Pion form factor and reactions $e^+e^- \to \omega \pi^0$ and $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$ at energies up to 2 – 3 GeV in the many-channel approach.

N. N. Achasov

Laboratory of Theoretical Physics, S. L. Sobolev Institute for Mathematics, 630090, Novosibirsk, Russian Federation

A. A. Kozhevnikov

Laboratory of Theoretical Physics, S. L. Sobolev Institute for Mathematics, and Novosibirsk State University, 630090, Novosibirsk, Russian Federation

(Dated: May 28, 2013)

Using the field-theory-inspired expression for the pion electromagnetic form factor $F_\pi$, a good description of the data in the range $-10 < s < 1 \text{ GeV}^2$ is obtained upon taking into account the pseudoscalar-pseudoscalar (PP) loops. When the vector-pseudoscalar (VP) and the axial vector-pseudoscalar (AP) loops are taken into account in addition to the PP ones, a good description of the BABAR data on the reaction $e^+e^- \to \omega \pi^0$ and $e^+e^- \to \pi^+\pi^-\pi^+\pi^-$. This task is also made, with the SND data on the $\omega\pi^0$ production and the BABAR data on the $\pi^+\pi^-\pi^+\pi^-$ production, both in $e^+e^-$ annihilation, upon taking into account $\rho(770)$ and the heavier $\rho(1450)$, $\rho(1700)$, and $\rho(2100)$ resonances. The problems with inclusion of the VP and AP loops are pointed out and discussed.

PACS numbers: 13.40.Gp, 12.40.Vv, 13.66.Bc, 14.40.Be

I. INTRODUCTION

Some time ago the present authors suggested a new expression for the electromagnetic form factor of the pion $F_\pi$ \[1\], which describes the data on the reaction $e^+e^- \to \pi^+\pi^-$ \[2,3\] restricted to the time-like region $4m_e^2 < s \leq 1 \text{ GeV}^2$. The expression takes into account the strong resonance mixing via common decay modes and the $\rho\omega$ mixing. It has both the correct analytical properties and the normalization condition $F_\pi(0) = 1$, and can be represented in the form:

$$
F_\pi(s) = \frac{1}{\Delta}(g_{\gamma\rho_1}, g_{\gamma\rho_2}, g_{\gamma\rho_3}, \ldots)
$$

where $i$ ($i = 1, 2, 3, \ldots$) counts the $\rho$-like resonance states $\rho_1 \equiv \rho(770)$, $\rho_2 \equiv \rho(1450)$, $\rho_3 \equiv \rho(1700)$, ..., the quantity $g_{\gamma V} = \frac{m_V^2}{g_V}$,

$$(V = \rho_1, \rho_2, \ldots, \omega)$$

is introduced in such a way that $eg_{\gamma V}$, where $e$ is the electric charge, is the $\gamma V$ transition amplitude. As usual, the coupling constant $g_V$ is calculated from the electronic width

$$
\Gamma_{\nu \to e^+e^-} = \frac{4\pi\alpha^2 m_\pi}{3g_V^2}
$$

of the resonance $V$. The quantities $g_{ij}/\Delta$ are the matrix elements of the matrix $G^{-1}$ given by Eq. 3.7 below, and $\Delta = \det G$. Ellipsis mean additional states like $\rho(2100)$ etc. It is assumed that the direct G-parity violating decay $\omega \to \pi^+\pi^-$ is absent, that is, $g_{\omega\pi\pi} = 0$. The quantity $\Pi_{\rho_1,\omega}$ is responsible for the $\rho\omega$ mixing. See Ref. \[1\] for more detail concerning Eq. 1.1. Notice that the expression similar to Eq. 1.1 was used earlier \[8\] for the description of those times data in the time-like domain, but it had a disadvantage that the normalization condition $F_\pi(0) = 1$ was satisfied only within the 20% accuracy.

Using the resonance parameters found from fitting the data \[4,5\], the continuation to the space-like region $s < 0$ was made, and the curve describing the behavior of $F_\pi(s)$ in the range $-0.2 \text{ GeV}^2 < s < 0 \text{ GeV}^2$ was obtained \[1\].
and compared with the data \[9\] in this interval of the momentum transfer squared. The space-like interval was further expanded to \(s = -10\) GeV\(^2\) in the subsequent work \[3\], and the comparison was made with the data \[10\,12\] existing in that interval. The basic ingredient in the above treatment is the inclusion of the pseudoscalar-pseudoscalar loops (PP), specifically, the \(\pi^+\pi^-\) and \(K\bar{K}\) ones. These contributions are dominant at the center-of-mass energy \(\sqrt{s} \leq 1\) GeV. Going to higher energies up to 3 GeV, of the reaction \(e^+e^- \rightarrow \pi^+\pi^-\), requires the inclusion of the vector-pseudoscalar (VP) and the axial-vector-pseudoscalar (AP) intermediate states. This is the aim of the present work. The particular VP state \(\omega\pi^0\) produced in \(e^+e^-\) annihilation was studied by SND team in Ref. \[13\], while the AP one of the type \(a_1\pi\) is the intermediate state in the reaction \(e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^-\) studied by BABAR \[14\]. An attempt to describe these reactions in the framework of the three-channel approach, taking into account the PP, VP, and AP intermediate states, is also undertaken in the present work.

The material is organized as follows. The polarization operators due to the VP and AP loops are calculated in Section IV. The expression for the PP polarization operator is reminded in the same section. The quantities for comparison with experimental data are discussed in Sec. III. The results of the data fitting are represented in Sec. IV. Section V contains the conclusion.

II. POLARIZATION OPERATORS DUE TO PSEUDOSCALAR-PSEUDOSCALAR, VECTOR PSEUDOSCALAR, AND AXIAL VECTOR PSEUDOSCALAR LOOPS

The final states \(\pi^+\pi^-\), \(\omega\pi^0\), and \(\pi^+\pi^-\pi^+\pi^-\) considered in the present work, have the isotopic spin \(I = 1\). Hence, they are produced in \(e^+e^-\) annihilation via the unit spin \(\rho\)-like intermediate states \(\rho_1 \equiv \rho(770)\), \(\rho_2 \equiv \rho(1450)\), \(\rho_3 \equiv \rho(1700)\), etc. These states have rather large widths and are mixed via their common decay modes. The effects of finite width and of the mixing are taken into account by means of the diagonal and non-diagonal polarization operators \(\Pi_{\rho_1,\rho_2}\) \[1\]. In particular, the effects of finite width appear in the inverse propagator of the resonance \(\rho_i\) via the replacement

\[
m^2_{\rho_i} - s \rightarrow m^2_{\rho_i} - s - \Pi_{\rho_1,\rho_2} \equiv D_i. \tag{2.1}\]

Indeed, according to unitarity relation, the particular contribution to the imaginary part of the diagonal polarization operator is due to the real intermediate state \(ab\):

\[
\Pi_{\rho_1,\rho_2}^{ab}(s) = \frac{1}{s} \int_{\tilde{m}_a + m_{l_2} - m_{l_1}}^{\infty} \sqrt{\Gamma_{\rho_1 \rightarrow ab}(s')} \sqrt{s'}(s' - s - i\varepsilon) ds',
\]

\[
\text{Im}\Pi_{\rho_1,\rho_2}^{ab}(s) = \sqrt{s} \sqrt{\Gamma_{\rho_1 \rightarrow ab}(s)}. \tag{2.2}\]

Hereafter, the quantity \(s\) is the energy squared. As is explained earlier \[1\], the dispersion relation written for the polarization operator divided by \(s\), automatically guarantees the correct normalization of the form factor \(F_\pi(0) = 1\). In the present work, the states which are taken into account are the PP states \(\pi^+\pi^-\), \(K^+K^- + K^0\bar{K}^0\) of the pair of pseudoscalar mesons, the vector pseudoscalar VP states \(\omega\pi^0\), \(K^+K^- + K^0\bar{K}^0\), and the axial-vector-pseudoscalar AP states \(a_1'(1260)\pi^+ + a_1(1260)\pi^+\), \(K_1(1270)\bar{K} + \text{c.c.}\), \(K_1(1400)\bar{K} + \text{c.c.}\). The polarization operators due to the PP loops are considered in detail elsewhere \[1\].

A. Pseudoscalar-pseudoscalar loop

The polarization operators due to the PP loop, both diagonal and non-diagonal, are represented in the form

\[
\Pi_{\rho_1,\rho_2}^{PP}(s) = g_{\rho_1,\rho_2}g_{\rho_1,\rho_2} \Pi_{\rho_1,\rho_2}^{PP}, \tag{2.3}\]

where \(\Pi^{PP} \equiv \Pi^{PP}(s, m_V, m_P) = \Pi_0^{PP} + \Pi_1^{PP}\), and

\[
\Pi_0^{PP} = \frac{s}{48\pi^2} \left[ 8m_P \left( \frac{1}{m_V^2} - \frac{1}{m^2} \right) + \frac{s}{4} \right] \times
\]

\[
\ln \left( \frac{1 + v_P(m_P^2)}{1 + v_P(m_V^2)} \right) \ln \theta(m_V - 2m_P) - \frac{2v_P^3(m_P^2)}{m_V^2} \arctan \left( \frac{\theta(2m_P - m_V)}{v_P} \right),
\]

\[
\Pi_1^{PP} = \frac{s}{48\pi^2} \left[ \frac{v_P^3(s)}{4} \left( \ln(1 + v_P(s)) - \ln(1 - v_P(s)) \right) \theta(s - 4m^2_P) + \frac{2v_P^3(s)}{\tilde{v}_P(s)} \theta(4m^2_P - s) \right] \theta(s), \tag{2.4}\]

and

\[
v_P(s) = \sqrt{1 - \frac{4m_P^2}{s}}, \tag{2.5}\]

\[
\tilde{v}_P(s) = \frac{4m_P^2}{s} - 1,
\]

\(\theta\) is the step function. The detailed discussion of \(\Pi^{PP}_{\rho_1,\rho_2}\) is given in Ref. \[1\].

B. Vector – pseudoscalar loop

The polarization operators due to the VP loop, both diagonal and non-diagonal, are represented in the form

\[
\Pi_{\rho_1,\rho_2}^{VP}(s) = g_{\rho_1,\rho_2}g_{\rho_1,\rho_2} \Pi_{\rho_1,\rho_2}^{VP}, \tag{2.5}\]
where the quantity $\Pi^{(VP)} \equiv \Pi^{(VP)}(s, m_{\rho}, m_{V}, m_{P})$ is calculated from the dispersion relation

$$\frac{\Pi^{(VP)}}{s} = \frac{1}{12\pi^2} \int_{(m_{V}+m_{P})^2}^{\infty} \frac{q_{VP}^3(s', m_{V}, m_{P})}{\sqrt{s'}(s' - s + i\varepsilon)} \times \left( \frac{s_0 + m_{\rho}^2}{s_0 + s'} \right) ds'. \quad (2.6)$$

The notations are as follows. The quantity

$$q_{ab}(s, m_{a}, m_{b}) = \left[ s^2 - 2(m_{a}^2 + m_{b}^2) + (m_{a}^2 - m_{b}^2)^2 \right]^{1/2} / 2\sqrt{s} \quad (2.7)$$

is the momentum of the particle $a$ or $b$ in the rest frame of the decaying particle with the invariant mass $s$; $m_{\rho}$, $m_{V}$ and $m_{P}$ are, respectively, the masses of the resonance $\rho$, the vector $V$ and pseudoscalar $P$ mesons propagating in the loop, $g_{\rho_{i}VP}$ is the coupling constant of the resonance $\rho_i$ with the VP state. It is well-known that the partial width of the decay $\rho_i \to VP$,

$$\Gamma_{\rho_iVP}(s) = \frac{g_{\rho_{i}VP}^2 q_{VP}^3(s, m_{V}, m_{P})}{12\pi},$$

grows with the energy increase. This growth spoils the convergence of the integral Eq. (2.6). This is the reason for appearance of the function $(s_0 + m_{\rho}^2)/(s' + m_{\rho}^2)$ in the integrand of Eq. (2.6). It suppresses the fast growth of the partial width and improves the convergence of the above integral at large $s'$. However, the integral still remains logarithmically divergent, and one should perform the subtraction of the real part $\text{Re}\Pi^{(VP)}_{\rho_{i}i}/s$ at $s = m_{\rho_i}^2$.

The expression for $\Pi^{(VP)}$ resulting from Eq. (2.6) can be represented in the form

$$\Pi^{(VP)} = \frac{1}{48\pi^2} \left[ \Pi_0^{(VP)} + \Pi_1^{(VP)} + \Pi_2^{(VP)} \right], \quad (2.8)$$

where

$$\Pi_0^{(VP)} = \frac{m_{\rho_i}^2 + s_0}{2s_0} \left\{ \frac{1}{s_0} - \frac{s_0}{m_{\rho_i}^2} \frac{m_{V}}{m_{P}} \ln \frac{m_{V}}{m_{P}} + \left( 1 - \frac{s_0}{m_{\rho_i}^2} \right) \frac{m_{V}}{m_{P}} \times \right. \left[ \frac{3}{2} \left(\frac{m_{\rho_i}^2 + m_{V}^2}{m_{\rho_i}^2 + m_{P}^2 + m_{V}^2} \right) \ln \frac{m_{V}}{m_{P}} + m_{V} + m_{P} \right] \left. - \frac{s}{2s_0} \frac{m_{\rho_i}^2 + s_0}{s + s_0} \left[ \left( m_{\rho_i}^2 + s_0 \right) \left( m_{\rho_i}^2 + s_0 \right) \right]^{3/2} \times \right. \left. \ln \frac{m_{\rho_i}^2 + s_0 + m_{\rho_i}^2}{m_{\rho_i}^2 + s_0 - m_{\rho_i}^2} \arctan \frac{m_{\rho_i}^2 - m_{\rho_i}^2}{m_{\rho_i}^2 - m_{\rho_i}^2} \theta(m_{\rho_i} - m_{P}) - \right.$$

$$\frac{1}{2} \left[ \left( m_{\rho_i}^2 - m_{\rho_i}^2 \right) \left( m_{\rho_i}^2 - m_{P}^2 \right) \right]^{3/2} \ln \frac{m_{\rho_i}^2 - m_{\rho_i}^2}{m_{\rho_i}^2 - m_{\rho_i}^2} \times \theta(m_{\rho_i} - m_{P}^2)) \right\},$$

$$\Pi_1^{(VP)} = \frac{s}{m_{\rho_i}^2} \left\{ \left( m_{\rho_i}^2 - m_{\rho_i}^2 \right) \left( m_{\rho_i}^2 - m_{P}^2 \right) \right\}^{3/2} \arctan \frac{m_{\rho_i}^2 - m_{\rho_i}^2}{m_{\rho_i}^2 - m_{\rho_i}^2} \theta(m_{\rho_i}^2 - m_{P}^2) -$$

$$\frac{1}{2} \left[ \left( m_{\rho_i}^2 - m_{\rho_i}^2 \right) \left( m_{\rho_i}^2 - m_{P}^2 \right) \right]^{3/2} \ln \frac{m_{\rho_i}^2 - m_{\rho_i}^2}{m_{\rho_i}^2 - m_{\rho_i}^2} \theta(m_{\rho_i}^2 - m_{P}^2) +$$

$$\arctan \frac{s - m_{\rho_i}^2}{m_{\rho_i}^2 - s} \theta(s - m_{\rho_i}^2) \theta(m_{\rho_i}^2 - s) + \left( \frac{\pi}{2} - \frac{1}{2} \ln \frac{s - m_{\rho_i}^2}{m_{\rho_i}^2 - s} \right) \theta(s - m_{\rho_i}^2), \quad (2.9)$$

while $m_{\pm} = m_{V} \pm m_{P}$, and $\theta$ is the usual step function. The dependence of $\text{Re}\Pi_{\rho_{i}i}^{(VP)}(s)$ on energy squared is shown in Fig. 1.

**C. Axial-vector – pseudoscalar loop**

The axial vector – pseudoscalar meson state $AP = a_1(1260)\pi$ is considered to be one of the states contribut-
ing to the four-pion production amplitude \[16\]. For soft pions, when taking into account the requirements of chiral symmetry, this amplitude and the corresponding partial width, are very complicated \[17–20\]. This prevents one from using the dispersion relation to obtain the contribution of the four pion state to the polarization operator of the state \(\rho\). In the present work, the simplest \(a_1\pi\) dominance model of the four-pion \[21\] production is used: \(e^+e^- \rightarrow \rho_i \rightarrow a_1\pi \rightarrow 4\pi\). The amplitude of the transition \(\rho_i \rightarrow a_\pi\) is chosen in the simplest form

\[
A(\rho_q \rightarrow a_{1k}\pi_p) = g_{\rho_i a_1\pi}[(\epsilon_{a_1}\epsilon_{\rho_i})(kq) - (\epsilon_{a_1},q)(\epsilon_{\rho_i},k)],
\]

where \(q, k,\) and \(p\) are, respectively, the four-momenta of the mesons \(\rho_i, a_1,\) and \(\pi,\) while \(\epsilon_{a_1}\) and \(\epsilon_{\rho_i}\) denote the polarization four-vectors of \(a_1\) and \(\rho_i\). The expression \[\text{(2.10)}\] is chosen on the grounds that it should be explicitly transverse in the \(\rho_i\) leg.

The polarization operators due to the AP loop, both diagonal and non-diagonal, are represented in the form

\[
\Pi_{\rho_i\rho_j}^{(AP)} = g_{\rho_i,AP}g_{\rho_j,AP}\Pi^{(AP)},
\]

where the quantity \(\Pi^{(AP)}(s, m_{\rho_i}, m_A, m_P)\) is calculated from the dispersion relation

\[
\frac{\Pi^{(AP)}}{s} = \frac{1}{48\pi^2} \int_{(m_A+m_P)^2}^{\infty} \frac{q_{AP}(s', m_A, m_P)}{(s')^{3/2}(s'-s-i\epsilon)} \times \frac{[s'+m_A^2-m_P^2]^3}{([s'+m_A^2-m_P^2]^2+2s'm_A^2)]},
\]

where \(m_\pm = m_A \pm m_P\).

\[\text{FIG. 1.} \text{ The dependence of } Re\Pi_{\rho_i\rho_j}^{(VP)}(s) \text{ on energy squared. The insertion shows a smaller region } -0.2 \text{ GeV}^2 \leq s \leq 0.8 \text{ GeV}^2 \text{ demonstrating the vanishing of } Re\Pi_{\rho_i\rho_j}^{(VP)}(s) \text{ at } s = 0 \text{ and } s = m_{\rho_i}^2.\]

\[\text{FIG. 2.} \text{ The dependence of } Re\Pi_{\rho_i\rho_j}^{(AP)}(s) \text{ on energy squared } s \text{ for the } AP = a_1(1260)\pi \text{ and } AP = K_1(1270)\bar{K} \text{ loops demonstrating discontinuity of } \frac{dRe\Pi^{(AP)}}{ds} \text{ at } s = (m_A + m_P)^2. \text{ The necessary parameters are taken from the } \pi^+\pi^-\pi^+\pi^- \text{ column of the Table II}.\]

\[
\Gamma_{\rho_i \rightarrow AP}(s) = \frac{g_{\rho_i,AP}^2}{48\pi s} \left([s + m_A^2 - m_P^2]^2 + 2s m_A^2\right) \times \frac{q_{AP}(s, m_A, m_P)}{\left(s_0 + m^2_{\rho_i}\right)} \left(s_0 + s\right).
\]

for the \(\rho_i \rightarrow AP\) decay width found from the effective vertex Eq. \[\text{(2.10)}\] is inserted into the integrand of the dispersion relation. The result of integration is represented in the form

\[
\Pi^{(AP)} = \frac{s}{96\pi^2} \left(m_+ \right)^3 \left(m_+^2 + s_0 \right) \left[ J(s) - \frac{m_0^2}{m_+^2 + s_0} \right],
\]

where

\[
J(s) = sf(s) \left[ \left(\frac{m_0^2}{s} - 1\right)^2 + \frac{2m_A^2}{m_+} \times \right.
\]

\[
\times \left(s + \frac{s_0}{m_0^2 + s_0}\right) \times \left[s + \frac{s_0}{m_0^2 + s_0}\right] \times \left[s + \frac{s_0}{m_0^2 + s_0}\right],
\]
The dependence of $\text{Re} \Pi$ with $\Pi$ of pseudoscalar (VP) a-agonal polarization operators are resonances $\rho(1540), \rho(1700), \ldots$ is the subject of current and future studies, the quark-antiquark model relations between their coupling constants are assumed:

$$g_{\rho_i, K^*} = \frac{1}{2} g_{\rho_i, \omega \pi} ,$$

with $\frac{1}{2} \Pi_{\rho_i, K^*} = \frac{1}{2} \Pi_{\rho_i, \omega \pi}$ given by Eq. (2.18). Second, it is the vector-pseudoscalar (VP) $\omega \pi$ and $K^* K^+ K^*$ loops:

$$\Pi_{\rho_i, \omega \pi} = g_{\rho_i, \omega \pi} [\Pi_{\omega \pi}(s, m_{\rho_i}, m_{\omega}) + \Pi_{\omega \pi}(s, m_{\rho_i}, m_{\omega}, m_{\pi}) + \Pi_{\omega \pi}(s, m_{\rho_i}, m_{K^*}, m_{K^*})],$$

with $\Pi_{\omega \pi}(s, m_{\rho_i}, m_{\rho_i}, m_{\omega}, m_{\pi})$ given by Eq. (2.18). Third, it is the axial vector-pseudoscalar (AP) $a_1 \pi^+$, $K_{1(1270)K^+}$ c.c. loops:

$$\Pi_{\rho_i, \omega \pi} = 2 g_{\rho_i, \omega \pi} [\Pi_{\omega \pi}(s, m_{\rho_i}, m_{\omega}, m_{\pi}) + \Pi_{\omega \pi}(s, m_{\rho_i}, m_{K_{1(1270)}}, m_{K^*})],$$

with $\Pi_{\omega \pi}(s, m_{\rho_i}, m_{\omega}, m_{\pi})$ given by Eq. (2.18). Although the nature of the higher resonances $\rho(1540), \rho(1700), \ldots$ is the subject of current and future studies, the quark-antiquark model relations between their coupling constants are assumed:

$$g_{\rho_i, K^*} = \frac{1}{2} g_{\rho_i, \omega \pi} ,$$

with $\frac{1}{2} \Pi_{\rho_i, K^*} = \frac{1}{2} \Pi_{\rho_i, \omega \pi}$ given by Eq. (2.18). Second, it is the vector-pseudoscalar (VP) $\omega \pi$ and $K^* K^+ K^*$ loops:

$$\Pi_{\rho_i, \omega \pi} = g_{\rho_i, \omega \pi} [\Pi_{\omega \pi}(s, m_{\rho_i}, m_{\omega}) + \Pi_{\omega \pi}(s, m_{\rho_i}, m_{\omega}, m_{\pi}) + \Pi_{\omega \pi}(s, m_{\rho_i}, m_{K^*}, m_{K^*})],$$

with $\Pi_{\omega \pi}(s, m_{\rho_i}, m_{\rho_i}, m_{\omega}, m_{\pi})$ given by Eq. (2.18). Third, it is the axial vector-pseudoscalar (AP) $a_1 \pi^+$, $K_{1(1270)K^+}$ c.c. loops:

$$\Pi_{\rho_i, \omega \pi} = 2 g_{\rho_i, \omega \pi} [\Pi_{\omega \pi}(s, m_{\rho_i}, m_{\omega}, m_{\pi}) + \Pi_{\omega \pi}(s, m_{\rho_i}, m_{K_{1(1270)}}, m_{K^*})],$$

with $\Pi_{\omega \pi}(s, m_{\rho_i}, m_{\rho_i}, m_{\omega}, m_{\pi})$ given by Eq. (2.18). Although the nature of the higher resonances $\rho(1540), \rho(1700), \ldots$ is the subject of current and future studies, the quark-antiquark model relations between their coupling constants are assumed:

$$g_{\rho_i, K^*} = \frac{1}{2} g_{\rho_i, \omega \pi} ,$$

Polarization operators which take into account three channels described above are the following. The full di-agonal polarization operators are

$$\Pi_{\rho_i, \omega \pi} = \Pi_{\rho_i, \omega \pi} + \Pi_{\rho_i, \omega \pi} + \Pi_{\rho_i, \omega \pi},$$

while the full non-diagonal ones $i, j \neq 1$ are

$$\Pi_{\rho_i, \omega \pi} = \Pi_{\rho_i, \omega \pi} + \Pi_{\rho_i, \omega \pi} + \Pi_{\rho_i, \omega \pi},$$

$$\Pi_{\rho_i, \omega \pi} = \Pi_{\rho_i, \omega \pi} + \Pi_{\rho_i, \omega \pi} + \Pi_{\rho_i, \omega \pi} + sa_{ij} \Pi_{\rho_i, \omega \pi}.$$
demand that Re$\Pi_{\rho_1,\rho_2,\ldots}(m_{\rho_i}^2) = 0$. This type of argument is not applicable in case of poor-studied resonances $\rho_2 = \rho(1450)$, $\rho_3 = \rho(1700)$, ... so the real parts of the non-diagonal polarization operators Re$\Pi_{\rho_i,\rho_j}$, $i \neq j$, may admit additional contributions parameterized here as $a_{ij}s$, with $s$ introduced in order to preserve the normalization $F_\omega(0) = 1$.

### III. QUANTITIES FOR COMPARISON WITH THE DATA

Three channels of $e^+e^-$ annihilation are considered in the present work. They are

$$e^+e^- \rightarrow \pi^+\pi^-,$$

(3.1)

$$e^+e^- \rightarrow \omega\pi^0,$$

(3.2)

and

$$e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^-.$$

(3.3)

The justification of the restriction to these reactions are given below in Sec. [IV]. Let us turn to the working expressions necessary for the comparison with the experimental data.

#### A. $\pi^+\pi^-$ production

In the present case, the relevant quantity is the so-called bare cross section of the reaction (3.1):

$$\sigma_{\text{bare}} = \frac{8\pi\alpha^2}{3s^{5/2}} |F_\pi(s)|^2 q_\pi^3(s) \left[1 + \frac{\alpha}{\pi} a(s)\right],$$

(3.4)

where $F_\pi(s)$ is the pion form factor Eq. (1.1), the quantity $a(s)$ takes into account the radiation by the final pions, and $\alpha = 1/137$ is the fine structure constant. The necessary discussion concerning the quantities in Eq. (3.4) are given elsewhere [1].

#### B. $\omega\pi^0$ production

The cross section of the reaction $e^+e^- \rightarrow \omega\pi^0$ is taken in the form

$$\sigma_{e^+e^-\rightarrow\omega\pi^0} = \frac{4\pi\alpha^2}{3s^{3/2}} |A_{e^+e^-\rightarrow\omega\pi^0}|^2 \times q_\omega^3(s, m_\omega, m_\pi),$$

(3.5)

where $q_\omega$ is given by Eq. (2.7),

$$A_{e^+e^-\rightarrow\omega\pi^0} = (g_{\gamma\rho_1}, g_{\gamma\rho_2}, g_{\gamma\rho_3}, \ldots)G^{-1} \times \begin{pmatrix} g_{\rho_1\omega} \\ g_{\rho_2\omega} \\ g_{\rho_3\omega} \\ \vdots \end{pmatrix},$$

(3.6)

is the amplitude of the reaction, and the matrix

$$G = \begin{pmatrix} D_1 & -\Pi_{12} & -\Pi_{13} & \cdots \\ -\Pi_{12} & D_2 & -\Pi_{23} & \cdots \\ -\Pi_{13} & -\Pi_{23} & D_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(3.7)

is introduced in order to take into account the strong mixing of the resonances $\rho_i$ [1]. Here, $\Pi_{ij} \equiv \Pi_{\rho_i,\rho_j}$ are the polarization operators. See Eqs. (2.22) and (2.23). The non-diagonal $i \neq j$ terms describe the mixing. Inverse propagators $D_i$ are given by Eq. (2.10).

#### C. $\pi^+\pi^-\pi^+\pi^-$ production

The width of the decay $\rho_i \rightarrow 2\pi^+2\pi^-$ in the model (2.10) is represented in the form

$$\Gamma_{\rho_i \rightarrow 2\pi^+2\pi^-}(s) = g_{\rho_i a_1}^2 W_{\pi^+\pi^-\pi^+\pi^-}(s),$$

(3.8)

where

$$W_{\pi^+\pi^-\pi^+\pi^-}(s) = \frac{1}{12\pi} \int \frac{(\sqrt{s} - m_\pi)^2}{(3m_\pi)^2} \rho_{a_1}(m^2) \times \left[s + m^2 - m_\pi^2\right]^2 \frac{2s}{2s} - m^2 \frac{q_{a_1\pi}(s, m^2, m_\pi^2)dm^2}{q_{a_1\pi}(s, m^2, m_\pi^2)dm^2}$$

(3.9)

and $q_{a_1\pi}$ is given by Eq. (2.7). The function

$$\rho_{a_1}(m^2) = \frac{m_{a_1} \Gamma_{a_1}/\pi}{(m^2 - m_{a_1}^2)^2 + m_{a_1}^2 \Gamma_{a_1}^2}$$

(3.10)

is introduced to take into account the large width of the intermediate $a_1$ resonance in a minimal way, by taking the limit of the fixed $a_1$ width. The mass and width of $a_1(1260)$ are determined from fitting the data on the reaction $e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^-$. The cross section of the reaction $e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^-$ is represented in the form
with $W_{\pi^+\pi^-\pi^+\pi^-}(s)$ given by Eq. (3.10), represents the effective phase space volume of four pions via smearing of the $a_1\pi$ phase space volume. The matrix $G$ is the matrix of inverse propagators, Eq. (3.7).

IV. RESULTS OF DATA FITTING

This section is devoted to the presentation of the results of fitting the data on the reactions $e^+e^-\rightarrow\pi^+\pi^-$, $e^+e^-\rightarrow\omega\pi^0$, and $e^+e^-\rightarrow\pi^+\pi^-\pi^+\pi^-$. Two possible schemes of fitting were used.

- Scheme 1. Three resonances $\rho(770) + \rho(1450) + \rho(1700)$ and the loops PP of pseudoscalar mesons are taken into account [1, 2].

When adding the vector-pseudoscalar (VP) and the axial-vector – pseudoscalar (AP) loops to the pion form factor one should also include also the VP and AP final states in consideration. These states manifest, respectively, in the reactions $e^+e^-\rightarrow\omega\pi^0$ and $e^+e^-\rightarrow\pi^+\pi^-\pi^+\pi^-$ which should also be treated in the present framework. Since the energies higher than 2 GeV are considered, the third heavy isovector resonance $\rho(2100)$ is added. Hence, the scheme with the resonances $\rho(770) + \rho(1450) + \rho(1700) + \rho(2100)$ and the PP, VP, and AP loops in the polarization operators is used. This is the scheme 2:

- Scheme 2. The data on the reactions $e^+e^-\rightarrow\pi^+\pi^-$ [2], $e^+e^-\rightarrow\omega\pi^0$ [12], and $e^+e^-\rightarrow\pi^+\pi^-\pi^+\pi^-$ [14] are fitted separately in the model which takes into account the resonances $\rho(770) + \rho(1450) + \rho(1700) + \rho(2100)$ side by side with allowing for the PP, VP, and AP loops in the polarization operators.

The parameters found from the fitting scheme 2 are listed in the Table 1. Let us comment on each of the three mentioned channels.

A. Fitting $e^+e^-\rightarrow\pi^+\pi^-$ data

When fitting the data on the reaction $e^+e^-\rightarrow\pi^+\pi^-$ at energies $\sqrt{s} \leq 1$ GeV in our previous publication [1–3], the fitting scheme 1 was used. There, the restriction to the PP loop was justifiable because of rather low energies under consideration. Using the resonance parameters found from fitting the data in the time-like region, the pion form factor $F_{\pi}(s)$ in the space-like region $s < 0$ was calculated up to $-s = Q^2 = 0.2$ GeV$^2$ and compared with available data [9]. The comparison with the data in the wider range up to $-s = Q^2 = 10$ GeV$^2$ was made in Ref. [9]. In the present work, we give the corresponding plot in Fig. 3 for the sake of completeness. The continuation to the spacelike domain in the fitting scheme 2 is discussed below.

The cross section of the reaction $e^+e^-\rightarrow\pi^+\pi^-$ fitted in the scheme 2 is shown in Fig. 4. As for the $\rho(770)$ resonance parameters are concerned, one can observe that in comparison with the fit in the scheme 1 [1–3], the bare mass of the resonance $\rho_1$ determined in the scheme 2 in the sensitive channel $e^+e^-\rightarrow\pi^+\pi^-$ is typically lower. Compare the table in Fig. 4 here and the table I in e.g., Ref. [1]. The same concerns the coupling constant $g_{\rho_1}$ which parameterizes the leptonic decay width Eq. (1.3). The coupling constant $g_{\rho_1\pi\pi}$ in the scheme 2 is greater than in the scheme 1. The above distinction can be qualitatively explained by the effect of renormalization of the coupling constants described in Ref. [1]. Indeed, as is shown in
TABLE I. The resonance parameters found from fitting the data on the reactions $e^{+}e^{-} \rightarrow \pi^{+}\pi^{-}$, $e^{+}e^{-} \rightarrow \pi^{+}\pi^{-}\pi^{+}\pi^{-}$, and $e^{+}e^{-} \rightarrow \omega\pi^{0}$, in the fitting scheme 2, see text. The parameter $\text{Re}\Pi_{\rho}$ is responsible parameterizes the $\omega\rho$ mixing, see Ref. 3 for more detail. The parameter $g_{\rho_{4}}$ is not given because it is fixed by the sum rule $\frac{g_{\rho_{4}\pi\pi}}{r_{1}} + \frac{g_{\rho_{4}\omega\pi}}{r_{2}} + \frac{g_{\rho_{4}\omega\omega}}{r_{3}} + \frac{g_{\rho_{4}\omega\gamma}}{r_{4}} = 1$ necessary for correct normalization $F_{\pi}(0) = 1$.

| parameter        | $e^{+}e^{-} \rightarrow \pi^{+}\pi^{-}$ | $e^{+}e^{-} \rightarrow \pi^{+}\pi^{-}\pi^{+}\pi^{-}$ | $e^{+}e^{-} \rightarrow \omega\pi^{0}$ |
|------------------|---------------------------------|-------------------------------------------------|---------------------------------|
| $m_{\rho_{1}}$ [MeV] | $766.9 \pm 0.2$                 | $764 \pm 1$                                    | $777$                                           |
| $g_{\rho_{1}\pi\pi}$ | $6.345 \pm 0.003$              | $7.99 \pm 0.01$                                | $6.21 \pm 0.03$                                |
| $g_{\rho_{1}}$           | $4.666 \pm 0.002$              | $5.817 \pm 0.003$                              | $4.52 \pm 0.02$                                |
| $g_{\rho_{1}\omega\pi}$ | $2.08 \pm 0.05$                | $0.254 \pm 0.005$                              | $-0.7 \pm 0.3$                                 |
| $m_{\omega}$ [MeV]          | $782.01 \pm 0.09$             | $782.01$                                       | $782.01$                                       |
| $\text{Re}\Pi_{\rho} \times 10^{3}$ [GeV] | $4.41 \pm 0.10$                | $-$                                            | $-$                                            |
| $m_{\rho_{2}}$ [MeV]          | $1307 \pm 4$                   | $1323 \pm 3$                                   | $1747 \pm 4$                                   |
| $g_{\rho_{2}\pi\pi}$         | $0.01 \pm 0.07$                | $1.71 \pm 0.01$                                | $0.1 \pm 0.1$                                  |
| $g_{\rho_{2}}$             | $> 10^{5}$                     | $10.94 \pm 0.02$                               | $200 \pm 0$                                    |
| $g_{\rho_{2}\omega\pi}$      | $-15.9 \pm 0.2$                | $-16.3 \pm 0.02$                               | $1.45 \pm 0.1$                                 |
| $m_{\rho_{3}}$ [MeV]          | $1491 \pm 2$                   | $1596 \pm 1$                                   | $1880 \pm 160$                                 |
| $g_{\rho_{3}\pi\pi}$         | $-3.13 \pm 0.05$               | $-3.42 \pm 0.01$                               | $38 \pm 4$                                    |
| $g_{\rho_{3}}$             | $600 \pm 200$                  | $12.14 \pm 0.05$                               | $420 \pm 40$                                   |
| $g_{\rho_{3}\omega\pi}$      | $4.10 \pm 0.10$                | $4.41 \pm 0.02$                                | $110 \pm 7$                                    |
| $g_{\rho_{3}\omega\gamma}$   | $2.00 \pm 0.02$                | $1.286 \pm 0.005$                              | $18 \pm 4$                                    |
| $m_{K_{1}(1270)}$ [MeV]      | $2008 \pm 8$                   | $1875 \pm 5$                                   | $2060 \pm 40$                                  |
| $g_{K_{1}(1270)}\pi\pi$      | $-3.49 \pm 0.02$               | $-0.56 \pm 0.001$                              | $-3.60 \pm 0.06$                               |
| $g_{K_{1}(1270)}\omega\pi$    | $-0.17 \pm 0.10$               | $-10.08 \pm 0.04$                              | $-19.0 \pm 0.3$                                |
| $g_{K_{1}(1270)}\omega\gamma$ | $0.50 \pm 0.08$                | $-0.311 \pm 0.004$                             | $0.4 \pm 0.2$                                  |
| $m_{\Omega_{c}(1600)}$ [MeV] | $1220 \pm 10$                  | $1184 \pm 1$                                   | $1230$                                         |
| $\Gamma_{\Omega_{c}(1600)}$  | $376 \pm 3$                    | $-312 \pm 0.001$                               | $0.2 \pm 0.2$                                  |
| $s_{0}$ [GeV$^{2}$]          | $4.31 \pm 0.04$                | $9.52 \pm 0.06$                                | $3.2 \pm 0.07$                                 |
| $a_{23}$                    | $-0.020 \pm 0.004$             | $0.120 \pm 0.001$                              | $0.58 \pm 0.11$                                |
| $a_{24}$                    | $-0.387 \pm 0.003$             | $0.56 \pm 0.002$                               | $-0.50 \pm 0.01$                               |
| $a_{34}$                    | $0.032 \pm 0.002$              | $-0.312 \pm 0.001$                             | $0.2 \pm 0.2$                                  |
| $\chi^{2}/N_{d.o.f}$        | $320/313$                       | $333/71$                                       | $48/19$                                        |

Ref. 1, the renormalization results in the substitutions

$$g_{\rho_{1}\pi\pi} \rightarrow Z_{\rho}^{-1/2} g_{\rho_{1}\pi\pi},$$

$$g_{\rho_{1}} \rightarrow Z_{\rho}^{1/2} g_{\rho_{1}},$$

where

$$Z_{\rho} = 1 + \frac{d\text{Re}\Pi_{\rho_{1}\rho_{1}}(s)}{ds} \bigg|_{s=m_{\rho_{1}}^{2}}.$$  

Eq. 4.1 means that the bare $g_{\rho_{1}\pi\pi}$ obtained from the fit is related to the “physical” one obtained from the visible peak, upon multiplying by $Z_{\rho}^{1/2}$, while the opposite is true for $g_{\rho_{1}}$. The contributions of the VP loop to $d\text{Re}\Pi_{\rho_{1}\rho_{1}}/ds$ near $s = m_{\rho_{1}}^{2}$, as is observed from Fig. 1, is positive and exceed the negative contribution from the PP loop. See Fig. 7 in Ref. 1. The same is true for the AP loop. As a result, one has $Z_{\rho} > 1$.

Although the energy behavior of the cross section up to $\sqrt{s} = 1.7$ GeV is described in the adopted model, including the dip near 1.5 GeV, one can see that the structure in the interval $2 - 2.5$ GeV demands, in all appearance, additional $\rho$–like resonances and/or intermediate states in the loops. Indeed, the contribution of $K_{1}(1400)$ states coupled solely to the resonance $\rho_{4}$, with the fitted coupling constant $g_{\rho_{4}K_{1}(1400)}$ and mass $m_{K_{1}(1400)}$, slightly improves the agreement in the interval $1.75 < \sqrt{s} < 2$ GeV. But it occurs at the expense of adding two additional free parameters $m_{K_{1}(1400)}$ and $g_{\rho_{4}K_{1}(1400)}$, with $m_{K_{1}(1400)} \approx 1505$ MeV. Moreover, the inclusion of the $K_{1}(1400)$ in the loops does not result in reproducing the peak near $\sqrt{s} = 2.3$ GeV. However, the magnitude of $\chi^{2}/N_{d.o.f}$ is reduced from 320/313 to 289/311.

The continuation to the space-like region $s < 0$ with the resonance parameters obtained in the region $s > 4m_{\pi}^{2}$ in the fitting scheme 2 with the VP and AP loops added, results in unwanted behavior of $F_{\pi}(s)$. See Fig. 4. Specifically, the curve goes through experimental points up to $s = -0.2$ GeV$^{2}$, but at larger values of $-s = Q^{2}$ one encounters infinities arising from the the Landau poles due to the VP and AP loops. As was pointed out in Ref. 1, the Landau pole is present even in the case of the PP loop, but its position is at $\sqrt{Q^{2}} \approx 90$ GeV, that is, it is far from accessible momentum transfers. In the case of the VP and AP loops the Landau poles appear in the region accessible to existing experiments 10 12.
FIG. 4. The cross section of the reaction $e^+e^- \rightarrow \pi^+\pi^-$. The data are [2], the curve is drawn using the resonance parameters of the scheme 2. The $\rho - \omega$ resonance region is shown in the insert.

because of both the large magnitude of coupling constant $g_{\rho_1\omega\pi} = 13.2$ GeV$^{-1}$ [22].

An important feature of the new expression for the pion form factor obtained in Ref. [1] which was not mentioned in that reference, is that it does not require [2] the commonly accepted Blatt – Weisskopf centrifugal factor [23]

$$C_\pi(k) = \frac{1 + R_\pi^2 k^2}{1 + R_\pi^2 k_R^2},$$

in the expression for $\Gamma_{\rho\pi\pi}(s)$ [16]. Here $k$ is the pion momentum at arbitrary energy while $k_R$ is its value at the resonance energy. The fact is that the usage of $R_\pi$ dependent centrifugal barrier penetration factor in particle physics (for example, in the case of the $\rho(770)$ meson [16]), results in the problem which is overlooked. Indeed,

the meaning of $R_\pi$ is that this quantity is the characteristic of the potential (or the $t$-channel exchange in field theory) resulting in the phase $\delta_{bg}$ of the potential scattering in addition to the resonance phase [23]. For example, in case of the $P$-wave scattering in the potential

$$U(r) = G\delta(r - R_\pi)$$

where the resonance scattering is possible, the background phase is

$$\delta_{bg} = -R_\pi k + \arctan(R_\pi k).$$

At the usual value of $R_\pi \sim 1$ fm, $\delta_{bg}$ is not small. However, in the $\rho$ meson region, the background phase shift $\delta_{bg}$ is negligible and the phase shift $\delta_1$ is completely determined by the resonance. See Fig. 8 in Ref. [1]. Therefore, the descriptions of the hadronic resonance distributions taking into account the parameter $R_\pi$, have a dubious character.

B. Fitting $e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^-$ data

The energy dependence of the cross section of the reaction $e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^-$ is shown in Fig. 6. One can see that at energies $\sqrt{s} > 1.75$ GeV the chosen scheme with three heavier rho-like resonances $\rho_{2,3,4}$ cannot reproduce the structures in the measured cross sec-
tion such as the bizarre sharp turn in the energy behavior followed by fluctuations. As in the case of the reaction such as the bizarre sharp turn in the energy behavior followed by fluctuations. As in the case of the reaction $e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^-$, the contributions of the VP loops $a_1(1260)\pi$ and $K_1(1270)\bar{K} + c.c.$ coupled to all $\rho_i$ resonances ($i = 1,2,3,4$) were invoked to explain the features above 1.75 GeV. The structures remain unexplained. As is seen from the Table I, the coupling constants of the $\rho_{1,2,3,4}$ meson found from fitting this channel, differ from those found from fitting the $\pi^+\pi^-$ one. Furthermore, the coupling constant $g_{\rho_1,\rho_2}$ found from fitting the channel $e^+e^- \rightarrow \pi^+\pi^-\pi^+\pi^-$, is suppressed in comparison with the naive chiral symmetry estimate $g_{a_1\rightarrow\rho_2\pi} \sim 1/2f_\pi \sim 5$ GeV$^{-1}$ [24], where $f_\pi = 92.4$ MeV is the pion decay constant. [Note the relation $g_{\rho_1,\rho_2} = g_{a_2\rightarrow\rho_2\pi}$ which is valid in the effective vertex approach Eq. (2.1)]. We believe that this difference is an artifact of the oversimplified $a_2\pi$ model and the price for the possibility of the analytical calculation of the VP and AP loops simulating the contributions of the multi-particle meson states in polarization operators. In the meantime, $g_{\rho_1,\rho_2} \approx 2.1$ GeV$^{-1}$ found from fitting the $e^+e^- \rightarrow \pi^+\pi^-$ channel, looks meaningful. For comparison, the estimates of $g_{\rho_{1,2,3,4}}$ in the model adopted in the present work are $\sim 6$ GeV$^{-1}$ and $\sim 4$ GeV$^{-1}$ extracted from $\Gamma_{\rho_1} \approx 0.6$ GeV and $0.3$ GeV [16], respectively. Note also that $\Gamma_{\rho_1\rightarrow\rho\pi} \sim 1$ GeV when evaluated in the generalized hidden local symmetry chiral model for $m_{\rho_1} \approx 1.2$ GeV [24], while the width of the visible peak in Fig. 6 is about 0.44 GeV.

C. Fitting $e^+e^- \rightarrow \omega\pi^0$ data

Quite recently, new data on the reaction $e^+e^- \rightarrow \omega\pi^0$ in the decay mode $\omega \rightarrow \pi^0\gamma$ were published by SND collaboration [13]. We use them in the fitting scheme 2. The resulting curve calculated with the parameters cited in the Table I is shown in Fig. 7.
V. CONCLUSION

The main goal of the present work is to describe the pion electromagnetic form factor $F_\pi(s)$ in all available energy range, using the expression obtained in Ref. [1]. This expression permits a good description of the data of SND, CMD-2, KLOE, BaBar on $\pi^+\pi^-$ production in $e^+e^-$ annihilation at $\sqrt{s} < 1$ GeV, describes the scattering kinematical domain up to $-s = Q^2 = 10$ GeV$^2$, and does not contradict the data on $\pi\pi$ scattering phase $\delta$. The goal of extending the description to the energies up to 3 GeV in the time-like domain is reached. Going to higher energies demands the inclusion of the vector – pseudoscalar (VP) and axial-vector – pseudoscalar (AP) loops. These loops contain the couplings of the $\rho$-like resonances with the states VP and AP and generate, in turn, the final states $\omega\pi^0$ and $\pi^+\pi^-\pi^+\pi^-$ in $e^+e^-$ annihilation. So, the consistency demands the treatment of these final states, too. As is shown in the present work, the energy behavior of the cross sections of the reactions $e^+e^-\rightarrow\omega\pi^0$ and $e^+e^-\rightarrow\pi^+\pi^-\pi^+\pi^-$ obtained in the adopted simplified model, does not contradict the data. The statistically poor description of the cross section of the reaction $e^+e^-\rightarrow\pi^+\pi^-\pi^+\pi^-$ is, probably, an artifact of the oversimplified model for its amplitude which ignores both the requirements of the chiral symmetry at the lower energies and a complicated intermediate states at higher energies. In the meantime, the continuation to the space-like domain of the expression for $F_\pi(s)$ with the contributions of the VP and AP loops meets the difficulty with the encountering the Landau poles. In all appearance, this is the consequence of the chosen parametrization of the vertex form factor which restricts the growth of the partial widths with the energy increase in a modest way. A more strong suppression would effectively suppress the coupling constants with the VP and AP states and, in turn, push the Landau zeros to higher spacelike momentum transfers.

We are grateful to M. N. Achasov for numerous discussions which stimulated the present work. The work is supported in part by Russian Foundation for Basic Research Grant no. 13-02-00039 and the Interdisciplinary project No 102 of Siberian Division of Russian Academy of Sciences.

References:

[1] N. N. Achasov and A. A. Kozhevnikov, Phys. Rev. D83, 113005 (2011). Erratum-ibid. D85, 019901 (2012).
[2] N. N. Achasov and A. A. Kozhevnikov, Nucl.Phys.Proc.Suppl. 225-227, 10 (2012).
[3] N. N. Achasov and A. A. Kozhevnikov, JETP Letters 96, 559 (2013) [Pis'ma v ZhETF 96, 627 (2012)].
[4] M. N. Achasov, et al., J.Exp.Theor.Phys. 101, 1053 (2005); Zh.Eksp.Teor.Fiz. 101, 1201 (2005) [arXiv:hep-ex/0506076v1].
[5] R. R. Akhmetshin, et al. (CMD–2 Collaboration), Phys.Lett.B648, 28 (2007) [arXiv:hep-ex/0610021v3].
[6] F. Ambrosino, et al. (KLOE Collaboration), Phys.Lett. B700, 102 (2011) [arXiv:1006.5313].
[7] B. Aubert, et al. (The BABAR Collaboration), Phys.Rev.Lett.103, 231801 (2009) [arXiv:0908.3589v1].
[8] N. N. Achasov and A. A. Kozhevnikov, Phys. Rev. D55, 2663 (1997).
[9] S. R. Amendolia, et al. Nucl.Phys.B277, 168 (1986).
[10] C. J. Bebek, et al. Phys.Rev.D17, 1693 (1978).
[11] T. Horn, et al. Phys.Rev.Lett.97., 192001 (2006).
[12] V. Tadevosyan, et al. Phys.Rev.C75, 055205 (2007).
[13] M. N. Achasov, et. al. Study of $e^+e^-\rightarrow\omega\pi^0\rightarrow\pi^0\pi^0\gamma$ in the energy range 1.05 – 2.00 GeV with SND. [arXiv:1303.5198 [hep-ex]].
[14] J. P. Lees, et al. (The BABAR Collaboration), Phys. Rev. D85, 112009 (2012).
[15] We use the opportunity to correct the misprint in the expression for $\Pi^{(PP)}_1$ in Refs. [2,3] where the braces were omitted when typesetting.
[16] J. Beringer, et al. (Particle Data Group), Phys. Rev. D86, 010001 (2012).
[17] N. N. Achasov and A. A. Kozhevnikov, Phys.Rev. D62, 056011 (2000).
[18] N. N. Achasov and A. A. Kozhevnikov, J.Exp.Theor.Phys. 91, 433 (2000) [Zh.Eksp.Teor.Fiz. 91, 499 (2000)].
[19] N. N. Achasov and A. A. Kozhevnikov, JETP Lett. 88, 1 (2008)[Pis’ma v ZhETF 88, 3 (2008)].
[20] N. N. Achasov and A. A. Kozhevnikov, Eur.Phys.J. A38, 61 (2008).
[21] R. R. Akhmetshin, et al. (CMD-2 Collab.), Phys.Lett. B466, 392 (1999).
[22] The coupling constant $g_{\rho\omega\pi}$ is extracted from the partial width of the decay $\omega(782)\rightarrow\pi^+\pi^-\pi^0$.
[23] J. M. Blatt and V. F. Weisskopf. Theoretical Nuclear Physics, (Wiley, New-York – London, 1952)
[24] N. N. Achasov and A. A. Kozhevnikov, Phys.Rev. D71, 034015 (2005).