DVORETZKY-TYPE THEOREM FOR AHLFORS
REGULAR SPACES

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Abstract. It is proved that for any $0 < \beta < \alpha$, any bounded Ahlfors
$\alpha$-regular space contains a $\beta$-regular compact subset that embeds biLip-
schitzly in an ultrametric with distortion at most $O(\alpha/(\alpha - \beta))$. The
bound on the distortion is asymptotically tight when $\beta \to \alpha$. The main
tool used in the proof is a regular form of the ultrametric skeleton the-
orem.

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1. Introduction

Fix a metric space $(X,d)$, a point $x \in X$ and a radius $r \in [0,\infty)$. The
corresponding closed ball is denoted $B(x,r) = B_d(x,r) = \{y \in X : d(y,x) \leq r\}$, and the corresponding open ball is denoted $B^o_d(x,r) = \{y \in X : d(y,x) < r\}$. A complete metric space $(X,d)$ is called Ahlfors $\alpha$-regular (or $\alpha$-regular for short), if there exists a Borel measure $\mu$ such that for all $x \in X$ and $r \in (0,\text{diam}(X))$,

\begin{equation}
  c r^\alpha \leq \mu(B_d(x,r)) \leq C r^\alpha.
\end{equation}

Here $C \geq c > 0$ are independent of $x$ and $r$. Ahlfors $\alpha$-regular space $X$ has, in particular, Hausdorff dimension $\dim_H(X) = \alpha$. For more information on Ahlfors regular spaces and their importance, see [10, 6, 8].

An ultrametric space is a metric space $(U, \rho)$ satisfying the strengthened
triangle inequality $\rho(x,y) \leq \max\{\rho(x,z),\rho(y,z)\}$ for all $x,y,z \in U$. Saying that $(X,d)$ embeds (biLipschitzly) with distortion $D \in [1,\infty)$ into an ultrametric space means that there exists an ultrametric $\rho$ on $X$ satisfying

\[ d(x,y) \leq \rho(x,y) \leq Dd(x,y) \]

for all $x,y \in X$. The ultrametric distortion

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of $X$ is the infimum over $D$ for which $X$ embeds in an ultrametric with distortion at most $D$.

In this paper we study regular (approximate) ultrametric subsets of Ahlfors regular spaces. Arcozzi et. al. [1, Theorem 1] proved that for every $0 < \beta < \alpha$, any bounded Ahlfors $\alpha$-regular space $X$ contains a $\beta$-regular subset $Y$. Matilla and Saaranen [15, Theorem 3.1] (see also [11, Corollary 5.2]) showed that $Y$ can be chosen to be both $\beta$-regular and biLipschitz embeddable in an ultrametric. In their proof, the ultrametric distortion of $Y$ is bounded above by $\exp(O(\alpha/(\alpha - \beta)))$. Here we prove a similar result with an exponentially improved bound on the ultrametric distortion.

**Theorem 1.1.** For every $0 < \beta < \alpha$, any bounded Ahlfors $\alpha$-regular space $X$ contains a $\beta$-regular compact subset $Y$ whose ultrametric distortion is $O(\alpha/(\alpha - \beta))$.

When $\beta \rightarrow \alpha$ the bound on the distortion in the Theorem 1.1 is asymptotically tight. A tight example is given in [18, Theorem 1.4]. Specifically, for any $\alpha > 0$ a compact metric $X_\alpha$ is constructed with the following properties. Its Hausdorff dimension is $\dim_H(X_\alpha) = \alpha$, and for any subset $S \subseteq X_\alpha$, the ultrametric distortion (indeed, even the Euclidean distortion) of $S$ is $\Omega(\alpha/(\alpha - \beta))$, where $\beta = \dim_H(S)$. Observing the space $X_\alpha$ from [18], it is clear that it is compact, self-similar and it satisfies the open-set condition of [10]. Therefore it is Ahlfors $\alpha$-regular (see [10, Theorem 5.3(1)]). Furthermore, when $\alpha = n \in \mathbb{N}$ is a natural number, $[0,1]^n \subset \mathbb{R}^n$ is also an asymptotically tight example: It is proved in [13] that for any $S \subseteq [0,1]^n$ of Hausdorff dimension $\beta = \dim_H(S)$, the ultrametric distortion of $S$ is at least $0.25 \cdot n/(n - \beta)$.

Results similar to Theorem 1.1 were previously obtained in other settings and were called Dvoretzky-type theorems for metric spaces and matric Ramsey theory, see [19, §8 & §9] and references therein. In particular, it is proved in [18, 17] that for every compact metric space $X$ of Hausdorff dimension $\alpha = \dim_H(X)$ and any $0 < \beta < \alpha$, there exists a closed subset $Y \subset X$ and a Borel measure $\nu$ supported on $Y$ such that $Y$ embeds in an ultrametric with distortion $O(\alpha/(\alpha - \beta))$, and $\nu$ satisfies $\nu(B(x,r)) \leq C r^\beta$ for any $x \in Y$ and $r \in (0, \text{diam}_d(Y))$ (which implies that $\dim_H(Y) \geq \beta$). Here we amend the arguments from [18, 17, 16] so that in the Ahlfors regular setting the measure $\nu$ would also satisfy $\nu(B(x,r)) \geq c r^\beta$.

The tool we use is a regular form of the ultrametric skeleton theorem [17], which apply more broadly to doubling spaces. A metric space $(X,d)$ is called $\lambda$-doubling if any bounded subset $Z \subseteq X$ can be covered by at most $\lambda$ sub-

\footnote{The embedding of $Y$ in [15] is into a Cantor set of $[0,1]^n$ for some $n > \beta$, and the distortion is $\sqrt{n} \exp(O(\alpha/(\alpha - \beta)))$. Their embedding is factored through a $\beta$-regular ultrametric. The $\sqrt{n}$ factor is the distortion of the straightforward embedding of the ultrametric in an $n$-dimensional Cantor set, which is irrelevant in our setting. However, the exponential dependence on $\alpha/(\alpha - \beta)$ seems inherent to their approach. As aside, we note that bounded $\beta$-regular ultrametrics can be embedded in $n$-dimensional Cantor set with a constant distortion, provided that $n \geq C \beta$, for some a constant $C > 1$. This can be done using binary error-correcting codes as in [13, Proposition 3].}
for some $\lambda \in \mathbb{N}$. A Borel measure $\sigma$ on $(X, d)$ is called $\kappa$-doubling measure if $\sigma(B_d(x, 2r)) \leq \kappa \cdot \sigma(B_d(x, r))$, for any $x \in X$ and $r > 0$. A complete metric space is doubling if and only if it has a doubling measure [20, 14]. Observe that an Ahlfors $\alpha$-regular space is doubling since the measure $\mu$ from (1) is a $(C2^\alpha/c)$-doubling measure.

**Theorem 1.2** (Regular ultrametric skeleton for doubling spaces). Let $(X, d)$ be a compact $\lambda$-doubling metric space, and let $\mu$ be a Borel probability measure on $X$. Then for every $t \in \mathbb{N}$ there exists a compact subset $S \subseteq X$ and a Borel probability measure $\nu$ supported on $S$ satisfying:

- The ultrametric distortion of $S$ is at most $16t$.
- For every $x \in X$ and $r \in [0, \infty)$, $\nu(B_d(x, r)) \leq 2^{2/t} \cdot \mu(B_d(x, C_1 tr)^{1-1/t}$.
- For every $y \in S$ and $r \in [0, \infty)$, there exists $x \in X$ such that $B_d(x, C_1 r/t) \subset B_d(y, r)$ and $\nu(B_d(y, r)) \geq (\lambda^{-2/t}/2) \cdot \mu(B_d(x, c_1 r/t))^{1-1/t}$.

Here $C_1, c_1 > 0$ are universal constants.

Informally speaking, the theorem above constructs for every doubling metric space and Borel measure $\mu$, an approximate ultrametric subset (a “skeleton”) with a measure $\nu$ that behaves “similarly to $\mu^{1-1/t}$”. The original ultrametric skeleton theorem [17] has applications for online algorithms, data-structures, probability theory, and geometric measure theory. See [17, §1] for more details. Theorem 1.2, when compared to the original ultrametric skeleton, has an additional lower bound (3) on $\nu$ in the conclusion, but it also has an additional doubling condition in the assumptions. See Section 6 for remarks about the necessity of the doubling assumption.

The construction of the ultrametric skeleton here follows the construction in [16], which in turn uses Bartal’s Ramsey decomposition lemma [3] as a key tool. Since we will use Bartal’s Ramsey decomposition lemma [3] in slightly more general form than in [3, 16], we rephrase and reprove it in Section 2. The proof of Theorem 1.2 is detailed in Section 4, after introducing in Section 3 an auxiliary structure of hierarchical nets which is needed to ensure the existence of the “small balls” in (3). Finally, the proof of Theorem 1.1, a rather straightforward corollary of Theorem 1.2, is presented in Section 5.

### 2. Bartal’s Ramsey decomposition lemma

Fix a compact metric measure space $(X, d, \mu)$, and $\Delta > 0$. For a Borel subset $A \subseteq X$, define

$$\mu^\Delta(A) = \sup_{a \in A} \mu(B_d(a, \Delta/4) \cap A).$$

Before delving into the details of Bartal’s lemma, we give a quick and intuitive explanation of it and its use in the proof of Theorem 1.2. See also [16, §1.2] for a similar explanation in a somewhat simpler setting. Roughly speaking, Corollary 2.3 below states that any compact metric measure space $(X, d, \mu)$ of diameter $\Delta = \text{diam}(X)$, contains two non-empty closed subsets
\[ P, Q \subset X \text{ that are separated, i.e., } d(P, Q) \geq \Delta/(8t) \text{ for some fixed } t \in \{2, 3,\ldots\}, \text{ and} \]
\[
\frac{\mu(P)}{\mu^{\Delta/2}(P)^{1/t}} + \frac{\mu(Q)}{\mu^{\Delta}(Q)^{1/t}} \geq \frac{\mu(X)}{\mu^{\Delta}(X)^{1/t}}.
\]
Recursive applications of (5) creates a hierarchy of subsets. The separation property implies that the "surviving" set of points, \(C\), forms a Cantor-like subset. I.e., \(C\) is biLipschitz equivalent to an ultrametric, and its ultrametric distortion is \(8t\). In doubling space, \(\mu^\Delta(X) \propto \mu(X)\), and therefore (5) implies, roughly speaking, that \(\xi = \mu/\mu^\Delta\) is a sub-measure controlled from above by \(\mu^{1-1/t}\). By "pruning" the set \(C\) a bit further we can obtain a subset \(S \subset C\) on which \(\xi\) is additive. To control \(\xi\) from below using \(\mu^{1-1/t}\), we should ensure that the subsets \(P\) and \(Q\) in (5) and all the subsets created from iterative applications of (5) contain "small" balls of \(X\). To achieve that, we do not actually apply (5) directly on subsets of \(X\), and instead opt to apply it on (a hierarchy of) metric nets \(\mathcal{N}\) of \(X\), where each of the net's points represents its Voronoi cell. Since the diameter of \(\mathcal{N}\) may be slightly smaller than the diameter of \(X\) when applying (5) on \(\mathcal{N}\), we actually use \(\Delta \approx \text{diam}(\mathcal{N}) + \text{"e"}\) instead of \(\Delta = \text{diam}(\mathcal{N})\), which explains why \(\Delta\) is a parameter in (5).

We next continue with rigorous definitions and arguments. The following properties of \(\mu^\Delta\) are straightforward (for a proof of (6) see [16]):

**Proposition 2.1.** Let \(0 < \delta \leq \Delta\), and \(A \subset C \subset X\). Then \(\mu^\delta(A) \leq \mu^\Delta(C)\).

If \(X\) is \(\lambda\)-doubling, and \(\Delta \geq \text{diam}(A)\), then
\[
\mu^\Delta(A) \leq \mu(A) \leq \lambda^2 \mu^\Delta(A).
\]

**Lemma 2.2.** Let \((X, d)\) be a compact metric space, and let \(\mu\) be a finite Borel measure on \(X\). For any compact subset \(\emptyset \neq Z \subset X\), \(0 < \Delta < 2 \text{diam}_d(Z)\) and integer \(t \in \{2, 3,\ldots\}\), there exist non-empty disjoint and compact subsets \(P, Q \subset Z\) that, denoting \(Q^c = Z \setminus Q\), satisfy \(d(P, Q) \geq \Delta/(8t)\), \(\text{diam}_d(Q^c) \leq \Delta/2\), \(\text{diam}_d(P) \leq \left(\frac{1}{2} - \frac{1}{4t}\right)\Delta\), and
\[
\mu(P) \geq \mu(Q^c) \cdot \left(\frac{\mu^{\Delta/2}(Q^c)}{\mu^{\Delta}(Z)}\right)^{1/t},
\]
where we use the convention \(0/0 = 0\) in (7).

**Proof.** The statement of the Lemma is vacuous when \(\mu(Z) = 0\), so we assume that \(\infty > \mu(Z) > 0\). With the convention \(0/0 = 0\), let \(x \in Z\) be a point that maximizes\(^2\)
\[
\frac{\mu(B(x, \Delta/8) \cap Z)}{\mu(B^o(x, \Delta/4) \cap Z)}.
\]

With this choice, \(\mu(B(x, \Delta/8) \cap Z) > 0\). For \(i \in \{0, 1,\ldots, t - 1\}\), let \(H_i = B(x, (1 + i/t)\Delta/8) \cap Z\), and also define \(H_t = B^o(x, \Delta/4) \cap Z\). Clearly there

\(^2\)Indeed, the maximum here and below exists. See [16, Remark 3] for a straightforward proof.
exists $i \in \{1, \ldots, t\}$ for which

\begin{equation}
\mu(H_i) \leq \mu(H_{i-1}) \cdot \left( \frac{\mu(H_i)}{\mu(H_0)} \right)^{1/t}.
\end{equation}

We then set $P = H_{i-1}$, $Q^c = B^0_i(x, (1 + i/t)\Delta/8) \cap Z$, and $Q = Z \setminus Q^c$. Observe that $P$ and $Q$ are closed, $\mu(P) > 0$, and $H_i \supseteq Q^c \supseteq P$. By the triangle inequality, for every $a \in P$, and $b \in Q$, $d(a, b) \geq d(b, x) - d(a, x) \geq \Delta/(8t)$. The measure satisfies

\begin{equation}
\mu(P) \geq \mu(Q^c) \cdot \left( \frac{\mu(B(x, \Delta/8) \cap Z)}{\mu(B^0(x, \Delta/4) \cap Z)} \right)^{1/t}.
\end{equation}

Let $u \in \overline{Q^c} \subseteq B(x, (1+ i/t)\Delta/8)$ the point that maximizes $\mu(B(u, \Delta/8) \cap \overline{Q^c})$, where $\overline{Q^c}$ is the topological closure of $Q^c$. Since $B(u, \Delta/8) \cap \overline{Q^c} \subseteq B(u, \Delta/8) \cap Z$, we have $\mu^{\Delta/2}(\overline{Q^c}) \leq \mu(B(u, \Delta/8) \cap Z)$. Also, $\mu^{\Delta}(Z) \geq \mu(B^0(u, \Delta/4) \cap Z)$. From the definition of $x$ we therefore have

\[
\frac{\mu(B(x, \Delta/8) \cap Z)}{\mu(B^0(x, \Delta/4) \cap Z)} \geq \frac{\mu(B(u, \Delta/8) \cap Z)}{\mu(B^0(u, \Delta/4) \cap Z)} \geq \frac{\mu^{\Delta/2}(Q^c)}{\mu^{\Delta}(Z)} \geq \frac{\mu^{\Delta/2}(Q^c)}{\mu^{\Delta}(Z)}.
\]

Applying the above inequality to (9), we obtain (7). \hfill \square

We use Lemma 2.2 via the following corollary.

**Corollary 2.3.** Let $(X, d)$ be a compact metric space and let $\mu$ be a Borel probability measure on $X$. For any closed subset $Z \subseteq X$, $\mu(Z) > 0$, and integer $t \in \{2, 3, \ldots\}$, there exist closed subsets $P, Q \subseteq Z$ such that $\mu(P) > 0$, $d(P, Q) \geq \Delta/(8t)$, $\text{diam}_d(Z \setminus Q) \leq \Delta/2$, $\text{diam}_d(P) \leq (\frac{1}{2} - \frac{1}{4t})\Delta$, and

\begin{equation}
\frac{\mu(P)}{\mu^{\Delta/2}(P)^{1/t}} + \frac{\mu(Q)}{\mu^{\Delta}(Q)^{1/t}} \geq \frac{\mu(Z)}{\mu^{\Delta}(Z)^{1/t}}.
\end{equation}

where $\mu^{\Delta}$ is defined in (4), and with the convention $0/0 = 0$.

**Proof.** Denote $Q^c = Z \setminus Q$. Applying Lemma 2.2 on $Z$,

\[
\frac{\mu(P)}{\mu^{\Delta/2}(P)^{1/t}} + \frac{\mu(Q)}{\mu^{\Delta}(Q)^{1/t}} \geq \frac{\mu(P)}{\mu^{\Delta/2}(Q^c)^{1/t}} + \frac{\mu(Q)}{\mu^{\Delta}(Z)^{1/t}} \geq \frac{\mu(Q^c)}{\mu^{\Delta/2}(Q^c)^{1/t}} \cdot \frac{\mu^{\Delta/2}(Q^c)^{1/t}}{\mu^{\Delta}(Z)^{1/t}} + \frac{\mu(Q)}{\mu^{\Delta}(Z)^{1/t}} = \frac{\mu(Z)}{\mu^{\Delta}(Z)^{1/t}}.
\]

The first inequality above is obtained using Proposition 2.1 and recalling that $P \subseteq Q^c$, and $Q \subseteq Z$. \hfill \square

3. **Net-trees and ultrametrics in compact spaces**

As explained at the beginning of Section 2, Bartal’s Ramsey decomposition lemma will be employed on an auxiliary hierarchy of metric nets, the existence of which should be considered a folklore. It is mostly similar to (the easy part of) Christ’s dyadic decomposition [5] for spaces with doubling measure, or to the net-tree [7] for finite spaces. For completeness, we provide here a self-contained treatment for compact spaces.
Definition 3.1 (Rooted trees). A rooted tree $T$ is a set of vertices with a distinguished vertex $r = r_T \in T$ called the root. Every vertex $u \in T \setminus \{r\}$, except $r$, has a unique parent $\psi(u) \in T$. The $k$-th ancestor $\psi^{(k)}(u)$ (if exists) of a vertex $u$ is defined inductively, $\psi^{0}(u) = u$, and $\psi^{(k+1)}(u) = \psi(\psi^{(k)}(u))$. The set of ancestors of $u$ is written as $\psi^{(*)}(u) = \{\psi^{(0)}(u), \psi^{(1)}(u), \psi^{(2)}(u), \ldots \}$. For every vertex $u \in T$, the sequence of ancestors $u = \psi^{(0)}(u), \psi^{(1)}(u), \ldots$ is finite and ends with the root $r$. If $u = \psi(v)$ then $v$ is called a child of $u$. The set of children of a vertex $u$ is denoted $\psi^{-1}(u)$ and must be finite and non-empty (i.e., there are no leaves). For $u \in T$, we denote by $T_u = \{v \in T : u \in \psi^{(*)}(v)\}$ the set of (weak) descendants of $u$ in $T$.

For every two vertices $u, v \in T$ we denote by $u \wedge v \in T$ their least common ancestor, i.e., $u \wedge v \in \psi^{(*)}(u) \cap \psi^{(*)}(v)$, and if $w \in \psi^{(*)}(u) \cap \psi^{(*)}(v)$ then $w \in \psi^{(*)}(u \wedge v)$.

A branch is an infinite sequence of vertices $b = (r = u_0, u_1, u_2, \ldots)$ that begins with the root and has $u_i = \psi(u_{i+1})$. The least common ancestor extends to branches: for two branches $x \neq y$, $x \wedge y$ is the deepest common vertex $v \in x \cap y$. The tree boundary $\bar{\partial}(u)$ of a vertex $u \in T$ is defined as the set of branches that contain $u$.

For a function $f : T \to X$, and a branch $b = (v_0, v_1, \ldots) \in \bar{\partial}(r)$, we sometimes use the notation $\lim_{i \to b} f(v) = \lim_{i \to \infty} f(v_i)$.

Definition 3.2 (subtree). Fix a tree $T$ and vertex $u \in T$. A subset $S \subset T$, is called a subtree of $T$ rooted at $u$ if the induced parent relation of $T$ on $S$ forms a rooted tree (according to Definition 3.1) whose root is $u$. For example, $T_u$ is a subtree of $T$ rooted in $u \in T$.

Definition 3.3 (Net-tree). Let $(X, d)$ be a metric space, and $\kappa \geq 1$. A $\kappa$-net-tree over $X$ is a rooted tree $N$ (in the sense of Definition 3.1) with the root being $r = \kappa N$, together with:

1. **Labels for vertices:** $\Delta : N \to [0, \infty)$.
2. **Representative points for vertices:** $x(\cdot) : N \to X$.
3. **Monotone labels:** $\Delta(u) \leq \Delta(\psi(u))$ for every $u \in N \setminus \{r\}$.
4. **Vanishing labels:** $\lim_{u \to b} \Delta(v) = 0$, for every $b \in \bar{\partial}(r)$.
5. **Covering:** For every vertex $u \in N$,
   $$\{x(w) : w \in T_u\} \subseteq B_d(x(u), \Delta(u))$$

6. **Packing:** For every $u, v \in N$, if $u \wedge v \notin \{u, v\}$ (i.e., no ancestor/descendant relation) and $\Delta(v) \leq \Delta(u)$, then
   $$B_d(x(u), \Delta(\psi(u))/\kappa) \cap \{x(w) : w \in T_v\} = \emptyset$$

We call $N$ a net-tree over $X$ if it is $\kappa$-net-tree over $X$ for some $\kappa \in [1, \infty)$.

For a vertex $u \in N$ with $\Delta(u) > 0$ and $\delta \in (0, \Delta(u)]$, we further define the $\delta$-descendants of $u$ as

$$\hat{\nabla}(u, \delta) = \hat{\nabla}_N(u, \delta) = \begin{cases} \{u\} & \delta = \Delta(u) \\ \{v \in T_u : \Delta(v) \leq \delta < \Delta(\psi(v))\} & \delta \in (0, \Delta(u)) \end{cases}$$
By a straightforward induction $\overline{\partial}(u, \delta)$ is finite.

**Proposition 3.4.** Let $N$ be a net-tree over $(X, d)$. For every branch $b \in \tilde{\partial}(r)$, \(\lim_{v \to b} x(v)\) exists.

**Proof.** Fix a branch \(b = (r = v_0, v_1, v_2, \ldots) \in \tilde{\partial}(r)\). Let \(D_n = \bigcap_{i=0}^{n} B_d(x(v_i), \Delta(v_i))\). \((D_n)_{n \geq 0}\) is a decreasing sequence of compact subsets with diameters that tend to 0. By the covering property, \(x(v_n) \in D_n \neq \emptyset\). Hence, by Cantor intersection theorem, \(\bigcap_{n=0}^{\infty} D_n = \bigcap_{n=0}^{\infty} B_d(x(v_i), \Delta(v_i))\) is a singleton which is the limit of \(x(v_n)\). □

The above proposition allows us to extend the notion of representative points to branches.

**Definition 3.5** (Boundary and Surjection). Let \(x(\cdot) : \tilde{\partial}_N(r) \to X\), defined by \(x(b) = \lim_{v \to b} x(v)\). The **boundary** of a vertex \(v\), \(\partial(v) = \partial_N(v) = \{x(b) : b \in \tilde{\partial}(v)\}\), is the set of points represented by the branches that contain \(v\). The image of the net-tree \(N\) is defined as the boundary of the root \(\text{Im}(N) = \partial_N(t_0)\). A net-tree \(N\) over a metric space \((X, d)\) is called **surjective** if \(\text{Im}(N) = X\).

**Proposition 3.6.** Further properties of a net-tree \(N\) over a compact space \((X, d)\):

(A) For every \(u \in N\), \(\partial(u) \subseteq B_d(x(u), \Delta(u))\).

(B) A subtree \(\mathcal{S} \subseteq N\) is also a net-tree.

(C) For every \(u \in N\), \(\partial(u)\) is compact. In particular, \(\text{Im}(N)\) is compact.

**Proof.** Item (A): Fix \(b \in \tilde{\partial}(u)\). By the covering property, for almost all \(v \in b\), \(x(v) \in B_d(x(u), \Delta(u))\), and since this ball is closed, we conclude that it is also contains \(\lim_{v \to b} x(v)\).

Item (B): A subtree is also a net-tree since the four required properties: monotone and vanishing labels, covering, and packing, are closed for subtrees.

Item (C): Next we prove that \(\partial(u)\) is compact. Assume \(\partial(u)\) is infinite (otherwise, it is trivially compact). Fix an infinite \(A \subseteq \partial(u)\). We should prove that \(A\) has an accumulation point in \(\partial(u)\). To achieve it, we construct an infinite non-increasing sequence of infinite subsets \(A = A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots\) and an infinite sequence of vertices \(u = v_1, v_2, v_3, \ldots \) in \(N\) satisfying \(A_i \subseteq \partial(v_i)\), and \(v_{i+1} \in \psi^{-1}(v_i)\). The construction is by induction. The base case \(v_1 = u\), \(A_1 = A \subseteq \partial(u)\) holds by assumption. Assume we have already defined \(A_1, \ldots, A_n\) and \(v_1, \ldots, v_n\) to satisfy the above when \(i \in \{1, \ldots, n-1\}\). Since \(\psi^{-1}(v_n)\) is finite and

\[
A_n = A_n \cap \partial(v_n) = \bigcup_{w \in \psi^{-1}(v_n)} (A_n \cap \partial(w))
\]

is infinite, there must be \(w \in \psi^{-1}(v_n)\) for which \(A_n \cap \partial(w)\) is infinite. Define \(v_{n+1} = w\) and \(A_{n+1} = A_n \cap \partial(v_{n+1})\). The sequence \((v_n)_n\) is a suffix of a unique branch \(b \in \tilde{\partial}(u)\). For every \(n \in \mathbb{N}\) we have, by the covering property,

\(\emptyset \neq A_n \subseteq A \cap \partial(v_n) \subseteq B(x(v_n), \Delta(v_n)) \ni x(b)\).
Hence, \( B_\delta(x(b), 2\Delta(v_n)) \cap A \supseteq A_n \neq \emptyset \). Since \( \Delta(v_n) \to 0 \), it means that \( x(b) \in \partial(u) \) is an accumulation point of \( A \).

The mapping \( x(\cdot) : \tilde{\partial}(r) \to X \) is not necessarily injective. In particular, it means that \( \partial(u) = \bigcup_{v \in \psi^{-1}(u)} \partial(v) \), but the union is not necessarily disjoint. We (partially) remedy this problematic aspect by using partial boundaries.

**Proposition 3.7.** Let \( N \) be a \( \kappa \)-net-tree over a metric space \( (X, d) \). There exists a partial boundary \( \diamond = \diamond_N : N \to 2^X \) with the following properties:

(A) \( \diamond(u) \subseteq \partial(u) \), for every \( u \in N \).

(B) \( \diamond(r) = \partial(r) = \text{Im}(N) \).

(C) The partial boundary of a vertex \( u \) is a disjoint union of the partial boundary of its children, i.e.

\[
\diamond(u) = \bigcup_{v \in \psi^{-1}(u)} \diamond(v).
\]

(D) Every partial boundary \( \diamond(u) \) is a Borel subset.

(E) \( B_\delta^o(x(u), \Delta(\psi(u))/\kappa) \cap \text{Im}(N) \subseteq \diamond(u) \), for every \( u \in N \).

**Proof.** The construction of the partial boundary is done inductively on the net-tree from the root downward. First we define \( \diamond(r) = \partial(r) = \text{Im}(N) \).

Assume inductively that \( \diamond(u) \subseteq \partial(u) \) has been defined and is a Borel set. Let \( \psi^{-1}(u) = \{v_1, \ldots, v_m\} \) be the children of \( u \) in some arbitrary order. Define

\[
\diamond(v_i) = \diamond(u) \cap (\partial(v_i) \setminus (\partial(v_1) \cup \ldots \cup \partial(v_{i-1}))).
\]

Obviously \( \diamond(v_i) \subseteq \partial(v_i) \). Since \( \diamond(u) \) is a Borel set, and so are \( \partial(v_1), \ldots, \partial(v_i) \) (which are closed according to Proposition 3.6), we conclude that \( \diamond(v_i) \) is also a Borel set. Equality (11) follows immediately.

Lastly, we prove Item (E). Fix \( u \in N \), and let

\[
B = \left\{ v \in N : u \land v \notin \{u, v\} \text{ and } \Delta(v) < \Delta(u) \leq \Delta(\psi(v)) \right\}.
\]

We claim that \( \tilde{\partial}(r) = \tilde{\partial}(u) \cup \bigcup_{b \in B} \tilde{\partial}(v) \). Indeed, fix an arbitrary branch \( b \in \tilde{\partial}(r) \). If \( u \leq b \) we are done. Otherwise let \( v \in b \) be the first vertex on \( b \) for which \( \Delta(v) < \Delta(u) \) (there must be such a vertex because the labels along \( b \) vanish). By definition, \( v \) is not the root and \( \Delta(\psi(v)) \geq \Delta(u) \). From the monotonicity of the labels, \( v \) is not a strict ancestor of \( u \), but it is also not a descendant of \( u \) (otherwise, \( u \in b \)). Hence \( u \land v \notin \{u, v\} \) and so \( v \in B \).

From that and an inductive application of equality (11) we deduce that \( \text{Im}(N) = \diamond(u) \cup \bigcup_{v \in B} \diamond(v) \). By the packing property of \( N \),

\[
B_\delta^o(x(u), \Delta(\psi(u))/\kappa) \cap \left( \bigcup_{v \in B} \diamond(v) \right) \subseteq B_\delta^o(x(u), \Delta(\psi(u))/\kappa) \cap \left( \bigcup_{v \in B} \partial(v) \right) = \emptyset.
\]

Therefore, \( B_\delta^o(x(u), \Delta(\psi(u))/\kappa) \cap \text{Im}(N) \subseteq \diamond(u) \). \( \diamond(u) = \bigcup_{v \in \tilde{\partial}(u, \delta)} \diamond(v) \).
Proposition 3.6 stated that the image of a net-tree is a compact metric space. The next proposition states that the reverse is also true.

**Proposition 3.8.** Every compact metric space has a surjective 20-net-tree over it.

**Proof.** Fix a compact metric space \((X, d)\). Since \(X\) is bounded, by rescaling we may assume without loss of generality that \(\text{diam}(X) = 1\). Let \(\tau = 1/4\).

Recall that \(\delta\)-net of a metric space \((X, d)\) is a subset \(N \subseteq X\) such that \(d(x, y) \geq \delta\) for every pair of distinct points \(x, y \in N\), and \(d(x, N) < \delta\) for every \(x \in X\). Let \((N_\ell)_{\ell=0}^\infty\) be a sequence of \(\delta_\ell\)-nets of \(X\) where \(\delta_\ell = (1-\tau)\tau^\ell\).

For every \(\ell \in \{0, 1, 2, \ldots\}\), and \(a \in N_\ell\), we define a unique vertex \(v_{\ell,a}\). Define \(\ell(v_{\ell,a}) = \ell\), and \(x(v_{\ell,a}) = a\). When \(\ell > 0\) the parent of a vertex \(v_{\ell,a}\) is \(v_{\ell-1,c} = \psi(v_{\ell,a})\), where \(c \in N_{\ell-1}\) is the closest point in \(N_{\ell-1}\) to \(a\), breaking ties arbitrarily.

Fix \(u, v \in N\). If \(\psi(v) = u\) then by the construction above,

\[
d(x(v), x(u)) = \min_{h \in N(u)} d(x(v), h) \leq (1 - \tau)\tau^{\ell(u)} = \tau^{\ell(u)} - \tau^{\ell(v)}.
\]

By induction and the triangle inequality, \(d(x(u), x(v)) \leq \tau^{\ell(u)} - \tau^{\ell(v)}\) when \(u \in \psi^{(s)}(v)\). Define \(\Delta(u) = \tau^{\ell(u)}\), and denote \(B_u = B_d(x(u), \Delta(u))\). Using the triangle inequality and the above bound on the distance between representatives, we conclude that if \(u \in \psi^{(s)}(v)\) then \(B_v \subseteq B_u\). This, in particular, implies the covering property of \(N\).

Next, we prove surjectivity — \(\text{Im}(N) = X\). Fix \(x \in X\), and for \(m \geq \ell \geq 0\) let

\[
N_m^\ell(x) = \{ \psi^{(m-\ell)}(v_{m,a}) \in N : v_{m,a} \in N, d(x, a) \leq \tau^m \}.
\]

Observe that for a fixed \(\ell \geq 0\), \((N_m^\ell(x))_{m=\ell}^\infty\) is a sequence of finite, non-increasing and non-empty sets. Indeed, \(N_m^\ell(x) \subseteq N_\ell\) which is finite. It is also non-empty: By construction \(N_m\) is \((1 - \tau)\tau^m\)-net, and hence there exists a point \(a \in N_m\) for which \(d(x, a) \leq (1 - \tau)\tau^m \leq \tau^m\), and therefore \(\psi^{(m-\ell)}(v_{m,a}) \in N_m^\ell(x)\). We are left to prove that \(N_{m+1}^\ell(x) \subseteq N_m^\ell(x)\) for every \(m \geq \ell\). Fix \(u \in N_{m+1}^\ell(x)\). There exist \(v_{m+1,a} \in N\) such that \(d(a, x) \leq \tau^{m+1}\), and \(u = \psi^{(m+1-\ell)}(v_{m+1,a})\), so \(x \in B_{v_{m+1,a}}\). Denote \(\psi(v_{m+1,a}) = v_{m,c}\). As we observed above \(B_{v_{m+1,a}} \subseteq B_{v_{m,c}}\), and hence \(d(x, c) \leq \tau^m\). This means that \(u \in N_m^\ell(x)\). As this is true for every \(u \in N_{m+1}^\ell(x)\), we conclude that \(N_{m+1}^\ell(x) \subseteq N_m^\ell(x)\).

We can thus define \(N_\ell^\infty(x) = \bigcap_{m=\ell}^\infty N_m^\ell(x) \neq \emptyset\). \(N_\ell^\infty(x)\) clearly satisfies the following two properties: For every \(u \in N_\ell^\infty(x)\) there exists \(v \in N_{\ell+1}^\infty(x)\) such that \(u = \psi(v); \) and \(d(x, x(u)) \leq \tau^\ell\). Hence we can construct a branch \(b = (v_0, v_1, \ldots)\) by choosing \(v_0 = r \in N_0^\infty(x) = \{r\}\), and \(v_i+1 \in N_{i+1}^\infty(x)\) a child of \(v_i\). From the above, \(\lim_{i \to \infty} d(x(v_i), x) = 0\), and hence \(x(b) = x\).

Lastly, we prove the packing property of \(N\). Let \(u, v \in N\) such that \(u \wedge v \notin \{u, v\}\), and \(\ell(v) \geq \ell(u)\). Let \(w \in N\) such that \(x(w) \in B_d(x(u), \Delta(\psi(u))/20) = B_d(x(u), \tau^{\ell(u)}/5)\). Assume towards a contradiction that \(w \in T_u\). Assume first that \(\ell(u) = \ell(v)\). Since \(u \wedge v \notin \{u, v\}\), we have \(v \neq u\). Clearly we can rule out \(w = v\), since this directly contradicts the net-tree construction. Let
\(v' \in \psi^{(*)}(w) \cap \psi^{-1}(v)\), be the child of \(v\) which is also an ancestor of \(w\). Then, by the assumption and the covering property,
\[
d(x(u), x(v')) \leq d(x(u), x(w)) + d(x(v'), x(w)) \leq \left(\frac{1}{2} + \tau\right) \tau^{\ell(u)} < \frac{1}{2} \tau^{\ell(u)}.
\]
By the net property and the triangle inequality
\[
d(x(v), x(v')) \geq d(x(v), x(u)) - d(x(u), x(v')) > \tau^{\ell(u)} - \frac{1}{2} \tau^{\ell(u)} = \frac{1}{2} \tau^{\ell(u)},
\]
which contradicts the construction of \(N\) in which
\[
d(x(v'), x(v)) \leq d(x(v'), x(u)).
\]
This proves \(B_d(x(u), \Delta(\psi(u))/20) \cap \{x(w) : w \in \mathbb{T}_v\} = \emptyset\) when \(\ell(u) = \ell(v)\).

If \(\ell(v) > \ell(u)\), then let \(\tilde{v} \in \psi^{(*)}(v)\) be an ancestor of \(v\) such that \(\ell(\tilde{v}) = \ell(u)\). Since \(v \land u \notin \{u, v\}\), necessarily \(u \neq \tilde{v}\). So from the above, \(B_d(x(u), \Delta(\psi(u))/20) \cap \{x(w) : w \in \mathbb{T}_{\tilde{v}}\} = \emptyset\). Combining it with the assumption that \(v \in \psi^{(*)}(\tilde{v})\) is an ancestor of \(\tilde{v}\), we conclude that in this case too, \(B_d(x(u), \Delta(\psi(u))/20) \cap \{x(w) : w \in \mathbb{T}_v\} = \emptyset\).

For compact ultrametrics some net-trees have more structure:

**Lemma 3.9.** Fix a net-tree \(\mathbb{T}\) with the root \(r_{\mathbb{T}} \in \mathbb{T}\) over a compact metric space \((X, d)\). The space \((U, \rho)\), where \(U = \partial \mathbb{T}\) and \(\rho(a, b) = \Delta(a \land b)\), is a compact ultrametric. Furthermore, \(\mathbb{T}\) is also surjective 1-net-tree over \((U, \rho)\) and the mapping \(x(\cdot) : (U, \rho) \to (X, d)\) is 2-Lipschitz.

In the reverse direction, fix a compact ultrametric \((U, \rho)\). There exists a surjective 1-net-tree \(\mathbb{T}\) over \(U\) for which \(\rho(x(a), x(b)) = \Delta(a \land b)\), for every \(a, b \in \partial \mathbb{T}\).

Furthermore, \(\mathcal{O}_\mathbb{T} = \{\partial \mathbb{T}(u) : u = r_{\mathbb{T}}\text{ or } \Delta(u) < \Delta(\psi(u))\}\) is the set of open balls in \((U, \rho)\) and the set of closed balls which are not singleton.

**Proof.** We begin with the first statement. Observe that in a tree, for any three branches \(a, b, c \in \partial \mathbb{T}\) \(a \land b\) or \(b \land c\) (or both) must be a weak ancestor of \(a \land c\), and from the monotonicity of the labels,
\[
\rho(a, c) = \Delta(a \land c) \leq \max\{\Delta(a \land b), \Delta(b \land c)\} = \max\{\rho(a, b), \rho(b, c)\}.
\]
Hence \(\rho\) is an ultrametric. Observe that \(\mathbb{T}\) is a surjective net-tree over \((U, \rho)\), so by Proposition 3.6, \(U\) is compact. The 2-Lipschitz property of \(x(\cdot)\) follows from the covering property of \(\mathbb{T}\) over \(X\).

The second and third statements are proved in [17, §2].

The above representation of (compact) ultrametrics as boundaries of net-trees is well known and is discussed in the literature using assortment of terminologies. Examples are *hierarchical well-separated trees* (in, e.g., [4, §3.1]), *boundary of the visual metric over trees* (in, e.g., [1, §1]), and *the end space of trees* (in [9, §1] and references therein).

4. **Proof of the ultrametric skeleton theorem**

The result of iterative applications of Corollary 2.3 on a net-tree of a given compact space, is a hierarchy described in the following lemma. For \(A \subseteq X\), denote by \(\overline{A}\) the topological closure of \(A\).
Lemma 4.1. Fix a compact $\lambda$-doubling metric space $(X, d)$, a finite Borel measure $\mu$ on $X$, and $t \in \{2, 3, 4, \ldots \}$. Then, there exists a net-tree $T$ with the root $r \in T$ and Borel subsets $C_u \subseteq X$ associated with every $u \in T$, having the following properties.

(A) $C_r = X$.
(B) $C_u \subseteq C_{\psi(u)}$ for every $u \in T \setminus \{r\}$.
(C) $\Delta(u) = \text{diam}_d(C_u)$.
(D) If $u, v \in T$, $u \land v \notin \{u, v\}$, then $d(C_u, C_v) \geq \Delta(u \land v)/(16t)$.
(E) For every branch $b \in \bar{\partial}(r)$, $\bigcap_{v \in b} \overline{C_v} = \{x(b)\}$.
(F) $\bar{\partial}(u) \subseteq \overline{C_u}$, for every $u \in T$.
(G) $B_d(x(u), c\Delta(\psi(u))/t) \subseteq C_u$, for some universal $c > 0$, and for every $u \in T \setminus \{r\}$.

In particular, defining $\rho(x(a), x(b)) = \Delta(a \land b)$ for $a, b \in \bar{\partial}(r)$,

(H) $\rho$ is an ultrametric on $\text{Im}(T)$ satisfying $d(x, y) \leq \rho(x, y) \leq 16t \cdot d(x, y)$, for every $x, y \in \text{Im}(T)$.

Lastly, there exists a function $\xi : T \to [0, \infty)$ satisfying:

(I) $\xi$ is sub-additive on $T$. I.e., for every vertex $u \in T$, $\xi(u) \leq \sum_{v \in \psi^{-1}(u)} \xi(v)$.
(J) For every $u \in T$,

\[ \mu^{1-1/t}(C_u) \leq \xi(u) \leq \lambda^{2/t} \mu^{1-1/t}(C_u). \]

Lemma 4.1 is very similar to a corresponding lemma in [16], but with the crucial addition of Item (G), which adds a significant technical complication to the proof.

Proof of Lemma 4.1. Fix a surjective 20-net-tree $N$ over $X$ with the root $r_N$, whose existence was established in Proposition 3.8. The rooted tree $T$, the clusters associated with it, and their representative points are defined inductively from top to bottom. For every vertex $u \in T$, instead of defining $x(u)$ and $C_u$ directly on $X$, we will define "tilde versions", $\tilde{C}_u \subseteq N$ and $\tilde{x}(u) \in \tilde{C}_u$ using vertices of the net-tree $N$. We will maintain by induction that $\tilde{C}_u$ is finite. Using the tilde versions, we define $\tilde{x}(u) = x_N(\tilde{x}(u))$, and $\tilde{C}_u = \bigcup_{x \in \tilde{C}_u} \Diamond_N(x)$. Define $\Delta(u) = \text{diam}_d(C_u)$ (which satisfies Item (C)).

The cluster associated with the root of $T$, $r \in T$, is $\tilde{C}_r = \{r_N\}$, and its representative is $\tilde{x}(r) = r_N$, which satisfies Item (A).

Assume next that the vertex $u \in T$, the associated cluster $\tilde{C}_u$, and its representative $\tilde{x}(u)$ were defined. If $\Delta(u) = 0$ (i.e., $C_u$ is a singleton), we simply define a new vertex $v \in T$ whose parent is $u = \psi(v)$, $\tilde{C}_v = \{\tilde{v}\}$, $\tilde{x}(v) = \tilde{v}$, and $\Delta(v) = 0$.

Next we assume that $\Delta(u) > 0$, (i.e., $|C_u| > 1$). Let $s_u = 4^{|\log_4 \Delta(u)|} / (64t)$ and

\[ \tilde{Z}_u = \bigcup_{y \in \tilde{C}_u} \tilde{\Diamond}_N(y, s_u). \]

Since (by the inductive hypothesis) $\tilde{C}_u$ is finite and so are $\tilde{\Diamond}_N(y, s_u)$ for $y \in \tilde{C}_u$, their union, $\tilde{Z}_u$, is also finite.
Informally speaking, both $\tilde{Z}_u$ and $\tilde{C}_u$ represent the cluster $C_u \subset X$. They both represent $C_u$ as a disjoint union of partial boundaries of $N$'s vertices, but at different scales. The scale of the vertices in $\tilde{C}_u$ is $s_{\psi(u)}$, while the scale of the vertices in $\tilde{Z}_u$ is $s_u$.

Observe that necessarily $|\tilde{Z}_u| > 1$ because, by the covering property, for every vertex $x \in \tilde{Z}_u$, $\text{diam}_d(\triangle_N(x)) \leq \Delta(v)/(64t) < \text{diam}_d(C_v)$. Let $\tilde{d}_u$ be a metric on $\tilde{Z}_u$ defined as $\tilde{d}_u(x, y) = d(x_N, x_N(y))$. Since $(\tilde{Z}_u, \tilde{d})$ is a metric induced by a subset of $X$ it is also $\lambda$-doubling. We also define a (discrete) measure $\tilde{\mu}_u$ on $\tilde{Z}_u$ as $\tilde{\mu}_u(\{x\}) = \mu(\triangle_N(x))$. Since the partial boundaries of unrelated vertices in $N$ are disjoint, we have for any non-root vertex $u \in T$, $\mu(C_u) = \tilde{\mu}_{\psi(u)}(\tilde{C}_u) = \tilde{\mu}_u(\tilde{Z}_u)$.

Apply Corollary 2.3 on $(\tilde{Z}_u, \tilde{d}_u, \tilde{\mu}_u)$ with $\Delta = \Delta(u) + 12s_u$, and let $\tilde{P}, \tilde{Q} \subset \tilde{Z}_u$ the resulting subsets. Define new vertices $v$ and $w$ as the children of $u$ in $T$. Define $\tilde{C}_v = \tilde{P}$ and $\tilde{C}_w = \tilde{Q}$. This, in particular, satisfies Item (B). Define $\bar{x}(v) \in \tilde{C}_v, \bar{x}(w) \in \tilde{C}_w$ arbitrary net-tree vertices in $\tilde{C}_v$ and $\tilde{C}_w$, respectively. We have thus finished describing the inductive construction of $T$. We are left to check that $T$ is indeed a net-tree over $X$ and prove the rest of the properties of $T$.

Item (D): Fix a pair of vertices $v, w \in T$ for which $v \cap w \notin \{v, w\}$. Denote $u = v \cap w$, and let $v' \in \psi^{-1}(u) \cap \psi^{(s)}(w)$, $w' \in \psi^{-1}(u) \cap \psi^{(t)}(w)$ be the children of $u$ which are ancestors of $v, w$ (respectively). By Corollary 2.3,

$$d(\bar{C}_v', \bar{C}_w') \geq \text{diam}_d(\bar{C}_u)/8t.$$  

By the covering property of the net-tree $N$,

$$\Delta(u) = \text{diam}_d(C_u) \leq \text{diam}_d(\bar{C}_u) + 2s_u \leq \text{diam}_d(\bar{C}_u) + \Delta(u)/(32t).$$

Similarly, by the covering property and the triangle inequality

$$d(C_v, C_w) \geq \bar{d}(\bar{C}_v', \bar{C}_w') - 2s_u \geq \bar{d}(\bar{C}_v', \bar{C}_w') - \Delta(u)/(32t).$$

Concatenating the last three inequalities above,

$$d(C_v, C_w) \geq d(C_v, C_w) \geq \Delta(u)/(16t),$$

which proves Item (D).

Item (E): Fix a branch $b = (r = v_0, v_1, v_2, \ldots) \in \partial T(r)$. $\overline{C}_{v_i}$ is compact and $C_{v_{i+1}} \subseteq C_{v_i}$. Obviously $\Delta(v_i) = \text{diam}(C_{v_i}) = \text{diam}(\overline{C}_{v_i})$. We claim that $\Delta(v_i) \geq 0$. Indeed, $\Delta(v_i)$ is non-increasing. Suppose towards a contradiction that there exists $\varepsilon > 0$ such that $\Delta(v_i) \geq \varepsilon$, for every $i \in \mathbb{N}$. If there exists $v_i$ which has only one child then by the construction above, $C_{v_i}$ is a singleton, its descendants have only one child, and by the definition of $\Delta(\cdot)$, $\Delta(v_j) = 0$ for $j \geq i$. So assume now that every vertex $v_i$ has two children. Let $u_i$ be the “other child” of $v_i$, i.e., the child of $v_i$ such that $u_i \neq v_{i+1}$. Observe that for $i < j$, $u_i \cap u_j = v_i$. Therefore, for every $i \neq j$,

$$d(C_{u_i}, C_{u_j}) \geq \Delta(v_{\min\{i,j\}})/(16t) \geq \varepsilon/(16t),$$
which contradicts the compactness of \( X \). Hence, \( \lim_{t \to \infty} \Delta(v_t) = 0 \). By Cantor’s intersection theorem, \( \bigcap_{t \in \mathbb{N}} C_{v_t} \) is a singleton. Since, by construction \( x(v_t) \in C_{v_t} \), we have \( x(b) = \lim_{t \to \infty} x(v_t) \in \bigcap_{t \in \mathbb{N}} C_{v_t} \) is the unique element.

Item (F): The assertion \( \partial \subseteq \overline{C}_u \) is an immediate corollary of Item (E).

Item (G): Assume \( u \neq r_T \). By construction, \( C_u \supseteq \bigtriangleup_{N}(\bar{x}(u)) \), where (again by the construction) \( \bar{x}(u) \in \bar{C}_u \subseteq \bar{Z}_{\psi(u)} \subseteq \bigwedge_{N}(\bar{r}_N, s_{\psi(u)}) \). This means that \( \Delta_N(\psi_N(\bar{x}(u))) \geq \Delta(\psi(u))/(256t) \). Therefore, by the packing property of the net-tree \( N \) (Proposition 3.7, Item (E)), we have \( C_u \supseteq \bigtriangleup_{N}(\bar{x}(u)) \supseteq B_{d}^u(x(u), \Delta_N(\psi_N(\bar{x}(u)))/20) \supseteq B_d(x(u), \Delta(\psi(u))/(5121t)) \).

Item (H): Fix \( a, b \in \partial_T(r) \). Since \( x(a), x(b) \in \overline{C}_{a \wedge b} \), we have \( d(x(a), x(b)) \leq \text{diam}_d(C_{a \wedge b}) = \rho(x(a), x(b)) \). On the other hand, denote \( v = a \cap \psi^{-1}(a \wedge b) \), \( w = b \cap \psi^{-1}(a \wedge b) \). Since \( x(a) \in \overline{C}_v \) while \( x(b) \in \overline{C}_w \), we have \( d(x(a), x(b)) \leq d(C_v, C_w) = \Delta(a \wedge b)/(16t) = \rho(x(a), x(b))/(16t) \).

This in particular implies that \( T \) is a \((16t)\)-net-tree over \( X \).

We are left to define \( \xi : T \to [0, \infty) \), and prove that it is sub-additive and approximates \( \mu^{1-1/t}(C_u) \). Define \( \mu^* = \mu_{\Delta(u)+12s_u} \), and

\[
\xi(u) = \frac{\mu(C_u)}{\mu^*(\bar{C}_u)^{1/t}}.
\]

If \( u \) has only one child, then, by construction, \( C_u \) is a singleton and \( \xi(u) = \mu^{1-1/t}(C_u) \). In this case the sub-additivity is trivial. So we assume now that \( u \) has two children. Let \( v \) be the child of \( u \) associated with \( P \) in Corollary 2.3, and \( w \) the child associated with \( Q \). By Inequality (10) of Corollary 2.3, we have

\[
(13) \quad \xi(u) = \frac{\mu_{\Delta(u)+12s_u} \bar{C}_u}{\mu_{\Delta(u)+12s_u} \bar{C}_u}^{1/t} \leq \frac{\mu(C_v)}{\mu_{\Delta(u)+2+6s_u} \bar{C}_v}^{1/t} + \frac{\mu(C_w)}{\mu_{\Delta(u)+12s_u} \bar{C}_w}^{1/t}.
\]

**Claim 4.2.**

\[
(14) \quad \mu_{\Delta(u)+12s_u}^*(\bar{C}_v) \leq \mu_{\Delta(u)+2+6s_u}^*(\bar{C}_v), \quad \mu_{\Delta(u)+12s_u}^*(\bar{C}_w) \leq \mu_{\Delta(u)+12s_u}^*(\bar{C}_w)
\]

**Proof of Claim 4.2.** We begin with the inequality on the left. By Corollary 2.3,

\[
(15) \quad \Delta(v) = \text{diam}_d(C_v) \leq \text{diam}_d(\bar{C}_v) + 2s_u
\]

\[
\leq \left( \frac{1}{2} - \frac{1}{4t} \right)(\Delta(u) + 12s_u) + 2s_u \leq \Delta(u)/2 + 6s_u - \Delta(u)/(4t) + 2s_u \leq \Delta(u)/2 - 8s_u.
\]

By construction \( s_v \leq s_u \), and their ratio is a multiple of 4. So either \( s_v = s_u \) or \( s_v \leq s_u/4 \). Assume first that \( s_v = s_u \). In this case, by definition, \( \bar{Z}_v = \bar{C}_v \), and so

\[
\mu_{\Delta(u)+12s_u}^*(\bar{Z}_v) = \mu_{\Delta(u)+12s_u}^*(\bar{C}_v) = \mu_{\Delta(u)+12s_u}^*(\bar{C}_v)
\]

\[
\leq \mu_{\Delta(u)/2-8s_u+12s_u}^*(\bar{C}_v) \leq \mu_{\Delta(u)/2+6s_u}^*(\bar{C}_v).
\]
Next, assume that \( s_v \leq s_u/4 \). Fix \( x \in \tilde{Z}_v \). If \( y \in B_{\tilde{d}_u}(x, (\Delta(v) + 12s_v)/4) \), then
\[
\tilde{d}_u(x, y) \leq \frac{\Delta(v) + 12s_v}{4} \leq \frac{\Delta(u)/2 - 8s_u + 12s_v}{4} \leq \frac{\Delta(u)/2 - 3s_u}{4}.
\]
By the construction of \( \tilde{Z}_v \), there exists a unique \( x' \in \tilde{C}_v \cap \psi_N^{(s)}(x) \). By the covering property of \( x' \), \( d(x_N(x), x_N(x')) \leq \Delta_N(x') \leq s_u \). Similarly, there exists a unique \( y' \in \tilde{C}_v \cap \psi_N^{(s)}(y) \) and it satisfies \( d(x_N(y), x_N(y')) \leq s_u \). So \( y' \in B_{\tilde{d}_u}(x', (\Delta(u)/2 - 3s_u)/4 + 2s_u) \). This implies
\[
\tilde{\mu}_w(B_{\tilde{d}_u}(x, (\Delta(v) + 12s_v)/4) \cap \tilde{Z}_v) \leq \tilde{\mu}_w(B_{\tilde{d}_u}(x', (\Delta(u)/2 + 5s_u)/4) \cap \tilde{C}_v),
\]
which implies \( \mu^*_w(\tilde{Z}_v) \leq \tilde{\mu}_u(\tilde{Z}_v) \).

We move to prove the right inequality of (14). Similar to the argument above, either \( s_w = s_u \) or \( s_w \leq s_u/4 \). Assume first that \( s_u = s_w \). In this case, by definition, \( \tilde{Z}_w = \tilde{C}_w \), and so
\[
\mu^*_w(\tilde{Z}_w) = \tilde{\mu}_w(\tilde{Z}_w) = \tilde{\mu}_u(\tilde{C}_w) \leq \tilde{\mu}_u(\tilde{C}_w).
\]
Next, assume that \( s_w \leq s_u/4 \). Fix \( x \in \tilde{Z}_w \) and \( y \in B_{\tilde{d}_w}(x, (\Delta(w) + 12s_w)/4) \). Then
\[
\tilde{d}_w(x, y) \leq \frac{\Delta(w) + 12s_w}{4} \leq \frac{\Delta(u) + 3s_u}{4}.
\]
By an argument similar to the above, there exist \( x' \in \tilde{C}_w \cap \psi_N^{(s)}(x) \), \( y' \in \tilde{C}_w \cap \psi_N^{(s)}(y) \) for which \( d(x_N(x), x_N(x')) \leq s_u \), and \( d(x_N(y), x_N(y')) \leq s_u \). So \( y' \in B_{\tilde{d}_w}(x, (\Delta(u) + 3s_u)/4 + 2s_u) \). This implies
\[
\tilde{\mu}_w(B_{\tilde{d}_w}(x, (\Delta(u) + 12s_u)/4) \cap \tilde{Z}_w) \leq \tilde{\mu}_w(B_{\tilde{d}_w}(x', (\Delta(u) + 11s_u)/4) \cap \tilde{C}_w),
\]
which implies \( \mu^*_w(\tilde{Z}_w) \leq \tilde{\mu}_w(\tilde{Z}_w) \). This finishes the proof of Claim 4.2.

Applying (14) on (13), we obtain the sub-additivity of \( \xi \):
\[
\xi(u) \leq \frac{\mu(C_u)}{\mu^*_w(\tilde{Z}_w)^{1/t}} + \frac{\mu(C_w)}{\mu^*_w(\tilde{Z}_w)^{1/t}} = \xi(v) + \xi(w).
\]
This proves Item (I).

Item (J): Lastly, we prove Inequality (12). Recall that \((\tilde{Z}_u, \tilde{d}_u)\) is also \( \lambda \)-doubling, so using Inequality (6),
\[
\lambda^{-2} \mu(C_u) = \lambda^{-2} \tilde{\mu}_u(\tilde{Z}_u) \leq \mu^*_w(\tilde{Z}_u) \leq \tilde{\mu}_u(\tilde{Z}_u) = \mu(C_u).
\]
Thus, Inequality (12) follows. \( \square \)

One of the outputs of Lemma 4.1 is \( \xi \), which is essentially a sub-additive premeasure on the skeleton \( T \) controlled by \( \mu^{1-1/t} \) from above and from below. The next lemma trims \( T \) and “lowers” \( \xi \) to make \( \xi \) additive.

**Lemma 4.3.** Fix \( \delta \in (0, 1] \), a rooted tree \( T \) with the root \( r \in T \), and \( \xi : T \to [0, \infty) \). Assume that \( \xi \) is sub-additive, i.e., \( \xi(u) \leq \sum_{\psi^{-1}(u)} \xi(v) \), for every \( u \in T \). Further, assume that for any vertex \( v \in T \) and an ancestor \( u \in \psi^{(s)}(v) \), \( \xi(u) \geq \delta \xi(v) \). Then there exists a subtree \( S \subseteq T \) with the same
root $r$, and additive mapping $\sigma: S \to [0, \infty)$ for which $\sigma(r) = \xi(r)$, and $\sigma$ is a $(\delta/2)$-approximation of $\xi$ on $S$. I.e., for every $u \in S$,

\begin{equation}
\xi(u) \geq \sigma(u) - \frac{\delta}{2} \xi(u), \quad \sigma(u) = \sum_{v \in S} \min_{\psi^{-1}(u)} \sigma(v).
\end{equation}

Proof. We use an inductive argument on $T$, starting from the root, to decide which vertices to keep in $S$, and how to change $\xi$. Formally, the proof is by induction on the depth of the vertices in $T$.

During the inductive process we maintain for each vertex of $T$ a status in $S$ as one of the following: “pending”, “retained” (in $S$), and “deleted” (from $S$). The function $\sigma$ is constructed during the process, and at each point during the process, $\sigma$ is defined on the retained vertices and their non-deleted children. In the process we maintain the invariants that the set of vertices which are either “retained” or “pending” form a subtree rooted at $r$, and the retained vertices form a (graph theoretic) connected component containing the root $r$. Furthermore, if a retained vertex $u \in S$ has one or more non-deleted sibling or $u = r$, then $\xi(u) \geq \sigma(u) > \xi(u)/2$. Otherwise $(u \neq r$ and $|\psi^{-1}(\psi(u)) \cap S| = 1)$, $\xi(u) \geq \sigma(u) \geq \delta \xi(u)/2$.

We begin with all vertices of $T$ in a “pending” status. The mapping $\sigma$ is defined only on the root, $\sigma(r) = \xi(r)$. In the inductive step, let $u \in T$ be a vertex in “pending” status with minimal depth among the pending vertices. The vertex $u$ is either $r$ (if it is the first step of the inductive process), or its parent is already “retained”. Either way, by the inductive invariant, $\sigma(u)$ is already set. The status of $u$ is changed to “retained”.

Let $L \subseteq \psi^{-1}(u)$ be a minimal subset (with respect to containment) of the children of $u$ such that $\sigma(u) \leq \sum_L \xi(v)$ (such a subset exists from the sub-additivity of $\xi$, and the inductive assumption $\sigma(u) \leq \xi(u)$). We label all the vertices in $\psi^{-1}(u) \setminus L$ and their descendants as “deleted” (i.e., they will not be part of $S$).

If $|L| > 1$, then from the minimality condition of $L$, we have

\[
\frac{\sum_L \xi(v)}{2} \leq \sum_L \xi(v) - \min_L \xi(v) < \sigma(u).
\]

Hence $\sum_L \xi(v) < 2\sigma(u)$, and we define for every $v \in L$,

\[
\sigma(v) := \frac{\sigma(u)}{\sum_L \xi(v)} : \xi(v) \in (\xi(v)/2, \xi(v)].
\]

This, in particular, means that, $\sigma(u) = \sum_L \sigma(v)$.

If $L = \{v\}$ (a singleton), we set $\sigma(v) = \sigma(u)$. The additivity condition is trivially true. Let $w \in \psi^{(*)}(u)$ be the lowest ancestor of $u$ which is either $r$ or having at least two non-deleted children. Let $\{y\} = \psi^{(*)}(u) \cap \psi^{-1}(w)$ be a (weak) ancestor of $u$ which is a child of $w$. From the inductive hypothesis about the additivity of $\sigma$, $\sigma(v) = \sigma(u) = \sigma(y)$. Also from the inductive hypothesis, $\sigma(y) > \xi(y)/2$; and from the hypothesis of the lemma, $\xi(y) \geq \delta \xi(v)$. Hence, $\sigma(v) > \delta \xi(v)/2$.

Since every vertex of $T$ is of finite depth and the set of vertices at the same or smaller depth is also finite, this process classifies all the vertices
of \( T \) as either “deleted” or “retained”. From the inductive invariant, the retained vertices forms a subtree \( S \), and \( \sigma \) is defined on \( S \) and satisfies the assertions of the lemma. \( \square \)

**Proof of Theorem 1.2.** First observe that when \( t = 1 \) the assertion is trivial: We set \( S = \{ x \} \) for some \( x \in X \), and \( \mu = \delta_x \) the dirac delta measure on \( x \). Henceforth, we therefore assume \( t \in \{2,3,\ldots\} \).

An application of Lemma 4.1 on \( X \) results in a net-tree \( T \), clusters \( \{ C_u \}_{u \in T} \), and a sub-additive \( \xi : T \to [0, \infty) \) satisfying (12). Observe that if \( u, v \in T, u \in \psi(x)(v) \) then

\[
\xi(u)^{(12)} \geq \mu^{1-1/t}(C_u) \geq \mu^{1-1/t}(C_v)^{(12)} \geq \lambda^{-2/t} \xi(v).
\]

Thus, \( T \) and \( \xi \) satisfy the conditions of Lemma 4.3 with \( \delta = \lambda^{-2/t} \). Application of Lemma 4.3 on \( T \) results in a subtree \( S \), and additive \( \sigma : S \to [0, \infty) \) that satisfies

\[
(17) \quad \lambda^{-2/t} \mu^{1-1/t}(C_u)/2 \leq \lambda^{-2/t} \xi(u)/2 \leq \sigma(u) \leq \xi(u) \leq \lambda^{2/t} \mu^{1-1/t}(C_u),
\]

for every \( u \in S \).

Define \( S = \text{Im}(S) \subseteq \text{Im}(T) \). The induced ultrametric on \( S \) is the restriction of \( \rho \) to \( S \).

Recall that a (measure-theoretic) semi-ring in \( X \) is a collection of subsets \( S \subseteq 2^X \) satisfying (i) \( \emptyset \in S \); (ii) if \( A, B \in S \) then \( A \cap B \in S \); (iii) if \( A, B \in S \) then there exist \( n \geq 0 \) and \( A_1, \ldots, A_n \in S \) pairwise disjoint such that \( A \setminus B = \bigcup_{i=1}^n A_i \). By Lemma 3.9, \( \{ \emptyset \} \cup \{ \partial_T(v) \}_{v \in T} \) is a semi-ring consisting of all the open balls in \( (U, \rho) \). Furthermore and \( \nu(\partial_S(v)) = \sigma(v) \) is a pre-measure on that semi-ring. By Carathéodory extension theorem (see [12, Theorem 1.53]), \( \nu \) can be extended to a measure on the \( \sigma \)-algebra generated by \( \{ \partial_T(v) \}_{v \in T} \), which is the \( \sigma \)-algebra of Borel sets of \( (S, \rho) \). Since the metrics \( d|_S \) and \( \rho|_S \) are topologically equivalent, and \( S \) is a Borel (closed) set of \( (X, d) \), \( \nu \) can be extended to Borel measure on \( X \) by simply define \( \nu(A) = \nu(A \cap S) \) on every Borel \( A \subseteq X \).

We next prove (2). Fix \( x \in X \) and \( r \geq 0 \). If \( B_d(x, r) \cap S = \emptyset \), then \( \nu(B_d(x, r)) = 0 \) and there is nothing to prove. Otherwise, let \( y \in B_d(x, r) \cap S \), so

\[
B_d(x, r) \cap S \subseteq B_d(y, 2r) \cap S \subseteq B_{\rho|_S}(y, 32tr).
\]

Since \( B_{\rho|_S}(y, 32tr) \) is a closed \( (S, \rho|_S) \) ball, by Lemma 3.9 there exists some \( v \in S \) such that \( B_{\rho|_S}(y, 32tr) = \partial_S(v) \). In particular \( \text{diam}_d(C_v) = \Delta(v) \leq 32tr \). Observe that

\[
B_{\rho|_S}(y, 32tr) = \partial_S(v) \subseteq \overline{C_v} \subseteq B_d(y, 32tr) \subseteq B_d(x, (32t + 1)r).
\]

Hence

\[
\nu(B_d(x, r)) \leq \nu(\partial_S(v)) = \sigma(v) \leq \lambda^{2/t} \mu(C_v)^{1-1/t} \leq \lambda^{2/t} \mu(B_d(x, (32t + 1)r))^{1-1/t}.
\]

We are left to prove (3). Let \( y \in S \) and \( r \geq 0 \),

\[
B_d(y, r) \cap S \supseteq B_{\rho|_S}(y, r),
\]
By Lemma 3.9 there exists some \( v \in S \) such that \( \partial_S(v) = B_{\rho_S}(y, r) \). Let \( v \in S \) be such a vertex with the smallest depth in \( S \). In particular \( \Delta(v) \leq r \) which implies that \( C_v \subseteq B_d(y, r) \). Furthermore, either:

- The vertex \( v \) is the root. In which case \( B_d(x(v), r/t) \subseteq X = C_v \).
- There exists \( z \in \partial_S(\psi(v)) \setminus B_{\rho_S}(y, r) \). In this case, \( z, y \in \partial_S(\psi(v)) \), and \( \rho(z, y) > r \), and so \( \Delta(\psi(v)) > r \). From Item (G) of Lemma 4.1, we also have

\[
B_d(x(v), c\Delta(\psi(v))/t) \subseteq C_v \subseteq B_d(y, r),
\]

for some universal constant \( c > 0 \).

We conclude:

\[
\nu(B_d(y, r)) \geq \nu(\partial_S(v)) = \sigma(v) \geq (\lambda^{-2/t}/2) \cdot \mu(C_v)^{1-1/t}
\]

\[
\geq (\lambda^{-2/t}/2) \cdot \mu(B_d(x(v), cr/t))^{1-1/t}.
\]

\[ \square \]

5. Proof of the Dvoretzky-type theorem

Using Theorem 1.2 we prove the following Dvoretzky-type result for Ahlfors regular spaces.

**Lemma 5.1.** For every \( t \in \{2, 3, \ldots\} \), \( \alpha > 0 \), and compact Ahlfors \( \alpha \)-regular space \((X, d)\), there exists a compact \(((1-1/t)\alpha)\)-regular subset \( Y \subseteq X \) whose ultrametric distortion is at most \( O(t) \).

**Proof.** Let \( \mu \) be the measure satisfying (1) for \( X \). By Theorem 1.2, there exists a compact subset \( Y \subseteq X \) which embeds with distortion \( O(t) \) in an ultrametric and a measure \( \nu \) supported on \( Y \) such that for any \( x \in X \) and \( r \in (0, \text{diam}_d(X)) \), there exists \( x' \in X \) satisfying

\[
\nu(B_d(x, r)) \leq \lambda^{2/t}(\mu(B_d(x, C_1tr)))^{1-1/t}
\]

\[
\leq \lambda^{2/t}C(C_1tr)^{(1-1/t)\alpha} = A_{\lambda,t} \cdot r^{1-1/t},
\]

\[
\nu(B_d(x, r)) \geq 0.5\lambda^{-2/t}(\mu(B_d(x', cr/t)))^{1-1/t}
\]

\[
\geq 0.5\lambda^{-2/t}c(c_1r/t)^{1-1/t} = a_{\lambda,t} \cdot r^{1-1/t}.
\]

Since \( \nu \) is supported on \( Y \), it means that \( Y \) is \(((1-1/t)\alpha)\)-regular. \[ \square \]

The lemma above extracts an Ahlfors \( \beta \)-regular approximate ultrametric subset for a sequence of \( \beta \)'s as close as we wish to \( \alpha \), but not at any \( \beta < \alpha \). To obtain an Ahlfors \( \beta \)-regular subset for arbitrary \( \beta < \alpha \), we use the following lemma from [1]. Here we give a quick and self-contained proof based on Lemma 4.3.

**Lemma 5.2 ([1, Theorem 1.4]).** Let \((U, \rho)\) be a compact Ahlfors \( \alpha \)-regular ultrametric, and \( \beta \in (0, \alpha) \). Then there exists a \( \beta \)-regular subset \( S \subseteq U \).

**Proof.** Assume \( \beta < \alpha \) (when \( \beta = \alpha \) there is nothing to prove). Let \( T \) be a net-tree representing \( U \) according to Lemma 3.9. Denote \( \xi : T \to [0, \infty) \), \( \xi(v) = \mu(\partial_T(v)) \). Since \( \mu \) is a measure, \( \xi \) is additive and monotone. I.e.,
there exists a compact subset of Lemma 1.1. This means that is similar to the one employed in [16, M. MENDEL]. We conclude that is additive and  exists an Ahlfors (1 − 1/2)-regular compact subset that is additive and  ≥  / 2, for any  ∈ . Define  = 1*( ) ⊆ .

It follows from Lemma 3.9 that {∂\(\mathcal{S}(v)\)}\(v\in\mathcal{S}\) is a (measure-theoretic) semi-ring consisting of all the open balls in \((\mathcal{S}, \rho|_\mathcal{S})\). Thus,  =  is a pre-measure on that semi-ring. By Carathéodory extension theorem,  can be extended to a measure on the σ-algebra generated by \{∂\(\mathcal{S}(u)\}\}_{u∈\mathcal{S}}\), which are the Borel sets of \((\mathcal{S}, \rho|_\mathcal{S})\).

Fix a closed ball  =  ⊆ , where  ∈ ,  ∈ (0, diam\(\mathcal{S}(S)\)). By Lemma 3.9, there exists  ∈ such that  =  =  =  and let  be such vertex of least depth. I.e., Either  =  or  < Δ (v) ≤ Δ (ψ(v)). Observe that  ∈  and  =  ⊆  \(\mathcal{S}(x, r)\). Therefore,

\[\nu(\mathcal{S}(x, r)) ≤ \xi(\beta/\alpha) = \mu(\partial\mathcal{T}(v))^{\beta/\alpha} ≤ \mu(\mathcal{S}(x, r))^{\beta/\alpha} ≤ C^{\beta/\alpha} \cdot r^\beta,\]

In the other direction, assume first that  =  =  \(\mathcal{T}(v)\). Then

\[\nu(\mathcal{S}(x, r)) ≥ 0.5 \cdot \xi(r)^{\beta/\alpha} = 0.5 \cdot \mu(\partial\mathcal{T}(v))^{\beta/\alpha} ≥ 0.5 \cdot \mu(U)^{\beta/\alpha} ≥ 0.5c^{\beta/\alpha} \cdot r^\beta.\]

If  ≠  =  =  \(\mathcal{T}(v)\), then by the choice of  = ∊ \(\mathcal{S}(\psi(v))\) such that  >  and hence

\[\partial\mathcal{T}(\psi(v)) \supseteq \mathcal{S}(x, r) \supset \partial\mathcal{T}(v).\]

By Lemma 3.9 this means that  =  = \(\mathcal{T}(v)\), and therefore,

\[\nu(\mathcal{S}(x, r)) ≥ 0.5 \cdot \xi(\beta/\alpha) = 0.5 \cdot \mu(\partial\mathcal{T}(v))^{\beta/\alpha} ≥ 0.5 \cdot \mu(\mathcal{S}(x, r))^{\beta/\alpha} ≥ 0.5c^{\beta/\alpha} \cdot r^\beta.\]

We conclude that  is \(\beta\)-regular. □

Proof of Theorem 1.1. Fix a bounded Ahlfors \(\alpha\)-regular space \((X, d)\). \(X\) must be compact, see [6, Corollary 5.2]. Fix  = (0, \(\alpha\)). Let  = \(\lceil \frac{\alpha}{\alpha - \beta} \rceil \in \{2, 3, \ldots\}\). Observe that \((1 - 1/t)\alpha ≥ \beta\) and  \(t ≤ 2\alpha / \alpha.\) By Lemma 5.1 there exists an Ahlfors ((1 − 1/t)\(\alpha\))-regular compact subset  ⊆  such that \((Z, d)\) embeds in an ultrametric \((Z, \rho)\) with distortion  = \(O(\alpha/\alpha - \beta)\). Ahlfors regularity is invariant of biLipschitz isomorphisms, and therefore \((Z, \rho)\) is also ((1 − 1/t)\(\alpha\))-regular. By Lemma 5.2 there exists a compact subset \(Y ⊆ Z\) such that \((Y, \rho|_Y)\) is \(\beta\)-regular. Since is \((Y, d|_Y)\) is biLipschitz isomorphic to \((Y, \rho|_Y)\) (with distortion \(O(\alpha/(\alpha - \beta))\)), it is also \(\beta\)-regular. □

6. Remarks

The basic approach used here for proving Theorem 1.2 is similar to the one employed in [16]. In order to obtain the lower bound (3) on \(\nu\), we amended that approach as follows:

a) We made sure that the clusters created in the hierarchical decomposition contain balls of the underlying spaces (Item (G) of Lemma 4.1). We achieved it by applying Corollary 2.3 not directly on the space  = \(X\), but on a hierarchy of the nets (the net-tree) that ensured that each point of the net carries with it a porportional ball.
b) Lemma 4.1 produced a sub-additive “premeasure” $\xi$, while we need an additive premeasure. This issue was addressed in [17, 16] by artificially lowering $\xi$ down the net-tree. The resulting measure $\nu$ can not be bounded from below by $\mu^{1-1/t}$. Here we used a different tactic in Lemma 4.3: The bulk of the slack in the sub-additivity is eliminated by trimming the net-tree. The lowering of $\xi$ is done only sparingly for fine-tuning.

Both steps do not seem to be particularly tied to the construction from [16], and might as well work with the more general construction of ultrametric skeleton from [18, 17]. That is, an appropriate version of Theorem 1.2 may also hold without the doubling assumption. We did not pursue this direction further here.

The lower bound (3) on $\nu$ in Theorem 1.2 is qualitatively weaker than the upper bound (2). Some of it is clearly unavoidable: the universal quantifier can not be “for every $y \in X$”, since ultrametric subsets must have “gaps”, like the interval $[0.4, 0.6]$ in the Cantor set (as a subset of $X = [0, 1]$). It is not clear whether the two balls in (3) can be concentric. I.e., whether it is possible to obtain the statement: “For every $y \in S$ and $r \in [0, \infty)$, $\nu(B_d(y, r)) \geq c \cdot \mu(B_d(y, c r))^{1-1/t}$.”

Theorem 1.2 is phrased for general finite Borel measures, but is applied in the proof of Theorem 1.1 only to regular measures, which are, in particular, doubling measures. For doubling measures, one could conceivably replace the use of the auxiliary net-tree in the proof of Lemma 4.1 with Christ’s dyadic-decomposition (see [5, Theorem 11] or [2, Theorem A]) to simplify the exposition of the proof of Theorem 1.1.

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References

[1] Nicola Arcozzi, Alessandro Monguzzi, and Maura Salvatori. Ahlfors regular spaces have regular subspaces of any dimension, 2019. arXiv:1912.02055.

[2] Nicola Arcozzi, Richard Rochberg, Eric T. Sawyer, and Brett D. Wick. Potential theory on trees, graphs and Ahlfors-regular metric spaces. Potential Anal., 41(2):317–366, 2014. arXiv:1010.4788, doi:10.1007/s11118-013-9371-8.

[3] Yair Bartal. Advances in metric ramsey theory and its applications, 2021. arXiv:2104.03484.

[4] Yair Bartal, Nathan Linial, Manor Mendel, and Assaf Naor. On metric Ramsey-type phenomena. Ann. of Math. (2), 162(2):643–709, 2005. arXiv:math/0406353, doi:10.4007/annals.2005.162.643.

[5] Michael Christ. A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math., 60/61(2):601–628, 1990. doi:10.4064/cm-60-61-2-601-628.

[6] Guy David and Stephen Semmes. Fractured fractals and broken dreams, volume 7 of Oxford Lecture Series in Mathematics and its Applications. The Clarendon Press, Oxford University Press, New York, 1997. Self-similar geometry through metric and measure.
[7] Sariel Har-Peled and Manor Mendel. Fast construction of nets in low-dimensional metrics and their applications. *SIAM J. Comput.*, 35(5):1148–1184, 2006. arXiv:cs/0409057, doi:10.1137/S009753970446281.

[8] Juha Heinonen. *Lectures on analysis on metric spaces*. Universitext. Springer-Verlag, New York, 2001. doi:10.1007/978-1-4613-0131-8.

[9] Bruce Hughes. Trees and ultrametric spaces: a categorical equivalence. *Adv. Math.*, 189(1):148–191, 2004. doi:10.1016/j.aim.2003.11.008.

[10] John E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981. doi:10.1512/iumj.1981.30.30055.

[11] Esa Järvenpää, Maarit Järvenpää, Antti Käenmäki, Tapio Rajala, Sari Rogovin, and Ville Suomala. Packing dimension and Ahlfors regularity of porous sets in metric spaces. *Math. Z.*, 266(1):83–105, 2010. doi:10.1007/s00209-009-0555-2.

[12] Achim Klenke. *Probability theory*. Universitext. Springer-Verlag London, Ltd., London, 2008. A comprehensive course, Translated from the 2006 German original. doi:10.1007/978-1-84800-048-3.

[13] James R. Lee, Manor Mendel, and Mohammad Moharrami. On the Hausdorff dimension of ultrametric subsets in $\mathbb{R}^n$. *Fund. Math.*, 218(3):285–290, 2012. arXiv:1205.2094, doi:10.4064/fm218-3-5.

[14] Jouni Luukkainen and Eero Saksman. Every complete doubling metric space carries a doubling measure. *Proc. Amer. Math. Soc.*, 126(2):531–534, 1998. doi:10.1090/S0002-9939-98-04201-4.

[15] Pertti Mattila and Pirjo Saaranen. Ahlfors-David regular sets and bilipschitz maps. *Ann. Acad. Sci. Fenn. Math.*, 34(2):487–502, 2009. URL: http://www.acadsci.fi/mathematica/Vol34/MattilaSaaranen.html, arXiv:0809.4877.

[16] Manor Mendel. A simple proof of dvoretzky-type theorem for hausdorff dimension in doubling spaces, 2021. arXiv:2104.11944.

[17] Manor Mendel and Assaf Naor. Ultrametric skeletons. *Proc. Natl. Acad. Sci. USA*, 110(48):19256–19262, 2013. arXiv:1112.3416, doi:10.1073/pnas.1202500109.

[18] Manor Mendel and Assaf Naor. Ultrametric subsets with large Hausdorff dimension. *Invent. Math.*, 192(1):1–54, 2013. arXiv:1106.0879, doi:10.1007/s00222-012-0402-7.

[19] Assaf Naor. An introduction to the Ribe program. *Jpn. J. Math.*, 7(2):167–233, 2012. arXiv:1205.5993, doi:10.1007/s11537-012-1222-7.

[20] A. L. Vol’berg and S. V. Konyagin. On measures with the doubling condition. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(3):666–675, 1987. doi:10.1070/im1988v030n03abeh001034.

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