Donnelly–Glaberson Instability Exciting Kelvin Waves in Atomic Bose-Einstein Condensates

Hiromitsu Takeuchi and Makoto Tsubota
Department of Physics, Osaka City University, Sumiyoshi-Ku, Osaka 558-8585, Japan.
E-mail: hiromitu@sci.osaka-cu.ac.jp

Abstract. The Donnelly–Glaberson (DG) instability causes the amplification of Kelvin waves on vortex lines in superfluid $^4$He and $^3$He-B at finite temperatures due to mutual friction between vortex lines and the normal fluid component. In this letter, we theoretically show that the DG instability is possible in atomic Bose-Einstein condensates even at $T = 0$ K without the normal component. The DG instability in atomic condensates can lead to observe the dispersion relation of Kelvin waves, vortex reconnections and quantum turbulence.

1. Introduction
It is known that quantized vortices play an important role in the dynamics in superfluid. Kelvin waves, which deform the line into a helix propagating along itself, are a fundamental motion of the vortex lines. Kelvin waves were first discussed in a classical inviscid fluid in the 19th century [1], and recently in superfluid $^4$He, $^3$He-B [2], and atomic Bose-Einstein condensates (BECs) [3, 4, 5]. These years it is pointed out that the physics of Kelvin waves is indispensable for understanding quantum turbulence [2]. The atomic BECs are useful to study Kelvin waves since we can observe directly them in experiments [4], and enable us to analyze them from the microscopic viewpoint with the Gross–Pitaevskii (GP) and the Bogoliubov–de Gennes (BdG) models [3, 5, 6]. This is contrast to helium superfluids, where, because vortex core size is of the order of angstrom, it is difficult to detect Kelvin waves experimentally and vortex dynamics is mainly discussed in a phenomenological model [7]. Thus we expect that the study of Kelvin waves in atomic BECs could provide new physics from the microscopic point of view.

In this work, we theoretically study the Donnelly–Glaberson (DG) instability [8] in atomic BECs with the GP and the BdG models. We show that the DG instability can occur even at $T = 0$ K in atomic BECs. The DG instability at $T = 0$ K can be expressed as superradiance which causes the amplification of excitations in superfluid through the Landau instability [9]. The DG instability in atomic BECs gives the possibility of observing the dispersion relation of Kelvin waves, vortex reconnections and quantum turbulence.

2. Formulation
Let us consider a singly quantized vortex in a condensate rotating with a frequency $\Omega = \Omega \hat{z}$ in a harmonic trapping potential $V_t = M \omega_t^2 \rho^2 / 2$ with the cylindrical coordinates $(\rho, \theta, z)$, where $M$ and $\omega_t$ are the atomic mass and trapping frequency, respectively. Here we assume axisymmetrical and periodic systems along the rotation axis for simplicity. We add an external object moving helically with velocity $V_{obj} = V \hat{z} + \Omega \hat{z} \times \mathbf{r}$, where $V$ and $\Omega$ are positive constants. In this case,
it is convenient to describe the condensate wave function $\Psi$ by the time-dependent GP equation in the co-moving frame where the object is at rest:

$$i \frac{\partial}{\partial t} \Psi = (-\frac{1}{2} \nabla^2 + \frac{1}{2} \rho^2 + g|\Psi|^2 - \mu - V \hat{\rho} - \Omega \hat{z}) \Psi,$$

where $\mu$ is the chemical potential, $\hat{\rho} = -i \nabla_z$, $\hat{z} = -i (r \times \nabla)_z$. The units of energy, length, and time are given as $\hbar \omega_1$, $b_\perp = \sqrt{\hbar/M \omega_1}$, and $\omega_1^{-1}$, respectively. The normalization is done as $\int d\rho d\theta \int_{-L/2}^{L/2} dz |\Psi(\rho)|^2 = 1$, where $L$ is the periodicity along the $z$-axis. The atomic interaction is characterized by $g = 4\pi aN/b_\perp > 0$, where $a$ and $N$ are the s-wave scattering length and the particle number in the system, respectively. We assume a small interaction between the object and the condensate, and neglect the corresponding term in Eq. (1) for simplicity.

In the co-moving frame we can write the axisymmetric wave function $\Psi_0 = \psi_0(\rho)e^{i\theta}$ in the stationary state with a straight vortex line along the $z$-axis. In this case the GP Eq. (1) is reduced to

$$\left[ -\frac{1}{2} \left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{1}{\rho^2} \right) + \frac{1}{2} \rho^2 + g|\psi_0|^2 - \mu - \Omega \right] \psi_0 = 0. \quad (2)$$

The chemical potential $\mu$ can be defined as

$$\mu \equiv \mu(V, \Omega) = \mu(0, 0) - \Omega, \quad (3)$$

where $\mu(V, \Omega)$ and $\mu(0, 0)$ are the chemical potential in the co-moving frame and the laboratory frame, respectively. We see that $\psi_0$ and $\mu(0, 0)$ are independent of $V$ and $\Omega$.

A Kelvin wave is expressed as one of the Bogoliubov collective modes in atomic BECs [3, 5, 6]. Now we write the condensate wave function with a collective mode as $\Psi = \Psi_0 + \delta \Psi_{k,l}$. A collective mode with a frequency $\omega$ can be written as

$$\delta \Psi_{k,l} = e^{i\theta} [u_{k,l}(\rho)e^{ikz+i\theta-\omega t} - v_{k,l}^*(\rho)e^{-i(kz+i\theta-\omega t)}], \quad (4)$$

where $k$ and $l$ refer to the wave number and angular quantum number of the excitation along the $z$-axis, respectively. Here, we consider only the lowest modes along the radial direction. The wave functions are normalized as $\int dV(u_{k,l}^* u_{k',l'} - v_{k,l}^* v_{k',l'}) = \eta \delta_{k,k'} \delta_{l,l'}$, where $\eta > 0$ and $\delta_{l,l'}$ is the Kronecker delta. The modes can be divided into three groups; Kelvin waves with $l = -1$ [3, 5, 6], varicose waves with $l = 0$ [7], and surface waves with $l \neq -1, 0$. For the Kelvin modes, the vortex core for the total wave function $\Psi = \Psi_0 + \delta \Psi_{k,-1}$ is shifted from the $z$-axis since the maximum amplitude of the wave functions $\delta \Psi_{k,-1}$ is located on the axis [3, 5, 6]. The shift distance is increased with the amplitude $\eta^{1/2}$ for the Kelvin mode.

Linearizing the GP Eq. (1) with Eq. (4), we obtain the reduced BdG equations

$$(\omega + kV + i\Omega) \begin{pmatrix} u_{k,l} \\ v_{k,l} \end{pmatrix} = \begin{pmatrix} \hat{h}_+ & -g \psi_0^2 \\ g\psi_0^2 & -\hat{h}_- \end{pmatrix} \begin{pmatrix} u_{k,l} \\ v_{k,l} \end{pmatrix}, \quad (5)$$

where $\hat{h}_\pm = -\frac{1}{2} [(d^2/d\rho^2) + (d/d\rho)(d/d\rho) - ((l \pm 1)^2/\rho^2) - k^2] + 2g|\psi_0|^2 - \mu(0, 0)$. Analogous to Eq. (3), we can define the frequency $\omega$ as

$$\omega \equiv \omega_{k,l}(V, \Omega) = \omega_{k,l}(0, 0) - kV - i\Omega, \quad (6)$$

where $\omega_{k,l}(V, \Omega)$ and $\omega_{k,l}(0, 0)$ are the dispersion relation in the co-moving frame and the laboratory frame, respectively. Since Eqs. (5) is independent of the parameters $V$ and $\Omega$ with Eq. (6), the wave functions $u_{k,l}$ and $v_{k,l}$ are independent too for $\eta = \text{const.}$
3. Landau instability exciting Kelvin waves

The single-vortex states become unstable due to the Landau instability when there is at least one mode with negative frequency. Figure 1 shows the dispersion relation $\omega_{k,l}(0,0)$ in the laboratory frame for various $l$ by numerically solving GP Eq. (2) and BdG Eqs. (5) for $g_{2D} \equiv g/L = 500$. The frequency for the surface waves with $l < -1$ are not shown in Fig.1 since the frequencies for these modes are always higher than for the Kelvin mode $l = -1$ and not important in the discussion. The single-vortex states are unstable for $\Omega = 0$ since Kelvin waves have negative frequencies for small wave numbers. It was investigated in Ref. [10] that the single-vortex states can be stabilized for $\Omega < \omega_{0,1}(0,0)$ when $V = 0$. Here the critical frequencies $\Omega_L$ and $\Omega_U$ are defined as $\Omega_L \equiv -\omega_{0,-1}(0,0)$ and $\Omega_U \equiv \min (\omega_{0,l}(0,0)/l)$, respectively. When $\Omega$ is fixed to a value in the region $\Omega_L < \Omega < \Omega_U$, the frequency $\omega_{k,l}(V,\Omega)$ becomes negative by increasing $V$ as shown by Eq. (6). The critical velocity $V_l$ for the mode with $l$ is given as

$$V_l(\Omega) = \min_k \left( \frac{\omega_{k,l}(0,0) - l\Omega}{k_l} \right) = \frac{\omega_{k,l}(0,0) - l\Omega}{k_l},$$

where $k_l$ is the critical wave number. The mode with $k_l$ and $l$ is spontaneously radiated due to Landau instability when the velocity exceeds the critical velocity $V_l$. In particular, the Kelvin waves are spontaneously excited when $V > V_{-1}$. This is the onset of the DG instability at $T = 0$ K due to the Landau instability. The form of the critical velocity $V_{-1} = \min_k \left( \frac{\omega_{k,-1}(0,0) + \Omega}{k} \right)$ is the same as that of the DG criterion obtained with a phenomenological model in superfluid helium in the presence of a vortex line [7].

To observe the DG instability, $V_{-1}$ must be smaller than other critical velocities; $V_{-1} < V_{l\neq -1}$. Figure 2 shows $V_l$ obtained from the results in Fig.1 with Eq. (7). The critical velocity $V_0$ for varicose waves is typically higher than that for Kelvin waves. While the critical velocity $V_{-1}$ for Kelvin waves monotonically increases with $\Omega$, $V_{l>0}$ for surface waves decreases. The increase in $\Omega$ reduces $V_{l>0}$ for some $l$ to be equal to $V_{-1}$ at $\Omega = \Omega_M$. Thus the condition $V_{-1} < V_{l\neq -1}$ is satisfied in the region $\Omega_L < \Omega < \Omega_M$. The frequency $\Omega_M$ is always between $\Omega_L$ and $\Omega_U$. The dependence of $\Omega_L$ and $\Omega_U$ on $g$ is described in Ref. [10].
4. After the Landau instability

Finally we shall briefly mention the dynamics after the spontaneous radiation of Kelvin modes due to the Landau instability. When $V > V_{-1}$ for $\Omega_L < \Omega < \Omega_M$ the stationary state $\Psi_0$ is no longer the local minimum of the thermodynamic potential $K(V, \Omega) = \int d\mathbf{r} \left[ \Psi^* \left( -\frac{1}{2} \nabla^2 + \frac{1}{2} \rho^2 + g|\Psi|^2 - \mu - V \hat{p}_z - \Omega \hat{l}_z \right) \Psi \right]$. The radiated Kelvin modes should be amplified for decreasing $K(V, \Omega)$ just after the Landau instability. This kind of spontaneous radiation and amplification due to the thermodynamic instability is called superradiance [9]. The amplification of the Kelvin modes could make the vortex shift large enough to observe the Kelvin wave in experiments. If the critical wave number $k_{-1}$ for $V = V_{-1}$ is observed, we will confirm the dispersion relation $\omega_{k_{-1},-1}(0,0) = k_{-1}V_{-1} - \Omega$.

It is also interesting to study the DG instability in vortex lattices in atomic BECs. When the vortex shift distance due to the Kelvin mode becomes to the order of the interspace between vortices, vortex reconnections will occur. As discussed in superfluid helium [11], the DG instability in vortex lattices can lead to a dense vortex tangle, namely quantum turbulence. The similar scenario is expected in atomic BECs too. The instability gives us a possibility to realize quantum turbulence in atomic BECs [12].

5. Conclusion

In conclusion, we theoretically reveal that the DG instability is possible in atomic BECs at $T = 0$ K. The DG instability is represented by the Landau instability. The DG instability can lead to the observation of the dispersion relation of Kelvin waves, vortex reconnections, and quantum turbulence. The optical laser potential, which causes the Landau instability [13, 14], could be used for observing the DG instability in experiments. The details on the dynamics after the Landau instability of Kelvin waves have been discussed in Ref. [15].

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References

[1] Thomson W 1880, Phil Mag. 10, 155.
[2] Vinen W F 2006 J. Low Temp. Phys. 145 7.
[3] Isoshima T and Machida K 1999 Phys. Rev. A 59, 2203.
[4] Bretin V, Rosenbusch P, Chevy F, Shlyapnikov G V, and Dalibard J 2003 Phys. Rev. Lett. 90, 100403.
[5] Fetter A L 2004 Phys. Rev. A 69, 043617.
[6] Pitaevskii L P 1961 Zh. Eksp. Teor. Fiz. 40, 646 [1961 Sov. Phys. JETP 13, 451].
[7] Donnelly R J 1991 Quantized Vortices in Helium II (Cambridge University Press).
[8] Glaberson W J, Johnson W W and Ostermeier R M 1974 Phys. Rev. Lett. 33, 1197.
[9] Bekenstein J D and Schiffer M 1998 Phys. Rev. D 58, 064014.
[10] Isoshima T and Machida K 1999 J. Phys. Soc. Jpn. 68, 487.
[11] Tsubota M, Araki T, and Barenghi C F 2003 Phys. Rev. Lett. 90, 205301.
[12] Kobayashi M and Tsubota M 2007 Phys. Rev. A 76, 045603.
[13] Raman C, M. Köhl, R. Onofrio, D. S. Durfee, C. E. Kuklewicz, Z. Hadzibabic, and W. Ketterle 1999 Phys. Rev. Lett. 83, 2502.
[14] Miller D E, J. K. Chin, C. A. Stan, Y. Liu, W. Setiawan, C. Sanner, and W. Ketterle 2007 Phys. Rev. Lett. 99, 070402.
[15] H. Takeuchi, K. Kasamatsu, and M. Tsubota, e-print arXiv:0812.0863.