UNIPOTENT ELEMENTS AND TWISTING IN LINK HOMOLOGY

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Abstract. Let $\mathcal{U}$ be the unipotent variety of a complex reductive group $G$. Fix opposed Borel subgroups $B_\pm \subseteq G$ with unipotent radicals $U_\pm$. The map that sends $x_+x_- \mapsto x_+x_-x_+^{-1}$ for all $x_\pm \in U_\pm$ restricts to a map from $U_+U_- \cap gB_+$ into $\mathcal{U} \cap gB_+$, for any $g$. We conjecture that the restricted map forms half of a homotopy equivalence between these varieties, and thus, induces a weight-preserving isomorphism between their compactly-supported cohomologies. Noting that the map is equivariant with respect to certain actions of $B_+ \cap gB_+g^{-1}$, we prove for type $A$ that an equivariant analogue of this isomorphism exists. Curiously, this follows from a certain duality in Khovanov–Rozansky homology, a tool from knot theory.

1. Introduction

1.1. Let $G$ be a complex, connected, reductive algebraic group. Let $B_+$ and $B_-$ be opposed Borel subgroups of $G$, and let $U_\pm$ be the unipotent radical of $B_\pm$. For instance, if $G$ is the group of invertible $n \times n$ matrices $GL_n$, then we can choose $U_+$, resp. $U_-$, to be the subgroup of upper-triangular, resp. lower-triangular, matrices with 1’s along the diagonal. Every element of $U_+U_-$ can be written uniquely in the form $x_+x_-$, where $x_\pm \in U_\pm$.

Let $\mathcal{U} \subseteq G$ be the closed subvariety of unipotent elements. With the notation above, there is a map $\Phi : U_+U_- \to \mathcal{U}$ defined by

$$\Phi(x_+x_-) = x_+x_-x_+^{-1}.$$  

Note that $U_+U_-$ is an affine space, whereas $\mathcal{U}$ is usually singular. Nonetheless, in the analytic topology, the sets $U_+U_-(\mathbb{C})$ and $\mathcal{U}(\mathbb{C})$ are homotopy equivalent, as they are both contractible.

For any $g \in G(\mathbb{C})$, the map $\Phi$ restricts to a map $\Phi_g : \mathcal{U}_g \to \mathcal{U}_g$, where

$$\mathcal{U}_g = \mathcal{U} \cap gB_+,$$

$$\mathcal{V}_g = U_+U_- \cap gB_+.$$

Note that $\mathcal{U}_g, \mathcal{V}_g, \Phi_g$ only depend on the coset $gB_+$. We endow $\mathcal{U}_g(\mathbb{C})$ and $\mathcal{V}_g(\mathbb{C})$ with the analytic topology. The new idea proposed in this note is:

**Conjecture 1.** The map $\Phi_g$ defines half of a homotopy equivalence between $\mathcal{U}_g(\mathbb{C})$ and $\mathcal{V}_g(\mathbb{C})$.

The simplest case is $g \in B_+(\mathbb{C})$. Here, $\mathcal{U}_g(\mathbb{C}) = U_+(\mathbb{C}) = \mathcal{V}_g(\mathbb{C})$ and $\Phi_g$ is a retract from the affine space $U_+(\mathbb{C})$ onto the point corresponding to the identity of $G(\mathbb{C})$. In general, neither $\mathcal{U}_g(\mathbb{C})$ nor $\mathcal{V}_g(\mathbb{C})$ will be contractible, as we will show in Section 3 through examples.
We emphasize that for an arbitrary map of varieties $\Phi : Y \to X$ that induces a homotopy equivalence on $\mathbb{C}$-points, there may not exist a nonconstant map of varieties $X \to S$ such that $\Phi$ induces a homotopy equivalence of fibers $Y_s(\mathbb{C}) \to X_s(\mathbb{C})$ for every $s \in S(\mathbb{C})$. For instance, take $\Phi$ to be the projection from a quadric cone onto its axis of symmetry. Both total spaces are contractible, so $\Phi$ is automatically a homotopy equivalence. For the map of fibers over $s$ to be a homotopy equivalence as well, $X_s(\mathbb{C})$ must contract onto the origin of $X(\mathbb{C}) = \mathbb{C}$. Since $X \to S$ is nonconstant, some $s$ must violate this condition.

1.2. Some motivation for Conjecture 1 comes from classical results about finite groups of Lie type. To state them, let us implicitly replace $G$ with its split form over a finite field $F$ of good characteristic.

In [S, Thm. 15.1], Steinberg showed the identity $|U(F)| = |U_+U_-(F)|$. In [Ka, §4], Kawanaka showed an identity equivalent to $|U_g(F)| = |V_g(F)|$; (1.1) see Remark 22. More recently, Lusztig has given a new proof of (1.1) in [L].

Conjecture 1 essentially implies (1.1). Indeed, once one checks that $U_g$ and $V_g$ have the same dimension, Conjecture 1 implies that $\Phi_g$ induces an isomorphism between the rational, compactly-supported cohomologies of $U_g$ and $V_g$. Since $\Phi_g$ is algebraic, this isomorphism must match their weight filtrations in the sense of mixed Hodge theory. One can further check that both sides of (1.1) are polynomial functions of $|F|$, so by the results explained in [Kat], the virtual weight polynomials of $U_g$ and $V_g$ specialize to their $F$-point counts.

Our main result is evidence for an equivariant analogue of Conjecture 1. Let $T = B_+ \cap B_-$, so that $B_+ = TU_+ \simeq T \rtimes U_+$. The map $\Phi$ transports the $B_+$-action on $U_+U_-$ defined by

$$tu \cdot x_+x_- = (tx_+t^{-1})(tx_-t^{-1})$$

for all $(t, u) \in T \times U_+$ onto the $B_+$-action on $\mathcal{U}$ by left conjugation. Setting

$$H_g = B_+ \cap gB_+g^{-1},$$

we find that these $B_+$-actions restrict to $H_g$-actions on $\mathcal{V}_g$ and $\mathcal{U}_g$, respectively. The $B_+$-equivariance of $\Phi$ thus restricts to $H_g$-equivariance of $\Phi_g$. Though we do not have a general theorem about $\Phi_g$ itself, we prove:

**Theorem 2.** If $G$ is a split reductive group of type $A$ over $F$, then for all $g \in G(F)$, there is an isomorphism of bigraded vector spaces

$$\text{gr}^W H_{c,H_g}^\bullet(\mathcal{U}_g, \bar{Q}_\ell) \simeq \text{gr}^W H_{c,H_g}^\bullet(\mathcal{V}_g, \bar{Q}_\ell),$$

where $H_{c,H_g}^\bullet(\cdot, \bar{Q}_\ell)$ denotes $H_g$-equivariant, compactly-supported $\ell$-adic cohomology and $W$ denotes its weight filtration.

The proof in Sections 4–5 uses ideas from the rather different world of low-dimensional topology, as we now explain.
Let \( \mathcal{X}_g \) be the variety of Borel subgroups of \( G \) in generic position with respect to both \( B_+ \) and \( gB_+g^{-1} \). In Section 4, we will construct an isomorphism \( \mathcal{X}_g \rightarrow \mathcal{V}_g \) that transports the \( H_g \)-action on \( \mathcal{X}_g \) by left conjugation to the \( H_g \)-action on \( \mathcal{V}_g \) described above.

The variety \( \mathcal{X}_g \) is closely related to the so-called braid varieties that have been studied recently by several authors, including [M, SW, CGGS, GL]. To be more precise: Let \( W \) be the Weyl group of \( G \), and let \( Br^+_W \) be the positive braid monoid of \( W \). By definition, \( Br^+_W \) is generated by elements \( \sigma_w \) for each \( w \in W \) modulo \( \sigma_{ww'} = \sigma_w \sigma_{w'} \) whenever \( \ell(ww') = \ell(w) + \ell(w') \), where \( \ell \) is the Bruhat length function on \( W \). A braid variety is a configuration space of tuples of Borels, where the relative positions of cyclically consecutive Borels have constraints determined by a fixed element \( \beta \in Br^+_W \). Note that there is a central element \( \pi \in Br^+_W \) known as the full twist and given by \( \pi = \sigma_{w_0}^2 \), where \( w_0 \) is the longest element of \( W \). In the notation of [T, Appendix B], the variety \( \mathcal{X}_g \) is isomorphic to the braid variety attached to \( \sigma_w \pi \), where \( w \) is the relative position of the pair \((B_+, gB_+g^{-1})\).

In [T], it was shown that the weight filtration on the equivariant, compactly-supported cohomology of the braid variety of \( \beta \) encodes a certain summand of a certain triply-graded vector space attached to \( \beta \), known as its \textsc{HOMFLYPT} or Khovanov–Rozansky (KR) homology. When \( W \) is the symmetric group \( S_n \), the braid \( \beta \) represents the isotopy class of a topological braid on \( n \) strands, and the KR homology of \( \beta \) is an isotopy invariant of the link closure of \( \beta \), up to grading shifts \([KR,Kh]\).

At the same time, when \( w \) is the relative position of \((B_+, gB_+g^{-1})\), it turns out that \( \mathcal{U}_g \) is closely related to another variety attached to \( \sigma_w \) in [T]. Just as \( \mathcal{X}_g \) encodes the “highest \( a \)-degree” of the KR homology of \( \sigma_w \pi \), so \( \mathcal{U}_g \) encodes the “lowest \( a \)-degree” of the KR homology of \( \sigma_w \). For \( W = S_n \), Gorsky–Hogancamp–Mellit–Nagakane have established an isomorphism between these bigraded vector spaces for general braids \( \beta \), not just \( \sigma_w \), which they deduce from an analogue of Serre duality for the homotopy category of Soergel bimodules for \( S_n \) \([GHMN]\). Our Theorem 2 follows from this isomorphism.

This proof suggests that we regard Conjecture 1 as a geometric realization of the (purely algebraic) Serre duality of \([GHMN]\). Moreover, it suggests an extension of Conjecture 1 with positive braids in place of elements of \( W \). We state the extended conjecture in Section 5.

The isomorphism of \textit{ibid.} categorifies an earlier identity of Kálmán, relating the bivariate HOMFLY series of the link closures of \( \beta \) and \( \beta \pi \). In [T], we generalized Kálmán’s theorem to the braids associated with arbitrary finite Coxeter groups. (We expect, but do not show, that \([GHMN]\) admits a similar generalization.) For Weyl groups, we also interpreted our result as an identity of \( F \)-point counts. In Section 6, we review these results, and explain how they recover (1.1).

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2. Reductions

In this section, we collect lemmas about easy reductions and special cases of the conjecture.

Lemma 3. If $G = G_1 \times G_2$, where $G_1, G_2$ are again reductive, then Conjecture 1 holds if and only if it holds with $G_1$ in place of $G$ and with $G_2$ in place of $G$.

Proof. We can factor $U_\pm = U_{+,1} \times U_{+,2}$ and $\mathcal{U} = \mathcal{U}_{1} \times \mathcal{U}_{2}$ and $\Phi = \Phi^{(1)} \times \Phi^{(2)}$, where $U_{\pm,i}, \mathcal{U}_{i}, \Phi^{(i)}$ are the analogues of $U_\pm, \mathcal{U}, \Phi$ with $G_i$ in place of $G$.

Lemma 4. Both $U_g$ and $V_g$ are contained within the derived subgroup $G^{\text{der}} \subseteq G$. In particular, we can assume $G$ is semisimple in Conjecture 1.

Proof. It is enough to show $U \subseteq G^{\text{der}}$, because in that case, we also have $U_+ U_- \subseteq G^{\text{der}} G^{\text{der}} = G^{\text{der}}$. Suppose instead that $U \not\subseteq G^{\text{der}}$. Then $G/G^{\text{der}}$ contains nontrivial unipotent elements, because homomorphisms of algebraic groups preserve Jordan decompositions [Mi, §9.21]. But $G/G^{\text{der}}$ is isogeneous to the center of $G$ by [Mi, Ex. 19.25], so it is a torus, in which the only unipotent element is the identity.

Lemma 5. The map $\Phi_g$ is unchanged, up to composition with maps that induce homeomorphisms on $\mathbb{C}$-points, when we replace $G$ with the adjoint group $G^{\text{ad}} = G/Z(G)$ and $g$ with its image in $G^{\text{ad}}$. In particular, Conjecture 1 is equivalent to its analogue where we replace $G$ with any quotient by a central isogeny, and $g$ with its image in that quotient.

Proof. Let $U_\pm$ and $\mathcal{U}$ be the respective analogues of $U_\pm$ and $\mathcal{U}$ with $G^{\text{ad}}$ in place of $G$. Again using the preservation of Jordan decomposition, and the fact that central elements of $G$ are semisimple, we can check that the quotient map $G \to G^{\text{ad}}$ restricts to maps $U_\pm \to U_\pm$ and $\mathcal{U} \to \mathcal{U}$ that give rise to bijections on field-valued points. Writing $\Phi : U_+ U_- \to \mathcal{U}$ for the analogue of $\Phi$, we see that the diagram

\[
\begin{array}{ccc}
U_+U_- & \xrightarrow{\Phi} & \mathcal{U} \\
\downarrow \scriptstyle U & & \downarrow \scriptstyle \mathcal{U} \\
\bar{U}_+\bar{U}_- & \xrightarrow{\bar{\Phi}} & \bar{\mathcal{U}}
\end{array}
\]

commutes. Now the result follows.

The following lemma is motivated by the Bruhat decomposition

\[
G = \coprod_{w \in W} U_w wB_+,
\]

where $w \mapsto \hat{w}$ is any choice of set-theoretic lift from $W \simeq N_G(T)/T$ into $N_G(T)$. Note that $\hat{w}B_+$ only depends on $w$ because $T \subseteq B_+$.

Lemma 6. If Conjecture 1 holds for some $g \in G(\mathbb{C})$, then it holds with $ug$ in place of $g$, for all $u \in U_+(\mathbb{C})$. In particular, if Conjecture 1 holds for all $g \in N_G(T)(\mathbb{C})$, then it holds in general.
Proof. We can factor $\Phi_{ug}$ as a composition

$$\gamma_{ug} \xrightarrow{=} \gamma_g \xrightarrow{\Phi_g} \mathbb{U}_g \xrightarrow{=} \mathbb{U}_{ug},$$

where the first arrow is left multiplication by $u^{-1}$, and the last arrow is left conjugation by $u$. Since these are both isomorphisms of varieties, we get the first assertion of the lemma. The second follows from the first via Bruhat.

If $P_+$ and $P_-$ are opposed parabolic subgroups of $G$ containing $B_+$ and $B_-$, respectively, and $L = P_+ \cap P_-$ is their common Levi subgroup, then we write $U_{\pm, P}$ for the unipotent radical of $P_{\pm}$, and write $B_{\pm, L}, U_{\pm, L}, \mathbb{U}_L$ for the analogues of $B_\pm, U_\pm, \mathbb{U}$ with $L$ in place of $G$. Thus we have

$$P_\pm = LU_{\pm, P} \simeq L \times U_{\pm, P},$$
$$B_\pm = B_{\pm, L}U_{\pm, P} \simeq B_{\pm, L} \times U_{\pm, P},$$
$$U_\pm = U_{\pm, L}U_{\pm, P} \simeq U_{\pm, L} \times U_{\pm, P}.$$

If $g \in L$, then we write $\mathbb{U}_{L,g}, \gamma_{L,g}, \Phi_{L,g}$ for the analogues of $\mathbb{U}_g, \gamma_g, \Phi_g$ with $L$ in place of $G$.

Lemma 7. Let $P_{\pm} \supseteq B_\pm$ be opposed parabolic subgroups of $G$, and let $L = P_+ \cap P_-$. Then for all $g \in L(C)$, we have isomorphisms of algebraic varieties

$$\mathbb{U}_g \simeq \mathbb{U}_{L,g}U_{+, P} \simeq \mathbb{U}_{L,g} \times U_{+, P},$$
$$\gamma_g \simeq \gamma_{L,g}U_{+, P} \simeq \gamma_{L,g} \times U_{+, P}.$$

In particular, if $g \in G(C)$ belongs to a Levi subgroup of $G$, then in Conjecture 1, we can replace $G$ with that Levi subgroup.

Proof. Since the decomposition $P_+ \simeq L \times U_{+, P}$ preserves Jordan decompositions, we have $\mathbb{U} \cap P_+ = \mathbb{U}_LU_{+, P} \simeq \mathbb{U}_L \times U_{+, P}$. Intersecting with $gB_+$, we get

$$\mathbb{U} \cap gB_+ = (\mathbb{U}_L \cap gB_{+, L})U_{+, P} \simeq (\mathbb{U}_L \cap gB_{+, L}) \times U_{+, P}.$$

Next, $U_+U_- \cap gB_+ \subseteq gB_+ \subseteq P_+$ and $U_- \cap P_+ = U_{-, L} \cap P_+$ together imply $U_+U_- \cap gB_+ = U_+U_{-, L} \cap gB_+$, from which

$$U_+U_- \cap gB_+ = U_+U_{-, L}U_{+, P} \cap gB_{+, L}U_{+, P} = (U_+U_{-, L} \cap gB_{+, L})U_{+, P} \simeq (U_+U_{-, L} \cap gB_{+, L}) \times U_{+, P}.$$

So it remains to prove the last assertion in the lemma. For this, it is more convenient to use the decomposition $\gamma_g = U_{+, P}\gamma_{L,g} \simeq U_{+, P} \times \gamma_{L,g}$. For all $(x, y_+, y_-) \in U_{+, P} \times U_{+, L} \times U_{-, L}$, observe that

$$\Phi_g(xy+y_-) = xxy_+y_-^{-1}x^{-1} = \text{Ad}_x(\Phi_{L,g}(y_+y_-)),$$

where $\text{Ad}_x(u) = xux^{-1}$. A choice of deformation retract from $U_{+, P}(C)$ onto $\{1\}$ induces a homotopy from $\text{Ad}_x : \mathbb{U}_g \to \mathbb{U}_g$ onto $\text{id} : \mathbb{U}_g \to \mathbb{U}_g$, which in turn
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induces a homotopy from \( \Phi_g : \mathcal{V}_g \rightarrow \mathcal{U}_g \) onto the composition

\[
\mathcal{V}_g \xrightarrow{p_{L,g}} \mathcal{V}_{L,g} \xrightarrow{\Phi_{L,g}} \mathcal{U}_{L,g} \xrightarrow{i_{L,g}} \mathcal{U}_g,
\]

where \( p_{L,g} : \mathcal{V}_g \rightarrow \mathcal{V}_{L,g} \) is the retract induced by the projection \( U_+, P(C) \rightarrow \{1\} \) and \( i_{L,g} : \mathcal{U}_{L,g} \rightarrow \mathcal{U}_g \) is the section induced by the inclusion \( \{1\} \rightarrow U_+, P(C) \). Therefore, \( \Phi_{L,g} \) being half of a homotopy equivalence is equivalent to \( \Phi_g \) being half of a homotopy equivalence.

\[\square\]

**Lemma 8.** Conjecture 1 holds for \( g \in B_+(C) \).

As observed in the introduction, this can be proved by computing \( \mathcal{U}_g(C), \mathcal{V}_g(C) \), and \( \Phi_g \) directly. Alternatively:

Proof. Since \( \mathcal{U}_g, \mathcal{V}_g, \Phi_g \) only depend on \( gB_+ \), we can assume \( g = 1 \). Then Lemma 7 reduces us to the case \( G = T \), where \( \mathcal{U}_g = \{1\} = \mathcal{V}_g \). \[\square\]

3. Examples

3.1. By Lemmas 3–5, it suffices to check Conjecture 1 for one representative \( G \) from each central isogeny class of semisimple algebraic group with connected Dynkin diagram. Moreover, by Lemma 6, it suffices to fix a lift \( w \mapsto \dot{w} \) from \( W \) into \( N_G(T) \) and check Conjecture 1 for cosets of the form \( gB_+ = \dot{w}B_+ \). In this section, we settle the conjecture completely for \( G \in \{\text{SL}_2, \text{SL}_3\} \), and check one further case in which \( G = \text{Sp}_4 \).

Without loss of generality, we can always take \( U_+, \text{resp.} \ U_- \), to be the group of unipotent upper-triangular, \( \text{resp.} \) unipotent lower-triangular, matrices in \( G \). To produce defining equations for \( \mathcal{U}_- \), we use the coefficients of the map that sends \( g \in G \) to its characteristic polynomial. We write \( e \) for the identity element of \( W \).

3.2. The Group \( \text{SL}_2 \). We have

\(
\mathcal{U} = \{g \in \text{SL}_2 \mid \text{tr}(g) = 2\}.
\)

The map \( \Phi : U_+U_- \rightarrow \mathcal{U} \) is

\[
\Phi \left( \begin{pmatrix} 1 & b \\ 1 & b' \end{pmatrix} \right) = \begin{pmatrix} 1 + bb' & -b^2b' \\ b' & 1 - bb' \end{pmatrix}.
\]

We can write \( W = \{e, w_0\} \). By Lemmas 6 and 8, it suffices to check \( gB_+ = \dot{w}_0B_+ \).

The varieties \( \mathcal{U}_g \) and \( \mathcal{V}_g \) are

\[
\mathcal{U}_g = \left\{ \begin{pmatrix} -X \\ 2 \end{pmatrix} \mid X \neq 0 \right\},
\]

\[
\mathcal{V}_g = \left\{ \begin{pmatrix} b' \\ -1 \end{pmatrix} \mid b' \neq 0 \right\}.
\]

The coordinates define isomorphisms \( \Phi_g : \mathcal{U}_g \rightarrow \mathbf{G}_m \) and \( \mathcal{V}_g \rightarrow \mathbf{G}_m \). The map \( \Phi_g \) is an isomorphism of varieties, corresponding to setting \( X = b' \).
3.3. The Group SL₃. Let \( \Lambda^2(g) \) denote the exterior square of a matrix \( g \). From the identity \( 2 \text{tr}(\Lambda^2(g)) = \text{tr}(g)^2 - \text{tr}(g^2) \), we have

\[
\mathcal{U} = \left\{ g \in \text{SL}_3 \left| \begin{array}{l}
\text{tr}(g) = 3, \\
\text{tr}(\Lambda^2(g)) = 3
\end{array} \right\} = \left\{ g \in \text{SL}_3 \left| \begin{array}{l}
\text{tr}(g) = 3, \\
\text{tr}(g)^2 = 3
\end{array} \right\}.
\]

The map \( \Phi : U_+ U_- \to \mathcal{U} \) is

\[
\Phi \left( \begin{pmatrix}
1 & a & b \\
1 & c & 1 \\
1 & b' & c'
\end{pmatrix} \right) = \begin{pmatrix}
1 + aa' + bb' & a' + b^2b' - bcc' + a^2a' + abc' \\
a' + cb' & b' & c' - ab' \\
b' & c' & 1 - bb' - cc' + abc'
\end{pmatrix}.
\]

We can write \( W = \{ e, s, t, ts, st, w_0 \} \), where \( s \) and \( t \) are the simple reflections. The simple reflections lift to elements of \( N_G(T) \) contained in proper Levi subgroups of \( G \), so by the SL₂ case and Lemmas 4, 6, and 7, it remains to consider \( gB_+ = \tilde{w}B_+ \) for \( w \in \{ ts, st, w_0 \} \). In what follows, we choose \( \tilde{s}, \tilde{t} \) so that

\[
\tilde{s}B_+ \subseteq \left\{ \begin{pmatrix}
* & * \\
* & *
\end{pmatrix} \right\} \quad \text{and} \quad \tilde{t}B_+ \subseteq \left\{ \begin{pmatrix}
* & * \\
* & *
\end{pmatrix} \right\}.
\]

3.3.1. If \( w = ts \), then the varieties \( \mathcal{U}_g \) and \( \mathcal{V}_g \) are

\[
\mathcal{U}_g = \left\{ \begin{pmatrix}
Y & C \\
\frac{1}{\sqrt{Z}} & -\frac{1}{Z}(3 + \frac{C}{\sqrt{Z}})
\end{pmatrix} \left| \begin{array}{l}
Y, Z \neq 0
\end{array} \right\},
\]

\[
\mathcal{V}_g = \left\{ \begin{pmatrix}
-\frac{1}{\alpha'} & b \\
-\frac{a'}{c} & c
\end{pmatrix} \left| \begin{array}{l}
a', c \neq 0
\end{array} \right\}.
\]

The coordinates define isomorphisms \( (Y, Z, C) : \mathcal{U}_g \to \mathbb{G}^2_m \times \mathbb{A}^1 \) and \( (c, a', b) : \mathcal{V}_g \to \mathbb{G}^2_m \times \mathbb{A}^1 \). The map

\[
\Phi_g(x_+ x_-) = \begin{pmatrix}
-\frac{1}{\alpha'} & b + \frac{c}{\alpha'} \\
-\frac{a'}{c} & c
\end{pmatrix}
\]

is an isomorphism of varieties, corresponding to \( (Y, Z, C) = (-\frac{1}{\alpha'}, c, b + \frac{c}{\alpha'}) \).

3.3.2. If \( w = st \), then the varieties are

\[
\mathcal{U}_g = \left\{ \begin{pmatrix}
X & A \\
Y & -\frac{1}{3}(3 - 3A + A^2)
\end{pmatrix} \left| \begin{array}{l}
X, Y \neq 0
\end{array} \right\},
\]

\[
\mathcal{V}_g = \left\{ \begin{pmatrix}
a' + \frac{b}{\alpha'} & b \\
1 & c
\end{pmatrix} \left| \begin{array}{l}
b, a' \neq 0
\end{array} \right\}.
\]

The coordinates define isomorphisms \( (X,Y,A) : \mathcal{U}_g \xrightarrow{\sim} G^2_m \times A^1 \) and \( (b,a',c) : \mathcal{V}_g \xrightarrow{\sim} G^2_m \times A^1 \). The map

\[
\Phi_g(x+x_-) = \begin{pmatrix}
\frac{2}{\nu \nu} + \frac{c}{\nu \nu} - (ba' + \frac{c}{\nu \nu} + c) \\
1 - \frac{c}{\nu \nu}
\end{pmatrix}
\]

is an isomorphism of varieties, corresponding to \( (X,Y,A) = (a', \frac{1}{\nu \nu}, 1 + \frac{c}{\nu \nu}) \).

3.3.3. If \( w = w_0 \), then the varieties are

\[
\mathcal{U}_g = \left\{ \left( \begin{array}{ccc}
\frac{-XZ}{A} & Z \\
X & 3 + \frac{1}{XZ}
\end{array} \right) \left| \begin{array}{c}
X, Z \neq 0, \\
(1 + \frac{1}{XZ})^3 + \frac{AC}{XZ} = 0
\end{array} \right\}
\right.
\]

\[
\mathcal{V}_g = \left\{ \left( \begin{array}{ccc}
b \\
1 + c'c \\
c'
\end{array} \right) \left| \begin{array}{c}
b, b' \neq 0, \\
1 + bb' + (bb')(cc') = 0
\end{array} \right\}
\right.
\]

The coordinates define isomorphisms

\[
\mathcal{U}_g \xrightarrow{\sim} \{(X,Z,A,C) \in G^2_m \times A^2 \mid (1 + \frac{1}{XZ})^3 + \frac{AC}{XZ} = 0\},
\]
\[
\mathcal{V}_g \xrightarrow{\sim} \{(b,b',c,c') \in G^2_m \times A^2 \mid 1 + bb' + (bb')(cc') = 0\}.
\]

The map

\[
\Phi_g(x+x_-) = \begin{pmatrix}
\frac{2}{\nu \nu} + \frac{c}{\nu \nu} - (ba' + \frac{c}{\nu \nu} + c) \\
1 - \frac{c}{\nu \nu}
\end{pmatrix}
\]

corresponds to setting \( (X,Z,A,C) = (b, b, (1 + bb')c', (1 + \frac{1}{\nu \nu})c) \). Note that \( \mathcal{U}_g \) and \( \mathcal{V}_g \) are not isomorphic as varieties.

**Proposition 9.** For \( G = \text{SL}_3 \) and \( gB_+ = w_0B_+ \), the map \( \Phi_g \) is neither injective nor surjective on \( \mathbb{C} \)-points, but does define half of a homotopy equivalence.

**Proof.** Let

\[
\mathcal{U}_g^\dagger = \{(u,A,C) \in G_m \times A^2 \mid AC = -(1 + u)(1 + \frac{1}{u})^2\},
\]
\[
\mathcal{V}_g^\dagger = \{(u,c,c') \in G_m \times A^2 \mid cc' = -(1 + \frac{1}{u})\}.
\]

Let \( \Phi_g^\dagger : \mathcal{V}_g^\dagger \to \mathcal{U}_g^\dagger \) be the map \( \Phi_g^\dagger(u,c,c') = (u, (1 + u)c', (1 + \frac{1}{u})c) \). Then \( \Phi_g \) is a pullback of \( \Phi_g^\dagger \), so it suffices to show the claim of the proposition with \( \mathcal{U}_g^\dagger, \mathcal{V}_g^\dagger, \Phi_g^\dagger \) in place of \( \mathcal{U}_g, \mathcal{V}_g, \Phi_g \).

Observe that \( \Phi_g^\dagger \) preserves \( u \in G_m \). Over the subvariety of \( G_m \) where \( u \neq -1 \), the fibers of \( \mathcal{U}_g^\dagger \) and \( \mathcal{V}_g^\dagger \) are copies of \( G_m \): say, via the coordinates \( A \) and \( c' \). In these coordinates, \( \Phi_g^\dagger \) amounts to rotating \( G_m \) by \( 1 + u \). Over the point \( u = -1 \), the fibers are copies of the transverse intersection of two lines. Altogether, \( \mathcal{U}_g^\dagger(C) \) and \( \mathcal{V}_g^\dagger(C) \) are both homotopic to pinched tori, and \( \Phi_g^\dagger \) induces a self-map of the pinched torus that preserves its longitude and top homology. Thus \( \Phi_g^\dagger \) fits into a homotopy equivalence. It is neither injective nor surjective because \( \Phi_g^\dagger(-1, c, c') = (-1, 0, 0) \).
3.4. The Group $\text{Sp}_4$. We set $\text{Sp}_4 = \{g \in \text{GL}_4 \mid g^*Jg = J\}$, where

$$J = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}.$$  

For $\text{Sp}_4$, the only nontrivial coefficients of the characteristic polynomial are $\text{tr}(g) = \text{tr}(\Lambda^3(g))$ and $\text{tr}(\Lambda^2(g))$, so similarly to $\text{SL}_3$, we have:

$$U = \left\{ g \in \text{Sp}_4 \mid \begin{array}{c} \text{tr}(g) = 4, \\ \text{tr}(\Lambda^2(g)) = 4 \end{array} \right\} = \left\{ g \in \text{Sp}_4 \mid \text{tr}(g^2) = 4 \right\}.$$  

The map $\Phi : U \rightarrow U$ is

$$\Phi = \begin{pmatrix} 1 & a & b + ad & c \\ 1 & 2d & b - ad & c \\ 1 & -a & b' + a'd' & 2d' \\ 1 & c' & b' - a'd' - a' & 1 \end{pmatrix} \begin{pmatrix} 1 & a' \\ a' & 1 \\ b' + a'd' & 2d' \\ c' & b' - a'd' - a' \end{pmatrix}.$$  

where we set

$$f_1 = ba'd' + ada'd',$$

$$g_1 = aa' + bb' + cc' + adb',$$

$$g_2 = -aa' + bb' + 4dd' - abc' - 3adb' + a^2 dc'$$

and

$$f_{1,2} = -(c + ab + a^2d)a'd',$$

$$g_{1,2} = 2bd' + cb' - a(aa' + bb' + cc' - 2dd' + adb'),$$

$$f_{1,3} = -ca' - b(aa' + bb' + cc' + 4dd') - 2cdb' + ad(aa' + cc' - 4dd' + adb'),$$

$$g_{1,3} = -(b^2 - 2cd - a^2d^2)a'd',$$

$$f_{2,1} = 2da'd',$$

$$g_{2,1} = a' + ba' + 2db' - adc',$$

$$f_{3,1} = b' - ac',$$

$$g_{3,1} = a'd'$$

and

$$h_{1,4} = -(2aa' + 2bb' + cc') - 2b^2d' - 2ad(cb' + 2bd' + add'),$$

$$h_{2,3} = -2aa' + 2d(aa' - 2bb' - 4dd') - b^2c' + ad(2bc' + 4db' - adc'),$$

$$h_{3,2} = 2d' - 2ab' + a^2c',$$

$$h_{4,1} = c'.$$
We can write \( W = \{ e, s, t, ts, st, sts, tst, w_0 \} \), where \( s \) and \( t \) are the simple reflections. By the \( SL_2 \) case and Lemmas 4, 6, and 7, it remains to consider \( gB_+ = \dot{w}B_+ \) for \( w \in \{ ts, st, sts, tst, w_0 \} \).

Below, we will only check \( w = sts \). Without loss of generality, we can assume

\[
stsB_+ = \left\{ \begin{pmatrix} Y & 2YD & Y(B - AD) \\ -X & -XA & -X(B + AD) \end{pmatrix} \middle| X, Y \neq 0 \right\}.
\]

The varieties \( U_g \) and \( V_g \) are

\[
U_g = \left\{ \begin{pmatrix} \frac{1}{X} & 2YD & Y(B - AD) \\ -1 - XA & 4Y - \frac{1}{Y} \end{pmatrix} \middle| \begin{array}{l} X, Y \neq 0, \\ XA(Y(B - AD) - \frac{1}{Y}(B + AD)) = \frac{1}{Y^2}(1 - Y)^4 \end{array} \right\},
\]

\[
V_g = \left\{ \begin{pmatrix} \frac{1}{1 + aa'} & 2(1 + aa')d - c(a')^2 & ca' - 2ad \\ 1 + aa' & -a & \frac{1}{c} \end{pmatrix} \middle| c, 1 + aa' \neq 0 \right\}.
\]

The coordinates define isomorphisms

\[
U_g \sim \left\{ (X, Y, A, B, D) \in G_m^2 \times A^3 \middle| XA(Y(B - AD) - \frac{1}{Y}(B + AD)) = \frac{1}{Y^2}(1 - Y)^4 \right\},
\]

\[
V_g \sim \{ (c, a, d, a') \in G_m \times A^3 \mid 1 + aa' \neq 0 \}.
\]

The map

\[
\Phi_g(x, x^{-1}) = \begin{pmatrix} \frac{c}{1 + aa'} & 2ada'(\frac{2 + aa'}{1 + aa'}) - c(a')^2 & 2a^2da' - \frac{a^2c(a')^3}{1 + aa'} \\ \frac{1}{c} & \frac{a^2a'}{c(1 + aa')} & \frac{1}{a^2a'} \\ \frac{1}{1 + aa'} & \frac{2da'(1 + aa' + \frac{1}{1 + aa'}) - \frac{1}{2}a^2c(a')^3}{1 + aa'} & 3 - aa' - \frac{1}{1 + aa'} \end{pmatrix}
\]

corresponds to setting

\[
\begin{pmatrix} X \\ Y \\ A \\ B \\ D \end{pmatrix} = \begin{pmatrix} \frac{1}{c} \\ \frac{1}{1 + aa'} \\ -\frac{a^2a'}{1 + aa'} \\ a^2da'(1 + aa' + \frac{1}{1 + aa'}) - \frac{1}{2}a^2c(a')^3 \\ ada'(2 + aa') - \frac{1}{2}c(a')^2(1 + aa') \end{pmatrix}.
\]

**Proposition 10.** For \( G = Sp_4 \) and \( gB_+ = stsB_+ \), the map \( \Phi_g \) is neither injective nor surjective on \( \mathbb{C} \)-points, but does define half of a homotopy equivalence.
Proof. The map $\Phi_g$ fits into a commutative diagram

$$
\begin{array}{ccc}
\Psi_g & \xrightarrow{\Phi_g} & \mathcal{U}_g \\
\downarrow & & \downarrow \\
\Psi\Psi & \xrightarrow{\Phi_g} & \mathcal{U}_g
\end{array}
$$

where the new varieties are

- $\mathcal{U}_g^+ = \{(X, Y, A, B, -B_+) \in \mathbb{G}_m^2 \times \mathbb{A}^3 \mid XA(YB_+ - \frac{B_+}{Y}) = \frac{1}{Y^2}(1 - Y)^4\}$,
- $\mathcal{V}_g^+ = \{(c, u, a_1, a_2, d') \in \mathbb{G}_m^2 \times \mathbb{A}^3 \mid (u - 1)^4 = a_1a_2\},$

and the new maps are

$$
\Phi_g^+ (c, u, a_1, a_2, d') = \left(\frac{c}{u}, \frac{1}{u}, -\frac{a_1}{u}, 2ud' - ca_2, \frac{2a_2}{u}\right),
$$

$$
\Psi_g (X, Y, A, B, D) = (X, Y, A, B - AD, B + AD),
$$

$$
\Psi^+ (c, a, d, a') = (c, 1 + aa', a^2a', a^2(a')^3, a^2da').
$$

The map $\Phi_g^+$ is algebraically invertible, so to show that $\Phi_g$ induces a homotopy equivalence, it remains to study the topology of the maps $\Psi_g$ and $\Psi_g^+$.

To show that $\Psi_g$ induces a homotopy equivalence, we first note that it preserves $(X, A) \in \mathbb{A}^2$. Over the subvariety of $\mathbb{A}^2$ where $A \neq 0$, it is invertible. Over the line $A = 0$, the defining equations of $\mathcal{U}_g$ and $\mathcal{V}_g^+$ both simplify to $(1 - Y)^4 = 0$, which has the unique solution $Y = 1$ over $\mathbb{C}$, so over this line, the fibers of $\mathcal{U}_g(\mathbb{C})$ and $\mathcal{V}_g^+(\mathbb{C})$ are contractible, being copies of $\mathbb{C}^2$.

To show that $\Psi_g^+$ induces a homotopy equivalence, it suffices to show that for the map from $\{(a, a') \in \mathbb{A}^2 \mid aa' \neq -1\}$ into $\{(u, a_1, a_2) \in \mathbb{G}_m \times \mathbb{A}^2 \mid (u - 1)^4 = a_1a_2\}$ that sends $(a, a') \mapsto (1 + aa', a^2a', a^2(a')^3)$. Indeed, this map restricts to an isomorphism from the subvariety where $aa' \neq 0$ onto the subvariety where $a_1a_2 \neq 0$, and collapses the subvariety where $aa' = 0$ onto the point $(u, a_1, a_2) = (1, 0, 0)$. The set of $\mathbb{C}$-points in the domain where $aa' = 0$ is contractible, and the set of $\mathbb{C}$-points in the target where $a_1a_2 = 0$ admits a deformation retract onto $\{(1, 0, 0)\}$. □

4. Configurations of Flags

4.1. In this section, we relate $\mathcal{U}_g$ and $\mathcal{V}_g$ to varieties that were studied in [T]. Henceforth, we fix any field $\mathbb{F}$ of good characteristic for $G$, and replace $G$ with its split form over $\mathbb{F}$. We also assume that $B_+$ is defined over $\mathbb{F}$.

Let $\mathcal{B}$ be the flag variety of $G$, i.e., the variety that parametrizes its Borel subgroups. As these subgroups are all self-normalizing and conjugate to one another, there is an isomorphism of varieties:

$$
G/B_+ \xrightarrow{\sim} \mathcal{B}, \quad xB_+ \mapsto xB_+x^{-1}
$$
It transports the $G$-action on $G/B_+$ by left multiplication to the $G$-action on $\mathcal{B}$ by left conjugation.

The orbits of the diagonal $G$-action on $\mathcal{B} \times \mathcal{B}$ can be indexed by the Weyl group $W$. The closure order on the orbits corresponds to the Bruhat order on $W$ induced by the Coxeter presentation. For all $(B_1, B_2) \in \mathcal{B} \times \mathcal{B}$ and $w \in W$, we write $B_1 \overset{w}{\to} B_2$ to indicate that $(B_1, B_2)$ belongs to the $w$th orbit, in which case we say that it is in relative position $w$. In particular, note that $B_+ \overset{w_0}{\to} B_-$ because $B_- = \dot{w}_0 B_+ \dot{w}_0^{-1}$.

Under the Bruhat decomposition, (4.1) restricts to an isomorphism

\begin{equation}
U_+ \dot{w} B_+ / B_+ \simeq \{ B \in \mathcal{B} \mid B_+ \overset{w_0}{\to} B \},
\end{equation}

Each side is isomorphic to an affine space of dimension $\ell(w)$, where $\ell : W \to \mathbb{Z}_{\geq 0}$ is the Bruhat length function.

4.2. Fix $g \in G(F)$. Recall that $H_g = B_+ \cap gB_+ g^{-1}$ acts on $\mathcal{V}_g = U_+ U_- \cap gB_+$ according to (1.2). Let

\[ \mathcal{X}_g = \{ B \in \mathcal{B} : B_+ \overset{w_0}{\to} B \overset{w_0}{\to} gB_+ g^{-1} \}, \]

and let $H_g$ act on $\mathcal{X}_g$ by left conjugation. We will prove:

**Proposition 11.** There is a $H_g$-equivariant isomorphism of varieties $\mathcal{X}_g \to \mathcal{V}_g$.

We give the proof in two steps. For convenience, we set $\mathcal{Y}_g = gU_+ \dot{w}_0 B_+ / B_+ \subseteq G/B_+$. Let $H_g$ act on $\mathcal{Y}_1 \cap \mathcal{Y}_g$ by left multiplication.

**Lemma 12.** The isomorphism (4.2) for $w = w_0$ restricts to an $H_g$-equivariant isomorphism $\mathcal{Y}_1 \cap \mathcal{Y}_g \simeq \mathcal{X}_g$.

**Proof.** Recall that (4.2) is $G$-equivariant. Under the action of an element $x$, the $w = w_0$ case is transported to an isomorphism

\[ \mathcal{Y}_x \simeq \{ B \in \mathcal{B} \mid x B_+ x^{-1} \overset{w_0}{\to} B \}. \]

On the right-hand side, the direction of the arrow $\overset{w_0}{\to}$ can be reversed because $w_0^{-1} = w_0$. Now take the fiber product of the isomorphisms for $x = 1$ and $x = g$ over the isomorphism (4.1).

In what follows, recall that via the decomposition $B_+ \simeq U_+ \times T$, any element of $B_+$ can be written as $ut$ for some uniquely determined $(u, t) \in U_+ \times T$. As a consequence, we also get a decomposition $\dot{w}_0 B_+ = U_- \dot{w}_0 T = U_- T \dot{w}_0$.

**Lemma 13.** The map

\[ \mathcal{V}_g = U_+ U_- \cap gB_+ \rightsquigarrow \mathcal{Y}_1 \cap \mathcal{Y}_g \]

\[ x_+ x_- \overset{\text{gut}}{\to} x_+ x_- t^{-1} \dot{w}_0 B_+ = g u t \dot{w}_0 B_+ \]

is an $H_g$-equivariant isomorphism of varieties.

**Proof.** Let $\mathcal{V}_g' = U_+ \dot{w}_0 B_+ \cap g U_+ \dot{w}_0$. Since the map

\[ \mathcal{V}_g \to \mathcal{V}_g' \]

\[ x_+ x_- \overset{\text{gut}}{\to} x_+ x_- t^{-1} \dot{w}_0 = g u t \dot{w}_0 \]
is an isomorphism, it remains to show that the map \( f : \nu'_g \to \mathcal{Y}_1 \cap \mathcal{Y}_g \) given by

\[
\nu'_g \to \nu'_g B_+ \to (\nu'_g B_+)/B_+ = \mathcal{Y}_1 \cap \mathcal{Y}_g
\]

is bijective on \( R \)-points for every \( \mathbf{F} \)-algebra \( R \). For convenience, we suppress \( R \) in the notation below.

Let \( y B_+ \in (\nu'_g B_+)/B_+ \). Then we can write \( y = u \tilde{w}_0 b = gu' \tilde{w}_0 b' \) for some \( u, u' \in U_+ \) and \( b, b' \in B_+ \). Therefore, \( y B_+ = f(y(b')^{-1}) \), where \( y(b')^{-1} = gu' \tilde{w}_0 \in \nu'_g \). This proves \( f^{-1}(y B_+) \) is nonempty. We claim that \( f^{-1}(y B_+) \) contains only one point. Recall that the map \( U_+ \to U_+ \tilde{w}_0 B_+/B_+ \) that sends \( v \mapsto v \tilde{w}_0 B_+ \) is an isomorphism. Thus, \( v \neq u \) implies \( v \tilde{w}_0 B_+ \neq u \tilde{w}_0 B_+ \). We deduce that

\[
f^{-1}(y B_+) = f^{-1}(u \tilde{w}_0 B_+)
\]

\[
\subseteq u \tilde{w}_0 B_+ \cap gU_+ \tilde{w}_0
\]

\[
\simeq \tilde{w}_0^{-1} g^{-1} u \tilde{w}_0 B_+ \cap U_+.
\]

But the intersection of \( U_+ \) with any coset of \( B_+ \) contains only one point. \( \square \)

4.3. Let \( \mathcal{G}_w, \mathcal{U}_w, \mathcal{X}_w \) be the varieties defined by

\[
\mathcal{G}_w = \{(B', B'') \in B \times B \mid B' \overset{w}{\to} B''\},
\]

\[
\mathcal{U}_w = \{(u, B') \in \mathcal{U} \times B \mid B' \overset{w}{\to} uB'u^{-1}\},
\]

\[
\mathcal{X}_w = \{(B, B', B'') \in B \times B \times B \mid B' \overset{w}{\to} B \overset{w}{\to} B'' \overset{w}{\to} B\}.
\]

Let \( G \) act on these varieties by (diagonal) left conjugation. We regard \( \mathcal{U}_w \) and \( \mathcal{X}_w \) as varieties over \( \mathcal{G}_w \) via the \( G \)-equivariant maps \((u, B') \mapsto (B', uB'u^{-1}) \) and \((B, B', B'') \mapsto (B', B'') \), respectively.

Let \( H_g \) act on \( G \) by right multiplication. For any variety \( X \) with an \( H_g \)-action, let \( H_g \) act diagonally on \( X \times G \), and let \( G \) act on \( (X \times G)/H_g \) by left multiplication on the second factor. Finally, fix a prime \( \ell > 0 \) invertible in \( \mathbf{F} \), so that we can form the equivariant \( \ell \)-adic compactly-supported cohomology groups

\[
H^*_c(\mathcal{G}_w\mathcal{X}_w) \simeq H^*_c((X \times G)/H_g).
\]

With these conventions, we have:

**Proposition 14.** If \( B_+ \overset{w}{\to} gB_+g^{-1} \), then there are \( G \)-equivariant isomorphisms

\[
(\mathcal{U}_g \times G)/H_g \xrightarrow{\sim} \mathcal{U}_w,
\]

\[
(\mathcal{X}_g \times G)/H_g \xrightarrow{\sim} \mathcal{X}_w.
\]

In particular, they induce isomorphisms on compactly-supported cohomology:

\[
H^*_c, H^*_c(\mathcal{U}_g, \mathcal{X}_w) \xrightarrow{\sim} H^*_c(\mathcal{U}_w, \mathcal{X}_w),
\]

\[
H^*_c, H^*_c(\mathcal{X}_g, \mathcal{X}_w) \xrightarrow{\sim} H^*_c(\mathcal{X}_w, \mathcal{X}_w).
\]

**Proof.** The maps \((\mathcal{U}_g \times G)/H_g \to \mathcal{U}_w \) and \((\mathcal{X}_g \times G)/H_g \to \mathcal{X}_w \) are

\[
[u, x] \mapsto (xux^{-1}, xB_+ x^{-1}),
\]

\[
[B, x] \mapsto (xBx^{-1}, xB_+ x^{-1}, xgB_+ g^{-1} x^{-1}),
\]
To show that they are isomorphisms: Observe that $G$ acts transitively on $G_w$, and the stabilizer of $(B_+, gB_+g^{-1})$ is precisely $H_g$. The preimage of this point in $\mathcal{U}_w$, resp. $\mathcal{X}_w$, is $\mathcal{U}_g$, resp. $\mathcal{X}_g$. Therefore, the maps above are the respective pullbacks to $\mathcal{U}_w$ and $\mathcal{X}_w$ of the isomorphism $G/H_g \xrightarrow{\sim} G_w$ that sends $xH_g \mapsto (xB_+x^{-1}, xgB_+g^{-1}x^{-1})$. \hfill $\square$

Note that when $F = C$, the maps on cohomology in Proposition 14 preserve weight filtrations because the maps that induce them are algebraic.

5. Khovanov–Rozansky Homology

5.1. In this section, we prove Theorem 2 by way of more general constructions motivated by knot theory.

Let $Br^+_W$ be the positive braid monoid of $W$. It is the monoid freely generated by a set of symbols $\{\sigma_w\}_{w \in W}$, modulo the relations $\sigma_{ww'} = \sigma_w \sigma_{w'}$ for all $w, w' \in W$ such that $\ell(ww') = \ell(w) + \ell(w')$. The full twist is the element $\pi = \sigma_{w_0} \in Br^+_W$.

For all $\beta = \sigma_{w_1} \cdots \sigma_{w_k} \in Br^+_W$, we set

$$
\mathcal{U}(\beta) = \{(u, B_1, \ldots, B_k) \in \mathcal{U} \times B^k \mid u^{-1}B_1u \xrightarrow{w_1} B_1 \xrightarrow{w_2} \cdots \xrightarrow{w_k} B_k\},
$$

$$
\mathcal{X}(\beta) = \{(B_1, \ldots, B_k) \in B^k \mid B_1 \xrightarrow{w_1} B_1 \xrightarrow{w_2} \cdots \xrightarrow{w_k} B_k\}.
$$

Let $G$ act on these varieties by left conjugation. We regard $\mathcal{U}(\beta)$ and $\mathcal{X}(\beta)$ as varieties over $G_w$, where $w = w_1 \cdots w_k \in W$, via the equivariant maps $(u, (B_1, \ldots, B_k)) \mapsto (B_k, B_1)$ and $(B_1, \ldots, B_k) \mapsto (B_k, B_1)$, respectively. Deligne showed that up to isomorphism over $B \times B$, these varieties only depend on $\beta$, not on the sequence of elements $w_i$. His full result describes the extent to which the isomorphism can be pinned down uniquely; see [D] for details.

In particular, we have equivariant identifications

$$
\mathcal{U}_w \xrightarrow{\sim} \mathcal{U}(\sigma_w),
$$

$$
\mathcal{X}_w \xrightarrow{\sim} \mathcal{X}(\sigma_w \pi)
$$

via $(u, B_1) = (u, B')$ and $(B_1, B_2, B_3) = (B', B'', B)$.

5.2. If $W$ is the symmetric group on $n$ letters, denoted $S_n$, then the group completion of $Br_W$ is the group of topological braids on $n$ strands, denoted $Br_n$. Any braid can be closed up end-to-end to form a link: that is, an embedding of a disjoint union of circles into 3-dimensional space. Thus there is a close relation between isotopy invariants of links and functions on the groups $Br_n$.

In [KR], Khovanov and Rozansky introduced a link invariant valued in triply-graded vector spaces. Its graded dimension can be written as a formal series in variables $a, g, t$. In [Kh], Khovanov showed how to construct it in terms of class functions on the groups $Br_n$, and more precisely, in terms of functors on monoidal additive categories attached to the groups $S_n$. When we set $t = -1$, the Khovanov–Rozansky invariant of a link specializes to its so-called HOMFLYPT series, and Khovanov’s functors specialize to class functions originally introduced by Jones and Ocneanu.
The positive braid monoid $B_{r+}^W$ and its group completion $B_{r+}^W$ can actually be defined for any Coxeter group $W$, not just Weyl groups. In [G], Y. Gomi extended the construction of Jones–Ocneanu to finite Coxeter groups. There is a similar extension of Khovanov's construction, up to a choice of a (faithful) representation on which $W$ acts as a reflection group.

Fix such a representation $V$. For any braid $\beta \in B_{r+}^W$, we write $\text{HHH}_V(\beta)$ to denote the Khovanov–Rozansky (KR) homology of $\beta$ with respect to $V$. We will use the grading conventions in [T], so that

$$P_V(\beta) = (at)^{[\beta]}a^{-\dim(V)}\sum_{i,j,k}(a^2q^{\frac{1}{2}}t)^{\dim(V)-i}q^{\frac{1}{2}}t^{-k}\dim \text{HHH}_{V}^{i+j+k}(\beta)$$

is an isotopy invariant of the link closure of $\beta$. In the case where $W = S_n$, taking $V$ to be the $(n-1)$-dimensional reflection representation yields what is usually called reduced KR homology and denoted $\text{HHH}$, while taking $V$ to be the $n$-dimensional permutation representation yields what is usually called unreduced KR homology and denoted $\text{HHH}$. They are related by

$$P(\beta) = \overline{P}(\beta),$$

where $P$ and $\overline{P}$ denote the series $P_V$ for these respective choices of $V$.

Henceforth, let $r = \dim(V)$ and $N = \dim(W)$. The results below are [T, Cor. 4] and [GHMN, Thm. 1.9].

**Theorem 15.** Suppose that $W$ is the Weyl group of a split reductive group $G$ over $\mathbf{F}$ with root lattice $\Phi$, and that $V = \mathbb{Z}\Phi \otimes \mathbb{Z} \mathbf{Q}$. Then for any $\beta \in B_{r+}^W$, we have isomorphisms

$$\text{gr}_{j}^{W}H_{c,G}^{j+k+2r}(\mathcal{U}(\beta), \mathcal{Q}_c) \simeq \text{HHH}_{V}^{0,j+k}(\beta),$$

$$\text{gr}_{j}^{W}H_{c,G}^{j+k+2(r+N)}(\mathcal{F}(\beta), \mathcal{Q}_c) \simeq \text{HHH}_{V}^{r+j,k}(\beta)$$

for all $j, k$.

**Theorem 16** (Gorsky–Hogancamp–Mellit–Nakagane). For any integer $n \geq 1$ and $\beta \in B_{r+}^n$, we have

$$\text{HHH}^{0,j+k}(\beta) \simeq \text{HHH}^{r+j,k}(\beta \pi)$$

for all $j, k$.

**Proof of Theorem 2.** We must have $B_{r+}^W \xrightarrow{w} gB_{r+}^W g^{-1}$ for some $w \in W$. Combining Proposition 11, Proposition 14, and Theorem 15, we get isomorphisms

$$\text{gr}_{j}^{W}H_{c,H}^{j+k+2n}(\mathcal{U}_g, \mathcal{Q}_c) \simeq \text{HHH}_{V}^{0,j+k}(\sigma_w),$$

$$\text{gr}_{j}^{W}H_{c,H}^{j+k+2(n+N)}(\mathcal{F}_g, \mathcal{Q}_c) \simeq \text{HHH}_{V}^{r+j,k}(\sigma_w \pi),$$

where $V = \mathbb{Z}\Phi \otimes \mathbb{Z} \mathbf{Q}$ and $\Phi$ is the root lattice of $G$.

If $G = \text{GL}_n$, then $V$ is the permutation representation of $S_n$. So in this case, $\text{HHH}_V = \text{HHH}$, and we are done by Theorem 16. Finally, we bootstrap from $\text{GL}_n$ to any other split reductive group of type $A$ using Lemmas 4 and 5. $\square$
5.3. Theorems 15–16 suggest the following generalization of Conjecture 1.

**Conjecture 17.** For any $\beta \in Br^+_W$, there is a homotopy equivalence between $\mathcal{U}(\beta)(C)$ and $\mathcal{X}(\beta\pi)(C)$ that matches the weight filtrations on their compactly-supported cohomology.

**Remark 18.** It would be desirable to generalize the map of stacks $[Y_g/H_g] \to [\mathcal{U}_g/H_g]$ that arises from $\Phi_g$ to an explicit map $[\mathcal{X}(\beta\pi)/G] \to [\mathcal{U}(\beta)/G]$ for any positive braid $\beta$. Due to the inexplicit nature of Lemma 13, we have not yet found such a generalization.

6. **Point Counts over Finite Fields**

6.1. For any braid $\beta \in Br_n$, we write $\hat{\beta}$ to denote its link closure. The reduced HOMFLYPT series $P(\hat{\beta})$ is related to the KR homology of $\beta$ by

$$P(\hat{\beta}) = P(\beta)|_{t \to -1}. $$

This is an element of $\mathbb{Z}[q^{\frac{1}{2}}][q^{-\frac{1}{2}}][a^{\pm 1}]$. We write $[a^{\beta}]P(\hat{\beta})$ to denote the coefficient of $a^\beta$ in $P(\hat{\beta})$, viewed as an element of $\mathbb{Z}[q^{\frac{1}{2}}][q^{-\frac{1}{2}}]$.

If $\beta = \sigma_{s_1} \cdots \sigma_{s_\ell}$, where the elements $s_1, \ldots, s_\ell \in W$ are all simple reflections, then we set $|\beta| = \ell$. This number only depends on $\beta$. Theorem 16 then specializes to the following result from [K].

**Theorem 19 (Kálmán).** For any integer $n \geq 1$ and $\beta \in Br_n$, we have

$$[a^{\beta-n+1}]P(\hat{\beta}) = [a^{\beta+n-1}]P(\beta\pi).$$

In [T, §8], we generalized Kálmán’s result from $Br_n$ to $Br_W$. In this section, we review the statement, then explain its relation to the point-counting identity (1.1).

6.2. Let $H_W$ be the Iwahori–Hecke algebra of $W$. For our purposes, $H_W$ is the quotient of the group algebra $\mathbb{Z}[q^{\pm \frac{1}{2}}][Br_W]$ by the two-sided ideal

$$\langle (\sigma_s - q^{\frac{1}{2}})(\sigma_s + q^{-\frac{1}{2}}) | \text{simple reflections } s \rangle.$$

For any element $\beta \in Br_W$, we abuse notation by again writing $\beta$ to denote its image in $H_W$.

The sets $\{\sigma_w\}_{w \in W}$ and $\{\sigma_w^{-1}\}_{w \in W}$ are bases for $H_W$ as a free $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-module. Let $\tau^\pm : H_W \to \mathbb{Z}[q^{\pm \frac{1}{2}}]$ be the $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-linear functions defined by:

$$\tau^\pm(\sigma_w) = \begin{cases} 1 & w = e \\ 0 & w \neq e \end{cases}$$

For $W = S_n$, comparing $\tau^\pm$ with the Jones–Ocneanu trace on $H_W$ shows that

$$[a^{\beta\pm (n-1)}]P(\hat{\beta}) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{(n-1)}(-1)^{|\beta|}\tau^\pm(\beta)$$

for all $\beta \in Br_n$. Therefore the following result from [T, §8] generalizes Kálmán’s theorem to arbitrary $W$.

**Theorem 20.** For any finite Coxeter group $W$ and braid $\beta \in Br_W$, we have

$$\tau^- (\beta) = \tau^+ (\beta\pi).$$
6.3. We return to the setting of Section 5, so that $W$ is the Weyl group of $G$. Under the hypotheses of Theorem 15, the following identities from loc. cit. relate Theorem 20 to point counting:

$$\left| U^{(\beta)}(F) \right| \left| G(F) \right| = (q - 1)^{-r} (q^{\frac{1}{2}})^{\tau^-} (\beta),$$

$$\left| \mathcal{X}^{(\beta)}(F) \right| \left| G(F) \right| = (q - 1)^{-r} (q^{\frac{1}{2}})^{\tau^+} (\beta).$$

Together they imply:

**Corollary 21.** Keep the hypotheses of Theorem 15. Then for any $\beta \in Br_W$, we have

$$\left| U^{(\beta)}(F) \right| \left| G(F) \right| = \left| U^{(\beta \pi)}(F) \right| \left| G(F) \right|.$$
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