Spin-disordered superfluid state for spin-1 bosons with fractional spin and statistics

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We study a strongly correlated spin-1 Bose gas in 2D space by using the projective construction. A spin-disordered superfluid state is constructed and proposed as a candidate competing with the conventional polar condensate when interaction is antiferromagnetic. This novel state has a nontrivial topological order whose low energy excitations carry fractional spin, charge, and statistics. The spin excitations become gapless only at the edge and are described by level-1 SU(2)spin Kac-Moody algebra. The edge state is identical to the edge state of the chiral spin liquid or the right moving sector of spin-1/2 chain.

So far most studies of ultra cold alkali atomic gases have been focused on Bose-Einstein condensation (BEC) in weakly interacting dilute limit (for review, see for instance [4]). By contrast, a strongly correlated Bose gas may lead us to interesting novel phenomena. While it is unclear whether one can achieve a stable strongly correlated gas in 3D by simply cranking up the scattering length without collapsing the system, a 2D gas with even relatively weak interaction can be strongly correlated in nature. The argument is based on the renormalization group analysis [2] that showed that the interaction in 2D is marginally irrelevant only in a dilute limit specified by \( \ln(1/na^2) \gg 1 \), where \( n \) is the particle density and \( a \) can be thought of the interaction range or the scattering length. The double logarithm imposes a more strict condition of the validity of the dilute limit in 2D than the familiar \( na^2 \ll 1 \) in 3D. For generic BEC system such as the disk condensate of \(^{23}\text{Na}\) of Ref. [3] or a microelectronic chip of condensed \(^{87}\text{Rb}\) atoms [1], \( \ln(1/na^2) \) is of order \( 1 \). These systems are, at least, not weakly correlated. Now the question is that what is the ground state for those not-weakly correlated 2D boson gas? In this letter, we shall show that a new class of 2D superfluid can emerge due to the strong correlation.

a. Proposed spin-disordered superfluid. Let us consider a strongly correlated gas of spin-1 bosons \( \phi_m \) \((m = 0, \pm 1)\) in a 2D homogeneous space with generic two-body interactions \( H_{\text{int}} = \int d^2r d^2r' \sum_n \mathcal{H}_{\text{int}}\) with \( \mathcal{H}_{\text{int}} = \frac{1}{2}V_F(r - r')C^{SS_2}_{m'n}C^{SS_2}_{m} \phi_m^\dagger(r)\phi_{m'}^\dagger(r')\phi_{m'}(r')\phi_m(r) \)

\[ (1) \]

where \( C^{SS_2}_{m'n} = \langle SS_2 | 11; mnm'n \rangle \) are the Clebsch-Gordan coefficient for total spin \( S = 1 + 1 \). In the alkali atomic gases, the interactions for all three spin channels \( S_\sigma, S = 0, 1, 2 \) are commonly short-ranged. The Hamiltonian has a phase and spin rotation symmetry: \( U(1)_{\text{charge}} \times SU(2)_{\text{spin}} \). The Hamiltonian derived by Ho [1] is the same as \( \mathcal{H}_{\text{int}} = \frac{1}{2}(c_0\hat{n}^2 + c_2\hat{S}^2) \) where \( \hat{n} \) and \( \hat{S} \) are the density and spin operators, corresponds to a special form of \( \mathcal{H}_{\text{int}} \) with \( \mathcal{H}_0 = \langle \langle \phi_m^\dagger(\phi_m) \rangle = (c_0 - c_2)\delta(r - r'), \mathcal{V}_i = (c_0 - c_2)\delta(r - r') \) and \( \mathcal{V}_2(r - r') = (c_0 + c_2)\delta(r - r'). \) The case of antiferromagnetic spin interaction (\( \mathcal{V}_2 > \mathcal{V}_0 \)) or correspondingly \( c_2 > 0 \) is somewhat intriguing. Mean field theories [5, 6] realize that the (coherent or fragmented) polar condensate is energetically favored for this case. We note that the polar condensate, no matter coherent or fragmented, is a spin ordered state, as it directly condenses single bosons \((\phi_m \neq 0)\) and therefore always breaks SU(2)spin symmetry in the thermodynamical limit. For spin-1 bosons in an optical lattice, Demler and Zhou speculated possible spin disordered (liquid) phases and fractionalized spin excitations for arbitrary dimensions \[7\]. But, a question remains: Is there a spin-disordered superfluid competing with the polar condensate?

We have found such a spin-disordered state. It can be visualized as a gas of closed (connected) clusters constituted by various number of bosons; the size \( l \) can take up the value of 2 (dimer), 3 (triangle), \ldots, up to \( N \) for an \( N \) boson system (see Fig. 1). Each cluster is a spin singlet and is totally symmetric among all bosons within the cluster. Mathematically, an \( l \)-cluster is described by \( C_l(\{r_i, m_i\}_{i=1\ldots l}) = \prod_{i=1}^l g(r_{i} - r_{i+1}) \text{tr} \prod_{i=1}^l e^{i\epsilon_{m_i}} \) (with \( r_{l+1} \equiv r_1 \) identified), where \( g(r) \) is a pairing wavefunction of \( p_x + ip_y \)-symmetry (thus odd in \( r \)), \( \epsilon \) is a \( 2 \times 2 \) antisymmetric matrix, and \( \epsilon_{m_\alpha} \equiv \langle SS_2 | 1m_\alpha \rangle \frac{1}{2} \). The state is a superposition of all possible cluster configurations specified by \( \{n_i\} \) with \( n_i \) the number of \( l \)-clusters. The wavefunction for the component of the state with \( N \) bosons is, apart from an overall factor,

\[ \Psi_N^\beta(\{r_i, m_i\}_{i=1}^N) \sim \mathcal{F} \sum \{n_i\} 2^{N_\epsilon} \prod_{p=1}^{N_\epsilon} (l_p - 1)! \times C_{l_p}(\{r_{i}^l, m_{i}^l\}_{i=1}^{l_p}), \]

\[ (2) \]

where \( N_\epsilon = \sum n_i \) is the total number of clusters, \( p \) is a label used to sort all \( N_\epsilon \) clusters, and \( l_p \) is the size of the \( p \)-th cluster. In this labelling, the boson spins and

FIG. 1: A cluster configuration for \( N = 11 \) spin-1 bosons with arrows indicating spin. ‘Dotted line’ = \( g(r_i - r_j) \).
coordinates \(\{r_i, m_i\}_{i=1}^{N}\) are one-to-one mapped onto the set \(\{r_j^p, n_j^p\}_{p=1..N; i=1..p}\). The wavefunction is symmetrized for \(N\) bosons, as indicated by \(\mathcal{S}\).

Table 1 compares the interaction energies of the two states. One sees that the spin-disordered state we proposed has a lower interaction energy when \(V_2\) is positively large and short ranged. Another important feature is that wherever two bosons in the same spin state (except \(m = 0\), say located at \(r\) and \(r'\), approach each other, \(\Psi^s_N\) vanishes as \(\sim (r - r')^2\), as implied in Table 1. Readers who are not interested in the technical details may skip now to the end of the paper for other interesting physical properties.

### b. Projective construction: a theoretical technique.

Unlike the usually studied, weakly interacting dilute limit, a Bose gas of strong repulsion cannot be treated by conventional perturbation theory. How to derive the groundstate wavefunction is obviously challenging. Furthermore, even if one has such a posposed wavefunction as the \(\Psi^s_N\) above, it is still difficult to extract the low energy excitation properties, since it is a highly ‘entangled’ state. The analogue of Gross-Pitaevskii equation does not exist yet. To make some progress towards such a difficult task, we borrow the ‘projective construction’ method from quantum Hall studies, which is a standard, successful approach in parallel with the Laughlin wavefunction approach (see Ref. 1 and references therein). In this spirit, we introduce two spin-\(\frac{1}{2}\) fermions, \(\psi_{\alpha a}\), each with two (physical) spins \(\alpha = \uparrow, \downarrow\) and two fictitious “colors” labelled by \(a, b\). The color is necessary to furnish a minimal spin-1 bosonic representation at every spacetime point. The boson can then be represented by

\[
\phi_{\alpha a}(r) = \psi_{\alpha a}(r)\psi_{\beta b}(r)\epsilon_{ab}\epsilon^{\alpha\beta}_{\alpha\beta}.
\]

Note that the color degree of freedom is non-physical. All physical states or operators are required invariant under a local SU(2) color transformation: \(\psi_{\alpha a}(r) \rightarrow W(r)_{ab}\psi_{\alpha a}(r)\), where \(W(r)\) is an SU(2) color matrix. One can quickly check that the boson operator \(\phi_{\alpha a}\) defined in (3) is indeed a color singlet, perfectly invariant under above transformation. Eq. (3) allows us to construct physical boson many-body wave function \(|\Psi^s\rangle\) from the unphysical fermion many-body wave function \(|\Psi^f\rangle\) by projecting it into the “color” singlet sector. In a mathematical equation, that means

\[
|\Psi^b(r_1, m_1; r_2, m_2; ...) = \langle 0 \prod_{\alpha a} \phi_{\alpha a}(r_1)|\Psi^f\rangle,
\]

whose explicit form is precisely \(\Psi^s_N\) of Eq. (2).

The relationship (3) suggests that the low energy effective theory of our spin-1 boson system can either be described in terms of the boson operator \(\phi_{\alpha a}\), or equivalently, in terms of the fermion operator \(\psi_{\alpha a}\). Unfortunately, for a strongly correlated 2D system, there is no known rigorous way to derive the effective theory from a microscopic Hamiltonian like (3). One usually first writes down a most natural form of it on symmetry grounds, checks its stability against interactions in low energy limit, and finally compares it with experiments. In this spirit, the effective theory for the state \(\Psi^b(r_1, m_1; r_2, m_2; ...)\) is described, in the fermion description, by a theory of independent fermions coupled to color SU(2) gauge fields. The gauge field is denoted as \(A_\mu = \frac{1}{2}\sigma^\mu a^\mu_{a b}, l = 1, 2, 3\), where \(\sigma^\mu\) are the Pauli matrices generating the SU(2) color algebra. The gauge fields are introduced to project out the unphysical colored excitations (3). The effective theory is then

\[
\mathcal{L} = i\bar{\psi}_{\alpha a}^\dagger(D_0)_{ab}\psi_{\beta b} + \frac{1}{2M}\bar{\psi}_{\alpha a}^\dagger(D\cdot D)_{ab}\psi_{\beta b} + \text{all symmetry allowed interactions},
\]

where \((D_\mu)_{ab} = \delta_{ab}\partial_\mu - i(A_\mu)_{ab}\) (for notation, see Ref. 1) are the covariant derivatives. From the fermion effective theory, we can study various fermion states, which, after the projection (3), lead to various physical boson states.

To see which fermion states are likely to appear as the ground state, we need to consider the interactions between the fermions. Interactions can be either originated from the boson-boson interactions or dynamically generated by gauge interactions. As an example of non-Abelian gauge theory, the study of QCD (4) shows that the Yang-Mills gauge fluctuations can generate a strong attractive interaction between quarks due to the instanton effect, which leads to quark confinement. In our case, the gauge fields are used to mediate a strong attractive interaction between color-opposite fermions, since by definition (3) the gauge interaction is supposed to bind two fermions locally into a colorless boson. Therefore, the strong SU(2) gauge interaction naturally leads to color-singlet Cooper pairing.

Let us consider two simplest color-singlet parings:

\[
\langle \psi_{\alpha a}(r)\psi_{\beta b}(0) = \begin{cases} \epsilon_{ab}\epsilon^{\alpha\beta}_{\alpha\beta}R_s(|r|), & (s, \text{spin triplet}) \\ \epsilon_{ab}\epsilon_{\alpha\beta}R_p(|r|)(x + iy), & (p_{x+iy}, \text{spin singlet}) \end{cases}
\]

where \(\bar{m}\) can be 0 or 1. Both \(s\)- and \(p_{x+iy}\)-wave states produce a full gap on the Fermi surface. \(R_{s,p}(|r|)\) are complex functions of \(|r|\), generically expected to monotonically fall off exponentially at large distance.

### c. s-wave pairing: conventional BEC phases.

Conventional BEC phases are easily recovered through the
s-wave paring channel. In the limit of strong confinement, $R_s(|r|)$ becomes a delta function $\sim \delta(r)$.

**Polar condensate.** This is a special kind of spin nematic state. Fermion confinement occurs in spin-1, $m = 0$ channel. Here, the pairing in Eq. (7) reduces to $\langle \psi_{a\alpha}(r)\psi_{b\beta}(r) \rangle = \sqrt{m^2} \epsilon_{\alpha\beta} \epsilon_{\alpha\beta} = 0$ where one may think of $\rho$ related to the condensed boson density. The resulting state is nothing but the so-called polar condensate with $\langle \phi_{m=0} \rangle \neq 0$. This state breaks both $U(1)_{\text{charge}}$ and SU(2)$_{\text{spin}}$ invariance, and was considered to be favored if the spin interaction is antiferromagnetic ($V_2 \gtrsim V_0$) [I].

**Ferromagnetic condensate.** This case is the same as the polar condensate except that the fermions are confined into the $m = 1$ (or equivalently $-1$) channel. The order parameter becomes $\langle \psi_{a\alpha}(r)\psi_{b\beta}(r) \rangle = \sqrt{m^2} \epsilon_{\alpha\beta} \epsilon_{\alpha\beta} = 1$ which corresponds to the ferromagnetic condensate. Like the polar condensate, it breaks both $U(1)_{\text{charge}}$ and SU(2)$_{\text{spin}}$ invariance. This state is presumably favored if the spin interaction is ferromagnetic, $V_2 \lesssim V_0$.

d. $p_{x+iy}$-state: topological superfluid. The spin-disordered $p_{x+iy}$-wave pairing energetically competes with the polar condensate for a large positive $V_2$ (see Table. [I]). Assuming the state to exist, its low energy effective theory can be routinely constructed, simply based on the broken (physical) symmetries without relying on microscopic details. We find the effective Lagrangian of the state,

$$\mathcal{L}_{\text{eff}} = \frac{\epsilon}{2(2\pi)^3} \int d^3k \text{Tr} \ln \mathcal{D}(k - A)$$

where the ‘Tr’ is over the internal space of $\vec{r} \otimes \vec{\tau} \otimes \vec{\sigma}$. With the $\mathcal{D}$ matrix given in (10), a straightforward calculation gives

$$\mathcal{L}_{\text{eff}} = \frac{\sigma}{2} e^\mu \rho \text{tr} \{ A_{\mu} A_{\rho} - \frac{1}{4} g_{\mu\rho} A_{\sigma} A_{\tau} \} + \cdots , $$

where $Q$ is a (topological) winding number (Q = 1) and the ‘…’ stands for higher derivatives, including a Maxwell term. Now the ‘$\text{tr}$’ in (13) is over only the color gauge indices. The effective theory of gauge bosons is thus a level 1 (non-Abelian) SU(2) Chern-Simons theory. All gauge excitations gain a dynamically generated topological mass (i.e., gapped). The gauge interaction becomes short ranged. So the effective theory we found in (6) is stable.

e. Physical properties of the $p_{x+iy}$-state. After obtaining the low energy effective theory, we are ready to study the measurable properties of the state. First the $p_{x+iy}$-state is a superfluid which does not break spin rotation symmetry. The only gapless excitation is the superfluid mode described by the phase of $\Delta$. All spin excitations have a finite energy gap. Due to the Chern-Simons term, the SU(2) gauge field is not confining. Thus the excitations described by $\psi$ (or $\eta$) have a finite energy gap (instead of infinite energy gap). Those excitations carry spin-1/2 and one half of boson charge! They also have a semion statistics (i.e., the statistical angle is $\theta = \pi/2$, right between boson and fermion), as implied by the level-1 SU(2) Chern-Simons terms. The spin rotation symmetry implies that $\langle \phi_m \rangle = 0$ for all $m$. However, two-boson and three-boson operators both can have finite expectation values. In short distance, $\langle \phi_m(z_1) \phi_m(z_2) \rangle \sim -4\sqrt{\delta (z_1 - z_2)^2 C_{m,m'}^{00}}$ and $\langle \phi_m(z_1) \phi_m(z_2) \phi_m(z_3) \rangle \sim 8\sqrt{2 (z_1 - z_2) (z_2 - z_3) (z_3 - z_1) C_{m,m',m''}}$ where $z = x + iy$. The spin-disordered superfluid has an unusual off-diagonal long range order. We see that the minimum vortex has one unit of quantized vorticity. Our $p_{x+iy}$ spin-disordered state breaks the parity and time reversal symmetry. The total angular momentum of the ground state is $h$ per boson. (Note such a total angular momentum is equal to the total angular
momentum of usual boson superfluid with one vortex at its center.) Thus, spinning the bosons may help to create the $p_{x+iy}$ spin-disordered superfluid.

\textit{f. Edge excitations} In the $\eta$ bases, the mean-field fermion Hamiltonian (described by (1) with $A_\mu$ being set to zero) contains four identical $2 \times 2$ blocks: $H = \hat{h}(\vec{k}) \cdot \vec{\sigma}$. In each block, the function $\hat{h}(\vec{k})$ defines a mapping from the k-space to $S^2$ with a winding number 1. This non-trivial winding number leads to a unit Hall conductance [13, 14]. Since each block contributes a unit Hall conductance, it leads to an edge state similar to the one from $\nu = 1$ quantum Hall state [15] as required by gauge invariance [16, 17, 18]. Such an edge state can be described by one chiral fermion field $\lambda_{a\alpha}$ (one for each block). Therefore, if we ignore the SU(2) gauge fluctuations, the mean-field edge state is described by the following effective theory: $\lambda_{a\alpha}^\dagger i(\partial_t - vt)\lambda_{a\alpha}$. Only two of the four $\lambda$'s are independent because $\eta_{a\alpha}$ (see (3)) are Majorana fermions and each $\lambda_{a\alpha}$ is obtained as a linear combination of the two components of spinor $\eta_{a\alpha}$. Therefore, the mean-field edge state contains only two independent branches of chiral fermions. Obviously the mean-field edge effective theory has SU(2)$_{\text{color}}$ and SU(2)$_{\text{spin}}$ symmetries generated by $\tau^l \otimes \tau^0$ and $\tau^0 \otimes \tau^l$, $l = 1, 2, 3$, respectively. Having the SU(2)$_{\text{color}} \times$ SU(2)$_{\text{spin}}$ symmetry and a central charge $c = 2$ (i.e., two branches of chiral fermions), we find that the mean-field edge state is described by SU(2)$_{\text{color}} \times$ SU(2)$_{\text{spin}}$ Kac-Moody current algebra of level 1. After including the SU(2)$_{\text{color}}$ gauge fluctuations to go beyond the mean-field theory, the edge effective theory becomes

$$\lambda_{a\alpha}^\dagger i \left[ (\partial_t - iA_0) - v(\partial_x - iA_x) \right]_{ab} \lambda_{b\alpha}. \quad (14)$$

The effect of SU(2)$_{\text{color}}$ gauge fields is to remove the SU(2)$_{\text{color}}$ sector of the Kac-Moody algebra from the low energy spectrum [19]. Thus the physical edge state of the $p_{x+iy}$-state is described by a level-1 SU(2)$_{\text{spin}}$ Kac-Moody algebra. The physical edge state is identical to the right moving sector of spin-1/2 chain. Despite the finite gap in the bulk, the spin excitation is gapless at the edge. The operator that creates the gapless spin-1/2 quasiparticle on the edge is given by the spin-1/2 primary field $V_\alpha(x,t)$ in the SU(2)$_{\text{spin}}$ Kac-Moody algebra which has a scaling dimension 1/4. The quasiparticle propagator has a form $\langle V_\alpha(x,t) V_\alpha^\dagger(0) \rangle \sim (x - vt)^{-1/2}$. The boson operator $\phi_m$ on the edge becomes the spin-1 primary field which is the spin current operator on the edge. The boson propagator on the edge is given by $\langle \phi_m(x,t) \phi_m^\dagger(0) \rangle \sim (x - vt)^{-2}$. (The boson propagator is short ranged in the bulk due to the finite spin gap.) This will lead to a non-linear I-V curve $I \propto |V|^3$ for boson tunneling between two edges. The spin-1/2 quasiparticles can tunnel between two edges separated by a bulk $p_{x+iy}$ state. The tunneling I-V curve has a form $I \propto |V|^{-1}$ in the weak tunneling limits. Finally, we briefly mention that the polar condensate has gapless spin excitations in the bulk whereas the $p_{x+iy}$ spin-disordered superfluid has a gapless spin excitation only at the edge. A dramatic difference can be seen in the spin susceptibility by NMR experiments.

\textit{g. Conclusions} A two-dimensional boson gas in ultra-cold alkali atomic systems can be strongly correlated. A spin-1 boson system can have a very interesting spin-disordered superfluid state, which carries a non-trivial topological order [18]. Such possibility is interesting, since they might exhibit some of the spin liquid phases that have been long theoretically speculated in the context of high $T_c$ superconductors but never been clearly identified by experiments. We believe that the alkali atomic gases may provide the first controlled laboratory to check those speculated theories and enrich our understanding of the strongly correlated systems. In fact the $p_{x+iy}$-state is closely related to the chiral spin state [20].

The two states have the same bulk effective theory described by level-1 SU(2)$_{\text{color}}$ Chern-Simons theory [21] and the same edge effective theory described by level-1 SU(2)$_{\text{spin}}$ Kac-Moody algebra [15]. The spin sector of the two states are identical.

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is given by a topological invariant, \( Q = \frac{1}{8\pi} \epsilon_{ij} \int d^2 k \hat{\mathbf{h}} \cdot (\partial \hat{\mathbf{h}}/\partial k_i \times \partial \hat{\mathbf{h}}/\partial k_j) \), \( \hat{\mathbf{h}} \equiv \hat{\mathbf{h}} / |\hat{\mathbf{h}}| \). Inserting the \( \hat{\mathbf{h}}(\mathbf{k}) \) functions defined in (11) gives \( Q = 1 \).

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