i^\alpha = X_j^\beta\) equals zero unless \(i = j\) and \(\alpha = \beta\). Thus, with probability one \(\{X_i^\alpha\}\) consists of \(nN\) distinct points in \([0, 1]\).

Define the indicator function \(k : [0, 1] \times [0, 1] \rightarrow \{0, 1\}\) with \(k(x, y) = 1\) if \(x > y\) and \(= 0\) otherwise. Hence, \(k(X_i, X_j)\) is a Bernoulli random variable with expectation \(P(X_i > X_j)\) which is greater than \(\frac{1}{2} + \epsilon\) if \((i, j) \in R\).

For each \(i \neq j\)

\[
\sum_{\alpha, \beta} k(X_i^\alpha, X_j^\beta) = |\{(\alpha, \beta) \in [N] \times [N] : X_i^\alpha > X_j^\beta\}|.
\]

Consider the random variable

\[
Z_{i,j} = \frac{1}{N^2} \sum_{\alpha, \beta} k(X_i^\alpha, X_j^\beta) - P(X_i > X_j) = \frac{1}{N^2} \sum_{\alpha, \beta} Z_{i,j}^{\alpha,\beta},
\]

(5.3)

with \(Z_{i,j}^{\alpha,\beta} = k(X_i^\alpha, X_j^\beta) - E(k(X_i^\alpha, X_j^\beta))\).
Thus, the expectation \( E(Z_{i,j}) = 0 \). To compute the variance = \( E(Z_{i,j}^2) \), we recall that the variance of a Bernoulli random variable and the covariance of two Bernoulli random variables are each bounded by \( \frac{1}{4} \), since \( p(1-p) \) has its maximum at \( p = \frac{1}{2} \).

\[
E(Z_{i,j}^2) = \frac{1}{N^4} \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2} E(Z_{i,j}^{\alpha_1,\beta_1} \cdot Z_{i,j}^{\alpha_2,\beta_2})
\]

There are \( N^2 \) terms with \( \alpha_1 = \alpha_2 \) and \( \beta_1 = \beta_2 \), \( N^2(N-1) \) terms with \( \alpha_1 = \alpha_2 \) and \( \beta_1 \neq \beta_2 \) and \( N^2(N-1) \) terms with \( \alpha_1 \neq \alpha_2 \) and \( \beta_1 = \beta_2 \). Each of the terms is bounded by \( \frac{1}{4} \). The remaining terms are all zero by independence. It follows that the variance of \( Z_{i,j} \) is bounded by \( \frac{1}{2N} \).

By Chebyshev’s Inequality (see, e.g. [1] Theorem 10.1.11)

\[
P(|Z_{i,j}| > \epsilon) \leq \frac{1}{2N\epsilon^2}.
\]

In \( R \) there are \( n(n-1)/2 \) pairs \((i,j)\). Hence, the probability that \( P(|Z_{i,j}| > \epsilon) \) for some pair \((i,j)\) \( \in R \) is bounded by \( \frac{n^2}{4N\epsilon^2} \).

If \( N > \frac{n^2}{4\epsilon^2} \), then there is positive probability such that, for all \((i,j)\) \( \in R \), \( |Z_{i,j}| \leq \epsilon \). In that case for every \((i,j)\) \( \in R \)

\[
\frac{1}{N^2} \sum_{\alpha,\beta} k(X_i^\alpha, X_j^\beta) > \frac{1}{2}.
\]

Thus, when \( N > \frac{n^2}{4\epsilon^2} \), there exist \( X_i^\alpha \in [0,1] \) distinct and so that \( |Z_{ij}| \leq \epsilon \) for every pair \((i,j)\) \( \in R \). Let \( a_1 < \cdots < a_{nN} \) list the values of the \( X_i^\alpha \)'s in order. Thus, \( a_p < a_q \) if and only if \( p < q \). Let

\[
A_i = \{ p : a_p = X_i^\alpha \text{ for some } \alpha \in [N] \}.
\]

From (5.2) and (5.6) it follows that, for all \((i,j)\) \( \in R \), \( A_i \rightarrow A_j \).

Because there are only finitely many tournaments on \([n]\), we can, as before, choose \( M \) so that if \( N \) is greater than \( M \), then every tournament on \([n]\) occurs as the tournament of a regular \( n \)-partition of \([nN]\).

There are partition constructions analogous to the \( n \)-tuple constructions of (3.23).

Let \( \mathcal{A} = \{ A_1, \ldots, A_n \} \) be a regular \( n \)-partition of \([Nn]\), let \( \pi \) be a permutation of \([n]\) and \( M \) be a positive integer. Define \( \mathcal{A}^\pi = \{ A_1^\pi, \ldots, A_n^\pi \} \), \( \mathcal{A} = \{ A_1, \ldots, A_n \} \), \( \mathcal{A}^{(M)} = \{ A_1^{(M)}, \ldots, A_n^{(M)} \} \), regular \( n \)-partitions on
[Nn], on [Nn] and on [MNNn], respectively, by
\[
A_i^* = \{Nn - a + 1 : a \in A_i\},
\]
\[
A_i^\pi = A_{i-1},
\]
\[
A_i^{(M)} = \{(Q - 1)Nn + a : a \in A_i, Q \in [M]\},
\]
for \(i \in [n]\).

It is easy to check that
\[
R[A^*] = R[A]^{-1}, \quad R[A^\pi] = \pi R[A], \quad R[A^{(M)}] = R[A].
\]

6. Appendix: An Alternative Proof

In this section we present an alternative proof of Theorem 2.1.

**Definition 6.1.** We call a sequence \(\{p_0, p_1, \ldots\}\) of nonzero, continuously differentiable elements of \(C([-1, 1])\) a special sequence when it satisfies the following properties:

- (Even) All the \(p_i\)'s are even functions \((p_i(-t) = p_i(t)\) for all \(t \in [-1, +1]\)) and \(p_0 = 1\).
- (Orthogonal) For all \(i \neq j,\)
  \[
  \int_{-1}^{1} p_i(t) \cdot p_j(t) dt = 2 \int_{0}^{1} p_i(t) \cdot p_j(t) dt = 0.
  \]
- (Bounded) \(\|p_i\| \leq \frac{1}{2}\) for all \(i > 0\).
- (Boundary Values) \(p_i(\pm 1) = 0\) for all \(i > 0\).

To construct an example, recall that the Legendre polynomials \(\ell_n : n = 0, 1, \ldots\) define an orthogonal sequence on \([-1, +1]\) with \(\ell_0 = 1\) and \(\ell_n(1) = 1\) for all \(n\). In addition, they consist of only even power nonzero terms when \(n\) is even and so define even functions for even \(n\). Thus, we obtain a special sequence by choosing \(p_0 = 1\) and for \(i > 0, \ p_i = C_i[\ell_{4i} - \ell_{4i-2}]\) with the positive constant \(C_i\) chosen small enough to obtain the boundedness condition.

Given a special sequence \(\{p_0, p_1, \ldots\}\), we define the associated sequence \(\{g_1, g_2, \ldots\}\) by
\[
g_i(t) = \int_{0}^{t} [1 + p_i(s)] ds \quad \text{for } t \in [-1, 1].
\]
Observe that \( g_i \) is odd because \( 1 + p_i \) is even. Since \( g'_i = 1 + p_i \geq \frac{1}{2} \), the function \( g_i \) is increasing. Since \( p_i \) is orthogonal to 1, \( g_i(1) = 1 \). Thus, each \( g_i \in S_{00} \). Since \( \frac{3}{2} \geq g_i' \geq \frac{1}{2} \) it follows that \( 2 \geq (g_i^{-1})' \geq \frac{2}{3} \).

Define \( C_0 \) to be the set of \( \xi \in C \) such that

- \( \xi(\pm 1) = 0 \).
- \( \int_{-1}^{1} \xi(t) \, dt = 0 \).
- \( \xi \) is continuously differentiable.

For \( \xi \in C_0 \) and \( |z| \) sufficiently small in \( \mathbb{R} \), \( H_i(\xi, z) = g_i^{-1} + z\xi \) has a derivative with \( 1/3 < H_i(\xi, z)'(t) < 3 \) for \( t \in [-1, 1] \). In addition, \( H_i(\xi, z)(\pm 1) = \pm 1 \). Thus each such \( H_i(\xi, z) \in S_0 \).

Observe first that

\[
(6.2) \quad \int_{-1}^{1} g_j(H_i(\xi, 0)(t)) \, dt = \int_{-1}^{1} g_j(g_i^{-1}(t)) \, dt = 0,
\]

because \( g_j \circ g_i^{-1} \in S_{00} \).

Furthermore,

\[
\frac{d}{dz}|_{z=0} \int_{-1}^{1} g_j(H_i(\xi, z))(t) \, dt = \int_{-1}^{1} g'_j(g_i^{-1}(t)) \cdot \xi(t) \, dt
\]

\[
= \int_{-1}^{1} g'_j(s)g_i'(s)\xi(g_i(s)) \, ds = \int_{-1}^{1} [1 + p_j(s)]\eta(s) \, ds,
\]

with

\[
(6.4) \quad \eta(s) = g_i'(s)\xi(g_i(s)) \quad \text{and so} \quad \\
\xi(t) = \eta(g_i^{-1}(t)) \div g_i'(g_i^{-1}(t)) = \eta(g_i^{-1}(t)) \cdot (g_i^{-1})'(t).
\]

Notice that

\[
(6.5) \quad \int_{-1}^{1} \eta(s) \, ds = \int_{-1}^{1} g_i'(s)\xi(g_i(s)) \, ds = \int_{-1}^{1} \xi(t) \, dt.
\]

Hence,

\[
(6.6) \quad \int_{-1}^{1} \eta(s) \, ds = 0 \iff \int_{-1}^{1} \xi(t) \, dt = 0.
\]

We now prove the following which implies Theorem 2.1.

**Theorem 6.2.** Let \( \{p_0, p_1, \ldots \} \) be a special sequence in \( C([-1, 1]) \) with associated sequence \( \{g_1, g_2, \ldots \} \) in \( S_{00} \). If \( R \) is a tournament on \( [n] \) and \( \epsilon > 0 \), then there exists an \( n \)-tuple \( f = (f_1, \ldots, f_n) \in (S_0)^n \) such that

- For \( i \in [n] \), \( ||f_i - g_i|| < \epsilon \).
- For \( i \in [n], t \in [-1, 1], 1/3 < f_i'(t) < 3 \).
• For \( i, j \in [n], \) with \( i \neq j, \)
\[
f_j \circ f_i^{-1} \in \mathcal{G}_+ \iff (i, j) \in R.
\]

**Proof.** By induction on \( n. \) With \( n = 1 \) the only tournament is empty and we can let \( f_1 = g_1. \)

For the inductive step, assume that \( n > 1 \) and define

\[
\eta(s) = \sum_{j \in R(n)} p_j(s) - \sum_{j \in R^{-1}(n)} p_j(s)
\]

(6.7)

\[
\xi(t) = \eta(g_n^{-1}(t)) \div g_n'(g_n^{-1}(t)) = \eta(g_n^{-1}(t)) \cdot (g_n^{-1})'(t).
\]

From (6.6) it follows that that \( \xi \in \mathcal{C}_{1,0}. \)

So for \( j = 1, \ldots, n − 1 \) the orthogonality assumptions imply

\[
\int_{-1}^{1} [1 + p_j(s)]\eta(s) \, ds = \begin{cases} 
\int_{-1}^{1} p_j(s)^2 \, ds > 0 & \text{for } j \in R(n), \\
-\int_{-1}^{1} p_j(s)^2 \, ds < 0 & \text{for } j \in R^{-1}(n).
\end{cases}
\]

(6.8)

There exists \( \epsilon_1 \) with \( \epsilon > \epsilon_1 > 0 \) so that with \( |z| < \epsilon_1, H_n(\xi, z) \in \mathcal{G}_0 \)
with derivative between 1/3 and 3 on \([-1, 1]\). From (6.2) and (6.3) it follows that we can choose \( z \) with \( 0 < z < \epsilon_1 \) so that with \( f_n^{-1} = H_n(\xi, z) \) we have

\[
\int_{-1}^{1} g_j(f_n^{-1}(t)) \, dt \begin{cases} 
> 0 & \text{for } j \in R(n), \\
< 0 & \text{for } j \in R^{-1}(n).
\end{cases}
\]

(6.9)

That is,

\[
g_j(f_n^{-1}) \in \mathcal{G}_+ \quad \text{for } j \in R(n), \quad \text{and} \quad g_j(f_n^{-1}) \in \mathcal{G}_- \quad \text{for } j \in R^{-1}(n).
\]

(6.10)

Because \( \mathcal{G}_\pm \) are open sets, there exists \( \delta \) with \( 0 < \delta < \epsilon \) such that \( ||f_j - g_j|| < \delta \) for \( j \in [n − 1] \) implies

\[
f_j(f_n^{-1}) \in \mathcal{G}_+ \quad \text{for } j \in R(n), \quad \text{and} \quad f_j(f_n^{-1}) \in \mathcal{G}_- \quad \text{for } j \in R^{-1}(n).
\]

(6.11)

We apply the inductive hypothesis to the restricted tournament \( \mathcal{R} = R|[n − 1] \) and we obtain \( f_j \) with \( ||f_j - g_j|| < \delta \) for \( j \in [n − 1] \) such that for \( j, k \in [n − 1], f_k \circ f_j^{-1} \in \mathcal{G}_+ \) if and only if \( (j, k) \in \mathcal{R}. \)

From (6.11) it follows that \( \{f_1, \ldots, f_{n−1}, f_n\} \) is the required list. 

\( \square \)
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