SPECTRAL ACTION BEYOND THE STANDARD MODEL

Thomas SCHÜCKER\footnote{Unité Propre de Recherche 7061} and Sami ZOUZOU\footnote{also at Université de Provence, schucker@cpt.univ-mrs.fr, szouzou2000@yahoo.fr}

Abstract

We rehabilitate the $M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ model of electro-magnetic, weak and strong forces as an almost commutative geometry in the setting of the spectral action.

PACS-92: 11.15 Gauge field theories
MSC-91: 81T13 Yang-Mills and other gauge theories

CPT-01/P.4239
hep-th/yymmxxxx
1 Introduction

The associative algebra \( \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \) allows to interpret the standard model of electromagnetic, weak and strong forces as an almost commutative geometry \([1, 2]\). As such they are naturally unified with gravity by the spectral action \([3, 4]\). In \([5]\) we have tried to slightly increase the algebra to \( \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \) in the setting without gravity. This model suffered from light Higgs scalars generated by noncommutative geometry. In the present paper we reconsider the model including gravity.

2 The fermion masses

In his approach without gravity, Connes derives the Higgs field from noncommutative geometry as a fluctuation of a Yang-Mills connection: the Higgs is an anti-Hermitian 1-form in a noncommutative differential calculus with exterior derivative \( \delta \) generated by a self-adjoint Dirac operator \( D \),

\[
H = \sum_j \{ a_{0j} \delta a_{1j} + J a_{0j} \delta a_{1j} J^{-1} \} = \sum_j \{ (-i) \rho(a_{0j})[D, \rho(a_{1j})] + J (-i) \rho(a_{0j})[D, \rho(a_{1j})] J^{-1} \}, \tag{1}
\]

where the \( a_{0j} \) and \( a_{1j} \) are elements of the associative algebra \( \mathcal{A} \), \( \rho \) is a faithful representation of \( \mathcal{A} \) on the Hilbert space \( \mathcal{H} \) of the fermions and \( J \) is the anti-unitary charge conjugation operator. The Higgses carry the usual affine group representation of connections

\[
^uH = uHu^{-1} + u\delta u^{-1} = \rho(u)H\rho(u)^{-1} - i\rho(u)[D, \rho(u)^{-1}], \tag{2}
\]

where \( u \) is a unitary, \( u \in U(\mathcal{A}) \).

We take the algebra

\[
\mathcal{A} = \mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \ni (b, a, c). \tag{3}
\]

The Hilbert space is copied from the Particle Physics Booklet \([6]\),

\[
\mathcal{H}_L = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}) , \tag{4}
\]

\[
\mathcal{H}_R = (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}) . \tag{5}
\]

In each summand, the first factor denotes weak isospin doublets or singlets, the second denotes \( N \) generations, \( N = 3 \), and the third denotes colour triplets or singlets. Let us choose the
following basis of the Hilbert space, counting fermions and antifermions independently, $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_c^L \oplus \mathcal{H}_c^R = \mathbb{C}^{90}$:

$$
\begin{pmatrix}
(u \\ d)
\end{pmatrix}_L, \begin{pmatrix}
c \\ t \\ b
\end{pmatrix}_L, \begin{pmatrix}
\nu_e \\ \nu_\mu \\ \nu_\tau
\end{pmatrix}_L, \begin{pmatrix}
\mu
\end{pmatrix}_L, \begin{pmatrix}
\tau
\end{pmatrix}_L;
$$

$$
\begin{pmatrix}
u_e \\ \nu_\mu \\ \nu_\tau
\end{pmatrix}_L,
\begin{pmatrix}
\mu
\end{pmatrix}_L, \begin{pmatrix}
\tau
\end{pmatrix}_L.
$$

This is the current eigenstate basis, the representation $\rho$ acting on $\mathcal{H}$ by

$$
\rho(b, a, c) := \begin{pmatrix}
\rho_L & 0 & 0 & 0 \\
0 & \rho_R & 0 & 0 \\
0 & 0 & \tilde{\rho}_L^c & 0 \\
0 & 0 & 0 & \tilde{\rho}_R^c
\end{pmatrix}
$$

with

$$
\rho_L(a) := \begin{pmatrix}
a \otimes 1_N \otimes 1_3 & 0 \\
0 & a \otimes 1_N
\end{pmatrix},
\rho_R(b) := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \bar{b}_1 \otimes 1_N & 0 & 1_N \otimes 1_3 \\
0 & 0 & 1_N \otimes 1_3 & 0 \\
0 & 0 & 0 & \bar{b}_1
\end{pmatrix},
$$

$$
\rho_L^c(b, c) := \begin{pmatrix}
1_2 \otimes 1_N \otimes c & 0 \\
0 & \bar{b}_1 \otimes 1_N
\end{pmatrix},
\rho_R^c(b, c) := \begin{pmatrix}
1_N \otimes c & 0 & 0 \\
0 & 1_N \otimes c & 0 \\
0 & 0 & \bar{b}_1\n\end{pmatrix}.
$$

The apparent asymmetry between particles and antiparticles – the former are subject to weak, the latter to strong interactions – disappears after application of the spin lift which involves the charge conjugation

$$
J = \begin{pmatrix}
0 & 1_{15N} \\
1_{15N} & 0
\end{pmatrix} \circ \text{complex conjugation}.
$$

For the sake of completeness, we record the chirality as matrix

$$
\chi = \begin{pmatrix}
-1_{8N} & 0 & 0 & 0 \\
0 & 1_{7N} & 0 & 0 \\
0 & 0 & -1_{8N} & 0 \\
0 & 0 & 0 & 1_{7N}
\end{pmatrix}.
$$

The Dirac operator

$$
\mathcal{D} = \begin{pmatrix}
0 & \mathcal{M} & 0 & 0 \\
\mathcal{M}^* & 0 & 0 & 0 \\
0 & 0 & \mathcal{M}^* & 0 \\
0 & 0 & 0 & \mathcal{M}
\end{pmatrix}
$$

2
is constructed from the fermionic mass matrix of the standard model,
\[
\mathcal{M} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \otimes M_u \otimes 1_3 + \begin{pmatrix}
0 & 0 \\
0 & 1 \\
\end{pmatrix} \otimes M_d \otimes 1_3 + \begin{pmatrix}
0 \\
0 \\
\end{pmatrix} \otimes M_e,
\]
(12)
with
\[
M_u := \begin{pmatrix}
m_u & 0 & 0 \\
0 & m_c & 0 \\
0 & 0 & m_t \\
\end{pmatrix},
M_d := C_{KM} \begin{pmatrix}
m_d & 0 & 0 \\
0 & m_s & 0 \\
0 & 0 & m_b \\
\end{pmatrix},
\]
(13)
and
\[
M_e := \begin{pmatrix}
m_e & 0 & 0 \\
0 & m_\mu & 0 \\
0 & 0 & m_\tau \\
\end{pmatrix},
\]
(14)
with \(C_{KM}\) denoting the Cabibbo-Kobayashi-Maskawa matrix.

The intersection form
\[
\cap_{ij} := \text{tr} \left[ \chi \rho(p_i) J \rho(p_j) J^{-1} \right],
\]
(15)
with a set of minimal projectors \(p_j\) in \(\mathcal{A}\) is non-degenerate and Poincaré duality holds. Indeed, we have \(p_1 = (1, 0, 0), p_2 = \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0\right), p_3 = \left(0, 0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)\), and
\[
\cap = N \begin{pmatrix}
2 & -1 & 2 \\
-1 & 0 & -1 \\
2 & -1 & 0 \\
\end{pmatrix}.
\]
(16)
Note that for Dirac instead of Weyl neutrinos in all three generations, \(\mathcal{H} = \mathbb{C}^{96}\), Poincaré duality would fail:
\[
\cap = N \begin{pmatrix}
4 & -1 & 2 \\
-1 & 0 & -1 \\
2 & -1 & 0 \\
\end{pmatrix}.
\]
(17)
Since Majorana masses are excluded in Connes’ noncommutative geometry, we arrive at the same conclusion as in the \(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})\) version: Poincaré duality allows at most two out of three massive neutrinos. For completeness we recall the intersection form of the quaternionic version with Weyl or Dirac neutrinos:
\[
N \begin{pmatrix}
2 & -2 & 2 \\
-2 & 0 & -2 \\
2 & -2 & 0 \\
\end{pmatrix} \quad \text{or} \quad N \begin{pmatrix}
4 & -2 & 2 \\
-2 & 0 & -2 \\
2 & -2 & 0 \\
\end{pmatrix}.
\]
(18)
Let us come back to our example and compute the Higgs representation. It is made of two isospin doublets, colour singlets $h_1$ and $h_2$,

$$H = i \begin{pmatrix} 0 & \rho_L(h)\mathcal{M} & 0 & 0 \\ \mathcal{M}^*\rho_L(h^*) & 0 & 0 & 0 \\ 0 & 0 & \mathcal{M}^*\rho_L(h^*) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(19)

$$h = (h_1 \ h_2) = \begin{pmatrix} h_{11} \\ h_{21} \\ h_{12} \\ h_{22} \end{pmatrix} \in M_2(\mathbb{C}).$$

(20)

The Higgs being a fluctuation of the Yang-Mills connection, the Higgs Lagrangian is derived from the Yang-Mills Lagrangian in noncommutative geometry [1, 2]. This computation leads to the three pieces, that are added by hand in the conventional theory: the kinetic term with its minimal coupling to the Yang-Mills fields, the Higgs potential with its spontaneous symmetry breaking and the Yukawa couplings to the fermions. The ground state of the Higgs potential is $H_0 = 0$ and in terms of the scalar variable $\Phi = H - i\mathcal{D}$, that transforms homogeneously, $\rho(u)\Phi\rho(u)^{-1}$, the ground state, that gives masses to the fermions, is the Dirac operator $\Phi_0 = -i\mathcal{D}$. This justifies the identification of Dirac operator and fermionic mass matrix above. The ground state also gives masses to the physical scalars and the above choice has neutral and charged ones, that are too light to withstand confrontation with experiment.

In his second approach, Connes uses the same noncommutative geometry to derive Yang-Mills connections and their Higgs scalars as fluctuations of the gravitational field [3]. Now the Higgs is computed as:

$$H = \sum_j \{ \rho(u_j)[\mathcal{D}, \rho(u_j)^{-1}] + J \rho(u_j)[\mathcal{D}, \rho(u_j)^{-1}] J^{-1} \}, \quad u_j \in U(\mathcal{A}).$$

(21)

In our example, $U(\mathcal{A}) = U(1) \times U(2) \times U(3)$, we get the same result as without gravity: two isospin doublets of scalars, $\varphi_1$ and $\varphi_2$ in the homogeneous notation.

The Higgs being a fluctuation of the metric, the Higgs Lagrangian is now derived from the noncommutative version of general relativity, the spectral action [4]. This computation leads again to the three pieces, kinetic term with its minimal coupling, Higgs potential and Yukawa couplings. The Higgs potential breaks again the gauged group $U(\mathcal{A})$ spontaneously. Here however, it is not true in general that the ground state is given by the Dirac operator, $\Phi_0 = -i\mathcal{D}$. This holds true for the standard model with algebra $\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The first counter example with $\Phi_0 \neq -i\mathcal{D}$ is due to Girelli [7] with $\mathcal{A} = \mathbb{H} \oplus \mathbb{H}$. We will show that replacing the quaternions $\mathbb{H}$ by complex $2 \times 2$ matrices in the standard model also spoils this precious property. We say precious because it allows for different masses within the same irreducible fermion multiplet.
In our example, writing the two complex doublets $\varphi_1$, $\varphi_2$ as a complex $2 \times 2$ matrix $\varphi$ like we did before with its inhomogeneous counter part $h$ in equation (20), the Higgs potential takes the simple form,

$$V(\varphi) = \lambda \text{tr} \left[ (\Phi^*\Phi)^2 \right] - \frac{1}{2} \mu^2 \text{tr} [\Phi^*\Phi] = 4\lambda \text{tr} \left[ (\mathcal{M}\mathcal{M}^*\rho_L(\varphi^*\varphi))^2 \right] - 2\mu^2 \text{tr} [\mathcal{M}\mathcal{M}^*\rho_L(\varphi^*\varphi)],$$

with positive parameters $\lambda$ and $\mu$. For simplicity let us put all fermion masses to zero except for the top and bottom mass that we take different and let us put the Cabibbo-Kobayashi-Maskawa matrix to one. Then the ground state is

$$\varphi_0 = \frac{\mu}{2\sqrt{\lambda}} \begin{pmatrix} m_t^{-1} & 0 \\ 0 & m_b^{-1} \end{pmatrix}.$$  

But this means that the spontaneous symmetry breaking induces identical top and bottom masses and the input values $m_t$ and $m_b$ in the initial Dirac operator have nothing to do with fermion masses.

3 Spin lifts with central extensions

In the spectral action setting our model has three shortcomings, $m_t = m_b$, light physical scalars and two additional $U(1)$ bosons, which are anomalous and also too light. In this section we centrally extend the spin lift, that is necessary to fluctuate the gravitational field. We motivate these extensions by imposing the lift to be double-valued and show that the most economical such extension solves all three shortcomings. Indeed it recuperates the standard model as Connes derives it from the algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

3.1 Commutative geometry

To fluctuate the gravitational field Connes proceeds in two steps. In the first step, he reformulates Einstein’s derivation of general relativity from Riemannian geometry in the algebraic language of his geometry by carefully avoiding to use the commutativity of the underlying algebra $\mathcal{A} = C^\infty(M)$ of differentiable functions on spacetime $M$. Here the key ingredient is a group homomorphism $L$ that maps every general coordinate transformation $\varphi$ of spacetime to a gauged Lorentz transformation acting on Dirac spinors. This homomorphism generalizes the spin lift $SO(3) \rightarrow SU(2)$ of quantum mechanics to the special and general relativistic setting. The local form of this double-valued homomorphism is spelled out in [8]. The algebraic formulation of coordinate transformations is the group of automorphisms of the algebra, $\text{Aut}(\mathcal{A})$. Note that although $\mathcal{A} = C^\infty(M)$ is commutative, its automorphism group, $\text{Aut}(\mathcal{A})=\text{Diff}(M)$ is
highly nonAbelian. The algebraic definition of the gauged spin group is what Connes calls the group of automorphisms lifted to the Hilbert space

$$\text{Aut}_H(A) := \{ U \in \text{End}(H), \quad UU^* = U^*U = 1, \quad UJ = JU, \quad U\chi = \chi U, \quad i_U \in \text{Aut}(\rho(A)) \}, \quad (24)$$

with $i_U(x) := UxU^{-1}$. In Riemannian geometry the Hilbert space consists of square integrable Dirac spinors, the chirality is $\chi = \gamma_5$ and $J$ is the charge conjugation of Dirac spinors. The first three properties say that a lifted automorphism $U$ preserves probability, charge conjugation and chirality. The fourth, called covariance property is related to the locality requirement of field theory. It allows to define the projection $p : \text{Aut}_H(A) \rightarrow \text{Aut}(A)$ by

$$p(U) = \rho^{-1}i_U\rho. \quad (25)$$

Of course we demand that the lift respects the projection, $p(L(\varphi)) = \varphi$.

Einstein shows that the gravitational field is coded in a Riemannian metric and Connes shows that the Riemannian metric is coded in its Dirac operator $\bar{\partial}$. Furthermore starting from the flat Dirac operator $\bar{\partial}_0$ one gets a curved one $\bar{\partial} = L(\varphi)\bar{\partial}_0L(\varphi)^{-1}$ by fluctuating with a general coordinate transformation $\varphi$. The spectral action is simply the trace of $\bar{\partial}$ properly regularized and it reproduces the Einstein-Hilbert action. Centrally extending $L$ to the group of unitaries $U(A) = MU(1)$ yields gravity coupled to Maxwell’s electromagnetism. However welcome, this central extension is optional in the sense that it does not change the degree of valuedness of the lift $L$.

In the second step, Connes repeats his derivation of general relativity for almost commutative geometries, that is tensor products of the infinite dimensional, commutative algebra $C^\infty(M)$ with finite dimensional, noncommutative algebras. It is in this precise context that he derives some very special Yang-Mills-Higgs models by fluctuating the metric.

### 3.2 Central extensions of finite geometries

We may restrict ourselves to the finite dimensional, ‘internal’ part where central extensions of the spin lift are readily available [9]. Let $\mathcal{A}$ be a real, associative involution algebra with unit, that admits a faithful * representation $\rho$. In finite dimensions, a simple such algebra is a real, complex or quaternion matrix algebra, $\mathcal{A} = Mn(\mathbb{R}), \quad Mn(\mathbb{C})$ or $Mn(\mathbb{H})$, represented irreducibly on the Hilbert space $\mathcal{H} = \mathbb{R}^n, \quad \mathbb{C}^n$ or $\mathbb{C}^{2n}$. In the first and third case, the representations are the fundamental ones, $\rho(a) = a, \quad a \in \mathcal{A}$, while $Mn(\mathbb{C})$ has two non-equivalent irreducible representations on $\mathbb{C}^n$, the fundamental one, $\rho(a) = a$ and its complex conjugate $\rho(a) = \bar{a}$. In the general case we have sums of simple algebras and sums of irreducible representations as the model under consideration. With this application in mind, we concentrate on complex matrix
algebras $M_n(\mathbb{C})$ in this section. Anyhow, $M_n(\mathbb{R})$ and $M_n(\mathbb{H})$, do not have central unitaries close to the identity. In the following it will be important to separate the commutative and noncommutative parts of the algebra:

$$\mathcal{A} = \mathbb{C}^M \oplus \bigoplus_{k=1}^{N} M_{n_k}(\mathbb{C}) \ni a = (b_1, \ldots, b_M, c_1, \ldots, c_N), \quad n_k \geq 2. \quad (26)$$

Its group of unitaries is

$$U(\mathcal{A}) = U(1)^M \times \bigtimes_{k=1}^{N} U(n_k) \ni u = (v_1, \ldots, v_M, w_1, \ldots, w_N) \quad (27)$$

and its group of central unitaries

$$U_c(\mathcal{A}) := U(\mathcal{A}) \cap \text{center}(\mathcal{A}) = U(1)^{N+M} \ni u_c = (v_{c1}, \ldots, v_{cM}, w_{c1}, \ldots, w_{cN}). \quad (28)$$

The component of the automorphism group $\text{Aut}(\mathcal{A})$, that is connected to the identity, is the group of inner automorphisms, $\text{Aut}(\mathcal{A})^e = \text{In}(\mathcal{A})$. There are additional, discrete automorphisms, the complex conjugation and, if there are identical summands in $\mathcal{A}$, their permutations. These discrete automorphisms do not concern us here. An inner automorphism is of the form $i_u(a) = uau^{-1}$ for some unitary $u \in U(\mathcal{A})$. Multiplying $u$ with a central unitary $u_c$ of course does not affect the inner automorphism $i_{ucu} = i_u$. Note that this ambiguity distinguishes between ‘harmless’ central unitaries $v_{c1}, \ldots, v_{cM}$ and the others, $w_{c1}, \ldots, w_{cN}$, in the sense that

$$\text{In}(\mathcal{A}) = U^n(\mathcal{A})/U^{nc}(\mathcal{A}), \quad (29)$$

where we have defined the group of noncommutative unitaries

$$U^n(\mathcal{A}) := \bigtimes_{k=1}^{N} U(n_k) \ni w \quad (30)$$

and $U^{nc}(\mathcal{A}) := U^n(\mathcal{A}) \cap U^c(\mathcal{A}) \ni w_c$. The map

$$i : U^n(\mathcal{A}) \longrightarrow \text{In}(\mathcal{A})$$

$$(1, w) \longmapsto i_w \quad (31)$$

has kernel $\text{Ker} i = U^{nc}(\mathcal{A})$.

The lift of an inner automorphism to the Hilbert space has a natural form $[2]$, $L = \hat{L} \circ i^{-1}$ with

$$\hat{L}(w) = \rho(1, w) J \rho(1, w) J^{-1}. \quad (32)$$
It satisfies \( p(\hat{L}(w)) = i(w) \). If the kernel of \( i \) is contained in the kernel of \( \hat{L} \) then the lift is well defined, as e.g. for \( A = \mathbb{H}, U^{nc}(\mathbb{H}) = \mathbb{Z}_2. \)

\[
\begin{array}{ccc}
\text{Aut}_H(A) & \overset{p}{\twoheadrightarrow} & L \\
\downarrow & & \downarrow \ell \\
\text{In}(A) & \overset{i}{\hookleftarrow} & U^n(A) \overset{\det}{\approx} U^{nc}(A)
\end{array}
\]

For more complicated real or quaternionic algebras, \( U^{nc}(A) \) is finite and the lift \( L \) is multi-valued with a finite number of values. For noncommutative, complex algebras, their continuous family of central unitaries can not be eliminated except for very special representations and we face a continuous infinity of values. The solution of this problem is to extend \( \hat{L} \) by the harmful central unitaries \( w_c \in U^{nc}(A) \):

\[
\ell(w_c) = \rho\left( \prod_{j_1=1}^{N} (w_{cj_1})^{q_1j_1}, \ldots, \prod_{j_M=1}^{N} (w_{cj_M})^{q_Mj_M}, \prod_{j_{M+1}=1}^{N} (w_{cj_{M+1}})^{q_{M+1}j_{M+1}1_{n_1}}, \ldots, \prod_{j_{M+N}=1}^{N} (w_{cj_{M+N}})^{q_{M+N}j_{N}1_{n_N}} \right) \text{J}_\rho(...)^{-1} \tag{34}
\]

with the \((M+N)\times N\) matrix of charges \( q_{kj} \), charge because in the commutative case there is only one (harmless) \( U(1) \) and \( q \) is the electric charge. We allow multi-valued group homomorphisms, \( q_{kj} \in \mathbb{Q} \). The general extension satisfies indeed \( p(\ell(w_c)) = 1 \in \text{In}(A) \) for all \( w_c \in U^{nc}(A) \).

Having adjoined the harmful, continuous central unitaries, we may now streamline our notations and write the group of inner automorphisms as

\[
\text{In}(A) = \left( \prod_{k=1}^{N} \mathbb{Z}_{n_k} \right) / \Gamma \ni [w_\varphi \equiv [(w_{\varphi 1}, \ldots, w_{\varphi N})] \mod \gamma. \tag{35}
\]

\( \Gamma \) is the discrete group

\[
\Gamma = \prod_{k=1}^{N} \mathbb{Z}_{n_k} \ni (z_11_{n_1}, \ldots, z_N1_{n_N}), \quad z_k = \exp[-m_k2\pi i/n_k], \quad m_k = 0, \ldots, n_k - 1. \tag{36}
\]

The quotient is factor by factor. This way to write inner automorphisms is convenient for complex matrices, but not available for real and quaternionic matrices. Equation (29) remains the general characterization of inner automorphisms.

The lift \( L(w_\varphi) = (\hat{L} \circ i^{-1})(w_\varphi) \) is multi-valued with, depending on the representation, up to \( |\Gamma| = \prod_{j=1}^{N} n_j \) values. More precisely the multi-valuedness of \( L \) is indexed by the elements of
the kernel of the projection \( p \) restricted to the image \( L(\text{In}(\mathcal{A})) \). Depending on the choice of the charge matrix \( q \), the central extension \( \ell \) may reduce this multi-valuedness. Extending harmless central unitaries is useless for any reduction. With the multi-valued group homomorphism

\[
(h_\varphi, h_c) : U^n(\mathcal{A}) \longrightarrow \text{In}(\mathcal{A}) \times U^{n_\text{c}}(\mathcal{A})
\]

\[
(w_j) \\ \longmapsto (\varphi w_j, w_{cj}) = ((w_j(\det w_j)^{-1/n_j}, (\det w_j)^{1/n_j})
\]

we can write the two lifts \( L \) and \( \ell \) together in closed form \( \mathbb{L} : U^n(\mathcal{A}) \rightarrow \text{Aut}_H(\mathcal{A}) \):

\[
\mathbb{L}(w) = L(h_\varphi(w)) \ell(h_c(w))
\]

\[
= \rho \left( \prod_{j=1}^{N} (\det w_{j_1})^{q_{j_1}}, ..., \prod_{j_M=1}^{N} (\det w_{j_M})^{\hat{q}_{Mj_M}},
\right.
\]

\[
\left. \begin{array}{c}
w_1 \prod_{j_{M+1}=1}^{N} (\det w_{j_{M+1}})^{\hat{q}_{M+1,j_{M+1}}}, ..., w_N \prod_{j_{N+M}=1}^{N} (\det w_{j_{N+M}})^{\hat{q}_{N+M,j_{N+M}}} \end{array} \right) 
\]

\[
\times J \rho(...) J^{-1}.
\] (38)

We have set

\[
\hat{q} := \begin{pmatrix} q - (0_{M \times N}) \\ 1_{N \times N} \end{pmatrix} \left( \begin{array}{c} n_1 \\ \vdots \\ n_N \end{array} \right)^{-1}.
\] (39)

Due to the phase ambiguities in the roots of the determinants, the extended lift \( \mathbb{L} \) is multi-valued in general. We will impose it to be double-valued because this is the valuedness of the lift in the commutative case, extended or not. In addition we will impose the extended lift to be even, \( \mathbb{L}(-u) = \mathbb{L}(u) \), which translates into conditions on the charges, conditions that depend on the details of the representation \( \rho \). This property is motivated from the observation that the conjugation \( i \) is even. Also, in the case where the associative algebra \( \mathcal{A} \) has no commutative part \( \mathbb{C} \), and consequently no harmless \( U(1) \), the homomorphism \( \hat{L} \) is even.

### 3.3 The \( M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \) Example

The algebra has two harmful \( U(1) \)s and the representation \( (133) \) yields the extended lift,

\[
\mathbb{L}(w_1, w_2) = \text{diag} \left( \begin{array}{c}
w_1 \otimes 1_N \otimes w_2 (\det w_1)^{\hat{q}_{21} + \hat{q}_{31}} (\det w_2)^{\hat{q}_{22} + \hat{q}_{32}} \\
w_1 \otimes 1_N (\det w_1)^{-\hat{q}_{11} + \hat{q}_{21}} (\det w_2)^{-\hat{q}_{12} + \hat{q}_{22}} \\
1_N \otimes w_2 (\det w_1)^{\hat{q}_{11} + \hat{q}_{31}} (\det w_2)^{\hat{q}_{12} + \hat{q}_{32}} \\
1_N \otimes w_2 (\det w_1)^{-\hat{q}_{11} + \hat{q}_{31}} (\det w_2)^{-\hat{q}_{12} + \hat{q}_{32}} \\
1_N (\det w_1)^{-2\hat{q}_{11}} (\det w_2)^{-2\hat{q}_{12}} \\
\text{complex conjugate} \end{array} \right),
\] (40)
with $w_1 \in U(2)$, $w_2 \in U(3)$. This lift is even in the following two cases: $\tilde{q}_{12}$ even and both $\tilde{q}_{32}$ and $\tilde{q}_{23}$ odd or $\tilde{q}_{12}$ is odd and both $\tilde{q}_{23}$ and $\tilde{q}_{32}$ are even. For the lift to be double-valued the entries of the first column of $\tilde{q}$ must be quarter integers. More precisely the most general double-valued and even lift has a charge matrix

$$
\tilde{q} = \begin{pmatrix}
  z_1 + k_1/4 & 2z_4 + 1 + z_5 \\
  z_2 + k_2/4 & z_5 \\
  z_3 + k_3/4 & 2z_6 + z_5
\end{pmatrix},
$$

(41)

with six integers $z_j$. The three $k_j$ are either from the set $\{0,2\}$ or from $\{1,3\}$, the three $k_j$ cannot be all 0 and they cannot simultaneously take the value 2. The corresponding matrix $q$ is,

$$
q = \begin{pmatrix}
  2z_1 + k_1/2 & 6z_4 + 3 + 3z_5 \\
  2z_2 + 1 + k_2/2 & 3z_5 \\
  2z_3 + k_3/2 & 6z_6 + 3z_5 + 1
\end{pmatrix}.
$$

(42)

The number of $U(1)$s in the image of the extended lift, the rank of $q$, is equal to one or two. Let us take the minimal choice, rank one. Then the $k_j$ are even. The cheapest solution is

$$
\tilde{q} = \begin{pmatrix}
  0 & 1 \\
  -1/2 & 0 \\
  0 & 0
\end{pmatrix}, \quad q = \begin{pmatrix}
  0 & 3 \\
  0 & 0 \\
  0 & 1
\end{pmatrix}
$$

(43)

The image of the extended lift is $L(U^n(A)) = [U(1) \times SU(2) \times SU(3)]/[Z_2 \times Z_3]$ and $L$ is precisely the fermionic group representation of the standard model.

In section 2 the Higgs was obtained by fluctuating the Dirac operator with all unitaries, equation (21). In the language of central extensions this corresponds to including harmless $U(1)$s and taking $\tilde{q}$ to be the $(M+N) \times (M+N)$ zero matrix. This maximally extended lift is single-valued, even and of rank three. If instead we take the double-valued, even lift $L$ of rank one defined by the charge matrix (43) and fluctuate the Dirac operator with $L(U^n(A))$ then we obtain only one doublet of Higgs scalars and a fermionic mass matrix, that coincides with the Dirac operator. In fact we reproduce Connes’ version of the standard model on the nose.

3.4 Anomalies

Since it coincides with the fermionic representation of the standard model, the extended lift of (43) is free of Yang-Mills and mixed gravitational-Yang-Mills anomalies [10]. To spell out these conditions for a general extended lift $\mathbb{L}(w)$, equation (38), we need its infinitesimal version, $\mathbb{L}(X)$ defined as

$$
\mathbb{L}(1 + X) = 1 + \mathbb{L}(X) + O(X^2), \quad X = (X_1, ..., X_N) \in \bigoplus_{k=1}^N u(n_k).
$$

(44)
The lift $L$ is free of Yang-Mills anomalies if $\text{tr} \left[ \mathbb{L}(X)^3 \chi \varepsilon \right]$ vanishes for all $X$ and it is free of mixed gravitational-Yang-Mills anomalies if $\text{tr} \left[ \mathbb{L}(X) \chi \varepsilon \right]$ vanishes for all $X$. We denote by $\varepsilon$ the projector on the particle space, $\varepsilon = \text{diagonal} (\rho_L(1), \rho_R(1), 0, 0)$. For our model, $L$ is anomaly free if and only if $q_{11} = 3q_{31}$, $q_{12} = 3q_{32}$, $q_{21} = q_{22} = 0$. In particular the lift is of rank one and its charge matrix can be chosen to yield a double-valued, even lift,

$$
\tilde{q} = \begin{pmatrix}
3z_3 & 6z_6 + 1 \\
-1/2 & 0 \\
z_3 & 2z_6
\end{pmatrix}.
$$

We remark that the ‘bizzare’ solution \cite{11} where only the right-handed quarks have non-vanishing hypercharges is not a central extension.

\section*{4 So what?}

Since nine years Daniel Kastler asks the question whether the unimodularity condition can be imposed before computing the Higgs representation. In terms of spin lifts and fluctuations of the metric, his question has a precise meaning and a natural answer. Different central extensions do change in general both the number of Yang-Mills bosons and the number of Higgs bosons, while different unimodularity conditions did not modify the number of Higgs bosons. However in the standard model based on the algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, the Higgs representation is the same for all central extensions.

Recently Connes, Moscovici and Kreimer discovered a subtle link between a noncommutative generalization of the index theorem and perturbative quantum field theory. This link is a Hopf algebra relevant to both theories \cite{12}. On the other hand, Connes proposes to consider a Hopf algebraic generalization of the spin group, in particular quantum $su(2)$ at third root of unity. Indeed this Hopf algebra has $M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ as quotient by its nilradical \cite{2, 13}.

It is a pleasure to acknowledge José Gracia-Bondía and Bruno Iochum’s friendly advice.

\section*{References}

[1] A. Connes, Noncommutative Geometry, Academic Press (1994)

[2] A. Connes, Noncommutative geometry and reality, J. Math. Phys. 36 (1995) 6194

[3] A. Connes, Gravity coupled with matter and the foundation of noncommutative geometry, hep-th/9603053, Comm. Math. Phys. 155 (1996) 109
[4] A. Chamseddine & A. Connes, *The spectral action principle*, hep-th/9606001, Comm. Math. Phys. 186 (1997) 731

[5] I. Pris & T. Schücker, *Non-commutative geometry beyond the standard model*, hep-th/9604113, J. Math. Phys. 38 (1997) 2255

[6] The Particle Data Group, *Particle Physics Booklet* and http://pdg.lbl.gov

[7] F. Girelli, *Left-right symmetric models in noncommutative geometry?*, hep-th/0011123, Lett. Math. Phys. in press

[8] T. Schücker, *Spin group and almost commutative geometry*, hep-th/0007047

[9] S. Lazzarini & T. Schücker, *A farewell to unimodularity*, hep-th/0104038, Phys. Lett B 510 (2001) 277

[10] E. Alvarez, J. M. Gracia-Bondía & C. P. Martín, *Anomaly cancellation and the gauge group of the Standard Model in Non-Commutative Geometry*, hep-th/9506115, Phys. Lett. B 364 (1995) 33

[11] J. A. Minahan, P. Pamond & R. C. Warner, *Comment on anomaly cancellation in the standard model*, Phys. Rev. D 41 (1990) 715

[12] A. Connes & H. Moscovici, *Hopf Algebra, cyclic cohomology and the transverse index theorem*, Comm. Math. Phys. 198 (1998) 199

D. Kreimer, *On the Hopf algebra structure of perturbative quantum field theories*, q-alg/9707029, Adv. Theor. Math. Phys. 2 (1998) 303

A. Connes & D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. 1. The Hopf algebra structure of graphs and the main theorem*, hep-th/9912002, Comm. Math. Phys. 210 (2000) 249

A. Connes & D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem. 2. the beta function, diffeomorphisms and the renormalization group*, hep-th/0003188, Comm. Math. Phys. 216 (2001) 215

[13] D. Kastler, *Notes on Sl_q(2) at third root of unity*, in the proceedings of the Workshop on Quantum Groups, Palermo, 1997, eds.: Marc Rosso et al, Nova Science, 1999

D. Kastler, *Noncommutative geometry and basic physics*, Lect. Notes Phys. 543 (2000) 131