An Improvement of Cover/El Gamal’s Compress-and-Forward Relay Scheme

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Abstract

The compress-and-forward relay scheme developed by (Cover and El Gamal, 1979) is improved with a modification on the decoding process. The improvement follows as a result of realizing that it is not necessary for the destination to decode the compressed observation of the relay; and even if the compressed observation is to be decoded, it can be more easily done by joint decoding with the original message, rather than in a successive way. An extension to multiple relays is also discussed.

I. Introduction

The relay channel, originally proposed in [1], models a communication scenario where there is a relay node that can help the information transmission between the source and the destination, as shown in Fig. 1. Two fundamentally different relay strategies were developed in [2], which, depending on whether the relay decodes the information or not, are generally known as decode-and-forward and compress-and-forward respectively. The compress-and-forward relay strategy is used when the relay cannot decode the message sent by the source, but still can help by compressing and forwarding its observation to the destination.

![Fig. 1. The relay channel.](image-url)
In the compress-and-forward coding scheme developed in [2], the relay first compresses its observation $Y_1$ into $\hat{Y}_1$, and then forwards this compressed version to the destination via $X_1$. This compression is generally necessary since the destination may not be able to completely recover $Y_1$. Instead, the compressed version $\hat{Y}_1$ can be recovered, as long as the following constraint is satisfied:

$$I(X_1; Y) > I(Y_1; \hat{Y}_1|X_1, Y).$$

(1)

Then, based on $\hat{Y}_1$ and $Y$, the destination can decode the original message $X$ if the rate

$$R < I(X; \hat{Y}_1, Y|X_1).$$

(2)

In this paper, we propose a modification of this compress-and-forward coding scheme by realizing that it is not necessary to recover $\hat{Y}_1$ since the original problem is to decode $X$ only; and even if $\hat{Y}_1$ is to be decoded, it can be done by jointly decoding $\hat{Y}_1$ and $X$, instead of successively decoding $\hat{Y}_1$ and then $X$.

We will show that without decoding $\hat{Y}_1$, the constraint (1) is not needed, and the achievable rate is more generally given by

$$R < I(X; \hat{Y}_1, Y|X_1) - \max\{0, I(Y_1; \hat{Y}_1|X_1, Y) - I(X_1; Y)\}. $$

(3)

Obviously, any rate satisfying (1)-(2) also satisfies (3). However, it remains a question whether there are interesting channel models where (3) is strictly larger than (1)-(2). This problem will not be addressed here. Instead, we point out an immediate advantage of (3) over (1)-(2). For (1)-(2), the relay needs to know the value of $I(Y_1; \hat{Y}_1|X_1, Y)$ in order to decide on the appropriate compressed version $\hat{Y}_1$ to choose. This requires the knowledge of the channel dynamics from $X$ to $Y$, which may be difficult to obtain for the relay, e.g., in wireless communications. However, this is not necessary for (3), where the relay can choose any version $\hat{Y}_1$ that is sufficiently close to $Y_1$, since $\hat{Y}_1$ is not to be decoded.

What if we also want to decode $\hat{Y}_1$? It turns out that by jointly decoding $\hat{Y}_1$ and $X$, the constraint (1) is not necessary; instead, we need a less strict inequality as the following:

$$I(X_1; Y) > I(Y_1; \hat{Y}_1|X_1, Y, X)$$

(4)

where, obviously, the difference from (1) is the additional information provided by $X$. 

II. THE SINGLE RELAY CASE

Formally, the single-relay channel depicted in Fig. 1 can be denoted by

\[(\mathcal{X} \times \mathcal{X}_1, p(y, y_1|x, x_1), \mathcal{Y} \times \mathcal{Y}_1)\]

where, \(\mathcal{X}\) and \(\mathcal{X}_1\) are the transmitter alphabets of the source and the relay respectively, \(\mathcal{Y}\) and \(\mathcal{Y}_1\) are the receiver alphabets of the destination and the relay respectively, and a collection of probability distributions \(p(\cdot, \cdot|x, x_1)\) on \(\mathcal{Y} \times \mathcal{Y}_1\), one for each \((x, x_1) \in \mathcal{X} \times \mathcal{X}_1\). The interpretation is that \(x\) is the input to the channel from the source, \(y\) is the output of the channel to the destination, and \(y_1\) is the output received by the relay. The relay sends an input \(x_1\) based on what it has received:

\[x_1(t) = f_t(y_1(t-1), y_1(t-2), \ldots), \quad \text{for every time } t, \tag{5}\]

where \(f_t(\cdot)\) can be any causal function. Note that a one-step time delay is assumed in (5) to account for the signal processing time at the relay.

Theorem 2.1: For the single-relay channel depicted in Fig. 1 by the modified compress-and-forward coding scheme, a rate \(R\) is achievable if it satisfies

\[R < I(X; \hat{Y}_1, Y|X_1) - \max\{0, I(Y_1; \hat{Y}_1|X_1, Y) - I(X_1; Y)\} \tag{6}\]

for some \(p(x)p(x_1)p(y_1|y_1, x_1)\). In addition, the compressed version \(\hat{Y}_1\) can be decoded if

\[I(X_1; Y) > I(Y_1; \hat{Y}_1|X_1, Y, X). \tag{7}\]

In the modified scheme, the codebook generation and encoding process is exactly the same as that in the proof of Theorem 6 of [2]. The modification is only on the decoding process at the destination: i) The destination finds the unique \(X\) sequence that is jointly typical with the \(Y\) sequence received, and also with a \(\hat{Y}_1\) sequence from the specific bin sent by the relay via \(X_1\); ii) If the \(\hat{Y}_1\) sequence is to be decoded, the destination finds the unique pair of \(X\) sequence and \(\hat{Y}_1\) sequence from the specific bin that are jointly typical with the \(Y\) sequence received.

III. EXTENSION TO MULTIPLE RELAYS

An extension of Cover/El Gamal’s compress-and-forward coding scheme to multiple relays was presented in [3]. We can also extend the modified scheme to multiple relays.

A multiple-relay channel is depicted in Fig. 2 which can be denoted by

\[(\mathcal{X} \times \mathcal{X}_1 \times \cdots \times \mathcal{X}_n, p(y, y_1, \ldots, y_n|x, x_1, \ldots, x_n), \mathcal{Y} \times \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n)\]
where, $\mathcal{X}, \mathcal{X}_1, \ldots, \mathcal{X}_n$ are the transmitter alphabets of the source and the relays respectively, 
$\mathcal{Y}, \mathcal{Y}_1, \ldots, \mathcal{Y}_n$ are the receiver alphabets of the destination and the relays respectively, and a 
collection of probability distributions $p(\cdot, \cdot, \ldots, | x, x_1, \ldots, x_n)$ on $\mathcal{Y} \times \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$, one for 
each $(x, x_1, \ldots, x_n) \in \mathcal{X} \times \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$. The interpretation is that $x$ is the input to the channel 
from the source, $y$ is the output of the channel to the destination, and $y_i$ is the output received 
by the $i$-th relay. The $i$-th relay sends an input $x_i$ based on what it has received:

$$x_i(t) = f_{i,t}(y_i(t-1), y_i(t-2), \ldots), \quad \text{for every time } t,$$

where $f_{i,t}(\cdot)$ can be any causal function.

![Fig. 2. A multiple-relay channel.](image)

Before presenting the achievable result, we introduce some simplified notations. Denote the 
set $\mathcal{N} = \{1, 2, \ldots, n\}$, and for any subset $S \subseteq \mathcal{N}$, let $X_S = \{X_i, i \in S\}$, and use similar 
notations for other variables. We have the following achievable result.

**Theorem 3.1:** For the multiple-relay channel depicted in Fig. 2 by the modified compress-
and-forward coding scheme, a rate $R$ is achievable if for some 

$$p(x)p(x_1) \cdots p(x_n)p(y_1|x_1) \cdots p(y_n|x_n),$$

there exists a rate vector $\{R_i, i = 1, \ldots, n\}$ satisfying 

$$\sum_{i \in S_1} R_i < I(X_{S_1}; Y|X_{S_1^c})$$

for any subset $S_1 \subseteq \mathcal{N}$, such that for any subset $S \subseteq \mathcal{N}$,

$$R < I(X; \hat{Y}_N, Y|X_N) - H(\hat{Y}_S|\hat{Y}_{S^c}, Y, X_N) + \sum_{i \in S} H(\hat{Y}_i|Y_i, X_i) + \sum_{i \in S} R_i.$$  

In addition, a subset of the compressed version $\hat{Y}_D$ for some $D \subseteq \mathcal{N}$ can be decoded, if for any 
$S \subseteq \mathcal{N}$ with $S \cap D \neq \emptyset$,

$$H(\hat{Y}_S|\hat{Y}_{S^c}, Y, X_N) - \sum_{i \in S} H(\hat{Y}_i|Y_i, X_i) < \sum_{i \in S} R_i.$$
It is easy to check that Theorem 3.1 implies Theorem 2.1 by noting the Markov Chain 
\((X, Y) \rightarrow (X_1, Y_1) \rightarrow \hat{Y}_1.\)

IV. FURTHER IMPROVEMENT

Furthermore, we can even consider joint decoding with \(X_N.\) Then the constraint (9) is not necessary for the decoding of \(X_N,\) with the help of \(X\) and \(\hat{Y}_N\) from the previous block. For this, we have the following achievability result.

**Theorem 4.1:** For the multiple-relay channel depicted in Fig. 2, a rate \(R\) is achievable if for some 
\[ p(x)p(x_1) \cdots p(x_n)p(\hat{y}_1|y_1, x_1) \cdots p(\hat{y}_n|y_n, x_n), \]
there exists a rate vector \(\{R_i, i = 1, \ldots, n\}\) such that for any \(S_1 \subseteq S \subseteq N,\)
\[ R < I(X; \hat{Y}_N, Y|X_N) - H(\hat{Y}_S|\hat{Y}_{S^c}, Y, X_N) + \sum_{i \in S} H(\hat{Y}_i|Y_i, X_i) + \sum_{i \in S \setminus S_1} R_i + I(X_{S_1}; Y|X_{S_1^c}) \quad (12) \]
and
\[ H(\hat{Y}_S|\hat{Y}_{S^c}, Y, X, X_N) - \sum_{i \in S} H(\hat{Y}_i|Y_i, X_i) - \sum_{i \in S \setminus S_1} R_i - I(X_{S_1}; Y|X_{S_1^c}) < 0. \quad (13) \]

In addition, a subset of the compressed version \(\hat{Y}_D\) for some \(D \subseteq N\) can be decoded, if for any 
\(S \subseteq N\) with \(S \cap D \neq \emptyset,\)
\[ H(\hat{Y}_S|\hat{Y}_{S^c}, Y, X, X_N) - \sum_{i \in S} H(\hat{Y}_i|Y_i, X_i) < \sum_{i \in S} R_i. \quad (14) \]

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