LOCAL LIMIT THEOREM FOR THE MAXIMUM OF A RANDOM WALK IN THE HEAVY-TRAFFIC REGIME

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ABSTRACT. Consider a family of $\Delta$-latticed aperiodic random walks $\{S^{(a)}, 0 \leq a \leq a_0\}$ with increments $X^{(a)}_i$ and non-positive drift $-a$. Suppose that
\[ \sup_{a \leq a_0} E[(X^{(a)})^2] < \infty \quad \text{and} \quad \sup_{a \leq a_0} E[\max\{0, X^{(a)}\}^{2+\varepsilon}] < \infty \]
for some $\varepsilon > 0$. Assume that $X^{(a)} \xrightarrow{w} X^{(0)}$ as $a \to 0$ and denote by $M^{(a)} = \max_{k \geq 0} S^{(a)}_k$ the maximum of the random walk $S^{(a)}$. In this paper we provide the asymptotics of $P(M^{(a)} = y\Delta)$ as $a \to 0$ in the case, when $y \to \infty$ and $ay = O(1)$.

This asymptotics follows from a representation of $P(M^{(a)} = y\Delta)$ via a geometric sum and a uniform renewal theorem, which is also proved in this paper.

1. Introduction and statement of results

Let $\{S^{(a)}, a \in [0, a_0]\}$ denote a family of random walks with drift $-a \leq 0$ and increments $X^{(a)}_i$, that is,
\[ S^{(a)}_0 := 0, \quad S^{(a)}_n := \sum_{i=1}^n X^{(a)}_i, \quad n \geq 1. \]

We shall assume that $X^{(a)}_1, X^{(a)}_2, \ldots$ are independent copies of a random variable $X^{(a)}$. In the case $a = 0$ we write $S$, $X_i$ and $X$ instead of $S^{(0)}$, $X^{(0)}_i$ and $X^{(0)}$ respectively. Assume that, as $a \to 0$,
\[ X^{(a)} \xrightarrow{w} X \quad (1) \]
and
\[ \sup_{a \in [0, a_0]} E[|X^{(a)}|^2] < \infty \quad \text{and} \quad \sup_{a \in [0, a_0]} E[\max\{0, X^{(a)}\}^{2+\varepsilon}] < \infty \quad (2) \]
for some $a_0, \varepsilon > 0$. If $a > 0$, the random walk $S^{(a)}$ drifts to $-\infty$ and the total maximum
\[ M^{(a)} := \max_{k \geq 0} S^{(a)}_k \]
is finite almost surely. However, as $a \to 0$, $M^{(a)} \to \infty$ in probability. From this fact arises the natural question how fast $M^{(a)}$ grows as $a \to 0$. The first result concerning this question goes back to Kingman [17], who considered the case when $|X|$ has an exponential moment and proved that, as $a \to 0$,
\[ P(M^{(a)} > y) \sim e^{-2ay/\sigma^2} \quad (3) \]
for all fixed values $y \geq 0$, where $\sigma^2 = \text{Var}(X)$ denotes the variance of the increments in the case of zero-drift. Prokhorov [10] extended this result to the case that the increments have finite variance. Kingman and Prokhorov had a motivation for examining $M^{(a)}$ that comes from queueing theory: It is well known that a stationary distribution of the waiting time of a customer in a single-server first-come-first-served (GI/GI/1) queue coincides with that of the maximum of a corresponding random walk. In the context of queueing theory, the limit $a \to 0$ means that the traffic load tends to 1. Thus, the question on the distribution of $M^{(a)}$ may be seen as the question on the growth rate of a stationary waiting-time distribution in a GI/GI/1 queue. This is one of the most important questions in queueing theory and is usually referred to as heavy-traffic analysis.

Another interesting question is whether (3) remains valid, if we do not fix the value $y$, but consider $y = y(a) \to \infty$ as $a \to 0$ sufficiently slow. Olvera-Cravioto, Blanchet and Glynn [9] showed that, if the increments possess regular varying tails with index $r > 2$, there exists a critical value $y(a) \approx \sigma^2 (r - 2) a^{-1} \ln a^{-1/2}$, under which the heavy traffic approximation holds. Denisov and Kugler [5] (see also [2]) identified the critical value for general subexponential distributions, e.g. $y(a) \approx a^{-1/(1-\gamma)}$ in the Weibull case, where $\gamma \in (0, 1)$ is the parameter of the Weibull distribution.

In this paper we assume that $X^{(a)}$ possesses a $\Delta$-lattice distribution, that means there exists some $\Delta > 0$ such that $P(X \in \Delta \mathbb{Z}) = 1$ and $\Delta$ is the maximal positive number with this property. Let us assume without loss of generality that $\Delta$ is an integer. Our main result is a local limit theorem for the probability $P(M^{(a)} = y\Delta)$ as $a \to 0$ for $y$ such that $y \to \infty$ and $ay = O(1)$ under the assumption that the increments possess an aperiodic lattice distribution with zero-shift. The main idea for our proof is to find a representation of the probability $P(M^{(a)} = y\Delta)$ as a geometric sum and to derive and apply a uniform renewal theorem to find the asymptotic behaviour of this sum. This uniform renewal theorem will be a generalization of a result attained by Nagaev [8].

It is worth mentioning that the approach used in this paper is similar to the method used in [2], where the authors use the well-known representation of $P(M^{(a)} > y)$ as a geometric sum of independent random variables (see for example [1]) and a uniform renewal theorem from [3] to establish the asymptotic behaviour of $P(M^{(a)} > y)$ as $a \to 0$ and $y \to \infty$ for subexponential distributions. In [3] there is also a uniform renewal theorem used to develop asymptotic expansions of the distribution of a geometric sum.

We now state our main result.

**Theorem 1.** Assume that (1) and (2) hold and suppose that $X^{(a)}$ possesses an aperiodic $\Delta$-lattice distribution for a small enough. Then, as $a \to 0$,

$$P(M^{(a)} = y\Delta) \sim \frac{2a\Delta}{\sigma^2} \exp \left\{ -\frac{2ay\Delta}{\sigma^2} \right\}$$

uniformly for all $y$ such that $y \to \infty$ and $ay = O(1)$ as $a \to 0$.

In the non-local case, it is known (see for example Wachtel and Shneer [12]) that one only needs to assume $\lim_{a \to 0} \text{Var}X^{(a)} = \sigma^2 \in (0, \infty)$ and a Lindeberg-type condition

$$\lim_{a \to 0} E[(X^{(a)})^2; |X^{(a)}_1| > K/a] = 0 \quad \text{for all } K > 0$$
to establish (3). This means that we must make stronger assumptions to establish our local result than it is needed in the non-local case.

Obviously, Theorem II restates the heavy traffic asymptotics (3): As $a \to \infty$,

$$
P(M^{(a)} \geq y\Delta) = \sum_{x=y}^{\infty} P(M^{(a)} = x\Delta) \sim \frac{2a\Delta}{\sigma^2} \sum_{x=y}^{\infty} e^{-2ax\Delta/\sigma^2}
$$

$$
= \frac{2a\Delta}{\sigma^2} \frac{e^{-2ay\Delta/\sigma^2}}{1 - e^{-2ay\Delta/\sigma^2}} \sim e^{-2ay\Delta/\sigma^2}
$$

for all $y$ such that $y \to \infty$ and $ay = O(1)$ as $a \to 0$.

2. Uniform renewal theorem

In this section we prove a modification of Theorem 1 in Nagaev [8] which is, unlike the uniform renewal theorem from Nagaev, even uniform in the expected value. This renewal theorem is the key to the proof of our main result.

Consider a family of non-negative $\Delta$-latticed and aperiodic random variables \{\text{\(Z^{(b)}, b \in I\)}\} with $E[Z^{(b)}] = b > 0$ and a non-empty set $I \subseteq \mathbb{R}$ that contains at least one accumulation point. Denote by $F^{(b)}$ the distribution function of $Z^{(b)}$ and by $F^{(b)}_k$ the $k$-fold convolution of $F^{(b)}$ with itself. Let

$$
H(x, b, A) = \sum_{k=0}^{\infty} A^k F^{(b)}_k (x), \quad A > 0.
$$

In renewal theory one usually studies the asymptotic behavior of $H(x + h, b, 1) - H(x, b, 1), h > 0$. However, the case $A \neq 1$ is also of great interest. Nagaev’s motivation for studying the case $A \neq 1$ comes from branching processes, since there arises a need for an asymptotic representation for $H(x + h, b, A) - H(x, b, A)$ as $x \to \infty$ with an estimate for the remainder term which is uniform in $A$. For our purposes we seek a representation for $H(x + h, b, A) - H(x, b, A)$ as $x \to \infty$ and the estimate for the remainder shall be uniform in $A$ and $b$. Assume that there exists some $s > 1$ such that

$$
\sup_{b \in I} E[(Z^{(b)})^s] < \infty. \quad (5)
$$

Put

$$
f^{(b)}_{\Delta} = F^{(b)}((k\Delta) - F^{(b)}((k - 1)\Delta), \quad f^{(b)}_y(z) = \sum_{k=0}^{y} f^{(b)}_{\Delta} k,
$$

$$
\mu_{y}^{(b)}(z) = f^{(b)}_{y}(z) = \sum_{k=1}^{y} k f^{(b)}_{\Delta} z^{k-1}.
$$

Proposition 2. Let $\lambda^{(b)}_y(A)$ be the real non-negative root of the equation $Af^{(b)}_y(z) = 1$. Assume that \[ holds for some $s > 1$. Then, there exists a positive constant $\alpha$ for every accumulation point $b_0$ of $I$, such that

$$
\sum_{k=1}^{\infty} A^k \left( F^{(b)}_k (y\Delta) - F^{(b)}_k ((y - 1)\Delta) \right) = \frac{\lambda^{(b)}_y (A))^{-y-1}}{A \mu^{(b)}_y (\lambda^{(b)}_y (A))} + o(y^{-\min\{1,s-1\}} \ln y) \quad (6)
$$

uniformly in $b \in I \cap \{b \in I : |b - b_0| \leq \alpha\}$ and $A y \leq A \leq 1$, where

$$
A y = 1 - C/y \quad (7)
$$

with a fixed positive number $C$. 


2.1. Proof of the uniform renewal theorem. Although the uniform renewal theorem is a generalization of Theorem 1 in Nagaev [8], the main idea of the proof is the same. However, for reasons of completeness, we give the whole proof.

Let us assume without loss of generality $\Delta = 1$, $I = [0, b_1]$ with $b_1 > 0$ and that $y$ is sufficiently large in this section, even if it is not explicitly mentioned. Throughout the following $\int_a^b g(x) dF(b)(x)$ is to be interpreted as $\int_a^{b+} g(x) dF(b)(x)$.

**Lemma 3.** Assume that \([5]\) holds for some $I$ and $s > 1$. Put $\mu^{(b)} = E[Z^{(b)}]$, $b \in I$, and $U_y(\delta) = \{z : 1 \leq |z| \leq e^{h_y}, |\arg z| \leq \delta\}$ for some $h_y = O(1/y)$. Then,

$$\lim_{\delta \to 0} \lim_{b \to \infty} \sup_{b \in I, z \in U_y(\delta)} |\mu^{(b)}(z) - \mu^{(b)}| = 0.$$  \hspace{1cm} (8)

**Proof.** First of all,

$$|\mu^{(b)}(z) - \mu^{(b)}| = \left| \int_0^y xz^{x-1}dF(b)(x) - \int_0^\infty xdF(b)(x) \right| \leq \int_0^y x|z^{x-1} - 1|dF(b)(x) + \int_y^\infty xdF(b)(x). \hspace{1cm} (9)$$

When $x, |z| \geq 1$, one can easily see by Taylor’s approximation that $|z^{x-1} - 1| \leq x|z - 1||z|^x$. Using this estimate we obtain for all $z \in U_y(\delta)$ and $N \leq y$,

$$\int_0^N x|z^{x-1} - 1|dF(b)(x) \leq |z - 1| \int_0^N x^2|z|^x dF(b)(x) \leq |z - 1|e^{h_y} \int_0^N x^2dF(b)(x) \leq N^2|z - 1|e^{h_yy}.$$

Further, a straightforward trigonometric calculation shows that for $\delta$ sufficiently small,

$$|z - 1| \leq |z - e^{i\arg z}| + |1 - e^{i\arg z}| = |z| - 1 + \sqrt{2(1 - \cos(\arg z))} \leq e^{h_y} - 1 + 2\delta$$

for all $z \in U_y(\delta)$ and hence, as $y \to \infty$,

$$\int_0^N x|z^{x-1} - 1|dF(b)(x) \leq e^{h_y}N^2(e^{h_y} - 1 + 2\delta) = e^{h_y}N^2(2\delta + h_y + o(h_y))$$

uniformly in $b \in I$. At the same time, for $z \in U_y(\delta)$, assumption \([5]\) and $h_y = O(1)$ imply that there exists an absolute number $K > 0$ such that for all $N \leq y$,

$$\int_N^y x|z^{x-1} - 1|dF(b)(x) \leq (1 + e^{h_y}) \int_N^y xdF(b)(x) \leq \frac{1 + e^{h_y}}{N} \int_N^\infty x^s dF(b)(x) \leq KN^{1-s}$$

and by setting $N = (2\delta + h_y)^{-1/3}$ and choosing $K_1$ such that $e^{h_y} \leq K_1$ (which is possible due to the assumption $h_y = O(1/y)$), we attain

$$\int_0^y x|z^{x-1} - 1|dF(b)(x) \leq e^{h_y}(2\delta + h_y)^{1/3} + K(2\delta + h_y)^{(s-1)/3} + o(h_y) \leq 2^{1/3}K_1^3\delta^{1/3} + K2^{(s-1)/3}\delta^{(s-1)/3} + o(1) \hspace{1cm} (10)$$
Thus, uniformly in $b \in I$ as $y \to \infty$. Plugging the (10) into (9) and using (5) once more, we conclude

$$|\mu_{y}^{(b)}(z) - \mu^{(b)}| \leq 2^{1/3}K_{1}\delta^{1/3} + K2^{(s-1)/3}\delta^{(s-1)/3} + o(1)$$

uniformly in $b \in I$ as $y \to \infty$.

Lemma 4. Assume that (3) holds for some $I$ and $s > 1$. Then, for large enough $y$, $\lambda_{y}^{(b)}(A) < e^{h_{y}}$ for all $A_{y} \leq A \leq 1$ and $b \in I$, where $A_{y} = 1 - C/y$ with some constant $C > 0$ and $h_{y} = C_{1}/(\mu^{(0)}y)$ with $C_{1} > C\mu^{(0)}/\inf_{b \in I}\mu^{(b)}$.

Proof. We want to estimate the difference $\lambda_{y}^{(b)}(A) - 1$. First of all, by regarding the definition of $\lambda_{y}^{(b)}(A)$,

$$\int_{0}^{1} \left((\lambda_{y}^{(b)}(A))^{x} - 1\right) dF^{(b)}(x) = f_{y}^{(b)}(\lambda_{y}^{(b)}(A)) - \int_{0}^{1} dF^{(b)}(x)$$

$$= \frac{1}{A} - 1 + \int_{y}^{\infty} dF^{(b)}(x) = \frac{1 - A}{A} + \int_{y}^{\infty} dF^{(b)}(x).$$

Further, $\lambda_{y}^{(b)}(A) \geq 1$ for $A \leq 1$ and therefore by the binomial formula,

$$(\lambda_{y}^{(b)}(A))^{x} - 1 \geq x(\lambda_{y}^{(b)}(A) - 1), x \geq 0.$$

Thus, uniformly in $b \in I$,

$$(\lambda_{y}^{(b)}(A) - 1) \int_{0}^{y} x dF^{(b)}(x) \leq \int_{0}^{y} \left((\lambda_{y}^{(b)}(A))^{x} - 1\right) dF^{(b)}(x)$$

$$= \frac{1}{A} - 1 + \int_{y}^{\infty} dF^{(b)}(x) = \frac{1 - A}{A} + O(y^{-s}),$$

where we used (3) in the last line. The condition $A_{y} \leq A \leq 1$ implies that $1 - A \leq C/y$, hence

$$\frac{1}{A} = 1 + \frac{1 - A}{A} = 1 + O\left(\frac{1}{y}\right)$$

and consequently

$$\frac{1 - A}{A} \leq \frac{C}{A_{y}} = \frac{C}{y} + O\left(\frac{1}{y^{2}}\right).$$

(13)

From the inequalities (12), (13) and (5) we conclude that

$$\lambda_{y}^{(b)}(A) - 1 \leq \frac{C/\mu^{(b)} - \int_{y}^{\infty} xdF^{(b)}(x)}{\mu^{(b)} - \int_{y}^{\infty} xdF^{(b)}(x)} = \frac{C/(\mu^{(b)}y)}{1 - O(y^{-s})} + O(y^{-2}) + O(y^{-s})$$

$$= \frac{C}{\mu^{(b)}y} + O(y^{-2}) + O(y^{-s}) < \frac{C_{1}}{\mu^{(b)}y}$$

uniformly in $b \in I$ for all $y$ large enough. Therefore, since $e^{x} - 1 \geq x$ for all $x > 0$, $\lambda_{y}^{(b)}(A) < e^{h_{y}}$ uniformly in $A_{y} \leq A \leq 1$ and $b \in I$, if $y$ is sufficiently large. □

Lemma 5. Assume that (3) holds for some $I$ and $s > 1$. Put $h_{y} = C_{1}/(\mu^{(0)}y)$ with a constant $C_{1} > C\mu^{(0)}/\inf_{b \in I}\mu^{(b)}$. Then, there exists some $b_{2} > 0$ such that for $y$ large enough, $A_{y}^{(b)}(z) - 1$ has no other zeros in the disc $|z| < e^{h_{y}}$ apart from $\lambda_{y}^{(b)}(A)$ and this holds uniform in $A_{y} \leq A \leq 1$ and $0 \leq b \leq b_{2}$. □
Proof. First of all, for all $|z| \leq e^{h_y},$

$$|\mu^{(b)}_{y}(z)| \leq \int_{0}^{y} x|z|^{x-1}dF^{(b)}(x) \leq e^{h_y} \varphi^{(b)}.$$

Using in addition $h_y = O(1)$ and \([6]\), we conclude

$$\sup_{y, b \leq b_1, |z| \leq e^{h_y}} |\mu^{(b)}_{y}(z)| < \infty.$$  \hspace{1cm} (14)

Therefore,

$$\lim_{y \to \infty} \sup_{b \leq b_1, 0 \leq \varphi \leq 2\pi} \sup_{0 \leq y \leq e^{h_y}} \left| f^{(b)}_{y}(re^{i\varphi}) - f^{(b)}_{y}(e^{i\varphi}) \right|$$

$$= \lim_{y \to \infty} \sup_{b \leq b_1, 0 \leq \varphi \leq 2\pi} \left| \mu^{(b)}_{y}(e^{i\varphi}) \right| \left| re^{i\varphi} - e^{i\varphi} \right| = 0.$$ \hspace{1cm} (15)

On the other hand,

$$\lim_{y \to \infty} \sup_{b \leq b_1, 0 \leq \varphi \leq 2\pi} \left| f^{(b)}_{y}(e^{i\varphi}) - f^{(b)}_{\infty}(e^{i\varphi}) \right| \leq \lim_{y \to \infty} \sup_{b \leq b_1, 0 \leq \varphi \leq 2\pi} \int_{y}^{\infty} |e^{i\varphi x}|dF^{(b)}(x)$$

$$= \lim_{y \to \infty} \sup_{b \leq b_1} F^{(b)}(y) = 0.$$ \hspace{1cm} (16)

As $b \to 0$, $F^{(b)}(\cdot) \to F^{(0)}(\cdot)$ in the sense of Definition 3 from chapter VIII.1 in Feller \([6]\) and $F^{(0)}$ is not defective because of \([5]\). Obviously, $u_\varphi(\cdot) = e^{i\varphi}$ is equicontinuous with $|u_\varphi| = 1 < \infty$. Hence, by a corollary in chapter VIII.1 in Feller \([6]\),

$$\int_{0}^{\infty} e^{i\varphi x} dF^{(b)}(x) \to \int_{0}^{\infty} e^{i\varphi x} dF^{(0)}(x)$$ \hspace{1cm} (17)

uniformly in $0 \leq \varphi \leq \pi$ as $b \to 0$.

Now, let us first consider values of $z$ in the circle $|z| < e^{h_y}$ that are not in the vicinity of $\lambda^{(b)}_{y}(A)$. Due to Lemma \([11]\) these values can be characterized as those values that satisfy $|z| < e^{h_y}$ and $\delta \leq |\arg z| \leq \pi, \delta > 0$. It is

$$\sup_{\delta \leq \varphi \leq \pi} \left| f^{(0)}_{\infty}(e^{i\varphi}) \right| = \sup_{\delta \leq \varphi \leq \pi} \left| \int_{0}^{\infty} e^{i\varphi x} dF^{(0)}(x) \right| < \sup_{\delta \leq \varphi \leq \pi} \int_{0}^{\infty} |e^{i\varphi x}| dF^{(0)}(x) = 1.$$

Combining the latter inequality with \([17]\), we conclude that there exists some $b_2 > 0$ (assume without loss of generality $b_2 \leq b_1$) such that

$$\sup_{b \leq b_2} \sup_{\delta \leq \varphi \leq \pi} \left| f^{(b)}_{\infty}(e^{i\varphi}) \right| < 1$$

and since this inequality is strict,

$$m(\delta) := \inf_{b \leq b_2} \inf_{A \leq A \leq h_y} \inf_{0 \leq \delta \leq \varphi \leq \pi} \left| Af^{(b)}_{\infty}(e^{i\varphi}) - 1 \right| > 0.$$ \hspace{1cm} (18)

By combining \([15]\), \([16]0\) and \([15]0\), we conclude that for large enough $y$ and $A \in \mathcal{A}_y$,

$$\inf_{b \leq b_2} \inf_{1 \leq \delta \leq e^{h_y} \varphi \leq 2\pi} \left| Af^{(b)}_{y}(re^{i\varphi}) - 1 \right| > \frac{m(\delta)}{2} > 0.$$ \hspace{1cm} (19)

On the basis of \([16]0\) we can assert that if $Af^{(b)}_{y}(z) - 1$ has a zero $\lambda^{(b)}_{y}(A)$ in the disc $|z| \leq e^{h_y}$ differing from $\lambda^{(b)}_{y}(A)$, then $\lambda^{(b)}_{y}(A)$ will lie outside the region $\{ z : 1 \leq |z| \leq e^{h_y}, |\arg z| \geq \delta \}$ and this holds uniformly in $b \in [0, b_2]$ and $A \in \mathcal{A}_y$. 
Next, consider the region $U_y(\delta) = \{ z : 1 \leq |z| \leq e^{h_y}, |\arg z| < \delta \}$. Observe that Taylor’s formula implies

$$Af_y^{(b)}(z) - 1 = Af_y^{(b)}(z) - Af_y^{(b)}(\lambda_y^{(b)}(A)) \geq Af_y^{(b)}(\lambda_y^{(b)}(A))(z - \lambda_y^{(b)}(A)).$$

This inequality plus the equicontinuity of $f_y^{(b)}(z)$ imply the existence of a $\delta_1(b, A) > 0$ such that $|Af_y^{(b)}(z) - 1|$ has no other zeros in the disc $|z - \lambda_y^{(b)}(A)| \leq \delta_1(b, A)$ apart from $\lambda_y^{(b)}(A)$. Therefore,

$$\tilde{m}(\delta_2) := \inf_{b \leq b_2} \inf_{A \in \mathfrak{A}_y} \inf_{z : |z - \lambda_y^{(b)}(A)| \leq \delta_2} |Af_y^{(b)}(z) - 1| > 0,$$

where $\delta_2 = \inf_{b \leq b_2} \inf_{A \in \mathfrak{A}_y} \delta_1(b, A) > 0$. Observe that $\lambda_y^{(b)}(A) \geq 1$ for $A \leq 1$ and $\lambda_y^{(b)}(A) < e^{h_y}$ by Lemma $\text{[4]}$. Hence, for $\delta$ small enough, say $\delta \leq \delta_3$, the region

$$\bigcup_{b \leq b_2} \bigcup_{A \in \mathfrak{A}_y} \{ z : |z - \lambda_y^{(b)}(A)| \leq \delta_1 \}$$

covers $U_y(\delta)$ and that means $\tilde{\lambda}_y^{(b)}(A)$ cannot lie in the region $\{ z : 1 - \varepsilon_0 \leq |z| \leq e^{h_y}, |\arg z| < \delta_3 \}$. Setting $\delta = \delta_3$ in $\text{[14]}$ we conclude that $\tilde{\lambda}_y^{(b)}(A)$ cannot lie in the annulus $1 \leq |z| \leq e^{h_y}$. Since $|\tilde{\lambda}_y^{(b)}(A)| \geq 1$ for all $A \leq 1$, we finally obtain that $\tilde{\lambda}_y^{(b)}(A)$ does not lie in the disc $|z| \leq e^{h_y}$, so $\lambda_y^{(b)}(A)$ is the only root of the equation $Af_y^{(b)}(A) = 1$ in the disc $|z| \leq e^{h_y}$ and this holds uniformly in $b \leq b_2$ and $A_y \leq A \leq 1$. \hfill $\Box$

Proof of Proposition $\text{[2]}$. Let $\gamma_y$ be a circle of radius $r_y = e^{h_y}$ with $h_y = C_1/(\mu^{(0)} y)$, $C_1 \geq \mu^{(0)} + C\mu^{(0)}/\inf_{k \leq b_1} \mu^{(b)}$ and $C$ from $\text{[7]}$. Then, according to Lemma $\text{[3]}$ and Lemma $\text{[5]}$ there exists some $b_2 > 0$ such that for all $0 \leq b \leq b_2$ and $A \in \mathfrak{A}_y$, the function $1 - Af_y^{(b)}(z)$ is zero in the disc $|z| \leq e^{h_y}$, if and only if $z = \lambda_y^{(b)}(A)$. Hence, the Residue theorem states that

$$\frac{1}{2\pi i} \int_{\gamma_y} \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)} \, dz = \text{Res} \left( \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)}, \lambda_y^{(b)}(A) \right) + \text{Res} \left( \frac{z^{-y-1}}{1 - Af_y^{(b)}(z)}, 0 \right).$$

(20)

for $0 \leq b \leq b_2$ and $A \in \mathfrak{A}_y$.

In the following denote by $C_n(f(z))$, $n \geq 1$, the coefficient of $z^n$ in the Taylor series of the function $f(z)$. An easy calculation shows that

$$A^n(f^{(b)}(z)) = A^n \sum_{j=1}^{\infty} \left( F^{(b)}(j) - F^{(b)}(j-1) \right) z^j$$

and consequently, by changing the order of summation, it is not hard to see that

$$\sum_{k=1}^{\infty} A^k \left( F^{(b)}(n) - F^{(b)}(n-1) \right) = C_n \left( \frac{1}{1 - Af^{(b)}(z)} \right).$$

On the other hand, when $n \leq y$,

$$C_n \left( \frac{1}{1 - Af^{(b)}(z)} \right) = C_n \left( \frac{1}{1 - Af^{(b)}(z)} \right).$$
and thus, for $n \leq y$,

$$\sum_{k=1}^{\infty} A^k \left( F_k(b)(n) - F_k(b)(n-1) \right) = C_n \left( \frac{1}{1 - A\lambda_y^{(b)}(z)} \right). \quad (21)$$

Regarding (21) with $n = y$, one can easily verify

$$\text{Res} \left( \frac{z^{-y-1}}{1 - A\lambda_y^{(b)}(z)}, 0 \right) = \sum_{k=1}^{\infty} A^k \left( F_k(b)(y) - F_k(b)(y-1) \right).$$

The pole of the function $z^{-y-1}/(1 - A\lambda_y^{(b)}(z))$ in $z = \lambda_y^{(b)}(A)$ is of order 1. Therefore, it is not hard to see that

$$\text{Res} \left( \frac{z^{-y-1}}{1 - A\lambda_y^{(b)}(z)}, \lambda_y^{(b)}(A) \right) = - \frac{\lambda_y^{(b)}(A)^{-y-1}}{A\lambda_y^{(b)}(\lambda_y^{(b)}(A))}$$

and by combining the latter results we obtain

$$\sum_{k=1}^{\infty} A^k \left( F_k(b)(y) - F_k(b)(y-1) \right) = \frac{\lambda_y^{(b)}(A)^{-y-1}}{A\lambda_y^{(b)}(\lambda_y^{(b)}(A))} + \frac{1}{2\pi i} \int_{\gamma_y} \frac{z^{-y-1}}{1 - A\lambda_y^{(b)}(z)} dz.$$

It remains to show that under the conditions of Proposition 2

$$\frac{1}{2\pi i} \int_{\gamma_y} \frac{z^{-y-1}}{1 - A\lambda_y^{(b)}(z)} dz = o \left( y^{-\min\{1, -1\} \ln y} \right) \quad (22)$$

uniformly in $b \leq b_2$ and $A_\nu \leq A \leq 1$. Let

$$\varphi_y^{(b)}(z) = A(f_y^{(b)}(z) - f_y^{(b)}(r_y)) - A\mu_y^{(b)}(r_y)(z - r_y),$$

$$\psi_y^{(b)}(z) = 1 - A\mu_y^{(b)}(r_y)(z - r_y).$$

Then, the following identity holds:

$$\frac{1}{1 - A\lambda_y^{(b)}(z)} - \frac{1}{\psi_y^{(b)}(z)} = \frac{\varphi_y^{(b)}(z) - 1 + A\lambda_y^{(b)}(z)}{(1 - A\lambda_y^{(b)}(z))\psi_y^{(b)}(z)} = \frac{\varphi_y^{(b)}(z)}{(1 - A\lambda_y^{(b)}(z))\psi_y^{(b)}(z)}.$$ \quad (23)

Let $\varepsilon > 0$, $\gamma_y(\varepsilon) = \gamma_y \cap U_y(\varepsilon)$ and let $\overline{\gamma_y}(\varepsilon)$ be the complement of $\gamma_y(\varepsilon)$ with respect to $\gamma_y$. By (23),

$$\int_{\gamma_y} \frac{z^{-y-1}}{1 - A\lambda_y^{(b)}(z)} dz = \int_{\gamma_y} \frac{z^{-y-1}}{\psi_y^{(b)}(z)} dz + \int_{\overline{\gamma_y}(\varepsilon)} \frac{z^{-y-1}\varphi_y^{(b)}(z)}{(1 - A\lambda_y^{(b)}(z))\psi_y^{(b)}(z)} dz$$

$$+ \int_{\overline{\gamma_y}(\varepsilon)} \frac{z^{-y-1}\varphi_y^{(b)}(z)}{(1 - A\lambda_y^{(b)}(z))\psi_y^{(b)}(z)} dz.$$

Using (23) once again, the last integral of the latter identity can be rewritten as

$$\int_{\overline{\gamma_y}(\varepsilon)} \frac{z^{-y-1}\varphi_y^{(b)}(z)}{(1 - A\lambda_y^{(b)}(z))\psi_y^{(b)}(z)} dz = - \int_{\overline{\gamma_y}(\varepsilon)} \frac{z^{-y-1}}{\psi_y^{(b)}(z)} dz + \int_{\gamma_y(\varepsilon)} \frac{z^{-y-1}}{1 - A\lambda_y^{(b)}(z)} dz.$$

Hence,

$$\int_{\gamma_y} \frac{z^{-y-1}}{1 - A\lambda_y^{(b)}(z)} dz = I_1^{(b)}(y) + \sum_{j=2}^{4} I_j^{(b)}(y, \varepsilon), \quad (24)$$
where

\[ I_1^{(b)}(y) = \int_{\gamma_y} \frac{z^{-y-1}}{\psi_y(z)} \, dz, \quad I_2^{(b)}(y, \varepsilon) = \int_{\gamma_y(\varepsilon)} \frac{z^{-y-1} r_y^{(b)}(z)}{(1 - A f_y^{(b)}(z)) \psi_y(z)} \, dz, \]

\[ I_3^{(b)}(y, \varepsilon) = -\int_{\gamma_y(\varepsilon)} \frac{z^{-y-1}}{\psi_y(z)} \, dz, \quad I_4^{(b)}(y, \varepsilon) = \int_{\gamma_y(\varepsilon)} \frac{z^{-y-1}}{1 - A f_y^{(b)}(z)} \, dz. \]

To calculate \( I_1^{(b)} \) let us examine integrals of the form

\[ \int_{|z|=c^2} \frac{z^{-n}}{dz + h} \, dz, \quad (25) \]

where \( n > 0, d, h \in \mathbb{C} \) and \( |h| < c^2|d| \). For \( |h| < c^2|d| \), the function \( z^{-n}/(dz + h) \) has exactly two singularities in the disc \( |z| \leq c^2 \), one in 0 and the other in \( -h/d \). Consequently the Residue theorem states that

\[ \int_{|z|=c^2} \frac{z^{-n}}{dz + h} \, dz = \text{Res} \left( \frac{z^{-n}}{dz + h}, 0 \right) + \text{Res} \left( \frac{z^{-n}}{dz + h}, \frac{h}{d} \right). \]

The pole in \( z = 0 \) has order \( n \), hence

\[ \text{Res} \left( \frac{z^{-n}}{dz + h}, 0 \right) = (-1)^{n-1} d^{n-1} h^{-n} \]

and the pole in \( z = -h/d \) is of order 1, thus

\[ \text{Res} \left( \frac{z^{-n}}{dz + h}, \frac{h}{d} \right) = (-1)^n d^{n-1} h^{-n}. \]

Therefore,

\[ \int_{|z|=c^2} \frac{z^{-n}}{dz + h} \, dz = [(-1)^{n-1} + (-1)^{n}]d^{n-1} h^{-n} = 0. \quad (26) \]

By the equicontinuity of \( \mu_y^{(b)}(\cdot) \), the result from \( \text{(1)} \), Lemma 3 and Lemma 4 as \( y \to \infty \),

\[ f_y^{(b)}(r_y) - f_y^{(b)}(\lambda_y^{(b)}(A)) = (r_y - \lambda_y^{(b)}(A))\mu_y^{(b)}(\lambda_y^{(b)}(A)) + o(r_y - \lambda_y^{(b)}(A)) \]

\[ = (r_y - \lambda_y^{(b)}(A))\mu_y^{(b)} + o(r_y - \lambda_y^{(b)}(A)) \quad (27) \]

uniformly in \( b \leq b_2 \) and \( A \in \mathfrak{A}_y \). By virtue of Lemma 3 and the definition of \( C_1, |\lambda_y^{(b)}(A)| \leq e^{h_y^{-1}/y} \) and consequently

\[ r_y - \lambda_y^{(b)}(A) \geq e^{h_y} (1 - e^{-1/y}) \]

\[ = (1 + h_y + o(y^{-1}))(y^{-1} + o(y^{-1})) = y^{-1} + o(y^{-1}) \]

uniformly in \( b \leq b_2 \) and \( A \in \mathfrak{A}_y \). By plugging this results into \( \text{(27)} \),

\[ 1 - A f_y^{(b)}(r_y) \leq -\frac{A \mu_y^{(b)}}{y} + o \left( \frac{1}{y} \right) < 0 \quad (28) \]

for \( y \) large enough. Now put \( h = 1 - A f_y^{(b)}(r_y) + A \mu_y^{(b)}(r_y) r_y \) and \( d = -A \mu_y^{(b)}(r_y) \). Then, since \( A \mu_y^{(b)}(r_y) r_y \geq A \mu_y^{(b)}(1) \neq o(1) \), we obtain by virtue of \( \text{(28)} \),

\[ |h| \leq A \mu_y^{(b)}(r_y) r_y = |d|r_y \]
and consequently by \(26\),
\[
I_1^{(b)}(y) = \int_{\gamma_y} \frac{z^{-y-1}}{\psi_y^{(b)}(z)} dz = 0. \tag{29}
\]

Let us now consider \(I_2^{(b)}\). Clearly,
\[
I_2^{(b)}(y, \varepsilon) = \frac{y}{\pi} \int_{\varepsilon}^{2\pi-\varepsilon} \frac{\varphi_y^{(b)}(r_y e^{it})}{1 - Af_y^{(b)}(r_y e^{it})} e^{-i y t} dt.
\]

Taibleson \[11\] estimate for Fourier coefficients states that for any function \(f\) with bounded variation on \([0, 2\pi]\) and \(f(x) \sim \sum_{n=\infty}^{\infty} c_n e^{inx}\) as \(x \to \infty\), it is
\[
|c_n| \leq \frac{2\pi \text{var}(f)}{n},
\]
where \(\text{var}\) denotes the variation of \(f\), defining this to be the sum of the variations of the real and the imaginary parts. Hence,
\[
I_2^{(b)}(y, \varepsilon) = O \left( \frac{1}{\pi} \text{var} \left( \frac{\varphi_y^{(b)}(z)}{1 - Af_y^{(b)}(z)\psi_y^{(b)}(z)} \right) \right). \tag{30}
\]

The variation of \(\omega_y^{(b)}(z) := \varphi_y^{(b)}(z)/(1 - Af_y^{(b)}(z)\psi_y^{(b)}(z))\) on \(\gamma_y(\varepsilon)\) can be rewritten as follows:
\[
\text{var} \left( \omega_y^{(b)}(z) \right) = \text{var} \left( \text{Re}(\omega_y^{(b)}(z)) \right) + \text{var} \left( \text{Im}(\omega_y^{(b)}(z)) \right)
= \int_{\gamma_y(\varepsilon)} \left( \left| \frac{d}{dl} \text{Re}(\omega_y^{(b)}(z)) \right| + \left| \frac{d}{dl} \text{Im}(\omega_y^{(b)}(z)) \right| \right) dl,
\]
where \(dl\) is the differential of the arc along \(\gamma_y(\varepsilon)\). Due to the binomial formula,
\[
\left( \left| \frac{d}{dl} \text{Re}(\omega_y^{(b)}(z)) \right| + \left| \frac{d}{dl} \text{Im}(\omega_y^{(b)}(z)) \right| \right)^2 \leq 2 \left( \left| \frac{d}{dl} \text{Re}(\omega_y^{(b)}(z)) \right|^2 + \left| \frac{d}{dl} \text{Im}(\omega_y^{(b)}(z)) \right|^2 \right)
= 2 \left| \frac{d}{dz} \omega_y^{(b)}(z) \right|^2
\]
and thus,
\[
\text{var} \left( \frac{\varphi_y^{(b)}(z)}{1 - Af_y^{(b)}(z)\psi_y^{(b)}(z)} \right) \leq \sqrt{2} \int_{\gamma_y(\varepsilon)} \left| \frac{d}{dz} \frac{\varphi_y^{(b)}(z)}{1 - Af_y^{(b)}(z)\psi_y^{(b)}(z)} \right| dz
\]
\[
\leq \sqrt{2} \left( \int_{\gamma_y(\varepsilon)} \left| \frac{\psi_y^{(b)}(z)\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z)\psi_y^{(b)}(z))^2} \right| dz + \int_{\gamma_y(\varepsilon)} \left| \frac{A\mu_y^{(b)}(z)\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z))^2\psi_y^{(b)}(z)} \right| dz \right.
\]
\[
\left. + \int_{\gamma_y(\varepsilon)} \left| \frac{\varphi_y^{(b)}(z)}{(1 - Af_y^{(b)}(z)\psi_y^{(b)}(z))} \right| dz \right) = \sqrt{2}(I_{21}^{(b)} + I_{22}^{(b)} + I_{23}^{(b)}). \tag{31}
\]
Let us bound the terms appearing in the integrands of the integrals from the latter inequality. Using the definition of the complex absolute value, an easy calculation
shows that
\[ |Af_y^{(b)}(z) - 1|^2 = A^2|f_y^{(b)}(z) - f_y^{(b)}(r_y)|^2 + |Af_y^{(b)}(r_y) - 1|^2 \]
\[ - 2A(Af_y^{(b)}(r_y) - 1)\text{Re}(f_y^{(b)}(r_y) - f_y^{(b)}(z)). \]
By the equicontinuity and Lemma 3 as \( y \to \infty \),
\[ |f_y^{(b)}(r_y) - f_y^{(b)}(z)| = |r_y - z|\mu^{(b)}(z) + o(r_y - z) \geq (1 - \delta)\mu^{(b)}|z - r_y| \] (32)
and
\[ |f_y^{(b)}(r_y) - f_y^{(b)}(z)| = |r_y - z|\mu^{(b)}(z) + o(r_y - z) \leq (1 + \delta)\mu^{(b)}|z - r_y| \] (33)
uniformly in \( b \leq b_2 \) and \( z \in U_y^{(b)}(\delta) \), if \( \delta \) is small enough. Further, for all \( z \in U_y(\delta) \) with \( \delta \) sufficiently small,
\[ \text{Re}(r_y - z) = \sin(\arg z)|z - r_y| \leq \delta|z - r_y|. \]
By the virtue of (28), (32) and (33),
\[ |Af_y^{(b)}(z) - 1|^2 \geq |1 - Af_y^{(b)}(r_y)|^2 + (1 - \delta)(\mu^{(b)})^2A^2|z - r_y|^2 \]
\[ - 2\delta(1 + \delta)\mu^{(b)}(A|f_y^{(b)}(r_y) - 1|)|z - r_y| \]
and by the binomial formula for \( \delta \) sufficiently small,
\[ 2\mu^{(b)}(A|f_y^{(b)}(r_y) - 1|)|z - r_y| \leq (\mu^{(b)})^2A^2|z - r_y|^2 + |1 - Af_y^{(b)}(r_y)|^2. \]
Therefore, again by the binomial formula,
\[ |Af_y^{(b)}(z) - 1|^2 \geq (1 - \delta - \delta(1 + \delta)) \left[ |1 - Af_y^{(b)}(r_y)|^2 + (\mu^{(b)})^2A^2|z - r_y|^2 \right] \]
\[ \geq \frac{1 - \delta - \delta(1 + \delta)}{2} \left[ |1 - Af_y^{(b)}(r_y)| + A\mu^{(b)}|z - r_y| \right]^2. \]
Since \( \delta \) can be chosen arbitrary small, one can especially choose \( \delta \) so small that
\( 1 - \delta - \delta(1 + \delta) \geq 1/2. \) Thus,
\[ |Af_y^{(b)}(z) - 1| \geq \frac{|Af_y^{(b)}(r_y) - 1|}{2} + \frac{A\mu^{(b)}|z - r_y|}{2} \] (34)
uniformly in \( b \leq b_2 \) and \( z \in U_y^{(b)}(\delta) \). We proceed analogously to bound |\( \psi_y^{(b)}(z) \)| for \( z \in U_y^{(b)}(\delta) \) from below. It is \( \text{Re}(r_y - z) \leq |z - r_y| \) and by virtue of Lemma 4
\( \mu_y^{(b)}(r_y) \in [(1 - \delta_1)\mu^{(b)}, (1 + \delta_1)\mu^{(b)}] \) for \( y \) large enough. Consequently, one can easily see that for \( \delta_2 \) small enough,
\[ |\psi_y^{(b)}(z)|^2 = |1 - f_y^{(b)}(r_y)|^2 + A^2\left(\mu_y^{(b)}(r_y)\right)^2|z - r_y|^2 \]
\[ - 2A(f_y^{(b)}(r_y) - 1)\mu_y^{(b)}(r_y)\text{Re}(r_y - z) \]
\[ \geq \frac{1 - \delta_2}{2} \left[ |1 - f_y^{(b)}(r_y)| + A\mu^{(b)}|z - r_y| \right]^2 \] (35)
for all \( \delta_2 \leq 1/2. \) Hence,
\[ |\psi_y^{(b)}(z)| \geq \frac{|1 - Af_y^{(b)}(r_y)|}{4} + \frac{A\mu^{(b)}|z - r_y|}{4}. \] (36)
On the other hand, one can easily see that for every \( z \) on \( \gamma_y(\varepsilon) \) with \( \varepsilon \) sufficiently small,

\[
|z - y| \geq |e^{i \arg z} - 1| = \sqrt{\sin^2(\arg z) + (1 - \cos(\arg z))^2} \\
= \sqrt{2 - 2\cos(\arg z)} \geq \frac{\arg z}{2},
\]

(37)

where we used \( \cos \varphi \leq 1 - \varphi^2/8 \) in the last inequality. Combining inequalities (\ref{eq:28}) and (\ref{eq:37}) with (\ref{eq:34}), we obtain

\[
|1 - A f_y^{(b)}(z)| \geq \frac{A \mu(y)}{4} \left( \frac{1}{y} + |\arg z| \right),
\]

(38)

for \( b \leq b_2 \) and \( z \in U_y^{(b)}(\delta) \). The inequalities (\ref{eq:28}), (\ref{eq:37}) and (\ref{eq:36}) provide

\[
|\psi_y^{(b)}(z)| \geq \frac{A \mu(y)}{8} \left( \frac{1}{y} + |\arg z| \right)
\]

and, moreover, an easy calculation shows

\[
|\psi_y^{(b)}(z)| = A \mu(y)(y) \leq e^{h_y y} A \mu(y).
\]

(40)

For \( z \in U_y(y) \),

\[
|f_y^{(b)}(z)| \leq \left\{ \begin{array}{ll}
e^{h_y y} E(Z^{(b)})^2 & : s \geq 2, \\
e^{h_y y^2 s} E(Z^{(b)})^s & : 1 < s < 2,
\end{array} \right.
\]

and consequently,

\[
\frac{\varphi_y^{(b)}(z)}{|z - y|^{2y \max(0, 2 - s)}} \sim \frac{\varphi_y^{(b)}(z)}{2|z - y| y^{\max(0, 2 - s)} \sim \frac{A f_y^{(b)}(z)}{2y^{\max(0, 2 - s)}} = O(1)
\]

as \( y \to \infty \). By virtue of (\ref{eq:57}),

\[
|e^{i \arg z}| = \sqrt{2 - 2\cos(\arg z)} \leq \arg z,
\]

if \( \arg z \) is sufficiently small. Hence, if \( \arg z \) is is sufficiently small,

\[
\varphi_y^{(b)}(z) = O(y^{\max(0, 2 - s)}|z - y|^2) = O(y^{\max(0, 2 - s)} \arg^2(z))
\]

(41)

and

\[
\varphi_y^{(b)}(z) = O(y^{\max(0, 2 - s)}|z - y|) = O(y^{\max(0, 2 - s)}|\arg(z)|)
\]

(42)

uniformly in \( b \leq b_2 \) and \( A \in \mathfrak{A}_y \). Considering (\ref{eq:38}), (\ref{eq:39}), (\ref{eq:40}), (\ref{eq:41}) and \( h_y y = O(1) \) provides

\[
|f_y^{(b)}| \leq r_y \int_{-\varepsilon}^{\varepsilon} \frac{d\psi_y^{(b)}(r_y e^{it})}{|\psi_y^{(b)}(r_y e^{it})|} dt = O\left( y^{\max(0, 2 - s)} \int_{0}^{\varepsilon} \frac{t^2}{(y^{-1} + t)^3} dt \right)
\]

uniformly in \( b \leq b_2 \) and \( A \in \mathfrak{A}_y \). Moreover,

\[
\int_{0}^{\varepsilon} \frac{t^2}{(y^{-1} + t)^3} dt = \int_{1/y}^{\varepsilon + 1/y} \frac{(w - y^{-1})^2}{w^3} dw \\
\sim \ln(\varepsilon + y^{-1}) - \ln(y^{-1}) = \ln(1 + \varepsilon y) \sim \ln(y)
\]

(43)

and therefore, uniformly in \( b \leq b_2 \) and \( A \in \mathfrak{A}_y \),

\[
|f_y^{(b)}| = O(y^{\max(0, 2 - s)} \ln y).
\]

(44)
In analogy, by additionally taking into account that \( \mu_y(b) \leq 2 \mu(b) \) due to Lemma 3 for \( y \) large enough, one can easily see that

\[
I_{22}^{(b)} = O(y_{\max{0.2-s}} \ln y)
\]  

and furthermore, by regarding (44),

\[
I_{23}^{(b)} = O \left( y_{\max{0.2-s}} \int_0^\varepsilon \frac{t}{(y^{-1} + t)^2} dt \right) = O(y_{\max{0.2-s}} \ln y).
\]  

Finally, plugging (44), (45) and (46) into (31) we attain

\[
\text{In analogy to (31), one can show that}
\]

\[
\var_{z \in \gamma_y(\varepsilon)} \frac{\varphi_y(b)(z)}{(1 - A_f y(b))\psi_y(b)(z)} = O(y_{\max{0.2-s}} \ln y)
\]

and hence by (30),

\[
|I_2^{(b)}(y, \varepsilon)| = o(y_{\max{-1,-(s-1)}} \ln y)
\]

uniformly in \( b \leq b_2 \) and the admissible values of \( A \). Next, we draw our attention to the integral \( I_3^{(b)} \).

\[
I_3^{(b)}(y, \varepsilon) = -i r_y^{-y} \int_{\varepsilon \leq |t| \leq \pi} \frac{e^{-iyt}}{\psi_y(b)(r_y e^{it})} dt.
\]  

To bound this integral we use Taibleson's estimate for Fourier coefficients again:

\[
\int_{\varepsilon \leq |t| \leq \pi} \frac{e^{-iyt}}{\psi_y(b)(r_y e^{it})} dt = O \left( \frac{1}{y \var_{z \in \gamma_y(\varepsilon)} \frac{1}{\psi_y(b)(z)} \right)
\]  

In analogy to (41), one can show that

\[
\var_{z \in \gamma_y(\varepsilon)} \frac{1}{\psi_y(b)(z)} \leq \sqrt{2} \int_{\gamma_y(\varepsilon)} \left| \frac{d}{dz} \frac{1}{\psi_y(b)(z)} \right| dz \leq \sqrt{2} \int_{\gamma_y(\varepsilon)} \frac{|\psi_y(b)(z)|}{|\psi_y(b)(z)|^2} dz.
\]

By (40),

\[
|\psi_y(b)(z)|^2 \geq A^2(\mu(b))^2 \geq \frac{A^2(\mu(b))^2}{16} |z - r_y|^2 \geq \frac{A^2(\mu(b))^2}{16} \varepsilon^2,
\]

where we used that \( |z - r_y| > \varepsilon \) for all \( z \in \gamma_y(\varepsilon) \). Therefore, by (40),

\[
\var_{z \in \gamma_y(\varepsilon)} \frac{1}{\psi_y(b)(z)} = O(1)
\]

and consequently by combining this result with (48), (49) and \( h_y y = O(1) \),

\[
|I_3^{(b)}(y, \varepsilon)| = O \left( \frac{1}{y} \right)
\]  

uniform in \( b \leq b_2 \) and \( A \in A_y \). It remains to consider \( I_4^{(b)} \).

\[
|I_4^{(b)}(y, \varepsilon)| = \left| i r_y^{-y} \int_{\varepsilon \leq |t| \leq \pi} \frac{e^{-iyt}}{1 - A_f y(b)(r_y e^{it})} dt \right| = O \left( \frac{1}{y \var_{z \in \gamma_y(\varepsilon)} \frac{1}{1 - A_f y(b)(z)}} \right).
\]  

(51)
Further, by (14) and (19),

\[
\frac{\text{var}}{\tau_y(z)} \frac{1}{1 - Af_y(b)(z)} \leq \sqrt{2} \int_{\tau_y(z)} d\tau_y(z) \frac{1}{1 - Af_y(b)(z)} |d\tau_y(z)| = O(1)
\]

and consequently

\[
I_4(b, y, \varepsilon) = O\left(\frac{1}{y}\right).
\]

Finally, by plugging the results attained in (29), (47), (50) and (52) into (24), we get

\[
\int_{\gamma_y} z^{-y-1} dz = o(y^{\max\{-1, -(s-1)\}} \ln y) + o(y^{-(s-1)}) + O(y^{-1})
\]

\[
= o(y^{-\min\{1, s-1\}} \ln y)
\]

uniformly in \(0 \leq b \leq b_2\) and \(A_y \leq A \leq 1\).

3. Proof of the local limit theorem

Put \(\tau^{(a)} = 0\) and define recursively for \(i \geq 1\) the \(i\)-th strict ascending ladder epoch of the random walk \(S^{(a)}\) and its corresponding ladder height by

\[
\tau^{(a)}_{+, i} := \min\{k \geq \tau^{(a)}_{+, i-1} : S^{(a)}_k > S^{(a)}_{\tau^{(a)}_{+, i-1}}\} \quad \text{and} \quad \chi^{(a)}_i = S^{(a)}_{\tau^{(a)}_{+, i}} - S^{(a)}_{\tau^{(a)}_{+, i-1}}.
\]

In the case \(i = 1\) we write \(\tau^{(a)}_{+, 1}\) and \(\chi^{(a)}_1\) instead of \(\tau^{(a)}_{+, 1}\) and \(\chi^{(a)}\) respectively and, if additionally \(a = 0\), we write \(\tau^{(b)}_+\) and \(\chi^{(b)}\) instead of \(\tau^{(b)}_{+, 1}\) and \(\chi^{(b)}\) respectively. Define random variables \(Z_i^{(a)}\) as iid copies of a random variable \(Z^{(a)}\) with

\[
\mathbb{P}(Z^{(a)} \in \cdot) = \mathbb{P}(\chi^{(a)}_1 \in \cdot | \tau^{(a)}_{+, 1} < \infty).
\]

Denote by \(\theta := \min\{k \geq 0 : S^{(a)}_k = M^{(a)}\}\) the first time the random walk reaches its maximum. Then,

\[
\mathbb{P}(M^{(a)} = y \Delta) = \sum_{n=1}^{\infty} \mathbb{P}(M^{(a)} = y \Delta, \theta = n).
\]

We further define \(M^{(a)} := \max_{k \leq n} S^{(a)}_k\) and \(\theta_n := \min\{k \leq n : S^{(a)}_k = M^{(a)}\}\). By the Markov property,

\[
\mathbb{P}(M^{(a)} = y \Delta, \theta = n) = \mathbb{P}(S^{(a)}_n = y \Delta, \theta_n = n) \mathbb{P}(\tau^{(a)}_+ = \infty).
\]

Hence the following representation holds for the maximum:

\[
\mathbb{P}(M^{(a)} = y \Delta) = \mathbb{P}(\tau^{(a)}_+ = \infty) \sum_{n=1}^{\infty} \mathbb{P}(S^{(a)}_n = y \Delta, \theta_n = n).
\]

Clearly,

\[
\mathbb{P}(S^{(a)}_n = y \Delta, \theta_n = n) = \mathbb{P}(S^{(a)}_n = y \Delta, n \text{ is a strict ascending ladder epoch})
\]

\[
= \sum_{k=1}^{\infty} \mathbb{P}(\chi^{(a)}_1 + \chi^{(a)}_2 + \cdots + \chi^{(a)}_k = y \Delta, \tau^{(a)}_{+, 1} + \tau^{(a)}_{+, 2} + \cdots + \tau^{(a)}_{+, k} = n).
\]

\[
(54)
\]
Denote the distribution function of $Z^{(a)}$ by $F^{(\mu^{(a)})}$, where $\mu^{(a)} = E[Z^{(a)}]$, and let $F^{(\mu^{(a)})}_k$ be the k-fold convolution of $F^{(\mu^{(a)})}$ with itself. Then, by using (55), changing the order of summation and using the Markov property,

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} P(S_n^{(a)} = y \Delta, \theta_n = n) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} P(\chi_1^{(a)} + \chi_2^{(a)} + \cdots + \chi_k^{(a)} = y \Delta, \tau_{+1}^{(a)} + \tau_{+2}^{(a)} + \cdots + \tau_{+k}^{(a)} = n)$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} P(\chi_1^{(a)} + \chi_2^{(a)} + \cdots + \chi_k^{(a)} = y \Delta | \tau_{+k}^{(a)} < \infty) P(\tau_{+k}^{(a)} < \infty)$$

$$= \sum_{k=1}^{\infty} A^k \left( F_k^{(\mu^{(a)})}(y \Delta) - F_k^{(\mu^{(a)})}((y - 1) \Delta) \right)$$

(56)

with $A = P(\tau_{+}^{(a)} < \infty)$. Combining results (54) and (56) we attain

$$P(M^{(a)} = y \Delta) = P(\tau_{+}^{(a)} = \infty) \sum_{k=1}^{\infty} A^k \left( F_k^{(\mu^{(a)})}(y \Delta) - F_k^{(\mu^{(a)})}((y - 1) \Delta) \right).$$

(57)

Next, we want to use Proposition 2 to determine the asymptotic behaviour of the sum on the right hand side of the latter equality. Therefore, let us first show that under the assumptions of Theorem 1,

$$Z^{(a)} \Rightarrow Z^{(0)}$$

(58)

as $a \to 0$. It is known that

$$P(\tau_{+}^{(a)} < \infty) \sim P(\tau_{+} < \infty) = 1.$$ 

(59)

Thus, as $a \to 0$,

$$P(Z^{(a)} > x) = \frac{P(\chi^{(a)} > x, \tau_{+}^{(a)} < \infty)}{P(\tau_{+}^{(a)} < \infty)} \sim P(\chi^{(a)} > x, \tau_{+}^{(a)} < \infty)$$

and, on the other hand, (1) and (59) imply that for every $R > 0$, as $a \to 0$,

$$P(\chi^{(a)} > x, R < \tau_{+}^{(a)} < \infty) \leq P(R < \tau_{+}^{(a)} < \infty)$$

$$= P(\tau_{+}^{(a)} < \infty) - P(\tau_{+}^{(a)} \leq R) \sim P(\tau_{+} > R).$$

Further, by using (1) and the continuous mapping theorem,

$$P(\chi^{(a)} > x, \tau_{+}^{(a)} \leq R) = \sum_{k=0}^{R-1} P(S_{k+1}^{(a)} > x, \max_{1 \leq l \leq k} S_l^{(a)} \leq 0)$$

$$\sim \sum_{k=0}^{R-1} P(S_{k+1} > x, \max_{1 \leq l \leq k} S_l \leq 0) = P(\chi > x, \tau_{+} \leq R)$$

as $a \to 0$. Thus,

$$\limsup_{a \to 0} P(Z^{(a)} > x) \leq P(\chi > x, \tau_{+} \leq R) + P(\tau_{+} > R)$$

and by letting $R \to \infty$ we conclude

$$\limsup_{a \to 0} P(Z^{(a)} > x) \leq P(\chi > x, \tau_{+} < \infty) = P(Z^{(0)} > x).$$
On the other side, the above calculations give
\[ \liminf_{a \to 0} P(Z^{(a)} > x) \geq \liminf_{a \to 0} P(\chi^{(a)} > x, \tau_+^{(a)} \leq R) = P(\chi > x, \tau_+ \leq R) \]
and by letting \( R \to \infty \),
\[ \liminf_{a \to 0} P(Z^{(a)} > x) \geq P(\chi > x, \tau_+ < \infty) = P(Z^{(0)} > x). \]
This means that (58) holds under our assumptions.

Due to relation (16) of Chow \[4\] there exists a constant \( C \) such that
\[
E[(S^{(a)}_\tau)^{1+\varepsilon}; \tau_+ < \infty] \leq C \int_0^\infty \frac{u^{2+\varepsilon}}{E[|S^{(a)}_\tau| \wedge u]} dP(\max\{0, X^{(a)}\} < u)
\]
Obviously,
\[
E[|S^{(a)}_\tau| \wedge u] \geq E[|S^{(a)}_\tau| \wedge \Delta] \geq P(S_1^{(a)} < 0) > 0
\]
for all \( u \geq \Delta \) and therefore
\[
E[(S^{(a)}_\tau)^{1+\varepsilon}; \tau_+ < \infty] \leq \frac{C}{P(S_1^{(a)} < 0)} \int_0^\infty u^{2+\varepsilon} dP(\max\{0, X^{(a)}\} < u).
\]
Hence, by virtue of (2),
\[
\sup_{a \leq a_0} E[(Z^{(a)})^{1+\varepsilon}] < \infty. \tag{60}
\]
The convergence from (58) combined with (60) implies
\[
\mu^{(a)} \to \mu^{(0)} \tag{61}
\]
as \( a \to 0 \) by dominated convergence. It is known that for all \( a > 0 \) the stopping time \( \tau_+^{(a)} \) is infinite with positive probability and that
\[ P(\tau_+^{(a)} = \infty) = 1/E[\tau_+^{(a)}], \tag{62} \]
where \( \tau_+^{(a)} = \min\{k \geq 1 : S_k^{(a)} \leq 0\} \) is the first weak descending ladder epoch.
Totally analogously to (60), one can use (15) from Chow \[4\] to show that the existence of the second moment in assumption (2) implies \( \sup_{a \leq a_0} E[S^{(a)}_{\tau_+^{(a)}}] < \infty. \)
Hence, one can use dominated convergence to show that
\[ E[S^{(a)}_{\tau_+^{(a)}}] \to E[S_{\tau_+^{(0)}}] \]
as \( a \to 0 \). Thus, using (62), the known identity
\[ \frac{\sigma^2}{2} = -\mu^{(0)} E[S_{\tau_+^{(0)}}] \tag{63} \]
and Wald’s identity imply that
\[ P(\tau_+^{(a)} = \infty) = \frac{1}{E[\tau_+^{(a)}]} \sim \frac{a}{-E[S_{\tau_+^{(0)}}]} \sim \frac{2a\mu^{(0)}}{\sigma^2}. \tag{64} \]
The assumption \( ay = O(1) \) implies the existence of a constant \( C \) such that \( y \leq C/a \). Therefore, by (64),
\[ P(\tau_+^{(a)} < \infty) \geq 1 - \frac{3C\mu^{(0)}}{\sigma^2 y} \tag{65} \]
for a small enough. Summing up the results from \( \text{(61)} \) and \( \text{(63)} \), this means that we can apply Proposition 2 for \( I = \{\mu^{(a)} : 0 \leq a \leq a_0\} \) with \( a_0 > 0 \) small enough, \( A_y = 1 - 3C\mu^{(0)}/(\sigma^2y) \), \( A = \mathbb{P}(r_+^{(a)} < \infty) \) and \( s = 1 + \varepsilon \). Hence,

\[
\sum_{k=1}^{\infty} A^k \left( F_k^{(\mu^{(a)})}(y\Delta) - F_k^{(\mu^{(a)})}((y - 1)\Delta) \right) = \frac{(\lambda^{(a)}_y(A))^{-1}}{A\mu^{(a)}_y(\lambda^{(a)}_y(A))} + o(y^{-\min(1,\varepsilon)} \ln y)
\]

and consequently, by combining equations \( \text{(67)} \), \( \text{(66)} \) and the fact that \( 1 - A = O(a) \), we attain

\[
\mathbb{P}(M^{(a)} = y\Delta) = (1 - A) \frac{(\lambda^{(a)}_y(A))^{-1}}{A\mu^{(a)}_y(\lambda^{(a)}_y(A))} + o(y^{-\min(1,\varepsilon)} \ln y).
\]

Let us now determine \( \lambda^{(a)}_y(A) \) and \( \mu^{(a)}_y(\lambda^{(a)}_y(A)) \). Write \( \lambda_y\Delta \) and \( \mu_y(\lambda_y\Delta) \) instead of \( \lambda^{(a)}_y(A) \) and \( \mu^{(a)}_y(\lambda^{(a)}_y(A)) \) respectively for abbreviation and put \( \lambda_y\Delta = e^{\theta_y\Delta} \).

According to the definition of \( \lambda_y\Delta \), we want to find \( \theta_y\Delta \) such that

\[
\mathbb{E}[\exp(\theta_y\Delta Z^{(a)}/\Delta); Z^{(a)} \leq y\Delta] = \frac{1}{A}.
\]

It turns out we don’t need an exact solution for this equation and it is sufficient to determine \( \theta_y \) such that

\[
\mathbb{E}[\exp(\theta_y\Delta Z^{(a)}/\Delta); Z^{(a)} \leq y\Delta] = \frac{1}{A} + O(y^{-1-\varepsilon}).
\]

By Taylor’s formula,

\[
\mathbb{E}[\exp(\theta_y\Delta Z^{(a)}/\Delta); Z^{(a)} \leq y\Delta]
\]

\[
= 1 + \frac{\theta_y\Delta \mu^{(a)}_y}{\Delta} - \mathbb{P}(Z^{(a)} > y\Delta) - \frac{\theta_y\Delta}{\Delta} \mathbb{E}[Z^{(a)}; Z^{(a)} > y\Delta]
\]

\[
+ \frac{\theta^2_y\Delta}{2\Delta^2} \mathbb{E}[(Z^{(a)})^2 \exp\{\gamma\theta_y\Delta Z^{(a)}/\Delta\}; Z^{(a)} \leq y\Delta]
\]

with some random \( \gamma \in (-\infty, 1) \). We restrict ourselves to \( \theta_y\Delta \) such that \( \theta_y\Delta = O(1/y) \). Then, \( \text{(60)} \) implies

\[
\mathbb{P}(Z^{(a)} > y\Delta) + \frac{\theta_y\Delta}{\Delta} \mathbb{E}[Z^{(a)}; Z^{(a)} > y\Delta] = O(y^{-1-\varepsilon})
\]

and

\[
\frac{\theta^2_y\Delta}{2\Delta^2} \mathbb{E}[(Z^{(a)})^2 \exp\{\gamma\theta_y\Delta Z^{(a)}\}; Z^{(a)} \leq y\Delta]
\]

\[
= O \left( \theta^2_y\Delta \mathbb{E}[(Z^{(a)})^2; Z^{(a)} \leq y\Delta] \right) = O(y^{-1-\varepsilon}).
\]

This means that to find \( \theta_y \) that suffices \( \text{(69)} \), it is sufficient to choose \( \theta_y \) such that

\[
1 + \frac{\theta_y\Delta \mu^{(a)}_y}{\Delta} = \frac{1}{A} + O(y^{-1-\varepsilon})
\]

or

\[
\theta_y\Delta = \frac{(1 - A)\Delta}{A\mu^{(a)}_y} + O(y^{-1-\varepsilon}).
\]

Consequently,

\[
\lambda_y\Delta = \exp \left\{ \frac{(1 - A)\Delta}{A\mu^{(a)}_y} + O(y^{-1-\varepsilon}) \right\}.
\]
Further,
\[
\mu_{y\Delta}^{(a)}(\lambda_{y\Delta}) = \sum_{k=1}^{y} k f^{(a)}_{k \Delta} \frac{1}{\Delta \lambda_{y\Delta}} E[Z^{(a)} \exp\{\theta_{y\Delta} Z^{(a)}/\Delta\} ; Z^{(a)} \leq y\Delta]
\]
\[
= \frac{1}{\Delta \lambda_{y\Delta}} \left\{ E[Z^{(a)}, Z^{(a)} \leq y\Delta] + \frac{\theta_{y\Delta}}{\Delta} E[(Z^{(a)})^{2} \exp\{\tilde{\gamma}_{y\Delta} Z^{(a)}/\Delta\} ; Z^{(a)} \leq y\Delta] \right\}
\]
for some random \( \tilde{\gamma} \in (-\infty, 1] \). For all \( \theta_{y\Delta} = O(1/y) \) the result (60) gives
\[
E[(Z^{(a)})^{2} \exp\{\tilde{\gamma}_{y\Delta} Z^{(a)}/\Delta\} ; Z^{(a)} \leq y\Delta] = O(y^{1-\varepsilon})
\]
and
\[
E[Z^{(a)} ; Z^{(a)} \leq y\Delta] = \mu^{(a)} + O(y^{-\varepsilon}).
\]
Consequently,
\[
\mu_{y\Delta}^{(a)}(\lambda_{y\Delta}) = \frac{\mu^{(a)}}{\Delta \lambda_{y\Delta}} + O(y^{-\varepsilon}). \tag{71}
\]
Plugging the results from (70) and (71) into the right hand side of (67), we obtain by regarding \( 1 - A = O(a) \),
\[
P(M^{(a)} = y\Delta)
\]
\[
\begin{align*}
&= \frac{(1 - A)\Delta}{A \mu^{(a)} + O(y^{-\varepsilon})} \exp \left\{ -\frac{(1 - A)y\Delta}{A \mu^{(a)}} + O(y^{-\varepsilon}) \right\} + o(ay^{-\min\{1, \varepsilon\}} \ln y) \\
&= \frac{(1 - A)\Delta}{A \mu^{(a)} + O(y^{-\varepsilon})} \exp \left\{ -\frac{(1 - A)y\Delta}{A \mu^{(a)}} \right\} + o(ay^{-\min\{1, \varepsilon\}} \ln y) \\
&= \frac{(1 - A)\Delta}{A \mu^{(a)}} \exp \left\{ -\frac{(1 - A)y\Delta}{A \mu^{(a)}} \right\} + o(ay^{-\min\{1, \varepsilon\}} \ln y) + O(ay^{-\varepsilon}) \tag{72}
\end{align*}
\]
uniformly for all \( y \) such that \( ay = O(1) \) as \( a \to 0 \). Here, we applied Taylor’s formula in the last line. As a consequence of (59), (61) and (64),
\[
\frac{1 - A}{A \mu^{(a)}} = \frac{2a}{\sigma^{2}} + o(a)
\]
and hence, by plugging this result into (72), we finally obtain
\[
P(M^{(a)} = y\Delta) \sim \frac{2a\Delta}{\sigma^{2}} \exp \left\{ -\frac{2ay\Delta}{\sigma^{2}} \right\}
\]
uniformly for all \( y \) such that \( y \to \infty \) and \( ya = O(1) \) as \( a \to 0 \).

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