High energy modifications of blackbody radiation and dimensional reduction

Viqar Husain,1,* Sanjeev S. Seahra,1,† and Eric J. Webster2,‡

1Department of Mathematics and Statistics, University of New Brunswick, Fredericton, NB, Canada E3B 5A3
2Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

Quantization prescriptions that realize generalized uncertainty relations (GUP) are motivated by quantum gravity arguments that incorporate a fundamental length scale. We apply two such methods, polymer and deformed Heisenberg quantization, to scalar field theory in Fourier space. These alternative quantizations modify the oscillator spectrum for each mode, which in turn affects the blackbody distribution. We find that for a large class of modifications, the equation of state relating pressure \( P \) and energy density \( \rho \) interpolates between \( P = \rho/3 \) at low \( T \) and \( P = 2\rho/3 \) at high \( T \), where \( T \) is the temperature. Furthermore, the Stefan-Boltzman law gets modified from \( \rho \propto T^4 \) to \( \rho \propto T^{5/2} \) at high temperature. This suggests an effective reduction to 2.5 spacetime dimensions at high energy.

I. INTRODUCTION

The inherent difficulties in constructing a quantum theory of gravity have led many to believe that conventional physical theories need to be significantly altered on very small scales. For example, string theory predicts extra dimensions of exceedingly small size, while loop quantum gravity posits that standard Schrödinger quantization is merely an approximation to polymer quantization valid on super-Planckian scales. A naturally interesting question is how exotic small scale physics might affect quantum field theory (QFT). A number of authors have looked at this issue in the context of Hawking radiation [1–3], the two-point function and wave propagation of a free scalar field [4–6], and the trans-Planckian problem for the generation of primordial perturbations during inflation [7–31].

In this work we are interested in how exotic high energy physics may impact the behaviour of blackbody radiation at high temperature. This problem has been considered before: Effects of the deformation of the quantum commutator algebra between position and momentum on a scalar gas have been investigated in the classical limit [32], and by employing modified field quantization in Fourier space [33]. Possible ramifications of spacetime non-commutativity on the blackbody spectrum were discussed in [34]. The dependence of the thermal characteristics of the gas on the existence of non-compact [35] and compact [36] extra dimensions has been reported. Finally in [37], the combined effects of modified dispersion relations and extra dimensions were studied.

We restrict our attention to high-energy effects that modify the energy levels of field quanta. It is well-known that if one considers the Fourier transform of a non-interacting scalar field, the Hamiltonian governing the evolution of the Fourier coefficients is that of a collection of decoupled one-dimensional simple harmonic oscillators (SHOs). All the familiar results concerning the quantum behaviour of a free scalar field can be derived by quantizing these oscillators using ordinary quantum mechanics [5, 6]. Hence one of the most straightforward ways to incorporate high-energy modifications into QFT is to apply non-Schrödinger quantization techniques to these oscillators, as is done in Refs. [3, 5, 6, 31, 33].

The most important consequence for statistical mechanics is that these alternate quantization schemes will change the energy spectrum of each Fourier mode. That is, the energy level spacing is no longer \( \hbar \), but rather a more complicated function. This information can be directly used to calculation the partition function of the scalar gas at finite temperature, which in turn specifies all its thermal properties [38].

After reviewing the oscillator description of QFT in §II, we investigate the implications of generic modifications to the oscillator energy levels in §III assuming that that for large wavenumbers, energy eigenvalues scale like \( k^\alpha \) rather than \( k \). This implies that the internal energy \( U \) of the gas is \( \propto T^{1+3/\alpha} \) and its equation of state is \( P = \frac{\alpha}{3} \rho \), where \( P \) is the pressure and \( \rho \) is the density.

In §IV, we consider several specific models that result in modifications of the oscillator energy levels. One class of such models involves modifying the commutator of the field amplitude \( \hat{\phi}_k \) and the field momentum \( \hat{p}_k \) in Fourier space as follows:

\[
[\hat{\phi}_k, \hat{p}_k] = i \quad \mapsto \quad [\hat{\phi}_k, \hat{p}_k] = if(\hat{\pi}_k/M_*)
\]

where \( M_* \) is an energy scale that indicates the threshold for exotic physics. Finding the energy levels of the Fourier modes is equivalent to finding the energy levels of a SHO quantized according to the commutator \([\hat{x}, \hat{p}] = if(\hat{\beta} \hat{p})\), where \( \beta \) is a dimensional constant. The SHO energy levels for the \( f(x) = 1 + x^2 \) case were found in [39], while the \( f(x) = 1 - x^2 \) case was considered in [6].

The leading order low temperature corrections to blackbody radiation due to the choice \( f = 1 + x^2 \) were reported in [33]. (In this paper, we do not restrict ourselves to the low temperature regime.) In §IV A, we give details of the energy level calculations for the choices \( f(x) = 1 \pm x \) and summarize the results in Table I.

* husain@unb.ca
† sseahra@unb.ca
‡ ewebster@uwaterloo.ca
We then explore a different class of new physics suggested by the background independent (or “polymer”) approach to quantization that is deployed in loop quantum gravity (LQG) [40]. The principal difference between this approach and traditional quantization is that operators corresponding to the momentum do not exist due to the unconventional choice of Hilbert space, but operators corresponding to the exponentiated momentum are well defined. The novel features of this approach to quantization have been considered in a number of quantum mechanical systems [45–51], quantum cosmology [48–51], and quantum field theory [4, 5, 52–55].

The statistical mechanics of a polymer quantized harmonic oscillator has also been discussed [56], but our results go further. In §IVB, we review the basic properties of a polymer oscillator and present its energy eigenvalues [5, 41]. An important point is that the polymer quantization scheme for this system carries with it an ambiguity related to the choice of a super-selected sector of the Hilbert space (this issue is discussed in detail in Refs. [41, 51]). Hence, we obtain several possible energy spectra based on how this ambiguity is resolved.

In §VI we numerically calculate the thermal properties of a scalar gas using the modified energy levels presented in §IV. We find that some of the modifications lead to UV divergent integrals for thermodynamic quantities. All of the other modifications discussed share the property that the energy eigenvalues scale as $k^2$ for large $k$, which by the arguments of §III lead to the high temperature Stefan-Boltzmann law $\mathcal{U} \propto T^{5/2}$ and equation of state $P = 2 \rho/3$. We confirm these expectations numerically. We summarize and discuss our results in §VI.

II. OSCILLATOR DESCRIPTION OF QUANTUM FIELD THEORY

We consider the quantum field theory of a scalar field in Minkowski space time. The Hamiltonian is

$$H_\phi = \int d^3x \left[ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 \right].$$

(2)

Here, $\pi$ is the momentum conjugate to $\phi$ such that $\{\phi(t, x), \pi(t, y)\} = \delta^{(3)}(x - y)$. We decompose $\phi$ and $\pi$ into Fourier modes as follows:

$$\phi(t, x) = \frac{1}{\sqrt{V}} \sum_k \phi_k(t) e^{ik \cdot x},$$

$$\pi(t, x) = \frac{1}{\sqrt{V}} \int d^3x \ e^{-ik \cdot x} \pi(t, x),$$

(3)

with a similar expansion for $\pi(t, x)$; $V$ is the fiducial volume used in our box normalization

$$V = \int d^3x.$$

(4)

After a suitable redefinition of the independent modes to enforce that $\phi$ is real, the Hamiltonian is

$$H_\phi = \sum_k H_k = \sum_k \left[ \frac{\pi_k^2}{2} + \frac{k^2}{2} \phi_k^2 \right].$$

(5)

with the Poisson bracket $\{\phi_k, \pi_{k'}\} = \delta_{k,k'}$.

We now quantize the classical system (5). The structure of the Hamiltonian is that of a collection of decoupled harmonic oscillators labelled by $k$. Hence, the quantum state of the system will be

$$|\psi\rangle = \bigotimes_k |\psi_k\rangle,$$

(6)

where each of the $|\psi_k\rangle$ satisfy the Schrödinger equation

$$i\partial_t |\psi_k\rangle = \hat{H}_k |\psi_k\rangle,$$

$$\hat{H}_k = \frac{1}{2} \pi_k^2 + \frac{k^2}{2} \phi_k^2.$$

(7)

Since the energy eigenstates $|n_k\rangle$ of $\hat{H}_k$ form a basis, we can express $|\psi_k\rangle$ as

$$|\psi_k\rangle(t) = \sum_{n=0}^{\infty} e^{-iE_n \beta} |n_k\rangle,$$

(8)

where

$$\hat{H}_k |n_k\rangle = E_{n,k} |n_k\rangle.$$

(9)

In standard Schrödinger quantization, the solution to the eigenvalue problem (9) is the well-known energy eigenstates of the simple harmonic oscillator with $E_n = k(n + 1/2)$.

For alternative quantizations, this spectrum is modified for modes with $k \gtrsim M_*$, where $M_*$ is an energy scale associated with quantum gravity. To parametrize such effects, let us introduce a dimensional quantity

$$g = \frac{k}{M_*},$$

(10)

such that the oscillator spectrum takes the scaling form

$$\frac{E_{n,k}}{M_*} = \varepsilon_n(g),$$

(11)

where the functions $\varepsilon_n(g)$ will depend of the specific class of high-energy modification being considered. The correct low energy physics is recovered if

$$\varepsilon_n(g) \approx g(n + 1/2), \quad g \ll 1.$$  

(12)

III. STATISTICAL MECHANICS

A. Formalism

Let us now turn our attention to the statistical properties of the field at a finite temperature $T$. The partition function for a given $k$ mode is

$$Z_k(\beta) = \sum_{n=0}^{\infty} e^{-\beta(E_{n,k} - E_{0,k})},$$

(13)
where
\[ \beta = \frac{1}{k_B T}, \]
and \( k_B \) is Boltzmann’s constant. The average energy in the mode (above the ground state) is then
\[ E_k(\beta) = -\frac{\partial}{\partial \beta} \ln Z_k(\beta). \]  

In three spatial dimensions, the total internal energy \( U \) of the gas is
\[ U = \frac{V}{\pi^2} \int_0^\infty dk \, k^2 E_k(\beta). \]

These formulae may be converted to dimensionless units by introducing
\[ \tilde{\beta} = \beta M_* . \]

Then the energy is given by
\[ U = V M_*^4 \int_0^\infty dg \, I(g, \tilde{\beta}), \]
where the dimensionless intensity is
\[ I(g, \tilde{\beta}) = -\frac{g^2}{\pi^2} \frac{\partial}{\partial \beta} \ln \left( \sum_{n=0}^\infty e^{-\beta \Delta \varepsilon_n(g)} \right), \]
and
\[ \Delta \varepsilon_n(g) = \varepsilon_n(g) - \varepsilon_0(g). \]

The entropy \( S \) and pressure \( P \) of the gas is obtained from \( U \) in the standard way. Define the energy and entropy densities by
\[ \rho(T) = \frac{U}{V}, \quad s(T) = \frac{S}{V}. \]

Then the thermodynamic identities
\[ \left( \frac{\partial S}{\partial U} \right)_V = \frac{1}{T}, \quad \left( \frac{\partial S}{\partial T} \right)_V = \left( \frac{\partial S}{\partial U} \right)_V \left( \frac{\partial U}{\partial T} \right)_V, \]
yield the following formula for the entropy density
\[ s(T) = \frac{\rho(T)}{T} + \int_0^T \frac{\rho(\theta)}{\theta^2} d\theta , \]
where we have assumed
\[ s(0) = 0, \quad \lim_{T \to 0^+} \frac{\rho}{T} = 0. \]

Using the Maxwell relation
\[ \left( \frac{\partial S}{\partial V} \right)_T = \left( \frac{\partial P}{\partial T} \right)_V, \]
and noting that the pressure is intensive, we obtain
\[ P(T) = \int_0^T s(\theta) d\theta. \]

To summarize, the formulae in this subsection give the energy \( U \), entropy \( S \) and pressure \( P \) of an scalar gas in terms of the temperature \( T \) and volume \( V \), using the energy eigenvalues \( \varepsilon_n(g) \) of modified oscillators corresponding to different \( k \) modes of the scalar field. All other thermodynamic quantities are similarly determined.

### B. Low and high temperature limits

In the low temperature limit \( \tilde{\beta} \gg 1 \), the exponential functions appearing in (19) are highly damped for \( g \gtrsim 1 \); hence, we can use low-\( g \) approximation for \( \Delta \varepsilon_n \). Then the correct low energy limit
\[ \Delta \varepsilon_n \approx ng, \quad g \ll 1. \]
gives the usual blackbody spectrum at low temperature,
\[ I = \frac{1}{\pi^2} \frac{g^3}{e^{\beta g} - 1} = \frac{1}{M_*^2 \pi^2} \frac{k^3}{e^{\beta k} - 1}, \]
and Stefan-Boltzmann law
\[ U = \frac{\pi^2}{15} V k_B T^4. \]

The expressions (23) and (26) then yield familiar results
\[ s = \frac{4}{3} \frac{\rho}{T}, \quad P = \frac{1}{3} \rho. \]

In the high-temperature limit \( \tilde{\beta} \ll 1 \), we cannot write down a closed form expression for the intensity without knowing the energies \( \varepsilon_n(g) \). However, we would expect the integrand in (18) to be peaked at some high wavenumber \( g \gg 1 \). Furthermore, it is reasonable to assume that the energy differences scale as some power of \( g \) in this limit:
\[ \Delta \varepsilon_n \approx g^a f_n, \quad g \gg 1, \]
where the numbers \( f_n \) are independent of \( g \). Under such an assumption, we find
\[ U = KVT^{\frac{3}{2} + 1}, \]
where
\[ K = \kappa k_B^{\frac{1}{2} + 1} M_*^{\frac{3}{2} - \frac{3}{2}}, \]
and
\[ \kappa = \frac{1}{\alpha \pi^2} \int_0^\infty dx \, x^{3/2} \left( -\frac{\partial}{\partial x} \ln \sum_{n=0}^\infty e^{-f_n x} \right). \]
is a dimensionless constant which depends on the precise form of \( f_n \). At high temperatures, the integrals in (23) and (26) are dominated by the high \( T \) behaviour of \( \rho(T) \), which give

\[
s = \left( 1 + \frac{\alpha}{3} \right) \rho \frac{\beta}{T}, \quad P = \frac{\alpha}{3} \rho. \tag{35}
\]

In other words, the equation of state is \( P/\rho = \alpha/3 \) at high temperature. A similar analysis was carried out in [57] in the context of holography.

IV. ALTERNATIVE QUANTIZATIONS OF THE OSCILLATOR

In the previous section, we saw how the oscillator energy spectrum (7) fixes the thermal properties of the scalar gas. In this section we will see how the this spectrum is modified by GUP and polymer quantizations as a prelude to deriving its effects on the blackbody distribution.

A. GUP quantization

Several approaches to quantum gravity suggest that one should modify the fundamental commutator \([\hat{x}, \hat{p}] = i\) of quantum mechanics at high energies, which leads to deformed uncertainty relations. In terms of the QFT oscillator variables of §II, we write these as

\[
[\hat{\phi}_k, \hat{\pi}_k] = i f \left( \frac{\hat{\pi}_k}{M_*^{1/2}} \right), \tag{36}
\]

where \( f(0) = 1 \) in order to recover the standard commutator for \( \pi_k \ll M_*^{1/2} \). We introduce a basis of momentum eigenstates \(|\pi_k \rangle\) and express the Hamiltonian eigenstates \(|n_k \rangle\) as wavefunctions:

\[
|\pi_k \rangle_{n_k} = \psi_n(\pi_k). \tag{37}
\]

In order to faithfully reproduce the above commutator, we take the action of \( \hat{\phi}_k \) and \( \hat{\pi}_k \) on these wavefunctions as:

\[
|\pi_k \rangle_{\hat{\pi}_k} = f(\pi_k/M_*^{1/2}) \partial_{\pi_k} \psi_n(\pi_k), \tag{38a}
\]

\[
|\pi_k \rangle_{\hat{\phi}_k} = \pi_k \psi_n(\pi_k). \tag{38b}
\]

In this representation the eigenvalue equation (9) reads

\[
E_{n,k} \psi_n(\pi_k) = \left\{ \frac{\pi_k^2}{2} - \frac{k^2}{2} f \left( \frac{\hat{\pi}_k}{M_*^{1/2}} \right) \frac{\partial}{\partial \pi_k} \right\} \psi_n(\pi_k). \tag{39}
\]

Let us change variables according to

\[
\pi_k = M_*^{1/2} P(z), \quad z = \int_0^z \frac{du}{f(u)}; \tag{40}
\]

and define the following dimensionless quantities

\[
g = \frac{k}{M_*}, \quad \Psi_n(z) = \psi_n(P(z)), \quad \kappa_n = \frac{E_{n,k}}{g^2 M_*}. \tag{41}
\]

Then the eigenvalue equation reads

\[
\kappa_n \Psi_n(z) = \left[ -\frac{1}{2} \frac{\partial^2}{\partial z^2} + V(z) \right] \Psi_n(z), \tag{42}
\]

where the potential is given by

\[
V(z) = \frac{P^2(z)}{2g^2}. \tag{43}
\]

**Case A0: \( f(x) = 1 \)**

This is the standard result that follows from the Heisenberg algebra

\[
[\hat{\phi}_k, \hat{\pi}_k] = i. \tag{44}
\]

It leads to the eigenvalue equation

\[
\kappa_n \Psi_n(z) = \left[ -\frac{1}{2} \frac{\partial^2}{\partial z^2} + \frac{z^2}{2g^2} \right] \Psi_n(z), \tag{45}
\]

where \( \pi_k = M_*^{1/2} \). Its solutions are given in terms of Hermite polynomials and are summarized in Table I.

**Case AI: \( f(x) = 1 + x^2 \)**

For the modified commutator

\[
[\hat{\phi}_k, \hat{\pi}_k] = i \left( 1 + \frac{\hat{\pi}_k^2}{M_*^{1/2}} \right), \tag{46}
\]

the eigenvalue equation is

\[
\kappa_n \Psi_n(z) = \left[ -\frac{1}{2} \frac{\partial^2}{\partial z^2} + \frac{\tan^2(z)}{2g^2} \right] \Psi_n(z), \tag{47}
\]

where \( \pi_k = M_*^{1/2} \tan z. \)

This eigenvalue problem is analytically solvable in terms of ultraspherical (Gegenbauer) polynomials [58] for eigenfunctions satisfying Dirichlet boundary conditions:

\[
\Psi_n(\pm\pi/2) = 0, \tag{48}
\]

and inner product

\[
(\Psi_n, \Psi_m)_{\pi/2} \equiv \int_{-\pi/2}^{\pi/2} \Psi_n^*(z) \Psi_m(z) dz = \delta_{m,n}. \tag{49}
\]

Explicit formulae for the energy eigenfunctions and eigenvalues are given in Table I, and they are plotted in Figures 1 and 2, respectively. The eigenvalues have the asymptotic behaviour

\[
\varepsilon_n(g) \approx \begin{cases}
g(n + \frac{1}{2}), \quad g \ll 1, \\
\frac{1}{2} g^2 (n + 1)^2, \quad g \gg 1.
\end{cases} \tag{50}
\]
It is therefore evident that the usual result at low $g$ is recovered. At high $g$, the energy differences are
\[ \Delta \varepsilon_n(g) \approx \frac{1}{2} g^2 n(n + 2) = g^2 f_n. \]  
(51)

Case AII: $f(x) = 1 - x^2$ without cut-off

The modified commutator is
\[ [\hat{\phi}_k, \hat{\pi}_k] = i \left( 1 - \frac{\pi_k^2}{M_\ast} \right), \]  
(52)
and the eigenvalue equation takes the form
\[ \kappa_n \Psi_n(z) = \left[ -\frac{1}{2} \frac{\partial^2}{\partial z^2} + \frac{\tanh^2(z)}{2g^2} \right] \Psi_n(z), \]  
(53)
with $\pi_k = M_\ast^{1/2} \tanh z$. As in Case AII, this eigenvalue problem has known solutions in terms of ultraspherical polynomials for eigenfunctions satisfying Dirichlet boundary conditions at infinity.

\[ \lim_{z \to \pm \infty} \Psi_n(z) = 0, \]  
(54)
and inner product $(\Psi_n, \Psi_m)_{\infty} = \delta_{m,n}$. Explicit formulae for the eigenfunctions and energy eigenvalues are given in Table I, and are plotted in Figures 1 and 2, respectively.

Unlike the previous cases considered, the potential in the Schrödinger equation (53) only supports a finite number of bound states, which do not form a complete energy eigenfunction basis (unless the potential is put in a box). The is a direct consequence of the finite height of the potential appearing in (53), as can be seen in Figure 1.

Case AIII: $f(x) = 1 - x^2$ with cut-off

Since the energy eigenfunction basis of Case AII is not complete, the formulae of §III are not applicable and we cannot directly obtain the blackbody spectrum. However, we can recover a complete eigenfunction basis if we impose Dirichlet boundary conditions on a finite boundary, i.e.
\[ \Psi_n(\pm \lambda) = 0. \]  
(55)
where $\lambda$ is an adjustable dimensionless (box) parameter. In this case the exact solution of (53) may be written in terms of associated Legendre functions as in Table I. It can be shown that the energy eigenvalues $\varepsilon_n$ are related to the roots $\nu_n$ of the equation
\[ |P_{j}^{\nu_n}(\tanh \lambda)|^2 = |P_{j}^{\nu_n}(-\tanh \lambda)|^2, \]  
(56)
via the relation
\[ \varepsilon_n = \frac{1}{2} (g^2 \nu_n + 1). \]  
(57)
Note that $\nu_n$ can be either real or imaginary, corresponding to eigenmodes with energy greater or less than $M_\ast$, respectively. Unfortunately, (56) is not exactly solvable for $\nu_n$, but it is possible to obtain the energies numerically for a given value of $\lambda$. Examples of numerically obtained eigenfunctions and eigenvalues are shown in Figures 1 and 2 for $\lambda = 5$.

Before we move one, it may be useful to interpret the boundary condition (55) in terms of the original field variables. It is easy to see that the boundary condition implies
\[ |\pi_k| \leq M_\ast^{1/2} \tanh \lambda. \]  
(58)
Hence by enforcing (55), we are effectively imposing a cut-off for the field momentum $\pi_k$. We will see below that this cut-off is necessary to tame UV divergences in quantities such as the internal energy of the scalar gas at finite temperature.

B. Polymer quantization

As described elsewhere [31, 41], there is no explicit realization of the momentum operator $\pi_k$ in polymer quantum mechanics. However, we can write an “effective” momentum operator as
\[ \hat{\pi}_k^* = \frac{\hat{U}_\lambda - \hat{U}_\lambda^\dagger}{2i\lambda_\ast}, \]  
(59)
where $\lambda_\ast \equiv M_\ast^{-1/2}$ is a fixed parameter with dimensions of (mass)$^{-1/2}$, and $\hat{U}_\lambda$ is an operator which induces translations of magnitude $\lambda$ in the field amplitude $\phi_k$:
\[ [\hat{\phi}_k, \hat{U}_\lambda] = -i\hat{U}_\lambda \phi_k, \quad \hat{U}_\lambda^\dagger \phi_k \hat{U}_\lambda = \phi_k - \lambda \mathbf{1}. \]  
(60)
We now introduce a basis $|\pi_k\rangle$ and express the Hamiltonian eigenstates $|\psi_n\rangle$ as wavefunctions:
\[ \langle \pi_k | \psi_n \rangle = \psi_n(\pi_k). \]  
(61)
The action of $\hat{\phi}_k$ and $\hat{U}_\lambda$ on these wavefunctions is
\[ \langle \pi_k | \hat{\phi}_k | \psi_n \rangle = i \partial_\pi \psi_n(\pi_k), \]  
(62a)
\[ \langle \pi_k | \hat{U}_\lambda | \psi_n \rangle = \exp(i\lambda \pi_k) \psi_n(\pi_k). \]  
(62b)
The energy eigenvalue equation (9) becomes
\[ E_{n,k} \psi_n(\pi_k) = \left[ \sin^2(\lambda_n \pi_k) k^2 \frac{\partial^2}{\partial \pi_k^2} - \frac{\kappa_n}{2} \frac{\partial^2}{\partial \pi_k^2} \right] \psi_n(\pi_k), \]  
(63)
The inner product is
\[ \int_{-\pi/2\lambda_\ast}^{\pi/2\lambda_\ast} \psi_m^* (\pi_k) \psi_n(\pi_k) d\pi_k = \delta_{m,n}. \]  
(64)
To solve the eigenvalue problem, it is convenient to transform to dimensionless quantities:
\[ g = \lambda_\ast^2 k = kM_\ast^{-1}, \quad \pi_k = \sqrt{M_\ast} \pi, \]  
\[ \psi_n(\pi_k) = M_\ast^{-1/4} \Psi_n(z), \quad E_{n,k} = g^2 M_\ast \kappa_n. \]  
(65)
In terms of these, the eigenvalue problem becomes the Mathieu equation. A complete basis of eigenfunctions is obtained numerically. Also, the normalization constants $\Psi_n^{(\pm)}$ refer to Mathieu characteristic value functions; and $\zeta = 1/4g^2$.

\[
\kappa_n \Psi_n(z) = \left[ -\frac{1}{2} \frac{\partial^2}{\partial z^2} + V(z) \right] \Psi_n(z), \quad (66)
\]

where the potential is

\[
V(z) = \frac{\sin^2 z}{2g^2}. \quad (67)
\]

We see that the dimensionless quantities and Schrödinger equation arising from this analysis are analogous to the definitions (41) and the ODE (42); in both we have a similar quantum mechanics problem.

This differential equation is simply related to the well-known Mathieu equation. A complete basis of eigenfunctions of definite periodicity can be written down in terms of elliptic cosine $\sigma_n$ and elliptic sine $\sigma_n$, functions, while the energy eigenvalues are given by the Mathieu characteristic value functions $A_n$ and $B_n$.

To completely specify the eigenvalue problem, we must impose boundary conditions on $\psi_n$ at $z = \pm \pi/2$. There is some ambiguity in this depending on the lattice offset one uses to specify a super-selected Hilbert space; we consider three possible choices which are summarized in Table 1. The energy eigenfunctions and eigenvalues are plotted in Figures 1 and 2, respectively. All three classes of boundary condition recover the usual energy levels of the simple harmonic oscillator in the $g \to 0$ limit,

\[
\epsilon_n \approx g(n + 1/2), \quad g \ll 1. \quad (68)
\]

However at high $g$, the eigenvalues are sensitive to the boundary conditions; in particular, the energy differences obey

\[
\Delta \epsilon_n \approx g^2 \left\{ \begin{array}{ll}
\frac{1}{2} \left[ \frac{n(n+1)}{2} \right], & \text{case BI}, \\
\frac{1}{2} \left[ \frac{n(n+1)}{2} \right] + \frac{1}{3g^2} \delta_{n,1}, & \text{case BII}.
\end{array} \right. \quad (69)
\]

Here $[ \cdot \cdot \cdot ]$ and $\lceil \cdot \cdot \cdot \rceil$ are the ceiling and floor functions, respectively. We note that $\Delta \epsilon_n \approx g^2 f_n$ for cases BI and BII, where the numbers $\{f_n\}_{n=0}^\infty$ do not depend on $g$.

V. NUMERIC CALCULATIONS OF BLACKBODY SPECTRA

In this section, we analyze the thermal properties of scalar field gases governed by the modified oscillator spectra discussed in the previous section.
FIG. 1. Potentials and energy eigenfunctions in the polymer quantization case. We have taken \( g = 0.15 \) for Cases AI–AIII and \( g = 0.225 \) for Cases BI–BIII. Also, we take \( \ell = 5 \) for Case AIII.

A. Cases AI, AIII, BI and BII

These scenarios are characterized by a complete basis of energy eigenfunctions. The energy differences satisfy

\[
\Delta \varepsilon_n \approx \begin{cases} g n, & g \ll 1, \\ g^2 f_n, & g \gg 1. \end{cases}
\]

(70)

where the numbers \( f_n \) are independent of \( g \). According to the general discussion of §III B, we expect the total energy to obey

\[
\mathcal{U} \propto \begin{cases} T^4, & k_B T \ll M_*, \\ T^{5/2}, & k_B T \gg M_*. \end{cases}
\]

(71)

This result is confirmed numerically by calculating the dimensionless intensity (19) using (18), and then integrating to get \( \mathcal{U} \). We approximate the infinite sum in equation (19) by truncating at some large value of \( n \). The cutoff is determined by demanding that the fractional change in the sum induced by an additional term is than small tolerance (which we take to be \( \sim 10^{-6} \)).

Figure 3 shows the intensity of blackbody radiation at low, moderate and high temperatures for several deformed oscillator spectra; for comparison the standard result for the conventional oscillator spectrum is case A0. Figure 4 shows the internal energy density \( \rho \) as a function of temperature; i.e., the Stefan-Boltzmann Law. We see that the usual \( \mathcal{U} \propto T^4 \) behaviour is recovered at low temperature, whereas at high temperature it matches the analytic expectation \( \mathcal{U} \propto T^{5/2} \).

From the numerically obtained energy density, Eqns. (23) and (26) give the entropy density and effective equa-
FIG. 2. Energy eigenvalue spectra for the various modified field theories considered. For case AIII, we take $\ell = 5$.

FIG. 3. Blackbody radiation spectra for various scalar field gases at low (left), moderate (centre) and high (right) temperature. In all situations, the spectra match the standard A0 result at small wavenumbers $k/M_* \ll 1$. At low temperature there is close agreement near the peak of the intensity distributions, which implies the thermodynamic properties of the system are virtually indistinguishable. At moderate and high temperatures the peak degeneracy is lifted to some degree, although cases A1, BI and BII remain very close together. At high temperature, the intensity of the standard case A0 is several orders of magnitude greater than the others for high $g$. This reflects the fact energy differences scale as $g^2$ for the modified cases; i.e., they are larger than in the A0 case.

tion of state $w = P/\rho$ as functions of temperature. These are shown in Figures 5 and 6, respectively; we see that the equation of state interpolates between $P = 1/3 \rho$ at low $T$ and $P = 2/3 \rho$ at high $T$, consistent with the low and high temperature approximations in §III B.

B. Case AII

This case involves a Schrödinger equation with potential $\propto \tanh^2 z$ and Dirichlet boundary conditions at infinity. We can view this scenario as the $\ell \to \infty$ limit of case AIII. In this case, we can use the Bohr-Sommerfeld quantization condition to estimate the energy eigenvalues.
FIG. 4. Stefan-Boltzmann law for various scalar field gases with modified oscillator spectra. Here, the energy density is defined by \( \rho = U/V \), where \( U \) is the total energy and \( V \) is the volume. In the bottom panel, we show the ratio of the modified energy densities to the conventional (case A0) result \( \rho_0 = \pi^2 k^4 T^4 / 15 \).

For the ground state, the exact result from Table I is
\[
E_{0,k} = \frac{k}{4} \left( \sqrt{4 + \frac{k^2}{M_*}} - \frac{k}{M_*} \right). \tag{74}
\]
For \( \ell \to \infty \) the energy levels are closely spaced so sums over \( n \) are well approximated by integrals. Substituting these energy eigenvalues into (13) and (15) yield the average energy in a \( k \) mode is
\[
\bar{E}_k(\beta) = \frac{1}{2 \beta} \left[ 1 + O\left( \frac{\beta M^3}{k^2} \right) \right]. \tag{75}
\]
For high \( k \), this is the familiar “particle in a box” result that the average energy is a function of temperature only. Since \( \bar{E}_k(\beta) \) approaches a constant, the integral for the internal energy (16) is UV divergent. It would seem that this particular class of modification does not lead to a viable finite temperature thermal system. The divergence of the internal energy as \( \ell \to \infty \) seems to be consistent with the finite \( \ell \) curves in Figure 4, where \( \rho \) for \( \ell = 10 \) is about an order of magnitude larger than for \( \ell = 5 \) at high temperature.

C. Case BIII

This case differs from the other polymer quantization cases (BI and BII) in that the energy differences satisfy
\[
\Delta \varepsilon_1 \approx \frac{1}{4}, \quad \Delta \varepsilon_n \propto g^2, \quad g \gg 1, \quad n \geq 2. \tag{76}
\]
That is, \( \Delta \varepsilon_1 \) remains finite in the \( g \to \infty \) limit. From this, it follows that at small wavelengths the intensity (19) satisfies
\[
I \propto g^2, \quad \tilde{\beta} g^2 \gg 1. \tag{77}
\]
This in turn implies that the integral for the internal energy (18) is UV divergent. As for Case AII, it would seem that this particular class of modification does not lead to a viable finite temperature thermal system.

VI. DISCUSSION

We applied alternative quantization methods that come with a fundamental scale to the thermodynamics of a scalar gas. The various quantization prescriptions change the microscopic energy levels through potentials and boundary conditions (as shown in Fig. 1), which in turn impact the thermodynamics.

We obtained two classes of results, one where the internal energy of the gas is divergent, and the other where it is finite. In the latter case the UV behaviour is the same for all the quantization prescriptions considered.

The two cases in the first category are the polymer quantized scalar field with π-anti periodic boundary conditions (case III), and where the commutator of Fourier space pause space variables is modified to

\[ [\hat{\phi}_k, \hat{\pi}_k] = i \left( 1 - \frac{\pi_k^2}{M_*} \right), \]  

with no additional bounds on the expectation value of \( \pi_k^2 \) (case AII).

All the other cases considered—which include (78) with a momentum cutoff \( |\pi_k| < M_* \)—share the property that the energy levels of Fourier oscillators scale as \( k^2/M_* \) for \( k \gg M_* \). This implies that at high temperature, the internal energy is proportional to \( T^{5/2} \) and the equation of state is \( P = 2\rho/3 \).

The reason for this generic high temperature behaviour stems from the nature of the modified oscillator potentials governing each of the Fourier modes (cf. Figure 1). For \( k \gg M_* \), these potentials and boundary conditions all reproduce the energy levels of a non-relativistic particle in a box, i.e. \( E_{n,k} \propto k^2 \) (modulo some degeneracies in the polymer case). This is the key feature that leads to the same high \( T \) characteristics. We expect that any modified quantization that leads to a potential that looks like a square well for \( k \gg M_* \) will give similar results.

For blackbody radiation in \( d \) spatial dimensions, the Stefan-Boltzmann law is \( U \propto T^{d-1} \). Thus one can use the temperature scaling of the internal energy as a measure of the effective dimensionality of space. In our case, this gives a curious fractional effective space dimension \( d = 3/2 \) (or spacetime dimension 5/2) at high temperature.

It is interesting in this context to note that various approaches to quantum gravity give indications that the effective spacetime dimension at high energy is closer to two than four. These results are summarized in [59]. Could it be that there is dimensional reduction in quantum field theory in the UV with alternative quantizations? Such a scenario is not envisaged in conventional effective field theory, where the dimension of spacetime is fixed at the outset at any energy scale.

It may be interesting to couple the modified scalar fields discussed in this paper to Friedmann-Roberstson-Walker cosmologies to see if the gradual “phase change” of the equation of state from \( w = 1/3 \) to \( w = 2/3 \) has an impact on the high energy radiation phase of the universe’s expansion. It is known that big bang nucleosynthesis requires that the universe have equation of state \( P = \rho/3 \) at temperature of \( T \sim 1 \text{MeV} \). This places a rather weak constraint of \( M_* \gtrsim 1 \text{MeV} \) on the new physics energy scale [60].

These are among the many interesting questions concerning the physical implications of quantizations prescriptions that come with both \( h \) and a mass scale \( M_* \).

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