Relativistic Diffusions

Jacques FRANCHI and Yves LE JAN

March 2004

Abstract

The purpose of this note is to introduce and study a relativistic motion whose acceleration, in proper time, is given by a white noise. We begin with the flat case of special relativity, continue with the case of general relativity, and finally consider more closely the example of the Schwarzschild space.

A detailed and completed version of this work is in progress.

1 Introduction

Several authors, mathematicians and physicists as well, have been interested for a long time in studying a stochastic relativistic process.

In the present note, we consider diffusions, that is to say continuous Markov processes.

We start, in Section 2 below, with the flat case of Minkowski space $M_{1,d} \equiv \mathbb{R}^{1,d}$, and therefore with the Brownian motion of its unit pseudo-sphere, integrated then to yield the only true relativistic diffusion, according to [D1]. We get then in a simple way its asymptotic behavior, hopefully simplifying the point of view of [D3].

In Section 3 below, we present an extension of the preceding construction to the framework of general relativity, that is to say of a generic Lorentz manifold. For that, we define first a process at the level of pseudo-orthonormal frames, with Brownian noise only in the vertical directions, and then see that this diffusion projects into a diffusion on the pseudo-unit tangent bundle. The infinitesimal generator is the sum of the vertical Laplacian with the vector field generating the geodesic flow.

In our last Section 4, we deal in detail with the Schwarzschild space. This is the most classical model for the complement in $\mathbb{R}^{1,3}$ of some central body, star or black hole; see for example [DF-C], [F-N], or [S]. In this setting, the relativistic diffusion projects on a three-dimensional diffusion. The choice of coordinates is suggested by the integration of the geodesic flow.

We show that almost surely, our diffusion hits the central body, or wanders out to infinity, both events occurring with positive probability; and that in the second case it goes away in some random asymptotic direction, asymptotically with the velocity of light.
2 A relativistic diffusion in Minkowski space

Let $M_{1,d}$ be the Minkowski space of dimension $d + 1$, and $\mathbb{H}^d$ the positive time part of its pseudo-unit sphere, which is a representation of the $d$-dimensional hyperbolic space. We identify the unitary tangent bundle $T^1M_{1,d}$ with $M_{1,d} \times \mathbb{H}^d$, and we define on it the relativistic diffusion $(\xi_s, B_s)$ by $\xi_s := \xi_0 + \sigma \int_0^s B_t \, dt$, where $B$ denotes the Brownian motion on $\mathbb{H}^d$ (started from some $B_0 \in \mathbb{H}^d$), $\sigma \in \mathbb{R}^*$, and $\xi_0$ is a fixed point in $M_{1,d}$.

Its infinitesimal generator is $B \frac{\partial}{\partial \xi} + \frac{\sigma^2}{2} \Delta^H_B$, where $\Delta^H$ denotes the hyperbolic Laplacian. It is invariant in law under any Lorentz transformation. In any fixed frame the time coordinate $\xi_0$ is strictly increasing and the velocity $\left( v^j_s := \frac{\dot{\xi}^j_s}{\xi^0_s} \right)_{1 \leq j \leq d}$ is bounded by $1$, the velocity of light.

From the asymptotic behavior of the hyperbolic Brownian motion, it is easy to see that there exists almost surely some $\theta \in \mathbb{S}^{d-1}$ such that $v^j_s$ and $\xi^j_s/\xi^0_s$ converge towards $\theta^j$ for $1 \leq j \leq d$ and $s \to +\infty$. So our diffusion process almost surely wanders out to infinity, asymptotically in one direction and with the velocity of light.

3 Extension to General Relativity

Let us start with a given pseudo-Riemannian manifold $\mathcal{M}$, whose metric tensor has signature $(1,d)$ at each point. Let $SO_{1,d}\mathcal{M}$ denote its principal bundle of pseudo-orthonormal frames, which has its fibers modelled on the special Lorentz group.

Let $H_0, \ldots, H_d$ and $\{V_{kl} \mid 0 \leq k < l \leq d\}$ denote the canonical horizontal and vertical vector fields on $SO_{1,d}\mathcal{M}$, and denote by $\{w^j_s \mid 1 \leq j \leq d\}$ $d$ independent real Wiener processes. The Stratonovitch stochastic differential equation

\[ (*) \quad du_s = H_0(u_s)ds + \sigma \sum_{j=1}^d V_{0j}(u_s) \circ dw^j_s \]

has for each $u_0 \in SO_{1,d}\mathcal{M}$ a unique solution which is a diffusion on $SO_{1,d}\mathcal{M}$ with infinitesimal generator $H_0 + \frac{\sigma^2}{2} \sum_{j=1}^d V_{0j}^2$.

Let $\pi$ denote the canonical projection from $SO_{1,d}\mathcal{M}$ onto $\mathcal{M}$, and $\pi_1$ denote the canonical projection from $SO_{1,d}\mathcal{M}$ onto the unitary tangent bundle $T^1\mathcal{M}$, which we identify with $SO_{1,d}\mathcal{M}/SO_d$. Set $\xi_s := \pi(u_s)$, and $u_s = (\xi_s; e_0(s), \ldots, e_d(s))$.

Let $D$ denotes the covariant differential along the curves: in local coordinates $(\xi^i, e^j_i)$,
with \( e_j = e^k_j \frac{\partial}{\partial x^k} \), it writes \((De_j)^k = de_j^k + \Gamma^k_{ij}e^i_j d\xi^j\). The equation \((*)\) writes equivalently

\[
\dot{\xi}_s = e_0(s) \quad ; \quad De_0(s) = \sigma \sum_{j=1}^d e_j(s) \circ dw^j_s \quad ; \quad De_j(s) = \sigma e_0(s) \circ dw^j_s \quad \text{for} \quad 1 \leq j \leq d.
\]

The stochastic flow defined by \((*)\) commutes with the action of \(SO_d\) on \(SO_{1,d}M\), and therefore the projection \((\xi_s, \dot{\xi}_s) = \pi_1(u_s)\) is a diffusion on \(T^1M\). When \(M = M_{1,d}\), it coincides with the diffusion defined in section 2 above (see Lemma 1 below).

So we got the following general existence result for our relativistic diffusion \((\xi_s, \dot{\xi}_s)\).

**Theorem 1** The \(SO_{1,d}M\)-valued Stratonovitch stochastic differential equation

\[
(*) \quad du_s = H_0(u_s) ds + \sigma \sum_{j=1}^d V_{0j}(u_s) \circ dw^j_s
\]

defines a diffusion \((\xi_s, \dot{\xi}_s) := \pi_1(u_s)\) on \(T^1M\), whose infinitesimal generator is \(L_0 + \frac{\sigma^2}{2} \Delta_v\), \(L_0\) denoting the generator of the geodesic flow and \(\Delta_v\) denoting the vertical Laplacian.

**4 Example of the Schwarzschild space**

This is the most classical model for the complement of a spherical central body, star or black hole; see for example [DF-C], [F-N], [S].

We take \(M := \{\xi = (t, r, \theta) \in \mathbb{R} \times [R, +\infty[ \times \mathbb{S}^2\}\), where \(R \in \mathbb{R}_+\) is a parameter of the central body, endowed with the radial pseudo-metric:

\[
(1 - \frac{R}{r}) dt^2 - (1 - \frac{R}{r})^{-1} dr^2 - r^2|d\theta|^2.
\]

The coordinate \(t\) represents the time (multiplied by the velocity of light \(c\)), and \(r\) the distance from the origin.

In spherical coordinates \(\theta = (\varphi, \psi) \in [0, \pi] \times (\mathbb{R}/2\pi\mathbb{Z})\), we have \(|d\theta|^2 = d\varphi^2 + \sin^2\varphi d\psi^2\), and the non-vanishing Christoffel symbols are:

\[
\Gamma^r_{rt} = -\Gamma^r_{rt} = \frac{R}{2(r-R)} ; \quad \Gamma^r_{tt} = \frac{R(r-R)}{2r^3} ; \quad \Gamma^r_{\varphi\varphi} = R - r ; \quad \Gamma^r_{\psi\psi} = (R-r) \sin^2\varphi ;
\]

\[
\Gamma^r_{r\varphi} = \Gamma^r_{r\psi} = r^{-1} ; \quad \Gamma^\varphi_{\varphi\psi} = -\sin\varphi \cos\varphi ; \quad \Gamma^\varphi_{\psi\psi} = \cotg\varphi.
\]

The Ricci tensor vanishes, the space \(\mathcal{M}\) being empty. The limiting case \(R = 0\) is the flat case of special relativity, considered in section 2. There is no other radial pseudo-metric in \(\mathcal{M}\) which satisfies these constraints.

**Lemma 1** The case \(R = 0\) of the Schwarzschild space is actually the case of special relativity studied in Section 4.
Proof Using Theorem 1 we just have to to check that the vertical Laplacian is actually the hyperbolic Laplacian. Now, owing to the triviality of the tangent bundle, this vertical Laplacian is here merely the restriction to the unit pseudo-sphere $\mathbb{H}^3$ of the Laplacian on the Lorentz group, induced by the “boost” transformations, which is indeed the Laplacian of the hyperboloid $\mathbb{H}^3$. ∗

If $(\xi_s, \dot{\xi}_s) = \pi_1(u_s)$ is the relativistic diffusion of Section 3, we easily see that $(r_s, T_s := \dot{r}_s, U_s := |\dot{\theta}_s|)$ is an autonomous diffusion, with lifetime $D := \min \{ s > 0 \mid r_s = R \}$, and infinitesimal generator

$$T \frac{\partial}{\partial r} + \sigma^2 T U \frac{\partial^2}{\partial T \partial U} + \frac{\sigma^2}{2} (T^2 + 1 - \frac{R}{r}) \frac{\partial^2}{\partial T^2} + \frac{\sigma^2}{2} (U^2 + 1 - \frac{R}{r}) \frac{\partial^2}{\partial U^2} +$$

$$+ \left( \frac{3\sigma^2}{2} T + (r - \frac{3}{2} R) U^2 - \frac{R}{2 \sqrt{r}} \right) \frac{\partial}{\partial T} + \left( \frac{3\sigma^2}{2} U - \frac{2TU}{r} + \frac{\sigma^2}{2 \sqrt{r}} \right) \frac{\partial}{\partial U} .$$

Note that the unit pseudo-norm relation writes $(1 - \frac{R}{r_s})(\dot{r}_s)^2 - (1 - \frac{R}{r_s})^{-1} T_s^2 = r_s^2 U_s^2 + 1$.

If we set (according for example to ([F-N], 4.4) in the deterministic case):

$$a_s := (1 - R/r_s) \dot{r}_s , \quad \text{and} \quad b_s := r_s^2 U_s ,$$

we get $T^2 = a^2 - (1 - R/r)(1 + b^2/r^2)$, and the following:

Proposition 1 The process $(r_s, a_s, b_s, T_s)$ is a degenerate diffusion, with lifetime $D := \min \{ s > 0 \mid r_s = R \}$, and infinitesimal generator

$$L := T \frac{\partial}{\partial r} + \frac{\sigma^2}{2} \left( a - 1 + \frac{R}{r} \right) \frac{\partial^2}{\partial a^2} + \frac{\sigma^2}{2} \left( b^2 + r^2 \right) \frac{\partial^2}{\partial b^2} + \sigma^2 ab \frac{\partial^2}{\partial a \partial b}$$

$$+ \frac{3\sigma^2}{2} a \frac{\partial}{\partial a} + \left( \frac{3\sigma^2}{2} b + \frac{\sigma^2 r^2}{2b} \right) \frac{\partial}{\partial b} + \sigma^2 a T \frac{\partial}{\partial a T} + \sigma^2 b T \frac{\partial}{\partial b T}$$

$$+ \frac{\sigma^2}{2} \left( T^2 + 1 - \frac{R}{r} \right) \frac{\partial^2}{\partial T^2} + \left( \frac{3\sigma^2}{2} T + (r - \frac{3}{2} R) \frac{b^2}{r^4} - \frac{R}{2r^2} \right) \frac{\partial}{\partial T} .$$

Equivalently, we have the following system of stochastic differential equations:

$$dr_s = T_s ds , \quad dT_s = dM_s^1 + \frac{3\sigma^2}{2} T_s ds + (r_s - \frac{3}{2} R) \frac{b_s^2}{r_s^4} ds - \frac{R}{2r_s^2} ds ,$$

$$da_s = dM_s^a + \frac{3\sigma^2}{2} a_s ds , \quad db_s = dM_s^b + \frac{3\sigma^2}{2} b_s ds + \frac{\sigma^2 r_s^2}{2 b_s} ds ,$$

with quadratic covariation matrix of the local martingale $(M^a, M^b, M^1)$ given by

$$K_s' := \sigma^2 \begin{pmatrix}
\frac{a_s^2}{r_s^4} & \frac{a_s b_s}{r_s^4} & a_s T_s \\
\frac{a_s b_s}{r_s^4} & \frac{b_s^2 + r_s^2}{r_s^4} & b_s T_s \\
\frac{a_s T_s}{r_s^4} & \frac{b_s T_s}{r_s^4} & (T_s^2 + 1 - \frac{R}{r_s})
\end{pmatrix}.$$
Corollary 1 The process \((r_s, b_s, T_s)\) is a diffusion, with lifetime \(D\) and infinitesimal generator
\[
\mathcal{L}' := T \frac{\partial}{\partial r} + \frac{\sigma^2}{2} (b^2 + r^2) \frac{\partial^2}{\partial b^2} + \frac{\sigma^2}{2b} (3b^2 + r^2) \frac{\partial}{\partial b} + \sigma^2 b T \frac{\partial^2}{\partial b \partial T} \\
+ \frac{\sigma^2}{2} (T^2 + 1 - \frac{R}{r}) \frac{\partial^2}{\partial T^2} + \left( \frac{3 \sigma^2}{2} T + (r - \frac{3}{2} R \frac{b^2}{r^4} - \frac{R}{2r^2}) \right) \frac{\partial}{\partial T}.
\]
We have the following result on the behavior of the coordinate \(a_s\).

Lemma 2 There exist a standard real Brownian motion \(w_s\), and a real process \(\eta_s\), almost surely converging in \(\mathbb{R}\) as \(s \nearrow D\), such that \(|a_s| = \exp(\sigma^2 s + \sigma w_s + \eta_s)\) for all \(s \in [0, D]\). In particular \(a_s\) almost surely cannot vanish, so that we can assume \(a_s > 0\) for all \(s \in [0, D]\).

Proof Proposition[] above shows that \((a_s^2 - 1) \sigma^2 ds \leq (dM_s^2) \leq a_s^2 \sigma^2 ds\), for \(0 \leq s < D\).
So that we have almost surely (as \(s \to \infty\), when \(D = \infty\)):
\[
\log |a_s| - \log |a_0| = 3 \sigma^2 s/2 - \frac{1}{2} \int_0^s a_t^{-2} (dM_t^2) + \int_0^s a_t^{-1} dM_t^2 \geq \sigma^2 s + \int_0^s a_t^{-1} dM_t^2 \\
= \sigma^2 s + o \left( \int_0^s a_t^{-2} (dM_t^2) \right) = \sigma^2 s + o(s).
\]
Since \((1 - \frac{R}{r_s}) \leq a_s^2\), this implies \(\int_0^D (1 - \frac{R}{r_s}) a_s^{-2} ds < \infty\) almost surely.

For some real Brownian motion \(w\), the process \(\eta\) being defined by the formula in the statement, we have:
\[
d\eta_s = d(\log |a_s|) - \sigma^2 ds - \sigma dw_s = \frac{1}{2} (1 - R/r_s) a_s^{-2} \sigma^2 ds + \left( \sqrt{1 - (1 - R/r_s)a_s^{-2}} - 1 \right) \sigma dw_s,
\]
and then for any \(s < D\):
\[
\eta_s = \eta_0 + \frac{\sigma^2}{2} \int_0^s (1 - R/r_t) a_t^{-2} dt - \sigma \int_0^s \frac{(1 - R/r_t) a_t^{-2}}{1 + \sqrt{1 - (1 - R/r_t) a_t^{-2}}} dw_t,
\]
which converges almost surely in \(\mathbb{R}\) as \(s \nearrow D\), since almost surely for all \(s \in ]0, D[\):
\[
\left\langle \int_0^s \frac{(1 - R/r_t) a_t^{-2}}{1 + \sqrt{1 - (1 - R/r_t) a_t^{-2}}} dw_t \right\rangle \leq \int_0^s \left( (1 - R/r_t) a_t^{-2} \right)^2 dt < \int_0^D (1 - R/r_t) a_t^{-2} ds < \infty.
\]

The deterministic case \(\sigma = 0\) corresponds to the case of the geodesic flow, so we recover that the functionals \(a\) and \(b\) are constants of motions.

There are five types of timelike geodesics:
- running from \(R\) to \(+\infty\), or in the opposite direction;
- running from $R$ to $R$;
- running from $+\infty$ to $+\infty$;
- running from $R$ to some $R_1$ or from $R_1$ to $+\infty$, or idem in the opposite direction;
- bounded geodesics.

In the stochastic case $\sigma \neq 0$ we have the following.

**Theorem 2**  
1) For any initial condition, the radial process $(r_s)$ almost surely reaches $R$ within a finite time $D$ or goes to $+\infty$ as $s \to +\infty$ (equivalently as $t(s) \to +\infty$).
2) Both events in 1) above occur with positive probability, from any initial condition.
3) Conditionally to the event $\{D = \infty\}$ of non-reaching the central body, the Schwarzschild diffusion $(\xi_s, \dot{\xi}_s)$ goes almost surely to infinity in some random asymptotic direction of $\mathbb{R}^3$, asymptotically with the velocity of light.

In the proof of this theorem, below, we shall use the following very simple lemma.

**Lemma 3** Let $M_s$ be a continuous local martingale, and $A_s$ a process such that
\[
\liminf_{s \to \infty} A_s/(\langle M \rangle)_s > 0 \text{ almost surely on } \{\langle M \rangle_\infty = \infty\}.
\]
Then \[
\lim_{s \to \infty} (M_s + A_s) = +\infty \text{ almost surely on } \{\langle M \rangle_\infty = \infty\}.
\]

**Proof** Writing $M_s = W(\langle M \rangle_s)$, for some real Brownian motion $W$, we find almost surely some $\varepsilon > 0$ and some $s_0 \geq 0$ such that $A_s \geq 2\varepsilon \langle M \rangle_s$ and $|M_s| \leq \varepsilon \langle M \rangle_s$ for $s \geq s_0$. Whence $M_s + A_s \geq \varepsilon \langle M \rangle_s$ for $s \geq s_0$. \(\Box\)

**Proof of Theorem 2** We prove successively the 3 assertions of the statement.

1) **Almost sure convergence on $\{D = \infty\}$ of $r_s$ to $\infty$.**

This delicate proof will be split into six parts.

Let us denote by $A$ the set of paths with infinite lifetime $D$ such that the radius $r_s$ remains bounded. We have to show that it is negligible for any initial condition $x = (r, b, T) = (r_0, b_0, T_0)$ belonging to the state space $[R, \infty] \times \mathbb{R}_+ \times \mathbb{R}$.

The cylinder $\{r = 3R/2\}$ plays a remarquable rôle in Schwarzschild geometry. In particular, it contains light lines. We see in the following first part of proof that we have to deal with this cylinder.

(i) $r_s$ must converge to $3R/2$, almost surely on $A$.

Let us apply Itô’s formula to $Y_s := (1 - \frac{3R}{2r_s}) \frac{T_s}{a_s}$:
\[
Y_s = M_s + \frac{3R}{2} \int_0^s \frac{T^2_t}{a_t r^3_t} dt + \int_0^s \left(1 - \frac{3R}{2r_t}ight)^2 \frac{b^2_t}{a_t r^3_t} dt - \sigma^2 \int_0^s \left(1 - \frac{R_t}{r_t}ight) \frac{Y^2_t}{a_t^2} dt - \int_0^s \left(1 - \frac{3R}{2r_t}ight) \frac{R}{2a_t r^3_t} dt,
\]
with some local martingale $M$ having quadratic variation:

$$\langle dM_s \rangle = (1 - \frac{3R}{2r_s})^2 (1 - \frac{R}{r_s}) \left( 1 - \frac{\sigma^2}{a_s^2} \right) a_s^2 ds \leq \sigma^2 a_s^{-2} ds.$$  

Now observe from the unit pseudo-norm relation (Property 1) that $T_s/a_s$ and $Y_s$ are bounded by 1. Hence Lemma 2 implies that the last two terms in the expression of $Y_s$ above have almost surely finite limits as $s \to \infty$. Idem for $\langle M_s \rangle$, and then for $M_s$. Moreover the two remaining bounded variation terms in the expression of $Y_s$ above increase. As a consequence, we get that $Y_s$, $\int_0^s \frac{T_i^2}{a_t r_t^2} dt$, and $\int_0^s (1 - \frac{3R}{2r_s})^2 \frac{b_t^2}{a_t r_t^3} dt$ converge almost surely in $\mathbb{R}$ as $s \to \infty$. So does also $\int_0^s \frac{dt}{a_t r_t^2}$.

Now using that $\frac{a}{r^2} \leq \left( a + \frac{R}{r} \left( \frac{b^2}{a} + \frac{1}{a} \right) \right) r^{-2} = \frac{T^2}{a r^2} + \frac{b^2}{ar^4} + \frac{1}{ar^2}$, we deduce that almost surely

$$\int_0^\infty (1 - \frac{3R}{2r_t})^2 \frac{d}{dt} (\frac{1}{r_t}) dt = \int_0^\infty (1 - \frac{3R}{2r_t})^2 \frac{|T_t|}{r_t^2} dt \leq \int_0^\infty (1 - \frac{3R}{2r_t})^2 \frac{a_t}{r_t^3} dt < \infty.$$  

This implies the almost sure convergence of $\left( 1 - \frac{3R}{2r_s} + \frac{3R^2}{4r_s^2} \right) / r_s$, and therefore of $(1/r_s)$. Since $\lim_{s \to \infty} (1/r_s)$ cannot be 0 on $A$, we have necessarily $\lim_{s \to \infty} r_s = 3R/2$ almost surely on $A$, from the convergence of $\int_0^\infty (1 - \frac{3R}{2r_t})^2 \frac{a_t}{r_t^3} dt$.  

(ii) $b_s/a_s$ converges to $3R\sqrt{3}/2$, and $T_s/b_s$ goes to 0, almost surely on $A$.

Indeed, Itô’s formula gives (for some real Brownian motion $w$)

$$\frac{b_s^2}{a_s^2} = \frac{b_0^2}{a_0^2} + 2\sigma \int_0^s \frac{b_s}{a_s} \sqrt{\frac{r_s^2}{b_s^2} - \frac{1 - \frac{R}{r_s}}{a_s^2}} dw_s + 2\sigma^2 \int_0^s \frac{r_s^2}{a_s^2} ds - 3\sigma^2 \int_0^s (1 - \frac{R}{r_s}) \frac{b_s^2}{a_s^4} ds.$$  

Since by the unit pseudo-norm relation we have $\frac{b_s^2}{a_s^2} < r_s^2/(1 - \frac{R}{r_s})$, whence $b_s^2/a_s^2$ bounded on $A$, the above formula and Lemma 2 imply the almost sure convergence of $b_s^2/a_s^2$ on $A$. Indeed the bounded variation terms converge, and as $b_s^2/a_s^2$ is positive, the martingale part has to converge also. Using the unit pseudo-norm relation again, we deduce that

$$\frac{T_s^2}{a_s^2} = 1 - (1 - \frac{R}{r_s}) \left( \frac{b_s^2}{a_s^2 r_s^2} + \frac{1}{a_s^2} \right)$$  

has also to converge, necessarily to 0, since otherwise we would have an infinite limit for $T_s$, which is clearly impossible on $A$. The value of the limit of $b_s/a_s$ follows now directly from this and from (i).

(iii) We have almost surely on $A$: $\int_0^\infty (r_t - \frac{3R}{2})^2 b_t^2 dt < \infty$, and $\int_0^\infty T_t^2 dt < \infty$.  

7
Let us write Itô’s formula for $Z_s := \left(r_s - \frac{3R}{2}\right)T_s = \frac{1}{2} \frac{d}{ds} \left(r_s - \frac{3R}{2}\right)^2$:

$$Z_s = Z_0 + M_s + \frac{3R^2}{2} (r_s - r_0) + \int_0^s T_t^2 dt + \int_0^s (r_t - \frac{3R}{2}) b_t^2 \frac{dt}{r_t^4} - \frac{R}{2} \int_0^s (r_t - \frac{3R}{2}) \frac{dt}{r_t^2},$$

where $M_s$ is a local martingale having quadratic variation given by:

$$\langle M \rangle_s = \sigma^2 \int_0^s (r_t - \frac{3R}{2})^2 \left(1 - \frac{R}{r_t} + T_t^2\right) dt.$$

Note that if $\langle M \rangle_\infty = \infty$, then by (ii) above $\lim_{s \to \infty} \int_0^s (r_t - \frac{3R}{2})^2 b_t^2 \frac{dt}{r_t^4} / \langle M \rangle_s = \infty$. Note moreover that in this case $\int_0^s |r_t - \frac{3R}{2}| \frac{dt}{r_t^4} \leq \sqrt{\int_0^s (r_t - \frac{3R}{2})^2 b_t^2 \frac{dt}{r_t^4}} \times \sqrt{\int_0^s \frac{dt}{r_t^4}}$ is also negligible with respect to $\int_0^s (r_t - \frac{3R}{2})^2 b_t^2 \frac{dt}{r_t^4}$.

On the other hand, we must have $\liminf_{s \to \infty} |Z_s| = 0$ on $A$.

Therefore we deduce from Lemma 3 that necessarily $\langle M \rangle_\infty < \infty$, and then that $M_s$ converges, almost surely on $A$.

Using again that $\liminf_{s \to \infty} |Z_s| = 0$, we deduce the almost sure boundedness and convergence on $A$ of $\int_0^\infty T_t^2 dt$ and of $\int_0^\infty (r_t - \frac{3R}{2})^2 b_t^2 \frac{dt}{r_t^4}$.

(iv) $(r_s - \frac{3R}{2})^2 b_s$ and $T_s^2 / b_s$ go to 0 as $s \to \infty$, almost surely on $A$.

Indeed, on one hand we deduce from (iii) that (for some real Brownian motion $W$)

$$(r_s - \frac{3R}{2})^2 b_s = \sigma W \left[ \int_0^s (r_t - \frac{3R}{2})^4 (b_t^2 + r_t^2) dt \right] + 2 \int_0^s (r_t - \frac{3R}{2}) T_t b_t dt + \frac{\sigma^2}{2} \int_0^s (r_t - \frac{3R}{2})^2 (3b_t + \frac{r_t}{b_t}) dt$$

has to converge almost surely on $A$ as $s \to \infty$, necessarily to 0 since it is integrable with respect to $s$.

On the other hand we have for some real Brownian motion $W'$, by Itô formula:

$$\frac{T_s^2}{b_s} = \frac{T_0^2}{b_0} + \sigma W' \left[ \int_0^s \left( \frac{T_t^4}{b_t^2} + r_t^2 \frac{T_t^4}{b_t^4} + 4(1 - \frac{R}{r_t}) \frac{T_t^2}{b_t^2} \right) dt \right] + \frac{\sigma^2}{2} \int_0^s \frac{T_t^2}{b_t} dt + 2 \int_0^s (r_t - \frac{3R}{2}) T_t b_t \frac{dt}{r_t^4}$$

$$- \int_0^s \frac{R T_t}{r_t^4} b_t dt + \sigma^2 \int_0^s \left( 1 - \frac{R}{r_t} \right) \frac{dt}{b_t} + \frac{\sigma^2}{2} \int_0^s \frac{r_t^2 T_t^2}{b_t^4} dt.$$

Recall from (i) that $\frac{T_t}{b_t} \to 0$ and that $b_t \sim \frac{3R^2}{2} a_t$. Thus using (iii) we see easily that all integrals in the above formula converge. Hence we deduce the almost sure convergence of $s \mapsto T_s^2 / b_s$ on $A$, necessarily to 0, since it is integrable.

(v) It is sufficient to show that $\int_0^\infty |r_t - \frac{3R}{2}| |T_t| b_t^2 dt < \infty$, and that $\int_0^\infty T_t^4 dt < \infty$, almost surely on $A$.  

Indeed, assuming that these 2 integrals are finite, Itô’s formula shows that we have for some real Brownian motion $W''$:

$$T_s^2 = T_0^2 + 2\sigma W'' \left[ \int_0^s \left( T_t^2 + 1 - \frac{R}{r_t} \right) T_t^2 \, dt \right] + 4\sigma^2 \int_0^s T_t^2 \, dt + 2 \int_0^s (r_t - \frac{3R}{2}) T_t b_t \frac{dt}{r_t^2}$$

$$+ \sigma^2 \int_0^s (1 - \frac{R}{r_t}) \, dt - R \int_0^s \frac{T_t^2}{r_t^2} \, dt$$

$$= \gamma_s + \sigma^2 \int_0^s (1 - \frac{R}{r_t}) \, dt - R \int_0^s \frac{T_t}{r_t} \, dt = \gamma_s + \int_0^s \left[ \frac{1}{3} + \frac{2}{3r_t} \left( r_t - \frac{3R}{2} \right) \right] \, dt + \frac{R}{r_s} - \frac{R}{r_0} = \gamma'_s + s/3,$$

where $\gamma, \gamma'$ are bounded converging processes on $A$. Whence $\lim_{s \to \infty} T_s^2 = \infty$ almost surely on $A$, which with (iii) above implies that $A$ must be negligible.

(vi) **End of the proof of the convergence of $r_s$ to \( \infty \)** on \( \{ D = \infty \} \).

By Schwarz inequality, the first bound in (v) above will follow from $\int_0^s T_t^2 b_t \, dt < \infty$ and from $\int_0^s (r_t - \frac{3R}{2}) b_t^2 \, dt < \infty$. Now these two terms appear in the Itô expression for $Z_s^1 := (r_s - \frac{3R}{2}) T_s b_s$:

$$Z_s^1 = Z_0^1 + M_s^1 + \frac{\sigma^2}{2} \int_0^s [8 + \frac{r_t^2}{b_t^2}] Z_t^1 \, dt + \int_0^s T_t^2 b_t \, dt + \int_0^s (r_t - \frac{3R}{2}) b_t^3 \frac{dt}{r_t^3} - \frac{R}{r_t^2} \int_0^s (r_t - \frac{3R}{2}) b_t \frac{dt}{r_t^2},$$

with a local martingale $M_s^1$ having quadratic variation:

$$\langle M^1 \rangle_s = \sigma^2 \int_0^s (r_t - \frac{3R}{2}) b_t^2 \times \left( 1 - \frac{R}{r_t} + \left[4 + r_t^2 b_t^{-2} \right] T_t^2 \right) \, dt.$$

Note that by Schwarz inequality, (iii) above implies that $\int_0^\infty |Z_t^1| \, dt < \infty$, and then that $\int_0^s [8 + \frac{r_t^2}{b_t^2}] Z_t^1 \, dt$ is bounded and converges, almost surely on $A$, as $s \to \infty$.

Using the first assertion of (iv), observe that $\lim_{s \to \infty} \int_0^s \frac{(r_t - \frac{3R}{2}) b_t^3 \, dt}{r_t^2} + \int_0^s T_t^2 b_t \, dt = \infty$ if $\langle M^1 \rangle_\infty = \infty$.

Note moreover that in this case

$$\left| \int_0^s (r_t - \frac{3R}{2}) b_t \frac{dt}{r_t^2} \right| \leq \sqrt{\int_0^s \left( r_t - \frac{3R}{2} \right)^2 b_t^3 \frac{dt}{r_t^2} \times \int_0^s \frac{dt}{b_t}}$$

is also negligible with respect to $\int_0^s (r_t - \frac{3R}{2})^2 b_t^3 \frac{dt}{r_t^3} + \int_0^s T_t^2 b_t \, dt$.

Therefore we deduce from Lemma 3 and from the integrability of $t \mapsto |Z_t^1|$, that necessarily $\langle M^1 \rangle_\infty < \infty$, and then that $M_s^1$ has to converge, almost surely on $A$. 9
Hence $Z^1$ must have a limit almost surely on $A$, which must be 0, owing to the integrability of $Z^1$. This forces clearly $\int_0^\infty \left( r_t - \frac{3 \sigma^2}{2} \right)^2 b_t^2 \frac{dt}{r_t^4} + \int_0^\infty T_t^2 b_t dt$ to be finite, almost surely on $A$, showing the first bound in (v) above.

Finally, the integrability of $T_t^2 b_t$ and the second convergence of (iv) imply the second bound in (v) above: $\int_0^\infty T_t^4 dt < \infty$ almost surely on $A$.

This concludes the proof of the first assertion in Theorem 3.

2) $r_s \to R$ and $r_s \to \infty$ occur both with positive probability, from any initial condition.

Let us use the support theorem of Stroock and Varadhan (see for example ([I-W], Theorem VI.8.1)) to show that the diffusion $(r, b, T)$ of Corollary 1 is irreducible. Since we can decompose further the equations given in Property 1 for $(r, b, T)$, using a standard Brownian motion $(w, \beta, \gamma) \in \mathbb{R}^3$, as follows:

$$dr_s = T_s \, ds, \quad db_s = b_s \, dw_s + r_s \, d\beta_s + \frac{3 \sigma^2}{2} \, b_s \, ds + \frac{\sigma^2 r_s^2}{2 b_s} \, ds,$$

$$dT_s = T_s \, dw_s + \sqrt{1 - \frac{R}{r_s}} \, d\gamma_s + \frac{3 \sigma^2}{2} T_s \, ds + (r_s - \frac{3}{2} R) \frac{b_s^2}{r_s^3} \, ds - \frac{R}{2 b_s^2} \, ds,$$

we see that trajectories moving the coordinate $b$ without changing the others, and trajectories moving the coordinate $T$ without changing the others, belong to the support of $(r, b, T)$. Moreover we see that there are timelike geodesics, and then trajectories in the support, which link $r$ to $r'$, and then considering the velocities also, which link say $(r, b', T')$ to $(r', b'', T'')$. So, for given $(r, b, T)$ and $(r', b', T')$ in the state space, we can, within the support of $(r, b, T)$, move $(r, b, T)$ to $(r', b'', T'')$, then $(r, b', T'')$ to $(r', b', T''')$, and finally move $(r', b', T''')$ to $(r', b', T')$, thereby showing the irreducibility of $(r, b, T)$.

This implies that it is enough to show that for large enough $r_0, T_0$, the convergence to $\infty$ occurs with probability $\geq 1/2$, and that for $r_0$ close enough from $R$ and $T_0$ negative enough, the convergence to $R$ occurs with probability $\geq 1/2$ as well. Now this can be done by a classical supermartingale argument using the process $1/|T_s|$, stopped at some hitting time. Indeed we see from Property 1 that

$$\frac{1}{|T_s|} + \int_0^s \left( \frac{\sigma^2}{2} T_t^2 - \sigma^2 \left(1 - \frac{R}{r_t}\right) - \frac{R T_t}{2 r_t^2} + (2 r_t - 3 R) \frac{b_t^2 T_t}{r_t^4} \right) \frac{dt}{|T_t|^3}$$

is a local martingale.

Take first $r_0 \geq 3 R/2$, $T_0 \geq 4 + \frac{3 \sigma^2}{R^2}$, and $\tau := \inf \left\{ s > 0 \left| T_s = 2 + \frac{3 \sigma^2}{R^2} \right. \right\}$: $r_s$ increases on $\{0 \leq s < \tau\}$ and then we see that $1/|T_{s \wedge \tau}|$ is a supermartingale, which implies that

$$(2 + \frac{3 \sigma^2}{R^2})^{-1} \mathbb{P}(\tau < \infty) \leq \liminf_{s \to \infty} \mathbb{E}(\frac{1}{|T_{s \wedge \tau}|} 1_{\{r_s < \infty\}}) \leq \liminf_{s \to \infty} \mathbb{E}(\frac{1}{|T_{s \wedge \tau}|}) \leq \mathbb{E}(\frac{1}{T_0}) \leq (4 + \frac{3 \sigma^2}{R^2})^{-1},$$

and then that $\mathbb{P}(\lim_{s \to \infty} r_s = +\infty) \geq \mathbb{P}(\tau = \infty) \geq 1/2$.
Conversely take $r_0 \leq 3R/2$, $T_0 \leq -2$, and $\tau' := \inf \{ s > 0 \mid T_s = -\sqrt{2} \}$: $r_s$ decreases on $\{ 0 \leq s < \tau' \}$ and then we see that $1/|T_{s \wedge \tau'}|$ is a supermartingale, which implies that
$$2^{-1/2} \mathbb{P}(\tau' < \infty) \leq \lim_{s \to \infty} \mathbb{E}(\frac{1}{|T_{s \wedge \tau'}|} \mathbf{1}_{(\tau' < \infty)}) \leq \lim_{s \to \infty} \mathbb{E}(\frac{1}{|T_x|}) \leq \mathbb{E}(\frac{1}{|T_0|}) \leq 1/2,$$
and then that $\mathbb{P}(D < \infty) \geq \mathbb{P}(\tau' = \infty) \geq 1/\sqrt{2}$.

This concludes the proof of the second assertion in Theorem 2.

3) Existence of an asymptotic direction for the Schwarzschild diffusion, on $\{ D = \infty \}$.

We want to generalize the observation made in Section 2 for $R = 0$. Recall from Lemma 2 that it does not matter for this asymptotic behavior whether we consider the trajectories as function of $s$ or of $t(s)$ (i.e., as viewed from a fixed point).

We shall use Section 2 and Lemma 1, to proceed by comparison between the flat Minkowski case $R = 0$ and the Schwarzschild case $R > 0$.

Let us split this proof into four parts.

(i) We have $\int_0^\infty \frac{a_t}{r_t^2} dt < \infty$ and $\int_0^\infty \frac{U_t}{r_t} dt < \infty$, almost surely on $\{ D = \infty \}$.

We know from 1) above that $r_s \to \infty$ almost surely on $\{ D = \infty \}$.

The very beginning of this proof remains valid: Using (1,i) again, we have almost surely
$$\int_0^\infty \frac{T_t^2}{a_t r_t^2} dt \text{ and } \int_0^\infty (1 - \frac{3R}{2r_t})^2 \frac{b_t^2}{a_t r_t^2} dt \text{ finite, whence } \int_0^\infty \frac{b_t^2}{a_t r_t^2} dt \text{ finite, and then, since }$$
$$\frac{a}{r^2} \leq \frac{T^2}{a r^2} + \frac{b^2}{a r^4} + \frac{1}{a r^4}, \text{ also } \int_0^\infty \frac{a_t}{r_t^2} dt \text{ finite, almost surely on } \{ D = \infty \}.$$

Now by the unit pseudo-norm relation, we have $\frac{U}{r} = \frac{b}{r^3} \leq \frac{a}{r^2 \sqrt{1 - \frac{R}{r}}}$, whence
$$\int_0^\infty \frac{U_t}{r_t} dt \text{ finite, almost surely on } \{ D = \infty \}.$$

(ii) The perturbation of the Christoffel symbols due to $R$ is $O(r^{-2})$.

Recall from the beginning of Section 4 the values of the Christoffel symbols $\Gamma^i_{jk}$. Denote by $\tilde{\Gamma}^i_{jk}$ the difference between these symbols and their analogues for $R = 0$, which is a tensor, has only five non-vanishing components in spherical coordinates, and then is easily computed in Euclidian coordinates ($x_1 = r \sin \phi \cos \psi; x_2 = r \sin \phi \sin \psi; x_3 = r \cos \phi$) : we find
$$\tilde{\Gamma}^i_{x_jx_k} = \frac{\partial x_i}{\partial r} \times \left( \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_k} \Gamma^r_{rr} + \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} \Gamma^r_{\phi\phi} + \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_k} \Gamma^r_{\psi\psi} \right)$$
$$= \frac{x_i}{r} \times \left( -\frac{R}{2r(r-R)} \frac{x_j x_k}{r} + R \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k} + R \sin^2 \phi \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_k} \right) \mathcal{O}(r^{-2})$$

11
since $|\frac{\partial \varphi}{\partial x_j}| \leq 1/r$ and $|\frac{\partial \varphi}{\partial x_i}| \leq 1/(r \sin \varphi)$. The same is valid directly for the remaining components $\hat{\Gamma}_{t,i}^x$ and $\hat{\Gamma}_{x,t}^y$.

(iii) The stochastic parallel transport converges, almost surely on \( \{ D = \infty \} \).

Let us denote by $\xi^i_j(s)$ the matrix carrying out the inverse parallel transport along the $C^1$ curve \( (\xi^i| 0 \leq s' \leq s) \), in the global pseudo-Euclidean coordinates \( (t, x_1, x_2, x_3) \).

We have

\[
\frac{d}{ds} \xi^i_j(s) = \xi^i_j(s) \times \Gamma_{j,k}^i(\xi^i_s) \times \xi^i_s,
\]

so that, using (ii) above and $|T_s| \leq a_s$:

\[
\tilde{\xi}^i_j(s) = \int_0^s \mathcal{O}(r_{v}^{-2} \times |\xi_v|) \, dv = \int_0^s \mathcal{O}(|v| + |v| + r_v |\dot{v}|) r_v^{-2} \, dv = \int_0^s \mathcal{O}(2 \frac{a_v}{r_v} + \frac{U_v}{r_v}) \, dv,
\]

we can conclude by using (i) above, that the Schwarzschild stochastic parallel transport (like its inverse $\tilde{\xi}^i_j(s)$) admits a finite limit as $s \to \infty$, almost surely on \( \{ D = \infty \} \).

(iv) End of the proof.

Let us consider $\eta_s := \tilde{\xi}^i_j(s) \dot{\xi}_s$, for $s \geq 0$.

This is a continuous process living on the fixed unit pseudo-sphere $T^1_{\xi_0} \mathcal{M}$.

Recall from Section 3 that (for $0 \leq \ell \leq 3$)

\[
d\tilde{\xi}_s^\ell = \sigma \sum_{k=1}^{3} e_k^\ell(s) \circ dw^k_s - \Gamma_{ij}^\ell(x_s) \dot{\xi}_s^i \dot{\xi}_s^j ds.
\]

Therefore we get

\[
d\tilde{\eta}_s^\ell = \sigma \sum_{k=1}^{3} \xi^\ell_m(s) e_k^m(s) \circ dw^k_s - \tilde{\xi}^\ell_{m} \Gamma_{ij}^m(x_s) \dot{\xi}_s^i \dot{\xi}_s^j ds + (d \tilde{\xi}^\ell(s)) \xi_s^j
\]

\[
= \sigma \sum_{k=1}^{3} \xi^\ell_m(s) e_k^m(s) \circ dw^k_s - \xi^\ell_m \Gamma_{ij}^m(x_s) \dot{\xi}_s^i \dot{\xi}_s^j ds + (\xi^\ell_m(s) \Gamma_{ij}^m(x_s) \dot{\xi}_s^i) \xi_s^j
\]

\[
= \sigma \sum_{k=1}^{3} \xi^\ell_m(s) e_k^m(s) \circ dw^k_s = \sigma \sum_{k=1}^{3} \tilde{e}_k(s) \circ dw^k_s,
\]

where $\tilde{e}_k(s) := \frac{\tilde{\xi}^\ell(s)}{\xi^\ell_m(s)} e_k(s)$, for $1 \leq k \leq 3$ and $s \geq 0$.

Observe that, for any $s \geq 0$, \( (\tilde{\eta}_s, \tilde{e}_1(s), \tilde{e}_2(s), \tilde{e}_3(s)) \) constitutes a pseudo-orthonormal basis of the fixed unit pseudo-sphere $T^1_{\xi_0} \mathcal{M}$. Hence we find that the velocity process $\eta$ defines a hyperbolic Brownian motion on this unit pseudo-sphere $T^1_{\xi_0} \mathcal{M}$. 

12
Now, according to Lemma 1 and Section 2, we know that $\frac{\eta_s}{a_s}$ converges almost surely as $s \to \infty$ towards $(1, 1, \hat{\theta}_\infty)$ (in coordinates $(t, r, \theta)$), for some random $\hat{\theta}_\infty \in S^2$, exponentially fast.

Using $(iii)$ above, we deduce that $\frac{\dot{\xi}_s}{a_s}$ converges almost surely as $s \to \infty$ towards $(1, 1, \hat{\theta}_\infty)$, for some random $\hat{\theta}_\infty \in S^2$. This means also that the Schwarzschild diffusion seen from a fixed point, that is to say the implicit trajectory $Z := (t_s \mapsto (r_s, \theta_s))$, sees almost surely its velocity $dZ_t/dt$ converging towards $(1, \hat{\theta}_\infty)$, 1 being here the velocity of light.

Moreover, this shows a posteriori that $T_s$ goes to $+\infty$ and that $r_s \sim a_s$ as $s \to \infty$, and then that the convergences in $(i)$ and in $(iii)$ above occur exponentially fast, so that it must be the same for the convergences of $\frac{\dot{\xi}_s}{a_s}$ and $dZ_t/dt$. This allows to integrate, to get finally the generalization of Section 2 to the Schwarzschild diffusion.

This ends the whole proof of Theorem 2. □

REFERENCES

[DF-C] De Felice F., Clarke C.J.S. Relativity on curved manifolds.
Cambridge surveys on mathematical physics, Cambridge university press, 1990.

[D1] Dudley R.M. Lorentz-invariant Markov processes in relativistic phase space.
Arkiv för Matematik 6, n° 14, 241-268, 1965.

[D2] Dudley R.M. Asymptotics of some relativistic Markov processes.
Proc. Nat. Acad. Sci. USA n° 70, 3551-3555, 1973.

[F-N] Foster J., Nightingale J.D. A short course in General Relativity.
Longman, London 1979.

[I-W] Ikeda N., Watanabe S. Stochastic differential equations and diffusion processes.
North-Holland Kodansha, 1981.

[S] Stephani H. General Relativity. Cambridge university press, 1990.

Jacques FRANCHI: Université Louis Pasteur, I.R.M.A., 7 rue René Descartes, 67084 Strasbourg cedex. France. franchi@math.u-strasbg.fr

Yves LE JAN: Université Paris Sud, Mathématiques, Bâtiment 425, 91405 Orsay. France. yves.lejan@math.u-psud.fr

13