A parameterized approximation algorithm for the Multiple Allocation $k$-Hub Center

Marcelo P. L. Benedito, Lucas P. Melo, and Lehilton L. C. Pedrosa

Institute of Computing, University of Campinas, Brazil
{mplb,lehilton}@ic.unicamp.br, lucaspradomelo@gmail.com

Abstract. In the Multiple Allocation $k$-Hub Center (MA$k$HC), we are given a connected edge-weighted graph $G$, sets of clients $C$ and hub locations $H$, where $V(G) = C \cup H$, a set of demands $D \subseteq C^2$ and a positive integer $k$. A solution is a set of hubs $H \subseteq H$ of size $k$ such that every demand $(a, b)$ is satisfied by a path starting in $a$, going through some vertex of $H$, and ending in $b$. The objective is to minimize the largest length of a path. We show that finding a $(3-\varepsilon)$-approximation is NP-hard already for planar graphs. For arbitrary graphs, the approximation lower bound holds even if we parameterize by $k$ and the value $r$ of an optimal solution. An exact FPT algorithm is also unlikely when the parameter combines $k$ and various graph widths, including pathwidth. To confront these hardness barriers, we give a $(2+\varepsilon)$-approximation algorithm parameterized by treewidth, and, as a byproduct, for unweighted planar graphs, we give a $(2+\varepsilon)$-approximation algorithm parameterized by $k$ and $r$. Compared to classical location problems, computing the length of a path depends on non-local decisions. This turns standard dynamic programming algorithms impractical, thus our algorithm approximates this length using only local information. We hope these ideas find application in other problems with similar cost structure.

Keywords: parameterized approximation algorithm · hub location problem · treewidth

1 Introduction

In the classical location theory, the goal is to select a set of centers or facilities to serve a set of clients [25,10,26,12]. Usually, each client is simply connected to the closest selected facility, so that the transportation or connection cost is minimized. In several scenarios, however, the demands correspond to connecting a set of pair of clients. Rather than connecting each pair directly, one might select

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a set of hubs that act as consolidation points to take advantage of economies of scale [30,8,23,31]. In this case, each origin-destination demand is served by a path starting at the origin, going through one or more selected hubs and ending at the destination. Using consolidation points reduces the cost of maintaining the network, as a large number of goods is often transported through few hubs, and a small fleet of vehicles is sufficient to serve the network [9].

Many hub location problems have emerged through the years, that vary depending on the solution domain, whether it is discrete or continuous; on the number of hub stops serving each demand; on the number of selected hubs, and so on [1,16]. Central to this classification is the nature of the objective function: for median problems, the objective is to minimize the total length of the paths serving the demands, while, for center problems, the objective is to find a solution whose maximum length is minimum. In this paper, we consider the Multiple Allocation $k$-Hub Center (MA$k$HC), which is a center problem in the one-stop model [29,42], where clients may be assigned to multiple hubs for distinct demands, and whose objective is to select $k$ hubs to minimize the worst connection cost of a demand.

Formally, an instance of MA$k$HC is comprised of a connected edge-weighted graph $G$, sets of clients $C$ and hub locations $H$, where $V(G) = C \cup H$, a set of demand pairs $D \subseteq C^2$ and a positive integer $k$. The objective is to find a set of hubs $H \subseteq H$ of size $k$ that minimizes $\max_{(a,b) \in D} \min_{h \in H} d(a, h) + d(h, b)$, where $d(u, v)$ denotes the length of a shortest path between vertices $u$ and $v$. In the decision version of MA$k$HC, we are also given a non-negative number $r$, and the goal is to determine whether there exists a solution of value at most $r$.

This problem is closely related to the well-known $k$-Center [26,24], where, given an edge-weighted graph $G$, one wants to select a set of $k$ vertices, called centers, so that the maximum distance from each vertex to the closest center is minimized. In the corresponding decision version, one also receives a number $r$, and asks whether there is a solution of value at most $r$. By creating a demand $(u, u)$ for each vertex $u$ of $G$, one reduces $k$-Center to MA$k$HC, thus MA$k$HC can be seen as a generalization of $k$-Center. In fact, MA$k$HC even generalizes the $k$-Supplier [27], that is a variant of $k$-Center whose vertices are partitioned into clients and locations, only clients need to be served, and centers must be selected from the set of locations.
For NP-hard problems, one might look for an $\alpha$-approximation, that is a polynomial-time algorithm that finds a solution whose value is within a factor $\alpha$ of the optimal. For $k$-CENTER, a simple greedy algorithm already gives a 2-approximation, that is the best one can hope for, since finding an approximation with smaller factor is NP-hard [24]. Analogously, there is a best-possible 3-approximation for $k$-SUPPLIER [27]. These results have been extended to MAkHC as well, which also admits a 3-approximation [39]. Later, we prove this approximation factor is tight, unless P = NP.

An alternative is to consider the problem from the perspective of parameterized algorithms, that insist on finding an exact solution, but allow running times with a non-polynomial factor that depends only on a certain parameter of the input. More precisely, a decision problem with parameter $w$ is fixed-parameter tractable (FPT) if it can be decided in time $f(w) \cdot n^{O(1)}$, where $n$ is the size of the input and $f$ is a function that depends only on $w$. Feldmann and Marx [19] showed that $k$-CENTER is W[1]-hard for planar graphs of constant doubling dimension when the parameter is a combination of $k$, the highway dimension and the pathwidth of the graph. Blum [5] showed that the hardness holds even if we additionally parameterize by the skeleton dimension of the graph. Under the assumption that FPT $\neq$ W[1], this implies that $k$-CENTER does not admit an FPT algorithm for any of these parameters, even if restricted to planar graphs of constant doubling dimension.

Recently, there has been interest in combining techniques from parameterized and approximation algorithms [36,18]. An algorithm is called a parameterized $\alpha$-approximation if it finds a solution within factor $\alpha$ of the optimal value and runs in FPT time. The goal is to give an algorithm with improved approximation factor that runs in super-polynomial time, where the non-polynomial factors of the running time are dependent on the parameter only. Thus, one may possibly design an algorithm that runs in FPT time for a W[1]-hard problem that, although it finds only an approximate solution, has an approximation factor that breaks the known NP-hardness lower bounds.

For $k$-CENTER, Demaine et al. [14] give an FPT algorithm parameterized by $k$ and $r$ for planar and map graphs. All these characteristics seem necessary for an exact FPT algorithm, as even finding a $(2 - \epsilon)$-approximation with $\epsilon > 0$ for the general case is W[2]-hard for parameter $k$ [17]. If we remove the solution value $r$
and parameterize only by $k$, the problem remains $W[1]$-hard if we restrict the instances to planar graphs [19], or if we add structural graph parameters, such as the vertex-cover number or the feedback-vertex-set number (and thus, also treewidth or pathwidth) [32].

To circumvent the previous barriers, Katsikarelis et al. [32] provide an efficient parameterized approximation scheme (EPAS) for $k$-Center with different parameters $w$, i.e., for every $\epsilon > 0$, one can compute a $(1 + \epsilon)$-approximation in time $f(\epsilon, w) \cdot n^{O(1)}$, where $w$ is either the cliquewidth or treewidth of the graph. More recently, Feldmann and Marx [19] have also given an EPAS for $k$-Center when it is parameterized by $k$ and the doubling dimension, which can be a more appropriate parameter for transportation networks than $r$.

*Our results and techniques* We initiate the study of MA$k$HC under the perspective of parameterized algorithms. We start by showing that, for any $\epsilon > 0$, there is no parameterized $(3 - \epsilon)$-approximation for MA$k$HC when the parameter is $k$, the value $r$ is bounded by a constant and the graph is unweighted, unless FPT = $W[2]$. For planar graphs, finding a good constant-factor approximation remains hard in the polynomial sense, as we show that it is NP-hard to find a $(3-\epsilon)$-approximation for MA$k$HC in this case, even if the maximum degree is 3.

To challenge the approximation lower bound, one might envisage an FPT algorithm by considering an additional structural parameter, such as vertex-cover and feedback-vertex-set numbers or treewidth. However, this is unlikely to lead to an exact FPT algorithm, as we note that the hardness results for $k$-Center [32,19,5] extend to MA$k$HC. Namely, we show that, unless FPT = $W[1]$, MA$k$HC does not admit an FPT algorithm when parameterized by a combination of $k$, the highway and skeleton dimensions and the pathwidth of the graph, even if restricted to planar graphs of constant doubling dimension; or when parameterized by $k$ and the vertex-cover number. Instead, we aim at finding an approximation with factor strictly smaller than 3 that runs in FPT time.

In this paper, we present a $(2 + \epsilon)$-approximation for MA$k$HC parameterized by the treewidth of the graph, for $\epsilon > 0$. The running time of the algorithm is $O^*((tw/\epsilon)^{O(tw)})$, where polynomial factors in the size of the input are omitted. Moreover, we give a parameterized $(2 + \epsilon)$-approximation for MA$k$HC when the input graph is planar and unweighted, parameterized by $k$ and $r$. 
Our main result is a non-trivial dynamic programming algorithm over a tree decomposition, that follows the spirit of the algorithm by Demaine et al. [14]. We assume that we are given a tree decomposition of the graph and consider both $k$ and $r$ as part of the input. Thus, for each node $t$ of this decomposition, we can guess the distance from each vertex in the bag of $t$ to its closest hub in some (global) optimal solution $H^*$. The subproblem is computing the minimum number of hubs to satisfy each demand in the subgraph $G_t$, corresponding to $t$.

Compared to $k$-CENTER and $k$-SUPPLIER, however, MA$k$HC has two additional sources of difficulty. First, the cost to satisfy a demand cannot be computed locally, as it is the sum of two shortest paths, each from a client in the origin-destination pair to some hub in $H^*$ that satisfies that pair. Second, the set of demand pairs $\mathcal{D}$ is given as part of the input, whereas every client must be served in $k$-CENTER or in $k$-SUPPLIER. If we knew the subset of demands $D^*_t$ that are satisfied by some hub in $H^* \cap V(G_t)$, then one could solve every subproblem in a bottom-up fashion, so that every demand would have been satisfied in the subproblem corresponding to the root of the decomposition.

Guessing $D^*_t$ leads to an FPT algorithm parameterized by $tw$, $r$ and $|\mathcal{D}|$, which is unsatisfactory as the number of demands might be large in practice. Rather, for each node $t$ of the tree decomposition, we compute deterministically two sets of demands $D_t, S_t \subseteq \mathcal{D}$ that enclose $D^*_t$, that is, that satisfy $D_t \subseteq D^*_t \subseteq D_t \cup S_t$. By filling the dynamic programming table using $D_t$ instead of $D^*_t$, we can obtain an algorithm that runs in FPT time on parameters $tw$ and $r$, and that finds a 2-approximation.

The key insight for the analysis is that the minimum number of hubs in $G_t$ that are necessary to satisfy each demand in $D_t$ by a path of length at most $r$ is a lower bound on $|H^* \cap V(G_t)|$. At the same time, the definition of the set of demands $S_t$ ensures that each such demand can be satisfied by a path of length at most $2r$ using a hub that is close to a vertex in the bag of $t$. This is the main technical contribution of the paper, and we believe that these ideas might find usage in algorithms for similar problems whose solution costs have non-local components.

Using only these ideas, however, is not enough to get rid of $r$ as a parameter, as we need to enumerate the distance from each vertex in a bag to its closest hub. A common method to shrink a dynamic programming table with large integers is storing only an approximation of each number, causing the solution value to be
computed approximately. This eliminates the parameter $r$ from the running time, but adds a term $\epsilon$ to the approximation factor. This technique is now standard [34] and has been applied multiple times for graph width problems [14,20,32,4].

Specifically, we employ the framework of approximate addition trees [34]. For some $\delta > 0$, we approximate each value $\{1, \ldots, r\}$ of an entry in the dynamic programming table by an integer power of $(1 + \delta)$, and show that each such value is computed by an addition tree and corresponds to an approximate addition tree. By results in [34], we can readily set $\delta$ appropriately so that the number of distinct entries is polynomially bounded and each value is approximated within factor $(1 + \epsilon)$.

**Related work** The first modern studies on hub location problems date several decades back, when models and applications were surveyed [37,38]. Since then, most papers focused on integer linear programming and heuristic methods [1,16]. Approximation algorithms were studied for the single allocation median variant, whose task is to allocate each client to exactly one of the given hubs, minimizing the total transportation cost [28,2,22]. Later, constant-factor approximation algorithms were given for the problem of, simultaneously, selecting hubs and allocating clients [3]. The analogous of MA$k$HC with median objective was considered by Bordini and Vignatti [7], who presented a $(4\alpha)$-approximation algorithm that opens $\left(\frac{2\alpha}{2\alpha - 1}\right)k$ hubs, for $\alpha > 1$.

There is a single allocation center variant that asks for a two-level hub network, where every client is connected to a single hub and the path satisfying a demand must cross a given network center [41,35]. Chen et al. [11] give a $\frac{5}{3}$-approximation algorithm and showed that finding a $(1.5 - \epsilon)$-approximation, for $\epsilon > 0$, is NP-hard. This problem was shown to admit an EPAS parameterized by the treewidth [4] and, to our knowledge, is the first hub location problem studied in the parameterized setting.

**Organization** The remainder of the paper is organized as follows. Section 2 introduces basic concepts and describes the framework of approximate addition trees. Section 3 shows the hardness results for MA$k$HC in both classical and parameterized complexity. Section 4 presents the approximation algorithm parameterized by treewidth, which is analyzed in Section 5. Section 6 presents the final remarks. The case of planar graphs is considered in Appendix A.
2 Preliminaries

An $\alpha$-approximation algorithm for a minimization problem is an algorithm that, for every instance $I$ of size $n$, has running time $n^{O(1)}$ and outputs a solution of value at most $\alpha \cdot \text{OPT}(I)$, where $\text{OPT}(I)$ is the optimal value of $I$. A parameterized algorithm for a parameterized problem is an algorithm that, for every instance $(I, k)$, has running time $f(k) \cdot n^{O(1)}$, where $f$ is a computable function that depends only on the parameter $k$, and decides $(I, k)$ correctly. A parameterized problem that admits a parameterized algorithm is called fixed-parameter tractable, and the set of all such problems is denoted by FPT. Finally, a parameterized $\alpha$-approximation algorithm for a (parameterized) minimization problem is an algorithm that, for every instance $I$ and corresponding parameter $k$, has running time $f(k) \cdot n^{O(1)}$ and outputs a solution of value at most $\alpha \cdot \text{OPT}(I)$. For a complete exposition, we refer the reader to [40,13,36].

We adopt standard graph theoretic notation. Given a graph $G$, we denote the set of vertices and edges as $V(G)$ and $E(G)$, respectively. For $S \subseteq V(G)$, the subgraph of $G$ induced by $S$ is denoted as $G[S]$ and is composed by the vertices of $S$ and every edge of the graph that has both endpoints in $S$.

A tree decomposition of a graph $G$ is a pair $(T, X)$, where $T$ is a tree and $X$ is a function that associates a node $t$ of $T$ to a set $X_t \subseteq V(G)$, called bag, such that:

(i) $\bigcup_{t \in V(T)} X_t = V(G)$;
(ii) for every $(u, v) \in E(G)$, there exists $t \in V(T)$ such that $u, v \in X_t$;
(iii) for every $u \in V(G)$, the set $\{t \in V(T) : u \in X_t\}$ induces a connected subtree of $T$.

The width of a tree decomposition is $\max_{t \in V(T)} |X_t| - 1$ and the treewidth of $G$ is the minimum width of any tree decomposition of the graph. Also, for a node $t \in V(T)$, let $T_t$ be the subset of nodes that contains $t$ and all its descendants, and define $G_t$ as the induced subgraph of $G$ that has $\bigcup_{t' \in T_t} X_{t'}$ as the set of vertices.

Dynamic programming algorithms over tree decompositions often assume that the decomposition has a restricted structure. In a nice tree decomposition of $G$, $T$ is a binary tree and each node $t$ has one of the following types:

(i) leaf node, which has no child and $X_t = \emptyset$;
(ii) introduce node, which has a child $t'$ with $X_t = X_{t'} \cup \{u\}$, for $u \notin X_{t'}$;
(iii) forget node, which has a child $t'$ with $X_t = X_{t'} \setminus \{u\}$, for $u \in X_{t'}$;
(iv) join node, which has children $t'$ and $t''$ with $X_t = X_{t'} = X_{t''}$.

Given a tree decomposition $(T, X)$ of width $tw$, there is a polynomial-time algorithm that finds a nice tree decomposition of the same width and $O(tw \cdot |V(G)|)$ nodes [33]. Moreover, we may assume without loss of generality that our algorithm receives as input a nice tree decomposition of $G$ whose tree has height $O(tw \cdot \log |V(G)|)$, using the same arguments as discussed in [6,4].

### 2.1 Approximate addition trees

An addition tree is an abstract model that represents the computation of a number by successively adding two other previously computed numbers.

**Definition 1.** An addition tree is a full binary tree such that each leaf $u$ is associated to a non-negative integer input $y_u$, and each internal node $u$ with children $u'$ and $u''$ is associated to a computed number $y_u := y_{u'} + y_{u''}$.

One can replace the sum with some operator $\oplus$, which computes each such sum only approximately, up to an integer power of $(1 + \delta)$, for some parameter $\delta > 0$. The resulting will be an approximate addition tree. While the error of the approximate value can pile up as more operations are performed, Lampis [34] showed that, for some $\epsilon > 0$, as long as $\delta$ is not too large, the relative error can bounded by $1 + \epsilon$. Figure 1 illustrates an addition tree and the corresponding approximate addition tree.

![Example calculation in both types of Trees.](image-url)
Definition 2. An approximate addition tree with parameter $\delta > 0$ is a full binary tree, where each leaf $u$ is associated to a non-negative integer input $z_u$, and each internal node $u$ with children $u'$ and $u''$ is associated to a computed value $z_u := z_{u'} \oplus z_{u''}$, where $a \oplus b := 0$ if both $a$ and $b$ are zero, and $a \oplus b := (1 + \delta)^{\lceil \log_{1+\delta} (a+b) \rceil}$, otherwise.

For simplicity, here we defined only a deterministic version of the approximate addition tree, since we can assume that the height of the tree decomposition is bounded by $O(\text{tw} \cdot \log |V(G)|)$. For this case, Lampis showed the following result.

Theorem 1 ([34]). Given an approximate addition tree of height $\ell$, if $\delta < \frac{\epsilon}{2\ell}$, then, for every node $u$ of the tree, we have $\max \{ \frac{z_u}{y_u}, \frac{y_u}{z_u} \} < 1 + \epsilon$.

2.2 Preprocessing

For an instance of MAKHC and a demand $(a, b) \in D$, define $G_{ab}$ as the induced subgraph of $G$ with vertex set $V(G_{ab}) = \{ v \in V(G) : d(a, v) + d(v, b) \leq r \}$.

Notice that if a solution $H$ has a hub $h \in V(G_{ab})$, then the length of a path serving $(a, b)$ that crosses $h$ is at most $r$. In this case, we say that demand $(a, b)$ is satisfied by $h$ with cost $r$. Thus, in an optimal solution $H^*$ of MAKHC, for every $(a, b) \in D$, the set $H^* \cap V(G_{ab})$ must be non-empty.

Also, if there is $v \in V(G)$ such that $d(a, v) + d(v, b) > r$ for every $(a, b) \in D$, then $v$ does not belong to any $(a, b)$-path of length at most $r$, and can be safely removed from $G$. From now on, assume that we have preprocessed $G$ in polynomial time, such that for every $v \in V(G)$,

$$\min_{(a,b)\in D} d(a, v) + d(v, b) \leq r.$$ 

Moreover, we assume that each edge has an integer weight and that the optimal value, OPT, is bounded by $O(\frac{1}{\epsilon}|V(G)|)$, for a given constant $\epsilon > 0$. If not, then we solve another instance for which this holds and that has optimal value $\text{OPT'} \leq (1 + \epsilon)\text{OPT}$ using standard rounding techniques [40]. It suffices finding a constant-factor approximation of value $A \leq 3\text{OPT}$ [39], and defining a new distance function such that $d'(u, v) = \left\lceil \frac{3|V(G)|}{\epsilon A} d(u, v) \right\rceil$. 


3 Hardness

Next, we observe that approximating MAkHC is hard, both in the classical and parameterized senses. First, we show that approximating the problem by a factor better than 3 is NP-hard, even if the input graph is planar and unweighted. This result strengthens the previous known lower bound and matches the approximation factor of the greedy algorithm [39].

**Theorem 2.** For every $\epsilon > 0$, if there is a $(3 - \epsilon)$-approximation for MAkHC when $G$ is an unweighted planar graph, then $P = NP$.

**Proof.** We present a reduction from **Vertex Cover** (VC), whose task is to find a subset of $k$ vertices that contains at least one endpoint of every edge of the graph. More specifically, we consider a particular version of the problem.

**Claim.** **Vertex Cover** is NP-hard even if the input graph is planar, triangle-free and has maximum degree 3.

**Proof.** We self-reduce the problem from the case the input graph is planar and with maximum degree 3, which is known to be NP-hard [21]. Given an instance $(G, k)$ of vertex cover, create another instance $(G', k')$, where $G'$ is obtained by subdividing each edge of $G$ in three parts, and $k' = k + |E(G)|$. Let $u_e$ and $v_e$ be new vertices added for the subdivision of an edge $e = (u, v) \in E(G)$ and that are incident with $u$ and $v$, respectively.

Assume $S$ is a vertex cover for $G$ with size $k$, and build a vertex cover $S'$ for $G'$ as follows. Initialize $S'$ with a copy of $S$ and, for each edge $e = (u, v)$ of $G$, add $v_e$ to $S'$, if $u \in S$, and add $u_e$, otherwise. Note that $S'$ is a vertex cover of $G'$ of size $k'$. For the other direction, assume $S'$ is a vertex cover of $G'$ with size $k'$, and define $S = S' \setminus \{u_e, v_e : e \in E(G)\}$. If, for some edge $(u_e, v_e) \in G'$, both $u_e$ and $v_e$ are in $S'$, then $S' \setminus \{u_e\} \cup \{u\}$ is a vertex cover of $G'$. Thus, assume for every such edge $(u_e, v_e)$, either $u_e$ or $v_e$ is in $S'$. It follows that $S$ is a vertex cover of $G$ of size $k$.  

Given an instance $(G, k)$ of VC, build an instance $(G, C, H, D, k)$ of MAkHC, where $C = H = V(G)$ and $D = E(G)$. Observe that there exists a vertex cover $S$ of size $k$ in $G$ if, and only if, the solution $S$ for MAkHC has value 1. Suppose that the optimal value is greater than 1, then it would have to be at least 3, since
the graph has no triangles. Then, for \( \epsilon > 0 \), a \((3 - \epsilon)\)-approximation for \( \text{MAkHC} \) can decide whether the optimal value is 1, thus deciding whether there is a vertex cover of size \( k \) in \( G \).

From this reduction, one may observe that the previous theorem holds even for the case where the maximum degree is 3 and the optimal value is bounded by 3.

To find a better approximation guarantee, one might resource to a parameterized approximation algorithm. The natural candidates for parameters of \( \text{MAkHC} \) are the number of hubs \( k \) and the value \( r \) of an optimal solution. The next theorem states that this choice of parameters does not help, as it is \( \text{W}[2] \)-hard to find a parameterized approximation with factor better than 3, when the parameter is \( k \), the value \( r \) is bounded by a constant and \( G \) is unweighted.

**Theorem 3.** For every \( \epsilon > 0 \), if there is a parameterized \((3 - \epsilon)\)-approximation for \( \text{MAkHC} \) with parameter \( k \), then \( \text{FPT} = \text{W}[2] \). This holds even for the particular case of \( \text{MAkHC} \) with instances \( I \) such that \( \text{OPT}(I) \leq 6 \).

**Proof.** The theorem will follow by a reduction from **Hitting Set** (HS), which is known to be \( \text{W}[2] \)-hard [15]. We show that a \((3 - \epsilon)\)-approximation for \( \text{MAkHC} \) can decide the instance of HS, implying that \( \text{FPT} = \text{W}[2] \). Remember that in HS, we are given a set \( U \), a family of sets \( \mathcal{F} \subseteq 2^U \) and an integer \( k \), and the objective is to decide whether there exists a set \( H \subseteq U \) of size \( k \) that intersects every set in \( \mathcal{F} \).

Given an instance \( I = (U, \mathcal{F}, k) \) of HS, we build an instance \( I' = (G, \mathcal{C}, \mathcal{H}, \mathcal{D}, k) \) of \( \text{MAkHC} \): for each element \( e \in U \), create a vertex \( h_e \) in \( G \) and add it to \( \mathcal{H} \); for each set \( S \in \mathcal{F} \), create vertices \( u_S \) and \( v_S \) in \( G \), add them to \( \mathcal{C} \), create a demand \((u_S, v_S)\) in \( \mathcal{D} \) and connect \( u_S \) and \( v_S \) to vertices \( \{h_e : e \in S\} \).

Consider a hitting set \( H \) of size \( k \), and let \( H' = \{h_e : e \in H\} \) be a set of hubs of size \( k \). This set of hubs satisfies every demand in \( \mathcal{D} \) with cost 2, since for every \( S \in \mathcal{F} \), there is \( e \in S \cap H \) and thus \( h_e \in H' \). In the other direction, consider a set of hubs \( H' \) of size \( k \) that satisfies every demand in \( \mathcal{D} \) with cost 2, and let \( H = \{e : h_e \in H'\} \) be a set of elements of size \( k \). For each set \( S \in \mathcal{F} \), there exists a corresponding demand \((u_S, v_S)\) in \( \mathcal{D} \) that is satisfied by a hub \( h_e \in H' \) with cost 2. Since the length of this path is 2, \( h_e \) must be a neighbor of \( u_S \) and \( v_S \) in \( G \), then \( e \in S \cap H \). It follows that \( H \) is a hitting set for \( I \).
We have shown that $I$ is a yes-instance if, and only if, the optimal value of $I'$ is 2. Now, if the optimal value of $I'$ is greater than 2, then it would have to be at least 6. Indeed, if a demand $(u_S, v_S)$ is satisfied by a hub $h_e \in H'$ with cost greater than 2, then $h_e$ is not a neighbor of $u_S$. But $G$ is bipartite and $u_S$ and $h_e$ are at different parts, then $d(u_S, h_e) \geq 3$. Analogously, we have $d(v_S, h_e) \geq 3$, and thus $d(u_S, h_e) + d(v_S, h_e) \geq 6$. We conclude that a $(3 - \epsilon)$-approximation can decide whether the optimal value of $I'$ is 2, thus deciding whether $I$ is a yes-instance. □

Due to the previous hardness results, a parameterized algorithm for MA$k$HC must consider different parameters, or assume a particular case of the problem. In this paper, we focus on the treewidth of the graph, that is one of the most studied structural parameters [13], and the particular case of planar graphs. This setting is unlikely to lead to an (exact) FPT algorithm, though, as the problem is W[1]-hard, even if we combine these conditions. The next theorem follows directly from a result of Blum [5], since MA$k$HC is a generalization of $k$-Center.

**Theorem 4.** Even on planar graphs with edge lengths of constant doubling dimension, MA$k$HC is W[1]-hard for the combined parameter $(k, pw, h, \kappa)$, where $pw$ is the pathwidth, $h$ is the highway dimension and $\kappa$ is the skeleton dimension of the graph.

**Proof.** Given an instance $I = (G, k)$ of $k$-Center, we build an instance $I' = (G, C, H, D, k)$ of MA$k$HC where $C = H = V(G)$ and $D = \{(u, u): u \in V(G)\}$. Now, note that there is a solution of value at most $r$ for $I$ if, and only if, there is a solution of value $2r$ for $I'$. The theorem follows, as we do not change the graph or the number of hubs $k$. □

Note that MA$k$HC inherits other hardness results of $k$-Center by Katsikarelis et al. [32], thus it is W[1]-hard when parameterized by a combination of $k$ and the vertex-cover number.

Recall that the treewidth is a lower bound on the pathwidth, thus the previous theorem implies that the problem is also W[1]-hard for planar graphs when parameterized by a combination of $k$ and tw. To circumvent these hardness results, in Section 4, we give a $(2 + \epsilon)$-approximation algorithm for MA$k$HC for arbitrary graphs that is parameterized by tw, breaking the approximation barrier of 3. In Appendix A, we complement this result with a $(2 + \epsilon)$-approximation for unweighted planar graphs parameterized by $k$ and $r$. 
4 The algorithm

In this section, we give a \((2+\epsilon)\)-approximation parameterized only by the treewidth. In what follows, we assume that we receive a preprocessed instance of \(MA_kHC\) and a nice tree decomposition of the input graph \(G\) with width \(tw\) and height bounded by \(O(tw \cdot \log |V(G)|)\). Also, we assume that \(G\) contains all edges connecting pairs \(u, v \in X_t\) for each node \(t\). Moreover, we are given an integer \(r\) bounded by \(O((1/\epsilon)|V(G)|)\). Our goal is to design a dynamic programming algorithm that computes the minimum number of hubs that satisfy each demand with a path of length \(r\). The overall idea is similar to that of the algorithm for \(k\)-Center by Demaine et al. [14], except that we consider a tree decomposition, instead of a branch decomposition, and that the computed solution will satisfy demands only approximately.

Consider some fixed global optimal solution \(H^*\) and a node \(t\) of the tree decomposition. Let us discuss possible candidates for a subproblem definition. The subgraph \(G_t\) corresponding to \(t\) in the decomposition contains a subset of \(H^*\) that satisfies a subset \(D^*_t\) of the demands. The shortest path serving each demand with a hub of \(H^* \cap V(G_t)\) is either completely contained in \(G_t\), or it must cross some vertex of the bag \(X_t\). Thus, as in [14], we guess the distance \(i\) from each vertex \(u\) in \(X_t\) to the closest hub in \(H^*\), and assign “color” \(\downarrow i\) to \(u\) to mean that the corresponding shortest path is in \(G_t\), and color \(\uparrow i\) to mean otherwise. Since the number of demands may be large, we cannot include \(D^*_t\) as part of the subproblem definition. For \(k\)-Center, if the shortest path serving a vertex in \(G_t\) crosses a vertex \(u \in X_t\), then the length of this path can be bounded locally using the color of \(u\), and the subproblem definition may require serving all vertices. For MA\(k\)HC, however, there might be demands \((a, b)\) such that \(a\) is in \(G_t\), while \(b\) is not, thus the coloring of \(X_t\) is not sufficient to bound the length of a path serving \((a, b)\).

Instead of guessing \(D^*_t\), for each coloring \(c\) of \(X_t\), we require that only a subset \(D_t(c)\) must be satisfied in the subproblem, and they can be satisfied by a path of length at most \(2r\). Later, we show that the other demands in \(D^*_t\) are already satisfied by the hubs corresponding to the coloring of \(X_t\). More specifically, we would like to compute \(A_t(c)\) as the minimum number of hubs in \(G_t\) that satisfy each demand in \(D_t(c)\) with a path of length at most \(2r\) and that respect the distances given by \(c\).
Since we preprocessed the graph in Section 2, there must be a hub in $H^*$ to each vertex of $X_t$ at distance at most $r$. Thus, the number of distinct colorings to consider for each $t$ is bounded by $r^{O(tw)}$. To get an algorithm parameterized only by $tw$, we need one more ingredient: in the following, the value of each color is stored approximately as an integer power of $(1 + \delta)$, for some $\delta > 0$. Later, using the framework of approximate addition trees, for any constant $\epsilon > 0$, we can set $\delta$ such that the number of subproblems is bounded by $O^*((tw/\epsilon)^{O(tw)})$, and demands are satisfied by a path of length at most $(1 + \epsilon)2r$.

The set of approximate colors is

$$\Sigma = \{\downarrow 0\} \cup \{\uparrow i, \downarrow j : j \in \mathbb{Z}_{\geq 0}, i = (1 + \delta)^j, i \leq (1 + \epsilon)r \}.$$ A coloring of $X_t$ is represented by a function $c : X_t \to \Sigma$. For each coloring $c$, we compute a set of demands that are “satisfied” by $c$.

**Definition 3.** Define $S_t(c)$ as the set of demands $(a, b)$ for which there exists $u \in X_t$ with $c(u) \in \{\uparrow i, \downarrow i\}$ and such that $d(a, u) + 2i + d(u, b) \leq (1 + \epsilon)2r$.

The intuition is that a demand $(a, b) \in S_t(c)$ can be satisfied by a hub close to $u$ by a path of length at most $(1 + \epsilon)2r$. Also, we compute a set of demands that must be served by a hub in $G_t$ by the global optimal solution.

**Definition 4.** Define $D_t(c)$ as the set of demands $(a, b)$ such that $(a, b) \notin S_t(c)$ and either: (i) $a, b \in V(G_t)$; or (ii) $a \in V(G_t)$, $b \notin V(G_t)$ and there is $h \in V(G_{ab}) \cap V(G_t)$ such that $d(h, V(G_{ab}) \cap X_t) > r/2$.

We will show in Lemmas 4 and 5 that $D_t(c) \subseteq D^*_t \subseteq D_t(c) \cup S_t(c)$, thus we only need to take care of demands in $D_t(c)$ in the subproblem. Formally, for each node $t$ of the tree decomposition and coloring $c$ of $X_t$, our algorithm computes a number $A_t(c)$ and a set of hubs $H \subseteq \mathcal{H} \cap V(G_t)$ of size $A_t(c)$ that satisfies the conditions below.

(C1) For every $u \in X_t$, if $c(u) = \downarrow i$, then there exists $h \in H$ and a shortest path $P$ from $u$ to $h$ of length at most $i$ such that $V(P) \subseteq V(G_t)$;

(C2) For every $(a, b) \in D_t(c)$, $\min_{h \in H} d(a, h) + d(h, b) \leq (1 + \epsilon)2r$.

If the algorithm does not find one such set, then it assigns $A_t(c) = \infty$. We describe next how to compute $A_t(c)$ for each node type.
For a leaf node $t$, we have $V(G_t) = \emptyset$, then $H = \emptyset$ satisfies the conditions, and we set $A_t(c_\emptyset) = 0$, where $c_\emptyset$ denotes the empty coloring.

For an introduce node $t$ with child $t'$, let $u$ be the introduced vertex, such that $X_t = X_{t'} \cup \{u\}$. Let $I_t(c)$ be the set of colorings $c'$ of $X_{t'}$ such that $c'$ is the restriction of $c$ to $X_{t'}$ and, if $c(u) = \downarrow i$ for some $i > 0$, there is $v \in X_{t'}$ with $c'(v) = \downarrow j$ such that $i = d(u,v) \oplus j$. Note that this set is either a singleton or is empty. If $I_t(c)$ is empty, discard $c$. Define:

$$A_t(c) = \min_{c' \in I_t(c): \exists d_t(c) \subseteq D_{t'}(c')} \begin{cases} A_{t'}(c') + 1 & \text{if } c(u) = \downarrow 0, \\ A_{t'}(c') & \text{otherwise.} \end{cases}$$

If $H'$ is the solution corresponding to $A_{t'}(c')$, we output $H = H' \cup \{u\}$ if $c(u) = \downarrow 0$, or $H = H'$ otherwise.

For a forget node $t$ with child $t'$, let $u$ be the forgotten vertex, such that $X_t = X_{t'} \setminus \{u\}$. Let $F_t(c)$ be the set of colorings $c'$ of $X_{t'}$ such that $c$ is the restriction of $c'$ to $X_t$ and, if $c'(u) = \uparrow i$, then there is $v \in X_t$ such that $c(v) = \uparrow j$ and $i = d(u,v) \oplus j$. If $F_t(c)$ is empty, discard $c$. Define:

$$A_t(c) = \min_{c' \in F_t(c): \exists d_t(c) \subseteq D_{t'}(c') \cup S_{t'}(c')} A_{t'}(c').$$

We output as solution the set $H = H'$, where $H'$ corresponds to the solution of the selected subproblem in $t'$.

For a join node $t$ with children $t'$ and $t''$, we have $X_t = X_{t'} = X_{t''}$. Let $J_t(c)$ be the set of pairs of colorings $(c',c'')$ of $X_t$ such that, for every $u \in X_t$, when $c(u)$ is $\downarrow 0$ or $\uparrow i$, then $c'(u) = c''(u) = c(u)$; else, if $c(u)$ is $\down i$, then $(c'(u),c''(u))$ is either $(\up i, \down i)$ or $(\down i, \up i)$. If $J_t(c)$ is empty, discard $c$. Define:

$$A_t(c) = \min_{(c',c'') \in J_t(c): \exists d_t(c) \subseteq D_{t'}(c') \cup D_{t''}(c'')} A_{t'}(c') + A_{t''}(c'') - h(c),$$

where $h(c)$ is the number of vertices $u$ in $X_t$ such that $c(u) = \down 0$. We output a solution $H = H' \cup H''$, where $H'$ and $H''$ are the solutions corresponding to $t'$ and $t''$, respectively.
In the next lemma, we show that the algorithm indeed produces a solution of bounded size that satisfies both conditions.

**Lemma 1.** If $A_t(c) \neq \infty$, then the algorithm outputs a set $H \subseteq \mathcal{H} \cap V(G_t)$, with $|H| \leq A_t(c)$, that satisfies (C1) and (C2).

**Proof.** We prove the lemma by induction on the height of node $t$, thus assume the lemma holds for nodes below $t$. For leaves, the algorithm outputs an empty set, satisfying both conditions.

For an introduce node $t$ with child $t'$ and $u \in X_t \setminus X_{t'}$, let $H'$ be the solution corresponding to $t'$ with coloring $c'$. Since $c'$ is the restriction of $c$ to $X_{t'}$, condition (C1) is satisfied for every $v \in X_{t'}$, by induction. If $c(u) = \downarrow 0$, then it is satisfied for $u$, since, in this case, $u \in H$. Else, if $c(u) = \downarrow i$ for $i > 0$, then it is also satisfied, since in this case there is $v \in X_{t'}$ with $c'(v) = \downarrow j$ such that $i = d(u, v) \oplus j$. Condition (C2) is satisfied as well, since $D_t(c) \subseteq D_{t'}(c')$, and $H'$ satisfies (C2).

For a forget node $t$ with child $t'$ and $u \in X_{t'} \setminus X_t$, we have that $H = H'$, where $H'$ is the solution to $t'$ corresponding to some coloring $c'$ of $X_{t'}$. Since $c$ is the restriction of $c'$ to $X_t$, $H$ satisfies (C1) by induction. For (C2), let $(a, b) \in D_t(c)$, and remember that $D_t(c) \subseteq D_{t'}(c') \cup S_{t'}(c')$. If $(a, b) \in D_{t'}(c')$, then this demand is satisfied by $H$ with cost at most $(1 + \epsilon)2r$. Else, $(a, b) \in S_{t'}(c')$, but $(a, b) \notin S_t(c)$. Thus, for the forgotten vertex $u$, we have $c'(u) \in \{\uparrow i, \downarrow i\}$ and $d(a, u) + 2i + d(u, b) \leq (1 + \epsilon)2r$. We consider two cases:

- If $c'(u) = \downarrow i$, then, since $H$ satisfies (C1), there is $h \in H$ such that the distance from $u$ to $h$ is at most $i$. Thus condition (C2) is satisfied, because

$$d(a, h) + d(h, b) \leq d(a, u) + 2i + d(u, b) \leq (1 + \epsilon)2r.$$  

- If $c'(u) = \uparrow i$, there is $v \in X_t$ with $c(v) = \uparrow j$ and $i = d(u, v) \oplus j$. We get

$$d(a, v) + 2j + d(v, b) \leq d(a, u) + 2(d(u, v) + j) + d(u, b) \leq d(a, u) + 2i + d(u, b) \leq (1 + \epsilon)2r,$$

where we used $d(u, v) + j \leq d(u, v) \oplus j = i$ in the second inequality. But this means that $(a, b) \in S_t(c)$, which is a contradiction.
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For a join node $t$ with children $t'$ and $t''$, let $H'$ and $H''$ be solutions for the subproblems at $t'$ and $t''$ corresponding to the selected pair of colorings $c'$ and $c''$. We claim that $H = H' \cup H''$ satisfies both conditions. For $(C1)$, note that if $c(u) = \downarrow i$, for some $u \in X_t$, then $c'(u) = \downarrow i$ or $c''(u) = \downarrow i$. For $(C2)$, note that, for $(a, b) \in D_t(c)$, we have $(a, b) \in D_t(c') \cup D_t(c'')$, and thus this demand is satisfied with cost at most $(1 + \epsilon)2r$ by a vertex in $H'$ or $H''$. 

Let $t_0$ be the root of the tree decomposition and $c_\emptyset$ be the empty coloring. Since the bag corresponding to the root node is empty, we have $S_{t_0}(c_\emptyset) = \emptyset$ and thus $D_{t_0}(c_\emptyset) = D$. Therefore, if $A_{t_0}(c_\emptyset) \leq k$, Lemma 1 implies that the set of hubs $H$ computed by the algorithm is a feasible solution that satisfies each demand with cost at most $(1 + \epsilon)2r$. In the next section, we bound the size of $H$ by the size of the global optimal solution $H^*$.

5 Analysis

For each node $t$ of the tree decomposition, we want to show that the number of hubs computed by the algorithm for some coloring $c$ of $X_t$ is not larger than the number of hubs of $H^*$ contained in $G_t$, that is, we would like to show that $A_t(c) \leq |H^* \cap V(G_t)|$ for some $c$. If the distances from each vertex $u \in X_t$ to its closest hub in $H^*$ were stored exactly, then the partial solution corresponding to $H^*$ would induce one such coloring $c^*_t$, and we could show the inequality for this particular coloring. More precisely, for each $u \in V(G)$, let $h^*(u)$ be a hub of $H^*$ such that $d(u, h^*(u))$ is minimum and $P^*(u)$ be a corresponding shortest path. Assume that each $P^*(u)$ is obtained from a shortest path tree to $h^*(u)$ and that it has the minimum number of edges among the shortest paths. The signature of $H^*$ corresponding to a partial solution in $G_t$ is a function $c^*_t$ on $X_t$ such that

$$c^*_t(u) = \begin{cases} \downarrow d(u, h^*(u)) & \text{if } V(P^*(u)) \subseteq V(G_t), \\ \uparrow d(u, h^*(u)) & \text{otherwise}. \end{cases}$$

Since distances are stored approximately as integer powers of $(1 + \delta)$, the function $c^*_t$ might not be a valid coloring. Instead, we show that the algorithm considers a coloring $\tilde{c}_t$ with roughly the same values of $c^*_t$ and that its values are computed by approximate addition trees. We say that an addition tree and an
approximate addition tree are corresponding if they are isomorphic and have the same input values. Also, recall that a coloring $c$ of $X_t$ is discarded by the algorithm if the set $I_t(c)$, $F_t(c)$ or $J_t(c)$ corresponding to $t$ is empty.

**Lemma 2.** Let $\ell_{t_0}$ be the height of the tree decomposition. There exists a coloring $\bar{c}_t$ that is not discarded by the algorithm and such that, for every $u \in X_t$, the values $c_t^*(u)$ and $\bar{c}_t(u)$ are computed, respectively, by an addition tree and a corresponding approximate addition tree of height at most $2\ell_{t_0}$.

**Proof.** A partial addition tree is a pair $(T, p)$, where $T$ is an addition tree and $p$ is a leaf of $T$. The vertex $p$ represents a subtree that computes a pending value $x_p$, and may be replaced by some other (partial) addition tree that computes this value.

For some node $t$, let $\ell_t$ be the height of $t$ and define $U_t$ as the set of vertices $u \in X_t$ such that $c_t^*(u) = \uparrow i$ for some $i$. We say that a vertex $v \in V(G_t) \setminus U_t$ is $t$-complete according to the following cases:

- if $V(P^*(v)) \subseteq V(G_t)$ and $v \in X_t$, then $d(v, h^*(v))$ is computed by an addition tree of height at most $\ell_t$;
- if $V(P^*(v)) \subseteq V(G_t)$ and $v \notin X_t$, then $d(v, h^*(v))$ is computed by an addition tree of height at most $2\ell_t$;
- if $V(P^*(v)) \not\subseteq V(G_t)$, then $d(v, h^*(v))$ is computed by a partial addition tree $(T, p)$ of height at most $\ell_t$ such that $x_p = d(w, h^*(w))$ for some $w \in U_t$.

We will show by induction on the height of $t$ that every $v \in V(G_t) \setminus U_t$ is $t$-complete. The claim holds trivially for leaves, thus suppose that $t$ is not a leaf.

Assume $t$ is an introduce node with child $t'$, and let $u$ be the introduced vertex. Since $U_{t'} \subseteq U_t$, if $v \in V(G_{t'}) \setminus U_{t'}$, then $v$ is $t$-complete by the induction hypothesis. Else, we have $v = u$ and, since $v \notin U_t$, $c_t^*(v) = \downarrow d(v, h^*(v))$. Thus, there is $w \in X_{t'} \setminus U_{t'}$ such that $c_{t'}^*(w) = \downarrow d(w, h^*(w))$ and $d(v, h^*(v)) = d(v, w) + d(w, h^*(w))$. Since $d(w, h^*(w))$ can be computed by an addition tree of height at most $\ell_{t'}$, this implies that $d(v, h^*(v))$ can be computed by an addition tree of height at most $\ell_{t'} + 1 \leq \ell_t$.

Now, assume $t$ is a forget node with child $t'$, and let $u$ be the forgotten vertex. Since $V(G_t) = V(G_{t'})$, if $V(P^*(v)) \subseteq V(G_{t'})$, then $v$ is $t$-complete by the induction hypothesis. Otherwise, by the induction hypothesis, $d(v, h^*(v))$ is computed by a
partial addition tree \((T, p)\) of height at most \(\ell_v\) such that \(x_p = d(w', h^*(w'))\) for some \(w' \in U_v\). If \(w' \in U_t\), then \(v\) is \(t\)-complete. So, assume \(w' \in U_v \setminus U_t\), which implies that \(w'\) is the forgotten vertex \(u\) and \(c^*_p(u) = \uparrow d(u, h^*(u))\). Thus, \(P^*(u)\) crosses some vertex \(w \in U_t\) such that \(d(u, h^*(u)) = d(u, w) + d(w, h^*(w))\). It follows that \(d(u, h^*(u))\) can be computed by a partial addition tree \((T_u, p_u)\) of height 1 such that \(x_{p_u} = d(w, h^*(w))\). Therefore, we can replace the vertex \(p\) by the subtree \(T_u\), and the height of \(T\) becomes at most \(\ell_v + 1 \leq \ell_t\).

Finally, assume \(t\) is a join node with children \(t'\) and \(t''\), and recall that \(X_t = X_{t'} = X_{t''}\). If \(v \in X_t\), then \(V(P^*(v)) \subseteq V(G_{t'})\) or \(V(P^*(v)) \subseteq V(G_{t''})\), because \(X_t\) induces a clique and \(P^*(v)\) is a shortest path with minimum number of edges. Thus, \(v\) is \(t\)-complete by the induction hypothesis. Otherwise, \(v \in V(G_t) \setminus X_t\). Assume \(v \in V(G_{t'}) \setminus X_t\), as the other case is analogous. By the induction hypothesis for \(t'\), \(d(v, h^*(v))\) is computed by a partial addition tree \((T', p)\) of height at most \(\ell_v\) such that \(x_p = d(w, h^*(w))\) for some \(w \in U_{t'}\). If \(w \in U_t\), then \(v\) is \(t\)-complete. Thus, assume \(w \notin U_t\), which implies that \(V(P^*(w)) \subseteq V(G_t)\). Again, since \(X_t\) induces a clique, \(P^*(w)\) is included in \(V(G_{t'})\) or \(V(G_{t''})\), but since \(w \in U_{t'}\), we have \(V(P^*(w)) \subseteq V(G_{t'})\). It follows that \(c^*_p(w) = \downarrow d(w, h^*(w))\). By the induction hypothesis for \(t''\), \(d(w, h^*(w))\) is computed by an addition tree \(T''\) of height at most \(\ell_{t''}\). Therefore, we can replace the vertex \(p\) by the subtree \(T''\), and the height of \(T'\) becomes at most \(\ell_v + \ell_{t''} \leq 2\ell_t\). This completes the induction.

For the root node \(t_o\), we have \(X_{t_o} = \emptyset\), thus for every \(v \in V(G)\), the distance \(d(v, h^*(v))\) is computed by an addition tree \(T_v\) of height at most \(2\ell_{t_o}\). Let \(\bar{T}_v\) be the approximate addition tree corresponding to \(T_v\), and define \(\bar{d}(v)\) as the output of \(\bar{T}_v\). For every node \(t\), and \(u \in X_t\), if \(c^*_t(u) = \downarrow d(u, h^*(u))\), define \(\bar{c}_t(u) = \downarrow \bar{d}(u)\); else, define \(\bar{c}_t(u) = \uparrow \bar{d}(u)\). By repeating the arguments above, and replacing the addition operator by \(\oplus\), one can show that, for every \(t\), the coloring \(\bar{c}_t\) is not discarded by the algorithm. \(\blacksquare\)

By setting \(\delta = \epsilon/(2\ell_{t_o} + 1)\), Theorem 1 implies the next lemma.

**Lemma 3.** For every \(u \in X_t\), if \(c^*_t(u) \in \{\uparrow i, \downarrow i\}\) and \(\bar{c}_t(u) \in \{\uparrow j, \downarrow j\}\), then \(j \leq (1 + \epsilon)i\).

Recall that \(H^*\) is a fixed global optimal solution that satisfies each demand with cost \(r\). Our goal is to bound \(A_t(\bar{c}_t) \leq |H^* \cap V(G_t)|\) for every node \(t\), thus we
would like to determine the subset of demands $D_t^*$ that are necessarily satisfied by hubs $H^* \cap V(G_t)$ in the subproblem definition. This is made precise in the following.

**Definition 5.** $D_t^* = \{(a, b) \in D : \min_{h \in H^* \setminus V(G_t)} d(a, h) + d(h, b) > r\}$.

Since the algorithm cannot determine $D_t^*$, we show that, for each node $t$, it outputs a solution $H$ for the subproblem corresponding to $A_t(\bar{c}_t)$ that satisfies every demand in $D_t(\bar{c}_t)$. In Lemma 4, we show that every demand in $D_t(\bar{c}_t)$ is also in $D_t^*$, as, otherwise, there could be no solution with size bounded by $|H^* \cap V(G_t)|$. Conversely, we show in Lemma 5 that a demand in $D_t^*$ that is not in $D_t(\bar{c}_t)$ must be in $S_t(\bar{c}_t)$, thus all demands are satisfied.

**Lemma 4.** $D_t(\bar{c}_t) \subseteq D_t^*$.

**Proof.** Let $(a, b) \in D_t(\bar{c}_t)$ and consider an arbitrary hub $h^* \in H^*$ that satisfies $(a, b)$ with cost $r$. We will show that $h^* \in V(G_t)$, and thus $(a, b) \in D_t^*$. For the sake of contradiction, assume that $h^* \in V(G) \setminus V(G_t)$.

First we claim that $d(h^*, V(G_{ab}) \cap X_t) > r/2$. If not, then let $u \in V(G_{ab}) \cap X_t$ be a vertex with $\bar{c}_t(u) \in \{\uparrow, \downarrow\}$ such that $d(u, h^*) \leq r/2$. Because the closest hub to $u$ has distance at least $i/(1 + \epsilon)$, we have $i \leq (1 + \epsilon)d(u, h^*) \leq (1 + \epsilon)r/2$, but since $u \in V(G_{ab})$, this implies that $(a, b) \in S_t(\bar{c}_t)$, and thus $(a, b) \notin D_t(\bar{c}_t)$. Then, it follows that indeed $d(h^*, V(G_{ab}) \cap X_t) > r/2$.

Now we show that it cannot be the case that $a, b \in V(G_t)$. Suppose that $a, b \in V(G_t)$. Consider the shortest path from $a$ to $h^*$, and let $u$ be the last vertex of this path that is in $V(G_t)$. Since $X_t$ separates $V(G_t) \setminus X_t$ from $V(G) \setminus V(G_t)$, it follows that $u \in X_t$. From the previous claim, $d(h^*, u) > r/2$, and thus $d(h^*, a) > r/2$. Analogously, $d(h^*, b) > r/2$, but then $d(a, h^*) + d(h^*, b) > r$, which contradicts the fact that $h^*$ satisfies $(a, b)$ with cost $r$. This contradiction comes from supposing that $a, b \in V(G_t)$. Thus, either $a$ or $b$ is not in $V(G_t)$.

Assume without loss of generality that $a \in V(G_t)$ and $b \notin V(G_t)$. From the definition of $D_t(\bar{c}_t)$, we know that there exists $h \in V(G_{ab}) \cap V(G_t)$ such that $d(h, V(G_{ab}) \cap X_t) > r/2$. Let $P$ be a path from $a$ to $b$ crossing $h^*$ with length at most $r$. Similarly, since $h \in V(G_{ab})$, there exists a path $Q$ from $a$ to $b$ crossing $h$ with length at most $r$. Let $u$ be the last vertex of $P$ with $u \in X_t$, and let $v$ be the last vertex of $Q$ with $v \in X_t$ (see Figure 2). Concatenating $P$ and $Q$ leads to
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![Diagram](image)

**Fig. 2.** Closed walk formed by \( P \) and \( Q \).

a closed walk of length at most \( 2r \). This walk crosses \( u, h^* \), \( v \) and \( h \), and thus

\[
2r \geq d(a, h^*) + d(h^*, b) + d(a, h) + d(h, b) \\
= d(u, h^*) + d(h^*, v) + d(v, h) + d(h, u) \tag{1}
\]

where we used the fact that each term in (1) is greater than \( r/2 \). This is a contradiction, so \( h^* \in V(G_t) \) and then \( (a, b) \in D_t^* \).

**Lemma 5.** \( D_t^* \subseteq D_t(\tilde{c}_t) \cup S_t(\tilde{c}_t) \).

**Proof.** Let \( (a, b) \in D_t^* \). Assume \( (a, b) \notin S_t(\tilde{c}_t) \), as otherwise we are done. If \( a, b \in V(G_t) \), we have \( (a, b) \in D_t(\tilde{c}_t) \). Thus, suppose without loss of generality that \( a \in V(G_t) \) and \( b \notin V(G_t) \). Since \( (a, b) \in D_t^* \), there is \( h^* \in H^* \cap V(G_{ab}) \cap V(G_t) \). Let \( u \in V(G_{ab}) \cap X_t \) with \( \tilde{c}_t(u) \in \{\uparrow i, \downarrow i\} \) for some \( i \). Because \( (a, b) \notin S_t(\tilde{c}_t) \), we have \( i > (1 + \epsilon)r/2 \). But the distance from \( u \) to the closest hub in \( H^* \) is at least \( i/(1 + \epsilon) \), thus \( i \leq (1 + \epsilon)d(u, h^*) \). It follows that \( d(u, h^*) > r/2 \). Therefore, \( h^* \in V(G_{ab}) \cap V(G_t) \) and \( d(h^*, V(G_{ab}) \cap X_t) > r/2 \), and then \( (a, b) \in D_t(\tilde{c}_t) \). \( \square \)

Before bounding the number of hubs opened by the algorithm, we prove some auxiliary results.

**Lemma 6.** If \( t \) is an introduce node with child \( t' \), then \( D_t(\tilde{c}_t) \subseteq D_{t'}(\tilde{c}_{t'}) \).

**Proof.** We claim that \( D_t^* \setminus D_{t'}^* \subseteq S_t(\tilde{c}_t) \). Let \( u \) be introduced vertex, and note that \( V(G_t) \setminus V(G_{t'}) = \{u\} \). If \( (a, b) \in D_t^* \setminus D_{t'}^* \), by definition, we know that \( \min_{h \in H^* \setminus V(G_t)} d(a, h) + d(h, b) > r \), but \( \min_{h \in H^* \setminus V(G_{t'})} d(a, h) + d(h, b) \leq r \). This can only happen if \( u \in H^* \), so \( \tilde{c}_t(u) = \downarrow 0 \), and then \( (a, b) \in S_t(\tilde{c}_t) \). \( \square \)
Since $D_t^v \subseteq D_t^*$ and $S_t(\bar{c}_v) \subseteq S_t(\bar{c}_t)$, the claim implies $D_t^* \setminus S_t(\bar{c}_t) \subseteq D_t^* \setminus S_t(\bar{c}_v)$. Using Lemmas 4 and 5, we get

$$D_t(\bar{c}_t) \subseteq D_t^* \setminus S_t(\bar{c}_t) \subseteq D_t^* \setminus S_t(\bar{c}_v) \subseteq D_t(\bar{c}_v).$$

Lemma 7. If $t$ is a forget node with child $t'$, then $D_t(\bar{c}_t) \subseteq D_t(\bar{c}_v) \cup S_t(\bar{c}_v)$.

Proof. In this case $V(G_t) = V(G_v)$, thus $D_t^* = D_v^*$. Using Lemmas 4 and 5,

$$D_t(\bar{c}_t) \subseteq D_t^* = D_v^* \subseteq D_v(\bar{c}_v) \cup S_v(\bar{c}_v).$$

Lemma 8. If $t$ is a join node with children $t'$ and $t''$, then $D_t(\bar{c}_t) \subseteq D_v(\bar{c}_v) \cup D_v(\bar{c}_v)$.

Proof. We claim that $D_t^* \setminus (D_t^v \cup D_v^*) \subseteq S_t(\bar{c}_v)$. Let $(a, b) \in D_t^* \setminus (D_t^v \cup D_v^*)$ and suppose, for a contradiction, that $(a, b) \notin S_t(\bar{c}_t)$. Then, for every $h^* \in H^*$ and $u \in V(G_{ab}) \cap X_t$, we have $d(h^*, u) > r/2$. Since $(a, b) \in D_t^*$, but $(a, b) \notin D_v^*$, there is $h' \in H^* \cap V(G_v) \setminus X_t$ that satisfies $(a, b)$. Similarly, there is $h'' \in H^* \cap V(G_v) \setminus X_t$. Now, $h', h'' \in V(G_{ab})$, but the diameter of $G_{ab}$ is at most $r$, thus $d(h', h'') \leq r$. Since $X_t$ separates $h'$ and $h''$, there is $u \in X_t$ with $d(h', u) + d(u, h'') \leq r$. Thus, either $d(h', u) \leq r/2$ or $d(h'', u) \leq r/2$, a contradiction. This implies $(a, b) \in S_t(\bar{c}_t)$.

Observe that $D_v^* \cup D_v^* \subseteq D_t^*$ and $S_t(\bar{c}_v) = S_v(\bar{c}_v) = S_v(\bar{c}_v)$. Combining with Lemmas 4 and 5, we get

$$D_t(\bar{c}_t) \subseteq D_t^* \setminus S_t(\bar{c}_t) \subseteq (D_v^* \setminus S_v(\bar{c}_v)) \cup (D_v^* \setminus S_v(\bar{c}_v)) \subseteq D_v(\bar{c}_v) \cup D_v(\bar{c}_v).$$

Combining Lemma 2 and Lemmas 6–8, we can show that the algorithm does not open too many hubs.

Lemma 9. $A_t(\bar{c}_t) \leq |H^* \cap V(G_t)|$.

Proof. Assume the lemma holds for the children of $t$. For a leaf node, the output set is empty, and the inequality is satisfied trivially.

Let $t$ be an introduce node with child $t'$ and $u \in X_t \setminus X_v$. From Lemmas 2 and 6, we know that $\bar{c}_v \in I_t(\bar{c}_t)$ and $D_t(\bar{c}_t) \subseteq D_v(\bar{c}_v)$. Thus, if $\bar{c}_t(u) = 0$, we have $u \in H^*$ and $A_t(\bar{c}_t) = A_v(\bar{c}_v) + 1 \leq |H^* \cap V(G_v)| + 1 = |H^* \cap V(G_t)|$. Otherwise, $A_t(\bar{c}_t) = A_v(\bar{c}_v) \leq |H^* \cap V(G_v)| = |H^* \cap V(G_t)|$. 

Let \( t \) be a forget node with child \( t' \) and \( u \in X_t \setminus X_t \). From Lemmas 2 and 7, we know that \( \tilde{c}_v \in F_t(\tilde{c}_t) \) and \( D_t(\tilde{c}_t) \subseteq D_t(\tilde{c}_v) \cup S_t(\tilde{c}_v) \). Thus, \( A_t(\tilde{c}_t) \leq A_t(\tilde{c}_v) \leq |H^* \cap V(G_v)| = |H^* \cap V(G_t)| \).

Let \( t \) be a join node with children \( t' \) and \( t'' \). From Lemmas 2 and 8, we know that \( (\tilde{c}_v, \tilde{c}_w) \in J_t(\tilde{c}_t) \) and \( D_t(\tilde{c}_t) \subseteq D_t(\tilde{c}_v) \cup D_t(\tilde{c}_w) \). Let \( H' \) and \( H'' \) be the output solutions corresponding to \( t' \) and \( t'' \), respectively. We have

\[
|H| = |H'| + |H''| - |H' \cap H''| \\
\leq A_t(\tilde{c}_v') + A_t(\tilde{c}_w') - h(\tilde{c}_t) \\
\leq |H^* \cap V(G_v)| + |H^* \cap V(G_w)| - h(\tilde{c}_t) \\
= |H^* \cap V(G_t)|. \]

Now we can state the main result.

**Theorem 5.** For every \( \epsilon > 0 \), there is a parameterized \((2 + \epsilon)\)-approximation algorithm for \textsc{MAkHC} running in time \( O^*((tw/\epsilon)^{O(tw)}) \).

**Proof.** Consider a preprocessed instance \((G, C, H, D, k)\) of \textsc{MAkHC}, in which the optimal value \( \text{OPT} \) is an integer bounded by \( O(\frac{1}{\epsilon}|V(G)|) \). We run the dynamic programming algorithm for each \( r = 1, 2, \ldots \), and output the first solution with no more than \( k \) hubs. Next, we show that the dynamic programming algorithm either correctly decides that there is no solution of cost \( r \) that opens \( k \) hubs, or finds a solution of cost \((1+\epsilon)2r\) that opens \( k \) hubs. Thus, when the main algorithm stops, \( r \leq \text{OPT} \), and the output is a \((2 + \epsilon')\)-approximation, for a suitable \( \epsilon' \).

Assume \( H^* \) is a solution that satisfies each demand with cost \( r \) with minimum size. Recall \( t_0 \) is the root of the tree decomposition and \( c_\emptyset \) is the coloring of an empty bag. If \( A_{t_0}(c_\emptyset) \leq k \), then Lemma 1 states that the dynamic programming algorithm outputs a set of hubs \( H \) of size at most \( k \) that satisfies each demand in \( D_{t_0}(c_\emptyset) = D \) with cost \((1+\epsilon)2r\). Otherwise, \( k < A_{t_0}(c_\emptyset) \), and Lemma 9 implies \( k < A_{t_0}(c_\emptyset) \leq |H^* \cap V(G_{t_0})| = |H^*| \). Thus, by the minimality of \( H^* \), there is no solution of cost \( r \) that opens \( k \) hubs.

Finally, we bound the running time. Let \( n = |V(G)| \). The tree decomposition has \( O(tw \cdot n) \) nodes and, for each node \( t \), the number of colorings is \( |\Sigma|^{O(tw)} \). Also, each recurrence can be computed in time \( O^*(|\Sigma|^{O(tw)}) \). Since \( r = O(\frac{1}{\epsilon} n) \),
and $\delta = \Theta(\frac{\epsilon}{\text{tw} \cdot \log n})$, the size of $\Sigma$ is

$$|\Sigma| = \mathcal{O}\left(\frac{\log r}{\log(1 + \delta)}\right) = \mathcal{O}\left(\frac{\log n + \log(1/\epsilon)}{\delta}\right)$$

$$= \mathcal{O}\left((\frac{\text{tw}/\epsilon}{\log^2 n + \log n \log(1/\epsilon)})\right) = \mathcal{O}\left((\frac{\text{tw}/\epsilon}{\log^2 n})\right).$$

Notice that $\mathcal{O}(\log^{O(\text{tw})} n) = \mathcal{O}^*(2^{O(\text{tw})})$, thus the total running time is bounded by $\mathcal{O}^*\left(|\Sigma|^{\mathcal{O}(\text{tw})}\right) = \mathcal{O}^*\left((\frac{\text{tw}/\epsilon}{\log^2 n})\right).$ □

### 6 Final remarks

Our parameterized $(2 + \epsilon)$-approximation algorithm circumvents hardness barriers coming from both classical and parameterized complexity theories. Improving on the 3-approximation is NP-hard and, as we note, W[2]-hard even if we take $r$ as a constant and parameterize by $k$. Thus, since we drop $k$ as parameter and take $r$ as part of the input, parameterizing by treewidth is a necessary condition of the algorithm to break the 3-approximation lower bound. Approximating is also necessary, as the problem on planar graphs is W[1]-hard for pathwidth and several other parameters.

These results are analogous to $k$-CENTER, which has a 2-approximation lower bound and does not admit an FPT algorithm. Unlike $k$-CENTER, however, we left open whether MA$k$HC admits an EPAS when parameterized by treewidth. The challenge seems to be the non-locality of the paths serving the demands, thus established techniques are not sufficient to tackle this issue. In this paper, we show how to compute a special subset of demands that must be served locally for each subproblem. We hope this technique may be of further interest. A possible direction of research is to consider the single allocation variant in the two-stop model, which is a well-studied generalization of MA$k$HC [16,3].

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A The planar case

In this section, we give a \((2 + \varepsilon)\)-approximation algorithm parameterized by \(k\) and \(r\), when the input is restricted to unweighted planar graphs. This algorithm can be seen as another way to challenge the approximation lower bound presented in Section 3. Indeed, by Theorem 3, finding a \((3 - \varepsilon)\)-approximation parameterized by \(k\) and \(r\) is \(W[2]\)-hard for unweighted graphs, even when \(r\) is a constant. Thus, we restrict the input to planar graphs, but get a better approximation factor.

The algorithm is built upon the bidimensionality framework and follows the arguments for \(k\)-Center by Demaine et al. [14]. In the following, let \((G, C, H, D, k, r)\) be a positive instance of MA\(_k\)HC such that \(G\) is an unweighted planar graph.

Lemma 10. If \(G\) has a \((\rho \times \rho)\)-grid as minor, then \(\rho \leq \sqrt{k}(2r + 1) + 2r\).

Proof. We begin with a series of definitions.

Let \(F\) be a \((\rho \times \rho)\)-grid, where

\[
V(F) = \{1, \ldots, \rho\} \times \{1, \ldots, \rho\} \quad \text{and} \\
E(F) = \{((x, y), (x', y')) : |x - x'| + |y - y'| = 1\}.
\]

Let \(V_{\text{ext}}\) be the set of vertices of \(F\) whose degrees are smaller than 4. We assume the vertices of \(V_{\text{ext}}\) belong to the external face of some embedding of \(F\) and call the other faces internal.

Let \(V_{\text{int}}\) be the set of vertices of \(F\) that have distance at least \(r\) from every vertex in \(V_{\text{ext}}\). Note that \(V_{\text{int}}\) induces a subgraph \(F[V_{\text{int}}]\) that is a subgrid of \(F\) with \(|V_{\text{int}}| = (\rho - 2r)^2\).

Define \(\delta((x, y), (x', y')) = \max\{|x - x'|, |y - y'|\}\) and let \(d_F((x, y), (x', y'))\) be the distance from \((x, y)\) to \((x', y')\) in \(F\).

Let \(J\) be the supergraph of \(F\) with the same set of vertices and with the additional set of (diagonal) edges:

\[
\{((x, y), (x + 1, y + 1)), ((x, y + 1), (x + 1, y)) : 1 \leq x, y < \rho\}.
\]

Let \(R\) be a subgraph of \(J\) and let \(d_R(u, v)\) be the length of a shortest path from \(u\) to \(v\) in \(R\). Observe that, for every \(u, v \in V(R)\), we have \(\delta(u, v) \leq d_R(u, v)\). Define \(N'_R(u) = \{v : d_R(u, v) \leq \ell\}\).
For a \((x, y) \in V(F)\) and an integer \(\ell\), define

\[
B_\ell((x, y)) = \{(x', y') : \delta((x, y), (x', y')) \leq \ell\}.
\]

Now, consider a sequence of edge contractions and removals which transforms \(G\) into a minor isomorphic to \(F\) using a maximal number of edge contractions. Let \(H\) be the result of applying only the contractions of that sequence to \(G\), and consider an embedding of \(H\) in the plane that corresponds to an embedding of \(F\). Partition the edges of \(H\) in three sets: the edges that occur in \(F\), the set \(E_1\) that connect non-adjacent vertices of an internal face of \(F\), and the set \(E_2\) with all other edges. Note that edges in \(E_2\) are only incident with vertices in \(V_{\text{ext}}\).

Call \(R\) the graph we obtain by adding edges \(E_1\) to \(F\), and note that \(R\) is a subgraph of \(J\). Then, for a vertex \(u\) of \(R\) and an integer \(\ell\), we have that \(N_\ell^R(u) \subseteq B_\ell(u)\). Observe that the set of edges of \(H\) is \(E(R) \cup E_2\). For a vertex \(u \in V_{\text{int}}\), we claim that \(N_{\ell}^H(u) \subseteq B_{\ell}(u)\). This holds because paths of length at most \(\ell\) starting at a vertex of \(V_{\text{int}}\) do not use edges of \(E_2\) and, as a consequence, \(N_{\ell}^H(u) = N_{\ell}^R(u)\).

Let \(S\) be a solution for the instance of \(\text{MA}_{k}^\text{HC}\). Observe that the distance between every client and a hub of \(S\) is at most \(r\), since every vertex is in some set \(V(G_{ab})\). Also, note that, for vertices \(u \text{ and } v \) of \(G\) associated with vertices \(u' \text{ and } v' \) of \(H\), \(d_H(u', v') \leq d_G(u, v)\), as \(H\) is obtained from \(G\) using only edge contractions.

Define a set of vertices:

\[
Y = V_{\text{int}} \cap \{(2r + 1)i + r + 1, (2r + 1)j + r + 1 : i, j \in \mathbb{Z}_{\geq 0}\}.
\]

The size of this set is \(|Y| \geq \left[\frac{\theta - 2r}{2r + 1}\right]^2 \geq \left(\frac{\theta - 2r}{2r + 1}\right)^2\).

For distinct \(y, y' \in Y\), we have \(B_r(y) \cap B_r(y') = \emptyset\). Also, there must exist a hub in \(S\) that is associated with some vertex in \(N^r_H(y) \subseteq B_r(y)\). Therefore, each \(y \in Y\) is associated to one unique hub in \(S\), and finally,

\[
k \geq |Y| \geq \left(\frac{\theta - 2r}{2r + 1}\right)^2.
\]

\textbf{Corollary 1.} \(\text{tw}(G) \leq 6\sqrt{k}(2r + 1) + 12r + 1\).
Proof. Robertson, Seymour and Thomas\textsuperscript{1} prove that, if $G$ has no $(\rho+1)\times(\rho+1)$-grid as a minor, then $\text{tw}(G) \leq 6(\rho + 1) - 5$. Let $\rho$ be the largest integer for which $G$ has a $(\rho \times \rho)$-grid as a minor. Then, using Lemma 10, we have that $\text{tw}(G) \leq 6\sqrt{k}(2r + 1) + 12r + 1$.

Using the previous bound and Theorem 5, we get the main result of this section.

**Theorem 6.** For every $\epsilon > 0$, there is a parameterized $(2 + \epsilon)$-approximation algorithm for MA$k$HC when the parameters are $k$ and $r$, and the input graph is unweighted and planar.

Notice that a version of the dynamic programming algorithm presented in Section 4 that stores distances exactly is a 2-approximation parameterized by $\text{tw}$ and $r$. Thus, for MA$k$HC with unweighted planar graphs, there is actually a 2-approximation algorithm parameterized by $k$ and $r$.

\textsuperscript{1} Theorem 6.2, N. Robertson, P. Seymour, and R. Thomas. Quickly excluding a planar graph. Journal of Combinatorial Theory, Series B, 1994.