Human Operator Modeling and Lie-Derivative Based Control

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Abstract

The motivation behind mathematically modeling the human operator is to help explain the response characteristics of the complex dynamical system including the human manual controller. In this paper, we present two approaches to human operator modeling: classical linear control approach and modern nonlinear control approach. The latter one is formalized using both fixed and adaptive Lie-Derivative based controllers.

Keywords: Human operator, linear control, nonlinear control, Lie derivative operator
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1 Introduction

Despite the increasing trend toward automation, robotics and artificial intelligence (AI) in many environments, the human operator will probably continue for some time to be integrally involved in the control and regulation of various machines (e.g., missile–launchers, ground vehicles, watercrafts, submarines, spacecrafts, helicopters, jet fighters, etc.). A typical manual control task is the task in which control of these machines is accomplished by manipulation of the hands or fingers. As human–computer interfaces evolve, interaction techniques increasingly involve a much more continuous form of interaction with the user, over both human–to–computer (input) and computer–to–human (output) channels. Such interaction could involve gestures, speech and animation in addition to more ‘conventional’ interaction via mouse, joystick and keyboard. This poses a problem for the design of interactive systems as it becomes increasingly necessary to consider interactions occurring over an interval, in continuous time.
The so-called *manual control theory* developed out of the efforts of feedback control engineers during and after the World War II, who required models of human performance for continuous military tasks, such as tracking with anti-aircraft guns [2]. This seems to be an area worth exploring, firstly since it is generally concerned with systems which are controlled in continuous time by the user, although discrete time analogues of the various models exist. Secondly, it is an approach which models both system and user and hence is compatible with research efforts on ‘synthetic’ models, in which aspects of both system and user are specified within the same framework. Thirdly, it is an approach where continuous mathematics is used to describe functions of time. Finally, it is a theory which has been validated with respect to experimental data and applied extensively within the military domains such as avionics.

The premise of manual control theory is that for certain tasks, the performance of the human operator can be well approximated by a describing function, much as an inanimate controller would be. Hence, in the literature frequency domain representations of behavior in continuous time are applied. Two of the main classes of system modelled by the theory are *compensatory* and *pursuit* systems. A system where only the error signal is available to the human operator is a compensatory system. A system where both the target and current output are available is called a pursuit system. In many pursuit systems the user can also see a portion of the input in advance; such tasks are called *preview tasks* [3].

A simple and widely used model is the ‘crossover model’ [9], which has two main parameters, a *gain* $K$ and a *time delay* $\tau$, given by the transfer function in the Laplace transform $s$ domain

$$H = \frac{K e^{-\tau s}}{s}.$$  

Even with this simple model we can investigate some quite interesting phenomena. For example consider a compensatory system with a certain delay, if we have a low gain, then the system will move only slowly towards the target, and hence will seem sluggish. An expanded version of the crossover model is given by the transfer function [1]

$$H = \frac{K (T_L s + 1)e^{-(\tau s + \alpha/s)}}{(T_I s + 1)(T_N s + 1)},$$

where $T_L$ and $T_I$ are the lead and lag constants (which describe the *equalization* of the human operator), while the first–order lag $(T_N S + 1)$ approximates the neuromuscular lag of the hand and arm. The expanded term $\alpha/s$ in the time delay accounts for the ‘phase drop’, i.e., increased lags observed at very low frequency [4].

Alternatively if the gain $K$ is very high, then the system is very likely to overshoot the target, requiring an adjustment in the opposite direction, which may in turn overshoot, and so on. This is known as ‘oscillatory behavior’. Many more detailed models have also been developed; there are ‘anthropomorphic models’, which have a cognitive or physiological basis. For example the ‘structural model’ attempts to reflect the structure of the human, with central nervous system, neuromuscular and vestibular components [3]. Alternatively there is the ‘optimal control modeling’ approach, where algorithmic models which very closely match empirical data are used, but which do not have any direct relationship or explanation in terms of human neural and cognitive architecture [10]. In this model, an operator is assumed to perceive a vector of displayed quantities and must exercise control to minimize a *cost functional* given by [11]

$$J = E\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T [q_i y_i^2(t) + \sum_i (r_i u_i^2(t) + g_i \dot{u}_i^2(t))]dt \},$$

3
which means that the operator will attempt to minimize the expected value $E$ of some weighted combination of squared display error $y$, squared control displacement $u$ and squared control velocity $\dot{u}$. The relative values of the weighting constants $q_i, r_i, g_i$ will depend upon the relative importance of control precision, control effort and fuel expenditure.

In the case of manual control of a vehicle, this modeling yields the ‘closed–loop’ or ‘operator–vehicle’ dynamics. A quantitative explanation of this closed–loop behavior is necessary to summarize operator behavioral data, to understand operator control actions, and to predict the operator–vehicle dynamic characteristics. For these reasons, control engineering methodologies are applied to modeling human operators. These ‘control theoretic’ models primarily attempt to represent the operator’s control behavior, not the physiological and psychological structure of the operator [6,7]. These models ‘gain in acceptability’ if they can identify features of these structures, ‘although they cannot be rejected’ for failing to do so [8].

One broad division of human operator models is whether they simulated a continuous or discontinuous operator control strategy. Significant success has been achieved in modeling human operators performing compensatory and pursuit tracking tasks by employing continuous, quasi–linear operator models. Examples of these include the crossover optimal control models mentioned above.

Discontinuous input behavior is often observed during manual control of large amplitude and acquisition tasks [9,11,12,13]. These discontinuous human operator responses are usually associated with precognitive human control behavior [9,14]. Discontinuous control strategies have been previously described by ‘bang–bang’ or relay control techniques. In [13], the authors highlighted operator’s preference for this type of relay control strategy in a study that compared controlling high–order system plants with a proportional versus a relay control stick. By allowing the operator to generate a sharper step input, the relay control stick improved the operators’ performance by up to 50 percent. These authors hypothesized that when a human controls a high–order plant, the operator must consider the error of the system to be dependent upon the integral of the control input. Pulse and step inputs would reduce the integration requirements on the operator and should make the system error response more predictable to the operator.

Although operators may employ a bang–bang control strategy, they often impose an internal limit on the magnitude of control inputs. This internal limit is typically less than the full control authority available [9]. Some authors [16] hypothesized that this behavior is due to the operator’s recognition of their own reaction time delay. The operator must tradeoff the cost of a switching time error with the cost of limiting the velocity of the output to a value less than the maximum.

A significant amount of research during the 1960’s and 1970’s examined discontinuous input behavior by human operators and developed models to emulate it [14,17,18,19,20,21,22,23,24]. Good summaries of these efforts can be found in [23,11,9] and [6,7]. All of these efforts employed some type of relay element to model the discontinuous input behavior. During the 1980’s and 1990’s, pilot models were developed that included switching or discrete changes in pilot behavior [26,27,28,29,12,13].

Recently, the so-called ‘variable structure control’ techniques were applied to model human operator behavior during acquisition tasks [6,7]. The result was a coupled, multi–input model replicating the discontinuous control strategy. In this formulation, a switching surface was the mathematical representation of the human operator’s control strategy. The performance of the variable strategy model was evaluated by considering the longitudinal control of an aircraft during the visual landing task.
In this paper, we present two approaches to human operator modeling: classical linear control approach and modern nonlinear Lie-Derivative based control approach.

2 Classical Control Theory versus Nonlinear Dynamics and Control

In this section we review classical feedback control theory (see e.g., [30, 4, 31]) and contrast it with nonlinear and stochastic dynamics (see e.g., [32, 33, 34]).

2.1 Basics of Kalman’s Linear State–Space Theory

Linear multiple input–multiple output (MIMO) control systems can always be put into Kalman canonical state–space form of order \( n \), with \( m \) inputs and \( k \) outputs. In the case of continual time systems we have state and output equation of the form

\[
\frac{dx}{dt} = A(t)\,x(t) + B(t)\,u(t), \quad (1)
\]

\[
y(t) = C(t)\,x(t) + D(t)\,u(t),
\]

while in case of discrete time systems we have state and output equation of the form

\[
x(n+1) = A(n)\,x(n) + B(n)\,u(n), \quad (2)
\]

\[
y(n) = C(n)\,x(n) + D(n)\,u(n).
\]

Both in (1) and in (2) the variables have the following meaning:

- \( x(t) \in \mathbb{X} \) is an \( n \)-vector of state variables belonging to the state space \( \mathbb{X} \subset \mathbb{R}^n \);
- \( u(t) \in \mathbb{U} \) is an \( m \)-vector of inputs belonging to the input space \( \mathbb{U} \subset \mathbb{R}^m \);
- \( y(t) \in \mathbb{Y} \) is a \( k \)-vector of outputs belonging to the output space \( \mathbb{Y} \subset \mathbb{R}^k \);
- \( A(t) : \mathbb{X} \rightarrow \mathbb{X} \) is an \( n \times n \) matrix of state dynamics;
- \( B(t) : \mathbb{U} \rightarrow \mathbb{X} \) is an \( n \times m \) matrix of input map;
- \( C(t) : \mathbb{X} \rightarrow \mathbb{Y} \) is an \( k \times n \) matrix of output map;
- \( D(t) : \mathbb{U} \rightarrow \mathbb{Y} \) is an \( k \times m \) matrix of input–output transform.

Input \( u(t) \in \mathbb{U} \) can be empirically determined by trial and error; it is properly defined by optimization process called Kalman regulator, or more generally (in the presence of noise), by Kalman filter (even better, extended Kalman filter to deal with stochastic nonlinearities).

2.2 Linear Stationary Systems and Operators

The most common special case of the general Kalman model (1), with constant state, input and output matrices (and relaxed boldface vector–matrix notation), is the so–called stationary linear model

\[
\dot{x} = Ax + Bu, \quad y = Cx. \quad (3)
\]

The stationary linear system (3) defines a variety of operators, in particular those related to the following problems:

1. regulators,
2. end point controls,
3. servomechanisms, and
4. repetitive modes (see [39]).
2.2.1 Regulator Problem and the Steady State Operator

Consider a variable, or set of variables, associated with a dynamical system. They are to be maintained at some desired values in the face of changing circumstances. There exist a second set of parameters that can be adjusted so as to achieve the desired regulation. The effecting variables are usually called inputs and the affected variables called outputs. Specific examples include the regulation of the thrust of a jet engine by controlling the flow of fuel, as well as the regulation of the oxygen content of the blood using the respiratory rate.

Now, there is the steady state operator of particular relevance for the regulator problem. It is
\[ y_\infty = -CA^{-1}Bu_\infty, \]
which describes the map from constant values of \( u \) to the equilibrium value of \( y \). It is defined whenever \( A \) is invertible but the steady state value will only be achieved by a real system if, in addition, the eigenvalues of \( A \) have negative real parts. Only when the rank of \( CA^{-1}B \) equals the dimension of \( y \) can we steer \( y \) to an arbitrary steady state value and hold it there with a constant \( u \). A nonlinear version of this problem plays a central role in robotics where it is called the inverse kinematics problem (see, e.g., [40]).

2.2.2 End Point Control Problem and the Adjustment Operator

Here we have inputs, outputs and trajectories. In this case the shape of the trajectory is not of great concern but rather it is the end point that is of primary importance. Standard examples include rendezvous problems such as one has in space exploration.

Now, the operator of relevance for the end point control problem, is the operator
\[ x(T) = \int_0^T \exp[A(T - \sigma)] Bu(\sigma) d\sigma. \]
If we consider this to define a map from the \( mD \ L_2 \) space \( L_2^n [0, T] \) (where \( u \) takes on its values) into \( \mathbb{R}^m \) then, if it is an onto map, it has a Moore–Penrose (least squares) inverse
\[ u(\sigma) = B^T \exp[A^T(T - \sigma)] \left( W[0, T]\right)^{-1} \left( x(T) - \exp(AT) x(0) \right), \]
with the symmetric positive definite matrix \( W \), the controllability Gramian, being given by
\[ W[0, T] = \int_0^T \exp[A(T - \sigma)] B B^T \exp[A^T(T - \sigma)] d\sigma. \]

2.2.3 Servomechanism Problem and the Corresponding Operator

Here we have inputs, outputs and trajectories, as above, and an associated dynamical system. In this case, however, it is desired to cause the outputs to follow a trajectory specified by the input. For example, the control of an airplane so that it will travel along the flight path specified by the flight controller.

Now, because we have assumed that \( A, B \) and \( C \) are constant
\[ y(t) = C \exp(At) x(0) + \int_0^t C \exp[A(T - \tau)] Bu(\tau) d\tau, \]
and, as usual, the Laplace transform can be used to convert convolution to multiplication. This brings out the significance of the Laplace transform pair.
\[ C \exp(At)B \iff C(Is - A)^{-1}B \] (4)

as a means of characterizing the input–output map of a linear model with constant coefficients.

### 2.2.4 Repetitive Mode Problem and the Corresponding Operator

Here again one has some variable, or set of variables, associated with a dynamical system and some inputs which influence its evolution. The task has elements which are repetitive and are to be done efficiently. Examples from biology include the control of respiratory processes, control of the pumping action of the heart, control of successive trials in practicing a athletic event.

The relevant operator is similar to the servomechanism operator, however the constraint that \( u \) and \( x \) are periodic means that the relevant diagonalization is provided by Fourier series, rather than the Laplace transform. Thus, in the Fourier domain, we are interested in a set of complex matrices

\[ G(iw_i) = C(iw_i - A)^{-1}B, \quad w_i = 0, w_0, 2w_0, ... \]

More general, but still deterministic, models of the input–state–output relation are afforded by the **nonlinear affine model** (see, e.g., [41])

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + g(x(t)) u(t), \\
y(t) &= h(x(t));
\end{align*}
\]

and the still more general **fully nonlinear model**

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)), \\
y(t) &= h(x(t)).
\end{align*}
\]

### 2.2.5 Feedback Changes the Operator

No idea is more central to automatic control than the idea of feedback. When an input is altered on the basis of the difference between the actual output of the system and the desired output, the system is said to involve feedback. Man made systems are often constructed by starting with a basic element such as a motor, a burner, a grinder, etc. and then adding sensors and the hardware necessary to use the measurement generated by the sensors to regulate the performance of the basic element. This is the essence of **feedback control**. Feedback is often contrasted with open loop systems in which the inputs to the basic element is determined without reference to any measurement of the trajectories. When the word feedback is used to describe naturally occurring systems, it is usually implicit that the behavior of the system can best be explained by pretending that it was designed as one sees man made systems being designed [39].

In the context of linear systems, the effect of feedback is easily described. If we start with the stationary linear system (3) with \( u \) being the controls and \( y \) being the measured quantities, then the effect of feedback is to replace \( u \) by \( u - Ky \) with \( K \) being a matrix of feedback gains. The closed–loop equations are then

\[
\dot{x} = (A - BK) x + Bu, \quad y = Cx.
\]
Expressed in terms of the Laplace transform pairs \[4\], feedback effects the transformation
\[(C \exp(At)B; C(I\sigma - A)^{-1}B) \rightarrow C \exp(A - BK)B; C(I\sigma - A + BK)^{-1}B.\]

Using such a transformation, it is possible to alter the dynamics of a system in a significant way. The modifications one can effect by feedback include influencing the location of the eigenvalues and consequently the stability of the system. In fact, if \(K\) is \(m\) by \(p\) and if we wish to select a gain matrix \(K\) so that \(A-BK\) has eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\), it is necessary to insure that
\[\det \left( \begin{array}{cc} C(I\sigma - A)^{-1}B & -I \\ I & K \end{array} \right) = 0, \quad i = 1, 2, \ldots, n.\]

Now, if \(CB\) is invertible then we can use the relationship \(C\dot{x} = CAx + CBu\) together with \(y = Cx\) to write \(\dot{y} = CAx + CBu\). This lets us solve for \(u\) and recast the system as
\[
\begin{align*}
\dot{x} &= (A - B(CB)^{-1}CA)x + B(CB)^{-1}\dot{y}, \\
u &= (CB)^{-1}\dot{y} - (CB)^{-1}CAx.
\end{align*}
\]

Here we have a set of equations in which the roles of \(u\) and \(y\) are reversed. They show how a choice of \(y\) determines \(x\) and how \(x\) determines \(u\) \[39\].

### 2.3 Stability and Boundedness

Let a time–varying dynamical system may be expressed as
\[
\dot{x}(t) = f(t, x(t))
\]
where \(x \in \mathbb{R}^n\) is an \(n\)D vector and \(f: \mathbb{R}^+ \times D \rightarrow \mathbb{R}^n\) with \(D = \mathbb{R}^n\) or \(D = B_h\) for some \(h > 0\), where \(B_h = \{x \in \mathbb{R}^n : |x| < h\}\) is a ball centered at the origin with a radius of \(h\). If \(D = \mathbb{R}^n\) then we say that the dynamics of the system are defined globally, whereas if \(D = B_h\) they are only defined locally. We do not consider systems whose dynamics are defined over disjoint subspaces of \(\mathbb{R}\). It is assumed that \(f(t, x)\) is piecemeal continuous in \(t\) and Lipschitz in \(x\) for existence and uniqueness of state solutions. As an example, the linear system \(\dot{x}(t) = Ax(t)\) fits the form of \(5\) with \(D = \mathbb{R}^n\) \[38\].

Assume that for every \(x_0\) the initial value problem
\[
\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0,
\]
possesses a unique solution \(x(t, t_0, x_0)\); it is called a solution to \(5\) if \(x(t, t_0, x_0) = x_0\) and \(\frac{\partial}{\partial t} x(t, t_0, x_0) = f(t, x(t, t_0, x_0))\) \[38\].

A point \(x_0 \in \mathbb{R}^n\) is called an equilibrium point of \(5\) if \(f(t, x_0) = 0\) for all \(t \geq 0\). An equilibrium point \(x_0\) is called an isolated equilibrium point if there exists an \(\rho > 0\) such that the ball around \(x_0\), \(B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}\), contains no other equilibrium points besides \(x_0\) \[38\].

The equilibrium \(x_0 = 0\) of \(5\) is said to be stable in the sense of Lyapunov if for every \(\epsilon > 0\) and any \(t_0 \geq 0\) there exists a \(\delta(\epsilon, t_0) > 0\) such that \(|x(t, t_0, x_0)| < \epsilon\) for all \(t \geq t_0\) whenever \(|x_0| < \delta(\epsilon, t_0)\) and \(x(t, t_0, x_0) \in B_h(x_0)\) for some \(h > 0\). That is, the equilibrium is stable if when the system \(5\) starts close to \(x_0\), then it will stay close to it. Note that stability is a property of an equilibrium, not a system. A system is
stable if all its equilibrium points are stable. Stability in the sense of Lyapunov is a local property. Also, notice that the definition of stability is for a single equilibrium $x_e \in \mathbb{R}^n$ but actually such an equilibrium is a trajectory of points that satisfy the differential equation in (5). That is, the equilibrium $x_e$ is a solution to the differential equation (5), $x(t, t_0, x_0) = x_e$ for $t \geq 0$. We call any set such that when the initial condition of (5) starts in the set and stays in the set for all $t \geq 0$, an invariant set. As an example, if $x_e = 0$ is an equilibrium, then the set containing only the point $x_e$ is an invariant set, for (5) [38].

If $\delta$ is independent of $t_0$, that is, if $\delta = \delta(\epsilon)$, then the equilibrium $x_e$ is said to be uniformly stable. If in (5) $f$ does not depend on time (i.e., $f(x)$), then $x_e$ being stable is equivalent to it being uniformly stable. Uniform stability is also a local property.

The equilibrium $x_e = 0$ of (5) is said to be asymptotically stable if it is stable and for every $t_0 \geq 0$ there exists $\eta(t_0) > 0$ such that $\lim_{t \to \infty} |x(t, t_0, x_0)| = 0$ whenever $|x_0| < \eta(t_0)$. That is, it is asymptotically stable if when it starts close to the equilibrium it will converge to it. Asymptotic stability is also a local property. It is a stronger stability property since it requires that the solutions to the ordinary differential equation converge to zero in addition to what is required for stability in the sense of Lyapunov.

The equilibrium $x_e = 0$ of (5) is said to be uniformly asymptotically stable if it is uniformly stable and for every $\epsilon > 0$ and and $t_0 \geq 0$, there exist a $\delta_0 > 0$ independent of $t_0$ and $\epsilon$, and a $T(\epsilon) > 0$ independent of $t_0$, such that $|x(t, t_0, x_0) - x_e| \leq \epsilon$ for all $t \geq t_0 + T(\epsilon)$ whenever $|x_0 - x_e| < \delta(\epsilon)$. Again, if in (5) $f$ does not depend on time (i.e., $f(x)$), then $x_e$ being asymptotically stable is equivalent to it being uniformly asymptotically stable. Uniform asymptotic stability is also a local property.

The set $X_\delta \subset \mathbb{R}^n$ of all $x_0 \in \mathbb{R}^n$ such that $|x(t, t_0, x_0)| \to 0$ as $t \to \infty$ is called the domain of attraction of the equilibrium $x_e = 0$ of (5). The equilibrium $x_e = 0$ is said to be asymptotically stable in the large if $X_\delta \subset \mathbb{R}^n$. That is, an equilibrium is asymptotically stable in the large if no matter where the system starts, its state converges to the equilibrium asymptotically. This is a global property as opposed to the earlier stability definitions that characterized local properties. This means that for asymptotic stability in the large, the local property of asymptotic stability holds for $B_h(x_e)$ with $h = \infty$ (i.e., on the whole state-space).

The equilibrium $x_e = 0$ is said to be exponentially stable if there exists an $\alpha > 0$ and for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $|x(t, t_0, x_0)| \leq e^{-\alpha(t-t_0)}$, whenever $|x_0| < \delta(\epsilon)$ and $t \geq t_0 \geq 0$. The constant $\alpha$ is sometimes called the rate of convergence. Exponential stability is sometimes said to be a ‘stronger’ form of stability since in its presence we know that system trajectories decrease exponentially to zero. It is a local property; here is its global version. The equilibrium point $x_e = 0$ is exponentially stable in the large if there exists $\alpha > 0$ and for any $\beta > 0$ there exists $\epsilon(\beta) > 0$ such that $|x(t, t_0, x_0)| \leq \epsilon(\beta)e^{-\alpha(t-t_0)}$, whenever $|x_0| < \beta$ and $t \geq t_0 \geq 0$.

An equilibrium that is not stable is called unstable.

Closely related to stability is the concept of boundedness, which is, however, a global property of a system in the sense that it applies to trajectories (solutions) of the system that can be defined over all of the state-space [38].

A solution $x(t, t_0, x_0)$ of (5) is bounded if there exists a $\beta > 0$, that may depend on each solution, such that $|x(t, t_0, x_0)| < \beta$ for all $t \geq t_0 \geq 0$. A system is said to possess Lagrange stability if for each $t_0 \geq 0$ and $x_0 \in \mathbb{R}^n$, the solution $x(t, t_0, x_0)$ is bounded. If an equilibrium is asymptotically stable in the large or exponentially stable in the large then the system for which the equilibrium is defined is also Lagrange stable (but not necessarily vice versa). Also, if an equilibrium is stable, it does not imply that
the system for which the equilibrium is defined is Lagrange stable since there may be a way to pick \( x_0 \) such that it is near an unstable equilibrium and \( x(t, t_0, x_0) \to \infty \) as \( t \to \infty \).

The solutions \( x(t, t_0, x_0) \) are uniformly bounded if for any \( \alpha > 0 \) and \( t_0 \geq 0 \), there exists a \( \beta(\alpha) > 0 \) (independent of \( t_0 \)) such that if \( |x_0| < \alpha \), then \( |x(t, t_0, x_0)| < \beta(\alpha) \) for all \( t \geq t_0 \geq 0 \). If the solutions are uniformly bounded then they are bounded and the system is Lagrange stable.

The solutions \( x(t, t_0, x_0) \) are said to be uniformly ultimately bounded if there exists some \( B > 0 \), and if corresponding to any \( \alpha > 0 \) and \( t_0 > 0 \) there exists a \( T(\alpha) > 0 \) (independent of \( t_0 \)) such that \( |x_0| < \alpha \) implies that \( |x(t, t_0, x_0)| < B \) for all \( t \geq t_0 + T(\alpha) \). Hence, a system is said to be uniformly ultimately bounded if eventually all trajectories end up in a \( B \)-neighborhood of the origin.

### 2.4 Lyapunov’s Stability Method

A. M. Lyapunov invented two methods to analyze stability [38]. In his indirect method he showed that if we linearize a system about an equilibrium point, certain conclusions about local stability properties can be made (e.g., if the eigenvalues of the linearized system are in the left half plane then the equilibrium is stable but if one is in the right half plane it is unstable).

In his direct method the stability results for an equilibrium \( x_e = 0 \) of (5) depend on the existence of an appropriate Lyapunov function \( V : D \to \mathbb{R} \) where \( D = \mathbb{R}^n \) for global results (e.g., asymptotic stability in the large) and \( D = B_h \) for some \( h > 0 \), for local results (e.g., stability in the sense of Lyapunov or asymptotic stability). If \( V \) is continuously differentiable with respect to its arguments then the derivative of \( V \) with respect to \( t \) along the solutions of (5) is

\[
\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x).
\]

As an example, suppose that (5) is autonomous, and let \( V(x) = x^T P x \) where \( x \in \mathbb{R}^n \) and \( P = P^T \). Then, \( \dot{V}(x) = \partial_x f(t, x) = \dot{x}^T P x + x^T P \dot{x} = 2x^T P \dot{x} \) [38].

Lyapunov’s direct method provides for the following ways to test for stability. The first two are strictly for local properties while the last two have local and global versions.

- **Stable**: If \( V(t, x) \) is continuously differentiable, positive definite, and \( \dot{V}(t, x) \leq 0 \), then \( x_e = 0 \) is stable.

- **Uniformly stable**: If \( V(t, x) \) is continuously differentiable, positive definite, decrescent\(^1\) and \( V(t, x) \leq 0 \), then \( x_e = 0 \) is uniformly stable.

- **Uniformly asymptotically stable**: If \( V(t, x) \) is continuously differentiable, positive definite, and decrescent, with negative definite \( \dot{V}(t, x) \), then \( x_e = 0 \) is uniformly asymptotically stable (uniformly asymptotically stable in the large if all these properties hold globally).

- **Exponentially stable**: If there exists a continuously differentiable \( V(t, x) \) and \( c, c_1, c_2, c_3 > 0 \) such that

\[
\begin{align*}
    c_1 |x|^c & \leq V(t, x) \leq c_2 |x|^c, \\
    \dot{V}(t, x) & \leq -c_3 |x|^c,
\end{align*}
\]

\(^1\) A \( C^0 \)-function \( V(t, x) : \mathbb{R}^+ \times B_h \to \mathbb{R} \) (defined on \( [0, \infty) \)) is said to be decrescent if there exists a strictly increasing function \( \gamma \) defined on \( [0, r) \) for some \( r > 0 \) (defined on \( [0, \infty) \)) such that \( V(t, x) \leq \gamma(|x|) \) for all \( t \geq 0 \) and \( x \in B_h \) for some \( h > 0 \).
for all $x \in B_h$ and $t \geq 0$, then $x_e = 0$ is exponentially stable. If there exists a continuously differentiable function $V(t, x)$ and Equations (6) and (7) hold for some $c, c_1, c_2, c_3 > 0$ for all $x \in \mathbb{R}^n$ and $t \geq 0$, then $x_e = 0$ is exponentially stable in the large [38].

2.5 Nonlinear and Impulse Dynamics of Complex Plants

In this section we give two examples of nonlinear dynamical systems that are beyond reach of the classical control theory.

2.5.1 Hybrid Dynamical Systems of Variable Structure

Consider a hybrid dynamical system of variable structure, given by $n$-dimensional ODE (see [46])

$$\dot{x} = f(t, x),$$ (8)

where $x = x(t) \in \mathbb{R}^n$ and $f = f(t, x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$. Let the domain $G \subset \mathbb{R}^+ \times \mathbb{R}^n$, on which the vector-field $f(t, x)$ is defined, be divided into two subdomains, $G^+$ and $G^-$, by means of a smooth $(n-1)$-manifold $M$. In $G^+ \cup M$, let there be given a vector-field $f^+(t, x)$, and in $G^- \cup M$, let there be given a vector-field $f^-(t, x)$. Assume that both $f^+ = f^+(t, x)$ and $f^- = f^-(t, x)$ are continuous in $t$ and smooth in $x$. For the system (8), let

$$f = \begin{cases} f^+ & \text{when } x \in G^+ \\ f^- & \text{when } x \in G^- \end{cases}.$$

Under these conditions, a solution $x(t)$ of ODE (8) is well-defined while passing through $G$ until the manifold $M$ is reached.

Upon reaching the manifold $M$, in physical systems with inertia, the transition

from $\dot{x} = f^-(t, x)$ to $\dot{x} = f^+(t, x)$

does not take place instantly on reaching $M$, but after some delay. Due to this delay, the solution $x(t)$ oscillates about $M$, $x(t)$ being displaced along $M$ with some mean velocity.

As the delay tends to zero, the limiting motion and velocity along $M$ are determined by the linear homotopy ODE

$$\dot{x} = f^0(t, x) \equiv (1 - \alpha) f^-(t, x) + \alpha f^+(t, x),$$ (9)

where $x \in M$ and $\alpha \in [0, 1]$ is such that the linear homotopy segment $f^0(t, x)$ is tangential to $M$ at the point $x$, i.e., $f^0(t, x) \in T_x M$, where $T_x M$ is the tangent space to the manifold $M$ at the point $x$.

The vector-field $f^0(t, x)$ of the system (9) can be constructed as follows: at the point $x \in M$, $f^-(t, x)$ and $f^+(t, x)$ are given and their ends are joined by the linear homotopy segment. The point of intersection between this segment and $T_x M$ is the end of the required vector-field $f^0(t, x)$. The vector function $x(t)$ which satisfies (8) in $G^-$ and $G^+$, and (9) when $x \in M$, can be considered as a solution of (8) in a general sense.

However, there are cases in which the solution $x(t)$ cannot consist of a finite or even countable number of arcs, each of which passes through $G^-$ or $G^+$ satisfying (8), or moves along the manifold $M$ and satisfies the homotopic ODE (9). To cover such cases, assume that the vector-field $f = f(t, x)$ in ODE (8) is a Lebesgue-measurable function in a domain $G \subset \mathbb{R}^+ \times \mathbb{R}^n$, and that for any closed bounded domain $D \subset G$ there exists
a summable function $K(t)$ such that almost everywhere in $D$ we have $|f(t, x)| \leq K(t)$. Then the absolutely continuous vector function $x(t)$ is called the generalized solution of the ODE (8) in the sense of Filippov (see [46]) if for almost all $t$, the vector $\dot{x} = \dot{x}(t)$ belongs to the least convex closed set containing all the limiting values of the vector field $f(t, x^*)$, where $x^*$ tends towards $x$ in an arbitrary manner, and the values of the function $f(t, x^*)$ on a set of measure zero in $\mathbb{R}^n$ are ignored.

Such hybrid systems of variable structure occur in the study of nonlinear electric networks (endowed with electronic switches, relays, diodes, rectifiers, etc.), in models of both natural and artificial neural networks, as well as in feedback control systems (usually with continuous–time plants and digital controllers/filters).

2.5.2 Impulse Dynamics of Kicks and Spikes

The Spike Function. Recall that the Dirac’s $\delta$–function (also called the impulse function in the systems and signals theory) represents a limit of the Gaussian bell–shaped curve

$$g(t, \alpha) = \frac{1}{\sqrt{\pi \alpha}} e^{-t^2/\alpha} \quad \text{(with parameter } \alpha \to 0) \quad (10)$$

where the factor $1/\sqrt{\pi \alpha}$ serves for the normalization of (10),

$$\int_{-\infty}^{+\infty} \frac{dt}{\sqrt{\pi \alpha}} e^{-t^2/\alpha} = 1, \quad (11)$$

i.e., the area under the pulse is equal to unity. In (10), the smaller $\alpha$ the higher the peak. In other words,

$$\delta(t) = \lim_{\alpha \to 0} \frac{1}{\sqrt{\pi \alpha}} e^{-t^2/\alpha}, \quad (12)$$

which is a pulse so short that outside of $t = 0$ it vanishes, whereas at $t = 0$ it still remains normalized according to (11). Therefore, we get the usual definition of the $\delta$–function:

$$\delta(t) = 0 \text{ for } t \neq 0,$$

$$\int_{-\epsilon}^{+\epsilon} \delta(t) \, dt = 1, \quad (13)$$

where $\epsilon$ may be arbitrarily small. Instead of centering the $\delta$–pulse around $t = 0$, we can center it around any other time $t_0$ so that (13) is transformed into

$$\delta(t - t_0) = 0 \text{ for } t \neq t_0,$$

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t - t_0) \, dt = 1. \quad (14)$$

Another well–known fact is that the integral of the $\delta$–function is the Heaviside’s step function

$$H(T) = \int_{-\infty}^{T} \delta(t) \, dt = \begin{cases} 
0 & \text{for } T < 0 \\
1 & \text{for } T > 0 \\
\frac{1}{2} & \text{for } T = 0
\end{cases}. \quad (15)$$

Now we can perform several generalizations of the relation (15). First, we have

$$\int_{-\infty}^{T} \delta(ct - t_0) \, dt = \begin{cases} 
0 & \text{for } T < t_0/c \\
1/c & \text{for } T > t_0/c \\
\frac{1}{2c} & \text{for } T = t_0/c
\end{cases}. \quad (16)$$
More generally, we can introduce the so–called phase function $\phi(t)$, (e.g., $\phi(t) = ct - t_0$) which is continuous at $t = t_0$ but its time derivative $\dot{\phi}(t) \equiv \frac{d\phi(t)}{dt}$ is discontinuous at $t = t_0$ (yet positive, $\dot{\phi}(t) > 0$), and such that

$$
\int_{-\infty}^{T} \delta(\phi(t)) \dot{\phi}(t) \, dt = \begin{cases} 
0 & \text{for } T < t_0 \\
1/\dot{\phi}(t_0) & \text{for } T > t_0 \\
1/\phi(t_0) & \text{for } T = t_0
\end{cases}.
$$

Finally, we come the the spike function $\delta(\phi(t))\dot{\phi}(t)$, which like $\delta$–function represents a spike at $t = t_0$, such that the normalization criterion\(^1\) is still valid,

$$
\int_{t_0-\epsilon}^{t_0+\epsilon} \delta(\phi(t))\dot{\phi}(t) \, dt = 1.
$$

**Deterministic Delayed Kicks.** Following Haken\(^{47}\), we consider the mechanical example of a soccer ball that is kicked by a soccer player and rolls over grass, whereby its motion will be slowed down. In our opinion, this is a perfect model for all ‘shooting–like’ actions of the human operator.

We start with the Newton’s (second) law of motion, $m\ddot{v} = force$, and in order to get rid of superfluous constants, we put temporarily $m = 1$. The force on the r.h.s. consists of the damping force $-\gamma v(t)$ of the grass (where $\gamma$ is the damping constant) and the sharp force $F(t) = s\delta(t - \sigma)$ of the individual kick occurring at time $t = \sigma$ (where $s$ is the strength of the kick, and $\delta$ is the Dirac’s ‘delta’ function). In this way, the single–kick equation of the ball motion becomes

$$
\ddot{v} = -\gamma v(t) + s\delta(t - \sigma),
$$

with the general solution

$$
v(t) = sG(t - \sigma),
$$

where $G(t - \sigma)$ is the Green’s function\(^2\)

$$
G(t - \sigma) = \begin{cases} 
0 & \text{for } t < \sigma \\
e^{-\gamma(t-\sigma)} & \text{for } t \geq \sigma
\end{cases}.
$$

Now, we can generalize the above to $N$ kicks with individual strengths $s_j$, occurring at a sequence of times $\{\sigma_j\}$, so that the total kicking force becomes

$$
F(t) = \sum_{j=1}^{N} s_j\delta(t - \sigma_j).
$$

\(^2\)This is the Green’s function of the first order system\(^{16}\). Similarly, the Green’s function

$$
G(t - \sigma) = \begin{cases} 
0 & \text{for } t < \sigma \\
(t - \sigma)e^{-\gamma(t-\sigma)} & \text{for } t \geq \sigma
\end{cases}
$$

corresponds to the second order system

$$
\left(\frac{d}{dt} + \gamma\right)^2 G(t - \sigma) = \delta(t - \sigma).
$$
In this way, we get the multi–kick equation of the ball motion

\[ \dot{v} = -\gamma v(t) + \sum_{j=1}^{N} s_j \delta(t - \sigma_j), \]

with the general solution

\[ v(t) = \sum_{j=1}^{N} s_j G(t - \sigma_j). \quad (17) \]

As a final generalization, we would imagine that the kicks are continuously exerted on the ball, so that kicking force becomes

\[ F(t) = \int_{t_0}^{T} s(\sigma) \delta(t - \sigma) d\sigma \equiv \int_{t_0}^{T} d\sigma F(\sigma) \delta(t - \sigma), \]

so that the continuous multi–kick equation of the ball motion becomes

\[ \dot{v} = -\gamma v(t) + \int_{t_0}^{T} s(\sigma) \delta(t - \sigma) d\sigma \equiv -\gamma v(t) + \int_{t_0}^{T} d\sigma F(\sigma) \delta(t - \sigma), \]

with the general solution

\[ v(t) = \int_{t_0}^{T} d\sigma F(\sigma) G(t - \sigma) = \int_{t_0}^{T} d\sigma F(\sigma) e^{-\gamma(t-\sigma)}. \quad (18) \]

**Random Kicks and Langevin Equations.** We now denote the times at which kicks occur by \( t_j \) and indicate their direction in a one–dimensional game by \((\pm 1)_j\), where the choice of the plus or minus sign is random (e.g., throwing a coin). Thus the kicking force can be written in the form

\[ F(t) = s \sum_{j=1}^{N} \delta(t - t_j)(\pm 1)_j, \quad (19) \]

where for simplicity we assume that all kicks have the same strength \( s \). When we observe many games, then we may perform an average \(< \ldots >\) over all these different performances,

\[ < F(t) > = s < \sum_{j=1}^{N} \delta(t - t_j)(\pm 1)_j >. \quad (20) \]

Since the direction of the kicks is assumed to be independent of the time at which the kicks happen, we may split (20) into the product

\[ < F(t) > = s < \sum_{j=1}^{N} \delta(t - t_j) > < (\pm 1)_j >. \]

As the kicks are assumed to happen with equal frequency in both directions, we get the cancellation

\[ < (\pm 1)_j > = 0, \]

which implies that the average kicking force also vanishes,

\[ < F(t) > = 0. \]
In order to characterize the strength of the force \( \mathbf{19} \), we consider a quadratic expression in \( \mathbf{F} \), e.g., by calculating the correlation function for two times \( t, t' \),

\[
<F(t)F(t')> = s^2 < \sum_j \delta(t - t_j)(\pm1j) \sum_k \delta(t' - t_k)(\pm1k) .
\]

As the ones for \( j \neq k \) will cancel each other and for \( j = k \) will become 1, the correlation function becomes a single sum

\[
<F(t)F(t')> = s^2 < \sum_j \delta(t - t_j)\delta(t' - t_k) > ,
\]

which is usually evaluated by assuming the Poisson process for the times of the kicks.

Now, proper description of random motion is given by Langevin rate equation, which describes the Brownian motion: when a particle is immersed in a fluid, the velocity of this particle is slowed down by a force proportional to its velocity and the particle undergoes a zig–zag motion (the particle is steadily pushed by much smaller particles of the liquid in a random way). In physical terminology, we deal with the behavior of a system (particle) which is coupled to a heat bath or reservoir (namely the liquid). The heat bath has two effects:

1. It decelerates the mean motion of the particle; and
2. It causes statistical fluctuation.

The standard Langevin equation has the form

\[
\dot{v} = -\gamma v(t) + F(t),
\]

where \( F(t) \) is a fluctuating force with the following properties:

1. Its statistical average \( \mathbf{20} \) vanishes; and
2. Its correlation function \( \mathbf{21} \) is given by

\[
<F(t)F(t')> = Q\delta(t - t_0),
\]

where \( t_0 = T/N \) denotes the mean free time between kicks, and \( Q = s^2/t_0 \) is the random fluctuation.

The general solution of the Langevin equation \( \mathbf{22} \) is given by \( \mathbf{18} \).

The average velocity vanishes, \( < v(t) > = 0 \), as both directions are possible and cancel each other. Using the integral solution \( \mathbf{18} \) we get

\[
<v(t)v(t')> = < \int_{t_0}^t d\sigma \int_{t_0}^{t'} d\sigma' F(\sigma)F(\sigma')e^{-\gamma(t-\sigma)}e^{-\gamma(t'-\sigma')} > ,
\]

which, in the steady–state, reduces to

\[
<v(t)v(t')> = \frac{Q}{2\gamma}e^{-\gamma(t-\sigma)},
\]

and for equal times

\[
<v(t)^2> = \frac{Q}{2\gamma}.
\]
If we now repeat all the steps performed so far with \( m \neq 1 \), the final result reads

\[
<v(t)^2> = \frac{Q}{2\gamma m}.
\]

(24)

Now, according to thermodynamics, the mean kinetic energy of a particle is given by

\[
\frac{m}{2} <v(t)^2> = \frac{1}{2}k_B T,
\]

(25)

where \( T \) is the (absolute) temperature, and \( k_B \) is the Boltzman’s constant. Comparing (24) and (25), we obtain the important Einstein’s result

\[
Q = 2\gamma k_B T,
\]

which says that whenever there is damping, i.e., \( \gamma \neq 0 \), then there are random fluctuations (or noise) \( Q \). In other words, fluctuations or noise are inevitable in any physical system. For example, in a resistor (with the resistance \( R \)) the electric field \( E \) fluctuates with a correlation function (similar to (23))

\[
<E(t)E(t')> = 2Rk_B T \delta(t - t_0).
\]

This is the simplest example of the so–called dissipation–fluctuation theorem.

3 Nonlinear Control Modeling of the Human Operator

In this section we present the basics of modern nonlinear control, as a powerful tool for controlling nonlinear dynamical systems.

3.1 Graphical Techniques for Nonlinear Systems

Graphical techniques preceded modern geometrical techniques in nonlinear control theory. They started with simple plotting tools, like the so–called ‘tracer plot’. It is a useful visualization tool for analysis of second order dynamical systems, which just adds time dimension to the standard 2D phase portrait. For example, consider the damped spring governed by

\[
\ddot{x} = -k\dot{x} - x, \quad x(0) = 1.
\]

Its tracer plot is given in Figure 1. Note the stable asymptote reached as \( t \to \infty \).

The most important graphical technique is the so–called describing function analysis.

3.1.1 Describing Function Analysis

Describing function analysis extends classical linear control technique, frequency response analysis, for nonlinear systems [37]. It is an approximate graphical method mainly used to predict limit cycles in nonlinear ODEs.

For example, if we want to predict the existence of limit cycles in the classical Van der Pol’s oscillator given by

\[
\ddot{x} + \alpha(x^2 - 1) \dot{x} + x = 0,
\]

(26)
Figure 1: Tracer plot of the damped spring.

Figure 2: Feedback interpretation of the Van Der Pol oscillator (after [37]). Here $p$ is a (linear) differentiator, and $\Pi$ a (nonlinear) multiplicator.

we need to rewrite (26) as a linear unstable low–pass block and a nonlinear block (see Figure 2). In this way, using the nonlinear block substitution, $w := -\dot{x}x^2$, we get

$$\ddot{x} - \alpha \dot{x} + x = \alpha w,$$

or

$$x(p^2 - \alpha p + 1) = \alpha w,$$

or just considering the transfer function from $w$ to $x$,

$$\frac{x}{w} = \frac{\alpha}{p^2 - \alpha p + 1}.$$ 

Now, if we assume that the Van der Pol oscillator does have a limit cycle with a frequency of

$$x(t) = A \sin(wt),$$

so $\dot{x} = Aw \cos(wt)$, therefore the output of the nonlinear block is

$$z = -x^2 \ddot{x} = -A^2 \sin^2(wt) Aw \cos(wt)$$

$$= -\frac{A^3w}{2} (1 - \cos(2wt)) \cos(wt)$$

$$= -\frac{A^3w}{4} (\cos(wt) - \cos(3wt)).$$
Note how \( z \) contains a third harmonic term, but this is attenuated by the low-pass nature of the linear block, and so does not affect the signal in the feedback. So we can approximate \( z \) by

\[
z \approx \frac{A^3}{4} w \cos(w t) = \frac{A^2}{4} \frac{d}{dt} (-A \sin(w t)).
\]

Therefore, the output of the nonlinear block can be approximated by the quasi-linear transfer function which depends on the signal amplitude, \( A \), as well as frequency. The frequency response function of the quasi-linear element is obtained by substituting \( p \equiv s = iw \),

\[
N(A, w) = \frac{A^2}{4} (iw).
\]

Since the system is assumed to contain a sinusoidal oscillation,

\[
x = A \sin(w t) = G(iw) z = G(iw) N(A, w) (-x),
\]

where \( G(iw) \) is the transfer function of the linear block. This implies that,

\[
\frac{x}{-x} = -1 = G(iw) N(A, w),
\]

so

\[
1 + \frac{A^2(iw)}{4} \frac{\alpha}{(iw)^2 - \alpha(iw) + 1} = 0,
\]

which solving gives,

\[
A = 2, \quad \omega = 1,
\]

which is independent of \( \alpha \). Note that in terms of the Laplace variable \( p \equiv s \), the closed loop characteristic equation of the system is

\[
1 + \frac{A^2(iw)}{4} \frac{\alpha}{p^2 - \alpha p + 1} = 0,
\]

whose eigenvalues are

\[
\lambda_{1,2} = -\frac{1}{8} \alpha (A^2 - 4) \pm \sqrt{\frac{\alpha^2 (A^2 - 4)^2}{64} - 1}.
\]

Corresponding to \( A = 2 \) gives eigenvalues of \( \lambda_{1,2} = \pm i \) indicating an existence of a limit cycle of amplitude 2 and frequency 1 (see Figure 3). If \( A > 2 \) eigenvalues are negative real, so stable, and the same holds for \( A < 2 \). The approximation of the nonlinear block with \( (A^2/4)(iw) \) is called the describing function. This technique is useful because most limit cycles are approximately sinusoidal and most linear elements are low-pass in nature. So most of the higher harmonics, if they existed, are attenuated and lost.

### 3.2 Feedback Linearization

The idea of feedback linearization is to algebraically transform the nonlinear system dynamics into a fully or partly linear one so that the linear control techniques can be applied. Note that this is not the same as a conventional linearization using Jacobians. In this subsection we will present the modern, geometrical, Lie-derivative based techniques for exact feedback linearization of nonlinear control systems.
3.2.1 The Lie Derivative and Lie Bracket in Control Theory

Recall that given a scalar function \( h(x) \) and a vector–field \( f(x) \), we define a new scalar function, \( \mathcal{L}_f h := \nabla h f \), which is the Lie derivative of \( h \) w.r.t. \( f \), i.e., the directional derivative of \( h \) along the direction of the vector \( f \) (see [4, 5]). Repeated Lie derivatives can be defined recursively:

\[
\begin{align*}
\mathcal{L}_f^0 h &= h, \\
\mathcal{L}_f^i h &= \mathcal{L}_f \left( \mathcal{L}_f^{i-1} h \right) = \nabla \left( \mathcal{L}_f^{i-1} h \right) f, \quad \text{for } i = 1, 2, \ldots
\end{align*}
\]

Or given another vector–field, \( g \), then \( \mathcal{L}_g \mathcal{L}_f h(x) \) is defined as

\[
\mathcal{L}_g \mathcal{L}_f h = \nabla \left( \mathcal{L}_f h \right) g.
\]

For example, if we have a control system

\[
\begin{align*}
\dot{x} &= f(x), \\
y &= h(x),
\end{align*}
\]

with the state \( x = x(t) \) and the output \( y \), then the derivatives of the output are:

\[
\begin{align*}
\dot{y} &= \frac{\partial h}{\partial x} \dot{x} = \mathcal{L}_f h, \quad \text{and} \\
\ddot{y} &= \frac{\partial \mathcal{L}_f h}{\partial x} \dot{x} = \mathcal{L}_f^2 h.
\end{align*}
\]

Also, recall that the curvature of two vector–fields, \( g_1, g_2 \), gives a non–zero Lie bracket, \([g_1, g_2]\) (see Figure 4). Lie bracket motions can generate new directions in which the system can move.
In general, the Lie bracket of two vector–fields, $f(x)$ and $g(x)$, is defined by

$$[f, g] := ad_f g := \nabla g f - \nabla f g := \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g,$$

where $\nabla f := \frac{\partial f}{\partial x}$ is the Jacobian matrix. We can define Lie brackets recursively,

$$ad^0_f g = g,$$
$$ad^i_f g = [f, ad^{i-1}_f g],$$

for $i = 1, 2, ...$

Lie brackets have the properties of bilinearity, skew–commutativity and Jacobi identity.

For example, if

$$f = \begin{pmatrix} \cos x_2 \\ x_1 \end{pmatrix}, \quad g = \begin{pmatrix} x_1 \\ 1 \end{pmatrix},$$

then we have

$$[f, g] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos x_2 \\ x_1 \end{pmatrix} - \begin{pmatrix} 0 & -\sin x_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos x_2 + \sin x_2 \\ -x_1 \end{pmatrix}.$$
3.2.3 Algorithm for Exact Feedback Linearization

We want to find a nonlinear compensator such that the closed-loop system is linear (see Figure 5). We will consider only affine SISO systems of the type (27), i.e., \( \dot{x} = f(x) + g(x) u, \) \( y = h(x) \), and we will try to construct a control law of the form

\[
u = p(x) + q(x) v, \tag{28}
\]

where \( v \) is the setpoint, such that the closed-loop system

\[
\dot{x} = f(x) + g(x) p(x) + g(x) q(x) v, \\
y = h(x),
\]

is linear from command \( v \) to \( y \).

The main idea behind the feedback linearization construction is to find a nonlinear change of coordinates which transforms the original system into one which is linear and controllable, in particular, a chain of integrators. The difficulty is finding the output function \( h(x) \) which makes this construction possible.

We want to design an exact nonlinear feedback controller. Given the nonlinear affine system, \( \dot{x} = f(x) + g(x), \) \( y = h(x) \), we want to find the controller functions \( p(x) \) and \( q(x) \). The unknown functions inside our controller (28) are given by:

\[
p(x) = -\left( L^r f h(x) + \beta_1 L^{r-1} f h(x) + ... + \beta_{r-1} L f h(x) + \beta_r h(x) \right), \\
q(x) = 1 L g L^{r-1} f h(x), \tag{29}
\]

which are comprised of Lie derivatives, \( L f h(x) \). Here, the relative order, \( r \), is the smallest integer \( r \) such that \( L g L^{r-1} f h(x) \neq 0 \). For linear systems \( r \) is the difference between the number of poles and zeros.

To obtain the desired response, we choose the \( r \) parameters in the \( \beta \) polynomial to describe how the output will respond to the setpoint, \( v \) (pole–placement).

\[
\frac{d^ry}{dt^r} + \beta_1 \frac{d^{r-1}y}{dt^{r-1}} + ... + \beta_{r-1} \frac{dy}{dt} + \beta_r y = v.
\]

Here is the proposed algorithm [41, 43, 37]:

1. Given nonlinear SISO process, \( \dot{x} = f(x, u) \), and output equation \( y = h(x) \), then:
2. Calculate the relative order, \( r \).
3. Choose an \( r \)th order desired linear response using pole–placement technique (i.e., select \( \beta \)). For this could be used a simple \( r \)th order low–pass filter such as a Butterworth filter.
4. Construct the exact linearized nonlinear controller (29), using Lie derivatives and perhaps a symbolic manipulator (Mathematica or Maple).
5. Close the loop and obtain a linear input–output black–box (see Figure 5).
6. Verify that the result is actually linear by comparing with the desired response.

3.3 Controllability

3.3.1 Linear Controllability

A system is \textit{controllable} if the set of all states it can reach from initial state \( x_0 = x(0) \) at the fixed time \( t = T \) contains a ball \( B \) around \( x_0 \). Again, a system is \textit{small time locally controllable} (STLC) iff the ball \( B \) for \( t \leq T \) contains a neighborhood of \( x_0 \).

In the case of a linear system in the standard state–space form

\[
\dot{x} = Ax + Bu,
\]

(30)

where \( A \) is the \( n \times n \) state matrix and \( B \) is the \( m \times n \) input matrix, all controllability definitions coincide, i.e.,

\[
0 \rightarrow x(T),
\]

\[
x(0) \rightarrow 0,
\]

\[
x(0) \rightarrow x(T),
\]

where \( T \) is either fixed or free.

\textit{Rank condition} states: System (30) is controllable iff the matrix

\[
W_n = (B AB ... A^{n-1}B)
\]

has full rank.

In the case of nonlinear systems the corresponding result is obtained using the formalism of Lie brackets, as Lie algebra is to nonlinear systems as matrix algebra is to linear systems.

3.3.2 Nonlinear Controllability

Nonlinear MIMO–systems are generally described by differential equations of the form (see [41][42][43]):

\[
\dot{x} = f(x) + g_i(x) u^i, \quad (i = 1, ..., n),
\]

(31)

defined on a smooth \( n \)–manifold \( M \), where \( x \in M \) represents the state of the control system, \( f(x) \) and \( g_i(x) \) are vector–fields on \( M \) and the \( u^i \) are control inputs, which belong to a set of \textit{admissible controls}, \( u^i \in U \). The system (31) is called \textit{driftless}, or \textit{kinematic}, or \textit{control linear} if \( f(x) \) is identically zero; otherwise, it is called a \textit{system with drift}, and the vector–field \( f(x) \) is called the \textit{drift term}. The flow \( \phi_t^x(x_0) \) represents the solution of the differential equation \( \dot{x} = g(x) \) at time \( t \) starting from \( x_0 \). Geometrical way to understand the \textit{controllability} of the system (31) is to understand the geometry of the vector–fields \( f(x) \) and \( g_i(x) \).

\[^3\text{The above definition of controllability tells us only whether or not something can reach an open neighborhood of its starting point, but does not tell us how to do it. That is the point of the trajectory generation.}\]
**Example: Car–Parking Using Lie Brackets.** In this popular example, the driver has two different transformations at his disposal. He can turn the steering wheel, or he can drive the car forward or back. Here, we specify the state of a car by four coordinates: the \((x, y)\) coordinates of the center of the rear axle, the direction \(\theta\) of the car, and the angle \(\phi\) between the front wheels and the direction of the car. \(L\) is the constant length of the car. Therefore, the configuration manifold of the car is 4D, \(M := (x, y, \theta, \phi)\).

Using (31), the driftless car kinematics can be defined as:

\[
\dot{x} = g_1(x) u_1 + g_2(x) u_2, \quad (32)
\]

with two vector–fields \(g_1, g_2 \in X^k(M)\).

The infinitesimal transformations will be the vector–fields

\[
g_1(x) \equiv \text{DRIVE} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} + \frac{\tan \phi}{L} \frac{\partial}{\partial \theta} \equiv \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \\ \frac{1}{L} \tan \phi \end{pmatrix},
\]

and

\[
g_2(x) \equiv \text{STEER} = \frac{\partial}{\partial \phi} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\]

Now, \text{STEER} and \text{DRIVE} do not commute; otherwise we could do all your steering at home before driving off on a trip. Therefore, we have a Lie bracket

\[
[g_2, g_1] \equiv [\text{STEER, DRIVE}] = \frac{1}{L \cos^2 \phi} \frac{\partial}{\partial \theta} \equiv \text{ROTATE}.
\]

The operation \([g_2, g_1] \equiv \text{ROTATE} \equiv [\text{STEER,DRIVE}]\) is the infinitesimal version of the sequence of transformations: steer, drive, steer back, and drive back, i.e.,

\[
\{\text{STEER, DRIVE, STEER}^{-1}, \text{DRIVE}^{-1}\}.
\]

Now, \text{ROTATE} can get us out of some parking spaces, but not tight ones: we may not have enough room to \text{ROTATE} out. The usual tight parking space restricts the \text{DRIVE} transformation, but not \text{STEER}. A truly tight parking space restricts \text{STEER} as well by putting your front wheels against the curb.

Fortunately, there is still another commutator available:

\[
[g_1, [g_2, g_1]] \equiv [\text{DRIVE, [STEER, DRIVE]}] = [(g_1, g_2), g_1] \equiv [\text{DRIVE, ROTATE}] = \frac{1}{L \cos^2 \phi} \left( \sin \theta \frac{\partial}{\partial x} - \cos \theta \frac{\partial}{\partial y} \right) \equiv \text{SLIDE}.
\]

The operation \([[g_1, g_2], g_1] \equiv \text{SLIDE} \equiv [\text{DRIVE,ROTATE}]\) is a displacement at right angles to the car, and can get us out of any parking place. We just need to remember to steer, drive, steer back, drive some more, steer, drive back, steer back, and drive back:

\[
\{\text{STEER, DRIVE, STEER}^{-1}, \text{DRIVE, STEER, DRIVE}^{-1}, \text{STEER}^{-1}, \text{DRIVE}^{-1}\}.
\]

We have to reverse steer in the middle of the parking place. This is not intuitive, and no doubt is part of the problem with parallel parking.
Thus from only two controls $u_1$ and $u_2$ we can form the vector fields \( \text{DRIVE} \equiv g_1, \text{STEER} \equiv g_2, \text{ROTATE} \equiv [g_2, g_1], \text{and SLIDE} \equiv [[g_1, g_2], g_1], \) allowing us to move anywhere in the configuration manifold $M$. The car kinematics $\dot{x} = g_1 u_1 + g_2 u_2$ is thus expanded as:

\[
\begin{pmatrix}
\dot{x} \\
\dot{y} \\
\dot{\phi}
\end{pmatrix} = \text{DRIVE} \cdot u_1 + \text{STEER} \cdot u_2 \\
\begin{pmatrix}
\cos \theta \\
\sin \theta \\
\frac{1}{L} \tan \phi
\end{pmatrix} \cdot u_1 + \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} \cdot u_2.
\]

The parking theorem says: One can get out of any parking lot that is larger than the car.

The Unicycle Example. Now, consider the unicycle example (see Figure 6). Here we have

\[
g_1 = \begin{pmatrix}
\cos x_3 \\
\sin x_3 \\
0
\end{pmatrix}, \quad g_2 = \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix},
\]

\[
[g_1, g_2] = \begin{pmatrix}
\sin x_3 \\
-\cos x_3 \\
0
\end{pmatrix}.
\]

The unicycle system is full rank and therefore controllable.

### 3.3.3 Controllability Condition

Nonlinear controllability is an extension of linear controllability. The nonlinear SIMO system

\[
\dot{x} = f(x) + g(x) u
\]

is controllable if the set of vector fields

\[
\{g, [f, g], ..., [f^{n-1}, g]\}
\]

is independent.

For example, for the kinematic kar system of the form (32), the nonlinear controllability criterion reads: If the Lie bracket tree:

\[
g_1, g_2, [g_1, g_2], [[g_1, g_2], g_1], [[g_1, g_2], g_1], [[[g_1, g_2], g_1], g_1], [[[g_1, g_2], g_1], g_2],
\]

\[
[[[g_1, g_2], g_2], g_1], [[[g_1, g_2], g_2], g_2], ...
\]
– has full rank then the system is controllable [41, 44, 42]. In this case the combined input
\[
(u_1, u_2) = \begin{cases}
(1, 0), & t \in [0, \varepsilon] \\
(0, 1), & t \in [\varepsilon, 2\varepsilon] \\
(-1, 0), & t \in [2\varepsilon, 3\varepsilon] \\
(0, -1), & t \in [3\varepsilon, 4\varepsilon]
\end{cases}
\]
gives the motion \(x(4\varepsilon) = x(0) + \varepsilon^2 [g_1, g_2] + O(\varepsilon^3)\), with the flow given by
\[
F_t^{[g_1, g_2]} = \lim_{n \to \infty} \left( F_{t/n}^{-g_2} F_{t/n}^{-g_1} F_{t/n}^{g_2} F_{t/n}^{g_1} \right)^n.
\]

3.4 Adaptive Lie–Derivative Control

In this subsection we develop the concept of machine learning in the framework of Lie–derivative control formalism (see [3.2.1] above). Consider an \(n\)-dimensional, SISO system in the standard affine form (27), rewritten here for convenience:
\[
\dot{x}(t) = f(x) + g(x) u(t), \quad y(t) = h(x),
\]
(33)

As already stated, the feedback control law for the system (33) can be defined using Lie derivatives \(\mathcal{L}_f h\) and \(\mathcal{L}_g h\) of the system’s output \(h\) along the vector–fields \(f\) and \(g\).

If the SISO system (33) is a relatively simple (quasilinear) system with relative degree \(r = 1\), it can be rewritten in a quasilinear form
\[
\dot{x}(t) = \gamma_i(t) f_i(x) + d_j(t) g_j(x) u(t),
\]
(34)
where \(\gamma_i (i = 1, \ldots, n)\) and \(d_j (j = 1, \ldots, m)\) are system’s parameters, while \(f_i\) and \(g_j\) are smooth vector–fields.

In this case the feedback control law for tracking the reference signal \(y_R = y_R(t)\) is defined as (see [41, 44])
\[
u = -\frac{-\mathcal{L}_f h + \dot{y}_R + \alpha (y_R - y)}{\mathcal{L}_g h},
\]
(35)
where \(\alpha\) denotes the feedback gain.

Obviously, the problem of reference signal tracking is relatively simple and straightforward if we know all the system’s parameters \(\gamma_i(t)\) and \(d_j(t)\) of (34). The question is can we apply a similar control law if the system parameters are unknown?

Now we have much harder problem of adaptive signal tracking. However, it appears that the feedback control law can be actually cast in a similar form (see [43, 45]):
\[
\tilde{u} = -\frac{-\mathcal{L}_f \hat{h} + \dot{y}_R + \alpha (y_R - y)}{\mathcal{L}_g \hat{h}},
\]
(36)
where Lie derivatives \(\mathcal{L}_f h\) and \(\mathcal{L}_g h\) of (35) have been replaced by their estimates \(\hat{\mathcal{L}}_f h\) and \(\hat{\mathcal{L}}_g h\), defined respectively as
\[
\hat{\mathcal{L}}_f h = \hat{\gamma}_i(t) \mathcal{L}_f h, \quad \hat{\mathcal{L}}_g h = \hat{d}_j(t) \mathcal{L}_g h,
\]

Relative degree equals the number of differentiations of the output function \(y\) required to have the input \(u\) appear explicitly. Technically, the system (33) is said to have relative degree \(r\) at the point \(x^0\) if (see [31, 44])
(i) \(\mathcal{L}_g \mathcal{L}_f^k h(x) = 0\) for all \(x\) in a neighborhood of \(x^0\) and all \(k < r - 1\), and
(ii) \(\mathcal{L}_g \mathcal{L}_f^{r-1} h(x^0) \neq 0\),
where \(\mathcal{L}_f^k h\) denotes the \(k\)th Lie derivative of \(h\) along \(f\).
in which $\hat{\gamma}_i(t)$ and $\hat{d}_j(t)$ are the estimates for $\gamma_i(t)$ and $d_j(t)$.

Therefore, we have the straightforward control law even in the uncertain case, provided that we are able to estimate the unknown system parameters. Probably the best known parameter update law is based on the so-called Lyapunov criterion (see [43]) and given by

$$\dot{\psi} = -\gamma \epsilon W,$$

(37)

where $\psi = \{\gamma_i - \hat{\gamma}_i, d_j - \hat{d}_j\}$ is the parameter estimation error, $\epsilon = y - y_R$ is the output error, and $\gamma$ is a positive constant, while the matrix $W$ is defined as:

$$W = [W_1^TW_2^T]^T,$$ with

$$W_1 = \begin{bmatrix} \mathcal{L}_{f_1}h \\ \vdots \\ \mathcal{L}_{f_n}h \end{bmatrix}, \quad W_2 = \begin{bmatrix} \mathcal{L}_{g_1}h \\ \vdots \\ \mathcal{L}_{g_m}h \end{bmatrix},$$

$$-\dot{\mathcal{L}}_f h + \dot{y}_R + \alpha (y_R - y), \quad \frac{\mathcal{L}_g h}{\dot{y}_R}.$$

The proposed adaptive control formalism (36–37) can be efficiently applied whenever we have a problem of tracking a given signal with an output of a SISO–system (33–34) with unknown parameters.

4 Conclusion

In this paper we have presented two approaches to the human operator modeling: linear control theory approach and nonlinear control theory approach, based on the fixed and adaptive versions of a single-input single output Lie-Derivative controller. Our future work will focus on the generalization of the adaptive Lie-Derivative controller to MIMO systems. It would give us a rigorous closed–form model for model–free neural networks.

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