Connecting the Breit Frame to the Infinite Momentum Light Front Frame: How $G_E$ turns into $F_1$

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Abstract

We investigate the connection between the Breit and infinite momentum frames and show that when the nucleon matrix element of the time component of the electromagnetic current, which yields $G_E$ in the Breit frame, is boosted to the infinite momentum (or light-front) frame, the quantity $F_1$ is obtained.

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I. INTRODUCTION

A tremendous amount of successful experimental effort has been devoted to measuring the electromagnetic form factors of the nucleons (see the reviews [1, 2, 3, 4]). The experiments have achieved unprecedented accuracy, but the interpretation of the form factors in terms of charge or magnetization densities has been clouded by the need to understand the relativistic motion of the target as a whole and of the ultrarelativistic motion of the light $u$ and $d$ quarks moving within.

The standard interpretation follows from the fact that the nucleon helicity-flip matrix element of the time component of the electromagnetic current density, when evaluated in the Breit frame [in which initial momentum of the nucleon is antiparallel to that of the incident virtual photon ($\mathbf{q}$)], yields $G_E$. Thus $G_E$ is the matrix element of the charge density under the stated conditions. This connection has been used to imply that the charge density is the three-dimensional Fourier transform of $G_E$. However, the initial and final nucleons have different momenta and therefore have different wave functions. The separation between relative and center of mass variables that occurs under nonrelativistic dynamics does not occur for rapidly moving constituents or targets. Thus the initial and final wave functions are related by a boost that generally depends on interactions.

This difficulty can be surmounted by using an infinite momentum frame analysis, with the Drell-Yan condition that $q^+ = (q^0 + q^3) = 0$, where the infinite momentum component of the nucleon is in the $z$ direction. In this case, one obtains a model-independent transverse density that is the two-dimensional Fourier transform of the $F_1$ form factor [5, 6, 7, 8, 9].

One consequence of using this model-independent formalism is that the central value of the transverse charge density of the neutron is negative [8], in seeming contradiction to the long held notion that the center of the neutron is positively charged. This contradiction may arise simply from working in the infinite momentum frame to extract the transverse density. Our purpose here is to examine the connection between using the Breit and infinite momentum frames to compute the matrix element of the charge density operator.

We generalize the usual Breit frame formalism to include the use of arbitrary spin directions for the initial and final states in Sec. II. These states are related to those defined by the use of light-cone spinors in Sec. III, and the use of the infinite momentum frame is discussed in Sec. IV. The remaining section is used for a brief summary and discussion.

II. GENERALIZED BREIT FRAME FORMALISM

The form factors are defined by the matrix element of the electromagnetic current operator $J^\mu(x)$ as

$$\langle f | J^\mu(0) | i \rangle = \bar{u}_f(p') \left( \gamma^\mu F_1(Q^2) + ig^{\mu\nu}q_\nu \frac{F_2(Q^2)}{2M} \right) u_i(p),$$

where $q = p' - p$ is the momentum transfer, which is space-like such that $-q^2 \equiv Q^2 > 0$, $f$ indicates final state, and $i$ indicates initial state. The Sachs form factors [8], defined as

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4M^2} F_2(Q^2)$$
$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

have the common interpretation of being related to the charge and magnetization density of the nucleon, respectively.
Consider the Breit frame (Frame A), with a nucleon with an incoming four-momentum of \( p_A^\mu = (E_0, p_0, 0, 0) \) and an outgoing four-momentum of \( p'_A^\mu = (E_0, -p_0, 0, 0) \), leading to \( q^\mu = p'_A^\mu - p_A^\mu = (0, -2p_0, 0, 0) \) and \(-q^2 = Q^2 = 4p_0^2\). The relevant Dirac spinors for this Breit frame analysis can be written as

\[
\begin{align*}
    u_{\pm x}(p_A) &= \frac{1}{\sqrt{E_0 + M}}(p_A + M) \begin{pmatrix} \chi_{\pm x} \\ 0 \end{pmatrix} \\
    \bar{u}_{\pm x}(p_A) &= \frac{1}{\sqrt{E_0 + M}}(\chi_{\pm x}^\dagger 0)(p_A + M),
\end{align*}
\]

where \( \chi_{\pm x} = 1/\sqrt{2}(1, \pm 1) \) is the two-component spinor in the \( \pm x \) direction, and the normalization is \( \bar{u}(p)u(p) = 2M \).

We compose the incoming state as a yet-to-be-determined state

\[
    u_{iA}(p_A) = a \ u_{+x}(p_A) + b \ u_{-x}(p_A),
\]

and likewise the final state is

\[
    \bar{u}_{fA}(p'_A) = c \ \bar{u}_{+x}(p'_A) + d \ \bar{u}_{-x}(p'_A),
\]

with the constraints

\[
\begin{align*}
|a|^2 + |b|^2 &= 1 \\
|c|^2 + |d|^2 &= 1,
\end{align*}
\]

maintaining the normalization condition.

When the current of Eq. (1) is evaluated between these initial and final states, one finds

\[
\begin{align*}
    \langle f_A|J^0(0)|i_A \rangle &= 2M(ac + bd) \ G_E(Q^2) \\
    \langle f_A|J^1(0)|i_A \rangle &= 0 \\
    \langle f_A|J^2(0)|i_A \rangle &= -2i(ad + bc)p \ G_M(Q^2) \\
    \langle f_A|J^3(0)|i_A \rangle &= 2(ad - bc)p \ G_M(Q^2).
\end{align*}
\]

We see that it is possible to achieve a matrix element of the \( J^0 \) component that is proportional to \( G_E(Q^2) \) by using a multitude of sets of initial and final states in the Breit frame, not simply the typical helicity-flip elements. This is why we use the term “generalized” Breit frame formalism.

We shall see that a convenient choice of the coefficients of the vectors is to use

\[
a = \frac{\sqrt{E_0 + M}}{\sqrt{2E_0}}, \quad b = -\frac{\sqrt{E_0 - M}}{\sqrt{2E_0}}, \quad c = \frac{\sqrt{E_0 + M}}{\sqrt{2E_0}}, \quad d = \frac{\sqrt{E_0 - M}}{\sqrt{2E_0}}.
\]

This choice has the property that, for large values of \( Q^2 \) (or \( E_0 \)), the initial and final states correspond to spins in the \(-z\) and \(+z\) directions. Moreover, Eq. (8) becomes

\[
    \langle f_A|J^0(0)|i_A \rangle = \frac{2M^2}{E_0} \ G_E(Q^2).
\]
The use of Eq. (12) yields the spinors
\[
\begin{align*}
   u_{iA}(p_A) &= \frac{1}{2\sqrt{E_0}} \begin{pmatrix}
   E_0 + M - p_0 \\
   E_0 + M + p_0 \\
   E_0 - M + p_0 \\
   -E_0 + M + p_0
\end{pmatrix}, \\
   \bar{u}_{fA}(p'_A) &= \frac{1}{2\sqrt{E_0}} \begin{pmatrix}
   E_0 + M + p_0, E_0 + M - p_0, -E_0 + M + p_0, E_0 - M + p_0
\end{pmatrix},
\end{align*}
\]
(14)
Next we boost these in the z direction to a new frame, so the incoming and outgoing states both have some momentum \( p_z \) in the \( z \) direction. This is achieved by boosting the spinors with the appropriate boost matrix, given by the formula [10]
\[
S = \exp(-\frac{i\omega_i}{2\sigma_{0i}}) = \frac{1}{\sqrt{2E_0}} \begin{pmatrix}
   \sqrt{E + E_0} & 0 & \sqrt{E - E_0} & 0 \\
   0 & \sqrt{E + E_0} & 0 & -\sqrt{E - E_0} \\
   \sqrt{E - E_0} & 0 & \sqrt{E + E_0} & 0 \\
   0 & -\sqrt{E - E_0} & 0 & \sqrt{E + E_0}
\end{pmatrix},
\]
(15)
where \( \mathbf{v} \) is the boost velocity \( \mathbf{v} = \gamma z \hat{z} \), \( E = \sqrt{p_z^2 + p_0^2 + M^2} \) is the energy of the initial and final states after boost, and the rapidity \( \omega_i = \tanh^{-1}(|\mathbf{v}|/M) \). After this boost, \( q^\mu \) remains unchanged while \( p_A^\mu \) of the incoming state is boosted to \( p_B^\mu = (E, p_0, 0, p_z) \). Likewise, \( p_B^\mu = (E, -p_0, 0, p_z) \). We denote this frame as Frame B. The incoming Breit frame spinor Eq. (14), when boosted with the matrix \( S \) (15), is
\[
S u_{iA}(p_A) = \frac{1}{2\sqrt{p_B^+}} \begin{pmatrix}
   p_B^+ + M - p_0 \\
   p_B^+ + M + p_0 \\
   p_B^- - M + p_0 \\
   -p_B^- + M + p_0
\end{pmatrix},
\]
(16)
where \( p^\pm = p^0 \pm p^3 \) for any four-momentum \( p^\mu \), and we have used the identity \( p_B^+ p_B^- = E^2 - p_z^2 = p_0^2 + M^2 = E_0^2 \). Note also that \( q^+ = q^0 + q^3 = 0 \), a condition that can be chosen for any space-like virtual momentum transfer.

### III. LIGHT CONE SPINORS

We can construct the “light-cone” spinors [11, 12, 13] using the formula
\[
u_{\uparrow x}(p) = \frac{1}{\sqrt{2p^+}}(p^+ + M)\gamma^+(\chi_{+x})^0,
\]
(17)
with an analogous definition for \( \nu_{\downarrow x} \), and
\[
\bar{\nu}_{\uparrow x}(p') = \frac{1}{\sqrt{2p^+}}(\chi_{+x}^0)\gamma^+(p'^+ + M),
\]
(18)
with \( \gamma^+ = \gamma^0 + \gamma^3 \). Using these definitions, we obtain the explicit representations
\[
u_{\uparrow x}(p_B) = \frac{1}{2\sqrt{p_B^+}} \begin{pmatrix}
   p_B^+ + M - p_0 \\
   p_B^+ + M + p_0 \\
   p_B^- - M + p_0 \\
   -p_B^- + M + p_0
\end{pmatrix}, \quad \nu_{\downarrow x}(p_B) = \frac{1}{2\sqrt{p_B^+}} \begin{pmatrix}
   p_B^+ + M + p_0 \\
   -p_B^- - M + p_0 \\
   p_B^- - M - p_0 \\
   p_B^+ - M + p_0
\end{pmatrix},
\]
(19)
and

\[ \bar{u}_{lx}(p_B') = \frac{1}{2\sqrt{p_B^+}} \begin{pmatrix} p_B^+ + M + p_0, & p_B^+ + M - p_0, & -p_B^+ + M + p_0, & -p_B^+ + M + p_0 \end{pmatrix}, \]

\[ \bar{u}_{ix}(p_B') = \frac{1}{2\sqrt{p_B^+}} \begin{pmatrix} p_B^+ + M - p_0, & -p_B^+ - M - p_0, & -p_B^+ + M - p_0, & p_B^+ + M + p_0 \end{pmatrix}. \] (20)

The Breit frame spinors \((14)\), when boosted with the matrix \(S\) \((15)\), can now be decomposed in this basis. It is clear from Eq. \((16)\) and Eq. \((19)\) that

\[ S u_{i_A}(p_A) = u_{\uparrow x}(p_B). \] (21)

Boosting on \(\bar{u}_{f_A}(p_A')\) from Eq. \((14)\) gives

\[ \bar{u}_{f_A}(p_A') \rightarrow \bar{u}_{f_A}(p_A')\gamma_0 S \gamma_0 = \bar{u}_{\uparrow x}(p_B'). \] (22)

Additionally, examining Eq. \((14)\) further, we see that in the Breit frame \(p_A^+ = p_A^0 + p_A^3 = E_0\) and that we can write the spinors in Eq. \((14)\) as

\[ u_{i_A}(p_A) = \frac{1}{2\sqrt{p_A^+}} \begin{pmatrix} p_A^+ + M - p_0 \\ p_A^+ + M + p_0 \\ p_A^+ - M + p_0 \\ -p_A^+ + M + p_0 \end{pmatrix} = u_{\uparrow x}(p_A) \]

\[ \bar{u}_{f_A}(p_A') = \bar{u}_{\uparrow x}(p_A'). \] (23)

Thus the choice of \(a, b, c, d\) of Eq. \((12)\) corresponds to using “light-cone” spinors in the Breit frame (Frame A) that are boosted to the “light-cone” spinors in the boosted frame (Frame B).

**IV. EVALUATION IN THE INFINITE MOMENTUM FRAME (IMF)**

In the infinite momentum frame the charge density operator \(J^0\) becomes \(J^+ = J^0 + J^3\). This is obtained using a Lorentz transformation with the velocity taken to be infinitesimally close to unity and noting that the “\(\gamma\)” factor is absorbed into the change in the \(z\) coordinate \([14]\).

We evaluate the matrix element of \(J^+\) in Frame B to find

\[ \langle f_B|J^+(0)|i_B \rangle = 2p_B^+ F_1(Q^2). \] (24)

Note that this relationship is independent of the boost parameter \(p_z\) and remains the same when we take the IMF limit. However, it is only in the IMF that the operator \(J^0\) becomes \(J^+\). If we boost to the IMF with infinite momentum \(p_z\), helicity spinors are now defined with the spin aligned along the \(z\) direction, because the momentum of the spinor in the \(x\) direction is negligible compared to the large \(z\) momentum. Because \(u_{\uparrow x}(p_B) = \sqrt{2}(u_{\uparrow x}(p_B) + u_{\downarrow x}(p_B))\) as seen from the definition in Eq. \((17)\), we see that a matrix element in the IMF frame corresponding to a two-dimensional charge density can be formed with a linear combination of IMF helicity spinors, instead of only the expected helicity-non-flip matrix elements.
The results Eq. (13) and Eq. (24) are our central results. They show that a Breit frame matrix element of $J^0$ yielding $G_E$ is converted to a matrix element of $J^+$ that yields $F_1$ in the IMF.

Furthermore, we can attempt to begin with solely helicity spinors in the IMF and determine what matrix elements they boost back to in the Breit frame. It is also true that

$$u_{\uparrow z}(p_A) \rightarrow S u_{\downarrow z}(p_B) = u_{\uparrow z}(p_B)$$

$$\bar{u}_{\downarrow z}(p'_A) \rightarrow \bar{u}_{\uparrow z}(p'_B) \gamma_0 S^\dagger \gamma_0 = \bar{u}_{\uparrow z}(p'_B).$$

From here we can construct typical helicity matrix elements in the IMF and determine what they should look like in the Breit frame. For the helicity non-flip in the IMF, which involves transitions between two states with spin aligned along the z axis, we have

$$\langle f_B'|J^+|i_B' \rangle = 2p_B^+ F_1,$$

which corresponds to the matrix elements in the Breit frame,

$$\langle f_A'|J^0|i_A' \rangle = 2M \frac{M}{E_0} G_E$$

$$\langle f_A'|J^1|i_A' \rangle = 0$$

$$\langle f_A'|J^2|i_A' \rangle = -i Q G_M$$

$$\langle f_A'|J^3|i_A' \rangle = \frac{Q^2}{2E_0} G_M.$$ (27)

If we consider the helicity flip element in the IMF, we have

$$\langle f_B''|J^+|i_B'' \rangle = p_B^+ \frac{Q}{M} F_2,$$

which corresponds to the matrix elements in the Breit frame,

$$\langle f_A''|J^0|i_A'' \rangle = -M \frac{Q}{E_0} G_E$$

$$\langle f_A''|J^1|i_A'' \rangle = 0$$

$$\langle f_A''|J^2|i_A'' \rangle = 0$$

$$\langle f_A''|J^3|i_A'' \rangle = M \frac{Q}{E_0} G_M.$$ (30)

V. DISCUSSION

The key result that we obtain is that a Breit frame matrix element of $J^0$ yielding $G_E$, Eq. (13), is converted via a boost, Eq. (15), to a matrix element of $J^+$ that yields $F_1$, Eq. (24), in the infinite momentum frame. The use of the infinite momentum frame, along with the Drell-Yan condition, $q^+ = 0$, allows the extraction of a transverse density as the two-dimensional Fourier transform of $F_1$ [5, 6, 7, 8, 9]. The present result establishes a connection between the standard Breit frame procedure involving $G_E$ and the infinite momentum frame procedure involving $F_1$, which is related to the transverse density. Future work will be concerned with determining whether or not a connection between the rest frame charge density and the transverse density can be established.
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