On Maximal Extensions of Nilpotent Lie Algebras

V. V. Gorbatsevich

Received April 5, 2022; in final form, September 8, 2022; accepted September 12, 2022

ABSTRACT. Extensions of finite-dimensional nilpotent Lie algebras, in particular, solvable extensions, are considered. Some properties of maximal extensions are proved. A counterexample to L. Šnobl’s conjecture concerning the uniqueness of maximal solvable extensions is constructed.

Key Words: nilpotent Lie algebra, solvable Lie algebra, extension, splitting.

DOI: 10.1134/S0016266322040037

This paper studies finite-dimensional Lie algebras with given nilradical. We refer to a Lie algebra $L$ whose nilradical coincides (up to isomorphism) with a given nilpotent Lie algebra $N$ as an extension of the Lie algebra $N$. If $L$ is solvable, then we call it a solvable extension.

Such extensions may be useful in various fields of mathematics and physics. In particular, the description of solvable Lie groups and algebras as nilpotent extensions is useful in the study of symmetries of differential and other equations and of representations of solvable Lie groups. In physics solvable Lie algebras and groups are used in string theory and other theories of elementary particles, as well as in certain multidimensional theories in cosmology.

We will assume that the base field $k$ of the Lie algebras under consideration is of characteristic 0; sometimes we also assume it to be algebraically closed. In studying Lie algebras over the field $k$, we use the notion of a toral Lie subalgebra or, in other words, an Abelian Lie subalgebra consisting of semisimple (i.e., completely reducible in a connected representation) elements. It is well known that, for an algebraically closed field of characteristic 0, all maximal toral Lie subalgebras in an arbitrary Lie algebra are conjugate to each other. This is no longer the case for open fields; thus, the simple Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ has two nonconjugate (one-dimensional) toral Lie subalgebras, one compact (isomorphic to $\mathfrak{so}(2)$) and the other splitting (isomorphic to the Lie subalgebra of diagonal matrices with trace 0).

General information about Lie algebras used in this paper can be found, e.g., in [1] and properties of Lie algebras, e.g., in [2]. Given a Lie algebra $L$, by $\text{Der}(L)$ we denote the Lie algebra of its derivations and by $\text{IDer}(L)$, the Lie algebra of its inner derivations (induced by the action of the Lie algebra on itself by means of the adjoint representation $\text{ad}_L$). The Lie algebra $\text{IDer}(L)$ is an ideal in $\text{Der}(L)$ isomorphic to $L/Z(L)$, where $Z(L)$ is the center of the Lie algebra $L$. The quotient algebra $O\text{Der}(N) = \text{Der}(N)/\text{IDer}(N)$ parameterizes the set of outer derivations (although they do not generally form a Lie algebra).

Let $N$ be a nilpotent Lie algebra over the field $k$. Consider all finite-dimensional Lie algebras $L$ over $k$ with nilradical isomorphic to $N$. Clearly, such Lie algebras $L$ exist, and there are infinitely many of them. For example, such are all Lie algebras $N \oplus S$ being direct (or even semidirect) sums of the Lie algebra $N$ and an arbitrary semisimple Lie algebra $S$. Note that in this example the connected action of the Lie algebra $S$ on the nilradical $N$ is trivial. In general, the direct sum of an arbitrary Lie algebra and any semisimple Lie algebra has the same nilradical as the original Lie algebra.

Yet another way of constructing Lie algebras, even solvable ones, with given nilradical $N$ is as follows. Consider the Lie algebra $\text{Der}(N)$ of derivations of a Lie algebra $N$. It is known to be algebraic (the nilpotency of $N$ does not matter). Like any algebraic Lie algebra, the Lie algebra $\text{Der}(N)$ has a Chevalley decomposition $\text{Der}(N) = P + U$ into a semidirect sum of a maximal reductive Lie subalgebra $P$ and a nilradical $U$ (consisting of nilpotent elements). The reductive Lie subalgebra $P$ (for an algebraically closed field, it is unique up to conjugation in $\text{Der}(N)$) uniquely...
decomposes into a direct sum $S \oplus T$ of a semisimple Lie algebra $S$ and a central Lie subalgebra $T$ (which is toral, i.e., consists of semisimple elements). If $T_S$ is a Cartan Lie subalgebra in $S$ (or, which is the same thing in the case under consideration, a maximal Abelian Lie subalgebra consisting of semisimple elements), then $T_S \oplus T$ is a maximal toral subalgebra of $\text{Der}(N)$ (that is, a maximal Abelian Lie subalgebra of $\text{Der}(N)$ consisting of semisimple elements). Consider the Lie algebra $L = (T_S \oplus T) + N$, namely, the semidirect sum of the Lie subalgebra $T_S \oplus T$ and the ideal $N$ with respect to the natural action of the Lie subalgebra $T_S \oplus T \subset \text{Der}(N)$ on $N$ by derivations. Clearly, $L$ is solvable and its nilradical is isomorphic to the original nilpotent Lie algebra $N$. It is also easy to see that the Lie algebra $L$ thus constructed contains any splittable extension $R$ of the nilpotent Lie algebra $N$. In [3] all solvable splittable Lie algebras $R$ are described up to isomorphism. The description is based on the consideration of the actions of certain finite groups on the set of all toral Lie subalgebras in Malcev splittings.

Similarly, considering arbitrary reductive Lie subalgebras of $\text{Der}(N)$ instead of $T_S \oplus T$ and their semidirect sums with $N$, we obtain generally unsolvable Lie algebras with nilradical isomorphic to $N$.

Of interest (from the negative point of view) is the case of characteristically nilpotent Lie algebras (that is, algebras whose Lie algebras of derivations are nilpotent). It is easy to see that such Lie algebras have no “nontrivial” (i.e., not isomorphic to the original Lie algebras) solvable extensions.

In this paper we are interested in maximal Lie algebras $L$ with given nilradical $N$. As a special but very important case, we often consider solvable Lie algebras $R$ with given nilradical. Since there are very many such Lie algebras $L$ and $R$, we distinguish maximal ones among them. Maximal can be understood in two different senses, by inclusion and by dimension. A Lie algebra maximal by inclusion is not contained in or even isomorphic to its proper subalgebra in any other Lie algebra with the same property (in our case, having the same nilradical). A Lie algebra is maximal by dimension if its dimension is highest among those of all Lie algebras with the property under consideration.

If we consider arbitrary (not only solvable) finite-dimensional Lie algebras with given nilradical, then, clearly, there exist no algebras maximal in any of these senses, because the addition of an arbitrary semidirect summand to a Lie algebra $L$ does not change its nilradical. For this reason, we distinguish a special class of Lie algebras with given nilradical, which we will call exact by analogy with the terminology of [4]. Namely, we say that a Lie algebra $L$ with nilradical $N$ is exact if the kernel of the adjoint action of its semisimple part $S$ on $N$ is trivial (as is known, this is equivalent to the triviality of the kernel of the action of $S$ on the radical $R$ of the Lie algebra $L$). It is easy to see that a Lie algebra $L$ is exact if and only if it cannot be decomposed into a direct sum of a semisimple Lie algebra and some other Lie algebra (whose nilradical is obviously isomorphic to $N$). If we consider only solvable Lie algebras with given nilradical, then the exactness assumption is not needed. We will refer to extensions of a nilpotent Lie algebra which are exact Lie algebras as exact extensions.

Generally, the two notions of maximality mentioned above are different even for solvable extensions. The simplest (of lowest possible dimension) example was communicated to the author by the anonymous referee, to whom the author is sincerely grateful.

Consider the three-dimensional solvable Lie algebra $L$ over an arbitrary field $k$ of characteristic 0 being the semidirect sum $k^+ \sigma k^2$ with respect to the homomorphism $\phi: k \rightarrow \text{gl}_2(k)$ such that $\phi(1) = J_2(1)$ is the second-order Jordan block with eigenvalue 1.

Let us show that this Lie algebra $L$ is an inclusion-maximal extension of the Abelian Lie algebra $k^2$. Suppose that this is not true and let $L'$ be a solvable extension of $k^2$ strictly containing the Lie algebra $L$. Consider the image $A$ of the Lie algebra $L'$ under the action on the ideal $k^2$ determined by the adjoint representation $\text{ad}_{L'}$ in an appropriate basis of $L'$ (note that $k^2$ is the nilradical of both Lie algebras $L$ and $L'$). The image of $L$ is one-dimensional and spanned by the matrix $J_2(1)$. Clearly, $A$ is an Abelian Lie subalgebra in the Lie algebra $\text{gl}_2(k)$, and its dimension must be higher than 1. Moreover, it must contain the matrix $J_2(1)$ and therefore be contained in the centralizer of this matrix, which, as is easy to see, is a two-dimensional Abelian Lie subalgebra. From dimensional considerations, the Lie subalgebra $A$ coincides with this centralizer; in particular, it contains the
nilpotent component $J_2(0)$ of the matrix $J_2(1)$. Therefore, the dimension of the nilradical of $A$ is 3. But this contradicts the definition of a solvable extension, according to which the nilradical of an extension must coincide with the original nilpotent Lie algebra. Hence there exists no Lie algebra $L'$ with the properties specified above. Thus, our Lie algebra $L$ is an inclusion-maximal extension of the Lie algebra $k^2$. But it is not dimension-maximal, because the two-dimensional Abelian Lie algebra $k^2$ has a solvable extension of dimension 4: it corresponds to the choice in $\text{gl}_2(k)$ of a two-dimensional splittable torus (consisting of all diagonal matrices). In what follows, we will mainly be interested in dimension-maximal extensions.

In this paper we use the notion of a splitting of a Lie algebra, in particular, that introduced by Malcev in [3]. It is also described in [5] (in a slightly different form). It should be mentioned that Malcev’s fundamental paper [3] has been undeservedly forgotten. There are very few references to it in the literature. This paper is usually mentioned only in connection with the fact that it reduces the classification of complex solvable Lie algebras to that of complex nilpotent Lie algebras. This is not quite true, because, as shown in that paper, such a reduction still requires describing all orbits of certain linear actions of certain nilpotent Lie groups, which is a nontrivial problem. On the other hand, the paper contains many important results concerning solvable (and not only solvable) Lie algebras.

**Definition 1.** A Lie algebra $L$ is said to be **split** if it is decomposed as $L = S + T + U$, where $U$ is its nilradical, $S$ is its semisimple part, $T$ is an Abelian Lie subalgebra centralizing the Lie subalgebra $S$, and, for any $X \in T$, the linear operator $\text{ad}_L(X)$ is nontrivial and semisimple (i.e., $T$ is a toral Lie subalgebra).

An embedding of a Lie algebra into a split Lie algebra, as well as this split Lie algebra itself, is called a **splitting** of the given Lie algebra.

It follows from the definition of a split Lie algebra $L$ that $T \cap U = \{0\}$ (because the action of the elements of $T \cap U$ on $L$ is semisimple, nilpotent, and hence trivial) and that $T$ is isomorphic to a subalgebra in the Lie algebra $\text{Der}(U)$ of derivations of the Lie algebra $U$.

Note that all algebraic Lie algebras are splittable, i.e., admit splittings (this follows from the Chevalley decomposition; see, e.g., [2]). The very notion of a split Lie algebra is, in a sense, an extension of the notion of an algebraic Lie algebra.

**Definition 2.** Let $L$ be a Lie algebra. Its **Malcev splitting** is an embedding $\alpha: L \hookrightarrow M(L)$ in a split algebra $M(L) = S + T + U$ under which $M(L)$ is a semidirect sum of the subalgebra $T$ and the ideal $\alpha(L)$ and $\alpha(L) + U = M(L)$.

In his paper Malcev also imposed the minimality condition on the split Lie algebra containing the given Lie algebra. In what follows, we refer to such a splitting as a **Malcev splitting**. In the construction described above this condition is not imposed. The dimension of a Malcev splitting depends only on that of the given Lie algebra and of its nilradical (see below). Note that if the Lie algebra $L$ itself is split, then it coincides with its own splitting, while its Malcev splitting is of higher dimension.

In [5] it was proved that a Malcev splitting exists for an arbitrary finite-dimensional Lie algebra $L$ over a field of characteristic 0 (for complex Lie algebras and Malcev’s interpretation of a splitting, this was first proved by Malcev in [3]) and is unique (up to a naturally understood isomorphism of splittings).

Below we briefly describe the construction of a Malcev splitting for a Lie algebra over the field $k$ (it is given in more detail in [5] for the parallel notion of a splitting of a Lie group).

Let $L = S + R$ be the Levi decomposition of a Lie algebra $L$. Consider the adjoint representation $\text{ad}_L: L \rightarrow \text{gl}(L)$. We set $L^* = \text{ad}_L(L)$ and denote by $\langle L^* \rangle$ the algebraic closure of the Lie subalgebra $L^*$ in $\text{gl}(L)$ (that is, the smallest algebraic Lie subalgebra containing $L^*$). Since the Lie algebra $\langle L^* \rangle$ is algebraic, it admits the Chevalley decomposition $\langle L^* \rangle = S^* + T^* + U^*$, where $U^*$ is a nilpotent radical, the Lie subalgebra $S^*$ is semisimple (this is the Levy factor), and the Lie subalgebra $T^*$ is Abelian and consists of semisimple (i.e., completely reducible over $k$) elements. Moreover, $\langle L^* \rangle$ is contained in $\text{Der}(L)$, because the Lie subalgebra $\text{Der}(L)$ is obviously algebraic and $L^*$ is contained in it. Consider the semidirect sum $W^* = S^* + U^*$. We have $\langle L^* \rangle = T^* + W^*$, where $T^* \cap W^* = \{0\}$.

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Let \( f: T^* + W^* \to T^* \) be the natural epimorphism with kernel \( W^* \), and let \( T = f(L^*) \). Clearly, \( T \subset \text{Der}(L) \); therefore, we can form the semidirect sum \( T + L \) with respect to the natural action of \( T \) on \( L \). We refer to the resulting Lie algebra \( T + L \) (into which the Lie algebra \( L \) is naturally embedded) as the Malcev splitting of \( L \) and denote it by \( M(L) \). The dimension of the Lie algebra \( M(L) \) is \( \dim L + \dim T \), and \( \dim T = \dim R/N \) (where \( N \) and \( R \) are, respectively, the nilradical and the radical of the Lie algebra \( L \)).

In [5] it was proved that this construction yields a version of splitting. To be more precise, in that paper this was proved for arbitrary connected Lie groups; our consideration of Lie algebras corresponds to the case where these Lie groups are simply connected. Note that the Malcev splitting for a given Lie algebra is unique up to a naturally defined isomorphism of splittings [5].

There are also other constructions of splitting for Lie algebras, e.g., those based on a fixed exact linear finite-dimensional representation of a Lie algebra (which always exists by Ado’s theorem). It seems somewhat surprising that rather artificial nonfunctorial splitting constructions (including that going back to Malcev and then modified by L. Auslender; see [1]) give the same result (up to isomorphism). This is explained by a more complicated but functorial (in a certain sense) splitting construction going back to Mostow and Hochschild’s paper [6], which is based on representation theory and uses infinite-dimensional Lie algebras. In [6] the construction was presented at the level of Lie groups, but it can easily be adapted to the case of Lie algebras. It is based on the consideration of representing functions of Lie groups and Lie algebras, that is, functions whose orbits under the action of some representation are contained in finite-dimensional subspaces. This construction leads to the notion of a splitting in terms of infinite-dimensional Lie algebras and has not been widespread. However, it this construction which reveals the essence of the uniqueness of the Malcev and similar splittings.

In [7] it was proved that, for a solvable Lie algebra \( R \) with nilradical \( N \), the dimension of the quotient Lie algebra \( R/N \) (which equals \( \dim T \), as shown above) does not exceed that of the Abelian Lie algebra \( N/[N,N] \) (in [7] a slightly different terminology was used and only Lie algebras over \( \mathbb{R} \) or \( \mathbb{C} \) were considered). Below we prove a generalization of this statement.

Note that the Lie algebra \( L/N \) is always reductive. We denote its toral rank (the maximum dimension of its toral Lie subalgebra) by \( r_t(L/N) \).

**Theorem 1.** Let \( L \) be an exact Lie algebra over a field of characteristic 0, and let \( N \) be its nilradical. Then \( r_t(L/N) \leq \dim N/[N,N] \).

If \( L \) is solvable, then the exactness assumption is redundant. In this case, the rank of the toral (Abelian) Lie algebra \( L/N \) is equal to its dimension.

**Proof.** Consider the Malcev splitting for the Lie algebra \( L = S + R \). We have \( M(L) = T + L \), where \( T \) is a toral Lie subalgebra of \( \text{Der}(L) \).

In what follows, we use the well-known fact that, for a nilpotent Lie algebra \( N \), the minimum number of generators is \( \dim(N/[N,N]) \). This can be expressed as follows: if \( V \) is a subspace of \( N \) complementary to \( [N,N] \), then it generates the Lie algebra \( N \).

The adjoint action of the reductive subalgebra \( S \oplus T \subset M(L) \) on \( L \) preserves \( N \) (as a characteristic ideal in \( L \)) and the ideal \( [N,N] \). Thus, since the action of the subalgebra \( S \oplus T \) on \( N \) is reductive, it follows that it generates an action on some subspace \( V \subset N \) complementary to \( [N,N] \) in \( N \). If the action on \( V \) of some nonzero \( X \in S \oplus T \) is trivial, then so is its action on the whole Lie algebra \( N \) (because \( V \) generates \( N \)). On the other hand, the set of all elements of \( S \oplus T \subset M(L) \) whose action on \( V \) is trivial is an ideal in \( S \oplus T \subset M(L) \). But the ideals in \( S \oplus T \) obviously have the form \( S' \oplus T' \), where \( S' \) is an ideal in \( S \) and \( T' \) is a subalgebra in \( T \).

Suppose that, for the kernel of the above action, the ideal \( S' \) is nontrivial. Then, since \( V \) generates the whole Lie algebra \( N \), it follows that the Lie algebra \( L \) is not exact, which contradicts the assumption. Therefore, \( S' = \{0\} \) and the kernel of the action lies in \( T \). However, by the construction of the Malcev splitting, this can happen only if \( T' = \{0\} \). Thus, we have proved that the homomorphism of the Lie algebra \( T \oplus S \to \text{gl}(V) \) is exact, that is, its kernel is trivial. Obviously, this implies \( r_t(L/N) \leq \dim(N/[N,N]) \).
The assertion of the theorem concerning the case of solvable \( L \) immediately follows from the above considerations.

We proceed to maximal extensions of nilpotent Lie algebras. We begin with the case of solvable extensions.

In [7] (see also [8]) Šnobl conjectured that all dimension-maximal solvable extensions of a complex nilpotent Lie algebra \( N \) are semidirect sums of maximal toral Lie subalgebras \( T \subset \text{Der}(N) \) and the given Lie algebra \( N \); in particular, the dimension-maximal extension of every nilpotent Lie algebra is unique up to isomorphism. This conjecture was confirmed in a number of particular cases (a list of related papers can be found in [7]). For example, it is true for all nilpotent Lie algebras of dimension \( \leq 6 \). It is also true for certain classes of nilpotent Lie algebras of arbitrary dimension. However, as shown below, in the general case, this conjecture is false; we will give an example of a nilpotent Lie algebra of dimension 8 and two nonisomorphic dimension-maximal solvable extensions of this algebra. Thus, there may be several different maximal extensions. Moreover, if \( \dim \text{ODer}(N) > 1 \) (which is the case for many characteristically nilpotent Lie algebras), then, apparently, it is possible to construct continuum many distinct solvable extensions by analogy with the example given above. Thus, presumably, the number of nonisomorphic maximal solvable extensions of a nilpotent Lie algebra is either 0 (for characteristically nilpotent Lie algebras), 1, or infinite (moreover, in the latter case, there are continuum many of such extensions).

Note that the counterpart of Šnobl’s conjecture in [7] for the case of a non-algebraically closed field \( k \) is false for a very simple reason. The point is that, for such fields, maximal toral Lie subalgebras are very often not conjugate to each other. This makes it possible to easily construct counterexamples to Šnobl’s conjecture. For instance, \( N = \mathbb{R}^2 \) has two dimension-maximal solvable extensions (they correspond to two toral Lie subalgebras). These are the semidirect sum \((\text{so}(2) \oplus C) + \mathbb{R}^2\) (where \( C \) is the subalgebra of scalar matrices) and the Lie algebra \( \mathbb{R}^2 + \mathbb{R}^2 \) (being a semidirect sum as well) corresponding to the splittable two-dimensional Abelian Lie subalgebra in \( \text{gl}_2(\mathbb{R}) \) formed by all diagonal matrices.

Let \( L \) be an arbitrary finite-dimensional Lie algebra. Note that if \( L \) is nilpotent, then it has nontrivial outer derivations (these are, for example, some of the so-called central derivations; see, e.g., [1]). A solvable Lie algebra does not necessarily have outer derivations; for example, this is the case for the two-dimensional solvable Lie algebra \( r_2 \) (given by the commutation relation \([X, Y] = Y\) in a basis \( X, Y \)). Semisimple Lie algebras never have outer derivations.

If a Lie algebra \( L \) is nilpotent, then so is the Lie algebra \( \text{IDer}(L) \). Therefore, the intersection of the maximal toral Lie subalgebra \( T \subset \text{Der}(L) \) with \( \text{IDer}(L) \) is trivial. If a Lie algebra is semisimple and defined over an algebraically closed field \( k \), then any its maximal toral Lie subalgebra is a maximal Abelian subalgebra; it is called a Cartan subalgebra. But for a nonsimple Lie algebra, the structure of a Cartan subalgebra may be more complex: by definition it is nilpotent and coincides with its normalizer. In particular, a maximal toral Lie subalgebra in a nonsimple Lie algebra is not always a maximal Abelian Lie subalgebra, and its centralizer may be larger than this subalgebra itself. For example, a Cartan subalgebra of a Lie algebra of the form \( S \oplus N \), where \( S \) is a semisimple Lie algebra and \( N \) is a nilpotent Lie algebra, is the nilpotent Lie algebra \( T \oplus N \).

Now we proceed to the construction of a counterexample to Šnobl’s conjecture stated above. We assume the base field \( k \) to be an algebraically closed field \( k \) of characteristic 0.

Let \( N_7 \) be a seven-dimensional nilpotent Lie algebra for which the Lie algebra \( \text{Der}(N_7) \) is nilpotent as well (such Lie algebras are said to be characteristically nilpotent; as is known, all of them are of dimension 7 or higher). To be specific, consider the seven-dimensional Lie algebra constructed by Favre in [9]. It is given by the relations

\[
[X_1, X_i] = X_{i+1} \quad (2 \leq i \leq 6),
\]

\[
[X_3, X_2] = X_6, \quad [X_4, X_2] = [X_5, X_2] = X_7, \quad [X_4, X_3] = -X_7
\]

in a basis \( X_1, \ldots, X_7 \).

We mention that there exists a list of all characteristically nilpotent Lie algebras of dimension 7: it consists of seven isolated Lie algebras and one one-parameter family [10].
Consider the nilpotent Lie algebra \( N = k \oplus N_7 \), where \( k \) is a one-dimensional Abelian nilpotent Lie algebra. Clearly, the Lie algebra \( \text{Der}(k) \) is isomorphic to the toral one-dimensional Lie algebra \( \text{gl}_1(k) \).

We have to describe the Lie algebra \( \text{Der}(N) \). For this purpose, we use the description of the Lie algebras of derivations for direct sums of Lie algebras, which was given in [11]. In our case, \( N \) has two direct summands, \( k \) and \( N_7 \). Therefore, according to [11], \( \text{Der}(N) \) can be represented as a sum of four subspaces:

\[
\text{Der}(N) = (\text{Der}(k) \oplus \text{Der}(N_7)) + (\text{Der}(k, N_7) + \text{Der}(N_7, k)),
\]

where \( \text{Der}(k, N_7) \) is the set of linear mappings \( k \to Z(N_7) \) to the center \( Z(N_7) \) (which is non-trivial due to the nilpotency of the Lie algebra \( N_7 \)) and \( \text{Der}(N_7, k) \) is the set of linear mappings \( N_7/[N_7, N_7] \to k \). Here we concretized the general statement of [11] for the particular case under consideration.

Note that a maximal reductive Lie subalgebra in \( \text{Der}(N) \) is the direct sum of maximal reductive subalgebras in the algebras of derivations of the Lie algebras \( k \) and \( N_7 \) (obviously, they are algebraic and, in particular, splittable). This observation was made in [4] for the general case.

We are interested in the Lie subalgebra \( \text{Der}(k) \oplus \text{Der}(N_7) \) of \( \text{Der}(N) \). The Lie algebra \( \text{Der}(k) \) is a toral Lie algebra (we denote it by \( T \)). Let \( X \in T \) be a nonzero vector and fix an outer external derivation \( d \) in \( \text{Der}(N_7) \) (as mentioned above, nilpotent Lie algebras always have outer derivations). To be specific, we take the central derivation which sends \( X_1 \) to the generator \( X_7 \) of the (one-dimensional) center and the other basis elements to 0. Clearly, the subalgebra \( T \) commutes with \( d \). Consider two solvable extensions of the nilpotent Lie algebra \( N \), \( R_1 = r_2 \oplus N_7 \), which is obtained by deriving \( r \), and \( R_2 \), which is the extension corresponding to deriving \( X + d \). Obviously, these extensions are maximal. However, these two Lie algebras are not isomorphic to each other. Indeed, the Lie algebra \( R_1 \) is split and hence coincides with its Malcev splitting (and is of dimension 9), while, as is easy to see, the dimension of the Malcev splitting of \( R_2 \) equals 10. Thus, the Malcev splittings of the Lie algebras \( R_1 \) and \( R_2 \) are of different dimensions, and therefore these algebras are not isomorphic in view of the uniqueness of Malcev splitting. We conclude that the Lie algebra \( N = k \oplus N_7 \) has two dimension-maximal solvable nonisomorphic extensions.

There is yet another, purely computational, method for proving that the Lie algebras \( R_1 \) and \( R_2 \) are nonisomorphic, which was communicated to the author by B. Omirov. Calculating the Lie algebras of derivations of the Lie algebras \( R_1 \) and \( R_2 \), e.g., for a Favre nilpotent Lie algebra, we find that the dimensions of these algebras of derivations are different: they equal 13 and 12, respectively. The calculation can be performed with the aid of mathematical software including blocks of symbolic computations (the author used the GAP package).

The example constructed above shows that Šnobl’s conjecture is generally false. Quite similar examples can be constructed for certain nilpotent Lie algebras which decompose into direct sums of a nontrivial nilpotent Lie algebra (we used a one-dimensional one) and a characteristically nilpotent one (such algebras exist in the dimension 7 and all higher dimensions).

We conclude with a few words about arbitrary exact extensions of nilpotent Lie algebras. Recall that there are no maximal (either by inclusion or by dimension) extensions of nilpotent Lie algebras, unless they are assumed to be exact; namely, any nilpotent Lie algebra has extensions of arbitrarily high dimension.

Suppose that a Lie algebra \( L = S + R \) is an exact extension of a nilpotent Lie algebra \( N \). Consider the Malcev splitting \( M(L) = S + T + U \). We denote the semisimple part of the Lie algebra \( L \) by the same letter \( S \) as the semisimple part of the Lie algebra \( M(L) \), because, as shown in [3] and [5], these two semisimple parts are isomorphic to each other. Obviously, the dimension-maximal exact extensions of the Lie algebra \( N \) have the form \( S + T + N \) (where \( S + T \) is a maximal reductive Lie subalgebra in \( \text{Der}(N) \)). It is natural to call them standard extensions of the nilpotent Lie algebra \( N \). As to solvable extensions, in some cases, there exist not only these standard maximal extensions but also nonstandard exact maximal extensions.
Acknowledgments

The author is grateful to B. Omirov for useful discussions of the results and to the referee for pointing out mistakes in the first version of the paper.

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V. V. Gorbatsevich
Moscow, Russia
E-mail: vgorvich@yandex.ru

Translated by O. V. Sipacheva