Gauge dependence identities for color superconducting QCD

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Using generalized Nielsen identities a formal proof is given that the fermionic quasiparticle dispersion relations in a color superconductor are gauge independent. This turns out to involve gluonic tadpoles which are calculated to one-loop order in a two-flavor color superconductor. Regarding the appearance of gluon tadpoles, we argue that in QCD the color superconducting phase is automatically color neutral.

I. INTRODUCTION

It is well known that sufficiently cold and dense quark matter is a color superconductor \[1, 2\]. At asymptotic densities, when the QCD coupling constant is small, it is possible to investigate the properties of a color superconductor using mostly perturbative techniques based on the fundamental Lagrangian of QCD, because the dominant interaction leading to the formation of Cooper pairs is single-gluon exchange, which is attractive in the color-antitriplet channel. The correct leading order result for the gap has been first obtained by Son \[3\],

\[
\phi = \frac{b_0}{g^2\mu} \exp\left(-\frac{c}{g}\right) \left[1 + \mathcal{O}(g)\right],
\]

with \( c = 3\pi^2/\sqrt{2} \). This result has been confirmed, and the constant \( b_0 \) has been determined \[4, 5, 6, 7\], taking into account also contributions from the quark self energy \[8, 9\].

The gap equation is usually derived from the Schwinger-Dyson equation, where some approximations are made such as the neglect of vertex corrections. In \[10\] it has been shown that large gauge dependences occur in such calculations of the value of the gap for \( g > g_c \sim 0.8 \). In \[11\] it has been shown that the corresponding gauge dependent terms appear in the gap equation at sub-sub-leading order, though gauge dependences arise already at sub-leading order if the gap parameter is not evaluated on the quasiparticle mass shell. Recently, the gap equation has been considered in a non-local gauge where the quark self energy and vertex corrections vanish \[12\] to include higher-order corrections to the prefactor, exploiting the expected gauge independence of the gap as a physical quantity.

In view of all these efforts, it seems desirable to have a formal proof of gauge independence of the relevant quantities. In this Note we shall consider the fermionic quasiparticle dispersion relations in a color superconductor, which are modified by the appearance of a gap which dynamically breaks color symmetry. Such a proof is given in section 2 of the present paper as a straightforward generalization of a similar proof for finite temperature QCD \[13\]. It turns out that in order to formulate this proof one has to allow for the appearance of gluon tadpoles in a color superconducting phase. We demonstrate the possible existence of such tadpoles in section 3 by a one-loop calculation in the example of an \( N_f = 2 \) color superconductor, discussing also briefly the issue of color neutrality. We argue that in QCD (in contrast to NJL models) the color superconducting phase is automatically color neutral.

Our notations follow mostly those of Pisarski and Rischke, Refs. \[4, 14, 15\].
II. GAUGE INDEPENDENCE OF FERMIONIC DISPERSION RELATIONS

We consider the inverse quark propagator in the Nambu-Gor’kov formalism \cite{1, 9, 16},

\[
S^{-1} = \left( \frac{Q + \mu \gamma_0 + \Sigma}{\Phi^+} - \frac{\Phi^-}{Q - \mu \gamma_0 + \Sigma} \right),
\]

(2)

where \( \Phi^\pm \) are the gap functions, related by \( \Phi^- (Q) = \gamma_0 [\Phi^+ (Q)]^\dagger \gamma_0 \), and \( \Sigma (Q) = C [\Sigma (-Q)]^T C^{-1} \) with the charge conjugation matrix \( C \). Flavor and fundamental color indices are suppressed in (2).

This inverse propagator is the momentum space version of the second derivative of the effective action,

\[
\frac{\delta^2 \Gamma}{\delta \Psi (x) \delta \Psi (y)} \bigg|_{\psi = \bar{\psi} = A_i^a = 0, A_0^a = \bar{A}_0^a},
\]

(3)

where \( \Psi = (\psi, \bar{\psi})^T \), \( \bar{\Psi} = (\bar{\psi}, \bar{\bar{\psi}}) \), and \( \bar{A}_0^a \) is the expectation value of \( A_0^a \), which we allow to be non-vanishing (see section 3).

It should be noted that the doubling of fermionic fields in terms of \( \Psi \) and \( \bar{\Psi} \) is just a notational convenience here; the effective action itself should be viewed as depending only on either \( (\psi, \bar{\psi}) \) or the set \( \Psi = (\psi, \bar{\psi})^T \).

The gauge dependence identity for the effective action (generalized Nielsen identity, see appendix) follows from considering a completely arbitrary variation of the gauge fixing function. It can be written as

\[
\delta \Gamma = \int dx \left( \frac{\delta \Gamma}{\delta \psi (x)} \delta X (\psi (x)) - \frac{\delta \Gamma}{\delta \bar{\psi} (x)} \delta X (\bar{\psi} (x)) \right) + \frac{\delta \Gamma}{\delta A^a (x)} \delta X^{a \mu} (x),
\]

(4)

where the various \( \delta X \) are defined in (A3).

Eq. (4) can be cast in a more compact form using the DeWitt notation for the fermions,

\[
\delta \Gamma = \Gamma_{ij} \delta X^i + \int dx \frac{\delta \Gamma}{\delta A^a (x)} \delta X^{a \mu} (A^a (x)),
\]

(5)

where \( i = (\psi (x), \bar{\psi} (x))^T \), \( \bar{i} = (\bar{\psi} (x), \bar{\bar{\psi}} (x))^T \), and the comma denotes functional derivation. Taking the second derivative of (5), setting \( \psi = \bar{\psi} = A_i^a = 0 \), \( A_0^a = \bar{A}_0^a \), and using the fact that

\[
\frac{\delta \Gamma}{\delta A^a_0} \bigg|_{\psi = \bar{\psi} = A_i^a = 0, A_0^a = \bar{A}_0^a} = 0,
\]

(6)

we obtain a gauge dependence identity for the inverse propagator \( \bar{A}^{a 0} \),

\[
\delta \Gamma_{\bar{i} \bar{j}} = -\Gamma_{k \bar{i}} \delta X^k_{\bar{j}} + \Gamma_{\bar{k} \bar{j}} \delta X^k_{\bar{j}} + \int dx \frac{\delta \Gamma_{\bar{i} \bar{j}}}{\delta A^a (x)} \delta X^{a 0} (A^a (x)).
\]

(7)

Up to this point, our functional relations are completely general and apply also to the case of inhomogeneous condensates which may be realized in the so-called LOFF phase (see \cite{17} for a recent review). In this Note we do not attempt to cover the complications this case may add to the question of gauge independence, but continue by assuming translational invariance. This allows us to introduce \( \delta \bar{A}^{a 0} := -\delta X^{a 0} (A^a (x) = 0) \) and to write (6) as

\[
\delta \Gamma_{\bar{i} \bar{j}} = -\Gamma_{k \bar{i}} \delta X^k_{\bar{j}} - \delta X^k_{\bar{j}} \Gamma_{\bar{k} \bar{j}} - \frac{\partial \Gamma_{\bar{i} \bar{j}}}{\partial A^a (x)} \delta \bar{A}^{a 0}.
\]

(8)
Furthermore, we can transform eq. (8) into momentum space,

$$
\delta \Gamma_{ij}(Q) + \delta \tilde{A}^{a0} \frac{\partial \Gamma_{ij}(Q)}{\partial A^{a0}} = -\Gamma_{kj}(Q)\delta X^k_{\bar{i}j}(Q) - \delta X^k_{\bar{i}j}(Q)\Gamma_{ik}(Q),
$$

(9)

where the indices $i$ and $\bar{i}$ from now on comprise only color, flavor, Dirac and Nambu-Gor’kov indices. Using the fact that

$$
\delta \det M = (\det M) \text{Tr}[M^{-1}\delta M]
$$

(10)

for any matrix $M$, we obtain from eq. (9)

$$
\delta \det(\Gamma_{ij}) + \delta \tilde{A}^{a0} \frac{\partial}{\partial A^{a0}} \det(\Gamma_{ij}) \equiv \delta_{\text{tot}} \det(\Gamma_{ij}) = -\det(\Gamma_{ij})[\delta X^k_{\bar{i}k} + \delta X^k_{\bar{k}i}].
$$

(11)

The left hand side of this identity is the total variation \cite{11,18} of the determinant of the inverse quark propagator, with the first term corresponding to the explicit variation of the gauge fixing function, and the second term coming from the gauge dependence of $\tilde{A}^0_a$.

Since the determinant is equal to the product of the eigenvalues, eq. (11) implies that the location of the singularities of the quark propagator is gauge independent, provided the singularities of $\delta X^k_{\bar{i}k}$ do not coincide with those of the quark propagator. As in \cite{12} one may argue that $\delta X$ is 1PI up to a full ghost propagator, and up to gluon tadpole insertions (see Ref. \cite{12} for the explicit diagrammatic structure). The singularities of the ghost propagator are not correlated to the singularities of the quark propagator. Their fundamental difference is in fact enhanced by medium effects—e.g., there exist no hard-thermal/dense-loop (HTL/HDL) \cite{20} corrections to vertex functions involving ghosts, whereas all physical degrees of freedom acquire HTL/HDL self-energy corrections.

Gauge independence of the zeros of the inverse fermion propagator then follows provided that also the 1PI parts of $\delta X$ have no singularities coinciding with the singularities of the propagator. An important caveat in fact comes from massless poles in the unphysical degrees of freedom of the gauge boson propagator, which are typical in covariant gauges and which can give rise to spurious mass shell singularities as encountered in the case of hot QCD in \cite{21}. But, as was pointed out in \cite{22}, these apparent gauge dependences are avoided if the quasiparticle mass-shell is approached with a general infrared cut-off such as finite volume, and this cut-off lifted only in the end, i.e., after the mass-shell limit has been taken. The tadpoles do not introduce any singularities.

The determinant which appears in (11) is taken with respect to color, flavor, Dirac and Nambu-Gor’kov indices. The determinant in Nambu-Gor’kov space may be evaluated explicitly by noting that

$$
\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(\mathcal{D}A - \mathcal{D}BD^{-1}C)
$$

(12)

for arbitrary $n \times n$ matrices $A, B, C, D$ (with $\mathcal{D}$ invertible). Hence we obtain from \cite{2}

$$
\det(\Gamma_{ij}) = \det \left[ (Q - \mu \gamma_0 + \bar{\Sigma})(Q + \mu \gamma_0 + \Sigma) \\
- (Q - \mu \gamma_0 + \bar{\Sigma})\Phi^{-1} (Q - \mu \gamma_0 + \bar{\Sigma})^{-1} \Phi^+ \right].
$$

(13)

The inverse of the matrix of which the determinant is taken here appears, of course, in the ordinary quark propagator, which is obtained by inverting \cite{2},

$$
G^+(Q) = \left[ (Q - \mu \gamma_0 + \bar{\Sigma})(Q + \mu \gamma_0 + \Sigma) \\
- (Q - \mu \gamma_0 + \bar{\Sigma})\Phi^{-1} (Q - \mu \gamma_0 + \bar{\Sigma})^{-1} \Phi^+ \right]^{-1} (Q - \mu \gamma_0 + \bar{\Sigma}).
$$

(14)
At leading order, when the quark self energy can be neglected, eqs. (11) and (13) imply that the gap function is gauge independent on the quasiparticle mass shell. It should be noted, however, that at higher orders only dispersion relations obtained from (13), which also include the quark self energy $\Sigma$, can be expected to be gauge independent.

**III. GLUON TADPOLE AND COLOR NEUTRALITY**

As we have seen, the gauge dependence identities (11) involve gluon tadpoles in a non-trivial manner.

In a color superconductor global color symmetry is dynamically broken by the diquark condensate. This means that there is no symmetry which forbids the existence of gluon tadpoles. Let us consider the one-loop tadpole diagram,

$$\mathcal{T}^a = \left( -\frac{g}{2} \right) \int \frac{d^4 Q}{i(2\pi)^4} \text{Tr}_{D,c,f,NG} [\hat{\Gamma}_0^a \mathcal{S}(Q)].$$

Here $\mathcal{S}(Q)$ is the quark propagator in the Nambu-Gor'kov basis, whose inverse is given by (suppressing color and flavor indices)

$$S^{-1} = \begin{pmatrix} Q + \mu \gamma_0 & \Phi^- \\ \Phi^+ & Q - \mu \gamma_0 \end{pmatrix}$$

with $\Phi^- = \gamma_0 (\Phi^+)^\dagger \gamma_0$. The quark-gluon vertex $\hat{\Gamma}_0^a$ is given by

$$\hat{\Gamma}_0^a = \begin{pmatrix} \Gamma_0^a & 0 \\ 0 & \bar{\Gamma}_0^a \end{pmatrix},$$

with $\bar{\Gamma}_0^a = \gamma_0 T^a$ and $\Gamma_0^a = -\gamma_0 (T^a)^T$. The trace in (15) has to be taken with respect to Dirac, color, flavor and Nambu-Gor'kov indices.

The gap $\Phi^+$ is a matrix in Dirac, color and flavor space. We consider as an example an $N_f = 2$ color superconductor and take the following ansatz for the gap $\Phi^+$:

$$\Phi^+_{fg,ij}(Q) = \epsilon_{fg} \epsilon_{ij3} (\phi^+(P^+ - P^-) + \phi^-(P^- - P^+)),$$

where $f, g$ are flavor indices and $i, j$ are fundamental color indices. The $P$’s are the projection operators introduced in (18). We have assumed for simplicity that the right-handed and left-handed gap functions are equal up to a sign (18).

First we evaluate the trace over Nambu-Gor'kov space which gives

$$\mathcal{T}^a = \left( -\frac{g}{2} \right) \int \frac{d^4 Q}{i(2\pi)^4} \text{Tr}_{D,c,f} [\hat{\Gamma}_0^a \mathcal{G}^+(Q) + \bar{\Gamma}_0^a \mathcal{G}^-(Q)],$$

where $\mathcal{G}^\pm$ is obtained from inverting (10) [14],

$$\mathcal{G}^\pm_{fg,ij} = \delta_{fg} [\delta_{ij} - \delta_{i3} \delta_{j3}] G^\pm + \delta_{i3} \delta_{j3} G^\pm_0],$$

with [14]

$$G^\pm(Q) = \sum_{e=\pm} \frac{q_0 \mp (\mu - eq)}{q_0^2 - (\mu - eq)^2 - |\phi^e|^2 \Lambda^\pm e(q) \gamma_0},$$

$$G^\pm_0(Q) = \sum_{e=\pm} \frac{q_0 \mp (\mu - eq)}{q_0^2 - (\mu - eq)^2} \Lambda^\pm e(q) \gamma_0.$$
In order to obtain an order of magnitude estimate, we make the approximation space we get

\[ T^a = g(T^a)_{33} \int \frac{d^4Q}{i(2\pi)^4} \text{Tr}_D[\gamma_0(G^+ - G^- - G_0^+ + G_0^-)]. \] (23)

Assuming that \( \phi^- \simeq 0 \) [4] and that \( \phi^+ \) has negligible four-momentum dependence in the vicinity of the quasiparticle pole we obtain

\[ T^a \simeq -4g(T^a)_{33} \int \frac{d^4Q}{i(2\pi)^4} \left( \frac{\mu - q}{q_0^2 - (\mu - q)^2 - |\phi^+/|^2} \right) \]
\[ \simeq \frac{g}{\pi^2}(T^a)_{33} \int_0^\infty dq \frac{q^2(\mu - q)}{\sqrt{(\mu - q)^2 + |\phi^+/|^2}} - \frac{1}{|\mu - q|}. \] (24)

In order to obtain an order of magnitude estimate, we make the approximation \( \phi^+(q) \simeq \phi_0^+ \theta(2\mu - q) \) with \( \phi_0^+ = \text{const.} \) [14]. Then the \( q \)-integration can be readily performed, with the result

\[ T^a \simeq -\frac{2g}{\pi^2}(T^a)_{33} \mu (\phi_0^+)^2 \ln \left( \frac{\phi_0^+}{2\mu} \right) + \mathcal{O}(g\mu\phi_0^+)^2, \quad (T^a)_{33} = -\delta^{a3}/\sqrt{3}. \] (25)

This result is in fact of order \( \mu\phi^2 \), because \( \ln(\phi/(2\mu)) \) is of order 1/g. At this order it cannot be excluded that there are cancellations from higher-loop contributions which have been neglected so far [14]. In particular it is still possible that \( T^a \equiv 0 \), although this seems to be unlikely as there is no symmetry which forces \( T^a \) to be exactly zero.

To address the question of color neutrality [24, 25, 26, 27] we consider the partition function

\[ \exp(-\Omega/T) = \int D\varphi \exp(-S[\varphi]), \] (26)

where \( \varphi \) denotes the set of all fields, and \( S[\varphi] \) is the QCD action (including gauge fixing terms and ghosts). Following the argument given in [28] it is easy to see that the system described by this partition function is color neutral, at least if one chooses a gauge fixing which does not involve \( A_0^a \), for instance Coulomb gauge:

The fields \( A_0^a \) appear in the action as Lagrange multipliers for the Gauss law constraint [29]. Therefore in the path integral the integration over the zero-momentum modes \( A_0^a, \bar{\rho} = 0 \) produces delta functions, \( \delta(N_a) \), where \( N_a \) are the color charges. This means that only color neutral field configurations contribute to the partition function, q.e.d.

The gluon tadpole obtained in [25] may in fact be interpreted as an effective chemical potential for the color number eight of the order

\[ g m_D^{-2} \mu \phi^2 \sim \phi^2/(g\mu), \]

where \( m_D \propto g\mu \) is the corresponding leading-order Debye mass [14]. It may be noted that the chemical potential \( \mu_8 \) which has been found by requiring color neutrality in an NJL model is also proportional to \( \phi^2 \) [26]. We emphasize however that whereas in NJL models color neutrality has to be imposed as an additional condition, color neutrality is guaranteed automatically in QCD by the integration over the \( A_0^a \) zero-momentum modes.

**IV. CONCLUSIONS**

We have presented a formal proof of gauge independence for the fermionic quasiparticle dispersion relations in color superconducting QCD. As long as the quark self energy can be neglected,
this implies gauge independence of the gap function on the quasiparticle mass shell. In general, however, only the singularities of the propagator, which involve also the gauge-dependent quark self energy, can be expected to be gauge independent. At higher orders, gauge parameter variations may also affect the temporal component of the gluon tadpole, which, as we have shown in a simple example, can be nonvanishing in a color superconductor. While a nonvanishing gluon tadpole may be viewed as an effective chemical potential for color charge, we have argued that color neutrality is an automatic consequence of QCD, if no explicit chemical potentials for color are introduced.

Acknowledgments

We would like to thank R. Pisarski and D. Rischke for useful correspondence. This work has been supported by the Austrian Science Foundation FWF, project no. P16387-N08.

APPENDIX A: GAUGE DEPENDENCE IDENTITY FOR THE EFFECTIVE ACTION

An arbitrary gauge theory is defined by an action $S_{\text{inv}}[\varphi]$ which is invariant under (infinitesimal) gauge transformations

$$
\delta \varphi^i = D^i_\alpha[\varphi] \delta \xi^\alpha.
$$

(A1)

In order to quantize the theory it is necessary to fix the gauge freedom, for instance with a quadratic gauge-breaking term,

$$
S[\varphi] = S_{\text{inv}}[\varphi] + \frac{1}{2} F^\alpha_\beta[\varphi] F^\alpha_\beta[\varphi].
$$

(A2)

By performing a certain (non-local) gauge transformation on the path-integration variable $\varphi$, it can be shown that an infinitesimal variation of the gauge fixing function, $\delta F^\beta$, induces a change in the effective action as follows,

$$
\delta \Gamma[\bar{\varphi}] = -\Gamma, i[\bar{\varphi}] \langle D^i_\alpha[\varphi] G^\alpha_\beta[\varphi] \delta F^\beta[\varphi] \rangle[\bar{\varphi}] =: \Gamma, i[\bar{\varphi}] \delta X^i[\bar{\varphi}],
$$

(A3)

where $G^\alpha_\beta[\varphi]$ is the ghost propagator in a background field $\varphi$. This gauge dependence identity is a generalized version [13, 30] of the Nielsen identity [18, 19, 31].

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1 We use the DeWitt notation, where a Latin index comprises all discrete and continuous field labels, and a Greek index comprises group and space-time indices.
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