Derivations of the trigonometric $BC_n$ Sutherland model by quantum Hamiltonian reduction

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Abstract

The $BC_n$ Sutherland Hamiltonian with coupling constants parametrized by three arbitrary integers is derived by reductions of the Laplace operator of the group $U(N)$. The reductions are obtained by applying the Laplace operator on spaces of certain vector valued functions equivariant under suitable symmetric subgroups of $U(N) \times U(N)$. Three different reduction schemes are considered, the simplest one being the compact real form of the reduction of the Laplacian of $GL(2n, \mathbb{C})$ to the complex $BC_n$ Sutherland Hamiltonian previously studied by Oblomkov.
1 Introduction

The family of Calogero-Sutherland type many-body models is very important both in physics and mathematics, as is amply demonstrated in the reviews [1, 2, 3, 4, 5, 6]. In this paper we focus on the group theoretic derivation of the trigonometric Sutherland models introduced by Olshanetsky and Perelomov [7] in correspondence with the crystallographic root systems. The Hamiltonian of the model associated with the roots system $\mathcal{R}$ is given by

$$H_{\mathcal{R}} = -\frac{1}{2}\Delta + \frac{1}{4} \sum_{\alpha \in \mathcal{R}} |\alpha|^2 \mu_\alpha (\mu_\alpha + 2\mu_{2\alpha} - 1) \frac{\sin^2(\alpha \cdot q)}{\sin^2(\alpha \cdot q)},$$

(1.1)

where $\Delta$ is the Laplacian on the Euclidean space of the roots and the $\mu_\alpha$ are arbitrary real constants depending only on the lengths of the roots, with $\mu_{2\alpha} := 0$ if $2\alpha \notin \mathcal{R}$. In the original $A_{n-1}$ case the model was solved by Sutherland [8]. An interesting general observation [9] is that the radial part of the Laplace operator of any compact Riemannian symmetric space is always conjugate to a Sutherland operator (1.1) built on the root system of the symmetric space, with coupling constants determined by the multiplicities of the roots. This observation showed the algebraic integrability of the resulting Hamiltonians $H_{\mathcal{R}}$ at (small) finite sets of coupling constants and inspired later developments. The integrability, and exact solvability in terms of a triangular structure, was first established for the models (1.1) in full generality by Heckman and Opdam [10, 11]. Their technique is based on differential-reflection operators belonging to the Hecke algebraic generalization of harmonic analysis [2, 12].

The Hecke algebraic approach is very powerful, but it is still desirable to treat as many cases of the models (1.1) in group theoretic terms as possible. Important progress in this direction was achieved by Etingof, Frenkel and Kirillov [13] who worked out the quantum mechanical version of the classical Hamiltonian reduction due to Kazhdan, Kostant and Sternberg [14] and thereby showed that the $A_{n-1}$ Sutherland Hamiltonian arises as the restriction of the Laplace operator of $SU(n)$ to certain vector valued spherical functions. A spherical function $F$ on $SU(n)$ with values in the $SU(n)$ module $V$ satisfies the equivariance condition $F(gxg^{-1}) = g \cdot F(x)$ and thus it is uniquely determined by its restriction to the maximal torus $T < SU(n)$. It is easily seen that the restricted function $f = F|_T$ must vary in the zero-weight subspace $V^T$ and the action of the Laplace operator of $SU(n)$ on $F$ can be expressed by the action of a scalar differential operator on $f$ whenever $\dim(V^T) = 1$. This latter condition singles out the symmetric tensorial powers $V = S^{kn}(\mathbb{C}^n)$ ($k \in \mathbb{Z}_{\geq 0}$) and their duals among the irreducible highest weight representations of $SU(n)$, and the resulting scalar differential operator turns out to be the Sutherland operator $H_{A_{n-1}}$ with coupling parameter $\mu_\alpha = k + 1$.

The above arguments cannot be extended to the simple Lie groups beyond $SU(n)$, since in general they do not admit non-trivial highest weight representations with multiplicity one for the zero weight\(^1\). However, taking any compact connected Lie group $Y$, there exist other nice actions of certain subgroups of $Y \times Y$ on $Y$ for which one can try to generalize the above arguments. Indeed [17], if $G$ is the fixed point set of an involution of $Y \times Y$, then every orbit of the natural action of $G$ on $Y$ can be intersected by a toral subgroup $A < Y$. Therefore the $G$-equivariant functions on $Y$ with values in a representation $V$ of $G$ give rise to $V^K$-valued

\(^1\) The only exceptions [15, 16] are the defining representation of $SO(2n+1)$ and the 7-dimensional representation of $G_2$. In the former case we have checked that the reduced Laplacian gives a decoupled system.
functions on \( A \), where \( K \) is the isotropy group of the generic elements of \( A \). Moreover, if \( \dim(V^K) = 1 \), then the application of the Laplace operator of \( Y \) on \( C^\infty(Y,V)^G \) may induce a scalar Sutherland operator. The group actions just alluded to are called Hermann actions. They received a lot of attention in differential geometry (see e.g. \[17\], \[18\] and references therein), but their use for the construction of integrable systems still has not been explored systematically.

The goal of this paper is to explain that certain Hermann actions on \( Y = U(N) \) permit derivations of the \( BC_n \) Sutherland Hamiltonian from the Laplacian of \( U(N) \). The derivations that we present are partly motivated by an earlier derivation found in the complex holomorphic setting in \[19\], and by our previous paper \[20\] where we discussed how the classical mechanical version of the trigonometric \( BC_n \) model with three arbitrary coupling constants can be obtained by reducing the free particle moving on the group \( U(N) \). Taking for \( \mathcal{R} \) the root system

\[
BC_n = \{ \epsilon_i \pm \epsilon_j, \pm \epsilon_k, \pm 2 \epsilon_k | i, j, k \in \{1, \ldots, n\}, i \neq j \},
\]

with orthonormal vectors \( \{ \epsilon_i \} \), and introducing new coupling parameters \( a, b, c \) by the definition

\[
\mu_{\epsilon_i \pm \epsilon_j} := a + 1, \quad \mu_{\epsilon_k} := b - c, \quad \mu_{2 \epsilon_k} := c + \frac{1}{2},
\]

the Hamiltonian (1.1) reads

\[
H_{BC_n} = -\frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial q_j^2} + \sum_{1 \leq k < l \leq n} \left( \frac{a(a+1)}{\sin^2(q_k - q_l)} + \frac{a(a+1)}{\sin^2(q_k + q_l)} \right) + \frac{1}{2} \sum_{j=1}^{n} \frac{b^2 - \frac{1}{4}}{\sin^2(q_j)} + \frac{1}{2} \sum_{j=1}^{n} \frac{c^2 - \frac{1}{4}}{\cos^2(q_j)}.
\]

In fact, we shall obtain this Hamiltonian with arbitrary non-negative integers \( a, b \) and \( c \) as a reduction of the Laplace operator of \( U(N) \). More precisely, we shall present 3 different derivations, for which \( N = 2n, N = 2n + 1 \) or \( N = 2n + 2 \).

There is considerable conceptual overlap between this paper and the above-mentioned work \[19\] of Oblomkov, who related the eigenfunctions of the holomorphic \( BC_n \) Sutherland operator to vector valued spherical functions on the group \( GL(N, \mathbb{C}) \). If we replace \( GL(N, \mathbb{C}) \) by \( U(N) \), then Oblomkov’s construction leads to our construction in the most important \( N = 2n \) case. However, there are also different cases considered in \[19\] and in this paper even after such replacement, and the language and the techniques used are rather different. In fact, we shall obtain the results by applying a recently developed general framework of quantum Hamiltonian reduction under polar group actions \[21\]. We shall raise interesting open questions, too, and to facilitate their future investigation we describe our analysis in a self-contained manner.

The organization of the article is as follows. In the next section we recall the necessary notions and results concerning quantum Hamiltonian reductions of the Laplace operator on a Riemannian manifold that admits generalized polar coordinates adapted to the symmetry group in the sense of \[22\]. In section 3 we specialize to Hermann actions on a compact Lie group \( Y \), and describe those Hermann actions on \( Y = U(N) \) that are expected to lead to \( BC_n \) Sutherland models if the representation of the symmetry group \( G < Y \times Y \) is chosen appropriately. The key part of the paper is section 4, where we confirm the above expectation for three infinite families of cases. In section 5 we summarize the results, further discuss the comparison with \[19\] and formulate open questions. There is also an appendix containing background material.
2 Quantum Hamiltonian reduction under polar actions

We here collect general definitions and results that will be used subsequently. Our main purpose is to explain that formula (2.14) characterizes the reductions of the Laplace operator of a Riemannian manifold under so-called polar actions [22] of compact symmetry groups. The exposition is restricted to the necessary minimum, for more details see [21] and references therein.

Let $Y$ be a smooth, connected, complete Riemannian manifold with metric $\eta$. Consider the Laplace operator $\Delta_Y$ corresponding to $\eta$. For a smooth function $F$, in local coordinates $\{y^\mu\}$ on $Y$ one has $\Delta_Y F = |\eta|^{-\frac{1}{2}} \partial_\mu (|\eta|^{\frac{1}{2}} \partial^\mu F)$ with $|\eta| := \det(\eta_{\mu,\nu})$. The restriction of $\Delta_Y$ onto the space of the complex-valued compactly supported smooth functions,

$$\Delta_Y^0 := \Delta_Y|_{C^\infty_c(Y)} : C^\infty_c(Y) \to C^\infty_c(Y), \quad (2.1)$$

is an essentially self-adjoint linear operator of the Hilbert space $L^2(Y, d\mu_Y)$, where $\mu_Y$ denotes the measure generated by the Riemannian volume form, locally defined by $|\eta|^{\frac{1}{2}} \prod_\mu dy^\mu$. Suppose that a compact Lie group $G$ acts on $(Y, \eta)$ by isometries. The action is given by a smooth map

$$\phi : G \times Y \to Y, \quad (g, y) \mapsto \phi(g, y) = g.y \quad (2.2)$$

such that $\phi_g^* \eta = \eta$ for every $g \in G$. The measure $\mu_Y$ inherits the $G$-invariance and therefore the Hilbert space $L^2(Y, d\mu_Y)$ naturally carries a continuous unitary representation of $G$. This in turn is unitarily equivalent to an orthogonal direct sum, $L^2(Y, d\mu_Y) \cong \bigoplus_\rho M_\rho \otimes \bar{V}_\rho$, where $(\rho, V_\rho)$ runs over a complete set of pairwise inequivalent irreducible unitary representations of $G$, $\bar{\rho}$ denotes the contragredient of the representation $\rho$, and $M_\rho$ is a ‘multiplicity space’ on which $G$ acts trivially. Correspondingly, the self-adjoint scalar Laplace operator, $\bar{\Delta}_Y^0$, which by definition is the closure of $\Delta_Y^0$ (2.1), can be decomposed as $\bar{\Delta}_Y^0 \cong \bigoplus_\rho \bar{\Delta}_\rho \otimes \text{id}_{V_\rho}$, where $\bar{\Delta}_\rho$ is a self-adjoint operator on the Hilbert space $M_\rho$. The system $(M_\rho, \bar{\Delta}_\rho)$ is called the reduction of the system $(L^2(Y, d\mu_Y), \bar{\Delta}_Y^0)$ having the symmetry type $\bar{\rho}$.

In order to present a convenient model of $(M_\rho, \bar{\Delta}_\rho)$, consider now an irreducible unitary representation $(\rho, V)$ of $G$, where $V$ is a finite dimensional complex vector space with inner product $(\cdot, \cdot)_V$. By simply acting componentwise, the differential operator $\Delta_Y^0$ extends onto the complex vector space of the $V$-valued compactly supported smooth functions, $C^\infty_c(Y, V)$. This gives the essentially self-adjoint operator

$$\Delta_Y^0 : C^\infty_c(Y, V) \to C^\infty_c(Y, V) \quad (2.3)$$

of the Hilbert space $L^2(Y, V, d\mu_Y)$. Because of the $G$-symmetry of the metric $\eta$, the set

$$C^\infty_c(Y, V)^G := \{ F \mid F \in C^\infty_c(Y, V), \ F \circ \phi_g = \rho(g) \circ F \ \ (\forall g \in G) \} \quad (2.4)$$

of the $V$-valued, compactly supported $G$-equivariant smooth functions is an invariant linear subspace of $\Delta_Y^0$. Moreover, the restriction of $\Delta_Y^0$ (2.3) onto $C^\infty_c(Y, V)^G$,

$$\Delta_\rho := \Delta_Y^0|_{C^\infty_c(Y, V)^G} : C^\infty_c(Y, V)^G \to C^\infty_c(Y, V)^G, \quad (2.5)$$

of the Hilbert space $L^2(Y, V, d\mu_Y)$.
is a densely defined, symmetric, essentially self-adjoint linear operator on the Hilbert space $L^2(Y,V,d\mu_Y)^G$ of the square-integrable $G$-equivariant functions. It is not difficult to demonstrate the unitary equivalence

$$(M_\rho,\tilde{\Delta}_\rho) \cong (L^2(Y,V,d\mu_Y)^G,\tilde{\Delta}_\rho) \quad \text{with} \quad V := V_\rho, \quad (2.6)$$

where $\tilde{\Delta}_\rho$ denotes the closure of $\Delta_\rho$ in $(2.5)$. It is convenient for many purposes to use the realization of the reduced quantum system furnished by $L^2(Y,V,d\mu_Y)^G$.

Particularly simple cases of the reduction arise if the reduced configuration space $Y_{\text{red}} := Y/G$ is a smooth manifold, although this happens very rarely. However, restricting to the $Y/G$ realization of the reduced quantum system furnished by $L$.

The restriction of functions appearing in the definition $(2.8)$ gives rise to a linear isomorphism $\text{Fun}(\Sigma,V^K) \cong C^\infty_c(\Sigma,V^K) \hookrightarrow L^2(Y,V,d\mu_Y)^G$. This induces a scalar product on $\text{Fun}(\Sigma,V^K)$ making it a pre-Hilbert space whose closure satisfies the Hilbert space isomorphism $\text{Fun}(\Sigma,V^K) \cong L^2(Y,V,d\mu_Y)^G$. Next, consider the Lie algebra $G := \text{Lie}(G)$ and its subalgebra $K := \text{Lie}(K)$. Fix a $G$-invariant positive definite scalar product, $B_\Sigma$, on $G$ and
thereby determine the orthogonal complement $K^\perp$ of $K$ in $G$. For any $ξ ∈ G$ denote by $ξ^\perp$ the associated vector field on $Y$. Then at each point $Q ∈ Q$ the linear map $K^\perp ∋ ξ ↦ ξ_Q ∈ T_QY$ is injective, and the inertia operator $J(Q) ∈ \text{End}(K^\perp)$ can be defined by the requirement

$$η_Q(ξ_Q^\perp, ζ_Q^\perp) = B_ξ(ξ, J(Q)ζ), \quad \forall ξ, ζ ∈ K^\perp. \tag{2.10}$$

Note that $J(Q)$ is symmetric and positive definite with respect to $B_ξ|_{K^\perp×K^\perp}$. By choosing dual bases $\{T_α\}, \{T^α\} ⊂ K^\perp$, that is, $B_ξ(T_α, T_β) = δ_β^α$, we let

$$b_α,β(Q) := B_ξ(T_α, J(Q)T_β), \quad b^{α,β}(Q) := B_ξ(T^α, J(Q)^{-1}T^β). \tag{2.11}$$

The $G$-orbit $G.Q ⊂ Y$ through any point $Q ∈ Q$ is an embedded submanifold of $Y$ and by its embedding it inherits a Riemannian metric, $η_{G.Q}$. Thus we can define the smooth density function $δ : Q → (0, ∞)$ by

$$δ(Q) := \text{volume of the Riemannian manifold } (G.Q, η_{G.Q}), \tag{2.12}$$

where the volume is understood with respect to the measure, $μ_{G.Q}$, belonging to $η_{G.Q}$. It is easy to see that

$$δ(Q) = C|\det(J(Q))|^\frac{1}{2} = C|\det(b_α,β(Q))|^\frac{1}{2}, \tag{2.13}$$

with some constant $C > 0$. In the following proposition, quoted from [21], $ρ'$ denotes the representation of $G$ corresponding to the representation $ρ$ of $G$.

**Proposition 2.1** Let us consider a polar $G$-action using the above notations. Then the reduced system (2.0) associated with an admissible irreducible unitary representation $(ρ, V)$ of $G$ can be identified with the pair $(L^2(\tilde{Σ}, V^K, dμ_Σ), Δ_{red})$, where

$$Δ_{red} = Δ_Σ - δ^{-\frac{1}{2}}Δ_Σ(δ^{\frac{1}{2}}) + b^{α,β}(T_α)ρ(T_β) \tag{2.14}$$

with domain $D(Δ_{red}) = δ^{\frac{1}{2}}\text{Fun}(\tilde{Σ}, V^K)$ is a densely defined, symmetric, essentially self-adjoint operator on the Hilbert space $L^2(\tilde{Σ}, V^K, dμ_Σ)$.

The above statement results by calculating the action of $Δ_Y$ on the $V$-valued equivariant functions in (2.8) with the aid of polar coordinates, using also the Hilbert space identifications

$$\overline{\text{Fun}}(\tilde{Σ}, V^K) \cong L^2(Y, V, dμ_Y)^G \cong L^2(\tilde{Σ}, V^K, dμ_Σ). \tag{2.15}$$

The last equality follows by integrating out the ‘angular’ coordinates in the scalar product of equivariant functions. One also uses the unitary map $U : L^2(\tilde{Σ}, V^K, δdμ_Σ) → L^2(\tilde{Σ}, V^K, dμ_Σ)$ defined by $U : f ↦ δ_\frac{1}{2}f$.

The first term in (2.14) corresponds to the kinetic energy of a particle moving on $(Y_{red}, η_{red}) \cong (\tilde{Σ}, η_Σ)$ and the rest represents potential energy if $\dim(V^K) = 1$. The second term of (2.14) is always potential energy, which is constant in some cases. We refer to this term as the ‘measure factor’. It represents a significant difference between the outcomes of the corresponding classical and quantum Hamiltonian reductions [21]. If $\dim(V^K) > 1$, then one says that the reduced system contains internal ‘spin’ degrees of freedom and then the third term of (2.14) encodes ‘spin-dependent potential energy’.
3 Examples of polar actions on compact Lie groups

From now we take the ‘unreduced configuration space’ \( Y \) to be a compact, connected, real Lie group endowed with a bi-invariant metric \( \eta \), induced by a positive definite, \( Y \)-invariant bilinear form \( \mathcal{B}_Y \) of the Lie algebra \( \mathfrak{y} := \text{Lie}(Y) \). For the reduction group \( G \) one may choose any symmetric subgroup of the direct product group \( Y \times Y \), that is,

\[
(Y \times Y)_{0}^{\sigma} \leq G \leq (Y \times Y)^{\sigma},
\]

where \((Y \times Y)^{\sigma}\) stands for the fixed-point set of some involutive automorphism \( \sigma \in \text{Inv}(Y \times Y) \), and \((Y \times Y)_{0}^{\sigma}\) is the connected component of the identity in \((Y \times Y)^{\sigma}\). The group \( G \) acts on \( Y \) by the map

\[
\phi: G \times Y \to Y, \quad (g_{L}, g_{R}, y) \mapsto \phi(g_{L}, g_{R})(y) := g_{L}yg_{R}^{-1}.
\]

The group actions of this form are often called Hermann actions. Under mild conditions, which hold in the examples below, these are polar actions in the sense of [22]. In fact, the sections are provided by certain toral subgroups \( A \subset Y \). Thus the sections are flat in the induced metric, which is the characteristic property of the so-called hyperpolar actions [17]. In the simplest special case \( \sigma(y_{1}, y_{2}) = (y_{2}, y_{1}) \), \( G = Y_{\text{diag}} = \{ (y, y) \mid y \in Y \} \cong Y \) and (3.2) is just the adjoint action of \( Y \) on itself, for which the sections are the maximal tori of \( Y \).

3.1 Hermann actions associated with pairs of involutions

The reductions that we study later arise from the following construction. Let \( \sigma_{L}, \sigma_{R} \in \text{Inv}(Y) \) be two involutions of \( Y \), and let \( Y_{L}, Y_{R} \leq Y \) be corresponding symmetric subgroups of \( Y \),

\[
(Y_{I})_{0}^{\sigma_{I}} \leq Y_{I} \leq Y_{I}^{\sigma_{I}} \quad (I \in \{L, R\}).
\]

We suppose that the scalar product \( \mathcal{B}_Y \) is invariant under both \( \sigma_{L} \) and \( \sigma_{R} \) and introduce \( \sigma \in \text{Inv}(Y \times Y) \) by \( \sigma(y_{1}, y_{2}) := (\sigma_{L}(y_{1}), \sigma_{R}(y_{2})) \). Then

\[
G := Y_{L} \times Y_{R}
\]

is a symmetric subgroup of \( Y \times Y \) and equation (3.2) defines a hyperpolar Hermann action of \( G \) on \( Y \). The classification of the inequivalent pairs of involutions \((\sigma_{L}, \sigma_{R})\) has been worked out by Matsuki [24]. We assume for simplicity that the two involutions \( \sigma_{L} \) and \( \sigma_{R} \) commute with each other, which holds for the large majority of cases in the classification. Subsequently, the induced Lie algebra involutions are denoted by the same letters \( \sigma_{L} \) and \( \sigma_{R} \).

Now, with the aid of the subspaces

\[
\mathcal{Y}_{I}^{\sigma_{I}, \pm} \, = \, \ker(\sigma_{I} \mp \text{Id}_{Y}) \subset \mathcal{Y} \quad (I \in \{L, R\}) \quad \text{and} \quad \mathcal{Y}^{\pm} \, = \, \mathcal{Y}^{\sigma_{L}, \pm} \cap \mathcal{Y}^{\sigma_{R}, \pm} \subset \mathcal{Y}
\]

we obtain the orthogonal decomposition

\[
\mathcal{Y} = \mathcal{Y}^{++} \oplus \mathcal{Y}^{+-} \oplus \mathcal{Y}^{-+} \oplus \mathcal{Y}^{--},
\]

\footnote{A toral subgroup \( A < Y \) is a connected and closed Abelian subgroup. It is the closedness of the relevant subgroups that requires some conditions. If \( \mathcal{Y} \) is semi-simple, then a sufficient condition is to take \( \mathcal{B}_Y \) as a multiple of the Killing form [17].}
which gives also a $\mathbb{Z}_2 \times \mathbb{Z}_2$-gradation of $\mathcal{Y}$. The Lie algebra of the symmetric subgroup $Y_I \leq Y$ is $\text{Lie}(Y_I) \cong \mathcal{Y}^{\sigma_I \cdot \cdot \cdot}$ ($I \in \{L,R\}$). Then, we choose a maximal Abelian subalgebra $A$ in $\mathcal{Y}^{--}$ and also define $A := \exp(A)$, which is a toral subgroup of $Y$. According to an important theorem proved in [25, 26], the Lie group $Y$ admits the generalized Cartan decomposition

$$Y = Y_L A Y_R.$$  

(3.7)

This means that every element of $Y$ can be written as a product of the elements of the subgroups in (3.7). Recalling the definition of the Hermann action (3.2) for $G = Y_L \times Y_R$, equation (3.7) says that the subgroup $A$ intersects every $G$-orbit. Moreover, it does so orthogonally at every intersection point, and thus $A$ provides a section for the $G$-action in the sense of [22]. Below $\tilde{A}$ denotes a connected component of the regular part of the section $A$.

Let us introduce the subgroups $Y_{LR} := Y_L \cap Y_R \leq Y$ and

$$M := \{ g \mid g \in Y_{LR}, \ gag^{-1} = a \ (\forall a \in A) \} \leq Y_{LR}.$$  

(3.8)

Their Lie algebras are

$$\text{Lie}(Y_{LR}) \cong \text{Lie}(Y_L) \cap \text{Lie}(Y_R) \cong \mathcal{Y}^{\sigma_L \cdot \cdot \cdot} \cap \mathcal{Y}^{\sigma_R \cdot \cdot \cdot} = \mathcal{Y}^{++},$$  

$$\mathcal{M} := \text{Lie}(M) = \{ X \mid X \in \mathcal{Y}^{++}, \ \text{ad}_X(q) = 0 \ (\forall q \in A) \},$$  

(3.9)

(3.10)

where $\text{ad}_X$ is defined by the Lie bracket on $\mathcal{Y}$. It can be shown that the centralizer of the section $A = \exp(A)$ (the isotropy subgroup of the elements of $\tilde{A}$) is now furnished by

$$K = M_{\text{diag}} = \{ (g,g) \mid g \in M \} \leq G.$$  

(3.11)

To specialize the inertia operator $J$ defined in (2.10), we introduce a $G$-invariant scalar product on the Lie algebra

$$\mathcal{G} = \text{Lie}(G) = \text{Lie}(Y_L \times Y_R) \cong \text{Lie}(Y_L) \oplus \text{Lie}(Y_R) \cong \mathcal{Y}^{\sigma_L \cdot \cdot \cdot} \oplus \mathcal{Y}^{\sigma_R \cdot \cdot \cdot}$$  

(3.12)

by the formula

$$\mathcal{B}_\mathcal{G}((\xi_L, \xi_R), (\zeta_L, \zeta_R)) := \mathcal{B}_\mathcal{Y}(\xi_L, \zeta_L) + \mathcal{B}_\mathcal{Y}(\xi_R, \zeta_R), \quad \forall (\xi_L, \xi_R), (\zeta_L, \zeta_R) \in \mathcal{G}.$$  

(3.13)

This induces the decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{K}^\perp$, where $\mathcal{K} = \text{Lie}(K)$. By using the decomposition $\mathcal{Y} = \mathcal{M} \oplus \mathcal{M}^\perp$ defined by $\mathcal{B}_\mathcal{Y}$, we also introduce the subspaces

$$\mathcal{K}_o^\perp := \{ (X, -X) \mid X \in \mathcal{M} \} \subset \mathcal{K}^\perp,$$  

$$\mathcal{K}_e^\perp := \{ (\xi_L, \xi_R) \mid \xi_L, \xi_R \in \mathcal{M}^\perp \cap \mathcal{Y}^{++} \} \subset \mathcal{K}^\perp,$$  

$$\mathcal{K}_o^\perp := \{ (\zeta_L, \zeta_R) \mid \zeta_L \in \mathcal{Y}^{+-}, \zeta_R \in \mathcal{Y}^{-+} \} \subset \mathcal{K}^\perp,$$  

(3.14)

(3.15)

(3.16)

which yield the orthogonal decomposition

$$\mathcal{K}^\perp = \mathcal{K}_o^\perp \oplus \mathcal{K}_e^\perp \oplus \mathcal{K}_o^\perp.$$  

(3.17)

Now consider the vector field $\xi^\sharp = (\xi_L, \xi_R)^\sharp$ on $Y$ associated with $\xi = (\xi_L, \xi_R) \in \mathcal{G}$ by means of the $G$-action. At an arbitrary point $e^q \in A$ ($q \in A$) of the section $A$ we find

$$\xi^\sharp e^q = (\xi_L, \xi_R)^\sharp e^q = (dL_{e^q})_e (\xi_R - e^{-\text{ad}_q}(\xi_L)) \in T_{e^q}Y,$$  

(3.18)

where $L_y$ denotes the left-translation on $Y$ by group element $y \in Y$. Simply by plugging (3.18) into the definition (2.10), routine algebraic manipulations lead to the following result.
Lemma 3.1  Equation (3.14) is a decomposition of $K_\perp$ into invariant subspaces of the inertia operator $J(e^q)$ at any point $e^q \in A$. One has $J(e^q)|_{K_\perp} = 2\text{Id}_{K_\perp}$ and, writing $\xi = (\xi_L, \xi_R) \in G$ as a 2-component column vector with components $\xi_L$ and $\xi_R$, the action of $J(e^q)$ on $K_\perp$ and $K_\perp^\perp$ is encoded by the matrices

$$J(e^q)|_{K_\perp} = \begin{bmatrix} 1 & -\cosh(\text{ad}_q) \\ -\cosh(\text{ad}_q) & 1 \end{bmatrix}, \quad J(e^q)|_{K_\perp^\perp} = \begin{bmatrix} 1 & -\sinh(\text{ad}_q) \\ \sinh(\text{ad}_q) & 1 \end{bmatrix}.$$

(3.19)

For the inverse of $J(e^q)$ one has $J(e^q)^{-1}|_{K_\perp} = \frac{1}{2}\text{Id}_{K_\perp}$ together with

$$J(e^q)^{-1}|_{K_\perp} = -\begin{bmatrix} \sinh^{-2}(\text{ad}_q) & \cosh(\text{ad}_q)\sinh^{-2}(\text{ad}_q) \\ \cosh(\text{ad}_q)\sinh^{-2}(\text{ad}_q) & \sinh^{-2}(\text{ad}_q) \end{bmatrix}|_{K_\perp^\perp},$$

(3.20)

$$J(e^q)^{-1}|_{K_\perp^\perp} = \begin{bmatrix} \cosh^{-2}(\text{ad}_q) & \sinh(\text{ad}_q)\cosh^{-2}(\text{ad}_q) \\ -\sinh(\text{ad}_q)\cosh^{-2}(\text{ad}_q) & \cosh^{-2}(\text{ad}_q) \end{bmatrix}|_{K_\perp^\perp}.$$

(3.21)

3.2  A family of two involutions on $U(N)$

For our later purpose we now focus on the unitary group

$$Y := U(N) = \{ y \mid y \in GL(N, \mathbb{C}), \quad y^\dagger y = 1_N \}.$$

(3.22)

We equip the Lie algebra

$$\mathcal{Y} := u(N) = \{ X \mid X \in \mathfrak{gl}(N, \mathbb{C}), \quad X^\dagger + X = 0 \}$$

(3.23)

with the scalar product

$$B_y(X, Z) := -\text{tr}(XZ), \quad \forall X, Z \in u(N).$$

(3.24)

To any pair $(m, n) \in \mathbb{Z}_{\geq 0}^2$ with $m \geq n$ and $m + n = N$ we associate the block-matrix

$$I_{m,n} := \text{diag}(1_m, -1_n) = \begin{bmatrix} 1_m & 0 \\ 0 & -1_n \end{bmatrix} \in U(N),$$

(3.25)

and the involutive inner automorphism

$$\theta_{m,n} : U(N) \rightarrow U(N), \quad y \mapsto \theta_{m,n}(y) := I_{m,n}yI_{m,n}^{-1}. $$

(3.26)

The fixed-point set of $\theta_{m,n}$ is

$$U(N)^{\theta_{m,n}} = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a \in U(m), b \in U(n) \right\} \cong U(m) \times U(n).$$

(3.27)

Note that $U(N)^{\theta_{m,n}}$ is connected. The induced Lie algebra involution operates as

$$\theta_{m,n}(X) = I_{m,n}X^{-1}_{m,n}, \quad \forall X \in u(N).$$

(3.28)
Using the block-matrix realization

\[ u(N) = \left\{ \begin{bmatrix} A & C \\ -C^\dagger & B \end{bmatrix} \middle| A \in u(m), B \in u(n), C \in \mathbb{C}^{m \times n} \right\}, \tag{3.29} \]

the eigenspaces \( u(N)^{\theta_{m,n,\pm}} \) are

\[
u(N)^{\theta_{m,n,+}} = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \middle| A \in u(m), B \in u(n) \right\}, \quad \nu(N)^{\theta_{m,n,-}} = \left\{ \begin{bmatrix} 0 & C \\ -C^\dagger & 0 \end{bmatrix} \middle| C \in \mathbb{C}^{m \times n} \right\}. \tag{3.30} \]

Now we take two pairs \((m, n), (r, s) \in \mathbb{Z}_+^2\) with the additional requirements \(m \geq r \geq s \geq n\) and \(m + n = r + s = N\), and consider the commuting involutions

\[
\sigma_L := \theta_{r,s} \quad \text{and} \quad \sigma_R := \theta_{m,n}. \tag{3.31}
\]

The corresponding symmetric subgroups \( Y_L, Y_R \leq Y \) are

\[
U(N)_L := U(N)^{\sigma_L} \cong U(r) \times U(s) \quad \text{and} \quad U(N)_R := U(N)^{\sigma_R} \cong U(m) \times U(n). \tag{3.32}
\]

The partition \( N = n + (r - n) + (s - n) + n \) leads to a \( 4 \times 4 \) block-matrix decomposition of any \( N \times N \) matrix in general. (Of course, if \( r = n \) or \( s = n \), then the block-matrix decomposition contains fewer blocks.) That is, any matrix \( X \in \mathbb{C}^{N \times N} \) can be written as

\[
X = \begin{bmatrix}
X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} \\
X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} \\
X_{3,1} & X_{3,2} & X_{3,3} & X_{3,4} \\
X_{4,1} & X_{4,2} & X_{4,3} & X_{4,4}
\end{bmatrix},
\tag{3.33}
\]

where the entries \( X_{i,j} \) are themselves matrices, \( X_{1,1} \in \mathbb{C}^{n \times n}, X_{1,2} \in \mathbb{C}^{n \times (r-n)}, X_{1,3} \in \mathbb{C}^{n \times (s-n)}, X_{1,4} \in \mathbb{C}^{n \times n}, \) etc. Then for the Lie group \( Y_{LR} = Y_L \cap Y_R \) we have

\[
U(N)_{LR} = \left\{ \begin{bmatrix}
a_{1,1} & a_{1,2} & 0 & 0 \\
a_{2,1} & a_{2,2} & 0 & 0 \\
0 & 0 & a_{3,3} & 0 \\
0 & 0 & 0 & a_{4,4}
\end{bmatrix} \middle| \begin{bmatrix}
a_{1,1} & a_{1,2} \\
a_{2,1} & a_{2,2} \\
0 & 0 \\
0 & 0
\end{bmatrix} \in U(r), a_{3,3} \in U(s-n), a_{4,4} \in U(n) \right\}. \tag{3.34}
\]

Therefore \( U(N)_{LR} \cong U(r) \times U(s-n) \times U(n) \) and the Lie algebra \( \text{Lie}(U(N)_{LR}) = u(N)^{++} \) is isomorphic to \( u(r) \oplus u(s-n) \oplus u(n) \). In our case the subspace \( \mathcal{Y}^{--} \) in \( \{3.35\} \) reads

\[
u(N)^{--} = \left\{ \begin{bmatrix}
0 & 0 & 0 & A_{1,4} \\
0 & 0 & 0 & A_{2,4} \\
0 & 0 & 0 & 0 \\
-A_{1,4}^\dagger & -A_{2,4}^\dagger & 0 & 0
\end{bmatrix} \middle| A_{1,4} \in \mathbb{C}^{n \times n}, A_{2,4} \in \mathbb{C}^{(r-n) \times n} \right\}. \tag{3.35}
\]

To proceed, we define the diagonal \( n \times n \) matrix

\[
q := \text{diag}(q_1, q_2, \ldots, q_n) \in \mathbb{R}^{n \times n} \tag{3.36}
\]
for any real $n$-tuple $(q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$, and we also set

\[
q := \begin{bmatrix}
0 & 0 & 0 & q \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-q & 0 & 0 & 0
\end{bmatrix} \in u(N)^{-}. \quad (3.37)
\]

Then the set of matrices

\[
\mathcal{A} := \{q \mid (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n\} \subset u(N)^{-}
\]

is a maximal Abelian subalgebra in $u(N)^{-}$. A basis of the dual space $\mathcal{A}^*$ is given by the functionals

\[
\epsilon_k : \mathcal{A} \to \mathbb{R}, \quad q \mapsto \epsilon_k(q) := q_k.
\]

The corresponding subgroup $A = \exp(\mathcal{A})$ has the form

\[
A = \left\{ e^q = \begin{bmatrix}
\cos(q) & 0 & 0 & \sin(q) \\
0 & 1_{r-n} & 0 & 0 \\
0 & 0 & 1_{s-n} & 0 \\
-\sin(q) & 0 & 0 & \cos(q)
\end{bmatrix} \mid (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n \right\}. \quad (3.40)
\]

If $\mathbb{T}(n)$ denotes the diagonally embedded standard torus in $U(n)$, then it is straightforward to show that the subgroup $M$ (3.8) is now furnished by

\[
M = \left\{ \begin{bmatrix}
a & 0 & 0 & 0 \\
b & 0 & 0 & 0 \\
c & 0 & 0 & 0 \\
a & 0 & 0 & a
\end{bmatrix} \mid a \in \mathbb{T}(n), b \in U(r-n), c \in U(s-n) \right\}. \quad (3.41)
\]

Note that $M$ is connected, and therefore so is the centralizer $K = M_{\text{diag}}$ of the section $A$. Moreover, we have the identifications

\[
K \cong M_{\text{diag}} \cong M \cong \mathbb{T}(n) \times U(r-n) \times U(s-n) \cong U(1)^n \times U(r-n) \times U(s-n). \quad (3.42)
\]

It is shown in [26] (page 63) that the closed, connected subset

\[
A_+ := \left\{ e^q \mid 0 \leq q_1 \leq q_2 \leq \ldots \leq q_n \leq \frac{\pi}{2} \right\} \subset A \quad (3.43)
\]

intersects each orbit of $G = U(N)^{s_L} \times U(N)^{s_R}$ under the action (3.2) precisely once. Note also that matrix exponentiation provides a bijection from

\[
\mathcal{A}_+ := \left\{ q \mid 0 \leq q_1 \leq q_2 \leq \ldots \leq q_n \leq \frac{\pi}{2} \right\} \subset \mathcal{A} \quad (3.44)
\]

onto $A_+$. By inspecting the isotropy subgroup $G_{e^q} \leq G$ for $e^q \in A_+$, we find that $G_{e^q} = K$ if and only if $q \in \mathcal{A}_+$, where $\mathcal{A}_+$ denotes the connected open subset

\[
\mathcal{A}_+ := \left\{ q \mid 0 < q_1 < q_2 < \ldots < q_n < \frac{\pi}{2} \right\} \subset \mathcal{A}_+. \quad (3.45)
\]

We can conclude from the above that the subset $\hat{A} := \exp(\mathcal{A}_+)$ provides a connected component for the regular part of the section $A$. Regarding the components $q_k$ in (3.45) as global coordinates on $\hat{A}$, for the Laplace operator $\Delta_{\hat{A}}$ defined by the induced metric we obtain

\[
\Delta_{\hat{A}} = \frac{1}{2} \sum_{k=1}^{n} \frac{\partial^2}{\partial q_k^2}. \quad (3.46)
\]
3.3 Diagonalization of the inertia operator

We continue the study of the examples \((3.31)\) by presenting a basis of \(J(e^q)\) \((3.14)\) for any \(q \in \mathbb{A}_+\) in \((3.15)\). We then use this basis to compute the density \(\delta^\perp\) that enters the second term of the reduced Laplacian \((2.11)\). Note that \(\delta^\perp\) could be found also by the specialization of general formulae available for two commuting involutions \([25, 2]\), but we need to fix a basis for the evaluation of the third term of \((2.11)\), which will be performed later.

We start by defining an orthonormal basis (ONB) in the space \(\mathcal{M}^\perp \cap u(N)^{++}\), which (due to \((3.34)\) and \((3.41)\)) has the form

\[
\mathcal{M}^\perp \cap u(N)^{++} = \left\{ \begin{bmatrix} X_{1,1} & X_{1,2} & 0 & 0 \\
-\frac{X_{1,2}}{X_{1,1}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{X_{4,4}}{X_{1,2}} & 0 & 0 & X_{4,4} \end{bmatrix} \middle| \begin{array}{ll}
X_{1,1}, X_{4,4} \in u(n), & (X_{1,1} + X_{4,4})_{\text{diag}} = 0, \\
X_{1,2} \in \mathbb{C}^{n \times (r-n)} & \end{array} \right\}.
\]

(3.47)

If \(r = n\), then there are no off-diagonal blocks, and in general \(\dim(\mathcal{M}^\perp \cap u(N)^{++}) = n(2r - 1)\). For all \(1 \leq j \leq n\) we let

\[
E^j_{2\varepsilon_j} := \frac{i}{\sqrt{2}} \begin{bmatrix}
E_{jj} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -E_{jj} & 0 \\
\end{bmatrix},
\]

and for all \(1 \leq k < l \leq n\) we define

\[
E^r_{\varepsilon_k + \varepsilon_l} := \frac{1}{2} \begin{bmatrix}
E_{kl} - E_{lk} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & E_{lk} - E_{kl} \\
\end{bmatrix}, \quad E^l_{\varepsilon_k + \varepsilon_l} := \frac{i}{2} \begin{bmatrix}
E_{kl} + E_{lk} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -E_{kl} - E_{lk} \\
\end{bmatrix},
\]

(3.48)

\[
E^l_{\varepsilon_k - \varepsilon_l} := \frac{1}{2} \begin{bmatrix}
E_{kl} - E_{lk} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & E_{lk} - E_{kl} \\
\end{bmatrix}, \quad E^l_{\varepsilon_k - \varepsilon_l} := \frac{i}{2} \begin{bmatrix}
E_{kl} + E_{lk} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & E_{kl} + E_{lk} \\
\end{bmatrix}.
\]

(3.49)

For all \(1 \leq j \leq n\) and \(1 \leq d \leq r - n\) we set

\[
E^r_{\varepsilon_j} := \frac{1}{\sqrt{2}} \begin{bmatrix}
0 & E_{jd} & 0 & 0 \\
-E_{dj} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad E^l_{\varepsilon_j} := \frac{i}{\sqrt{2}} \begin{bmatrix}
0 & E_{jd} & 0 & 0 \\
E_{dj} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

(3.50)

The superscripts \(i\) and \(r\) refer to purely imaginary and to real matrices, respectively, and the elementary matrices \(E_{ab}\) are always understood to be of the correct size as dictated by \((3.33)\). The set of matrices

\[
\{E^D_{\alpha}\}_{\alpha, D} := \{E^r_{\varepsilon_k + \varepsilon_l}, E^l_{\varepsilon_k + \varepsilon_l}\}_{1 \leq k < l \leq n} \cup \{E^i_{2\varepsilon_j}\}_{j=1}^n \cup \{E^r_{\varepsilon_j}, E^l_{\varepsilon_j}\}_{1 \leq j \leq n, 1 \leq d \leq r - n}.
\]

(3.51)
forms an ONB in $\mathcal{M}^+ \cap \mathfrak{u}(N)^{++}$. Here $D$ is an ‘index of degeneration’ and $\alpha$ runs over the positive roots $\mathcal{R}_+$ for the root system $C_n$ or $BC_n$. More precisely,

$$\mathcal{R}_+ = \begin{cases} 
\mathcal{R}_+(C_n) & \text{if } r = n, \\
\mathcal{R}_+(BC_n) & \text{if } r > n.
\end{cases}$$  \hspace{1cm} (3.52)

One can easily verify the relations

$$(\text{ad}_q)^2 E^D_{\alpha} = -\alpha(q)^2 E^D_{\alpha}.$$  \hspace{1cm} (3.53)

Next, we deal with the subspaces $\mathfrak{u}(N)^{+-}$ and $\mathfrak{u}(N)^{-+}$ given by

$$\mathfrak{u}(N)^{+-} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & X_{3,4} & 0 \\
0 & 0 & -X_{3,4} & 0 \end{bmatrix} \mid X_{3,4} \in \mathbb{C}^{(s-n)\times n} \right\},$$  \hspace{1cm} (3.54)

$$\mathfrak{u}(N)^{-+} = \left\{ \begin{bmatrix} 0 & 0 & X_{1,3} & 0 \\
0 & 0 & X_{2,3} & 0 \\
-X_{1,3} & -X_{2,3} & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix} \mid X_{1,3} \in \mathbb{C}^{n\times(s-n)}, X_{2,3} \in \mathbb{C}^{(r-n)\times(s-n)} \right\}. $$  \hspace{1cm} (3.55)

Note that both $\mathfrak{u}(N)^{+-}$ and $\mathfrak{u}(N)^{-+}$ are trivial if $s = n$. In general, $\dim(\mathfrak{u}(N)^{+-}) = 2n(s-n)$ and $\dim(\mathfrak{u}(N)^{-+}) = 2r(s-n)$. For all $1 \leq j \leq n$ and $1 \leq d \leq s-n$ we define

$$\tilde{E}_{t,j}^{r,d} := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & E_{d(j)} & 0 \\
0 & 0 & -E_{jd} & 0 \end{bmatrix}, \quad \tilde{E}_{t,j}^{i,d} := \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix},$$  \hspace{1cm} (3.56)

$$\tilde{F}_{t,j}^{r,d} := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & -E_{jd} & 0 \\
0 & 0 & 0 & 0 \\
E_{dj} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{F}_{t,j}^{i,d} := \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & E_{dj} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}. $$  \hspace{1cm} (3.57)

For all $1 \leq c \leq r - n$ and $1 \leq d \leq s-n$ we introduce

$$\tilde{F}_{0,c,d}^{r} := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & E_{cd} & 0 \\
0 & 0 & 0 & 0 \\
0 & -E_{dc} & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{F}_{0,c,d}^{i} := \frac{i}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & E_{cd} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{bmatrix}. $$  \hspace{1cm} (3.58)

The set of matrices

$$\{\tilde{E}_{t,j}^{D}\}_{j,D} := \{\tilde{E}_{t,j}^{r,d}, \tilde{E}_{t,j}^{i,d}\}_{1 \leq j \leq n, 1 \leq d \leq s-n}$$  \hspace{1cm} (3.59)

forms an ONB in $\mathfrak{u}(N)^{++}$. The set of matrices

$$\{\tilde{F}_{t,j}^{D}\}_{j,D} := \{\tilde{F}_{t,j}^{r,d}, \tilde{F}_{t,j}^{i,d}\}_{1 \leq j \leq n, 1 \leq d \leq s-n}$$  \hspace{1cm} (3.60)
together with the set
\[
\{ \tilde{F}_0^D \}_D := \{ \tilde{F}_{r,c}^D, \tilde{F}_0^D \}_{1 \leq c \leq r-n, 1 \leq d \leq s-n}
\] (3.61)
form an ONB in \( \mathfrak{u}(N)^{-+} \). They verify the relations
\[
\text{ad}_q(\tilde{F}_0^D) = q_j \tilde{F}_j^D, \quad \text{ad}_q(\tilde{F}_0^D) = -q_j \tilde{F}_j^D, \quad \text{ad}_q(\tilde{F}_0^D) = 0.
\] (3.62)

Now we compute the matrix of \( J \) and of \( J^{-1} \) on the invariant subspaces in (3.17). First, choose an arbitrary ONB \( \{L_j\}_{j=1}^{\dim(M)} \) in \( \mathcal{M} \). Then the vectors
\[
\tilde{L}_j := \frac{1}{\sqrt{2}} (L_j, -L_j) \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} L_j \\ -L_j \end{bmatrix}
\] (3.63)
yield an ONB in \( \mathcal{K}^\perp_\alpha \). The matrix entries of \( J(e^q)|_{\mathcal{K}^\perp_\alpha} \) and \( J(e^q)^{-1}|_{\mathcal{K}^\perp_\alpha} \) read
\[
\mathcal{B}_{g}(\tilde{L}_k, J(e^q)\tilde{L}_l) = 2 \delta_{k,l}, \quad \mathcal{B}_{g}(\tilde{L}_k, J(e^q)^{-1}\tilde{L}_l) = \frac{1}{2} \delta_{k,l}.
\] (3.64)

Second, upon introducing the vectors
\[
V_\alpha^D := \frac{1}{\sqrt{2}} \begin{bmatrix} E_\alpha^D \\ F_\alpha^D \end{bmatrix}, \quad W_\alpha^D := \frac{1}{\sqrt{2}} \begin{bmatrix} E_\alpha^D \\ -F_\alpha^D \end{bmatrix},
\] (3.65)
we obtain an ONB in \( \mathcal{K}^\perp_\alpha \), and by applying (3.19) on these vectors we get
\[
J(e^q)V_\alpha^D = \frac{1}{\sqrt{2}} \begin{bmatrix} (1 - \cosh(\text{ad}_q))E_\alpha^D \\ (1 - \cosh(\text{ad}_q))F_\alpha^D \end{bmatrix}, \quad J(e^q)W_\alpha^D = \frac{1}{\sqrt{2}} \begin{bmatrix} (1 + \cosh(\text{ad}_q))E_\alpha^D \\ -(1 + \cosh(\text{ad}_q))F_\alpha^D \end{bmatrix}.
\] (3.66)

We find from the relations (3.53) that \( \cosh(\text{ad}_q)E_\alpha^D = \cos(\alpha(q))E_\alpha^D \), and then elementary trigonometric identities yield
\[
J(e^q)V_\alpha^D = 2 \sin^2 \left( \frac{\alpha(q)}{2} \right) V_\alpha^D, \quad J(e^q)W_\alpha^D = 2 \cos^2 \left( \frac{\alpha(q)}{2} \right) W_\alpha^D.
\] (3.67)

Therefore the only nontrivial matrix entries of \( J(e^q)|_{\mathcal{K}^\perp_\alpha} \) and \( J(e^q)^{-1}|_{\mathcal{K}^\perp_\alpha} \) are the following ones:
\[
\mathcal{B}_{g}(V_\alpha^D, J(e^q)V_\alpha^D) = 2 \sin^2 \left( \frac{\alpha(q)}{2} \right), \quad \mathcal{B}_{g}(W_\alpha^D, J(e^q)W_\alpha^D) = 2 \cos^2 \left( \frac{\alpha(q)}{2} \right),
\]
\[
\mathcal{B}_{g}(V_\alpha^D, J(e^q)^{-1}V_\alpha^D) = \frac{1}{2 \sin^2 \left( \frac{\alpha(q)}{2} \right)}, \quad \mathcal{B}_{g}(W_\alpha^D, J(e^q)^{-1}W_\alpha^D) = \frac{1}{2 \cos^2 \left( \frac{\alpha(q)}{2} \right)}.
\] (3.68)

Third, by introducing
\[
\tilde{V}_{\epsilon_j}^D := \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{E}_{\epsilon_j}^D \\ \tilde{F}_{\epsilon_j}^D \end{bmatrix}, \quad \tilde{W}_{\epsilon_j}^D := \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{E}_{\epsilon_j}^D \\ -\tilde{F}_{\epsilon_j}^D \end{bmatrix}, \quad \tilde{Z}_0^D := \begin{bmatrix} 0 \\ \tilde{F}_0^D \end{bmatrix},
\] (3.69)
we obtain an ONB in \( \mathcal{K}^\perp_\alpha \), and the application of (3.19) on these basis vectors gives
\[
J(e^q)\tilde{V}_{\epsilon_j}^D = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{E}_{\epsilon_j}^D - \sinh(\text{ad}_q)\tilde{F}_{\epsilon_j}^D \\ \sinh(\text{ad}_q)\tilde{E}_{\epsilon_j}^D + \tilde{F}_{\epsilon_j}^D \end{bmatrix}, \quad J(e^q)\tilde{W}_{\epsilon_j}^D = \frac{1}{\sqrt{2}} \begin{bmatrix} \tilde{E}_{\epsilon_j}^D + \sinh(\text{ad}_q)\tilde{F}_{\epsilon_j}^D \\ \sinh(\text{ad}_q)\tilde{E}_{\epsilon_j}^D - \tilde{F}_{\epsilon_j}^D \end{bmatrix}.
\] (3.70)
By using the relations (3.62) we see that
\[ J(e^q)\tilde{V}^D_{\epsilon j} = (1 + \sin(q_j))\tilde{V}^D_{\epsilon j}, \quad J(e^q)\tilde{W}^D_{\epsilon j} = (1 - \sin(q_j))\tilde{W}^D_{\epsilon j}. \] (3.71)

Since \( J(e^q)\tilde{Z}^D_0 = \tilde{Z}^D_0 \), we conclude that the only nontrivial matrix entries of \( J(e^q)|_{\mathcal{K}^\perp} \) and its inverse \( J(e^q)^{-1}|_{\mathcal{K}^\perp} \) are the following ones:

\[
\begin{align*}
\mathcal{B}_g(\tilde{V}^D_{\epsilon j}, J(e^q)\tilde{V}^D_{\epsilon j}) &= 1 + \sin(q_j), & \mathcal{B}_g(\tilde{W}^D_{\epsilon j}, J(e^q)\tilde{W}^D_{\epsilon j}) &= 1 - \sin(q_j), \\
\mathcal{B}_g(\tilde{V}^D_{\epsilon j}, J(e^q)^{-1}\tilde{V}^D_{\epsilon j}) &= \frac{1}{1 + \sin(q_j)}, & \mathcal{B}_g(\tilde{W}^D_{\epsilon j}, J(e^q)^{-1}\tilde{W}^D_{\epsilon j}) &= \frac{1}{1 - \sin(q_j)}, \\
\mathcal{B}_g(\tilde{Z}^D_0, J(e^q)\tilde{Z}^D_0) &= 1, & \mathcal{B}_g(\tilde{Z}^D_0, J(e^q)^{-1}\tilde{Z}^D_0) &= 1. \end{align*}
\] (3.72)

**Lemma 3.2** By using the identification \( \tilde{\Sigma} := \tilde{A} = \exp(\tilde{A}_+) \) with \( \tilde{A}_+ \) in (3.43), the second term of the reduced Laplacian (2.14) is given by

\[
\delta^{-\frac{r}{2}}\Delta_{\tilde{A}}(\delta^\perp) = \frac{(m-n)(r-s)}{2} \sum_{j=1}^n \frac{1}{\sin^2(q_j)} + \frac{4(s-n)^2 - 1}{2} \sum_{j=1}^n \frac{1}{\sin^2(2q_j)} - \frac{n(3m^2 + n^2 - 1)}{6}. \] (3.73)

**Proof.** Consider the function

\[
\mathcal{J} := \prod_{1 \leq k < l \leq n} \left[ \sin(q_k - q_l) \sin(q_k + q_l) \right]^\nu_k \prod_{j=1}^n \left[ \sin(q_j) \right]^{\nu_1} \prod_{j=1}^n \left[ \sin(2q_j) \right]^{\nu_2}, \] (3.74)

where the domain of the variables \( q_1, q_2, \ldots, q_n \) is such that all \( \sin \) functions are positive and \( \nu, \nu_1, \nu_2 \in \mathbb{R} \) are arbitrary parameters. Recall from [9] the identity

\[
\mathcal{J}^{-1} \sum_{a=1}^n \frac{\partial^2 \mathcal{J}}{\partial q_a^2} = \nu(\nu - 1) \sum_{1 \leq k < l \leq n} \left( \frac{1}{\sin^2(q_k - q_l)} + \frac{1}{\sin^2(q_k + q_l)} \right) + \nu_1(\nu_1 + 2\nu_2 - 1) \sum_{j=1}^n \frac{1}{\sin^2(q_j)} + 4\nu_2(\nu_2 - 1) \sum_{j=1}^n \frac{1}{\sin^2(2q_j)} \] (3.75)

\[- n \left[ (\nu_1 + 2\nu_2)^2 + 2\nu_1(\nu_1 + 2\nu_2)(n - 1) + \frac{2}{3} \nu^2(n - 1)(2n - 1) \right].
\]

By calculating \( \det(J(e^q)) \) using the above basis of \( \mathcal{K}^\perp \), it is easily obtained from (2.13) that \( \delta^\perp(e^q) \propto \mathcal{J}(q_1, q_2, \ldots, q_n) \) with

\[
\nu = 1, \quad \nu_1 = r - s, \quad \nu_2 = s - n + \frac{1}{2}. \] (3.76)

Taking into account (3.40), the required statement follows immediately. \( Q.E.D. \)

The subsequent formula is obtained by direct substitution since we have determined the matrix elements of \( J(e^q)^{-1} \) (cf. (2.11)). It will be used in Section 4, when we shall further inspect the reduced Laplace operator (2.14) in interesting cases.
Lemma 3.3 In terms of the above notations, the third term of the reduced Laplacian (2.14) takes the following form:

\[
\begin{align*}
&b^{\alpha,\beta}\rho(T_\alpha)\rho(T_\beta) = \frac{1}{2} \sum_{1 \leq j \leq \dim(M)} \rho(\hat{L}_j)^2 + \frac{1}{2} \sum_{j=1}^{n} \left( \frac{\rho(V_{2i,j})^2}{\sin^2(q_j)} + \frac{\rho(W_{2i,j})^2}{\cos^2(q_j)} \right) \\
&+ \frac{1}{2} \sum_{1 \leq k < l \leq n} \left( \frac{\rho(V_{e_k-e_l})^2}{\sin^2\left(\frac{\pi}{2} - \frac{\theta_{k-l}}{2}\right)} + \frac{\rho(W_{e_k-e_l})^2}{\cos^2\left(\frac{\pi}{2} - \frac{\theta_{k-l}}{2}\right)} \right) \\
&+ \frac{1}{2} \sum_{1 \leq k < l \leq n} \left( \frac{\rho(V_{e_k+e_l})^2}{\sin^2\left(\pi - \frac{\theta_{k-l}}{2}\right)} + \frac{\rho(W_{e_k+e_l})^2}{\cos^2\left(\pi - \frac{\theta_{k-l}}{2}\right)} \right) \\
&+ \frac{1}{2} \sum_{j=1}^{n} \sum_{d=1}^{r-n} \left( \frac{\rho(V_{t,j}^d)^2 + \rho(V_{t,j}^{\prime d})^2}{\sin^2\left(\frac{\pi}{2} - \frac{\theta_{t,j}}{2}\right)} + \frac{\rho(W_{t,j}^d) + \rho(W_{t,j}^{\prime d})^2}{\cos^2\left(\frac{\pi}{2} - \frac{\theta_{t,j}}{2}\right)} \right) \\
&+ \sum_{j=1}^{n} \sum_{d=1}^{s-n} \left( \frac{\rho(\tilde{V}_{t,j}^d)^2 + \rho(\tilde{V}_{t,j}^{\prime d})^2}{1 + \sin(q_j)} + \frac{\rho(\tilde{W}_{t,j}^d) + \rho(\tilde{W}_{t,j}^{\prime d})^2}{1 - \sin(q_j)} \right) \\
&+ \sum_{c=1}^{n} \sum_{d=1}^{r-s-n} \left( \rho(\tilde{Z}_{0,c,d}^d) + \rho(\tilde{Z}_{0,c,d}^{\prime d})^2 \right). \quad (3.77)
\end{align*}
\]

4 \textbf{BC}_n \textbf{ Sutherland models from the KKS ansatz}

In this section we study interesting examples of the quantum Hamiltonian reduction based on the Hermann action (3.2) on \( Y = U(N) \) associated with the involutions (3.31). The reductions correspond to certain \textsc{uirreps} \( \rho \) of the symmetry group \( G = U(N)_L \times U(N)_R = (U(r) \times U(s)) \times (U(m) \times U(n)) \). (4.1)

To describe them, we now briefly summarize our notations for the \textsc{uirreps} of \( U(n) \), for arbitrary \( n \). (See also Appendix A.) First, we have the \textsc{uirrep} \( (\Pi_\lambda, V_{\lambda}) \) of \( SU(n) \) in correspondence to any highest weight \( \lambda \in P_+(SU(n)) \), that can be written as \( \lambda = \sum_{i=1}^{n-1} a_i \varpi_i \) using the fundamental weights \( \varpi_i \) and integers \( a_i \in \mathbb{Z}_{\geq 0} \). A label \( \mu_n(\lambda) \in \{0, 1, \ldots, n-1\} \) is attached to the highest weight \( \lambda \) by the congruence relation

\[
\mu_n(\lambda) \equiv \sum_{k=1}^{n-1} ka_k \pmod{n} \quad \text{for} \quad \lambda = \sum_{i=1}^{n-1} a_i \varpi_i. \quad (4.2)
\]

It enters the equality \( \Pi_\lambda(e^{i\varphi}1_n) = e^{i\varphi \mu_n(\lambda)}\text{Id}_{V_\lambda} \). Then, for any \( k \in \mathbb{Z} \), the representation \( \Pi_\lambda \) of \( SU(n) \) extends to the representation \( \rho_{(k,\lambda)} \) of \( U(n) \) defined by

\[
\rho_{(k,\lambda)}(g) = \xi^{nk+\mu_n(\lambda)}\Pi_\lambda(g), \quad \forall \xi \in U(1), \forall g \in SU(n). \quad (4.3)
\]

Up to equivalence, all \textsc{uirreps} of \( U(n) \) are obtained in this way. The notation makes sense even for \( n = 1 \), by putting \( P_+(SU(1)) := \{0\} \), and we have \( \rho_{(k,0)}(g) = (\det g)^k \) (\( \forall g \in U(n) \)).
By letting \( \rho_{(k,\lambda)} \) and \( \pi_\lambda \) stand for the infinitesimal version of the representations \( \rho_{(k,\lambda)} \) and \( \Pi_\lambda \), respectively, we have

\[
\rho_{(k,\lambda)}(Z) = \pi_\lambda \left( Z - \frac{\text{tr}(Z)}{n} I_n \right) + (\mu_\lambda + nk) \frac{\text{tr}(Z)}{n} \text{Id}_{V_\lambda}, \quad \forall Z \in \mathfrak{u}(n). \tag{4.4}
\]

We use the notations \( \pi_\lambda^{(n)} \), \( V_\lambda^{(n)} \), \( \rho_{(k,\lambda)}^{(n)} \) etc. when considering various values of \( n \) simultaneously.

The UIRREPS of the direct product group \( G \) \eqref{4.1} have the form

\[
\rho = \left( \rho_{(k^1,\lambda^1_L)}^{(r)} \boxtimes \rho_{(k^2,\lambda^2_L)}^{(s)} \right) \boxtimes \left( \rho_{(k^1,\lambda^1_R)}^{(m)} \boxtimes \rho_{(k^2,\lambda^2_R)}^{(n)} \right), \tag{4.5}
\]

where \( \lambda^1_L, \lambda^2_L, \lambda^1_R, \lambda^2_R \) are the highest weights and \( k^1_L, k^2_L, k^1_R, k^2_R \in \mathbb{Z} \) according to \eqref{4.3}. The main problem is to find the UIRREPS \( (\rho, V) \) for which

\[
\dim(V^K) = 1, \tag{4.6}
\]

where \( K = M_{\text{diag}} < G \) is given by \eqref{3.32}. We investigate this problem by adopting the ansatz that one of the 4 constituent representations in \eqref{4.5} has the form \( \rho_{(k,a_1=1)}^{(l)} \) \( (l \in \{ r, s, m, n \}) \) and the other 3 constituent representations are one-dimensional. More exactly, \( \rho_{(k,a_1=1)}^{(l)} \) will be used for a factor of the maximal size, \( l = \max\{ r, s, m, n \} \). We call this assumption the KKS ansatz, since it eventually originates from the seminal paper by Kazhdan, Kostant and Sternberg \[14\].

The usefulness of this assumption is also supported by results in \[13, 19, 20\]. The key property is that all weight-multiplicities of \( \rho_{(k,a_1=1)}^{(l)} \) are equal to one. The analysis of the condition \eqref{4.6} is the easiest if the group \( K \) \eqref{3.42} is Abelian, which happens in the following cases:

- case I: \( m = r = s = n, \ N = 2n \),
- case II: \( m = r = n + 1, \ s = n, \ N = 2n + 1 \),
- case III: \( m = n + 2, \ r = s = n + 1, \ N = 2n + 2 \).

Next we describe the simplest case I in detail, then present the essential points for the other two cases. The complex holomorphic analogue of case I was studied in \[19\]; and the results are consistent. The other two cases of our KKS ansatz have not been investigated before.

Remark: The reader may wonder why we take \( l = \max\{ r, s, m, n \} \) in our KKS ansatz in cases II and III. In fact, we previously studied \( [20] \) and unpublished work) the classical Hamiltonian reductions of the free particle on \( U(N) \) based on the symmetry group \( (4.1) \) by using a minimal coadjoint orbit of positive dimension for any one of the 4 factors and one-point orbits for the other 3 factors. We found that this leads to the classical \( BC_n \) Sutherland model with three independent coupling constants only in the three cases mentioned above, and only if the minimal coadjoint orbit of positive dimension, \( 2(l - 1) \) for \( U(l) \), is associated with a factor of maximal size. The connection to quantum Hamiltonian reduction is clear from the relation between the coadjoint orbits of \( U(l) \) of dimension \( 2(l - 1) \) and the representations \( \rho_{(k,a_1=1)}^{(l)} \) (and their contragredients), which follows for example from geometric quantization.
4.1 Case I: \( m = r = s = n, \ N = 2n \)

Now \( \sigma_L = \sigma_R = \theta_{n,n} \) and \( U(N)_L = U(N)_R \cong U(n) \times U(n) \). The decomposition (3.33) of any matrix in \( \mathbb{C}^{N \times N} \) simplifies to a two by two block form with all 4 blocks having size \( n \times n \). We look for admissible UIRREPS \( \rho \) of \( G \) (4.11) by adopting the KKS ansatz

\[
\rho := \left( \rho^{(n)}_{(k_L^1,a_1 \varpi_1)} \boxtimes \rho^{(n)}_{(k_R^1,0)} \right) \boxtimes \left( \rho^{(n)}_{(k^1_R,0)} \boxtimes \rho^{(n)}_{(k^0_R,0)} \right),
\]

where \( a_1 \in \mathbb{Z}_{\geq 0}, k_L^1, k_R^1, k_R^0, k_R^2 \in \mathbb{Z} \) and the representation space is identified as

\[
V \equiv V^{(n)}_{a_1 \varpi_1}.
\]

Note that any element \( X \in \mathcal{G} \cong u(N)^{\sigma_L,+} \oplus u(N)^{\sigma_R,+} \) of the symmetry algebra \( \mathcal{G} \) can be realized as a pair \( X = (X_L, X_R) \) with \( X_L, X_R \in u(N)^{\sigma_L,+} = u(N)^{\sigma_R,+} = u(n) \oplus u(n) \). So, for any \( X \in \mathcal{G} \) we have the refined decomposition

\[
X = (X_L, X_R) = ((X^1_L, X^2_L), (X^1_R, X^2_R)),
\]

where \( X^1_L, X^2_L, X^1_R, X^2_R \in u(n) \) and as block-matrices

\[
(X^1_L, X^2_L) := \begin{bmatrix}
X^1_L & 0 \\
0 & X^2_L 
\end{bmatrix}, \quad (X^1_R, X^2_R) := \begin{bmatrix}
X^1_R & 0 \\
0 & X^2_R 
\end{bmatrix}.
\]

With these notations, the formula of the Lie algebra representation corresponding to (4.7) reads

\[
\rho^\prime(X) = \pi^{(n)}_{a_1 \varpi_1} (X^1_L - \frac{\text{tr}(X^1_L)}{n}1_n) + (k_L^1 + \frac{\mu_n(a_1 \varpi_1)}{n}) \text{tr}(X^1_L) + \text{tr}(k_R^1 X_L^2 + k_R^1 X_R^1 + k_R^2 X_R^2 ) \text{Id}_V.
\]

Lemma 4.1 The KKS ansatz (4.7) defines admissible UIRREPS of \( G \) satisfying \( \dim(V^K) \neq 0 \) if and only if \( k_L^1 + k_R^2 + k_R^1 + k_R^2 = 0 \) and \( a_1 = \gamma n \) with some \( \gamma \in \mathbb{Z}_{\geq 0} \). In these cases \( \dim(V^K) = 1 \). Using the bosonic oscillator realization of \( V \) (4.8) described in Appendix A, \( V^K \) has the form

\[
V^K \cong V^{(n)}_{\gamma n \varpi_1} [0] = \text{span}_\mathbb{C} \{ |\gamma, \gamma, \ldots, \gamma \rangle \}.
\]

Proof. The isotropy subalgebra is \( \mathcal{K} = \mathcal{M}_{\text{diag}} = \{ X = (X_0, X_0) \ | \ X_0 \in \mathcal{M} \} \), where \( \mathcal{M} \) can be parametrized as

\[
\mathcal{M} = \left\{ X_0 = \begin{bmatrix}
H + i x 1_n & 0 \\
0 & H + i x 1_n 
\end{bmatrix} \ \bigg| \ H \in \mathcal{H}^{(n)}_{x}, \ x \in \mathbb{R} \right\}.
\]

That is, for the components of any \( X \in \mathcal{K} \) we have the parametrization

\[
X^1_L = X^2_L = X^1_R = X^2_R = H + i x 1_n.
\]

Thus, using equation (4.11), for any \( v \in V^{(n)}_{a_1 \varpi_1} \) and \( X \in \mathcal{K} \) we can write

\[
\rho(X)v = \pi^{(n)}_{a_1 \varpi_1} (H)v + i x (\mu_n(a_1 \varpi_1) + n(k_L^1 + k_R^2 + k_R^1 + k_R^2)) v.
\]
In the same manner, the equalities we get
\[ \pi^{(n)}_{a_1\bar{\omega}_1}(H)v = 0 \quad (\forall H \in i\mathcal{T}^{(n)}_{a_1}) \quad \text{and} \quad \mu_n(a_1\bar{\omega}_1) + n(k_L^1 + k_L^2 + k_R^1 + k_R^2) = 0. \] (4.16)

Therefore \( V^K = V^0 = V_{a_1\bar{\omega}_1}^{(n)}[0] \), provided that \( \mu_n(a_1\bar{\omega}_1) + n(k_L^1 + k_L^2 + k_R^1 + k_R^2) = 0 \). It is easy to see that \( V_{a_1\bar{\omega}_1}^{(n)}[0] \neq \{0\} \) if and only if \( a_1 = \gamma n \) for some \( \gamma \in \mathbb{Z}_{\geq 0} \). Since \( \mu_n(\gamma n\bar{\omega}_1) = 0 \) by (4.12), the requirement \( k_L^1 + k_L^2 + k_R^1 + k_R^2 = 0 \) then also follows from (4.16). Finally, note that by using the oscillator realization of \( V_{\gamma n\bar{\omega}_1}^{(n)} \) one has the second equality in (4.12). \( \text{Q.E.D.} \)

In what follows we make use of the basis of \( \mathcal{K}^\perp \) constructed in subsection 3.3. In the present case this is given by the basis \( \{ V^a_{\alpha}, W^a_{\alpha} \}_{a \in \{r,i\}, \alpha \in \mathbb{R}_+} \) of \( \mathcal{K}^\perp \), together with the basis \( \{ \hat{L}_j \} \) of \( \mathcal{K}^\perp_\alpha \) defined according to (3.63) by using the following orthonormal basis \( \{ L_j \}_{j=1}^n \) of \( \mathcal{M} \):

\[ L_j := \frac{i}{\sqrt{2}} \begin{bmatrix} E_{jj} & 0 \\ 0 & -E_{jj} \end{bmatrix} \in \mathcal{M} \quad (1 \leq j \leq n). \] (4.17)

**Lemma 4.2** In the case of the KKS ansatz (4.17) subject to the conditions of Lemma 4.1 the third term in the reduced Laplacian (2.14) gives

\[
\begin{align*}
&\quad b^{a_3\beta} \rho(T_\alpha)J_\rho(T_\beta) = -\frac{1}{2} n(k_L^1 + k_L^2)^2 - \gamma(\gamma + 1) \sum_{1 \leq k < l \leq n} \left( \frac{1}{\sin^2(q_k - q_l)} + \frac{1}{\sin^2(q_k + q_l)} \right) \\
&\quad - \frac{(k_L^1 + k_R^2)^2 - (k_L^2 + k_R^1)^2}{2} \sum_{j=1}^n \frac{1}{\sin^2(q_j)} - 2(k_L^2 + k_R^1)^2 \sum_{j=1}^n \frac{1}{\sin^2(2q_j)}. \quad (4.18)
\end{align*}
\]

**Proof.** Note that in the present case only the first 4 sums occur in the formula (3.77). Recalling that \( \mu_n(\gamma n\bar{\omega}_1) = 0 \) and utilizing formula (4.11) for \( \rho \), we can calculate the action of the various terms. For example, since

\[ \hat{L}_j = \frac{1}{\sqrt{2}} (L_j, -L_j) = \frac{i}{2} \left( (E_{jj}, E_{jj}), (-E_{jj}, -E_{jj}) \right), \] (4.19)

we get

\[ \rho(\hat{L}_j) = \frac{i}{2} \left( \pi^{(n)}_{\gamma n\bar{\omega}_1} \left( E_{jj} - \frac{1}{n} 1_n \right) + (k_L^1 + k_L^2 - k_R^1 - k_R^2)I_{\mathcal{V}} \right). \] (4.20)

The action of \( \rho(\hat{L}_j) \) on \( V^K \) can be easily calculated in the bosonic oscillator picture. Since \( \pi^{(n)}_{\gamma n\bar{\omega}_1} \left( E_{jj} - \frac{1}{n} 1_n \right) \mid \gamma, \gamma, \ldots, \gamma \rangle = 0, \) and since \( k_R^2 = -k_L^1 - k_L^2 - k_R^1, \) it follows that on the subspace \( V^K = \text{span}_{\mathbb{C}} \{ \mid \gamma, \gamma, \ldots, \gamma \rangle \} \) the operator \( \rho(\hat{L}_j) \) acts as the scalar \( \rho(\hat{L}_j) = i(k_L^1 + k_L^2). \)

In the same manner, the equalities \( \rho(V^r_{\bar{e}_k}) = i(k_L^1 + k_L^2) \) and \( \rho(W^r_{\bar{e}_k}) = -i(k_L^1 + k_L^2) \) hold on \( V^K \). Furthermore, we have on \( V \)

\[
\begin{align*}
\rho(V^r_{\bar{e}_k}) &= \rho(W^r_{\bar{e}_k}) = \rho(V^r_{\bar{e}_k} + \epsilon_l) = \rho(W^r_{\bar{e}_k} + \epsilon_l) = \frac{1}{2\sqrt{2}} \left( \pi^{(n)}_{\gamma n\bar{\omega}_1} (E_{kk}) - \pi^{(n)}_{\gamma n\bar{\omega}_1} (E_{lk}) \right), \quad (4.21) \\
\rho(V^i_{\bar{e}_k} - \epsilon_i) &= \rho(W^i_{\bar{e}_k} - \epsilon_i) = \rho(V^i_{\bar{e}_k} + \epsilon_i) = \rho(W^i_{\bar{e}_k} + \epsilon_i) = \frac{i}{2\sqrt{2}} \left( \pi^{(n)}_{\gamma n\bar{\omega}_1} (E_{kk}) + \pi^{(n)}_{\gamma n\bar{\omega}_1} (E_{lk}) \right). \quad (4.22)
\end{align*}
\]
Next, \( \forall k, l \in \{1, 2, \ldots, n\}, k \neq l \), we obtain
\[
\pi_{\gamma \neq \pi_1}^{(n)}(E_k) \pi_{\gamma \neq \pi_1}^{(n)}(E_k) |\gamma, \gamma, \ldots, \gamma\rangle = b_k^\dagger b_l^\dagger b_k b_l |\gamma, \gamma, \ldots, \gamma\rangle = \gamma(\gamma + 1) |\gamma, \gamma, \ldots, \gamma\rangle. \tag{4.23}
\]
The above equations imply that on \( V^K \)
\[
\rho(\hat{L}_j)^2 = -(k_L^1 + k_L^2)^2, \quad \rho(\hat{V}_{2\epsilon_j})^2 = -(k_L^1 + k_R^2)^2, \quad \rho(\hat{W}_{2\epsilon_j})^2 = -(k_L^2 + k_R^2)^2, \tag{4.24}
\]
\[
\rho(\hat{V}_{\alpha})^2 + \rho(\hat{V}_{\alpha}^1)^2 = \rho(\hat{W}_{\alpha})^2 + \rho(\hat{W}_{\alpha}^1)^2 = -\frac{1}{2} \gamma(\gamma + 1) \quad \text{for} \quad \alpha = \epsilon_k \pm \epsilon_l, \quad k \neq l.
\tag{4.25}
\]
Now (4.18) results by substitution into (3.77), using obvious trigonometric identities. \( Q.E.D. \)

The following proposition is obtained by putting together the statements of equation (3.46), Lemma 3.2 and Lemma 4.2.

**Proposition 4.3** Under the KKS ansatz (4.7) the general formula (2.14) gives the following result for the reduction of the Laplace operator of \( U(N) \):
\[
- \Delta_{\text{red}} = H_{BCN} + \frac{1}{2} n(k_L^1 + k_L^2)^2 - \frac{1}{6} n(2n - 1)(2n + 1), \tag{4.26}
\]
where \( H_{BCN} \) is the Sutherland Hamiltonian (1.1) with the coupling parameters defined by
\[
a \equiv \gamma, \quad b \equiv |k_L^1 + k_L^2|, \quad c \equiv |k_L^2 + k_R^2| \quad \text{for} \quad k \neq l \in \mathbb{Z}\]

Remark: By varying \( \gamma, k_L^1, k_L^2, k_R^1 \), the coupling parameters \( a, b, c \) in (4.3) can take arbitrary non-negative integer values. As further discussed in Section 5, Proposition 4.3 follows also from the results of Oblomkov [19].

### 4.2 Case II: \( m = r = n + 1, \quad s = n, \quad N = 2n + 1 \)

In this case \( \sigma_L = \sigma_R = \theta_{n+1,n} \) and correspondingly \( U(N)_L = U(N)_R \cong U(n + 1) \times U(n) \). We consider the following ansatz for the UIRREP \((\rho, V)\) of the symmetry group \( G (4.1) \),
\[
\rho := \left( \rho^{(n+1)}_{(k_L^1, a_1 \pi_1)} \boxtimes \rho^{(n)}_{(k_L^2, 0)} \right) \boxtimes \left( \rho^{(n+1)}_{(k_R^1, 0)} \boxtimes \rho^{(n)}_{(k_R^2, 0)} \right), \tag{4.28}
\]
where \( a_1 \in \mathbb{Z}_{\geq 0}, \quad k_L^1, k_L^2, k_R^1, k_R^2 \in \mathbb{Z} \) and the carrier space is identified as \( V \equiv V^{(n+1)}_{a_1 \pi_1} \). Similarly to (4.3), any \( X \in G \cong u(N)^{\sigma_L \perp} \oplus u(N)^{\sigma_R \perp} \) can be realized as a pair \( X = (X_L, X_R) \) with \( X_L, X_R \in u(N)^{\sigma_L \perp} = u(N)^{\sigma_R \perp} \cong u(n + 1) \oplus u(n) \). So, we write \( X \in G \) as \( X = (X_L, X_R) = ((X_L^1, X_L^2), (X_R^1, X_R^2)) \) with \( X_L^1, X_L^2 \in u(n + 1), \quad X_R^1, X_R^2 \in u(n) \). Then (4.28) implies the formula
\[
\rho(\hat{X}) = \pi^{(n+1)}_{a_1 \pi_1} \left( X_L^1 - \frac{\text{tr}(X_L^1)}{n+1} 1_{n+1} \right)
+ \left( \frac{k_L^1 + \mu_{n+1}(a_1 \pi_1)}{n+1} \right) \text{tr}(X_L^1) + k_L^2 \text{tr}(X_L^2) + k_R^1 \text{tr}(X_R^1) + k_R^2 \text{tr}(X_R^2) \right) \text{Id}_V. \tag{4.29}
\]
Lemma 4.4 The KKS ansatz (4.28) yields admissible UIRREPS of $G$ if and only if $\exists \gamma, \tilde{\gamma} \in \mathbb{Z}_{\geq 0}$ such that the parameters $k_1^L, k_2^L, k_1^R, k_2^R \in \mathbb{Z}$, and $\alpha_1 \in \mathbb{Z}_{\geq 0}$ satisfy the conditions
\begin{equation}
\alpha_1 = \gamma n + \tilde{\gamma}, \quad k_2^L + k_2^R = \tilde{\gamma} - \gamma, \quad k_1^L + k_1^R = R - (\tilde{\gamma} - \gamma),
\end{equation}
where $\tilde{\gamma} - \gamma = Q + (n + 1)R$ with uniquely determined $Q = Q(\gamma, \tilde{\gamma}) \in \{0, 1, \ldots, n\}$ and $R = R(\gamma, \tilde{\gamma}) \in \mathbb{Z}$. If these conditions hold, then $\dim(V^K) = 1$ and $V^K$ is given by
\begin{equation}
V^K \cong V_{a_1\omega_1}^{(n+1)}[\gamma e_1 + \gamma e_2 + \cdots + \gamma e_n + \gamma e_{n+1}] = \text{span}_\mathbb{C}\{[\gamma, \gamma, \ldots, \gamma, \tilde{\gamma}]\},
\end{equation}
where the last equality refers to the bosonic oscillator realization of $V_{a_1\omega_1}^{(n+1)}$.

Proof. For the isotropy subalgebra we have $\mathcal{K} = \mathcal{M}_{\text{diag}} = \{X = (X_0, X_0) | X_0 \in \mathcal{M}\}$, where
\begin{equation}
\mathcal{M} = \left\{ X_0 = i\begin{bmatrix} D & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & D \end{bmatrix} \middle| D = \text{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{R}^{n \times n}, \omega \in \mathbb{R} \right\}.
\end{equation}
So, for any $X \in \mathcal{K}$ we have $X_L = X_R = X_0$, and
\begin{equation}
X_1^L = X_1^R = i\begin{bmatrix} D & 0 \\ 0 & \omega \end{bmatrix}, \quad X_2^L = X_2^R = iD.
\end{equation}
Now, for each $\phi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in \mathbb{R}^n$ we let $\bar{\phi} := \sum_{j=1}^n \varphi_j$, and consider the traceless Cartan elements
\begin{equation}
H_\varphi := \text{diag}(\varphi_1, \varphi_2, \ldots, \varphi_n, -\bar{\phi}) \in \mathcal{H}_\mathbb{R}^{(n+1)}, \quad \tilde{H}_\varphi := \text{diag}(\varphi_1, \varphi_2, \ldots, \varphi_n) - \frac{1}{n} \bar{\phi} \mathbb{1}_n \in \mathcal{H}_\mathbb{R}^{(n)}.
\end{equation}
Then the components of $X \in \mathcal{K}$ can be parametrized as
\begin{equation}
X_1^L = X_1^R = iH_\varphi + i\mathbb{1}_{n+1}, \quad X_2^L = X_2^R = i\tilde{H}_\varphi + i \left( x + \frac{1}{n} \bar{\phi} \right) \mathbb{1}_n,
\end{equation}
where $\varphi \in \mathbb{R}^n$ and $x \in \mathbb{R}$. From (4.29) it follows that $\forall v \in V_{a_1\omega_1}^{(n+1)}$ we have
\begin{equation}
\rho(X)v = \pi_{a_1\omega_1}^{(n+1)}(iH_\varphi)v + i(k_1^L + k_2^R)\bar{\varphi}v + i(x + \frac{1}{n} \bar{\phi})\mathbb{1}_n \cdot v.
\end{equation}
Clearly $\rho(X)v = 0 \ (\forall X \in \mathcal{K})$ if and only if
\begin{equation}
\pi_{a_1\omega_1}^{(n+1)}(H_\varphi)v = -(k_2^L + k_2^R)\bar{\varphi}v \ (\forall \varphi \in \mathbb{R}^n),
\end{equation}
and $\mu_{a_1\omega_1}^{(n+1)}(a_1\omega_1) + (n + 1)(k_1^L + k_1^R) + n(k_2^L + k_2^R) = 0$. Note that $\bar{\varphi} = \sum_{j=1}^n \varphi_j = \sum_{j=1}^n \epsilon_j(H_\varphi)$, so after introducing the shorthand notations
\begin{equation}
\kappa_1 := k_1^L + k_1^R \in \mathbb{Z} \quad \text{and} \quad \kappa_2 := k_2^L + k_2^R \in \mathbb{Z},
\end{equation}
we conclude that
\begin{equation}
V^K = V^K \cong V_{a_1\omega_1}^{(n+1)}[-\kappa_2 \sum_{j=1}^n \epsilon_j],
\end{equation}
where
\begin{equation}
\kappa_2 \in \mathbb{Z}.\]
provided that \(\mu_{n+1}(a_1 \varpi_1) + (n + 1)\kappa_1 + n\kappa_2 = 0\). Our next goal is to identify the weight space 
\[V_{a_1 \varpi_1}^{(n+1)}[-\kappa_2(e_1 + e_2 + \cdots + e_n)].\] 
Recall that 
\[-\kappa_2(e_1 + e_2 + \cdots + e_n) \in W_{a_1 \varpi_1}^{(n+1)}\] if and only if 
\[\exists (l_1, l_2, \ldots, l_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1} \text{ with } l_1 + l_2 + \cdots + l_{n+1} = a_1,\] 
such that
\[-\kappa_2(e_1 + e_2 + \cdots + e_n) = \sum_{j=1}^{n+1} l_j e_j = \sum_{j=1}^{n} (l_j - l_{n+1})e_j. \tag{4.40}\]

Since the functionals \(e_1, e_2, \ldots, e_n\) are linearly independent, we end up with the requirement
\[l_1 = l_2 = \cdots = l_n = l_{n+1} - \kappa_2.\]
For the free parameters we choose \(\gamma := l_1\) and \(\tilde{\gamma} := l_{n+1}\), then the parameters \(\kappa_2 = k_L^2 + k_R^2\) and \(a_1\) have to obey the equations \(\kappa_2 = \tilde{\gamma} - \gamma\) and \(a_1 = \gamma n + \tilde{\gamma}\). Noted that under these assumptions we have
\[V_{a_1 \varpi_1}^{(n+1)}[-\kappa_2(e_1 + e_2 + \cdots + e_n)] = V_{a_1 \varpi_1}^{(n+1)}[\gamma e_1 + \gamma e_2 + \cdots + \gamma e_n + \tilde{\gamma} e_{n+1}] = \text{span}_\mathbb{C}\{[\gamma, \gamma, \ldots, \gamma, \tilde{\gamma}]\}. \tag{4.41}\]

Now let us express the value of the label \(\mu_{n+1}(a_1 \varpi_1) \in \{0, 1, \ldots, n\}\) in terms of \(\gamma\) and \(\tilde{\gamma}\).
Recalling \((4.23)\), we can write
\[\mu_{n+1}(a_1 \varpi_1) = \mu_{n+1}((\gamma n + \tilde{\gamma}) \varpi_1) \equiv \gamma n + \tilde{\gamma} \equiv \tilde{\gamma} - \gamma \pmod{(n + 1)}. \tag{4.42}\]

Notice that \(\exists! Q = Q(\gamma, \tilde{\gamma}) \in \{0, 1, \ldots, n\}\) and \(\exists! R = R(\gamma, \tilde{\gamma}) \in \mathbb{Z}\) such that 
\[
\tilde{\gamma} - \gamma = Q + (n + 1)R,
\]
thereby the previous congruence relation translates into the equation 
\(\mu_{n+1}(a_1 \varpi_1) = Q\). Plugging this equation into the requirement 
\(\mu_{n+1}(a_1 \varpi_1) + (n + 1)\kappa_1 + n\kappa_2 = 0\), we get
\[0 = Q + (n + 1)\kappa_1 + n(Q + (n + 1)R) = (n + 1)(\tilde{\gamma} - \gamma - R + \kappa_1), \tag{4.43}\]
therefore we end up with the additional constraint \(k_L^1 + k_R^1 = \kappa_1 = R - (\tilde{\gamma} - \gamma)\). \(Q.E.D.\)

Observe from Lemma 4.4 that \(k_L^1, k_R^2 \in \mathbb{Z}\) and \(\gamma, \tilde{\gamma} \in \mathbb{Z}_{\geq 0}\) can be taken as free parameters that label the admissible cases of the KKS ansatz \((4.28)\). By proceeding like in subsection 4.1, it is matter of straightforward substitutions to specialize the reduced Laplacian \((2.11)\) to our case. In this way we found the following result.

**Proposition 4.5** Under the KKS ansatz \((4.28)\) with parameters satisfying \((4.30)\) the Laplace operator of \(U(N)\) reduces to
\[
- \Delta_{\text{red}} = H_{BC_a} + \frac{1}{2}n(k_L^1 + k_R^2)^2 + (k_R^1)^2 - \frac{1}{3}n(n + 1)(2n + 1), \tag{4.44}\]
where \(H_{BC_a}\) is given by \((1.4)\) with the coupling parameters determined in terms of the arbitrary parameters \(k_L^1, k_R^2 \in \mathbb{Z}\) and \(\gamma, \tilde{\gamma} \in \mathbb{Z}_{\geq 0}\) according to
\[a \equiv \gamma, \quad b \equiv \gamma + \tilde{\gamma} + 1, \quad c \equiv |\tilde{\gamma} - \gamma + k_R^1 - k_R^2|. \tag{4.45}\]

**Remark:** The non-negative integer coupling parameters \(a, b, c\) that arise in this case satisfy the condition \(b \geq a + 1\).
4.3 Case III: \( m = n + 2 \), \( r = s = n + 1 \), \( N = 2n + 2 \)

Now the fixpoint subgroups of the two different involutions \( \sigma_L = \theta_{n+1,n+1} \) and \( \sigma_R = \theta_{n+2,n} \) are \( U(N)_L \cong U(n + 1) \times U(n + 1) \) and \( U(N)_R \cong U(n + 2) \times U(n) \). We consider the reductions associated with UIRREPS (\( \rho, V \)) of \( G \) (4.4) having the form

\[
\rho := \left( \rho_{(k^1_L, 0)}^{(n+1)} \boxtimes \rho_{(k^2_L, 0)}^{(n+1)} \right) \boxtimes \left( \rho_{(k^1_R, a_1 a_1)}^{(n+2)} \boxtimes \rho_{(k^2_R, 0)}^{(n)} \right),
\]

where \( a_1 \in \mathbb{Z}_{\geq 0} \) and \( k^1_L, k^2_L, k^1_R, k^2_R \in \mathbb{Z} \), and the representation space is identified as \( V \equiv V_{a_1 a_1}^{(n+2)} \).

Any \( X \in G \) is a pair \( X = (X_L, X_R) \) with \( X_L \in U(n + 1) \oplus u(n + 1) \) and \( X_R \in U(n + 2) \oplus u(n) \), and we may further write \( X_L = (X^1_L, X^2_L) \) and \( X_R = (X^1_R, X^2_R) \), where now \( X^1_L, X^2_L \in U(n + 1) \) and \( X^2_R \in u(n) \). Then the \( G \)-representation can be written as

\[
\rho(X) = \pi_{a_1 a_1}^{(n+2)}(X^1_R - \frac{\text{tr}(X^1_R)}{n + 2}1_{n+2}) + \left[ k^1_L \text{tr}(X^1_L) + k^2_L \text{tr}(X^2_L) + \left( k^1_R + \frac{\mu_{n+2}(a_1 a_1)}{n + 2} \right) \text{tr}(X^1_R) + k^2_R \text{tr}(X^2_R) \right] \text{Id}_V.
\]

**Lemma 4.6** The KKS ansatz (4.46) yields admissible UIRREPS if and only if \( \exists \gamma, \tilde{\gamma}, \check{\gamma} \in \mathbb{Z}_{\geq 0} \) and \( k \in \mathbb{Z} \) such that the parameters \( k^1_L, k^2_L, k^1_R, k^2_R \in \mathbb{Z} \) and \( a_1 \in \mathbb{Z}_{\geq 0} \) satisfy the conditions

\[
a_1 = \gamma n + \tilde{\gamma} + \check{\gamma}, \quad k^1_L = k, \quad k^2_L = \check{\gamma} - \gamma + k, \quad k^1_R = R - \tilde{\gamma} - k, \quad k^2_R = \check{\gamma} - \gamma - k,
\]

where \( a_1 = Q + (n + 2)R \) with uniquely determined \( Q = Q(\gamma, \tilde{\gamma}, \check{\gamma}) \in \{0, 1, \ldots, n + 1\} \) and \( R = R(\gamma, \tilde{\gamma}, \check{\gamma}) \in \mathbb{Z} \). If the above conditions are met, then \( \dim(V^K) = 1 \) and concretely

\[
V^K = V_{a_1 a_1}^{(n+2)}[\gamma e_1 + \gamma e_2 + \cdots + \gamma e_n + \tilde{\gamma} e_{n+1} + \check{\gamma} e_{n+2}] = \text{span}_\mathbb{C}\{[\gamma, \gamma, \ldots, \gamma, \tilde{\gamma}, \check{\gamma}]\},
\]

where the last equality refers to the bosonic oscillator realization of \( V_{a_1 a_1}^{(n+2)} \).

**Proof.** For the isotropy subalgebra we have \( \mathcal{K} = \mathcal{M}_{\text{diag}} = \{ X = (X_0, X_0) \mid X_0 \in \mathcal{M} \} \), where

\[
\mathcal{M} = \left\{ X_0 = \begin{bmatrix} D & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & D \end{bmatrix} \mid D = \text{diag}(d_1, d_2, \ldots, d_n) \in \mathbb{R}^{n \times n}, \omega, \tilde{\omega} \in \mathbb{R} \right\}.
\]

Any \( X = (X_L, X_R) \in \mathcal{K} \) satisfies \( X_L = X_R = X_0 \), and therefore it has the components

\[
X^1_L = i \begin{bmatrix} D & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}, \quad X^2_L = i \begin{bmatrix} \tilde{\omega} & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & \tilde{\omega} \end{bmatrix}, \quad X^1_R = i \begin{bmatrix} D & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{bmatrix}, \quad X^2_R = iD.
\]

For any real \((n + 1)\)-tuple \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{n+1}) \in \mathbb{R}^{n+1} \) we let \( \tilde{\varphi} := \sum_{j=1}^{n+1} \varphi_j, \check{\varphi} := \sum_{j=1}^{n} \varphi_j \), and introduce the traceless matrices

\[
H_\varphi := \text{diag}(\varphi_1, \varphi_2, \ldots, \varphi_{n+1}, -\tilde{\varphi}), \quad H^2_R := \text{diag}(\varphi_1, \varphi_2, \ldots, \varphi_n) - \frac{\check{\varphi}}{n}1_n,
\]

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We then write the components of $X \in K$ in the form

$$X_1^1 = iH_\phi + i x 1_{n+1}, \quad X_2^1 = iH_R^1 + i \left(x + \frac{\bar{\varphi}}{n+1}\right) 1_{n+1}. \quad (4.54)$$

$$X_1^2 = iH_L^1 + i \left(x + \frac{\bar{\varphi}}{n+1}\right) 1_{n+1}, \quad X_2^2 = iH_R^2 + i \left(x - \frac{\bar{\varphi} + 1}{n+1}\right) 1_{n+1}. \quad (4.55)$$

From (4.47) it follows that for any $v \in V_{a_1 \omega_1}^{(n+2)}$ and $X \in K$ we have

$$\rho(X)v = \pi^{(n+2)}_{a_1 \omega_1}(iH_\phi)v + i(k_1^L \varphi - k_2^L \bar{\varphi} + k_2^R \bar{\varphi})v + i(x(\mu_{n+2}(a_1 \omega_1) + (n + 2)k_1^R) + (n + 1)(k_1^L + k_2^L) + nk_2^R)v. \quad (4.56)$$

Clearly $\rho(X)v = 0$ ($\forall X \in K$) if and only if

$$\pi^{(n+2)}_{a_1 \omega_1}(H_\phi)v = (k_2^L \varphi - k_1^L \bar{\varphi} + k_2^R \bar{\varphi})v \quad (\forall \varphi \in \mathbb{R}^n), \quad (4.57)$$

and

$$\mu_{n+2}(a_1 \omega_1) + (n + 2)k_1^R + (n + 1)(k_1^L + k_2^L) + nk_2^R = 0. \quad (4.58)$$

Since

$$k_2^L \varphi + k_1^L \bar{\varphi} + k_2^R \bar{\varphi} = -(k_1^L + k_2^R)(e_1 + e_2 + \cdots + e_n)(H_\phi) + (k_2^L - k_1^L)e_{n+1}(H_\phi), \quad (4.59)$$

we obtain from (4.57) that we must have

$$V^K = V_{a_1 \omega_1}^{(n+2)}[-(k_1^L + k_2^R)(e_1 + e_2 + \cdots + e_n) + (k_2^L - k_1^L)e_{n+1}] \quad (4.60)$$

It is easy to see (cf. Appendix A) that the weight space in (4.60) is non-trivial if and only if $\exists (l_1, l_2, \ldots, l_{n+2}) \in \mathbb{Z}_{\geq 0}^{n+2}$ with $l_1 + l_2 + \cdots + l_{n+2} = a_1$, such that

$$-(k_1^L + k_2^R)(e_1 + e_2 + \cdots + e_n) + (k_2^L - k_1^L)e_{n+1} = \sum_{j=1}^{n+1} (l_j - l_{n+2})e_j. \quad (4.61)$$

We set

$$\gamma := l_1, \quad \tilde{\gamma} := l_{n+1}, \quad \hat{\gamma} := l_{n+2}, \quad k := k_2^L. \quad (4.62)$$

Then (4.61) requires $l_1 = l_2 = \cdots = l_n = \gamma$ and $\tilde{\gamma} - \gamma = k + k_2^R$ with $\hat{\gamma} - \gamma = k_1^L - k$. So, regarding $\gamma, \tilde{\gamma}, \hat{\gamma} \in \mathbb{Z}$ and $k \in \mathbb{Z}$ as free parameters, we see that the other parameters have to obey the relations

$$k_2^L = \tilde{\gamma} - \gamma + k, \quad k_2^R = \hat{\gamma} - \gamma - k, \quad a_1 = \gamma n + \tilde{\gamma} + \hat{\gamma}. \quad (4.63)$$

To satisfy the remaining condition (4.58), we now define $Q = Q(\gamma, \tilde{\gamma}, \hat{\gamma}) \in \{0, 1, \ldots, n+1\}$ and $R = R(\gamma, \tilde{\gamma}, \hat{\gamma}) \in \mathbb{Z}$ by the equality

$$a_1 = \gamma n + \tilde{\gamma} + \hat{\gamma} = Q + (n + 2)R. \quad (4.64)$$

Then (4.58) translates into the condition $k_1^R = R - \hat{\gamma} - k$, which completes the proof. Q.E.D.

Further direct calculations yield the explicit form of the reduced Laplacian (2.14).

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Proposition 4.7  Under the KKS ansatz (4.46) parametrized by arbitrary $\gamma, \tilde{\gamma}, \hat{\gamma} \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}$ according to Lemma 4.6, the reduced Laplacian of $U(N)$ satisfies $-\Delta_{\text{red}} = H_{B_{C_n}} + C$ with the constant

$$C = -\frac{1}{6}n(4n^2 + 12n + 11) + \frac{1}{2}n(2k + \tilde{\gamma} - \hat{\gamma})^2 + (\tilde{\gamma} + k)(\tilde{\gamma} + k + 1) + (\hat{\gamma} - k)(\hat{\gamma} - k + 1)$$

(4.65)

and coupling parameters given in the notation (1.4) by

$$a \equiv \gamma, \quad b \equiv \gamma + \tilde{\gamma} + 1, \quad c \equiv \gamma + \hat{\gamma} + 1.$$  

(4.66)

Remark: The integer coupling parameters $a, b, c$ arising in this case satisfy $b, c \geq a + 1$.

5 Discussion

We here summarize the results, discuss the related work [19] and point out open problems.

In this paper we applied the formalism of quantum Hamiltonian reduction under polar group actions to study the reductions of the Laplace operator of $U(N)$ by means of the Hermann action (3.2) of the symmetry group $G = (U(r) \times U(s)) \times (U(m) \times U(n))$ with $N = m + n = r + s$. We concentrated on the 3 series of cases for which the centralizer of the corresponding section, the group $K = M_{\text{diag}}$ (3.42), is Abelian. We built the representation $(\rho, V)$ of the symmetry group that enters the definition of the reduction by using as building blocks in (4.5) one-dimensional representations and a symmetric power of the defining representation of the ‘largest’ factor of $G$. In the framework of this ‘KKS ansatz’ we determined all cases for which the reduction is consistent (that is dim($V^K$) $\neq 0$), and saw also that in these admissible cases dim($V^K$) = 1. We then calculated the explicit formula of the reduced Laplacian by specializing equation (2.14), and found that up to an additive constant it yields the $B_{C_n}$ Sutherland Hamiltonian (1.4) with coupling parameters given as follows:

- case I: $a, b, c \in \mathbb{Z}_{\geq 0}$,
- case II: $a, b, c \in \mathbb{Z}_{\geq 0}$ with $b \geq a + 1$,
- case III: $a, b, c \in \mathbb{Z}_{\geq 0}$ with $b, c \geq a + 1$.

The dependence of the additive constant and of the coupling parameters $a, b, c$ on the parameters of the respective representation $(\rho, V)$ is given by the 3 propositions formulated in section 4.

The above results show that case I, which is the simplest case, covers all integral values of the coupling parameters $a, b, c$ and the other two cases allow for alternative group theoretic descriptions of the $B_{C_n}$ model at proper subsets of the integral coupling parameters. This state of affairs could not be foreseen before performing the analysis of the different reduction schemes. Observe also that if $b = c$, then the Hamiltonian (1.4) becomes of type $C_n$, but the $B_n$ and $D_n$ type Sutherland models do not arise from (1.4) at any values of the integers $a, b, c$. This is in contrast with the corresponding classical Hamiltonian reduction [20], which covers all coupling constants of the classical $B_{C_n}$ model, and is due to the never vanishing second
term of the ‘measure factor’ given by (3.73). The measure factor represents a kind of quantum anomaly since it gives the difference between the naive quantization of the reduced classical Hamiltonian and the outcome of the corresponding quantum Hamiltonian reduction [21].

In case I, our analysis is consistent with the results of Oblomkov [19], who studied reductions of the Laplace operator of $GL(m + n, \mathbb{C})$ using the symmetry group

$$G^C := (GL(m, \mathbb{C}) \times GL(n, \mathbb{C})) \times (GL(m, \mathbb{C}) \times GL(n, \mathbb{C})), \quad m \geq n. \quad (5.1)$$

In fact, in case I our reduction is nothing but the compact real form of the reduction studied in [19] for $m = n$. For the $m > n$ cases of the symmetry group (5.1) a generalization of the KKS ansatz was employed in [19], which was found to yield the complex version of the $BC_n$ Sutherland Hamiltonian (1.4) with integer coupling parameters subject to the restriction $c \geq b - (m - n) \geq 0$. Thus the coupling parameters obtained for $m > n$ form a proper subset of those obtained for $m = n$, and this proper subset is different from those that we derived in our cases II and III. For clarity we note that the KKS ansatz (4.28) that we adopted in case II was motivated by the corresponding classical reduction [20], and it does not correspond to the ansatz used in [19] for $m - n = 1$. It is not clear to us how the classical analogues of the $m > n$ reductions of [19] work.

Of course, the reductions can be applied also to the differential operators associated with the higher Casimirs. This can be used to explain the complete integrability of the $BC_n$ Sutherland model and to derive the spectra as well as the form of the joint eigenfunctions of the corresponding commuting Hamiltonians at the pertinent values of the coupling constants from representation theory [19].

We stress that the general method that we applied in our analysis can be used also to study other problems in the future. For example, one may try to determine all possible values of the coupling constants of the Sutherland models (1.1) that may result as reductions of the Laplacian of a compact Lie group in general. This is closely related to the open problem concerning the classification of the Hermann actions and representations $(\rho, V)$ of symmetric subgroups $G$ (3.1) such that the condition $\dim(V^K) = 1$ holds for the centralizer $K < G$ of the section. In all such cases the reduced Laplace operator (2.13) is expected to provide a many-body model that can be solved by the group theoretic method because of its very origin.

Besides the trigonometric real form that we considered, the complex $BC_n$ Sutherland model admits the well known hyperbolic real form and other physically very different real forms associated with two types of particles [27, 28]. The derivation of the hyperbolic model by quantum Hamiltonian reduction can be done similarly to the present work, but starting from $U(n, n)$ instead of $U(2n)$ (in case I) taking the Cartan involution both for $\sigma_L$ and for $\sigma_R$ (see also [20]). The models with two types of particles pose a more difficult problem. At the classical level, it can be seen from [28] that to derive them one needs to take the Cartan involution of $U(n, n)$ for $\sigma_L$ and a different involution for $\sigma_R$ that has a non-compact fixpoint subgroup. Therefore the corresponding quantum Hamiltonian reduction would require some modifications of the method used in this paper, which need further investigation.
A Some representation theoretic facts

In this appendix we gather some basic facts in order to fix the notations used in Section 4.

\section*{A.1 On the UIRREPS of SU(n) and U(n)}

Since the Lie group SU(n) is compact, connected and simply-connected, there is a one-to-one correspondence between the UIRREPS (\(\Pi, V\)) of SU(n) and the finite dimensional complex IRREPS (\(\pi, V\)) of \(\mathfrak{sl}(n, \mathbb{C}) = \mathfrak{su}(n)^{\mathbb{C}}\). In the complex simple Lie algebra \(\mathfrak{sl}(n, \mathbb{C})\) we have the Cartan subalgebra \(\mathcal{H}\) consisting of diagonal matrices, and use also the real Cartan subalgebra

\[ \mathcal{H}_{\mathbb{R}} := \{H \mid H \in \mathfrak{sl}(n, \mathbb{C}), \quad H \text{ is diagonal with real entries} \} \subset \mathcal{H}. \quad (A.1) \]

The functionals \(\{e_i\}_{i=1}^n \subset \mathcal{H}^*\) are defined by the formula \(e_i(H) := H_{ii} (H \in \mathcal{H})\). The roots with respect to \(\mathcal{H}\) form the set \(\mathcal{R} := \{e_i - e_j | 1 \leq i, j \leq n, i \neq j\} \subset \mathcal{H}^*\) and we fix the root vectors \(E_{e_i - e_j} := E_{ij}\). The set of positive roots is \(\mathcal{R}_+ := \{e_i - e_j | 1 \leq i < j \leq n\}\) and the simple roots are \(\alpha_i := e_i - e_{i+1} (1 \leq i \leq n - 1)\). Let \(\omega_i = \sum_{k=1}^{n} e_k \in \mathcal{H}^*\) \((1 \leq i \leq n - 1)\) denote the fundamental weights. The equivalence classes of the IRREPS of \(\mathfrak{sl}(n, \mathbb{C})\) can be uniquely labeled by the highest (dominant integral) weights, which are the elements of

\[ P_+(SU(n)) = \{a_1 \omega_1 + a_2 \omega_2 + \cdots + a_{n-1} \omega_{n-1} | a_1, a_2, \ldots, a_{n-1} \in \mathbb{Z}_{\geq 0}\} \cong \mathbb{Z}_{\geq 0}^{n-1}. \quad (A.2) \]

Now take an \(\mathfrak{sl}(n, \mathbb{C})\) IRREP \(\pi_\lambda, V_\lambda\) of highest weight \(\lambda \in P_+(SU(n))\). To any linear functional \(\nu \in \mathcal{H}^*\) we associate the weight space

\[ V_\lambda[\nu] := \bigcap_{H \in \mathcal{H}} \ker (\pi_\lambda(H) - \nu(H) \text{Id}_{V_\lambda}) \subset V_\lambda, \quad (A.3) \]

and we also define the set of weights \(\mathcal{W}_\lambda := \{\nu \mid \nu \in \mathcal{H}^*, V_\lambda[\nu] \neq \{0\}\}\). Then we have the weight space decomposition \(V_\lambda = \bigoplus_{\nu \in \mathcal{W}_\lambda} V_\lambda[\nu]\). Note that \(\lambda \in \mathcal{W}_\lambda\) and \(\dim(V_\lambda[\lambda]) = 1\), so we can write \(V_\lambda[\lambda] = \mathbb{C} v_\lambda\) with some highest weight vector \(v_\lambda\). The characteristic property of the non-zero vector \(v_\lambda\) is that \(\pi_\lambda(E_\alpha) v_\lambda = 0\) holds for all \(\alpha \in \mathcal{R}_+\). The IRREP \((\pi_\lambda, V_\lambda)\) of \(\mathfrak{sl}(n, \mathbb{C})\) induces the UIRREP \((\Pi_\lambda, V_\lambda)\) of SU(n) by the requirement \(\Pi_\lambda(e^X) = e^{\pi_\lambda(X)}\) for all \(X \in \mathfrak{su}(n)\). The corresponding scalar product on \(V_\lambda\) can be defined by fixing the norm of \(v_\lambda\) and requiring the anti-hermiticity of \(\pi_\lambda(X)\) for all \(X \in \mathfrak{su}(n)\).

The UIRREPS of U(n) are usually parametrized by the set

\[ P_+(U(n)) = \{m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n \mid m_1 \geq m_2 \geq \cdots \geq m_n\}. \quad (A.4) \]

The representation \(\rho_m\) of U(n) may be defined as the extension of the representation \(\Pi_\lambda\) of SU(n) < U(n) characterized by the properties

\[ \lambda = \sum_{i=1}^{n-1} (m_i - m_{i+1}) \omega_i \quad \text{and} \quad \rho_m(\xi 1_n) = \xi^{m_1 + \cdots + m_n} \text{Id}_{V_\lambda} \quad \forall \xi \in U(1). \quad (A.5) \]

In the main text we use a slightly different parametrization by pairs \((k, \lambda) \in \mathbb{Z} \times P_+(SU(n))\). The correspondence is given by the relation \(m_1 + \cdots + m_n = \mu_n(\lambda) + kn\), as is seen from the comparison between (A.5) and (4.2) and (4.3).
A.2 On the bosonic oscillator realization of \((\pi_{m\varpi_1}, V_{m\varpi_1})\)

Fix an integer \(n \geq 2\) and to each \(n\)-tuple \((l_1, l_2, \ldots, l_n) \in \mathbb{Z}_{\geq 0}^n\) associate a ‘symbol’ \(|l_1, l_2, \ldots, l_n\rangle\). Let \(\mathcal{F}\) denote the complex vector space generated by these symbols,

\[
\mathcal{F} := \bigoplus_{(l_1, l_2, \ldots, l_n) \in \mathbb{Z}_{\geq 0}^n} \mathbb{C}|l_1, l_2, \ldots, l_n\rangle.
\]

(A.6)

Endow \(\mathcal{F}\) with the scalar product \((\ , \)\) for which the vectors \(|l_1, l_2, \ldots, l_n\rangle\) satisfy

\[
(|l_1, l_2, \ldots, l_n\rangle, |l_1', l_2', \ldots, l_n'\rangle) = \delta_{l_1,l_1'}\delta_{l_2,l_2'}\cdots\delta_{l_n,l_n'},
\]

(A.7)

and introduce the annihilation and creation operators \(b_i\) and \(b_i^\dagger\) (\(1 \leq i \leq n\)) on \(\mathcal{F}\) by

\[
b_i|l_1, l_2, \ldots, l_n\rangle := \begin{cases} \sqrt{t_i}|l_1, l_2, \ldots, l_i - 1, \ldots, l_n\rangle & \text{if } l_i \geq 1, \\ 0 & \text{if } l_i = 0, \end{cases}
\]

(A.8)

\[
b_i^\dagger|l_1, l_2, \ldots, l_n\rangle := \sqrt{l_i + 1}|l_1, l_2, \ldots, l_i + 1, \ldots, l_n\rangle.
\]

(A.9)

Then \(b_i^\dagger\) is the adjoint of \(b_i\), and one has the commutation relations

\[
[b_i, b_j] = 0, \quad [b_i^\dagger, b_j^\dagger] = 0, \quad [b_i, b_j^\dagger] = \delta_{i,j}\text{Id}_\mathcal{F}.
\]

(A.10)

The ‘bosonic Fock space’ \(\mathcal{F}\) decomposes as the orthogonal direct sum \(\mathcal{F} = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} \mathcal{F}_m\) with

\[
\mathcal{F}_m := \text{span}_\mathbb{C}\{|l_1, l_2, \ldots, l_n\rangle \mid (l_1, l_2, \ldots, l_n) \in \mathbb{Z}_{\geq 0}^n, \quad l_1 + l_2 + \cdots + l_n = m\}.
\]

(A.11)

Now consider the linear map \(\psi: \mathfrak{gl}(n, \mathbb{C}) \to \text{End}(\mathcal{F})\) defined on the standard basis \(\{E_{ij}\}_{1 \leq i, j \leq n}\) of \(\mathfrak{gl}(n, \mathbb{C})\) by

\[
\psi(E_{ij}) := b_i^\dagger b_j.
\]

(A.12)

Then \((\psi, \mathcal{F})\) is a representation of \(\mathfrak{gl}(n, \mathbb{C})\) and the subspace \(\mathcal{F}_m\) is invariant under \(\psi\). The map

\[
\psi_m: \mathfrak{gl}(n, \mathbb{C}) \to \text{End}(\mathcal{F}_m), \quad X \mapsto \psi_m(X) := \psi(X)|_{\mathcal{F}_m}
\]

(A.13)

provides a finite dimensional representation of the Lie algebra \(\mathfrak{gl}(n, \mathbb{C})\). By restricting \(\psi_m\) to the subalgebra \(\mathfrak{sl}(n, \mathbb{C}) < \mathfrak{gl}(n, \mathbb{C})\), we end up with a finite dimensional representation \((\psi_m, \mathcal{F}_m)\) of \(\mathfrak{sl}(n, \mathbb{C})\). The set of weights of the representation \((\psi_m, \mathcal{F}_m)\) is

\[
\mathcal{W}_m := \left\{ \sum_{i=1}^n l_i e_i \left| (l_1, l_2, \ldots, l_n) \in \mathbb{Z}_{\geq 0}^n, \quad l_1 + l_2 + \cdots + l_n = m \right\}
\]

(A.14)

and the weight space \(\mathcal{F}_m[\nu] \subset \mathcal{F}_m\) corresponding to weight \(\nu = \sum_{i=1}^n l_i e_i \in \mathcal{W}_m\) takes the form

\[
\mathcal{F}_m[l_1 e_1 + l_2 e_2 + \cdots + l_n e_n] = \mathbb{C}|l_1, l_2, \ldots, l_n\rangle.
\]

(A.15)

Note that each weight space is one-dimensional. The representation \((\psi_m, \mathcal{F}_m)\) contains the (up to rescaling) unique highest weight vector \(v_m := |m, 0, \ldots, 0\rangle\), with weight \(m\varpi_1 = me_1 \in \mathcal{W}_m\). This shows that \((\psi_m, \mathcal{F}_m)\) is equivalent to the IRREP \((\pi_{m\varpi_1}, V_{m\varpi_1})\). We identify these \(\mathfrak{sl}(n, \mathbb{C})\) (and the naturally corresponding \(\mathfrak{su}(n)\)) representations in the proofs presented in Section 4.

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