Realisations of $W_3$ Symmetry

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ABSTRACT

We perform a systematic investigation of free-scalar realisations of the Zamolodchikov $W_3$ algebra in which the operator product of two spin-three generators contains a non-zero operator of spin four which has vanishing norm. This generalises earlier work where such an operator was required to be absent. By allowing this spin-four null operator we obtain several realisations of the $W_3$ algebra both in terms of two scalars as well as in terms of an arbitrary number $n$ of free scalars. Our analysis is complete for the case of two-scalar realisations.
1 Introduction

In recent years, there has been a lot of activity in the study of extended conformal symmetries, better known under the name “W-symmetries”. These symmetries constitute extensions of the Virasoro algebra which are generically denoted by “W-algebras”. W-symmetries can be used to clarify the structure of conformal field theory. They also occur as a “natural” symmetry in a variety of physical models. Another approach is to use W-symmetries for the construction of higher-spin extensions of two-dimensional gravity (“W-gravity”) or new string models (“W-strings”).

In view of the above-mentioned applications, it is important to have a good understanding of all possible realisations of W-symmetries. The simplest example of a W-algebra is the $W_3$-algebra of [1] which contains, in addition to the spin-two Virasoro generator $T$, a spin-three generator $W$.

Using the language of Operator Product Expansions (OPE), the algebra is given by

$$\begin{align*}
    T(z)T(w) &= \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular part}, \\
    T(z)W(w) &= \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + \text{regular part}, \\
    W(z)W(w) &= \frac{c}{3(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
    &\quad + \frac{3}{10} \frac{\partial^2 T(w)}{(z-w)^2} + \frac{1}{15} \frac{\partial^3 T(w)}{z-w} \\
    &\quad + \frac{16}{22 + 5c} \left( \frac{2\Lambda(w)}{(z-w)^2} + \frac{\partial \Lambda(w)}{z-w} \right) + \text{regular part},
\end{align*}$$

with $\Lambda = (TT) - \frac{3}{10} \partial^2 T$. The round brackets in $(TT)$ indicate a natural normal ordering in terms of the Laurent modes of the generators (see, e.g. [2]). The first equation in (1) gives the Virasoro algebra, while the second equation expresses the fact that $W$ is a primary field of spin three. The last equation tells us that the OPE of two spin-three generators gives rise to the conformal family of the unit operator. The particular coefficients arising in this equation can all be fixed by the requirement of conformal invariance.
Note that in (1) we have used a particular normalisation of the $W$-generator, i.e. $<WW> = c/3$, in agreement with the common convention.

In order to construct $W$-algebras and to obtain realisations of them one can follow different strategies. One approach is to develop a specific construction procedure, like e.g. the Miura transformation of [3] or the coset construction of [1]. Another approach is to start from an Ansatz for the OPE’s of a given set of abstract generators and to require closure of the algebra (see e.g. [3]). Alternatively, one could start from an Ansatz for the generators of the $W$-algebra in terms of scalar fields and then impose closure. This has been the approach of [3, 7, 8], where a systematic search for free field realisations of the $W_3$ algebra was undertaken. In particular, starting from certain Ansätze for the generators [4, 8], the following $n$-scalar realisation was found [8]:

$$T = \frac{1}{2}(A_0A_0) + \sqrt{2}a_0A'_0 + T_\mu,$$

where $A_0$ is the derivative of a free scalar field, i.e. $A_0 \equiv \partial \phi_0$. The other $n-1$ scalars are represented by $T_\mu$ which commutes with $A_0$ and satisfies a Virasoro algebra with central charge given by $c_\mu = \frac{1}{2}c + \frac{1}{2}$. The parameter $a_0$ is the background charge and is related to the central charge parameter $c$ via $c = 2(1 - 16a_0^2)$. The resulting realisation coincides for $n = 2$ with the Fateev-Zamolodchikov (FZ) two-scalar realisation [3]. It can be viewed as a natural generalisation of the FZ realisation to an arbitrary number $n$ of scalar fields. Note that in the definition of the nonlinear term $(TT)$ in the $W_3$ algebra we use a normal ordering in terms of the Laurent modes of the generators. A normal ordering of this term with respect to the modes of the free scalar fields was considered in [1].

The aim of this letter is to generalise the analysis of [3, 4, 8] by allowing spin-four null operators in the operator product of two spin-three generators. To be more precise, instead of the third equation in (1) we require that the following OPE holds:

$$W(z)W(w) = \text{as in (1)} + \frac{V(w)}{(z-w)^2} + \frac{1}{2}\partial V(w) (z-w),$$

where

$$T = \frac{1}{2}(A_0A_0) + \sqrt{2}a_0A'_0 + T_\mu,$$

$$W = -\frac{1}{3}(A_0A_0A_0) - \sqrt{2}a_0(A_0A'_0) - \frac{2}{3}a_0^2A''_0 + 2(A_0T_\mu) + \sqrt{2}a_0T'_\mu,$$
where $V$ is a spin-four null operator, i.e. $<VV>=0$. Of course, strictly speaking, the algebra corresponding to (3) is not the same as the $W_3$ algebra given in (1). However, since $V$ is a null operator, it can only generate other null fields in its OPE. The full set of null operators constitutes an ideal of the algebra. It is therefore consistent to set all these null operators equal to zero and one thus obtains a representation of the $W_3$ algebra.

Realisations of $W$-symmetries modulo null fields have been considered before in the literature. For instance, they occur in the coset construction of [4] and also, in a supersymmetric context, in [10]. More recently, in [11, 12], such realisations were obtained, for specific values of the central charge, from a certain nonlinear $W_\infty$ algebra [12] based upon the coset $SL(2,R)/U(1)$. This algebra is related to the parafermion current algebra of [13]. From a somewhat different point of view, extensions of the $W_3$ algebra with null generators have occurred recently in a study of certain singular contractions of $W$-algebras [14].

In our analysis of the $W_3$ algebra, we have restricted ourselves in the following two ways. First of all, we only consider spin-four null operators. In principle, one could also allow for spin-two null operators in the OPE of two spin-three generators. However, since in most formulations of $W$-algebras every spin occurs only once, it is less natural to allow for spin-two operators in addition to the Virasoro generator. Secondly, we only consider free field realisations. We will not consider the inclusion of vertex operators in the Ansatz as was done in [15].

2 Ansätze

Our starting point is the following free field Ansatz for the spin-two and spin-three generators of the $W_3$-algebra [4, 8]:

$$T = \frac{1}{2}g_{ij}(A^iA^j) + \sqrt{2}a_iA^{i'},$$

$$W = \frac{1}{3}d_{ijk}(A^iA^jA^k) + 2\sqrt{2}e_{ij}(A^iA^{i'}) + 2f_iA^{i''},$$

where $A^i \equiv \partial\phi^i$ and the $\phi^i$ ($i = 0, \ldots, n-1$) represent a set of $n$ free scalar fields and $g_{ij}, a_i, d_{ijk}, e_{ij}$ and $f_i$ are yet undetermined coefficients. The $A^i$
satisfy the OPE

\[ A^i(z)A^j(w) = \frac{g^{ij}}{(z-w)^2} + \text{regular part}, \quad (6) \]

where \( g^{ij} \) is the inverse of \( g_{ij} \). Our conventions are slightly different from those of [8]. Note that with the above Ansatz the spin-two generator \( T(z) \) already satisfies the Virasoro algebra with central charge \( c = n - 24a_i a^i \).

Following [8], we split the \( n \)-component index \( i \) into “0” and an \((n-1)\)-component index \( \mu \) and take the coefficients \( d_{ijk} \) to be

\[ d_{000} = s , \; d_{0\mu\nu} = -sg_{\mu\nu} , \quad (7) \]

where the parameter \( s \) is fixed by the choice of normalisation of the \( W \) generator. The expression for the \( d \)-coefficients is a solution to

\[ d_{(ij}^m d_{kl)m} = s^2 g_{(ij} g_{kl)} . \quad (8) \]

The latter equation guarantees the closure of the classical version \( w_3 \) of the \( W_3 \) algebra [7]. In the analysis of [8], equations for the unknown coefficients in (4) and (5) were found by demanding that the generators satisfy the \( W_3 \) algebra given in (1), i.e. without spin-four null operators. It was subsequently shown that these equations were solved by the \( n \)-scalar realisation given in (2).

We now consider the same Ansatz (4, 5), but instead require that the generators satisfy the \( W_3 \) algebra modulo a spin-four null operator as indicated in (3). This allows us to take the following less restrictive Ansatz for the coefficients \( d_{ijk} \):

\[ d_{000} = s , \; d_{0\mu\nu} = tg_{\mu\nu} , \quad (9) \]

with \( s \) and \( t \) free parameters (although one of them may be fixed by choosing a normalisation for \( W \)).

The following three equations have to be satisfied in order that \( W \) is primary w.r.t. \( T \):

\[ d^j_{ji} - 24e_{ij}a^i + 12f_i = 0 , \quad (10) \]
\[ 2e_{(ij)} - d_{ijk}a^k = 0 , \quad (11) \]
\[ 3f_i - 2a^j e_{ji} = 0 . \quad (12) \]
For more details, see [8]. On the fourth order pole of the OPE of \( W \) with itself a primary spin-two operator shows up besides a multiple of the energy momentum tensor. We require that this operator vanishes because we want \( T \) to be the only spin-two operator in the algebra. This leads to the following equation

\[
d_i^{kl}d_{jkl} + 12d_{ijk}f^k - 24e_i^k e_{jk} = \frac{3}{2c}N_3g_{ij}.
\] (13)

In (13) \( N_3 \) is the norm of the operator \( W \), which we prefer not to fix for the moment:

\[
N_3 \equiv < WW > = \frac{2}{3}
\left[ d_{ijk}d^{ijk} - 72e_{ij}e^{ij} - 48e_{ij}e^{ij} + 720f_if^i \right].
\] (14)

In [8] two more equations were used, which guaranteed the vanishing of a primary spin-four operator \( V \) in the OPE of \( W(z)W(w) \). Instead, we will allow such a spin-four operator, but only if it is null. This requirement leads to one more, rather complicated, equation which we have given in Appendix A. We will refer to this equation as the spin-four equation.

We conclude that the full set of equations that has to be satisfied by the Ansatz (4), (5) and (9) is given by equations (10-13) and the spin-four equation which can be found in Appendix A. The general analysis of these equations is rather complicated. Among the solutions one should of course find, as a special case, those of [8] which are characterized by taking \( s = -t \) in (4) and \( V \equiv 0 \), i.e. no spin-four null operator. We will now discuss the new solutions we obtained.

### 3 Solutions

Our strategy is to first solve equations (10-13) and afterwards impose the the spin-four equation. It is convenient to distinguish between the two cases corresponding to \( a_0 \neq 0 \) and \( a_0 = 0 \). From now on we will take \( t = 1 \) as a choice of normalisation. Note that in general this differs from the standard normalisation  \( < WW > = c/3 \). For \( a_0 \neq 0 \) (case I) we find:

\[
\begin{align*}
e_{00} &= \frac{1}{2}sa_0, & e_{\mu 0} &= 0, & e_{0\mu} &= a_\mu, \\
e_{(\mu\nu)} &= \frac{1}{2}a_0g_{\mu\nu}, & e_{[\mu\nu]} &= 0, \\
f_0 &= \frac{1}{3}sa_0^2, & f_\mu &= a_0a_\mu.
\end{align*}
\] (15)
Besides the Romans solution, corresponding to \( s = -1 \), these equations have the following other solutions as well:

\[
\begin{align*}
I \\
2a_0^2 &= \frac{s - 2}{2(s - 3)}, \quad a_\mu a^\mu = \frac{-3s^2 + 4s + 3 + n(s - 3)}{24(s - 3)}, \\
c &= 3s - 7,
\end{align*}
\]  

(16) 

where the parameter \( s \) is still undetermined. We now substitute these solutions into the spin-four equation. It turns out that this equation is satisfied only for the values \( s = \frac{7}{3}, \frac{5}{3}, -1, \frac{5}{2} \) and \( \frac{13}{5} \). For \( s = \frac{7}{3} \) and \( s = \frac{5}{2} \), corresponding to \( c = 0 \) and \( c = \frac{1}{2} \), respectively, \( W \) turns out to be a null field as well, and we will not consider these cases further. For \( s = -1 \) we get the Romans solution for \( c = -10 \). The two new solutions we find are given by \( s = \frac{5}{3} \) (\( c = -2 \)) and \( s = \frac{13}{5} \) (\( c = \frac{4}{5} \)). In appendix B the explicit form of these realisations is given for \( n = 2 \).

We note that the \( c = \frac{4}{5} \) realisation has an imaginary background charge \( a_0 \). In order to obtain real coefficients in the realisation it is necessary to perform the redefinitions \( A_0 \to iA_0 \) and \( W \to iW \). The result is a “non-compact” realisation where the quadratic \( A_0 \) part in \( T \) has a minus sign.

A general feature of the case I solutions is that the \( W_3 \) generators take on the form

\[
\begin{align*}
T &= \frac{1}{2}(A_0 A_0) + \sqrt{2}a_0 A_0' + T_\mu, \\
W &= \frac{1}{3}s(A_0 A_0 A_0) + \sqrt{2}s a_0 (A_0 A_0') + \frac{2}{3}a_0^2 A_0'' + 2(A_0 T_\mu) + \sqrt{2}a_0 T_\mu',
\end{align*}
\]  

(18) 

(19)

where \( T_\mu \) is the energy momentum tensor corresponding to the \( n - 1 \) fields \( A_\mu \) with central charge

\[
c_\mu = -s(1 - 8a_0^2).
\]  

(20)

The total central charge is

\[
c = c_0 + c_\mu = 1 - 24a_0^2 - s(1 - 8a_0^2).
\]  

(21)

So there is one scalar that appears explicitly in (18, 19), and the rest enters only via their energy momentum tensor \( T_\mu \). This situation also occurs in the Romans solution (2). We note that for both the Romans solution (2) at \( c = -2 \) as well as the case I \( c = -2 \) solution given in (18, 19), \( T_\mu \) is null and the \( A_0 \) part becomes the one scalar realisation of \( W_3 \) (10).
We next consider solutions of eqs. (10-13) for $a_0 = 0$ (case II). From equations (10-13) we deduce that

\[
\begin{align*}
II \\
\epsilon_{\mu} &= 0, \quad \epsilon_{\mu0} + \epsilon_{\rho0} = \alpha_{\mu}, \\
\epsilon_{\mu} \alpha^\mu &= \frac{1}{4} \alpha_{\mu} \alpha^\mu + \frac{1}{33} (s + n - 1), \\
\epsilon_{(\mu\nu)} &= 0, \quad \epsilon_{[\mu\nu]} \alpha^\nu = 0, \quad \epsilon_{[\mu\nu]} \epsilon_0^\nu = 0, \\
f_0 &= \frac{1}{4} \alpha_{\mu} \alpha^\mu - \frac{1}{38} (s + n - 1), \quad f_\mu = 0.
\end{align*}
\] (22)

Furthermore, the background charges and the central charge are given by

\[
II \\
a_{\mu} \alpha^\mu &= \frac{1}{24} (n - 3s + 7), \quad c = 3s - 7.
\] (23)

We also obtain expressions for the contractions $\epsilon_{0\mu} \epsilon_{0\mu}$ and $\epsilon_{[\mu\nu]} \epsilon_{[\mu\nu]}$. Since they are rather involved we will not give them here. We still have to impose the spin-four equation. We were able to simplify this equation only for $n = 2$ and have not analysed it for general values of $n$. For $n = 2$ the spin-four equation becomes, rewritten in terms of $c$ using (23):

\[
< V V > = \frac{16c(2 + c)(7 + c)(10 + c)^2(-\frac{1}{2} + c)(-4 + 5c)}{27(-2 + c)^2(22 + 5c)} = 0.
\] (24)

From the series of roots of (24) the values $c = 0, -7, 1/2$ make $W$ a null field as well, and for $c = -10$ ($s = -1$) we get a FZ realisation. The new solutions occur again for $c = -2$ and $c = 4/5$. The case II $c = -2$ and $c = 4/5$ realisations also appear in (12) as specific truncations of a non-linear $W_\infty$ algebra. They can also be derived from the second realisation mentioned in a footnote of the paper by Fateev and Zamolodchikov [6]. The explicit form of the solutions can be found in Appendix B.

Unlike the case I realisations the case II realisations are not of the form (18, 19), i.e. there exists no SO(2) redefinition of the fields such that (18, 19) is obtained. It is therefore not clear whether these solutions can be generalised to $n \geq 2$ scalars.

4 Generalisations

We now discuss generalisations of the case I and case II realisations. First, consider the $c = -2$ one-scalar realisation of (10):

\[
T_0 = \frac{1}{2} (A_0 A_0) + \frac{1}{2} A_0',
\] (25)
\[ W_0 = -\frac{2}{3}(A_0 T_0) - \frac{1}{6} T'_0. \] (26)

We now add an extra energy momentum tensor, denoted by \( \tilde{T} \), to the above system that commutes with \( A_0 \) and which is null. We then make the following Ansatz for \( W \):

\[ T = T_0 + \tilde{T}, \] (27)

\[ W = W_0 + d_1 (A_0 \tilde{T}) + d_2 \tilde{T}' . \] (28)

Since \( \tilde{T} \) commutes with \( T_0 \) the total central charge is given by \( c = -2 \). The requirement that \( W \) is primary w.r.t. \( T \) can be shown to imply \( d_1 = 4 d_2 \).

Next, in order to get rid of a primary spin-two field in the OPE \( W(z)W(w) \), which occurs in addition to \( T \), the following quadratic equation has to be satisfied:

\[ 20 d_2^2 - 4 d_2 - 3 = 0 \] (29)

with roots \( 1/2 \) and \( -3/10 \). If we represent \( \tilde{T} \) in terms of \( n - 1 \) scalar fields \( (n \geq 2) \) then we obtain for \( d_2 = 1/2 \) the Romans realisation at \( c = -2 \) and for \( d_2 = -3/10 \) the case I \( c = -2 \) realisation (cf. (19)). Note that if, in the above example, we do not modify \( W_0 \) (i.e. \( d_1 = d_2 = 0 \) in (28)), the algebra also closes modulo null operators. However, in this case also a spin-two null operator is present in \( W(z)W(w) \).

We now perform the same procedure starting from the Romans realisation (2) for arbitrary \( c \). Again we add a null energy momentum tensor \( \tilde{T} \) to the generators in such a way that they remain primary. We thus obtain

\[ T = T_0 + T_\mu + \tilde{T}, \] (30)

\[ W = W_0 + 2(A_0 T_\mu) + \sqrt{2} a_0 T'_\mu + d[(A_0 \tilde{T}) + \frac{1}{2} \sqrt{2} a_0 \tilde{T}'], \] (31)

\[ T_0 = \frac{1}{2} (A_0 A_0) + \sqrt{2} a_0 A'_0, \] (32)

\[ W_0 = -\frac{1}{3} [2(A_0 T_0) + \sqrt{2} a_0 T'_0]. \] (33)

The central charges are given by (take \( s = -1 \) in (20) and (21))

\[ c_0 = 1 - 24 a_0^2 = \frac{3}{2} c - \frac{1}{2}, \] (34)

\[ c_\mu = 1 - 8 a_0^2 = \frac{1}{4} c + \frac{1}{2}. \] (35)
The requirement that the additional spin-two primary field that occurs in $W W W$ vanishes, now leads to the following equation
\[-2 + \frac{1}{2}d^2 + a_0^2(-\frac{3}{2}d^2 - 2d + 10) = 0, \tag{36}\]
with roots
\[d = 2, \quad d = -2\frac{1 - 5a_0^2}{1 - 3a_0^2}. \tag{37}\]

For the $d = 2$ solution, $\tilde{T}$ can be absorbed into $T_{\mu}$ and we obtain the Romans realisation. The second solution for $d$ does not fit within the Ansatz (9), which is why we did not find this solution before. For $c = -2$, $a_0^2 = 1/8$, $T_{\mu}$ is null and can be consistently put to zero, and we find that for this value of $c$ the second solution reduces to the case I $c = -2$ realisation.

In principle, one could generalise the case II realisations from $n = 2$ to arbitrary $n$ by the same procedure. One adds a null field $\tilde{T}$ to $T_0$ and adds $d_1(A_0\tilde{T}) + d_2(A_1\tilde{T}) + d_3\tilde{T}'$ to $W_0$. Making $W$ primary fixes one parameter, and the spin-two absence implies a quadratic equation in the two remaining parameters. We have not attempted to investigate systematically the solutions to this equation.

### 5 Comments

We have performed a systematic investigation of free-field realisations of the $W_3$ algebra where we allow in the OPE of two spin-three generators a spin-four null field. Our starting point was a free field Ansatz for the generators. Closure of the algebra then led to a set of equations for the coefficients occurring in the Ansatz. We analyzed these equations and gave several solutions to them. Besides the Romans solution (see (2)), we found further two-scalar solutions (case II) as well as n-scalar solutions (case I and the second solution of eqs. (30-33)).

Since we used a specific Ansatz, our analysis is not exhaustive. Only in the case of two-scalar realisations were we able to verify that our analysis is complete. Besides the FZ realisation we found four more realisations whose explicit form can be found in Appendix B. Two of these solutions also occur in the work of [8, 11, 12]. It would be interesting to see whether the other two solutions could be understood from other construction procedures as well.
Finally, one could consider the classical limit of our results. In the case of the Romans realisation one obtains in this limit a realisation of a classical version $w_3$ of the $W_3$ algebra. This is consistent with the fact that the Ansatz of [7, 8] satisfies the identity (8) which guarantees the closure of the classical $w_3$ algebra [7]. Our Ansatz does not satisfy (8) and therefore, to obtain closure in the classical limit, one should include the whole ideal of null operators generated by the spin-four operator $V$. For the case II solutions, this leads to the classical limit of the nonlinear $W_\infty$ algebra of [12]. It would be interesting to see which classical algebras the case I realisations lead to.

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### A The Spin-Four Equation

To determine the spin-four equation mentioned in section 2, we must first calculate the expression for the spin-four operator $V$. This expression can be found from eq. (3.12) in [8] by subtracting the descendents of the energy-momentum tensor. Next, it is a straightforward exercise to calculate the norm of $V$ and require it to be zero. We thus find the following spin-four equation:

$$< VV > = 24S_{ijkl}^{ijkl} + 30S_{i}^{ikl}S_{jkl}^{j} - 280S_{i}^{ikl}S_{j}^{j}S_{k}^{m}a_{l}a_{m}$$

$$-60\sqrt{2}S_{i}^{ikl}T_{kl}a^{m} + 24\sqrt{2}S_{i}^{ikl}T_{mkl}a^{m}$$
\[\begin{align*}
+ \frac{560}{3} \sqrt{2} S^i_{\ jkl} (T^m_{nml} + 2T^m_{lmm}) a_k a^m a^n - 12 T^{ijk} T_{ijk} + \\
- 16 T^{ijk} T_{ikj} + 60 T_{ijk} T^{ij} a^k a^l - 48 T_{ijk} T^l_{ji} a^k a^l \\
+ \frac{328}{3} T_{kij} T^i_{jkl} a^k a^l + \frac{104}{3} T_{kij} T^i_{jkl} a^k a^l \\
- \frac{560}{9} (T^m_{ijm} T^m_{kl} + 4T^m_{imj} T^m_{kl} + 4T^m_{imj} T^m_{k l}) a^l a^j a^k a^l = 0,
\end{align*}\]

where \( S \) and \( T \) are given by

\[S_{ijkl} = d_{\ (ij} m d_{kl)m} - \frac{24N_3}{c(22 + 5c)} g_{(ij gkl)},\]

\[T_{ijk} = 4\sqrt{2} \left( -2d_{ij} l c_{[kl]} + 2c_{(i} d_{j)kl} - \frac{24N_3}{c(22 + 5c)} g_{ij a_k} \right).\]

### B Two-scalar realisations

For the case of two scalars \((n = 2)\) we find all possible realisations of \( W_3 \) that close modulo a non-zero spin-four null field. We find four different realisations. Firstly, the case \( c = -2 \) realisation is given by

\[T = \frac{1}{2} (A_0 A_0) + \frac{1}{2} (A_1 A_1) + \frac{1}{2} A'_0 + \frac{1}{6} \sqrt{3} A'_1,\]

\[W = \frac{5}{9} (A_0 A_0 A_0) + \frac{5}{6} (A_0 A'_0) + \frac{5}{36} A''_0 + \frac{1}{3} \sqrt{3} (A_0 A'_1) + \frac{1}{2} (A_1 A'_1) + \frac{1}{12} \sqrt{3} A''_1.\]

Secondly, the case \( c = 4/5 \) realisation in a real basis is given by

\[T = -\frac{1}{2} (A_0 A_0) + \frac{1}{2} (A_1 A_1) + \frac{1}{2} \sqrt{6} A'_0 + \frac{2}{5} \sqrt{10} A'_1,\]

\[W = \frac{13}{15} (A_0 A_0 A_0) - \frac{13}{10} \sqrt{6} (A_0 A'_0) + \frac{13}{10} A''_0 + \frac{4}{5} \sqrt{10} (A_0 A'_1) + \frac{1}{2} \sqrt{6} (A_1 A'_1) + \frac{1}{5} \sqrt{60} A''_1.\]

The above realisations are obtained from (18, 19) by substituting the appropriate values for the parameters and by realising \( T^\mu \) in terms of \( A_1 \), the derivative of the scalar field \( \phi_1 \).
Next, the case II $c = -2$ realisation is given by:

$$T = \frac{1}{2} (A_0 A_0) + \frac{1}{2} (A_1 A_1) + \frac{1}{3} \sqrt{3} A'_1,$$

$$W = \frac{5}{9} (A_0 A_0 A_0) + (A_0 A_1 A_1) + \frac{1}{2} \sqrt{3} (A_0 A'_1) + \frac{1}{6} \sqrt{3} (A'_0 A_1) + \frac{1}{18} A''_0.$$

Finally, the case II $c = 4/5$ realisation is given by

$$T = \frac{1}{2} (A_0 A_0) + \frac{1}{2} (A_1 A_1) + \frac{1}{10} \sqrt{10} A'_1,$$

$$W = \frac{13}{15} (A_0 A_0 A_0) + (A_0 A_1 A_1) + \frac{1}{2} \sqrt{10} (A_0 A'_1) - \frac{3}{10} \sqrt{10} (A'_0 A_1) - \frac{1}{10} A''_0.$$

For the $n = 2$, $c = -2$ realisations $<WW> = \frac{-25}{9}$. The $c = 4/5$ realisations have $<WW> = \frac{52}{75}$ (case I) and $<WW> = \frac{-52}{75}$ (case II).

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