Injective Semimodules - Revisited*

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Abstract

Injective modules play an important role in characterizing different classes of rings (e.g. Noetherian rings, semisimple rings). Some semirings have no non-zero injective semimodules (e.g. the semiring of non-negative integers). In this paper, we study some of the basic properties of the so called e-injective semimodules introduced by the first author using a new notion of exact sequences of semimodules. We clarify the relationships between the injective semimodules, the e-injective semimodule, and the i-injective semimodules through several implications, examples and counter examples. Moreover, we provide partial results for the so called Embedding Problem (of semimodules in injective semimodules).

Introduction

Semirings (defined, roughly, as rings not necessarily with subtraction) generalize both rings and distributive bounded lattices. Semirings and their semimodules (defined, roughly, as modules not necessarily with subtraction) have many applications in Mathematics, Computer Science and Theoretical Science (e.g., [HW1998], [Gla2002], [LM2005]). Our main reference on semirings and their applications is Golan’s book [Go1999], and Our main reference in rings and modules is [Wis1991].

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The notion of injective objects can be defined in any category relative to a suitable factorization system. Injective semimodules have been studied intensively (see [Gla2002] for details). Recently, several papers established homological characterizations of special classes of semirings using (cf., [KNT2009], [Ili2010], [KN2011], [Abu2014], [KNZ2014], [AIKN2015], [IKN2017], [AIKN2018]). For example, left (right) V-semirings, all of whose congruence-simple left (right) semimodules are injective have been completely characterized in [AIKN2015].

In addition to the classical notions of injective semimodules over a semiring, several other notions were considered in the literature, e.g. the so called \(i\)-injective semimodules [Alt2003] and the \(k\)-injective semimodules [KNT2009]. One reason for the interest of such notions is the phenomenon that assuming that all semimodules of a given semiring \(S\) to be injective forces the semiring to be a (semisimple) ring (cf. [Ili2010, Theorem 3.4]). Using a new notion of exact sequences of semimodules over a semiring, Abuhlail [Abu2014-CA] introduced a homological notion of exactly injective semimodules (\(e\)-injective semimodules for short) assuming that an appropriate \(\text{Hom}\) functors preserve short exact sequences. Such semimodules were called initially uniformly injective semimodules and used in [Abu2014-SF] under the name normally injective semimodules; the terminology \(e\)-injective semimodules was used first in [AIKN2018].

The paper is divided into three sections.

In Section 1, we collect some basic definitions, examples and preliminaries used in this paper. In particular, we recall the definition and basic properties of exact sequences in the sense of Abuhlail [Abu2014].

In Section 2, we investigate mainly the \(e\)-injective semimodules over a semiring and clarify their relationships with the injective semimodules and the \(i\)-injective semimodules. In Lemma 2.12 and Proposition 2.14 we provide homological detailed proofs of the fact that the class of injective left semimodules is closed under retracts and direct products. It was shown in [AIKN2018, Proposition-Example 4.6.] that, for an additively idempotent division semiring \(D\), the class of \(e\)-injective \(D\)-semimodules is strictly larger than the class of injective \(D\)-semimodules. Subsection 2.1 is devoted to showing that for the semiring \(S := M_2(\mathbb{R}^+)\), the class of \(S\)-\(i\)-injective left semimodules is strictly larger than the class of \(S\)-\(e\)-injective left \(S\)-semimodules: Lemma 2.18 shows that all left \(S\)-semimodules are \(S\)-\(I\)-injective, while Example 2.19 provides a left \(S\)-semimodule which is not \(S\)-\(e\)-injective.

In Section 3, we investigate the so called Embedding Problem. While every module over a ring \(R\) can be embedded in an injective semimodules, and a module \(M\) is injective if \(M\) is \(R\)-injective (using the Baer’s Criterion), any semiring whose category of semimodules has both of these nice properties is a ring [Ili2008, Theorem 3]. Call a left \(S\)-semimodule \(c\)-\(i\)-injective if it is \(M\)-\(i\)-injective for every cancellative left \(S\)-semimodule \(M\). We prove in Theorem 3.18 that every left \(S\)-semimodule can be embedded subtractively in a \(c\)-\(i\)-injective left \(S\)-semimodule.
1 Preliminaries

In this section, we provide the basic definitions and preliminaries used in this work. Any notions that are not defined can be found in our main reference [Gol1999]. We refer to [Wis1991] for the foundations of Module and Ring Theory.

Definition 1.1. ([Gol1999]) A semiring is a datum \((S, +, 0, \cdot, 1)\) consisting of a commutative monoid \((S, +, 0)\) and a monoid \((S, \cdot, 1)\) such that \(0 \neq 1\) and

\[
\begin{align*}
    a \cdot 0 &= 0 = 0 \cdot a \text{ for all } a \in S; \\
    a(b + c) &= ab + ac \text{ and } (a + b)c = ac + bc \text{ for all } a, b, c \in S.
\end{align*}
\]

A semiring \(S\) with \((S, \cdot, 1)\) a commutative monoid is called a commutative semiring.

A semiring \(S\) with \(a + a = a\) for all \(a \in S\) is said to be an additively idempotent semiring. A semiring with no non-zero zero-divisors is called entire. We set \(V(S) := \{s \in S \mid s + t = 0 \text{ for some } t \in S\}\).

If \(V(S) = \{0\}\), then we say that \(S\) is zerosumfree.

Examples 1.2. ([Gol1999])

- Every ring is a cancellative semiring.
- Any distributive bounded lattice \(L = (L, \lor, 1, \land, 0)\) is a commutative idempotent semiring and \(1\) is an infinite element of \(L\).
- The sets \((\mathbb{Z}^+, +, 0, \cdot, 1)\) (resp. \((\mathbb{Q}^+, +, 0, \cdot, 1)\), \((\mathbb{Q}^+, +, 0, \cdot, 1)\)) of non-negative integers (resp. non-negative rational numbers, non-negative real numbers) is a commutative cancellative semiring which is not a ring.
- \(M_n(S)\), the set of all \(n \times n\) matrices over a semiring \(S\), is a semiring.
- \(\mathbb{B} := \{0, 1\}\) with \(1 + 1 = 1\) is an additively idempotent semiring called the Boolean semiring.
- \(\mathbb{R}_{\text{max}} := (\mathbb{R} \cup \{-\infty\}, \max, -\infty, +, 0)\) is an additively idempotent semiring.

1.3. [Gol1999] Let \(S\) and \(T\) be semirings. The categories \(\mathcal{SM}\) of left \(S\)-semimodules with morphisms the \(S\)-linear maps, \(\mathcal{SM}_T\) of right \(S\)-semimodules with morphisms the \(T\)-linear maps, and \(\mathcal{SM}_T\) of \((S, T)\)-bisemimodules with morphisms the \(S\)-linear \(T\)-linear maps are defined as for left (right) modules and bimodules over rings. The set of cancellative elements of a (bi)semimodule \(M\) is

\[
K^+(M) := \{m \in M \mid m + m_1 = m + m_2 \implies m_1 = m_2 \text{ for any } m_1, m_2 \in M\};
\]

and we say that \(M\) is cancellative, if \(K^+(M) = M\). We write \(L \leq_S M\) to indicate that \(L\) is a subsemimodule of the \(S\)-semimodule \(M\).
Example 1.4. The category of $\mathbb{Z}^+$-semimodules is nothing but the category of commutative monoids.

Example 1.5. ([Gol1999, page 150, 154]) Let $S$ be a semiring, $M$ be a left $S$-semimodule and $L \leq_S M$. The **subtractive closure** of $L$ is defined as

$$\mathcal{L} := \{ m \in M \mid m + l = l' \text{ for some } l, l' \in L \}. \quad (2)$$

One can easily check that $\mathcal{L} = \text{Ker}(\pi : M \longrightarrow M/L)$, where $\pi$ is the canonical projection. We say that $L$ is **subtractive**, if $L = \mathcal{L}$. We say that $M$ is a **subtractive semimodule**, if every $S$-subsemimodule $L \leq_S M$ is subtractive.

Following [BHJK2001], we use the following definitions.

**Definition 1.6.** Let $S$ be a semiring. A left $S$-semimodule $M$ is **ideal-simple**, if $0$ and $M$ are the only $S$-subsemimodules of $M$.

1.7. (cf., [AHS2004]) The category $\mathcal{S}_{SM}$ of left semimodules over a semiring $S$ is a closed under homomorphic images, subobjects and arbitrary products (i.e. a variety in the sense of Universal Algebra). In particular, $\mathcal{S}_{SM}$ is **complete**, i.e. has all limits (e.g., direct products, equalizers, kernels, pullbacks, inverse limits) and **cocomplete**, i.e. has all colimits (e.g., direct coproducts, coequalizers, cokernels, pushouts, direct colimits). For the construction of the pullbacks and the pushouts, see [AN].

1.8. Let $M$ be a left $S$-semimodule. We say that $N \leq_S M$ is a

- **retract** of $M$, if there exists a (surjective) $S$-linear map $\theta : M \longrightarrow N$ and an (injective) $S$-linear map $\psi : N \longrightarrow M$ such that $\theta \circ \psi = \text{id}_N$;
- **direct summand** of $M$, if there exists $L \leq_S M$ such that $M = L \oplus N$.

**Exact Sequences**

Throughout, $(S, +, 0, \cdot, 1)$ is a semiring and, unless otherwise explicitly mentioned, an $S$-module is a left $S$-semimodule.

**Definition 1.9.** A morphism of left $S$-semimodules $f : L \longrightarrow M$ is

- **$k$-normal**, if whenever $f(m) = f(m')$ for some $m, m' \in M$, we have $m + k = m' + k'$ for some $k, k' \in \text{Ker}(f)$;
- **$i$-normal**, if $\text{Im}(f) = \overline{f(L)} := \{ m \in M \mid m + l \in L \text{ for some } l \in L \}$.
- **normal**, if $f$ is both $k$-normal and $i$-normal.

**Remark 1.10.** Among others, Takahashi ([Tak1981]) and Golan [Gol1999] called $k$-normal (resp., $i$-normal, normal) $S$-linear maps $k$-regular (resp., $i$-regular, regular) morphisms. We changed the terminology to avoid confusion with the regular monomorphisms and regular epimorphisms in Category Theory which have different meanings when applied to categories of semimodules.
The following technical lemma is helpful in several proofs in this and forthcoming related papers.

**Lemma 1.11.** ([AN]) Let $L \xrightarrow{f} M \xrightarrow{g} N$ be a sequence of semimodules.

(1) Let $g$ be injective.

(a) $f$ is $k$-normal if and only if $g \circ f$ is $k$-normal.

(b) If $g \circ f$ is $i$-normal (normal), then $f$ is $i$-normal (normal).

(c) Assume that $g$ is $i$-normal. Then $f$ is $i$-normal (normal) if and only if $g \circ f$ is $i$-normal (normal).

(2) Let $f$ be surjective.

(a) $g$ is $i$-normal if and only if $g \circ f$ is $i$-normal.

(b) If $g \circ f$ is $k$-normal (normal), then $g$ is $k$-normal (normal).

(c) Assume that $f$ is $k$-normal. Then $g$ is $k$-normal (normal) if and only if $g \circ f$ is $k$-normal (normal).

There are several notions of exactness for sequences of semimodules. In this paper, we use the relatively new notion introduced by Abuhlail:

**Definition 1.12.** ([Abu2014, 2.4]) A sequence

$$ L \xrightarrow{f} M \xrightarrow{g} N $$

of left $S$-semimodules is **exact**, if $g$ is $k$-normal and $f(L) = \text{Ker}(g)$.

1.13. We call a sequence of $S$-semimodules $L \xrightarrow{f} M \xrightarrow{g} N$

- **proper-exact** if $f(L) = \text{Ker}(g)$ (exact in the sense of Patchkoria [Pat2003]);
- **semi-exact** if $\overline{f(L)} = \text{Ker}(g)$ (exact in the sense of Takahashi [Tak1981]);
- **quasi-exact** if $\overline{f(L)} = \text{Ker}(g)$ and $g$ is $k$-normal (exact in the sense of Patil and Doere [PD2006]).

1.14. We call a (possibly infinite) sequence of $S$-semimodules

$$ \cdots \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} M_{i+2} \xrightarrow{f_{i+2}} \cdots $$

- **chain complex** if $f_{j+1} \circ f_j = 0$ for every $j$;
- **exact** (resp., proper-exact, semi-exact, quasi-exact) if each partial sequence with three terms $M_j \xrightarrow{f_j} M_{j+1} \xrightarrow{f_{j+1}} M_{j+2}$ is exact (resp., proper-exact, semi-exact, quasi-exact).

A **short exact sequence** (or a **Takahashi extension** [Tak1982b]) of $S$-semimodules is an exact sequence of the form

$$ 0 \xrightarrow{} L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{} 0 $$

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Remark 1.15. In the sequence (3), the inclusion \( f(L) \subseteq \text{Ker}(g) \) forces \( f(L) \subseteq \overline{f(L)} \subseteq \text{Ker}(g) \), whence the assumption \( f(L) = \text{Ker}(g) \) guarantees that \( f(L) = \overline{f(L)} \), i.e. \( f \) is \( i \)-normal. So, the definition puts conditions on \( f \) and \( g \) that are dual to each other (in some sense).

The follows examples show some of the advantages of the new definition of exact sequences over the old ones:

**Lemma 1.16.** Let \( L, M \) and \( N \) be \( S \)-semimodules.

1. \( 0 \longrightarrow L \xrightarrow{f} M \) is exact if and only if \( f \) is injective.
2. \( M \xrightarrow{g} N \longrightarrow 0 \) is exact if and only if \( g \) is surjective.
3. \( 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \) is semi-exact and \( f \) is normal if and only if \( L \simeq \text{Ker}(g) \).
4. \( 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \) is exact if and only if \( L \simeq \text{Ker}(g) \) and \( g \) is \( k \)-normal.
5. \( L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \) is semi-exact and \( g \) is normal if and only if \( N \simeq M / f(L) \).
6. \( L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \) is exact if and only if \( N \simeq M / f(L) \) and \( f \) is \( i \)-normal.
7. \( 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \) is exact if and only if \( L \simeq \text{Ker}(g) \) and \( N \simeq M / L \).

**Corollary 1.17.** The following assertions are equivalent:

1. \( 0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \) is an exact sequence of \( S \)-semimodules;
2. \( L \simeq \text{Ker}(g) \) and \( N \simeq M / f(L) \);
3. \( f \) is injective, \( f(L) = \text{Ker}(g) \), \( g \) is surjective and \( (k-) \)normal.

   \( \text{In this case, } f \text{ and } g \text{ are normal morphisms.} \)

**Remark 1.18.** An \( S \)-linear map is a monomorphism if and only if it is injective. Every surjective \( S \)-linear map is an epimorphism. The converse is not true in general; for example the embedding \( \iota : \mathbb{Z}^+ \hookrightarrow \mathbb{Z} \) is an epimorphism of \( \mathbb{Z}^+ \)-semimodules.

**Proposition 1.19.** (cf., [Bor1994, Proposition 3.2.2]) Let \( \mathcal{C}, \mathcal{D} \) be arbitrary categories and \( \mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{D} \xrightarrow{\mathcal{G}} \mathcal{C} \) be functors such that \((\mathcal{F}, \mathcal{G})\) is an adjoint pair.

1. \( \mathcal{F} \) preserves all colimits which turn out to exist in \( \mathcal{C} \).
2. \( \mathcal{G} \) preserves all limits which turn out to exist in \( \mathcal{D} \).

**Corollary 1.20.** Let \( S, T \) be semirings and \( \tau F_S \) a \((T,S)\)-bisemimodule.
(1) \( \text{Hom}_T(\_ , G) : \tau \text{SM} \rightarrow \text{SM}_S \) converts all colimits in to limits.

(2) For every family of left \( T \)-semimodules \( \{ Y_\lambda \}_{\lambda \in \Lambda} \), we have a canonical isomorphism of right \( S \)-semimodules
\[
\text{Hom}_T\left( \bigoplus_{\lambda \in \Lambda} Y_\lambda , G \right) \cong \prod_{\lambda \in \Lambda} \text{Hom}_T( Y_\lambda , G).
\]

(3) For any directed system of left \( T \)-semimodules \( (X_j, \{ f_{jj'} \})_J \), we have an isomorphism of right \( S \)-semimodules
\[
\text{Hom}_T(\lim_{\rightarrow} X_j , G) \cong \lim_{\leftarrow} \text{Hom}_T(X_j, G).
\]

(4) \( \text{Hom}_T(\_ , G) \) converts coequalizers into equalizers;

(5) \( \text{Hom}_T(\_ , G) \) converts cokernels into kernels.

**Proof.** By [KN2011], \((G \otimes_S \_ , \text{Hom}_T(G, \_))\) is an adjoint pair of covariant functors, where
\[
G \otimes_S \_ : \text{SM}_S \rightarrow \tau \text{SM} \text{ and } \text{Hom}_T(G, \_) : \tau \text{SM} \rightarrow \text{SM}_S.
\]
It follows directly from Proposition 1.19 that \( \mathcal{G} := \text{Hom}_T(G, \_) \) preserves limits, whence the contravariant functor \( \text{Hom}_T(\_ , G) : \tau \text{SM} \rightarrow \text{SM}_S \) converts colimits to limits. In particular, \( \text{Hom}_T(\_ , G) \) converts direct coproducts (resp. coequalizers, cokernels, pushouts, direct colimits) to direct products (resp. equalizers, kernels, pullbacks, inverse limits).\( \blacksquare \)

Corollary 1.20 allows us to improve [Tak1982a, Theorem 2.6].

**Proposition 1.21.** Let \( \tau G_S \) be a \((T, S)\)-bisemimodule and consider the functor \( \text{Hom}_T(\_ , G) : \tau \text{SM} \rightarrow \text{SM}_S \). Let
\[
L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0 \tag{5}
\]
be a sequence of left \( T \)-semimodules and consider the sequence of right \( S \)-semimodules
\[
0 \rightarrow \text{Hom}_T(N, G) \xrightarrow{(g, G)} \text{Hom}_T(M, G) \xrightarrow{(f, G)} \text{Hom}_T(L, G) \tag{6}
\]

(1) If \( M \xrightarrow{g} N \rightarrow 0 \) is exact and \( g \) is normal, then \( 0 \rightarrow \text{Hom}_T(N, G) \xrightarrow{(g, G)} \text{Hom}_T(M, G) \) is exact and \((g, G)\) is normal.

(2) If (5) is semi-exact and \( g \) is normal, then (6) is proper-exact (semi-exact) and \((g, G)\) is normal.

(3) If (5) is exact and \( \text{Hom}_T(\_ , G) \) converts i-normal morphisms into k-normal ones, then (6) is exact.
Proof. (1) The following implications are clear: \( M \xrightarrow{g} N \longrightarrow 0 \) is exact \( \implies g \) is surjective \( \implies (g,G) \) is injective \( \implies 0 \longrightarrow \text{Hom}_T(N,G) \xrightarrow{(g,G)} \text{Hom}_T(M,G) \) is exact. Assume that \( g \) is normal and consider the exact sequence of \( S \)-semimodules

\[
0 \longrightarrow \text{Ker}(g) \xrightarrow{1} M \xrightarrow{g} N \longrightarrow 0.
\]

Notice that \( N \cong \text{Coker}(i) \). By Corollary 1.20, \( \text{Hom}_T(−,G) \) converts cokernels into kernels, we conclude that \((g,G) = \text{ker}((f,G)) \) whence normal.

(2) Apply Lemma 1.16 (5): \( L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0 \) is semi-exact and \( g \) is normal \( \iff M \cong \text{Coker}(f) \). Since the contravariant functor \( \text{Hom}_T(−,G) \) converts cokernels into kernels, it follows that \( \text{Hom}_T(N,G) = \text{Ker}((f,G)) \) which is in turn equivalent to (6) being semi-exact and \((g,G)\) being normal. Notice that

\[
(g,G)\text{Hom}_S(N,G) = (g,G)(\text{Hom}_S(N,G)) = \text{Ker}((f,G)),
\]

i.e. (6) is proper-exact (whence semi-exact).

(3) This follows immediately from “2” and the assumption on \( \text{Hom}_T(−,G) \).

2 Injective Semimodules

There are several notions of injectivity for a semimodule \( M \) over a semiring \( S \) which coincide if it were a module over a ring. In this section, we consider some of these and clarify the relationships between them. In particular, we investigate the so called \( e \)-injective semimodules which turn to coincide with the so called normally injective semimodules (both notions introduced by Abuhlail and called uniformly injective semimodules in [Abu2014-CA, 1.25, 1.24], the terminology “\( e \)-injective” was first used in [AIKN2018]. We also clarify their relationships with injective semimodules [Gol1999] and \( i \)-injective semimodules [Alt2003].

As before, \((S,+,0,\cdot,1)\) is a semiring and, unless otherwise explicitly mentioned, and \( S \)-module is a left \( S \)-semimodule. Exact sequences here are in the sense of Abuhlail [Abu2014] (see Definition 1.12).

Definition 2.1. ([Abu2014-CA, 1.24]) Let \( M \) be a left \( S \)-semimodule. A left \( S \)-semimodules \( J \) is \( M-e \)-injective, if the contravariant functor

\[
\text{Hom}_S(−,J) \colon \text{S}\text{SM} \longrightarrow \mathbb{Z}^+\text{SM}
\]

transfers every short exact sequence of left \( S \)-semimodules

\[
0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0
\]

into a short exact sequence of commutative monoids

\[
0 \longrightarrow \text{Hom}_S(N,J) \longrightarrow \text{Hom}_S(M,J) \longrightarrow \text{Hom}_S(L,J) \longrightarrow 0.
\]

We say that \( J \) is \( e \)-injective, if \( J \) is \( M-e \)-injective for every left \( S \)-semimodule \( M \).
2.2. Let $I$ be a left $S$-semimodule.

For a left $S$-semimodule $M$, we say that $I$ is

$M$-injective [Gol1999, page 197] if for every injective $S$-linear map $f : L \rightarrow M$ and any $S$-linear map $g : L \rightarrow I$, there exists an $S$-linear map $h : M \rightarrow I$ such that $h \circ f = g$;

![Diagram](image)

$M$-i-injective [Alt2003] if for every normal monomorphism $f : L \rightarrow M$ and any $S$-linear map $g : L \rightarrow I$, there exists an $S$-linear map $h : M \rightarrow I$ such that $h \circ f = g$;

![Diagram](image)

normally $M$-injective [Abu2014-CA, 1.24] if for every normal monomorphism $f : L \rightarrow M$ and any $S$-linear map $g : L \rightarrow I$, there exists an $S$-linear map $h : M \rightarrow I$ such that $h \circ f = g$

![Diagram](image)

and whenever an $S$-linear map $h' : M \rightarrow I$ satisfies $h' \circ f = g$, there exist $S$-linear maps $h_1, h_2 : M \rightarrow I$ such that $h_1 \circ f = 0 = h_2 \circ f$ and $h + h_1 = h' + h_2$.

We say that $I$ is injective (resp., $i$-injective, normally injective) if $I$ is $M$-injective (resp., $M$-i-injective, normally $M$-projective) for every left $S$-semimodule $M$.

**Proposition 2.3.** Let $I$ be a left $S$-semimodule.

1. Let $M$ be a left $S$-semimodule. Then $I$ is $M$-e-injective if and only if $I$ is normally $M$-injective.

2. $SI$ is $e$-injective if and only if $SI$ is normally injective.

**Proof.** We only need to prove (1). Let $M$ be a left $S$-semimodule.

$(\Rightarrow)$ Assume that $I$ is $M$-e-injective. Let $L \leq_S M$ be a subtractive $S$-subsemimodule. By Lemma 1.16, we have a short exact sequence of left $S$-semimodules

$$0 \rightarrow L \rightarrow M \rightarrow M/L \rightarrow 0$$  \hspace{1cm} (7)
where \( t \) is the canonical embedding and \( \pi \) is the canonical projection. By our assumption, the contravariant functor \( \text{Hom}_S(\cdot, I) : S\text{SM} \rightarrow \mathbb{Z}^{S\text{SM}} \) preserves exact sequences, whence the following sequence of commutative monoids

\[
0 \rightarrow \text{Hom}_S(M/L, I) \xrightarrow{(\pi, I)} \text{Hom}_S(M, I) \xrightarrow{(t, I)} \text{Hom}_S(L, I) \rightarrow 0
\]

is exact. In particular, \( (t, I) : \text{Hom}_S(M, I) \rightarrow \text{Hom}_S(L, I) \) is a normal epimorphism, i.e. \( I \) is normally \( M \)-injective.

\( (\Leftarrow) \) Let

\[
0 \rightarrow L \xrightarrow{f} M \xrightarrow{r} N \rightarrow 0 \tag{8}
\]

be an exact sequence of left \( S \)-semimodules. Applying the contravariant functor \( \text{Hom}_S(\cdot, I) \) to (8) it follows by Lemma 1.21 (2) and our assumption that the following sequence of commutative monoids

\[
0 \rightarrow \text{Hom}_S(N, I) \xrightarrow{(g, I)} \text{Hom}_S(M, I) \xrightarrow{(f, I)} \text{Hom}_S(L, I) \rightarrow 0 \tag{9}
\]

is exact, i.e. \( gI \) is injective. \( \blacksquare \)

The proof of the following result is similar to that of [Alt2003, Theorem 3.7]

**Proposition 2.4.** Let

\[
L \xrightarrow{f} M \xrightarrow{g} N \tag{10}
\]

be a sequence of left \( S \)-semimodules, \( I \) a left \( S \)-semimodule and consider the sequence

\[
\text{Hom}_S(N, I) \xrightarrow{(g, I)} \text{Hom}_S(M, I) \xrightarrow{(f, I)} \text{Hom}_S(L, I) \tag{11}
\]

of commutative monoids.

1. If (10) is exact with \( g \) normal and \( I \) is \( i \)-injective, then (11) is proper-exact.
2. If (10) is exact with \( g \) normal and \( I \) is \( e \)-injective, then (11) is exact and \( (g, I) \) is normal.
3. If (10) is exact and \( I \) is injective, then (11) is proper exact.

**Proof.** By Corollary 1.17, we have a short exact sequence of left \( S \)-semimodules

\[
0 \rightarrow \ker(g) \xrightarrow{1} M \xrightarrow{\pi} M/\ker(g) \rightarrow 0
\]

where \( 1 \) and \( \pi \) are the canonical \( S \)-linear maps. Since (10) is proper exact, \( f(M) = \ker(g) \) and \( M/\ker(g) = M/f(M) \simeq \text{Coker}(f) \). By the Universal Property of Kernels, there exists a unique \( S \)-linear map \( \tilde{f} : L \rightarrow \ker(g) \) such that \( t \circ \tilde{f} = f \) (and \( \tilde{f} \) is surjective). On the other hand, by the
Universal Property of Cokernels, there exists a unique $S$-linear map $\tilde{g} : M/\text{Ker}(g) \rightarrow N$ such that $\tilde{g} \circ \pi = g$. So, we have a commutative diagram of left $S$-semimodules

$$
\begin{array}{ccccccccc}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & \text{Ker}(g) & \xrightarrow{i} & M & \xrightarrow{\pi} & M/\text{Ker}(g) & \rightarrow & 0 \\
& & 0 & \downarrow{f} & \xrightarrow{g} & N & \xrightarrow{\tilde{g}} & 0 \\
& & 0 & \downarrow{\pi} & \xrightarrow{\text{Ker}(g)} & 0 & \downarrow{\text{g}} & \\
& & & & & & & & \\
\end{array}
$$

(12)

Applying the contravariant functor $\text{Hom}_S(-,I)$, we get the sequence

$$
0 \rightarrow \text{Hom}_S(M/\text{Ker}(g),I) \xrightarrow{(\pi,I)} \text{Hom}_S(M,I) \xrightarrow{(i,I)} \text{Hom}_S(\text{Ker}(g),I) \rightarrow 0
$$

(13)

and we obtain the commutative diagram

$$
\begin{array}{ccccccccc}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
0 & \xrightarrow{(\tilde{f},I)} & \text{Hom}_S(M/\text{Ker}(g),I) & \xrightarrow{(\pi,I)} & \text{Hom}_S(M,I) & \xrightarrow{(i,I)} & \text{Hom}_S(\text{Ker}(g),I) & \rightarrow & 0 \\
& & 0 & \downarrow{(g,I)} & \xrightarrow{(\tilde{g},I)} & \text{Hom}_S(N,I) & \rightarrow & & \\
& & \text{Hom}_S(L,I) & \xrightarrow{(f,I)} & \text{Hom}_S(M/\text{Ker}(g),I) & \rightarrow & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
$$

(14)

of commutative monoids.

1. Since $g$ is $k$-normal, we conclude that $\tilde{g}$ is injective. Moreover, $\pi$ is surjective, and $g = \tilde{g} \circ \pi$ is normal, whence $\tilde{g}$ is normal by Lemma 1.11 (2). Since $S I$ is $i$-injective, the sequence (13) is proper-exact and $(\tilde{g},I)$ is surjective (see Proposition 1.21 (2)). It follows that

$$
\text{Ker}((f,I)) = \text{Ker}((\tilde{f},I) \circ (i,I)) = \text{Ker}((i,I)) \quad ((\tilde{f},I) \text{ is injective})
$$

$$
= \text{im}((\pi,I)) \quad ((13) \text{ is proper exact})
$$

$$
= \text{im}((\pi,I) \circ (\tilde{g},I)) \quad ((\tilde{g},I) \text{ is surjective})
$$

$$
= \text{im}((g,I)).
$$

2. By (1), the sequence (11) is proper-exact. Since $(\tilde{f},I)$ is injective and $(i,I)$ is $k$-normal, it follows by Lemma 1.11 (1-a) that $(f,I) = (\tilde{f},I) \circ (i,I)$ is $k$-normal. Consequently, (11) is exact.

Notice that, moreover, $(\pi,I)$ is a normal monomorphism and $(\tilde{g},I)$ is $i$-normal, whence $(g,I) = (\pi,I) \circ (\tilde{g},I)$ is normal by Lemma 1.11 (1-c).
(3) The proof is similar to that of (1). Notice that by our assumption (10) is exact; in particular, $g$ is $k$-normal, which is need to show that $\tilde{g}$ is injective, whence $(\tilde{g}, I)$ is surjective since $\mathcal{S}I$ is injective. ■

**Theorem 2.5.** Let $M$ be a left $S$-Semimodule. The following are equivalent for a left $S$-semimodule $I$:

1. $\mathcal{S}I$ is normally $M$-injective;
2. $\mathcal{S}I$ is $M$-e-injective;
3. For every exact sequence of left $S$-semimodules (10) with $g$ normal, the induced sequence of commutative monoids (11) is exact and $(g, I)$ is normal.

**Proof.** (1) $\Rightarrow$ (2) This follows by Proposition 2.3.
(2) $\Rightarrow$ (3) This follows by Proposition 2.4.
(3) $\Rightarrow$ (1) This follows directly by applying the assumption to the exact sequences of left $S$-semimodules the form $0 \rightarrow M \xrightarrow{g} N$ with $g$ normal. ■

Using Propositions 1.21 and 2.4, one can easily recover the following characterizations of injective and $i$-injective semimodules proved in [Alt2003] which inspired our characterizations of $e$-injective semimodules in Theorem 2.5.

**Theorem 2.6.** Let $M$ be a left $S$-Semimodule. The following are equivalent for a left $S$-semimodule $I$:

1. $\mathcal{S}I$ is $M$-i-injective;
2. for every proper-exact sequence of left $S$-semimodules (8) in which $f$ is normal and $g$ is $k$-normal, the induced sequence of commutative monoids (9) is proper-exact;
3. for every proper-exact sequence of left $S$-semimodules (10) with $g$ normal, the induced sequence of commutative monoids (11) is proper-exact.

**Theorem 2.7.** Let $M$ be a left $S$-Semimodule. The following are equivalent for a left $S$-semimodule $I$:

1. $\mathcal{S}I$ is $M$-injective;
2. for every proper-exact sequence of left $S$-semimodules (8) in which $f$ is $k$-normal and $g$ is normal, the induced sequence of commutative monoids (9) is proper-exact.
3. for every exact sequence of left $S$-semimodules (10), the induced sequence of commutative monoids (11) is proper-exact.
It follows directly from the definitions that, for any semiring \( S \) and any left \( S \)-semimodule \( M \), the class \( \mathcal{I}_S^i(M) \) of \( M \)-\textit{i-injective} left \( S \)-semimodules contains both the class \( \mathcal{I}_S(M) \) of \textit{injective} left \( S \)-semimodules and the class \( \mathcal{I}_S^e(M) \) of \( S \)-\textit{e-injective} left \( S \)-semimodules, i.e.

\[
\mathcal{I}_S(M) \cup \mathcal{I}_S^e(M) \subseteq \mathcal{I}_S^i(M).
\] (15)

While every \textit{projective} semimodule is \textit{e-projective} (see [AIKN2018]), it is not evident that every \textit{injective} semimodule is \textit{e-injective} if the base semiring is arbitrary. However, we have a partial result:

**Proposition 2.8.** ([AIKN2018, Theorem 4.5]) \textit{Let} \( S \) \textit{be an additively idempotent semiring. Then every injective left \( S \)-semimodule is e-injective.}

The following examples show that the converse of Proposition 2.8 is not true in general.

**Example 2.9.** ([AIKN2018, 4.6]) Let \( D \) be an \textit{additively idempotent} division semiring (e.g., \( D = \mathbb{B} \), the Boolean semiring). Then \( D \) has an \( e \)-injective left \( S \)-semimodule which is not injective.

We illustrate first that \textit{relative injectivity} and \textit{relative e-injectivity} are \textit{not} related even when the base semiring is commutative and additively idempotent. The following example shows that the relative version of Proposition 2.8 is not valid, i.e. relative injectivity does not guarantee relative \( e \)-injectivity.

**Example 2.10.** Consider the semiring \( S := \{0, 1, a\} \) [Alt2003] with addition and multiplication given by

| +  | 0 | 1 | a |
|----|---|---|---|
| 0  | 0 | 1 | a |
| 1  | 1 | 1 | 1 |

| \cdot  | 0 | 1 | a |
|-------|---|---|---|
| 0     | 0 | 0 | 0 |
| 1     | 0 | 1 | a |
| a     | a | 1 | a |

We show that \( S \) is \( S \)-injective but not \( S \)-\( e \)-injective. The ideals of \( S \) are \{0\}, \{0, a\}, and \( S \). Clearly, \{0, a\} is subtractive, whence \( S \) is a \textit{subtractive} semiring and our example shows that the inclusion \( \mathcal{I}_S^e(S) \subseteq \mathcal{I}_S^i(S) = \mathcal{I}_S(S) \) is strict.

**Claim I:** \( S \) is \( S \)-injective.

We need to consider only the canonical embedding \( \{0, a\} \xrightarrow{1} S \) and the \( S \)-linear map

\[
\varphi : \{0, a\} \rightarrow S, \ 0 \mapsto 0 \text{ and } a \mapsto a.
\] (16)

Notice that \( \varphi(a) \neq 1 \) : if so, then \( 1 = \varphi(a) = \varphi(a \cdot 1) = a\varphi(1) \), i.e. \( a \) has a multiplicative inverse, a contradiction. Notice that \( \varphi \) can be extended to an \( S \)-linear map through \( id_S \). It follows that \( S \) is \( S \)-injective.

**Claim II:** \( S \) is \textit{not} \( S \)-\( e \)-injective.

The \( S \)-linear map (16) can be extended through another \( S \)-linear map, namely

\[
\tilde{h} : S \rightarrow S, \ 0 \mapsto 0, \ a \mapsto a, \ 1 \mapsto a.
\]
However, the only $S$-linear map $h : S \to S$ such that $(h \circ \iota)(\{0, a\}) = 0$ is the $h = 0 :$ indeed, $h(1) = a$ implies $0 = h(a) = h(a \cdot 1) = ah(1) = a \cdot a = a$, a contradiction; and $h(1) = 1$ implies $0 = h(a) = h(a \cdot 1) = ah(1) = a \cdot 1 = a$, a contradiction. So, we cannot find $h_1, h_2 \in Hom_S(S, S)$ such that $id_S + h_1 = \tilde{h} + h_2$. Consequently, $S$ is not $S$-e-injective. ■

The following example shows that relative e-injectivity does not guarantee relative injectivity. In fact, we given an example for which

$$\mathcal{S}(S) \subseteq \mathcal{I}^e_S(S) = S M = \mathcal{I}^e_S(S).$$

**Example 2.11.** Consider the commutative additively idempotent semiring $S := (\mathbb{Z}^+, \max, 0, \cdot, 1)$. Then $S$ has no non-trivial proper subtractive ideals, whence every $S$-semimodule is $S$-e-injective ($S$-i-injective). By [Alt2003, Example 2.7], $S$ is not $S$-injective. In particular, our example shows that the inclusion $\mathcal{S}(S) \subseteq \mathcal{I}^e_S(S)$ is strict.

Next, we provide detailed homological proofs rather than compact categorical ones of the facts that, for a given left $S$-semimodule $M$, the class of $M$-e-injective semimodules is closed under retracts and direct products (cf., [AIKN2018, Corollary 3.3]).

**Proposition 2.12.** (1) Let $M$ be a left $S$-semimodule. Every retract of a left $M$-e-injective $S$-semimodule is $M$-e-injective.

(2) A retract of an e-injective $S$-semimodule is e-injective.

2.13. We need to prove (1) only.

Let $J$ be an $M$-e-injective left $S$-semimodule and $I$ a retract of $J$ along with $S$-linear maps $t : I \to J$ and $\pi : J \to I$ such that $\pi \circ t = id_I$. Let $f : L \to M$ be a normal $S$-monomorphism and $g : L \to I$ be an $S$-linear map.

$$
\begin{array}{ccc}
0 & \longrightarrow & L \\
\downarrow & & \downarrow f \\
I & \downarrow \pi & \longrightarrow I \\
& & \downarrow h^* \\
& & \downarrow t \\
& & \downarrow \pi \\
J & & \\
\end{array}
$$

Since $J$ is $M$-e-injective, there is an $S$-linear map $h^* : M \to J$ such that $h^* \circ f = t \circ g$. Consider $h := \pi \circ h^*$. Then we have

$$
h \circ f = (\pi \circ h^*) \circ f = \pi \circ (h^* \circ f) = \pi \circ (t \circ g) = (\pi \circ t) \circ g = id_I \circ g = g.
$$

Suppose that $h' : M \to I$ is an $S$-linear map such that $h' \circ f = g$. Notice that $t \circ h' \circ f = t \circ g$. Since $J$ is $M$-e-injective, there exist $S$-linear maps $h_1', h_2^* : M \to J$ such that $h_1' \circ f = 0 = h_2^* \circ f$. 

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and \( h^* + h_1^* = t \circ h' + h_2^* \).

Consider \( h_1 := \pi \circ h_1^* \) and \( h_2 := \pi \circ h_2^* \). Then we have, for \( i = 1, 2 \), \( h_i \circ f = \pi \circ h_i^* \circ f = \pi \circ 0 = 0 \). Moreover, we have

\[
\begin{align*}
    h + h_1 & = \pi \circ t \circ h + \pi \circ h_1^* \\
          & = \pi \circ (t \circ h + h_1^*) \\
          & = \pi \circ (t \circ h' + h_2^*) \\
          & = \pi \circ t \circ h' + \pi \circ h_2^* \\
          & = h' + h_2^*.
\end{align*}
\]

**Proposition 2.14.** Let \( M \) be a left \( S \)-semimodule and \( \{ J_\lambda \}_{\lambda \in \Lambda} \) be a collection of left \( S \)-semimodules. Then \( \prod_{\lambda \in \Lambda} J_\lambda \) is \((M)\)-e-injective if and only if \( J_\lambda \) is \( M\)-e-injective for every \( \lambda \in \Lambda \).

**Proof.** Let \( J := \prod_{\lambda \in \Lambda} J_\lambda \) and, for each \( \lambda \in \Lambda \), let \( t_\lambda : J_\lambda \rightarrow J \) and \( \pi_\lambda : J \rightarrow J_\lambda \) be the canonical \( S \)-linear maps.

(\( \Rightarrow \)) For each \( \lambda \in \Lambda \), we have \( \pi_\lambda \circ t_\lambda = id_{J_\lambda} \), i.e. \( J_\lambda \) is a retract of \( J \). The result follows from Lemma 2.12.

(\( \Leftarrow \)) Assume that \( J_\lambda \) is \( M\)-e-injective for every \( \lambda \in \Lambda \). Let \( f : L \rightarrow M \) be a normal monomorphism and \( g : L \rightarrow J \) an \( S \)-linear map.

Since \( J_\lambda \) is \( M\)-e-injective for each \( \lambda \in \Lambda \), there is an \( S \)-linear map \( h_\lambda^* : M \rightarrow J_\lambda \) such that \( h_\lambda^* \circ f = \pi_\lambda \circ g \). By the **Universal Property of Direct Products**, there exists an \( S \)-linear map

\[
h : M \rightarrow J, \ m \mapsto \prod_{\lambda \in \Lambda} (t_\lambda \circ h_\lambda^*)(m).
\]

Notice that for every \( l \in L \), we have

\[
(h \circ f)(l) = \prod_{\lambda \in \Lambda} (t_\lambda \circ h_\lambda^*)(f(l)) = \prod_{\lambda \in \Lambda} (t_\lambda \circ \pi_\lambda)(g(l)) = g(l).
\]
Suppose that there exists an $S$-linear map $h' : M \to J$ such that $h' \circ f = g$. It follows that $\pi_\lambda \circ h' \circ f = \pi_\lambda \circ g$ for every $\lambda \in \Lambda$. Since $J_\lambda$ is $M$-e-injective, there exist $S$-linear maps $h^*_1, h^*_2 : M \to J$ such that $h^*_1 \circ f = 0 = h^*_2 \circ f$ and $h^*_1 + h^*_2 = \pi_\lambda \circ h' + h^*_2$.

For $i = 1, 2$, there exists by the Universal Property of Direct Products an $S$-linear map

$$h_i : M \to J, \quad m \mapsto \prod_{\lambda \in \Lambda} (t_\lambda \circ h^*_{i\lambda})(m).$$

For $i = 1, 2$ and every $l \in L$ we have

$$(h_i \circ f)(l) = \prod_{\lambda \in \Lambda} (t_\lambda \circ h^*_{i\lambda})(f(l)) = \prod_{\lambda \in \Lambda} t_\lambda(0) = 0.$$  

Moreover, we have for every $m \in M$:

$$(h + h_1)(m) = \prod_{\lambda \in \Lambda} (t_\lambda \circ \pi_\lambda \circ h)(m) + \prod_{\lambda \in \Lambda} (t_\lambda \circ h^*_{1\lambda})(m)$$

$$= \prod_{\lambda \in \Lambda} (t_\lambda \circ (\pi_\lambda \circ h + h^*_1))(m)$$

$$= \prod_{\lambda \in \Lambda} (t_\lambda \circ (h^*_2 + h^*_1))(m)$$

$$= \prod_{\lambda \in \Lambda} (h^*_2 + t_\lambda \circ h^*_1)(m)$$

$$= (h^*_2 + h_2)(m).$$

**Lemma 2.15.** Let

$$0 \to L \overset{p}{\to} M \overset{q}{\to} N \to 0$$

be a short exact sequence of left $S$-semimodules. If a left $S$-semimodule $J$ is $M$-e-injective, then $J$ is $L$-e-injective and $N$-e-injective.

**Proof.** Let $J$ be a left $S$-semimodule.

**Step I:** $J$ is $L$-e-injective.

Let $f : K \to L$ be a normal monomorphism and $g : K \to J$ an $S$-linear map. Clearly, $p \circ f$ is a normal monomorphism.
Since $J$ is $M$-e-injective, there exists an $S$-linear map $h^* : M \to J$ such that $h^* \circ p \circ f = g$. Consider $h := h^* \circ p : L \to J$. Then $h \circ f = h^* \circ p \circ f = g$.

Suppose now that $h' : L \to J$ is an $S$-linear map such that $h' \circ f = g$. Since $p : L \to M$ is a normal monomorphism and $J$ is $M$-e-injective, there exists an $S$-linear map $\tilde{h} : M \to J$ such that $\tilde{h} \circ p = h'$. Since $\tilde{h} \circ p = h^* \circ p \circ f = g$, there exist $S$-linear maps $h_1^*, h_2^* : M \to J$ such that $h_1^* \circ p \circ f = 0 = h_2^* \circ p \circ f$ and $h^* + h_1^* = \tilde{h} + h_2^*$. Considering $h_1 := h_1^* \circ p$ and $h_2 := h_2^* \circ p$, we have $h_1 \circ f = h_1^* \circ p \circ f = 0 = h_2^* \circ p \circ f = h_2 \circ f$ and

$$h + h_1 = h^* \circ p + h_1^* \circ p = (h^* + h_1^*) \circ p = (\tilde{h} + h_2^*) \circ p = \tilde{h} \circ p + h_2^* \circ p = h' + h_2.$$

**Step II:** $J$ is $N$-e-injective.

Let $f : K \to N$ be a normal monomorphism and $g : K \to J$ an $S$-linear map.

Let $(U; q', f')$ be a pullback of $(q, f)$ (see [Tak1982b, 1.7]). Clearly, $f'$ is a normal $S$-monomorphism. Since $J$ is $M$-e-injective, there exists an $S$-linear map $h^* : M \to J$ such that $h^* \circ f' = g \circ q'$. Let $n \in N$. Since $q$ is surjective, there exists $m_n \in M$ such that $n = q(m_n)$. Define

$$h : N \to J, \quad n \mapsto h^*(m_n).$$

**Claim:** $h$ is well-defined.

Suppose that there exists another $m \in M$ such that $q(m) = n = q(m_n)$. Since $q$ is $k$-normal, there exist $m_1, m_2 \in \text{Ker}(q)$ such that $m + m_1 = m_n + m_2$. Since $m_1, m_2 \in \text{Ker}(q)$, $(m_1, 0), (m_2, 0) \in U$ and so for $i = 1, 2$ we have $h^*(m_i) = (h^* \circ f')(m_i, 0) = (g \circ q')(m_i, 0) = g(0) = 0$, whence $h^*(m) = h^*(m_n)$. Thus $h$ well defined as a map. Clearly, $h$ is $S$-linear. Moreover, for every $k \in K$ we have $f(k) = q(m_{f(k)})$ for some $m_{f(k)} \in M$, thus $(m_{f(k)}, k) \in U$ and it follows that

$$(h \circ f)(k) = (h \circ f \circ q')(m_{f(k)}, k) = (h \circ q \circ f')(m_{f(k)}, k) = (h^* \circ f')(m_{f(k)}, k) = (g \circ q')(m_{f(k)}, k) = g(k),$$

i.e. $h \circ f = g$.

Suppose that there exists an $S$-linear map $h' : N \to J$ such that $h' \circ f = g$. Notice that $h' \circ q \circ f' = h' \circ f \circ q' = g \circ q'$. Since $J$ is $M$-e-injective, there exist $h_1^*, h_2^* : M \to J$ such that $h_1^* \circ f' = 0 = h_2^* \circ f'$ and $h^* + h_1^* = h' \circ q + h_2^*$. 

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Let \( n \in \mathbb{N} \). Since \( q \) is surjective, there exists \( m_n \in M \) such that \( q(m_n) = n \). Define

\[
h_1 : N \rightarrow J, \ n \mapsto h_1^*(m_n) \quad \text{and} \quad h_2 : N \rightarrow J, \ n \mapsto h_2^*(m_n).
\]

One can prove as above that \( h_1 \) and \( h_2 \) are well-defined. It is clear that both \( h_1 \) and \( h_2 \) are \( S \)-linear.

Notice that for every \( k \in K \), we have \((m_{f(k)}, k) \in U\) whence, for \( i = 1, 2 \), we have

\[
(h \circ f)(k) = (h \circ f \circ q)(m_{f(k)}, k) = (h \circ q \circ f')(m_{f(k)}, k)
\]

Moreover, for every \( n \in N \), we have

\[
(h + h_1)(n) = h(n) + h_1(n) = h^*(m_n) + h_1^*(m_n)
\]

\[= (h' \circ q + h_2^*)(m_n) = h'(n) + h_2(n)
\]

\[= (h' + h_2)(n).
\]

**Remark 2.16.** The converse of Lemma 2.15 is not true in general as will be shown in Example 2.20.

### 2.1 A Counter Example

This subsection is devoted to studying the left self-injectivity of \( S := M_2(\mathbb{R}^+) \). We show in particular that \( \mathcal{I}(S) = \mathcal{S} \mathcal{M} \) and that the inclusion \( \mathcal{I}(S) \not\subseteq \mathcal{I}(\mathcal{S}) \) is strict.

**Lemma 2.17.** The only non-trivial proper subtractive left ideals of \( S \) are

\[
E_1 = \text{Span}\left( \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\} \right) = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\}
\]

\[
E_2 = \text{Span}\left( \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \right) = \left\{ \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \mid c, d \in \mathbb{R}^+ \right\}
\]

\[
N_r = \left\{ \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\}, \ r \in \mathbb{R}^+ \setminus \{0\}.
\]

**Proof.** We give the proof is three steps.

**Step 1:** \( E_1, E_2 \) and \( N_r \) are subtractive left ideals of \( S \).

Notice that \( E_1 \) is a left ideal of \( S \), since for every \( a, b, c, d, p, q, r, s \in \mathbb{R}^+ \) we have

\[
\begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} + \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix} = \begin{bmatrix} pa + qb + c & 0 \\ ra + sb + d & 0 \end{bmatrix} \in E_1.
\]

Moreover, \( E_1 \) is subtractive since

\[
\begin{bmatrix} p & q \\ r & s \end{bmatrix} + \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} c & 0 \\ d & 0 \end{bmatrix}
\]
implies \( q = 0 = s \), i.e. \( \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in E_1 \). Similarly, we have \( E_2 \) is a subtractive ideal.

For any non-zero \( r \in \mathbb{R}^+ \), \( N_r \) is a left ideal since

\[
\begin{bmatrix} k & l \\ m & n \end{bmatrix} \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} + \begin{bmatrix} rc & c \\ rd & d \end{bmatrix} = \begin{bmatrix} r(ka+lb+c) & ka+lb+c \\ r(ma+nb+d) & ma+nb+d \end{bmatrix} \in N_r
\]

for all \( a, b, c, d, k, l, m, n \in \mathbb{R}^+ \). Moreover, \( N_r \) is subtractive since

\[
\begin{bmatrix} k & l \\ m & n \end{bmatrix} + \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} = \begin{bmatrix} rc & c \\ rd & d \end{bmatrix} \in N_r
\]

implies \( c = a + k/r = a + l \) and \( d = b + m/r = b + n \), whence \( k = rl \) and \( m = rn \), i.e. \( \begin{bmatrix} k & l \\ m & n \end{bmatrix} \in N_r \).

**Step II:** The only subtractive left ideal containing \( E_1 \), \( E_2 \) or \( N_r \) for some \( r \in \mathbb{R}^+ \) strictly is \( I = S \).

Let \( I \) be a subtractive left ideal of \( M_2(\mathbb{R}^+) \).

**Case 1:** \( E_1 \not\subset I \).

In this case, there exists \( \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in I \) such that \( q \neq 0 \) or \( s \neq 0 \), which implies \( \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} \in I \) as \( \begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} \in I \) and

\[
\begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix} + \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in I
\]

If \( q \neq 0 \), then

\[
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1/q & 0 \end{bmatrix} \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} \in I.
\]

If \( s \neq 0 \), then

\[
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1/s \end{bmatrix} \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix} \in I.
\]

Either way \( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in I \), whence \( E_2 \not\subset I \) and \( I = S \).

**Case 2:** \( E_2 \not\subset I \). One can show, in a way similar to that of Case 1, that \( I = S \).

**Case 3:** \( N_r \not\subset I \) for some \( r \in \mathbb{R}^+ \setminus \{0\} \).

In this case, there exists some \( \begin{bmatrix} k & l \\ m & n \end{bmatrix} \in I \) with \( k \neq rl \) or \( m \neq rn \). Assume, without loss of generality, that \( k < rl \). Then \( k + p = rl \) for some \( p \in \mathbb{R}^+ \setminus \{0\} \). Thus \( \begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} \in I \) or \( \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \in I \) for some \( q \in \mathbb{R}^+ \) as

\[
\begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} + \begin{bmatrix} k & l \\ m & n \end{bmatrix} = \begin{bmatrix} rl & l \\ rn & n \end{bmatrix} \in I
\]
or
\[
\begin{bmatrix}
p & 0 \\
0 & q \\
\end{bmatrix}
+ \begin{bmatrix}
k & l \\
m & n \\
\end{bmatrix}
= \begin{bmatrix}
rl & l \\
m & m/r \\
\end{bmatrix}
\in I.
\]

Thus
\[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
1/p & 0 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
p & 0 \\
0 & q \\
\end{bmatrix}
\in I
\]
or
\[
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
1/p & 0 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
p & 0 \\
0 & q \\
\end{bmatrix}
\in I.
\]

Either way we have \(\begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}\in I\), whence \(E_1 \subseteq I\) and so \(I = S\).

**Step III.** Let \(I\) be any non-zero subtractive left ideal of \(S\). Then \(\begin{bmatrix}
k & l \\
m & n \\
\end{bmatrix}\in I\setminus\{0\}\) for some \(k,l,m,n \in \mathbb{R}^+\).

**Case 1:** \(k \neq 0\). In this case, we have
\[
\begin{bmatrix}
1/k & 0 \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
k & l \\
m & n \\
\end{bmatrix}
= \begin{bmatrix}
1 & l/k \\
0 & 0 \\
\end{bmatrix}
\in I.
\]

If \(l = 0\), then \(\begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}\in I\), whence \(E_1 \subseteq I\). Otherwise, \(\begin{bmatrix}
k/l & 1 \\
0 & 0 \\
\end{bmatrix}\in I\), whence \(N_{k/l} \subseteq I\). In either case, it follows by Step II that \(I \in \{E_1,N_{k/l},S\}\).

**Case 2:** \(l \neq 0\). In this case, we have
\[
\begin{bmatrix}
0 & 0 \\
1/l & 0 \\
\end{bmatrix}
\begin{bmatrix}
k & l \\
m & n \\
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
k/l & 1 \\
\end{bmatrix}
\in I.
\]

If \(k = 0\), then \(\begin{bmatrix}
0 & 0 \\
0 & 1 \\
\end{bmatrix}\in I\), whence \(E_2 \subseteq I\). Otherwise \(N_{m/n} \subseteq I\). In either case, it follows by Step II that \(I \in \{E_2,N_{k/l},S\}\).

**Case 3:** \(m \neq 0\). In this case, we have
\[
\begin{bmatrix}
0 & 1/m \\
0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
k & l \\
m & n \\
\end{bmatrix}
= \begin{bmatrix}
1 & n/m \\
0 & 0 \\
\end{bmatrix}
\in I.
\]

If \(n = 0\), then \(\begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}\in I\), whence \(E_1 \subseteq I\). Otherwise, \(\begin{bmatrix}
m/n & 1 \\
0 & 0 \\
\end{bmatrix}\in I\), whence \(N_{m/n} \subseteq I\). In either case, it follows by Step II that \(I \in \{E_1,N_{m/n},S\}\).

**Case 4:** \(n \neq 0\). In this case, we have
\[
\begin{bmatrix}
0 & 0 \\
0 & 1/n \\
\end{bmatrix}
\begin{bmatrix}
k & l \\
m & n \\
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
m/n & 1 \\
\end{bmatrix}.
\]

If \(m = 0\), then \(\begin{bmatrix}
0 & 0 \\
0 & 1 \\
\end{bmatrix}\in I\), whence \(E_2 \subseteq I\). Otherwise, \(N_{m/n} \subseteq I\). In either case, it follows by Step II that \(I \in \{E_2,N_{m/n},S\}\).■
Lemma 2.18. Every left $S$-semimodule is $S$-i-injective.

Proof. Let $M$ be a left $S$-semimodule, $f : N \to S$ a normal $S$-monomorphism, and $g : N \to M$ an $S$-linear map. Then $f(N)$ is a subtractive left ideal of $S$, whence $f(N) \in \{0, E_1, E_2, S\}$ or $f(N) = N_r$ for some $r \in \mathbb{R}^+ \backslash \{0\}$.

Case I: $f(N) = 0$. In this case, choose $h = 0 : S \to M$. Clearly, $g = h \circ f$.

Case II: $f(N) = S$. In this case, $f$ is an $S$-isomorphism. Choose $h = g \circ f^{-1}$, whence $g = h \circ f$.

Case III: $f(N) = E_1$. In this case, there exists a unique $n_0 \in N$ such that

$$f(n_0) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$  

Consider the $S$-linear map

$$h : S \to M, \begin{bmatrix} p & q \\ r & s \end{bmatrix} \mapsto \begin{bmatrix} p & q \\ r & s \end{bmatrix} g(n_0).$$

Let $n \in N$. It follows that

$$f(n) = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} f(n_0) = f \left( \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} n_0 \right)$$

for some $a, b \in \mathbb{R}^+$. Since $f$ is injective, $n = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} n_0$. It follows that

$$(h \circ f)(n) = h(f(n)) = h \left( \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} g(n_0) = g \left( \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} n_0 \right) = g(n).$$

Case IV: $f(N) = E_2$. The proof is similar to Case III.

Case V: $f(N) = N_r$ for some $r \in \mathbb{R}^+ \backslash \{0\}$. In this case, there exists a unique $n_0 \in N$ such that

$$f(n_0) = \begin{bmatrix} 1 & 1/r \\ 0 & 0 \end{bmatrix}.$$  

Define

$$h : S \to M, \begin{bmatrix} j & k \\ l & m \end{bmatrix} \mapsto \begin{bmatrix} j & k \\ l & m \end{bmatrix} g(n_0).$$

For every $n \in N$, we have

$$f(n) = \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} = \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} f(n_0) = f \left( \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} n_0 \right).$$
for some \(a, b \in \mathbb{R}^+\). Since \(f\) is injective, \(n = \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} n_0\) and so
\[
(h \circ f)(n) = h(f(n)) = h\left(\begin{bmatrix} ra & a \\ rb & b \end{bmatrix}\right) = \begin{bmatrix} ra & a \\ rb & b \end{bmatrix} g(n_0) = g\left(\begin{bmatrix} ra & a \\ rb & b \end{bmatrix} n_0\right) = g(n).
\]

We are now ready to provide an example of an \(S\)-\(i\)-injective semimodule which is not \(S\)-\(e\)-injective.

**Example 2.19.** The left \(S\)-semimodule
\[
N_1 = \left\{ \begin{bmatrix} a & a \\ b & b \end{bmatrix} \mid a, b \in \mathbb{R}^+ \right\}
\]
(17)
is \(S\)-\(i\)-injective but not \(S\)-\(e\)-injective.

**Proof.** Let \(\iota : N_1 \to S\) be an embedding and \(id : N_1 \to N_1\) be the identity map. Since \(N_1\) is subtractive, \(\iota\) is a normal \(S\)-monomorphism. Let \(h_1, h_2 : S \to N_1\) with
\[
h_1\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) = \begin{bmatrix} p & p \\ r & r \end{bmatrix} \quad \text{and} \quad h_2\left(\begin{bmatrix} p & q \\ r & s \end{bmatrix}\right) = \begin{bmatrix} q & q \\ s & s \end{bmatrix}.
\]
Then
\[
(h_1 \circ \iota)\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right) = h_1\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right) = \begin{bmatrix} a & a \\ b & b \end{bmatrix} = id\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right)
\]
and
\[
(h_2 \circ \iota)\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right) = h_2\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right) = \begin{bmatrix} a & a \\ b & b \end{bmatrix} = id\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}\right).
\]
Suppose that there exist \(k_1, k_2 : S \to N_1\) such that \(k_1 \circ \iota = 0 = k_2 \circ \iota\) and \(h_1 + k_1 = h_2 + k_2\). Write
\[
k_1\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} l & m \\ n & o \end{bmatrix} \quad \text{and} \quad k_2\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} p & q \\ r & s \end{bmatrix}
\]
for some $k, l, m, n, o, p, q, r, s \in \mathbb{R}^+$. Then
\[
\begin{align*}
  k_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} l & m \\ n & o \end{bmatrix}, \\
  k_2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} p & q \\ r & s \end{bmatrix}
\end{align*}
\]
for every $a, b, c, d \in \mathbb{R}^+$. It follows that
\[
0 = (k_1 \circ \iota) \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} l & m \\ n & o \end{bmatrix} = \begin{bmatrix} l+n & m+o \\ 0 & 0 \end{bmatrix},
\]
which implies that $l = m = n = o = 0$ as 0 is the only element of $\mathbb{R}^+$ which has additive inverse. So,
\[
0 = (k_2 \circ \iota) \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = k_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p+r & q+s \\ 0 & 0 \end{bmatrix},
\]
which implies that $p = q = r = s = 0$ as 0 is the only element of $\mathbb{R}^+$ which has additive inverse. Thus $k_1 = 0 = k_2$, a contradiction with $h_1 + k_1 = h_2 + k_2$ as $h_1 \neq h_2$. Hence, $N_1$ is not $S$-e-injective.

The following example shows that the converse of Lemma 2.15 is not true in general.

**Example 2.20.** Consider the short exact sequence
\[
0 \to E_1 \xrightarrow{\iota_{E_1}} S \xrightarrow{\pi_{E_2}} E_2 \to 0
\]
of left $S$-semimodules. Then $N_1$ is $E_1$-e-injective and $E_2$-e-injective but not $S$-e-injective.

**Proof.** Let $f : M \to E_1, g : M \to N_1$ be $S$-linear maps where $f$ is a normal monomorphism. If $f = 0$, then we are done. If $f \neq 0$, then $f$ is an isomorphism as $E_1$ is ideal-simple. Define $h = g \circ f^{-1}$. Then $h \circ f = (g \circ f^{-1}) \circ f = g$. If $h' : E_1 \to N_1$ is an $S$-linear map satisfies $h' \circ f = g$, then $h' = h' \circ (f \circ f^{-1}) = g \circ f^{-1} = h$. Hence $N_1$ is $E_1$-e-injective. Similarly, $N_1$ is $E_2$-e-injective. However, $N_1$ is not $S$-e-injective as shown in Example 2.19.

## 3 The Embedding Problem

It is well-known that the category of left (right) modules over a ring $R$ has enough injectives, *i.e.* every left (right) $R$-module $M$ can be embedded in an injective left (right) $R$-module (e.g., $E(M)$, the *injective hull* of $M$). This is true only for the left (right) semimodules over some special semirings which are not rings (e.g., the *additively idempotent semirings* [Wan1994], [Gol1999, Corollary 17.34]). In [Ili2016], Il’in conjectured that a semiring $S$ has the property that every left (right) $S$-semimodule has an *injective hull* if and only if $S$ is *additively regular* (i.e. for every $a \in S$, there exists some $x \in S$ such that $a + x + a = a$). In fact, the situation over some semirings can be extremely bad:
Lemma 3.1. If $S$ is an entire, cancellative, zero-sumfree semiring, then the only injective left $S$-semimodule is $\{0\}$ (cf., [Gol1999, Proposition 17.21]).

Example 3.2. The category of commutative monoids (i.e., $\mathbb{Z}^+$-semimodules) has no non-zero injective objects.

Another significant difference is that Baer’s Criterion (a left module $M$ over a ring $R$ is injective if $M$ is $R$-injective) is not valid for semimodules over arbitrary semirings (which are not rings).

Definition 3.3. Let $S$ be a semiring. We say that the category $\mathcal{S} \text{SM}$ has enough injectives (resp. enough $e$-injectives, enough $i$-injectives), if every left $S$-semimodule can be embedded in an injective (resp. $e$-injective, $i$-injective) left $S$-semimodule.

Lemma 3.4. ([Ili2008, Theorem 3]) If $\mathcal{S}S$ satisfies the Baer’s criterion and $\mathcal{S} \text{SM}$ has enough injectives, then $S$ is a ring.

Proposition 3.5. Let $S$ be a semiring. If $\mathcal{S} \text{SM}$ has enough $e$-injectives, then every injective left $S$-semimodule is $e$-injective.

Proof. Let $I$ be an injective left $S$-semimodule. By assumption, there is an embedding $I \hookrightarrow E$, where $\mathcal{S}E$ is $e$-injective. Since $\mathcal{S}I$ is injective, there exists and $S$-linear map $\pi : E \rightarrow I$ such that $\pi \circ t = id_I$. It follows that $\mathcal{S}I$ is a retract of an $e$-injective left $S$-semimodule, whence $e$-injective by Proposition 2.12.■

Proposition 3.6. (compare with [AIKN2018, Theorem 4.5]) Let $S$ be an additively idempotent semiring. Then $\mathcal{S} \text{SM}$ has enough $e$-injectives, and every injective left $S$-semimodule is $e$-injective.

3.7. We define a left $S$-semimodule $N$ to be divisible, if for every $s \in S$, which is not a zero divisor, there exists for every $n \in N$ some $m_n \in N$ such that $sm_n = n$. As in the case of modules over a ring, every injective semimodule over a semiring is divisible.

The proof of the following observation is similar to that in the case of modules over rings [Wis1991, 16.6].

Lemma 3.8. Every $S$-injective left $S$-semimodule is divisible.

Proof. Let $N$ be an injective left $S$-semimodule and $n \in N$. Let $s \in S$ be a non-zero-divisor. Claim: there exists $m_n \in N$ such that $sm_n = n$. Consider the canonical embedding $0 \rightarrowtail Ss \rightarrowtail S$ and the $S$-linear map $h : Ss \rightarrowtail N, ts \mapsto tn$.

By our assumption, $N$ is $S$-injective, whence there exists an $S$-linear map $g : S \rightarrowtail N$ such that $g \circ t = h$. Let $m_n := g(1_S)$. Then we have $n = h(s) = (g \circ t)(s) = g(s) = g(s \cdot 1_S) = sg(1_S) = sm_n$.■

The converse of Lemma 3.8 is not true in general as the following example shows.
Example 3.9. \( \mathbb{Q} \) is a divisible commutative monoid which is not injective.

3.10. Let \( R \) be a ring. Every left \( R \)-module can be embedded in an injective module \( \text{Hom}_\mathbb{Z}(R,D) \), (cf., [Gri2007, page 407, 421]). For a semiring \( S \), we prove that every left \( S \)-semimodule can be embedded into \( \text{Hom}_{\mathbb{Z}^+}(S,D) \) for some divisible commutative monoid \( D \). However, it is unknown whether \( \text{Hom}_{\mathbb{Z}^+}(S,D) \) is necessarily \( e \)-injective.

**Lemma 3.11.** Every commutative monoid can be embedded subtractively in a divisible commutative monoid.

**Proof.** Let \( B \) be a commutative monoid. Then there exists a surjective morphism of monoids \( f : \mathbb{Z}^+(\Lambda) \rightarrow B \) for some index set \( \Lambda \). Let \( g \) be the embedding of \( \mathbb{Z}^+(\Lambda) \) into \( \mathbb{Q}^+(\Lambda) \). Let \( (g', f'; P) \) be a pushout of \( (f, g) \) (see [AN, Theorem 2.3, Corollary 2.4]).

\[
\begin{array}{ccc}
\mathbb{Q}^+(\Lambda) & \xrightarrow{f'} & P \\
\downarrow^g & & \downarrow^{g'} \\
\mathbb{Z}^+(\Lambda) & \xrightarrow{f} & B
\end{array}
\]

Notice that \( g' \) is subtractive since \( g \) is subtractive. Moreover, the commutative monoid \( P \) is divisible since for every \( n \in \mathbb{Z}^+ \) and \( p \in P \) we have \( (p) = (q) \in \mathbb{Q}^+ \) such that \( p = f'(n) \) and \( nq = q' \). Thus \( nf'(q') = f'(nq) = f'(q) \).

Let \( C := \{ q \in \mathbb{Q}^+ | 0 \leq q < 1 \} \). Then \( P \oplus C^{(\Lambda)} \) is a commutative monoid with
\[
(b, (c)) + (b', (c')) = (b + b' + f((c + c'))), (c + c' - (c + c'))
\]

The map
\[
g^* : B \rightarrow P \oplus C^{(\Lambda)} \quad b \mapsto (b, 0)
\]

is a \( \mathbb{Z}^+ \)-monomorphism. The map
\[
f^* : \mathbb{Q}^+(\Lambda) \rightarrow B \oplus C^{(\Lambda)}, \quad (q) \mapsto (f((\lfloor q \rfloor)), (q - \lfloor q \rfloor))
\]

is a \( \mathbb{Z}^+ \)-homomorphism. Since \( f^* \circ g = g^* \circ f \), there exists, by the *Universal Property of Pushouts*, a unique map \( \phi : P \rightarrow B \oplus C^{(\Lambda)} \) such that \( \phi \circ f' = f^* \) and \( \phi \circ g' = g^* \). Since \( g^* \) is injective, \( g' \) is injective. Hence \( g' : B \rightarrow P \) is a normal \( \mathbb{Z}^+ \)-monomorphism from \( B \) into the divisible commutative monoid \( P \). \( \blacksquare \)
Lemma 3.12. Every left $S$-semimodule can be embedded into $\text{Hom}_{\mathbb{Z}^+}(S,D)$ for some divisible commutative monoid $D$.

Proof. Let $M$ be a left $S$-semimodule. By Lemma 3.11 there exists a normal monomorphism of commutative monoids $\mu : M \to D$ for some divisible commutative monoid $D$. Consider the canonical $S$-linear map

$$\epsilon : M \to \text{Hom}_{\mathbb{Z}^+}(S,D), \ m \mapsto [s \mapsto \mu(sm)].$$

Suppose that $\epsilon(m) = \epsilon(m')$ for some $m, m' \in M$. Then, in particular, $\epsilon(m)(1_S) = \epsilon(m')(1_S)$, i.e. $\mu(m) = \mu(m')$. Since $\mu$ is injective, we conclude that $m = m'$. ■

The embedding into an injective $R$-module (where $R$ is a ring) implies a nice result in the category of $R$-modules: an $R$-module $P$ is projective if and only if $P$ is $J$-projective for every injective $R$-module $J$ [Gri2007, page 411]. For semimodules, we have so far the following implication.

Proposition 3.13. Let $\gamma : T \to S$ be a morphism of semirings and $M$ a left $S$-semimodule. If $TA$ is $TM$-$i$-injective, then $\text{Hom}_T(S,A)$ is $S$-$M$-$i$-injective.

Proof. Let $\iota : K \to M$ be a normal $S$-monomorphism and $f : K \to \text{Hom}_T(S,A)$ an $S$-linear map.

Recall the canonical isomorphism of commutative monoids

$$\text{Hom}_S(K,\text{Hom}_T(S,A)) \cong \text{Hom}_T(K,A), \ f \mapsto [k \mapsto f(k)(1_S)].$$

Consider the $T$-linear map $\theta_{KA}(f) : K \to A$.

Since $\iota : K \to M$ is also a normal $T$-monomorphism and $TA$ is $M$-$i$-injective, there exists a $T$-linear map $h : M \to A$ such that $h \circ \iota = \theta_{KA}(f)$. Notice that $\theta_{MA}^{-1}(h) : M \to \text{Hom}_T(S,A)$ is $S$-linear and that for all $k \in K$ and every $s \in S$ we have

$$((\theta_{MA}^{-1}(h) \circ \iota)(k))(s) = \theta_{MA}^{-1}(h)(st(k)) = h(st(k)) = (h \circ \iota)(sk) = \theta_{KA}(f)(sk) = f(sk)(1_S) = (sf(k))(1_S) = f(k)(1_S \cdot s) = f(k)(s).$$

Hence, $\text{Hom}_T(S,A)$ is $M$-$i$-injective as a left $S$-semimodule. ■
The following result is a combination of Proposition 3.13 and [AIKN2018, Corollary 3.5].

**Corollary 3.14.** Let $\gamma : T \rightarrow S$ be a morphism of semirings. The functor

$$\text{Hom}_T(S,-) : \text{SM} \rightarrow \text{SM}$$

preserves injective, $e$-injective and $i$-injective objects.

**Lemma 3.15.** Every divisible commutative monoid is $\mathbb{Z}^+$-$i$-injective.

**Proof.** Let $D$ be a divisible commutative monoid, $f : I \rightarrow \mathbb{Z}^+$ a normal monomorphism of commutative monoids and $g : I \rightarrow D$ a morphism of commutative monoids. Since $f(I)$ is subtractive, $f(I) = k\mathbb{Z}^+$ for some $k \in \mathbb{Z}^+$. Let $i_0 \in I$ be such that $f(i_0) = k$ and notice that $i_0$ is unique as $f$ is injective. By our choice, $D$ is divisible and so there exists $d \in D$ such that $kd = g(i_0)$. The map

$$h : \mathbb{Z}^+ \rightarrow D, n \mapsto nd$$

is a well-defined morphism of monoids. Moreover, for every $i \in I$, we have $f(i) = nk$ for some $n \in \mathbb{Z}^+$ whence $i = ni_0$ as $f$ is injective. It follows that $f(i) = f(ni_0) = nf(i_0) = nk$, and so

$$(h \circ f)(i) = h(nk) = h(n)k = ndk = ng(i_0) = g(ni_0) = g(i).$$

It follows that $hf = g$. ■

**Definition 3.16.** We say that a left $S$-semimodule $I$ is $c$-injective (resp. $c$-$e$-injective, $c$-$i$-injective), if $I$ is $M$-injective (resp., $M$-$e$-injective, $M$-$i$-injective) for every cancellative left $S$-semimodule $M$.

**Proposition 3.17.** Every divisible commutative monoid is $c$-$i$-injective.

**Proof.** Let $D$ be a divisible commutative monoid, $N$ a cancellative left $S$-semimodule, $f : M \rightarrow N$ a normal $\mathbb{Z}^+$-monomorphism and $g : M \rightarrow J$ a morphism of commutative monoids.

$$0 \rightarrow M \xrightarrow{f} N \xrightarrow{g} J$$

Define

$$\mathcal{S} = \{(A, \alpha) : A \subseteq \mathbb{Z}^+, M \subseteq A, \alpha : A \rightarrow J \text{ with } \alpha(m) = g(m) \forall m \in M\}.$$ 

Notice that $\mathcal{S}$ is not empty, since $(M, g) \in \mathcal{S}$. Define an order on $\mathcal{S}$ as follows:

$$(A, \alpha) \leq (B, \beta) \iff A \subseteq B \text{ and } \beta(a) = \alpha(a) \forall a \in A.$$ 

Let $((A, \alpha))_{\Lambda}$ be a chain in $\mathcal{S}$. Set $A := \bigcup_{\lambda \in \Lambda} A_{\lambda}$ and define $\alpha : A \rightarrow J$ such that, if $x \in A_{\lambda}$, then $\alpha(x) = \alpha_{\lambda}(x)$. Notice that $\alpha$ is well-defined, thus the chain has an upper bound $(A, \alpha)$. By Zorn’s Lemma, $\mathcal{S}$ has a maximal element $(C, \gamma)$.

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Claim: If $A \neq N$, then $(A, \alpha) \in \mathcal{S}$ is not maximal.

Let $(A, \alpha) \in \mathcal{S}$ with $A \subseteq N$. Choose $b \in N \setminus A$ and set $B := A + \mathbb{Z}^+b$. Notice that $L := \{ r \in \mathbb{Z}^+ | rb \in A \}$ is an ideal of $\mathbb{Z}^+$ and

$$\kappa : L \rightarrow J, \ r \mapsto \alpha(rb)$$

is a morphism of monoids. By Lemma 3.15 there exists a morphism of monoids $\chi : \mathbb{Z}^+ \rightarrow J$ such that $\chi(r) = \alpha(rb) \ \forall \ r \in L$. Define

$$\beta : B \rightarrow J, \ a + rb \mapsto \alpha(a) + \chi(r).$$

We claim that $\beta$ is well-defined. Suppose that $a + rb = a' + r'b$ for some $r \in L$ and $a \in A$. Assume, without loss of generality, that $r' > r$, whence $r' = r + \tilde{r}$ for some $\tilde{r} \in \mathbb{Z}^+$. It follows that $a + rb = a' + r'b = a' + r + \tilde{r}b$, whence $a = a' + \tilde{r}b$ as $N$ is cancellative. It follows that

$$\beta(a' + r'b) = \beta((a' + r\tilde{b}) + rb) = \alpha(rb + a') + \chi(r) = \alpha(a) + \chi(r) = \beta(a + rb).$$

Thus $\beta$ is well-defined as morphism of monoids with $\beta(a) = \alpha(a) \ \forall \ a \in A$. Thus $(A, \alpha)$ is not maximal in $\mathcal{S}$. It follows that there exists a morphism of monoids $h : N \rightarrow J$ such that $(N, h)$ is maximal in $\mathcal{S}$. Clearly, $h : N \rightarrow J$ such that $h \circ f = g$.

The following result is, in some sense, a generalization of the fact (mentioned without proof in [Gol1999, 17.35]) that any cancellative semimodule over semiring can be embedded in a $c$-injective module. While $c$-injectivity is formally weaker than $c$-injectivity, our result works for arbitrary, not necessarily cancellative, semimodules over semirings.

**Theorem 3.18.** Every left $S$-semimodule can be embedded as a subtractive subsemimodule of a $c$-injective left $S$-semimodule.

**Proof.** Let $M$ be a left $S$-semimodule. By Lemma 3.12, $M$ can be embedded as a subtractive subsemimodule of the left $S$-semimodule $\text{Hom}_{\mathbb{Z}^+}(S, D)$ for some divisible commutative monoid $D$. Let $N$ be a cancellative left $S$-semimodule; then $N$ is, in particular, a cancellative commutative monoid. By Proposition 3.17, $D$ is an $N$-injective $\mathbb{Z}^+$-semimodule, whence $\text{Hom}_{\mathbb{Z}^+}(S, D)$ is $N$-$i$-injective by Proposition 3.13.

The following examples shows one of the advantages of Theorem 3.18.

**Example 3.19.** Let $S$ be an entire, cancellative, zerosumfree semiring. By Theorem 3.18, every left $S$-semimodule $L$ can be embedded subtractively in a $c$-$i$-injective left $S$-semimodule. On the other hand, if $L \neq 0$, then $L$ cannot be embedded in an injective $S$-semimodule since the only injective left $S$-semimodule is $\{0\}$ (by Lemma 3.1). This is the case, in particular, for $S := \mathbb{Z}^+$.

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