A Counting Proof of the Graham Pollak Theorem

Sundar Vishwanathan
Department of Computer Science and Engineering
Indian Institute of Technology, Bombay
Powai, Mumbai
India 400076
sundar@cse.iitb.ernet.in

Abstract

We give a counting based proof of the Graham-Pollak theorem.

1 Introduction

A spectacular application of linear algebra to prove a combinatorial statement is the Graham-Pollak theorem [3]. The theorem states that the edge set of the complete graph $K_n$ cannot be written as the disjoint union of $n-2$ complete bipartite graphs. The original proof used Sylvester’s law of inertia. See [2, 4, 5] for other short proofs. These proofs seem to use linear algebra inherently. Combinatorialists have often commented that a combinatorial proof for the theorem is not known. See, for example comments regarding the problem in [6, 1].

In this note we show that the linear algebra proof of [5], for instance, can be explained combinatorially. The first observation is that the linear algebra part of the proof can be replaced by a pigeon-hole argument. The other (minor) observation in this note is that with this in place, one can explain the slick calculations in the usual linear algebra proofs of this theorem with a longer explicit bijective argument.

2 The Proof

Theorem 1 (Graham-Pollak) Suppose that $K_n$ is obtained as the edge-disjoint union of $m$ complete bipartite graphs. Then $m \geq n - 1$.

Proof: For a contradiction, consider a covering of $K_n$ by complete bipartite graphs $(L_i, R_i) : 1 \leq i \leq n - 2$. The vertex set is identified with $[n]$.

Consider a labeling of the $n$ vertices, $\sigma : [n] \to [k]$ where $k > n^n$. We associate a pattern with $\sigma$, as an $n-1$ tuple, where the $i$th entry of the tuple, for $i < n-1$, is given by the sum of the $\sigma$ values of the vertices in $L_i$; that is the $i$th entry is $\Sigma_{j \in L_i} \sigma(j)$. The $n-1$th entry is the sum of the $\sigma$ values of all vertices. The number of possible patterns is at most $(kn)^{n-1}$. The total number of labelings is $k^n$. Hence,
since $k$ is large enough, there are two distinct labelings with the same pattern, say $\sigma_1$ and $\sigma_2$. Define $\tau = \sigma_1 - \sigma_2$. Note that for each $1 \leq i \leq n-2$, $\Sigma_{j \in L_i} \tau(j) = 0$. Also $\Sigma_{j=1}^n \tau(j) = 0$ and $\tau$ is non-zero on at least one vertex.

Consider the following equality:

$$\left(\Sigma_{j=1}^n \tau(j)\right)^2 = \Sigma_{j=1}^n (\tau(j))^2 + 2\Sigma_{i<j} \tau(i)\tau(j)$$

The left hand side is zero. The first term in the right hand side is non-zero. For a contradiction we will show that the second term is zero. Because we have a disjoint cover of $K_n$,

$$\Sigma_{i<j} \tau(i)\tau(j) = \Sigma_{i=1}^n (\Sigma_{j \in L_i} \tau(j))(\Sigma_{k \in R_i} \tau(k)).$$

But the right hand side is zero since for each $i$, $\Sigma_{j \in L_i} \tau(j) = 0$.  

3 Explaining the Calculations

The last part of the proof; defining $\tau$ and the calculations following it seem rather mysterious. Especially the use of the fact that if the sum of squares is zero then each term should be zero. We give an explicit bijection to explain this phenomenon. We continue the proof after defining $\sigma$.

We consider two graphs $H$ and $H'$ defined below. Both have the same vertex set $W$ which is partitioned into $2n$ non-empty parts: $W = V_1 \cup V'_1 \cup V_2 \cup V'_2 \cdots V_n \cup V'_n$. We will require $\Sigma_{i=1}^n |V_i| = \Sigma_{i=1}^n |V'_i| = N$ (say.) $|V_i|$ could be any positive integer. The edge set of $H$ is as follows: $\{uv : u \in V_i, v \in V'_j, i \neq j\} \cup \{uv : u \in V_i', v \in V'_j, i \neq j\}$. Note that $H$ is the union of two disjoint complete $n$-partite graphs, each on $N$ vertices. The edge set of $H'$ is as follows: $\{uv : u \in V_i, v \in V'_j, i \neq j\}$. Hence $H'$ is a bipartite graph. The key observation is this.

Lemma 2 If the number of edges in $H$ and $H'$ are the same then $|V_i| = |V'_i|$ for all $1 \leq i \leq n$.

Proof. (Sketch.) We first claim that one can prove by an explicit bijection the fact that the number of edges of $K_p \cup K_q$ plus $\min\{p, q\}$ is at least the number of edges in $K_{pq}$. They are equal iff $p = q$ or $q = p+1$, assuming $q$ is larger. To see the bijection, assume $p < q$, consider two sets of vertices of size $p$ (say $P$) and $q$ (say $Q$). Map an edge $(i, j)$ (for $i \in P, j \in Q, i \neq j$) of $K_{pq}$ to an edge in $K_p \cup K_q$ as follows. If $i < j \leq p$ then to $ij$ in $K_p$. Otherwise to $ij$ in $K_q$. Note that this is one to one. Also note that the edges $ij$ with both $i, j$ greater than $p$ are not in the range of the map. Hence the number of edges in $K_p \cup K_q$ plus $\frac{p+q}{2}$ is at least the number of edges in $K_{pq}$. And they are equal only if $p = q$.

If we take the complement of each component of $H$ with respect to the complete graph $K_N$, then we get disjoint cliques of size $p_i$ and $q_i$ for each $i$. If we take the
complement of $H'$, with respect to $K_{N,N}$, then we get disjoint copies of $K_{p_i,q_i,s}$ for each $i$. We will use these complements in the next paragraph.

If the number edges in $H$ and $H'$ are both $m$, then the number of edges in the complement of $H$ is $2\binom{N}{2} - m$ and that in the complement of $H'$ is $N^2 - m$. That is the former plus $N$ equals the latter. However, by the argument in the first paragraph, if $p_i \neq q_i$ for any one $i$, then by adding the inequality for the number of edges for each $i$, we infer that the number of edges in the complement of $H$ plus $N$ is strictly larger than the latter, a contradiction.

Consider a disjoint cover of the edges of $K_n$ by bipartite graphs $(A_j, B_j) : 1 \leq j \leq k$. Here $k$ is not restricted. We define $H, H'$ as a disjoint union of $n$ sets of vertices, the $i$th (called $V_i$ and $V_i'$ in $H$ and $H'$ respectively) corresponding to some number of copies of the $i$th vertex of $K_n$. Now, suppose that for each $1 \leq j \leq k$, $\sum_{p \in A_j} |V_p| = \sum_{p \in A_j} |V_p'|$ then we claim that the number of edges in $H$ and $H'$ are the same. Indeed, we can cover the edges of $H$ by the following bipartite graphs; two for each $1 \leq j \leq k$:

$$(\cup_{p \in A_j} V_p, \cup_{q \in B_j} V_q)$$
$$(\cup_{p \in A_j} V_p', \cup_{q \in B_j} V_q')$$

Similarly we can cover the edges of $H'$ by the following bipartite graphs, two for each $j$:

$$(\cup_{p \in A_j} V_p, \cup_{q \in B_j} V_p')$$
$$(\cup_{p \in A_j} V_p', \cup_{q \in B_j} V_q).$$

It can be seen that, for each $j$, the total number of edges in the top two bipartite graphs covering edges in $H$, is equal to the total number of edges in the bottom two bipartite graphs. This implies that the number of edges in $H$ and $H'$ are equal. By the lemma this implies that $|V_p| = |V_p'|$, for every $p$.

To use the above, to yield a contradiction for the Graham Pollak result, we use $\sigma_1$ and $\sigma_2$ constructed earlier, to define these two graphs, with $|V_p| \neq |V_p'|$ for at least one $p$. □

The main observation resulting in this note is that the following fact, which follows from linear algebra, can also be proved by a pigeon hole argument.

**Lemma 3** Let $A$ be an $m \times n$ integer matrix, with $m < n$. Then, there are positive integer vectors $x_1$ and $x_2 \neq x_1$ such that $Ax_1 = Ax_2$.

Essentially, if the domain is restricted to a large enough (finite) set, then the range can be made smaller than the domain and hence two domain elements map to the same point in the range. Other theorems where the linear algebra part used is the lemma above can be proved using the pigeon hole principle.
4 On Proofs.

What constitutes a combinatorial proof and what is a linear algebra (or topological) proof are questions faced by mathematicians for some time now. It is quite conceivable that different people have different notions! In the proof in this note we do not use the notion of a field, which seems to be necessary for linear algebra based proofs. The Steinitz exchange lemma and Gaussian Elimination, both use the concept of inverses in the field.

The proof in this note however constructs intermediate structures of large size. But implicitly so do the linear algebra proofs. Even if we work over the rationals, for the intermediate values, the numerator and denominator may be as large as the determinant of an $n \times n$ $0-1$ matrix each of which can be as large as $n^{O(n)}$. We insist that these have to be written in unary. For a graph on $n$ vertices we either construct graphs on $n^n$ vertices or use labels of size $n^n$ (again assuming labels are written in unary.) Typically, one sees this phenomenon of using large numbers for the proof in Ramsey Theory and not so much in other areas of Combinatorics. Call a proof an effective combinatorial proof if the size of the proof (assuming that intermediate labels are written in unary) is polynomial in the input size. (Here the input is an explicit description of $K_n$.) Finding an effective combinatorial proof for the Graham Pollak theorem is a nice open problem. This method will give us an effective proof for a worse bound, by restricting $k$ to be a polynomial in $n$. A bound of $n/2$ by an effective combinatorial proof which does not use counting would be a nice first question to solve. Note that a bound of $n/2$ can be proved using linear algebra over $\mathbb{F}_2$, but this uses counting with parity and/or exact counting with large numbers.

References

[1] M. Aigner and G. M. Zeigler. Proofs from The Book. Third Edition. Springer Verlag, 2004.

[2] L. Babai and P. Frankl. Linear Algebra Methods in Combinatorics. Preliminary Version 2, Department of Computer Science, The University of Chicago, September 1992.

[3] R. L. Graham and H. O. Pollak. On embedding graphs in squashed cubes. In: Graph Theory and Appl., Springer Lecture Notes in Math. 303, pp. 99-110, 1972.

[4] G. W. Peck. A new proof of a theorem of Graham and Pollak. Discrete Math., 49, pp. 327–328,1984.

[5] H. Tverberg. On the decomposition of $K_n$ into complete bipartite graphs. J. Graph Theory, 6, pp. 493–494,1982.
[6] J. H. van Lint and R. M. Wilson. A Course in Combinatorics. Cambridge University Press, Jan 29 1993.