Local gauge theory and coarse graining

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Abstract. Within the discrete gauge theory which is the basis of spin foam models, I study the problem of macroscopically faithful coarse graining. Macroscopic data is identified; it contains the holonomy evaluation along a discrete set of loops and the homotopy classes of certain maps. When two configurations share this data they are related by a local deformation. The interpretation is that such configurations differ by “microscopic details”. In many cases the homotopy type of the relevant maps is trivial for every connection; two important cases in which the homotopy data is composed by a set of integer numbers are: (i) a two dimensional base manifold and structure group \( U(1) \), (ii) a four dimensional base manifold and structure group \( SU(2) \). These cases are relevant for spin foam models of two dimensional gravity and four dimensional gravity respectively. This result suggests that if spin foam models for two-dimensional and four-dimensional gravity are modified to include all the relevant macroscopic degrees of freedom—the complete collection of macroscopic variables necessary to ensure faithful coarse graining—, then they could provide appropriate effective theories at a given scale.

1. Connections (modulo gauge) in terms of local holonomy variables

Let \((E, \pi, M)\) be a principal \( G \)-bundle over a \( d \)-dimensional base space \( M \) and \( A_\pi \) be the space of connections on it.

In order to introduce a concept of measuring scale and locality we use a smooth triangulation \( \Delta \) of \( M \); alternatively, we can use a cellular decomposition that refines a smooth triangulation. This decomposition is inherited by the bundle.

**Figure 1.** (a) The smooth \( G \)-bundles \( \pi_{\nu_1} \) and \( \pi_{\nu_2} \) intersect at the smooth \( G \)-bundle \( \pi_\tau \).
(b) Paths and loops in \( \mathcal{P}(\nu) \).
(c) \( \text{Sd}(\nu) \), the baricentric subdivision of \( \nu \). We also show \( s \), the support of \( B \) in the example.
Definition 1  Decomposition of the bundle (see figure 1.a):

- The principal $G$-bundle over $M$, $(E_{\Delta}, \pi_\Delta, M_\Delta)$, is called $\Delta$-smooth if its restriction to $\pi_\Delta^{-1}$ for each simplex $\nu$ of the triangulation is the restriction of a smooth bundle to an open neighborhood of $\pi_\Delta^{-1}$.
- $A^{\infty}_{\pi_\nu}$ is the space of smooth connections in the bundle $\pi_\tau = \pi|_{\tau \in (|\Delta|, \phi)}$ (restrictions to $\pi_\tau$ of connections which are smooth in an open neighborhood of the subbundle $\pi_\tau$ of $\pi$, where $\tau$ is a simplex in the triangulation).
- $\tilde{\times}_{\tau \in (|\Delta|, \phi)} A^{\infty}_{\pi_\tau}$ is the subset of the cartesian product defined by the compatibility condition $\sigma \subset \tau \Rightarrow A_\sigma = A_\tau|_{\sigma}$ for every two simplices of the triangulation. We will write $A^\infty_\pi = \tilde{\times}_{\tau \in (|\Delta|, \phi)} A^{\infty}_{\pi_\tau}$.
- $A^{\infty}_{\Delta}/G^{\infty}_{\Delta, \pi}$ is the group of $\Delta$-smooth bundle equivalence maps which consists of $\Delta$-smooth bundle maps which induce the identity map on the base space $M$. We will write $A^\infty_\pi/G^{\infty}_{\Delta, \pi} = A^{\infty}_{\Delta}/G^{\infty}_{\Delta, \pi}$.

Now I present local holonomy data for the bundle and the connection. It is the application of a construction of Barrett and Kobayashi to the subbundles over the simplices.

In simplex $\nu$ of the triangulation choose its baricenter $C\nu$ as base point, and consider the space $\mathcal{L}(C\nu, \nu)$ of piecewise smooth oriented $C\nu$-based loops modulo reparametrization and modulo retracing. It turns out that $\mathcal{L}(C\nu, \nu)$ is a group. Once we have identified $G$ with $\pi^{-1}(C\nu)$ the holonomy of a given connection $\mathcal{L}(C\nu, \nu)$ is a group element $\text{Hol}(l, A) \in G$. Moreover, if we fix the connection the holonomy map gives us a group homomorphism

$$\text{Hol}_A \equiv \text{Hol}(-, A) : \mathcal{L}(C\nu, \nu) \to G.$$  

Gauge transformations act on holonomies only at the base point and the action is by conjugation.

Holonomy evaluations are specified by points of the space $A/G^{\infty}_{\Delta, \pi}$, the space of connections modulo the group of gauge transformations whose restriction to the fiber $\pi^{-1}(C\nu)$ is the identity. Holonomy evaluations characterize the bundle over $\nu$ (modulo bundle equivalence) and the connection (modulo gauge); this is an application of a theorem of Barrett [1].

Theorem 1 (Barrett, Kobayashi)  Let $H^\nu : \mathcal{L}(C\nu, \nu) \to G$ be a group homomorphism which is smooth\(^1\). Then, there is a differentiable principal $G$-bundle $(E_{\nu}, \pi_{\nu}, \nu)$, a point $b \in \pi^{-1}(C\nu)$, and a connection $\mathcal{A}_\nu \in A_{\pi_{\nu}}$ such that $H^\nu = \text{Hol}_A$. The bundle and the connection are unique up to a bundle equivalence transformation.

The theorem does not only state the existence of the bundle and the connection; its proof is constructive. Here is a sketch of the main idea. The total space is given by

$$E_{\nu} = G \times \mathcal{P}_{C\nu} / \sim_{\text{Hol}^\nu}$$

where $\mathcal{P}_{C\nu}$ is the space of paths in $\nu$ (modulo reparametrization and retracing) whose source is $C\nu$. The idea behind the equivalence relation is that a pair $(g, p) \in G \times \mathcal{P}_{C\nu}$ gives a point on the fiber over $t(p)$, the target of $p$; $g$ specifies a point on the fiber at the beginning of the path and it is parallel transported to the end of the path. One only has to acknowledge that

\(^1\) The definition of smoothness is axiom H3 in Barrett’s article, stating that any smooth finite dimensional family of loops leads to a smooth submanifold in $G$ after composition with $H^\nu$. See [1] for a detailed explanation.
(g₁, p₁) and (g₂, p₂) determine the same point if t(p₁) = t(p₂) and the parallel transports agree, \( \text{PT}_{p₁} g₁ = \text{PT}_{p₂} g₂ \). This last condition can also be written as \( g₁ H' (p₂⁻¹ \circ p₁) = g₂ \).

If this theorem is applied to each simplex of the smooth triangulation, the connection and the bundle will be described by means of holonomies. However, a collection of such holonomy evaluations for each simplex will be related to a bundle and a connection on it only if certain compatibility conditions hold. The pasting of descriptions over the different simplices needs extra data. The extra data glues the bundle over a simplex to the bundle over any simplex that intersect it.

Consider two simplices of the smooth triangulation such that \( \tau \subset \nu \). Thus, for each point \( x \in \tau \) the above construction gives two constructions of \( \pi_x : E_{\tau|x} \) and \( E_{\nu|x} \). The glue for these bundles is again parallel transport. There are two groups \( L(C\tau) \) and \( L(C\nu, \nu) \); and there are also paths in \( \nu \) between \( C\tau \) and \( C\nu, P^\nu_{\tau\nu} \). The three spaces can be considered together forming a semigroup; two of its elements can be composed only if the target of the first is the same as the source of the second. Consider all the subsimplices of \( \nu, \tau_i \subset \nu \). The semigroup of paths in \( \nu \) which have sources and targets at \( C\nu \) or any of the \( C\tau_i \) will be called \( \mathcal{P}(\nu) \) (see figure 1.b).

Notice that this semigroup contains \( \mathcal{L}(C\nu, \nu) \) and \( \mathcal{L}(C\tau_i, \tau_i) \) for each \( \tau_i \subset \nu \). Once the fibers over \( C\nu \) and all the \( C\tau_i \)'s have been identified with \( G \), parallel transport is described by assignment of group elements, and parallel transport is a semigroup homomorphism

\[
\text{PT}^\nu : \mathcal{P}(\nu) \to G.
\]

\( \text{PT}^\nu \) contains all the information in \( H^\nu \) and each of the \( H^\tau \); in addition, it induces a further equivalence relation that glues the bundle over \( \nu \) with the bundles over each \( \tau_i \). For example, \((g₁, p₁) \in E_{\nu|x} \) is equivalent to \((g₂, p₂) \in E_{\tau|x} \) if \( g₁ \text{PT}^\nu (p₂⁻¹ \circ p₁) = g₂ \).

One can extend this construction to glue the bundles over all the simplices of the smooth triangulation to obtain a local version of theorem of Barrett and Kobayashi. The ingredients are \( \mathcal{P}(\Delta) \) –the semigroup associated to the smooth triangulation of the manifold- and a parallel transport map which will be called simply PT. \( \mathcal{P}(\Delta) \) is generated by composing paths (modulo reparametrization and retracing) contained in some simplex of the triangulation and with source and target being baricenters of simplices of the triangulation. PT is required to be a \( \Delta \)-smooth semigroup homomorphism. The smoothness condition means that its restriction to paths contained in any simplex is the restriction of a map on paths contained in an open neighborhood of the simplex which is smooth in the sense of Barrett [1]. A detailed study of the abelian case was presented at [2].

**Theorem 2** Let \( M \) be a manifold, \( \Delta \) a smooth triangulation of it, and let \( \text{PT} : \mathcal{P}(\Delta) \to G \) be a \( \Delta \)-smooth semigroup homomorphism. Then, there is a \( \Delta \)-smooth \( G \)-bundle \((E, \pi_\Delta, M)\), a collection of points \( \{b_\nu \in \pi⁻¹(C\nu)\}_{\nu \in \Delta} \), and a connection \( A \in A_{\pi_\Delta} \) such that \( \text{PT} = \text{PT}_A \). The bundle and the connection are unique up to a bundle equivalence transformation.

## 2. Macroscopic variables and faithful coarse graining

The last theorem describes the bundle and the connection from a parallel transport map. Since this theorem also applies to manifolds with boundary, it extends Theorem 1 gluing the bundles over each simplex using the parallel transport data of paths that link the centers of different simplices.

Notice that since bundles over discs are trivial and simplices have the topology of the disc, Theorem 1 applied to each simplex constructs a trivial bundle and a connection on it; that is, the bundle over a given simplex is essentially independent of the parallel transport map used to build it. Different parallel transport maps may yield inequivalent bundles only because they glue these **standard building blocks** in inequivalent ways chosen by the parallel transport data.
Each $\mathcal{P}(\nu)$ is decomposed into subsemigroups that intersect each other as dictated by the simplicial structure of triangulation

$$\nu_1 \cap \nu_2 = \tau \Rightarrow \mathcal{P}(\nu_1) \cap \mathcal{P}(\nu_2) = \mathcal{P}(\tau).$$

This property lets us treat one simplex at a time, and see the non trivial gluing just be seeing how the bundle over a simplex is glued to the bundles over its boundary faces (its lower dimensional subsimplices).

If $\tau \subset \partial \nu$ then Theorem 2 provides $\pi_\nu$, $\pi_\tau$ and a gluing map $I_{\tau \nu} : \pi_\tau \to \pi_\nu$ determined by $\text{PT}_\nu$. We will show that there is macroscopic data that can be extracted from $\text{PT}_\nu$ assuring that if $\text{Data}_\Delta(\text{PT}_1) = \text{Data}_\Delta(\text{PT}_2)$ then $I_{\tau \nu} : \pi_\tau \to \pi_\nu(\text{PT}_1)$ is equivalent to $I_{\tau \nu} : \pi_\tau \to \pi_\nu(\text{PT}_2)$. A realization of this goal in the abelian case was presented at [2].

Below is an example of inequivalent gluings. It shows that $\text{Data}_\Delta(\text{PT}_\nu)$ must include some data a part from a set of parallel transport evaluations along a discrete collection of paths and loops.

**Example:**

Let $G = U(1)$ and $M \sim S^2$ with $\Delta$ a smooth triangulation of it; also let $\nu \in \Delta$ be a triangle and $\tau \in \Delta$ be one of its one dimensional faces. The baricentric subdivision of $\nu$, $\text{Sd}(\nu)$, is shown in figure 1.c.

In $S^2$ we have a parallel transport map $\text{PT}_1$ such that $\text{PT}_1^\nu$ is a flat parallel transport (all holonomies are the identity). In an appropriate local trivialization over $\nu$, $\text{PT}_1^\nu$ would be described by a connection one form $A_1 = 0$. We have a second parallel transport map $\text{PT}_2$ such that for every $\delta \in \Delta$ such that $\delta \neq \nu$ we have $\text{PT}_2^\delta = \text{PT}_1^\delta$.

Let us use the trivialization over $\nu$ according to which $A_1 = 0$. $\text{PT}_2^\nu$ described in the same trivialization is characterized by a connection one form $A_2$ determined by a curvature two form $\Omega$ which is zero everywhere except for the compact support $s \subset \nu$ shown in figure 1.c. The size of $\Omega$, is adjusted in such a way that $\int_{\partial s} \Omega = 2\pi$, where $D$ is the triangle in $\text{Sd}(\nu)$ containing $s$.

All the gluings between bundles over simplices determined by $\text{PT}_2$ agree with those determined by $\text{PT}_1$, except for the gluing of the bundles over $\nu$ and $\tau$ and $\nu$ and $\nu'$ (the triangle which shares $\tau$ with $\nu$). However, the induced total bundles $\pi_{\Delta,1}$ and $\pi_{\Delta,2}$ are inequivalent since their Euler numbers differ by 1.

Notice that the parallel transport maps $\text{PT}_1$ and $\text{PT}_2$ agree when evaluated on any path contained in the 1-skeleton of $\text{Sd}(\Delta)$. Thus, this discrete set of parallel transport evaluations is not capable of detecting the difference between $\text{PT}_1$ and $\text{PT}_2$. It is clear that given any other choice of discrete set of paths, with an appropriate choice of $s$ we can fabricate a parallel transport map $\text{PT}_2$ which agrees with $\text{PT}_1$ when restricted to the chosen paths while inducing an inequivalent bundle.

The gluing between $\pi_\nu$ and $\pi_\tau$ is a change of trivialization of $\pi_\tau$, from the one constructed using $\text{PT}_\nu$ to the one constructed using $\text{PT}_\nu$. As any other change of trivialization it is described by an assignment $g : \tau \to U(1)$. In the case of the $\text{PT}_1$ gluing the assignment is $g(x) = \text{id}$ for all $x \in \tau$, while the $\text{PT}_2$ gluing (which needs to agree at $\partial \tau$) winds around $U(1)$ once. For a study of the abelian case see [2].

Now we describe (up to scale $\Delta$) $\{\{\pi_\nu \to A \pi_\tau\}_{\tau \subset \partial \nu} (\text{mod. equivalence})\} 
\subset A/\text{G}_{\pi_\nu}$ in terms of parallel transport maps. The macroscopic data needed for such description is $\text{Data}_\Delta(\text{PT}_\nu) = (\text{PT}_\Delta^\nu, W^\nu)$ where

$$\text{PT}_\Delta^\nu : \mathcal{P}_\Delta(\nu) \to G$$

is the restriction of $\text{PT}_\nu$ to the discrete semigroup of paths $\mathcal{P}_\Delta(\nu) \subset \mathcal{P}(\nu)$ consisting of paths that fit in the 1-skeleton of $\text{Sd}(\nu)$. And,

$$W^\nu \in \{\text{Homotopy class of map } T_{\partial \nu, \tau} : \tau \to G, \text{ with } T|_{\partial \nu} \text{ fixed}\}_{\tau \subset \partial \nu},$$

where the gluing map $T_{\partial \nu, \tau}$ is constructed form $\text{PT}_\nu$ following a variation of Barrett’s construction [1]. The result is the following:

**Theorem 3 (Faithful coarse graining)** The data $\text{Data}_\Delta(\text{PT}_\nu)$ characterizes

- the gluing maps $\{\pi_\nu \to_{\text{PT}_\nu} \pi_\tau\}_{\tau \subset \partial \nu}$ up to equivalence and
the connection modulo gauge $A \in \mathcal{A}/\mathcal{G}_{\pi_\nu}$ up to a microscopical deformation;

where $A, A' \in \mathcal{A}/\mathcal{G}_{\pi_\nu}$ are said to agree up to a microscopical deformation if they can be deformed to the same (singular) semigroup homomorphism by a homotopy of semigroup homomorphisms which preserves $\text{Data}_\Delta(\text{PT}^\nu)$.

In the cases relevant for two-dimensional and four-dimensional euclidean gravity ([dim$M = 2$, $G = U(1)$] and [dim$M = 4$, $G = SU(2)$] respectively) $W^\nu$ is characterized by a set of integers, one per codimension one face of $\nu$. For many other cases, like the one relevant for three-dimensional euclidean gravity ([dim$M = 3$, $G = SU(2)$]), $W^\nu$ is trivial.

This result suggests that if the current spin foam models for two-dimensional and four-dimensional gravity are modified to include all the relevant macroscopic degrees of freedom –the complete collection of macroscopic variables necessary to ensure faithful coarse graining–, then they could provide appropriate effective theories at a given scale.

A spin foam model for two-dimensional gravity which incorporates extra data regarding the bundle structure was given in [3]. In the same reference a naive spin foam model which ignored the bundle data was shown to yield unphysical results. An explicit relation between the work presented here and [3] will be presented elsewhere.

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3. References

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