Calculating $B_K$ using HYP staggered fermions

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We give an update on our calculation of $B_K$ using HYP-smeared valence staggered quarks. We have results for $B_K$ at tree-level on several coarse MILC lattices ($a \approx 0.12$ fm) and one of the fine lattices ($a \approx 0.09$ fm), using 10 light valence quarks ranging down to $m_{\text{phys}}/10$. We have generalized staggered chiral perturbation theory to our mixed action setup, and outline the results. We explain our present fitting strategy, and give some preliminary results.
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1. Introduction

An accurate result for the kaon $B$-parameter is important both for its phenomenological impact \([1]\) and as a bellwether of the success in incorporating chiral symmetry and controlling systematics in lattice calculations. Calculations using several different fermion methods are underway, with the present best result using domain-wall fermions \([2]\). We are pursuing a calculation using improved staggered fermions. This has the advantage of being computationally cheap, but the challenge of dealing with the effects of taste-breaking in a context where there is operator mixing \([3]\).

We use the standard staggered action with HYP-smeared links for our valence fermions, which reduces taste-breaking by a factor of 3 compared to asqtad quarks \([4,5]\). We use the MILC lattices generated with 2+$1$ flavors of asqtad sea quarks. For the coarse MILC lattices, on which we focus here, the resultant taste-breaking in the pion masses, while reduced compared to asqtad quarks, remains large enough that we must use the standard power-counting of staggered chiral perturbation theory, in which $a^2 \approx p^2$. The complications that this introduces have been explained in Ref. \([3]\).

Our method for calculating $B_K$ using wall sources is explained in Ref. \([6]\). For each lattice we use 10 valence quark masses running from $\approx m_{\text{phys}}$ down to $\approx m_{\text{phys}}/10$ in equal steps, and we calculate $B_K$ and $m_K$ for the Goldstone taste for all 55 quark-mass combinations. Results for one-loop matching factors using the mixed action are not yet complete so we use tree-level matching. For this, and other reasons to be explained, all results obtained here should be regarded as very preliminary. They are essentially our first pass at fitting the data, which we are using to inform subsequent fitting and to determine where improvements in statistics are needed.

2. Staggered chiral perturbation theory for a mixed action

Staggered chiral perturbation theory ($S\chi$PT) \([7,8]\) incorporates discretization errors into the chiral expansion, and in particular includes the effects of taste-breaking and rooting. For our lightest kaons, such effects are comparable to those coming from the explicit chiral symmetry breaking due to quark masses, and thus enter $S\chi$PT at LO. A major effect in $S\chi$PT is that chiral loops (which begin at next-to-leading-order [NLO]) must be evaluated with the masses of the pions of the appropriate tastes, rather than a common mass. For $B_K$, which involves an insertion of the weak Hamiltonian, one must also deal with mixing between operators having different tastes. This leads to a significantly larger number of unknown coefficients multiplying NLO terms than are present in the continuum, as explained in Ref. \([3]\).

The $S\chi$PT analysis of Ref. \([3]\) does not apply directly to our set-up, however, because we use a mixed action. Following the methods developed in other mixed-action contexts \([2]\), we have generalized the results of Ref. \([3]\) to our setup. This turns out to be straightforward. Here we only give a summary—further details will be presented in Ref. \([10]\).

There are three classes of effects resulting from using a mixed action. The first, which corresponds to the short-distance parts of sea-quark loops, is simply that the coefficients multiplying various terms in the $S\chi$PT expression for $B_K$ will change. Since these coefficients were previously unknown, however, this has no practical impact—one trades one set of unknown coefficients for another.
The second class comes from loop diagrams involving mixed pions—those composed, say, of a valence (HYP) quark and a sea (asqtad) antiquark. These, however, are absent for $B_K$ at NLO.

The third class involves loop diagrams with pions composed of sea-quarks alone. For $B_K$ at NLO these loops all include “hairpin” vertices, for this is the only way in which the long-distance part of sea-quark loops can enter. The effect of using a mixed action then boils down to the need to distinguish between three types of hairpin vertices—valence-valence, valence-sea and sea-sea—all of which would be the same if the same valence and sea quarks were being used. It turns out that this has an impact only for tastes $V$ and $A$ (the taste singlet hairpin coupling to a particle which is being integrated out anyway), so that one ends up with 6 hairpin parameters (3 for each of two tastes) instead of 2. We name the hairpin parameters $\delta'_{ss}^B$, $\delta'_{sv}^B$, and $\delta'_{sv}^B$, with $B = A$ or $V$, and the superscript indicating the types of quark involved. They appear at NLO in the combinations

$$
\delta_{MA1}^B = \frac{(\delta_{ss}^B)^2}{\delta_{ss}^B}, \quad \delta_{MA2}^B = \left[ \delta_{ss}^B \delta_{ss}^B - (\delta_{sv}^B)^2 \right] / \delta_{ss}^B. \tag{2.1}
$$

These only enter in the expressions for non-degenerate quarks—those for degenerate quarks in the kaon do not involve hairpin vertices. Specifically, in the contribution denoted $M^PQ_{disc}$ in Ref. [3], and given in eq. (50) of that paper, one must, for both $B = V$ and $A$, make the substitution

$$
\delta_B^s \to \delta_{MA1}^B, \quad \delta_{MA2}^B \to \delta_{ss}^B. \tag{2.2}
$$

and add the following new term

$$
\alpha^2 \delta_{MA2}^B \frac{2 C^B_X + C^B_Z}{\pi^2 f^4} \left( 2 \frac{\ell(X_B) - \ell(Y_B)}{Y_B - X_B} + \bar{\ell}(X_B) + \bar{\ell}(Y_B) \right). \tag{2.3}
$$

Here we use the notation $X_B$ ($Y_B$) for the mass-squared of the flavor non-singlet valence pion with taste $B$ and composition $\bar{x}x$ ($\bar{y}y$), where the kaon itself has the composition $\bar{x}y$ (and taste $P$). In the notation of Ref. [3] $X_B = m^2_{X_B}$ and $Y_B = m^2_{Y_B}$. The functions $\ell$ and $\bar{\ell}$ are chiral logarithms, and are defined in Ref. [3].

The original hairpins, $\delta_{ss}^B$, enter through their (unchanged) contributions to the masses of the flavor-singlet mesons, $\eta_B$ and $\eta'_B$.

The hairpin parameters are a measure of taste-symmetry breaking, and we expect them to satisfy a similar hierarchy to that we observe in the pion spectrum, namely $\delta_{ss}^B / \delta_{ss}^s \approx 1/3$. In words, hairpins correspond to quark-antiquark pairs communicating through intermediate gluons, and the taste-breaking component is, by construction, reduced for our HYP-smeared valence quarks. Based on this argument we also expect that $(\delta_{sv}^B / \delta_{ss}^B)^2 \approx 1/3$. Combining these expectations we find

$$
\delta_{MA1}^B \approx \delta_{ss}^s / 3 \quad \text{and} \quad \delta_{MA2}^B \approx 0. \tag{2.4}
$$

If these expectations are accurate, then using a mixed action has essentially no impact on chiral fitting, because the new terms proportional to $\delta_{MA2}^B$ [eq. (2.3)] can be dropped, and the terms proportional to $\delta_{MA1}^B$ were present anyway with an unknown coefficient.

3. Fitting strategy

The NLO expression for $B_K$ in partially-quenched mixed-action $S\chi$PT takes the form

$$
B_K = \sum_{i=1}^{16} c_if_i, \tag{3.1}
$$
in which \( f_i \) are known functions and \( c_i \) are coefficients to be determined. Of the 16 unknown coefficients, 1 \((c_0)\) appears at LO (and is the value of \( B_K \) when \( a \to 0 \) and then the chiral limit is taken), 4 are the NLO low-energy constants (LECs) present in the continuum, and the remaining 11 are LECs due to lattice artifacts and truncated perturbative matching. For degenerate quarks these numbers reduce to 9 coefficients = 1 LO + 3 continuum LECs + 5 “lattice LECs”. This counting is for a single lattice spacing—the dependence of the artifacts on \( a \) involves a mix of \( a^2, \alpha^2 \) and \( \alpha^2 a^2 \) \([\text{3}]\).

The functions \( f_i \) depend on the masses of the flavor-non-singlet valence pions of all tastes and all compositions \((\bar{x}, \bar{y} \text{ and } \bar{y})\), which we determine as part of our calculation \([\text{5}]\). In addition, we need the masses of the sea-quark \( \ell \ell \) and \( \bar{s}s \) flavor non-singlet pions with taste \( V, A \) and \( I \), and of the flavor singlet \( \eta_B \) and \( \eta_B^* \) for \( B = V \) and \( A \). These we take from the results of the MILC collaboration (including the axial and vector hairpin vertices \( \delta_B^{(a)} \) \([\text{1}]\)). The final inputs we need are \( a \) (we use the MILC values) and \( f \) (which, for the moment, we simply set to \( f = 132 \text{MeV} \)).

Although we have 10 degenerate and 45 non-degenerate data points on each lattice, a direct fit to eq. (5.1) is difficult, since many of the fit functions are similar. As a first stage, therefore, we use only a few representative “lattice” contributions, while keeping all the continuum terms. We also use Bayesian priors to constrain some terms, since we know their orders of magnitude.

The fit functions we use are, firstly,

\[
f_1 = 1 + \frac{3}{8f^2G} [\mathcal{M}_{\text{conn}} + \mathcal{M}_{\text{disc}}]_{\text{continuum}},
\]

where \( G = m_{\pi}^2 \), \( \mathcal{M}_{\text{conn,disc}} \) come from quark connected/disconnected 1-loop diagrams and are given in Ref. \([\text{1}]\), and the superscript indicates that we keep only the contributions from chiral operators present in the continuum. We do, however, include taste breaking in the pion masses in these contributions. We set the scale in the chiral logarithm to \( \mu = 1 \text{GeV} \). Next we include the continuum analytic terms (with the \( \chi \text{PT} \) scale set to \( \Lambda = 1 \text{GeV} \)):

\[
f_2 = \frac{G}{\Lambda^2}, \quad f_3 = \frac{(G/\Lambda^2)^2}{\mathcal{M}_{\text{conn}}}, \quad f_5 = \frac{(X_P - Y_P)^2}{(G\Lambda^2)},
\]

Note that \( f_3 \) is a NNLO contribution, but is needed to fit our data up to the highest quark masses.

Finally, we include three representative discretization terms. We use the contribution to \( \mathcal{M}_{\text{conn}} \) containing taste-\( T \) pions, which contains two parts with independent coefficients \([\text{3}]\):

\[
f_6 = \frac{3}{8fG} \left\{ \ell(X_T) + \ell(Y_T) - 2\ell(m_{\pi,T}^2) \right\}
\]

and the remainder which we call \( f_4 \). We also include the likely dominant contribution to \( \mathcal{M}_{\text{disc}} \), which is that proportional to \( \delta_A^{MA1} \). We set \( c_7 = (2C^2 + C_{X}^2) \alpha_s \delta_A^{MA1}/(2\pi^2 f^2) \) and \( f_7 \) is the coefficient of this term in eq. (50) of Ref. \([\text{3}]\). Note that \( f_{5,6,7} \) contribute only for non-degenerate quarks.

Based on the power-counting of Ref. \([\text{3}]\), the coefficients should have the magnitudes:

\[
c_1 \approx c_2 \approx c_3 \approx c_5 \sim O(1)
\]

\[
c_4 \approx c_6 \approx \Lambda_{\text{QCD}}^2(a\Lambda_{\text{QCD}})^2 = 0.003 \quad \text{to} \quad \Lambda_{\text{QCD}}^2 \alpha_s^2 = 0.01 \text{GeV}^2 \text{ on the coarse lattices. } \quad \text{(3.4)}
\]

\[
c_7 \approx \Lambda_{\text{QCD}}^4(a\Lambda_{\text{QCD}})^2 = 0.0003 \quad \text{to} \quad \Lambda_{\text{QCD}}^4 \alpha_s^2 = 0.001 \text{GeV}^4 \text{ on the coarse lattices.}
\]
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Figure 1: Fits of $B_K$ (with degenerate quarks) vs. $m_K^2$ on the MILC coarse lattices with $am_\ell = 0.01$ and $am_s = 0.05$. The fits are to continuum PQ$\chi$PT (left) and S$\chi$PT (right), as described in the text.

Here, we use $\Lambda_{QCD} \approx 0.3$ GeV and $\alpha_s = \alpha_{\overline{MS}}(\mu = 1.6\textrm{GeV}) \approx 0.36$. The smallness of $c_{4,6,7}$ is somewhat offset by the fact that $f_{4,6,7}$ are logarithmically divergent in the chiral limit.

4. Examples of fits

We first try fitting to the degenerate data without the lattice terms, so that only $c_{1-3}$ are non-zero. We compare using only the Goldstone-kaon mass in loops (“PQ$\chi$PT” fit) to using the appropriate combination of all tastes (“S$\chi$PT”). The resulting fits are shown in Fig. 1. The S$\chi$PT fit is significantly better, because its smaller curvature at small quark masses more accurately represents our data. This is the familiar “softening” of the chiral logarithms caused by the heavier masses of non-Goldstone taste pions.

| fit type | $c_1$   | $c_2$   | $c_3$   | $c_4$   | $c_5$   | $c_6$   | $c_7$     | $\chi^2$/dof |
|----------|---------|---------|---------|---------|---------|---------|----------|---------------|
| D-T4     | 0.39(1) | -.01(1) | 0.89(17)|         |         |         |          | .05(11)      |
| D-BT4    | 0.390(3)| -.011(7)| 0.90(4) | .0001(4)|         |         |          | .07(16)      |
| ND-BT4   | 0.390(2)| -.011(1)| 0.90(2) | .0001(2)| .13(7)  | -.011(7)| .0018(8) | .06(7)       |
| ND-T2    | 0.31(7) | .6(5)   | 0.23(50)| .002(2) | .04(4)  | -.005(4)| .0010(5) | .03(2)       |

Table 1: Fitting parameters. Fits are described in the text.

The degenerate data itself shows no indication of a logarithmically divergent contribution (as would be produced by $f_4$). In a first attempt to quantify this, we try and make use of the fact that the dominant contribution from $f_4$ is for the lightest few kaon masses. Thus we drop the lightest mass point from the S$\chi$PT degenerate fit, giving a fit we name “D-T4”, whose parameters we list in Table 1. They have the expected magnitudes (except perhaps for $c_2$, but this is scale dependent,

\footnote{We fit using an uncorrelated $\chi^2$, with errors in fit parameters determined by jackknife. Thus only the relative goodness of fit can be estimated, but not the absolute goodness of fit.}
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Figure 2: $\Delta B_K$ vs. $m_K^2$, including degenerate (red) and non-degenerate (blue) data.

and becomes $\approx 0.4$ if $\mu = 0.77$ GeV instead of $\mu = 1$ GeV. We then do a fit to all 10 degenerate points including $f_4$, but with $c_{1-3}$ constrained, using the values and errors from D-T4 as Bayesian priors [12]. This fit (“D-BT4” in Table 1), has a very small $c_4$, with the $f_4$ term making no more than a 1% contribution. Finally, we fit to the full data set (55 points) now including $f_{5-7}$, but with $c_{1-4}$ constrained using the results from fit D-BT4. The resulting fit we call “ND-BT4”. We have also done an unconstrained fit using all 7 $f_i$ to all 55 data points—fit “ND-T2” in the Table.

The difference between fits ND-BT4 and ND-T2 indicates the size of the present uncertainty in the coefficients. While this is substantial, it is encouraging that the coefficients have sizes roughly consistent with the estimates [3,4]. Furthermore, when we use the fit form to determine our best estimate for the continuum $B_K$ for physical kaon masses, we obtain consistent values: 0.76(6) and 0.67(4). It is also encouraging that the overall fits looks reasonable. We illustrate this in Fig. 2, where we compare the residuals, $\Delta B_K(x) = B_K(x) - f(x)$ for a continuum PQ$\chi$PT fit ($c_{4,6,7} = 0$, Goldstone kaon masses only) to the S$\chi$PT fit (ND-BT4). We notice a significant improvement in the fitting quality using S$\chi$PT.

We have repeated this analysis on four other coarse MILC ensembles and one fine ensemble. Some results are collected in Table 2. All we can conclude at this stage is that there is a rough consistency between different coarse ensembles, and that discretization errors are not enormous. We stress again that these results use tree-level matching and so are very preliminary.

5. Conclusion

We have taken the first stab at fitting our mixed-action results for $B_K$. We clearly have a lot of work to do to control the systematic errors (and at this stage cannot quote a continuum result with all errors estimated). It is important to keep in mind, however, that the main goal of

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$^2$The relatively large size of $c_7$ in fit ND-BT4 is possible, in part, because of a cancellation with the $c_6$ contribution, and it may be better to directly constrain both these coefficients to have smaller magnitudes.

$^3$These values are obtained by setting $c_{4,6,7} = 0$. We stress, however, that they still contain those discretization errors that are absorbed into $c_1$, since they are based on fits at a single lattice spacing.
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| $a$ (fm) | $am_1/am_s$ | geometry | ens   | $B_K$( tree, ND-BT4) | $B_K$( tree, ND-T2) |
|---------|-------------|----------|-------|----------------------|----------------------|
| 0.12    | 0.03/0.05   | $20^3 \times 64$ | 564   | 0.83(6)              | 0.63(4)              |
| 0.12    | 0.02/0.05   | $20^3 \times 64$ | 486   | 0.72(6)              | 0.71(4)              |
| 0.12    | 0.01/0.05   | $20^3 \times 64$ | 671   | 0.76(7)              | 0.67(4)              |
| 0.12    | 0.007/0.05  | $20^3 \times 64$ | 651   | 0.87(5)              | 0.62(3)              |
| 0.12    | 0.005/0.05  | $24^3 \times 64$ | 509   | 0.80(4)              | 0.66(3)              |
| 0.09    | 0.0062/0.031| $28^3 \times 96$ | 995   | 0.72(4)              | 0.62(3)              |

Table 2: Comparison of tree-level $B_K$ from different ensembles (preliminary)

the fitting is to provide a reasonable extrapolation formula to the physical quark masses. We are less interested in the coefficients themselves (except for $c_0$, which can be compared to results from large $N_c$ approaches). We are also investigating fits based on (mixed-action staggered) SU(2) chiral perturbation theory, which appear to be much simplified because discretization terms are of NNLO. We also expect that the more extensive data on the fine lattices which we are presently collecting should be more straightforward to fit. Finally, we are improving our statistics on several of the coarse and fine ensembles.

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