Power-series summability methods in de Branges–Rovnyak spaces

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Abstract. We show that there exists a de Branges–Rovnyak space $H(b)$ on the unit disk containing a function $f$ with the following property: even though $f$ can be approximated by polynomials in $H(b)$, neither the Taylor partial sums of $f$ nor their Cesàro, Abel, Borel or logarithmic means converge to $f$ in $H(b)$.

A key tool is a new abstract result showing that, if one regular summability method includes another for scalar sequences, then it automatically does so for certain Banach-space-valued sequences too.

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1. Introduction

We denote by $\mathbb{D}$ the open unit disk and $\text{Hol}(\mathbb{D})$ the space of holomorphic functions on $\mathbb{D}$. Polynomial approximation in Banach spaces of holomorphic functions on the unit disk has attracted attention recently. In [12], the authors proved that, for a Banach space of holomorphic functions $X$ on the open unit disk having the approximation property and containing a dense set of polynomials, there exist linear bounded operators $T_n : X \to X$ such that, for each $f \in X$, the functions $T_n(f)$ are polynomials and $T_n(f) \to f$ in the norm of $X$. The authors called the aforementioned sequence $(T_n)$ a linear polynomial approximation scheme.

For some spaces, the operators $T_n$ are explicitly known. For example, if $X = H^2$, the Hardy space, then the bounded linear operators can be chosen to be the $n$-th partial sums $s_n[f]$ of the Taylor expansion of a function $f(z) =$
\[ \sum_{n \geq 0} a_n z^n \] \text{ belonging to } H^2. This is an easy consequence of the definition of the norm in \( H^2 \). When \( X = H^p \), the Hardy spaces with \( 1 < p < \infty \), we can still choose the partial sums \( s_n[f] \) of a function \( f \in H^p \) as a linear polynomial approximation scheme. The proof is more elaborate and is based on a result of Riesz on the boundedness of the Hilbert transform. The reader is referred to [10, p. 108]. When \( X = A(\mathbb{D}) \), the disk algebra, due to a slight variant of du Bois-Reymond’s theorem that establishes the existence of a continuous function on the unit circle whose Fourier series diverges at one point, the partial sums \( s_n[f] \) do not converge to \( f \) in the norm of \( A(\mathbb{D}) \). Instead, we can use the Cesàro means of order 1 of the Taylor expansion of a function \( f \in A(\mathbb{D}) \), defined as

\[ \sigma_n[f] := \frac{1}{n+1} \sum_{k=0}^{n} s_k[f]. \]

This is essentially Fejér’s theorem. This procedure also works for certain other spaces, for example the Hardy space \( H^1 \) and the weighted Dirichlet spaces \( D_\omega \) for superharmonic weights \( \omega : \mathbb{D} \to (0, \infty) \).

However there are some spaces for which we do not explicitly know the linear polynomial approximation scheme. One such family of spaces are the de Branges–Rovnyak spaces \( \mathcal{H}(b) \), where \( b \in H^\infty \) is a non-extreme point of the unit ball of \( H^\infty \). Despite the fact that the set of polynomials is dense in \( \mathcal{H}(b) \), the authors of [7] showed that, for certain choices of \( b \), the partial sums \( s_n[f] \) and the Cesàro means \( \sigma_n[f] \) may fail to converge in \( \mathcal{H}(b) \) to the initial function \( f \). Therefore the attention turned to other linear summability methods that do not give a polynomial approximation, but have a better chance to approximate the function \( f \) in the norm of \( \mathcal{H}(b) \), namely the dilates of \( f \), which are in fact the Abel means of the partial sums \( s_n[f] \) :

\[ f_r(z) := \sum_{n \geq 0} a_n r^n z^n = (1 - r) \sum_{n \geq 0} s_n[f](z) r^n \quad (r \in [0, 1], z \in \mathbb{D}) \]

for \( f(z) = \sum_{n \geq 0} a_n z^n \). Nevertheless, they showed that even this summability method, which includes all the Cesàro summability methods of order \( \alpha > -1 \), may fail for de Branges–Rovnyak spaces with non-extreme symbols \( b \). Furthermore, the same authors showed in the same article that there is a constructive way to obtain a polynomial approximation of a given function \( f \in \mathcal{H}(b) \) for any non-extreme point \( b \) of the unit ball of \( H^\infty \). However, the procedure is highly non-linear and it does not correspond to a polynomial approximation scheme.

In this article, we study another summability method which includes the Abel method : the logarithmic method. Its means are defined as

\[ L_r[f](z) := \frac{r}{\log \left( \frac{1}{1-r} \right)} \sum_{n \geq 0} \frac{1}{n+1} s_n[f](z) r^n \tag{1.1} \]

for \( f(z) = \sum_{n \geq 0} a_n z^n \) and for \( r \in [0, 1] \), \( z \in \mathbb{D} \). It was introduced by Borwein in [2] through a power-series method.
If \( f \in \mathcal{H}(b) \), it is well known that \( f_r \in \mathcal{H}(b) \). We will show that also \( L_r[f] \in \mathcal{H}(b) \). However, our main result reveals that \( L_r[f] \) may diverge.

**Theorem 1.1.** There exist a non-extreme point \( b \) of the unit ball of \( H^\infty \) and a function \( f \in \mathcal{H}(b) \) such that
\[
\lim_{r \to 1^-} \|L_r[f]\|_b = \infty.
\]

Theorem 1.1 is proved in §4. It is a consequence of the example already constructed in [7] to show that \( s_n[f] \) and \( \sigma_n[f] \) may diverge in \( \mathcal{H}(b) \), together with an integral formula that links the dilates of a function to its logarithmic means.

As a corollary, we obtain the following result concerning generalized Abel methods \( A^\alpha_r \), where \( \alpha > -1 \) and where the generalized Abel means \( A^\alpha_r[f] \) are defined by the following expression:
\[
A^\alpha_r[f](z) := (1 - r)^{1+\alpha} \sum_{n \geq 0} \frac{n + \alpha}{\alpha} s_n[f](z)r^n \quad (r \in [0, 1)).
\]

**Corollary 1.2.** For every \( \alpha > -1 \), there exist a non-extreme point \( b \) of the unit ball of \( H^\infty \) and a function \( f \in \mathcal{H}(b) \) such that \( A^\alpha_r[f] \not\to f \) in \( \mathcal{H}(b) \) as \( r \to 1^- \).

This corollary is a consequence of an abstract result in functional analysis. It enables us to compare the summability of a sequence of elements in a Banach space with respect to two summability methods, based on the inclusion of one summability method in the other for scalar sequences. This theorem is proved in §5, and Corollary 1.2 is deduced in §6.

## 2. Sequence-to-function summability methods

We start by defining some terminology in summability theory. Our main references are Hardy [8] and Boos [1]. Throughout this section, we let \( X \) denote a Banach space over the complex numbers \( \mathbb{C} \) and \( \| \cdot \|_X \) be its norm. We denote by \( c(X) \) the space of convergent sequences in \( X \).

A **sequence-to-function summability method** \( K \) is given by a sequence \((k_n)_{n \geq 0}\) of functions \( k_n : [0, R) \to \mathbb{C} \) where \( R \in (0, \infty) \). The \( K \)-**means** of a sequence of vectors \( x := (x_n)_{n \geq 0} \subset X \) are defined by the following formal series
\[
K_r[x] := \sum_{n \geq 0} k_n(r)x_n \quad (r \in [0, R)).
\]

We say that a sequence \( x := (x_n)_{n \geq 0} \subset X \) is **\( K \)-summable** or **summable by the method** \( K \) if the series defining \( K_r[x] \) converges for every \( r \in [0, R) \) and, moreover, \( K_r[x] \) converges in norm to some \( y \in X \) as \( r \to R^- \). We say that a sequence-to-function summability method \( K \) is **regular** if, whenever \( X \) is a Banach space and \( (x_n)_{n \geq 0} \in c(X) \), then \( (x_n)_{n \geq 0} \) is \( K \)-summable and \( \lim_{r \to R^-} K_r[x] = \lim_{n \to \infty} x_n \). Necessary and sufficient conditions for
a sequence-to-function summability method to be regular are given in the following theorem, which is a slight modification of Theorem 5 in [8, p. 49].

**Theorem 2.1.** Let $X$ be a Banach space. A sequence-to-function summability method $K$ is regular if and only if the following conditions hold:

- there exists an $R_0 \in [0, R)$ such that the function $r \mapsto \sum_{n \geq 0} |k_n(r)|$ is uniformly bounded on $[R_0, R)$;
- for each $n \geq 0$, we have $k_n(r) \to 0$ as $r \to R^-$;
- the function $k(r) := \sum_{n \geq 0} k_n(r)$ converges to $1$ as $r \to R^-$.  

Proof. By considering the homeomorphism $r \mapsto \log \left( \frac{R}{R-r} \right)$ from $[0, R)$ onto $[0, \infty)$, we may restrict our attention to the situation on the interval $[0, \infty)$. Suppose that the method $K$ is regular. Since $\mathbb{C}$ embeds isometrically into $X$ and the functions $k_n(r)$ are complex-valued, $K$ is regular for the space of convergent complex-valued sequences. Then the result follows from the classical case $X = \mathbb{C}$ (see Theorem 5 in [8, p. 49]).

If the conditions are satisfied, then elementary estimates using the conditions show that $K_r[x] \to \lim_{n \to \infty} x_n$ as $r \to R^-$ for any sequence $(x_n)_{n \geq 0} \in c(X)$. □

Let $K$ and $H$ be two sequence-to-function summability methods. We say that $K$ is included in $H$, denoted by $K \subseteq H$, if, whenever $(x_n)_{n \geq 0}$ is a $K$-summable sequence in a Banach space $X$, then $(x_n)_{n \geq 0}$ is also $H$-summable and

$$
\lim_{r \to R^-} H_r[x] = \lim_{r \to R^-} K_r[x].
$$

If $K \subseteq H$ and $H \subseteq K$, we say that the two summability methods are equivalent. If $X = \mathbb{C}$, all of the aforementioned definitions will be preceded by the word “scalar”. For example, when we write “the summability method $K$ is scalar-equivalent to the summability method $H$”, this means that they are equivalent when applied to scalar-valued sequences, that is $(x_n)_{n \geq 0} \subset \mathbb{C}$.

We now give some examples of sequence-to-function summability methods.

**2.1. Matrix summability methods**

Let $R = \infty$ and, for each $n \geq 0$, let the function $k_n$ be constant on each interval $[m, m+1)$, where $m \geq 0$ is an integer. Then, for each $r \in [m, m+1)$, the expression of the $K$-mean becomes

$$
K_r[x] = \sum_{n \geq 0} k_n(r)x_n = \sum_{n \geq 0} k_n(m)x_n.
$$

Hence the method given by the sequence of functions $k_n$ can be viewed as an infinite matrix $(k_n(m))_{m,n \geq 0}$. In these circumstances, we call the summability method a matrix summability method. The necessary and sufficient conditions for a matrix summability method to be regular are attributed to Silverman and Toeplitz (see [8, p. 43]). The conditions in Theorem 2.1 now become:

- there is a number $M > 0$ such that $\sum_{n \geq 0} |k_n(m)| \leq M$ for every $m \geq 0$;
• for each $n \geq 0$, $\lim_{m \to \infty} k_n(m) = 0$;
• we have $\sum_{n \geq 0} k_n(m) \to 1$, as $m \to \infty$.

There is a generalization of the Silverman-Toeplitz Theorem, due to Robinson, to matrix summability methods that are given by a matrix $(C_{m,n})_{m,n \geq 0}$, where $C_{m,n}$ are bounded linear operators on $X$ (see [13, Theorem IV]).

2.2. Power-series summability methods

Let $p(r) := \sum_{n \geq 0} p_n r^n$ be a power series with a radius of convergence $R_p > 0$, where $p_0 > 0$ and $p_n \geq 0$ for $n \geq 1$. Following §3.6 of [1], we say that a sequence $x := (x_n)_{n \geq 0} \subset X$ is summable by the power-series method $(p)$, or is $P$-summable, if the series

$$P_r[x] := \frac{1}{p(r)} \sum_{n \geq 0} p_n x_n r^n$$

converges for each $r \in [0,R)$ and there exists a $y \in X$ such that

$$P_r[x] \to y \ (r \to R^-).$$

According to Theorem 2.1, a power-series method $(p)$ is regular if and only if $p(r) \to \infty$ as $r \to R^-$. The Abel summability method is a special case of the power-series method $(p)$ with $p(r) = (1-r)^{-1}$ and $r \in [0,1)$. The $A_r$-means are

$$A_r[x] = (1-r) \sum_{n \geq 0} x_n r^n \ (0 \leq r < 1).$$

In this paper, we apply a power-series method to the sequence of partial sums $(s_n[f])_{n \geq 0}$ of the Taylor expansion of a function $f \in Hol(\mathbb{D})$. The expression of the means defined by a power-series method $(p)$ are

$$P_r[f](z) := \frac{1}{p(r)} \sum_{n \geq 0} p_n s_n[f](z) r^n,$$

where $p$ has a radius of convergence $R_p \geq 1$. Since $|s_n[f](z)| \leq C(R) R^n$ for any $R > 1$ and some constant $C(R) > 0$, the function $P_r[f]$ is well-defined for each $z \in \mathbb{D}$. Also, the series defining $P_r[f]$ converges uniformly on compact subsets of $\mathbb{D}$, and thus it defines a function holomorphic on all of $\mathbb{D}$.

A useful power-series method is the logarithmic method. As we shall see later in the paper, this power-series method is convenient because it contains many other summability methods. The logarithmic method is the power-series method associated with the power series

$$l(r) := \sum_{n \geq 0} \frac{r^n}{n+1} = \frac{1}{r} \log \frac{1}{1-r} \ (0 \leq r < 1).$$

Thus, the expression of the logarithmic mean of the partial sums $s_n[f]$ is

$$L_r[f](z) := \frac{r}{\log \frac{1}{1-r}} \sum_{n \geq 0} \frac{s_n[f](z)}{n+1} r^n \ (0 \leq r < 1).$$
We will also study the generalized Abel methods. Given a number \( \alpha > -1 \), the generalized Abel means of order \( \alpha \) are associated with the power series

\[
a^\alpha_r(\alpha) := \sum_{n \geq 0} \binom{n + \alpha}{\alpha} r^n = \frac{1}{(1 - r)^{1+\alpha}} \quad (r \in [0, 1)).
\]

Applied to the partial sums \( (s_n[f])_{n \geq 0} \) of the Taylor series of \( f \in \text{Hol}(\mathbb{D}) \), the expression of the mean \( A^\alpha_r[f] \) is

\[
A^\alpha_r[f](z) := (1 - r)^{\alpha+1} \sum_{n \geq 0} \binom{n + \alpha}{\alpha} s_n[f](z)r^n.
\]

If \( \alpha < \beta \), then the generalized Abel method of order \( \beta \) is scalar-included in the generalized Abel method of order \( \alpha \) (see [2, Theorem 2]). When \( \alpha = 1 \), we obtain the classical Abel means. As we mention earlier in the paper, the expression of \( A^1_r[f] \) can be rearranged to give

\[
A^1_r[f](z) = \sum_{n \geq 0} a_n r^n z^n,
\]

which is the dilate \( f_r \) of \( f \in \text{Hol}(\mathbb{D}) \). We present a pointwise relation between the logarithmic means and the dilates of a function. It will be used later in §4 and its proof is straightforward.

**Lemma 2.2.** For any \( f \in \text{Hol}(\mathbb{D}) \), \( r \in [0, 1) \) and \( z \in \mathbb{D} \), we have

\[
L_r[f](z) = \frac{1}{\log \frac{1}{1-r}} \int_0^r \frac{f_t(z)}{1-t} \, dt.
\]

**3. Background on de Branges–Rovnyak spaces**

Let \( \phi \in L^\infty(\mathbb{T}) \). The Toeplitz operator \( T_\phi : H^2 \to H^2 \) with symbol \( \phi \) is defined as

\[
T_\phi f := P_+(\phi f) \quad (f \in H^2),
\]

where \( P_+ : L^2(\mathbb{T}) \to H^2 \) is the orthogonal projection of \( L^2(\mathbb{T}) \) onto \( H^2 \). This is clearly a bounded operator with \( \| T_\phi \| \leq \| \phi \|_{L^\infty(\mathbb{T})} \). (In fact, \( \| T_\phi \| = \| \phi \|_{\infty} \) by a theorem of Brown and Halmos, but we do not need this here.) The adjoint of \( T_\phi \) is \( T_\phi^* \). If \( \phi \in H^\infty \), then \( T_\phi \) is simply the operator of multiplication by \( \phi \). We now introduce the de Branges–Rovnyak space associated to a function \( b \in H^\infty \), where \( \| b \|_{\infty} \leq 1 \). This is Sarason’s definition taken from [14].

**Definition 3.1.** Let \( b \in H^\infty \) with \( \| b \|_{\infty} \leq 1 \). The associated de Branges–Rovnyak space, denoted by \( \mathcal{H}(b) \), is the range space \((I - T_b T_b^*)^{1/2}H^2 \) equipped with the following inner product

\[
\langle (I - T_b T_b^*)^{1/2} f, (I - T_b T_b^*)^{1/2} g \rangle_b := \langle f, g \rangle_2,
\]

where \( f, g \in H^2 \ominus \ker(I - T_b T_b^*)^{1/2} \).
The definition of the inner product $\langle \cdot, \cdot \rangle_b$ makes the operator $(I - T_b T_b^*)^{1/2} : H^2 \to H^2$ a partial isometry from $H^2$ onto $\mathcal{H}(b)$. Its cousin, the space $\mathcal{H}(\overline{b})$, is defined similarly by interchanging the roles of $b$ with $\overline{b}$ in the above definition.

There is a close relationship between $\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$. This is the content of the next theorem.

**Theorem 3.2.** [14, §II-2]. A function $f \in H^2$ belongs to $\mathcal{H}(b)$ if and only if $T_b f$ belongs to $\mathcal{H}(\overline{b})$. In this case, we have that

$$\|f\|_b^2 = \|f\|_2^2 + \|T_b f\|_b^2.$$

The structure of a de Branges–Rovnyak space depends strongly on whether $b$ is an extreme or a non-extreme point of the unit ball of $H^\infty$.

**Theorem 3.3.** [7, Theorem 2.2]. Let $b \in H^\infty$ with $\|b\|_\infty \leq 1$. The following statements are equivalent:

i) $b$ is a non-extreme point of the unit ball of $H^\infty$;

ii) $\mathcal{H}(b)$ contains all functions holomorphic in a neighbourhood of $\overline{D}$;

iii) $\mathcal{H}(b)$ contains all polynomials;

iv) polynomials are dense in $\mathcal{H}(b)$.

From now on, we shall simply say that $b$ is “extreme” or “non-extreme” to indicate that $b$ is correspondingly an extreme or non-extreme point of the unit ball of $H^\infty$. The function $b$ is non-extreme if and only if log$(1 - |b|^2) \in L^1(\mathbb{T})$ (see [6, Theorem 7.9]). In this case, there exists a unique outer function $a \in H^\infty$, normalized so that $a(0) > 0$, such that $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$. We call $(b, a)$ the Pythagorean pair associated to $b$.

There is a useful characterization of $\mathcal{H}(b)$ when $b$ is non-extreme.

**Theorem 3.4.** [14, §IV-1]. Let $b$ be non-extreme, let $(b, a)$ be the corresponding Pythagorean pair, and let $f \in H^2$. Then $f \in \mathcal{H}(b)$ if and only if $T_b f \in T_a H^2$. In this case, there exists a unique function $f^+ \in H^2$ such that $T_b f = T_a f^+$, and

$$\|f\|_b^2 = \|f\|_2^2 + \|f^+\|_2^2.$$

The authors of [5] obtained an explicit formula for the $\mathcal{H}(b)$-norm of functions $f$ holomorphic in a neighbourhood of $\overline{D}$. Notice that, if $(b, a)$ is a pair, then $\phi := b/a \in N^+$, the Smirnov class.

**Theorem 3.5.** [5, Theorem 4.1] Let $(b, a)$ be a pair, and let $\phi := b/a$, say $\phi(z) = \sum_{n \geq 0} c_n z^n$. Let $f$ be holomorphic in a neighbourhood of $\overline{D}$ with expansion $f(z) = \sum_{n \geq 0} a_n z^n$. Then the series $\sum_{j \geq 0} a_{j+n} \overline{c}_j$ converges absolutely for each $n$ and

$$\|f\|_b^2 = \sum_{n \geq 0} |a_n|^2 + \sum_{n \geq 0} \left| \sum_{j \geq 0} a_{j+n} \overline{c}_j \right|^2. \quad (3.1)$$

We remark two consequences of this result that will be useful in what follows.
Corollary 3.6. Let $b$ be non-extreme and let $f \in \mathcal{H}(b)$. Then, for each $R > 1$, the partial sums of the Taylor series of $f$ satisfy $\|s_n[f]\|_b = O(R^n)$ as $n \to \infty$.

Proof. Fix $S$ with $1 < S^2 < R$. Let $(b, a)$ be the Pythagorean pair corresponding to $b$, and let $\phi := b/a$. Since both $f$ and $\phi$ are holomorphic on $\mathbb{D}$, their respective Taylor coefficients satisfy $a_j = O(S^j)$ and $c_j = O(S^j)$ as $j \to \infty$. Feeding this information into the formula (3.1), applied to $s_N[f]$ in place of $f$, we obtain

$$\|s_N[f]\|^2_b = O((N + 1)S^{2N}) + O((N + 1)^3S^{4N}) \quad (N \to \infty).$$

This implies that $\|s_N[f]\|_b = O(R^N)$. □

Corollary 3.7. Let $(b, a)$ be a pair, and let $\phi := b/a$, say $\phi(z) = \sum_{j \geq 0} c_j z^j$. Let $f(z) = \sum_{n \geq 0} a_n z^n$ be a function in $H^2$ and let $r \in [0, 1)$. Then there is a constant $C(\phi, r)$, which depends only on $\phi$ and $r$, such that

$$\|f_r\|^2_b \leq C(\phi, r)\|f\|^2_{H^2}.$$

The dependence $r \mapsto C(\phi, r)$ can be chosen to be increasing.

Proof. Since $f_r(z) := \sum_{n \geq 0} a_n r^n z^n$, using formula (3.1), we get

$$\|f_r\|^2_b = \sum_{n \geq 0} r^n |a_n|^2 + \sum_{n \geq 0} \left| \sum_{j \geq 0} r^j a_{j+n} \overline{c}_j \right|^2 \leq \|f\|^2_{H^2} + \sum_{n \geq 0} \left( \sum_{j \geq 0} r^j |a_{j+n}| |c_j| \right)^2.$$

By the Cauchy-Schwarz inequality, we obtain

$$\|f_r\|^2_b \leq \|f\|^2_{H^2} + \sum_{n \geq 0} \left( \sum_{j \geq 0} r^j |a_{j+n}| \right) \left( \sum_{j \geq 0} r^j |c_j| \right)^2 = \|f\|^2_{H^2} + \sum_{n \geq 0} r^n \| (S^*)^n (f_{\sqrt{r}}) \|^2_{H^2} \left( \sum_{j \geq 0} r^j |c_j| ^2 \right),$$

where $S^*$ is the backward shift operator on $H^2$ and $(S^*)^n$ is the $n$-fold composition of $S^*$. Since $\|S^*\| \leq 1$, we get

$$\|f_r\|^2_b \leq \|f\|^2_{H^2} + \sum_{n \geq 0} r^j |c_j|^2 \left( \sum_{j \geq 0} r^j |a_{j+n}|^2 \right) \|f_{\sqrt{r}}\|^2_{H^2},$$

where $\sum_{j \geq 0} r^j |c_j|^2 < \infty$ since $\phi \in \text{Hol}(\mathbb{D})$. Finally, since $\|f_{\sqrt{r}}\|^2_{H^2} \leq \|f\|^2_{H^2}$, we obtain

$$\|f_r\|^2_b \leq C(\phi, r)\|f\|^2_{H^2},$$

as desired, where $C(\phi, r) := 1 + \sum_{n \geq 0} r^j |c_j|^2/(1-r)$. It is easy to see from the definition of the constant $C(\phi, r)$ that the function $r \mapsto C(\phi, r)$ is increasing on $[0, 1)$. □
We end this section by stating an estimate obtained in the core of the proofs of Theorem 3.1 and Theorem 3.6 in [7]. This example plays a central role in [7], and will be equally important for us here. The exact choices of $b$ and $f \in \mathcal{H}(b)$ do not matter in our situation, so we will just state the estimate as follows.

**Theorem 3.8.** [7, Theorems 3.1 and 3.6]. There exist a non-extreme $b$ and a function $f \in \mathcal{H}(b)$ such that $(f_r)_+(0)$ is non-negative for all $r \in [0,1)$ and, moreover, $(f_r)_+(0) \to +\infty$ as $r \to 1^-$.  

4. Divergence of the logarithmic means

To prove that the logarithmic means diverge in the $\mathcal{H}(b)$-norm, we use another expression of the logarithmic means in terms of the classical Abel means or, equivalently, in terms of the dilates $f_r$ of a function $f \in \text{Hol}(\mathbb{D})$.

The first step toward this formula is the following lemma.

**Lemma 4.1.** Let $b$ be non-extreme, let $f \in \mathcal{H}(b)$ and let $r \in (0,1)$. The application $F : [0,r] \to \mathcal{H}(b)$, defined by $F(t) := f_t$, is continuous from $[0,r]$ into $\mathcal{H}(b)$.

**Proof.** By Corollary 3.6, for any $R > 1$ there exists a finite positive constant $C > 0$, which depends on $R$ and $f$, such that

$$\|s_n[f]\|_b \leq CR^n.$$  

By choosing $R$ so that $Rr < 1$, we can ensure that the series

$$(1 - t) \sum_{n \geq 0} s_n[f]t^n \quad (0 \leq t \leq r)$$

converges in $\mathcal{H}(b)$-norm to some function $g^{(t)} \in \mathcal{H}(b)$ on $[0,r]$. The dilates $f_t$ can be expressed pointwise as

$$f_t(z) = (1 - t) \sum_{n \geq 0} s_n[f](z)t^n \quad (z \in \mathbb{D}),$$

and therefore we must have that $g^{(t)} = f_t$, since convergence in $\mathcal{H}(b)$ implies pointwise convergence.

The continuity of $F$ now follows from the scalar continuity of each map $t \mapsto t^n(1 - t)$ on $[0,r]$, $n \geq 0$. $\square$

For the next lemma, we show that the integral formula (2.1) linking the logarithmic means to the Abel means is also valid in $\mathcal{H}(b)$. We use the Bochner integral as the definition of the vector-valued integral. For background on this topic, we refer the reader to [9, Chapter 3, §1].

**Lemma 4.2.** Let $b$ be non-extreme and let $f \in \mathcal{H}(b)$, say $f(z) = \sum_{n \geq 0} a_n z^n$. Then, for each $r \in [0,1)$, we have $L_r[f] \in \mathcal{H}(b)$ and

$$L_r[f] = \frac{1}{\log \frac{1}{1-r}} \int_0^r \frac{f_t}{1 - t} dt.$$
Proof. Fix $r \in [0, 1)$. From Lemma 4.1, the function $t \mapsto \frac{f_t}{t}$ is continuous from $[0, r]$ into $\mathcal{H}(b)$, and so its Bochner integral is well-defined.

By Corollary 3.6 again, for any $R > 1$, we have that $\|s_n[f]\|_b \leq CR^n$, where $C$ is a positive constant depending only on $R$ and $f$. Let $R > 1$ be chosen so that $RR < 1$.

Firstly, we compute an upper bound for the series defining $L_r[f]$: \[
\frac{r}{\log \frac{1}{1-r}} \sum_{n \geq 0} \frac{\|s_n[f]\|_b}{n+1} r^n \leq C \frac{r}{\log \frac{1}{1-r}} \sum_{n \geq 0} \frac{(rR)^n}{n+1} = C \frac{\log \frac{1}{1-r}}{R \log \frac{1}{1-r}}.
\]

Therefore the series $\frac{r}{\log \frac{1}{1-r}} \sum_{n \geq 0} \frac{\|s_n[f]\|_b}{n+1} r^n$ converges absolutely in $\mathcal{H}(b)$ and it defines a function in $\mathcal{H}(b)$, say $g^{(r)} \in \mathcal{H}(b)$. Since convergence in $\mathcal{H}(b)$ implies pointwise convergence, we must also have \[
g^{(r)}(z) = \frac{r}{\log \frac{1}{1-r}} \sum_{n \geq 0} \frac{s_n[f](z)}{n+1} r^n \quad (z \in \mathbb{D}),
\]
which gives $g^{(r)} = L_r[f] \in \mathcal{H}(b)$.

Secondly, we have
\[
\frac{r}{\log \frac{1}{1-r}} \sum_{n \geq 0} \frac{s_n[f]}{n+1} r^n = \frac{1}{\log \frac{1}{1-r}} \sum_{n \geq 0} \int_0^r s_n[f] t^n dt.
\]

The series $\sum_{n \geq 0} s_n[f] t^n$ is absolutely and uniformly convergent in $\mathcal{H}(b)$ on $[0, r]$. Therefore, the order of summation and integration can be interchanged and we get
\[
L_r[f] = \frac{1}{\log \frac{1}{1-r}} \int_0^r \sum_{n \geq 0} s_n[f] t^n dt = \frac{1}{\log \frac{1}{1-r}} \int_0^r \frac{f_t}{1-t} dt,
\]
where the last equality comes from the fact that $f_t = (1-t) \sum_{n \geq 0} s_n[f] t^n$. \qed

Using this last lemma and the estimate in Corollary 3.7, we obtain the following result. We consider the logarithmic means $L_r$ as linear operators on $\mathcal{H}(b)$ defined by $L_r(f) := L_r[f]$ for each $f \in \mathcal{H}(b)$.

**Corollary 4.3.** Let $b$ be non-extreme. Then, for every $r \in [0, 1)$, the linear map $L_r : \mathcal{H}(b) \to \mathcal{H}(b)$ is a bounded linear operator with $\|L_r\| \leq \sqrt{C(\phi, r)}$, where $C(\phi, r)$ is the same constant as in Corollary 3.7. In fact we even have
\[
\|L_r(f)\|_b \leq \sqrt{C(\phi, r)} \|f\|_{H^2} \quad (f \in \mathcal{H}(b)). \tag{4.1}
\]

**Proof.** Let $r \in [0, 1)$ and $f \in \mathcal{H}(b)$. Then, from Corollary 3.7, we have \[
\frac{1}{\log \left(\frac{1}{1-r}\right)} \int_0^r \frac{\|f_t\|_b}{1-t} dt \leq \frac{\|f\|_{H^2}}{\log \left(\frac{1}{1-r}\right)} \int_0^r \frac{\sqrt{C(\phi, t)}}{1-t} dt,
\]
and, since $t \mapsto C(\phi, t)$ is increasing on $[0, r)$, the above expression is
\[
\leq \frac{C(\phi, r) \|f\|_{H^2}}{\log \left(\frac{1}{1-r}\right)} \int_0^r \frac{1}{1-t} dt \leq \sqrt{C(\phi, r)} \|f\|_{H^2}.
\]
In combination with Lemma 4.2, this establishes (4.1). Since $\|f\|_{H^2} \leq \|f\|_b$, it follows that $L_r$ is a bounded operator on $\mathcal{H}(b)$ with $\|L_r\| \leq \sqrt{C(\phi, r)}$. □

Now we can exploit the integral formula of $L_r[f]$ to express $(L_r[f])^+$ in terms of a certain integral involving $(f_t)^+$.

**Theorem 4.4.** For any $f \in \mathcal{H}(b)$ and $r \in (0, 1)$, we have that

$$(L_r[f])^+ = \frac{1}{\log \frac{1}{1-r}} \int_0^r \frac{(f_t)^+}{1-t} \, dt. \quad (4.2)$$

**Proof.** Define $F(t) := \frac{f_t}{1-t}$. This is a continuous function from $[0, r]$ into $\mathcal{H}(b)$, by Lemma 4.1. The fact that $F(t)$ is continuous from $[0, r]$ into $\mathcal{H}(b)$, combined with the formula for the norm in $\mathcal{H}(b)$ in terms of $f$ and $f^+$, imply that the mapping $t \mapsto \frac{(f_t)^+}{1-t}$ is continuous from $[0, r]$ into $H^2$. Therefore, it is Bochner-integrable on $[0, r]$. Thus, since $T_\pi : H^2 \to H^2$ and $T_\pi : H^2 \to H^2$ are bounded operators, we get

$$T_\pi \left( \left( \log \frac{1}{1-r} \right)^{-1} \int_0^r \frac{(f_t)^+}{1-t} \, dt \right) = \left( \log \frac{1}{1-r} \right)^{-1} \int_0^r \frac{T_\pi(f_t)^+}{1-t} \, dt = \left( \log \frac{1}{1-r} \right)^{-1} \int_0^r \frac{T_\pi f_t}{1-t} \, dt = T_\pi \left( \left( \log \frac{1}{1-r} \right)^{-1} \int_0^r \frac{f_t}{1-t} \, dt \right).$$

Thus, by the uniqueness of $(L_r[f])^+$, we get formula (4.2). □

Now, we can prove our main result.

**Proof of Theorem 1.1.** Choose $b$ and $f$ as in Theorem 3.8. Let $A > 0$ and choose $r_0 \in (0, 1)$ so that

$$(f_t)^+(0) \geq A \quad (r_0 < t < 1).$$

Convergence in $H^2$ implies pointwise convergence on $\mathbb{D}$. Therefore, by Theorem 4.4, the following equality holds:

$$(L_r[f])^+ = \frac{1}{\log \frac{1}{1-r}} \int_0^r \frac{(f_t)^+(0)}{1-t} \, dt.$$

Fix $r \in [0, 1)$. Splitting the integral in two parts, from 0 to $r_0$ and from $r_0$ to $r$, we have

$$(L_r[f])^+(0) \geq \frac{\log \frac{1-r_0}{1-r}}{\log \frac{1}{1-r}} A + \frac{1}{\log \frac{1}{1-r}} \int_0^{r_0} \frac{(f_t)^+(0)}{1-t} \, dt.$$

Taking the lim inf as $r \to 1^-$, we get

$$\liminf_{r \to 1^-} (L_r[f])^+(0) \geq A.$$
Since $A$ was arbitrary, $\liminf_{r \to 1^{-}} (L_r[f])^+(0) = \infty$. By the expression of the norm of $L_r[f]$ in $\mathcal{H}(b)$, it follows that
$$\|L_r[f]\|_b \geq |(L_r[f])^+(0)|$$
which implies that $\liminf_{r \to 1^{-}} \|L_r[f]\|_b = \infty$. This concludes the proof of the theorem.

\section{Scalar-inclusion and summability in Banach spaces}

Before analyzing the consequences of Theorem 1.1 on other power-series methods, we take a little detour to prove an abstract theorem on scalar-inclusion of two sequence-to-function summability methods. This theorem will be the main ingredient to prove Corollary 1.2.

\textbf{Theorem 5.1.} Let $K$ and $H$ be two regular sequence-to-function summability methods. Let $X$ and $Y$ be Banach spaces, and let $S : X \to Y$ and $S_n : X \to Y$ ($n \geq 0$) be bounded linear operators. Suppose that:
\begin{itemize}
  \item $S_n(x) \to S(x)$ for all $x \in W$, where $W$ is a dense subset of $X$;
  \item $(S_n(x))_{n \geq 0}$ is $K$-summable to $S(x)$ for all $x \in X$;
  \item $K$ is scalar-included in $H$.
\end{itemize}
Then $(S_n(x))_{n \geq 0}$ is $H$-summable to $S(x)$ for all $x \in X$.

\textbf{Proof.} Let $(k_n)_{n \geq 0}, (h_n)_{n \geq 0} : [0, R) \to \mathbb{C}$ be the functions defining the summability methods $K$ and $H$ respectively. We need to prove that, for each $x \in X$:
\begin{enumerate}
  \item[(i)] $\sum_{n \geq 0} h_n(r)S_n(x)$ converges in $Y$ for all $r \in [0, R)$;
  \item[(ii)] $\|\sum_{n \geq 0} h_n(r)S_n(x) - S(x)\|_Y \to 0$ as $r \to R^-$.
\end{enumerate}

We begin with (i). Fix $r \in [0, R)$. Given $x \in X$, the sequence $(S_n(x))_{n \geq 0}$ is $K$-summable to $S(x)$. By linearity and continuity, for each $\phi \in Y^*$, the sequence $(\phi(S_n(x)))$ is $K$-summable to $\phi(S(x))$. As $K$ is scalar-included in $H$, it follows that $(\phi(S_n(x)))$ is also $H$-summable to $\phi(S(x))$. In particular, the series $\sum_{n \geq 0} h_n(r)\phi(S_n(x))$ converges in $\mathbb{C}$. Hence
$$\sup_{m \geq 0} \left| \sum_{n=0}^{m} h_n(r)\phi(S_n(x)) \right| < \infty \quad (x \in X, \ \phi \in Y^*).$$

In other words
$$\sup_{m \geq 0} \left| \phi\left( \sum_{n=0}^{m} h_n(r)S_n(x) \right) \right| < \infty \quad (x \in X, \ \phi \in Y^*).$$

Applying the Banach–Steinhaus theorem twice (once for $\phi$ and once for $x$), we obtain that
$$M := \sup_{m \geq 0} \left\| \sum_{n=0}^{m} h_n(r)S_n \right\| < \infty,$$
where now the norm is the operator norm.

(5.1)
Given $x \in X$ and $\epsilon > 0$, choose $w \in W$ such that $\|x - w\|_X < \epsilon/M$. As $S_n(w) \to S(w)$ and $H$ is a regular summability method, the sequence $(S_n(w))$ is $H$-summable to $S(w)$. In particular, the series $\sum_{n \geq 0} h_n(r)S_n(w)$ converges in $Y$. It follows that $\|\sum_{m_2}^{m_2} h_n(r)S_n(w)\|_Y < \epsilon$ for all large enough $m_1, m_2$. For all such $m_1, m_2$, we then have

$$
\left\| \sum_{n = m_1}^{m_2} h_n(r)S_n(x) \right\|_Y \leq \left\| \sum_{n = m_1}^{m_2} h_n(r)S_n(x - w) \right\|_Y + \left\| \sum_{n = m_1}^{m_2} h_n(r)S_n(w) \right\|_Y
$$

$$
\leq \left\| \sum_{n = m_1}^{m_2} h_n(r)S_n \right\|_X \|x - w\|_X + \epsilon
$$

$$
\leq 2M(\epsilon/M) + \epsilon = 3\epsilon.
$$

This shows that the series $\sum_{n = 0}^{\infty} h_n(r)S_n(x)$ is Cauchy, and therefore it converges in $Y$, thereby completing the proof of (i).

Now we turn to (ii). For each $r \in [0, R)$, define an operator $S_r^H : X \to Y$ by

$$
S_r^H(x) := \sum_{n \geq 0} h_n(r)S_n(x) \quad (x \in X).
$$

By (i) the series converges, so $S_r^H$ is well-defined and linear. Furthermore, it follows easily from (5.1) that $S_r^H$ is a bounded linear operator from $X$ into $Y$. As we saw in (i), for each $x \in X$ and $\phi \in Y^*$, the sequence $\phi(S_n(x))$ is $H$-summable to $\phi(S(x))$, in other words

$$
\phi(S_r^H(x)) \to \phi(S(x)) \quad (r \to R^-).
$$

(5.2)

We want to prove that $S_r^H(x) \to S(x)$ as $r \to R^-$. To do so, let $(r_j)_{j \geq 0}$ be a sequence in $[0, R)$ such that $r_j \to R^- \quad (j \to \infty)$. We will prove that $S_{r_j}^H(x) \to S(x)$ as $j \to \infty$. By (5.2), we have $\phi(S_{r_j}^H(x)) \to \phi(S(x))$ as $j \to \infty$ for each $x \in X$ and $\phi \in Y^*$. By the Banach–Steinhaus theorem, again applied twice, it follows that

$$
N := \sup_{j \geq 0} \|S_{r_j}^H\| < \infty.
$$

Given $x \in X$ and $\epsilon > 0$, choose $w \in W$ such that

$$
\|x - w\|_X < \epsilon/\max\{N, \|S\|\}.
$$

By regularity of $H$, we have

$$
S_{r_j}^H(w) \to S(w) \quad (j \to \infty).
$$

Hence $\|S_{r_j}^H(w) - S(w)\|_Y < \epsilon$ for all $j$ sufficiently large. For all such $j$, we then have

$$
\|S_{r_j}^H(x) - S(x)\|_Y \leq \|S_{r_j}^H(x - w)\|_Y + \|S_{r_j}^H(w) - S(w)\|_Y + \|S(w - x)\|_Y
$$

$$
\leq N(\epsilon/N) + \epsilon + \|S\|(\epsilon/\|S\|) = 3\epsilon.
$$

We conclude that $S_{r_j}^H(x) \to S(x)$ as $j \to \infty$, completing the proof of (ii).
6. Consequences for other power-series methods

Our final goal is to prove Corollary 1.2. Recall from §2 that the generalized Abel means of order $\alpha > -1$ applied to a function $f \in \text{Hol}(\mathbb{D})$ are

$$A^\alpha_r[f](z) := (1 - r)^{1+\alpha} \sum_{n \geq 0} \binom{n + \alpha}{\alpha} s_n(z) r^n \quad (z \in \mathbb{D}, \, r \in [0, 1)).$$

Let $f \in \mathcal{H}(b)$ and let $r \in [0, 1)$. By Corollary 3.6, for any $R > 1$, we have $\|s_n[f]\|_b \leq CR^n$ for some constant $C$ depending only on $R$ and $f$. From this, it is easy to see that the series defining $A^\alpha_r[f]$ converges absolutely to some function $g^{(r)} \in \mathcal{H}(b)$ and, since convergence in $\mathcal{H}(b)$ implies pointwise convergence, we get $A^\alpha_r[f] = g^{(r)} \in \mathcal{H}(b)$. Thus, we may view the generalized Abel mean as a linear operator on $\mathcal{H}(b)$ defined by

$$A^\alpha_r(f) := A^\alpha_r[f] \quad (f \in \mathcal{H}(b), \, r \in [0, 1)).$$

By the closed-graph theorem, it is a bounded linear operator on $\mathcal{H}(b)$.

To use Theorem 5.1, we need the following relation between the generalized Abel methods and the logarithmic method. This result can be found in [3, §5].

**Theorem 6.1.** All generalized Abel methods of order $\alpha$, with $\alpha > -1$, are scalar-included in the logarithmic method.

We are now ready to prove Corollary 1.2.

**Proof of Corollary 1.2.** Let $b$ be as in Theorem 1.1. Suppose, if possible, that, for every $f \in \mathcal{H}(b)$,

$$A^\alpha_r(f) \to f, \quad (r \to 1^-).$$

To apply Theorem 5.1, let $X = Y = \mathcal{H}(b)$, let $W$ be the set of polynomials, let $R = 1$, let $K$ and $H$ be the generalized Abel method of order $\alpha > -1$ and the logarithmic method respectively, let $S_n := s_n$, the operator that maps each function to the $n$-th partial sum of its Taylor expansion, and let $S := I$ be the identity on $\mathcal{H}(b)$.

By Theorem 6.1, the Abel method of order $\alpha > -1$ is scalar-included in the logarithmic method. The logarithmic method is also a regular sequence-to-function summability method because the function $\frac{1}{r} \log(\frac{1}{1-r}) \to \infty$ as $r \to 1^-$. Moreover, by Theorem 3.3, the set of polynomials $W$ is dense in $\mathcal{H}(b)$ since $b$ is non-extreme.

It remains to verify the first condition of Theorem 5.1, that is, $s_n(p) \to p$ for any $p \in W$. This is clear since, for $p \in W$, with

$$p(z) = \sum_{n=0}^{N} a_n z^n \quad (z \in \mathbb{D}),$$

we have $s_n(p) = s_n[p] = p$ if $n \geq N$.

Therefore, by Theorem 5.1, we infer that $(s_n(f))_{n \geq 0}$ is summable by the logarithmic method for every $f \in \mathcal{H}(b)$. This contradicts Theorem 1.1.
Therefore, there exists a function \( f \in \mathcal{H}(b) \) such that \( A_r^n(f) \not
rightarrow f \) as \( r \to 1^- \).

In fact, Corollary 1.2 generalizes to a whole family of power methods. This generalization is a consequence of the following inclusion theorem due to Borwein.

**Theorem 6.2.** [3. Theorem A] Let \( p(x) = \sum_{n \geq 0} p_n x^n \) be a power series with a radius of convergence \( R_p > 0 \). Let \( q(x) = \sum_{n \geq 0} q_n x^n \) be another power series with a radius of convergence \( R_q = R_p \). Suppose that there exist an integer \( N \), a finite signed measure \( \mu \) on the interval \([0,1]\) and a number \( \delta \in (0,1) \) such that, for all \( n \geq N \),

1. \( p_n = q_n \int_0^1 t^n \, d\mu(t) \);
2. \( \int_0^1 t^n \, d\mu(t) \geq \delta \int_0^1 t^n \, |d\mu(t)| \).

Then the power-series method \((q)\) is scalar-included in the power-series method \((p)\).

Consider the power-series method defined by the power series \( p(r) := \sum_{n \geq 0} p_n r^n \) with a radius of convergence \( R_p = 1 \). We suppose that, for the coefficients \( (p_n)_{n \geq 0} \), there exist an integer \( N \), a finite signed measure \( \mu \) on \([0,1]\) and a number \( \delta \in (0,1) \) such that, for every \( n \geq N \),

\[
(A) \quad \frac{1}{n+1} = p_n \int_0^1 t^n \, d\mu(t);
(B) \quad \frac{1}{n+1} \geq \delta p_n \int_0^1 t^n \, |d\mu(t)|.
\]

As an explicit example, the coefficients \( p_n := \binom{n+\alpha}{\alpha} \) with \( \alpha > -1 \) satisfy these requirements (see [3] for details).

Recall that, applied to the partial sums of the Taylor expansion of \( f(z) = \sum_{n \geq 0} a_n z^n \), the means defined by the power-series method \((p)\) are

\[
P_r[f] := \frac{1}{p(r)} \sum_{n \geq 0} p_n s_n[f] r^n \quad (0 \leq r < 1).
\]

Using formula (3.1) again, we can prove that the series defining \( P_r[f] \) converges, in \( \mathcal{H}(b) \), to some function \( g^{(r)} \in \mathcal{H}(b) \) and that \( P_r[f] = g^{(r)} \in \mathcal{H}(b) \).

Also, by the closed-graph theorem, the linear operator \( P_r : \mathcal{H}(b) \to \mathcal{H}(b) \), defined by \( P_r(f) := P_r[f] \), is bounded for each \( r \in [0,1] \).

**Theorem 6.3.** Let \( p(r) := \sum_{n \geq 0} p_n r^n \) be a power series with a radius of convergence \( R_p = 1 \). Suppose that the coefficients \( (p_n)_{n \geq 0} \) satisfy the conditions \((A)\) and \((B)\) above, with \( N \geq 0 \) an integer, \( \mu \) a finite signed measure on \([0,1]\) and \( \delta \in (0,1) \). Then there exist a non-extreme point \( b \in H^\infty \) and a function \( f \in \mathcal{H}(b) \) such that \( P_r[f] \not
rightarrow f \) in \( \mathcal{H}(b) \) as \( r \to 1^- \).

**Proof.** The proof is similar to Corollary 1.2. The only difference is that we use Theorem 6.2 instead of Theorem 6.1 to show that the power-series method defined by \((p)\) is scalar-included in the logarithmic method. \( \square \)
There are plenty of other summability methods which are included in the logarithmic method that we did not treat here. For example, in [4], it was shown that the generalized Borel method $B^{\alpha,\beta}_x$, with its means defined by

$$B^{\alpha,\beta}_x[f] := \sum_{n\geq N} \frac{s_n[f]}{\Gamma(\alpha n + \beta)} x^{\alpha n + \beta - 1},$$

where $x > 0$, $\alpha > 0$, $\beta \in \mathbb{R}$ and $N$ is a non-negative integer such that $\alpha N + \beta > 1$, is scalar-included in the logarithmic method. With similar techniques, we can prove that $B^{\alpha,\beta}_x[f] \in \mathcal{H}(b)$ for each $f \in \mathcal{H}(b)$ and, by using Theorem 5.1, we can prove that there exist a non-extreme $b$ and a function $f \in \mathcal{H}(b)$ such that $B^{\alpha,\beta}_x[f] \not\to f$ in $\mathcal{H}(b)$ as $x \to \infty$.

7. An open problem

We have studied some classes of sequence-to-function summability methods in this article and we proved that none of them works in general for de Branges–Rovnyak spaces. Moreover, in [11], the authors showed that there exist a Hilbert space of analytic functions $\mathcal{H}$ continuously embedded in $\text{Hol}(\mathbb{D})$ and a function $f \in \mathcal{H}$ such that $f$ lies outside the closed linear span of $\{s_n[f] : n \geq 0\}$. In particular, no summability methods applied to the partial sums $s_n[f]$ would work to make the Taylor series summable to $f$ in $\mathcal{H}$. This motivates the following problem.

Problem 7.1. Does there exist a non-extreme function $b$ and a function $f \in \mathcal{H}(b)$ such that $f$ lies outside the closed linear span of $\{s_n[f] : n \geq 0\}$?

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