Quantum information-flow, concretely, and axiomatically

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Abstract
These lecture notes survey joint work with Samson Abramsky. I will somewhat informally discuss the main results of the papers [2,4,5,6,13,14] in a pedestrian not too technical way. These include:

• ‘The logic of entanglement’ [14], that is, the identification and abstract axiomatization of the ‘quantum information-flow’ which enables protocols such as quantum teleportation. To this means we define strongly compact closed categories which abstractly capture the behavioral properties of quantum entanglement.

• ‘Postulates for an abstract quantum formalism’ [5] in which classical information-flow (e.g. token exchange) is part of the formalism. As an example, we provide a purely formal description of quantum teleportation and prove correctness in abstract generality. In this formalism types reflect kinds, contra the essentially typeless von Neumann formalism [29]. Hence even concretely this formalism manifestly improves on the usual one.

• ‘Towards a high-level approach to quantum informatics’ [2]. Indeed, the above discussed work can be conceived as aiming to solve:

\[
\begin{array}{c}
\text{von Neumann quantum formalism} \\
\sim \\
\text{high-level language}
\end{array}
\]
\[
\begin{array}{c}
\text{low-level language}
\end{array}
\]

1 What? When? Where? Why?

First of all, for us ‘quantum’ stands for the concepts (both operational and formal) which had to be added to classical physics in order to understand observed phenomena such as the structure of the spectral lines in atomic spectra, experiments exposing non-local correlations, seemingly \(4\pi\) symmetries, etc. While the basic part of classical mechanics deals with the (essentially) reversible unitary dynamics of physical systems, quantum required adding the notions of measurement and (possibly non-local) correlations to the discussion. The corresponding mathematical formalism was considered to have reached its maturity in von Neumann’s book [29]. However!

The quantum teleportation protocol. The quantum teleportation protocol [10] involves three qubits \(\alpha, \beta\) and \(\gamma\) and two spatial regions \(A\) (for “Alice”) and \(B\) (for “Bob”). Qubit \(\alpha\) is in a state \(|\phi\rangle\) and located in \(A\). Qubits \(\beta\) and \(\gamma\) form an ‘EPR-pair’, that is, their joint state is \(|00\rangle + |11\rangle\). We assume that these qubits are initially in \(B\) e.g. Bob created them. After spatial relocation so that \(\alpha\) and \(\beta\) are located in \(A\), while \(\gamma\) is positioned in \(B\), or in other words, “Bob sends qubit \(\beta\) to Alice”, we can start the actual teleportation of qubit \(\alpha\).

*Howard Barnum, Rick Blute, Sam Braunstein, Vincent Danos, Ross Duncan, Peter Hines, Martin Hyland, Prakash Panangaden, Peter Selinger and Vlatko Vedral provided feedback. Samson Abramsky and Mehrnoosh Sadrzadeh read this manuscript.
Alice performs a Bell-base measurement $M_{\text{Bell}}$ on $a$ and $b$ at $A$, that is, a measurement such that each projector in the spectral decomposition of the corresponding self-adjoint operator projects on one of the one-dimensional subspaces spanned by a vector in the Bell basis:

\[
\begin{align*}
b_1 &= |00\rangle + |11\rangle \quad & b_2 &= |01\rangle + |10\rangle \quad & b_3 &= |00\rangle - |11\rangle \quad & b_4 &= |01\rangle - |10\rangle \\
&= \frac{1}{\sqrt{2}} \quad & & & & & = \frac{-1}{\sqrt{2}}
\end{align*}
\]

We will omit scalar multiples from now on. This measurement may be of the ‘destructive’ kind. Alice observes the outcome of the measurement and “sends these two classical bits ($x \in B^2$) to Bob”. Depending on which classical bits he receives Bob then performs one of the unitary transformations

\[
\begin{align*}
\beta_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \beta_2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \beta_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \beta_4 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\end{align*}
\]

on $c$. $\beta_1, \beta_2, \beta_3$ are all self-inverse while $\beta_4^{-1} = -\beta_4$. The final state of $c$ proves to be $|\phi\rangle$ as well.

**Where does “it” flow?** Consider this quantum teleportation protocol. In this process continuous data is transmitted from Alice to Bob while only using a two-bit classical channel. So where does the ‘additional information’ flow? The quantum formalism does not tell us in an explicit manner. Clearly it has something to do with the nature of quantum compoundness, but, what exactly? Note that this reasonably simple protocol was only discovered some 60 years after von Neumann’s formalism. *Wouldn’t it be nice to have a formalism in which inventing quantum teleportation would be an undergraduate exercise?*

**Where are the types?** While in the lab measurements are applied to physical systems, application of the corresponding self-adjoint operator $M : \mathcal{H} \to \mathcal{H}$ to the vector $\psi \in \mathcal{H}$ which represents the system’s state, hence yielding $M(\psi)$, does not reflect how the state changes during the act of measurement! The actual change is $\psi \mapsto P_i(\psi)$ for spectral decomposition $M = \sum_i a_i P_i$, where $a_i$ is the outcome of the measurement. In addition to this change of state a measurement involves provision of data to ‘the observer’ cf. teleportation where this data determines the choice of the unitary correction. This contradicts what the corresponding types seem to indicate. The same argument goes for the composite of two self-adjoint operators which in general is not self-adjoint while measurements can be performed sequentially in the lab. *Wouldn’t it be nice if types reflect kinds?*

**Much worse even, where is the classical information and its flow?** Indeed, the problem regarding types is directly connected to the fact that in von Neumann’s formalism there is no place for storage, manipulation and exchange of the classical data obtained from measurements. *We want a quantum formalism which allows to encode classical information and its flow, and hence also one which has enough types to reflect this!*
**What is the true essence of quantum?**  John von Neumann himself was the first to look for this, teaming up with the ‘king of lattices’ Garrett Birkhoff [11]. It is fair to say that as an attempt to understand ‘the whole of quantum mechanics’ this particular ‘quantum logic’ program has failed. While it provided a much better understanding of quantum superposition and the superselection rules (for a survey try to get hold of Piron’s [26] and Varadarajan’s [28] books), it failed at teaching us anything about quantum entanglement, and definitely didn’t teach us anything on how quantum and classical information interact. So lattices don’t seem to be capable of doing the job. Which mathematical setting provides an abstract quantum formalism, and its corresponding logic?

2  The logic of entanglement

A mathematics exercise. The ‘Where does “it” flow?’ question was addressed and solved in [13, 14]. But the result challenges quantum mechanics’ faithfulness to vector spaces! We start by playing a quiz testing the reader’s knowledge on the Hilbert space tensor product. Consider the situation depicted below where all boxes represent bipartite projectors on one-dimensional subspaces of Hilbert spaces $H_i \otimes H_j$, that is, linear maps $P_{\Xi} : H_i \otimes H_j \rightarrow H_i \otimes H_j :: \Phi \mapsto \langle \Psi_{\Xi} | \Phi \rangle \cdot \Psi_{\Xi}$

with $\Psi_{\Xi} \in H_i \otimes H_j$ and $|\Psi_{\Xi}| = 1$ so $P_{\Xi}(\Psi_{\Xi}) = \Psi_{\Xi}, \phi_{in} \in H_1, \phi_{out} \in H_5, \Phi_{in} \in H_2 \otimes H_3 \otimes H_4 \otimes H_5$ and hence $\Psi_{in}, \Psi_{out} \in H_1 \otimes H_2 \otimes H_3 \otimes H_4 \otimes H_5$,

(up to a scalar multiple is ok!) In algebraic terms this means solving

$$k \cdot \zeta \left( \phi_{in} \otimes \Phi_{in} \right) = \Psi_{\Pi} \otimes \Psi_{\Pi} \otimes \phi_{out}$$

in the unknown $\phi_{out}$ for $k \in \mathbb{C}$ and

$$\zeta := (P_{\Pi} \otimes P_{\Pi} \otimes 1_5) \circ (1_1 \otimes P_{\Pi} \otimes 1_{4,5}) \circ (1 \otimes P_{\Pi} \otimes 1_{4,5}) \circ (1_1 \otimes P_{\Pi} \otimes 1_{4,5}) \circ (1_1 \otimes P_{\Pi} \otimes 1_{4,5}) \circ (1_1 \otimes P_{\Pi} \otimes 1_{4,5}) \circ (1_1 \otimes P_{\Pi} \otimes 1_{4,5})$$

where $1_i$ is the identity on $H_i$ and $1_{ij}$ is the identity on $H_i \otimes H_j$.

At first sight this seems a randomly chosen nasty problem without conceptual significance. But it is not! Observe that bipartite vectors $\Psi \in H_1 \otimes H_2$ are in bijective correspondence with linear maps $f : H_1 \rightarrow H_2$ through matrix representation in bases $\{e_i^{(1)}\}_{i}$ and $\{e_j^{(2)}\}_{j}$ of $H_1$ and $H_2$,

$$\Psi = \sum_{ij} m_{ij} \cdot e_i^{(1)} \otimes e_j^{(2)} \Leftrightarrow \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kn} \end{pmatrix} \Leftrightarrow f : e_i^{(1)} \mapsto \sum_j m_{ij} \cdot e_j^{(2)}$$
or in bra-ket/qu-nit notation,
\[ \sum_{ij} m_{ij} \langle i | j \rangle = \sum_{ij} m_{ij} | i \rangle \otimes | j \rangle \]

This correspondence lifts to an isomorphism of vector spaces. As an example, the (non-normalized) EPR-state corresponds to the identity
\[ | 00 \rangle + | 11 \rangle \xrightarrow{\sim} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \xrightarrow{\sim} 1 = \langle 0 | - \rangle \cdot | 0 \rangle + \langle 1 | - \rangle \cdot | 1 \rangle . \]

In fact, the correspondence between \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) and anti-linear maps is a more natural one, since it is independent on the choice of a base for \( \mathcal{H}_1 \),
\[ \sum_{ij} m_{ij} | i \rangle \otimes | j \rangle \xrightarrow{\sim} \sum_{ij} m_{ij} \langle - | i \rangle \cdot | j \rangle , \]
or equivalently, the correspondence between \( \mathcal{H}_1^* \otimes \mathcal{H}_2 \) and linear maps, where \( \mathcal{H}_1^* \) is the vector space of linear functionals \( \varphi : \mathcal{H}_1 \to \mathbb{C} \) which arises by setting \( \varphi := \langle \psi | - \rangle \) for each \( \psi \in \mathcal{H}_1 \). We will ignore this for now (see [13] for a detailed discussion) and come back to this issue later.

Since we can now ‘represent’ vectors \( \Psi \in \mathcal{H}_i \otimes \mathcal{H}_j \) by linear functions of type \( \mathcal{H}_i \to \mathcal{H}_j \), and hence also the projectors \( P \) which appear in the above picture, we can redraw that picture as

\[ \Psi_{\text{out}} := \Psi_{\text{in}} \xrightarrow{\sim} f_1 \xrightarrow{\sim} f_3 \xrightarrow{\sim} f_5 \xrightarrow{\sim} f_4 \xrightarrow{\sim} f_6 \xrightarrow{\sim} f_8 \xrightarrow{\sim} f_7 \xrightarrow{\sim} f_9 \xrightarrow{\sim} f_1 \phi_{\text{in}} \]

where now \( \Psi_{\text{in}} \xrightarrow{\sim} f_1 \) and \( \Psi_{\text{in}} \xrightarrow{\sim} f_3 \), and the arrows \( \xrightarrow{\sim} f_i \) specify the domain and the codomain of the functions \( f_i \), and, I should mention that the new (seemingly somewhat random) numerical labels of the functions and the direction of the arrows are well-chosen (since, of course, I know the answer to the quiz question). We claim that, provided \( k \neq 0 \) (see [13]),
\[ \phi_{\text{out}} = (f_8 \circ f_7 \circ f_6 \circ f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)(\phi_{\text{in}}) \]

(up to a scalar multiple), and we also claim that this is due to the fact that we can draw a ‘line’ of which the allowed passages through a projector are restricted to

that is, if the line enters at an input (resp. output) of a bipartite box then it has to leave by the other input (resp. output) of that box (note the deterministic nature of the path). In other words

**Permitted are:**

![Permitted Diagrams]
Forbidden are:

\[ \phi \]
\[ \phi \text{in} \]
\[ \phi \text{out} = (f_8 \circ \ldots \circ f_1)(\phi_{\text{in}}) \]

what results in:

When we follow this line, we first pass through the box labeled \( f_1 \), then the one labeled \( f_2 \) and so on until \( f_8 \). Hence it seems “as if” the information flows from \( \phi_{\text{in}} \) to \( \phi_{\text{out}} \) following that line and that the functions \( f_i \) labeling the boxes act on this information. Also, \( \phi_{\text{out}} = (f_8 \circ \ldots \circ f_1)(\phi_{\text{in}}) \) does not depend on the input of the projectors at \( \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \otimes \mathcal{H}_5 \) and, more importantly, the order in which we apply the projectors does not reflect the order in which \( f_1, \ldots, f_8 \) are applied to \( \phi_{\text{in}} \) in the expression \( (f_8 \circ \ldots \circ f_1)(\phi_{\text{in}}) \). Doesn’t this have a somewhat ‘acausal’ flavor to it?

**The logic of quantum entanglement.** We claim that the above purely mathematical observation exposes a quantum information-flow. It suffices to conceive the projectors \( P_\Xi \) as appearing in the spectral decompositions of self-adjoint operators \( M_\Xi := \sum \alpha_\Xi,i \cdot P_\Xi,i \) representing quantum measurements, that is, for some \( i \) we have \( P_\Xi = P_\Xi,i \) (hence the outcome of the measurement represented by \( M_\Xi \) is \( \alpha_\Xi,i \)). As an example, consider

\[ \phi_{\text{out}} = (1 \circ 1)(\phi_{\text{in}}) = \phi_{\text{in}} \]

where, since all labeling functions are identities, both projectors project on the EPR-state. Since the first projector corresponds to ‘preparing an EPR-state’, this picture seems to provide us with a teleportation protocol,
However, physically we cannot implement $P_{EPR}$ on its own ‘with certainty’.

But $P_{EPR}$ is part of Bell-base measurement together with three other projectors. We denote the corresponding labeling functions by $\beta_2, \beta_3, \beta_4$. The grey boxes below denote unitary transformations. We have

\[
\phi_{\text{out}} = \phi_{\text{in}}
\]

where $\gamma_i \circ \beta_i$ has to be the identity so $\gamma_i = \beta_i^{-1}$. These four pictures together yield the full teleportation protocol!

The classical communication is encoded in the fact that in each picture the unitary correction $\gamma_i$ depends on $\beta_i$, that is, the measurement outcome. Hence the classical communication does not contribute to the transmission of the data, it only distributes the knowledge about ‘which of the four pictures is actually taking place’.

To conclude this paragraph we stress that the functional labels are not actual physical operations but only arise in the above discussed mathematical isomorphism. Further, in the generic example

\[
\phi_{\text{out}} = (f_2 \circ f_1)(\phi_{\text{in}})
\]

the order of the physical operations is opposite to the order in which their labels apply to the input state in the expression $(f_2 \circ f_1)(\phi_{\text{in}})$. Algebraically,$^1$

\[
k \cdot \zeta(\phi_{\text{in}} \otimes \Phi_{\text{in}}) = \Psi_{f_i} \otimes (f_2 \circ f_1)(\phi_{\text{in}}) \quad \text{for} \quad \zeta = (P_{f_1} \otimes 1) \circ (1 \otimes P_{f_2})
\]

with $\Psi_f \leftrightarrow f$ and $P_f(\Psi_f) = \Psi_f$ as a new notation. Slightly simpler,

\[
(P_{f_1} \otimes 1)(\phi_{\text{in}} \otimes \Psi_{f_2}) = \Psi_{f_i} \otimes (f_2 \circ f_1)(\phi_{\text{in}}),
\]

by conceiving the first projector as a state. Furthermore, the above discussed $\ast$ in $\mathcal{H}_1^* \otimes \mathcal{H}_2$ which is necessary to have a base-independent correspondence with linear functions ‘is not a bug but a feature’, it actually witnesses (by means of a phase conjugation) the fact that the line changes its temporal direction every time it passes a projector box (see [13]). Using the same line of thought it is also easy to reconstruct other protocols such as logic-gate teleportation [17] and entanglement swapping [31], and, the quantum information-flow interpretation also extends to multipartite projectors. We refer the reader to [13, 14] for details on this. Then we asked:

“Are these information-flow features specifically related to the Hilbert space structure? Or to ...”

---

$^1$ The pictures really look much better than the formulas, don’t they?
Sets, relations and the cartesian product. Doesn’t sound very ‘quantum’ you say? Let’s see. We make the following substitutions in the above:

- Hilbert space \( \mathcal{H} \) \( \leadsto \) set \( X \)
- linear function \( f \) \( \leadsto \) relation \( R \)
- tensor product \( \otimes \) \( \leadsto \) cartesian product \( \times \)

Can we also translate projectors to this world of relations? Observe that for projectors on one-dimensional subspaces, which take the general form \( P_\psi = \langle \psi \mid - \rangle \cdot \langle - \mid \psi \rangle : \mathcal{H} \to \mathcal{H} \), we have \( \psi \otimes \psi \leftrightarrow \langle \psi \mid - \rangle \cdot \langle - \mid \psi \rangle \).\(^2\) That is, projectors correspond with symmetric pure tensors. By analogy we define a projector of type \( X \to X \) as \( A \times A \subseteq X \times X \) in the world of relations.\(^3\) Hence \( R \times R \subseteq (X \times Y) \times (X \times Y) \) with \( R \subseteq (X \times Y) \) is a bipartite projector in the world of relations which we denote by \( \mathcal{P}_R \) in analogy with \( P_f \). Since for the identity relation \( I \subseteq X \times X \) we have \( x_1 x_2 \leftrightarrow x_1 = x_2 \) and since

\[
P_R := R \times R = \{(x_1, y_1), (x_2, y_2) \in (X \times Y) \times (X \times Y) \mid x_1 y_1, x_2 y_2 \},
\]

for \( R_1 \subseteq X \times Y \) and \( R_2 \subseteq Y \times Z \) we have

\[
(x_1, y_1, z_1)(1_X \otimes P_{R_2})(x_2, y_2, z_2) \leftrightarrow y_1 R_2 z_1 \quad \text{and} \quad y_2 R_2 z_2, \quad \text{and} \quad x_1 = x_2,
\]

\[
(x_2, y_2, z_2)(P_{R_1} \otimes 1_Y)(x_3, y_3, z_3) \leftrightarrow x_2 R_1 y_2 \quad \text{and} \quad x_3 R_1 y_3, \quad \text{and} \quad z_2 = z_3.
\]

Setting \( s_{in} := x_1, s_{out} := z_3 \) and using the underlined expressions,

\[
(s_{in}, y_1, z_1)((P_{R_2} \otimes 1) \circ (1 \otimes P_{R_2}))(x_3, y_3, s_{out})
\]

entails \( s_{in}(R_2 \circ R_1)s_{out} \). (we invite the reader to make a picture of this) But this is not an accident!

3 The abstract algebra of entanglement

Categories for physical systems. Which abstract structure do Hilbert spaces and relations share? First of all, the above construction would not work if instead of relations we had taken functions. The importance of considering appropriate maps indicates that we will have to consider categories. As theoretical computer scientists know, categories are not just a language, nor metamathematics, nor hyper abstraction. They are mathematical objects in their own right which arise very naturally in ‘real situations’. E.g. one takes the state spaces of the systems under consideration to be the objects, and (physical) operations on these systems to be morphisms (including a skip operation), the axioms of a category are then satisfied by the mere fact that operations can be composed. We denote by \( \text{Rel} \) the category of sets and relations, by \( \text{Set} \) the category of sets and functions, by \( \text{FdHilb} \) finite dimensional (complex) Hilbert spaces and linear maps, and more generally, by \( \text{FdVec}_K \) finite dimensional vector spaces over a field \( K \).

If instead of the cartesian product we would have considered disjoint union on sets, again things wouldn’t have worked out. Also in the quantum case the use of the tensor product is crucial. All this indicates that we want some specific bifunctor \( \boxtimes \) to live on our category, \( \times \) on \( \text{Rel} \) and \( \otimes \) on \( \text{FdVec}_K \). Intuitively, we think of a bifunctor as an operation which allows to combine systems, and also the operations thereon, and, the bifunctoriality property has a clear physical interpretation: if \( S_1 \) and \( S_2 \) are distinct physical entities, when performing operation \( O_1 \) on \( S_1 \) and \( O_2 \) on \( S_2 \), the order in which we perform \( O_1 \) and \( O_2 \) doesn’t matter. One typically thinks of local operations on spatially separated systems.

In categories, elements of an object \( A \) can be thought of as morphisms \( q : I \to A \) where \( I \) is a unit for the bifunctor, i.e. \( A \boxtimes I \approx I \boxtimes A \approx A \). In \( (\text{FdHilb}, \otimes) \) we have \( I := \mathbb{C} \), and indeed, maps \( q : \mathbb{C} \to \mathcal{H} \) are in bijective correspondence with \( \mathcal{H} \) itself, by considering \( q(1) \in \mathcal{H} \). In \( (\text{Set}, \times) \) and \( (\text{Rel}, \times) \) we have \( I := \{\ast\} \), i.e., a singleton. In \( (\text{Set}, \times) \) maps \( q : \{\ast\} \to X \) are in bijective correspondence with elements of \( X \) by considering

\(^2\) Again we ignore un-naturality, that is, the slight base-dependency.

\(^3\) Recall that a relation of type \( X \to Y \) is a subset of \( X \times Y \) (cf. its ‘graph’).
$q(\star) \in X$. But not in \(\text{Rel}, \times\)! Morphisms \(q \subseteq \{\star\} \times X\) now correspond to all subsets of \(X\), which can be thought of as superpositions of the individual elements.\(^4\)

We want not only a unit \(I\) for \(\boxdot\), but a full symmetric monoidal structure, that is, we want the following natural isomorphisms\(^5\)

\[
\lambda_A : A \simeq I \boxtimes A \quad \rho_A : A \simeq A \boxtimes I \quad \sigma_{A,B} : A \boxtimes B \simeq B \boxtimes A
\]

\[
\alpha_{A,B,C} : A \boxtimes (B \boxtimes C) \simeq (A \boxtimes B) \boxtimes C.
\]

Note here that we do not require \(\boxdot\)-projections \(p_{A,B} : A \boxtimes B \to A\) nor \(\boxdot\)-diagonals \(\Delta_A : A \to A \boxtimes A\) to exist. More precisely, we don’t want them to exist, and this will be guaranteed by a piece of structure we shall introduce.

In physical terms this non-existence means no-cloning\(^6\) and no-deleting\(^7\). In categorical terms it means that \(\boxdot\) is not a categorical product.\(^5\) In logical terms this means that we are doing linear logic\(^8\) as opposed to classical logic. In linear logic we are not allowed to copy and delete assumptions, that is, \(A \land B \Rightarrow A\) and \(A \Rightarrow A \land A\) are not valid.

**Compact closure and information-flow.** Crucial in the analysis of the quantum information-flow was \(H_1^* \otimes H_2 \simeq H_1 \to H_2\). In categorical terms, making sense of \(H_1 \to H_2\) requires the category to be closed.\(^7\) To give sense to the \(*\) we require it to be *-autonomous\(^9\),\(^8\) and finally, requiring \(H_1^* \otimes H_2 \simeq H_1 \to H_2\) implies that the category is compact closed\(^10\). In logical terms this means that we have the multiplicative fragment of linear logic, with negation, and where conjunction is self-dual, that is, it coincides with disjunction — indeed, you read this correct, \(A \land B \simeq A \lor B\).

But we will follow a different path which enables us to use less categorical jargon. This path is known in category theory circles as Australian\(^1\) or Max Kelly\(^2\) style category theory. Although this style is usually conceived (even by category theoreticians) as of an abstract\(^\infty\) nature, in our particular case, it’s bull’s-eye for understanding the quantum information-flow.\(^9\)

In\(^21\) a category \(C\) is defined to be compact closed iff for each object \(A\) three additional pieces of data are specified, an object denoted \(A^*\), a morphism \(\eta_A : I \to A^* \boxtimes A\) called unit and a morphism \(\epsilon_A : A \boxtimes A^* \to I\) called counit, which are such that the diagram

\[
\begin{align*}
A & \cong A \boxtimes I \\
& \cong I \boxtimes A \\
& \cong I \boxtimes A \\
A & \cong A \boxtimes I \\
\end{align*}
\]

and the same diagram for \(A^*\) both commute. Although at first sight this diagram seems quite intangible, we shall see that this diagram perfectly matches the teleportation protocol. Both \((\text{Rel}, \times)\) and \((\text{FdVec}_{K}, \otimes)\) are compact closed, respectively for \(X^* := X, \eta_X = \{(\star, (x, x)) \mid x \in X\}\) and \(\epsilon_X = \{(x, x, \star) \mid x \in X\}\), and, for \(V^*\) the dual vector space of linear functionals, for \(\{e_i\}_{i=1}^{n}\) being the base of \(V^*\) satisfying \(\epsilon_i(e_j) = \delta_{ij}\),

\[
\eta_V : 1 \mapsto \sum_{i=1}^{n} \bar{e}_i \otimes e_i \quad \text{and} \quad \epsilon_V : e_i \otimes \bar{e}_j \mapsto \delta_{ij}.
\]

\(^4\)Compare this to ‘superposition’ in lattice theoretic terms: an atomic lattice has superposition states if the join of two atoms has additional atoms below it (e.g. cf. \(\mathbb{B}\)).

\(^5\)A categorical isomorphism is a morphism \(f : A \to B\) with an inverse \(f^{-1} : B \to A\), that is, \(f \circ f^{-1} = 1_A\) and \(f^{-1} \circ f = 1_B\). A natural isomorphism is a strong notion of categorical isomorphism. For vector spaces it essentially boils down to ‘base independent’, e.g. there exists a natural isomorphism of type \((H_1^* \otimes H_2) \to (H_1 \to H_2)\) but not one of type \((H_1 \otimes H_2) \to (H_1 \to H_2)\), where we treat \(H_1 \to H_2\) as a Hilbert space.

\(^6\)See below where we discuss biproducs.

\(^7\)For a monoidal category to be closed indeed means that we can ‘internalize’ morphism sets \(A \to B\) as objects, also referred to as the category having exponentials. Typically, one thinks of \(\boxdot\) as conjunction and of this internalization as implication.

\(^8\)\(*\)-autonomy means that there exists an operation \(*\) on the monoidal category from which the internalization of morphism sets follows as \((A \boxtimes B^*)^*,\) cf. classical logic where we have \(A \Rightarrow B = \neg A \lor B = \neg (A \land \neg B)\) by the De Morgan rule.

\(^9\)When we spell out this alternative definition of compact closure it indeed avoids much of the categorical jargon. But it also has a very elegant abstract formulation in terms of bicategories: a compact closed category is a symmetric monoidal category in which, when viewed as a one-object bicategory, every one-cell \(A\) has a left adjoint \(A^*\).
(if $V$ has an inner-product, $e_i := (e_i | -)$) Note that $\eta_V(1)$ can be thought of as an abstract generalization of the notion of an EPR-state.

Given the name and coname of a morphism $f : A \to B$, respectively

$$\begin{align*}
\Rightarrow f^\triangledown &: (1 \boxtimes f) \circ \eta_A : I \to A^* \boxtimes B \quad \text{and} \quad \triangleleft f,^\triangledown &: \epsilon_A \circ (f \boxtimes 1) : A \boxtimes B^* \to I,
\end{align*}$$

one can prove the **Compositionality Lemma** ([5] §3.3), diagrammatically,

$$\begin{array}{c}
\begin{array}{ccc}
A & \cong & A \boxtimes I \\
\downarrow & & \downarrow \\
\cong & \cong & A \boxtimes (B^* \boxtimes C) \\
\downarrow & & \downarrow \\
C & \cong & I \boxtimes C
\end{array}
\end{array}$$

for $f_1 : A \to B$ and $f_2 : B \to C$. This lemma generalizes the defining diagram of compact closedness since $\eta_A = \triangledown 1_A$ and $\epsilon_A = \triangleleft 1_A$ (cf. EPR-state $\cong 1$). The careful reader will have understood the picture by now,

$$\begin{array}{c}
\begin{array}{c}
\triangleleft f_1 \downarrow
\end{array}
\begin{array}{c}
\Rightarrow f_2 \triangledown
\end{array}
\end{array}$$

hence it seems as if there is an information flow through names and conames,

$$\begin{array}{c}
\begin{array}{c}
\triangleleft f_1 \downarrow
\end{array}
\begin{array}{c}
\Rightarrow f_2 \triangledown
\end{array}
\end{array}$$

Are we really there yet? We actually have two things, names and conames, and names act as `the output of a bipartite projector’ while conames act as `the input of a bipartite projector’. The obvious thing to do is to glue a coname and a name together in order to produce a bipartite projector.

$$\begin{array}{c}
\begin{array}{c}
\triangleleft f_1 \downarrow
\end{array}
\begin{array}{c}
\Rightarrow f_2 \triangledown
\end{array}
\end{array}$$

However, we have a type-mismatch.

$$P_f := \Rightarrow f^\triangledown \circ \triangleleft f,^\triangledown : A \boxtimes B^* \to A^* \boxtimes B$$

To solve this problem we need a tiny bit of extra structure. This bit of extra structure will capture the idea of *complex conjugation*. When conceiving elements as *Dirac-kets*, it will provide us with a notion of *Dirac-bra*. We will introduce strong compact closure, metaphorically,

$$\begin{array}{c}
\text{strong compact closure} \sim \text{sesquilinear inner-product space}
\end{array}$$

$$\begin{array}{c}
\text{compact closure} \sim \text{vector space}
\end{array}$$

**Strong compact closure, inner-products and projectors.** The assignment $A \rightarrow A^*$ which arises as part of the definition of compact closure actually extends to one on morphisms,

$$\begin{array}{c}
\begin{array}{c}
\cong & \cong & \cong \\
B^* & I \boxtimes B^* & (A^* \boxtimes A) \boxtimes B^*
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
\cong & \cong & \cong \\
A^* \boxtimes I & A^* \boxtimes (B \boxtimes B^*)
\end{array}
\end{array}$$
We can then set 
that is, the Hilbert space is a strong compact closed structure. But for relational converses, the same goes for any compact closed category where we discuss these scalars and contravariant functors.

For vector spaces the matrix of $f^*$ is the transposed of the matrix of $f$ when taking $\{e_i\}_{i=1}^n$ as base for $V^*$ given base $\{e_i\}_{i=1}^n$ of $V$. For relations $R^*$ is the relational converse of $R$. One verifies that $(\cdot)^*: C \rightarrow C$ is a contravariant functor, that is $(f_1 \circ f_2)^* = f_2^* \circ f_1^*$, and that there exists a natural isomorphisms $A^{**} \simeq A$ and $(A \boxtimes B)^* \simeq A^* \boxtimes B^*$.

**Definition 3.1** A strongly compact closed category [3] is a compact closed category for which $A = A^{**}$ and for which the assignment $A \mapsto A^*$ and $(A \boxtimes B)^* \simeq A^* \boxtimes B^*$ has also an involutive covariant functorial extension, which commutes with the compact closed structure.

We set $f \mapsto f_*$ for this functorial extension. For each morphism $f: A \rightarrow B$ we define its adjoint and a bipartite projector as 

$$f^\dagger := (f^*)_*: B \rightarrow A$$

and now we can define bipartite projectors to be 

$$P_f := f^\dagger \circ (f^\dagger)^\dagger = f^\dagger \circ \omega_f: A^* \boxtimes B \rightarrow A^* \boxtimes B,$$

and we call an isomorphism $U: A \rightarrow B$ unitary iff $U^{-1} = U^\dagger$. An abstract notion of inner-product also emerges. Given elements $\psi, \phi: I \rightarrow A$ we set $\langle \psi | \phi \rangle := \psi^\dagger \circ \phi \in C(I, I)$ where $C(I, I)$ are the morphisms of type $I \rightarrow I$ — we discuss these scalars in more detail below. We can now prove the usual defining properties of adjoints and unitarity in abstract generality,

$$\langle f^\dagger \circ \psi | \phi \rangle_B = \langle f^\dagger \circ \psi \rangle_B \circ \phi = \psi^\dagger \circ f \circ \phi = \langle \psi | f \circ \phi \rangle_A,$$

$$\langle U \circ \psi | U \circ \varphi \rangle_B = (U^\dagger \circ U \circ \psi | \varphi \rangle_A = \langle \psi | \varphi \rangle_A.$$

When calling $\psi: I \rightarrow A$ a ket, then $\psi^\dagger: I \rightarrow A$ is the corresponding bra and the scalar $\phi^\dagger \circ \psi: I \rightarrow I$ is a bra-ket. Hence strong compact closure provides a nice and juicy lump of Hilbert space — see [3] §7 and [6] §2 for details.

The category $(\text{Rel}, \times)$ is trivially strongly compact closed for $R_* := R$, so $R^\dagger = R^*$, that is, adjoints are relational converses. The same goes for any compact closed category where $A^* = A$. For $(\text{FdVec}_K, \otimes)$ we don’t have $V^* = V$, nor does the above defined compact closed structure satisfy $V^{**} = V$, so it cannot be extended to a strong compact closed structure. But for $\mathbb{K} := \mathbb{R}$, finite-dimensional real inner-product spaces are strongly compact closed for $V := V^*$ and $\epsilon_V := \langle - | - \rangle$, and for $\mathbb{K} := \mathbb{C}$, our main category $(\text{FdHilb}, \otimes)$ is also strongly compact closed when we take $H^*$ to be the conjugate space, that is, the Hilbert space with the same elements as $H$ but with $\alpha \bullet_H^* \phi := \overline{\alpha} \bullet_H \phi$ as scalar multiplication and $\langle \phi | \psi \rangle_{H^*} := \langle \psi | \phi \rangle_H$ as (sesquilinear) inner-product. We can then set $\epsilon_H: H \otimes H^* 

\rightarrow I: \psi \otimes \phi \mapsto \langle \phi | \psi \rangle$. One verifies that we recover the usual notion of adjoint, that is, the conjugate transpose, where $(\cdot)^*$ provides transposition while $(\cdot)_*$ provides complex conjugation.

Let us end this paragraph by saying that most things discussed above extend to infinite dimensional settings when using ideas from [3].
A note on categorical traces. This paragraph slightly diverges from our story line, but we do want to mention that much of the inspiration for [13, 12] emerged from [4] where we studied the physical realization of ‘abstract traces’ [19], which generalize traditional feedback traces [18]. It turns out that both on (Rel, ×) and (FdVecK, ⊗), due to compact closure, the trace also admits a feedback-loop type interpretation, but a linear ‘only-use-once’ one. Please consult [6] for more details and some nice pictures.

4 Beyond von Neumann’s axiomatics

Biproducts. Strong compact closure provides a serious lump of Hilbert space, but we need some additional types which enable to encode classical information and its flow in our quantum formalism. They will capture ‘gluing pictures together’ and ‘distributing the knowledge on in which picture we are’ (cf. §2). To this means we use biproducts, that is, objects A ⊸ B which both are the product and the coproduct for A and B, and corresponding induced morphisms f ⊸ g: A ⊸ B → C ⊸ D for f: A → C and g: B → D. Contrary to , biproducts go (by definition) equipped with projections pij: ⊸ iAi → Aj, also with injections qij: Aj → ⊸ jAi, and with pairing and copairing operations, ⟨fij⟩i: A ⊸ iAi and [fij]: ⊸ jAi → A, for morphisms fi: A → Ai and gi: Ai → A with coinciding domain and codomain respectively. From these we can construct diagonals and codiagonals, ΔA := ⟨1A, 1A⟩: A → A ⊸ A and ∇A := [1A, 1A]: A ⊸ A → A. This ‘non-linear’ ⊸-structure encodes that there is no difference between looking at two pictures separately, or together — the components of a compound quantum system cannot be considered separately, hence is linear.

We take the projections and injections such that they work nicely together with the strong compact closure by setting qij = pij (and hence p′ ij = qij). Of crucial importance for us is the distributivity of over ,[10] that is, there is a natural isomorphism

\[ \text{DIST}: A \boxtimes (B_1 \boxplus B_2) \cong (A \boxtimes B_1) \boxplus (A \boxtimes B_2). \]

For (Rel, ×) the disjoint union + provides a biproduct structure with inclusion as injections. For (FdHilb, ⊗) the direct sum ⊠ provides a biproduct structure with coordinate projections as projections.

Categorical quantum mechanics. We define a quantum formalism relative to any strongly compact closed category with some biproducts.

i. We take state spaces to be objects which do not involve explicit biproducts and use ⊸ to describe compound systems. The basic data unit is a state space Q which is unitary isomorphic to I ⊸ I, which in the case of (Rel, ×, +) where I\boxplus = {∗} + {∗} yields the boolean type and in the case of (FdHilb, ⊗, ⊠) where I\boxplus = C ⊠ C yields the qubit type.

ii. Explicit biproducts express ‘different pictures’ due to distinct measurement outcomes, they enable to encode classical data. The distributivity isomorphism DIST expresses exchange of classical data! (see below)

iii. We have already defined bipartite projectors. To turn them into a measurement we need to glue a complete family of mutually orthogonal ones to each other. More generally, we define a spectral decomposition to be a unitary morphism U: A → ⊸ iAi. We define the corresponding non-destructive measurement to be the copairing

\[ \langle P_i \rangle_i: A \rightarrow \boxplus iAi \quad \text{where} \quad P_j = \pi_j^\dagger \circ \pi_j: A \rightarrow A \quad \text{for} \quad \pi_j = p_j \circ U \]

with pij: ⊸ iAi → Ai the projections for the biproduct ⊸ iAi. As shown in [5], these general projectors Pij: A → A are self-adjoint, mutually orthogonal, and their sum is 1A — we discuss the sum of morphisms below.

When the spectral decomposition is of type A → ⊸ iI the corresponding measurement is non-degenerated. We call such a spectral decomposition, which by the defining property of products can be rewritten as ⟨πi⟩i: A → ⊸ iI, a non-degenerated destructive measurement. For an explicit definition of an abstract Bell-base measurement, or any other measurement which allows teleportation, we refer to [5]. Isolated reversible dynamics is unitary.

iv. The passage from a non-degenerated non-destructive measurement to a destructive one involves dropping \( \psi_i := \pi_i^\dagger \) : I → A. We conceive such a component as a preparation. Hence a non-destructive measurement decomposes in ⟨πi⟩i, which gives the measurement’s outcome, and \( \psi_i \), which gives the state ‘after the collapse’ (cf. von Neumann’s projection postulate).

---

[10] Which follows by closedness of and being a coproduct.
Abstract quantum teleportation. The righthandside of the diagram

\[
\begin{align*}
Q & \overset{(1 \otimes 1^Q) \circ \rho_Q}{\longrightarrow} Q \\
Q \otimes (Q^* \otimes Q) & \cong \text{spatial relocation} \\
(Q \otimes Q^*) \otimes Q & \overset{\text{Bell-base measurement}}{\longrightarrow} (1^Q, \beta_i^1) \otimes Q \\
(1^Q)_i^4 & \overset{\text{classical communication}}{\longrightarrow} (\oplus^4_i \lambda_Q^{-1}) \circ \text{DIST} \\
\oplus^4_i \beta_i^{-1} & \rightarrow \text{unitary correction}
\end{align*}
\]

gives a complete description of the teleportation protocol. The lefthandside expresses the intended behavior (obtaining an identity in each of the four pictures). In [5] we proved correctness, the diagram commutes!

Abstract presentations and proofs of correctness of logic gate teleportation [17] and entanglement swapping [31] can be found in [5].

Immediately after the Bell-base measurement the type is \((\oplus^4_i \beta_i^{-1}) \otimes Q\) where \(\oplus^4_i \beta_i^{-1}\) represents the four different measurement outcomes. However, these four pictures only exist ‘locally’. After distributing this information,

\[
\begin{align*}
(\oplus^4_i \beta_i^{-1}) \otimes Q & \overset{\text{DIST}}{\longrightarrow} (\oplus^4_i \beta_i^{-1}) \otimes (1 \otimes Q) \\
\oplus^4_i \beta_i^{-1} & \overset{\text{classical communication}}{\longrightarrow} (\oplus^4_i \beta_i^{-1}) \otimes Q
\end{align*}
\]

there are four different pictures ‘globally’. Hence we can apply the appropriate unitary correction \(\beta_i^{-1} : Q \rightarrow Q\) in each picture, that is, \(\oplus^4_i \beta_i^{-1}\).

The spectrum of a measurement \(\langle P_i \rangle_i\) is the index set \(\{i\}_i\), which for example could encode locations in physical space. Since for teleportation we assume to work with spatially located particles, that is, there are no spatial superpositions, the associativity natural isomorphism allows to encode spatial association (i.e. proximity) in a qualitative manner.

Scalars, normalization, probabilities and the Born rule. Up to now one might think that the abstract setting is purely qualitative (whatever that means anyway). But it is not! The scalars \(C(I, I)\) of any monoidal category \(C\) have a commutative composition [21], that is, a multiplication.

If the biproduct \(I \oplus I\) exists, we can define a sum of scalars \(s, s' : I \rightarrow I\) as

\[
s + s' := \nabla_1 \circ (s \oplus s') \circ \Delta_1 : I \rightarrow I
\]

and one shows that the above defined multiplication distributes over this sum and that there is a zero \(O_I : I \rightarrow I\). Hence we obtain an abelian semiring.\(^{11}\)

\(^{11}\)That is, a field except that there are no inverses for addition nor for multiplication.
Furthermore, each scalar \( s : I \to I \) induces a natural transformation

\[
s_A : \lambda_A^{-1} \circ (s \otimes 1_A) \circ \lambda_A : A \to A
\]

for each object \( A \), which allows us to define \textit{scalar multiplication} as \( s \cdot f := f \circ s_A \) for \( f : A \to B \), where \( f \circ s_A = s_B \circ f \) by naturality, that is, \textit{morphisms preserve scalar multiplication}.

Since we have an inner-product (which, of course, is scalar valued) we can now talk about \textit{normalization} e.g. an element \( \psi : I \to A \) is normalized iff \( \psi^\dagger \circ \psi = 1_1 \).\(^{12}\) Besides the special scalars \( 1_1 \) and \( 0_1 \) there are many others, those which satisfy \( s^\dagger = s \), those of the form \( s^1 \otimes s \), those which arise from inner-products of normalized elements, and the latter multiplied with their adjoint, in \( (FdHilb, \otimes, +) \) respectively being \( 1_0 \) and \( 0_0 \), the positive reals \( \mathbb{R}^+ \), the unit disc in \( \mathbb{C} \) and the unit interval \([0, 1]\).

Consider now the basic protocol of (non-destructively) measuring a state

\[
\begin{array}{c}
\psi \\
\rightarrow
\end{array}
\begin{array}{c}
A
\end{array}

(\langle P_i \rangle_{i=1}^{i=n} \oplus_1^{i=1} A).
\]

If we look at one component of the biproduct, i.e., one picture,

\[
\begin{array}{c}
\psi \\
\rightarrow
\end{array}
\begin{array}{c}
A
\end{array}

\pi_i

\begin{array}{c}

s_i \in \mathbb{C}(I, I)
\end{array}

\begin{array}{c}
\rightarrow
\end{array}

\begin{array}{c}
\psi^{\dagger}
\end{array}

\begin{array}{c}
\rightarrow
\end{array}

\begin{array}{c}
A
\end{array},
\]

we discover a special scalar of the \textquoteleft unit disc type\textquoteright. One verifies that

\[
\text{PROB}(P_i, \psi) := s_i^{\dagger} \circ s_i \quad \text{satisfies} \quad \sum_{i=1}^{i=n} \text{PROB}(P_i, \psi) = 1_1,
\]

hence these \( [0, 1] \) type scalars \text{PROB}(P_i, \psi) provide an abstract notion of \textit{probability} \( [5] \). Moreover, using our abstract inner-product one verifies that \( \text{PROB}(P_i, \psi) = \langle \psi \mid P_i \circ \psi \rangle \), that is, we prove the \textit{Born rule}.

\textbf{Mixing classical and quantum uncertainty.} This section comprises a \textit{proposal} for the abstract status of density matrices. Having only one page left, we need to be brief. In the von Neumann formalism density matrices are required for two reasons: \textbf{i}, to describe part of a larger (compound) system, say \textit{ontic density matrices}, and, \textbf{ii}, to describe a system about which we have incomplete knowledge, say \textit{epistemic density matrices}. Hence ontic density matrices arise by considering one component of an element of the name type, \( \lceil \xi \rceil : I \to A_1 \boxtimes A_2 \) for \( \xi : A_1^1 \to A_2 \). In order to produce epistemic density matrices, consider the situation of a measurement, but we extract the information concerning the actual outcome from it, that is, we do the converse of distributing classical data,

\[
\begin{array}{c}
\phi \\
\rightarrow
\end{array}
\begin{array}{c}
A
\end{array}

\langle P_i \rangle_i \\
\oplus_i A

\oplus_i \lambda_Q \\
\oplus_i (I \boxtimes A)

\begin{array}{c}
\rightarrow
\end{array}

\begin{array}{c}
(\boxtimes I) \boxtimes A
\end{array}.
\]

This results again in an element of the name type, \( \lceil \omega \rceil : I \to (\boxtimes I) \boxtimes A \) for \( \omega : (\boxtimes I)^* \to A \). Metaphorically one could say that the \textit{classical data is entangled with the quantum data}. Since our formalism allows both to encode classical data and quantum data there is no need for a separate density matrix formalism as it is the case for the von Neumann formalism.

One verifies that the principle of \textit{no signalling faster than light} still holds for the name type in the abstract formalism, that is, operations locally on one component will not alter the other, provided there is no classical data exchange. But there can be a passage from ontic to epistemic e.g.

\[
I \to A_1 \boxtimes A_2 \quad \sim \quad I \to (\boxtimes I) \boxtimes A_2
\]

when performing the measurement \( \langle \pi_i \rangle_i \boxtimes 1_{A_2} : A_1 \boxtimes A_2 \to (\boxtimes I) \boxtimes A_2 \). For epistemic density matrices this means that the classical data and the quantum data are truly distinct entities.

\(^{12}\) A discussion of normalization of projectors can be found in \( [6] \).
Using Lemma 7.6 of [5] one verifies that \( \omega : \oplus I \rightarrow A^\ast \) is given by \( \omega = [s_i \bullet (\pi^\ast_i \circ u_1)]_i \) where \( s_i := \pi_i \circ \phi \) and \( u_1 : I \simeq I^\ast \) (a natural transformation which exists by compact closure). Hence \( \omega \) and hence also \( \Gamma \omega \) is determined by a list of orthogonal (pure) states \( (\pi^\ast_i : I \rightarrow A)_i \) and a list of scalars \( (s^1_i : I \rightarrow A)_i \) all of the unit disc type — compare this to the orthogonal eigenstates of a standard Hilbert space density matrix and the corresponding eigenvalues which all are of the \([0, 1]\) type.

So we can pass from pure states \( \phi : I \rightarrow I \) to density matrices by ‘plugging in an ancilla’, which either represents classical data (epistemic) or which represents an external part of the system (ontic). The other concepts that can be derived from basic quantum mechanics by ‘acting on part of a bigger system’ (non-isolated dynamics, generalized measurements, etc.) can also be defined abstractly, e.g. generalized measurements as

\[
\langle f_i \rangle_{i=1}^n : A \rightarrow \oplus_i A \quad \text{with} \quad \sum_{i=1}^n f_i \circ f_i = 1_A,
\]

while abstract analogous of theorems such as Naimark’s can be proven. Of course, many things remain to be verified such as abstract analogous of Gleason’s theorem. I might have something to add to this in my talk. :)
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