Hawking Radiation in the Laboratory

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We propose an experimental model using the Laval nozzle of a sonic analogue of Hawking radiation. We derive the power spectrum of the outgoing wave emitted from the vicinity of the sonic horizon instead of the created particle number. Our treatment is based on classical theory, and it should make experiments easier to implement. This experimental feasibility is a great advantage of our model.

Black hole evaporation, so-called Hawking radiation, is one of the most surprising predictions in theoretical physics.\(^1\),\(^2\) Despite the importance of Hawking radiation, its actual observation in the universe is considered to be very difficult. For this reason, it seems quite natural to seek experiments in the laboratory that can simulate this interesting phenomenon. In addition, Unruh showed that there exists a sonic analogue of black holes. We speculate that something like Hawking radiation may be observed in the evaporation of these analogous structures.\(^3\),\(^4\)

Of course, black hole evaporation is a quantum phenomenon. Thus, in the case of the corresponding laboratory experiments, we should maintain the quantum coherency of the entire system to observe processes of particle creation and changes of vacuum caused by the existence of the horizon. With this in mind, several experiments employing superfluid \(^3\)!He have been proposed.\(^5\),\(^6\) However, the supersonic motion of this fluid along the walls of the apparatus would cause the collapse of superfluidity and quantum coherence, because vortices may be created near the boundaries. Additionally, even in the an experimental system\(^6\) designed to overcome this problem, no evidence of Hawking radiation has yet been reported.

In consideration of the points raised above, we propose an experiment to observe a classical analogue of Hawking radiation,\(^7\) i.e. the power spectrum of an outgoing wave emitted from the vicinity of the horizon. Since we are only concerned with classical wave propagation in a fluid flow that has a sonic horizon, this experiment has nothing to do with quantum particle creation. Thus, we cannot detect the emission of particles corresponding to the evaporation of a black hole. However, Hawking radiation possesses another striking feature, a thermal spectrum. It can be shown that the power spectrum of an outgoing wave obeys the Planck distribution\(^7\) and a role corresponding to that of the particle number in the usual scenario of black hole evaporation. Because the behavior we describe does not require a quantum

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coherence of the system, the experiment we propose is much easier to carry out than the previous ones using $^3$He.

We first give an outline of the sonic analogue of the black hole in an ordinary fluid. We begin with a perfect fluid, which obeys

$$\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla p(\rho), \quad \frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) = 0,$$

(1)

where $\rho$, $p$ and $\vec{v}$ are the density, the pressure and the velocity of the fluid. Moreover, we assume the case of an adiabatic and rotation-free fluid, implying $p = C_\rho \rho^\gamma$, $\gamma = C_p/C_v$, $\nabla \times \vec{v} = 0$ (i.e. $\vec{v} = \nabla \Phi$), where $C_p$ and $C_v$ are constant pressure and constant volume specific heat respectively, and $\Phi$ is the velocity potential. The Euler equation (1) can be integrated, yielding the Bernoulli’s equation,

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} \vec{v} \cdot \vec{v} + h(\rho) = 0.$$

(2)

Here, $h(\rho)$ is the enthalpy and $c_s = \sqrt{(dp/d\rho)_{ad}} = \sqrt{\gamma p/\rho}$ is the local velocity of sound. Then, we consider $\rho$ and $\Phi$ to be perturbated forms written $\rho = \rho_0 (1 + \psi)$ and $\Phi = \Phi_0 + \phi$ where $\rho_0$ and $\Phi_0$ are the density and velocity potential of the background flow. After substituting these into (1) and (2), we obtain the following equations for the perturbations:

$$\frac{\partial \phi}{\partial t} + \vec{v}_0 \cdot \nabla \phi + c_s^2 \psi = 0,$$

(3)

$$\frac{\partial \psi}{\partial t} + \vec{v}_0 \cdot \nabla \psi + \Delta \phi + \nabla (\ln \rho_0) \cdot \nabla \phi = 0.$$

(4)

The background flow obeys the same equations (1), but with $\rho_0$ and $v_0$ replacing $\rho$ and $v$. Eliminating $\psi$ in the Eqs. (3) and (4), we derive the equation for the velocity potential $\phi$ as

$$\left[ \frac{1}{\rho_0} \left( \frac{\partial}{\partial t} + \nabla \cdot \vec{v}_0 \right) \rho_0 c_s^2 \left( \frac{\partial}{\partial t} + \vec{v}_0 \cdot \nabla \right) - \frac{1}{\rho_0} \nabla (\rho_0 \nabla) \right] \phi = 0.$$

(5)

This equation can be interpreted as describing a scalar field $\phi$ on the background geometry with the metric

$$g_{\mu\nu} = \frac{\alpha \rho_0}{c_s} \begin{pmatrix} - (c_s^2 - \vec{v}_0 \cdot \vec{v}_0) & -v_0^i \\ -v_0^i & \delta_{ij} \end{pmatrix},$$

(6)

where $\alpha$ is a constant. In order to understand the correspondence between this geometry and a black hole, let us consider one-dimensional flow with velocity $\vec{v}_0 = (v_0(x), 0, 0)$. In the $t-x$ plane, the metric can be reduced to the form

$$ds^2 = \frac{\alpha \rho_0}{c_s} \left[ - \left( 1 - \frac{v_0(x)^2}{c_s(x)^2} \right) c_s(x)^2 d\tau^2 + \frac{dx^2}{1 - \frac{v_0(x)^2}{c_s(x)^2}} \right],$$

(7)

where we have introduced the new time coordinate $\tau$ defined by $d\tau = dt + \frac{v_0 dx}{c_s^2 - v_0^2}$. This implies that the sonic point, at which $v_0(x) = c_s(x)$, may play the role of the
horizon. By the same argument as in the case of the original Hawking radiation, it can be shown that this fluid model describes Hawking radiation with a Planckian distribution, and we can derive the “surface gravity” which determines the temperature of the radiation as

$$\kappa = \left. \frac{d(c_s - v_0)}{dx} \right|_{\text{sonic horizon}}.$$  \hfill (8)

Here we propose a model using a Laval nozzle\(^8\) that realizes the situation discussed above. Consider an axi-symmetric tube with cross section \(A(x)\), where \(x\) is the coordinate along the tube. When \(A(x)\) changes only slightly along \(x\), we can regard the flow to be uniform in cross section and one dimensional. For stationary background flow along the Laval nozzle, from the Eqs. (1) and (2) we can derive a relation between the velocity of the flow and the area of the cross section

$$\left(M^2 - 1\right) \frac{dv_0}{v_0} = \frac{dA}{A},$$  \hfill (9)

where \(M = v_0/c_s\) is the Mach number. This suggests that subsonic flow will be accelerated when the cross section becomes smaller, and supersonic flow will be accelerated when the cross section increases. In this paper, we consider a tube that has narrow throat near the center of the tube and increases at the ends. Such a tube is called a “Laval nozzle”. In a Laval nozzle, \(A(x)\) decreases along the direction of flow \((dA < 0)\) in the region upstream from the throat and increases \((dA > 0)\) in the region downstream (Fig.1). Therefore, if we can realize \(M = 1\) in the throat region, we can realize supersonic flows in the downstream region and a sonic horizon emerges in the throat. Suppose that there exist an asymptotic region in the upstream of the flow where the fluid is at rest, i.e. \(v_0 = 0\) and the pressure and density are given by \(p_0 = p_u\), \(\rho_0 = \rho_u\). It is obvious that this region can be related to an asymptotically flat spacetime region in the corresponding black hole spacetime. Then Bernoulli’s equation (2) becomes

$$\frac{1}{2} v_0^2 + \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} = \frac{\gamma}{\gamma - 1} \frac{p_u}{\rho_u},$$  \hfill (10)

which determines the velocity of the fluid as a function of the density \(\rho_0/\rho_u\).

Figure 2 displays profiles of the pressure and Mach number along the nozzle whose shape is depicted in Fig.1.\(^8\) Under the condition that the inlet pressure at \(x = 0\) is fixed, the nature of the flow is determined by the value of the outlet pressure at \(x = 3\). We note that a sonic horizon at the throat can be formed without any fine-tuning, so that the horizon is stable with respect to changes in the parameter values of the nozzle. Furthermore, even in the unsteady situation, experimental and numerical studies of time-dependent flow within the nozzle\(^9\) show that the flow is settled to the above static one which is predicted by the Bernoulli equation (10).

Next, let us consider the fluctuations of the fluid. Substituting \(\phi(t,x) = e^{-i\omega t} \times e^{i \int k(x) \, dx}\) into Eq. (5) and using the WKB approximation, we obtain

$$\left(c_s^2 - v_0^2\right) k^2 - 2 \omega v_0 k - \omega^2 = 0,$$  \hfill (11)
Fig. 2. Profiles of the pressure $p$ and Mach number $M$ of the background flows along the Laval nozzle (Fig.1) in the case of $\gamma = 7/5$ . The inlet, throat and outlet of the nozzle are located at $x = 0,1,$ and $3,$ respectively. The vertical dashed-line indicates the shock surface. Above the critical value of the outlet pressure (case c), the flow along the nozzle is completely subsonic (cases a and b). Contrastingly, the sonic horizon ($M = 1$ at $x = 1$) and regular supersonic flow are realized for sufficiently low pressure at the exit (case e). In the intermediate case (case d), a shock is formed in the upstream region, which does not affect the flow near the sonic horizon at $x = 1$.

whose solutions are given by $k = \omega/(c_s - v_0) \equiv k_{\text{out}}$ and $k = -\omega/(c_s + v_0) \equiv k_{\text{in}}$. The outgoing solution $k_{\text{out}}$ represents a wave that propagates upstream against the background flow. In the asymptotic region ($v_0 \sim 0$), this mode is a usual outgoing plane wave and should be regarded as an observable. However, we note that this solution corresponds to a wave that marginally escapes from the sonic horizon $v_0 = c_s$, so that the behavior near the horizon differs drastically from that in the asymptotic region. Contrastingly, $k_{\text{in}}$ is an ingoing wave that propagates downstream and goes through the horizon almost senselessly. Hereafter, we are mainly concerned with the outgoing solution $k_{\text{out}}$.

To proceed with the investigation, we assume that the background flow of the fluid satisfies $\rho_0/\rho_u = g(x/L)$ and $p_0/p_u = g^\gamma(x/L)$, where $L$ is the characteristic length scale of the Laval nozzle. Although the function $g$ is determined by the actual shape of the Laval nozzle, its precise form is not necessary in the subsequent analysis. We specify only the locations of the throat and the asymptotic region as follows: The throat is located at $x_{\text{th}}$ (where “th” means “throat”), which satisfies $g(x_{\text{th}}/L) = (2/\gamma + 1)/2(\gamma - 1)$, and the asymptotic region in the upstream region is given by $x \to +\infty, v_0 \to 0$. Under these conditions and using Eq. (10), we obtain

$$\frac{c_s - v_0}{c_{su}} = g \frac{\gamma - 1}{\gamma - 3/2} \left( \frac{x}{L} \right) - \sqrt{\frac{2}{\gamma - 1} \left( 1 - g^{\gamma - 1} \left( \frac{x}{L} \right) \right)}.$$

When we introduce a coordinate $z$ near the sonic point $x_{\text{th}}$ as $x = x_{\text{th}} + z$, we can calculate the surface gravity using Eq. (8),

$$\kappa = \frac{g'_{\text{th}}}{L} \left( \frac{\gamma + 1}{2} \right)^{(\gamma - 3)/2(\gamma - 1)}, \quad g'_{\text{th}} = \frac{dg(y)}{dy} \Bigg|_{y=(x_{\text{th}}/L)}.$$

Thus, we obtain the following expression for the outgoing mode of the perturbations
near the sonic horizon, $x \sim x_{th}$

$$\phi^{\text{out}}_\omega = \exp \left( i \frac{\omega}{c_s \kappa} \ln z \right) = z^{i\omega/c_s \kappa}. \quad (14)$$

We note that this wave behaves as a usual out-going plane wave

$$\phi^{\text{out}}_\omega = \exp \left( i \frac{\omega}{c_s u} x \right) \quad (15)$$

in the asymptotic region, i.e. $x \to +\infty$.

In Fig. 3, we give a comparison between the usual black hole spacetime and the fluid flow in the Laval nozzle. In the scenario of Hawking radiation,\textsuperscript{1), 2) 10}) a normal mode corresponding to the vacuum, i.e. a positive frequency part, is prepared in the past null infinity $I^-$. Here, we consider the evolution of this mode toward the past horizon $H^-$. The existence of the black hole region and the future horizon $H^+$ do not affect this evolution so that a quantum state on $H^-$ is nothing more than the vacuum state in $I^-$. An observer prepares the vacuum state in the future null infinity $I^+$, which is determined by a positive frequency normal mode in $I^+$, and he defines a particle in the vacuum. As in the previous case, we investigate the evolution of the mode in $I^+$ \textit{backward in time} toward the past horizon $H^-$. In this case, however, the future horizon $H^+$ significantly affects the evolution of the mode. Thus, a comparison between the normal modes in $I^-$ and $I^+$ in the same place $H^-$ shows that the future vacuum in $I^+$ differs from the past one in $I^-$, and the thermal emission of particle can be observed in the future null infinity $I^-$. We note that the propagation of the perturbation in the background flow in the Laval nozzle possesses an essential feature of Hawking radiation in black hole space-time. The outgoing plane wave (15) in the asymptotic region, which corresponds to the normal mode in the future null infinity $I^+$, is affected significantly by the sonic horizon, and therefore, its behavior near the sonic horizon is drastically changed, as described by (14). It is well known that the logarithmic behavior of the mode (14) results in the thermal property of the spectrum of the created particles and its temperature is related to the value of the surface gravity $\kappa$. This means that if we could realize the ideal system in which the quantum coherence is being maintained perfectly, we would be able to simulate Hawking radiation in the fluid in a Laval

Fig. 3. Comparison between the black hole spacetime and the flow of the Laval nozzle.
nozzle. However, it seems that this would be a very difficult experiment to carry out with a sonic horizon, because interaction of the fluid with the wall of the apparatus would cause the vortex formation and destroy the quantum coherence of the system.

Because of the difficulty described above, we treat our problem classically and look for the classical analogue of Hawking radiation. We prepare an outgoing wave at the past (sonic) horizon $H^{-}$ as

$$\Psi_{\Omega}^{in} = \exp(i\Omega z)$$

which mimics the vacuum state function in the past null infinity $I^{-}$. Suppose that we observe this wave in the asymptotic region $I^{+}$ in terms of a power spectrum instead of the particle number. For this purpose, we evaluate the Fourier components

$$f(\omega) = \int_{0}^{\infty} dz \, \Psi_{\Omega}^{in} \phi_{\omega}^{out} = -i\Omega^{-1} \Gamma(1 - i\omega/c_{s}\kappa) e^{-\pi\omega^{2}/c_{s}\kappa}$$

of the prepared wave $\Psi_{\Omega}^{in}$ with respect to the observed wave in the asymptotic region $\phi_{\omega}^{out}$. Finally, we obtain the power-spectrum that would be observed upstream:

$$|f(\omega)|^2 \propto \frac{2\pi \omega}{c_{s}\kappa} \frac{1}{e^{2\pi\omega/c_{s}\kappa} - 1}.$$  

This is the Planckian distribution, which can be regarded as the classical analogue of Hawking radiation. From Eqs.(13) and (18), the spectrum is characterized by a wavelength $\lambda \sim 1/\kappa$.

Of course, our proposal does not represent a full quantum simulation of Hawking radiation. However, the nature of the wave propagation near the horizon in the presently considered system is similar to that in the case of Hawking radiation. In particular, we should be able to observe a Planckian distribution, which is one of the important features of Hawking radiation. One of the main advantages of the present system is that actually carrying out an experiment and observing the power spectrum would be much easier than the previous proposal. In the present case, the sonic horizon is believed to be stable. In addition to experimental and numerical results that indicates this stability, our result (11) itself, which indicates the non-existence of growing modes in the perturbation equation (5), can be recognized as proof of the stability. Taking into account these advantages, we regard our proposal as a first step toward simulation of Hawking radiation in the laboratory.

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