WELL-POSEDNESS OF EINSTEIN’S EQUATION WITH REDSHIFT DATA

CHRISTOPHER J. WINFIELD

Abstract. We study the solvability of a system of ordinary differential equations derived from null geodesics of the LTB metric with data given in terms of a so-called redshift parameter. Data is introduced along these geodesics by the luminosity distance function. We check our results with luminosity distance depending on the cosmological constant and with the well-known FRW model.

INTRODUCTION

Resulting from the Lemaître-Tolman-Bondi metric

\[ ds^2 = -dt^2 + \frac{R'(t, r)^2 dr^2}{1 + 2E(r)} + R(t, r)^2 d\Omega^2 \]

1991 Mathematics Subject Classification. 83F05, 34A34.

Key words and phrases. redshift parameter, ordinary differential equations, LTB cosmology, luminosity distance.

The author thanks the Mathematics Departments of UW-Madison and UW-Oshkosh and the UW-Madison Physics Department for the use of their resources during the course of the present research. The author would particularly like to thank Prof. Daniel Chung of the UW-Madison Physics Department for his conversations and suggestions leading to the present article.
are so-called symmetric dust solutions to the Einstein equation given by

\begin{equation}
\left( \frac{\dot{R}}{R} \right)^2 = \frac{2E}{R^2} + \frac{2M}{R^3}
\end{equation}

\begin{equation}
\rho(t, r) = \frac{M'(r)}{R(t, r) \dot{R}(t, r)}
\end{equation}

(c.f. [17]) for some suitable \( \rho \) (energy density) where superscript \( t \) and \( \cdot \) denote partial derivatives with respect to \( r \) and \( t \), respectively. Setting \( \sigma \overset{\text{def}}{=} \text{sgn}\dot{R} \), \( \delta \overset{\text{def}}{=} \text{sgn}R' \), \( A \overset{\text{def}}{=} B\sqrt{1+2E} \) and \( B \overset{\text{def}}{=} \sigma\sqrt{2E+2M/R} \), we study resulting system

\begin{align}
\frac{dr}{dz} &= \frac{\sqrt{1+2E}}{(1+z)\partial^2_t, R(t, r)} = \frac{A}{E' + M'/R - MR'/R^2} \\
\frac{dt}{dz} &= \frac{-|R'|}{(1+z)\partial^2_t, R(t, r)} = \frac{-BR'\delta}{E' + M'/R - MR'/R^2},
\end{align}

taken along null geodesics of (0.1). Here data is given for the function \( R \), prescribing values \( R(t(z), r(z)) \) along curves given by (0.4) and (0.5). As a result, corresponding solutions of this system provide maps

\begin{equation}
(E(r), D_L(z), R_0(r)) \rightarrow (r(z), t(z), M(r(z))
\end{equation}

as introduced in [6] which we study in some detail in this article.

As an application of our analysis, we will consider data given in the form

\begin{equation}
R(t(z), r(z)) = \frac{D_L(z)}{(1+z)^2} = \frac{\int_1^{1+z} I(y)dy}{1+z}
\end{equation}

for \( D_L(z) = (1+z)\int_1^{1+z} I(y)dy \) with \( I(y) = 1/\sqrt{\Omega_{\Lambda} + (1-\Omega_{\Lambda})y^2} \) for a real parameter \( 0 \leq \Omega_{\Lambda} \leq 1 \). Here, \( D_L \) is generally referred to as "luminosity distance"
and, in particular models, $\Omega_\Lambda = \frac{\Lambda}{3H^2_0}$ is directly proportional to the so-called "cosmological constant" $\Lambda$ [3, 5]. For further details of the physical and mathematical derivation of the present problem, the author recommends the aforementioned articles along with [7, 9, 10, 15, 19, 14] - to name but a few.

This work is physically motivated by competing cosmological theories in explaining certain observations of matter distribution and cosmic inflation. Such theories include those of "dark energy" [8], certain metric perturbations from the FRW model [11, 16, 13], radial inhomogeneities of the unperturbed LTB model (via $E(r)$, $R_0(r)$ and $M(r)$), and the cosmological constant (here via $D_L$) - with our work involving the later two. Here, we study the map (0.6) mostly on purely mathematical grounds, presenting a framework of analysis and, in a special case, estimates on resulting functions $M$ in terms of $z$. Furthermore, we test our results for certain functions $D_L$, $E$, and $R_0$, arising from various FRW-type models, and study singularities of $M$ as indications of (in-) compatibility of these models.

1. Singularities

From the Chain Rule (c.f. equation (14) [6]) we observe that $R'$ takes the form

$$R' = \mathcal{F}(R, R_0, R'_0, E, E', M, t) + M'\mathcal{G}(R, R_0, E, E', M).$$
With \( \dot{R} = \sigma \sqrt{2E + 2M/R} \) and \( R_0(r) \stackrel{\text{def}}{=} R(r, t_0) \) for a fixed \( t_0 > 0 \), we restrict \( R, R_0, t > 0, M \geq -ER, E > 0 \) and set

\[
J(R, R_0, M, E, t) \stackrel{\text{def}}{=} \sqrt{2}(t - t_0) - \sigma \int_{R_0}^{R} \frac{\tau}{\tau E + M} d\tau = 0
\]

solutions of which define smooth manifolds \( \mathcal{O}^\pm \) depending on constant \( \sigma = \pm 1 \), respectively.

We introduce notation: For a given function \( f = f(t, r) \), depending implicitly or explicitly on \((t, r)\), we will denote \( f[z] \stackrel{\text{def}}{=} f(t(z), r(z)) \) and, with slight abuse of notation, set \( \frac{df}{dz} \stackrel{\text{def}}{=} \frac{df[z]}{dz} \). We now set

\[
(1.2) \quad R' = \mathcal{F} + \mathcal{G} \frac{dM/dz}{dr/dz}
\]

where, from the chain rule, with subscript denoting the associated partial derivative,

\[
(1.3) \quad -(\partial_R J) \mathcal{F} = E' \partial_E J + R_0' \partial_{R_0} J
\]

\[
(1.4) \quad -(\partial_R J) \mathcal{G} = \partial_M J.
\]

with \( \xi \stackrel{\text{def}}{=} M/R, \xi^2 \stackrel{\text{def}}{=} M/R_0, \xi^\sharp \stackrel{\text{def}}{=} R_0/R, \) and

\[
(1.5) \quad J_R = -\sigma/\sqrt{E + \xi} \quad J_{R_0} = \sigma/\sqrt{E + \xi^\sharp} \quad J_M = -\frac{\sigma}{2} \int_{1}^{b} \frac{\nu^{1/2} d\nu}{(E\nu + \xi)^{3/2}} \quad J_E = -\frac{\sigma R}{2} \int_{1}^{b} \frac{\nu^{3/2} d\nu}{(E\nu + \xi)^{3/2}}
\]
WELL-POSEDNESS OF EINSTEIN’S EQUATION

Substituting (1.2) into equations (0.4) and (0.5), we obtain

\begin{equation}
(1.6) \quad \left( E' - \frac{M}{R^2} F \right) \frac{dr}{dz} + \frac{dM}{dz} = \frac{M dM}{dz} G = A
\end{equation}

\begin{equation}
(1.7) \quad \left( E' - \frac{M}{R^2} F \right) \frac{dr}{dz} + \frac{dM}{dz} = \left( \frac{dM}{dz} - \frac{M dM}{dz} G \right) \frac{dM}{dz} \frac{dt}{dz} = -\delta \cdot B \cdot \left( F + \frac{dM}{dz} \frac{dr}{dz} G \right) \frac{dr}{dz}
\end{equation}

We then substitute

\begin{equation}
\left( E' - \frac{M}{R^2} F \right) \frac{dr}{dz} = A - \left( \frac{dM}{dz} - \frac{M dM}{dz} G \right)
\end{equation}

so that equation (1.7) becomes

\begin{equation}
(1.8) \quad A \frac{dt}{dz} = -\delta \cdot B \cdot \left( F \frac{dr}{dz} + \frac{dM}{dz} G \right).
\end{equation}

Equation (1.8) can be verified by equations (0.2) and (0.3).

Now, equations (1.6), (1.8), and (1.2) along with the Chain Rule result in the following system:

\begin{align*}
\left( E' - \frac{M}{R^2} F \right) \frac{dr}{dz} + \left( \frac{1}{R} - \frac{MG}{R^2} \right) \frac{dM}{dz} = & A \\
\delta B J \frac{dr}{dz} + A \frac{dt}{dz} + \delta B G \frac{dM}{dz} = & 0 \\
F \frac{dr}{dz} + \sigma \sqrt{(2E + 2M/R)} \frac{dt}{dz} + \varphi \frac{dM}{dz} = & \frac{dR}{dz}
\end{align*}

which we may write in matrix form as

\begin{equation}
(1.9) \quad \mathcal{U} \frac{dX}{dz} = \mathbf{Y}
\end{equation}
for

\[
\mathcal{U} \overset{\text{def}}{=} \begin{pmatrix}
E' - \frac{M}{R^2} F & 0 & \frac{1}{R} - \frac{MG}{R^2} \\
\delta BF & A & \delta GB \\
F & \sigma \sqrt{2E + 2M/R} & G
\end{pmatrix}
\]

\[
\vec{X} = \begin{pmatrix} r \\ t \\ M \end{pmatrix}, \quad \vec{Y} = \begin{pmatrix} A \\ 0 \\ \frac{4R}{\pi^2} \end{pmatrix}.
\]

We check the invertibility of \( \mathcal{U} \) as we compute

\[
\det \mathcal{U} = (E' - \frac{M}{R^2} F)(AG - \sigma \delta GB \sqrt{2E + 2M/R})
\]

\[
+ \left( \frac{1}{R} - \frac{MG}{R^2} \right)(\delta BF \sqrt{2E + 2M/R} - FA)
\]

\[
= B(E'G - \frac{F}{R})(\sqrt{1 + 2E} - \sigma \delta \sqrt{2E + 2M/R})
\]

From these computations we conclude

**Proposition 1.10.** Suppose that \( E, R > 0 \) with \( \partial_t R, \partial_r R, \partial_{tt} R \neq 0 \). Then, \( \mathcal{U}^{-1} \)

is a smooth function of \( R, R_0, E, E', \text{ and } M \) except for the following cases:

Either

1.) both \( \delta = \sigma \) and \( R = 2M \); or,

2.) \( E'R G = F \).

We may extend the domain of \( \mathcal{U} \) to include \(-1/2 < E < 0\), say, but for simplicity we impose the above hypothesis throughout the rest of this section.

We continue with
Proposition 1.11. Suppose that for some $z^* > 0$, $\delta[z^*] = \sigma[z^*]$ and that
\[ \frac{dR(z)}{dz}|_{z=z^*} = 0. \] Then, the matrix $U[z]$ is singular at $z = z^*$.

Proof. We have from the Chain Rule and equations (0.4) and (0.5) that
\[
\frac{dR}{dz} = R' \frac{dr}{dz} + R_\tau \frac{dt}{dz}
= R' \frac{\sqrt{1 + 2E}}{(1 + z)\partial_{r,t}^2R} - R_\tau \frac{|R'|}{(1 + z)\partial_{r,t}^2R}
= R' \frac{\sqrt{1 + 2E} - \delta\sigma \sqrt{2E + 2M/R}}{(1 + z)\partial_{r,t}^2R}
\]
By our hypotheses on the partial derivatives of $R$ we may conclude
\[
\left(\sigma\delta \sqrt{2E + 2M/R}\right)[z^*] = \left(\sqrt{1 + 2E}\right)[z^*]
\]
With $\sigma = \delta$ at $z = z^*$, we have that $\sigma\delta = \delta^2 = 1$ and that $\sqrt{2E + 2M/R} = \sqrt{1 + 2E}$ so that $2M[z^*] = R[z^*]$. Then from Proposition 1.10 we see that $\det U[z^*] = 0$. □

We note that the type of singularity of item 1) of Proposition 1.10 appears analogous to that of the well-known "Schwarzschild" singularity: It is not yet clear here if this is merely an artifact of the specific model or if such singularities are removable by passing to alternate coordinate systems or metrics (c.f. §31 [14], §6.4 [18]), taking us beyond the scope of the present article.

We may interpret item 2) of Proposition 1.10 in terms of the tangent bundles $T\mathcal{O}^\pm$ (resp.) of manifolds obtained from (1.1). We may consider the transformation $\phi^\pm : \mathbb{R} \times (0, +\infty) \to \mathcal{O}^\pm$ given by $\phi^\pm(t, r) \stackrel{\text{def}}{=} (R(t, r), R_0(r), E(r), M(r), t)$
and $d\phi$ as a push forward, to interpret corresponding solutions to
\[ E'R\partial_M J = E'\partial_E J + R'_0\partial_{R_0} J \]
as subsets $\mathcal{M}^\pm$ of $T\mathcal{D}^\pm$ (resp.) in coordinate form. Let $\mathcal{T}$ denote the set $(\Omega^+ \setminus \pi\mathcal{M}^+) \cup (\Omega^- \setminus \pi\mathcal{M}^-)$ where $\pi$ denotes the natural projection $\pi : TM \to \mathcal{M}$ of a manifold $\mathcal{M}$.

By calculating $U^{-1}\vec{Y}$ from (1.9) with $R_z \overset{\text{def}}{=} \frac{dR|z}{dz}$, we arrive at the following system of ordinary differential equations:

\[
\begin{align*}
\frac{dr}{dz} &= \frac{GAR}{GE'R - F} - \frac{R_z \cdot (MG - R)\sqrt{1 + 2E}}{R \cdot (\delta \sqrt{2E} + 2M/R - \sqrt{1 + 2E})(GE'R - F)} \\
\frac{dt}{dz} &= \frac{\delta \cdot R_z}{\delta \sqrt{2E} + 2M/R - \sqrt{1 + 2E}} \\
d\frac{dM}{dz} &= -FRA \frac{R_z \cdot (R^2 E' - FM)\sqrt{1 + 2E}}{R \cdot (\delta \sqrt{2E} + 2M/R - \sqrt{1 + 2E})(GE'R - F)}
\end{align*}
\]

We are ready to state

**Proposition 1.14.** The matrix $U$ is non-singular for $R \neq 2M$ provided $(R, R_0, E, M, t) \in \mathcal{T}$. Indeed, if for some $z_0 > 0$, these conditions hold for $(R, R_0, E, M, t)[z]|_{z=z_0}$, then the system of equations (1.13) has a unique $C^\infty$ solution $\vec{X}[z]$ in some open interval containing $z_0$.

**Proof.** It is clear that the elements of $U$ are continuously differentiable where $\det U$ is non-zero. The result follows by applying standard theory of ordinary differential equations [4]. □
To further investigate the solvability of the system (1.13), we compute

\[
GE'R - F = \frac{E'R \cdot (J_E/R - J_M) + R'_0 J_R}{J_R} = \sqrt{E + \xi} \left( \frac{RE'}{2} \int_1^b \frac{s^{1/2}(s-1)ds}{(Es + \xi)^{3/2}} - \frac{R'_0}{\sqrt{E + \xi^2}} \right).
\]

Since \( \frac{\nu}{(Ev + \xi)^{3/2}} \leq \frac{4}{27E\xi^2} \) we find

\[
\int_1^b \frac{\nu^{1/2}(\nu - 1)d\nu}{(Ev + \xi)^{3/2}} \leq \frac{2}{\xi\sqrt{27E}} \int_1^b (\nu - 1)d\nu = \frac{(b - 1)^2}{\xi\sqrt{27E}}
\]

Lacking any other simplifying assumptions, we thus obtain strong criteria for local solvability:

**Proposition 1.16.** System (1.13) is locally solvable at any point of \( \mathfrak{D}^\pm \) where \( \delta \neq \sigma \) or where \( 2M = R \) if either of the following holds:

1) \( \text{sgn} E' \neq \text{sgn} R'_0 \)

2) \( \left| \frac{E'(R_0 - R_0^2)}{2M \sqrt{27E}} \right| < \left| \frac{R'_0 \sqrt{R_0}}{\sqrt{E(R_0 + M)}} \right| \)

Indeed, given \( r_0, t_0, M_0 > 0 \) and smooth \( E, R, R'_0 > 0 \), the system (1.13) has on an open interval \( I \ni z_0 \) a unique solution satisfying

\[
\vec{X}(z_0) = \begin{pmatrix} r_0 \\ t_0 \\ M_0 \end{pmatrix}
\]
2. Decoupled equations: A Case of Constant $E$

We consider the case of constant $E > 0$ in which we can rescale $M$ and $R$ to assume the case $E = 1$, retaining

\begin{equation}
\left(\frac{\dot{R}}{R}\right)^2 = \frac{2}{R^2} + \frac{2M}{R^3}
\end{equation}

Here, equations (1.13) reduce to

\begin{align*}
\frac{dr}{dz} &= \frac{-GAR}{F} + \frac{R_z \cdot (1 - \frac{M}{R}G)\sqrt{3}}{(\sqrt{3} - \sigma \delta \sqrt{2 + 2M/R})F} \\
\frac{dt}{dz} &= \frac{-\delta \cdot R_z}{\sqrt{3} - \sigma \delta \sqrt{2 + 2M/R}} \\
\frac{dM}{dz} &= RA + \frac{R_z \frac{M}{R} \sqrt{3}}{\sqrt{3} - \sigma \delta \sqrt{2 + 2M/R}}
\end{align*}

with $A = \frac{\sigma \sqrt{3} \sqrt{1 + M/R}}{1+\xi}$.

For the remainder of the section we assume that $E, \sigma, \delta \equiv 1$ and denote by $\mathcal{T}_1$ the corresponding subset of $\mathcal{T}$. Then, $R(t, r) < R_0(r) \forall t < t_0$. And, for $h$ and $\xi$ as above, we obtain

\begin{align*}
\partial_M J &= -\frac{1}{2} \int_1^h \frac{1}{\sqrt{\nu + \xi}} \frac{1}{\sqrt{\nu + \xi}} d\nu \\
\partial_{R_0} J &= \frac{1}{\sqrt{1 + \xi^2}} \quad \partial R J = -\frac{1}{\sqrt{1 + \xi}}
\end{align*}
with $\xi \geq \xi^2$ and $\mathfrak{h} \geq 1$, so that the following hold:

\begin{align*}
0 \leq J_1(r, z, \xi) &\overset{\text{def}}{=} -\frac{G}{\mathcal{F}} = \frac{\sqrt{1 + \xi^2}}{2R_0'} \int_1^h \sqrt{\frac{\nu}{\nu + \xi \nu + \xi}} d\nu \\
&\leq \frac{\sqrt{\mathfrak{h}(1 + \xi^2)}}{R_0'} \left( \frac{1}{\sqrt{1 + \xi}} - \frac{1}{\sqrt{\mathfrak{h} + \xi}} \right) \leq \sqrt{\frac{\mathfrak{h}}{2R_0'(1 + \xi)}},
\end{align*}

\begin{align*}
0 < \frac{1}{\mathcal{F}} = \frac{1}{R_0'} \sqrt{1 + \xi^2} &\overset{\text{def}}{=} J_2(r, z, \xi) \leq 1/R_0' ;
0 < \frac{1 - \xi G}{\mathcal{F}} = J_2 + \xi J_1 \leq \frac{1 + \sqrt{\mathfrak{h}}/2}{R_0'} .
\end{align*}

Our change of variables leads to

\[ \frac{d\xi}{dz} = \frac{dM}{dz}/R - Rz\xi/R \]

with $A = \frac{\sqrt{6\sqrt{1 + \xi}}}{1 + z}$ whereby the system (1.13) now reduces further to

\begin{align*}
\frac{dr}{dz} &= \frac{RJ_1\sqrt{6\sqrt{1 + \xi}}}{1 + z} + \frac{\sqrt{3}Rz \cdot (J_2 + \xi J_1)}{\sqrt{3} - \sqrt{2 + 2\xi}} \\
\frac{dt}{dz} &= \frac{-Rz}{\sqrt{3} - \sqrt{2 + 2\xi}} \\
\frac{d\xi}{dz} &= \frac{\sqrt{6\sqrt{1 + \xi}}}{1 + z} + \xi \frac{R_0}{R} \left( \frac{\sqrt{3} \sqrt{1 + \xi}}{\sqrt{3} - \sqrt{2 + 2\xi}} \right).
\end{align*}

Here, we note that the equation for $\frac{d\xi}{dz}$ decouples from the others, allowing for $\xi$ to be solved for explicitly in $z$. Then, with the solution to $\xi(z)$ in hand, both $\mathcal{I}_1$ and $\mathcal{I}_2$ depend only on $z$ and $r$ whereby the remaining equations are then decoupled.

We give estimates for the system (2.4) assuming uniform bounds on $R$, $M/R$, $R_z$, $R_0$, and $R_0'$. We suppose the following bounds hold for $0 < z_0 \leq z \leq z_1$ and
\(0 < r, M, t\) on some compact sets (to be determined): \(\xi \leq \xi^*\) with \(|2\xi - 1| \geq \epsilon\)

\(> 0; \rho_{min} \leq R \leq \rho_{max}; |R_{z}| \leq \lambda; 1 < b \leq b^*;\) and, \(|R_0'| \geq r > 0.\) Here, applying

\[\frac{dr}{dz} \leq \sqrt{3} \rho_{max} \sqrt{b^*/2 + (1 + \sqrt{b^*/2})M_1} \quad \text{def} = M_2\]

\[\frac{d\xi}{dz} \leq \sqrt{3}(1 + \xi^*)(\sqrt{2} + M_1 \xi^*/\rho_{min}) \quad \text{def} = M_3\]

\(\text{Let } M = \max_j \{M_j\}_{j=1}^3 \text{ and suppose } r_0, t_0, \text{ and } M_0/R[z_0] \text{ def } \xi_0 \neq 1/2 \text{ satisfy the restrictions on } (r, t, \xi) \text{ for some } 0 < \xi_0 < \xi^* \text{ as above with}\)

\[\vec{X}_0 = \vec{X}(z_0) = \begin{pmatrix} r_0 \\ t_0 \\ M_0 \end{pmatrix}\]

For an interval \(I\) of the form \(0 \leq z_0 \leq z \leq z_1,\) the following now results from standard theory of differential equations [4]:

**Proposition 2.7.** For \(z_0 \geq 0,\) the system (2.2) is solvable on an interval of the form \(I = \{z | z_0 \leq z \leq z_1\}\) provided that the conditions (2.6) and (2.5) hold for \(\vec{X} \text{ in subset of } T_1 \text{ given by } |(\vec{X} - \vec{X}_0)_j| \leq b : j = 1, 2, 3 \text{ for some constant } b < 1/M.\) Here, a unique solution may be computed by the method of successive approximations.
Proof. We may apply Theorem 3.1, Chapt. 1 [4]: The conditions assure Lipshitz continuity of the right-hand sides of (2.4) and that both \( z - z_0 \) and \( |\vec{X} - \vec{X}_0|/\mathfrak{M} \) are bounded above by \( |z_1 - z_0| \), so that the result follows. \( \square \)

Recalling that we set \( E \equiv 1 \), we will suppose for the rest of the section that \( R(z), R'_0[z] \) are smooth and positive for \( z > 0 \). For some of our analysis below we will suppose also that

\[(2.8) \quad R[z] > Cz|R_z|.
\]

holds on some real interval. We now present our estimates on \( M[z] \) depending on \( R[z] \) and initial conditions given by \( \xi_0 \equiv \xi(r(z_0), z_0) \).

**Theorem 1.** Suppose that (2.8) holds on some interval \( I = [z_0, z_1) \subset \mathbb{R}^+ \).

Then the following statements hold for some constants \( 0 < c_1 < 1/2 < c_2 \), each depending on the choice of \( C \):

1) If \( 0 < \xi_0 < 1/2 \) and \( R_z < 0 \), then \( M[z] \leq c_1 R[z] \) holds on \( I \).

2) If \( \xi_0 > 1/2 \) and \( R_z > 0 \), then \( M[z] \leq c_2 R[z] \) on \( I \).

**Proof.** Let us choose \( C < 1/2 \) and set \( \Delta_\xi \equiv \sqrt{3} - \sqrt{2 + 2\xi} \). In case 1) we use the estimate \( 1/\Delta_\xi \geq \sqrt{3}/(1 - 2\xi) \) for \( 0 < \xi < 1/2 \) so that from (2.4)

\[
\frac{d\xi}{dz} < \frac{\sqrt{3}}{z} \sqrt{1 + \xi} \left( 1 - \frac{1}{C} \frac{\xi}{1 - 2\xi} \right).
\]

Here, \( \frac{d\xi}{dz} < 0 \) for \( 1/2 > \xi > \xi_1^* \equiv C/(2C + 1) \). Let \( c_1 = \max\{\xi_0, \xi_1^*\} \).
In case 2) we note that \( 1/\Delta \xi \leq -\sqrt{3}/(2\xi - 1) \) for \( \xi > 1/2 \). We find

\[
\frac{d\xi}{dz} < \frac{\sqrt{6}}{2} \sqrt{1 + \xi} \left( 1 - \frac{1}{C 2\xi - 1} \right)
\]

and \( \frac{d\xi}{dz} < 0 \) for \( 1/2 < \xi < \xi_* \) \( \text{def} = C/(2C - 1) \). Let \( c_2 = \max\{\xi_0, \xi_*\} \). □

**Theorem 2.** Suppose \( R_z < 0 \) on \( I = [z_0, \infty) \) with \( z_0 > 0 \) and \( \xi_0 > 1/2 \). Then, for \( \rho = \sqrt{3/2} \), there are positive constants \( \alpha, c_3 \) and \( c_4 \) so that the following holds on \( I \):

\[
c_3 \left( (R[z])^{-(\rho - 1)} + R[z] \ln \left( \frac{1 + z}{1 + z_0} \right) \right) \leq M[z] \leq c_4 \left( 1 + \ln(1 + z) \right) (R[z])^{(\rho - 1/2)}
\]

**Proof.** We first note that since \( R_z/\Delta \xi > 0 \) on \( I \), we find from (2.4) that \( \frac{d\xi}{dz} > 0 \).

Let us set \( K_\xi = -\sqrt{3}/\Delta \xi \), noting that \( \rho = \sqrt{3/2} < K_\xi \leq K_\xi_0 \) for \( \xi > 1/2 \) is decreasing as function of \( \xi \) and, in turn, also as a function of \( z \). Recalling that \( R_z < 0 \), we find

\[
\frac{d\xi}{dz} + \rho \xi \frac{R_z}{R[z]} \geq \frac{d\xi}{dz} + K_\xi \xi \frac{R_z}{R[z]} \geq \frac{\sqrt{6}}{\Delta \xi_0} \sqrt{1 + \xi_0} \leq \sqrt{6} \sqrt{1 + \xi_0} \frac{R_0[z]}{1 + z};
\]

\[
\frac{d}{dz} (\xi R^\rho) \geq \sqrt{6} \sqrt{1 + \xi_0} R^\rho[z] \frac{R_0[z]}{1 + z}.
\]

Now, by the monotonicity of \( R[z] \),

\[
\xi[z] \geq R^{-\rho}[z] \left( \xi_0 R^\rho[z_0] + \sqrt{6} \sqrt{1 + \xi_0} \int_{z_0}^z \frac{R_0[s]}{1 + s} ds \right)
\]

\[
\geq R^{-\rho}[z] \left( \xi_0 R^\rho[z_0] + R^\rho[z] \sqrt{6} \sqrt{1 + \xi_0} \int_{z_0}^z \frac{ds}{1 + s} \right)
\]

After multiplying through by \( R \), it is clear that we may choose \( c_3 \leq \min\{\xi_0(R[z_0])^\rho, \sqrt{6} \sqrt{1 + \xi_0}\} \).
Now, let us set $\rho_0 \overset{\text{def}}{=} K_{\xi_0}$ and $q_0 \overset{\text{def}}{=} (\xi_0 + 1)/\xi_0$. Then, for obvious substitution defining $\xi[s]$,

$$\frac{d\xi}{dz} + \rho_0 \xi \frac{R_z}{R[z]} \leq \sqrt{6(1 + \xi)};$$

$$\frac{d}{dz} (\xi R^\omega) \leq \sqrt{6(1 + \xi)} \frac{R_{\rho_0} [z]}{1 + z}$$

$$\xi[z] \leq R^{-\rho_0} [z] \left( \xi_0 R^\rho [z_0] + \sqrt{6} \int_{z_0}^{z} \frac{\sqrt{1 + \xi[s]} R^\rho [s]}{1 + s} ds \right)$$

$$\leq R^{-\rho_0} [z] \left( \xi_0 R^\rho [z_0] + R^\rho [z_0] \sqrt{6(1 + \xi[z])} \int_{z_0}^{z} \frac{ds}{1 + s} \right)$$

$$\sqrt{\frac{\xi[z]}{q_0}} < \frac{\xi[z]}{\sqrt{1 + \xi[z]}} \leq R^{-\rho_0} [z] \left( \frac{R^\rho [z_0] \sqrt{\xi_0}}{\sqrt{1 + \xi_0}} + \sqrt{6} R^\rho [z_0] \int_{z_0}^{z} \frac{ds}{1 + s} \right)$$

$$\xi[z] < q_0 R^{2\rho} [z_0] R^{-2\rho} [z] \left( 1 + \sqrt{6} \int_{0}^{z} \frac{ds}{1 + s} \right)^2$$

noting that $\xi^2/(1 + \xi) \geq \xi/q_0$. Choosing $c_4 \geq q_0 6 R^{2\rho} [z_0]$, the result follows by multiplying through by $R$. \hfill \Box

We see that Theorems 1 and 2 can apply for $R[z] = R_{\Omega_\Lambda}[z]$ (modulo a rescaling factor) as above for certain values of $\Omega_\Lambda$: We denote by $I_{\Omega_\Lambda}^{\pm}$ the subset of $(0, \infty)$ for which $\pm R_z > 0$ and we replace $C$ by $C^\pm$ in the case that (2.8) holds, respectively.

Remark 2.9. For $R_{\Omega_\Lambda}[z]$ as in (0.7) we find that when $\Omega_\Lambda = 1$ there is to every interval of the form $(0, z_2)$, an associated $C^+$ depending on $z_2 > 0$. Moreover, for every $0 \leq \Omega_\Lambda < 1$ there is a $z_\Lambda > 0$ where for every positive $z^\pm$ with $z^\pm \geq z_\Lambda$...
there is a $C^\pm$ associated to $(0, z^\pm)$ and $(z^-, \infty)$, respectively. [The singularities
$z_\Lambda$ will be discussed in further detail in Section 3.]

Proof. For $\Omega_\Lambda = 1$ we find $I^+_1 = (0, \infty)$ with $zR_z/R[z] = 1/(z+1)$. For $0 \leq \Omega_\Lambda < 1$, it is not difficult to show that $z/R[z]$ is bounded from below on $(0, \infty)$ by a
positive constant, depending $\Omega_\Lambda$. Therefore, the sign of $zR_z/R[z]$ depends on that
of $R_z$. For $\Omega_\Lambda < 1$ we find that the sign of $R_z$ is same as that of $(z + 1)I(z + 1) -
\int_1^{z+1} I(y)dy$ which is a monotonically decreasing function of $z$ with a unique
positive root $z_\Lambda > 0$, depending on $\Omega_\Lambda$. So, $I^+_{\Omega_\Lambda} = (0, z_\Lambda)$ and $I^-_{\Omega_\Lambda} = (z_\Lambda, \infty)$. Hence, for $z^\pm$ as above, there are positive constants $C^\pm$ so that $zR_z(z)/R[z] >
\pm C^\pm$ on intervals $(0, z^\pm)$ and $(z^-, \infty)$, respectively. □

In a certain case of interest, we find that for certain initial conditio
ons the growth of $M[z]$ roughly follows that of a power function for large $z$.

**Corollary 2.10.** In the case of Theorem 2 we have for $R = R_{\Omega_\Lambda}$ with $0 \leq
\Omega_\Lambda < 1$ that, given $M_0 > 2R[z_0] > 0$ and $z_0 > z_\Lambda$, for any $\alpha > 0$ there are
positive constants $k_1$ and $k_2$ so that for $\rho = \sqrt{3/2}$,

$$k_1\rho^{\alpha-1} \leq M[z] \leq k_2\rho^{2\rho-1+\alpha}$$

on $I = [z_0, \infty)$. 
Proof. It is not difficult to show that to any such $\Omega$ there are positive constants $C_1$ and $C_2$ so that

$$C_1/z < R_{\Omega} \frac{z}{z} < C_2/z$$

holds on $I$. The result immediately follows by Theorem 2.

We may also conclude

**Corollary 2.11.** If either case 1) or 2) of Theorem 1 holds on $I = [z_0, z_1]$, then $r(z)$ and $t(z)$ are both solvable on $I$. Moreover, $r(z)$ is strictly increasing and $t(z)$ is strictly decreasing on $I$.

Proof. We find that $\frac{dt}{dz}$ and $\frac{dr}{dz}$ are smooth functions of $z$ since $\xi \neq 1/2$ is smooth. By inspection, we find that $\frac{dt}{dz} < 0$ on $I$ so that, by our assumption on $\sigma$, we see for $h$ as in (2.3) that $h \geq 1$, increasing with $z$. Then, $I_2 + \xi I_1 > 0$ for $z \in I$ and, hence, from (2.4) we see that $\frac{dr}{dz} > 0$ for $z \in I$.

We note finally that these results are consistent with physical interpretation where $t$ is interpreted as "look-back" time from an observer at $r = 0$ with a (locally) expanding universe (c.f. [5, 12]).

3. **Study of Singularities, part A: Critical points depending on $\Omega$**

We now consider how singularities may depend on the parameter $\Omega$ for $R[z] = R_{\Omega_\Lambda}[z]$. As in Proposition 1.11, a singularity arises at $z = z_\Lambda$ where

$$[R_z]_{z=z_\Lambda} = \frac{(1 + z_\Lambda) \cdot I(1 + z_\Lambda) - \int_1^{1+z_\Lambda} I(y)dy}{(1 + z_\Lambda)^2} = 0.$$
Proposition 3.2. The values $z_\Lambda$ satisfy $z_\Lambda \geq 1.25$, increasing as a continuous function of $\Omega_\Lambda$ in the domain $0 \leq \Omega_\Lambda < 1$. Moreover, there are positive constants $c_1, c_2$ and $c_3$ so that

\[
\left[ c_1 \ln \left( \frac{1}{1 - \Omega_\Lambda} \right) + c_2 \right]^{1/4} \leq z_\Lambda + 1 \leq c_3 \frac{1}{1 - \Omega_\Lambda},
\]

$\forall \Omega_\Lambda$. Hence, $z_\Lambda \to +\infty$ as $\Omega_\Lambda \to 1$.

Proof. It is not difficult to show from (3.1) that $z_\Lambda|_{\Omega_\Lambda=0} = 1.25$ and that $z_\Lambda > 0 \forall \Omega_\Lambda$. Now, let us set $q \overset{\text{def}}{=} 1 + z_\Lambda$ and note that (3.1) gives $q\mathcal{I}(q) = \int_1^q \mathcal{I}(y)dy$. Implicit differentiation now gives

\[
q \frac{dq}{d\Omega_\Lambda} \frac{\partial \mathcal{I}(q)}{\partial \Omega_\Lambda} = \int_1^q \frac{\partial \mathcal{I}(y)}{\partial \Omega_\Lambda} dy - q \frac{\partial \mathcal{I}(q)}{\partial \Omega_\Lambda}.
\]

Applying $q\mathcal{I}^3(q) = \mathcal{I}^2(q) \int_1^q \mathcal{I}(y)dy$ on the second term, right-hand side, we compute

\[
Q(q) \frac{dq}{d\Omega_\Lambda} = \frac{1}{\mathcal{I}^3(q)} \int_1^q \mathcal{I}(y)K(y, q)dy
\]

where $K(y, q) \overset{\text{def}}{=} \mathcal{I}^2(q)(q^3 - 1) - \mathcal{I}^2(y)(y^3 - 1)$ and $Q(q) \overset{\text{def}}{=} 3q^3(1 - \Omega_\Lambda)$. Since $\mathcal{I}^2(y)(y^3 - 1)$ is strictly increasing as a function of $y \geq 1$, we find $K(y, q) > 0$ for $1 \leq y < q$. Thus, $\frac{dq}{d\Omega_\Lambda} > 0 \forall \Omega_\Lambda$ and hence $q \geq 2.25 \forall \Omega_\Lambda$. For $k_1 \overset{\text{def}}{=} \frac{\mathcal{I}(y)K(y, 2.25)dy}{\int_1^2 \mathcal{I}(y)K(y, 2.25)dy}$ we estimate

\[
Q(q) \frac{dq}{d\Omega_\Lambda} \geq \frac{3k_1/4}{\mathcal{I}^3(q)} \geq 3k_1/4
\]

so that

\[
4 \int_{2.25}^q y^3 dy \geq k_1 \int_0^{\Omega_\Lambda} \frac{dx}{1-x},
\]
and our choices of $c_1$ and $c_2$ are clear since

$$q^4 \geq k_1 \ln \left( \frac{1}{1 - \Omega} \right) + 2.25^4$$

Next we note that

$$q = \frac{1}{I(q)} \int_1^q I(y) dy \geq \sqrt{(1 - \Omega)} q^2 \int_1^q \frac{dy}{\sqrt{1 + y^3}}$$

so that $\sqrt{q} \leq k_2 \sqrt{1 - \Omega}$ with $1/k_2 \overset{\text{def}}{=} \int_1^{2.25} \frac{dy}{\sqrt{1 + y^3}}$. We choose $c_3 = k_2^2$ and we are done.

With a broad range of values $z_{\Omega,\Lambda}$, bound by the estimates of Proposition 3.2, one may expect difficulties in applying the present work to cosmological models - with singularities $z_{\Omega,\Lambda}$ well within observed redshift values [3, 5, 6]. However, some such singularities may conceivably be of type 0/0 if both $R_z$ and $\sqrt{1 + 2E[z]} - \sqrt{2M[z]/R[z] + 2E[z]}$ were to have zeros of identical order, rendering the singularities, in some sense, removable. We demonstrate such a case in the next section.

4. Study of Singularities, part B: FRW Model

Using solutions from the well-known Freedman-Robertson-Walker model, we analyze our map $(E, D_L, R_0) \rightarrow (r, t, M)$ and study singularities of the system (1.13) and their dependence on $\Omega$. We restrict the map as follows: We fix the function $E(r)$ and restrict $D_L(z)$ and $R_0(r)$ to certain one-parameter classes in
the pre-image space; and, we fix the function $M(r)$ in the image space. Here, we consider data given by $R[z] = R_{\Omega}[z]$ as in (0.7) and we set

\[(4.1) \quad E = \frac{r^2}{2}, M = \frac{r^3}{2}, R_0 = cM/E = cr\]

for parameter $c > 0$. Following [1], we have $R(r, t) = r \cdot a(t)$ where for some (real) parameter $\eta$ with $k_c \overset{\text{def}}{=} \sqrt{c + c^2}$,

\[(4.2) \quad a(t) = \frac{\cosh \eta - 1}{2} + (c \cosh \eta + k_c \sinh \eta) \overset{\text{def}}{=} \mathfrak{F}_c(\eta)\]

\[t = \frac{\sinh \eta - \eta}{\sqrt{2}} + \sqrt{2}(c \sinh \eta + k_c \cosh \eta) \overset{\text{def}}{=} \mathfrak{G}_c(\eta)\]

Here, $\eta$ is known as "conformal time" which in our case depends on $a$ and $t$ by $\eta(t) = \int_t^{z(t)} \frac{dt}{\sqrt{2k_c \sqrt{x(t)}}}$. We note that $\mathfrak{F}_c$ and $\mathfrak{G}_c$ are each invertible for $\eta$ on an open interval containing 0. In particular, $\mathfrak{F}_c$ is invertible for $\eta > -\arctanh(2k_c/(1+2c))$ and $\mathfrak{G}_c$ is invertible where $a > 0$, so that $a(t) = \mathfrak{F}_c \circ \mathfrak{G}_c^{-1}(t)$ indeed holds for $t$ in a neighborhood containing $k_c$. Moreover, using (0.5) and setting $c = a(t_0)$ with $t_0 \overset{\text{def}}{=} t(z_0) = \sqrt{2k_c}$ for some $z_0 > 0$,

\[\frac{dt}{dz} = \frac{-a(t)}{(1 + z)a(t)}; \quad a[z] = a(t(z)) = c \frac{1 + z_0}{1 + z}.\]
Given $R[z]$, we find, indirectly, the resulting solutions of (1.13):

\[(4.3) \quad t(z) = \Theta_c(\tilde{\omega}_c^{-1}(a[z]))\]
\[r(z) = R[z]/a[z] = \frac{\int_1^{1+z} I(y)dy}{(1 + z_0)c}\]
\[M[z] = \frac{1}{2} \left( \frac{\int_1^{1+z} I(y)dy}{(1 + z_0)c} \right)^3\]

As for the relevance of this case to physical models, we note that the associated energy density $\rho[z]$ is a smooth function on $(0, \infty)$.

We are ready to state

**Theorem 3.** For any given $0 \leq \Omega \leq 1$ and $z_0 > 0$ there is a smooth function $R_0(r)$ so that $E(r)$, $M(r)$ as in (4.1) and $R[z] = R_\Omega[z]$, the system (1.13) with initial conditions

\[(4.4) \quad \vec{X}(z_0) = \begin{pmatrix} R[z_0]/c \\ \sqrt{2k_c} \\ (R[z_0]/c^3)/2 \end{pmatrix}\]

has a smooth solution $\vec{X}$ on an open interval $I \ni z_0$.

**Proof.** For those $\eta$ where the solutions (4.2) hold we also have $R' = a(t) > 0$ and $\tilde{R}' = \frac{da}{dt} = \frac{da}{d\eta} \tilde{\omega}_c^{-1}(a(\eta)) > 0$. Since the initial conditions hold for $\eta = \tilde{\omega}_c^{-1}(a(\eta_0)) = 0$, (4.3) also holds for $\eta$ in some interval containing 0. From continuity arguments we see there is also some open interval $I \ni z_0$ on which such solutions $\vec{X}(z)$ in turn hold. \(\square\)
We note that the above method provides no solutions for \(r(z)\) and \(M[z]\) in the case \(R_0 \equiv 0\) unless more data is prescribed, such as asymptotic conditions for the ratio \(R/c\) in terms \(z\) and \(z_0\) (c.f. Example A, p. 5 [6]). Moreover, we note that singularities may occur in the form \(R = 2M\) and/or \(\dot{a} = 0\) away from \(z_0\) so that we may not arbitrarily extend the domain \(I\) of the solution via Proposition 1.14.

We may apply Proposition 1.14 in regards to uniqueness of solution: To rule out one type of singularity, we compute \(E'G - F\) via (1.15). First, we set \(\xi = r/(2a(t))\) and \(\xi_0 = r/(2c)\) and compute

\[
\frac{E'}{J_R} \cdot (J_E - RJ_M) = \sqrt{E + \xi R} - \frac{c^{1/2}(s - 1)ds}{(Es + \xi)^{3/2}}
\]

\[
= -\frac{1}{2} \int_0^c \frac{\sqrt{\tau}(\tau - a(t))}{(\tau + 1)^{3/2}} d\tau \sqrt{a(t) + 1} \leq 0
\]

for \(c, a(t) > 0\). We now compute,

\[
\frac{R_0 J_{R_0}}{J_R} = -c \cdot \sqrt{\frac{R_0}{ER_0 + M}} \sqrt{\frac{R}{ER + M}}
\]

\[
= -c \cdot \sqrt{\frac{c}{c + 1}} \sqrt{\frac{a(t) + 1}{a(t)}}
\]

which is strictly negative. Therefore, \(E'G - F < 0\) and we have ruled out case 2) of Proposition 1.10. Knowing also that \(\dot{R}[z]_{z=z_0} \neq 0\) in this case we state

**Theorem 4.** The solutions of Theorem 3 are unique for \(z_0 \neq z_\Lambda\).

The solutions (1.13) stand in glaring contrast to the result of Proposition 1.10: Indeed, we note that the right-hand sides of equations (0.4) and (0.5) under the
conditions of Theorem 3 have no positive singularities \( z = z_\Lambda \) as \( E' + M'/R - MR'/R^2 = r + r/a > 0 \); yet, we find that the determinant of \( U \) in (1.9) vanishes at \( z = z_\Lambda \). Since Theorem 3 applies in the case \( z_0 = z_\Lambda \), one may suspect that these singularities are, in some sense, removable - so we shall see in remainder of this section.

We give specific cases, depending on \( R_0 \), in which the solutions \( \vec{X} \) can be smoothly extended across singularities \( z = z_\Lambda \). For such solutions to be valid, it suffices that \( \dot{a}[z] > 0 \) is smooth, that (3.1) holds, and that as in (1.12) \( R[z_\Lambda] = 2M[z_\Lambda] \) (or perhaps as smooth extensions defined at \( z_\Lambda \)). Then,

\[
R^2[z_\Lambda] = \frac{(1 + z_0)c)^3}{(1 + z_\Lambda)^3}
\]

and, hence,

\[
I(1 + z_\Lambda) = R[z_\Lambda] = \frac{(1 + z_0)c)^{3/2}}{(1 + z_\Lambda)^{3/2}}.
\]

From this we obtain the corresponding value of \( c \) by which we define

\[
(4.5) \quad c_\Lambda \overset{\text{def}}{=} \frac{1 + z_\Lambda}{(1 + z_0)(\Omega_\Lambda + (1 - \Omega_\Lambda)(1 + z_\Lambda)^3)^{1/3}}.
\]

We are ready to state

**Theorem 5.** Under the hypothesis of Theorem 3 for every \( z_0 > 0 \) and \( 0 \leq \Omega_\Lambda < 1 \) there is a smooth \( R_0(r) \) for which the resulting solution \( \vec{X}(z) \) with initial conditions (4.4) can be uniquely extended to be of class \( C^\omega((0, \infty)) \).
Proof. We take \( R_\theta (r) = c_\alpha r \) for \( c_\alpha \) as in (4.5). Using (4.2) and following the Chain Rule formula

\[
\dot{a}(t(z)) = \frac{\frac{da[z]}{dz}}{\sqrt{2} \tilde{a}(\eta(z)) \frac{d\eta[z]}{dz}} = \frac{\frac{da[z]}{dz}}{\sqrt{2} \alpha[z] \frac{d\eta[z]}{dz}},
\]

it suffices to show that \( \eta[z] \) is smooth and that \( \frac{dn}{dz} \) is strictly positive on \((0, \infty)\). To do this, we set

\[
\eta[z] = -\int \frac{dt[s]}{\sqrt{2} a[s]},
\]

with \( dt[z] \equiv \frac{dt[z]}{dz} dz \), and we proceed to analyze the integral. We may write

\[
\frac{dt}{dz} = H(z) \frac{-R_s}{2M - R} = H(z) \frac{-R_s}{r(z)(r(z)^2 - a(z))}
\]

for some real-valued function \( H > 0 \), analytic for \( z > 0 \). Here \( r^2 - a \) is an increasing function which vanishes at \( z_\Lambda \) and is of the same sign as that of \(-R_s\) \( \forall z > 0 \).

Now, we check the behavior of \( \frac{dt}{dz} \) near the singularity, applying analyticity arguments as follows: Using (4.3) and (3.1) we compute

\[
\left[ \frac{d^2 R[z]}{dz^2} \right]_{z=z_\Lambda} = -\frac{3(1 - \Omega_\Lambda)}{2} (1 + z_\Lambda)^3(1 + z_\Lambda) < 0
\]

\[
\left[ \frac{dM[z]}{dz} \right]_{z=z_\Lambda} = \frac{3I(1 + z_\Lambda)}{2(1 + z_\Lambda)} \equiv \mathcal{M}_\Lambda > 0;
\]

and, in turn, we find that

\[
2M[z] - R[z] = 2\mathcal{M}_\Lambda \cdot (z - z_\Lambda) + \mathcal{P}(z)(z - z_\Lambda)^2
\]
for some analytic function $\mathcal{P}$. Here, $\frac{d^2 R[z]}{dz^2} < 0$ on a neighborhood of $z_\Lambda$ where $R_z$ has a zero of order exactly 1. We therefore find that the following limit exists as we compute:

$$
\lim_{z \to z_\Lambda} \frac{-dR}{dz} \frac{2M[z] - R[z]}{2dM[z]} = \left[ -\frac{d^2 R[z]}{dz^2} \right]_{z = z_\Lambda} = \frac{(1 - \Omega_\Lambda)(1 + z_\Lambda)I(1 + z_\Lambda)}{2} > 0
$$

We may conclude therefore that $\frac{R}{2M[z] - R[z]}$ extends to an analytic, positive-valued function on $(0, \infty)$ and, hence, $C^\omega((0, \infty)) \ni \frac{d\eta[z]}{dz} > 0$. Therefore, $\hat{a}[z]$ is well-defined and is non-zero; and, moreover, $\eta[z]$ is of class $C^\omega((0, \infty))$. The uniqueness follows since Theorem 3 applies to any open interval not containing $z_\Lambda$. 

**Remark 4.8.** Following the calculations in the proof of Theorem 5, we note that any other choice of positive $c \neq c_\Lambda$ leads to a singularity of order one at $z = z_\Lambda$ for $\dot{R}'[z] = \dot{a}[z]$ as evident in (4.6) and (4.7). This gives singularities in equations (0.4), (0.5), and (1.2), and renders the resulting system (1.13) invalid at $z_\Lambda$.

**Remark 4.9.** Our FRW model is consistent with the construction of $R[z]$ as in [5] where $\frac{R_0}{R} = 1 + z$ with no prescribed value of $c$. Moreover, our choice of $c = c_\Lambda$ is optimal in assuring the largest possible domain of $C^\omega$-solvability.

**Discussion**

We make several concluding comments and a conjecture: First, we note that in the case of Theorem 5 the various right-hand sides of the system (1.13) can each
be written in the form $A(z) + B(z)\frac{R}{R[z] - 2M[z]}$ for smooth functions $A$ and $B$. Thus, the arguments for the smooth extension of $\frac{dr}{dz}$ beyond the critical points $z_\Lambda$ of $R[z]$ also apply to $\frac{dr}{dz}$ (also, of course with circular reasoning, to $\frac{dM}{dz}$). However, in the general mapping scheme (0.6), we have no way to predict the order of the zeros of $\frac{dM}{dz}$ nor any a priori justification to expect these singularities to be removable - not even as we fix our choice of $R[z] = R_{\Omega_\Lambda}[z]$.

Second, one may interpret the removability or non-existence of such singularities as indication of compatibility of the corresponding models as one imposes $R[z]$ on a model that prescribes $E$ and $R_0$. (Here the LTB model would be said to ‘mimic’ the given cosmological constant model, c.f. [2].) Applying such criteria to Remark 4.8 one does not expect every LTB model to be compatible with such a cosmological-constant model (at least not for $z$ near $z_\Lambda$). However, from Theorem 5 we do find, as a check of our analysis, that the cosmological-constant models for $0 \leq \Omega_\Lambda < 1$ are each compatible with at least one LTB/FRW model: Our choice of $R_0$ identifies an optimal FRW model, in the sense of Remark 4.9.

Finally, one conjectures that these removable, 0/0-type singularities may yet lead to instability of numerical solutions of the system (1.13) (but here at certain finite $z$ (!) c.f. §IV [6]). Such investigations are beyond the scope of the present work.

References

[1] H. Alnes, M. Amarzguioui, O. Gron, Phys. Rev., D73, 083519, 2006.
[2] A. Aguirre, Z. Haiman, Cosmological Constant or Intergalactic Dust? Constraints from the Cosmic Far-Infrared Background, *ApJ*, 532:28-36, 2000.

[3] M.N. Celerier, Do we really see a cosmological constant in supernova data?, *A& A*, 2, 2008.

[4] E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[5] S. Carroll, W. Press, E. Turner, The cosmological constant, *ARA A*, 30, 1992.

[6] D. Chung, A. Romano, Mapping luminosity-redshift relationship to LTB cosmology, *Phys. Rev.*, D74, 10103507, 2006.

[7] K. Enqvist, Lemaitre-Tolman-Bondi model, *Gen. Rel. Grav.* 40:451-466, 2008.

[8] J. Freiman, Lectures on Dark Energy and Cosmic Acceleration, *AIP Conf. Proc.*, vol. 1057, pp. 87-124, 2008.

[9] J. Islam, *An Introduction to Mathematical Cosmology*, Cambridge University Press, 2002.

[10] J. Kristian, R. Sachs, Observations in cosmology, *ApJ*, 143, 1966.

[11] E.W. Kolb, S. Matarrese, A. Riotto, On cosmic acceleration without dark energy, *NewJ.Phys.* 8:322, 2006.

[12] E. Kolb, M. Turner, *The Early Universe*, Addison Westly, 1990.

[13] J.W. Mofit, Late-time Inhomogeneity and Acceleration Without Dark Energy, *JCAP*, 0605, 001 (2006)

[14] *Gravitation*, C. Misner, K. Thorne, J. Wheeler, W.H. Freeman and Company, 1973.

[15] H. Partovi, B. Mashhoon, Toward verification of large-scale homogeneity in cosmology, *ApJ* 276, 1984.

[16] S. Rasanen, Backreaction in the Lemaitre-Tolman-Bondi model, *JCAP*, 0411:010, 2004
[17] H. Stepani, D. Kramer, M. MacCallum, C. Hoenselaers, E. Herlt, *Exact Solutions of Einstein’s Field Equations*, Cambridge University Press, 2003.

[18] R. Wald, *General Relativity*, University of Chicago Press, 1984.

[19] S. Weinberg, *Gravitation and Cosmology*, Wiley and Sons, Inc., 1972.

E-mail address: winfield@madscitech.org