A stochastic Gronwall’s inequality in random time horizon and its application to BSDE

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Received: date / Accepted: date

Abstract In this paper, we introduce and prove a stochastic Gronwall’s inequality in (unbounded) random time horizon. As an application, we prove a comparison theorem for backward stochastic differential equation (BSDE for short) with random terminal time under stochastic monotonicity condition.

Keywords Gronwall’s inequality, stochastic, random time horizon, backward stochastic differential equation, comparison

Mathematics Subject Classification (2010) MSC 60E15 · MSC 60H20

1 Introduction

Gronwall’s inequality is a handy tool to derive many useful results such as uniqueness, comparison, boundness, continuous dependence and stability in the theory of differential and integral equations. It was first introduced by Gronwall [5] as a differential form and the integral inequality was proposed by Bellman [1]. Since then, many researchers studied the various types of generalizations of this inequality motivated by the development of the differential and integral equations ([2], [4], [6]). Among such generalizations, we are concerned with the stochastic version of Gronwall’s inequality.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which a $d$-dimensional Brownian motion $B = (B_t)_{t \geq 0}$ is defined. Let $(\mathcal{F}_t)_{t \geq 0}$ be the right-continuous completion of the natural filtration generated by $B$, that is, $\mathcal{F}_t := \sigma \{ B_{s \leq t} \}$ and argument them by $P$-null sets.

In Wang and Fan [9], they first proved the following stochastic Gronwall’s inequality.

**Proposition 1** Let $c > 0$, $T > 0$ and an $(\mathcal{F}_t)_{t \geq 0}$-progressive measurable process $a : \Omega \times [0,T] \to \mathbb{R}^+$ satisfy $\int_0^T a(t) \, dt \leq M$, $\mathbb{P}$-a.s. for some constant $M > 0$. If an $(\mathcal{F}_t)_{t \geq 0}$-progressive measurable process $x : \Omega \times [0,T] \to \mathbb{R}$ satisfies

$$
\mathbb{E}\left[ \sup_{t \in [0,T]} x(t) \right] < +\infty, \quad x(t) \leq c + \mathbb{E}\left[ \int_t^T a(s)x(s) \, ds \right| \mathcal{F}_t], \quad t \in [0,T].
$$

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then, for each $t \in [0, T]$,

$$x(t) \leq c \cdot E[e^{\int_0^T a(s) \, ds} | \mathcal{F}_t], \quad \mathbb{P} - a.s.$$  

If the random processes $a(t)$ and $x(t)$ are deterministic functions in above proposition, then we reach the well-known Gronwall’s inequality as follows.

**Corollary 1** If $a(t)$ and $x(t)$ are two non-negative (deterministic) functions defined on $[0, T]$ which satisfy

$$x(t) \leq c + \int_t^T a(s)x(s) \, ds, \quad t \in [0, T].$$

then, for each $t \in [0, T]$,

$$x(t) \leq c \cdot e^{\int_0^t a(s) \, ds}.$$  

In this paper, we study the complete version of Gronwall’s inequality in stochastic sense. More precisely, in above Proposition 1, the constants $c$ and $T$ are replaced by a random variable and (unbounded) stopping time, respectively and the integral of $a(t)$ is not assumed to be essentially bounded.

We use methods by the martingale representation and random time change to prove the main inequality. Due to the type of the proposed inequality, the application to the stochastic differential (or integral) equation with stochastic coefficients defined up to random time (more precisely, stopping time) is naturally considered. We give the proof of a comparison theorem of $L^p$-solutions to backward stochastic differential equation (BSDE for short) with random terminal time and stochastic coefficients by using the stochastic Gronwall’s inequality in random time horizon, effectively.

2 Notations

Let $p > 1$ and $\tau$ be an $(\mathcal{F}_t)_{t \geq 0}$-stopping time. That is, $\forall t \geq 0, \{ \omega | \tau(\omega) \leq t \} \in \mathcal{F}_t$. Throughout all the paper, $| \cdot |$ means the standard euclidean norm. We put $A(t) := \int_0^t a(s) \, ds$, where $a(s)$ is a non-negative progressive-measurable process. The symbols $E[\cdot]$ and $E[\cdot | \mathcal{F}_t]$ denote the expectation and conditional expectation (with respect to $\mathcal{F}_t$), respectively.

- $L^p_0(\mathcal{F}_t)$ is the set of real-valued $\mathcal{F}_t$-measurable random variables $\xi$ such that

$$E[|\xi|^{p/2} A(t)] < +\infty.$$

- $H^p_{t, \theta}(\mathbb{R})$ is the set of real-valued càdlàg, adapted processes $Y$ such that

$$E \left[ \sup_{0 \leq s \leq \tau} e^{p/2 A(t)} |Y_s|^p \right] < \infty.$$

- $H^p_{t, d}(\mathbb{R})$ is the set of real-valued càdlàg, adapted processes $Y$ such that

$$E \left[ \left( \int_0^t a(t) e^{\frac{p}{2} A(t)} |Y_t|^p \, dt \right)^{p/2} \right] < \infty.$$

- $M^p_{t, b}(\mathbb{R}^{1 \times d})$ is the set of predictable processes $Z$ with values in $\mathbb{R}^{1 \times d}$ such that

$$E \left[ \left( \int_0^t e^{\theta A(t)} |Z_t|^2 \, ds \right)^{p/2} \right] < \infty.$$

- For $m, n \in \mathbb{R}$, $m \land n := \min\{m, n\}$ and $m^+ := \max\{m, 0\}$. 
3 Main inequality

**Theorem 1** Let $p > 1, l \geq 0$ be constants and $q$ be a constant such that $1/p + 1/q = 1$. Let $\tau$ be an $(\mathcal{F}_t)_{t \geq 0}$-stopping time and $\xi$ be a non-negative random variable. Let $a(t)$ and $x(t)$ be non-negative progressive measurable processes. Assume that $a(t) > \varepsilon$ for some constant $\varepsilon > 0$ and $x(t)$ belongs to $H^{p,a}_{loc}(\mathbb{R})$. We further assume that $\xi$ belongs to $L^p_0(\mathbb{R})$ for some constant $0 \leq \theta \leq 1$ satisfying $\mathbb{E}[\xi^2(2\theta-\theta^2A(\tau))] < \infty$, where $A(t) := \int_0^t a(s) \, ds$.

If $x(t) \leq \mathbb{E}[\xi + l \int_0^t a(s)x(s) \, ds | \mathcal{F}_t]$, $P-a.s.,$ then, we have $\mathbb{P} - a.s.$,

$$x(t) \leq \mathbb{E}[\xi, \int_0^t a(s)x(s) \, ds | \mathcal{F}_t].$$

**Proof** Define the process $X(t) := \mathbb{E}[\xi + l \int_0^t a(s)x(s) \, ds | \mathcal{F}_t]$, then it follows from the assumptions that $x(t) \leq X(t)$. Let us put $\eta := \xi + l \int_0^t a(s)x(s) \, ds$. Then,

$$\mathbb{E} \left[ \left( \int_0^t a(s)x(s) \, ds \right)^p \right] \leq \mathbb{E} \left[ \left( \int_0^t a(s)\xi(s) e^{\theta A(s)/2} \, ds \right)^p \right] < \infty.$$

Therefore,

$$\mathbb{E}[|\eta|^p] \leq 2^{p-1} \left( \mathbb{E}[|\xi|^p] + l^p \mathbb{E} \left[ \left( \int_0^t a(s)x(s) \, ds \right)^p \right] \right) < \infty.$$

That is, $\eta \in L^p_0(\mathcal{F}_\tau)$. By the martingale representation theorem (see [7], page 116, Theorem 2.42), there exists a process $Z$ satisfying $Z \in M^p_{loc}(\mathbb{R}^{1 \times d})$ for any $T > 0$ such that $\mathbb{P} - a.s.$,

$$\mathbb{E}[\eta | \mathcal{F}_t] = \mathbb{E}[\eta] + \int_0^{t \wedge T} Z_s \, dB_s.$$

So,

$$X(t) = \mathbb{E}[\eta | \mathcal{F}_t] - l \mathbb{E} \left[ \int_0^{t \wedge \tau} a(s)x(s) \, ds | \mathcal{F}_t \right] = \mathbb{E}[\eta] + \int_0^{t \wedge \tau} Z_s \, dB_s - l \int_0^{t \wedge \tau} a(s)x(s) \, ds.$$

Moreover, we have the backward version: for all $T > 0$,

$$X(t) = X(T \wedge \tau) - \int_{t \wedge \tau}^{T \wedge \tau} Z_s \, dB_s + l \int_{t \wedge \tau}^{T \wedge \tau} a(s)x(s) \, ds.$$

with $X(\tau) = \xi$.

Or equivalently,

$$X(t) = \xi - \int_{t \wedge \tau}^{T \wedge \tau} Z_s \, dB_s + l \int_{t \wedge \tau}^{T \wedge \tau} a(s)x(s) \, ds. \tag{1}$$

Now we shall show that $Z \in M^p_{loc}(\mathbb{R}^{1 \times d})$. First, we introduce a certain random time change. Due to the fact that the process $A_t := \int_0^t a(s) \, ds$ is strictly increasing and continuous, we can define its reverse denoted by $C_s := A^{-1}(s)$. Then a family of stopping times, $\{C_s\}, s \geq 0$ is an $(\mathcal{F}_t)_{t \geq 0}$-random time change (see [8] for systematic study of random time change).

Set $\tilde{\mathcal{F}}_t := \mathcal{F}_{C_t}$, then $(\tilde{\mathcal{F}}_t)_{t \geq 0}$ is a new filtration. For any adapted process $X$, we assume that $\tilde{X}$ means the time-changed process, that is, $\tilde{X} = X_{C_t}$. We also define $\tilde{\tau} := A(\tau)$ and
\[ W_t := \int_0^t \sqrt{a(s)} dB_s. \] Note that \( W = (W^1, W^2, \ldots, W^d) \) is a \( d \)-dimensional \( (\mathcal{F}_t)_{t \geq 0} \)-Brownian motion by Levy’s characterization theorem. In fact, for each \( i \in \{1, d\}, \)

\[
< W^i, W^i > = \int_0^t \alpha(s) d < B^i, B^i > = \int_0^t \alpha(s) d < \mathcal{B}^i, \mathcal{B}^i > = \frac{\alpha(s) C_t}{\alpha(t)} = \frac{\alpha(t)}{\alpha(t)} \int_0^t \alpha(s) ds = \Lambda(C_t) = t.
\]

From the properties of (stochastic) integral with respect to time change,

\[
\int_{\tau \setminus \tau}^\tau \alpha(s) x(s) ds = \int_{A(\tau \setminus \tau)}^{A(\tau)} x(C_s) ds = \int_{A(\tau \setminus \tau)}^\tau \bar{x}(s) ds,
\]

\[
\int_{\tau \setminus \tau}^\tau Z_s dB_s = \int_{A(\tau \setminus \tau)}^\tau \bar{Z}_s dB_s = \int_{A(\tau \setminus \tau)}^\tau \alpha(s)^{-1/2} \bar{Z}_s dW_s.
\]

So, we get the following expression with respect to \( (\mathcal{F}_t)_{t \geq 0} \).

\[
\bar{X}(t) = \xi - \int_{\tau \setminus \tau}^\tau \bar{x}(s)^{-1/2} \bar{Z}_s dW_s + i \int_{\tau \setminus \tau}^\tau \bar{x}(s) ds.
\]

According to [3], Proposition 3.2, for all \( \tilde{T} > 0 \),

\[
E \left[ \sup_{t \in [0, \tilde{T} \setminus \tau]} e^{\theta t} \bar{X}(t)^p + \left( \int_0^{\tilde{T} \setminus \tau} \frac{e^{\theta t}}{\alpha(t)} |\bar{Z}_t|^2 dt \right)^{p/2} \right] 
\leq c(p) \cdot E \left[ e^{p/2 (\tilde{T} \setminus \tau)} \bar{X}(\tilde{T} \setminus \tau)^p + \left( \int_0^{\tilde{T} \setminus \tau} i e^{\theta t} \bar{x}(s) ds \right)^p \right]
\]

for some constant \( c(p) \). Sending \( \tilde{T} \to +\infty \), Fatou’s Lemma ensures that

\[
E \left[ \sup_{t \in [0, \tau]} e^{\theta t} \bar{X}(t)^p + \left( \int_0^\tau \frac{e^{\theta t}}{\alpha(t)} |\bar{Z}_t|^2 dt \right)^{p/2} \right] 
\leq c(p) \cdot E \left[ e^{p/2} \bar{X}(\tau)^p + \left( i \int_0^\tau e^{\theta t} \bar{x}(s) ds \right)^p \right].
\]

And we see that

\[
\sup_{t \in [0, \tau]} e^{\theta t} \bar{X}(t)^p = \sup_{t \in [0, \tau]} e^{p/2 \Lambda(t)} X(t)^p,
\]

\[
\int_0^\tau e^{\theta t} \bar{x}(s) ds = \int_0^\tau e^{\theta A(s)/2} x(s) dA(s) = \int_0^\tau a(s)e^{\theta A(s)/2} x(s) ds,
\]

\[
\int_0^\tau \frac{e^{\theta t}}{\alpha(t)} |\bar{Z}_t|^2 dt = \int_0^\tau \frac{e^{\theta A(t)}}{\alpha(t)} |\bar{Z}_t|^2 dt A(t) = \int_0^\tau e^{\theta A(t)} |\bar{Z}_t|^2 dt.
\]

Therefore, we deduce

\[
E \left[ \sup_{t \in [0, \tau]} e^{p/2 \Lambda(t)} X(t)^p + \left( \int_0^\tau e^{\theta A(t)} |\bar{Z}_t|^2 dt \right)^{p/2} \right] 
\leq c(p) \cdot E \left[ e^{p/2 \Lambda(\tau)} \bar{X}(\tau)^p + \left( \int_0^\tau d(s)e^{\theta A(s)/2} x(s) ds \right)^p \right] < \infty. \quad (2)
\]
So we have proved that \( Z \in M^{p}_{t,\theta}(\mathbb{R}^{1 \times d}) \). Applying Ito’s formula to (1), we get

\[
X(t)e^{A(t)\wedge \tau} = \xi e^{A(t)} + I \int_{t}^{\tau} a(s)e^{A(s)\wedge \tau}X(s) - X(s)\, ds - \int_{t}^{\tau} e^{A(s)}Z_s dB_s.
\]

From \( x(t) \leq X(t) \), it follows that

\[
X(t)e^{A(t)\wedge \tau} \leq \xi e^{A(t)} - \int_{t}^{\tau} e^{A(s)}Z_s dB_s.
\]

From the expression (2), Burkholder-Davis-Gundy’s inequality (BDG inequality for short) and Young’s inequality, we deduce

\[
\mathbb{E}\left[ \sup_{t \geq 0} \int_{0}^{t} e^{A(s)}Z_s dB_s \right] \leq c \cdot \mathbb{E}\left[ \left( \int_{0}^{\tau} e^{2A(s)}|Z_s|^2\, ds \right)^{1/2} \right]
\]

\[
= c \cdot \mathbb{E}\left[ \left( \int_{0}^{\tau} e^{(\theta + 2\theta)}|Z_s|^2\, ds \right)^{1/2} \right]
\]

\[
\leq c \cdot \mathbb{E}\left[ \left( \int_{0}^{\tau} |Z_s|^2\, ds \right)^{1/2} \right]
\]

\[
\leq \frac{c}{p} \mathbb{E}\left[ \left( \int_{0}^{\tau} |Z_s|^2\, ds \right)^{p/2} \right] + c \mathbb{E}\left[ \mathbb{E}\left[ \left( \int_{0}^{\tau} e^{2(\theta - \theta)A(s)}\, ds \right)^{p/2} \right] \right].
\]

Thus, \( M(t) = \int_{0}^{t} e^{A(s)}Z_s dB_s \) is the uniformly integrable martingale. Taking conditional expectations with respect to \( \mathcal{F}_t \) on both sides of (3), we get

\[
X(t)e^{A(t)\wedge \tau} = \mathbb{E}[X(t)e^{A(t)\wedge \tau}|\mathcal{F}_t] \leq \mathbb{E}[\xi e^{A(t)}|\mathcal{F}_t].
\]

So, we have \( x(t) \leq X(t) \leq \mathbb{E}[\xi e^{A(t)\wedge \tau}|\mathcal{F}_t] \), which is the desired result.

Remark 1 If \( A(t) = \int_{0}^{t} a(s)\, ds \leq M, \mathbb{P} - a.s. \) for some \( M \geq 0 \), then

\[
\mathbb{E}\left[ \left( \int_{0}^{t} a(t)e^{A(t)}x(t)\, dt \right)^p \right] \leq \mathbb{E}\left[ \left( A(t) \cdot \sup_{0 \leq s \leq t} e^{2A(s)}x(t) \right)^p \right]
\]

\[
\leq M^p \cdot \mathbb{E}\left[ \sup_{0 \leq s \leq t} e^{p/2A(s)}x(t)^p \right].
\]

So, \( x(t) \in H^{p,\theta}(\mathbb{R}) \) holds whenever \( x(t) \in H^{p,\theta}(\mathbb{R}) \). Therefore, in Theorem 1, we can give an alternative assumption such that \( x(t) \in H^{p,\theta}(\mathbb{R}) \), instead of \( x(t) \in H^{p,\theta}(\mathbb{R}) \).

4 Application

In this section we show a comparison principle of \( L^p \)-solutions of BSDEs with random terminal time under stochastic monotonicity condition on generator. Let us consider the following one-dimensional BSDE with random terminal time.

\[
Y_t = \xi + \int_{t \wedge \tau}^{\tau} f(s, Y_s)\, ds - \int_{t \wedge \tau}^{\tau} Z_s dB_s,
\]

where \( \xi \) is an \( \mathcal{F}_\tau \)-measurable random variable.
where the terminal time $\tau$ is an $(\mathcal{F}_t)_{t \geq 0}$-stopping time, the terminal value $\xi$ is an $\mathcal{F}_t$-measurable random variable and the generator $f : \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{1 \times d} \rightarrow \mathbb{R}$ is $(\mathcal{F}_t)_{t \geq 0}$-progressive measurable. The solution of BSDE (4) is a pair $(Y_t, Z_t)_{t \geq 0}$ of adapted processes such that $Y_t = \xi$ and $Z_t = 0$, $\mathbb{P}$-a.s. for $t \geq \tau$ and for any $T \geq 0$,
\[
Y_{t\wedge T} = Y_{T\wedge T} + \int_{t\wedge T}^{T\wedge T} f(s, Y_s) \, ds - \int_{t\wedge T}^{T\wedge T} Z_s \, dB_s, \quad 0 \leq t \leq T.
\]

For the convenience, we characterize the BSDE (4) by a triple $(\tau, \xi, f)$.

**Theorem 2** Let $p > 1$ and consider two BSDEs with data $(\tau, \xi^1, f^1)$ and $(\tau, \xi^2, f^2)$. Let $(Y^1, Z^1)$ and $(Y^2, Z^2)$ be two solutions of the BSDE (4) corresponding to $(\tau, \xi^1, f^1)$ and $(\tau, \xi^2, f^2)$, respectively. Suppose that $f$ is stochastic monotone in $y$, that is, there exists a non-negative, progressively process $a(t)$ such that for all $y, y' \in \mathbb{R}$:
\[
(y - y')(f^1(t, y) - f^1(t, y')) \leq a(t)(y - y')^2.
\]

We assume that $a_t > \varepsilon$ for some $\varepsilon > 0$. Set $A(t) := \int_0^t a_s \, ds$. Let $q$ be a constant such that $1/p + 1/q = 1$. Suppose that $(Y^i, Z^i), i = 1, 2$ belongs to $H_{\tau, \theta}^{p, 1}(\mathbb{R}) \times M_{\tau, 0}^{\sigma}(\mathbb{R}^{1 \times d})$ for some $\theta \in \mathbb{R}$ such that $\mathbb{E}[e^{\theta(\tau - T)} A(T)] < +\infty$.

If $\xi^1 \leq \xi^2$ and $f^1(t, Y^2_t) \leq f^2(t, Y^2_t)$, then $Y^1_t \leq Y^2_t$, $\mathbb{P}$-a.s.

**Proof** Define $\tilde{Y} := Y^1 - Y^2, \tilde{Z} := Z^1 - Z^2, \tilde{\xi} := \xi^1 - \xi^2$, then
\[
\tilde{Y}_t = \tilde{\xi} + \int_0^t [f^1(s, Y^1_s) - f^2(s, Y^2_s)] \, ds - \int_0^t \tilde{Z}_s \, dB_s.
\]

By the virtue of Ito-Tanaka’s formula (see Exercise VI.1.25 in [8] for details), we have
\[
\tilde{Y}_t^+ = \tilde{\xi}^+ + \int_0^t 1_{Y^1_s > 0} [f^1(s, Y^1_s) - f^2(s, Y^2_s)] \, ds - \int_0^t 1_{\tilde{Y}^+_s > 0} \tilde{Z}_s \, dB_s - \frac{1}{2} \tilde{L}^0_t
\]
\[
\leq \tilde{\xi}^+ + \int_0^t 1_{Y^1_s > 0} [f^1(s, Y^1_s) - f^2(s, Y^2_s)] \, ds - \int_0^t 1_{\tilde{Y}^+_s > 0} \tilde{Z}_s \, dB_s,
\]

where $\tilde{L}^0_t$ is the local time of semi-martingale $\tilde{Y}_t$, it is increasing and $\tilde{L}^0_0 = 0$.

Using this and the assumptions of the theorem, we get
\[
\tilde{Y}_t^+ \leq \tilde{\xi}^+ + \int_0^t 1_{Y^1_s > 0} [f^1(s, Y^1_s) - f^1(s, Y^1_s)] \, ds - \int_0^t 1_{\tilde{Y}^+_s > 0} \tilde{Z}_s \, dB_s
\]
\[
= \tilde{\xi}^+ + \int_0^t 1_{Y^1_s > 0} \frac{Y^1_s}{\sqrt{\varphi}} [f^1(s, Y^1_s) - f^1(s, Y^1_s)] \, ds - \int_0^t 1_{\tilde{Y}^+_s > 0} \tilde{Z}_s \, dB_s
\]
\[
\leq \tilde{\xi}^+ + \int_0^t a \tilde{Y}_s^+ \, ds - \int_0^t 1_{Y^1_s > 0} \tilde{Z}_s \, dB_s.
\]

Since $\int_0^{t\wedge T} 1_{Y^1_s > 0} \tilde{Z}_s \, dB_s, t \geq 0$ is a martingale, we obtain
\[
\tilde{Y}_t^+ \leq \mathbb{E} \left[ \tilde{\xi}^+ + \int_0^t a \tilde{Y}_s^+ \, ds \mid \mathcal{F}_t \right].
\]

From $\xi^1 \leq \xi^2$, it follows that $(\tilde{\xi}^+)^p = 0$. Theorem 1 yields that $\tilde{\xi}^+ = 0$. Hence, $Y^1_t \leq Y^2_t$. 
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