LIE ALGEBROIDS GENERATED BY COHOMOLOGY OPERATORS

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Abstract. By studying the Frölicher-Nijenhuis decomposition of cohomology operators (that is, derivations $D$ of the exterior algebra $\Omega(M)$ with $\mathbb{Z}$–degree 1 and $D^2 = 0$), we describe new examples of Lie algebroid structures on the tangent bundle $TM$ (and its complexification $T^CM$) constructed from pre-existing geometric ones such as complex, product or tangent structures. We also describe a class of Lie algebroids on tangent bundles associated to idempotent endomorphisms with nontrivial Nijenhuis torsion.

1. Introduction

In this paper, we present an algebraic approach to the study of the relationship between Lie algebroids and cohomology operators which is based on the Frölicher-Nijenhuis calculus (various approaches to this issue can be found, for example, in [6, 12, 16, 17]). Our idea comes from the relation between the exterior differential and the Lie bracket of vector fields on a manifold $M$. As a first-order differential operator on the (sections of) exterior algebra, $\Omega(M) = \Gamma\Lambda T^*M$, the exterior differential can be characterized by giving its action on generators:

$$
\begin{align*}
    df(X) &= Xf, \\
    d\alpha(X, Y) &= X\alpha(Y) - Y\alpha(X) - \alpha([X, Y]),
\end{align*}
$$

where $f \in \mathcal{C}^\infty(M)$, $\alpha \in \Omega(M)$, $X, Y \in \mathcal{X}(M)$ are arbitrary. The action is then extended to the whole of $\Omega(M)$ as a derivation of $\mathbb{Z}$–degree 1. Notice that the cohomology property $d^2 = 0$ is equivalent to the Jacobi identity for $[.,.]$.

This setup allows us to reverse the definitions. Starting from the exterior differential

1991 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. Lie algebroids, cohomology operators, product structures, complex structures, tangent structures, sprays.

The second author was partially supported by a CONACyT project CB-179115.
we will recall how, given a Lie algebra \( \mathfrak{L}(M) \), their bracket is the unique element \([X, Y] \in \mathfrak{L}(M)\) such that,
\[
\alpha([X, Y]) = X\alpha(Y) - Y\alpha(X) - d\alpha(X, Y),
\]
for any \( \alpha \in \Omega^1(M) \). In particular, if \( \alpha = df \), for \( f \in C^\infty(M) \), the preceding formula along with \( d^2 = 0 \) gives
\[
[X, Y]f = df([X, Y]) = Xdf(Y) - Ydf(X) = X(Yf) - Y(Xf),
\]
so the defined bracket certainly coincides with the Lie one. Again, the Jacobi identity is readily seen to be equivalent to the coboundary condition \( d^2 = 0 \).

The correspondence between brackets and cohomology operators can be extended to the setting of Lie algebroids. Recall that a Lie algebroid on a manifold \( M \) consists of a vector bundle \( E \) over \( M \), together with a vector bundle map \( q : E \to TM \) over \( M \) (called the anchor map), and a Lie bracket on sections \([ , ] : \Gamma E \times \Gamma E \to \Gamma E\) satisfying the Leibniz rule
\[
[A, fB] = f[A, B] + qA(f)B,
\]
for all \( A, B \in \Gamma E, f \in C^\infty(M) \). It can be thought as a replacement of the tangent bundle \( TM \), joint with the Lie bracket on vector fields, by the new bundle \( E \) and the bracket \([ , ]\) on its sections. For a comprehensive reference on Lie algebroids, see [17]. The relevance of Lie algebroids in Mechanics is explained in detail in works such as [23, 4, 18] and references therein.

We first describe a class of Lie algebroids arising from an idempotent endomorphism with nontrivial Nijenhuis torsion, and then construct new examples of Lie algebroids. More precisely, the paper is organized as follows: Section 2 of this paper contains a brief résumé of the Frölicher-Nijenhuis calculus for derivations of the exterior algebra over a vector bundle. In section 3 we will recall how, given a Lie algebroid \( E \) on \( M \), one can construct a cohomology operator on \( \Gamma E^*\), and study its Frölicher-Nijenhuis decomposition. It turns out that this decomposition can be described within the framework of Lie algebroid connections [10, 17, 2]. In section 4 we will follow the reverse path, constructing Lie algebroids on the tangent bundle \( E = TM \) from a given cohomology operator \( D \) on \( \Omega(M) \). Our main tool here will be again the Frölicher-Nijenhuis decomposition of derivations, which we apply in section 5 to the case of idempotent endomorphisms with nonzero Nijenhuis decomposition of the tangent bundle. In section 6, on the framework of our approach, we discuss Lie algebroids associated to regular foliations. Finally, sections 7 to 9 are devoted to introducing new examples of Lie algebroids canonically associated to a generalized foliation, and to complex, product and tangent manifolds (the latter, with the aid of a connection defined through a semispray).

2. The Frölicher-Nijenhuis decomposition

The proofs of the results stated in this section are slight generalizations of those corresponding to the real case, which can be found, e.g., in [19] or [15].

Let \( M \) be a differential manifold with tangent bundle \( TM \). Its complexified bundle is then \( T_C M = TM \oplus iTM \). Complex vector fields are of the form \( Z = X + iY \), with \( X, Y \in \Gamma TM \), and complex 1-forms are constructed as the duals \( \Omega^1_{C}(M) = \Gamma(T^*_C M) \simeq (\Gamma T^*M) \oplus i(\Gamma T^*M) \simeq \Omega^1(M) \oplus i\Omega^1(M) \). By taking exterior products we obtain the complex k-forms \( \Omega^k_{C}(M) = \Gamma(\Lambda^k T^*_C M) \simeq \Omega^k(M) \oplus i\Omega^k(M) \),
where $\Gamma$ of degree $k$ and, for any homogeneous form $\alpha$, exists a unique vector-valued complex form $i\alpha$. Insertions are tensorial so, applying proposition $\Omega_{\mathbb{C}}(M)$ is again a derivation $[D_1, D_2] = D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1$ where $|D_1|$, $|D_2|$ denote the degree of $D_1$, $D_2$, respectively.

Any complex vector-valued $(k+1)$-form $K \in \Omega^{k+1}_{\mathbb{C}}(M; TM)$, defines a derivation of degree $k$, called the insertion of $K$ and denoted $i_K$: if $\omega \in \Omega^{k}_{\mathbb{C}}(M)$ and $Z_j \in \Gamma(TM \oplus \iota TM)$, with $j = 1, ..., k + p$, then

$$i_K \omega(Z_1, \ldots, Z_{k+p}) = \sum_{\sigma \in S_{k+1,p-1}} \text{sgn}(\sigma) \omega(K(Z_{\sigma(1)}, \ldots, Z_{\sigma(k+1)}), Z_{\sigma(k+2)}, \ldots, Z_{\sigma(k+p)}),$$

where $S_{k+1,p-1}$ are shuffle permutations and sgn denotes the signature.

A derivation $D \in \text{Der} \Omega_{\mathbb{C}}(M)$ such that it vanishes on functions, $D(f) = 0$ for all $f \in \Omega^0_{\mathbb{C}}(M)$, is called a tensorial (or algebraic) derivation. They all are insertions.

**Proposition 2.1.** Let $D \in \text{Der} \Omega_{\mathbb{C}}(M)$ be a tensorial derivation. Then, there exists a unique vector-valued complex form $K \in \Omega^{k+1}_{\mathbb{C}}(M; TM)$ such that $D = i_K$.

The $K$ in this proposition can be obtained simply by making $D$ act on complex valued $1$–forms $df$. If $K \in \Omega^{k+1}_{\mathbb{C}}(M; TM)$, $L \in \Omega^{l+1}_{\mathbb{C}}(M; TM)$, their corresponding insertions are $i_K \in \text{Der} \Omega_{\mathbb{C}}(M)$, $i_L \in \text{Der} \Omega_{\mathbb{C}}(M)$, so their graded commutator is again a derivation $[i_K, i_L] \in \text{Der} \Omega_{\mathbb{C}}(M)$. Moreover, this new derivations is obviously tensorial so, applying proposition 2.1, there exists a unique vector-valued complex $(k + l + 1)$–form determined by it.

**Definition 2.2.** Given $K \in \Omega^{k+1}_{\mathbb{C}}(M; TM)$, $L \in \Omega^{l+1}_{\mathbb{C}}(M; TM)$, their Richardson-Nijenhuis bracket is defined as the element $[K, L]_{\mathbb{R}N} \in \Omega^{k+l+1}_{\mathbb{C}}(M; TM)$ such that

$$[i_K, i_L] = i_{[K, L]_{\mathbb{R}N}}.$$

The bracket of vector fields on $M$ can be clearly extended by $\mathbb{C}$–linearity to complex vector fields; the same occurs with the exterior differential $d$, which can be extended by $\mathbb{C}$–linearity to a derivation $d \in \text{Der} \Omega_{\mathbb{C}}(M)$ (we will use the same notation for $d$ and this extension). Then, given a $K \in \Omega^{k+1}_{\mathbb{C}}(M; TM)$, as $i_K \in \text{Der} \Omega_{\mathbb{C}}(M)$ and $d \in \text{Der} \Omega_{\mathbb{C}}(M)$, their graded commutator will be again a derivation, called the Lie derivative along $K$,

$$L_K := [i_K, d] = i_K \circ d - (-1)^k d \circ i_K.$$

As a consequence of the Jacobi identity for the graded commutator of derivations, and the nilpotency of $d$, we get the following.
Proposition 2.3. For any \( K \in \Omega^k_C(M; TM) \), the graded commutator of the derivations \( \mathcal{L}_K \) and \( d \) vanishes.

Thus, there exist tensorial derivations, of type \( \mathfrak{i}_K \) for \( K \in \Omega^k_C(M; TM) \), and derivations commuting with the exterior differential \( d \), such as Lie derivatives. The Frölicher-Nijenhuis decomposition theorem states that any other derivation is a sum of two of these.

Theorem 2.4 (Frölicher-Nijenhuis). Let \( D \in \text{Der}^k \Omega_C(M) \). Then, there exists a unique couple \( (K, L) \), \( K \in \Omega^k_C(M; TM) \) and \( L \in \Omega^{k+1}_C(M; TM) \), such that

\[
D = \mathcal{L}_K + \mathfrak{i}_L.
\]

A useful consequence is the following result.

Corollary 2.5. Let \( D \in \text{Der}^k \Omega_C(M) \). Then

1. \( D \) is tensorial if and only if \( K = 0 \).
2. \( D \) commutes with \( d \) if and only if \( L = 0 \).

Example 1. The Frölicher-Nijenhuis decomposition of the exterior differential is

\[
d = \mathcal{L}_\text{Id},
\]

(where \( \text{Id} : TM \to TM \) is the identity morphism) as it should be for a derivation that commutes with \( d \).

If \( K \in \Omega^k_C(M; TM) \) and \( L \in \Omega^l_C(M; TM) \), then \( [\mathcal{L}_K, \mathcal{L}_L] \) is again a derivation and it commutes with \( d \), because of the Jacobi identity. Thus, according to Corollary 2.5, it must be of the form \( \mathcal{L}_R \in \text{Der}^{k+l}(\Omega_C(M; TM)) \) for a unique \( R \in \Omega^{k+l}(\Omega_C(M; TM)) \).

Definition 2.6. Given \( K \in \Omega^k_C(M; TM) \) and \( L \in \Omega^l_C(M; TM) \), their Frölicher-Nijenhuis bracket is the unique element \( [K, L]_{FN} \in \Omega^{k+l}_C(M; TM) \) such that

\[
[\mathcal{L}_K, \mathcal{L}_L] = [K, L]_{FN}.
\]

Along with the Frölicher-Nijenhuis bracket we have the notion of Nijenhuis torsion: if \( N \in \Omega^1_C(M; TM) \) is a vector-valued 1–form, it is defined as the vector-valued 2–form \( T_N(X, Y) = [NX, NY] - N[NX, Y] - N[X, NY] + N^2[X, Y] \), for any \( X, Y \in \mathcal{X}(M) \).

3. From Lie algebroids to cohomology operators

Suppose that \( (E, q, [\, , \,]) \) is a Lie algebroid. Let us define an operator \( D : \Gamma(\Lambda^p E^*) \to \Gamma(\Lambda^{p+1} E^*) \) by putting

\[
\begin{cases}
Df(A) = qA(f) \\
D\alpha(A, B) = qA(\alpha(B)) - qB(\alpha(A)) - \alpha([A, B]),
\end{cases}
\]

for any \( f \in \mathcal{C}^\infty(M), \alpha \in \Gamma(\Lambda^1 E^*), A, B \in E \), and extending its action to \( \Gamma(\Lambda^\bullet E^*) \) as a derivation of \( \mathbb{Z} \)-degree 1.

Proposition 3.1. \( D \) is a cohomology operator, that is,

\[
D^2 = \frac{1}{2} [D, D] = 0,
\]

where \([\, , \,] \) denotes the graded commutator of derivations.
This is a very well-known construction (see chapter 7 in [17], for instance), sometimes called the De Rham differential of the Lie algebroid (because one recovers the usual exterior differential $d$, when $E = TM$ endowed with the Lie bracket on vector fields and the identity as anchor map). The operator $D$ is a derivation of $\Gamma(\Lambda^*E^*)$, so it can be decomposed à la Frölicher-Nijenhuis. Let us see how this can be done in a particular, but instructive, case.

**Example 2.** When $E = TM$, we have a Lie algebroid $(TM, q, [\ , \ ])$ and a derivation $D \in \text{Der}\Omega(M)$. Recall (see [11]) that given a bilinear operation $\circ$ on the sections of a vector bundle $E$ (such as $[\ , \ ]$ on $\Gamma TM$), and a bundle endomorphism $\varphi$ (such as $q : \Gamma TM \to \Gamma TM$), the contracted bracket of the operation is given by $A \circ_{\varphi} B = \varphi(X) \circ Y + X \circ \varphi(Y) - \varphi(X \circ Y)$, for $A, B \in E$. In our case, the contracted bracket of the Lie bracket of vector fields by the anchor map of the algebroid is

$$[X, Y]_q = [qX, Y] + [X, qY] - q[X, Y].$$

Now, a computation using the Frölicher-Nijenhuis decomposition $D = \mathcal{L}_K + i_L$, where $K \in \Omega^1(M; TM)$ and $L \in \Omega^2(M; TM)$, shows that

$$K = q,$$

and

$$L(X, Y) = [X, Y]_q - [X, Y].$$

Indeed, we have that $K$ is given by

$$KX(f) = Df(X) = qX(f),$$

for each $f \in C^\infty(M)$, $X \in \mathcal{X}(M)$, and also

$$(D - \mathcal{L}_q)(\alpha)(X, Y) = (i_L\alpha)(X, Y) = \alpha(L(X, Y)).$$

This is enough to characterize $i_L$, as it is determined by its action on $1$–forms (on functions it vanishes by definition): Developing the left-hand side in (5),

$$(D\alpha)(X, Y) = qX(\alpha(Y)) - qY(\alpha(X)) - \alpha([X, Y]),$$

and, from the definition of $\mathcal{L}_q\alpha$,

$$(\mathcal{L}_q\alpha)(X, Y) = qX(\alpha(Y)) - qY(\alpha(X)) + \alpha([qX, Y]) + \alpha([qY, X]) + \alpha(q[X, Y]),$$

thus:

$$\alpha(L(X, Y)) = \alpha([qX, Y] + [X, qY] - q[X, Y] - [X, Y]).$$

As this is valid for arbitrary $\alpha \in \Omega^1(M)$, $X, Y \in \mathcal{X}(M)$, we get (4).

The question now is how to obtain an analog result for an arbitrary Lie algebroid $(E, q, [\ , \ ])$, where $D \in \text{Der}\Gamma(\Lambda^*E^*)$ and we do not have a Lie derivative operator at our disposal. We can proceed by choosing a connection $\nabla$ on $E$; then, a proof analogous to that of the classical Frölicher-Nijenhuis theorem (see [20]), shows that $D$ must decompose itself as

$$D = \nabla_K + i_L$$

for certain tensor fields $K \in \Gamma(E^* \otimes TM)$ and $L \in \Gamma(\Lambda^2E^* \otimes E)$. We briefly describe the proof of this result here, as we will make use of some of its intermediate steps below.
Theorem 3.2. Let \( \pi : F \to M \) be a smooth vector bundle over the differential manifold \( M \). Let \( D \in \text{Der}(\Lambda^k F^*) \), and let \( \nabla \) be a linear connection on \( F^* \). Then, there exist unique tensor fields \( K \in \Gamma(F \otimes TM) \) and \( L \in \Gamma(\Lambda^k F \otimes F^*) \), such that

\[
D = \nabla_K + i_L.
\]

Proof. Let \( \alpha^1, \ldots, \alpha^k \in \Gamma(F^*) \) be smooth sections. The map \( f \to (Df)(\alpha^1, \ldots, \alpha^k) \), where \( f \in C^\infty(M) \), is a derivation so it defines a vector field on \( M \), which we denote by \( K(\alpha^1, \ldots, \alpha^k) \). The map from \( \Gamma(F^*) \times \cdots \times \Gamma(F^*) \to \mathfrak{X}(M) \) defined by \( (\alpha^1, \ldots, \alpha^k) \mapsto K(\alpha^1, \ldots, \alpha^k) \) is \( C^\infty(M) \)-linear and skew-symmetric, therefore it defines a section \( K \in \Gamma(\Lambda^k F \otimes TM) \) that satisfies \( \nabla f = Df \) for every \( f \in C^\infty(M) \).

The operator \( D \circ \nabla \) is a derivation of degree \( k \) that acts trivially on \( C^\infty(M) \); therefore it is a \( C^\infty(M) \)-linear endomorphism of \( \Gamma(\Lambda^k F \otimes F^*) \) which is determined by its action on the sections of degree 1. Then, if \( s \in \Gamma(F) \), the map \( s \to (D \circ \nabla)s \) defines a morphism from \( \Gamma(F) \) into \( \Gamma(\Lambda^{k+1} F \otimes F^*) \) such that \( (D \circ \nabla)s = i_L s \). The operator \( i_L \) is a derivation of degree \( k \) that acts trivially on \( C^\infty(M) \) and as \( D \circ \nabla \) on the sections of \( F \). Then, \( D = \nabla_K + i_L \). \( \square \)

The decomposition (6) above follows, of course, taking \( F = E^* \). From the proof, we get that \( K : \Gamma E \to \Gamma TM \) is such that, for any \( A \in \Gamma E \) and \( f \in C^\infty(M) \),

\[
K(A)(f) = Df(A) = qA(f),
\]

so \( K = q \), as before.

The following result will be useful when doing explicit computations.

Lemma 3.3. Let \( K : E \to TM \) be a vector bundle morphism, and let \( \nabla \) be a connection on \( E \). If \( a \in \Gamma E^* \) and \( A, B \in \Gamma E \), then,

\[
(\nabla_K a)(A, B) = KA(a(B)) - a(\nabla_K A B) - KB(a(A)) + a(\nabla_K B A).
\]

Proof. Consider a basis of sections of \( \Gamma E \), \( \{s_a\}_{a=1}^r \) (with \( r = \text{rank} E \)), and let \( \{\eta^b\}_{b=1}^m \) be the dual basis. Also, let \( \{\partial_i\}_{i=1}^m \) be a local basis of local vector fields on \( M \) (with \( \dim M = m \)). Then, we have the local expressions \( A = A^a s_a \), \( B = B^b s_b \), and \( K = K^i_\alpha \eta^c \otimes \partial_j \). We can write

\[
\nabla_K = \nabla_{K^i_\alpha} \eta^c \otimes \partial_j = K^i_\alpha \eta^c \otimes \nabla_{\partial_j},
\]

and compute

\[
(\nabla_K a)(A, B) = (K^i_\alpha \eta^c \otimes \nabla_{\partial_j} a)(A, B)
\]

\[
= K^i_\alpha (\eta^c(A) \nabla_{\partial_j} a)(B) - \eta^c(B) (\nabla_{\partial_j} a)(A)
\]

\[
= K^i_\alpha (A^c(\nabla_{\partial_j} a(B)) - a(\nabla_{\partial_j} B)) - B^c(\nabla_{\partial_j} a(A)) - a(\nabla_{\partial_j} A))
\]

\[
= \nabla_{K^i_\alpha A^c \partial_j} a(B) - K^i_\alpha A^c a(\nabla_{\partial_j} B) - \nabla_{K^i_\alpha B^c \partial_j} a(A) + K^i_\alpha B^c a(\nabla_{\partial_j} A).
\]

Notice that \( KA = K^i_\alpha A^c \partial_j \), so we can rearrange the preceding expression, using also the \( C^\infty(M) \)-linearity of \( a \), as

\[
(\nabla_K a)(A, B) = \nabla_{KA}(a(B)) - a(\nabla_{KA} B) - \nabla_{KB}(a(A)) - a(\nabla_{KB} A)
\]

\[
= KA(a(B)) - KB(a(A)) + a(\nabla_{KB} A - \nabla_{KA} B).
\]

Now, we can find \( L \) in the decomposition (6). Let us start with:

\[
(D - \nabla_q)(a)(A, B) = (i_L a)(A, B) = a(L(A, B)).
\]
Now, on the one hand,
\[(D\alpha)(A, B) = qA(\alpha(B)) - qB(\alpha(A)) - \alpha([A, B])\],
and on the other, from lemma 3.3
\[\nabla_q \alpha(A, B) = qA(\alpha(B)) - \alpha(\nabla_q A) - qB(\alpha(A)) + \alpha(\nabla_q B)A\].
Thus,
\[\alpha(L(A, B)) = (D - \nabla_q)(\alpha)(A, B) = \alpha(\nabla_q A - \nabla_q B - [A, B])\],
hence:
\[(9)\]
\[L(A, B) = \nabla_q A - \nabla_q B - [A, B]\].

**Remark 1.** Let \((E, q, [\cdot, \cdot])\) be a Lie algebroid over \(M\). Let \(V \to M\) another vector bundle over \(M\). An \(E\)-connection on \(V\), \(\delta\), is an \(\mathbb{R}\)-bilinear map \(\delta : \Gamma E \times \Gamma V \to \Gamma V\), whose action is denoted \(\delta(A, s) = \delta_A s\), such that, for any \(f \in C^\infty(M)\),
\[
\delta f_A s = f \delta_A s \\
\delta_A (fs) = (qA)(fs) + f \delta_A s.
\]
An \(E\)-connection is simply an \(E\)-connection on \(E\) itself (see [10, 17, 2] for references on Lie algebroid connections and their applications). In this case, it is possible to define the torsion of \(\delta\) as the \(E\)-valued 2-form \(\text{Tor} \delta \in \Omega^2(E; E)\) given by the usual expression, but using the Lie algebroid bracket instead of the Lie one:
\[\text{Tor} \delta(A, B) := \delta_A B - \delta_B A - [A, B]\].
Notice that each linear connection \(\nabla\) on \(E\) determines an \(E\)-connection \(\delta\nabla\). Simply define
\[(10)\]
\[\delta_{\nabla}^A B := \nabla qAB\].
The torsion of this connection is then
\[\text{Tor} \delta_{\nabla}(A, B) = \nabla qAB - \nabla qBA - [A, B]\],
which is precisely our expression (9).

We summarize these calculations in the following result.

**Theorem 3.4.** Let \((E, q, [\cdot, \cdot])\) be a Lie algebroid and \(D \in \text{Der} \Gamma(\Lambda^*E^*)\) be the derivation defined in (2). Then, given any linear connection \(\nabla\) on \(E\), \(D\) can be written as
\[D = \nabla_q + \iota_{L^\nabla}\],
where \(L^\nabla \in \Gamma(\Lambda^2 E^* \otimes E)\), the torsion of the \(E\)-connection \(\delta\nabla\), is given by
\[L^\nabla(A, B) = \nabla qAB - \nabla qBA - [A, B]\],
for \(A, B \in \Gamma E\).

An equivalent formulation of this result can be obtained in the particular case of a Lie algebroid on the tangent bundle, which offers an alternative interpretation of the contracted bracket \([X, Y]_q\) in terms of the torsion of the \(TM\)-connection \(\delta\nabla\) in (10).
Observe that if \( E = TM \), then, given any symmetric linear connection \( \nabla \) on \( TM \), the torsion of the corresponding \( TM \)-connection \( \delta \nabla \) is given by
\[
\text{Tor} \delta \nabla = L \nabla (X, Y) = \nabla_q X Y - \nabla q Y X - [X, Y] = [X, Y]_q + (\nabla_Y q)(X) - (\nabla_X q)(Y) - [X, Y],
\]
for any \( X, Y \in \mathcal{X}(M) \).

Just apply that \( \nabla \) is torsionless, and the properties of any covariant derivative.

4. From cohomology operators to Lie algebroids

Suppose we have a smooth vector bundle \( \pi : E \to M \), and a derivation \( D \in \text{Der}(\Lambda^\bullet E^\ast) \) such that \( D^2 = 0 \). Then, we can define a mapping \( q : \Gamma E \to \Gamma TM \) as follows,
\[
q(A)(f) := Df(A),
\]
for any \( f \in C^\infty(M), A \in \Gamma E \).

Let us define also a bracket on the sections of \( E \) in the following way: if \( A, B \in \Gamma E \), then their bracket \([A, B]\) is the section of \( E \) characterized by
\[
\alpha([A, B]) = D(\alpha(B))(A) - D(\alpha(A))(B) - d(\alpha(A, B)),
\]
for any \( \alpha \in \Gamma E^\ast \). The following result is well-known.

Proposition 4.1. The triple \((E, q, [\cdot, \cdot])\) is a Lie algebroid.

Remark 2. It follows from proposition 3.1 and proposition 4.1, that there exists a one to one correspondence between Lie algebroids on \( E \) and cohomology operators in \( \Gamma(\Lambda E^\ast) \), that is, derivation \( D \) of degree one on \( \Gamma(\Lambda E^\ast) \) such that \( D^2 = 0 \). For various formulations of this results see, for example, [16] and [6].

Let us delve into the structure of this Lie algebroid when \( E = TM \). In this case, last proposition can be reformulated: Let \( D \) be a derivation, \( D \in \text{Der}(\Lambda^\bullet T^\ast M) \), such that \( D^2 = 0 \), then it has a Frölicher-Nijenhuis decomposition \( D = L_K + \iota_K \), where \( K \in \Omega^1(M; TM) \) and \( L \in \Omega^2(M; TM) \). Then, the triple \((E, q, [\cdot, \cdot])\) is a Lie algebroid with
\[
q = K
\]
and
\[
[X, Y] = [X, Y]_K - L(X, Y),
\]
where the contracted bracket is given in (3).

Formula (11) can be deduced by observing that \( D \) is precisely the operator associated to the Lie algebroid by the construction in the previous section. In particular, that tells us that the distribution \( \text{Im}K \) is involutive, because (as the anchor map of a Lie algebroid is a Lie algebra morphism) \([KX, KY] = K[X, Y] \in \text{Im}K\). Notice (see [11]) that when the Nijenhuis torsion of \( K \) vanishes, then \([\cdot, \cdot]_K \) is a Lie bracket.

Taking the action of \( K \) on both sides of (11), we get
\[
KL(X, Y) = K[KX, Y] + K[X, KY] - K^2[X, Y] - K[X, Y].
\]
But, as \( K = q \) is a Lie algebroid anchor, it is a Lie algebra morphism, so
\[
[KX, KY] - K[KX, Y] - K[X, KY] + K^2[X, Y] = -KL(X, Y),
\]
or, more succinctly, in terms of the Nijenhuis torsion of \( K \),
\[
T_K(X, Y) = -KL(X, Y).
\]
This expression allows us to prove the following result: Let \( D \in \text{Der}^1\Omega(M) \) be such that \( D^2 = 0 \) and its Frölicher-Nijenhuis decomposition has the form \( D = \mathcal{L}_K \) (i.e., \( L = 0 \)).

**Proposition 4.2.** The derivation \( D = \mathcal{L}_K \) induces a Lie algebroid structure on \((\text{sections of}) \ T M\) with anchor map \( q = K \) and bracket

\[
[X, Y]_K = [KX, Y] + [X, KY] - K[X, Y],
\]

if and only if \( K \) is integrable (i.e, \( T_K = 0 \)).

**Proof.** The condition \( T_K = 0 \) follows from (12) above. On the other hand, the sufficiency is guaranteed by the identities \( D^2 = \frac{1}{2}[D, D] \) and \([\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_{[K, L]} \) and by proposition 4.1. □

**Remark 3.** The ‘if’ part of this proposition was given as theorem 3.7 in [12] and as exercise 40 in [5].

Recall that, given a bundle map \( N : TM \to TM \), when \( T_N = 0 \), \( N \) is called a Nijenhuis tensor. As every Nijenhuis tensor induce a Lie algebroid, then, any complex structure \((K^2 = -I)\), tangent structure \((K^2 = 0)\) or product structure \((K^2 = I)\), defines a Lie algebroid on \( TM \). Let us see what happens when a term \( L \neq 0 \) is non trivial. Still demanding that \( D = \mathcal{L}_K + iL \) be of square zero; we will do this in two steps, assuming in the first one that \( K \) is invertible. If \( K : \Gamma TM \to \Gamma TM \) is an invertible endomorphism, the vector-valued 2–form \( L \in \Omega^2(M; TM) \) is determined by

\[
L(X, Y) := -K^{-1}T_K(X, Y),
\]

for any \( X, Y \in \Gamma TM \), and then, the bracket \( [\ , \ ] : \Gamma TM \to \Gamma TM \) is

\[
[X, Y] := [X, Y]_K + K^{-1}T_K(X, Y),
\]

for all \( X, Y \in \Gamma TM \).

**Lemma 4.3.** The bracket (14) satisfies

\[
[X, Y] = K^{-1}[KX, KY].
\]

**Proof.** It is a straightforward computation:

\[
[X, Y] = [X, Y]_K + K^{-1}T_K(X, Y) = [KX, Y] + [X, KY] - K[X, Y] + K^{-1}(KX, KY) - K[KX, Y] - K[X, KY] + K^2[X, Y]) = K^{-1}[KX, KY]
\]

□

**Proposition 4.4.** The triple \((TM, K, [\ , \ ])\), with the bracket defined in (14), is a Lie algebroid.

**Proof.** That \((\Gamma TM, [\ , \ ])\) is a Lie algebra is immediate (Jacobi’s identity results by applying Lemma 4.3), and \( K \) is a morphism of \( C^\infty(M)\)–modules, so we only need to check the Leibniz rule:

\[
[X, fY] = [X, fY]_K + K^{-1}T_K(fX, Y) = [KX, fY] + [X, fKY] - K[X, fY] + fK^{-1}T_K(X, Y) = f[X, Y]_K + KX(f)Y + X(f)KY - X(f)KY + fK^{-1}T_K(X, Y) = f[X, Y] + KX(f)Y.
\]
Every Lie algebroid induced by an invertible endomorphism is isomorphic to the trivial Lie algebroid:

**Proposition 4.5.** The Lie algebroids \((TM, K, [~,~])\) (given by proposition 4.4) and the trivial one \((TM, Id, [~,~])\) are isomorphic.

**Proof.** The desired isomorphism is \(\phi = K^{-1} : \Gamma TM \rightarrow \Gamma TM\), since
\[
K \circ \phi = K \circ K^{-1} = Id,
\]
and
\[
\phi([X, Y]) = K^{-1}([X, Y]) = K^{-1}([K \circ K^{-1}(X), K \circ K^{-1}(Y)]) = [\phi(X), \phi(Y)]
\]
by lemma 4.3. 

To summarize, we have the following result which, in particular, applies to almost-complex or almost-product structures.

**Theorem 4.6.** Let \(K : TM \rightarrow TM\) be an invertible bundle endomorphism. Then, a derivation of the form
\[
D = \mathcal{L}_K + \iota_L \in \text{Der}^1\Omega(M),
\]
(with \(L \in \Omega^2(M; TM)\)) has vanishing square if and only if
\[
L = -K^{-1}T_K.
\]
In this case, the Lie algebroid structure on \(TM\) determined by \(D\) is isomorphic to the trivial one.

In the next step, we consider the general situation of a derivation of \(Z\)-degree 1, \(D = \mathcal{L}_K + \iota_L\), where \(K\) is not necessarily invertible. We ask ourselves under which conditions on \(K\) and \(L\), it defines a Lie algebroid structure. From the Frölicher-Nijenhuis decomposition \(D = \mathcal{L}_K + \iota_L\) and the general properties of the Frölicher-Nijenhuis and Richardson-Nijenhuis brackets (see [15, 19, 21]), we get:

\[
D^2 = \frac{1}{2}[D, D] = \frac{1}{2}[\mathcal{L}_K + \iota_L, \mathcal{L}_K + \iota_L]
\]
\[
= \frac{1}{2}\mathcal{L}[K, K]_{FN} + \iota[K, L]_{FN} + \mathcal{L}_L K + \frac{1}{2}[L, L]_{RN},
\]
so \(D^2 = 0\) is equivalent to the conditions

\[
\begin{cases}
\frac{1}{2}[K, K]_{FN} + \iota L K = 0 \\
[K, L]_{FN} + \frac{1}{2}[L, L]_{RN} = 0.
\end{cases}
\]

The first of these is already known, it comes from the imposition that the Lie algebroid bracket be of the form \([X, Y] = [X, Y]_K - L(X, Y)\) and is nothing more than equation (12). A straightforward computation shows that, in fact, it is also equivalent to \(D^2 f = 0\), for any \(f \in C^\infty(M)\). The second condition is much more difficult to satisfy. In the following sections we will construct some solutions out of structures on the tangent bundle of a manifold \(M\) with geometric relevance.
5. Lie algebroids associated to idempotent endomorphisms

Let $TM \to M$ be the tangent bundle over the connected manifold $M$, and let $N : TM \to TM$ be an idempotent endomorphism (that is, such that $N^2 = N$). Then, $N$ has locally constant rank and $\ker N, \text{Im } N$ are vector (sub)bundles such that $TM = \ker N \oplus \text{Im } N$ (see [14]). We will assume that $\text{Im } N$ is an involutive distribution and write down proofs in the real case for simplicity, but the results are valid in the complex case as well.

Lemma 5.1. Under the above assumptions, the following holds:

\begin{equation}
N[NX, NY] = [NX, NY].
\end{equation}

Proof. As $\text{Im } N$ is involutive, there exists a $Z \in \Gamma TM$ such that $[NX, NY] = NZ$, and by applying $N$ to both sides of this equation,

$N[NX, NY] = N^2 Z = NZ = [NX, NY].$

Consider now the Nijenhuis torsion of $N$ (1). In our case, it can be rewritten as

$T_N(X, Y) = [NX, NY] - N[NX, Y] - N[X, NY] + N[X, Y].$

By applying $N$ to both sides, we get

$NT_N(X, Y) = N[NX, NY] - N^2[NX, Y] - N^2[X, NY] + N^2[X, Y],
\end{equation}

and, using lemma 5.1 along with the property $N^2 = N$,

\begin{equation}
NT_N(X, Y) = [NX, NY] - N[NX, Y] - N[X, NY] + N[X, Y] = T_N(X, Y).
\end{equation}

This suggest to define the vector-valued 2–form

\begin{equation}
L := -T_N.
\end{equation}

The pair $(N, L)$ is a solution to equations (15).

Proposition 5.2. Let $N : TM \to TM$ be an idempotent endomorphism with $\text{Im } N$ involutive, and $L \in \Omega^2(M; TM)$ defined by (18). Then $NL = -T_N$, also

\begin{equation}
\begin{cases}
[N, L]_{FN} = 0, \\
[L, L]_{RN} = 0
\end{cases}
\end{equation}

Proof. The first affirmation is a direct consequence of definition (18).

As $L = -\frac{1}{2}[N, N]_{FN}$, and $(\Omega^*(M; TM), [, ]_{FN})$ is a graded Lie algebra (see [19]), we have

$[N, L]_{FN} = -\frac{1}{2}[N, [N, N]_{FN}]_{FN} = 0.$

Moreover, as $L \in \Omega^2(M; TM)$, it follows that $[L, L]_{RN} \in \Omega^3(M; TM)$, $i_L \in \text{Der}^1\Omega(M)$, and $i_{[L, L]_{RN}} \in \text{Der}^2\Omega(M)$; thus, $i_{[L, L]_{RN}}$ is completely characterized by its action on 0–forms (smooth functions) and 1–forms. But $i_{[L, L]_{RN}}$ is a tensorial derivation, so it vanishes on $C^\infty(M)$. Now, given an $\alpha \in \Omega^1(M)$ it is
\[ i_{[L,L]_{RN}} \in \Omega^2(M) \text{ so, whenever } X,Y,Z \in \Gamma TM, \]
\[ (i_{[L,L]_{RN}} \alpha)(X,Y,Z) = \alpha([L,L]_{RN}(X,Y,Z)) \]
\[ = 2i_L^2 \alpha(X,Y,Z) \]
\[ = 2 \odot i_L \alpha(L(X,Y), Z) \]
\[ = 2 \odot \alpha(L(L(X,Y), Z)) \]
\[ = 2 \odot \alpha(T_N(T_N(X,Y), Z)) = 0, \]
(here \( \odot \) denotes cyclic sum in \((X,Y,Z)\) because
\[ T_N(T_N(X,Y), Z) \]
\[ = [NT_N(X,Y), NZ] - N[NT_N(X,Y), Z] - N[T_N(X,Y), NZ] + N^2[T_N(X,Y), Z] \]
\[ = N[NT_N(X,Y), NZ] - N[T_N(X,Y), NZ] - N[T_N(X,Y), NZ] + N[T_N(X,Y), Z] \]
\[ = N[T_N(X,Y), NZ] - N[T_N(X,Y), Z] - N[T_N(X,Y), NZ] + N[T_N(X,Y), Z] = 0, \]
where we have used lemma 5.1, equation (17), and the property \( N^2 = N. \) \( \square \)

Combining (15) with proposition 5.2, we arrive at the following result

**Theorem 5.3.** Let \( N \in \Omega^1(M; TM) \) be such that \( N^2 = N \) and \( \text{Im } N \) is involutive. Then, defining \( L \) as in (18), the derivations given by \( D_1 = \mathcal{L}_N + i_L \), and \( D_2 = i_L \in \text{Der}^1 \Omega(M) \) are cohomology operators, that is, \( D_1^2 = 0 = D_2^2. \)

Note that when \( N \) is a Nijenhuis tensor, automatically \( \text{Im } N \) is involutive, since in this case \([N(X), N(Y)] = N([N(X), Y] + [X, N(Y)] - N([X, Y]))\) for every \( X,Y \in \Gamma TM. \) And if it holds that \( \text{Ker } N \) and \( \text{Im } N \) are involutive, it follows that \( N \) is Nijenhuis tensor.

**Corollary 5.4.** Let \( N \in \Omega^1(M; TM) \) be such that \( N^2 = N \) and \( T_N = 0. \) Then, the derivation given by \( D = \mathcal{L}_{Id - N} \in \text{Der}^1 \Omega(M) \) is a cohomology operator, that is, \( D^2 = 0. \)

**Proof.** First we observe that \((Id - N)^2 = Id - N, \) moreover using the identity \([Id, N]_{FN} = - [N, Id]_{FN}, \) we get
\[ T_{Id - N} = [Id - N, Id - N]_{FN} \]
\[ = [Id, Id]_{FN} - [N, Id]_{FN} - [Id, N]_{FN} + [N, N]_{FN} \]
\[ = 0 \]
Then simply apply theorem 5.3. \( \square \)

Applying formula (11), the Lie algebroid induced by the operator \( D_1 = \mathcal{L}_N + i_L \) of Theorem 5.3 can be explicitly described as follows.

**Theorem 5.5.** Let \( N \in \Omega^1(M; TM) \) be such that \( N^2 = N \) and \( \text{Im } N \) is involutive. Then, there exists a Lie algebroid structure \((TM, q, [,], [\cdot,\cdot])\) with anchor map \( q = N \) and bracket
\[ [X,Y] = [X,Y]_N + T_N(X,Y). \]

We remark that the bracket on \( TM \) constructed this way, is given by the sum of the deformed bracket \([\cdot,\cdot]_N\) and the torsion \( T_N. \) Another useful expression for this bracket is
\[ [X,Y] = [NX, NY] + (Id - N)([NX, Y] + [X, NY]). \]
6. Relation with the foliated exterior differential

Let \( \mathcal{F} \) be a regular foliation on a manifold \( M \) and \( T\mathcal{F} \subset TM \) its tangent bundle. Denote by \( j : T\mathcal{F} \hookrightarrow TM \) the canonical injection. To any such a foliation we can associate a Lie algebroid in two equivalent ways.

(a) The space of tangent sections \( \Gamma(T\mathcal{F}) \) is closed with respect to the Lie bracket of vector fields on \( M \), by the integrability of \( \mathcal{F} \), so it carries a natural (foliated) bracket \([\cdot,\cdot]_\mathcal{F}\) which is just the restriction of the Lie bracket to \( \mathcal{F} \). Then,

\[
(E = T\mathcal{F}, [\cdot,\cdot]_\mathcal{F}, q = j : T\mathcal{F} \hookrightarrow TM),
\]

is a Lie algebroid, called the Lie algebroid of the foliation \( \mathcal{F} \) (see, for example, [10]). In this case the corresponding cohomology operator is the foliated exterior differential \( d_\mathcal{F} : \Gamma(\Lambda^pT^*\mathcal{F}) \rightarrow \Gamma(\Lambda^{p+1}T^*\mathcal{F}) \), which gives rise to the foliated DeRham cohomology.

(b) Let \( H \) be an Ehresmann connection on \( (M, \mathcal{F}) \) complementary to \( T\mathcal{F} \), that is, \( TM = H \oplus \mathcal{F} \). Denote by \( \gamma \) the associated projection on \( \mathcal{F}, \gamma : H \oplus \mathcal{F} \rightarrow \mathcal{F} \), so \( H = \ker \gamma \) and \( \mathcal{F} = \Im \gamma \). It follows that \( \gamma \) is a vector-valued 1-form, and then the curvature of the connection is defined as the Nijenhuis torsion

\[
R = \frac{1}{2}[\gamma, \gamma]_{FN} = T_\gamma.
\]

On the whole \( TM \) we can construct the Lie algebroid structure given by Theorem 5.5. In this case, the explicit expression of the bracket can be easily computed from (20):

\[
\begin{align*}
[X, X']_\gamma &= 0, \text{ for all } X, X' \in H \\
[X, Y]_\gamma &= [X, Y] - \gamma[X, Y], \text{ for all } X \in H, Y \in \Gamma(T\mathcal{F}) \\
[Y, X]_\gamma &= [Y, X] - \gamma[Y, X], \text{ for all } X \in H, Y \in \Gamma(T\mathcal{F}) \\
[Y, Y']_\gamma &= [Y, Y'], \text{ for all } Y, Y' \in \Gamma(T\mathcal{F}).
\end{align*}
\]

The anchor map, as we know, is given by \( q = \gamma \). Obviously, this algebroid structure coincides with the previous one when restricted to \( \Gamma(T\mathcal{F}) \), independently of the chosen connection \( \gamma \).

In what follows, we offer an interpretation of the foliated cohomology within the framework described in item (b). The splitting induced by \( \gamma \), \( TM = H \oplus T\mathcal{F} \), define a bigrading on \( \Omega(M) \). A differential form \( \omega \) on a foliated manifold is said to be of type \( (p, q) \) if it has degree \( p + q \) and \( \omega(X_1, \ldots, X_{p+q}) = 0 \) whenever the arguments contain more than \( q \) vector fields in \( \Gamma(T\mathcal{F}) \), or more than \( p \) in \( H \). According to this bigrading, we have a decomposition of the exterior differential on \( M \),

\[
d = d_{1,0} + d_{-1,1} + d_{0,1},
\]

(see [22]), where \( d_{i,j} : \Omega^{p,q}(M) \rightarrow \Omega^{p+i,q+j} \).

**Remark 4.** Denoting by \( h : \Gamma(\Lambda T^*\mathcal{F}) \rightarrow \Gamma(\Lambda H^0) \) the natural identification, we notice that \( d_{0,1} \) is a \( \gamma \)-dependent extension of the foliated exterior differential, that is,

\[
(d_{0,1} \circ h) \omega = (h \circ d_\mathcal{F}) \omega,
\]

for all \( \omega \in \Gamma(\Lambda T^*\mathcal{F}) \).
The Frölicher-Nijenhuis decomposition of the operators appearing in the decomposition (21) can be readily computed from the results in section 2:

\[ d_{1,0} = \mathcal{L}_{\text{Id} - \gamma} + i_{2R} \]
\[ d_{0,1} = \mathcal{L}_{\gamma} - i_{R} \]
\[ d_{2,-1} = -i_{R}. \]

From (21) and \( d^2 = 0 \), it follows that

\[ d^2_{0,1} = 0 = d^2_{2,-1}. \]

Moreover, the derivation \( d_{1,0} \) is a cohomology operator if and only if the curvature of the connection \( \gamma \) vanishes.

**Theorem 6.1.** The cohomology operator associated to the Lie algebroid on \( \mathcal{T}M \) described in item (b) above, is given by

\[ D = d_{0,1}. \]

Moreover, the Lie algebroid of the foliation \( \mathcal{F} \) and the restriction to \( T\mathcal{F} \) of the Lie algebroid associated to a connection \( \gamma \) on \( \mathcal{T}M \), coincide. Thus, the complexes associated to the cohomology operators \( d_{\mathcal{F}} \) and \( d_{0,1}|_{\Gamma(\Lambda^H)} \) are isomorphic.

**Proof.** The first statement is just a consequence of Theorem 3.1 and example 2. The second one follows from preceding results, along with remark 4. \( \square \)

**Remark 5.** Notice that, once this equivalence has been established, given the Lie algebroid structure \( N : \mathcal{T}M \to \mathcal{T}M \), with \( N \) idempotent, we could define the associated foliated cohomology independently of any regularity condition on \( \mathcal{F} \), just using the decomposition

\[ d = \mathcal{L}_{\text{Id}} = (\mathcal{L}_{\text{Id} - \gamma} + 2i_{R}) + (\mathcal{L}_{\gamma} - i_{R}) - i_{R}. \]

## 7. Complex Lie algebroids associated to complex structures

Recall that an almost-complex structure on a manifold \( M \) is a vector-valued 1-form \( J \in \Omega^1(M; \mathcal{T}M) \) such that \( J^2 = -\text{Id}_{\mathcal{T}M} \). In order to diagonalize the endomorphism \( J \), we work in the complexified tangent bundle \( \mathcal{T}^CM \), and extend by \( \mathbb{C} \)-linearity all real endomorphisms and differential operators on \( \mathcal{T}M \) (with a little abuse of notation, we will denote these extensions by the same symbols as their real counterparts). The Lie bracket of vector fields can also be extended by \( \mathbb{C} \)-linearity.

The almost-complex structure \( J \) is said to be integrable if its Nijenhuis torsion vanishes, that is, if for every \( X, Y \in \mathcal{X}(M) \),

\[ T_J(X,Y) = \frac{1}{2}[J,J]_{FN}(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] = 0. \]

The Newlander-Nirenberg theorem states that this happens if and only if \( M \) has he structure of a complex manifold.

From the condition \( J^2 = -\text{Id}_{\mathcal{T}M} \), we know that \( J \) has the eigenvalues \( \pm i \). If we define the projection operators

\[ p^\pm := \frac{1}{2} (\text{Id} \mp iJ) : \mathcal{T}^CM \to \mathcal{T}^CM, \]
we get the usual properties
\[(p^\pm)^2 = p^\pm
\]
\[p^+ + p^- = \Id_{T^\pm M}
\]
\[p^+ \circ p^- = 0 = p^- \circ p^+,
\]
which determine the subbundle decomposition \(T^\pm M = T^+ M \oplus T^- M\), where
\[\Gamma T^\pm M = \{ Z \in \Gamma T^\pm M : JZ = \pm iZ \}.
\]
The elements of \(\Gamma T^+ M\) are called holomorphic vector fields, and those of \(\Gamma T^- M\) anti-holomorphic. Notice that \(\text{Im}^\pm = T^\pm M\), and \(\ker p^\pm = T^\mp M\), and that if \(Z \in \Gamma T^+ M\), then its complex conjugate \(\bar{Z} \in \Gamma T^- M\) (and vice versa).

We will need the following technical results.

**Lemma 7.1.** Let \(J \in \Omega^1(M; TM)\) be an almost-complex structure on \(M\), and let \(T_J\) be its Nijenhuis torsion. If \(T_J = 0\), then its complex extension also satisfies \(\mathcal{T}_J = 0\).

**Proof.** Let \(Z = X + iY, W = X' + iY'\). A straightforward computation shows that
\[
\mathcal{T}_J(Z, W) = [JZ, JW] - J[JZ, W] - J[Z, JW] - [Z, W]
\]
\[= T_J(X, X') - T_J(Y, Y') + i(T_J(Y, X') + T_J(X, Y'))
\]
\[= 0.
\]
\[\Box
\]

**Proposition 7.2.** An almost-complex structure \(J \in \Omega^1(M; TM)\) is integrable if and only if \([T^+ M, T^+ M] \subset T^+ M\), that is, the distribution defined by the holomorphic vector fields is involutive.

**Proof.** Consider first the case of \(J \in \Omega^1(M; TM)\) integrable. By the preceding lemma, if \(T_J = 0\), then also \(\mathcal{T}_J = 0\). If \(Z, W\) are holomorphic vector fields, we have \(JZ = iZ\) and \(JW = iW\), so
\[
0 = \mathcal{T}_J(Z, W) = [iZ, iW] - J[iZ, W] - J[Z, iW] - [Z, W]
\]
\[= -[Z, W] - iJ[Z, W] - iJ[Z, W] - [Z, W]
\]
\[= -2([Z, W] + iJ[Z, W]),
\]
that is \(J[Z, W] = -\frac{1}{2}[Z, W] = i[Z, W]\), so \([Z, W]\) is holomorphic. Reciprocally, assume that \([Z, W]\) is a holomorphic vector field whenever \(Z, W\) are. Given arbitrary vector fields \(X, Y \in \Gamma TM\), we can rewrite them in the form \(X = Z + \bar{Z}, Y = W + \bar{W}\) for some \(Z, W \in \Gamma T^+ M\). Then, it is readily proved that
\[
\mathcal{T}_J(Z, W) = \mathcal{T}_J(Z, \bar{W}) = 0,
\]
and
\[
T_J(X, Y) = T_J(Z, W) + T_J(\bar{Z}, \bar{W}).
\]
But for holomorphic vector fields we already know that \(T_J(Z, W) = -2([Z, W] + iJ[Z, W])\), and, as \([Z, W]\) is holomorphic by assumption, \(J[Z, W] = i[Z, W]\). Therefore \(T_J(Z, W) = -2([Z, W] + i^2[Z, W]) = 0\). A similar calculation shows that \(T_J(\bar{Z}, \bar{W}) = 0\), and we conclude that \(J\) is integrable.

\[\Box
\]
We are now ready to construct a complex Lie algebroid canonically associated to a complex manifold. Recall (see [24]) that a complex Lie algebroid over the manifold \(M\) is given by a complex vector bundle \(E\) over \(M\) endowed with a complex
Lie algebra structure on its space of sections $\Gamma E$, together with a bundle map $q : E \to T^C M$ (the anchor map) satisfying the Leibniz rule

$$[A, fB] = f[A, B] + (qA)(f)B,$$

for any $A, B \in \Gamma E$, and $f : M \to \mathbb{C}$ smooth. The main example of a complex Lie algebroid is given by the inclusion $q : T^+ M \hookrightarrow T^C M$ of the subbundle of holomorphic vector fields on a complex manifold. Notice that the vector bundle in this case, $E = T^+ M$, is not $T^C M$.

**Remark 6.** We want to make use of the property $(p^+)^2 = p^+$ of the projection operator $p^+ = \frac{1}{2}(\text{Id} - iJ)$, and the results in section 5 to construct an algebroid where $p^+$ will be the anchor map, and the whole $T^C M$ the corresponding bundle. However, as $\text{Im} p^+$ must be involutive, proposition 7.2 forces us to restrict ourselves to complex manifolds.

Our main result is, then, the following.

**Theorem 7.3.** Let $J \in \Omega^1(M; TM)$ be a complex structure on the manifold $M$. There exists a complex Lie algebroid structure on $M$, $(T^C M, q, [~, ~])$, where the anchor map is $q = p^+ = \frac{1}{2}(\text{Id} - iJ)$, and the bracket

$$[Z, W] = [Z, W]_{p^+} = \frac{1}{2}([Z, W] - i[Z, W]_J).$$

**Proof.** First of all, notice that given a complex structure $J$ on $M$, there is a relation

$$\mathcal{T}_{p^+} = -\frac{1}{4} \mathcal{T}_J,$$

which can be proved by a direct computation. According to the results on section 5, the idempotent endomorphism $p^+$ will induce a Lie algebroid $T^C M \to T^C M$ where the anchor map is precisely $q = p^+$, and the bracket $[Z, W] = [Z, W]_{p^+} - \mathcal{T}_{p^+}$. However, the above comment implies that $\mathcal{T}_{p^+} = 0$, so the bracket is simply $[Z, W] = [Z, W]_{p^+}$.

A further straightforward computation shows that

$$[Z, W]_{p^+} = \frac{1}{2}([Z, W] - i[Z, W]_J).$$

□

Notice that this Lie algebroid is different to that obtained by applying proposition 4.2, where the resulting bracket would be $[Z, W] = [Z, W]_J$ (that is, the $\mathbb{C}$–linear extension of $[, ,]_J$ to $T^C M$).

**Remark 7.** The proof extends trivially to the case of an endomorphism $J \in \Omega^1(M; TM)$ such that $J^2 = -\epsilon^2 \text{Id}_{TM}$ with $\epsilon \in \mathbb{R} - \{0\}$.

### 8. Lie Algebroids Associated to Product Structures

Consider now an almost-product structure on the manifold $M$, that is, a vector bundle endomorphism $P : TM \to TM$ such that $P^2 = \text{Id}_{TM}$. It has locally constant rank and it defines two projection operators associated to its eigenvalues $\lambda = \pm 1$, $p^\pm := \frac{1}{2}(\text{Id} \pm P)$. We have a decomposition of the tangent bundle analogous to that of the complex case, $TM = T^+ M \oplus T^- M$, where, this time, $T^\pm M = \text{Imp}^\pm$.

A basic result is that the complementary distributions $T^\pm M$ are integrable if and only if $P$ is integrable, in the sense of having vanishing Nijenhuis torsion (see [9]).
that is, $T^\pm M$ are integrable if and only if $P$ is a product structure. In this case, both $T^\pm M$ are involutive, as we will assume. A simple computation shows that

$$T_p^\pm = \frac{1}{4} T_P,$$

thus, in the case of a product structure we have $T_p^- = 0$.

Of course, given a product structure on $M$ we can construct a Lie algebroid, induced by the operator $D = \mathcal{L}_P$, applying proposition 4.2. The bracket is then the contracted one,

$$[X, Y] = [X, Y]_P.$$

As we know, the Lie algebroid obtained in this way is isomorphic to the trivial one on $TM$ (recall proposition 4.5). The results in section 5 show that we can define another Lie algebroid structure.

**Theorem 8.1.** Let $P \in \Omega^1(M; TM)$ be a product structure on the manifold $M$. There exists a Lie algebroid structure on $M$, $(TM, q, [\ [, ]])$, where the anchor map is $q = p^- = \frac{1}{2} (\text{Id} - P)$, and the bracket

$$[X, Y] = [X, Y]_p^- = \frac{1}{2} ([X, Y] - [X, Y]_P).$$

**Proof.** The statement is actually a corollary to theorem 5.3. The last equality is just a trivial computation. □

**Remark 8.** The theorem applies also to the case of an operator $P \in \Omega^1(M; TM)$ such that $P^2 = \epsilon^2 \text{Id}_{TM}$, with $\epsilon \in \mathbb{R} - \{0\}$.

9. **The Lie Algebroid Associated to a Tangent Structure and a Connection**

Let us recall some facts about the geometry of the tangent bundle (see [13]). Let $\pi : TM \rightarrow M$ be the tangent bundle of a manifold $M$, and let $\tau : TT M \rightarrow TM$ be the second tangent bundle. Then, we have a short exact sequence of vector bundles

$$0 \rightarrow TM \times_M TM \xrightarrow{j} TT M \xrightarrow{j} TM \times_M TM \rightarrow 0,$$

where

$$\iota(u, w) = \left. \frac{d}{dt} (u + tw) \right|_{t=0}$$

is the natural injection, and $j = (\tau, \pi)$.

The vertical sub-bundle $\mathcal{V}TM = \ker \pi_*$ can be expressed in either form

$$\text{Im} \iota = \mathcal{V}TM = \ker j,$$

and the vector-valued form $J \in \Omega^1(TM; TT M)$ defined by

$$J = \iota \circ j,$$

is called the vertical endomorphism. It is immediate to prove the following properties:

$$J^2 = 0$$

$$\ker J = \mathcal{V}TM = \text{Im} J$$

$$T_J = \frac{1}{2} [J, J]_{FN} = 0.$$
Thus, $J$ defines a (canonical) tangent structure on $TM$. It has locally constant rank equal to $\dim M$. We also have the (canonical) Liouville vector field on $TM$, $C \in \mathcal{X}(TM)$, defined as

$$C = \iota \circ \Delta,$$

where $\Delta : TM \to TM \times_M TM$ is the diagonal $\Delta(v) = (v, v)$. The Liouville vector field has the property

$$\mathcal{L}_C J = -J.$$

A (nonlinear) connection $\Gamma$ on $TM$ is a vector-valued form $\Gamma \in \Omega^1(TM; TT M)$ such that

$$J \circ \Gamma = -\Gamma \circ J.$$

A basic property of a connection is that it defines an almost product structure on $TM$, that is,

$$\Gamma^2 = \text{Id}_{TT M},$$

in such a way that the eigenbundle corresponding to the eigenvalue $\lambda = -1$ is precisely the vertical distribution. In other words, if we define the projectors

$$h = \frac{1}{2}(\text{Id}_{TT M} + \Gamma),$$

$$v = \frac{1}{2}(\text{Id}_{TT M} - \Gamma),$$

we have $\mathcal{V}TM = \text{Im} v$, and $TT M = \text{Im} v \oplus \text{Im} h$.

Notice that the vertical distribution is integrable (Im $v$ is involutive), while, in general, the horizontal distribution is not (in fact, a necessary and sufficient condition for this is the vanishing of the curvature of the connection). Thus, we expect to be able to apply the results of section 5 in order to construct a Lie algebroid structure where the vector bundle is $E = TT M$, and the anchor map $v$.

**Theorem 9.1.** Let $M$ be a manifold, and $J$ the canonical tangent structure on $TM$. Let $\Gamma \in \Omega^1(TM; TT M)$ be a connection on $TM$. Then, there exists a Lie algebroid structure on $TM$, $(TT M, q, \{\cdot, \cdot\})$, where the anchor map is $q = v = \frac{1}{2}(\text{Id}_{TT M} - \Gamma)$ and the bracket

$$[A, B] = \frac{1}{2}(\{A, B\} - [A, B]_\Gamma) + \frac{1}{4}T_\Gamma(A, B).$$

**Proof.** This is just a corollary to theorem 5.3. Simply notice that, as a quick calculation shows,

$$T_\nu(A, B) = \frac{1}{4}T_\Gamma(A, B),$$

for all $A, B \in \mathcal{X}(TM)$, and

$$[A, B]_\nu = [vA, B] + [A, vB] - v[A, B]$$

$$= \frac{1}{2}([A - \Gamma A, B] + [A, B - \Gamma B] - [A, B] + \Gamma[A, B])$$

$$= \frac{1}{2}([A, B] - \Gamma A, B] - [A, \Gamma B] + \Gamma[A, B])$$

$$= \frac{1}{2}([A, B] - [A, B]_\Gamma).$$

$\square$
The easiest method for obtaining connections on $TM$ is to choose a semispray. A semispray (or a second-order differential equation, see [1]) on $M$ is a vector field on $TM$, $S \in \mathcal{X}(TM)$, such that it is also a section of the second tangent bundle $T(TM) \to TM$. Vector fields $S \in \mathcal{X}(TM)$ which are semisprays can be characterized with the aid of the canonical structures on $TM$ as those satisfying

$$J \circ S = C.$$  

It is well known (see [1, 9]) that for any semispray $S$, the endomorphism $\Gamma = -\mathcal{L}_S J$ is a connection on $TM$, with associated projectors

$$h = \frac{1}{2}(\text{Id}_{TM} - \mathcal{L}_S J),$$

$$v = \frac{1}{2}(\text{Id}_{TM} + \mathcal{L}_S J).$$

**Remark 9.** This intimate relation between semisprays and connections has been used in Physics to study such topics as the inverse problem for Lagrangian dynamics [7], degenerate Lagrangian systems [3], or the geometry and mechanics of higher order tangent bundles [8].

We have the following result.

**Theorem 9.2.** Let $M$ be a manifold, and $J$ the canonical tangent structure on $TM$. Let $S \in \mathcal{X}(TM)$ be a semispray on $M$. Then, there exists a Lie algebroid structure on $T(TM)$, $(T(TM), q, \{\cdot, \cdot\})$, where the anchor map is $q = v = \frac{1}{2}(\text{Id}_{TTM} + \mathcal{L}_S J)$ and the bracket

$$[A, B] = \frac{1}{2} ([A, B] - \bigcirc_{A, B, S} [A, \mathcal{L}_S B]_J) + \frac{1}{4} T_{\mathcal{L}_S J}(A, B).$$

**Proof.** Rewrite (22) taking into account (23) so, for any $X \in \mathcal{X}(TM)$,

$$\mathcal{L}_S J)X = [S, JX] - J[S, X]$$

$$= [S, X]_J - [JS, X]$$

$$= [S, X]_J - [C, X].$$

That is, $\Gamma X = [C, X] - [S, X]_J$. A long but otherwise straightforward computation leads to (24).

**Acknowledgments**

The authors express their gratitude to J. V. Beltrán and J. Monterde (both at the Universitat de València, Spain), for many useful discussions.

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