\((\omega, \rho)\)-PERIODIC SOLUTIONS OF ABSTRACT
INTEGRO-DIFFERENTIAL IMPULSIVE EQUATIONS ON
BANACH SPACE

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Abstract. In this paper, we investigate the existence and uniqueness of \((\omega, \rho)\)-
periodic solutions for a class of the abstract impulsive integro-differential equa-
tions on Banach space.

1. Introduction and preliminaries

The class of \((\omega, c)\)-periodic functions was introduced and investigated by E. Al-
varez et al. in \cite{5}-\cite{6}. This type of periodicity naturally arises in the solution \(y(t)\)
of Mathieu’s equations \(y'' + ay = 2q\cos(2t)y\). In \cite{1}, the authors studied the exis-
tence and uniqueness of \((\omega, c)\)-periodic solutions for semilinear evolution equations
\(u' = Au + f(t, u)\) in complex Banach spaces. The notion of \((\omega, c)\)-periodicity was
generalized in \cite{12}, by M. Fečkan, K. Liu and J. Wang, who considered \((\omega, T)\)-
periodic solutions for this class of semilinear evolution equations, where \(T\) is linear
isomorphism on a Banach space \(X\). For some other generalizations of this concept
see \cite{13}.

On the other side, the impulsive differential equations describe evolution pro-
cesses characterized by the fact that at certain moments they experience a change
of state abruptly, i.e., these processes are subject to short-term perturbations whose
duration is negligible compared with the duration of the whole process (see \cite{2}-\cite{3},
\cite{7}-\cite{8}, \cite{10}, \cite{14}, \cite{15}, \cite{20}, \cite{26}, \cite{38}, \cite{41}). Many biological phenomena involving
thresholds, bursting rhythm models in medicine and biology, optimal control
models in economics, pharmacokinetics and frequency modulated systems, do ex-
hibit impulsive effects, \cite{19}. Therefore, the interest for investigating the qualitative
features of the solutions of these impulsive systems is quite big. In \cite{21}, the \((\omega, c)\)-
periodic solutions of impulsive differential systems with coefficient of matrices were
investigated, while the authors of \cite{24} used the fixed point theorems in order to
clarify certain results concerning the existence and uniqueness of \((\omega, c)\)-periodic
solutions for nonlinear impulsive differential equations. The authors of \cite{12} es-
tablished results on the existence and uniqueness of \((\omega, T)\)-periodic solutions for
impulsive linear and semilinear problems. Furthermore, there are many papers on
periodic solutions for periodic system on infinite-dimensional spaces (see \cite{41}, \cite{22},
\cite{29}, \cite{31}); we also refer to \cite{10} and \cite{25}, where the integro-differential systems

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on finite and infinite-dimensional Banach space are investigated. The existence of piecewise continuous mild solutions and optimal control of integro-differential systems is presented in [41]. In [36]-[37], the integro-differential impulsive periodic systems on infinite-dimensional spaces are discussed. To our best knowledge, the existence and uniqueness of \((\omega, c)\)-periodic solutions for integro-differential systems \((c \in \mathbb{C}, c \neq 0)\) have not been extensively studied.

As a continuation of the investigations on \((\omega, \mathbb{T})\)-periodic solutions for linear and semilinear problems, and periodic solutions for integro-differential impulsive periodic systems, we consider here \((\omega, \rho)\)-periodic solutions of impulsive differential equations as a generalization of the previous concepts (see [1], [3]-[7], [9], [11]-[12], [17]-[18], [21]-[23], [25], [29]-[31], [34]-[40]). The main aim of this paper is to present results concerning the existence and uniqueness of \((\omega, \rho)\)-periodic solutions for certain classes of abstract semilinear integro-differential impulsive equations, considered on a infinite-dimensional pivot Banach space \(X\).

The organization of paper can be described briefly as follows. After recalling some preliminary results and definitions from the theory of \((\omega, c)\)-periodic functions and strongly continuous semigroups of bounded operators, we present some results on the solutions of the nonhomogeneous linear impulsive equations and certain useful estimates for the further investigations. In the last section, we use the Banach fixed point theorem and the Schauder fixed point theorem to prove the existence and uniqueness of the \((\omega, \rho)\)-solutions for semilinear integro-differential equations under our considerations.

### 1.1. Preliminaries

Let \(I = \mathbb{R}\) or \(I = [0, \infty)\). By \((X, \| \cdot \|)\) is denoted a complex Banach space. The abbreviations \(C_b(I : X)\) and \(\mathcal{C}(K : X)\), where \(K\) is a non-empty compact subset of \(\mathbb{R}\), stand for the spaces of bounded continuous functions \(I \mapsto X\) and continuous functions \(K \mapsto X\), respectively. Both spaces are Banach ones endowed with the sup-norm. The space of \(X\)-valued piecewise continuous functions on \(I\) is given by

\[
\mathcal{PC}(I : X) = \{ y : I \to X : y \in \mathcal{C}((t_i, t_{i+1}] : X), \quad t_i \neq 0, \text{ for all } i \in \mathbb{N}, y(t_i^-) = y(t_i), \text{ and } y(t_i^+) \text{ exist for any } i \in \mathbb{N} \},
\]

where the symbols \(y(t_i^-)\) and \(y(t_i^+)\) denote the left and the right limits of the function \(y(t)\) at the point \(t = t_i, i \in \mathbb{N}\), respectively. Let us recall that \(\mathcal{PC}(I : X)\) is a Banach space endowed with the sup-norm.

For an operator family \((T(t))_{t \geq 0}\) of a bounded linear operators on a Banach space \(X\) it is said that is a strongly continuous semigroup of bounded linear operators (shortly \(C_0\) semigroup) if and only if:

(i) For all \(t, s \geq 0\), we have \(T(t+s) = T(t)T(s)\);

(ii) \(T(0) = E\), the identity operator on \(X\);

(iii) For every \(x \in X\), we have \(\lim_{t \to 0} T(t)x = x\).

Let \(\rho : X \to X\). A function \(f : [0, \infty) \to X\) is called \((\omega, \rho)\)-periodic function (see [12]) if and only if there is a real number \(\omega > 0\) such that \(f(t + \omega) = \rho f(t)\) for all \(t \geq 0\). By \(\Phi_{\omega, \rho}\) we denote the set of all piecewise continuous and \((\omega, \rho)\)-periodic functions, i.e.,

\[
\Phi_{\omega, \rho} = \{ y : y \in \mathcal{PC}([0, \infty) : X) \text{ and } y(\cdot + \omega) = \rho y(\cdot) \}.
\]
We continue the investigations started in [12] by studying the \((\omega, \rho)\)-periodic solutions of the following abstract integro-differential impulsive equation

\[
\begin{align*}
\dot{y}(t) &= Ay(t) + f \left( t, y(t), \int_0^t g(t, s, y(t)) \, ds \right), \quad t \neq \tau_k, \ k \in \mathbb{N}; \\
\Delta y|_{t=\tau_k} &= B_k y(t) + d_k, \quad k \in \mathbb{N},
\end{align*}
\]

where \(A\) is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators \((T(t))_{t \geq 0}\), and \(B_k\) is a bounded linear operator on \(X\) for all \(k \in \mathbb{N}\).

In this paper, we consider the following assumptions:

(A1) \(A\) is the infinitesimal generator of a strongly continuous semigroup of bounded operators \((T(t))_{t \geq 0}\) in \(X\). The operators \(B_k, k \in \mathbb{N}\) are bounded linear operators and \(T(t)B_k = B_k T(t)\), for all \(k \in \mathbb{N}\), \(t \geq 0\).

(A2) The constants \(d_k\) and the time sequence \(\tau_k > 0\) are such that \(B_{k+m} = B_k\), \(d_{k+m} = \rho d_k\), \(\tau_{k+m} = \tau_k + \omega\), \(k \in \mathbb{N}\), for some fixed \(m = i(0, \omega)\), where by \(i(0, s)\) is denoted the number of impulsive points between \([0, s]\).

(A3) \(\rho : X \to X\) is a linear isomorphism and \(\rho A = A \rho\), \(\rho B_k = B_k \rho\) for all \(k \in \mathbb{N}\).

(A4) The operator \(\rho - T(\omega) \prod_{k=1}^m (E + B_k)\) is injective.

(A5) For all \(t \geq 0\) and \(y \in X\), we have

\[
f \left( t + \omega, \rho y, \rho \int_0^t g(t, s, y) \, ds \right) = \rho f \left( t, y, \int_0^t g(t, s, y) \, ds \right).
\]

(A6) For all \(t \geq s \geq 0\) and \(y \in X\), it holds

\[
g(t + \omega, s, \rho y) = \rho g(t, s, y).
\]

(A7) Let \(f : [0, \infty) \times X \times X \to X\) and the function \(t \mapsto (t, x, y)\) be measurable for all \((x, y) \in X \times X\). For every \(\nu > 0\), there exists \(L_f(\nu) > 0\) such that for almost all \(t \geq 0\) and all \(x_1, x_2, y_1, y_2 \in X\) with \(|x_1|, |x_2|, |y_1|, |y_2| \leq \nu\), we have

\[
|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_f(\nu) \left( |x_1 - x_2| + |y_1 - y_2| \right).
\]

Set \(D := \{(t, s) \geq 0 \times [0, \infty) : 0 \leq s \leq t\}\). The function \(g : D \times X \to X\) is continuous, and for each \(\nu > 0\) there exists \(L_g(\nu) > 0\) such that for each \((t, s) \in D\) and for each \(x, y \in X\) with \(|x|, |y| \leq \nu\), we have

\[
|g(t, s, x) - g(t, s, y)| \leq L_g(\nu) \|x - y\|.
\]

(A8) There are constants \(\alpha, \beta \geq 0\) such that

\[
\left\| f \left( t, y(t), \int_0^t g(t, s, y(t)) \, ds \right) \right\| \leq \alpha + \beta \|y\|,
\]

for any \(t \geq 0\).

(A9) Let \(M \geq 1\) and \(\gamma \in \mathbb{R}\) be such that \(|T(t)| \leq Me^{\gamma t}\) for all \(t \geq 0\).

(A10) \(X\) is finite-dimensional.
2. Nonhomogeneous Linear Impulsive Problem

Of concern is the following equation

\begin{equation}
 y'(t) = f(t, y(t), \int_0^t g(t, s, y(s)) \, ds)
\end{equation}

accompanied with the conditions

\begin{equation}
 f(t + \omega, \rho y, \rho \int_0^{t+\omega} g(t, s, y(s)) \, ds) = \rho f(t, y, \int_0^t g(t, s, y(s)) \, ds)
\end{equation}

and

\begin{equation}
 g(t + \omega, s, \rho y) = \rho g(t, s, y).
\end{equation}

We need the following auxiliary lemma:

**Lemma 2.1.** Let $f$ be continuous and locally Lipschitzian in last two coordinates. If

\begin{equation}
 f(t + \omega, \rho y(t), \rho \int_0^{t+\omega} g(t, s, y(t)) \, ds) = \rho f(t, y(t), \int_0^t g(t, s, y(s)) \, ds), \quad t \geq 0,
\end{equation}

then $y(t)$ is a solution of equation

\begin{equation}
 y'(t) = f(t, y(t), \int_0^t g(t, s, y(s)) \, ds)
\end{equation}

satisfying $t \in \Phi_{\omega, \rho}$ if and only if we have

\[ y(\omega) = \rho y(0). \]

**Proof.** Let $y \in \Phi_{\omega, \rho}$. By the definition of $\Phi_{\omega, \rho}$, we have $y(t + \omega) = \rho y(t)$ for all $t \geq 0$. Put $t = 0$, so $y(\omega) = \rho y(0)$. Now, let $y(\omega) = \rho y(0)$. Set $x(t) := \rho^{-1} y(t + \omega)$, $t \geq 0$. Using (2.2) and (2.3), we have

\[
x'(t) = \rho^{-1} y'(t + \omega)
\]

\[
= \rho^{-1} f(t + \omega, y(t + \omega), \int_0^{t+\omega} g(t + \omega, s, y(t + \omega)) \, ds)
\]

\[
= \rho^{-1} f(t + \omega, \rho \rho^{-1} y(t + \omega), \int_0^{t+\omega} g(t + \omega, s, \rho \rho^{-1} y(t + \omega)) \, ds)
\]

\[
= \rho^{-1} f(t + \omega, \rho x(t), \int_0^{t+\omega} g(t, s, \rho x(t)) \, ds)
\]
Additionally,

\[ x(0) = \rho^{-1} y(\omega) = y(0). \]

Now, both \( y(t) \) and \( x(t) \), \( t \geq 0 \), satisfy (2.2) and \( y(0) = x(0) \). From the uniqueness of the solutions, we conclude that \( x(t) = y(t) \), so \( y(t + \omega) = \rho y(t), \ t \geq 0 \). \( \square \)

At the very beginning of our work, we consider the homogeneous linear impulsive evolution equation.

**Lemma 2.2.** Let (A1)–(A3) hold. Then the homogeneous linear impulsive evolution equation

\[ \begin{cases} \dot{y}(t) = Ay(t), & t \neq \tau_k, \ k \in \mathbb{N} \\ \Delta y|_{t=\tau_k} = B_k y(t) + d_k, & k \in \mathbb{N}, \end{cases} \tag{2.4} \]

has a solution \( y \in \Phi_{\omega, \rho} \) if and only if \( y(\omega) = \rho y(0) \) or

\[ (\rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k)) y(0) = \sum_{0 < \tau_i < \omega} T(\omega - \tau_i) \prod_{k=1}^{i(\tau_i, \omega)} (E + B_k) d_i. \]

**Proof.** The direct implication is obvious. We prove the opposite direction. Let us assume that \( y(\omega) = \rho y(0) \). The solution of (2.4), with \( y(0) \), for any \( t \neq \tau_k, \ k \in \mathbb{N} \) is given by the formula (22, (2.21)):

\[
\begin{align*}
y(t) &= T(t) \prod_{k=1}^{i(0,t)} (E + B_k) y(0) + \int_{0}^{t} T(t - \tau) \prod_{k=1}^{i(\tau, t)} (E + B_k) f(\tau) d\tau \\
&\quad + \sum_{0 < \tau_i < t} T(t - \tau_i) \prod_{k=1}^{i(\tau_i, t)} (E + B_k) d_i, \quad t \geq 0.
\end{align*}
\]

For every \( t \geq 0 \) and \( t \neq \tau_k \) for any \( k \in \mathbb{N} \), we have:

\[
\begin{align*}
y(t + \omega) &= T(t + \omega) \prod_{k=1}^{i(0,t+\omega)} (E + B_k) y(0) + \sum_{0 < \tau_i < t+\omega} T(t + \omega - \tau_i) \prod_{k=1}^{i(\tau_i, t+\omega)} (E + B_k) d_i \\
&= T(t) T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \cdot \prod_{k=1}^{i(0,\omega)} (E + B_k) y(0) \\
&\quad + \sum_{0 < \tau_i < t+\omega} T(t) T(\omega - \tau_i) \prod_{k=1}^{i(\tau_i, t+\omega)} (E + B_k) \cdot \prod_{k=1}^{i(\tau_i, t+\omega)} (E + B_k) d_i
\end{align*}
\]
\[ f \]

\[ \text{Lemma 2.3.} \]

\[ \text{Proof.} \]

Using the formula \([28, (2.21)]\) and Lemma 2.1, we obtain

\[ H(t, \tau) = \begin{cases} 
T(t) \prod_{k=1}^{i(t, \tau)} (E + B_k) 
& \text{for } \tau \leq t \leq \omega, \\
+ \sum_{0 < \tau_k < t} T(\omega - \tau) \prod_{k=1}^{i(\tau_k, \omega)} (E + B_k)d_i 
& \text{for } 0 < \tau < t, \\
+ \sum_{\omega < \tau_k < t} T(\omega - \tau) \prod_{k=1}^{i(\tau_k, \omega)} (E + B_k)d_i 
& \text{for } t < \tau < \omega. 
\end{cases} \]

Next, we consider the \((\omega, \rho)\)-periodic solutions of the following problem:

\[ \begin{aligned} 
& \dot{y}(t) = Ay(t) + f(t), \quad t \neq \tau_k, \quad k \in \mathbb{N} \\
& \Delta y(t) = B_k y(t) + d_k, \quad k \in \mathbb{N}, 
\end{aligned} \]  

where \( f \in C([0, \infty) : X) \) and \( f \) is an \((\omega, \rho)\)-periodic function.

**Lemma 2.3.** Let \((A1)-(A4)\) hold. Then the \((\omega, \rho)\)-periodic solution \( y \in \Psi = PC([0, \omega] : X) \) of \((2.2)\) is given by

\[ y(t) = \int_0^\omega H(t, \tau) f(\tau) d\tau + \sum_{i=1}^m H(t, \tau_i) d_i, \]

where the function \( H(\cdot, \cdot) \) is given by

\[ H(t, \tau) = \begin{cases} 
T(t) \prod_{k=1}^{i(t, \tau)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(\tau, \omega)} (E + B_k) \right)^{-1} T(\omega - t) \prod_{k=1}^{i(\tau, \omega)} (E + B_k) + E 
& \text{for } \tau \leq t \leq \omega, \\
+ \sum_{0 < \tau_k < t} T(\omega - \tau) \prod_{k=1}^{i(\tau_k, \omega)} (E + B_k)d_i 
& \text{for } 0 < \tau < t, \\
+ \sum_{\omega < \tau_k < t} T(\omega - \tau) \prod_{k=1}^{i(\tau_k, \omega)} (E + B_k)d_i 
& \text{for } t < \tau < \omega. 
\end{cases} \]

**Proof.** Using the formula \([28, (2.21)]\) and Lemma 2.1, we obtain

\[ y(\omega) = T(\omega) \prod_{k=1}^{i(\omega, \omega)} (E + B_k)y_0 + \int_0^\omega T(\omega - \tau) \prod_{k=1}^{i(\tau, \omega)} (E + B_k)f(\tau) d\tau 
+ \sum_{i=1}^m T(\omega - \tau_i) \prod_{k=1}^{i(\tau_i, \omega)} (E + B_k)d_i = \rho y_0. \]
Hence,

\[
y_0 = \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} + \left( \int_0^\omega T(\omega - \tau) \prod_{k=1}^{i(0,\omega)} (E + B_k) f(\tau) d\tau \right)
+ \sum_{i=1}^m T(\omega - \tau_i) \prod_{k=1}^{i(\tau,\omega)} (E + B_k)d_i,
\]

so the solution of (2.5) can be written as

\[
y(t) = T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \left( \int_0^\omega T(\omega - \tau) \prod_{k=1}^{i(0,\omega)} (E + B_k) f(\tau) d\tau \right)
+ \sum_{i=1}^m T(t - \tau_i) \prod_{k=1}^{i(\tau,\omega)} (E + B_k)d_i
+ \sum_{0 < \tau_i < t} T(t - \tau_i) \prod_{k=1}^{i(\tau,\omega)} (E + B_k)d_i
\]

\[
= \int_0^t T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \left( \int_0^\omega T(\omega - \tau) \prod_{k=1}^{i(0,\omega)} (E + B_k) f(\tau) d\tau \right)
+ \sum_{i=1}^m T(t) \prod_{k=1}^{i(0,\omega)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \left( \int_0^\omega T(\omega - \tau) \prod_{k=1}^{i(0,\omega)} (E + B_k) d\tau \right)
+ \sum_{0 < \tau_i < t} T(t - \tau_i) \prod_{k=1}^{i(\tau,\omega)} (E + B_k)d_i
\]

\[
= \int_0^t T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \left( \int_0^\omega T(\omega - \tau) \prod_{k=1}^{i(0,\omega)} (E + B_k) f(\tau) d\tau \right)
+ \sum_{i=1}^m T(t) \prod_{k=1}^{i(0,\omega)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \left( \int_0^\omega T(\omega - \tau) \prod_{k=1}^{i(0,\omega)} (E + B_k) d\tau \right)
+ \sum_{0 < \tau_i < t} T(t - \tau_i) \prod_{k=1}^{i(\tau,\omega)} (E + B_k)d_i
\]

\[
= \int_0^t T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \left( \int_0^\omega T(\omega - \tau) \prod_{k=1}^{i(0,\omega)} (E + B_k) f(\tau) d\tau \right)
+ \sum_{i=1}^m T(t) \prod_{k=1}^{i(0,\omega)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \left( \int_0^\omega T(\omega - \tau) \prod_{k=1}^{i(0,\omega)} (E + B_k) d\tau \right)
+ \sum_{0 < \tau_i < t} T(t - \tau_i) \prod_{k=1}^{i(\tau,\omega)} (E + B_k)d_i
\]

\[
= \int_0^t T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \left( \int_0^\omega T(\omega - \tau) \prod_{k=1}^{i(0,\omega)} (E + B_k) f(\tau) d\tau \right)
+ \sum_{i=1}^m T(t - \tau_i) \prod_{k=1}^{i(\tau,\omega)} (E + B_k)d_i
\]
\[ \times T(t-\tau) \prod_{k=1}^{i(\tau,t)} (E+B_k)f(\tau) \, d\tau + \int_0^{\omega} T(t) \prod_{k=1}^{i(0,t)} (E+B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E+B_k) \right)^{-1} T(\omega-\tau) \prod_{k=1}^{i(\tau,\omega)} (E+B_k)f(\tau) \, d\tau + \sum_{0<\tau_i<t} \left( T(t) \prod_{k=1}^{i(0,t)} (E+B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E+B_k) \right)^{-1} T(\omega-\tau_i) \prod_{k=1}^{i(\tau_i,\omega)} (E+B_k) + E \right) \times T(t-\tau_i) \prod_{k=1}^{i(\tau_i,t)} (E+B_k) d_i \]

\[ + \sum_{0<\tau_i<\omega} T(t) \prod_{k=1}^{i(0,t)} (E+B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E+B_k) \right)^{-1} T(\omega-\tau_i) \prod_{k=1}^{i(\tau_i,\omega)} (E+B_k) d_i \]

\[ = \int_0^{\omega} H(t,\tau)f(\tau) \, d\tau + \sum_{i=1}^{m} H(t,\tau_i) d_i. \]

As a consequence of the previous result, we can state the following:

**Lemma 2.4.** Let (A1)–(A4) hold, and let for each \( t \geq 0 \) we have

\[ \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E+B_k) \right)^{-1} T(t) = T(t) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E+B_k) \right)^{-1}. \]

Then the unique \((\omega,\rho)\)-periodic solution \( y \in \Psi \) of (2.5) is given by

\[ y(t) = \int_0^{\omega} H(t,\tau)f(\tau) \, d\tau + \sum_{i=1}^{m} H(t,\tau_i) d_i, \]

where \( H(\cdot,\cdot) \) is defined by

\[ H(t,\tau) := \begin{cases} 
\rho \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E+B_k) \right)^{-1} T(t-\tau) \prod_{k=1}^{i(\tau,t)} (E+B_k), & 0 < \tau < t; \\
T(t+\omega-\tau) \prod_{k=1}^{i(0,t)+i(\tau,\omega)} (E+B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E+B_k) \right)^{-1}, & t \leq \tau < \omega.
\end{cases} \]

**Proof.** By the foregoing, the unique solution of considered problem is given by

\[ y(t) = T(t) \prod_{k=1}^{i(0,t)} (E+B_k)y_0 + \int_0^{t} T(t-\tau) \prod_{k=1}^{i(\tau,t)} (E+B_k)f(\tau) \, d\tau + \sum_{0<\tau_i<t} T(t-\tau_i) \prod_{k=1}^{i(\tau_i,t)} (E+B_k) d_i. \]
Using Lemma 2.1, we obtain

\[ y(\omega) = T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) y_0 + \int_0^{\omega} T(\omega - \tau) \prod_{k=1}^{i(\tau,\omega)} (E + B_k) f(\tau) d\tau \]

\[ + \sum_{i=1}^{m} T(\omega - \tau_i) \prod_{k=1}^{i(\tau_i,\omega)} d_i = \rho y_0. \]

Now,

\[ y_0 = \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \left( \int_0^{\omega} T(\omega - \tau) \prod_{k=1}^{i(\tau,\omega)} (E + B_k) d(\tau) \right) \]

\[ + \sum_{i=1}^{m} T(\omega - \tau_i) \prod_{k=1}^{i(\tau_i,\omega)} d_i, \]

Hence, the solution of (2.5) can be written as

\[ y(t) = \]

\[ = T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \left( \int_0^{\omega} T(\omega - \tau) \prod_{k=1}^{i(\tau,\omega)} (E + B_k) d(\tau) \right) \]

\[ + \sum_{i=1}^{m} T(t - \tau_i) \prod_{k=1}^{i(\tau_i,t)} (E + B_k) d_i \]

\[ = \int_0^{\omega} T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} T(\omega - \tau) \prod_{k=1}^{i(\tau,\omega)} (E + B_k) d(\tau) \]

\[ + \sum_{0 < \tau_i < t} T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} T(\omega - \tau_i) \prod_{k=1}^{i(\tau_i,\omega)} (E + B_k) d_i \]

\[ + \sum_{t \leq \tau_i < \omega} T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} T(\omega - \tau_i) \prod_{k=1}^{i(\tau_i,\omega)} (E + B_k) d_i. \]
Lemma 2.5. Let

\[ T(t) = \frac{\pi(t,t)}{\tau} \prod_{k=1}^{i(\tau,t)} (E + B_k) \]

for any \( \tau \). Then

\[ \sum_{i=1}^{m} \| H(t, \tau_i) \| d_i \leq C_1 \]

\[ M \max \left\{ \prod_{k=1}^{i(\tau,t)} (E + B_k), 1 \right\} \cdot \max \left\{ e^{2\gamma_\omega}, 1 \right\} \cdot \left( M \right) \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(\tau,t)} (E + B_k) \right)^{-1} \right\| + 1 \]

\[ \times \sum_{1<i<m} e^{(\omega-\gamma_\omega)} \| d_i \|, \quad \gamma > 0; \]

\[ M \max \left\{ \prod_{k=1}^{i(\tau,t)} (E + B_k), 1 \right\} \left( M \right) \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(\tau,t)} (E + B_k) \right)^{-1} \right\| + 1 \]

\[ \times \sum_{1<i<m} \| d_i \|, \quad \gamma \leq 0, \]

for any \( t \in [0, \omega] \).

Proof. We have

\[ \sum_{i=1}^{m} \| H(t, \tau_i) \| \cdot \| d_i \| = \sum_{0<\tau_i<t} \| H(t, \tau_i) \| \cdot \| d_i \| + \sum_{t\leq\tau_i<\omega} \| H(t, \tau_i) \| \cdot \| d_i \| , \]
so that

\[
\sum_{i=1}^{m} \| H(t, \tau_i) \| \cdot \| d_i \|
\]

\[
\leq \sum_{0 < \tau_i < t} \left( \| T(t) \| \prod_{k=1}^{i(0,t)} (E + B_k) \| \cdot \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \|^{-1} \cdot \| T(\omega - \tau_i) \| \right.
\]

\[
\times \left. \left| \prod_{k=1}^{i(0,t)} (E + B_k) \| \cdot \| (\rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \|^{-1} \cdot \| T(\omega - \tau_i) \| \cdot \left| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right| \cdot \| d_i \| \right)
\]

\[
\leq \sum_{0 < \tau_i < \omega} \| T(t) \| \cdot \left| \prod_{k=1}^{i(0,t)} (E + B_k) \right| \cdot \left| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right| \cdot \left| \prod_{k=1}^{i(0,\omega)} M e^{\gamma t} \cdot \prod_{k=1}^{i(0,\omega)} (E + B_k) \right| \cdot \| d_i \|
\]

\[
\leq \left| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right| \cdot \| d_i \| + \sum_{0 < \tau_i < \omega} \| T(t - \tau_i) \| \cdot \left| \prod_{k=1}^{i(0,\omega)} M e^{\gamma (t - \tau_i)} \cdot \prod_{k=1}^{i(0,\omega)} (E + B_k) \right| \cdot \| d_i \|
\]

\[
\leq \left| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right| \cdot \| d_i \| + \sum_{0 < \tau_i < \omega} M e^{\gamma (t - \tau_i)} \cdot \| d_i \|
\]

\[
\leq M \max \left\{ \left| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right|, 1 \right\} \cdot \left| \prod_{k=1}^{i(0,\omega)} \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right) \right|^{-1}
\]

\[
\times \sum_{0 < \tau_i < \omega} M e^{\gamma (t - \tau_i)} \cdot \| d_i \|
\]

for any \( t \in [0, \omega] \). We will consider separately the following two cases: \( \gamma > 0 \) and \( \gamma \leq 0 \). For \( \gamma > 0 \), we have

\[
\sum_{i=1}^{m} \| H(t, \tau_i) \| \cdot \| d_i \|
\]

\[
\leq M \max \left\{ \left| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right|, 1 \right\} \cdot \left| \prod_{k=1}^{i(0,\omega)} \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right) \right|^{-1}
\]

\[
\times \sum_{0 < \tau_i < \omega} M e^{\gamma (t - \tau_i)} \cdot \| d_i \|
\]
Under an additional condition, we can state the following modified version of previous lemma:

Lemma 2.6. Let (A1)–(A4) hold, and let

$$\left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \quad T(t) = T(t) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1}, \quad t \geq 0.$$
Then
\[
\sum_{i=1}^{m} \| H(t, \tau_i) d_i \| \leq C_1' \equiv \begin{cases} 
M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\} \\
\times \max \left\{ \| T \|, e^{\gamma} \right\} \cdot \sum_{0<\tau_i<\omega} e^{\gamma(\omega-\tau_i)} \| d_i \|, \gamma > 0; \\
M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\} \\
\times \max \left\{ \| T \|, 1 \right\} \cdot \sum_{1<i<m} \| d_i \|, \gamma \leq 0.
\end{cases}
\]

Proof. Using (A9), we obtain
\[
\sum_{i=1}^{m} \| H(t, \tau_i) \| \cdot \| d_i \| = \sum_{0<\tau_i<t} \| H(t, \tau_i) \| \cdot \| d_i \| + \sum_{t\leq\tau_i<\omega} \| H(t, \tau_i) \| \cdot \| d_i \|
\leq \sum_{0<\tau_i<t} \| T \| \cdot \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \| T(t - \tau_i) \| \cdot \left( \prod_{k=1}^{i(\tau_i,t)} (E + B_k) \right)^{-1} \| d_i \|
+ \sum_{t\leq\tau_i<\omega} \| T(t + \omega - \tau_i) \| \cdot \left( \prod_{k=1}^{i(0,t)+i(\tau_i,\omega)} (E + B_k) \right)^{-1} \| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \| d_i \|
\leq M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\}
\times \left( \sum_{0<\tau_i<t} \| T \| e^{\gamma(\tau_i)} \| d_i \| + \sum_{t\leq\tau_i<\omega} e^{\gamma(t + \omega - \tau_i)} \| d_i \| \right),
\]
for all \( t \in [0, \omega] \). The following two cases emerge: \( \gamma > 0 \) and \( \gamma \leq 0 \). For \( \gamma > 0 \), we have
\[
\sum_{i=1}^{m} \| H(t, \tau_i) \| \cdot \| d_i \|
\leq M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\}
\times \left( \sum_{0<\tau_i<t} \| T \| e^{\gamma(\omega - \tau_i)} \| d_i \| + \sum_{t\leq\tau_i<\omega} e^{\gamma(2\omega - \tau_i)} \| d_i \| \right)
\leq M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\}
\times \max \left\{ \| T \|, e^{\gamma\omega} \right\} \cdot \sum_{0<\tau_i<\omega} e^{\gamma(\omega - \tau_i)} \| d_i \|.\]
For $\gamma \leq 0$, we have
\[
\sum_{i=1}^{m} \left\| H(t, \tau_i) \right\| \cdot \left\| d_i \right\| \leq M \left\{ \left( \rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k) \right)^{-1} \max \left\{ \left\| \prod_{k=1}^{i(0, \omega)} (E + B_k) \right\|, 1 \right\} \right. \\
\times \left. \left( \sum_{0<\tau_i<t} \|T\| \cdot \left\| d_i \right\| + \sum_{t<\tau_i<\omega} \left\| d_i \right\| \right) \right\}
\leq \left\{ \left( \rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k) \right)^{-1} \max \left\{ \left\| \prod_{k=1}^{i(0, \omega)} (E + B_k) \right\|, 1 \right\} \right. \\
\times \left. \max \{\|T\|, 1\} \cdot \sum_{1<i<m} \left\| d_i \right\| \right\}.
\]

This completes the proof of lemma. $\square$

We also need the following lemma:

**Lemma 2.7.** Let (A1)-(A4) hold, and let $t \in (0, \omega)$. Then
\[
\int_{0}^{\omega} \left\| H(t, \tau) \right\| \, d\tau \leq C_2
\]
\[
= \begin{cases} 
M \max \left\{ \left\| \prod_{k=1}^{i(0, \omega)} (E + B_k)^2 \right\|, 1 \right\} \left( \rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k) \right)^{-1} e^{\gamma \omega} \\
+ \max \left\{ \left\| \prod_{k=1}^{i(0, \omega)} (E + B_k) \right\|, 1 \right\} M e^{\gamma \omega} - 1, & \gamma \neq 0; \\
M \max \left\{ \left\| \prod_{k=1}^{i(0, \omega)} (E + B_k)^2 \right\|, 1 \right\} \left( \rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k) \right)^{-1} \right\} \\
+ \max \left\{ \left\| \prod_{k=1}^{i(0, \omega)} (E + B_k) \right\|, 1 \right\} M \omega, & \gamma = 0.
\end{cases}
\]
Proof. We have

\[
\int_0^\omega \|H(t, \tau)\| \, d\tau \\
= \int_0^t \left\| T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} T(\omega - t) \prod_{k=1}^{i(t,\omega)} (E + B_k) + E \right\| \, d\tau \\
\times T(t - \tau) \prod_{k=1}^{i(\tau,t)} (E + B_k) \, d\tau \\
+ \int_0^t \left\| T(t) \prod_{k=1}^{i(0,t)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} T(\omega - t) \prod_{k=1}^{i(t,\omega)} (E + B_k) \right\| \, d\tau \\
\leq \int_0^t \|T(t)\| \left\| \prod_{k=1}^{i(0,t)} (E + B_k) \right\| \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \|T(\omega - \tau)\| \times \prod_{k=1}^{i(\tau,t)} (E + B_k) \, d\tau \\
+ \int_0^t \|T(t)\| \left\| \prod_{k=1}^{i(0,t)} (E + B_k) \right\| \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \prod_{k=1}^{i(t,\omega)} (E + B_k) \, d\tau \\
\leq M^2 \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k)^2 \right\| , 1 \right\} \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \int_0^\omega e^{\gamma(\omega + t - \tau)} \, d\tau \\
+ M \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\| , 1 \right\} \int_0^t e^{\gamma(t - \tau)} \, d\tau.
\]

Now, we have two subcases: \( \gamma \neq 0 \) and \( \gamma = 0 \). For \( \gamma \neq 0 \), we have

\[
\int_0^\omega \|H(t, \tau)\| \, d\tau \\
\leq M^2 \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k)^2 \right\| , 1 \right\} \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \int_0^\omega e^{\gamma(\omega + t - \tau)} \, d\tau \\
+ M \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\| , 1 \right\} \int_0^t e^{\gamma(t - \tau)} \, d\tau
\]

Then we have

\[
\leq M^2 \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\} \cdot \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \frac{e^{\gamma(t+\omega)} - e^{\gamma t}}{\gamma} \\
+ M \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\} \cdot \frac{e^{\gamma t} - 1}{\gamma} \\
\leq \left\{ M \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\} \cdot \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| e^{\gamma \omega} \\
+ \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\} \right\} \frac{e^{\gamma \omega} - 1}{\gamma}.
\]

For \( \gamma = 0 \), we have

\[
\int_{0}^{\omega} \| H(t, \tau) \| d\tau \leq M^2 \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\} \cdot \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \int_{0}^{\omega} e^{\gamma (t+\tau)} d\tau \\
+ M \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\} \cdot \int_{0}^{t} e^{\gamma (t-\tau)} d\tau \\
\leq \left\{ M \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\} \cdot \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \\
+ \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\} \right\} M \omega.
\]

With an additional condition from the previous lemma, we can state the following:

**Lemma 2.8.** Let (A1)-(A4) hold, and let

\[
\left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} T(t) = T(t) \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1}, \quad t \geq 0.
\]

Then we have

\[
\int_{0}^{\omega} \| H(t, \tau) \| d\tau \leq C_2' \equiv \left\{ \begin{array}{ll}
M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| & \times \max \{\|T\|, 1\} \frac{e^{\gamma \omega} - 1}{\gamma}, \quad \gamma \neq 0; \\
M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| & \times \max \{\|T\|, 1\} \omega, \quad \gamma = 0,
\end{array} \right.
\]

\[\leq C_2' \equiv \left\{ \begin{array}{ll}
M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| & \times \max \{\|T\|, 1\} \frac{e^{\gamma \omega} - 1}{\gamma}, \quad \gamma \neq 0; \\
M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| & \times \max \{\|T\|, 1\} \omega, \quad \gamma = 0,
\end{array} \right.\]
for any $t \in (0, \omega)$.

Proof. We have

$$
\int_0^\omega \|H(t, \tau)\| \, d\tau
\leq \int_0^t \|T\| \cdot \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \cdot \left\| T(t - \tau) \right\| \cdot \left\| \prod_{k=1}^{i(\tau, t)} (E + B_k) \right\| \, d\tau
+ \int_t^\omega \|T(t + \omega - \tau)\| \cdot \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\| \cdot \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \, d\tau
\leq M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \cdot \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\}
\times \left( \int_0^t \|T\| e^{\gamma(t-\tau)} \, d\tau + \int_t^\omega e^{\gamma(t+\omega-\tau)} \, d\tau \right).
$$

Now, we have two subcases: $\gamma \neq 0$ and $\gamma = 0$. For $\gamma \neq 0$, we have

$$
\int_0^\omega \|H(t, \tau)\| \, d\tau
\leq M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \cdot \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\}
\times \left( \int_0^t \|T\| e^{\gamma(t-\tau)} \, d\tau + \int_t^\omega e^{\gamma(t+\omega-\tau)} \, d\tau \right)
\leq M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \cdot \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\}
\times \left( \|T\| \frac{e^{\gamma t} - 1}{\gamma} + \frac{e^{\gamma \omega} - e^{\gamma t}}{\gamma} \right)
\leq M \left\| \left( \rho - T(\omega) \prod_{k=1}^{i(0,\omega)} (E + B_k) \right)^{-1} \right\| \cdot \max \left\{ \left\| \prod_{k=1}^{i(0,\omega)} (E + B_k) \right\|, 1 \right\}
\times \max \{ \|T\|, 1 \} \left( \frac{e^{\gamma t} - 1}{\gamma} \right),
$$

for any $t \in (0, \omega)$.
For $\gamma = 0$, we have

$$\int_0^\omega \|H(t, \tau)\| d\tau$$

$$\leq M \left( \rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k) \right)^{-1} \max \left\{ \|i(0, \omega)\|, 1 \right\} \max \{\|T\|, 1\} \omega.$$

\[ \Box \]

3. ($\omega, \rho$)-Periodic Solutions of Integro-Differential Impulsive Problem

In this section, we consider the following abstract integro-differential impulsive equation:

$$\begin{cases}
\dot{y}(t) = Ay(t) + f\left(t, y(t), \int_0^t g(t, s, y(s)) ds\right), & t \neq \tau_k, \ k \in \mathbb{N}; \\
\Delta y|_{t=\tau_k} = B_k y(t) + d_k, & k \in \mathbb{N}.
\end{cases}$$

We are going to prove, by using the Banach fixed point theorem and the Schauder fixed point theorem, that under certain conditions, we have the existence and uniqueness of solutions of (1.1).

**Theorem 3.1.** Let (A1)–(A7) and (A9) hold. If $0 < LC_2 < 1$, where $L = \left( L_f(\nu) + M_1 L_g(\nu) \right)$ and $M_1$ is certain constant, then the equation (1.1) has a unique $(\omega, \rho)$-periodic solution $y \in \Phi_{\omega, \rho}$, satisfying

$$\|y\| \leq \frac{\|f\|_0 C_2 + C_1}{1 - LC_2}.$$

where

$$\|f\|_0 := \left\| f\left(\cdot, 0, \int_0^\tau g(\cdot, s, 0) ds\right) \right\|_\infty = \max_{t \in [0, \omega]} \left\| f\left(t, 0, \int_0^t g(t, s, 0) ds\right) \right\|.$$

**Proof.** Note that, if $y \in \Phi_{\omega, \rho}$, then $f(\cdot, y, \int_0^\tau g(\cdot, s, y) ds) \in \Phi_{\omega, \rho}$. Keeping in mind Lemma 2.1 and Lemma 2.3, we need to solve the fixed point problem

$$y(t) = \int_0^\omega H(t, \tau) f\left(\tau, y(\tau), \int_0^\tau g(\tau, s, y(s)) ds\right) d\tau + \sum_{i=1}^m H(t, \tau_i) d_i, \quad t \in [0, \omega].$$

We define the operator $R$ on the space $\Psi$ as

$$(Ry)(t) := \int_0^\omega H(t, \tau) f\left(\tau, y(\tau), \int_0^\tau g(\tau, s, y(s)) ds\right) d\tau + \sum_{i=1}^m H(t, \tau_i) d_i.$$
We will show that $R : \Psi \to \Psi$ is a contraction mapping. Let $y_1, y_2 \in \Psi$. Using Lemma 2.7 we obtain:

$$
\|(Ry_1)(t) - (Ry_2)(t)\| \\
\leq \int_0^{\omega} \|H(t, \tau)\| \cdot \left\| f\left(\tau, y_1(\tau), \int_0^{\tau} g(\tau, s, y_1(s)) \, ds\right) - f\left(\tau, y_2(\tau), \int_0^{\tau} g(\tau, s, y_2(s)) \, ds\right) \right\| \, d\tau \\
\leq \int_0^{\omega} \|H(t, \tau)\| \cdot L_f(\nu) \left(\|y_1 - y_2\| + \left\| \int_0^{\tau} g(\tau, s, y_1(s)) - g(\tau, s, y_2(s)) \, ds \right\| \, d\tau \right) \\
\leq \int_0^{\omega} \|H(t, \tau)\| \cdot L_f(\nu) \left(\|y_1 - y_2\| + L_g(\nu) \int_0^{\tau} \|y_1 - y_2\| \, ds \right) \\
\leq (L_f(\nu) + M_1 L_g(\nu)) \cdot \|y_1 - y_2\| \cdot \int_0^{\omega} \|H(t, \tau)\| \, d\tau \\
\leq LC_2 \cdot \|y_1 - y_2\|,
$$

where $L := L_f(\rho) + M_1 L_g(\rho)$. So,

$$
\|(Ry_1) - (Ry_2)\| \leq LC_2 \|y_1 - y_2\|, \quad 0 < LC_2 < 1.
$$

Hence, the uniqueness of the solution of \((1.1)\) follows by the Banach contraction mapping principle. Moreover, we have

$$
\|y\| = \|Ry\| \\
\leq \int_0^{\omega} \|H(t, \tau)\| \cdot \left\| f\left(\tau, y(\tau), \int_0^{\tau} g(\tau, s, y(s)) \, ds\right) \right\| \, d\tau + \sum_{i=1}^{m} \|H(t_i) d_i\| \\
\leq \int_0^{\omega} \|H(t, \tau)\| \cdot \left\| f\left(\tau, y(\tau), \int_0^{\tau} g(\tau, s, y(s)) \, ds\right) - f\left(\tau, 0, \int_0^{\tau} g(\tau, s, 0) \, ds\right) \right\| \, d\tau
$$
Proof. We consider the operator $R$ defined like in the proof of the previous theorem on $B_l = \{ y \in \Psi : \|y\| \leq l \}$ and the constant $l$ is given by $l = \frac{\alpha C_2 + C_1}{1 - \beta C_2}$. We are going to prove the statement of the theorem in the following steps:

**Step 1.** We show that $R(B_l) \subset B_l$. Let $y \in B_l$ be arbitrary, and let $t \in [0, \omega]$. Then we have:

$$\|Ry(t)\| \leq \int_0^\omega \|H(t, \tau)\| \cdot \left\| f \left( \tau, y(\tau), \int_0^\tau g(t, s, y(s)) \, ds \right) \right\| \, d\tau + \sum_{i=1}^m \|H(t, \tau_i)d_i\|$$

$$\leq \beta \int_0^\omega \|H(t, \tau)\| \cdot \|y(\tau)\| \, d\tau + \alpha \int_0^\omega \|H(t, \tau)\| \, d\tau$$

$$+ \sum_{i=1}^m \|H(t, \tau_i)\| \cdot \|d_i\| \leq \beta C_2 \|y\| + \alpha C_2 + C_1 = l,$$

implying $\|Ry(t)\| \leq l$, so $R(B_l) \subset B_l$ for any $t \in [0, \omega]$. Here the constants $\alpha$ and $\beta$ are from the assumption (A8).

**Step 2.** We prove that the operator $R$ is continuous on $B_l$. Let $(y_n)$ be a Cauchy sequence such that $y_n \to y$, when $n \to \infty$ in $B_l$. Let

$$f_n = f \left( t, y_n(t), \int_0^t g(t, s, y_n(s)) \, ds \right), \quad t \in [0, \omega] \quad \text{and} \quad f = f \left( t, y(t), \int_0^t g(t, s, y(s)) \, ds \right).$$
It is clear that \( f_n \to f \), when \( y_n \to y \), for any \( t \in [0, \omega] \). Now,

\[
\| (R y_n)(t) - (R y)(t) \| \\
\leq \int_0^\omega \| H(t, \tau) \| \cdot \| f\left(t, y_n(t), \int_0^t g(t, s, y_n(s)) \, ds \right) - f\left(t, y(t), \int_0^t g(t, s, y(s)) \, ds \right) \| \, d\tau \\
\leq \int_0^\omega \| H(t, \tau) \| \cdot \| f_n - f \| \, d\tau \leq C_2 \| f_n - f \|.
\]

Hence, \( R \) is continuous operator on \( B_1 \).

**Step 3.** We show that \( R(B_1) \) is relatively compact set. Since \( R(B_1) \subset B_1 \), it follows that \( R(B_1) \) is uniformly bounded. Now, we show that the operator \( R \) is an equicontinuous operator. For any \( t_1, t_2 \in [0, \omega] \) and \( y \in B_1 \), using (A8), we have

\[
\|(R y)(t_2) - (R y)(t_1)\| \\
\leq \int_0^\omega \| H(t_2, \tau) - H(t_1, \tau) \| \cdot \| f\left(\tau, y(\tau), \int_0^\tau g(\tau, s, y(s)) \, ds \right) \| \, d\tau \\
+ \sum_{i=1}^m \| H(t_2, \tau_i) - H(t_1, \tau_i) \| \cdot \| d_i \| \\
\leq (\alpha + \beta \| y \|) \int_0^\omega \| H(t_2, \tau) - H(t_1, \tau) \| \, d\tau + \sum_{i=1}^m \| H(t_2, \tau - i) - H(t_1, \tau_i) \| \cdot \| d_i \|.
\]

We have:

\[
\| H(t_2, \tau) - H(t_1, \tau) \| = \begin{cases} \\
\left\| T(t_2) \prod_{k=1}^{i(0, t_2)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k) \right)^{-1} T(\omega - \tau) \\
- T(t_1) \prod_{k=1}^{i(0, t_1)} (E + B - k) \left( \rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k) \right)^{-1} \\
\times T(\omega - \tau) \prod_{k=1}^{i(\tau, \omega)} (E + B_k) - T(t_1 - \tau) \prod_{k=1}^{i(\tau, t_1)} (E + B_k) \right\|, \\
0 < \tau < t_1 < t_2; \\
\left\| T(t_2) \prod_{k=1}^{i(0, t_2)} (E + B_k) \left( \rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k) \right)^{-1} T(\omega - \tau) \\
- T(t_1) \prod_{k=1}^{i(0, t_1)} (E + B - k) \left( \rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k) \right)^{-1} \\
\times \prod_{k=1}^{i(0, \omega)} (E + B_k) - T(t_1) \prod_{k=1}^{i(0, \omega)} (E + B_k) \left( \rho - T(\omega) \right) \\
\prod_{k=1}^{i(\tau, \omega)} (E + B_k) \right\|, \quad t_1 < t_2 < \tau < \omega.
\end{cases}
\]
If \(0 < \tau < t_1 < t_2\), then we have

\[
\|H(t_2, \tau) - H(t_1, \tau)\| \\
\leq \left\| T(t_2) \prod_{k=1}^{i(0, t_2)} (E + B_k) + T(t_2 - \tau) \prod_{k=1}^{i(\tau, t_2)} (E + B_k)T(\omega - \tau) \right. \\
\times \prod_{k=1}^{i(\tau, \omega)} (E + B_k) + T(t_2 - \tau) \prod_{k=1}^{i(\tau, t_2)} (E + B_k) \\
- T(t_1) \prod_{k=1}^{i(0, t_1)} (E + B_k) + T(t_2 - \tau) \prod_{k=1}^{i(\tau, t_1)} (E + B_k) \\
\times T(\omega - \tau) \prod_{k=1}^{i(\tau, \omega)} (E + B_k) - T(t_1 - \tau) \prod_{k=1}^{i(\tau, t_1)} (E + B_k) \right\| \\
\leq \|T(t_2) - T(t_1)\| \cdot \left\| \prod_{k=1}^{i(0, \omega)} (E + B_k)^2 \right\| \cdot \left\| (\rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k))^{-1} \right\| \\
\times Me^{\gamma(\omega - \tau)} + \|T(t_2 - \tau) - T(t_1 - \tau)\| \cdot \left\| \prod_{k=1}^{i(0, \omega)} (E + B_k) \right\| .
\]

Using (A10), we obtain \(\|H(t_2, \tau) - H(t_1, \tau)\| \to 0\) as \(t_2 \to t_1\).

If \(t_1 < t_2 < \tau < \omega\), we have

\[
\|H(t_2, \tau) - H(t_1, \tau)\| \\
= \left\| T(t_2) \prod_{k=1}^{i(0, t_2)} (E + B_k) \left(\rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k)\right)^{-1} T(\omega - \tau) \right. \\
\times \prod_{k=1}^{i(\tau, \omega)} (E + B_k) - T(t_1) \prod_{k=1}^{i(0, t_1)} (E + B_k) \\
\times \left(\rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k)\right)^{-1} T(\omega - \tau) \prod_{k=1}^{i(\tau, \omega)} (E + B_k) \right\| \\
\leq \|T(t_2) - T(t_1)\| \cdot \left\| \prod_{k=1}^{i(0, \omega)} (E + B_k)^2 \right\| \\
\times \left\| (\rho - T(\omega) \prod_{k=1}^{i(0, \omega)} (E + B_k))^{-1} \right\| Me^{\gamma(\omega - \tau)}.\]

Again, using (A10), we obtain that \(\|H(t_2, \tau) - H(t_1, \tau)\| \to 0\) as \(t_2 \to t_1\). Therefore, for any \(t_1, t_2 \in [0, \omega]\), we have \(H(t_2, \tau) \to H(t_1, \tau)\), when \(t_2 \to t_1\). This implies that \(\|(Ry)(t_2) - (Ry)(t_1)\| \to 0\) when \(t_2 \to t_1\), hence the operator \(R\) is an equicontinuous operator.

**Step 4.** We show that \(R\) maps a bounded set into a precompact set in \(X\). We
consider the approximate operator $R_\varepsilon$ on $B_1$ as following
\[
(R_\varepsilon)(t) = \int_0^\omega H(t - \varepsilon, \tau)f(\tau, y(\tau), \int_0^\tau g(\tau, s, y(\tau))\,ds)\,d\tau + \sum_{i=1}^m H(t - \varepsilon, \tau_i)d_i, \quad t \in [0, \omega].
\]

Let $K = \{(Ry)(t) : t \in [0, \omega]\}$ and
\[
K_\varepsilon = T(\varepsilon)\{(R_\varepsilon y)(t) : t \in [0, \omega]\}, \quad \varepsilon \in (0, \omega).
\]

Since $K$ is bounded and (A10) holds, we obtain that $K_\varepsilon$ is precompact. Next,
\[
\|(R_\varepsilon y)(t) - (Ry)(t)\|
\leq \int_0^\omega \|H(t, \tau) - H(t - \varepsilon, \tau)\| \left\| f\left(\tau, y(\tau), \int_0^\tau g(\tau, s, y(\tau))\,ds\right)\right\|\,d\tau
\]
\[
+ \sum_{i=1}^m \|H(t, \tau_i) - H(t - \varepsilon, \tau_i)\| \cdot \|d_i\|
\]
\[
\leq (\alpha + \beta l) \int_0^\omega \|H(t, \tau) - H(t - \varepsilon, \tau)\|\,d\tau
\]
\[
+ \sum_{i=1}^m \|H(t, \tau_i) - H(t - \varepsilon, \tau_i)\| \cdot \|d_i\|.
\]

Since for any $t_1, t_2 \in (0, \omega)$, $H(t_2, \tau) \to H(t_1, \tau)$ as $t_2 \to t_1$, we have $\|(R_\varepsilon y)(t) - (Ry)(t)\| \to 0$ as $\varepsilon \to 0$. Hence, $K$ can be approximated by a precompact set $K_\varepsilon$ with arbitrary accuracy. So, $K$ itself is a precompact set in $X$, i.e., $R$ maps a bounded set into a precompact set. The Arzelà-Ascoli theorem implies the compactness of the operator $R$, so the statement of the theorem follows by applying the Schauder fixed point theorem. \(\square\)

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