Notes on The Connectivity of Cayley Coset Digraphs

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Abstract. Hamidoune’s connectivity results [11] for hierarchical Cayley digraphs are extended to Cayley coset digraphs and thus to arbitrary vertex transitive digraphs. It is shown that if a Cayley coset digraph can be hierarchically decomposed in a certain way, then it is optimally vertex connected. The results are obtained by extending the methods used in [11]. They are used to show that cycle-prefix graphs [5] are optimally vertex connected. This implies that cycle-prefix graphs have good fault tolerance properties.

1 Introduction

Good interconnection networks for parallel computing usually have the following properties [2, 1]: They are symmetric, so that each node has the same view of the network. There are simple routing methods for finding paths from one node to another. The number of edges is small. The maximum distance between two nodes is small. The network can be easily constructed in 2 or 3 dimensions. The network is fault tolerant.

This motivates the study of vertex transitive, small degree and diameter, optimally connected digraphs. Sabidussi [15] showed that the class of vertex transitive digraphs is the same as that of Cayley coset digraphs. For that reason, there has been a substantial effort to construct and analyze Cayley coset digraphs with good interconnection network properties [7, 5, 1].

The connectivity properties of vertex transitive graphs were studied by Watkins [16] and Mader [12, 13]; and of digraphs by Hamidoune [8, 11, 10]. Mader [13] showed that connected vertex transitive digraphs have optimal edge

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connectivity. The first general results on the vertex connectivity of connected vertex transitive graphs were obtained by Mader \cite{12} and Watkins \cite{16}. They show that every connected edge and vertex transitive graph has optimal vertex connectivity (i.e. the vertex connectivity is the same as the degree). Mader \cite{12,13} shows that every connected vertex transitive graph without \(K_4\) is optimally vertex connected. He also shows that every connected edge transitive graph is optimally vertex connected. This work was extended to Cayley digraphs by Hamidoune \cite{8,9,10,11}. In \cite{10}, the abelian Cayley digraphs without \(K_4\) which are not optimally vertex connected are characterized. In \cite{11}, it is shown that connected \textit{hierarchical} Cayley digraphs are optimally vertex connected. A similar result is obtained in Baumslag \cite{7} using more direct methods.

The main tool for obtaining connectivity results in vertex transitive graphs is the concept of an \textit{atom}, which, briefly, is a minimal part of the graph with connectivity many neighbors. In this note, atoms are used for proving vertex connectivity results for Cayley coset digraphs. The main result is Theorem \cite{10}, which generalizes Proposition 3.1 of \cite{11}. This result is used to obtain a hierarchical decomposition result for Cayley coset digraphs, which as a corollary yields the result of Hamidoune \cite{11} and Baumslag \cite{7} that connected hierarchical Cayley graphs are optimally vertex connected. The main result is applied to show that the cycle-prefix graphs which were proposed as interconnection networks in \cite{5} are optimally vertex connected.

This note is organized as follows. Cayley coset digraphs are defined in Section \ref{sec:CayleyCosetDigraphs}, which also contains some elementary observations on Cayley coset digraphs. Vertex connectivity is discussed in detail in Section \ref{sec:VertexConnectivity}. Most of this section follows closely the methods described in \cite{11} but generalizes them to Cayley coset digraphs. In Section \ref{sec:HierarchicalDecomposition}, the main result is applied to hierarchical Cayley coset digraphs. Part of Hamidoune’s main result of \cite{11} is obtained as a corollary. Additionally it is proven that cycle-prefix graphs are optimally vertex connected. Finally, for completeness, Mader’s results \cite{13} on the edge connectivity of vertex transitive graphs and their proofs are given in Section \ref{sec:EdgeConnectivity}.

### 2 Cayley coset digraphs

Knowledge of basic group and graph theory is assumed (see for example Herstein \cite{4} and Tutte \cite{6}). All structures are assumed to be finite. If \(G\) is a group and \(F\) is a union of left cosets \(gH\) of a subgroup \(H\) of \(G\), then \(F/H\) denotes the set of left cosets of \(H\) in \(F\). We have \(\bigcup(F/H) = F\). \(\langle H_1,H_2,\ldots \rangle\) denotes the subgroup of \(G\) generated by \(H_1,H_2,\ldots\).

A digraph \(G\) is \textit{vertex transitive} iff the automorphisms of \(G\) act transitively on the set of vertices of \(G\). The \textit{transpose} of a digraph \(G\), denoted by
Given a group \( G \), a subgroup \( H \) of \( G \) and a set of generators \( S \subseteq G \setminus H \), the Cayley coset digraph \( \mathcal{G}(G, H, S) \) (or \( \mathcal{G} \) if \( G, H \) and \( S \) are clear from context) is obtained as follows: the set \( V(\mathcal{G}) \) of vertices of \( \mathcal{G} \) is given by the set of left cosets \( G/H = \{gH \mid g \in G\} \) of \( H \) in \( G \), and the set \( E(\mathcal{G}) \) of edges of \( \mathcal{G} \) consists of the ordered pairs \((gH, g'H)\) with \( gH \cap g'H \neq \emptyset \) for some \( s \in S \). The graph \( \mathcal{G} \) is vertex transitive. A transitive group of automorphisms of \( \mathcal{G} \) acting on \( V(\mathcal{G}) \) consists of the maps \( \varphi_g : g'H \mapsto gg'H \). Every vertex transitive digraph is a Cayley coset digraph, as is shown in [15].

A digraph \( \mathcal{G} \) is a Cayley digraph iff \( H = \{e\} \), where \( e \) is the identity of \( G \). A Cayley (coset) graph is a symmetric Cayley (coset) digraph (i.e. if \((x, y) \in E(\mathcal{G})\) then \((y, x) \in E(\mathcal{G}))\).

For \( s \in S \), let
\[
E_s = \{(g_1H, g_2H) \mid g_1Hs \cap g_2H \neq \emptyset\}
\]
be the set of edges induced by \( s \). The next lemma shows that we can assume that \( S \) consists of distinct representatives of the double cosets \( HgH \) in \( G \).

**Lemma 2.1** The following are equivalent:

(i) The edge \((g_1H, g_2H)\) is in \( E_s \).

(ii) \( g_1^{-1}g_2 \) is in \( HsH \).

(iii) \( g_1HsH \supseteq g_2H \).

**Proof.** We have \((g_1H, g_2H) \in E_s\) iff \( Hs \cap g_1^{-1}g_2H \neq \emptyset \) iff \( g_1^{-1}g_2 \in HsH \) iff \( g_2 \in g_1HsH \) iff \( g_2H \subseteq g_1HsH = g_1HsH \). \( \square \)

Lemma 2.1 implies that \( E_s \cap E_{s'} = \emptyset \) unless \( HsH = Hs'H \) in which case \( E_s = E_{s'} \).

**Assumption.** From now on we assume that the generators are representatives from distinct double cosets of \( H \).

A digraph \( G \) is strongly connected iff for every \( a, b \in V(G) \), there is a path from \( a \) to \( b \). Since only strong connectivity is considered in this note, the word “strongly” will be omitted. Note that for vertex transitive digraphs, strong connectivity is equivalent to weak connectivity, i.e. in a vertex transitive digraph, if there is a path from \( a \) to \( b \), then there is one from \( b \) to \( a \).

**Lemma 2.2** \( \mathcal{G} \) is connected iff \( (H, S) = G \).
Proof. By vertex transitivity, it suffices to check that for every \( g \in G \), there is a path from \( H \) to \( gH \) in \( \mathcal{G} \). A path is a sequence \( H, g_1H, g_2H, \ldots, gH \) where \( g_1H \subseteq Hs_1H, \ g_2H \subseteq g_1Hs_2H, \ldots \) with \( s_i \in S \). There is such a sequence ending at \( gH \) iff every element of \( gH \) can be expressed as a product of elements of \( H \) and generators. The result follows.

Lemma 2.2 implies that the components of \( \mathcal{G} \) are determined by the left cosets of \( \langle H, S \rangle \), i.e. each component is of the form \( (g(H, S))/H \).

Assumption. We assume that \( H \) and \( S \) generate \( G \) or equivalently, that \( \mathcal{G} \) is connected.

By Lemma 2.2, the set of neighbors of \( H \) due to \( s \) is given by \( HsH/H \). Thus, the contribution of \( E_s \) to the degree \( d \) of \( \mathcal{G} \) is determined by the index

\[
d_s = |HsH/H|.
\]

Lemma 2.3 \( d = \sum_{s \in S} d_s \).

Proof. The result follows from the discussion above and from the assumption that the generators come from distinct double cosets of \( H \).

For \( S' \subseteq S \), let \( d_{S'} = \sum_{s \in S'} d_s \).

The numbers \( d_s \) can be computed using the following elementary result from group theory:

Lemma 2.4 \( |HsH/H| = |H/(H \cap sHs^{-1})| \).

3 Vertex connectivity

Definitions. Let \( G \) be a digraph. The vertex connectivity of \( G \) is the smallest number of vertices that need to be removed from \( G \) so that the digraph induced on the remaining vertices is not connected. The vertex connectivity of \( G \) is denoted by \( \kappa(G) \) or simply \( \kappa \) if the graph is clear from the context.

For \( A \subseteq V(G) \), let \( N_G(A) \) denote the set of neighbors of \( A \), where

\[
N_G(A) = \{ x \in V(G) \setminus A \mid \exists y \in A \text{ such that } (y, x) \in E(G) \}.
\]

The subscript is omitted if it is clear which graph is being considered. \( A \) is a part of \( G \) iff \( V(G) \setminus (A \cup N(A)) \) is non-empty. \( A \) is an atom iff \( A \) is a minimum size
part of $G$ with the property that $|N(A)| = \kappa$. Note that this definition differs slightly from the one given by Hamidoune [3], who defines atoms as minimal size parts $A$ of $G$ or $G^*$ satisfying $|N_G(A)| = \kappa$ or $|N_{G^*}(A)| = \kappa$, respectively.

The only digraphs without atoms are the complete digraphs where every pair of vertices is an edge. Assume that $G$ is not complete. The next lemmas are used to show that the atoms of $G$ partition $G$, provided that they are small enough.

**Lemma 3.1** Let $G$ be a digraph on $n$ vertices of vertex connectivity $\kappa$. Let $A$ be an atom of $G$ and $B$ a part of $G$ with $|N(B)| = \kappa$. If $V(G) \setminus (A \cup N(A) \cup B \cup N(B))$ is non-empty, then either $A \cap B = \emptyset$ or $A \subseteq B$.

**Proof.** Suppose that $A \cap B$ and $A \setminus B$ are both non-empty. We show that $A \cup B$ is a part with $|N(A \cup B)| < \kappa$ to derive a contradiction. Since $V(G) \setminus (A \cup B \cup N(A \cup B)) = V(G) \setminus (A \cup N(A) \cup B \cup N(B))$, it follows that $A \cup B$ is a part. We have

$$N(A \cup B) = (N(B) \setminus A) \cup (N(A) \setminus (B \cup N(B))).$$

To obtain the contradiction, it suffices to show that $|N(A) \setminus (B \cup N(B))| < |N(B) \cap A|$. This follows from Lemma 3.2 which is proved next.  

**Lemma 3.2** Let $G$ be a digraph with $A$ an atom and $B$ a part of $G$. If $A \cap B$ and $A \setminus B$ are both non-empty, then $|N(A) \setminus (B \cup N(B))| < |N(B) \cap A|$.

**Proof.** Since $A$ is an atom, $|N(A \cap B)| > |N(A)|$. Using the fact that $N(A \cap B)$ is included in $(N(B) \cap A) \cup (N(A) \cap (B \cup N(B)))$, we can deduce

$$|N(A)| < |(N(B) \cap A) \cup (N(A) \cap (B \cup N(B)))|$$

$$= |N(B) \cap A| + |N(A) \cap (B \cup N(B))|$$

$$= |N(B) \cap A| + |N(A)| - |N(A) \setminus (B \cup N(B))|.$$ 

This gives the result.

**Lemma 3.3** Let $G$ be a digraph on $n$ vertices of vertex connectivity $\kappa$. If the atoms of $G$ have size at most $(n - \kappa)/2$, then any two distinct atoms of $G$ are disjoint.
Proof. By Lemma 3.1 it suffices to show that if $A$ and $B$ are distinct intersecting atoms of $G$ of size at most $(n - \kappa)/2$, then $V(G) \setminus (A \cup N(A) \cup B \cup N(B)) \neq \emptyset$. The set $A \cup N(A) \cup B \cup N(B)$ is the disjoint union of $A \cup B$, $N(A) \setminus (B \cup N(B))$, $N(A) \cap N(B)$ and $N(B) \setminus (A \cup N(A))$. Thus

$$\left| A \cup N(A) \cup B \cup N(B) \right| = \left| A \cup B \right| + \left| N(A) \cap N(B) \right| + \left| N(A) \setminus (B \cup N(B)) \right| + \left| N(B) \setminus (A \cup N(A)) \right| \leq n - \kappa + \left| N(A) \cap N(B) \right| + \left| N(A) \setminus (B \cup N(B)) \right| + \left| N(B) \setminus (A \cup N(A)) \right|.$$ 

To bound the last sum, we use Lemma 3.2 to obtain

$$2 \left[ \left| N(A) \setminus (B \cup N(B)) \right| + \left| N(B) \setminus (A \cup N(A)) \right| \right] < \left| N(A) \cap B \right| + \left| N(B) \cap A \right| + \left| N(A) \setminus (B \cup N(B)) \right| + \left| N(B) \setminus (A \cup N(A)) \right| + 2\left| N(A) \cap N(B) \right| = \left| N(A) \right| + \left| N(B) \right| = 2\kappa.$$ 

It follows that $\left| A \cup N(A) \cup B \cup N(B) \right| < n - \kappa + \kappa = n$, which gives the result.

**Lemma 3.4** Let $G$ be a digraph of vertex connectivity $\kappa$ which is not complete. Then either $G$ or $G^*$ has an atom of size at most $(n - \kappa)/2$.

**Proof.** Let $A$ be an atom of $G$. Then $V(G) \setminus (A \cup N(A))$ is a part of $G^*$. Since $\left| N(A) \right| = \kappa$, the result follows.

**Assumption.** From now on we assume that $\mathcal{G}$ has an atom of size at most $(n - \kappa)/2$. To see that this assumption does not restrict the generality of the results to be shown, it suffices to apply Lemma 3.4 and note that the properties used to prove the results are preserved if $\mathcal{G}$ is replaced by $\mathcal{G}^*$. In particular, note that the vertex connectivity of $\mathcal{G}^*$ is the same as the vertex connectivity of $\mathcal{G}$. Furthermore, the generating set $S^{-1}$ of $\mathcal{G}^*$ has the same associated degrees and also consists of distinct double coset representatives.

**Lemma 3.5** With the given assumption, the atoms of $\mathcal{G}$ partition the vertices of $\mathcal{G}$. The automorphisms of $\mathcal{G}$ induce permutations of the atoms and each atom is a vertex transitive induced subgraph of $\mathcal{G}$.
Proof. The automorphic image of an atom is an atom. By transitivity of \( \mathcal{G} \), the atoms cover \( \mathcal{G} \). Since distinct atoms are disjoint, they partition the vertices of \( \mathcal{G} \). This also implies that the automorphisms of \( \mathcal{G} \) induce permutations of the atoms. Since \( \mathcal{G} \) is transitive, each atom is transitive. 

Let \( A_0 \) be the atom containing \( H \). Let \( S_0 \) be the set of generators \( s \in S \) such that \( s \in \bigcup A_0 \). Let \( S_1 = S \setminus S_0 \).

**Lemma 3.6** The subset \( \bigcup A_0 \) of \( G \) is the subgroup \( \langle H, S_0 \rangle \) generated by \( H \) and \( S_0 \). The edges induced on \( A_0 \) by \( \mathcal{G} \) are given by \( E_{S_0} \cap (A_0 \times A_0) \).

**Proof.** To see that \( \bigcup A_0 \) is a subgroup, let \( g \in \bigcup A_0 \) and consider the automorphism \( \phi_g \). Since \( \phi_g(H) \in A_0 \), \( \phi_g(A_0) = A_0 \). This implies that \( \bigcup A_0 \) is closed under multiplication by \( g \). That \( \bigcup A_0 \) is a subgroup follows by arbitrariness of \( g \).

If \( g_1 H \) and \( g_2 H \) are in \( A_0 \) and there is an edge from \( g_1 H \) to \( g_2 H \) in \( \mathcal{G} \) induced by \( s \), then \( g_2 H \subseteq g_1 H s H \). In particular, there are \( h_1, h_2 \in H \) such that \( g_2 = g_1 h_2 h_2 \). This gives \( h_1^{-1} g_1^{-1} g_2 h_2^{-1} = s \). Since \( \bigcup A_0 \) is a subgroup of \( G \), \( s \in \bigcup A_0 \), so that \( (g_1 H, g_2 H) \in E_{S_0} \).

Note that \( A_0 \) is connected, for otherwise any part of \( A_0 \) with outdegree zero in \( A_0 \) is a smaller part of \( \mathcal{G} \) with at most \( \kappa \) neighbors in \( \mathcal{G} \). Lemma 2.2 implies that \( \bigcup A_0 = \langle H, S_0 \rangle \).

**Example 3.7** Consider again the assumption that \( \mathcal{G} \) has an atom of size at most \((n - \kappa)/2\). There are Cayley graphs which do not satisfy this assumption and where the atoms do not partition the set of vertices. As an example consider the group \( S_n \) of permutations on \( n \geq 4 \) vertices. Using cycle notation for permutations, let \( a = (12) \) and \( b = (123\ldots n) \). Then \( \langle a, b \rangle = S_n \). Let \( \mathcal{G} = \mathcal{G}(S_n, \{(1)\}, \{a, b, ba\}) \). Then \( S_n = \langle a, b \rangle = \langle a, ba \rangle = \langle b, ba \rangle \). Therefore the only candidates for atoms of \( \mathcal{G} \) or \( \mathcal{G}^\ast \) are \( H_1 = \langle \rangle \), \( H_2 = \langle a \rangle \), \( H_3 = \langle b \rangle \) and \( H_4 = \langle ba \rangle \). Let \( N_i = N(H_i) \) and \( N_i^\ast = N_{\mathcal{G}^\ast}(H_i) \). We have

\[
N_1 = \{a, b, ba\} \\
N_1^\ast = \{a, b^{-1}, ab^{-1}\} \\
N_2 = \{b, ba, ab, aba\} \\
N_2^\ast = \{b^{-1}, ab^{-1}\}.
\]

Since \( |H_3| = n \), \( |N_3| \geq n \) and \( |N_3^\ast| \geq n \). Since \( ba \) is a cycle of length \( n - 1 \), \( |N_4| \geq n - 1 \) and \( |N_4^\ast| \geq n - 1 \). Since \( |N_2^\ast| = 2 \), \( \kappa = 2 \). However, none of the
connected subgroups of $S_n$ have 2 neighbors. The atom containing the identity of $G^*$ is given by $H_2$.

Lemma 3.4 shows that to check the vertex connectivity of $G$ it suffices to check the number of neighbors of each subgroup generated by $H$ and a subset of $S$. Since an atom is never the whole graph, the next result is immediate.

**Corollary 3.8** Suppose that for each $s \in S$, $H$ and $s$ generate $G$. Then $G$ is optimally vertex connected.

Except when the degree of $G$ is 1, the size of atoms is strictly smaller than the degree $d$ of $G$. This is a consequence of the next lemma.

**Lemma 3.9** $|N(A_0)| = |(\bigcup A_0)S_1H/H| \geq \max(|A_0|, d_{S_1})$ and $|N(A_0)|$ is a multiple of $|A_0|$.

**Proof.** Let $s \in S_1$. Then $HsH \cap (\bigcup A_0) = \emptyset$, for otherwise $s$ induces an edge in $A_0$. Since $\bigcup A_0$ is a subgroup, it follows that $((\bigcup A_0)HsH) \cap (\bigcup A_0) = \emptyset$. The set $((\bigcup A_0)HsH) = (\bigcup A_0)sH$ is the union of the neighbors of $A_0$ reachable by an edge in $E_s$. Thus the union of the neighbors of $A_0$ is given by $(\bigcup A_0)S_1H$. This implies the first identity. The inequality is obtained by observing that $d_{S_1} = |HsH/H|$. Since $(\bigcup A_0)S_1H$ is a union of right cosets of $(\bigcup A_0)$, $|N(A_0)|$ is a multiple of $|A_0/H| = |A_0|$.

The following theorem generalizes Proposition 3.1 of [11]:

**Theorem 3.10** Let $R_1$ and $R_2$ be a partition of $S$ such that the group $G' = \langle H, R_1 \rangle$ does not contain any members of $R_2$. Let $G' = \mathcal{G}(G', H, R_1)$. Suppose that if $r, s \in R_2$ and $G'G' = G'sG'$, then $\langle H, r \rangle = \langle H, s \rangle$. Then $\kappa(G) \geq \min(|V(G')|, \kappa(G') + d_{R_2})$.

**Proof.** The atom $A_0$ of $\mathcal{G}$ satisfies one of the following cases:

1. $\bigcup A_0 \subset G'$,
2. $\bigcup A_0 \supseteq G'$,
3. $\bigcup A_0$ and $G'$ are incomparable.

Consider case 1. If $r \in R_2$, then $(\bigcup A_0)rH$ is disjoint from $G'$, because otherwise $r \in G'$. Thus the neighbors of $A_0$ due to $R_2$ are disjoint from $\mathcal{G}'$.
and since \(|(\cup A_0)rH/H| \geq |A_0|\), there are at least \(\max(|A_0|, d_{R_2})\) many such neighbors. It follows that

\[ |N(A_0)| \geq |N_{G'}(A_0)| + \max(|A_0|, d_{R_2}). \]

Either \(|N_{G'}(A_0)| \geq \kappa(G')\) or \(A_0 \cup N_{G'}(A_0) = V(G')\), and we are done.

Consider case 2. Since the number of neighbors of \(A_0\) is at least \(|A_0|\), trivially \(|N_{G'}(A_0)| \geq V(G')\).

Consider case 3. Let \(A_0 = A_0 \cap V(G')\). Let \(R_{21} = \{s \in R_2 \mid s \notin \cup A_0\}\). Let \(R_{22} = \{s \in R_2 \mid s \in \cup A_0\}\). Observe that for \(r \in R_{21}\) and \(s \in R_{22}\), \(r\) and \(s\) come from distinct double cosets of \(G'\). Otherwise, by assumption, \(\langle H, r \rangle = \langle H, s \rangle\), which would imply that either both \(r\) and \(s\) are in \(\cup A_0\), or both \(r\) and \(s\) are not in \(\cup A_0\). Thus \(G'R_{21}G'\) and \(G'R_{22}G'\) are disjoint. We have \((\cup A)R_{22}H \subseteq (\cup A_0 \setminus V(G'))\), so that the set \(N_1 = (\cup A)R_{22}HR_{11}H/H\) consists of neighbors of \(A_0\) not in \(G'\). The set \(N_2 = (\cup A)R_{21}H/H\) also consists of neighbors of \(A_0\) not in \(G'\). Since \((\cup A)R_{22}HR_{11}H \subseteq G'R_{22}G'\) and \((\cup A)R_{21}H \subseteq G'R_{21}G'\), \(N_1\) and \(N_2\) are disjoint. It follows that

\[ |N_{G}(A_0)| \geq |N_{G'}(A) + |N_1|| + |N_2|. \]

If \(N_{G'}(A) \cup A = V(G')\), then using \(|N_1| \geq |A|\) we get \(|N_{G}(A_0)| \geq |V(G')|\). If not, then using \(|N_1| \geq d_{R_{22}}\) and \(|N_2| \geq d_{R_{21}}\) gives

\[ |N_{G}(A_0)| \geq |N_{G'}(A)| + d_{R_{22}} + d_{R_{21}} \geq \kappa(G') + d_{R_2}, \]

as desired.

**Corollary 3.11** Let \(\{S_1, \ldots, S_k\}\) be a partition of \(S\). Define \(G_i = \langle H, S_1, \ldots, S_i \rangle\) and \(d_i = d_{S_1} + \ldots + d_{S_i}\). Suppose that the following hold:

1. The \(G_i\) are distinct.
2. If \(r, s \in S_{i+1}\) and \(G_1rG_i = G_isG_i\), then \(\langle H, r \rangle = \langle H, s \rangle\).
3. \(\kappa(G(G_i, H, S_1)) = d_1\).
4. \(|G_i/H| \geq d_{i+1}|.\)

Then \(\kappa(G) = d_S\).
Proof. It suffices to apply Theorem 3.10 to each step in the tower of Cayley coset digraphs $G(G_i, H, S_i)$.

The next corollary shows how to replace the restriction on the size of the groups $G_i$ by conditions on the degrees induced by the partition of the generators.

Corollary 3.12 Let $\{S_1, \ldots, S_k\}$ be a partition of $S$. Define $G_i = \langle H, S_1, \ldots, S_i \rangle$ and $d_i = d_{S_1} + \ldots + d_{S_i}$. Suppose that the following hold:

1. The $G_i$ are distinct.
2. If $r, s \in S_{i+1}$ and $G_i r G_i = G_i s G_i$, then $\langle H, r \rangle = \langle H, s \rangle$.
3. $\kappa(G(G_1, H, S_1)) = d_1$.
4. $|G_1/H| \geq d_2$.
5. For all $i$, $d_{S_{i+1}} \leq d_i$.

Then $\kappa(G) = d_S$.

Proof. It suffices to show by induction that $|G_i/H| \geq d_{i+1}$ and apply Corollary 3.11. Since $G_{i-1}$ is a proper subgroup of $G_i$, $|G_i/H| \geq 2|G_{i-1}/H| \geq 2d_i$. Since $d_{S_{i+1}} \leq d_i$, the result follows.

4 Applications

Hierarchical Cayley coset digraphs.

Definition. $G$ is a quasi-minimal or hierarchical Cayley coset digraph iff there is an ordering $\{s_1, \ldots, s_k\}$ of the generators of $G$ such that the subgroups $\langle H, s_1, s_2, \ldots, s_i \rangle$ are distinct. $G$ is minimal iff for no $S' \subset S$, $\langle H, S' \rangle = G$.

Corollary 3.12 can be simplified for hierarchical Cayley coset digraphs.

Theorem 4.1 Let $G$ be hierarchical with generators ordered by $\{s_1, \ldots, s_k\}$. Let $d_i = d_{S_1} + \ldots + d_{S_i}$ and $G_i = \langle H, s_1, \ldots, s_i \rangle$. Suppose that for each $i$, $d_{S_{i+1}} \leq d_i$ and $|G_1/H| \geq d_2$. Then $\kappa(G) = d(G)$.
Proof. Condition 1 of Corollary 3.12 is satisfied by the definition of hierarchical Cayley coset digraphs. For the partition of $S$ into singletons, condition 2 is trivially satisfied. A Cayley coset digraph generated by a single generator is optimally vertex connected (Corollary 3.8), so that condition 3 is satisfied. Conditions 4 and 5 are satisfied by assumption. ■

Corollary 4.2 The assumption that $|G_1/H| \geq d_2$ in the statement of Theorem 4.1 can be replaced by the assumption that $Hs_1^{-1}H \neq Hs_1H$.

Proof. The assumption that $Hs_1^{-1}H \neq Hs_1H$ is equivalent to the assumption that there are no cycles of length two in $G_1$. This implies $|V(G_1)| \geq 2d_1 + 1 > d_2$. ■

The fact that hierarchical Cayley digraphs are optimally vertex connected can now be easily shown.

Corollary 4.3 (Baumslag [7], Hamidoune [11]) Hierarchical Cayley digraphs are optimally vertex connected.

Proof. Let $G$ be a hierarchical Cayley digraph with generators ordered by $\{s_1, \ldots, s_k\}$. Since $|\langle s_1 \rangle| \geq 2$ and $d_{s_i} = 1$ for each $i$, the result is immediate by Theorem 4.1. ■

Part of the more general result in Hamidoune [11] also follows and was obtained by Hamidoune using a restricted version of Theorem 3.10.

Theorem 4.4 (Hamidoune [11]) Let $G = G(G, \{e\}, S \cup S')$, where $S' \subseteq S^{-1}$ and the elements of $S'$ have order at least three. Assume that $G(G, \{e\}, S)$ is hierarchical with the elements of $S$ ordered by $S = \{s_1, \ldots, s_k\}$. If $|\langle s_1, s_2 \rangle| \neq 4$ then $G$ is optimally vertex connected.

Proof. Partition $S \cup S'$ into the sets defined by: $S_1 = \{s_1, s_2\} \cup (\{s_1^{-1}, s_2^{-1}\} \cap S')$ and for $i > 1$, $S_i = \{s_i, s_{i+1}\}$ if $s_{i+1}^{-1} \notin S'$ and $S_i = \{s_i, s_i^{-1}, s_{i+1}\}$ otherwise. Let $d_i$ and $G_i$ be defined as in Corollary 3.12. We have $2 \leq d_1 \leq 4$, and for each $i \geq 1$, $d_{S_i} \leq 2$. To apply Corollary 3.12 it suffices to check that $G_1 = G(G_1, \{e\}, S_1)$ is optimally vertex connected and $|G_1| \geq d_2$.

If $G_1$ is not optimally vertex connected, then the atom $A$ of $G_1$ is given by either $\langle s_1 \rangle$ or $\langle s_2 \rangle$. Let $s_i \notin A$ and $s_j \in A (\{i, j\} = \{1, 2\})$. We can assume that $2 \leq |A| \leq 3$, for otherwise $|V(A)| \geq |A| \geq 4 \geq d_1$. Similarly, since $|A|$ divides $|N(A)|$, we can assume that $|N(A)| = |A|$. Suppose that $s_i^{-1} \in S'$ so that $s_i \neq s_i^{-1}$. Then the neighbors of $A$ are given by expressions of the form
$s_j s_i$ and $s_j s_i^{-1}$. Since $|N(A)| = |A|$, for some $l \neq l'$, $s_j s_i = s_j s_i^{-1}$. This implies that $s_i^2 = s_j^{l-l'}$ so that $A \cup N(A) = G_1$, contradicting the assumption that $A$ is an atom. Thus $s_i^{-1} \notin S'$. This implies that $d_1 \leq 3$ so that $|A| = 2$. But this implies that $s_j^{-1} \notin S'$, whence $d_1 = 2$ contradicting the assumption that $A$ is a nontrivial atom.

By assumption, $G_1$ has a proper nontrivial subgroup and $|G_1| \neq 4$. This implies that $G_1 \geq 6 \geq d_2$.

Hamidoune [11] continues the analysis of the proof of Theorem 4.4 to show that if $G$ is as in the statement of this theorem and $G$ is not optimally vertex connected, then $k \geq 3$, $\kappa(G) = |S \cup S'| - 1$, $s_i^2 = s_1$ for $i > 1$, $s_1^2 = 1$ and $S' = (S \setminus \{s_1\})^{-1}$.

**Cycle-prefix graphs.** The cycle-prefix graphs (CP-graphs) are Cayley coset digraphs defined on the group $S_n$ of permutations of $[n] = \{1, \ldots, n\}$. They were proposed as interconnection networks with good degree and diameter properties in [3]. If $\pi$ is a permutation which maps $i$ to $\pi_i$, then we write $\pi = \pi_1 \pi_2 \ldots \pi_n$. Application of permutations is on the right, so that $i \pi = \pi_i$. Composition is defined by $i(\pi \sigma) = (i \pi) \sigma$. The cycle-prefix permutations $\gamma(k)$ with $2 \leq k \leq n$ are the permutations which cyclically permute $\{1, \ldots, k\}$ to the right and leave other numbers fixed. Thus $\gamma(k) = k 12 \ldots (k - 1)(k + 1) \ldots (n - 1)n$. Let $H_k$ be the subgroup of $S_n$ consisting of the permutations $\pi$ with $\pi_i = i$ for $i \leq n-k$. For $1 \leq k \leq n - 1$, let $CP(n, k) = G(S_n, H_k, \{\gamma(2), \ldots, \gamma(n-k)\})$. Then $CP(n, k)$ is a Cayley coset digraph of degree $n - 1$. The degrees induced by the generators in $CP(n, k)$ are given by $d_{\gamma(j)} = 1$ for $2 \leq i \leq n-k$ and $d_{\gamma(n-k+1)} = k$.

**Theorem 4.5** $CP(n, k)$ is optimally vertex connected.

**Proof.** The digraphs $CP(n, k)$ are hierarchical with generators ordered by $\{\gamma(2), \ldots, \gamma(n-k+1)\}$. If $k = n-1$, then $CP(n, k)$ is the complete digraph on $n$ vertices and we are done. If $k = 1$, then $CP(n, k)$ is a hierarchical Cayley digraph and is optimally vertex connected by Corollary 4.3. Assume that $1 < k < n - 1$. Let $G' = \langle H, \gamma(2), \ldots, \gamma(n-k) \rangle$ and $G' = G(G', H, \{\gamma(2), \ldots, \gamma(n-k)\})$. Then $G'$ is isomorphic to $CP(n-k, 1)$ and $|G'| = (n-k)!$. This follows from the fact that $\gamma(j)$ is in the normalizer of $H$ for $j \leq n-k$. Thus $G'$ is optimally vertex connected. If $(n-k)! \geq n-1$, Theorem 3.10 can be applied to show that $CP(n, k)$ is optimally vertex connected. Assume that $(n-k)! < n-1$. Then $k \geq n/2$, because $(n+1)/2! \geq n-1$ for all $n \geq 1$. Following the proof of Theorem 3.10, suppose that $A$ is a nontrivial atom of $CP(n, k)$ with $|A| > 1$ and $|N(A)| < n - 1$. Note that $\langle H, \gamma(n-k+1) \rangle = S_n$ which implies that the edges due to $\gamma(n-k+1)$ are not in $A$ and $A \subseteq G'$. For any subgroup $F$ of $G'$ which contains $H$, the number of neighbors of $F/H$ due to $\gamma(n-k+1)$ is
\[ F/H \mid k. \] Proof: \( F \gamma(n - k + 1)H \) is given by the disjoint union \( \bigcup_{l=0}^{k-1} F_l \), where \( F_l \) is the set of permutations defined by

\[ F_l = \{ \pi \mid \pi_1 = n - l \text{ and } \exists \sigma \in F \text{ such that for } 2 \leq i \leq n - k, \pi_i = \sigma_{i-1} \}. \]

Note that for each member \( \sigma \) of \( F \), \( \sigma_{n-k} \) is determined by the \( \sigma_1 \ldots \sigma_{n-k-1} \). Hence \( F_l = F \) which gives \( |F\gamma(n - k + 1)H/H| = |F/H|k. \) As a result, \( |A| < n/k \) and since \( |A| > 1 \) this implies that \( k < n/2 \), contrary to assumption. 

\section{Edge connectivity}

For completeness we include the result that vertex transitive digraphs are optimally edge connected. This result and its proof (for the undirected case) are due to Mader [13].

\textbf{Definition.} The edge connectivity of a digraph \( G \) is the smallest number of edges that need to be removed so that the resulting digraph is not connected. The edge connectivity of \( G \) is denoted by \( \lambda(G) \).

\textbf{Theorem 5.1} Every Cayley coset digraph has edge connectivity equal to its degree.

\textbf{Proof.} Let \( \lambda \) be the edge connectivity of \( G \). We start by showing that there is a notion of atom applicable to edge connectivity. An \( e \)-atom of \( G \) is a minimal subset of the vertices of \( G \) with exactly \( \lambda \) outgoing edges. Let \( N_e(A) \) denote the set of edges leaving \( A \). Let \( E(A) \) denote the set of edges included in \( A \).

\textbf{Lemma 5.2} If \( A \) is an \( e \)-atom and \( B \subset V(G) \) with \( |N_e(B)| = \lambda \), then \( A \subseteq B \) or \( A \cap B = \emptyset \) or \( A \cup B = V(G) \).

\textbf{Proof.} Suppose that \( A \cap B \neq \emptyset \) and \( A \cup B \neq V(G) \).

\[
\begin{align*}
N_e(A \cap B) &= (N_e(A) \cap E(B)) \cup (N_e(A) \cap N_e(B)) \cup (N_e(B) \cap E(A)), \\
N_e(A \cup B) &= (N_e(A) \setminus (E(B) \cup N_e(B))) \cup N_e(B) \setminus E(A),
\end{align*}
\]

where the unions are disjoint. This gives

\[
\begin{align*}
|N_e(A \cup B)| &= |N_e(A) \setminus (E(B) \cup N_e(B))| + |N_e(B) \setminus E(A)| \\
&= |N_e(A) \setminus (E(B) \cup N_e(B))| - |N_e(B) \cap E(A)| + |N_e(B)| \\
&= |N_e(A) \setminus (E(B) \cup N_e(B))| - |N_e(B) \cap E(A)| + \lambda.
\end{align*}
\]
Since \(|N_e(A \cup B)| \geq \lambda|,
\[|N_e(A) \setminus (E(B) \cup N_e(B))| \geq |N_e(B) \cap E(A)|.\]

Hence
\[|N_e(A \cap B)| \leq |N_e(A) \cap E(B)| + |N_e(A) \cap N_e(B)| + |N_e(A) \setminus (E(B) \cup N_e(B))|.\]

Since the right-hand side of this expression is \(|N_e(A)| = \lambda|), minimality of \(A\) requires that \(A \subseteq B\).

Lemma 5.2 implies that the observations about atoms of \(G\) apply to e-atoms of \(G\). In particular, provided that the size of an e-atom is at most \(n/2\), distinct e-atoms are disjoint, so that they form blocks under the group of automorphisms of \(G\). Again, either \(G\) or \(G^*\) satisfies the size condition, so without loss of generality, assume that an atom has at most \(n/2\) elements.

Let \(A_0\) be the e-atom which contains \(H\). Then \(\bigcup A_0\) is a subgroup of \(G\). We can partition the generators as before into the set \(S_0\) of generators in \(\bigcup A_0\) and \(S_1\) of generators outside \(\bigcup A_0\). In this case the analysis is simple: \(A_0\) contains at least \(d_{S_0} + 1\) members each of which has at least \(d_{S_1}\) edges going outside of \(A_0\). Thus \(|N_e(A_0)| \geq d_{S_1}(d_{S_0} + 1) \geq d_{S_0} + d_{S_1} = d_S\). Thus \(|A_0| = 1\) and \(\lambda = |N_e(A_0)| = d_S\). 

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