Transfer matrix method in systems with topological insulators

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Abstract. In the frame of axion electrodynamics a 4×4 transfer matrix method was developed in order to describe the electrodynamic properties of systems containing topological insulators, which are characterized by non trivial topological properties related to time reversal symmetry. Analytical expressions were obtained for the components of the transfer matrix for the interface between a conventional dielectric and a topological insulator, which generalize the relations obtained in conventional electrodynamics for the description of the transmission of TE and TM polarized modes. It was shown that such matrix couples the TE and TM modes, where the terms which are related to such coupling are linear with respect to the fine structure constant. Analytical expressions were obtained for the changes induced by the non trivial topology in the reflection coefficients from TM and TE incident waves and for the rotation of the polarization planes of reflected and transmitted waves.

1. Introduction

Topological insulators (TI) [1-6] belong to a class of solids that have strong spin-orbit coupling and are insulating in the bulk, but have an odd number of gapless conducting surface states protected against time-reversal invariant disorder. When the symmetry is broken in a 3D TI (by proximity to a magnetic material or by an externally applied magnetic field), a topological magnetoelectric effect arises, which is associated to a polarization charge induced by a magnetic induction \( B \) and a magnetization induced by an electric field \( E \). In the context of classical electrodynamics, the description of the topological magnetoelectric effect contains a term of the form \( \theta(E \times B) \) (where the topological axion parameter \( \theta \) takes values that are integer multiples of \( \pi \)), leading to the following modified constitutive relations between the electric displacement \( D \), the magnetic field \( H \) and the fields \( E, B \) [7]:

\[
D = \varepsilon E - \varepsilon_0 \alpha \frac{\theta}{\pi} c B, \quad cH = \frac{cB}{\mu} + \alpha \frac{\theta}{\pi} \frac{E}{\mu_0}.
\]

In the above expressions \( \varepsilon, \varepsilon_0 \) and \( \mu, \mu_0 \) are the dielectric permittivities and magnetic permeabilities of the topological insulator and vacuum, respectively and \( \alpha \) is the fine-structure constant.

The axion coupling results in the modification of the usual laws of electromagnetic wave propagation [8]: in particular, light reflected off an interface where the topological parameter \( \theta \) jumps experiences a nontrivial Kerr rotation of its polarization and the light transmitted through the interface also undergoes a nontrivial Faraday rotation; in both cases the rotation of the polarization plane induced by the non trivial topology is a linear function of the jump of the topological term. The accumulation of the polarization rotation effect occurring in multilayer walls made of alternating...
layers of a topological insulator and a normal insulator can enhance the Casimir repulsion between two parallel multilayer walls [9, 10]. Additionally, axion coupling introduces changes in the spectra of collective excitations arising at topologically nontrivial interfaces, such as plasmons and surface plasmon-polaritons in nanostructured topological insulators (thin films, multilayers, and nanoribbons) [11-14].

Transfer matrix formalism [15, 16] has been used extensively for the description of wave propagation in a wide range of physical systems, including photonic crystals and metamaterials. For systems containing topological insulators some results have been reported for the calculation of the reflection coefficients of an electromagnetic wave reflecting on the interface with a jump of the topological term [9, 10]. It is the aim of this communication to give a detailed derivation of the analytical expressions of the transfer matrix for the interface between a conventional dielectric and a topological insulator, in order to generalize the relations obtained in conventional electrodynamics for the description of the transmission of TE and TM polarized modes.

2. Model and general relations
The system consists of two semi-infinite homogeneous isotropic TI media with dielectric permittivities \( \varepsilon^{(1)}, \varepsilon^{(2)} \), magnetic permeabilities \( \mu^{(1)}, \mu^{(2)} \), topological parameters \( \theta^{(1)}, \theta^{(2)} \), and occupying the regions \( z<0, z>0 \), respectively. Because of the homogeneity of the system in the \( x \) and \( y \) directions, the \( y \)-components of the electric (\( E \)) and magnetic (\( H \)) fields at the different regions can be taken in the form (where the superscripts \( +, - \) indicate the forward and backward directions, respectively)

\[
\begin{align*}
\begin{pmatrix}
E_x^{(i)}(x,z,t) \\
H_y^{(i)}(x,z,t)
\end{pmatrix} &=
\begin{pmatrix}
E_x^{(i)} \\
H_y^{(i)}
\end{pmatrix}
\exp \left[ i \theta^{(i)}(x,z,t) \right]
\begin{pmatrix}
E_x^{(i)} \\
H_y^{(i)}
\end{pmatrix}
\exp \left[ i \theta^{(i)}(x,z,t) \right], z<0 \\
\begin{pmatrix}
E_x^{(i)}(x,z,t) \\
H_y^{(i)}(x,z,t)
\end{pmatrix} &=
\begin{pmatrix}
E_x^{(i)} \\
H_y^{(i)}
\end{pmatrix}
\exp \left[ i \theta^{(i)}(x,z,t) \right]
\begin{pmatrix}
E_x^{(i)} \\
H_y^{(i)}
\end{pmatrix}
\exp \left[ i \theta^{(i)}(x,z,t) \right], z>0.
\end{align*}
\]

Here \( \theta^{(i)}(x,z,t) = k_x x \pm k_y z - \omega t \), \( k_x^{(i)} = \frac{\omega}{c} \sqrt{\varepsilon^{(i)} \mu^{(i)} - k_y^2}, i=1, 2 \). The remaining components of the fields can be found with the aid of conventional Lorentz-Maxwell equations.

Note that by matching the phases of the plane waves (2a, 2b) at \( z=0 \), it follows that the laws of reflection and refraction remain valid even at the interface where the topological parameter \( \theta \) jumps [8]. The continuity of the components \( E_x, E_y, H_x, D_z \) at the surface \( z=0 \) leads to the following relation between the amplitudes of the electric and magnetic fields:

\[
\begin{pmatrix}
1 \\
1 \\
\mu^{(1)} \\
\mu^{(2)} \\
\varepsilon^{(1)} \\
\varepsilon^{(2)} \\
0 \\
0 \\
\mu^{(1)} \\
\mu^{(2)}
\end{pmatrix}
\begin{pmatrix}
k_x^{(1)} \\
k_x^{(2)} \\
\alpha \theta^{(1)} \\
\alpha \theta^{(2)} \\
\alpha \theta^{(1)} \\
\alpha \theta^{(2)} \\
0 \\
0 \\
\mu^{(1)} \\
\mu^{(2)}
\end{pmatrix}
\begin{pmatrix}
E_x^{(i)} \\
E_y^{(i)} \\
H_x^{(i)} \\
D_z^{(i)}
\end{pmatrix}
= \begin{pmatrix}
E_x^{(i)} \\
E_y^{(i)} \\
H_x^{(i)} \\
D_z^{(i)}
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
\mu^{(1)} \\
\mu^{(2)} \\
\varepsilon^{(1)} \\
\varepsilon^{(2)} \\
0 \\
0 \\
\mu^{(1)} \\
\mu^{(2)}
\end{pmatrix}
\begin{pmatrix}
k_x^{(1)} \\
k_x^{(2)} \\
\alpha \theta^{(1)} \\
\alpha \theta^{(2)} \\
\alpha \theta^{(1)} \\
\alpha \theta^{(2)} \\
0 \\
0 \\
\mu^{(1)} \\
\mu^{(2)}
\end{pmatrix}
\begin{pmatrix}
E_x^{(i)} \\
E_y^{(i)} \\
H_x^{(i)} \\
D_z^{(i)}
\end{pmatrix}.
\]
\[
\left\{ \begin{array}{cccc}
1 & 1 & 0 & 0 \\
\frac{k_z^{(2)}}{\mu^{(2)}} & \frac{k_z^{(2)}}{\mu^{(2)}} & -\alpha \theta^{(2)} & -\alpha \theta^{(2)} \\
\alpha \theta^{(2)} & \alpha \theta^{(2)} & 0 & 0 \\
0 & 0 & -\frac{k_x^{(2)}}{\mu^{(2)}} & -\frac{k_x^{(2)}}{\mu^{(2)}} \\
\end{array} \right\} \left\{ \begin{array}{c}
E^{(2+)}_x \\
E^{(2-)}_x \\
E^{(2+)}_y \\
E^{(2-)}_y \\
\end{array} \right\} = \left\{ \begin{array}{c}
E^{(2+)}_{x'} \\
E^{(2-)}_{x'} \\
E^{(2+)}_{y'} \\
E^{(2-)}_{y'} \\
\end{array} \right\}.
\] (3)

Therefore, the wave function vectors \( \Xi^{(1)} = (E^{(1+)}_x, E^{(1-)}_x, -\mu^{(1)} c^{(1)} H^{(1+)}_y, -\mu^{(1)} c^{(1)} H^{(1-)}_y) \), \( \Xi^{(2)} = (E^{(2+)}_x, E^{(2-)}_x, -\mu^{(2)} c^{(2)} H^{(2+)}_x, -\mu^{(2)} c^{(2)} H^{(2-)}_x) \) are related by \( \Xi^{(1)} = M^{(21)} \Xi^{(2)} \), where the elements of the transfer matrix \( M^{(21)} \) are given by

\[
M_{11}^{(21)} = M_{22}^{(21)} = \frac{1}{2} \left( 1 + z^{(2)}(s) \right),
M_{12}^{(21)} = M_{21}^{(21)} = \frac{1}{2} \left( 1 - z^{(2)}(s) \right),
\] (4a)

\[
M_{13}^{(21)} = M_{24}^{(21)} = -M_{23}^{(21)} = -M_{14}^{(21)} = \frac{1}{2} \sqrt{\frac{\varepsilon^{(1)} \mu^{(1)}}{\varepsilon^{(2)} \mu^{(2)}}} \delta_{12},
\] (4b)

\[
M_{33}^{(21)} = M_{44}^{(21)} = \frac{1}{2} \sqrt{\frac{\varepsilon^{(1)} \mu^{(1)}}{\varepsilon^{(2)} \mu^{(2)}}} \left( 1 + z^{(1)}(\rho) \right),
M_{34}^{(21)} = M_{43}^{(21)} = \frac{1}{2} \sqrt{\frac{\mu^{(1)} c^{(1)}}{\mu^{(2)} c^{(2)}}} \left( 1 - z^{(1)}(\rho) \right),
\] (4c)

\[
M_{31}^{(21)} = M_{42}^{(21)} = M_{41}^{(21)} = M_{24}^{(21)} = -\frac{1}{2} \sqrt{\frac{\varepsilon^{(1)} \mu^{(1)}}{\mu^{(2)} c^{(2)}}} \delta_{12},
\] (4d)

\[
z^{(2)}(s) = \frac{\mu^{(2)} k_z^{(1)}}{\mu^{(1)} k_z^{(2)}},
\delta^{(2)}(\rho) = \frac{\varepsilon^{(2)} k_z^{(1)}}{\varepsilon^{(1)} k_z^{(2)}},
\delta_{12} = \frac{\alpha(\theta^{(1)} - \theta^{(2)})}{\pi}.
\] (4e)

Note that \( \frac{1}{2} \left( (1 + z^{(2)}(s))(1 - z^{(2)}(s))(1 + z^{(2)}(\rho))(1 - z^{(2)}(\rho)) \right) \) is the transfer matrix of pure transversal electric (TE), whereas \( \frac{1}{2} \left( (1 + z^{(2)}(s))(1 - z^{(2)}(s))(1 + z^{(2)}(\rho))(1 - z^{(2)}(\rho)) \right) \) is the transfer matrix of pure transversal magnetic (TM) modes arising in the absence of topological term [15]. Note also that the terms associated to the topological parameters \( \theta^{(1)} \), \( \theta^{(2)} \), couple the TE and TM modes. Such terms depend on the jump \( \theta^{(1)} - \theta^{(2)} \) of the topological parameter at the considered surface and they are linear with respect to the fine structure constant.

3. Results and discussion

General relations (4a-c) can be used for the description of several electrodynamic properties of systems containing TI media. Let us consider the case when the region \( z < 0 \) of the space is vacuum, with \( \varepsilon^{(1)} = \varepsilon_0 \), \( \mu^{(1)} = \mu_0 \), and \( \theta^{(1)} = 1 \), whereas the region \( z > 0 \) is occupied by a TI material with \( \mu^{(2)} = \mu_0 \), \( \theta^{(2)} = \pi \) and refraction index \( n^{(2)} = \sqrt{\varepsilon^{(2)} / \varepsilon_0} \).

Suppose a TE mode electromagnetic wave with amplitude \( E^{(1+)}_y \), which is incident from the region \( z < 0 \) with an incident angle \( \vartheta_i \); in the region \( z > 0 \) we have only forward fields. Then
\( (E^+(1), E^-(1), 0, -\mu(1) c(1) H_y^+) \), \( (E^+(2), 0, -\mu(2) c(2) H_y^+, 0) \) and the relation \( \Xi(1) = M^{(1)} \Xi(1) \) leads to the following expressions for the amplitudes of the reflected and transmitted TE and TM modes:

\[
E_y^+ = r_{s1}^{(21)} E_y^+ + E_y^+ = [M_{11}^{(21)} + M_{12}^{(21)} r_{s2}^{(21)} - M_{14}^{(21)} r_{sp}^{(21)}] E_y^+, \tag{5a}
\]

\[
H_y^+ = \sqrt{\mu} \left( \frac{c(1)}{c(2)} \right) r_{sp}^{(21)} E_y^+, \quad H_y^+ = \sqrt{\mu} \left( \frac{c(2)}{c(1)} \right) \left[ M_{31}^{(21)} - M_{32}^{(21)} r_{s2}^{(21)} + M_{34}^{(21)} r_{sp}^{(21)} \right] E_y^+, \tag{5b}
\]

\[
r_{sp}^{(21)} = M_{21}^{(21)} M_{44}^{(21)} - M_{24}^{(21)} M_{42}^{(21)} \left( \frac{M_{22}^{(21)} M_{24}^{(21)} + M_{24}^{(21)} M_{42}^{(21)} - M_{22}^{(21)} M_{44}^{(21)}}{M_{22}^{(21)} M_{24}^{(21)} + M_{42}^{(21)} M_{44}^{(21)}} \right) \tag{5c}
\]

In the expressions (5c) \( r_{sp}^{(21)} \) is the reflection amplitude of an incident TE mode which is reflected in a TE (TM) mode.

In a similar way, for a TM mode electromagnetic wave with amplitude \( H_y^{+} \) the amplitudes of the reflected and transmitted TM and TE modes are given by:

\[
H_y^+ = r_{pp}^{(21)} H_y^+ + H_y^+ = \sqrt{\mu} \left( \frac{c(1)}{c(2)} \right) r_{pp}^{(21)} H_y^+, \tag{6a}
\]

\[
E_y^+ = \sqrt{\mu} \left( \frac{c(2)}{c(1)} \right) r_{pp}^{(21)} H_y^+, \quad E_y^+ = \sqrt{\mu} \left( \frac{c(1)}{c(2)} \right) \left[ M_{12}^{(21)} r_{sp}^{(21)} - M_{13}^{(21)} + M_{14}^{(21)} r_{pp}^{(21)} \right] H_y^+, \tag{6b}
\]

\[
r_{pp}^{(21)} = M_{21}^{(21)} M_{42}^{(21)} - M_{22}^{(21)} M_{41}^{(21)} \left( \frac{M_{22}^{(21)} M_{41}^{(21)} + M_{42}^{(21)} M_{41}^{(21)} - M_{22}^{(21)} M_{42}^{(21)} M_{41}^{(21)}}{M_{22}^{(21)} M_{42}^{(21)} + M_{42}^{(21)} M_{41}^{(21)}} \right), \tag{6c}
\]

In (6c) \( r_{pp}^{(21)} \) is the reflection amplitude of an incident TM mode which is reflected in a TM (TE) mode.

Note that \( R^{(21)} = \left( r_{s1}^{(21)}, r_{s2}^{(21)}, r_{sp}^{(21)}, r_{pp}^{(21)} \right) \) is the reflection matrix describing the relation between the incident and the reflection fields at the single interface of TI from the medium 1 to medium 2 [9, 10].

By taking into account the explicit relations (4a-d) for the elements of the transfer matrix, the reflection coefficients of the considered modes are

\[
R_{pp}^{(21)}(\delta_{12}) = \left| r_{pp}^{(21)} \right|^2 \approx R_p \left[ 1 + \frac{4 \zeta_{(s)}^{(21)} \zeta_{(p)}^{(21)}}{1 + \zeta_{(s)}^{(21)}} \right] \delta_{12}^2, \tag{7a}
\]

\[
R_{sp}^{(21)}(\delta_{12}) = \left| r_{sp}^{(21)} \right|^2 \approx R_p \left[ 1 + \frac{4 \zeta_{(p)}^{(21)} \zeta_{(s)}^{(21)}}{1 + \zeta_{(p)}^{(21)}} \right] \delta_{12}^2, \tag{7b}
\]

\[
R_{sp}^{(21)}(\delta_{21}) = \left| r_{sp}^{(21)} \right|^2 \approx R_p \left[ 1 + \frac{4 \zeta_{(s)}^{(21)} \zeta_{(p)}^{(21)}}{1 + \zeta_{(p)}^{(21)}} \right] \delta_{21}^2, \tag{7c}
\]

where \( R_p = \left( \frac{1 - \zeta_{(s)}^{(21)} \zeta_{(p)}^{(21)}}{1 + \zeta_{(s)}^{(21)}} \right) \) are the reflection coefficients of TE and TM waves in conventional dielectrics [17].
It can be seen that the changes in reflection coefficients $\delta R_{pp}^{(2)}$, $\delta R_{ss}^{(2)}$ which are induced by the non trivial topology vary as quadratic functions of the topological term. Although these corrections are small, they show some special features, which are evident from Fig. 1:

\begin{enumerate}
  \item The magnitude of the change $\delta R_{jl}^{(2)}$ of the reflection coefficient for TE modes is a slowly varying function of the incident angle $\theta_1$ and it vanishes only at the incidence angle $\theta_1 = \pi / 2$.
  \item For TM modes the change $\delta R_{pp}^{(2)}$ vanishes additionally at an incidence angle $\tilde{\theta} < \pi / 2$ and it has an extremum at $\tilde{\theta} < \theta_1 < \pi / 2$. The positions of both the zero and the extremum of $\delta R_{pp}^{(2)}$ shifts toward $\pi / 2$ as well as the refraction index of the TI media increases.
\end{enumerate}
(iii) As well as the refraction index contrast increases, the correction $R_{pp}^{(21)}$ becomes greater than $R_{sp}^{(21)}$ in a wide interval of incidence angles.

It is worthy to mention that $R_{pp}^{(21)}(\delta_{12})$ given by (7a) vanishes at the incidence angle given by

$$\delta_{B_p} = \tan^{-1}\left(1 - \frac{2n^{(1)}n^{(2)}}{(n^{(1)} + n^{(2)})^2}\delta_{12}\right),$$

which reduces to the Brewster angle $\delta_{B_p} = \tan^{-1}\left(\frac{n^{(1)}}{n^{(2)}}\right)$ in the absence of jump in the axion term. The transmission angle $\delta^{(2)} = \sin^{-1}\left(\frac{n^{(1)}}{n^{(2)}}\sin\delta_{B_p}\right)$ is, therefore,

$$\delta^{(2)} = \frac{\pi}{2} - \delta_{B_p} + \frac{2n^{(1)}n^{(2)}}{(n^{(1)} + n^{(2)})^2}\delta_{12}.$$

This result indicates that for the incidence of a TM wave with incidence angle $\delta_{B_p}$, the propagation vector of the reflected wave is no longer perpendicular to that of the transmitted wave, which is in agreement with [8].

The expressions for the 4×4 transfer matrix (4a-d) can be used to obtain the nontrivial Kerr (Faraday) rotation angle $\nu_{K}$ ($\nu_{F}$) of light reflected (transmitted) off an interface with jump in the topological axion parameter. For an incident TM wave these rotation angles are given by

$$\tan(\nu_{K}) = \frac{E_{y}^{1-}}{E_{x}^{1-}} = -\frac{\omega}{c_{1}k_{z}^{(1)}} \frac{r_{pp}}{r_{pp}}, \quad \tan(\nu_{F}) = \frac{E_{y}^{2+}}{E_{x}^{2+}} = \frac{\omega}{c_{2}k_{z}^{(2)}} \frac{M_{13}r_{pp} - M_{14}r_{pp} + M_{32}r_{pp} + M_{33}r_{pp}}{(-M_{32}r_{pp} + M_{33}r_{pp})}.$$ (10)

Note that these magnitudes are linear functions of the parameter $\delta_{12} = \frac{\alpha(\delta^{(1)} - \delta^{(2)})}{\pi}$ [12, 18], but their behavior depend additionally on the relation between the incident $\delta^{(i)}$ and the Brewster $\delta_{B_p}$ angles, as is illustrated in Fig. 2 for the parameter relations of Fig. 1b. Note that the magnitude of $\nu_{K}$ increases for $\delta^{(i)} < \delta_{B_p}$ whereas it decreases for $\delta_{B_p} < \delta^{(i)} < \pi/2$ and vanishes at $\delta^{(i)} = \pi/2$; additionally, the direction of rotation of the polarization plane of the reflected light changes when the angle of the incident TM wave passes through the Brewster angle. On the other hand, the direction of rotation of the polarization plane of the transmitted light is conserved in the whole interval $0 < \delta^{(i)} < \pi/2$, but $\nu_{F}$ changes as a non-monotonous function of the incident angle $\delta^{(i)}$ and vanishes at $\delta^{(i)} = \pi/2$. 


4. Conclusion
Some electrodynamic properties of systems containing topological insulators were described on the basis of the analytical expressions obtained in the frame of axion electrodynamics for the components of the transfer matrix for the interface between a conventional dielectric and a topological insulator. TE and TM modes are coupled by matrix terms which are linear with respect to the jump of the axion topological parameter. Analytical expressions were obtained for the changes induced by the non trivial topology in the reflection coefficients from TM and TE incident waves and for the rotation of the polarization planes of reflected and transmitted waves.

The obtained relations can be applied for the description of more complex geometries, by extending the well-known techniques developed for conventional dielectrics [15, 16].

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