Positive solutions of semilinear elliptic problems in unbounded domains with unbounded boundary

Solutions positives de problèmes semi-linéaires elliptiques dans des domaines non bornes ayant frontière non borné

Giovanna Cerami a, Riccardo Molle b,*, Donato Passaseo c

a Dipartimento di Matematica, Politecnico di Bari, Campus Universitario, Via Orabona 4, 70125 Bari, Italy
b Dipartimento di Matematica, Università di Roma “Tor Vergata”, Via della Ricerca Scientifica n° 1, 00133 Roma, Italy
c Dipartimento di Matematica “E. De Giorgi”, Università di Lecce, P.O. Box 193, 73100 Lecce, Italy

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Abstract

This paper is concerned with the existence and multiplicity of positive solutions of the equation $-\Delta u + u = u^{p-1}$, $2 < p < 2^* = \frac{2N}{N-2}$, with Dirichlet zero data, in an unbounded smooth domain $\Omega \subset \mathbb{R}^N$ having unbounded boundary. Under the assumptions:

(h1) $\exists \tau_1, \tau_2, \ldots, \tau_k \in \mathbb{R}^+ \setminus \{0\}, 1 \leq k \leq N - 2$, such that

$$ (x_1, x_2, \ldots, x_N) \in \Omega \iff (x_1, \ldots, x_i, x_i + \tau_i, \ldots, x_N) \in \Omega, \forall i = 1, 2, \ldots, k, $$

(h2) $\exists R \in \mathbb{R}^+ \setminus \{0\}$ such that $\mathbb{R}^N \setminus \Omega \subset \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N: \sum_{j=k+1}^{N} x_j^2 \leq R^2\}$

the existence of at least $k + 1$ solutions is proved.

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Résumé

Dans cet article on étudie l’existence et la multiplicité de solutions positives pour l’équation $-\Delta u + u = u^{p-1}$, $2 < p < 2^* = \frac{2N}{N-2}$, avec la condition de Dirichlet $u = 0$ sur $\partial \Omega$. Le domaine $\Omega \subset \mathbb{R}^N$ est non borné et $\partial \Omega$ est non borné aussi. En supposant que les conditions

(h1) $\exists \tau_1, \tau_2, \ldots, \tau_k \in \mathbb{R}^+ \setminus \{0\}, 1 \leq k \leq N - 2$, tels que

$$ (x_1, x_2, \ldots, x_N) \in \Omega \iff (x_1, \ldots, x_i, x_i + \tau_i, \ldots, x_N) \in \Omega, \forall i = 1, 2, \ldots, k, $$

(h2) $\exists R \in \mathbb{R}^+ \setminus \{0\}$ tel que $\mathbb{R}^N \setminus \Omega \subset \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N: \sum_{j=k+1}^{N} x_j^2 \leq R^2\}$

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* Corresponding author.
E-mail addresses: cerami@poliba.it (G. Cerami), molle@mat.uniroma2.it (R. Molle).

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soient vérifiées, on démontre que le problème possède au moins $k+1$ solutions.

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1. Introduction and statement of the results

In this paper we are concerned with the existence and the multiplicity of solutions to

\[
(P) \begin{cases}
-\Delta u + u = u^{p-1} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u \in H^1_0(\Omega)
\end{cases}
\]

where $2 < p < 2^* = \frac{2N}{N-2}$ and $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is an unbounded smooth domain with $\partial \Omega$ unbounded.

Problem $(P)$ has a variational structure: its solutions can be found looking for positive functions that are critical points of the functional

\[
E(u) = \int_\Omega (|\nabla u|^2 + u^2) \, dx
\]

constrained to lie on the manifold

\[
V = \{u \in H^1_0(\Omega) : |u|_{L^p(\Omega)} = 1\}.
\]

However, the usual variational techniques (minimization, minimax methods) cannot be applied straightly, because of the lack of compactness, due to the unboundedness of $\Omega$. Indeed, since the embedding $j : H^1_0(\Omega) \to L^p(\Omega)$ is continuous, but not compact, the manifold $V$ is not closed for the weak $H^1_0$-topology and, moreover, the basic Palais–Smale condition is not satisfied by $E$ at every energy level. Furthermore, as we shall see, the situations, one has to face, are strongly depending on the shape of the domain in which $(P)$ is considered, so the corresponding technical difficulties can be considerably different.

When $\mathbb{R}^N \setminus \Omega$ is a ball the existence of a solution to $(P)$ can be, quite easily, proved (see [13]), by minimizing $E$ on the manifold $V_r = \{u \in H_r(\Omega) : |u|_{L^p(\Omega)} = 1\}$, $H_r(\Omega)$ being the subspace of $H^1_0(\Omega)$ consisting of spherically symmetric functions, that, as well known [21], embeds compactly in $L^p(\Omega)$. Analogous devices can be used when $\mathbb{R}^N \setminus \Omega$ is bounded and it is “nearly” spherical, in a suitable sense, or enjoys of some other kind of symmetry [10] and, moreover, when $\Omega$ is a “strip-like” domain [12].

The question becomes more difficult when $\mathbb{R}^N \setminus \Omega$ has no symmetry properties. Indeed a classical, by now, result [13] states that, for a very large class of unbounded domains, those satisfying the condition: $\exists \bar{x} \in \mathbb{R}^N : (v(x), \bar{x}) \geq 0 \ \forall x \in \partial \Omega$, $(v(x), \bar{x}) \not\equiv 0$, $(P)$ admits only the trivial solution $u \equiv 0$. Moreover, even when $\mathbb{R}^N \setminus \Omega$ is bounded, problem $(P)$ is not easy to handle and cannot be solved by minimization: the infimum of $E$ on $V$ equals the infimum of $\|u\|^2_{H^1(\mathbb{R}^N)}$ on the manifold $\{u \in H^1(\mathbb{R}^N) : |u|_{L^p(\mathbb{R}^N)} = 1\}$ and is not achieved [4]. Nevertheless, in this case, a careful analysis of the Palais–Smale sequences behaviour [4] has made possible an estimate of the energy levels in which the compactness is saved and to give some answers to the existence and multiplicity questions for $(P)$. The existence of a positive nonminimizing solution to $(P)$ has been proved (see [4,2]) by minimax methods, and furthermore multiplicity results have been obtained, by using subtle geometric and topological arguments, in [6–8,19].

When both $\Omega$ and $\mathbb{R}^N \setminus \Omega$ are unbounded the compactness situation can be even more complex. Indeed, when $\mathbb{R}^N \setminus \Omega$ is bounded, the above mentioned result states that a Palais–Smale sequence either converges strongly to its weak limit or differs from it by a finite number of sequences that are noting but normalized solutions of the limit problem

\[
(P_\infty) \begin{cases}
-\Delta u + u = u^{p-1} & \text{in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N)
\end{cases}
\]
“travelling to infinity” and infinitely far away each other. When $\Omega$ and $\mathbb{R}^N \setminus \Omega$ are unbounded and invariant with respect to a group of translations, it is not difficult to understand that, besides the above described behaviour, a noncompact Palais–Smale sequence can also look like a solution of $(P)$ (normalized in $L^p(\Omega)$) “travelling to infinity”, of course by means of translations that leave $\Omega$ invariant. This happens, for instance, when $\Omega$ is the complement of a cylinder: $\Omega = \mathbb{R}^N \setminus \{(x_1, \ldots, x_N) \in \mathbb{R}^N: \sum_{j=k+1}^{N} x_j^2 \leq R^2\}$, where $1 \leq k \leq N - 2$ and $R > 0$; in fact, if $u$ solves $(P)$, then $u(x - y')/|u|_{L^p(\Omega)}$, with $y' = (x_1', 0, \ldots, 0)$ and $x_1' \to +\infty$ is a noncompact Palais–Smale sequence at the same energy level of the critical point, of $E$ on $V$, $u/|u|_{L^p(\Omega)}$. Moreover, we remark that even worse phenomena can occur, in fact unbounded domains with unbounded boundary exist such that the Palais–Smale condition for the related energy functional $E$ can fail at every energy level (see [17]).

The question of the existence of solutions of $(P)$ when $\Omega$ is an unbounded domain having unbounded boundary is still very partially investigated. Most of the known existence results concern domains bounded in some co-ordinates (strip-like, cylinders) (see [23] and references therein), while only recently some existence results have been proved under suitable condition at infinity on $\Omega$ and on $\partial \Omega$ [18,17].

The research, we present here, deals with problem $(P)$ when $\Omega$ is an unbounded “exterior” domain having unbounded boundary. Precisely, we suppose that $\Omega$ satisfies the following assumptions:

$$(h_1)$$ there exist $k$ positive real numbers $\tau_1, \tau_2, \ldots, \tau_k$, $1 \leq k \leq N - 2$, such that

$$(x_1, x_2, \ldots, x_N) \in \Omega \iff (x_1, \ldots, x_{i-1}, x_i + \tau_i, \ldots, x_N) \in \Omega, \ \forall i = 1, 2, \ldots, k;

(h_2)$$ there exists $R \in \mathbb{R}$, $R > 0$, such that

$$\mathbb{R}^N \setminus \Omega \subset \left\{ (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N: \sum_{j=k+1}^{N} x_j^2 \leq R^2 \right\}.

The main result we obtain is contained in the following

**Theorem 1.1.** Let $\Omega$ be a smooth domain verifying conditions $(h_1)$ and $(h_2)$. Then problem $(P)$ has at least $(k + 1)$ solutions, $u_1, u_2, \ldots, u_{k+1}$, nonequivalent, in the sense that $\forall i \neq j$, $i, j = 1, \ldots, k + 1$, does not exist $(h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k$ such that $u_i(x_1, \ldots, x_N) = u_j(x_1 + h_1 \tau_1, x_2 + h_2 \tau_2, \ldots, x_k + h_k \tau_k, x_{k+1}, \ldots, x_N)$.

We stress the fact that, dropping assumption $(h_1)$, Theorem 1.1 is not true and moreover $(P)$ could not have any solution, as one easily understands considering $\Omega = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N: |(x_2, \ldots, x_N)| \leq f(x_1)\}$, where $f: \mathbb{R} \to \mathbb{R}$ is a bounded smooth function such that $0 < \inf_{\mathbb{R}} f < \sup_{\mathbb{R}} f < \text{H}, \ H \in \mathbb{R}$ and $f'(t) > 0 \ \forall t \in \mathbb{R}$. $\Omega$ is a domain satisfying the above mentioned nonexistence condition, our condition $(h_2)$, but not condition $(h_1)$. Moreover we point out that if in assumptions $(h_1), (h_2)$ we have $k = N - 1$ then Theorem 1.1 does not hold, in general. Consider, for example, $D = \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^N: |x_1| > R\}, R > 0$: since no solution exists on half-spaces, problem $(P)$ has no solution on $D$. Indeed, the proof we carry on does not work when $k = N - 1$, because in such a case Lemma 4.7 is not true, with $q = 2$.

On the contrary, a simple example of a domain to which Theorem 1.1 applies, giving the existence of at least two solutions, is obtained considering $\Omega = \{x \in \mathbb{R}^3: \text{dist}(x, \mathcal{E}) > \text{H}, \ H \in \mathbb{R}^+ \setminus \{0\}, \text{with} \ \mathcal{E} = \{(x_1, \cos x_1, \sin x_1): x_1 \in \mathbb{R}\}.

We, also, must observe that, in domains verifying the assumptions of Theorem 1.1, but having richer symmetry properties (as the complement of a cylinder), solutions nonequivalent in the sense of Theorem 1.1 can be identified by a different kind of translation. Because of this we explicitly state the following corollary, for which, on the other hand, we have an independent proof, simpler than that of Theorem 1.1.

**Theorem 1.2.** Let $\Omega$ be a smooth domain verifying conditions $(h_1)$ and $(h_2)$. Then problem $(P)$ has at least one solution.

The paper is organized as follows: Sections 2 and 3 are devoted to build the variational framework for the study, namely, in Section 2, after some remarks, the necessary equivariant, Ljusternik–Schnirelmann type, theory is exposed, while in Section 3 the compactness question is studied. Section 4 contains some basic asymptotic estimates and in Section 5 the proofs of Theorems 1.1 and 1.2 are displayed.
2. Notations, preliminary remarks, equivariant theory recalls

Throughout the paper we make use of the following notations:

- \( L^p(D) \), \( 1 \leq p < +\infty \), \( D \subseteq \mathbb{R}^N \), denotes a Lebesgue space; the norm in \( L^p(D) \) is denoted by \( | \cdot |_{p,D} \);
- \( H^1_0(D) \), \( D \subseteq \mathbb{R}^N \), and \( H^1(\mathbb{R}^N) \equiv H^1_0(\mathbb{R}^N) \) denote the Sobolev spaces obtained, respectively, as closure of \( C_0^\infty(D) \), and \( C_0^\infty(\mathbb{R}^N) \), with respect to the norms

\[
\| u \|_D = \left[ \int_D (|\nabla u|^2 + u^2) \, dx \right]^{1/2}, \quad \| u \|_{\mathbb{R}^N} = \left[ \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx \right]^{1/2};
\]

- if \( D_1 \subset D_2 \subseteq \mathbb{R}^N \) and \( u \in H^1_0(D_1) \), we denote also by \( u \) its extension to \( D_2 \) obtained setting \( u \equiv 0 \) outside \( D_1 \). Hence for all \( u \in H^1_0(\Omega) \)

\[
E(u) = \int_{\Omega} (|\nabla u|^2 + u^2) \, dx = \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx;
\]

- the generic point \( x = (x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_N) \in \mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k} \) is denoted by \( (x_1, x_2, \ldots, x_k, x'') \) where \( x'' = (x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-k} \), \( k \) being the number that appears in assumption \( (h_1) \), we put also \( |x'| = (\sum_{j=k+1}^N x_j^2)^{1/2} \);

- \( B(y, r) \) denotes the open ball of \( \mathbb{R}^N \), having radius \( r \) and centred at \( y \).

We set

\[
m := \inf\{ \| u \|_{\mathbb{R}^N}^2 : u \in H^1(\mathbb{R}^N), \| u \|_{p,\mathbb{R}^N} = 1 \}. \tag{2.1}
\]

The infimum in (2.1) is achieved (see [21,5]) by a positive function \( \omega \), that is unique modulo translation [16] and radially symmetric about the origin, decreasing when the radial co-ordinate increases and such that

\[
\lim_{|x| \to +\infty} D^s \omega(x)|x|^{(N-1)/2} e^{|x|} = d_s > 0, \quad s = 0, 1 \tag{2.2}
\]

(see [15] and [5]).

The following proposition shows that, on the contrary, \( (P) \) cannot be solved by minimization.

**Proposition 2.1.** Setting

\[
m_\Omega := \inf\{ E(u) : u \in V \} \tag{2.3}
\]

the relation

\[
m_\Omega = m \tag{2.4}
\]

holds and the minimization problem (2.3) has no solution.

**Proof.** Since we may consider \( H^1_0(\Omega) \) as a subspace of \( H^1(\mathbb{R}^N) \),

\[
m_\Omega \geq m.
\]

To prove that the equality holds, let us consider the sequence \( (\omega_{y_n})_{n \in \mathbb{N}} \) defined by

\[
\omega_{y_n}(x) := \frac{\varphi(x) \omega(x - y_n)}{|\varphi(x) \omega(x - y_n)|_{p,\Omega}}, \quad x \in \Omega,
\]

where, \( \forall n \in \mathbb{N}, y_n = ((y_n)_1, (y_n)_2, \ldots, (y_n)_k, y'_n) \in \Omega, \lim_{n \to +\infty} |y'_n| = +\infty, \omega \) is the function realizing (2.1) and \( \varphi \in C_0^\infty(\mathbb{R}^N, [0, 1]) \) is a cut-off function defined by \( \varphi(x_1, x_2, \ldots, x_k, x'') = \tilde{\varphi}(|x'|), \tilde{\varphi} : \mathbb{R}^+ \to [0, 1] \) being a \( C^\infty \) non-decreasing function such that \( \tilde{\varphi}(s) = 0 \) \( \forall s \leq R, \tilde{\varphi}(s) = 1 \) \( \forall s \geq R + 1 \) (where \( R \) is the number defined in assumption \( (h_2) \)). Using (2.2) it is not difficult to verify that

\[
\begin{align*}
\text{(a)} \lim_{n \to +\infty} \| \varphi(x) \omega(x - y_n) - \omega(x - y_n) \|_{p,\mathbb{R}^N} &= 0, \\
\text{(b)} \lim_{n \to +\infty} \| \varphi(x) \omega(x - y_n) - \omega(x - y_n) \|_{\mathbb{R}^N} &= 0
\end{align*}
\tag{2.5}
\]
hence
\[
\lim_{n \to +\infty} E(\omega_{yn}) = m. \tag{2.6}
\]
Let us now assume \( u^* \in V \) exists so that \( E(u^*) = m \), then by the uniqueness of the family of functions realizing (2.1)
\[ u^*(x) = \omega(x - y^*) \]
for some \( y^* \in \mathbb{R}^N \).
This is impossible because \( \omega(x) > 0 \ \forall x \in \mathbb{R}^N \) and \( \mathbb{R}^N \setminus \overline{\Omega} \neq \emptyset \). \( \square \)

From the above result we can deduce, also, some useful estimates on the \( L^p \) norm of a critical point and a lower bound for the energy of a changing sign critical point of \( E \) on \( V \).

**Corollary 2.2.** Let \( \bar{u} \) be a nontrivial solution of
\[
(P_\mu) \quad \begin{cases}
-\Delta u + u = \mu |u|^{p-2} u & \text{in } \mathcal{D}, \\
u \in H^1_0(\mathcal{D})
\end{cases}
\]
with either \( \mathcal{D} = \Omega \) or \( \mathcal{D} = \mathbb{R}^N \). Then
\[
|\bar{u}|_{p,\mathcal{D}} \geq \left( \frac{m}{\mu} \right)^{1/(p-2)}.
\tag{2.7}
\]

**Proof.** By (2.1), (2.3) and (2.4) we have
\[
m |\bar{u}|^2_{p,\mathcal{D}} \leq \|\bar{u}\|^2_{\mathcal{D}},
\]
and being \( \bar{u} \) solution of \( (P_\mu) \)
\[
\|\bar{u}\|^2_{\mathcal{D}} = \mu |\bar{u}|^p_{p,\mathcal{D}}.
\]
Thus
\[
|\bar{u}|^{p-2}_{p,\mathcal{D}} \geq \frac{m}{\mu}
\]
and (2.7) follows. \( \square \)

**Corollary 2.3.** Let \( \bar{u} \) be a critical point of \( E \) on \( V \). If \( E(\bar{u}) \in (m, 2^{1-2/p} m) \) then \( \bar{u} \) does not change sign.

**Proof.** Let us assume \( E(\bar{u}) = \mu \), \( |\bar{u}|_{p,\Omega} = 1 \), \( \bar{u} = \bar{u}^+ - \bar{u}^- \) and \( \bar{u}^+ \not\equiv 0 \), \( \bar{u}^- \not\equiv 0 \). Then, taking into account that \( \bar{u} \) solves \( (P_\mu) \) in \( \Omega \) and using (2.3),(2.4), we obtain
\[
m |\bar{u}^\pm|^2_{p,\Omega} \leq \|\bar{u}^\pm\|^2_{\Omega} = \mu |\bar{u}^\pm|^p_{p,\Omega}
\]
hence
\[
1 = |\bar{u}|^p_{p,\Omega} = |\bar{u}^+|^p_{p,\Omega} + |\bar{u}^-|^p_{p,\Omega} \geq 2 \left( \frac{m}{\mu} \right)^{p/(p-2)}
\]
that implies
\[
\mu \geq 2^{1-2/p} m. \quad \square
\]

**Remark 2.4.** Obviously, the same conclusion holds true for every changing sign critical point of \( \|u\|^2_{\mathbb{R}^N} \) on \( \{u \in H^1(\mathbb{R}^N): |u|_{p,\mathbb{R}^N} = 1 \} \) and every normalized changing sign solution of \( (P_\mu) \) in \( \Omega \) or in \( \mathbb{R}^N \).

We recall, now, some facts about equivariant critical points theory, that is the needful topological framework for our research.

Let \( X \) be a normed space and \( G \) a topological group. The action of \( G \) on \( X \) is a continuous map
\[
G \times X \xrightarrow{\text{def}} X, \quad [g, u] \mapsto gu
\]
verifying the conditions
(i) $1 \cdot u = u$,
(ii) $(gh)u = g(hu)$,
(iii) $u \mapsto gu$ is linear.

The action of $G$ on $X$ is said isometric if
\[ \|gu\| = \|u\| \quad \forall g \in G \forall u \in X. \]

A set $A \subseteq X$ is $G$-invariant if $gA = A$ for all $g \in G$.

Two elements $w, z \in X$ are $G$-equivalent if $g(w) = z$ for some $g \in G$. We denote by $[w]$ the orbit of $w$, i.e. the subspace $\{gw: g \in G\}$, and by $X/G$ the orbit space, i.e. the quotient space obtained by identifying each orbit to a point.

Two sequences $(w_n)_{n \in \mathbb{N}}, (z_n)_{n \in \mathbb{N}}, w_n, z_n \in X$ are $G$-equivalent if, for all $n \in \mathbb{N}$, $w_n$ is $G$-equivalent to $z_n$, in other words, a sequence $(g_n)_{n \in \mathbb{N}}$ exists so that $g_n \in G$ and $g_n w_n = z_n$.

A map $f: Y \to T$, $Y \subseteq X$ $G$-invariant set, $T$ normed space, is $G$-invariant if
\[ f \circ g(y) = f(y) \quad \forall g \in G \forall y \in Y. \]

A map $f: Y \to X$, $Y \subseteq X$ $G$-invariant set, is $G$-equivariant if
\[ g \circ f = f \circ g \quad \forall g \in G. \]

**Definition 2.5.** Let $A, B, Y, B \subset A \subseteq Y$ be closed $G$-invariant subsets of a normed space $X$ on which a topological group $G$ acts. The $G$-equivariant category of $A$ in $Y$, relative to $B$, denoted by $\text{cat}_G^Y(A, B)$, is the least integer $l$ such that there exist $(l+1)$ closed $G$-invariant subsets of $Y$, $C_0, C_1, \ldots, C_l$ and $(l+1)$ maps $h_j \in \mathcal{C}(C_j \times [0, 1], Y)$, such that
\[ \begin{align*}
(a) \quad & A \subseteq \bigcup_{j=0}^l C_j, \quad B \subseteq C_0; \\
(b) \quad & \begin{cases}
(i) \quad h_j(c, t) \text{ is } G\text{-equivariant} \quad \forall t \in [0, 1], \quad j = 0, 1, \ldots, l; \\
(ii) \quad h_j(c, 0) = c \quad \forall c \in C_j, \quad j = 0, 1, \ldots, l; \\
(iii) \quad h_0(c, 1) \in B \quad \forall c \in C_0; \quad h_0(B, t) \subset B \quad \forall t \in [0, 1]; \\
(iv) \quad \forall j = 1, 2, \ldots, l \exists w_j \in Y \text{ such that } h_j(c, 1) \in [w_j] \forall c \in C_j.
\end{cases}
\end{align*} \]

If such a number does not exist, we say that the $G$-equivariant category of $A$ in $Y$ relative to $B$ is $+\infty$.

**Definition 2.6.** Let $M$ be a $C^1$, $G$-invariant manifold embedded in an Hilbert space $H$ on which the topological group $G$ acts isometrically. Let $F \in \mathcal{C}^1(M, \mathbb{R})$ a $G$-invariant functional. The functional $F$ satisfies the $G$-Palais–Smale condition, briefly $(PS)^G$, at the level $c$ if for every sequence $(u_n)_n, u_n \in M$, such that
\[ F(u_n) \xrightarrow{n \to +\infty} c, \quad \nabla F(u_n) \xrightarrow{n \to +\infty} 0 \]
there exists a sequence $(v_n)_n$, $G$-equivalent to $(u_n)_n$, relatively compact.

The following Ljusternik–Schnirelmann type theorem provides a lower bound for the number of critical points of an invariant functional in suitable ranges of its values.

**Theorem 2.7.** Let $H$ be an Hilbert space on which the topological group $G$ acts isometrically. Let be $M \subseteq H$ a $G$-invariant $C^{1,1}$-manifold and $F \in \mathcal{C}^{1,1}(M, \mathbb{R})$ a $G$-invariant functional. Put for any $c \in \mathbb{R}$
\[ F^c = \{ u \in M: F(u) \leq c \}, \]
\[ K^c = \{ u \in M: F(u) = c, \ (\nabla F)(u) = 0 \}. \]

Consider $-\infty < a < b < +\infty$, and assume $K^a = \emptyset = K^b$ and that $F$ satisfies the $(PS)^G$ for all $c \in [a, b]$. Then $F$ has at least $\text{cat}_G^{F^b}(F^b, F^a)$ critical points that are not $G$-equivalent and to which there correspond critical levels lying in $(a, b)$.
Proof. Since $F$ is $G$-invariant, we have for all $u \in M$, $\forall g \in G$

$$F'(gu)[v] = \lim_{t \to 0} \frac{F(u + tg^{-1}v) - F(u)}{t}$$
$$= F'(u)[g^{-1}v].$$

Thus, being the action of $G$ isometric, we obtain

$$(\nabla F(gu), v) = (\nabla F(u), g^{-1}v) = (g\nabla F(u), v) \quad \forall g \in G, \forall u \in M$$

and we deduce that $\nabla F: M \to H$ is an equivariant map. Furthermore, because of the $(PS)^G$ condition, if $F$ has not critical values in the interval $[\alpha, \beta]$, there exists a positive number $\delta > 0$ for which

$$\|\nabla F(u)\| > \delta \quad \forall u \in F^{-1}([\alpha, \beta]).$$

(2.8)

Indeed, if (2.8) were false, a sequence $(u_n)_n, u_n \in M$, would exist so that

$$F(u_n) \xrightarrow{n \to +\infty} c \in [\alpha, \beta],$$
$$\nabla F(u_n) \xrightarrow{n \to +\infty} 0.$$

Thus, a sequence $(g_n)_n, g_n \in G$, would exist so that, passing eventually to a subsequence, $g_nu_n \xrightarrow{n \to +\infty} v \in M$. Hence we infer

$$F(v) = \lim_{n \to +\infty} F(g_nu_n) = \lim_{n \to +\infty} F(u_n) = c,$$

$$\|\nabla F(v)\| = \lim_{n \to +\infty} \|\nabla F(g_nu_n)\| = \lim_{n \to +\infty} \|g_n\nabla F(u_n)\| = \lim_{n \to +\infty} \|\nabla F(u_n)\| = 0$$

contradicting the nonexistence of critical values in $[\alpha, \beta]$.

Therefore, using well known methods (see e.g. [24] Lemmas 1.14 and 3.1), a number $\varepsilon > 0$ and a continuous deformation $\eta \in C([0, 1] \times M, M)$ can be constructed so that

(i) $\eta(t, \cdot) \quad \forall t \in [0, 1]$ is a $G$-equivariant homeomorphism of $M$;

(ii) $\eta(0, u) = u, \quad \forall u \in M$;

(iii) $\eta(t, u) = u, \quad \forall t \in [0, 1]$ if $u \notin F^{-1}[\alpha - \varepsilon, \beta + \varepsilon]$;

(iv) $\eta(1, F^\beta) \subset F^\alpha$;

(v) $F(\eta(\cdot, u))$ is nonincreasing $\forall u \in M$.

Then, the conclusion follows, by applying classical arguments of the generalized Lusternik–Schnirelmann theory (see [24] Theorem 5.19). 

\[ \square \]

Remark 2.8. The same result could be also proved supposing $H$ Banach, instead of Hilbert, space and under weaker regularity assumptions on $F$.

We end this section pointing out that, in our setting, a noncompact group of translations, $G$, acting on $\mathbb{R}^N$ and, in turn, on $H^1(\mathbb{R}^N)$ and $H^1_0(\Omega)$ is considered. Namely, for all $h \equiv (h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k$, we define $T_h: \mathbb{R}^N \to \mathbb{R}^N$, by

$$T_h(x) = T_h(x_1, x_2, \ldots, x_k, x') := (x_1 + \tau_1h_1, x_2 + \tau_2h_2, \ldots, x_k + \tau_kh_k, x')$$

hence we say that $x, y \in \mathbb{R}^N$ are equivalent if and only if there exists $(h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k$ such that $y = (x_1 + \tau_1h_1, x_2 + \tau_2h_2, \ldots, x_k + \tau_kh_k, x')$.

Clearly $\Omega$ is invariant under the action of $G$.

Analogously, for all $h \equiv (h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k$, we define $T_h: H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$, by

$$T_h(u)(x) := u(T_h(x)) = u(x_1 + \tau_1h_1, x_2 + \tau_2h_2, \ldots, x_k + \tau_kh_k, x')$$

and we say that $u, v \in H^1(\mathbb{R}^N)$ are equivalent if and only if there exists $(h_1, h_2, \ldots, h_k) \in \mathbb{Z}^k$ such that

$$v(x_1, x_2, \ldots, x_k, x') = u(x_1 + \tau_1h_1, x_2 + \tau_2h_2, \ldots, x_k + \tau_kh_k, x').$$

We remark that the action on $H^1(\mathbb{R}^N)$ of the group $G := \{T_h : h \in \mathbb{Z}^k \}$ is isometric, that $H^1_0(\Omega)$ can be seen as an invariant subspace of $H^1(\mathbb{R}^N)$, $V$ is an invariant manifold in $H^1_0(\Omega)$ and the functional $E$ is invariant.
3. A compactness result

The purpose of this section is to show that there exists an energy interval in which the compactness of the functional $E$ is saved.

The result we prove is stated in the following

**Proposition 3.1.** The functional $E$ satisfies the $G$-Palais–Smale condition on $V$ at every level $c \in (m, 2^{1-2/p}m)$.

**Proof.** Let $(u_n)_n$ be a Palais–Smale sequence for $E$ constrained on $V$, e.g.

\[
\begin{align*}
\begin{array}{l}
(a) |u_n|_{p, \Omega} = 1, \\
(b) \lim_{n \to +\infty} E(u_n) = c, \\
(c) \lim_{n \to +\infty} \nabla E|_V (u_n) = 0
\end{array}
\end{align*}
\]  

(3.1)

and assume

\[c \in (m, 2^{1-2/p}m).\]  

(3.2)

By definition of $E$, (3.1)(b) implies that $(u_n)_n$ is bounded in $H^1_0(\Omega)$, so there exists $u_0 \in H^1_0(\Omega)$ such that, up to a subsequence,

\[
\begin{align*}
\begin{array}{l}
(a) u_n \rightharpoonup u_0 \text{ weakly in } H^1_0(\Omega) \text{ and in } L^p(\Omega), \\
(b) u_n(x) \to u_0(x) \text{ a.e. in } \Omega.
\end{array}
\end{align*}
\]  

(3.3)

By (3.1)(c), there exists a sequence $(\mu_n)_n, \mu_n \in \mathbb{R}$, such that

\[
\begin{align*}
\left( \nabla E|_V (u_n), w \right) = & \int_\Omega \left[ (\nabla u_n, \nabla w) + u_n w \right] dx - \mu_n \int_\Omega |u_n|^{p-2} u_n w = o(1) \|w\|_\Omega \quad \forall w \in H^1_0(\Omega)
\end{align*}
\]  

(3.4)

and, in view of (3.1)(a), (b), setting in (3.4) $w = u_n$, we deduce

\[\lim_{n \to +\infty} \mu_n = c.\]  

(3.5)

Hence $u_0$ solves

\[
\begin{align*}
\begin{array}{l}
-\Delta u + u = c|u|^{p-2}u \quad \text{in } \Omega, \\
u \in H^1_0(\Omega).
\end{array}
\end{align*}
\]  

(3.6)

Set now

\[v_n(x) = \begin{cases} 
(u_n - u_0)(x), & x \in \Omega, \\
0, & x \in \mathbb{R}^N \setminus \Omega.
\end{cases}\]  

Then, by (3.3)(a),

\[v_n \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R}^N) \text{ and } L^p(\mathbb{R}^N)\]  

(3.7)

and

\[\|v_n\|_{\mathbb{R}^N}^2 = \|u_n\|_{L^2}^2 - \|u_0\|_{L^2}^2 + o(1).\]  

(3.8)

Furthermore, by (3.3)(b), the Brezis–Lieb theorem can be applied and it gives

\[|v_n|_{p, \mathbb{R}^N}^p = |u_n|_{p, \Omega}^p - |u_0|_{p, \Omega}^p + o(1).\]  

(3.9)

Let us suppose, now, $\|v_n\|_{\mathbb{R}^N} \not\to 0$ strongly (otherwise we are done). So, up to a subsequence, $\|v_n\|_{\mathbb{R}^N} \geq k_0 > 0$ $\forall n \in \mathbb{N}$, for some $k_0 \in \mathbb{R}$. Then, using (3.4), (3.6), (3.8), (3.9), we deduce that $k_1 \in \mathbb{R}$ exists such that $|v_n|_{p, \mathbb{R}^N}^p \geq k_1 > 0$.

Let us decompose, now, $\mathbb{R}^N$ into $N$-dimensional hypercubes $Q_l$, having unitary sides and vertices with integer co-ordinates, and put for all $n \in \mathbb{N}$

\[d_n = \max_{l \in \mathbb{N}} |v_n|_{p, Q_l}.\]
We claim that $\gamma \in \mathbb{R}$, $\gamma > 0$, exists such that (up to a subsequence)

$$d_n \geq \gamma > 0 \quad \forall n \in \mathbb{N}.$$  

(3.10)

Indeed

$$0 < k_1 \leq |v_n|_{p,\mathbb{R}^N}^p = \sum_{i \in \mathbb{N}} |v_i|_{p,\mathbb{R}^N}^p \leq \max_{i \in \mathbb{N}} |v_i|_{p,\mathbb{R}^N}^{p-2} \sum_{i \in \mathbb{N}} |v_i|^2_{p,\mathbb{R}^N} \leq d_n^{p-2-2k_2} \sum_{i \in \mathbb{N}} \|v_n\|^2_{Q_i} \leq d_n^{p-2-2k_2} \|v_n\|^2_{\mathbb{R}^N},$$

$k_2 \in \mathbb{R}^+ \setminus \{0\}$ independent of $i$. Thus, in view of (3.1) and (3.8), (3.10) follows.

Let us call, for all $n \in \mathbb{N}$, $y_n$ the centre of an hypercube $Q_n$ in which $|v_n|_{p,\mathbb{R}^N} = d_n$.

If $(y_n)_n$ were bounded, then, passing eventually to a subsequence, we could assume that the $y_n$, for all $n$, belong to the same cube $\hat{Q}$, and, hence, that they coincide. Thus, in $\hat{Q}$ we would have, for all $n$, $|v_n|_{p,\hat{Q}} \geq \gamma > 0$ and, on the other hand, $\|vn\|_{\hat{Q}} \leq \|vn\|_{\mathbb{R}^N} \leq k_3$; as a consequence, by the Rellich Theorem, $(v_n)_n$ would converge strongly in $L^p(\hat{Q})$ to a nonzero function, contradicting (3.7). Therefore

$$|y_n|_{n \to +\infty} \to +\infty.$$

Let us, now, call $\tilde{v}_0$ the weak limit, in $H^1(\mathbb{R}^N)$, of the sequence $\tilde{v}_n(x) := v_n(x + y_n)$. Arguing as before in the hypercube $\hat{Q}$ centred at the origin and having unitary sides, we conclude that $\tilde{v}_0 \neq 0$. Moreover, as a consequence of (3.4), (3.5), $\tilde{v}_0$ is a weak solution, on its domain $\mathcal{D}$, of $-\Delta u + u = c|u|^{p-2}u$ and, since $|y_n| \to +\infty$ and $\Omega$ satisfies (h2), we deduce $\mathcal{D} = \mathbb{R}^N$, when dist$(y_n, \mathbb{R}^N \setminus \Omega)_n \to +\infty$, $\mathcal{D} = \Omega$ (up to a translation) when dist$(y_n, \mathbb{R}^N \setminus \Omega_n)$ is bounded.

Now, we claim that

$$\begin{cases}
\text{(a) } u_0 = 0, \\
\text{(b) } \tilde{v}_0 \text{ does not change sign,} \\
\text{(c) } \tilde{v}_n \underset{n \to +\infty}{\to} \tilde{v}_0 \text{ strongly in } H^1_0(\mathcal{D}).
\end{cases}$$

(3.11)

Equality (3.11)(a) follows by observing that (3.8) implies

$$\|u_n\|^2_{\Omega} \geq \|u_0\|^2_{\Omega} + \|\tilde{v}_0\|^2_{\Omega} + o(1),$$

(3.12)

thus, if $u_0$ were not zero, taking into account that $\tilde{v}_0 \neq 0$ and that Corollary 2.2 applies to both $u_0$ and $\tilde{v}_0$, we would infer

$$E(u_n) = \|u_n\|^2_{\Omega} \geq 2m \cdot \left(\frac{m}{c}\right)^{2/(p-2)} + o(1)$$
and, then

$$c \geq 2^{1-2/p} m$$

contradicting (3.2).

Assertion (3.11)(b) is a direct consequence of Remark 2.4 and of the arguments of Corollary 2.3.

Let us prove, then, (3.11)(c). Let us assume, by contradiction, that $\tilde{v}_n \not\to \tilde{v}_0$ strongly. Then, setting $w_n(x) := (\tilde{v}_n - \tilde{v}_0)(x)$, $w_n(x) \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$, and $w_n(x) \to 0$ strongly in $H^1(\mathbb{R}^N)$. So, we can repeat step by step the argument before applied to $(v_n)_n$, concluding that a sequence of points $(z_n)_n, z_n \in \mathbb{R}^N, |z_n|_{n \to +\infty} \to +\infty$, and a nonzero function, $\tilde{w}_0$, exist such that

$$\tilde{w}_n(x) := w_n(x + z_n) \rightharpoonup \tilde{w}_0(x) \quad \text{weakly in } H^1_0(\mathcal{D}),$$

$\mathcal{D}$ being either $\mathbb{R}^N$ or $\Omega$, and $\tilde{w}_0(x)$ being a solution of $-\Delta u + u = c|u|^{p-2}u$ in $\mathcal{D}$. Furthermore the inequality

$$\|\tilde{w}_n\|^2_{\mathbb{R}^N} = \|w_n\|^2_{\mathbb{R}^N} = \|u_n\|^2_{\Omega} - \|\tilde{v}_0\|^2_{\Omega} + o(1)$$
holds, thus
\[ \|u_n\|_D^2 \geq \|v_0\|_D^2 + \|w_0\|_D^2 + o(1) \]
and, then,
\[ c \geq 2^{1-2/p}m \]
follows, contradicting (3.2) and giving (3.11)(c). Now, (3.11) and (3.9) imply \(|v_0| = |w_0| = 0\) on \(D\). Hence, if \(D = \mathbb{R}^N\), the uniqueness of the positive regular solutions to \(-\Delta u + u = c|u|^{p-2}u\) in \(\mathbb{R}^N\) implies \(c = E(v_0) = m\), contradicting (3.2). Therefore, \(D = \Omega\) and \(0 < \text{dist}(y_n, \mathbb{R}^N \setminus \Omega) < H\) for some \(H \in \mathbb{R}^+ \setminus \{0\}\). Let us consider the sequence \((h_n)n=1\infty=(h_n,1,h_n,2,\ldots,h_n,k)\in\mathbb{Z}^k\) such that
\[ \tau_1h_{n,i} \leq y_{n,i} < \tau_i(h_n,i + 1), \quad i = 1, 2, \ldots, k, \quad n \in \mathbb{N}, \]
and define
\[ u^*_n(x_1, x_2, \ldots, x_k, x') := u_n(x_1 + \tau_1h_{n,1}, \ldots, x_k + \tau_kh_{n,k}, x'). \]
The sequence \((u^*_n)n=1\infty\) is \(G\)-equivalent to \((u_n)n=1\infty\) and converges strongly in \(H^1_0(\Omega)\) to a function \(u^*\) that is a nontrivial critical point of \(E\) on \(V\).

### 4. Useful tools and basic estimates

For what follows we need to introduce a barycenter type function. For all \(u \in L^p(\mathbb{R}^N)\) we set
\[ \tilde{u}(x) = \frac{1}{|B(x, 1)|} \int_{B(x, 1)} |u(y)| \, dy \quad \forall x \in \mathbb{R}^N, \]
\(|B(x, 1)|\) denoting the Lebesgue measure of \(B(x, 1)\), and
\[ \hat{u}(x) = \left[ \tilde{u}(x) - \frac{1}{2} \max_{\mathbb{R}^N} \tilde{u}(x) \right]^+ \quad \forall x \in \mathbb{R}^N; \]
we, then, define \(\beta : L^p(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N\) by
\[ \beta(u) = \frac{1}{|\tilde{u}|_p^p} \int_{\mathbb{R}^N} x(\hat{u}(x))^p \, dx. \quad (4.1) \]
We point out that \(\beta\) is well defined for all \(u \in L^p(\mathbb{R}^N) \setminus \{0\}\), because \(\tilde{u} \neq 0\) and has compact support, that \(\beta\) is continuous and
\[ \begin{cases} (a) & \beta(u(x - y)) = \beta(u(x)) + y \quad \forall u \in L^p(\mathbb{R}^N) \setminus \{0\}, \forall y \in \mathbb{R}^p, \\ (b) & \beta(\omega(x)) = 0. \end{cases} \quad (4.2) \]

**Remark 4.1.** We stress the fact that the above barycenter map has been introduced some years ago by the first and the third author ([9], pages 265–266).

This map has been useful in many situations; in fact, it has been used also in [6,17,18] and recently, with a very slight modification, in [3] (actually, in [3] the definition of barycenter map is introduced as a new one and the definition given in [9] is not quoted).

We set, for all \(r \in \mathbb{R}^+\),
\[ D_r := \{ (x_1, x_2, \ldots, x_k, x') \in \mathbb{R}^N : |x'| < r \} \]
and
\[ B_r := \inf \{ E(u) : u \in V, \beta(u) \in D_r \}, \quad (4.3) \]
so, in particular,
\[ D_0 := \{ (x_1, x_2, \ldots, x_k, x') \in \mathbb{R}^N : x' = 0 \} \]
and
\[ B_0 := \inf \{ E(u) : u \in V, \ (\beta(u))' = (\beta(u))_{k+1}, (\beta(u))_{k+2}, \ldots, (\beta(u))_N ) = 0 \} . \] (4.4)

We remark that
\[ B_0 \geq B_r \ \forall r > 0. \] (4.5)

In what follows, for every \( y = (y_1, y_2, \ldots, y_k, y') \in \mathbb{R}^N \), we set
\[ z_y = (y_1, y_2, \ldots, y_k, 1, 0, \ldots, 0) \in \mathbb{R}^N \]
and we denote by \( S, \Sigma \) and \( \Lambda \), respectively, the sets
\[ S := \{(x_1, x_2, \ldots, x_k, x') \in \mathbb{R}^N : |x'| = 2\}, \] (4.6)
\[ \Sigma := S + z_0, \] (4.7)
\[ \Lambda := \{ \sigma y + (1 - \sigma)z_y : y \in \Sigma, \sigma \in [0, 1] \}. \] (4.8)

For every \( \rho > 0 \) we define the operator
\[ \Psi_\rho : \Sigma \times [0, 1] \rightarrow V \]
by
\[ \Psi_\rho[y, \sigma](x) = \frac{\varphi(x)[(1 - \sigma)\omega(x - \rho y) + \sigma \omega(x - \rho z_y)]}{|\varphi(x)[(1 - \sigma)\omega(x - \rho y) + \sigma \omega(x - \rho z_y)]|_{p, \Omega}}, \] (4.9)
where \( \varphi \in C^\infty(\mathbb{R}^N, [0, 1]) \) is the cut-off function introduced in Proposition 2.1.

We note that
\[ \Psi_\rho[y, 0](x) = \frac{\varphi(x)\omega(x - \rho y)}{|\varphi(x)\omega(x - \rho y)|_{p, \Omega}} = \omega_{\rho y}(x). \] (4.10)

For every \( \rho > 0 \) we consider, also, the map
\[ \xi_\rho : \Sigma \times [0, 1] \rightarrow \rho \Lambda \]
defined by
\[ \xi_\rho[y, \sigma] = \rho((1 - \sigma)y + \sigma z_y). \] (4.11)

**Proposition 4.2.** Let \( B_r \) be the numbers defined in (4.3). Then for all \( r \in \mathbb{R}^+ \), there exists \( \mu_r \in \mathbb{R} \), such that
\[ B_r \geq \mu_r > m. \] (4.12)

**Proof.** Clearly, for all \( r \geq 0, B_r \geq m \); to prove (4.12) we argue by contradiction and we assume that \( B_r = m \) for some \( \hat{r} \geq 0 \). Hence, a sequence \((u_n)_n\) must exist such that \( u_n \in V \) and
\[ \begin{cases} (i) \beta(u_n) \in D_{\hat{r}} \ \forall n \in \mathbb{N}, \\ (ii) \lim_{n \rightarrow +\infty} E(u_n) = m. \end{cases} \] (4.13)

Then, by the uniqueness of the minimizers family of (2.1), a sequence of points \((y_n)_n, y_n \in \mathbb{R}^N\), and a sequence of functions \((\chi_n)_n, \chi_n \in H^1(\mathbb{R}^N)\) exist so that, passing eventually to a subsequence, still denoted by \((u_n)_n\),
\[ \begin{cases} (i) u_n(x) = \omega(x - y_n) + \chi_n(x) \ \forall x \in \mathbb{R}^N, \\ (ii) \lim_{n \rightarrow +\infty} \chi_n(x) = 0 \ \text{in} \ H^1(\mathbb{R}^N) \ \text{and in} \ L^p(\mathbb{R}^N) \end{cases} \] (4.14)
(see also [4] Lemma 3.1). Therefore, by (4.2), (4.14) (i) and the continuity of \( \beta \), the relation
\[ |\beta(u_n) - y_n|_{n \rightarrow +\infty} = 0 \]
holds and, together with (4.13)(i), implies that the sequence \((y_n)_n\) is bounded. Hence, either the sequence \((y_n)_n\) is bounded, or it is unbounded, but, in view of the assumption \((h_1)\), it can be replaced by an equivalent sequence, still
denoted by \((y_n)_n\), contained in a bounded set; so, passing eventually to a subsequence, we conclude that \(y_n \rightarrow \bar{y}\). Thus, either \((u_n)_n\) or a \(G\)-equivalent sequence, still denoted by \((u_n)_n\), satisfies
\[
\lim_{n \rightarrow +\infty} u_n(x) = \omega(x - \bar{y}).
\]
Then, by (4.13)(ii),
\[
m = \lim_{n \rightarrow +\infty} E(u_n) = \int_{\Omega} \left( |\nabla \omega(x - \bar{y})|^2 + (\omega(x - \bar{y}))^2 \right) dx
\]
that is impossible, because \(\omega(x)\) realizes (2.1), \(\omega > 0\) in \(\mathbb{R}^N\) and \(\mathbb{R}^N \setminus \overline{\Omega} \neq \emptyset\), so the statement follows. \(\square\)

**Lemma 4.3.** Let \(\Sigma, \Psi_\rho, B_0\) be as defined, respectively, in (4.7), (4.9), (4.4). Then there exists \(\tilde{\rho} \in \mathbb{R}\) such that for all \(\rho \geq \tilde{\rho}\)
\[
B_0 \leq \max_{\Sigma \times [0,1]} E(\Psi_\rho[y, \sigma]).
\]  \hspace{1cm} (4.15)

**Proof.** Taking into account (2.2), (4.2), (4.10), (2.5)(a), it is not difficult to verify that
\[
\lim_{\rho \rightarrow +\infty} |\beta \circ \Psi_\rho[y, 0] - \rho y|_{\mathbb{R}^N} = 0 \quad \forall y \in \Sigma.
\]  \hspace{1cm} (4.16)

Thus, for \(\rho\) large enough, \(\beta \circ \Psi_\rho(\Sigma \times \{0\})\) is homotopically equivalent in \(\mathbb{R}^N \setminus D_0\) to \(\rho \Sigma\) and, then, there exists \((\hat{y}, \hat{\sigma}) \in \Sigma \times [0,1]\) such that \((\beta \circ \Psi_\rho)[\hat{y}, \hat{\sigma}] \in D_0\), so
\[
B_0 \leq E(\Psi_\rho[\hat{y}, \hat{\sigma}]) \leq \max_{\Sigma \times [0,1]} E(\Psi_\rho[y, \sigma]),
\]
as desired. \(\square\)

**Corollary 4.4.** Let \(\Sigma, \Psi_\rho\) and \(\tilde{\rho}\) as in Lemma 4.3. Let \(B_r, r \in \mathbb{R}^+, \) as defined (4.3). Then for all \(\rho \geq \tilde{\rho}\)
\[
B_r \leq \max_{\Sigma \times [0,1]} E(\Psi_\rho[y, \sigma]).
\]  \hspace{1cm} (4.17)

**Proof.** Inequality (4.17) is an immediate consequence of (4.15) and (4.5). \(\square\)

Next step is to establish some, crucial, asymptotic estimates on the energy of \(\Psi_\rho(\Sigma \times \{0\})\) and of \(\Psi_\rho(\Sigma \times [0,1])\). To this end, we need, first, to recall some known results:

**Lemma 4.5.** For all \(a, b \in \mathbb{R}^+\), for all \(p \geq 2\), the relation
\[
(a + b)^p \geq a^p + b^p + (p - 1)(a^{p-1}b + ab^{p-1})
\]
holds true.

**Lemma 4.6.** Let \(g \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\) and \(h \in C(\mathbb{R}^N)\) be radially symmetric functions satisfying for some \(\alpha \geq 0, b \geq 0, \gamma \in \mathbb{R}\)
\[
\lim_{|x| \rightarrow +\infty} g(x) \exp(\alpha |x|)|x|^b = \gamma,
\]
\[
\int_{\mathbb{R}^N} |h(x)| \exp(\alpha |x|)(1 + |x|^b) dx < +\infty.
\]
Then
\[
\lim_{|y| \rightarrow +\infty} \left( \int_{\mathbb{R}^N} g(x + y)h(x) dx \right) \exp(\alpha |y|)|y|^b = \gamma \int_{\mathbb{R}^N} h(x) \exp(-\alpha x_1) dx
\]
holds.
The proof of Lemma 4.5 can be found in [8], while the proof of Lemma 4.6 is in [1]. Now, we state, in the following lemma, a basic, preliminary, asymptotic relation.

**Lemma 4.7.** Let be \( k \in \mathbb{N} \), \( 1 \leq k \leq N - 2 \) and \( h \in C(\mathbb{R}^N, \mathbb{R}) \) such that \( h(x_1, x_2, \ldots, x_k, x') = \tilde{h}(x') \) with \( \tilde{h} \in C_0(\mathbb{R}^{N-k}, \mathbb{R}) \), then the relation

\[
\lim_{\rho \to +\infty} \sup_{\Sigma \times [0,1]} \left( \int_{\mathbb{R}^N} h(x)[(1 - \sigma)\omega(x - \rho y) + \sigma \omega(x - \rho z_y)]^q \, dx \right) \cdot \rho^{(N-1)/2} \exp(2\rho) = 0
\]

(4.18)

holds for all \( q \geq 2 \).

**Proof.** Since \( \tilde{h} \) has compact support, there exists \( \hat{R} > 0 \) such that \( \text{supp} \tilde{h} \subset \{w \in \mathbb{R}^{N-k}: |w| < \hat{R}\} = B_{N-k}(0, \hat{R}) \), so \( h(x) = 0 \) if \( x' > \hat{R} \), furthermore \( \max_{\mathbb{R}^N} h = \max_{\mathbb{R}^{N-k}} \tilde{h} = +\infty \).

In what follows we can, also, suppose \( \rho > \max(1, \hat{R}) \), hence, observing that, for all \( y \in \Sigma \), \( |y| \geq 1 \) and that \( \omega \) is radially decreasing when the radial co-ordinate increases, we deduce that \( \forall x \in \mathbb{R}^N \), for which \( |x'| < \hat{R} \), \( \omega(x - \rho \frac{y}{|y|}) \geq \omega(x - \rho y) \).

Thus we have

\[
\left| \int_{\mathbb{R}^N} h(x)[(1 - \sigma)\omega(x - \rho y) + \sigma \omega(x - \rho z_y)]^q \, dx \right|
\]

\[
\leq c_1 \max_{\mathbb{R}^N} |h| \int_{\mathbb{R}^N} \left( \left[ \omega\left(x - \rho \frac{y}{|y|}\right) \right]^q + \left[ \omega(x - \rho z_y) \right]^q \right) \, dx,
\]

\( c_1 \in \mathbb{R}^+ \setminus \{0\} \). So we must show

\[
\lim_{\rho \to +\infty} \left( \int_{\mathbb{R}^k} \int_{B_{N-k}(0, \hat{R})} (\omega(x - \rho \frac{y}{|y|})^q \, dx_{k+1} \cdots dx_N) \, dx_1 \cdots dx_k \right) \cdot \rho^{(N-1)/2} \exp(2\rho) = 0
\]

(4.19)

with both \( v = \frac{y}{|y|} \), \( y \in \Sigma \) and \( v = z_y, y \in \Sigma \).

Let us evaluate (4.19) when \( v = \frac{y}{|y|}, y \in \Sigma \).

Without any loss of generality, we can assume \( v = (0, 0, \ldots, 0, v') \), |\( v' \)| = 1. Taking, again, advantage of the behaviour of \( \omega \) and of its asymptotic decay, we infer, for large values of \( \rho \),

\[
\int_{\mathbb{R}^k} \left( \int_{B_{N-k}(0, \hat{R})} (\omega(x - \rho \frac{y}{|y|})^q \, dx_{k+1} \cdots dx_N) \, dx_1 \cdots dx_k \right)
\]

\[
\leq c_2 \int_{\mathbb{R}^k} (\omega(x_1, x_2, \ldots, x_k, (\hat{R} - \rho)v'))^q \, dx_1 \cdots dx_k
\]

\[
\leq c_3 \int_{\mathbb{R}^k} \left[ \frac{1}{((\rho - \hat{R})^2 + \sum_{i=1}^{k} x_i^2)^{(N-1)/2}} \cdot \frac{1}{e^{((\rho - \hat{R})^2 + \sum_{i=1}^{k} x_i^2)^{1/2}}} \right]^q \, dx_1 \cdots dx_k,
\]

\( c_2, c_3 \in \mathbb{R}^+ \setminus \{0\} \).

So, setting \( \hat{x} = (x_1, x_2, \ldots, x_k) \), to obtain (4.19), we have to show that, for all \( q \geq 2 \)

\[
\lim_{\rho \to +\infty} \int_{\mathbb{R}^k} \frac{\rho}{[(\rho - \hat{R})^2 + ||\hat{x}\|^2]^{q/2}} \left( \frac{1}{e^{((\rho - \hat{R})^2 + ||\hat{x}\|^2)^{1/2}-2\rho}} \right)^{(N-1)/2} \, \frac{1}{e^{q((\rho - \hat{R})^2 + ||\hat{x}\|^2)^{1/2}-2\rho}} \, d\hat{x} = 0.
\]

(4.20)

When \( q > 2 \), (4.20) follows at once because

\[
\left( \frac{\rho}{[(\rho - \hat{R})^2 + ||\hat{x}\|^2]^{q/2}} \right)^{(N-1)/2} \cdot \frac{1}{e^{q((\rho - \hat{R})^2 + ||\hat{x}\|^2)^{1/2}-2\rho}} \leq c_4 \frac{1}{e^{c_5 \rho^2 + ||\hat{x}\|^2/2}},
\]

\( c_4, c_5 \in \mathbb{R}^+ \setminus \{0\} \).
Let us, then, consider the case $q = 2$. Clearly, being $\hat{R}$ fixed, (4.20) easily comes once we show that

$$\lim_{\rho \to +\infty} \int_{\mathbb{R}^k} \left( \frac{\rho}{\rho^2 + |\hat{x}|^2} \right)^{(N-1)/2} \cdot \frac{e^{2\rho}}{e^{2(\rho^2 + |\hat{x}|^2)^{1/2}}} \, d\hat{x} = 0. \quad (4.21)$$

Now

$$\int_{\mathbb{R}^k} \left( \frac{\rho}{\rho^2 + |\hat{x}|^2} \right)^{(N-1)/2} \cdot \frac{e^{2\rho}}{e^{2(\rho^2 + |\hat{x}|^2)^{1/2}}} \, d\hat{x} = \frac{1}{\rho^{(N-1)/2}} \int_{\mathbb{R}^k} \left( \frac{1}{1 + |\hat{x}|^2} \right)^{(N-1)/2} \cdot \frac{1}{e^{2\rho(1+(|\hat{x}|^2)^{1/2}) - 1}} \, d\hat{x}$$

and for all $\hat{x} \neq 0$

$$\lim_{\rho \to +\infty} \left( \frac{1}{1 + |\hat{x}|^2} \right)^{(N-1)/2} \cdot \frac{\rho^{(2k+1-N)/2}}{e^{2\rho(1+(|\hat{x}|^2)^{1/2}) - 1}} = 0.$$

Furthermore, when $k \leq \frac{N-1}{2}$

$$\left( \frac{1}{1 + |\hat{x}|^2} \right)^{(N-1)/2} \cdot \frac{\rho^{(2k+1-N)/2}}{e^{2\rho(1+(|\hat{x}|^2)^{1/2}) - 1}} \leq \left( \frac{1}{1 + |\hat{x}|^2} \right)^{(N-1)/2}$$

for large $\rho$, while, when $k > \frac{N-1}{2}$, taking into account that

$$\max_{t \in \mathbb{R}^+} \frac{t^\alpha}{e^{ct}} = \left( \frac{\alpha}{c} \right)^{1/c} \cdot \frac{1}{c^{\alpha}}, \quad \alpha > 0,$$

we deduce

$$\left( \frac{1}{1 + |\hat{x}|^2} \right)^{(N-1)/2} \cdot \frac{\rho^{(2k+1-N)/2}}{e^{2\rho(1+(|\hat{x}|^2)^{1/2}) - 1}} \leq \left( \frac{2k+1-N}{4e} \right)^{(2k+1-N)/2} \left( \frac{1}{1 + |\hat{x}|^2} \right)^{(2k+1-N)/2} \left( \frac{1}{1 + |\hat{x}|^2} \right)^{(N-1)/2} \quad (4.23)$$

and, since $k \leq N - 2$, the right-hand sides of (4.22) and (4.23) belong to $L^1(\mathbb{R}^k)$. Thus, by the Lebesgue Theorem, (4.21) is true. As a consequence, (4.19) holds true when $v = \frac{y}{|y|}, \; y \in \Sigma$. The argument when $v = z_y, \; y \in \Sigma$ is quite analogous, so the statement is proved. $\square$

We are, now, ready to state and prove the main energy asymptotic estimates.

**Proposition 4.8. The relations**

\begin{align*}
\lim_{\rho \to +\infty} \max \{ E(\Psi_\rho[y, 0]) \} &= m, \quad (4.24) \\
\max_{\Sigma \times [0,1]} E(\Psi_\rho[y, \sigma]) &< 2^{1-2/p} m \quad \text{for } \rho \text{ large enough} \quad (4.25)
\end{align*}

hold.

**Proof.** In view of (4.7) and of the fact that $\forall y \in \Sigma, \lim_{\rho \to +\infty} |\rho y'| = +\infty$, arguing as for proving (2.6) in Proposition 2.1, it is easy to show that

$$\lim_{\rho \to +\infty} E(\Psi_\rho[y, 0]) = m \quad \forall y \in \Sigma,$$

so (4.24) follows.

In order to prove (4.25), let us put
\[ N_{\rho}[y, \sigma] = \| \varphi(x)[(1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y)] \|_{\mathbb{R}^N}^2 \]
\[ D_{\rho}[y, \sigma] = \varphi(x)[(1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y)]^p_{\rho, \mathbb{R}^N} \]

for all \( \rho > 0 \), for all \((y, \sigma) \in \Sigma \times [0, 1]\).

We have
\[
N_{\rho}[y, \sigma] = \int_{\mathbb{R}^N} \left( \varphi(x)^2 \left[ |\nabla((1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y))|^2 \right. \right.
+ \left. (1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y) \right] \, dx
+ \int_{\mathbb{R}^N} \left( |\nabla\varphi(x)|^2 \left[ (1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y) \right] \right. \, dx
+ \frac{1}{2} \int_{\mathbb{R}^N} \left( (\varphi(x))^2, (1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y) \right) \, dx
\leq \int_{\mathbb{R}^N} \left[ |\nabla((1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y))|^2 + (1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y) \right] \, dx
+ \int_{\mathbb{R}^N} \left( |\nabla\varphi(x)|^2 - \frac{1}{2} \Delta(\varphi(x))^2 \right) \left[ (1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y) \right] \, dx
\]
\[
=(1 - \sigma)^2 + \sigma^2 \int_{\mathbb{R}^N} \left[ |\nabla\omega(x)|^2 + (\omega(x))^2 \right] \, dx + 2m\sigma(1 - \sigma) \int_{\mathbb{R}^N} \omega(x - \rho y)^p \omega(x - \rho z y) \, dx
- \int_{\mathbb{R}^N} (\varphi \Delta \varphi)((1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y))^2 \, dx. \tag{4.26}
\]

Now, setting
\[
\varepsilon_{\rho} := \int_{\mathbb{R}^N} \omega(x - \rho y)^p \omega(x - \rho z y) \, dx = \int_{\mathbb{R}^N} \omega(x - \rho y)^p \omega(x - \rho z y) \, dx, \tag{4.27}
\]
in view of (2.2), by applying Lemma 4.6, we get
\[
\lim_{\rho \to \infty} \varepsilon_{\rho} \left[(2\rho)^{(N-1)/2} \exp(2\rho) \right] = \tilde{c} > 0. \tag{4.28}
\]

Therefore, using Lemma 4.7, we obtain
\[
N_{\rho}[y, \sigma] \leq [(1 - \sigma)^2 + \sigma^2]m + 2\sigma(1 - \sigma)m\varepsilon_{\rho} + o(\varepsilon_{\rho}).
\]

On the other hand, using Lemmas 4.5 and 4.7, we deduce
\[
D_{\rho}[y, \sigma] = \int_{\mathbb{R}^N} \left[ (1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y) \right]^p \, dx
+ \int_{\mathbb{R}^N} (\varphi^p - 1)[(1 - \sigma)\omega(x - \rho y) + \sigma\omega(x - \rho z y)]^p \, dx
\geq [(1 - \sigma)^p + \sigma^p] \omega^p_{\rho, \mathbb{R}^N} + (p - 1)[(1 - \sigma)^{p-1}\sigma + \sigma^{p-1}(1 - \sigma)]\varepsilon_{\rho} + o(\varepsilon_{\rho})
\]
where \( \varepsilon_{\rho} \) is defined in (4.27).

Hence
We claim that there exists a critical level $c$. Then, by Proposition 3.1, using standard arguments (as displayed in Theorem 2.7), a positive number $\hat{\rho}$ exists such that for all $\rho > \hat{\rho}$ the inequalities

$$m < \max_{\Sigma} E(\Psi_\rho[y,0]) < B_0 \leq \max_{\Sigma \times [0,1]} E(\Psi_\rho[y,\sigma]) < 2^{1-2/p} m \tag{5.1}$$

hold and, moreover, $\beta \circ \Psi_\rho[\Sigma \times \{0\}]$ is homotopically equivalent in $\mathbb{R}^N \setminus D_0$ to $\rho \Sigma$. So let us fix $\rho > \hat{\rho}$ and set

$$A := \max_{\Sigma} E(\Psi_\rho[y,0]),$$

$$\mathcal{L} := \max_{\Sigma \times [0,1]} E(\Psi_\rho[y,\sigma]).$$

We claim that there exists a critical level $c^* \in [B_0, \mathcal{L}]$. Arguing by contradiction, let us assume

$$\{u \in V: E(u) \in [B_0, \mathcal{L}], \nabla E|_V(u) = 0\} = \emptyset.$$

Then, by Proposition 3.1, using standard arguments (as displayed in Theorem 2.7), a positive number $\delta \in (0, B_0 - A)$ and a continuous function

$$\eta: E^{\mathcal{L}} \rightarrow E^{B_0 - \delta} \tag{5.2}$$

can be found so that

$$\eta(u) = u \quad \forall u \in E^{B_0 - \delta}. \tag{5.3}$$

5. Proof of the results

In what follows for all $c \in \mathbb{R}$ we set

$$E^c = \{u \in V: E(u) \leq c\},$$

$$(E^c)^+ = \{u \in E^c: u \geq 0 \text{ a.e. in } \Omega\},$$

$$(E^c)^- = \{u \in E^c: u \leq 0 \text{ a.e. in } \Omega\}.$$
Now, let us define $\mathcal{H}: \Sigma \times [0, 1] \to \mathbb{R}^N$ by

$$\mathcal{H}(y, \sigma) := \beta \circ \eta(\Psi_{\rho}[y, \sigma]).$$

By (5.3) and the choice of $\delta$, $\mathcal{H}([y, 0]) = \beta(\Psi_{\rho}[y, 0])$, thus, by the choice of $\rho > \tilde{\rho}$, $\mathcal{H}([\Sigma \times \{0\}])$ is homotopically equivalent in $\mathbb{R}^N \setminus D_0$ to $\rho \Sigma$; moreover $\mathcal{H}$ is continuous, so a point $(\tilde{y}, \tilde{\sigma}) \in \Sigma \times [0, 1]$ must exist so that

$$\beta \circ \eta(\Psi_{\rho}[\tilde{y}, \tilde{\sigma}]) \in D_0.$$  

This is impossible, because, by (5.2)

$$\mathcal{H}(\Sigma \times [0, 1]) \cap D_0 = \emptyset.$$  

Therefore, the claim is proved and there exists a critical point $u^*$, of $E$ on $V$, such that $E(u^*) = c^*$. By Corollary 2.3 we can assume $u^* \geq 0$, so $v^* = (c^*)^{1/(p-2)}u^* \geq 0$ solves $(P)$ and, by the maximum principle $v^* > 0$. \hfill \square

**Proof of Theorem 1.1.** By Propositions 4.2, 4.8, Corollary 4.4 and (4.16) of Lemma 4.3, a $\tilde{\rho} \in \mathbb{R}, \tilde{\rho} > 0$, exists such that for all $\rho > \tilde{\rho}$

$$m < \max_{\Sigma} E(\Psi_{\rho}[y, 0]) < B_{3R} \leq \max_{\Sigma \times [0, 1]} E(\Psi_{\rho}[y, \sigma]) < 2^{1-2/p}m,$$

$R$ being as in assumption (h2), and

$$\left| \beta \circ \Psi_{\rho}(y, 0) \right| > \frac{\rho}{2} \quad \forall y \in \Sigma.$$

(5.5)

So, let us fix $\rho > \max(\tilde{\rho}, 6R)$ and set

$$\mathcal{A} := \max_{\Sigma} E(\Psi_{\rho}[y, 0]),$$

$$\mathcal{L} := \max_{\Sigma \times [0, 1]} E(\Psi_{\rho}[y, \sigma]).$$

Let us choose then $\hat{A} \in [\mathcal{A}, B_{3R})$ and $\hat{L} \in [\mathcal{L}, 2^{1-2/p}m)$ such that

$$\{ u \in V: E(u) = \hat{A}, \nabla E|_{V}(u) = 0 \} = \emptyset,$$

$$\{ u \in V: E(u) = \hat{L}, \nabla E|_{V}(u) = 0 \} = \emptyset.$$  

Let us remark that, if one of these choices were not possible, we would be done, because the functional $E$ constrained on $V$ would have infinitely many critical values, to which there would correspond infinitely many solutions of $(P)$.

Now, taking into account Proposition 3.1, we can apply Theorem 2.7 to the functional $E$, on $V$, subject to the action of the group $\mathcal{G}$. So, we deduce that $E$ possesses at least $\text{cat}G_{E, \hat{L}}(E^{1}, E^{\hat{A}})$ not $G$-equivalent critical points, to which there correspond critical values lying in $(\hat{A}, \hat{L})$.

Now, to conclude the proof, we just need to show that

$$\text{cat}G_{E, \hat{L}}(E^{\hat{L}}, E^{\hat{A}}) \geq 2 \text{cat}G_{\rho A}(\rho \Lambda, \rho \Sigma).$$

(5.6)

Indeed, being true (5.6), we can infer the existence of at least $2 \text{cat}G_{\rho A}(\rho \Lambda, \rho \Sigma)$ not $G$-equivalent critical points of $E$ on $V$, $u_i$, that, since $E(u_i) \in (\hat{A}, \hat{L}) \subset (m, 2^{1-2/p}m)$, by Corollary 2.3, do not change sign. Hence, in view of the maximum principle, the existence of at least $\text{cat}G_{\rho A}(\rho \Lambda, \rho \Sigma)$ positive not $G$-equivalent solutions of $(P)$ follows. The argument is, then, completed by observing that, by applying Corollary 7.6(ii) in [11] (see also [14]), we obtain

$$\text{cat}G_{\rho A}(\rho \Lambda, \rho \Sigma) = k + 1$$

(5.7)

(for the reader’s convenience, a proof of (5.7) is given in Appendix A).

Let us prove now (5.6). To this end, we show that

$$\begin{cases}
\text{(a) } \text{cat}G_{(E^{\hat{L}})^+}(E^{\hat{L}})^+, (E^{\hat{A}})^+) \geq \text{cat}G_{\rho A}(\rho \Lambda, \rho \Sigma), \\
\text{(b) } \text{cat}G_{(E^{\hat{L}})^-}(E^{\hat{L}})^-, (E^{\hat{A}})^-) \geq \text{cat}G_{\rho A}(\rho \Lambda, \rho \Sigma),
\end{cases}$$

(5.8)
we remark, in fact, that \((E^+_{\hat{L}})^+\) and \((E^+_{\hat{L}})^-\) belong to disjoint connected components of \(E^+_{\hat{L}}\), because \(E^+_{\hat{L}} < 2^{1/2/r}m\).

Obviously (5.8)(a) is true when \(\text{cat}^G((E^+_{\hat{L}})^+, (E^+_{\hat{A}})^+) = +\infty\), hence let us assume

\[
\text{cat}^G((E^+_{\hat{L}})^+, (E^+_{\hat{A}})^+) = l \in \mathbb{N};
\]

this means \(l\) is the least number for which there exist \((l + 1)\) closed \(G\)-invariant sets \(T_i \subset (E^+_{\hat{L}})^+, i = 0, 1, \ldots, l, (l + 1)\) continuous maps \(\theta_i : T_i \times [0, 1] \to (E^+_{\hat{L}})^+, i = 0, 1, \ldots, l,\) and \(l\) points \(w_i \in (E^+_{\hat{L}})^+, i = 1, 2, \ldots, l,\) such that

\[
\begin{cases}
(E^+_{\hat{L}})^+ = \bigcup_{i=0}^l T_i, (E^+_{\hat{A}})^+ \subset T_0, \\
\theta_i(\cdot, t) \text{ is } G\text{-invariant } \forall t \in [0, 1], \quad i = 0, 1, \ldots, l, \\
\theta_i(u, 0) = u \quad \forall u \in T_i, \quad i = 0, 1, \ldots, l, \\
\theta_i(u, 1) \in \{w_i\} \quad \forall u \in T_i, \quad i = 1, 2, \ldots, l, \\
\theta_0(u, 1) \in (E^+_{\hat{A}})^+ \quad \forall u \in T_0, \\
\theta_0(u, t) \in (E^+_{\hat{A}})^+ \quad \forall u \in (E^+_{\hat{A}})^+, \forall t \in [0, 1].
\end{cases}
\]

(5.9)

Now, we consider

\[
K_i := (\xi_\rho \circ \Psi^{-1}_\rho)(T_i), \quad i = 0, 1, \ldots, l,
\]

\(\Psi_\rho\) and \(\xi_\rho\) being the maps defined in (4.10) and (4.11) respectively. We remark that \(K_i\) are \(G\)-invariant subsets of \(\mathbb{R}^N\) and, by (5.4) and (5.9),

\[
K_i \subset \rho \Lambda, \quad \bigcup_{i=0}^l K_i = \rho \Lambda, \quad \rho \Sigma \subset K_0.
\]

(5.10)

Let us denote for all \(x \in \mathbb{R}^N \setminus D_0, x = (x_1, x_2, \ldots, x_k, x'),\) by \(\Pi(x)\) the unique point belonging to \(\rho \Sigma\) and to the half line containing \(x\) and having origin at \((x_1, x_2, \ldots, x_k, 0)\).

We define, then, for \(i = 1, 2, \ldots, l\)

\[
\lambda_i : K_i \times [0, 1] \to \rho \Lambda
\]

by

\[
\lambda_i(x, t) = \begin{cases}
(1 - 2t)x + 2th \circ \beta(\Psi_\rho \circ \xi^{-1}_\rho(x)), & 0 \leq t \leq \frac{1}{2}, \\
h \circ \beta \circ \theta_i(\Psi_\rho \circ \xi^{-1}_\rho(x), 2t - 1), & \frac{1}{2} \leq t \leq 1,
\end{cases}
\]

where

\[
h(x_1, x_2, \ldots, x_k, x') = \begin{cases}
(x_1, x_2, \ldots, x_k, x') & \text{if } |x'| \leq |(\Pi(x))'| \text{ or } x' = 0 \\
\Pi(x) & \text{if } |x'| \geq |(\Pi(x))'|
\end{cases}
\]

and we define

\[
\lambda_0 : K_0 \times [0, 1] \to \rho \Lambda
\]

by

\[
\lambda_0(x, t) = \begin{cases}
(1 - 3t)x + 3t\tilde{h}(x), & 0 \leq t \leq \frac{1}{3}, \\
\tilde{h}(3tx + (3t - 1)\beta(\Psi_\rho \circ \xi^{-1}_\rho(x))), & \frac{1}{3} \leq t \leq \frac{2}{3}, \\
\tilde{h} \circ \beta \circ \theta_0(\Psi_\rho \circ \xi^{-1}_\rho(x), 3t - 2), & \frac{2}{3} \leq t \leq 1,
\end{cases}
\]

where \(\tilde{h} : \mathbb{R}^N \to \mathbb{R}^N\) is the map defined by

\[
\tilde{h}(x_1, \ldots, x_k, x') = \begin{cases}
(x_1, \ldots, x_k, x') & \text{if } |x'| \leq R, \\
(x_1, \ldots, x_k, \left[R + \frac{|(\Pi(x))'| - R}{R}(|x'| - R)\right]x') & \text{if } R \leq |x'| \leq 2R, \\
\Pi(x) & \text{if } |x'| > 2R.
\end{cases}
\]
We remark that the maps $\lambda_i$, $i = 0, 1, \ldots, l$, are well defined: indeed, even if $\xi_{\rho}^{-1}$ can contain more than one element, $\Psi_\rho \circ \xi_{\rho}^{-1}(x)$ is uniquely determined; moreover the $\lambda_i$, $i = 0, 1, \ldots, l$, turn out to be continuous maps having the following properties

$$\lambda_i(t, \cdot) \text{ is } G\text{-equivariant} \quad \forall t \in [0, 1], \; i = 0, 1, \ldots, l,$$

$$\lambda_0(x, 0) = x \quad \forall x \in K_0,$$

$$\lambda_0(x, 1) \in \rho \Sigma \quad \forall x \in \rho \Sigma,$$

$$\lambda_{0}(x, t) \in \rho \Sigma \quad \forall x \in \rho \Sigma, \; \forall t \in [0, 1].$$

Relations (5.10) and (5.11) imply

$$\text{cat}^G_{\rho \Lambda}(\rho \Lambda, \rho \Sigma) \leq l,$$

so (5.8)(a) is proved. An analogous argument gives (5.8)(b), completing the proof. □

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Appendix A

Proposition A.1. Let $\Sigma$ and $\Lambda$ as in (4.7) and (4.8), respectively. Then, for every $\rho > 0$,}

$$\text{cat}^G_{\rho \Lambda}(\rho \Lambda, \rho \Sigma) = k + 1.$$ (A.1)

Proof. If we set $B = (S^1)^k \times D_{N-k}$ and $S = (S^1)^k \times S^{N-k-1}$, then (A.1) is equivalent to $\text{cat}(B, S) = k + 1$.

It is well known that $\text{cat}(B) = k + 1$ (see [20] or [22]), so, by definition of category, $\text{cat}(B, S) \leq \text{cat}(B) = k + 1$ follows.

In order to show that the reverse inequality holds, let us observe that the cohomology algebra $H^*(B)$ is an exterior algebra on $k$ one-dimensional generators and, moreover, the relative cohomology algebra $H^*(B, S)$ is a free $H^*(B)$-module on a $(N-k)$-dimensional generator. Thus,

$$(\widetilde{H}^*(B))^k \cdot H^*(B, S) \neq 0,$$

where $\widetilde{H}^*(B)$ is the reduced cohomology, $(\widetilde{H}^*(B))^k$ stands for the $n$-th power and the product is the cup product. As a consequence, by using Corollary 7.6(ii) in [11] (see, also, [14]) $\text{cat}(B, S) \geq k + 1$ follows. □

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