Non-distributive positive logic as a fragment of first-order logic over semilattices

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Abstract
We characterise non-distributive positive logic as the fragment of a single-sorted first-order language that is preserved by a new notion of simulation called a meet-simulation. Meet-simulations distinguish themselves from simulations because they relate pairs of states from one model to single states from another. En route to this result we use a more traditional notion of simulations and prove a Hennessy-Milner style theorem for it, using an analogue of modal saturation called meet-compactness.

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1 Introduction
The celebrated Van Benthem characterisation theorem characterises normal modal logic as the bisimulation-invariant fragment of first-order logic [3]. More precisely, it states that a first-order formula with one free variable is equivalent to the standard translation of a modal formula if and only if it is invariant under bisimulations. Similar theorems have been proven for non-normal modal logics, such as monotone modal logic [14], neighbourhood logic [15], and instantial neighbourhood logic [13]. For the latter three we only consider equivalence over particular classes of first-order structures, namely those corresponding to the modal logic semantics. Apart from modal extensions of classical propositional logic, other propositional logics and their modal extensions can be viewed as fragments of first-order logic as well. Examples include (bi-)intuitionistic logic [21, 19, 2] and positive modal logic [19, 7]. All such theorems are instances of (model-theoretic) preservation theorems, see for example [8, Section 5.2] or [9].

Sometimes “invariant under bisimulations” is replaced with “preserved by simulations.” This allows one to exclude negations from the characterisation. Indeed, if a formula is invariant under bisimulations then so is its negation, but preservation by simulations does not imply preservation of its negation.

The goal of the present paper is to characterise non-distributive positive logic [20, 22, 10] as a fragment of first-order logic. It has recently been noted that non-distributive positive logic can be given frame semantics by means of meet-semilattices with a valuation, with filters serving as denotations of formulae [11, 5]. Conjunctions are interpreted as usual, while \( \varphi \lor \psi \) holds at a state \( w \) if there are two states \( v \) and \( u \) satisfying \( \varphi \) and \( \psi \), respectively, such that their meet lies below \( w \). This non-standard interpretation of disjunctions prevents distributivity.

Since meet-semilattices are in particular partially ordered sets, they can be viewed as interpreting structures for the first-order language with one binary predicate. Indeed, the partial order underlying a meet-semilattice then serves as the interpretation of the binary relation. Guided by this observation, we translate non-distributive positive logic into the first-order logic with one binary predicate and a unary predicate for each proposition letter. This translation is similar to the one in [5].

To establish this characterisation result, we have to define an appropriate analogue of simulations. The main challenge is to exclude classical (locally evaluated) disjunctions from being preserved by this notion. Ordinary simulations do not suffice because they preserve all locally evaluated monotone connectives. Instead, we introduce meet-simulations. These differ from ordinary simulations because they relate pairs of states from one model to states of another. This modification, it turns out, allows us to prevent preservation of classical disjunctions. We prove the following characterisation theorem:
A first-order formula \( \alpha(x) \) is preserved by meet-simulations if and only if it is equivalent to a \( \neg(x = x) \) or to the standard translation of a non-distributive positive formula over the class of meet-semilattices.

Observe that we can replace \( \neg(x = x) \) with any contradiction, such as \( x \land \neg x \), so it is possible to work in a first-order language without equality. To eliminate reference to \( \neg(x = x) \) entirely, we finally suggest an adaptation of meet-simulations to meet-\( \omega \)-simulations. The special feature of meet-\( \omega \)-simulations is that they relate finite subsets of states of one model to states of another. With this adaptation, we obtain:

A first-order formula \( \alpha(x) \) is preserved by meet-\( \omega \)-simulations if and only if it is equivalent to the standard translation of a non-distributive positive formula over the class of meet-semilattices.

**Related work.** The idea of using (bi)simulations to relate pairs of states to pairs of states appeared before in [12], where it is used to characterise fragments of the calculus of binary relations, and in [1] in the context of paths on data trees.

**Structure of the paper.** In Section 2 we recall the language and semantics of non-distributive positive logic. In Section 3 we summarise some basic first-order logic required for the results in this paper, and we define the standard translation for non-distributive positive logic.

The contributions of this paper start in Section 4, where we define simulations between models and show that states related by a simulation are logically inclusive (that is, the theory of the former is included in the theory of the latter). In Section 5 we give a Hennessy-Milner style theorem for simulations. We first prove that the finite models form a Hennessy-Milner class. Taking stock of the proof, we define an analogue of modal saturation called meet-compactness. We then prove that \( \omega \)-saturated models are meet-compact, and that the meet-compact models form a Hennessy-Milner class.

Subsequently, in Section 6 we work towards a Van Benthem style characterisation and point out why simulations do not admit such a result. We then introduce meet-simulations in Section 7 and use the results from Section 6 to prove the characterisation theorem announced above. Finally, in Section 8 we show how to adapt the notion of a meet-simulation to prevent preservation of classical contradictions. We conclude in Section 9.

## 2 Non-distributive positive logic

Denote by \( L \) the language of positive logic, i.e. the language generated by the grammar

\[
\varphi ::= p \mid \top \mid \bot \mid \varphi \land \varphi \mid \varphi \lor \varphi,
\]

where \( p \) ranges over some arbitrary but fixed set \( \text{Prop} \) of proposition letters. One can define a logic of consequence pairs of \( L \)-formula whose algebraic semantics is given by lattices, see [5, Section 3.1].

Formulæ from \( L \) can be interpreted in meet-semilattices with a valuation [11]. The intuition behind this is that the collection of filters of a meet-semilattice is closed under (arbitrary) intersections. Therefore they form a complete lattice, but disjunctions are not given by unions. This gives rise to a non-standard interpretation of disjunctions which prevents distributivity. We call the resulting frames and models \( L_1 \)-frames, to distinguish them from the slightly different \( L \)-frames used in [5], see Remark 2.3.

In this paper, by a *meet-semilattice* we mean a partially ordered set in which every finite subsets has a greatest lower bound, called its *meet*. The meet of \( w \) and \( v \) is denoted by \( w \land v \), reserving the symbol \( \land \) for conjunctions of formulæ. The empty meet is the top element, denoted by \( 1 \). If \((W, 1, \land)\) is a meet-semilattice then we write \( \preceq \) for the partial order given by \( w \preceq v \) if \( w \land v = w \). A *filter* of a meet-semilattice \((W, 1, \land)\) is a subset \( F \) of \( W \) which is upward closed under \( \preceq \) and closed under finite meets. Filters are nonempty because they contain the empty meet, \( 1 \).

**2.1 Definition.** An *\( L_1 \)-frame* is a meet-semilattice \((W, 1, \land)\). An *\( L_1 \)-model* is an \( L_1 \)-frame \((W, 1, \land)\) together with a valuation \( V \) that assigns to each proposition letter \( p \in \text{Prop} \) a filter \( V(p) \) of \((W, 1, \land)\).
The interpretation of \( \varphi \in L \) at a state \( w \) of an \( L_1 \)-model \( M = (W, 1, \lambda, V) \) is defined recursively via

\[
M, w \models p \quad \text{iff} \quad w \in V(p)
\]

\[
M, w \models \top \quad \text{always}
\]

\[
M, w \models \bot \quad \text{iff} \quad w = 1
\]

\[
M, w \models \varphi_1 \land \varphi_2 \quad \text{iff} \quad M, w \models \varphi_1 \text{ and } M, w \models \varphi_2
\]

We denote the truth set of \( \varphi \) by \( \llbracket \varphi \rrbracket_M := \{ w \in W \mid M, w \models \varphi \} \). The theory of a state \( w \in W \) is defined by \( \text{th}_M(w) := \{ \varphi \in L \mid M, w \models \varphi \} \). If \( w \) and \( w' \) are states in \( M \) and \( M' \), respectively, then we write \( M, w \rightarrow M', w' \) if \( \text{th}_M(w) \subseteq \text{th}_{M'}(w') \).

It can be shown that the truth set of every formula in any \( L_1 \)-model \( M \) is a filter in the underlying \( L_1 \)-frame. In particular, the truth set of a formula of the form \( \varphi_1 \lor \varphi_2 \) is the smallest filter containing both \( \llbracket \varphi_1 \rrbracket_M \) and \( \llbracket \varphi_2 \rrbracket_M \), and can also be given by \( \{ w \in W \mid u \land v \not\leq w \text{ for some } u \in \llbracket \varphi_1 \rrbracket_M \text{ and } v \in \llbracket \varphi_2 \rrbracket_M \} \).

The following notion of morphism is taken from [1], Definition 3.4.9.

2.2 Definition. An \( L_1 \)-morphism between \( L_1 \)-frames \( (W, 1, \lambda) \) and \( (W', 1', \lambda') \) is a function \( f : W \rightarrow W' \) that preserves all finite meets, and satisfies for all \( w \in W \) and \( u', v' \in W' \):

- \( f(w) = 1' \) if \( w = 1 \);
- If \( u' \land v' \not\leq f(w) \), then there exist \( u, v \in W \) such that \( u' \not\leq f(u) \) and \( v' \not\leq f(v) \) and \( u \land w \not\leq w \).

The second condition can be depicted as follows:

An \( L_1 \)-morphism between \( L_1 \)-models \( (W, 1, \lambda, V) \) and \( (W', 1', \lambda', V') \) is an \( L_1 \)-morphism \( f \) between the underlying frames such that \( V(p) = f^{-1}(V'(p)) \) for all \( p \in \text{Prop} \).

2.3 Remark. In [5] a variation of the semantics presented above is used where meet-semilattices are only assumed to have binary meets. As a consequence, they need not have a top element, and filters are allowed to be empty. An advantage of this approach is that the truth set of \( \bot \) is empty, so that models contain no inconsistent state. The drawback is that it complicates interpretation of \( \varphi_1 \lor \varphi_2 \), cf. [5, Definition 3.6]. This, in turn, affects the definition of truth-preserving morphisms between models.

In this paper we choose to allow an inconsistent state \( 1 \) satisfying \( \bot \) because it simplifies the definitions of simulations and meet-simulations.

3 First-order translation

We define the first-order language we work with and the standard translation of \( L \) into this language. Then we characterise the class of first-order structures corresponding to \( L_1 \)-models and we recall the definition of \( \omega \)-saturation.

3.1 Definition. Let \( \text{FOL} \) be the single-sorted first-order language which has a unary predicate \( P_p \) for every proposition letter \( p \in \text{Prop} \), and a binary predicate \( R \). To avoid confusion with the interpretation of disjunctions from \( L \), we denote classical disjunctions like the one in \( \text{FOL} \) by \( \lor \).

Models for \( \text{FOL} \) are denoted by \( M, N \) (as opposed to \( M, N \) for \( L_1 \)-frames). Intuitively, the relation symbol of our first-order language accounts for the poset structure of \( L_1 \)-frames. If \( x, y \) and \( z \) are variables,
then we can express that $x$ is the meet of $y$ and $z$ in the ordering induced by the interpretation of $R$ using a first-order sentence. To streamline notation we abbreviate this as follows:

$$\text{ismeet}(x, y, z) := (x R y) \land (x R z) \land \forall x'((x'Ry \land x'Rz) \implies x'Rx).$$

We are now ready to define the standard translation.

3.2 Definition. Let $x$ be a variable. Define the standard translation $st_x : \text{L} \to \text{FOL}$ recursively via

$$st_x(p) = P_x,$$

$$st_x(T) = (x = x),$$

$$st_x(\bot) = \forall y(yRx),$$

$$st_x(\varphi \land \psi) = st_x(\varphi) \land st_x(\psi),$$

$$st_x(\varphi \lor \psi) = \exists x' \exists y \exists z (\text{ismeet}(x', y, z) \land x'Rx \land st_y(\varphi) \land st_z(\psi)).$$

While one often sees the standard translation of $\top$ defined as $x = x$, it is not strictly necessary: any tautology with free variable $x$ suffices. Thus, we can also work in a first-order language without equality.

Every $L_1$-model $M = (W, 1, \land, V)$ gives rise to a first-order structure for FOL. Indeed, we define the interpretation of $R$ as $\preceq$, and the interpretation of the unary predicates is given via the valuations of the proposition letters. We write $M^\circ$ for the $L_1$-model $M$ conceived of as a first-order structure for FOL. The following proposition is an adaptation of [5, Proposition 3.26]. Satisfaction of FOL-formulae in a first-order structure is defined as usual.

3.3 Proposition. For every $L_1$-model $M = (W, 1, \land, V)$, state $w \in W$ and formula $\varphi \in \text{L}$ we have

$$M, w \models \varphi \iff M^\circ \models st_x(\varphi)[w].$$

Proof. We use induction on the structure of $\varphi$. If $\varphi = p$ or $\varphi = \top$ then the statement is obvious. If $\varphi = \bot$ then we have $M, w \models \varphi$ if $w = 1$ if $M^\circ \models \forall y(yRw)$ iff $M^\circ \models st_x(\bot)[w]$. If $\varphi$ is of the form $\varphi_1 \lor \varphi_2$ then the inductive step is routine. If $\varphi = \varphi_1 \lor \varphi_2$ and $M, w \models \varphi_1 \lor \varphi_2$ then there exist $u, v \in W$ such that $u \land v \preceq w$ and $M, u \models \varphi_1$ and $M, v \models \varphi_2$. Taking $x' = u \land v$, $y = u$ and $z = v$, this witnesses $M^\circ \models \exists x' \exists y \exists z (\text{ismeet}(x', y, z) \land x'Rx \land st_y(\varphi_1) \land st_z(\varphi_2))[w], \quad (1)$

so that $M^\circ \models st_x(\varphi_1 \lor \varphi_2)[w]$. Conversely, validity of (1) entails the existence of suitable states $u$ and $v$ witnessing $M, w \models \varphi_1 \lor \varphi_2$. □

Clearly not every structure for FOL is of the form $M^\circ$. We can classify the ones that are.

3.4 Definition. Let FSL be the class of first-order structures for FOL that satisfy the following axioms:

(M1) $\forall x(xRx)$

(M2) $\forall x \forall y (xRy \land yRx \implies x = y)$

(M3) $\forall x \forall y \forall z (xRy \land yRz \implies xRz)$

(M4) $\forall x \forall y \forall z (xRx \land zRy \land \forall z'(z'Rx \land z'Ry \implies z'Rz))$

(M5) $\exists x \forall y (yRx)$

(M6) $(\exists w Pw) \land \forall x \forall y \forall z (\text{ismeet}(x, y, z) \implies ((Py \land Pz) \leftrightarrow Px))$

Here $P$ ranges over all unary predicates of FOL.

Axioms (M1) to (M5) state that the interpretation of $R$ should be a partial order. The fourth one adds that this partial order should have binary greatest lower bounds and (M5) stipulates a top element. Finally, we have an axiom for each unary predicate stating that its interpretation should be a filter in the meet-semilattice induced by the domain and the interpretation of $R$.

3.5 Proposition. A structure for FOL is isomorphic to $M^\circ$ for some $L_1$-model $M$ iff it is in FSL.

Proof. The direction from left to right is easy. Conversely, let $M = (W, I(R), \{I(P_p) \mid p \in \text{Prop}\})$ be a first-order structure that satisfies all axioms from Definition 3.4. Then $I(R)$ is a partial order on $W$ with binary greatest lower bounds and a top element. Denote the latter by $1 \in W$ and the greatest lower bound of $w, v \in W$ by $w \land v$. Define a valuation $V$ by $V(p) = I(P_p) \subseteq W$. Then by (M6) $V$ assigns to each proposition letter a filter. Therefore $M_\circ := (W, 1, \land, V)$ is an $L_1$-model. It is easy to see that the identity on $W$ yields an isomorphism between $M$ and $(M_\circ)^\circ$. □
Finally, we recall basic properties of $\omega$-extensions needed for the proof of the characterisation theorem. Let $\mathcal{M} = (W,I(R),\{I(P) \mid p \in \text{Prop}\})$ be a FOL-structure. For a set $A \subseteq W$, the $A$-expansion $\text{FOL}[A]$ of FOL is obtained by extending FOL with new constants $a$ for each $a \in A$. $\mathcal{M}_A$ is the expansion of $\mathcal{M}$ to a structure for $\text{FOL}[A]$ where $a$ is interpreted as $a \in W$.

3.6 Definition. A FOL-model $\mathcal{M} = (W,I(R),\{I(P) \mid p \in \text{Prop}\})$ is called $\omega$-saturated if for all finite $A \subseteq W$ and every collection $\Gamma(x_1, \ldots, x_n)$ of FOL-formulae with a finite number $n$ of free variables the following holds: if $\Gamma(x)$ is finitely satisfiable in $\mathcal{M}_A$, then it is satisfiable in $\mathcal{M}_A$.

3.7 Remark. The usual definition of $\omega$-saturation uses only one free variable. However, we may equivalently assume a finite number of variables, see e.g. [8, Definition 2.3.6]. We need the formulation with multiple variables in Lemma 5.5 below.

Using e.g. ultraproducts, one can show that every FOL-model has an $\omega$-saturated elementary extension [8]. We denote this extension of $\mathcal{M}$ by $\mathcal{M}^*$, and the image of a state $w$ under the extension is denoted by $w^*$. Moreover, if $\mathcal{M} \in \text{FSL}$ then its $\omega$-saturated elementary extension is also in FSL, since validity of the axioms from Definition 3.4 is preserved under elementary extensions.

## 4 Simulations

In this section we define simulations between $L_1$-models. Simulations only preserve truth of formulae in one direction, that is, if $S$ is a simulation and $(w,w') \in S$ then every formula satisfied at $w$ is also satisfied at $w'$. This prevents preservation of negations, and hence has been used to characterise the negation-free part of classical normal modal logic [19]. Since the language of non-distributive positive logic does not have negations, this is a good starting point for our attempt to characterise it.

However, by its nature, simulations preserve all monotone connectives, including classical (locally evaluated) disjunctions. So this approach is bound to fail, since the collection of first-order formulae preserved by simulations is closed under classical disjunctions. Simulations are still worth investigating because they provide a stepping stone towards the Van Benthem style characterisation we are after. In fact, in Section 6 we will prove that a first-order formula is preserved by simulations if and only if it is the classical disjunction of standard translations of formulae in $L$. We extend this to a proper characterisation in Sections 7 and 8 using the notion of a meet-simulation.

4.1 Definition. Let $\mathfrak{M} = (W,1,\wedge,V)$ and $\mathfrak{M}' = (W',1',\wedge',V')$ be two $L_1$-models. An $L_1$-simulation from $\mathfrak{M}$ to $\mathfrak{M}'$ is a relation $S \subseteq W \times W'$ such that for all $(w,w') \in S$:

$(S_1)$ If $w \in V(p)$ then $w' \in V'(p)$, for all $p \in \text{Prop}$;

$(S_2)$ If $w = 1$ then $w' = 1'$;

$(S_3)$ If $v,u \in W$ are such that $v \wedge u \preceq w$, then there exist $v',u' \in W'$ such that $(v,v') \in S$ and $(u,u') \in S$ and $v' \wedge' u' \preceq' w'$.

We call $w \in W$ and $w' \in W'$ $L_1$-similar if there is an $L_1$-simulation $S$ between $\mathfrak{M}$ and $\mathfrak{M}'$ such that $(w,w') \in S$. This is denoted by $\mathfrak{M},w \rightarrow \mathfrak{M}',w'$.

It is straightforward to see that the collection of $L_1$-simulations between two models is closed under arbitrary unions. Therefore we have:

4.2 Proposition. Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two $L_1$-models.

1. The collection of $L_1$-simulations between $\mathfrak{M}$ and $\mathfrak{M}'$ forms a complete lattice.

2. The relation of $L_1$-similarity between $\mathfrak{M}$ and $\mathfrak{M}'$ is again an $L_1$-simulation.

There are several generic examples of $L_1$-simulations.

4.3 Example. Let $\mathfrak{M} = (W,1,\wedge,V)$ be an $L_1$-model. Then the relations $S_1 = \{(w,w) \mid w \in W\}$ and $S_2 = \{(w,v) \in W \mid w \preceq v\}$ are $L_1$-simulations from $\mathfrak{M}$ to itself.

$L_1$-morphisms between models also give rise to $L_1$-simulations.
4.4 Example. Let \( \mathfrak{M} = (W, 1, \lambda, V) \) and \( \mathfrak{M}' = (W', 1', \lambda', V') \) be \( L_1 \)-models and \( f : \mathfrak{M} \to \mathfrak{M}' \) an \( L_1 \)-morphism. Then the following variations on the graph of \( f \) are \( L_1 \)-simulations:

\[
\text{Gr } f := \{(w, f(w)) \in W \times W' \mid w \in W\},
\]

\[
\text{Gr}^f f := \{(w, w') \in W \times W' \mid f(w) \neq f'(w')\}.
\]

Taking \( f = \text{id} : \mathfrak{M} \to \mathfrak{M} \) we obtain \( S_1 \) and \( S_2 \) from Example 4.3 as \( \text{Gr } f \) and \( \text{Gr}^f f \).

As expected, \( L_1 \)-similarity implies logical inclusion.

4.5 Proposition. If \( \mathfrak{M}, w \models \mathfrak{M}', w' \) then \( \mathfrak{M}, w \models \mathfrak{M}', w' \).

Proof. Let \( \varphi \in L \) be such that \( \mathfrak{M}, w \models \varphi \). We show by induction on the structure of \( \varphi \) that \( \mathfrak{M}', w' \models \varphi \). The case for \( \varphi = \top \) is trivial, and if \( \varphi = p \in \text{Prop} \) or \( \varphi = \bot \) then the result follows from \( (S_1) \) and \( (S_2) \), respectively. The inductive step for \( \varphi = \varphi_1 \land \varphi_2 \) is straightforward.

Suppose \( \mathfrak{M}, w \models \varphi_1 \lor \varphi_2 \). Then there are \( u, v \in W \) such that \( \mathfrak{M}, u \models \varphi_1 \) and \( \mathfrak{M}, v \models \varphi_2 \) and \( u \lor v \models w \). Using \( (S_3) \) we can find \( u', v' \in W' \) such that \( (u, u') \in S \) and \( (v, v') \in S \) and \( u' \lambda' v' \models w' \). By the induction hypothesis \( \mathfrak{M}', u' \models \varphi_1 \) and \( \mathfrak{M}', v' \models \varphi_2 \), which implies \( \mathfrak{M}', w' \models \varphi_1 \lor \varphi_2 \).

5 A Hennessy-Milner style theorem

While Proposition 4.5 guarantees that \( L_1 \)-similarity implies logical inclusion, the opposite (i.e. logical inclusion implies \( L_1 \)-similarity) need not be true. Classes on which \( L_1 \)-similarity and logical inclusion coincide are often called Hennessy-Milner classes, after the authors who proved an analogous result for classical normal modal logic [17]. This section is devoted to finding Hennessy-Milner classes.

5.1 Definition. A class \( \mathcal{C} \) of \( L_1 \)-models is called a Hennessy-Milner class if for all \( \mathfrak{M}, \mathfrak{M}' \in \mathcal{C} \) and states \( w, w' \) in them we have

\[
\mathfrak{M}, w \models \mathfrak{M}', w' \quad \text{iff} \quad \mathfrak{M}, w \models \mathfrak{M}', w'.
\]

We start by proving that the class of finite models is a Hennessy-Milner class. This will serve as inspiration for an analogue of modal saturation which we formulate in terms of compactness in a topology, and a more general Hennessy-Milner style result.

5.2 Proposition. The class of finite \( L_1 \)-models is a Hennessy-Milner class.

Proof. Let \( \mathfrak{M} = (W, 1, \lambda, V) \) and \( \mathfrak{M}' = (W', 1', \lambda', V') \) be two finite \( L_1 \)-models. We claim that

\[
S := \{(w, w') \in W \times W' \mid \mathfrak{M}, w \models \mathfrak{M}', w' \}
\]

is an \( L_1 \)-simulation. Together with Proposition 4.5 this proves the proposition. Item \((S_1)\) holds by definition of \( S \), and \((S_2)\) follows from the fact that \( 1 \) and \( 1' \) are the only elements satisfying \( \bot \).

Let \((w, w') \in S \) and let \( v, \bar{v} \in W \) be such that \( v \land u \models w \). Suppose towards a contradiction that \((S_3)\) is not satisfied. Then for each \((v', u') \in M(w') := \{(v', u') \in W' \times W' \mid v' \land u' \models w' \}\) either

- \((v, v') \notin S \), so there exists a formula \( \varphi \) that is satisfied at \( v \) but not at \( v' \); or
- \((u, u') \notin S \), so there exists a formula \( \psi \) that is satisfied at \( u \) but not at \( u' \).

For each \((v', u') \in M(w')\) select a \( \varphi \) or \( \psi \) as specified. Let \( \Phi \) be the set of all such \( \varphi \) and \( \Psi \) of all such \( \psi \). Let \( \bar{\varphi} := \bigwedge \Phi \) and \( \bar{\psi} := \bigwedge \Psi \) (taking to empty conjunction to be \( \top \)). Then \( v \) and \( u \) satisfy \( \bar{\varphi} \) and \( \bar{\psi} \), respectively. On the other hand, for each \((v', u') \in M(w')\) either \( v' \) does not satisfy \( \bar{\varphi} \) or \( u' \) does not satisfy \( \bar{\psi} \). Therefore

\[
\mathfrak{M}, w \models \bar{\varphi} \lor \bar{\psi} \quad \text{but} \quad \mathfrak{M}', w' \not\models \bar{\varphi} \lor \bar{\psi}.
\]

This contradicts the assumption that \((w, w') \in S \). So \((S_3)\) must hold and \( S \) is an \( L_1 \)-simulation.

If we try to apply this proof to infinite models we may encounter infinite sets \( \Phi \) and \( \Psi \). So our analogue of modal saturation should remedy this. Indeed, we need a compactness property that allows us to reduce infinite \( \Phi \) and \( \Psi \) to finite sets.
5.3 Definition. Let \( \mathfrak{M} = (W, 1, \lambda, V) \) be an \( \mathbb{L}_1 \)-model. Then we denote by \( \tau_V \) the topology on \( W \) generated by the clopen subbase
\[
\{ \langle \varphi \rangle^{\mathfrak{M}} | \varphi \in \mathbb{L} \} \cup \{ W \setminus \langle \varphi \rangle^{\mathfrak{M}} | \varphi \in \mathbb{L} \}.
\]
The \( \mathbb{L}_1 \)-model \( \mathfrak{M} \) is called meet-compact if for all \( w \in W \) the set \( M(w) := \{(v, u) \in W \times W | v \wedge u \leq w\} \) is a compact subset of \( (W, \tau_V) \times (W, \tau_V) \).

5.4 Example. Every finite monotone \( \mathbb{L}_1 \)-model is meet-compact. Evidently, the finiteness entails the compactness requirement.

As another example, we show that \( \omega \)-saturated \( \mathbb{L}_1 \)-models are meet-compact. An \( \mathbb{L}_1 \)-model \( \mathfrak{M} \) is called \( \omega \)-saturated if \( \mathfrak{M}^\omega \) is \( \omega \)-saturated (see Definition 3.6).

5.5 Lemma. If \( \mathfrak{M} = (W, 1, \lambda, V) \) is \( \omega \)-saturated, then it is meet-compact.

Proof. Let \( w \in W \). We need to show that \( M(w) = \{(u, v) \in W \times W | u \wedge v \leq w\} \) is compact in \( (W, \tau_V) \times (W, \tau_V) \). By the Alexander subbase theorem, it suffices to show that every open cover of subbasic opens has a finite subcover. A subbase for the topology on \( (X, \tau_V) \times (X, \tau_V) \) can be given by the collection of open squares \( a \times b \), where \( a \) and \( b \) range over the subbase for \( \tau_V \), that is, over the truth sets of formulae and their complements. So suppose
\[
\begin{align*}
M(w) & \subseteq \bigcup \{ \langle \varphi_i \rangle^{\mathfrak{M}} \times \langle \psi_j \rangle^{\mathfrak{M}} | i \in I \} \cup \bigcup \{ \langle \varphi_i \rangle^{\mathfrak{M}} \times (X \setminus \langle \psi_j \rangle^{\mathfrak{M}}) | j \in J \} \\
& \cup \bigcup \{(X \setminus \langle \varphi_i \rangle^{\mathfrak{M}}) \times \langle \psi_j \rangle^{\mathfrak{M}} | k \in K \} \cup \bigcup \{(X \setminus \langle \varphi_i \rangle^{\mathfrak{M}}) \times (X \setminus \langle \psi_j \rangle^{\mathfrak{M}}) | \ell \in L \}
\end{align*}
\]
where \( I, J, K, L \) are index sets. Let \( A = \{ w \} \) and
\[
\Gamma(x, y, z) = \{ xRw \} \cup \{ \text{ismeet}(x, y, z) \}
\]
\[
\cup \bigcup \{ \neg \text{st}_y(\varphi_i) \land \neg \text{st}_z(\psi_j) | i \in I \} \\
\cup \bigcup \{ \neg \text{st}_y(\varphi_j) \land \text{st}_z(\psi_j) | j \in J \} \\
\cup \bigcup \{ \text{st}_y(\varphi_k) \land \neg \text{st}_z(\psi_k) | k \in K \} \\
\cup \bigcup \{ \text{st}_y(\varphi_\ell) \land \text{st}_z(\psi_\ell) | \ell \in L \}.
\]
Then \( \Gamma(x) \) is not satisfiable in \( \mathfrak{M}^\omega \), because if it were then there is a state \( x \in W \) such that \( xRw \) which is the meet of a pair \( (y, z) \) that is not in the open cover given in (2). Since \( \mathfrak{M}^\omega \) is \( \omega \)-saturated, this implies that \( \Gamma(x) \) is not finitely satisfiable in \( \mathfrak{M}^\omega_A \), which by a reverse argument yields a finite subcover of (2).

5.6 Theorem. The meet-compact \( \mathbb{L}_1 \)-models form a Hennessy-Milner class.

Proof. Let \( \mathfrak{M} = (W, 1, \lambda, V) \) and \( \mathfrak{M}' = (W', 1', \lambda', V') \) be two meet-compact \( \mathbb{L}_1 \)-models. We claim that
\[
S = \{(w, w') \in W \times W' | \mathfrak{M}, w \sim \mathfrak{M}', w' \}
\]
is an \( \mathbb{L}_1 \)-simulation. Together with Proposition 4.5 this proves the theorem.

Item (S1) is satisfied by definition, and (S2) follows from the fact that 1 and 1’ are the only elements satisfying \( \perp \). To prove (S3), assume \( (w, w') \in S \) and \( u, v \in W \) are such that \( u \wedge v \leq w \). Suppose there are no \( u', v' \in W' \) such that \( u' \wedge v' \leq w' \) and \( (u, w') \in S \) and \( (v, v') \in S \). Proceeding as in the proof of Proposition 5.2, we obtain (potentially infinite) sets \( \Phi \) and \( \Psi \). Now observe that we have an open cover
\[
M(w') \subseteq \bigcup_{\varphi \in \Phi} \left( (W' \setminus \langle \varphi \rangle^{\mathfrak{M}'}) \times W' \right) \cup \bigcup_{\psi \in \Psi} \left( W' \times (W' \setminus \langle \psi \rangle^{\mathfrak{M}'}) \right).
\]
By assumption \( M(w') \) is compact, so we can find a finite subcover indexed by finite sets \( \Phi' \subseteq \Phi \) and \( \Psi' \subseteq \Psi \). Define \( \bar{\varphi} := \bigwedge \Phi' \) and \( \bar{\psi} := \bigwedge \Psi' \). The remainder of the proof is the analogous to Proposition 5.2.
6 Towards characterisation

In this section we characterise the first-order L₁-simulation-invariant fragment of FOL over the class FSL of first-order structures corresponding to L₁-models.

6.1 Definition. A first-order formula α(x) with one free variable x is preserved by L₁-simulations if

\[\mathcal{M}^x \models \alpha(x)[w] \implies \mathcal{M}^x \models \alpha(x)[v]\]

whenever \(\mathcal{M}, w \models \mathcal{N}, v\), for all L₁-models \(\mathcal{M}, \mathcal{N}\).

We cannot yet characterise the L₁-simulation-preserving formulae as the language \(\mathcal{L}\), because we can find first-order formulæ \(\alpha(x)\) that are preserved by L₁-simulations but not equivalent to the standard translation of a formula in \(\mathcal{L}\). The next example gives such an \(\alpha(x)\).

6.2 Example. Let \(p\) and \(q\) be proposition letters, and \(P\) and \(Q\) their corresponding unary predicates. Consider the first-order formula \(\alpha(x) := Px \lor Qx\) (recall that \(\lor\) denotes classical disjunction). If \(\mathcal{M}, w \models \mathcal{N}, v\) and \(\mathcal{M}^x \models \alpha(x)[w]\), then \(\mathcal{M}, w \models p\) or \(\mathcal{M}, w \not\models q\), which by (S₁) implies that \(\mathcal{M}, v \models p\) or \(\mathcal{M}, v \not\models q\). Hence \(\mathcal{M} \models \alpha(x)[v]\). So \(\alpha(x)\) is preserved by L₁-simulations.

We give a model \(\mathcal{M}\) where the collection of states \(w\) such that \(\mathcal{M}^w \models \alpha(x)[w]\) is not a filter. Since the interpretation of any formula \(\varphi \in \mathcal{L}\) yields a filter, it then follows from Proposition 3.3 that \(\alpha(x)\) is not equivalent to the standard translation of a formula in \(\mathcal{L}\). Consider the model \(\mathcal{M} = (W, 1, \lambda, V)\), where \(W = \{ u, v, 1 \}\) is ordered by \(w \ll u \ll v\) and \(w \not\ll v\). Let \(V(p) = \{ u, 1 \}\) and \(V(q) = \{ v, 1 \}\). Then \(\mathcal{M}\) is an L₁-model and the set \(\{ w \in W \mid \mathcal{M}^w \models \alpha(x)[w] \}\) = \{ \(u, v, 1\) \} is not a filter in \((W, 1, \lambda)\) (see Figure 1).

![Figure 1: The model \(\mathcal{M} = (W, 1, \lambda, V)\) from Example 6.2.](image)

Along the lines of [6, Theorem 2.68], we characterise the first-order formulæ preserved by simulations as those formulæ in FOL equivalent to the disjunction of standard translations of \(\mathcal{L}\)-formulæ.

6.3 Theorem. A first-order formulæ \(\alpha(x) \in \text{FOL}\) with one free variable \(x\) is equivalent over FSL to a formula of the form \(\text{st}_x(\varphi_1) \lor \cdots \lor \text{st}_x(\varphi_n)\), where \(\varphi_i \in \mathcal{L}\), if and only if it is preserved by L₁-simulations.

Proof. The left-to-right implication follows from Proposition 4.5. For the converse, let \(\alpha(x)\) be a formulæ preserved by L₁-simulations. For \(\varphi_1, \ldots, \varphi_n \in \mathcal{L}\), abbreviate \(\text{st}_x[\varphi_1, \ldots, \varphi_n] := \text{st}_x(\varphi_1) \lor \cdots \lor \text{st}_x(\varphi_n)\). (We take the empty disjunction to be the formula \((x = x)\) in FOL.) Define

\[\text{MOC}(\alpha) = \{ \text{st}_x[\varphi_1, \ldots, \varphi_n] \mid \alpha(x) \models_{\text{FSL}} \text{st}_x[\varphi_1, \ldots, \varphi_n] \text{ and } n \in \omega \text{ and } \varphi_i \in \mathcal{L}\}.\]

It suffices to prove that \(\text{MOC}(\alpha) \models_{\text{FSL}} \alpha(x)\), because a compactness argument then yields the existence of a finite subset \(\text{MOC}'(\alpha) := \{ \text{st}_x[\varphi_1, \ldots, \varphi_n_1], \ldots, \text{st}_x[\varphi_k, \ldots, \varphi_n_k] \} \subseteq \text{MOC}(\alpha)\) that entails \(\alpha(x)\).

Since also \(\alpha \models_{\text{FSL}} \text{st}_x[\varphi_1, \ldots, \varphi_n]\) by definition, we then find that \(\alpha(x)\) is equivalent over FSL to

\[\text{st}_x[\varphi_1, \ldots, \varphi_n] \land \cdots \land \text{st}_x[\varphi_k, \ldots, \varphi_n] = \bigvee \{ \text{st}_x[\varphi_1, \ldots, \varphi_n] \mid \psi(i) \in \{1, \ldots, n_1\} \text{ for all } i \in \{1, \ldots, k\} \}
\]

\[= \bigvee \{ \text{st}_x[\varphi_1, \ldots, \varphi_n] \mid \psi(i) \in \{1, \ldots, n_i\} \text{ for all } i \in \{1, \ldots, k\} \}
\]

For the first equality above we use distributivity in FOL. The resulting formulæ is the finite disjunction of standard translations of formulæ in \(\mathcal{L}\), because \(\varphi_1, \ldots, \varphi_n \in \mathcal{L}\) whenever \(\varphi_1, \ldots, \varphi_n \in \mathcal{L}\).

So let \(\mathcal{M} \in \text{FSL}\) and assume \(\mathcal{M} \models \text{MOC}(\alpha)[w]\). Let \(\mathcal{M}_w = (W, 1, \lambda, V)\). For \(w \in W\), let

\[\neg \text{th}(w) := \{ \varphi \in \mathcal{L} \mid \mathcal{M}_w, w \not\models \varphi \}.\]

(3)
We claim that \( \{ \alpha(x) \} \cup \{ \neg \text{st}_x(\varphi) \mid \varphi \in \neg \text{th}(w) \} \) is satisfiable. Suppose not, then there exists a finite number of formulae \( \varphi_1, \ldots, \varphi_m \in \neg \text{th}(w) \) such that \( \alpha(x) \models \neg (\text{st}_x(\varphi_1) \land \cdots \land \text{st}_x(\varphi_m)) \). This implies \( \alpha(x) \models \text{st}_x(\varphi_1) \lor \cdots \lor \text{st}_x(\varphi_m) \). But then \( \text{st}_x(\varphi_1) \lor \cdots \lor \text{st}_x(\varphi_m) \in \text{MOC}(\alpha) \), by assumption \( \text{M} \models \text{st}_x(\varphi_1) \lor \cdots \lor \text{st}_x(\varphi_m)[w] \). This implies that \( \text{M} \models \text{st}_x(\varphi_i)[w] \) for one of the \( \varphi_i \), and hence \( \text{M}_o, w \vDash \varphi_i \), which contradicts the assumption that \( \varphi_i \notin \text{th}(w) \). So the set in (3) is satisfiable.

Thus, we can find a structure \( N \in \text{FSL} \) and a state \( v \) in the domain of \( N \) such that \( N \models \alpha(x)[v] \) and \( N_o, v \vDash \forall \varphi \) for all \( \varphi \notin \text{th}(w) \). This implies

\[
N_o, v \vDash \text{M}_o, w. \tag{4}
\]

Recall that we write \( \text{M}^* \) and \( \text{N}^* \) for \( \omega \)-saturated elementary extensions of \( \text{M} \) and \( \text{N} \), and \( w^* \) and \( v^* \) for the images of \( w \) and \( v \). As a consequence of Lemma 5.5 the models \( \text{M}^*_o \) and \( \text{N}^*_o \) are meet-compact. Since elementary embeddings preserve truth of formulae, (4) and Theorem 5.6 imply

\[
(\text{N}^*)_o, v^* \vDash (\text{M}^*)_o, w^*.
\]

Now \( N \models \alpha(x)[v] \) implies \( \text{N}^* \models \alpha(x)[v^*] \). Since \( \alpha(x) \) is assumed to be preserved by \( L_1 \)-simulations, we find \( \text{M}^*_o \models \alpha(x)[w^*] \). Invariance of truth of formulae under elementary embeddings gives \( \text{M} \models \alpha(x)[w] \). □

6.4 Example. We investigate what goes wrong in the proof of Theorem 6.3 if we try to eliminate the additional classical disjunctions by taking

\[
\text{MOC}(\alpha) = \{ \text{st}_x(\varphi) \mid \alpha(x) \models_{\text{FSL}} \text{st}_x(\varphi), \varphi \in L \}.
\]

Let \( \alpha(x) = p_x \lor Q_x \). Since \( p \lor q \) is the smallest \( L \)-formula whose truth set contains both \( p \) and \( q \) (in any model), any formula \( \varphi \in L \) such that \( \alpha(x) \models_{\text{FSL}} \text{st}_x(\varphi) \) must be implied by \( p \lor q \) (in the sense that truth of \( p \lor q \) implies truth of \( \varphi \)). Taking \( M \) as in Example 6.2, we find \( M \vDash p \lor q \) hence \( \text{MOC}(\alpha) \models \text{st}_{x}(p \lor q)[w] \).

Since we argued that \( \text{st}_{x}(p \lor q) \) is the smallest formula in \( \text{MOC}(\alpha) \), we have \( \text{MOC}(\alpha) \models \text{st}_{x}(p \lor q)[x] \). But at the same time both \( p \) and \( q \) are in \( \neg \text{th}(w) \). Then supposing that \( \{ \alpha(x) \} \cup \{ \neg \text{st}_{x}(\varphi) \mid \varphi \in \neg \text{th}(w) \} \) is not satisfiable, we obtain \( \alpha \models_{\text{FSL}} \text{st}_{x}(p) \lor \text{st}_{x}(q) \). This does not yield a contradiction, because \( \text{st}_{x}(p) \lor \text{st}_{x}(q) \) is not in \( \text{MOC}(\alpha) \). Hence the proof fails.

7 Characterisation using meet-simulations

Making use of Theorem 6.3, we now prove a Van Benthem style characterisation theorem for \( L \). As we have seen, the main challenge is to prevent classical disjunctions from being preserved. We do so by adapting the notion of an \( L_1 \)-simulation to that of a meet-simulation. Instead of relating states with states, meet-simulations relate pairs of states from one model to single states of another.

7.1 Definition. Let \( M = (W, 1, \cdot, V) \) and \( M' = (W', 1', \cdot', V') \) be two \( L_1 \)-models. A meet-simulation from \( M \) to \( M' \) is a relation \( T \subseteq (W \times W) \times W' \) such that for all \( (w_1, w_2, w') \in T \):

\[
\begin{align*}
(M_1) & \quad \text{If } w_1 \in V(p) \text{ and } w_2 \in V(p) \text{ then } w' \in V'(p), \text{ for all } p \in \text{Prop}; \\
(M_2) & \quad \text{If } w_1 = w_2 = 1 \text{ then } w' = 1'; \\
(M_3) & \quad \text{If } u_1, u_2, v_2 \in W \text{ are such that } u_1 \land v_1 \preceq u_2 \text{ and } u_2 \land v_2 \preceq w_2, \text{ then there exist } v', u' \in W' \text{ such that } (u_1, u_2, u') \in T \text{ and } (v_1, v_2, v') \in T \text{ and } v' \land u' \preceq v'.
\end{align*}
\]

7.2 Remark. Instead of relating pairs of states from one model to states of another, we can also define meet-simulations as relations between three (possibly distinct) models. The definition above is then obtained as the special case where the first two models are the same. The results in this section work for either definition of meet-simulation.

We give some examples of meet-simulations.

7.3 Example. For any \( L_1 \)-model \( M = (W, 1, \cdot, V) \), the relation

\[
T = \{ (w_1, w_2, w_3) \in W \times W \times W \mid w_1 \land w_2 \preceq w_3 \}
\]

is a meet-simulation on \( M \). Let us verify this. Condition (M_1) follows from the fact that proposition letters are interpreted as filters, so if both \( w_1 \) and \( w_2 \) satisfy \( p \in \text{Prop} \), then so does \( w_1 \land w_2 \) and everything above it. Condition (M_2) follows immediately from the definition. For (M_3), suppose \( (w_1, w_2, w_3) \in T, u_1 \land v_1 \preceq w_1 \text{ and } u_2 \land v_2 \preceq w_2 \). Then \( u_3 := u_1 \land u_2 \text{ and } v_3 := v_1 \land v_2 \) witness truth of (M_3), as \( u_3 \land v_3 = u_1 \land u_2 \land v_1 \land v_2 \preceq u_1 \land u_2 \land v_1 \land v_2 \preceq w_1 \land w_2 \preceq w_3 \).
7.4 Example. Let $\mathcal{M} = (W, 1, \alpha, V)$ and $\mathcal{M}' = (W', 1', \alpha', V')$ be two $L_1$-models and $f : \mathcal{M} \to \mathcal{M}'$ an $L_1$-morphism. Then the following relations are meet-simulations:

\[
\begin{align*}
T_1 &= \{(w_1, w_2, w') \in W \times W \times W' \mid f(w_1, w_2) = w'\} \\
T_2 &= \{(w_1, w_2, w') \in W \times W \times W' \mid f(w_1, w_2) \not\cong w'\} \\
T_3 &= \{(w'_1, w'_2, w') \in W' \times W' \times W \mid w'_1 \not\cong w'_2 \not\cong f(w)\}
\end{align*}
\]

We show that $T_2$ is a meet-simulation. The verification for the other two is similar. Let $(w_1, w_2, w') \in T_2.$

(M1) Suppose $w_1, w_2 \in V(p)$ for some proposition letter $p.$ Then $w_1 \land w_2 \in V(p)$ because $V(p)$ is a filter of $(W, 1, \alpha).$ By definition of an $L_1$-morphism this implies that $f(w_1, w_2) \in V'(p).$ Using the fact that $V'(p)$ is a filter of $(W', 1', \alpha')$ we find $w' \in V'(p).

(M2) If $w_1 = w_2 = 1$ then $1' = f(1 \land 1) \not\cong w',$ which implies $w' = 1'.$

(M3) Suppose that $u_1, v_1, u_2, v_2 \in W$ are such that $u_1 \land v_1 \not\cong w_1$ and $u_2 \land v_2 \not\cong w_2.$ Set $u' := f(u_1 \land u_2)$ and $v' := f(v_1 \land v_2).$ Then we have $(u_1, u_2, u') \in T_2$ and $(v_1, v_2, v') \in T_2$ by definition, and

\[
\begin{align*}
u' \not\cong v' &= f(u_1 \land u_2) \not\cong f(v_1 \land v_2) = f(u_1 \land v_1 \land u_2 \land v_2) \not\cong f(w_1 \land w_2) \not\cong w'.
\end{align*}
\]

We define preservation of a first-order formula with one free variable by meet-simulations as follows.

7.5 Definition. A first-order formula $\alpha(x)$ with one free variable $x$ is said to be preserved by meet-simulations if for every meet-simulation $T$ between $\mathcal{M}$ and $\mathcal{M'}$ with $(w_1, w_2, w') \in T$ we have:

\[
\text{if } \mathcal{M} \models \alpha(w_1) \text{ and } \mathcal{M}' \models \alpha(w_2) \text{ then } (\mathcal{M}')^o \models \alpha(w').
\]

7.6 Example. Let $\mathcal{M} = (W, 1, \alpha, V)$ be the model from Example 6.2 and $\alpha(x) = Px \uparrow Qx.$ By Example 7.3 the tuple $(u, v, w)$ is related by a meet-simulation. We have seen that $\mathcal{M}^o \models \alpha(x)[u]$ and $\mathcal{M}'^o \models \alpha(x)[v].$ But clearly we do not have $\mathcal{M}'^o \models \alpha(x)[w].$ So $\alpha(x)$ is not preserved by meet-simulations.

7.7 Proposition. Let $T$ be a meet-simulation between $\mathcal{M}$ and $\mathcal{M'}, (w_1, w_2, w') \in T$ and $\varphi \in L.$

If $\mathcal{M}, w_1 \models \varphi$ and $\mathcal{M'}, w_2 \models \varphi$ then $\mathcal{M'}, w' \models \varphi.$

Proof. The proof by induction on the structure of $\varphi$ is analogous to the proof of Proposition 4.5. □

In the remainder of this section we work towards the desired characterisation theorem. The following proposition allows us to exploit the result from Section 6.

7.8 Proposition. Let $\mathcal{M} = (W, 1, \alpha, V)$ and $\mathcal{M}' = (W', 1', \alpha', V')$ be $L_1$-models, $S \subseteq W \times W'$ and

\[
T_S = \{(w_1, w_2, w') \in W \times W \times W' \mid (w_1, w') \in S \text{ or } (w_2, w') \in S\}.
\]

Then $S$ is an $L_1$-simulation between $\mathcal{M}$ and $\mathcal{M'}$ if and only if $T_S$ is a meet-simulation between them.

Proof. Suppose $S$ is an $L_1$-simulation. We verify that $T_S$ is a meet-simulation. Let $(w_1, w_2, w') \in T_S$ and assume without loss of generality that $(w_1, w') \in S.$ If $w_1 \in V(p)$ and $w_2 \in V(p)$ then $(S_1)$ implies that $w' \in V'(p),$ so that $(M_1)$ is satisfied. If $(1, 1, w') \in T_S$ then $(1, w') \in S$ so by $(S_2)$ $w' = 1',$ proving $(M_2).$ Lastly, if $v_1 \land v_2 \not\cong w_1$ and $v_2 \land w_2 \not\cong w_2$ then using $(S_3)$ we can find $v', u' \in W'$ such that $v' \not\cong u' \not\cong w'$ and $(v_1, v') \in S$ and $(u_1, u') \in S.$ This implies $(v_1, v', u') \in T_S$ and $(u_1, u', u') \in T_S,$ so that $(M_3)$ holds.

For the reverse direction, assume that $T_S$ is a meet-simulation. Using the fact that for each $(w, w') \in S$ we have $(w, w, w') \in T_S$ we can show that $(M_1)$ implies $(S_1),$ $(M_2)$ implies $(S_2),$ and $(M_3)$ implies $(S_3).$ □

7.9 Corollary. If $\alpha(x)$ is preserved by meet-simulations, then it is also preserved by simulations.

Proof. Let $S$ be a simulation between $\mathcal{M}$ and $\mathcal{M}'$ such that $(w, w') \in S$ and $\mathcal{M} \models \alpha(x)[w].$ Constructing $T_S$ from $S$ as in Proposition 7.8 we find a meet-simulation $T_S$ such that $(w, w, w') \in T_S.$ The assumption that $\alpha(x)$ is preserved by meet-simulations now entails $\mathcal{M}' \models \alpha(w'[w]).$ □

7.10 Lemma. Let $\alpha(x)$ be a first-order formula with one free variable that is preserved by meet-simulations. Let $\mathcal{M} = (W, I(R), \{I(P)p \mid p \in \text{Prop})\}$ be a first order structure in $\text{FSL}$ and $\mathcal{M}_o = (W, 1, \alpha, V).$ Then the set $[[\alpha(x)]]^\mathcal{M} := \{w \in W \mid \mathcal{M} \models \alpha(x)[w]\}$ is either empty or a filter of $(W, 1, \alpha).$
Proof. If there are no \( w \in W \) such that \( M \models \alpha(x)[w] \) then \( \{\alpha(x)\}^M \) is empty. Assume \( \{\alpha(x)\}^M \neq \emptyset \). Define \( T = \{(w,v,u) \in W \mid (w \land v) \preceq u\} \). We have seen in Example 7.3 that this is a meet-simulation. Suppose \( M \models \alpha(x)[w] \) and \( (w,v) \in T \). Then \((w,v) \in T \) and since \( \alpha(x) \) is assumed to be preserved by meet-simulations we find \( M \models \alpha(x)[v] \). This implies that \( \{\alpha(x)\}^M \) is closed under \( \land \) and \( \preceq \), and that \( \{\alpha(x)\}^M \) is upward closed under the partial order \( \preceq \). Next suppose \( M \models \alpha(x)[w] \) and \( M \models \alpha(x)[v] \). Then \((w,v) \in T \) so \( M \models \alpha(x)[w \land v] \). This implies that \( \{\alpha(x)\}^M \) is closed under binary meets. So \( \{\alpha(x)\}^M \) is a filter of \((W,1,\land)\).

7.11 Theorem. Let \( \alpha(x) \in \text{FOL} \) be a formula with one free variable \( x \). Then \( \alpha(x) \) is equivalent over FSL to \( \neg(x \neq x) \) or to the standard translation of an L-formula iff it is preserved by meet-simulations.

Proof. Since \( \alpha(x) \) is preserved by meet-simulations, it is also preserved by simulations. Therefore Theorem 6.3 implies that we can find \( \varphi_1, \ldots, \varphi_n \) such that
\[
\alpha(x) \equiv_{\text{FSL}} \text{st}_x(\varphi_1) \lor \cdots \lor \text{st}_x(\varphi_n).
\]

If \( n = 0 \) then \( \alpha(x) \equiv_{\text{FSL}} \neg(x = x) \). Suppose \( n \geq 1 \). We claim that \( \alpha(x) \equiv_{\text{FSL}} \text{st}_x(\varphi_1 \lor \cdots \lor \varphi_n) \). It is easy to see that \( \text{st}(\varphi_1) \lor \cdots \lor \text{st}(\varphi_n) \equiv_{\text{FSL}} \text{st}(\varphi_1 \lor \cdots \lor \varphi_n) \), and hence \( \alpha(x) \equiv_{\text{FSL}} \text{st}_x(\varphi_1 \lor \cdots \lor \varphi_n) \). So it suffices to prove \( \text{st}_x(\varphi_1 \lor \cdots \lor \varphi_n) \models_{\text{FSL}} \alpha(x) \).

Let \( M \) be a FOL-structure and \( M_x = (W,1,\land,\lor,\forall) \). Then \( M_x \models \text{st}_x(\varphi_1 \lor \cdots \lor \varphi_n) \) for all \( i \in \{1, \ldots, n\} \). Now Lemma 7.10 implies \( M \models \alpha(x)[w] \).

8 Preventing preservation of classical contradictions

The characterisation result in Theorem 7.11 requires manual treatment of the case where \( \alpha(x) \) is equivalent to \( \neg(x = x) \). This is necessary because the standard translation of \( \bot \in L \) is \( \text{st}_x(\bot) = \forall y(yRx) \), rather than the usual \( \neg(x = x) \). This, in turn, is a consequence of having an inconsistent state in \( L_1 \)-models.

In this final section we suggest a method to resolve this imperfection by adapting the definition of a meet-simulation. Rather than a relation between pairs of states from one model and single states of another model, we relate finite subsets of one model to single states of the other.

We denote by \( \mathcal{P}_wW \) the collection of finite subsets of \( W \), and call the adapted notion of a meet-simulation a meet-simulation.

8.1 Definition. A meet-simulation \( T \) between \( \mathcal{M} = (W_1,1,\land,\lor,\forall) \) and \( \mathcal{M}' = (W_1',1',\land,\lor,\forall) \) is a relation \( T \) between \( \mathcal{P}_wW \) and \( W' \), such that for all \( (X,w') \in T \):
\[
\begin{align*}
(M'_1) & \quad \text{If } w \in V(p) \text{ for all } w \in X, \text{ then } w' \in V'(p), \text{ for each } p \in \text{Prop}; \\
(M'_2) & \quad \text{If } w = 1 \text{ for all } w \in X, \text{ then } w' = 1'; \\
(M'_3) & \quad \text{If for each } w \in X \text{ the elements } u_w, v_w \in W \text{ are such that } u_w \lor v_w \preceq w, \text{ then there exist } v', w' \in W' \text{ such that } (\{w \mid w \in X\}, w') \in T \text{ and } (\{v_w \mid w \in X\}, v') \in T \text{ and } v' \lor w' \preceq v'.
\end{align*}
\]

Meet-simulations satisfy each of these conditions, so we have:

8.2 Lemma. Every meet-simulation is a meet-simulation.

A routine induction on the structure of \( \varphi \) proves the following analogue of Propositions 4.5 and 7.7.

8.3 Proposition. Let \( T \) be a meet-simulation between \( \mathcal{M} = (W_1,1,\land,\lor,\forall) \) and \( \mathcal{M}' = (W_1',1',\land,\lor,\forall) \). Then for each \( (X,w') \in T \) and every formula \( \varphi \in L \):
\[
\text{if } \mathcal{M}, w \models \varphi \text{ for all } w \in X, \text{ then } \mathcal{M}', w' \models \varphi.
\]

8.4 Definition. A FOL-formula \( \alpha(x) \) is said to be preserved by meet-simulations if for every meet-simulation \( T \) between models \( \mathcal{M} \) and \( \mathcal{M}' \) and all \( (X,w') \in T \) we have:
\[
\text{if } \mathcal{M} \models \alpha(x)[w] \text{ for all } w \in X, \text{ then } \mathcal{M}' \models \alpha(x)[w']
\]

The crucial observation of this section is that classical contradictions are not preserved by meet-simulations. This is shown in the following example.
8.5 Example. Let $\beta(x) \in \text{FOL}$ be a contradiction, such as $\neg(x = x)$. Consider any $L_1$-model $\mathcal{M} = \langle W, 1, \land, V \rangle$. Then $T = \{\langle \emptyset, 1 \rangle\}$ is a meet-simulation on $\mathcal{M}$. We vacuously have $\mathcal{M} \models \beta(x)[w]$ for every \( w \in \emptyset \), but not $\mathcal{M} \models \beta(x)[1]$. So $\beta(x)$ is not preserved by meet\_simulations.

8.6 Theorem. Let $\alpha(x) \in \text{FOL}$ be a formula with one free variable $x$. Then $\alpha(x)$ is equivalent over FSL to the standard translation of an $L$-formula iff it is preserved by meet\_simulations.

Proof. The direction from left to right follows from combining Propositions 3.3 and 8.3. For the converse, suppose that $\alpha(x)$ is preserved by meet\_simulations. Since every meet-simulation is a meet\_simulations by Lemma 8.2, $\alpha(x)$ is also preserved by meet\_simulations. It then follows from Theorem 7.11 that $\alpha(x)$ is either a contradiction or equivalent over FSL to the standard translation of a $L$-formula. Since Example 8.5 shows that $\alpha(x)$ cannot be a contradiction, the theorem follows.

8.7 Remark. Inspection of the proofs above shows that we can replace $\mathcal{P}_2\mathcal{W}$ in Definition 8.1 by $\mathcal{P}_{\leq 2}\mathcal{W}$, the collection of subsets of $\mathcal{W}$ containing at most two states.

9 Conclusion

We have characterised non-distributive positive logic as a fragment of first-order logic with one binary predicate and unary a predicate for each proposition letter. We first used $L_1$-simulations to approach the characterisation. Subsequently, we observed that classical disjunctions of standard translations of formulae are still preserved by $L_1$-simulations. To remedy this we introduced meet-simulations between models, which relate pairs of elements of a model to single states of a second model. These allowed us to give a Van Benthem style characterisation theorem for non-distributive first-order logic.

It would be worth investigating meet-simulations further. In particular, it is not clear how the composition of meet-simulations should be defined and whether this results in a meet-simulation again. We also wonder if the largest meet-simulation on a model yields a congruence, hence a quotient model.

Another potential avenue for further research is to extend the results in this paper to other logics. Obvious choices are modal extensions of non-distributive lattice logic, such as the one studied in [10].

Additionally, there may be connections with (bi)simulations for team semantics of dependence logics [4].

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