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A Mathematical Model for Behavioral Changes by Pair Interactions and Its Relation to Game Theory

Abstract

A mathematical model for behavioral changes by pair interactions (i.e. due to direct contact) of individuals is developed. Three kinds of pair interactions can be distinguished: Imitative processes, avoidance processes, and compromising processes. Representative solutions of the model for two different interacting subpopulations are illustrated by computational results.

The equations of game theory are shown to result for a special case of imitative processes. Moreover, a stochastic version of game theory is formulated. It allows the derivation of equations for the most probable or the expected distribution of behavioral strategies and of (co)variance equations. The knowledge of the (co)variances is necessary for the calculation of the reliability of game theoretical descriptions.

The use and application of the introduced equations is illustrated by concrete examples. Especially, computational results for the selforganization of social conventions by competition of two equivalent strategies are presented.

1 Introduction

This paper treats a mathematical model for the change of the fraction $P(i,t)$ of individuals who show a certain behavior $i$. Models of this kind are of great interest for a quantitative understanding or prognosis of social developments. For the description of the competition or cooperation of populations there already exist game theoretical approaches (see, for example, Mueller (1990), Axelrod (1984), von Neumann and Morgenstern (1944), Luce and Raiffa (1957)).

However, the model developed in this paper shows to be more general, since it includes as special cases

- not only the game dynamical equations (Hofbauer and Sigmund (1988)), but also

- the logistic equation (Verhulst (1845), Pearl (1924), Helbing (1992)),

- the Gravity model (Ravenstein (1876), Zipf (1946)),

- the Lotka-Volterra equations (Lotka (1920, 1956), Volterra (1931), Hofbauer (1981), Goel et. al. (1971), Hallam (1986), Goodwin (1967)), and

- the quantitative social models of Weidlich and Haag (Weidlich & Haag (1983, 1988), Weidlich (1991)).

This model assumes behavioral changes to occur with a certain probability per time unit, called the transition rate. The transition rate is decomposed into

- a rate describing spontaneous behavioral changes, and

- a rate describing behavioral changes due to pair interactions of individuals.

Three different kinds of pair interactions can be distinguished:

- First, imitative processes, which describe the tendency to take over the behavior of another individual.

- Second, avoidance processes, causing an individual to change the behavior if meeting another individual with the same behavior.
• Third, *compromising processes*, which describe the readiness to change the behavior to a new one when meeting an individual with another behavior.

Representative solutions of the model are illustrated by computer simulations. By distinguishing several *subpopulations* \( a \), different *types* of behavior can be taken into account.

As one would expect, there is a connection of this model with the game dynamical equations. In order to establish this connection, the transition rates have to be taken in a special way which depends on the *expected success* of the behavioral strategies. The essential effect is given by imitative processes.

A stochastic formulation of the game dynamical equations shows that the ordinary game dynamical equations result as equations for the *most probable* behavioral distribution or as *approximate mean value* equations.

For the approximate mean values corrections can be calculated. These corrections depend on the (co)variances \( \sigma_{ij} \) of the numbers \( n_i \) of individuals who show a certain behavior \( i \). The calculation of the (co)variances is also useful to determine the reliability of game theoretical descriptions.

An example of two equivalent competing strategies serves as an illustration of the game dynamical equations and their stochastic version. It allows the description of the *selforganization* of a behavioral convention.

### 2 The master equation

Suppose, we have a social system with \( N \) individuals. These individuals can be divided into \( A \) *subpopulations* \( a \) consisting of \( N_a \) individuals, i.e.,

\[
\sum_{a=1}^A N_a = N.
\]

By subpopulations different social groups (e.g. blue and white collars) or different characteristic *types* of behavior are distinguished.

The \( N_a \) individuals of each subpopulation \( a \) are distributed over several *states*

\[
i \in \{1, \ldots, S\},
\]

which represent the *behavior* or the *(behavioral) strategy* of an individual. If the *occupation number* \( n_i^a \) denotes the number of individuals of subpopulation \( a \) who show the behavior \( i \), we have the relation

\[
\sum_{i=1}^S n_i^a = N_a. \quad (1)
\]

Let

\[
n := (n_1^1, \ldots, n_i^a, \ldots, n_S^A)
\]

be the vector consisting of all occupation numbers \( n_i^a \). This vector is called the *socioconfiguration*, since it contains all information about the distribution of the \( N \) individuals over the states \( i \). \( P(n, t) \) shall denote the *probability* to find the socioconfiguration \( n \) at time \( t \). This implies

\[
0 \leq P(n, t) \leq 1 \quad \text{and} \quad \sum_n P(n, t) = 1.
\]

If transitions from socioconfiguration \( n \) to \( n' \) occur with a probability of \( P(n', t + \Delta t | n, t) \) during a short time interval \( \Delta t \), we have a *(relative) transition rate* of

\[
w(n', n; t) := \lim_{\Delta t \to 0} \frac{P(n', t + \Delta t | n, t)}{\Delta t}.
\]

The *absolute* transition rate of changes from \( n \) to \( n' \) is the product \( w(n', n; t)P(n, t) \) of the probability \( P(n, t) \) to have the configuration \( n \) and the *(relative) transition rate* \( w(n', n; t) \) if having the configuration \( n \). Whereas the *inflow* into \( n \) is given as the sum over all absolute transition rates of changes from an arbitrary configuration \( n' \) to \( n \), the *outflow* from \( n \) is given as the sum over all absolute transition rates of changes from \( n \) to another configuration \( n' \). Since the temporal change of the probability \( P(n, t) \) is determined by the inflow into \( n \) reduced by the outflow from \( n \), we find the *master equation*
\[ \frac{d}{dt} P(n, t) = \text{inflow into } n \]
- outflow from \( n \) 
\[ = \sum_{n'} w(n, n'; t) P(n'; t) \]
\[ - \sum_{n'} w(n', n; t) P(n, t) \]  
(2)

(see Haken (1983)), which is a stochastic equation.

It shall be assumed that two processes contribute to a change of the socioconfiguration \( n \):

- Individuals may change their behavior \( i \) spontaneously and independently of each other to another behavior \( i' \) with an individual transition rate \( \bar{w}_a(i', i; t) \). These changes correspond to transitions of the socioconfiguration from \( n \) to

\[ n_{i,i'}^a := (n_1^a, \ldots, (n_i^a + 1), \ldots, (n_i^a - 1), \ldots, n_3^a) \]

with a configurational transition rate \( w(n_{i,i'}^a, n; t) = n_i^a \bar{w}_a(i', i; t) \), which is proportional to the number \( n_i^a \) of individuals who can change the behavior \( i \).

- An individual of subpopulation \( a \) may change the behavior from \( i \) to \( i' \) during a pair interaction with an individual of a subpopulation \( b \) who changes the behavior from \( j \) to \( j' \). Let transitions of this kind occur with a probability \( \bar{w}_{ab}(i', j'; i, j; t) \) per time unit. The corresponding change of the socioconfiguration from \( n \) to

\[ n_{i',j'}^{ab} := (n_1^a, \ldots, (n_i^a + 1), \ldots, (n_i^a - 1), \ldots, (n_j^b - 1), \ldots, n_3^b) \]

leads to a configurational transition rate \( w(n_{i',j'}^{ab}, n; t) = n_i^a n_j^b \bar{w}_{ab}(i', j'; i, j; t) \), which is proportional to the number \( n_i^a n_j^b \) of possible pair interactions between individuals of subpopulations \( a \) resp. \( b \) who show the behavior \( i \) resp. \( j \). (Exactly speaking—in order to exclude self-interactions—\( n_i^a n_i^b \bar{w}_{aa}(i', j'; i, i; t) \) has to be replaced by \( n_i^a (n_i^a - 1) \bar{w}_{aa}(i', j'; i, i; t) \), if \( P(n, t) \) is not negligible where \( n_i^a \gg 1 \) does not hold, and \( \sum_{j'} \bar{w}_{aa}(i', j'; i; i; t) \ll \bar{w}_a(i', i; t) \) is invalid.)

The resulting configurational transition rate \( w(n', n; t) \) is given by

\[ w(n', n; t) := \begin{cases} n_i^a \bar{w}_a(i', i; t) & \text{if } n' = n_{i,i'}^a \\ n_i^a n_j^b \bar{w}_{ab}(i', j'; i, j; t) & \text{if } n' = n_{i',j'}^{ab} \\ 0 & \text{otherwise.} \end{cases} \]  
(3)

As a consequence, the explicit form of the master equation (2) is

\[ \frac{d}{dt} P(n, t) = \sum_{a,i,i'} \left[ (n_i^a + 1) \bar{w}_a(i, i'; t) P(n_{i,i'}^a, t) \\ - n_i^a \bar{w}_a(i', i; t) P(n, t) \right] \]
\[ + \frac{1}{2} \sum_{a,i,i',b,j,j'} \left[ (n_i^a + 1)(n_j^b + 1) \times \bar{w}_{ab}(i, j; i', j'; t) P(n_{i',j'}^{ab}, t) \\ - n_i^a n_j^b \bar{w}_{ab}(i', j'; i, j; t) P(n, t) \right] \]

(see Helbing (1992a)).

3 Most probable and expected behavioral distribution

Because of the great number of possible socioconfigurations \( n \), the master equation for the determination of the configurational distribution \( P(n, t) \) is usually difficult to solve (even with a computer). However,

- in cases of the description of single or rare social processes the most probable behavioral distribution

\[ P_a(i, t) := \frac{\hat{n}_i^a(t)}{N_a} \]

is the quantity of interest, whereas
in cases of frequently occuring social processes the interesting quantity is the expected behavioral distribution

\[ P_a(i, t) := \frac{\langle n_i^a \rangle_t}{N_a} . \]

Equations for the most probable occupation numbers \( \tilde{n}_i^a(t) \) can be deduced from a Langevin equation for the development of the socioconfiguration \( \mathbf{n}(t) \). For the mean values \( \langle n_i^a \rangle_t \) of the occupation numbers \( n_i^a \) only approximate closed equations can be derived. A measure for the reliability (or representativity) of \( \tilde{n}_i^a(t) \) and \( \langle n_i^a \rangle_t \) with respect to the possible temporal developments of \( n_i^a(t) \) are the (co)variances \( \sigma_{ij}^a(t) \) of \( n_i^a(t) \).

### 3.1 Mean value and (co)variance equations

The mean value of a function \( f(\mathbf{n}, t) \) is defined by

\[ \langle f(\mathbf{n}, t) \rangle_t \equiv \langle f(\mathbf{n}, t) \rangle := \sum_n f(\mathbf{n}, t)P(\mathbf{n}, t) . \]

It can be shown that the mean values of the occupation numbers \( f(\mathbf{n}, t) = n_i^a \) are determined by the equations

\[ \frac{d\langle n_i^a \rangle_t}{dt} \approx \langle m_i^a(\mathbf{n}, t) \rangle \]

(4)

with the drift coefficients

\[ m_i^a(\mathbf{n}, t) := \sum_{n'} (n_i^{a'} - n_i^a)w(n', \mathbf{n}; t) \]

\[ = \sum_{n'} \left[ \overline{w}(i, i'; t)n_i^a + \overline{w}(i', i; t)n_i^a \right] \]

(5)

and the effective transition rates

\[ \overline{w}(i', i; t) := \overline{w}_a(i', i; t) \]

\[ + \sum_b \sum_{j'} \sum_b \overline{w}_{ab}(i', j'; i, j; t)n_j^b \]

(6)

(see Helbing (1992a)). Obviously, the contributions \( \overline{w}_{ab}(i', j'; i, j; t)n_j^b \) due to pair interactions are proportional to the number \( n_j^b \) of possible interaction partners.

### 3.1.1 Approximate mean value and (co)variance equations

Equations (5) are no closed equations, since they depend on the mean values \( \langle n_i^a n_j^b \rangle \), which are not determined by (5). However, if the configurational distribution \( P(\mathbf{n}, t) \) has only small (co)variances

\[ \sigma_{ij}^{ab} := \langle (n_i^a - \langle n_i^a \rangle)(n_j^b - \langle n_j^b \rangle) \rangle = \langle n_i^a n_j^b \rangle - \langle n_i^a \rangle \langle n_j^b \rangle , \]

(7)

we find in first order Taylor approximation the approximate mean value equations

\[ \frac{\partial \langle n_i^a \rangle_t}{\partial t} \approx \left\langle m_i^a(\mathbf{n}, t) \right\rangle 

+ \sum_{b, j} \left( \langle n_j^b \rangle - \langle n_j^b \rangle \right) \frac{\partial m_i^a(\mathbf{n}, t)}{\partial \langle n_j^b \rangle} \]

\[ = m_i^a(\mathbf{n}, t) . \]

(8)

In many cases, the initial configuration \( \mathbf{n}_0 \) at time \( t_0 \) is known by a measurement, i.e., the initial distribution is

\[ P(\mathbf{n}, t_0) = \delta_{\mathbf{n}\mathbf{n}_0} , \]

where the Kronecker function \( \delta_{xy} \) is defined by

\[ \delta_{xy} := \left\{ \begin{array}{ll} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y . \end{array} \right. \]

As a consequence, the (co)variances \( \sigma_{ij}^{ab} \) vanish at time \( t_0 \) and remain small during a certain time interval. For the temporal development of \( \sigma_{ij}^{ab} \), the equations

\[ \frac{d\sigma_{ij}^{ab}}{dt} = \left\langle m_{ij}^{ab}(\mathbf{n}, t) \right\rangle 

+ \left\langle (n_i^a - \langle n_i^a \rangle)m_j^b(\mathbf{n}, t) \right\rangle 

+ \left\langle (n_j^b - \langle n_j^b \rangle)m_i^a(\mathbf{n}, t) \right\rangle \]

(9)

can be found. Here,

\[ m_{ij}^{ab}(\mathbf{n}, t) := \sum_{n'} (n_i^n - n_i^a)(n_j^n - n_j^b)w(n', \mathbf{n}; t) \]
leads to the equations

\[ \approx \frac{\partial n_i^a}{\partial t} \approx m_i^a(\langle n \rangle, t) \]

\[ + \frac{1}{2} \sum_{c,k} \sigma_{ik}^{ab} \frac{\partial^2 m_i^a(\langle n \rangle, t)}{\partial (n_k^c) \partial (n_k^c)} (13) \]

and the corrected (co)variance equations

\[ \frac{d\sigma_{ij}^{ab}}{dt} \approx m_{ij}^{ab}(\langle n \rangle, t) \]

\[ + \frac{1}{2} \sum_{c,k} \sigma_{ik}^{cd} \frac{\partial^2 m_{ij}^{ab}(\langle n \rangle, t)}{\partial (n_k^c) \partial (n_k^c)} + \sigma_{jk}^{bc} \frac{\partial m_{ij}^{ab}(\langle n \rangle, t)}{\partial (n_k^c)} (14) \]

For increasing (co)variances, a better approximation of (12), (13) should be taken. A second order Taylor approximation results in the corrected mean value equations

\[ \frac{\partial(n_i^a)}{\partial t} \approx m_i^a(\langle n \rangle, t) \]

\[ + \frac{1}{2} \sum_{c,k} \sigma_{ik}^{ab} \frac{\partial^2 m_i^a(\langle n \rangle, t)}{\partial (n_k^c) \partial (n_k^c)} (13) \]

and the corrected (co)variance equations

\[ \frac{d\sigma_{ij}^{ab}}{dt} \approx m_{ij}^{ab}(\langle n \rangle, t) \]

\[ + \frac{1}{2} \sum_{c,k} \sigma_{ik}^{cd} \frac{\partial^2 m_{ij}^{ab}(\langle n \rangle, t)}{\partial (n_k^c) \partial (n_k^c)} + \sigma_{jk}^{bc} \frac{\partial m_{ij}^{ab}(\langle n \rangle, t)}{\partial (n_k^c)} (14) \]

Note, that the corrected mean value equations explicitly depend on the (co)variances \( \sigma_{ij}^{ab} \), i.e., on the fluctuations due to the stochasticity of the processes described! They cannot be solved without solving the (co)variance equations. However, the calculation of the (co)variances is always recommendable, since they are a measure for the reliability (or representativity) of the mean value equations.

A comparison of exact, approximate and corrected mean value and variance equations is given in figures 1 to 3. A criterium for the validity of (13) and (14) resp. (13) and (14) are the relative central moments

\[ C_m(i_1, \ldots, i_m; t) := \frac{\langle (n_{i_1}^{a_1} - \langle n_{i_1}^{a_1} \rangle) \cdots (n_{i_m}^{a_m} - \langle n_{i_m}^{a_m} \rangle) \rangle}{\langle n_{i_1}^{a_1} \rangle \cdots \langle n_{i_m}^{a_m} \rangle} \]

Whereas the approximate equations (13), (14) already fail, if

\[ |C_m| \leq 0.04 \quad (15) \]

is violated for \( m = 2 \) (compare to (12), (13)), the corrected equations (13), (14) only presuppose (15) for \( 2 < m \leq l \) with a certain value \( l \) (see Helbing (1992a)). However, even the corrected equations (13), (14)
become useless, if the probability distribution $P(n, t)$ becomes multimodal.

Figures 1 to 3 show computational results corresponding to the example of section 5.3. Exact mean values $\langle n_1 \rangle$ and variances $\sigma_{11}$ are represented by solid lines, whereas approximate results according to (8), (11) are represented by dotted lines, and corrected results according to (13), (14) by broken lines. As expected, the corrected mean value equations yield better results than the approximate mean value equations.

**Figure 1:** Exact (—), approximate (⋯) and corrected (– –) mean values (upper curves) and variances (lower curves) for a small configurational distribution $P(n, t)$: Both, approximate and corrected equations are applicable.

**Figure 2:** As figure 1, but for a broad configurational distribution: The corrected equations still yield useful results, whereas the approximate equations already fail, since the variances are not negligible.

**Figure 3a:** As figure 1 but for a multimodal configurational distribution: Not only the approximate but also the corrected equations fail after a certain time interval. However, whereas the approximate mean value and variance become unreliable already for $t > 1$, the corrected mean value and variance remain valid until $t > 3$.

**Figure 3b:** The relative central moments are a criterion for the validity of the approximate resp. the corrected mean value and (co)variance equations: If $|C_2|$ (—) exceeds the value 0.04, the approximate equations fail, whereas the corrected equations fail, if $|C_3|$ (—) or $|C_4|$ (⋯) exceed the value 0.04.

### 3.2 Equations for the most probable behavioral distribution

The master equation (8) can be reformulated in terms of a Langevin equation (see Helbing (1992)):

$$\frac{d}{dt} n_i^0(t) \equiv m_i^0(n, t) + \text{fluctuations.} \quad (16)$$

The Langevin equation (16) describes the behavior of the socioconfiguration $n(t)$ in dependence of process immanent fluctuations.
(that are determined by the diffusion coefficients \( m^{ab}_{ij} \)). As a consequence,

\[
\frac{d}{dt} \hat{n}^a_i(t) \overset{N \gg 1}{=} m^{a}_i(\hat{\mathbf{n}}, t) \tag{17}
\]

are the equations for the most probable occupation numbers \( \hat{n}^a_i(t) \). The equations (17) look exactly like the approximate mean value equations (8). Therefore, if \( N \gg 1 \), the approximate mean value equations (8) have an interpretation even for great variances, since they also describe the most probable behavioral distribution.

4 Kinds of pair interactions

The pair interactions

\[
i', j' \leftarrow i, j
\]

of two individuals of subpopulations \( a \) resp. \( b \) who change their behavior from \( i \) resp. \( j \) to \( i' \) resp. \( j' \) can be completely classified according to the following scheme:

\[
\begin{align*}
    &i, i \leftarrow i, i \quad (i \neq j) \quad (1) \\
    &i, j \leftarrow i, j \quad (i \neq j) \quad (2) \\
    &j, j \leftarrow i, j \quad (i \neq j) \quad (3) \\
    &i, j' \leftarrow i, j \quad (i \neq j, j' \neq j, j' \neq i) \quad (4)
\end{align*}
\]

For the transition rates corresponding to these kinds of interaction processes the following plausible form shall be assumed (see Helbing (1992)):

\[
\bar{w}_{ab}(i', j'; i, j; t) := \bar{\nu}_{ab}(t) \times \begin{cases} 
    p^{1}_{ab}(i'|i; t) & \text{if } i' = j \text{ and } j' = j \\
    p^{2}_{ba}(j'|j; t) & \text{if } j' = i \text{ and } i' = i \\
    0 & \text{if } i' = j \text{ and } j' \neq j \\
    0 & \text{if } j' = i \text{ and } i' \neq i \\
    p^{3}_{ab}(i'|i; t) & \text{if } i' \neq j \text{ and } j' \neq i \\
    p^{4}_{ba}(j'|j; t) & \text{otherwise (} k \in \{2,3\} \text{)}. 
\end{cases}
\tag{18}
\]

Here,

\[
\nu_{ab}(t) := N_b \bar{\nu}_{ab}(t)
\]

is the contact rate between an individual of subpopulation \( a \) with individuals of subpopulation \( b \). \( p^{k}_{ab}(j'|i; t) \) is the probability of an
individual of subpopulation $a$ to change the behavior from $i$ to $j$ during a pair interaction of kind $k$ with an individual of subpopulation $b$, i.e.,

$$\sum_j p^k_{ab}(j|i; t) = 1.$$ 

Let us assume

$$p^k_{ab}(j|i; t) := f^k_{ab}(t) R_a(j, i; t),$$

where $f^k_{ab}(t)$ is a measure for the frequency of pair interactions of kind $k$ between individuals of subpopulation $a$ and $b$, and $R_a(j, i; t)$ is a measure for the readiness of individuals belonging to subpopulation $a$ to change the behavior from $i$ to $j$ during a pair interaction. Inserting the rate (18) of pair interactions into (8) and using the conventions

$$w_a(i', i; t) := \bar{w}_a(i', i; t),$$

$$w_{ab}(i', j'; i, j; t) := N_b \bar{w}_{ab}(i', j'; i, j; t),$$

$$\nu^k_{ab}(t) := \nu_{ab}(t) f^k_{ab}(t),$$

$$P_a(i, t) := \frac{\langle n^a_i \rangle}{N_a},$$

we arrive at the approximate mean value equations

$$\frac{d}{dt} P_a(i, t) = \sum_{i'} \left[ w^a(i, i'; t) P_a(i', t) - w^a(i', i; t) P_a(i, t) \right]$$

(20) (see (8), (3)) with the mean transition rates

$$w^a(i, i'; t) := w_a(i, i'; t) + R_a(i, i'; t)$$

$$\times \sum_b \left[ \left( \nu^1_{ab}(t) - \nu^3_{ab}(t) \right) P_b(i, t) + \left( \nu^2_{ab}(t) - \nu^3_{ab}(t) \right) P_b(i', t) + \nu^3_{ab}(t) \right]$$

(21)

(if $N_a \gg 1$; see HELBING (1992, 1992b)). The mean transition rates include contributions of spontaneous behavioral changes, and of behavioral changes due to pair interactions (i.e., of imitative, avoidance and compromising processes). (20), (21) are BOLTZMANN-like equations (see BOLTZMANN (1964), HELBING (1992a)).

Due to (1), (19), and $0 \leq n_i^a \leq N_a$ we have the relations

$$\sum_i P_a(i, t) = 1 \quad \text{and} \quad 0 \leq P_a(i, t) \leq 1.$$

Therefore, $P_a(i, t)$ can be interpreted as the fraction of individuals within subpopulation $a$ who show the behavior $i$. With respect to the total population, the fraction $P(i, t)$ of individuals with behavior $i$ is given by

$$P(i, t) = \frac{\langle n_i \rangle}{N} = \frac{\sum_a \langle n_i^a \rangle}{N_a} = \sum_a \frac{N_a}{N} P_a(i, t).$$

4.1 Computer simulations

For an illustration of the BOLTZMANN-like equations (20), (21) we shall assume to have two subpopulations ($A = 2$), and three different behaviors ($S = 3$). With

$$R_a(i', i; t) := \frac{e^{U_a(i', t) - U_a(i, t)}}{D_a(i', i; t)},$$

(22)

(see WEIDLICH and HAAG (1988), HELBING (1992)) the readiness $R_a(i', i; t)$ for an individual of subpopulation $a$ to change the behavior from $i$ to $i'$ will be the greater, the greater the difference of the utilities $U_a(i, t)$ of behaviors $i'$ and $i$ is, and the smaller the incompatibility ("distance")

$$D_a(i', i; t) = D_a(i, i'; t) > 0$$

between the behaviors $i$ and $i'$ is.

In the following computer simulations $D_a(i', i; t) \equiv 1$ has been taken. For both subpopulations the preferred behavior, i.e., the behavior with the greatest utility $U_a(i, t)$ is represented by a solid line, whereas the behavior with the lowest utility is represented by a dotted line, and the behavior with medium utility by a broken line. Figures 1a to 11 show the effects of imitative processes ($w^1_{ab}(t) \equiv 1, \nu^2_{ab}(t) \equiv 0 \equiv \nu^3_{ab}(t)$), of avoidance processes ($w^2_{ab}(t) \equiv 1, \nu^1_{ab}(t) \equiv 0 \equiv \nu^3_{ab}(t)$), resp. of compromising and imitative processes ($\nu^3_{ab}(t) \equiv 1 \equiv \nu^1_{ab}(t), \nu^2_{ab}(t) \equiv 0$)
a) for equal behavioral preferences \((U_1(1) = c = U_2(1), U_1(2) = 0 = U_2(2), U_1(3) = -c = U_2(3))\), and

b) for different behavioral preferences \((U_1(1) = c = U_2(2), U_1(2) = 0 = U_2(1), U_1(3) = -c = U_2(3))\).

In more complicated cases, there are also oscillatory or chaotic behavioral changes possible, as illustrated in figures 7 (see HELBING (1992b, 1992)) and 8 (see HELBING (1992)).

**Figure 4a:** Effect of imitative processes for two subpopulations preferring the same behavior \((c = 0.5)\): Only the fraction of the preferred behavior (—) is increasing. The other behaviors vanish in the course of time.

**Figure 4b:** Effect of imitative processes for two subpopulations preferring different behaviors \((c = 0.5)\): The preferred behavior (—) becomes the dominating one in each subpopulation, but the behavior which is preferred in the other subpopulation (—) can also convince a certain fraction of individuals. A behavior which is not preferred by any subpopulation (⋯) vanishes.

**Figure 5a:** Effect of avoidance processes for two subpopulations preferring the same behavior \((c = 1)\): The fraction of the preferred behavior (—) is limited, since the subpopulations avoid to show the same behavior. As a consequence, the other behaviors are also used by a certain fraction of individuals.

**Figure 5b:** Effect of avoidance processes for two subpopulations preferring different behaviors \((c = 1)\): The fraction of the preferred behavior (—) wins a greater majority in comparison with figure 5a, since the situations of avoidance are reduced.

**Figure 6a:** Effect of compromising and imitative processes for two subpopulations preferring the same behavior \((c = 0.5)\): Only the preferred behavior (—) survives, since a readiness for compromises is not necessary.
Figure 6b: Effect of compromising and imitative processes for two subpopulations preferring different behaviors ($c = 0.5$): Most of the individuals show the preferred behavior (—), but a certain fraction of individuals also decides for a compromise (···).

Figure 7a: Oscillations are one possible effect of imitative processes. For $S = 5$ different behaviors, the oscillatory changes look quite irregular without a short-term periodicity.

Figure 7b: Phase portrait of oscillatory changes between $S = 5$ behaviors having the shape of a torus: A long-term periodicity is indicated by the closeness of the curve.

Figure 8: Phase portrait of the scaled variables $y_i(\tau) := \alpha_i P(i, \beta t)$ representing chaotic changes of the behavioral fractions $P(i, t)$.

5 Game dynamical equations

In game theory, $i$ denotes a (behavioral) strategy. Let $E_a(i, t)$ be the expected success of a strategy $i$ for an individual of subpopulation $a$, and

$$\langle E_a \rangle := \sum_i E_a(i, t) P_a(i, t)$$

the mean expected success. If the relative increase

$$\frac{dP_a(i, t)}{dt}$$

of the fraction $P_a(i, t)$ is assumed to be proportional to the difference $[E_a(i, t) - \langle E_a \rangle]$ between the expected and the mean expected success, one obtains the game dynamical equations

$$\frac{d}{dt} P_a(i, t) = \nu_a(t) P_a(i, t) \left[ E_a(i, t) - \langle E_a \rangle \right].$$

(23)

That means, the fractions of strategies with an expected success that exceeds the average $\langle E_a \rangle$ are growing, whereas the fractions of the remaining strategies are falling. For the expected success $E_a(i, t)$, one often takes the form

$$E_a(i, t) := \sum_b \sum_j A_{ab}(i, j; t) P_b(j, t),$$

(24)

where $A_{ab}(i, j; t)$ have the meaning of pay-offs. We shall assume

$$A_{ab}(i, j; t) := r_{ab}(t) E_{ab}(i, j; t)$$
with
\[
\frac{\nu_{ab}(t)}{\sum_c \nu_{ac}(t)},
\]
where \( r_{ab}(t) \) is the relative contact rate of an individual of subpopulation \( a \) with individuals of subpopulation \( b \), and \( E_{ab}(i, j; t) \) is the success of strategy \( i \) for an individual of subpopulation \( a \) during an interaction with an individual of subpopulation \( b \) who uses strategy \( j \). Since \( r_{ab}(t)P_b(j, t) \) is the relative contact rate of an individual of subpopulation \( a \) with individuals of subpopulation \( b \) who use strategy \( j \), \( E_a(i, t) \) is the mean (or expected) success of strategy \( i \) for an individual of subpopulation \( a \) in interactions with other individuals.

By inserting (24) and
\[
\langle E_a \rangle = \sum_{i'} \sum_{b, j} P_a(i', t)A_{ab}(i', j; t)P_b(j, t)
\]
into (23), one obtains the explicit form
\[
\frac{d}{dt}P_a(i, t) = \nu_a(t)P_a(i, t) \left[ \sum_{b, j} A_{ab}(i, j; t)P_b(j, t) - \sum_{i'} \sum_{b, j} P_a(i', t)A_{ab}(i', j; t)P_b(j, t) \right]
\]
(25)
of the game dynamical equations. (25) is a continuous formulation of game theory (see Hofbauer and Sigmund (1988)). Equations of this kind are very useful for the investigation and understanding of the competition or cooperation of individuals (see, e.g., Mueller (1990), Hofbauer and Sigmund (1988), Schuster et. al. (1981)).

A slightly generalized form of (23),
\[
\frac{d}{dt}P_a(i, t) = \sum_{i'} \left[ w_a(i, i'; t)P_a(i', t) - w_a(i', i; t)P_a(i, t) \right] + \nu_a(t)P_a(i, t) \left[ E_a(i, t) - \langle E_a \rangle \right],
\]
(26a)
is also known as selection mutation equation (Hofbauer and Sigmund (1988)): (26a) can be understood as effect of a selection (if \( E_a(i, t) \) is interpreted as fitness of strategy \( i \)), and (26a) can be understood as effect of mutations. Equation (26) is a powerful tool in evolutionary biology (see Eigen (1971), Fisher (1930), Eigen and Schuster (1979), Hofbauer and Sigmund (1988), Feistel and Ebeling (1989)). In game theory, the mutation term could be used for the description of trial and error behavior or of accidental variations of the strategy.

### 5.1 Connection between BOLTZMANN-like and game dynamical equations

One expects that there must be a connection between the BOLTZMANN-like equations (20), (21) and the game dynamical equations (26), since they are both quantitative models for behavioral changes. A comparison of (20), (21) with (26) shows, that both models can become identical only under the conditions
\[
\nu_{ab}^1(t) = \nu_a(t)\delta_{ab},
\]
\[
\nu_{ab}^2(t) = 0,
\]
\[
\nu_{ab}^3(t) = 0.
\]
(27)
That means, the game dynamical equations include spontaneous and imitative behavioral changes, but they exclude avoidance and compromising processes.

In order to make the analogy between the game dynamical and the BOLTZMANN-like equations complete the following assumptions have to be made:

- In interactions with other individuals the expected success
\[
E_a(i, t) = \sum_{b, j} \sum_c \nu_{ab}(t)E_{ab}(i, j; t)P_b(j, t)
\]
(28)
of a strategy is evaluated. This is possible, since an individual is able to determine the quantities \( \nu_{ab}(t) \), \( P_b(j, t) \) and
$E_{ab}(i, j; t)$: An individual of subpopulation $a$ meets individuals of subpopulation $b$ with a contact rate of $\nu_{ab}(t)$. With a probability of $P_b(j, t)$, the individuals of subpopulation $b$ use the strategy $j$. During interactions with individuals of subpopulation $b$ who use the strategy $j$, an individual of subpopulation $a$ has a success of $E_{ab}(i, j; t)$ if using the strategy $i$.

- In interactions with individuals of the same subpopulation an individual tends to take over the strategy of another individual, if the expected success would increase: If an individual who uses a strategy $i$ meets another individual of the same subpopulation who uses a strategy $j$, they will compare their expected success’ $E_a(i, t)$ resp. $E_a(j, t)$ (by observation or exchange of their experiences). The individual with strategy $i$ will imitate the other’s strategy $j$ with a probability $p_{ab}^t(j|i; t)$ that is growing with the expected increase

$$\Delta_{ji}E_a := E_a(j, t) - E_a(i, t)$$

of success. If a change of strategy would imply a decrease of success ($\Delta_{ji}E_a < 0$), the individual will not change the strategy $i$. Therefore, the readiness for replacing the strategy $i$ by $j$ during an interaction within the same subpopulation can be assumed to be

$$R_a(j, i; t) := \max \left( E_a(j, t) - E_a(i, t), 0 \right), \quad (29)$$

where $\max(x, y)$ is the maximum of the two numbers $x$ and $y$. However, due to different criteria for the grade of success, the expected success of a strategy $i$ will usually be varying with the subpopulation $a$ (i.e., $E_a(i, t) \neq E_b(i, t)$ for $a \neq b$). As a consequence, an imitative behavior of individuals belonging to different subpopulations is not plausible, and we shall assume

$$f_{ab}^t(t) := \delta_{ab},$$

that means,

$$\nu_{ab}^t(t) = \nu_{aa}(t)\delta_{ab}.$$ 

Inserting (27), (28) and (24) into the BOLTZMANN-like equations (26), (21), the game dynamical equations (26) result as a special case, since

$$\max \left( E_a(i, t) - E_a(j, t), 0 \right) - \max \left( E_a(j, t) - E_a(i, t), 0 \right) = E_a(i, t) - E_a(j, t).$$

### 5.2 Stochastic version of the game dynamical equations

Applying the formalism of section 3, a stochastic version of the game dynamical equations can easily be formulated. This is given by the master equation (3) with the configurational transition rates (3) and

$$w_{ab}(i', j'; i, j; t) := N_b \tilde{w}_{ab}(i', j'; i, j; t) := \nu_a(t)\delta_{ab}\tilde{R}_a(i, j; t)\delta_{ij} \delta_{jj'}(1 - \delta_{ij}) + \nu_a(t)\delta_{ab}\tilde{R}_a(j, i; t)\delta_{ij} \delta_{jj'}(1 - \delta_{ij}),$$

where

$$\tilde{R}_a(j, i; t) := \max \left( \hat{E}_a(j, t) - \hat{E}_a(i, t), 0 \right)$$

and

$$\hat{E}_a(i, t) := \sum_b \sum_j A_{ab}(i, j; t) \frac{n_b}{N_b}$$

(compare to FEISTEL and EBELING (1989), EBELING and FEISTEL (1982), EBELING et. al. (1990)). A comparison with (3), (20), (21) shows, that the ordinary game dynamical equations (26) are the approximate mean value equations of this special master equation. Therefore, they can only be interpreted as mean value equations as long as the (co)variances $\sigma_{ij}^{ab}$ are small (see (12)). Otherwise they describe the most probable behavioral distribution (see sect. 3.2).

### 5.3 Selforganization of behavioral conventions by competition between equivalent strategies

This section gives an illustration of the methods and results derived in sections 3 and 5.2.
As an example, we shall consider a case with one subpopulation only ($A = 1$), and, therefore, omit the index $a$ in the following. Let us suppose the individuals choose between two equivalent strategies $i \in \{1, 2\}$, i.e., the payoff matrix $\mathbf{A}(t)$ shall be symmetrical:

$$\mathbf{A}(t) \equiv \left( A(i, j; t) \right) := \begin{pmatrix} A + B & B \\ B & A + B \end{pmatrix}. \quad (30)$$

According to the relation

$$n_1 + n_2 = N$$

(see (11)), the fraction $P(2, t) = 1 - P(1, t)$ is already determined by $P(1, t)$. By scaling the time,

$$\nu(t) \equiv 1$$

can be presupposed. For the spontaneous change of strategies due to trial and error we shall assume the transition rates

$$w(j, i; t) := W. \quad (31)$$

A situation of the above kind is the avoidance behavior of pedestrians (see Helbing (1991)): In pedestrian crowds with two opposite directions of movement, the pedestrians have sometimes to avoid each other in order to exclude a collision. For an avoidance maneuver to be successful, both pedestrians concerned have to pass the respective other pedestrian either on the right hand side or on the left hand side. Otherwise, both pedestrians have to stop (see figure 9). Therefore, both strategies (to pass pedestrians on the right hand side or to pass them on the left hand side) are equivalent, but the success of a strategy grows with the number $n_i$ of individuals who use the same strategy. In the payoff matrix (11) we have $A > 0$, then.

The game dynamical equations (26) corresponding to (18), (31) have the explicit form

$$\frac{d}{dt} P(i, t) = -2 \left( P(i, t) - \frac{1}{2} \right) \times \left[ W + A P(i, t) \left( P(i, t) - 1 \right) \right]. \quad (32)$$

According to (32), $P(i) = 1/2$ is a stationary solution. This solution is stable only for

$$\kappa := 1 - \frac{4W}{A} < 0,$$

i.e., if spontaneous strategy changes due to trial and error (the “mutations”) are dominating. For $\kappa > 0$ the stationary solution $P(i) = 1/2$ is unstable, and the game dynamical equations (18) can be rewritten in the form

$$\frac{d}{dt} P(i, t) = -2 \left( P(i, t) - \frac{1}{2} \right) \times \frac{1 + \sqrt{\kappa}}{2} \times \left( P(i, t) - \frac{1 - \sqrt{\kappa}}{2} \right).$$

That means, for $\kappa > 0$ we have two additional stationary solutions $P(i) = (1 + \sqrt{\kappa})/2$ and $P(i) = (1 - \sqrt{\kappa})/2$, which are stable. Depending on the random initial condition $P(i, t_0)$, one strategy will win a majority of $100 \cdot \sqrt{\kappa}$ percent. This majority is the greater, the smaller the rate $W$ of spontaneous strategy changes is.

At the critical point $\kappa = \kappa_0 := 0$ there appears a phase transition. This can be seen best in figures 10 to 12, where the distribution $P(n, t) \equiv P(n_1, n_2; t) = P(n_1, N - n_1; t)$ loses its unimodal form for $\kappa > 0$. As a consequence of the phase transition, one strategy is preferred, i.e. a behavioral convention develops.
Figure 10: Probability distribution $P(n,t) \equiv P(n_1,N-n_1;t)$ of the socioconfiguration $n = (n_1,N-n_1)$ for two equivalent competing strategies. Mutation dominated region ($\kappa < 0$): Since $P(n_1,N-n_1;t)$ has, after a certain time interval, one maximum at $n_1 = N/2$, each strategy will most probably be used by about one half of the individuals.

Figure 11: As figure 10, but for the critical point $\kappa = 0$: The broadness of the probability distribution $P(n_1,N-n_1;t)$ indicates critical fluctuations, i.e., a phase transition.

Figure 12: As figure 10, but after the phase transition ($\kappa > 0$): The configurational distribution $P(n_1,N-n_1;t)$ becomes multimodal with maxima that are symmetrical with respect to $N/2$, because of the equivalence of the strategies. Due to the maxima at $n_1 > N/2$ and $n_2 = N-n_1 > N/2$, one of the strategies will very probably win a majority of users. This implies the selforganization of a behavioral convention.

Figure 13: As figure 11, but with a modified ansatz for the readiness $R_a(j,i;t)$ to change the behavior from $i$ to $j$, which does not produce a crease of $P(n_1,N-n_1;t)$ at $N/2$. 
The crease of $P(n_1, N-n_1; t)$ at $n_1 = N/2 = \frac{n_2}{2}$ is a consequence of the crease of the function $\hat{R}_a(j,i; t) = \max(\hat{E}_a(j,t) - \hat{E}_a(i,t), 0)$. It can be avoided by using the modified ansatz

$$\hat{R}_a(j,i; t) := \frac{e^\hat{E}_a(j,t) - \hat{E}_a(i,t)}{D_a(j,i; t)}$$

(compare to (24)), which also shows a phase transition for $\kappa = 0$ (see figure 13).

6 Summary and Conclusions

A quite general model for behavioral changes has been developed, which takes into account spontaneous changes and changes due to pair interactions. Three kinds of pair interactions have been distinguished: imitative, avoidance and compromising processes. The game dynamical equations result for a special case of imitative processes. They can be interpreted as equations for the most probable behavioral distribution or as approximate mean value equations of a stochastic formulation of game theory. In order to determine the reliability (or representativity) of game dynamical descriptions, one has to evaluate the corresponding (co)variance equations.

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References

Axelrod, R. (1984), The Evolution of Cooperation. New York: Basic Books

Boltzmann, L. (1964), Lectures on Gas Theory. Berkeley: University of California

Ebeling, W. / Engel, A. / Feistel, R. (1990), Physik der Evolutionsprozesse. Berlin: Akademie-Verlag

Ebeling, W. / Feistel, R. (1982), Physik der Selbstorganisation und Evolution. Berlin: Akademie-Verlag

Eigen, M. (1971), The selforganization of matter and the evolution of biological macromolecules. Naturwissenschaften 58, 465

Eigen, M. / Schuster, P. (1979), The Hypercycle. Berlin: Springer

Feistel, R. / Ebeling, W. (1989), Evolution of Complex Systems. Dordrecht: Kluwer Academic

Fisher, R. A. (1930), The Genetical Theory of Natural Selection. Oxford: Oxford University

Goodwin, R. M. (1969), A growth cycle. In: Feinstein, C. H. (ed.), Socialism, Capitalism and Economic Growth. Cambridge: Cambridge University Press. Revised version in: Hunt, E. K. / Schwarz, J. G. (eds.), A Critique of Economic Theory. Harmondsworth: Penguin, pp. 442-449

Goel, N. S. / Maitra, S. C. / Montroll, E. W. (1971), Reviews of Modern Physics 43, 231-276

Haken, H. (1983), Synergetics. An Introduction. Berlin: Springer, pp. 79-83

Hallam, Th. G. (1986), Community dynamics in a homogeneous environment. In: Hallam, Th. G. / Levin, S. A. (eds.) Mathematical Ecology. Berlin: Springer, pp. 241-285

Helbing, D. (1991), A mathematical model for the behavior of pedestrians. Behavioral Science 36, 298-310

Helbing, D. (1992), Stochastische Methoden, nichtlineare Dynamik und quantitative Modelle sozialer Prozesse. Universität Stuttgart: Dissertation

Helbing, D. (1992a), Interrelations between stochastic equations for systems with pair interactions. Physica A 181, 29-52

Helbing, D. (1992b), A mathematical model for attitude formation by pair interactions. Behavioral Science 37, 190-214

Hofbauer, J. (1981), On the occurrence of limit cycles in the Lotka-Volterra equation. Nonlinear Analysis TMA 5, 1003-1007

Hofbauer, J. / Sigmund, K. (1988), The Theory of Evolution and Dynamical Systems.
Lotka, A. J. (1920), Proc. Nat. Acad. Sci. U.S. 6, 410

Lotka, A. J. (1956), Elements of Mathematical Biology. New York: Dover

Luce, R. D. / Raiffa, H. (1957), Games and Decisions. New York: Wiley

Mueller, U. (ed.) (1990), Evolution und Spieltheorie. München: Oldenbourg

von Neumann, J. / Morgenstern, O. (1944), Theory of Games and Economic Behavior. Princeton: Princeton University Press

Pearl, R. (1924), Studies in Human Biology. Baltimore: Williams & Wilkins

Ravenstein, E. (1876), The birthplaces of the people and the laws of migration. The Geographical Magazine III, 173-177, 201-206, 229-233

Schuster, P. / Sigmund, K. / Hofbauer, J. / Wolff, R. (1981), Selfregulation of behavior in animal societies. Biological Cybernetics 40, 1-25

Verhulst, P. F. (1845), Nuov. Mem. Acad. Roy. Bruxelles 18, 1

Volterra, V. (1931), Leçons sur la théorie mathématique de la lutte pour la vie. Paris: Gauthier-Villars

Weidlich, W. / Haag, G. (1983), Concepts and Models of a Quantitative Sociology. The Dynamics of Interacting Populations. Berlin: Springer

Weidlich, W. / Haag, G. (1988), Interregional Migration. Berlin: Springer

Weidlich, W. (1991), Physics and social science—The approach of synergetics. Physics Reports 204, 1-163

Zipf, G. K. (1946), The P1P2/D hypothesis on the intercity movement of persons. American Sociological Review 11, 677-686