Efficient unitary designs with nearly time-independent Hamiltonian dynamics

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We provide new constructions of unitary $t$-designs for general $t$ on one qudit and $N$ qubits, and propose a design Hamiltonian, a random Hamiltonian of which dynamics always forms a unitary design after a threshold time, as a basic framework to investigate randomising time evolution in quantum many-body systems. The new constructions are based on recently proposed schemes of repeating random unitaries diagonal in mutually unbiased bases. We first show that, if a pair of the bases satisfies a certain condition, the process on one qudit approximately forms a unitary $t$-design after $O(t)$ repetitions. We then construct quantum circuits on $N$ qubits that achieve unitary $t$-designs for $t = o(N^{1/2})$ using $O(t N^2)$ gates, improving the previous result using $O(t^{10} N^2)$ gates in terms of $t$. Based on these results, we present a design Hamiltonian with periodically changing two-local spin-glass-type interactions, leading to fast and relatively natural realisations of unitary designs in complex many-body systems.

I. INTRODUCTION

Random quantum processes play important roles in quantum information processing, as one of the fundamental primitives in quantum Shannon theory [1–8] and as a useful resource to demonstrate quantum supremacy in many protocols [9–17]. In recent years, random processes have been also revealed to be the key to understanding fundamental physics in complex quantum systems, leading to new developments in quantum thermodynamics [18–20] (see Ref. [21] for a comprehensive review), the black hole information science [22–27] and strongly correlated many-body physics [28–30]. In quantum systems, random processes are often represented by random unitaries drawn uniformly at random according to the Haar measure, referred to as Haar random unitaries. However, when a system consists of a large number of particles, it is highly inefficient to implement Haar random unitaries, implying that they rarely appear in natural systems composed of many particles especially when the interactions are local. This fact has lead to the research area on finite-degree approximations of Haar random unitaries, so-called unitary designs [31–33], and their efficient implementations [34–44]. A unitary $t$-design is called exact when it simulates all the first $t$ moments of Haar random unitaries and approximate when the simulations are with errors.

Traditionally, unitary $t$-designs have been studied for small $t$. In particular, unitary 2-designs were intensely studied [31, 38, 41, 42] due to the facts that they are useful in important tasks, such as decoupling [3, 8] and randomised benchmarking [6, 12], and that the Clifford group is an exact unitary 2-design [31]. Later, the Clifford group on qubits was also shown to be a unitary 3-design but not to be a 4-design [43, 44]. For $t \geq 4$, a few applications are known (e.g. state discrimination [13], quantum speed-ups in query complexity [14] and compressed sensing [15, 16]), but they are of potential importance when strong large deviation bounds are needed. So far, only a couple of efficient implementations for $t \geq 4$ are known to the best of our knowledge. One is to use a classical tensor product expander and the Fourier transformation, forming approximate unitary $t$-designs for $t = O(N/\log N)$ by using poly($N$) quantum gates [39]. The other is to use local random quantum circuits composed of random two-qubit gates applied onto neighbouring qubits, which achieves approximate unitary $t$-designs for $t = \text{poly}(N)$ using $O(t^{10} N^2)$ gates [40].

Despite these implementations of unitary designs by quantum circuits, there exists a certain
gap between the constructions and physically feasible dynamics in quantum many-body systems. The constructions require a finely structured circuit \[39\] or the use of randomly varying interactions \[40\], while dynamics in physically feasible many-body systems is typically not structured and is generated by a Hamiltonian, which may slightly fluctuate over time but should be based on time-independent one. Indeed, if we interpret local random quantum circuits on \(N\) qubits in terms of Hamiltonian dynamics, the interactions should be changed uniformly at random \(O(t^{10}N)\) times before the dynamics achieves unitary \(t\)-designs. Due to its dependence on the number of particles, the total Hamiltonian should be highly time-dependent and may not be so physically feasible in large systems, resulting in a lack of solid basis of a number of studies of fundamental phenomena in many-body systems based on random dynamics \[18–27\]. There is also an increasing demand from the black hole information science and quantum chaos to fully understand microscopic dynamics of randomisation, where so-called scrambling has been intensely studied \[22–30\]. As scrambling is a weak variant of unitary designs, studying natural Hamiltonians generating unitary designs will bring better understandings in the context. Further, implementations of unitary designs by Hamiltonian dynamics are of practical importance, helping experimental realisations of designs, as any quantum circuit is fundamentally implemented by engineering Hamiltonians.

In this paper, we provide new constructions of unitary \(t\)-designs and propose a design Hamiltonian, a random Hamiltonian of which dynamics forms a unitary design at any time after a threshold time. The constructions are based on the scheme of repeating random unitaries diagonal in mutually unbiased bases \[42, 45–47\]. We first show that the process on one qudit achieves unitary \(t\)-designs after \(O(t)\) repetitions if a pair of the two bases satisfies a certain condition, which is met by a pair of any basis and its Fourier basis and that of the Pauli-\(X\) and -\(Z\) bases. As the construction works for any space, it will be useful to implement unitary designs in a subspace, such as a symmetric subspace, which is known to demonstrate quantum supremacy in metrology \[17\]. We then focus on random diagonal unitaries in the Pauli-\(X\) and -\(Z\) bases on \(N\) qubits and investigate how to approximate them efficiently by quantum circuits. By mapping this problem to a combinatorial problem called a local permutation check problem, which can be further reduced to a special type of constrained problems in extremal algebraic theory \[48, 49\], we prove that an approximate unitary \(t\)-design for \(t = o(N^{1/2})\) can be achieved by using \(O(tN^2)\) gates. In terms of \(t\), this drastically improves the previous result \[40\], which uses \(O(t^{10}N^2)\) gates, and is essentially optimal. As higher-designs are useful to improve the performance of any applications of lower-designs due to their large deviation bounds \[50\], this construction will contribute to improving the performance of any applications of designs \[11, 17\]. Finally, we introduce design Hamiltonians and present a nearly time-independent one with spin-glass-type interactions, where the interactions should vary only \(O(t)\) times to achieve unitary \(t\)-designs. As a simple consequence, the design Hamiltonians quickly saturate the so-called out-of-time-ordered correlators \[28–30\] to the Haar averaged values, suggesting a close relation between the design Hamiltonians and quantum chaos. We also propose a conjecture about the timescale for a natural design Hamiltonian to generate unitary designs, which can be seen as a generalisation of the fast scrambling conjecture \[23\].

The paper is organised as follows. In Section II, we introduce necessary notation and explain definitions of several random unitaries. All the main results are summarised in Section III, of which proofs are provided in Section IV. We conclude and discuss possible future directions in Section V. Small lemmas and propositions presented in the paper are proven in Appendices.

II. PRELIMINARIES

We use the following standard asymptotic notation. Let \(f(n)\) and \(g(n)\) be functions on \(\mathbb{R}^+\). We say \(f(n) = O(g(n))\) if there exist \(c, n_0 > 0\) such that \(f(n) \leq cg(n)\) for all \(n \geq n_0\). When there exist
respectively. For a superoperator \( C \) (and the difference between a quantum TPE and a unitary \( \eta, t \)), we denote by \( [i, j] \) a sequence of numbers from \( i \) to \( j \), \( [i, i + 1, \ldots, j - 1, j] \).

Let \( \mathcal{H} \) be a Hilbert space and \( \mathcal{B}(\mathcal{H}) \) be a set of bounded operators on \( \mathcal{H} \). We use several norms of operators and superoperators. For operators, we use the operator norm \( \| \cdot \|_\infty \) and the \( p \)-norm \( (p \geq 1) \) defined by \( |X|_\infty := \max_i x_i \), where \( \{x_i\} \) are the singular values of \( X \), and \( |X|_p := (\text{tr}|X|^p)^{1/p} \), respectively. For a superoperator \( C : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \), we use a family of superoperator norms \( \|C\|_{q \to p} \) \((q, p \geq 1)\) and the diamond norm \( \|C\|_\diamond \) defined by

\[
\|C\|_{q \to p} = \sup_{X \neq 0} \frac{|C(X)|_p}{|X|_q}, \quad |C|_\diamond := \sup_k \|C \otimes \text{id}_k\|_{1 \to 1},
\]

respectively, where \( \text{id}_k \) is the identity map acting on a Hilbert space of dimension \( k \).

The following are the definitions of Haar random unitaries, random diagonal-unitaries, and unitary t-designs.

**Definition 1 (Haar random unitaries)** Let \( \mathcal{U}(d) \) be a unitary group of degree \( d \), and \( \mathcal{H} \) be the Haar measure (i.e. the unique unitarily invariant probability measure) on \( \mathcal{U}(d) \). A Haar random unitary \( U^H \) is a \( \mathcal{U}(d) \)-valued random variable distributed according to the Haar measure, \( U^H \sim \mathcal{H} \).

**Definition 2 (Random diagonal-unitaries \([45]\))** Let \( E = \{|k\}\}_{k \in [0, d - 1]} \) be an orthogonal basis in a Hilbert space \( \mathcal{E} \), with dimension \( d \). Let \( \mathcal{D}_E(d) \) be the set of \( d \times d \) unitaries diagonal in the basis \( E \). Let \( \mathcal{D}_E \) denote a probability measure on \( \mathcal{D}_E(d) \) induced by a uniform probability measure on the parameter space \([0, 2\pi)^d\). A random diagonal-unitary in the basis of \( E \), \( \mathcal{D}_E \), is a \( \mathcal{D}_E(d) \)-valued random variable distributed according to \( \mathcal{D}_E \), \( \mathcal{D}_E \sim \mathcal{D}_E \).

To define a unitary t-design \((t \in \mathbb{Z}^+)\), let \( G_{U \sim \nu}^{(t)}(X) \) be a superoperator given by \( G_{U \sim \nu}^{(t)}(X) := \mathbb{E}_{U \sim \nu}[U^{\otimes t} X U^{\dagger \otimes t}] \) for any \( X \in \mathcal{B}(\mathcal{H}^{\otimes t}) \), where \( \mathbb{E}_{U \sim \nu} \) represents an average over a random unitary \( U \) according to a probability measure \( \nu \). An \( \epsilon \)-approximate unitary t-design is then defined as follows.

**Definition 3 (An \( \epsilon \)-approximate unitary t-design \([32, 37]\))** A random unitary \( U \) is called an \( \epsilon \)-approximate unitary t-design if \( \|G_{U \sim \nu}^{(t)} - G_{U \sim \mathcal{H}}^{(t)}\|_\diamond \leq \epsilon \).

The following is a trivial but useful lemma about unitary designs.

**Lemma 4** If \( U \) is an \( \epsilon \)-approximate unitary t-design, then for any random unitary \( V \) independent of \( U \), \( UV \) and \( VU \) are also \( \epsilon \)-approximate unitary t-designs.

The proof is straightforward and is given in Appendix A. We also use the quantum tensor product expander introduced in Ref. [52].

**Definition 5 (Quantum tensor product expander (quantum TPE) \([52]\))** Let \( \nu \) be a probability measure on \( \mathcal{U}(d) \). Then \( \nu \) is a quantum \((\eta, t)\)-TPE if

\[
\|\mathbb{E}_{U \sim \nu}[U^{\otimes t}] - \mathbb{E}_{U \sim \mathcal{H}}[U^{\otimes t}]\|_\infty \leq \eta,
\]

where \( \eta < 1 \), \( U^{\otimes t} := U^{\otimes t} \otimes U^{\dagger \otimes t} \), and \( U^* \) is a complex conjugation of \( U \).

Note that this definition is equivalent to

\[
\|G_{U \sim \nu}^{(t)} - G_{U \sim \mathcal{H}}^{(t)}\|_{2 \to 2} \leq \eta,
\]

and the difference between a quantum TPE and a unitary t-design is just the norm used in their definitions. The fact that iterating quantum \((\eta, t)\)-TPE yields an approximate unitary t-design is often used in the literature \([39, 40]\), which is formally stated in the following theorem (a proof is given in Appendix B for completeness).

**Theorem 6** Let \( \nu \) be a quantum \((\eta, t)\)-TPE. Then iterating the TPE \( \ell \geq \frac{1}{\log 1/\eta} \log \frac{d^t}{\epsilon} \) times results in an \( \epsilon \)-approximate unitary t-design.
III. MAIN RESULTS

Here, we present a summary of our three main results. We first provide implementations of approximate unitary designs on one qudit in Subsection III A. In Subsection III B, we consider $N$-qubit systems and show that $\epsilon$-approximate unitary $t$-designs can be implemented by quantum circuits with length $O(N(tN + \log 1/\epsilon))$. Finally, in Subsection III C, we propose design Hamiltonians and provide a design Hamiltonian with two-body interactions that achieves unitary designs in a short time.

A. One qudit case

Let us introduce a Fourier-type pair of bases, which is important in our result.

Definition 7 A pair of orthogonal bases $(E, F)$ in a $d$-dimensional Hilbert space is called a Fourier-type pair of bases if each element in $F = \{|\alpha\rangle_F\}_{\alpha \in [0, d-1]}$ is expanded in the basis of $E = \{|k\rangle_E\}_{k \in [0, d-1]}$ as follows:

$$|\alpha\rangle_F = \frac{1}{\sqrt{d}} \sum_{k \in [0, d-1]} e^{i\theta_{k\alpha}} |k\rangle_E,$$

where the phases $\theta_{k\alpha} \in [0, 2\pi)$ satisfy the condition that $\forall k, l, \alpha \in [0, d-1], \theta_{k+l,\alpha} = \theta_{k\alpha} + \theta_{l\alpha}$. In the index of $\theta$, $+$ should be an additive operation with respect to which $[0, d-1]$ is an additive group.

The following are two important examples of Fourier-type pairs of bases (see Appendix C for the proof).

Lemma 8 The following pairs of bases are Fourier-type;

1. any orthogonal basis $\{|k\rangle\}_{k \in [0, d-1]}$ and its Fourier basis $\{d^{-1/2} \sum_k \omega^{\alpha k} |k\rangle\}_{\alpha \in [0, d-1]}$, where $\omega$ is a $d$th root of unity.
2. the Pauli-$X$ and Pauli-$Z$ bases on $N$ qubits.

Our first main result is that, for Fourier-type pairs $(E, F)$ of bases, a product of a random diagonal-unitary in the basis of $E$ and that in the basis of $F$ is a quantum TPE.

Theorem 9 (Main Result 1) Let $d = \Omega(t^2 t!)$ and $(E, F)$ be a Fourier-type pair of bases. For independent random diagonal-unitaries $D^E$ and $D^F$ in the basis of $E$ and $D^F$ in the basis of $F$, $D^E D^F D^E$ is a quantum $(\eta, t)$-TPE with $\eta$ given by

$$\eta = \frac{(1 + t^2) t!^2 + t^2}{d} + O\left(\frac{t^4 t!^2}{d^2}\right).$$

A proof is given in Sec. IV B. From Theorems 6 and 9 and noticing that applying two random diagonal-unitaries in the same basis is equivalent to applying one random diagonal-unitary in that basis, we immediately obtain the following corollary.
Corollary 10 Let \((E, F)\) be a Fourier-type pair of bases and assume that \(d = \Omega(t^2t^2)\). A random unitary \(D[\ell] := (\prod_{i=1}^\ell D_i^E D_i^F) D_0^E\), where \(D_i^E\) and \(D_i^F\) are independent random diagonal unitaries in the basis of \(E\) and \(F\), respectively, is an \(\epsilon\)-approximate unitary \(t\)-design if

\[
\ell \geq \frac{1}{\log d - 2 \log t} \left( t \log d + \log 1/\epsilon \right),
\]

up to the leading order of \(d\) and \(t\).

This construction of designs works for any space, which is not necessarily a whole tensor-product space, and will be useful when we need designs in certain subspaces. This is the case for instance in quantum metrology, where it was recently shown that almost any random symmetric states are useful to demonstrate quantum supremacy [17]. As unitary designs in the symmetric subpace are needed for generating such random states, our construction will help the demonstration of quantum supremacy in metrology. Another interesting instance is an experimental simulation of thermalisation phenomena in quantum many-body systems, which occurs when Haar random unitaries or unitary designs are applied onto the system and the environmental system [18–20]. The temperature of the system is determined by the total energy in the system and the environment, so unitary designs should act on the subspace with restricted energy. Our construction is suited in this situation because a pair of position and momentum bases of pseudo-particles with fixed energies forms a Fourier-type pair of bases and may be physically feasible to deal with. Random diagonal-unitaries also have a clear physical interpretation as they are considered to be idealised dynamics by random time-independent Hamiltonians.

Our result should be also compared with the result in Ref. [37], where an implementation of approximate unitary \(t\)-designs was given based on the iterations of classical tensor product expanders and the Fourier transformation. The number of iterations in the implementation is approximately \(t \log d\). As our result requires approximately only \(t\) iterations when \(t \ll d\), our construction may seem more efficient. This is however simply a consequence of the fact that random diagonal-unitaries use more randomness than the classical tensor product expander.

B. \(N\) qubits case

We now focus on \(N\)-qubit systems. In particular, we consider applying random diagonal-unitaries in the Pauli-\(X\) and -\(Z\) bases. From Corollary 10 repeating these random diagonal-unitaries yields an \(\epsilon\)-approximate unitary \(t\)-design if the number \(\ell\) of repetitions satisfies

\[
\ell \geq \frac{1}{N - 2 \log_2(t)} \left( tN + \log_2 1/\epsilon \right),
\]

as long as \(t! \approx O(2^{N/2})\). However, this construction is inefficient because an exact implementation of random diagonal-unitaries by quantum circuits requires an exponential number of local gates. Thus, we need to find efficient implementations of “approximate” random diagonal-unitaries by quantum circuits. As the Pauli-\(X\) and -\(Z\) bases are related by the Hadamard transformation, it suffices to consider those only in the Pauli-\(Z\) basis.

We especially study the following family of random diagonal circuits. Let \(I = \{I_i\}\) be a set of \(I_i \subset [1, N]\). At the \(i\)th step of the circuit, we apply a random diagonal gate \(\text{diag}_Z\{e^{i\varphi_0}, \ldots, e^{i\varphi_{|I_i|} - 1}\}\) onto the qubits located in \(I_i\), where the gate is diagonal in the Pauli-\(Z\) basis and the phases \(\varphi_k\) \((k \in [0, 2^{|I_i|} - 1])\) are chosen independently and uniformly at random from \([0, 2\pi]\) every step. We refer to \(|I|\) as the length of the circuit. As the circuit is fully specified by \(I\), we denote it by \(RDC(I)\).
The problem of approximating random diagonal-unitaries in the Pauli-Z basis by $RDC(\mathcal{I})$ is related to an elementary combinatorial problem, which may be of interest in its own right. We first introduce the combinatorial problem here, and then show the connection to the original problem.

Let $K$ and $K'$ be $t \times N$ matrices with elements in $\{0, 1\}$. For given $s \in [1, t]$ and $I \subset [1, N]$, we denote a subsequence $(K_{s,m})_{m \in I}$ of the $s$th row of $K$ by $K_{s,I}$, and a set $\{K_{s,I}\}_{s \in [1, t]}$ of such subsequences over all $s$ by $K_I$. We use the same notation also for $K'$. Let $\Omega$ be a canonical map that rearranges the subsequences $K_I$ in ascending order, where the subsequences are regarded as binary numbers. For $\mathcal{I} = \{I\}$, we say that $K$ is an $\mathcal{I}$-local permutation of $K'$ if $\forall I \in \mathcal{I}$, $\Omega(K_I) = \Omega(K'_I)$. In particular, we say $K$ is a row permutation of $K'$ if $\Omega(K_I) = \Omega(K'_I)$ for $I = [1, N]$, which simply implies that a set of rows of $K$ is a permutation of that of $K'$. In the following, we denote by $\mathcal{I}_r$ a set of all subsets in $[1, N]$ with $r$ elements. Using this notation, we define local permutation check problems as follows.

**Definition 11 (Local permutation check problems)** Let $K$ and $K'$ be $t \times N$ matrices with elements in $\{0, 1\}$. For a given $\mathcal{I} = \{I_i\}$ ($I_i \subset [1, N]$), the task of the $\mathcal{I}$-local permutation check problem is to count the number of pairs $(K, K')$ such that $K$ is not a row permutation but an $\mathcal{I}$-local permutation of $K'$. We denote the number of such pairs by $\Lambda(\mathcal{I})$. In particular, for $\mathcal{I}_r$, we call the problem an $r$-local permutation check problem and denote the number of pairs by $\Lambda_r$.

The following lemma provides the connection between the $\mathcal{I}$-local permutation check problem and implementations of quantum TPEs by $RDC(\mathcal{I})$ (see Appendix 4 for the proof).

**Lemma 12** Let $2^N = \Omega(2^t t!)$. For a given $\mathcal{I} = \{I_i\}$ where $I_i \subset [1, N]$, iterating $RDC(\mathcal{I})$ and the Hadamard transformation $H_N$ on $N$ qubits, such as $RDC(\mathcal{I})H_NRDC(\mathcal{I})H_NRDC(\mathcal{I})$ (see Fig. 4 as well), yields a quantum $(\tilde{\eta}, t)$-TPE where

$$\tilde{\eta} \leq \eta + 3t^2 \Lambda(\mathcal{I}) \frac{\Lambda(\mathcal{I})}{2^t N} + \left( \frac{\Lambda(\mathcal{I})}{2^t N} \right)^2,$$

where

$$\eta = \frac{(1+t^2)t^2+t^2}{2^N} + O(t^4 t^2 \frac{t^2}{2^N}).$$

To obtain our second main result, $RDC(\mathcal{I}_2)$ and the 2-local permutation check problem are of particular importance. Due to the result in Ref. [53], we know that $\Lambda_2 = 0$ for $t \leq 3$. When $t \geq 4$, the problem can be rephrased as an extremal problem under dimension constraints, which is a constrained problem in extremal algebraic theory [48, 49]. By solving the problem, we obtain the following key lemma (see Sec. 12 for the proof).

**Lemma 13** For the 2-local permutation check problem, it holds that $\Lambda_2 \leq 2^{2t^2+(t-1)N}$.
It immediately follows from Lemmas 12 and 13 that iterating $RDC(\mathcal{I}_2)$ and the Hadamard transformation is a quantum $(\tilde{\eta}, t)$-TPE, where

$$\tilde{\eta} \leq 2^{2t^2+2-N}t! + O(t^2t!^22^{-N}),$$

from which we obtain an efficient implementation of a unitary $t$-design due to Theorem 6. Furthermore, we can use the same technique as in Ref. 52 of reducing the number of random bits necessary for the implementation. To simulate up to the $t$th order moments of $RDC(\mathcal{I}_2)$, each two-qubit gate in $RDC(\mathcal{I}_2)$ can be replaced with

$$(\text{diag}\{1, e^{i\varphi_1}\} \otimes \text{diag}\{1, e^{i\varphi_2}\})\text{diag}\{1, 1, 1, e^{i\vartheta}\},$$

where $\varphi_1$ and $\varphi_2$ are chosen independently from a set $\{2\pi m/(t+1) : m \in [0, t]\}$ uniformly at random, and $\vartheta$ is chosen from a $\{2\pi m/(\lfloor t/2 \rfloor +1) : m \in [0, \lfloor t/2 \rfloor]\}$. When those discrete gates are used instead of continuous gates in $RDC(\mathcal{I}_2)$, we denote the circuit by $RDC_{\text{disc}}^{(t)}(\mathcal{I}_2)$, where one two-qubit gate requires $2\log(t+1) + \log(\lfloor t/2 \rfloor + 1) < 3\log(t+1)$ random bits. Together with all of these results, we obtain our second main result.

**Theorem 14 (Main Result 2)** Let $t = o(N^{1/2})$. Then, iterating $RDC_{\text{disc}}^{(t)}(\mathcal{I}_2)$ and the Hadamard transformation on $N$ qubits such as $(RDC_{\text{disc}}^{(t)}(\mathcal{I}_2)H_N)^{2\ell}RDC_{\text{disc}}^{(t)}(\mathcal{I}_2)$ yields an $\epsilon$-approximate unitary $t$-design if

$$\ell \geq t + \frac{1}{N} \log_2 1/\epsilon,$$

up to the leading order of $d$ and $t$. The total number of two-qubit gates and random bits are given by

$$\text{# of two-qubit gates} = O(N(tN + \log_2 1/\epsilon)),$$

$$\text{# of random bits} = O((\log_2 t)N(tN + \log_2 1/\epsilon)),$$

respectively.

In terms of $t$, Theorem 14 drastically improves the previous result using $O(t^9N(tN + \log_2 1/\epsilon))$ two-qubit gates [40] and is essentially optimal when the design is defined on a finite set of unitaries. This is because the support of a unitary $t$-design should contain at least $O(2^{2N})$ unitaries [51]. Thus, when each gate in a random quantum circuit is chosen from a finite set, the scaling of the length necessary for the circuit achieving a $t$-design cannot be substantially better than linear in $t$. In practical uses of unitary designs such as decoupling [5-8] and randomised benchmarking [9-12], unitary 2-designs are known to be sufficient, which can be achieved more efficiently than our construction if one uses a Clifford circuit [41]. However, unitary 4-designs are needed in a few applications [12, 16], which cannot be achieved by any Clifford circuit [43]. Moreover, higher-designs are generally more useful than lower-designs because they have stronger large deviation bounds [50], which are finite approximations of the concentration of measure for Haar random unitaries stating that values of any slowly varying function on a unitary group are likely to be almost constant if the dimension is large [55]. This implies that using higher-designs in any applications of unitary designs results in better performance. As our implementation provides a shorter quantum circuit for $t$-designs than the existing ones [37, 40], where the scaling in terms of $t$ is likely to be optimal, it contributes to improve the performance of quantum protocols using unitary designs [1, 17].

This construction of approximate designs also has advantages from the viewpoint of experimental realisations. As emphasized in Ref. 12, 17, the quantum circuits repeating $RDC(\mathcal{I}_2)$ or
RDC$^{(t)}_{\text{disc}}(I_2)$ and the Hadamard transformation are divided into a constant number of commuting parts. Indeed, only non-commuting parts are the Hadamard parts. Because the gates in each commuting part do not have any temporal order, they can be applied simultaneously in experimental realisations, making the implementations more robust. Hence, the commuting structure of our construction may help reducing the practical time and increasing the robustness of the implementations.

C. Hamiltonian dynamics and unitary designs

In the last decade, unitary designs were revealed to be the key to understanding typical phenomena in complex quantum many-body systems [18–27], in most of which the dynamics is assumed to be so random that it can be described by unitary designs. This assumption may be reasonable as a first approximation. However, due to the lack of full understanding of natural microscopic dynamics generating unitary designs, it is not clear to what extent the assumption can be justified. Most recently, the idea of scrambling was introduced in the black hole information science [22, 23]. The main concern there is the fast scrambling conjecture, stating that the shortest time necessary for natural dynamics to scramble many-body systems scales logarithmically with the system size [23–27]. The conjecture arises from the thought experiment concerning the black hole evaporation and the no-cloning theorem [23], but has been also studied intensely in connection with quantum chaos [28–30]. So far, several inequivalent definitions of scrambling were proposed [23, 25, 27]. Although they are useful for clarifying the relations between scrambling and other notions of randomisation such as unitary designs and the out-of-time-ordered (OTO) correlators diagnosing quantum chaos [28–30], there does not seem to be consensus on a rigorous mathematical definition of scrambling.

Here, we introduce design Hamiltonians as a unifying framework for studying randomising operations by physically natural Hamiltonian dynamics. In terms of the design Hamiltonians, we generalise the fast scrambling conjecture and propose a natural design Hamiltonian conjecture. We then construct a design Hamiltonian, where the interactions are needed to be changed only a few times to achieve unitary designs. This is in a sharp contrast to the Hamiltonian dynamics based on local random quantum circuits [40], which will be elaborated later.

We start with the definition of $k$-local Hamiltonians.

**Definition 15 (k-local Hamiltonians [51])** Let $\Lambda_j \subset [1, N]$ such that $|\Lambda_j| \leq k$ and $\Lambda_i \neq \Lambda_j$ if $i \neq j$. A $k$-local Hamiltonian $H$ on $N$ qubits is one in the form of $H = \sum_i H_i$, where each term $H_i$ may depend on time, acts non-trivially only on the qubits in $\Lambda_i$ and satisfies $\|H_i\|_\infty \leq 1$. We denote a set of all $k$-local Hamiltonians by $\mathcal{H}_k$.

The interactions of $k$-local Hamiltonians are not necessarily geometrically local on lattice systems. They are rather interpreted as interactions on a given graph, where each vertex represents a particle. To normalise the time scale of the dynamics, we also assume that the strength of each local interaction is bounded. In the following, to avoid confusion, we always use a small $t$ and capital $T$ for $t$-designs and time, respectively. Denoting by $U_H(T) := T \exp[-i \int_0^T ds H(s)]$, where $T \exp$ is the time-ordered exponential, the time evolution operator at time $T$ generated by a possibly time-dependent Hamiltonian $H$, we now introduce a $t$-design Hamiltonian with $k$-local interactions as follows.

**Definition 16 (An $\epsilon$-approximate $t$-design Hamiltonian with $k$-local interactions)** Let $\mathcal{H}_k^{(t)} \subset \mathcal{H}_k$ and $\text{Ham}_k^{(t)}$ be a probability measure on $\mathcal{H}_k^{(t)}$. If there exists $T_0 > 0$ such that, $\forall T \geq T_0$, a random unitary $U_H(T)$ generated by $H \sim \text{Ham}_k^{(t)}$ is an $\epsilon$-approximate unitary $t$-design, the random
Hamiltonian $H$ is called an $\varepsilon$-approximate $t$-design Hamiltonian with $k$-local interactions. We also call the time $T_0$ a design time of $H$.

Note that, in this sense, there is no design Hamiltonian on a finite ensemble of time-independent Hamiltonians. Due to the Poincaré recurrence theorem \[56\], the time-evolution operator generated by a time-independent Hamiltonian is in the neighbourhood of the identity operator at the recurrence time. Although the time-evolution operators generated by other Hamiltonians are possibly not the identity at the recurrence time of one Hamiltonian, by multiplying the recurrence time of all Hamiltonians and multiplying an additional integer, we can always find the time where all operators are close to the identity. Hence, at that time, an ensemble of time-evolution operators does not form unitary designs. However, this problem can be avoided if we consider time-dependent Hamiltonians or a continuous ensemble of time-independent Hamiltonians. We can also relax the condition of $\forall T \geq T_0$ to most of the time after $T_0$. For simplicity, in this paper, we define the design Hamiltonian as in Definition 16.

As our main purpose is to find physically natural Hamiltonians generating unitary designs, we are most interested in the design Hamiltonians which are not finely structured, are time-independent and are with geometrically local interactions. In addition, we may further require that, due to the fast scrambling conjecture, the design time scales logarithmically with the system size, which may depend on $t$. Thus, we arrive at the following conjecture.

**Conjecture (Natural design Hamiltonian conjecture)** There exist $\varepsilon$-approximate $t$-design Hamiltonians on $N$ qubits that satisfy the following three conditions:

1. the interactions are geometrically local,
2. the interactions are all time-independent,
3. the design time is given by $O(t \log N)$, which may also depend on $\varepsilon$.

The conjecture is based on an established language of unitary designs and so will be helpful to explore randomising operations in physically natural systems in a mathematically rigorous manner. We note that the conjecture is not only of theoretical interest but also of practical importance because, by applying such a random Hamiltonian onto a system, a unitary design will be spontaneously obtained. Most importantly, there is no need to change the interactions and no fine control of time is required. This will drastically simplify the implementations of unitary designs in experiments, also resulting in the simplification of many quantum protocols \[1–17\].

The construction of designs by local random quantum circuits \[40\] can be naturally translated into design Hamiltonians: a set of Hamiltonians with neighbouring two-body interactions is a $t$-design Hamiltonian if the interactions vary randomly and independently at every time step. Such varying interactions can be considered to be fluctuations induced by white noise on two-body interactions \[57\]. This design Hamiltonian $H_{\text{rand}}$ satisfies the first condition of the conjecture, as it uses only neighbouring interactions, but not the second and the third ones. Indeed, to achieve a unitary $t$-design by the dynamics of $H_{\text{rand}}$, the interactions should be changed $O(t^{10} N)$ times uniformly at random. This is far from time-independent and takes much longer than $O(t \log N)$.

Here, we concern more on the second condition of the conjecture and provide a design Hamiltonian $H_{XZ}$ based on Theorem 14. We start with introducing a parameter set $\mathcal{P}_t$ by

$$
\mathcal{P}_t = \left\{ \frac{m}{2 \lfloor t/2 \rfloor + 1} : m \in [-\lfloor t/2 \rfloor, \lfloor t/2 \rfloor] \right\},
$$

(14)
Interactions
All-to-all two-body interactions

Time-dependence
Highly dependent

Design time
$O(t^{10}N)$

| Design Hamiltonian | Interactions          | Time-dependence | Design time |
|--------------------|-----------------------|----------------|-------------|
| $H_{\text{rand}}$  | Nearest neighbour interactions | Highly dependent | $O(1)$       |
| $H_{\text{XZ}}$    | All-to-all two-body interactions | Nearly time-independent | $O(t)$        |

TABLE I. A comparison of design Hamiltonians, $H_{\text{rand}}$ and $H_{\text{XZ}}$, in terms of the three conditions of the natural design Hamiltonian conjecture. The design time of $H_{\text{XZ}}$ is much shorter than that of $H_{\text{rand}}$ both in terms of $t$ and $N$. Although the improvement in terms of $t$ is generic to $H_{\text{XZ}}$, that in terms of $N$ is simply due to its all-to-all interactions (see the main text).

where $[x]$ is the floor function. We then define finite sets of commuting Hamiltonians:

$$
\mathcal{S}_{\text{Z}}^{(t)} := \left\{ - \sum_{j<k} J_{ik} Z_j \otimes Z_k - \sum_j B_j Z_j \right\}, \quad \mathcal{S}_{\text{X}}^{(t)} := \left\{ - \sum_{j<k} J_{ik} X_j \otimes X_k - \sum_j B_j X_j \right\}
$$

These types of disordered Hamiltonians are similar to those in many-body localised systems [58–60], while interactions typically decay with increasing distance in such systems. Finally, we introduce a notation $\in \mathbb{R}$ which implies that the left-hand side is drawn uniformly at random from the set in the right-hand side.

Our third result is given as follows (see Sec. [IV.D] for the proof).

**Corollary 17 (Main result 3)** Let $t = o(N^{1/2})$ and $\mathcal{S}_{\text{XZ}}^{(t)}$ be a set of 2-local time-dependent Hamiltonians in the form of

$$
H_{\text{XZ}}(T) = \begin{cases} 
H_{\text{Z}}^{(m)} & \text{if } 2m\pi \leq T < (2m + 1)\pi, \\
H_{\text{X}}^{(m)} & \text{if } (2m + 1)\pi \leq T < 2(m + 1)\pi,
\end{cases}
$$

where $T$ denotes time, and $H_{\text{W}}^{(m)} \in \mathcal{S}_{\text{W}}^{(t)}$ for any $m = 0, 1, \cdots$ ($W = X, Z$). Then, the random Hamiltonian $H_{\text{XZ}} \in \mathcal{S}_{\text{XZ}}^{(t)}$ is an $\epsilon$-approximate $t$-design Hamiltonian. The design time of $H_{\text{XZ}}$ is given by $(2t + 1 + \frac{2\epsilon}{N} \log 1/\epsilon)\pi$.

Corollary 17 implies that the random Hamiltonian $H_{\text{XZ}}$ quickly generates the time evolution which can be hardly distinguished from completely random one. Most notably, the design time is $O(t)$ and independent of the system size. As a simple consequence, any correlation functions at time $T$ in the system described by such a Hamiltonian quickly converges to the Haar averaged values. One of the important instances is the 2t-point OTO correlator, which is expected to diagnose quantum chaos and has been studied in strongly correlated systems [28–30]. As the 2t-point OTO correlators are polynomials of a unitary with degree $t$, their values in the system of a random Hamiltonian $H_{\text{XZ}}$ are $\epsilon$-close to the Haar random averages when $T \gtrsim (2t + 1 + \frac{2\epsilon}{N} \log 1/\epsilon)\pi$. Furthermore, due to the large deviation bounds for unitary designs [50], this implies that almost any Hamiltonian in $\mathcal{S}_{\text{XZ}}^{(t)}$ saturates the 2t-point OTO correlators to the Haar random averages in a short time irrespective of the system size. As the OTO correlators are saturated in quantum chaotic systems [27], our result indicates a close connection between the Hamiltonians in $\mathcal{S}_{\text{XZ}}^{(t)}$ and quantum chaos, which also suggests that the framework of design Hamiltonians may be useful to investigate the dynamics in quantum chaotic systems.

We make a comment on the short design time $O(t)$ of $H_{\text{XZ}}$, which is significantly faster than the design time $O(t^{10}N)$ of $H_{\text{rand}}$ in terms of both $t$ and $N$. However, we should note that, although the improvement in terms of $t$ is intrinsic to $H_{\text{XZ}}$, that in terms of $N$ is rather due to the all-to-all interactions.
interactions of $H_{XZ}$. Such interactions may naturally appear in cavity QED \[61–63\] due to the cavity modes mediating long-range interactions, and unitary designs may possibly be realised in a short time. Nevertheless, for the fair comparison with $H_{\text{rand}}$, the realisation of all-to-all interactions by neighbouring ones should be taken into account. This can be achieved if every particle travels all corners of the system and interacts with all the other particles, taking $O(N)$ time. Hence, when the interactions are neighbouring, the actual time for $H_{XZ}$ to achieve unitary designs is considered to be $O(tN)$.

Unfortunately, both design Hamiltonians $H_{\text{rand}}$ and $H_{XZ}$ do not satisfy all three conditions of the natural design Hamiltonian conjecture. However, we believe that two design Hamiltonians $H_{\text{rand}}$ and $H_{XZ}$ and supports from the original fast scrambling conjecture \[23–27\] provide substantial evidences for the natural design Hamiltonian conjecture.

IV. PROOFS

In this section, we provide proofs of theorems and lemmas given in Section III. We first introduce additional notation and useful lemmas in Subsection IV A. The proof of our first main result, Theorem 9, is given in Subsection IV B. We prove Lemma 13 in Subsection IV C which is the key lemma to obtain our second main result, and conclude this section by showing Corollary 17 about design Hamiltonians in Subsection IV D.

A. Additional notation and lemmas

Let $E = \{|k\rangle_E\}_{k \in [0,d-1]}$ and $F = \{|\alpha\rangle_F\}_{\alpha \in [0,d-1]}$ be orthogonal bases in $\mathcal{H}$. As we deal with $t$ copies of the Hilbert space, $\mathcal{H}^\otimes t$, we denote $[0,d-1]^t$ by $\mathcal{N}$ and introduce bases $\{|k\rangle_W\}_{k \in \mathcal{N}}$ ($W = E, F$) in $\mathcal{H}^\otimes t$, where $|k\rangle_W = \bigotimes_{s=1}^t |k_s\rangle_W$, $k = (k_1, \cdots, k_t)^T \in \mathcal{N}$, and $T$ represents the transpose. In the following, we always label the basis $E$ and $F$ by Latin and Greek alphabets, respectively, and do not write the subscript $E$ and $F$ explicitly.

Let $S_t$ be a permutation group of degree $t$. For $\pi \in S_t$, we denote $(k_{\pi(1)}, \cdots, k_{\pi(t)})^T$ by $k_\pi$, and define a state $|\Psi_\pi\rangle \in \mathcal{H}^\otimes 2t$ by

\[|\Psi_\pi\rangle := I \otimes V(\pi)|\Phi\rangle\]

\[= \frac{1}{d^{t/2}} \sum_{k \in \mathcal{N}} |k, k^*_\pi\rangle\]

\[= \frac{1}{d^{t/2}} \sum_{\alpha \in \mathcal{N}} |\alpha, \alpha^*_\pi\rangle,\]  \hspace{1cm} (18)  \hspace{1cm} (19)  \hspace{1cm} (20)

where $V(\pi)$ is a unitary representation of $\pi$, $|\Phi\rangle$ is the maximally entangled state between the first $\mathcal{H}^\otimes t$ and the second $\mathcal{H}^\otimes t$, $|k, k^*_\pi\rangle = |k\rangle \otimes (|k^*_\pi\rangle)^*$ and $|\alpha, \alpha^*_\pi\rangle = |\alpha\rangle \otimes (|\alpha^*_\pi\rangle)^*$. Note that $|\Psi_\pi\rangle$ and $|\Psi_\pi\rangle$ are not necessarily orthogonal due to the permutation element. We denote $|\Psi_\pi\rangle \langle \Psi_\pi|$ simply by $\Psi_\pi$.

We also introduce three subspaces in $\mathcal{H}^\otimes 2t$,

$\mathcal{H}_E = \text{span}\{|k, k^*_\pi\} : k \in \mathcal{N}, \pi \in S_t\}$, \hspace{1cm} (21)

$\mathcal{H}_F = \text{span}\{|\alpha, \alpha^*_\pi\} : \alpha \in \mathcal{N}, \pi \in S_t\}$, \hspace{1cm} (22)

$\mathcal{H}_0 = \text{span}\{|\Psi_\pi\} : \pi \in S_t\}$. \hspace{1cm} (23)

Obviously, $\mathcal{H}_E \supseteq \mathcal{H}_0$ and $\mathcal{H}_F \supseteq \mathcal{H}_0$. The projectors onto the subspaces $\mathcal{H}_E$, $\mathcal{H}_F$ and $\mathcal{H}_0$ are denoted by $P_E$, $P_F$ and $P_0$, respectively. We further introduce an equivalent relation $\sim_k$ ($k \in \mathcal{N}$)
in $S_t$ such that $\pi \sim_k \sigma$ if and only if $k_\pi = k_\sigma$. We then introduce a set $S^k_t$ of representative elements in equivalence classes by $\sim_k$. Using this notation, the projectors $P_E$ and $P_F$ are explicitly given by

\begin{align}
P_E &= \sum_{k \in \mathbb{N}} \sum_{\pi \in S^k_t} |k, k_\pi^* \rangle \langle k, k_\pi^*|, \\
P_F &= \sum_{\alpha \in \mathbb{N}} \sum_{\pi \in S^\alpha_t} |\alpha, \alpha_\pi^* \rangle \langle \alpha, \alpha_\pi^*|.
\end{align}

We have the following lemmas for these projectors.

**Lemma 18 (Ref. [40, 53])** For Haar random unitaries $U$, random diagonal-unitaries $D^E$ in the basis of $E$, and those $D^F$ in the basis of $F$, the following hold

\begin{align}
E_{U \sim \mathcal{H}[U^{\otimes t,t}]} &= P_0, \\
E_{D^E \sim D^E}&[(D^E)^{\otimes t,t}] = P_E, \\
E_{D^F \sim D_F}&[(D^F)^{\otimes t,t}] = P_F.
\end{align}

**Lemma 19 (Ref. [40])** It holds that $\|P_0 - \sum_{\pi \in S_t} \Psi_\pi\|_\infty \leq \frac{\ell^2}{d}$.

### B. Proof of Theorem 9

We now prove Theorem 9. Due to the independence of random diagonal unitaries $D^E$, $D'^E$ and $D^F$ and Lemma 18 we have

\begin{equation}
\|E_{D^E, D'^E \sim D^E, D^F \sim D_F}[(D^E D'^E D^F)^{\otimes t,t}] - E_{U \sim \mathcal{H}[U^{\otimes t,t}]}\|_\infty = \|P_E P_F P_E - P_0\|_\infty,
\end{equation}

which is bounded from above as follows:

\begin{align}
\|P_E P_F P_E - P_0\|_\infty &\leq \|P_E P_F P_E - \sum_{\pi \in S_t} \Psi_\pi\|_\infty + \|P_0 - \sum_{\pi \in S_t} \Psi_\pi\|_\infty \\
&\leq \|P_E (P_F - \sum_{\pi \in S_t} \Psi_\pi) P_E\|_\infty + \frac{\ell^2}{d},
\end{align}

where we have used the triangular inequality in the first line, the fact that $|\Psi_\pi\rangle \in \mathcal{H}_0 \subset \mathcal{H}_E$ and Lemma 19 in the second line. Using the fact that the operator norm for Hermitian operators is bounded from above by the row norm, defined by $\max_j \sum_i |A_{ij}|$ for an Hermitian operator $A$, we have

\begin{equation}
\|E_{D^E, D'^E \sim D^E, D^F \sim D_F}[(D^E D'^E D^F)^{\otimes t,t}] - E_{U \sim \mathcal{H}[U^{\otimes t,t}]}\|_\infty \leq C,
\end{equation}

where

\begin{equation}
C = \max_{l \in \mathbb{N}} \max_{\sigma \in S_l^1} \sum_{k \in \mathbb{N}} \sum_{\chi \in \mathbb{N}} |\langle l, l_\pi^* | P_F - \sum_{\pi \in S_t} \Psi_\pi |k, k_\chi^*\rangle| + \frac{\ell^2}{d}.
\end{equation}

Note that it suffices to consider only vectors in $\mathcal{H}_E^{\otimes t,t}$ when we compute the first term of Eq. (31), which is because the operator is sandwiched by the projector $P_E$. In the following, we evaluate $C$. 
Substituting $|\Psi_\alpha\rangle = \frac{1}{\sqrt{d^d}} \sum_{m\in N} |m, m_\alpha^\ast\rangle$, the second term is given by

$$\langle l, l_\sigma^\ast | \sum_{\pi\in S_t} \Psi_\pi | k, k_\chi^\ast\rangle = \frac{1}{d^t} \sum_{\pi\in S_t} \delta_{l_\pi, l_\sigma} \delta_{k_\pi, k_\chi}. \quad (34)$$

On the other hand, using an explicit form of $P_F$ given in Eq. (25), the first term can be expanded to be

$$\langle l, l_\sigma^\ast | P_F | k, k_\chi^\ast\rangle = \sum_{\alpha\in N} \sum_{\pi\in S_t^{\alpha}} \langle l|\alpha\rangle \langle\alpha|k\rangle \langle k_{\pi^{-1}\alpha}\chi|\alpha\rangle \langle\alpha|l_{\pi^{-1}\alpha}\sigma\rangle. \quad (35)$$

Since a pair of the bases $(E, F)$ is a Fourier-type pair, it satisfies for any $l, k, \alpha \in [0, d - 1]$ that $\langle l|\alpha\rangle \langle k|\alpha\rangle = \langle l + k|\alpha\rangle / d^{1/2}$, where $l + k \in [0, d - 1]$ as $[0, d - 1]$ is an additive group with respect to $+$. Denoting $(l_1 + k_1, \cdots, l_t + k_t)^T$ by $l + k$, we have

$$\langle l, l_\sigma^\ast | P_F | k, k_\chi^\ast\rangle = \frac{1}{d^t} \sum_{\alpha\in N} \sum_{\pi\in S_t^{\alpha}} \langle l + k_{\pi^{-1}\alpha}\chi|\alpha\rangle \langle\alpha|k + l_{\pi^{-1}\alpha}\sigma\rangle \quad (36)$$

$$= \frac{1}{d^t} \sum_{\alpha\in N} \left( \sum_{\pi\in S_t} - \sum_{\pi\in S_t \setminus S_t^{\alpha}} \right) \langle l + k_{\pi^{-1}\alpha}\chi|\alpha\rangle \langle\alpha|k + l_{\pi^{-1}\alpha}\sigma\rangle \quad (37)$$

$$= \frac{1}{d^t} \left( \sum_{\pi\in S_t} \delta_{l_\pi + k_{\pi^{-1}\alpha}\chi, k_\alpha + l_{\pi^{-1}\alpha}\sigma} - M_{l,k} \right), \quad (38)$$

where $M_{l,k} = \sum_{\alpha\in N} \sum_{\pi\in S_t \setminus S_t^{\alpha}} \langle l + k_{\pi^{-1}\alpha}\chi|\alpha\rangle \langle\alpha|k + l_{\pi^{-1}\alpha}\sigma\rangle$, and we used $\sum_{\alpha\in N} |\alpha\rangle \langle\alpha| = I_{H \otimes t}$. Hence, using the triangular inequality, we obtain

$$|\langle l, l_\sigma^\ast | P_F - \sum_{\pi\in S_t} \Psi_\pi | k, k_\chi^\ast\rangle| = \frac{1}{d^t} \left| \sum_{\pi\in S_t} \left( \delta_{l_\pi + k_{\pi^{-1}\alpha}\chi, k_\alpha + l_{\pi^{-1}\alpha}\sigma} - \delta_{1_\pi, l_\sigma} \delta_{k_\pi, k_\chi} \right) - M_{l,k} \right| \quad (39)$$

$$\leq \frac{1}{d^t} \left| \sum_{\pi\in S_t} \left( \delta_{l_\pi + k_{\pi^{-1}\alpha}\chi, k_\alpha + l_{\pi^{-1}\alpha}\sigma} - \delta_{1_\pi, l_\sigma} \delta_{k_\pi, k_\chi} \right) \right| + \frac{1}{d^t} |M_{l,k}|. \quad (40)$$

An upper bound of $|M_{l,k}|$ can be obtained from the fact that the bases $E$ and $F$ are mutually unbiased, leading to

$$|M_{l,k}| \leq \frac{1}{d^t} \sum_{\alpha\in N} |S_t \setminus S_t^{\alpha}|. \quad (41)$$

As $|S_t \setminus S_t^{\alpha}|$ depends only on how many different elements $\alpha$ contains, of which number we denote by $k$, and the number of every different element $\alpha_i$ in $\alpha$, denoted by $s_i$, we replace the summation with that over $k$ and obtain

$$\sum_{\alpha\in N} |S_t \setminus S_t^{\alpha}| = \sum_{k=1}^{t} \binom{d}{k} g^{(k)}(t), \quad (42)$$

where the binomial coefficient counts the number of possible choices of $k$ different numbers from $[0, d - 1]$, and $g^{(k)}(t)$ is the function that depends only on $k$ and $t$ given by

$$g^{(k)}(t) = \sum_{(s_1, \cdots, s_k)} \frac{t!}{s_1! \cdots s_k!} \left( t - \frac{t!}{s_1! \cdots s_k!} \right). \quad (43)$$
Here, the second summation is taken over all possible \((s_1, \cdots, s_k)\) such that \(\forall i \in [1, k] \ s_i \in [1, t]\) and \(\sum_{i=1}^k s_i = t\). For a fixed \(k\), the number of such combinations is simply given by \(\binom{t-1}{k-1}\). For \(k = t\), \(s_i = 1\) for all \(i \in [1, k]\) and so, \(g(t)(t) = 0\). For the remaining terms \(g^{(k)}(t)\) \((k \in [1, t-1])\), we use an upper bound given by

\[
g^{(k)}(t) \leq \binom{t-1}{k-1} \frac{t^2}{4},
\]

which is optimal when \(k = t - 1\). Substituting these, we obtain

\[
\sum_{\alpha \in \mathcal{N}} \left| S_t \setminus S_t^\alpha \right| \leq \frac{t^2}{4} \sum_{k=1}^{t-1} \binom{d}{k} \binom{t-1}{k-1} = \frac{t^2}{4} \left( \binom{d}{t} + \binom{d}{t-1} \right),
\]

where the last line is obtained due to the Vandermonde’s identity. Since \(d = \Omega(t^2)\), an upper bound is obtained such as

\[
\sum_{\alpha \in \mathcal{N}} \left| S_t \setminus S_t^\alpha \right| \leq t^2 d^{-1} + O(t^4 d^{-2}),
\]

which leads to

\[
\left| (1, l, 0 \mid P_F - \sum_{\pi \in S_t} \Psi_\pi \mid k, k_\chi) \right| \leq \frac{1}{d^t} \sum_{\pi \in S_t} \left( \delta_{l+k_{s-1} \pi \chi, k+l_{s-1} - \pi \sigma} - \delta_{l+k_{s-1} \pi \chi, k+l_{s-1} - \pi \sigma} \right) + \frac{t^2}{d^t+1} + O \left( \frac{t^4}{d^{t+2}} \right).
\]

Substituting this into \(C\), the following upper bound can be obtained:

\[
C \leq \frac{t^2(t^2 + 1)}{d} + \frac{1}{d^t} \max_{l \in \mathcal{N}} \max_{\sigma \in S_t^1} \sum_{\chi \in \mathcal{N}} \sum_{\pi \in S_t} \left( \delta_{l+k_{s-1} \pi \chi, k+l_{s-1} - \pi \sigma} - \delta_{l+k_{s-1} \pi \chi, k+l_{s-1} - \pi \sigma} \right) + O \left( \frac{t^4}{d^2} \right)
\]

\[
= \frac{t^2(t^2 + 1)}{d} + \frac{1}{d^t} \max_{l \in \mathcal{N}} \max_{\sigma \in S_t^1} \sum_{\chi \in \mathcal{N}} \sum_{\pi \in S_t} \left( \delta_{l+k_{s-1} \pi \chi, k+l_{s-1} - \pi \sigma} - \delta_{l+k_{s-1} \pi \chi, k+l_{s-1} - \pi \sigma} \right) + O \left( \frac{t^4(t^2 + 1)}{d^2} \right)
\]

\[
\leq \frac{t^2(t^2 + 1)}{d} + \frac{1}{d^t} \max_{l \in \mathcal{N}} \max_{\sigma \in S_t^1} \sum_{\chi \in \mathcal{N}} \sum_{\pi \in S_t} \left( \sum_{\chi \neq \pi} \delta_{l+k_{s-1} \pi \chi, k+l_{s-1} - \pi \sigma} \right) + O \left( \frac{t^4(t^2 + 1)}{d^2} \right),
\]

where the second line is due to a fact that the term in the modulus is non-negative because, when the second term is one, the first term is also one, and the last line is obtained by using a fact that the first and the second term cancel each other when \(\chi = \pi\) and by dropping negative terms when \(\chi \neq \pi\). For the delta function \(\delta_{l+k_{s-1} \pi \chi, k+l_{s-1} - \pi \sigma}\), we have

\[
\delta_{l+k_{s-1} \pi \chi, k+l_{s-1} - \pi \sigma} = 1 \iff \forall s \in [1, t], \quad l_s + k_{s-1} \pi \chi(s) = k_s + l_{s-1} - \pi \sigma(s)
\]

When \(\chi \neq \pi\), there exists at least one pair \((s, s')\) \((s \neq s' \in [1, t])\) such that \(\pi(s') = \chi(s)\). Hence, \(k_{s'} = k_s + l_{s-1} - \pi \sigma(s)\) should be at least satisfied for the delta function to be non-zero. Thus, the number of \(k\), for which the delta function is non-zero, is at most \(d^{t-1}\). Based on this observation, we obtain

\[
\max_{l \in \mathcal{N}} \max_{\sigma \in S_t^1} \sum_{\pi \in S_t} \sum_{\chi \neq \pi} \sum_{\pi \in S_t} \delta_{l+k_{s-1} \pi \chi, k+l_{s-1} - \pi \sigma} \leq t^2 d^{t-1}.
\]
Substituting this into Eq. (51), we obtain an upper bound of $C$, leading to
\begin{equation}
|\mathbb{E}_{D^E,D^F \sim D_E,D^F \sim D_F}[D^E D^F D^E U_{\otimes t}^t] - \mathbb{E}_{U \sim H}[U_{\otimes t}^t]|_\infty \leq \frac{(1 + t^2)\mu^2 + t^2}{d} + O \left( \frac{t^4 \mu^2}{d^2} \right).
\end{equation}
This concludes the proof.

C. Proof of Lemma 13

We first provide a key lemma to prove Lemma 13. The lemma is seen as a constrained problem in extremal algebraic theory \cite{48,49}. The proof is given in Appendix E.

**Lemma 20** Let $O$ be an orthogonal matrix acting on the Euclidean space $\mathbb{R}^t$, which contains a hypercube $\{0,1\}^t$. If there exists a set $S \subset \{0,1\}^t$ such that $OS \subset \{0,1\}^t$ and $|S| > 2^t - 1$, then $O$ is a permutation matrix.

Now, we prove Lemma 13, i.e. $\Lambda_2 = |L_2| \leq 2^{2^t + (t-1)N}$. Here, $L_2$ is a set of pairs $(K, K')$, where $K$ is a 2-local but not a row permutation of $K'$.

**Proof (Lemma 13)** Throughout the proof, we denote the column vectors of $K$ and $K'$ by $\vec{k}_i$ and $\vec{k}'_i$, respectively, for $i \in [1, N]$. The 2-local permutation condition is equivalent to the following:
\begin{equation}
\forall i,j \in [1, N], \quad \vec{k}_i \cdot \vec{k}_j = \vec{k}'_i \cdot \vec{k}'_j, \tag{55}
\end{equation}
where \cdot is the usual Euclidean inner product. This is because the conditions for $i = j$ imply that the number of 1’s in $\vec{k}_i$ and that in $\vec{k}'_i$ should be the same, and those for $i \neq j$ imply that the number of 11 in $K_{i,j}$ is equal to that in $K'_{i,j}$. These conditions together correspond to the necessary and sufficient conditions for the pair $(K, K')$ to be 2-local permutations. Moreover, Eq. (55) implies that the Gram matrix of a set $\{\vec{k}_i : i \in [1, N]\}$ of column vectors is the same as that of $\{\vec{k}'_i : i \in [1, N]\}$. Hence, $\text{span}\{\vec{k}_i : i \in [1, N]\}$ has the same dimension as $\text{span}\{\vec{k}'_i : i \in [1, N]\}$. It also follows that there exists a partial isometry $O$ that satisfies $O \vec{k}_i = \vec{k}'_i$ for any $i \in [1, N]$, i.e. $OK = K'$. If the partial isometry is restricted to its support, it is an orthogonal matrix as the elements of the vectors are in $\{0,1\}$, and it is not a permutation operator due to the assumption that $K$ is not a row permutation of $K'$.

We will now construct a set $O$ of orthogonal matrices on $\mathbb{R}^t$ that satisfies
\begin{equation}
\forall (K, K') \in L_2, \exists O \in O \text{ such that } OK = K'. \tag{56}
\end{equation}
This can be done as follows. Let $s := 2^t$ and $[0, s - 1]^{\leq t}$ be the set of $s$-ary strings of length $t$ or smaller. We describe a procedure for defining a set $S_2 \subset [0, s - 1]^{\leq t}$ and orthogonal matrices $O_b$ for $b \in S_2$, such that $S_2$ is a prefix code and that $O := \{O_b : b \in S_2\}$ satisfies Eq. (56). Our construction starts with $S_2 = \emptyset$ and is recursive in terms of the rank $\kappa$ of the partial isometry obtained from $(K, K')$. We repeat the subroutine described below from $\kappa = t$ to $\kappa = 1$ by decreasing $\kappa$ one by one. In the subroutine, we first choose $(K, K') \in L_2$ that defines a partial isometry with rank $\kappa$. We pick up an arbitrary set of independent column vectors $\{\vec{k}_{im}\}_{m=1}^{\kappa}$ in $K$ and those $\{\vec{k}'_{im}\}_{m=1}^{\kappa}$ in $K'$. These vectors can be converted to an $s$-ary string $b = (2^t k_{i1} + k'_{i1}, 2^t k_{i2} + k'_{i2}, \ldots, 2^t k_{in} + k'_{in})$ of length $\kappa$ by regarding each vector as a binary number with length $t$. If $b$ is a prefix of a string $b' \in S_2$, then the orthogonal matrix $O_{b'}$ satisfies $O_{b'}K = K'$ because, on the support of the partial isometry obtained from $(K, K')$, the action of $O_{b'}$ is the same as that of the isometry by construction. Otherwise, we append $b$ to $S_2$, and define an orthogonal matrix $O_b$ as an arbitrary
extension of the partial isometry. The subroutine is run for all \((K, K') \in L_2\) with a partial isometry 
of rank \(\kappa\).

Eventually, we obtain a set \(\mathcal{O}\) of orthogonal matrices on \(\mathbb{R}^t\). Importantly, it does not contain a permutation matrix and, by construction, \(|\mathcal{O}| = |S_2| \leq 2^{2^t}\). Introduce a set \(L_2(\mathcal{O})\) by \(\{(K, OK) : K, OK \in \{0, 1\}^{tN}\}\) for a given orthogonal matrix \(O \in \mathbb{R}^t\), we have \(L_2 \subset \bigcup_{O \in \mathcal{O}} L_2(\mathcal{O})\), leading to

\[
\Lambda_2 \leq \sum_{O \in \mathcal{O}} |L_2(O)| \leq |\mathcal{O}| \max_{O \in \mathcal{O}} |L_2(O)| \leq 2^{2^t} \max_{O \in \mathcal{O}} |L_2(O)|. 
\]

As each pair of columns of \((K, OK)\) is independent, \(|L_2(O)|\) for \(O \in \mathcal{O}\) is bounded from above by

\[
|L_2(O)| \leq \left(\max_{O \in \mathcal{O}} \{|\vec{k} \in \{0, 1\}^t : O\vec{k} \in \{0, 1\}^t|\}\right)^N.
\]

As \(O \in \mathcal{O}\) is on \(\mathbb{R}^t\) and is not a permutation matrix, from the contraposition of Lemma [20] we obtain

\[
\max_{O \in \mathcal{O}} \{|\vec{k} \in \{0, 1\}^t : O\vec{k} \in \{0, 1\}^t|\} \leq 2^{t-1}.
\]

Thus, we have \(\Lambda_2 \leq 2^{2^t + (t-1)N}\), and conclude the proof.

Finally, we note that the upper bound given in Lemma [13] is unlikely to be tight in terms of \(t\) because an upper bound \(|\mathcal{O}|\) given by \(2^{2^t}\) in the proof is far from optimal. This is observed from a fact that \(|\mathcal{O}| = |S_2|\) but \(S_2\) does not contain all strings with length \(t\). More concretely, we provide instances for a small \(t\). From the result in Ref. [52], we know that, for any pair \((K, K')\), \(K\) is a row permutation of \(K'\) if and only if \(K\) is a \([(\log_2 t) + 1]\)-local permutation of \(K'\). Hence, the smallest \(t\) making the 2-local permutation check problem non-trivial is 4. In this case, we can show that, if \(K\) is a 2-local but not a row permutation of \(K'\), the four rows of \(K\) and those of \(K'\) can be rearranged independently, resulting in \(K_\pi\) and \(K'_\sigma\) respectively \((\pi, \sigma \in S_4)\), such that a pair of the \(i\)th column of \(K_\pi\) and that of \(K'_\sigma\) are in the set \(C_0 \cup C_1\) \((\forall i \in [1, N])\), where

\[
C_0 = \{((0, 0, 0, 0)^T, (0, 0, 0, 0)^T), ((1, 1, 1, 1)^T, (1, 1, 1, 1)^T), ((0, 0, 1, 1)^T, (0, 0, 1, 1)^T), ((1, 1, 0, 0)^T, (1, 1, 0, 0)^T), ((1, 0, 1, 0)^T, (1, 0, 1, 0)^T), ((0, 1, 0, 1)^T, (0, 1, 0, 1)^T)\}, \\
C_1 = \{((0, 1, 1, 0)^T, (0, 1, 0, 1)^T), ((1, 0, 1, 0)^T, (1, 0, 1, 0)^T)\}.
\]

Note that \(K_\pi\) is still a 2-local permutation but is not a row permutation of \(K'_\sigma\). For \(K_\pi\) not being a row permutation of \(K'_\sigma\), all pairs of columns should not be chosen only from \(C_0\) or only from \(C_1\). Taking the number of choices of \(\pi\) and \(\sigma\) into account, we have

\[
\Lambda_2 \leq t!^2(\{|C_0| + |C_1|\}^N - |C_0|^N - |C_1|^N) < t!^28^N,
\]

which corresponds to \(t!^22^{(t-1)N}\) for \(t = 4\). For this reason, we conjecture that the optimal bound should be given by \(f(t)2^{(t-1)N}\) where \(f(t) = O(\text{poly}(t))\), which we have confirmed to hold if \(t \leq 7\) in the similar way to the case of \(t = 4\). If this conjecture is true, Theorem [14] works for \(t = o(N/\log N)\) instead of \(t = o(N^{1/2})\).
D. Proof of Corollary 17

We prove Corollary 17. To remind the statement, let \( \mathcal{H}_{XZ}^{(t)} \) be a set of 2-local time-dependent Hamiltonians in the form of

\[
H_{XZ}(T) = \begin{cases} 
H_{Z}^{(m)} & \text{if } 2m\pi \leq T < (2m + 1)\pi, \\
H_{X}^{(m)} & \text{if } (2m + 1)\pi \leq T < 2(m + 1)\pi,
\end{cases}
\]

(65)

over \( H_{W}^{(m)} \in \mathcal{H}_{W}^{(t)} \) (\( W = X, Z \) and \( m = 0, 1, \ldots \)), which are defined in Eqs. (15) and (16). Then, we show that \( \forall T \geq (2t + 1 + \frac{2}{N} \log 1/\epsilon)\pi \), a random unitary \( U_{XZ}(T) = T \exp[-i \int_{0}^{T} ds H_{XZ}(s)] \) generated by \( H_{XZ}(T) \in \mathbb{R}^{\mathcal{H}_{XZ}^{(t)}} \) at time \( T \) is an \( \epsilon \)-approximate unitary \( t \)-design.

Proof (Corollary 17) In the proof, we denote \( e^{-i\tau H_{W}^{(m)}} \) by \( U_{W}^{(m)}(\tau) \) (\( W = X, Z \)). As both Hamiltonians are composed of commuting terms, they are simply given by

\[
e^{-i\tau H_{X}^{(m)}} = \prod_{k<k'} e^{i\tau J_{kk'}^{(m)} X_{k} \otimes X_{k'}} \prod_{k} e^{i\tau B_{k}^{(m)} X_{k}} \quad \text{and} \quad e^{-i\tau H_{Z}^{(m)}} = \prod_{k<k'} e^{i\tau J_{kk'}^{(m)} Z_{k} \otimes Z_{k'}} \prod_{k} e^{i\tau B_{k}^{(m)} Z_{k}}.
\]

(66)

We first consider a random unitary \( U_{XZ}(T_{\ell}) \) at time \( T_{\ell} = (2\ell + 1)\pi \) (\( \ell = 1, 2, \ldots \)). Using the above notation, it is given by

\[
U_{XZ}(T_{\ell}) = U_{Z}^{(\ell+1)}(\pi) \prod_{m=\ell}^{\ell+1} U_{X}^{(m)}(\pi) U_{Z}^{(m)}(\pi).
\]

(67)

We take an average of \( U_{XZ}(T_{\ell}) \) over \( H_{XZ} \in \mathbb{R}^{\mathcal{H}_{XZ}^{(t)}} \), which is equivalent to take the average of the right-hand side of Eq. (67) over all parameters \( J_{kk'}^{(m)}, B_{k}^{(m)}, B_{k'}^{(m)} \in \mathcal{P}_{t} \), where the parameter sets are given in Eq. (14). Because it holds that

\[
\left( e^{i\tau J_{kk'}^{(m)} Z_{k} \otimes Z_{k'}} e^{i\tau B_{k}^{(m)} Z_{k}} \otimes e^{i\tau B_{k'}^{(m)} Z_{k'}} \right)^{\otimes t, t}
= \left( e^{i\tau (J_{kk'}^{(m)} + B_{k}^{(m)} + B_{k'}^{(m)})} \right)^{\otimes t, t}
\]

(68)

\[
= \left( e^{-2\pi i(J_{kk'}^{(m)} + B_{k}^{(m)} + B_{k'}^{(m)})} \right)^{\otimes t, t}
\]

(69)

\[
= \text{diag} \{ e^{-2\pi i(J_{kk'}^{(m)} + B_{k}^{(m)} + B_{k'}^{(m)})} \} \otimes t, t,
\]

(70)

after some simple calculation, the averaged one can be shown to satisfy

\[
\mathbb{E}_{J_{kk'}^{(m)}, B_{k}^{(m)}, B_{k'}^{(m)} \in \mathcal{P}_{t}} \left[ e^{i\tau J_{kk'}^{(m)} Z_{k} \otimes Z_{k'}} e^{i\tau B_{k}^{(m)} Z_{k}} \otimes e^{i\tau B_{k'}^{(m)} Z_{k'}} \right] = \mathbb{E}_{\varphi_{1}, \varphi_{2}, \varphi} \left[ \left( \text{diag} \{ 1, e^{i\varphi_{1}} \} \otimes \text{diag} \{ 1, e^{i\varphi_{2}} \} \right) \text{diag} \{ 1, 1, 1, e^{i\varphi} \} \right] \]

(71)

where the average in the right-hand side is taken over \( \varphi_{1} \) and \( \varphi_{2} \), which are chosen independently from a set \( \{ 2\pi m/(t+1) : m \in [0, t] \} \) uniformly at random, and \( \varphi \) chosen from a \( \{ 2\pi m/([t/2] + 1) : m \in [0, [t/2]] \} \). As a similar relation holds for \( X \) Hamiltonians, we can observe that \( U_{XZ}(T_{\ell}) \) is equivalent to \( (RDC^{(t)}_{\text{disc}}(I_{2})H_{N})^{2T} RDC^{(t)}_{\text{disc}}(I_{2}) \), where \( H_{N} \) is the Hadamard transformation on \( N \) qubits. Hence, it follows from Theorem 13 that \( U_{XZ}(T_{\ell}) \) is an \( \epsilon \)-approximate unitary \( t \)-design if \( \ell \geq t + \frac{1}{\epsilon} \log 1/\epsilon \).

To complete the proof, consider the time \( T \) satisfying \( T_{\ell} < T < T_{\ell+1} \) where \( \ell \geq t + \frac{1}{\epsilon} \log 1/\epsilon \). Because the time evolution operator from time \( T_{\ell} \) to time \( T \) is independent of the one before \( T_{\ell} \), it follows from Lemma 1 that \( U_{XZ}(T) \) is also an \( \epsilon \)-approximate unitary \( t \)-design. ■
V. CONCLUSION AND DISCUSSIONS

In this paper, we have presented new constructions of unitary $t$-designs and proposed design Hamiltonians as a general framework to investigate randomising operations in complex quantum many-body systems. The new constructions are based on repetitions of random diagonal-unitaries in mutually unbiased bases. We have first shown that, if the bases are Fourier-type, approximate unitary $t$-designs can be achieved on one qudit after $O(t)$ repetitions. We have then constructed quantum circuits on $N$ qubits that achieve approximate unitary $t$-designs using $O(tN^2)$ gates, which drastically improves the previous result \[40\] in terms of $t$. The dependence on $t$ is essentially optimal amongst designs with finite supports. The circuits were obtained by solving an interesting combinatorial problem called the local permutation check problem, showing an interesting connection between combinatorics and efficient implementations of designs. Based on these results, we have provided a design Hamiltonian, which changes the interactions only a few times to generate designs. This result supports the natural design Hamiltonian conjecture and is also practically important as it simplifies the experimental implementations of unitary designs.

Our approach of studying unitary designs and randomising operations in physically natural systems opens a lot of interesting questions. The following are a few questions concerning unitary designs:

1. In one-qudit systems, is it possible to implement unitary $t$-designs by repeating random diagonal-unitaries in any non-trivial pairs of bases? If so, how many repetitions are sufficient for the implementations?

2. What is the best strategy of the local permutation check problems?

3. What is the most efficient implementation by quantum circuits that approximate random diagonal-unitaries in the Pauli-$Z$ basis?

4. Explore applications of unitary $t$-designs for $t \geq 4$.

Regarding the question 1, we have found that repeating random diagonal-unitaries in non-trivial pairs of bases achieves a unitary 1-design if any vector in one basis is not orthogonal to any vector in the other basis. Although this non-orthogonality condition may not be necessary, we expect that, for arbitrary non-trivial pairs of bases satisfying the non-orthogonality condition, the process eventually achieves unitary $t$-designs. The questions 2 and 3 are related each other due to Lemma \[12\]. In this paper, we have considered only 2-local permutation check problems. However, if there exists a set $\mathcal{I} = \{I\}$ such that $\Lambda(\mathcal{I}) = O(2(t-1)^N)$ and $|I| = \text{constant}$ for all $I \in \mathcal{I}$, then we can implement approximate unitary $t$-designs using $O(t|\mathcal{I}|)$ quantum gates. Hence, finding a better strategy for the local permutation check problems immediately results in a faster implementation of unitary designs. It is also desirable to directly search efficient quantum circuits approximating random diagonal-unitaries in the $Z$ basis because Lemma \[12\] may not be tight. Finally, it is important to find applications of unitary $t$-designs for large $t$. A possible and promising direction is to further explore large deviation bounds for unitary designs as mentioned in Section \[III.B\].

We also list a few open questions about design Hamiltonians from the physical point of view:

I Prove or disprove the natural design Hamiltonian conjecture.

II What are the exact relations between natural design Hamiltonians and various definitions of scrambling or OTO correlators?

III If a design Hamiltonian is defined on a finite ensemble of Hamiltonians, how many Hamiltonians are need?

IV What are the static features of design Hamiltonians such as thermal or quantum phases?
The question I is the most interesting one, where we could use the methods developed in the random matrix theory [64]. A natural candidate of design Hamiltonians satisfying all the three conditions of the conjecture may be \( H_{\text{localGUE}} = \sum_{i,j} h_{ij} \) where each local term \( h_{ij} \) is drawn randomly and independently from the so-called Gaussian unitary ensemble [64] and the summation is taken over all neighbouring qubits. We expect that \( H_{\text{localGUE}} \) generates a unitary design after some time although it may also be possible that it does not due to the many-body localization. The question II is important to clarify the roles of design Hamiltonians in the black hole information science and quantum chaos. As design Hamiltonians are based on unitary designs, it suffices to investigate explicit relations between unitary designs and scrambling or the OTO correlators. The question III is not only of theoretical interest but also of practical importance because it determines the number of random bits necessary to construct design Hamiltonians. To address this question, it is needed to relax the definition of design Hamiltonians to exclude the Poincaré recurrence time as we have mentioned in Section [III C]. Finally, as design Hamiltonians are certain types of disordered Hamiltonians, it is natural to expect that they have special static properties, which is the question IV. A static property of the above random Hamiltonian \( H_{\text{localGUE}} \) was numerically studied from the viewpoint of distributions in a state space, and evidences of phase transitions were obtained [65]. However, as \( H_{\text{localGUE}} \) is not yet shown to be a design Hamiltonian and no time-independent design Hamiltonians have been found yet, it would be more realistic to start with investigating static properties of the ingredient Hamiltonian \( H_Z \) of \( H_{XZ} \), which has similarity to many-body localised systems, and their dependence on \( t \).

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Appendix A: Proof of Lemma 3

Here, a simple proof of Lemma 3 is given.
Proof As an ε-approximate unitary $t$-design $U$ satisfies $\|G(t)_{U \sim \nu} - G(t)_{U \sim H}\|_\circ \leq \epsilon$, we have
\[
\|G(t)_{V \circ} \circ G(t)_{U \sim \nu} - G(t)_{U \sim H}\|_\circ = |G(t)_{V \circ} \circ (G(t)_{U \sim \nu} - G(t)_{U \sim H})|_\circ \\
\leq |G(t)_{V \circ}| \|G(t)_{U \sim \nu} - G(t)_{U \sim H}\|_\circ \\
\leq \epsilon, 
\]
where we used the unitary invariance of the Haar measure in the first line, and a fact that $G(t)_V$ is a completely-positive and trace-preserving map in the last line. This implies that $\|VU\|$ is also an ε-approximate unitary $t$-design. The proof for $UV$ is similarly obtained. \[\square\]

Appendix B: Proof of Theorem 6

Here, we provide a proof of Theorem 6 which follows almost directly from the following simple lemma.

Lemma 21 For any unitary $U$, it holds that $U^{\otimes t, t} = P_0 + (I - P_0)U^{\otimes t, t}(I - P_0)$.

Proof Using $|\Psi_\pi\rangle = I \otimes V(\pi) |\Phi\rangle$, we have for any $\pi \in S_t$ that
\[
U^{\otimes t, t} |\Psi_\pi\rangle = U^{\otimes t} \otimes U^{\otimes t} V(\pi) |\Phi\rangle \\
= U^{\otimes t} \otimes V(\pi) U^{\otimes t} |\Phi\rangle \\
= U^{\otimes t} U^{\dagger \otimes t} \otimes V(\pi) |\Phi\rangle \\
= I \otimes V(\pi) |\Phi\rangle \\
= |\Psi_\pi\rangle ,
\]
where we have used the fact that $U^{\otimes t}$ commutes with $V(\pi)$ in the second line and the property of the maximally entangled state in the third line. This implies that $(I - P_0)U^{\otimes t, t} P_0 = 0$. Replacing $U$ with $U^\dagger$ in Eq. (B5), we also have $(I - P_0)U^{\dagger \otimes t, t} P_0 = 0$, implying $P_0 U^{\otimes t, t} (I - P_0) = 0$. Hence, we obtain $U^{\otimes t, t} = P_0 + (I - P_0)U^{\otimes t, t}(I - P_0)$. \[\square\]

Proof (Theorem 6) To prove Theorem 6 let $\nu$ be a quantum $(\eta, t)$-TPE, satisfying
\[
|E_{U \sim \nu}[U^{\otimes t, t}] - E_{U \sim H}[U^{\otimes t, t}]|_\infty \leq \eta. 
\]
Applying Lemma 21 to all the unitaries in $E_{U \sim \nu}[U^{\otimes t, t}]$, we have
\[
E_{U \sim \nu}[U^{\otimes t, t}] = P_0 + (I - P_0)E_{U \sim \nu}[U^{\otimes t, t}](I - P_0).
\]
Due to Lemma 18 which reads $P_0 = E_{U \sim H}[U^{\otimes t, t}]$, the quantum TPE $\nu$ satisfies that
\[
|(I - P_0)E_{U \sim \nu}[U^{\otimes t, t}](I - P_0)|_\infty \leq \eta. 
\]
Let $\nu^\ell$ be a measure corresponding to that of the $\ell$ iterations of the quantum TPE $\nu$. Then,
\[
\|G_{U \sim \nu^\ell} - G_{U \sim H}\|_2 \leq d\|E_{U \sim \nu}[U^{\otimes t, t}] - E_{U \sim H}[U^{\otimes t, t}]|_\infty \\
\leq d\|E_{U \sim \nu}[U^{\otimes t, t}](I - P_0)^\ell|_\infty \\
\leq d\|E_{U \sim \nu}[U^{\otimes t, t}](I - P_0)^\ell|_\infty \\
\leq d\eta^\ell. 
\]
Here, the first line is due to the inequality that $\|\mathcal{E}\|_\infty \leq D\|\mathcal{E}\|_{2-\epsilon}$ for any superoperators $\mathcal{E}$ acting on a $D$-dimensional system, the third line is obtained due to the independence of the measure of each iteration, the fourth line is from Eq. (B7), and the last line is from Eq. (B8). This implies that $\ell$ iterations of a quantum $(\eta,t)$-TPE is an $\epsilon$-approximate unitary $t$-design if $d^2\eta^\ell \leq \epsilon$.

**Appendix C: Proof of Lemma 8**

Here, we prove Lemma 8 about the Fourier-type bases.

**Proof (Lemma 8)** When a pair of two bases is that of arbitrary basis and its Fourier basis, it is clear that $\theta_{k\alpha} = \frac{2\pi k\alpha}{d}$ and the additive operation in the index is given by an addition modulo $d$. It can be easily checked that $[0,d-1]$ is an additive group with respect to the modular addition.

When the pair is given by the Pauli-X and -Z bases, using the binary representation such as $\alpha = \alpha_1 \cdots \alpha_N$ ($\forall j \in [1,N], \alpha_j \in \{0,1\}$), the Pauli-X and -Z bases can be represented by

\[
|\alpha\rangle_X = \bigotimes_{j=1}^{N} |\alpha_j\rangle_X, \quad |k\rangle_Z = \bigotimes_{j=1}^{N} |k_j\rangle_Z,
\]

respectively. Using a fact that $Z(k_j|\alpha_j\rangle_X = X(\alpha_j|k_j\rangle_Z)$ is equal to $1/\sqrt{2}$ if $(\alpha_j, k_j) = (0,0), (0,1), (1,0)$ and is equal to $-1/\sqrt{2}$ if $(\alpha_j, k_j) = (1,1)$, we have $\theta_{k\alpha} = \pi \sum_{j=1}^{N} \delta_{k_j,0} \delta_{\alpha_j,1}$, leading to

\[
\exp[i(\theta_{k\alpha} + \theta_{l\alpha})] = \exp[i\pi \sum_{j=1}^{N} (\delta_{k_j,1} + \delta_{l_j,1})\delta_{\alpha_j,1}]
\]

\[= \exp[i\pi \sum_{j=1}^{N} \delta_{k_j+l_j,1}\delta_{\alpha_j,1}]
\]

\[= \exp[i\theta_{k\oplus l,\alpha}],
\]

where $\oplus$ is a bitwise XOR and is the additive operation in the index, making $[0,d-1]$ an additive group.

**Appendix D: Proof of Lemma 12**

Here, we prove Lemma 12 which connects the achievability of quantum-TPE with random diagonal circuits and the local permutation check problem.

**Proof (Lemma 12)** Let $RDC(\mathcal{I})$ be the probability measure of $RDC(\mathcal{I})$. We denote the averaged operators $\mathbb{E}_{D^2 \sim RDC(\mathcal{I})}[(DZ)^{\otimes_l}]$ and $\mathbb{E}_{D^2 \sim DZ}[(DZ)^{\otimes_l}]$ by $Q_Z$ and $P_Z$, respectively. There exists a projector $R_Z$ diagonal in the Pauli-Z basis such that $Q_Z = P_Z + R_Z$ because $Q_Z P_Z = P_Z Q_Z = P_Z$ and $Q_Z$ is a projector diagonal in the Pauli-Z basis. Denoting $H_N^{\otimes_l} Q_Z H_N^{\otimes_l}$ by $Q_X$, where $H_N := H^{\otimes N}$ is the Hadamard transformation on $N$ qubits, and similarly decomposing it into $P_X + R_X$ ($P_X := H_N^{\otimes_l} P_Z H_N^{\otimes_l}$ and $R_X := H_N^{\otimes_l} R_Z H_N^{\otimes_l}$), we have

\[
\|Q_Z Q_X Q_Z - P_0\|_\infty = \|P_Z P_X P_Z - P_0 + R_Z P_X P_Z + Q_Z Q_X P_Z + Q_Z R_X P_Z + Q_Z R_X R_Z\|_\infty
\]

\[
\leq \|P_Z P_X P_Z - P_0\|_\infty + 2\|P_X R_Z\|_\infty + \|R_X P_Z\|_\infty + \|R_X R_Z\|_\infty
\]

\[
\leq \eta + 2\|P_X R_Z\|_\infty + \|R_X P_Z\|_\infty + \|R_X R_Z\|_\infty,
\]

where we used Theorem 9 in the last line.
We denote by \( W_Z \) a set of \((k_1, k_2) \in \mathcal{N} \times \mathcal{N}\) such that \( \langle k_1, k_2 \rangle R_Z |k_1, k_2\rangle = 1 \). Using an upper bound of the operator norm by the row norm and using the fact that \( |\langle l_1, l_2 \rangle P_X |k_1, k_2\rangle| = (\text{tr} P_X)/2^{2N} \leq t!/2^{tN} \) for any \((k_1, k_2)\) and \((l_1, l_2)\), we obtain

\[
\| R_X P_Z \|_\infty = \| P_X R_Z \|_\infty \leq \max_{(l_1, l_2) \in W_Z} \sum_{(k_1, k_2) \in W_Z} |\langle l_1, l_2 \rangle P_X |k_1, k_2\rangle| \leq \frac{t!}{2^{tN}} |W_Z|.
\]

Similarly, we have

\[
\| R_X R_Z \|_\infty \leq \max_{(l_1, l_2) \in W_Z} \sum_{(k_1, k_2) \in W_Z} |\langle l_1, l_2 \rangle R_X |k_1, k_2\rangle| \leq \left( \frac{|W_Z|}{2^{tN}} \right)^2.
\]

Substituting Eqs. (D4) and (D5) into Eq. (D3), we obtain

\[
\| Q_Z Q_X Q_Z - P_0 \|_\infty \leq \eta + 3t! \left( \frac{|W_Z|}{2^{tN}} \right)^2 + \left( \frac{|W_Z|}{2^{tN}} \right)^2.
\]

We finally show that \(|W_Z| = \Lambda(I)\). Note that \( \Lambda(I) \) is the number of \((K, K') \in \{0, 1\}^t N \times \{0, 1\}^t N\) such that \( K \) is not a row permutation but is an \( I \)-local permutation of \( K' \). We first express each \( k_s \in k \) in binary such as \( k_s = k_{s1} \cdots k_{sN} \) and define a \( t \times N \) matrix \( K \) with elements in \( \{0, 1\} \) corresponding to \( k \),

\[
K := \begin{pmatrix}
k_{11} & k_{12} & \cdots & k_{1N} \\
: & : & \ddots & : \\
k_{t1} & k_{t2} & \cdots & k_{tN}
\end{pmatrix},
\]

where \( k_{sm} \in \{0, 1\} \). Using this notation and noting that the \( Z \)-basis is real, the state \(|k, k'\rangle\) is expressed as \(|K, K'\rangle\). A random diagonal gate in \( RDC(I) \) applied on qubits in \( I \in I \) corresponds to, after taking the tensor product and the average, an projector onto span\(|K, K'\rangle : \Omega(K_I) = \Omega(K'_I)\rangle\), where \( \Omega \) is a canonical map that rearranges \( |I| \)-bit sequences \( \{K_s,I\}_{s \in [1,t]} \) in ascending order. Thus, we have

\[
\langle K, K' | Q_Z | K, K' \rangle = \begin{cases} 
1 & \text{if } \forall I \in I, \ \Omega(K_I) = \Omega(K'_I), \\
0 & \text{otherwise}.
\end{cases}
\]

Note that the off-diagonal elements of \( Q_Z \) is always zero because it is diagonal in the \( Z \) basis. We also have

\[
\langle K, K' | P_Z | K, K' \rangle = \begin{cases} 
1 & \text{if } K \text{ is a row permutation of } K', \\
0 & \text{otherwise}.
\end{cases}
\]

From these two equations, it is clear that \( R_Z = Q_Z - P_Z \) satisfies that \( \langle K, K' | R_Z | K, K' \rangle = 1 \) if and only if \( K \) is not a row permutation but is an \( I \)-local permutation of \( K' \). Otherwise \( \langle K, K' | R_Z | K, K' \rangle = 0 \). This implies \(|W_Z| = \Lambda(I)\).

**Appendix E: Proof of Lemma 20**

Here, we prove Lemma 20.
**Proof** (Lemma 20) Let $i \in [1, t]$ and $\vec{e}_i$ be a vector with elements in $\{0, 1\}$ where only the $i$th element is 1:

$$\vec{e}_i = (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0)^T. \quad (E1)$$

Then, for any $i$, there exists a vector $\vec{v}_i \in \{0, 1\}^t$ such that both $\vec{v}_i$ and $\vec{v}_i + \vec{e}_i$ are contained in $S$. This is for the following reason: if there is no such pair of $\vec{v}_i$ and $\vec{v}_i + \vec{e}_i$, it implies that a pair of vectors, which have different values only at the $i$th element, is not contained in $S$. This results in $|S| \leq 2^{t-1}$, which is in contradiction to the assumption that $|S| > 2^{t-1}$.

As $\vec{v}_i + \vec{e}_i \in S \subset \{0, 1\}^t$, the $i$th element of $\vec{v}_i$ is 0. Hence, we have $\vec{v}_i \cdot \vec{e}_i = 0$, implying that

$$O\vec{e}_i \cdot O(\vec{v}_i + \vec{e}_i) = \vec{e}_i \cdot \vec{v}_i + \vec{e}_i \cdot \vec{e}_i = \vec{e}_i \cdot \vec{e}_i = 1. \quad (E2)$$

It is also trivial that $O\vec{e}_i \cdot O\vec{e}_i = 1$ and that $O\vec{e}_i \in \{-1, 0, 1\}^t$, which follows from an identity $O\vec{e}_i = O(\vec{v}_i + \vec{e}_i) - O\vec{v}_i$ and a fact that both $O(\vec{v}_i + \vec{e}_i)$ and $O\vec{v}_i$ are in $\{0, 1\}^t$. From these three relations and again $O(\vec{v}_i + \vec{e}_i) \in \{0, 1\}^t$, we conclude

$$O\vec{e}_i = \vec{e}_j = (0, \ldots, 0, 1, 0, \ldots, 0)^T \quad (E3)$$

for some $j \in [1, t]$. Because $O$ is invertible, this implies that $O$ is a permutations matrix. \qed