Dynamic analysis of a plankton–herbivore state-dependent impulsive model with action threshold depending on the density and its changing rate

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Abstract A plankton–herbivore state-dependent impulsive model with nonlinear impulsive functions and action threshold including population density and rate of change is proposed. Since the use of action threshold makes the model have complex phase set and pulse set, we adopt the Poincaré map as a tool to study its complex dynamics. The Poincaré map is defined on the phase set and its properties in different situations are analyzed. Furthermore, the periodic solution of model is discussed, including the existence and stability conditions of the order-1 periodic solution and the existence of the order-k (k ≥ 2) periodic solutions. Compared with the fixed threshold in the existing literature, our results show that the use of action threshold is more practical, which is conducive to the sustainable development of population and makes people obtain more economic benefits. The analysis method used in this paper can study the complex dynamics of the model more comprehensively and deeply.

Keywords State-dependent impulsive model · Action threshold · Poincaré map · Periodic solution

1 Introduction

Plankton is at the bottom of the marine food chain. They are widely distributed and highly reproductive, forming the foundation of the marine food web. In recent years, human activities have seriously interfered with the ecosystem, and overexploitation of marine organisms has become a common problem. To better understand these interactions, more and more scholars pay attention to the plankton-related models, for example, a plankton-fish model under stochastic fluctuations was proposed and studied in [1], a phytoplankton–zooplankton model with harvesting was proposed in [2], the interactions between plankton and other species and effects of environmental changes were studied in [3], a random nutrition–plankton food chain model with nutritional recovery was proposed by [4], a stochastic model of phytoplankton–zooplankton in impulsive pollution environment was proposed by [5], in references [6–8] researchers investigated the pattern dynamics.

For a herbivore–plankton model with cannibalism, the existence and stability of order-1 periodic solution were discussed in [9], where the model was a system of Impulsive Differential Equations (IDEs), which has more complex and abundant dynamics, but can fully take into account the impact of instantaneous mutations on the state and more deeply reflect the law of things changing. IDEs are widely used in biology, medicine, control theory and other fields [10–13]. In biology, the impulsive differential equation can be proposed to incorporate the possible changes in the population into
the research model, and more information can be found from [14, 15]. In biology, the state-dependent feedback control means that the control will be implemented only when the population reaches a given threshold level [16–18]. This threshold control strategy is applied to solve many practical biological problems, such as the influence between biological populations [19–21], integrated pest management [22–24], infectious disease control [25–27], fishery harvesting [4, 5, 28–30], etc. Therefore, it is of great practical significance to establish the corresponding mathematical model to describe and study the state-dependent feedback control strategy. The impulsive semi-dynamical system or the state-dependent impulsive differential equation can characterize threshold control strategies very naturally, and a comprehensive analysis of the equations can reveal many important biological conclusions.

In controlling actual biological populations, we focus on when to take control and how to control them. The state-dependent feedback control as mentioned above is carried out on the biological population density when it reaches a fixed threshold in [10, 31]. Thus, the impulsive set is a straight line, which makes the calculation relatively simple. But from the perspective of biology, the fixed threshold cannot be combined with the actual situation of the biological population to determine when to control the biological population. There are two cases in practice: (a) the population density is small but the rate of change is high, which often occurs in the early stage of population growth; (b) the population density is large, but the rate of change is small. Therefore, the threshold of the model needs to contain both population density and change rate, i.e., the proportionally dependent action threshold. In the process of releasing and fishing the biological population, different factors need to be integrated to make the control methods adopted more in line with the development law of biological populations. Therefore, the adopted state-dependent impulsive model needs to consider both the selection of threshold and impulsive function.

In [9] authors only performed a pulse control when the fixed threshold is reached, and the impulsive function used is relatively simple. The more complex dynamic properties of the model are also not studied due to the difficulties in analysis. Now based on the work of [9] and our above analysis, we will propose a plankton–herbivore state-dependent impulsive model with action threshold and nonlinear impulsive function:

\[
\begin{align*}
  \frac{dp(t)}{dt} &= p(t) \left( k - \frac{h(t)}{1 + \frac{p(t)}{\eta}} \right), \\
  \frac{dh(t)}{dt} &= h(t) \left( \frac{p(t)}{1 + p(t)} - ah(t) \right), \\
  p(t^+) &= p(t) + \frac{\tau}{1 + \eta p(t)}, \\
  h(t^+) &= h(t) - \frac{\sigma h^2(t)}{h(t) + \mu}.
\end{align*}
\]

where \( p \) and \( h \) represent the plankton and herbivore density, respectively, \( k \) denotes the biological carrying capacity and \( a \) indicates the density-dependent influence of herbivores. The parameters \( u_1, v_1 \) and \( W \) are positive values and \( u_1 + v_1 = 1 \). When the amount of plankton reaches the action threshold \( u_1 p + v_1 \frac{dp}{dt} = W \), actions will be taken to control the plankton and herbivore populations. We assume that the model is dimensionless.

In previous studies, one of the main assumptions is that the fishing rate is a linear function that depends on the population density, and the release amount is a constant. However, the implementation of control strategies should often depend on the current population of the population and be affected by the limited natural resources, that is, plankton. Both the release amount and the catch rate of herbivores should be based on the saturation function of the population. Therefore, it is more reasonable to introduce the nonlinear factors of limited natural resources into the model. We will use the following nonlinear impulse functions related to the density of biological populations and fishing rates.

\[
\alpha = \frac{\tau}{1 + \eta p(t)} \quad \text{and} \quad \beta = -\frac{\sigma h^2(t)}{h(t) + \mu}.
\]

These two equations indicate how to implement comprehensive control measures that depend on the dynamic threshold level \( u_1 p + v_1 \frac{dp}{dt} = W \). That is to say, plankton should be released immediately, and herbivores should be caught to make the density reach \( h(t) - \frac{\sigma h^2(t)}{h(t) + \mu} \) and \( p(t) + \frac{\tau}{1 + \eta p(t)} \), respectively, where \( \tau \) is the maximum number of plankton released, \( \eta \) denotes the morphological parameter. \( \sigma \) indicates the maximum catch rate of herbivores and \( \mu \) represents its half-saturation constant. The nonlinear release factor \( \alpha \) is a simple decreasing function of \( p(t) \), which means that the release amount of plankton depends on its density. Because of some practical factors, such as limited resources, the release of plankton does not exceed \( \tau \). The use of the action threshold makes the pulse set
and the phase set become two curves, and the nonlinear pulse function complicates the model. Further, the dynamic properties of model become very complicated. The global dynamics of the state-dependent impulsive model at this time deserve our in-depth and comprehensive study.

From [9] or straightforward analysis, we obtain that when system (1) has no impulsive effect it becomes

\[
\begin{align*}
\frac{dp(t)}{dt} &= p(t) \left( k - p(t) - \frac{h(t)}{1 + p(t)} \right), \\
\frac{d\phi(t)}{dt} &= h(t) \left( \frac{p(t)}{1 + p(t)} - ah(t) \right),
\end{align*}
\]

which has nullclines:

\[L_1 : h = (k - p)(1 + p), \quad L_2 : h = \frac{p}{a(1 + p)},\]

and \(O(0, 0)\) is the boundary equilibrium point, which is unstable and \(E_0(k, 0)\) is a saddle point. System (2) has an internal equilibrium point \(E^*(p^*, h^*)\), where \(p^* = a(k - p^*)(1 + p^*)^2, h^* = \frac{a\mu}{a(1 + p^*)^2}\), and when \(k < 2p^* + 2\), the \(E^*\) is stable; when \(k > 2p^* + 2\), \(E^*\) is unstable, and the occurrence of the a Hopf bifurcation makes a stable limit cycle exist when \(k = 2p^* + 2\).

In our study here, we consider the case where \(E^*\) is stable. Thus, in the rest of discussion we assume that \(k < 2p^* + 2\).

2 Definition of the Poincaré map

For biologically meaningful, we restrict our discussion to

\[R^2_+ = \{(p, h) : p \geq 0, h \geq 0\} .\]

Since system (1) uses an action threshold that includes both population density and rate of change, the impulsive set and phase set are two curves. According to \(u_1 p + v_1 \frac{dp}{dt} = W\) and the first equation of system (1), we get

\[h = (k - p - \frac{W - u_1 p}{v_1 p})(1 + p),\]

denoted by \(L_M\). From \(p^* = p + \frac{\tau}{1 + \eta p}\), we obtain

\[p = \frac{\eta p^* - 1 + \sqrt{\eta^2(p^*)^2 + 2\eta p^* - 4\eta \tau + 1}}{2\eta} .\]

Therefore,

\[
h = \left( k - p - \frac{W - u_1 p}{v_1 p} \right)(1 + p) = \left( k - \eta p^* - 1 + \sqrt{\eta^2(p^*)^2 + 2\eta p^* - 4\eta \tau + 1} \right) \frac{1}{2\eta} W - u_1 \eta p^* - 1 + \sqrt{\eta^2(p^*)^2 + 2\eta p^* - 4\eta \tau + 1} \right) .
\]

Let

\[h^* = h(1 - \frac{\sigma h}{h + \mu}) = \phi(p^*),\]

denoted by \(L_N\). In what follows, unless special instructions, the initial point \(S_{\phi}^+ = (p_{\phi}^*, h_{\phi}^*)\) is on the curve \(L_N\). Since the trajectory starting from \(L_N\) may not reach \(L_M\) depending on the initial values, the specific ranges of the impulsive set and the phase set can be obtained according to different positional relations between the trajectory and the equilibrium point (see Fig. 1).

2.1 Case \((C_1)\): \(p^* \leq p_M^*\)

Let \(p_M^*\) be the horizontal coordinate of curve \(L_M\) under \(h = h^* = \frac{p^*}{a(1 + p^*)^2}.\) Since \(p^* \leq p_M^*\), there must be a point \(T_1(p_{T_1}, h_{T_1})\) on \(L_N\) such that \(\Gamma_{T_1}\) and \(L_N\) are tangent. And the trajectory \(\Gamma_{T_1}\) intersects with the \(L_M\) at point \(T_2(p_{T_2}, h_{T_2})\) (Fig. 1a). Thus, the trajectory of the starting point from the phase set must reach the impulsive set at some time. Therefore, in this case, the range of the impulsive set and phase set is as follows:

\[M_1 = \{(p, h) | p \geq p_{T_2}, h \geq h_{T_3}\}\]

and

\[N_1 = \{(p^+, h^+) | p^+ \geq p_{T_2} + \frac{\tau}{1 + \eta p_{T_2}}, h^+ \geq h_{T_2} - \frac{h_{T_3}^2}{\eta p_{T_2}^* + \mu}\} .\]

respectively, where \((p, h)\) is a point on the curve \(L_M : h = (k - p - \frac{W - u_1 p}{v_1 p})(1 + p)\) and \((p^+, h^+)\) is a point on the curve \(L_N\).

2.2 Case \((C_2)\): \(p^* > p_M^*\)

In this case, since \(p^* > p_M^*\), there must exist a point \(A(p_A, h_A)\) on impulse set, which makes the trajec-
Fig. 1 The trajectories of system (1) under different cases, where the parameter values are $k = 2.4, \eta = 5, \tau = 1.8, \mu = 0.4, \sigma = 0.8, a = 0.43, a = 0.1, b = 0.48, a = 0.14, \text{and } c = 0.43, a = 0.14$.

Let $\Gamma_A$ be the trajectory tangent to $L_M$. Thus, the trajectory $\Gamma_A^-$ and $L_1$ intersect at point $E_1(p_{E_1}, h_{E_1})$. Let $p_{N_1}$ be the horizontal coordinate of curve $L_N$ under $h = h_{E_1}$. We have the following two different situations:

(I) $p_{E_1} < p_{N_1}$. If $p_{E_1} < p_{N_1}$, the situation is similar to the Case $(C_1)$. At this time $\Gamma_A$ and $L_N$ have no intersection and all points starting from the phase set will eventually hit the impulsive set. There is a point $B_1(p_{B_1}, h_{B_1})$ on $L_N$ such that the trajectory $\Gamma_{B_1}$ is tangent to $L_N$. And $\Gamma_{B_1}$ reaches the impulsive set at point $B_2(p_{B_2}, h_{B_2})$ (Fig. 1b). Therefore, the range of the impulsive set is:

$$M_2 = \{(p, h) | p \geq p_{B_2}, h \geq h_{B_2}\}.$$

(II) $p_{E_1} > p_{N_1}$. If $p_{E_1} > p_{N_1}$, the curve $\Gamma_A^-$ intersects the line $L_N$ at points $D_1(p_{D_1}, h_{D_1})$ and $D_2(p_{D_2}, h_{D_2})$, where $h_{D_1} < h_{D_2}$. There is no intersection between the system trajectory and the impulsive set whose initial point is between $D_1$ and $D_2$ on $L_N$ at this case (Fig. 1c). So we give the following range of impulsive set:

$$M_3 = \{(p, h) | p \geq p_A, h \geq h_A\}.$$
and the range of the phase set is
\[ N_3 = \left\{ (p^+, h^+) | p^+ \in \left[ p_A + \frac{\eta}{1 + \eta p_A}, p_D \right], \right. \]
\[ \left. \cup \{p_{D_2}, +\infty\}, h^+ \in \left[ h_A - \frac{\sigma h_A^2}{h_A + \mu}, h_D \right], \right. \]
\[ \left. \cup \{h_{D_2}, +\infty\} \right\}. \]

According to the above discussion, the Poincaré map is constructed. Given an initial point \( S_0^+ (p^+_0, h^+_0) \) \( \in \mathbb{N} \), the trajectory starting from \( S_0^+ \) can be expressed as
\[ \Gamma(t, t_0, S_0^+) = \Gamma(p(t, t_0, (p^+_0, h^+_0)), h(t, t_0, (p^+_0, h^+_0))). \]

For \( \forall S_i^+(p_i^+, h_i^+) \in \mathbb{N} \), the trajectory from \( S_i^+ \) reaches the impulsive set at point \( S_{i+1}(p_{i+1}, h_{i+1}) \) elapsed time \( \bar{i} \). We express this process as
\[ \Gamma(\bar{i}, p_i^+, h_i^+) = \Gamma(\bar{i}, p_{i+1}, h_{i+1}) = \Gamma(p_{i+1}, h_{i+1}), \]
where \( h_{i+1} = \left( k - p_{i+1} - \frac{\eta}{\eta + (p_{i+1})} \right) (1 + p_{i+1}) \) and \( h_{i+1} = h^+(\bar{i}, p_{1+i}, h_i^+) \). According to the Cauchy–Lipschitz Theorem, \( h_{i+1} \) is only expressed by \( h_i^+ \). Define \( h_{i+1} = \xi(h_i^+) \). Then the expression of Poincaré map is
\[ h_{i+1} = h_{i+1} - \frac{\sigma h_i^2}{h_i^+ + \mu} = \xi(h_i^+) - \frac{\sigma \xi(h_i^+)^2}{\xi(h_i^+)} + \mu = Q_M(h_i^+). \]

3 The natures of Poincaré map

We now explore the nature of the Poincaré map \( Q(Z) \), which are summarized in the following theorem. The details are omitted here, and the proof of the theorem can be found in article [32].

**Theorem 3.1** Assume \( k < 2p^* + 2 \) and \( p^* \leq p_M^* \). The main properties of the Poincaré map are listed (Fig. 2):

(i) \( Q(Z) \) is defined on \( (0, +\infty) \), and it is monotonically decreasing on \( (0, h_{D_1}) \) and monotonically increasing on \( [h_{D_1}, +\infty) \). Furthermore, \( Q(Z) \) reaches its maximum value at \( h_{D_1} \) and no maximum value.

(ii) \( Q(Z) \) is continuously differentiable in \( (0, +\infty) \), and admits a unique fixed point.

When \( p^* > p_M^* \) and \( p_{E_1} < p_{N_1} \), the natures of Poincaré map is similar to Case (C1). Therefore, next we discuss the situation where \( p_{E_1} > p_{N_1} \).

**Theorem 3.2** Assume \( k < 2p^* + 2 \) and \( p^* > p_M^* \) and \( p_{E_1} > p_{N_1} \). The main properties of the Poincaré map are listed (Fig. 3):

(i) \( Q(Z) \) is defined on \( (0, h_{D_1}) \cup [h_{D_2}, +\infty) \). On \( (0, h_{D_1}) \) the map \( Q(Z) \) decreases and on \( [h_{D_2}, +\infty) \) the map \( Q(Z) \) increases.

(ii) \( Q(Z) \) is continuously differentiable in its domain.

(iii) If \( Q(h_{D_1}) < h_{D_1} \), \( Q(Z) \) has a unique fixed point on \( (0, h_{D_1}) \). If \( Q(h_{D_1}) > h_{D_1} \), \( Q(Z) \) has no fixed point.

**Proof** (i) When \( p_{E_1} > p_{N_1} \), we know the trajectory \( \Gamma_A \) and the phase set \( L_N \) intersect at two points \( D_1 \) and \( D_2 \), and \( \Gamma_A \) is tangent to the phase set \( L_M \). For \( \forall S_i^+(p_i^+, h_i^+) \in \mathbb{N} \), if \( h_i^+ \in (h_{D_1}, h_{D_2}) \), the trajectory starting from the phase set cannot reach the impulsive set. Therefore, \( Q(Z) \) is well defined when \( Z \in (0, h_{D_1}) \cup [h_{D_2}, +\infty) \). So Poincaré map is meaningful in the interval \( (0, h_{D_1}) \cup [h_{D_2}, +\infty) \).

Suppose there are two points \( S_{k_1}^+(p_{k_1}^+, h_{k_1}^+) \) and \( S_{k_2}^+(p_{k_2}^+, h_{k_2}^+) \) on the phase set. If \( h_{k_1}^+, h_{k_2}^+ \in [h_{D_2}, +\infty) \), let \( h_{k_1}^+ < h_{k_2}^+ \). From the trajectory of system (1) we get through time \( i \), \( \Gamma(i, p_i^+, h_i^+) = \Gamma(h_{D_1}, p_{i+1}, h_{i+1}) \). Since the uniqueness of the solution we have \( h_{k_1+1} < h_{k_2+1} \). From the expression of Poincaré map \( Q(Z) \), we get \( Q(h_{k_1+1}) < Q(h_{k_2+1}) \). So \( Q(Z) \) is monotonically increasing on \( [h_{D_2}, +\infty) \).
Fig. 2 The Poincaré map $Q(Z)$ and fixed point in case $(C_1)$, where the parameter values are $k = 2.4$, $a = 0.1$, $W = 0.43$, $\eta = 5$, $\tau = 1.8$, $u_1 = 0.9$, $v_1 = 0.1\ a\mu = 0.4$, $\sigma = 0.8\ b\mu = 0.8$, $\sigma = 0.2$

Fig. 3 The Poincaré map $Q(Z)$ and its fixed point in case $(C_2)(II)$, where the parameter values are $k = 2.4$, $a = 0.14$, $W = 0.43$, $\eta = 5$, $\tau = 1.8$, $u_1 = 0.9$, $v_1 = 0.1\ a\mu = 0.4$, $\sigma = 0.8\ b\mu = 0.8$, $\sigma = 0.3$
When \( h_{k1}^+, h_{k2}^- \in (0, D_1) \) and \( h_{k1}^+ < h_{k2}^- \), the trajectory starts from point \( S_{k1}^- S_{k2}' \) through the isometric line \( L_1 \) and intersects the \( L_N \) at the point \( S_{k1}'(p_{k1}', h_{k1}^-) \) and \( S_{k2}'(p_{k2}', h_{k2}^-) \). It is easy to get that \( h_{k1}^+ = h_{k2}^- \). Then \( Q_M(h_{k1}^-) > Q_M(h_{k2}^-) \) from the uniqueness of the solution. So \( Q(Z) \) is monotonically decreasing on \([0, h_{D1}]\).

(ii) According to Eq. (6), we obtain that the function \( g(p, h) \) is continuously differentiable. Therefore, from the continuous differentiability theorem of differential equations, Poincaré map \( Q(Z) \) is continuously differentiable on \([0, h_{D1}] \cup [h_{D2}, +\infty)\) in the first quadrant.

(iii) The curve \( \gamma_1 \) from \( D_1 \) reaches the impulse set at point \( A \), and then arrives in phase set at point \( A' \). If \( Q(h_{D1}) < h_{D1} \), then \( \gamma_A = Q(h_{D1}) \in (0, D_1) \) and \( h_{k1}^+ < h_{D1} \). Since \( Q(Z) \) is monotonically decreases on \([0, h_{D1}]\), then \( Q(h_{k1}^+) = h_{D1} \). We get there exists \( \tilde{h} \in (h_{D1}^-, h_{D1}) \) such that \( Q(\tilde{h}) = \bar{h} \) from the zero point existence theorem and the continuous differentiability of \( Q(Z) \). Therefore, \( Q(Z) \) has a unique fixed point on \([0, h_{D1}]\).

If \( Q(h_{D1}) > h_{D2} \), for any \( h_i \in [h_{D2}, +\infty) \), the trajectory from point \( h_i \) reaches the \( L_i \) at point \( h_{i+1} \). Then \( S_{i+1}'(p_{i+1}', h_{i+1}^-) \) is pulsed to \( (p_{i+1}', h_{i+1}^-) \). According to the trajectory of system (1), we easily obtain that \( h_i < h_{i+1} < h_{i+1}^- \). Therefore, there is no \( h_i = \infty \) such that \( Q(h_i) = \bar{h} \).



4 Order-k periodic solution

From the previous section we know that the system has a unique fixed point, which indicates that model (1) has a unique order-1 periodic solution.

Note: When \( \tau = 0, p'(t^+) = p(t) \), the trajectory of any initial point on the phase set will reach the impulsive set. After the impulse, the trajectory is a vertical line and cannot return to the phase set. Therefore, the order-1 periodic solution does not exist. When \( \sigma = 0, h'(t^+) = h(t) \), the trajectory is a horizontal straight line after the impulse. When the initial point is below the intersection of the isolinic line and the phase set, the system does not have an order-1 periodic solution. When the initial point is above the intersection of the isolinic line and the phase set, the system may have an order-1 periodic solution.

**Theorem 4.1** In case \((C_1)\), system (1) has a unique order-1 periodic solution. Furthermore, we obtain that

(i) If \( Q(h_{T1}) > h_{T1} \), the order-1 periodic solution is globally stable;

(ii) If \( Q(h_{T1}) < h_{T1} \), the order-1 periodic solution is globally stable when \( Q^2(h_{0}^+) < h_{0}^+ \) for any \( h_{0}^+ \).

**Proof** From (6), we get \( h_{i+1}^+ = \xi(h_{i}^+) - \frac{\sigma h_{i}^+}{\xi(h_{i}^+)+\mu} = Q(h_{i}^+) \). Therefore, the trajectory with \( S_{0}^+(p_{0}, h_{0}^+) \in N \) as the initial point intersects the impulsive set at the point \( S_{1}(p_{1}, h_{1}) \), and will arrive at point \( S_{1}^+(p_{1}^+, h_{1}^+) = h_{D1} \) through the action of the impulse. We express this process as \( Q(h_{1}^+) = \xi(h_{0}^+) - \frac{\sigma h_{0}^+}{\xi(h_{0}^+)+\mu} = h_{1}^+ \). Through the similar process we have \( Q(h_{1}^+) = Q(h_{2}^+) \). Further we get \( h_{2}^+ = Q(h_{1}^+) = h_{k}^+ \).

(i) In case \((C_1)\), from Theorem 3.1 we get if \( Q(h_{T1}) > h_{T1} \), the Poincaré map \( Q(Z) \) exists unique fixed point \( h \in [h_{T1}, +\infty) \). So we have the following two situations:

If \( h_{0}^+ \in (h_{T1}, h] \), \( Q(Z) \) monotonically increasing in this interval. Therefore, \( h = Q(h_{T1}) > Q(h_{0}^+) \). Further through the trajectory and the nature of \( Q(Z) \) we get

\[ h = Q(h_{T1}) > Q(h_{0}^+) \]

Therefore, \( h > Q(h_{0}^+) > Q^2(h_{0}^+) > \ldots > h_{T1} \), which represents that \( Q^k(h_{0}^+) \) increases monotonically and \( \lim_{n \to +\infty} Q^k(h_{0}^+) = h \).

If \( h_{0}^+ \in (h, +\infty) \), through similar analysis we obtain that

\[ h_{0}^+ > Q(h_{0}^+) > Q(h_{1}^+) > Q(h_{2}^+) \]

Further, we have

\[ h_{0}^+ > Q(h_{0}^+) > Q^k(h_{0}^+) > Q^k(h_{1}^+) > \ldots > h_{T1} \]

Therefore, \( Q(h_{0}^+) \) decreases monotonically and \( \lim_{n \to +\infty} Q^k(h_{0}^+) = h \). The order-1 periodic solution is globally stable if \( Q(h_{T1}) > h_{T1} \).
(ii) Sufficient condition: We know when \( Q(h_{T_{1}}) < h_{T_{1}}, \) \( Q(Z) \) has unique fixed point \( \tilde{h} \in (0, h_{T_{1}}] \) from Theorem 3.1. And \( Q(Z) \) monotonically decreasing on \((0, h_{T_{1}}]\). Therefore, if \( Q^{2}(h_{0}^{+}) < h_{0}^{+} \) for any \( h_{0}^{+} \in [\tilde{h}, h_{T_{1}}]\), then
\[
h_{T_{1}} > h_{0}^{+} > Q^{2}(h_{0}^{+}) > \tilde{h} > Q(h_{0}^{+}) > Q(h_{T_{1}}).
\]
Further, we derive that
\[
h_{T_{1}} > Q^{2k}(h_{0}^{+}) > \tilde{h} > Q^{2k+1}(h_{0}^{+}) > Q(h_{T_{1}}).
\]
Therefore, we have \( \lim_{k \to \infty} Q^{2k}(h_{0}^{+}) = \lim_{k \to \infty} Q^{2k+1}(h_{0}^{+}) = \tilde{h} \) from monotone bounded theorem. Then, the order-1 periodic solution is globally stable.

(iii) Necessary condition: When the order-1 periodic solution \( \tilde{h} \) is globally stable, we assume there exists \( \tilde{h}_{1}^{+} \in (\tilde{h}, h_{T_{1}}] \) such that \( Q^{2}(\tilde{h}_{1}^{+}) \geq \tilde{h}_{1}^{+} \). Then, there exists a \( \tilde{h}_{2}^{+} \in (\tilde{h} - \varepsilon, \tilde{h} + \varepsilon) \) and \( Q^{2}(\tilde{h}_{2}^{+}) < \tilde{h}_{2}^{+} \) from the stability of \( \tilde{h} \), where \( \varepsilon \) is a small enough number. Since Poincaré map is continuous, then there is at least a \( \tilde{h} \) between \( \tilde{h}_{1}^{+} \) and \( \tilde{h}_{2}^{+} \) which makes \( Q^{2}(\tilde{h}^{+}) = \tilde{h}^{+} \). At this time, system (1) has order-2 periodic solution which is inconsistent with the known conditions. Therefore, if the order-1 periodic solution is globally stable, then \( Q^{2}(h_{0}^{+}) < h_{0}^{+} \) for any \( h_{0}^{+} \in [\tilde{h}, h_{T_{1}}] \).

\( \square \)

**Theorem 4.2** In case (C2) (II), when \( Q(h_{D_{1}}) < h_{D_{1}} \) and \( Q^{2}(h_{0}^{+}) < h_{0}^{+} \) for \( \forall h_{0}^{+} \in (\tilde{h}, h_{D_{1}}] \), system (1) exists an order-1 periodic solution which is globally stable.

**Proof** According to Theorem 3.2, we get that Poincaré map \( Q(Z) \) exists a unique fixed point if \( Q(h_{D_{1}}) < h_{D_{1}} \) and has no fixed point if \( Q(h_{D_{1}}) > h_{D_{1}} \). So the order-1 periodic solution for system exists in \((0, h_{D_{1}}]\).

For \( \forall h_{0}^{+} \in (\tilde{h}, h_{D_{1}}] \), let \( Q^{k}(h_{0}^{+}) = h_{k}^{+} \). According to \( Q(Z) \) that is monotonically decreasing on \((0, h_{D_{1}}]\), then we get \( Q(h_{D_{1}}) \leq h_{1}^{+} < \tilde{h} \). If \( Q^{2}(h_{0}^{+}) < h_{0}^{+} \), then \( \tilde{h} < h_{2}^{+} < h_{0}^{+} < h_{D_{1}} \). By further deduction we get
\[
Q(h_{D_{1}}) \leq h_{1}^{+} < h_{3}^{+} < \ldots < h_{2k+1}^{+} < \tilde{h} < h_{2k}^{+} < \ldots < h_{4}^{+} < h_{2}^{+} < h_{0}^{+} < h_{D_{1}}.
\]
Then the order-1 periodic solution \( Q(h) = \tilde{h} \) is globally stable if \( Q(h_{D_{1}}) < h_{D_{1}} \) and \( Q^{2}(h_{0}^{+}) < h_{0}^{+} \) for \( \forall h_{0}^{+} \in (\tilde{h}, h_{D_{1}}] \).

\( \square \)

**Theorem 4.3** In case (C1), if \( Q(h_{T_{1}}) < h_{T_{1}} \) and \( Q^{2}(h_{T_{1}}) < h_{T_{1}} \), then the stable order-1 or order-2 periodic solution exists for system (1).

**Proof** For initial point \( S_{0}^{+}(p_{0}^{+}, h_{0}^{+}) \) in the phase set, where \( p_{0}^{+} > 0 \) and \( h_{0}^{+} > 0 \), we have a positive integer \( k \) such that \( h_{k}^{+} = Q^{k}(h_{0}^{+}) \) holds after the pulse. When \( h_{0}^{+} > h_{T_{1}} \), \( Q(Z) \) is no fixed point on \([h_{T_{1}}, +\infty) \) and monotonically increasing in this interval from Theorem 3.1. So there exists an integer \( q \) such that \( h_{q}^{+} < h_{T_{1}} \) and \( h_{q}^{+} < h_{T_{1}} \). It follows that \( h_{q}^{+} = Q(h_{q}^{+} - 1) > Q(h_{T_{1}}) \), then \( Q(h_{T_{1}}) < h_{q}^{+} < h_{T_{1}} \).

When \( 0 < h_{0}^{+} \leq h_{T_{1}} \), \( Q(Z) \) is monotonically decreasing in \((0, h_{T_{1}}] \) and \( Q(h_{T_{1}}) < h_{T_{1}} \), then, we get \( Q(h_{0}^{+}) \leq h_{T_{1}} \). From further analysis there exists an integer \( q \) and \( Q(h_{T_{1}}) < h_{q}^{+} < h_{T_{1}} \).

According to \( Q(Z) \) that decreases monotonically and \( Q^{2}(Z) \) that increases monotonically on \([Q(h_{T_{1}}), h_{T_{1}}]\), we get the following
\[
[Q(h_{T_{1}}), h_{T_{1}}] = [Q(h_{T_{1}}), Q^{2}(h_{T_{1}})] \subset [Q(h_{T_{1}}), h_{T_{1}}] .
\]
Therefore, the periodic solution only needs to be studied in interval \([Q(h_{T_{1}}), h_{T_{1}}]\). Let \( Q(h_{0}^{+}) = h_{1}^{+} \neq h_{0}^{+} \) and \( Q^{2}(h_{0}^{+}) = h_{2}^{+} \neq h_{0}^{+} \), then system (1) has an order-1 or order-2 periodic solution. We analyze the four situations as follows:

(i) If \( Q(h_{T_{1}}) \leq h_{2}^{+} < h_{0}^{+} < h_{1}^{+} \leq h_{T_{1}} \), then \( Q(h_{T_{1}}) \leq Q^{2}(h_{0}^{+}) < h_{0}^{+} < Q(h_{0}^{+}) \leq h_{T_{1}} . \)

From the monotonicity of \( Q(Z) \) we obtain
\[
\begin{align*}
& h_{1}^{+} = Q(h_{0}^{+}) < Q(Q^{2}(h_{0}^{+})) = Q^{3}(h_{0}^{+}) = h_{3}^{+}, \quad \text{and} \quad h_{2}^{+} = Q(Q^{2}(h_{0}^{+})) < Q(Q(h_{0}^{+})) = Q^{2}(h_{0}^{+}) = h_{2}^{+} .
\end{align*}
\]

By mathematical induction, the relation is obtained as follows
\[
\begin{align*}
Q(h_{T_{1}}) & \leq Q^{2k+2}(h_{0}^{+}) < Q^{2k}(h_{0}^{+}) \\
& < \ldots < Q^{2}(h_{0}^{+}) < h_{0}^{+} < Q(h_{0}^{+}) \\
& < \ldots < Q^{2k-1}(h_{0}^{+}) < Q^{2k+1}(h_{0}^{+}) < h_{T_{1}} .
\end{align*}
\]

(ii) If \( Q(h_{T_{1}}) \leq h_{0}^{+} < h_{2}^{+} < h_{1}^{+} \leq h_{T_{1}} \), then \( Q(h_{T_{1}}) \leq h_{0}^{+} < Q(h_{T_{1}}) < Q^{2}(h_{0}^{+}) \leq Q(h_{0}^{+}) \leq Q(h_{T_{1}}) \). In this case, we get: \( h_{3}^{+} = Q^{2}(h_{0}^{+})Q^{3}(h_{0}^{+}) = Q^{3}(h_{0}^{+}) = h_{3}^{+} \). Furthermore, we get
\[
\begin{align*}
Q(h_{T_{1}}) & \leq Q(h_{0}^{+}) < \ldots < Q^{2k-1}(h_{0}^{+}) \\
& < Q^{2k+1}(h_{0}^{+}) < \ldots < Q^{2k+2}(h_{0}^{+}) \\
& < Q^{2k}(h_{0}^{+}) < \ldots < Q^{2}(h_{0}^{+}) < h_{0}^{+} < h_{T_{1}} .
\end{align*}
\]
(iii) If \( Q(h_{T_1}) \leq h_1^+ < h_2^+ < h_0^+ \leq h_{T_1} \). After the similar discussion we have

\[
Q(h_{T_1}) \leq Q(h_0^+) < \ldots < Q^{2k-1}(h_0^+)
\]
\[
< Q^{2k+1}(h_0^+) < \ldots < Q^{2k+2}(h_0^+)
\]
\[
< Q^k(h_0^+) < \ldots < Q^2(h_0^+) < h_0^+ < h_{T_1}.
\]

(iv) If \( Q(h_{T_1}) \leq h_1^+ < h_0^+ < h_2^+ \leq h_{T_1} \). Through the similar derivation we get

\[
Q(h_{T_1}) \leq \ldots < Q^{2k+1}(h_0^+) < Q^{2k-1}(h_0^+)
\]
\[
< \ldots < Q(h_0^+) < h_0^+ < Q^2(h_0^+)
\]
\[
< \ldots < Q^2(h_0^+) < Q^{2k+2}(h_0^+) \leq h_{T_1}.
\]

After analysis, the sequence \( Q^{2k}(h_0^+) \) and \( Q^{2k+1}(h_0^+) \) is bounded in situation (ii) and (iii). Thus, there exist values \( \bar{h} \) such that

\[
\lim_{k \to \infty} Q^{2k}(h_0^+) = \bar{h}
\]

or exists different values \( \bar{h}_1, \bar{h}_2 \) such that

\[
\lim_{k \to \infty} Q^{2k}(h_0^+) = \bar{h}_1 \quad \text{and} \quad \lim_{k \to \infty} Q^{2k+1}(h_0^+) = \bar{h}_2,
\]

in situation (i) and (iv), only the latter case holds. Therefore, system (1) has an order-1 or order-2 periodic solution.

**Theorem 4.4** In case \( C_1 \), if \( Q(h_{T_1}) = h_{T_2}^+ < h_{T_1} \) and \( Q^2(h_{T_1}) < h_{m_1}^+ \), where \( h_{m_1}^+ = \min(h^+, Q(h^+)) = h_{T_1} \), then the system has an order-3 periodic solution.

**Proof** When \( Q(h_{T_1}) = h_{T_2}^+ < h_{T_1} \), the \( Q(Z) \) has a unique fixed point \( \bar{h} \in (Q(h_{T_1}), h_{T_2}) \). According to the continuity of \( Q(Z) \) and \( Q(\bar{h}) = \bar{h} \), there exist \( h_{m_1}^+ \in (0, \bar{h}) \) such that \( Q(h_{m_1}^+) = h_{T_1} \). Let \( G(h) = Q^3(h) - h \).

Then, from the continuity of \( G(h) \) and the expression of \( Q(Z) \), we get

\[
Q^3(h_{m_1}^+) = Q^3(h_{T_1}) < h_{m_1}^+ \quad \text{and} \quad Q^3(0) > 0
\]

so \( G(h_{m_1}^+) < 0 \) and \( G(0) > 0 \).

Therefore, there is \( \bar{h} \) in the interval \( (0, h_{m_1}^+) \) such that \( Q^3(\bar{h}) = \bar{h} \). Then, system (1) has an order-3 periodic solution. \( \square \)
Remark 4.1 Same as the proof method of Theorem 4.4, we can get when \( Q^{k-1}(h_{T_k}) < h_{m_1}^+ \), the order-k \((k > 2)\) periodic solution exist, and also Theorem 4.5.

**Theorem 4.5** In case \((C_2)(II)\), if \( Q(h_{D_1}) = h_A^+ < h_{D_1} \) and \( Q^2(h_{D_2}) < h_{m_2}^+ \), where \( h_{m_2}^+ = \min\{h^+, F_M(h^+) = h_{D_1}\} \), then an order-3 periodic solution exists.

### 5 Numerical simulations and discussion

In this section, we numerically verify the theoretical results obtained in the previous section. To this end, we make \( k = 2.4, a = 0.1, W = 0.43, \eta = 5, \tau = 1.8, \mu = 0.4, \sigma = 0.8, \) and \( u_1 = 0.9, v_1 = 0.1 \).

When \( u_1 = 1, v_1 = 0 \) in the action threshold, the impulsive set and phase set are both straight lines, which is similar to most of the previous studies (Fig. 4a). At this time, the threshold is only related to the population density of the plankton. If we change the parameters in the action threshold as shown in Fig. 4b and c, the impulsive set and phase set become two curves. And the degree of curvature of the curve changes with the change of \( u_1 \) and \( v_1 \). We get the population density and rate of change to determine the size of the action threshold, but system (1) still has an order-1 periodic solution.

The change of the period \( T \) under different thresholds is numerically analyzed as shown in Fig. 5. We get that the control period under the fixed threshold is longer than the action threshold. And with the increase in parameter \( a \), the period increases rapidly. When the action threshold is used, the control period decreases with parameter \( v_1 \) increases. As a result, herbivores can
be caught in time so that people get more economic benefits.

For case \(C_1\), we set \(k = 2.4, a = 0.1, W = 0.43, \eta = 5, \mu = 0.4, \tau = 1.8, \sigma = 0.8, u_1 = 0.9, v_1 = 0.1.\) The system starts from the point \((p_0, h_0) = (1, 0.25)\), such as Fig. 6a shown. Figure 6b and c, respectively, represents the time series of plankton and herbivore, respectively. Through the blue trajectory we get that the system has a stable order-1 periodic solution after the pulse. Here, the red trajectory indicates the solution of system (1) without pulses. The comparison indicates that the density of the plankton and herbivore can be maintained in a stable range under the state-dependent impulsive feedback control with action threshold. And when impulsive control is not adopted, the number of the plankton and herbivores rises in a short time and then drops to a small value. For case \(C_2\)(II), the trajectory of system (1) starting from the point \((p_0, h_0) = (1.002, 1.094)\) as shown in Fig. 7. When \(Q(h_{D_1}) < h_{D_1}\), we obtain that system (1) tends to the order-1 periodic solution (Fig. 7a), and no order-1 periodic solution when \(Q(h_{D_2}) > h_{D_2}\) (Fig. 7b).

6 Conclusion

In this paper, we proposed a herbivore–plankton state-dependent impulsive model with action threshold and nonlinear impulsive function. We thoroughly studied the complex dynamics of the model and illustrated the influence of the action threshold on the system dynamics. The control form adopted in this paper is more in line with the development law of biological population, which makes herbivores harvest in time and obtain more economic benefits.

Compared with the existing work, for example [9], the innovation of this article is:

1. The paper uses the action threshold determined by change rate and population density. The fixed threshold is a special case of the action threshold, and the action threshold is an extension of the fixed threshold. In this sense, our results generalize the
results in [9]. Due to the integration of various factors affecting the population, the control method adopted in this paper is more general and the results obtained are more realistic.

(2) Reference [9] uses the method of successor function to prove the existence of order-1 periodic solution, while this paper introduces Poincaré map to study the properties of the model more deeply. For example: Poincaré map monotonicity, continuous differentiability, fixed point.

(3) The dynamic characteristics of the model are studied more comprehensively and in detail. Ref. [9] only discusses the existence, uniqueness, and stability of the order-1 periodic solution. On this basis, this paper gives the necessary and sufficient conditions for the stability of the order 1 periodic solution and the existence conditions of the order-k(k ≥ 2) periodic solution.

Although some good results have been obtained in this article, the research content is not perfect. We will conduct research from the following aspects in the future.

(1) This article only considers the impact of state-dependent impulse control on the dynamic system of the biological model. However, in real life, there are other types of disturbances, and the changes of biological systems under other disturbances will be studied in the future.

(2) The research results and research methods used in this paper are based on two-dimensional systems, and it is hoped that this research method will be further extended to three-dimensional and more than three-dimensional biological systems in the future.

(3) This analysis technique and method are used to study the models under different release strategies and other models in different application fields.

(4) Nonlinear impulse feedback control leads to the complexity of the impulse set and phase set. Therefore, the Poincaré map determined by the impulse point sequence has more complex properties. Considering other biological models with dynamic thresholds, it provides a more comprehensive qualitative model for the nonlinear impulse dynamic system. It enriches the theoretical analysis of impulsive semi-dynamic systems.

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References

1. Mukhopadhyay, B., Bhattacharyya, R.: Role of gestation delay in a plankton-fish model under stochastic fluctuations. Math. Biosci. 215(1), 26–34 (2008)
2. Lv, Y., Pei, Y., Gao, S., Li, C.: Harvesting of a phytoplankton–zooplankton model. Nonlinear Anal. Real World Appl. 11(5), 3608–3619 (2010)
3. Zhang, T., Liu, X., Meng, X., Zhang, T.: Spatio-temporal dynamics near the steady state of a planktonic system. Comput. Math. Appl. 78(12), 4490–4505 (2018)
4. Yu, X., Yuan, S., Zhang, T.: Asymptotic properties of stochastic nutrient-plankton food chain models with nutrient recycling. Nonlinear Anal. Hybrid Syst. 34, 209–225 (2019)
5. Yu, X., Yuan, S., Zhang, T.: Survival and ergodicity of a stochastic phytoplankton-zooplankton model with toxin-producing phytoplankton in an impulsive polluted environment. Appl. Math. Comput. 347, 249–264 (2019)
6. Jia, D., Zhang, T., Yuan, S.: Pattern dynamics of a diffusive toxin producing phytoplankton-zooplankton model with three-dimensional patch. Int. J. Bifurc. Chaos 29, Article Number: 1930011 (2019)
7. Yan, S., Jia, D., Zhang, T., Yuan, S.: Pattern dynamics in a diffusive predator-prey model with hunting cooperation. Chaos Solitons Fractals 130, Article Number: 109428 (2020)
8. Peng, Y., Li, Y., Zhang, T.: Global bifurcation in a toxin producing phytoplankton–zooplankton system with prey-taxis. Nonlinear Anal. Real World Appl. 61 Article Number: 103326 (2021)
9. Fang, D., Pei, Y., Lv, Y., Chen, L.: Periodicity induced by state feedback controls and driven by disparate dynamics of a herbivore-plankton model with cannibalism. Nonlinear Dyn. 90(5), 1–16 (2017)
10. Tang, S., Tang, B., Wang, A., Xiao, Y.: Holling II predator-prey impulse semi-dynamic model with complex poincaré map. Nonlinear Dyn. 81(3), 1575–1596 (2015)
11. Bainov, D.D., Simeonov, P.S.: Impulsive Differential Equation: Periodic Solutions and Applications. Pergamon Press Inc, Oxford (2015)
12. Li, D., Cheng, H., Liu, Y.: Dynamic analysis of beddington-deangelis predator-prey system with nonlinear impulse feedback control. Complexity (2019)
13. Wang, F., Zhang, X.: Adaptive finite time control of nonlinear systems under time-varying actuator failures. IEEE Trans. Syst. Man Cybern. Syst. 1–8 (2018)
14. Ciesielski, K.: On stability in impulsive dynamical systems. Bull. Pol. Acad. Sci. Math. 52(84), 81–91 (2010)
15. Bonotto, E.M., Federson, M.: Limit sets and the Poincare–Bendixson theorem in impulsive semidynamical systems. J. Differ. Equ. 244(9), 2334–2349 (2008)
16. Baek, Hunki: The dynamics of a predator-prey system with state-dependent feedback control. Abstr. Appl. Anal. 2012, 1–17 (2012)
17. Yang, J., Tang, S.: Holling type II predator-prey model with nonlinear pulse as state-dependent feedback control. J. Comput. Appl. Math. 291, 225–241 (2016)
18. Liu, H., Cheng, H.: Dynamic analysis of a prey-predator model with state-dependent control strategy and square root response function. Adv. Differ. Equ. 2018(1), 63 (2018)
19. Li, T., Zhao, W.: Periodic solution of a neutral delay Leslie predator-prey model and the effect of random perturbation on the smith growth model. Complexity 2020, 15 (2020)
20. Li, Y., Li, Y., Liu, Y., Cheng, H.: Stability analysis and control optimization of a prey-predator model with linear feedback control. Discrete Dyn. Nat. Soc. 2018, 12 (2018)
21. Shi, Z., Cheng, H., Liu, Y., Li, Y.: A cydia pomonella integrated management predator-prey model with smith growth and linear feedback control. IEEE Access 7(1), 126066–126076 (2019)
22. Wang, Y., Cheng, H., Li, Q.: Dynamic analysis of wild and sterile mosquito release model with Poincaré map. Math. Biosci. Eng. 6(16), 7688–7706 (2019)
23. Shi, Z., Cheng, H., Wang, Y.: Optimization of an integrated feedback control for a pest management predator-prey model. Math. Biosci. Eng. 16(6), 7963–7981 (2019)
24. Xu, C., Yuan, S., Zhang, T.: Average break-even concentration in a simple chemostat model with telegraph noise. Nonlinear Anal. Hybrid Syst. 29, 373–382 (2018)
25. Qi, H., Leng, X., Meng, X., Zhang, T.: Periodic solution and ergodic stationary distribution of Seis dynamical systems with active and latent patients. Qual. Theory Dyn. Syst. 18(2), 347–369 (2019)
26. Zhang, T., Wang, J., Li, Y., Jiang, Z., Han, X.: Dynamics analysis of a delayed virus model with two different transmission methods and treatments. Adv. Differ. Equ. 2020(1), 1 (2020)
27. Wang, W., Lai, X.: Global stability analysis of a viral infection model in a critical case. Math. Biosci. Eng. 17, 1442–1449 (2020)
28. Li, D., Liu, Y., Cheng, H.: Dynamic complexity of a phytoplankton-fish model with the impulsive feedback control by means of Poincaré map. Complexity (2020)
29. Jiang, Z., Zhang, W., Zhang, J., Zhang, T.: Dynamical analysis of a Phytoplankton–Zooplankton system with harvesting term and Holling III functional response. Int. J. Bifurc. Chaos 28(13), 1850162 (2018)
30. Zhong, Z., Pang, L., Song, X.: Optimal control of phytoplankton-fish model with the impulsive feedback control. Nonlinear Dyn. 88(3), 2003–2011 (2017)
31. Yang, J., Tan, Y.: Effects of pesticide dose on Holling II predator-prey model with feedback control. J. Biol. Dyn. 12(1), 527–550 (2018)
32. Wang, Y., Cheng, H., Li, Q.: Dynamical properties of a herbivore-plankton impulsive semidyamnic system with eating behavior. Complexity 2020, 1–15 (2020)

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