Convergence of Linear Bregman ADMM for Nonconvex and Nonsmooth Problems with Nonseparable Structure

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The alternating direction method of multipliers (ADMM) is a very effective method for solving two-block separable convex problems and its convergence is well understood. When either the involved number of blocks is more than two, or there is a nonconvex function, or there is a nonseparable structure, ADMM or its directly extend version may not converge. In this paper, we proposed an ADMM-based algorithm for nonconvex multiblock optimization problems with a nonseparable structure. We show that any cluster point of the iterative sequence generated by the proposed algorithm is a critical point, under mild condition. Furthermore, we establish the strong convergence of the whole sequence, under the condition that the potential function satisfies the Kurdyka–Łojasiewicz property. This provides the theoretical basis for the application of the proposed ADMM in the practice. Finally, we give some preliminary numerical results to show the effectiveness of the proposed algorithm.

1. Introduction

In this paper, we consider the following possibly nonconvex and nonsmooth optimization problem:

\[
\begin{aligned}
\min & \quad \sum_{i=1}^{N-1} f_i(x_i) + g(x_1, \ldots, x_{N-1}, y), \\
\text{s.t.} & \quad \sum_{i=1}^{N-1} A_i x_i + By = b,
\end{aligned}
\]

where \( x_i \in \mathbb{R}^{n_i} \) (\( i = 1, 2, \ldots, N - 1 \)), \( y \in \mathbb{R}^n \) are variables vectors, \( g: \mathbb{R}^{l} \to \mathbb{R} (l = n_1 + n_2 + \cdots + n_{N-1} + n) \) is differentiable, and each \( f_i: \mathbb{R}^{n_i} \to \mathbb{R} \cup \{\infty\} \) is proper and lower semicontinuous, \( A_i \in \mathbb{R}^{m \times n_i} \) (\( i = 1, 2, \ldots, N - 1 \)), \( B \in \mathbb{R}^{m \times n} \) are given matrix, and \( b \in \mathbb{R}^m \).

The alternating direction method of multipliers (ADMM) is a very effective method for solving the convex two-block optimization problem [1, 2]. A natural idea is to extend ADMM to solve problem (1). However, ADMM or its directly extend version may not converge, when either the involved number of blocks is more than two, or there is a nonconvex function, or there is a nonseparable structure.

Recently, there have been a few developments on it, e.g., [3–13].

Hong et al. [6] considered the sharing and consensus problem and showed that the classical ADMM converges to the set of stationary solutions, only provided that the penalty parameter in the augmented Lagrangian is chosen to be sufficiently large. Li and Pong [8] studied the convergence of ADMM for some special two-block nonconvex models, where one of the matrices A and B is an identity matrix. Wang et al. [9, 10] studied the convergence of the nonconvex Bregman ADMM algorithm, which includes ADMM as a special case. Wang et al. [11] studied the convergence of the ADMM for nonconvex nonsmooth optimization with a nonseparable structure. Guo et al. [4, 5] studied the convergence of classical ADMM for two-block and multiblock nonconvex models where one of the matrices is an identity matrix. Yang et al. [13] studied the convergence of the ADMM for a nonconvex optimization model which come from the background/foreground extraction.

The purpose and the main contribution of this paper is to propose and prove the convergence of a new variant ADMM
for nonconvex coupled problems (1). The novelty of this paper can be summarized as follows:

(1) Compared to the existing literature, the model in this paper is more general. There is no nonseparable structure in the models considered by [4–10, 12, 13]. Wang et al. [11] considered two scenarios. If
\[ g(x_1, \ldots, x_{N-1}, y) = g_1(x_1, \ldots, x_{N-1}) + h(y), \]
then (1) is the scenario 1 in [11]. If \( f_i(x) \equiv 0 \), for \( i = 1, 2, \ldots, N - 1 \), then (1) becomes the scenario 2 in [11]. Furthermore, in this paper, the matrices \( A_i (i = 1, 2, \ldots, N - 1) \) and \( B \) are possibly not full column or row rank.

(2) The proposed algorithm combines linearization technology with regularization technology. Linearization technology and regularization technology can effectively reduce the difficulty of the solving subproblems.

The rest of this paper is organized as follows. In Section 2, some basic concepts and necessary preliminaries for further analysis are summarized. In Section 3, we propose the algorithm and analyze the convergence of it for 3-block nonconvex and nonsmooth coupled problems. Finally, some conclusions are made in Section 4.

2. Preliminaries

\( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space, \( \mathbb{R} \cup \{+\infty\} \) denotes the extended real number set, and \( \mathbb{N} \) denotes the natural number set. The image space of a matrix \( Q \in \mathbb{R}^{m \times n} \) is defined as \( \text{Im} Q = \{Qx : x \in \mathbb{R}^n\} \). \( P_Q(\cdot) \) denotes the Euclidean projection onto \( \text{Im} Q \). If matrix \( Q \neq 0 \), let \( \mu_Q \) denote the smallest positive singular value of the matrix \( QQ^T \). \( \| \cdot \| \) represents the Euclidean norm. \( \text{dom}(f) = \{ x \in \mathbb{R}^n : f(x) < +\infty \} \) is the domain of a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \). \( \langle x, y \rangle = x^T y = \sum_{i=1}^{n} x_i y_i \). \[ \| x \| = \| y \| = \sum_{i=1}^{m} \log (x_i)^2 \]. If \( S = \emptyset \), we set \( d(x, S) = +\infty \) for all \( x \in \mathbb{R}^n \). For a point-to-set mapping \( F \), its graph is defined by \( \text{Graph} F = \{(x, y) : y \in F(x)\} \).

**Definition 1** (see [14]). Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) be a proper function. If there exists \( \delta > 0 \) such that
\[
f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y) - \frac{(\delta \lambda (1 - \lambda))}{2} \| x - y \|^2,
\]
for all \( \lambda \in (0, 1) \) and \( x, y \in \text{dom} f \), then \( f \) is called strongly convex with modulus \( \delta \).

**Definition 2** (see [15]). For a convex differential function \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \), the associated Bregman distance is defined as
\[
\Delta_\phi(x, y) = \phi(x) - \phi(y) - \langle \nabla \phi(y), x - y \rangle,
\]
for all \( x, y \in \mathbb{R}^n \).

The Bregman distance plays an important role in iterative algorithms. The Bregman distance share many similar nice properties of the Euclidean distance. However, the Bregman distance is not a metric, since it does not satisfy the triangle inequality nor symmetry. Some examples of Bregman distance include [16]

(i) Classical Euclidean distance: if \( f(x) = \| x \|^2 \), then \( \Delta_\phi(x, y) = \| x - y \|^2 \).

(ii) Itakura–Saito distance: if \( f(x) = \sum_{i=1}^{m} x_i \log(x_i) \), then \( \Delta_\phi(x, y) = \sum_{i=1}^{m} \log(x_i/y_i) - \sum_{i=1}^{m} (x_i - y_i) \).

(iii) Mahalanobis distance: if \( f(x) = \| x \|_Q^2 = x^T Qx \) with \( Q \) a symmetric positive definite matrix, then \( \Delta_\phi(x, y) = \| x - y \|_Q^2 \).

Let us now collect some useful properties about Bregman distance.

**Proposition 1** (see [15]). Let \( \phi \) be differentiable and strongly convex function with modulus \( \delta \), then

(i) \( \Delta_\phi(x, y) \geq 0 \) and \( \Delta_\phi(x, y) = 0 \) if and only if \( x = y \).

(ii) \( \Delta_\phi(x, y) \geq (\delta/2) \| x - y \|^2 \) for all \( x \) and \( y \).

The following notations and definitions are quite standard and can be founded in [14, 17].

**Definition 3.** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) be a proper lower semicontinuous function.

(i) The Fréchet subdifferential, or regular subdifferential, of \( f \) at \( x \in \text{dom} f \) is
\[
\partial f(x) = \left\{ x^* : \lim_{y \rightarrow x} \inf_{y^*} \frac{f(y) - f(x) - \langle x^*, y - x \rangle}{\| y - x \|} \geq 0 \right\}.
\]
When \( x \notin \text{dom} f \), we set \( \partial f(x) = \emptyset \).

(ii) The limiting subdifferential, or simply the subdifferential, of \( f \) at \( x \in \text{dom} f \), written \( \partial f(x) \), is defined as
\[
\partial f(x) = \left\{ v^* \in \mathbb{R}^n : \exists x^k \rightarrow x, \text{ s.t. } f(x^k) \rightarrow f(x), v^k \in \partial f(x^k), v^k \rightarrow v^* \right\}.
\]

(iii) A point that satisfies \( 0 \notin \partial f(x) \) is called a critical point or a stationary point of the function \( f \). The set of critical points of \( f \) is denoted by \( \text{crit} f \).

The following proposition collects some properties of the subdifferential.

**Proposition 2** (see [17]). Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) and \( g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\} \) be proper lower semicontinuous functions. Then, the following holds:
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(i) \( \tilde{f}(x) \subseteq \partial f(x) \) for each \( x \in \text{dom } f \). Moreover, the first set is closed and convex, while the second is closed and not necessarily convex.

(ii) Let \( (x^k, u^k) \in \text{Graph } \tilde{f} \) be a sequence such that it converges to \( (x^*, u^*) \). If \( \lim_{k \to +\infty} f(x^k) \neq f(x^*) \), then \( (x^*, u^*) \in \text{Graph } \tilde{f} \).

(iii) If \( x \in \mathbb{R}^n \) is a local minimizer of \( f \) then \( 0 \in \tilde{f}(x) \).

(iv) If \( g: \mathbb{R}^n \to \mathbb{R} \) is continuous differentiable, then \( \tilde{f}(f + g)(x) = \partial f(x) + \nabla g(x) \).

The Lagrangian function of \( (1) \), with multiplier \( \lambda \in \mathbb{R}^m \), is defined as

\[
L(x_1, \ldots, x_{N-1}, y, \lambda) = f_1(x_1) + \cdots + f_{N-1}(x_{N-1}) + g(x_1, \ldots, x_{N-1}, y) - \lambda_1 A_1 x_1 + \cdots + A_{N-1} x_{N-1} + By - b.
\]

(6)

**Definition 4.** If \( u^* = (x_1^*, \ldots, x_{N-1}^*, y^*, \lambda^*) \) such that

\[
A_1 x_1^* + \cdots + A_{N-1} x_{N-1}^* + By^* - b = 0,
\]

then \( u^* \) is called a critical point or stationary point of the Lagrange function \( L(x_1, \ldots, x_{N-1}, y, \lambda) \).

A very important technique to prove the strong convergence of the ADMM for nonconvex optimization problems relies on the assumption that the benefit function satisfying Kurdyka-Łojasiewicz property (KL property) [18–21]. There are many functions which satisfy this inequality. Especially, when the function belongs to some functional classes, e.g., semialgebraic, real subanalytic, and log-exp (see [22–24]). It is often elementary to check that such an inequality holds.

For notational simplicity, we use \( \Phi_{\eta} (\eta > 0) \) to denote the set of concave functions \( \varphi: [0, \eta) \to (0, +\infty) \) such that

(i) \( \varphi(0) = 0 \)

(ii) \( \varphi \) is continuous differentiable on \( (0, \eta) \) and continuous at 0

(iii) \( \varphi'(s) > 0, \forall s \in (0, \eta) \).

The KL property can be described as follows.

**Definition 5** (see [18–21]) (KL property). Let \( f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a proper lower semicontinuous function. If there exists \( \eta \in (0, +\infty) \), a neighborhood \( U \) of \( x^* \), and a function \( \varphi \in \Phi_{\eta} \), such that for all \( x \in U \cap \{ f(x^*) < f(x^*) + \eta \} \), it holds that

\[
\varphi'(f(x) - f(x^*))d(0, \partial f(x)) \geq 1,
\]

then \( f \) is said to have the KL property at \( x^* \).

**Lemma 1** (see [22]) (unified KL property). Suppose that \( f: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is a proper lower semicontinuous function and \( \Omega \) is a compact set. If \( f(x) \equiv f^* \) for all \( x \in \Omega \) and satisfies the KL property at each point of \( \Omega \). Then, there exist \( \varepsilon > 0, \eta > 0 \), and \( \varphi \in \Phi_{\eta} \) such that

\[
\varphi'(f(x) - f^*)d(0, \partial f(x)) \geq 1,
\]

for all \( x \in \{ x \in \mathbb{R}^n : d(x, \Omega) \varepsilon \} \cap \{ f^* < f < f^* + \eta \}.

**Lemma 2** (see [25]) (Descent lemma). Let \( h: \mathbb{R}^n \to \mathbb{R} \) be a continuous differentiable function where gradient \( \forall h \) is Lipschitz continuous with the modulus \( l_h > 0 \), then for any \( x, y \in \mathbb{R}^n \), we have

\[
|h(x) - h(y) - \langle \nabla h(x), y-x \rangle | \leq \frac{l_h}{2} \| y - x \|^2.
\]

**Lemma 3** (see [26]). Let \( Q \in \mathbb{R}^{m \times p} \) be a nonzero matrix and let \( \mu_Q \) denote the smallest positive eigenvalue of \( QQ^T \). Then, for every \( u \in \mathbb{R}^p \), there holds

\[
\| P_Q u \| \leq \frac{1}{\sqrt{\mu_Q}} \| Q u \|.
\]

3. Algorithm and Convergence

For the convenience of analysis, we only consider the case of \( N = 2 \). The obtained results could naturally be generalized to the case of \( N > 2 \). Thus, in the rest of this paper, we consider the following nonconvex and nonsmooth 3-block optimization problem:

\[
\min \quad f_1(x_1) + f_2(x_2) + g(x_1, x_2, y),
\]

s.t. \( A_1 x_1 + A_2 x_2 + By = b \),

(12)

where \( f_i: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is proper and lower semicontinuous but possibly nonconvex, \( g: \mathbb{R}^{n_1 + n_2 + n_3} \to \mathbb{R} \) is differentiable, \( A_1 \in \mathbb{R}^{m \times n_1} \), \( A_2 \in \mathbb{R}^{m \times n_2} \), and \( b \in \mathbb{R}^m \).

In this paper, we present the following algorithm for (12).

**Algorithm 1.** LBADMM: start with \( (x_1^0, x_2^0, y^0) \) and \( \lambda^0 \). With the given iteration point \( w^k = (x_1^k, x_2^k, y^k, \lambda^k)^T \), the new iteration point \( w^{k+1} = (x_1^{k+1}, x_2^{k+1}, y^{k+1}, \lambda^{k+1})^T \) is given as follows:
\[
\begin{align*}
 \begin{cases}
 x_1^{k+1} \in \arg\min \bigg\{ f_1(x_1) + (x_1 - x_1^k)^T \nabla_x g(x_1^k, x_2^k, y^k) + \frac{\beta}{2} \left\| A_1 x_1 + A_2 x_2^k + B y^k - b - \frac{\lambda^k}{\beta} \right\|^2 + \Delta_{\varphi_1}(x_1, x_1^k) \bigg\}, \\
 x_2^{k+1} \in \arg\min \bigg\{ f_2(x_2) + (x_2 - x_2^k)^T \nabla_x g(x_1^k, x_2^k, y^k) + \frac{\beta}{2} \left\| A_1 x_1^{k+1} + A_2 x_2 + B y^k - b - \frac{\lambda^{k+1}}{\beta} \right\|^2 + \Delta_{\varphi_2}(x_2, x_2^k) \bigg\}, \\
 y^{k+1} = \arg\min \bigg\{ g(x_1^{k+1}, x_2^{k+1}, y) + \frac{\beta}{2} \left\| A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y - b - \frac{\lambda^{k+1}}{\beta} \right\|^2 + \Delta_{\varphi}(y, y^k) \bigg\}, \\
 \lambda^{k+1} = \lambda^k - \beta (A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y^{k+1} - b),
\end{cases}
\end{align*}
\]

where \(\Delta_{\varphi_1}(x_1, x_1^k), \Delta_{\varphi_2}(x_2, x_2^k),\) and \(\Delta_{\varphi}(y, y^k)\) are the Bregman distances associated with \(\varphi_1, \varphi_2,\) and \(\varphi,\) respectively.

**Remark 1.** Due to the different structures of the problem, the algorithm in this paper is different from the existing algorithms. In order to make use of the properties of differentiable blocks and simplify the calculation of each iteration, we linearize the differentiable part in the \(x_1\) and \(x_2\) subproblems. If the function \(g(x_1, x_2, y)\) is only related to the variable \(y,\) that is, \(g(x_1, x_2, y) = h(y),\) then the algorithm LBADMM will become the Bregman ADMM in [9, 10]. Different from [9, 10], we do not assume \(B\) is full row rank.

In this section, we always assume that the sequence \(\{u^k = (x_1^k, x_2^k, y^k, \lambda^k)^T\}\) is generated by algorithm LBADMM. Let \(\mu = (1/\sqrt{\mu_B})\), where \(\mu_B\) denotes the smallest positive eigenvalue of \(B^T B\).

**Assumption 1**

(i) \(\nabla g\) is \(l_g\)-Lipschitz continuous, i.e., \(\|\nabla g(u) - \nabla g(v)\| \leq l_g \|u - v\|\) for all \(u, v \in \mathbb{R}^n\)

(ii) \(B \neq 0\) and \(\text{Im} B^{(b)} = \text{Im} A\)

(iii) \(\nabla \varphi_1, \nabla \varphi_2, \nabla \varphi\) are Lipschitz continuous with the modulus \(l_{\varphi_1, \varphi_2, \varphi}\), respectively

(iv) \(\varphi_1, \varphi_2, \varphi\) strongly convex with the modulus \(\delta, \delta, \delta,\) and \(\delta, \delta, \delta > 0\), respectively

(v) \(\beta > \max\{6 \mu^2 (2 l_g^2 + l_g^2) / \delta, (6 \mu^2 l_g^2 / \delta - 1) \}, \) \(6 \mu^2 l_g^2 / \delta - 1\)

\[
\|B^T \lambda^{k+1} - B^T \lambda^k\|^2 = \|\nabla g(x_1^{k+1}, x_2^{k+1}, y^{k+1}) - \nabla g(x_1^k, x_2^k, y^k)\|^2 + \|\nabla \varphi(y^{k+1}) - \nabla \varphi(y^k)\|^2 + \|\nabla \varphi(y^{k+1}) - \nabla \varphi(y^k)\|^2
\]

\[
\leq 3 \left( \|\nabla g(x_1^{k+1}, x_2^{k+1}, y^{k+1}) - \nabla g(x_1^k, x_2^k, y^k)\|^2 + \|\nabla \varphi(y^{k+1}) - \nabla \varphi(y^k)\|^2 + \|\nabla \varphi(y^{k+1}) - \nabla \varphi(y^k)\|^2 \right)
\]

The following lemma establishes the relationship between the dual variable and the original variables.

**Lemma 4.** For each \(k \in \mathbb{N},\)

\[
\|\lambda^{k+1} - \lambda^k\|^2 \leq 3 \mu_B^2 \left( \|\lambda^{k+1} - \lambda^k\|^2 \right)
\]

Proof. By Assumption 1 (ii) and Lemma 3, we have

\[
\|\lambda^{k+1} - \lambda^k\| \leq \frac{1}{\sqrt{\mu_B}} \|B^T (\lambda^{k+1} - \lambda^k)\|
\]

From the optimality condition of \(y\)-subproblem in (14) yields

\[
0 = \nabla_y g(x_1^{k+1}, x_2^{k+1}, y^{k+1}) - B^T \lambda^k + \beta B^T (A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y^{k+1} - b)
\]

Taking into account \(\lambda^{k+1} = \lambda^k - \beta (A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y^{k+1} - b),\) one has

\[
B^T \lambda^{k+1} = \nabla_y g(x_1^{k+1}, x_2^{k+1}, y^{k+1}) + \nabla \varphi(y^{k+1}) - \nabla \varphi(y^k)
\]

Thus,

\[
\begin{align*}
\|B^T (\lambda^{k+1} - \lambda^k)\|^2 & = \|\nabla_y g(x_1^{k+1}, x_2^{k+1}, y^{k+1}) - \nabla_y g(x_1^k, x_2^k, y^k) + \nabla \varphi(y^{k+1}) - \nabla \varphi(y^k) + \nabla \varphi(y^{k+1}) - \nabla \varphi(y^k)\|^2 \\
& \leq \left( \|\nabla_y g(x_1^{k+1}, x_2^{k+1}, y^{k+1}) - \nabla_y g(x_1^k, x_2^k, y^k)\|^2 + \|\nabla \varphi(y^{k+1}) - \nabla \varphi(y^k)\|^2 + \|\nabla \varphi(y^{k+1}) - \nabla \varphi(y^k)\|^2 \right)^2 \\
& \leq 3 \left( \|\nabla_y g(x_1^{k+1}, x_2^{k+1}, y^{k+1}) - \nabla_y g(x_1^k, x_2^k, y^k)\|^2 + \|\nabla \varphi(y^{k+1}) - \nabla \varphi(y^k)\|^2 + \|\nabla \varphi(y^{k+1}) - \nabla \varphi(y^k)\|^2 \right)^2
\end{align*}
\]
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It follows from the abovementioned formula and (17) that
\[
\|\lambda^{k+1} - \lambda^k\|^2 \leq \mu^2 \|B^T (\lambda^{k+1} - \lambda^k)\|^2
\]
\[
\leq 3\mu^2 (\ell_g^1 \|x_1^{k+1} - x_1^k, x_2^{k+1} - x_2^k, y^{k+1} - y^k\|^2
\]
\[+ \ell_g^2 \|y^{k+1} - y^k\|^2 + \ell_g^3 \|y^{k-1} - y^{k-2}\|^2)
\]
\[
\leq 3\mu^2 (\ell_g^1 \|x_1^{k+1} - x_1^k\|^2 + \|x_2^{k+1} - x_2^k\|^2)
\]
\[+ 3\mu^2 (\ell_g^2 \|y^{k+1} - y^k\|^2 + \ell_g^3 \|y^{k-1} - y^{k-2}\|^2).
\]
\]
\[
(19)
\]

The proof is completed. □

The augmented Lagrangian function with multiplier \(\lambda\) of (12) is defined as
\[
L_\beta(x_1, x_2, y, \lambda) = L(x_1, x_2, y, \lambda) + \frac{\beta}{2} \|A_1 x_1 + A_2 x_2 + B y - b\|^2,
\]
\[
(20)
\]
where \(L(x_1, x_2, y, \lambda)\) is the Lagrangian function of (12). Let
\[
\tilde{L}(x_1, x_2, y, \lambda, \bar{y}) = L_\beta(x_1, x_2, y, \lambda) + \frac{3\mu^2 \ell_g^2}{\beta} \|y - \bar{y}\|^2.
\]
\[
(21)
\]

Let
\[
\bar{u}^k = (x_1^k, x_2^k, y, \lambda^k), w^k = (x_1^k, x_2^k, y, \lambda^k), u^k
\]
\[= (x_1^k, x_2^k, y^k),
\]
\[
r_k = A_1 x_1^k + A_2 x_2^k + B y^k - b.
\]

The following lemma implies the monotonicity of the sequence \(\{\tilde{L}(\bar{u}^k)\}_{k \in \mathbb{N}}\).

\[
\textbf{Lemma 5. For each } k \in \mathbb{N},
\]
\[
\tilde{L}(\bar{u}^{k+1}) \leq \tilde{L}(\bar{u}^k) - \sigma \left( \|x_1^{k+1} - x_1^k\|^2 + \|x_2^{k+1} - x_2^k\|^2 + \|y^{k+1} - y^k\|^2 \right),
\]
\[
(23)
\]

where
\[
\sigma = \min \left\{ \left( \frac{\delta \phi}{2} \right) - \left( \frac{3\mu^2 (2\ell_g^1 + \ell_g^2)}{\beta} \right), \left( \frac{\delta \psi_2 - l_g}{2} \right) \right\}.
\]
\[
(24)
\]

\[
\textbf{Proof. From (17), we have}
\]
\[
f_1(x_1^{k+1}) + \langle x_1^{k+1} - x_1^k, \nabla_x g(u^k) \rangle - \langle \lambda^k, A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b \rangle + \frac{\beta}{2} \|A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b\|^2
\]
\[
\leq f_1(x_1^k) - \langle \lambda^k, r_k \rangle + \frac{\beta}{2} \|r_k\|^2 - \nabla \phi(x_1^{k+1}, x_1^k),
\]
\[
f_2(x_2^{k+1}) + \langle x_2^{k+1} - x_2^k, \nabla_x g(u^k) \rangle - \langle \lambda^k, A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b \rangle + \frac{\beta}{2} \|A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b\|^2
\]
\[
\leq f_2(x_2^k) - \langle \lambda^k, A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b \rangle + \frac{\beta}{2} \|A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b\|^2 - \nabla \psi(x_2^{k+1}, x_2^k),
\]
\[
g(u^{k+1}) - \langle \lambda^k, r_{k+1} \rangle + \frac{\beta}{2} \|r_{k+1}\|^2 + \nabla \psi(y^{k+1}, y^k)
\]
\[
\leq g(x_1^{k+1}, x_2^{k+1}, y^k) - \langle \lambda^k, A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b \rangle + \frac{\beta}{2} \|A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b\|^2.
\]
\[
(25)
\]
Adding up the abovementioned three formulas, we have
\[
\begin{align*}
&f_1(x_1^{k+1}) + f_2(x_2^{k+1}) + g(u^{k+1}) - \langle \lambda^k, r_{k+1} \rangle + \frac{\beta}{2}r_{k+1}^2 \\
\leq f_1(x_1^k) + f_2(x_2^k) + (x_1^{k+1}, x_2^{k+1}, y^k)g - \langle \lambda^k, r_k \rangle + \frac{\beta}{2}r_k^2 \\
- \left[ \langle x_1^{k+1} - x_1^k, \nabla_x g(u^k) \rangle + \langle x_2^{k+1} - x_2^k, \nabla_x g(u^k) \rangle \right] \\
- \Delta_{\phi}(x_1^{k+1}, x_1^k) - \Delta_{\phi}(x_2^{k+1}, x_2^k) - \Delta_{\phi}(y^{k+1}, y^k), \\
\end{align*}
\] (26)

and hence
\[
\begin{align*}
&f_1(x_1^{k+1}) + f_2(x_2^{k+1}) + g(u^{k+1}) - \langle \lambda^k, r_{k+1} \rangle + \frac{\beta}{2}r_{k+1}^2 \\
\leq f_1(x_1^k) + f_2(x_2^k) + g(u^k) - \langle \lambda^k, r_k \rangle + \frac{\beta}{2}r_k^2 \\
+ g(x_1^{k+1}, x_2^{k+1}, y^k) - g(u^k) - \langle (x_1^{k+1} - x_1^k, x_2^{k+1} - x_2^k, 0), \nabla g(u^k) \rangle \\
- \Delta_{\phi}(x_1^{k+1}, x_1^k) - \Delta_{\phi}(x_2^{k+1}, x_2^k) - \Delta_{\phi}(y^{k+1}, y^k), \\
\end{align*}
\] (27)

that is,
\[
\begin{align*}
&L_\phi(x_1^{k+1}, x_2^{k+1}, y^{k+1}, \lambda^k) \\
\leq& L_\phi(u^k) + g(x_1^{k+1}, x_2^{k+1}, y^k) - g(u^k) \\
&- \langle (x_1^{k+1} - x_1^k, x_2^{k+1} - x_2^k, 0), \nabla g(u^k) \rangle \\
&- \Delta_{\phi}(x_1^{k+1}, x_1^k) - \Delta_{\phi}(x_2^{k+1}, x_2^k) - \Delta_{\phi}(y^{k+1}, y^k). \\
\end{align*}
\] (28)

From Lemma 2, Assumption 1 (iv), and Proposition 1, we obtain
\[
\begin{align*}
L_\phi(x_1^{k+1}, x_2^{k+1}, y^{k+1}, \lambda^k) \\
\leq& L_\phi(u^k) - \frac{\delta_\phi}{2}\|x_1^{k+1} - x_1^k\|^2 - \frac{\delta_\phi}{2}\|x_2^{k+1} - x_2^k\|^2 \\
&- \frac{\delta_\phi}{2}\|y^{k+1} - y^k\|^2 + \frac{3\mu^2\beta}{2}\|y^k - y^{k-1}\|^2 \\
&+ \frac{3\mu_2\beta}{\|x_1^{k+1} - x_1^k\|^2 + \|x_2^{k+1} - x_2^k\|^2} \leq L_\phi(u^k) - \left[ \frac{\delta_\phi}{2} - \frac{3\mu_2\beta}{\|x_1^{k+1} - x_1^k\|^2} \right] \|x_1^{k+1} - x_1^k\|^2 - \left[ \frac{\delta_\phi}{2} - \frac{3\mu_2\beta}{\|x_2^{k+1} - x_2^k\|^2} \right] \|x_2^{k+1} - x_2^k\|^2 \\
&- \left[ \frac{3\mu_2^2\beta}{\|x_1^{k+1} - x_1^k\|^2 + \|x_2^{k+1} - x_2^k\|^2} \right] \|y^{k+1} - y^k\|^2 + \frac{3\mu_2^2\beta}{\|y^k - y^{k-1}\|^2}, \\
\end{align*}
\] (29)

Recall that
\[
\begin{align*}
L_\phi(u^{k+1}) = L_\phi(x_1^{k+1}, x_2^{k+1}, y^{k+1}, \lambda^k) + \langle \lambda^k - \lambda_{k+1}, r_{k+1} \rangle \\
= L_\phi(x_1^{k+1}, x_2^{k+1}, y^{k+1}, \lambda^k) + \frac{1}{\beta}\|\lambda^k - \lambda_{k+1}\|^2. \\
\end{align*}
\] (30)

Adding up (71) and (72), we have
\[
\begin{align*}
L_\phi(u^{k+1}) &\leq L_\phi(u^k) - \frac{\delta_\phi}{2}\|x_1^{k+1} - x_1^k\|^2 - \frac{\delta_\phi}{2}\|x_2^{k+1} - x_2^k\|^2 \\
&- \frac{\delta_\phi}{2}\|y^{k+1} - y^k\|^2 + \frac{3\mu^2\beta}{2}\|y^k - y^{k-1}\|^2 \\
&+ \frac{3\mu_2\beta}{\|x_1^{k+1} - x_1^k\|^2 + \|x_2^{k+1} - x_2^k\|^2} \leq L_\phi(u^k) - \left[ \frac{\delta_\phi}{2} - \frac{3\mu_2\beta}{\|x_1^{k+1} - x_1^k\|^2} \right] \|x_1^{k+1} - x_1^k\|^2 - \left[ \frac{\delta_\phi}{2} - \frac{3\mu_2\beta}{\|x_2^{k+1} - x_2^k\|^2} \right] \|x_2^{k+1} - x_2^k\|^2 \\
&- \left[ \frac{3\mu_2^2\beta}{\|x_1^{k+1} - x_1^k\|^2 + \|x_2^{k+1} - x_2^k\|^2} \right] \|y^{k+1} - y^k\|^2 + \frac{3\mu_2^2\beta}{\|y^k - y^{k-1}\|^2}, \\
\end{align*}
\] (31)

Together with (14), we obtain
which implies that
\[
L(\hat{w}^{k+1}) \leq L(\hat{w}^k) + \frac{3\mu^2 l_2}{\beta}\|y^k - y^k\|^2 - \left[ \frac{\delta_v - l_\phi}{2} - \frac{3\mu^2 l_2}{\beta} \right] \|x_k^{k+1} - x_k\|^2
- \left[ \frac{\delta_v - l_\phi}{2} - \frac{3\mu^2 l_2}{\beta} \right] \|x_k^{k+1} - x_k\|^2
- \left[ \frac{3\mu^2(2l_2 + l_3)}{\beta} \right] \|y^k - y^k\|^2.
\]

(33)

That is, (23) holds. □

Remark 2. From Assumption 1 (iv), we have \(\delta_v - l_\phi > 0\) and \(\delta_v - l_\phi > 0\). Furthermore, from Assumption 1 (v), we have \(\sigma > 0\).

Lemma 6. If the sequence \(\{w^k = (x_1^k, x_2^k, y^k, \lambda^k)^T\}\) is bounded, then
\[
\sum_{k=0}^{\infty} \|w^{k+1} - w^k\|^2 < +\infty.
\]
(34)

Proof. Since \(\{w^k\}\) is bounded, the sequence \(\{\hat{w}^k\}\) is bounded and there exists a subsequence \(\{\hat{w}^k\}\) such that \(\lim_{j \to \infty} (\hat{w}^k) = \hat{w}^*\). Since \(f_1, f_2\) are lower semicontinuous and \(g\) is Lipschitz differentiable, the function \(\hat{L}(\cdot)\) is lower semicontinuous, which leads to
\[
\liminf_{j \to \infty} \hat{L}(\hat{w}^j) \geq \hat{L}(\hat{w}^*).
\]
(35)

Thus, \(\{\hat{L}(\hat{w}^k)\}\) is bounded from below. From Lemma 5, \(\{\hat{L}(\hat{w}^k)\}\) is nonincreasing. Thus, \(\{\hat{L}(\hat{w}^k)\}\) is convergent. Furthermore, \(\{\hat{L}(\hat{w}^k)\}\) is also convergent and \(\hat{L}(\hat{w}^k) \geq \hat{L}(\hat{w}^*)\) for each \(k\). By Lemma 5, we have

\[
\sigma \left( \|x_{k+1}^1 - x_k^1\|^2 + \|x_{k+1}^2 - x_k^2\|^2 + \|y_{k+1} - y_k\|^2 \right) \leq \hat{L}(\hat{w}^k) - \hat{L}(\hat{w}^{k+1}).
\]
(36)

From the abovementioned formula, we obtain
\[
\sigma \sum_{k=1}^{L} \left( \|x_{k+1}^1 - x_k^1\|^2 + \|x_{k+1}^2 - x_k^2\|^2 + \|y_{k+1} - y_k\|^2 \right)
\leq \hat{L}(\hat{w}^1) - \hat{L}(\hat{w}^{k+1}).
\]
(37)

Note that \(\sigma > 0\) and the arbitrariness of \(t\), we obtain
\[
\sum_{k=1}^{\infty} \|w^{k+1} - w^k\|^2 < +\infty.
\]
(38)

In view of (14), we have
\[
\sum_{k=0}^{\infty} \|\lambda^{k+1} - \lambda^k\|^2 < +\infty.
\]
(39)

Thus,
\[
\sum_{k=0}^{\infty} \|w^{k+1} - w^k\|^2 < +\infty.
\]
(40)

□

Lemma 7. There exists \(\delta > 0\) such that
\[
d(0, \partial\hat{L}(\hat{w}^{k+1})) \leq \delta a_k, \quad \text{for all} \quad k \in \mathbb{N},
\]
(41)

where
\[
a_k = \|x_{k+1}^1 - x_k^1\| + \|x_{k+1}^2 - x_k^2\| + \|y_{k+1} - y_k\| + \|x_{k+1} - x_k^1\|
+ \|x_{k+1} - x_{k+1}^1\| + \|y_{k+1} - y_{k+1}^1\|.
\]
(42)

Proof. From the definition of \(\hat{L}(\hat{w})\), we have

\[
\partial_{x_1} \hat{L}(\hat{w}^{k+1}) = \partial f_1(x_1^{k+1}) + \nabla_{x_1} g(u^{k+1}) - A_1^1 \hat{\lambda}^{k+1} + \beta A_1^1 r_{k+1},
\]

\[
\partial_{x_2} \hat{L}(\hat{w}^{k+1}) = \partial f_2(x_2^{k+1}) + \nabla_{x_2} g(u^{k+1}) - A_2^1 \hat{\lambda}^{k+1} + \beta A_2^1 r_{k+1},
\]

\[
\partial_{y} \hat{L}(\hat{w}^{k+1}) = \nabla_y g(u^{k+1}) - B^2 \hat{\lambda}^{k+1} + \beta B^2 r_{k+1} + \frac{6\mu^2 l_2}{\beta} (y^{k+1} - y^k),
\]

\[
\partial_{\hat{L}(\hat{w}^{k+1})} = - \frac{6\mu^2 l_2}{\beta} (y^{k+1} - y^k),
\]

\[
\partial_{\lambda} \hat{L}(\hat{w}^{k+1}) = \frac{1}{\beta} (\hat{\lambda}^{k+1} - \hat{\lambda}^k).
\]
(43)
From (14) and the optimality conditions, one has

\[
\begin{align*}
0 & \in \partial f_1(x_1^{k+1}) + \nabla_x g(u^k) - A_1^T \lambda^k + \beta A_1^T (A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b) + \nabla \varphi_1(x_1^{k+1}) - \nabla \varphi_1(x_1^k), \\
0 & \in \partial f_2(x_2^{k+1}) + \nabla_x g(u^k) - A_2^T \lambda^k + \beta A_2^T (A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y^k - b) + \nabla \varphi_2(x_2^{k+1}) - \nabla \varphi_2(x_2^k), \\
0 & = \nabla_y g(u^{k+1}) - B^T \lambda^k + \beta B^T (A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y^{k+1} - b) + \nabla \phi(y^{k+1}) - \nabla \phi(y^k), \\
\lambda^{k+1} & = \lambda^k - \beta (A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y^{k+1} - b).
\end{align*}
\]

That is,

\[
\begin{align*}
A_1^T \lambda^{k+1} - \nabla_x g(u^k) + \beta A_1^T (A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b) + \nabla \varphi_1(x_1^{k+1}) - \nabla \varphi_1(x_1^k), \\
A_2^T \lambda^{k+1} - \nabla_x g(u^k) + \beta A_2^T (A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y^k - b) + \nabla \varphi_2(x_2^{k+1}) - \nabla \varphi_2(x_2^k), \\
\nabla_y g(u^{k+1}) - B^T \lambda^k + \beta B^T (A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y^{k+1} - b) + \nabla \phi(y^{k+1}) - \nabla \phi(y^k), \\
\lambda^{k+1} & = \lambda^k - \beta (A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y^{k+1} - b).
\end{align*}
\]

From (43) and (45), we have

\[
\begin{align*}
\left( \rho_1^k, \rho_2^k, \rho_3^k, \rho_4^k, \rho_5^k \right)^T & \in \partial I(\bar{u}^{k+1}),
\end{align*}
\]

where

\[
\begin{align*}
\rho_1^k & = A_1^T (\lambda^k - \lambda^{k+1}) + \nabla_x g(u^{k+1}) - \nabla_x g(u^k) + \beta A_1^T (A_1 x_1^{k+1} + A_2 x_2^k + B y^k - b) + \nabla \varphi_1(x_1^{k+1}) - \nabla \varphi_1(x_1^k), \\
\rho_2^k & = A_2^T (\lambda^k - \lambda^{k+1}) - \nabla_x g(u^k) + \beta A_2^T (A_1 x_1^{k+1} + A_2 x_2^{k+1} + B y^k - b) + \nabla \varphi_2(x_2^{k+1}) + \nabla \varphi_2(x_2^k), \\
\rho_4^k & = \beta B^T (\lambda^k - \lambda^{k+1}) + \frac{6\mu^2 r_0^2}{\beta} (y^{k+1} - y^k) + \nabla \phi(y^{k+1}) - \nabla \phi(y^k), \\
\rho_5^k & = -\frac{6\mu^2 r_0^2}{\beta} (y^{k+1} - y^k), \\
\rho_3^k & = \frac{1}{\beta} (\lambda^k - \lambda^{k+1}).
\end{align*}
\]

Thus,

\[
d(0, \partial I(\bar{u}^{k+1})) \leq \left\| \left( \rho_1^k, \rho_2^k, \rho_3^k, \rho_4^k, \rho_5^k \right)^T \right\|.
\]

It follows from Assumption 1 and Lemma 4 that there exists a \( \delta > 0 \) such that

\[
d(0, \partial I(\bar{u}^{k+1})) \leq \delta a_k, \quad \text{for all} \ k \in \mathbb{N}.
\]

The following theorem shows that the algorithm LBADMM has global convergence.

**Theorem 1.** Let \( \{\bar{w}^k\} \) denote the cluster point set of \( \{\bar{w}^k\} \), then

(i) \( \bar{S}(\bar{w}^k) \) is a nonempty compact set, and

\[
\lim_{k \to \infty} d(\bar{w}^k, \bar{S}(\bar{w}^k)) = 0.
\]
(ii) If \((x_1^*, x_2^*, y^*, \lambda^*, \tilde{y}^*) \in S(\{\bar{w}_k\})\), then \((x_1^*, x_2^*, y^*, \lambda^*) \in \text{crit}(L)\).

(iii) \(\bar{L}(\cdot)\) is finite and constant on \(S(\{\bar{w}_k\})\) and equal to
\[
\bar{L}(\bar{w}_k) = \lim_{k \to \infty} \bar{L}(\bar{w}_k).
\]

**Proof**

(i) By the definition of \(S(\{\bar{w}_k\})\), it is trivial.

(ii) Let \(\bar{w}^* = (x_1^*, x_2^*, y^*, \lambda^*, \tilde{y}^*) \in S(\{\bar{w}_k\})\), then there exists a subsequence \(\{\bar{w}_k^*\}\) of \(\{\bar{w}_k\}\) converging to \(\bar{w}^*\). Since \(\|w^{k+1} - w^k\| \to 0 (k \to +\infty)\),
\[
\lim_{k \to \infty} \bar{w}_k^* = \bar{w}^*.
\]

(iii) From (33) and Lemma 5, we have
\[
\lim_{j \to \infty} \bar{L}(\bar{w}_k^{j+1}) = \bar{L}(\bar{w}^*) = L(w^*).
\]

From (55) and the descent of \(\{\bar{L}(\bar{w}_k)\}\), we obtain
\[
\lim_{k \to \infty} \bar{L}(\bar{w}_k) = L(w^*).
\]

The following theorem is the main result of this paper.

**Theorem 2** (strong convergence). Suppose that Assumption 1 holds, \(\bar{L}(\bar{w})\) satisfies the KL property at each point of \(S(\{\bar{w}_k\})\), then

(i) \(\sum_{k=0}^{\infty} \|u^{k+1} - u^k\| < +\infty\),

(ii) \(\{u^k\}\) converges to a critical point of \(L(\cdot)\).

**Proof.** From Theorem 1, we have \(\lim_{k \to \infty} \bar{L}(\bar{u}_k) = \bar{L}(\bar{w}^*)\) for all \(\bar{w}^* \in S(\{\bar{w}_k\})\). We consider two cases.

(i) If there exists an integer \(k_0\) such that \(\bar{L}(\bar{u}_{k_0}) = \bar{L}(\bar{w}^*)\). From Lemma 5, we have
\[
\sigma \left( \left\| x_1^{k+1} - x_1^k \right\|^2 + \left\| x_2^{k+1} - x_2^k \right\|^2 + \left\| y^{k+1} - y^k \right\|^2 \right) \leq \bar{L}(\bar{u}_k) - \bar{L}(\bar{u}_{k_0}) \leq L_p(\bar{u}_{k_0}) - L_p(\bar{w}^*) = 0,
\]
for all \(k > k_0\).

(57)

Thus, for any \(k > k_0\), we have \(x_1^{k+1} = x_1^k, x_2^{k+1} = x_2^k, y^{k+1} = y^k\). Hence, for any \(k > k_0 + 1\), it follows that \(\bar{w}_k^{k+1} = \bar{w}_k^*\) and the assertion holds.

(ii) Assume that \(\bar{L}(\bar{u}_k) > \bar{L}(\bar{w}^*)\) for all \(k \in \mathbb{N}\). Since \(\lim_{k \to \infty} d(\bar{u}_k, S(\{\bar{w}_k\})) = 0\), it follows that for any given \(\varepsilon > 0\), there exists \(k_1 > 0\), such that \(d(\bar{w}_k, S(\{\bar{u}_k\})) < \varepsilon\) for all \(k > k_1\). Since \(\lim_{k \to \infty} \bar{L}(\bar{u}_k) = \bar{L}(\bar{w}^*)\), for given \(\eta > 0\), there exists
\[ k_2 > 0, \text{ such that } L(\bar{\omega}) < \bar{\omega}^* + \eta \text{, for all } k > k_2. \]

Consequently, when \( k > \bar{k} = \max\{k_1, k_2\}, \)
\[
d(\bar{\omega}^k, S(\{\bar{\omega}^k\})) < \epsilon, L(\bar{\omega}^k) < \bar{\omega}^k \leq L(\bar{\omega}^*) + \eta. \quad (58)\]

Since \( S(\{\bar{\omega}^k\}) \) is a nonempty compact set and \( L(\cdot) \) is constant on \( S(\{\bar{\omega}^k\}) \), applying Lemma 1, we have
\[
\phi'(L(\bar{\omega}^k) - L(\bar{\omega}^*))d(0, \partial L(\bar{\omega}^k)) \geq 1, \quad \text{for all } k > \bar{k}. \quad (59)\]

From Lemma 7, one has
\[
\frac{1}{\phi'(L(\bar{\omega}^k) - L(\bar{\omega}^*))} \leq d(0, \partial L(\bar{\omega}^k)) \leq \delta a_k, \quad \text{for all } k > \bar{k}. \quad (60)\]

From the concavity of \( \phi \), we have
\[
\phi(L(\bar{\omega}^k) - L(\bar{\omega}^*)) - \phi(L(\bar{\omega}^k) - L(\bar{\omega}^*) - \phi(L(\bar{\omega}^k) - L(\bar{\omega}^*) < 0, \quad (61)\]

Thus, associating with Lemma 5 and \( \phi'(L(\bar{\omega}^k) - L(\bar{\omega}^*)) > 0, \) we have
\[
s(\|x^{k+1}_1 - x^k_1\|^2 + \|x^{k+1}_2 - x^k_2\|^2 + \|y^{k+1} - y^k\|^2) \\
\leq \frac{d(\bar{\omega}^k) - \phi(L(\bar{\omega}^k) - L(\bar{\omega}^*)) < 0, \quad (62)\]

For convenience, we set \( \Delta_{\rho, \eta} = \phi(L(\bar{\omega}^k) - L(\bar{\omega}^*)) - \phi(L(\bar{\omega}^k) - L(\bar{\omega}^*)). \)
Thus,
\[
\|x^{k+1}_1 - x^k_1\|^2 + \|x^{k+1}_2 - x^k_2\|^2 + \|y^{k+1} - y^k\|^2 \leq \frac{\delta}{\alpha} \Delta_{\rho, \eta}, \quad (63)\]

That is,
\[
3\left(\|x^{k+1}_1 - x^k_1\|^2 + \|x^{k+1}_2 - x^k_2\|^2 + \|y^{k+1} - y^k\|^2\right) \\
\leq \frac{\sqrt{3\left(\|x^{k+1}_1 - x^k_1\|^2 + \|x^{k+1}_1 - x^k_2\|^2 + \|y^{k+1} - y^k\|^2\right)}}{\epsilon} \leq \frac{\sqrt{2\delta}}{\alpha} \Delta_{\rho, \eta}, \quad \text{for all } k > \bar{k}. \quad (64)\]

By the fact \( 2\sqrt{\alpha} \leq \alpha + \beta (\alpha, \beta > 0), \) we obtain
\[
2\sqrt{\alpha} \delta \leq \alpha + \beta (\alpha, \beta > 0), \quad (65)\]

which along with (64) yields
\[
\sum_{k=k+1}^{m} 3\left(\|x^{k+1}_1 - x^k_1\|^2 + \|x^{k+1}_2 - x^k_2\|^2 + \|y^{k+1} - y^k\|^2\right) \leq a_k + \frac{2\delta}{\alpha} \Delta_{\rho, \eta}. \quad (66)\]

Summing up the abovementioned formula for \( k = \bar{k} + 1, \ldots, m \) yields
\[
\sum_{k=k+1}^{m} 3\left(\|x^{k+1}_1 - x^k_1\|^2 + \|x^{k+1}_2 - x^k_2\|^2 + \|y^{k+1} - y^k\|^2\right) \leq \sum_{k=k+1}^{m} \left[ a_k + \frac{2\delta}{\alpha} \Delta_{\rho, \eta} \right]. \quad (67)\]

Notice that \( \phi(L(\bar{\omega}^{m+1}) - L(\bar{\omega}^*)) > 0; \) thus,
\[
\sum_{k=k+1}^{m} \left(\|x^{k+1}_1 - x^k_1\|^2 + \|x^{k+1}_2 - x^k_2\|^2 + \|y^{k+1} - y^k\|^2\right) \\
\leq b_k + \frac{2\delta}{\alpha} \left(\phi(L(\bar{\omega}^{k+1}) - L(\bar{\omega}^*)) - \phi(L(\bar{\omega}^{m+1}) - L(\bar{\omega}^*))\right) \\
\leq b_k + \frac{2\delta}{\alpha} \phi(L(\bar{\omega}^{k+1}) - L(\bar{\omega}^*)), \quad (68)\]

where
\[
b_k = \|\bar{x}_1^{k+1} - x^k_1\|^2 + \|\bar{x}_2^{k+1} - x^k_2\|^2 + \|\bar{y}^{k+1} - y^k\|^2. \quad (69)\]

Thus,
\[
\sum_{k=0}^{\infty} \left(\|x^{k+1}_1 - x^k_1\|^2 + \|x^{k+1}_2 - x^k_2\|^2 + \|y^{k+1} - y^k\|^2\right) < +\infty. \quad (70)\]

By Lemma 4, one has \( \sum_{k=0}^{\infty} \|x^{k+1}_1 - x^k_1\|^2 < +\infty. \) Furthermore, \( \sum_{k=0}^{\infty} \|x^{k+1}_2 - x^k_2\|^2 < +\infty. \) Consequently \( \{\bar{u}^k\} \) is a Cauchy sequence. The assertion then follows immediately from Theorem 1. \( \square \)

Remark 3. In this section, the main conclusions are based on the boundedness assumption of the sequences \( \{\bar{u}^k = (x^{k}_1, x^{k}_2, y^{k})^T\}. \) The following conclusion shows that we only need to assume that the sequence \( \{u^k = (x^{k}_1, x^{k}_2, y^{k})^T\} \) is bounded.

**Proposition 3.** If \( \lambda^0 \in \text{Im}B \) and the sequence \( \{u^k = (x^{k}_1, x^{k}_2, y^{k})^T\} \) is bounded, then \( \{\lambda^k\} \) is bounded.

**Proof.** From (17), one has
\[ \|B^T\lambda^{k+1}\|^2 = \|\nabla_y g(x^{k}, x^{\lambda}, y^{k+1}) + \nabla \phi(y^{k+1}) - \nabla \phi(y^{k})\|^2 \]
\[ \leq \left( \|\nabla_y g(x^{k}, x^{\lambda}, y^{k+1})\|^2 + l_0 \|y^{k+1} - y^{k}\|^2 \right) \]
\[ \leq 2 \left( \|\nabla_y g(x^{k}, x^{\lambda}, y^{k})\|^2 + l_0 \|y^{k+1} - y^{k}\|^2 \right). \]

(71)

Since \( \{u^k = (x^k, x^\lambda, y^k, \lambda^k)^T\} \) is bounded, \( \{B^T\lambda^k\} \) is bounded. By Assumption 1 (ii) and Lemma 3, we have
\[ \|\lambda^{k+1}\| \leq \frac{1}{\sqrt{\mu B^2}} \|B^T\lambda^{k+1}\|. \]

(72)

Thus, \( \{\lambda^k\} \) is bounded.

Next, we present a sufficient condition of boundedness of the sequence \( \{u^k\} \), which is similar with Lemma 8 in [4].

**Lemma 8.** Let \( \{u^k = (x^k, x^\lambda, y^k, \lambda^k)\} \) be the sequence generated by Algorithm 1. Suppose that \( \lambda^0 \in \text{Im}B \) and there exists \( \theta > 0 \) such that
\[ \inf_u \left\{ g(u) - \theta \|\nabla_y g(u)\|^2 \right\} = \mathcal{G} > -\infty. \]

(73)

If
\[ \lim_{\|u\| \to \infty} \inf_{u \in C} \left[ f_1(x_1) + f_2(x_2) + g(u) \right] = \infty, l_g I_0 \beta \frac{2\mu^2}{\theta}, \]

(74)

then \( \{u^k\} \) is bounded.

**Proof.** From Lemma 5, we know that
\[ \bar{L}(x_1, x_2, y^k, \lambda^k, y^{k+1}) \leq \bar{L}(x_1, x_2, y^k, \lambda^k, y^{k+1}). \]

(75)

Then, combining with \( B^T\lambda^k = \nabla_y g(u^{k+1}) + \nabla \phi(y^{k+1}) - \nabla \phi(y^{k+1}) \), we obtain
\[ \bar{L}(x_1, x_2, y^k, \lambda^k, y^{k+1}) \]
\[ \geq f_1(x_1) + f_2(x_2) + g(u^k) - \langle \lambda^k, r_k \rangle + \frac{\beta}{2} \|r_k\|^2 \]
\[ + \frac{3\mu^2 l}{\beta} \|y^k - y^{k-1}\|^2 \]
\[ = f_1(x_1) + f_2(x_2) + g(u^k) - \frac{1}{2\beta} \|\lambda^k\|^2 + \frac{\beta}{2} \|r_k\|^2 - \frac{\lambda^k}{\beta} \]
\[ + \frac{3\mu^2 l}{\beta} \|y^k - y^{k-1}\|^2. \]

(76)

Note that \( \lambda^{k+1} \leq 2\mu \|B^T\lambda^k\| \leq 2\mu \|\nabla_y g(u^k)\|^2 + \frac{\mu^2}{\beta} \|y^k - y^{k-1}\|^2 \), we have
\[ \bar{L}(x_1, x_2, y^k, \lambda^k, \lambda^k) \]
\[ \geq \frac{1}{2} \left[ f_1(x_1^*) + f_2(x_2^*) + g(u^*) \right] + \frac{1}{2} \left[ f_1(x_1^*) + f_2(x_2^*) \right]
\[ + g(u^*) \right] - \frac{1}{2} \theta \|\nabla_y g(u^*)\|^2 \]
\[ \cdot \left( \frac{\mu^2}{\beta} \|\nabla_y g(u^*)\|^2 + \frac{\beta}{2} \|r_k\|^2 - \frac{\lambda^k}{\beta} \right) \]
\[ + \frac{3\mu^2 l}{\beta} \|y^k - y^{k-1}\|^2. \]

(77)

Under the assumptions, one can easily observe that \( \{x^k_1, x^k_2, y^k, \|\nabla_y g(u^k)\|^2\} \) and \( \{\beta/2r_k - \lambda^k/\beta^2\} \) are all bounded. Boundedness of \( \{\lambda^k\} \) follows from Proposition 3. Therefore, \( \{u^k\} \) is bounded.

\[ \square \]

4. Numerical Results

In compressed sensing, a fundamental problem is recovering an \( n \)-dimensional sparse signal \( x \) from a set of \( m \) incomplete measurements. In such a case one needs to find the sparsest solution of a linear system, which can be modeled as
\[ \min_{x_0} \|x_0\| \]
\[ \text{s.t.} \quad Dx = b, \]

(78)

where \( D \in \mathbb{R}^{m \times n} \) is the measurement matrix, \( b \in \mathbb{R}^m \) is the observed data, \( c > 0 \) is a regularization parameter, and \( \|x\|_0 \) denotes the number of nonzero elements of \( x \). In general, the abovementioned models are NP-hard. In order to overcome such a difficulty, one can relax the \( l_0 \) regularization to \( l_{1/2} \) regularization. And, some scholars generally solve the following problems instead of solving problem (78) [10, 27]:
\[ \min_{x_0} \|x_0\|^{1/2} + \frac{1}{2} \|y\|^2, \]
\[ \text{s.t.} \quad Dx - y = b, \]

(79)

where
\[ \|x\|^{1/2} = \left( \sum_{i=1}^n |x_i|^{1/2} \right)^2. \]

(80)

Based on (79), we construct the following problems:
\[ \min_{x_0} \|x_0\|^{1/2} + \frac{1}{2} \|x_2\|^2 + \frac{1}{2} \|B_1 x_1 + B_2 x_2 + y\|^2, \]
\[ \text{s.t.} \quad A_1 x_1 + A_2 x_2 + y = b. \]

(81)

In order to verify the effectiveness of the algorithm LBADMM, we now focus on applying the algorithm LBADMM to solve the nonconvex optimization problem (81). Applying the algorithm LBADMM to problem (81) with \( \phi_i(x_i) = (1/2) x_i^2 \) for \( i = 1, 2 \) and \( \phi(y) = (\mu_3/2) \|y\|^2 \), we have
\(\lambda_{2}\) Complexity

\[\begin{array}{ccccccc}
1500 & 1500 & 1500 & 94 & 3.942479 & 402.69 \\
1500 & 1500 & 2000 & 96 & 4.012569 & 650.11 \\
2000 & 1500 & 2000 & 94 & 3.594733 & 636.09 \\
2000 & 1500 & 2000 & 95 & 5.213782 & 523.09 \\
2000 & 1500 & 2000 & 95 & 5.904308 & 346.26 \\
1500 & 1500 & 2000 & 96 & 7.22839 & 514.67 \\
1500 & 1500 & 2000 & 96 & 17.289544 & 727.81 \\
1500 & 1500 & 2000 & 96 & 21.769152 & 1242.77 \\
1500 & 1500 & 2000 & 96 & 32.752034 & 924.24 \\
1500 & 1500 & 2000 & 96 & 32.752034 & 924.24 \\
1500 & 1500 & 2000 & 96 & 58.176851 & 1138.52 \\
6000 & 6000 & 6000 & 114 & 84.718137 & 1035.05 \\
6000 & 6000 & 6000 & 113 & 104.601905 & 1366.83
\end{array}\]

The numerical results are reported in Table 1. The codes were written by matlab R2016a, the computer running the program is configured as Windows 10 system, Inter (R) Core (TM) i7-6500U 2.5 GHz CPU, 8 GB memory. We report the number of reasonable termination criterion is that the residual must be small, so we choose the stop criterion as

\[\|r_k\|_2 \leq \sqrt{m} 10^{-4}.\]

The numerical results are reported in Table 1. The codes were written by matlab R2016a, the computer running the program is configured as Windows 10 system, Inter (R) Core (TM) i7-6500U 2.5 GHz CPU, 8 GB memory. We report the number of iterations ("Iter"), the computing time in seconds ("Time") and the objective function value ("f-val"). Numerical results show that the Algorithm LBADMM is stable and effective.

A part of computational results are presented in Figures 1–3. In each figure, we plot the trend of the objective function value ("f-val"). Numerical results show that the Algorithm LBADMM is stable and effective.
value ("objective-value") and the trend of the residual defined by $\|rk\| = \|A_1x_1^k + A_2x_2^k + y^k - b\|$ ("\|r\|_2^2").

5. Conclusions

We propose a new algorithm called linear Bregman ADMM for the three-blocks optimization problem with the nonseparable structure. The proposed algorithm integrates the basic ideas of the linearization technology and regularization technology. We show that any cluster point of the sequence generated by the proposed algorithm is a critical point. Under the condition that the potential function satisfies the Kurdyka-Łojasiewicz property and the penalty parameter is larger than a constant, the strong convergence of the algorithm is proved. Preliminary numerical results show that the algorithm LBADMM is stable and effective.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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