Basic Properties of Coherent–Squeezed States
Revisited

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Abstract

In this paper we treat coherent-squeezed states of Fock space once more and study some basic properties of them from a geometrical point of view.

Since the set of coherent-squeezed states \(|\alpha, \beta\rangle\ | \alpha, \beta \in \mathbb{C}\) makes a real 4-dimensional surface in the Fock space \(\mathcal{F}\) (which is of course not flat), we can calculate its metric.

On the other hand, we know that coherent-squeezed states satisfy the minimal uncertainty of Heisenberg under some condition imposed on the parameter space \(|\alpha, \beta\rangle\), so that we can study the metric from the viewpoint of uncertainty principle. Then we obtain a surprising simple form (at least to us).

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We also make a brief review on Holonomic Quantum Computation by use of a simple model based on nonlinear Kerr effect and coherent-squeezed operators.

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1 Introduction

We start with general theory of quantum harmonic oscillator. For example see [1], [2] for a standard textbook of Quantum Mechanics. Let \( \{a^\dagger, a, N = a^\dagger a\} \) be a generator of the Heisenberg algebra whose relations are given by

\[
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger] = 1
\]

(1.1)

where 1 is the identity. For the vacuum \( |0\rangle \) given by

\[
a|0\rangle = 0
\]

(1.2)

we can define the \( n \)-th state by

\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle
\]

(1.3)

for \( n \geq 0 \). Then it is easy to see

\[
a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle, \quad N|n\rangle = n|n\rangle.
\]

(1.4)

Moreover, we can prove both the orthogonality condition and the resolution of unity

\[
\langle m|n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1.
\]

(1.5)

Therefore, we can define the Fock space

\[
\mathcal{F} = \left\{ \sum_{n=0}^{\infty} c_n |n\rangle \in \mathbb{C}^\infty \mid \sum_{n=0}^{\infty} |c_n|^2 < \infty \right\},
\]

(1.6)
which is a kind of Hilbert space in the mathematical sense. Then the inner product of the space is given by
\[ \langle \sum_{n=0}^{\infty} c_n |n\rangle | \sum_{n=0}^{\infty} d_n |n\rangle \rangle = \sum_{n=0}^{\infty} \bar{c}_n d_n. \tag{1.7} \]

On this space we obtain matrix representations of \( \{a^\dagger, a, N\} \) like
\[
\begin{pmatrix}
0 & 1 \\
0 & \sqrt{2} \\
0 & \sqrt{3} \\
0 & \ldots \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 \\
\sqrt{2} & 0 \\
\sqrt{3} & 0 \\
\ldots & \ldots \\
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
1 \\
2 \\
3 \\
\ldots \\
\end{pmatrix}
\]

by use of (1.4).

Now, let us define coherent states and squeezed states which play an important role in Quantum Optics in the following. For details see for example [3], [4].

For \( \alpha \in \mathbb{C} \) a coherent state \( |\alpha\rangle \) is defined by the equation
\[ a|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.9} \]

The state is rewritten as
\[ |\alpha\rangle = e^{\alpha a^\dagger - \bar{\alpha} a}|0\rangle. \tag{1.10} \]

This expression is in our opinion better than (1.9). Next, if we define operators
\[
K_+ = \frac{1}{2}(a^\dagger)^2, \quad K_- = \frac{1}{2}a^2, \quad K_3 = \frac{1}{2} \left( N + \frac{1}{2} \right) \tag{1.11}
\]
then it is not difficult to see
\[ [K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3. \] (1.12)
This is called the \( su(1, 1) \) algebra, see for example [5].

For \( \beta \in \mathbb{C} \) a squeezed state \( |\beta\rangle \) is defined by
\[
|\beta\rangle = e^{\beta K_+ - \bar{\beta} K_-} |0\rangle
\] (1.13)
which corresponds to (1.10) not (1.9).

As a result, a coherent-squeezed state is defined as
\[
|\alpha, \beta\rangle = e^{\beta K_+ - \bar{\beta} K_-} e^{\alpha a^\dagger - \bar{\alpha} a} |0\rangle
\] (1.14)
for \( \alpha \in \mathbb{C} \) and \( \beta \in \mathbb{C} \) (we use the notation \( |\alpha, \beta\rangle \) in place of \( |(\alpha, \beta)| \) for simplicity). We set \( CS \) a set of all coherent-squeezed states like
\[
CS = \{|\alpha, \beta\rangle \in \mathcal{F} | \alpha, \beta \in \mathbb{C} \}.
\] (1.15)
This set (manifold) is our target in the paper and we want to study some basic properties from a geometrical point of view.

In last in this section, we treat the uncertainty principle by Heisenberg. The position operator \( \hat{q} \) and momentum operator \( \hat{p} \) are expressed as
\[
\hat{q} = \sqrt{\frac{\hbar}{2\omega}} (a^\dagger + a), \quad \hat{p} = i \sqrt{\frac{\omega \hbar}{2}} (a^\dagger - a)
\] (1.16)
in terms of \( \{a^\dagger, a\} \). Then the variances of \( \hat{q} \) and \( \hat{p} \) are defined by
\[
(\Delta q)^2 = \langle \chi | \hat{q}^2 |\chi\rangle - \langle \chi | \hat{q} |\chi\rangle^2,
\]
\[
(\Delta p)^2 = \langle \chi | \hat{p}^2 |\chi\rangle - \langle \chi | \hat{p} |\chi\rangle^2
\] (1.17)
for any normalized \( |\chi\rangle \in \mathcal{F} \) (we write \( \Delta q \) in place of \( (\Delta q)_\chi \) for simplicity) and the uncertainty principle is given by
\[
(\Delta q)(\Delta p) \geq \frac{\hbar}{2}.
\] (1.18)

We are interested in a state \( |\chi\rangle \) giving the minimal value
\[
(\Delta q)(\Delta p) = \frac{\hbar}{2}.
\] (1.19)
In the next section we set \( |\chi\rangle = |\alpha, \beta\rangle \) and look for \( \{\alpha, \beta\} \) giving the minimal uncertainty (1.19).
2 Coherent-Squeezed States and Uncertainty Principle

In this section we prove that the coherent-squeezed state $|\alpha, \beta\rangle$ (1.14) gives the minimal uncertainty (1.19) if we choose $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{R}$. However, this is well-known, see for example [4].

From (1.17) we must calculate the variances of $\hat{q}$ and $\hat{p}$ respectively. The result is

$$
(\Delta q)^2 = \frac{\hbar}{2\omega}\left(\cosh|\beta| + \beta \frac{\sinh|\beta|}{|\beta|}\right)\left(\cosh|\beta| + \beta \frac{\sinh|\beta|}{|\beta|}\right),
$$

$$
(\Delta p)^2 = \frac{\hbar\omega}{2}\left(\cosh|\beta| - \beta \frac{\sinh|\beta|}{|\beta|}\right)\left(\cosh|\beta| - \beta \frac{\sinh|\beta|}{|\beta|}\right). 
$$

(2.1)

It is very interesting that these have nothing to do with $\alpha$. Some calculation gives

$$
(\Delta q)^2(\Delta p)^2 = \left(\frac{\hbar}{2}\right)^2 \left\{1 - (\beta - \bar{\beta})^2 \left(\frac{\sinh 2|\beta|}{2|\beta|}\right)^2\right\}. 
$$

(2.2)

From this we have

$$
(\Delta q)^2(\Delta p)^2 = \left(\frac{\hbar}{2}\right)^2 \implies (\Delta q)(\Delta p) = \frac{\hbar}{2} 
$$

(2.3)

if $\beta = \bar{\beta}$ ($\beta$ is real).

As a minimal uncertainty surface we set

$$
\widetilde{CS} = \{|\alpha, \beta\rangle \in \mathcal{F} | \alpha \in \mathbb{C}, \beta \in \mathbb{R}\}. 
$$

(2.4)

3 Inner Product of Coherent-Squeezed States

In this section we give a formula of inner product of coherent-squeezed states. Our method is based on [6], [7] or [8] (review paper)\(^1\)

The formula is well-known for coherent states. For $\alpha$ and $\alpha'$ it is easy to see

$$
\langle \alpha | \alpha' \rangle = e^{\left\{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha'|^2 + \bar{\alpha}\alpha'\right\}}. 
$$

(3.1)

\(^1\) The method developed in this section is powerful and convenient, see [9], [10] for a further application.
Therefore, our target is to calculate the inner product (1.7)

\[ \langle \alpha, \beta | \alpha', \beta' \rangle \] (3.2)

to find the inner product \( \alpha, \alpha' \in \mathbb{C} \) and \( \beta, \beta' \in \mathbb{C} \). This is not trivial as shown in the following.

The inner product (3.2) can be written as

\[ \langle \alpha, \beta | \alpha', \beta' \rangle = \langle 0 | e^{-(\beta K_+ \bar{\beta} K_-)} e^{\bar{\beta} K_+ \beta K_-} | \alpha' \rangle \]

by use of coherent states (1.10).

First of all, we must calculate the (product) operator

\[ e^{-(\beta K_+ \bar{\beta} K_-)} e^{\bar{\beta} K_+ \beta K_-} \]

explicitly. For the purpose let us recall the relations (1.12) once more

\[ [K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3, \quad K_- = K_+^\dagger. \]

If we set \( \{k_+, k_-, k_3\} \) as

\[ k_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad k_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad k_3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

then it is very easy to check the relations

\[ [k_3, k_+] = k_+, \quad [k_3, k_-] = -k_-, \quad [k_+, k_-] = -2k_3, \quad k_- \neq k_+^\dagger. \]

That is, \( \{k_+, k_-, k_3\} \) are generators of the Lie algebra \( su(1,1) \). We show by \( SU(1,1) \) the corresponding Lie group, which is a typical noncompact group.

Since \( SU(1,1) \) is contained in the special linear group \( SL(2; \mathbb{C}) \), we assume that there exists an infinite dimensional unitary representation \( \rho : SL(2; \mathbb{C}) \rightarrow U(\mathcal{F}) \) (group homomorphism) satisfying

\[ d\rho(k_+) = K_+, \quad d\rho(k_-) = K_-, \quad d\rho(k_3) = K_3. \]
Then we have
\[ e^{\gamma K_+ - \bar{\gamma} K_-} = e^{\gamma d\rho(k_+) - \bar{\gamma} d\rho(k_-)} = e^{d\rho(\gamma k_+ - \bar{\gamma} k_-)} = \rho \left( e^{\gamma k_+ - \bar{\gamma} k_-} \right) \] (by definition)
\[ \equiv \rho \left( e^A \right) \]
where
\[ A = \gamma k_+ - \bar{\gamma} k_- = \begin{pmatrix} 0 & \gamma \\ \bar{\gamma} & 0 \end{pmatrix} \implies e^A = \begin{pmatrix} \cosh |\gamma| & \gamma \sinh |\gamma| \\ \bar{\gamma} \sinh |\gamma| & \cosh |\gamma| \end{pmatrix}. \]
Therefore, some calculation gives
\[ e^{-(\beta K_+ - \bar{\beta} K_-)} e^{\beta' K_+ - \bar{\beta'} K_-} = \rho \left( \begin{pmatrix} \cosh |\beta| & -\beta \frac{\sinh |\beta|}{|\beta|} \\ \frac{-\beta \sinh |\beta|}{|\beta|} & \cosh |\beta| \end{pmatrix} \right) \rho \left( \begin{pmatrix} \cosh |\beta'| & \beta' \frac{\sinh |\beta'|}{|\beta'|} \\ \frac{-\beta' \sinh |\beta'|}{|\beta'|} & \cosh |\beta'| \end{pmatrix} \right) \]
\[ = \rho \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \]
where
\[ a \equiv a(\beta, \beta') = \cosh |\beta| \cosh |\beta'| - \frac{\beta' \bar{\beta}}{|\beta||\beta'|} \sinh |\beta| \sinh |\beta'|, \]
\[ b \equiv b(\beta, \beta') = \frac{\beta'}{|\beta'|} \cosh |\beta| \sinh |\beta'| - \frac{\beta}{|\beta|} \sinh |\beta| \cosh |\beta'|, \]
\[ c \equiv c(\beta, \beta') = \frac{\beta}{|\beta'|} \cosh |\beta| \sinh |\beta'| - \frac{\bar{\beta}}{|\beta|} \sinh |\beta| \cosh |\beta'| = \bar{b}, \]
\[ d \equiv d(\beta, \beta') = \cosh |\beta| \cosh |\beta'| - \frac{\bar{\beta} \beta'}{|\beta||\beta'|} \sinh |\beta| \sinh |\beta'| = \bar{a} \]
because \( \rho \) is a group homomorphism. For the latter convenience we show
\[ b = -\frac{\beta}{|\beta|} \tanh |\beta| + \frac{\beta'}{|\beta'|} \tanh |\beta'| \]
\[ \frac{b}{d} = \frac{-\frac{\beta}{|\beta|} \tanh |\beta| + \frac{\beta'}{|\beta'|} \tanh |\beta'|}{1 - \frac{\beta' \beta}{|\beta||\beta'|} \tanh |\beta| \tanh |\beta'|}, \]
\[ c = -\frac{\beta}{|\beta|} \tanh |\beta| + \frac{\beta'}{|\beta'|} \tanh |\beta'| \]
\[ \frac{c}{d} = \frac{-\frac{\beta}{|\beta|} \tanh |\beta| + \frac{\beta'}{|\beta'|} \tanh |\beta'|}{1 - \frac{\beta' \beta}{|\beta||\beta'|} \tanh |\beta| \tanh |\beta'|} = \frac{\bar{b}}{\bar{d}}. \]
Next, by use of the Gauss decomposition formula (in \( SL(2; \mathbb{C}) \))
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \left( ad - bc = 1 \right)
\]
we obtain
\[
\rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \rho \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \left( ad - bc = 1 \right)
\]
\[
= \rho \left( e^{bK_+} e^{-2 \log d_{K_3} e^{-\frac{c}{d}K_-}} \right)
\]
\[
= \rho \left( e^{bK_+} \rho \left( e^{-2 \log d_{K_3}} \right) \rho \left( e^{-\frac{c}{d}K_-} \right) \right)
\]
\[
= e^{bK_+} e^{-2 \log d_{K_3} e^{-\frac{c}{d}K_-}}
\]
where we have used again that \( \rho \) is a group homomorphism. Therefore, our formula is

Formula
\[
e^{-(\beta K_+ - \bar{\beta}K_-)} e^{\beta' K_+ - \bar{\beta}' K_-} = e^{bK_+} e^{-2 \log d_{K_3} e^{-\frac{c}{d}K_-}}.
\]

However, the proof of this formula is not complete because we assumed that \( \rho \) is a group homomorphism. We complete the proof as follows (see section 3.2 of [8]). For \( t \geq 0 \) we set
\[
F(t) = e^{-t(\beta K_+ - \bar{\beta}K_-)} e^{t(\beta' K_+ - \bar{\beta}' K_-)},
\]
\[
G(t) = e^{f(t)K_+} e^{g(t)K_3} e^{h(t)K_-}
\]
where
\[
f(t) = \frac{b(t)}{d(t)} = \frac{-\frac{\beta}{|\beta|} \tanh(t|\beta|) + \frac{\beta'}{|\beta'|} \tanh(t|\beta'|)}{1 - \frac{\beta}{|\beta|} \tanh(t|\beta|) \tanh(t|\beta'|)},
\]
\[
g(t) = -2 \log d(t) = -2 \log \left( \cosh(t|\beta|) \cosh(t|\beta'|) - \frac{\bar{\beta} \beta'}{|\beta||\beta'|} \sinh(t|\beta|) \sinh(t|\beta'|) \right),
\]
\[
h(t) = \frac{c(t)}{d(t)} = \frac{-\frac{\beta}{|\beta|} \tanh(t|\beta|) + \frac{\bar{\beta}' \beta'}{|\beta||\beta'|} \tanh(t|\beta'|)}{1 - \frac{\beta}{|\beta|} \tanh(t|\beta|) \tanh(t|\beta'|)}.
\]
Then a hard calculation gives
\[
F(0) = G(0) = 1 \quad \text{and} \quad F'(t) = G'(t),
\]

\(^2\text{we don't know such an explicit construction}\)
so that we complete the proof required

\[ F(t) = G(t) \implies F(1) = G(1) \implies \text{Formula.} \]

Let us continue our work. From the formula

\[
\langle \alpha, \beta | \alpha', \beta' \rangle = \langle \alpha | e^{-\beta K_+ - \beta K_-} e^{\beta K_+ - \beta K_-} | \alpha' \rangle = \langle \alpha | e^{\frac{\beta}{2} K_+} e^{-2 \log d K_3} e^{-\frac{\beta}{2} K_-} | \alpha' \rangle = \langle \alpha | e^{-2 \log d K_3} | \alpha' \rangle e^{\frac{\beta}{2} \frac{2}{3} \frac{1}{2} \alpha'^2} = \langle \alpha | e^{-\log d N} | \alpha' \rangle e^{\frac{\beta}{2} \frac{2}{3} \frac{1}{2} \alpha'^2 - \frac{1}{4} \log d}
\]

where we have used equations

\[
K_- | \alpha' \rangle = \frac{1}{2} a^2 | \alpha' \rangle = \frac{\alpha'^2}{2} | \alpha' \rangle, \quad \langle \alpha | K_+ = \frac{\alpha^2}{2} \langle \alpha |, \quad K_3 = \frac{1}{2}(N + \frac{1}{2}).
\]

Therefore, we have only to calculate the term \( \langle \alpha | e^{-\log d N} | \alpha' \rangle \). For the purpose we set

\[
f(t) = \langle \alpha | e^{-t \log d N} | \alpha' \rangle
\]

for \( t \geq 0 \). It is \( f(0) = \langle \alpha | \alpha' \rangle \) and

\[
f'(t) = - \log d \langle \alpha | e^{-t \log d N} N | \alpha' \rangle
\]

\[
= - \log d \langle \alpha | e^{-t \log d N} a\dagger | \alpha' \rangle \alpha'
\]

\[
= - \log d \langle \alpha | e^{-t \log d N} a^\dagger e^{t \log d N} e^{-t \log d N} | \alpha' \rangle \alpha'
\]

\[
= - \log d \langle \alpha | e^{-t \log d} a\dagger e^{-t \log d N} | \alpha' \rangle \alpha'
\]

\[
= - \log d e^{-t \log d} \bar{\alpha} \alpha' \langle \alpha | e^{-t \log d N} | \alpha' \rangle
\]

\[
= - \log d e^{-t \log d} \bar{\alpha} \alpha' f(t).
\]

That is, we have

\[
f(0) = \langle \alpha | \alpha' \rangle \quad \text{and} \quad f'(t) = - \log d e^{-t \log d} \bar{\alpha} \alpha' f(t).
\]

The solution is given by

\[
f(t) = \langle \alpha | \alpha' \rangle e^{\bar{\alpha} \alpha' (e^{-t \log d} - 1)}
\]
and we have

\[ f(1) = \langle \alpha | \alpha' \rangle e^{\bar{\alpha} \bar{\alpha} \left( e^{-\log d} - 1 \right)} = \langle \alpha | \alpha' \rangle e^{\bar{\alpha} \bar{\alpha} \left( \frac{1}{d} - 1 \right)}. \]

As a result we obtain

\[ \langle \alpha, \beta | \alpha', \beta' \rangle = \langle \alpha | \alpha' \rangle \exp \left\{ \frac{b \bar{\alpha}^2}{d} - \frac{\bar{b} \bar{\alpha'}^2}{d} + \left( 1 - \frac{1}{d} \right) \bar{\alpha} \bar{\alpha'} \right\}. \]

Note With respect to \( d \) it may be better to rewrite as

\[ d = \cosh |\beta| \cosh |\beta'| \left\{ 1 - \frac{\bar{\beta} \bar{\beta'}}{|\beta||\beta'|} \tanh |\beta| \tanh |\beta'| \right\}. \]

Let us summarize the result as

**Theorem I**

\[ \langle \alpha, \beta | \alpha', \beta' \rangle = \langle \alpha | \alpha' \rangle \langle \beta | \beta' \rangle \exp \left\{ \frac{b \bar{\alpha}^2}{d} - \frac{\bar{b} \bar{\alpha'}^2}{d} + \left( 1 - \frac{1}{d} \right) \bar{\alpha} \bar{\alpha'} \right\}. \]
Then the (induced) metric $dl^2$ is given by

$$
dl^2 = \langle \delta | \varphi \rangle |\delta|\varphi\rangle - |\langle \varphi | \delta \rangle |\varphi\rangle |^2, \tag{4.1}\n$$

where $\langle \cdot | \cdot \rangle$ is the inner product in \textbf{(3.2)}. See (2.13) in \textbf{[11]} for details.

Some algebra gives

$$
dl^2 = 1 - |\langle \varphi | \varphi + d\varphi \rangle|^2 = 1 - |\langle \varphi | \varphi + d\varphi \rangle|^2, \tag{4.2}\n$$

so we have only to calculate the term $\langle \varphi | \varphi + d\varphi \rangle \equiv \langle \varphi | \varphi + d\varphi \rangle$ (bra-ket).

Let us list our results (we omit the calculation because it is very complicated).

### 4.1 Coherent States

This case is very easy. From \textbf{[3.1]} the result is simply

$$
dl^2 = 1 - |\langle \alpha | \alpha + d\alpha \rangle|^2 = d\alpha d\bar{\alpha} = (d\alpha_1)^2 + (d\alpha_2)^2 \tag{4.3}\n$$

if we write $d\alpha = d\alpha_1 + id\alpha_2$ ($d\bar{\alpha} = d\alpha_1 - id\alpha_2$). See §2 in \textbf{[11]}.

### 4.2 Squeezed States

This case is not easy. First, we introduce the notation

$$
K = \frac{\cosh(|\beta|) \sinh(|\beta|)}{|\beta|} = \frac{\sinh(2|\beta|)}{2|\beta|}
$$

for later convenience. We must calculate

$$
dl^2 = 1 - |\langle \beta | \beta + d\beta \rangle|^2
$$

and from \textbf{[5.3]} the result is

$$
dl^2 = \frac{1}{8} \frac{\bar{\beta}^2}{|\beta|^2} (1 - K^2)(d\beta)^2 + \frac{2}{8} (1 + K^2)d\beta d\bar{\beta} + \frac{1}{8} \frac{\beta^2}{|\beta|^2} (1 - K^2)(d\bar{\beta})^2
$$

$$
= \frac{1}{8} \left\{ 2(1 + K^2) + \left( \frac{\beta^2}{|\beta|^2} + \frac{\bar{\beta}^2}{|\beta|^2} \right) (1 - K^2) \right\} (d\beta_1)^2 - \\
\frac{2i}{8} \left( \frac{\beta^2}{|\beta|^2} - \frac{\bar{\beta}^2}{|\beta|^2} \right) (1 - K^2)d\beta_1 d\beta_2 + \\
\frac{1}{8} \left\{ 2(1 + K^2) - \left( \frac{\beta^2}{|\beta|^2} + \frac{\bar{\beta}^2}{|\beta|^2} \right) (1 - K^2) \right\} (d\beta_2)^2 \tag{4.4}\n$$
if we write $d\beta = d\beta_1 + id\beta_2$ ($d\bar{\beta} = d\beta_1 - id\beta_2$). The determinant of the metric is

$$|g| = \frac{1}{4} K^2 (> 0).$$

If $\beta$ is real ($\beta = \bar{\beta} \Rightarrow \beta = \beta_1$) then we have

$$dl^2 = \frac{1}{2} (d\beta_1)^2.$$  \hspace{1cm} (4.5)

That is, the metric is extremely simple.

4.3 Coherent-Squeezed States

This case is very and very hard. First, we introduce the notations

$$K_1 = \frac{\sinh(|\beta|)}{|\beta|}, \quad K_2 = \frac{\sinh(2|\beta|)}{2|\beta|}, \quad K_4 = \frac{\sinh(4|\beta|)}{4|\beta|}$$

and

$$X = \beta - 3\bar{\beta} - 2\beta \cosh(2|\beta|) + 2(\beta + 2\bar{\beta})K_2 - (\beta + \bar{\beta})K_4,$$
$$Y = -\beta + \bar{\beta} - 2\bar{\beta}K_2 + (\beta + \bar{\beta})K_4,$$
$$Z = \beta + \bar{\beta} + 2\beta \cosh(2|\beta|) - 2\beta K_2 - (\beta + \bar{\beta})K_4$$

and

$$F = -1 + 4|\beta|^2 + 2 \cosh(2|\beta|) - \cosh(4|\beta|),$$
$$G = 1 + 4|\beta|^2 - 2 \cosh(2|\beta|) + \cosh(4|\beta|).$$

Let us note down (very) important equations

$$X + 2Y + Z = 0 \quad \text{and} \quad F + G = 8|\beta|^2.$$  \hspace{1cm} (4.6)

We must calculate

$$dl^2 = 1 - |\langle \alpha, \beta | \alpha + d\alpha, \beta + d\beta \rangle|^2$$

\hspace{1cm} (We used MATHEMATICA in order to calculate the metric. The calculation by “hand” may be almost impossible.)
and from (3.6) the result is

\[
\begin{align*}
\mathrm{d}t^2 &= \mathrm{d}\alpha \mathrm{d}\bar{\alpha} + \\
&\quad \frac{1}{8} \left\{ \frac{\bar{\beta}^2}{|\beta|^2} (1 - K_2^2) + \frac{Z\alpha^2 + F|\alpha|^2 + X\bar{\alpha}^2}{\beta^2} \right\} (\mathrm{d}\beta)^2 + \\
&\quad \frac{2}{8} \left\{ (1 + K_2^2) + \frac{Y\alpha^2 + G|\alpha|^2 + Y\bar{\alpha}^2}{|\beta|^2} \right\} \mathrm{d}\beta \mathrm{d}\bar{\beta} + \\
&\quad \frac{1}{8} \left\{ \frac{\beta^2}{|\beta|^2} (1 - K_2^2) + \frac{X\alpha^2 + F|\alpha|^2 + Z\bar{\alpha}^2}{\beta^2} \right\} (\mathrm{d}\bar{\beta})^2 + \\
&\quad \frac{1}{2} \left\{ \frac{\alpha\bar{\beta}}{|\beta|^2} (1 - K_2) - \bar{\alpha}\bar{\beta}K_1^2 \right\} \mathrm{d}\alpha \mathrm{d}\beta + \\
&\quad \frac{1}{2} \left\{ \alpha(1 + K_2) + \bar{\alpha}\bar{\beta}K_1^2 \right\} \mathrm{d}\alpha \mathrm{d}\bar{\beta} + \\
&\quad \frac{1}{2} \left\{ \bar{\alpha}(1 + K_2) + \alpha\bar{\beta}K_1^2 \right\} \mathrm{d}\bar{\alpha} \mathrm{d}\beta + \\
&\quad \frac{1}{2} \left\{ \frac{\bar{\alpha}\bar{\beta}}{|\beta|^2} (1 - K_2) - \alpha\bar{\beta}K_1^2 \right\} \mathrm{d}\bar{\alpha} \mathrm{d}\bar{\beta}.
\end{align*}
\]

(4.7)

Let us rewrite this into a real matrix form. By setting \( \mathrm{d}\alpha = \mathrm{d}\alpha_1 + i\mathrm{d}\alpha_2 \) (\( \mathrm{d}\bar{\alpha} = \mathrm{d}\alpha_1 - i\mathrm{d}\alpha_2 \)) and \( \mathrm{d}\beta = \mathrm{d}\beta_1 + i\mathrm{d}\beta_2 \) (\( \mathrm{d}\bar{\beta} = \mathrm{d}\beta_1 - i\mathrm{d}\beta_2 \)) we have

**Theorem II**

\[
\begin{align*}
\mathrm{d}t^2 &= (\mathrm{d}\alpha_1, \mathrm{d}\alpha_2, \mathrm{d}\beta_1, \mathrm{d}\beta_2) \\
&\quad \begin{pmatrix}
1 & 0 & g_{13} & g_{14} \\
0 & 1 & g_{23} & g_{24} \\
g_{13} & g_{23} & g_{33} & g_{34} \\
g_{14} & g_{24} & g_{34} & g_{44}
\end{pmatrix}
\begin{pmatrix}
\mathrm{d}\alpha_1 \\
\mathrm{d}\alpha_2 \\
\mathrm{d}\beta_1 \\
\mathrm{d}\beta_2
\end{pmatrix}
\end{align*}
\]

(4.8)
$$g_{33} = \frac{1}{8} \left\{ 2(1 + K_2^2) + \frac{\beta^2 + \bar{\beta}^2}{|\beta|^2} (1 - K_2^2) + 2 \frac{Y \alpha^2 + G|\alpha|^2 + Y \bar{\alpha}^2}{|\beta|^2} \right. \\
\left. + \frac{Z \alpha^2 + F|\alpha|^2 + X \bar{\alpha}^2}{|\beta|^2} \right\},$$

$$g_{44} = \frac{1}{8} \left\{ 2(1 + K_2^2) - \frac{\beta^2 + \bar{\beta}^2}{|\beta|^2} (1 - K_2^2) + 2 \frac{Y \alpha^2 + G|\alpha|^2 + Y \bar{\alpha}^2}{|\beta|^2} \right. \\
\left. - \frac{Z \alpha^2 + F|\alpha|^2 + X \bar{\alpha}^2}{|\beta|^2} \right\},$$

$$g_{43} = g_{34} = \frac{i}{8} \left\{ - \frac{\beta^2 - \bar{\beta}^2}{|\beta|^2} (1 - K_2^2) + \frac{Z \alpha^2 + F|\alpha|^2 + X \bar{\alpha}^2}{|\beta|^2} - \frac{X \alpha^2 + F|\alpha|^2 + Z \bar{\alpha}^2}{|\beta|^2} \right\},$$

$$g_{13} = g_{31} = \frac{1}{4} \left\{ \frac{\alpha \beta^2 + \bar{\alpha} \beta^2}{|\beta|^2} (1 - K_2) + (\alpha + \bar{\alpha})(1 + K_2) - (\alpha - \bar{\alpha})(\beta - \bar{\beta}) K_1^2 \right\},$$

$$g_{24} = g_{42} = \frac{1}{4} \left\{ - \frac{\alpha \beta^2 + \bar{\alpha} \beta^2}{|\beta|^2} (1 - K_2) + (\alpha + \bar{\alpha})(1 + K_2) + (\alpha + \bar{\alpha})(\beta + \bar{\beta}) K_1^2 \right\},$$

$$g_{14} = g_{41} = \frac{i}{4} \left\{ \frac{\alpha \beta^2 - \bar{\alpha} \beta^2}{|\beta|^2} (1 - K_2) - (\alpha - \bar{\alpha})(1 + K_2) + (\alpha - \bar{\alpha})(\beta + \bar{\beta}) K_1^2 \right\},$$

$$g_{23} = g_{32} = \frac{i}{4} \left\{ \frac{\alpha \beta^2 - \bar{\alpha} \beta^2}{|\beta|^2} (1 - K_2) + (\alpha - \bar{\alpha})(1 + K_2) + (\alpha + \bar{\alpha})(\beta - \bar{\beta}) K_1^2 \right\}. \quad (4.9)$$

The determinant of the metric is

$$|g| \equiv \begin{vmatrix} E & A \\ A^t & B \end{vmatrix} = |B - A^tA| = \\
\begin{vmatrix} g_{33} - (g_{13}^2 + g_{23}^2) & g_{34} - (g_{13}g_{14} + g_{23}g_{24}) \\ g_{34} - (g_{13}g_{14} + g_{23}g_{24}) & g_{44} - (g_{14}^2 + g_{24}^2) \end{vmatrix}.$$  

In spite of our very effort we could not obtain a compact form of $|g|$, so we present

**Problem** Give a compact form to $|g|$.

We will report some mathematical experiments by use of MATHEMATICA in another paper, [12].

## 5 Special Case : $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{R}$

In this section, let us study the metric by restricting to the case $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{R}$ (see Section II).

For simplicity we set $\beta = \beta_1$. Then, the metric $(4.8)$ becomes a dramatic simple form
Theorem III

\[ dl^2 = (d\alpha_1, d\alpha_2, d\beta) \begin{pmatrix} 1 & 0 & \frac{\alpha + \bar{\alpha}}{2} \\ 0 & 1 & i\frac{\beta - \bar{\beta}}{2} \\ \frac{\alpha + \bar{\alpha}}{2} & i\frac{\beta - \bar{\beta}}{2} & \frac{1+4|\alpha|^2}{2} \end{pmatrix} \begin{pmatrix} d\alpha_1 \\ d\alpha_2 \\ d\beta \end{pmatrix} \]  

(5.1)

by use of the equations (4.6).

It is very interesting (mysterious) that there is no \( \beta \)-dependence on \( g = (g_{ij}) \). Therefore, the determinant of the metric is simply

\[ |g| = \frac{1+2|\alpha|^2}{2} = \frac{1}{2} + |\alpha|^2. \]

6 Holonomic Quantum Computation

In this section let us make a brief review of Holonomic Quantum Computation for readers by making use of a simple model based on coherent-squeezed operators. See for example [13], [14] and [15].

From (1.14) we set

\[ O(\alpha, \beta) = V(\beta)U(\alpha) \equiv e^{\beta K_+ - \bar{\beta} K_-} e^{\alpha a^\dagger - \bar{\alpha} a} \]  

(6.1)

for \( \alpha, \beta \in \mathbb{C} \). Of course, this operator is unitary and called a coherent-squeezed operator in the following.

Next, we define an effective Hamiltonian representing a nonlinear Kerr effect

\[ H_0 = \hbar \omega (a^\dagger)^2 a^2 = \hbar \omega N(N - 1). \]  

(6.2)

If we set

\[ \mathcal{C} = \text{Vect}_\mathbb{C}\{|0\}, |1\} \]

then we have

\[ H_0 \mathcal{C} = O. \]

The space of eigenstates corresponding to 0 is two dimensional, which will become a qubit space. In order to construct a geometric method of Quantum Computation we utilize this (degenerate) space.
By use of the operator (6.1) we can define a family of Hamiltonians by

\[ H_{(\alpha, \beta)} = \mathcal{O}(\alpha, \beta) H_0 \mathcal{O}(\alpha, \beta)\dagger. \]  

(6.3)

Then \( \mathcal{O}(\alpha, \beta) \mathcal{C} \) becomes

\[ H_{(\alpha, \beta)} \mathcal{O}(\alpha, \beta) \mathcal{C} = \mathcal{O}(\alpha, \beta) H_0 \mathcal{C} = \mathcal{O}, \]

so that we have a family of two dimensional vector spaces in \( \mathcal{F} \) parametrized by \((\alpha, \beta)\). This is just a vector bundle over \( \mathbb{C} \times \mathbb{C} \).  

\[ \mathcal{O}(\alpha, \beta) \mathcal{C} \]

\( \mathbb{C} \times \mathbb{C} \)

From this, a canonical connection form \( \mathcal{A} \) is given by

\[ \mathcal{A} = \mathcal{O}(\alpha, \beta)\dagger d\mathcal{O}(\alpha, \beta) \equiv \mathcal{O}\dagger d\mathcal{O} \quad (\implies \mathcal{A}\dagger = -\mathcal{A}) \]  

(6.4)

where

\[ d = \frac{d\alpha}{\partial \alpha} + \frac{d\beta}{\partial \beta} + \frac{d\bar{\alpha}}{\partial \bar{\alpha}} + \frac{d\bar{\beta}}{\partial \bar{\beta}} \]

and therefore the curvature form \( \mathcal{F} \) becomes

\[ \mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = d\mathcal{O}\dagger \wedge d\mathcal{O} + \mathcal{O}\dagger d\mathcal{O} \wedge \mathcal{O}\dagger d\mathcal{O}. \]  

(6.5)

Next, let us define the holonomy operator. Let \( \gamma \) be a loop in the space \( \mathbb{C} \times \mathbb{C} \) starting from \((0, 0)\) like

\[ \gamma : [0, 1] \rightarrow \mathbb{C} \times \mathbb{C} \text{ (differentiable)}, \quad \gamma(0) = \gamma(1) = (0, 0). \]

\[ \text{To be precise, it is a little ambiguous, see [14] for more details.} \]
For a loop $\gamma$ the holonomy (operator) $\Gamma$ is defined by the path–ordered exponential integral (along $\gamma$) like

$$\Gamma(\gamma) = P \exp \left( \int_{\gamma} A \right) \in U(2).$$

(6.6)

See the following figure.

Now we are in a position to calculate these quantities. However, we don’t repeat them in the paper. Read [13], [14] and [15] in detail.

7 Concluding Remarks

In the paper we revisited coherent-squeezed states and calculated the inner product and gave the metric induced from the Fock space (a kind of Hilbert space).

Its metric form is very complicated but becomes extremely simple by considering the special case satisfying the minimal uncertainty of Heisenberg.

We also reviewed a non-abelian holonomic Quantum Computation by taking a simple model performed by coherent-squeezed operators.

We would like to emphasize that coherent-squeezed states or coherent-squeezed operators play a central role in Quantum Optics or Mathematical Physics, so that we expect young researchers to generalize our results widely.
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