Some Characterizations of Bloch Functions

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Abstract. We define Bloch-type functions of $C^1(D)$ on the unit disk of complex plane $C$ and characterize them in terms of weighted Lipschitz functions. We also discuss the boundedness of a composition operator $C_\phi$ acting between two Bloch-type spaces. These obtained results generalize the corresponding known ones to the setting of $C^1(D)$.

Key Words: Bloch space, majorant, composition operator.

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1 Introduction

Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disk of the complex plane $\mathbb{C}$, and $\mathcal{C}^1(D)$ be the set of all complex-valued functions having continuous partial derivatives on $D$. For $\alpha > 0$, a function $f \in \mathcal{C}^1(D)$ is called $\alpha$-Bloch if

$$\|f\|_\alpha = \sup_{z \in D} (1 - |z|^2)^\alpha (|f_z(z)| + |f_\bar{z}(z)|) < \infty.$$ 

It is readily seen that the set of all $\alpha$-Bloch functions on $D$ is a Banach space $\mathcal{B}_\alpha$ with the norm $\|f\|_{\mathcal{B}_\alpha} = |f(0)| + \|f\|_\alpha$.

Let $\omega : [0, +\infty) \to [0, +\infty)$ be an increasing function with $\omega(0) = 0$, we say that $\omega$ is a majorant if $\omega(t)/t$ is non-increasing for $t > 0$ (cf. [4]). Following [5], given a majorant $\omega$ and $\alpha > 0$, the $\omega$-$\alpha$-Bloch space $\mathcal{B}_{\omega,\alpha}$ consists of all functions $f \in \mathcal{C}^1(D)$ such that

$$\|f\|_{\omega,\alpha} = \sup_{z \in D} \omega((1 - |z|^2)^\alpha) (|f_z(z)| + |f_\bar{z}(z)|) < \infty$$

and the little $\omega$-$\alpha$-Bloch space $\mathcal{B}_{\omega,0}$ consists of the functions $f \in \mathcal{B}_{\omega,\alpha}$ such that

$$\lim_{|z| \to 1^-} \omega((1 - |z|^2)^\alpha) (|f_z(z)| + |f_\bar{z}(z)|) = 0.$$  

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For $0 < \alpha \leq 1$, the weighted hyperbolic metric $ds_\alpha$ of $D$, introduced in [1] is defined as
\[ ds_\alpha^2 = \frac{|dz|^2}{(1-|z|^2)^{2\alpha}}. \]

Suppose that $\gamma(t)$ ($0 \leq t \leq 1$) is a continuous and piecewise smooth curve in $D$. Then the length of $\gamma(t)$ with respect to the weighted hyperbolic metric $ds_\alpha$ is equal to
\[ L_{h_\alpha}(\gamma) = \int_{\gamma} ds_\alpha = \int_0^1 \frac{|\gamma'(t)|}{(1-|\gamma(t)|^2)^\alpha} dt. \]

Consequently, the associated distance between $z$ and $w$ in $D$ is
\[ h_\alpha(z, w) = \inf\{ L_{h_\alpha}(\gamma) : \gamma(0) = z, \gamma(1) = w \}, \]
where $\gamma$ is a continuous and piecewise smooth curve in $D$. Note that $h_1 (\alpha = 1)$ is the hyperbolic distance on $D$.

Let $s, t \geq 0$ and $f$ be a continuous function in $D$. If there exists a constant $C$ such that
\[ (1-|z|^2)^s(1-|w|^2)^t|f(z) - f(w)| \leq C|z-w| \quad (\text{resp.} \quad \leq Ch_\alpha(z, w)), \]
for any $z, w \in D$, then we say that $f$ is a weighted Euclidian (resp. hyperbolic) Lipschitz function of indices $(s, t)$. In particular, when $s = t = 0$, we say that $f$ is a Euclidian (resp. hyperbolic) Lipschitz function (cf. [12]).

In the theory of function spaces, the relationship between Bloch spaces and (weighted) Lipschitz functions has attracted much attention. For instance, in 1986, Holland and Walsh [7] established a classical criterion for analytic Bloch space in the unit disc $D$ in terms of weighted Euclidian Lipschitz functions of indices $(\frac{1}{2}, \frac{1}{2})$. Ren and Tu [13] extended the criterion to the Bloch space in the unit ball of $\mathbb{C}^n$, Li and Wulan [8], Zhao [15] characterized holomorphic $\alpha$-Bloch space in terms of
\[ (1-|z|^2)^{\beta}(1-|w|^2)^{\alpha-\beta}|f(z) - f(w)|/|z-w|. \]

In [16, 17], Zhu investigated the relationship between Bloch spaces and Bergman Lipschitz functions and proved that a holomorphic function belongs to Bloch space if and only if it is Bergman Lipschitz. For the related results of harmonic functions, we refer to [2, 3, 5, 6, 12, 14] and the references therein.

Motivated by the known results mentioned above, we consider the corresponding problems in the setting of $\mathcal{C}^1(D)$ in this paper. In Section 2, we collect some known results that will be needed in the sequel. The main results and their proofs are presented in Sections 3 and 4.

Throughout this paper, constants are denoted by $C$, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant $C$ such that $B/C \leq A \leq CB$. 
2 Several lemmas

In this section, we introduce some notations and recall some known results that we need later.

For each $a \in D$, the Möbius transformation $\varphi_a : D \to D$ is defined by

$$\varphi_a = \frac{a-z}{1-\overline{a}z}, \quad z \in D.$$  

If $a, z \in D$ and $r \in (0,1)$, we define the pseudo-hyperbolic disk with center $a$ and radius $r$ as

$$E(a, r) = \{ z \in D : \rho(a, z) = |\varphi_a(z)| < r \}.$$  

A straightforward calculation shows that $E(a, r)$ is a Euclidean disk with center at

$$\left( \frac{1-|a|^2}{2} \right) a \quad \text{and the radius} \quad \frac{1-|a|^2}{1-|a|^2 r^2}.$$  

The following lemma is proved in [17].

**Lemma 2.1.** Let $r \in (0,1)$, $w \in E(a, r)$. Then we have

$$(1-|a|^2) \asymp (1-|w|^2) \asymp |1-\overline{a}w|.$$  

The following lemma is useful for us.

**Lemma 2.2** (see [5]). Let $\omega(t)$ be a majorant and $s \in (0,1)$, $v \in (1,\infty)$. Then for $t \in (0,\infty)$,

$$\omega(st) \geq s\omega(t), \quad \omega(vt) \leq v\omega(t).$$  

As applications of Lemmas 2.1 and 2.2, we have

**Lemma 2.3.** Let $r \in (0,1)$, $w \in E(a, r)$ and $\omega(t)$ be a majorant. Then

$$\omega((1-|a|^2)) \asymp \omega((1-|w|^2)).$$  

3 Bloch functions

Let $f$ be a harmonic Bloch mapping in the unit disc $D$. In [3], Colonna proved that the Bloch constant $B_f$ of $f$ equals to its Bloch semi-norm, i.e.,

$$B_f = \sup_{z,w \in D, z \neq w} \frac{|f(z) - f(w)|}{h_1(z,w)} = \sup_{z \in D} (1-|z|^2)(|f_z(z)| + |f_z(z)|),$$  

where $h_1$ is the hyperbolic distance in $D$.

In this section, we first characterize the space $B^a$ in terms of weighted hyperbolic Lipschitz condition and generalize Colonna’s result to the setting of $c^1(D)$. 


**Theorem 3.1.** Let \( f \in C^1(\mathcal{D}) \) and \( 0 < a \leq 1 \). Then \( f \in \mathcal{B}^a \) if and only if there is a constant \( C > 0 \) such that
\[
|f(z) - f(w)| \leq Ch_a(z,w), \quad z, w \in \mathcal{D}.
\]
Moreover, we have
\[
\|f\|_a = \sup_{z, w \in \mathcal{D}, z \neq w} \frac{|f(z) - f(w)|}{h_a(z,w)}
\]
for all \( f \in \mathcal{B}^a \).

**Proof.** We first prove the sufficiency. For any \( z, w \in \mathcal{D} \), from the definition of \( h_a(z,w) \), we assume that \( \gamma(s) \) is the geodesic between \( z \) and \( w \) (parametrized by arc-length) with respect to \( h_a \). Since \( h_a(\gamma(0), \gamma(s)) = s \), we have
\[
|f(z) - f(w)| \leq Cs.
\]
Dividing both sides by \( s \) and then letting \( s \to 0 \) in the above inequality gives
\[
(|f_z(z)| + |f_z(w)|)|\gamma'(0)| \leq C.
\]
From the minimal length property of geodesics,
\[
h_a(\gamma(0), \gamma(s)) = \int_0^s \frac{|\gamma'(t)|}{(1 - |\gamma(t)|^2)^{a/2}} dt = s, \quad 0 < s < \epsilon,
\]
we obtain that
\[
\lim_{s \to 0} \frac{1}{s} \int_0^s \frac{|\gamma'(t)|}{(1 - |\gamma(t)|^2)^{a/2}} dt = \frac{|\gamma'(0)|}{(1 - |\gamma(0)|^2)^{a/2}} = 1.
\]
It follows that \((1 - |z|^2)^a (|f_z(z)| + |f_z(w)|) \leq C \) and hence \( f \in \mathcal{B}^a \) with
\[
(1 - |z|^2)^a (|f_z(z)| + |f_z(w)|) : z \in \mathcal{D} \} \leq \sup \left\{ \frac{|f(z) - f(w)|}{h_a(z,w)} : z \neq w \right\}.
\]
For the conversely, we assume that \( f \in \mathcal{B}^a \). Let \( z, w \in \mathcal{D} \) and \( \gamma(t) (0 \leq t \leq 1) \) be a smooth curve from \( z \) to \( w \). Then
\[
|f(z) - f(w)| = \left| \int_0^1 \frac{df}{dt}(\gamma(t)) dt \right|
\leq \int_0^1 (|f_z(\gamma(t))| + |f_z(\gamma(t))|)|\gamma'(t)|dt
\leq \|f\|_a \int_0^1 \frac{|\gamma'(t)|}{(1 - |\gamma(t)|^2)^a} dt
\leq \|f\|_a h_a(\gamma(t)).
\]
Taking the infimum over all piecewise continuous curves connecting \( z \) and \( w \), we conclude that
\[
|f(z) - f(w)| \leq \|f\|_a h_a(z,w)
\]
for all \( z, w \in \mathcal{D} \). This completes the proof. \( \square \)
In the following, we characterize the spaces $B_{\alpha,\omega}^0$, $B_{\alpha,\omega}^1$ in terms of Euclidean weighted Lipschitz functions.

**Theorem 3.2.** Let $r \in (0,1)$, $f \in C^1(D)$. Then $f \in B_{\alpha,\omega}^0$ if and only if

$$K = \sup_{w \in E(z,r), z \neq w} \omega((1-|z|^2)^a) \frac{|f(z) - f(w)|}{|z-w|} < \infty.$$  

**Proof.** Sufficiency. Let $f \in C^1(D)$. For $z \in D$, we have

$$\sup_{w \in E(z,r), z \neq w} \frac{|f(z) - f(w)|}{|z-w|} \leq \frac{K}{\omega((1-|z|^2)^a)}.$$  

By letting $w \to z$, we obtain that

$$|f_z(z)| + |f_z(z)| \leq \frac{K}{\omega((1-|z|^2)^a)},$$  

from which we see that $f \in B_{\alpha,\omega}^0$.

Conversely, let $f \in B_{\alpha,\omega}^0$ and for any $w \in E(z,r), z \neq w$,

$$|f(z) - f(w)| = \left| \int_0^1 \frac{df}{ds}(sz + (1-s)w)ds \right|$$

$$
\leq |z-w| \int_0^1 \left( \left| \frac{\partial f}{\partial s}(sz + (1-s)w) \right| + \left| \frac{\partial f}{\partial w}(sz + (1-s)w) \right| \right)ds
$$

$$
\leq C |z-w| \|f\|_{\omega,a} \int_0^1 \frac{ds}{\omega((1-|sz + (1-s)w|^2)^a)}
$$

$$
\leq \frac{C |z-w|}{\omega((1-|z|^2)^a)},
$$

where the last inequality follows from Lemma 2.3. Thus,

$$\sup_{w \in E(z,r), z \neq w} \omega((1-|z|^2)^a) \frac{|f(z) - f(w)|}{|z-w|} < \infty.$$  

The proof of Theorem 3.2 is completed. \qed

A similar result is true for the little Bloch-type spaces.

**Theorem 3.3.** Let $r \in (0,1)$, $f \in B_{\alpha,\omega}^1$. Then $f \in B_{\alpha,\omega,0}^1$ if and only if

$$\lim_{|z| \to 1} \sup_{w \in E(z,r), z \neq w} \omega((1-|z|^2)^a) \frac{|f(z) - f(w)|}{|z-w|} = 0.$$  

The proof is almost the same as the one of Theorem 3.2 in [13]. Thus we omit it here.

**Remark 3.1.** When $\omega(t) = t$, Li and Wulan [8] obtained the analogues of Theorems 3.2 and 3.3 for holomorphic Bloch space on the unit ball of $\mathbb{C}^n$. 

4 Composition operators

Let $\phi$ be a holomorphic self-mapping of $\mathbb{D}$. The composition operator $C_\phi$, induced by $\phi$ is defined by $C_\phi(f) = f \circ \phi$ for $f \in \mathcal{O}(\mathbb{D})$. During the past few years, composition operators have been studied extensively on spaces of holomorphic functions on various domains in $\mathbb{C}$ and $\mathbb{C}^n$, see e.g., [9, 10, 18]. In this section, we discuss the boundedness of composition operators between Bloch spaces of $\mathbb{C}$.

Theorem 4.1. Let $\alpha, \beta > 0$ and $\phi$ be a holomorphic self-mapping of $\mathbb{D}$. Then $C_\phi : B^\alpha \to B^\beta$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |\phi'(z)|}{(1 - |\phi(z)|^2)^\alpha} < \infty.$$  \hspace{1cm} (4.1)

Proof. First suppose that

$$L = \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |\phi'(z)|}{(1 - |\phi(z)|^2)^\alpha}.$$

For $f \in B^\alpha$ and $z \in \mathbb{D}$, we have

$$(1 - |z|^2)^\beta (|C_\phi(f)_z| + |C_\phi(f)_\overline{z}|) = (1 - |z|^2)^\beta (|(|f \circ \phi)_z(z)| + |(f \circ \phi)_\overline{z}(z)|)$$

$$= (1 - |z|^2)^\beta |\phi'(z)|(|f_z(\phi(z))| + |f_{\overline{z}}(\phi(z))|)$$

$$\leq L(1 - |\phi(z)|^2)^\alpha (|f_z(\phi(z))| + |f_{\overline{z}}(\phi(z))|)$$

$$\leq C \|f\|_\alpha$$

and

$$|f(\phi(0))| \leq C \|f\|_\alpha.$$

Hence $C_\phi : B^\alpha \to B^\beta$ is bounded.

For the converse, assume that $C_\phi : B^\alpha \to B^\beta$ is a bounded operator with

$$\|C_\phi(f)\|_\beta \leq C \|f\|_\alpha$$

for all $f \in B^\alpha$. Fix a point $z_0 \in \mathbb{D}$ and let $w = \phi(z_0)$. If $\alpha \neq 1$, consider the function $f_w(z) = (1 - wz)^{1-\alpha} - 1$. Then it is easy to check that $f_w \in B^\alpha$. The boundedness of $C_\phi$ implies that

$$\frac{(1 - |z|^2)^\beta |\phi'(z)|}{|1 - \overline{w}\phi(z)|^\alpha} \leq C.$$

In particular, take $z = z_0$, we get

$$\frac{(1 - |z_0|^2)^\beta |\phi'(z_0)|}{(1 - |\phi(z_0)|^2)^\alpha} \leq C.$$

Since $z_0$ is arbitrary, the result follows.

If $\alpha = 1$, we only need to consider the function $f_w(z) = \ln(1/(1 - \overline{w}z))$. Following a discussion similar to the above, it can be proved that (1) holds. The proof of Theorem 4.1 is completed. \hfill $\square$
Recall that the classical Schwarz-Pick Lemma in the unit disk gives that for a holomorphic self-mapping $\phi$ of $D$, $(1 - |z|^2)|\phi'(z)| \leq 1 - |\phi(z)|^2$ holds for all $z \in D$. As an application of this result, it is easy to derive the following corollary.

**Corollary 4.1.** Let $\phi$ be a holomorphic self-mapping of $D$. Then $C_\phi : B^1 \to B^1$ is bounded.

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