Stable Self-Interacting Pais-Uhlenbeck Oscillator

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Abstract

It is shown that the interacting Pais-Uhlenbeck oscillator necessarily leads to a description with a Hamiltonian that contains positive and negative energies associated with two oscillators. Descriptions with a positive definite Hamiltonians, considered by some authors, can hold only for a free Pais-Uhlenbeck oscillator. We demonstrate that the solutions of a self-interacting Pais-Uhlenbeck oscillator are stable on islands in the parameter space, as already observed in the literature. If we slightly modify the system, by considering a sine interaction term, and/or by taking unequal masses of the two oscillators, then the system is stable on the continents that extend from zero to infinity in the parameter space. Therefore, the Pais-Uhlenbeck oscillator is quite acceptable physical system.

1 Introduction

The ordinary gravity, described by the Einstein-Hilbert action (containing the curvature scalar \( R \)), is not renormalizable. The higher derivative gravity, with \( R + R^2 \), is renormalizable. But higher derivative theories are considered as problematic, because according the the Ostrogradski formalism \[1\] they contain negative energies which, according to the wide spread belief, automatically imply instabilities at the classical and quantum level. It is often stated that at the quantum level such a theory implies negative probabilities due to ghosts, and is therefore not unitary. Whether a theory implies negative probabilities and positive energies, or vice versa, positive probabilities and negative energies, depends on choice of vacuum and corresponding creation and annihilations operators \[2\]–\[8\].

A model for a higher derivative theory is the Pais-Uhlenbeck oscillator \[9\]. It has been studied by many authors, because understanding the issues concerning its stability, could pave the way towards quantum gravity. Smilga \[10\]–\[12\] has found that there are islands of stability of the classical solutions of the interacting Pais-Uhlenbeck (PU) oscillator. An example of an unconditionally stable interacting system was also found \[12\]. This system, which is a non linear extension of the PU oscillator, is a close relative of a supersymmetric higher-derivative system \[13\]. Recently, Mostafazadeh \[14\] has found a Hamiltonian formulation of the PU oscillator that yields a stable and unitary quantum system. Other authors \[15\]–\[19\] have also arrived at the positive definite Hamiltonians for the PU oscillator. A procedure with a PT symmetric Hamiltonian without ghosts and negative energies in the spectrum has been considered in Refs. \[20\]–\[24\]. In this paper we show that the descriptions of
Ref. [14]–[19] hold for a free PU oscillator only, but not for a self-interacting one. In the latter case one has to describe the PU oscillator by the second order Lagrangian and employ the Ostrogradski formalism. Then, as it is well known, the PU oscillator can be written as a system of two coupled oscillators, one with positive and the other one with negative energy. Stability issues arise, because the energy can flow from one to the other oscillator in such a way that their kinetic energies escape into positive and negative infinity, respectively, while the total energy of the system remains constant and finite.

In this paper we first show explicitly that, in general, a self-interacting PU oscillator cannot be described by a positive definite Hamiltonian. Then we consider numerical solutions of the PU oscillator with the quartic self-interaction. Unless the coupling constant or the initial velocity is too high, the (classical) system is stable. So we indeed have islands of stability as observed by Smilga [10]–[12], and recently by [25]. Then we consider two modifications of the PU oscillator. (i) We replace the quartic interaction term with a term that contains the forth power of sine. We show numerically and analytically that such a modification gives infinite continents of stability. (ii) Instead of taking equal masses of the two oscillators, we consider the case in which the masses are different. Such a modified system is in fact just like the PU oscillators, only the coefficients in front of the terms are changed, and a non-linear term is added. Again we obtain stability for the vast range of parameters. Moreover, regardless of how high the initial velocity is, the solution is stable. Such behavior of classical solution implies that the quantum system is stable as well [25].

2 The Pais-Uhlenbeck oscillator as a system of two oscillators

The Pais-Uhlenbeck oscillator is given by the 4th order differential equation

\[
\left( \frac{d^2}{dt^2} + \omega_1^2 \right) \left( \frac{d^2}{dt^2} + \omega_2^2 \right) x = 0, \tag{1}
\]

which gives

\[
x^{(4)} + (\omega_1^2 + \omega_2^2) \ddot{x} + \omega_1^2 \omega_2^2 x = 0. \tag{2}
\]

As observed by Mostafazadeh [14], this can be written as the system of two oscillators\footnote{We use here a different notation for coefficients.}

\[
\ddot{x} + \mu_1 x - \rho_1 y = 0, \tag{3}
\]

\[
\ddot{y} + \mu_2 y - \rho_2 x = 0, \tag{4}
\]

where \( \mu_1, \mu_2, \rho_1, \rho_2 \) are real constants.
From Eq. (3) we have $y = (1/\rho_1)(\ddot{x} + \mu_1 x)$. Inserting this into Eq. (4), we obtain

$$x^{(4)} + (\mu_1 + \mu_2)\ddot{x} + (\mu_1 \mu_2 - \rho_1 \rho_2) x = 0.$$  \hspace{1cm} (5)

Comparison of the latter equation with (2) gives the relations

$$\mu_1 + \mu_2 = \omega_1^2 + \omega_2^2$$  \hspace{1cm} (6)

$$\mu_1 \mu_2 - \rho_1 \rho_2 = \omega_1^2 \omega_2^2.$$  \hspace{1cm} (7)

The solutions is

$$\omega_{1,2}^2 = \frac{1}{2}(\mu_1 + \mu_2) \pm \frac{1}{2} \sqrt{(\mu_1 + \mu_2)^2 - 4(\mu_1 \mu_2 - \rho_1 \rho_2)}.$$  \hspace{1cm} (8)

Let us now find possible Lagrangians corresponding to the equations of motion (3),(4).

**Case I.**

Assuming the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}(\mu_1 x^2 + \mu_2 y^2 - 2\rho_1 xy),$$  \hspace{1cm} (9)

we obtain the equations of motion (3),(4), if

$$\rho_2 = \rho_1.$$  \hspace{1cm} (10)

Then from Eq. (8) we have

$$\omega_{1,2}^2 = \frac{1}{2}(\mu_1 + \mu_2) \pm \frac{1}{2} \sqrt{(\mu_1 - \mu_2)^2 + 4\rho_1^2}.$$  \hspace{1cm} (11)

We see that for a large range of the coefficients $\mu_1, \mu_2, \rho_1$, the squared frequencies $\omega_1^2$ and $\omega_2^2$ are positive. Then $\omega_1, \omega_2$ are real, in which case we have oscillating motion.

The Hamiltonian is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(\mu_1 x^2 + \mu_2 y^2 - 2\rho_1 xy),$$  \hspace{1cm} (12)

where $p_x = \partial L/\partial \dot{x} = \dot{x}$, and $p_y = \partial L/\partial \dot{y} = \dot{y}$. By performing a rotation in the $(x, y)$-space,

$$\begin{align*}
  x' &= x \cos \alpha + y \sin \alpha \\
  y' &= -x \sin \alpha + y \cos \alpha
\end{align*}$$  \hspace{1cm} (13)

with the accompanying rotation of momenta,

$$\begin{align*}
  p_{x'} &= p_x \cos \alpha + p_y \sin \alpha \\
  p_{y'} &= -p_x \sin \alpha + p_y \cos \alpha,
\end{align*}$$  \hspace{1cm} (14)

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the Hamiltonian (12) can be diagonalized. By comparing the new Hamiltonian

\[ H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(ax'^2 + by'^2), \]  

(15)

with the old one (12), we obtain the system of three equations for the unknowns \( a, b, \alpha \):

\begin{align*}
    a \cos^2 \alpha + b \sin^2 \alpha &= \mu_1 \\
    a \sin^2 \alpha + b \cos^2 \alpha &= \mu_2 \\
    (a - b) \cos \alpha \sin \alpha &= \rho_1.
\end{align*}

(16)

The solution is

\begin{align*}
    a &= \frac{1}{2}(\mu_1 + \mu_2) + \frac{1}{2}\sqrt{(\mu_1 - \mu_2)^2 + 4\rho_1^2} = \omega_1^2. \\
    b &= \frac{1}{2}(\mu_1 + \mu_2) - \frac{1}{2}\sqrt{(\mu_1 - \mu_2)^2 + 4\rho_1^2} = \omega_2^2. \\
    \cos 2\alpha &= \frac{\mu_1 - \mu_2}{\sqrt{(\mu_1 - \mu_2)^2 + 4\rho_1^2}}.
\end{align*}

(17) (18) (19)

The \( a, b \) are just equal to the squared frequencies \( \omega_1^2, \omega_2^2 \) of the PU oscillator. This can be directly verified by inserting the expressions (16) into the equations of motion (5) and using (10).

In the new coordinates, the system is described by the Lagrangian

\[ L = \frac{1}{2}(\dot{x}'^2 + \dot{y}'^2) - \frac{1}{2}(\omega_1^2 x'^2 + \omega_2^2 y'^2), \]  

(20)

and the Hamiltonian

\[ H = \frac{1}{2}(\dot{x}'^2 + \dot{y}'^2) + \frac{1}{2}(\omega_1^2 x'^2 + \omega_2^2 y'^2). \]  

(21)

The energy of this system is positive. It is remarkable that when we diagonalize the \( L \) and \( H \) for a system of two oscillators (3) and (4), we obtain two different frequencies, \( \omega_1 \) and \( \omega_2 \), that correspond to those occurring in the PU oscillators.

**Case II.**

Alternatively, we may assume that the Lagrangian is

\[ L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - \frac{1}{2}(\mu_1 x^2 - \mu_2 y^2 - 2\rho_1 xy), \]  

(22)

This gives the equations of motion (3) and (4) if

\[ \rho_2 = -\rho_1. \]  

(23)
Inserting this into Eq. (8), we obtain
\[ \omega_{1,2}^2 = \frac{1}{2}(\mu_1 + \mu_2) \mp \frac{1}{2}\sqrt{(\mu_1 - \mu_2)^2 - 4\rho_1^2}. \] (24)

The frequencies \( \omega_1, \omega_2 \) are real if \((\mu_1 - \mu_2)^2 > 4\rho_1^2\) and \(\mu_1 + \mu_2 > \sqrt{(\mu_1 - \mu_2)^2 - 4\rho_1^2}\).

The Hamiltonian is
\[ H = \frac{1}{2}(p_x^2 - p_y^2) + \frac{1}{2}(\mu_1x^2 - \mu_2y^2 - 2\rho_1xy). \] (25)

By performing the hyperbolic rotation in the \((x, y)\)-space,
\[ x' = x \cosh \alpha + y \sinh \alpha \]
\[ y' = x \sinh \alpha + y \cosh \alpha \] (26)

with the accompanying rotation of momenta,
\[ p_{x'} = p_x \cosh \alpha + p_y \sin \alpha \]
\[ p_{y'} = p_x \sinh \alpha + p_y \cosh \alpha, \] (27)

the Lagrangian (22) and the Hamiltonian (25) become
\[ L = \frac{1}{2}(\dot{x}'^2 - \dot{y}'^2) - \frac{1}{2}(\omega_1^2x'^2 - \omega_2^2y'^2), \] (28)
\[ H = \frac{1}{2}(x'^2 - y'^2) + \frac{1}{2}(\omega_1^2x'^2 - \omega_2^2y'^2). \] (29)

Again, the diagonalized Lagrangian and Hamiltonian contain the frequencies \( \omega_1, \omega_2 \) of the PU oscillator.

Now we have the relations
\[ \omega_1^2 \cosh^2 \alpha - \omega_2^2 \sinh^2 \alpha = \mu_1 \] (30)
\[ -\omega_1^2 \sinh^2 \alpha + \omega_2^2 \cosh^2 \alpha = \mu_2 \] (31)
\[ (\omega_1^2 - \omega_2^2) \cosh \alpha \sinh \alpha = -\rho_1. \] (32)

The energy of the system is either positive or negative, depending on which degree of freedom is more excited.

Cases I and II show that the PU oscillator can be described as a system of two oscillators whose Hamiltonian is either (21) or (29). Case I means positive definite Hamiltonian, whereas Case II means indefinite Hamiltonian.

3 Self-interacting PU oscillator

3.1 Equations of motion and the Lagrangian

We have seen that the PU oscillator can be described as a system of two oscillators (3) and (4) that can be written in the explicit uncoupled form
\[ \ddot{x}' + \omega_1^2x' = 0 \] (33)
\[ \ddot{y} + \omega_2^2 y' = 0 \quad (34) \]

For real \( \omega_1, \omega_2 \), this is an oscillating system, regardless of whether for the corresponding Lagrangian we take (20) or (28). Both Lagrangians are equally good for describing the PU oscillator \[15, 18\].

If we include an interaction between the \( x' \) and \( y' \), then energy can be transferred between those two degrees of freedom. Then it does matter which Lagrangian we take.

(i) Let us first consider the following Lagrangian that is an extension of (20) (Case I):

\[ L = \frac{1}{2}(\dot{x}'^2 + \dot{y}'^2) - \frac{1}{2}(\omega_1^2 x'^2 + \omega_2^2 y'^2) - \frac{\lambda}{4}(x' + y')^4 \quad (35) \]

The corresponding equations of motions are

\[ \ddot{x}' + \omega_1^2 x' + \lambda (x' + y')^3 = 0 \quad (36) \]
\[ \ddot{y}' + \omega_2^2 y' + \lambda (x' + y')^3 = 0 \quad (37) \]

Introducing the new coordinates

\[ u =\frac{x' + y'}{\sqrt{2}}, \quad v =\frac{x' - y'}{\sqrt{2}}, \quad (38) \]

we have

\[ L = \frac{1}{2}(\dot{u}^2 + \dot{v}^2) - \frac{1}{4}[(\omega_1^2 + \omega_2^2)(u^2 + v^2) + 2(\omega_1^2 - \omega_2^2)uv] - \lambda u^4 \quad (39) \]

\[ \ddot{u} + \mu_1 u - \rho_1 v + 4\lambda u^3 = 0 \quad (40) \]
\[ \ddot{v} + \mu_2 v - \rho_1 u = 0, \quad (41) \]

where

\[ \mu_1 = \mu_2 = \frac{1}{2}(\omega_1^2 + \omega_2^2), \quad -\rho_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2). \quad (42) \]

Eliminating \( u \), we obtain the 4th order differential equation for \( v \):

\[ v^{(4)} + (\mu_1 + \mu_2)\ddot{v} + (\mu_1 \mu_2 - \rho_1^2)v + 4\lambda \rho_1 (\ddot{v} + \mu_2 v)^3 = 0, \quad (43) \]

which is just that of the PU oscillator with an extra non linear term.

Similarly, by eliminating \( v \), we obtain

\[ u^{(4)} + (\mu_1 + \mu_2)\ddot{u} + (\mu_1 \mu_2 - \rho_1^2)u + 4\mu_2 \lambda u^3 + 4\lambda \frac{d^2}{dt^2} (u^3) = 0, \quad (44) \]

which is also the PU oscillator with a non-linear term.

(ii) Let us now consider the Lagrangian that is an extension of (28) (Case II):

\[ L = \frac{1}{2}(\dot{x}'^2 - \dot{y}'^2) - \frac{1}{2}(\omega_1^2 x'^2 - \omega_2^2 y'^2) - \frac{\lambda}{4}(x' + y')^4. \quad (45) \]
The corresponding equations of motions are now
\[ \ddot{x}' + \omega_1^2 x' + \lambda (x' + y')^3 = 0 \quad (46) \]
\[ \ddot{y}' + \omega_2^2 y' - \lambda (x' + y')^3 = 0 \quad (47) \]
Notice the minus sign in the second equation.
In the new variables \( u, v \), defined in Eq. (38), we have
\[ L = \dot{u} \dot{v} - \frac{1}{4}[(\omega_1^2 - \omega_2^2)(u^2 + v^2) + 2(\omega_1^2 + \omega_2^2)uv] - \lambda u^4 \quad (48) \]
\[ \ddot{u} + \mu_1 u - \rho_1 v = 0 \quad (49) \]
\[ \ddot{v} + \mu_2 v - \rho_1 u + 4\lambda u^3 = 0, \quad (50) \]
where \( \mu_1, \mu_2 \) and \( \rho_1 \) are given in Eqs. (42). By eliminating \( v \), we obtain
\[ u^{(4)} + (\mu_1 + \mu_2) \ddot{u} + (\mu_1 \mu_2 - \rho_1^2) u + 4\rho_1 \lambda u^3 = 0. \quad (51) \]
Using (42) and introducing \( \Lambda = 2(\omega_1^2 - \omega_2^2)\lambda \), the latter equation reads
\[ u^{(4)} + (\omega_1^2 + \omega_2^2) \ddot{u} + \omega_1 \omega_2^2 u - \Lambda u^3 = 0. \quad (52) \]
Now we obtain the equation of motion for the PU oscillator with a self-interaction term. The second order Lagrangian that gives the fourth order equation of motion (52) is that of the PU oscillator with a quartic self-interaction term:
\[ L = \frac{1}{2} [\ddot{u}^2 - (\omega_1^2 + \omega_2^2) \dot{u}^2 + \omega_1 \omega_2^2 u^2] + \frac{1}{4} \Lambda u^4. \quad (53) \]
Notice that if \( \omega_1 = \omega_2 \), then \( \rho_1 = 0 \), and \( v \) in (49) cannot be expressed in terms of \( u \). Consequently, in the case \( \omega_1 = \omega_2 \), the system (49), (50) does not give the equation (51) for the PU oscillator.
The Ostrogradski second order formalism then leads us to the phase space Lagrangian [17, 22, 23] that is equivalent to (53),
\[ L = p_u \dot{u} + p_q \dot{q} - H, \quad (54) \]
where
\[ H = p_u q + \frac{1}{2} [p_q^2 + (\omega_1^2 + \omega_2^2) q^2 - \omega_1 \omega_2^2 u^2] + \frac{1}{4} \Lambda u^4. \quad (55) \]
The latter Hamiltonian can be transformed into
\[ H' = \frac{1}{2}(p_{x'}^2 - p_{y'}^2) + \frac{1}{2}(\omega_1^2 x'^2 - \omega_2^2 y'^2) + \frac{\lambda}{4}(x' + y')^4. \quad (56) \]
The phase space Lagrangian
\[ L' = p_{x'} \dot{x'} + p_{y'} \dot{y'} - H' \quad (57) \]
is equivalent to (45). This can be directly seen by using the equations of motion \( p_x' = \dot{x}' \) and \( p_y' = \dot{y}' \), and eliminating \( p_x', p_y' \) from (57).

Therefore, the correct procedure is to start from the 4th order self-interacting Lagrangian (53) and to employ the Ostrogradski formalism. The Hamiltonian so obtained can be positive or negative. The procedures discussed in Refs. [14], and also in Refs. [15]–[19], have limited validity, because they do not consider an interaction term. They are valid descriptions of the PU oscillator in the absence of an interaction, but not if one switches on an interaction.

3.2 Solutions

The Lagrangian of the form (45) is usually considered as unsuitable for physics, because it implies indefinite Hamiltonian, with positive and negative energy states. The interaction term that mixes the two types of states leads to instabilities. But as pointed out in Refs. [10]–[12],[25], there exist islands of stability. We show this explicitly by solving numerically the equations of motion (46), (47). In Fig. 1 there are examples of such calculations, done by MATHEMATICA. In all examples we take \( \omega_1^2 = 1 \) and \( \omega_2^2 = 1.5 \)

![Figure 1: Solutions of Eqs. (46) (47) for different values of the coupling constant \( \lambda \) and different initial conditions. Left and middle: the trajectories in the \((x',y')\) space. Right: The kinetic energy \( \dot{x}'^2/2 \) as function of time. The oscillations within the envelope are so fine that they fill the diagram.](image)

We see that the system is stable for sufficiently small coupling constant \( \lambda \) and the
initial velocity $\dot{x}'(0), \dot{y}'(0)$. If $\lambda$ is too high, the system is unstable (Fig. 2, up).

Similarly, the system is unstable at too high velocities (Fig. 2, down). Close to the critical value of $\lambda$, the system seems to be stable for long time, but then it escapes into infinity (Fig. 3). A similar behaviour occurs close to the critical value of the initial velocity.

Figure 2: Up: By increasing the $\lambda$, the system becomes unstable. The trajectory and the kinetic energy escape into infinity. Down: Similarly, by increasing the initial velocity, the system also becomes unstable.

Figure 3: At certain values of $\lambda$ and the initial conditions, the system behaves stably for a long time, before it finally escapes to infinity. The total energy $E_{tot}$ remains constant within the numerical error.
By just slightly decreasing the coupling constant from that of figure 3, $\lambda = 0.02299$, to $\lambda = 0.0229$, the system appears to be stable. We checked its stability up to $t = 2664$, but we do not plot the solutions here, in order to not crowd the paper with too many figures.

The interaction potential $\frac{\lambda}{4}(x' + y')^4$ runs into infinity. More realistically, it should not run into infinity, but there should be a cutoff. As a more realistic coupling term let us consider $\frac{\lambda}{4}\sin^4(x' + y')$, that leads to the Lagrangian \(\text{[53]}\) in which $u^4$ is replaced by $\sin^4 u$. The equations of motion are then

\[
\ddot{x}' + \omega_1^2 x' + \lambda \sin^3 (x' + y') \cos (x' + y') = 0, \quad (58)
\]

\[
\ddot{y}' + \omega_2^2 y' - \lambda \sin^3 (x' + y') \cos (x' + y') = 0 \quad (59)
\]

Such system is stable at all values of $\lambda > 0$ and initial velocity. We have checked this by performing many numerical runs. In (Fig. 4) we give two examples of numerical solutions. Later we will demonstrate also analytically why the solutions of the system \(\text{[58]}, \text{[59]}\) are stable.

Figure 4: Solutions to the equations of motion \(\text{[58]}, \text{[59]}\), in which the quartic interaction $\frac{\lambda}{4}(x' + y')^4$ is replaced by $\frac{\lambda}{4}\sin^4(x' + y')$. The system is now stable for all positive values of $\lambda$.

Another possible generalization is in replacing the Lagrangian \(\text{[45]}\) with

\[
L = \frac{1}{2}(m_1 \dot{x}'^2 - m_2 \dot{y}'^2) - \frac{1}{2}(\omega_1^2 x'^2 - \omega_2^2 y'^2) - \frac{\lambda}{4}(x' + y')^4 \quad (60)
\]
where \( m_1 \) and \( m_2 \) are now two different “masses”. In terms of the variables \( u, v \), we have
\[
L = \frac{1}{2} \left[ m(\dot{u}^2 + \dot{v}^2) + 2M \dot{u} \dot{v} + \rho_1 (u^2 + v^2) - 2\mu_1 uv \right] - \lambda u^4, \tag{61}
\]
where
\[
m = \frac{1}{2}(m_1 - m_2), \quad M = \frac{1}{2}(m_1 + m_2) \tag{62}
\]
The equations of motion are now
\[
m \ddot{u} + M \ddot{v} - \rho_1 u + \mu_1 v + 4\lambda u^3 = 0 \tag{63}
\]
\[
m \ddot{v} + M \ddot{u} - \rho_1 v + \mu_1 u = 0. \tag{64}
\]
The corresponding 4th order equation is
\[
u^{(4)}(M^2 - m^2) + 2uuM(\mu_1 M + \rho_1 m) + uM(\mu_1^2 - \rho_1^2) + 4M \rho_1 \lambda u^3 - 4\rho_1 \lambda u^3 \frac{d^2}{dt^2} (u^3) = 0 \tag{65}
\]
This is a deformed version of the equation \([51]\) for the interacting PU oscillator. By taking \( m = 0, M = 1, \) we obtain the ordinary PU oscillator of Eq. \([51]\).

Examples of numerical solutions to the equations of motion
\[
m_1 \ddot{x}' + \omega_1^2 x' + \lambda(x' + y')^3 = 0, \tag{66}
\]
\[
m_2 \ddot{y}' + \omega_2^2 y' - \lambda(x' + y')^3 = 0, \tag{67}
\]
derived from the Lagrangian \([60]\), are given in Fig. 5. Whilst in the case of equal masses, \( m_1 = m_2 = 1, \) the system is unstable at \( \lambda = 0.03 \) and higher, we see that for different masses, \( m_1 < m_2, \) the system is stable regardless of the values of \( \lambda > 0 \) and initial velocities. This has been confirmed in many numerical runs that we have done. Only a small sample is shown in Fig. 5. That for unequal masses the system becomes stable we previously observed in Ref. \([26]\), where we studied an analogous system of two oscillators, but with a different coupling term, namely, \( \frac{1}{4}(x^2 - y^2)^2, \) which is a special case of that considered in Ref. \([25]\). However, such a coupling term does not correspond to a quartic self-interaction of the PU oscillator, because the coupling term \( \lambda u^4 \) in \([48]\) is then replaced by \( \lambda u^2 v^2, \) which does not lead to Eq. \([52]\), but to a more complicated equation with non-linear terms.

To see why the system with different masses is stable, let us inspect the equations of motion \([66], [67]\). We already know that at small values of \( \lambda, \) the system is stable. At large values of \( \lambda, \) we can neglect the terms \( \omega_1^2 x' \) and \( \omega_2^2 y'. \) Equations of motion are then
\[
\ddot{x}' + \frac{1}{m_1} \lambda(x' + y')^3 = 0, \tag{68}
\]
\[
\ddot{y}' - \frac{1}{m_2} \lambda(x' + y')^3 = 0. \tag{69}
\]
\[ \lambda=5, \ m_1=0.7, \ m_2=1.3 \]
\[ \dot{x}'(0)=0.3, \ y'(0)=1 \]
\[ \dot{x}'(0)=4, \ y'(0)=-0.5 \]
\[ \omega_1=1, \ \omega_2=\sqrt{1.5} \]

\[ \lambda=500, \ m_1=0.7, \ m_2=1.3 \]
\[ \dot{x}'(0)=0.3, \ y'(0)=1 \]
\[ \dot{x}'(0)=4, \ y'(0)=0.5 \]
\[ \omega_1=1, \ \omega_2=\sqrt{1.5} \]

\[ \lambda=5, \ m_1=0.7, \ m_2=1.3 \]
\[ \dot{x}'(0)=0.3, \ y'(0)=1 \]
\[ \dot{x}'(0)=40, \ y'(0)=55 \]
\[ \omega_1=1, \ \omega_2=\sqrt{1.5} \]

\[ \lambda=5, \ m_1=0.7, \ m_2=1.3 \]
\[ \dot{x}'(0)=0.3, \ y'(0)=1 \]
\[ \dot{x}'(0)=4, \ y'(0)=-0.5 \]
\[ \omega_1=\omega_2=0 \]

Figure 5: Solutions of Eqs. (66)-(67) for different values of the coupling constant \( \lambda \) and different initial conditions. We show here the kinetic energy \( \dot{x}'^2/2 \) as function of time.

Taking the sum and the difference of the latter equations, we obtain

\[ \ddot{\xi} + \left( \frac{1}{m_1} - \frac{1}{m_2} \right) \lambda \xi^3 = 0, \tag{70} \]

\[ \ddot{\eta} + \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \lambda \xi^3 = 0, \tag{71} \]

where \( \xi = x' + y' \) and \( \eta = x' - y' \).

In the case of unequal masses, \( m_1 < m_2, \ \lambda > 0 \), Eq. (70) describes the quartic oscillator with the potential \( \frac{1}{4} \xi^4(1/m_1 - 1/m_2) \), which has stable, oscillatory solutions. Then, Eq. (71) also has stable, oscillatory, solutions. Stability is maintained in the presence of the terms \( \omega_1^2 x' \) and \( \omega_2^2 y' \).

In the case of equal masses, \( m_1 = m_2 \), Eq. (70) becomes \( \ddot{\xi} = 0 \), with the solution \( \xi = \xi_0 + c_1 t \). Then the general solution of (71) is a runaway function

\[ \eta = -\frac{2}{m_1} \frac{\lambda}{20} (\xi_0 + c_1 t)^5 + c_2 t. \tag{72} \]

In the presence of the terms \( \omega_1^2 x \) and \( \omega_2^2 y \), the above runaway behavior is modulated by oscillations.

Solutions to the system described by the Lagrangian (60) are stable, if \( m_1 < m_2, \ \lambda > 0 \). If \( m_1 = m_2 \), then the solutions are stable at sufficiently small \( \lambda \), whereas at higher values of \( \lambda \), they are unstable. (Fig. 2).
If instead of \( \frac{1}{4}(x' + y') \) we take the interaction term \( \frac{1}{4}\sin^4(x' + y') \), we have stability even in the case \( m_1 = m_2 \). The equations of motion are then

\[
\ddot{\xi} = 0, \quad \ddot{\eta} + \frac{2}{m_1}\lambda \sin^3 \xi \cos \xi = 0,
\]

the general solution being

\[
\xi = \xi_0 + c_1 t, \quad \dot{\xi} = -\frac{2}{m_1} \frac{1}{4c_1} \lambda \sin^4 (\xi_0 + c_1 t),
\]

\[
\eta = -\frac{2}{m_1} \frac{1}{128c_1^2} \lambda [12(\xi_0 + c_1 t) - 8 \sin (2(\xi_0 + c_1 t)) + \sin (4(\xi_0 + c_1 t))]
\]

This solution is stable in the sense that the velocity and the kinetic energy remain finite. The coordinates \( \xi, \eta \), or equivalently, \( x', y' \), proceed with time, on average linearly, into infinity. The velocity thus oscillates around a constant velocity\(^3\). If we include into the potential also the terms \( \frac{1}{2}\omega_1^2 x'^2 \) and \( \frac{1}{2}\omega_1^2 y'^2 \), then the coordinates do not escape into infinity, but they oscillate.

### 4 Discussion

It has been shown by some authors [14, 19] (see also [15]–[18]) that the Pais-Uhlenbeck oscillator can be described as a system of two degrees of freedom with positive definite Hamiltonian. We point out that this holds for the free PU oscillator only and that one cannot include a coupling term such that the system would be equivalent to the PU oscillator with a quartic or similar self-interaction term. The interacting Pais-Uhlenbeck oscillator must be described, as usually, by the second order Lagrangian. The Ostrogradski formalism then leads to the indefinite Hamiltonian, with positive and negative energies. An equivalent system is that of two oscillators described by the equations of motion (46),(47), derived from the Lagrangian (45). We have studied numerical solutions to the latter system for various coupling constants \( \lambda \) and initial velocities. Solutions are stable below a critical value of \( \lambda \) and initial velocity. We then considered two modifications of the Lagrangian that drastically increase the range of stability.

Firstly, we replace the quartic interaction term \( \frac{1}{4}(x' + y')^4 \), that runs into infinity, with the term \( \frac{1}{4}\sin^4(x' + y') \) that is finite for all \( x', y' \). Then, instead of islands of stability, we obtain the continent of stability that extends into infinity in the space of the parameter \( \lambda \) and initial conditions. Fig. 4 shows that now the system is stable even at \( \lambda = 5 \), whereas with the quartic interaction it was unstable already at \( \lambda = 0.03 \). We have done many numerical runs with higher values of \( \lambda \), even with \( \lambda = 500 \), and

\[^3\] In the quantized theory, to such modulated uniform motion there corresponds a modulated traveling wave, or uniformly moving wave packet.
the solutions were always stable. By inspecting the equations of motion, we also found analytically that such interacting system is indeed stable for any positive $\lambda$, and for any initial velocity.

Secondly, we replace the kinetic term $\frac{1}{2}(\dot{x}'^2 - \dot{y}'^2)$ with $\frac{1}{2}(m_1\dot{x}'^2 - m_2\dot{y}'^2)$, and consider the case in which the “masses” $m_1$ and $m_2$ are different. If $m_1 < m_2$, $\lambda > 0$, and $\omega_1^2 \leq \omega_2^2$, the system is stable for all finite positive values of $\lambda$ and for all finite positive or negative initial velocities $\dot{x}'(0), \dot{y}'(0)$. Analogously, the system is stable if $m_1 > m_2$, $\lambda < 0$, and $\omega_1^2 \geq \omega_2^2$.

Our findings invalidate the generally held belief that the Pais-Uhlenbeck oscillator in the presence of an interaction is unstable, and therefore problematic. There are vast regimes of stability that hold for all initial velocities. This has consequences for the quantum PU oscillator. Namely, stability of a classical system does not necessarily imply stability of the corresponding quantum system, because the latter system can tunnel through a potential barrier and then roll down the potential. But if a classical system remains stable, regardless of how high is the initial velocity, then also the quantum system is stable. We conclude that the Pais-Uhlenbeck oscillator with a suitable self-interaction is quite acceptable from the physical point of view. Since the PU oscillator is a toy model for higher derivative gravity, we expect that also the negative energy problems of the latter theory could be resolved along similar lines as investigated in this paper.

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