Why and how to add direction to a quantum walk

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Abstract
We formalize the treatment of directed (or chiral) quantum walks using Hermitian adjacency matrices, bridging two developing fields of research in quantum information and spectral graph theory. We display results and simulations which highlight the conceptual differences between having directions encoded in the Hamiltonians or not. This leads to a construction of a new type of quantum phenomenon: zero transfer between pairs of sites in a connected coupled network, which is only possible in the directed model we study. Our main result is a description of several families of directed cycles that admit zero transfer.

1 Introduction

The interplay between graph theory and quantum information has been widely investigated [1, 2, 4, 8, 10]. In the context of quantum walks, this is caused by the very natural way in which a graph models the quantum system defined by Hamiltonians which are symmetric and real and encode couplings between qubits [11, 12]. The amount of works devoted to understand this connection is proportional to the extensive existing literature of spectral graph theory, most of which is devoted to study of how the spectral properties of the adjacency matrix or versions of the Laplacian matrix for undirected graphs relate to quantum properties [3, 15, 16].

Recently, there has been a surged interest in using Hermitian complex matrices, which very conveniently provide models for arc direction in directed graphs [24]. From the graph theoretic point of view, several works have focused on studying how
the combinatorics of arc direction is manifested in spectral properties of Hermitian adjacency matrices [18]. This motivates the seek of which quantum information phenomenon can be achieved in this more general setting of Hermitian quantum walks which in contrast are not available for the restricted undirected model [9, 17].

Notably, the possibility of setting up a system in which a qubit state hops from one site to any other at different times with perfect fidelity and with no time-dependent control on the Hamiltonian can only be achieved with Hermitian complex Hamiltonians [13].

Achieving state transfer over large distances in finite qubit networks has also been a task pursued since early works [6, 7]. This has been found possible with real symmetric Hamiltonians only upon modulating the couplings with high energy [19], and unfortunately allowing for complex weightings is not helpful, as we verify in Sect. 2.

Investigations of directed quantum walks on chains (paths) and rings (cycles) have appeared recently [28]; special attention has been paid to the possibility of shielding part of the network to the state evolution initiated somewhere else. According to Sett et al. [26], this could show as a possible way to indirectly measure decoherence in systems which we cannot control the coupling between all of the qubits. This is our main result: we show in Sect. 4.2 how to achieve zero transfer at all times between antipodal pairs of sites in ring networks upon adding certain complex weights. Although this phenomenon was described earlier in the literature, its connection with the combinatorial properties of the graph was absent. This generalizes known examples and displays an interesting connection between some old theorems, opening avenues for future exploration.

The paper is organized as follows. We begin our treatment formalizing our Hamiltonian model in Sect. 2, showing how it decomposes into invariant subspaces corresponding to the $k$-excitation subspaces, and how the block corresponding to the 1-excitation subspace is precisely the Hermitian adjacency matrix that appears in [18]. We also add simulations with interesting results for the complete graphs (all qubits coupled). Then, we introduce and show the result mentioned above: zero transfer in cycles in Sect. 4.2.

## 2 Hamiltonian for directed quantum walks

### 2.1 Setting

The continuous-time quantum walk can be defined by a state $|\psi(t)\rangle$ which varies according to the time $t$. This state is encoded in a set of $n$ qubits corresponding to $\mathbb{C}^{2^n}$, and its evolution is governed by the Schrödinger’s equation

$$i \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle,$$  \hspace{1cm} (1)

where the Hamiltonian $H$ is a self-adjoint operator acting on this space, and we have set $\hbar = 1$. This operator might encode the Laplacian matrix or the adjacency matrix of a given graph, understanding that $H$ defines the kinetic energy of a free particle.
Solving this equation for time-independent $H$ and initial state $|\psi(0)\rangle$, and including real constants into the parameter $t$, we obtain the solution

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle.$$  

(2)

Certain choices for $H$ allow for a block decomposition in which each block corresponds to the subspace spanned by the global states where precisely $k$ qubits are at $|1\rangle$, and the remaining at $|0\rangle$. Thus, there are $n+1$ subspaces, one for each $k \in \{0, \ldots, n\}$, each of dimension $\binom{n}{k}$ [25]. If $H$ is the XY-Hamiltonian (apparently also called XX in some texts), the block corresponding to $k = 1$ coincides with the adjacency matrix of the underlying graph that describes the couplings. If $H$ is the Heisenberg Hamiltonian, one observes the Laplacian matrix of the underlying graph in the same $k = 1$ block [11].

The XY-Hamiltonian is defined in terms of two-body interactions determined by the edge set of the graph. Given a graph $G = (V, E)$, defined by the vertex set $V$ and edge set $E$, the vertices correspond to qubits, and we can write the Hamiltonian matrix as:

$$H = \frac{1}{2} \sum_{ab \in E(G)} X_a X_b + Y_a Y_b,$$

where $X_k$ corresponds to the operator which acts as the Pauli matrix $X$ onto the qubit in position $k$, and analogously for $Y_k$ and the Pauli matrix $Y$. It is possible to input real weights multiplying $(X_a X_b + Y_a Y_b)$ and still obtain the block decomposition described above, with the Hamiltonian corresponding to the 1-excitation subspace now being the real symmetric weighted adjacency matrix of a graph.

### 2.2 Encoding direction

A directed graph can be conveniently represented by means of an Hermitian adjacency matrix. For example, if there is a directed arc from $a$ to $b$, the $(a, b)$ entry can be set to $i$ and therefore the $(b, a)$ entry to $-i$. In this model, it is possible to replace a pair of arcs of opposing direction between a pair of vertices by an undirected edge of weight $+1$ in the Hermitian adjacency matrix. If instead of $i$, one chooses $e^{i\pi/3}$ to encode direction, then the weight for an undirected edge is exactly the sum of the weights of a pair of opposing direction arcs. This convenient fact was explored in [18], in a context unrelated to quantum walks.

Graphs for which direction of an edge is encoded in the adjacency matrix upon the use of complex numbers of norm 1 are also known as complex unit gain graphs.

A natural question at this point is whether there is a Hamiltonian model (preferably defined in terms of 2-body interactions only) whose action onto the 1-excitation subspace corresponds precisely to a general Hermitian adjacency matrix. The answer is affirmative (see, for instance, [28]). We add a proof for reference.

**Theorem 2.1** Let $M$ be an Hermitian matrix with entries $m_{a,b}$, with rows and columns indexed by the vertex set of a graph on $n$ vertices and $X_a, Y_a, Z_a$ be the $X, Y,$ and
Z Pauli matrices acting onto qubit in position a. Then, $M$ is a block of the $2^n \times 2^n$ matrix $H$, defined as:

$$H = \frac{1}{2} \sum_{a \neq b} \Re(m_{ab})(X_a X_b + Y_a Y_b) + \Im(m_{ab})(X_a Y_b - X_b Y_a)$$

$$+ \frac{1}{2} \sum_a m_{aa}(I - Z_a)$$

Moreover, $M$ corresponds to the action of $H$ onto the subspace spanned by $|a\rangle = |0 \cdots 01\cdots 0\rangle$, where the 1 appears in the ath position, for all $a \in V(G)$.

**Proof** Assume $\Sigma$ is an alphabet of \{0, 1\} strings with length equals to $V(G)$. Each string is associated with the number of spin up and down, respectively, in an arbitrary quantum system. Fix a string $S \subset \Sigma$ with 1s proportional to the number of qubits in the state $|1\rangle$, that is, if the system starts with $k$ qubits in the state $|1\rangle$, then $S$ also has $k$ 1s. Note that

$$X_a X_b |S\rangle = |S \oplus \{a, b\}\rangle,$$

$$Y_a Y_b |S\rangle = -(1)^{|S \cap \{a, b\}|} |S \oplus \{a, b\}\rangle,$$

$$X_a Y_b |S\rangle = i(1)^{|S \cap \{a\}|} |S \oplus \{a, b\}\rangle,$$

where $S \oplus T$ denotes the symmetric difference of sets. Then,

$$X_a X_b + Y_a Y_b |S\rangle = \left(1 - (-1)^{|S \cap \{a, b\}|}\right) |S \oplus \{a, b\}\rangle$$

$$= \begin{cases} 2|S \oplus \{a, b\}\rangle & \text{if } |S \cap \{a, b\}| = 1, \\ 0 & \text{otherwise} \end{cases}.$$  

(6)

The same reasoning used in (6) applies for all other terms in the Hamiltonian $H$. Hence, we get the following results:

$$\frac{1}{2}(X_a X_b + Y_a Y_b)|S\rangle = \begin{cases} |S \oplus \{a, b\}\rangle & \text{if } |S \cap \{a, b\}| = 1, \\ 0 & \text{otherwise}. \end{cases}$$  

(7)

$$\frac{1}{2}(X_a Y_b - X_b Y_a)|S\rangle = \begin{cases} i|S \oplus \{a, b\}\rangle & \text{if } a \in S, b \notin S, \\ -i|S \oplus \{a, b\}\rangle & \text{if } a \notin S, b \in S, \\ 0 & \text{otherwise}. \end{cases}$$  

(8)

$$\frac{1}{2}(I - Z_a)|S\rangle = \begin{cases} |S\rangle & \text{if } a \in S, \\ 0 & \text{otherwise}. \end{cases}$$  

(9)

It is immediate to verify that $H|S\rangle$ is a linear combination of $\{|U\rangle : U \subset \Sigma, |U| = |S|\}$, and that $M$ represents the action of $H$ over the subspace determined by the subsets of size 1.

The theorem above guarantees that our model is physical, and can be constructed upon the use of 2-site interactions. From here on, we shall assume the underlying
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Fig. 1 Directed cycle graph with six vertices

directed graph of the network has been given along with a function $\alpha : E \rightarrow [0, 2\pi)$ so that the Hermitian adjacency matrix we consider is

$$H_\alpha = \sum_{(a, b) \in E(G)} e^{i\alpha(a, b)} |a\rangle \langle b| + e^{-i\alpha(b, a)} |b\rangle \langle a|,$$

(10)

where we impose the constraint $\alpha(a, b) = \alpha(b, a)$, and Fig. 1 illustrates this definition for a directed cycle graph.

2.3 Constant weight

Extensive work has been done for when the directed arcs are all encoded with constant complex weight equal to $i$ (see for instance [9, 13, 17]). We show at least one more general case that can be reduced to this.

We consider the definition of a Hamiltonian as defined in Eq. (10), and we set a constant weight to every edge in the graph with a fixed $\alpha$. In this way, every edge can be written as two arcs, and we split into two matrices. The first matrix $B^{V(G) \times V(G)}$ consists of a 01 matrix with 1 in position $(a, b)$ for every arc $(a, b) \in E(G)$, and 0 elsewhere; the second matrix will be its transpose. Finally, we write the Hamiltonian as:

$$H_\alpha = e^{i\alpha} B + e^{-i\alpha} B^T.$$

(11)

Figure 1 represents this decomposition where we may consider the matrix $B$ as the red solid line, and $B^T$ as the blue dashed line. We may expand $e^{i\alpha}$ to find

$$H_\alpha = \cos(\alpha)[B + B^T] + \sin(\alpha)[i (B - B^T)].$$

(12)
Note in particular that $[B + B^T]$ is the adjacency matrix of the underlying undirected graph, whereas $[i(B - B^T)]$ is precisely equal to the Hermitian adjacency matrix for which constant weight $i$ has been chosen to encode direction. Even though the Hamiltonian is so nicely decomposed, the transition matrix of the quantum walk might not, as generally $B$ and $B^T$ do not commute. If they do, however, which corresponds to the case where $B$ is a normal matrix, the transition matrix of the quantum walk, as in (2), will be given as:

$$e^{itH_\alpha} = e^{it\cos(\alpha)(B + B^T)} e^{it\sin(\alpha)(i(B - B^T))},$$

where each factor in the product is the transition matrix of a quantum walk on a suitable (Hermitian) Hamiltonian.

Standard examples of normal matrices $B$ that will provide interesting cases are the sums of circulant matrices (which can all be simultaneously diagonalizable with Fourier coefficients). For instance, if $G$ is a complete graph (all vertices are connected) on $n$ vertices, with $n$ odd, it is possible to write

$$A(G) = \sum_i B_i + B_i^T,$$  \hspace{1cm} (13)

where each $B_i$ is a circulant matrix. As an example, suppose $B_i$ is a $n \times n$ matrix with 1 in position $i$ in the first line, the following line is the permutation of $i$ and $i + 1$, the following is the permutation of $i$ and $i + 2$, and so on. This example gives the matrix associated with a complete graph while keeping commutativity of all $B_i$ and $B_i^T$. Therefore, certain directions of the edges of the complete graph given by weights $e^{i\alpha}$ can be cast into the above framework.

In the next subsection, we display an interesting phenomenon for when direction of the edges of the complete graph cannot be interpreted as such, suggesting that freedom in choosing the weights $\alpha$ might lead to interesting cases.

### 2.4 A first example: quantum walk on a complete graph

We show how the presence of weights changes the transition probability in complete graphs. Let $K_n$ denote the complete graph on vertex set $\{1, ..., n\}$. We use $|i\rangle$ to denote the indicator function of vertex $i$. It corresponds to the global state that assigns some state to the qubit at $i$, and orthogonal states to the remaining qubits. Thus, the Hamiltonian is

$$H = H_\alpha = \sum_{(i,j) \in E(K_n)} e^{i\alpha} |i\rangle\langle j| + e^{-i\alpha} |j\rangle\langle i| = e^{i\alpha} B + e^{-i\alpha} B^T,$$

where $B$ is the all ones upper triangular matrix (with 0 diagonal). Notice that, in this case, $B$ and $B^T$ do not commute which results in the impossibility of decomposing the matrix in a similar fashion to Eq. (13). Therefore, the interest in directed graph for transport properties might go beyond those that are easily decomposed. The graph in Fig. 2 depicts the value of
Fig. 2 Dynamics of a continuous-time quantum walk on $K_4$ and $K_6$ with different values of $\alpha$

$$P_{0\to 1}(t) = |\langle 1 | e^{itH} | 0 \rangle|^2,$$

for increasing $t$, considering the cliques $K_4$ and $K_6$, and different choices of $\alpha$, where $2\pi$ is the undirected model.

This suggests that direction can be a useful tool to increase the probability of state transfer. Xu [27] showed that for complete undirected graphs, there is a strong localization of the walker in the initial vertex of the quantum walk. Hence, this gives an indication that this type of direction in quantum walks opens a new venue of exploration for transport properties in graphs with strong localization such as the complete and star graphs.

### 3 Direction on trees

The typical example when studying quantum walks is for the Hamiltonian to be defined based on a linear chain of nearest neighbour interacting qubits. More generally, for the purposes of this section, we assume the underlying graph is a tree (an acyclic connected graph). We show below that adding arbitrary directions given by complex numbers of absolute value 1 affects the transition matrix of the quantum walk in a very predictable manner (in explicit contrast to what occurs if weights put to edges have absolute value different than 1).

The following lemma is elementary, but its conclusion is relevant to our context.

**Lemma 3.1** Let $H_\alpha$ be the adjacency matrix of a directed tree $T$ on $n$ vertices, where each arc $(a, b)$ has received weight $e^{i\alpha(a,b)}$ (recall that the $(b, a)$ of $H_\alpha$ is thus equal to $e^{-i\alpha(a,b)}$). Then, there is a diagonal matrix $D$, that obeys $D^\dagger D = I$, so that

$$D^\dagger H_\alpha D = H_0,$$

(14)
where $H_0$ is the adjacency matrix of an undirected underlying tree.

**Proof** The proof goes by induction on the number of vertices. The base case for a tree on 2 vertices is trivial. Assume $a$ is a leaf of $T$, connected to $b$ (say by an arc $(a, b)$). Let $E$ be the $(n-1) \times (n-1)$ diagonal matrix that gives $E^\dagger H_\alpha(T-a)E = H_0(T-a)$. Let $D$ be obtained from $E$ upon appending one diagonal entry corresponding to vertex $a$ of $T$, so that

\[ D_{aa} = E_{bb} \cdot e^{-i \alpha(a, b)}. \]

It is immediate to verify that $|D_{aa}| = 1$, and that $D^\dagger H_\alpha(T)D = H_0(T)$. \hfill \Box

As a consequence, we have that

\[ e^{itH_\alpha} = D e^{itH_0} D^\dagger. \]

This shows that if the initial state of the walk is of the form $|a\rangle$ for some vertex $a$ of the tree, then adding direction does not affect the probabilities that this state is observed elsewhere in the tree after a given time.

Kubota et al. [22] essentially showed that when $\alpha$ is constant, the matrices are similar (though they did not explicitly used diagonal similarity, which leads to equivalence of walks). The proof we present above is different from theirs, but we believe the method they use to analyse graphs with a given size of shortest cycle should also work for when $\alpha$ is possibly non-constant.

4 Quantum walk on a cycle

4.1 Setting

Let us consider the case where $\alpha$ assumes the same value to each arc and the conjugate to another. The Hamiltonian of cycle, based on its adjacency matrix, is defined by

\[ H_\alpha = \sum_{x=0}^{N-1} e^{i \alpha(x+1, x)} |x + 1\rangle \langle x| + e^{-i \alpha(x, x+1)} |x\rangle \langle x + 1| \]  

(15)

as we can see its representation in Fig. 1, where we consider a cyclic boundary condition by performing addition modulo $N$, and $\alpha(u, v) = \alpha(v, u)$

This model of quantum walk can produce new interference patterns and it can help to find new transport properties. In order to study this behaviour, we consider the initial condition

\[ |\psi(0)\rangle = |0\rangle. \]  

(16)

Figure 3 depicts the behaviour of the quantum walk considering over a cycle graph with 26 vertices and the same $\alpha$ for all of the edges, and the initial condition in (16).
This figure illustrates how this parameter $\alpha$ affects the dynamics and, depending on its value, gives a more centralized behaviour or certain tendency to right or left side over the line.

4.2 Zero transfer in even cycles

Given a graph, extensive work has been done in studying when an input state at a given qubit can be transferred to another with maximum (perfect state transfer, see [19, 21] for some surveys) or almost maximum probability (pretty good state transfer, see [5, 20] for some recent work). The somewhat symmetrical problem of asking when the probability of transfer between two vertices is 0 seems to have received less attention. See, for instance, [28]. If this probability is required to be constant equal to 0 for all times, the phenomenon has been called zero transfer, and was studied in [26].

Given a Hamiltonian $H_\alpha$ for a finite graph, as we defined in Sect. 2, we say that zero transfer occurs between vertices $a$ and $b$ if

$$|\langle b | e^{itH_\alpha} | a \rangle| = 0,$$

(17)

for all $t$. The aim of this section is to prove that there is zero transfer between antipodal vertices in cycle graphs with $2m$ vertices if the product of the weights following a direction of the cycle is equal to $-1$. This is typically achieved, for example, if $k$ of its arcs have been directed in the same direction with weight $e^{i\alpha}$ where $\alpha = \pi/k$, and the other remaining arcs have been left unaltered. This result strongly generalizes the known examples of zero transfer for when one of the edges is signed with $-1$ (see [26]).

The following widely known application of the Laplace expansion for the determinant will be useful to us.
Fig. 4 Dynamics of a continuous-time quantum walk on a cycle graph with initial condition equals $|0\rangle$ and $\alpha = \pi/k$

**Lemma 4.1** Let $Q_n$ be a tridiagonal matrix given by

$$Q_n = \begin{pmatrix} a_1 & b_1 \\
                   c_1 & a_2 & b_2 \\
                       & \ddots & \ddots & \ddots \\
                       & \ddots & \ddots & b_{n-1} \\
                       & \cdots & \cdots & c_{n-1} & a_n \end{pmatrix}, \quad (18)$$

and define $Q'_k$ to be the principal $k \times k$ block of $Q_n$ containing its first $k$ rows and columns. Then, denoting $\det(Q'_k) = f_k$, we have

$$f_k = a_k f_{k-1} - c_{k-1} b_{k-1} f_{k-2}, \quad (19)$$

where $f_{-1} = 0$ and $f_0 = 1$.

A direct consequence, that will be useful later on, occurs when $a_k = 0$ and $b_k = c_k = 1$ for all $k$, which leads to

$$f_{2k} = (-1)^k, \quad (20)$$
$$f_{2k+1} = 0. \quad (21)$$

Before moving on, we recall the definition of the Chebyshev polynomial of first kind, $T_n(x)$, and second kind, $U_n(x)$, as

**Definition 4.2** The Chebyshev polynomial of first kind is defined as:

$$T_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2k} (1 - x^{-2})^k, \quad (22)$$
while the Chebyshev polynomials of second kind is

\[
U_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} (2k)^{n-2k}.
\] (23)

Those two polynomials are related by the following known properties (see [23]):

\[
T_n(x) = \frac{1}{2} \left( U_n(x) - U_{n-2}(x) \right), \quad \text{(24)}
\]

\[
T_{2n}(x) = 2T_n^2(x) - 1. \quad \text{(25)}
\]

We are now ready to prove the following

**Theorem 4.3** Let \( H_\alpha \) be the adjacency matrix of a cycle with \( n = 2m \) vertices, with vertices \( v_1, \ldots, v_n \), where \( v_j \) is adjacent to \( v_{j\pm 1} \). Suppose

\[
\sum_{j=1}^{n} \alpha(j, j + 1) = \pi.
\]

Then, the characteristic polynomial \( \phi_{H_\alpha} \) is given by

\[
\phi_{H_\alpha} = \phi_H + 4, \quad \text{(26)}
\]

where \( \phi_H \) is the characteristic polynomial of an undirected cycle with \( 2m \) vertices. Additionally, \( \phi_{H_\alpha} \) can be decomposed as:

\[
\phi_{H_\alpha} = (2T_m(x/2))^2. \quad \text{(27)}
\]

**Proof** The Leibniz formula for the determinant gives that

\[
\phi_{H_\alpha} = \sum_\sigma (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^{2k} (xI - H_\alpha)_{i\sigma(i)}, \quad \text{(28)}
\]

where the sum corresponds to all permutations \( \sigma \) of the set \{1, 2, \ldots, 2m\}. Notice that the permutation will only give a non-zero entry of \( (xI - H_\alpha) \) if it maps \( i \) to a neighbour. Therefore, we can rewrite

\[
\phi_{H_\alpha} = \sum_{D \subseteq V(G)} x^{\left| D \right|} (-1)^{n-\left| D \right|} \det(H_\alpha \setminus D), \quad \text{(29)}
\]

where the sum runs over all subsets \( D \) of the vertices of the graph, and \( (H_\alpha \setminus D) \) denotes the matrix \( H_\alpha \) with rows and columns corresponding to \( D \) removed.

It is easy to see that subgraphs obtained for \( |D| < n \) are all paths or multiple disconnected paths where some edges might have been weighted. As we saw on
Lemma 3.1, all these possibly weighted directed paths are similar (via a diagonal matrix) to their undirected unweighted counterparts. Therefore,

\[ \phi_{H_\alpha} = \phi_H - \det(H) + \det(H_\alpha). \quad (30) \]

To compute the determinant of \( H_\alpha \), it suffices to observe the following decomposition, which follows the Laplace expansion (see, for instance, [14]):

\[
\det(H_\alpha) = - (H_\alpha)_{12}(H_\alpha)_{21} \det((H_\alpha)\setminus\{1, 2\}) + (-1)^{n+1} \cdot 2 \cdot \prod_{j=1}^{n} (H_\alpha)_{j\ j+1} - (H_\alpha)_{1n}(H_\alpha)_{n1} \det((H_\alpha)\setminus\{1, n\}) \quad (31)
\]

From the given weights, Lemma 3.1 and from (20), we have

\[
(H_\alpha)_{12}(H_\alpha)_{21} = (H_\alpha)_{1n}(H_\alpha)_{n1} = 1,
\det((H_\alpha)\setminus\{1, 2\}) = \det((H_\alpha)\setminus\{1, n\}) = (-1)^m,
\prod_{j} (H_\alpha)_{j\ j+1} = -1.
\]

If instead we were computing the determinant of \( H \), the only difference would have been that \( \prod_{j} (H)_{j\ j+1} = 1 \). Therefore,

\[ \phi_{H_\alpha} = \phi_H + 4. \quad (32) \]

Also following from (31), we have

\[ \phi_H = \phi_{P_n} - \phi_{P_{n-2}} - 2, \quad (33) \]

where \( \phi_{P_n} \) is the characteristic polynomial of the adjacency matrix of the path graph with \( n = 2m \) vertices; hence, from (20), we have

\[
\det(H) = \begin{cases} 
0 & \text{if } m \text{ is even}, \\
-4 & \text{if } m \text{ is odd}.
\end{cases} \quad (34)
\]

Also, \( \phi_{P_n} \) can be defined in terms of the Chebyshev polynomials as:

\[ \phi_{P_n} = U_n(x/2). \quad (35) \]

Using that in the expression for \( \phi_{H_\alpha} \) and the properties presented in the definition of the Chebyshev polynomials we have

\[ \phi_{H_\alpha} = (2T_m(x/2))^2. \quad (36) \]
A direct consequence of this theorem is

**Corollary 4.4** Let $H_\alpha$ be the adjacency matrix of a cycle with $n = 2m$ vertices where $\alpha$ satisfies the hypothesis in Theorem 4.3. Then, there is zero transfer for any time $t$ between any vertex $a \in \{1, ..., n\}$ and its antipodal vertex $(a + m) \pmod{n}$.

**Proof** Take a vertex $a$, and consider its indicator vector $|a\rangle$. The minimal polynomial of $H_\alpha$ on $|a\rangle$ is the monic polynomial $p(x)$ of smallest degree so that $p(H_\alpha)|a\rangle = 0$. It is well known that $p(x)$ divides the minimal polynomial of the matrix $H_\alpha$, and this latter polynomial has degree equal to the number of distinct eigenvalues of $H_\alpha$. From Theorem 4.3, this number is at most $m$. Therefore, $H_\alpha^\ell|a\rangle$, for $\ell \geq m$, is a linear combination of $H_\alpha^k|a\rangle$ for $0 \leq k \leq m - 1$. Because the combinatorial distance between $a$ and $(a + m)$ is $m$, it follows that $\langle a + m | H_\alpha^k | a \rangle = 0$ for all $0 \leq k \leq m - 1$, and therefore $\langle a + m | H_\alpha^\ell | a \rangle = 0$ for all $\ell \geq 0$. This immediately implies that for all idempotents $E_r$ in the spectral decomposition of $H_\alpha$, this corresponding entry related to $a$ and $(a + m)$ is $0$, and therefore,

$$|\langle a + m | e^{itH_\alpha} | a \rangle|^2 = |\sum_r e^{it\theta_r} \langle a + m | E_r | a \rangle|^2 = 0. \quad (37)$$

Notice that as a consequence of the above corollary, the states $|a\rangle$ and $|a + m\rangle$ belong to invariant subspaces. To see that, it is enough to verify that $\{H_\alpha^k|a\rangle\}_{k \geq 0}$ generate an invariant subspace, and $|a + m\rangle$ is orthogonal to this subspace. Because $H$ is Hermitian, if $U$ is an invariant subspace, then $U^\perp$ is also.

In Fig. 5, we display the probability of transfer between antipodal vertices in $C_{10}$ when half of their arcs have received the indicated weights, while the other half is maintained with weight 1. To show our results, the values of $\alpha$ are changed and it is possible to see that zero transfer disappears if $\alpha \neq \pi/m$.

This section elucidated why complex exponential and $-1$ weights are possible for the occurrence of zero transfer. The results show how the weights change the characteristic polynomial of the Hamiltonian which results in a factorization with squares of Chebyshev polynomials. As a consequence, all eigenvalues of the Hamiltonian are degenerate. This suggest that eigenspaces of dimension bigger than one are essential.
for the occurrence of zero transfer when the phenomenon is analysed from a combinatorial perspective. Additionally, it indicates a possible equivalence between quantum walks in graphs with complex weights and $\pm 1$ real weights. It might be an interest topic of research when and for which family of graphs this equivalence appears.

Kubota et al. [22] computed the characteristic polynomial of cycles that have received constant weights on some of their arcs, but they did not showed this connection to zero transfer.

5 Conclusion

We have shown how to obtain zero transfer in more general directed cycles than previously observed in the literature. We showed a connection between energy degeneracies of the Hamiltonian and the occurrence of zero transfer. We believe that a desirable research target is a full characterization of graphs that admit it. Our result gives a possible path for its complete characterization and the connection with the spectral properties of the adjacency matrix of the graph.

We have also displayed interesting examples of directed complete graphs whose transfer probability between two vertices approaches one, as opposed to their undirected counterparts. This suggests that further investigation on the possible features directed arcs bring is imperative, specially for graphs with strong localization properties.

Finally, we displayed the viability of the proposed formalization as a physical realizable Hamiltonian with only two-body interactions.

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Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no competing interests to declare that are relevant to the content of this article. All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript. The authors have no financial or proprietary interests in any material discussed in this article.

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