Low Temperature Expansion of Matrix Models

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We show how to expand the free energy of a matrix model coupled to arbitrary matter in powers of the matter coupling constant. Concentrating on $\nu$ uncoupled Ising models—which have central charge $\nu/2$—we work out the expansion to sixth order for $\nu = 1, 2,$ and $3$. Analyzing the series by the ratio method, we exhibit the spin-ordering phase transition. We discuss the limit $\nu \to \infty$, which is especially clear in the low temperature expansion; we prove that in this limit the dependence of the model on $\nu$ becomes trivial.

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1. Introduction

One of the unsolved questions in matrix models [1] is, how do you introduce matter? This is an important question. If the matrix model is viewed as two-dimensional euclidean quantum gravity, the matter is the statistical system which is coupled to the random surface. If the matrix model is viewed as bosonic string theory, the matter represents the target space in which the string propagates. And at least one phase of nonabelian gauge theory can be reduced to a matrix model with matter [2].

A general framework for coupling matter to matrix models has not been found. The only exactly solved cases are the open chain of matrices, which gives unitary models with central charge $c < 1$, and the one-dimensional case. The region $c > 1$ remains almost completely unknown, although it is easy to formulate matrix models with any central charge. Recently there have been a number of attempts to develop alternate schemes that one hopes could deal with matter in this region [3,4]. The method and results of [3] are promising, but one is nervous about modeling random surfaces out of about ten squares or triangles, which are rather unwieldy, especially if the surfaces are to be branched polymers.

In the study of conventional spin models, low and high temperature expansions have proven valuable. Maybe such expansions can prove useful for spin systems coupled to lattices as well. Before Mehta solved matrix chains [5], Itzykson and Zuber [6] suggested expanding the two-matrix model in low temperature series. This suggestion has been forgotten, but the problem of coupling $c > 1$ matter to a fluctuating surface remains unsolved. Here we develop a low temperature expansion for matrix models. In each order $n$ in the matter coupling constant, we get arbitrarily large surfaces in which all the links join equal spins, except for $n$ “bridge” links. These bridges connect the several “blobs,” open surfaces on which all spins are frozen equal; two blobs connected by one or more bridges have unequal spins. This is the random surface analog of the ordinary low temperature expansion. There are two advantages of this expansion: one can expand a system with arbitrary central charge to rather high (or infinite) order in the cosmological constant, obtaining surfaces which are—hopefully—close to the continuum; and one can use it to study the $c \rightarrow \infty$ limit.
2. The expansion

The partition function of a matrix model coupled to arbitrary matter can be written

\[ Z_Q(g,a) = \int \prod_i D\phi_i \ e^{-\text{Tr} \left[ \sum_i V(\phi_i) - \sum_{ij} Q_{ij} \phi_i \phi_j \right]} \]  

(2.1)

where \( \phi_i \) are hermitian \( N \times N \) matrices. We consider cubic models, with \( V(\phi) = \phi^2/2 + g\phi^3/\sqrt{N} \). The type of matter is encoded by the number of matrices in the model and the matter coupling matrix \( Q \), which depends on \( a = e^{-\beta} \), the matter coupling constant.

Here we specialize to \( \nu \) uncoupled Ising models. They are “uncoupled” only in the bare action, though; each one interacts with the fluctuating surface and they thus interact among themselves. At the critical point of such a model, the central charge of the matter is \( \nu/2 \). For \( \nu \) Ising models, we need \( 2^\nu \) matrices to represent every combination of spins at each site. The coupling matrix \( Q \) can be considered as the connection matrix of the target space graph; the graph is a \( \nu \)-dimensional hypercube, in which the connection strength between any two vertices is \( a \) raised to the power of the dimension of the lowest simplex which contains both vertices. For one or two Ising models, for example, the matrices can be written\(^\dagger\):

\[ Q^{(1)} = \frac{1}{2} \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix} \quad Q^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & a & a & a^2 \\ a & 0 & a^2 & a \\ a & a^2 & 0 & a \\ a^2 & a & a & 0 \end{pmatrix} \]  

(2.2)

In the spherical limit, the free energy of these models is

\[ F^{(\nu)}(g,a) = \lim_{N \to \infty} \frac{1}{N^2} \log \frac{Z^{(\nu)}(g,a)}{Z^{(\nu)}(g,0)} \]  

(2.3)

(which we normalize for later convenience). The problem is to expand \( F^{(\nu)} \) in powers of \( a \). We begin by expanding \( Z^{(\nu)} = z^{(0)} + z^{(1)}_1 a + z^{(2)}_2 a^2 + \cdots \); of course, we will generate both disconnected and connected terms, but the former will be canceled by the logarithm. Consider \( z^{(1)}_1 \):

\[ z^{(1)}_1 = \langle \text{Tr} \phi_1 \phi_2 \rangle = \langle \phi_3^\alpha \rangle \langle \phi_\beta^\beta \rangle, \]  

(2.4)

where in the last expression the averages are with respect to a single matrix model. By expressing the traces in terms of components, we have reduced the calculation to two surfaces (blobs), each of which has uniform spins, with one link (bridge) joining the two.

\(^\dagger\) A superscript in parentheses will refer to \( \nu \).
Similarly, $z_2$ will contain contributions from two blobs joined by two bridges, or three blobs joined in an open chain. Obviously, we can repeat this procedure for all multi-matrix model averages, yielding contractions of one-matrix model Green’s functions.

The object of interest, then, is the one-matrix model average tensor $\langle \phi^{\alpha_1}_{\beta_1} \cdots \phi^{\alpha_n}_{\beta_n} \rangle$. Because of the $\phi \mapsto U^\dagger \phi U$ symmetry, it can only depend on $\delta^{\alpha_i}_{\beta_j}$, and we must keep separate the upper and lower indices. In general,

$$
\langle \phi^{\alpha_1}_{\beta_1} \cdots \phi^{\alpha_n}_{\beta_n} \rangle = \sum_{\pi \in \Pi_n} \lambda_{\pi, \pi} T^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n}(\pi)
$$

(2.5)

$$
T^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n}(\pi) = \delta^{\alpha_1}_{\beta_1} \cdots \delta^{\alpha_n}_{\beta_n}
$$

where $\Pi_n$ is the set of permutations of $n$ objects. Because the matrix components commute, many of the coefficients in (2.5) are equal. We can characterize this by a mapping from permutations to the partitions of the integer $n$, $f : \Pi_n \to P_n$, where $f(\pi)$ is the set of the lengths of the cycles of permutation $\pi$; if $f(\pi) = f(\pi')$, then $\lambda_\pi = \lambda_{\pi'}$. Therefore we define

$$
\langle \phi^{\alpha_1}_{\beta_1} \cdots \phi^{\alpha_n}_{\beta_n} \rangle = \sum_{p \in P_n} \kappa_{n, p} \left[ (T^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n}(\pi_1) + T^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n}(\pi_2) + \cdots) \right]
$$

(2.6)

$$
f(\pi_1) = f(\pi_2) = \cdots = p
$$

For $n = 3$, for example, we have

$$
\langle \phi^{\alpha_1}_{\beta_1} \phi^{\alpha_2}_{\beta_2} \phi^{\alpha_3}_{\beta_3} \rangle = \kappa_{3, 1} \delta^{\alpha_1}_{\beta_1} \delta^{\alpha_2}_{\beta_2} \delta^{\alpha_3}_{\beta_3} + \kappa_{3, 2} \left( \delta^{\alpha_1}_{\beta_1} \delta^{\alpha_2}_{\beta_3} \delta^{\alpha_3}_{\beta_2} + \delta^{\alpha_1}_{\beta_3} \delta^{\alpha_2}_{\beta_2} \delta^{\alpha_3}_{\beta_1} + \delta^{\alpha_1}_{\beta_2} \delta^{\alpha_2}_{\beta_1} \delta^{\alpha_3}_{\beta_3} \right)
$$

(2.7)

$$
+ \kappa_{3, 3} \left( \delta^{\alpha_1}_{\beta_3} \delta^{\alpha_2}_{\beta_1} \delta^{\alpha_3}_{\beta_2} + \delta^{\alpha_1}_{\beta_2} \delta^{\alpha_2}_{\beta_3} \delta^{\alpha_3}_{\beta_1} \right)
$$

(for a given $n$, the $\kappa$’s will be numbered in lexicographic order of the partitions).

The task now is to calculate the coefficients $\kappa_{n, p}(g)$. (The reader may wonder why we do not simply use one-matrix model connected Green’s functions [7] to represent the blobs. We wish that this were possible, but—despite one’s intuition—blobs that make up a planar surface may themselves be counted by nonplanar Green’s functions.) The most straightforward method is to contract (2.6) with the various tensors $T^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_n}(\pi)$, obtaining a closed set of linear equations for $\kappa_{n, p}$, where the inhomogeneous terms are one-matrix averages of products of traces. This becomes quite cumbersome, though, when $n$ gets large. Another method is to contract (2.6) with $\Lambda^{\beta_1}_{\alpha_1} \cdots \Lambda^{\beta_n}_{\alpha_n}$, where $\Lambda$ is some $N \times N$ hermitian tensor; this gives

$$
\langle (\text{Tr } \Lambda \phi)^n \rangle = \sum_{p \in P_n} \mu_{n, p} \kappa_{n, p} \text{Tr } \Lambda^{p_1} \text{Tr } \Lambda^{p_2} \cdots
$$

(2.8)
where \( p_1, p_2, \ldots \) are elements of the partition \( p \), and \( \mu_{n,p} \) is the number of different permutations \( \pi \in \Pi_n \) such that \( f(\pi) = p \).

The averages on the left hand side of (2.8) can be calculated by expanding the external field integral

\[
\mathcal{Z}(g, \Lambda) = \int D\phi \ e^{-\text{Tr}[V(\phi) - \Lambda \phi]}
\]

in powers of \( \Lambda \). Fortunately, this integral has been computed in the spherical limit by Kazakov and Kostov \(^8\) and by Gross and Newman \(^9\) using loop equations.\(^\dagger\) Using Gross and Newman’s notation,

\[
\mathcal{F}(g, \Lambda) = \lim_{N \to \infty} \frac{1}{N^2} \log \mathcal{Z}(g, \Lambda) = -\frac{1}{2N^2} \sum_{a,b} \log(\mu_a + \mu_b) - \frac{1}{6g}(\sigma_2 - x) + \frac{2}{\sqrt{27g}}\sigma_3
\]

\[+ \sigma_1 \sigma_{-1} + \sqrt{\frac{g}{48}} \sigma_1^3 - \frac{1}{108g^2} - \frac{1}{4} \log 3g \]

(2.10)

where \( \mu_a = \sqrt{\lambda_a + x}, \sigma_k = \frac{1}{N} \sum_a 1/(\lambda_a + x)^{k/2} \), \( \lambda_a \) are the eigenvalues of \( \Lambda \), and \( x \) satisfies the equation

\[
x = 1/(12g) - \sqrt{3g}\sigma_1(x).
\]

(2.11)

We first expand \( x = x_0 + x_1 \text{Tr} \Lambda + x_2(\text{Tr} \Lambda)^2 + x_22 \text{Tr} \Lambda^2 + \cdots \); the correct root of (2.11) has \( x_0 = 1/(12g) - 6g + \cdots \). Plugging \( x \) into (2.10) and comparing with (2.8), we can calculate the required coefficients. There is only one subtlety: we do not want Green’s functions where two external legs are connected directly to each other, as this would “short circuit” that surface; \( i.e. \), the number of dissident links would be one less than required. Since the Green’s functions generated by \( \mathcal{F} \) are connected, the problem only occurs in second order; therefore we must subtract the constant term from that Green’s function before we exponentiate \( \mathcal{F} \) to obtain \( \mathcal{Z} \). Proceeding confidently, we can now read off the coefficients. The first few are:

\[
\kappa_{1,1} = \left( -\frac{1}{6g} + \frac{1}{4x_0} + \frac{\sqrt{x_0}}{3g} \right) \sqrt{N}
\]

\[
\kappa_{2,1} = \frac{1}{16x_0^2 N} + \left( -\frac{1}{6g} + \frac{1}{4x_0} + \frac{\sqrt{x_0}}{3g} \right)^2 N
\]

\[
\kappa_{2,2} = \frac{1}{2\sqrt{3g^2x_0}} - \frac{1}{16x_0^2} - 1
\]

(2.12)

\(^\dagger\) This integral seems to be much harder for a quartic than for a cubic potential. That is the reason why we use a cubic potential in this work.
and \( x_0 \) satisfies \( x_0 = 1/(12g) - \sqrt{3g/x} \).

Equipped with the coefficients \( \kappa \), we can directly evaluate the free energies (2.3) for any value of \( \nu \). The \( a^0 \) term is of course just \( 2^\nu \) times the free energy of the one-matrix model [7]. In the Appendix, we give \( F^{(\nu)} \) through order \( a^3 \) where the coefficients are given for arbitrary \( \nu \) as exact functions of \( \nu \) and \( g \); in other words, we have included arbitrarily large surfaces. Beyond that, we have calculated \( F^{(1)} \), \( F^{(2)} \), and \( F^{(3)} \) through order \( a^6 \), where the series coefficients have been expanded to order \( g^{32} \), meaning that we have included surfaces made of up to 32 triangles. We will present a brief analysis of the series in the next section, but first we will give a summary of the difficulties encountered in the calculation, and how we can check it.

When one expands the interaction term in \( Z^{(\nu)} \) in powers of \( a \), in each order one gets large numbers of graphs in target space, which is a \( \nu \)-dimensional hypercube for \( \nu \) Ising models. The number of graphs increases rapidly with both \( \nu \) and the order in \( a \). Many of the graphs are isomorphic, and so give the same results. The choice is between generating many labeled graphs, or many fewer unlabeled graphs. In the latter case (which is what we did up to order 3), however, one has to solve the difficult combinatorical problem of how many ways there are to embed each unlabeled graph in the hypercubic target space. The other major computational difficulty is in expanding the external field integral (2.9); one could, perhaps, speed things up by dropping the quadratic term from \( V(\phi) \) and then shifting \( \phi \) to induce it [10].

We can check \( F^{(1)} \), of course, against the exact result [11]. For more than one Ising model, we can regroup the series into powers of \( g \), and compare with small-surface expansions similar to the ones developed by Brézin and Hikami [3]. This is a series to much higher order in \( a \) but much lower order in \( g \) than what we have. One may, of course, also use the low temperature expansion to check the small-surface expansion.

3. Series Analysis

To exhibit critical behavior, we perform a series analysis modeled after those in [3]. Having expanded to sixth order, we do not expect to calculate the critical exponents, but we will learn something about the critical behavior. We first regroup the series into powers of \( g \). The coefficient \( A_n \) of \( g^n \) is then the sum of the matter partition functions on all the
n-th order graphs, i.e., surfaces of n triangles; we know it only to order $a^6$, though. If we had calculated it to all orders in $a$, its asymptotic behavior at large $n$ would be

$$A_n \approx g_c^{-n} n^{\zeta}$$

(3.1)

where the exponent $\zeta = \gamma_{str} - 3$. From the exact solution for $\nu = 1$ one finds [11] that $\zeta(a) = -7/2$, the pure gravity value, for all $0 \leq a \leq 1$ except the critical point $a^{*\langle 1 \rangle} = (2\sqrt{7} - 1)/27 \approx 0.1589$, which is where the spin ordering phase transition takes place and where $\zeta = -10/3$. This is how we will look for the phase transition.

To calculate $\zeta(c)$, we will use a ratio method [12]. We define $r_n^{[1]} = A_n / A_{n-2}$, $q_n^{[1]} = n(r_n^{[1]} - r_n^{[2]})/2$, and for $u > 1$,

$$r_n^{[u]} = \frac{nr_n^{[u-1]} - (n - 2u + 2)r_n^{[u-1]}}{2p - 2}$$
$$q_n^{[u]} = \frac{nq_n^{[u-1]} - (n - 2u + 2)q_n^{[u-1]}}{2p - 2}$$
$$s_n^{[u]} = r_n^{[u]}/q_n^{[u]}$$

(3.2)

Naively, the asymptotic behavior as $n \to \infty$ should be

$$r_n^{[u]} \approx g_c^{-2} [1 + O(n^{-u})]$$
$$s_n^{[u]} \approx \zeta [1 + O(n^{-u})]$$

(3.3)

This is true, however, only if there are no confluent singularities. Another difficulty with the method is that as $u$ gets large, the coefficient of the $n^{-u}$ term can get large as well.

We start with one Ising model. In fig. [1] we plot $s_n^{[u]} \approx \zeta(a)$ from the sixth-order series for various iterations of the ratio method, $u = 3, \ldots, 7$. The known value of $a^{*\langle 1 \rangle}$ is shown as a vertical line. At $a = 0$, $\zeta$ is very close to its exact value for pure gravity, $-7/2$; for $u = 4$, for example, we have $\zeta \approx -3.4991$. These graphs are not exactly what we would have expected; the peak, for one, is much too high. Nonetheless, fig. [1] does give qualitative evidence for the spin-ordering phase transition. Moreover, as we increase $u$, we obtain increasingly accurate values for $a^*$ (the closest one being 0.1547 at $u = 6$),† as well as lower peaks. The ratio approximants seem to deteriorate, however, after $u = 4$, developing a second peak. We will therefore use a compromise value, $u = 4$.

† One can also get very good estimates for $a^*$ from the peak in the specific heat, $C \approx \frac{d^2}{da^2} \log g^*$. 

6
In figs. 2–4, we plot $\zeta^{(\nu)}(a)$ for $\nu = 1, 2,$ and 3, keeping terms up to orders 3–6 in $a$. The right-hand side of the graphs is given only for completeness, as there is no reason to trust the expansion when $a$ is not small; if we had expanded to all orders in $a$, the curve would come back to $-7/2$ at $a = 1$, because at infinite temperature each spin fluctuates independently. Two things can be claimed with certainty: the plots are showing evidence for the spin-ordering phase transition which becomes stronger and more realistic as we expand to higher orders in $a$; and the critical point gets closer to zero as the central charge increases.

4. The $\nu \to \infty$ limit

As can be seen for the first three orders from the expression for the free energy in the Appendix, apart from an overall normalization, the coefficient of $a^n$ in the free energy is an $n$-th degree polynomial in $\nu$. This property, although not a priori obvious, can be shown to hold to all orders. The easiest way to see it is through the small-surface expansion \[3\]. There, the coefficient of $g^m$ in the free energy is a sum of the (one Ising model) partition functions of all surfaces of area $m$, with each partition function raised to the power $\nu$. Each of those partition functions is a $3m/2$-degree polynomial in $a$, so its $n$-th power cannot have any higher power of $\nu$ than $\nu^n$. This proves the assertion.

We can see immediately that the asymptotic dependence of $F^{(\nu)}$ on $\nu$ becomes trivial as $\nu$ (and therefore the central charge) approaches $\infty$, since $\nu$ will then simply be a multiplicative renormalization for $a$. The critical exponents of the system will be independent of $\nu$, and the critical temperature will have the asymptotic behavior

$$a^{*\nu} \to \frac{a_{\infty}}{\nu} \quad \text{as} \quad \nu \to \infty. \quad (4.1)$$

This is already approximately true in our results for up to $\nu = 3$. In fig. 3 we plot $a^{*\nu}$ for $\nu = 1, 2,$ and 3, showing the plausibility of the asymptotics $(4.1)$. It is also intriguing to note that in the third-order expression for the free energy in the Appendix, the graphs which appear as coefficients of $\nu^n$ in order $n$ are all trees. If this behavior can be shown to hold in all orders, this will prove that surfaces in the $c \to \infty$ limit are branched polymers.
5. Discussion

In this letter, we have shown how to expand a matrix model coupled to arbitrary matter in powers of the matter coupling constant. We have shown that already at order $a^6$ one can qualitatively observe the spin-ordering phase transition, with good quantitative results for the critical coupling. The immediate extension to this work would be to expand to higher orders (in $a$ and $g$), and to refine the series analysis. The reader should know that the present work was carried out with rather simple algorithms programmed in the useful but very slow *Mathematica* language, running on an overloaded Iris 4D/480S; altogether, the calculations took about a day of real time to execute. For conventional spin models, the low temperature series are typically known to order 20 or 30, which gives accurate results for exponents; perhaps one could push the present expansion as far?

Another possibility is to expand other thermodynamic quantities than the free energy. The magnetic susceptibility series, for instance, is known to converge rapidly. It is not hard to add a magnetic field to a matrix model \[11\]. One could, of course, also experiment with different types of matter such as Potts models.

Finally, the limit $\nu \to \infty$ is worth studying. This limit becomes quite clear in the context of the low temperature expansion: one must calculate the $\nu^n$ coefficient of the $a^n$ term. Many of the calculations leading to the low temperature series drastically simplify in this limit. The combinatorics on a hypercube, for example, is much simpler when its dimension goes to infinity. There exist predictions for behavior of a random surface embedded in $D \to \infty$ dimensions \[13\] which can be tested. In this way, perhaps we could learn whether the central charge $c \to \infty$ limit is universal.

Appendix

Here we give the free energy for arbitrary $\nu$ to first three orders.† $\kappa_{n,p,\ell}$ is the coefficient

† As in eq. (2.3), the expression here does not include the homogeneous (pure gravity) term.
of the $N^\ell$ term in $\kappa_{n,p}$.

\[
\frac{F}{2^\nu} = \frac{1}{2} \kappa_{1,1,\frac{1}{2}}^2 \nu a + \\
\frac{1}{4} \left[ \left( \kappa_{2,1,\frac{1}{2}}^2 + 2 \kappa_{1,1,1}^2 \kappa_{2,1,-1} + 2 \kappa_{1,1,1}^2 \kappa_{2,2,0} \right) \nu^2 \\
\left( \kappa_{2,2,0} - \kappa_{1,1,\frac{1}{2}}^2 \right) \nu \right] a^2 + \\
\frac{1}{12} \left[ \kappa_{1,1,\frac{1}{2}}^2 \left( 1 + 6 \left( \kappa_{2,1,-1} + \kappa_{2,2,0} \right) + 6 \left( \kappa_{2,1,-1} + \kappa_{2,2,0} \right)^2 \\
+ 2 \kappa_{1,1,\frac{1}{2}} \left( \kappa_{3,1,-\frac{1}{2}} + 3 \kappa_{3,2,-\frac{1}{2}} + 2 \kappa_{3,3,-\frac{1}{2}} \right) \nu^3 + \\
+ 3 \kappa_{1,1,\frac{1}{2}} \left( 2 \kappa_{2,2,0} \kappa_{3,2,-\frac{1}{2}} + 4 \kappa_{2,2,0} \kappa_{3,3,-\frac{1}{2}} - \kappa_{1,1,\frac{1}{2}} \right) \nu^2 + \\
2 \left( \kappa_{1,1,\frac{1}{2}} + \kappa_{3,3,-\frac{1}{2}} \right) \nu \right] a^3 + O \left( c^4 \right)
\]

(1)

The $\kappa_{n,p}$ for $n \leq 0$ were given in the text; the third-order coefficients are:

\[
\kappa_{3,1} = \left( \frac{3g + 2x_0 \sqrt{3gx_0}}{8x_0^3(3g - 4x_0^3)} \right) N^{-5/2} + 3 \kappa_{1,1,\frac{1}{2}} \kappa_{2,1,-1} N^{-1/2} + \kappa_{1,1,\frac{1}{2}}^3 N^{3/2}
\]

(2)

\[
\kappa_{3,2} = - \frac{N^{-3/2}}{16x_0^3} + \kappa_{1,1,\frac{1}{2}} \kappa_{2,2,0} N^{1/2}
\]

\[
\kappa_{3,3} = \left( \frac{1}{32x_0^3} - \frac{1}{8x_0 \sqrt{3gx_0}} \right) N^{-1/2}
\]
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Figure Captions

Fig. 1. $\zeta(a)$ for $\nu = 1$ (one Ising model), different iterations of the ratio method: $u = 3, \ldots, 7$. The vertical line marks the known critical point.

Fig. 2. $\zeta(a)$ for $\nu = 1$, to orders $a^3, a^4, a^5$, and $a^6$

Fig. 3. Same as fig. 2. $\nu = 2$

Fig. 4. Same as fig. 2. $\nu = 3$

Fig. 5. The critical point $a^{*}(\nu)$ for $\nu = 1, 2, \text{ and } 3$
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Figure 1
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