Stabilization of a Rigid Body Payload With Multiple Cooperative Quadrotors

This paper presents the full dynamics and control of arbitrary number of quadrotor unmanned aerial vehicles (UAVs) transporting a rigid body. The rigid body is connected to the quadrotors via flexible cables where each flexible cable is modeled as a system of arbitrary number of serially connected links. It is shown that a coordinate-free form of equations of motion can be derived for the complete model without any simplicity assumptions that commonly appear in other literature, according to Lagrangian mechanics on a manifold. A geometric nonlinear controller is presented to transport the rigid body to a fixed desired position while aligning all of the links along the vertical direction. A rigorous mathematical stability proof is given and the desirable features of the proposed controller are illustrated by numerical examples and experimental results.

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1 Introduction

Quadrotor UAVs are being considered for various missions such as Mars surface exploration, search and rescue, and particularly payload transportation. There are various applications for aerial load transportation such as usage in construction, military operations, emergency response, or delivering packages. Load transportation with UAVs can be performed using a cable or by grasping the payload [1,2]. There are several limitations for grasping a payload with UAVs such as in situuations where the landing area is inaccessible, or, it transporting a heavy/bulk object by multiple quadrotors (Fig. 1).

Load transportation with the cable-suspended load has been studied traditionally for a helicopter [3,4] or for small UAVs such as quadrotor UAVs [5–7].

In most of the prior works, the dynamics of aerial transportation has been simplified due to the inherent dynamic complexities. For example, it is assumed that the dynamics of the payload is considered completely decoupled from quadrotors, and the effects of the payload and the cable are regarded as arbitrary external forces and moments exerted to the quadrotors [8–10], thereby making it challenging to suppress the swinging motion of the payload actively, particularly for agile aerial transportsations.

Recently, the coupled dynamics of the payload or cable has been explicitly incorporated into control system design [11]. In particular, a complete model of a quadrotor transporting a payload modeled as a point mass, connected via a flexible cable is presented, where the cable is modeled as serially connected links to represent the deformation of the cable [12,13]. In these studies, the payload simplified and considered as a point mass without the attitude and the moment of inertia. In another study, multiple quadrotors transporting a rigid body payload have been studied [14], but it is assumed that the cables connecting the rigid body payload and quadrotors are always taut. These assumptions and simplifications in the dynamics of the system reduce the stability of the controlled system, particularly in rapid and aggressive load transportation where the motion of the cable and payload is excited nontrivially.

The other critical issue in designing controllers for quadrotors is that they are mostly based on local coordinates. Some aggressive maneuvers are demonstrated at Ref. [15] based on Euler angles. However, they involve complicated expressions for trigonometric functions, and they exhibit singularities in representing quadrotor attitudes, thereby restricting their ability to achieve complex rotational maneuvers significantly. A quaternion-based feedback controller for attitude stabilization was shown in Ref. [16]. By considering the Coriolis and gyroscopic torques explicitly, this controller guarantees exponential stability. Quaternions do not have singularities but, as the three-sphere double-covers the special orthogonal group, one attitude may be represented by two antipodal points on the three-sphere. This ambiguity should be carefully resolved in quaternion-based attitude control systems, otherwise they may exhibit unwinding, where a rigid body unnecessarily rotates through a large angle even if the initial attitude error is small [17]. To avoid these, an additional mechanism to lift attitude onto the unit-quaternion space is introduced [18].

Recently, the dynamics of a quadrotor UAV is globally expressed on the special Euclidean group, SE(3), and nonlinear control systems are developed to track outputs of several flight modes [19]. Several aggressive maneuvers of a quadrotor UAV are demonstrated based on a hybrid control architecture, and a nonlinear robust control system is also considered in Refs. [20,21]. As they are directly developed on the special orthogonal group, complexities, singularities, and ambiguities associated with
minimal attitude representations or quaternions are completely avoided. The proposed control system is particularly useful for rapid and safe payload transportation in complex terrain, where the position of the payload should be controlled concurrently while suppressing the deformation of the cables.

Compared with the prior work of the authors in Refs. [22–24] and other existing studies, this paper is the first study considering a complete model which includes a rigid body payload with attitude, arbitrary number of quadrotors, and flexible cables. Also, this study presents a control system to stabilize the rigid body at desired position. Geometric nonlinear controllers are developed for the presented model. More explicitly, we show that the rigid body payload is asymptotically transported into a desired location, while aligning all of the links along the vertical direction corresponding to a hanging equilibrium.

In short, new contributions and the unique features of the dynamics model and control system proposed in this paper compared with other studies are as follows: (i) it is developed for the full dynamic model of arbitrary number of multiple quadrotor UAVs on SE(3) transporting a rigid body connected via flexible cables, including the coupling effects between the translational dynamics and the rotational dynamics on a nonlinear manifold, (ii) the control systems are developed directly on the nonlinear configuration manifold in a coordinate-free fashion. Thus, singularities of local parameterization are completely avoided to generate agile maneuvers in a uniform way, (iii) a rigorous Lyapunov analysis is presented to establish stability properties without any timescale separation assumption or singular perturbation, (iv) an integral control term is proposed to guarantee asymptotical convergence of tracking error variables in the presence of unstructured uncertainties in both rotational and translational dynamics, and (v) the proposed algorithm is validated with experiments for payload transportation with multiple cooperative quadrotor UAVs. A rigorous and complete mathematical analysis for multiple quadrotor UAVs transporting a payload on SE(3) with experimental validations for payload transportation maneuvers is unprecedented.

This paper is organized as follows. A dynamic model is presented and the problem is formulated at Sec. 2. Control systems are constructed at Secs. 3 and 4, which are followed by numerical examples in Sec. 5. Finally, experimental results are presented in Sec. 6.

2 Problem Formulation

Consider a rigid body with the mass \( m_0 \in \mathbb{R} \) and the moment of inertia \( J_i \in \mathbb{R}^{3 \times 3} \), being transported with \( n \) arbitrary number of quadrotors as shown in Fig. 2. The location of the mass center of the rigid body is denoted by \( x_0 \in \mathbb{R}^3 \), and its attitude is given by \( R_0 \in \text{SO}(3) \), where the special orthogonal group is given by \( \text{SO}(3) = \{ R \in \mathbb{R}^{3 \times 3} | R^T R = I, \det(R) = 1 \} \). We choose an inertial frame \( \{e_1, e_2, e_3\} \) and body-fixed frame \( \{b_1, b_2, b_3\} \) attached to the payload. We also consider a body-fixed frame attached to the \( i \)-th quadrotor \( \{b_{i1}, b_{i2}, b_{i3}\} \). In the inertial frame, the third axes \( e_3 \) points downward along the gravity and the other axes are chosen to form an orthonormal frame.

The mass and the symmetric moment of inertia of the \( i \)-th quadrotor are denoted by \( m_i \in \mathbb{R} \) and \( J_i \in \mathbb{R}^{3 \times 3} \), respectively. The cables connecting each quadrotor to the rigid body is modeled as an arbitrary numbers of links for each quadrotor with varying masses, \( m_{ij} \), and lengths, \( l_{ij} \). The direction of the \( i \)-th link of the \( j \)-th quadrotor, measured outward from the \( j \)-th quadrotor toward the payload is defined by the unit vector \( q_{ij} \in S^2 \), where \( S^2 = \{ q \in \mathbb{R}^3 | ||q|| = 1 \} \). The mass and length of that link is denoted with \( m_{ij} \) and \( l_{ij} \), respectively. The number of links in the cable connected to the \( i \)-th quadrotor is defined as \( n_i \).

The configuration manifold for this system is given by \( \text{SO}(3) \times \mathbb{R}^3 \times \text{SO}(3)^n \times (S^2)^{2n_i} \).

The \( i \)-th quadrotor can generate a thrust force of \(- f_i R_i e_3 \in \mathbb{R}^3\) with respect to the inertial frame, where \( f_i \in \mathbb{R} \) is the total thrust magnitude of the \( i \)-th quadrotor. It also generates a moment \( M_i \in \mathbb{R}^3 \) with respect to its body-fixed frame. Also, we define \( \Delta_i \) and \( \Delta_R \in \mathbb{R}^3 \) as fixed disturbances applied to the \( i \)-th quadrotor’s translational and rotational dynamics, respectively. It is also assumed that an upper bound of the infinite norm of the uncertainty is known

\[
\| \Delta_i \|_\infty \leq \delta
\]

for a positive constant \( \delta \).

Throughout this paper, \( \lambda_m(A) \) and \( \lambda_M(A) \) denote the minimum eigenvalue and the maximum eigenvalue of a square matrix \( A \), respectively, and \( \lambda_m \) and \( \lambda_M \) are the minimum eigenvalue and the maximum eigenvalue of the inertia matrix \( J \), i.e., \( \lambda_m = \lambda_m(J) \) and \( \lambda_M = \lambda_M(J) \). The two-norm of a matrix \( A \) is denoted by \( \| A \| \). The standard dot product is denoted by \( x \cdot y = x^T y \) for any \( x, y \in \mathbb{R}^n \).

2.1 Lagrangian. The kinematics equations for the links, payload, and quadrotors are given by

\[
\dot{q}_{ij} = \omega_{ij} \times q_{ij} = \hat{\omega}_{ij} q_{ij}
\]

\[
\dot{R}_0 = R_0 \Omega_0
\]

\[
\dot{R}_i = R_i \Omega_i
\]

where \( \omega_{ij} \in \mathbb{R}^3 \) is the angular velocity of the \( j \)-th link in the \( i \)-th quadrotor, \( \Omega_0 \in \mathbb{R}^3 \) is the angular velocity of the payload and \( \Omega_i \in \mathbb{R}^3 \) is the angular velocity of the \( i \)-th quadrotor, expressed with respect to the corresponding body-fixed frame. The hat map \( \hat{\cdot} : \mathbb{R}^3 \rightarrow \text{SO}(3) \) is defined by the condition that \( \hat{x}y = x \times y \) for all \( x, y \in \mathbb{R}^3 \). More explicitly, for a vector \( a = [a_1, a_2, a_3]^T \in \mathbb{R}^3 \), the matrix \( \hat{a} \) is given by

\[
\hat{a} = \begin{bmatrix}
0 & -a_3 & a_2 \\
a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{bmatrix}
\]
This identifies the Lie algebra $\text{SO}(3)$ with $\mathbb{R}^3$ using the vector cross product in $\mathbb{R}^3$. The inverse of the hat map is denoted by the \text{vee} map, $\nu : \text{SO}(3) \rightarrow \mathbb{R}^3$.

The position of the $i$th quadrotor is given by
\[ x_i = x_0 + R_0 p_i - \sum_{a=1}^{n} l_{ai} q_{ia} \]
(6)

where $p_i \in \mathbb{R}^3$ is the vector from the center of mass of the rigid body to the point that $i$th cable is connected to the rigid body. Similarly, the position of the $j$th link in the cable connecting the $i$th quadrotor to the rigid body is given by
\[ x_{ij} = x_0 + R_0 p_j - \sum_{a=1}^{n} l_{aij} q_{ia} \]
(7)

We derive equations of motion according to Lagrangian mechanics. Total kinetic energy of the system is given by
\[ T = \frac{1}{2} m_0 ||\dot{x}_0||^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{2} m_{ij} ||\dot{x}_{ij}||^2 + \frac{1}{2} \sum_{i=1}^{n} m_i ||\dot{x}_i||^2 \]
\[ + \frac{1}{2} \sum_{i=1}^{n} \Omega_i \cdot J_i \Omega_i + \frac{1}{2} \Omega_0 \cdot J_0 \Omega_0 \]
(8)

The gravitational potential energy is given by
\[ V = -m_0 g e_3 \cdot x_0 - \sum_{i=1}^{n} m_i g e_3 \cdot x_i - \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} g e_3 \cdot x_{ij} \]
(9)

where it is assumed that the unit vector $e_3$ points downward along the gravitational acceleration as shown at Fig. 2. The corresponding Lagrangian of the system is $L = T - V$.

2.2 Euler–Lagrange Equations. Coordinate-free form of Lagrangian mechanics on the two-sphere $S^2$ and the special orthogonal group $\text{SO}(3)$ for various multibody systems has been studied in Refs. [25,26]. The key idea is representing the infinitesimal variation of $R_i \in \text{SO}(3)$ in terms of the exponential map
\[ \delta R_i = \frac{d}{d\eta} \bigg|_{\eta=0} R_i \exp(\bar{\eta} q_i) = R_i \bar{\eta} \]
(10)

for $\eta \in \mathbb{R}^3$. The corresponding variation of the angular velocity is given by $\delta \Omega_i = \bar{\eta} \times \dot{\Omega}_i$. Similarly, the infinitesimal variation of $q_j \in S^2$ is given by
\[ \delta q_j = \bar{\zeta}_j \times q_j \]
(11)

for $\zeta_j \in \mathbb{R}^3$ satisfying $\bar{\zeta}_j \cdot q_j = 0$. Using these, we obtain the following Euler–Lagrange equations.

Proposition 1. The equations of motion for the proposed payload transportation system are as follows:

\[ M_f \ddot{x}_0 = \sum_{j=1}^{n} \sum_{i=1}^{n} M_{0ij} l_{0i} \dot{q}_{ij} - \sum_{i=1}^{n} M_f R_0 \bar{p}_i \dot{\Omega}_0 \]
\[ = M_f g e_3 + \sum_{i=1}^{n} (-f_i R_i e_3 + \Delta_{i}) - \sum_{i=1}^{n} M_f R_0 \bar{p}_i \dot{\Omega}_0 \]
(12)

\[ J_0 \ddot{\Omega}_0 + \sum_{i=1}^{n} M_f R_i \bar{p}_i \dot{\Omega}_0 = \sum_{j=1}^{n} \sum_{i=1}^{n} M_{0ij} l_{0i} \bar{p}_i \dot{q}_{ij} \]
\[ = \sum_{i=1}^{n} \bar{p}_i R_i^T (-f_i R_i e_3 + M_f g e_3 + \Delta_{i}) - \dot{\Omega}_0 J_0 \Omega_0 \]
(13)
and submatrices of $P$ matrix are also defined as

$$ P_{ij} = M_T g e_3 + \sum_{i=1}^{n} (-f_i R_i e_3 + \Delta_i) - \sum_{i=1}^{n} M_T R_i \tilde{\Omega}_i \tilde{\Omega}_i^T $$

$$ P_{\alpha k} = -\tilde{\Omega}_{k}^T (\tilde{\Omega}_0 + \sum_{i=1}^{n} \tilde{\rho}_i \tilde{R}_i^T (M_T g e_3 - f_i R_i e_3 + \Delta_i) $$

$$ P_{ij} = -\tilde{R}^{-1}_j (-f_j R_j e_3 + M_{0ij} g e_3 + \Delta_n) + M_{0ij} \tilde{\rho}_j \tilde{R}_i \tilde{\Omega}_i \tilde{\Omega}_i^T $$

$$ + M_{0ij} \tilde{\rho}_j \tilde{R}_i \tilde{\Omega}_i \tilde{\Omega}_i^T $$

Proof. See Appendix A.

These equations are derived directly on a nonlinear manifold. The dynamics of the payload, flexible cables, and quadrotors are considered explicitly, and they avoid singularities and complexities associated to local coordinates.

3 Control System Design for Simplified Dynamic Model

3.1 Control Problem Formulation. Let $x_0 \in \mathbb{R}^3$ be the desired position of the payload. The desired attitude of the payload is considered to be $R_0 = I_{3x3}$, and the desired direction of links is aligned along the vertical direction. The corresponding location of the $i$th quadrotor at this desired configuration is given by

$$ x_{ai} = x_{ai} + \rho_i - \sum_{i=1}^{n} l_i e_3 $$

(24)

We wish to design control forces $f_i$ and control moments $M_i$ of quadrotors such that this desired configuration becomes asymptotically stable.

3.2 Simplified Dynamic Model. Control forces for each quadrotor are given by $-f_i R_i e_3$ for the given equations of motion (12)–(15). As such, the rotational dynamics is underactuated. The total thrust magnitude of each quadrotor can be arbitrary chosen, but the direction of the thrust vector is always along the third body-fixed axis, represented by $R_i e_3$. But, the rotational attitude dynamics of the quadrotors are fully actuated, and they are not affected by the translational dynamics of the quadrotors or the dynamics of links.

Based on these observations, in this section, we simplify the model by replacing the $-f_i R_i e_3$ term by a fictitious control input $u_i \in \mathbb{R}^3$, and design an expression for $u_i$ to asymptotically stabilize the desired equilibrium. In other words, we assume that the attitude of the quadrotor can be instantaneously changed. Also, $\Delta_i$ are ignored in the simplified dynamic model. The effects of the attitude dynamics are incorporated in Sec. 4.

3.3 Linear Control System. The control system for the simplified dynamic model is developed based on the linearized equations of motion. At the desired equilibrium, the position and the attitude of the payload are given by $x_0$ and $R_0 = I_3$, respectively. Also, we have $q_{ij} = e_3$ and $R_{ij} = I_3$. In this equilibrium configuration, the control input for the $i$th quadrotor is $u_i = -f_i R_i e_3$

(25)

where the total thrust is $f_0 = (M_T + (m_0/n)) g$.

The variation of $x_0$ is given by

$$ \delta x_0 = x_0 - x_{0i} $$

(26)

and the variation of the attitude of the payload is defined as

$$ \delta R_0 = R_0 \tilde{\eta}_0 = \tilde{\eta}_0 $$

for $\eta_0 \in \mathbb{R}^3$. The variation of $q_{ij}$ can be written as

$$ \delta q_{ij} = \xi_{ij} e_3 $$

(27)

where $\xi_{ij} \in \mathbb{R}^3$ with $\xi_{ij} \cdot e_3 = 0$. The variation of $\omega_j$ is given by $\delta \omega_j \in \mathbb{R}^3$ with $\delta \omega_j \cdot e_3 = 0$. Therefore, the third element of each $\xi_{ij}$ and $\delta \omega_j$ for any equilibrium configuration is zero, and they are omitted in the following linearized equations. The state vector of the linearized equation is composed of $C^T \xi_{ij} \in \mathbb{R}^3$, where $C = e_1, e_2, e_3 \in \mathbb{R}^{3\times 3}$. The variation of the control input $\delta u_i = u_i - u_{ai}$.

Proposition 2. The linearized equations of the simplified dynamic model are given by

$$ M_S + G_X = B \delta u + g(x, \dot{x}) $$

(28)

where $g(x, \dot{x})$ corresponds to the higher-order term and the state vector $x \in \mathbb{R}^{3n}$ with $D_x = 6 + 2 \sum_{i=1}^{n} \| n_i \|$. This is given by

$$ x = [ \delta x_0, \delta \eta_0, C^T \xi_{11}, C^T \xi_{21}, \ldots, C^T \xi_{n1} ] $$

and $\delta u = [ \delta u_{i1}, \delta u_{i2}, \ldots, \delta u_{in} ]^T \in \mathbb{R}^{3n \times 1}$. The matrix $M \in \mathbb{R}^{3n \times D_x}$ are defined as

$$ M_{\alpha k} = \begin{bmatrix} M_{T1} & M_{01} & M_{01} & \cdots & M_{01} \\ M_{01} & M_{01} & M_{01} & \cdots & M_{01} \\ M_{01} & M_{01} & M_{01} & \cdots & M_{01} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{01} & M_{01} & M_{01} & \cdots & M_{01} \end{bmatrix} $$

(29)

and the submatrix $M_{qq} \in \mathbb{R}^{2n \times 2n}$ is given by

$$ M_{qq} = \begin{bmatrix} M_1 & M_1 & \cdots & M_1 \\ M_1 & M_1 & \cdots & M_1 \\ \vdots & \vdots & \ddots & \vdots \\ M_1 & M_1 & \cdots & M_1 \end{bmatrix} $$

(30)

and the submatrix $M_{qq} \in \mathbb{R}^{2n \times 2n}$ is given by

$$ M_{qq} = \begin{bmatrix} M_1 & M_1 & \cdots & M_1 \\ M_1 & M_1 & \cdots & M_1 \\ \vdots & \vdots & \ddots & \vdots \\ M_1 & M_1 & \cdots & M_1 \end{bmatrix} $$

The matrix $G \in \mathbb{R}^{3n \times D_x}$ is defined as

$$ G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & G_{01} & 0 & 0 & 0 & 0 \\ 0 & 0 & G_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & G_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & G_n \end{bmatrix} $$

where $G_{01} = \sum_{i=1}^{n} (m_0/n) g R_i e_3$ and the submatrices $G_i \in \mathbb{R}^{2n \times 2n}$ are

$$ G_i = \text{diag} \left[ \begin{bmatrix} -M_T - m_0/n + M_{ij} \end{bmatrix} g e_3 \right] $$

The matrix $B \in \mathbb{R}^{D_x \times 3n}$ is given by

$$ B = \begin{bmatrix} I_3 & I_3 & \cdots & I_3 \\ \tilde{\rho}_1 & \tilde{\rho}_2 & \cdots & \tilde{\rho}_n \\ B_B & 0 & 0 & 0 \\ 0 & B_B & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & B_B \end{bmatrix} $$

where $B_B = [-C^T \tilde{e}_3, C^T \tilde{e}_3, \ldots, C^T \tilde{e}_3]^T$. 

Transactions of the ASME
Proof. See Appendix B.

We present the following proportional-derivative-type control system for the linearized dynamics:

$$\delta u_i = -K_i \delta x - K_i \dot{x}$$  \hspace{1cm} (32)

for controller gains $K_i, K_{ii} \in \mathbb{R}^{3 \times D_n}$. Provided that Eq. (28) is controllable, we can choose the combined controller gains $K = [K_1^T, \ldots, K_n^T]^T$, $K_S = [K_{i1}^T, \ldots, K_{in}^T]^T \in \mathbb{R}^{3 \times D_n}$ such that the equilibrium is asymptotically stable for the linearized equations [27]. Then, the equilibrium becomes asymptotically stable for the nonlinear Euler–Lagrange equation. The controlled linearized system can be written as

$$\dot{z}_1 = \Lambda z_1 + \mathcal{B}(\mathbf{x}, \dot{\mathbf{x}})$$  \hspace{1cm} (33)

where $z_1 = [\mathbf{x}, \dot{\mathbf{x}}]^T \in \mathbb{R}^{2D_n}$ and

$$\Lambda = \begin{bmatrix} 0 & -M^{-1}(G + B K_d) \\ -M^{-1} B K_S & -M^{-1} \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}$$  \hspace{1cm} (34)

We choose $K_d$ and $K_S$ such that $\Lambda \in \mathbb{R}^{2D_n \times 2D_n}$ is Hurwitz. Then for any positive definite matrix $Q \in \mathbb{R}^{2D_n \times 2D_n}$, there exist a positive definite and symmetric matrix $P \in \mathbb{R}^{2D_n \times 2D_n}$ such that $\Lambda^T P + P \Lambda = -Q$ according to Ref. [27, Theorem 3.6].

### 4 Control System Design for the Full Dynamic Model

The control system designed at Sec. 3 is based on a simplifying assumption that each quadrotor can generate a thrust along any direction. In the full dynamic model, the direction of the thrust for each quadrotor is parallel to its third body-fixed axis always. In this section, the attitude of each quadrotor is controlled such that the third body-fixed axis becomes parallel to the direction of the ideal control force designed in Sec. 3. Also in the full dynamics model, we consider the $A_{m}$ in the control design and introduce a new integral term to eliminate the disturbances and uncertainties. The central idea is that the attitude $R_i$ of the quadrotor is controlled such that its total thrust direction $-R_i e_3$, corresponding to the third body-fixed axis, asymptotically follows the direction of the fictitious control input $u_i$. By choosing the total thrust magnitude properly, we can guarantee local asymptotical stability for the full dynamic model.

Let $A_i \in \mathbb{R}^3$ be the ideal total thrust of the $i$th quadrotor that locally asymptotically stabilize the desired equilibrium, defined as

$$A_i = u_{i_d} + \delta u_i = -K_i \delta x - K_i \dot{x} - K_i \text{sat}_e(e_i) + u_{i_d}$$  \hspace{1cm} (35)

where $f_d \in \mathbb{R}$ and $u_{i_d} \in \mathbb{R}^3$ are the total thrust and control input of each quadrotor at its equilibrium, respectively, and the following integral term $e_i \in \mathbb{R}^3$ is added to eliminate the effect of disturbance $A_{m}$ in the full dynamic model:

$$e_i = \int_0^t (P \mathcal{B}(\mathbf{x}, \dot{\mathbf{x}})) d\tau$$  \hspace{1cm} (36)

where $K_i = [k_i1, k_i1, k_i1, k_i1, \ldots, k_i1, k_i1] \in \mathbb{R}^{3 \times D_n}$ is an integral gain. For a positive constant $\sigma \in \mathbb{R}$, a saturation function $\text{sat}_e : \mathbb{R} \rightarrow [-\sigma, \sigma]$ is introduced as

$$\text{sat}(y) = \begin{cases} \sigma & \text{if } y > \sigma \\ y & \text{if } -\sigma \leq y \leq \sigma \\ -\sigma & \text{if } y < -\sigma \end{cases}$$

If the input is a vector $y \in \mathbb{R}^n$, then the above saturation function is applied element by element to define a saturation function $\text{sat}_e(y) : \mathbb{R}^n \rightarrow [-\sigma, \sigma]^n$ for a vector.

From the direction of the third body-fixed axis of the $i$th quadrotor, namely, $b_{3i} \in \mathbb{S}^2$, is given by

$$b_{3i} = \frac{A_i}{||A_i||}$$  \hspace{1cm} (37)

This provides a two-dimensional constraint on the three-dimensional desired attitude of each quadrotor, such that there remains one degree-of-freedom. To resolve it, the desired direction of the first body-fixed axis $b_{1i}(t) \in \mathbb{S}^2$ is introduced as a smooth function of time. Due to the fact that the first body-fixed axis is normal to the third body-fixed axis, it is impossible to follow an arbitrary command $b_{1i}(t)$ exactly. Instead, its projection onto the plane normal to $b_{3i}$ is followed, and the desired direction of the second body-fixed axis is chosen to constitute an orthonormal frame [23].

More explicitly, the desired attitude of the $i$th quadrotor is given by

$$R_i = \begin{bmatrix} (\hat{b}_{3i})^2 b_{1i} \\ (\hat{b}_{3i})^2 b_{1i} \\ b_{3i} \end{bmatrix}$$  \hspace{1cm} (38)

which is guaranteed to be an element of $\text{SO}(3)$. The desired angular velocity is obtained from the attitude kinematics equation, $\Omega_i = (R_i^T \dot{R}_i)^T \in \mathbb{R}^3$.

Define the tracking error vectors for the attitude and the angular velocity of the $i$th quadrotor as

$$e_k = \frac{1}{2} (R_i^T R_i - R_i^T R_i)^T, \quad e_{\Omega} = \Omega_i - R_i^T \dot{R}_i \Omega_i$$  \hspace{1cm} (39)

and a configuration error function on $\text{SO}(3)$ as follows:

$$\Psi_i = \frac{1}{2} \text{tr} [I - R_i^T R_i]$$  \hspace{1cm} (40)

The thrust magnitude is chosen as the length of $u_i$ projected on to $-R_i e_3$, and the control moment is chosen as a tracking controller on $\text{SO}(3)$

$$f_i = -A_i : R_i e_3$$  \hspace{1cm} (41)

$$M_i = -k_{er} e_{\Omega} - k_{o} e_{\Omega} - k_i e_i + (R_i^T R_i \Omega_i)^T J R_i^T R_i \Omega_i + J R_i^T \dot{R}_i \Omega_i$$  \hspace{1cm} (42)

where $k_{er}, k_{o}$, and $k_i$ are positive constants and the following integral term is introduced to eliminate the effect of fixed disturbance $A_{m}$:

$$e_i = \int_0^t e_{\Omega}(\tau) + c_2 e_k(\tau) d\tau$$  \hspace{1cm} (43)

where $c_2$ is a positive constant. Stability of the corresponding controlled systems for the full dynamic model can be studied by showing the error due to the discrepancy between the desired direction $b_{3i}$ and the actual direction $R_i e_3$.

Proposition 3. Consider control inputs $f_i, M_i$ defined in Eqs. (41) and (42). There exist controller parameters and gains such that, (i) the zero equilibrium of tracking error is stable in the sense of Lyapunov; (ii) the tracking errors $e_k, e_{\Omega}, x, \dot{x}$ asymptotically converge to zero as $t \rightarrow \infty$; and (iii) the integral terms $e_i$ and $e_{\Omega}$ are uniformly bounded.

Proof. See Appendix C.

By utilizing geometric control systems for quadrotor, we show that the hanging equilibrium of the links can be asymptotically stabilized while translating the payload to a desired position and attitude. The control systems proposed explicitly consider the coupling effects between the cable/load dynamics and the quadrotor dynamics. We presented a rigorous Lyapunov stability analysis to establish stability properties without any timescale separation.

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Fig. 3 Stabilization of a rigid body connected to multiple quadrotors. (a) Payload position (\(x_0: \text{solid}, \ x_0^d: \text{dotted}\)), (b) payload velocity (\(v_0: \text{solid}, \ v_0^d: \text{dotted}\)), (c) payload angular velocity \(\Omega_0\), (d) quadrotors angular velocity errors \(\epsilon_{\Omega_i}\), (e) payload attitude error \(\Psi_0\), (f) quadrotors attitude errors \(\Psi_i\), (g) quadrotors total thrust inputs \(f_i\), and (h) direction error \(\epsilon_q\) and angular velocity error \(\epsilon_{\omega}\) for the links.
assumptions or singular perturbation, and a new nonlinear integral control term is designed to guarantee robustness against unstructured uncertainties in both rotational and translational dynamics.

5 Numerical Example

We demonstrate the desirable properties of the proposed control system with numerical examples. Two cases are presented. At the first case, a payload is transported to a desired position from the ground. The second case considers stabilization of a payload with large initial attitude errors.

5.1 Stabilization of the Rigid Body. Consider four quadrotors \((n = 4)\) connected via flexible cables to a rigid body payload. Initial conditions are chosen as

\[
x_0(0) = [1.0, 4.8, 0.0]^T \text{ m}, \quad v_0(0) = 0_{3 \times 1} \\
q_{ij}(0) = e_3, \quad o_{ij}(0) = 0_{3 \times 1}, \quad R_i(0) = I_{3 \times 3}, \quad \Omega_i(0) = 0_{3 \times 1} \\
R_0(0) = I_{3 \times 3}, \quad \Omega_0 = 0_{3 \times 1}
\]

The desired position of the payload is chosen as

\[
x_0(t) = [0.44, \ 0.78, \ -0.5]^T \text{ m} \tag{44}
\]

The mass properties of quadrotors are chosen as

\[
m_i = 0.755 \text{ kg} \\
J_i = \text{diag}[0.557, \ 0.557, \ 1.05] \times 10^{-2} \text{ kg/m}^2 \tag{45}
\]

and the attitude control gains of each quadrotor are defined with the following expressions:

\[
k_{R_i} = \omega_{\text{eq}}^2 \|J_i\| \\
k_{\Omega_i} = 2 \omega_{\text{eq}} \xi_\omega \|J_i\|
\]

where considering \(\omega_{\text{eq}} = 8\) and \(\xi_\omega = 0.7\), the attitude control gains are given by \(k_{R_i} = 0.6724\) and \(k_{\Omega_i} = 0.1177\) for all quadrotors and \(k_2 = k_3 = 0.01\). The linear controller gains which are \(K_x\) and \(K_\omega\) are calculated using the linear quadratic regulator method which each has a dimension of \(12 \times 46\). The payload is a box with mass \(m_0 = 0.5\) kg, and its length, width, and height are 0.6, 0.8, and 0.2 m, respectively. Each cable connecting the rigid body to the \(i\)th quadrotor is considered to be \(n_i = 5\) rigid links. All the links have the same mass of \(m_{n_i} = 0.01\) kg and length of \(l_{n_i} = 0.15\) m. Each cable is attached to the following points of the payload:

\[
\rho_1 = [0.3, \ -0.4, \ -0.1]^T \text{ m}, \quad \rho_2 = [0.3, \ 0.4, \ -0.1]^T \text{ m} \\
\rho_3 = [-0.3, \ -0.4, \ -0.1]^T \text{ m}, \quad \rho_4 = [-0.3, \ 0.4, \ -0.1]^T \text{ m}
\]

Numerical simulation results are presented at Figs. 3 and 4, which shows the position and velocity of the payload and its tracking errors. We have also presented the link direction and link angular velocity errors defined as

\[
e_q = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \|q_{ij} - e_q\| \tag{46}
\]

\[
e_{\omega} = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \|o_{ij}\| \tag{47}
\]

5.2 Payload Stabilization With Large Initial Attitude Errors. In the second case, we consider large initial errors for the attitude of the payload and quadrotors. Initially, the rigid body is tilted about its \(b_1\) axis by 30 deg, and the initial direction of the links is chosen such that two cables are curved along the horizontal direction. The initial conditions are given by

\[
x_0(0) = [2.4, \ 0.8, \ -1.0]^T, \quad v_0(0) = 0_{3 \times 1} \\
\omega_{ij}(0) = 0_{3 \times 1}, \quad \Omega_i(0) = 0_{3 \times 1} \\
R_0(0) = R_i(30 \text{ deg}), \quad \Omega_0 = 0_{3 \times 1}
\]

where \(R_i(30 \text{ deg})\) denotes the rotation about the first axis by 30 deg. The initial attitude of quadrotors is chosen as

\[
R_1(0) = R_i(-35 \text{ deg}), \quad R_2(0) = I_{3 \times 3} \\
R_3(0) = R_i(-35 \text{ deg}), \quad R_4(0) = I_{3 \times 3}
\]

The mass properties of quadrotors and cables are chosen as previous example. The payload mass is \(m = 1.0\) kg, and its length, width, and height are 1.0, 1.2, and 0.2 m, respectively. Each cable is attached to the following points of the payload:

\[
\rho_1 = [0.5, \ -0.6, \ -0.1]^T \text{ m}, \quad \rho_2 = [0.5, \ 0.6, \ -0.1]^T \text{ m} \\
\rho_3 = [-0.5, \ -0.6, \ -0.1]^T \text{ m}, \quad \rho_4 = [-0.6, \ 0.6, \ -0.1]^T \text{ m}
\]

Figure 5 illustrates the tracking errors, and the total thrust of each quadrotor. Snapshots of the controlled maneuvers are also illustrated at Fig. 6. It is shown that the proposed controller is able to stabilize the payload and cables at their desired configuration even from the large initial attitude errors.
Fig. 5  Stabilization of a payload with multiple quadrotors connected with flexible cables. (a) Payload position ($x_0$: solid, $x_0^d$: dotted), (b) payload velocity ($v_0$: solid, $v_0^d$: dotted), (c) payload angular velocity $\Omega_0$, (d) quadrotors angular velocity errors $\omega_1, \omega_2$, (e) payload attitude error $\Psi_0$, (f) quadrotors attitude errors $\Psi_1, \Psi_2$, (g) quadrotors total thrust inputs $f_i$, and (h) direction error $e_{\phi}$ and angular velocity error $e_{\omega}$ for the links.
6 Experiment

In this section, an experimental setup is described and the proposed geometric nonlinear controller is validated with experiments.

6.1 Hardware Description. The quadrotor UAV developed at the flight dynamics and control laboratory at the George Washington University is shown at Fig. 7(a), and its parameters are the same as described as Sec. 5. The angular velocity is measured from inertial measurement unit (IMU) and the attitude is obtained from IMU data. Position of the UAV is measured from motion capture system (Vicon) and the velocity is estimated from the measurement. Ground computing system receives the Vicon data and sends it to the UAV via XBee. The Gumstix is adopted as micro computing unit on the UAV. The flight control software has three main threads, namely, Vicon thread, IMU thread, and control thread. The Vicon thread receives the Vicon measurement and estimates linear velocity of the quadrotor. In IMU thread, it receives the IMU measurement and estimates the attitude. The last thread handles the control outputs at each time step. Also, control outputs are calculated at 120 Hz which is fast enough to run any kind of aggressive maneuvers. Information flow of the system is illustrated in Fig. 8.

We have developed an accurate CAD model as shown in Fig. 9 to identify several parameters of the quadrotor, such as moment of inertia and center of mass. Furthermore, a precise rotor calibration is performed for each rotor, with a custom-made thrust stand as shown in Fig. 7(b) to determine the relation between the command in the motor speed controller and the actual thrust. For various values of motor speed commands, the corresponding thrust is measured, and those data are fitted with a second-order polynomial.

6.2 Stabilizing a Rod With Two Quadrotors. As a special case of rigid body payload, we consider two quadrotors and a rod as shown in Fig. 10.

Two quadrotors are commanded to hover at a fixed position initially while holding the rod directly under them. Then, we utilize
Fig. 11 Benchmark: quadrotor position control system [21]. (a) Rod’s actual and desired positions, $x_0, x_{0d}$, (b) quadrotor’s positions, $x_0, x_1$, (c) first cable’s direction, $q_1$, and (d) second cable’s direction, $q_2$.

Fig. 12 Proposed controller: two quadrotors with rigid body payload. (a) Rod’s actual and desired positions, $x_0, x_{0d}$, (b) quadrotor’s positions, $x_1, x_2$, (c) first cable’s direction, $q_1$, and (d) second cable’s direction, $q_2$. 
a wire to pull the payload and release it as a disturbance. The performance of the proposed controller is then compared to the situation where there is no active controller to stabilize the payload.

Both cables have length of \( l_1 = 1.3 \text{ m} \). The mass and the length of the rod are \( m_0 = 0.52 \text{ kg} \) and \( l_0 = 2.05 \text{ m} \), respectively. Each quadrotor has mass of \( m_1 = m_2 = 0.755 \text{ kg} \) and the following moment of inertia:

\[
J_1, J_2 = \begin{bmatrix} 5.5711 & 0.0618 & -0.0251 \\ 0.0617 & 5.5757 & 0.0101 \\ -0.0250 & 0.0100 & 1.0505 \end{bmatrix} \times 10^{-2} \text{ kgm}^2.
\]

The vectors \( x_1, x_2 \) denote the position of each quadrotor, and \( x_{11}, x_{22} \) correspond to the positions of the two end points of the rod, which are all measured by the Vicon motion capture system. Then, the direction of each cable is obtained as follows:

\[
q_1 = \frac{x_{11} - x_1}{\|x_{11} - x_1\|}, \quad q_2 = \frac{x_{22} - x_2}{\|x_{22} - x_2\|}, \quad q_0 = \frac{x_{22} - x_{11}}{\|x_{22} - x_{11}\|}.
\]

6.2.1 Benchmark. First, the dynamics of the cables and payload are ignored and a geometric nonlinear controller is used for each quadrotor to keep quadrotors hovering at the fixed position for comparison [21].

The results of the experiment are presented in Fig. 11, which shows the direction of cables, the position of the payload, and the position of the quadrotors. It is shown that after the payload is released, there are continued, large oscillations of the payload and cables.

6.2.2 Proposed Dynamical System and Controller. Next, quadrotors are initially commanded to hover at the fixed position using the geometric nonlinear controller. The payload is pulled with a wire up to 30 deg angle as the above case, and then released. At the moment of release, the proposed controller that considers the dynamics of the cable and the payload is engaged.

Figure 12 illustrates the corresponding results of the experiment. The vertical dotted line indicates the time when geometric nonlinear controller is switched with the proposed controller. As shown in this figure, the proposed controller reduces and eliminates the oscillations of the cables and payload effectively, compared with the previous case. Snapshots of the controlled maneuvers are also illustrated in Fig. 13.

7 Conclusions

In this paper, Euler–Lagrange equations are derived for the complete model of multiple quadrotor UAVs transporting a rigid body connected via flexible cables. These are developed in a remarkably compact form, which allows us to choose an arbitrary number and any configuration of the links and an arbitrary number of quadrotors. Then, a geometric nonlinear controller is constructed to stabilize the quadrotors and the payload. A rigorous Lyapunov stability analysis is also presented to illustrate the stability properties. Numerical simulations and experimental results are presented to the efficacy of the proposed control system.

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Nomenclature

\( \{b_1, b_2, b_3\} = \) body-fixed frame of the payload

\[\text{Fig. 13 Snapshots of experiments.}^2\]

\[\text{2A short video is available at https://youtu.be/u65GqIl2skY.}\]
\{b_1, b_2, b_3\} = \text{body-fixed frame of the } i\text{th quadrotor}
\{e_1, e_2, e_3\} = \text{inertial frame}
g \in \mathbb{R} = \text{gravitational acceleration}
J_i \in \mathbb{R}^{3 \times 3} = \text{inertia matrix of the } i\text{th quadrotor}
m_i \in \mathbb{R} = \text{mass of the } i\text{th quadrotor}
m_0 \in \mathbb{R} = \text{mass of the payload}
n = \text{number of quadrotors}
n_i = \text{number of links in the } i\text{th cable}
R_i \in \text{SO}(3) = \text{attitude of the } i\text{th quadrotor}
\text{SE}(3) = \text{special Euclidean group}
\text{SO}(3) = \text{special orthogonal group}
v_i \in \mathbb{R}^3 = \text{velocity of the } i\text{th quadrotor}
x_i \in \mathbb{R}^3 = \text{position of the } i\text{th quadrotor}
\Omega_i \in \mathbb{R}^3 = \text{angular velocity of the } i\text{th quadrotor}

Appendix A: Proof for Proposition 1

A.1 Kinetic Energy

The kinetic energy of the whole system is composed of the kinetic energy of quadrotors, cables, and the rigid body, as

\[ T = \frac{1}{2} m_0 \|x_0\|^2 + \sum_{i=1}^{n} \frac{1}{2} m_i \|x_i\|^2 + \frac{1}{2} \sum_{i=1}^{n} m_i \|\dot{x}_i\|^2 + \frac{1}{2} \sum_{i=1}^{n} \Omega \cdot J_i \Omega + \frac{1}{2} \Omega_0 \cdot J_0 \Omega_0 \]  
(A1)

Substituting the derivatives of Eqs. (6) and (7) into the above expression, we have

\[ T = \frac{1}{2} m_0 \|\dot{x}_0\|^2 + \sum_{i=1}^{n} \frac{1}{2} m_i \|\dot{x}_i + \dot{R}_0 \rho_i - \sum_{a=d+1}^{n} l_a \dot{q}_{aw}\|^2 + \frac{1}{2} \sum_{i=1}^{n} \Omega \cdot J_i \Omega + \frac{1}{2} \Omega_0 \cdot J_0 \Omega_0 \]  
(A2)

We expand the above expression as follows:

\[ T = \frac{1}{2} \left( m_0 \|\dot{x}_0\|^2 + \sum_{i=1}^{n} \sum_{j=d}^{n} m_i \|\dot{x}_i\|^2 + \sum_{i=1}^{n} m_i \|\dot{x}_0\|^2 \right) + \frac{1}{2} \sum_{i=1}^{n} \left( \sum_{j=d}^{n} m_j \|\dot{R}_0 \rho_i\|^2 + m_i \|\dot{R}_0 \rho_i\|^2 \right) \]

\[ + \frac{1}{2} \sum_{i=1}^{n} \left( \sum_{j=d}^{n} m_{ij} \dot{x}_0 \cdot \dot{R}_0 \rho_i + m_{ij} \dot{x}_0 \cdot \dot{R}_0 \rho_i \right) \]

\[ + \frac{1}{2} \sum_{i=1}^{n} \left( \sum_{j=d}^{n} m_j \|l_a \dot{q}_{aw}\|^2 + m_j \|l_a \dot{q}_{aw}\|^2 \right) - \sum_{i=1}^{n} \left( \sum_{j=d}^{n} m_{ij} \dot{x}_0 \cdot \sum_{a=d}^{n} l_a \dot{q}_{aw} + \dot{x}_0 \cdot \sum_{a=d}^{n} l_a \dot{q}_{aw} \right) \]

\[ - \sum_{i=1}^{n} \left( \sum_{j=d}^{n} m_{ij} \dot{R}_0 \rho_i \cdot \sum_{a=d}^{n} l_a \dot{q}_{aw} + m_{ij} \dot{R}_0 \rho_i \cdot \sum_{a=d}^{n} l_a \dot{q}_{aw} \right) \]

\[ + \frac{1}{2} \sum_{i=1}^{n} \left( \Omega \cdot J_i \Omega + \frac{1}{2} \Omega_0 \cdot J_0 \Omega_0 \right) \]

\[ = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} m_i \|\dot{x}_i\|^2 + \frac{1}{2} \sum_{i=1}^{n} m_i \|\dot{x}_i\|^2 + m_0 \|\dot{x}_0\|^2 + \sum_{i=1}^{n} \frac{1}{2} m_i \|\dot{x}_i\|^2 + \frac{1}{2} \sum_{i=1}^{n} m_i \|\dot{x}_i\|^2 + \frac{1}{2} \sum_{i=1}^{n} \Omega \cdot J_i \Omega + \frac{1}{2} \Omega_0 \cdot J_0 \Omega_0 \]

\[ = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} m_i \|\dot{x}_i\|^2 + \frac{1}{2} \sum_{i=1}^{n} m_i \|\dot{x}_i\|^2 + \frac{1}{2} \sum_{i=1}^{n} \Omega \cdot J_i \Omega + \frac{1}{2} \Omega_0 \cdot J_0 \Omega_0 \]

\[ = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} m_i \|\dot{x}_i\|^2 + \frac{1}{2} \sum_{i=1}^{n} \Omega \cdot J_i \Omega + \frac{1}{2} \Omega_0 \cdot J_0 \Omega_0 \]

and substituting Eqs. (16) and (17), it is rewritten as

\[ T = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{2} m_i \|\dot{x}_i\|^2 + \frac{1}{2} \sum_{i=1}^{n} \Omega \cdot J_i \Omega + \frac{1}{2} \Omega_0 \cdot J_0 \Omega_0 \]

A.2 Potential Energy

We can derive the potential energy expression by considering the gravitational forces on each part of system as given

\[ V = -m_0 g e_3 \cdot x_0 - \sum_{i=1}^{n} m_i g e_3 \cdot x_i - \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} g e_3 \cdot x_0 \]

(A5)

Using Eqs. (6) and (7), we obtain

\[ V = -m_0 g e_3 \cdot x_0 - \sum_{i=1}^{n} m_i g e_3 \cdot (x_0 + R_0 \rho_i - \sum_{a=d+1}^{n} l_a q_{aw}) \]

\[ - \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} g e_3 \cdot (x_0 + R_0 \rho_i - \sum_{a=d+1}^{n} l_a q_{aw}) \]

(A6)

and utilizing Eq. (17), we can simplify the potential energy as

\[ V = -M_T g e_3 \cdot x_0 - \sum_{i=1}^{n} M_i g e_3 \cdot R_0 \rho_i + \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} l_{ij} q_{ij} \cdot e_3 \]

(A7)

A.3 Derivatives of Lagrangian

We develop the equation of motion for the Lagrangian \( L = T - V \). The derivatives of the Lagrangian are given by

\[ D_i L = M_T \dot{x}_0 + \sum_{i=1}^{n} M_i \dot{R}_0 \rho_i - \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} \dot{l}_{ij} q_{ij} \]

(A8)

\[ D_i L = M_T g e_3 \]

(A9)

\[ D_q L = \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} \dot{l}_{ij} q_{ij} - \sum_{i=1}^{n} M_{ij} \dot{l}_{ij} (x_0 + R_0 \rho_i) \]

(A10)

\[ D_q L = -\sum_{i=1}^{n} M_{ij} \dot{l}_{ij} e_3 \]

(A11)

where \( D_i \) denote the derivative with respect to \( \dot{x}_0 \), and other derivatives are defined similarly. We also have

\[ D_{\Omega_0} L = J_0 \Omega_0 + \sum_{i=1}^{n} M_T \dot{R}_0 R_i \tilde{\rho}_0 \tilde{\dot{x}_0} \]

\[ - \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} \dot{l}_{ij} \tilde{R}_0 R_i \tilde{\dot{q}_{ij}} - \sum_{i=1}^{n} M_i \tilde{\dot{\rho}_0}^2 \Omega_0 \]

(A12)

which can be rewritten as

\[ D_{\Omega_0} L = J_0 \Omega_0 + \sum_{i=1}^{n} \tilde{\dot{\rho}_0}^T \left( M_T \dot{x}_0 - \sum_{i=1}^{n} M_{ij} \dot{l}_{ij} q_{ij} \right) \]

(A13)
where \( J_0 \) is defined as
\[
J_0 = J_0 - \sum_{i=1}^{n} M_{ii} \ddot{\rho}_i^2
\]  
(A14)
The derivative with respect to \( \Omega \) is simply given by
\[
D_{\Omega_i} L = \sum_{j=1}^{n} J_i \dot{\Omega}_j
\]  
(A15)
The derivative of the Lagrangian with respect to \( R_0 \) along \( \delta R_0 = R_0 \delta \) is given by
\[
D_{R_0} L \cdot \delta R_0 = \sum_{i=1}^{n} M_{ii} \ddot{\rho}_i (\ddot{\hat{\rho}}_i + \dot{\Omega}_i \dot{\rho}_i + \dot{\rho}_i - \Omega_i \ddot{\rho}_i) + M_{ij} \ddot{\rho}_j \dot{\rho}_i + M_{ij} \dot{\rho}_i \ddot{\rho}_j
\]  
(A16)
which can be rewritten as
\[
D_{R_0} L \cdot \delta R_0 = d_{R_0} \cdot \eta_0
\]  
(A17)where
\[
d_{R_0} = \sum_{i=1}^{n} \left( \left( \ddot{\hat{\rho}}_i (M_{ii} \ddot{\rho}_i) + \sum_{j=1}^{n} M_{ij} \ddot{\rho}_j \dot{\rho}_i + M_{ij} \dot{\rho}_i \ddot{\rho}_j \right) \right)
\]  
(A18)

### A.4 Lagrange–d’Alembert Principle
Consider \( \delta \theta = \int_{t_0}^{t_1} L \) be the action integral. Using the equations derived in Sec. A.3, the infinitesimal variation of the action integral can be [9] written as
\[
\delta \theta = \int_{t_0}^{t_1} \left( \frac{d}{dt} D_{R_0} L - D_{\theta_i} L \right) dt - \int_{t_0}^{t_1} \left( \frac{d}{dt} D_{\dot{\theta}_i} L \right) dt
\]  
(A21)
\[
\frac{d}{dt} D_{\Omega_i} + \Omega_i \times D_{\dot{\Omega}_i} - d_{R_0} = \sum_{i=1}^{n} \ddot{\rho}_i R_0^T \left( u_i + \Delta u_i \right)
\]  
(A22)
\[
\ddot{\theta}_i \frac{d}{dt} D_{\theta_i} L - \dot{\theta}_i D_{\dot{\theta}_i} L = -l_i \ddot{\theta}_i \left( u_i + \Delta u_i \right)
\]  
(A23)
\[
\frac{d}{dt} D_{\theta_i} L + \Omega_i \times D_{\dot{\theta}_i} L = M_i + \Delta \theta_i
\]  
(A24)
Substituting the derivatives of Lagrangian into Eqs. (A21)–(A24) and rearranging, the equations of motion are given by Eqs. (12)–(15). We utilize the following equation to obtain explicit expression for \( \ddot{\theta}_i \). The first term on the left-hand side of Eq. (14) can be written as:
\[
\sum_{j=1}^{n} M_{ij} \dot{\theta}_j \ddot{\theta}_j = \sum_{j=1}^{n} M_{ij} \dot{\theta}_j \ddot{\theta}_j + M_{ij} \dot{\theta}_j \ddot{\theta}_j
\]  
(A25)
As \( \dot{\theta}_j \ddot{\theta}_j = 0 \), we have \( \dot{\theta}_j \ddot{\theta}_j + \ddot{\theta}_j = 0 \), using this, we have
\[
-\dot{\theta}_j \ddot{\theta}_j = -(\dot{\theta}_j \ddot{\theta}_j + \ddot{\theta}_j)
\]  
(A26)
Substituting the above expression in Eq. (A25)
\[
\sum_{j=1}^{n} M_{ij} \dot{\theta}_j \ddot{\theta}_j = \sum_{j=1}^{n} M_{ij} \dot{\theta}_j \ddot{\theta}_j = \sum_{j=1}^{n} M_{ij} \dot{\theta}_j \ddot{\theta}_j - M_{ij} \ddot{\theta}_j l_i \ddot{\theta}_j
\]  
(A27)
where the last two terms on the right-hand side of the above expression are utilized to construct \( N_{\dot{\theta}_i} \) and \( P_{\dot{\theta}_i} \), respectively.

### Appendix B: Proof for Proposition 2
The variations of \( x \) and \( q \) are given by Eqs. (26) and (27). From the kinematics equation \( \dot{q}_j = \omega_j \times q_j \) and
\[
\delta q_j = \dot{\xi}_j \times e_3 = \delta \omega_j \times e_3 + 0 + \delta \omega_j \times e_3 \times (\dot{\xi}_j \times e_3)
\]  
(B1)
Since both sides of the above equation is perpendicular to \( e_3 \), this is equivalent to \( \dot{q}_j \times e_3 = \delta \omega_j \times e_3 \), which yields
\[
\dot{\xi}_j = \delta \omega_j \times e_3 - (\dot{\xi}_j \times e_3) e_3 = \delta \omega_j - (\dot{\xi}_j \times e_3)
\]  
(B2)
Since \( \dot{\xi}_j \times e_3 = 0 \), we have \( \dot{\xi}_j \times e_3 = 0 \). As \( e_3 \cdot \delta \omega_j = 0 \) from the constraint, we obtain the linearized equation for the kinetics equation of the link as
\[
\dot{\xi}_j = \delta \omega_j
\]  
(B3)
The infinitesimal variation of \( R_0 \in SO(3) \) in terms of the exponential map
\[
\delta R_0 = \frac{d}{d\xi} |_{\xi=0} R_0 \exp(\hat{\xi} \theta_0) = R_0 \delta \theta_0
\]  
(B2)
for \( \delta \theta_0 \in \mathbb{R}^3 \). Substituting these into Eqs. (12)–(14), and ignoring the higher-order terms, we obtain the following sets of linearized equations of motion:
\[
M_f \dot{\dot{\xi}}_0 - \sum_{i=1}^{n} M_{fi} \dot{\rho}_i \dot{\delta \theta}_0 + \sum_{i=1}^{n} \sum_{j=1}^{n} M_{ij} \dot{\theta} j \dot{\theta} j \hat{c}_\theta j = \sum_{i=1}^{n} \delta \theta_i
\]  
(B3)
\[
\begin{align*}
\sum_{i=1}^{n} M_{T} \ddot{\mathbf{p}}_i \delta \mathbf{v}_0 + \mathbf{J}_0 \delta \mathbf{\Omega}_0 + \sum_{i=1}^{n} M_{00} l_{ij} \ddot{\mathbf{e}}_i C^{T} \mathbf{\hat{z}}_{ij} \\
+ \sum_{i=1}^{n} \mathbf{m}_i g \ddot{\mathbf{e}}_i \eta_0 = \sum_{i=1}^{n} \mathbf{\dot{p}}_i \mathbf{\dot{u}}_i 
\end{align*}
\]
\[\text{B4}\]
\[
- M_{00} C^{T} \dot{\mathbf{e}}_3 \delta \mathbf{v}_0 + M_{00} C^{T} \ddot{\mathbf{e}}_3 \mathbf{\dot{p}}_i \delta \mathbf{\Omega}_0 + \sum_{k=1}^{n} M_{00} l_{ik} \mathbf{\dot{d}}_2 \left( C^{T} \mathbf{\hat{z}}_{ij} \right) \\
= - C^{T} \dot{\mathbf{e}}_3 \mathbf{\dot{u}}_i + \left( - M_{T} - \frac{m_i}{n} + M_{00} \right) g e_3 \mathbf{\dot{d}}_2 \left( C^{T} \mathbf{\hat{z}}_{ij} \right) \quad \text{B5}
\]
\[
\dot{\eta}_i = \delta \mathbf{\Omega}_i, \quad \dot{\eta}_0 = \delta \mathbf{\Omega}_0, \quad J_0 \delta \mathbf{\Omega}_0 = \delta \mathbf{M}_i
\]
\[\text{B6}\]

which can be written in a matrix form as presented in Eq. (28). We used \( C^{T} \mathbf{\hat{z}} C = -I_2 \) to simplify these derivations.

Appendix C: Proof for Proposition 3

We first show stability of the rotational dynamics of each quadrotor, and later, it is combined with the stability analysis for the remaining parts.

C.1 Attitude Error Dynamics

Here, the attitude error dynamics for \( e_{Ri} \), \( \varepsilon_{Ri} \) are derived and we find conditions on control parameters to guarantee the stability. The time-derivative of \( J_0 \varepsilon_{Ri} \) can be written as
\[ J_0 \mathbf{\dot{e}}_i = \left( J_0 \varepsilon_{Ri} + d_i \right) \cdot \mathbf{\dot{e}}_i - k_R \mathbf{e}_R - k_{\varepsilon_{Ri}} \mathbf{e}_{\varepsilon_{Ri}} - k_{ei} \mathbf{e}_i + \Delta \mathbf{\dot{R}}_i \quad \text{C1}\]

where \( d_i = \left( 2J_i - \text{tr}\left( J_i \dot{\mathbf{R}}_i \right) \mathbf{\dot{R}}_i \right) \in \mathbb{R}^3 \) [21]. The important property is that the first term of the right-hand side is normal to \( e_{Ri} \), and it simplifies the subsequent Lyapunov analysis.

C.2 Stability for Attitude Dynamics

Define a configuration error function on \( \mathbf{SO}(3) \) as follows:
\[ \Psi_i = \frac{1}{2} \text{tr} \left[ I - R_i^T R_i \right] \quad \text{C2}\]

We introduce the following Lyapunov function:
\[ \mathcal{V}_2 = \sum_{i=1}^{n} \mathcal{V}_{2i} \quad \text{C3}\]

where
\[ \mathcal{V}_{2i} = \frac{1}{2} \mathbf{e}_{\varepsilon_{Ri}} \cdot J_0 \mathbf{\dot{e}}_i + k_k \Psi_i \left( R_i \mathbf{R}_i \right) + c_2 \mathbf{e}_{\varepsilon_{Ri}} \cdot \mathbf{e}_{\varepsilon_{Ri}} + \frac{1}{2} k_{\varepsilon_{Ri}} \mathbf{e}_{\varepsilon_{Ri}} \cdot \mathbf{\dot{R}}_i \quad \text{C4}\]

Consider a domain \( D_2 \) given by
\[ D_2 = \{ \left( R_i, \varepsilon_{Ri} \right) \in \mathbf{SO}(3) \times \mathbb{R}^3 | \Psi_i \left( R_i, \varepsilon_{Ri} \right) < \psi_2 < 2 \} \quad \text{C5}\]

In this domain, we can show that \( \mathcal{V}_2 \) is bounded as follows [21]:
\[ z_2^T M_{22} z_2 + \frac{k_j}{2} \mathbf{\dot{e}}_i - \frac{\Delta \mathbf{R}_i}{k_j} \mathbf{\dot{R}}_i \leq \mathcal{V}_2 \leq z_2^T M_{21} z_2 + \frac{k_j}{2} \mathbf{\dot{e}}_i - \frac{\Delta \mathbf{R}_i}{k_j} \mathbf{\dot{R}}_i \quad \text{C6}\]

where \( z_2 = ||\mathbf{e}_k||, ||\mathbf{e}_{\varepsilon_{Ri}}|| \in \mathbb{R}^2 \) and matrices \( M_{21}, M_{22} \in \mathbb{R}^{2 \times 2} \) are given by

\[ M_{21} = \frac{1}{2} \begin{bmatrix} k_R & -c_2 \lambda_M \\ -c_2 \lambda_M & \lambda_M \end{bmatrix} \quad \text{C7}\]
\[ M_{22} = \frac{1}{2} \begin{bmatrix} 2k_R & c_2 \lambda_M \\ c_2 \lambda_M & c_2 \lambda_M \end{bmatrix} \]

The time derivative of \( \mathcal{V}_2 \) along the solution of the controlled system is given by
\[ \dot{\mathcal{V}}_2 = \sum_{i=1}^{n} -k_R \mathbf{e}_{\varepsilon_{Ri}} \cdot \mathbf{\dot{e}}_i + k_k \mathbf{e}_{\varepsilon_{Ri}} \cdot \mathbf{\dot{R}}_i + c_2 \mathbf{e}_{\varepsilon_{Ri}} \cdot \mathbf{\dot{R}}_i + c_2 \mathbf{e}_{\varepsilon_{Ri}} \cdot \mathbf{\dot{R}}_i \quad \text{C8}\]

We have \( \dot{\mathbf{e}}_i = c_2 \mathbf{e}_{\varepsilon_{Ri}} + \mathbf{\dot{R}}_i \). Substituting Eq. (C1), the above equation becomes
\[ \dot{\mathcal{V}}_2 = \sum_{i=1}^{n} -k_R \mathbf{e}_{\varepsilon_{Ri}} \cdot \mathbf{\dot{e}}_i + c_2 \mathbf{e}_{\varepsilon_{Ri}} \cdot \mathbf{\dot{R}}_i + c_2 \mathbf{e}_{\varepsilon_{Ri}} \cdot \mathbf{\dot{R}}_i + c_2 \mathbf{e}_{\varepsilon_{Ri}} \cdot \mathbf{\dot{R}}_i \quad \text{C9}\]

We have \( \mathbf{e}_{\varepsilon_{Ri}} \leq 1, \mathbf{\dot{e}}_i \leq \mathbf{e}_{\varepsilon_{Ri}} \) [28], and choose a constant \( B_2 \) such that \( ||\dot{d}_i|| \leq B_2 \). Then, we obtain
\[ \mathcal{V}_2 \leq - \sum_{i=1}^{n} z_2^T W_2 z_2 \quad \text{C10}\]

where the matrix \( W_2 \in \mathbb{R}^{2 \times 2} \) is given by
\[ W_2 = \begin{bmatrix} c_2 k_R & -c_2 \left( k_R + B_2 \right) \\ -c_2 \left( k_R + B_2 \right) & k_R - c_2 \lambda_M \end{bmatrix} \]

The matrix \( W_2 \) is a positive definite matrix if
\[ c_2 < \min \left\{ \sqrt{c_2 \lambda_M}, \frac{4k_R}{8k_R \lambda_M + (k_R + B_2)^2} \right\} \]

This implies that
\[ \mathcal{V}_2 \leq - \sum_{i=1}^{n} \lambda_m \left( W_2 \right) ||z_2||^2 \quad \text{C11}\]

which shows stability of the attitude dynamics of quadrotors.

C.3 Error Dynamics of the Payload and Links

We derive the tracking error dynamics and a Lyapunov function for the translational dynamics of a payload and the dynamics of links. Later, it is combined with the stability analyses of the rotational dynamics. From Eqs. (12), (28), (35), and (41), the equation of motion for the controlled dynamic model is given by
\[ \mathbf{M}_i \mathbf{\ddot{x}} + \mathbf{G}_i \mathbf{x} = \mathbf{B} \left( u - u_d \right) + g(\mathbf{x}, \mathbf{\dot{x}}) + \mathbf{B} \Delta \mathbf{t} \quad \text{C12}\]

where \( \Delta \mathbf{t} \in \mathbb{R}^{2 \times 1} \) is
\[ \Delta \mathbf{t} = \begin{bmatrix} \Delta t_1 & \Delta t_2 & \cdots & \Delta t_n \end{bmatrix}^T \quad \text{C13}\]

and
\[ u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad u_d = \begin{bmatrix} -\left( M_{1f} + \frac{m_0}{n} \right) r_3 \\ -\left( M_{2f} + \frac{m_0}{n} \right) r_3 \\ \vdots \\ -\left( M_{df} + \frac{m_0}{n} \right) r_3 \end{bmatrix} \]
and \( g(x, \dot{x}) \) corresponds to the higher-order terms. As \( u_i = -f_i R_i e_3 \) for the full dynamic model, \( \delta u = u - u_i \) is given by

\[
\delta u = \begin{bmatrix}
-f_1 R_1 e_3 + \left( M_{TT} + \frac{m_0}{n} \right) g e_3 \\
-f_2 R_2 e_3 + \left( M_{TT} + \frac{m_0}{n} \right) g e_3 \\
\vdots \\
f_a R_a e_3 + \left( M_{TT} + \frac{m_0}{n} \right) g e_3
\end{bmatrix}
\]  

(C13)

The subsequent analyses are developed in the domain \( D_1 \)

\[
D_1 = \{(x, \dot{x}, R, \psi_0) \in \mathbb{R}^{3a} \times \mathbb{R}^{3a} \times \mathrm{SO}(3) \times \mathbb{R}^3 \mid \Psi < \psi_i < 1 \}
\]

(C14)

In the domain \( D_1 \), we can show that

\[
\frac{1}{2} ||e_k||^2 \leq \Psi(R_i, R_i) \leq \frac{1}{2 - \psi_i} ||e_k||^2
\]

(C15)

Consider the quantity \( e_3^T R_i^T R_i e_3 \), which represents the cosine of the angle between \( b_k = R_i e_3 \) and \( b_k = R_i e_3 \). Since \( \Psi(R_i, R_i) \) represents the cosine of the eigen-axis rotation angle between \( R_i \) and \( R_i \), we have \( e_3^T R_i^T R_i e_3 \geq 1 - \Psi(R_i, R_i) > 0 \) in \( D_1 \). Therefore, the quantity \( 1/e_3^T R_i^T R_i e_3 \) is well-defined. We add and subtract \( (f_3/e_3^T R_i^T R_i e_3) R_i e_3 \) to the right-hand side of Eq. (C13) to obtain

\[
\delta u = \begin{bmatrix}
f_1 e_3^T R_i^T R_i e_3 - X_1 + \left( M_{TT} + \frac{m_0}{n} \right) g e_3 \\
f_2 e_3^T R_i^T R_i e_3 - X_2 + \left( M_{TT} + \frac{m_0}{n} \right) g e_3 \\
\vdots \\
f_a e_3^T R_i^T R_i e_3 - X_a + \left( M_{TT} + \frac{m_0}{n} \right) g e_3
\end{bmatrix}
\]

(C16)

where \( X_i \in \mathbb{R}^3 \) is defined by

\[
X_i = f_i e_3^T R_i^T R_i e_3 \left( \left( e_3^T R_i^T R_i e_3 \right) R_i e_3 - R_i e_3 \right)
\]

(C17)

Using

\[
-f_i e_3^T R_i^T R_i e_3 - R_i e_3 = -\frac{\|A_i\| ||R_i e_3|| \cdot R_i e_3}{e_3^T R_i^T R_i e_3} \cdot \frac{A_i}{\|A_i\|} = A_i
\]

(C18)

Equation (C16) becomes

\[
\delta u = \begin{bmatrix}
A_1 - X_1 + \left( M_{TT} + \frac{m_0}{n} \right) g e_3 \\
A_2 - X_2 + \left( M_{TT} + \frac{m_0}{n} \right) g e_3 \\
\vdots \\
A_a - X_a + \left( M_{TT} + \frac{m_0}{n} \right) g e_3
\end{bmatrix}
\]

(C19)

Substituting Eq. (35) into the above equation, Eq. (C10) becomes

\[
M \dot{x} + Gx = B(-K_{e}x - K_{s}x - X - K_z \sigma(e_k) + \Delta_i) + g(x, \dot{x})
\]

(C20)

where \( X = [X_1^T, X_2^T, \ldots, X_n^T]^T \in \mathbb{R}^{3n} \). This can be rearranged as

\[
\dot{x} = -(M^{-1} G + M^{-1} B K_z) x - (M^{-1} B K_z) \dot{x} - M^{-1} B \Delta_i
\]

(C21)

Using the definitions for \( A, B, \) and \( z_i \) presented before, the above expression can be rearranged as

\[
\dot{z}_i = A z_i + B (-B X + g(x, \dot{x}) - B K_z \sigma(e_k) + B \Delta_i)
\]

(C22)

### C.4 Lyapunov Candidate for Simplified Dynamics

From the linearized control system developed at Sec. 3, we use matrix \( P \) to introduce the following Lyapunov candidate for translational dynamics:

\[
V_1 = z_i^T P z_1 + 2 \int_{\rho_q} (B K_z \sigma(e_k) - B \Delta_i) \cdot d\mu
\]

(C23)

The last integral term of the above equation is positive definite about the equilibrium point \( e_k = p_{eq} \) where

\[
p_{eq} = \left[ \frac{A_i}{k_z}, 0, 0, \ldots \right]
\]

(C24)

If \( \delta < k_z \sigma \), considering the fact that \( \sigma(x) = y \) if \( y < \sigma \). The time derivative of the Lyapunov function using the Leibniz integral rule is given by

\[
\dot{V}_1 = z_i^T P z_1 + z_i^T P z_1 + 2 e_k \cdot (B K_z \sigma(e_k) - B \Delta_i)
\]

(C25)

Since \( e_i^T = ((PB)^T z_1)^T = z_1^T P B \) from Eq. (36), the above expression can be written as

\[
\dot{V}_1 = z_i^T P z_1 + z_i^T P z_1 + 2 e_i^T P B (B K_z \sigma(e_k) - B \Delta_i)
\]

(C26)

Substituting Eq. (33) into Eq. (C26), it reduces to

\[
\dot{V}_1 = z_i^T (A^T P + P A) z_1 + 2 e_i^T P B (-B X + g(x, \dot{x})
\]

(C27)

Let \( e_i = 2 \|P B \|_2 \in \mathbb{R} \) and using \( A^T P + P A = -Q \), we have

\[
\dot{V}_1 \leq z_i^T Q z_1 + e_i^T ||X|| + 2 e_i^T P B g(x, \dot{x})
\]

(C28)

The second term on the right-hand side of the above equation corresponds to the effects of the attitude tracking error on the translational dynamics. We find a bound of \( X_i \), defined at Eq. (C17), to show stability of the coupled translational dynamics and rotational dynamics in the subsequent Lyapunov analysis. Since

\[
f_i = ||A_i|| (e_i^T R_i^T R_i e_3)
\]

(C29)

we have

\[
||X_i|| \leq ||A_i|| ||(e_i^T R_i^T R_i e_3) R_i e_3 - R_i e_3||
\]

(C30)

The last term \( ||(e_i^T R_i^T R_i e_3) R_i e_3 - R_i e_3|| \) represents the sine of the angle between \( b_3 = R_i e_3 \) and \( b_3 = R_i e_3 \), since \( b_3 \cdot b_3 = b_3 \cdot b_3 = b_3 \times (b_3 \times b_3) \). The magnitude of the attitude error vector, \( ||e_k|| \), represents the sine of the eigen-axis rotation angle between \( R_i \) and \( R_i \). Therefore, \( ||(e_i^T R_i^T R_i e_3) R_i e_3 - R_i e_3|| \leq ||e_k|| \) in \( D_1 \). It follows that:

\[
||e_i^T R_i^T R_i e_3 R_i e_3 - R_i e_3|| \leq ||e_k|| = \sqrt{\Psi_i^2 (2 - \Psi_i)}
\]

(C31)

\[
\leq \left\{ \sqrt{\Psi_i (2 - \Psi_i)} \leq \frac{\dot{z}_i}{z_i} \right\} < 1
\]

(C31)
therefore
\[ \|X_i\| \leq \|A_i\|\|e_0\| \leq \|A_i\|\|z_i\| \] (C32)

We find an upper bound for
\[ A_i = -K_s x - K_s \dot{x} - K_s \operatorname{sat}(e_x) + u_i \] (C33)

We define \(K_{\text{max}}, K_{\text{z}} \in \mathbb{R}\)
\[ K_{\text{max}} = \max(\|K_s\|, \|K_s\|) K_{\text{z}} = \|K_s\| \]
by defining \(\|u_i\| \leq B_0\), the upper bound of \(A_i\) is given by
\[ \|A_i\| \leq K_{\text{max}}(\|x\| + \|\dot{x}\|) + \sigma K_{\text{z}} + B_1, \] (C34)
\[ \leq 2K_{\text{max}}\|z_i\| + (B_1 + \sigma K_{\text{z}}) \] (C35)

Using the above steps, we can show that
\[ \|X\| \leq \sum_{i=1}^{n} (2K_{\text{max}}\|z_i\| + (B_1 + \sigma K_{\text{z}}))\|e_R\| \leq (2K_{\text{max}}\|z_i\| + (B_1 + \sigma K_{\text{z}}))z \] where \(x = \sum_{i=1}^{n} z_i\). Then, we can simplify Eq. (C28) as
\[ \dot{V}_i \leq -\left(\hat{\lambda}_{\text{min}}(Q) - 2c_1 K_{\text{max}} z\right)\|z_i\|^2 + \sum_{i=1}^{n} c_1(B_1 + \sigma K_{\text{z}})\|z_i\|\|e_R\| + 2z^T P B g(x, \dot{x}) \] (C37)

C.5 Lyapunov Candidate for the Complete System

Let \( \mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 \) be the Lyapunov function for the complete system. The time derivative of \( \mathcal{V} \) is given by
\[ \dot{\mathcal{V}} = \dot{\mathcal{V}}_1 + \dot{\mathcal{V}}_2 \] (C38)

Substituting Eqs. (C37) and (C9) into the above equation
\[ \dot{\mathcal{V}} \leq -\left(\hat{\lambda}_{\text{min}}(Q) - 2c_1 K_{\text{max}} z\right)\|z_i\|^2 + 2z^T P B g(x, \dot{x}) + \sum_{i=1}^{n} c_1(B_1 + \sigma K_{\text{z}})\|z_i\|\|e_R\| - \sum_{i=1}^{n} \lambda_{\text{min}}(W_2)\|z_i\|^2 \] (C39)

and using \(\|e_R\| \leq \|z_i\|\), it can be written as
\[ \dot{\mathcal{V}} \leq -\left(\hat{\lambda}_{\text{min}}(Q) - 2c_1 K_{\text{max}} z\right)\|z_i\|^2 + 2z^T P B g(x, \dot{x}) + \sum_{i=1}^{n} c_1(B_1 + \sigma K_{\text{z}})\|z_i\|\|z_i\| - \sum_{i=1}^{n} \lambda_{\text{min}}(W_2)\|z_i\|^2 \] (C40)

The \(2z^T P B g(x, \dot{x})\) term in the above equation is indefinite. But, the function \(g(x, \dot{x})\) satisfies
\[ \|g(x, \dot{x})\| \to 0 \quad \text{as} \quad \|z_i\| \to 0 \]

Then, for any \(\gamma > 0\), there exists \(r > 0\) such that
\[ \|g(x, \dot{x})\| < \gamma \|z_i\| \quad \forall \|z_i\| < r \]

Therefore,
\[ 2z^T P B g(x, \dot{x}) \leq 2\gamma \|P\|_2 \|z_i\|^2 \] (C41)

Substituting the above inequality into Eq. (C40)
\[ \dot{\mathcal{V}} \leq -\left(\hat{\lambda}_{\text{min}}(Q) - 2c_1 K_{\text{max}} z\right)\|z_i\|^2 + \sum_{i=1}^{n} c_1(B_1 + \sigma K_{\text{z}})\|z_i\|\|z_i\| - \sum_{i=1}^{n} \lambda_{\text{min}}(W_2)\|z_i\|^2 \] (C42)

and rearranging
\[ \dot{\mathcal{V}} \leq -\sum_{i=1}^{n} \left(\hat{\lambda}_{\text{min}}(Q) - 2c_1 K_{\text{max}} z\right)\|z_i\|^2 - c_1(B_1 + \sigma K_{\text{z}})\|z_i\|\|z_i\| + \lambda_{\text{min}}(W_2)\|z_i\|^2 \] (C43)
\[ + 2\gamma \|P\|_2 \|z_i\|^2 \]

we obtain
\[ \dot{\mathcal{V}} \leq -\sum_{i=1}^{n} \left(\hat{\lambda}_{\text{min}}(W_1) - 2\gamma \|P\|_2\right)\|z_i\|^2 \] (C44)

where \(z_i = \|(z_2, z_3)\| \in \mathbb{R}^2\) and
\[ W_i = \left[ \begin{array}{c} \hat{\lambda}_{\text{min}}(Q) - 2c_1 K_{\text{max}} z \quad -c_1(B_1 + \sigma K_{\text{z}}) \\ -c_1(B_1 + \sigma K_{\text{z}}) \\ 2 \lambda_{\text{min}}(W_2) \end{array} \right] \] (C45)

By using \(\|z_i\| \leq \|z_i\|\), we obtain
\[ \dot{\mathcal{V}} \leq -\sum_{i=1}^{n} \left(\hat{\lambda}_{\text{min}}(W_i) - 2\gamma \|P\|_2\right)\|z_i\|^2 \] (C46)

Choosing \(\gamma < n(\hat{\lambda}_{\text{min}}(W_i))/2\|P\|_2\) and
\[ \hat{\lambda}_{\text{min}}(W_2) > n c_1(B_1 + \sigma K_{\text{z}}) \] (C47)

ensures that \(\dot{\mathcal{V}}\) is negative semidefinite. This implies that the zero equilibrium of tracking errors is stable in the sense of Lyapunov and \(\dot{\mathcal{V}}\) is nonincreasing. Therefore, all of error variables \(z_1, z_2, z_3\) and integral control terms \(e_1, e_2\) are uniformly bounded. Also, from Lasalle–Yoshizawa theorem [27, Theorem 3.4], we have \(z_i \to 0\) as \(t \to \infty\).

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