Periodic and solitary wave solutions to the Fornberg-Whitham equation

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Abstract

In this paper, new travelling wave solutions to the Fornberg-Whitham equation

\[ u_t - u_{xxt} + u_x + uu_x = uu_{xxx} + 3u_xu_{xx} \]

are investigated. They are characterized by two parameters. The expressions of the periodic and solitary wave solutions are obtained.

Key words: Fornberg-Whitham equation, solitary wave, periodic wave

1 Introduction

Recently, Ivanov [1] investigated the integrability of a class of nonlinear dispersive wave equations

\[ u_t - u_{xxt} + \partial_x(\kappa u + \alpha u^2 + \beta u^3) = \nu u_x u_{xx} + \gamma uu_{xxx}, \]  

(1.1)

where and \( \alpha, \beta, \gamma, \kappa, \nu \) are real constants.

The important cases of Eq.(1.1) are:

The hyperelastic-rod wave equation

\[ u_t - u_{xxt} + 3uu_x = \gamma(2u_xu_{xx} + uu_{xxx}), \]  

(1.2)

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has been recently studied as a model, describing nonlinear dispersive waves in cylindrical compressible hyperelastic rods [2]-[7]. The physical parameters of various compressible materials put $\gamma$ in the range from -29.4760 to 3.4174 [2,4].

The Camassa-Holm equation
\[ u_t - u_{xxt} + 3uu_x = 2u_xu_xx + uu_{xxx}, \] (1.3)
describes the unidirectional propagation of shallow water waves over a flat bottom [8,9]. It is completely integrable [1] and admits, in addition to smooth waves, a multitude of travelling wave solutions with singularities: peakons, cuspons, stumpons and composite waves [9]-[12]. The solitary waves of Eq.(1.2) are smooth if $\kappa > 0$ and peaked if $\kappa = 0$ [9,10]. Its solitary waves are stable solitons [13,14], retaining their shape and form after interactions [15]. It models wave breaking [16]-[18].

The Degasperis-Procesi equation
\[ u_t - u_{xxt} + 4uu_x = 3u_xu_xx + uu_{xxx}, \] (1.4)
models nonlinear shallow water dynamics. It is completely integrable [1] and has a variety of travelling wave solutions including solitary wave solutions, peakon solutions and shock waves solutions [20]-[27].

The Fornberg-Whitham equation
\[ u_t - u_{xxt} + u_x + uu_x = uu_{xxx} + 3u_xu_xx, \] (1.5)
appeared in the study qualitative behaviors of wave-breaking [28]. It admits a wave of greatest height, as a peaked limiting form of the travelling wave solution [29], $u(x,t) = A \exp(-\frac{1}{2} |x - \frac{4}{3}t|)$, where $A$ is an arbitrary constant. It is not completely integrable [1].

The regularized long-wave or BBM equation
\[ u_t - u_{xxt} + u_x + uu_x = 0, \] (1.6)
and the modified BBM equation
\[ u_t - u_{xxt} + u_x + 3u^2u_x = 0, \] (1.7)
have also been investigated by many authors [30]-[38].
Many efforts have been devoted to study Eq. (1.2)-(1.4), (1.6) and (1.7), however, little attention was paid to study Eq. (1.5). In [39], we constructed two types of bounded travelling wave solutions \( u(\xi)(\xi = x-ct) \) to Eq. (1.5), which are defined on semifinal bounded domains and called kink-like and antikink-like wave solutions. In this paper, we continue to study the travelling wave solutions to Eq. (1.5). Following Vakhnenko and Parkes’s strategy in [19], we obtain some periodic and solitary wave solutions \( u(\xi) \) to Eq. (1.5) which are defined on \((-\infty, +\infty)\). The travelling wave solutions obtained in this paper are obviously different from those obtained in our previous work [39]. To the best of our knowledge, these solutions are new for Eq. (1.5). Our work may help people to know deeply the described physical process and possible applications of the Fornberg-Whitham equation.

The remainder of the paper is organized as follows. In Section 2, for completeness and readability, we repeat Appendix A in [19], which discuss the solutions to a first-order ordinary differential equation. In Section 3, we show that, for travelling wave solutions, Eq. (1.5) may be reduced to a first-order ordinary differential equation involving two arbitrary integration constants \( a \) and \( b \). We show that there are four distinct periodic solutions corresponding to four different ranges of values of \( a \) and restricted ranges of values of \( b \). A short conclusion is given in Section 4.

2 Solutions to a first-order ordinary differential equation

This section is due to Vakhnenko and Parkes (see Appendix A in [19]). For completeness and readability, we repeat it in the following.

Consider solutions to the following ordinary differential equation

\[
(\varphi \varphi_\xi)^2 = \epsilon^2 f(\varphi),
\]  

(2.1)

where

\[
f(\varphi) = (\varphi - \varphi_1)(\varphi - \varphi_2)(\varphi_3 - \varphi)(\varphi_4 - \varphi),
\]

(2.2)

and \( \varphi_1, \varphi_2, \varphi_3, \varphi_4 \) are chosen to be real constants with \( \varphi_1 \leq \varphi_2 \leq \varphi \leq \varphi_3 \leq \varphi_4 \).

Following [40] we introduce \( \zeta \) defined by

\[
\frac{d\xi}{d\zeta} = \frac{\varphi}{\epsilon},
\]

(2.3)
so that Eq. (2.1) becomes

\[(\varphi_\zeta)^2 = f(\varphi).\]  \hspace{1cm} (2.4)

Eq. (2.4) has two possible forms of solution. The first form is found using result 254.00 in [41]. Its parametric form is

\[
\begin{cases} 
\varphi = \frac{\varphi_3 - \varphi_1 n \text{sn}^2(w|m)}{1 - n \text{sn}^2(w|m)}, \\
\xi = \frac{1}{\sqrt{p}} (w\varphi_1 + (\varphi_2 - \varphi_1)\Pi(n;w|m)), 
\end{cases}
\]  \hspace{1cm} (2.5)

with \(w\) as the parameter, where

\[m = \frac{(\varphi_3 - \varphi_2)(\varphi_1 - \varphi_1)}{(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_1)}, \quad p = \frac{1}{2} \sqrt{(\varphi_4 - \varphi_2)(\varphi_3 - \varphi_1)}, \quad w = p\zeta, \]  \hspace{1cm} (2.6)

and

\[n = \frac{\varphi_3 - \varphi_2}{\varphi_3 - \varphi_1}. \]  \hspace{1cm} (2.7)

In (2.5) \(\text{sn}(w|m)\) is a Jacobian elliptic function, where the notation is as used in Chapter 16 of [42]. \(\Pi(n;w|m)\) is the elliptic integral of the third kind and the notation is as used in Section 17.2.15 of [42].

The solution to (2.1) is given in parametric form by (2.5) with \(w\) as the parameter. With respect to \(w\), \(\varphi\) in (2.5) is periodic with period \(2K(m)\), where \(K(m)\) is the complete elliptic integral of the first kind. It follows from (2.5) that the wavelength \(\lambda\) of the solution to (2.1) is

\[\lambda = \frac{2}{\varepsilon p} |\varphi_1 K(m) + (\varphi_2 - \varphi_1)\Pi(n|m)|, \]  \hspace{1cm} (2.8)

where \(\Pi(n|m)\) is the complete elliptic integral of the third kind.

When \(\varphi_3 = \varphi_4\), \(m = 1\), (2.5) becomes

\[
\begin{cases} 
\varphi = \frac{\varphi_3 - \varphi_1 n \tanh^2 w}{1 - n \tanh^2 w}, \\
\xi = \frac{1}{\varepsilon} (\frac{w\varphi_3}{p} - 2 \tanh^{-1}(\sqrt{n}\tanh w)). 
\end{cases}
\]  \hspace{1cm} (2.9)

The second form of solution of (2.5) is found using result 255.00 in [41]. Its
parametric form is

\[
\begin{align*}
\phi &= \frac{\varphi_1 - \varphi_4 \operatorname{nsn}^2(w|m)}{1 - \operatorname{nsn}^2(w|m)}, \\
\xi &= \frac{1}{\varepsilon p}(w \varphi_4 - (\varphi_4 - \varphi_3)\Pi(n; w|m)),
\end{align*}
\]

where \(m, p, w\) are as in (2.6), and

\[
n = \frac{\varphi_3 - \varphi_2}{\varphi_4 - \varphi_2}.
\]

The solution to (2.1) is given in parametric form by (2.10) with \(w\) as the parameter. The wavelength \(\lambda\) of the solution to (2.1) is

\[
\lambda = \frac{2}{\varepsilon p}|\varphi_4 K(m) - (\varphi_4 - \varphi_3)\Pi(n|m)|.
\]

When \(\varphi_1 = \varphi_2, \ m = 1\), (2.10) becomes

\[
\begin{align*}
\phi &= \frac{\varphi_1 - \varphi_4 n \tanh^2 w}{1 - n \tanh^2 w}, \\
\xi &= \frac{1}{\varepsilon}(\frac{w \varphi_2}{p} + 2 \tanh^{-1}(\sqrt{n} \tanh w)).
\end{align*}
\]

3 Periodic and solitary wave solutions to Eq.(1.5)

Eq.(1.5) can also be written in the form

\[
(u_t + uu_x)_{xx} = u_t + uu_x + u_x.
\]

Let \(u = \varphi(\xi) + c\) with \(\xi = x - ct\) be a travelling wave solution to Eq.(3.1), then it follows that

\[
(\varphi \varphi_\xi)_{\xi \xi} = \varphi \varphi_\xi + \varphi_\xi.
\]

Integrating (3.2) twice with respect to \(\xi\), we have

\[
(\varphi \varphi_\xi)^2 = \frac{1}{4}(\varphi^4 + \frac{8}{3} \varphi^3 + a \varphi^2 + b),
\]

where \(a\) and \(b\) are two arbitrary integration constants.
Eq. (3.3) is in the form of Eq. (2.1) with \( \varepsilon = \frac{1}{2} \) and \( f(\varphi) = (\varphi^4 + \frac{8}{3}\varphi^3 + a\varphi^2 + b) \). For convenience we define \( g(\varphi) \) and \( h(\varphi) \) by

\[
f(\varphi) = \varphi^2 g(\varphi) + b, \quad \text{where } g(\varphi) = \varphi^2 + \frac{8}{3}\varphi + a, \tag{3.4}
\]

\[
f'(\varphi) = 2\varphi h(\varphi), \quad \text{where } h(\varphi) = 2\varphi^2 + 4\varphi + a, \tag{3.5}
\]

and define \( \varphi_L, \varphi_R, b_L, \) and \( b_R \) by

\[
\varphi_L = -\frac{1}{2}(2 + \sqrt{4 - 2a}), \quad \varphi_R = -\frac{1}{2}(2 - \sqrt{4 - 2a}), \tag{3.6}
\]

\[
b_L = -\varphi_L^2 g(\varphi_L) = \frac{a^2}{4} - 2a + \frac{8}{3} + \frac{2}{3}(2 - a)\sqrt{4 - 2a}, \tag{3.7}
\]

\[
b_R = -\varphi_R^2 g(\varphi_R) = \frac{a^2}{4} - 2a + \frac{8}{3} - \frac{2}{3}(2 - a)\sqrt{4 - 2a}. \tag{3.8}
\]

Obviously, \( \varphi_L, \varphi_R \) are the roots of \( h(\varphi) = 0 \).

In the following, suppose that \( a < 2 \) and \( a \neq 0 \) such that \( f(\varphi) \) has three distinct stationary points: \( \varphi_L, \varphi_R, 0 \) and comprise two minimums separated by a maximum. Under this assumption, Eq. (3.3) has periodic and solitary wave solutions that have different analytical forms depending on the values of \( a \) and \( b \) as follows:

1. \( a < 0 \)

In this case \( \varphi_L < 0 < \varphi_R \) and \( f(\varphi_L) < f(\varphi_R) \). For each value \( a < 0 \) and \( 0 < b < b_R \) (a corresponding curve of \( f(\varphi) \) is shown in Fig.1(a)), there are periodic inverted loop-like solutions to Eq. (3.3) given by (2.5) so that \( 0 < m < 1 \), and with wavelength given by (2.8); see Fig.2(a) for an example.

The case \( a < 0 \) and \( b = b_R \) (a corresponding curve of \( f(\varphi) \) is shown in Fig.1(b)) corresponds to the limit \( \varphi_3 = \varphi_4 = \varphi_R \) so that \( m = 1 \), and then the solution is an inverted loop-like solitary wave given by (2.9) with \( \varphi_2 \leq \varphi < \varphi_R \) and

\[
\varphi_1 = -\frac{1}{6}(2 + 3\sqrt{4 - 2a} + 2\sqrt{4 + 6\sqrt{4 - 2a}}), \tag{3.9}
\]

\[
\varphi_2 = -\frac{1}{6}(2 + 3\sqrt{4 - 2a} - 2\sqrt{4 + 6\sqrt{4 - 2a}}), \tag{3.10}
\]

see Fig.3(a) for an example.
(2) $0 < a < \frac{16}{9}$

In this case $\varphi_L < \varphi_R < 0$ and $f(\varphi_L) < f(0)$. For each value $0 < a < \frac{16}{9}$ and $b_R < b < 0$ (a corresponding curve of $f(\varphi)$ is shown in Fig.1(c)), there are periodic hump-like solutions to Eq.(3.3) given by (2.5) so that $0 < m < 1$, and with wavelength given by (2.8); see Fig.2(b) for an example.

The case $0 < a < \frac{16}{9}$ and $b = 0$ (a corresponding curve of $f(\varphi)$ is shown in Fig.1(d)) corresponds to the limit $\varphi_3 = \varphi_4 = 0$ so that $m = 1$, and then the solution can be given by (2.9) with $\varphi_1$ and $\varphi_2$ given by the roots of $g(\varphi) = 0$, namely

$$\varphi_1 = -\frac{4}{3} - \frac{1}{3} \sqrt{16 - 9a}, \quad \varphi_2 = -\frac{4}{3} + \frac{1}{3} \sqrt{16 - 9a}. \quad (3.11)$$

In this case we obtain a weak solution, namely the periodic upward-cusp wave

$$\varphi = \varphi(\xi - 2j\xi_m), (2j - 1)\xi_m < \xi < (2j + 1)\xi_m, \quad j = 0, \pm 1, \pm 2, \cdots, \quad (3.12)$$

where

$$\varphi(\xi) = (\varphi_2 - \varphi_1 \tanh^2(\xi/4)) \cosh^2(\xi/4), \quad (3.13)$$

and

$$\xi_m = 4 \tanh^{-1} \sqrt{\frac{\varphi_2}{\varphi_1}}. \quad (3.14)$$
see Fig. 3(b) for an example.

Fig. 2. Periodic solutions to Eq. (3.3) with $0 < m < 1$. (a) $a = -50, b = 233$ so $m = 0.7885$; (b) $a = 1.5, b = -0.05$ so $m = 0.6893$; (c) $a = \frac{16}{9}, b = -0.1$ so $m = 0.8254$; (d) $a = 1.9, b = -0.24$ so $m = 0.6121$.

Fig. 3. Solutions to Eq. (3.3) with $m = 1$. (a) $a = -50, b = 374.1346$; (b) $a = 1.5, b = 0$; (c) $a = \frac{16}{9}, b = 0$; (d) $a = 1.9, b = -0.2010$.

(3) $a = \frac{16}{9}$

In this case $\varphi_L < \varphi_R < 0$ and $f(\varphi_L) = f(0)$. For $a = \frac{16}{9}$ and each value $b_R < b < 0$ (a corresponding curve of $f(\varphi)$ is shown in Fig. 1(e)), there are periodic hump-like solutions to Eq. (3.3) given by (2.10) so that $0 < m < 1$, and with wavelength given by (2.12); see Fig. 2(c) for an example.
The case $a = \frac{16}{9}$ and $b = 0$ (a corresponding curve of $f(\varphi)$ is shown in Fig.1(f)) corresponds to the limit $\varphi_1 = \varphi_2 = \varphi_L = -\frac{4}{3}$ and $\varphi_3 = \varphi_4 = 0$ so that $m = 1$. In this case neither (2.5) nor (2.10) is appropriate. Instead we consider Eq.(3.3) with $f(\varphi) = \frac{1}{4}\varphi^2(\varphi + \frac{4}{3})^2$ and note that the bound solution has $-\frac{4}{3} < \varphi \leq 0$. On integrating Eq.(3.3) and setting $\varphi = 0$ at $\xi = 0$ we obtain a weak solution

$$\varphi = \frac{4}{3} \exp\left(-\frac{1}{2} |\xi|\right) - \frac{4}{3},$$

(3.15)

i.e. a single peakon solution with amplitude $\frac{4}{3}$, see Fig.3(c).

(4) $\frac{16}{9} < a < 2$

In this case $\varphi_L < \varphi_R < 0$ and $f(\varphi_L) > f(0)$. For each value $\frac{16}{9} < a < 2$ and $b_R < b < b_L$ (a corresponding curve of $f(\varphi)$ is shown in Fig.1(g)), there are periodic hump-like solutions to Eq.(3.3) given by (2.10) so that $0 < m < 1$, and with wavelength given by (2.12); see Fig.2(d) for an example.

The case $\frac{16}{9} < a < 2$ and $b = b_L$ (a corresponding curve of $f(\varphi)$ is shown in Fig.1(h)) corresponds to the limit $\varphi_1 = \varphi_2 = \varphi_L$ so that $m = 1$, and then the solution is a hump-like solitary wave given by (2.13) with $\varphi_L < \varphi \leq \varphi_3$ and

$$\varphi_3 = \frac{1}{6}(-2 + 3\sqrt{4 - 2a} - 2\sqrt{4 - 6\sqrt{4 - 2a}}),$$

(3.16)

$$\varphi_4 = \frac{1}{6}(-2 + 3\sqrt{4 - 2a} + 2\sqrt{4 - 6\sqrt{4 - 2a}}),$$

(3.17)

see Fig.3(d) for an example.

On the above, we have obtained expressions of parametric form for periodic and solitary wave solutions $\varphi(\xi)$ to Eq.(3.3). So in terms of $u = \varphi(\xi) + c$, we can get expressions for the periodic and solitary wave solutions $u(\xi)$ to Eq.(1.5).

4 Conclusion

In this paper, we have found expressions for new travelling wave solutions to the Fornberg-Whitham equation. These solutions depend, in effect, on two parameters $a$ and $m$. For $m = 1$, there are inverted loop-like ($a < 0$), single peaked ($a = \frac{16}{9}$) and hump-like ($\frac{16}{9} < a < 2$) solitary wave solutions. For
\[ m = 1, 0 < a < \frac{16}{9} \text{ or } 0 < m < 1, a < 2 \text{ and } a \neq 0, \] there are periodic hump-like wave solutions.

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