New Proofs of the Basel Problem using Stochastic Processes

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Abstract

The number \( \pi^2/6 \) is involved in the variance of several distributions in statistics. At the same time it holds \( \sum_{k=1}^{\infty} k^{-2} = \pi^2/6 \), which solves the famous Basel problem. We first provide a historical perspective on the Basel problem, and second show how to generate further proofs building on stochastic processes.

Keywords Euler; Wiener process; Brownian bridge; Karhunen-Loève expansion.

1 Introduction

The number \( \pi^2/6 \) solves the so-called Basel problem in that \( \sum_{k=1}^{\infty} k^{-2} = \pi^2/6 \); the first proof was given by Leonhard Euler. A statistician encounters this number in several places. First, \( \pi^2/6 \) equals the variance of the famous standard Gumbel (or extreme value) distribution, see Gumbel (1941). Second, it shows up in the limiting variance of the appropriately normalized estimator from the so-called log-periodogram regression. This estimator of the fractional order of integration is sometimes called GPH estimator after the seminal paper by Geweke and Porter-Hudak (1983), which

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was recently celebrated by a special issue of the *Journal of Time Series Analysis*, see Nielsen and Hualde (2019); confer also Hurvich, Deo, and Brodsky (1998, Thm. 1). Third, the inverse of this number, \( \frac{6}{\pi^2} \), amounts to the variance in the limiting distribution of the maximum likelihood estimator of the order of integration of so-called fractionally integrated noise, see Hassler (2019, Coro. 8.1).

In this note, we show how to generate further proofs of \( \sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6} \) using Karhunen-Loève expansions of the Wiener process and of related processes. The next section briefly reviews the history of the Basel problem until the year 1735. Section 3 provides a short review and a classification of the multitude of earlier proofs. Section 4 uses a technique from the theory of stochastic processes to produce new proofs. The final section provides some concluding remarks.

## 2 Basel problem

According to Eneström (1911), it was Pietro Mengoli\(^1\) (1625/26-1686) of Bologna in Italy who posed the problem to determine the value of \( \sum_{k=1}^{\infty} k^{-2} \), for which we nowadays write \( \zeta(2) \) with Riemann zeta function \( \zeta(s) = \sum_{k=1}^{\infty} k^{-s} \) for complex \( s \) with \( \text{Re}(s) > 1 \). It was clear that \( \zeta(2) \) is finite for the following reason. By partial fractions the series of reciprocals of triangular numbers “telescopes”,

\[
T_N := \sum_{k=1}^{N} \frac{2}{k(k+1)} = 2 \sum_{k=1}^{N} \left( \frac{1}{k} - \frac{1}{k+1} \right) = 2 \left( 1 - \frac{1}{N+1} \right) \to 2
\]

as \( N \to \infty \). Because of \( 2k^2 \geq k^2 + k \) it follows that \( \zeta(2) \leq 2 \). In fact, Eneström (1911, p. 144-145) gave three different proofs for \( T_N \to 2 \) originally published by Mengoli in 1650. It was Jakob Bernoulli (1654-1705) from the city of Basel in Switzerland who popularized such results (*Tractatus de seriebus infinitis*, 1689), apparently without being aware of the previous work by Mengoli. We quote his treatise as Bernoulli (1713) published posthumously together with his famous *Ars

\(^1\)Born in 1625 according to Boyer (1968), while Eneström (1911) dates his year of birth to 1626. In 1650 he published *Novae Quadraturae Arithmetica, sev De Additione Fractionum*, which seems to have been largely forgotten until the rediscovery by Eneström.
Conjectandi. His tractatus is organized in 60 propositions. Bernoulli (1713, Prop. XV) established convergence of \( \sum_{k=1}^{N} \frac{2^k}{k(k+1)} \), and he proved in Prop. XVI that the harmonic series \( \sum_{k=1}^{N} \frac{1}{k} \) does not converges, but he failed in Prop. XVII to determine the value of \( \zeta(2) \); Bernoulli (1713, p. 254) wrote the request that he would be much obliged if someone found what had escaped his efforts and communicated it to him. Hence, the Basel problem was there and resisted the efforts of mathematicians of this time. In 1705, Johann Bernoulli (1667-1748) took the position of his late brother Jakob at the University of Basel, see Merian (1860) for a history of the mathematical Bernoulli dynasty. Leonhard Euler (1707-1783) graduated at the University of Basel in 1724 at the age of 17 as one of Johann Bernoulli’s students, see the biographical sketch in Ayoub (1974, p. 1068). The Basel problem remained unsolved until Euler (1735) provided the following formula.

**Theorem** \( \zeta(2) = \frac{\pi^2}{6} \).

We briefly review a selection of earlier proofs next.

### 3 Earlier proofs

Euler’s first and famous proof built on a factorization of the Taylor expansion of the sine function, see e.g. the exposition by Ayoub (1974, Sect. 4). According to Ayoub (1974, p. 1077), Daniel Bernoulli (1700-1782, son of Johann Bernoulli, see again Merian (1860)) did not doubt Euler’s result, but had two concerns with respect to the validity of the arguments leading there. As early as 1738, Nicolaus Bernoulli (1687-1759, a nephew of Johann and Jacob Bernoulli) presented quite a different idea of proof, which nowadays is rarely associated with him. It was published in Volume 7 of the Memoirs of the Imperial Academy of Sciences in St. Petersburg (Commentarii Academiae Scientiarum Imperialis Petropolitanae). The title page of this volume, which by the way contains 10 papers by Euler, carries three year dates: 1734/1735 and 1740. The first ones are the years when the papers were presented to the Academy, and the latter date refers to the year of printing; we follow the international custom and quote Euler’s paper as Euler (1735).

\[^3\] Just like Euler’s original paper it was published in the Memoirs of the Imperial Academy of Sciences in St. Petersburg, Volume 10. This time the title page carries two year dates, 1738 and 1747, where 1747 is the year of printing; we refer to the paper as Bernoulli (1738). We do not
(1738) defined the sum over reciprocals of odd squares as $Z_N$, and the alternating series $q_N$:

$$Z_N = \sum_{k=0}^{N} \frac{1}{(2k+1)^2} \quad \text{and} \quad q_N = \sum_{k=0}^{N} \frac{(-1)^k}{(2k+1)}.$$  

Bernoulli’s idea was to square $q_N$, such that $Z_N = q_N^2 + y_N$, where $y_N$ is implicitly defined. Then he argued that $q_N \to \pi/4$ and $y_N \to \pi^2/16$ as $N \to \infty$, which amounts to

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$  \hfill (1)

Obviously, this solves the Basel problem since

$$\zeta(2) - \frac{1}{4} \zeta(2) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$  \hfill (2)

Similarly, with more rigorous arguments but without referring to Bernoulli (1738), Estermann (1947, p. 12) established $Z_N = 2q_N^2 + u_N$, $u_N \to 0$, which proves (1) again; see also Knopp (1951, p. 322-324).

Euler himself added a second proof in 1743 that builds on (1) with (2), although it does not rely on squaring $q_N$ but rather on an expansion of the arcsine function. Stäckel (1907) called this paper a “forgotten treatise by Leonhard Euler” and provided a historical perspective including a reprint of the French paper. For Euler’s second proof in a nutshell we refer to Kimble (1987) or footnote 63 in Knopp (1951, p. 376); see also Choe (1987), where the last author does not seem to be aware of reproducing Euler’s second proof.

Ever since Euler cracked the Basel problem, many other proofs appeared. Some of them also culminate in establishing (1), e.g., in chronological order: Giesy (1972), Hofbauer (2002), Harper (2003), Ivan (2008), Marshall (2010), Hirschhorn (2011), Muzaffar (2013) and Ritelli (2013). More proofs that do not establish (1) in order to show $\zeta(2) = \frac{\pi^2}{6}$ were given by Knopp and Schur (1918), Yaglom and Yaglom.

\footnote{The title is Démonstration de la somme the cette Suite $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \text{etc.}$ published in Volume 2 of a journal called somehow surprisingly Journal littéraire d’Allemagne, de Suisse et du Nord.}

\footnote{This proof can also be found in English in Knopp (1951, p. 266, 267).}
(1953), Matsuoka (1961), Stark (1969), and Papadimitriou (1973). A note following Papadimitriou (1973) says, that this paper was translated from a Greek manuscript, and that the proof coincides with the one given in Norwegian by Holme (1970); we checked that it is actually identical with the one by Yaglom and Yaglom (1953)\footnote{For simplicity, we quote Yaglom and Yaglom (1953) with an English translation of the title although the paper is in Russian.}. Further, more or less elementary proofs have been published by Apostol (1983), Beukers, Kolk, and Calabi (1993), Kortram (1996), Borwein, Borwein, and Dilcher (1989), Passare (2008), Daners (2012), Xu and Zhou (2014) and Lord (2016). This list does not include proofs that rely on Fourier analysis or proofs that establish more generally closed form results for $\zeta(2n)$, $n \in \mathbb{N}$. Therefore the list is far from being complete.

There are two recent proofs that stand out against all other ones using a probabilistic approach: Pace (2011) and Holst (2013). In the next section we give additional proofs for $\zeta(2) = \frac{\pi^2}{6}$ that are rooted in probability theory, too, relying on results from the theory of stochastic processes.

4 Karhunen-Loève expansions and new proofs

Consider a stochastic zero mean process $X(t)$, $t \in [0, 1]$, and assume that it is Gaussian and continuous in quadratic mean with positive definite covariance kernel $k(s, t) = \operatorname{E}(X(s)X(t))$ that is symmetric and continuous. Then $X(t)$ is endowed with a Karhunen-Loève (KL) expansion,

$$X(t) = \sum_{j=1}^{\infty} \lambda_j^{-1/2} f_j(t) Z_j,$$

where $Z_j$, $j \in \mathbb{N}$, is a sequence of independent standard normal random variables, and $\lambda_j$ and $f_j(t)$ are the eigenvalues and eigenfunctions of $k(s, t)$ satisfying the

\footnote{This proof is similar to the one by Matsuoka (1961), but more straightforward.}
following Fredholm integral equation, see Loève (1978) for a discussion:

\[ f(t) = \lambda \int_0^1 k(s, t)f(s)ds. \]

It is worth noting that \( f_j(t), j \in \mathbb{N} \), form an orthonormal base for \( L^2 \). Itô and Nisio (1968) established that the sum \( \sum_{j=1}^{N} \lambda_j^{-1/2} f_j(t) Z_j \) converges a.s. uniformly to \( X(t), N \to \infty \). From (3) we have

\[ k(t, t) = \sum_{j=1}^{\infty} \frac{f_j^2(t)}{\lambda_j}, \tag{4} \]

see also Mercer’s Theorem, e.g. Tanaka (1996, Thm. 5.2). For three related processes we will use (1) to show \( \zeta(2) = \frac{\pi^2}{6} \). Proofs that are similar in spirit can be established using Hochstadt (1973, Coro. 2, p. 92).

Let \( W(t) \) denote a standard Brownian motion or Wiener process with kernel \( k(s, t) = \min(s, t) \). Further, \( W^\mu \) and \( W^\tau \) are demeaned and detrended Wiener processes, respectively; they are defined as the orthogonal component of the projection on a constant or on a linear time trend:

\[ W^\mu(t) = W(t) - \int_0^1 W(s)ds, \]

\[ W^\tau(t) = W(t) + (6t - 4) \int_0^1 W(s)ds + (6 - 12t) \int_0^1 sW(s)ds. \]

For their kernels it is straightforward to obtain

\[ k^\mu(s, t) = \min(s, t) - (s + t) + \frac{1}{2}(s^2 + t^2) + \frac{1}{3}, \]

\[ k^\tau(s, t) = \min(s, t) - \frac{11}{10}(s+t) + 2(s^2+t^2) - (s^3 + t^3) - 3(st^2 + ts^2) + 2(st^3 + ts^3) + \frac{2}{5}st + \frac{2}{15}, \]

where the latter result can be found e.g. in Ai, Li, and Liu (2012, Lemma 2.1). Further, the eigenstructure has been characterized as follows.
Lemma The eigenvalues and eigenfunctions of $k$, $k^\mu$ and $k^\tau$ are $(j = 1, 2, \ldots)$

$$
\lambda_j = (j - 1/2)^2 \pi^2, \quad \lambda_j^\mu = j^2 \pi^2, \quad \lambda_j^\tau = \begin{cases} 
(j + 1)^2 \pi^2, & j = 2n - 1 \\
4z_{3/2,j/2}^2, & j = 2n
\end{cases},
$$

where $z_{3/2,n}$ are the positive roots of the Bessel function $J_{3/2}$ of the first kind, and

$$
f_j(t) = \sqrt{2} \sin((j - 1/2)\pi t), \quad f_j^\mu(t) = \sqrt{2} \cos(j\pi t),
$$

where $\Lambda_j$ is given in Hosseinkouchack and Hassler (2016, eq. (22)) and can be reduced to $\Lambda_j = 2 \left( \sin \frac{\sqrt{\lambda_j}}{2} \right)^{-2}$.

Proof Hosseinkouchack and Hassler (2016) derive the eigenstructure of the kernels of demeaned and detrended Ornstein-Uhlenbeck processes; $W^\mu$ and $W^\tau$ are embedded as special cases, see Hosseinkouchack and Hassler (2016, Remark 1 and 2). In particular, $\lambda^\mu$ and $f^\mu$ are also given by Beghin, Nikitin, and Orsingher (2005, p. 2495), and Ai et al. (2012, Thm. 1) derive $\lambda^\tau$, however, without giving $f^\tau$. The case of the standard Wiener process is textbook knowledge.

With the Lemma at hand, new proofs of $\zeta(2) = \frac{\pi^2}{6}$ are obvious.

Proof 1: From (4) we obtain for $k(t, t) = t$ with $\lambda_j$ and $f_j$ that

$$
t = 8 \sum_{j=1}^{\infty} \frac{\sin^2 [(j - 1/2) \pi t]}{(2j - 1)^2 \pi^2}.
$$

Evaluation for $t = 1$ amounts to (11), which completes the proof by (2).

Proof 2: From (4) we obtain for $k^\mu(1, 1)$ by the Lemma that $\frac{1}{3} = \frac{2}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{j^2}$, which is the required result.

Proof 3: Note from the Lemma that $f_j^\tau(1/2) = 0$ for even $j$. From (4) it hence follows for $k^\tau(1/2, 1/2)$ that $\frac{1}{12} = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{\zeta(2)}{2\pi^2}$, which proves the result.
5 Concluding remarks

More proofs of the Basel problem can be produced following the route tackled here. All one needs is the expansion from (4) for some stochastic process. We employed expansions for the Wiener process, for the demeaned Wiener process and for the detrended Wiener process. As a further example, one may consider the so-called Brownian bridge, for which the required expansion can be found in Shorack and Wellner (1986, pp. 213-214).

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