Matrix Model with Superconformal Symmetry

Michiaki TAKAMA

Research Institute for Mathematical Sciences,

Kyoto University, Kyoto, 606 Japan

February 1992

Abstract

A matrix model is presented which leads to the discrete “eigenvalue model” proposed recently by Alvarez-Gaumé et.al. for 2D supergravity (coupled to superconformal matters).
Through the recent advances in understanding 2D quantum gravity, it has been confirmed that the nonperturbative formulation of the theory is successfully defined in terms of matrix models[1]. In these models, the integral over matrices realizes the random sum of the triangulations of two dimensional surfaces. Encouraged by these successes, we incline to proceed to the supersymmetric version of the theory. However, in the case of discretized gravity, its supersymmetrization perplexes us with the difficulty of imaging what is meant by “triangulations of super-surfaces”. In formulating matrix models to get supersymmetry[2], most of the attempts that have been made so far thus deal with the target space supersymmetry and not the world-surface supersymmetry,( excepting phenomenological approaches by means of the super-extension of soliton equations[3]).

Very recently, Alvarez-Gaumé et.al.[4] proposed a discrete model which seems probably to provide a discrete version of 2D supergravity. Taking the “planar” limit, they showed that the model reproduces the string susceptibility and the spectrum of anomalous dimensions of the $(2, 4m)$-minimal superconformal models coupled to 2D supergravity. Since their model is presented in the form of “eigenvalue model”, its interpretation such as triangulated super-surfaces is impossible as it stands. In this letter, we lift their model to a matrix model. The resulting model, however, has a peculiar form which seems not so easy to deal with. We have not yet had an answer whether such an interpretation mentioned above is in fact possible.

The idea to construct a model is very simple. As expounded in ref.[4], the guiding principle is the super-Virasoro structure of the partition function. Let us consider a super-pair of $N \times N$ hermitian matrix $(\Phi, \Theta)$, whose matrix elements $\phi_{ij}, \theta_{ij}$ $1 \leq i, j \leq N$ are Grassmann even and odd variables respectively.\#1 When $\Phi$ is diagonal, we denote it by $\Phi = \Lambda = \left(\lambda_1 \cdots \lambda_N\right)$. In usual matrix models, Virasoro structure can be recognized by the change of the variables $1 + \delta_{\varepsilon}^{(n)} : \Phi \rightarrow \Phi + \varepsilon \Phi^{n+1}$ and as a result the partition function obeys the Virasoro constraints[5]. By analogy, we consider the transformation, with an odd constant parameter $\zeta$:

$$1 + \delta_{\zeta}^{(n-\frac{1}{2})} : (\Phi, \Theta) \rightarrow (\Phi + \Theta \Phi^n \zeta, \Theta + \Phi^n \zeta) \ .$$

\#1 The hermiticity of $\Theta$ would be imposed such that $\overline{\theta_{ij}} \zeta = \overline{\zeta} \theta_{ij} = \theta_{ji} \zeta$ with $\zeta$ being a real odd constant $(\overline{\zeta} = \zeta)$, hence $\theta_{ij} = -\theta_{ji}$. Throughout this letter, we deal with $\phi_{ij}$ and $\theta_{ij}$ as independent $N^2 + N^2$ variables for simplicity.
The generators of the transformation is written as \( \tilde{G}_{n-\frac{1}{2}} = -Tr \Phi^n (\partial_\Theta - \Theta \partial_\Phi) \) with \((\partial_\Phi)_{ij}, (\partial_\Theta)_{ij} = \partial_{\phi_{ji}}\) . Obviously, only when \((\Phi, \Theta)\) is guaranteed in a proper way to satisfy the conditions \([\Phi, \Theta] = 0\) and \(\Theta^2 = 0\), the generators \(\tilde{G}_{n-\frac{1}{2}}, n \geq 0\) form the super-Virasoro algebra with \(\tilde{L}_n = -Tr(\Phi^{n+1} \partial_\Phi + \frac{n+1}{2} \Theta \Phi^n \partial_\Theta)\) \(\ n \geq -1\).

We define the partition function of our model as follows:

\[
Z_N = \int d\mu(\Phi, \Theta) e^{-\beta Tr V(\Phi, \Theta)}, \tag{2}
\]

with

\[
d\mu(\Phi, \Theta) = d^{N^2} \Phi d^{N^2} \Theta F_N(\Phi, \Theta) d^{N^2} \Theta \equiv \prod_{i=1}^{N} d\theta_{ii} \prod_{1 \leq i < j \leq N} d\theta_{ij} d\theta_{ji} \tag{3}
\]

From the above argument, we require the super-function \(F_N(\Phi, \Theta)\) to have the following properties:

(A) \(F_N\) is invariant under the adjoint action of the \(U(N)\) group to \((\Phi, \Theta)\), i.e.

\[
F_N(\Phi, \Theta) = F_N(U\Phi, U\Theta) = (U\Phi U^\dagger, U\Theta U^\dagger) .
\]

(B) Under the multiplication by \(F_N\), it is ensured that \([\Phi, \Theta] = 0\) and \(\Theta^2 = 0\) in the integrand.

(C) \(F_N\) is invariant under the transformation (1).

From the requirement (A), the measure \(d\mu(\Phi, \Theta)\) becomes invariant under the adjoint action of \(U(N)\). As usual, the \(U(N)\) integration yields the volume factor of the unitary group, which we drop. The partition function (2) is then given by the integral:

\[
\int d^N \Lambda d^{N^2} \Theta \Delta(\Lambda)^2 F_N(\Lambda, \Theta) e^{-\beta Tr V(\Lambda, \Theta)},
\]

\[
\Delta(\Lambda) = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j) . \tag{4}
\]

The requirement (B) and (C) are the sufficient conditions for \(\tilde{G}_{n-\frac{1}{2}}\) and \(\tilde{L}_{n-1}\) \(\ (n \geq 0)\) to form (the half of) the super-Virasoro algebra. Although the potential \(V\) is assumed to be a polynomial of the matrices \(\Phi\) and \(\Theta\), only the terms up to the linear order with respect to \(\Theta\) contribute because of (B).

Let us start seeking the expression of \(F_N\). For (A) and (B), it is necessary and sufficient to find a manifestly \(U(N)\)-invariant super-function \(G_N(\Phi, \Theta)\) such that

\[
G_N(\Lambda, \Theta) \propto \prod_{i < j} \theta_{ij} \theta_{ji} .
\]
We find the following as such a function:

\[ G_N(\Phi, \Theta) = \frac{1}{(\frac{N}{2})!} \left[ Tr \Phi \Theta^2 \right]^{(\frac{N}{2})} \]  

(5)

In fact,

\[ G_N(\Lambda, \Theta) = \Delta(\Lambda) \prod_{i<j} \theta_{ij} \theta_{ji} \]  

(6)

Here we make some comments on \( G_N \). The super-function \( G_N \) is essentially

\[ \delta^{N^2}(\Xi) = \prod_{i,j} \Xi_{ij} \], with \( \Xi = [\Phi, \Theta] \). This expression is manifestly \( U(N) \)-invariant, but equals to zero in practice, since there are \( N \) linear relations among the \( N^2 \) variables \( \Xi_{ij} \).

These linear dependency are described by introducing \( U(N) \)-invariant odd variables \( \xi_1 \cdots, \xi_N \) as follows:

\[ \xi_j = \Sigma_{j-\frac{1}{2}} \Phi, \Xi \equiv \frac{1}{j!} (\partial_{\xi_j}) \partial_{\xi_j} \det \{ I + t (\Phi + \xi \Xi) \} \bigg|_{t=0} \]  

(7)

with \( \xi \) being an odd parameter. In fact, if the commutator \( [\Phi, \Theta] \) is substituted for \( \Xi \) in (7), these variables become automatically zero. One finds

\[ (\prod_i \partial_{\xi_i}) \prod_{i,j} \Xi_{ij} \bigg|_{\Xi=[\Phi,\Theta]} = G_N(\Phi, \Theta) \]  

(8)

Having regard to (4) and (6), we write \( F_N(\Lambda, \Theta) \) as

\[ F_N(\Lambda, \Theta) = \frac{1}{\Delta(\Lambda)^3} f_N(\Lambda, \Theta_{diag}) \Delta(\Lambda) \prod_{i<j} \theta_{ij} \theta_{ji} \]  

(9)

where \( \Theta_{diag} \) is a matrix of the diagonal part of \( \Theta \), i.e. \( \Theta_{diag} = \left( \theta_{11} \cdots \theta_{NN} \right) \). The requirement(A) can be established if we can rewrite the factor \( \frac{1}{\Delta(\Lambda)^3} f_N(\Lambda, \Theta_{diag}) \) to a \( U(N) \)-invariant form. We postpone concerning this point, and assume it for a while.

As to the requirement(C), it is sufficient to investigate the invariance at \( \Phi = \Lambda \), since

\[ F_N(\Phi + \hat{\Theta} \Phi^n \zeta, \hat{\Theta} + \Phi^n \zeta) = F_N(\Lambda + \Theta \Lambda^n \zeta, \Theta + \Lambda^n \zeta) \]

with \( (\Lambda, \Theta) = (U \Phi, U \hat{\Theta}) \), provided that \( F_N \) is \( U(N) \)-invariant. For \( \Lambda \) generic, i.e. \( \lambda_i \neq \lambda_j \) for \( i \neq j \), the matrix \( \Lambda + \Theta \Lambda^n \zeta \) can be diagonalized as follows

\[ (I + \Omega \zeta)^{-1}(\Lambda + \Theta \Lambda^n \zeta) (I + \Omega \zeta) = \Lambda + \Theta_{diag} \Lambda^n \zeta \]  

(10)
Ω_{ij} = \begin{cases} 
0 & \text{for } i = j \\
\frac{\theta_{ij}\lambda_j^n}{\lambda_j - \lambda_i} & \text{for } i \neq j 
\end{cases} \quad (11)

Hence we have

$$F_N(\Lambda + \Theta\Lambda^n\zeta, \Theta + \Lambda^n\zeta) = F_N(\Lambda + \Theta_{diag}\Lambda^n\zeta, \Theta + (\{\Omega, \Theta\} + \Lambda^n)\zeta) \quad (12)$$

and here

$$\{\Omega, \Theta\}_{ij} = \sum_k (\theta_{ik} - \theta_{kj}) (\theta_{ii} + \theta_{jj}) \lambda_k^n - \sum_k (\theta_{ik} + \theta_{kj}) \lambda_k^n \quad . \quad (13)$$

Consequently, (C) is equivalent to the claim for $F_N(\Lambda, \Theta)$ to be invariant under the following transformation:

$$1 + \hat{\delta}_\zeta^{(n - \frac{1}{2})} : \begin{cases} 
\lambda_i \rightarrow \lambda_i + \theta_{ii}\lambda_i^n\zeta \\
\theta_{ij} \rightarrow \theta_{ij} + \{\Omega, \Theta\}_{ij} + \delta_{ij}\lambda_i^n\zeta 
\end{cases} \quad . \quad (14)$$

Then we see that

$$\hat{\delta}_\zeta^{(n - \frac{1}{2})} \prod_{i<j} \lambda_i^n - \lambda_j^n \theta_{ij} \zeta = \sum_{i<j} (\lambda_i^n + \lambda_j^n) (\theta_{ii} - \theta_{jj}) \prod_{i<j} \lambda_i^n - \lambda_j^n \theta_{ij} \zeta \quad , \quad (15)$$

$$\hat{\delta}_\zeta^{(n - \frac{1}{2})} \log \Delta (\Lambda)^{-2} = -2 \sum_{i<j} \frac{(\lambda_i^n\theta_{ii} - \lambda_j^n\theta_{jj})}{\lambda_i - \lambda_j} \zeta \quad . \quad (16)$$

Accordingly, the invariance of $F_N$ requires

$$\hat{\delta}_\zeta^{(n - \frac{1}{2})} \log f_N(\Lambda, \Theta_{diag}) = \sum_{i<j} \frac{(\lambda_i^n - \lambda_j^n)}{\lambda_i - \lambda_j} (\theta_{ii} + \theta_{jj}) \zeta \quad . \quad (17)$$

Note that the function $f_N(\Lambda, \Theta_{diag})$ is defined modulo $\theta_{ij}$ ($i \neq j$). Taking this into account, the left hand side of (17) is written as $\zeta \sum_i \lambda_i^n(\partial_{\theta_{ii}} - \theta_{ii}\partial_{\lambda_i}) \log f_N$. Hence we obtain

$$- \sum_i \lambda_i^n(\partial_{\theta_{ii}} - \theta_{ii}\partial_{\lambda_i}) \log f_N(\Lambda, \Theta_{diag}) = \sum_{i\neq j} \frac{(\lambda_i^n - \lambda_j^n)}{\lambda_i - \lambda_j} \theta_{ij} \quad . \quad (18)$$

This is the same equation that led to the eigenvalue model of ref.[4]. The equation (18) is easily integrated and its unique solution (up to a multiplicative constant) is

$$f_N(\Lambda, \Theta_{diag}) = \prod_{i<j} (\lambda_i - \lambda_j - \theta_{ii}\theta_{jj}) \quad . \quad (19)$$
The condition (C) may seem rather stronger requirement. In order to ensure the super-Virasoro structure, it suffices that \( \hat{\delta}^{(n-\frac{1}{2})} F_N(\Lambda, \Theta) \) becomes a product of \( \prod_{i<j} \theta_{ij} \theta_{ji} \) and a factor which can be compensated by properly differentiating the integrand \( e^{-\beta \text{Tr} V} \) with respect to the coupling constants. Note that, however, the expression in (17) takes the form which can not be obtained by such differentiations for \( n = 0 \).

We return to the requirement (A). From (19), we see that

\[
\frac{1}{\Delta(\Lambda)^3} f_N(\Lambda, \Theta_{\text{diag}}) = \frac{1}{\prod_{i<j} \{ (\lambda_i - \lambda_j)^2 + (\lambda_i - \lambda_j)\theta_{ii}\theta_{jj} \}} .
\]

The denominator of the right hand side is a symmetric super-polynomial of the variables \( (\lambda_i, \theta_{ii}) \quad 1 \leq i \leq N \), namely, a super-polynomial invariant under the permutation \( (\lambda_i, \theta_{ii}) \leftrightarrow (\lambda_j, \theta_{jj}) \). We can also define for the super case elementary symmetric super-polynomials \( \sigma_{\frac{j}{2}} \quad j = 1, \cdots, 2N \):

\[
\sigma_r = \sigma_r(\Lambda) \equiv \text{usual elementary polynomial of } r-\text{th order } , \\
\sigma_{r-\frac{1}{2}} \equiv \sum_{i=1}^{N} \theta_{ii} \partial_{\lambda_i} \sigma_r = \Sigma_{r-\frac{1}{2}}(\Lambda, \Theta_{\text{diag}}) \text{ defined by (7)} ,
\]

\[
r = 1, 2, \cdots, N .
\]

Generalizing the argument of the usual case, one can prove that any symmetric super-polynomial is expressed as a super-polynomial in \( \sigma_{\frac{j}{2}} \)'s. Let us introduce another type of symmetric super-polynomials defined by

\[
s_r = \sum_{i=1}^{N} \lambda_i^r \quad \text{and} \quad s_{r-\frac{1}{2}} = \sum_{i=1}^{N} \theta_{ii} \lambda_i^{r-1} .
\]

Then, one finds the generalized Newton’s formulae:

\[
n\sigma_n - \sigma_{n-1}s_1 + \cdots + (-1)^r \sigma_{n-r}s_r + \cdots + (-1)^n s_n = 0 \\
\sigma_{n-\frac{1}{2}} - \sigma_{n-1}s_{\frac{1}{2}} + \cdots + (-1)^r \sigma_{n-r}s_{r-\frac{1}{2}} + \cdots + (-1)^n s_{n-\frac{1}{2}} = 0 .
\]

These formulae enable us to write \( \sigma_{\frac{j}{2}} \)'s in terms of \( s_{\frac{j}{2}} \)'s. We also remark \( s_r = \text{Tr} \Phi_r \) and \( s_{r-\frac{1}{2}} = \text{Tr} \Theta_{\text{diag}} \Lambda^{r-1} = \text{Tr} \Theta \Lambda^{r-1} \). From the above, we understand that the expression (20) is rewritten as

\[
\frac{1}{\Delta(\Lambda)^3} f_N(\Lambda, \Theta_{\text{diag}}) = \frac{1}{E_N(\Phi, \Theta) \big|_{\Phi=\Lambda}} .
\]
where $E_N(\Phi, \Theta)$ is a certain super-polynomial in $s_\frac{1}{2}(\Phi, \Theta)$'s:

$$E_N(\Phi, \Theta) = \sum_{j} \sum_{k} \sum_{\nu=N(N-1)} \mu_{j_1, \ldots, j_\mu; k_1, \ldots, k_2, \nu} s_{j_1} \cdots s_{j_\mu} s_{k_1+\frac{1}{2}} \cdots s_{k_2+\frac{1}{2}} \cdots s_{\nu+\frac{1}{2}}. \quad (25)$$

We have now reached the conclusion that the preceding requirements (A), (B) and (C) determine uniquely the desired measure $d\mu(\Phi, \Theta)$ and that the partition function (2) is reduced to that of the eigenvalue model in [4]. The explicit form of the model depends heavily on the matrix size $N$ and seems somewhat ugly, especially due to the factor $E_N(\Phi, \Theta)^{-1}$. (In spite of the appearance, the matrix integral is well defined, as can be seen from the reduced integral over the eigenvalues.) For instance, in the most simple case $N = 2$, our model is the following:

$$Z_2 = \int d^4 \Phi d^4 \Theta \frac{Tr \Phi \Theta^2}{2 Tr \Phi^2 - (Tr \Phi)^2 + Tr \Phi \Theta \cdot Tr \Theta} e^{-\beta Tr V(\Phi, \Theta)}. \quad (26)$$

The problem is whether the matrix integral of the model bears the translation into the random sum of the triangulated super-surfaces by means of the graphical expansion. At this stage, it is far from obvious and needs more study.

We conclude this letter with a remark on the super-Virasoro constraints. The partition function (2) is obviously invariant under the shift of the variables (1). Following ref.[4], let us write the potential $V$ as

$$V(\Phi, \Theta) = \sum_{k \geq 0} \left( g_k \Phi^k + \xi_{k+\frac{1}{2}} \Theta \Phi^k \right). \quad (27)$$

The change of the integrand is then given by

$$-\zeta \sum_{k \geq 0} \left( k g_k \partial \xi_{n-\frac{1}{2}+k} + \xi_{k+\frac{1}{2}} \partial g_{n+k} \right) e^{-\beta Tr V}. \quad (28)$$

Because the function $F_N$ is invariant under the shift, the change of the measure $d\mu$ comes only from the Jacobian:

$$\frac{\partial(\Phi + \Theta \Phi^n \zeta, \Theta + \Phi^n \zeta)}{\partial(\Phi, \Theta)} = 1 - \zeta \sum_{k=0}^{n-1} Tr \Theta \Phi^k \cdot Tr \Phi^{n-1-k} \quad (29)$$

Hence the super-Virasoro constraints[4] on the partition function follows:

$$G_{n-\frac{1}{2}} Z_N = 0 \quad n \geq 0 \quad (30)$$
with

\[ G_{n-\frac{1}{2}} = \sum_{k \geq 0} \left( k g_k \partial_{\xi_{n-\frac{1}{2}+k}} + \xi_{k+\frac{1}{2}} \partial g_{n+k} + \frac{1}{\beta^2} \sum_{k=0}^{n-1} \partial_{\xi_{k+\frac{1}{2}}} \partial g_{n-1-k} \right). \]  \hspace{1cm} (31)

References

[1] E. Brézin and V.A. Kazakov, Phys. Lett. 236B(1990)144; M.R. Douglas and S.H. Shenker, Nucl. Phys. B335(1990)635; D.J. Gross and A.A. Migdal, Nucl. Phys. B340(1990)333.

[2] J. Alfaro and P.H. Damgaard, Phys. Lett. 222B(1989)429; A. Miković and W. Siegel, Phys. Lett. 240B(1990)363; S. Bellucci, T.R. Govindrajan, A. Kumar and R.N. Oerter, Phys. Lett. 249B(1990)49; J. Ambjørn and S. Varsted, Phys. Lett. 257B(1991)305; E. Marinari and G. Parisi, Phys. Lett. 240B(1990)375; M. Karliner and A.A. Migdal, Mod. Phys. Lett. A5(1990)2565; A. Dabholkar, Rutgers preprint RU-91-20(1991); J. Gonzalez, Phys. Lett. 255B(1991)367; S. Nojiri, Prog. Theor. Phys. 85(1991)671; A. Jevicki and J.P. Rodrigues, Phys. Lett. 268B(1991)53.

Supermatrix models were studied in G. Gilbert and M. Perry, Nucl. Phys. B364(1991)734; L. Alvarez-Gaumé and J.L. Mañes, Mod. Phys. Lett. A6(1991)2039; S.A. Yost, U. Florida preprint UFIFT-HEP-91-12.

[3] P. Di Francesco, J. Distler and D. Kutasov, Mod. Phys. Lett. A5(1990)2135; M. Awada, U. Florida preprints UFIFT-HEP-90-18 and -29.

[4] L. Alvarez-Gaumé, H. Itoyama, J.L. Mañes and A. Zadra, “Superloop Equations and Two Dimensional Supergravity”, CERN preprint CERN-TH.6329/91.

[5] A. Mironov and A. Morozov, Phys. Lett. 252B(1990)47; Y. Matsuo, unpublished.

As to the Virasoro constraints after taken the double scaling limit, see M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6(1991)1385; R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B348(1991)435.