Fair and Fast Tie-Breaking for Voting

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Abstract

We introduce a notion of fairest tie-breaking for voting w.r.t. two widely-accepted fairness criteria: anonymity (all voters being treated equally) and neutrality (all alternatives being treated equally). We proposed a polynomial-time computable fairest tie-breaking mechanism, called most-favorable-permutation (MFP) breaking, for a wide range of decision spaces, including single winners, $k$-committees, $k$-lists, and full rankings. We characterize the semi-random fairness of commonly-studied voting rules with MFP breaking, showing that it is significantly better than existing tie-breaking mechanisms, including the commonly-used lexicographic and fixed-agent mechanisms.

1 Introduction

Voting is a popular method for collective decision making, in which agents report their preferences as rankings over a set of alternatives, and then a voting rule is applied to make a collective decision. Various notions of fairness were proposed and used to guide the design and analysis of voting rules. Unfortunately, the ANR impossibility theorem, which is “among the most well-known results in social choice theory” [Ozkes and Sanver, 2021], illustrates a fundamental incompatibility of two widely-accepted and natural fairness criteria: anonymity, which requires all voters to be treated equally, and neutrality, which requires all alternatives to be treated equally, when the voting rule is resolvable, meaning that a single decision must be made.

Despite this impossibility, many voting rules were designed and applied in practice, by first designing an irresolute voting rule, where ties are allowed, that satisfies anonymity and neutrality, and then applying a tie-breaking mechanism when ties occur. For example, the lexicographic tie-breaking chooses the (unique) winner that has the highest rank w.r.t. a fixed order over alternatives from the co-winners; the fixed-agent tie-breaking uses a fixed agent’s ranking to break ties.

However, little was know about how to break ties optimally, so that the resolute rule satisfies anonymity and neutrality to the greatest possible extent. Such rules are, therefore, fairest w.r.t. anonymity and neutrality. Recently, Xia [2020] pointed out that lexicographic tie-breaking and fixed-agent tie-breaking are from being optimal, and proposed a new tie-breaking rule that is asymptotically optimal under a large class of semi-random models for even number of agents. Nevertheless, the following question remains largely open.

How can we optimally break ties to satisfy anonymity and neutrality?

The challenges are two-fold: first, it is unclear how optimality should be defined, especially when no distributional information about the profiles are available; and second, even if a fairest tie-breaking mechanism can be defined, it may not be efficiently computable.

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1.1 Our contributions

Our main **conceptual contribution** is a general definition of *fairest* tie-breaking mechanisms for a wide range of voting scenarios, where the decision space can be single winners, $k$-committees, $k$-lists, or rankings over all alternatives. In fact, such fairest tie-breaking mechanisms admit a neat group-theoretic characterization as shown in Lemma 1. Our main **technical contributions** are threefold. First, Lemma 1 naturally leads to characterizations of the existence of ANR rules—voting rules that satisfy anonymity, neutrality, and resolvability—when the decision space is $k$-committees (Proposition 1) or $k$-lists (Proposition 2), which were not previously known. Second, we propose a polynomial-time fairest tie-breaking mechanism called most-favorable-permutation (MFP) breaking (Algorithm 1 and Theorem 1). MFP breaking is based on the extension of a lexicographic order over the alternatives to the histograms, and then using a permutation that optimally permutes the histogram of the given profile to break ties. Third, we characterize the semi-random likelihood of violations of ANR for commonly-studied voting rules under MFP breaking (Theorems 2 and 3), showing that the likelihood converges to 0 much faster than that under lexicographic or fixed-agent tie-breaking.

**Technical innovations.** Our work builds on notation and principles in *algebraic voting theory* [Daugherty et al., 2009], and extends them to general decision spaces. While some ideas in the proof of Lemma 1 have appeared in previous work [Bubboloni and Gori, 2014, Doğan and Giritligil, 2015, Xia, 2020], we were not aware of a similar statement as Lemma 1 that works for a wide range of decision spaces. The generality of Lemma 1 allows us to easily characterize the existence of ANR $k$-committee rules (Proposition 1), which was a topic of study in multiple papers but still remained open. Our characterization of ANR $k$-list rules (Proposition 2) is new. Our MFP tie-breaking mechanisms, despite being simple and natural, address the computational challenge pointed out in previous work [Bubboloni and Gori, 2021]. Therefore, we view the surprising simplicity in Lemma 1 and MFP breaking an advantage. The semi-random analyses in Theorems 2 and 3 are done by non-trivial applications of the polyhedral approach developed in [Xia, 2020, 2021].

1.2 Related work and discussions

Existence of ANR rules. A series of previous work focused on characterizing the existence of ANR rules by conditions on $m$ (the number of alternatives) and $n$ (the number of agents). For choosing a winner, Moulin [1983] proved that an ANR rule exists if and only if $m$ cannot be represented as a sum of $n$’s non-trivial (i.e., $>1$) divisors. Moulin [1983] also proved that an ANR rule that also satisfies Pareto efficiency exists if and only if $m$ is smaller than $n$’s smallest non-trivial divisor, which is equivalent to gcd($m, n$) $= 1$, where gcd means greatest common divisor. Campbell and Kelly [2015] pointed out that sometimes ANR comes at the cost of other desirable properties. Doğan and Giritligil [2015] characterized ANR rules that also satisfy monotonicity. Ozkes and Sanver [2021] investigated the existence of rules that satisfy anonymity, resolvability, and a weaker notion of neutrality. For choosing a ranking, Bubboloni and Gori [2014] proved that an ANR rule exists if and only if gcd($m, n$) $= 1$. Bubboloni and Gori [2015] considered a setting where voters are divided into groups, and investigated a new and weaker notion of in-group anonymity. For choosing a $k$-committee, Bubboloni and Gori [2016] proved that an ANR rule that satisfies Pareto efficiency exists if and only if gcd($m, n$) $= 1$. Bubboloni and Gori [2021] provided sufficient conditions for the existence of ANR rules that satisfy weaker notions of anonymity and neutrality. However, the fundamental question of the existence of ANR $k$-committee rules (without being required to satisfy any other properties) remained open, which is resolved by our Proposition 1. To the best of our knowledge, no previous work characterized ANR rules for choosing a $k$-list, which is addressed by our Proposition 2.

Tie-breaking mechanisms. Doğan and Giritligil [2015] proposed to use an ANR ranking rule to break ties for single-winner rules, when gcd($m, n$) $= 1$. Bubboloni and Gori [2016] characterized conditions for the existence of a tie-breaking mechanism to preserve weaker versions of anonymity and neutrality. Doğan and Giritligil [2015] proposed a mechanism to reduce the number of winners to minimize the “irresoluteness outlook”, i.e., the set of possible sizes of the winners under all profiles. Bubboloni and Gori [2021] proposed tie-breaking mechanisms for $k$-committees and rankings, that are based on defining a system of “representatives” for each equivalent class of profiles (see the proof of our Lemma 1 for a similar high-level idea). They also highlighted the computational chal-
lenges in their approach: (using our notation) “finding explicitly a system of representatives is in general a complex problem. However, that can be managed for small values of $m$ and $n$, that is, when the size $(m^n)$ of $L(A)^n$ is not too large”. Xia [2020] showed that the lexicographic and fixed-agent tie-breaking mechanisms are far from optimal under a large class of semi-random models, proposed the most-popular-singleton-ranking (MPSR) tie-breaking mechanism, and proved that it is asymptotically optimal for even $n$’s. It remained an open question of whether ANR can be optimally preserved by some polynomial-time tie-breaking mechanism, which is address by our MFP breakings (Algorithm 1 and Theorem 1). Our MFP breakings achieve better fairness than MPSR, because MFP breakings are fairest for every anonymous and neutral irresolute rule, every $m$ and $n$, and every profile.

**Ties in voting.** Many commonly-studied voting rules are initially defined as irresolute rules without explicitly specifying the tie-breaking mechanism. In real-world elections, despite the common belief that ties are rare [Campbell and Witcher, 2015], ties indeed happened, and sometimes it is unclear how they are resolved, because the tie-breaking mechanism was unspecified. The likelihood of ties were investigated in a series theoretical and simulation-based work initiated by Beck [1975], and previous work has establishing an $\Theta\left(\frac{1}{\sqrt{n}}\right)$ likelihood under a wide range of semi-random models [Xia, 2021], which are more general than the i.i.d. uniform distribution, known as the Impartial Culture in social choice. Complexity and algorithmic aspects of tie-breaking have been investigated in a series of work, for STV [Conitzer et al., 2009, Wang et al., 2019], ranked pairs [Brill and Fischer, 2012], and Baldwin and Coombs [Mattei et al., 2014]. Another line of work investigates the complexity of strategic manipulation under different tie-breaking mechanisms [Obraztsova et al., 2011, Obraztsova and Elkind, 2011, Aziz et al., 2013, Obraztsova et al., 2013]. Our work is different, because we aim at designing fast and fairest (w.r.t. ANR) tie-breaking mechanisms.

## 2 Preliminaries

Let $A = [m] = \{1, \ldots, m\}$ denote the set of $m \geq 2$ alternatives. Let $L(A)$ denote the set of all linear orders over $A$. There are $n \in \mathbb{N}$ agents, each of which uses a linear order to represent his or her preferences, called a vote. The vector of $n$ agents’ votes, denoted by $P$, is called a (preference) profile, sometimes called an $n$-profile. Let $D$ denote the decision space. Commonly-studied decision spaces are: $D = A$ (choosing a single winner); $D = A_k$ (choosing a $k$-committee), where $1 \leq k \leq m - 1$ and $A_k$ denote the set of all size-$k$ subsets of $A$; $D = \tilde{L}(A)$ (choosing a ranking). In this paper we also study $D = \tilde{L}_k(A)$ (choosing a $k$-list), where $\tilde{L}_k(A)$ is the set of all linear orders over all sets in $A_k$, which is motivated by ranking top $k$ items in recommender systems. Clearly $A = A_1$ and $L(A) = \tilde{L}_m(A)$.

For any profile $P$, let $\text{Hist}(P) \in \mathbb{Z}_+^{m^0}$ denote the anonymized profile of $P$, also called the histogram of $P$, which contains the total weight of every linear order in $L(A)$ according to $P$. Let $\mathcal{V}_{m,n}$ denote the set of histograms of all $n$-profiles over $m$ alternatives. An irresolute voting rule $\bar{\tau}$ is a mapping from a profile to a non-empty subset of $D$. A resolute voting rule is an irresolute rule that always chooses a single decision. Next, we recall the definitions of $k$-committee (which includes single-winner) and $k$-list (which includes ranking) versions of positional scoring rules.

**Positional scoring rules.** An irresolute positional scoring rule $\bar{\tau}_s$ is characterized by a scoring vector $\bar{s} = (s_1, \ldots, s_m)$ with $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$. For any alternative $a$ and any linear order $R \in L(A)$, we let $s(R, a) \triangleq s_i$, where $i$ is the rank of $a$ in $R$. Given a profile $P$, let $s(R, a) \triangleq \sum_{R \in P} s(R, a)$ denote the total score of $a$. When $D = A_k$, $\bar{\tau}_s$ chooses the set of all size-$k$ subsets $\tilde{A}$ such that for any $b \in (A \setminus \tilde{A})$, the score of $b$ is no more than the score of any alternative in $\tilde{A}$. When $D = \tilde{L}_k(A)$, $\bar{\tau}_s$ chooses all linearizations of all winning $k$-committees in the non-increasing order of their total scores. Special positional scoring rules include plurality, whose scoring vector is $(1, 0, \ldots, 0)$, Borda, whose scoring vector is $(m - 1, m - 2, \ldots, 0)$, and veto, whose scoring vector is $(1, \ldots, 1, 0)$.

**Tie-breaking mechanisms.** Many commonly-studied resolute voting rules are defined as the result of a tie-breaking mechanism on the outcome of an irresolute rule. A tie-breaking mechanism $f$ is a mapping from a profile $P$ and a non-empty set $D \subseteq D$ to a single decision in $D$. For example, when $D = A$, the lexicographic tie-breaking breaks ties in favor of alternatives ranked higher.
w.r.t. a predefined ranking; and the fixed-agent tie-breaking break ties using a pre-defined agent’s ranking. Let \( \mathcal{R}_f \) denote the resolute rule obtained from \( \mathcal{R} \) by applying \( f \). That is, for any profile \( P \), \( \mathcal{R}_f(P) = \{ f(P, \mathcal{R}(P)) \} \).

**Anonymity, neutrality, and resolvability.** For any irresolute rule \( \mathcal{R} \) and any profile \( P \), we define \( \text{ANO}(\mathcal{R}, P) \) if all agents in \( P \) are treated equally under \( \mathcal{R} \), i.e., for any profile \( P' \) with \( \text{Hist}(P') = \text{Hist}(P) \), we have \( \mathcal{R}(P') = \mathcal{R}(P) \); otherwise \( \text{ANO}(\mathcal{R}, P) = 0 \). We define \( \text{NEU}(\mathcal{R}, P) \) if all decisions are treated equally w.r.t. permutations among alternatives, i.e., for any permutation \( \sigma \) over \( A \), we have \( \nu(\sigma(P)) = \sigma(\nu(P)) \), where \( \sigma \) is naturally extended to \( P \) and the decision spaces \( D \) studied in this paper (i.e., \( k \)-committees and \( k \)-list), which is formally defined in Definition 1; otherwise \( \text{NEU}(\mathcal{R}, P) = 0 \). Notice that a resolute rule outputs a single decision, which may not be a single alternative—for example for \( k \)-committee rules with \( k \geq 2 \), a decision is a set of \( k \) alternatives. If \( \text{ANO}(\mathcal{R}, P) = 1 \) (respectively, \( \text{NEU}(\mathcal{R}, P) = 1 \)), then we say that \( \mathcal{R} \) satisfies anonymity (respectively, neutrality or resolvability) at \( P \). We further define \( \text{ANR}(\mathcal{R}, P) \) if and only if \( \mathcal{R} \) satisfies anonymity, neutrality, and resolvability at \( P \). Given \( n \), we say that \( \mathcal{R} \) satisfies anonymity (respectively, neutrality, resolvability, or \( \text{ANR} \)) if and only if for all \( n \)-profiles \( P \), we have \( \text{ANO}(\mathcal{R}, P) = 1 \) (respectively, \( \text{NEU}(\mathcal{R}, P) = 1 \), \( \text{ANR}(\mathcal{R}, P) = 1 \)).

**Definition 1 (Permutations).** Let \( S_A \) denote the permutation group over \( A \). Any permutation \( \sigma \in S_A \) can be naturally extended to \( k \)-committees, \( k \)-lists, profiles, and histograms over \( A \) in the following way. For any \( k \)-committee \( C = \{ a_1, \ldots, a_k \} \), let \( \sigma(C) = \{ \sigma(a_1), \ldots, \sigma(a_k) \} \); for any \( k \)-list \( R = \{ a_1 \prec \cdots \prec a_k \} \), let \( \sigma(R) = \{ \sigma(a_1) \prec \cdots \prec \sigma(a_k) \} \); for any profile \( P = (R_1, \ldots, R_n) \), let \( \sigma(P) = (\sigma(R_1), \ldots, \sigma(R_n)) \); and for any histogram \( \vec{v} \in V_{m,n} \), let \( \sigma(\vec{v}) \) be the histogram such that for every ranking \( R \), \( \sigma(\vec{v})[R] = \vec{v}[\sigma^{-1}(R)] \), where \( \sigma(\vec{v})[R] \) is the value of \( R \)-component in \( \sigma(\vec{v}) \).

According to Definition 1, any permutation \( \sigma \in S_A \) on \( V_{m,n} \) can be equivalently defined as follows. For any \( \vec{v} \in V_{m,n} \), let \( P_{\vec{v}} \) denote an arbitrary profile such that \( \text{Hist}(P_{\vec{v}}) = \vec{v} \). Then, we define \( \sigma(\vec{v}) = \text{Hist}(\sigma(P_{\vec{v}})) \).

In group-theoretic terms, Definition 1 defines group actions of \( S_A \) on \( k \)-committees, \( k \)-lists, profiles, and histograms, respectively, formally defined as follows. Basic definitions and notation about group theory can be found in Appendix A.

**Definition 2 (Group actions).** A group \( G \) acts on a set \( X \), if every \( g \in G \) can be viewed as a permutation on \( X \), such that (1) for all \( g_1, g_2 \in G \) and all \( x \in X \), we have \( g_1(g_2(x)) = g_1 \circ g_2(x) \), where \( \circ \) is the operation in \( G \); and (2) let \( Id \in G \) denote the identity, then for all \( x \in X \), we have \( Id(x) = x \).

We will frequently use the notion of stabilizer, orbit, and fixed point that are based on group actions.

**Definition 3 (Stabilizer, orbit, and fixed point).** For any group \( G \) that acts on \( X \) and any \( x \in X \), let \( \text{Stab}_G(x) \subseteq G \) denote the stabilizer of \( x \), i.e., the set of elements \( g \in G \) of which \( x \) is a fixed point, i.e., \( g(x) = x \). Let \( \text{Orbit}_G(x) \triangleq \{ g(x) : g \in G \} \) denote the orbit of \( x \). Let \( \text{Fix}_G(X) \triangleq \{ x \in X : \forall g \in G \, g(x) = x \} \) denote the set of fixed points. To simplify notation, the subscript \( G \) is omitted when \( G = S_A \).

**Example 1.** Let \( m = 5 \), \( n = 10 \), \( D = A \), and \( P_{10} \) denote a 10-profile as follows.

| Ranking | 13245 | 23145 | 34125 | 34215 | 43125 | 43215 |
|---------|-------|-------|-------|-------|-------|-------|
| # in Hist(\( P_{10} \)) | 1 | 1 | 2 | 2 | 2 | 2 |

In the table, 13245 represents the ranking \( 1 \succ 2 \succ 3 \succ 4 \succ 5 \). It is not hard to verify that \( \text{Hist}(P_{10}) \) is a fixed point of the permutation \((1,2)\), which exchanges 1 and 2. Let \( \sigma_1 = (1,3)(2,4) \), which maps 34125 to 23145. Let \( \sigma_2 = (1,3,2,4) \), which maps 43125 to 23145. Then, \( \sigma_1(\text{Hist}(P_{10})) = \text{Hist}(\sigma_1(P_{10})) \) and \( \sigma_2(\text{Hist}(P_{10})) = \text{Hist}(\sigma_2(P_{10})) \) consist of the following votes.

| Ranking | 13245 | 23145 | 34125 | 34215 | 43125 | 43215 |
|---------|-------|-------|-------|-------|-------|-------|
| # in Hist(\( \sigma_1(P_{10}) \)) | 2 | 2 | 2 | 1 | 0 | 1 |
| # in Hist(\( \sigma_2(P_{10}) \)) | 2 | 2 | 2 | 0 | 1 | 0 |
We have $\text{Hist}(P_{10}) \neq \sigma_1(\text{Hist}(P_{10}))$ and $\text{Hist}(P_{10}) \neq \sigma_2(\text{Hist}(P_{10}))$. Therefore, $\sigma_1 \notin \text{STAB}(\text{Hist}(P_{10}))$ and $\sigma_2 \notin \text{STAB}(\text{Hist}(P_{10}))$, based on the group action of $S_A$ on $\mathbb{Y}_{m,n}$ (Definition 1). In fact, $\text{STAB}(\text{Hist}(P_{10})) = \{\text{Id}, (1, 2)\}$ and $\text{FIXED}_{\text{STAB}(\text{Hist}(P_{10}))}(A) = \{3, 4, 5\}$.

### 3 Fairest Tie-Breaking: Characterization and Definition

In this section, we give a positive answer to the following question.

Can we define fairest tie-breaking w.r.t. anonymity and neutrality for a wide range of $D$?

We show in Lemma 1 below that the intrinsic source of ANR impossibility at a profile $P$ is the lack of fixed-point winners, defined as follows.

**Definition 4 (Fixed-point winners and $\mathcal{P}^{\mathcal{T}}_{m,n}$).** For any $D \subseteq D$ and any profile $P$, let $\text{FW}(P, D)$ denote the fixed-point winners,

$$\text{FW}(P, D) \triangleq D \cap \text{FIXED}_{\text{STAB}(\text{Hist}(P))}(D)$$

For any anonymous and neutral irresolute rule $\mathcal{T}$, define

$$\mathcal{P}^{\mathcal{T}}_{m,n} \triangleq \{P \in \mathbb{L}(A)^n : \text{FW}(P, \mathcal{T}(P)) = \emptyset\}$$

That is, given a profile $P$ and a set of winners $D$, a decision is a fixed-point winner if and only if it is a winner (i.e., in $D$) as well as a fixed point of $\text{STAB}(\text{Hist}(P))$ (i.e., in $\text{FIXED}_{\text{STAB}(\text{Hist}(P))}(D)$). $\mathcal{P}^{\mathcal{T}}_{m,n}$ is the set of profiles $P$ that has no fixed-point winners under $\mathcal{T}$.

**Example 2 (Fixed-point winners).** Continuing the setting of Example 1, let $\mathcal{T} = \text{Veto}$ denote the veto rule. Recall that $D = A$. We have $\text{Veto}(P_{10}) = \{1, 2, 3, 4\}$, which means that

$$\text{FW}(P_{10}, \text{Veto}(P_{10})) = \{1, 2, 3, 4\} \cap \text{FIXED}_{\text{STAB}(\text{Hist}(P_{10}))}(A) = \{3, 4\} \neq \emptyset$$

Therefore, $P_{10} \notin \mathcal{P}^{\text{Veto}}_{5,10}$.

We say that a rule $\mathcal{T}'$ refines another rule $\mathcal{T}$, if for all profiles $P$, $\mathcal{T}'(P) \subseteq \mathcal{T}(P)$. In this case $\mathcal{T}'$ is called a refinement of $\mathcal{T}$.

**Lemma 1.** Let $\mathcal{T}$ be an anonymous and neutral rule over $D$ that $S_A$ acts on. For any refinement $\mathcal{T}'$ of $\mathcal{T}$ and any $P \in \mathcal{P}^{\mathcal{T}}_{m,n}$, $\text{ANR}(\mathcal{T}', P) = 0$. There exists a refinement $r^*$ of $\mathcal{T}$ such that for every $P \notin \mathcal{P}^{\mathcal{T}}_{m,n}$, $\text{ANR}(r^*, P) = 1$.

**Proof.** For any $P \in \mathcal{P}^{\mathcal{T}}_{m,n}$, suppose for the sake of contradiction that there exists a refinement $r$ of $\mathcal{T}$ that satisfies anonymity, neutrality, and resolvability at $P$. Let $r(P) = \{d\}$. Then, according to the definition of $\mathcal{P}^{\mathcal{T}}_{m,n}$, $d$ is not a fixed point of $\text{STAB}(\text{Hist}(P))$, which means that there exists a permutation $\sigma \in \text{STAB}(\text{Hist}(P))$ such that $\sigma(d) \neq d$. Because $\text{Hist}(\sigma(P)) = \text{Hist}(P)$, by anonymity, we have $r(\sigma(P)) = r(P) = \{d\}$. On the other hand, by neutrality, we have $r(\sigma(P)) = \{\sigma(d)\} \neq \{d\}$, which is a contradiction.

Next, we define a refinement $r^*$ that satisfies ANR at every $P \notin \mathcal{P}^{\mathcal{T}}_{m,n}$. Consider the partition of $\mathbb{Y}_{m,n}$ into orbits under group actions $S_A$. Let $\sim$ denote this equivalence relationship, i.e., for any pair of histograms $\vec{v}_1, \vec{v}_2 \in \mathbb{Y}_{m,n}$, we write $\vec{v}_1 \sim \vec{v}_2$ if and only if there exists a permutation $\sigma \in S_A$ such that $\vec{v}_1 = \sigma(\vec{v}_2)$. Then, for each set in the partition of $\mathbb{Y}_{m,n}$, we fix a “representative” histogram $\vec{v}$ and define $r^*(\vec{v})$ to consist of an arbitrary but fixed decision in $\mathcal{T}(\vec{v})$ that is also a fixed point of $\text{STAB}(\vec{v})$. For any profile $P$, let $\sigma$ denote the permutation such that $\sigma(\text{Hist}(P))$ is a representative $\vec{v}$, and then define $\mathcal{T}(P) = \sigma^{-1}(\mathcal{T}(\vec{v}))$. It is not hard to verify that $\text{ANR}(r^*, P) = 1$.}

While the idea of “representatives” used in the proof of Lemma 1 has been considered in previous characterizations of ANR impossibility theorems [Bubboloni and Gori, 2016, Xia, 2020], we are not aware of a similar result that characterize the intrinsic impossibility of ANR for general decision space $D$ that $S_A$ acts on.

Lemma 1 is useful in two ways. The first is conceptual—Lemma 1 enables the definition of “fairest” tie-breakings (or refinements) w.r.t. ANR, formally defined as follows, where $\mathcal{T}_f$ is the refinement of $\mathcal{T}$ obtained by applying a tie-breaking mechanism $f$. 

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Definition 5 (Fairest tie-breakings). Given a voting rule \( \overline{f} \) that satisfies anonymity and neutrality, a tie-breaking mechanism \( f \) is fairest (w.r.t. ANR), if for every \( P \notin \mathcal{P}_{m,n}^{f} \), ANR(\( \overline{f}, P \)) = 1.

We emphasize that the notion of “fairest” in this paper is defined w.r.t. anonymity and neutrality. Notice that the proof of Lemma 1 only guarantees the existence of a fairest tie-breaking mechanism for \( \overline{f} \). It is unclear whether any fairest tie-breaking mechanism can be computed in polynomial time as commented in [Bubboloni and Gori, 2021]. Moreover, it is unclear whether a fairest tie-breaking mechanism for one rule is also fairest for another. We will give positive answers to both questions in the next section.

The second usage of Lemma 1 is technical—it can be applied to obtain (new) characterizations of the existence of ANR rules for \( D \in \{ A_{k}, L_{k}(A) : 1 \leq k \leq m \} \). To present the characterizations, we introduce the following notation. For any \( n \in \mathbb{N} \), let \( \text{Div}_{n} \) denote the set of all divisors of \( n \), including 1, and let \( \text{Div}_{n}^{*} \equiv \bigcup_{t=1}^{\infty} (\text{Div}_{n})_{t}^{*} \). For any \( 1 \leq k \leq m \), let \( \text{Div}_{m,n}^{k} \subseteq \text{Div}_{n}^{*} \) denote the set of vectors \( \vec{d} \in \text{Div}_{n}^{*} \) such that (1) \( \vec{d} \cdot \vec{1} = m \), and (2) \( \vec{d} \) does not have a subvector whose sum is \( k \). Specifically, \( \text{Div}_{m,n}^{1} \) consists of vectors of \( n \)’s non-trivial divisors that sum up to \( m \). For example, \( \text{Div}_{5,6}^{1} = \{(2,3), (3,2)\} \). As another example, \( \text{Div}_{6}^{2} = \emptyset \), because \( \text{Div}_{6} = \{1, 2, 3, 6\} \), and in every way \( m = 6 \) is represented as the sum of \( n \)’s divisors, it has a subvector whose sum is 2.

Proposition 1. Given \( m, n, \) and \( k \leq m \), there exists a \( k \)-committee rule that satisfies anonymity, neutrality, and resolvability if and only if \( \text{Div}_{m,n}^{k} = \emptyset \).

Proof sketch. By Lemma 1, where \( \overline{f} \) is the rule that always chooses all \( k \)-committees, the existence of an ANR \( k \)-committee rule is equivalent to \( \mathcal{P}_{m,n}^{f} = \emptyset \). It is not hard to verify that the latter is equivalent to the following condition, by taking the perspective of the stabilizer of profiles in \( \mathcal{P}_{m,n}^{f} \).

Condition 1. There does not exist a permutation group \( U \) over \( A \) such that (i) \( \lvert U \rvert \) is a divisor of \( n \), and (ii) \( D \) does not contain a fixed point under \( U \).

In the rest of the proof, we show that when \( D = A_{k} \), Condition 1 is equivalent to \( \text{Div}_{m,n}^{k} = \emptyset \).

For the \( \Rightarrow \) direction, suppose for the sake of contradiction that Condition 1 holds but \( \text{Div}_{m,n}^{k} \neq \emptyset \). We derive a contradiction by explicitly constructing \( U \) as follows. Choose any \( \vec{d} = (d_{1}, \ldots, d_{t}) \in \text{Div}_{m,n}^{k} \). Let \( U \) be the permutation group generated by

\[
\sigma_{\vec{d}} \equiv (1, \ldots, d_{1}) (d_{1} + 1, \ldots, d_{1} + d_{2}) \cdots (m - d_{t} + 1, \ldots, m)
\]

Part (i) of Condition 1 clearly holds for \( U \), because \( \lvert U \rvert = \text{lcm}(\vec{d}) \) and \( \vec{d} \in \text{Div}_{n}^{*} \). Because no subvector of \( \vec{d} \) sum up to \( k \), for any \( A \in A_{k} \), there must exist an alternative \( a \in A \) such that \( \sigma_{\vec{d}}(a) \notin A \), which means that Part (ii) of Condition 1 holds for \( U \). Therefore, Condition 1, which requires that no \( U \) satisfies both parts, does not hold.

For the \( \Leftarrow \) direction, suppose for the sake of contradiction that \( \text{Div}_{m,n}^{k} = \emptyset \) but Condition 1 does not hold, i.e., there exists a permutation group \( U \) that satisfies the two parts. Let \( \vec{d} = (d_{1}, \ldots, d_{T}) \) denote a vector that consists of the sizes of orbits over \( A \) under \( U \). Due to the orbit-stabilizer theorem, see e.g., [Rosenberger et al., 2010, Theorem 13.14], for every \( t \leq T \), \( d_{t} \) is a divisor of \( \lvert U \rvert \). Recall from part (i) of Condition 1 that \( \lvert U \rvert \) is a divisor of \( n \). Therefore, \( d_{t} \) is a divisor of \( n \), which means that \( \vec{d} \in \text{Div}_{n}^{*} \). If \( \vec{d} \) contains a subvector that sum up to \( k \), then we let \( A \) denote the \( k \) alternatives in the corresponding orbits. It follows that \( A \) is a fixed point of \( U \), which contradicts part (ii) of Condition 1. Therefore, \( \vec{d} \) does not contain a subvector that sum up to \( k \), which means that \( \vec{d} \in \text{Div}_{m,n}^{k} \). This contradicts the assumption that \( \text{Div}_{m,n}^{k} = \emptyset \). The full proof can be found in Appendix B.

Proposition 2. Given \( m, n, \) and \( k \leq m \), there exists a \( k \)-list rule that satisfies anonymity, neutrality, and resolvability if and only if for every \( m' \) with \( m - k + 1 \leq m' \leq m \), \( \text{Div}_{m,m'} = \emptyset \).

We note that \( \text{Div}_{m,m'} = \emptyset \) if and only if \( m' \) cannot be represented as a sum of \( n \)'s non-trivial (i.e., \( > 1 \)) divisors. The proof is similar to the proof of Proposition 1 and can be found in Appendix B.
4 Fairest Tie-Breaking: Fast Computation

In this section, we give a positive answer to the following question.

*Can we compute a fairest tie-breaking in polynomial time?*

As shown in the proof of Lemma 1, the key step in defining a fairest tie-breaking is to identify and fix a “representative” histogram for each equivalent class, choose a fixed-point winner for each representative histogram, and then apply permutations to extend the decision to other histograms in the same equivalent class. The main challenge is that the number of equivalent classes is exponentially large in \( m \), which means that pre-computing the representatives for tie-breaking takes exponential time. Our solution is conceptually simple and natural—we define a priority order over all histograms by extending a ranking \( \succ \) over \( A \) lexicographically, let the representative of an equivalent class be the histogram that is highest in the priority order, and choose a decision based on an extension of \( \succ \) to \( D \) from the fixed-point winners (Definition 4).

**Definition 6 (Priority order \( \succ \)).** Let \( \succ \in \mathcal{L}(A) \). We extend \( \succ \) to \( \mathcal{L}(A) \), such that rankings are compared lexicographically w.r.t. their top-ranked alternatives, second-ranked alternatives, etc. We extend \( \succ \) to \( \mathcal{V}_{m,n} \), such that the coordinates of vectors in \( \mathcal{V}_{m,n} \) are ordered w.r.t. \( \succ \) on \( \mathcal{L}(A) \), and then vectors in \( \mathcal{V}_{m,n} \) are compared lexicographically, favoring vectors with higher values in more important (earlier) coordinates. We extend \( \succ \) to \( D = A_k \) (respectively, \( D = \mathcal{L}_k(A) \)), such that size-\( k \) sets (respectively \( k \)-lists) of alternatives are compared w.r.t. their top-ranked alternatives, second-ranked alternatives, etc.

In the remainder of this paper, we use \( \succ = [1 \succ \cdots \succ m] \) by default, and the coordinates in \( \text{Hist}(P) \) are ordered lexicographically according to \( \succ \). We emphasize that in general \( \succ \) can be any ranking over \( A \).

**Example 3.** Continuing the setting of Example 1, among the 8 types of rankings in \( \sigma_1(P_{10}) \) and \( \sigma_2(P_{10}) \), we have \( 12345 \succ 12435 \succ 21345 \succ 21435 \succ 31425 \succ 31245 \succ 41325 \succ 42315 \). Therefore, \( \sigma_1(P_{10}) \succ \sigma_2(P_{10}) \), because the top four non-zero coordinates of \( \text{Hist}(\sigma_1(P_{10})) \) and \( \text{Hist}(\sigma_2(P_{10})) \) are the same (two for \( 12345 \), \( 12435 \), \( 21345 \), and \( 21435 \)), and the remaining two non-zero coordinates of \( \text{Hist}(\sigma_1(P_{10})) \), i.e., \( \{31425, 41325\} \), are more preferred to the remaining two non-zero coordinates of \( \text{Hist}(\sigma_2(P_{10})) \), i.e., \( \{32415, 42315\} \).

Given a profile \( P \), we define a special permutation that maps \( \text{Hist}(P) \) to a histogram that is ranked as high as possible according to \( \succ \).

**Definition 7 (Most-favorable permutations (MFP)).** For any profile \( P \), let

\[
\text{MFP}(P) \triangleq \arg \max_{\sigma \in S_A} \text{Hist}(\sigma(P))
\]

We note that \( \arg \max \) is defined to maximize the priority in \( \succ \) instead of maximizing the rank number in \( \succ \) (which corresponding to minimizing priority). We are now ready to define a fairest tie-breaking mechanism, which uses a “backup” tie-breaking mechanism when \( \text{FW}(P, D) = \emptyset \).

**Definition 8 (MFP tie-breaking).** Given a tie-breaking mechanism \( f \), a profile \( P \), and \( D \subseteq D \), we choose any \( \sigma \in \text{MFP}(P) \) and define

\[
\text{MFP}_{f}(P, D) \triangleq \begin{cases} 
\arg \max_{d \in \text{FW}(P, D)} f(\sigma(d)) & \text{if } \text{FW}(P, D) \neq \emptyset \\
\text{otherwise}
\end{cases}
\]

In words, if \( \text{FW}(P, D) \neq \emptyset \), which means that it is possible to break ties in \( D \) to achieve ANR at \( P \), MFP\(_f \) first picks an arbitrary permutation \( \sigma \in \text{MFP}(P) \), and then chooses a fixed point \( d \in D \) of \( \text{STAB}(\text{Hist}(P)) \), such that \( \sigma(d) \) has the highest priority in \( \succ \) as defined in Definition 6. Otherwise MFP\(_f \) applies the backup tie-breaking mechanism \( f \). Proposition 3 in Appendix C confirms that MFP\(_f \) is well-defined, by proving that the choice of \( \sigma \) does not matter.

**Example 4 (MFP and MFP\(_f \)).** Continuing the setting of Examples 2 and 3, let \( \sigma^*_1 \triangleq (1, 4, 2, 3) \). Then, \( \text{MFP}(P_{10}) = \{\sigma_1, \sigma^*_1\} \) and \( \sigma^*_1(P_{10}) = \sigma^*_1(P_{10}) \). Let \( \text{Veto} \) denote the veto rule. For any backup tie-breaking mechanism \( f \), we have \( \text{MFP}_{f}(P_{10}, \text{Veto}(P_{10})) = \text{MFP}_{f}(P_{10}, \{1, 2, 3, 4\}) = \{3\} \), because \( \sigma_1(3) = \sigma^*_1(3) = 1 \succ 2 = \sigma_1(4) = \sigma^*_1(4) \).

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Next, we present a polynomial-time algorithm for computing MFP. For any profile \( P \) and any ranking \( R \), let \( [\text{Hist}(P)]_R \) denote the \( R \)-component of \( \text{Hist}(P) \), i.e., the multiplicity of \( R \) in \( P \). A ranking \( R \) is a most popular ranking in \( P \), if it has the highest multiplicity in \( P \). Let \( \text{MPR}(P) \triangleq \arg \max_R [\text{Hist}(P)]_R \) denote the set of all most popular rankings in \( P \).

We use the list form to store \( \text{Hist}(P) \), which contains the rankings that appear in \( P \) in the decreasing order w.r.t. \( \succ \), and for each such ranking \( R \), we also store \( [\text{Hist}(P)]_R \). In other words, we do not list zero entries so that \( \text{Hist}(P) \) takes \( O(nm \log m) \) space to store. Moreover, comparing histograms of two profiles w.r.t. \( \succ \) takes polynomial time by scanning down the lists.

**Algorithm 1:** Compute MFP

1. Let \( \sigma_j \) be the permutation such that \( \sigma_j(R_j) = R_\ell \). Compute

\[
\text{STAB}[\text{Hist}(P)] = \bigcap_{1 \leq \ell \leq n} \{ \sigma_j : 1 \leq \ell \leq n \text{ such that } [\text{Hist}(P)]_{R_j} = [\text{Hist}(P)]_{R_\ell} \} \quad (1)
\]

2. Compute FW\((P, D)\), and if FW\((P, D) = \emptyset\), then return \( f(P, D) \).

3. for every most popular ranking \( R \in \text{MPR}(P) \), compute \( R \in \mathcal{S}_A \) such that \( R(R) = \succ \).

4. Let \( R^* \in \arg \max_{R \in \text{MPR}(P)} R(\text{Hist}(P)) \).

5. return \( \arg \max_{d \in \text{FW}(P, D)} \sigma_{R^*}(d) \).

We note that in step 5, decisions in FW\((P, D)\) are chosen w.r.t. \( R^* \), which is extended to \( D \) according to Definition 6. For all \( D \) studied in this paper, this step is equivalent to choosing the highest-ranked decision in FW\((P, D)\), after applying \( \sigma_{R^*} \), w.r.t. \( \succ = [1 \succ \cdots \succ n] \).

**Example 5 (Execution of Algorithm 1).** Continuing the setting of Example 4, we run Algorithm 1 on \( P_{10} \) and \( \tau = \text{Veto} \) in this example. In step 1, the right hand side of Equation (1) for \( j = 1 \) is \( \{Id, (1, 2)\} \), because only 23145 (in addition to 13245 itself) has the same multiplicity. Recall from Example 1 that \( \text{STAB}[\text{Hist}(P_{10})] = \{Id, (1, 2)\} \). In step 2, we have FW\((P_{10}, \text{Veto}(P_{10})) = \{3, 4\} \) as shown in Example 2. In step 3, there are four most popular rankings: \{34125, 34215, 43125, 43215\}. Let \( \sigma_1^* \triangleq (1, 4)(2, 3) \). It is not hard to verify that \( \sigma_34125 = \sigma_1, \sigma_34215 = \sigma_1, \sigma_43125 = \sigma_2 \), and \( \sigma_43215 = \sigma_2 \). Then in step 4, we have \( \sigma_1(P_{10}) = \sigma_1^*(P_{10}) \) and \( \sigma_2(P_{10}) = \sigma_2^*(P_{10}) \). Recall from Example 3 that \( \sigma_1(P_{10}) \succ \sigma_2(P_{10}) \). Therefore, we can choose \( R^* = 34125 \). Finally, in step 5, \( \sigma_{R^*}(3) = 1 \succ 2 = \sigma_{R^*}(4) \), which means that \( \text{Veto}_{\text{MFP}}(P_{10}) = \{3\} \).

The correctness and computational efficiency is confirmed by the following theorem, which works for any \( D \in \{\mathcal{A}_k, \mathcal{E}_k(\mathcal{A}) : 1 \leq k \leq m\} \).

**Theorem 1.** Algorithm 1 runs in polynomial time and computes MFP, which is a fairest tie-breaking mechanism for any anonymous and neutral voting rule.

The proof can be found in Appendix C.

## 5 Semi-Random Fairness after MFP Breaking

It follows from the optimality of MFP breaking (Theorem 1) that it is fairer (w.r.t. anonymity and neutrality) than any non-fairest tie-breaking mechanisms, especially the common-used lexicographic breaking and fixed-agent breaking. Therefore, the question for this section is:

*How much better MFP breakings are compared to other non-fairest tie-breaking mechanisms?*

We answer this question under the general semi-random framework [Xia, 2020], which was inspired by and resembles the smoothed analysis [Spielman and Teng, 2009], by characterizing the semi-random fairness (or more precisely, unfairness) of commonly-studied voting rules after applying MFP breaking for \( D = \mathcal{A} \). For any (irresolute) rule \( \tau \), we let \( \tau_{\text{MFP}} \) denote the resolute rule obtained

\[1\text{In fact, for any } D \text{ that } \mathcal{S}_A \text{ acts on, Algorithm 1 computes MFP, which is a fairest tie-breaking mechanism. If comparing decisions in } D \text{ according to the extension of } \succ \text{ takes polynomial time (which holds for the decision spaces studied in this paper and the extension defined in Definition 6), then Algorithm 1 runs in polynomial time as well.} \]
from $\tau$ by applying a MFP breaking. Following [Xia, 2020], we define the semi-random likelihood for ANR to be violated as follows.

**Definition 9.** Given a set $\Pi$ of distributions over $\mathcal{L}(\mathcal{A})$, any voting rule $r$, and any $n \in \mathbb{N}$, we define

$$\overline{\text{ANR}}_{\text{max}}(r, n) \triangleq \sup_{\pi \in \Pi_n} \Pr_{P \sim \pi}(\neg \text{ANR}(r, P))$$

That is, $\overline{\text{ANR}}_{\text{max}}(r, n)$ is the maximum expected violation of ANR of $r$, where the expectation is taken over randomly generate profile $P$ from an adversarially chosen distribution $\pi$. We note that while the votes in $P$ are independently generated, distributions in $\pi$ can be arbitrarily correlated. In this paper, we make the following assumptions on $\Pi$, following [Xia, 2020].

**Assumption 1.** We assume that (1) $\Pi$ is strictly positive, meaning that there exists $\epsilon > 0$ such that all probabilities in all distributions in $\Pi$ is larger than $\epsilon$; (2) $\Pi$ is a closed set in $\mathbb{R}^{m!}$; and (3) the convex hull of $\Pi$ contains $\pi_{\text{uni}}$, which is the uniform distribution over $\mathcal{L}(\mathcal{A})$.

While Assumption 1 sounds technical, it is indeed satisfied by many commonly studied models, such as single-agent Mallows and single-agent Plackett-Luce as shown in [Xia, 2020, Example 2 in Appendix A]. Assumption 1 is also satisfied by the Impartial Culture, which corresponds to the case $\Pi = \{\pi_{\text{uni}}\}$. Therefore, results in this section (Theorems 2 and 3) hold for Impartial Culture. It turns out that the semi-random-violation of ANR is related to two values defined as follows.

**Definition 10.** For any $d \in \text{Div}_n$, let $\text{lcm}(d)$ denote the least common multiple of components of $d$. We define $d_{\text{min}} \triangleq \min\{d \in \text{Div}_n : 2 \leq d \leq m\}$ and $\ell_{\text{min}} \triangleq \min\{\text{lcm}(d) : d \in \text{Div}_{m,n}\}$.

If $\text{Div}_n \cap \{2, \ldots, m\} = \emptyset$ (respectively, $\text{Div}_{m,n} = \emptyset$), then we let $d_{\text{min}} = -\infty$ (respectively, $\ell_{\text{min}} = -\infty$).

We note that $d_{\text{min}}$ and $\ell_{\text{min}}$ only depend on $m$ and $n$; if $d_{\text{min}} = -\infty$ (which is equivalent to $\gcd(m!, n) = 1$), then $\ell_{\text{min}} = -\infty$ (which is equivalent to $m$ cannot be represented as a sum of $n$’s non-trivial divisors); and if $\ell_{\text{min}} \neq -\infty$, then $2 \leq d_{\text{min}} \leq \ell_{\text{min}}$.

**Example 6 ($\ell_{\text{min}}$ and $d_{\text{min}}$).** If $m = 4$, $n = 5$, then $d_{\text{min}} = \ell_{\text{min}} = -\infty$; if $m = 3$, $n = 10$, then $d_{\text{min}} = 2$ and $\ell_{\text{min}} = -\infty$; if $m = 5$, $n = 6$, then $d_{\text{min}} = 2$ and $\ell_{\text{min}} = 6$.

**Theorem 2 (Bounds on semi-random violation of ANR after MFP breaking).** For any fixed $m \geq 2$ and $\Pi$ that satisfies Assumption 1, there exist a constant $C_1 > 0$ and a constant $C_2 > 0$, such that for any voting rule $\tau$ with $\mathcal{D} = \mathcal{A}$ that satisfies anonymity and neutrality, any MFP breaking, and any $n \geq 1$,

$$C_1 \cdot \begin{cases} 0 & \text{if } \ell_{\text{min}} = -\infty \text{ and } \frac{d_{\text{min}} - 1}{n} \cdot \frac{m!}{n} > \overline{\text{ANR}}_{\text{max}}(\tau_{\text{MFP}}, n) \\ \frac{\ell_{\text{min}}}{n} - \frac{(\ell_{\text{min}} - 1) \cdot \frac{m!}{n}}{n} & \text{otherwise} \end{cases} \leq \overline{\text{ANR}}_{\text{max}}(\tau_{\text{MFP}}, n) \leq C_2 \cdot \begin{cases} 0 & \text{if } d_{\text{min}} = -\infty \\ \frac{d_{\text{min}} - 1}{n} \cdot \frac{m!}{n} & \text{otherwise} \end{cases}$$

The proof can be found in Appendix D. Next, we prove that the lower and upper bound in Theorem 2 are asymptotically tight. More precisely, the lower bound can be achieved by MFP applied to the irresolve rule that always chooses all alternatives as the co-winners, denoted by $\tau'$. The upper bound is achieved by MFP applied to a large class of voting rules that satisfies anonymity, neutrality, and two additional properties called $\delta$-unanimity and canceling-out defined as follows.

**Definition 11 ($\delta$-unanimity).** For any $0 \leq \delta \leq 1$, a voting rule $\tau$ with $\mathcal{D} = \mathcal{A}$ satisfies $\delta$-unanimity, if for any set of alternatives $A \subseteq \mathcal{A}$ and any $n$-profile $P$ such that $A$ are ranked in the top positions in at least $(1 - \delta)n$ votes of $P$, we have $\tau(P) \subseteq A$.

Clearly, Pareto efficiency, which requires that the winner is not ranked lower than another decision by all agents, is equivalent to 0-unanimity. The larger $\delta$ is, the weaker $\delta$-unanimity is as a desirable property for voting rules. In other words, a voting rule $\tau$ that satisfies $\delta$-unanimity would satisfy $\delta'$-unanimity for all $\delta' \leq \delta$.

**Definition 12 (Canceling-out).** An anonymous voting rule $\tau$ with $\mathcal{D} = \mathcal{A}$ satisfies canceling-out, if for any profile $P$, $\tau(P) = \tau(P \cup \mathcal{L}(\mathcal{A}))$.

That is, votes in $\mathcal{L}(\mathcal{A})$ cancel out each other under $\tau$. Proposition 4 in Appendix D proves that many commonly studied voting rules, including positional scoring rules with $s_1 > s_2 > \cdots > s_m$ and others defined in Appendix A, satisfy $\delta$-unanimity (for some $0 < \delta < 1$ that does not depend on $n$) and canceling-out.
\textbf{Theorem 3 (Tightness of bounds).} For any fixed $m \geq 2$, any $\Pi$ that satisfies Assumption 1, any MFP, and any $n \geq 1$, let $\mathcal{P}'$ denote the rule that always outputs $\mathcal{A}$.

\begin{align*}
\text{Tightness of lower bound: } & \max_{\mathcal{P}' \in \mathcal{P}} \left( \frac{\max_{\pi \in \Pi} (\mathcal{P}'_{\text{MFP}}(\pi, n)) - \min_{\mathcal{A} \in \mathcal{A}} (\mathcal{A}_n)}{\min_{\mathcal{A} \in \mathcal{A}} (\mathcal{A}_n)} \right) \text{ if } \ell_{\min} = -\infty \\
& \Theta \left( \frac{1}{n^{\min - 1}} \right) \text{ otherwise}
\end{align*}

For any $\mathcal{P}$ with $D = \mathcal{A}$ that satisfies anonymity, neutrality, canceling-out, and $\delta$-unanimity for some fixed $\delta > 0$,

\begin{align*}
\text{Tightness of upper bound: } & \max_{\mathcal{P}' \in \mathcal{P}} \left( \frac{\max_{\pi \in \Pi} (\mathcal{P}'_{\text{MFP}}(\pi, n)) - \min_{\mathcal{A} \in \mathcal{A}} (\mathcal{A}_n)}{\min_{\mathcal{A} \in \mathcal{A}} (\mathcal{A}_n)} \right) \text{ if } d_{\min} = -\infty \\
& \Theta \left( \frac{1}{n^{\min - 1}} \right) \text{ otherwise}
\end{align*}

The proof can be found in Appendix D.

\textbf{Discussions.} The tightness of upper bound part in Theorem 3 implies that many commonly studied voting rules, including all positional scoring rules with $s_1 > \cdots > s_m$ and all rules mentioned in Proposition 4, are the least fair when ties are broken optimally, though it still converges to 0 at a fast rate. This can be viewed as a form of efficiency-fairness tradeoff, because $\delta$-unanimity can be viewed as a form of efficiency. Notice that when $d_{\min} \neq -\infty$, we have $d_{\min} \geq 2$. Therefore, the $\max_{\mathcal{A} \in \mathcal{A}} (\mathcal{A}_n)$ upper bound, which is smaller than $\Theta \left( \frac{1}{n^{\min - 1}} \right)$, is much smaller than the $\Omega(n^{-0.5})$ lower bound for positional scoring rules under the commonly-used lexicographic and fixed-agent tie-breaking mechanisms [Xia, 2020, Proposition 1].

Theorems 2 and 3 together provide a quantitative extension of Moulin [1983]’s ANR impossibility theorem and Doğan and Giritligil [2015]’s tie-breaking theorem. Specifically, when $n$ cannot be represented as the sum of $n$’s non-trivial divisors, we have $\ell_{\min} = -\infty$, which means that there exists a voting rule that satisfies ANR, and the tightness of lower bound in Theorem 3 implies that there exists such a rule that can be computed in polynomial time. When $\gcd(m!, n) = 1$, we have $d_{\min} = -\infty$, which means that any anonymous and neutral rule $\mathcal{P}$ can be made to satisfy ANR by tie-breaking, as proved by Doğan and Giritligil [2015]. The tightness of lower bound in Theorem 3 implies that there exists such tie-breaking mechanisms (MFP breakings) that can be computed in polynomial time, which was unknown previously, especially in [Doğan and Giritligil, 2015].

\section{Summary and Future Work}

We introduced the notion of fairest tie-breaking w.r.t. two classical and widely-accepted fairness criteria, i.e., anonymity and neutrality, and proposed a class of polynomial-time fairest tie-breaking mechanisms, called MFP breakings, for a large class of decision space. We also characterized the semi-random likelihood of violation of fairness after applying an MFP breaking to commonly-studied voting rules with $D = \mathcal{A}$. There are many directions for future work. One interesting question is how to deal with other types of preference structures, for example $k$-committee, $k$-list, or approval preferences. While Lemma 1 and MFP breaking can be naturally extended to such preference structures, it is unclear whether neat characterizations like Propositions 1 and 2 can be obtained. Another interesting question is to analyze other notions of fairness and other desirable properties. Semi-random likelihood of other choices of $D$ is also an interesting direction for future work.

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A Additional Preliminaries

A.1 Other commonly-studied voting rules

Weighted Majority Graphs. For any profile \( P \) and any pair of alternatives \( a, b \), let \( P[a \succ b] \) denote the total weight of votes in \( P \) where \( a \) is preferred to \( b \). Let \( \text{WMG}(P) \) denote the weighted majority graph of \( P \), whose vertices are \( A \) and whose weight on edge \( a \rightarrow b \) is \( w_P(a, b) = P[a \succ b] - P[b \succ a] \).

A voting rule is said to be weighted-majority-graph-based (WMG-based) if its winners only depend on the WMG of the input profile. We consider the following commonly-studied WMG-based rules. For each rule, we will define its \( D \) weighted-majority-graph-based (WMG-based) version for \( 1 \leq k \leq m - 1 \), and its \( \mathcal{A}_k \) version chooses all (unordered) alternatives involved in its \( \mathcal{L}_k(A) \) version. Recall that \( D = \mathcal{A} \) is a special case of \( D = \mathcal{L}_k(A) \), where \( k = 1 \); and \( D = \mathcal{L}(A) \) is a special case of \( D = \mathcal{L}_k(A) \), where \( k = m \).

- Copeland. The Copeland rule is parameterized by a number \( 0 \leq \alpha \leq 1 \), and is therefore denoted by Copeland\(_\alpha \), or \( \text{Cd}_\alpha \) for short. For any fractional profile \( P \), an alternative \( a \) gets 1 point for each other alternative it beats in their head-to-head competition, and gets \( \alpha \) points for each tie. Copeland\(_\alpha \), for \( D = \mathcal{L}_k(A) \) chooses all linear extensions of weak orders of top \( k \) alternatives w.r.t. their Copeland scores. In other words, for any profile \( P \), \( a_1 \succ \cdots \succ a_k \in \text{Cd}_\alpha(P) \) if and only if (1) for any \( k_1, k_2 \leq k \) such that \( a_{k_1} \succ a_{k_2} \), the Copeland score of \( a_{k_1} \) is no more than the Copeland score of \( k_1 \); and (2) for any \( k' \succ k \), the Copeland score of \( a_{k'} \) is no more than the Copeland score of \( k_2 \).

- Maximin. For each alternative \( a \), its min-score is defined to be \( \min_{b \in A} w_P(a, b) \). Maximin for \( D = \mathcal{L}_k(A) \) chooses all linear extensions of weak orders of top \( k \) alternatives w.r.t. their minimin scores.

- Ranked pairs. Given a profile \( P \), we fix edges in WMG\((P)\) one by one in a non-increasing order w.r.t. their weights (and sometimes break ties), unless it creates a cycle with previously fixed edges. After all edges (with positive, 0, or negative weights) are considered, the fixed edges represent a linear order over \( A \) and the ranked top-\( k \) alternatives are chosen. We consider all tie-breaking methods to define the ranked top-\( k \) alternatives. This is known as the parallel-universes tie-breaking (PUT) [Conitzer et al., 2009].

- Schulze. For any directed path in the WMG, its strength is defined to be the minimum weight on any single edge along the path. For any pair of alternatives \( a, b \), let \( s(a, b) \) be the highest weight among all paths from \( a \) to \( b \). Then, we write \( a \succeq b \) if and only if \( s(a, b) \geq s(b, a) \), and the strict version of this binary relation, denoted by \( \succ \), is transitive [Schulze, 2011]. The Schulze rule for \( D = \mathcal{L}_k(A) \) chooses ranked top-\( k \) alternatives in all linear extensions of \( \succeq \).

STV. The single transferable vote (STV) rule has \( m - 1 \) rounds. In each round, the alternative with the lowest plurality score is eliminated. STV for \( D = \mathcal{L}_k(A) \) chooses ranked top-\( k \) alternatives according to the inverse elimination order. Like ranked pairs, PUT is used to select the winning \( k \)-lists.

A.2 Basics of group theory

A group is a set \( G \) equipped with a binary operation \( \circ \) that satisfies the following conditions: (i) for any \( g_1, g_2 \in G \), we have \( g_1 \circ g_2 \in G \); (ii) there exists an identity element, denoted by \( \text{Id} \in G \), such that for all \( g \in G \), we have \( g \circ \text{Id} = \text{Id} \circ g = g \); (iii) any \( g \in G \) has an inverse, denoted by \( g^{-1} \in G \), such that \( g \circ g^{-1} = \text{Id} \); (iv) the operation \( \circ \) is associative, which means that for any \( g_1, g_2, g_3 \in G \), \((g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)\).

Let \( |G| \) denote the number of elements in \( G \), also called the order of \( G \). A group \( H \) is a subgroup of another group \( G \), denoted by \( H \leq G \), if \( H \subseteq G \) and the operation for \( H \) is the restriction of the operation for \( G \) on \( H \).

For any \( g \in G \) in a group and any \( K \in \mathbb{N} \), we let \( g^K \triangleq g \circ \cdots \circ g \). The symmetric group over \( A = [m] \), denoted by \( S_A \), is the set of all permutations over \( A \). A permutation \( \sigma \) that maps each \( a \in A \) to \( \sigma(a) \) can be represented in two ways.
• Two-line form: \( \sigma \) is represented by a \( 2 \times m \) matrix, where the first row is \((1, 2, \ldots, m)\) and the second row is \((\sigma(1), \sigma(2), \ldots, \sigma(m))\).

• Cycle form: \( \sigma \) is represented by non-overlapping cycles over \( \mathcal{A} \), where each cycle \((a_1, \ldots, a_K)\) represents \( a_{i+1} = \sigma(a_i) \) for all \( i \leq K - 1 \), and with \( a_1 = \sigma(a_K) \).

For example, all permutations in \( S_3 \) are represented in two-line form and cycle form respective in the Table 1.

| Two-line | Cycle |
|----------|-------|
| \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} | (1) or Id |
| \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} | (1,2) |
| \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} | (2,3) |
| \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} | (1,3) |
| \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} | (1,2,3) |
| \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} | (1,3,2) |

Table 1: \( S_3 \) with \( m = 3 \).

A permutation group \( G \) over \( \mathcal{A} \) is a subgroup of \( S_\mathcal{A} \), i.e., \( G \subseteq S_\mathcal{A} \). For any vector \( \vec{d} \) of non-negative integers that sum up to \( m \), we define a special permutation \( \sigma_{\vec{d}} \) and a permutation group \( G_{\vec{d}} \) generated by \( \sigma_{\vec{d}} \) as follows.

**Definition 13.** For any \( m \) and non-negative integer vector \( \vec{d} \) such that \( \vec{d} \cdot \vec{1} = m \), we define

\[
\sigma_{\vec{d}} \triangleq (1, \ldots, d_1)(d_1 + 1, \ldots, d_1 + d_2)\cdots(m - d_1 + 1, \ldots, m) \quad \text{and} \quad G_{\vec{d}} \triangleq \{ (\sigma_{\vec{d}})^K : K \in \mathbb{N} \}
\]

It follows that \( |G| = \text{lcm}(\vec{d}) \), where we recall that \( \text{lcm}(\vec{d}) \) is the least common multiple of \( d_1, \ldots, d_t \).

### B Materials for Section 3

**Proposition 1.** Given \( m, n \), and \( k \leq m \), there exists a \( k \)-committee rule that satisfies anonymity, neutrality, and resolvability if and only if \( \text{Div}^k_{m,n} = \emptyset \).

**Proof.** We first provide a necessary and sufficient condition for the existence of ANR rules under general \( D \) that \( S_\mathcal{A} \) acts on, then apply it to the \( D = \mathcal{A}_k \) case as described in the proposition. Let \( \mathcal{T}_D \) denote the irresolute rule that always chooses \( D \). To simplify notation, let \( \mathcal{P}_D^{\mathcal{A}} \) denote \( \mathcal{T}_D \).

By Lemma 1, the existence of an ANR rule over \( D \) is equivalent to \( \mathcal{P}_D^{\mathcal{A}} = \emptyset \). We prove that \( \mathcal{P}_D^{\mathcal{A}} = \emptyset \) is equivalent to the following condition, by taking the perspective of the stabilizer (which is a permutation group over \( \mathcal{A} \)) of histograms of profiles in \( \mathcal{P}_D^{\mathcal{A}} \).

**Condition 1*. There does not exist a permutation group \( U \subseteq S_\mathcal{A} \) such that (i) \( \text{FIXED}_U(V_{m,n}) \neq \emptyset \) and (ii) \( \text{FIXED}_U(D) = \emptyset \).

Part (i) says that \( V_{m,n} \) contains a fixed point under \( U \), and part (ii) says that \( D \) does not contain a fixed point under \( U \).

**Claim 1.** For any \( D \) that \( S_\mathcal{A} \) acts on, any \( m \), and any \( n \), \( \mathcal{P}_D^{\mathcal{A}} = \emptyset \) if and only if Condition 1* holds.

**Proof.** We prove the contrapositive: \( \mathcal{P}_D^{\mathcal{A}} \neq \emptyset \) if and only if Condition 1* does not hold. We emphasize that the latter (Condition 1* does not hold) is equivalent to that there exists \( U \subseteq S_\mathcal{A} \) that satisfies part (i) and (ii) in Condition 1*.

\( \Rightarrow \): Choose an arbitrary \( P \in \mathcal{P}_D^{\mathcal{A}} \) and let \( U = \text{STAB}_D(\text{Hist}(P)) \). This means that \( \text{Hist}(P) \) is a fixed point of \( U \), which satisfies part (i) of Condition 1*. Due to the definition of \( \mathcal{P}_D^{\mathcal{A}} \), we have \( \mathcal{T}_D(\mathcal{D}) \cap \text{FIXED}_U(\text{Hist}(P))(\mathcal{D}) = \emptyset \), which means that \( \mathcal{T}_D \cap \text{FIXED}_U(\text{Hist}(P))(\mathcal{D}) = \emptyset \), or in other words, \( \text{FIXED}_U(\mathcal{D}) = \emptyset \). Therefore, part (ii) of Condition 1* also holds.

\( \Leftarrow \): Let \( U \subseteq S_\mathcal{A} \) denote an arbitrary permutation group that satisfies (i) and (ii) in Condition 1* and let \( P \) denote a profiles such that \( \text{Hist}(P) \in V_{m,n} \) is a fixed point of \( U \). It follows that \( U \subseteq \text{STAB}(\text{Hist}(P)) \) (or more precisely, \( U \) is a subgroup of \( \text{STAB}(\text{Hist}(P)) \)), i.e., \( U \subseteq \text{STAB}(\text{Hist}(P)) \). Therefore, by part (ii) of Condition 1*, we have

\[
\text{FIXED}_U(\text{Hist}(P))(\mathcal{D}) \subseteq \text{FIXED}_U(\mathcal{D}) = \emptyset
\]
Therefore $P \in \mathcal{P}_{m,n}^D$, which means that $\mathcal{P}_{m,n}^D \neq \emptyset$.

This completes the proof of Claim 1. \hfill \Box

Next, we prove that for any $1 \leq k \leq m$, when $D = A_k$, Condition 1* is equivalent to Condition 1 in the proof sketch in the main text, i.e.:

**Condition 1.** There does not exist a permutation group $U$ over $A$ such that (i) $|U|$ is a divisor of $n$, and (ii) $D$ does not contain a fixed point under $U$.

**Claim 2.** Condition 1* is equivalent to Condition 1.

*Proof.* We prove the contraposition: the set of permutation groups $U^* \leq S_A$ that satisfies (i) and (ii) in Condition 1* (i.e., Condition 1* is not satisfied) is the set of permutation groups $U \leq S_A$ that satisfies (i) and (ii) in Condition 1 (i.e., Condition 1 is not satisfied).

$\Rightarrow$: Let $U^*$ denote an arbitrary permutation group that satisfies (i) and (ii) in Condition 1*. Because $U^*$ also acts on $L(A)$ (see Definition 1), we consider the sizes of orbits in $L(A)$ under $U^*$. It follows that every such orbit has the same size $|U^*|$, because for any $R \in L(A)$ and any $\sigma_1, \sigma_2 \in U^*$, we have $\sigma_1(R) \neq \sigma_2(R)$. Recall that according to part (i) of Condition 1*, there exists $\vec{v} \in V_{m,n}$ that is a fixed point of $U^*$. This means that for any pair of rankings $R_1, R_2$ in the same orbit under $U^*$ (or equivalently, there exists $\sigma \in U^*$ such that $R_1 = \sigma(R_2)$), we have $[\vec{v}]_{R_1} = [\vec{v}]_{R_2}$, i.e., the multiplicities of $R_1$ and $R_2$ in $\vec{v}$ are the same. This means that $|U^*|$ is a divisor of $n$, and proves that part (i) of Condition 1 holds. Part (ii) of Condition 1 is naturally satisfied. Therefore, $U^*$ satisfies (i) and (ii) in Condition 1.

$\Leftarrow$: Let $U$ denote an arbitrary permutation group that satisfies (i) and (ii) in Condition 1. Fix an arbitrary ranking $\vec{v} \in V_{m,n}$ to be the histogram such that for every $\sigma \in U$, $[\vec{v}]_{\sigma(R)} = \frac{n}{|U|}$. It is not hard to verify that $\vec{v}$ is a fixed point of $U$, which proves that part (i) of Condition 1* holds, and part (ii) in Condition 1* naturally holds. \hfill \Box

Finally, we prove that when $D = A_k$, Condition 1 is equivalent to $\text{Div}_{m,n}^k = \emptyset$. For the $\Rightarrow$ direction, suppose for the sake of contradiction that Condition 1 holds but $\text{Div}_{m,n}^k \neq \emptyset$. Choose an arbitrary $\vec{d} \in \text{Div}_{m,n}^k$. We derive a contradiction by considering $G_{\vec{d}}$ defined in Definition 13. Part (i) of Condition 1 clearly holds for $G_{\vec{d}}$ because $|G_{\vec{d}}| = \text{lcm}(\vec{d})$ and $\vec{d} \in \text{Div}_{m,n}^k$. Because no subsequence of $\vec{d}$ sum up to $k$, for any $A \in A_k$, there must exist an alternative $a \in A$ such that $\sigma_A(a) \notin A$, which means that Part (ii) of Condition 1 holds for $U$. Therefore, Condition 1, which requires that no $U$ satisfies both parts, does not hold.

For the $\Leftarrow$ direction, suppose for the sake of contradiction that $\text{Div}_{m,n}^k = \emptyset$ but Condition 1 does not hold, i.e., there exists a permutation group $U$ that satisfies the two parts. Let $\vec{d} = (d_1, \ldots, d_T)$ denote a vector that consists of the sizes of orbits over $A$ under $U$. Due to the orbit-stabilizer theorem, see e.g., [Rosenberger et al., 2010, Theorem 13.1.4], for every $t \leq T$, $d_t$ is a divisor of $|U|$. Recall from part (i) of Condition 1 that $|U|$ is a divisor of $n$. Therefore, $d_t$ is a divisor of $n$, which means that $\vec{d} \in \text{Div}_{m,n}^k$. If $\vec{d}$ contains a subvector that sum up to $k$, then we let $A$ denote the $k$ alternatives in the union of the orbits in $A$ that corresponds to the subvector. It follows that $A$ is a fixed point of $U$, which contradicts part 1 (ii) of Condition 1. Therefore, $\vec{d}$ does not contain a subvector whose components sum up to $k$, which means that $\vec{d} \in \text{Div}_{m,n}^k$. This contradicts the assumption that $\text{Div}_{m,n}^k = \emptyset$. \hfill \Box

**Proposition 2.** Given $m$, $n$, and $k \leq m$, there exists a $k$-list rule that satisfies anonymity, neutrality, and resolvability if and only if for every $m'$ with $m - k + 1 \leq m' \leq m$, $\text{Div}_{m,m'}^1 = \emptyset$.

*Proof.* Recall that $\text{Div}_{m,m'}^1$ is the set of non-negative integer vectors $\vec{d}$ whose components are nontrivial divisors $n$ and $\vec{d} \vec{d}' \vec{d}' = m'$. Like the proof of Proposition 1, it is not hard to verify that when $D = L_k(A)$, Condition 1* is equivalent to Condition 1. Therefore, it suffices to prove that Condition 1 (where $D = L_k(A)$) is equivalent to $\forall, m - k + 1 \leq m' \leq m, \text{Div}_{m,m'}^1 = \emptyset$. For the $\Rightarrow$ direction,
suppose for the sake contradiction that Condition 1 holds but there exists \( m - k + 1 \leq m' \leq m \) such that \( \text{Div}_{n,m'} \neq \emptyset \). Choose an arbitrary \( \vec{d} \in \text{Div}_{n,m'} \). We derive a contradiction by considering \( G_{\vec{d}} \) defined in Definition 13. Part (i) of Condition 1 clearly holds, because \( |G_{\vec{d}}| = \text{lcm}(\vec{d}) \) and \( \vec{d} \in \text{Div}_{n,m'}^* \).

Notice that an alternative \( a \) is a fixed point of \( G_{\vec{d}} \) in \( \mathcal{A} \) if and only if its orbit only consists itself. For any \( R \in \mathcal{L}_k(\mathcal{A}) \), there exists an alternative \( a \) that appears in both \( R \) and is not a fixed point of \( G_{\vec{d}} \) (because the total number of fixed points under \( U \) is strictly less than \( k \)). This means that \( R \) is not a fixed point of \( G_{\vec{d}} \) in \( \mathcal{L}_k(\mathcal{A}) \). Therefore, \( G_{\vec{d}} \) satisfies part (ii) of Condition 1, which means that Condition 1 does not hold, which is a contradiction.

For the \( \Leftarrow \) direction, suppose for the sake of contradiction that for every \( m - k + 1 \leq m' \leq m \), \( \text{Div}_{n,m'} = \emptyset \) but Condition 1 does not hold, i.e., there exists a permutation group \( U \) that satisfies the parts two. Let \( \vec{d} = (d_1, \ldots, d_T) \) denote a vector that consists of the sizes of the partition of \( \mathcal{A} \) into orbits under \( U \). Like the proof of Proposition 1, due to the orbit-stabilizer theorem, for every \( t \leq T \), \( d_t \) is a divisor of \(|U|\), which means that \( \vec{d} \in \text{Div}_{n}^* \). The total number of 1 components in \( \vec{d} \) is no more than \( k \), because otherwise let \( R \) denote a ranking over any subset of \( k \) alternatives that correspond to the 1 components (in other words, they are fixed points under \( U \)), we have that \( R \) is a fixed point of \( U \). This means that the sum of components of \( d \) that are strictly larger than 1 is between \( m - k + 1 \) and \( m \), which contradicts the assumption that for every \( m - k + 1 \leq m' \leq m \), \( \text{Div}_{n,m'} = \emptyset \). This concludes the proof of the \( \Leftarrow \) direction. \( \square \)

C Materials for Section 4

Proposition 3. For any profile \( P \), any \( \sigma_1, \sigma_2 \in \text{MFP}(P) \), and any fixed point \( d \in \text{FIXED\text{-}STAB}(\text{Hist}(P))(D) \), we have \( \sigma_1(d) = \sigma_2(d) \).

Proof. Suppose for the sake of contradiction that \( \sigma_1(d) \neq \sigma_2(d) \). Notice that Hist(\( \sigma_1(P) \)) = Hist(\( \sigma_2(P) \)). Therefore, Hist(\( \sigma_2^{-1} \circ \sigma_1(P) \)) = Hist(\( \sigma_2^{-1} \circ \sigma_2(P) \)) = Hist(\( \sigma_2(P) \)), which means that \( \sigma_2^{-1} \circ \sigma_1 \in \text{STAB}(\text{Hist}(P)) \). Because \( \sigma_2^{-1} \circ \sigma_1(d) \neq d \), \( d \) is not a fixed point of \( \text{STAB}(\text{Hist}(P)) \), which is a contradiction. \( \square \)

Proposition 3 implies that all \( \sigma \in \text{MFP}(P) \) maps any fixed point \( d \in \text{FIXED\text{-}STAB}(\text{Hist}(P))(D) \) to the same decision. Therefore, the choice of \( \sigma \) in Definition 8 does not matter.

Theorem 1. Algorithm 1 runs in polynomial time and computes \( \text{MFP}_f \), which is a fairest tie-breaking mechanism for any anonymous and neutral voting rule.

Proof. We first verify that Algorithm 1 correctly computes an MFP breaking. (1) holds because \( \sigma \in \text{STAB}(\text{Hist}(P)) \) if and only if for every ranking \( R \), \( \text{Hist}(P)|_R = \text{Hist}(P)_{\sigma(R)} \). Let \( R^* = \arg \max_{R \in \text{MPR}(P)} \text{Hist}(\sigma_R(P)) \). To verify that \( \sigma_{R^*} \) is indeed a highest-priority permutation, for the sake of contradiction suppose there exists \( \sigma \in \mathcal{S}_A \) such that \( \text{Hist}(\sigma(P)) \triangleright \text{Hist}(\sigma_{R^*}(P)) \). This means that \( \text{Hist}(\sigma(P))|_{\triangleright} \geq \text{Hist}(\sigma_{R^*}(P))|_{\triangleright} > 0 \), where \( \text{Hist}(\sigma(P))|_{\triangleright} \) is the \( \triangleright \) coordinate of \( \text{Hist}(\sigma(P)) \), or in other words, the multiplicity of ranking \( \triangleright = [1 \triangleright \cdots \triangleright m] \) in \( \sigma(P) \). Therefore, \( \sigma^{-1}(\triangleright) \) must be a most popular ranking in \( P \). This contradicts the maximality of \( R^* \).

Next, we verify that Algorithm 1 runs in polynomial time in \( m, n, \) and \(|D|\). (1) takes \( O(mn^3) \) time. Step 2 takes \( \text{poly}(mn)|D| \) time, because it only need to verify whether every \( d \in D \) is a fixed point of \( \text{STAB}(\text{Hist}(P)) \). In step 4, computing Hist(\( \sigma_R(P) \)) (in the list form) for each \( R \) takes \( O(mn + m \log n) \) time, and each comparison when computing the \( \arg \max_{R \in \text{MPR}(P)} \) takes \( \text{poly}(mn) \) time, which means that the overall time for step 4 is \( \text{poly}(mn) \). Step 5 takes \( \text{poly}(m|D|) \) time.

Finally, we prove that MFP breaking is a fairest tie-breaking mechanism by proving that for every \( P \notin \mathcal{P}^*_m \), \( \text{ANO}(\mathcal{T}_{\text{MFP}}, P) = \text{NEU}(\mathcal{T}_{\text{MFP}}, P) = \text{RES}(\mathcal{T}_{\text{MFP}}, P) = 1 \). Recall that \( \mathcal{T}_{\text{MFP}} \) is the resolute rule obtained from \( \mathcal{T} \) by MFP breaking. Intuitively, this is true because Algorithm 1 does not depend on the identity of agents or the decisions (alternatives). Formally, we have the following proof.
\(\tau_{MFP}\) satisfies anonymity at \(P\). It suffices to prove that for any profile \(P'\) with \(Hist(P') = Hist(P)\), \(MFP(P) = MFP(P')\). This follows after noticing that for any permutation \(\sigma \in S_A\),

\[
Hist(\sigma(P)) = \sigma(Hist(P)) = \sigma(Hist(P')) = Hist(\sigma(P'))
\]

More precisely, any \(\sigma \in S_A\) that maximizes \(Hist(\sigma(P))\) according to \(\succ\) would also maximize \(Hist(\sigma(P'))\).

\(\tau_{MFP}\) satisfies neutrality at \(P\). Suppose for the sake of contradiction that \(\tau_{MFP}(P) = \{a\}\) and there exists a permutation \(\sigma \in S_A\) such that \(\tau_{MFP}(\sigma(P)) = \{b\}\), where \(b \neq a\). Let \(\sigma_a \in MFP(P)\) and \(\sigma_b \in MFP(\sigma(P))\) denote any pair of permutations. We show that \(P\) and \(\sigma(P)\) are “similar” by proving the following two properties.

1. \(Hist(\sigma_a(P)) = Hist(\sigma_b(\sigma(P)))\). Suppose for the sake of contradiction that this does not hold and w.l.o.g. \(Hist(\sigma_a(P)) \succ Hist(\sigma_b(\sigma(P)))\). Then, notice that \(Hist(\sigma_a(P)) = Hist(\sigma_a \circ \sigma^{-1}(\sigma(P)))\), which means that \(\sigma_a \circ \sigma^{-1}\) maps \(Hist(\sigma(P))\) to a profile that is ranked higher than \(\sigma_b(\sigma(\sigma(P)))\). This contradicts the optimality of \(\sigma_b\), i.e., \(\sigma_b \in MFP(P)\).

2. \(\sigma(a) \in FW(\sigma(P), \sigma(\tau(P)))\). Suppose for the sake of contradiction that \(\sigma(a) \notin FW(\sigma(P), \sigma(\tau(P)))\). Because \(\sigma(a) \in \sigma(\tau(P))\), we must have that \(\sigma(a)\) is a fixed point under \(\text{STAB}(Hist(\sigma(P)))\), which means that there exists \(\sigma' \in \text{STAB}(Hist(\sigma(P)))\) such that \(Hist(\sigma'(\sigma(P))) = Hist(\sigma(P))\) and \(\sigma'(\sigma(a)) \neq \sigma(a)\). Let \(\sigma^* = \sigma^{-1} \circ \sigma' \circ \sigma\), we have \(Hist(\sigma^*(P)) = Hist(P)\) and \(\sigma^*(a) \neq a\), which means that \(a\) is not a fixed point under \(\text{STAB}(Hist(P))\), which is a contradiction.

It follows from (i) that \(\sigma_b \circ \sigma(P)\) is the most-preferred histogram (which is the same as \(Hist(\sigma_a(P))\)) among permuted histograms according to \(\succ\) defined in Definition 6. Therefore, \(\sigma_b \circ \sigma \in MFP(P)\). Notice that because \(b \in FW(\sigma(P), \sigma(\tau(P)))\), we have \(\sigma^{-1}(b) \in FW(P, \tau(P))\) according to (ii), where we switch the roles of \(a\) and \(b\). Also recall that \(b \neq \sigma(a)\). Therefore, \(\sigma^{-1}(b) \neq a\). Because of the optimality of \(a\) and Proposition 3, \(a\) has a higher priority than \(\sigma^{-1}(b)\) for any permutation in \(MFP(P)\), especially \(\sigma_b \circ \sigma\). Therefore, \(\sigma_b(\sigma(a)) = \sigma_b \circ \sigma(a) \succ \sigma_b \circ \sigma(\sigma^{-1}(b)) = \sigma_b(b)\), which contradicts the optimality of \(b\), as \(\sigma(a) \in FW(\sigma(P), \sigma(\tau(P)))\) according to (ii).

\(\tau_{MFP}\) is resolute at \(P\). This part follows after the definition of \(\tau_{MFP}\). \(\square\)

D Materials for Section 5

**Theorem 2 (Bounds on semi-random violation of ANR after MFP.)** For any fixed \(m \geq 2\) and \(\Pi\) that satisfies Assumption 1, there exist a constant \(C_1 > 0\) and a constant \(C_2 > 0\), such that for any voting rule \(\tau\) with \(D = A\) that satisfies anonymity and neutrality, any MFP breaking, and any \(n \geq 1\),

\[
\begin{align*}
C_1 \frac{\ell_{\min} = -\infty}{\max_{\min_{\text{ANR}}(\tau_{MFP}, n)}} \leq \text{ANR}_{\Pi, n} & \leq C_2 \frac{\ell_{\min} = -\infty}{\max_{\min_{\text{ANR}}(\tau_{MFP}, n)}} \\
\text{if } d_{\min} = -\infty & \text{otherwise}
\end{align*}
\]

**Proof.** Recall from Lemma 1 that the source of ANR impossibility is \(P_{m,n}\). Due to the optimality of MFP proved in Theorem 1, for any \(\pi \in \Pi^n\), we have

\[
\Pr_{P \sim \pi}(\text{ANR}(\tau_{MFP}, n) = 1) = \Pr_{P \sim \pi}(P \in P_{m,n}^\tau)
\]

Recall that \(P \in P_{m,n}^\tau\) holds if and only if \(FW(P, \tau(P)) = \tau(P) \cap \text{FIXED}_{\text{STAB}(Hist(P))}(A) = \emptyset\). Therefore, if \(\text{FIXED}_{\text{STAB}(Hist(P))}(A) = \emptyset\), then \(P \in P_{m,n}^\tau\); and for every \(P \in P_{m,n}^\tau\), we have \(\text{FIXED}_{\text{STAB}(Hist(P))}(A) \neq A\), which is equivalent to \(\text{STAB}(Hist(P)) \neq \{\text{Id}\}\). Consequently,

\[
\{P \in \mathcal{L}(A)^n : \text{FIXED}_{\text{STAB}(Hist(P))}(A) = \emptyset\} \subseteq P_{m,n}^\tau \subseteq \{P \in \mathcal{L}(A)^n : \text{STAB}(Hist(P)) \neq \{\text{Id}\}\}
\]
Notice that the left hand side is $\mathcal{P}_{n,n}^{A}$, which we recall is the set corresponding to the rule that always chooses $A$, denoted by $\tau_{A}$. The right hand side is equivalent to $\{P \in \mathcal{L}(A)^{n} : |\text{STAB}(\text{Hist}(P))| > 1\}$. Therefore,

$$-	ext{ANR}_{H}^{\max} (\mathcal{P}_{A}, n) \leq -\text{ANR}_{H}^{\max} (\mathcal{P}_{\text{MFP}}, n) \leq \sup_{\pi \in \Pi^{n}} \text{Pr}_{\sim \pi} (|\text{STAB}(\text{Hist}(P))| > 1)$$ (2)

**Upper bound of Theorem 2.** We take the polyhedral approach [Xia, 2020, 2021] to prove the upper bound as follows.

$$\sup_{\pi \in \Pi^{n}} \text{Pr}_{\sim \pi} (|\text{STAB}(\text{Hist}(P))| > 1) = \begin{cases} 0 & \text{if } d_{\min} = -\infty \\ \Theta \left( n \left( \frac{1}{d_{\min}} - 1 \right) \frac{m}{2} \right) & \text{otherwise} \end{cases}$$ (3)

The polyhedral approach has the following two steps. First, we represent the histograms of profiles in $\{P \in \mathcal{L}(A)^{n} : |\text{STAB}(\text{Hist}(P))| > 1\}$ as unions of finitely many polyhedra; then in the second step, we apply [Xia, 2021, Theorem 1] to prove (3).

We define the polyhedron $H^{U}$ to represent histograms of profile that are fixed points of $U$ in $V_{m,n}$.

**Definition 14 ($H^{U}$).** For any permutation group $U \subseteq S_{A}$, let $H^{U} \triangleq \{ \bar{x} \in \mathbb{R}^{m} : \mathbf{A}^{U} \times (\bar{x}) ^ {\top} \leq (b^{U}) ^ {\top} \}$, where $\mathbf{A}^{U}$ is the matrix such that for each $\sigma \in U$ and every $R \in \mathcal{L}(A)$, there is a row $\bar{e}_{R} - \bar{e}_{\sigma(R)}$, where $\bar{e}_{R}$ is the binary vector that takes 1 at the $R$-th component and takes 0 otherwise; $b^{U} \triangleq \bar{0}$.

For any polyhedron $H = \{ \bar{x} : \mathbf{A} \times (\bar{x}) ^ {\top} \leq (b) ^ {\top} \}$, let $H_{\leq 0} = \{ \bar{x} : \mathbf{A} \times (\bar{x}) ^ {\top} \leq (0) ^ {\top} \}$ denote its recess cone, or characteristic cone. It is not hard to verify that $H^{U} = H_{\leq 0}$, because $b^{U} = \bar{0}$. Let $\dim(H)$ denote the dimension of $H$, i.e., the dimension of the smallest affine space that contain $H$, or equivalently, the maximum number of affinely independent points in $H$ minus 1. We say that a polyhedron $H$ is active at $n \in \mathbb{N}$, if it contains a non-negative integer vector $\bar{x}$ with $\bar{x} \cdot \bar{1} = n$.

We will apply [Xia, 2021, Theorem 1], which is included below for completeness. In the theorem, $\tilde{X}_{\bar{x}}$ is the histogram of $n$ independent random variables over $[q]$ distributed as in $\bar{x}$. $H_{\bar{x}}^{\leq 0}$ is the set of non-negative integer vectors in $H$ whose components sum up to $n$. $\text{CH}(H)$ is the convex hull of $H$.

**[Xia, 2021, Theorem 1].** Given any $q \in \mathbb{N}$, any closed and strictly positive $\Pi$ over $[q]$, and any polyhedron $H$ characterized by an integer matrix $\mathbf{A}$, for any $n \in \mathbb{N},$

$$\sup_{\pi \in \Pi^{n}} \text{Pr}_{\sim \pi} \left( \tilde{X}_{\bar{x}} \in H \right) = \begin{cases} 0 & \text{if } H_{\bar{x}}^{\leq 0} = \emptyset \\ \exp ( - \Theta(\pi(n)) ) & \text{if } H_{n}^{\leq 0} \neq \emptyset \text{ and } H_{\leq 0} \cap \text{CH}(\Pi) = \emptyset \\ \Theta \left( n \frac{\dim(H_{\leq 0})}{2} \right) & \text{otherwise (i.e. } H_{n}^{\leq 0} \neq \emptyset \text{ and } H_{\leq 0} \cap \text{CH}(\Pi) \neq \emptyset \text{)} \end{cases}$$

$$\inf_{\pi \in \Pi^{n}} \text{Pr}_{\sim \pi} \left( \tilde{X}_{\bar{x}} \in H \right) = \begin{cases} 0 & \text{if } H_{\bar{x}}^{\leq 0} = \emptyset \\ \exp ( - \Theta(\pi(n)) ) & \text{if } H_{n}^{\leq 0} \neq \emptyset \text{ and } H_{\leq 0} \cap \text{CH}(\Pi) = \emptyset \\ \Theta \left( n \frac{\dim(H_{\leq 0})}{2} \right) & \text{otherwise (i.e. } H_{n}^{\leq 0} \neq \emptyset \text{ and } H_{\leq 0} \cap \text{CH}(\Pi) \subseteq H_{\leq 0} \text{)} \end{cases}$$

We will apply the sup part of [Xia, 2021, Theorem 1], where $q = m!$, $\tilde{X}_{\bar{x}} = \text{Hist}(P)$, and $P \sim \pi$. The following claim characterizes some properties of $H^{U}$ that will be used later in the proof.

**Claim 3 (Properties of $H^{U}$).** For any permutation group $U \subseteq S_{A}$,

(i) for any $\bar{x} \in \mathbb{R}^{m}$, $\bar{x} \in H^{U}$ if and only if $\bar{x}$ is a fixed point of $U$;

(ii) $\dim(H^{U}) = \frac{m!}{|\Pi|}$;

(iii) for any $n \in \mathbb{N}$, $H^{U}$ is active at $n$ if and only if $|U|$ divides $n$.

**Proof.** (i) follows after the definition of $H^{U}$. 

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To prove (ii), we note that \( b^U = \emptyset \) and for any pair of rankings \( R_1, R_2 \), if \( R_2 = \sigma(R_1) \) for some \( \sigma \in U \), then \( R_1 = \sigma^{-1}(R_2) \) and \( \sigma^{-1} \in U \), which means that the implicit equalities of \( A^U \) are themselves.

**Definition 15.** Let \( \equiv_U \) denote the equivalence relationship over \( L(A) \) induced by orbits under \( U \). \( \equiv_U \) partitions \( L(A) \) into \( \frac{|U|}{|U|} \) orbits \( O_1, \ldots, O_{\frac{|U|}{|U|}} \), each of which has \( |U| \) linear orders.

In other words, for any pair of linear orders \( R_1, R_2, R_1 \equiv_U R_2 \) if and only if there exists \( \sigma \in U \) such that \( R_1 = \sigma(R_2) \). It is not hard to verify that \( \equiv \) is an equivalence relation over \( L(A) \). Therefore, \( \equiv \) partitions \( L(A) \) into \( \frac{|U|}{|U|} \) orbits, each of which has \( |U| \) linear orders.

For (iii), notice that each orbit \( O_t \) has the same number of rankings, i.e., \( |U| \). Therefore, \( |U| \) divides \( n \). Then, let \( P \) denote the profile where there are \( \frac{|U|}{|U|} \) copies of each ranking in \( O_t \). It is not hard to verify that \( \text{Hist}(P) \in \mathcal{H}^U \). This proves (iii) and completes the proof of Claim 3.

Notice that for any fixed \( m \), the total number of permutation groups over \( A \) can be viewed as a constant. Therefore, according to Claim 3 (i), it suffices to prove that for every permutation group \( U \leq S_A \) with \( U \neq \{\text{Id}\} \) (or equivalently, \( |U| > 1 \)).

\[
\sup_{\pi \in \Pi^n} \mathbb{P}_{\rho \sim \pi}(\text{Hist}(P) \in \mathcal{H}^U) = \begin{cases} 0 & \text{if } d_{\text{min}} = -\infty \\ \Omega \left(n! \left(\frac{1}{m_{\text{min}} - 1}\right)^{\frac{m_{\text{min}}}{m_{\text{min}}}}\right) & \text{otherwise} \end{cases}
\]  

(4)

For the \( d_{\text{min}} = -\infty \) case of (4), we prove that \( \mathcal{H}^U \) is inactive. Suppose for the sake of contradiction that \( \mathcal{H}^U \) is active for some \( U \leq S_A \) with \( |U| > 1 \). Let \( P^* \) denote an \( n \)-profile such that \( \text{Hist}(P^*) \in \mathcal{H}^U \). By Claim 3 (iii), \( |U| \) is a divisor of \( n \). Because \( U \neq \{\text{Id}\} \), there exists an orbit \( O_t \) in \( A \) under \( U \), as defined in Definition 15, such that \( |O_t| \geq 2 \). Therefore, according to the orbit-stabilizer theorem, \( |O_t| \) (which is no more than \( m \)) is a non-trivial divisor of \( |U| \), which is a divisor \( n \). This contradicts the assumption that \( d_{\text{min}} = -\infty \).

For the \( d_{\text{min}} \neq -\infty \) case of (4), notice that if \( \mathcal{H}^U \) is active, then \( |U| > 1 \) and \( |U| \) is a divisor of \( n \) (Claim 3 (iii)). This means that \( |U| \geq d_{\text{min}} \). The upper bound follows after applying [Xia, 2021, Theorem 1] to \( \mathcal{H}^U \) by noticing that \( \pi_{\text{uni}} \in \text{CH}(\Pi) \) (according to Assumption 1), \( \pi_{\text{uni}} \in \mathcal{H}^U \), and \( \text{dim} \left( \mathcal{H}^U_{\geq 0} \right) = \frac{m_{\text{min}}!}{m_{\text{min}}} \leq \frac{m_{\text{min}}!}{|U| \cdot m_{\text{min}}} \).

**Lower bound of Theorem 2.** The \( \ell_{\text{min}} = -\infty \) case trivially holds. When \( \ell_{\text{min}} \neq -\infty \), let \( \vec{d}^* \in \text{Div}_{m,n}^1 \) denote a vector such that \( \text{lcm}(\vec{d}^*) = \ell_{\text{min}} \). Recall from Definition 13 that \( G_{\vec{d}^*} \) is the permutation group generated from \( \sigma_{\vec{d}^*} \). It follows that \( \text{FIXED}_{G_{\vec{d}^*}}(A) = \emptyset \). Therefore,

\[
\{P \in L(A)^n : \text{STAB}(\text{Hist}(P)) = G_{\vec{d}^*} \} \subseteq \{P \in L(A)^n : \text{FIXED}_{\text{STAB}(\text{Hist}(P))}(A) = \emptyset \}
\]

Because of Claim 3 (i), it suffices to prove

\[
\sup_{\pi \in \Pi^n} \mathbb{P}_{\rho \sim \pi}(\text{Hist}(P) \in \mathcal{H}^{G_{\vec{d}^*}}) = \Omega \left(n! \left(\frac{1}{m_{\text{min}} - 1}\right)^{\frac{m_{\text{min}}}{m_{\text{min}}}}\right) \]

(5)

Because \( \vec{d}^* \in \text{Div}_{m,n}^1 \), we have \( \text{lcm}(\vec{d}^*) \mid n \). Therefore, by Claim 3 (iii), \( \mathcal{H}^{G_{\vec{d}^*}} \) is active at \( n \). Then, (5) follows after applying [Xia, 2021, Theorem 1] to \( \mathcal{H}^{G_{\vec{d}^*}} \) and noticing that \( \text{dim} \left( \mathcal{H}^{G_{\vec{d}^*}}_{\geq 0} \right) = \frac{m_{\text{min}}!}{m_{\text{min}}} \) (by Claim 3 (ii)).

**Proposition 4.** For any fixed \( m \), any positional scoring rules with \( s_1 > s_2 > \cdots > s_m \), STV, maximin, Copeland, ranked pairs, and Schulze satisfy canceling-out and \( \delta \)-unanimity for some \( 0 < \delta < 1 \).

**Proof.** The proof for cancelling-out follows straightforwardly after the definitions of the rules. For \( \delta \)-unanimity, let \( A \subseteq A \) denote a set of alternatives and let \( P \) denote an \( n \)-profile where alternatives in \( A \) are ranked at the top positions in at least \( (1 - \delta)n \) votes.
Proof for positional scoring rules with \( s_1 > \cdots > s_m \). Let \( \eta \) denote the smallest difference between adjacent scores, i.e., \( \eta = \min_{1 \leq i \leq m-1} (s_i - s_i) \). For any \( \delta < \frac{1}{m(m-1)} \), we have \( \eta(1-\delta) > \delta(s_1 - s_m) \), which means that for any \( a \in A \) and any \( b \in A \setminus A \), the score difference between \( a \) and \( b \) is at least \( \eta(1-\delta)n - \delta(s_1 - s_m)n > 0 \), which implies that the rule satisfies \( \delta \)-unanimity.

Proof for STV. For any \( \delta < \frac{1}{m+1} \), in every round of STV, the plurality score of any alternative \( b \in A \setminus A \) is no more than \( \frac{n}{m+1} \), which is strictly smaller than the average plurality score. Therefore, no alternative in \( A \setminus A \) can be a co-winner.

Proof for maximin. For any \( \delta < \frac{1}{m} \), the min score of any alternative \( b \in A \setminus A \) is no more than \( -\frac{n(m-2)}{m} \). Next, we prove that there exists \( a \in A \) whose min score is strictly more than \( -\frac{n(m-2)}{m} \). Suppose for the contradiction this is not true, i.e., the min score of any \( a \in A \) is no more than \( -\frac{n(m-2)}{m} \). Then, there exists a subset of \( A \), w.l.o.g. \( \{1, \ldots, t\} \), such that \( w_P(2,1) \leq -\frac{n(m-2)}{m} \), \( w_P(3,2) \leq -\frac{n(m-2)}{m} \), \ldots, \( w_P(t,t-1) \leq -\frac{n(m-2)}{m} \), and \( w_P(1,t) \leq -\frac{n(m-2)}{m} \). For any profile \( P^* \), we define

\[
W_{p^*} \triangleq w_{P^*}(2,1) + w_{P^*}(3,2) + \cdots + w_{P^*}(t,t-1) + w_{P^*}(1,t)
\]

Then, we have \( W_p < -n(m-2) \). However, for every ranking \( R \), we have \( W(R) \geq -(t-2) \geq -(m-2) \), which means that \( W_p = \sum_{R \in P} W(R) \geq -n(m-2) \), a contradiction.

Proof for Copeland. For any \( \delta < \frac{1}{m} \), the weight on the edge in WMG from any alternative \( a \in A \) to any alternative \( b \in A \setminus A \) is positive. Therefore, the Copeland score of \( a \) is at least \( m - |A| \) and the Copeland score of \( b \) is no more than \( m - |A| - 1 \), which means that the winners must be a subset of \( A \).

Proof for ranked pairs. The proof is similar to the proof for Copeland. For any \( \delta < \frac{1}{3} \), the edge \( a \rightarrow b \) from any \( a \in A \) to any \( b \in A \setminus A \) has strictly positive weight in WMG. These edges will be fixed during the execution of ranked pairs, because if this is not true, then there exists \( c \in A \setminus A \) such that \( c \rightarrow a \) is fixed earlier, which means that it has positive weight, a contradiction.

Proof for Schulze. The proof is similar to the proof for Copeland and ranked pairs. For any \( \delta < \frac{1}{3} \), the edge \( a \rightarrow b \) from any \( a \in A \) to any \( b \in A \setminus A \) has strictly positive weight in WMG. Therefore, \( s[a,b] > 0 \). We also have \( s[b,a] < 0 \), because any path from \( b \) to \( a \) must contain an edge from an alternative in \( A \setminus A \) to an alternative in \( A \). Therefore, \( s[a,b] > s[b,a] \), which means that \( b \) is not a co-winner.

\[ \square \]

**Theorem 3 (Tightness of bounds.)** For any fixed \( m \geq 2 \), any \( \Pi \) that satisfies Assumption 1, any MFP, and any \( n \geq 1 \),

Tightness of lower bound: \( \underline{\text{ANR}}_{\Pi}^{\max} (\text{MFP}_{\tau_A}, n) = \begin{cases} 0 & \text{if } \ell_{\min} = -\infty \\ \Theta \left( n \left( \frac{1}{\ell_{\min} - 1} \right)^{\frac{m}{2}} \right) & \text{otherwise} \end{cases} \)

For any \( \tau \) over \( D = A \) that satisfies anonymity, neutrality, canceling-out, and \( \delta \)-unanimity for some fixed \( \delta > 0 \),

Tightness of upper bound: \( \overline{\text{ANR}}_{\Pi}^{\max} (\text{MFP}, n) = \begin{cases} 0 & \text{if } \ell_{\min} = -\infty \\ \Theta \left( n \left( \frac{1}{\ell_{\min} - 1} \right)^{\frac{m}{2}} \right) & \text{otherwise} \end{cases} \)

**Proof.** We first prove the **tightness of the lower bound.** Recall that \( \mathcal{P}_{m,n}^A \) is the set of profiles \( P \) such that \( \text{STAB}(P) \) has no fixed point in \( A \). According to the optimality of MFP (Theorem 1), it suffices to prove

\[
\sup_{\pi \in \Pi} \Pr_{P \sim \pi} (P \in \mathcal{P}_{m,n}^A) = \begin{cases} 0 & \text{if } \ell_{\min} = -\infty \\ \Theta \left( n \left( \frac{1}{\ell_{\min} - 1} \right)^{\frac{m}{2}} \right) & \text{otherwise} \end{cases}
\]
The proof is done by applying [Xia, 2021, Theorem 1] to the representation of $\mathcal{P}_{m,n}^A$ as the union of multiple polyhedra $\mathcal{H}^U$ as in [Xia, 2020]. More precisely, let $U_m$ denote the set of all permutation groups $U$, such that $U$ has no fixed point in $A$. We define the following union of polyhedra:

$$C_A \triangleq \bigcup_{U \in U_m} \mathcal{H}^U$$

It is not hard to verify that $P \in \mathcal{P}_{m,n}^A$ if and only if $\text{Hist}(P) \in C_A$.

**Claim 4.** For any $U \in U_m$ and $n \geq 2$, if $\mathcal{H}^U$ is active at $n$, then $\ell_{\min} \neq -\infty$ and $|U| \geq \ell_{\min}$.

**Proof.** We define $\vec{d}_U$ to be a vector that consists of the sizes of orbits in $A$ under $U$. For any $U \in U_m$, because each alternative is mapped to a different alternative under some permutation in $U$, each component of $\vec{d}_U$ is at least 2 and $\vec{d}_U \cdot \mathbf{1} = m$. Because $\mathcal{H}^U$ is active, we have that $|U|$ is a divisor of $n$ according to Claim 3 (iii). Therefore, $\vec{d}_U$ represents a way to write $m$ as the sum of $n$’s non-trivial divisors, which means that $\ell_{\min} \neq -\infty$. This also means that lcm($\vec{d}_U$) $\geq \ell_{\min}$.

Recall that lcm($\vec{d}_U$) $\leq |U|$. Therefore, for any $U \in U_m$ such that $\mathcal{H}^U$ is active at $n$, we have $|U| \geq \ell_{\min}$. $\square$

The $\ell_{\min} = -\infty$ case of (6) follows after Claim 4, which implies that when $\ell_{\min} = -\infty$, for any $U \in U_m$, $\mathcal{H}^U$ is not active at $n$.

To prove the $\ell_{\min} \neq -\infty$ case of (6), because $|U_m|$ can be viewed as a constant that does not depend on $n$, it suffices to prove that when $\ell_{\min} \neq -\infty$, for every $U \in U_m$, the following equation holds:

$$\sup_{\pi \in \Pi} \Pr_{P \sim \pi}(\text{Hist}(P) \in \mathcal{H}^U) = O \left( n^{\frac{1}{\ell_{\min}} - 1} \right)$$

This follows after the application of [Xia, 2021, Theorem 1] to $\mathcal{H}^U$ and noticing that $\frac{1}{|U|} \leq \frac{1}{\ell_{\min}}$ according to Claim 4.

Next, we prove the tightness of the upper bound. We will define a polyhedron $\mathcal{H}^*$ such that for every profile $P$ such that $\text{Hist}(P)$ is in $\mathcal{H}^*$, we have $\pi(P) = [d_{\min}] = \{1, \ldots, d_{\min}\}$ and none of $[d_{\min}]$ is a fixed point of $\text{STAB}(\text{Hist}(P))$, which means that $P \in \mathcal{P}_{m,n}^F$. Then, we apply [Xia, 2021, Theorem 1] to prove that when $d_{\min} \neq -\infty$,

$$\sup_{\pi \in \Pi} \Pr_{P \sim \pi}(\text{Hist}(P) \in \mathcal{H}^*) = \Omega \left( n^{\frac{1}{\ell_{\min}} - 1} \right),$$

which lower bounds $\max_{\pi \in \Pi} \text{ANR}_{\pi,\text{MFP},n}$.

For any $R \in \mathcal{L}(A)$, define Surplus$_R \in \mathbb{R}^{m!}$ as follows

$$\forall R' \in \mathcal{L}(A), [\text{Surplus}_R]_{R'} \triangleq \begin{cases} 1 - \frac{m!}{d_{\min}}, & \text{if } R' = R \\ \frac{m!}{d_{\min}}, & \text{otherwise} \end{cases}$$

It follows that for any $n$-profile $P$, $\text{Surplus}_R \cdot \text{Hist}(P) = [\text{Hist}(P)]_R - \frac{m!}{d_{\min}}$, or in other words, Surplus$_R \cdot \text{Hist}(P)$ is surplus of number of $R$ votes in $P$ relative to the average votes of $\frac{m!}{d_{\min}}$. Notice that Surplus$_R \cdot \text{Hist}(P)$ may not be an integer, and it may be negative.

Let $(d_{\min})$ denote the vector that consists of a single value $d_{\min}$. Recall from Definition 13 that $G(d_{\min})$ is the permutation group generated by $\sigma(d_{\min})$, the cyclic permutation $1 \rightarrow 2 \rightarrow \cdots \rightarrow d_{\min} \rightarrow 1$. Define $L_1 \subseteq L(A)$ to be the set of rankings where $[d_{\min}]$ are ranked at the top positions; let $R^* \in L_2$ denote an arbitrary but fixed ranking not in $L_1$; and define $L_2$ to rankings not in $L_1$ that cannot be obtained from $R^*$ by applying any permutation in $G(d_{\min})$. It follows that

$$\mathcal{L}(A) = L_1 \cup L_2 \cup \{\sigma(R^*) : \sigma \in G(d_{\min})\}$$

We are now ready to define $\mathcal{H}^*$ using the notation introduced above. Let $\mathcal{H}^*$ be polyhedron obtained from $\mathcal{H}^{U_{(d_{\min})}}$ by adding the following linear constraints.

$$\forall R \in L_1 \cup L_2, (\text{Surplus}_{R^*} - \text{Surplus}_R) \cdot \vec{x} \leq 0 \quad (8)$$

$$\left[1 - \frac{\delta}{2d_{\min}} \times \frac{m!}{d_{\min}} \times \sum_{R \in L_2} (\text{Surplus}_R - \text{Surplus}_{R^*}) - \sum_{R \in L_1} (\text{Surplus}_R - \text{Surplus}_{R^*}) \right] \cdot \vec{x} \leq 0 \quad (9)$$

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(8) is equivalent to requiring that $\vec{x}_{R^*}$ is a smallest component in $\vec{x}$, which implies that $\text{Surplus}_{R^*} \cdot \vec{x} \leq 0$. (9) implies that, after subtracting $\text{Surplus}_{R^*} \cdot \vec{x}$ from all components of $\vec{x}$, the total weights of rankings in $L_1$ at least $(1 - \delta)$ multiplied by the total weights of all rankings.

Therefore, for every $n$-profile $P^*$, if $\text{Hist}(P^*) \in \mathcal{H}^*$, then $[d_{\min}] \cap \text{FIXED}_{\mathcal{H}^*}(\text{Hist}(P^*))(1) = \emptyset$, because of constraints in $\mathcal{H}^{G(d_{\min})}$; and $\mathcal{T}(P^*) = [d_{\min}]$, because after $[\text{Hist}(P^*)]_{R^*}$ copies of $L(A)$ are removed, the winners (which are still $\mathcal{T}(P^*)$ due to canceling-out) are a subset of $[d_{\min}]$ (due to $\delta$-unanimity). This means that $P^* \in \mathcal{P}_{m,n}$. Consequently, 

$$\overline{\text{ANR}}_{\mathcal{H}^*}(\mathcal{T}_{\mathcal{MFP}}, n) \geq \sup_{\vec{x} \in \mathcal{P}_n} \Pr_{\vec{x} \sim \vec{x}}(\text{Hist}(P) \in \mathcal{H}^*)$$

(7) follows after the application of the sup part of [Xia, 2021, Theorem 1] to $\mathcal{H}^*$, based on the following observations:

- $\mathcal{H}^*$ is active at $n$. To see this, we construct a profile $P^*$ that consists of $n$ copies of $\{\sigma_{[d_{\min}]}^t(1 \succ \cdots \succ m) : 1 \leq t \leq d_{\min}\}$. It is not hard to verify that Hist$(P) \in \mathcal{H}^*$.

- CH$(\Pi) \cap \mathcal{H}_{\leq 0}^* \neq \emptyset$. This follows after observing that $\pi_{\text{uni}} \in \text{CH}(\Pi) \cap \mathcal{H}_{\leq 0}^* \neq \emptyset$ (notice that $\text{Surplus}_{R^*} \cdot \pi_{\text{uni}} = \text{Surplus}_{R^*} \cdot \vec{1} = 0$).

- $\dim(\mathcal{H}_{\leq 0}^*) = \frac{m!}{d_{\min}}$. Recall that Claim 3 (ii) implies that $\dim(\mathcal{H}_{\leq 0}^{G(d_{\min})}) = \frac{m!}{G(d_{\min})!} = \frac{m!}{d_{\min}}$. Let $P^*$ be the profile defined above to prove that $\mathcal{H}^*$ is active at $n$. Then, Hist$(P^*) \in \mathcal{H}_{\leq 0}^*$ and (8) and (9) are strict. Therefore, $\dim(\mathcal{H}_{\leq 0}^*) = \dim(\mathcal{H}_{\leq 0}^{G(d_{\min})}) = \frac{m!}{d_{\min}}$.

This completes the proof of the tightness of the upper bound and therefore concludes the proof of Theorem 3. \qed