Two-cylinder entanglement entropy under a twist

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Abstract. We study the von Neumann and Rényi entanglement entropy (EE) of the scale-invariant theories defined on the tori in $2+1$ and $3+1$ spacetime dimensions. We focus on the spatial bi-partitions of the torus into two cylinders, and allow for twisted boundary conditions along the non-contractible cycles. Various analytical and numerical results are obtained for the universal EE of the relativistic boson and Dirac fermion conformal field theories (CFTs), the fermionic quadratic band touching and the boson with $z=2$ Lifshitz scaling. The shape dependence of the EE clearly distinguishes these theories, although intriguing similarities are found in certain limits. We also study the evolution of the EE when a mass is introduced to detune the system from its scale-invariant point, by employing a renormalized EE that goes beyond a naïve subtraction of the area law. In certain cases we find the non-monotonic behavior of the torus EE under RG flow, which distinguishes it from the EE of a disk.

Keywords: entanglement entropies, entanglement in extended quantum systems, conformal field theory
## Contents

1. **Introduction** 3  
   1.1. Summary of the main results 6  

2. **General properties of torus entanglement in two dimensions** 8  
   2.1. The thin slice limit 8  
   2.2. The thin torus limit 9  
   2.3. The wide torus limit 9  

3. **Two-cylinder entropy for relativistic free field theory** 10  
   3.1. The shape dependence 11  
   3.2. Thin torus limit and semi-infinite cylinder 13  
       3.2.1. Massive case 15  

4. **$z = 2$ free bosons and fermions with a twist** 17  
   4.1. $z = 2$ free bosons 17  
       4.1.1. Two-cylinder entropy on a cylinder 17  
       4.1.2. Two-cylinder entropy on a torus 20  
   4.2. Fermionic quadratic band touching 22  

5. **Free relativistic theories in 3 + 1 dimensions** 23  
   5.1. Massless case 24  
       5.1.1. The periodic case $\lambda_1 = \lambda_2 = 0$ 26  
   5.2. The massive case 26  

6. **Conclusions** 28  

**Acknowledgments** 30  

**Appendix A. Useful functions and identities** 30  

**Appendix B. Various zeta functions** 31  
B.1. Hurwitz zeta function 31  
B.2. The Epstein zeta function in $d = 1$ 31  
B.3. Epstein zeta function in $d = 2$ 32  

**Appendix C. Free boson partition function on the cylinder and torus** 33  
C.1. Torus 33  
       C.1.1. The periodic boundary conditions in $x$- and $y$-directions 33  
       C.1.2. Twisted boundary condition 35  
C.2. Open cylinder 35  
       C.2.1. The periodic in the $y$-direction 35  
       C.2.2. The twisted boundary condition in the $y$-direction 36  

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1. Introduction

The entanglement entropy (EE) is a non-local quantity, which, when properly treated, can capture the long-distance properties of correlated quantum many-body states and quantum field theories. Given its non-local nature, the behavior of the EE is not as well understood as the correlators of the local operators in quantum field theory, and in particular, how they are related to each other. Moreover, due to the non-local character of the EE it can capture global properties which may not be accessible from the measurements of local operators.

To date, the most complete and detailed understanding of the behavior of the EE is in the $1+1$-dimensional conformal field theories (CFTs), which describe the universal behavior of systems at quantum criticality. In these CFTs, the von Neumann EE of a single interval in the ground state has a universal logarithmic dependence on the length of the interval. The prefactor of the logarithm yields the central charge $c$ for the underlying CFT, which is the most important quantity for characterizing the theory $[1-4]$. Furthermore, a related quantity involving the mutual information of two disjointed intervals depends not only on the central charge, but also on the operator content of the CFT $[5]$. These results, which connect the central charge of a CFT to the scaling of the EE, suggest that this scaling may be related to the definitions of a central charge in higher dimensions $[6]$, and to a generalization of the Zamolodchikov $c$-theorem $[7]$ for renormalization group flows.

Much less is known about the behavior of the EE in higher-dimensional field theories. In space dimensions $d > 1$, the von Neumann EE of a local field theory in a finite but macroscopically large region of space satisfies the ‘area law’ and scales with the size of the boundary of the observed region $[8, 9]$. In local quantum field theory this behavior, which is reminiscent of the area law of the Bekenstein–Hawking black hole entropy $[10, 11]$, is governed by a cutoff-dependent non-universal prefactor, and any universal behavior of the EE (independent of the UV cutoff) should be present in the subleading corrections to the area law.

Beyond the area law, many of the general results for the scaling of the EE in higher-dimensional CFTs are known primarily from the holographic EE of Ryu and Takayanagi $[12, 13]$. In higher-dimensional theories, the geometry of the entangling regions (and the topology of space) plays a richer role. In $3+1$ dimensions, this richer structure allows for universal subleading logarithmic terms in the EE, even for smooth surfaces $[13-15]$. Such terms are present if the entangling regions are spheres $[16, 17]$, and have a universal coefficient related to the two central charges of the trace anomaly of the energy-momentum tensor $[6]$.

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References 39

Appendix D. The numerical method for calculating EE in free systems

D.1. Free scalar field theory ................................ 37

D.1.1. The relativistic boson in $2+1d$. The Hamiltonian for the relativistic boson in $2+1d$ is ........................................ 37

D.2. The numerical method for the free Dirac fermion in $2+1$ dimensions . . . 38

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In this paper, we will consider the finite universal terms in the scaling of the EE both in $2 + 1$ and $3 + 1$ dimensions, where not many results are available. In particular, we will study the variation of this finite universal term under twist boundary conditions. From general arguments for the local field theories, relativistic or not, we know that the leading term obeys the area law. In $2 + 1$ dimensions, for an entangling region with a smooth boundary, one expects on dimensional grounds that the subleading term should be a finite scale-invariant function of the aspect ratio(s) of the entangling region. The first example of this behavior was found in topological phases of matter in $2 + 1$ dimensions, which are states with large-scale entanglement. In topological phases, and in topological quantum field theories such as Chern–Simons gauge theory [18], the finite term of the von Neumann EE is a constant which is given in terms of topological invariants of the underlying topological field theory, and on the topology of the observed region [19–21].

For a $2 + 1$-dimensional scale-invariant system, the EE of a region in the ground state satisfies the area law with a finite subleading correction, which, in general, depends on the shape of the region. This problem has been investigated in several field theories. One such theory is the quantum Lifshitz model [22], which is a free compactified scalar field in $2 + 1$ dimensions with a dynamic critical exponent $z = 2$ that describes the quantum (multi) critical point of generalized quantum dimer models in two dimensions [23–26]. In this model, the von Neumann EE has a finite universal subleading term, which depends on the compactification radius of the scalar field and of the aspect ratio (and the geometry) of the entangling surface [27–34]. A logarithmic dependence on the size of the region was found for entangling surfaces with corners (or cusps) [27, 35].

For relativistic CFTs in $2 + 1$ dimensions, this scaling of the EE has been shown to hold within the $\epsilon$-expansion for an $O(N)$ scalar field at the Wilson–Fisher fixed point for a partition of the $(3 - \epsilon)$-dimensional space into two half-spaces with a planar entangling surface [36]. In the limit $N \to \infty$, the von Neumann EE was recently obtained for general entangling regions [37]. Moreover, for a circular entangling surface, the universal subleading term of the von Neumann EE for a disk, which in this context is usually called $-F$, is a universal constant which behaves much like the central charge in a $1 + 1$-dimensional CFT, in that there is an appropriately defined function $F(R)$ which monotonically decreases along the renormalization group flow from the UV to the IR and agrees with the constant value of $F_{UV,IR}$ at the respective fixed points. This result is known as the $F$-theorem [38, 39], and in its strong form, it was postulated within the AdS/CFT correspondence [40]. The weaker version $F_{UV} > F_{IR}$ was first proposed in a holographic context [41] and subsequently verified to hold within the supersymmetric models [42] and various other non-supersymmetric models [43–45]. In contrast to the case of smooth boundaries, for entangling regions with corners, a logarithmic dependence on the size of the region is also found in explicit computations in relativistic models in $2 + 1$ dimensions such as the $O(2)$ and $O(3)$ scalars at their Wilson–Fisher fixed points [46, 47], the free massless scalar and the free massless Dirac fields [48, 49], and by the AdS/CFT correspondence [50]. A general argument was given for the coefficient of this log for opening angles close to $\pi$ in any relativistic CFT [51–53].

Here, we will consider the ground state of the scale-invariant theories in $2 + 1$ and $3 + 1$ dimensions on a torus and investigate the scaling of entanglement when the torus is partitioned into two cylinders. In $2 + 1$ dimensions, the torus has circumferences $L_x$ and $L_y$ and the observed cylindrical region $A$ has the length $L_A$, as shown in figure 1.
The Rényi EE for the cylindrical region $A$, of aspect ratio $u = L_A/L_x$, satisfies the area law with a finite subleading correction $[28, 33, 54–57]$,

$$S_n = \alpha \frac{2L_y}{\epsilon} - J_n(u; b) + \ldots$$

(1.1)

where $n$ is the Rényi index, and $b = L_x/L_y$. The leading area law term arises from short-range entanglement localized to the boundary between $A$ and $B$, and is proportional to the length of the boundary, $2L_y$. The coefficient $\alpha$ is non-universal, being cutoff-dependent. We will discuss the behavior of the finite universal term $J_n(u; b)$ in several theories on a torus with twisted boundary conditions, which is allowed by the non-trivial topology.

We emphasize that $J_n(u; b)$ is a universal scaling function that depends on the two aspect ratios: $b = L_x/L_y$ and $u = L_A/L_x$, and varies from theory to theory. It thus acts as a non-trivial fingerprint for quantum systems. This has been studied in several model systems and there are at least three analytical expressions for it. One expression was derived in the quantum Lifshitz model describing a compact free boson with the dynamics exponent $z = 2$ $[30, 32, 33]$. The ground state of this model has the conformal invariance in a spatial direction and therefore $J_n(u, b)$ can be analytically constructed in terms of the partition functions for two-dimensional CFTs. An explicit expression on the torus has been given for $n \geq 2$ $[33]$. The other expression was derived holographically using the Ryu–Takayanagi formula, which is valid within the AdS/CFT correspondence, and is expected to yield the EE for certain strongly interacting CFTs $[12, 13, 55]$. The third instance $[56]$, which is the simplest, is obtained within a toy theory called the extensive mutual information model $[58, 59]$. Although these three expressions are different, they share similar properties and take the same scaling form in various limits. The $J$-function can also be computed numerically for some simple theories $[37, 55–57]$. The $n = 2$ scaling function has also been determined by quantum Monte Carlo simulations at the quantum critical point of the two-dimensional Ising model in a transverse field by Inglis and Melko $[54]$. Although the quantum critical behavior of this model is described by the relativistic real scalar field at the Wilson–Fisher fixed point, Inglis and Melko found that the finite term of the second Rényi entropy is (surprisingly) well approximated by the scaling function of the quantum Lifshitz model, derived by Stephan and co-workers $[33]$.

In this paper, we study both analytically and numerically the behavior of the scaling function $J(u; b)$ for two massless relativistic models in $2+1$ dimensions, the
relativistic massless scalar field (which we refer to as the ‘free boson model’) and the massless Dirac field. We also study a free massless boson model with the dynamical exponent $z = 2$ and a free fermion massless model with the dynamical exponent $z = 2$ (the quadratic band touching model [60]), both also in $2 + 1$ dimensions. Although all four models are massless, and hence define scale invariant systems in $2 + 1$ dimensions, they describe very different types of fixed points. Indeed, in $2 + 1$ dimensions the free relativistic massless scalar field is unstable in the IR, and under a $\lambda \phi^4$ perturbation flows to the Wilson–Fisher fixed point. Similarly, the free massless Dirac field is an IR stable fixed point and, as such, it defines a stable phase of matter. In contrast, both $z = 2$ theories, the free boson and the quadratic band touching fermion model, describe systems at marginality (they are ‘asymptotically free’, and hence perturbatively renormalizable). These differences make the comparisons of the scaling functions interesting. In particular, here we focus on the toroidal geometry with twisted boundary conditions, investigate the IR flows of the EE in these geometries and compare its behavior in the free (non-compact) complex scalar field as well as the free (also non-compact) Dirac fermion.

We will not discuss here the interesting case of the compact relativistic boson, which should be regarded as a Goldstone boson of a spontaneously broken $U(1)$ symmetry, and hence describes a theory at the IR stable fixed point. The entanglement properties of this model have only been discussed in the literature analytically for the geometries of disks [61, 62] and numerically for cylinders [63]. Likewise, we will not discuss the von Neumann EE for the compactified quantum Lifshitz model on a torus. While its entanglement properties have been extensively discussed in several geometries, for the torus the only analytical results available are for Rényi entropies with $n \geq 2$ and not the von Neumann case, $n = 1$ [33].

1.1. Summary of the main results

In this paper, we study the function $J_n(u; b)$ in several free field theories. For the free relativistic boson CFT and the Dirac fermion CFT in $2 + 1$ dimensions, although the complete analytical form for $J_n(u; b)$ is unknown, at around $u = 1/2$, $J_n$ can still be analytically obtained in the thin torus limit [36]. As $b$ increases, $J_n(1/2)$ converges to a constant which only depends on the boundary condition in the $y$-direction. Here, $J_n(1/2)$ is the shorthand notation for $J_n(1/2; b \to \infty)$. We also extend this calculation to $3 + 1$ dimensions and calculate $J_n(1/2)$ in the thin torus limit for various boundary conditions. We verify our analytical expression by performing a numerical calculation on the lattice models.

We further explore the monotonicity of $J_n(1/2)$ and the connection with the F-theorem, which states that for the $2 + 1$ dimensional relativistic CFT, the subleading correction term of the von Neumann EE for a disk is a universal constant and can serve as an RG monotone [17, 39, 42, 45]. We explicitly add a mass term in these free systems and define some renormalized EE so that it is equal to $J(1/2)$, when the mass is zero and approaches zero as the mass increases. However, no matter how we define the renormalized EE, it cannot be an RG monotone. This is because as we change the boundary condition, $J_n(1/2)$ can take both positive and negative values. A similar issue happens in $3 + 1$ dimensions, where $J_n(1/2)$ changes sign as we change the boundary condition.
We also study the $z = 2$ free boson model in $2 + 1$ dimensions with twisted boundary conditions. Since the ground state wave function has conformal invariance, we can obtain the complete $J(u, b)$ by applying the replica trick method directly to the ground state wave function. In the wide torus limit $b \to 0$, $J_n(u)$ can be exactly mapped to the corner correction $J(\theta)$ defined on an infinite plane. In the thin torus limit, $J_n(1/2)$ takes the same form as that for a free boson CFT and a Dirac fermion CFT up to a prefactor. For completeness, we also study the fermionic quadratic band touching model [60], which is a critical fermionic system with $z = 2$, and analyze the $J_n(u)$ function numerically in various limits. The main difference from the other three models in $2 + 1$d is that $J_n(1/2)$ is always equal to zero for any $b$.

For these $2 + 1$ dimensional non-interacting scale-invariant models, $J_n(u)$ in the thin torus limit $L_y \to 0$ reads

$$2 \gamma_n = \begin{cases} 
\frac{1}{3} (1 + \frac{1}{n}) \log(2 \sin(\pi \lambda)), & \text{free complex boson CFT} \\
\frac{1}{6} (1 + \frac{1}{n}) \log(2 \sin(\pi \lambda)), & \text{Dirac fermion CFT} \\
2 \log(2 \sin(\pi \lambda)), & z = 2 \text{ complex boson} \\
0, & \text{fermionic quadratic band touching}
\end{cases}$$

(1.2)

where $\lambda \in (0, 1)$ denotes the twist along the $y$-direction and in the bosonic models, it is defined as follows

$$\phi(x, y + L_y) = e^{i 2 \pi \lambda} \phi(x, y).$$

(1.3)

In the fermionic models, $\lambda$ can be defined in the same way; $\lambda = 1/2$ corresponds to anti-periodicity. Here, $\gamma_n$ is (minus) the universal EE of a semi-infinite cylindrical region obtained by bi-partioning an infinite cylinder. For the first three theories, $\gamma_n$ has the same dependence on the twist $\lambda$ up to a prefactor and will diverge in the limit $\lambda \to 0$. Interestingly, the complex boson value of $\gamma_n$ is twice that of the Dirac fermion, and the quadratic band touching has a vanishing universal contribution in this limit. We note in passing that in all four theories, $\gamma_n$ vanishes identically for the special twist parameter $\lambda = 1/6$.

For the $3 + 1$ dimensional complex scalar CFT, in the thin torus limit with $L_y, L_z \to 0$,

$$2 \gamma_n^{3d} = \frac{1}{6} \left(1 + \frac{1}{n}\right) \log \left( \frac{\theta \left[ \frac{\lambda_2 - \frac{1}{2}}{\lambda_1 - \frac{1}{2}} \right] (\tau) \theta \left[ \frac{\lambda_2 - \frac{1}{2}}{-\lambda_1 + \frac{1}{2}} \right] (\tau)}{\eta(\tau)} \right)$$

(1.4)

where $\tau = i r = i L_y / L_z$ is the modular parameter of each of the two boundaries of $A$, which are 2-tori. $\theta[\alpha](\tau)$ is given in terms of a Jacobi theta function, equation (B.16).

The structure of this paper is as follows. We first discuss the two cylinder EE for relativistic free field theory with a twisted boundary condition in $2 + 1$ dimensions in section 3. We focus on the thin torus limit and define the renormalized EE. Then we compute the two-cylinder EE defined on the cylinder and the torus for $z = 2$ free boson theory with a twisted boundary condition in section 4. We consider various limits in both cases, and for comparison, we also numerically study the fermionic quadratic band touching model. We further study the two-cylinder EE for relativistic free-field theory with a twisted boundary condition in $3 + 1$ dimensions in section 5. We summarize and
conclude in section 6. The appendices are devoted to details of the calculations and techniques used in this paper.

2. General properties of torus entanglement in two dimensions

We review the basic properties of the universal torus function $J_n(u; b)$. Our present discussion is concerned with the thermodynamic limit of $J_n$, where lattice effects have been extrapolated away. First, the Rényi entanglement entropies of the cylindrical regions $A$ and $B$ are equal, since we work with the ground state, which is pure. This leads to the reflection symmetry: $J_n(1-u) = J_n(u)$. Further, the strong subadditive property of the EE implies [56] that $J_1(u)$ is a decreasing convex function of $u$ on the interval $(0, 1/2]$, for any value of $b$ and any choice of boundary conditions. For a fixed aspect ratio $b$, $J_1(1/2)$ is thus the smallest value of $J_1(u)$. These properties can be clearly seen in figures 4 and 5. We now consider various limits where we can make exact statements.

2.1. The thin slice limit

An important limit is the so-called thin slice limit $u \to 0$, figure 2(a), where the universal term diverges as

$$J_n(u \to 0; b) = \frac{\kappa_n}{u}, \quad (2.1)$$

where $\kappa_n$ is a universal coefficient which characterizes the theory. Interestingly, this is the same coefficient that dictates the universal Rényi entropy of a long strip living in the infinite plane: $S_{\text{strip}} = BL/\delta - \kappa L/w$, where $w$ is the strip’s width and $L$ is a scale used to regulate the large-distance divergence. Alternatively, one can consider the EE per unit length $S_{\text{strip}}/L$. The reason for the appearance of the same $\kappa$ in both geometries is that the boundary conditions along the $x$ and $y$ cycles of the torus will not influence the universal EE as $u \to 0$ (we assume, as is generically the case, that there are no zero modes in the compactified geometry).
2.2. The thin torus limit

As the name suggests, we take $L_y \to 0$, while keeping $L_A, L_x$ fixed, figure 2(b). This implies that the aspect ratio diverges $b \to \infty$, while $0 < u < 1$ remains fixed. In this case, the universal Rényi entropy will tend to a constant,

$$J_n(u; b \to \infty) = 2\gamma_n,$$

(2.2)

where $\gamma_n$ is independent of all length scales. This saturation comes about because we consider generic theories/boundary conditions which preclude the zero modes, so that the theory on the torus possesses a large gap $\sim 1/L_y$, and thus becomes insensitive to the length scales $L_x, L_A \gg L_y$. Furthermore, the universal part of the EE cannot depend on $L_y$ because no other length scale remains to form a dimensionless ratio (we work with the groundstate of a scale-invariant system). In contrast, when the zero modes are present (non-generic), $\gamma_n$ will depend on the scales $L_x, L_A$ and the short-distance cutoff $\epsilon$, as we illustrate in sections 3 and 5 with the free boson and fermion with periodic boundary conditions.

2.3. The wide torus limit

We take the opposite limit, $b \to 0$, by sending $L_y$ to infinity, but again keeping $L_A, L_x$ fixed, see figure 2(c). In this case, the EE will be dominated by the diverging length scale $L_y$. By translation invariance along the $y$-direction, $J_n$ is expected to scale extensively with $L_y$:

$$J_n(u; b \to 0) = L_y \cdot \frac{f_n(u)}{L_x} = \frac{1}{b} \cdot f_n(u)$$

(2.3)

where independent of $b$ we have made $f_n(u)$ dimensionless by factorizing $1/L_x$. The growth of $|J_n|$ with decreasing $b$ can be observed in figures 4 and 6 for the free boson and Dirac fermion CFTs, respectively. Furthermore, in a holographic calculation [55] for an interacting CFT, it was found that the relation $J_1 = f_1(u)/b$ is exactly obeyed for all $b < 1$. The $J$-function can be generalized to $3+1$ dimensional theories defined on a three-torus, as shown in figure 3 [56] In this case, $J_n$ depends on the aspect ratio of the subsystem $A$ and $u = L_A/L_x$, as well as on the two aspect ratios of the torus: $b_1 = L_x/L_y$ and $b_2 = L_y/L_z$. In the thin slice limit $u \to 0$, $J_n(u) \to 1/(u^2 b_1 b_2)$ [56], while in the thin torus limit $b_1, b_2 \to \infty$ and $J_n$ will saturate to $\gamma^{3d}_n(L_y/L_z)$, which only depends on the aspect ratio of the boundary of region $A$. 

Figure 3. A three-dimensional torus, with opposite faces of the box being identified. Region $A$ is a cylinder of length $L_A$, with two boundaries (blue/shaded).
3. Two-cylinder entropy for relativistic free field theory

The continuum Hamiltonian for the relativistic boson in $2 + 1$ d is

$$H = \frac{1}{2} \int \! \! dx \! \! dy \left( \Pi(x, y)^2 + (\nabla \phi(x, y))^2 + m^2 \phi(x, y)^2 \right),$$

(3.1)

where the integral is over the $L_x \times L_y$ torus. Since the $y$-direction is compactified into a circle of length $L_y$, we can decompose $\phi(x, y)$ into discrete Fourier modes in the $y$-direction

$$\phi(x, y) = \sum_{k_y} \frac{e^{-i k_y y}}{\sqrt{L_y}} \phi_{k_y}(x)$$

(3.2)

with quantized momentum $k_y$:

$$k_y = \frac{2\pi (p + \lambda)}{L_y},$$

(3.3)

where $p \in \mathbb{Z}$ and $\lambda$ parametrize the twist along the $y$-direction. In other words, we have

$$\phi(x, y + L_y) = e^{i2\pi \lambda} \phi(x, y).$$

(3.4)

The twist in the $x$-direction is $\lambda_x$. For a real scalar field, $\phi(x, y)$ is a Hermitian operator, so $\lambda, \lambda_x$ can only take the values $0$ or $1/2$, which correspond to periodic and anti-periodic boundary conditions, respectively. For a real bosonic field, the Fourier modes need to satisfy $\phi_{k_y}^\dagger(x) = \phi_{-k_y}(x)$, therefore equation (3.1) can be rewritten as

$$H = \sum_{k_y = -\infty}^{\infty} \frac{1}{2} \int \! \! dx \left[ \Pi_{k_y}^\dagger(x) \Pi_{k_y}(x) + (\partial_x \phi_{k_y}^\dagger(x))(\partial_x \phi_{k_y}(x)) + (m^2 + k_y^2) \phi_{k_y}^\dagger(x) \phi_{k_y}(x) \right].$$

(3.5)

Notice that only the $k_y \geq 0$ modes are independent degrees of freedom.

$$\phi_{k_y}(x) = \frac{1}{\sqrt{2}}(\phi_{k_y,1} + i\phi_{k_y,2}).$$

(3.6)

The Hamiltonian becomes

$$H = \sum_{k_y = -\infty}^{\infty} \frac{1}{4} \int \! \! dx \left[ \Pi_{k_y,1}^2(x) + (\partial_x \phi_{k_y,1})^2 + (m^2 + k_y^2)(\phi_{k_y,1})^2 \right]$$

$$= \sum_{k_y \geq 0} \frac{1}{2} \int \! \! dx \left[ \Pi_{k_y,1}^2(x) + (\partial_x \phi_{k_y,1})^2 + (m^2 + k_y^2)(\phi_{k_y,1})^2 \right]$$

$$+ \sum_{k_y > 0} \frac{1}{2} \int \! \! dx \left[ \Pi_{k_y,2}^2(x) + (\partial_x \phi_{k_y,2})^2 + (m^2 + k_y^2)(\phi_{k_y,2})^2 \right]$$

(3.7)

where we use $\phi_{k_y,I} = \phi_{-k_y,I}$ ($I = 1, 2$), and the component $\phi_{0,2} = 0$. Note that the $k_y = 0$ mode exists only when $\lambda = 0$. This Hamiltonian consists of a sum of decoupled $1 + 1$ d free bosons, with the effective mass $\sqrt{m^2 + k_y^2}$. 

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Similarly, for Dirac fermions in $2 + 1d$, the Hamiltonian on the torus can be expressed as a sum over an infinite number of $1 + 1$-dimensional massive Dirac fermions with a mass given by $\sqrt{m^2 + k_y^2}$.

### 3.1. The shape dependence

We bi-partition the torus into two cylinders, as shown in figure 1, and calculate the entanglement entropy between them. Since a full analytical treatment of the shape dependence for the torus EE of relativistic bosons and the Dirac fermion is difficult, we compute the EE numerically by using the method explained in appendix D. By exploiting the decomposition into the decoupled $(1 + 1)$-dimensional massive chains described above, we are able to work with very large lattices. A similar analysis was previously done for the free Dirac fermion and free boson CFT, but only the quantity $-S(u) + S(1/2) = J(u) - J(1/2)$ was obtained, and only for a limited set of boundary conditions [55–57]. Here, we determine the full $J(u; b)$, including the $J(1/2; b)$ part, by subtracting the area law contribution from $S$. The result is shown in figures 4 and 5 for the boson and fermion, respectively. In these figures, two sets of boundary conditions are used: $(\lambda_x, \lambda) = (0, 1/2)$ and $(1/2, 1/2)$, where 0 means periodic while 1/2 is anti-periodic. As mentioned in the introduction, $J_1(u)$ is a decreasing convex function of $u$ on the interval $(0, 1/2]$, for any value of $b$ and any choice of boundary conditions. This follows from the strong subadditivity property of the EE. For a fixed aspect ratio $b$, the minimum

**Figure 4.** (a) The universal EE on the torus $J_1(u)$ for the non-interacting real boson CFT for various $b = L_x/L_y$ with anti-periodic boundary conditions (BCs) in the $y$-direction and periodic BCs in the $x$-direction: $(\lambda_x, \lambda) = (0, 1/2)$. The result is doubled for a complex boson. We use $L_x = 400$ lattice sites in the $x$-direction. (b) $J_1(u)$ with anti-periodic BCs in both directions: $\lambda_x = \lambda = 1/2$. The other parameters are the same as in (a). Since $J_1(u)$ is symmetric around $u = 1/2$, we only show $u \in (0, 1/2]$ in (a) and $u \in [1/2, 1)$ in (b). The inset shows the difference, $\Delta J_1(u)$, between the two choices of BCs.
value of $J_1(u)$ is thus $J_1(1/2)$. $J_n(1/2; b)$ is plotted as a function of $b$ in figure 6 for both the fermion and boson, and the Rényi indices $n = 1, 2$.

An important limit is the so-called thin slice limit $u \rightarrow 0$, where the universal term diverges as

$$J_1(1/2; b) \rightarrow 0.2310 \quad \text{and} \quad J_2(1/2; b) \rightarrow 0.174.$$
\[ J_n(u \to 0; b) = \frac{\kappa_n}{u} \]  

(3.8)

where \( \kappa_n \) is a universal coefficient characterizing the theory. Interestingly, this is the same coefficient that dictates the universal Rényi entropy of the long thin strip. The point is that the boundary conditions along the \( x \)- and \( y \)-directions will not alter the value of \( \kappa \) (we assume that neither boundary condition is simultaneously periodic to avoid the zero mode).

### 3.2. Thin torus limit and semi-infinite cylinder

We examine another important limit—the thin torus—obtained by sending \( b \to \infty \), while the ratio \( u \) remains fixed. In this case, the universal Rényi entropy saturates to a pure constant when \( 0 < \lambda < 1 \),

\[ J_n(u; b \to \infty) \to 2\gamma_n(\lambda), \]  

(3.9)

where \( \gamma_n \) is the subleading term in the EE for the bi-partition of an infinite cylinder into semi-infinite cylinders, as explained in the introduction. We note that the universal constant \( \gamma_n \) only depends on the twist along the \( y \)-direction, \( \lambda \). This becomes manifest in figures 4 and 5, where the left panels have \( \lambda_x = 0 \), while the right ones have \( \lambda_x = 1/2 \). Their differences, \( \Delta J_1(u) \), for the two choices of \( x \)-boundary conditions are shown in the insets. Indeed, at large \( b \), \( J_1(u) \) becomes insensitive to \( \lambda_x \), and \( \Delta J_1(u) \approx 0 \).

We can analytically compute \( \gamma_n \) for all \( n \) using the 1d decomposition given above, equation (3.7). For a finite interval of length \( L_A \), each 1 + 1d massive chain, with an effective mass of \( \sqrt{m^2 + k_y^2} \), contributes an EE

\[ S_{n}^{1d}(k_y) = -\frac{1}{12} \left( 1 + \frac{1}{n} \right) \log \left[ (m^2 + k_y^2)\epsilon^2 \right], \]  

(3.10)

where the length of the interval is taken to be much larger than the inverse mass, \( L_A \gg 1/\sqrt{m^2 + k_y^2} \); \( \epsilon \) is the UV cutoff, and \( n \) is the Rényi index. The total EE is then

\[ S_n = -\sum_{k_y} \frac{1}{12} \left( 1 + \frac{1}{n} \right) \log \left[ (m^2 + k_y^2)\epsilon^2 \right]. \]  

(3.11)

We first consider the massless case, \( m = 0 \),

\[ S_n = -\sum_{p \in \mathbb{Z}} \frac{1}{6} \left( 1 + \frac{1}{n} \right) \left( \log |2\pi(p + \lambda)| + \log \frac{\epsilon}{L_y} \right) \]  

(3.12)

where \( k_y = 2\pi(p + \lambda)/L_y \). \( \lambda \) denotes the twisted boundary condition in the \( y \)-direction, \( \phi(x, y + L_y) = e^{2\pi i \lambda} \phi(x, y) \). This expression can be regularized and we get

\[ S_n = \alpha \frac{L_y}{\epsilon} - 2\gamma_n. \]  

(3.13)

By using the Hurwitz zeta function regularization method discussed in appendix B.1, we have for \( 0 < \lambda < 1 \)
\[ 2\gamma_n = \sum_{p \in \mathbb{Z}} \frac{1}{6} \left( 1 + \frac{1}{n} \right) \log |p + \lambda| = \frac{1}{6} \left( 1 + \frac{1}{n} \right) \log (2 \sin(\pi \lambda)). \]  

(3.14)

For the real boson field, \( \lambda \) can only be 0 or 1/2. When \( \lambda = 1/2 \), \( 2\gamma_1 = \frac{\log 2}{3} = 0.231 \). This matches the numerical results shown in figure 6. For the Dirac fermion, \( \lambda \) can take values between 0 and 1, which corresponds to the arbitrary twisted boundary condition and can be realized by coupling the Dirac fermion to a \( U(1) \) gauge field. In this case, \( \gamma_n \) takes the same formula shown in equation (3.14). For a complex boson, \( \lambda \in (0, 1) \) and \( \gamma_n \) is twice as large, \( \gamma_n \) for a real boson field or a Dirac fermion field. \( \gamma_n \) in the thin torus limit takes the same form for both the \( \lambda = 1 \) free boson and Dirac fermion model.

Strictly speaking, equation (3.14) only works for \( 0 < \lambda < 1 \). As \( \lambda \to 0 \), the effective mass of the \( k_y = 0 \) \((1 + 1)d \) mode vanishes and the use of equation (3.14) will yield an incorrect result. This issue can be resolved by separately treating the massive modes \( k_y \neq 0 \) and the zero mode with \( k_y = 0 \). The \( k_y = 0 \) zero mode contributes a subleading term to the EE that will depend on the aspect ratio \( u = L_A/L_x \), as well as on \( L_x/\epsilon \), where \( \epsilon \) is the UV cutoff. In addition, this subleading term will depend on the twist along the \( x \)-direction, \( \lambda_x \). Since the zero mode is not suppressed by a \( \sim 1/L_y \) gap, it is sensitive to the entire geometry, including the boundary conditions along the ‘long direction’, \( x \). For the special case of periodic boundary conditions along \( x \), \( \lambda_x = 0 \), we obtain

\[ 2\gamma_n = \frac{1}{6} \left( 1 + \frac{1}{n} \right) \log \left( \frac{L_x}{\pi \epsilon} \sin \left( \frac{\pi L_A}{L_x} \right) \right). \]  

(3.15)

The massive modes, after regularization, contribute a finite non-universal constant.

Equation (3.15) shows a \( \log(1/\epsilon) \) divergence, and is the classic ‘chord-length’ expression for the EE of a single interval in a \( 1 + 1 \) d CFT on a circle [3]. The scaling in equation (3.15) was numerically observed for a massless Dirac fermion on a thin torus in [55]. We note that when a small twist along \( y \) is introduced, \( \lambda > 0 \), an order \( \lambda \) mass is induced for the zero mode that cuts off the \( \log(L_A/\epsilon) \) divergence to \( \log(1/\lambda) \).

At \( \lambda = 0 \), but for generic \( \lambda_x \), the answer for \( \gamma_n \) is expected to differ from equation (3.15). Indeed, in the case of a real \((1 + 1)d \) boson with anti-periodic boundary conditions \( \lambda_x = 1/2 \), numerical calculations on the lattice [57] have shown deviations from equation (3.15). More recently, the Rényi entropies for a free boson CFT with the Rényi index \( n \neq 1 \) were analytically computed as a function of \( \lambda_x \) by Shiba [64], and they differ from equation (3.15); see the added note after the conclusions.

Finally, before moving to the massive case, we want to emphasize a subtle difference between our analytical and numerical calculation. For the free boson model in equation (3.7), we expressed the Hamiltonian in the continuum, hence we have a sum over a set of discrete and yet infinite number of Fourier modes in equation (3.11). Alternatively, we can consider a lattice version of the free boson Hamiltonian defined on the torus, given in equation (D.8). The effective mass for each \( 1 + 1 \) d chain then changes to \( \sqrt{2 - 2\cos(k_y)} + m^2 \) (where the lattice spacing is set to unity), and we only need to sum over a finite number of momenta. In the limit of large lattices that we consider, our results agree with the analytical calculations performed in the continuum. In the thin torus limit, a detailed comparison between the analytical and lattice results is presented in figure 6, where good agreement is found.
3.2.1. Massive case. We now consider the effect of adding a mass to the theory, leading to \( m^2 \phi^2 \) term in the Lagrangian. We note that for periodic boundary conditions \( (\lambda = 0) \), a finite mass needs to be introduced to cure the divergence of \( \gamma_n \), equation (3.14). Therefore, we need to calculate this infinite sum

\[
g(\lambda, mL_y) = \sum_{p=-\infty}^{\infty} \log \left[ (mL_y)^2 + (2\pi)^2 (p + \lambda)^2 \right].
\]  

(3.16)

This can be regularized by using the Epstein zeta function regularization method (see appendix B.2), and we have

\[
g(\lambda, mL_y) = \log \left[ 2 \cosh(mL_y) - 2 \cos(2\pi\lambda) \right].
\]  

(3.17)

This bare result was also computed in [37] and [65]. Naively, the universal EE would then be proportional to this quantity. However, we notice a strange property: equation (3.17) diverges linearly in the large mass limit, \( g(\lambda, mL_y \gg 1) \sim mL_y + \ldots \). This is at odds with the intuition that the infinite mass fixed point, which describes a state without spatial entanglement, should have \( \gamma = 0 \). This divergence points to a deeper problem with our naive procedure to extract the universal EE by subtracting the area law contribution. Indeed, when the theory is away from the massless fixed point, the additional length scale \( 1/m \) renders the subtraction procedure ambiguous. A similar situation occurs when considering the EE of a region \( A \) in infinite flat space instead of a torus. For example when \( A \) is a perfect disk living in \( \mathbb{R}^2 \), the universal subleading term in the EE is often called \( -F \). A statement called the \( F \)-theorem has been shown for relativistic theories [38]: \( F \) decreases monotonically along an RG flow. If we thus have a UV CFT that flows into an IR one, the following inequality will hold \( F_{\text{UV}} > F_{\text{IR}} \).

Now, for a free massive boson, a naive computation of \( F \) by subtraction of the area law leads to linear divergence \( F \sim -mR \) at large mass. Here, \( R \) is the radius of the disk. This is exactly analogous to the divergence that we have seen in equation (3.17). However, the divergence of \( F \) is at odds with the \( F \)-theorem since the infinite mass fixed point is trivial, and has \( F = 0 \). The cure for the disk is known: one needs to consider a renormalized EE, \( F(R) = -S + R \partial S/\partial R \), where \( S \) is the full EE (computed using any given regulator) [38, 40]. At conformal fixed points, \( F = F \) agrees with the expected CFT value, while \( F(R) \) decreases monotonically along an RG flow linking two fixed points.

Going back to our situation on the torus, we can define a renormalized \( \gamma \) as follows [66]:

\[
2\tilde{\gamma}(L_y) = -S + L_y \frac{\partial S}{\partial L_y},
\]  

(3.18)

where \( S \) is evaluated here in the thin torus limit \( b \to \infty \) (this can be naturally generalized beyond the thin torus limit to all \( u, b [66] \)). Equation (3.18) is the one-to-one analog of the renormalized disk EE used in the \( F \)-theorem, discussed above. In contrast to the \( F \)-theorem, however, there is no proof that this quantity is the ‘natural’ one to consider—in the sense of being an RG monotone, say. Our motivation for using equation (3.18) for defining the EE away from a fixed point is its simplicity, and its connection to the disk EE prescription (which is also non-unique when \( A \) is not a disk). In the case of the massive complex boson, we thus find:

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Two-cylinder entanglement entropy under a twist

\[
2\tilde{\gamma}_n(mL_y) = \frac{1}{6} \left( 1 + \frac{1}{n} \right) \left[ g(\lambda, mL_y) - L_y \frac{\partial g(\lambda, mL_y)}{\partial L_y} \right] \\
= \frac{1}{6} \left( 1 + \frac{1}{n} \right) \left\{ \log[2 \cosh(mL_y) - 2 \cos(2\pi\lambda)] - \frac{mL_y \sinh(mL_y)}{\cosh(mL_y) - \cos(2\pi\lambda)} \right\}.
\]

In the small and large \( t = mL_y \) limits, the renormalized EE becomes

\[
2\tilde{\gamma}_1(mL_y) = \frac{t^2}{48 \sin^2(\pi\lambda)} + \frac{t^4(2 + \cos(2\pi\lambda))}{384 \sin^4(\pi\lambda)} + O(t^6)
\]

\[
\tilde{\gamma}_1(t \to 0) = \gamma_1 - \frac{t^2}{48 \sin^2(\pi\lambda)} + O(t^4)
\]

\[
\tilde{\gamma}_1(t \to \infty) = -\frac{\cos(2\pi\lambda)}{6} t e^{-t} + \ldots
\]

respectively. We see that the leading correction at small \( t \) is always negative, i.e. \( \tilde{\gamma}_n \) initially decreases under the RG flow, irrespective of \( \lambda \). Interestingly, a similar result was found in the context of strongly interacting holographic CFTs at \( n = 1 \) [66]. It would be interesting to investigate the RG flow of \( \tilde{\gamma} \) in the vicinity of more general UV fixed points (weak detuning regime), in the same spirit as was done for the EE of a disk [67]. Second, we observe that \( \tilde{\gamma}_n \) decays to zero exponentially fast in the deep IR limit \( t \to \infty \), equation (3.21), as shown in figure 7. However, as shown in that figure, it is not a monotonically decreasing function for all values of the twist \( \lambda \). Indeed, equation (3.21) shows that \( \tilde{\gamma}_n \) becomes strictly increasing at large \( t \) when \( 0 < \lambda < 1/4 \). The non-monotonicity could have been anticipated, because when \( m = 0 \), \( \gamma_n \) can be strictly negative for certain values of \( \lambda \), and we know that at the trivial infinite mass fixed point it will vanish, \( \tilde{\gamma}_n = 0 \). Interestingly, at \( \lambda = 1/6 \), \( \tilde{\gamma}_n \) is zero at both fixed points, \( mL_y = 0 \) and \( \infty \). However, it is a non-trivial function along the RG flow that interpolates between these two fixed points, as shown in figure 7.
4. 
4.1. \( z = 2 \) free bosons

The Hamiltonian for a free boson field \( \phi \) with a dynamical scaling exponent \( z = 2 \) is
\[
H = \int d^2x \left( \frac{1}{2} \left( \Pi^2 + [\nabla^2 \phi]^2 \right) \right),
\]
where \( \Pi \) is its conjugate canonical momentum. When \( \phi \) is compact (i.e. \( \phi \equiv \phi + 2\pi R \)), this model is a quantum Lifshitz model and describes the quantum dimer model at a critical point [22]. There have been a lot of references discussing EE in this model [27–34]. Here we study the entanglement of the non-compact version, focusing on the subleading universal term on the torus, when there is a twist in the \( y \)-direction. The ground state wavefunction of equation (4.1) has a simple and elegant form
\[
|\psi\rangle = \frac{1}{\sqrt{Z}} \int [d\phi] e^{-\frac{1}{2} S[\phi]} |\phi\rangle.
\]
Here \( Z \) is the partition function of the free boson CFT in a two-dimensional Euclidean spacetime, and \( S[\phi] \) is the corresponding Euclidean action,
\[
Z = \int [d\phi] e^{-S[\phi]}, \quad S[\phi] = \frac{1}{2} \int d^2 x \ (\nabla \phi)^2.
\]

For the above wave function, if \( \phi \) is non-compact, by using the replica trick directly on the wave function, we have [27]
\[
\text{Tr} \rho^n_A = \left( \frac{Z(A)Z(B)}{Z(A \cup B)} \right)^{n-1}
\]
where \( Z(A) \) and \( Z(B) \) are the free boson partition functions on region \( A \) and \( B \), respectively, with a Dirichlet boundary condition \( \phi = 0 \) on the boundary. \( Z(A \cup B) \) is the boson partition function on the entire space, with the same boundary conditions as those imposed on the \( 2 + 1 \) dimensional theory. The Rényi entanglement entropy is then
\[
S_n = -\log \left( \frac{Z(A)Z(B)}{Z(A \cup B)} \right),
\]
which is independent of the Rényi index \( n \). Below we consider the entire system \( A \cup B \) to be an open cylinder and torus, respectively.

4.1.1. Two-cylinder entropy on a cylinder. We first study the groundstate on the open cylinder and calculate the two-cylinder EE with a twisted boundary condition in the \( y \)-direction, which is also denoted by \( \lambda \) and is defined through \( \phi(x, y + L_y) = e^{2\pi i \lambda} \phi(x, y) \).
Two-cylinder entanglement entropy under a twist

We impose a Dirichlet boundary condition with $\phi = 0$ on both ends of the entire cylinder $A \cup B$. As shown in figure 8, $Z(A)$, $Z(B)$ and $Z(A \cup B)$ are all partition functions defined on the cylinder. For the real boson, the only possible twisted boundary condition corresponds to $\lambda = 1/2$. Under this twist, the partition function of the cylinder (after regularization) is equal to

$$Z = \sqrt{2} \frac{\eta(\frac{1}{2})}{\eta^2(\frac{1}{2})} = \sqrt{2} \frac{\eta(2\tau)}{\eta(4\tau)}$$

(4.6)

where $\tau = iL_x/L_y$ is the modular parameter and $u = L_A/L_x$. The explicit forms of $\eta(\tau)$, $\theta_2(\tau)$ and $\theta_4(\tau)$ are shown in appendix A. According to equation (4.5), the subleading correction $-J_n(u)$ of EE is equal to

$$J_n(u) = \frac{1}{2} \log \left( \frac{2\eta(2u\tau)\eta(2(1-u)\tau)\theta_4(2\tau)}{\eta(2\tau)\theta_2(2u\tau)\theta_4(2(1-u)\tau)} \right).$$

(4.7)

In the thin cylinder limit $\tau \to \infty$, if the ratio $u$ takes a finite value, the cylinder partition function $Z \to \sqrt{2}$ and therefore $J_n(u) = \log(\sqrt{2} \times \sqrt{2}/\sqrt{2}) = \log(\sqrt{2})$.

For a complex boson, $\lambda \in (0, 1)$. The partition function on the cylinder (after regularization) under a general twist is equal to

$$Z = e^{\pi i (\lambda - \frac{1}{2})} (1 - e^{-2\pi i \lambda}) \frac{\eta(\frac{1}{2})}{\theta[\lambda - \frac{1}{2}, -\frac{1}{2}]} \frac{\theta[1/2]}{\theta[1/2]}$$

$$= (1 - e^{-2\pi i \lambda}) \frac{\eta(2\tau)}{\theta[\lambda - \frac{1}{2}, -\frac{1}{2}]} (2\tau).$$

(4.8)

The explicit form of the $\theta[a, b] (\tau)$ function is given in equation (A.1). According to equation (4.5), the universal EE contribution for the two-cylinder bi-partition of the open cylinder in the case of a complex boson is

Figure 8. The bipartition of a cylindrical geometry. We impose Dirichlet boundary conditions $\phi = 0$ at both ends. The universal part of the EE is $-\mathcal{J}$, which is generally distinct from the analogous quantity defined on a torus, $-J$.

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Two-cylinder entanglement entropy under a twist

\[ \mathcal{J}_n(u) = \log \left( \frac{\eta(2u\tau)\eta(2(1-u)\tau)\theta\left[\frac{\lambda-\frac{1}{2}}{2}\right](2\tau)}{\eta(2\tau)\theta\left[\frac{\lambda-\frac{1}{2}}{2}\right](2u\tau)\theta\left[\frac{\lambda-\frac{1}{2}}{2}\right](2(1-u)\tau)} \right) + \log \left( 1 - e^{-2\pi i \lambda} \right) \quad (4.9) \]

where \( \mathcal{J}_n(u) \) is written here in terms of the theta and Dedekind \( \eta \) functions, and is fairly non-trivial. In the following sections, we examine the various limits of \( \mathcal{J}_n \).

**The thin slice limit.** The partition function for cylinder \( A \) of the complex boson is equal to

\[ Z(A) = e^{\frac{\pi}{12}w} \prod_{n=1}^{\infty} \frac{1}{(1 - e^{2\pi i \lambda} e^{\frac{2\pi i n}{u}})(1 - e^{-2\pi i \lambda} e^{\frac{2\pi i n}{u}})} \quad (4.10) \]

If we keep the ratio \( b = L_x/L_y \) finite in the thin slice limit \( u \to 0 \), \( Z(A) \to e^{\frac{\pi}{12}w} \). The remaining contribution to \( S_n \) is \( \log[Z(B)/Z(A \cup B)] \approx 0 \), and therefore we have

\[ \mathcal{J}_n(u \to 0) = \frac{\pi c}{24ub} = \frac{\pi c L_y}{24 L_A} \quad (4.11) \]

where \( c \) is the central charge for the classical two dimensional conformal field theory. For the complex boson we discuss here, it is equal to \( c = 2 \), while for the real boson, \( c = 1 \). Notice that in this limit, \( \mathcal{J}_n(u) \) is independent of the twists in the \( x \)- and \( y \)-directions.

**The thin cylinder limit.** We take \( L_y \to 0 \) at fixed \( u \), which means \( \tau \to i \infty \). Then the first term in equation (4.9) is equal to \( \log(e^{\pi i (\lambda - \frac{1}{2})}) \). When combined with the second term in \( S_n \), we find that the EE saturates to a constant that is independent of any length scale, \( \gamma_n \):

\[ \gamma_n = \log \left( (1 - e^{-2\pi i \lambda}) e^{\pi i (\lambda - \frac{1}{2})} \right) = \log \left[ 2 \sin(\pi \lambda) \right] \quad (4.12) \]

which is exactly the same \( \gamma_n \) that appears in the universal EE of a thin torus, equations (1.2) and (4.15) below. However, in the torus case, the EE tends to \( 2 \gamma_n \) instead because of the presence of two boundaries. The agreement, up to this trivial factor of two, between the open-cylinder and torus geometries in the \( L_y \to 0 \) limit, follows because the EE is insensitive to degrees of freedom far from the entanglement cut, where distances ought to be compared to the limiting length scale \( L_y \). We note that the \( \lambda \)-dependence of \( \gamma_n \) for the \( z = 2 \) boson is the same as for the \( 2 + 1 \)-dimensional free boson and Dirac fermion CFTs, as given in equation (3.14), up to an overall prefactor.

The \( \mathcal{J} \)-function in equation (4.9) can be compared with that for the quantum Lifshitz model, which takes this form on the open cylinder [34]:

\[ \mathcal{J}_1^{QLM}(u) = \log \left( \frac{\eta(2\tau)}{\eta(2\tau u)\eta(2\tau(1-u))} \right) - \frac{1}{2} \log[2u(1-u)|\tau|] + W(\tau,R) \quad (4.13) \]

where \( W(\tau,R) \) is a \( u \)-independent term coming from the zero mode sector for the compact boson with a compactification radius \( R \). The boson is assumed here to be periodic along \( y \). In the thin slice limit, \( \mathcal{J}_1^{QLM}(u \to 0) = \pi/(24ub) \), which is the same as \( \mathcal{J}_1 \) of the non-compact \( z = 2 \) boson, equation (4.9). In the thin cylinder limit, \( L_y \to 0 \), \( \mathcal{J}_1^{QLM} \) tends to a shape-independent constant, \( -\log(\sqrt{4\pi R}) + \frac{1}{2} \), which only depends on \( R \) [28, 34].
As \( R \to \infty \), one recovers the non-compact boson answer. Indeed \( J_{QLM}^{1} \to -\infty \) is the same as \( \gamma_1 \) in the non-compact case in the limit \( \lambda \to 0 \), equation \((4.12)\).

**4.1.2. Two-cylinder entropy on a torus.** If the total system is on a torus, there are two boundaries between \( A \) and \( B \). By using the partition function of the torus \( Z(A \cup B) \), equation \((C.13)\), we can write down the \( J \)-function for a two-cylinder EE defined on the torus,

\[
J_n(u) = \log \left( \frac{\eta(2u\tau)\eta(2(1-u)\tau)\theta \left[ \frac{-\lambda - \frac{1}{2}}{-\lambda + \frac{1}{2}} \right] (\tau)\theta \left[ \frac{-\lambda + \frac{1}{2}}{-\lambda - \frac{1}{2}} \right] (\tau)}{\eta^2(\tau)\theta \left[ \frac{\lambda - \frac{1}{2}}{-\frac{1}{2}} \right] (2u\tau)\theta \left[ \frac{\lambda + \frac{1}{2}}{-\frac{1}{2}} \right] (2(1-u)\tau)} \right) + 2 \log \left( 1 - e^{-2\pi i\lambda} \right)
\]

where \( \lambda_x \) and \( \lambda \) represent the twists in the \( x \)- and \( y \)-directions, respectively. \( \quad \quad \quad \quad \quad \quad \quad \quad (4.14) \)

**Thin torus limit.** In the thin torus limit, \( J_n(u) \) approaches a constant \( 2\gamma_n \) when \( \lambda > 0 \), which is independent of \( \lambda_x \), and is twice the value of the corresponding coefficient for a bipartition of an infinite cylinder into two equal halves:

\[
2\gamma_n = 2 \log \left[ (1 - e^{-2\pi i\lambda})e^{\pi i(\lambda - \frac{1}{2})} \right] = 2 \log \left( 2 \sin(\pi \lambda) \right).
\]

The overall factor of two arises because of the two boundaries between \( A \) and \( B \). Therefore, the \( z = 2 \) massless free boson, the free boson CFT and the Dirac fermion CFT all have the same twist dependence but with different overall prefactors. We numerically obtain \( 2\gamma_1 \) by calculating \( J_1(1/2) \) on the lattice at a large \( b \) for these three
models and the results are shown in figure 9. For $\lambda$ close to $1/2$, the numerical results agree perfectly with the analytical expression for $2\gamma_1$. As $\lambda$ decreases, the Dirac fermion and free boson CFT start to show deviations from $2\gamma_1$. This is because as $\lambda$ decreases, the $1+1$-dimensional modes around $k_y=0$ have smaller effective masses, and larger $b$ and $L_y$ are therefore required to obtain better agreement.

*The wide torus limit.* We now consider the limit that is opposite to that of the thin torus, namely the wide torus: $L_y/L_x \to \infty$ (figure 2(c)). If $u$ is finite, we have

$$\log Z(A) + \log Z(B) = \pi c \frac{1}{24 b u (1 - u)}.$$  \hspace{1cm} (4.16)

This result is obtained by using the derivation in equation (C.19). In contrast, the partition function $Z(A \cup B)$ on the entire torus will contribute a $u$-independent term. In the limit $b \to 0$, it is

$$\log Z(A \cup B) = \frac{\pi c}{6b}.$$ \hspace{1cm} (4.17)

Therefore we have

$$J_n(u) = \frac{\pi c}{24b} \frac{1}{u(1 - u)} - \frac{\pi c}{6b} \hspace{1cm} (4.18)$$

where for a real bosonic field, $c = 1$, while $c = 2$ for the complex boson. Equation (4.18) can be re-written as

$$J_n(u) = \frac{2\pi}{b} a_n(\theta), \hspace{1cm} \theta = 2\pi u,$$ \hspace{1cm} (4.19)

where

$$a_n(\theta) = \frac{c}{12} \frac{(\theta - \pi)^2}{\theta (2\pi - \theta)}.$$ \hspace{1cm} (4.20)

We have used the natural mapping between the cylinder’s normalized length $u$, and the angle $\theta = 2\pi u$. Actually, $a_n(\theta)$ is the coefficient of the logarithmic term in the EE for the $z=2$ boson, which arises in the presence of a sharp corner of opening angle $\theta$ in the entangling surface [27]. This is because the infinite cylinder can be mapped to the infinite plane through a conformal transformation. The transformation maps an infinite strip of width $u$ living on the cylinder to a wedge of angle $\theta = 2\pi u$ on the plane. To make this clear, we can also directly calculate the corner correction of EE for the $z=2$ boson model. Since $\log Z$ for a wedge with an opening angle $\theta$ is equal to [68]

$$\log Z = -\frac{c\theta}{24\pi} \left(1 - \frac{\pi^2}{\theta^2}\right) \log(L/\epsilon).$$ \hspace{1cm} (4.21)

Following [27], one readily finds that the contribution to the entanglement entropies follows from adding the corner contribution to the free energies of regions $A$ (with angle $\theta$) and $B$ (with angle $2\pi - \theta$), leading to the result that the Rényi entropies have a subleading logarithmic contribution associated with a corner of angle $\theta$ of the form

$$S_n = B_n L - a_n(\theta) \log(L/\epsilon) + \ldots$$ \hspace{1cm} (4.22)
where the corner coefficient for the $z = 2$ boson is given in equation (4.20). We emphasize that this direct relation between the wide-torus limit of $J_n(u)$ and the corner coefficient $a_n(\theta)$ arises here because the ground state is invariant under the infinite group of the two-dimensional conformal transformations [22]. This is not the case for general critical theories like the gapless boson and Dirac fermion CFTs.

The case $\lambda = 0$ must be treated separately, just as was done for CFTs; we leave such analysis for the future.

4.2. Fermionic quadratic band touching

The fermionic quadratic band touching (QBT) model describes free fermions with a quadratic energy dispersion ($z = 2$). The low-energy Hamiltonian for the QBT takes the form

$$H = \int \frac{d^2k}{(2\pi)^2} \Psi^\dagger(k) \begin{pmatrix} k_x^2 - k_y^2 & -2ik_xk_y \\ 2ik_xk_y & -k_x^2 + k_y^2 \end{pmatrix} \Psi(k),$$

where we have defined the two-component spinor $\Psi(k) = (\psi_1(k), \psi_2(k))^T$. We have set the band-curvature scale $M$ to unity, $k_i k_j / M \to k_i k_j$, as it will play no role in our discussion. In contrast to the Dirac fermion, this model corresponds to a critical point not a critical phase, and has a finite density of states at the band touching point [60].

The QBT model naturally satisfies the area law due to the absence of an extended Fermi surface. On the square torus, $b = 1$, the $u$-dependence of the correction $J_1(u) - J_1(1/2)$ has been numerically studied in [55]. Unlike for the $z = 2$ free boson model, it is not known whether the ground state for the QBT model is connected to a two-dimensional CFT, and therefore the analytical methods used to study the $z = 2$ free boson cannot be applied.

We numerically study the $u$ and $b$ dependence of $J_n(u; b)$ and find that the different Rényi indices lead to similar behavior. Therefore, we will only focus on $n = 1, 2$. Different from the other three models we have studied in $2 + 1$ dimensions, we find that when $A$ covers half the torus, $J_n(1/2; b)$ vanishes for arbitrary boundary conditions along $x$ and $y$:

$$J_n(1/2; b) = 0.$$  \hfill (4.24)

This is true for all aspect ratios $b$. In particular, taking the thin torus limit $b \to \infty$, equation (4.24) implies that $\gamma_n = 0$. This behavior at $u = 1/2$ distinguishes the QBT from all the critical theories that have been studied so far, and it would be interesting to understand the physical reasons underlying equation (4.24). As shown in figure 10, as $u \to 1/2$, all the curves indeed approach zero.

In the wide torus limit, $b \to 0$, we previously found that the $z = 2$ boson has $\lim_{b \to 0} b J_n(1/2; b) = 0$, which also holds for the QBT by virtue of equation (4.24). In contrast, the non-interacting boson and Dirac fermion CFTs have $J_n(1/2) \sim 1/b$ as $b \to 0$. This is also the case for certain CFTs described by the AdS/CFT holographic correspondence [55]. For $u \neq 1/2$, when $b \ll 1$, the QBT has $J_n(u) \sim 1/b$, in accordance with our general argument. In the inset of figure 10, we plot $J_1(1/4)$ and $J_2(1/4)$ as a function of $1/b$ to illustrate this fact.
In the thin slice slice limit $u \to 0$, taken at fixed $b$, we obtain the expected divergence \cite{55, 56}
\[ J_n(u) = \frac{\kappa_n}{bu} = \frac{\kappa_n L_y}{L_A}, \]
with $\kappa_1 = 0.182$, $\kappa_2 = 0.263$. \hfill (4.25)

We first note that $\kappa_2 > \kappa_1$, which is distinct from the behavior exhibited by the free boson and Dirac fermion CFTs, where $\kappa_n$ decreases with $n$ for $n = 1, 2, 3, 4, \infty$ \cite{69}. The value equation (4.25) can be compared with those of the free Dirac fermion \cite{48, 69}: $\kappa_{1\text{Dirac}} = 0.0722$ and $\kappa_{2\text{Dirac}} = 0.0472338$, which are 2.5 and 5.6 smaller than for the QBT, respectively. This is in line with the heuristic expectation that $\kappa$ is a measure of the gapless degrees of freedom, since a QBT can be split into a pair of Dirac fermions at different momenta. The splitting can be accomplished by adding a term of the form $A\sigma_z$ to the Hamiltonian equation (4.23).

5. Free relativistic theories in $3 + 1$ dimensions

We now consider the free relativistic complex boson, and the four-component Dirac fermion in three spatial dimensions. Let us consider their groundstate on the three dimensional torus $L_x \times L_y \times L_z$, and take region $A$ to be the cylinder $L_A \times L_y \times L_z$, as shown in figure 3. The entanglement entropies will take the following form:
\[ S_n = \alpha_n \frac{2L_y L_z}{\epsilon^2} - J_n(u; b_1, b_2) + \ldots \]
with the two aspect ratios being $b_1 = L_x/L_y$ and $b_2 = L_z/L_z$. We note that there is no logarithm because the entangling surface is not curved. $J_n$ will also depend on the boundary conditions along the three non-contractible cycles. The function $J_1(u; b_1, b_2) - J_1(1/2; b_1, b_2)$ was studied in \cite{56} for the free complex boson at fixed
boundary conditions and $b_1 = b_2$. Here we focus on the thin torus limit of the full $J_n(u; b_1, b_2)$, where $L_y$ and $L_z$ tend toward zero at fixed $L_x$ and $L_A$:

$$
\lim_{L_y, L_z \to 0} J_n(u; b_1, b_2) = 2\gamma_n^{3d}(r), \quad r = L_y/L_z. 
$$

(5.2)

We shall study the general twist dependence. In the thin torus limit, the Rényi entropies can be obtained by Fourier transforming along $y$ and $z$, which maps the problem to a sum over massive 1d bosons/fermions. Since $L_y, L_z \ll L_A, L_x$, we have

$$
S_n = -\sum_{k_x, k_y} \frac{1}{6} \left(1 + \frac{1}{n}\right) \log[(k_y^2 + k_z^2 + m^2)e^2]
$$

(5.3)

where we have used the $1 + 1$-dimensional expression for the EE [70], setting the Virasoro central charge to $c = 2$ since we work with a complex boson or a four-component Dirac fermion. The momenta are quantized as follows: $k_y = 2\pi (n_1 + \lambda_1)/L_y$, $k_z = 2\pi (n_2 + \lambda_2)/L_z$, $n_1, n_2 \in \mathbb{Z}$ and $\lambda_1, \lambda_2 \in (0, 1)$. The case $\lambda_1 = \lambda_2 = 0$ requires special care, and we treat it separately below. To extract the universal subleading correction from the EE, we need to evaluate the double series

$$
g(mL_y; \lambda_1, r) = \sum_{n_1, n_2 = -\infty}^{\infty} \log \left[\left(\frac{mL_y}{2\pi}\right)^2 + (n_1 + \lambda_1)^2 + r^2(n_2 + \lambda_2)^2\right]
$$

(5.4)

where $r = L_y/L_z$. We consider the massless $m = 0$ and massive $m \neq 0$ cases separately.

### 5.1. Massless case

For the conformal state, $m = 0$, the double series of equation (5.4) becomes

$$
g(0; \lambda_1, r) = \sum_{n_1, n_2 = -\infty}^{\infty} \log \left[(n_1 + \lambda_1)^2 + r^2(n_2 + \lambda_2)^2\right].
$$

(5.5)

As shown in appendix B.3, this series is given, after regularization, by the expression

$$
g(0; \lambda_1, r) = \log \left(\frac{\theta \left[\frac{\lambda_1 - \frac{1}{2}}{\lambda_1 + \frac{1}{2}}\right](\tau)}{\eta(\tau)} \cdot \frac{\theta \left[\frac{\lambda_2 - \frac{1}{2}}{\lambda_2 + \frac{1}{2}}\right](\tau)}{\eta(\tau)}\right).
$$

(5.6)

where $\tau = ir = iL_y/L_z$ is the modular parameter of the toroidal boundary. We have assumed $\lambda_1, \lambda_2 > 0$, i.e. we exclude the simultaneous periodic boundary conditions along $y$ and $z$. The $\lambda_1 = \lambda_2 = 0$ case requires special care, as we discuss below. Therefore, in the massless case, the subleading universal term of the EE is (for $\lambda_1, \lambda_2 > 0$)

$$
2\gamma_n^{3d} = \frac{1}{6} \left(1 + \frac{1}{n}\right) \log \left(\frac{\theta \left[\frac{\lambda_1 - \frac{1}{2}}{\lambda_1 + \frac{1}{2}}\right](\tau)}{\eta(\tau)} \cdot \frac{\theta \left[\frac{\lambda_2 - \frac{1}{2}}{\lambda_2 + \frac{1}{2}}\right](\tau)}{\eta(\tau)}\right).
$$

(5.7)

In the special case $\lambda_1 = \lambda_2$ and $r = 1$, $\gamma_n^{3d}$ was derived in [36] for the complex boson, where they used the replica trick method, and expressed the result in terms of the first Jacobi theta function. Equation (5.7) reduces to their answer when $\lambda_1 = \lambda_2$, as can be seen by using the relation between $\theta^{[n]}(\tau)$ and the first Jacobi theta function.
Two-cylinder entanglement entropy under a twist

\[ \theta_1, \text{ equation (B.16)}. \] We should emphasize that our result in equation (5.7) is valid for arbitrary values of the twists \( \lambda_1 \) and \( r \), and that it also holds for the Dirac fermion.

Our setup is symmetric under the interchange of the \( y \)- and \( z \)-directions, which means that \( J_n(1/2) \) is invariant under the simultaneous exchanges \( r \leftrightarrow 1/r \) and \( \lambda_1 \leftrightarrow \lambda_2 \). This symmetry can be seen by taking \( \tau \to -1/\tau \), \( \lambda_1 \to \lambda_2 \) and \( \lambda_2 \to \lambda_1 \) in equation (5.6), and using the relations of equation (A.3), \( \eta(-1/\tau) = \sqrt{-1/\pi} \eta(\tau) \) and \( \theta(\alpha)[-1/\tau] = \sqrt{-1/\tau} e^{2\pi \alpha \beta} \theta(\beta)[-\alpha] \).

Since the closed form answer equation (5.7) is somewhat opaque, we find it useful to examine special limits where \( \gamma^{3d} \) simplifies. For instance when the boundary becomes very elongated in one direction, say \( r = L_y/L_z \to \infty \), we find

\[
2\gamma^{3d}_n = - \left( 1 + \frac{1}{n} \right) \frac{2\pi r}{3} \left[ \frac{1}{2} \left( \lambda_2 - \frac{1}{2} \right)^2 - \frac{1}{24} \right].
\] (5.8)

Thus, \( \gamma^{3d}_n \) diverges linearly with \( r \) and only depends on the twist in the (short) \( z \) direction, \( \lambda_2 \). This asymptotic behavior is seen in figure 11. Conversely, in the opposite limit, \( r \to 0 \), we find instead

\[
2\gamma^{3d}_n = - \left( 1 + \frac{1}{n} \right) \frac{2\pi \lambda_1}{3r} \left[ \frac{1}{2} \left( \lambda_1 - \frac{1}{2} \right)^2 - \frac{1}{24} \right].
\] (5.9)

In this case, it only depends on the twist in the \( y \)-direction and is independent of \( \lambda_2 \). We note that equation (5.9) can be obtained from equation (5.8) by using the symmetry that interchanges the \( y, z \)-directions. The \( 1/r \) divergence is shown in figure 11.

Figure 11 shows the full result for \( \gamma^{3d}_1 \) as a function of \( r \) and different values of \( \lambda_1 \) and \( \lambda_2 \). Notice that \( \gamma_1 \) can be both negative and positive depending on the value of the

\[ https://doi.org/10.1088/1742-5468/aa668a \]
twists $\lambda_1$ and $\lambda_2$. We see that there is an apparent divergence as $\lambda_1, \lambda_2 \to 0$, which we now turn to by examining the periodic case $\lambda_1 = \lambda_2 = 0$.

5.1.1. The periodic case $\lambda_1 = \lambda_2 = 0$. We can no longer use equation (5.3) because of the zero mode $k_y = k_z = 0$. The zero mode will contribute a term dependent on $u = L_A/L_x$ and $L_x/\epsilon$ where $\epsilon$ is the short distance cutoff, as well as on the twist in the $x$-direction $\lambda_x$. This is exactly the same as for the $2 + 1$d CFTs, and is a consequence of the momentum decomposition of the EE in the free CFTs: $S_n = \sum_{k_y,k_z} S^{1d}_n(k_y,k_z)$, where the transverse momenta determine the effective masses of the $1d$ modes, omitting the dependence on the twist parameters. When $\lambda_1 = \lambda_2 = 0$, the $1d$ mode with $k_y = k_z = 0$ has zero mass and will dominate the EE. It is exactly the same zero mode encountered in $2 + 1$d, which implies that

$$\gamma_n^{1d}|_{\lambda_1=\lambda_2=0} = \gamma_n^{1d}|_{\lambda=0} + \ldots$$

where the dots denote a constant independent of $L_x, \epsilon, \lambda_x$. For periodic boundary conditions along $x$ and $\lambda_x = 0$, we have, see equation (3.15),

$$\gamma_n^{1d}|_{\lambda_1=\lambda_2=0} = \frac{1}{6} \left( 1 + \frac{1}{n} \right) \log \left( \frac{L_x}{\pi \epsilon} \sin \left( \frac{\pi L_A}{L_x} \right) \right) + \ldots$$

(5.10)

For $\lambda_x > 0$, the Rényi entropies for a free boson CFT in $1 + 1$d were analytically computed as a function of $\lambda_x$ by Shiba [64], and these have to be used instead of (5.11). For the Dirac fermion, the answer is not known at present. Finally, we note that by turning on a small $\lambda_1, \lambda_2 > 0$, the apparent logarithmic divergence $\log(1/\epsilon)$ will be cut off, leading to the $\log(\lambda_{1,2})$ scaling seen above.

5.2. The massive case

We turn on a finite mass $m > 0$, and we work in the thin torus limit $L_y, L_z \ll L_A, L_x$, studying $\gamma^{3d}$ as a function of $mL_y$ and $mL_z$, and the twists. This means that we keep $mL_y, L_z$ order unity. This is equivalent to working on the semi-infinite cylinder with infinite $L_y$ and $L_A$ (where we would get $\gamma^{3d}$ instead of $2\gamma^{1d}$). To evaluate the double sum over the modes—equation (5.4)—we can first sum over $n_1$. According to equation (B.9), we have

$$g(mL_y; \lambda_i, r) = g_1(mL_y; \lambda_i, r) + g_2(mL_y; \lambda_i, r)$$

(5.12)

where $g_1$ is given by

$$g_1(mL_y; \lambda_i, r) = 2\pi \sum_{n_2 \in \mathbb{Z}} \sqrt{\left( \frac{mL_y}{2\pi} \right)^2 + r^2(n_2 + \lambda_2)^2},$$

(5.13)

and $g_2$ by

$$g_2(mL_y; \lambda_i, r) = \sum_{n_2 \in \mathbb{Z}} \log \left[ (1 - e^{2\pi i \lambda_1 - 2\sqrt{t}})(1 - e^{-2\pi i \lambda_1 - 2\sqrt{t}}) \right]$$

(5.14)

with $t = (mL_y/2\pi)^2 + r^2(n_2 + \lambda_2)^2$. $g_2$ is finite but $g_1$ is divergent and needs to be further regularized. Using the derivation in appendix B.2, we can separate the divergent and finite contributions to $g_1$.

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Two-cylinder entanglement entropy under a twist

\[
g_1 = -\frac{(mL_y)^2}{4\pi r} \Gamma(-1) - \frac{(mL_y)^2}{\pi r} \sum_{p \neq 0} e^{2\pi i p \lambda_1} \frac{K_1(z)}{z} \tag{5.15}
\]

where \( z = \frac{mL_y}{r} |p| \), \( \Gamma(-1) = \infty \) and \( K_\nu(z) \) is the modified Bessel function of the second kind.

Just as was the case in \( 2+1 \) dimensions, we cannot simply subtract the area law in order to obtain a sensible answer for \( \gamma_{3d}^{\alpha} \) when the mass is finite. Working in the thin torus limit, we consider the one-parameter family of renormalized EEs [66]:

\[
2\gamma_{n}^{(\alpha)}(mL_y; r) = \frac{1}{2} \left[ \alpha L_y^2 \frac{\partial^2 S_n(L_y)}{\partial L_y^2} + (1 - \alpha) L_y \frac{S_n(L_y)}{\partial L_y} - 2S_n(L_y) \right] \tag{5.16}
\]

which is the most general form that ensures a cancellation of the area law, is linear in \( S_n \), and includes terms up to the second order in \( L_y \) derivatives. The linear case in derivatives, \( \alpha = 0 \), is the natural analog of the \( 2+1 \) dimensional renormalized EE used above, equation (3.18). The aspect ratio \( r = L_y/L_z \) is kept as a fixed constant to ensure that the entangling geometry does not change shape as we probe the entanglement along the RG flow parametrized by \( mL_y \). At a scale-invariant fixed point, equation (5.16) reduces to \( 2\gamma_{3d}^{\alpha}(r) \). In \( \gamma_{n}^{(\alpha)} \), the divergent term in \( g_1 \) will cancel and we have

\[
2\gamma_{n}^{(\alpha)}(mL_y) = -\frac{1}{12} \left( 1 + \frac{1}{n} \right) \left[ \alpha L_y^2 \frac{\partial^2 (\tilde{g}_1 + g_2)}{\partial L_y^2} + (1 - \alpha) L_y \frac{\partial (\tilde{g}_1 + g_2)}{\partial L_y} - 2(\tilde{g}_1 + g_2) \right] \tag{5.17}
\]

where \( \tilde{g}_1 \) is the finite part of \( g_1 \). We evaluate this expression numerically and plot it in figure 12 for \( \alpha = 0, 1/2 \) and \( r = 1 \). In the deep IR limit, \( mL_y \rightarrow \infty \), \( \gamma_{1}^{(\alpha)} \) eventually approaches zero. The fact that \( \gamma = 0 \) at the trivial gapped IR fixed point has been discussed in [71]. In most cases, it does not monotonically decrease with \( mL_y \). Actually, in the massless limit, \( \gamma_{n}^{(\alpha)}(mL_y) \) reduces to equation (5.7) by construction. Clearly, it

Figure 12. The renormalized von Neumann EE \( \gamma_{1}^{(\alpha)} \), (a) for \( \alpha = 0 \) and (b) \( \alpha = 1/2 \), given in equation (5.17), for a complex boson/four-component Dirac fermion of mass \( m \) in \( 3+1 \) dimensions. It is a scaling function of \( mL_y \), and is shown for four choices of the twists \( \lambda_{1,2} \), and a fixed torus aspect ratio \( r = 1 \).
is not possible to define $\gamma^{(\alpha)}_1$ as a monotonically decreasing function since at $mL_y = 0$ (IR), as shown in figure 13, in the massless limit $\gamma^{3d}_1$ can already take either positive or negative values as a function of the twists $\lambda_1$ and $\lambda_2$. In the special cases $\lambda_1 \simeq 0.319$ and $\lambda_2 = 0$, $\gamma^{3d}$ is strictly equal to zero at the massless point, but we find that no matter how we tune $\alpha$, $\gamma^{(\alpha)}_1(mL_y)$ becomes finite at $mL_y > 0$, which means that it cannot be monotonic since it also approaches zero as $mL_y \to \infty$. Although $\gamma^{(\alpha)}_1$ is generally not a monotonic function along the RG flow, it is nevertheless a simple and non-trivial entanglement measure that is universal (independent of the UV cutoff).

6. Conclusions

In this paper, we investigated the subleading correction term $J_n(u)$ of a two-cylinder EE for several scale-invariant-free systems in both 2 + 1 dimensions and 3 + 1 dimensions. We numerically studied $J_n(u, b)$ under various twisted boundary conditions in 2 + 1 dimensions. In the thin torus limit, we found that $J_n(u, b)$ converges to some constant $2\gamma_n$ which only depends on the twist and is independent of any length scale of the system. We further evaluated this constant analytically for the Dirac fermion CFT, the free boson CFT and the massless $z = 2$ free boson system, and find that they take the same form, as a function of the twist, up to a prefactor. This is in contrast with the fermionic quadratic band touching model, where $\gamma_n$ is always equal to zero. We further extended our calculations to 3 + 1 dimensions and obtained the analytical expression for $\gamma^{3d}_n$ in the thin torus limit for both a free boson CFT and a Dirac fermion CFT with twisted boundary conditions.

The finite constant terms in the EE contain non-trivial and universal information on the properties of the scale-invariant field theories. In addition to providing new
insights about these fixed-point theories, they may also constrain the allowed RG flows between two fixed points after we deform away from the UV theory. Indeed a long-standing problem is to find possible generalizations of the celebrated Zamolodchikov \( c \)-theorem of the \( 1+1 \)-dimensional field theories, which showed that under the action of a relevant perturbation, the relativistic RG flow always connects two fixed points with decreasing values of the conformal central charge \( c \), and that this change is described by a monotonically decreasing \( c \)-function along the RG flow [7].

One of the problems faced in this endeavor is to find a quantity which will play a role similar to that of the central charge of the trace anomaly of the energy momentum tensor [6]. This quantity is only unique in \( 1+1 \)-dimensional CFTs, and in \( 3+1 \) dimensions there are already two such ‘central charges’, named \( c \) and \( a \) respectively. In odd spacetime dimensions there are no such quantities associated with the energy-momentum tensor. The discovery that the central charge \( c \) in \( 1+1 \)-dimensional CFTs determines the coefficient of the logarithmic term of the EE, as well as the demonstration that EE provides a new RG monotone, has turned the problem of finding generalizations of Zamolodchikov’s theorem into that of finding similar behavior for EE in higher dimensional theories.

In \( 3+1 \) dimensional relativistic theories, an \( a \)-theorem for the \( a \) central charge has been established [72]. Here \( a \) enters the logarithmic corrections to the area law term of the EE for a spherical entangling surface. In this case, the actual \( c \)-function has not been proven to be related to an EE along the RG-flow; instead a certain integrated cross section, related to a four point function of the trace of the stress tensor, plays this role. The analog of this result in \( 2+1 \) dimensional relativistic CFTs is the \( F \)-theorem [38, 42, 43], associated with the finite universal corrections to the area law for circular entangling surfaces and of an associated function defined with respect to the EE, which decreases monotonically along the RG flows. There are further distinctions between the \( a \)-theorem and the \( F \)-theorem in that the latter is not obviously related to the properties of any known local operator and \( F \) can count topological degrees of freedom [19, 20], while these do not show up in \( a \).

In this paper we have investigated the RG flow of the similar finite universal quantities which exist in the EE for cylindrical cuts of \( 2+1 \)-dimensional theories on a torus. The goal here was to search for an RG monotone in the possible parameter space (including twisted boundary conditions and various geometries). The quantities we studied in this paper, just like \( F \), have a more non-local origin as they are also able to capture the contributions of non-local degrees of freedom, such as those in topological field theories [21] and in scale-invariant theories with compactified fields [28, 34, 61, 62]. However, we only ended up ruling out various reasonable possibilities since the quantities we study do not obey the sought after monotonicity property. Early attempts at generalizing the Zamolodchikov \( c \)-theorem using the thermodynamic entropy density (which is a natural physical quantity used to estimate the number of degrees of freedom) also failed to exhibit monotonically decreasing behavior [73, 74].

Moving forward, one could now expand the search for RG monotones to other geometries and entangling cuts. There are several reasons why we think this would be a useful pursuit. From a relativistic point of view, the existence of the \( c \)-functions is on fairly sound footing in \( 1+1 \), \( 2+1 \) and \( 3+1 \) dimensions—albeit without a unified picture. Perhaps some new entanglement entropy quantity will provide such a
unification. Similarly for non-relativistic theories, positive results on the \( c \)-functions are non-existent and filling this gap would likely have deep consequences.

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Added Note. After submission of this paper, a preprint [64] of Noburo Shiba appeared, giving the analytical calculation of the Rényi entanglement entropies in a free boson CFT on a circle in \( 1 + 1 \)d as a function of the twist \( \lambda_x \). This is a lower dimensional analog of our present analysis in \( 2 + 1 \)d and \( 3 + 1 \)d. There exists a close connection between our results and the \( 1 + 1 \)d result of Shiba in the thin torus limit, as explained in sections 3 and 5.

Appendix A. Useful functions and identities

We define the following function

\[
\theta \left[ \alpha \beta \right] (\tau) = \eta(\tau) e^{2\pi i \alpha \beta} q^{\frac{\alpha^2 - \beta^2}{8\pi i}} \prod_{n=1}^{\infty} \left( 1 + q^{n+\alpha-\frac{1}{2}} e^{2\pi i \beta} \right) \left( 1 + q^{n-\alpha-\frac{1}{2}} e^{-2\pi i \beta} \right) \tag{A.1}
\]

where \( q = e^{2\pi i \tau} \), and \( \eta(\tau) \) is the Dedekind eta function:

\[
\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{A.2}
\]

Under the \( S \) transformation, \( \tau \rightarrow -1/\tau \), we have

\[
\theta \left[ \alpha \beta \right] (-1/\tau) = \sqrt{-i\tau} e^{2\pi i \alpha \beta} \theta \left[ \beta \alpha \right] (\tau). \tag{A.3}
\]

We also define

\[
\theta_2(\tau) \equiv \theta \left[ \frac{1}{2} \right] (\tau) = 2q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n)^2,
\]

\[
\theta_4(\tau) \equiv \theta \left[ \frac{1}{2} \right] (\tau) = \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{n-\frac{1}{2}})^2 \tag{A.4}
\]
which are obtained from the full theta Jacobi functions \( \theta_{\nu} (z, \tau) \) by setting \( z = 0 \). We note that \( \theta \left[ \left[ \frac{a}{b} \right] \right] (\tau) \) can be expressed in terms of the theta Jacobi function \( \theta_1 (z, \tau) \), as shown in equation (B.16).

**Appendix B. Various zeta functions**

The details of these zeta functions can be found in [75].

**B.1. Hurwitz zeta function**

The Hurwitz zeta function is defined in this way,

\[
\zeta (s, \lambda) = \sum_{n=0}^{\infty} \frac{1}{(n + \lambda)^s}.
\]

This is the Hurwitz zeta function and it satisfies

\[
\zeta' (0, \lambda) = -\sum_{n=0}^{\infty} \log (n + \lambda) = \log \Gamma (\lambda) - \frac{1}{2} \log (2\pi).
\]

Therefore we can regularize the infinite sum

\[
g(\lambda) = \sum_{n=-\infty}^{\infty} \log |n + \lambda| = \log \prod_{n=0}^{\infty} (n + \lambda) \prod_{n=0}^{\infty} [n + (1 - \lambda)]
\]

\[
= -\zeta'(0, \lambda) - \zeta'(0, 1 - \lambda)
\]

\[
= -\log \Gamma(\lambda) + \frac{1}{2} \log (2\pi) - \log \Gamma(1 - \lambda) + \frac{1}{2} \log (2\pi)
\]

\[
= -\log [\Gamma(\lambda)\Gamma(1 - \lambda)] + \log (2\pi)
\]

\[
= \log (2\sin (\pi \lambda))
\]

where in the last step, we used \( \Gamma (\lambda)\Gamma (1 - \lambda) = \pi / \sin (\pi \lambda) \).

**B.2. The Epstein zeta function in \( d = 1 \)**

Define

\[
f(s, \lambda, c, \alpha) = \sum_{n=\infty}^{\infty} \left[ \frac{1}{c + \alpha (n + \lambda)^2} \right]^s
\]

\[
= \frac{1}{\Gamma (s)} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dt \, t^{s-1} e^{-t(c + \alpha (n + \lambda)^2)}
\]

\[
= \frac{\sqrt{\pi/\alpha}}{\Gamma (s)} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dt \, t^{s-\frac{3}{2}} e^{-\frac{\alpha t^2}{\pi} - ct + 2\pi i n \lambda}
\]

where we use
Two-cylinder entanglement entropy under a twist

\[ \Gamma(s) = \int_0^\infty dx \ x^{s-1} e^{-x} \]

\[ \sum_{n \in \mathbb{Z}} e^{-\pi a n^2 + b n} = \frac{1}{\sqrt{\alpha}} \sum_{k \in \mathbb{Z}} e^{-\frac{x}{\alpha}(k + \frac{b}{\pi \alpha})^2} \]  
\( \text{with } a = \frac{x}{\alpha} \text{ and } b = 2\pi i \lambda. \)

The \( n = 0 \) part gives

\[ f^{(n=0)}(s, \lambda, c, \alpha) = \frac{\sqrt{\pi / \alpha} \Gamma(s - \frac{1}{2})}{\Gamma(s)} \]  
\( \text{For } n \neq 0, \text{ we have} \)

\[ f^{(n \neq 0)}(s, \lambda, c, \alpha) = \frac{\sqrt{\pi / \alpha}}{\Gamma(s)} \sum_{n \neq 0} e^{2\pi i n \lambda} \int_0^\infty dt \frac{e^{-t/2}}{c^s t^{s-\frac{3}{2}}} e^{-t - \frac{x^2}{2\alpha \pi}} \]

\[ = \frac{2e^{-s+1/2} \sqrt{\pi / \alpha}}{\Gamma(s)} \sum_{n \neq 0} e^{2\pi i n \lambda} (\frac{x}{2})^{s-\frac{1}{2}} K_v(z) \]  
\( \text{where } K_v(z) \text{ is the modified Bessel function of the second kind} \)

\[ K_v(z) = \frac{1}{2} \left( \frac{z}{2} \right)^v \int_0^\infty dt \ e^{-t - \frac{z^2}{4t} t^{v-1}} \]  
\( \text{where } v = -s + \frac{1}{2} \text{ and } z = 2\sqrt{c/\alpha \pi |n|}. K_v(z) = \sqrt{\frac{\pi}{2z}} e^{-z} + \ldots \) Around \( s = 0, \Gamma(s) \sim 1/s. \)

We expand \( f(s, \lambda, c, \alpha) \) around \( s = 0, \)

\[ f(s \to 0, \lambda, c, \alpha) = \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \left( \frac{\sqrt{\pi}}{(c/\alpha)^{\frac{1}{2}}} + s \sum_{n \neq 0} e^{2\pi i n \lambda} \frac{1}{|n|} e^{-2\sqrt{c/\alpha \pi |n|}} \right) \]

\[ = -2\pi s(c/\alpha)^{\frac{1}{2}} - s \log \left[ (1 - e^{2\pi i \lambda - 2\sqrt{c/\alpha \pi}})(1 - e^{-2\pi i \lambda - 2\sqrt{c/\alpha \pi}}) \right] \]

\( \text{where we use } \Gamma(-1/2) = -2\sqrt{\pi} \text{ and } \log(1 - x) = -\sum_{k=1}^\infty x^k/k. \)

\subsection{B.3. Epstein zeta function in } \textbf{d = 2}

We define the following double series

\[ g(\lambda) = \sum_{n_1, n_2 \in \mathbb{Z}} \log \left[ (n_1 + \lambda_1)^2 + r^2(n_2 + \lambda_2)^2 \right]. \]  
\( \text{To regularize the above equation, we define} \)

\[ f(s, \lambda) = \sum_{n_1, n_2 \in \mathbb{Z}} \left[ \frac{1}{(n_1 + \lambda_1)^2 + r^2(n_2 + \lambda_2)^2} \right]^s. \]  
\( \text{Here, we first sum over } n_1. \text{ According to the result in equation (B.9), around } s = 0, \text{ we have} \)

\[ f(s \to 0, \lambda) = \sum_{n_2 = -\infty}^\infty \frac{\Gamma(s - \frac{1}{2}) \sqrt{\pi}}{c^{s-\frac{1}{2}}} - \sum_{n_2 = -\infty}^\infty s \log \left[ (1 - e^{2\pi i \lambda_1 - 2\sqrt{c/\pi}})(1 - e^{-2\pi i \lambda_1 - 2\sqrt{c/\pi}}) \right] \]

\( \text{https://doi.org/10.1088/1742-5468/aa668a} \)
where $\sqrt{c} = |n_2 + \lambda_2|r$.

Hence

$$f(s \to 0, \lambda) = -s \sum_{n_2=-\infty}^{\infty} \log \left[ (1 - e^{2\pi i \lambda_1 - 2r(n_2 + \lambda_2)})(1 - e^{-2\pi i \lambda_1 - 2r(n_2 + \lambda_2)\pi}) \right]$$

$$+ \frac{r \sqrt{\pi} \Gamma(s - \frac{1}{2})}{\Gamma(s)} \left[ \sum_{n_2=0}^{\infty} \frac{1}{(n_2 + \lambda_2)^{2s-1}} + \sum_{n_2=0}^{\infty} \frac{1}{(n_2 + 1 - \lambda_2)^{2s-1}} \right]$$

$$= -s \sum_{n_2 > 0} \log \left[ (1 - e^{2\pi i \lambda_1 - 2r(n_2 + \lambda_2 - 1)})\pi)(1 - e^{-2\pi i \lambda_1 - 2r(n_2 - \lambda_2)}) \right]$$

$$\times (1 - e^{-2\pi i \lambda_1 - 2r(n_2 + \lambda_2 - 1)})\pi - 2r s \pi \left[ \zeta(2s - 1, \lambda_2) + \zeta(2s - 1, 1 - \lambda_2) \right]$$

$$= -s \sum_{n_2 > 0} \log \left[ (1 - e^{2\pi i \lambda_1 - 2r(n_2 + \lambda_2 - 1)})\pi)(1 - e^{-2\pi i \lambda_1 - 2r(n_2 - \lambda_2)}) \right]$$

$$\times (1 - e^{-2\pi i \lambda_1 - 2r(n_2 + \lambda_2 - 1)})\pi + 2r s \pi (\lambda_2^2 - \lambda_2 + \frac{1}{3}) \tag{B.13}$$

where $\zeta(s, x)$ is the Hurwitz zeta function

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}, \quad \zeta(-1, x) = -\frac{x^2 - x + \frac{1}{6}}{2}. \tag{B.14}$$

Therefore

$$g(\lambda) = -f'(s, \lambda)|_{s=0} = \log \left\{ \frac{\theta \left[ \frac{\lambda_2 - \frac{1}{2}}{\lambda_1 - \frac{1}{2}} \right](\tau)}{\eta(\tau)} \left[ \frac{\lambda_2 - \frac{1}{2}}{\lambda_1 + \frac{1}{2}} \right](\tau) \right\} \tag{B.15}$$

where $\tau = ir$, $g = e^{2\pi ir}$. The above theta-function in the above equation is related to the first elliptic theta function in the following way:

$$\frac{\theta \left[ \frac{\lambda_2 - \frac{1}{2}}{\lambda_1 - \frac{1}{2}} \right](\tau)}{\eta(\tau)} = e^{2\pi i \frac{1}{2}}(\lambda_2 - \frac{1}{2})^{\frac{1}{2}} e^{\pi i r}(\lambda_2^2 - \lambda_2 + \frac{1}{4}) \prod_{n=1}^{\infty} \left( 1 - q^{n-1} e^{2\pi i (\lambda_1 + \lambda_2 \tau)} \right) \left( 1 - q^n e^{-2\pi i (\lambda_1 + \lambda_2 \tau)} \right)$$

$$= i e^{2\pi i \frac{1}{2}}(\lambda_2 - \frac{1}{2})^{\frac{1}{2}} e^{\pi i r}(\lambda_2^2 + \lambda_2) \prod_{n=1}^{\infty} \left( 1 - q^{n-1} e^{2\pi i z} \right) \left( 1 - q^n e^{-2\pi i z} \right)$$

$$= i e^{2\pi i \frac{1}{2}}(\lambda_2 - \frac{1}{2})^{\frac{1}{2}} e^{\pi i r}(\lambda_2^2 + \lambda_2) \frac{\theta_1(z, \tau)}{\eta(\tau)} \tag{B.16}$$

where $z = - (\lambda_1 + \lambda_2 \tau)$, and $\theta_1(z, \tau)$ is the first Jacobi theta function.

**Appendix C. Free boson partition function on the cylinder and torus**

**C.1. Torus**

*C.1.1. The periodic boundary conditions in $x$ and $y$-directions.* Here we follow the method described in chapter 10.2 of [65]. The partition function for a non-compact free boson on the torus without the zero-mode is

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Two-cylinder entanglement entropy under a twist

\[ Z = \int [d\varphi] \sqrt{A} \delta \left( \int d^2x \varphi \varphi_0 \right) \exp \left( -\frac{1}{2} \int d^2x (\nabla \varphi)^2 \right) \]

\[ = \sqrt{A} \int \prod_n dc_n \exp \left( -\frac{1}{2} \sum_n \omega_n c_n^2 \right) \]

\[ = \sqrt{A} \prod_n \left( \frac{2\pi}{\omega_n} \right)^{1/2} \] (C.1)

where \( A = L_x L_y \) is the area of the torus, and \( \varphi_0 = 1/\sqrt{A} \) is the normalized eigenfunction of the zero-mode. We have expanded the field in terms of the eigenfunctions of the Laplacian operator, \( \varphi = \sum_n c_n \phi_n(x) \), with the corresponding eigenvalues \( \omega_n \). We define

\[ G(s) = \sum_{n, \omega_n \neq 0} \frac{1}{\omega_n^s} \] (C.2)

which satisfies

\[ \frac{d}{ds} G(s) = -\sum_n \log(\omega_n) \frac{1}{\omega_n^s}. \] (C.3)

Therefore we have

\[ Z(\tau) = \sqrt{A} \exp \left( \frac{1}{2} G'(0) \right). \] (C.4)

The eigenvalues \( \omega_{n,m} \) are labeled by \( k_x \) and \( k_y \),

\[ \omega_{n,m} = k_x^2 + k_y^2 \] (C.5)

with \( k_x = \frac{2\pi n}{L_x} \) and \( k_y = \frac{2\pi m}{L_y} \). Hence

\[ \left| \frac{2\pi L_x}{A} \right|^{2s} G(s) = \sum_{(m,n) \neq (0,0)} \frac{1}{|m + n\tilde{\tau}|^{2s}} \]

\[ = \sum_{m \neq 0} \frac{1}{|m|^{2s}} + \sum_{n \neq 0} \left( \sum_m \frac{1}{|m + n\tilde{\tau}|^{2s}} \right) \]

\[ = 2\zeta(2s) + \sum_{n \neq 0} \left( \sum_m \frac{1}{|m + n\tilde{\tau}|^{2s}} \right) \] (C.6)

where \( \zeta(z) \) is the Riemann \( \zeta \) function, \( \tilde{\tau} = -1/\tau \) and \( \tau = iL_x/L_y \). Here \( \tau \) is a pure imaginary number. For the second term, using the result in section B.2, when \( s \to 0 \), we have

\[ \sum_{n \neq 0} \sum_m \frac{1}{|m + n\tilde{\tau}|^{2s}} = -s \sum_{n \neq 0} \left\{ 2\pi |n| \text{Im}\tilde{\tau} + \log \left[ (1 - e^{-2\pi |n| \text{Im}\tilde{\tau}})(1 - e^{-2\pi |n| \text{Im}\tilde{\tau}}) \right] \right\} \]

\[ = -2 \log |\eta(\tilde{\tau})|^2 \] (C.7)

where we use \( \zeta(-1) = -1/12 \). The \( \eta(\tau) \) function is defined as

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\[ \eta(\tau) = q^{\frac{1}{2\pi}} \prod_{n=1}^{\infty} (1 - q^n) \] (C.8)

where \( q = e^{2\pi i \tau} \).

After including the first term in \( G(s) \) and using \( \zeta(0) = -\frac{1}{2} \), we have

\[ G'(0) = -2 \log \left( \sqrt{A \text{Im} \tilde{\tau}} |\eta(\tilde{\tau})| \right). \] (C.9)

Therefore, the free boson partition function is

\[ Z_{\text{bos}}(\tau) = \sqrt{A} \exp \left( \frac{1}{2} G'(0) \right) = \frac{1}{\sqrt{\text{Im} \tilde{\tau}} |\eta(\tilde{\tau})|}. \] (C.10)

In the last step, we used the \( S \) transformation of the \( \eta(\tau) \) function.

**C.1.2. Twisted boundary condition.** If we impose a twisted boundary condition in the \( x \)- or \( y \)-direction, the eigenvalues \( \omega_{m,n} \) will be modified,

\[ \omega_{n,m} = \left( \frac{2\pi(n + \lambda_1)}{L_x} \right)^2 + \left( \frac{2\pi(m + \lambda_2)}{L_y} \right)^2 \] (C.11)

where \( \lambda_{1,2} \) are the twists along the \( x \)- and \( y \)-directions. After considering the twisted boundary condition, there is no zero mode anymore and we have

\[ \left| \frac{2\pi L_x}{A} \right|^{2s} G(s) = \sum_{m,n \in \mathbb{Z}^2 \setminus \{m + \lambda_2 + (n + \lambda_1) \tilde{\tau} \} \setminus \mathbb{Z}^2} \frac{1}{m + \lambda_2 + (n + \lambda_1) \tilde{\tau}}. \] (C.12)

Using the results in section B.2, we have

\[ \log Z = -\frac{1}{2} \log \left\{ \frac{\theta \left[ \frac{\lambda_1 - \frac{1}{2}}{\lambda_2 - \frac{1}{2}} \right] (\tilde{\tau}) \theta \left[ \frac{\lambda_1 - \frac{1}{2}}{-\lambda_2 + \frac{1}{2}} \right] (\tilde{\tau})}{\eta(\tilde{\tau}) \eta(\tilde{\tau})} \right\} \]

\[ = -\frac{1}{2} \log \left\{ \frac{\theta \left[ \frac{-\lambda_2 - \frac{1}{2}}{-\lambda_1 + \frac{1}{2}} \right] (\tau) \theta \left[ \frac{-\lambda_2 + \frac{1}{2}}{-\lambda_1 + \frac{1}{2}} \right] (\tau)}{\eta(\tau) \eta(\tau)} \right\}. \] (C.13)

To obtain this result, we perform an \( S \) transformation \( \tau \rightarrow -1/\tau \) in the second step.

**C.2. Open cylinder**

**C.2.1. The periodic in the \( y \)-direction.** For the cylinder, if we impose a Dirichlet boundary at two boundaries with \( \phi(x = 0) = \phi(x = L_x) = 0 \), only half of the modes will remain,

\[ \left| \frac{2\pi L_x}{A} \right|^{2s} G(s) = \sum_{m \in \mathbb{Z}, n > 0} \frac{1}{m + \frac{n\tau}{2}} \] (C.14)

\[ = \sum_{n > 0} \left( \sum_{m \in \mathbb{Z}} \frac{1}{m + \frac{n\tau}{2}} \right). \]
Using the zeta function regularization technique, we expand $G(s)$ around $s = 0$,

$$G(s) = \frac{1}{6}s\pi \frac{\text{Im}\tilde{\tau}}{2} - s \prod_{n>0} \log \left[ \left(1 - e^{2\pi i n \frac{\tilde{\tau}}{2}}\right)^2 \right] + \cdots = -s \log \left( \left| \eta(\tilde{\tau}/2) \right|^2 \right) + \cdots \quad \text{(C.15)}$$

where the dots denote higher order terms in $s$.

Hence

$$Z_{\text{cyl}}(\tau) = \exp \left( \frac{1}{2}G'(0) \right) = \frac{1}{\eta(-\frac{1}{2\tau})} = \frac{1}{\sqrt{-2i\tau \eta(2\tau)}}. \quad \text{(C.16)}$$

### C.2.2. The twisted boundary condition in the $y$-direction.

In this case, we need to calculate

$$\left| \frac{2\pi L_x}{A} \right|^{2s} G(s) = \sum_{m \in \mathbb{Z}, n>0} \frac{1}{|m + \lambda + \frac{n\tau}{2}|^{2s}} = \sum_{n>0} \left( \sum_{m \in \mathbb{Z}} \frac{1}{|m + \lambda + \frac{n\tau}{2}|^{2s}} \right). \quad \text{(C.17)}$$

Expanding $G(s)$ around $s = 0$, we have

$$G(s) = \frac{1}{6}s\pi \frac{\text{Im}\tilde{\tau}}{2} - s \prod_{n>0} \log \left[ \left(1 - e^{2\pi i \lambda + 2\pi i n \frac{\tilde{\tau}}{2}}\right)(1 - e^{-2\pi i \lambda + 2\pi i n \frac{\tilde{\tau}}{2}}) \right]. \quad \text{(C.18)}$$

Therefore the partition function is

$$Z = \tilde{q}^{-1/24} \prod_{n=1}^{\infty} \sqrt[2]{\frac{1}{(1 - q^n)(1 - e^{2\pi i \lambda} q^n)}} \quad \text{where } \tilde{q} = e^{2\pi i \left( \frac{-1}{L_y} \right)} \text{ and } q = e^{2\pi i \tau}.$$

For instance, if we impose an anti-periodic boundary condition in the $y$-direction with $\lambda = 1/2$, the partition function becomes

$$Z = \sqrt{2} \sqrt[2]{\frac{\eta(-1/2\tau)}{\theta_2(-1/2\tau)}} = \sqrt{2} \sqrt[2]{\frac{\eta(2\tau)}{\theta_4(2\tau)}} = \sqrt{2} q^{\frac{1}{8}} \prod_{n=1}^{\infty} \frac{1}{(1 - q^{2n-1})}. \quad \text{(C.20)}$$
Appendix D. The numerical method for calculating EE in free systems

D.1. Free scalar field theory

The discrete free boson Hamiltonian defined on a lattice reads

\[ H = \frac{1}{2} \left( \sum_{i=1}^{N} \Pi_i^2 + \sum_{i,j=1}^{N} \phi_i K_{ij} \phi_j \right) \]  \hspace{1cm} (D.1)

where \( K_{ij} \) is a discretized version of \(-\nabla^2 + m^2\). The operators satisfy the canonical commutation relations

\[ [\phi_i, \Pi_j] = i \delta_{ij}, \quad [\phi_i, \phi_j] = 0, \quad [\Pi_i, \Pi_j] = 0. \]  \hspace{1cm} (D.2)

The ground state for this Hamiltonian is

\[ |\Psi\rangle = \mathcal{N} \sum_{\varphi} e^{-\frac{1}{2} \sum_{ij} \varphi_i K_{ij}^{1/2} \varphi_j} |\varphi\rangle \]  \hspace{1cm} (D.3)

where \( \varphi = \{\varphi_i\} \) denotes a field configuration, \( \phi_i |\varphi\rangle = \varphi_i |\varphi\rangle \), and \( \mathcal{N} \) is a normalization. We use this wavefunction to calculate the entanglement entropy. For a free boson system, there is a very efficient numerical method for calculating the entanglement entropy. Below is a short summary of the method, while the details can be found in [48].

The correlation functions for the ground state are

\[ \langle \phi_i \phi_j \rangle = \frac{1}{2} K_{ij}^{-1/2} \equiv X_{ij} \]
\[ \langle \pi_i \pi_j \rangle = \frac{1}{2} K_{ij}^{1/2} \equiv P_{ij}. \]  \hspace{1cm} (D.4)

The von Neumann EE for subsystem \( A \) can be calculated by using these correlation functions,

\[ S_1(A) = \sum_{\ell} \left( \nu_\ell + \frac{1}{2} \right) \log \left( \nu_\ell + \frac{1}{2} \right) - \left( \nu_\ell - \frac{1}{2} \right) \log \left( \nu_\ell - \frac{1}{2} \right) \]  \hspace{1cm} (D.5)

where the \( \nu_\ell \) are eigenvalues of \( C = \sqrt{X_A P_A} \), and \( X_A \) and \( P_A \) are the correlation functions defined on subregion \( A \). Similarly, the Rényi EE for \( n > 0 \) reads

\[ S_n = \sum_{\ell} \frac{1}{n-1} \left[ \log \left( \nu_\ell + \frac{1}{2} \right)^n - \left( \nu_\ell - \frac{1}{2} \right)^n \right]. \]  \hspace{1cm} (D.6)

D.1.1. The relativistic boson in 2 + 1d. The Hamiltonian for the relativistic boson in 2 + 1d is

\[ H = \frac{1}{2} \int d^2 x \left[ \Pi^2 + (\nabla \phi)^2 + m^2 \phi^2 \right]. \]  \hspace{1cm} (D.7)

The corresponding discrete lattice Hamiltonian (on the torus) is

\[ H = \frac{1}{2} \sum_{i,j} \left[ \Pi_{ij}^2 + (\phi_{i+1,j} - \phi_{i,j})^2 + (\phi_{i,j+1} - \phi_{i,j})^2 + m^2 \phi_{ij}^2 \right]. \]  \hspace{1cm} (D.8)
In the momentum space, the Hamiltonian becomes
\[ H = \frac{1}{2} \int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 \Pi(k_1) \Pi(-k_1) \phi(k) \phi(-k). \]

The two point correlation functions are
\[ \langle \phi_{i,j} \phi_{i+n_1,j+n_2} \rangle = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 \frac{\cos(k_1 n_1) \cos(k_2 n_2)}{\sqrt{4 - 2 \cos(k_1) - 2 \cos(k_2) + m^2}} \]
\[ \langle \pi_{i,j} \pi_{i+n_1,j+n_2} \rangle = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} dk_1 \int_{-\pi}^{\pi} dk_2 \cos(k_1 n_1) \cos(k_2 n_2) \sqrt{4 - 2 \cos(k_1) - 2 \cos(k_2) + m^2} \]
and we can use these correlation functions to construct the \( C \) matrix and calculate EE.

**D.2. The numerical method for the free Dirac fermion in 2 + 1 dimensions**

Here, we follow the method in [77] and show a simple example for the massless Dirac fermion. The Hamiltonian in the momentum space is
\[ H_D = \int_{BZ} \frac{d^2k}{(2\pi)^2} \Psi^\dagger(k) H_D(k) \Psi(k) \]
where \( \Psi^\dagger(k) = (\psi^\dagger_k, d_k^\dagger) \) is a two-component spinor, BZ stands for the first Brillouin zone, \(-\pi < k_x \leq \pi \) and \(-\pi < k_y \leq \pi \). The one-particle lattice Dirac Hamiltonian \( H_D(k) \) takes the form
\[ H_D(k) = h_1 \sigma_x + h_3 \sigma_z \]
with \( h_1 = \cos(k_x) \) and \( h_3 = \cos(k_y) \). The Dirac points are at \((\pm \pi/2, \pm \pi/2)\) and there are four Dirac cones in the first Brillouin zone.

\( H_D(k) \) can be diagonalized by a unitary transformation \( V^{-1} H_D V = M \), where \( M \) is the diagonal matrix with the eigenvalues \( E(k)_\pm = \pm \sqrt{\cos(k_x)^2 + \cos(k_y)^2} \). The \( V \) matrix equals
\[ V = \frac{1}{\sqrt{(2 \cos(k_y)^2 + 2 \cos(k_x)^2 - 2 \cos(k_y) \sqrt{\cos(k_y)^2 + \cos(k_x)^2})}} \times \begin{pmatrix} \cos(k_x) & -\cos(k_y) + \sqrt{\cos(k_y)^2 + \cos(k_x)^2} \\ -\cos(k_y) + \sqrt{\cos(k_y)^2 + \cos(k_x)^2} & -\cos(k_x) \end{pmatrix} \]

For the ground state with the lower band fully filled, the correlation functions in momentum space equal
\[ \langle c_k^\dagger c_k \rangle = V_{12}^2, \quad \langle d_k^\dagger d_k \rangle = V_{22}^2, \quad \langle c_k^\dagger d_k \rangle = V_{12} V_{22}. \]

Using the above equations, we can construct the correlation function matrix \( C_E \) for the subsystem \( A \), and therefore the von Neumann EE is
\[ S_{vN} = - \sum \nu_l \log \nu_l + (1 - \nu_l) \log(1 - \nu_l) \]

https://doi.org/10.1088/1742-5468/aa668a
where $\nu_{\ell}$ is the eigenvalue for $C_{E}$. Similarly, the Rényi entropy is

$$S_n = \frac{1}{1-n} \sum_{\ell} \log [(1 - \nu_{\ell})^n + \nu_{\ell}^n].$$

(D.16)

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