Quantum radar

Lorenzo Maccone and Changliang Ren
1. Dip. Fisica and INFN Sez. Pavia, University of Pavia, via Bassi 6, I-27100 Pavia, Italy
2. Center for Nanofabrication and System Integration, Chongqing Institute of Green and Intelligent Technology, Chinese Academy of Sciences, Chongqing 400714, People’s Republic of China

We propose a quantum metrology protocol for the localization of a non-cooperative point-like target in three-dimensional space, by illuminating it with electromagnetic waves. It employs all the spatial degrees of freedom of $N$ entangled photons to achieve an uncertainty in localization that is $\sqrt{N}$ times smaller for each spatial direction than what could be achieved by $N$ independent photons.

Quantum metrology is a set of procedures that increase the precision in the estimations of parameters by employing quantum effects such as entanglement or squeezing. By entangling $N$ different probes, typical protocols achieve a $\sqrt{N}$ decrease in the statistical noise over what would be achievable without entanglement. Here we will present a quantum metrology protocol for a radar. Radar stands for RAdio Detection And Ranging, so the bare minimum for a protocol to qualify as such is that it is able to detect a target and return its position relative to the receiver. However, previous quantum radar protocols fail this requirement as they can only discriminate whether the target is present or not, and they give a quantum advantage only in the presence of a rather specific thermal noise model. Other protocols still are unable to provide both detection and position of the target with enhanced precision. Here we will present a quantum metrology protocol for a radar. Instead our protocol returns both and does not require the target to cooperate. It achieves a $N^{3/2}$ decrease in the uncertainty volume of the target position over what could be achieved with $N$ independent photons of the same spatial bandwidth, namely a $\sqrt{N}$ decrease in uncertainty along each of the three spatial dimensions. The main drawbacks of our protocol are the difficulty in creating the required entangled state of the electromagnetic field and its sensitivity to noise. Regarding the first problem, according to current technologies, we discuss how at least the case of $N = 2$ can be experimentally realized through spontaneous parametric down-conversion under a tightly focused pulse pump based on type II noncritical phase matching. Regarding the second, known techniques can be adapted here, leading to a reduction of the impact of noise with a slight decrease in performance.

The main idea of our protocol is to combine a three-dimensional generalization of the one-dimensional quantum localization protocol of with a free-space propagation analysis of the signal from target to receiver. The use of all the spatial degrees of freedom of the entangled photons allows three dimensional localization.

The paper’s outline follows. To simplify the discussion, we will first present the case of two photons, and then give the $N$-photon general protocol. We start with the case of two maximally-entangled photons. Then we show that, while a reduction of entanglement entails a reduction of precision, it also decreases the transverse dimensions of the required detector. We conclude by providing some modifications of the protocol that strengthen the protocol against the effects of noise.

The protocol allows a receiver to find her position relative to an uncooperating target object that is illuminated with a suitable entangled state of light composed of $N$ entangled photons, see Fig. 1. To this aim the receiver measures their arrival position and arrival time on a transverse plane at her location. Consider $N = 2$ first. The joint probability of photodetection, namely of finding the two photons at times $t_1$ and $t_2$ and at positions $\vec{r}_1$, $\vec{r}_2$ (two-dimensional transverse vectors) is

$$p(t_1, \vec{r}_1; t_2, \vec{r}_2) \propto |\langle 0|E^+(t_1, \vec{r}_1)E^+(t_2, \vec{r}_2)|\psi_2\rangle|^2,$$

where $|0\rangle$ is the vacuum state, $|\psi_2\rangle$ is the state of the two photons (we work in the Heisenberg picture where the
operators evolve from an initial time $t_0$) and (e.g. \textsuperscript{14})

$$E^+(t, \vec{r}) \equiv \int d\vec{k}_3 \ g(\vec{k}_3, t, \vec{r}) \ a(\vec{k}_3) \ e^{i\omega(t-t_0)}, \quad (2)$$

where $g$ is the transfer function (defined below) between the object plane (at the target’s position) and the image plane (at the position of the receiver), $a(\vec{k}_3)$ is the electromagnetic field annihilation operator for the mode with wave vector $\vec{k}_3$. As customary, we will employ the far-field approximation, valid when the object-receiver distance is sufficiently large. In this case, the longitudinal component of the wave vector is much larger than the transverse components: $k_z^2 + k_\perp^2 \ll |\vec{k}|^2$, where $|\vec{k}| = (k_z^2 + k_\perp^2 + k_\parallel^2)^{1/2} = \omega/c$, with $\omega$ the light’s frequency. So the $\vec{k}_3$ integral can be approximated as

$$\int d\vec{k}_3 = \int \frac{d\omega}{c^2} \frac{dk}{\sqrt{1 - \frac{k_z^2 + k_\perp^2}{\omega^2}}} \simeq \frac{1}{c} \int d\omega \ dk, \quad (3)$$

with $\vec{k}$ the two dimensional transverse wave vector $\vec{k} = (k_x, k_y)$. Then, Eq. (2) can be replaced by

$$E^+(t, \vec{r}) \simeq \int d\omega \ dk \ g(\vec{k}, \vec{r}) \ a(\omega, \vec{k}) \ e^{i\omega(t-t_0)}, \quad (4)$$

where the longitudinal component contributes only with a phase factor which measures the longitudinal distance $z = c(t-t_0)$ that the light travels from the source to the target, and back to the detector, and where the free-space (transverse) transfer function is

$$g(\vec{k}, \vec{r}) \equiv \int d\vec{r}_0 \ A(\vec{r}_0) \ e^{i\vec{k} \cdot (\vec{r}_0 - \vec{r})}, \quad (5)$$

where $A$ is the object transfer function and the integral is over the (transverse) object plane, namely $\vec{r}_0, \vec{r}$ are two-dimensional transverse vectors. We will consider a point-like reflective object which reflects only the photons that impinge on its position $\vec{r}_p$. The other photons are lost. This situation is described by a transfer function which has value $a$ in the vicinity of $\vec{r}_p$ in the object plane, and value zero elsewhere in the object plane, namely $A(\vec{r}_0) \propto \delta(\vec{r}_0 - \vec{r}_p)$. Slightly more general situations can be considered, but it is not possible to perform more complex imaging with entangled light since the transfer function $g$ of any imaging apparatus is more complex than $\delta(\vec{r}_0 - \vec{r}_p)$ and the photon correlations in $(\delta(\vec{r}_0 - \vec{r}_p)$ below) will prevent the formation of a discernible image. For quantum radar applications, we are only interested in free-space propagation, described by $(\delta(\vec{r}_0 - \vec{r}_p)$ and in detection and ranging, rather than imaging.

The necessary entangled two-photon state, produced at the initial time $t_0$, in the far-field approximation, is

$$|\psi_2\rangle \equiv \int d\omega \ d\vec{k} \ \psi(\omega, \vec{k})(a^\dagger(\omega, \vec{k})|^0\rangle \quad (6)$$

where $a^\dagger(\omega, \vec{k})$ creates a photon with frequency $\omega$ and transverse wave vector $\vec{k}$, $\psi$ is the biphoton’s spatiotemporal wavefunction and we omit the normalization since it is a non-normalizable state as all EPR states $\textsuperscript{15}$. It is a maximally-entangled state in three different degrees of freedom: $k_x, k_y$ and $\omega$ (we will drop this assumption later). We must also suppose that at the receiver’s location there is a negligible probability of seeing the photons that are not scattered by the object, namely $|\psi_2\rangle$ is an approximation of the electromagnetic field valid only in the object’s vicinity. In practice this can be implemented by requiring that the longitudinal component of the wave vector $\vec{k}$ is directed away from the detector (which is implicit in the far field approximation).

Replacing these quantities into Eq. (1), we find

$$p(t_1, \vec{r}_1; t_2, \vec{r}_2) \propto |\tilde{\psi}(t_1 + t_2 - 2t_0, \vec{r}_1 + \vec{r}_2 - 2\vec{r}_p)|^2, \quad (7)$$

where $\tilde{\psi}(t, \vec{r}) = \int d\omega \ d\vec{k} \ \psi(\omega, \vec{k}) \ e^{i(\omega t + \vec{k} \cdot \vec{r})}$ is the Fourier transform of $\psi(\omega, \vec{k})$. This implies that the average time of the arrival is equal to the transit time of the signal from its production at $t_0$ to its detection at $t$, and that the average arrival transverse position is equal to the object’s transverse position. The statistical noise of these two quantities is given by half the standard deviation of $|\tilde{\psi}|^2$ in time and in position. Indeed, the left-hand-side of $(\delta(\vec{r}_0 - \vec{r}_p)$ can also be written as $|\psi(2(t + t_2 - t_0), 2(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_p))|^2$. Hence, the standard deviation of the average time of arrival gains a factor of $1/2$, and similarly for each of the two components of the average position.

Naturally, we must compare this result to what one can obtain using two unentangled photons with the same spectral characteristics. Consider then a single photon in the state

$$|\psi_1\rangle = \int d\omega \ d\vec{k} \ \psi(\omega, \vec{k}) \ a^\dagger(\omega, \vec{k})|0\rangle, \quad (8)$$

with the same spectrum $\psi(\omega, \vec{k})$ as in $(\delta(\vec{r}_0 - \vec{r}_p)$. The probability of detecting it at time $t$ at transverse position $\vec{r}$ on the screen is

$$p(t, \vec{r}) \propto |\langle 0| E^+(t, \vec{r}) |\psi_2\rangle|^2 \propto |\tilde{\psi}(t, \vec{r})|^2 \quad (9)$$

In other words, by using the two-photon entangled state $|\psi_2\rangle$ we reduced the statistical noise of the time of arrival and of the transverse position by a half with respect to what one would have obtained from a single photon $|\psi_1\rangle$ with identical spectral function $\psi(\omega, \vec{k})$. Clearly, a fair comparison must be between the two-photon entangled strategy and an unentangled strategy that uses two unentangled photons $|\psi_1\rangle \otimes |\psi_1\rangle$. If each of the unentangled photons provide an error equal to the standard deviation of $|\psi|^2$, the standard deviation of the average time of arrival gains a factor of $1/\sqrt{2}$ (because the variance of the sum is the sum of variances), and similarly for each of the two components of the average position. Thus, using a biphoton entangled state $|\psi_2\rangle$ one obtains a net gain equal to the square root $\sqrt{2}$ of the number of photons in the resolution along each of the three spatial directions with respect to a strategy that employs two unentangled photons $|\psi_1\rangle$. 

It is easy now to extend the above discussion to an arbitrary number of photons: the joint probability of detecting them at time $t_j$ at transverse position $\vec{r}_j$ is

$$p(t_j, \vec{r}_j) = |\langle 0 | \prod_j E^+(t_j, \vec{r}_j) | \psi_N \rangle|^2 \propto |\tilde{\psi}(\sum_j t_j - N t_0, \sum_j \vec{r}_j - N \vec{r}_p)|^2,$$

(10)

if one uses a far-field $N$-photon entangled state

$$|\psi_N\rangle \equiv \int d\omega_1 d\vec{k} \psi(\omega, \vec{k})(a_1^\dagger(\omega, \vec{k}))^N |0\rangle.$$

Clearly, (10) gives a distribution that has a standard deviation for each position component and for the time of arrival that is $\sqrt{N}$ times smaller than the standard deviation obtained by averaging $N$ unentangled photons in the state $|\psi_1\rangle$, with arrival probability (11).

We now discuss the feasibility of the experiment. For the state $|\psi_N\rangle$, the arrival time $t_j$ and position $\vec{r}_j$ of each photon is completely random. In fact, consider the case $N=2$: $|\psi_2\rangle$ can be written also as

$$|\psi_2\rangle = \int dt_1 d\vec{r}_1 dt_2 d\vec{r}_2 \tilde{\psi}(t_1 + t_2, \vec{r}_1 + \vec{r}_2)$$

$$\times a^\dagger(t_1, \vec{r}_1) a^\dagger(t_2, \vec{r}_2) |0\rangle,$$

(12)

where we introduced into (11) the operator $a^\dagger(t, \vec{r}) \propto \int d\omega d\vec{k} a^\dagger(\omega, \vec{k}) e^{i(\omega t + \vec{k} \cdot \vec{r})}$ that creates a photon at time $t$ and transverse position $\vec{r}$. Each of the two photons in (12) taken by themselves can arrive at any time and at any position, since the time and position difference have uniform probability amplitude. It is only the time and position sums (or averages) that are peaked. Indeed, the probability (7) depends only on the sums $t_1 + t_2$ and $\vec{r}_1 + \vec{r}_2$, so that the differences $t_1 - t_2$ and $\vec{r}_1 - \vec{r}_2$ must be uniformly distributed.

So, there are two main practical issues with this protocol. On one hand, it is very demanding to produce the maximally-entangled states (6) and (11). On the other hand, the complete randomness in arrival times and positions requires an infinite measurement time and transverse screen. Both these problems can be overcome by reducing the amount of entanglement among photons. This, of course, will reduce the resolution gain, but it will still allow for a better-than-classical enhancement. Again, for the sake of illustration, we will consider the case $N=2$ first, and then extend to arbitrary $N$.

Consider the partially-entangled two-photon state

$$|\phi_2\rangle \equiv \int d\omega d\vec{k} d\omega_d d\vec{k}_d \psi(\omega, \vec{k}) \gamma(\omega, \vec{k}) \xi(\omega, \vec{k}) \times$$

$$a^\dagger(\omega, \vec{k}) a^\dagger(\omega + \omega_d, \vec{k} + \vec{k}_d) |0\rangle,$$

(13)

where $\omega_d$ and $\vec{k}_d$ are the frequency difference and transverse wave vector divergence between the two photons, governed by the probability amplitudes $\gamma$ and $\xi$ respectively. The state $|\phi_2\rangle$ is normalized and tends to $|\psi_2\rangle$ in the limit when $\gamma$ and $\xi$ tend to delta functions $\gamma \to \delta(\omega_d)$, $\xi \to \delta(\omega_d)$. Replacing $|\psi_2\rangle$ with $|\phi_2\rangle$ into (11), we find

$$p(t_1, \vec{r}_1; t_2, \vec{r}_2) \propto |\tilde{\psi}(t_1 + t_2 - 2t_0, \vec{r}_1 + \vec{r}_2 - 2\vec{r}_p)|^2 \times$$

$$|\tilde{\gamma}(t_2 - t_0) \xi(\vec{r}_2 - \vec{r}_p) + \tilde{\gamma}(t_1 - t_0) \xi(\vec{r}_1 - \vec{r}_p)|^2,$$

(14)

where $\tilde{\gamma}$ and $\tilde{\xi}$ are the Fourier transforms of $\gamma$ and $\xi$. In the limit in which $\gamma$ and $\xi$ are deltas, then $\tilde{\gamma}$ and $\tilde{\xi}$ are uniform, so the second line of (14) is a constant and we reobtain the maximally entangled result of (7). The opposite limit of constant $\gamma$ and $\xi$ corresponds to the case in which one photon has spectrum $\psi$ and the other photon has infinite temporal and spatial bandwidth. In this case, $|\tilde{\gamma}|^2$ and $|\tilde{\xi}|^2$ are deltas and we obtain

$$p(t_1, \vec{r}_1; t_2, \vec{r}_2) \propto |\tilde{\psi}(t_1 - t_0, \vec{r}_1 - \vec{r}_p)|^2 \delta(\vec{r}_2 - \vec{r}_p) \delta(t_2 - t_0)$$

$$+ |\tilde{\psi}(t_2 - t_0, \vec{r}_2 - \vec{r}_p)|^2 \delta(\vec{r}_1 - \vec{r}_p) \delta(t_1 - t_0),$$

which is the joint distribution one expects when one photon (the one with infinite bandwidth) determines the position of the object exactly, whereas the other finds it with probability (9).

In the intermediate case (11) in which the differences $\omega_d$ and $\vec{k}_d$ have finite bandwidth, there are two competing effects: on one hand the average time $\langle \delta t \rangle = \omega_d$ and average position $\langle \delta \vec{r} \rangle = \vec{k}_d$ have a distribution that is wider than $\psi$, so these quantities are determined with a lower resolution than the maximally entangled case. On the other hand, the marginal distributions of the times $t_j$ and positions $\vec{r}_j$ are not uniform anymore: in the first term of the second line of (14) the distance between $t_2$ and $t_0$ cannot be much larger than the standard deviation of $|\tilde{\gamma}|^2$, and thus also the distance between $t_1$ and $t_0$ cannot be too large, since $t_1 + t_2 - 2t_0$ has a width governed by $|\tilde{\psi}|^2$. Analogously the distance between $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_p$ is limited by the standard deviations of $|\tilde{\xi}|^2$ and $|\tilde{\gamma}|^2$, and similar considerations apply to the second term. In essence, each of the photon’s time of arrival $t_j$ and transverse position $\vec{r}_j$ is limited (in contrast to the maximally entangled case), but the spread in their averages is dominated by the product between $|\tilde{\psi}|^2$ and $|\tilde{\gamma}|^2$. For such non-maximal entangled states Eq. (13), the standard deviation of the average time of arrival gains a factor of $\lambda$ with $1/2 \leq \lambda \leq 1$, and similarly for each of the two components of the average position. When the bandwidth of $\xi$ and $\gamma$ is larger than that of $\psi$, $\lambda \leq 1/\sqrt{2}$, it will always achieve a better-than-classical enhancement both in time and transverse positions.

The $N$-photon extension for the non-maximally entangled state is now straightforward: use the state
$|\phi_N\rangle \equiv \int d\omega \, d\vec{k} \prod_j d\omega_j \, d\vec{k}_j \, \psi(\omega, \vec{k}) \, \gamma(\omega_1) \cdots \gamma(\omega_N) \, \xi(\vec{k}_1) \cdots \xi(\vec{k}_N) \, a^\dagger(\omega, \vec{k}) \, a^\dagger(\omega + \omega_1, \vec{k} + \vec{k}_1) \cdots a^\dagger(\omega + \omega_N, \vec{k} + \vec{k}_N)|0\rangle$

to calculate

$$p(\{t_j, \vec{r}_j\}_{j=1\ldots N}) \propto \langle 0 \mid \prod_j E^+(t_j, \vec{r}_j) |\phi_N\rangle^2 \propto |\bar{\psi}(\sum_j t_j - Nt_0, \sum_j \vec{r}_j - N\vec{r}_p) \prod_{j \neq j} \gamma(t_n - t_0) \xi(\vec{r}_n - \vec{r}_p)|^2 ,$$

(15)

for which considerations analogous to the case $N=2$ seen above apply. In the intermediate case $|\phi_2\rangle$ in which the differences $\{\omega_n\}$ and $\{k_n\}$ have finite bandwidth, there are two competing effects: on one hand the average time $\sum_j t_j$ and average position $\sum_j \vec{r}_j$ have a distribution that is wider than $\bar{\psi}$, so these quantities are determined with a lower resolution than the maximally entangled case. In essence, each of the photon’s time of arrival $t_j$ and transverse position $\vec{r}_j$ is limited, but the spread in their averages is dominated by the product between $|\bar{\psi}|^2$ and $\prod_n \gamma_n \xi_n|^2$. For the non-maximal entangled states $|\phi_N\rangle$, the standard deviation of the average time of arrival gains a factor of $\lambda$ with $1/N \leq \lambda \leq 1$, and similarly for each of the two components of the average position. When the bandwidth of $\xi$ and $\gamma$ is larger than that of $\bar{\psi}$, $\lambda \leq 1/\sqrt{N}$, it will always achieve a better-than-classical enhancement both in time and transverse positions.

The ideal state $|\psi_N\rangle$ and $|\phi_N\rangle$ for arbitrary $N$ is actually a state that is positively correlated both in frequency and transverse momentum. For $N=2$, the state $|\psi_2\rangle$ has been experimentally realized under a tightly focused pulsed pump based on type II noncritical phase matching [10]. Pulsed pumping can provide the bandwidth for the frequency correlation ($\omega_1 = \omega_2$), and a tightly focused process can modulate the transverse momentum correlation ($k_1 = k_2$). According to the ideal phase matching relation [10], a maximal positively-correlated momentum source requires an infinitely long crystal ($L \to +\infty$) and extremely narrow beam waist $w_0 \to 0$, where $w_0$ is waist radius of pump at the entrance to the crystal. $\theta(z)$ represents the variation of the pump’s phase, which depends on the propagation length and the confocal length of the pump, where the confocal length of the pump is $b = w_0^2 k_p$, and $k_p$ is the pump wave vector. We define a focal parameter $\chi = \frac{L}{b}$. When $\chi \ll 1$, i.e., $L \ll b$, the pump is considered to be collimated where the effect of phase $\theta$ can be neglected, it leads to generate a state which possesses a negatively-correlated momentum from spontaneous parametric down-conversion. While $\chi \gg 1$, i.e., $L \gg b$, the effect of phase $\theta$ plays an important role, which leads to generate a state which possesses a positively-correlated momentum from spontaneous parametric down-conversion. Hence, to obtain positive-correlation in momentum, the requirement $\chi \gg 1$ should be satisfied, which can be realized by increasing the length of crystal $L$ and decreasing the beam waist $w_0$. In realistic situations, these ideal requirements are not met and one obtains the partially-entangled states $|\phi_2\rangle$ and $|\phi_N\rangle$, which are the ones required for our proposal. One can generate a relatively high quality non-maximally entangled state by using centimeter-sized periodically poled materials and feasible focal parameters of the pump. As illustrated in [10], taking the SPDC process in a KTiOPO4 (KTP) crystal as the example, a relatively high quality positively-correlated entangled state can be generated with the crystal length $L = 5cm$ and beam waist $w_0 = 10\mu m$ ($\chi = 35.81$). Moreover, optical superlattice technologies seem to have great potential in generating high quality sources [17].

We now briefly consider the effect of noise. The maximally entangled protocol is extremely sensitive to noise, as typically happens in quantum metrology: the loss of a single photon will render all the other $N-1$ ones completely useless for the estimation, since their times and positions of arrival are completely random. This is the typical scenario in quantum metrology in the presence of noise [18, 19], but many different strategies that reduce the effect of noise at the cost of a slight decrease in resolution have been proposed. For example, the non-maximally entangled state $|\phi_N\rangle$ is more robust to the loss of photons: the photons that do arrive, still contain partial information on the object position. As another example, the strategies proposed in [11] can be adapted to the current case. The main idea is that one divides the $N$ photons into subsets of $M$ entangled photons and then entangles these subsets among each other (a nested strategy). Then if one photon is lost, only the photons of its subset become useless, while those of the other subsets can still attain a better-than-classical resolution. Other possible strategies involve the use of quantum error correcting codes [21] or the use of ancillary systems that do not participate to the estimation procedure [22].

In conclusion, we have proposed a quantum estimation protocol to estimate the location of a target in three dimensions with a precision increase equal to the square root of the number of photons employed, when compared to the best unentangled strategy using photons with equal spectral characteristics. In this paper we have focused on entanglement among photons, but quantum squeezing would allow a similar enhancement [23]. As a future application, one might consider the extension of the protocol to the localization in four-dimensional space.
time to determine the spatial location and the time of an event. Unfortunately such extension is nontrivial because in electromagnetic waves the spatial and temporal degrees of freedom are connected (they are constrained by being a solution to a wave equation). So one would need a further, independent, degree of freedom to use as a clock, in addition to the photon’s spatial degrees of freedom that we used here.

L.M. acknowledges funding from Unipv, “Blue sky” project - grant n. BSR1718573; C.L.R. acknowledge the funding from National key research and development program (No. 2017YFA0305200), the Youth Innovation Promotion Association (CAS) (No. 2015317), the National Natural Science Foundation of China (No. 11605205), the Natural Science Foundation of Chongqing (No. cstc2015jcyjA00021, cstc2018jcyjAX0656), the Entrepreneurship and Innovation Support Program for Chongqing Overseas Returnees (No.cx2017134, No.cx2018040), the fund of CAS Key Laboratory of Microwave Magnetic Resonance, and the fund of CAS Key Laboratory of Quantum Information.

[1] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced measurements: being the standard quantum limit, Science 306, 1330 (2004); V. Giovannetti, S. Lloyd, L. Maccone, Advances in quantum metrology, Nature Phot. 5, 222 (2011).
[2] V. Giovannetti, S. Lloyd, L. Maccone, Quantum metrology, Phys. Rev. Lett. 96, 010401 (2006).
[3] S. L. Braunstein, C. M. Caves, G. J. Milburn, Generalized uncertainty relations: Theory, examples, and Lorentz invariance, Ann. Phys. 247, 135-173 (1996); S. L. Braunstein, C. M. Caves, Statistical distance and the geometry of quantum states. Phys. Rev. Lett. 72, 3439 (1994).
[4] R. Demkowicz-Dobrzański, M. Jarzyna, J. Kodyński, Quantum Limits in Optical Interferometry, Progress in Optics, 60, 345 (2015) also at [arXiv:1405.7703] (2014).
[5] M. G. A. Paris, Quantum estimation for quantum technology, Int. J. Quantum Inf. 7, 125 (2009).
[6] M. Lanzagorta, J. Uhlmann, Quantum radar. San Rafael, California (USA): Morgan and Claypool,66 (2012).
[7] S. Lloyd, Enhanced Sensitivity of Photodetection via Quantum Illumination, Science 321, 1463 (2008); S.-H. Tan, B.I. Erkmen, V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, S. Pirandola, J. H. Shapiro, Quantum Illumination with Gaussian States, Phys. Rev. Lett. 101, 253601 (2008); S. Lloyd, Quantum afterlife Scientific American, February (2009).
[8] M. Lanzagorta, Quantum Radar, Synthesis Lectures on Quantum Computing (Morgan & Claypool Publ., 2011).
[9] J. F. Smith III, Quantum interferometer and radar theory based on N00N, M and M or linear combinations of entangled states, in Proc. Quantum information and computation VIII, New York, 770201 (2010); Malik, O. S. Magana-Loaiza, and R. W. Boyd, Quantum-secure imaging, Appl. Phys. Lett. 101, 241103 (2012); D. Luong, C. W. Sandbo Chang, A. M. Vadiraj, A. Damini, C. M. Wilson, B. Balaji, Receiver Operating Characteristics for a Prototype Quantum Two-Mode Squeezing Radar, [arXiv:1903.00101] (2019).
[10] V. Giovannetti, S. Lloyd, and L. Maccone, Positioning and clock synchronization through entanglement, Phys. Rev. A 65, 022309 (2002).
[11] Changliang Ren, H.F. Hofmann, Clock synchronization using maximal multipartite entanglement, Phys. Rev. A 86, 014301 (2012).
[12] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum-enhanced positioning and clock synchronization, Nature 412, 417 (2001).
[13] L. Mandel, E. Wolf, Optical Coherence and Quantum Optics (Cambridge Univ. Press, Cambridge, 1995).
[14] Y. Shih, Quantum imaging, IEEE J. Sel. Top. Quantum Electron. 13 (4), 1016 (2007).
[15] A. Einstein, B. Podolsky, N. Rosen, Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?, Phys. Rev. 47, 777 (1935).
[16] W. Liu, P.-X. Chen, C.-Z. Li, and J.-M. Yuan, Preparation and identification of two-photon positively-momentum-correlated entangled states, Phys. Rev. A 79, 061802 (2009); H. D. L. Pires and M. P. van Exter, Observation of near-field correlations in spontaneous parametric down-conversion, Phys. Rev. A 79, 041801(2009); S. Yun, P. Xu, J. S. Zhao, Y. X. Gong, Y. F. Bai, J. Shi, and S. N. Zhu, Generation of positively-momentum-correlated biphotons from spontaneous parametric down-conversion, Phys. Rev. A 86, 023852 (2012); M. Zhong, P. Xu, L. Lu, AND S. Zhu, Experimental realization of positively momentum correlated photon pairs from a long periodically poled lithium tantalate crystal, JOSA. B 32, 002081 (2015).
[17] X. Q. Yu et al., Transforming Spatial Entanglement Using a Domain-Engineering Technique, Phys. Rev. Lett. 101, 233601 (2008); H. Y. Leng, X. Q. Yu, Y. X. Gong, P. Xu, Z. D. Xie, H. Jin, C. Zhang and S. N. Zhu, On-chip steering of entangled photons in nonlinear photonic crystals, Nature Commun. 2, 429 (2011).
[18] R. Demkowicz-Dobrzański, J. Koldyński, M. Guta, The elusive Heisenberg limit in quantum-enhanced metrology, Nature Comm. 3, 1063 (2012).
[19] B. M. Escher, R. L. de Matos Filho, L. Davidovich, The elusive Heisenberg limit in quantum-enhanced metrology, Nature Phys. 4, 406 (2011).
[20] S.M. Barnett D.T. Pegg, On the Hermitian Optical Phase Operator, J. Mod. Opt. 36, 7 (1988).
[21] E. M. Kessler, I. Lovchinsky, A. O. Sushkov, and M. D. Lukin, Phys. Rev. Lett. 112, 150802 (2014); W. Dur, M. Skotiniotis, F. Fröwis, and B. Kraus, Phys. Rev. Lett. 112, 080801 (2014); G. Arrad, Y. Vinkler, D. Aharonov and A. Retzker, Phys. Rev. Lett. 112, 150801 (2014).
[22] R. Demkowicz-Dobrzański, L. Maccone, Using entanglement against noise in quantum metrology, Phys. Rev. Lett. 113, 250801 (2014).
[23] L. Maccone, A. Riccardi, Squeezing metrology, [arXiv:1907.07482] (2019).