Parasupersymmetry and $\mathcal{N}$-fold Supersymmetry in Quantum Many-Body Systems I. General Formalism and Second Order

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Abstract

We propose an elegant formulation of parafermionic algebra and parasupersymmetry of arbitrary order in quantum many-body systems without recourse to any specific matrix representation of parafermionic operators and any kind of deformed algebra. Within our formulation, we show generically that every parasupersymmetric quantum system of order $p$ consists of $\mathcal{N}$-fold supersymmetric pairs with $\mathcal{N} \leq p$ and thus has weak quasi-solvability and isospectral property. We also propose a new type of non-linear supersymmetries, called quasi-parasupersymmetry, which is less restrictive than parasupersymmetry and is different from $\mathcal{N}$-fold supersymmetry even in one-body systems though the conserved charges are represented by higher-order linear differential operators. To illustrate how our formulation works, we construct second-order parafermionic algebra and three simple examples of parasupersymmetric quantum systems of order 2, one is essentially equivalent to the one-body Rubakov–Spiridonov type and the others are two-body systems in which two supersymmetries are folded. In particular, we show that the first model admits a generalized 2-fold superalgebra.

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I. INTRODUCTION

Concept of symmetry has played a central role in the development of modern theoretical physics and mathematical science. It may be almost certain that there is an underlying symmetry if a system under consideration exhibits a significant property that is not shared in general cases. Thus, a discovery of a new symmetry enlarges our ability and possibility to describe new phenomena both in the physical nature and mathematical models. Even in the case when some physical requirements prohibit any physically relevant model which has a certain kind of symmetry, it can motivate us to consider another kind of symmetry. For instance, the no-go theorem by Coleman and Mandula [1], which explains the failure of earlier attempts to unify the space-time Poincaré symmetry and the approximate flavor symmetry within a larger Lie algebra, promoted the study of supersymmetric theories. Although the latter attempts arrived at another no-go theorem shown by Haag, Lopuszański, and Sohnius [2], so far there has been, to the best of our knowledge, no other no-go theorems which can be applicable to any kind of symmetry outside Lie superalgebra. Indeed, this fact has motivated to investigate novel field theoretical models with new kind of symmetry such as weak supersymmetry [3], cubic supersymmetry [4, 5], and so on.

Generally speaking, however, it is extremely difficult to construct such a new quantum field theory in higher-dimensional space-time. But if some new symmetry can be realized in higher-dimensional models, we can always consider it in one space-time dimension, namely, in quantum mechanical models. It means that if any quantum mechanical systems cannot admit a certain symmetry, nor can quantum field theoretical models (except for such a kind of symmetry which becomes trivial only in one dimension). Hence, quantum mechanical models provide a good touchstone to examine whether some symmetry has the possibility to be realized in higher-dimensional theories. Furthermore, they also provide a toy model to investigate non-trivial aspects which one can hardly do in field theoretical models. In fact, the latter was the reason why Witten introduced supersymmetric quantum mechanics in order to acquire insight into non-perturbative aspects of supersymmetric quantum field theory [6, 7]. We note that this strategy was also employed in the study of weak supersymmetry and it was shown that some field theoretical models with weak supersymmetry in one dimension reduces to $N$-fold supersymmetric quantum systems with $N = 2$ [3].

The research field of $N$-fold supersymmetry in quantum mechanics was initiated as a naive higher-derivative generalization of the representation of supercharges [8]. Later, its true appreciation in connection with supersymmetric quantum field theory was given by the proof of the equivalence to (weak) quasi-solvability [9]. That is, it was shown that quasi-solvability, which is a less restrictive concept than quasi-exact solvability [10, 11], is a one-dimensional analog and, in a sense, a generalization of the perturbative non-renormalization theorems in supersymmetric quantum field theory. In fact, this connection naturally explains the disappearance of leading divergence of perturbation series not only for the ground state but also for a finite number of excited states of $N$-fold supersymmetric quantum models, irrespective of whether $N$-fold supersymmetry is dynamically broken or not [12, 13]. We note that it can provide an implication in higher-dimensional theories; the connection between weak and $N$-fold supersymmetries implies that characteristic aspects of weak-supersymmetric quantum field theory, if exists, have much intimate relation to those of $N$-fold supersymmetric quantum mechanics. Thus, we can again recognize the importance of investigating symmetry in quantum mechanical systems.

In this article, we would like to focus on parasupersymmetry. It is, roughly speaking,
symmetry between bosons and parafermions, and is first proposed in one-body quantum
mechanics [14]. However, its characteristic features have been less understood than those
of $\mathcal{N}$-fold supersymmetry. One of the reasons stems from the fact that parasupersymmetric
quantum mechanics has been usually formulated in terms of matrix representations of
parafermions [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33,
34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52]. Once a spe-
cific matrix representation is introduced, one can calculate any product of parafermions
which is not defined in the original parafermionic algebra. As a result, it is difficult to
appreciate which part of the results is generic and which part of the results depends on
a specific choice of representations. Another reason is that the mathematical relations
among parasuperalgebras and various types of deformed oscillator algebras including $q$-
deformed ones have attracted much more attention in the recent development of the re-
search field [53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63] (see also the references cited in
Refs. [61, 63]).

Considering the situations described above, we first propose an elegant formulation of
parafermionic algebra of arbitrary order without recourse to any specific representation of
parafermionic operators. It does not introduce any new type of operators which do not exist
in the fermionic case and does not involve any kind of deformed algebra. With the aid of
solely this algebra, we then formulate parasupersymmetric quantum many-body systems
of arbitrary order in a generic way and then derive the general aspects of parasupersymmetry.
In particular, we show generically that every parasupersymmetric quantum systems of order
$p$ consists of $\mathcal{N}$-fold supersymmetric pairs with $\mathcal{N} \leq p$. From this result, we propose a
generalization of parasupersymmetry which we would call quasi-parasupersymmetry. Then
we examine the second-order case in detail.

We organize the article as follows. In the next section, we first define parafermionic op-
erators and a linear space on which they act. Then, we propose postulates of parafermionic
algebra. In Section III after reviewing the definition of parasupersymmetry, we formulate
in a generic way parasupersymmetry of arbitrary order in quantum many-body systems
in terms of the parafermionic operators just defined in Section II. Using the parafermionic
algebra, we derive the conditions for the systems under consideration to have parasupersym-
mety of arbitrary order in terms of component parasupercharges and Hamiltonians. From
them we show that every parasupersymmetric system consists of $\mathcal{N}$-fold supersymmetric
pairs with $\mathcal{N} \leq p$ and thus has weak quasi-solvability and isospectral property. Observing
these results, we propose a generalization of parasupersymmetry in Section IV. Then, to
see more explicitly how the general formalism developed in Sections II and III works, in
Section V we construct systematically parafermionic algebra of order 2 solely based on the
postulates proposed in Section II. With the aid of this parafermionic algebra of order 2, we
investigate second-order parasupersymmetric quantum systems and exhibit three simple ex-
amples of such systems in Section VI. Finally in the last section, we summarize and discuss
the results obtained in this paper and possible future issues.

II. PARAFERMICIONIC ALGEBRA

First of all, let us define parafermionic algebra of order $p (\in \mathbb{N})$. It is an associative algebra
composed of the identity operator $I$ and two parafermionic operators $\psi^-$ and $\psi^+$ of order $p$
which satisfy the nilpotency:

\[(\psi^-)^p \neq 0, \quad (\psi^+)^p \neq 0, \quad (\psi^-)^{p+1} = (\psi^+)^{p+1} = 0. \tag{2.1}\]

Hence, we immediately have \(2p + 1\) non-zero elements \(\{I, \psi^-, \ldots, (\psi^-)^p, \psi^+, \ldots, (\psi^+)^p\}\). We call them the fundamental elements of parafermionic algebra of order \(p\). Parafermionic algebra is characterized by anti-commutation relation \[A, B\} = AB + BA\ and commutation relation \([A, B] = AB - BA\ among the fundamental elements. When \(p = 1\), the algebra must reduce to the ordinary fermionic one:

\[\{\psi^-, \psi^-\} = \{\psi^+, \psi^+\} = 0, \quad \{\psi^-, \psi^+\} = I. \tag{2.2}\]

The nilpotency \(2.1\) for arbitrary \(p\) is a trivial generalization of the first fermionic algebra in Eq. \(2.2\). Thus, the next problem is how to generalize the second relation in Eq. \(2.2\). In this paper, we propose the following relation as a generalization of it to arbitrary order \(p\):

\[\{\psi^-, \psi^+\} + \{(\psi^-)^2, (\psi^+)^2\} + \cdots + \{(\psi^-)^p, (\psi^+)^p\} = pI. \tag{2.3}\]

Before proceeding further with defining the algebra, we shall next define parafermionic Fock spaces \(V_p\) of order \(p\) on which the parafermionic operators act. The latter space is \((p + 1)\) dimensional and its \(p + 1\) bases \(|k\rangle\ (k = 0, \ldots, p)\) are defined by

\[\psi^-|0\rangle = 0, \quad |k\rangle = (\psi^+)^k|0\rangle, \quad \psi^-|k\rangle = |k-1\rangle \quad (k = 1, \ldots, p). \tag{2.4}\]

That is, \(\psi^-\) and \(\psi^+\) act as annihilation and creation operators of parafermions, respectively. The state \(|0\rangle\) is called the parafermionic vacuum. The subspace spanned by each state \(|k\rangle\ (k = 0, \ldots, p)\) is called the \(k\)-parafermionic subspace and is denoted by \(V_p^{(k)}\). The adjoint vector \(|k\rangle\) of each \(|k\rangle\) is introduced as a linear operator which maps every vector in \(V_p\) into a complex number as follows:

\[\langle k|l\rangle = \langle 0|(\psi^-)^k|l\rangle, \quad \langle 0|l\rangle = \langle 0|(\psi^+)^l|0\rangle = \delta_{0,l} \quad (k, l = 0, \ldots, p). \tag{2.5}\]

By the definitions \(2.4\) and \(2.5\), we immediately have a bi-orthogonal relation:

\[\langle k|l\rangle = \delta_{k,l} \quad (k, l = 0, \ldots, p). \tag{2.6}\]

We can now define a set of projection operators \(\Pi_k : V_p \rightarrow V_p^{(k)}\ (k = 0, \ldots, p)\) which satisfy

\[\Pi_k|l\rangle = \delta_{k,l}|k\rangle, \quad \Pi_k \Pi_l = \delta_{k,l}\Pi_k, \quad \sum_{k=0}^{p} \Pi_k = I. \tag{2.7}\]

When \(p = 1\), the projection operators are given in terms of the fermionic operators by

\[\Pi_0 = \psi^-\psi^+, \quad \Pi_1 = \psi^+\psi^- \tag{2.8}\]

At this stage, however, we do not know how the projection operators are expressed in terms of the parafermionic operators for arbitrary order \(p > 1\). From the definitions \(2.4\) and \(2.7\),

\[\Pi_k\psi^+|l\rangle = \Pi_k|l + 1\rangle = \delta_{k,l+1}|k\rangle, \quad \psi^+\Pi_k|l\rangle = \delta_{k,l}\psi^+|k\rangle = \delta_{k,l}|k + 1\rangle, \quad \psi^-\Pi_k|l\rangle = \delta_{k,l}\psi^-|k\rangle = \delta_{k,l}|k - 1\rangle, \tag{2.9}\]
and we obtain
\[ \Pi_{k+1} \psi^+ = \psi^+ \Pi_k, \] (2.9)
where and hereafter we put \( \Pi_k \equiv 0 \) for all \( k < 0 \) and \( k > p \). Similarly, from the definitions (2.4) and (2.7), we obtain
\[ \psi^- \Pi_{k+1} = \Pi_k \psi^- . \] (2.10)

We now come back to the parafermionic algebra. Apparently, the relations (2.1) and (2.3) are not sufficient for the determination of the full algebra. To determine other multiplication relations we impose the following postulates:

i. First, the algebra must be consistent with Eq. (2.4). This requirement is indispensable for defining consistently the parafermionic Fock space \( V_p \). When \( p = 1 \), for instance, we have from the fermionic algebra (2.2)
\[ \psi^- |1\rangle = \psi^- \psi^+ |0\rangle = (I - \psi^+ \psi^-) |0\rangle = |0\rangle, \]
thus the algebra is indeed consistent with Eq. (2.4).

ii. Every projection operator \( \Pi_k (k = 0, \ldots, p) \) can be expressed as a polynomial of the fundamental elements of the corresponding order \( p \) so that the algebra is consistent with the definition (2.7).

iii. Every product of three fundamental elements can be expressed as a polynomial of at most second-degree in the fundamental elements. These formulas are called the multiplication law. For example, the multiplication law for the fermion operators (\( p = 1 \)) is given by
\[ \psi^- \psi^+ \psi^- = \psi^- , \quad \psi^+ \psi^- \psi^+ = \psi^+. \] (2.11)
We assume that the relations in Eq. (2.11) hold for parafermionic operators of any order \( p \). As a consequence of this assumption, we immediately obtain for all \( m, n \in \mathbb{N} \)
\[ (\psi^-)^m \psi^+ (\psi^-)^n = (\psi^-)^{m+n-1} , \quad (\psi^+)^m \psi^- (\psi^+)^n = (\psi^+)^{m+n-1}, \] (2.12)
which also hold for arbitrary order.

iv. We also assume that the following relations hold for arbitrary order:
\[ (\psi^-)^p \psi^+ \Pi_{p-1} = (\psi^-)^{p-1} \Pi_{p-1} , \quad \psi^+ (\psi^-)^p \Pi_p = (\psi^-)^{p-1} \Pi_p . \] (2.13)
In the case of \( p = 1 \), they are trivial; \( \psi^- \psi^+ \Pi_0 = \Pi_0 \) and \( \psi^+ \psi^- \Pi_1 = \Pi_1 \). As we will see later, the assumptions (2.11) and (2.13) turn to be crucial for a parasupersymmetric condition to make sense in quantum systems.

We note that every polynomial composed of the fundamental elements can be reduced to a polynomial of at most second-degree in them as a consequence of the third postulate and the associativity. Hence, together with the second postulate it means in particular every projection operators must be expressed as a polynomial of second-degree in the fundamental elements.

Finally, we introduce the quantity of parafermionic degree of operators as follows:
\[ \deg I = 0 , \quad \deg \psi^+ = 1 , \quad \deg \psi^- = p , \] (2.14)
\[ \deg AB \equiv \deg A + \deg B \quad (\text{mod} \ p + 1) . \] (2.15)
For example, \( \deg (\psi^+)^k = k \) and \( \deg (\psi^-)^k = p + 1 - k \ (k = 1, \ldots, p) \).
III. PARASUPERSYMMETRY

Parasupersymmetry of order 2 in quantum mechanics was first introduced by Rubakov and Spiridonov [14] and was later generalized to arbitrary order independently by Tomiya [35] and by Khare [36]. A different formulation for order 2 was proposed by Beckers and Debergh [19] and a generalization of the latter to arbitrary order was attempted by Chenaghlou and Fakhri [52]. Thus, we call them RSTK and BDCF formalism, respectively.

To define a $p^{th}$-order parasupersymmetric system, we first introduce a pair of parasupercharges $Q^\pm$ of order $p$ which satisfy

\[(Q^-)^p \neq 0, \quad (Q^+)^p \neq 0, \quad (Q^-)^{p+1} = (Q^+)^{p+1} = 0.\]  

A system $H$ is said to have parasupersymmetry of order $p$ if it commutes with the parasupercaracteres of order $p$

\[[Q^-, H] = [Q^+, H] = 0,\]  

and satisfies the non-linear relations in the RSTK formalism

\[\sum_{k=0}^{p} (Q^-)^{p-k} Q^+ (Q^-)^{k} = C_p (Q^-)^{p-1} H, \quad \sum_{k=0}^{p} (Q^+)^{p-k} Q^- (Q^+)^{k} = C_p H (Q^+)^{p-1},\]  

or in the BDCF formalism

\[\begin{align*}
\underbrace{[Q^-, \ldots, [Q^-, Q^-]]}_{(p-1) \text{ times}} \ldots &= (-1)^p C_p (Q^-)^{p-1} H, \\
\underbrace{[Q^+, \ldots, [Q^+, Q^+]]}_{(p-1) \text{ times}} \ldots &= C_p H (Q^+)^{p-1},
\end{align*}\]  

where $C_p$ is a constant. An apparent drawback of the BDCF formalism is that the relations (3.4) do not reduce to the ordinary supersymmetric anti-commutation relation $\{Q^-, Q^+\} = C_1 H$ when $p = 1$, in contrast to the RSTK relation (3.3). For this reason, we discard the BDCF formalism in this paper though its defect may be amended by, e.g., replacing all the commutators in (3.4) by anti-commutators, graded commutators $[A, B] = AB - (-1)^{\deg A \cdot \deg B} BA$, and so on.

An immediate consequence of the commutativity (3.2) is that each $n^{th}$-power of the parasupercaracteres $(2 \leq n \leq p)$ also commutes with the system $H$

\[[(Q^-)^n, H] = [(Q^+)^n, H] = 0 \quad (2 \leq n \leq p).\]  

Hence, every parasupersymmetric system $H$ satisfying (3.2) always has $2p$ conserved charges.

To realize parasupersymmetry in quantum mechanical systems, we usually consider a vector space $\mathfrak{F} \times V_p$ where $\mathfrak{F}$ is a linear space of complex functions such as the Hilbert space $L^2$ in Hermitian quantum theory and the Krein space $L^2_\mathcal{P}$ in $\mathcal{P}\mathcal{T}$-symmetric quantum theory [65, 66]. A parafermionic quantum system $H$ is introduced by

\[H = \sum_{k=0}^{p} H_k \Pi_k,\]  

where $H_k$ are the parasupercaracteres of order $p$.
where $H_k (k = 0, \ldots, p)$ are scalar Hamiltonians of $p$ variables acting on $\mathfrak{F}$:

$$H_k = -\frac{1}{2} \sum_{i=1}^{p} \frac{\partial^2}{\partial q_i^2} + V_k(q_1, \cdots, q_p) \quad (k = 0, \ldots, p). \quad (3.7)$$

Two parasupercharges $Q^\pm$ are defined by

$$Q^- = \sum_{k=0}^{p} Q^-_k \psi^+ \Pi_k, \quad Q^+ = \sum_{k=0}^{p} Q^+_k \Pi_k \psi^+, \quad (3.8)$$

where $Q^+_k (k = 0, \ldots, p)$ are first-order linear operators acting on $\mathfrak{F}$

$$Q^+_k = \sum_{i=1}^{p} w_{k,i}(q_1, \ldots, q_p) \frac{\partial}{\partial q_i} + W_k(q_1, \ldots, q_p) \quad (k = 0, \ldots, p), \quad (3.9)$$

and for each $k$, $Q^-_k$ is given by a certain ‘adjoint’ of $Q^+_k$, e.g., the (ordinary) adjoint $Q^-_k = (Q^+_k)^\dagger$ in the Hilbert space $L^2$, the $P$-adjoint $Q^-_k = P(Q^+_k)^\dagger P$ in the Krein space $L^2_P$, and so on. For all $k \leq 0$ and $k > p$ we put $Q^\pm_k \equiv 0$. When $p = 1$, the triple $(H, Q^-, Q^+)$ defined in Eqs. (3.6) and (3.8) becomes

$$H = \sum_{k=0}^{1} H_k \Pi_k = H_0 \psi^- \psi^+ + H_1 \psi^+ \psi^-, \quad (3.10)$$

$$Q^- = \sum_{k=0}^{1} Q^-_k \psi^- \Pi_k = Q^-_1 \psi^-, \quad (3.11)$$

$$Q^+ = \sum_{k=0}^{1} Q^+_k \Pi_k \psi^+ = Q^+_1 \psi^+, \quad (3.12)$$

where Eqs. (2.8) and (2.11) are used, and thus reduces to an ordinary supersymmetric quantum mechanical system [3]. The non-linear relation (3.3) together with the nilpotency (3.1) for $p = 1$ are just the anti-commutation relations between supercharges

$$\{Q^\pm, Q^\pm\} = 0, \quad \{Q^-, Q^+\} = C_1 H. \quad (3.13)$$

Hence, the parasupersymmetric quantum systems defined by Eqs. (3.1)–(3.9) provide a natural generalization of ordinary supersymmetric quantum mechanics.

We next note that the parasupercharges $Q^\pm$ defined by Eq. (3.8) already satisfy the nilpotency (3.1). This is an immediate consequence of the following formulas:

$$(Q^-)^n = \sum_{k=0}^{p} Q^-_{k-n+1} \cdots Q^-_{k-1} Q^-_k (\psi^-)^n \Pi_k \quad (n \in \mathbb{N}), \quad (3.14)$$

$$(Q^+)^n = \sum_{k=0}^{p} Q^+_k Q^+_k \cdots Q^+_k (\psi^+)^n \Pi_k \quad (n \in \mathbb{N}). \quad (3.15)$$
These formulas are easily proved by induction. We first note that they are identical with the defining equations (3.8) of $Q^\pm$ when $n = 1$. Suppose, for instance, Eq. (3.14) holds for a natural number $n (< p)$. Then, using Eqs. (2.7) and (2.10) we have,

$$
(Q^-)^{n+1} = \sum_{k,l=0}^{p} Q_{l-n+1}^+ \cdots Q_{l-1}^+ Q_k^- (\psi^-)^n \Pi_l \psi^- \Pi_k
$$

$$
= \sum_{k,l=0}^{p} Q_{l-n+1}^- \cdots Q_{l-1}^- Q_k^+ (\psi^-)^{n+1} \Pi_l+1 \Pi_k
$$

$$
= \sum_{k=0}^{p} Q_{k-n}^- \cdots Q_{k-2}^- Q_{k-1}^+ (\psi^-)^{n+1} \Pi_k.
$$

(3.16)

The latter formula is nothing but Eq. (3.14) with $n$ replaced by $n + 1$. Thus we have completed the proof of the formula (3.14). In the same way, we can prove the other formula (3.15). Then, the nilpotency (3.1) immediately follows from Eq. (2.1) and thus is always guaranteed by the definition (3.8).

The commutation relations of $H$ in (3.6) and $Q^\pm$ in (3.8) are calculated with the aid of Eqs. (2.7), (2.9), and (2.10) as

$$
[Q^-, H] = \sum_{k,l=0}^{p} Q_k^- H_l \psi^- \Pi_k \Pi_l - \sum_{k,l=0}^{p} H_l Q_k^- \Pi_l \psi^- \Pi_k
$$

$$
= \sum_{k=0}^{p} Q_k^- H_k \psi^- \Pi_k - \sum_{k=0}^{p} H_{k-1} Q_k^- \psi^- \Pi_k,
$$

(3.17)

$$
[Q^+, H] = \sum_{k,l=0}^{p} Q_k^+ H_l \Pi_k \psi^+ \Pi_l - \sum_{k,l=0}^{p} H_l Q_k^+ \Pi_l \Pi_k \psi^+
$$

$$
= \sum_{k=0}^{p} Q_k^+ H_{k-1} \Pi_k \psi^+ - \sum_{k=0}^{p} H_k Q_k^+ \Pi_k \psi^+.
$$

(3.18)

Hence, the commutativity (3.2) is satisfied if and only if

$$
H_{k-1} Q_k^- = Q_k^- H_k, \quad Q_k^+ H_{k-1} = H_{k-1} Q_k^+, \quad \forall k = 1, \ldots, p.
$$

(3.19)

That is, each pair of $H_{k-1}$ and $H_k$ must satisfy the intertwining relations with respect to $Q_k^-$ and $Q_k^+$. Furthermore, the commutation relations between $(Q^\pm)^n$ and $H$ ($2 \leq n \leq p$) are similarly calculated by using Eqs. (2.7), (2.9), (2.10), (3.14), and (3.15) as

$$
[(Q^-)^n, H] = \sum_{k=0}^{p} Q_{k-n+1}^- \cdots Q_{k-1}^- Q_k^- H_k (\psi^-)^n \Pi_k
$$

$$
- \sum_{k=0}^{p} H_{k-n} Q_{k-n+1}^- \cdots Q_{k-1}^- (\psi^-)^n \Pi_k.
$$

(3.20)

$$
[(Q^+)^n, H] = \sum_{k=0}^{p} Q_k^+ Q_{k-1}^+ \cdots Q_{k-n+1}^+ H_{k-n} \Pi_k (\psi^+)^n
$$

$$
- \sum_{k=0}^{p} H_k Q_k^+ Q_{k-1}^+ \cdots Q_{k-n+1}^+ \Pi_k (\psi^+)^n.
$$

(3.21)
Hence, from the commutativity (3.5) any pair of \( H_{k-n} \) and \( H_k \) (\( 1 \leq n \leq k \leq p \)) satisfies

\[
H_{k-n} Q_{k-n+1}^{-} \cdots Q_{k-1}^{-} Q_k^- = Q_{k-n+1}^{-} \cdots Q_{k-1}^{-} Q_k^- H_k, \tag{3.22a}
\]

\[
Q_k^+ Q_{k-1}^+ \cdots Q_{k-n+1}^+ H_{k-n} = H_k Q_k^+ Q_{k-1}^+ \cdots Q_{k-n+1}^+, \tag{3.22b}
\]

which means that \( H_{k-n} \) and \( H_k \) constitute a pair of \( \mathcal{N} \)-fold supersymmetry with \( \mathcal{N} = n \). The relations (3.22) can be also derived by repeated applications of Eq. (3.19). Since \( \mathcal{N} \)-fold supersymmetry is essentially equivalent to weak quasi-solvability [68, 69], parasupersymmetric quantum systems also possess weak quasi-solvability. To see the structure of weak quasi-solvability in the parasupersymmetric system \( H \) more precisely, let us first define

\[
\mathcal{V}_{n,k}^- = \ker(Q_{k-n+1}^- \cdots Q_k^-), \quad \mathcal{V}_{n,k}^+ = \ker(Q_k^+ \cdots Q_{k-n+1}^+) \quad (1 \leq n \leq k \leq p). \tag{3.23}
\]

By the definition (3.23), the vector spaces \( \mathcal{V}_{n,k}^\pm \) for each fixed \( k \) are related as

\[
\mathcal{V}_{1,k}^- \subset \mathcal{V}_{2,k}^- \subset \cdots \subset \mathcal{V}_{k,k}^-; \quad \mathcal{V}_{1,k}^+ \subset \mathcal{V}_{2,k}^+ \subset \cdots \subset \mathcal{V}_{k,k}^+. \tag{3.24}
\]

On the other hand, it is evident from the intertwining relations (3.22) that each Hamiltonian \( H_k \) (\( 0 \leq k \leq p \)) preserves vector spaces as follows:

\[
H_k \mathcal{V}_{n,k}^- \subset \mathcal{V}_{n,k}^- \quad (1 \leq n \leq k), \tag{3.25a}
\]

\[
H_k \mathcal{V}_{n,k+n}^+ \subset \mathcal{V}_{n,k+n}^+ \quad (1 \leq n \leq p-k). \tag{3.25b}
\]

From Eqs. (3.24) and (3.25), the largest space preserved by each \( H_k \) (\( 0 \leq k \leq p \)) is given by

\[
\mathcal{V}_{k,k}^- + \mathcal{V}_{p-k,p}^+ \quad (0 \leq k \leq p). \tag{3.26}
\]

Needless to say, each Hamiltonian \( H_k \) preserves the two spaces in Eq. (3.26) separately. The intertwining relations (3.19) and (3.22) ensure that all the component Hamiltonians \( H_k \) (\( k = 0, \ldots, p \)) of the system \( H \) are isospectral outside the sectors \( \mathcal{V}_{n,k}^\pm \) (\( 1 \leq n \leq k \leq p \)). The spectral degeneracy of \( H \) in these sectors depends on the form of each component of the parasupercscharges, \( Q_k^\pm \) (\( k = 1, \ldots, p \)), and its structure can be very complicated even in the case of second-order, see e.g. Refs. [68, 69].

In addition to those ‘power-type’ symmetries, every parasupersymmetric quantum system \( H \) defined in Eq. (3.6) can have ‘discrete-type’ ones. To see it, we first recall the basic fact that for each operator \( O \) which commutes with the system \( H \), the operator defined by \([O, Q^\pm] \) also commutes with \( H \). It is an immediate consequence of the Jacobi identity

\[
[[O, Q^\pm], H] = -[[H, O], Q^\pm] - [[Q^\pm, H], O] = 0. \tag{3.27}
\]

Then, it follows from the intertwining relations (2.9) and (2.10) that every \( p \)-th order parasupersymmetric quantum system \( H \) commutes with \((\psi^-)^n(\psi^+)^n\) and \((\psi^+)^n(\psi^-)^n\) (\( n = 1, \cdots, p \)), respectively; for instance,

\[
[H, (\psi^-)^n(\psi^+)^n] = \sum_{k=0}^{p} H_k \Pi_k (\psi^-)^n(\psi^+)^n - \sum_{k=0}^{p} H_k (\psi^-)^n(\psi^+)^n \Pi_k
\]

\[
= \sum_{k=0}^{p-n} H_k (\psi^-)^n \Pi_{k+n}(\psi^+)^n - \sum_{k=0}^{p-n} H_k (\psi^-)^n \Pi_{k+n}(\psi^+)^n = 0. \tag{3.28}
\]
Hence, if we define
\[ Q^\pm_{(n)} = \{(\psi^-)^n, (\psi^+)^n\}, Q^\pm \], \quad Q^\pm_{[n]} = \{[(\psi^-)^n, (\psi^+)^n], Q^\pm \} \quad (n = 1, \ldots, p), \quad (3.29) \]
all of \( Q^\pm_{(n)} \) and \( Q^\pm_{[n]} \) commute with \( H \), cf. Eq. (3.27):
\[ [Q^\pm_{(n)}, H] = [Q^\pm_{[n]}, H] = 0 \quad (n = 1, \ldots, p). \quad (3.30) \]
Explicitly, those conserved charges are expressed as
\[ Q^-_{(n)} = \sum_{k=0}^{p} Q^-_k [(\psi^-)^n, (\psi^+)^n] \Pi_k, \quad Q^-_{[n]} = \sum_{k=0}^{p} Q^-_k [(\psi^-)^n, (\psi^+)^n], (\psi^-) \Pi_k, \quad (3.31a) \]
\[ Q^+_{(n)} = \sum_{k=0}^{p} Q^+_k \Pi_k [(\psi^-)^n, (\psi^+)^n], \quad Q^+_{[n]} = \sum_{k=0}^{p} Q^+_k \Pi_k [(\psi^-)^n, (\psi^+)^n]. \quad (3.31b) \]
We note, however, that they are in general not linearly independent and we cannot determine the number of linearly independent conserved charges without the knowledge of parafermionic algebra of each order. For \( n = 1 \), the conserved charges \( Q^-_{(1)} \) and \( Q^+_{(1)} \) admit simpler expressions thanks to the assumption (2.11):
\[ Q^-_{(1)} = \sum_{k=0}^{p} Q^-_k [\psi^+, (\psi^-)^2] \Pi_k, \quad Q^-_{[1]} = 2Q^- - \sum_{k=0}^{p} Q^-_k \{\psi^+, (\psi^-)^2\} \Pi_k, \quad (3.32a) \]
\[ Q^+_{(1)} = \sum_{k=0}^{p} Q^+_k \Pi_k [\psi^-, (\psi^+)^2], \quad Q^+_{[1]} = \sum_{k=0}^{p} Q^+_k \Pi_k \{\psi^-, (\psi^+)^2\} - 2Q^+. \quad (3.32b) \]
From these expressions, we obtain for \( p = 1 \)
\[ Q^\pm_{(1)} = 0, \quad Q^-_{[1]} = 2Q^-, \quad Q^+_{[1]} = -2Q^+, \quad (3.33) \]
and thus there are no new conserved charges in supersymmetric quantum mechanics. As we will later show in Section VII in the case of second-order parasupersymmetric quantum systems they exactly corresponds to the additional charges and symmetry reported in Ref. [18].
The non-linear relations (3.3) can be also calculated in a similar way. Using Eqs. (2.7), (2.9), (2.10), (3.6), (3.8), and (3.14), we obtain
\[ (Q^-)^p Q^+ = Q^-_{l} \cdots Q^-_{p} Q^+ (\psi^-)^p \psi^+ \Pi_{p-1}, \quad (3.34) \]
\[ (Q^-)^{p-k} Q^+ (Q^-)^k = \sum_{l=p-k}^{p} Q^-_{l-p+2} \cdots Q^-_{l-k+1} Q^+_{l-k} \]
\[ \times Q^-_{l-k+1} \cdots Q^-_{l} (\psi^-)^{p-k} \psi^+ (\psi^-)^2 \Pi_{l} \quad (1 \leq k \leq p - 1), \quad (3.35) \]
\[ Q^+ (Q^-)^p = Q^+_{l} Q^-_{l-p} \psi^+ (\psi^-)^p \Pi_{p}, \quad (3.36) \]
\[ (Q^-)^{p-1} H = \sum_{l=p-1}^{p} Q^-_{l-p+2} \cdots Q^-_{l} H_l (\psi^-)^{p-1} \Pi_{l} \quad (p \geq 2). \quad (3.37) \]
Employing the assumptions \((2.12)\) and \((2.13)\), we conclude that the first non-linear relation in Eq. \((3.3)\) is satisfied if and only if the following two identities hold:

\[
Q_1^- \cdots Q_p^- Q_p^+ + \sum_{k=1}^{p-1} Q_1^- \cdots Q_{p-k}^- Q_{p-k}^+ Q_{p-k}^- \cdots Q_{p-1}^- = C_p Q_1^- \cdots Q_{p-1}^- H_{p-1}, \tag{3.38a}
\]

\[
\sum_{k=1}^{p-1} Q_2^- \cdots Q_{p-k+1}^- Q_{p-k+1}^+ Q_{p-k+1}^- \cdots Q_p^- + Q_1^+ Q_1^- \cdots Q_p^- = C_p Q_2^- \cdots Q_p^- H_p. \tag{3.38b}
\]

The conditions for the second non-linear relation in Eq. \((3.3)\) are apparently given by the ‘adjoint’ of Eqs. \((3.38)\). Here we see the crucial role played by the formulas \((2.12)\) and \((2.13)\): without them, the non-linear condition \((3.3)\) would just lead to the conclusion that every component appeared in Eqs. \((3.34)\)–\((3.37)\) directly come from the commutation relation \((3.2)\), leads us to consider physical consequences. \((3.1)\)–\((3.3)\) are too strong to obtain various non-trivial models having several intriguing consequences.

IV. QUASI-PARASUPERSYMMETRY

In the previous section, we have found that every pair of the component Hamiltonians possesses \(\mathcal{N}\)-fold supersymmetry with \(\mathcal{N} \leq p\) and thus has isospectral property and weak quasi-solvability. However, \(\mathcal{N}\)-fold supersymmetric systems constructed from \(\mathcal{N}\) repeated applications of first-order intertwining relations as in the present case (cf. Eq. \((3.19)\)) are well known to be quite restrictive in comparison with general \(\mathcal{N}\)-fold supersymmetric systems (see Ref. \([70]\) in the case of type A \(\mathcal{N}\)-fold supersymmetry). This naturally explains the fact that almost all the parasupersymmetric potentials so far found are shape-invariant types (see e.g. Refs. \([25, 38, 41, 44]\)). In this sense, we can say the conditions of parasupersymmetry \((3.1)\)–\((3.3)\) are too strong to obtain various non-trivial models having several intriguing physical consequences.

The above observation, if we take into account the fact that the strict first-order intertwining relations \((3.19)\) directly come from the commutation relation \((3.2)\), leads us to consider a less restrictive condition as follows. With a given pair of parastocharges \(Q^\pm\) of order \(p\) which satisfy the nilpotency \((5.1)\), a system \(H\) is said to have quasi-parasupersymmetry of order \((p,q)\) if there exists a natural number \(q\) (\(1 \leq q \leq p\)) such that \((Q^-)^q\) commutes with \(H\) and the non-linear constraint \((3.3)\) is satisfied. That is, it is characterized by the following algebraic relations:

\[
(Q^-)^p \neq 0, \quad (Q^+)^p \neq 0, \quad (Q^-)^{p+1} = (Q^+)^{p+1} = 0, \tag{4.1}
\]

\[
[(Q^-)^q, H] = [(Q^+)^q, H] = 0 \quad (1 \leq q \leq p), \tag{4.2}
\]

\[
\sum_{k=0}^{p} (Q^-)^{p-k} Q^+ (Q^-)^k = C_p (Q^-)^{p-1} H, \quad \sum_{k=0}^{p} (Q^+)^{p-k} Q^- (Q^+)^k = C_p H (Q^+)^{p-1}. \tag{4.3}
\]

By definition, quasi-parasupersymmetry of order \((p,q)\) reduces to (ordinary) parasupersymmetry when \(q = 1\). Thus, it can be regarded as a generalization of parasupersymmetry. A key ingredient of this new symmetry is that the commutativity \([(Q^-)^n, H] = 0\) for \(n < q\) is not necessarily fulfilled in contrast to (ordinary) parasupersymmetry. As a consequence,
only the less restrictive \( q \)th-order intertwining relations \((3.22)\) with \( n = q \) should be satisfied between every pair of \( H_{k-q} \) and \( H_k \) in the case of quasi-parasupersymmetry of order \((p,q)\). The ‘power-type’ conserved charges in this case are apparently given by

\[
[(Q^-)^n, H] = [(Q^+)^n, H] = 0 \quad (2 \leq n \leq \lfloor \frac{p}{q} \rfloor),
\]

(4.4)

where \([x]\) is the maximum integer which does not exceed \( x \), and thus the number of conserved charges is reduced to \( 2\lfloor \frac{p}{q} \rfloor \). It is evident that parasupersymmetry of order \( p \) always implies quasi-parasupersymmetry of order \((p,q)\) for all \( q = 1, \ldots, p \). We note, however, that quasi-parasupersymmetry of order \((p,q)\) does not necessarily imply that of order \((p,q+n)\) with \( n > 0 \). This fact can be easily understood from the difference of the conserved charges. For examples, the conserved charges in order \((p,2)\) are \((Q^\pm)^2\), \((Q^\pm)^4\), \ldots while those in order \((p,3)\) are \((Q^\pm)^3\), \((Q^\pm)^6\), \ldots, and thus they are different symmetries. This observation clearly indicates that for every pair of natural numbers \(q_1\) and \(q_2\) \((q_1, q_2 \leq p)\) quasi-parasupersymmetries of order \((p,q_1)\) and \((p,q_2)\) have common conserved charges \((Q^\pm)^n\) only when \(n(\leq p)\) is a common multiple of \(q_1\) and \(q_2\). The ‘discrete-type’ conserved charges (cf. Eq. (3.20)) are similarly defined.

We note that the components of conserved charges of this symmetry are given by higher-derivative linear differential operators and thus it looks like a kind of \( N \)-fold supersymmetry. In particular, the full algebra of quasi-parasupersymmetry of order \((p,p)\) includes

\[
\{ (Q^-)^p, (Q^-)^p \} = \{ (Q^+)^p, (Q^+)^p \} = 0, \quad [(Q^\pm)^p, H] = 0,
\]

(4.5)

where the components of \((Q^\pm)^p\) are \( p \)-th-order linear differential operators, and thus it resembles \( N \)-fold supersymmetry with \( N = p \). Indeed, in the case of a single variable the anti-commutator \( \{ (Q^-)^p, (Q^+)^p \} \) can be expressed as a polynomial of degree \( p \) in \( H \) \([9, 71]\) and thus the system exactly coincides with one-body \( N \)-fold supersymmetry with \( N = p \) if the system admits \( H_1 = \cdots = H_{p-1} \equiv 0 \). However, the latter cannot be the case in quasi-parasupersymmetry due to the first constraint on \( H_{p-1} \) in Eq. (3.38a), which comes from the non-linear constraint \((3.3)\), unless the l.h.s. of Eq. (3.38a) accidentally vanishes. Hence, despite the resemblance of algebras and the fact that the conserved charges are represented by \( N \)-th-order linear differential operators with \( N = p \), quasi-parasupersymmetry provides a new type of non-linear supersymmetries which is different from \( N \)-fold supersymmetry even in one-body systems.

V. PARA-FERMIONIC ALGEBRA OF ORDER 2

In this section, we shall construct parafermionic algebra of order 2 based on the postulates in Section II. The starting point is the relations \((2.1)\) and \((2.3)\) for \( p = 2 \):

\[
(\psi^-)^2 = (\psi^+)^2 = 0, \quad (5.1)
\]

\[
\{ \psi^-, \psi^+ \} + \{ (\psi^-)^2, (\psi^+)^2 \} = 2I. \quad (5.2)
\]

First, multiplying \((5.2)\) by two \( \psi^-\)s as \((\psi^-)^2 \times (5.2)\), \( \psi^- \times (5.2) \times \psi^-\), and \((5.2) \times (\psi^-)^2\), and applying the nilpotency \((5.1)\), we have

\[
(\psi^-)^2 \psi^+ \psi^- + (\psi^-)^2 (\psi^+)^2 (\psi^-)^2 = 2(\psi^-)^2, \quad (5.3a)
\]

\[
(\psi^-)^2 \psi^+ \psi^- + \psi^- \psi^+ (\psi^-)^2 = 2(\psi^-)^2, \quad (5.3b)
\]

\[
\psi^- \psi^+ (\psi^-)^2 + (\psi^-)^2 (\psi^+)^2 (\psi^-)^2 = 2(\psi^-)^2. \quad (5.3c)
\]
From the above set of equations, we obtain
\[(\psi^-)^2\psi^+\psi^- = \psi^-\psi^+(\psi^-)^2 = (\psi^-)^2(\psi^+)^2(\psi^-)^2 = (\psi^-)^2, \tag{5.4}\]
\[(\psi^-)^2\psi^+(\psi^-)^2 = 0. \tag{5.5}\]

We note that these formulas are consistent with the assumption \[(2.12). \]
Next, multiplying \[(5.2)\] by \(\psi^+\) and \(\psi^-\) as \(\psi^+\psi^-\times(5.2), \psi^+\times(5.2)\times\psi^-, \) and \((5.2)\times\psi^+\psi^-\), and applying the nilpotency \[(5.1)\] and the formula \[(5.4)\], we have
\[\psi^+(\psi^-)^2\psi^+ + \psi^+\psi^-\psi^+ + (\psi^+)^2(\psi^-)^2 = 2\psi^+\psi^-, \tag{5.6a}\]
\[\psi^+\psi^-\psi^+ + (\psi^+)^2(\psi^-)^2 + \psi^+(\psi^-)^2\psi^+ = 2\psi^+\psi^-, \tag{5.6b}\]
\[\psi^-\psi^+\psi^- + \psi^-\psi^+\psi^- + (\psi^+)^2(\psi^-)^2 = 2\psi^+\psi^-. \tag{5.6c}\]

From the above set of equations, we obtain
\[\psi^+(\psi^-)^2\psi^+ = \psi^+(\psi^-)^2(\psi^+)^2\psi^- = \psi^-\psi^+(\psi^-)^2\psi^- = \psi^+\psi^+\psi^-\psi^-. \tag{5.7}\]

Multiplying \[(5.7)\] by \(\psi^-\) from left or right, and applying the formula \[(5.4)\], we get
\[(\psi^-)^2(\psi^+)^2\psi^- = (\psi^-)^2\psi^+, \quad \psi^+(\psi^-)^2\psi^- = \psi^+(\psi^-)^2. \tag{5.8}\]

Next, we multiply Eq. \[(5.2)\] by \(\psi^-\) from left to have
\[(\psi^-)^2\psi^+ + \psi^-\psi^-\psi^- + \psi^-\psi^+(\psi^-)^2 = 2\psi^-. \tag{5.9}\]
Comparing Eqs. \[(5.8)\] and \[(5.9)\], and using the assumption \[(2.11)\], we get
\[\{\psi^+, (\psi^-)^2\} = \psi^-\psi^+\psi^- = \psi^-. \tag{5.10}\]

Finally, multiplying the second formula in Eq. \[(5.10)\] by \(\psi^+\) from left and right, we have
\[(\psi^+)^2(\psi^-)^2 + \psi^+(\psi^-)^2\psi^+ = \psi^+\psi^- + \psi^+(\psi^-)^2(\psi^+)^2 = \psi^+\psi^+. \tag{5.11}\]
Thus, we obtain the following formula:
\[\psi^+(\psi^-)^2\psi^+ = \psi^+\psi^- - (\psi^+)^2(\psi^-)^2 = \psi^+\psi^- - (\psi^-)^2(\psi^+)^2. \tag{5.12}\]

The second equality in Eq. \[(5.12)\] can be expressed as
\[\left[\psi^-, \psi^+\right] = [(\psi^-)^2, (\psi^+)^2]. \tag{5.13}\]

We note that all the formulas so far derived also hold when all the indices of + and − are interchanged since the original algebra \[(5.1)\] and \[(5.2)\] is invariant under the interchange of + and −. Thus, Eqs. \[(5.4), (5.5), (5.8), (5.10), \) and \[(5.12)\] respectively imply
\[(\psi^+)^2\psi^-\psi^+ = \psi^+\psi^-\psi^+ = (\psi^+)^2(\psi^-)^2 = (\psi^+)^2, \tag{5.14}\]
\[(\psi^+)^2\psi^-\psi^+ = 0, \tag{5.15}\]
\[(\psi^+)^2(\psi^-)^2\psi^+ = (\psi^+)^2\psi^+, \quad \psi^+(\psi^-)^2\psi^+ = \psi^+(\psi^-)^2, \tag{5.16}\]
\[\{\psi^-, (\psi^+)^2\} = \psi^+\psi^-\psi^+ = \psi^+, \tag{5.17}\]
\[\psi^-(\psi^+)^2\psi^- = \psi^-\psi^+ - (\psi^-)^2(\psi^+)^2 = \psi^+\psi^- - (\psi^+)^2(\psi^-)^2. \tag{5.18}\]
Similarly, the other projection operators $\Pi_0$ and $\Pi_2$ are given by

$$\Pi_0 = (\psi^-)^2(\psi^+)^2, \quad \Pi_2 = (\psi^+)^2(\psi^-)^2. \quad (5.25)$$
VI. SECOND-ORDER PARASUPERSYMMETRIC QUANTUM SYSTEMS

We can easily check with the aid of the multiplication law and the formula (5.23) that the operators $\Pi_i$ ($i = 0, 1, 2$) in Eqs. (5.24) and (5.25) satisfy the definition (2.7) for $p = 2$. For example,

$$\Pi_1^2 = \psi^+\psi^-\psi^+\psi^- - \psi^+\psi^- (\psi^+)^2 (\psi^-)^2$$

$$- (\psi^+)^2 (\psi^-)^2 \psi^+\psi^- + (\psi^+)^2 (\psi^-)^2 (\psi^+)^2 (\psi^-)^2$$

$$= \psi^+\psi^- - (\psi^+)^2 (\psi^-)^2 - (\psi^+)^2 (\psi^-)^2 + (\psi^+)^2 (\psi^-)^2 = \Pi_1,$$

$$\Pi_1\Pi_2 = \psi^+\psi^- (\psi^+)^2 (\psi^-)^2 - (\psi^+)^2 (\psi^-)^2 (\psi^+)^2 (\psi^-)^2$$

$$= (\psi^+)^2 (\psi^-)^2 - (\psi^+)^2 (\psi^-)^2 = 0,$$

and so on. The intertwining relations (2.9) and (2.10) can be easily checked as

$$\Pi_1\psi^+ = \psi^+\Pi_0 = \psi^- (\psi^+)^2, \quad \Pi_2\psi^+ = \psi^+\Pi_1 = (\psi^+)^2 \psi^-,$$

$$\psi^-\Pi_1 = \Pi_0\psi^- = (\psi^-)^2 \psi^+, \quad \psi^-\Pi_2 = \Pi_1\psi^- = \psi^+ (\psi^-)^2.$$

We also note that the second-order parasuperalgebra (5.19)–(5.23) is consistent with Eq. (2.4). From the relations (5.23) and (5.20), we have

$$\psi^- |1\rangle = \psi^- \psi^+ |0\rangle = (I - (\psi^+)^2 (\psi^-)^2) |0\rangle = |0\rangle,$$

$$\psi^- |2\rangle = \psi^- (\psi^+)^2 |0\rangle = (\psi^+ - (\psi^+)^2 \psi^-) |0\rangle = |1\rangle,$$

which are exactly Eq. (2.4). The assumption (2.13) for $p = 2$ is also satisfied as

$$(\psi^-)^2 \psi^+ \Pi_1 = \psi^- \Pi_1 = (\psi^-)^2 \psi^+, \quad \psi^+ (\psi^-)^2 \Pi_2 = \psi^- \Pi_2 = \psi^+ (\psi^-)^2.$$

Therefore, we have confirmed that all the postulates in Section III are fulfilled. Applying the algebra and multiplication law, we can derive the trilinear relations which characterize parafermionic statistics [72, 73] as

$$[\psi^-, [\psi^+, \psi^-]] = 2\psi^- \psi^+ \psi^- - \{\psi^+, (\psi^-)^2\} = \psi^-,$$

$$[\psi^+, [\psi^-, \psi^+]] = 2\psi^+ \psi^- \psi^+ - \{\psi^-, (\psi^+)^2\} = \psi^+.$$

(5.26)

(5.27)

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We are now in a position to construct second-order parasupersymmetric quantum systems by using the second-order parafermionic algebra just derived in the previous section. From Eqs. (5.24), (5.25), and the multiplication law, the triple $(H, Q^-, Q^+)$ in Eqs. (3.6) and (3.8) for $p = 2$ is given by

$$H = \sum_{k=0}^{2} H_k \Pi_k$$

$$= H_0 (\psi^-)^2 (\psi^+)^2 + H_1 (\psi^+ \psi^- - (\psi^+)^2 (\psi^-)^2) + H_2 (\psi^+)^2 (\psi^-)^2,$$

(6.1)

$$Q^- = \sum_{k=0}^{2} Q^-_k \psi^- \Pi_k = Q^-_1 (\psi^-)^2 \psi^+ + Q^-_2 \psi^+ (\psi^-)^2,$$

(6.2)

$$Q^+ = \sum_{k=0}^{2} Q^+_k \Pi_k \psi^+ = Q^+_1 \psi^- (\psi^+)^2 + Q^+_2 (\psi^+)^2 \psi^-.$$

(6.3)
We recall the fact that the above second-order parasupercharges (6.2) and (6.3) already satisfy the nilpotent condition (3.1) for \( p = 2 \), \((Q^-)^3 = (Q^+)^3 = 0\), as we have generically shown in Section III.

Now that we have had the second-order parafermionic algebra and multiplication law, we can reveal the true character of the ‘discrete-type’ symmetry mentioned in Section III. In the case of second-order, we can construct eight additional charges \( Q^\pm_{[n]} \) and \( Q^\pm_{[i]} \) \((n = 1, 2)\) defined by Eq. (3.29). Substituting Eqs. (5.20), (5.24), and (5.25) into Eq. (3.32) and using the multiplication law, we have

\[
Q^-_{[1]} = -Q^-_{[2]} = -Q_1^-(\psi^-)^2\psi^+ + Q_2^-\psi^+(\psi^-)^2, \quad Q^-_{[i]} = Q^-_{[2]} = Q^-,
\]

\[
Q^+_{[1]} = -Q^+_{[2]} = Q_1^+\psi^- (\psi^+)^2 - Q_2^+ (\psi^+)^2\psi^-, \quad Q^+_{[i]} = Q^+_{[2]} = -Q^+.
\]

Hence, there are essentially two new conserved charges \( Q^\pm_{[1]} \) obtained from \( Q^\pm \) by changing the relative signs between \( Q^\pm_1 \) and \( Q^\pm_2 \), respectively. Apparently, they have \( \mathbb{Z}_2 \) structure and that is the reason why we have called them ‘discrete-type’ symmetries. These conserved charges are identical with the ones first found in Ref. [2].

From Eqs. (3.19) and (3.38), the commutativity (5.2) and the non-linear constraints (3.3) for \( p = 2 \)

\[
(Q^-)^2 Q^+ + Q^- Q^+ Q^- + Q^+(Q^-)^2 = C_2 Q^- H, \quad (6.5a)
\]

\[
(Q^+)^2 Q^- + Q^+ Q^- Q^+ + Q^- (Q^+)^2 = C_2 HQ^+, \quad (6.5b)
\]

hold if and only if the following conditions

\[
H_0 Q^-_1 = Q^-_1 H_1, \quad H_1 Q^-_2 = Q^-_2 H_2, \quad (6.6)
\]

\[
Q^-_1 Q^-_2 Q^+_1 + Q^-_1 Q^+_1 Q^-_1 = C_2 Q^-_1 H_1, \quad (6.7)
\]

\[
Q^-_2 Q^-_2 Q^+_2 + Q^-_2 Q^+_2 Q^-_2 = C_2 Q^-_2 H_2, \quad (6.8)
\]

and their ‘adjoint’ relations

\[
Q^+_1 H_0 = H_1 Q^+_1, \quad Q^+_2 H_1 = H_2 Q^+_2, \quad (6.9)
\]

\[
Q^+_1 Q^+_2 Q^-_1 + Q^+_1 Q^-_2 Q^+_1 = C_2 H_1 Q^+_1, \quad (6.10)
\]

\[
Q^+_2 Q^+_1 Q^-_1 + Q^+_2 Q^-_2 Q^+_2 = C_2 H_2 Q^+_2, \quad (6.11)
\]

are satisfied. We note that when a solution to Eq. (6.7) or (6.10) is given by

\[
C_2 H_1 = Q^+_1 Q^-_1 + Q^-_2 Q^+_2, \quad (6.12)
\]

the conditions (6.8) and (6.11) become identical with the second intertwining relations in Eqs. (6.6) and (6.9), respectively. Thus, in this case it is sufficient to solve the following operator identities

\[
(C_2 H_0 - Q^-_1 Q^+_1)Q^-_1 = Q^-_1 Q^-_2 Q^+_2, \quad (6.13)
\]

\[
Q^-_2 (C_2 H_2 - Q^+_2 Q^-_2) = Q^+_1 Q^-_1 Q^-_2. \quad (6.14)
\]

In general, we do not need to solve the ‘adjoint’ conditions.

For the second-order case, we have one new quasi-parasupersymmetry, namely, that of order \( (2, 2) \). The conditions are given by Eqs. (6.6)–(6.11), but the first-order intertwining relations \( (6.6) \) and \( (6.9) \) are replaced by the second-order intertwining relations

\[
H_0 Q^-_1 Q^-_2 = Q^-_1 Q^-_2 H_2, \quad Q^+_2 Q^+_1 H_0 = H_2 Q^+_2 Q^+_1. \quad (6.15)
\]

In the followings, we will show three different representations for the system \((H_k, Q^\pm_k)\) which satisfies the condition \((6.6)–(6.11)\) for second-order parasupersymmetry.
A. One-Variable Representation

First, we shall realize a second-order parasupersymmetric quantum system of one degree of freedom. Let us put $C_2 = 4$ and

$$H_k = -\frac{1}{2} \partial^2 + V_k(q), \quad Q_k^\pm = \pm \partial + W_k(q), \quad (6.16)$$

where $\partial = dq/dq$. Substituting Eq. (6.16) into Eq. (6.7), we find that the condition (6.7) is satisfied if and only if

$$4V_1 = W_1' + W_1^2 - W_2' + W_2^2. \quad (6.17)$$

Hence, the general solution to (6.7) is given by Eq. (6.12). In this case we have

$$C_2 H_0 - Q_1^- Q_1^+ = -\partial^2 + 4V_0 + W_1' - W_1^2 \equiv -\partial^2 + 2\bar{V}_0. \quad (6.18)$$

Then, the condition (6.13) is equivalent to the following set of equations:

$$2\bar{V}_0 = -2W_1' - W_2 + W_2^2, \quad (6.19)$$
$$2\bar{V}_0' + 2W_1 \bar{V}_0 = -W_2'' - W_1 W_2' + W_1 W_2^2. \quad (6.20)$$

From them we immediately obtain

$$-W_2' + W_2^2 = W_1' + W_1^2 + 4C, \quad 2\bar{V}_0 = -W_1' + W_1^2 + 4C, \quad (6.21)$$

where $C$ is a constant. In terms of $H_i$ and $Q_i^\pm$ they are expressed as

$$Q_2^- Q_2^+ = Q_1^- Q_1^+ + 4C, \quad C_2 H_0 = 2Q_1^- Q_1^+ + 4C. \quad (6.22)$$

Substituting the first formula in Eq. (6.22) into the condition Eq. (6.14), we obtain

$$C_2 Q_2^- H_2 = 2Q_2^- Q_2^+ Q_2^- - 4CQ_2^- . \quad (6.23)$$

This identity holds if and only if

$$2V_2 = W_2' + W_2^2 - 2C. \quad (6.24)$$

Hence, we finally obtain

$$H_0 = \frac{1}{2} Q_1^- Q_1^+ + C, \quad H_1 = \frac{1}{2} Q_1^+ Q_1^- + C = \frac{1}{2} Q_2^- Q_2^+ - C, \quad H_2 = \frac{1}{2} Q_2^+ Q_2^- - C. \quad (6.25)$$

This second-order parasupersymmetric system is essentially the same as the one first constructed by Rubakov and Spiridonov in Ref. [14]. We also note that if we put

$$W_1(q) = W(q) + \frac{E(q)}{2}, \quad W_2(q) = W(q) - \frac{E(q)}{2}, \quad (6.26)$$

the Hamiltonians $H_0$ and $H_2$ are intertwined by

$$Q_2^- Q_1^+ = \left(\frac{d}{dq} + W(q) - \frac{E(q)}{2}\right) \left(\frac{d}{dq} + W(q) + \frac{E(q)}{2}\right), \quad (6.27)$$

which is exactly the component of type A 2-fold supercharge, and thus they constitute a pair of type A 2-fold supersymmetry [67, 70].
B. Two-Variable Representation I

If we consider the fact that each parasupercarg of order 2 has two independent components, it would be natural to expect that a second-order parasupersymmetry can be realized in quantum systems of two degrees of freedom. For this purpose, we put \( C_2 = 2 \) and

\[
H_k = -\frac{1}{2}\partial_1^2 - \frac{1}{2}\partial_2^2 + V_k(q_1, q_2), \quad Q_k^\pm = \pm\partial_k + W_k(q_1, q_2), \tag{6.28}
\]

where \( \partial_k = \partial/\partial q_k \). Substituting Eq. (6.28) into Eq. (6.7), we find that the condition (6.7) is satisfied if and only if

\[
2V_1 = (\partial_1 W_1) + W_1^2 - (\partial_2 W_2) + W_2^2, \tag{6.29}
\]

where \( (\partial f) = \partial f/\partial q_i \) and thus the general solution is again given by Eq. (6.12). In this case,

\[
C_2 H_0 - Q_1 Q_1^* = -\partial_2^2 + 2V_0 + (\partial_1 W_1) - W_1^2 \equiv -\partial_2^2 + 2\bar{V}_0, \tag{6.30}
\]

and the condition (6.13) is equivalent to the following set of equations:

\[
(\partial_2 W_1) = 0, \tag{6.31}
\]

\[
2\bar{V}_0 = -(\partial_2 W_2) + W_2^2, \tag{6.32}
\]

\[
2(\partial_1 \bar{V}_0) + 2W_1 \bar{V}_0 = -W_1(\partial_2 W_2) + W_1 W_2^2. \tag{6.33}
\]

From them we have

\[
2(\partial_1 \bar{V}_0) = -(\partial_1 \partial_2 W_2) + (\partial_1 W_2^2) = 0. \tag{6.34}
\]

On the other hand,

\[
C_2 H_2 - Q_2^+ Q_2^- = -\partial_1^2 + 2\bar{V}_2 - (\partial_2 W_2) - W_2^2 \equiv -\partial_1^2 + 2\bar{V}_2, \tag{6.35}
\]

and the condition (6.14) is equivalent to the following set of equations:

\[
(\partial_1 W_2) = 0, \tag{6.36}
\]

\[
2\bar{V}_2 = (\partial_1 W_1) + W_1^2, \tag{6.37}
\]

\[
-2(\partial_2 \bar{V}_2) + 2W_2 \bar{V}_2 = (\partial_1 W_1)W_2 + W_1^2 W_2. \tag{6.38}
\]

From them we have

\[
2(\partial_2 \bar{V}_2) = (\partial_2 \partial_1 W_1) + (\partial_2 W_1^2) = 0. \tag{6.39}
\]

Summarizing the above results, we obtain for the parasupercargs

\[
Q_k^\pm = \pm\partial_k + W_k(q_k) \quad (k = 1, 2), \tag{6.40}
\]

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Similarly, the condition (6.14) is satisfied if and only if
\[ \partial_1 W_1 + (\partial_2 W_1) = 0, \]
\[ 2V_0 = (\partial_1 W_2) - (\partial_2 W_2) + W_2^2, \]
\[ -(\partial_1^2 W_2) + (\partial_1 W_2^2) + (\partial_2^2 W_2) - (\partial_2 W_2^2) = 0. \]

Then, it is easy to check that the condition (6.13) is satisfied if and only if
\[ (\partial_1 W_1) + (\partial_2 W_1) = 0, \]
\[ 2V_0 = (\partial_1 W_2) - (\partial_2 W_2) + W_2^2, \]
\[ -(\partial_1^2 W_2) + (\partial_1 W_2^2) + (\partial_2^2 W_2) - (\partial_2 W_2^2) = 0. \]

C. Two-Variable Representation II

To construct a parasupersymmetric interacting two-body system, we put \( C_2 = 4 \) and
\[ H_k = \frac{1}{2} \partial_1^2 - \frac{1}{2} \partial_2^2 + V_k(q_1, q_2), \]  
(6.44a)
\[ Q_1^+ = \pm \partial_1 \mp \partial_2 + W_1(q_1, q_2), \]  
(6.44b)
\[ Q_2^+ = \pm \partial_1 \pm \partial_2 + W_2(q_1, q_2). \]  
(6.44c)

As in the previous two examples, the general solution to Eq. (6.7) is again given by Eq. (6.12). In this case we have
\[ C_2 H_0 - Q_1^- Q_1^+ = -\partial_1^2 - 2\partial_1 \partial_2 - \partial_2^2 + 4V_0 + (\partial_1 W_1) - (\partial_2 W_1) - W_1^2 \]
\[ \equiv -\partial_1^2 - 2\partial_1 \partial_2 - \partial_2^2 + 2V_0, \]
(6.45)
\[ C_2 H_2 - Q_2^- Q_2^+ = -\partial_1^2 + 2\partial_1 \partial_2 - \partial_2^2 + 4V_2 - (\partial_1 W_2) - (\partial_2 W_2) - W_2^2 \]
\[ \equiv -\partial_1^2 + 2\partial_1 \partial_2 - \partial_2^2 + 2V_2. \]
(6.46)

Then, it is easy to check that the condition (6.13) is satisfied if and only if
\[ (\partial_1 W_1) + (\partial_2 W_1) = 0, \]
\[ 2V_0 = -(\partial_1 W_2) - (\partial_2 W_2) + W_2^2, \]
\[ -(\partial_1^2 W_2) + (\partial_1 W_2^2) + (\partial_2^2 W_2) - (\partial_2 W_2^2) = 0. \]

Similarly, the condition (6.14) is satisfied if and only if
\[ (\partial_1 W_2) - (\partial_2 W_2) = 0, \]
\[ 2V_2 = (\partial_1 W_1) - (\partial_2 W_1) + W_1^2, \]
\[ (\partial_1^2 W_1) + (\partial_1 W_1^2) - (\partial_2^2 W_1) + (\partial_2 W_1^2) = 0. \]
We note that Eqs. (6.47) and (6.50) imply Eqs. (6.52) and (6.49), respectively. Summarizing
the results, we obtain for the parasupercharges

\begin{align*}
Q_1^\pm &= \pm \partial_1 \mp \partial_2 + W_1(q_1 - q_2), \\
Q_2^\pm &= \pm \partial_1 \pm \partial_2 + W_2(q_1 + q_2),
\end{align*}

(6.53)

(6.54)

and for the Hamiltonians

\begin{align*}
H_0 &= \frac{1}{4}(Q_1^- Q_1^+ + Q_2^- Q_2^+) \\
&= -\frac{1}{2}\partial_1^2 - \frac{1}{2}\partial_2^2 + \frac{1}{4} (-W'_1(q_-) + W_1(q_-)^2 - W'_2(q_+) + W_2(q_+)^2),
\end{align*}

(6.55)

\begin{align*}
H_1 &= \frac{1}{4}(Q_1^- Q_1^- + Q_2^- Q_2^-) \\
&= -\frac{1}{2}\partial_1^2 - \frac{1}{2}\partial_2^2 + \frac{1}{4} (W'_1(q_-) + W_1(q_-)^2 - W'_2(q_+) + W_2(q_+)^2),
\end{align*}

(6.56)

\begin{align*}
H_2 &= \frac{1}{4}(Q_1^- Q_1^+ + Q_2^+ Q_2^-) \\
&= -\frac{1}{2}\partial_1^2 - \frac{1}{2}\partial_2^2 + \frac{1}{4} (W'_1(q_-) + W_1(q_-)^2 + W'_2(q_+) + W_2(q_+)^2),
\end{align*}

(6.57)

where \(q_\pm = q_1 \pm q_2\) and \(W_k (k = 1, 2)\) are arbitrary differentiable functions. Hence, the
potential terms of the parasupersymmetric system in this case consist of the interactions
depending only on the relative coordinate \(q_1 - q_2\) and the external fields acting on the
center-of-mass coordinate \(q_1 + q_2\).

**D. Summary of the three examples**

In contrast to supersymmetric quantum systems, it is well-known that parasupersymmetric
ones do not necessarily have non-negative spectrum and do not generically admit a
generalization of the Witten index. These facts led to consider some additional non-linear
relations among \(H\) and \(Q^\pm\), in particular, some cases where the system \(H\) can be expressed
as a non-negative function of \(Q^\pm\) in closed form, see e.g. Refs. [30, 68, 69]. In the latter cases,
since the operator \(H\) is of parafermionic degree 0, each term in a function of \(Q^\pm\) should
contain the same number of \(Q^\pm\) and \(Q^\mp\). In the case of second-order parasupersymmetry,
non-trivial such monomials of the lowest degree in \(Q^\pm\) are as follows:

\begin{align*}
Q^- Q^+ &= Q_1^- Q_1^+ \Pi_0 + Q_2^- Q_2^+ \Pi_1, \\
Q^+ Q^- &= Q_1^+ Q_1^- \Pi_1 + Q_2^+ Q_2^- \Pi_2.
\end{align*}

(6.58)

(6.59)

Similarly, those of the next-to-lowest degree in \(Q^\pm\) which cannot be obtained by a function
of \(Q^- Q^+\) and \(Q^+ Q^-\) are as follows:

\begin{align*}
(Q^-)^2 (Q^+)^2 &= Q_1^- Q_1^+ Q_2^- Q_2^+ \Pi_0, \\
(Q^+)^2 (Q^-)^2 &= Q_2^+ Q_2^- Q_1^+ Q_1^- \Pi_2.
\end{align*}

(6.60)

(6.61)

Owing to the nilpotency \((Q^\pm)^3 = 0\), every function of \(Q^\pm\) which has zeroth parafermionic
degree can be expressed, in principle, as a function of the monomials (6.58)–(6.61).
Let us now consider the first example in Section VI A. Using the first relation in Eqs. (6.22) and (6.25), we obtain

\[
Q^- Q^+ = 2(H_0 - C)\Pi_0 + 2(H_1 + C)\Pi_1, \quad (6.62)
\]

\[
Q^+ Q^- = 2(H_1 - C)\Pi_1 + 2(H_2 + C)\Pi_2, \quad (6.63)
\]

\[
(Q^-)^2(Q^+) = 4(H_0^2 - C^2)\Pi_0, \quad (6.64)
\]

\[
(Q^+)^2(Q^-)^2 = 4(H_2^2 - C^2)\Pi_2. \quad (6.65)
\]

Hence, we can easily find a non-linear relation

\[
(Q^-)^2(Q^+) + Q^±(Q^±)^2Q^± + (Q^+)^2(Q^-)^2 = 4(H^2 - C^2). \quad (6.66)
\]

It is interesting to note that this non-linear relation can be regarded as a generalization of 2-fold superalgebra. Indeed, if we restrict the linear space \(S \times V_2\) on which the system \(H\) acts to \(S \times (V_0^0 + V_2^2)\) (cf. the definition between Eqs. (2.4) and (2.5)), we have

\[
\{(Q^-)^2, (Q^+)\} = 4(H^2 - C^2)\big|_{S \times (V_0^0 + V_2^2)}. \quad (6.67)
\]

This, together with the trivial (anti-)commutation relations

\[
\{(Q^-)^2, (Q^-)\} = \{(Q^+)^2, (Q^+)\} = \{(Q^±)^2, H\} = 0, \quad (6.68)
\]

constitutes a type of 2-fold superalgebra in the sector \(S \times (V_0^0 + V_2^2)\).

In the second and third examples in Sections VI B and VI C, the parasupersymmetric system \(H\) is given by

\[
C_2H = (Q_1^- Q_1^+ + Q_2^- Q_2^+ )\Pi_0 + (Q_1^- Q_1^+ + Q_2^- Q_2^+ )\Pi_1 \\
+ (Q_1^+ Q_1^- + Q_2^- Q_2^+)\Pi_2. \quad (6.69)
\]

Here we can observe a different situation from the previous one-body case. Since \(\Pi_0\)-component of the system \(H\) has the term \(Q_2^- Q_2^+\), \(\Pi_0\)-component of \(H^n\) always includes the term \((Q_2^- Q_2^+)^n\). However, the latter term in \(\Pi_0\)-component cannot be reproduced by any function of the operators in Eqs. (6.58) - (6.61). The same is true for the term \(Q_1^+ Q_1^-\) in \(\Pi_2\)-component. Hence, in the second and third cases we cannot express any function of \(H\) in terms of \(Q^±\).

Finally, we note that for all the choices (6.16), (6.28), and (6.44) in the three examples, quasi-parasupersymmetry of order (2, 2) does not produce any new result over parasupersymmetry of order 2. The reason is that in all the three cases the solution to the non-linear constraint (6.7) is given by Eq. (6.12). In this case, as we have already mentioned previously, the other non-linear constraint (6.8) is identical with the first-order intertwining relation between \(H_1\) and \(H_2\), namely, the second relation in Eqs. (6.6) and (6.9). As a result, the condition (6.15) for quasi-parasupersymmetry is equivalent to

\[
H_0 Q_1^- Q_2^- = Q_1^- H_1 Q_2^- , \quad Q_2^+ Q_1^+ H_0 = Q_2^+ H_1 Q_1^+ , \quad (6.70)
\]

which is very close to the first relation in Eqs. (6.3) and (6.9). Hence, in most of second-order cases, quasi-parasupersymmetry would be identical to parasupersymmetry.
VII. DISCUSSION AND SUMMARY

In this article, we have proposed the systematic construction of parafermionic algebra and parasupersymmetric quantum systems. Assuming relatively small number of assumptions, namely, Eqs. (2.3), (2.11), and (2.13) in addition to the nilpotency (2.1), we have shown that we can systematically construct the full set of parafermionic algebra and multiplication law, at least for second order, which are totally independent of any specific representation of parafermionic operators.

With the aid of the parafermionic algebra, we have formulated in the generic way parasupersymmetric quantum systems. Remarkably, we have shown that all the parasupersymmetric conditions, namely, the commutation relations (3.2) and the non-linear relations (3.3) can be expressed in closed form in terms of the component scalar Hamiltonians $H_k$ and the component parasupercharges $Q^k_{\pm}$. From those expressions we have found that every pair of the component Hamiltonians possesses $N$-fold supersymmetry with $N \leq q$ and thus has isospectral property and weak quasi-solvability. This means in particular that parasupersymmetric quantum field theory (for some attempts of construction, see, e.g., Refs. [74, 75]), if exists, should have characteristic features analogous to those of $N$-fold supersymmetry, as in the case of weak supersymmetry (cf. Section I). The fact that the parasupersymmetric condition is too strong has naturally led us to the generalization of parasupersymmetry, which we have called quasi-parasupersymmetry.

We have investigated second-order parasupersymmetric quantum systems in detail and exhibited the three simple examples of such systems, the first one is essentially equivalent to the original one-body system in Ref. [14] and the others are two-body systems in which two independent supersymmetries are folded. In particular, we have shown that the first model admits a generalization of 2-fold superalgebra. In all the three models, quasi-parasupersymmetry is equivalent to parasupersymmetry.

Constructions of higher-order parafermionic algebra and (quasi-)parasupersymmetric quantum systems are straightforward. An extensive study of higher-order cases would be reported in a subsequent publication [76].

In this article, we have restricted ourselves to ordinary Schrödinger operators (3.7) as representations of parasuperalgebra, there are of course other possible applications in various areas of physics and mathematical science. One of them is the application to quantum systems with position-dependent mass (PDM) described generically by von Roos operators [77]. It would be straightforward since $N$-fold supersymmetry has been already successfully formulated also for PDM quantum systems [78]. For instance, we can generalize the second example of second-order parasupersymmetry in Section VI B to PDM two-body quantum systems by choosing the component parasupercharges as

$$Q^k_{\pm} = m_k(q_k)^{-\frac{1}{2}} \left( \pm \frac{\partial}{\partial q_k} + W_k(q_k) - \frac{m'_k(q_k)}{4m_k(q_k)} \right) \quad (k = 1, 2). \quad (7.1)$$

As we have mentioned in Section I there have been various formulations of parasupersymmetry. Thus, it would be interesting and important to examine the relationship among them including the present formulation.

A generalization of the present formulation to several parafermionic variables $(\psi^-_1, \ldots, \psi^-_n, \psi^+_1, \ldots, \psi^+_n)$ is a challenging problem. The nilpotency (2.1) can be trivially generalized as $(\psi^-_i)^{p+1} = (\psi^+_i)^{p+1} = 0$ for all $i = 1, \ldots, n$. A naive generalization of the fermionic anti-commutation relation $\{\psi^-_i, \psi^+_j\} = \delta_{i,j} I$, which reduces to Eq. (2.3) when
\( n = 1 \) may be
\[
\{\psi^-_i, \psi^+_j\} + \{(\psi^-_i)^2, (\psi^+_j)^2\} + \cdots + \{(\psi^-_i)^p, (\psi^+_j)^p\} = p\delta_{i,j}I. \tag{7.2}
\]

It would be interesting to examine whether this kind of generalization to several parafermionic variables works well.

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