Abstract. In this paper we investigate some particular spin lattice (a higher dimensional generalization of a spin chain) related to Zamolodchikov model, in the limit when both sizes of the lattice tend to infinity. An infinite set of bilinear equations, describing a distribution of eigenvalues of infinite set of mutually commuting operators, is derived. The distribution for the maximal eigenvalues is obtained explicitly. The way to obtain the excitations is discussed.

Introduction

Integrability of Zamolodchikov-Baxter three dimensional spin model [2,1] is based on the existence of commutative set of transfer-matrices \( T(\theta_1, \theta_2, \theta_3) \),

\[
\left[ T(\theta_1, \theta_2, \theta_3), T(\theta_1, \theta_2', \theta_3') \right] = 0,
\]

where \( \theta_j \) are Zamolodchikov’s dihedral angles, and we understand \( T \) as an operator in the vertex formulation [3] of Zamolodchikov-Bazhanov-Baxter model [4] with two spin states. In contrast to two-dimensional integrable models, \( T \) is layer-to-layer transfer matrix [5], that means

- it is associated with a rectangular lattice with the size \( N \times M \), therefore matrix \( T \) has the dimension \( 2^NM \times 2^NM \),
- two parameters \( \theta_2 \) and \( \theta_3 \) are varied in eq. (1).

Matrix elements of \( R \)-matrix of the Zamolodchikov–Baxter model are not positively defined: it is the obstacle for the decent interpretation of it as the model of statistical mechanics. The quantum mechanical interpretation is preferable.

Relation (1) implies the existence of a set of commutative operators \( \{ t_{m,n}(\theta_1) \} \) such that

\[
[t_{m,n}(\theta_1), T(\theta_1, \theta_2, \theta_3)] = 0 \quad \forall \ \theta_2, \theta_3, m, n,
\]

i.e. the problem of diagonalization of \( T \) for any \( \theta_2, \theta_3 \) and the problem of simultaneous diagonalization of \( \{ t_{m,n} \} \) are equivalent. In what follows, we mean a determined definition of \( \{ t_{m,n} \} \) with \( 0 \leq m \leq M \) and \( 0 \leq n \leq N \) related to an auxiliary problem of the model.

It is well known, the Zamolodchikov model and its generalization – Bazhanov-Baxter model [4] – are related to the generalized chiral Potts model [6]. The set of \( \{ t_{m,n}(\theta_1) \} \) may be produced by the expansion of

1991 Mathematics Subject Classification. 37K15.

Key words and phrases. Integrable spin system, Zamolodchikov and Zamolodchikov-Bazhanov-Baxter models.

This work was supported by the Australian Research Council.
transfer matrices of the length $= M$ chain of Lax operators for cyclic representation of $\hat{U}_{q=-1}(\hat{s}l_N)$ (these $N$ and $M$ are exactly the size of the layer). Another scheme to produce the same set $\{t_{m,n}(\theta_1)\}$ was proposed in [7, 8]. This scheme is the invariant one from the point of view of $2 + 1$ dimensional integrability, in particular the $N \leftrightarrow M$ symmetry is evident in this scheme. All $t_{m,n}$ are simple polynomials in the algebra of observables, and besides they are evidently hermitian (i.e. the model is indeed a model of quantum mechanics). The invariant scheme we mention here as the system of interacting spins on a two dimensional lattice, or as the two dimensional spin lattice.

The eigenvalues of the set $\{t_{m,n}\}$ may be found as a solution of a system of second order equations. In the language of auxiliary transfer matrices for generalized chiral Potts, the system of second order equation is the complete set of fusion relations for fundamental transfer matrices. The reader may find the investigation and discussion of the fusion relations and Bethe Ansatz for Zamolodchikov model for $N = 3$ in [9, 10, 11]. In the direct 3D scheme the whole system of fusion relations is encoded into a single spectral equation [7, 11].

The physical problem is to find the spectrum of all $t_{m,n}(\theta_1)$ simultaneously. One way to solve this problem is the nested Bethe Ansatz for the fusion algebra, $M \to \infty$ and finite $N$. Suppose for a moment, somebody has succeeded in solving the nested Bethe Ansatz equation for arbitrary $N$ (i.e. in finding the limiting $M \to \infty$ densities of $N - 1$ Bethe Ansatz’s distributions of zeros) and then sends $N \to \infty$. From $2 + 1$ dimensional point of view such $1 + 1$ dimensional result would be related to the limit $N, M \to \infty$ with $\frac{N}{M} \to 0$, i.e. $N \leftrightarrow M$ symmetry would be lost – the model in this approach remains the $1 + 1$ dimensional model with infinite symmetry group. This approach would give a correct answer for a quantity independent on $N/M$.

Contrary to this, the spectral equation in the direct $2 + 1$ scheme is initially $N \leftrightarrow M$ invariant. In this paper the spectral equation is evaluated in the limit

$$\tag{3} N, M \to \infty, \quad \frac{N}{M} \to \zeta$$

where $\zeta$ in the non-singular aspect ratio for the layer. The main result of this paper is the exact distribution of the largest eigenvalues (the ground state) $t_{m,n} = f(m, n; \theta_1, \zeta)$ in the limit [9]. The other result is the limiting form of the spectral equation allowing one to describe (at least qualitatively) the gap-less excitations of the ground state.

This paper is organized as follows. In sections 1, 2 and 3 we formulate first the system of interacting spins, recall its finite $N \times M$ – volume spectral equation and make its leading term evaluation. Content of the first three sections is a repetition of [7, 8, 11, 12, 13]. Next, in the fourth section, we expose some preliminary numerical results for the spectrum of $t_{m,n}$ and discuss the main idea for the limiting [9] procedure. In the fifth section re-write the spectral equations in the thermodynamic limit $N, M \to \infty$. In the sixth section the qualitative analysis of the thermodynamical spectral equation is given, the distribution of the maximal eigenvalues of $t_{m,n}$ is obtained and the structure of excitations is discussed.
1. Formulation of the Spin Lattice System

All the ways to produce the set \( \{ t_{m,n} \} \), both via Lax operators for cyclic representation of \( U_{q=-1} (\hat{sl}_N) \) and via 3D linear problem [4], finally may be reformulated in the following combinatorial form.

Consider a square lattice with the size \( N \times M \) with periodical boundary conditions – exactly the layer of (1). Each vertex \( j \) of the lattice may be labelled by the pair of the indices \( j = (n, m) \), \( n \in \mathbb{Z}_N \), \( m \in \mathbb{Z}_M \). A local triplet of the Pauli matrices \( \sigma_x^{n,m}, \sigma_y^{n,m} \) and \( \sigma_z^{n,m} = i \sigma_x^{n,m} \sigma_y^{n,m} \) is assigned to each vertex.

Consider a set of non-self-intersecting paths on the periodic lattice with the following rules of bypassing a vertex and following factors \( \gamma_j \) associated with each variant of bypassing (note the multiplier \( \kappa \) in the third variant):

\[
\gamma_{n,m} = \sigma_x^{n,m} \quad \gamma_{n,m} = \sigma_y^{n,m} \quad \gamma_{n,m} = \kappa \sigma_z^{n,m}
\]

An example of such path for \( 4 \times 4 \) lattice is drawn below:

Any path \( \mathcal{P} \) has a homotopy class \( c(\mathcal{P}) = mA + nB \), where \( A \) is the cycle left to right and \( B \) is the cycle from bottom to top. In the other words, \( m \) is the horizontal winding number and \( n \) is the vertical winding number of the path \( \mathcal{P} \). The path at the example above has \( n = m = 1 \).

For fixed winding numbers \( n \) and \( m \) let

\[
(4) \quad J_{m,n}(\kappa) = \sum_{\mathcal{P} : c(\mathcal{P}) = mA + nB} \prod_{\mathcal{P}} \gamma_j
\]

be the sum of the products \( \prod_{\mathcal{P}} \gamma_j \) of \( \gamma \)-factors along a path \( \mathcal{P} \) for all possible paths with the given winding numbers. In particular, \( J_{0,0} \equiv 1 \). The winding numbers of \( J_{m,n} \) run \( 0 \leq m \leq M \) and \( 0 \leq n \leq N \). The reader should distinguish the periodical discrete coordinates \( (n,m) \in (\mathbb{Z}_N, \mathbb{Z}_M) \) of the algebra of observables and the winding numbers \( (m,n) \) labelling the operators \( J \). It is known [4, 5], operators \( J_{m,n} \) obey the following exchange relations:

\[
(5) \quad J_{m,n} J_{m',n'} = (-)^{nm'+n'm} J_{m',n'} J_{m,n}.
\]
It means, they can be quasi-diagonalized simultaneously:

\begin{equation}
J_{m,n}(\kappa) = i^{m+n} (\sigma^z)^m (\sigma^y)^n t_{m,n}(\kappa) , \quad [t_{m,n}(\kappa), t_{m',n'}(\kappa)] = 0 .
\end{equation}

Auxiliary matrices $\sigma^x$ and $\sigma^y$, $\sigma^x \sigma^y = -\sigma^y \sigma^x$, belong to the algebra of observables. They correspond to the homotopy classes $1 \cdot \mathcal{A} + 0 \cdot \mathcal{B}$ and $0 \cdot \mathcal{A} + 1 \cdot \mathcal{B}$. Without loss of generality one may fix

\begin{equation}
\sigma^x = \prod_n \sigma^n_{x,1} \quad \text{and} \quad \sigma^y = \prod_n \sigma^n_{y,1} .
\end{equation}

The key meaning of the auxiliary matrices is that they represent one extra degree of freedom of $\{J_{m,n}\}$ with respect to $\{t_{m,n}\}$. For any of $2^{NM-1}$ eigenstates of $\{t_{m,n}\}$ auxiliary $\sigma^x, \sigma^y$ are usual $2 \times 2$ Pauli matrices.

The set of $J_{m,n}$ (and $\{t_{m,n}\}$ as well) is the set of “integrals of motion” for Zamolodchikov model in its vertex formulation. Namely, the layer-to-layer transfer matrix of Zamolodchikov model $T(\theta_1, \theta_2, \theta_3)$ commutes with all $J_{m,n}$ for $\kappa = \tan \frac{\theta_1}{2}$ and arbitrary $\theta_2, \theta_3$. In this paper we prefer to call $t_{m,n}$ the moduli since in the classical limit they become the moduli of the classical spectral curve.

The advantage of the present quantum-mechanical formulation is that if $\kappa$ is real, all $J_{m,n}$ and $t_{m,n}$ are self-adjoint since the Pauli matrices are self-adjoint, therefore the model is evidently physical. An eigenstate of the model is defined by eigenvalues of all $t_{m,n}$ – one can label the eigenstates by the corresponding values of $\{t_{m,n}\}$. Our aim is to describe all eigenstates.

2. Finite size spectral equations

In this section we recall the functional equation for the set of $t_{m,n}$. For its rigorous derivation see [11].

Consider the following generating function:

\begin{equation}
J(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} (-)^{n+m+nm} x^m y^n J_{m,n} ,
\end{equation}

where $x$ and $y$ are generic complex numbers. In the basis of the auxiliary $\sigma$-matrices $J(x, y)$ is

\begin{equation}
J(x, y) = t_{0,0}(x, y) - \sigma^x t_{1,0}(x, y) - \sigma^y t_{0,1}(x, y) - \sigma^x t_{1,1}(x, y)
\end{equation}

where

\begin{equation}
t_{0,0}(x, y) = \sum_{m,n} x^{2m} y^{2n+2m} t_{2m,2n} , \quad t_{1,0}(x, y) = \sum_{m,n} (-)^{n} x^{2m+1} y^{2m+2n} t_{2m+1,2n} , \quad t_{0,1}(x, y) = \sum_{m,n} (-)^{n} x^{2m+1} y^{2n+1} t_{2m+1,2n+1} , \quad t_{1,1}(x, y) = \sum_{m,n} (-)^{n} x^{2m} y^{2n+1} t_{2m,2n+1} .
\end{equation}

The reader should not be confused by the notation $t_{\alpha,\beta}(x, y)$ with $\alpha, \beta = 0, 1$ and the set of $t_{m,n}$, $0 \leq m, n \leq M, N$. It is known, the complete Abelian algebra of $t_{m,n}$ is generated by the polynomial decomposition of

\begin{equation}
t_{0,0}(x, y)^2 - t_{1,0}(x, y)^2 - t_{0,1}(x, y)^2 - t_{1,1}(x, y)^2 = F(x^2, y^2) ,
\end{equation}

where

\begin{equation}
t_{0,0}(x, y) = x^{2m} y^{2n+2m} t_{2m,2n} , \quad t_{1,0}(x, y) = (-)^{n} x^{2m+1} y^{2m+2n} t_{2m+1,2n} , \quad t_{0,1}(x, y) = (-)^{n} x^{2m+1} t_{2m+1,2n+1} , \quad t_{1,1}(x, y) = (-)^{n} x^{2m} y^{2n+1} t_{2m,2n+1} .
\end{equation}
where

\[ F(\lambda^N, \mu^M) = \prod_{n=0}^{N-1} \prod_{m=0}^{M-1} (1 - \lambda e^{2\pi in/N} - \mu e^{2\pi im/M} - \kappa^2 \lambda \mu e^{2\pi i(n/N+m/M)}) , \]

is a polynomial of \( \lambda^N = x^2, \mu^M = y^2 \):

\[ F(x^2, y^2) = \sum_{P=0}^{M} \sum_{Q=0}^{N} x^{2P} y^{2Q} F_{P,Q} . \]

As it was mentioned in the introduction, equation (11) encodes the whole fusion algebra of auxiliary transfer matrices for \( U_{q=-1}(\hat{sl}_N) \), the reader may find its explanation for e.g. \( N = 3 \) in the appendix.

The right hand side of (11) may be re-written as

\[ \sum_{m,n} (-)^{m+n+mn+P} t_{m,n} t_{2P-m,2Q-n} . \]

Equation (11) is the principal solution of the model, in the same way as the Bethe-ansatz is the principal solution for the spin chains: the problem of diagonalization of \( N M \times N M \) matrices \( t_{m,n} \) is reduced to a system of \( N M \) algebraic equations.

\[ \sum_{m,n} (-)^{m+n+mn+P} t_{m,n} t_{2P-m,2Q-n} = F_{P,Q} . \]

3. The leading term and relation to Zamolodchikov model

Suppose, no one term is zero in the product (12). Then \( F \) in (11) is exponentially big, and one may definitely conclude [13],

\[ \text{Each of} \quad (t_{\alpha,\beta}(x, y))^2_{\alpha,\beta=0,1} \sim |F(x^2, y^2)| \sim e^{NM \Phi(\lambda, \mu, \kappa^2)} , \]

where \( x = \lambda^{N/2}, \mu = \mu^{M/2} \), and the integral

\[ g(\lambda, \mu; \kappa^2) = \lim_{N,M \to \infty} \frac{1}{NM} \log |F(X, Y)| = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi d\phi' \log |1 - \lambda e^{i\phi} - \mu e^{i\phi'} - \kappa^2 \lambda \mu e^{i(\phi+\phi')}| , \]

being parameterized by

\[ |\lambda| = \frac{\sin r_2}{\sin r_1}, \quad |\mu| = \frac{\sin r_3}{\sin r_1}, \quad \kappa^2 = \frac{\sin r_0 \sin r_1}{\sin r_2 \sin r_3} , \]

with \( r_j \) bounded by

\[ r_0 + r_1 + r_2 + r_3 = \pi \quad \text{and} \quad 0 \leq r_1 + r_2, r_1 + r_3, r_2 + r_3 < \pi , \]

has the value [13]

\[ g(\lambda, \mu; \kappa^2) = -\log 2 \sin r_1 + \sum_{j=0}^{3} \left( \frac{r_j}{\pi} \log 2 \sin r_j + \Phi(r_j) \right) , \]

where \( \Phi(r) \) is the polylogarithm [1]:

\[ \Phi(r) = \sum_{m=1}^{\infty} \frac{\sin(2mr)}{2\pi m^2} . \]

This value is closely related to Baxter’s result for the bulk free energy of the Zamolodchikov model [1][12][13].
Relation (16) gives the solution of (10) for the mean eigenvalue problem for \( t_{\alpha,\beta}(x, y) \): any eigenvalue of 
\[ \left( t_{\alpha,\beta}(x, y) \right)^2 \]
has this leading behavior. The answer (20) is unsatisfactory from the quantum mechanical point of view, it corresponds to the asymptotically infinite values of the spectral parameters \( x = \lambda N/2 \) and \( y = \mu M/2 \) and do not clarify the structure of eigenvalues of the whole set of \( t_{m,n} \).

4. Preliminary evaluation for finite \( N, M \)

We started the investigation of (15) with the numerical tests for relatively small \( N, M \) (up to \( N = M = 8 \)) and for simple choices of \( \kappa \) (\( \kappa = 0, 1 \)).

The principal observation for finite \( N, M \) is the following. Excluding \( t_{m,n} \) from (15) step-by-step, one comes to a final polynomial equation for a single \( t_{m,n} \): such polynomial equation is exactly the characteristic equation for the operator \( t_{m,n} \). Therefore, the system (15) and the problem of direct diagonalization of operators \( t_{m,n} \) are equivalent. In other words, any solution of equations (15) is indeed an eigenstate. For this reason we call (15) the complete Abelian algebra.

Note in addition the parity property: if a set \( \{ t_{m,n} \} \) solves the equation (15), then the set \( \{ \tilde{t}_{m,n} \} \),
\[ \tilde{t}_{2m+\alpha,2n+\beta} = \varepsilon_{\alpha,\beta} t_{2m+\alpha,2n+\beta} , \]
where \( \alpha, \beta = 0, 1 \) and \( \varepsilon_{\alpha,\beta} \) are four arbitrary signs, solves (15) as well. This ambiguity corresponds to the ambiguity of definition of the auxiliary \( \sigma^x, \sigma^y, \sigma^z \).

It is useful to visualize the domain of the indices of \( \{ t_{m,n} \} \) as the set \( \Pi \) of points \( (m, n) \) on the “momentum” plane:
\[ \Pi = \{(m, n)\} : \ 0 \leq n \leq N , \ 0 \leq m \leq M . \]
The domain of \( F_{P,Q} \) is the same, it is the Newton polygon for \( F(x^2, y^2) \). On the boundary of the rectangular \( \Pi \) the eigenvalues of \( t_{m,n} \) as well as the values of \( F_{P,Q} \) are simple. Just putting e.g. \( y = 0 \) in (9), one gets
\[ t_{0,0}(x, 0)^2 - t_{1,0}(x, 0)^2 = (1 - x^2)^M . \]
This equation defines all possible boundary eigenvalues \( t_{m,0} \). Subject of interest is the calculation of \( t_{m,n} \) in the middle of \( \Pi \).

Numerical calculations show that for all eigenstates the absolute values of \( t_{m,n} \) as well as the coefficients \( F_{m,n} \) grow significantly when \( (m, n) \) goes from the boundary of \( \Pi \) to its middle. One eigenstate (up to the parity equivalence (22)) is strictly separated from all others: absolute value of any its \( t_{m,n} \) is the maximal with respect to values of the same \( t_{m,n} \) for all other eigenstates. We will call it the ground state.

For a given eigenstate, especially for the ground state, the values of \( t_{m,n} \) are maximal in some \( (m, n) = (P_0, Q_0) \) in the middle of rectangular \( \Pi \). In the same point the coefficient \( F_{P_0,Q_0} \) has the maximal absolute value with respect to all other \( F_{P,Q} \). The observed feature of the ground state is that \( \frac{t_{P_0+\alpha, Q_0+n}}{t_{P_0,Q_0}} \) with \( |\alpha| \) and \( |n| \) being relatively small, depends essentially only on \( N/M \) and \( \kappa \) when \( N \) and \( M \) are big. The same asymptotical independence of \( N, M \) is valid for \( \frac{F_{P_0+\alpha, Q_0+n}}{F_{P_0,Q_0}} \) as well. Since \( F_{P,Q} \) takes the maximal value at
\((P_0, Q_0)\), expression (13) is the result of a competition between the domain of maximal values of \(F_{P,Q}\) and big (or small) values of \(\lambda^{MP} \mu^{NQ}\) accompanying \(F_{P,Q}\), sf. (13).

Another feature of \(\{t_{m,n}\}\) may be mentioned. We observed that the sets of signs of \(\{t_{m,n}\}\) for \((m, n)\) surrounding \((P_0, Q_0)\) are different (up to \(\pm \)) for different eigenstates.

These observations allow us to suggest an idea for evaluation of (15). Since both \(\{t_{m,n}\}\) and \(\{F_{P,Q}\}\) have a domain of dominance – the neighborhood of \((P_0, Q_0)\) in the middle of \(\Pi\), far from the boundary – one can move to \((P_0, Q_0)\) and concentrate on its neighborhood. The boundary of \(\Pi\) is far from \((P_0, Q_0)\), and in the limit \(N, M \to \infty\) the boundary goes to infinity, so that the domain of the dominance becomes the open \(\mathbb{Z}^2\).

It follows for finite \(N, M\), in the neighborhood of \((P_0 + m, Q_0 + n)\)

\[
|t_{P_0+m, Q_0+n}|^2 \sim |F_{P_0+m, Q_0+n}| \sim e^{N M (1 + \kappa^2)} \quad |m| \text{ and } |n| \text{ are small},
\]

so that singular at \(N, M \to \infty\) exponential is just the common factor for all eigenstates. Cancelling it, one does can evaluate the spectral equations (15) in the domain of dominance in the limit (8). This will be done in the next section.

5. Spectral equation in the thermodynamic limit

To rewrite equations (11) or (15) in the thermodynamic limit \(N, M \to \infty\) with \(\zeta = \frac{N}{M}\) being fixed, we need to introduce several notations.

Define parameters \(c\) and \(a\) via

\[
c = \cot \frac{a}{2} = \sqrt{\frac{1 + \kappa^2}{3 - \kappa^2}} \iff \kappa^2 = \frac{\sin \frac{3a}{2}}{\sin \frac{a}{2}}.
\]

At \(N, M \to \infty\) the middle point \((P_0, Q_0)\) is defined by

\[
M \cdot \left(1 - \frac{a}{\pi}\right) = P_0 - u_1, \quad N \cdot \left(1 - \frac{a}{\pi}\right) = Q_0 - u_2
\]

where \(P_0\) and \(Q_0\) are even integers while \(u_1\) and \(u_2\), \(-1 < u_1, u_2 \leq 1\), are fractional parts. If \(a\) is not a rational fraction of \(\pi\), both \(u_1\) and \(u_2\) are extra variables.

Define next the quadratic form parameterized in the terms of \(c\) and aspect ratio \(\zeta\):

\[
\Omega(p, q) = \frac{\pi}{2} \left(\zeta^2 - \frac{1 + c^2}{2c} p^2 + \frac{1 - c^2}{c} pq + \zeta^2 \frac{1}{2c} q^2\right).
\]

The \(N, M \to \infty\) limit of (15) is based on the following behavior of the coefficients \(F_{P,Q}\):

\[
F_{P_0 + p, Q_0 + q} = (-1)^{p+q+pq} e^{NM \Omega(0, \kappa^2)} \cdot e^{-\Omega(p+u_1, q+u_2)} \cdot f_0 \left(1 + \frac{f_1 + f_2 \Omega(p + u_1, q + u_2)}{V}\right) + \ldots,
\]

where

\[
\Omega(0, \kappa^2) \equiv \Omega(1, 1; \kappa^2) = \left(1 - \frac{3a}{2\pi}\right) \log \kappa^2 + 3 \Phi \left(\frac{a}{2}\right) - \Phi \left(\frac{3a}{2}\right).
\]

Coefficients \(f_0, f_1, f_2\) are some functions of \(\kappa^2, \zeta, u_1\) and \(u_2\) (a sketch derivation of (29) and the value of \(f_0\) will be given in the appendix).
Define $\tau_{m,n}$ as the fine structure of $t_{m,n}$,

\[ t_{P_0+Q_0+n} = \sqrt{f_0} e^{\frac{1}{2}NM_{P_0}(x^2)} \tau_{m,n} . \]

Here, according to the idea of the previous section, we have moved to the middle $(P_0, Q_0)$ and canceled common exponential factor. Substituting (29) and (31) into (15), cancelling the exponents and taking the limit $N, M \to \infty$, we come to the following equations for $\tau_{m,n}$,

\[ \sum_{m,n \in \mathbb{Z}} (-)^{m+n+mn} \tau_{P_0+Q_0+n} \tau_{P_0-Q_0-n} = e^{-\Omega(p+u_1, q+u_2)} . \]

The next substitution

\[ \tau_{m,n} = c_{m,n} e^{-\frac{1}{2}\Omega(m+u_1, n+u_2)} \]

transforms (32) into the free from $u_1, u_2$ form:

\[ \sum_{m,n \in \mathbb{Z}} (-)^{m+n+mn} e^{-\beta\Omega(n,m)} c_{P_0+Q_0+n} c_{P_0+Q_0-n} = 1 \quad \forall \ p, q \in \mathbb{Z} . \]

Equations (32) and (34) are two forms of (11) in the thermodynamical limit.

6. ANALYSIS OF (34)

For the analysis of (34), let us modify it slightly at the first:

\[ \sum_{m,n \in \mathbb{Z}} (-)^{m+n+mn} e^{-\beta\Omega(n,m)} c_{P_0+Q_0+n} c_{P_0+Q_0-n} = 1 \quad \forall \ p, q \in \mathbb{Z} , \]

where the cut-off parameter $\beta \geq 1$.

Consider for a moment $\zeta = 1$. In this case

\[ \Omega(m, n) = \frac{\pi}{4} (e^{-1}(m+n)^2 + e(m-n)^2) , \]

and we have two small parameters in (35).

\[ Q = e^{-\beta\pi/4c} \quad \text{and} \quad \tilde{Q} = e^{-\beta\pi\varepsilon/4} . \]

Equation (35) may be analyzed in the terms of the perturbative expansion with respect to $Q, \tilde{Q}$. The zero

order reads

\[ c_{P_0+Q_0+n}^2 + o(1) = 1 \Rightarrow c_{P_0+Q_0+n} = \varepsilon_{P_0+Q_0+n} (1 + o(1)) \]

where $\varepsilon_{P_0+Q_0+n} = (\pm)$ is the sign of $c_{P_0+Q_0+n}$. In the first non-trivial order,

\[ c_{P_0+Q_0+n} = \varepsilon_{P_0+Q_0+n} (1 + (\varepsilon_{P_0+1, Q_0+n} + \varepsilon_{P_0, Q_0-n} + \varepsilon_{P_0, Q_0+n+1})Q\tilde{Q} + \ldots) . \]

This procedure may be continued, the result is a series with respect to $Q$ and $\tilde{Q}$,

\[ c_{P_0+Q_0+n} = \varepsilon_{P_0+Q_0+n} \left( 1 + \sum_{m,n>0} \chi_{P_0+Q_0+n} \right) . \]
with coefficients \( \chi_{m,n}^{(m,n)} \) being sums of products of \( \varepsilon_{m,n} \) for \( (m,n) \) surrounding \((p,q)\): The first few nonzero \( \chi_{m,n}^{(m,n)} \) with \( m+n \leq 4 \) are

\[
\chi_{p,q}^{(1,1)} = \varepsilon_{p+1,q} \varepsilon_{p-1,q} + \varepsilon_{p,q-1} \varepsilon_{p,q+1},
\]

\[
\chi_{p,q}^{(2,2)} = \varepsilon_{p,q+1} \varepsilon_{p,q-1} \varepsilon_{p+1,q-1} \varepsilon_{p-1,q-1} + \varepsilon_{p+1,q} \varepsilon_{p-1,q-1} \varepsilon_{p,q-1} \varepsilon_{p-1,q-1} + \varepsilon_{p-1,q} \varepsilon_{p-1,q} \varepsilon_{p,q-1} \varepsilon_{p,q-1} - \varepsilon_{p-1,q} \varepsilon_{p,q} \varepsilon_{p,q-1} \varepsilon_{p,q-1} - 1 + \varepsilon_{p,q+1} \varepsilon_{p,q-1} \varepsilon_{p,q-1} \varepsilon_{p,q-1} - 1 + \varepsilon_{p,q+1} \varepsilon_{p,q-1} \varepsilon_{p,q-1} \varepsilon_{p,q-1} - 1 + \varepsilon_{p,q+1} \varepsilon_{p,q-1} \varepsilon_{p,q-1} \varepsilon_{p,q-1} - 1 + \varepsilon_{p,q+1} \varepsilon_{p,q-1} \varepsilon_{p,q-1} \varepsilon_{p,q-1} - 1 ,
\]

and

\[
\chi_{p,q}^{(4,0)} = \varepsilon_{p-1,q} \varepsilon_{p+1,q+1} , \quad \chi_{p,q}^{(0,4)} = \varepsilon_{p-1,q} \varepsilon_{p+1,q-1} .
\]

This procedure may be formulated for \( \Omega(m,n) \) with general \( \zeta \) as well.

**Conjecture 1.** If \( \beta > 1 \), all seria converge. Solution of \( \Omega \) is defined uniquely by the distribution of the signs \( \varepsilon \equiv \{ \varepsilon_{m,n} \} \).

Note, due to the parity structure of \( \Omega \), any distribution \( \{ \varepsilon_{m,n} \} \) is equivalent to \( \{ \varepsilon'_{m,n} = \varepsilon_{m,n}(\pm)^m(\pm)^n \} \), sf. \( \Omega \).

The homogeneous distribution \( \varepsilon_{m,n} = (+) \) is the distinguished one since in this case \( c_{m,n} = c_0 \), expression for \( c_0 \) follows from \( \Omega \) and matches the series form \( \Omega \).

\[
c_0 = \left( \sum_{m,n} (-)^{m+n} e^{-\beta \Omega(m,n)} \right)^{-1/2} .
\]

But if \( \beta \to 1 \), diverges as

\[
c_0 \approx \frac{1}{\sqrt{(\beta - 1)}} \chi
\]

for some \( \chi = \chi(c, \zeta) \). This divergence may be explained by the \( \frac{1}{NM} \) term in \( \Omega \): \( \beta = 1 + \frac{f_2}{NM} \) when \( N, M \to \infty \), so that asymptotically

\[
c_0 = \sqrt{\frac{NM}{f_2 \chi}} \sim \sqrt{NM} .
\]

The distribution \( \varepsilon_{m,n} = (+) \) and \( c_{m,n} = c_0 \sim \sqrt{NM} \) is the ground state according to the numerical tests.

If the signs \( \varepsilon_{m,n} \) vary for different \( m, n \) (even if only one sign is opposite to all the others), we have

**Conjecture 2.** The seria with inhomogeneous \( \varepsilon \) converge at \( \beta = 1 \).
We can explain $c_0 \sim \sqrt{NM}$ in a bit different way. Consider for instance the following distribution of the signs:

\begin{equation}
    \varepsilon_{p+m,q+n} = \begin{cases} 
        (+) & \text{if } \Omega(m,n) \leq \frac{\pi}{2}V, \\
        \text{randomly } (\pm) & \text{if } \Omega(m,n) > \frac{\pi}{2}V
    \end{cases}
\end{equation}

In this case a very rough estimation gives

\begin{equation}
    c_{p,q} \sim \sqrt{V}.
\end{equation}

Thus the finite-volume domain of positive signs on the infinite lattice is effectively equivalent to finite lattice, and the strip $\Omega(m,n) \sim \frac{\pi}{2}V$ plays the rôle of an effective boundary.

The homogeneous distribution $\varepsilon_{m,n} = (+)$ in a big volume $V$ gives evidently the maximal eigenvalues of the quantum mechanical model, any variation of the signs gives an excitation of the spectrum. A distribution of the signs $\varepsilon_{m_1,n_1} = \varepsilon_{m_2,n_2} = ... \varepsilon_{m_k,n_k} = (-)$ with $(m_1,n_1)...(m_k,n_k)$ inside $V$ and with all other $\varepsilon_{m,n} = (+)$ inside $V$, is a candidate for a $k$-particles state.

One particle state, $\varepsilon_{m_1,n_1} = (-)$ with all other $\varepsilon_{m,n} = (+)$ inside $V$, is described asymptotically by two continuous parameters $(\mu, \nu) = \left(\frac{m_1}{\sqrt{V}}, \frac{n_1}{\sqrt{V}}\right)$. We expect a “dispersion relation” in the form $\tau_{m,n} = \sqrt{V}$ a smooth function of $(\mu, \nu)$. The model evidently is gap-less.

The behavior (48) allows one to suggest a candidate for the Hamiltonian of the system:

\begin{equation}
    H = -\sum_{m,n} \tau_{m,n}^2 \equiv -\sum_{m,n} c_{m,n}^2 \exp^{-\Omega(m,n)},
\end{equation}

sf. (33). At the ground state $H \approx -h_0 V$, i.e. one can talk about the density energy $-h_0$ of the ground state, and the spectrum of $H$ describes bound states $-h_0 \leq H < 0$.

From the alternative point of view, one may consider the Hamiltonian

\begin{equation}
    H' = -H.
\end{equation}

For this Hamiltonian, the ground state corresponds to a random distribution of the signs – we can say nothing about it. Excitations are the islands of constant signs in the sea of random ones, and its maximal value is described by the finite energy density $+h_0$.

7. Discussion

The main results of this paper are the following. The ground state distribution of the moduli

\begin{equation}
    \tau_{p,q} = c_{p,q} \exp^{-\frac{1}{2}H(p,q)},
\end{equation}

related to the moduli $t_{m,n}$ by (61), is given by

\begin{equation}
    \tau_{p,q} = c_0 \exp^{-\frac{1}{2}H(p,q)}, \quad c_0 \sim \sqrt{V}.
\end{equation}
where $V$ is the volume of the system. Any excited state is uniquely defined by a distribution of the signs $\varepsilon_{p,q} = \text{sign of } c_{p,q}$, the set of $c_{p,q}$ is the solution of

$$
\sum_{m,n} (-)^{m+n+mn} e^{-\Omega(n,m)} c_{p-m,q-n}c_{p+m,q+n} = 1.
$$

The various problems of our approach are to be mentioned. At the first, we are still unable to evaluate explicitly for non-periodical distribution of signs. Even $f_2$ in the asymptotic or more exact estimation of are not known. Also it is not known yet how to express the momenta, corresponding to two orthogonal shifts of the periodical lattice, in the terms of $\{t_{m,n}\}$. Without the physical momenta, one hardly can interpret physically the dispersion relation conjectured above. The third problem is the calculation of the spectrum of $T(\theta_1, \theta_2, \theta_3)$. It is still unclear how the transfer matrix of Zamolodchikov-Baxter model is related to $\{t_{m,n}\}$ for generic $N, M$. Some discussion repeating is given in appendix.

**Acknowledgements** The author should like to thank V. Bazhanov, V. Mangazeev, M. Batchelor, X-W. Guan and all the Mathematical Physics group of the Department of Theoretical Physics of RSPhysSE for fruitful discussions.

**References**

[1] R. J. Baxter 1983 On Zamolodchikov’s solution of the tetrahedron equation *Commun. Math. Phys.* **88** 185-205

R. J. Baxter 1984 Partition function of the three-dimensional Zamolodchikov model *Phys. Rev. Lett.* **53** 1795

R. J. Baxter 1986 The Yang-Baxter equations and the Zamolodchikov model *Physica* **18D** 321-347

[2] A. B. Zamolodchikov, “Tetrahedron equations and integrable systems in three dimensions”, *JETP* **79** (1980) 641-664 (in russian)

A. B. Zamolodchikov, “Tetrahedron equations and the relativistic S matrix of straight strings in 2+1 dimensions”, *Commun. Math. Phys.* **79** (1981) 489-505

[3] S. M. Sergeev, V. V. Mangazeev and Yu. G. Stroganov, “Vertex reformulation of the Bazhanov – Baxter model”, *J. Stat. Phys.* **82** (1996) 31-50

[4] V. V. Bazhanov and R. J. Baxter, “New solvable lattice models in three dimensions”, *J. Stat. Phys.* **69** (1992) 453-485

[5] V. Bazhanov and Yu. Stroganov, “Conditions of commutativity of transfer-matrices on a multidimensional lattice”, *Theor. Math. Phys.* **52** (1982) 685-691

[6] V. V. Bazhanov, R. M. Kashaev, V. V. Mangazeev and Yu. G. Stroganov, “$Z_N \otimes n - 1$ generalization of the chiral Potts model”, *Comm. Math. Phys.* **138** (1991) 393–408

[7] S. Sergeev, “Quantum 2 + 1 evolution model”, *J. Phys. A: Math. Gen.* **32** (1999) 5693–5714

[8] S. M. Sergeev, “Auxiliary transfer matrices for three-dimensional integrable models”, *Theoretical and Mathematical Physics* **124** (2000) 391–409

[9] V. V. Bazhanov and R. M. Kashaev, “Cyclic $L$-operator related with a 3-state $R$-matrix”, *Comm. Math. Phys.* **136** (1991) 607–623

[10] H. E. Boos and V. V. Mangazeev, “Functional relations and nested Bethe ansatz for sl(3) chiral Potts model at $q^2 = -1$”, *J. Phys. A: Math. Gen.* **32** (1999) 3041-3054

H. E. Boos and V. V. Mangazeev, “Bethe ansatz for the three-layer Zamolodchikov model”, *J. Phys. A: Math. Gen.* **32** (1999) 5285-5298

H. E. Boos and V. V. Mangazeev, “Some exact results for the three-layer Zamolodchikov model”, *Nucl. Phys. B* **592** (2001) 597-626
One may show combinatorially, t cyclic representation in the quantum space and the fundamental representation $\pi$ is supposed, (54) is the co-vector representation. (56) where (55) $F$ Choose the particular value $N = 3$. The series (5) may be rewritten as the four-term sum

\[ J(x, y) = \sum_{n=0}^{N} \sum_{m=0}^{M} (-i)^{mn} (-x \sigma^x)^m (-y \sigma^y)^n t_{m,n} = \]

\[ \left( \sum_{m=0}^{M} (-x \sigma^x)^m t_{m,0} \right) - \left( \sum_{m=0}^{M} (ix \sigma^x)^m t_{m,1} \right) y \sigma^y + \left( \sum_{m=0}^{M} (x \sigma^x)^m t_{m,2} \right) y^2 - \left( \sum_{m=0}^{M} (-ix \sigma^x)^m t_{m,3} \right) y^3 \sigma^y \]

\[ \equiv t_0(x \sigma^x) - t_1(x \sigma^x) y \sigma^y + t_2(x \sigma^x) y^2 - t_3(x \sigma^x) y^3 \sigma^y. \]

One may show combinatorially, $t_k(x)$ is the $\mathcal{U}_{q=-1}(\hat{sl}_N)$ transfer matrix for the Lax operators with the cyclic representation in the quantum space and the fundamental representation $\pi_k$ in the auxiliary space. It is supposed, $\pi_0$ and $\pi_N$ are the scalar representations, $\pi_1$ is the vector representation etc. In $N = 3$ case $\pi_2$ is the co-vector representation.

Decomposition of $F(x^2, y^2)$ with respect to $y^2$ is following ($\omega = e^{2\pi i/3}$ and $\lambda^3 = x^2$):

\[ F(x^2, y^2) = \prod_{n=0}^{2} \left( (1 - \lambda \omega^n)^M - y^2 (1 + \kappa^2 \lambda \omega^n)^M \right) = A(x^2) - B(x^2) y^2 + C(x^2) y^4 - D(x^2) y^6, \]

where

\[ A(x^2) = (1 - x^2)^M, \quad D(x^2) = (1 + \kappa^6 x^2)^M, \]

\[ B(x^2) = (1 - x^2)^M \left( \frac{1 + \kappa^2 \lambda}{1 - \lambda} \right)^M + \left( \frac{1 + \kappa^2 \lambda \omega}{1 - \lambda \omega} \right)^M + \left( \frac{1 + \kappa^2 \lambda \omega^2}{1 - \lambda \omega^2} \right)^M, \]

\[ C(x^2) = (1 + \kappa^6 x^2)^M \left( \frac{1 - \lambda}{1 + \kappa^2 \lambda} \right)^M + \left( \frac{1 - \lambda \omega}{1 + \kappa^2 \lambda \omega} \right)^M + \left( \frac{1 - \lambda \omega^2}{1 + \kappa^2 \lambda \omega^2} \right)^M. \]

Equating now (51) in all orders of $y^2$, one comes at $y^0$ and $y^6$ to

\[ t_0(x)t_0(-x) = (1 - x^2)^M, \quad t_3(x)t_3(-x) = (1 + \kappa^6 x^2)^M. \]
For the generalized chiral Potts model the choice is prescribed:

\[(58) \quad t_0(x) = (1 - x)^M, \quad t_3(x) = (1 + i\kappa^3 x).\]

The orders \(y^2\) and \(y^4\) give

\[(59) \quad t_1(x) t_1(-x) = t_0(-x) t_2(x) + t_0(x) t_2(-x) + B(x^2),\]
\[(59) \quad t_2(x) t_2(-x) = t_3(-x) t_1(x) + t_3(x) t_1(-x) + C(x^2).\]

Relations (59) with (58) are exactly the fusion algebra for \(sl_3\), [9, 10].

**Appendix B. Asymptotic of \(F_{P,Q}\)**

Let us discuss briefly the derivation of (29). Taking into account (20), one may use the saddle point method for the estimation of \(F_{P,Q}\). Basically,

\[(60) \quad F_{P,Q} = \frac{1}{(2\pi i)^2} \oint \oint dX dY \frac{F(X,Y)}{X^P Y^Q}.\]

Let

\[(61) \quad \alpha_p = \frac{P\pi}{M}, \quad \alpha_q = \frac{Q\pi}{N}.\]

Then

\[(62) \quad \log \left( \frac{F(X,Y)}{X^P Y^Q} \right) \sim NM \left( g(\lambda, \mu; \kappa^2) - \frac{\alpha_p}{\pi} \log \lambda - \frac{\alpha_q}{\pi} \log \mu \right)\]

It has the extremum (minimum) with respect to \(\lambda, \mu\) (\(\kappa^2\) being fixed) at

\[(63) \quad r_0 + r_2 = \alpha_p, \quad r_0 + r_3 = \alpha_q.\]

The extremum value of \(g(\lambda, \mu; \kappa^2) - \frac{\alpha_p}{\pi} \log \lambda - \frac{\alpha_q}{\pi} \log \mu\) is

\[(64) \quad g(\alpha_p, \alpha_q; \kappa^2) = \frac{r_0}{\pi} \log \kappa^2 + \sum_{j=0}^{3} \Phi(r_j)\]

where the numbers \(r_j\) are to be calculated via

\[(65) \quad r_0 = \pi - \frac{a_1 + a_2 + a_3}{2}, \quad r_1 = \frac{a_2 + a_3 - a_1}{2}, \quad r_2 = \frac{a_3 + a_1 - a_2}{2}, \quad r_3 = \frac{a_1 + a_2 - a_3}{2},\]

and

\[(66) \quad a_2 = \pi - \alpha_p, \quad a_3 = \pi - \alpha_q, \quad a_1 = \arccos \left( \cos a_2 \cos a_3 + \frac{\kappa^2 - 1}{\kappa^2 + 1} \sin a_2 \sin a_3 \right).\]

The last equality is the solution of \(\kappa^2 = \frac{\sin r_0 \sin r_1}{\sin r_2 \sin r_3}\) with respect to \(a_1\). Therefore asymptotically

\[(67) \quad F_{P,Q} = (-)^{P+Q+PQ} f_0 \cdot \left( 1 + \frac{F'}{NM} + \ldots \right) \cdot e^{NMg(\alpha_p, \alpha_q; \kappa^2)}.\]

---

\(^1\) In details, \(\frac{\partial g}{\partial \lambda} = \frac{r_0 + r_2}{\pi}, \quad \frac{\partial g}{\partial \mu} = \frac{r_0 + r_3}{\pi}, \quad \kappa^2 \frac{\partial g}{\partial \kappa^2} = \frac{r_0}{\pi}.\)
Function $g(\alpha_p, \alpha_q; \kappa^2)$ has the maximum near $\alpha_p = \alpha_q = \pi - a$, where $a$ is defined by (26), and

$$g(\alpha_p, \alpha_q; \kappa^2) = g_0(\kappa^2) - \frac{1 + c^2}{4\pi c} (\delta\alpha_p^2 + \delta\alpha_q^2) - \frac{1 - c^2}{2\pi c} \delta\alpha_p \delta\alpha_q,$$

where $g_0(\kappa^2)$ is given by (34). Let further even integers $P_0, Q_0$ and real numbers $u_1, u_2$ are defined by (27). Then

$$\delta\alpha_p = \frac{\pi}{M} (p + u_1), \quad \delta\alpha_q = \frac{\pi}{N} (q + u_2).$$

Therefore, the leading term of (67) is

$$F_{P_0 + p, Q_0 + q} = (-)^{p+q+pq} f_0 \cdot e^{NM g_0(\kappa^2) - \Omega(p + u_1, q + u_2)},$$

where the quadratic form is given by (28).

The next order in (67), $F' = f_1 + f_2 \Omega(p + u_1, q + u_2)$, is the result of numerical tests.

**APPENDIX C. THETA-FUNCTIONS**

In the limit $M, N \to \infty$ the polynomial $F(X, Y)$ as well as the eigenstates of $t_{x, y}(x, y)$ for periodical distribution of the signs $\varepsilon_{m, n}$ become the theta-functions. In particular, equations (32) may be re-written in a theta-functions-like form:

$$\left(\sum x^{2m} y^{2n} \tau_{2m, 2n}\right)^2 - \left(\sum (-)^{m} x^{2m+1} y^{2n+1} \tau_{2m+1, 2n+1}\right)^2 - \left(\sum (-)^{m} x^{2m} y^{2n+1} \tau_{2m, 2n+1}\right)^2 - \left(\sum (-)^{m} x^{2m+1} y^{2n} \tau_{2m+1, 2n}\right)^2$$

$$= \sum_{p,q} (-)^{p+q+pq} e^{-\Omega(p + u_1, q + u_2)} x^{2p} y^{2q}.$$

Let us re-define $(x = e^{i\pi z_1}, y = e^{i\pi z_2})$. Then the theta-function-like seris

$$\tau_{\alpha, \beta}(z_1, z_2) = \sum_{m,n \in \mathbb{Z}} (-)^{\alpha m + \beta n} \tau_{2m + \alpha, 2n + \beta} e^{i(2m + \alpha)z_1 + i(2n + \beta)z_2},$$

stand for the transfer matrices.

It is helpful to discuss some properties of theta-functions. Let

$$\Theta_{u_1, u_2}^{(\beta)}(z_1, z_2) = \sum_{p,q} e^{-\beta \Omega(p + u_1, q + u_2) + 2\pi i p z_1 + 2\pi i q z_2}$$

for our particular quadratic form $\Omega$ (28). It has the general Jacobi transform property:

$$\Theta_{u_1, u_2}^{(\beta)}(z_1, z_2) = \frac{2}{\beta} e^{-2\pi i (z_1 u_1 + z_2 u_2)} \Theta_{z_2 - z_1}^{(4/\beta)}(-u_2, u_1).$$

The other $\theta$-function, related to $F$, is

$$F_{u_1, u_2}(z_1, z_2) = \sum_{p,q} (-)^{p+q+pq} e^{-\Omega(p + u_1, q + u_2) + 2\pi i z_1 + 2\pi i z_2}.$$
One can easily see,

\[ F_{u_1,u_2}(z_1,z_2) = \frac{1}{2} \left( \Theta_{u_1,u_2}^{(1)}(z_1 + x + \frac{1}{2}, z_2 + \frac{1}{2}) + \Theta_{u_1,u_2}^{(1)}(z_1 + x + \frac{1}{2}, z_2) + \Theta_{u_1,u_2}^{(1)}(z_1 + x + \frac{1}{2}, z_2 + \frac{1}{2}) - \Theta_{u_1,u_2}^{(1)}(z_1, z_2) \right) \]

\[ = \left( 2\Theta_{u_1,u_2}^{(4)}(2z_1, 2z_2) - \Theta_{u_1,u_2}^{(1)}(z_1, z_2) \right). \]

For the case \( u_1 = u_2 = 0 \) the polynomial identity \( F_{2N,2M}(x^2, y^2) = F_{N,M}(x, y)F_{N,M}(-x, y)F_{N,M}(y, x)F_{N,M}(-y, x) \) provides

\[ f_0F_{0,0}(z_1, z_2) = f_0^4F_{0,0}(\frac{z_1}{2}, \frac{z_2}{2})F_{0,0}(\frac{z_1 + 1}{2}, \frac{z_2}{2})F_{0,0}(\frac{z_1}{2}, \frac{z_2 + 1}{2})F_{0,0}(\frac{z_1 + 1}{2}, \frac{z_2 + 1}{2}). \]

The limit \( z_1, z_2 \to 0 \) gives \( f_0 \) for \( C_{21} \).

\[ f_0 = \sqrt{\frac{4}{f_0(0,0)})f_0(0,0)\frac{1}{f_0(0,0)(0,0)}}. \]

As well, the value of \( \chi \) for \( C_{21} \) follows from

\[ \sum_{m,n}(-1)^{m+n+m-n}e^{-\beta m(n,m)} = F_{0,0}(0,0) = \frac{1}{\beta} \Theta_{0,0}^{(1/\beta)} - \Theta_{0,0}^{(\beta)} \approx (1 - \beta)\chi \]

at \( \beta \to 1 \) with \( \chi = \Theta_{0,0}^{(1)} + 2\frac{\partial \Theta_{0,0}^{(\beta)}}{\partial \beta}|_{\beta=1}. \)

**APPENDIX D. EXAMPLES OF PERIODICAL DISTRIBUTION**

Here we give an example is a periodical distribution of the signs. Let

\[ \varepsilon_{2m+\alpha,2n+\beta} = \varepsilon_{\alpha,\beta} e^{i\pi(um+vn)} \]

with \( u, v = 0 \) or \( 1 \). Periodicity of \( \varepsilon_{m,n} \) provides the periodicity of the series expansions \( C_{21} \), and therefore

\[ c_{2m+\alpha,2n+\beta} = \varepsilon_{2m+\alpha,2n+\beta} C_{\alpha,\beta}. \]

Equation \( 43 \) gives

\[ c_{\alpha,\beta}^2\Theta_{0,0}^{(4/\beta)} - c_{1-\alpha,\beta}^2e^{i\pi u}\Theta_{0,0}^{(4/\beta)} - c_{\alpha,1-\beta}^2e^{i\pi v}\Theta_{0,0}^{(4/\beta)} - c_{1-\alpha,1-\beta}^2e^{i\pi(u+v)}\Theta_{0,0}^{(4/\beta)} = 1 \]

for all four choices of \( (\alpha, \beta) \), its solution is \( c_{0,0}^2 = c_{1,0}^2 = c_{0,1}^2 = c_{1,1}^2 \) (it follows as well from the careful analysis of the structure of \( \varepsilon_{m,n} \)-products in \( 10 \)), so that

\[ c_{2m+\alpha,2n+\beta} = \varepsilon_{\alpha,\beta} e^{i\pi(um+vn)} \left( \Theta_{0,0}^{(4/\beta)} - e^{i\pi u}\Theta_{0,0}^{(4/\beta)} - e^{i\pi v}\Theta_{0,0}^{(4/\beta)} - e^{i\pi(u+v)}\Theta_{0,0}^{(4/\beta)} \right)^{-1/2}. \]
Appendix E. Transfer matrix of Zamolodchikov-Bazhanov-Baxter model

In the last section we would like to describe the relation between (20) and Baxter’s free energy for Zamolodchikov’s model. We will refer to [12], where the inhomogeneous model was considered and divisor parameterization was used. Equations (231) in [12] look like

\[(84) \quad J(X) \cdot T = T \cdot J(X') = 0.\]

Here \(J(X)\) and \(J(X')\) are generating functions \([3]\), operator \(T\) is a modified transfer matrix for Zamolodchikov-Bazhanov-Baxter model (in general, the Pauli matrices may be replaced by the Weyl algebra generators at root of unity). It follows from (84), \(T\) up to a normalization is the product of algebraic supplements of \(J(X)\) and \(J(X')\).

In our particular case, \(J(X)\), \(J(X')\) and \(T\) after the quasi-diagonalization are 2×2 matrices (in the basis of the Pauli matrices). Transfer-matrix of Zamolodchikov’s model \(T(\theta_1, \theta_2, \theta_3)\), mentioned in the Introduction, is the trace of \(T\):

\[(85) \quad T = \text{Trace}_{2\times2} T.\]

Generating functions \(J(X)\) and \(J(X')\) stand for \(J(\lambda(X)^{N/2}, \mu(X)^{M/2}; \kappa^2)\) and \(J(\lambda(X')^{N/2}, \mu(X')^{M/2}; \kappa^2)\) in the present notations, where

\[(86) \quad \kappa^2 = \tan^2 \frac{\theta_1}{2} = \frac{\sin \beta_2 \sin \beta_3}{\sin \beta_0 \sin \beta_1}\]

is the \(\kappa\)-parameter in both \(J(X)\) and \(J(X')\), and explicit evaluations for \(\lambda\) and \(\mu\) from [12] to the terms of linear excesses \(\beta_j\) give

\[(87) \quad \lambda(X) = e^{-i(\beta_1 + \beta_2)} \frac{\sin \beta_0}{\sin \beta_3}, \quad \mu(X) = e^{i(\beta_0 + \beta_2)} \frac{\sin \beta_1}{\sin \beta_3},\]

and

\[(88) \quad \lambda(X') = e^{i(\beta_0 + \beta_3)} \frac{\sin \beta_1}{\sin \beta_2}, \quad \mu(X') = e^{-i(\beta_1 + \beta_3)} \frac{\sin \beta_0}{\sin \beta_2}.\]

It gives us the identification \(\{r_j\} = \{\text{a permutation of } \beta_j\}\) and relates (20) to Baxter’s answer for the partition function per site \(k\):

\[(89) \quad \log k = \text{normalization} + \sum_{j=0}^{3} \left(\frac{\beta_j}{2\pi} \log 2 \sin \beta_j + \Phi(\beta_j)\right).\]

The reader may see the discrepancy, \(\frac{\beta_j}{2\pi} \log 2 \sin \beta_j\) in [31] and \(\frac{\beta_j}{\pi} \log 2 \sin \beta_j\) in [20], it means that the normalization is not trivial – it comes from a certain variational principle.

Department of Theoretical Physics, Research School of Physical Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia

E-mail address: sergey.sergeev@anu.edu.au