HOOK LENGTHS AND 3-CORES

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Abstract. Recently, the first author generalized a formula of Nekrasov and Okounkov which gives a combinatorial formula, in terms of hook lengths of partitions, for the coefficients of certain power series. In the course of this investigation, he conjectured that \(a(n) = 0\) if and only if \(b(n) = 0\), where integers \(a(n)\) and \(b(n)\) are defined by

\[
\sum_{n=0}^{\infty} a(n)x^n := \prod_{n=1}^{\infty} (1 - x^n)^8,
\]

\[
\sum_{n=0}^{\infty} b(n)x^n := \prod_{n=1}^{\infty} \frac{(1 - x^{3n})^3}{1 - x^n}.
\]

The numbers \(a(n)\) are given in terms of hook lengths of partitions, while \(b(n)\) equals the number of 3-core partitions of \(n\). Here we prove this conjecture.

1. Introduction and Statement of Results

In their work on random partitions and Seiberg-Witten theory, Nekrasov and Okounkov [8] proved the following striking formula:

\[
F_z(x) := \sum_{\lambda} x^{\vert \lambda \vert} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{n=1}^{\infty} (1 - x^n)^{z-1}.
\]

Here the sum is over integer partitions \(\lambda\), \(\vert \lambda \vert\) denotes the integer partitioned by \(\lambda\), and \(\mathcal{H}(\lambda)\) denotes the multiset of classical hooklengths associated to a partition \(\lambda\). In a recent preprint, the first author [3] has obtained an extension of (1.1), one which has a specialization which gives the classical generating function

\[
C_t(x) := \sum_{n=0}^{\infty} c_t(n)x^n = \prod_{n=1}^{\infty} \frac{(1 - x^{tn})^t}{1 - x^n}
\]

for the number of \(t\)-core partitions of \(n\). Recall that a partition is a \(t\)-core if none of its hook lengths are multiples of \(t\).

In the course of his work, the first author [4] formulated a number of conjectures concerning hook lengths of partitions. One of these conjectures is related to classical identities of Jacobi. For positive integers \(t\), he compared the functions \(F_{z^2}(x)\) and \(C_t(x)\). If \(t = 1\), we obviously have that

\[
F_1(x) = C_1(x) = 1.
\]

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For $t = 2$, by two famous identities of Jacobi, we have
\[
F_4(x) = \prod_{n=1}^{\infty} (1 - x^n)^3 = \sum_{k=0}^{\infty} (-1)^k (2k + 1) x^{(k^2 + k)/2},
\]
\[
C_2(x) = \prod_{n=1}^{\infty} \frac{(1 - x^{2n})}{1 - x^n} = \sum_{k=0}^{\infty} x^{(k^2 + k)/2}.
\]
In both pairs of power series one sees that the non-zero coefficients are supported on the same terms. For $t = 3$, we then have
\[
F_9(x) = \sum_{n=0}^{\infty} a(n) x^n := \prod_{n=1}^{\infty} (1 - x^n)^8
\]
\[
= 1 - 8x + 20x^2 - 70x^4 + \cdots - 520x^{14} + 57x^{16} + 560x^{17} + 182x^{20} + \cdots
\]
and
\[
C_3(x) = \sum_{n=0}^{\infty} b(n) x^n := \prod_{n=1}^{\infty} \frac{(1 - x^{3n})^3}{1 - x^n}
\]
\[
= 1 + x + 2x^2 + 2x^4 + \cdots + 2x^{14} + 3x^{16} + 2x^{17} + 2x^{20} + \cdots.
\]
Remark. It is clear that $b(n) = c_3(n)$.

In accordance with the elementary observations when $t = 1$ and 2, one notices that the non-zero coefficients of $F_9(x)$ and $C_3(x)$ appear to be supported on the same terms. Based on substantial numerical evidence, the first author made the following conjecture.

**Conjecture 4.6.** (Conjecture 4.6 of [4])

Assuming the notation above, we have that $a(n) = 0$ if and only if $b(n) = 0$.

Remark. The obvious generalization of Conjecture 4.6 and the examples above is not true for $t = 4$. In particular, one easily finds that
\[
F_{16}(x) = 1 - 15x + 90x^2 - \cdots + 641445x^{52} + 1537330x^{54} + \cdots,
\]
\[
C_4(x) = 1 + x + 2x^2 + 3x^3 + \cdots + 5x^{52} + 8x^{53} + 10x^{54} + \cdots.
\]

The coefficient of $x^{53}$ vanishes in $F_{16}(x)$ and is non-zero in $C_4(x)$.

Here we prove that Conjecture 4.6 is true. We have the following theorem.

**Theorem 1.1.** Assuming the notation above, we have that $a(n) = 0$ if and only if $b(n) = 0$. Moreover, we have that $a(n) = b(n) = 0$ precisely for those non-negative $n$ for which $\text{ord}_p(3n + 1)$ is odd for some prime $p \equiv 2 \pmod{3}$.

Remark. As usual, $\text{ord}_p(N)$ denotes the power of a prime $p$ dividing an integer $N$. 

Remark. Theorem 1.1 shows that \( a(n) = b(n) = 0 \) in a systematic way. The vanishing coefficients are associated to primes \( p \equiv 2 \pmod{3} \). If \( n \equiv 1 \pmod{3} \) has the property that \( \text{ord}_p(n) \) is odd, then we have

\[
a \left( \frac{n-1}{3} \right) = b \left( \frac{n-1}{3} \right) = 0.
\]

For example, since \( \text{ord}_5(10) = 1 \equiv 1 \pmod{2} \), we have that \( a(3) = b(3) = 0 \).

As an immediate corollary, we have the following.

**Corollary 1.2.** For positive integers \( N \), we have that

\[
\sum_{\lambda \vdash N} \prod_{h \in H(\lambda)} \left( 1 - \frac{9}{h^2} \right) = 0
\]

if and only if there are no 3-core partitions of \( N \).

Theorem 1.1 implies that “almost all” of the \( a(n) \) and \( b(n) \) are 0. More precisely, we have the following.

**Corollary 1.3.** Assuming the notation above, we have that

\[
\lim_{X \to +\infty} \frac{\#\{0 \leq n \leq X : a(n) = b(n) = 0\}}{X} = 1.
\]

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2. **Proofs**

It is convenient to renormalize the functions \( a(n) \) and \( b(n) \) using the series

\[
A(z) = \sum_{n=1}^{\infty} a^*(n)q^n := \sum_{n=0}^{\infty} a(n)q^{3n+1} = q - 8q^4 + 20q^7 - 70q^{13} + 64q^{16} + 56q^{19} - 125q^{25} - 160q^{28} + \cdots.
\]

and

\[
B(z) = \sum_{n=1}^{\infty} b^*(n)q^n := \sum_{n=0}^{\infty} b(n)q^{3n+1} = q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + q^{25} + 2q^{28} + \cdots.
\]

Here we have that \( z \in \mathbb{H} \), the upper-half of the complex plane, and we let \( q := e^{2\pi i z} \). We make these changes since \( A(z) \) and \( B(z) \) are examples of two special types of modular forms (for background on modular forms, see \([1, 6, 7, 9]\)). The modularity of these two series follows easily from the properties of Dedekind’s eta-function

\[
\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n).
\]
The proofs of Theorem 1.1 and Corollary 1.3 shall rely on exact formulas we derive for the numbers $a^*(n)$ and $b^*(n)$.

2.1. Exact formulas for $a^*(n)$. The modular form $A(z)$ given by

$$A(z) = \eta(3z)^8 = \sum_{n=1}^{\infty} a^*(n)q^n$$

is in $S_4(\Gamma_0(9))$, the space of weight 4 cusp forms on $\Gamma_0(9)$. This space is one dimensional (see Section 1.2.3 in [9]). Therefore, every cusp form in the space is a multiple of $A(z)$. It turns out that $A(z)$ is a form with complex multiplication.

We now briefly recall the notion of a newform with complex multiplication (for example, see Chapter 12 of [6] or Section 1.2 of [9, 10]). Let $D < 0$ be the fundamental discriminant of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. Let $O_K$ be the ring of integers of $K$, and let $\chi_K := \left(\frac{D}{\cdot}\right)$ be the usual Kronecker character associated to $K$. Let $k \geq 2$, and let $c$ be a Hecke character of $K$ with exponent $k - 1$ and conductor $f_c$, a non-zero ideal of $O_K$. By definition, this means that

$$c : I(f_c) \rightarrow \mathbb{C}^\times$$

is a homomorphism, where $I(f_c)$ denotes the group of fractional ideals of $K$ prime to $f_c$. In particular, this means that

$$c(\alpha O_K) = \alpha^{k-1}$$

for $\alpha \in K^\times$ for which $\alpha \equiv 1 \mod f_c$. To $c$ we naturally associate a Dirichlet character $\omega_c$ defined, for every integer $n$ coprime to $f_c$, by

$$\omega_c(n) := \frac{c(nO_K)}{n^{k-1}}.$$

Given this data, we let

$$\Phi_{K,c}(z) := \sum_{a} c(a)q^{N(a)},$$

where $a$ varies over the ideals of $O_K$ prime to $f_c$, and where $N(a)$ is the usual ideal norm. It is known that $\Phi_{K,c}(z) \in S_k(|D| \cdot N(f_c), \chi_K \cdot \omega_c)$ is a normalized newform.

Using this theory, we obtain the following theorem.

**Theorem 2.1.** Assume the notation above. Then the following are true:

1. If $p = 3$ or $p \equiv 2 \pmod{3}$ is prime, then $a^*(p) = 0$.
2. If $p \equiv 1 \pmod{3}$ is prime, then

$$a^*(p) = 2x^3 - 18xy^2,$$

where $x$ and $y$ are integers for which $p = x^2 + 3y^2$ with $x \equiv 1 \pmod{3}$.

**Remark.** It is a classical fact that every prime $p \equiv 1 \pmod{3}$ is of the form $x^2 + 3y^2$. Moreover, there is a unique pair of positive integers $x$ and $y$ for which $x^2 + 3y^2 = p$. Therefore, the formula for $a^*(p)$ is well defined.
Proof. There is a form with complex multiplication in $S_4(\Gamma_0(9))$. Following the recipe above, it is obtained by letting $k = 4$, $Q(\sqrt{D}) = Q(\sqrt{-3})$ and $f_c := (\sqrt{-3})$. For primes $p$, the coefficients of $q^p$ in this form agree with the claimed formulas. Since $S_4(\Gamma_0(9))$ is one dimensional, this form must be $A(z)$. □

Using this theorem, we obtain the following immediate corollary.

**Corollary 2.2.** The following are true about $a^*(n)$.

1. If $m$ and $n$ are coprime positive integers, then
   \[ a^*(mn) = a^*(m)a^*(n). \]

2. For every positive integer $s$, we have that $a^*(3^s) = 0$.

3. If $p \equiv 2 \pmod{3}$ is prime and $s$ is a positive integer, then
   \[ a^*(p^s) = \begin{cases} 0 & \text{if } s \text{ is odd,} \\ (-1)^{s/2}p^{3s/2} & \text{if } s \text{ is even.} \end{cases} \]

4. If $p \equiv 1 \pmod{3}$ is prime and $s$ is a positive integer, then $a^*(p^s) \neq 0$. Moreover, we have that
   \[ a^*(p^s) \equiv (8x^3)^s \pmod{p}, \]
   where $p = x^2 + 3y^2$ with $x \equiv 1 \pmod{3}$.

Proof. Since $S_4(\Gamma_0(9))$ is one dimensional and since $a^*(1) = 1$, it follows that $A(z)$ is a normalized Hecke eigenform. Claim (1) is well known to hold for all normalized Hecke eigenforms.

Claim (2) follows by inspection since $a^*(n) = 0$ if $n \equiv 0, 2 \pmod{3}$.

To prove claims (3) and (4), we note that since $A(z)$ is a normalized Hecke eigenform on $\Gamma_0(9)$, it follows, for every prime $p \neq 3$, that
\[ a^*(p^s) = a^*(p)a^*(p^{s-1}) - p^3a^*(p^{s-2}). \]

If $p \equiv 2 \pmod{3}$ is prime, then Theorem 2.1 implies that
\[ a^*(p^s) = -p^3a(p^{s-2}). \]

Claim (3) now follows by induction since $a^*(1) = 1$ and $a^*(p) = 0$.

Suppose that $p \equiv 1 \pmod{3}$ is prime. By Theorem 2.1 we know that $a^*(p) \neq 0$. More importantly, we have that
\[ a^*(p) \equiv 8x^3 \pmod{p}, \]
where $p = x^2 + 3y^2$ with $x \equiv 1 \pmod{3}$. To see this, one merely observes that
\[ 2x^3 - 18xy^2 = 2x(x^2 - 9y^2) = 2x(x^2 - 3(p - x^2)) \equiv 8x^3 \pmod{p}. \]

Since $|x| \leq \sqrt{p}$ and is non-zero, it follows that $a^*(p) \equiv 8x^3 \neq 0 \pmod{p}$. By (2.5), we then have that
\[ a^*(p^s) \equiv a^*(p)a^*(p^{s-1}) \equiv 8x^3a^*(p^{s-1}) \pmod{p}. \]

By induction, it follows that $a^*(p^s) \equiv (8x^3)^s \pmod{p}$, which is non-zero modulo $p$. This proves claim (4). □
Example 2.3. Here we give some numerical examples of the formulas for \( a^*(n) \).

1) One easily finds that \( a^*(13) = -70 \). The prime \( p = 13 \) is of the form \( x^2 + 3y^2 \) where \( x = 1 \) and \( y = 2 \). Obviously, \( x = 1 \equiv 1 \pmod{3} \), and so Theorem 2.1 asserts that \( a^*(13) = 2 \cdot 1^3 - 18 \cdot 1 \cdot 2^2 = -70 \).

2) We have that \( a^*(13) = -70 \) and \( a^*(16) = 64 \). One easily checks that \( a^*(13 \cdot 16) = a^*(208) = -70 \cdot 64 = -4408 \). This is an example of Corollary 2.2 (1).

3) If \( p = 5 \) and \( s = 3 \), then Corollary 2.2 (3) asserts that \( a^*(5^3) = 0 \). If \( p = 5 \) and \( s = 4 \), then it asserts that \( a^*(5^4) = 5^6 = 15625 \). One easily checks both evaluations numerically.

4) Now we consider the prime \( p = 13 \equiv 1 \pmod{3} \). Since \( x = 1 \) and \( y = 2 \) for \( p = 13 \), Corollary 2.2 (4) asserts that \( a^*(13^s) \equiv 8^s \pmod{13} \). One easily checks that
\[
a^*(13) = -70 \equiv 8 \pmod{13}, \quad a^*(13^2) = 2703 \equiv 8^2 \pmod{13}, \quad a^*(13^3) = -35420 \equiv 8^3 \pmod{13}.
\]

2.2. Proof of Theorem 1.1 and Corollary 1.3. Before we prove Theorem 1.1, we recall a known formula for \( b(n) \) (also see Section 3 of [2]), the number of 3-core partitions of \( n \).

Lemma 2.4. Assuming the notation above, we have that
\[
B(z) = \sum_{n=1}^{\infty} b^*(n)q^n = \sum_{n=0}^{\infty} b(n)q^{3n+1} = \sum_{n=0}^{\infty} \sum_{d|3n+1} \left( \frac{d}{3} \right) q^{3n+1},
\]
where \( \left( \frac{\bullet}{3} \right) \) denotes the usual Legendre symbol modulo 3.

Proof. We have that \( B(z) = \eta(9z)^3/\eta(3z) \) is in \( M_1(\Gamma_0(9), \chi) \), where \( \chi := \left( \frac{-3}{\bullet} \right) \). The lemma follows easily from this fact. One may implement the theory of weight 1 Eisenstein series to obtain the desired formulas.

Alternatively, one may use the weight 1 form
\[
\Theta(z) = \sum_{n=0}^{\infty} c(n)q^n = q + 6q^4 + 12q^7 + 12q^{10} + 12q^{13} + 12q^{16} + 12q^{19} + 6q^{25} + \cdots.
\]
Using the theory of twists, we find that
\[
\tilde{\Theta}(z) = \sum_{n=1 \pmod{3}}^{\infty} c(n)q^n = 6q + 6q^4 + 12q^7 + 12q^{10} + 12q^{13} + 6q^{16} + 12q^{19} + 6q^{25} + \cdots.
\]
By dimensionality (see Section 1.2.3 of [2]) we have that \( B(z) = 6 \tilde{\Theta}(z) \). The claimed formulas for the coefficients follows easily from the fact that \( x^2 + xy + y^2 \) corresponds to the norm form on the ring of integers of \( \mathbb{Q}(\sqrt{-3}) \).
Example 2.5. The only divisors of primes $p \equiv 1 \pmod{3}$ are 1 and $p$, and so we have that $b^*(p) = 1 + \left(\frac{2}{p}\right) = 1 + \left(\frac{4}{p}\right) = 2$.

Proof of Theorem 1.1. The theorem follows immediately from Theorems 2.1, 2.2 and Lemma 2.4. One sees that the only $n \equiv 1 \pmod{3}$ for which $a^*(n) = 0$ are those $n$ for which $\text{ord}_p(n)$ is odd for some prime $p \equiv 2 \pmod{3}$. The same conclusion holds for $b^*(n)$. Using the fact that

$$a(n) = a^*(3n + 1) \quad \text{and} \quad b(n) = b^*(3n + 1),$$

the theorem follows. \[\square\]

Proof of Corollary 1.3. In a famous paper [11], Serre proved that “almost all” of the coefficients of a modular form with complex multiplication are zero. This implies that almost all of the $a^*(n)$ are zero. The result now follows thanks to Theorem 1.1. \[\square\]

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