Optimization of Reliability in Complex Systems Using an Artificial Neural Network Approach

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Abstract: Reliability-based design optimization (RBDO) is used for improving systems with vulnerabilities in structure factors, framework parameters, or both. RBDO incorporates a dependability examination that needs various computational battles, particularly for real-world issues. Here, a high performance Surrogate-Assisted RBDO approach is given. Computational insight and RBDO decomposition-based techniques are consolidated to build up a speedy RBDO. This creative methodology depends on the artificial neural networks (ANNs) as a surrogate model and Sequential Optimization and Reliability Assessment (SORA) strategy as a RBDO procedure. The issue in SORA is isolated into two segments of successive deterministic streamlining and dependability evaluation. In this paper, to upgrade the computational effectiveness and to extend the application scope of the SORA procedure, an Augmented SORA (ASORA) adaptation is proposed. In the created strategy, to recognize the latent probabilistic imperatives and separate the fulfilled limitations from reliability evaluation, a paradigm is utilized to diminish the related computational costs. Besides, the adjustments in the move vectors acquired for fulfilled requirements are controlled to get the estimation of zero for the following RBDO cycle. To exhibit the viability and exactness of the proposed techniques, some scientific models with various degrees of multifaceted nature and a viable designing model are fathomed and the outcomes are analyzed.

Keywords: Optimization; Reliability-Based Design; Computational Intelligence; Surrogate Model; Neural Networks

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1 INTRODUCTION

Enhancing Reliability-Based Design Optimization (RBDO) is majorly considered for obtaining an optimal design with a certain failure probability. It makes it possible to make a tradeoff between an increase in reliability and a decrease in cost [1]. The methods such as Reliability Index Approach (RIA) and Performance Measure Approach (PMA) are among the most commonly used techniques in this regard. The main problem in using these methods is their considerable computational burden caused by indeterminacy analysis trend. Therefore, as stated in the literature, classical methods involve low convergence or even non-convergence rates due to computational burden problems derived from reliability constraints calculation [2]. Theo et al. initially proposed an innovative concept known as PMA. In this method, solving an inverse reliability problem, the probability index is converted to the performance index and then a low-performance point is searched on the optimal reliability surface [4]. Since calculating the reliability index in RIA or Probabilistic Performance Measure (PPM) in PMA is fundamentally an optimization in PMA, the classical methods are based on double-loop optimization and communication algorithm between probabilistic calculations and optimization for solving the RBDO problems. Accordingly, the computational cost of double loops structures will be very considerable and overwhelming especially in the problem with a large number of probabilistic constraints or nonlinear functional functions. In order to resolve or modify these problems, various approximate methods, such as single-loop and segregated methods, have been developed. The Sequential Optimization and Reliability Assessment (SORA) is among the methods developed for improving the RBDO methods.

SORA, which was proposed by Deo and Chen [5], employs a single-loop approach including optimization and reliability assessment. Considering the fact that the reliability assessment and certain optimization are performed sequentially in SORA, certain optimization is required to assess the replaced constraints. Furthermore, using this method, the sensitivity and quantity of probabilistic constraints functions are calculated in MPP, in finding the reliability information that is conducted by inverse reliability analysis. In reliability-based design, direct integration always requires very heavy mathematical calculations. To solve this problem, Deo et al. [6] proposed a model of SORA for improving the efficiency in solving the Multidiscip-
plinary Design Optimization (MDO) problems based on reliability. The main idea in this method is the separation of reliability analysis from a determinate multidisciplinary design loop, which makes them applicable in the form of a sequential loop. Later, Cheo et al. [7] using methods of moving asymptotes could achieve an optimized SORA. They also provided an optimized SORA method using convex linearization. A different formulation of reliability-based SORA in the presence of fuzzy indeterminacies was provided by Li et al. In order to make the SORA process more efficient and extend it, Huang et al. [10] offered the enhanced SORA (ESORA) assuming both fixed and variable standard deviations for the random design variables and maintaining the sequential structure. Huang et al. [11] attempted to ensure the convergence stability of the optimal response in the SORA using an incremental shift strategy for constraining boundaries. To enhance the effectiveness of SORA and achieve more resistant results, Yie et al. proposed an approximate method of SORA for solving the RBDO problems. Using the MPP and PPM approximates in reliability assessment over SORA, their method obviates the need for evaluating functional functions in definitive optimization. Jiang et al. [13] proposed an RBDO for the reliability-based design of time-variable issues.

Despite numerous efforts in developing RBDO methods, enhancing the efficiency and reducing the computational costs are also the major challenges in this regard. In the present study, an augmented SORA (ASORA) is offered to further improve the computational efficiency in SORA in order to reduce the total computational cost of the RBDO process by preventing a reliability analysis for probabilistic constraints through minimizing the number of calls for probabilistic constraints. In addition, meta-heuristic methods are used to further enhance the computational efficiency and reduce the computational burden resulting from the probabilistic constraints calculation by replacing heavy probabilistic constraints with simple approximation functions. It is noteworthy that the application of computational intelligence, along with methods of experiment designing and meta-heuristic modeling, is a new and emerging field that has been subjected to the intense research in the last decade for solving the complex optimization problems such as RBDO [14].

Among the most common metamodeling methods are Polynomial Response Surface, Artificial Neural Networks, Kriging 10, and Support Vector Regression. Jane et al. [15] investigated the effectiveness of surrogate models in optimization problems under indeterminacies, using Kriging meta-heuristic model instead of the original model. Agarwal and Rinavd [16] and Youn and Chouvi [17] are among those who utilized the response surface meta-heuristic model for RBDO. Hyun and Chai [18] used the Momentary Method in RBDO together with Kriging surrogate model. Liu et al. [19] provided a comparative sampling method for RBDO using Support Vector Regression and Kriging. By combining the support vector regression and the neural network, Dai and Kao [20] provided a method for evaluating reliability in the structure design. In the present work, combining the metamodel of neural network and ASORA, a method known as Surrogate-Assisted ASORA (SA-ASORA) is proposed for solving the RBDO problems. In order to increase the precision of pre-built metamodel during the convergence process to a reliable point, the pre-built metamodel is updated in each iteration of the RBDO process.

To demonstrate the efficiency and accuracy of the proposed method, several mathematical problems with various complexity levels and the design of the route of a satellite carrier are solved as a practical engineering example with a large computational burden. Then, the results of the different methods are discussed.

## 2 Optimal Reliability-Based Design

Generally, a typical optimization problem is expressed as follows:

\[
\begin{align*}
\text{Minimizing} & \quad f(\tilde{d}, \tilde{p}) \\
\text{Relative to} & \quad g_j(\tilde{d}, \tilde{p}) \leq 0, j = 1, \ldots, n \\
\tilde{d}^l \leq \tilde{d} \leq \tilde{d}^u
\end{align*}
\]

(1)

Where \( () \) represents the target function, \( \tilde{d} \) is the vector of design variable, \( \tilde{p} \) denotes the vector of design parameters, \( g(.) \) represents the constraints functions, and \( \tilde{d}^l \) and \( \tilde{d}^u \) are the upper and lower limit of design variables vector, respectively. In the determinate optimization problem, the indeterminacies in the system cannot be directly considered. In the reliability-based optimization problem formulated as Eq. (2), determinate and indeterminate design variables and parameters are considered simultaneously in the optimization problem.

\[
\begin{align*}
\text{Minimizing} & \quad f(\tilde{x}, \tilde{d}, \tilde{p}) \\
\text{Relative to} & \quad g_j(\tilde{x}, \tilde{d}, \tilde{p}) \leq 0, j = 1, \ldots, n \\
\tilde{x}^l \leq \tilde{x} \leq \tilde{x}^u \\
\tilde{d}^l \leq \tilde{d} \leq \tilde{d}^u
\end{align*}
\]

(2)

where \( \tilde{x} \) is the vector of indeterminate design variables and \( \tilde{p} \) is the set of indeterminate parameters that are expressed by a probabilistic normal distribution with mean value (\( \mu \)) and covariance matrix (\( \sigma^2 \)). In these problems, the design point achieved by the optimization should be distanced enough in the space available to provide the required
reliability. In other words, the further from the optimal design point to the possible design space, the higher the reliability of the system; however, the system correspondingly moves away from the optimality. Hence, through the RBDO, there would be an acceptable trade-off between the reliability and optimality. So, to have the optimal reliability ($R$), it is desirable to select a point of the space where, under indeterminacies, the probability of violation of the probabilistic constraint is less than $1 - R$.

To achieve such results, the optimization problem expressed in Eq. (2) should be redefined for reliability-based optimization problems. This type of problems is defined as an indeterminate problem due to the existence of indeterminate design variables and parameters, the target function, and the probabilistic constraints functions.

$$
\text{Minimizing } f(\hat{x}, \hat{d}, \bar{p})
$$

Relative to

$$\Pr\left\{ g_j(\hat{x}, \hat{d}, \bar{p}) \leq 0 \right\} \geq R_j, j = 1 \ldots, n$$

$$\bar{d}^L \leq \hat{d} \leq \bar{d}^U$$

(3)

Where $R_j$ is the reliability of fulfillment of $j$th constraint that the failure probability of the $j$th can be expressed as $P_j$ by Eq. (4).

$$
\text{Minimizing } P_j(\hat{x}, \hat{d}, \bar{p}) = \int_{g_j(\hat{x}, \hat{d}, \bar{p})} \varphi(\hat{x}, \bar{p}) d\hat{x} d\bar{p}
$$

(4)

Where $P_j$ is the probability of failure and $\varphi$ is the probability density function. However, in this integral, finding an analytic expression for each constraint is impossible. Therefore, to estimate the integral of Eq. (4), a statistical approximation can be utilized based on sampling or optimization methods.

The main idea in reliability-based optimization methods is to determine the constraint boundary point of the problem with the least distance with a definite response. For this purpose, using Rosenblatt transitions [12], the system coordinate should be shifted from the real space ($X$) to the coordinate system into the normal space ($U$), where the normal standard random variables are determined using the mean value of 0 and standard deviation of the unit.

In this space, the upper surface, $g(\hat{x}, \hat{d}, \bar{p}) = 0$, or its equivalent $g_i(\bar{U}) = 0$ are estimated using the first-order approximation in MPP. In other words, MPP corresponds to the reliability index ($\beta_j$), which is obtained from the first-order approximation of $P_j = \Phi(-\beta_j)$. The common formulation of RBDO problem is as follows:

$$
\text{Minimizing } f(\hat{x}, \hat{d}, \bar{p})
$$

Relative to

$$\Pr\left\{ g_j(\hat{x}, \hat{d}, \bar{p}) \leq 0 \right\} \geq \beta_j, j = 1 \ldots, n$$

$$\bar{d}^L \leq \hat{d} \leq \bar{d}^U$$

(5)

where $f(.)$ is the target function, $\hat{d}$ represents the vector of the determinate design variables, $\hat{x}$ and $\bar{p}$ are respectively the variables vector and the indeterminate design parameters, $g_j(\hat{x}, \hat{d}, \bar{p})$ indicates the functions of probabilistic constraints, $\Phi(.)$ is the cumulative distribution function, and $\beta_j$ shows the considered reliability index for the $j$th probabilistic constraint.

The RIA and PMA double-loops are the most common methods for solving the RBDO problems. However, SORA is a more effective method in solving the RBDO problems and, as a result of converting the double-loops structures to the single-loop or series loop, the efficiency in solving the RBDO problems increases [22, 23].

2.1 Reliability index method

RIA technique is based on whether the probabilistic constraints are met through reliability index [23]. In this way, the outer loop of the RBDO problem is defined as follows:

$$
\text{Minimizing } f(\hat{d}, \bar{p})
$$

Relative to

$$\beta_j(\bar{U}) \geq \beta_j$$

$$\bar{d}^L \leq \hat{d} \leq \bar{d}^U$$

(6)

In this method, the internal loop optimization problem is defined for the reliability analysis in normal standard space as follows:

$$
\text{Minimizing } ||U||
$$

Relative to

$$g_j(\bar{U}, \hat{d}) = 0$$

(7)
Where $\vec{U}$ is the vector of indeterminate variables with normal distribution in the normal standard space and $G_i(.)$ is the probabilistic constraint defined in the normal standard space. Solving Eq. (7), MPP is obtained as the optimal point $u^*$ in the normal standard space and $u^*$ is defined in the real space.

2.2 Performance measurement method

Within PMA, reliable results are obtained through searching for the minimum value of the constraint function in case satisfying the target reliability index. The RBDO problem based on PMA is defined as follows:

Minimizing $f(\vec{d}, \vec{p})$
Relative to $G_{mj}(\vec{d}, \vec{p}) \leq 0$
$\vec{d}^l \leq \vec{d} \leq \vec{d}^u$ (8)

$G_{mj}$ is the maximum $j$th constraint. So, to assess the reliability, the internal loop optimization problem is defined in the standard normal space as follows:

Minimizing $G_{mj} = \max G_j(\vec{U})$
Relative to $\|\vec{U}\| = \beta_j^*$ (9)

The formulation of the reliability analysis in PMA is considered as the inverse reliability analysis to the RIA method. Thus, the result obtained from the PMA is expressed as the most probable inverse point (IMPP). Therefore, in the $U^*$ point obtained in PMA, the $G(U)$ constraint will have the least value.

2.3 Sequential Optimization and Reliability Assessment (SORA)

SORA employs a single-loop strategy with series optimization and reliability assessment. The optimization and reliability assessment are performed separately in each SORA cycle. Therefore, it is not necessary to assess the reliability of the optimization structure that facilitates the design process [24].

The main concept in SORA is to transfer and replace the boundary of the violated constraints in the possible direction and prevent the shift of the satisfied constraints. In this method, new MPP is obtained by reliability assessment based on MPP and the optimal points of the previous cycle. As a result, a transition vector similar to Eq. (10) is obtained. It should be considered that each probabilistic constraint has its own transition vector thereby specific MPP. When the obtained MPP is close enough to the optimal point resulting from the optimizer in the previous cycle, the transition vector in each cycle will tend to be zero, leading to the non-shift of the boundary of the constraint. Such a process will frequently be repeated in each cycle until meeting all the probabilistic constraints.

$$S_j^k = \vec{\mu}^k_x - \vec{X}_{MPP}^k$$ (10)

where $\mu^k_x$ is the vector for the mean value of the indeterminate variables and $X_{MPP}^k$ represents MPP related to $j$th probabilistic constraint in each cycle. Using SORA, the formulation of the optimization problem is expressed as follows.

Minimizing $\mu^k_x, d^k$
Relative to $G_j(\vec{d}^k, \vec{\mu}^k_x, \vec{\mu}_p^k) \leq 0$
$\vec{d}^l \leq \vec{d} \leq \vec{d}^u, \vec{\mu}^k_x \leq \vec{\mu}^k \leq \vec{\mu}^u$ (11)

where $\vec{d}^k$ is the vector for the determinate design variables, $\vec{\mu}^k_x$ is the vector for the mean indeterminate variables, and $\vec{\mu}_p$ is the mean value of indeterminate design parameters. Fig. 1 shows the flowchart of the SORA method.
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Fig. 1. Flowchart of the SORA method

3 DESIGN OPTIMIZATION THROUGH NEURAL NETWORKS META-HEURISTIC

A surrogate model is a mathematical estimate of a costly computational model or a complex simulation. In other words, the meta-heuristic model is a modeling a model [14]. Different mathematical formulas are used for surrogate models. Some of these formulas are suitable for general estimation and indeed represent the entire design space while others are appropriate for local estimation (a part of the design space). In addition, the accuracy of metamodels depends on several factors, such as dimensions, problem space, and the number of sampling points for the training of the metamodel. Making a metamodel for indeterminate design issues has its own complexity, as the space of the problem will be severely non-linear with increasing indeterminacies and the interactions between themes involved in the design. Hence, different strategies have been developed for metamodels applications, which are referred to as “metamodel management” [14]. An overview of the process of making a metamodel is presented in Fig. 2.

The multilayer neural network metamodel is one of the best surrogate models that by appropriate adjustment of its parameters can be very suitable for approximating very complex spaces compared to other metamodels [25]. Therefore, in the present study, the neural network metamodel is used adaptively such that in each repetition of the reliability-based design optimization process by SORA, the resulting design point is added to the pre-determined point for training the metamodel and then the metamodel is rebuilt, updated in the next replication, and used in the reliability analysis.

4 THE PROPOSED METHOD

As stated in the introduction section, the proposed method in this paper is based on the combination of the neural network metamodel with a new methodology for improving SORA performance, which is discussed in the following.
4.1 ASORA using surrogate-assisted ASORA (SA-ASORA) model

As expressed earlier, SORA separates the double-loop structures by making the reliability assessment cycle series and its optimization [24]. The key concepts related to SORA are the shift of the violated probabilistic constraints boundary and non-shift of the satisfied probabilistic constraints boundary. Although in SORA the vector variations in each cycle for the satisfied constraints will be zero or close to zero, as long as all probability constraints are not satisfied, a reliability analysis is iterated for all probabilistic constraints. However, it should be noted that the reliability analysis will impose computational burden for satisfied constraints. Moreover, being exactly zero is not realized for the transition vector variations obtained in each repetition for the satisfied probabilistic constraints. Since the reliability analysis is not performed only with MPP, it requires the points obtained in each replication in addition to the previous MPP replication [22]. In this study, ASORA was provided to prevent the impact of such elements and to reduce the number of recalls for probabilistic constraints functions. In ASORA, using Eq. (12), it can be evaluated which probabilistic constraint is still active or inactive. Inactiveness of the probabilistic constraint represents satisfying the related probabilistic constraint. Therefore, the variations in the vector of satisfied constraints will be zero (Eq. 13), and the next replication MPP will correspond to the current replication MPP (Eq. 14).

\[
G \left( -\beta \left( \frac{\nabla G(u_{MPP})}{\| \nabla G(u_{MPP}) \|} \right) \right) \geq 0 \tag{12}
\]

where \( u_{MPP}^{k-1} \) represents the previous replication MPP in the normal standard space, \( \nabla G \) is the gradient vector performance function at the point \( u_{MPP}^{k-1} \), and \( k \) indicates the \( k \)th replication.

\[
S_j^k = \tilde{S}_j^{k-1} + \Delta \tilde{s}, \Delta \tilde{s} = 0 \tag{13}
\]

\[
u_{MPP}^k = u_{MPP}^{k-1} \tag{14}
\]

\[
X_{MPP}^k = X_{MPP}^{k-1} \tag{15}
\]

where \( \Delta \tilde{s} \) is the transition vector which is zero for inactive performance function.

The points expressed for a system with two indeterminate design variable and three probabilistic constraints are presented in Fig. 3. According to this figure, the reliable MPP point is located on the boundary of the first and second constraints, after the constraint transition. It can be inferred from Fig. 3 that the MPP point is obtained only by the first and second probabilistic constraint transition and the ultimate result is not affected by the transition of the third probabilistic constraint.
This method reduces significantly the number of function recalls in reliability assessment. In addition, if the probabilistic constraints are still active, the related transition vector is determined based on SORA method and Eq. (10).

Fig. 3. Concept of the constraints boundary transition in ASORA method

To further modulate the computational load in reliability-based design, in the proposed method, the individual neural network metamodels was constructed by sampling the design space through Latin cubic design for target functions and constraints. Then, the resulted point in each ASORA replication is added to the initial extracted points set for training the metamodel and the metamodel is rebuilt for use in the next replication. Fig. 4 illustrates the flowchart of ASORA using SA-ASORA. In the indeterminate design, the range intended for sampling variables and indeterminate design parameters is given by Eq. (15). Here, the indeterminate design parameters are considered as the input in the modeling in addition to the indeterminate design variables.

\[ \mu_i - 3\sigma_i \leq X_i \leq \mu_i + 3\sigma_i, \quad i = 1, \ldots, n \]  

(16)

Fig. 5 presents the cubic Latin sampling points for a problem with two indeterminate design variables. To ensure the accuracy of the cubic Latin sampling, the minimum number of samples required for a problem corresponds to n design variable as [26]:

\[ n_s = \frac{(n+1)(n+2)}{2} \]  

(17)

where ns is the number of minimum required sampling of the problem space. It has to be noted that by increasing the sampling, the accuracy of the metamodel construction can be increased, while a compromise must be made within the sampling number and calculation burden. However, the excessive increase in the sampling rate results in an “over-adaptation” error in some metamodels, such as neural networks. To prevent this, the prepared metamodel should be tested in the designing process before being used on a set of random points of the problem space such that to ensure the accuracy of the metamodel quality throughout the design space.

Before using a metamodel in the design process, its quality should be assessed. One of the most commonly used criteria for this purpose is the mean squared error (MSE) defined in Eq. (17), which is also utilized in this paper.

\[ MSE = \frac{1}{N} \sum_{i=1}^{N} (f(x_i) - y_i)^2 \]  

(18)

where N represents the number of points used for metamodel assessment, f(.) is the main function value and y indicates the predicted value in terms of the input points. Finally, for each user, movies that have the highest score are proposed to the them.
Fig. 4. Flowchart of Surrogate-Assisted ASORA

Fig. 5. Characteristic sampling points for a two-variable problem
5 **Mathematical Examples**

In this section, the strategy proposed for design optimization was based on the reliability of several single-subject, multi-subject, and multi-target mathematical problems. Since the presented examples are the math functions with no computational burden, the number of recalls of the target functions and probabilistic constraints is presented in each method to compare the function of the different methods.

5.1 **Single-subject RBDO problems**

5.1.1 **Example 1**

Single-subject RBDO is a common problem in the RBDO methods that is defined as follows [27]:

Minimizing: \( f(\bar{d}) = d_1 + d_2 \)

Relative to \( \Pr(G_i(\bar{X}) > 0) \leq \Phi(-\beta_i), i = 1, \ldots, 8 \)

\[
G_1(\bar{X}) = \frac{x_1^2 + x_2^2}{20} - 1,
\]

\[
G_2(\bar{X}) = \frac{(x_1 + x_2 - 5)^2}{30} + \frac{(x_1 - x_2 - 12)^2}{120} - 1,
\]

\[
G_3(\bar{X}) = \frac{x_1^2 + 8x_4 + 5}{80} - 1,
\]

\[
\beta_1 = \beta_2 = \beta_3 = 3
\]

\( 0 \leq d_i \leq 10, i = 1, 2 \)

\( X_i \sim N(\mu_i, \sigma_i), i = 1, 2 \)

\( \bar{d}_0 = [3.1107, 2.0606] \)  

(19)

where \( d_i \) and \( X_i \) represent the determinate and indeterminate design parameters, respectively; \( F \) is the target function; \( G_i \) is the design constraints; \( \Phi \) is the cumulative distribution function; \( \beta \) is the reliability index; and \( \mu_i \) and \( \sigma_i \) denote the mean and standard deviations of indeterminate design variables, respectively. In this problem, the mean value is considered equal to the value of the corresponding determinate design variable and the standard deviation as 0.3. The reliability index for all probabilistic constraints is also assumed to be 3. All the random variables are independent with a normal distribution.

In order to increase the convergence speed and to achieve an inclusive optimal response, a surrogate model was built for the target function. Next, each problem constraint was updated using a neural network metamodel and used in the design process. To make the initial metamodel, 15 design points were extracted in the entire design space by a cubic Latin sampling method and the target function and the corresponding constraints were calculated for each value. Then, the set of these design points were used to train the metamodel neural network. The design space of the probabilistic constraints and the extracted points for the initial training of the metamodel are shown in Fig. 6. In each iteration of SA-ASORA, the points extracted obtained are added to train the metamodels. Via the new training points set, the metamodel is retrained to make new metamodel that is used in the next RBDO.

*Fig. 6. Design space and metamodels initial train points for Example 1*
In solving this problem, the optimal point obtained from determinate design was considered as the start point of the indeterminate design \(d^0\) in the design formulation. Table 1 presents the results of problem design defined through SA-ASORA. The results provided for RIA, PMA, and SORA are consistent with those present in [4], indicating the accuracy of the reliability-based design algorithms performance. The results show that in simple problems, SORA outperforms the SA-ASORA. However, due to the lower number of recalls of the target and constraint functions in SA-ASORA compared to the RIA and PMA, the present method will have better performance.

Table 1. Comparative RBDO results for Example 1

| Methods       | Replication function | Target function | Design parameters | Number of functions recall |
|---------------|----------------------|-----------------|-------------------|----------------------------|
| RIA           | -                    | 6.72            | (3.44, 3.28)      | 30                         | 2,805                      |
| PMA           | -                    | 6.72            | (3.44, 3.28)      | 35                         | 3,296                      |
| SORA          | 4                    | 6.72            | (3.44, 3.28)      | 60                         | 368                        |
| SA-ASORA      | 4                    | 6.72            | (3.44, 3.28)      | 125                        | 445                        |

2.1.5 Example 2

This problem includes 10 random variables and 8 probabilistic constraints with the reliability of 3 for each one [28]. The modeling of the problem was defined in Eq. (19). All the random variables are independent with a normal distribution. In the whole design space, 50 points were extracted via Latin cube sampling in which the initial surrogate models are created based on the target function and constraint. These points set used for training the initial metamodel will be updated in each SA-ASORA replication.

Minimizing: 

\[ f(\bar{d}) = d_1^2 + d_2^2 + d_3d_4 - 14d_1 - 16d_2 + (d_3 - 10)^2 + 4(d_4 - 5)^2 + (d_5 - 3)^2 + 2(d_6 - 1)^2 + 5d_7^2 + 7(d_8 - 11)^2 + 2(d_9 - 10)^2 + (d_{10} - 7)^2 + 45 \]

Relative to 

\[ G_1(\bar{X}) = \frac{\Pr\{G_1(\bar{X}) > 0\} - \phi(-\beta^*_1), i = 1, ..., 8}{105} - 1, \]

\[ G_2(\bar{X}) = -8X_1 + 8X_2 - 17X_3 + 2X_8 \]

\[ G_3(\bar{X}) = -8X_1 + 2X_2 - 5X_5 - 2X_{10} - 1 \]

\[ G_4(\bar{X}) = \frac{6X_1^2 + 8X_2 + (X_3 - 6)^2 - 2X_4 - 1}{40} - 1, \]

\[ G_5(\bar{X}) = \frac{5X_1^2 + 8X_2 + 2X_6 - 5X_8}{12} - 1, \]

\[ G_6(\bar{X}) = \frac{0.5X_1 - 6)^2 + 2X_2 - 6)^2 + 3X_3 - X_6}{12} - 1, \]

\[ G_7(\bar{X}) = X_7^2 + 2(X_2 - 2)^2 - 3X_1X_2 + 14X_5 - 6X_6, \]

\[ G_8(\bar{X}) = -3X_1 + 6X_2 + 12(X_3 - 0)^2 - 7X_10, \]

\[ \beta^*_1 = \ldots = \beta^*_8 = 3 \]

\[ 0 \leq d_i, i = 1, ..., 10 \]

\[ X_i \sim N(d_i, 0.02), i = 1, \ldots, 10 \]

\[ \bar{d}^0 = [2.17, 2.36, 8.77, 5.10, 0.99, 1.43, 1.32, 9.83, 8.28, 8.38] \] (20)

Starting from the determinate design point \((d^0)\) in the formulation, the second example was designed through different methods. The design results based on different methods are provided in Table 2. The results of RIA, PMA, and SORA are consistent with the findings presented in [8]. In all obtained results, the reliability for each probabilistic constraint is provided in the optimal point. In this example, the number of recalls of the target function is the same for both SORA and SA-ASORA. Moreover, less number of recalls can be observed for the functions in the SORA indicating the improved performance. Table 2 presents the better performance of SA-ASORA in solving the reliability-based complicated design problems compared to RIA, PMA, and SORA.

5.1.3 Example 3: Speed reducing

The last example of single-subject problems is the well-known speed reducing the problem, which is schematically illustrated in Fig. 7 [28]. This problem has 7 random variables and 11 probabilistic constraints. The target function is defined as the minimum weight and the constraints are related to constant bending and stress, longitudinal displacement, shaft stress, and geometry. The design parameters include the gear width \((X_1)\), the teeth size \((X_2)\), the number of teeth in the pinion girth \((X_3)\), the distance between the bearings \((X_4, X_5)\), and the diameter of axes \((X_6, X_7)\). The variables are considered as random and independent variables with a standard deviation of 0.005. The formulation of this problem is provided in Eq. (20).
Table 2. Comparative RBDO results for Example 2

| Methods | Number of replication | Target function variables | Number of function recalls |
|---------|-----------------------|---------------------------|---------------------------|
| RIA     | -                     | -                         | 240                       |
| PMA     | -                     | 27.7465                   | 40,450                    |
| SORA    | 2                     | (2.134, 2.33, 8.7, 5.102, 0.922, 1.445, 1.388, 9.809, 8.155, 8.475) | 347                       |
| SA-ASORA| 2                     | (2.134, 2.33, 8.7, 5.102, 0.922, 1.445, 1.388, 9.809, 8.155, 8.475) | 1,223                     |

Through Latin cubic design, 50 points are determined from the design space to use in constructing the initial metamodel with the desired precision of the constraints and the target function of the problem. Then, a single multi-layer neuronal network metamodel is built for each constraint and function. The created metamodels are run such that the points resulting from each SA-SORA replication are added to the training points, updated based on the new points in the metamodels, and then utilized in the succeeding RBDO loop. The optimal points of the determinate design, defined by $d^{(0)}$, are regarded as the start point of the indeterminate design trend. The results of different methods for this RBDO problem are compared in Table 3. In this table, the PMA and SORA results are consistent with the results of Kouvang et al. [10]. At the optimal points, the reliability was met for all probabilistic constraints. Table 3 shows the convergence of SA0ASORA in two iterations of the RBDO loop, indicating the less computational burden of this method compared to the other methods. As a result of using surrogate models instead of main target functions and constraints of the problem, the speed is higher in the present technique; however, it is negligible because of the simplicity of the functions.

![Image of a speed reducer](20]

Minimizing: $f(x) = 0.7854d_1d_2^2(3.3333d_3^2 + 14.9334d_4 - 43.0934) - 1.508d_1(d_4 + d_7) + 0.7854(d_4d_8 + d_5d_7)$

Relative to $Pr[G(X) > 0] \leq \Phi(-\beta),\dot{i} = 1, \ldots, 11$

\[
\begin{align*}
G_1(X) &= \frac{27}{X_1^2X_2X_3^2} - 1, \\
G_2(X) &= \frac{397.5}{X_1X_2X_3^2} - 1, \\
G_3(X) &= \frac{1.93X_3^3}{X_2X_3X_6^2} - 1, \\
G_4(X) &= \frac{1.93X_3^3}{X_2X_3X_7^2} - 1, \\
G_5(X) &= \frac{(745X_4)}{X_2X_3}^2 + 16.9 \times 10^6 \sqrt{0.1X_6^3 - 1100}
\end{align*}
\]
Furthermore, the number of probabilistic constraint recalls in SA-SORA declined significantly compared to the SORA, which directly affects the computational burden reduction. In comparison, the number of probabilistic constraint recalls in PMA is much more than other methods, although it requires only 75 target function recalls. Moreover, in this problem, RIA did not successfully converge to the optimal response.

5.2 Multi-objective RBDO problem

The multi-objective problem considered in this part is based on the reliability of a truss, schematically shown in Fig. 8 [29]. The purpose of presenting this double-objective problem is to demonstrate the effect of reliability on the possible responses set. This problem was solved for different reliability indices and the set of possible responses was provided by drawing the results as a beam front. In this way, the user is allowed to compromise between the reliability level and the set of possible responses.

In this problem, the design variables include cross-section diameter \(d\) and the height of the structure \(H\). The parameters include a vertical force \(P\), structure width \(B\), modulus of elasticity \(E\), and the thickness \(t\). The main objective of this problem is to minimize the volume and vertical displacement of the structure under tensile and bending constraints, the formulation of which is given in Eq. (21).

\[
\begin{align*}
G_6(\vec{X}) &= \bigg(\frac{745X_2}{X_2X_3}\bigg)^2 + 157.5 \times 10^6 \frac{0.1X_2^3 - 850}{1}, \\
G_7(\vec{X}) &= X_2X_3 - 40, \\
G_8(\vec{X}) &= 5 - \left(\frac{X_1}{X_2}\right), \\
G_9(\vec{X}) &= \left(\frac{X_1}{X_2}\right) - 12 \\
G_{10}(\vec{X}) &= \frac{1.5X_6 + 1.9}{X_4} - 1,
\end{align*}
\]

\[
\beta_1 = \ldots = \beta_7 = 3
\]

\[
\begin{align*}
2.6 \leq d_1 \leq 3.6 \\
0.7 \leq d_2 \leq 0.8 \\
17 \leq d_3 \leq 28 \\
7.3 \leq d_4 \leq 8.3 \\
7.3 \leq d_5 \leq 8.3 \\
2.9 \leq d_6 \leq 3.9 \\
5.0 \leq d_7 \leq 5.5 \\
X_i \sim N(d_i, 0.005), i = 1, \ldots, 7
\end{align*}
\]

\[
\bar{d}_0 = [3.5, 0.7, 17, 7.3, 7.72, 3.35, 5.29]
\]

Minimizing: \(f_1(d, H): volume = 2\pi dt\sqrt{B^2 + H^2}\)

| Methods  | Number of Replications | Target Function | Variables | Functions Recalls |
|----------|------------------------|-----------------|-----------|------------------|
|          |                        |                 |           |                  |
| RIA      | -                      | -               | -         |                  |
| PMA      | -                      | 2950.107        | (3.423, 0.7, 17.0, 7.3, 7.6765, 3.335, 5.271) | 75, 1,2450 |
| SORA     | 3                      | 3038.611        | 5.271     | 120, 1,506      |
| SA-SORA  | 2                      | 3038.612        | (3.576, 0.7, 17.0, 7.3, 7.754, 3.365, 5.301) | 120, 1,119 |
| ASORA    |                        |                 | (3.500, 0.7, 17.0, 7.3, 7.720, 3.350, 5.290) |                  |

In order to construct surrogate models for target functions and probabilistic constraints, the inputs and outputs required for the initial training of metamodels were provided through 30 samplings of the problem space. Here, the constructed metamodels play the role of the target function and probabilistic constraints in the SA-SORA for indeterminate design. The set of points used for the initial training of the metamodels will be updated based on the optimal point derived from each EBDO loop repetition, then the metamodels will be constructed for the subsequent replication. In this problem, NSGA-II multi-objective genetic algorithm is utilized as the optimizer. The beam front derived from the reliability-based design for the reliabilities of 2, 2.6, and 3 within various methods is illustrated in Fig. 9.
Relative to \( P \sim (150,5) \text{kN} \),
\( B \sim (750,10) \text{mm} \),
\( E \sim N(2.1 \text{e}5,5\text{e}3) \frac{N}{\text{mm}^2} \),
\( t \sim (2,0.4) \text{mm} \)
\( G_0(X): S \leq S_{\text{max}} \)
\( 20 \leq d \leq 80 \)
\( 200 \leq H \leq 1000 \)
\( S = \frac{P\sqrt{B^2 + H^2}}{2\pi dh} \)
\( S_{\text{crit}} = \frac{\pi^2 E(t^2 + d^2)}{8(B^2 + H^2)} \)
\( S_{\text{max}} = 400 \text{MPa} \)

As shown in Fig. 9, increasing the reliability index, the corresponding beam front becomes farther away from the beam front derived from the determinate design. As a result, by increasing the reliability, the designer will have limitations in selecting the possible designs. Therefore, achieving a high-reliability design will cost moving away from optimality, requiring the designer to compromise between the optimality criteria and reliability.

Since the criterion for stopping the optimizer of the utilized genetic algorithm is the number of generations, the number of recalls for target functions and constraints is the same in the optimization section of all methods. Therefore, in order to compare the performance of different methods, only the number of probabilistic constraints recalls is provided in the reliability analysis in Table 4. Since PMA and RIA are double-loops methods with the reliability assessment in each optimization replication, the number of probabilistic constraints in this method is not comparable with other sequential methods and is not presented in Table 4. Furthermore, based on Table 4, it is indicated that the number of probabilistic constraints recalls in reliability assessment in SA-ASORA is lower compared to the other methods. Consequently, the augmented method shows better performance in reducing the computational burden in solving RBDO problems.
6 Practical Engineering Example

Here, designing a launch vehicle was defined as an engineering practical example with a significant computational burden [25]. The problem was formulated into the RBDO problem and solved through augmented SA-ASORA.

The considered launch vehicle is a two-stage liquid-fuel launch vehicle in which two directions of aerodynamics and route simulation are exchanging information in its route design. Flight conditions including altitude, Mach, and angle of attack, and aerodynamic coefficients are among the points changing between these two issues. The aerodynamic coefficients are extracted by entering flight conditions and configuration of the system into the Missile Datcom software. To analyze the system performance, a flight simulation software with three degrees of freedom was codified. In this code, the transition motion equations of the device are formulated in the body machine and then the required performance parameters are obtained by integrating them and converting the required coordinates. In this simulation, the 1976 standard atmospheric model and elliptical ground model were utilized. Achieving mission objectives was also performed through offline guidance route design.

The mission of the launch vehicle is to consider certain load in a circular circuit of 200 km with a reliability of 90% for deviation from the orbital height in the presence of indeterminacies such as the trusts and dry mass of each stage in the system. The formulation of the problem is provided as an RBDO problem in Eq. (22).

Minimizing $f = \text{total mass}$

Relative to $\Pr(|H_t - H_d| \leq 0.5 H_d) \leq \Phi(-\beta)$

$\beta = 1.28$

$|V_t - V_d| \leq 10 \text{ms}^{-1}$

$|\gamma_t| \leq 1 \text{ deg}$

Indeterminacies:

$T_{1st\_S.} \sim \mathcal{N}(0,1000) \text{ N}$

$T_{2nd\_S.} \sim \mathcal{N}(0,100) \text{ N}$

First phase dry weight $\sim (0.70) \text{ kg}$

Second phase dry weight $\sim (0.10) \text{ kg}$

where the target function is the total mass of the system at the launching moment. $H_t$, $V_t$, and $\gamma_t$ are height, velocity, and route angle at the moment of load injection into the orbit, and optimal amounts of height and orbital speed, respectively. Trust error in the first and second phase ($T_{2nd\_S.}$ and $T_{1st\_S.}$) and dry mass error in the first and second phases (Dry Mass$_{2nd\_S.}$ and Dry Mass$_{1st\_S.}$) were considered as indeterminacies in this problem with a normal distribution.

The path design of the launch vehicles is based on the parametric adjustment of the pitch angle of the vehicle over the flight. Hence, in this problem, a launch vehicle’s pitch rate profile similar to Fig. 10 is assumed with the parameters
represented in “Table 5” as problem design variables. The variables of the launch vehicle’s pitch rate profile for the first phase include the coefficients of an order 3 polynomial, for which the upper and lower limits are obtained by fitting the curve to a pitch rate profile. The burning time of each stage of the system and the duration of the second stage engine shutdown after separation the first stage (the phase after separation of the first phase from the second) are other variables considered in this reliable trajectory design problem. By sampling the problem design space through a cubic Latin method, with 9 determinate design variables and 4 indeterminate design parameters, the inputs and outputs required for the initial training of the neural network metamodels were provided from the objective function and problem constraints. For this purpose, 2000 samplings were performed while checking the accuracy of the constructed models through 1000 random points throughout the problem space. NSGA-II multi-objective genetic algorithm was used in solving this problem. In this way, the set of points used for the initial training of the metamodels will be updated in each RBDO loop replication based on the optimal point derived from the optimizer. Then, the metamodels will be constructed for the next replication.

Fig. 10. A typical illustration of the launch vehicle’s pitch rate profile

Table 5. Design variables of launch vehicle trajectory design problem

| Variables                              | Unit | Upper limit | Lower limit |
|----------------------------------------|------|-------------|-------------|
| First stage burning time               | S    | 100         | 150         |
| Second stage burning time              | S    | 200         | 250         |
| Shut down duration within the stages   | S    | 0           | 60          |
| First stage pitch rate                 | rad/s| -2.5e-8     | -1.0e-8     |
|                                        |      | 3.0e-6      | 6.0e-6      |
|                                        |      | -5.0e-4     | -2.0e-4     |
| Second stage pitch rate                | rad/s| 1.0e-3      | 3.0e-3      |
|                                        |      | -0.006      | -0.002      |
|                                        |      | -0.002      | -0.0005     |

Table 6. Comparative RBDO results for launch vehicle trajectory design problem

| Method     | Replication Number | Total Mass (Kg) | Variables                                    | Constraint Recalls | Duration (hour) |
|------------|--------------------|-----------------|----------------------------------------------|--------------------|----------------|
| SORA       | 57                 | 21886           | (127.7, 202.6, 2.0, -1.9e-8, 4.3e-6, -3.55e-4, 0.0019, -0.0052, -9.64e-4) | 1960               | 7              |
| SA-ASORA   | 7                  | 21607           | -3.66e-4, 0.0012, -0.0052, -9.95e-4          | 159                | 1              |

Similar to the multi-objective RBDO problem, since the criterion for optimization stop is the number of generation, the number of the target functions and constraints recalls in the optimization section is the same for all methods. Therefore, the criterion for comparing the performance of different methods in solving this problem is just the number of probabilistic constraints recalls in the reliability assessment section. Moreover, in solving the mathematical examples in the previous section, it was observed that SORA and SA-ASORA methods outperform RIA and PMA particularly in solving complicated problems especially in terms of the computational burden. Hence, the reliability-based design problem of the launch vehicle route presented in this section was solved through SORA and SA-ASORA and the results were provided in Table 6. According to this table, a significant reduction is observed in the number of probabilistic constraints recalls in SA-ASORA compared to SORA. In addition, this reduction together with surrogate models instead of using the main function reduces the time needed for RBDO design run from several days to some hours.
7 Conclusion and future work

Sequential Optimization and Reliability Assessment (SORA) method, as an indeterminate reliability-based design, is one of the most effective single-loop methods for achieving the optimal response in RBDO problems. In the present article, an innovative method called Surrogate-Assisted SORA (SA-SORA) was proposed by integrating an augmented SORA (ASORA) and surrogate models instead of the main function. By not recalling the probabilistic constraints functions satisfied in the process of reliability assessment and accelerating the implementation of functions through surrogate models, this method resulted in the significant reduction in the computational struggles in the RBDO problems. Moreover, in the proposed technique, the optimal points derived from each RBDO replication is added to the metamodels training points and the metamodel is updated again. Accordingly, when the precision of the primitive metamodel is not high enough for the convergence of the optimizer, the amount of target functions and constraints at this point is corrected to prevent the wrong optimizer convergence to that point in the succeeding RBDO replication. Thus, the convergence of the optimizer to the points caused by the inaccuracy in the metamodel in those areas is avoided, which leads to an increase in the probability of reaching a comprehensive optimum.

The efficiency and feasibility of the augmented method were discussed by solving several problems of varying complexity. The results indicate a significant improvement in the performance of the augmented method, especially in dealing with complex and heavy engineering problems. Furthermore, according to the results of solving complex practical problems such as the launch vehicle path design presented in this paper, a reduction of more than 50% is observed in computational burden. This issue demonstrates the importance of the ASORA in solving multi-subject problems in which each issue has considerable computational burden alone and even in a determinate design.

Extending the presented method in larger dimensions and multi-objective RBDO problems with high-level analytical models are the objective for future work. Moreover, the proper metamodel management in guiding towards inclusive optimization and increasing their accuracy along with the use of major models are considered as the challenges for the current research and as the future trend for the authors of this article.

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