Syzygies of Prym and paracanonical curves of genus 8

Elisabetta Colombo, Gavril Farkas, Alessandro Verra, and Claire Voisin

Abstract. By analogy with Green’s Conjecture on syzygies of canonical curves, the Prym-Green conjecture predicts that the resolution of a general level \( p \) paracanonical curve of genus \( g \) is natural. The Prym-Green Conjecture is known to hold in odd genus for almost all levels. Probabilistic arguments strongly suggested that the conjecture might fail for level 2 and genus 8 or 16. In this paper, we present three geometric proofs of the surprising failure of the Prym-Green Conjecture in genus 8, hoping that the methods introduced here will shed light on all the exceptions to the Prym-Green Conjecture for genera with high divisibility by 2.

Keywords. Paracanonical curve; syzygy; genus 8; moduli of Prym varieties

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[Français]

Titre. Syzygies de Prym et courbes paracanoniques de genre 8

Résumé. Par analogie avec la conjecture de Green sur les syzygies des courbes canoniques, la conjecture de Prym-Green prédit que la résolution d’une courbe générale, paracanonical, de genre \( g \) et de niveau \( p \) est naturelle. Cette conjecture est connue en genre impair pour presque tout niveau. Des arguments probabilistes ont fortement suggéré qu’elle pourrait s’avérer fausse pour le niveau 2 en genre 8 et 16. Dans cet article, nous présentons trois démonstrations géométriques de la surprenante non-validité de la conjecture de Prym-Green en genre 8, en espérant que les méthodes introduites apporteront un éclairage nouveau sur toutes les exceptions à la conjecture de Prym-Green pour des genres divisibles par une grande puissance de 2.

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1. Introduction

By analogy with Green’s Conjecture on the syzygies of a general canonical curve [18], [19], the Prym-Green Conjecture, formulated in [10] and [3], predicts that the resolution of a paracanonical curve

$$\phi_{K_C \otimes \eta} : C \hookrightarrow \mathbb{P}^{g-2},$$

where $C$ is a general curve of genus $g$ and $\eta \in \text{Pic}^0(C)[\ell]$ is an $\ell$-torsion point is natural. For even genus $g = 2i + 6$, the Prym-Green Conjecture amounts to the vanishing statement

$$K_{i,2}(C, K_C \otimes \eta) = K_{i+1,1}(C, K_C \otimes \eta) = 0,$$

in terms of Koszul cohomology groups. Equivalently, the genus $g$ paracanonical level $\ell$ curve $C \subseteq \mathbb{P}^{g-2}$ satisfies the Green-Lazarsfeld property $(N_i)$. The Prym-Green Conjecture has been proved for all odd genera $g$ when $\ell = 2$, see [8], or $\ell \geq \sqrt{\frac{g+2}{2}}$, see [9]. For even genus, the Prym-Green Conjecture has been established by degeneration and using computer algebra tools in [3] and [4], for all $\ell \leq 5$ and $g \leq 18$, with two possible mysterious exceptions in level 2 and genus $g = 8, 16$ respectively. The last section of [3] provides various pieces of evidence, including a probabilistic argument, strongly suggesting that for $g = 8$, one has $\dim K_{1,2}(C, K_C \otimes \eta) = 1$, and thus the vanishing (1.1) fails in this case. It is tempting to believe that the exceptions $g = 8, 16$ can be extrapolated to higher genus, and that for genera $g$ with high divisibility by 2, there are genuinely novel ways of constructing syzygies of Prym-canonical curves waiting to be discovered. It would be very interesting to test experimentally the next relevant case $g = 24$. Unfortunately, due to memory and running time constraints, this is currently completely out of reach, see [3] and [7].

The aim of this paper is to confirm the expectation formulated in [3] and offer several geometric explanations for the surprising failure of the Prym-Green Conjecture in genus $g = 8$, hoping that the geometric methods described here for constructing syzygies of Prym-canonical curves will eventually shed light on all the exceptions to the Prym-Green Conjecture. We choose a general Prym-canonical curve of genus 8

$$\phi_{K_C \otimes \eta} : C \hookrightarrow \mathbb{P}^6,$$

with $\eta \otimes 2 = \mathcal{O}_C$. Set $L := K_C \otimes \eta$ and denote $I_{C,L}(k) := \text{Ker}\{\text{Sym}^k H^0(C, L) \to H^0(C, L \otimes k)\}$ for all $k \geq 2$. Observe that $\dim I_{C,L}(2) = \dim K_{1,1}(C, L) = 7$ and $\dim I_{C,L}(3) = 49$, therefore as $[C, \eta]$ varies in moduli, the multiplication map

$$\mu_{C,L} : I_{C,L}(2) \otimes H^0(C, L) \to I_{C,L}(3)$$

globalizes to a morphism of vector bundles of the same rank over the stack $\mathcal{R}_8$ classifying pairs $[C, \eta]$, where $C$ is a smooth curve of genus 8 and $\eta \in \text{Pic}^0[2] \setminus \{\mathcal{O}_C\}$.
Theorem 1. For a general Prym curve \([C, \eta] \in \mathcal{R}_8\), one has \(K_{1,2}(C, L) \neq 0\). Equivalently the multiplication map \(\mu_{C,L} : I_{C,L}(2) \otimes H^0(C, L) \to I_{C,L}(3)\) is not an isomorphism.

We present three different proofs of Theorem 1. The first proof, presented in Section 3 uses the structure theorem already pointed out in [3] for degenerate syzygies of paracanonical curves in \(P^6\). Precisely, if a paracanonical genus 8 curve \(\phi_{K_C \otimes \eta} : C \hookrightarrow P^6\), where \(\eta \neq O_C\), has a syzygy \(0 \neq \gamma \in K_{1,2}(C, K_C \otimes \eta)\) of sub-maximal rank (see Section 2 for a precise definition), then the syzygy scheme of \(\gamma\) consists of an isolated point \(p \in P^6 \setminus C\) and a residual septic elliptic curve \(E \subseteq P^6\) meeting \(C\) transversally along a divisor \(e\) of degree 14, such that if \(e\) is viewed as a divisor on \(C\) and \(E\) respectively, then

\[
\mu_C \in |K_C \otimes \eta^{\otimes 2}| \quad \text{and} \quad \mu_E \in |O_E(2)|. \tag{1.2}
\]

The union \(D := C \cup E \hookrightarrow P^6\), endowed with the line bundle \(O_D(1)\) is a degenerate spin curve of genus 22 in the sense of [5]. The locus of stable spin structures with at least 7 sections defines a subvariety of codimension 21 = \(\binom{7}{2}\) inside the moduli space \(\overline{S}_{22}\) of stable odd spin curves of genus 22. By restricting this condition to the locus of spin structures having \(D := C \cup_6 E\) as underlying curve, it turns out that one has enough parameters to realize this condition for a general \(C \subseteq P^6\) if and only if

\[
\dim |K_C \otimes \eta^{\otimes 2}| = 7,
\]

which happens precisely when \(\eta^{\otimes 2} \cong O_C\). Therefore for each Prym-canonical curve \(C \subseteq P^6\) of genus 8 there exists a corresponding elliptic curve \(E \subseteq P^6\) such that the intersection divisor \(E \cdot C\) verifies (1.2), which forces \(K_{1,2}(C, K_C \otimes \eta) \neq 0\).

The second and the third proofs involve the reformulation given in Section 2.B (see Proposition 5) of the condition that a paracanonical curve \(\phi_p : C \hookrightarrow P^6\) have a non-trivial syzygy. Precisely, if \(\phi_{L}(C)\) is scheme-theoretically generated by quadrics, then \(K_{1,2}(C, L) \neq 0\), if and only if there exists a quartic hypersurface in \(P^6\) singular along \(C \subseteq P^6\), which is not a quadratic polynomial in quadrics vanishing along \(C\), that is, it does not belong to the image of the multiplication map

\[
\text{Sym}^2 I_{C,L}(2) \to I_{C,L}(4).
\]

Equivalently, one has \(H^1(P^6, I_{C,P^6}^2(4)) \neq 0\).

The second proof presented in Section 4 uses intersection theory on the stack \(\mathcal{R}_8\). The virtual Koszul divisor of Prym curves \([C, \eta] \in \mathcal{R}_8\) having \(K_{1,2}(C, K_C \otimes \eta) \neq 0\), splits into two divisors \(\mathfrak{D}_1\) and \(\mathfrak{D}_2\) respectively, corresponding to the case whether \(C \subseteq P^6\) is not scheme-theoretically cut out by quadrics, or \(H^1(P^6, I_{C,P^6}^2(4)) \neq 0\) respectively. We determine the virtual classes of both closures \(\overline{\mathfrak{D}}_1\) and \(\overline{\mathfrak{D}}_2\). Using an explicit uniruled parametrization of \(\overline{\mathcal{R}}_8\) constructed in [11], we conclude that the class \([\overline{\mathfrak{D}}_2]\) \(\in CH^1(\mathcal{R}_8)\) cannot possibly be effective (see Theorem 20). Therefore, again \(K_{2,1}(C, K_C \otimes \eta) \neq 0\), for every Prym curve \([C, \eta] \in \mathcal{R}_8\).

The third proof given in Section 5 even though subject to a plausible, but still unproved transversality assumption, is constructive and potentially the most useful, for we feel it might offer hints to the case \(g = 16\) and further. The idea is to consider rank 2 vector bundles \(E\) on \(C\) with canonical determinant and \(h^0(C, E) = h^0(C, E(\eta)) = 4\). (Note that the condition that \(\eta\) is 2-torsion is equivalent to the fact that \(E(\eta)\) also has canonical determinant, which is essential for the existence of such nonsplit vector bundles, cf. [15].) By pulling back to \(C\) the determinantal quartic hypersurface consisting of rank 3 tensors in

\[
P \left( H^0(C, E)^{\vee} \otimes H^0(C, E(\eta))^{\vee} \right) \cong P^{15}
\]

under the natural map \(H^0(C, K_C \otimes \eta)^{\vee} \to H^0(C, E)^{\vee} \otimes H^0(C, E(\eta))^{\vee}\), we obtain explicit quartic hypersurfaces singular along the curve \(C \subseteq P^6\). Our proof that these are not quadratic polynomials
into quadrics vanishing along the curve, that is, they do not lie in the image of Sym\(^2 I_{C,L}(2)\) remains incomplete, but there is a lot of evidence for this.

The methods of Section 5 suggests the following analogy in the next case \(g = 16\). If \([C, \eta] \in \mathcal{R}_{16}\) is a Prym curve of genus 16, there exist vector bundles \(E\) on \(C\) with \(\det E \cong K_C\) and satisfying \(h^0(C, E) = h^0(C, E(\eta)) = 6\). Potentially they could be used to prove that \(K_{5,2}(C, K_C \otimes \eta) \neq 0\) and thus confirm the next exception to the Prym-Green Conjecture.

### 2. Syzygies of paracanonical curves of genus 8

Let \(C\) be a general smooth projective curve of genus 8. For a non-trivial line bundle \(\eta \in \text{Pic}^0(C)\), we shall study the paracanonical line bundle \(L := K_C \otimes \eta\). When \(\eta\) is a 2-torsion point, we speak of the Prym-canonical line bundle \(L\). For each paracanonical bundle \(L\), we have \(h^0(C, L) = 7\) and an induced embedding

\[
\phi_L : C \hookrightarrow \mathbb{P}^6.
\]

The goal is to understand the reasons for the non-vanishing of the Koszul group \(K_{1,2}(C, L)\) of a Prym-canonical bundle \(L\), as suggested experimentally by the results of [3], [4].

Let \(I_C(2) = I_{C,L}(2) \subseteq H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))\), respectively \(I_C(3) = I_{C,L}(3) \subseteq H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(3))\) be the ideal of quadrics, respectively cubics, vanishing on \(\phi_L(C)\). It is well-known that whenever \(L\) is projectively normal, the non-vanishing of the Koszul cohomology group \(K_{1,2}(C, L)\) is equivalent to the non-surjectivity of the multiplication map

\[
\mu_{C,L} : H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)) \otimes I_C(2) \rightarrow I_C(3).
\]

Note that

\[
\dim I_C(2) = \binom{8}{2} - 21 = 7, \quad \text{and} \quad \dim I_C(3) = \binom{9}{3} - 3 \cdot 14 + 7 = 49,
\]

respectively, so that the two spaces appearing in the map (2.3) have the same dimension. Denote by \(P_{14}^8\) the universal degree 14 Picard variety over \(\mathcal{M}_8\) consisting of pairs \([C, L]\), where \([C] \in \mathcal{M}_8\) and \(L \neq K_C\). The jumping locus

\[
\mathcal{R}_{\text{jump}} := \{[C, L] \in P_{14}^8 : K_{1,2}(C, L) \neq 0\}
\]

is a divisor. It turns out, cf. Theorem 5.3 of [3] and Proposition 8, that \(\mathcal{R}_{\text{jump}}\) splits into two components depending on the rank of the corresponding non-zero syzygy from \(K_{1,2}(C, L)\).

**Definition 2.** The rank of a non-zero syzygy \(\gamma = \sum_{i=0}^6 \ell_i \otimes q_i \in \text{Ker}(\mu_{C,L})\) is the dimension of the subspace \(\langle \ell_0, \ldots, \ell_6 \rangle \subseteq H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))\). The syzygy scheme \(\text{Syz}(\gamma)\) of \(\gamma\) is the largest subscheme \(Y \subseteq \mathbb{P}^6\) such that \(\gamma \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)) \otimes I_Y(2)\).

It is shown in [3], that \(\mathcal{R}_{\text{jump}}\) splits into divisors \(\mathcal{R}_{\text{jump}}^6\) and \(\mathcal{R}_{\text{jump}}^7\), depending on whether the syzygy \(0 \neq \gamma \in \text{Ker}(\mu_{C,L})\) has rank 6 or 7 respectively. By a specialization argument to irreducible nodal curves, it follows from [3] that \(\mathcal{R}_8 \not\subseteq \mathcal{R}_{\text{jump}}\). A direct, more transparent proof of this fact will be given in Proposition 13.

### 2.A. Paracanonical curves of genus 8 with special syzygies and elliptic curves

We summarize a few facts already stated or recalled in Section 5 of [3] concerning rank 6 syzygies of paracanonical curves in \(\mathbb{P}^6\). Very generally, let

\[
\gamma = \sum_{i=1}^6 \ell_i \otimes q_i \in H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1)) \otimes H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2))
\]
be a rank 6 linear syzygy among quadrics in \( \mathbb{P}^6 \). The linear forms \( \ell_1, \ldots, \ell_6 \) define a point \( p \in \mathbb{P}^6 \). Following Lemma 6.3 of [16], there exists a skew-symmetric matrix of linear forms \( A := (a_{ij})_{i,j=1,\ldots,6} \), such that
\[
q_i = \sum_{j=1}^{6} \ell_j a_{ij}.
\]

In the space \( \mathbb{P}^{20} \) with coordinates \( \ell_1, \ldots, \ell_6 \) and \( a_{ij} \) for \( 1 \leq i < j \leq 6 \), one considers the 15-dimensional variety \( X_6 \) defined by the 6 quadratic equations \( \sum_{j=1}^{6} \ell_j a_{ij} = 0 \), where \( i = 1, \ldots, 6 \) and by the cubic equation \( \text{Pfaff}(A) = 0 \) in the variables \( a_{ij} \). The original space \( \mathbb{P}^6 \) embeds in \( \mathbb{P}^{20} \) via evaluation. The syzygy scheme \( \text{Syz}(\gamma) \) is the union of the point \( p \) and of the intersection \( D \) of \( \mathbb{P}^6 \) with the variety \( X_6 \). It follows from Theorem 4.4 of [6], that for a general rank 6 syzygy \( \gamma \) as above, \( D \subseteq \mathbb{P}^6 \) is a smooth curve of genus 22 and degree 21 such that \( \mathcal{O}_D(1) \) is a theta characteristic.

In the case at hand, that is, when \( [C, L] \in \mathfrak{Ros}_{36} \), the curve \( D \) must be reducible, for it has \( C \) as a component. More precisely:

**Lemma 3.** For a general paracanonical curve \( C \subseteq \mathbb{P}^6 \) having a rank 6 syzygy, the curve \( D \) is nodal and consists of two components \( C \cup E \), where \( E \subseteq \mathbb{P}^6 \) is an elliptic septic curve. Furthermore, \( \mathcal{O}_D(2) = \omega_D \). The intersection \( e := C \cdot E \), viewed as a divisor on \( C \), satisfies \( e_C \in |\mathcal{O}_C(2) \otimes K_C^+) \), and as a divisor on \( E \), satisfies \( e_E \in |\mathcal{O}_E(2)| \).

**Remark 4.** Note that \( C \) is Prym-canonical or canonical if and only if \( e_C \in |K_C| \).

The construction above is reversible. Firstly, general element \( [C, L] \in \mathfrak{Ros}_{36} \) can be reconstructed as the residual curve of a reducible spin curve \( D \subseteq \mathbb{P}^6 \) of genus 22 containing an elliptic curve \( E \subseteq \mathbb{P}^6 \) with \( \text{deg}(E) = 7 \) as a component such that the union of \( D \) and some point \( p \in \mathbb{P}^6 \setminus E \) is the syzygy scheme of a rank 6 linear syzygy among quadrics in \( \mathbb{P}^6 \).

Furthermore, given a reducible spin curve \( D = C \cup E \subseteq \mathbb{P}^6 \) of genus 22 as above, that is, with \( \omega_D \cong \mathcal{O}_D(2) \), the genus 8 component \( C \) has a nontrivial syzygy of rank 6 involving the quadrics in the 6-dimensional subspace \( I_D(2) \subseteq I_C(2) \), see Lemma 29 for a proof of this fact.

### 2.B. Syzygies and quartics singular along paracanonical curves

We first discuss an alternative characterization of the non-surjectivity of the map \( \mu_{C,L} \):

**Proposition 5.** Assume the paracanonical curve \( \phi_L(C) \) is projectively normal and scheme-theoretically cut out by quadrics. Then \( K_{1,2}(C, L) \neq 0 \) if and only if there exists a degree 4 homogeneous polynomial on \( \mathbb{P}^6 \), which vanishes to order at least 2 along \( C \) but does not belong to the image of the multiplication map \( \text{Sym}^2 I_{C,L}(2) \to I_{C,L}(4) \).

**Proof.** We work on the variety \( X \to \mathbb{P}^6 \) defined as the blow-up of \( \mathbb{P}^6 \) along \( \phi_L(C) \). Let \( E \) be the exceptional divisor of the blow-up, and consider the line bundle \( H := \tau^* \mathcal{O}_{\mathbb{P}^6}(2)(-E) \) on \( X \). Its space of sections identifies to \( I_C(2) \), and our assumption that \( C \) is scheme-theoretically cut out by quadrics says equivalently that \( H \) is a globally generated line bundle on \( X \). The nonvanishing of \( K_{1,2}(C, L) \) is equivalent to the non-surjectivity of the multiplication map
\[
H^0(X, H) \otimes H^0(X, \tau^* \mathcal{O}(1)) \to H^0(X, H \otimes \tau^* \mathcal{O}(1)),
\]
where we use the identification
\[
H^0(X, H \otimes \tau^* \mathcal{O}(1)) = H^0(X, \tau^* \mathcal{O}(3)(-E)) = I_C(3).
\]
As $H$ is globally generated by its space $W := I_C(2)$ of global sections, the Koszul complex

$$0 \to \bigwedge^7 W \otimes \mathcal{O}_X(-7H) \to \ldots \to \bigwedge^2 W \otimes \mathcal{O}_X(-2H) \to W \otimes \mathcal{O}_X(-H) \to \mathcal{O}_X \to 0 \quad (2.5)$$

is exact. We now twist this complex by $\tau^* \mathcal{O}_{\mathbb{P}^6}(1)(H)$ and take global sections. The last map is then the multiplication map $(2.4)$. The successive terms of this twisted complex are

$$\bigwedge^i W \otimes \mathcal{O}_X(\tau^* \mathcal{O}(1))((-i + 1)H),$$

for $0 \leq i \leq 7$. The spectral sequence abutting to the hypercohomology of this complex, that is 0, has

$$E_{i,0}^{0,0} = \text{coker} \left\{ W \otimes H^0(X, \tau^* \mathcal{O}(1)) \to H^0(X, H \otimes \tau^* \mathcal{O}(1)) \right\} \quad (2.6)$$

and the terms $E_{1,i}^{i-1}$ for $i < -1$ are equal to $\bigwedge^{-i} W \otimes H^{-i-1}(X, \tau^* \mathcal{O}(1)((i + 1)H))$. Similarly, we have

$$E_{1,i}^{i-1} = \bigwedge^{-i} W \otimes H^{-i}(X, \tau^* \mathcal{O}(1)((i + 1)H)).$$

Lemma 6. (i) We have

$$E_{1,i}^{i,-i-1} = \bigwedge^{-i} W \otimes H^{-i-1}(X, \tau^* \mathcal{O}(1)((i + 1)H)) = 0, \quad (2.7)$$

for $-i - 1 = 5, \ldots, 1$.

(ii) For $-i - 1 = 6$, that is, $i = -7$, we have

$$E_{1}^{-7,6} = \bigwedge^7 W \otimes H^6(X, \tau^* \mathcal{O}(1)(-6H)) = \bigwedge^7 W \otimes I_C(4)_2, \quad (2.8)$$

where $I_C(4)_2 \subseteq I_C(4)$ is the set of quartic polynomials vanishing at order at least 2 along $C$, and

$$E_{1}^{-6,6} = \bigwedge^6 W \otimes H^6(X, \tau^* \mathcal{O}(1)(-5H)) = \bigwedge^6 W \otimes I_C(2)_2. \quad (2.9)$$

(iii) We have $E_{1,i}^{i,-i} = 0$, for $-6 < i < 0$.

Proof of Lemma 6. (i) We want equivalently to show that

$$H^\ell(X, \tau^* \mathcal{O}(1)(-\ell H)) = 0, \quad \text{when } \ell = 5, \ldots, 1.$$

Recall that $H = \tau^* \mathcal{O}(2)(-E)$. Furthermore,

$$K_X = \tau^* \mathcal{O}_{\mathbb{P}^6}(-7)(4E). \quad (2.10)$$

So we have to prove that

$$H^\ell(X, \tau^* \mathcal{O}(-2\ell + 1)(\ell E)) = 0, \quad \text{for } \ell = 5, \ldots, 1. \quad (2.11)$$

Examining the spectral sequence induced by $\tau$, and using the fact that

$$R^* \tau_* (\mathcal{O}_X(tE)) = 0$$
for $s \neq 0, 4$ and also for $s = 4, t \leq 4$, we see that for $1 \leq \ell \leq 4$,

$$
H^\ell(X, \tau^*\mathcal{O}(-2\ell + 1)(\ell E)) = H^\ell(\mathbb{P}^6, \mathcal{O}(-2\ell + 1) \otimes R^{0^\ell} \tau_\ast \mathcal{O}_X(\ell E)).
$$

For $1 \leq \ell \leq 4$, the right hand side is zero, because it is equal to $H^\ell(\mathbb{P}^6, \mathcal{O}(-2\ell + 1))$.

For $\ell = 5$, we have to compute the space $H^5(X, \tau^*\mathcal{O}(-9)(5E))$, which by Serre duality and by (2.10), is dual to the space

$$
H^1(X, \tau^*\mathcal{O}(2)(-E)) = H^1(\mathbb{P}^6, \mathcal{O}(2) \otimes \mathcal{I}_C) = 0.
$$

(ii) We have to compute the spaces $H^6(X, \tau^*\mathcal{O}(1)(-6H))$ and $H^6(X, \tau^*\mathcal{O}(1)(-5H))$. As $H := \tau^*\mathcal{O}(2)(-E)$, this is rewritten as $H^6(X, \tau^*\mathcal{O}(-11)(6E))$ and $H^6(X, \tau^*\mathcal{O}(-9)(5E))$ respectively. If we dualize using (2.10), we get

$$
H^6(X, \tau^*\mathcal{O}(-11)(6E))^\vee = H^6(X, \tau^*\mathcal{O}(4)(-2E)) = I_C(4)_2,
$$

$$
H^6(X, \tau^*\mathcal{O}(-9)(5E))^\vee = H^6(X, \tau^*\mathcal{O}(2)(-E)) = I_C(2).
$$

(iii) We have

$$
E^{i,-i}_1 = E^{i,-i}_1 = \bigwedge^i W \otimes H^{-i}(X, \tau^*\mathcal{O}(1)((i + 1)H)) = \bigwedge^i W \otimes H^{-i}(X, \tau^*\mathcal{O}(2i + 3)((-i - 1)E)).
$$

For $1 \leq -i \leq 5$, we have $R^s\tau_\ast \mathcal{O}_X((i - 1)E) = 0$ unless $s = 0$. Furthermore, we have

$$
R^0\tau_\ast \mathcal{O}_X((i - 1)E) = \mathcal{O}_{\mathbb{P}^6},
$$

so that

$$
H^{-i}(X, \tau^*\mathcal{O}(2i + 3)((i - 1)E)) = H^{-i}(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2i + 3)) = 0.
$$

\[\square\]

**Corollary 7.** Only one $E^{p,q}_2$-terms of this spectral sequence is possibly nonzero in degree $-1$, namely

$$
E^{7,6}_2 = \text{Ker}\left\{\bigwedge^7 W \otimes I_C(4)_2 \rightarrow \bigwedge^6 W \otimes I_C(2)^\vee \right\}. \tag{2.12}
$$

Furthermore, all the differentials $d_r$ starting from $E^{7,6}_2$ vanish for $2 \leq r < 7$.

Note that the map

$$
\bigwedge^7 W \otimes I_C(4)_2 \rightarrow \bigwedge^6 W \otimes I_C(2)^\vee
$$

is nothing but the transpose of the multiplication map

$$
W \otimes I_C(2) \rightarrow I_C(4)_2,
$$

up to trivialization of $\bigwedge^7 W$. It follows that

$$
(E^{7,6}_2)^\vee = \text{Coker}\left\{W \otimes I_C(2) \rightarrow I_C(4)_2 \right\}. \tag{2.13}
$$

Corollary 7 concludes the proof of the proposition since it implies that we have an isomorphism given by $d_7$ between (2.12) and (2.6), or a perfect duality between (2.12) and the cokernel (2.13). \[\square\]

Proposition 5 has the following consequence. Recall that $P^5_{8}$ is the moduli space of pairs $[C, L]$, with $C$ being a smooth curve of genus 8 and $L \neq K_C$ a paracanonical line bundle.
Proposition 8. The Koszul divisor $\mathfrak{Kos}_3$ of $P^4_8$ is the union of two divisors, one of them being the set of pairs $(C, L)$ such that $\phi_L(C)$ is not scheme-theoretically cut out by quadrics, the other being the set of pairs $(C, L)$ such that $H^1(P^6, I_C^k(4)) \neq 0$, or equivalently, such that there exists a quartic which is singular along $\phi_L(C)$ but does not lie in $\text{Sym}^2 I_C(2)$.

Proof. We first have to prove that the locus of pairs $(C, L)$ such that $\phi_L(C)$ is not scheme-theoretically cut-out by quadrics is contained in the divisor $\mathfrak{Kos}_3$. This is a consequence of the following lemmas:

Lemma 9. If $L \neq K_C$ is a projectively normal paracanonical line bundle on a curve of genus 8, then $\phi_L(C)$ is scheme-theoretically cut out by quadrics.

Proof of Lemma 9. We observe that the twisted ideal sheaf $I_C(3)$ is regular in Castelnuovo-Mumford sense. Indeed, we have

$$H^i(P^6, I_C(3-i)) = H^{i-1}(C, L^\otimes(3-i))$$

for $i \geq 2$, and the right hand side is obviously 0 for $i-1 \geq 2$, and also 0 for $i-1 = 1$ since $H^1(C, L) = 0$ because $L \neq K_C$ and $\deg L = 2g - 2$. For $i = 1$, we have

$$H^1(P^6, I_C(2)) = 0$$

by projective normality. Being regular, the sheaf $I_C(3)$ is generated by global sections. \hfill $\square$

Corollary 10. If $C, L$ are as above, and $C$ is not scheme-theoretically cut out by quadrics, then the multiplication map

$$I_C(2) \otimes H^0(P^6, \mathcal{O}_P(1)) \to I_C(3)$$

is not surjective.

To conclude the proof of the proposition, we just have to show that the sublocus of $P^4_8$ where $L$ is not projectively normal is not a divisor, since the statement of the proposition will be then an immediate consequence of Proposition 5. We argue along the lines of [12]. First of all, a line bundle $L$ of degree 14 is not generated by sections if and only if $L = K_C(-x + y)$ for some points $x, y \in C$. This determines a codimension 6 locus of $P^4_8$. Similarly $L$ is not very ample if and only if $L = K_C(-x - y + z + t)$, for some points $x, y, z, t$ of $C$, which is satisfied in a codimension 4 locus of $P^4_8$. Finally, assume $L$ is very ample but $\phi_L(C)$ is not projectively normal. Equivalently

$$\text{Sym}^2 H^0(C, L) \to H^0(C, L^\otimes 2)$$

is not surjective, which means that there exists a rank 2 vector bundle $F$ on $C$ which is a nontrivial extension

$$0 \longrightarrow K_C \otimes L^\vee \longrightarrow F \longrightarrow L \longrightarrow 0,$$

such that $h^0(C, F) = 7$. If $x, y, z \in C$, there exists a nonzero section $\sigma \in H^0(C, F)$ vanishing on $x, y$ and $z$, and thus $F$ is also an extension

$$0 \longrightarrow D \longrightarrow F \longrightarrow K_C \otimes D^\vee \longrightarrow 0,$$ (2.14)

where $D$ is a line bundle such that $h^0(C, D(-x - y - z)) \neq 0$, and $h^0(C, L \otimes D^\vee) \neq 0$. We thus have $h^0(C, D) + h^0(C, K_C \otimes D^\vee) \geq 7$ and $\text{Cliff}(D) \leq 2$. As $D$ is effective of degree at least 3, one has the following possibilities:

a) $h^0(C, K_C \otimes D^\vee) = 0$, and then $D = L$, which contradicts the fact that the extension (2.14) is not split;

b) $h^0(C, K_C \otimes D^\vee) = 1$ and $h^0(C, D) \geq 6$, and then $D = L(-x)$ and $h^0(K_C \otimes L^\vee(x)) \neq 0$, so $L = K_C(x - y)$, which happens in a locus of codimension at least 6 in $P^4_8$;

c) $D$ contributes to the Clifford index of $C$. As the locus of curves $[C] \in \mathcal{M}_8$ with $\text{Cliff}(C) \leq 2$ is of codimension 2 in $\mathcal{M}_8$, this situation does not occur in codimension 1. \hfill $\square$
We shall need later on the following result:

**Lemma 11.** Let \( \phi_L : C \to \mathbb{P}^6 \) be a projectively normal paracanonical curve of genus 8. If \( C \) is scheme-theoretically cut out by quadrics, the multiplication map

\[
\text{Sym}^2 I_{C,L}(2) \to I_{C,L}(4)
\]

is injective.

**Proof.** As the restriction map \( \phi^*_L : H^0(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(2)) \to H^0(C, L^2) \) is surjective, its kernel \( I_{C,L}(2) \) is of dimension 7. Let as before \( \tau : X \to \mathbb{P}^6 \) be the blow-up of \( \mathbb{P}^6 \) along \( \phi_L(C) \), and let \( E \) be its exceptional divisor. We view \( I_{C,L}(2) \) as \( H^0(X, \tau^*\mathcal{O}(2)(-E)) \) and our assumption is that \( I_{C,L}(2) \) generates the line bundle \( H := \tau^*\mathcal{O}(2)(-E) \) everywhere on \( X \). Thus \( I_{C,L}(2) \) provides a morphism

\[
\psi : X \to \mathbb{P}(I_{C,L}(2)).
\]

Now we have \( \deg c_1(H)^6 \neq 0 \) by Sublemma 12 below, and thus the morphism \( \psi \) has to be generically finite, hence dominant since both spaces have dimension 6. It is then clear that the pull-back map

\[
\psi^* : H^0(\mathbb{P}(I_{C,L}(2)), \mathcal{O}(2)) \to H^0(X, H^2)
\]

is injective. On the other hand, this morphism is nothing but the map (2.15). \( \square \)

**Sublemma 12.** With the same notation as above, we have

\[
\deg c_1(H)^6 = 8.
\]

**Proof.** We have

\[
c_1(H)^6 = \sum_i \binom{6}{i} (-2)^i h^i \cdot E^{6-i},
\]

where \( h := \tau^*c_1(\mathcal{O}_{\mathbb{P}^6}(1)) \), and

\[
h^i \cdot E^{6-i} = 0
\]

for \( i \neq 6, 1, 0 \). Furthermore

\[
h^6 = 1, \quad \text{and} \quad h \cdot E^5 = \deg \phi_L(C) = 14
\]

and \( E^6 = c_1(N_C) \). By adjunction formula

\[
\deg c_1(N_C) = 7\deg \phi_L(C) + \deg K_C = 8 \cdot 14.
\]

It follows that

\[
\deg c_1(H)^6 = 64 - 6 \cdot 28 + 8 \cdot 14 = 8,
\]

which proves (2.17). \( \square \)

Proposition 5 and Lemma 11 describe precisely the splitting of the Koszul divisor \( \text{Kosz} \) into the divisors \( \text{Kosz}_6 \) and \( \text{Kosz}_7 \) corresponding to paracanonical curves \( [C, L] \in P_{8}^{14} \) having a non-zero syzygy \( \gamma \in K_{1,2}(C, L) \) of rank 6 or respectively 7. Precisely, \( \text{Kosz}_6 \) is a unirational divisor (cf. [3] Theorem 5.3) consisting of those paracanonical curves \( C \subseteq \mathbb{P}^6 \) for which \( H^1(\mathbb{P}^6, T_C^2(4)) \neq 0 \). The divisor \( \text{Kosz}_7 \) consists of paracanonical curves \( C \subseteq \mathbb{P}^6 \) which are not scheme-theoretically cut out by quadrics.
3. First proof: reducible spin curves

3.A. The syzygy is degenerate

The first observation is the following result (already observed experimentally in [3]), which turns out to be useful for the description given below of the general paracanonical curve of genus 8 with nontrivial syzygies.

**Proposition 13.** Let $C \subseteq P^6$ be a smooth paracanonical curve of genus 8 and degree 14, scheme-theoretically generated by quadrics. Then a nontrivial syzygy

$$\gamma \in \text{Ker} \{ I_C(2) \otimes H^0(\mathcal{O}_{P^6}(1)) \to I_C(3) \}$$

must be degenerate, that is of rank at most 6.

**Proof.** We use the morphism

$$\psi : X \to P(I_C(2))$$

introduced in (2.16), where $\tau : X \to P^6$ is the blow-up of $C$ with exceptional divisor $E$, and $H := \tau^* \mathcal{O}_{P^6}(-2E)$. This gives us a morphism

$$(\tau, \psi) : X \to P^6 \times P^6$$

which is of degree 1 on its image, and the syzygy $\gamma$ induces a hypersurface $Y$ of bidegree $(1, 1)$ in $P^6 \times P^6$ containing the 6-dimensional variety $(\tau, \psi)(X)$. Assume to the contrary that $\gamma$ has maximal rank 7, or equivalently that $Y$ is smooth. Then by the Lefschetz Hyperplane Restriction Theorem, the restriction map $H^{10}(P^6 \times P^6, \mathbb{Z}) \to H^{10}(Y, \mathbb{Z})$ is surjective, so that $[(\tau, \psi)(X)]_Y \in H^{10}(Y, \mathbb{Z})$ is the restriction of a class $\beta \in H^{10}(P^6 \times P^6, \mathbb{Z})$, which implies that

$$[(\tau, \psi)(X)] = \beta \cdot [Y] \quad \text{in} \quad H^{12}(P^6 \times P^6, \mathbb{Z}),$$

(3.18)

where $[Y] \in H^2(P^6 \times P^6, \mathbb{Z})$ is the class of $Y$, that is $h_1 + h_2$, with $h_i$ for $i = 1, 2$ being the pull-backs of the hyperplane classes on each factor. Note that $H^{12}(P^6 \times P^6, \mathbb{Z})$ is the set of degree 6 homogeneous polynomials with integral coefficients in $h_1$ and $h_2$. We now have:

**Lemma 14.** An element $\alpha \in H^{12}(P^6 \times P^6, \mathbb{Z})$ is of the form $(h_1 + h_2) \cdot \beta$ if and only if it satisfies the condition

$$\sum_{i=0}^{6} (-1)^i h_1^i \cdot h_2^{6-i} \cdot \alpha = 0 \quad \text{in} \quad H^{24}(P^6 \times P^6, \mathbb{Z}) = \mathbb{Z}.$$  

(3.19)

**Proof of Lemma 14.** We have $(h_1 + h_2) \cdot (\sum_i (-1)^i h_1^i \cdot h_2^{6-i}) = 0$ in $H^{14}(P^6 \times P^6, \mathbb{Z})$, so one implication is obvious. That the two conditions are equivalent then follows from the fact that both conditions determine a saturated corank 1 sublattice of $H^{12}(P^6 \times P^6, \mathbb{Z})$. \hfill \Box

To conclude that $\gamma$ has to be degenerate, in view of Lemma 14, it suffices to prove that the class $[(\tau, \psi)(X)]$ does not satisfy (3.19). Since $(\tau, \psi)^* h_1 = c_1(H)$ and $(\tau, \psi)^* h_2 = 2c_1(H) - E$, it is enough to prove that

$$\sum_{i=0}^{6} (-1)^i c_1(H)^i \cdot (2c_1(H) - E)^{6-i} \neq 0,$$

which follows from the computations made in the proof of Sublemma 12. \hfill \Box
3.B. Syzygies and spin curves of genus 22 in \( \mathbb{P}^6 \)

Recall that \( \mathcal{S}_g \) denotes the moduli stack of odd stable spin curves of genus \( g \), see [5] for details. We start with a nodal genus 22 spin curve of the form \([D := C \cup E, \vartheta] \in \mathcal{S}_{22}^\varnothing\), where \( C \) is a smooth genus 8 curve, \( E \) is a smooth elliptic curve and \( e := C \cap E \) consists of 14 distinct points, thus \( p_a(D) = 22 \). Assume \( \vartheta \in \text{Pic}^{21}(D) \) verifies \( \vartheta^\oplus2 \cong \omega_D \), hence the restricted line bundles \( \vartheta_E \) and \( \vartheta_C \) have degrees 7 and 14 respectively. Furthermore, \( h^0(E, \vartheta_E) = 7 \), whereas \( h^0(C, \vartheta_C) = 7 \) if and only if \( \vartheta_C \not\cong K_C \).

The intersection divisor \( e \) on the two components of \( D \) is characterized by

\[
e_C \in |\vartheta_C^\oplus2 \otimes K_C| \quad \text{and} \quad e_E \in |\vartheta_E^\oplus2|.
\]

Note in particular that \( e_C \in |K_C| \) if and only if \( \vartheta_C^\oplus2 = K_C^\oplus2 \), that is \((C, \vartheta_C)\) is canonical or Prym canonical.

The line bundle \( \vartheta \) on \( D \) fits into the Mayer-Vietoris exact sequence:

\[
0 \rightarrow \vartheta \rightarrow \vartheta_C \oplus \vartheta_E \overset{r}{\rightarrow} \mathcal{O}_e(\vartheta) \rightarrow 0,
\]

where \( r \) is defined by the isomorphisms on the fibers of \( \vartheta_C \) and \( \vartheta_E \) over the points in \( e \). Given \( \vartheta_C \in \text{Pic}^{14}(C) \) with \( \vartheta_C^\oplus2 = K_C(e) \) and \( \vartheta_E \in \text{Pic}^7(E) \) with \( \vartheta_E^\oplus2 = \mathcal{O}_E(e) \), there is a finite number of stable spin curves \([D, \vartheta] \in \mathcal{S}_{22}^\varnothing\) such that the restrictions of \( \vartheta \) to \( C \) and \( E \) are isomorphic to \( \vartheta_C \) and \( \vartheta_E \) respectively. Passing to global sections in the Mayer-Vietoris sequence, we obtain the exact sequence:

\[
0 \rightarrow h^0(D, \vartheta) \rightarrow h^0(C, \vartheta_C) \oplus h^0(E, \vartheta_E) \overset{r}{\rightarrow} h^0(\mathcal{O}_e(\vartheta)) \rightarrow \cdots. \tag{3.20}
\]

Note that \( r \) is represented by a \( 14 \times 14 \) matrix and \( h^0(D, \vartheta) = 14 - \text{rk}(r) \). In the case of a reducible spin curve coming from the syzygy of a paracanonical genus 8 curve in \( \mathfrak{R}_{10}^{11,8} \), one has \( h^0(D, \vartheta) = \text{rk}(r) = 7 \).

3.C. Proof of Theorem 1 via reducible spin curves

Theorem 1 states that every Prym canonical curve of genus 8 has a syzygy of rank 6. First we observe the existence of such a curve having the generic behavior described in Lemma 3.

**Lemma 15.** There exists a curve \([C, \eta] \in \mathcal{R}_8\), whose Prym canonical model is scheme theoretically cut out by quadrics, and \( \mathcal{K}_{2,1}(C, K_C \otimes \eta) \) is 1-dimensional, generated by a syzygy \( \gamma \) of rank 6. The syzygy scheme of \( \gamma \) is the union of a point \( p \) and a nodal curve \( D = C \cup E \), such that \( E \) is a smooth elliptic curve of degree 7 and \( e := C \cdot E \in |K_C| \) consists of 14 mutually distinct points. Moreover, no cubic polynomial on \( \mathbb{P}^6 \) vanishes with multiplicity 2 along \( C \).

**Proof.** Examples of singular Prym canonical curves having all these properties have been produced in [3] Proposition 4.4 or [4]. A generic deformation in \( \mathcal{R}_8 \) of these singular examples will provide the required smooth Prym canonical curve. \( \square \)

**(First) proof of Theorem 1.** We denote by \( X \) the moduli space of elements \([C, \eta, x_1, \ldots, x_{14}]\), where \([C, \eta] \in \mathcal{R}_8\) is a smooth Prym curve of genus 8 and \( x_i \in C \) are pairwise distinct points such that \( x_1 + \cdots + x_{14} \in |K_C| \cong \mathbb{P}^7 \). Since the fibres of the forgetful map \( X \rightarrow \mathcal{R}_8 \) are 7-dimensional, it follows that \( X \) is an irreducible variety of dimension 28.

Let \( T \) be the locally closed parameter space of odd genus 22 spin curves having the form

\[
\left( [D := C \cup \{x_1, \ldots, x_{14}\}, E, \vartheta] : [C] \in \mathcal{M}_8, \sum_{i=1}^{14} x_i \in |K_C|, [E, x_1, \ldots, x_{14}] \in \mathcal{M}_{1,14}, \vartheta^\oplus2 = \omega_D \right).
\]
Observe that points in $T$, apart from the spin structure $[D, \vartheta] \in \mathfrak{S}_{22}$ also carry an underlying Prym structure $[C, \eta := K_C \otimes \vartheta_C^\vee] \in \mathcal{R}_8$, for $\vartheta_C \cong K_C\{x_1 + \cdots + x_{14}\} \cong K_C^{\otimes 2}$. One has an induced finite morphism $T \to X \times \mathcal{M}_{1,14}$, as well as a map $\mu : T \to \mathcal{R}_8$ forgetting the 14-pointed elliptic curve. It follows that $\dim T = \dim X + \dim \mathcal{M}_{1,14} = 42$. The locus

$$T_7 := \{[D, \vartheta] \in T : h^0(D, \vartheta) \geq 7\}$$

has the structure of a skew-symmetric degeneracy locus. Applying [13] Theorem 1.10, each component of $T_7$ has codimension at most $\binom{14}{2} = 21$ inside $T$, that is, $\dim(T_7) \geq \dim(\mathcal{R}_8)$.

By passing to a general 8-nodal Prym canonical curve $[C, \eta]$, following [3] Proposition 4.5, as well as Lemma 15, we have that $\dim K_{1,2}(C, K_C \otimes \eta) = 1$. In particular, the fibre $\mu^{-1}([C, \eta])$ contains an isolated point, which shows that $T_7$ is non-empty and has a component which maps dominantly under $\mu$ onto $\mathcal{R}_8$. Theorem 1 now follows.

**Remark 16.** The same construction can be carried out at the level of general paracanonical curves $[C, L] \in P_8^{14}$, where $L \in \text{Pic}^{14}(C) \setminus \{K_C\}$. The key difference is that we replace $T$ by a variety $T'$ parametrizing objects

$$\left([D := C \cup\{x_1, \ldots, x_{14}\}, E, \vartheta, L] : [C, x_1, \ldots, x_{14}] \in \mathcal{M}_{14,8}, L \in \text{Pic}^{14}(C) \setminus \{K_C\}, \sum_{i=1}^{14} x_i \in |L^{\otimes 2} \otimes K_C^\vee|, [E, x_1, \ldots, x_{14}] \in \mathcal{M}_{14,14}, \vartheta^{\otimes 2} = \omega_D\right).$$

Similarly, we have a morphism $\mu' : T' \to P_8^{14}$ retaining the pair $[C, L]$ alone. The main difference compared to the Prym canonical case is that now

$$\dim |L^{\otimes 2} \otimes K_C^\vee| = 6,$$

therefore $\dim(T') = \dim(P_8^{14}) + \dim(\mathcal{M}_{1,14}) + 6 = 49$. The degeneracy locus $T'_7 \subseteq T'$ defined by the condition $h^0(D, \vartheta) \geq 7$ has codimension 21 inside $T'$, that is,

$$\dim(T'_7) = 28 = \dim(P_8^{14}) - 1.$$

It follows that the image $\mu'(T'_7) \subseteq P_8^{14}$ has codimension 1, which is in accordance with $\mathfrak{R}_{\mathfrak{M} \mathfrak{S}_{16}}$ being a divisor in $P_8^{14}$.

## 4. Second proof: Divisor class calculations on $\overline{\mathcal{R}}_g$

Recall [10] that $\overline{\mathcal{R}}_g$ is the Deligne-Mumford moduli space of Prym curves of genus $g$, whose geometric points are triples $[X, \eta, \beta]$, where $X$ is a quasi-stable curve of genus $g$, $\eta \in \text{Pic}(X)$ is a line bundle of total degree 0 such that $\eta_E = \mathcal{O}_E(1)$ for each smooth rational component $E \subseteq X$ with $|E \cap X - E| = 2$ (such a component is said to be *exceptional*), and $\beta : \eta^{\otimes 2} \to \mathcal{O}_X$ is a sheaf homomorphism whose restriction to any non-exceptional component is an isomorphism. If $\pi : \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$ is the map dropping the Prym structure, one has the formula

$$\pi^*(\delta_0) = \delta_0 + \delta_0^{\vee} + 2\delta_0^{\text{ram}} \in CH^1(\overline{\mathcal{R}}_g), \quad (4.21)$$

where $\delta_0 := [\Delta_0]$, $\delta_0^{\vee} := [\Delta_0^{\vee}]$, and $\delta_0^{\text{ram}} := [\Delta_0^{\text{ram}}]$ are irreducible boundary divisor classes on $\overline{\mathcal{R}}_g$, which we describe by specifying their respective general points.

We choose a general point $[C_{xy}] \in \Delta_0 \subseteq \overline{\mathcal{M}}_g$ corresponding to a smooth 2-pointed curve $(C, x, y)$ of genus $g - 1$ and consider the normalization map $\nu : C \to C_{xy}$, where $\nu(x) = \nu(y)$. A general point
of $\Delta'_0$ (respectively of $\Delta''_0$) corresponds to a pair $[C_{xy}, \eta]$, where $\eta \in \text{Pic}^0(C_{xy})[2]$ and $\nu^*(\eta) \in \text{Pic}^0(C)$ is non-trivial (respectively, $\nu^*(\eta) = \mathcal{O}_C$). A general point of $\Delta''_0$ is a Prym curve of the form $(X, \eta)$, where $X := C \cup \{x, y\} \mathbb{P}^1$ is a quasi-stable curve with $p_a(X) = g$ and $\eta \in \text{Pic}^0(X)$ is a line bundle such that $\eta_{P_1} = \mathcal{O}_{P_1}(1)$ and $\eta_{P_2} = \mathcal{O}_C(-x - y)$. In this case, the choice of the homomorphism $\beta$ is uniquely determined by $X$ and $\eta$. In what follows, we work on the partial compactification $\tilde{R}_g \subseteq \mathcal{R}_g$ of $\mathcal{R}_g$ obtained by removing the boundary components $\pi^{-1}(\Delta_j)$ for $j = 1, \ldots, \lfloor \frac{g}{2} \rfloor$, as well as $\Delta''_0$. In particular, $CH^1(\tilde{R}_g) = \mathbb{Q}(\lambda, \delta'_0, \delta''_0, \delta''''_0)$.

For a stable Prym curve $[X, \eta] \in \tilde{R}_g$, set $L := \omega_X \otimes \eta \in \text{Pic}^{2g-2}(X)$ to be the paracanonical bundle. For $i \geq 1$, we introduce the vector bundle $\mathcal{N}_k$ over $\tilde{R}_g$, having fibres

$$\mathcal{N}_k[X, \eta] = H^0(X, L^\otimes k).$$

The first Chern class of $\mathcal{N}_k$ is computed in [10] Proposition 1.7:

$$c_1(\mathcal{N}_k) = \left(\frac{k}{2}\right)(12\lambda - \delta'_0 - 2\delta''_0) + \lambda - \frac{k^2}{4}\delta''''_0. \tag{4.22}$$

Then we define the locally free sheaves $\mathcal{G}_k$ on $\tilde{R}_g$ via the exact sequences

$$0 \rightarrow \mathcal{G}_k \rightarrow \text{Sym}^k \mathcal{N}_1 \rightarrow \mathcal{N}_k \rightarrow 0,$$

that is, satisfying $\mathcal{G}_k[X, \eta] := I_{X, L}(k) \subseteq \text{Sym}^k H^0(X, L)$. Using (4.22) one computes $c_1(\mathcal{G}_k)$. We also need the class of the vector bundle $\mathcal{G}$ with fibres

$$\mathcal{G}[X, \eta] = H^0(X, \omega_X^5 \otimes \eta^\otimes 4) = H^0(X, \omega_X \otimes L^\otimes 4).$$

**Lemma 17.** One has $c_1(\mathcal{G}) = 121\lambda - 10\delta'_0 - 24\delta''_0 \in CH^1(\tilde{R}_g)$.

**Proof.** We apply Grothendieck-Riemann-Roch to the universal Prym curve $f : \mathcal{C} \rightarrow \tilde{R}_g$. Denote by $L \in \text{Pic}(\mathcal{C})$ the universal Prym bundle, whose restriction to each Prym curve is the corresponding 2-torsion point, that is, $L|_{f^{-1}([X, \eta])} = \eta$, for each point $[X, \eta] \in \tilde{R}_g$. Since $R^1f_*((\omega_f^5 \otimes L^\otimes 4)) = 0$, we write

$$c_1(\mathcal{G}) = f_*\left(\left(1 + 5c_1(\omega_f) + 4c_1(L) + \frac{(5c_1(\omega_f) + 4c_1(L))^2}{2}\right) \cdot \left(1 - \frac{c_1(\omega_f)}{2} + \frac{c_1^2(\Omega_f^1) + [\text{Sing}(f)]}{12}\right)\right).$$

We use then the formulas $f_*([c_1^2(L)]) = -\delta'''_0/2$ and $f_*([c_1(L) \cdot c_1(\omega_f)]) = 0$ (see [10], Proposition 1.6) coupled with Mumford’s formula $f_*([c_1^2(\Omega_f^1) + [\text{Sing}(f)]) = 12\lambda$ as well with the identity

$$\kappa_1 := f_*([c_1^2(\omega_f)]) = 12\lambda - \delta'_0 - 2\delta''_0,$$

in order to conclude. \qed

The Koszul locus

$$Z_8 := \mathfrak{K}_{\mathcal{G}} \cap \mathcal{R}_8 = \left\{[C, \eta] \in \mathcal{R}_8 : K_{1,2}(C, K_C \otimes \eta) \neq 0\right\}$$

is a virtual divisor on $\mathcal{R}_8$, that is, the degeneracy locus of a map between vector bundles of the same rank over $\mathcal{R}_8$. If it is a genuine divisor (which we aim to rule out), the class of its closure in $\mathcal{R}_8$ is given by [3] Theorem F:

$$[Z_8] = 27\lambda - 4\delta'''_0 - 6\delta''''_0 \in CH^1(\tilde{R}_8).$$
Remark 18. Some of the considerations above can be extended to higher order torsion points. We recall that $\mathcal{R}_{g,\ell}$ is the moduli space of pairs $[C, \eta]$, where $C$ is a smooth curve of genus $g$ and $\eta \in \text{Pic}^0(C)$ is a non-trivial $\ell$-torsion point. It is then shown in [3] that the locus $\mathcal{Z}_{8,\ell} := \text{RQ}_{37} \cap \mathcal{R}_{8,\ell} \subseteq P^{14}_8$ is a divisor on $\mathcal{R}_{8,\ell}$ for each other level $\ell \geq 3$. The class of the compactification of $\mathcal{Z}_{8,\ell}$ is given by the following formula, see [3] Theorem F:

$$[\mathcal{Z}_{8,\ell}] = 27\lambda - 4\delta_0' - \sum_{a=1}^{\lfloor \frac{\ell}{2} \rfloor} 4(a^2 - a\ell + \ell^2)\delta_0^{(a)} \in C\mathcal{H}^1(\widetilde{\mathcal{R}}_{8,\ell}).$$

We refer to [3] Section 1.4, for the definition of the boundary divisor classes $\delta_0^{(a)}$, where $a = 1, \ldots, \lfloor \frac{\ell}{2} \rfloor$. If $\pi : \mathcal{R}_{g,\ell} \to \mathcal{M}_g$ is the map forgetting the level $\ell$ structure, then

$$\pi^*(\delta_0) = \delta_0' + \delta_0'' + \ell \sum_{\ell=1}^{\lfloor \frac{\ell}{2} \rfloor} \delta_0^{(a)}.$$

We fix now a genus 8 Prym-canonically embedded curve $\phi_L : C \hookrightarrow \mathbb{P}^6$. As usual, we denote the kernel bundle by $M_L := \Omega^1_{\mathbb{P}^1} \otimes (1)$, hence we have the exact sequence

$$0 \longrightarrow N_C^0 \otimes \mathbb{L}^4 \longrightarrow M_L \otimes \mathbb{L}^3 \longrightarrow K_C \otimes \mathbb{L}^4 \longrightarrow 0. \quad (4.23)$$

This can be interpreted as an exact sequence of vector bundles over $\mathcal{R}_8$. Denoting by $\mathcal{H}$ the vector bundle over $\mathcal{R}_8$ with fibres $H^0(C, N_C^0 \otimes \mathbb{L}^4)$, we compute using the previous formulas and the fact that $\text{rk}(N_1) = h^0(C, L) = 7$ and $\text{rk}(N_3) = h^0(C, L^3) = 35$:

$$c_1(\mathcal{H}) = 35c_1(N_1) + 7c_1(N_3) - c_1(N_4) - c_1(G) = 100\lambda - 5\delta_0' - \frac{53}{2} \delta_0^{\text{ram}}. \quad (4.24)$$

Thus $\mathcal{D}_1 = \text{RQ}_{37} \cap \mathcal{R}_8$ and $\mathcal{D}_2 = \text{RQ}_{36} \cap \mathcal{R}_8$. We have already seen in Proposition 5 that $K_{1,2}(C, L) \neq 0$ if and only if either $\phi_L(C) \subseteq \mathbb{P}^6$ is not scheme-theoretically cut out by quadrics, or else, $H^1(\mathbb{P}^6, \mathcal{I}_C^2(4)) \neq 0$. We write

$$\mathcal{Z}_8 = \mathcal{D}_1 + \mathcal{D}_2,$$

where

$$\mathcal{D}_1 \doteq \{ [C, \eta] \in \mathcal{R}_8 : \phi_L(C) \subseteq \mathbb{P}^6 \text{ isscheme-theoretically not cut out by quadrics} \}$$

and

$$\mathcal{D}_2 \doteq \{ [C, \eta] \in \mathcal{R}_8 : H^1(\mathbb{P}^6, \mathcal{I}_C^2(4)) \neq 0 \}.$$

We have already observed that $\text{dim } I_{C,L}(2) = 7$ and $\chi(\mathbb{P}^6, \mathcal{I}_C^2(4)) = 28$. If $\mathcal{Z}_8$ is a divisor, then $\mathcal{D}_2$ is a divisor as well and for $[C, \eta] \in \mathcal{R}_8 \setminus \mathcal{D}_2$, we have that

$$\text{dim Sym}^2 I_{C,L}(2) = \text{dim } I_{C,L}(4)_2 = 28.$$ 

Paying some attention to its definition, the divisor $\mathcal{D}_1$ can be thought as the degeneracy locus

$$\left\{ [C, \eta] \in \mathcal{R}_8 : \text{Sym}^2 I_{C,L}(2) \not\hookrightarrow I_{C,L}(4)_2 \right\},$$

which is an effective divisor on $\mathcal{R}_8$. We compute the class of this divisor:

**Theorem 19.** We have the following formulas:

$$[\mathcal{D}_1] = 7\lambda - \frac{1}{2} \delta_0' - \frac{3}{4} \delta_0^{\text{ram}} \in C\mathcal{H}^1(\widetilde{\mathcal{R}}_8)$$

and

$$[\mathcal{D}_2] = 20\lambda - \frac{7}{2} \delta_0' - \frac{21}{4} \delta_0^{\text{ram}} \in C\mathcal{H}^1(\widetilde{\mathcal{R}}_8).$$
Proof. We first globalize over \( \widehat{\mathcal{R}}_8 \) the following exact sequence:

\[
0 \longrightarrow I_{C,L}(4)_2 \longrightarrow I_{C,L}(4) \longrightarrow H^0(C, N^\vee_C \otimes L^{\otimes 4}) \longrightarrow H^1(\mathbb{P}^6, \widetilde{I}^2(4)) \longrightarrow 0.
\]

Denote by \( \mathcal{A} \) the sheaf on \( \widehat{\mathcal{R}}_8 \) supported along the divisor \( \mathcal{D}_2 \), whose fibre over a general point of that divisor is equal to to \( H^1(\mathbb{P}^6, \widetilde{I}^2(4)) \). There is a surjective morphism of sheaves

\[
\mathcal{H} \twoheadrightarrow \mathcal{A}
\]

and denote by \( \mathcal{G}'_4 \) its kernel. Since \( \mathcal{A} \) is locally free along \( \mathcal{D}_2 \) and \( \widehat{\mathcal{R}}_8 \) is a smooth stack, using the Auslander-Buchsbaum formula we find that \( \mathcal{G}'_4 \) is a locally free sheaf of rank equal to \( \text{rk}(H) = \chi(C, N^\vee_C(4L)) = 19 \cdot 7 \). Precisely, \( \mathcal{G}'_4 \) is an elementary transformation of \( \mathcal{H} \) along the divisor \( \mathcal{D}_2 \). Furthermore, \( c_1(\mathcal{G}'_4) = c_1(\mathcal{H}) - [\mathcal{D}_2] \).

The morphism \( \mathcal{G}_4 \to \mathcal{H} \) globalizing the maps \( I_{C,L}(4) \to H^0(C, N^\vee_C \otimes L^{\otimes 4}) \) factors through the subsheaf \( \mathcal{G}'_4 \) and we form the exact sequence:

\[
0 \longrightarrow \mathcal{G}_4^2 \longrightarrow \mathcal{G}_4 \longrightarrow \mathcal{G}'_4 \longrightarrow 0.
\]

The multiplication maps \( \text{Sym}^2 I_{C,L}(2) \to I_{C,L}(4)_2 \) globalize to a sheaf morphism

\[
\nu : \text{Sym}^2(\mathcal{G}_2) \to \mathcal{G}'_4
\]

between locally free sheaves of the same rank 28 over the stack \( \widehat{\mathcal{R}}_8 \). The degeneration locus of \( \nu \) is precisely the divisor \( \mathcal{D}_1 \). We compute:

\[
c_1(\text{Sym}^2(\mathcal{G}_2)) = 8c_1(\mathcal{G}_2) = 8(8c_1(N_1) - c_1(N_2)) = -40\lambda + 8(\delta'_0 + \delta^\text{ram}_0),
\]

and

\[
c_1(\mathcal{G}'_4^2) = 120c_1(N_1) - c_1(N_4) - c_1(\mathcal{H}) + [\mathcal{D}_2] = -53\lambda + 11\delta'_0 + \frac{25}{2}\delta^\text{ram}_0 + [\mathcal{D}_2].
\]

We obtain the relation \( [\mathcal{D}_1] - [\mathcal{D}_2] = -13\lambda + 3\delta'_0 + \frac{25}{2}\delta^\text{ram}_0 \). Since at the same time

\[
[\mathcal{D}_1] + [\mathcal{D}_2] = [\mathcal{Z}_8] = 27\lambda - 4\delta'_0 - 6\delta^\text{ram}_0,
\]

we solve the system and conclude. \( \square \)

We are now in a position to give a second proof of Theorem 1:

**Theorem 20.** The class \( \mathcal{D}_2 \) cannot be effective. It follows that \( \mathcal{Z}_8 = \mathcal{R}_8 \) and \( K_{1,2}(C, K_C \otimes \eta) \neq 0 \), for every Prym curve \( [C, \eta] \in \mathcal{R}_8 \).

Proof. We use the sweeping curve of the boundary divisor \( \Delta'_0 \) of \( \widehat{\mathcal{R}}_8 \) constructed via Nikulin surfaces in [11] Lemma 3.2: Precisely, through the general point of \( \Delta'_0 \) there passes a rational curve \( \Gamma \subseteq \Delta'_0 \), entirely contained in \( \mathcal{R}_8 \), having the following numerical characters:

\[
\Gamma \cdot \lambda = 8, \quad \Gamma \cdot \delta'_0 = 42, \quad \text{and} \quad \Gamma \cdot \delta^\text{ram}_0 = 8.
\]

We note that \( \Gamma \cdot \mathcal{D}_2 < 0 \). Writing \( \mathcal{D}_2 = \alpha \cdot \delta'_0 + E \), where \( \alpha \geq 0 \) and \( E \) is an effective divisor whose support is disjoint from \( \Delta'_0 \), we immediately obtain a contradiction. \( \square \)
The divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ can be defined in an identical manner at the level of each moduli space $\mathcal{R}_{g,\ell}$ of twisted level $\ell$ curves of genus $g$. As already pointed out, in the case $\ell \geq 3$ it follows from [3] Proposition 4.4 that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are actual divisors. Repeating the same calculations as for $\ell = 2$, we obtain the following formula on the partial compactification $\mathcal{R}_{g,\ell}$ of $\mathcal{R}_{g,\ell}$:

$$[\mathcal{D}_2] = 20\lambda - \frac{7}{2} \delta_0' - \sum_{a=1}^{\lfloor \frac{4}{\ell} \rfloor} \frac{1}{2\ell}(7\lambda^2 - 7\lambda \ell + 17\ell^2 - 20\ell)\delta_0^{(a)} \in CH^1(\mathcal{R}_{g,\ell}).$$

(4.25)

As an application, we mention a different proof of one of the main results from [1]:

**Theorem 21.** The canonical class of $\mathcal{R}_{g,\ell}$ is big for $\ell \geq 3$. It follows that $\mathcal{R}_{g,\ell}$ is a variety of general type for $\ell = 3, 4, 6$.

**Proof.** Using formula (4.25), it is a routine exercise to check that for $\ell \geq 3$ the canonical class computed in [3] Proposition 1.5

$$K_{\mathcal{R}_{g,\ell}} = 13\lambda - 2\delta_0' - (\ell + 1)\sum_{a=1}^{\lfloor \frac{4}{\ell} \rfloor} \delta_0^{(a)}$$

can be written as a positive combination of the big class $\lambda$ and the effective class $[\mathcal{D}_2]$, hence it is big. Arguing along the lines of [3] Remark 3.5, it is easy to extend this result to the full compactification $\mathcal{R}_{g,\ell}$ and deduce that $K_{\mathcal{R}_{g,\ell}}$ is big.

To conclude that $\mathcal{R}_{g,\ell}$ is of general type, one needs, apart from the bigness of the canonical class $K_{\mathcal{R}_{g,\ell}}$ of the moduli stack, a result that the singularities of the coarse moduli space $\mathcal{R}_{g,\ell}$ impose no adjunction conditions. This is only known for $2 \leq \ell \leq 6$, $\ell \neq 5$, see [2]. □

5. **Rank 2 vector bundles and singular quartics**

Our goal in this section is to propose a construction of syzygies of Prym canonical curves of genus 8. We also sketch the proof of the fact that these syzygies are nontrivial. We fix again a general element $[C, \eta] \in \mathcal{R}_8$ and set $L := K_C \otimes \eta$. According to Proposition 5, in order to prove that $K_{2,1}(C, L) \neq 0$, we have to produce quartic hypersurfaces in $\mathbb{P}^5$ which vanish at order at least 2 along $\phi_L(C)$, but do not lie in the image of the map $\text{Sym}^2 I_{C,L}(2) \to I_{C,L}(4)$. The goal of this section is to produce such quartics from rank 2 vector bundles on $C$. The (incomplete) proof that the quartics we construct are not in the image of $\text{Sym}^2 I_{C,L}(2)$ depends on an unproved general position statement (*), but there might be other approaches exploiting the fact that the hypersurfaces in question are determinantal.

The following construction produces quartics vanishing at order 2 along $C$. Let $E$ be a rank 2 vector bundle on $C$, with determinant $K_C$. Assume

$$h^0(C, E) = 4, \quad h^0(C, E(\eta)) = 4.$$

(5.26)

Setting $V_0 := H^0(C, E)$ and $V_1 := H^0(C, E(\eta))$, we have a natural map

$$V_0 \otimes V_1 \to H^0(C, L),$$

defined using evaluation and the following composite map:

$$H^0(E) \otimes H^0(E(\eta)) \to H^0(E \otimes E(\eta)) \cong H^0(\mathcal{E}_{\text{end}} E \otimes L) \xrightarrow{\text{Tr}} H^0(C, L).$$

(5.27)

This map gives dually a morphism

$$H^0(C, L)^\vee \to V_0^\vee \otimes V_1^\vee,$$

(which will be proved below to be injective for a general choice of $E$). We consider the quartic hypersurface $D_4$ on $\mathbb{P}(V_0^\vee \otimes V_1^\vee)$ parametrizing tensors of rank at most 3.
Lemma 22. The restriction $D_{4,E}$ of this quartic to $P(H^0(C,L)^\vee) \subseteq P(V_0^\vee \otimes V_1^\vee)$ is singular along the curve $C$.

Proof. The quartic $D_4$ is singular along the set $T_2 \subseteq P(V_0^\vee \otimes V_1^\vee)$ of tensors of rank at most 2. The quartic $D_{4,E}$ in $P(H^0(C,L)^\vee)$ is thus singular along $T_2 \cap P(H^0(C,L)^\vee)$, which obviously contains $C \subseteq P(H^0(C,L)^\vee)$, since at a point $p \in C$, the map $V_0 \otimes V_1 \to H^0(C,L)$ composed with the evaluation at $p$ factors through $E_p \otimes E(\eta)|_p$. □

By Brill-Noether theory, the variety $W_4^1(C)$ of degree 7 pencils on $C$ is 4-dimensional. There should thus exist finitely many elements $D \in W_4^1(C)$ with the property that

$$h^0(C,D) \geq 2, \quad h^0(C,D \otimes \eta) \geq 2. \quad (5.28)$$

We now have the following lemma:

Lemma 23. Let $[C,\eta] \in R_8$ be as above and $D \in W_4^1(C)$ satisfying (5.28). Then

(i) $h^0(C,D) = 2$ and $h^0(C,D \otimes \eta) = 2$. The multiplication map

$$\left( H^0(C,D) \otimes H^0(C,K_C \otimes D^\vee) \right) \oplus \left( H^0(C,D \otimes \eta) \otimes H^0(C,K_C \otimes D^\vee \otimes \eta) \right) \to H^0(C,K_C)$$

is surjective (in fact, an isomorphism).

(ii) The multiplication map

$$\left( H^0(C,D) \otimes H^0(C,K_C \otimes D^\vee \otimes \eta) \right) \oplus \left( H^0(C,D \otimes \eta) \otimes H^0(C,K_C \otimes D^\vee) \right) \to H^0(C,K_C(\eta))$$

is surjective.

Proof. This can be proved by a degeneration argument, for example by degenerating $C$ to the union of two curves of genus 4 meeting at one point. □

By Brill-Noether theory, the following corollary follows from (i) above:

Corollary 24. For $[C,\eta]$ as above, the set of pencils $D \in W_4^1(C)$ satisfying (5.28) is finite.

Given such a $D$, we form the rank 2 vector bundle

$$E = D \oplus (K_C \otimes D^\vee)$$
on $C$ which satisfies the conditions (5.26). The associated quartic is however not interesting for our purpose, due to the following fact:

Lemma 25. The quartic on $P(H^0(C,L)^\vee)$ associated to the vector bundle $D \oplus (K_C \otimes D^\vee)$ is the union of the two quadrics $Q_0$ and $Q_1$ associated respectively with the multiplication maps

$$H^0(D) \otimes H^0((K_C \otimes D^\vee)(\eta)) \to H^0(K_C(\eta)) \quad \text{and} \quad H^0(D(\eta)) \otimes H^0(K_C \otimes D^\vee) \to H^0(K_C(\eta)).$$

Both these quadrics contain $C$.

Proof. Indeed we have in this case

$$V_0 = H^0(C,E) = H^0(C,D) \oplus H^0(C,K_C \otimes D^\vee), \quad \text{respectively}$$

$$V_1 = H^0(C,E(\eta)) = H^0(C,D \otimes \eta) \oplus H^0(C,K_C \otimes D^\vee \otimes \eta).$$
Furthermore, it is clear that the map of (5.27) factors through the projection

$$V_0 \otimes V_1 \rightarrow \left( H^0(C, D) \otimes H^0(C, K_C \otimes D^\vee \otimes \eta) \right) \oplus \left( H^0(C, K_C \otimes D^\vee) \otimes H^0(C, D \otimes \eta) \right)$$

and induces on each summand the multiplication map. The quadric $Q_0$ is by definition associated with the multiplication map

$$\mu_0 : H^0(C, D) \otimes H^0(C, K_C \otimes D^\vee \otimes \eta) \rightarrow H^0(C, K_C \otimes \eta),$$

and is the set of elements $f$ in $\mathbf{P}(H^0(K_C \otimes \eta))^\vee$ such that $\mu_0(f)$ is a tensor of rank $\leq 1$. Similarly for $Q_1$, with $D$ being replaced with $D(\eta)$. Finally we use the fact that a tensor

$$(\mu_0^* f, \mu_1^* f) \in \left( H^0(C, D) \otimes H^0(C, K_C \otimes D^\vee \otimes \eta) \right) \oplus \left( H^0(C, K_C \otimes D^\vee) \otimes H^0(C, D \otimes \eta) \right)$$

has rank at most 3 if and only if one of $\mu_0^* f$ and $\mu_1^* f$ has rank at most 1. \hfill $\Box$

We now sketch the proof of the fact that for $C$ general of genus $8$ and $D \in W_2^3(C)$ satisfying (5.28), for a general deformation $E$ of the vector bundle $D \oplus (K_C \otimes D^\vee)$ satisfying $\det E \cong K_C$ and $h^0(C, E) = 4$, the associated quartic $D_{4, E}$ singular along $C$ is not defined by an element of $\text{Sym}^2 I_C(2)$. Combined with Proposition 5, this provides a third approach to Theorem 1. The proof of this fact rests on an unproven general position statement ($*$), so it is incomplete.

**Sketch of proof of the nontriviality of the syzygy.** The vector bundle $E$ is generated by sections, as it is a general section-preserving deformation of the vector bundle

$$D \oplus (K_C \otimes D^\vee)$$

which is generated by global sections, and similarly for $E(\eta)$. Along $C \subseteq \mathbf{P}(H^0(C, L)^\vee)$, then the rational map

$$\mathbf{P}(H^0(C, L)^\vee) \dashrightarrow \mathbf{P}(H^0(E)^\vee \otimes H^0(E(\eta))^\vee)$$

is well-defined and the image of $C$ is contained in the locus $T_{2, E}$ of tensors of rank exactly 2. In fact, the case of $D \oplus (K_C \otimes D^\vee)$ shows that this map is a morphism for general $E$ (one just needs to know that $H^0(C, K_C \otimes \eta)$ is generated by the two vector spaces $H^0(D) \otimes H^0(K_C \otimes D^\vee \otimes \eta)$ and $H^0(D \otimes \eta) \otimes H^0(K_C \otimes D^\vee)$ respectively, or rather their images under the multiplication map. Note that on $T_{2, E}$, there is a rank 2 vector bundle $M$ which restricts to $E$ on $C$.

In the case of the split vector bundle $E_{sp} = D \oplus (K_C \otimes D^\vee)$, Lemma 25 shows that the Zariski closure $\overline{T_{2, E_{sp}}}$ parameterizing tensors of rank $\leq 2$ in $\mathbf{P}(H^0(C, L)^\vee) \subseteq \mathbf{P}(V_0^\vee \otimes V_1^\vee)$ is equal to the singular locus of $D_{4, E_{sp}}$ and consists of the union of the two planes $P_0$, $P_1$ defined as the singular loci of the quadrics $Q_0$, $Q_1$ respectively, and the intersection $Q_0 \cap Q_1$. The locus $\overline{T_{2, E_{sp}}} \setminus T_{2, E_{sp}}$ is the locus where the tensor has rank 1, and this happens exactly along the two conics $P_0 \cap Q_1$ and $P_1 \cap Q_0$. The curve $C$ is contained in $Q_0 \cap Q_1$ and does not intersect $P_0 \cup P_1$. In particular, the rational map $\phi : \overline{\mathbf{P}^5} \dashrightarrow \mathbf{P}^6$ given by the linear system $I_C(2)$ is well defined along $P_0 \cup P_1$. We believe that the following general position statement concerning the two planes $P_i$ is true for general $C$ and $D$, $\eta$ as above.
\((\ast)\) The surfaces \(\phi(P_i)\) are projectively normal Veronese surfaces, generating a hyperplane \(\langle \phi(P_i) \rangle \subseteq P^6\). Furthermore, the surface \(\phi(P_0) \cup \phi(P_1) \subseteq P^6\) is contained in a unique quadric in \(P^6\), namely the union of the two hyperplanes \(\langle \phi(P_0) \rangle\) and \(\langle \phi(P_1) \rangle\).

We now prove that, assuming \((\ast)\), for a general vector bundle \(E\) as above, the associated quartic \(D_{4,E}\) singular along \(C\) is not defined by an element of \(\text{Sym}^2 I_C(2)\). As \(P_0, P_1\) are 2-dimensional reduced components of \(\overline{T_{2,E_{\text{sp}}}}\), hence of the right dimension, the theory of determinantal hypersurfaces shows that for general \(E\) as above, there is a reduced surface \(\Sigma_E \subseteq \overline{T_{2,E}}\) whose specialization when \(E = E_{\text{sp}}\) contains \(P_0 \cup P_1\). Let \(E \rightarrow C \times B\) be a family of vector bundles on \(C\) parameterized by a smooth curve \(B\), with general fiber \(E\) and special fiber \(E_{\text{sp}}\). Denote by \(\mathcal{E}_b\) the restriction of \(E\) to \(C \times \{b\}\). Property \((\ast)\) then implies that \(\phi(\Sigma_{\mathcal{E}_b})\) for general \(b \in B\) is contained in at most one quadric \(Q_{\mathcal{E}_b}\) in \(P^6\). We argue by contradiction and assume that the quartic \(D_{4,\mathcal{E}_b}\) is a pull-back \(\phi^{-1}(Q)\) for general \(b\). One thus must have \(Q = Q_{\mathcal{E}_b}\). Next, the determinantal quartic \(D_{4,\mathcal{E}_b}\) is singular along \(T_{2,\mathcal{E}_b}\), hence along \(\Sigma_{\mathcal{E}_b}\). Let \(b \mapsto q_{\mathcal{E}_b} \in \text{Sym}^2 I_C(2)\), where \(q_{\mathcal{E}_b}\) is a defining equation for the quadric \(Q_{\mathcal{E}_b}\). Then we find that the first order derivative of the family \(\phi^* q_{\mathcal{E}_b}\) at \(b_0\) also vanishes along \(\Sigma_{\mathcal{E}_0}\), hence it must be proportional to \(\phi^* q_{\mathcal{E}_0}\). We then conclude that the quadric \(Q_{\mathcal{E}_b}\) is in fact constant, and thus must be equal to the quadric \(Q_{E_{\text{sp}}}\). We now reach a contradiction by proving the following lemma.

**Lemma 26.** If the determinantal quartic \(D_{4,\mathcal{E}_b}\) is constant, equal to \(D_{\text{sp}} = Q_0 \cup Q_1\), then the vector bundle \(\mathcal{E}_b\) on \(C\) does not deform with \(b \in B\).

**Proof.** Denoting \(V_{0,0, b} := H^0(C, \mathcal{E}_b), V_{1,0, b} := H^0(C, \mathcal{E}_b(\eta))\), we have the multiplication map

\[
V_{0,0, b} \otimes V_{1,0, b} \rightarrow H^0(C, K_C \otimes \eta)
\]

which is surjective for generic \(b\) since it is surjective for \(\mathcal{E}_0 = D \oplus (K_C \otimes D^\vee)\) (see Lemma 23). The determinantal quartic \(D_{4,\mathcal{E}_b}\) is the vanishing locus of the determinant of the corresponding bundle map

\[
\sigma_b : V_{0,0, b} \otimes \mathcal{O}_{P(H^0(C, K_C \otimes \eta)^\vee)} \rightarrow V_{1,0, b} \otimes \mathcal{O}_{P(H^0(C, K_C \otimes \eta)^\vee)^\vee}(1)
\]

on \(P(H^0(C, K_C \otimes \eta)^\vee)^\vee\). We know that \(D_{4, \mathcal{E}_b} = Q_0 \cup Q_1\) for any \(b \in B\), where the quadrics \(Q_i\) are singular (of rank 4), but with singular locus \(P_i\) not intersecting \(C \subseteq Q_0 \cap Q_1\). The morphism \(\sigma_b\) has rank exactly 1 generically along each \(Q_i\) and the kernel of \(\sigma_{D_{4,b}}\) determines a line bundle \(K_{i,b}\) on its smooth locus \(Q_i \setminus P_i\). This line bundle is independent of \(b\) since \(\text{Pic}(Q_i \setminus P_i)\) has no continuous part. The restriction of \(K_{i,b}\) to \(C\) is thus constant. Finally, on the smooth part of \((Q_0 \cap Q_1)_{\text{reg}}\), the kernel \(\text{Ker}(\sigma)\) contains the two line bundles \(K_{i,b\mid Q_0 \cap Q_1}\). Restricting to \(C \subseteq (Q_0 \cap Q_1)_{\text{reg}}\), we conclude that \(\text{Ker} \sigma_{b\mid C}\) contains \(K_{i,0\mid C}\) for \(i = 0, 1\). For \(b = 0\), one has

\[
\text{Ker} \sigma_{0\mid C} = K_{0,0\mid C} \oplus K_{1,0\mid C}
\]

and this thus remains true for general \(b\). Finally, it follows from the construction and the fact that \(\mathcal{E}_b\) is generated by its sections that \(\text{Ker} \sigma_{b\mid C} = \mathcal{E}_b^\vee\), which finishes the proof. \(\square\)

6. Miscellany

6.A. Extra remarks on the geometry of paracanonical curves of genus 8 with a nontrivial syzygy

We now comment on an interesting rank 2 vector bundle appearing in our situation. Again, let \(\phi_L : C \rightarrow P^6\) be a paracanonical curve of genus 8. We assume \(L\) is scheme-theoretically cut out by quadrics. Denoting by \(N_C\) the normal bundle of \(C\) in the embedding in \(P^6\), we consider the natural
map $I_C(2) \otimes \mathcal{O}_C \to N_C^0 \otimes L^\otimes 2$ (which is surjective by our assumption) given by differentiation along $\phi_L(C)$, and let $F$ denote its kernel. We thus have the short exact sequence:

$$0 \to F \to I_C(2) \otimes \mathcal{O}_C \to N_C^0 \otimes L^\otimes 2 \to 0. \quad (6.30)$$

If $K_{1,2}(C, L) \neq 0$, the map $\mu : I_C(2) \otimes H^0(\mathbf{P}^6, \mathcal{O}(1)) \to I_C(3)$ is not surjective, hence not injective. A fortiori, the map

$$\overline{\mu} : I_C(2) \otimes H^0(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(1)) \to H^0(C, N_C^0 \otimes L^\otimes 3)$$

induced by (6.30) is not injective, so that $h^0(C, F(L)) \neq 0$. In fact, the equivalence between the statements $h^0(C, F(L)) \neq 0$ and $K_{1,2}(C, L) \neq 0$ follows from the same argument once we know that there is no cubic polynomial on $\mathbf{P}^6$ vanishing with multiplicity 2 along $C$.

We observe now that $F$ is a vector bundle of rank 2 on the curve $C$, with determinant equal to $\det N_C \otimes L^\otimes (-2) \cong K_C \otimes L^\otimes (-3)$. Hence if $F(L)$ has a nonzero section, assuming this section vanishes nowhere along $C$, then $F(L)$ is an extension of $K_C \otimes L^\vee$ by $\mathcal{O}_C$. This provides an extension class

$$e \in H^1(C, L \otimes K_C^\vee) = H^0(C, K_C^\otimes 2 \otimes L^\vee)^\vee. \quad (6.31)$$

Assume now $L \otimes K_C^\vee =: \eta$ is a nonzero 2-torsion element of Pic$^0(C)$. Then

$$e \in H^0(C, L)^\vee.$$

On the other hand, according to Theorem 20, there exists a nontrivial syzygy

$$\gamma = \sum_{i=1}^6 \ell_i \otimes q_i \in K_{1,2}(C, L) = \text{Ker}\left\{ H^0(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(1)) \otimes I_C(2) \to I_C(3) \right\},$$

which is degenerate by Proposition 13. As we saw already, it has in fact rank 6 for generic $[C, \eta]$, hence determines a nonzero element

$$f \in H^0(\mathbf{P}^6, \mathcal{O}_{\mathbf{P}^6}(1))^\vee = H^0(C, L)^\vee = H^1(C, K_C \otimes L^\vee) = H^1(C, L \otimes K_C^\vee), \quad (6.32)$$

which is well-defined up to a coefficient.

**Proposition 27.** The two elements $e$ and $f$ are proportional.

**Proof.** Equivalently, we show that the kernels of the two linear forms $e, f \in H^0(C, L)^\vee$ are equal. Viewing $\gamma$ as an element of $\text{Hom}(I_C(2)^\vee, H^0(C, L))$, we have $\text{Ker}(f) = \text{Im}(\gamma)$. On the other hand, the kernel of $e$ identifies with

$$\text{Im}\left\{ j : H^0(C, F \otimes L^\otimes 3 \otimes K_C^\vee) \to H^0(C, L) \right\},$$

where the map $j$ is obtained by twisting the exact sequence $0 \to \mathcal{O}_C \to F(L) \to K_C \otimes L^\vee \to 0$ by $K_C$. We have $F \otimes L^\otimes 3 \otimes K_C^\vee \cong F^\vee$ since $\det F \cong K_C \otimes L^\otimes (-3)$, hence there is a natural morphism

$$i^* : I_C(2)^\vee \otimes \mathcal{O}_C \to F^\vee \cong F(L^\otimes 3 \otimes K_C^\vee)$$

dual to the inclusion $F \hookrightarrow I_C(2) \otimes \mathcal{O}_C$ of (6.30). The proposition follows from the following claim:

**Claim.** The morphism $\alpha : I_C(2)^\vee \to H^0(C, L)$ is equal to $j \circ i^*$.

Forgetting about the last identification $F^\vee \cong F \otimes L^\otimes 3 \otimes K_C^\vee$, the claim amounts to the following general fact: For an evaluation exact sequence on a variety $X$

$$0 \to G \to W \otimes \mathcal{O}_X \to M \to 0$$

and for a section $s \in H^0(X, G(L)) = H^0(X, \text{Hom}(G^\vee, L))$ giving an element

$$s' \in \text{Ker}\left\{ W \otimes H^0(X, L) \to H^0(X, M \otimes L) \right\} \subseteq \text{Hom}(W^\vee, H^0(X, L)),$$

the induced map $s : H^0(X, G^\vee) \to H^0(X, L)$ composed with the map $W^\vee \to H^0(X, G^\vee)$ equals the map $s' : W^\vee \to H^0(X, L)$. \qed
6.B. Further properties

Using the exact sequence (6.30) in the general case of a genus 8 paracanonical curve \([C, L] \in P^4\), we obtain:

**Lemma 28.** A section \(s \in H^0(C, F(L)) \subseteq I_{C,L}(2) \otimes H^0(C, L) = \text{Hom}(I_{C,L}(2)^\vee, H^0(C, L))\) of rank 6, determines an element \(e \in [2L - K_C]\).

**Proof.** The multiplication by \(s \in H^0(F(L)) \subseteq I_{C,L}(2) \otimes H^0(C, L) = H^0(I_{C,L}(2)^\vee \otimes L)\) determines the natural maps \(F^\vee \rightarrow L\) and \(g_s : I_{C,L}(2)^\vee \otimes \mathcal{O}_C \rightarrow L\) sitting in the following diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \ker(g_s) & \rightarrow & I_{C,L}(2)^\vee \otimes \mathcal{O}_C & \rightarrow & L & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & 2L - K_C & \rightarrow & F^\vee & \rightarrow & L & \rightarrow & 0 ,
\end{array}
\]

where \(I_{C,L}(2)^\vee \otimes \mathcal{O}_C \rightarrow F^\vee\) is the dual of the natural inclusion of (6.30). Passing to global sections we get the inclusion \(H^0(\ker(g_s)) = \ker\{I_{C,L}(2)^\vee \rightarrow H^0(C, L)\} \hookrightarrow H^0(2L - K_C)\), which by hypothesis in 1-dimensional hence it defines an element \(e \in [2L - K_C]\). \(\square\)

Via the exact sequence (6.30) we can also show directly the following result that has been used in Section 3:

**Lemma 29.** If there is a spin curve \(D = C \cup E \rightarrow P^6\) of genus 22 and degree 21 containing the genus 8 paracanonical curve \([C, L]\) as in Lemma 3, then \(H^0(C, (F(L)) \neq 0\). If there is no cubic polynomial on \(P^6\) vanishing with multiplicity 2 along \(C\), then \(K_{1,2}(C, L) \neq 0\).

**Proof.** Let \(e = C \cap E\) and recall \(c_1(F) = -3L + K_C\) and \(\mathcal{O}_C(e) = 2L - K_C\). Note that \(I_D(2) \subseteq I_C(2)\) is 6-dimensional. Tensor then the first vertical exact sequence of the following diagram by \(L\) and pass to global sections.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{O}_C(-e) & \rightarrow & \mathcal{O}_C & \rightarrow & \mathcal{O}_C|_e & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}_C & \rightarrow & \mathcal{O}_C & \rightarrow & \mathcal{O}_C|_e & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I_{C,L}(2) \otimes \mathcal{O}_C & \rightarrow & \mathcal{I}_D/(\mathcal{I}_D \cap \mathcal{I}_C^2)(2) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & L^\vee & \rightarrow & I_{D,C}(2) \otimes \mathcal{O}_C & \rightarrow & \mathcal{I}_D/(\mathcal{I}_D \cap \mathcal{I}_C^2)(2) & \rightarrow & 0 ,
\end{array}
\]

\(\square\)

6.C. Nontrivial syzygies of paracanonical curves via vector bundles

We return to the proof of Theorem 20 given in Section 5. Consider now a general paracanonical curve \([C, K_C \otimes \eta] \in P^4\). For a rank 2 vector bundle on \(C\) of degree 14, with noncanonical determinant, the equation \(h^0(C, E) \geq 4\) imposes 16 conditions. Similarly, if \(\epsilon \in \text{Pic}^0(C)\), the equation \(h^0(C, E \otimes \epsilon) \geq 4\) imposes 16 conditions on the parameter space of \(E\). Given \(C\), there are \(29 = 4g - 3\) parameters for \(E\), and \(8 = g\) parameters for \(\epsilon\). It follows that we have at least a 5-dimensional family of pairs \((E, \epsilon)\), such that

\[h^0(C, E) \geq 4\quad\text{and}\quad h^0(C, E \otimes \epsilon) \geq 4.\] (6.33)

Furthermore, the construction of Section 5 (together with Proposition 5) shows that for a general triple \((C, E, \epsilon)\) as above, one has \(K_{2,1}(C, L) \neq 0\), where \(L := \det E \otimes \epsilon\). Assuming the map \((E, \epsilon) \mapsto L\)
is generically finite on its image, we constructed in this way a five dimensional family of paracanonical line bundles $L \in \text{Pic}^{14}(C)$ with a nontrivial syzygy: $K_{1,2}(C,L) \neq 0$. This family has the following property:

**Lemma 30.** If $L = \text{det} E \otimes \epsilon$, where $E$ satisfies (6.33), the line bundle $K_C^{\otimes 2} \otimes L^\vee$ satisfies the same property. The family above, which has dimension at least five, is thus invariant under the involution $L \mapsto K_C^{\otimes 2} \otimes L^\vee$ on $P_8^{14}$, whose fixed locus is the Prym moduli space $R_8$.

**Proof.** This follows from Serre duality, replacing $E$ with $E^\vee \otimes K_C$ and $E \otimes \epsilon$ by $E^\vee \otimes \epsilon^\vee \otimes K_C$ plus the fact that $\text{det} (E^\vee \otimes K_C) \otimes \epsilon^\vee \cong K_C^{\otimes 2} \otimes \text{det} E^\vee \otimes \epsilon^\vee$. 

One can ask in general the following question:

**Question 31.** Is the divisor $R_{[5]}$ on $P_8^{14}$ invariant under the involution $L \mapsto K_C^{\otimes 2} \otimes L^\vee$?

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