BRAIDS, THEIR PROPERTIES AND GENERALIZATIONS

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Abstract. In the paper we give a survey on braid groups and subjects connected with them. We start with the initial definition, then we give several interpretations as well as several presentations of these groups. Burau presentation for the pure braid group and the Markov normal form are given next. Garside normal form and his solution of the conjugacy problem are presented as well as more recent results on the ordering and on the linearity of braid groups. Next topics are the generalizations of braids, their homological properties and connections with the other mathematical fields, like knot theory (via Alexander and Markov theorems) and homotopy groups of spheres.

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1. INTRODUCTION

Braid groups describe intuitive concept of classes of continuous deformations of braids, which are collections of interwining strands whose endpoints are fixed. Mathematically they can be considered from various points of view. The first intuitive approach is formalized naturally as isotopy classes of a collection of $n$ connected curves (strings) in 3-dimensional space. This point of view is connected with the definition of braid group as the fundamental group of configuration space of $n$ points on a plane. Also braids can be interpreted as a mapping class group of a punctured disc and as a subgroup of automorphism group of a free group (subsection 3.3).

The present survey is organized as follows. In Section 2 we make some historical remark. Definition and general properties are considered in Section 3. Configuration spaces appear in Subsection 3.2. Connections with groups of automorphisms of free groups are given in Subsection 3.3. Presentations of the braid group which appeared quite recently are observed in Subsection 3.4. Section 4 is devoted to F. A. Garside’s classical work [91] and Section 5 to that of P. Dehornoy on ordering for braids. Representations and in particular linearity are discussed in Section 6. In Section 7 various generalization of braids are presented. Homological properties are observed in Section 8. In the last Section 9 we discuss connection with the knot theory given by the Alexander and Markov Theorems and with the homotopy groups of spheres.

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2. HISTORICAL REMARKS

Braids were rigorously defined by E. Artin [7] in 1925, although the roots of this natural concept are seen in the works of A. Hurwitz (117, 1891), R. Fricke and F. Klein (89, 1897) and even in the notebooks of K.-F. Gauss. E. Artin [7] gave the presentation of the braid group (see formulas (3.2) in Section 3) which is common now. Already in the book of Felix Klein [126] published in 1926 there appeared a chapter about braids. Essential topics about braids were also presented in the Reidemeister’s Knotentheorie [176] published in 1932.

In 30-ies there appeared a series of papers of Werner Burau [48], [49], [50] where he in particular gave the presentation of the pure braid group (see subsection 3.5) and introduced the Burau representation (subsection 6.1). Wilhelm Magnus in his work [139] published in 1934 established relations between braid groups and the mapping class groups. At the same time there appeared the work of A. A. Markov [150], which together with Alexander Theorem [3] builds a bijection between links and equivalence classes of braids. It became an essential ingredient in study of links and knots (in the work of V. F. R. Jones [121], for example). In 1936-37 were published the works of O. Zariski [201], [202], where he discovered connections between braid groups and the fundamental group of the complement of discriminant of the general polynomial

$$f_n(t) = a_0t^n + a_1t^{n-1} + \ldots + a_{n-1}t + a_n,$$
a point of view later rediscovered by V. I. Arnold [5]. Zariski also understood connections between braids and configuration spaces, gave the presentation of the braid group of the sphere, and studied the braid groups of Riemann surfaces. Amazingly and unfortunately these works of Zariski were not noticed by the specialists on braids and are not mentioned even in books and papers where the presentations of braids of surfaces are discussed.

In the beginning of 60-ies R. Fox and L. Neuwirth [86] and E. Fadell and L. Neuwirth [80] studied configuration spaces which turned out to be $K(\pi, 1)$-spaces and so give natural geometrical model of the classifying spaces for the braid groups. Later, V. I. Arnold [5] in this direction proved the first results on cohomologies of braids. The motivation for his study was a connection (which he discovered) with the problem of representing algebraic functions in several variables by superposing algebraic functions in fewer variables. Also, in 1969 V. I. Arnold completely described the cohomologies of pure braid groups [4].

In 1969 there appeared the publication of F. A. Garside’s work [91] where he suggests a new normal form of elements in the braid group and with its help gives a new solution of the word problem and also solves the conjugacy problem. In 1968 was published a two-page note of G. S. Makanin [143] where he sketches his algorithm for the solution of the conjugacy problem. The complete publication of Makanin’s work didn’t appear (as far as the author is aware).

In 70-ies the study of cohomologies of braids was continued independently and by different methods by D. B. Fuks [90] who determined mod 2 cohomologies, and F. R. Cohen [54], [55], [56] who described these homologies with coefficients in $\mathbb{Z}$ and in $\mathbb{Z}/p$ as modules over the Steenrod algebra.

In 1984-85 independently N. V. Ivanov [118] and J. McCarthy [152] proved the ”Tits-alternative” for the mapping class groups of surfaces and as a consequence it is true for the braid groups. Namely, they proved that every subgroup of the mapping class group either contains an abelian subgroup of finite index, or contains a non-abelian free group.

The question of whether braid groups are linear attracted significant attention. It was realized that the Burau representation is faithful for $Br_3$ [92], [142]. Then after a long break in 1991 J. A. Moody [154] proved that Burau representation is unfaithful for $n \geq 9$. This bound was improved to $n \geq 6$ by D. D. Long and M. Paton [138] and to $n = 5$ by S. Bigelow [28]. In 1999-2000 there appeared preprints of papers of D. Krammer [128], [129] and S. Bigelow [29] who proved that $Br_n$ is linear for all $n$ (using the other representation).

At the beginning of nineties P. Dehornoy [67], [68], [69] proved that there exits a left order in braid groups.

Interesting generalizations of braids were introduced in the work of E. Brieskorn [42]. The configuration space can also be considered as the orbit space of the complement of the complexification of the arrangement of hyperplanes corresponding to the Coxeter group $A_{n-1} = \Sigma_n$. Generalizing this approach for any finite Coxeter group, E. Brieskorn defined the so-called generalized braid groups which are also called Artin groups.

Another way of generalization is to consider braid groups in 3-manifolds, possibly with a boundary. The simplest examples are braids in handlebodies. A. B. Sossinsky [183] was the first who studied them. Such a group can be interpreted as the fundamental group of the configuration space of a plane without $g$ points where $g$ is the genus of the handlebody. The generalized braid group of type $C$ is isomorphic to the braid group in the solid torus.

Under the influence of the theory of Vassiliev-Goussarov (finite-type) invariants singular braids were introduced. The corresponding algebraic structures are the Baez–Birman monoid.
3. Definitions and General Properties

3.1. Systems of \( n \) curves in three-dimensional space and braid groups. First of all as it was already mentioned braids naturally arise as objects in 3-space. Let us consider two parallel planes \( P_0 \) and \( P_1 \) in \( \mathbb{R}^3 \), which contain two ordered sets of points \( A_1, ..., A_n \in P_0 \) and \( B_1, ..., B_n \in P_1 \). These points are lying on parallel lines \( L_A \) and \( L_B \) respectively. The space between the planes \( P_0 \) and \( P_1 \) we denote by \( \Pi \). Suppose that the point \( B_i \) is lying under the point \( A_i \), as a result of the orthogonal projection of the plane \( P_0 \) onto the plane \( P_1 \). Let us connect the set of points \( A_1, ..., A_n \) with the set of points \( B_1, ..., B_n \) by simple non-intersecting curves \( C_1, ..., C_n \) lying in the space \( \Pi \) and such that each curve meets only once each parallel plane \( P_t \) lying in the space \( \Pi \) (see Figure 3.1). This object is called a braid and the curves are called the strings of a braid. Usually braids are depicted by projections on the plane passing through the lines \( L_A \) and \( L_B \). This projection is supposed to be in general position so that there is only finite number of double points of intersection which are lying on pairwise different levels and intersections are transversal. The simplest braid \( \sigma_i \) (Fig. 3.2) corresponds to the transposition \((i, i + 1)\).

Let us introduce the following equivalence relation on the set of all braids with \( n \) strings and with fixed \( P_0, P_1, A_i \) and \( B_i \). It is defined by homeomorphisms \( h : \Pi \to \Pi \), identical on \( P_0 \cup P_1 \) and such that \( h(P_t) = P_t \). Braids \( \beta \) and \( \beta' \) are equivalent if there exists such a homeomorphism \( h \) that \( h(\beta) = \beta' \). On the set \( Br_n \) of equivalence classes under the considered relation the structure of a group introduces as follows. We put a copy \( \Pi' \) of the domain \( \Pi \) under the \( \Pi \) in such a way that \( P_0' \) coincides with \( P_1 \) and each \( A_i \) coincides with \( B_i \) and we glue braids \( \beta \) and

---

[10], [33] and the braid-permutation group by R. Fenn, R. Rimányi and C. Rourke [83], [84]. Various properties of these objects were studied [85], [203], [93], [94], [66], [119], [62], [104].
Figure 3.3.

β'. This gluing gives a composition of braids ββ' (Fig. 3.3). Unit element is the equivalence class containing a braid of n parallel intervals, the braid β⁻¹ inverse to β is defined by reflection of β with respect to the plane P₁/2. A string Cᵢ of a braid β connects the point Aᵢ with the pont Bᵢ defining the permutation Sβ. If this permutation is identical then the braid β is called pure. The map β → Sβ defines an epimorphism τₙ of the braid group Brₙ on the permutation group Σₙ with the kernel consisting of all pure braids:

(3.1) 1 → Pₙ → Brₙ → Σₙ → 1.

The following presentation of the braid group Brₙ with generators σᵢ, i = 1,...,n − 1 and two types of relations:

(3.2) \[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i, & \text{if } |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}
\end{align*}
\]

is the algebraic expression of the fact that any isotopy of braids can be broken down into “elementary moves” of two types that correspond to two types of relations.

If we add a vertical interval to the system of curves on Figure 3.1 we can get a canonical inclusion jₙ of the group Brₙ into the group Brₙ₊₁

\[
jₙ : Brₙ \to Brₙ₊₁.
\]

If the symmetric group Σₙ is given by its canonical presentation with generators sᵢ, i = 1,...,n − 1 and relations:

(3.3) \[
\begin{align*}
s_is_j &= s_j s_i, & \text{if } |i - j| > 1, \\
s_is_{i+1}s_i &= s_{i+1}s_is_{i+1} \\
s_i^2 &= 1
\end{align*}
\]

then the homomorphism τₙ is given by the formula

\[
\tauₙ(σᵢ) = sᵢ, \quad i = 1,\ldots,n - 1.
\]

It is possible to consider braids as classes of equivalence of braid diagrams which are generic projections of three dimensional braids on a plane. The classes of equivalence are defined by the Reidemeister moves depicted at Figure 3.4.
3.2. Braid groups and configuration spaces. If we look at the Figure 3.4, then this picture can be interpreted as a graph of a loop in the configuration space of $n$ points on a plane, that is the space of unordered sets of $n$ points on a plane, see Figure 3.5. So, it is possible to interpret the braid group as the fundamental group of the configuration space. Formally it is done as follows. The symmetric group $\Sigma_m$ acts on the Cartesian power $(\mathbb{R}^2)^m$ of the space $\mathbb{R}^2$:

$$w(y_1, \ldots, y_m) = (y_{w^{-1}(1)}, \ldots, y_{w^{-1}(m)}), \ w \in \Sigma_m.$$  

(3.4)  

Denote by $F(\mathbb{R}^2, m)$ the space of $m$-tuples of pairwise different points in $\mathbb{R}^2$:

$$F(\mathbb{R}^2, m) = \{ (p_1, \ldots, p_m) \in (\mathbb{R}^2)^m : p_i \neq p_j \text{ for } i \neq j \}.$$  

This is the space of regular points of our action. We call the orbit space of this action $B(\mathbb{R}^2, m) = F(\mathbb{R}^2, m)/\Sigma_m$ the configuration space of $n$ points on a plane. The braid group $Br_m$ is the fundamental group of configuration space

$$Br_m = \pi_1(B(\mathbb{R}^2, m)).$$

The pure braid group $P_m$ is the is the fundamental group of the space $F(\mathbb{R}^2, m)$. The covering

$$p : F(\mathbb{R}^2, m) \to B(\mathbb{R}^2, m)$$

defines the exact sequence:

$$1 \to \pi_1(F(\mathbb{R}^2, m)) \xrightarrow{p_*} \pi_1(B(\mathbb{R}^2, m)) \to \Sigma_m \to 1,$$

(3.5)  

which is equivalent to sequence (3.1).

It can be used for proving the canonical presentation of the braid group (3.2) as it is done, for example in the book of J. Birman [32].

Such considerations were made by R. Fox and L. Neuwirth [86].

3.3. Braid groups as automorphism groups of free groups and the word problem. Another important approach to the braid group bases on the fact that this group may be considered as a subgroup of the automorphism group of a free group.

Let $F_n$ be the free group of rank $n$ with the set of generators $\{x_1, \ldots, x_n\}$. Assume further that $\text{Aut} \ F_n$ is the automorphism group of $F_n$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure3.4.png}
\caption{Figure 3.4.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure3.5.png}
\caption{Figure 3.5.}
\end{figure}
We have the standard inclusions of the symmetric group $\Sigma_n$ and the braid group $Br_n$ into $\text{Aut} F_n$. For the braid group it may be described as follows. Let $\sigma_i \in \text{Aut} F_n, i = 1, 2, \ldots, n-1$, be given by the formula which describes its action on generators:

\begin{equation}
\begin{align*}
    x_i &\mapsto x_{i+1}, \\
x_{i+1} &\mapsto x_{i+1}^{-1}x_{i}x_{i+1}, \\
x_j &\mapsto x_j, j \neq i, i+1.
\end{align*}
\end{equation}

Let us define a map $\nu$ of the generators $\sigma_i, i = 1, \ldots, n-1$ of the braid group $Br_n$ to these automorphisms:

\begin{equation}
\nu(\sigma_i) = \sigma_i.
\end{equation}

**Theorem 3.1.** Formulas (3.7) define correctly a homomorphism $\nu : Br_n \to \text{Aut} F_n$, which is a monomorphism.

Theorem 3.1 gives a solution of the word problem for the braid groups. It was done first by E. Artin [7].

The free group $F_n$ is a fundamental group of a disc $D_n$ without $n$ points and the generator $x_i$ corresponds to a loop going around the $i$-th point. The braid group $Br_n$ is the mapping class group of a disc $D_n$ with its boundary fixed [32] and so it acts on the fundamental group of $D_n$. This action is described by the formulas (3.6) where $x_i$ correspond to the canonical loops on $D_n$ which form the generators of the fundamental group. Geometrically this action is depicted in the Figure 3.6.

### 3.4. Commutator subgroup and other presentations.

Let us define a homomorphism from braid group to integers by taking the sum of exponents of the entries of the generators $\sigma_i$ in the expression of any element of the group through these canonical generators:

\[ \text{deg} : Br_n \to \mathbb{Z}, \text{deg}(b) = \sum_j m_j, \text{where } b = (\sigma_{i_1})^{m_1} \cdots (\sigma_{i_k})^{m_k}. \]

**Proposition 3.1.** The homomorphism

\[ \text{deg} : Br_n \to \mathbb{Z} \]

gives the abelianization of the braid group and the commutator subgroup $Br'_n$ is characterized by the condition

\[ b \in Br'_n \text{ if and only if } \text{deg}(b) = 0. \]
Proof. Let \( a : Br_n \rightarrow A \) be a homomorphism to any other abelian group \( A \), then from the relations (3.2) we have:

\[
 a(\sigma_i)a(\sigma_{i+1})a(\sigma_i) = a(\sigma_{i+1})a(\sigma_i)a(\sigma_{i+1}).
\]

the commutativity of \( A \) gives that \( a(\sigma_{i+1}) = a(\sigma_i) \). This means that the homomorphism \( \text{deg} \) is universal. \( \square \)

Of course, there exist another presentations of the braid group. Let

\[
 \sigma = \sigma_1\sigma_2 \cdots \sigma_{n-1},
\]

then the group \( Br_n \) is generated by \( \sigma_1 \) and \( \sigma \) because

\[
 \sigma_{i+1} = \sigma^i\sigma_1\sigma^{-i}, \quad i = 1, \ldots, n - 2.
\]

The relations for the generators \( \sigma_1 \) and \( \sigma \) are the following

\[
\begin{align*}
(3.8) & \\
\{ & \\
\sigma_1 \sigma^i \sigma_1 \sigma^{-i} = \sigma^i \sigma_1 \sigma^{-i} \sigma_1 \text{ for } 2 \leq i \leq n/2, \\
\sigma^n = (\sigma \sigma_1)^{n-1}.
\}
\end{align*}
\]

This was observed by Artin in the initial paper [7].

An interesting series of presentations was given by V. Sergiescu [182]. For every planar graph he constructed a presentation of the group \( Br_n \), where \( n \) is the number of vertices of the graph, with generators corresponding to edges and relations reflecting the geometry of the graph. Artin’s presentation in this context corresponds to the graph consisting of the interval from 1 to \( n \) with the natural numbers (from 1 to \( n \)) as vertices and with segments between them as edges. For generalizations of braids graph presentations of these type were considered by P. Bellingeri and V. Vershinin [17, 21].

J. S. Birman; K. H. Ko; S. J. Lee [35] introduced the presentation with the generators \( a_{s,t} \) with \( 1 \leq s < t \leq n \) and relations

\[
\begin{align*}
(3.9) & \\
\{ & \\
a_{ts}a_{rq} = a_{rq}a_{ts} \text{ for } (t - r)(t - q)(s - r)(s - q) > 0, \\
a_{ts}a_{sr} = a_{tr}a_{ts} = a_{sr}a_{tr} \text{ for } 1 \leq r < s < t \leq n.
\}
\end{align*}
\]

The generators \( a_{s,t} \) are expressed by the canonical generators \( \sigma_i \) in the following form:

\[
(3.10) \quad a_{ts} = (\sigma_{t-1}\sigma_{t-2} \cdots \sigma_{s+1})\sigma_s(\sigma_{s+1}^{-1} \cdots \sigma_{t-2}^{-1}\sigma_{t-1}^{-1}) \text{ for } 1 \leq s < t \leq n.
\]

Geometrically the generators \( a_{s,t} \) are depicted in Figure 3.7.

The set of generators for braid groups was even enlarged in the work of Jean Michel [153] as follows. Let \( | \cdot | : \Sigma_n \rightarrow \mathbb{Z} \) be the length function on the symmetric group with respect to
the generators $s_i$: for $x \in \Sigma_n$, $|x|$ is the smallest natural number $k$ such that $x$ is a product of $k$ elements of the set $\{s_1, ..., s_{n-1}\}$. It is known ([11], Sect. 1, Ex. 13(b)) that two minimal expressions for an element of $\Sigma_n$ are equivalent by using only of the relations (3.2). This implies that the canonical projection $\tau_n : Br_n \to \Sigma_n$ has a unique set-theoretic section $r : \Sigma_n \to Br_n$ such that $r(s_i) = \sigma_i$ for $i = 1, ..., n-1$ and $r(xy) = r(x)r(y)$ whenever $|xy| = |x| + |y|$. Then the group $Br_n$ admits a presentation by generators $\{r(x) | x \in \Sigma_n\}$ and relations $r(xy) = r(x)r(y)$ for all $x, y \in \Sigma_n$ such that $|xy| = |x| + |y|$.

3.5. Presentation of the pure braid group and Markov normal form. Let $f(y_1, ..., y_m)$ be a word with (possibly empty) entries of $y_i$, where $y_i$ are some letters and $\epsilon$ may be $\pm 1$. If $y_i$ are elements of a group $G$ then $f(y_1, ..., y_m)$ will be considered as the corresponding element of $G$.

Let us define the elements $s_{i,j}$, $1 \leq i < j \leq m$, of the braid group $Br_m$ by the formula:

$$s_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_i^{1} \cdots \sigma_{j-1}.$$ 

These elements satisfy the following Burau relations ([18], [151], see also [135]):

$$
\begin{align*}
    s_{i,j}s_{k,l} &= s_{k,l}s_{i,j} \quad \text{for } i < j < k < l \text{ and } i < k < l < j, \\
    s_{i,j}s_{i,k}s_{j,k} &= s_{i,k}s_{j,k}s_{i,j} \quad \text{for } i < j < k, \\
    s_{i,k}s_{j,k}s_{i,j} &= s_{j,k}s_{i,j}s_{i,k} \quad \text{for } i < j < k, \\
    s_{i,k}s_{j,k}s_{l,k}^{-1}s_{i,j} &= s_{j,k}s_{l,k}^{-1}s_{i,j} \quad \text{for } i < j < k < l.
\end{align*}
$$

(3.11)

W. Burau and later A. A. Markov proved that the elements $s_{i,j}$ with the relations (3.11) give a presentation of the pure braid group $P_m$ [151]. The following formula is a consequence of the Burau relations and also belongs to A. A. Markov:

$$[s_{i,l}, s_{i,k}^\epsilon] = f(s_{1,l}, ..., s_{l-1,l}), \quad \epsilon = \pm 1, \quad k < l.$$ 

Let us define the elements $\sigma_{k,l}$, $1 \leq k \leq l \leq m$ by the formulas

$$\sigma_{k,k} = \epsilon,$$

$$\sigma_{k,l} = \sigma_{k}^{-1} \cdots \sigma_{l}^{-1}.$$ 

Let $P_m^k$ be the subgroup of $P_m$ generated by the elements $s_{i,j}$ with $k < j$.

Theorem 3.2. (A. A. Markov) (i) Every element of the group $Br_m$ can be uniquely written in the form

$$f_m(s_{1,m}, ..., s_{m-1,m}) \cdots f_j(s_{1,j}, ..., s_{j-1,j}) \cdots f_2(s_{1,2}) \sigma_{i,m} \cdots \sigma_{i,j} \cdots \sigma_{i,2}.$$ 

(3.13)

(ii) The factor group $P_m^k/P_m^{k-1}$ is the free group on free generators $s_{i,k+1}$, $1 \leq i \leq k$.

The form (3.13) is called the Markov normal form, it also gives the solution of the word problem for the braid groups.

4. Garside Normal Form, Center and Conjugacy Problem

Essential role in Garside work [91] plays the monoid of positive braids $Br_n^+$, that is the monoid which has a presentation with generators $\sigma_i$, $i = 1, ..., n$ and relations (3.2). In other words each element of this monoid can be represented as a word on the elements $\sigma_i$, $i = 1, ..., n$ with no entrances of $\sigma_i^{-1}$. Two positive words $A$ and $B$ in the alphabet $\{\sigma_i, (i = 1, ..., n - 1)\}$ will be said to be positively equal if they are equal as elements of $Br_n^+$. In this case we shall write $A \equiv B$. 

First of all Garside proves the following statement.

**Proposition 4.1.** In $Br_n^+$ for $i, k = 1, ..., n - 1$, given $\sigma_i A \doteq \sigma_k B$, it follows that

- if $k = i$, then $A \doteq B$,
- if $|k - i| = 1$, then $A \doteq \sigma_k \sigma_i Z, B \doteq \sigma_i \sigma_k Z$ for some $Z$,
- if $|k - i| \geq 2$, then $A \doteq \sigma_k Z, B \doteq \sigma_i Z$ for some $Z$.

The same is true for the right multiples of $\sigma_i$.

**Corollary 4.1.** If $A \doteq P, B \doteq Q, AXB \doteq PYQ$, $(L(A) \geq 0, L(B) \geq 0)$, then $X \doteq Y$. That is, monoid $Br_n^+$ is left and right cancellative.

Garside’s *fundamental word* $\Delta$ in the braid group $Br_{n+1}$ is defined by the formula:

$$
\Delta = \sigma_1 \ldots \sigma_n \sigma_1 \ldots \sigma_{n-1} \ldots \sigma_1 \sigma_2 \sigma_1.
$$

If we use Garside’s notation $\Pi_t \equiv \sigma_1 \ldots \sigma_t$, then $\Delta \equiv \Pi_{n-1} \ldots \Pi_1$.

For a positive word $W$ in $\sigma_i$, $i = 1, ..., n$ we say that $\Delta$ is a *factor* of $W$ or simply $W$ contains $\Delta$, if $W \doteq A\Delta B$ with $A$ and $B$ being arbitrary positive words, probably empty. If $W$ does not contain $\Delta$ we shall say $W$ is *prime to* $\Delta$.

Garside’s transformation of words $R$ is defined by the formula

$$
R(\sigma_i) \equiv \sigma_{n-1}.
$$

This gives the automorphism of $Br_n$ and the positive braid monoid $Br_n^+$.

**Proposition 4.2.** In $Br_n$

$$
\sigma_i \Delta \doteq \Delta R(\sigma_i).
$$

Geometrically this commutation is shown on Figure 4.1 ($\Delta \sigma_3 = \sigma_1 \Delta$).
Proposition 4.3. If $W$ is an arbitrary positive word in $Br_n^+$ such that either
\[ W \doteq \sigma_1 A_1 \doteq \sigma_2 A_2 \doteq \ldots \doteq \sigma_{n-1} A_{n-1}, \]
or
\[ W \doteq B_1 \sigma_1 \doteq B_2 \sigma_2 \doteq \ldots \doteq B_{n-1} \sigma_{n-1}, \]
then $W \doteq \Delta Z$ for some $Z$.

Proposition 4.4. The canonical homomorphism
\[ Br_n^+ \to Br_n \]
is a monomorphism.

Among positive words on the alphabet $\{\sigma_1 \ldots \sigma_n\}$ let us introduce a lexicographical ordering with the condition that $\sigma_1 < \sigma_2 < \cdots < \sigma_n$. For a positive word $W$ the base of $W$ is the smallest positive word which is positively equal to $W$. The base is uniquely determined. If a positive word $A$ is prime to $\Delta$, then for the base of $A$ the notation $\overline{A}$ will be used.

**Theorem 4.1.** (F. A. Garside) Every word $W$ in $Br_{n+1}$ can be uniquely written in the form $\Delta^m A$, where $m$ is an integer.

The form of a word $W$ established in this theorem we call the **Garside left normal form** and the index $m$ we call the **power** of $W$. The same way the **Garside right normal form** is defined and the corresponding variant of Theorem 4.1 is true. The Garside normal form also gives a solution to the word problem in the braid group.

**Theorem 4.2.** (F. A. Garside) The necessary and sufficient condition that two words in $Br_{n+1}$ are equal is that their Garside normal forms are identical.

Garside normal form for the braid groups was precised in the subsequent works of S. I. Adyan [1], W. Thurston [78], E. El-Rifai and H. R. Morton [76]. Namely, there was introduced the **left-greedy form** (in the terminology of W. Thurston [78])
\[ \Delta^t A_1 \ldots A_k, \]
where $A_i$ are the successive possible longest fragments of the word $\Delta$ (in the terminology of S. I. Adyan [1]) or positive permutation braids (in the terminology of E. El-Rifai and H. R. Morton [76]). Certainly, the same way the **right-greedy form** is defined. With the help of this form it proved that the braid group is biautomatic.

The center of the braid group was firstly found by W.-L. Chow [53]. Namely, as it follows from the presentation of braid groups with two generators $\sigma_1$ and $\sigma$ and relations (3.8) given in the subsection 3.1 the element $\sigma^n$ commutes with $\sigma \sigma_1$ and so with $\sigma_1$. Chow proved that it generates the center. Garside normal form gives an elegant proof of the following theorem.

**Theorem 4.3.** (i) When $n = 1$ the center of the group $Br_{n+1}$ is generated by $\Delta$.

(ii) When $n > 1$ the center of the group $Br_{n+1}$ is generated by $\Delta^2$.

Let $\alpha$ be a positive word such that $\Delta \doteq \alpha X$, where $X$ is an arbitrary positive word, probably empty. For any word $W$ in $Br_{n+1}$, the word $\alpha^{-1} W \alpha$, reduced to Garside normal form is called an $\alpha$-transformation of $W$.

For any word $W$ in $Br_{n+1}$ with the Garside normal form $\Delta^m \overline{A} \equiv W_1$ consider the following chains of $\alpha$-transformations: take all the $\alpha$-transformations of $W_1$ and let those which are of power $\geq m$ and which are distinct from each other be $W_2, W_3, \ldots, W_t$. Now repeat the process
for each of the words $W_2, W_3, \ldots, W_t$ in turn, denoting successively by $W_{t+1}, W_{t+2}, \ldots$, any new words occurring, the condition being always that each new word must be of power $\geq m$. Continue to repeat the process for every new distinct word arising, as the sequence $W_1, W_2, W_{t+2}, \ldots$, expands.

**Proposition 4.5.** The set $W_1, W_2, W_{t+2}, \ldots$, is finite.

Suppose that in the set $W_1, W_2, W_{t+2}, \ldots$, the highest power reached is $s$ and that the words of power $s$ form the subset $V_1, V_2, \ldots$. Then this set $V_1, V_2, \ldots$ is called the summit set of $W$.

**Theorem 4.4.** (F. A. Garside) Two elements $A$ and $B$ of the group $Br_{n+1}$ are conjugate if and only if their summit sets are identical.

J. S. Birman, K. H. Ko and S. J. Lee considered the word $\delta = a_{n(n-1)} \cdots a_{12}a_{21} = \sigma_n \cdots \sigma_2\sigma_1$, as a fundamental in their system of generators and proved that every element in $Br_n$ has a representative $W = \delta^j A_1A_2\cdots A_k$ with positive $A_i$ in a unique way in some sense. Based on this form they gave an algorithm for the word problem in $B_n$ which runs in time $(n^2)$ for a given word of length $m$.

5. Ordering of Braids

A group $G$ is said totally (or linearly) left (correspondingly right) ordered if it has a total order $<$ invariant by left (right) multiplication, i.e. if $a < b$, then $ca < cb$ for any $c \in G$. If this order is also invariant by right (left) multiplication, then the group $G$ is called ordered.

For any left ordered group $G$ denote by $P$ the set of positive elements $\{x \in G : x > 1\}$, then the set of negative elements is defined by the formula: $P^{-1} = \{x \in G : x \in P\}$. The total character of an order on $G$ is expressed by the partition

$$G = P \bigsqcup \{1\} \bigsqcup P^{-1}.$$ 

The invariance of multiplication is expressed by the inclusion $P^2 \subset P$, where $P^2$ is formed by products of couples of elements of $P$. Conversely, if there exist a subset $P$ of a group $G$ with the properties:

$$G = P \bigsqcup \{1\} \bigsqcup P^{-1}, \quad P^2 \subset P,$$

then $G$ is left ordered by the order defined by: $x < y$ if and only if $x^{-1}y \in P$. A group $G$ then is ordered if and only if $xPx^{-1} \subset P$ for all $x \in G$.

Let $i \in \{1, \ldots, n\}$ and a word $w$ on the alphabet $\{\sigma_1, \ldots, \sigma_n\}$ is expressed in the form

$$w_0\sigma_iw_1\sigma_i \cdots \sigma_iw_r,$$

where the subwords $w_0, \ldots, w_r$ are the words on the letters $\sigma_j^\pm$ with $j > i$. Then such a word is called $\sigma_i$-positive. This means that all entries of $\sigma_i^\pm$ in the word $w$ with $i$ minimal must be positive. If all such entries are negative then a word $w$ is called $\sigma_i$-negative. A braid of $Br_{n+1}$ is called $\sigma_i$-positive ($\sigma_i$-negative) if there exists its expression as a word on the standard generators which is $\sigma_i$-positive ($\sigma_i$-negative). A braid is called $\sigma$-positive ($\sigma$-negative) if it exists a number $i$, such that it is $\sigma_i$-positive ($\sigma_i$-negative).

**Theorem 5.1.** (P. Dehornoy) Every braid in $Br_{n+1}$ different from $1$ is either $\sigma$-positive or $\sigma$-negative.

**Corollary 5.1.** For all $n$ the braid group $Br_{n+1}$ is left ordered.
6. Representations

6.1. Burau representation. Let us map the generators of the braid group $Br_n$ to the following elements of the group $GL_n\mathbb{Z}[t, t^{-1}]$

\begin{equation}
\sigma_i \mapsto \begin{pmatrix}
E_{i-1} & 0 & 0 & 0 \\
0 & 1 - t & t & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & E_{n-i-1}
\end{pmatrix},
\end{equation}

where $E_i$ is the unit $i \times i$ matrix. The formula (6.1) defines correctly the representation of the braid group in $GL_n\mathbb{Z}[t, t^{-1}]$:

\[ r : Br_n \to GL_n\mathbb{Z}[t, t^{-1}], \]

which is called Burau representation [50].

**Theorem 6.1.** Burau representation is faithful for $n = 3$.

**Theorem 6.2.** (J. A. Moody; D. D. Long and M. Paton; S. Bigelow) Burau representation is unfaithful for $n \geq 5$.

The case $n = 4$ remains open.

6.2. Lawrence-Krammer representation. Consider the ring $K = \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ of Laurent polynomials in two variables $q, t$, and the free $K$-module

\[ V = \bigoplus_{1 \leq i < j \leq n} K x_{i,j}, \]

For $k \in \{1, 2, \ldots, n - 1\}$, define the action of the braid generators $\sigma_k$ on the basis of $V$ by the formula:

\begin{equation}
\sigma_k(x_{i,j}) = \begin{cases}
x_{i,j}, & k < i - 1 \text{ or } j < k; \\
x_{i-1,j} + (1 - q)x_{i,j}, & k = i - 1; \\
tq(q - 1)x_{i,i+1} + qx_{i+1,j}, & k = i < j - 1; \\
tq^2x_{i,j}, & k = i = j - 1; \\
x_{i,j} + tq^{k-i}(q - 1)^2x_{k,k+1}, & i < k < j - 1; \\
x_{i,j-1} + tq^{j-i}(q - 1)x_{j-1,j}, & k = j - 1; \\
(1 - q)x_{i,j} + qx_{i,j+1}, & k = j.
\end{cases}
\end{equation}

Direct computation shows that this defines a representation

\[ \rho_n : Br_n \to GL(V), \]

which was firstly defined by R. Lawrence [134] in topological terms and in the explicit form (6.2) by D. Krammer [129].

**Theorem 6.3.** (S. Bigelow [29], D. Krammer [129]) The representation

\[ \rho_n : Br_n \to GL(V) \]

is faithful for all $n \geq 1$. 
Remark 6.1. Actually, S Bigelow [29] proved this theorem for the representation $\rho_n$ characterized in homological terms and D. Krammer [129] proved the following. Let $K = \mathbb{R}[t^\pm 1]$, $q \in \mathbb{R}$, and $0 < q < 1$. Then the representation $\rho_n$ defined by (6.2) is faithful for all $n \geq 1$. This result implies Theorem 6.3: if a representation over $\mathbb{Z}[q^\pm 1, t^\pm 1]$ becomes faithful after assigning a real value to $q$, then it is faithful itself.

M. G. Zinno [204] established connection between Birman-Murakami-Wenzl algebra [40], [156] and the Lawrence-Krammer representation. Namely, he proved that the Lawrence-Krammer representation is identical to the irreducible representations of the Birman-Murakami-Wenzl algebra parametrized by Young diagrams of shapes $(n - 2)$ and $(1^{n-2})$. This means that the Young diagram in the case considered consists of one row (respectively of one column) only, with $n - 2$ boxes. It follows that Lawrence-Krammer representation is irreducible.

7. Generalizations of Braids

7.1. Configuration spaces of manifolds. The notion of configuration space of Subsection 3.2 can be naturally generalized for a configuration space of a manifold as follows. Let $Y$ be a connected topological manifold and let $W$ be a finite group acting on $Y$. A point $y \in Y$ is called \textit{regular} if its stabilizer $\{w \in W : wy = y\}$ is trivial, i.e., consists only of the unit of the group $W$. The set $\tilde{Y}$ of all regular points is open. Suppose that it is connected and nonempty. The subspace $\text{ORB}(Y, W)$ of the space of all orbits $\text{Orb}(Y, W)$ consisting of the orbits of all regular points is called the \textit{space of regular orbits}. There is a free action of $W$ on $\tilde{Y}$ and the projection $p : \tilde{Y} \to \tilde{Y}/W = \text{ORB}(Y, W)$ defines a covering. Let us consider the initial segment of the long exact sequence of this covering:

$$1 \to \pi_1(\tilde{Y}, y_0) \xrightarrow{p^*} \pi_1(\text{ORB}(Y, W), p(y_0)) \to W \to 1. \quad (7.1)$$

The fundamental group $\pi_1(\text{ORB}(Y, W), p(y_0))$ of the space of regular orbits is called the \textit{braid group} of the action of $W$ on $Y$ and is denoted by $Br(Y, W)$. The fundamental group $\pi_1(\tilde{Y}, y_0)$ is called the \textit{pure braid group} of the action of $W$ on $Y$ and is denoted by $P(Y, W)$. The spaces $\tilde{Y}$ and $\text{ORB}(Y, W)$ are path connected, so the pair of these groups is defined uniquely up to isomorphism and we may omit mentioning the base point $y_0$ in the notations.

For any space $Y$ the symmetric group $\Sigma_m$ acts on the Cartesian power $Y^m$ of the space $Y$ by the formulas (3.4). We denote by $F(Y, m)$ the space of $m$-tuples of pairwise different points in $Y$:

$$F(Y, m) = \{(p_1, ..., p_m) \in Y^m : p_i \neq p_j \text{ for } i \neq j\}.$$ 

This is the space of regular points of this action. In the case when $Y$ is a connected topological manifold $M$ without boundary and $\dim M \geq 2$, the space of regular orbits $\text{ORB}(M^m, \Sigma_m)$ is open, connected and nonempty. We call $\text{ORB}(M^m, \Sigma_m)$ the \textit{configuration space of the manifold} $M$ and denote by $B(M, m)$. The braid group $Br(M^m, \Sigma_m)$ is called the \textit{braid group on }$m$ \textit{strings of the manifold }$M$ and is denoted by $Br(m, M)$. Analogously, we call the group $P(M^m, \Sigma_m)$ the \textit{pure braid group on }$m$ \textit{strings of the manifold }$M$ and denote it by $P(m, M)$. These definitions of braid groups were given by R. Fox and L. Neuwirth [86].

7.2. Artin-Brieskorn braid groups. The braid groups are included in the series of so called generalized braid groups (this was their name in the work of E. Brieskorn of 1971 [42]), or Artin groups (so they were called by E. Brieskorn and K. Saito in the paper of 1972 [45]). They were defined by E. Brieskorn [42], so we call them Artin–Brieskorn groups.
Let $V$ be a finite dimensional real vector space ($\dim V = n$) with Euclidean structure. Let $W$ be a finite subgroup of $GL(V)$ generated by reflections. Let $M$ be the set of hyperplanes such that $W$ is generated by orthogonal reflections with respect to $M \in M$. We suppose that for every $w \in W$ and every hyperplane $M \in M$ the hyperplane $w(M)$ belongs to $M$.

The group $W$ is generated by the reflections $w_i = w_i(M_i), i \in I$, satisfying only the following relations

$$(w_iw_j)^{m_{i,j}} = e, \ i, j \in I,$$

where the natural numbers $m_{i,j} = m_{j,i}$ form the Coxeter matrix of $W$ from which the Coxeter graph $\Gamma(W)$ of $W$ is constructed [41]. We use the following notation of P. Deligne [73]: $\text{prod}(m;x,y)$ denotes the product $xyxy...$ ($m$ factors). The generalized braid group (or Artin–Brieskorn group) $Br(W)$ of $W$ [42, 73] is defined as the group with generators $\{s_i, i \in I\}$ and relations:

$$\text{prod}(m_{i,j}; s_i, s_j) = \text{prod}(m_{j,i}; s_j, s_i).$$

From this we obtain the presentation of the group $W$ by adding the relations:

$$s_i^2 = e; \ i \in I.$$

We see in Theorem [41] that this definition of the generalized braid group agrees with our general definition of a braid group of an action of a group $W$ (Subsection 7.1). We denote by $\tau_W$ the canonical homomorphism from $Br(W)$ to $W$. The classical braids on $k$ strings $Br_k$ are obtained by this construction if $W$ is the symmetric group on $k + 1$ symbols. In this case $m_{i,i+1} = 3$, and $m_{i,j} = 2$ if $j \neq i, i + 1$.

Classification of irreducible (with connected Coxeter graph) Coxeter groups is well known (see for example Theorem 1, Chapter VI, §4 of [Bo]). It consists of the three infinite series: $A$, $C$ (which is also denoted by $B$ because in the corresponding classification of simple Lie algebras two different series $B$ and $C$ have this group as their Weyl group) and $D$ as well as the exceptional groups $E_6, E_7, E_8, F_4, G_2, H_3, H_4$ and $I_2(p)$.

Now let us consider the complexification $V_C$ of the space $V$ and the complexification $M_C$ of $M \in M$. Let $Y_W = V_C - \bigcup_{M \in M} M_C$. The group $W$ acts freely on $Y_W$. Let $X_W = Y_W / W$ then $Y_W$ is a covering over $X_W$ corresponding to the group $W$. Let $y_0 \in A_0$ be a point in some chamber $A_0$ and let $x_0$ stand for its image in $X_W$. We are in the situation described in Subsection 7.1 in the definition of the braid group of the action of the group $W$. This braid group is defined as the fundamental group of the space of regular orbits of the action of $W$. In our case $ORB(V_C, W) = X_W$. So, the generalized braid group is $\pi_1(X_W, x_0)$. For each $j \in I$, let $\ell'_j$ be the homotopy class of paths in $Y_W$ starting from $y_0$ and ending in $w_j(y_0)$ which contains a polygon line with successive vertices: $y_0, y_0 + iy_0, w_j(y_0) + iy_0, w_j(y_0)$. The image $\ell_j$ of the class $\ell'_j$ in $X_W$ is a loop with base point $x_0$.

**Theorem 7.1.** The fundamental group $\pi_1(X_W, x_0)$ is generated by the elements $\ell_j$ satisfying the following relations:

$$\text{prod}(m_{j,k}; \ell_j, \ell_k) = \text{prod}(m_{k,j}; \ell_k, \ell_j).$$

This theorem was proved by E. Brieskorn [43].

The word problem and the conjugacy problem for Artin-Brieskorn groups were solved by E. Brieskorn and K. Saito [45] and P. Deligne [73]. The biautomatic structure of these groups was established by R. Charney [51].
In the case when $V$ is complex finite dimensional space and $W$ is a finite subgroup of $GL(V)$ generated by pseudo-reflections the corresponding braid groups were studied by M. Broué, G. Malle and R. Rouquier [47] and also by D. Bessis and J. Michel [27].

7.3. Braid groups of surfaces. Braid groups of a sphere $Br_n(S^2)$ also have simple geometric interpretation as a group of isotopy classes of braids lying in a layer between two concentric spheres. It has the presentation with generators $\delta_i$, $i = 1, ..., n - 1$, and relations:

$$
\begin{align*}
\delta_i \delta_j &= \delta_j \delta_i, \text{ if } |i - j| > 1, \\
\delta_i \delta_{i+1} \delta_i &= \delta_{i+1} \delta_i \delta_{i+1}, \\
\delta_1 \delta_2 \cdots \delta_{n-2} \delta_{n-1} \delta_n \cdots \delta_2 \delta_1 &= 1.
\end{align*}
$$

This presentation was first found by O. Zariski [201] in 1936 and then rediscovered by E. Fadell and J. Van Buskirk [81] in 1961.

Presentations of braid groups of all closed surfaces were obtained by G. P. Scott [181] but look rather complicated.

7.4. Braid groups in handlebodies. The subgroup $Br_{1,n+1}$ of the braid group $Br_{n+1}$ consisting of braids with the fixed first string can be interpreted also as the braid group in a solid torus. Here we study braids in a handlebody of the arbitrary genus $g$.

Let $H_g$ be a handlebody of genus $g$. The braid group $Br^n_g$ on $n$ strings in $H_g$ was first considered by A. B. Sossinsky [183]. Let $Q_g$ denote a subset of the complex plain $\mathbb{C}$, consisting of $g$ different points, $Q_g = \{z_1^0, ..., z_g^0\}$, say, $z_i^0 = i$. The interior of the handlebody $H_g$ may be interpreted as the direct product of the complex plain $\mathbb{C}$ without $g$ points: $\mathbb{C} \setminus Q_g$, and the open interval, for example, $(-1, 1)$:

$$
H_g = (\mathbb{C} \setminus Q_g) \times (-1, 1).
$$

The space $F(\mathbb{C} \setminus Q_g, n)$ can be interpreted as the complement of the arrangement of hyperplanes in $\mathbb{C}^{g+n}$ given by the formulas:

$$
H_{j,k} : z_j - z_k = 0 \text{ for all } j, k;
$$

$$
H^i_j : z_j = z_i^0 \text{ for } i = 1, ..., g; \ j = 1, ..., n.
$$

The braids in $Br^n_g$ are considered as lying between the planes with coordinates $z = 0$ and $z = 1$ and connecting the points $((g + 1, 0), ..., (g + n, 0))$. So $Br^n_g$ can be considered as a subgroup of the classical braid group $Br_{g+n}$ on $g + n$ strings such that the braids from $Br^n_g$ leave the first $g$ strings unbraided. In this subsection we denote by $\bar{\sigma}_j$ the standard generators of the group $Br_{g+n}$. Let $\tau_k$, $k = 1, 2, ..., g$, be the following braids:

$$
\tau_k = \sigma_g \sigma_{g-1} \cdots \sigma_{k+1} \sigma_k \sigma_{k+1}^{-1} \cdots \sigma_{g-1}^{-1} \sigma_g^{-1}.
$$

Such a braid is depicted in Figure 7.1. The elements $\tau_k$, $k = 1, 2, ..., g$, generate a free subgroup $F_g$ in the braid group $Br_{g+n}$. It follows for example from the Markov normal form that the elements $\tau_k$, $k = 1, 2, ..., g$, together with the standard generators $\sigma_{g+1}, ..., \sigma_{g+n-1}$ generate the group $Br^n_g$. So, the braid group in the handlebody $Br^n_g$ can be considered as a subgroup of $Br_{g+n}$, generated by two subgroups: $F_g$ and $Br^n_g$. Denote by $\sigma_1, ..., \sigma_{n-1}$ the standard generators of $Br_n$ considered as the elements of $Br^n_g$, $\sigma_i = \bar{\sigma}_{g+i}$, $i = 1, ..., n - 1$. So we have
the presentation of $Br_n^g$ with the generators $\tau_k$ and $\sigma_i$ and relations \cite{183}, \cite{190}, \cite{193}:

$$\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i \text{ if } |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
\tau_k \sigma_i &= \sigma_i \tau_k \text{ if } k \geq 1, \quad i \geq 2, \\
\tau_k \sigma_1 \tau_k \sigma_1 &= \sigma_1 \tau_k \sigma_1 \tau_k, \quad k = 1, 2, ..., g, \\
\tau_k \sigma_1^{-1} \tau_{k+l} \sigma_1 &= \sigma_1^{-1} \tau_{k+l} \sigma_1 \tau_k, \quad k = 1, 2, ..., g - 1; \quad l = 1, 2, ..., g - k.
\end{align*}$$

(7.3)

The relation of the fourth type in \eqref{7.3} is the relation of the braid group of type $B(C)$. The relations of the fifth type in \eqref{7.3} describe the interaction between the generators of the free group and their closest neighbour $\sigma_1$. Geometrically this is seen in Figure 7.2. If we introduce the new generators $\theta_k$, $k = 1, 2, ..., g - 1$; by the formulas:

$$\theta_k = \sigma_1^{-1} \tau_k \sigma_1$$
we obtain the “positive” presentation of the group $B^g_n$ with generators of the types $\sigma_i$, $\tau_k$, $\theta_k$ and relations:

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad &\text{if} \quad |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
\tau_k \sigma_i &= \sigma_i \tau_k \quad &\text{if} \quad k \geq 1, \quad i \geq 2, \\
\tau_k \sigma_1 \tau_k \sigma_1 &= \sigma_1 \tau_k \sigma_1 \tau_k, \quad k = 1, 2, ..., g, \\
\tau_k \theta_{k+l} &= \theta_{k+l} \tau_k, \quad k = 1, 2, ..., g - 1; \quad l = 1, 2, ..., g - k. \\
\sigma_i \theta_k &= \tau_k \sigma_i, \quad k = 1, 2, ..., g - 1.
\end{align*}
\]

(7.4)

There is a version of Markov Theorem [3.2] for the group $Br^g_n$ [190], [193].

7.5. **Braids with singularities.** Let $BP_n$ be the subgroup of $\text{Aut} F_n$, generated by both sets of the automorphisms $\sigma_i$ of (3.6) and $\xi_i$ of the following form:

\[
\begin{align*}
x_i \quad &\mapsto \quad x_{i+1}, \\
x_{i+1} \quad &\mapsto \quad x_i, \\
x_{j} \quad &\mapsto \quad x_{j}, \ j \neq i, \ i + 1,
\end{align*}
\]

This is the braid-permutation group. R. Fenn, R. Rimányi and C. Rourke proved [83], [84] that this group is given by the set of generators: $\{\xi_i, \sigma_i, \ i = 1, 2, ..., n - 1\}$ and relations:

\[
\begin{align*}
\xi_i^2 &= 1, \\
\xi_i \xi_j &= \xi_j \xi_i \quad &\text{if} \quad |i - j| > 1, \\
\xi_i \xi_{i+1} \xi_i &= \xi_{i+1} \xi_i \xi_{i+1}.
\end{align*}
\]

The symmetric group relations

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i \quad &\text{if} \quad |i - j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}.
\end{align*}
\]

The braid group relations

\[
\begin{align*}
\sigma_i \xi_j &= \xi_j \sigma_i \quad &\text{if} \quad |i - j| > 1, \\
\xi_i \xi_{i+1} \sigma_i &= \sigma_{i+1} \xi_i \xi_{i+1}, \\
\sigma_i \sigma_{i+1} \xi_i &= \xi_{i+1} \sigma_i \sigma_{i+1}.
\end{align*}
\]

The mixed relations

R. Fenn, R. Rimányi and C. Rourke also gave a geometric interpretation of $BP_n$ as a group of welded braids. At first they defined a welded braid diagram on $n$ strings as a collection of $n$ monotone arcs starting from $n$ points at a horizontal line of a plane (the top of the diagram) and going down to $n$ points at another horizontal line (the bottom of the diagram). The diagrams may have crossings of two types: 1) the same as usual braids as for example on the Figure 3.2 or 2) welds as depicted in Figure 7.3.

Composition of welded braid diagrams on $n$ strings is defined by stacking one diagram under the other. The diagram with no crossings or welds is the identity with respect to composition. So the set of welded braid diagrams on $n$ strings forms a semigroup which is denoted by $WD_n$.

R. Fenn, R. Rimányi and C. Rourke defined the allowable moves on welded braid diagrams. They consist of usual Reidemeister moves (Figure 3.4) and the specific moves depicted of Figures 7.4, 7.5, 7.6. The automorphisms of $F_n$ which lie in $BP_n$ can be characterized as
follows. Let $\pi \in \Sigma_n$ be a permutation and $w_i, \ i = 1, 2, \ldots, n$, be words in $F_n$. Then the mapping

$$x_i \mapsto w_i^{-1} x_{\pi(i)} w_i$$

determines an injective endomorphism of $F_n$. If it is also surjective, we call it an automorphism of permutation-conjugacy type. The automorphisms of this type comprise a subgroup of $\operatorname{Aut} F_n$ which is precisely $BP_n$.

The Baez–Birman monoid $SB_n$ or singular braid monoid [10], [33] is defined as the monoid with generators $g_i, g_i^{-1}, a_i, i = 1, \ldots, n - 1$, and relations

$$g_i g_j = g_j g_i, \text{ if } |i - j| > 1,$$
Geometrically the generators are defined by

\[ a_i a_j = a_j a_i, \text{ if } |i - j| > 1, \]
\[ a_i g_j = g_j a_i, \text{ if } |i - j| \neq 1, \]
\[ g_i g_{i+1} g_i = g_i g_{i+1} g_{i+1}, \]
\[ g_i g_{i+1} a_i = a_{i+1} g_i g_{i+1}, \]
\[ g_{i+1} g_i a_{i+1} = a_{i+1} g_i g_i, \]
\[ g_i g_i^{-1} = g_i^{-1} g_i = 1. \]

In pictures \( g_i \) corresponds to canonical generator of the braid group and \( a_i \) represents an intersection of the \( i \)th and \((i + 1)\)st strand as in Figure 7.7. More detailed geometric interpretation of the Baez–Birman monoid can be found in the article of J. Birman [33]. R. Fenn, E. Keyman and C. Rourke proved [32] that the Baez–Birman monoid embeds in a group \( SG_n \) which they called the singular braid group:

\[ SB_n \to SG_n. \]

So, in \( SG_n \) the elements \( a_i \) become invertible and all relations of \( SB_n \) remain true.

The analogue of the Birman-Ko-Lee presentation for the singular braid monoid was obtained in [199]. Namely, it was proved that the monoid \( SB_n \) has a presentation with generators \( a_{ts}, a_{ts}^{-1} \) for \( 1 \leq s < t \leq n \) and \( b_{pq} \) for \( 1 \leq p < q \leq n \) and relations

\[
\begin{align*}
    a_{ts} a_{rq} &= a_{rq} a_{ts} \quad \text{for } (t - r)(t - q)(s - r)(s - q) > 0, \\
    a_{ts} a_{sr} &= a_{tr} a_{ts} = a_{sr} a_{tr} \quad \text{for } 1 \leq r < s < t \leq n, \\
    a_{ts} a_{ts}^{-1} &= a_{ts}^{-1} a_{ts} = 1 \quad \text{for } 1 \leq s < t \leq n, \\
    a_{ts} b_{rq} &= b_{rq} a_{ts} \quad \text{for } (t - r)(t - q)(s - r)(s - q) > 0, \\
    a_{ts} b_{ts} &= b_{ts} a_{ts} \quad \text{for } 1 \leq s < t \leq n, \\
    a_{ts} b_{sr} &= b_{tr} a_{ts} \quad \text{for } 1 \leq r < s < t \leq n, \\
    a_{sr} b_{tr} &= b_{ts} a_{sr} \quad \text{for } 1 \leq r < s < t \leq n, \\
    a_{tr} b_{ts} &= b_{sr} a_{tr} \quad \text{for } 1 \leq r < s < t \leq n, \\
    b_{ts} b_{rq} &= b_{rq} b_{ts} \quad \text{for } (t - r)(t - q)(s - r)(s - q) > 0.
\end{align*}
\]

The elements \( a_{ts} \) are defined the same way as in [3.10] and the elements \( b_{pq} \) for \( 1 \leq p < q \leq n \) are defined by

\[ b_{qp} = (\sigma_q^{-1} \sigma_{q-2} \cdots \sigma_{p+1}) x_p (\sigma_{p+1}^{-1} \cdots \sigma_{q-2}^{-1} \sigma_{q-1}^{-1}) \quad \text{for } 1 \leq p < q \leq n. \]

Geometrically the generators \( b_{s,t} \) are depicted in Figure 7.8.
8. Homological Properties

8.1. Configuration spaces and $K(\pi, 1)$-spaces. Let $(q_i)_{i \in \mathbb{N}}$ be a fixed sequence of distinct points in the manifold $M$ and put $Q_m = \{q_1, \ldots, q_m\}$. We use $Q_{m,l} = (q_{l+1}, \ldots, q_{l+m}) \in F(M \setminus Q_l, m)$ as the standard base point of the space $F(M \setminus Q_l, m)$. If $k < m$ we define the projection \( \text{proj} : F(M \setminus Q_l, m) \to F(M \setminus Q_l, k) \) by the formula: \( \text{proj}(p_1, \ldots, p_m) = (p_1, \ldots, p_k) \).

The following theorems were proved by E. Fadell and L. Neuwirth [80].

**Theorem 8.1.** The triple \( \text{proj} : F(M \setminus Q_l, m) \to F(M \setminus Q_l, k) \) is a locally trivial fiber bundle with fibre \( \text{proj}^{-1}Q_{k,l} \) homeomorphic to \( F(M \setminus Q_{k+l}, m-k) \).

Consideration of the sequence of fibrations

\[
F(M \setminus Q_{m-1}, 1) \to F(M \setminus Q_{m-2}, 2) \to M \setminus Q_{m-2},
F(M \setminus Q_{m-2}, 2) \to F(M \setminus Q_{m-3}, 3) \to M \setminus Q_{m-3},
\]

leads to the following theorem.

**Theorem 8.2.** For any manifold $M$

\[
\pi_i(F(M \setminus Q_1, m - 1)) = \bigoplus_{k=1}^{m-1} \pi_i(M \setminus Q_k)
\]

for $i \geq 2$. If \( \text{proj} : F(M, m) \to M \) admits a section then

\[
\pi_i(F(M, m)) = \bigoplus_{k=0}^{m-1} \pi_i(M \setminus Q_k), \ i \geq 2.
\]

**Corollary 8.1.** If $M$ is the Euclidean $r$-space, then

\[
\pi_i(F(M, m)) = \bigoplus_{k=0}^{m-1} \pi_i(S^{r-1}_k \vee \cdots \vee S^{r-1}_k), \ i \geq 2.
\]

**Corollary 8.2.** If $M$ is the Euclidean 2-space, then $F(\mathbb{R}^2, m)$ is the $K(P_m, 1)$-space and $B(\mathbb{R}^2, m)$ is the $K(Br_m, 1)$-space.

Let $X_W$ be the space defined in Subsection 7.2.
Theorem 8.3. The universal covering of $X_W$ is contractible, and so $X_W$ is a $K(\pi; 1)$-space.

This theorem for the groups of types $C_n$, $G_2$ and $I_2(p)$, was proved by E. Brieskorn \[42\] in much the same way as E. Fadell and L. Neuwirth \[80\] proved Theorems 8.1, 8.2 and Corollary 8.2. For the groups of types $D_n$ and $F_4$ E. Brieskorn used this method with minor modifications. In general case Theorem 8.3 was proved by P. Deligne \[73\].

It follows from Theorem 8.2 that $F(\mathbb{C} \setminus Q_g, n)$ and $B(\mathbb{C} \setminus Q_g)$ are $K(\pi, 1)$-spaces, $\pi_1 B(\mathbb{C} \setminus Q_g) = Br_n^g$, so, $B(\mathbb{C} \setminus Q_g)$ can be considered as the classifying space of $Br_n^g$.

8.2. Cohomology of pure braid groups. Cohomology of pure braid groups were first calculated by V. I. Arnold \[3\]. The map

$$\phi : S^{n-1} \rightarrow F(\mathbb{R}^n, 2),$$

described by the formula $\phi(x) = (x, -x)$, is a $\Sigma_2$-equivariant homotopy equivalence. Denote by $A$ the generator of $H^{n-1}(F(\mathbb{R}^n, 2), \mathbb{Z})$ that is mapped by $\phi^*$ to the standard generator of $H^{n-1}(S^{n-1}, \mathbb{Z})$. For $i$ and $j$, such that $1 \leq i, j \leq m$, $i \neq j$, specify $\pi_{i,j} : F(\mathbb{R}^n, m) \rightarrow F(\mathbb{R}^n, 2)$ by the formula $\pi_{i,j}(p_1, ..., p_m) = (p_i, p_j)$. Put

$$A_{i,j} = \pi_{i,j}^*(A) \in H^{n-1}(F(\mathbb{R}^n, m), \mathbb{Z}).$$

It follows that $A_{i,j} = (-1)^nA_{j,i}$ and $A_{i,j}^2 = 0$. For $w \in \Sigma_m$ there is an action $w(A_{i,j}) = A_{w^{-1}(i), w^{-1}(j)}$, since $\pi_{i,j}w = \pi_{w^{-1}(i), w^{-1}(j)}$. Note also that under restriction to

$$F(\mathbb{R}^n \setminus Q_k, m - k) \cong \pi^{-1}(Q_k) \subset F(\mathbb{R}^n, m),$$

the classes $A_{i,j}$ with $1 \leq i, j \leq k$ go to zero since in this case the map $\pi_{i,j}$ is constant on $\pi^{-1}(Q_k)$.

Theorem 8.4. The cohomology group $H^*(F(\mathbb{R}^n \setminus Q_k, m - k), \mathbb{Z})$ is the free Abelian group with generators

$$A_{i_1,j_1}A_{i_2,j_2}...A_{i_s,j_s},$$

where $k < j_1 < j_2 < ... < j_s \leq m$ and $i_r < j_r$ for $r = 1, ..., s$.

The multiplicative structure and the $\Sigma_m$-algebra structure of $H^*(F(\mathbb{R}^n, m), \mathbb{Z})$ are given by the following theorem which is proved using the $\Sigma_3$-action on $H^*(F(\mathbb{R}^n, 3), \mathbb{Z})$.

Theorem 8.5. The cohomology ring $H^*(F(\mathbb{R}^n, m), \mathbb{Z})$ is multiplicatively generated by the square-zero elements

$$A_{i,j} \in H^{n-1}(F(\mathbb{R}^n, m), \mathbb{Z}), 1 \leq i < j \leq m,$$

subject only to the relations

$$(8.1) \quad A_{i,k}A_{j,k} = A_{i,j}A_{j,k} - A_{i,j}A_{i,k} \text{ for } i < j < k.$$

The Poincaré series for $F(\mathbb{R}^n, m)$ is the product $\prod_{j=1}^{m-1}(1 + jt^{n-1})$.

Remark 8.1. In the case of $\mathbb{R}^2 = \mathbb{C}$ the cohomology classes $A_{j,k}$ can be interpreted as the classes of cohomology of differential forms

$$\omega_{j,k} = \frac{1}{2\pi i} \frac{dz_j - dz_k}{z_j - z_k}.$$

E. Brieskorn calculated the cohomology of pure generalized braid groups \[42\] using ideas of V. I. Arnold for the classical case. Let $\mathcal{V}$ be a finite-dimensional complex vector space and $H_j \in \mathcal{V}$, $j \in I$ be the finite family of complex affine hyperplanes given by linear forms $l_j$. E. Brieskorn proved the following fact.
Theorem 8.6. The cohomology classes, corresponding to the holomorphic differential forms
\[ \omega_j = \frac{1}{2\pi i} \frac{dl_j}{l_j}, \]
generate the cohomology ring \( H^*(V \setminus \cup_{j \in I} H_j, \mathbb{Z}) \). Moreover, this ring is isomorphic to the \( \mathbb{Z} \)-subalgebra generated by the forms \( \omega_j \) in the algebra of meromorphic forms on \( V \).

The cohomologies of pure generalized braid groups are described as follows.

Theorem 8.7. (i) The cohomology group \( H_k(P(W), \mathbb{Z}) \) of the pure braid group \( P(W) \) with integer coefficients is a free Abelian group, and its rank is equal to the number of elements \( w \in W \) of length \( l(w) = k \), where \( l \) is the length considered with respect to the system of generators consisting of all reflections of \( W \).

(ii) The Poincaré series for \( H^*(P(W), \mathbb{Z}) \) is the product \( \prod_{j=1}^n (1 + m_j t) \), where \( m_j \) are the exponents of the group \( W \).

(iii) The multiplicative structure of \( H^*(P(W), \mathbb{Z}) \) coincides with the structure of algebra, generated by 1-forms described in the previous theorem.

8.3. Homology of braid groups. To study the cohomologies of the classical braid groups \( H^*(Br_n, \mathbb{Z}) \), V. I. Arnold [5] interpreted the space \( K(Br_n, 1) \cong B(\mathbb{R}^2, n) \) as the space of monic complex polynomials of degree \( n \) without multiple roots
\[ P_n(t) = t^n + z_1 t^{n-1} + ... + z_{n-1} t + z_n. \]
Using this idea he proved Theorems of finiteness, of recurrence and of stabilization. Homology with coefficients in \( \mathbb{Z}/2 \) were calculated by D. B. Fuks in the following theorems [90].

Theorem 8.8. The homology of the braid group on the infinite number of strings with coefficients in \( \mathbb{Z}/2 \) as a Hopf algebra is isomorphic to the polynomial algebra on infinitely many generators \( a_i, i = 1, 2, ... \); \( \deg a_i = 2^i - 1 \):
\[ H_*(Br_\infty, \mathbb{Z}/2) \cong \mathbb{Z}/2[a_1, a_2, ..., a_i, ...] \]
with the coproduct given by the formula:
\[ \Delta(a_i) = 1 \otimes a_i + a_i \otimes 1. \]

Theorem 8.9. The canonical inclusion \( Br_n \to Br_\infty \) induces a monomorphism in homology with coefficients in \( \mathbb{Z}/2 \). Its image is the subcoalgebra of the polynomial algebra \( \mathbb{Z}/2[a_1, a_2, ..., a_i, ...] \) with \( \mathbb{Z}/2 \)-basis consisting of monomials
\[ a_1^{k_1} ... a_i^{k_i} \] such that \( \sum_i k_i 2^i \leq n \).

Theorem 8.10. The canonical homomorphism \( Br_n \to BO_n, 1 \leq n \leq \infty \) induces a monomorphism (of Hopf algebras if \( n = \infty \))
\[ H_*(Br_n, \mathbb{Z}/2) \to H_*(BO_n, \mathbb{Z}/2). \]

F. R. Cohen calculated the homology of braid groups with coefficients \( \mathbb{Z}/p, p > 2 \) also as modules over the Steenrod algebra [54], [55], [56].
Later V. V. Goryunov [108], [109] applied the methods of Fuks and expressed the cohomologies of the generalized braid groups of types \( C \) and \( D \) in terms of the cohomologies of the classical braid groups.
9. CONNECTIONS WITH THE OTHER DOMAINS

9.1. Markov Theorem. Suppose a braid depicted in Figure 3.1 is placed in a cube. On the boundary of the cube join the point \(A_i\) to the point \(B_i\) by a mutually disjoint simple arc \(D_i\). Since our initial braid does not intersect the boundary of the cube except at the points \(A_1, \ldots, A_n\) and \(B_1, \ldots, B_n\) we obtain a link (or, in particular, a knot) i.e. a system of simple closed curves in \(\mathbb{R}^3\). A link obtained in such a manner is called the closure of the braid, see Figure 9.1.

Theorem 9.1. (J. W. Alexander) Any link can be represented by a closed braid.

The next step is to understand equivalence classes of braids which correspond to links. The following Markov Theorem gives an answer to this question. At first we define two types of Markov moves for braids.

**Type 1 Markov move** replaces a braid \(\beta\) on \(n\) strings by its conjugate \(\gamma \beta \gamma^{-1}\).

**Type 2 Markov move** replaces a braid \(\beta\) on \(n\) strings by the braid \(j_n(\beta)\sigma_n\) on \(n + 1\) strings or by \(j_n(\beta)\sigma_{n}^{-1}\) where \(j_n\) is the canonical inclusion of the group \(Br_n\) into the group \(Br_{n+1}\) (see Subsection 3.1)

\[
j_n : Br_n \rightarrow Br_{n+1}.
\]

**Theorem 9.2.** (A. A. Markov) Suppose that \(\beta\) and \(\beta'\) are two braids (not necessary with the same number of strings). Then, the closures of \(\beta\) and \(\beta'\) represent the same link if and only if \(\beta\) can be transformed into \(\beta'\) by means of a finite number of type 1 and type 2 Markov moves. Namely there exists the following sequence,

\[
\beta = \beta_0 \rightarrow \beta_1 \rightarrow \cdots \rightarrow \beta_m = \beta',
\]

such that, for \(i = 0, 1, \ldots, m - 1\), \(\beta_{i+1}\) is obtained from \(\beta_i\) by the application of a type 1 or 2 Markov moves or their inverses.

In other words, if we consider the disjoint union of all braid groups

\[
\prod_{n=1}^{\infty} Br_n,
\]
then the Markov moves of types 1 and 2 define the equivalence relation on this set \( \sim \) such that the quotient set

\[
\prod_{n=1}^{\infty} Br_n/ \sim
\]

is in one-to-one correspondence with isotopy classes of links.

There exist a lot of proofs of Markov Theorem, see for example the work of P. Traczyk [186].

9.2. **Homotopy groups of spheres and Makanin braids.** Consider the coordinate projections for the spaces \( F(M, m) \) where \( M \) is a manifold (see Subsection 7.1)

\[
d_i : F(M, n + 1) \to F(M, n), \quad i = 0, \ldots, n,
\]

defined by the formula

\[
d_i(p_1, \ldots, p_{i+1}, \ldots, p_{n+1}) = (p_1, \ldots, \hat{p}_{i+1}, \ldots, p_{n+1}).
\]

By taking the fundamental group the maps \( d_i \) induces group homomorphisms

\[
d_{i*} : P_{m+1}(M) \to P_m(M), \quad i = 0, \ldots, n.
\]

A braid \( \beta \in Br_{n+1} \) is called **Makanin** (smooth in the terminology of D. L. Johnson [120], Brunnian in the terminology of J. A. Berrick, F. R. Cohen, Y. L. Wong and J. Wu [22]) if \( d_i(\beta) = 1 \) for all \( 0 \leq i \leq n \). We call them Makanin, because up to our knowledge it was G. S. Makanin who first mentioned them [127], page 78, question 6.23. In other words the group of Makanin braids \( Mak_{n+1}(M) \) is given by the formula

\[
Mak_{n+1}(M) = \cap_{i=0}^{n} \ker(d_{i*} : P_{m+1}(M) \to P_m(M)).
\]

The canonical embedding of the open disc \( D^2 \) into the sphere \( S^2 \)

\[
f : D^2 \to S^2
\]

induces a group homomorphism

\[
f_* : Mak_n(D^2) \to Mak_n(S^2)
\]

where \( Mak_n(D^2) \) is the Makanin subgroup \( Mak_n \) of the classical braid group \( Br_n \). The group \( Mak_n \) is free [112, 120]. The following theorem is proved in [22].

**Theorem 9.3.** The is an exact sequence of groups

\[
1 \to Mak_{n+1}(S^2) \to Mak_n(D^2) \to Mak_n(S^2) \to \pi_{n-1}(S^2) \to 1
\]

for \( n \geq 5 \).

Here as usual \( \pi_k(S^2) \) denote the \( k \)-th homotopy group of the sphere \( S^2 \).

For instance, \( Mak_5(S^2) \) modulo \( Mak_5 \) is \( \pi_4(S^2) = \mathbb{Z}/2 \). The other homotopy groups of \( S^2 \) are as follows

\[
\pi_5(S^2) = \mathbb{Z}/2, \quad \pi_6(S^2) = \mathbb{Z}/12, \quad \pi_7(S^2) = \mathbb{Z}/2, \quad \pi_8(S^2) = \mathbb{Z}/2, \ldots
\]

Thus, up to certain range, \( Mak_n(S^2) \) modulo \( Mak_n \) are known by nontrivial calculation of \( \pi_*(S^2) \).
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