ABEL-TYPE WEIGHTED MEANS TRANSFORMATIONS INTO $\ell$

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ABSTRACT. Let $q_k = \binom{k + \alpha}{k}$ for $\alpha > -1$ and $Q_n = \sum_{k=0}^{n} q_k$. Suppose $A_q = \{a_{nk}\}$, where $a_{nk} = q_k / Q_n$ for $0 \leq k \leq n$ and 0 otherwise. $A_q$ is called the Abel-type weighted mean matrix. The purpose of this paper is to study these transformations as mappings into $\ell$. A necessary and sufficient condition for $A_q$ to be $\ell$-$\ell$ is proved. Also some other properties of the $A_q$ matrix are investigated.

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1. Introduction. Throughout this paper, we assume that $\alpha > -1$ and $Q_n$ is the partial sums of the sequence $\{q_k\}$, where $q_k$ is as above. Let $A_q = \{a_{nk}\}$. Then the Abel-type weighted mean matrix, denoted by $A_q$, is defined by

$$a_{nk} = \begin{cases} \frac{q_k}{Q_n} & \text{for } 0 \leq k \leq n, \\ 0 & \text{for } k > n. \end{cases} \quad (1.1)$$

The $A_q$ matrix is the weighted mean matrix that is associated with the Abel-type matrix introduced by M. Lemma in [5]. It is regular, indeed, totally regular.

2. Basic notation and definitions. Let $A = (a_{nk})$ be an infinite matrix defining a sequence summability transformation given by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad (2.1)$$

where $(Ax)_n$ denotes the $n$th term of the image sequence $Ax$. Let $y$ be a complex number sequence. Throughout this paper, we use the following basic notation and definitions:

(i) $c = \{\text{The set of all convergent complex sequences}\}$,
(ii) $\ell = \{y : \sum_{k=0}^{\infty} |y_k| < \infty\}$,
(iii) $\ell^p = \{y : \sum_{k=0}^{\infty} |y_k|^p < \infty\}$,
(iv) $\ell(A) = \{y : Ay \in \ell\}$,
(v) $G = \{y : y_k = O(r^k) \text{ for some } r \in (0,1)\}$,
(vi) $G_w = \{y : y_k = O(r^k) \text{ for some } r \in (0,w), 0 < w < 1\}$. 
**Definition 1.** If $X$ and $Y$ are sets of complex number sequences, then the matrix $A$ is called an $X$-$Y$ matrix if the image $Au$ of $u$ under the transformation $A$ is in $Y$, whenever $u$ is in $X$.

3. **Some basic facts.** The following facts are used repeatedly.

1. For any real number $\alpha > -1$ and any nonnegative integer $k$, we have
   \[
   \binom{k + \alpha}{k} \sim \frac{k^\alpha}{\Gamma(\alpha + 1)} \quad \text{as } k \to \infty. \tag{3.1}
   \]

2. For any real number $\alpha > -1$, we have
   \[
   \sum_{k=0}^{n} \binom{k + \alpha}{k} = \binom{n + \alpha + 1}{n}. \tag{3.2}
   \]

3. Suppose $\{a_n\}$ is sequence of nonnegative numbers with $a_0 > 0$, that
   \[
   A_n = \sum_{k=0}^{n} a_k \to \infty. \tag{3.3}
   \]

Let
   \[
   a(x) = \sum_{k=0}^{\infty} a_k x^k, \quad A(x) = \sum_{k=0}^{\infty} A_k x^k, \quad (3.4)
   \]

and suppose that
   \[
   a(x) < \infty \quad \text{for } 0 < x < 1. \tag{3.5}
   \]

Then it follows that
   \[
   (1 - x)A(x) = a(x) \quad \text{for } 0 < x < 1. \tag{3.6}
   \]

4. **The main results**

**Lemma 1.** If $A_\alpha$ is an $\ell$-$\ell$ matrix, then $1/Q \in \ell$.

**Proof.** By the Knopp-Lorentz theorem [4], $A_\alpha$ is an $\ell$-$\ell$ matrix implies that
   \[
   \sum_{k=0}^{\infty} |a_{n,0}| < \infty, \tag{4.1}
   \]

and consequently we have $1/Q \in \ell$. \hfill \Box

**Lemma 2.** We have that $1/Q \in \ell$ if and only if $\alpha > 0$.

**Proof.** By using (3.1), we have
   \[
   \frac{1}{Q_n} \sim \frac{\Gamma(\alpha + 2)}{n^{\alpha + 1}} \tag{4.2}
   \]

and hence the assertion easily follows. \hfill \Box
Lemma 3. If \(1/Q \in \ell\), then \(A_q\) is an \(\ell-\ell\) matrix.

Proof. By Lemma 2, we have \(\alpha > 0\). To show that \(A_q\) is an \(\ell-\ell\) matrix, we must show that the condition of the Knopp-Lorentz theorem [4] holds. Using (3.1), we have

\[
\sum_{n=0}^{\infty} |a_{nk}| = \left(\frac{k+\alpha}{k}\right) \sum_{n=k}^{\infty} \frac{1}{Q_n} \frac{1}{k} = \left(\frac{k+\alpha}{k}\right) \sum_{n=k}^{\infty} \frac{1}{n^{\alpha+1}}
\]

\[
\leq M_1 K^\alpha \sum_{n=k}^{\infty} \frac{1}{n^{\alpha+1}} \quad \text{for some } M_1 > 0, \quad (4.3)
\]

\[
\leq M_1 M_2 k^\alpha \int_k^{\infty} \frac{dx}{x^{\alpha+1}} \quad \text{for some } M_2 > 0,
\]

\[
= \frac{M_1 M_2}{\alpha}.
\]

Hence, by the Knopp-Lorentz theorem [4], \(A_q\) is an \(\ell-\ell\) matrix.

Theorem 1. The following statements are equivalent:

1. \(A_q\) is an \(\ell-\ell\) matrix;
2. \(1/Q \in \ell\);
3. \(\alpha > 0\).

Proof. The theorem easily follows by Lemmas 1, 2, and 3.

Remark 1. In Theorem 1, we showed that \(A_q\) is an \(\ell-\ell\) matrix if and only if \(1/Q \in \ell\). But the converse is not true in general for any weighted mean matrix \(W_p\) that corresponds to a sequence-to-sequence variant of the general \(J_p\) power series method of summability [1]. To see this, let

\[
p_k = (\ln(k+2))^\alpha, \quad \alpha > 1. \quad (4.4)
\]

We show that \(1/P \in \ell\) but \(W_p\) is not an \(\ell-\ell\) matrix. We have

\[
P_n = \sum_{k=0}^{n} (\ln(k+2))^\alpha
\]

\[
\sim \int_0^n (\ln(x+2))^\alpha \, dx \quad \text{(by [6, Thm. 1.20])}
\]

\[
\sim (n+2)(\ln(n+2))^\alpha,
\]

using integration by parts repeatedly. This yields

\[
\frac{1}{P_n} \sim \frac{1}{(n+2)(\ln(n+2))^\alpha} \quad (4.6)
\]

and by the condensation test, it follows that \(1/P \in \ell\).
Next, we show that $W_p$ is not an $\ell$-$\ell$ matrix by showing that the condition of the Knopp-Lorentz theorem [4] fails to hold. Using (4.6), it follows that

$$\sum_{n=0}^{\infty} |a_{nk}| = (\ln(k+2))^\alpha \sum_{n=k}^\infty \frac{1}{p_n} \geq M_1 (\ln(k+2))^\alpha \sum_{n=k}^\infty \frac{1}{(n+2)(\ln(n+2))^\alpha} \text{ for some } M_1 > 0 \tag{4.7}$$

$$\geq M_1 M_2 (\ln(k+2))^\alpha \int_k^\infty \frac{dx}{(x+2)(\ln(x+2))^\alpha} \text{ for some } M_2 > 0$$

Thus, we have

$$\sup_k \left\{ \sum_{n=0}^{\infty} a_{nk} \right\} = \infty,$$  

and hence $W_p$ is not an $\ell$-$\ell$ matrix.

**Corollary 1.** $A_Q$ is an $\ell$-$\ell$ matrix.

**Proof.** Since $Q_n = \left(\binom{n+\alpha+1}{n}\right)$ and $\alpha > -1$ implies that $\alpha + 1 > 0$, the assertion easily follows by Theorem 1.

**Corollary 2.** $A_q$ is an $\ell$-$\ell$ matrix if and only if $\lim_n (Q_n/nq_n) < 1$.

**Proof.** By Theorem 1, $A_q$ is an $\ell$-$\ell$ matrix implies that $\alpha > 0$, and as a consequence we have $1/(\alpha + 1) < 1$. Now using (3.1), we have

$$\lim_n \left( \frac{Q_n}{nq_n} \right) = \lim_n \frac{n^{\alpha+1} \Gamma(\alpha+1)}{\Gamma(\alpha+2)n^{\alpha+1}} = \frac{1}{\alpha + 1} < 1. \tag{4.9}$$

Conversely, if $\lim_n (Q_n/nq_n) < 1$, then it follows from (4.9) that $1/(\alpha + 1) < 1$ and consequently we have $\alpha > 0$, and hence, by Theorem 1, $A_q$ is an $\ell$-$\ell$ matrix.

**Corollary 3.** Suppose that $z_k = \left(\binom{k+\beta}{k}\right)$ and $\alpha < \beta$; then $A_z$ is an $\ell$-$\ell$ matrix whenever $A_q$ is an $\ell$-$\ell$ matrix.

**Proof.** The corollary follows easily by Theorem 1.

**Lemma 4.** If the Abel-type matrix $A_{\alpha,t}$ [5] is an $\ell$-$\ell$ matrix, then $A_{\alpha+1,t}$ is also an $\ell$-$\ell$ matrix.

**Proof.** By the Knopp-Lorentz theorem [4], $A_{\alpha,t}$ is an $\ell$-$\ell$ matrix implies that

$$\sup_k \left\{ \sum_{n=0}^{\infty} |a_{nk}| \right\} < \infty. \tag{4.10}$$

This is equivalent to

$$\sup_k \left\{ \binom{k+\alpha}{k} \sum_{n=0}^{\infty} t_n^k (1-t_n)^{\alpha+1} \right\} < \infty. \tag{4.11}$$
Now from (4.11), we can easily conclude that

\[
\sup_k \left\{ \left( \frac{k + \alpha + 1}{k} \right) \sum_{n=0}^{\infty} t_n^k (1-t_n)^{\alpha+2} \right\} < \infty. \tag{4.12}
\]

Hence, \( A_{\alpha+1,t} \) is an \( \ell \)-\( \ell \) matrix.

The next theorem compares the summability fields of the matrices \( A_q \) and \( A_{\alpha,t} \) [5].

**THEOREM 2.** If \( A_{\alpha,t} \) and \( A_q \) are \( \ell \)-\( \ell \) matrices, then \( \ell(A_q) \subseteq \ell(A_{\alpha,t}) \).

**PROOF.** Let \( x \in \ell(A_q) \). Then we show that \( x \in \ell(A_{\alpha,t}) \). Let \( y \) be the \( A_q \)-transform of the sequence \( x \). Then we have

\[
y_nQ_n = \sum_{k=0}^{n} q_kx_k. \tag{4.13}
\]

Now since \( y_nQ_n \) is the partial sums of the sequence \( q_x \), using (3.6) it follows that

\[
(1-t_n) \sum_{k=0}^{\infty} Q_ky_k t_n^k = \sum_{k=0}^{\infty} q_kx_k t_n^k. \tag{4.14}
\]

This yields

\[
(1-t_n)^{\alpha+2} \sum_{k=0}^{\infty} Q_ky_k t_n^k = (1-t_n)^{\alpha+1} \sum_{k=0}^{\infty} q_kx_k t_n^k, \tag{4.15}
\]

and as a consequence we have \((A_{\alpha+1,t}y)_n = (A_{\alpha,t}x)_n \). By Lemma 4, \( A_{\alpha,t} \) is an \( \ell \)-\( \ell \) matrix implies that \( A_{\alpha+1,t} \) is also an \( \ell \)-\( \ell \) matrix, and from the assumption that \( x \in \ell(A_q) \), it follows that \( y \in \ell \). Consequently, we have \( A_{\alpha+1,t}y \in \ell \) and this is equivalent to \( A_{\alpha,t}x \in \ell \). Thus, \( x \in \ell(A_{\alpha,t}) \) and hence our assertion follows. \( \square \)

**REMARK 2.** Theorem 2 gives an important inclusion result in the \( \ell \)-\( \ell \) setting that parallels the famous inclusion result that exists between the power series method of summability and its corresponding weighted mean in the \( c \)-\( c \) setting [1].

**LEMMA 5.** Suppose \( A = \{a_{nk}\} \) is an \( \ell \)-\( \ell \) matrix such that \( a_{nk} = 0 \) for \( k > n, m > s \) (both positive integers); then \( \ell(A^s) \subseteq \ell(A^m) \), where the interpretation for \( A^s \) and \( A^m \) is as given in [6, p. 28].

**THEOREM 3.** If \( B = A_q \) is an \( \ell \)-\( \ell \) matrix, then \( B^m \) is also an \( \ell \)-\( \ell \) matrix (for \( m \) a positive integer greater than 1.)

**PROOF.** Let \( x \in \ell \), \( B \) is an \( \ell \)-\( \ell \) matrix implies that \( x \in \ell(B) \). By Lemma 5, we have \( \ell(B) \subseteq \ell(B^m) \) and hence it follows that \( x \in \ell(B^m) \). Hence, \( B^m \) is an \( \ell \)-\( \ell \) matrix. \( \square \)

**REMARK 3.** Theorem 3 gives a result that goes parallel to a \( c \)-\( c \) result given on [6, Thm. 2.4, p. 28].

In Corollary 1, we showed that \( A_Q \) is an \( \ell \)-\( \ell \) matrix. Here, a question may be raised as to whether \( A_Q \) maps \( \ell^p \) into \( \ell \) for \( p > 1 \). But this is answered negatively by the following theorem.
**Theorem 4.** $A_Q$ does not map $\ell^p$ into $\ell$ for $p > 1$.

**Proof.** Let $A_Q = \{b_{nk}\}$. Note that if $A_{Q,\alpha}$ maps $\ell^p$ into $\ell$, then by [3, Thm. 2], we must have

$$\lim_{k \to \infty} \sum_{n=1}^{\infty} |b_{nk}| = 0.$$  \hspace{1cm} (4.16)

Let

$$R_n = \sum_{k=1}^{n} Q_k,$$  \hspace{1cm} (4.17)

then it follows that

$$\sum_{n=1}^{\infty} b_{nk} = \frac{(k + \alpha + 1)}{k} \sum_{n=k}^{\infty} \frac{1}{R_n} = \frac{(k + \alpha + 1)}{k} \sum_{n=k}^{\infty} \frac{1}{\sum_{n=1}^{n+\alpha+2} n} \geq M_1 k^{\alpha+1} \sum_{n=k}^{\infty} \frac{1}{n^{\alpha+2}} \text{ for some } M_1 > 0$$

$$\geq M_1 M_2 k^{\alpha+1} \int_{k}^{\infty} \frac{dx}{x^{\alpha+2}} \text{ for some } M_2 > 0$$

$$= \frac{M_1 M_2}{\alpha+1} > 0.$$  \hspace{1cm} (4.18)

Thus, it follows that

$$\lim_{k \to \infty} \sum_{n=1}^{\infty} |b_{nk}| > 0,$$  \hspace{1cm} (4.19)

and hence $A_Q$ does not map $\ell^p$ into $\ell$ for $p > 1$ by [3, Thm. 2].

Our next theorem has the form of an extension mapping theorem. It indicates that a mapping of $A_q$ from $G$ or $G_w$ into $\ell$ can be extended to a mapping of $\ell$ into $\ell$.

**Theorem 5.** The following statements are equivalent:

1. $A_q$ is an $\ell-\ell$ matrix;
2. $A_q$ is a $G-\ell$ matrix;
3. $A_q$ is a $G_w-\ell$ matrix.

**Proof.** Since $G$ is a subset of $\ell$ and $G_w$ a subset of $G$, (1)$\Rightarrow$(2)$\Rightarrow$(3) follow easily. The assertion that (3)$\Rightarrow$(1) follows by [7, Thm. 1.1] and Theorem 1.

**Corollary 4.** (1) $A_Q$ is a $G-\ell$ matrix.
(2) $A_Q$ a $G_w-\ell$ matrix.

**Proof.** Since $A_Q$ is an $\ell-\ell$ matrix by Corollary 1, the assertion follows by Theorem 5.

**Corollary 5.** (1) If $A_q$ is a $G-G$ matrix, then $A_q$ is an $\ell-\ell$ matrix.
(2) If $A_q$ is a $G_w-G_w$ matrix, then $A_q$ is an $\ell-\ell$ matrix.

**Proof.** The assertion follows easily by Theorem 5.
**Theorem 6.** $A_q$ is a $G$-$G$ matrix if and only if $1/Q \in G$.

**Proof.** If $A_q$ is a $G$-$G$ matrix, then the first column of $A_q$ is must in $G$. This gives $1/Q \in G$ since $a_{n,0} = q_0/Q_n$. Conversely, suppose $1/Q \in G$. Then $1/Q_n \leq M_1 r^n$ for $M_1 > 0$ and $r \in (0,1)$. Now let $u \in G$, say $|u_k| \leq M_2 t^k$ for some $M_2 > 0$ and $t \in (0,1)$. Let $Y$ be the $A_q$-transform of the sequence $u$. Then we have

$$|Y_n| \leq M_1 M_2 r^n \sum_{k=0}^{n} \binom{k + \alpha}{k} t^k < M_1 M_2 r^n (1-t)^{-(\alpha+1)} < M_3 r^n$$

for some $M_3 > 0$. (4.20)

Therefore, $Y \in G$ and hence it follows that $A_q$ is a $G$-$G$ matrix.

**Theorem 7.** $A_q$ is a $G_w$-$G_w$ matrix if and only if $1/Q \in G_w$.

**Proof.** The proof follows easily using the same steps as in the proof of Theorem 6 by replacing $G$ with $G_w$.

**Lemma 6.** If the Abel-type matrix $A_{\alpha,t}$ [5] is a $G$-$G$ matrix, then $A_{\alpha+1,t}$ is also a $G$-$G$ matrix.

**Proof.** By [5, Thm. 7], $A_{\alpha,t}$ is $G$-$G$ implies that $(1-t)^{\alpha+1} \in G$. But $(1-t)^{\alpha+1} \in G$ yields $(1-t)^{\alpha+2} \in G$, and hence by [5, Thm. 7], it follows that $A_{\alpha+1,t}$ is a $G$-$G$ matrix.

**Lemma 7.** If the Abel-type matrix $A_{\alpha,t}$ [5] is a $G_w$-$G_w$ matrix, then $A_{\alpha+1,t}$ is also a $G_w$-$G_w$ matrix.

**Proof.** The assertion easily follows by replacing $G$ with $G_w$ in the proof of Lemma 6.

**Theorem 8.** If $A_{\alpha,t}$ [5] and $A_q$ are $G$-$G$ matrices, then the $G(A_{\alpha,t})$ contains $G(A_q)$.

**Proof.** The proof easily follows using the same techniques as in the proof of Theorem 3 by replacing $\ell$ with $G$ and applying Lemma 6.

**Theorem 9.** If $A_{\alpha,t}$ [3] and $A_q$ are $G_w$-$G_w$ matrices, then $G_w(A_{\alpha,t})$ contains $G_w(A_q)$.

**Proof.** The proof easily follows using the same techniques as in the proof of Theorem 3 by replacing $\ell$ with $G_w$ and applying Lemma 7.

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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

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