An efficient single-step scheme for manipulating quantum information of two trapped ions beyond the Lamb-Dicke limit

L.F. Wei\textsuperscript{1,2} and Franco Nori\textsuperscript{1,3}\textsuperscript{*}

\textsuperscript{1}Frontier Research System, The Institute of Physical and Chemical Research (RIKEN), Wako-shi, Saitama, 351-0198, Japan
\textsuperscript{2}Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, P.R. China
\textsuperscript{3}Center of Theoretical Physics, Physics Department, Center for the Study of Complex Systems, University of Michigan, Ann Arbor, Michigan 48109-1120, USA

Based on the exact conditional quantum dynamics for a two-ion system, we propose an efficient single-step scheme for coherently manipulating quantum information of two trapped cold ions by using a pair of synchronous laser pulses. Neither the auxiliary atomic level nor the Lamb-Dicke approximation are needed.

Keywords: Quantum information; trapped cold ions; Lamb-Dicke limit.

PACS numbers: 03.65.Bz, 32.80.Pj, 89.70.+c

I. INTRODUCTION

The entanglement between different particles has recently become a focus of activity in quantum physics (see, e.g., \cite{1}), because of experiments on non-local features of quantum mechanics and the development of quantum information physics. Einstein-Podolsky-Rosen (EPR) entangled states with two particles have been employed not only to test Bell’s inequality \cite{2}, but also to realize quantum cryptography and quantum teleportation \cite{3}. Also, entanglement plays a central quantitative role in quantum parallelism \cite{4}. The demonstrations of quantum entanglement to date are usually based on various probabilistic processes, e.g., the generation of photon pairs in parametric down conversion \cite{5}. However, it is very difficult to generate the entanglement of larger numbers of particles, as the probability of randomly generating the appropriate conditions decreases exponentially with the number of particles. Intense activities are now focused on generating an entanglement of particles in a deterministic way, i.e., to produce a desired entangled state. For example, the entanglements of two and four trapped ions have been produced experimentally \cite{6}.

As first suggested by Cirac and Zoller \cite{7}, a very promising scenario for implementing a practical quantum information processor is the system of laser-cooled trapped ions, due to its long coherence time \cite{8}. Information in this system is stored in the spin states of an array of trapped cold ions and manipulated by using laser pulses. The ions are held apart from one another by their mutual Coulomb repulsion. Each ion can be individually addressed by focusing laser beams on the selected ion. The collective normal modes of oscillation shared by all of the ions form the information bus, through which all gate operations can be performed. In the past few years, several key features of the proposal in \cite{7}, including the production of entangled states and the implementation of quantum controlled operations between a pair of trapped ions, have already been experimentally demonstrated \cite{9,10,11,12,13,14,15,16,17}.

Also, several alternative theoretical schemes (see, e.g., \cite{11,12,13,14,15,16,17}) have been developed for overcoming various difficulties in realizing a practical ion-trap quantum information processor.

The Lamb-Dicke (LD) approximation is made in almost all of these schemes (see, e.g., \cite{11,12,13,14,15,16,17}), in order to simplify the treatment of the laser-ion interaction. This approximation requires that the coupling between the external and internal degrees of freedom of the ion is very weak, i.e., the spatial dimension of the motion of the ground state of the trapped ion should be much smaller than the effective wavelength of the applied laser field (see, e.g., \cite{18}). Thus, in the LD limit the interaction between the internal states $|s\rangle = \{|g\rangle, |e\rangle\}$ and the external motional harmonic oscillator states $\{|n\rangle; n = 0, 1, 2, \ldots\}$ of the ion can be expanded to the lowest order of the LD parameter $\eta_L$, where the usual Jaynes-Cummings (JC) or anti-JC-type model can be derived. However, the quantum motion of the trapped ions is not limited to the LD regime \cite{19,20,21}. Inversely, utilizing the laser-ion interaction beyond this limit could be helpful for reducing the noise in the trap and improving the cooling rate (see, e.g., \cite{22}). Therefore, it would be useful to implement the trapped-ion quantum information processing outside the LD regime. In fact, several schemes \cite{13,14} have been proposed to implement the quantum computation with trapped ions beyond the LD limit by sequentially applying a series of pulses.

In this work, we propose an alternative scheme for manipulating quantum information, e.g., realizing quantum controlled operations and generating entanglement, of two trapped ions beyond the LD limit by using a pair of synchronous laser pulses. The information bus, i.e., the center-of-mass (CM) vibrational quanta of the ions, for communicating different ions, may be either in its ground or an arbitrary excitation state. Neither the auxiliary atomic level nor the Lamb-Dicke approximation are needed in this work. The experimental realization of this simple approach is discussed.
II. CONDITIONAL QUANTUM DYNAMICS FOR TWO TRAPPED IONS DRIVEN BY TWO SYNCHRONOUS CLASSICAL LASER BEAMS BEYOND THE LAMB-DICKE LIMIT

The ion trap quantum information processor consists of a string of ions stored in a very cold linear radio-frequency trap. The motion of the ions, which are coupled together due to the Coulomb force between them, is quantum mechanical in nature. The ions are sufficiently separated apart (see, e.g. [20, 22]) to be easily addressed by different laser beams, i.e., each ion can be illuminated individually by a separate laser beam. The communication and logic operations between qubits are usually performed by exciting or de-exciting quanta of the collective vibration (i.e., the shared phonon) modes, which act as the information bus (see, e.g., [6, 13, 15]).

We consider an array of $N$ two-level cold ions of mass $M$ trapped in a one-dimensional harmonic potential of frequency $\nu$. The ions are able to perform small oscillations around their equilibrium position $z_{io} \equiv (i = 1, 2, ..., N)$, due to the repulsive Coulomb force between them. Each one of the ions is assumed to be individually addressed by a separate laser beam. We consider the case where an arbitrary pair (labeled by $j = 1, 2$) of trapped cold ions from the chain of $N$ trapped cold ions are illuminated independently by two weak classical laser beams. This is different from the scheme proposed in [23] of using Raman lasers to drive the ions. The Hamiltonian corresponding to our situation is

$$\hat{H}(t) = \hbar \omega_0 \sum_{j=1}^{2} \frac{\hat{\sigma}_{z,j}}{2} + \hbar \nu (\hat{a}^\dagger \hat{a} + \frac{1}{2}) + \sum_{l=1}^{N-1} \hbar \nu_l (\hat{b}_l^\dagger \hat{b}_l + \frac{1}{2})$$

$$+ \frac{\hbar}{2} \sum_{j=1}^{2} \left\{ \Omega_j \hat{\sigma}_{+,j} \exp \{ i [\eta_j (\hat{a}^\dagger + \hat{a}) + \sum_{l=1}^{N-1} \eta_{j,l} (\hat{b}_l^\dagger + \hat{b}_l) - \omega_j t - \phi_j ] \} + H.c. \right\}. \quad (1)$$

Here, $\nu$ and $\nu_l \ (l = 1, ..., N - 1)$ are the frequencies of the collective center-of-mass (CM) vibrational motion ($l = 0$) and the higher normal modes ($l \geq 1$) of the trapped ions, respectively; $\hat{a}^\dagger$ and $\hat{a}$ are the ladder operators of CM mode, while $\hat{b}_l^\dagger$ and $\hat{b}_l$ are the ladder operators of the higher normal modes. $\Omega_j \ (j = 1, 2)$ is the carrier Rabi frequency, which describes the coupling strength between the laser and the $j$th ion and is proportional to the strength of the applied laser. $\hat{\sigma}_{z,j}$ and $\hat{\sigma}_{+,j}$ are Pauli operators, $\hbar \omega_0$ is the energy separation of the two internal states $|g\rangle$ and $|e\rangle$ of the ion, and $\phi_j$ is the initial phase of the applied laser beam. The LD parameters $\eta_j$ and $\eta_{j,l}$ account for the coupling strength between the internal state of the $j$th ion and the vibrational states of the CM mode and the higher frequency modes, respectively. Expanding Eq. (1) in terms of creation and annihilation operators of the normal modes, we can rewrite the Hamiltonian of system in the interaction picture as

$$\hat{H} = \frac{\hbar}{2} \sum_{j=1,2} \left\{ \Omega_j \hat{\sigma}_{+,j} \hat{G}_j \left[ \exp \left( -\frac{\eta_j^2}{2} - i \phi_j \right) \right] \sum_{m,n=0}^{\infty} \frac{(i \eta_j)^{m+n} \hat{a}^m \hat{a}^n}{m! n!} \exp[i(m-n)\nu t + i \delta_j t] \right\} + H.c. \right\}, \quad (2)$$

with

$$\hat{G}_j = \prod_{l=1}^{N-1} \exp \left( -\frac{\eta_{j,l}^2}{2} \right) \sum_{m',n'=0}^{\infty} \frac{(i \eta_j)^{m'+n'} \hat{b}_{l'}^m \hat{b}_{l'}^n}{m'! n'!} \exp[i(m'-n')\nu_l t].$$

We assume that the frequencies of the applied lasers to be tuned resonantly on the same lower red-sidebands of the
center-of-mass (CM) vibrational mode, i.e., the frequencies \( \omega_j \) of the applied lasers are chosen to be \( \omega_j = \omega_0 - k_j \nu, k_j = 1, 2, \ldots \). Then, like the procedure described in [23, 24, 25], we make the usual rotating wave approximation (RWA) and have the following effective Hamiltonian

\[
\hat{H}_{\text{eff}} = \frac{\hbar}{2} \sum_{j=1,2} \left\{ \Omega_j \hat{F}_j \hat{\sigma}_{+j} \exp \left( -\frac{\eta_j^2}{2} - i \phi_j \right) \sum_{n=0}^{\infty} \frac{(i \eta_j)^{2n+k} \hat{a}^{\dagger n} \hat{a}^{n+k}}{n! (n+k)!} + H.c. \right\},
\]

for small \( k \) values. The operator function

\[
\hat{F}_j = \prod_{l=1}^{N-1} \exp[-(\eta_j^2/2)] \sum_{n=0}^{\infty} \frac{(i \eta_j)^{2n}}{(n)!^2} \hat{b}_i^{\dagger n} \hat{b}_i^n,
\]

involving the number operators related to the higher-\( l \) normal modes (\( l \geq 1 \), is irrelevant because we are considering the weak excitation regime (\( \Omega_j \ll \nu \), i.e., the intensity of the applied laser beams is assumed to be sufficiently weak). Thus, hereafter we neglect all off-resonant transitions and the excitations of the higher vibrational modes [23, 24, 25], i.e., let \( \hat{F}_j = \hat{I} \), and only label the CM mode excitations. We stress the following important fact: the effective Hamiltonian (3) reduces to that in previous works (e.g., [5, 9, 12, 17] under the approximations of the higher vibrational modes [23, 24, 25], i.e., let \( \hat{F}_j = \hat{I} \), and only label the CM mode excitations. We stress the following important fact: the effective Hamiltonian (3) reduces to that in previous works (e.g., [5, 9, 12, 17] under the usual LD approximation: \( (m+1)\eta_j^2 \ll 1 \), to the lowest order of the LD parameter \( \eta_j \). Here \( m \) is the occupation number of the Fock state of the CM vibrational quanta. Also, the Hamiltonian (3) reduces to that in [24] for \( k = 1 \).

In order to manipulate a pair of trapped ions outside the LD regime, we now wish to solve the quantum dynamical problem associated with the above Hamiltonian (3) without using the LD approximation. All operations presented below are based on this solution and do not involve quantum transitions to auxiliary atomic levels. Without loss of generality, the information bus (i.e., the CM vibrational mode of the ions) is assumed to be prepared beforehand in a pure quantum state, e.g., the Fock state \( |m\rangle \) with \( m < k \). During the time-evolution \( \hat{U}(t) = \exp(-it\hat{H}_{\text{eff}}/\hbar) \), the initial state \( |m\rangle |g_1\rangle |g_2\rangle \) is unchanged, i.e.,

\[
|m\rangle |g_1\rangle |g_2\rangle \xrightarrow{\hat{U}(t)} |m\rangle |g_1\rangle |g_2\rangle,
\]

since \( \hat{H}_{\text{eff}} |m\rangle |g_1\rangle |g_2\rangle = 0 \). Using the relations

\[
\hat{H}_{\text{eff}} |m\rangle |e_1\rangle |g_2\rangle = (-i)^k \hbar e^{i\phi_1} \alpha_1 |m + k\rangle |g_1\rangle |g_2\rangle,
\]

\[
\hat{H}_{\text{eff}} |m\rangle |g_1\rangle |e_2\rangle = (-i)^k \hbar e^{i\phi_2} \alpha_2 |m + k\rangle |g_1\rangle |g_2\rangle,
\]

we have the evolutions

\[
|m\rangle |g_1\rangle |e_2\rangle \xrightarrow{\hat{U}(t)} B_1(t) |m + k\rangle |g_1\rangle |g_2\rangle + B_2(t) |m\rangle |g_1\rangle |e_2\rangle + B_3(t) |m\rangle |e_1\rangle |g_2\rangle,
\]

\[
|m\rangle |e_1\rangle |g_2\rangle \xrightarrow{\hat{U}(t)} C_1(t) |m + k\rangle |g_1\rangle |g_2\rangle + C_2(t) |m\rangle |g_1\rangle |e_2\rangle + C_3(t) |m\rangle |e_1\rangle |g_2\rangle,
\]

where

\[
B_1(t) = (-i)^{k+1} e^{i\phi_2} \frac{2}{\chi^2} \sin(\chi t), \quad B_2(t) = \frac{(i \eta_j)^{2n+k}}{\chi^2} \frac{2}{\chi^2} \sin(\chi t),
\]

\[
B_3(t) = e^{-i(\phi_1 - \phi_2)} \alpha_1 \alpha_2 \frac{\chi^2}{(\chi^2 - 1)^{\frac{1}{2}}} e^{i(\phi_1 - \phi_2)} \frac{\chi^2}{(\chi^2 - 1)^{\frac{1}{2}}} \frac{2}{\chi^2} \sin(\chi t), \quad C_1(t) = \frac{(i \eta_j)^{2n+k}}{\chi^2} \frac{2}{\chi^2} \sin(\chi t),
\]

\[
C_2(t) = \frac{(i \eta_j)^{2n+k}}{\chi^2} \frac{2}{\chi^2} \sin(\chi t), \quad C_3(t) = \frac{(i \eta_j)^{2n+k}}{\chi^2} \frac{2}{\chi^2} \sin(\chi t).
\]

Here, \( \chi = \sqrt{\alpha_1^2 + \alpha_2^2}, \quad \chi^2 = \frac{\alpha_1^2 + \alpha_2^2}{\alpha_1^2 + \alpha_2^2}, \quad \Omega_j, \gamma_j, \Omega_j^{m,k}, \gamma_j^{m,k} \)

Finally, in order to obtain the time-evolution of the initial state \( |m\rangle |e_1\rangle |e_2\rangle \), we solve the Schrödinger equation \( i\hbar \partial \psi(t)/\partial t = \hat{H}_{\text{eff}} \psi(t) \) in the invariant subspace \( \{|m + 2k\rangle |g_1\rangle |g_2\rangle, |m + k\rangle |e_1\rangle |g_2\rangle, |m + k\rangle |g_1\rangle |e_2\rangle, |m\rangle |e_1\rangle |e_2\rangle\} \).
After a long but direct algebraic derivation, we obtain the exact time-evolution

$$
|m⟩⟨e_1⟩ + e_i(t) D(t) |m⟩⟨e_2⟩ + D_2(t) |m⟩⟨e_1⟩ + D_4(t) |m⟩⟨e_2⟩,
$$

where $D_1(t) = (i) 2k_1 e^{i(\phi_1 + \phi_2)} e^{i(\lambda_+ t - \lambda_- t)}$,
$D_2(t) = \frac{1}{\Delta} [\zeta_+ \cos(\lambda_- t) - \zeta_- \cos(\lambda_+ t)]$, $D_3(t) = (-i)e^{i\phi_2} \frac{\alpha\rho_3 + \gamma_3 \zeta_+}{\lambda_+ \zeta_+} \sin(\lambda_+ t)$, $D_4(t) = (-i) e^{i\phi_1} \frac{\alpha\rho_3 + \gamma_3 \zeta_+}{\lambda_+ \zeta_+} \sin(\lambda_- t)$.

To effectively manipulate two-qubit quantum information stored in two ions by applying a pair of simultaneous laser pulses, generally, the motion state entangles with the spin states during the dynamical evolution. We afterwards focus on how to decouple them and realize in one step beyond the LD limit: either a two-qubit controlled operation or an entanglement between the trapped ions. The state of the information bus (CM mode) remains in its initial state, which is not entangled with the qubits after the operations. This is achieved by properly setting up the controllable experimental parameters, e.g., the Lamb-Dicke parameters $\eta_j$, the efficient Rabi frequencies $\Omega_j$, the frequencies $\omega_j$ ($j = 1, 2$) and the duration of the applied synchronous pulses.

**III. MANIPULATION OF QUANTUM INFORMATION IN TWO TRAPPED COLD IONS**

Based on the conditional quantum dynamics for the two-qubit system derived in the previous section, we now show how to effectively manipulate two-qubit quantum information stored in two ions by applying a pair of simultaneous laser pulses. Generally, the motion state entangles with the spin states during the dynamical evolution. We afterwards focus on how to decouple them and realize in one step and beyond the LD limit: either a two-qubit controlled operation or an entanglement between the trapped ions. The state of the information bus (CM mode) remains in its initial state, which is not entangled with the qubits after the operations. This is achieved by properly setting up the controllable experimental parameters, e.g., the Lamb-Dicke parameters $\eta_j$, the efficient Rabi frequencies $\Omega_j$, the frequencies $\omega_j$ ($j = 1, 2$) and the duration of the applied synchronous pulses.

**A. Two-Qubit Controlled Operations**

As one of the simplest universal two-qubit quantum gates, the $C^Z$ logic operation between the 1st and the 2nd ions

$$
\hat{C}^Z_{12} = |g_1⟩⟨g_2| + |g_1⟩⟨e_2| + |e_1⟩⟨g_2| - |e_1⟩⟨e_2|,
$$

means that if the first ion is in the state $|g_1⟩$, the operation has no effect, whereas if the control qubit (first ion) is in the state $|e_1⟩$, the state of the second ion is rotated by the Pauli operator $\sigma_z$. The first qubit is the control qubit and the second one is the target qubit. The $C^Z$ operation is also known as controlled-rotation (CROT). It is seen from Eqs. (4-6) that the $\hat{C}^Z_{12}$ gate can be implemented exactly by a one-step operation, if the experimental parameters are set up so that the following conditions are simultaneously satisfied,

$$
\cos(\tau_z) = 1, \quad \cos(\lambda_+ \tau_z) = \cos(\lambda_- \tau_z) = -1.
$$

Here $\tau_z$ is the duration of the two applied synchronous pulses. Notice that the second condition in (8), on $\lambda_+$, is equivalent to requiring $|D_4(t)|^2 = 1$, which forces $|D_1(t)|^2 = |D_2(t)|^2 = |D_3(t)|^2 = 0$ due to normalization. The information bus remains in its initial state after the operation. Without loss of generality, we give some solutions of the conditional equations (8) for $m = 0$ in Table I.

We see from the Table I that both the small and large, both the negative and positive, values of the LD parameters may be chosen to satisfy the conditions (8) for realizing the desired two-qubit controlled gate. Our approach does not assume the LD approximation where $\eta_j \ll 1$ for $m = 0$. Thus, the present scheme can operate outside the LD regime and $\eta_j$ can be large.
According to previous works (see, e.g.,\cite{6,13}), it is known that an exact two-qubit gate $\hat{C}^Z$ surrounded by two one-qubit rotations on the target qubit can give rise to an exact two-qubit CNOT gate $\hat{C}^X$. The present work shows that the CNOT gate between different ions can be realized by using a \textit{three-step} pulse process, which is simpler than the previous schemes which usually employed either \textit{five} pulses [6,13], \textit{six} pulses [15] or \textit{seven} operations [16].

| $\Omega_2/\Omega_1$ | $k$ | $\eta_1 = \eta_2 = \eta$ | $\Omega_1 \tau_2$ | $\Omega_2/\Omega_1$ | $k$ | $\eta_1 = \eta_2 = \eta$ | $\Omega_1 \tau_2$ |
|---------------------|-----|--------------------------|------------------|---------------------|-----|--------------------------|------------------|
| 2.03951             | 1   | 1.93185                  | 18.5069          | 1.81182             | 1   | 2.30578                  | 37.5859          |
|                     |     | 0.517638                 | 12.2197          |                     |     | 0.253727                 | 137.757          |
|                     | 2   | $\pm0.915272$            | 14.1979          | 2                   | $\pm2.1702$          | 57.2117          |
|                     | 3   | $\pm2.67624$             | 39.2315          | 3                   | 0.859544              | 33.8887          |
|                     |     | 1.12532                  | 17.9115          |                     |     | 3.40669                  | 124.419          |
| 0.658331            | 1   | 1.76579                  | 56.5182          | 2                   | $\pm0.983608$        | 14.3592          |
|                     |     | 0.939131                 | 34.7414          |                     |     | $\pm2.65189$            | 40.9890          |
|                     | 2   | $\pm1.09276$             | 45.1673          | 3                   | 1.16543               | 18.4811          |
|                     |     | $\pm2.60881$             | 131.088          |                     |     | 2.63899                  | 26.2548          |
|                     | 3   | 1.23348                  | 58.6299          |                     |     | 3.11088                 | 62.2365          |
|                     |     | 2.53336                  | 80.4486          |                     |     | 3.33069                 | 102.936          |

TABLE I. A few solutions of equations (8) for $m = 0$. These parameters realize a $\hat{C}^Z$ or CROT gate between two trapped ions. Here, $\tau_2$ is the duration of the applied pulses, $\Omega_2$ and $\eta_j$ ($j = 1, 2$) are the effective Rabi frequencies and the Lamb-Dicke parameters, respectively. The frequencies of the applied pulses are equal, i.e., $k_1 = k_2 = k = 1, 2, 3$.

B. Two-Qubit Entangled States

Recently, the quantum entanglement of two and four trapped ions has been generated experimentally (see, e.g.,\cite{5,10}), although the operations are limited to the weak-coupling LD regime. We now show that the entangled states of two trapped ions can also be produced outside the LD limit. Indeed, the dynamical evolutions (4-6) clearly reveal that there are many ways to produce various deterministic entangled states of two trapped ions. For example, if the conditions

$$\alpha_j = \beta_j, \quad \phi_1 = \phi_2, \quad \alpha_1 \neq \alpha_2, \quad \cos(\chi \tau_2) = -1, \quad (9)$$

are satisfied, then two equal red-sideband pulses (i.e., $k_1 = k_2 = k > 0$) with frequencies $\omega_1 = \omega_2 = \omega_0 + k \nu$, applied to two ions individually and simultaneously, yield the following dynamical evolutions:

$$\begin{align*}
|\psi_{12}^-\rangle &= U|g_1\rangle|e_2\rangle - V|e_1\rangle|g_2\rangle, \\
|\psi_{12}^+\rangle &= -V|g_1\rangle|e_2\rangle + U|e_1\rangle|g_2\rangle,
\end{align*}$$

(10)

with

$$U = \frac{\alpha_1^2 - \alpha_2^2}{\alpha_1^2 + \alpha_2^2}, \quad V = \frac{2\alpha_1 \alpha_2}{\alpha_1^2 + \alpha_2^2}.$$

Therefore, entangled states $|\psi_{12}^\pm\rangle$ can be generated, in a \textit{single-step} process, by the dynamical evolution of the initial non-entangled states $|g_1\rangle|e_2\rangle$ or $|e_1\rangle|g_2\rangle$. We note that the degrees of entanglement for the above entangled states $|\psi_{12}^\pm\rangle$ are equivalent. They are

$$E = -U^2 \log_2 U^2 - V^2 \log_2 V^2.$$  

(11)

Here $E$ is the degree of entanglement defined \cite{26} as $E[|\psi\rangle] = -\sum_i C_i^2 \log_2 C_i^2$, for an general two-particle entangled pure state $|\psi(A,B)\rangle = \sum_i C_i |\alpha_i\rangle_A \otimes |\beta_i\rangle_B$, $\sum_i |C_i|^2 = 1$. The entangled state of two trapped ions realized experimentally in \cite{10} is not the maximally entangled state. Its degree of entanglement $E$ is 0.94. However, in principle, maximally
Obviously, the values of pulses with frequencies \( \nu = \alpha_2/\alpha_1 \), which is a function of the LD parameters \( \eta_j \), effective Rabi frequencies \( \Omega_j \), and the laser frequencies \( \omega_j \), \( (j = 1, 2) \). Note that \( E = 1 \) for \( \alpha_2/\alpha_1 = \sqrt{2} \pm 1 \), and \( E = 0 \) for \( \alpha_2/\alpha_1 = 1 \). For \( \alpha_2/\alpha_1 > \sqrt{2} + 1 \) the degree of entanglement \( E \) decreases when \( \alpha_2/\alpha_1 \) increases.

Inversely, it is seen from Eqs. (12-15) that entangled states can be generated deterministically by using \( \alpha_2/\alpha_1 \) and \( \eta_j \), \( (j = 1, 2) \). For \( \alpha_2/\alpha_1 = 1 \) the condition (9) is violated and thus \( E = 0 \).

Similarly,

\[
\alpha_2 = \frac{\Omega_2}{\Omega_1} \left( \frac{\eta_2}{\eta_1} \right)^{|k|} \exp \left( \frac{-\eta_2^2 - \eta_1^2}{2} \right). \tag{15}
\]

Obviously, the values of \( E \) depend on the choice of the experimental parameters \( \Omega_j \), \( \eta_j \) and \( k \) (thus \( \omega_j \), \( (j = 1, 2) \). For \( \alpha_2/\alpha_1 = 1 \) the condition (9) is violated and thus \( E = 0 \).

Similarly,

\[
\lim_{\alpha_2/\alpha_1 \to \infty} U = 1, \quad \lim_{\alpha_2/\alpha_1 \to \infty} V = 0, \quad \lim_{\alpha_2/\alpha_1 \to \infty} E = 0. \tag{16}
\]

Inversely, it is seen from Eqs. (12-15) that \( E = 1 \) for \( \alpha_2/\alpha_1 = \sqrt{2} \pm 1 \). This implies that two-qubit maximally entangled states can be generated deterministically by using a single-step operation beyond the LD limit. For example, if the experimental parameters are set up simply as \( \eta_1 = \eta_2 \), \( \Omega_2/\Omega_1 = \sqrt{2} \pm 1 \), the EPR state \( |\Psi_{12}^+\rangle \) can be generated by using a single-step synchronous red-sideband \( \pi \) pulses with frequencies \( \omega_1 = \omega_2 = \omega_0 + \nu \) and duration \( \tau_c = \pi/(\alpha_1 \sqrt{4 \pm 2 \sqrt{2}}) \).

**IV. CONCLUSIONS AND DISCUSSIONS**

Based on the conditional quantum dynamics for two-qubit system, we have shown that, under certain conditions, the quantum controlled gate or entanglement between a pair of trapped ions can be realized deterministically by *only a single-step operation*, performed by simultaneously applying two laser pulses to two ions. Each of the laser beams interacts with a single ion. Neither auxiliary atomic level nor Lamb-Dicke approximation are required during the operation. The CM mode of the ions always remains in its initial quantum state after the operation.

We now give a brief discussion on the experimental realization of the present scheme. For ion-trap quantum information processing, the information bus, i.e., the usual collective CM vibrational mode, must first be initialized in a pure quantum state, e.g., its ground state. Recently, the collective motion of two and four \(^9\)Be\(^+\) ions has been successfully cooled to its ground state \([13, 14]\). This is a further step towards realizing the ion trap quantum computer.

Although all relevant experiments are operated in the LD regime, we show that the coherent manipulation of two trapped cold ions may also be implemented outside the LD limit. According to the experiment reported in \([10]\), two \(^9\)Be\(^+\) ions spaced \( 2 \mu \text{m} \) apart can be confined along the axis of a linear Paul trap with an axial center-of-mass frequency of \( \nu = \omega_z \approx 7 \text{ MHz} \ll \omega_x, y \). The collective mode of motion of the ions has been successfully cooled to the ground state by Doppler cooling (see, e.g., \([19]\)). The qubit states are \( 2s^2 S_{1/2} |F = 2, m_F = 2\rangle = |g\rangle \) and \( 2s^2 S_{1/2} |F = 1, m_F = 1\rangle = |e\rangle \) separated in frequency by \( \omega_0/2\pi \approx 1.25 \text{ GHz} \). According to the above analysis, we note that the present theoretical scheme may be realized by current experiments.
experimental technologies by properly setting up the controllable parameters (e.g., $\eta_j$, $\Omega_j$, $\omega_j$) and the durations of the applied laser pulses. Indeed, it is seen from the formulae (see, e.g., [20, 25])

$$\eta_j = \cos \theta_j \sqrt{\frac{\hbar k_j^2}{2 M \nu}}, \quad \theta_j = \arccos \left( \frac{\kappa_j \cdot \vec{z}_j}{|\kappa_j|} \right).$$

(17)

that the LD parameter $\eta_j$ can be controlled by adjusting the wave vector $\vec{k}_j$ of the applied laser pulse. Obviously, $\eta_j$ can be positive or negative, depending on the values of $\theta_j$. Here, $M \nu$ is the total mass of the ion chain and $\theta_j$ is the angle between the laser beam and the $z$ axis. The efficient Rabi frequency [10] of the $j$th ion,

$$\Omega_j = \Omega_c J_0 \left( |\delta \kappa_j| \xi_j \right),$$

(18)

can be chosen properly for separate addressing and satisfying the parameter conditions (8), (9), and (12) for different operations. Here $J_0$ is the zero-order Bessel function and $\xi_j$ is the amplitudes of the motion along $x$ associated with ion $j$ [10]. Jefferts et al. [27] point out that applying a static electric field to push the ions along $x$ may control the amplitude of $\xi_j$ and thus the $\Omega_j$ of the ion micro-motion.

Finally, we note that the duration of the two applied simultaneous pulses for realizing the above quantum controlled operation is not much longer than that for other schemes (see, e.g., [5, 2, 12, 13]) operating in the LD regime. The shortest duration of the applied synchronous pulses for realizing the above manipulations of two trapped ions is about $10^{-4}$ seconds, of the same order of the gate speed operating in the LD regime [8] for $\Omega_1/2\pi \approx 225$ kHz [10]. Of course, to excite only the chosen sidebands of the CM mode, the spectral width of the applied laser pulse has to be sufficiently small. It might seem at first, from the above numerical results, that the present scheme for realizing the desired gate operation cannot be easily implemented, as the relevant experimental parameters should be set up accurately. However, this is not the case. Simple numerical analysis shows that the lowest probability of realizing the desired operation is still very high, even if the relevant parameters cannot be set up exactly. For example, the lowest probability of realizing the two-qubit $C^Z$ operation is up to 99.97% (99.49%), if the rate of the two efficient Rabi frequencies $\Omega_2/\Omega_1$ is roughly set up as 2.03 (2.0), which is 0.5% (1.9%) away from the exact solution of condition (8), see table I. Therefore, based on the current ion-trap technologies the proposed scheme may be realizable in the near future.

**Acknowledgments**

We acknowledge X. Hu for useful discussions, C. Monroe for comments on the manuscript, and the partial support of ARDA, AFOSR, and the US National Science Foundation grant No. EIA-0130383.