Generation of bounded invariants via stroboscopic set-valued maps

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2. Problematic and description of the method
3. Euler’s method and error bounds
4. Systems with bounded uncertainty
5. Van der Pol example
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Motivation

- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state.
  - dynamical systems have a direct impact on human development.

⇒ The importance of studying:

- synchronization
- behavior
- stability
Dynamical systems:  
- in which a function describes the time dependence of a point in a geometrical space.
- we only know certain observed or calculated states of its past or present state.
- dynamical systems have a direct impact on human development.

⇒ The importance of studying:
- synchronization
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- stability
Stability

- A dynamical system is **stable**, if small perturbations to the solution lead to a new solution that stays **close** to the original solution forever.
- A **stable** system produces a **bounded output** for a given **bounded input**.
An invariant

- The bounded output of some periodic stable system can be considered as an invariant from certain $t$.
- An invariant is an unchanged object after operations applied to it.
Problematic

Parametric system (with unfixed parameters)

Generate

Invariant

Compute

Stability analysis

Initial condition
Description of the method

- Given a differential system $\Sigma : \frac{dx}{dt} = f(x)$ of dimension $n$, an initial point $x_0 \in \mathbb{R}^n$, a real $\varepsilon > 0$, and a ball $B_0 = B(x_0, \varepsilon)$\(^1\)

- The center of each ball at time $t$ is the Euler approximate solution $\tilde{x}(t)$ of the system starting at $x_0$, and the radius is a function $\delta_\varepsilon(t)$ bounding the distance between $\tilde{x}(t)$ and an exact solution $x(t)$ starting at $B_0$.

\(^1\) $B(x_0, \varepsilon)$ is the set $\{ z \in \mathbb{R}^n \mid \| z - x_0 \| \leq \varepsilon \}$ where $\| \cdot \|$ denotes the Euclidean distance.
Description of the method

- Given a differential system $\Sigma : \frac{dx}{dt} = f(x)$ of dimension $n$, an initial point $x_0 \in \mathbb{R}^n$, a real $\varepsilon > 0$, and a ball $B_0 = B(x_0, \varepsilon)$.

  The tube can be described as $\bigcup_{t \geq 0} B(t)$ where $B(t) \equiv B(\tilde{x}(t), \delta_{\varepsilon}(t))$.

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Description of the method

Given a differential system $\Sigma : \frac{dx}{dt} = f(x)$ of dimension $n$, an initial point $x_0 \in \mathbb{R}^n$, a real $\varepsilon > 0$, and a ball $B_0 = B(x_0, \varepsilon)$\(^1\)

To find a *bounded invariant*, we look for a positive real $T$ such that $B((i+1)T) \subseteq B(iT)$ for some $i \in \mathbb{N}$. In case of success, the ball $B(iT)$ is guaranteed to contain the “stroboscopic” sequence $\{B(jT)\}_{j=i,i+1,\ldots}$ of sets $B(t)$ at time $t = iT, (i+1)T, \ldots$ and thus constitutes the sought bounded invariant set.

\(^1\) $B(x_0, \varepsilon)$ is the set $\{z \in \mathbb{R}^n | \|z - x_0\| \leq \varepsilon\}$ where $\| \cdot \|$ denotes the Euclidean distance.
Euler’s method and error bounds

Let us consider the differential system:

\[
\frac{dx(t)}{dt} = f(x(t)),
\]

with states \( x(t) \in \mathbb{R}^n \) and \( x_0 \) a given initial condition.

- \( \tilde{x}(t; y_0) \) denotes Euler’s approximate value of \( x(t) \) (defined by \( \tilde{x}(t; y_0) = y_0 + t \times f(y_0) \) for \( t \in [0, \tau] \), where \( \tau \) is the integration time-step).
Proposition

[LCDVCF17] Consider the solution $x(t; y_0)$ of $\frac{dx}{dt} = f(x)$ with initial condition $y_0$ and the approximate Euler solution $\tilde{x}(t; x_0)$ with initial condition $x_0$. For all $y_0 \in B(x_0, \varepsilon)$, we have:

$$\|x(t; y_0) - \tilde{x}(t; x_0)\| \leq \delta_\varepsilon(t).$$
### Definition

$\delta_\varepsilon(t)$ is defined as follows for $t \in [0, \tau]$:

- If $\lambda < 0$:
  
  $$
  \delta_\varepsilon(t) = \left( \varepsilon^2 e^{\lambda t} + \frac{C^2}{\lambda^2} \left( t^2 + \frac{2t}{\lambda} + \frac{2}{\lambda^2} \left( 1 - e^{\lambda t} \right) \right) \right)^{\frac{1}{2}}
  $$

- If $\lambda = 0$:
  
  $$
  \delta_\varepsilon(t) = \left( \varepsilon^2 e^t + C^2 \left( -t^2 - 2t + 2(e^t - 1) \right) \right)^{\frac{1}{2}}
  $$

- If $\lambda > 0$:
  
  $$
  \delta_\varepsilon(t) = \left( \varepsilon^2 e^{3\lambda t} + \frac{C^2}{3\lambda^2} \left( -t^2 - \frac{2t}{3\lambda} + \frac{2}{9\lambda^2} \left( e^{3\lambda t} - 1 \right) \right) \right)^{\frac{1}{2}}
  $$

where $C$ and $\lambda$ are real constants specific to function $f$, defined as follows:

$$
C = \sup_{y \in S} L \| f(y) \|,
$$
**Definition**

$L$ denotes the Lipschitz constant for $f$, and $\lambda$ is the “one-sided Lipschitz constant” (or “logarithmic Lipschitz constant” [AS14]) associated to $f$, i.e., the minimal constant such that, for all $y_1, y_2 \in S$:

$$\langle f(y_1) - f(y_2), y_1 - y_2 \rangle \leq \lambda \| y_1 - y_2 \|^2, \quad (H0)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors of $S$.

The constant $\lambda$ can be computed using a nonlinear optimization solver (e.g., CPLEX [Cpl09]) or using the Jacobian matrix of $f$.

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[AS14] Z. Aminzare and E. D. Sontag, “Contraction methods for nonlinear systems: A brief introduction and some open problems,” in *53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014*, 2014, pp. 3835–3847.

[Cpl09] I. I. Cplex, “V12. 1: User’s manual for cplex,” *International Business Machines Corporation*, vol. 46, no. 53, p. 157, 2009.
Systems with bounded uncertainty

A differential system with bounded uncertainty is of the form

\[
\frac{dx(t)}{dt} = f(x(t), w(t)),
\]

with \(t \in \mathbb{R}_{\geq 0}^n\), states \(x(t) \in \mathbb{R}^n\), and uncertainty \(w(t) \in \mathcal{W} \subset \mathbb{R}^n\) (\(\mathcal{W}\) is compact, i.e., closed and bounded).

- We suppose (see [LCADSC+17]) that there exist constants \(\lambda \in \mathbb{R}\) and \(\gamma \in \mathbb{R}_{\geq 0}\) such that, for all \(y_1, y_2 \in S\) and \(w_1, w_2 \in \mathcal{W}\):

  \[
  \langle f(y_1, w_1) - f(y_2, w_2), y_1 - y_2 \rangle \leq \lambda \|y_1 - y_2\|^2 + \gamma \|y_1 - y_2\| \|w_1 - w_2\| \quad (H1).
  \]

- Instead of computing \(\lambda\) and \(\gamma\) globally for \(S\), it is advantageous to compute them \textit{locally} depending on the subregion of \(S\) occupied by the system state during a considered interval of time.

[LCADSC+17] A. Le Coënt et al., “Distributed control synthesis using Euler’s method,” in Proc. of International Workshop on Reachability Problems (RP’17), ser. Lecture Notes in Computer Science, vol. 247, Springer, 2017, pp. 118–131.
**Proposition**

\( \delta_\varepsilon(t) \) is defined as follows for \( t \in [0, \tau] \):

\[
\begin{align*}
\text{if } \lambda < 0 : & \quad \delta_\varepsilon, \mathcal{W}(t) = \left( \frac{C^2}{-\lambda^4} \left( -\lambda^2 t^2 - 2\lambda t + 2 e^{\lambda t} - 2 \right) \\
& \quad + \frac{1}{\lambda^2} \left( \frac{C \gamma |\mathcal{W}|}{-\lambda} \left( -\lambda t + e^{\lambda t} - 1 \right) + \lambda \left( \frac{\gamma^2 (|\mathcal{W}|/2)^2}{-\lambda} (e^{\lambda t} - 1) + \lambda \varepsilon^2 e^{\lambda t} \right) \right) \right)^{1/2} \quad (1) \\

\text{if } \lambda > 0 : & \quad \delta_\varepsilon, \mathcal{W}(t) = \frac{1}{(3\lambda)^{3/2}} \left( \frac{C^2}{\lambda} \left( -9\lambda^2 t^2 - 6\lambda t + 2 e^{3\lambda t} - 2 \right) \\
& \quad + 3\lambda \left( \frac{C \gamma |\mathcal{W}|}{\lambda} \left( -3\lambda t + e^{3\lambda t} - 1 \right) + 3\lambda \left( \frac{\gamma^2 (|\mathcal{W}|/2)^2}{\lambda} (e^{3\lambda t} - 1) + 3\lambda \varepsilon^2 e^{3\lambda t} \right) \right) \right)^{1/2} \quad (2) \\

\text{if } \lambda = 0 : & \quad \delta_\varepsilon, \mathcal{W}(t) = \left( C^2 \left( -t^2 - 2t + 2 e^t - 2 \right) + \left( C \gamma |\mathcal{W}| \left( -t + e^t - 1 \right) \\
& \quad + \left( \gamma^2 (|\mathcal{W}|/2)^2 (e^t - 1) + \varepsilon^2 e^t \right) \right) \right)^{1/2} \quad (3)
\end{align*}
\]
Proposition

Suppose that, for some index $1 \leq j \leq n$, we have $m^j_+ < M^j_-$ where $m^j_+$ (resp. $M^j_-$) denotes the minimum (resp. maximum) of $\tilde{x}^j(t) + \delta_\varepsilon, W(t)$ (resp. $\tilde{x}^j(t) - \delta_\varepsilon, W(t)$) for $t \in [iT, (i+1)T]$. Then $B[iT, (i+1)T]$ contains no fixed point of $\Sigma'$. 
Consider the Van der Pol (VdP) system $\Sigma_p$ of dimension $n = 2$ with parameter $p \in \mathbb{R}$, and initial condition in $B_0 = B(x_0, \varepsilon)$ for some $x_0 \in \mathbb{R}^2$ and $\varepsilon > 0$ (see [BQ20]):

\[
\begin{align*}
\frac{du_1}{dt} &= u_2 \\
\frac{du_2}{dt} &= pu_2 - pu_1^2 u_2 - u_1
\end{align*}
\]  

[BQ20] J. B. van den Berg and E. Queirolo, “A general framework for validated continuation of periodic orbits in systems of polynomial ODEs,” Journal of Computational Dynamics, vol. 0, no. 2158-2491-2019-0-10, 2020, ISSN: 2158-2491. DOI: 10.3934/jcd.2021004.
Van der Pol System with uncertainty

Consider now the system $\Sigma'$ with uncertainty $w(\cdot) \in \mathcal{W}_0 = [-0.5, 0.5]$ and initial condition $x_0$:

$$\begin{cases} \frac{du_1}{dt} = u_2 \\ \frac{du_2}{dt} = (p_0 + w)u_2 - (p_0 + w)u_1^2u_2 - u_1 \end{cases}$$

(5)

with $p_0 = 1.1$. It is easy to see that each solution of $\Sigma_p$ with $p \in [p_0 - 0.5, p_0 + 0.5] = [0.6, 1.6]$ is a particular solution of system $\Sigma'$. 
Van der Pol System with uncertainty

VdP system with parameter $p_0 = 1.1$, uncertainty $|\mathcal{W}_0| = 0.5$, initial radius $\varepsilon_0 = 0.2$, initial point $x_0 = (1.7018, -0.1284)$, period $T_0 = 6.746$, time-step $\tau = 10^{-3}$.

- We have: $B((i_0 + 1)T_0) \subset B(i_0 T_0)$ for $i_0 = 3$.
- The minimum $m^1_+$ of the upper green curve $\tilde{u}_1(t) + \delta_\mathcal{W}(t)$ is less than the maximum $M^1_-$ of the lower green curve $\tilde{u}_1(t) - \delta_\mathcal{W}(t)$.
- Whatever the value of $p \in [p_0 - |\mathcal{W}_0|, p_0 + |\mathcal{W}_0|] = [0.6, 1.6]$, the solution of $\Sigma_p$ never converges to a point of $\mathbb{R}^n$.
- Since the size of the system is $n = 2$, it follows by Poincaré-Bendixson’s theorem that the solution of $\Sigma_p$ converges always towards a limit circle.
Consider now the system $\Sigma'$ with uncertainty $w(\cdot) \in \mathcal{W}_1 = [-0.2, 0.2]$ and initial condition $x_0$:

$$
\begin{aligned}
\frac{du_1}{dt} &= u_2 \\
\frac{du_2}{dt} &= (p_1 + w)u_2 - (p_1 + w)u_1^2u_2 - u_1
\end{aligned}
$$

with $p_1 = 0.4$. It is easy to see that each solution of $\Sigma_p$ with $p \in [p_1 - 0.2, p_1 + 0.2] = [0.2, 0.6]$ is a particular solution of system $\Sigma'$. 

VdP system with parameter $p_1 = 0.4$, uncertainty $|\mathcal{W}_1| = 0.2$, initial radius $\varepsilon_1 = 0.2$, initial point $x_0 = (1.7018, -0.1284)$, period $T_1 = 6.347$, time-step $\tau = 10^{-3}$.

- We have: $B((i_1 + 1) T_1) \subset B(i_1 T_1)$ for $i_1 = 3$.

- We have $m_+^1 < M_-^1$, this shows that whatever the value of $p \in [p_1 - |\mathcal{W}_1|, p_1 + |\mathcal{W}_1|] = [0.2, 0.6]$, the solution of $\Sigma_p$ never converges to a point of $\mathbb{R}^n$.

- It follows by Poincaré-Bendixson’s theorem that the solution of $\Sigma_p$ converges always towards a limit circle for any $p \in [0.2, 0.6]$ and initial condition in $B(x_0, \varepsilon_1)$. 

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Consider now the system $\Sigma'$ with uncertainty $w(\cdot) \in \mathcal{W}_2 = [-0.3, 0.3]$ and initial condition $x_0$:

\[
\begin{align*}
\frac{du_1}{dt} &= u_2 \\
\frac{du_2}{dt} &= (p_2 + w)u_2 - (p_2 + w)u_1^2u_2 - u_1
\end{align*}
\] (5)

with $p_2 = 1.9$. It is easy to see that each solution of $\Sigma_p$ with $p \in [p_2 - 0.3, p_2 + 0.3] = [1.6, 2.2]$ is a particular solution of system $\Sigma'$. 
We have: $B((i_2 + 1)T_2) \subset B(i_2 T_2)$ for $i_2 = 3$. 

We have $m_+^1 < M_-^1$, then whatever the value of $p \in [p_2 - |\mathcal{V}_2|, p_2 + |\mathcal{V}_2|] = [1.6, 2.2]$, the solution of $\Sigma_p$ never converges to a point of $\mathbb{R}^n$. 

It follows by Poincaré-Bendixson’s theorem that the solution of $\Sigma_p$ converges always towards a limit circle for any $p \in [1.6, 2.2]$ and initial condition in $B(x_0, \varepsilon_2)$.

VdP system with parameter $p_2 = 1.9$, uncertainty $|\mathcal{V}_2| = 0.3$, initial radius $\varepsilon_2 = 0.1$, initial point $x_0 = (1.7018, -0.1284)$, period $T_2 = 7.531$, time-step $\tau = 10^{-3}$. 
Conclusion and Perspectives

Conclusion

- We presented a simple method to generate a bounded invariant for a differential system.
- The method shows that the solutions never converge to an equilibrium point for a parameterized differential system.
- The method uses a very general criterion of inclusion of one set in another.

Perspectives

- Adapt the method to solve the convergence to a limit cycle for complex systems.
- Extend our method in order to account for such an analysis.
Z. Aminzare and E. D. Sontag, “Contraction methods for nonlinear systems: A brief introduction and some open problems,” in 53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014, 2014, pp. 3835–3847.

J. B. van den Berg and E. Queirolo, “A general framework for validated continuation of periodic orbits in systems of polynomial ODEs,” Journal of Computational Dynamics, vol. 0, no. 2158-2491-2019-0-10, 2020, ISSN: 2158-2491. DOI: 10.3934/jcd.2021004.

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A. Le Coënt, J. Alexandre Dit Sandretto, A. Chapoutot, L. Fribourg, F. De Vuyst, and L. Chamoin, “Distributed control synthesis using Euler’s method,” in Proc. of International Workshop on Reachability Problems (RP’17), ser. Lecture Notes in Computer Science, vol. 247, Springer, 2017, pp. 118–131.

A. Le Coënt, F. De Vuyst, L. Chamoin, and L. Fribourg, “Control synthesis of nonlinear sampled switched systems using Euler’s method,” in SNR, (Apr. 22, 2017), ser. EPTCS, vol. 247, Uppsala, Sweden, 2017, pp. 18–33. DOI: 10.4204/EPTCS.247.2.