Resilience Family of Receiver Operating Characteristic Curves

Ruhul Ali Khan

Abstract—A new semiparametric model of the receiver operating characteristic (ROC) curve based on the resilience family or proportional reversed hazard family is proposed, which is an alternative to the existing models. The resulting ROC curve and its summary indices (such as area under the curve and Youden index) have simple analytic forms. The partial likelihood method is applied to estimate the ROC curve. Moreover, the estimation methodologies of the resilience family of the ROC curve have been developed based on area under the curve estimators exploiting Mann–Whitney statistics and the Rojo approach. A simulation study has been carried out to assess the performance of all considered estimators. Real data from the American National Health and Nutrition Examination Survey has been analyzed in detail based on the proposed model and the usual binormal model prevalent in the literature. Real data in the context of brain injury-related biomarkers are also analyzed in order to compare our model with the Lehmann family of the ROC curves. Finally, we show that the proposed model may be applicable in the misspecification scenario through the Ducheme muscular dystrophy data.

Index Terms—Area under curve (AUC), Mann–Whitney statistics, receiver operating characteristic (ROC) curve, semiparametric model.

I. INTRODUCTION

In statistical decision theory, one of the important aspects is to construct a reasonable classification rule based on some characteristics. The receiver operating characteristic (ROC) curve represents the performance of a binary classifier. While the ROC curve originated in the analysis of radar signals (see[13]), it has been consequently adopted and extended in diversified fields, such as diagnostic medicine, psychology, banking, finance, biometrics, forensic sciences, and many more (see [35], [26], and [49] for an overview).

Suppose there are two populations—a “positive” population $P$ and a “negative” population $N$ together with a classification rule assumed to be continuous function $S(X)$ of the random vector $X$ of variables measured on each individual. Let $t$ be the value of the threshold $t \in \mathbb{R}$ such that an individual is allocated to a population $P$ if the classification score $s(x)$ exceeds $t$ and, otherwise, to population $N$ where $x$ is the observed value of $X$. Based on this classification rule, four combinations of true disease status and diagnostic results may occur, which are as follows:

1) false positive (FP),
2) true negative (TN),
3) false negative (FN), and
4) true positive (TP) status.

Among these combinations, FP and TP status are directly used in ROC formulation. The true positive rate (TPR) is the probability that an individual from positive group is correctly classified (sensitivity), and the false positive rate (FPR) is the probability that an individual is misclassified from negative group (1 - specificity). Thus, $FPR(t) = p(S > t|N) = 1 - F_0(t) = F_0(t)$ and $TPR(t) = p(S > t|P) = 1 - F(t) = F(t)$ when $S$ possesses distribution function $F_0$ and $F$ for the population $N$ and $P$, respectively. Then, the ROC curve graphs the tradeoff between $F(t)$ and $F_0(t)$ or equivalently is a plot of

$$R(t) = 1 - F(F_0^{-1}(1 - t))$$

against $t$ for $t \in [0, 1]$ and $F_0^{-1}(u) = \inf \{x : F_0(x) \geq u\}$. So, the ROC curve is defined as a plot of TPRs against FPRs resulting from a continuous classification function for various threshold values. In statistical testing hypotheses terminology, ROC curve is a plot of type I error on the $x$-axis and the corresponding power on the $y$-axis by varying the decision threshold values. There are several indices in the literature for summarizing the information given by an ROC curve. The area under the ROC curve, denoted by area under curve (AUC), is perhaps the most widely used summary index for the ROC curve and is defined by

$$AUC = \int_0^1 R(t) dt.$$  

Note that $TPR(t) \geq FPR(t)$. Hence, the plot of an ROC curve lies above the diagonal of the unit square, and consequently, the AUC takes value between 0.5 and 1, where 0.5 and 1 represent random classification and perfect classification, respectively. The Youden index, denoted by $J$, depicts the maximum difference between the TPR and the FPR, i.e.,

$$J = \max_t \{TPR(t) - FPR(t)\}$$

$$= \max_t \{sensitivity(t) + specificity(t) - 1\}$$

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since TPR = sensitivity and FPR = 1 - specificity. The threshold corresponding to the Youden index J is often taken to be the optimal classification threshold (see [48]). In the ROC curve analysis, the assumption of the binormal model is common where $F_0$ and $F$ are assumed to be normal distribution (see [10], and the references therein) since the ROC curve remains the same if the classification scores undergo a strictly increasing transformation. There are several parametric models for ROC curve in the literature, such as bi-gamma, bi-beta, bi-logistic, bi-lognormal, and bi-Rayleigh (see [11] for an overview). One of the simplest methods for estimating the ROC curve is the use of empirical ROC curve, which can be obtained by plugging in empirical estimates of $F_0$ and $F$ into (1). The empirical ROC curve uniformly converges to the theoretical curve and preserves many interesting properties of the empirical distribution function (see [17]). Zou et al. [50] used kernel density estimator for $F_0$ and $F$ to estimate the ROC curve, which overcomes the lack of smoothness of the empirical estimator. Lehmann family of ROC was proposed by Gönen and Heller [12], where they assumed that $F = F_0^\gamma$ for $\gamma \in (0,1)$. In the last three decades, several researchers have proposed so many methods for estimating the ROC curve and its functionals.

In this article, we propose a new semiparametric model of the ROC curve based on the resilience family or proportional reversed hazard, which is an alternative to the existing model, such as binormal model, empirical ROC, Lehmann family of ROC, nonparametric ROC, and bi-Rayleigh. The resulting ROC curve and its corresponding AUC and Youden index have simple analytic forms. This model performs very well when binormality assumption and frailty assumption fail. Moreover, the proposed model can also handle the misspecification case. We estimate the proposed ROC curve using partial likelihood method, Mann-Whitney method, and the Rojo approach. We analyze three real data with the existing models, and important implications are presented.

The rest of this article is organized as follows. In Section II, the ROC curve based on the resilience parameter family or alternatively, a proportional reverse hazards family, with underlying distribution $F_0$, has been proposed, which is an alternative to the existing models. The resulting ROC curve and its summary indices (such as AUC and Youden index) have simple analytic forms. Section III deals with estimation methodology of the ROC curve based on the proposed model. The partial likelihood method is applied to estimate the ROC curve. Moreover, the estimation methodologies of the resilience family of the ROC curve have been developed based on AUC estimators exploiting Mann-Whitney statistics, and the Rojo approach. Estimation procedure of the Youden index is also discussed. In Section IV, a simulation study has been carried out in order to assess the performance of all considered estimators. Section V deals with a brief discussion about the verification of the assumed model has been made. In Section VI, a real data has been analyzed based on the proposed model and existing models, and some remarks are made. Further, another real data in the context of brain injury-related biomarkers are also analyzed in order to compare our model with the Lehmann family of the ROC curves. We showed that the proposed model may be applicable in the misspecification scenario through Duchene muscular dystrophy data. Finally, Section VII concludes this article and provides some prospects of the present study.

II. PROPOSED MODEL

One of the important properties of the ROC curve is that the curve remains unaltered if the classification scores undergo a strictly increasing transformation. Based on the previous discussion, suppose $F_0$ and $F$ are associated with absolutely continuous random variables $X$ and $Y$ with respect to Lebesgue measure on the real line (after some suitable transformation, if necessary). If $F(t)$ is defined as

$$F(t) = [F_0'(t)]^\theta \quad \theta > 0$$

then $\theta$ is called a resilience parameter and $\{F(\cdot|\theta), \theta > 0\}$ is a resilience parameter family with underlying distribution $F_0$ (see [32]). It is interesting to note that the assumption (4) corresponds to the proportional reversed hazard rates (PRHR) assumption of the form $h(t) = \theta$, where $h_0 = f_0/F_0$ and $h = f/F$ are reversed hazard rate (RHR) of the random variables $X$ and $Y$, respectively (see [45], [5], [23], and [14]). For a very small interval, the probability of failure in the interval given failure before the end of the interval is approximated by the product of the RHR and the length of the interval. Kalbfleisch and Lawless [22] showed that RHR is important in the estimation of the survival function under left-censored data. Block et al. [6] discussed some properties of the RHR function while Nanda and Shaked [34] derived some interesting results, which compare order statistics in the RHR orders. Kundu and Gupta [27] provided two simple characterizations of the PRHR class of distributions based on some conditional expectation and conditional variance. Discrete life distributions with decreasing RHR were studied by Nanda and Sengupta [33]. Sengupta and Nanda [42] pointed out that PRHR is applicable where the proportional hazard (PH) model is inappropriate. Wang et al. [46] developed an estimation methodology for the confidence intervals of the family of PRHR distributions based on lower record values. A proportional cause-specific reversed hazards model was introduced by Sankaran and Anjana [41], and Fallah et al. [9] proposed statistical inference for component lifetime distribution from coherent system lifetime under a PRHR model. Baratnia and Doostparast [4] proposed a new statistical method for one-way classification, which is based on the RHR function of the response variable, and Khan et al. [25] studied the relationship between RHR and mean inactive time function. Ever since the inception of the RHR, due to its diverse applications, many researchers have contributed to it, and a vast number of publications on this topic have appeared. Here, I have tried only to mention some significant works regarding RHR with a brief literature survey. For application in medical studies of RHR, one may refer to [1] or [21].

There exist many well-known distributions, which belong to the resilience parameter family. Some of these are the Burr X distribution (see [7]), generalized exponential distribution (see [15]), generalized Rayleigh distribution (see [28]), generalized Gompertz distribution (see [8]), and exponential-type distribution (see [30]).
and the corresponding optimum cutoff point is \( t^* = F^{-1}_0\left(\frac{\theta}{1+\theta}\right) \).

The next section deals with estimation methodologies for \( \theta \), \( \tau \), and \( J \).

III. ESTIMATION OF THE RESILIENCE FAMILY OF ROC CURVES

Suppose \( X_m = (X_1, X_2, \ldots, X_m) \) and \( Y_n = (Y_1, Y_2, \ldots, Y_n) \) are independent samples from \( F_0 \) and \( F \), respectively, where \( F_0 \) and \( F \) are unknown. Let \( F_{0m} \) and \( F_n \) be the empirical distribution functions of \( X_m \) and \( Y_n \). Now, the estimation methodologies of \( \theta \) will be presented in the sequel.

A. Maximum Partial Likelihood Estimator

Let \( W_1 \leq W_2 \leq \cdots \leq W_{m+n} \) be the ordered observations combining \( X_i \), \( i = 1, 2, \ldots, m \), and \( Y_i \), \( i = 1, 2, \ldots, n \). Suppose \( x_p \) of the \( X_i \)'s and \( y_p \) of the \( Y_i \)'s are less than or equal to \( W_1 \), \( p = 1, 2, \ldots, m + n \). Now, the proportional reversed hazard regression model will be introduced to study the relationship in the presence of covariates. The proportional reversed hazard model for \( i \)th individual is defined by

\[
h_i(t | Z) = h_0(t) \exp(\beta z_i)
\]

where \( e^\beta = \theta \) and \( z_i \in \{0, 1\} \). Since the model considers the simple case of comparing positive group \( P \) and negative group \( N \), the covariate \( z_i \) is either 1 (if the individual is in the positive group) or 0 (if the individual is in the negative group). Now, note that

\[
\sum_{C_i(t)} h_0(t) \exp(\beta z_i) = \frac{\exp(\beta z_i)}{\sum_{C_i(t)} \exp(\beta z_i)}
\]

where \( C_i(t) \) is number of \( W_i \)'s less or equal to \( t \). Thus, the partial likelihood can be written as

\[
E L(\theta, t) = \prod_{i=1}^{m+n} \frac{\exp(\beta z_i)}{\sum_{C_i(t)} \exp(\beta z_i)}.
\]

After some simplification log of the partial likelihood yields

\[
l(\theta, t) = n \log \theta - \sum_{i=1}^{m+n} \log (x_i + y_i \theta).
\]

Thus, as a result, the estimate of the parameter \( \theta \) of the resilience ROC curve, denoted by \( \hat{\theta} \), can be obtained by solving the following equation numerically:

\[
\frac{n}{\hat{\theta}} - \sum_{i=1}^{m+n} \frac{y_i}{x_i + y_i \theta} = 0.
\]

For an example in \( R \), one may refer to “nleqslv” package (see [16]) to solve the equation since the solution of the abovementioned equation does not have an explicit form.

Now, we will proceed to test \( H_0 : \theta = 1 \) against \( \theta \neq 1 \) and construct \( 100(1 - \alpha) \% \) confidence interval based on the Wald’s test. The observed information \( I(\hat{\theta}) \) yields from

\[
I(\theta) = -\frac{\partial^2 l}{\partial \theta^2} = \frac{n}{\hat{\theta}^2} - \sum_{i=1}^{m+n} \frac{y_i^2}{(x_i + y_i \theta)^2}
\]
by plugging in $\hat{\theta}^{pl}$. The variance of $\hat{\theta}^{pl}$ is approximately $1/I(\hat{\theta}^{pl})$, and the standard error, $\text{S.E.}(\hat{\theta}^{pl}) = 1/\sqrt{I(\hat{\theta}^{pl})}$. Now, we use $\text{S.E.}(\hat{\theta}^{pl})$ to construct a normalized test statistic $Z_w = \frac{\hat{\theta}^{pl} - \theta}{\text{S.E.}(\hat{\theta}^{pl})}$ and reject $H_0$ if $|Z_w| > Z_{\alpha/2}$, where $Z_\alpha$ is the upper $\alpha$ quantile of the standard normal distribution. Thus, the $100(1-\alpha)\%$ confidence interval is given by $(\hat{\theta}^{pl} - Z_{\alpha/2} \cdot \text{S.E.}(\hat{\theta}^{pl}), \hat{\theta}^{pl} + Z_{\alpha/2} \cdot \text{S.E.}(\hat{\theta}^{pl}))$.

Remark 3.1: The estimator of AUC, $\hat{\tau}^{pl}$ is $\frac{\hat{\theta}^{pl}}{1+\hat{\theta}^{pl}}$ and using delta method, we obtain the $100(1-\alpha)\%$ confidence interval of $\tau$, which is given by $(\hat{\tau}^{pl} - Z_{\alpha/2} \cdot \frac{\text{S.E.}(\hat{\theta}^{pl})}{1+\hat{\theta}^{pl}}, \hat{\tau}^{pl} + Z_{\alpha/2} \cdot \frac{\text{S.E.}(\hat{\theta}^{pl})}{1+\hat{\theta}^{pl}})$.

Remark 3.2: From (9), the estimator for the Youden index is given by

$$J^{pl} = \left( 1 - \frac{1}{\hat{\theta}^{pl}} \right) \frac{\hat{\theta}^{pl}}{1+\hat{\theta}^{pl}}. \quad (16)$$

One can also find confidence interval for the Youden index using delta method.

Remark 3.3: It is possible that the calculated value of $\hat{\theta}^{pl}$ may be found to be less than 1 when $\theta$ is close to 1 and the sample size is small. In this situation, one can consider $\min \{ \hat{\theta}^{pl}, 1 \}$ as an estimator to obtain smaller bias and variance.

B. Estimation Based on the Mann–Whitney Statistics

The expression given in (7) suggests to propose an estimator of the AUC ($\tau$) using Mann–Whitney statistics and derive its asymptotic distribution exploiting U-statistics theory. Let the kernel be $h(x,y) = I(x < y)$, with expectation $\tau = P(X < Y)$ where $I(x < y) = 1$ if $x < y$, 0 otherwise. Then, the corresponding U-statistic is $\hat{\tau}^{MW} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} h(X_i, Y_j)$. \( \tag{17} \)

Assume that there is no ties. Then, $\hat{\tau}^{MW}$ is the Mann–Whitney statistics, which is an unbiased nonparametric estimator of $\tau$. Now, one can obtain the following theorem by considering the projection of $\hat{\tau}^{MW}$ viewed as a two-sample U-statistic [43, p. 193]. In this context, some of the examples are [3] and [29].

Theorem 3.1: Let $m, n \to \infty$ in such a way that $m/(m+n) \to p$ and $0 < p < 1$. Suppose that $\sigma_{10}^2 = P(X < Y, X' < Y') - P(X < Y) < \infty$ and $\sigma_{01}^2 = P(X < Y, Y' < Y') - P(X < Y) < \infty$. Then,

$$\sqrt{m+n} \left[ \hat{\tau}^{MW} - \tau \right] \xrightarrow{d} \mathcal{N}(0, \sigma^2) \text{ as min} \{m,n\} \to \infty$$

where $\sigma^2 = \frac{1}{p^2} \sigma_{10}^2 + \frac{1}{1-p^2} \sigma_{01}^2$.

Remark 3.4: It is worthwhile noting that the abovementioned theorem can also be obtained by applying [17, Th. 2.3].

In the context of the proposed model, the simplified expressions of $\sigma_{10}^2$ and $\sigma_{01}^2$ are as follows:

$$\sigma_{10}^2 = \frac{\theta}{(2 + \theta)(1 + \theta)^2} \quad \text{and} \quad \sigma_{01}^2 = \frac{\theta^2}{(1 + 2\theta)(1 + \theta)^2}.$$

Thus, Theorem 3.1 reduces to the following theorem.

Theorem 3.2: Let $m, n \to \infty$ in such a way that $m/(m+n) \to p, 0 < p < 1$. Then, for $\sigma^2 > 0$

$$\sqrt{m+n} \left[ \hat{\tau}^{MW} - \tau \right] \xrightarrow{d} \mathcal{N}(0, \sigma^2) \text{ as min} \{m,n\} \to \infty$$

where $\sigma^2 = \frac{1}{p^2} \sigma_{10}^2 + \frac{1}{1-p^2} \sigma_{01}^2$.

Remark 3.5: The $100(1-\alpha)\%$ confidence interval of $\tau$ is given by

$$\left[ \hat{\tau}^{MW} - \frac{Z_{\alpha/2}}{\sqrt{m+n}} \cdot \sigma^2(\hat{\tau}^{MW}), \hat{\tau}^{MW} + \frac{Z_{\alpha/2}}{\sqrt{m+n}} \cdot \sigma^2(\hat{\tau}^{MW}) \right]$$

since $\hat{\tau}^{MW}$ is a consistent estimator of $\tau$.

From relation (8), the estimator of $\theta$ will be proposed by plug-in $\hat{\tau}^{MW}$. Thus, the estimator of $\theta$ based on Mann–Whitney statistics is given by

$$\hat{\theta}^{MW} = \frac{1}{1 - mn/\sum_{i=1}^{m} \sum_{j=1}^{n} h(X_i, Y_j)}.$$

Remark 3.6: An application of Jensen inequality, and the fact that $g(x) = \frac{x}{1+x}$ is a convex function on $(0,1)$ yields $E(\hat{\theta}^{MW}) \geq \theta$. Thus, $\hat{\theta}^{MW}$ is a biased estimator for $\theta$ but $\hat{\tau}^{MW}$ is an unbiased estimator for $\tau$.

Now, an application of delta method yields the following theorem which establishes the asymptotic normality of $\hat{\theta}^{MW}$.

Theorem 3.3: Under the assumption of Theorem 3.2,

$$\sqrt{m+n} \left[ \hat{\theta}^{MW} - \theta \right] \xrightarrow{d} \mathcal{N}(0, \sigma^2) \text{ as min} \{m,n\} \to \infty$$

where $\sigma^2 = \frac{1}{p} \theta(1+\theta)^2 + \frac{1}{1-p} \theta^2(1+\theta)^2$.

Remark 3.7: Similarly as in Remark 3.5, the $100(1-\alpha)\%$ confidence interval of $\theta$ is given by

$$\left[ \hat{\theta}^{MW} - \frac{Z_{\alpha/2}}{\sqrt{m+n}} \cdot \sigma^2(\hat{\theta}^{MW}), \hat{\theta}^{MW} + \frac{Z_{\alpha/2}}{\sqrt{m+n}} \cdot \sigma^2(\hat{\theta}^{MW}) \right]$$

Remark 3.8: In this context, the proposed estimator for the Youden index $J^{MW}$ can be obtained by replacing $\hat{\theta}^{pl}$ by $\hat{\theta}^{MW}$ in the right-hand side expression of (16). Moreover, an application of delta method can also produce asymptotic normality and confidence interval for this estimator. For the sake of brevity, the presentation of the asymptotic normality theorem has been avoided.

C. Estimation Based on Rojo Approach

The resilience family of ROC curve satisfies $X \leq_{st} Y$, i.e., $F_0 \geq F$. This motivates us to estimate $F_0$ and $F$ by imposing order restricted condition. In this context, Lo [31] proposed the estimators of $F_0$ and $F$ as $F_0^{LU} = \max \{ F_0^{UL}, F_n \}$ and $F^{LU} = \min \{ F_0^{UL}, F_n \}$, respectively, since the estimators satisfy the constraint of interest. However, Rojo [40] pointed out some drawbacks of the estimators given in [31] and proposed new estimators. Thus, the estimators of $F_0$ and $F$ based on Rojo approach will be used in this section.
From (7), it is easy to obtain \( \tau = \int_0^1 F_0(F^{-1}(t))dt \) and the proposed estimator for \( \tau \) is given by
\[
\hat{\tau}^R = \int_0^1 F_{0mn}(F_{mn}^{-1}(t)) dt = \int_0^\infty F_{0mn}(t)dF_{mn}(t) \quad (18)
\]
where
\[
F_{0mn}(t) = \max\{F_0(t), P_{mn}(t)\}
\]
and \( P_{mn} \) is the empirical cumulative distribution function based on the combined samples, i.e.,
\[
P_{mn}(t) = \frac{m}{m+n} F_0(t) + \frac{n}{m+n} F_n(t).
\]

Now, the following theorem can be obtained using similar argument given in the proof of [19, Th. 2].

**Theorem 3.4:** Under the assumption of Theorem 3.2
\[
\sqrt{m+n} \left[ \hat{\tau}^R - \tau \right] \overset{d}{\to} \mathcal{N}(0, \sigma^2(\tau)) \quad \text{as} \quad \min(m, n) \to \infty
\]
where \( \sigma^2(\tau) = \frac{1}{m} \left(\frac{\tau(2-\tau)}{(2-\tau)^2} + \frac{1}{2} \right) \).

From relation (8), the estimator of \( \theta \) based on Rojo approach is given by \( \hat{\theta}^R = \frac{\hat{\tau}^R}{\tau} \), and an application of delta method yields the following theorem.

**Theorem 3.5:** Under the assumption of Theorem 3.3
\[
\sqrt{m+n} \left[ \hat{\theta}^R - \theta \right] \overset{d}{\to} \mathcal{N}(0, \sigma^2(\theta)) \quad \text{as} \quad \min(m, n) \to \infty
\]
where \( \sigma^2(\theta) = \frac{1}{m} \left(\frac{\theta(1+\theta)^2}{(1+\theta)^2} + \frac{1}{2} \right) \).

**Remark 3.9:** The 100\((1-\alpha)\)% confidence interval of \( \tau \) and \( \theta \) are given by
\[
\left[ \hat{\tau}^R - \frac{Z_{\alpha/2}}{\sqrt{m+n}} \cdot \sigma^2(\hat{\tau}^R), \hat{\tau}^R + \frac{Z_{\alpha/2}}{\sqrt{m+n}} \cdot \sigma^2(\hat{\tau}^R) \right]
\]
and
\[
\left[ \hat{\theta}^R - \frac{Z_{\alpha/2}}{\sqrt{m+n}} \cdot \sigma^2(\hat{\theta}^R), \hat{\theta}^R + \frac{Z_{\alpha/2}}{\sqrt{m+n}} \cdot \sigma^2(\hat{\theta}^R) \right]
\]
respectively.

**Remark 3.10:** In this context also, the proposed estimator for the Youden index \( \hat{J}^R \) can be obtained by replacing \( \hat{\theta}^R \) by \( \hat{\theta}^R \) in the right-hand side expression of (16).

**IV. SIMULATION STUDY**

In this section, the performance of the three proposed estimators will be investigated by means of a simulation study. For this purpose, we choose the well-known generalized exponential distribution (GED), which was introduced by Gupta and Kundu [15]. The cumulative distribution function (cdf) of GED is given by
\[
F(t) = (1 - e^{-\lambda t})^{\theta}, \quad t > 0 \quad (19)
\]
where \( \lambda > 0 \) and \( \theta > 0 \) are the scale and resilience parameters, respectively, and is denoted by GED(\( \lambda, \theta \)). Note that GED belongs to the resilience parameter family with underlying distribution \( F_0(t) = 1 - e^{-\lambda t} \), i.e., the exponential distribution. The values of \( \theta = 2, 4, 6 \) and \( \lambda = 1 \) were considered for the simulation study. For each value of the parameter \( \theta \), we generate random observations of size \((m, n) = (60, 60), (60, 80), \) and \((60, 100)\) from GED(1, 1) and GED(1, \( \theta \)). The proposed estimators were calculated for each pair of \((m, n)\). All the simulation study is performed using R (see [38]) on PC platform, and the results are reported based on 10 000 replications. The calculated values of \( \tau \) are 0.6667, 0.8, and 0.8571 and \( \lambda \) are 0.25, 0.4725, and 0.5824 corresponding to \( \theta = 2, 4, \) and 6.

In Table I, performance of different methods is investigated in terms of average value, standard deviation (SD), root mean square error (RMSE), and coverage probability.

**TABLE I**

| \( \theta \) | \( (m, n) \) | Method | Avg(\( \hat{\theta} \)) | SD(\( \hat{\theta} \)) | RMSE(\( \hat{\theta} \)) | Coverage probability | Avg(\( \hat{J} \)) | Avg(\( \hat{J}^R \)) |
|----------------|----------------|--------|-------------------------|----------------|------------------------|----------------------|----------------|----------------|
| \( (60, 60) \) | MW             | 2.0712 | 0.4835                  | 0.4887        | 0.9432                 | 0.6744               | 2.0672         | 0.4544         |
|                | Rojo           | 2.0759 | 0.4795                  | 0.4855        | 0.9482                 | 0.6749               | 2.0663         | 0.4564         |
|                |                |        |                         |               |                        |                      |                |                |
| \( (60, 80) \) | MW             | 2.0450 | 0.3872                  | 0.3898        | 0.9433                 | 0.6716               | 2.0529         | 0.4587         |
|                | Rojo           | 2.0663 | 0.4516                  | 0.4564        | 0.9389                 | 0.6735               | 2.0607         | 0.4584         |
|                |                |        |                         |               |                        |                      |                |                |
| \( (60, 100) \)| MW             | 2.0390 | 0.3696                  | 0.3716        | 0.9463                 | 0.6710               | 2.0575         | 0.4340         |
|                | Rojo           | 2.0607 | 0.4316                  | 0.4358        | 0.9355                 | 0.6730               | 2.0675         | 0.4378         |

Results are reported based on 10 000 replications for each GED model and method.
square error (RMSE), and coverage probability of \( \theta \). The reported coverage probabilities are based on significance level \( \alpha = 0.05 \). The estimated values of AUC and Youden index are also presented. For GED model, observe that the Partial likelihood method works slightly better than other methods and all the estimators overestimate the resilience parameter \( \theta \). The behavior of the estimators based on Mann–Whitney statistics and Rojo approach are almost same in this case. Another interesting observation is that all the estimators perform well for small values of \( \theta \).

Similarly, we also carried out a simulation study by considering two parameter Burr X distribution as a resilience parameter family with underlying distribution \( F_0(t) = 1 - e^{-(\lambda t)^2}, t > 0 \), i.e., the Rayleigh distribution with scale parameter \( \lambda > 0 \) [denoted by Rayleigh(\( \lambda \))]. Now, the cdf of the Burr X distribution is given by

\[
F(t) = \left( 1 - e^{-(\lambda t)^2} \right)^\theta, \quad t > 0
\]

and is denoted by Burr(\( \lambda, \theta \)). Now, the values of resilience parameter \( \theta = 3, 5, 7 \) and \( \lambda = 1 \) are considered for the simulation study. Random observations of size \((m, n) = (60, 60), (60, 80), \) and \((60, 100)\) were generated from (Rayleigh(1), Burr(1, \( \theta \))) for each value of the parameter \( \theta \). The proposed estimators were calculated for each pair of \((m, n)\). The calculated values of \( \tau \) are 0.7500, 0.8333, and 0.8750 and \( J \) are 0.3849, 0.5350, and 0.6197 corresponding to \( \theta = 3, 5, \) and 7, respectively. All the estimated values are in terms of average value, SD, RMSE, and coverage probability of \( \theta \), and are tabulated in Table II. In Table II, the performance of different methods are same as that of Table I.

V. GRAPHICAL REPRESENTATION FOR PRHR ASSUMPTION

This section deals with a graphical representation for checking PRHR assumption since it is necessary to verify the PRHR assumption for the samples in order to analyze real datasets. Suppose \( X_m = (X_1, X_2, \ldots, X_m) \) and \( Y_n = (Y_1, Y_2, \ldots, Y_n) \) are independent samples from \( F_0 \) and \( F \), respectively, where \( F_0 \) and \( F \) are unknown. Now, from the proposed model given in (6), one can recall that \( F(t) = [F_0(t)]^\theta \) and \( F_0(t) \geq F(t) \). Note that \( F(t) = [F_0(t)]^\theta \) implies \( \log \left( -\log (F(t)) \right) - \log \left( -\log (F_0(t)) \right) = \log \theta \). Let \( F_0(m) \) and \( F_n \) be the empirical distribution functions of \( X_m \) and \( Y_n \). At first, the verification of the usual stochastic dominance between \( F_0(m) \) and \( F_n \) via empirical plots should be made. After observing \( F_0(m) \geq F_n \), one should proceed to observe the plots of \( -\log (F_0(m)) \) and \( -\log (F_n) \), denoted by log–log plot. If the plot shows almost same difference between \( -\log (F_0(m)) \) and \( -\log (F_n) \), then the data may support the assumption given in (4) graphically.

VI. ANALYSIS OF REAL DATA

In the next sections, three real data have been analyzed in the sequel. Before going to analyze each real data, we will verify the PRHR assumption for the samples based on the graphical method discussed in the previous section.

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**Table II**

| \( \theta \) | Method | \( \text{Avg}(\hat{\theta}) \) | SD(\( \hat{\theta} \)) | RMSE(\( \hat{\theta} \)) | Coverage probability \( \hat{\tau} \) | \( \hat{J} \) |
|---|---|---|---|---|---|---|
| \((60, 60)\) | PI | 3.1089 | 0.6810 | 0.6896 | 0.9498 | 0.7566 | 0.3961 |
| | MW | 3.1315 | 0.7846 | 0.7955 | 0.9469 | 0.7580 | 0.3984 |
| | Rojo | 3.1336 | 0.7836 | 0.7948 | 0.9480 | 0.7581 | 0.3986 |
| \((60, 80)\) | PI | 3.0864 | 0.6301 | 0.6360 | 0.9478 | 0.7553 | 0.3939 |
| | MW | 3.1193 | 0.7474 | 0.7568 | 0.9375 | 0.7572 | 0.3972 |
| | Rojo | 3.1210 | 0.7466 | 0.7563 | 0.9380 | 0.7573 | 0.3974 |
| \((60, 100)\) | PI | 3.0783 | 0.5990 | 0.6041 | 0.9483 | 0.7548 | 0.3930 |
| | MW | 3.1098 | 0.7260 | 0.7342 | 0.9290 | 0.7567 | 0.3962 |
| | Rojo | 3.1113 | 0.7252 | 0.7337 | 0.9299 | 0.7568 | 0.3964 |

Results are reported based on 10 000 replications for each Burr X distribution and method.

A. Diabetes Data From NHANES

In this section, data from the American National Health and Nutrition Examination Survey (NHANES) in 2009–2010 has been analyzed. The data are available in the “NHANES” package of R statistical software [37]. The NHANES data include 75 variables with 5000 individuals of all ages. Here, “Age” as a demographic variable, body mass index (BMI) as a physical measurement, total cholesterol (TotChol), and diabetes as health variables have been considered in order to apply logistic regression as a binary classifier where diabetes is the qualitative response and the predictors are “age,” “bmi,” and “totchol.” After removing “NA(s)” in “Age,” “BMI,” “TotChol,” and “Diabetes,”

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[1][Online]. Available: http://www.cdc.gov/nchs/data/series/sr_02/sr02_162.pdf
the data contain information of 4209 individuals. Here, the data arising from log odds have been used as a classification score by considering diabetes as the “positive” population (diseased, denoted by 1), and those who did not suffer due to diabetes as “negative” population (nondiseased, denoted by 0).

One of the interesting properties of the ROC curve is that the ROC curve remains same if the classification scores undergo an increasing transformation. Thus, in order to transform all the values of the classification score in the positive domain, $\Phi(S)$ has been chosen as a transformation. The use of this transformation has been considered for estimation of the resilience ROC curve only. Now, in the abovementioned setup, a single binary covariate $z_i \in \{0, 1\}$, the empirical distribution functions, and the empirical $\log - \log$ functions are plotted in Fig. 2 [red for $z_i = 0$, and sky for $z_i = 1$]. The left-hand side figure shows that $F_0 \geq F_n$, and the right-hand side figure shows that the $\log ( - \log F(x) )$ functions differ by a constant. The scores produced by the classifier for the individuals in each group indicates the proportional reversed hazard model. Moreover, under the null hypothesis ($\theta = 1$), the computed value of the test statistic $Z_w$ is 17.0741, and the corresponding $p$-value ($\approx 0$) suggests the rejection of the null hypothesis.

Table III tabulates all the estimated summary index of the ROC curve including resilience parameter $\theta$. Table III shows that all the estimates based on MW statistic and Rojo approach are the same. Fig. 3 shows the ROC plots based on partial likelihood and Mann–Whitney statistics.

We have used “ROCit” package to estimate the values of the AUC based on existing method [24]. The estimated values are 0.843, 0.8455, and 0.8375 using empirical, binormal, and nonparametric methods. Fig. 4(a) represents all the ROC curves evaluated based on empirical method, binormal assumption, nonparametric method, and the proposed model. Note that in this figure the estimate of $\theta$ is chosen based on Mann–Whitney statistics. Fig. 4(a) shows that the proposed model performs quite impressively in spite of having a very simple analytic form.

The invariant property of ROC curve under an increasing transformation motivates us to normalize the data for obtaining better results. Thus, in order to make the data into a normal shape the Yeo–Johnson transformation has been used since it does not require the input data to be positive unlike Box–Cox transformation (see [47]). For this purpose, the recommended package in R is “bestNormalize” (see [36]).

![Fig. 2. Empirical distribution functions and log–log plot.](image1)

![Fig. 3. ROC plots based on PI method and Mann–Whitney statistics.](image2)

![Fig. 4(a). ROC plots based on partial likelihood and Mann–Whitney statistics.](image3)

![Fig. 5(a). Normalized binormal ROC, resilience ROC, and binormal ROC curve with corresponding Youden index point.](image4)

**TABLE III**

| Method | $\hat{\theta}$ (confidence interval) | AUC | $(\hat{r})$ | Youden index | $(\hat{j})$ |
|--------|--------------------------------------|-----|-------------|--------------|-------------|
| MW     | 4.8444 (4.2883, 5.4005)              | 0.8289 | 0.5264     | (0.3366, 0.8631) |
| Rojo   | 5.3707 (4.3091, 6.4322)              | 0.8430 | 0.5540     | (0.3193, 0.8733) |

Fig. 5(a) shows normalized binormal ROC, resilience ROC, and binormal ROC curve with corresponding Youden index point. Here, “normalized binormal ROC” and “binormal ROC” implies the binormal ROC curve with normalized data and without normalized data, respectively. Moreover, this figure contains the resilience ROC curve based on Mann–Whitney statistics or Rojo approach since the AUC is more closer to the other existing models. Note that Fig. 5(a) indicates that the resilience
B. Data in the Context of Brain Injury-Related Biomarkers

The main focus of this section is to compare the proposed model with the Lehmann family of the ROC curves. At this juncture, it is reasonable to consider the data of [44] since these data have been analyzed by Jokiel-Rokita and Topolnicki [19] for the Lehmann family of the ROC curves. Turck et al. [44] conducted a study for outcome prediction following aneurysmal subarachnoid hemorrhage (aSAH) using a combination of clinical scores together with brain injury-related biomarkers, 113 patients admitted within 48 h. After six months, based on the condition of patients, the outcome was categorized as good when the Glasgow outcome scale was greater than equal to four (41 observations) or poor (72 observations), otherwise. These data can be found in pROC package [39]. To illustrate the proposed method, I will also consider nucleoside diphosphate kinase A (NDKA) level as the marker. Jokiel-Rokita and Topolnicki [19] pointed out that the hypotheses of normality of the NDKA level distribution in any of two groups were rejected.

Fig. 6 shows that both the groups support the proportional hazard and proportional reversed hazard assumption. Table IV tabulates all the estimated summary indices of the resilience ROC curve. Note that the estimated values are quite closer for

![ROC Curve Comparison](image-url)
MW and Rojo method, while the lowest estimate was observed for partial likelihood method. Fig. 7 shows estimated Lehmann ROC curve and resilience ROC curve based on MW method. Moreover, this figure shows that the Lehmann ROC curve dominates resilience ROC curve before $t = 0.4244$, and the resilience ROC curve dominates the Lehmann ROC curve $t = 0.4244$. We can conclude that the measurement of NDKA level has strong evidence in order to predict aSAH.

C. Ducheme Muscular Dystrophy Data

In this section, our aim is to address a misspecification scenario. Ducheme muscular dystrophy (DMD) disease causes rapid progression of muscle degeneration of a child. It is a genetically transmitted disease from a mother to her child. It is a well-known muscular dystrophy, and there is no cure at all for this disease. Thus, it is very important to diagnose all affected females. Andrews and Herzberg [2] reported DMD data in [2, Table 38.1], which was collected during a program conducted at a hospital for sick children in Toronto. In our study, we consider one of the serum enzyme levels, i.e., creatine kinase, for 75 carriers, and 134 noncarriers (healthy females) as a biomarker. The following Fig. 8 indicates nonproportionality in the RHR. Consequently, we are interested to apply the proposed model under misspecification together with existing models. Here, we consider binormal ROC curve and empirical ROC curve for comparison purpose. Moreover, in order to make the data into a normal shape, the Yeo–Johnson transformation has been used for binormal ROC curve (see [47]). The AUCs for binormal ROC curve and empirical ROC curve are 0.8741 and 0.8679, respectively. All the estimated summary indices of the resilience ROC curves are tabulated in Table V. Table V shows that all the estimates based on MW statistic and Rojo approach are same as expected. Fig. 9 shows that based on MW statistic and Rojo approach, the resilience ROC curve is also able to handle misspecification scenario.
This article proposes a new semiparametric model of the ROC curve, which is an alternative to the existing models based on the resilience parameter family or alternatively, a proportional reverse hazards family, with underlying distribution \( F_0 \). The proposed model does not require a full parametric specification of the distribution of the scores produced by the binary classifier for the two reference populations. The resulting ROC curve and its summary indices (such as AUC and Youden index) have simple analytic forms. A brief discussion about the verification of the PRHR assumption has been discussed. The partial likelihood method is applied to estimate the ROC curve. This method has some advantages for covariate adjustment due to its regression representation since it is possible to formulate most practical ROC problems using a regression model. The estimation methodology has been developed for the AUC exploiting Mann–Whitney statistics, and the Rojo approach. Estimation procedure of the Youden index is also presented based on Shapiro–Wilks test, and the two population groups hold the PRHR assumption. In the proposed model, the expression of the ROC curve has very simple analytic form and all the summary indices can be calculated quickly. Thus, the resilience family of ROC curve can be regarded as a potential tool in ROC curve analysis.

This article proposes ROC curve analysis in the direction of resilience family of distributions for the first time. A lot of work needs to be carried out as a further research for the proposed model. Development of the Bayesian estimation methodology for the resilience parameter (assuming \( \theta \in (1, \infty) \)) could be considered. The problem of estimation of \( \theta \) for the resilience family of the ROC curves based on smoothed empirical distribution function or minimum distance method using the technique similar to that in works of Jokiel-Rokita and Polonicki [20] and Jokiel-Rokita and Topolnicki [20] is also of great interest. Moreover, it is well worth to extend the estimation methodologies for the resilience ROC curves under left-censored data since Kalbfeisch and Lawless [22] established that the property of RHR is important in the estimation of the survival function under left-censored data.

**VII. CONCLUSION**

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