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Mutaz Mohammad
Zayed University

Carlo Cattani
Università degli Studi della Tuscia Viterbo

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APPLICATIONS OF BI-FRAMELET SYSTEMS FOR SOLVING FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

MUTAZ MOHAMMAD*
Department of Mathematics
College of Natural and Health Sciences
Zayed University, Abu Dhabi, UAE
Mutaz.Mohammad@zu.ac.ae

CARLO CATTANI
Engineering School, DEIM, Tuscia University
Viterbo, Italy

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Abstract
Framelets and their attractive features in many disciplines have attracted a great interest in the recent years. This paper intends to show the advantages of using bi-framelet systems in the context of numerical fractional differential equations (FDEs). We present a computational method based on the quasi-affine bi-framelets with high vanishing moments constructed using the generalized (mixed) oblique extension principle. We use this system for solving some types of FDEs by solving a series of important examples of FDEs related to many mathematical applications. The quasi-affine bi-framelet-based methods for numerical FDEs show the advantages of using sparse matrices and its accuracy in numerical analysis.

Keywords: Fractional Differential Equations; Quasi-Affine System; Bi-Framelet; Mixed Oblique Extension Principle.

*Corresponding author.

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1. INTRODUCTION

In recent years, fractional differential equations (FDEs) have been widely used in the development of many modern problems in engineering practices and applied sciences. For example, it has applications in the modeling of earthquakes, economics, fluidic dynamic of viscoelastic materials, and many other disciplines, e.g. see Refs. [3–5].

Fractional calculus is a mathematical area that studies and analyzes the properties of the derivative and integration of a non-integer orders. In particular, this area is getting more attention from many researchers to develop new methods for solving differential equations involved by fractional order. Atangana is unique among fractalists in his ability to bring to bear new definitions, theory, and ideas on some of the most intractable issues on FDEs. He is the founder of the fractional calculus with nonlocal and non-singular kernels popular in applied mathematics today and has achieved and contributed significantly to the numerical and pure analysis of FDEs. Atangana et al. defined the well-known Atangana–Baleanu fractional derivative definition that describes the complicated problems related to the power and exponential laws and free of singularity [4,5].

Note that, in general, the exact solution of most the FDEs does not exist. Therefore, examining and developing new numerical methods is very important. For example, Laplace and Fourier techniques were proposed [6–8] Adomian decomposition method in Refs. [9–11], variational iteration method in Ref. [12], and other methods can be found in Refs. [13–15]. Interested readers should consult other references therein to have an extra knowledge of the other used methods.

In the context of numerical and computational mathematics, framelet systems have proven as a powerful tool on tackling issues related to the numerical and computational framework. The main aim of this work is to shed some lights on the benefits of using bi-framelets in the area of the FDEs. Some of the FDEs we consider are as follows:

\[ \begin{align*}
\mathcal{D}_t^\alpha u(t) + c(t)u(t) &= f(t), \\
\frac{d^k u}{dt^k} &= 0, \quad k = 0, 1, 2, \ldots, n - 1,
\end{align*} \]

for some \( u \in N \) where \( n - 1 < \alpha \leq n, c(t), f(t) \) are known square integrable functions, \( \mathcal{D}_t^\alpha \) is Caputo FDE (see Definition [13]) operator of \( u \), and \( u(t) \) is the unknown function to be approximated.

\[ \mathcal{D}_t^\alpha u(t) = \begin{cases} 
\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(x)}{(t-x)^{n-\alpha}} dx & \text{if } n - 1 < \alpha \leq n, \\
\frac{d^nu(t)}{dt^n} & \text{if } \alpha = n.
\end{cases} \]

\[ \begin{align*}
\mathcal{D}_t^\alpha u(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u(x)}{(t-x)^{n-\alpha}} dx, \\
\frac{d^nu(t)}{dt^n} &= 0
\end{align*} \]

Note that, \( \mathcal{D}_t^\alpha u(t) = 0, n \leq 1 < \alpha \leq n \).

\[ \mathcal{D}_t^\alpha u(t) = u(t) \]

for \( t > 0 \).

2. AN OVERVIEW OF QUASI-AFFINE BI-FRAMELET

Let us recall some definitions and facts to clarify the concept of bi-framelets. It is known that orthonormal bases are non-redundant systems, while their redundant setup are known as frames. The redundancy here is very useful in many applications such as the error recovery/correction in transmission of data. Frames are more general than orthogonal systems and provide better representations. They were introduced by Duffin and Schaeffer in Ref. [16]

A big development later on has been achieved in Refs. [17–20].

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Definition 2.1. Let \( L_2(\mathbb{R}) \) denote the space of all square integrable functions over \( \mathbb{R} \). A family of vectors \( \{ \phi^v, v \in I \} \subset L_2(\mathbb{R}) \) is a framelet for \( L_2(\mathbb{R}) \) if there exist constants \( C_1, C_2 > 0 \) such that
\[
C_1\|g\|^2 \leq \sum_{v \in I} |\langle g, \phi^v \rangle|^2 \leq C_2\|g\|^2, \quad \forall g \in L_2(\mathbb{R}).
\] (3)

If \( C_1 = C_2 = 1 \), it is called a Parseval frame. The importance of such tightness in the framelet system is that they provide a simple and better reconstructions for the elements of the space \( L_2(\mathbb{R}) \).

Framelet analysis, we say that the function \( \phi \) is a refinable function if
\[
\phi = \sum_k 2h_0(k)\phi(2 \cdot -k),
\] (4)
where \( h_0 \) is a finitely supported sequence (called the low pass filter of \( \phi \) ) such that
\[
h_0(t) = \sum_{k \in \mathbb{Z}} h_0(k)e^{-2\pi it}.
\]

Note that, in frequency domain, Eq. (4) can be written as
\[
\hat{\phi}(2 \cdot \cdot) = \hat{h}_0(\cdot)\hat{\phi}(\cdot),
\] (5)
where
\[
\hat{\phi}(t) = \int_{\mathbb{R}} \phi(x)e^{-2\pi it}dx
\]
is the Fourier transform of \( \phi \).

Definition 2.2. For a given refinable functions \( \phi, \psi \), let \( E = \{1, 2, 3, \ldots, r\} \). \( \Psi = \{\psi^1, \psi^2, \ldots, \psi^r\} \subset L_2(\mathbb{R}) \) and \( \hat{\Psi} = \{\hat{\psi}^1, \hat{\psi}^2, \ldots, \hat{\psi}^r\} \subset L_2(\mathbb{R}) \) where we say that \( X(\Psi), X(\hat{\Psi}) \), where

\[ X(\Psi) = \{\psi^j_k, j, k \in \mathbb{Z}, \ell \in E\}, \]
\[ X(\hat{\Psi}) = \{\hat{\psi}^j_k, j, k \in \mathbb{Z}, \ell \in E\}, \]
and
\[
\psi^j_k = \begin{cases} \frac{1}{2^{j/2}}\psi^j(2^{j}x-k) & \text{if } j \geq 0, \\ \frac{1}{2^{j/2}}\psi^j(2^{j}x-k) & \text{if } j < 0 \end{cases}
\] (6)

is a pair of quasi-affine bi-framelet system of \( L_2(\mathbb{R}) \) if both \( X(\Psi) \) and \( X(\hat{\Psi}) \) are framelet systems in \( L_2(\mathbb{R}) \) and
\[
\langle g, f \rangle = \sum_{\ell \in E, j, k} \langle \psi^j_{\ell, k}, f \rangle \langle g, \psi^j_{\ell, k} \rangle \quad \text{holds} \quad \forall g, f \in L_2(\mathbb{R}).
\] (7)

Note that, Eq. 3 is also satisfied for all elements of \( \Psi, \hat{\Psi} \) with finitely supported sequences called high pass filters, \( h_0[k] \), and \( \hat{h}_0[k], \ell \in E \), respectively.

For simplicity, we define \( \psi^0 = \phi \) and \( \hat{\psi}^0 = \hat{\phi} \). Thus, for \( \xi \in \mathbb{R} \), we have
\[
\psi^\ell(2 \cdot) = \hat{h}_0(\cdot)\hat{\phi}(\cdot),
\]
\[
\hat{\psi}^\ell(2 \cdot) = \hat{h}_0(\cdot)\hat{\phi}(\cdot), \quad \forall \ell \in E.
\] (8)

Therefore, given a quasi-affine bi-framelet system \( X(\Psi) \), we can find a subset of \( L_2(\mathbb{R}) \), \( X(\hat{\Psi}) \), such that (similar to Eq. 3)
\[
C_2^{-1}\|g\|^2 \leq \sum_{\ell \in E, j, k} |\langle g, \psi^j_{\ell, k} \rangle|^2 \leq C_2^{-1}\|g\|^2, \quad \forall g \in L_2(\mathbb{R}).
\] (9)

In particular, if \( \phi = \phi \) and \( \psi^\ell = \hat{\psi}^\ell, \forall \ell \in E \), then it is called a tight framelet system for \( L_2(\mathbb{R}) \). So, we have the following equation:
\[
\|g\|^2 = \sum_{\ell \in E, j, k} |\langle g, \psi^j_{\ell, k} \rangle|^2.
\]

From Eq. 4, for a bi-framelet system, we have
\[
\langle f, g \rangle = \iint f(x)g(t)2^{j/2}E_{j, k} dxdt,
\] (10)
where
\[
2^{j/2}E_{j, k}(x, t) = \sum_{\ell \in E, j, k} \psi^j_{\ell, k}(x)\psi^j_{\ell, k}(t).
\]

Thus, we have the following quasi-affine bi-framelet representation:
\[
g = \sum_{\ell \in E, j, k} a^j_{\ell, k}\psi^j_{\ell, k},
\] (11)
where
\[
a^j_{\ell, k} = \langle g, \psi^j_{\ell, k} \rangle.
\]

Note that, the coefficients \( a^j_{\ell, k} \) is not unique but it is one of the best choices for a better simulation. Hence, one can consider the following truncated representation from Eq. 11 for \( f \) :
\[
U_k g = \sum_{\ell \in E, j, k \geq J} a^j_{\ell, k}\psi^j_{\ell, k}.
\] (12)

Sparse representation for a smooth function is of interest in many applications. Therefore, to have such sparsity, it is crucial for \( h_0 \) to have high vanishing moments, where a function \( \psi \) has a vanishing moments of order \( s \) if
\[
\langle \psi^s, \psi^t \rangle = 0, \quad \text{for all } k = 0, 1, \ldots, s - 1.
\]

In literature, there are many principles to construct bi-framelets, such as the the mixed unitary
extension principles (MUEP) (see e.g. Refs. 33–35 and references therein). In this paper, we use the generalization of the MUEP, namely, the mixed oblique extension principle (MOEP) presented in Ref. 33. By doing this, we will present some examples of bi-framelet systems that have high vanishing moments. However, to have such property, it is required to have some required constraints on \( b_0[k], b_0[k] \). In short, for some trigonometric functions \( \Theta(\cdot) \), if

- \( \hat{\Theta}(\cdot)\hat{\phi}(0)\hat{\phi}(0) = 1 \), and
- \( \hat{\Theta}(\cdot)\hat{b}_0[k]/2 + \pi\alpha\hat{b}_0[k]/2 + \sum_{k \in \mathbb{E}} \hat{b}_0[k]/2 + \pi\alpha \) \( \hat{b}_0[k] \) is required to have some required constraints on \( b_0[k], b_0[k] \). Hence, the system \((X(\Psi), X(\bar{\Psi}))\) defined in Definition 22 forms a quasi-affine bi-framelet system of \( L_2(\mathbb{R}) \).

The MOEP provides a way to construct bi-framelets from refinable functions and it gives us a better approximation orders and reconstruction. Framelets have a great deal of use in many applications due to the features of redundancy (by increasing the number \( r \)) and many other properties (see e.g. Ref. 37). In this paper, we use analytic expressions of bi-framelets with high redundancy generated via the MOEP using some analytic refinable functions called B-splines. B-splines are of importance in harmonic theory and have wide range of use for many applications in approximation analysis. It is defined using the convolution product as follows.

**Definition 2.3 (Ref. 37).** For \( m \in \mathbb{N} \), the B-spline of order \( m \), \( B_m \), is defined as

\[
B_1 = \chi_{[0,1)}, \quad \text{and} \quad B_m = \int_0^1 B_{m-1}(-x) \, dx.
\]

Figure 1 shows the graphs of the B-splines \( B_m \) for \( m = 1, \ldots, 4 \). Note that, \( B_m \) is a piecewise function of polynomials \((B_m \in C^{m-1}(\mathbb{R}))\) and is refined as

\[
B_m(2\xi) = \hat{b}_0(\xi)B_m(\xi),
\]

where

\[
\hat{b}_0(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^m.
\]

**3. SYSTEMS OF QUASI-AFFINE BI-FRAMELETS**

In this section, we use the MOEP to construct quasi-affine bi-framelet systems using B-splines and use it to solve some examples of FDEs.

**System A (HAAR bi-framelet).** Consider the B-spline of order 1, \( B_1 = \chi_{[0,1)} \). Define

\[
\psi^1(\xi) = \frac{1208e^{i\xi}}{34157\pi e^{i\xi/2}} + \frac{3341807e^{i\xi/2}}{4726360\pi e^{i\xi/2}} - \frac{489e^{i\xi}}{1384\pi e^{i\xi}}.
\]

Hence, the system \((X(\Psi), X(\bar{\Psi}))\) is a bi-framelet for \( L_2(\mathbb{R}) \). Figure 1 shows the graphs of the generators, \( \psi^1 \) and \( \bar{\psi}^1 \) in the time domain.

**System B (Linear bi-framelet).** For \( m = 2 \), consider the linear B-spline for \( \phi \) and \( \bar{\phi} \) with the following filters:

\[
\hat{b}_0(\xi) = \frac{1}{2}(1 + e^{-i\xi}), \quad \hat{b}_1(\xi) = 1 - \cos \xi, \quad \hat{b}_2(\xi) = \frac{1}{2}(1 + e^{-i\xi}.
\]
Consider the (cubic) B-spline of order 4, \( \phi \), and the (linear) B-spline of order 2, \( \tilde{\phi} \), with the following filters:

\[
\hat{h}_0(\xi) = \frac{1}{16}(1 + e^{-\xi})^4, \quad \hat{\tilde{h}}_0(\xi) = \frac{1}{4}(1 + e^{-\xi})^2.
\]

Depending on the MOEP, we define explicitly in the time domain the generators \( \{\psi, \tilde{\psi}, \ell = 1, 2\} \), that generate a bi-framelet system, namely

\[
\psi^1(x) = \begin{cases} 
1 + 3x + 3x^2 + x^3 & \text{if } -1 \leq x < -1/2, \\
1/2(1 - 6x^2 - 6x^3) & \text{if } -1/2 \leq x < 0, \\
1/2(1 - 6x^2 + 6x^3) & \text{if } 0 \leq x < 1/2, \\
1 - 3x + 3x^2 - x^3 & \text{if } 1/2 \leq x < 1, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\tilde{\psi}^1(x) = \begin{cases} 
1/3(1 + 3x + 3x^2 + x^3) & \text{if } -1 \leq x < 0, \\
1/3(1 + 3x + 3x^2 - 8x^3) & \text{if } 0 \leq x < 1/2, \\
1/3(-1 + 15x^4) & \text{if } 1/2 \leq x < 1, \\
1/3(8 - 12x + 6x^2 - x^3) & \text{if } 1 \leq x < 2, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\psi^2(x) = \begin{cases} 
-2x & \text{if } 1/2 \leq x < 1, \\
-7 + 6x & \text{if } 1 \leq x < 3/2, \\
11 - 6x & \text{if } 3/2 \leq x < 2, \\
-5 + 2x & \text{if } 2 \leq x < 5/2, \\
0 & \text{otherwise},
\end{cases}
\]

\[
\tilde{\psi}^2(x) = \begin{cases} 
1/4(-1 - 2x) & \text{if } -1/2 \leq x < 1/2, \\
1/2(-5 + 8x) & \text{if } 1/2 \leq x < 1, \\
1/2(11 - 8x) & \text{if } 1 \leq x < 3/2, \\
1/4(-5 + 2x) & \text{if } 3/2 \leq x < 5/2, \\
0 & \text{otherwise}.
\end{cases}
\]
The graphs of these generators are shown in Fig. 4.

System D. Now let’s consider the B-spline of order four for both $\phi$ and $\tilde{\phi}$. Again, by applying the setup of the MOEP, we are able to find the generators explicitly as follows:

$$
\psi_1(x) = \begin{cases} 
\frac{8x^3}{3} & \text{if } 0 \leq x \leq 1/2, \\
-8/3(-64 - 48x - 12x^2 + x^3) & \text{if } 7/2 < x < 4, \\
8/3(-279 + 240x - 72x^2 + 7x^3) & \text{if } 3 \leq x \leq 7/2, \\
-8(-159 + 170x - 60x^2 + 7x^3) & \text{if } 5/2 < x < 3, \\
8/3(-9 + 26x - 24x^2 + 7x^3) & \text{if } 1 \leq x \leq 3/2, \\
-8/3(-1 + 6x - 12x^2 + 7x^3) & \text{if } 1/2 < x < 1, \\
8/3(-398 + 540x - 240x^2 + 35x^3) & \text{if } 2 \leq x \leq 5/2, \\
-8/3(-162 + 300x - 180x^2 + 35x^3) & \text{if } 3/2 < x < 2, \\
0 & \text{otherwise}, 
\end{cases}
$$

$$
\psi_2(x) = \begin{cases} 
1/3(2891 - 4050x + 1860x^2 - 280x^3) & \text{if } 2 \leq x \leq 5/2, \\
1/3(63 - 186x + 180x^2 - 56x^3) & \text{if } 1 \leq x \leq 3/2, \\
2079 - 1882x + 564x^2 - 56x^3 & \text{if } 3 \leq x \leq 7/2, \\
1/3(729 - 486x + 108x^2 - 8x^3) & \text{if } 4 \leq x \leq 9/2, \\
1/3(-1 + 6x - 12x^2 + 8x^3) & \text{if } 1/2 < x < 1, \\
1/3(-3367 + 2586x - 660x^2 + 56x^3) & \text{if } 7/2 < x < 4, \\
-231 + 442x - 276x^2 + 56x^3 & \text{if } 3/2 < x < 2, \\
1/3(-5859 + 6450x - 2340x^2 + 280x^3) & \text{if } 5/2 < x < 3, \\
0 & \text{otherwise}, 
\end{cases}
$$

Fig. 4 The graphs of the generators in System C.
We depict the graphs of these generators in Fig. 5.
4. QUASI-AFFINE BI-FRAMELET OPERATIONAL MATRIX OF FDES

In this section, we introduce the quasi-affine bi-framelet operational matrix of the FDEs defined in Eq. (1) using the function approximation given by Eq. (3).

For simplicity, we present the bi-framelet operational matrix using System A (Haar bi-framelet system $X((x_1, x_2))$). For other types of bi-framelet systems that generated by higher order of B-splines given in Sec. 3, the procedure will be the same.

Theorem 4.1. Suppose that $D^2_j U_n u(\cdot)$ and the system $(X(\Psi), X(\Psi))$ are obtained using System A based on the MOEP, and the functions $u^{(m)}(\cdot), x \in (0,1)$, $m \in N$ are continuous and bounded. Then, for some constant $M$, we have

\[
\|u - U_n u\|_2^2 \leq M 2^{-2n}.
\]

Proof. Based on the upper bound of the bi-framelet system using Bessel property, we have

\[
\|u - U_n u\|_2^2 \leq \sum_{j>2n} \sum_{k<2} |\langle u, \psi_{j,k}^1 \rangle|^2 + |\langle u, \psi_{j,k}^2 \rangle|^2.
\]

But,

\[
|\langle u, \psi_{j,k}^1 \rangle| + |\langle u, \psi_{j,k}^2 \rangle| \\
\leq \|u\|_\infty (\|\psi_{j,k}^1\|_1 + \|\psi_{j,k}^2\|_1) \\
= \|u\|_\infty 2^{-j/2}(\|\psi^1\|_1 + \|\psi^2\|_1).
\]

Therefore, we have

\[
\|u - U_n u\|_2^2 \leq \|u\|_\infty \left( \max_{j} \{\|\psi^1\|_1 + \|\psi^2\|_1\} \right) \\
\times \sum_{j>2n} \sum_{k<2} 2^{-j/2} |\langle u, \psi_{j,k}^1 \rangle| \\
+ |\langle u, \psi_{j,k}^2 \rangle|.
\]

Using the mean value theorem for integration, one can find $\xi_1, \xi_2$ such that $\xi_1 < \xi_2$ and that

\[
|\langle u, \psi_{j,k}^1 \rangle| + |\langle u, \psi_{j,k}^2 \rangle| \leq 2 \left( \int_{A} |u|dx - \int_{B} |u|du \right) \\
\leq 2^{-j/2} (\|u(\xi_1) - u(\xi_2)\|_1),
\]

where $A = 2^{-j}[k, k+0.5)$ and $B = 2^{-j}[k+0.5, k+1]$. Hence,

\[
\|u - U_n u\|_2^2 \leq \|u\|_\infty \left( \max_{x} \{\|\psi^1\|_1 + \|\psi^2\|_1\} \right) \\
\times \sum_{j>2n} \sum_{k<2} 2^{-j/2} (\|u(\xi_1) - u(\xi_2)\|_1) \\
\leq \|u\|_\infty 2^{-j/2} \max_{x} \{\|\psi^1\|_1 + \|\psi^2\|_1\} \\
\times (\|u(\xi_1) - u(\xi_2)\|_1) 2^{-2n}.
\]

Then, the result is concluded. \qed

Notice, Eq. (13) can be written as

\[
U_n g(t) = A^n_G u(t).
\]
where the Haar bi-framelet vector $A_n$ ($A_n^T$ is the transpose of the bi-framelet vector $A_n$) and Haar bi-framelet function vector $G_n(t)$ are given as

$$A_n = \left[ a_0, a_1, a_2, \ldots, a_{2^n-1}, a_{2^n-2}, \ldots, a_{2^n-2^n-1}, a_{2^n-2^n-2}, \ldots \right]^T,$$

$$G_n(t) = \left[ \psi_0^{2^n-1-n}(t), \psi_1^{2^n-1-n}(t), \ldots, \psi_{2^n-1-n}(t), \right]^T.$$

Consider the collocation points

$$t_i = \frac{i}{n_p}, \quad i = 1, \ldots, n_p,$$

where $n_p$ is the length of interval of $\psi_{1,k}$ that covers the compact support of the bi-framelet system at hand, for example, for the case of Haar bi-framelet system, $n_p = 2^{n+1}(2n + 1)$. Define the operational bi-framelet matrix as follows:

$$L_{n_p \times n_p}(t_i) = [G_n(t_i)]_{i=1}^{n_p}.$$

Now, considering the system defined by Eq. (11) and in order to solve it using a bi-framelet system via the discretized points, we substitute the truncated expansion given in Eq. (11) thus

$$A_n^T (G_n(t_i) + c(t) \Psi^n \Phi^n) (t_i)$$

$$= f(t_i), \quad i = 1, \ldots, n_p,$$

where $\Psi^n$ is the FIO defined by Eq. (12).

For $i = 1, \ldots, n_p$, Eq. (11) can be written as

$$L_{n_p \times n_p}^T (t_i) A_n = F_n(t_i),$$

where $F_n(t_i) = \{ f(t_i) \}_{i=1}^{n_p}$ is a matrix function vector of order $n_p \times 1$. The coefficient bi-framelet vector $A_n$ in Eq. (12), and so the approximated solution, can be calculated using, for example, Mathematica software.

5. NUMERICAL APPLICATIONS

In this section, we use the bi-framelet systems presented in Sec. 2 to illustrate and show the efficiency of the proposed method by using some examples of FDEs.

Example 1. Consider the following FDEs:

$$D^\alpha u(t) + c(t)u(t) = f(t),$$

where

$$\alpha = \frac{1}{3}, \quad c(t) = \sqrt{t},$$

$$f(t) = 1.10773 \sqrt{t} + \sqrt{t},$$

$$u(0) = 0.$$

The exact solution for this FDE is $u(t) = t$.

By applying the procedure in Sec. 4 and solving the resulting system, we obtain the approximated solution of the FDE using System A, for example, when $n = 12, 13$, respectively, we have

$$u_{12}(x) = \begin{cases} 4.44399 \times 10^{-16} \\ + 8.51319 \times 10^{-13} x, & \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$= -3.19679 \times 10^{-17} + 8.52429 \times 10^{-13} x, \quad \frac{1}{4} \leq x \leq \frac{1}{2},$$

$$8.52429 \times 10^{-13} x, \quad 0 < x \leq \frac{1}{4}.$$
We also present the matrix $L_{n_p \times n_p}(n = 1)$ in Eq. (15) using System A where

$$L_{n_p \times n_p} = \begin{bmatrix}
0.123526 & -0.000125493 & -0.000177473 & -0.000250985 & 0. \\
0.213044 & -0.000216492 & -0.000306166 & -0.000432984 & 0. \\
0.297806 & -0.000302682 & -0.000428058 & -0.000605365 & 0. \\
0.38092 & -0.000387215 & -0.000547605 & -0.000825178 & 0. \\
0.463514 & -0.000471231 & -0.000666422 & -0.00107392 & 0. \\
0.546115 & -0.000555266 & -0.000875295 & -0.001298082 & 0. \\
0.629008 & -0.000639607 & -0.000992684 & -0.00141703 & 0.280299 \\
0.712356 & -0.000724419 & -0.00112202 & -0.001505 & 0.479997 \\
0.79626 & -0.000809802 & -0.0012794 & -0.00168281 & 0.66531 \\
0.880779 & -0.000985818 & -0.0014365 & -0.0019695 & 0.26767 \\
0.965953 & -0.000982504 & -0.00159398 & -0.00206223 & 0.026871 \\
1.05180 & -0.00106988 & -0.00175219 & -0.00105799 & -0.180348
\end{bmatrix}.$$
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#### Table 1: Numerical Results for \( u_3 \) Using Systems A through D.

| \( x \) | Exact | System A | System B | System C | System D |
|-----|-------|---------|---------|---------|---------|
| 0.1 | 0.1   | 0.1005346898001 | 0.100034753855 | 0.1000667285001 | 0.1000667285001 |
| 0.2 | 0.2   | 0.2009938636138  | 0.2005291547472 | 0.2005291547472 | 0.2005291547472 |
| 0.3 | 0.3   | 0.3013952033910  | 0.3007135510809 | 0.300459425356  | 0.300034600 |
| 0.4 | 0.4   | 0.4017562049210  | 0.4006812946634 | 0.400627915311  | 0.40000776200 |
| 0.5 | 0.5   | 0.5020926227360  | 0.5009858396464 | 0.500100815290  | 0.50009400200 |
| 0.6 | 0.6   | 0.6024023892389  | 0.6001153773575 | 0.600027980715  | 0.60000776200 |
| 0.7 | 0.7   | 0.7026093828032  | 0.7002207678028 | 0.700121414444  | 0.70001744200 |
| 0.8 | 0.8   | 0.8026619455757  | 0.800314535807  | 0.800034600 |
| 0.9 | 0.9   | 0.9032226772900  | 0.900401982061  | 0.900034600 |
| 1.0 | 1.0   | 1.0034649540220  | 1.000461888702  | 1.000034600 |

#### Table 2: Error Results of Example 1 for Different Values on \( n \).

| \( n \) | System A | System B | System C | System D |
|-----|---------|---------|---------|---------|
| 1   | 5.12 × 10^{-2} | 4.84 × 10^{-4} | 2.25 × 10^{-5} | 6.45 × 10^{-6} |
| 2   | 3.85 × 10^{-3} | 1.02 × 10^{-5} | 5.65 × 10^{-6} | 2.67 × 10^{-7} |
| 3   | 1.93 × 10^{-5} | 2.01 × 10^{-5} | 2.67 × 10^{-7} | 1.97 × 10^{-9} |
| 4   | 3.75 × 10^{-6} | 1.97 × 10^{-7} | 5.01 × 10^{-9} | 2.20 × 10^{-10} |
| 5   | 8.24 × 10^{-7} | 1.83 × 10^{-9} | 3.56 × 10^{-10} | 3.73 × 10^{-12} |

The graphical and numerical results of the Systems A and B, for different values of \( n \), are presented in Figs. 6 and 7. The obtained results are in an excellent agreement with the exact solution. Also, the error bounds are controlled as detailed in Tables 1 and 2.

Fig. 8: Numerical and exact solution using System A for \( n = 1, 2 \), respectively.
Example 2. Consider the following FDE that generated from a Newtonian fluid application:

\[
\begin{aligned}
\frac{D_\alpha u(t)}{D t^\alpha} + c(t)u(t) &= f(t), \\
\alpha &= \frac{1}{4}, \\
c(t) &= 1, \\
f(t) &= -0.678274u^{\frac{1}{3}} - 0.5t^3 + 1.44699t^{\frac{1}{3}} + t^1, \\
u(0) &= 0.
\end{aligned}
\]

The exact solution for the given FDE is \(u(t) = \frac{1}{4}(2t^4 - t^3)\). Again, based on the numerical scheme presented in Sec. 4, we conclude the following numerical and graphical results based on System A through D in Tables 3 and 4, Figs. 8 and 9.

Figure 9 shows the convergence behavior of the numerical results obtained in Examples 1 and 2 based on the Systems A through D.

### Table 3 Numerical Results for \(u_{33}\) Using Systems A through D.

| \(n\) | Exact | System A | System B | System C | System D |
|------|-------|----------|----------|----------|----------|
| 0.1  | -0.0004 | -0.00041860817 | -0.00040356327 | -0.000402299 | -0.00040195754 |
| 0.2  | -0.0024 | -0.00256630153 | -0.00240147694 | -0.002387746 | -0.0024028943 |
| 0.3  | -0.0054 | -0.00527275646 | -0.00538612986 | -0.005499365 | -0.0054045671 |
| 0.4  | -0.0064 | -0.00618757695 | -0.00631210453 | -0.006457773 | -0.0063912865 |
| 0.5  | 0.0000 | 0.0001775883833 | 0.0001475958073 | 0.0001921993 | 0.000192759685 |
| 0.6  | 0.0216 | 0.02184414164 | 0.02069736728 | 0.02146863434 | 0.021647676728 |
| 0.7  | 0.0686 | 0.07125148395 | 0.063785488274 | 0.06271743145 | 0.068816851729 |
| 0.8  | 0.1536 | 0.15358430000 | 0.153324796767 | 0.15374446756 | 0.153953489864 |
| 0.9  | 0.2046 | 0.20359868673 | 0.201739974332 | 0.20057775655 | 0.2017556755 |
| 1.0  | 0.5000 | 0.498312385646 | 0.4993664893769 | 0.49958836553 | 0.500574441403 |

### Table 4 Error Results of Example 2 for Different Values on \(n\).

| \(n\) | System A | System B | System C | System D |
|------|----------|----------|----------|----------|
| 1    | 6.44 \times 10^{-2} | 1.35 \times 10^{-3} | 2.35 \times 10^{-4} | 7.67 \times 10^{-6} |
| 2    | 3.02 \times 10^{-3} | 6.45 \times 10^{-5} | 4.45 \times 10^{-6} | 0.88 \times 10^{-6} |
| 3    | 4.56 \times 10^{-4} | 6.73 \times 10^{-6} | 3.35 \times 10^{-7} | 2.55 \times 10^{-9} |
| 4    | 6.45 \times 10^{-5} | 7.33 \times 10^{-7} | 3.01 \times 10^{-8} | 5.68 \times 10^{-10} |
| 5    | 1.24 \times 10^{-5} | 0.46 \times 10^{-8} | 2.55 \times 10^{-10} | 7.75 \times 10^{-12} |
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We provide the approximate solution of this FDE based on system A for \( n = 2 \), namely

\[
\| u_2(x) \| = \begin{cases} 
0.00365168 - 0.0302517x, & 3 \leq x \leq 4 \\
0.00363541 - 0.0301866x, & 1 < x < \frac{5}{4} \\
0.00201247 - 0.0215892x, & \frac{5}{8} \leq x < \frac{5}{4} \\
-0.00051834 - 0.0155074x, & \frac{5}{16} < x < \frac{3}{8} \\
0.00054747 - 0.00978988x, & \frac{1}{16} < x < \frac{1}{8} \\
-0.0141255 + 0.0196224x, & \frac{3}{8} \leq x < \frac{7}{16} \\
-0.0413182 + 0.0837772x, & \frac{7}{16} \leq x \leq \frac{1}{2} \\
-0.0887531 + 0.176474x, & \frac{1}{2} < x < \frac{5}{8} \\
-0.162199 + 0.307218x, & \frac{5}{8} \leq x < \frac{11}{16} \\
-0.272311 + 0.483979x, & \frac{11}{16} \leq x < \frac{3}{4} \\
-0.427521 + 0.709302x, & \frac{3}{4} \leq x \leq \frac{13}{16} \\
-0.638948 + 0.991072x, & 0 < x \leq \frac{15}{16} \\
-0.001022x, & \frac{15}{16} \leq x \leq 1 \\
-0.917987 + 1.5345x, & \frac{13}{16} \leq x < \frac{21}{16} \\
-1.27776 + 1.7411x, & \frac{7}{8} \leq x < \frac{15}{16} \\
-1.74605 + 2.24487x, & \frac{15}{16} \leq x \leq 1.
\end{cases}
\]

Fig. 10 Convergence rate of Examples 1 and 2 respectively based on Systems A through D.

6. CONCLUSION
The purpose of this work is to develop an efficient method for solving FDEs. We derive a new numerical scheme for solving important types of FDEs based on the quasi-affine bi-framelet operational matrices of fractional integration formula.

The Systems A through D were generated using B-spline functions of different orders and based upon the popular OEP that increase the accuracy orders of the approximated function by increasing the vanishing moments property. The utilized numerical scheme can be used to solve various types of FDEs including the nonlinear case that will be considered in the future work. Two examples of FDEs have been considered to test the validity of the proposed scheme and demonstrate the powerfulness of the method. The problems have been reduced to solving a system of algebraic equations.

The obtained numerical and graphical results show an excellent agreement with the exact values and the theoretical estimation. Increasing the partial sums and the order of the B-splines being used to generate the systems result an increase in the accuracy and efficiency of the method.

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