Quantum Democracy Is Possible

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It is shown that, since an ultrafilter over an operator-algebraically finite (i.e. isomorphic to the lattice of projections of a finite Von Neumann algebra) quantum logic is not necessarily principal, Arrow’s Impossibility Theorem doesn’t extend to the quantum case.

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II. INTRODUCTION

The differences between Classical Physics and Quantum Physics may be conceptually interpreted as the fact that the algebraic probability spaces underlying Classical Physics are commutative while those underlying Quantum Physics are noncommutative.

Restricting the analysis to the lattice of projections of the involved Von Neumann algebras this is equivalent to the fact that in the classical case such a lattice is Boolean while in the quantum case it is only orthomodular.

Such a viewpoint constitutes the essence of Quantum Logic, a research field whose conceptual value in order to understand the counterintuitive peculiarities of Quantum Mechanics cannot be overestimated (see [1], [2], [3], [4], [5], [6] as to Quantum Logic, see [7], [8] as to Quantum Probability, and see [9], [10], [11], [12], [13] as to the deep link existing between Quantum Logic and Quantum Probability).

In a completely different research field, the mathematical formalization of a democratic voting system led Kenneth Arrow to prove his celebrated Impossibility Theorem stating that a perfectly democratic voting system doesn’t exist [14], [15].

Since unfortunately such a theorem has been sometimes used by the enemies of democracy to support totalitarianism [16] let us remind the following fundamental quotation by Winston Churchill:

*No one pretends that democracy is perfect or all-wise. Indeed it has been said that democracy is the worst form of government except for those that have been tried from time to time.*

In this paper we show how a lattice theoretic reformulation of Arrow’s Theorem allows to investigate what happens when one substitutes the underlying classical logic with a quantum one.

We show that, contrary to its classical counterpart, quantum democracy is possible.
III. ARROW’S IMPOSSIBILITY THEOREM

Let us suppose to have an electoral process in which the voters belonging to a (finite) set \(V\) have to express their preference among the elements of a (finite) set \(C\) of candidates.

Let us recall, with this regard, the following basic:

**Definition III.1**

**partial ordering over \(C\):**

a binary relation \(\succeq\) over \(C\) satisfying the following conditions:

1. reflexivity:

\[ c \succeq c \quad \forall c \in C \quad (3.1) \]

2. transitivity:

\[ (c_1 \succeq c_2 \text{ and } c_2 \succeq c_3 \Rightarrow c_1 \succeq c_3) \quad \forall c_1, c_2, c_3 \in C \quad (3.2) \]

3. identitivity:

\[ (c_1 \succeq c_2 \text{ and } c_2 \succeq c_1 \Rightarrow c_1 = c_2) \quad \forall c_1, c_2 \in C \quad (3.3) \]

**Definition III.2**

**total ordering over \(C\):**

a partial ordering over \(C\) such that:

\[ c_1 \succeq c_2 \text{ or } c_2 \succeq c_1 \quad \forall c_1, c_2 \in C \quad (3.4) \]

Let \(O(C)\) be the set of all the total orderings over \(C\).

Elections can then be formalized in the following way:

**Definition III.3**

**voting system with voters’ set \(V\) and candidates’ set \(C\):**

a map \(S : V \times O(C) \mapsto O(C)\)

Let \(S(V, C)\) be the set of all the voting systems with voters’ set \(V\) and candidates’ set \(C\).

Given \(S \in S(V, C)\):

**Definition III.4**

**\(S\) is democratic:**

it satisfies the following conditions:

1. Independence of Irrelevant Alternatives:

\[ (c_1 S(O_{v_1}, \ldots, O_{v_{|V|}}) c_2 \text{ is determined by } c_1 O_{v_1}, \ldots, c_1 O_{v_{|V|}} c_2) \quad \forall c_1, c_2 \in C, \forall O_{v_1}, \ldots, O_{v_{|V|}} \in O(C) \quad (3.5) \]

2. Positive Association of Individual Values:

\[ (c_1 \geq_{v_1} c_2 \text{ and } c_1 \geq_{v_2} c_2 \text{ and } c_1 \geq_{v_3} c_2 \text{ and } c_1 \geq_{S(O_{v_1}, O_{v_2}, O_{v_3})} c_2 \Rightarrow c_1 \geq_{S(O_{v_1}, O_{v_2}, O_{v_3})} c_2) \quad \forall O_{v_1}, O_{v_2}, O_{v_3} \in O(C), \forall v_1, v_2, v_3 \in V, \forall c_1, c_2 \in C \quad (3.6) \]

3. Citizen Sovereignty:

\[ \not\exists c_1, c_2 \in C : (c_1 \geq_{S(O_{v_1}, \ldots, O_{v_{|V|}})} c_2 \text{ and } O_{v_1}, \ldots, O_{v_{|V|}} \in O(C)) \quad (3.7) \]

4. Nondictatorship:

\[ \not\exists v \in V : (c_1 S c_2 = c_1 O_v c_2 \text{ and } \forall c_1, c_2 \in C, \forall O_v \in O(C)) \quad (3.8) \]

Let \(D(V, C)\) be the set of all the democratic voting systems with voters’ set \(V\) and candidates’ set \(C\).

Then:

**Theorem III.1**

**Arrow’s Impossibility Theorem:**

\[ D(V, C) = \emptyset \quad \forall V, C : |V| \in \mathbb{N}_+ \text{ and } |C| \in \mathbb{N}_+ \quad (3.9) \]
IV. QUANTUM DEMOCRACY

Let us recall that:

**Definition IV.1**

*partially ordered set:*

a couple \((S, \succeq)\) such that:

1. \(S\) is a set
2. \(\succeq\) is a partial ordering over \(S\)

Given a partially ordered set \((S, \succeq)\):

**Definition IV.2**

*meet over \((S, \succeq)\):*

a map \(\wedge : S \times S \rightarrow S\) such that:

\[
x \succeq x \wedge y \quad \forall x, y \in S \quad (4.1)
\]

\[
y \succeq x \wedge y \quad \forall x, y \in S \quad (4.2)
\]

\[(x \succeq z \text{ and } y \succeq z \Rightarrow x \wedge y \succeq z) \quad \forall x, y, z \in S \quad (4.3)
\]

**Definition IV.3**

*join over \((S, \succeq)\):*

a map \(\vee : S \times S \rightarrow S\) such that:

\[
x \vee y \succeq x \quad \forall x, y \in S \quad (4.4)
\]

\[
x \vee y \succeq y \quad \forall x, y \in S \quad (4.5)
\]

\[(z \succeq x \text{ and } z \succeq y \Rightarrow z \succeq x \vee y) \quad \forall x, y, z \in S \quad (4.6)
\]

**Definition IV.4**

*lattice:*

\((S, \succeq, \wedge, \vee)\) such that:

1. \((S, \succeq)\) is a partially ordered set
2. \(\wedge\) is a meet over \((S, \succeq)\)
3. \(\vee\) is a join over \((S, \succeq)\)

Given a lattice \(L := (S, \succeq, \wedge, \vee)\) let \(0\) be its lower bound and let \(1\) be its upper bound.

**Definition IV.5**

\(L\) is distributive:

\[
x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x, y, z \in S \quad (4.7)
\]

\[
x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in S \quad (4.8)
\]

**Definition IV.6**
orthocomplementation over $\mathcal{L}$:
a map $’ : S \mapsto S$ such that:
\[(x’)’ = x \ \forall x \in S \quad (4.9)\]
\[x’ \land x = 0 \ \forall x \in S \quad (4.10)\]
\[x’ \lor x = 1 \ \forall x \in S \quad (4.11)\]

Definition IV.7

orthocomplemented lattice:
a couple $(\mathcal{L},’)$ such that:
1. $\mathcal{L}$ is a lattice
2. $’$ is an orthocomplementation over $\mathcal{L}$

Definition IV.8

Boolean lattice:
a distributive orthocomplemented lattice

In the physical literature a Boolean lattice is usually called a classical logic.

Definition IV.9

modular lattice:
an orthocomplemented lattice $(\mathcal{L},’)$ such that:
\[(z \geq x \Rightarrow (x \lor y) \land z = x \lor (y \land z)) \ \forall x, y, z \in L \quad (4.12)\]

Definition IV.10

orthomodular lattice:
an orthocomplemented lattice $(\mathcal{L},’)$ such that:
\[(y \geq x \Rightarrow x \lor (x’ \land y) = y) \ \forall x, y \in L \quad (4.13)\]

Let us recall that:

Proposition IV.1

Booleanity $\Rightarrow$ modularity $\Rightarrow$ orthomodularity \quad (4.14)

orthomodularity $\Rightarrow$ modularity $\Rightarrow$ Booleanity \quad (4.15)

Definition IV.11

quantum logic:
a non-Boolean orthomodular lattice

Given two orthocomplemented lattices $(\mathcal{L}_1, \land_1, \lor_1, ’_1)$ and $(\mathcal{L}_2, \land_2, \lor_2, ’_2)$:

Definition IV.12
isomorphism of \((\mathcal{L}_1, \land_1, \lor_1, '1)\) and \((\mathcal{L}_2, \land_2, \lor_2, '2)\):

a bijective map \(i : \mathcal{L}_1 \mapsto \mathcal{L}_2\) such that:

\[
i(x \land_1 y) = i(x) \land_2 i(y) \quad \forall x, y \in \mathcal{L}_1 \quad (4.16)
\]

\[
i(x \lor_1 y) = i(x) \lor_2 i(y) \quad \forall x, y \in \mathcal{L}_1 \quad (4.17)
\]

\[
i((x)_1') = (i(x))_2' \quad \forall x \in \mathcal{L}_1 \quad (4.18)
\]

Then:

**Theorem IV.1**

*structure’s theorem about classical logics:*

**HP:**

\((\mathcal{L}, \land, \lor, ')\) Boolean lattice

**TH:**

\(\exists S\) set such that \((\mathcal{L}, \land, \lor, ')\) is isomorphic to \((\mathcal{P}(S), \cap, \cup, -)\)

where \(\mathcal{P}(S)\) is the power set of \(S\) and \(-\) denotes set theoretic complement.

**Theorem IV.2**

*structure’s theorem about orthomodular lattices:*

**HP:**

\((\mathcal{L}, \land, \lor, ')\) orthomodular lattice

**TH:**

\(\exists A\) Von Neumann algebra such that \((\mathcal{L}, \land, \lor, ')\) is isomorphic to \((\mathcal{P}(A), \land_A, \lor_A, 'A)\)

where \(\mathcal{P}(A)\) is the lattice of projections of \(A\) on which the join \(\lor_A\), the meet \(\land_A\) and the orthocomplementation \('A\) are defined in the usual operator-algebraic way.

Given an orthomodular lattice \(\mathcal{L}:\)

**Definition IV.13**

\(\mathcal{L}\) is operator-algebraically finite:

\(\mathcal{L}\) is isomorphic to the lattice of projections \(\mathcal{P}(A)\) of a finite Von Neumann algebra

Given a subset \(S\) of a lattice \(\mathcal{L}:

**Definition IV.14**

\(S\) is upper:

\[x \in S \text{ and } y \succeq x \Rightarrow y \in S \quad (4.19)\]

**Definition IV.15**
$S$ is lower:

$$x \in S \text{ and } x \succeq y \Rightarrow y \in S \quad (4.20)$$

**Definition IV.16**

*upper set generated by $S$:*

$$S \uparrow := \{ x \in \mathcal{L} : (\exists y \in S : x \succeq y) \} \quad (4.21)$$

**Definition IV.17**

*lower set generated by $S:*

$$S \downarrow := \{ x \in \mathcal{L} : (\exists y \in S : y \succeq x) \} \quad (4.22)$$

**Definition IV.18**

$S$ is a filter:

$S$ is upper and $(x \land y \in S \ \forall x, y \in S)$ \quad (4.23)

Given a filter $F$ of a lattice $\mathcal{L}$:

**Definition IV.19**

*F is a proper filter:*

$F$ is a proper subset of $\mathcal{L}$

Given a proper filter $F$ of a lattice $\mathcal{L}$:

**Definition IV.20**

*F is a principal filter:*

$$\exists x \in \mathcal{L} : F = \{ x \} \uparrow \quad (4.24)$$

**Definition IV.21**

*F is an ultrafilter:*

$$\forall F' \text{ filter in } \mathcal{L} : F' \supset F \quad (4.25)$$

Then:

**Theorem IV.3**

*About ultrafilters and Booleanity:

$(F \text{ ultrafilter over an operator-algebraically finite orthomodular lattice } \mathcal{L} \Rightarrow F \text{ is principal } ) \iff \mathcal{L} \text{ is a classical logic}$ \quad (4.26)

Let us remark that the fact that if $\mathcal{L}$ is an operator-algebraically finite quantum logic an ultrafilter is not necessarily principal may be appreciated considering the following counterexample: the quantum logic $\mathcal{P}(R)$ where $R$ is the hyperfinite $II_1$ factor [17].

Given a voting system $S$ with voters’ set $V$ and candidates’ set $C$ let $\land_{O_v}$ and $\lor_{O_v}$ be, respectively the meet operator and the join operator associated to the generic ordering $O_v$ of the generic voter $v \in V$.

In an analogous way let $\land_S$ and $\lor_S$ be, respectively, the meet operator and the join operator associated to $S$.

Then it may be easily verified that:

**Theorem IV.4**
democraticity in the logical formalism:
S is democratic if and only if the following conditions are satisfied:

1. Independence of Irrelevant Alternatives:

\[ c_1 \land S(\land O_{v_1}, \ldots, \land O_{v_{|V|}}, \lor O_{v_{|V|}}) c_2 \] is determined by \[ c_1 \land O_{v_1} c_2, c_1 \lor O_{v_1} c_2, \ldots, c_1 \land O_{v_{|V|}} c_2, c_1 \lor O_{v_{|V|}} c_2 \]

\[ \forall c_1, c_2 \in C, \forall O_{v_1}, \ldots, O_{v_{|V|}} \in \mathcal{O}(C) \quad (4.27) \]

2. Positive Association of Individual Values:

\[ (c_1 \lor O_{v_1} c_2 = c_1 \text{ and } c_1 \lor O_{v_2} c_2 = c_1 \text{ and } c_1 \lor S(\land O_{v_1}, O_{v_3}) c_2 = c_1) \Rightarrow c_1 \lor S(\land O_{v_1}, O_{v_2}, O_{v_3}) c_2 = c_1 \]

\[ \forall O_{v_1}, O_{v_2}, O_{v_3} \in \mathcal{O}(C), \forall v_1, v_2, v_3 \in V, \forall c_1, c_2 \in C \quad (4.28) \]

3. Citizen Sovereignty:

\[ \nexists c_1, c_2 \in C : (c_1 \lor S(\land O_{v_1}, \ldots, O_{v_{|V|}}) c_2 = c_1 \land O_{v_1}, \ldots, O_{v_{|V|}} \in \mathcal{O}(C)) \quad (4.29) \]

4. Nondictatorship:

\[ \nexists v \in V : (c_1 \lor S(\land O_{v_1}, \ldots, O_{v_{|V|}}) c_2 = c_1 \lor O_v c_2, \forall O_{v_1}, \ldots, O_{v_{|V|}} \in \mathcal{O}(C), \forall c_1, c_2 \in C) \quad (4.30) \]

Combining the theorem IV.1, the theorem IV.2 and the theorem IV.4 it appears then evident that, from a mathematical viewpoint, democratic voting systems are nothing but ultrafilters of operator-algebraically finite classical logics.
It appears then natural to introduce the following:

Definition IV.22

quantum democracy:

an ultrafilter in a quantum logic

We will denote the set of all quantum democracies as \( QD \).
Then:

Theorem IV.5

Existence theorem of quantum democracy:

\[ QD \neq \emptyset \quad (4.31) \]

PROOF:

Since a quantum logic is nondistributive, the theorem IV.3 implies that the fact that a quantum democracy is an ultrafilter over an operator-algebraically finite orthomodular lattice doesn’t imply that it is principal.
So the Nondictatorship condition is not violated. ■
[1] G. Birkhoff J. von Neumann. The Logic of Quantum Mechanics. In F. Brody T.Vamos, editor, *The Neumann Compendium*, pages 103–123. World Scientific, Singapore, 1995.
[2] E.G. Beltrametti G. Cassinelli. *The Logic of Quantum Mechanics*. Addison-Wesley Publishing Company, Reading (Massachusetts), 1981.
[3] D.W. Cohen. *An Introduction to Hilbert Space and Quantum Logic*. Springer, New York, 1989.
[4] K. Svozil. *Quantum Logic*. Springer, Singapore, 1998.
[5] P. Ptak S. Pulmannova. *Orthomodular Structures as Quantum Logics*. Kluwer Academic Publishers, Dordrecht, 1991.
[6] M. Dalla Chiara R. Giuntini R. Greechie. *Reasoning in Quantum Theory. Sharp and Unsharp Quantum Logics*. Kluwer Academic Publishers, Dordrecht, 2004.
[7] K.R. Parthasarathy. *An Introduction to Quantum Stochastic Calculus*. Birkhauser, Basel, 1992.
[8] P.A. Meyer. *Quantum Probability for Probabilists*. Springer, Berlin, 1995.
[9] D. Petz M. Redei. John Von Neumann and the Theory of Operator Algebras. In F. Brody T.Vamos, editor, *The Neumann Compendium*, pages 163–185. World Scientific, Singapore, 1995.
[10] M. Redei. *Quantum Logic in Algebraic Approach*. Kluwer Academic Publishers, Dordrecht, 1998.
[11] M. Redei. Von Neumann’s Concept of Quantum Logic and Quantum Probability. In M. Redei N. Stöltzner, editor, *John Von Neumann and the Foundations of Quantum Physics*, pages 153–176. Kluwer Academic Publisher, Dordrecht, 2001.
[12] P. Ptak S. Pulmannova. Quantum Logics as Underlying Structures of Generalized Probability Theory. In K. Engesser D.M. Gabbay D. Lehmann, editor, *Handbook of Quantum Logic and Quantum Structures*, pages 147–214. Elsevier, Amsterdam, 2007.
[13] J. Hamhalter. Quantum Structures and Operator Algebras. In K. Engesser D.M. Gabbay D. Lehmann, editor, *Handbook of Quantum Logic and Quantum Structures*, pages 285–334. Elsevier, Amsterdam, 2007.
[14] J.K. Hodge R.E. Klima. *The Mathematics of Elections. A Hands-On Approach*. American Mathematical Society, Providence, Rhode Island, 2005.
[15] A.D. Taylor. *Social Choice and the Mathematics of Manipulation*. Cambridge University Press, Cambridge, 2005.
[16] P. Odifreddi. Ultrafilters, Dictators and Gods. In C.S. Calude G. Păun, editor, *Finite Versus Infinite. Contributions to an Eternal Dilemma*, pages 239–246. Springer, London, 2000.
[17] V. Jones V.S. Sunder. *Introduction to Subfactors*. Cambridge University Press, Cambridge, 1997.