CUBULATING MAPPING TORI OF POLYNOMIAL GROWTH FREE GROUP AUTOMORPHISMS

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ABSTRACT. Let $\Phi : F \to F$ be a polynomially-growing automorphism of a finite-rank free group $F$. Then $G = F \rtimes \mathbb{Z}$ acts freely on a CAT(0) cube complex.

1. Introduction

The goal of this paper is to prove:

**Theorem A.** Let $F$ be a finite-rank free group and let $\Phi : F \to F$ be a polynomially-growing automorphism. Then $G = F \rtimes \mathbb{Z}$ acts freely on a CAT(0) cube complex.

We emphasize that the action of $G$ on the cube complex above has exotic properties: it is not in general metrically proper, and the cube complex is not in general locally finite or finite-dimensional.

Theorem A applies to natural examples coming from a diverse and well-studied class of groups, mapping tori of free-group automorphisms. There has been a wave of recent interest in this topic, whose broad appeal is illustrated by the many recent results; see e.g. [AKHR15, AKR15, BBC10, BMMV06, BG10, CL14, Cla15, CPI10, DR10, DKL13, DKL15, Kap14, DT16, KLI15, Lev09, Lus14, Mac00, Mac02, Rey10, Sch08].

Relative train track maps provide a fundamental tool for studying free-by-cyclic groups (see [BH92, BFH00, BFH05]) and classify them according to the dynamics of the monodromy. After replacing $\Phi$ with a power (thus replacing $G$ with a finite-index subgroup), $\Phi$ is represented by a so-called improved relative train track map in the sense of [BFH00]. In this paper, we focus on groups of the form $G = F \rtimes \mathbb{Z}$ where $\Phi$ is polynomially growing in the sense that there exists a polynomial $h$ so that $|\Phi^n(f)| \leq h(n)|f|$ for each $f \in F$. We call such a $G$ a polynomial free-by-$\mathbb{Z}$ group. The nature of the improved relative train track in the polynomial case shows that $G$ splits as an iterated HNN extension with cyclic edge groups at each stage. Specifically, $G = G_n$, where $G_{i+1} = \langle G_i, e_i \mid e_i t e_i^{-1} = p_i t \rangle$ where $p_i \in G_i$. This sequence can be arranged to terminate with $G_0 = \langle t \rangle$.

This sequence of splittings has been used to prove various results illuminating the geometry of polynomial free-by-$\mathbb{Z}$ groups. For example, in [Mac02], Macura studied the divergence function of these groups, a helpful quasi-isometry invariant, showing that the divergence of $G$ is $\sim x^{r+1}$, where
r is the order of the (polynomial) growth of Φ. In fact, it is implicit in \textit{Mac02} that G is \textit{thick} in the sense of \textit{BDM09} (this is made explicit in \textit{BH18} using the same splittings).

Several other theorems on the geometry of free-by-\mathbb{Z} groups illustrate the presence or absence of various nonpositive curvature properties. A free-by-\mathbb{Z} group is hyperbolic when it contains no \mathbb{Z}^2 subgroup, i.e. when F has no Φ–periodic nontrivial conjugacy class \textit{BF92} \textit{Bri00}. However, in the polynomial case, G is never hyperbolic (unless G \cong \mathbb{Z} which happens in the degenerate case where F is trivial). In fact, polynomial free-by-\mathbb{Z} groups need not be CAT(0), or even admit a proper semisimple action on a CAT(0) space, as shown by the following example of Gersten \textit{Ger94}:

\[ \langle a, b, c, t \mid a^t = a, b^t = ba, c^t = ca^2 \rangle \]

Nonetheless, polynomial free-by-\mathbb{Z} groups enjoy properties reminiscent of nonpositive curvature: Macura showed that they satisfy a quadratic isoperimetric inequality \textit{Mac00}, which was generalized by Bridson-Groves to all free-by-\mathbb{Z} groups \textit{BG10}. Both results use improved relative train track maps.

The story of free-by-\mathbb{Z} groups acting on CAT(0) cube complexes is arduous. If G = F \rtimes_Φ \mathbb{Z} is hyperbolic then G acts freely and cocompactly on a CAT(0) cube complex \textit{HW13} \textit{HW14}. When F has rank 2, the automorphism is represented by a surface group automorphism. In this situation, the mapping torus is either a hyperbolic manifold or a graph manifold with a single block which is virtually a product, so the fundamental group is virtually cocompactly cubulated (the hyperbolic case is done in \textit{Wis11}). Moreover, Button-Kropholler \textit{BK15} have gone further and shown directly that G is the fundamental group of a compact nonpositively curved 2-dimensional cube complex. However, in view of Haglund’s semisimplicity theorem \textit{Hag07}, Gersten’s polynomial free-by-\mathbb{Z} group cannot act freely on a proper CAT(0) cube complex.

In addition to being free-by-\mathbb{Z}, Gersten’s group is \textit{tubular}, i.e. it splits as a graph of groups with \mathbb{Z}^2 vertex groups and cyclic edge groups. There are many tubular groups that act freely on CAT(0) cube complexes, even though they cannot act metrically properly. A necessary and sufficient condition for acting freely was provided in \textit{Wis14}, and in particular, Gersten’s group satisfies this condition. Recently, Button characterized the free-by-\mathbb{Z} groups that are tubular, and applied the condition from \textit{Wis14} to see that every tubular free-by-\mathbb{Z} group acts freely on a CAT(0) cube complex \textit{But15}. Since tubular groups are thick of order ≤ 1 relative to the \mathbb{Z}^2 vertex groups, results of \textit{Mac02} \textit{BD14} show that if G = F \rtimes_Φ \mathbb{Z} is tubular, then Φ is linearly-growing.

Theorem A is motivated by many of the above results. Most directly, it generalizes the tubular/linear-growth case to arbitrary polynomial free-by-\mathbb{Z} groups and also provides a non-hyperbolic counterpart to the cubulation of hyperbolic free-by-\mathbb{Z} groups. It also broadens the agenda of \textit{Wis14}: instead of studying a single splitting of a group as a graph of groups with cyclic edge groups, we are instead studying a sequence of iterated cyclic HNN extensions, where the edge groups at each stage can involve stable letters from previous stages. Thus, although the initial framework of the proof of Theorem A is superficially similar to that used for tubular groups, our situation is significantly more complicated.

The main object in this paper is a space D consisting of a nonpositively curved cube complex with a cylinder attached along its boundary circles, which we may assume are local geodesics. The fundamental group of D is G_n, and the cube complex has fundamental group G_{n−1}. Attaching the cylinder performs the HNN extension.

The results of this paper contribute to the problem of cubulating a cyclic HNN extension of a cubulated group. When D is word-hyperbolic, this problem has a positive solution \textit{HW15}. However, this is not the case in general. For instance, the Baumslag-Solitar group BS(1, 2) \cong...
\[ e, a \mid eae^{-1} = a^2 \] cannot act freely on a \( \text{CAT}(0) \) cube complex \cite{Hag07}, and even assuming that there are no bad Baumslag-Solitar subgroups, there is even an example of an HNN extension of a nonpositively curved square complex with a special double cover which cannot act freely on a \( \text{CAT}(0) \) cube complex \cite{HW15} Example 8.7.

### 1.1. Summary of the proof.

Let \( G = F \ltimes \Phi \langle t \rangle \). We first reduce to the case where \( \Phi \) is represented by an improved relative train track map enabling us to decompose \( G \) as an iterated HNN extension \( G = G_n \cong \langle G_{n-1}, e_n \mid e_n t e_n^{-1} = p_n t \rangle \), where \( G_{n-1} \) is a polynomial free-by-cyclic group and \( p_n, t \in G_{n-1} \) (see Remark \[2.4\]). The stable letter \( e_n \) comes from the top stratum of the relative train track representative. We induct on \( n \).

The inductive hypothesis is not simply that \( G_{n-1} \) acts freely on a \( \text{CAT}(0) \) cube complex, but that \( G_{n-1} \) acts freely on a \( \text{CAT}(0) \) cube complex \( C_0^{n-1} \times C_1^{n-1} \), with the following properties. First, each element of \( G_n \), including \( t \), that will generate an edge-group in the future splitting of \( G_n \) acts with translation length 0 on \( C_2 \), while each of these elements have the same, positive, translation length on \( C_0^{n-1} \). In the base case, this is easily satisfied by taking \( C_0^{0} \) to be a line with \( \langle t \rangle \) acting by translations, and taking \( C_0^{0} \) to be a point.

In the inductive step, we attach cylinders to the quotient \( G^n \backslash C_0^{n-1} \), forming a multiple HNN extension conjugating \( p_i t \) to \( t \) for each \( p_i \) that already appears in \( G_{n-1} \). We use the equality of the translation lengths of the attaching circles to extend the hyperplanes of \( C_0^{n-1} \) to walls in \( G_n \). Using cubical small-cancellation theory, and discarding some walls, we modify this wallspace structure to get a new cube complex \( C_0^n \), equipped with a \( G_0^n \)-action where \( t \) fixes a point and each element of \( G^n \) that could generate an edge-group of a future splitting (these elements are necessarily hyperbolic on the Bass-Serre tree) also has translation length 0.

We also construct \( C_0^n \), with a \( G_n \)-action in which the above hyperbolic elements, and \( t \), have the same positive translation length. The idea is roughly to produce a virtual surjection from \( G_n \) to \( H \times \langle t \rangle \), where \( H \) is a free group, such that all future edge-groups get sent to elements of the form \( pt^d \), for some fixed \( d > 0 \).

We then apply Lemma \[3.4\], which provides a (non-free) action of \( H \times \langle t \rangle \) on a \( \text{CAT}(0) \) cube complex where each \( (p, t) \) acts freely and each \( pt^d \) has the same (positive) translation length as \( pt^d \). This lemma is of independent interest, and relies on canonical completions for maps of graphs. We thus have a virtual action of \( G_n \) on this cube complex, which we promote to an action of \( G_n \), with the desired translation lengths, in a standard way.

### 1.2. Problems.

We conclude with problems suggested by the proof of Theorem A.

#### 1.2.1. Metrically proper actions.

As we are dealing with \( \text{CAT}(0) \) cube complexes that may not be locally finite or finite-dimensional, we must distinguish between free actions and metrically proper actions. In view of Theorem A, it is reasonable to ask:

**Problem 1.** Which polynomial free-by-cyclic groups act metrically properly on \( \text{CAT}(0) \) cube complexes, and, slightly more strongly, which such groups act freely on proper \( \text{CAT}(0) \) cube complexes? For which polynomially-growing \( \Phi \) can one produce a free action of \( G \) on a finite-dimensional \( \text{CAT}(0) \) cube complex? Additionally, we wonder if there is a characterization of the polynomially-growing \( \Phi \) for which \( F \ltimes \Phi \mathbb{Z} \) (virtually) acts freely and cocompactly on a \( \text{CAT}(0) \) cube complex.

For tubular groups, a criterion for finite-dimensional cubulation was given by Woodhouse \cite{Woo15}, which is part of a proof that tubular groups acting freely on locally finite \( \text{CAT}(0) \) cube complexes
are virtually special \cite{Woo16a}. We wonder about the implications between the following statements for a polynomial free-by-\( \mathbb{Z} \) group \( G \):

- \( G \) acts metrically properly on a CAT(0) cube complex;
- \( G \) acts freely on a locally finite CAT(0) cube complex;
- \( G \) acts freely on a uniformly locally finite CAT(0) cube complex;
- \( G \) is virtually special.

The class of tubular free-by-\( \mathbb{Z} \) groups shows that among \( G \) with \( \Phi \) linearly-growing one can already find examples where no metrically proper action exists. Accordingly, an answer to Problem 1 must involve more subtle properties of \( \Phi \) than the growth. We suspect it has to do with the nature of the splitting of \( G \) as an iterated cyclic HNN extension \( G_i = \langle G_{i-1}, e_i | e_{i+1}t e_i^{-1} = f_i t \rangle \) for \( 0 \leq i \leq n \), where each \( f_i \in G_i \) involves the stable letter in the splitting of \( G_i-1 \), we are in a “hyperbolic-like” situation where it is possible that there is an easier cubulation, possibly giving rise to a nicer cube complex and a metrically proper action.

The cubulations should be forced to be nasty if the edge-groups \( \langle p_i t \rangle \) appear early in the hierarchy relative to \( i \). Accordingly, consider the following situation: \( G_0 \cong F_0 \times \langle t \rangle \) and \( G_{i+1} = \langle G_i, e_{i+1} | e_{i+1}t e_i^{-1} = f_i t \rangle \) for \( 0 \leq i \leq n \), where each \( f_i \in F_0 \). For the purposes of obtaining a metrically proper action of \( G = G_n \) on a cube complex, is this in some sense a worst-case scenario? For which choices of \( \{ f_i \} \) can one produce such an action?

1.2.2. Arbitrary mapping tori. Finally, the methodology used here is totally unrelated to the technique used to cubulate mapping tori of hyperbolic free group automorphisms in \cite{HW14} \cite{HW13}. However, we hope the methods will eventually be combined to resolve:

**Problem 2.** Show that every free-by-\( \mathbb{Z} \) group acts freely on a CAT(0) cube complex.

We do not expect that a free action on a locally infinite CAT(0) cube complex will shed a large amount of light on the structure of the free-by-\( \mathbb{Z} \) group, although there are some interesting consequences such as undistortedness of maximal finitely generated free abelian subgroups \cite{Woo16b}. It is nonetheless intriguing that groups in this natural class admit such actions.

**Organization of the paper.** Section 2 contains background on relative train track maps, actions on CAT(0) cube complexes, cubical small-cancellation theory, and a useful lemma about actions on trees with \( \mathbb{Z} \) edge-stabilizers. In Section 3 we construct the required cubical action of \( H \times \mathbb{Z} \), and the main lemmas used in the inductive step — i.e. the constructions of \( C^0_n \) and \( C^m_t \) — are proved in Section 4. Theorem 1 is finally proved in Section 5.

**Acknowledgment.** We thank the referee for numerous useful corrections.

2. Background

2.1. Relative train track maps for polynomial automorphisms. Throughout, \( F \) is a finitely-generated free group and \( \Phi \in \text{Aut}(F) \).

**Lemma 2.1.** Let \( G = F \rtimes_{\Phi} \langle t \rangle \), let \( f \in F \), let \( H = \langle ft \rangle \). Then \( H \leq G \) is separable.

**Proof.** The subgroup \( \langle ft \rangle \) is a retract of \( G \) for each \( f \in F \): the retraction sends each \( h \in F \) to 1 and sends \( t \) to \( ft \). Any retract of a residually finite group is separable: indeed, since \( G \) is residually finite, the profinite topology on \( G \) is Hausdorff, and retracts of Hausdorff spaces are closed. \( \square \)
Definition 2.2 (Polynomial growth). The automorphism $\Phi$ is polynomially growing if for each $f \in F$, the word length $|\Phi^k(f)|$ grows polynomially as $k \to \infty$.

We set up our proof of Theorem A using a result of [BFH00] saying that each element of $\text{Aut}(F)$ has a nonzero power represented by an improved relative train track map, a homotopy equivalence of graphs with numerous useful properties. The full statement can be found in [BFH00, Theorem 5.1.5], but we need just the following:

**Proposition 2.3** (Relative train tracks for polynomially-growing automorphisms). Let $F$ be a finitely-generated free group and let $\Phi : F \to F$ be a polynomially-growing automorphism. Then there exists $k > 0$, a connected graph $\Gamma$, and a homotopy equivalence $f : \Gamma \to \Gamma$, representing $\Phi^k$, with the following properties:

1. There is a sequence of ($f$–invariant) subgraphs $\emptyset = \Gamma_{-1} \subset \Gamma_0 \subset \Gamma_1 \cdots \subset \Gamma_r = \Gamma$ such that $\Gamma_i - \text{Int}(\Gamma_{i-1})$ consists of a single (closed) edge $E_i$ for $1 \leq i \leq r$;
2. $f(E_i) = E_iP_{i-1}$, where $P_{i-1}$ is a (possibly trivial) closed combinatorial path whose edges are in $\Gamma_{i-1}$ and whose basepoint is fixed by $f$. Thus $f(v) = v$ for all $v \in \Gamma^0$.

**Remark 2.4** (The multiple HNN extension). Our application of Proposition 2.3 will be as follows.

**Connecting the $\Gamma_i$:** First, it will be (notationally) convenient to work with connected $\Gamma_i$. Hence let $\hat{\Gamma}$ be obtained from $\Gamma$ by identifying all of the vertices, so that each edge $E$ of $\Gamma$ projects to a loop $\hat{E}$, and $f$ descends to a cellular homotopy equivalence $\bar{f} : \hat{\Gamma} \to \hat{\Gamma}$. Moreover, the image $\bar{\Gamma}_i$ of each $\Gamma_i$ is connected, and $\hat{\Gamma}_i$ is obtained from $\bar{\Gamma}_{i-1}$ by adding $\hat{E}_i$. For each $i$, we have $\bar{f}(\hat{E}_i) = \hat{E}_iP_i$, i.e. $\bar{f}$ satisfies the conclusion of Proposition 2.3.

Let $\Phi^k$ be the automorphism of $\pi_1\hat{\Gamma}$ induced by $\bar{f}$. We claim that $\pi_1\hat{\Gamma} \times_{\Phi^k} \mathbb{Z}$ embeds in $\pi_1\hat{\Gamma} \times_{\Phi^k} \mathbb{Z}$. Indeed, let $\hat{\Gamma}$ be the graph obtained by choosing an ordering $v_1, \ldots, v_s$ of the vertices of $\Gamma$ and adding edges $(v_1, v_2), \ldots, (v_{s-1}, v_s)$. The subgraph of $\hat{\Gamma}$ consisting of the union of the new edges is a path. Extend $f$ to a map $\bar{f} : \hat{\Gamma} \to \hat{\Gamma}$ by declaring that $\bar{f}$ sends each of these new edges to itself identically. Then $\hat{\Gamma}$ is homotopy-equivalent to $\Gamma$ via a deformation retraction collapsing the new edges, and this homotopy takes $\hat{f}$ to $\bar{f}$. On the other hand, $\bar{f}$ restricts on $\Gamma \subset \hat{\Gamma}$ to $f$. Now, for each new edge $e$, the mapping torus of $\hat{f}$ contains an annulus $e \times S^1$; cutting along the core curve of the annulus, for each such $e$, induces a graph of groups decomposition of $\pi_1\hat{\Gamma} \times_{\Phi^k} \mathbb{Z}$ whose edge groups are isomorphic to $\mathbb{Z}$ and whose vertex groups are isomorphic to the fundamental group of the mapping torus of $f$, i.e. $\pi_1\hat{\Gamma} \times_{\Phi^k} \mathbb{Z}$. Hence $\pi_1\hat{\Gamma} \times_{\Phi^k} \mathbb{Z} \leq \pi_1\hat{\Gamma} \times_{\Phi^k} \mathbb{Z}$.

**Why it is sufficient to work with connected $\Gamma_i$:** In the proof of Theorem 1, we will see that it suffices to produce a free action of $\pi_1\hat{\Gamma} \times_{\Phi^k} \mathbb{Z}$ on a $\text{CAT}(0)$ cube complex, where $k > 0$ is the power from Proposition 2.3. Hence it suffices to produce such an action for any supergroup of $\pi_1\hat{\Gamma} \times_{\Phi^k} \mathbb{Z}$, and in particular it is sufficient to work with $\pi_1\hat{\Gamma} \times_{\Phi^k} \mathbb{Z}$.

Moreover, for convenience, we can and shall take $\Gamma_0$ to be a single ($f$–invariant) vertex instead of $\emptyset$ (as in Proposition 2.3).

**The final multiple HNN extension:** Hence we can and shall assume that all relative train track maps in this paper have domains $\Gamma$ with each $\Gamma_i$ connected. Each stratum from the proposition thus yields a splitting as a multiple HNN extension with $\mathbb{Z}$ edge groups, so after replacing $G$ if necessary by a finite-index subgroup (obtained by replacing $\Phi$ by an appropriate positive power), we have a sequence $F_0 \times \langle t \rangle = G_0 \leq G_1 \leq \cdots \leq G_n = G$ of subgroups so that the following holds:

- For $0 \leq i \leq n$, we have $G_i = F_i \times \langle t \rangle$, where $F_i$ is a free group, and $G_0 = F_0 \times \langle t \rangle$. 

For $0 \leq i \leq n - 1$, the free group $F_i$ contains a finite multiset $P_i$ such that, for $1 \leq i \leq n$, the group $G_i$ splits as a graph of groups with $G_{i-1}$ the single vertex group, and $\mathbb{Z}$ edge groups, as follows:

$$G_i = \langle G_{i-1}, e_1^i, \ldots, e_n^i \mid e_j^i t(e_j^i)^{-1} = p_j^i t, \ 1 \leq j \leq n_i \rangle,$$

where $\{p_j^i : 1 \leq j \leq n_i\} = P_i \subset F_{i-1}$.

- For all $i \geq 1$, each $p \in P_{i+1} \subset G_i$ and each $pt$ with $p \in P_{i+1}$ acts as a hyperbolic isometry of the Bass-Serre tree $\mathcal{T}_i$ for the splitting of $G_i$ described above.

Since we have arranged for $\Gamma_0$ to be a single vertex, we can and shall take $F_0 = \{1\}$.

### 2.2. Cubical translation length

Let $C$ be a CAT(0) cube complex. Then $g \in \text{Aut}(C)$ is (combinatorially) elliptic if $g$ fixes a 0–cube and (combinatorially) hyperbolic if there is a bi-infinite combinatorial geodesic $\tilde{A}$ in $C$ so that $g$ stabilizes $\tilde{A}$ in the following way: there exists $\tau \in \mathbb{Z}$ so that, regarding $\tilde{A}$ as an isometric embedding $\tilde{A} : \mathbb{R} \to C^1$, we have $g \tilde{A}(t) = \tilde{A}(t + \tau)$ for all $t \in \mathbb{R}$. By Corollary 5.2 of [Hag07], the translation length $\tau$ is the same for any choice of axis $\tilde{A}$, and in fact $\tau$ is the minimal distance by which $g$ moves a 0–cube. By replacing $C$ by a cubical subdivision – i.e. by subdividing $C$ so that the hyperplanes become subcomplexes and each hyperplane is replaced by 2 parallel hyperplanes – we can assume that any fixed group $G \leq \text{Aut}(C)$ has the property that each $g \in G$ is either elliptic or hyperbolic [Hag07]. Indeed, it is shown in [Hag07] that the action of $G$ on $C$ has this property provided $G$ acts without inversions in hyperplanes in the sense that whenever $g \in G$ stabilizes a hyperplane, it stabilizes each of the associated halfspaces. Passing to the cubical subdivision of $C$ ensures that the action is without inversions.

**Notation 2.5** (Translation length). Let $C$ be a CAT(0) cube complex. If $g \in \text{Aut}(C)$ fixes a point, let $\|g\| = 0$. If $g \in \text{Aut}(C)$ nontrivially stabilizes a combinatorial geodesic $\tilde{A}$ in $C$, then $\|g\|$ denotes the number of $\langle g \rangle$–orbits of 1–cubes in $\tilde{A}$. Note that if $G$ acts on $C$ without inversions, then $\|g\|$ is well-defined for all $g \in G$. Since we will consider actions on multiple cube complexes, we adopt the following convention on subscripts, where it will not cause confusion: when $g$ is acting on $C_\varnothing$, then we use the notation $\|g\|_\varnothing$.

**Observation 2.6.** Suppose that $C_\varnothing$ and $C_\Diamond$ are CAT(0) cube complexes, and $G$ acts on $C_\varnothing$ and $C_\Diamond$. Let $C = C_\varnothing \times C_\Diamond$. Then, with respect to the product action $G \to \text{Aut}(C)$,

$$\|g\|_C = \|g\|_\varnothing + \|g\|_{\Diamond}$$

for all $g \in G$. Moreover, $\|g^d\|_\varnothing = |d| \|g\|_\varnothing$ for all $g \in G$ and $d \in \mathbb{Z}$.

### 2.3. Cutting

We now recall a special case of the notion of a wallspace, introduced in [HP98], and refer the reader to [HW] for a more detailed discussion.

A wallspace $(\hat{X}, \mathcal{W})$ consists of a metric space $\hat{X}$ with a set $\mathcal{W}$ of connected subspaces $\hat{W}$, called walls, so that $\hat{X} \setminus \hat{W}$ has exactly two components, called halfspaces, with the additional property that for all $x, y \in \hat{X}$, there are finitely many $\hat{W} \in \mathcal{W}$ such that $x, y$ lie in different components of $\hat{X} \setminus \hat{W}$ (in which case $\hat{W}$ separates $x, y$). An automorphism of $(\hat{X}, \mathcal{W})$ is an isometry $g : \hat{X} \to \hat{X}$ so that $g\hat{W} \in \mathcal{W}$ for $\hat{W} \in \mathcal{W}$; the group of automorphisms of $(\hat{X}, \mathcal{W})$ is denoted $\text{Aut}(\hat{X}, \mathcal{W})$.

**Definition 2.7** (Cutting, parallel). Let $(\hat{X}, \mathcal{W})$ be a wallspace and let $g \in \text{Aut}(\hat{X}, \mathcal{W})$. Let $A : \mathbb{R} \to \hat{X}$ be an embedding with $g$–invariant image so that $g \circ A : \mathbb{R} \to \hat{X}$ is increasing. Then $\hat{W} \in \mathcal{W}$ cuts $g$ if there exists $x \in \mathbb{R}$ so that $A((x, \infty))$ and $A((-\infty, x))$ lie in different halfspaces associated to $\hat{W}$. If no such $A$ exists, then the set of walls cutting $g$ is empty.
We call \( g, h \in \text{Aut}(\tilde{X}, \mathcal{W}) \) parallel if: each \( W \in \mathcal{W} \) cuts \( g \) if and only if \( W \) cuts \( h \).

The CAT(0) cube complex \( C = C(\tilde{X}, \mathcal{W}) \) dual to the wallspace \((\tilde{X}, \mathcal{W})\) was defined in \cite{Sag95}. Denoting by \( \tilde{W} \) the set of halfspaces, the 0–cubes of \( C \) are maps \( c : \mathcal{W} \to \tilde{W} \) sending each wall to one of the two associated halfspaces, subject to: \( c(\tilde{W}) \cap c(\tilde{W}') \neq \emptyset \) for \( \tilde{W}, \tilde{W}' \in \mathcal{W} \), and for all \( x \in \tilde{X} \), we have \(|\{\tilde{W} : x \not\in c(\tilde{W})\}| < \infty\). The 0–cubes \( c, c' \) are joined by a 1–cube if and only if there is a unique wall \( \tilde{W} \) with \( c(\tilde{W}) \neq c'(\tilde{W}) \), and higher-dimensional cubes are added when their 1–skeleta appear.

If \( G \) acts on \((\tilde{X}, \mathcal{W})\), then there is an induced \( G \)–action on \( C \) defined as follows: for \( g \in G \) and \( c \in C^0 \), we have \((gc)(\tilde{W}) = g \cdot c(g^{-1}\tilde{W})\) for all \( \tilde{W} \in \mathcal{W} \). This action clearly takes hyperplanes to hyperplanes, and there is a \( G \)–equivariant bijection from \( \mathcal{W} \) [respectively \( \tilde{W} \)] to the set of hyperplanes [respectively, halfspaces] in \( C \).

We now state two lemmas, the first of which is \cite[Lemma 2.1]{Wis14}. The second follows by considering hyperplanes intersecting axes in the dual cube complex.

**Lemma 2.8** (Cut-wall criterion). Let \( G \) act on a wallspace \((\tilde{X}, \mathcal{W})\) and suppose that each \( g \in G - \{1_G\} \) is cut by some wall in \( \mathcal{W} \). Then \( G \) acts freely on \( C(\tilde{X}, \mathcal{W}) \).

**Lemma 2.9** (Cut-walls and translation length in \( C(\tilde{X}, \mathcal{W}) \)). Let \( G \) act on the wallspace \((\tilde{X}, \tilde{W})\) and let \( g \in G \). Let \( \|g\|_\mathcal{W} \) be the number of \( \langle g \rangle \)–orbits of walls that cut \( g \). Then the translation length \( \|g\| \) of \( g \) on \( C \) is defined and \( \|g\| = \|g\|_\mathcal{W} \).

We remark that if \( g \in G \) has finite order, then \( g \) cannot be cut by a wall in \( \mathcal{W} \), so \( \|g\|_\mathcal{W} = 0 \). On the other hand, \( g \) must fix a point in \( C \) (see \cite{Hag07}) so \( \|g\| = 0 \). In this paper, the groups under consideration are torsion-free, so this situation only arises for \( g = 1 \).

### 2.4. Cubical small-cancellation theory

We review the following background from \cite{Wis11}. Throughout, \( \tilde{X} \) denotes a CAT(0) cube complex and \( X \) a nonpositively-curved cube complex. For a hyperplane \( \tilde{U} \) of \( \tilde{X} \), we denote by \( N(\tilde{U}) \) its carrier, i.e. the union of all closed cubes intersecting \( \tilde{U} \). We do the same for (immersed) hyperplanes in \( X \). The **systole** \( \|X\| \) is the infimal length of an essential combinatorial closed path in \( X \).

#### 2.4.1. Cubical presentations and pieces.

**Definition 2.10** (Cubical presentation). A **cubical presentation** \( \langle X | \{Y_i : i \in I\} \rangle \) consists of connected non-positively curved cube complexes \( X \) and \( \{Y_i\} \), and local isometries \( \{Y_i \to X : i \in I\} \). We use the notation \( X^* \) for the cubical presentation above. As a topological space, \( X^* \) is \( X \) with a cone on each \( Y_i \).

More background on cubical presentations appears in Section 3 of \cite{Wis11}. Since each \( Y_i \to X \) is a local isometry, it is \( \pi_1 \)–injective, and we have \( \pi_1X^* \cong \pi_1X/\langle \langle \pi_1Y_i : i \in I \rangle \rangle \). For each \( i \in I \), the cone over \( Y_i \) in \( X^* \) lifts to the universal cover \( \tilde{X}^* \), and accordingly \( Y_i \to X \to X^* \) lifts to \( Y_i \to \tilde{X}^* \).

**Definition 2.11** (Piece). A **cone-piece** of \( X^* \) in \( Y_i \) is a component of \( \tilde{Y}_i \cap g\tilde{Y}_j \), where \( g \in \pi_1(X) \), excluding the case where \( i = j \) and \( g \in \text{Stab}(\tilde{Y}_i) \). A **wall-piece** of \( X^* \) in \( Y_i \) is a component of \( \tilde{Y}_i \cap N(\tilde{U}) \), where \( \tilde{U} \) is a hyperplane that is disjoint from \( \tilde{Y}_i \). A **piece-path** in \( Y \) is a path in a piece of \( Y \).
2.4.2. Simplified B(6) condition.

**Definition 2.12** (C'(α) cubical presentation). The cubical presentation \( X^* = \langle X \mid \{Y_i : i \in I\} \rangle \) satisfies the C'(α) **condition** if \( |P| \leq \alpha|S| \) for each geodesic piece-path \( P \) and each essential closed path \( S \to Y_i \) with \( P \) a subpath of \( S \).

The following is a simplified form of the B(6) condition described in [Wis11]; see Definition 5.1 and Remark 5.2.

**Definition 2.13** (Simplified B(6)). Let \( X^* = \langle X \mid \{Y_i : i \in I\} \rangle \) be a cubical presentation satisfying the C'(1/14) condition. Then \( X^* \) satisfies the **B(6) condition** if the following hold:

1. for each \( i \in I \), there is an equivalence relation \( \sim_i \) on the hyperplanes of \( Y_i \).
2. for each \( i \in I \) and each pair of (possibly equal) hyperplanes \( U, V \) of \( Y_i \) with \( U \sim_i V \), the hyperplanes \( U, V \) do not cross or osculate (recall that hyperplanes \( U, V \) osculate if they are disjoint but have intersecting carriers);
3. for each \( i \in I \), and each \( \sim_i \)-class \( [U] \) of hyperplanes, \( W = \cup_{V \in [U]} V \) is a **wall**; the space \( Y_i - W \) consists of two subsurfaces \( \hat{Y}_i, \tilde{Y}_i \) such that \( \text{Cl}(\hat{Y}_i) \cap \text{Cl}(\tilde{Y}_i) = W \);
4. if \( P \to Y_i \) is a path that is the concatenation of at most 7 piece-paths and \( P \) starts and ends on the carrier \( N(U) \) of a wall then \( P \) is path-homotopic into \( N(U) \);
5. \( \text{Aut}(Y_i \to X) \) preserves the wallspace structure.

The walls in \( Y_i \) Definition 2.13 need not be connected, so differ from the walls discussed in Section 2.3.

The B(6) condition provides a wallspace structure on \( \hat{X}^* \) as follows: two hyperplanes of \( \hat{X}^* \) are elementary equivalent if they intersect a lift \( Y_i \to \hat{X}^* \) of some \( Y_i \) in equivalent hyperplanes. The transitive closure of this relation is an equivalence relation on the hyperplanes of \( \hat{X}^* \) and the union of the hyperplanes in each class is a wall. The CAT(0) cube complex dual to this wallspace is the cube complex associated to \( X^* \); observe that \( \pi_1X^* \) acts on this cube complex.

2.4.3. Free action of \( \pi_1X^* \) on the associated cube complex.

**Definition 2.14** (Proximate). Let \( \langle X \mid \{Y_i : i \in I\} \rangle \) be a cubical presentation satisfying Definition 2.13(1), (2), (3). A hyperplane \( U \) in \( Y_i \) is proximate to a 0–cube \( v \) of \( Y_i \) if there is a 1-cube \( u \) dual to \( U \) and a path \( AB \) that starts with \( u \) and ends with \( v \), and where each of \( A \) and \( B \) is either a 1-cube or lies in a piece. A wall \( W \) in \( Y_i \) is proximate to \( v \) if some hyperplane \( U \) of \( W \) is proximate to \( v \).

Given \( \langle X \mid \{Y_i : i \in I\} \rangle \) as in Definition 2.14 a set \( \mathcal{R} \) of hyperplanes of \( X \) is preferred if each \( Y_i \) has the following property: let \( W \) be a wall in \( Y_i \). Then either all hyperplanes of \( W \) map to a hyperplane of \( X \) belonging to \( \mathcal{R} \), or no hyperplane of \( W \) maps to a hyperplane in \( \mathcal{R} \). If \( \mathcal{R} \) is a preferred set of hyperplanes in \( X \), then a wall \( W \) of \( \hat{X}^* \) is preferred (with respect to \( \mathcal{R} \)) if some (and hence any) hyperplane in \( \hat{W} \) maps to a preferred hyperplane in \( X \). If \( \hat{W} \) is not preferred, then none of its constituent hyperplanes maps to a preferred hyperplane in \( X \).

The following is Theorem 5.40 in [Wis11]:

**Theorem 2.15.** Let \( \langle X \mid \{Y_i\} \rangle \) be a cubical presentation. Suppose that:

1. \( X^* \) satisfies the B(6) condition and has short innerpaths in the sense of [Wis11] Definition 3.53;
2. \( X \) contains a preferred set \( \mathcal{R} \) of hyperplanes.
(3) The following holds for each \( Y \in \{ Y_i \} \). Let \( \kappa \to Y \) be a geodesic with endpoints \( p,q \). Let \( w_1, w'_1 \) be hyperplanes of \( Y \) lying in the same wall and mapping under \( Y \to X \) to hyperplanes in \( \mathcal{R} \). Suppose that \( \kappa \) traverses a 1–cube dual to \( w_1 \) and either \( w'_1 \) is proximate to \( q \) or \( w'_1 \) is dual to a 1–cube traversed by \( \kappa \). Then there is a preferred wall \( w_2 \) in \( Y \) that separates \( p,q \) but is not proximate to \( p \) or \( q \).

Let \( g \in \pi_1 X^* \). Then one of the following holds:

1. there exists \( Y \in \{ Y_i \} \) so that \( g \in \text{Aut}(Y) \) for some lift of \( Y \) to \( \tilde{X}^* \);
2. \( g \) is cut by a preferred wall of \( \tilde{X}^* \);
3. \( g \) is the image of some \( \tilde{g} \in \pi_1 X \) that is not cut by a hyperplane of \( \tilde{X} \) mapping to a hyperplane in \( \mathcal{R} \);
4. \( g \in \pi_1 X^* \) has finite order.

In our applications, \( \langle X \mid \{ Y_i \} \rangle \) will satisfy the short innerpaths condition by [Wis11] Lemma 3.67] because of the metric small-cancellation condition. Therefore, it is not necessary to define short innerpaths.

We will apply Theorem 2.15 in the case where each \( Y_i \) is a specific CAT(0) cube complex mapping by a local isometry to \( X \). We also need Theorem 5.20 in [Wis11], i.e.:

**Theorem 2.16.** Let \( X^* \) be a B(6) presentation and let \( W \) be a wall in \( \tilde{X}^* \). If \( H_1, H_2 \) are hyperplanes in \( W \), and \( Y \) is a cone, then \( H_1 \cap Y \) and \( H_2 \cap Y \) lie in the same wall of \( Y \).

2.5. **Graphs of groups with cyclic edge groups.** The following lemma about groups acting on trees is fundamental in the proof of Theorem A.

**Lemma 2.17.** Let \( H \) act without inversions on a tree \( T \). Suppose that for each edge \( e \) of \( T \), the following hold:

1. the edge group \( H_e = \langle h_e \rangle \) is a maximal \( \mathbb{Z} \) subgroup;
2. if \( kh_k^{-1} = h_k^x \) for some \( k \in H, p,q \neq 0 \) then \( kh_k = h_k k \).

Let \( L \) be a combinatorial line in \( T \) with stabilizer \( K \leq H \). Then both of the following hold:

- every element of \( K \) either acts as a translation on \( L \) or fixes \( L \) pointwise;
- one of the following holds:
  - \( K \cong \{ 1 \} \);
  - \( K \cong \mathbb{Z} \) and \( K \) acts by translations on \( L \);
  - each \( k \in K \) centralizes \( h_e \) for each edge \( e \) of \( L \).

**Proof.** Denote by \( \text{Aut}(L) \) the group of combinatorial automorphisms of \( L \), so that the action of \( K \) on \( L \) gives a homomorphism \( K \to \text{Aut}(L) \).

Suppose that \( v \) is a vertex of \( L \) with incident edges \( e,e' \). Let \( H_e = \langle h_e \rangle \) and \( H_{e'} = \langle h_{e'} \rangle \). Suppose \( x \in K \) exchanges \( e,e' \). Then \( x^2 \in \langle h_e \rangle \), but \( x \not\in H_e \), contradicting maximality of the edge groups. Hence, since \( K \) does not stabilize an edge of \( L \) but invert its endpoints, \( \text{Im}(K \to \text{Aut}(L)) \) is trivial or infinite cyclic, which proves the first assertion.

Note that \( \text{Ker}(K \to \text{Aut}(L)) = \cap_{e \in \text{Edges}(L)} H_e \). Suppose that \( k \in K \) and \( h \in \text{Ker}(K \to \text{Aut}(L)) \). Then for each edge \( e \) of \( L \), there exists \( n \neq 0 \) so that \( h = h_e^n \). Since \( k \) acts as a translation on \( L \), we have that \( [k,h] \) stabilizes \( e \), i.e. there exists \( t \in \mathbb{Z} \) so that \( [k,h_e^t] = h_e^t \), i.e. \( kh_k^{h_k^{-1}} = h_k^{t+n} \). Hence \( [k,h_e] = 1 \) by hypothesis (2). Hence either \( k \) centralizes each \( h_e \) or \( \text{Ker}(K \to \text{Aut}(L)) = \{ 1 \} \). \( \square \)
The goal of this section is to prove Lemma 3.4. We need a preparatory definition and lemma.

**Definition 3.1 (Virtual translation length).** Let $G$ be a group and let $G' \leq G$ be a subgroup of index $m < \infty$ acting on a CAT(0) cube complex $C'$. Let $g \in G$. Let $r > 0$ be such that $g^r \in G'$, e.g. $r = m!$. The *virtual translation length* of $g$ is $\|g\|_{C'}^{\text{virt}} = \frac{1}{r} \|g^r\|_{C'}$. Note that this is independent of the choice of $r$.

**Lemma 3.2.** Let $G$ be a group. Let $G' \leq G$ be a finite-index subgroup acting on a CAT(0) cube complex $C'$. There exists a CAT(0) cube complex $C$ so that $G$ acts on $C$ and the following holds for all $a, b \in G$. Suppose that $\|h_1 ah_1^{-1}\|_{C'}^{\text{virt}} = \|h_2 bh_2^{-1}\|_{C'}^{\text{virt}}$ for all $h_1, h_2 \in G$. Then $\|a\|_C = \|b\|_C$.

**Proof.** Let $\{g_1, \ldots, g_m\}$ be a complete set of representatives of left cosets of $G'$ in $G$. By [Wis11, Lem 7.8], $G$ acts on a CAT(0) cube complex $C = \prod_i C'_{g_i}$, where each $C'_{g_i}$ is a copy of $C'$. Moreover, each subgroup $g_i G' g_i^{-1}$ stabilizes the $g_i$–coordinate and acts $C'_{g_i}$ via the given $G'$–action on $C'$.

For any $b \in G$, the translation length $\|b^m\|_C$ is the sum over $i$ of the number of $(b^m)$–orbits of hyperplanes in $C'_{g_i}$ that cut $b^m$. For each $i$, the number of $(b^m)$–orbits of hyperplanes in $C'_{g_i}$ that cut $b^m$ is the translation length in $C'$ of $g_i b^m g_i^{-1}$, so

$$\|b\|_C = \frac{1}{m!} \|b^m\|_C = \frac{1}{m!} \sum_{i=1}^m \|g_i b^m g_i^{-1}\|_{C'}^{\text{virt}} = \frac{m}{m!} \|b^m\|_{C'} = m \|b\|_{C'}^{\text{virt}}.$$

Similarly, $\|a\|_C = m \|a\|_{C'}^{\text{virt}}$. But $m \|b\|_{C'}^{\text{virt}} = m \|a\|_{C'}^{\text{virt}}$, so the conclusion follows. \hfill $\square$

We also need the following special case of [Wis14, Thm 4.5]:

**Lemma 3.3.** Let $A \cong \langle a, t \mid [a, t] \rangle$ and let $n \in \mathbb{N}$. Let $\mathcal{P} = \{a^m : 0 < |m| \leq n\}$. Then there exists a free action of $A$ on a CAT(0) cube complex $C$ so that for any $d \geq 1$, we have $\|t^d\|_C = \|pt^d\|_C$ for all $p \in \mathcal{P}$.

**Proof.** Consider the action of $A$ on $\mathbb{E}^2$ by affine isometries, with $a$ acting as a unit translation along $(1, 0)^T$ and $t$ as a unit translation along $(0, 1)^T$. Let $L_0 \subset \mathbb{E}^2$ be the line through the origin parallel to $(n, 1)^T$ and let $L_1$ be the line parallel to $(-n, 1)^T$. Then $L_0, L_1$ are geometric walls in $\mathbb{E}^2$, and thus $(\mathbb{E}^2, A \cdot L_0 \cup A \cdot L_1)$ is a geometric wallspace on which $A$ acts. Let $C$ be the dual cube complex.

Let $(x, y) \in \mathbb{Z}^2$ (so that $(x, y)$ corresponds to the element $a^x t^y \in A$). Then

$$\|a^x t^y\|_C = \left| \det \begin{pmatrix} x & n \\ y & 1 \end{pmatrix} \right| + \left| \det \begin{pmatrix} x & -n \\ y & 1 \end{pmatrix} \right| = |x - ny| + |x + ny|.$$

For any $x, y \in \mathbb{Z}$, we thus have that $\|a^x t^y\|_C = 0$ only if $x = ny$ and $x = -ny$, which is possible only if $x = y = 0$. Hence the action of $A$ on $C$ is free.

When $x = 0$ and $y = d$, we obtain $\|t^d\|_C = 2dn$. Next, consider the case where $x = m$ for $0 < |m| \leq n$ and $y = d$. Then $\|a^m t^d\|_C = |m - dn| + |m + dn|$. If $m > 0$, this yields $\|a^m t^d\|_C = dn - m + m + dn = 2dn$. (Here, we have used that $|m| \leq n \leq dn$.) If $m \leq 0$, this yields $\|a^m t^d\|_C = dn + |m| + dn - |m| = 2dn$. Hence $\|a^m t^d\|_C = \|t^d\|_C$ for $0 \leq |m| \leq n$, as required. \hfill $\square$

We can now state the main lemma of this section.

**Lemma 3.4.** Let $F$ be a free group and let $G = F \times \mathbb{Z}$. We regard $G$ as being presented by $\langle F, t \mid [f, t], f \in F \rangle$, so each element has the form $f^nt^n$ for some $f \in F, n \in \mathbb{Z}$. Let $\mathcal{P} \subset F$ be finite. Then for any $d \geq 1$, there exists a CAT(0) cube complex $C_{\mathcal{P}}$ and an action of $G$ on $C_{\mathcal{P}}$ so that:
\[
\|pt^d\|_\ominus = \|t^d\|_\ominus \text{ for all } p \in \mathcal{P}.
\]
\[
(p, t) \text{ acts freely for all } p \in \mathcal{P}.
\]

Proof. Roughly, the idea is to use various canonical completions associated to the \(p \in \mathcal{P}\) to dictate virtual retractions from \(G\) to various \(\mathbb{Z}^2\) subgroups, and apply Lemma 3.3 and Lemma 3.2 to produce, for each \(p\), an action on a CAT(0) cube complex \(C_p\) satisfying the conclusion. We finally let \(C_\ominus = \prod_{p \in \mathcal{P}} C_p\).

\textbf{Simplifying assumptions about } \mathcal{P}: By replacing \(\mathcal{P}\) with a finite superset, we may assume that there are elements \(p_1, \ldots, p_n \in F\), generating distinct maximal cyclic subgroups, and natural numbers \(n_i \geq 1\), so that \(\mathcal{P} = \{p_i^n : 1 \leq i \leq n, 0 < |m| \leq n_i\}\).

\textbf{Representative graphs:} Let \(B\) be a connected graph with \(\pi_1 B\) identified with \(F\), and let \(S\) be a circle with one vertex, with \(\pi_1 S\) identified with \(\langle t \rangle\). We will use the notation \(p_i\) to denote both the element \(p_i \in F\) and an immersed closed based path in \(B\) representing it; we likewise let \(t\) denote both the generator of \(\pi_1 S\) and a representative embedded closed path.

It suffices to prove the lemma in the case where \(\mathcal{P}\) contains at most one element in each conjugacy class, and we shall assume each \(p_i\) is a cyclically reduced path. Indeed, conjugate elements will have the same translation length.

\textbf{Initial finite cover:} Let \(\hat{B} \to B\) and \(\hat{S} \to S\) be connected finite regular covers so that:

\begin{enumerate}
  \item for each \(i\), each elevation \(\hat{p}_i \to \hat{B}\) of \(p_i \to B\) is injective;
  \item there exists \(D_0 \geq 1\) so that the degree of \(p_i^{m_0} \to \hat{p}_i^{m_0} \to \hat{B} \times \hat{S}\) of \(\hat{p}_i^{m_0} \to \hat{B} \times \hat{S}\) and all \(i, m\).
\end{enumerate}

Indeed, let \(\hat{B} \to B\) be a finite regular cover with the first property, which exists by separability of the \(\langle p_i \rangle\) in \(\pi_1 B\). For each elevation \((p_i^{m_0}t^d)\) of \(\hat{p}_i^{m_0} \to \hat{B} \times \hat{S}\), let \(\delta_{im}\) be the degree of \((p_i^{m_0}t^d)' \to \hat{p}_i^{m_0} \to \hat{B} \times \hat{S}\), and let \(\hat{S} \to S\) be a connected \(D_0\)-fold cover where \(D_0 = \text{lcm}\{\delta_{im}\}\). Hence the degree of \(\hat{p}_i^{m_0}\) is \(D_0\) since it is the same as its order in \((\pi_1 B/\pi_1 \hat{B}) \times (\pi_1 S/\pi_1 \hat{S})\), since the order \(D_0\) in the second factor is a multiple of the order \(\delta_{im}\) in the first factor.

\textbf{Canonical completions:} Fix \(i \leq n\). By item (1) above, each elevation \(\hat{p}_i \to \hat{B}\) of \(p_i\) is embedded. Consider the canonical completion \(C(\hat{p}_i \to \hat{B}) \to \hat{B}\), which is a finite cover admitting a retraction \(r_{ij} : C(\hat{p}_i \to \hat{B}) \to \hat{p}_i\). We also have a canonical completion \(C(\hat{p}_i \to \hat{B}) \to \hat{p}_i\). As explained in [Wis12 Section 4.5], the embedding \(\hat{p}_i \to \hat{B}\) induces an embedding \(C(\hat{p}_i \to \hat{B}) \to C(\hat{p}_i \to \hat{B})\) so that the canonical retraction \(C(\hat{p}_i \to \hat{B}) \to \hat{p}_i\) is the restriction of \(r_{ij}\).

Also, \(C(\hat{p}_i \to \hat{B}) = \hat{p}_i \cup \hat{p}_i\), where \(\hat{p}_i\) is a connected \(|\hat{p}_i|\)–fold cover of \(\hat{p}_i\) on which the canonical retraction map restricts to a map of degree \(-1\).

We now employ \(r_{ij} \times \text{id} : C(\hat{p}_i \to \hat{B}) \times \hat{S} \to \hat{p}_i \times \hat{S}\), whose target is an embedded torus in \(C(\hat{p}_i \to \hat{B}) \times \hat{S}\).

Let \((p_i^{m_0}t^d)\) be an elevation of \(\hat{p}_i^{m_0} \to \hat{B} \times \hat{S}\) that lies in the torus \(\hat{p}_i \times \hat{S}\). Then \((r_{ij} \times \text{id})\) homotopic to the path \(\hat{p}_i^{m_0} \to \hat{B}\), where \(m_0\) depends on \(\delta_{im}\) and the degree of \(\hat{p}_i \to \hat{p}_i\).

If \((p_i^{m_0}t^d)\) is homotopic into \(\hat{p}_i \times \hat{S}\), then \((r_{ij} \times \text{id})\) is homotopic to the path displayed below. We emphasize that the precise constant \(D_{mi}\) is immaterial. We use that \(D_{mi} \geq 1\) and \(D_{mi}\) is independent of the choice \(\hat{p}_i\) of elevation of \(p_i\) to \(\hat{B}\) because \(\hat{B} \to B\) is regular. The only important feature of the \(\hat{p}_i\) exponent is that it is nonzero. The path is:
Indeed, for each $\alpha_i$.

Moreover, for each $\alpha$, let $D = \gcd(m, |\hat{p}_{ij}| - 1)$.

Otherwise, if $(\hat{p}_k^{m_{Dk}})^n$ is an elevation of $p_k^{m_{Dk}}$ and either $k \neq i$ or $k = i$ but $(\hat{p}_k^{m_{Dk}})^n$ is not inside $C(\hat{p}_{ij} \to \hat{p}_{ij}) \times \hat{S}$, then $(r_{ij} \times \text{id})((\hat{p}_k^{m_{Dk}t^d})^n)$ is homotopic to $t^{D_0}E_{kij}^d$, where $E_{kij}$ depends on the length of the elevations of $p_k$ to $\hat{B}$ and on the degree of the cover $C(\hat{p}_{ij} \to \hat{B}) \to \hat{B}$.

The finite cover associated to $\hat{p}_{ij}$: Let $D = \text{lcm}\{D_0, \{D_{mi}\}_{mi}, \{E_{kij}\}_{kij}\}$. Let $\hat{S} \to \hat{S}$ be a connected $D$-fold cover. For each $i$ and each elevation $\hat{p}_{ij}$ of $p_i$ to $\hat{B}$, consider the finite cover $C(\hat{p}_{ij} \to \hat{B}) \times \hat{S} \to \hat{B} \times S$ and the retraction $r_{ij} \times \text{id} : C(\hat{p}_{ij} \to \hat{B}) \times \hat{S} \to \hat{p}_{ij} \times \hat{S}$. Then each elevation of each $\hat{p}_k^{m_{Dk}t^d}$ to this cover has one of the following two properties:

- it is sent by $r_{ij} \times \text{id}$ to $t^Kd$, where $K$ is a fixed integer depending on $D_0, D$;
- it lies in $C(\hat{p}_{ij} \to \hat{p}_{ij})$ and is sent by $r_{ij} \times \text{id}$ to a path homotopic to $\hat{p}_k^{m_{Dk}t^d}$, where $q$ is one of finitely many nonzero natural numbers.

Indeed, $\hat{S} \to S$ was chosen precisely so that the induced further covers of the above elevations have the same degrees in the $\hat{S}$ factor under the above retraction maps.

The cubical action associated to $\hat{p}_{ij}$: By Lemma 3.2, there is an action of $\pi_1(\hat{p}_{ij} \times \hat{S})$ on a CAT(0) cube complex $X_{ij}$ with the following properties. First, $(\hat{p}_{ij}, t^Kd)$ acts on $X_{ij}$ freely. Second, $t$ has positive translation length and $\|\hat{p}_j^q \cdot t^Kd\|_{X_{ij}} = \|t^Kd\|_{X_{ij}}$ for all values $q$ appearing above. Let $\rho_{ij} : \pi_1(\hat{p}_{ij} \times \hat{S}) \to \text{Aut}(X_{ij})$ be this action.

For each $i$, we have $a_i \in \mathbb{Z}$ so that for each $j$, there exists $h_{ij} \in G$ so that $\hat{p}_{ij}$ corresponds to $h_{ij}^a h_{ij}^{-1} \in G$. Consider the action $\rho_{ij} \circ (r_{ij} \times \text{id}) : \pi_1(C(\hat{p}_{ij} \to \hat{B}) \times \hat{S}) \to \text{Aut}(X_{ij})$ on the cube complex $X_{ij}$. By construction, $\langle h_{ij}^a h_{ij}^{-1}, t^Kd \rangle$ acts on $X_{ij}$ freely, and the translation length of each $h_{ij}^a h_{ij}^{-1} t^Kd$ coincides with that of $t^Kd$. Moreover, by construction, for the remaining $p_k^{m_{Dk}t^d}$, the virtual translation length of any conjugate $u$ of $p_k^{m_{Dk}t^d}$ is the same as that of $t^d$, since $u$ is sent by $r_{ij} \times \text{id}$ to $t^d$. Hence we can apply Lemma 3.2 to the action $\rho_{ij} \circ (r_{ij} \times \text{id}) : \pi_1(C(\hat{p}_{ij} \to \hat{B}) \times \hat{S}) \to \text{Aut}(X_{ij})$ on the cube complex $X_{ij}$ to obtain an action $\rho_{ij}^c$ of $F \times \mathbb{Z}$ on a CAT(0) cube complex $C_{ij}$ so that each $p_k^{m_{Dk}t^d}$ has the same translation length as $t^d$, for $k \leq n$ and $|m| \leq n_k$. Moreover, the freeness of the action on $X_{ij}$ ensures that $\langle h_{ij}^a h_{ij}^{-1}, t^d \rangle$ acts freely.

Construction of $C_{ij}$: We have constructed, for each $i \leq n$, an action $\alpha_i : F \times \mathbb{Z} \to \text{Aut}(C_{pi})$ on a CAT(0) cube complex $C_{pi}$, with the following properties:

- $\|p^{m_{Dk}t^d}\|_{C_{pi}} = \|t^d\|_{C_{pi}}$ for $0 < |m| \leq n_i$;
- $(\hat{p}_{ij}, t)$ acts freely on $C_{ij}$;
- for all $i \neq j$ and $|m| \leq n_i$, we have $\alpha_i(p^{m_{Dk}t^d}) = \alpha_i(t^d)$.

Indeed, for each $i$, let $C_{pi}$ be the cube complex $C_{ij}$ associated to the base elevation $\hat{p}_{ij}$ of $p_i$, and let $\alpha_i = \hat{p}_{ij}^\alpha$.

Now let $C_0 = \prod_{i=1}^n C_{pi}$, and let $\alpha : F \times \mathbb{Z} \to \text{Aut}(C)$ be the diagonal action induced by the actions $\alpha_1, \ldots, \alpha_n$. Then $(\hat{p}_i, t)$ acts freely on $C_0$ for each $i \leq n_i$, because it acts freely on at least one factor. Moreover, for each $i$ and $m \leq n_i$, we have that $\|p^{m_{Dk}t^d}\|_{C_{pi}} = \|t^d\|_{C_{pi}}$ and $\|p^{m_{Dk}t^d}\|_{C_{pj}} = \|t^d\|_{C_{pj}}$ when $j \neq i$. Hence $\|p^{m_{Dk}t^d}\|_{\hat{S}} = \|t^d\|_{C_{pi}} + \sum_{j \neq i} \|t^d\|_{C_{pj}} = \|t^d\|_{\hat{S}}$, as required.
The next lemma uses Lemma [3.4] and is applied at each step in the inductive proof of Theorem 1.

**Lemma 4.1 (Torus cubulation).** Let $G$ split as a finite graph of groups where each edge group is separable in $G$. Suppose there is a homomorphism $\psi : G \to \mathbb{Z}$ which is surjective on each edge-group in the splitting of $G$. Choose $t$ in some edge-group of $G$ with $\psi(t) = 1$. Let $\mathcal{P} \subseteq G$ be finite, with $\psi(p) = 0$ and $p$ acting hyperbolically on the Bass-Serre tree of the splitting of $G$, for all $p \in \mathcal{P}$.

Then there exists an action of $G$ on a CAT(0) cube complex $C_\circ$ so that $\|pt\|_\circ = \|t\|_\circ$ and, moreover, $\|p^nt\|_\circ > 0$ for all $p \in \mathcal{P}$ unless $m = n = 0$.

**Proof.** The idea is to build a finite-index subgroup $G'' \subseteq G$, a quotient $G'' \to F \times \mathbb{Z}$, where $F$ is free, so that a generator of each $(pt) \cap G''$ has the same nonzero image in $\mathbb{Z}$ as a fixed generator of $(t) \cap G''$, and then apply Lemma 3.4 and Lemma 3.2 to obtain the desired cubical action of $G$.

The first finite-index subgroup: Let $\Gamma$ be the underlying graph of the hypothesized splitting of $G$. Let $\mathcal{P} = \{p_{i,j}^m : 1 \leq i \leq n, j \in J_i\}$, where we have partitioned the elements according to maximal cyclic subgroups.

For each $i$, $p_{i,j}t$ have normal form $p_{i,j}t = e_1e_2\cdots e_ke_{k+1}$, where $e_1\cdots e_k$ is a closed path in $\Gamma$ and each $e_j$ belongs to a vertex group. Suppose for some $j$ that $e_j = e_{j+1}^{-1}$. Then, by separability of the edge-groups in $G$, there is a finite index subgroup of $G$ containing the terminal edge-group of $e_j$ (which is the initial edge-group of $e_{j+1}$) but not containing $a_{j+1}$. Repeating this procedure for each such backtrack and taking intersections of conjugates gives a finite-index normal subgroup $G' \leq G$ so that the following holds for all $i \leq n$ and all $m_j$: let $\Gamma'$ be the underlying graph of the induced splitting of $G'$ and let $q_{i,j}'$ generate the cyclic subgroup $\langle p_{i,j}^m t \rangle \cap G'$. Then $q_{i,j}'$ has normal form projecting to an immersed closed path in $\Gamma'$.

**Equalizing $t$-lengths:** For each $q_{i,j}'$, let $d_{i,j} = \psi(q_{i,j}')$, which is nonzero since $\psi(p_{i,j}^m t) = 1$. Let $D = \text{lcm}\{d_{i,j}\}$ and let $G''$ be the kernel of the map $G \xrightarrow{\psi} \mathbb{Z} \to \mathbb{Z}/D\mathbb{Z}$. For each $i, j$, let $q_{i,j}''$ generate $\langle p_{i,j}^m t \rangle \cap G''$. By construction, $\psi(q_{i,j}'') = D > 0$ for all $i, j$. Moreover, if $q \in G''$ is conjugate in $G$ to $q_{i,j}'$, then $\psi(q) = D$. Indeed, conjugate elements of $G$ have the same $\psi$-image.

**Conclusion:** Let $\Gamma''$ be the underlying graph of the induced splitting of $G''$. Consider the map $\rho : G'' \to \pi_1\Gamma'' \times \mathbb{Z}$ induced by $G'' \to \pi_1\Gamma''$ (projection to the underlying graph) and $\psi : G'' \to \mathbb{Z}$. We now regard $\pi_1\Gamma'' \times \mathbb{Z}$ as being presented by $\langle \pi_1\Gamma'', f \mid [f, f], f \in \pi_1\Gamma''\rangle$. Each $q_{i,j}''$ maps to some $\bar{q}_{i,j}'' \in \pi_1\Gamma'' - \{1\}$ and $D > 0$.

Lemma [3.4] provides a CAT(0) cube complex $C'$ and an action $\alpha : \pi_1\Gamma'' \times \mathbb{Z} \to \text{Aut}(C')$ so that the following hold for any $\bar{q} \in G''$ that is conjugate in $G$ to some $q_{i,j}'$:

- each $\langle \rho(\bar{q}), \bar{t} \rangle$ acts freely;
- $\|t^n\|_{C'} = \|\rho(\bar{q})\|_{C'} > 0$.

Hence the action $\alpha \circ \rho$ of $G''$ on $C'$ has the property that $\|t^n\|_{C'} = \|\bar{q}\|_{C'} > 0$ for all $\bar{q} \in G''$ conjugate in $G$ to some $q_{i,j}'$. We can now apply Lemma 3.2 to the action of $G''$ on $C'$ to obtain an action of $G$ on a CAT(0) cube complex $C_\circ$ with the desired properties. \qed

The following lemma, about cubulating a multiple HNN extension $G$ of a group $H$, takes as its input a pair of actions of $H$ on CAT(0) cube complexes, and returns a single action of $G$ on a CAT(0) cube complex. In practice, the pair of actions of $H$ will arise as follows: one will exist by induction, and the other will have been obtained by applying Lemma 4.1.
Lemma 4.2 (Cubulation with turns). Let the group $G$ decompose as a finite graph of groups with a single vertex group $H$ and $\mathbb{Z}$ edge groups. Let $\mathcal{P} \subset H$ be a finite set, and let $t \in H - \mathcal{P}$ generate an edge-group. Suppose each stable letter $e$ conjugates $t$ to $pt$ for some $p \in \mathcal{P}$ (and each $p \in \mathcal{P}$ appears in this way).

Suppose that $H$ acts on CAT(0) cube complexes $C_i$ and $C_\circ$ with the following properties:

- For each $p \in \mathcal{P}$, we have $\|t\|_{C_i} = \|pt\|_{C_i} = 0$.
- For each $p \in \mathcal{P}$, we have $\|pt\|_{C_\circ} = \|t\|_{C_\circ}$.
- The diagonal action of $H$ on $C_i \times C_\circ$ is free.

Finally, let $Q$ be a finite subset of $G$ consisting of elements acting on $T$ hyperbolically. Then $G$ acts on a CAT(0) cube complex $C_i^1$ with the following properties:

- $\|qt\|_{C_i^1} = 0$ for all $q \in Q$.
- For each $g \in H$, we have $\|g\|_{C_i^1} = 2\|g\|_{C_i}$. In particular, $\|t\|_{C_i^1} = 0$.
- For every $g \in G$ acting hyperbolically on $T$, either $g$ is conjugate into some $\langle q, t \rangle$, $q \in Q$ (and conjugate into $\langle qt \rangle$ if $[q, t] \neq 1$), or $\|g\|_{C_i^1} > 0$.

Proof. There are several steps.

The initial tree of spaces: Let $D = H \setminus (C_i \times C_\circ)$, which is a nonpositively-curved cube complex because the $H$-action on $C_i \times C_\circ$ is free. Let $\ell = \|t\|_{C_i \times C_\circ}$, and note that $\|pt\|_{C_i \times C_\circ} = \ell$ for all $p \in \mathcal{P}$.

For each $p \in \mathcal{P}$, let $\tilde{S}_p \to C_i \times C_\circ$ be a combinatorial geodesic axis for $pt$ in $C_i \times C_\circ$ and let $S_p \to D$ be the path given by $S_p = \langle pt \rangle \setminus \tilde{S}_p$. Let $S_t$ be defined analogously for $t$.

Let $B_p$ be the cylinder $S \times [-\frac{1}{2}, \frac{1}{2}]$, where $S$ is a cycle of length $\ell$, and for each $p$, attach $B_p$ to $D$ by gluing along $S \times \{ \pm \frac{1}{2} \}$ via $S \times \{ \pm \frac{1}{2} \} \to S_p \to D$ and $S \times \{ -\frac{1}{2} \}$ to $S_t \to D$. Let $Z$ denote the resulting space, and observe that $\pi_1 Z \cong G$. Let $\tilde{Z} \to Z$ be the universal cover, which decomposes as a tree of spaces with underlying tree $T$. The vertex spaces are the elevations of $D$ and the edge spaces are strips of the form $\tilde{S} \times [-\frac{1}{2}, \frac{1}{2}]$.

Walls: For each $p \in \mathcal{P}$, and each edge $e$ of $S_p$, let $m^+_e$ be the midpoint of $e$, which maps to an immersed hyperplane $H_e$ of $D$. Likewise, for each edge $f$ of $S_t$, let $m^-_f$ be the midpoint of $f$, which maps to an immersed hyperplane $H_f$ of $D$. Since $|S_p| = |S_t| = \ell$, we can choose for each $p$ a bijection $b_p : \text{Edges}(S_p) \to \text{Edges}(S_t)$ and, for each $e \in \text{Edges}(S_p)$, choose a properly embedded arc $\alpha_e$ in $B_p$ joining $m^+_e$ to $m^-_{b_p(e)}$ and intersecting $S \times \{ \pm 1/2 \}$ in exactly two points. Declare immersed hyperplanes $H, H'$ of $D$ to be elementary equivalent if they intersect some $B_p$ in points joined by some arc $\alpha_e$; taking the transitive closure of this relation gives an equivalence relation on the immersed hyperplanes of $D$. An immersed wall $W \to D$ is formed from the disjoint union of the immersed hyperplanes in an equivalence class by attaching each of these arcs $\alpha_e$ as follows: join the endpoints of $\alpha_e$ to the points in immersed hyperplanes to which they map. Each immersed wall is connected, and lifts to a wall in $\tilde{Z}$ that intersects each vertex space in either $\emptyset$ or a single hyperplane, and intersects each edge space in $\emptyset$ or a single arc.

Let $\mathcal{H}$ be the set of walls in $\tilde{Z}$ of the above type that contain at least one arc. Let $\mathcal{V}$ be the set of walls of the following two types:

- hyperplanes of vertex spaces that do not intersect edge spaces;
- vertical walls (isometric to $\mathbb{R}$) of the form $\rho^{-1}(S \times \{0\})$, where $\rho : \tilde{Z} \to Z$ is the universal covering map and $S \times \{0\}$ is the core curve of some $B_p$. 
Let $\mathcal{E}_1$ be the set of hyperplanes in $\mathcal{D}$ of the form $H \times C_\odot$, where $H$ is a hyperplane of $C_1$. Let $\mathcal{E}_\odot$ be the set of hyperplanes in $\mathcal{D}$ of the form $C_2 \times V$, where $V$ is a hyperplane of $C_\odot$. Since $t$ and each $pt$, and all of their conjugates, act elliptically on $C_1$, each wall $W \in \mathcal{H}$ intersects each vertex space in $\tilde{Z}$ in a unique hyperplane which is a translate of an element of $\mathcal{E}_\odot$. The remaining walls of $\tilde{Z}$, i.e. the elements of $\mathcal{V}$, are either vertical, or are translates of elements of $\mathcal{E}_1$, and are in particular confined to single vertex spaces (in the latter case) or edge spaces (in the former).

Let $\mathcal{V}^−$ and $\mathcal{V}^+$ be two copies of the set of walls in $\mathcal{V}$, so that we have $G$–equivariant bijections $b^\pm : \mathcal{V} \to \mathcal{V}^\pm$, each of which is the identity when $\mathcal{V}^\pm$ is viewed as a copy of $\mathcal{V}$. Since each $V \in \mathcal{V}$ has a product neighbourhood in $\tilde{Z}$, we can perturb so that $b^\pm(V)$ are parallel, disjoint geometric walls.

**Auxiliary cube complex $X$:** Let $\tilde{X}$ be the CAT(0) cube complex dual to the wallspace $(G, \mathcal{H} \sqcup \mathcal{V}^− \sqcup \mathcal{V}^+)$. Note that $G$ acts on $\tilde{X}$, and moreover this action is free. Indeed, each $\mathcal{T}$–hyperbolic element of $G$ is cut by a vertical wall. Each elliptic element is cut by some wall, because the action of $H$ on $C_2 \times C_\odot$ is free and each wall intersects each vertex space in at most one hyperplane of the vertex space (and each hyperplane of a vertex space extends to a wall in $\tilde{Z}$).

Note that $\tilde{X}$ decomposes as a tree of spaces whose underlying tree is again $\mathcal{T}$, whose vertex spaces are isomorphic to $C_2 \times C_\odot$, and whose edge-spaces are convex subcomplexes.

We use the notation $\mathcal{H}$ to refer to the hyperplanes of $\tilde{X}$ corresponding to walls of $\tilde{Z}$ belonging to $\mathcal{H}$, and do likewise for $\mathcal{V}^\pm$. Note that each hyperplane $V^+ \in \mathcal{V}^+$ is parallel to, and osculates with, a hyperplane $V^-$, where $V^\pm = b^\pm(V)$ for some wall $V \in \mathcal{V}$. In particular, the vertical walls come in parallel pairs whose elements osculate. (Here, hyperplanes in $\tilde{X}$ are *parallel* if they cross exactly the same hyperplanes.)

**The cubical presentation of $G$:** Let $\eta : \tilde{X} \to \mathcal{T}$ be the $G$–equivariant map sending each vertex-space to the corresponding vertex, and, for each edge-space $V$, collapsing the hyperplane-carrier $V \times [-1/2, 1/2]$ to the corresponding edge (which we identify with $[-1/2, 1/2]$) in the obvious way. For each $q \in \mathcal{Q}$, let $\tilde{Y}_q = \eta^{-1}(L_q)$, where $L_q$ is the axis in $\mathcal{T}$ for the hyperbolic element $qt$. Note that $\tilde{Y}_q$ is a convex subcomplex of $\tilde{X}$ decomposing as a tree of spaces with underlying tree $L_q \cong \mathbb{R}$ and vertex spaces the translates of $C_2 \times C_\odot$ corresponding to vertices of $L_q$. Hence the inclusion $\tilde{Y}_q \to \tilde{X}$ as a convex subcomplex descends to a local isometry $\tilde{Y}_q \to X$, and we define $X^*$ to be the resulting cubical presentation $\langle X \mid \{\tilde{Y}_q \mid q \in \mathcal{Q}\} \rangle$. Since each $\tilde{Y}_q$ is CAT(0), we have $\pi_1 X^* \cong \pi_1 X \cong G$.

**Computing $\text{Aut}(\tilde{Y}_q)$:** The group $\text{Aut}(\tilde{Y}_q)$ can be identified with $\text{Stab}_G(\tilde{Y}_q)$, which we denote by $K$. We claim first that either $K = \langle qt \rangle$ or, if $[q, t] = 1$, then $K = \langle q, t \rangle$, and $K \cong \langle q, t \mid [q, t] \rangle \cong \mathbb{Z}_2^2$.

Indeed, the map $G \to \text{Aut}(\mathcal{T})$ induced by $\eta$ restricts to an action $K \to \text{Aut}(L_q)$. Note that $qt \in K$, so $K \neq \{1\}$. Applying Lemma 2.17 shows that either $K \cong \mathbb{Z}$, in which case $K \cong \langle qt \rangle$ (since $qt$ generates a maximal cyclic subgroup), or each element of $K$ centralises the generator of any edge group corresponding to an edge in $L_q$.

In the latter case, the image of $K$ in $\text{Aut}(L_q)$ coincides with the image of $\langle qt \rangle$, and the kernel is equal to the intersection of the edge groups occurring along $L_q$. Hence $\text{Ker}(K \to \text{Aut}(L_q)) = \bigcap_{n \in \mathbb{Z}} (t)^n = \langle t \rangle$, since $qt$ and $t$ commute. Hence $K$ is generated by $q$ and $t$, as required.

**Metric small-cancellation and short inner paths:** Next, we claim that $\langle X \mid \{\tilde{Y}_q\} \rangle$ satisfies the $C'(1/n)$ cubical small-cancellation condition for any $n \in \mathbb{N}$. Indeed, since each relator is simply connected, there are no essential closed paths in any $\tilde{Y}_q$, so any metric small-cancellation condition is trivially satisfied. Hence, by [Wis11] Lemma 3.67, $\langle X \mid \{\tilde{Y}_q\} \rangle$ has short inner paths.
Declaring walls in \( \tilde{Y}_q \): Recall that \( \mathcal{V}^\pm \) is a collection of hyperplanes of \( \tilde{X} \) that are either edge-spaces, or are contained in a single vertex space and do not intersect any of the incident edge spaces. Hence, for each \( V \in \mathcal{V}^\pm \) intersecting \( \tilde{Y}_q \), we have that \( V \subset \tilde{Y}_q \) and \( \{(qt)^n V : n \in \mathbb{Z}\} \) is a set of pairwise disjoint hyperplanes.

Fix \( q \in Q \) and consider \( \tilde{Y}_q \subset \tilde{X} \). We declare an equivalence relation on the hyperplanes of \( \tilde{Y}_q \) as follows. First, if \( \tilde{H} = H \cap \tilde{Y}_q \) is a hyperplane, where \( H \in \mathcal{H} \), then \( H \) is declared to be unique in its equivalence class. Next, let \( M \) be a large integer, to be specified below. Let \( V \in \mathcal{V} \) be a (combinatorial) hyperplane contained in \( \tilde{Y}_q \), which corresponds to a pair \( V^-, V^+ \) of parallel osculating hyperplanes in \( \tilde{Y}_q \). We declare \( (qt)^n V^\pm \sim (qt)^{M+n} V^\pm \) for \( n \in \mathbb{Z} \). The equivalence relation \( \sim \) determines walls in \( \tilde{Y}_q \); each wall is the union of one or two hyperplanes.

We claim that this system of walls in \( \tilde{Y}_q \) is \( K \)-invariant. Indeed, if \( K \cong \langle qt \rangle \), then this holds by construction.

Hence suppose that \( q, t \) commute and \( K = \langle q, t \rangle \). Since the \( \sim \)-class of each \( H \in \mathcal{H} \) consists of a single hyperplane, and \( \mathcal{H} \) is \( G \)-invariant, it is clear that \( q^n t^m H \) is a wall for each \( m, n \). Next, let \( V \in \mathcal{V} \). If \( V \) is a vertical wall, then \( tV^+ = V^+ \) and \( t(qt)^M V^- = (qt)^M tV^- = (qt)^M V^- \), so the class \( \{V^+, (qt)^M V^-\} \) is preserved by \( t \). On the other hand, \( qV^+ = qtV^+ \), since \( t \) fixes all edges of \( L_q \), and \( q(qt)^M V^- = (qt)(qt)^M V^- \), so the equivalence class \( \{V^+, (qt)^M V^-\} \) is sent by \( q \) to \( \{qtV^+, (qt)(qt)^M V^-\} \), which is another \( \sim \)-class.

Now suppose that \( V \) lies in some vertex space \( D \). Without loss of generality, one of the edges of \( L_q \) incident to the vertex corresponding to \( D \) has edge group \( \langle t \rangle \). Consider the action of \( t \) on \( D \cong C_x \times C_\circ \). Recall that \( t \) acts trivially on \( C_x \), and that \( V \) must be a hyperplane of the form \( W \times C_\circ \) for some hyperplane \( W \) of \( C_x \) (since \( V \in \mathcal{V} \)). Hence \( tV = V \). We can now argue exactly as above to see that \( t \) stabilises the equivalence class \( \{V^+, (qt)^M V^-\} \). Thus the wallspace structure on \( \tilde{Y}_q \) provided by \( \sim \) is \( K \)-invariant, as required by the \( B(6) \) condition.

Verifying \( B(6) \): We now verify that the cubical presentation \( \langle X | \{\tilde{Y}_q\} \rangle \) satisfies the \( B(6) \) condition from Definition [2.13]. Indeed, we have already verified Definition [2.13] (1),(3),(5), and it is easy to see that (2) holds whenever \( M \) is sufficiently large to ensure that hyperplanes in the same \( \sim \)-class are separated by a hyperplane (e.g. by virtue of being separated by an edge space). We have also already verified the \( C'(1/14) \) condition required by the definition of \( B(6) \).

Hence it suffices to check that if \( P \to \tilde{Y}_q \) is a path that decomposes as the concatenation of at most 7 piece-paths, and \( P \) starts and ends on the carrier \( N(U) \) of a wall \( U \), then \( P \) is path-homotopic into \( U \). Since \( \tilde{Y}_q \) is simply connected, this amounts to showing that, if our initial choice of \( M \) was sufficiently large, and \( U \) is formed from two hyperplanes \( V^+, (qt)^M V^- \in V^+ \sqcup V^- \), and \( P \) travels from \( V^+ \) to \( (qt)^M V^- \), then \( P \) cannot be the concatenation of at most 7 pieces.

First, observe that any hyperplane \( \bar{W} \) of \( \tilde{X} \) that is not contained in \( \tilde{Y}_q \), but whose carrier intersects \( \tilde{Y}_q \), has the property that \( N(H) \cap \tilde{Y}_q \) is contained in a single vertex space of \( \tilde{X} \).

Next, we claim that there exists a constant \( N_0 \) so that each piece-geodesic in \( \tilde{Y}_q \) projects to a path in \( L_q \) of length at most \( N_0 \). Indeed, let \( q' \in Q \) and \( q \in G \) be such that \( \tilde{Y}_q \) contains a piece of \( g \tilde{Y}_{q'} \), and let \( p \) be a path in this piece. Then the projection \( \bar{p} \) of \( p \) to \( L_q \) is contained in the intersection \( L_q \cap gL_{q'} \) in \( T \), so it suffices to bound this intersection. Since \( \text{Aut}(\tilde{Y}_q) \) acts as \( \langle qt \rangle \) coboundedly on the line \( L_q \) (by Lemma [2.17]), and the same is true of the \( \text{Aut}(\tilde{Y}_{q'})^g \)-action on \( gL_{q'} \) (with \( q \) replaced by \( gqtg^{-1} \)), and \( Q \) is finite, there exists a constant \( N_0 \) so that either \( \text{diam}(L_p \cap L_q) \leq N_0 \), or \( L_p \cap L_q \) is unbounded, in which case \( L_p = L_q \) because \( L_p, L_q \) are convex in \( T \). In the latter case,
\(Y_q = gY_{q'}\), by definition, contradicting that there is a piece between these two subcomplexes (the existence of a piece requires them to be distinct). Now let \(M\) be chosen so that for all \(q' \in Q\) and all vertices \(v \in L_{q'}\), we have \(d_T((q)Mv, v) \gg TM_0\). Given this choice, we see that no \(P\) traveling from \(V^+\) to \((q)M V^-\) can decompose as the concatenation of at most 7 pieces. This completes the verification of the \(B(6)\) condition.

**Verifying the proximality hypothesis:** We now declare preferred walls and verify that the hypotheses of Theorem 2.15 are satisfied. For each \(q\), a wall in \(Y_q\) is preferred if it arises from a ∼-class containing two hyperplanes, i.e. if its constituent hyperplanes belong to \(V^\pm\) (as opposed to \(H\)). It remains to check that the third hypothesis of the theorem holds.

Fix \(q \in Q\) and let \(\kappa \to \tilde{Y}_q\) be a geodesic with endpoints \(x, y\). Let \(V^-, (q)M V^+\) be preferred hyperplanes in the same wall, and suppose that \(\kappa\) traverses a 1–cube dual to \(V^-\). Suppose, moreover, that either \(\kappa\) traverses a 1–cube dual to \((q)M V^-\), or \((q)M V^-\) is proximate to \(y\). We claim that there is a preferred wall in \(\tilde{Y}_q\) that separates \(x, y\) but is not proximate to either \(x\) or \(y\).

Indeed, first suppose that \((q)M V^-\) is proximate to \(y\). Then our bound of \(M_0\) on the projection of piece-paths to \(L_q\) ensures that \(d_T(y, (q)MV^-) \leq 2M_0 + 1\), while we can make our initial choice of \(M\) so that \(d_T(V^-, (q)MV^+) > 100M_0\) (here, given subsets \(A, B\) of \(\tilde{X}\), we denote by \(d_T(A, B)\) the distance in \(T\) between the projections of \(A, B\)). Hence \(d_T(x, y) > 90M_0\). In particular, there exists a vertical (combinatorial) hyperplane \(V_1\) so that the parallel, osculating vertical hyperplanes \(V^\pm_1\) separate \(x, y\), and \(d_T(V^\pm_1, x) > 80M_0\) and \(d_T(V^\pm_1, y) \in [5M_0, 10M_0]\). Consider the wall \(U\) arising from the equivalence class \(\{V^+_1, (q)MV^-_1\}\). Then \((q)MV^-\) lies at \(T\)-distance at least \(90M_0\) from \(y\), and also from \(x\), and does not intersect \(\kappa\). Thus \(U\) separates \(x, y\) and is not proximate to either (because proximality would require \(U\) to lie at \(T\)-distance at most \(2M_0 + 1\) from one of \(x\) or \(y\), which contradicts our choice of \(V_0\).

Next, suppose that \((q)M V^-\) is not proximate to \(y\), so that \(\kappa\) traverses 1–cubes dual to both \(V^+\) and \((q)MV^-\). Then, again, we have \(d_T(x, y) \geq 90M_0\) and we can argue exactly as above.

**Applying Theorem 2.15:** We have verified all of the hypotheses of Theorem 2.15, which thus gives the following conclusion about the action of \(G\) on the cube complex \(C^0\) dual to the walls in \(X^+\) arising from the above ∼ relation. Let \(g \in G\) be an infinite-order element. Then one of the following holds:

- \(g\) was not cut by a preferred hyperplane of \(\tilde{X}\). In this situation, \(g\) is necessarily \(T\)-elliptic, and \(\|g\|_{C^0} = \|g\|_{C^0_{\epsilon}}\).
- \(g\) is elliptic on \(T\), and is cut by a preferred wall. Since each wall intersects each vertex space in a unique hyperplane, and our construction doubled the hyperplanes in \(V\) by subdivision, we have \(\|g\|_{C^0} = \|g\|_{C^0_{\epsilon}} + 2\|g\|_{C^1_{\epsilon}}\).
- \(g\) is conjugate into \(\langle q, t \rangle\) for some \(q \in Q\). (In the case where \([g, t] \neq 1\), we actually have that \(g\) is conjugate into \(\langle qt \rangle\).) In this case, by construction and Theorem 2.16, \(g\) is not cut by any preferred wall, either because we have turned all preferred walls away from \(g\) (in the case where \(g\) is hyperbolic) or, if \(g\) is elliptic, because \(g\) must be conjugate to \(t\), which acts trivially on \(C^1_{\epsilon}\).
- \(g\) is hyperbolic, and is cut by a preferred wall.

**Conclusion:** Let \(C^1_{\epsilon}\) be the restriction quotient of \(C^0\) obtained by cubulating the wallspace whose underlying space is \(C^0_{\epsilon}\) and whose walls are the preferred walls. The above discussion shows
that the action of \( G \) on \( C^0_1 \) descends to an action on \( C^1_1 \) such that each \( g \) has translation length as described in the statement; this completes the proof. \( \square \)

5. Proof of Theorem 1

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** Let \( G = F \times F \langle t \rangle \), where \( \Phi : F \to F \) is a polynomial-growth automorphism. Replacing \( G \) with a finite-index subgroup (but keeping our notation), we can assume that \( \Phi \) has an improved relative train track representative, by Proposition 2.3. Passing to a finite-index subgroup is permitted since, if \( G \) has a finite-index subgroup acting freely on a CAT(0) cube complex, then \( G \) also admits such an action.

Hence, as discussed in Remark 2.4, we can assume (effectively, by passing to a supergroup of a finite index subgroup) that \( G \) admits a decomposition as a sequence of iterated \( \mathbb{Z} \) HNN extensions, as follows. First, we have \( G_0 \leq G_1 \leq \ldots \leq G_n = G \), where \( G_0 = \langle t \rangle \). Next, we have \( \{1\} = F_0 \leq F_1 \leq \ldots \leq F_n = F \), where \( G_0 = F_0 \times \langle t \rangle \), and the following holds. For each \( i \leq n \), we have

\[
F_i \times \langle t \rangle \cong G_i = \langle G_{i-1}, e^i_1, \ldots, e^i_{n_i} \mid p^i_j t = e^i_j (e^i_j)^{-1} \rangle,
\]

where each \( p^i_j \in F_{i-1} \leq G_{i-1} \) acts elliptically on the associated Bass-Serre tree \( T_i \), where the edge groups are conjugates of \( t \) and \( p^i_j t \). Let \( \mathcal{P}_i \subset F_{i-1} \) be the multiset of \( p^i_j \). Moreover, \( F_i \subset G_i \) contains \( \mathcal{P}_{i+1} \), and for each \( p \in \mathcal{P}_{i+1} \), the elements \( p, pt \) act hyperbolically on \( T_i \).

We now argue by induction on \( i \) that \( G_i \) acts freely on a CAT(0) cube complex. In fact, to support the induction, we will prove a stronger claim at each step.

**Base case, \( i = 0 \):** In the base case, \( F_0 = \{1\}, G_0 = \langle t \rangle \), and \( \mathcal{P}_i = \emptyset \) (i.e. the generators of \( F_1 \) that we will add at the next stage all commute with \( t \)). Let \( C^0_0 \) be the standard tiling of \( \mathbb{R} \) by \( 1 \)-cubes, on which \( G_0 \) acts freely, with \( t \) acting as a unit translation. Then, vacuously, we have \( \|p\|_\infty > 0 \) and \( \|t\|_\infty = \|pt\|_\infty \) for all \( p \in \mathcal{P}_1 \). Moreover, \( \|t\|_\infty > 0 \). Let \( C^0_1 \) be a single point, with \( G_0 \) acting trivially. Then, vacuously, each nontrivial \( g \in F_0 \) not conjugate into \( \langle p \rangle \), \( p \in \mathcal{P}_1 \) has positive translation length on \( C^0_2 \) (since there are no such \( g \)), and each \( p \in \mathcal{P}_1 \) has translation length 0, again vacuously.

Let \( C_0 = C^0_0 \times C^0_0 \), with \( G_0 \) acting diagonally. Then each nontrivial element of \( G_0 \) has positive translation length on at least one of the factors, namely \( C^0_0 \), so the action is free. Furthermore, \( \|pt\|_\infty = \|t\|_\infty \) for each \( p \in \mathcal{P}_1 \).

**Inductive step:** Suppose by induction that \( G_i \) acts on a CAT(0) cube complex \( C^i_1 \) so that \( \|t\|_{C^i_1} = \|tp^i_j\|_{C^i_1} = 0 \) for all \( j \leq n_i \) and every element not conjugate into \( \langle t \rangle \) or some \( \langle p^i_j, t \rangle \) has positive translation length. Then Lemma 4.1 provides an action of \( G_i \) on a CAT(0) cube complex \( C^i_0 \) so that \( \|p^i_{j+1} t\|_{C^i_0} = \|t\|_{C^i_0} > 0 \) and \( \|p^i_{j+1} t\|_{C^i_0} > 0 \) for all \( j \leq n_{i+1} \).

Indeed, in order to apply Lemma 4.1, we must verify that \( G_i \) satisfies all of the hypotheses of that lemma. Since \( G_i \) is of the form \( F_1 \times \Phi_i \langle t \rangle \), where \( \Phi_i : F_1 \to F_1 \) is an automorphism (obtained by restricting some positive power of \( \Phi \) to \( F_1 \leq F \)), we have a surjection \( \Phi_i : G_i \to \mathbb{Z} \) with \( \Phi(t) = 1 \) and \( \text{Ker} \Phi = F_1 \). In particular, \( p^i_{j+1} \in \text{Ker} \Phi \) for all \( j \). Moreover, \( G_i \) splits as a graph of groups, as in Remark 2.4 whose edge groups are of the form \( \langle t \rangle \) and \( \langle p^i_j t \rangle, j \leq n_i \), where each \( p^i_j \in F_{i-1} \). By Lemma 2.1, these subgroups are separable in \( G_i \). Hence we can apply Lemma 4.1.
By construction, the diagonal action of $G_i$ on $C_i \times C_i = C_i$ is free, since each nontrivial element acts hyperbolically on at least one of the factors. Lemma 4.2 now provides an action of $G_{i+1}$ on a CAT(0) cube complex $C_{i+1}$ with the following properties:

- For all $p_{j+1}^{j+1}$, we have that $\|p_{j+1}^{j+1}t\|_{C_{i+1}} = 0$.
- For all $T_{i+1}$-elliptic $g \in G_{i+1}$, we have $\|g\|_{C_{i+1}} = 2\|g\|_{C_i}$. In particular, $\|T\|_{C_{i+1}} = 0$.
- If $g \in G_{i+1}$ is $T_{i+1}$-hyperbolic and is not conjugate into $(p_{j+1}^{j+1}, t)$ for any $j$, then $\|g\|_{C_{i+1}} > 0$.

Now, our earlier discussion shows that we may apply Lemma 4.1 to $G_{i+1}$ to produce an action of $G_{i+1}$ on a CAT(0) cube complex $C_{i+1}$ so that the diagonal action of $G_{i+1}$ on $C_{i+1} = C_i^{i+1} \times C_i^{i+1}$ is free. In particular, $G_n$ acts freely on the CAT(0) cube complex $C_n$. This completes the proof. □

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