Approximate four-loop QCD corrections to the Higgs-boson production cross section

G. Das∗, S. Moch† and A. Vogt‡

∗Theoretische Physik 1, Naturwissenschaftlich-Technische Fakultät, Universität Siegen
Walter-Flex-Strasse 3, D–57068 Siegen, Germany

†II. Institut für Theoretische Physik, Universität Hamburg
Luruper Chaussee 149, D–22761 Hamburg, Germany

‡Department of Mathematical Sciences, University of Liverpool
Liverpool L69 3BX, United Kingdom

Abstract

We study the soft and collinear (SV) contributions to inclusive Higgs-boson production in gluon-gluon fusion at four loops. Using recent progress for the quark and gluon form factors and Mellin moments of splitting functions, we are able to complete the soft-gluon enhanced contributions exactly in the limit of a large number of colours, and to a sufficiently accurate numerical accuracy for QCD. The four-loop SV contributions increase the QCD cross section at 14 TeV by 2.7% and 0.2% for the standard choices \( \mu_R = m_H \) and \( \mu_R = m_H / 2 \) of the renormalization scale, and reduce the scale uncertainty to below ±3%. As by-products, we derive the complete \( \delta(1-x) \) term for the gluon-gluon splitting function at four loops and its purely Abelian contributions at five loops, and provide a numerical result for the single pole of the four-loop gluon form factor in dimensional regularization. Finally we present the closely related fourth-order coefficients \( D_4 \) for the soft-gluon exponentiation of Higgs-boson and Drell-Yan lepton-pair production.

∗goutam.das@uni-siegen.de
†sven-olaf.moch@desy.de
‡andreas.vogt@liverpool.ac.uk
The production of the Standard Model (SM) Higgs boson in proton-proton collisions and its subsequent decay are flagship measurements in run 2 of the Large Hadron Collider (LHC) \cite{1, 2}. The main production mechanism for $pp \to H + X$ is the gluon-gluon fusion (ggF) process. The corresponding inclusive cross section serves as a benchmark for the achieved accuracy, both in the LHC experiments and for theoretical research. The radiative corrections in Quantum Chromodynamics (QCD) for the ggF process are large and have motivated significant efforts to improve the precision of the predictions. The QCD corrections are currently known to the next-to-next-to-next-to-leading order (N$^3$LO) in the effective theory for a large top-quark mass, $m_t \gg m_H$ \cite{3, 4}, and to next-to-next-to-leading order (NNLO) in the full theory for Higgs-boson masses $m_H < \sim 2 m_t$ \cite{5–7}.

Near the production threshold, for $z = m_H^2/\hat{s}$ close to unity, where $m_H$ is the Higgs mass and $\sqrt{\hat{s}}$ the partonic center-of-mass energy, the QCD corrections to the ggF process are dominated by the well-known large logarithmic corrections. At $n$-th order they appear in the partonic cross section in the $\overline{\text{MS}}$ scheme as plus-distributions $D_k = [(1-z)^{-1} \ln^k (1-z)]_+$ with $2n-1 \geq k \geq 0$, while the virtual contributions lead to $\delta (1-z)$ terms. In Mellin $N$-space, where $N$ is the conjugate variable of $z$, the threshold logarithms read $\ln^k N$ with $2n \geq k \geq 1$, and the virtual contributions lead to a constant in $N$. The soft-virtual (SV) approximation to the partonic ggF cross section in $N$-space yields reliable predictions for the total Higgs production cross section, as has been demonstrated with comparisons to exact fixed-order results up to NNLO, see, e.g., Refs. \cite{8, 9}. In addition, Mellin $N$-space lends itself to an all-order exponentiation of threshold contributions up to next-to-next-to-next-to-leading logarithmic (N$^3$LL) accuracy and beyond \cite{10, 11}.

These facts motivate the derivation of approximate QCD corrections to the ggF process at four loops in the effective theory, which can be achieved thanks to recent progress in the computation of QCD corrections for related quantities at the four-loop level. This comprises results for specific colour contributions, including quartic group invariants, and the planar limit of quark and gluon form factors \cite{12–18}, correlators of Wilson lines \cite{19–21}, splitting functions for the evolution of parton distributions (PDFs) \cite{22–25}, and, related, the knowledge of a low number of Mellin moments for the structure functions in deep-inelastic scattering (DIS), see Ref. \cite{26}.

Taken together, this knowledge enables us to determine precise numerical results for the complete SV approximation of the ggF process at four loops as well as partial information on terms suppressed by a power $1/N$ in Mellin $N$-space using physical evolution kernels at the same order \cite{9}. The results are used to provide new predictions for the ggF cross section at the collision energy of 14 TeV, as planned for run 3 of the LHC. We also present the corresponding expression for the Drell-Yan (DY) process, $pp \to \gamma^* + X$, which is closely related to the ggF process in the threshold limit, and the $N^4$LL soft-gluon exponentiation coefficient $D_4$ for both processes. As by-products, we obtain a complete result for the so-called gluon virtual anomalous dimension, i.e. the $\delta (1-z)$ terms of the gluon-gluon splitting function at four loops in QCD, together with partial information at five loops, and we derive a numerical result for the single pole $1/\epsilon$ of the dimensionally regulated gluon form factor at four loops.

The effective coupling of the Higgs boson to partons is described by the Lagrangian
\begin{equation}
\mathcal{L}_{\text{eff}} = - \frac{1}{4\upsilon^2} C(M^2_R) H G^a_{\mu\nu} G^{a\mu\nu},
\end{equation}
where $\upsilon \simeq 246$ GeV is the Higgs vacuum expectation value in the SM and $G^a_{\mu\nu}$ denotes the gluon
field strength tensor. The inclusive hadronic cross section for Higgs-boson production at a center-of-mass energy \( E_{cm} = \sqrt{S} \) is given in standard QCD factorization by

\[
\sigma(S, m_H^2) = \tau \sum_{a,b} \int_0^1 \frac{dx_1}{x_1} \frac{dx_2}{x_2} f_{a/h_1}(x_1, \mu_F^2) f_{b/h_2}(x_2, \mu_F^2) \int_0^1 dz \delta\left( z - \frac{\tau}{x_1 x_2} \right) \times
\]

\[
\times \tilde{\sigma}_0^H c_{ab}^H(z, \alpha_s(\mu_R^2), m_H^2/\mu_R^2, m_H^2/\mu_F^2) ,
\]

where \( \tau = m_H^2/S \), and \( \mu_F, \mu_R \) are the mass-factorization and renormalization scales, and \( f_{a/h}(x, \mu_F^2) \) the PDFs of the proton. The expansion in the strong coupling \( \alpha_s \) of the large-\( m_t \) effective vertex for the Higgs coupling to gluons is included in \( \tilde{\sigma}_0^H \), viz

\[
\tilde{\sigma}_0^H = \frac{\pi C(\mu_R^2)^2}{8 n_A v^2} \quad \text{with} \quad C(\mu_R^2) = -\frac{\alpha_s(\mu_R^2)}{3\pi} \left\{ 1 + 11 \frac{\alpha_s(\mu_R^2)}{4\pi} + \ldots \right\} ,
\]

where \( n_A = (n_c^2 - 1) \) denotes the dimension of the adjoint representation of the \( SU(n_c) \) gauge group, and the matching coefficient \( C(\mu_R^2) \) is fully known up to N^4LO in QCD (\( n_c = 3 \)) \[27\] \[3\]. The coefficient functions \( c_{ab}^H \) are expanded in powers of \( a_s = \alpha_s(\mu_R^2)/(4\pi) \),

\[
c_{ab}^H(z, \alpha_s(\mu_R^2), m_H^2/\mu_R^2, m_H^2/\mu_F^2) = \sum_{n=0} c_{ab}^n H(n, z, m_H^2/\mu_R^2, m_H^2/\mu_F^2) ,
\]

where the leading order (LO) is \( c_{ab}^{H,(0)} = \delta_{aa} \delta_{bb} \delta(1-z) \) and the corrections to N^3LO have been computed in Refs. \[3\] \[4\] \[5\]. For \( c_{g g}^{H,(4)}(z) \), at N^4LO, seven of the eight plus-distributions of the SV approximation are known. The coefficients of \( \mathcal{D}_k \) for \( 7 \geq k \geq 2 \) can be inferred from Ref. \[8\] and have been written down in Eq. (16) of Ref. \[38\] and that of \( \mathcal{D}_1 \) is fixed by the results of Ref. \[39\] and has been given in Eq. (13) of Ref. \[40\]. An approximate result for the \( \mathcal{D}_0 \) term has been provided before in Eqs. (2.13) and (2.14) of Ref. \[9\].

Here we present a new result for the latter coefficient for a general gauge group. The relevant Casimir invariants for \( SU(n_c) \) are \( C_A = n_c, C_F = (n_c^2 - 1)/(2n_c) \) and

\[
\frac{d_{A}^{abcd} d_{A}^{abcd}}{n_A} = \frac{1}{24} n_c^2(n_c^2 + 36) , \quad \frac{d_{F}^{abcd} d_{A}^{abcd}}{n_A} = \frac{1}{48} n_c(n_c^2 + 6) .
\]

With the recent progress at four loops on the pole structure of the QCD form factors, on splitting functions and on Mellin moments for DIS structure functions, and following the same procedure as employed for DIS structure functions in Ref. \[11\], the \( \mathcal{D}_0 \) term in \( c_{g g}^{H,(4)}(z) \) can now be given as

\[
c_{g g}^{H,(4)} \bigg|_{\mathcal{D}_0} = C_A \left( -\frac{40498399}{2187} + \frac{28613426}{729} \zeta_2 + \frac{10995352}{81} \zeta_3 + \frac{2598712}{81} \zeta_4 - \frac{7252952}{27} \zeta_2 \zeta_3 
\right.
\]

\[
+ \frac{3411280}{9} \zeta_5 - \frac{656216}{3} \zeta_2 \zeta_3 + \frac{1019381}{9} \zeta_6 - \frac{293488}{3} \zeta_5 \zeta_4 - 561344 \zeta_2 \zeta_5 + 986440 \zeta_7 .
\]

\[4\]See Refs. \[28\] \[30\] for previous work up to N^3LO in QCD.

\[5\]See Refs. \[31\] \[32\] and \[35\] \[37\] for previous work at lower orders in QCD.
which, in the normalization of Eq. (4), admit the perturbative expansion as

\[ \frac{1}{12} f_{4,dF^{abcd},A_A} + d_{A_A}^{abcd}A_A \left( -2 f_{4,dF^{abcd},A_A} + C_A^3n_f \right) + C_A^3n_f \left( \frac{17665315}{2916} - \frac{10870138}{729} \right) \]

\[ \frac{3234580}{81} \zeta_3 = \frac{364960}{81} \zeta_4 + \frac{520864}{9} \zeta_2 \zeta_3 - \frac{2216816}{27} \zeta_5 + \frac{263864}{9} \zeta_3^2 - \frac{578258}{27} \zeta_6 \]

\[ - b_{4,n_fC_A^3}^q - \frac{2 b_{4,n_fC_A^3}^q}{b_{4,dF^{abcd},A_A}} \left( \frac{2798861}{486} - \frac{39658}{9} \zeta_2 \right) \]

\[ - \frac{367508}{27} \zeta_3 = \frac{130640}{27} \zeta_4 + \frac{17120}{9} \zeta_2 \zeta_3 + \frac{21904}{9} \zeta_5 + \frac{34064}{3} \zeta_3^2 - 98 \zeta_6 + 4 b_{4,n_fC_A^3}^q \]

\[ + C_A C_F n_f \left( \frac{-27949}{54} - \frac{632}{9} \zeta_2 + \frac{2240}{9} \zeta_3 + \frac{668}{9} \zeta_4 + \frac{1024}{3} \zeta_2 \zeta_3 - \frac{7744}{3} \zeta_5 \right) \]

\[ - 736 \zeta_3^2 + \frac{29336}{9} \zeta_6 + 4 b_{4,n_fC_A^3}^q \right) + n_f \frac{d_{A_A}^{abcd}A_A}{n_A} \left( 768 - \frac{9088}{3} \zeta_2 + \frac{10624}{9} \zeta_3 \right) + \frac{1600}{3} \zeta_4 - 256 \zeta_2 \zeta_3 + \frac{43520}{9} \zeta_5 - \frac{2432}{3} \zeta_3^2 - \frac{2368}{9} \zeta_6 + 4 b_{4,dF^{abcd},A_A} \]

\[ - 2 q_{n_f}^2 \left( \frac{154315}{2916} - \frac{1171400}{729} \zeta_2 - \frac{176624}{81} \zeta_3^2 - \frac{1168}{9} \zeta_4 + \frac{71200}{27} \zeta_2 \zeta_3 - \frac{34592}{9} \zeta_5 \right) \]

\[ - C_A C_F n_f^2 \left( \frac{155083}{243} - \frac{5600}{9} \zeta_2 - \frac{4784}{9} \zeta_3 - \frac{160}{3} \zeta_4 + \frac{1280}{3} \zeta_2 \zeta_3 - 32 \zeta_5 \right) \]

\[ + C_A n_f^2 \left( \frac{10432}{2187} - \frac{3200}{81} \zeta_2 - \frac{3680}{81} \zeta_3 + \frac{112}{9} \zeta_4 \right). \]

Here the term \( f_{4,dF^{abcd},A_A} \) is related to the eikonal anomalous dimension of the four-loop quark form factor, i.e., to the single pole in dimensional regularization, cf. Ref. [11]. The expressions \( b_{4,dF^{abcd},A_A} \), \( b_{4,n_fC_A^3}^q \) and \( b_{4,n_fC_A^3}^q \) denote the four-loop coefficients of the respective colour factor in the quark virtual anomalous dimension \( B_q \), i.e., the coefficient of \( \delta(1-z) \) in the quark-quark splitting function \( P_{qq} \). At \( n \)-th order, expanding in powers of \( a_s \equiv \alpha_s(\mu_R^2)/(4\pi) \) analogous to Eq. (4), the flavor-diagonal splitting functions \( P_{ii} \) admit the large-\( z \) expansion \([41, 42]\) as

\[ P_{ii}^{(n-1)}(z) = A_i D_0 + B_i^{(1)} \delta(1-z) + \ldots, \quad i = q, g, \]

where the coefficients \( A_i \) are the well-known lightlike cusp anomalous dimensions. The quantities \( f_{4,dF^{abcd},A_A}^q \), \( b_{4,dF^{abcd},A_A}^q \), \( b_{4,n_fC_A^3}^q \) and \( b_{4,n_fC_A^3}^q \) are not known analytically. They drop out in the large-\( n_c \) limit of Eq. (6). Precise numerical estimates, i.e., \( b_{4,dF^{abcd},A_A}^q = -143.6 \pm 0.2 \), \( b_{4,n_fC_A^3}^q = -455.247 \pm 0.005 \) and \( b_{4,n_fC_A^3}^q = 80.780 \pm 0.005 \) have been given in Ref. [11] together with \( f_{4,dF^{abcd},A_A}^q = -100 \pm 100 \).

While the large-\( n_c \) limit of Eq. (6) is exact, the above general expression uses one assumption on the relation of quark and gluon form factors \( f^q \) and \( f^g \) in QCD in dimensional regularization, which, in the normalization of Eq. (4), admit the perturbative expansion as

\[ f^p = 1 + a_s \left( \frac{1}{2\varepsilon^2} A_i^p - \frac{1}{2\varepsilon^2} G_i^p \right) + O(a_s^2) \quad i = q, g, \]

see e.g. Refs. [11, 43] for the higher orders.
The logarithms of $f^q$ and $f^g$ contain double and single poles in $\varepsilon$, the former being controlled by their respective cusp anomalous dimensions, $A^q$ and $A^g$, cf. Eq. (7), which exhibit generalized Casimir scaling through four loops, see, e.g. [18,24]. The single poles on the other hand, which are proportional to functions $G^p_n(\varepsilon)$ at $n$-th order, are controlled by the collinear anomalous dimensions and can be converted to appropriate eikonal (Wilson line) quantities, after separation of the virtual anomalous dimensions $B^q$ and $B^g$, cf. Eq. (7), and terms proportional to the QCD $\beta$-function. In detail (see, e.g. [11,43]), the functions $G^p_n(\varepsilon)$ satisfy the following relations to five loops

\[
\begin{align*}
G^1_0 &= 2(B^0_1 - \delta_{ps} \beta_0) + f^p_0 + \varepsilon f^p_{01}, \\
G^2_0 &= 2(B^0_2 - 2\delta_{ps} \beta_1) + (f^p_2 + \beta_0 f^p_{01}) + \varepsilon f^p_{02}, \\
G^3_0 &= 2(B^0_3 - 3\delta_{ps} \beta_2) + (f^p_3 + \beta_1 f^p_{01} + \beta_0 f^p_{02}) + \varepsilon f^p_{03}, \\
G^4_0 &= 2(B^0_4 - 4\delta_{ps} \beta_3) + (f^p_4 + \beta_2 f^p_{01} + \beta_1 f^p_{02} + \beta_0 f^p_{03}) + \varepsilon f^p_{04}, \\
G^5_0 &= 2(B^0_5 - 5\delta_{ps} \beta_4) + (f^p_5 + \beta_3 f^p_{01} + \beta_2 f^p_{02} + \beta_1 f^p_{03} + \beta_0 f^p_{04}) + \varepsilon f^p_{05},
\end{align*}
\]

where the functions $f^p_{0n}(\varepsilon)$ at $n$ loops are polynomials in $\varepsilon$ and $\beta_n$ are the coefficients of the QCD $\beta$-function normalized as in Eq. (4), i.e., $\beta(a_s) = -\beta_0 a_s^2 - \ldots$ with $\beta_0 = 11/3 C_A - 2/3 n_f$.

The eikonal anomalous dimensions $f^q$ and $f^g$ of these Wilson line quantities for quarks and gluons exhibit the same maximal non-Abelian colour structure as the cusp anomalous dimensions, a fact verified explicitly at lower fixed orders [43,44] and generalized in Ref. [45]. Hence we assume here that also the expressions for $f^q$ and $f^g$ are related by generalized Casimir scaling at four loops (and beyond), in complete analogy to the cusp anomalous dimensions, $A^q$ and $A^g$, see also the recent work [46].

This leads immediately to the expression for the full colour dependence of the four-loop gluon virtual anomalous dimension $B^g_4$ as

\[
B^g_4 = \frac{C^4_A}{n_A} \left( b^g_{4,4} c^4_A \right) + \frac{d^A_{abcd} d^A_{abcd}}{n_A} \left( b^g_{4,abcd,abcd} \right) + n_f C^3_A \left( -\frac{8075}{108} - \frac{6155}{54} \frac{\zeta_2}{27} - \frac{22714}{27} \frac{\zeta_3}{18} + \frac{7789}{18} \zeta_4 \right. \\
+ \frac{1874}{9} \zeta_2 \zeta_3 + \frac{919}{9} \zeta_5 - \frac{1268}{3} \frac{\zeta_2}{27} - \frac{1777}{54} \frac{\zeta_3}{27} + \frac{1777}{54} \frac{\zeta_6}{27} - \frac{1}{4} b^q_{4,n_f} C^3_A - \frac{1}{2} b^g_{4,n_f} C^4_A - \frac{1}{4} b^g_{4,d^A_{abcd,abcd}} \right) \\
\left. + n_f C^2_A C_F \left( \frac{23566}{243} + \frac{4198}{27} \zeta_2 + \frac{8854}{27} \zeta_3 + \frac{27269}{27} \zeta_4 - \frac{2744}{9} \zeta_2 \zeta_3 + \frac{6712}{9} \zeta_5 + \frac{1928}{3} \zeta_2 \zeta_3 - \frac{2879}{9} \zeta_6 + \frac{2948}{9} \zeta_3 + \frac{204}{3} \zeta_2 \zeta_3 \\
- \frac{912}{9} \zeta_5 - \frac{224}{9} \zeta_2 \zeta_3 + \frac{6434}{9} \zeta_6 + b^g_{4,n_f} C_F \right) + n_f C^2_A \left( -\frac{2723}{27} + \frac{1952}{9} \frac{\zeta_2}{3} - \frac{2368}{3} \zeta_2 \right) \\
+ \frac{1312}{3} \zeta_3 + \frac{1016}{3} \zeta_4 + \frac{544}{3} \zeta_2 \zeta_3 - \frac{1520}{3} \zeta_5 + \frac{1496}{9} \zeta_6 + b^g_{4,d^A_{abcd,abcd}} + n_f C^2_A \left( \frac{1352}{81} + \frac{37}{27} \zeta_2 \right) \\
+ \frac{289}{27} \zeta_3 + \frac{200}{27} \zeta_4 - \frac{32}{9} \zeta_2 \zeta_3 - \frac{8}{9} \zeta_5 + n_f C_A C_F \left( \frac{3910}{243} + \frac{160}{9} \zeta_3 \right) + n_f C_F \left( \frac{338}{27} - \frac{176}{9} \zeta_3 \right) \\
\left. + n_f d^A_{abcd} d^A_{abcd} \left( -\frac{704}{9} + \frac{512}{3} \zeta_3 \right) + n_f C_F \left( \frac{154}{243} \right) \right),
\]

(10)
where the \( n_f^3 \)-dependent terms agree with Ref. [22]. In addition, we have checked that a numerical fit for the \( d_F^{abcd} d_A^{abcd} \) term in the gluon-gluon splitting function [24] to the known Mellin moments nicely confirms the value quoted in Eq. (10). Altogether, we take this as strong indications on the correctness of the assumption made in the derivation of Eqs. (6) and (10). The remaining unknowns can be determined numerically as \( b^g_4, c_A^4 = 1098 \pm 20 \) and \( b^g_{4, d_F^{abcd} d_A^{abcd}} = -1125.6 \pm 1.0 \) from the Mellin moments \( N = 2, 4, 6 \) and 8 obtained via DIS structure functions, see Ref. [25].

We also note that the purely Abelian (QED) contributions in \( B_4^g \) coincide with the respective terms in the four-loop QCD \( \beta \)-function [47, 48]. This concerns the colour factors \( n_f^2 \! \! C_F^3, n_f^2 \! \! C_F^2, n_f^2 \! \! C_F \) and \( d_F^{abcd} d_A^{abcd} \) and is a direct consequence of the generalized Casimir scaling of \( f^q \) and \( f^g \), which implies that \( f_4^g \) must have only non-Abelian colour factors (terms proportional to \( C_A^4 \), \( d_F^{abcd} d_A^{abcd} \) or \( d_A^{abcd} d_A^{abcd} \)). This reproduces a pattern already observed for \( B_4^g \) up to third order, \( n \leq 3 \), see [49], for all terms \( n_f^2 \! \! C_F^{n-k} \) with \( 1 \leq k \leq n-1 \). At four loops, the colour factors \( n_f^2 \! \! C_F^3 \) and \( d_F^{abcd} d_A^{abcd} \) are unique in the single pole \( 1/\varepsilon \) of the gluon form factor, cf. in \( G_4^g \) in Eq. (9), and therefore, can be related directly to those in \( \beta_3 \) and, hence, \( B_4^g \). The other two colour factors in \( G_4^g, n_f^2 C_F^2 \) and \( n_f^2 C_F \), do also receive contributions from lower orders. For instance \( n_f^2 C_F^2 \) terms are generated from \( \beta_1 f_{02}^g \), but cancel in extraction of \( B_4^g \) from \( G_4^g \).

With the help of Eqs. (9) and (10) we obtain the single pole in \( \varepsilon \) in the gluon form factor \( f_4^g \) at four loops as

\[
\begin{align*}
\left. f_4^g \right|_{\varepsilon} &= C_A^4 \left( \frac{-746918615}{104976} + \frac{595199}{216} \xi_2 + \frac{8305667}{1458} \xi_3 + \frac{975575}{972} \xi_4 + \frac{39811}{81} \xi_2 \xi_3 - \frac{781411}{405} \xi_5 \\
&\quad + \frac{41335}{54} \frac{\xi_3 \xi_4 - 272338}{81} \xi_2 \xi_5 + \frac{739783}{144} \xi_6 - \frac{14629}{45} \xi_2 \xi_5 + \frac{563669}{126} \xi_7 + \frac{1}{192} f_4^q, d_F^{abcd} d_A^{abcd} \\
&\quad - \frac{1}{4} C_A^4 \right) + \frac{d_A^{abcd} d_A^{abcd}}{n_A} \left( -\frac{80}{9} + \frac{704}{3} \xi_3 - \frac{1}{8} f_4^q, d_F^{abcd} d_A^{abcd} - \frac{1}{4} b^g_4, d_A^{abcd} d_A^{abcd} \right) + O(n_f) \\
&= C_A^4 \left( -1084.7 \pm 5.5 \right) + \frac{d_A^{abcd} d_A^{abcd}}{n_A} \left( 567.3 \pm 12.8 \right) + n_f \text{ terms}, \quad (11)
\end{align*}
\]

where all \( n_f \)-dependent contributions have been given analytically in Ref. [18].

The observed relation between the gluon virtual anomalous dimension \( B_5^g \) and the corresponding coefficient of the \( \beta \)-function for purely Abelian terms leads to new predictions at five loops. Using the expression for the \( \beta \)-function for a general gauge group at five loops [50, 52] (the QCD result was obtained before in Ref. [53]), one deduces for the splitting function \( P_{gg} \) in Eq. (7)

\[
\begin{align*}
B_5^g &= n_f^2 C_F \left( \frac{-4157}{12} - 64 \xi_3 \right) + n_f^2 C_F^2 \left( -\frac{2509}{18} - \frac{536}{3} \xi_3 + \frac{1160}{3} \xi_5 \right) \\
&\quad + n_f^2 C_F \frac{d_F^{abcd} d_A^{abcd}}{n_A} \left( \frac{4160}{3} + \frac{5120}{3} \xi_3 - \frac{12800}{3} \xi_5 \right) + n_f^3 C_F^2 \left( -\frac{4961}{324} + \frac{952}{27} \xi_3 - \frac{44}{3} \xi_4 \right) \\
&\quad + n_f^3 \frac{d_F^{abcd} d_F^{abcd}}{n_A} \left( -\frac{1760}{9} - \frac{1312}{3} \xi_3 + 128 \xi_4 + \frac{640}{3} \xi_5 \right) + n_f^4 C_F \left( -\frac{107}{486} - \frac{8}{27} \xi_3 \right) \\
&\quad + \text{ terms with } C_A, d_A^{abcd} d_A^{abcd}, d_F^{abcd} d_A^{abcd}.
\end{align*}
\]
Predictions for the purely Abelian part of the gluon form factor at five loops are possible, e.g., for the $n_t C_f^4$ in $f^8$, which can be read off from Eq. (12) using $G_3^g$ in Eq. (9), while finite terms of $f^8$ at lower orders are needed for other colour structures.

Beyond the SV approximation, predictions for the ggF cross section are possible using physical evolution kernels \[9\,54\,55\]. In $z$-space, this concerns terms enhanced as $\ln^k (1-z)$ with $2n - 1 \geq k \geq 1$ at $n$-th order, or equivalently power suppressed contributions $(\ln^k N)/N$ in Mellin $N$-space. Such next-to-leading power threshold effects have also been studied in Refs. \[56\,63\]. At $N^4$LO, these subleading terms in $c_{gg}^{H,4}(z)$ can be obtained from the physical evolution kernel $K_{gg}$, which one can define by re-expressing Eq. (2) as dimensionless ‘structure functions’ $f_{ab}$, i.e.,

$$
\sigma(S, m_T^2) = \sum_{a,b} \tilde{s}_0^H f_{ab}.
$$

The kernel $K_{gg}$ and its perturbative expansion for a scale choice $\mu_F = m_H$ are then given in terms of the splitting function $P_{gg}$, the $\beta$-function and the gluon coefficient function $c_{gg}^{H}$ by

$$
\frac{d}{d \ln m_T^2} f_{gg} = \left\{ 2P_{gg}(s) + \beta(s) \frac{dc_{gg}^{H}(a_s)}{da_s} \otimes (c_{gg}^{H}(a_s))^{-1} \right\} \otimes f_{gg}
$$

where $\otimes$ denotes the usual Mellin convolution and $\beta(a_s)$ the $\beta$-function as defined below Eq. (9).

The key feature of the kernel $K_{gg}$ is its single-logarithmic enhancement. In $z$-space, this implies at $n$-th order that all terms $\ln^k (1-z)$ with $2n - 1 \geq k \geq n+1$ have to cancel in Eq. (13) to all orders in $(1-z)$, which leads to predictions for coefficient function $c_{gg}^{H}$, cf. Ref. \[9\]. In Mellin $N$-space the leading large-$N$ logarithms of the sub-dominant $N^{-1}$ contributions in $K_{gg}$ take the simple form,

$$
K_{gg}^{(1)} \bigg|_{N^{-1}} = - \left( 8 \beta_0 C_A + 32 C_A^2 \right) \ln N + O(1),
$$

$$
K_{gg}^{(2)} \bigg|_{N^{-1}} = - \left( 16 \beta_0^2 C_A + 112 \beta_0 C_A^2 \right) \ln^2 N + O(\ln N),
$$

$$
K_{gg}^{(3)} \bigg|_{N^{-1}} = - \left( 32 \beta_0^3 C_A + \frac{896}{3} \beta_0^2 C_A^2 \right) \ln^3 N + O(\ln^2 N),
$$

$$
K_{gg}^{(4)} \bigg|_{N^{-1}} = - \left( 64 \beta_0^4 C_A + \frac{5384}{3} \beta_0^3 C_A^2 \right) \ln^4 N + O(\ln^3 N),
$$

where the first three lines follow from the known coefficient functions $c_{gg}^{H,(n)}$ up to $N^3$LO. The expression for $K_{gg}^{(4)}$ contains an unknown coefficient $\zeta_{gg}^{(4)}$, to be determined at $N^4$LO by explicit computations (the corresponding coefficient for DIS has been obtained in Refs. \[57\,58\,6\]). Eq. (14) predicts the following next-to-leading power threshold terms in the four-loop gluon coefficient function $c_{gg}^{H,4}$ for the ggF process at the scale $\mu_R = \mu_F = m_H$,

$$
c_{gg}^{H,4}(z) = c_{gg}^{H,4}(z) \bigg|_{SV} - \frac{4096}{3} C_A^4 \ln^7 (1-z) + \left\{ \frac{19712}{3} C_A^4 + \frac{3584}{3} C_A^3 \beta_0 \right\} \ln^6 (1-z)
$$

\[6\] We note that the pattern of the ratios of the lower order coefficients is $112/32 = 7/2$ and $(896/3)/112 = 8/3$. A generalization of this pattern leads to an estimate of $\zeta_{gg}^{(4)} = 670 \pm 300$ with a conservative numerical uncertainty, which has a sub-percent effect on the $N^{-1} \ln^4 N$ coefficient in Eq. (16) below.
\[ \kappa_n \alpha \text{ inside the SV and SV results and the approximation based on the SV} \]

\[ N \text{ six decimals. The quoted uncertainty in the coefficient of ln functions differ from the approximation based on the SV} + \]

\[ N \text{ at NNLO throughout. The prefactor } \alpha \text{ as well as the value of the strong coupling constant } \lambda \text{ taken order-independent throughout.} \]

\[ m_n \text{ band spanned by the SV and SV taken order-independent at NNLO throughout. The prefactor } \alpha \text{ as well as the value of the strong coupling constant } \lambda \text{ are completely negligible.} \]

\[ \text{The constant-N contribution } g_{0,4} \text{ has been estimated in Ref.} \ [9] \text{ by three Padé approximants which yield a fairly wide spread of values suggesting } \kappa_4 g_{0,4} = 65 \pm 65. \]

\[ \text{The } N^4 \text{LO coefficient function in the SV approximation, together with the sub-dominant } N^{-1} \text{ contributions in Mellin } N \text{-space, can be expected to provide a reliable approximation of the exact result. As pointed out earlier, the exact Mellin } N \text{-space result at lower orders resides inside the band spanned by the SV and SV}+N^{-1} \text{ terms at moderately large } N. \]

\[ \text{This is shown in Fig.} \ 1 \text{ (left) at } N^3 \text{LO (for corresponding NLO and NNLO plots see Fig.} \ 1 \text{ of Ref.} \ [9]). \]

\[ \text{The exact coefficient functions differ from the approximation based on the SV}+N^{-1} \text{ terms by } 0.44\% \text{ at NLO, } 0.83\% \text{ at } N^2 \text{LO and } 1.15\% \text{ at } N^3 \text{LO at } N = 12. \]

\[ \text{For smaller } N \text{ values the difference between the exact results and the approximation based on the SV}+N^{-1} \text{ terms is larger, however they always remain inside the SV and SV}+N^{-1} \text{ band. At } N^4 \text{LO, see Fig.} \ 1 \text{ (right), the SV approximation of Eq.} \ (16) \text{ is shown including the } N \text{-independent constant } g_{0,4} \text{ and the known } 1/N \text{ terms as specified in Eq.} \ (16). \]

\[ \text{The predictions for the ggF cross sections at the collision energy of } 14 \text{ TeV use a Higgs mass} \]

\[ m_H = 125 \text{ GeV, an on-shell top quark mass } m_t = 172.5 \text{ GeV, } n_f = 5 \text{ light flavors at scales of order } m_H^2, \text{ and truncated all exact numbers to six decimals. The quoted uncertainty in the coefficient of ln } \lambda \text{ stems entirely from } f_4^{(q)} d_{f}^{abcd} d_{f}^{abcd}, \]

\[ \text{as the uncertainties in the values of } b_4^{(q)} d_{f}^{abcd} d_{f}^{abcd}, b_4^{(q)} n_f C_A^2 C_A, \text{ and } b_4^{(q)} n_f C_A^2 \text{ are completely negligible.} \]

\[ \text{The constant-N contribution } g_{0,4} \text{ has been estimated in Ref.} \ [9] \text{ by three Padé approximants which yield a fairly wide spread of values suggesting } \kappa_4 g_{0,4} = 65 \pm 65. \]

\[ \text{The } N^4 \text{LO coefficient function in the SV approximation, together with the sub-dominant } N^{-1} \text{ contributions in Mellin } N \text{-space, can be expected to provide a reliable approximation of the exact result. As pointed out earlier, the exact Mellin } N \text{-space result at lower orders resides inside the band spanned by the SV and SV}+N^{-1} \text{ terms at moderately large } N. \]

\[ \text{This is shown in Fig.} \ 1 \text{ (left) at } N^3 \text{LO (for corresponding NLO and NNLO plots see Fig.} \ 1 \text{ of Ref.} \ [9]). \]

\[ \text{The exact coefficient functions differ from the approximation based on the SV}+N^{-1} \text{ terms by } 0.44\% \text{ at NLO, } 0.83\% \text{ at } N^2 \text{LO and } 1.15\% \text{ at } N^3 \text{LO at } N = 12. \]

\[ \text{For smaller } N \text{ values the difference between the exact results and the approximation based on the SV}+N^{-1} \text{ terms is larger, however they always remain inside the SV and SV}+N^{-1} \text{ band. At } N^4 \text{LO, see Fig.} \ 1 \text{ (right), the SV approximation of Eq.} \ (16) \text{ is shown including the } N \text{-independent constant } g_{0,4} \text{ and the known } 1/N \text{ terms as specified in Eq.} \ (16). \]

\[ \text{The predictions for the ggF cross sections at the collision energy of } 14 \text{ TeV use a Higgs mass} \]

\[ m_H = 125 \text{ GeV, an on-shell top quark mass } m_t = 172.5 \text{ GeV, } n_f = 5 \text{ active quark flavors and the PDF sets ABMP16} \ [64] \text{ and MMHT2014} \ [65] \text{ using the lhapdf} \ [66] \text{ interface. The PDF sets and as well as the value of the strong coupling constant } \alpha_s \text{ corresponding to the respective PDF set are taken order-independent at NNLO throughout. The prefactor } C(\mu_R^2) \text{ in Eq.} \ (3) \text{ is improved with the full top-mass dependence of the Born cross section. The results up to } N^3 \text{LO are computed with the program } iHixs \ [67] \text{ which directly provides the cross sections in this rescaled effective field theory.} \]
The residual uncertainty in the SV approximation of the ggF cross section due to the coefficients $f_{4q}^{abcd}$ and $g_{04}$ in Eq. (6) and Eq. (16), which are currently least constrained, is practically negligible: the 100% error on $g_{04}$ leads to an uncertainty of 0.3% for the cross section, while the 100% error on $f_{4q}^{abcd}$ is completely negligible. The virtual anomalous dimension $B_g^4$ only appears in the scale-dependent terms at N$^4$LO and its contribution vanishes for the central scale choice $\mu = m_H$. For the scale setting $\mu = m_H/2$ a change below 0.002% is observed for the cross-section at 14 TeV LHC due to the numerical uncertainty in $B_g^4$. Thus, precise predictions at N$^4$LO are now possible for all relevant kinematics and scale choices.

The impact of the SV corrections at N$^4$LO is shown in Fig. 2(left) in a range of center-of-mass energies for two different choices of renormalization scale, $\mu = m_H$ and $\mu = m_H/2$, keeping always $\mu_F = m_H$. The corresponding $K$-factors, defined as the ratio of the SV corrections at N$^4$LO over the exact N$^3$LO result, are displayed in the lower panel. The SV correction at N$^4$LO increases the cross-section at the LHC with 14 TeV by 1.41 pb for the scale $\mu = m_H$ and by 0.08 pb for the scale $\mu = m_H/2$. The $K$-factor shows little dependence on the collision energy in the entire range of center-of-mass energies displayed in Fig. 2(left). For the scale choice $\mu = m_H/2$, which is closer to the point of minimal sensitivity, the effect of higher order corrections is indeed small, leading to a $K$-factor close to unity, cf. also Ref. [9]. At the scale $\mu = m_H$ a $K$-factor of 1.027 is obtained for the LHC at 14 TeV.

In Fig. 2(right) we show the dependence of the cross section for the $E_{CM} = 14$ TeV on the renormalization scale $\mu$. The $\mu$ dependence indeed decreases order by order in perturbation theory up to N$^4$LO. In the range $m_H/4 \leq \mu \leq 2m_H$ the effect of the $\mu$ variation decreases drastically.
Figure 2: Left panel: The contribution of the SV approximation at N^4LO to the ggF cross section in proton-proton collisions as a function of the center-of-mass energy E_{CM} for \( \mu_F = m_H \) at two renormalization scales, \( \mu_R = m_H \) and \( \mu_R = m_H/2 \), using the MMHT2014 PDFs [65]. The corresponding K-factors with respect to N^3LO are shown in the lower panel. Right panel: The ggF cross section up to N^4LO with a variation of the renormalization scale \( \mu_R \) for the LHC at 14 TeV using the same PDFs and \( \mu_F = m_H \).

from \( \pm 27\% \) at NLO and \( \pm 14\% \) at NNLO to \( \pm 5\% \) at N^3LO, while it amounts to less than \( \pm 3\% \) at N^4LO. Also here the factorization scale is kept fixed, at \( \mu_F = m_H \), since beyond NNLO only flavour non-singlet results have published for the QCD splitting functions \( P_{ik}(z) \) [22, 23, 68], and PDFs fits have been limited to NNLO so far.

The uncertainty in the predicted ggF cross sections due to the truncation of the perturbation series is now, at N^4LO, smaller than that due to the use of different sets of PDFs and corresponding different values of the strong coupling constant \( \alpha_s \). For \( \sqrt{S} = 14 \) TeV, \( m_H = 125 \) GeV, the central scale \( \mu_R = m_H \), and including the PDF uncertainties at N^3LO, one obtains

\[
\sigma|_{N^4LO} = 49.6 \pm 0.5 \text{ pb}, \quad \sigma|_{N^4LO} = 50.8 \text{ pb, ABMP16}, \\
\sigma|_{N^3LO} = 52.3 \pm 0.8 \text{ pb}, \quad \sigma|_{N^3LO} = 53.8 \text{ pb, MMHT2014}, \quad (17)
\]

where the spread in predictions is due to different values of the strong coupling constant at NNLO corresponding to the different PDF sets used, i.e., \( \alpha_s(M_Z) = 0.1147 \) for ABMP16 and \( \alpha_s(M_Z) = 0.1180 \) for MMHT2014, and due to different gluon PDFs in the relevant kinematic range. These are consequences of different choices for the theoretical framework and assumptions on parameters used in the respective global fits, see Ref. [69], which lead to systematic shifts that are often significantly larger than the PDF and \( \alpha_s(M_Z) \) uncertainties associated to individual PDF sets.

Due to the universality of threshold dynamics for colourless final states in hadronic collisions, relevant formulae for the Drell-Yan process, \( pp \to \gamma^* + X \), can be easily obtained from the above
considerations, using Eq. (2) with the replacement \( \tilde{\sigma}_0^H c_{ab}^H \rightarrow \tilde{\sigma}_0^\gamma c_{ab}^{DY} \) with

\[
\tilde{\sigma}_0^\gamma = 4\pi \alpha^2 / (3Q^2 n_c) .
\]  (18)

Here \( \alpha \) is the fine-structure constant of QED and \( Q^2 \) the virtuality of the produced photon \( \gamma \). The coefficient functions \( c_{ab}^{DY} \) enjoy a perturbative expansive analogous to Eq. (4) with the leading order normalization \( c_{ab}^{DY, (0)} = \delta_{aq} \delta_{bq} \delta(1-z) \). The coefficients of \( D_k \) for \( 7 \geq k \geq 2 \) of the four-loop term \( c_{q\bar{q}}^{DY, (4)} \) can be found in Eq. (6) of Ref. [38] and that of \( D_1 \) in Eq. (14) of Ref. [40].

We are now in the position to present the four-loop \( D_0 \) term for the DY process. It is given by

\[
\left. c_{q\bar{q}}^{DY, (4)} \right|_{D_0} = \ldots
\]

\[
C_F^2 \left( 32704 \zeta_3 + 113152 \zeta_2 \zeta_3 - 196608 \zeta_5 - 15360 \zeta_2^2 - 491520 \zeta_2 \zeta_5 - 195584 \zeta_3 \zeta_4 + 983040 \zeta_7 \right) + C_F^2 C_A \left( \frac{206444}{27} - \frac{32740}{9} \zeta_2 - \frac{746878}{9} \zeta_3 + \frac{146768}{9} \zeta_4 - \frac{1011088}{9} \zeta_2 \zeta_3 + 274432 \zeta_5 - \frac{484192}{9} \zeta_3 + \frac{356048}{3} \zeta_6 - 73728 \zeta_2 \zeta_5 + 76000 \zeta_3 \zeta_4 \right)
\]

\[
+ \frac{C_F^2 C_A^2}{729} \left( \frac{15086188}{729} - \frac{12535492}{729} \zeta_2 + \frac{3043898}{81} \zeta_3 + \frac{2522080}{81} \zeta_4 - \frac{2968640}{27} \zeta_2 \zeta_3 + \frac{1046528}{9} \zeta_5 - \frac{82592}{3} \zeta_3 \zeta_4 + \frac{30184}{3} \zeta_6 + 3072 \zeta_2 \zeta_5 + \frac{60944}{3} \zeta_3 \zeta_4 \right) + C_F C_A \left( \frac{28325071}{2187} + \frac{5761670}{243} \zeta_2 + \frac{867584}{27} \zeta_3 - \frac{150632}{9} \zeta_4 - \frac{119624}{9} \zeta_2 \zeta_3 - 49840 \zeta_5 - \frac{4664}{3} \zeta_2 \zeta_3 + 832 \zeta_2 \zeta_5 + 1440 \zeta_3 \zeta_4 + 3400 \zeta_7 + \frac{1}{12} f_{4, d_F^{abcd}, d_A}^{q} \right) + \frac{d_{F}^{abcd} d_{A}^{abcd}}{n_F} \left( -2 f_{4, d_F^{abcd}, d_A}^{q} \right)
\]

\[
+ \frac{C_F^3 n_f}{54} \left( \frac{80221}{27} - \frac{25744}{27} \zeta_2 + \frac{95936}{9} \zeta_3 - \frac{11492}{9} \zeta_4 + \frac{189824}{9} \zeta_2 \zeta_3 - \frac{130624}{3} \zeta_5 + \frac{106336}{3} \zeta_3 \zeta_4 + \frac{160840}{9} \zeta_6 + 4 b_{4, n_f c_F}^{q} \right) + C_F^2 C_A n_f \left( \frac{955285}{1458} + \frac{3057110}{729} \zeta_2 - 1222648 \zeta_3 + \frac{1261168}{9} \zeta_4 + \frac{306400}{9} \zeta_2 \zeta_3 - 376165 \zeta_5 + \frac{11728}{3} \zeta_2 \zeta_3 - \frac{164}{3} \zeta_6 + 4 b_{4, n_f c_F}^{q} \right)
\]

\[
+ C_F C_A^2 n_f \left( \frac{10761379}{2916} - \frac{2418814}{243} \zeta_2 - \frac{948884}{81} \zeta_3 + \frac{213280}{27} \zeta_4 + \frac{28064}{9} \zeta_2 \zeta_3 - 29552 \zeta_5 - \frac{9736}{9} \zeta_2 \zeta_3 - \frac{32930}{27} \zeta_6 - \frac{1}{12} b_{4, d_F^{abcd}, d_A}^{q} \right)
\]

\[
+ n_f \frac{d_{F}^{abcd} d_{A}^{abcd}}{n_F} \left( -768 - \frac{9088}{3} \zeta_2 + \frac{10624}{9} \zeta_3 + \frac{1600}{3} \zeta_4 - 256 \zeta_2 \zeta_3 + \frac{43520}{9} \zeta_5 - \frac{2432}{3} \zeta_5 + \frac{2368}{9} \zeta_6 + 4 b_{4, d_F^{abcd}, d_A}^{q} \right) + C_F^2 n_f \left( -\frac{142769}{729} - \frac{99184}{729} \zeta_2 + \frac{113456}{81} \zeta_3 + \frac{23200}{27} \zeta_4 \right)
\]

\[
+ \frac{79360}{27} \zeta_2 \zeta_3 + \frac{33056}{9} \zeta_5 + C_F C_A n_f \left( \frac{898033}{2916} + \frac{293528}{243} \zeta_2 + \frac{87280}{81} \zeta_3 - \frac{1744}{3} \zeta_4 \right)
\]

\[
- \frac{608}{9} \zeta_2 \zeta_3 + \frac{608}{3} \zeta_5 \right) + C_F n_f^3 \left( \frac{10432}{2187} - \frac{3200}{81} \zeta_2 - \frac{3680}{81} \zeta_3 + \frac{112}{9} \zeta_4 \right) ,
\]  (19)
where the quartic Casimirs are normalized by the dimension of the fundamental representation of the $SU(n_c)$ gauge group, $n_F = n_c$. As Eq. (6), this result is exact in the large-$n_c$ limit and has an amply sufficient numerical accuracy for all phenomenological applications in QCD.

The next-to-leading power threshold terms for the DY process can also be derived with the help of the corresponding physical evolution kernel $K_{qq}$, which exhibits the same simple form in Mellin $N$-space as Eq. (14) for the leading large-$N$ logarithms of the $N^{-1}$ contributions with the obvious replacement $C_A \to C_F$, see Ref. [9] for further details.

Finally, by combining our new result (6) with Eq. (2.13) in Ref. [9], and proceeding analogously with its Drell-Yan counterpart (19), we can derive the four-loop coefficient $D_4$ for the soft-gluon exponentiation of inclusive Higgs production via $ggF$ and DY lepton-pair production. The two results are, as expected, related by generalized Casimir scaling [24] which reduces to standard ‘numerical’ $C_A/C_F$ lower-order Casimir scaling in the their exact large-$n_c$ limit:

$$D_4 =$$

$$C_I C_A \left( -\frac{28325071}{2187} + \frac{5761670}{243} \right) \zeta_2 + \frac{867584}{27} \zeta_3 - \frac{150632}{9} \zeta_4 - \frac{119624}{9} \zeta_5 - \frac{49840}{9} \zeta_6 + \frac{41789}{9} \zeta_7 + \frac{1}{12} f_{d_4}^{abcd} d_{abcd}$$

$$- \frac{d_{abcd} d_{I}^{abcd}}{n_I} \left( 2 f_{d_4}^{abcd} d_{abcd} + n_f C_I C_A \left( \frac{10761379}{2916} - \frac{2418814}{243} \zeta_2 - \frac{948884}{81} \zeta_3 + \frac{213280}{27} \zeta_4 \right)$$

$$+ \frac{28064}{9} \zeta_2 \zeta_3 - \frac{29552}{27} \zeta_5 - \frac{9736}{9} \zeta_3^2 + \frac{32930}{27} \zeta_6 - \frac{1}{12} b_{\epsilon_4}^{abcd} d_{abcd} - \frac{2 b_{\epsilon_4}^{abcd} C_A}{n_f} - \frac{b_{\epsilon_4}^{abcd} C_A}{n_f} \right)$$

$$+ n_f C_I C_F C_A \left( \frac{2149049}{486} - \frac{56222}{27} \zeta_2 - \frac{8932}{9} \zeta_3 - \frac{113360}{27} \zeta_4 + \frac{3808}{9} \zeta_5 + \frac{21904}{9} \zeta_6 \right)$$

$$+ \frac{6800}{3} \zeta_2^2 - \frac{1436}{9} \zeta_6 + 4 b_{\epsilon_4}^{abcd} C_A \right) + n_f C_I C_F \left( -\frac{27949}{54} - 632 \zeta_2 + \frac{2240}{9} \zeta_3 + 668 \zeta_4$$

$$+ \frac{1024}{3} \zeta_2 \zeta_3 - \frac{7744}{3} \zeta_5 - 736 \zeta_3^2 + \frac{29336}{9} \zeta_6 + 4 b_{\epsilon_4}^{abcd} C_A \right)$$

$$+ \frac{d_{abcd} d_{I}^{abcd}}{n_I} \left( \frac{10624}{9} \zeta_3 + \frac{1600}{3} \zeta_4 - 256 \zeta_2 \zeta_3 + \frac{43520}{9} \zeta_5 - \frac{2432}{3} \zeta_2 \zeta_3 - \frac{2368}{9} \zeta_6 + 4 b_{\epsilon_4}^{abcd} d_{abcd} \right)$$

$$+ n_f^2 C_I C_A \left( -\frac{898033}{2916} + \frac{293528}{243} \zeta_2 + \frac{87280}{81} \zeta_3 - \frac{1744}{3} \zeta_4 - \frac{608}{9} \zeta_5 \right)$$

$$+ n_f^2 C_F \left( -\frac{110059}{243} + \frac{384}{27} \zeta_2 + \frac{10768}{81} \zeta_3 + \frac{160}{3} \zeta_4 - 256 \zeta_2 \zeta_3 + \frac{32}{3} \zeta_5 \right)$$

$$+ C I n_f^2 C_F \left( -\frac{10432}{2187} - \frac{3200}{81} \zeta_2 + \frac{3680}{81} \zeta_3 + \frac{112}{9} \zeta_4 \right)$$

$$+ C I n_f^2 \left( -\frac{10432}{2187} - \frac{3200}{81} \zeta_2 + \frac{3680}{81} \zeta_3 + \frac{112}{9} \zeta_4 \right)$$

with $C_I = C_F, d_{abcd}^{abcd} = d_{abcd}^{abcd}$ and $n_I = n_F$ for the DY case, and $C_I = C_A, d_{abcd}^{abcd} = d_{abcd}^{abcd}$ and $n_I = n_A$ for Higgs production. The lower-order coefficients can be found in Eqs. (33) - (35) of Ref. [8].

With this result, and the approximate values of Ref. [68] for the small effect of the five-loop cusp anomalous dimensions $A_5$, all ingredients are now available for extending the soft-gluon exponentiation to the next-to-next-to-next-to-leading logarithmic ($N^4$LL) accuracy. The corresponding function $g_5$ can be inferred from the DIS result in Eq. (2.9) of Ref. [11] as described below Eq. (3.6) of Ref. [10].
Using recent progress on related fourth-order quantities, we have been able to determine the final soft-gluon enhanced contribution (6) to the N^4LO coefficient function for inclusive Higgs-boson production in gluon-gluon fusion in the heavy-top limit, and the corresponding result (19) for the Drell-Yan process \( pp \rightarrow \gamma^* + X \). These results also fix the respective N^4LL coefficients \( D_4 \) for the soft-gluon exponentiation (20) which are related by the same fourth-order generalization of the well-known Casimir scaling observed before in the cusp anomalous dimensions, now completely known at this order [21]. Our results are exact in the limit of a large number of flavours \( n_c \). Their colour-factor decomposition in full QCD involves a few quantities which are known only numerically at this point. The resulting uncertainties are practically negligible as can be seen from the \( \ln N \) coefficient in Eq. (16) above.

We have employed the latter Mellin \( N \)-space results to add the N^4LO soft + virtual (SV) corrections to the known complete N^3LO results [3,4] for the LHC at 14 TeV. With the effect of the only uncomputed quantity, the soft-gluon coefficient \( g_{04} \) for this process, being well below 1%, we find that the cross sections are enhanced by 2.7% for the scale choice \( \mu_R = m_H \), while the results are almost unchanged for \( \mu_R = 0.5 m_H \). It should be noted that these values refer to the not entirely realistic case of an order-independent \( \alpha_s \)-value and PDFs at \( \mu = m_H \). The renormalization-scale variation, estimated using the interval \( 0.25 m_H \leq \mu_R \leq 2 m_H \), is reduced from about 5% at N^3LO to less than 3% at N^4LO. Based on similar calculations at lower orders, we definitely expect that difference between the present \( N \)-space SV approximation and the complete N^4LO coefficient function will amount to well below 1% of the total cross section.

As by-products of our analysis, we have derived the expression (10) for the four-loop gluon virtual anomalous dimension (and determined the corresponding purely Abelian contributions at five loops), and provided a sub-percent accurate value (11) for the hitherto unknown \( 1/\epsilon \) coefficient of the matter-independent contribution to the four-loop gluon form factor for which the \( n_f \)-terms have been recently computed in Ref. [18].

Acknowledgements

G.D. thanks M.C. Kumar and V. Ravindran for useful discussions. S.M. acknowledges useful communication with B. Mistlberger on Ref. [4]. The algebraic computations have been done with the latest version of the symbolic manipulation system FORM [70,71]. This work has been supported by the Deutsche Forschungsgemeinschaft (DFG) under grant number MO 1801/2-1, and by the COST Action CA16201 PARTICLEFACE supported by European Cooperation in Science and Technology (COST). The research of G.D. is supported by the DFG within the Collaborative Research Center TRR 257 (“Particle Physics Phenomenology after the Higgs Discovery”).

References

[1] CMS, A.M. Sirunyan et al., Eur. Phys. J. C79, 421 (2019), arXiv:1809.10733
[2] ATLAS, G. Aad et al., Phys. Rev. D101, 012002 (2020), arXiv:1909.02845
[3] C. Anastasiou, C. Duhr, F. Dulat, F. Herzog, B. Mistlberger, Phys. Rev. Lett. 114, 212001 (2015), arXiv:1503.06056
[40] T. Ahmed, M. Mahakhud, N. Rana, and V. Ravindran, Phys. Rev. Lett. 113, 112002 (2014), [arXiv:1404.0360v1](http://arxiv.org/abs/1404.0360v1) (version 1).

[41] G.P. Korchemsky, Mod. Phys. Lett. A4, 1257 (1989).

[42] S. Albino and R.D. Ball, Phys. Lett. B513, 93 (2001), [arXiv:hep-ph/0011133](http://arxiv.org/abs/hep-ph/0011133).

[43] S. Moch, J.A.M. Vermaseren, and A. Vogt, Phys. Lett. B625, 245 (2005), [arXiv:hep-ph/0508055](http://arxiv.org/abs/hep-ph/0508055).

[44] V. Ravindran, J. Smith, and W.L. van Neerven, Nucl. Phys. B704, 332 (2005), [arXiv:hep-ph/0408315](http://arxiv.org/abs/hep-ph/0408315).

[45] L.J. Dixon, L. Magnea, and G.F. Sterman, JHEP 08, 022 (2008), [arXiv:0805.3515](http://arxiv.org/abs/0805.3515).

[46] G. Falcioni, E. Gardi, and C. Milloy, JHEP 11, 100 (2019), [arXiv:1909.00697](http://arxiv.org/abs/1909.00697).

[47] T. van Ritbergen, J.A.M. Vermaseren, and S.A. Larin, Phys. Lett. B400, 379 (1997), [arXiv:hep-ph/9701390](http://arxiv.org/abs/hep-ph/9701390).

[48] A. Vogt, S. Moch, and J.A.M. Vermaseren, Nucl. Phys. B569, 129 (2004), [arXiv:hep-ph/0404111](http://arxiv.org/abs/hep-ph/0404111).

[49] F. Herzog, B. Ruijl, T. Ueda, J.A.M. Vermaseren, and A. Vogt, JHEP 02, 090 (2017), [arXiv:1701.01404](http://arxiv.org/abs/1701.01404).

[50] T. Luthe, A. Maier, P. Marquard, and Y. Schröder, JHEP 10, 166 (2017), [arXiv:1709.07718](http://arxiv.org/abs/1709.07718).

[51] K.G. Chetyrkin, G. Falcioni, F. Herzog and J.A.M. Vermaseren, JHEP 10, 179 (2017), [arXiv:1709.08541](http://arxiv.org/abs/1709.08541).

[52] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. 118, 082002 (2017), [arXiv:1606.08659](http://arxiv.org/abs/1606.08659).

[53] S. Moch and A. Vogt, JHEP 11, 099 (2009), [arXiv:0909.2124](http://arxiv.org/abs/0909.2124).

[54] G. Soar, S. Moch, J.A.M. Vermaseren, and A. Vogt, Nucl. Phys. B832, 152 (2010), [arXiv:0912.0369](http://arxiv.org/abs/0912.0369).

[55] G. Grunberg and V. Ravindran, JHEP 10, 055 (2009), [arXiv:0902.2702](http://arxiv.org/abs/0902.2702).

[56] G. Grunberg, Phys. Lett. B 687, 405 (2010), [arXiv:0911.4471v5](http://arxiv.org/abs/0911.4471v5).

[57] A.A. Almasy, G. Soar and A. Vogt, JHEP 03, 030 (2011), [arXiv:1012.3352](http://arxiv.org/abs/1012.3352).

[58] A.A. Almasy, N.A. Lo Presti and A. Vogt, JHEP 01, 028 (2016), [arXiv:1511.08612](http://arxiv.org/abs/1511.08612).

[59] V. Del Duca, E. Laenen, L. Magnea, L. Vernazza, and C.D. White, JHEP 11, 057 (2017), [arXiv:1706.04018](http://arxiv.org/abs/1706.04018).

[60] N. Bahjat-Abbas, D. Bonocore, J. Sinninghe Damsté, E. Laenen, L. Magnea, L. Vernazza and C.D. White, JHEP 11, 002 (2019), [arXiv:1905.13710](http://arxiv.org/abs/1905.13710).

[61] M. Beneke, M. Garny, S. Jaskiewicz, R. Szafron, L. Vernazza and J. Wang, JHEP 01, 094 (2020), [arXiv:1910.12685](http://arxiv.org/abs/1910.12685).

[62] M. Beneke, A. Broggio, S. Jaskiewicz and L. Vernazza, [arXiv:1912.01585](http://arxiv.org/abs/1912.01585).

[63] S. Alekhin, J. Blümlein, S. Moch, and R. Placakyte, Phys. Rev. D96, 014011 (2017), [arXiv:1701.05838](http://arxiv.org/abs/1701.05838).

[64] L.A. Harland-Lang, A.D. Martin, P. Motylinski, and R.S. Thorne, Eur. Phys. J. C75, 204 (2015), [arXiv:1412.3989](http://arxiv.org/abs/1412.3989).

[65] A. Buckley et al., Eur. Phys. J. C75, 132 (2015), [arXiv:1412.7420](http://arxiv.org/abs/1412.7420).

[66] F. Dulat, A. Lazopoulos, and B. Mistlberger, Comput. Phys. Commun. 233, 243 (2018), [arXiv:1802.00827](http://arxiv.org/abs/1802.00827).

[67] F. Herzog, S. Moch, B. Ruijl, T. Ueda, J. Vermaseren, A. Vogt, Phys. Lett. B790, 436 (2019), [arXiv:1812.11818](http://arxiv.org/abs/1812.11818).

[68] A. Accardi et al., Eur. Phys. J. C76, 471 (2016), [arXiv:1603.08906](http://arxiv.org/abs/1603.08906).

[69] J.A.M. Vermaseren, [arXiv:math-ph/0010025](http://arxiv.org/abs/math-ph/0010025).

[70] B. Ruijl, T. Ueda, and J.A.M. Vermaseren, (2017), [arXiv:1707.06453](http://arxiv.org/abs/1707.06453).