ON THE CONJUGACY PROBLEM IN GROUP 
F/N₁ ∩ N₂.

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Аннотация. Let N₁ (resp., N₂) be the normal closure of a finite 
symmetrized set R₁ (resp., R₂) of a finitely generated free group 
F = F(A). It is well-known that if Rᵢ satisfies the condition C(6), 
then the conjugacy problem is solvable in F/Nᵢ. In the present 
paper we prove that if R₁ ∪ R₂ satisfies the condition C(6) and the 
presentation ⟨A | R₁, R₂⟩ is atorical, then the conjugacy problem 
is solvable in F/N₁ ∩ N₂. In particular, if R₁ ∪ R₂ satisfies the 
condition C(7) then the conjugacy problem is solvable in F/N₁ ∩ N₂. 

Bibliography: 13 items.

INTRODUCTION.

Let F = F(A) be a free group generated by a finite alphabet A. Let 
N₁ (resp., N₂) be the normal closure of non-empty finite set Rᵢ (resp., 
R₂) of elements of F. Assume that Rᵢ (i = 1, 2) is symmetrized, i.e., 
all elements of Rᵢ are cyclically reduced and for any r of Rᵢ all cyclic 
permutations of r and r⁻¹ also belong to Rᵢ.

We will use the following notations. Denote graphic (letter-by-letter) 
equality of words u, v ∈ F by u ≡ v. Denote free equality by u = v. 
If words u, v ∈ F present equal elements in a group H, we will write: 
u = v in H.

If two words u, v ∈ F are equal both in the group F/N₁ and in the 
group F/N₂, then they are evidently equal in the group F/N₁ ∩ N₂. It is 
natural to ask whether the conjugation of words u and v in F/N₁ ∩ N₂ 
follows from their conjugation both in F/N₁ and F/N₂? The answer is 
obviously negative. As an example showing that one can consider the 
free group F = F(a, b, c), the sets R₁ = {a±1}, R₂ = {b±1} and the 
words u ≡ c²ba, v ≡ cbca.

The aim of this paper is to find out conditions on R₁ and R₂ such that 
the solvability of the conjugacy problem in F/N₁ ∩ N₂ follows from the 
solvability of the conjugacy problem in F/N₁ and F/N₂.

Note that this problem is naturally associated with subdirect products. 
Indeed, one can consider F/N₁ ∩ N₂ as a subgroup of the direct product 
of F/N₁ and F/N₂, and F/N₁ ∩ N₂ is a subdirect product of F/N₁ and
 Conversely, given a subdirect product $H$ of groups $G_1$ and $G_2$, there exist normal subgroups $N_1$ and $N_2$ of some free group $F$ such that $F/N_i \cong G_i$ $(i=1,2)$ and $F/N_1 \cap N_2 \cong H$.

In turn, subdirect products of two groups are closely associated to the fibre product construction in the category of groups. Recall that, associated to each pair of short exact sequences of groups $1 \rightarrow L_i \rightarrow G_i \psi_i \rightarrow Q \rightarrow 1$, $i=1,2$, one has the fibre product $H = \{(x,y) \in G_1 \times G_2 | \psi_1(x) = \psi_2(y)\}$. It is shown in [1] that a subgroup $H \leq G_1 \times G_2$ is a subdirect product of $G_1$ and $G_2$ if and only if there is a group $Q$ and surjections $\psi_i : G_i \rightarrow Q$ such that $H$ is the fibre product of $\psi_1$ and $\psi_2$.

The question about the solvability of the conjugacy problem for subdirect products has been already considered for some groups (see, for example, [1, 2, 3]). Thus in the paper of C.F. Miller [2] there is an example of a fibre product in which $G_1 = G_2$ are non-abelian finitely generated free groups, $L_1 = L_2$, $\psi_1 = \psi_2$, $Q$ is a finitely presented group with undecidable word problem, and the conjugacy problem in $H$ is unsolvable. So the natural question, whether the solvability of the conjugacy problem in $F/N_1 \cap N_2$ always follows from the solvability of the conjugacy problem both in $F/N_1$ and in $F/N_2$, has negative answer. Since $Q$ is isomorphic to $F/N_1 N_2$, it follows from the example of C.F. Miller that the solvability of the word problem in $F/N_1 N_2$ is necessary for the solvability of the conjugacy problem in $F/N_1 \cap N_2$.

To formulate the main result of the paper, we recall the definitions of some geometrical objects, called pictures. Pictures were introduced in [4, 5]. These objects are a very useful tool in combinatorial group theory, and can be used in a variety of different ways (see, for example, [6, 7] and references in these papers).

Let $N$ be the normal closure of a symmetrized set $R$ of the free group $F(A)$.

A picture $P$ over the presentation $\hat{G} = \langle A \mid R \rangle$ on an oriented surface $T$ is a finite collection of "vertices" $V_1, ..., V_n \in T$, together with a finite collection of simple pairwise disjoint connected oriented "edges" $E_1, ..., E_m \in T \setminus \{\{V_1, ..., V_n\} \cup \partial T\}$ labelled by words of $F(A)$. But these edges need not all connect two vertices. An edge may connect a vertex to a vertex (possibly coincident), a vertex to $\partial T$, or $\partial T$ to $\partial T$. Moreover, some edges need have no endpoints at all, but be circles disjoint from the rest of $P$, such edges are called edges-circles.

In the paper we will only ever consider such paths on $T$, each of which doesn’t pass through any vertex and intersects the edges of $P$ only finitely many times (moreover, if a path intersects an edge then
it crosses it, and doesn’t just touch it). If we travel along an oriented path \( \gamma \) in the positive direction, we encounter a succession of edges \( E_{i_1}, \ldots, E_{i_k} \) labeled by \( g_{i_1}, \ldots, g_{i_k} \) respectively. These labels form the word \( g_{i_1}^{\varepsilon_{i_1}} \cdots g_{i_k}^{\varepsilon_{i_k}} \), where \( \varepsilon_{i_j} \in \{1, -1\} \) is a local intersection index of \( E_{i_j} \) and \( \gamma \). This word will be called the word along the path \( \gamma \) (or the label of \( \gamma \)) and denoted by \( \text{Lab}^+(\gamma) \). The subword \( g_{i_j}^{\varepsilon_{i_j}} (j = 1, \ldots, k) \) will be called the contribution of \( E_{i_j} \) in the label of \( \gamma \). Travelling along \( \gamma \) in the negative direction gives the word \( \text{Lab}^-(\gamma) \equiv \text{Lab}^+(\gamma) - 1 \).

If a path \( \gamma \) is closed, consider a point \( p \) on \( \gamma \) not belonging to any edge of \( P \). The word along \( \gamma \) read from \( p \) will be denoted by \( \text{Lab}_p^+(\gamma) \) or by \( \text{Lab}_p^-(\gamma) \) (depending on the direction of travelling along \( \gamma \)). Changing the disposition of \( p \) we obtain the same word up to cyclic permutation. We will denote the word along the path \( \gamma \) by \( \text{Lab}(\gamma) \) when the disposition of \( p \) and the direction of reading will not be essential.

For each vertex \( V \) of \( P \) consider a circle \( \Sigma \) of a small radius with center at \( V \) and a point \( p \in \Sigma \) not lying on any edge of \( P \). The word \( \text{Lab}_p^+(\Sigma) \) is called the label of the vertex \( V \). To complete the definition of the picture over the presentation \( \hat{G} = \langle A \mid R \rangle \) on the surface \( T \) it remains to require that the labels of all vertices in \( P \) belong to \( R \).

Below we will consider pictures on a surface \( T \), where \( T \) is a torus (torical pictures), an annulus (annulus pictures) or a disk (planar pictures).

For a planar picture the boundary label of the picture is the word \( \text{Lab}_p^+(\bar{\Sigma}) \), where \( \bar{\Sigma} \) is a circle near the boundary of the disk \( T \) and \( \bar{p} \in \bar{\Sigma} \) is a point not belonging to any edge.

The following result is well-known (use Theorem 11.1 [9] and dualise):

**Lemma 1.** Let \( W \) be a non-empty word on the alphabet \( A \). Then \( W \) represents the identity of the group \( \hat{G} = F/N \) if and only if there is a planar picture over the presentation \( \langle A \mid R \rangle \) of \( \hat{G} \) with the boundary label \( W \).

A dipole is two distinct vertices \( V_1 \) and \( V_2 \) of \( P \) connected by an edge \( E \) if there exists a simple path \( \psi \) joining points \( p_1 \) and \( p_2 \) on the circles \( \Sigma_1 \) and \( \Sigma_2 \) around these vertices, passing along \( E \) and not crossing any edge or vertex such that \( \text{Lab}_{p_1}^+(\Sigma_1) = \text{Lab}_{p_2}^-(\Sigma_2) \) in \( F \).

A presentation \( \hat{G} = \langle A \mid R \rangle \) is called atorical (see, for example, [9]) if every connected torical picture over \( \hat{G} = \langle A \mid R \rangle \) having at least one vertex contains a dipole.

The following theorem 1 will be proved in Section 1.
Theorem 1. Let $F$ be a free group generated by a finite alphabet $A$, $N_1$ (resp., $N_2$) be the normal closure of non-empty finite symmetrized set $R_1$ (resp., $R_2$) of elements of $F$.

Let the following conditions hold for the group $G_i = F/N_i$ ($i = 1, 2$):

1.1. The conjugacy problem is solvable in $G_i$.

1.2. In $G_i$, there exists an algorithm allowing for a reduced word $x \in F$, $x \neq 1$, to determine all $z \in F$ such that $x \in \langle z \rangle$ in $G_i$, and the number of such distinct elements $z$ of $G_i$ is finite.

Let the following conditions hold for the group $G = F/N_1N_2$:

2.1. The membership problem for a cyclic subgroup is solvable in $G$.

2.2. The presentation $G = \langle A \mid R_1 \cup R_2 \rangle$ is atorical.

Then the conjugacy problem is solvable in $F/N_1 \cap N_2$.

Note that Condition 2.2 of Theorem 1 provides the equality $N_1 \cap N_2 = [N_1, N_2]$ for disjoint $R_1$ and $R_2$ (see, for example, [10, 11]).

Recall the definition of small cancelation conditions $C(k)$ ([8]), used in Theorem 2 below. A nontrivial freely reduced word $b$ in $F$ is called a piece with respect to $R$ if there exist two distinct elements $r_1$ and $r_2$ in $R$ that both have $b$ as maximal initial segment, i.e. $r_1 \equiv bc_1$ and $r_2 \equiv bc_2$.

Let $k$ be a positive integer. $R$ is said to satisfy the small cancelation condition $C(k)$, if no element of $R$ can be written as a reduced product of fewer than $k$ pieces.

Using the notations of Theorem 1, we have:

Theorem 2. If $R_1 \cup R_2$ is a set satisfying the condition $C(6)$ and the presentation $G = \langle A \mid R_1 \cup R_2 \rangle$ is atorical, then the conjugacy problem is solvable in $F/N_1 \cap N_2$.

Proof of Theorem 2. Let us show that Theorem 2 follows from Theorem 1. Since $R_1 \cup R_2$ satisfies the condition $C(6)$, the subsets $R_1$ and $R_2$ also satisfy the condition $C(6)$. Therefore Condition 1.1 for $G_1$ and $G_2$ follows from Theorem 7.6 [8]; Condition 2.1 follows from Theorem 1 [12]; Condition 1.2 can be deduced from Theorem 1 [12], Theorem 2 [12] and (if there is an element of finite order in $G_i$) Theorem 1.4 [13] with regard to Theorem 13.3 [9].

It is well known that the condition $C(7)$ is sufficient for atoricty (the proof of it is similar to Theorem 13.3 [9]). So by Theorem 2 (using the notations of Theorem 1) we have the following:

Corollary 1. If $R_1 \cup R_2$ satisfies the condition $C(7)$, then the conjugacy problem is solvable in $F/N_1 \cap N_2$. 
The author is grateful to N.V. Bezverkhnii and A. Muranov for useful conversations during preparation of this paper and, most particular, A. Minasyan, who showed the author the example of C.F. Miller and the relationship between the present work and subdirect and fibre products.

1. Deduction of Theorem 1 from Assertion 1.

Below, no mentioning it explicitly, we will use the fact that Condition 1.1 (resp., 2.1) of Theorem 1 leads to the solvability of the word problem in $G_i$, $i \in \{1, 2\}$ (resp., $G$).

Let $u$ and $v$ be two reduced words of $F$. For each $i \in \{1, 2\}$ by Condition 1.1 of Theorem 1 there exists an algorithm which decides whether $u$ and $v$ present conjugated elements in $G_i = F/N_i$. If $u$ and $v$ turn out to be not conjugated in $G_i$ for at least one of the $i$, then $u$ and $v$ are not conjugated in $F/N_1 \cap N_2$. Hence further assume that for each $i \in \{1, 2\}$, $u$ and $v$ are conjugated by $h_i \in F$ in $G_i$. Therefore the word $h_i^{-1}uh_i v^{-1}$ is equal to the identity in $G_i$.

By Condition 1.1 of Theorem 1 the word $h_1^{-1}uh_1 v^{-1}$ can be effectively represented with defining relations $R_1$ of $G_1$ in the form $\prod_{s=1}^{m_1} g_{1,s} r_{1,s} g_{1,s}^{-1}$, where $r_{1,s} \in R_1$, $g_{1,s} \in F$. By this representation construct a planar picture $P_1$ over the presentation $G_1 = \langle A \mid R_1 \rangle$ with the boundary label equal to $h_1^{-1}uh_1 v^{-1}$ so that the edges of $P_1$ are labelled by letters. In addition on the boundary $\partial P_1$ of $P_1$ fix four points $a_1, b_1, c_1, d_1$ not belonging to any edge and dividing $\partial P_1$ into four subpaths so that the labels of the subpaths $[a_1, b_1], [b_1, c_1], [c_1, d_1], [d_1, a_1]$ are identically equal to $h_1^{-1}, u, h_1, v^{-1}$ respectively. Pasting together the subpaths $[a_1, b_1]$ and $[d_1, c_1]$ of $P_1$, we obtain an annulus picture $\overline{P_1}$ with the two boundary circles formed by $[b_1, c_1]$ with the label $u$ and $[d_1, a_1]$ with the label $v^{-1}$. The pasted points $b_1, c_1$ (resp., $d_1, a_1$) give a point $(bc)_1$ (resp., $(ad)_1$). The pasted subpaths $[a_1, b_1]$ and $[d_1, c_1]$ form a subpath $Con_j$.

Similarly changing the index 1 by the index 2 in the notation, by the word $h_2^{-1}u^{-1}h_2 v$ construct an annulus picture $\overline{P_2}$ over the presentation $G_2 = \langle A \mid R_2 \rangle$ with the two boundary circles formed by $[b_2, c_2]$ with the label $u^{-1}$ and $[d_2, a_2]$ with the label $v$.

Pasting together $\overline{P_1}$ over $\overline{P_2}$ by their boundaries we obtain a picture $P$ on the torus $T$ over the presentation $G = \langle A \mid R_1 \cup R_2 \rangle$. The pasted circles $[d_1, a_1]$ and $[a_2, d_2]$ (resp., $[b_1, c_1]$ and $[c_2, b_2]$) form a circle $Equ$ (resp., $\overline{Equ}$). The pasted points $(ad)_1$ and $(ad)_2 ((bc)_1$ and $(bc)_2$) form a point $p_u$ ($p_v$). By $Con_j$ denote a circle formed by the pasted subpaths $Con_j_1$ and $Con_j_2$. The circles $Equ$ and $\overline{Equ}$ will be called the equators. The points $p_u, p_v$ will be called the poles.
So $\text{Lab}_{p_u}(\overline{\text{Equ}})$ is equal to $v^{-1}$ or $v$, $\text{Lab}_{p_v}(\overline{\text{Equ}})$ is equal to $u$ or $u^{-1}$ depending on the direction of travelling along $\overline{\text{Equ}}$ and $\overline{\text{Equ}}$. Fix the positive direction of travelling along the equators so that $\text{Lab}_{p_v}(\overline{\text{Equ}})$ is equal to $v$, $\text{Lab}_{p_u}(\overline{\text{Equ}})$ is equal to $u$.

The equators $\overline{\text{Equ}}$ and $\overline{\text{Equ}}$ divide the torus $T$ into two annulus (corresponding to $P_1$ and $P_2$). The annulus containing the vertices with labels from $R_1$ (resp., $R_2$) will be called the $R_1$-annulus (resp., the $R_2$-annulus).

In the sequel we will use admissible moves to transform the picture $P$ on $T$. A move is called admissible if after the move,

(i) $\text{Lab}_{p_u}(\overline{\text{Equ}})$ (resp., $\text{Lab}_{p_v}(\overline{\text{Equ}})$) is replaced by a word equal to $v$ (resp., $u$) to within elements from $N_1 \cap N_2$;

(ii) all generalized vertices with labels from $N_1$ are only in the $R_1$-annulus, all generalized vertices with labels from $N_2$ are only in the $R_2$-annulus, where a generalized vertex is a vertex (one can consider it as a ‘small’ planar picture) with the label equal to an arbitrary (not-necessary reduced) word of $N_1$ (or $N_2$);

(iii) the equators and $\text{Conj}$ remain unchanged.

**Assertion 1.** Let the presentation $G = \langle A \mid R_1 \cup R_2 \rangle$ be atorical (Condition 2.2 of Theorem 1). Then there exists a finite sequence of admissible moves of $P$, at the end of which the labels of the equators will have one of the following form:

(1) $\text{Lab}_{p_u}(\overline{\text{Equ}}) = \alpha(\omega'\nu_1\nu_2)\alpha^{-1}$, $\text{Lab}_{p_v}(\overline{\text{Equ}}) = \beta(\omega'\nu_1\nu_2)\beta^{-1}$;

(2) $\text{Lab}_{p_u}(\overline{\text{Equ}}) = \alpha(\omega^k\nu_1\omega^{-l}\nu_2)\alpha^{-1}$, $\text{Lab}_{p_v}(\overline{\text{Equ}}) = \beta(\omega^k\nu_1\nu_2)\beta^{-1}$;

where $\nu_i \in N_i$, $\alpha, \beta, \omega, \omega' \in F$, $l, k \in \mathbb{Z}$, $l \neq 0$ can be determined by $P$ at the end.

To prove Theorem 1 let us use Assertion 1, which will be proved in Section 3. By Assertion 1 we have two possibilities for representation of $u$ and $v$ to within elements from $N_1 \cap N_2$. If $u$ and $v$ have the form (1), they are evidently conjugated in $F/N_1 \cap N_2$ by the word $h = \alpha^{-1}\beta$. Consider the case, when $u$ and $v$ have the form (2). The following notations will be used:

$$\tilde{u} = \alpha^{-1}u\alpha, \quad \tilde{v} = \beta^{-1}v\beta;$$

$\text{Roots}_{G_1}(\tilde{v}) = \{c \in F \mid \exists s = s(c) \in \mathbb{Z} : \tilde{v} = c^s \text{ v } G_1\}$;

$\text{Roots}_{G_2}(\tilde{v}) = \{d \in F \mid \exists t = t(d) \in \mathbb{Z} : \tilde{v} = d^t \text{ v } G_2\}$. 
Lemma 2. Let the presentation $G = \langle A \mid R_1 \cup R_2 \rangle$ be atorical (Condition 2.2 of Theorem 1) and $u = \alpha(\omega^k \nu_1 \omega^{-1} \nu_2 \omega^l)\alpha^{-1}$, $v = \beta(\omega^k \nu_1 \nu_2)\beta^{-1}$ in $F/N_1 \cap N_2$. Then $u$ and $v$ are conjugated in $F/N_1 \cap N_2$ if and only if there exist $c \in \text{Roots}_{G_1}(\widetilde{v})$, $d \in \text{Roots}_{G_2}(\widetilde{v})$, $\bar{s}, \bar{t} \in \mathbb{Z}$ with $0 \leq \bar{s} < s(c), 0 \leq \bar{t} < t(d)$ such that $d^{-\bar{t}}c^{-\bar{s}}\omega^l$ belongs to the cyclic subgroup $\langle \widetilde{v} \rangle$ of $G$.

Proof of Lemma 2. Assume that there exists a word $h \in F$ such that the equality $u = h^{-1}vh$ holds in $F/N_1 \cap N_2$. Then the equality $u = h^{-1}vh$ holds both in $F/N_1$ and $F/N_2$. It is clear that $u$ and $v$ are conjugated by $h$ if and only if $\widetilde{u}$ and $\widetilde{v}$ are conjugated by the word $x$, where $x = \alpha^{-1}h\beta$. Hence further we will consider $\widetilde{u}$ and $\widetilde{v}$ and investigate $x$.

In $G_1 = F/N_1$, $\bar{u} = \omega^{-1}(\omega^k \nu_2)\omega^l$ and $\bar{v} = \omega^k \nu_2$. Since $\bar{u} = x^{-1}\bar{v}x$ in $G_1$, we have that $\omega^lx^{-1}$ and $\bar{v}$ commute in $G_1$. By Condition 2.2 of Theorem 1 the presentation $G = \langle A \mid R_1 \cup R_2 \rangle$ is atorical, hence, the presentation $G_1 = \langle A \mid R_1 \rangle$ is also atorical. By Theorem 13.5 [9] it follows that there exists $c \in \text{Roots}_{G_1}(\widetilde{v})$ such that $\bar{v} = c^s$, $\omega^lx^{-1} = c^{m_1}$ in $G_1$ for some $s = s(c), m_1 \in \mathbb{Z}$. On the other hand, $\bar{u}$ and $\bar{v}$ are equal to $\omega^k \nu_1$ in $G_2 = F/N_2$. Since $\bar{u} = x^{-1}\bar{v}x$ in $G_2$, we have that $x^{-1}$ and $\bar{v}$ commute in $G_2$. By Condition 2.2 of Theorem 1 and Theorem 13.5 [9] there exists $d \in \text{Roots}_{G_2}(\widetilde{v})$ such that $\bar{v} = d^d$, $x^{-1} = d^{m_2}$ in $G_2$ for some $t = t(d), m_2 \in \mathbb{Z}$.

It follows from the equalities $\omega^lx^{-1} = c^{m_1}$ in $G_1$ and $x^{-1} = d^{m_2}$ in $G_2$ that $\omega^l = c^{m_1}d^{-m_2}$ in $G = F/N_1N_2$. Since $\bar{v} = c^s$ in $G_1$, $\bar{v} = d^d$ in $G_2$, we have $\omega^l = c^d\bar{v}^p$ in $G$ for $0 \leq \bar{s} < s, 0 \leq \bar{t} < t$ and some integer $p$, that is, $d^{-\bar{t}}c^{-\bar{s}}\omega^l = \bar{v}^p$ in $G$.

Conversely, suppose $d^{-\bar{t}}c^{-\bar{s}}\omega^l = \bar{v}^p$ in $G = F/N_1N_2$. Let us prove that $u$ and $v$ are conjugated in $F/N_1 \cap N_2$. Since $d^{-\bar{t}}c^{-\bar{s}}\omega^l = \bar{v}^p$ in $G$, the word $\bar{v}^{-p}d^{-\bar{t}}c^{-\bar{s}}\omega^l$ is represented in the form $\bar{v}_2\bar{v}_1^{-1}$ for some words $\bar{v}_i \in N_i$ ($i = 1, 2$). Therefore we have the equality $c^{-\bar{s}}\omega^l \bar{v}_1 = d^p\bar{v}^p\bar{v}_2$ in $F$. Let us verify that we can take $c^{-\bar{s}}\omega^l \bar{v}_1 = d^p\bar{v}^p\bar{v}_2$ as $x$. Indeed, in $G_1$ we have

$$x^{-1}\bar{v}x = x^{-1}c^s x = \omega^{-1}c^s c^{-\bar{s}} \omega^l = \omega^{-1}c^s \omega^l = \omega^{-1}\bar{v}\omega^l = \omega^{-1}(\omega^k \nu_2)\omega^l = \bar{u}.$$

In $G_2$ we have

$$x^{-1}\bar{v}x = x^{-1}d^d x = \bar{v}^{-p}d^{-\bar{t}}d^d \bar{v}^p = \bar{v}^{-p}\bar{v}^p = \bar{v} = \bar{u}.$$

Hence, $x^{-1}\bar{v}x = \bar{u}$ in $F/N_1 \cap N_2$. Therefore $u = h^{-1}vh$ in $F/N_1 \cap N_2$ for $h = \alpha x\beta^{-1}$.\[\blacksquare\]

By Lemma 2 we get the following algorithm.

By the word $\bar{v}$ determine finite sets $\text{Roots}_{G_1}(\bar{v})$, $\text{Roots}_{G_2}(\bar{v})$ (it is possible by Condition 1.2 of Theorem 1). For each $c \in \text{Roots}_{G_1}(\bar{v})$ and
Using Condition 1.1 of Theorem 1, find the numbers \( s = s(c), t = t(d) \in \mathbb{Z} \) with the least absolute values such that \( \tilde{v} = c^s \) in \( G_1 \) and \( \tilde{v} = d^t \) in \( G_2 \). Using Condition 2.1 of Theorem 1, verify whether there exists an integer \( p \) such that \( d^{-t}c^{-s}w = \tilde{v}^p \) in \( G = F/N_1N_2 \) for some integers \( \bar{s}, \bar{t} \) with \( 0 \leq \bar{s} < s(c), 0 \leq \bar{t} < t(d) \). If such \( p \) is found, express \( \tilde{v} - p d^{-t}c^{-s}w \) with defining relations \( R_1 \cup R_2 \) of \( G \) (it is possible by Condition 2.1 of Theorem 1) and represent \( \tilde{v} - p d^{-t}c^{-s}w \) in the form \( \tilde{v}_2\tilde{v}_1^{-1} \), where \( \tilde{v}_i \in N_i (i = 1, 2) \). One can take \( \alpha c^{-s}w \tilde{v}_1 \beta^{-1} \) as a word \( h \) conjugating \( u \) and \( v \). If for any \( c \in \text{Roots}_{G_1}(\tilde{v}) \) and \( d \in \text{Roots}_{G_2}(\tilde{v}) \) there is no such \( p \), conclude that \( u \) and \( v \) are not conjugated in \( F/N_1 \cap N_2 \). So Theorem 1 is proved.

2. Admissible moves using in the proof of Assertion 1.

Below any domain \( M \subset T \) homeomorphic to the square \( \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, -1 < y < 1 \} \) together with vertices and parts of edges belonging to \( M \) will be called a map. For a given path (an edge) on the torus \( T \), any part of the path (the edge) homeomorphic to \( \{x \in \mathbb{R} \mid -1 \leq x \leq 1 \} \) will be called a segment of the path (of the edge). We will say that a domain on the torus contains nothing, if it does not contain poles, vertices and segments of edges of \( P \). We will say that a domain on the torus contains absolutely nothing, if it contains nothing and there is no point from \( \text{Equ} \cup \overline{\text{Equ}} \cup \text{Conj} \) in it.

1) Isotopy.

An isotopy of the picture \( P \) is defined by replacing \( P \) by a picture \( F_t(P) \), where \( F_t : T \times [0, 1] \to T \times [0, 1] \) is a continuous isotopy of the torus \( T \) such that

(i) \( F_t \) leaves fixed all vertices and the both poles, i.e. for each \( t \in [0, 1] \) and each vertex \( V_i \), \( F_t(V_i) = V_i \), \( F_t(p_u) = p_u \), \( F_t(p_v) = p_v \);

(ii) for each \( t \in [0, 1] \) and each edge \( E_j \) the intersection of \( F_t(E_j) \) and \( \text{Equ} \), \( \overline{\text{Equ}} \), \( \text{Conj} \) consists of a finite number of points, moreover, if \( \text{Equ} \), or \( \overline{\text{Equ}} \), or \( \text{Conj} \) intersects \( F_t(E_j) \), then it crosses it, and doesn’t just touch it.
An isotopy of $P$ is an admissible move because either it corresponds to a succession of free insertions or free deletions in $\text{Lab}_{p_v}^+(\text{Equ})$ and $\text{Lab}_{p_u}^-(\text{Equ})$ or it does not change $\text{Lab}_{p_v}^+(\text{Equ})$ and $\text{Lab}_{p_u}^-(\text{Equ})$ at all (see Fig.1).

2) Deletion of a superfluous loop (this is a particular case of isotopy). Let $\text{Equ}$ (resp., $\overline{\text{Equ}}$, $\text{Conj}$) intersect any edge $E$ in two points, which divide $\text{Equ}$ (resp., $\overline{\text{Equ}}$, $\text{Conj}$) into two parts so that one of these parts $\zeta$ does not intersect any edge and does not contain the poles. By $\vartheta$ denote the segment of $E$ between these points. If a disk on the torus $T$ encircled by the circle $\zeta \sqcup \vartheta$ contains absolutely nothing inside, then $\vartheta$ is called a superfluous loop. It is clear that superfluous loops do not contribute to the corresponding equatorial label (considered as an element of the free group). Therefore superfluous loops can be removed (see Fig.1).

3) Bridge moves.
Assume that a map $M$ contains absolutely nothing except for two segments of edges $\{x = -1/2, -1 < y < 1\}$ and $\{x = 1/2, -1 < y < 1\}$, which are contrariwise oriented and labelled by the same word $g$. A transformation of $P$ is called a bridge move if it does not change $P$ out of $M$ and change $P$ inside $M$ as is shown on Fig.2. A bridge move is an admissible move because it does not change the equatorial labels.
4) **Uniting of edges.**

Let $E_1$ and $E_2$ be two edges-circles with labels $g_1$ and $g_2$, which side by side intersect $\overline{\text{Equ}}$, $\overline{\text{Equ}}$ and $\text{Conj}$ and bound on the torus an annulus, containing nothing, or $E_1$ and $E_2$ be two edges with labels $g_1$ and $g_2$, which join the same vertices, side by side intersect $\overline{\text{Equ}}$, $\overline{\text{Equ}}$ and $\text{Conj}$ and encircle on the torus a disk, containing nothing. Remove $E_2$. If $E_1$ and $E_2$ had the same orientation, label $E_1$ by $g_1g_2$ or $g_2g_1$, otherwise label $E_1$ by $g_1g_2^{-1}$ or $g_2^{-1}g_1$. The label for $E_1$ should be chosen so that the contribution of this label to the equatorial labels remains the same as the contribution of the both edges $E_1$ and $E_2$. We will assume that the multiplication of $g_1$ and $g_2^{\pm 1}$ is free.

5) **Cutting of complete dipoles.**

A complete $R_1$-dipole is a dipole $D_1$ such that the labels of its vertices is equal to $r_1^{\pm 1} \in R_1 \setminus R_2$ and its vertices are joined by a single edge $E_1$ with the label $r_1$.

Consider a map $M$ in the $R_1$-annulus such that $M$ contains absolutely nothing except for a segment of $E_1$: $\{x = 0\}$, starting at the point $(0, -1)$ and ending at the point $(0, 1)$. Cut out $M$ from $P$ and paste a new map $M'$ instead of $M$. The new map $M'$ contains absolutely nothing except for two vertices $V'$, $V''$ with the labels $r_1$, $r_1^{-1}$ and two edges one of which starts at $(0, -1)$ and ends at $V'$, and the other one starts at $V''$ and ends at $(0, 1)$. As a result one has two complete $R_1$-dipoles instead of one. This move is admissible because it does not change the equatorial labels.

Similarly one can define a complete $R_2$-dipole whose vertices are labelled by $r_2^{\pm 1} \in R_2 \setminus R_1$ and a corresponding move performed in the $R_2$-annulus. Similarly one can define a complete mixed dipole whose vertices are labelled by $r^{\pm 1} \in R_1 \cap R_2$.

6) **Conjugation of dipoles.**

Let $n_1 \in N_1$. A generalized $N_1$-dipole with the label $n_1$ is two generalized
vertices with the labels \( n_1^{\pm 1} \) and a single edge with the label \( n_1 \), joining them. For example, a complete \( R_1 \)-dipole is generalized one labelled by \( r_1 \in R_1 \subset N_1 \).

Let \( D_1 \) be a generalized \( N_1 \)-dipole with the label \( n_1 \) and \( C \) be an edge-circle with a label \( n_1 \), encircling on \( T \) a disk, containing nothing except for \( D_1 \). In addition the edge of \( D_1 \) and the edge-circle \( C \) side by side intersect \( \text{Equ} \), \( \text{Equ} \) and \( \text{Conj} \) and contribute \( (fn_1f^{-1})^{\pm 1} \in N_1 \) to the labels of \( \text{Equ} \), \( \text{Equ} \) and \( \text{Conj} \). Remove \( C \) and label the edge of \( D_1 \) by \( f_{n_1}f^{-1} \) and its generalized vertices by \( (f_{n_1}f^{-1})^{\pm 1} \). This move does not change the equatorial labels, hence it is admissible.

Similarly one can define a generalized \( N_2 \)-dipole and a corresponding move of it.

7) Deletion of a dipoles and an edge-circles not intersecting the equators. If a generalized dipole or an edge-circle does not intersect \( \text{Equ} \) and \( \text{Equ} \), then it does not contribute to \( \text{Lab}^+_{p_v}(\text{Equ}) \) and \( \text{Lab}^+_{p_u}(\text{Equ}) \). Hence remove it.

8) Conjugation of a pole.

Consider the pole \( p_v \) (everything is similar for \( p_u \)). Let \( C \) be an edge-circle with a label \( g \in F \) and \( C \) encircle on \( T \) a disk containing absolutely nothing except for \( p_v \), only one segment of \( \text{Equ} \) and only one segment of \( \text{Conj} \). The union of \( C \) and \( p_v \) is called a conjugated pole \( p_v \). The pole \( p_v \) itself will be considered as a conjugated pole (encircled by an edge-circle \( C \) with the label equal to the identity of the free group). If a conjugated pole \( p_v \) is surrounded in the same way by an edge-circle \( C \), then unite \( C \) and \( \bar{C} \). This move does not change the equatorial labels, hence it is admissible.

9) Deletion of a one-sided dipole.

Let the edge of a generalized \( N_1 \)-dipole \( D_1 \) (everything is similar for a generalized \( N_2 \)-dipole) with the label \( n_1 \in N_1 \) do not intersect \( \text{Conj} \) and intersect only one of the equators (for definiteness, \( \text{Equ} \)) and only at two points. Then \( D_1 \) is called a one-sided \( N_1 \)-dipole.

There exists a closed disk \( O \) containing absolutely nothing except for \( D_1 \) and two segments \([s_1,s_2]\) and \([t_1,t_2]\) of \( \text{Equ} \), where the points \( s_1, s_2, t_1, t_2 \) belong to \( \partial O \cap \text{Equ} \). Note that the labels of \([s_1,s_2]\) and \([t_1,t_2]\) are equal to \( n_1 \) and \( n_1^{-1} \) respectively, i.e., to the labels of \( D_1 \). In addition either \([s_2,t_1]\) or \([t_2,s_1]\) does not contain the pole. For definiteness let us assume that it is \([s_2,t_1]\). The points \( s_2, t_1 \) divide \( \partial O \) into two segments. By \( \varrho \) denote such of them which contains no points of the \( R_1 \)-annulus. Then the closed path \([s_2,t_1]\cup \varrho \) encircles a planar picture over the presentation \( G = \langle A \mid R_2 \rangle \). By Lemma 1 the label \( n_2 \) of \([s_2,t_1]\cup \varrho \) belongs to \( N_2 \). Since no edges intersect \( \varrho, n_2 \)
is the label of $[s_2, t_1]$. So the label of $[s_1, s_2] \cup [s_2, t_1] \cup [t_1, t_2]$ is equal to $n_1n_2n_1^{-1}$. Remove $D_1$ from $P$. The label of $[s_1, s_2] \cup [s_2, t_1] \cup [t_1, t_2]$ becomes equal to $n_2$. This move does not change $\text{Lab}^+_{pu}(\text{Equ})$ to within $n_1n_2n_1^{-1}n_2^{-1} \in N_1 \cap N_2$. Hence this move is admissible.

10) **Permutation of two-sided dipoles.**

Let the edge of a generalized $N_1$-dipole $D_1$ with the label $n_1 \in N_1$ do not intersect $\text{Conj}$ and intersect each of the equators $\text{Equ}$ and $\text{Equ}$ exactly at one point. Then $D_1$ is called a two-sided $N_1$-dipole.

There is an open disk $O_1$ containing absolutely nothing except for $D_1$ and two segments $[s_1, t_1] \in \text{Equ}$ and $[q_1, p_1] \in \text{Equ}$, where the points $s_1, t_1$ belong to $\partial O_1 \cap \text{Equ}$, the points $q_1, p_1$ belong to $\partial O_1 \cap \text{Equ}$. Note that the labels of $[s_1, t_1]$ and $[q_1, p_1]$ are equal to $n_1$ and $n_1^{-1}$ respectively, i.e., to the labels of $D_1$.

Similarly one can define a **two-sided $N_2$-dipole**. Substituting 2 instead of 1 in the above notations for the two-sided $N_1$-dipole, one gets the same notations for a two-sided $N_2$-dipole.

Now let both a two-sided $N_1$-dipole $D_1$ and a two-sided $N_2$-dipole $D_2$ be in $P$. The points $s_1, s_2, t_1, t_2$ divide $\text{Equ}$ into four segments. Assume that one of them (say $[t_1, s_2]$) does not intersect any edge and does not contain the pole. Then the label of the segment $\sigma = [s_1, t_1] \cup [t_1, s_2] \cup [s_2, t_2]$ is equal to $n_1n_2$. Permute the segments $[s_1, t_1]$ and $[s_2, t_2]$ (see Fig. 3).

![Fig. 3](image)

After this move, the label of the new segment $\sigma$ become equal to $n_2n_1$. This move is admissible, since after it $\text{Lab}^+_{pu}(\text{Equ})$ is not changed to within the word $n_2^{-1}n_1^{-1}n_2n_1 \in [N_1, N_2]$.

Similarly one can define the same move for $\text{Equ}$.

11) **Moving of an edge over a dipole or a pole.**

Let $X$ be a generalized $N_1$- or $N_2$-dipole (resp., a conjugated $p_u$ or $p_v$ pole), $O_1, O_2, O_3$ be three closed disks on the torus $T$ containing nothing
except for $X$ such that $O_3 \subset O_2 \setminus \partial O_2, O_2 \subset O_1 \setminus \partial O_1$. Let $E$ be an edge with a label $g \in F$ such that $E \cap O_1 = \emptyset$ and there exists a simple path $\gamma$ joining points $o \in E$ and $o_3 \in \partial O_3$, intersecting $E$ and $\partial O_1, \partial O_2$ exactly at one point, not intersecting other edges, the equators and $Conj$ and not passing through any vertex. Thus $Lab^+(\gamma) = g$. Put in $P$ two contrariwise oriented edge-circles $C_1 = \partial O_1$ and $C_2 = \partial O_2$ labelled by $g \in F$ so that $Lab^+(\gamma)$ becomes identically equal to $gg^{-1}g$.

Apply the bridge move to $E$ and $C_1$, the conjugation to $X$ and $C_2$. It is clear that this move is admissible, because either it corresponds to an insertion of inverse words in $Lab^+_p(Equ)$ and $Lab^+_q(Equ)$, or it does not change $Lab^+_p(Equ)$ and $Lab^+_q(Equ)$ at all.

3. Proof of Assertion 1.

**STEP 1. Extraction of complete dipoles.**

Since the presentation $G = \langle A \mid R_1 \cup R_2 \rangle$ is atorical, there exists a dipole $D$ in the picture $P$ on the torus $T$, i.e., there exists a couple of vertices $V_1$ and $V_2$ with mutually inverse labels $r$ and $r^{-1}$ such that $V_1$ and $V_2$ are connected by an edge $\rho$. Applying the bridge moves no more than $|r| - 1$ times, we obtain that all edges go from $V_1$ to $V_2$ side by side in a parallel way to $\rho$ and $\rho$ remains unchanged. Unite these edges. This makes the dipole $D$ complete.

Now the picture $P$ consists of two disjoint subpictures $P_1 \sqcup P_2$, one of which (say $P_1$) contains nothing except for the complete dipole $D$. The subpicture $P_2$ is a picture on the torus over presentation $G = \langle A \mid R_1 \cup R_2 \rangle$. Besides $P_2$ contains two fewer vertices than $P$. Repeating the above procedure for $P_2$, and so on, we will eventually reduce $P$ to $mV/2$ complete dipoles and edge-circles, where $mV$ is the number of vertices in $P$.

**STEP 2. A move after which dipoles do not intersect $Conj$.**

After Step 1 the picture $P$ consists of edges-circles, complete $R_1$-, $R_2$-dipoles and complete mixed dipoles. If the edges of some complete dipoles do not intersect the equators, remove these complete dipoles. Also remove complete mixed dipoles from $P$. This changes the equatorial labels by elements from $R_1 \cap R_2 \subset N_1 \cap N_2$. Now $P$ contains only complete $R_1$-, $R_2$-dipoles and edges-circles.

**Operation 1.** Consider a complete $R_1$-dipole $D$ (the case of a complete $R_2$-dipole is similar). Let its edge intersect $Conj_1$ at points $o_1, \ldots, o_\tilde{m}$. Near by $o_i$ ($i = 1, \ldots, \tilde{m}$) cut $D$ into three complete dipoles $D_1, D_2, D_3$, one of which ($D_2$) lies in the $R_1$-annulus as a whole and its edge intersects $Conj_1$ exactly at one point (at $o_i$). Remove $D_3$ from $P$. Repeating the same procedure to each of $\tilde{m}$ intersections, instead of one
complete $R_1$-dipole $D$, we obtain $\tilde{m} + 1$ complete $R_1$-dipoles, neither of which intersects $Conj_1$.

Apply Operation 1 to each of complete $R_1$- and $R_2$-dipoles. This gives that the edges of the complete $R_1$-dipoles do not intersect $Conj_1$ and the edges of the complete $R_2$-dipoles do not intersect $Conj_2$.

**Operation 2.** Consider $Conj_1$ (the case of $Conj_2$ is similar). It can be intersected only by the edges of complete $R_2$-dipoles and by edges-circles. Let $\rho_1, ..., \rho_{\tilde{m}}$ be edges-circles not conjugating the poles and edges of complete $R_2$-dipoles such that $\rho_1, ..., \rho_{\tilde{m}}$ intersect $Conj_1$, and we encounter them in the order $\rho_1, ..., \rho_{\tilde{m}}$ if we start at the conjugated pole $p_u$ and travel along $Conj_1$ to the conjugated pole $p_v$. Starting with $\rho_1$, move consecutively each edge $\rho_i$ over the conjugated pole $p_u$. This gives that the edges of the complete $R_2$-dipoles intersect only $Conj_2$.

Apply Operation 1 to these complete $R_2$-dipoles.

After Operation 2 applying to $Conj_1$ and $Conj_2$, the picture $P$ consists of edges-circles and only of complete $R_1$- and $R_2$-dipoles $D_1, ..., D_{\tilde{m}}$ not intersecting $Conj$. For each $D_i$, by $m_i$ denote the number of intersections of the equators and the edge of $D_i$. Note that $m_i$ is even. One can assume that $m_i > 0$, otherwise remove $D_i$ from $P$. If $m_i > 2$, cut $D_i$ into $m_i/2$ complete dipoles each of which intersects the equators exactly at two points. This move applying to each $D_i$ makes all dipoles either one-sided or two-sided. Remove all one-sided dipoles from $P$.

**STEP 3. Getting rid of contractible edges-circles.**

After Step 2 the picture $P$ consists just of two-sided dipoles and edges-circles. Call an edge-circle *contractible*, if it divides the torus into two parts one of which is homeomorphic to a disk. This part will be called the *interior* of the edge-circle.

After the deletions of superfluous loops from the equators and $Conj$ each contractible edge-circle $C$ belongs to one of the following types.

I) The interior of $C$ contains absolutely nothing.

II) The interior of $C$ contains nothing except for just one two-sided dipole or just one conjugated pole.

III) In the interior of $C$, there are at least two two-sided dipoles, or at least one two-sided dipole and at least one conjugated pole, or the both conjugated poles.

Remove all edges-circles of Type I from $P$. Apply the conjugation of dipoles or the conjugation of poles to all edges-circles of Type II. Now just $\tilde{n}$ edges-circles of Type III remain in $P$.

Call an edge-circle of Type III *minimal*, if there is no other edges-circles of Type III in its interior.
Operation 3. Let $C$ be a minimal edge-circle of Type III with $\tilde{m}_1$ two-sided dipoles and $\tilde{m}_2$ conjugated poles in its interior, $\tilde{m}_1 + \tilde{m}_2 \geq 2$. It is clear that by the isotopy and the $\tilde{m}_1 + \tilde{m}_2 - 1$ bridge moves, $C$ can be reduced to $\tilde{m}_1 + \tilde{m}_2$ edges-circles of Type II. Apply the conjugation to each of these edges-circles of Type II.

Operation 3 gives a picture $P$ with one fewer edges-circles. Hence after no more than $\tilde{m}$ applications of Operation 3, the picture $P$ will contain just non-contractible edge-circles and two-sided dipoles.

STEP 4. Uniting of non-contractible edges-circles.

After Step 3 the picture $P$ contains just non-contractible edges-circles $Z_1, ..., Z_m$ and two-sided dipoles. Cutting out one of the edges-circles ($Z_1$) from the torus converts the torus to a surface $\Omega$ homeomorphic to an annulus. In $\Omega$ any closed simple not contractible path ($Z_i, i \neq 1$) is homotopic to the boundary ($Z_1$) and to any other closed simple not contractible path ($Z_j, j \neq 1, i$) disjoint with it. The edges-circles $Z_2, ..., Z_m$ divide $\Omega$ into $m$ disjoint parts $\Omega_1, ..., \Omega_m$ each homeomorphic to an annulus. Assume that the edges-circles $Z_2, ..., Z_m$ are numbered so that $\Omega_1$ are bounded by $Z_1$ and $Z_2$, $\Omega_2$ are bounded by $Z_2$ and $Z_3$, ..., $\Omega_m$ are bounded by $Z_m$ and $Z_1$.

Consider $\Omega_1$. If there are conjugated poles or two-sided dipoles in $\Omega_1$, apply the isotopy and move $Z_2$ over these dipoles and poles to transpose these poles and dipoles from $\Omega_1$ to $\Omega_2$, and to approach $Z_2$ and $Z_1$ to each other so that $Z_1$ and $Z_2$ become parallel and side by side intersect the equator and $\text{Conj}$. Repeat the same procedure for each $\Omega_i, i = 2, ..., m - 1$ to transpose conjugated poles and generalized dipoles from $\Omega_i$ to $\Omega_{i+1}$. We will eventually obtain that all conjugated poles and two-sided dipoles of $P$ are in $\Omega_m$ and $Z_1, ..., Z_m$ are parallel and side by side intersect the equators and $\text{Conj}$. Unite $Z_1, ..., Z_m$. This gives a single edge-circle $Z$.

STEP 5. Disposition of two-sided dipoles in the order.

Above the orientation on the equators was fixed. If we start at $p_v$ and travel once around $\text{Equ}$ in the positive direction, we encounter a succession of edges of dipoles $D_1, ..., D_s$ intersecting $\text{Equ}$. We say that an $N_2$-dipole $D_i$ and an $N_1$-dipole $D_j$ form the inversion on $\text{Equ}$, if $i < j$, otherwise they form the order on $\text{Equ}$. In the same way one can define the inversion and the order on $\text{Equ}$. We will say that one circuit along $\text{Equ}$ (resp., $\text{Equ}$) in the positive direction starting at $p_v$ (resp., $p_a$) is a movement from the left to the right.

The edge-circle $Z$ is divided by the equators into segments. Two-sided $R_2$-pieces (resp., two-sided $R_1$-pieces) are such of these segments.
that do not intersect $Conj$ and lie in the $R_2$-annulus (resp., in the $R_1$-annulus) at the whole, starting on one of the equators and ending on the other one.

**Lemma 3.** There exists a finite succession of admissible moves that disposes all edges of two-sided $N_1$-dipoles in the $R_2$-annulus on the left side of the two-sided $R_2$-pieces of $Z$.

**Proof of Lemma 3.** If there are no two-sided $N_1$-dipole or two-sided $R_2$-pieces in $P$, there is nothing to prove. Otherwise let $m$ be the minimal number of transpositions to get all edges of two-sided $N_1$-dipoles on the left side of the two-sided $R_2$-pieces of $Z$. If $m = 0$, there is nothing to prove. Otherwise consider the rightest two-sided $N_1$-dipole $D$ which has a two-sided $R_2$-piece $\rho$ on the left such that there are no other $N_1$-dipoles or two-sided $R_2$-pieces between $D$ and $\rho$. Move $\rho$ over $D$ to the right of $D$. This decreases $m$ by 1. Now use induction on $m$. ■

The edges of two-sided $N_1$-dipoles consecutively intersect $Equ$ (resp., $\overline{Equ}$). For a given two-sided $N_1$-dipole, let $o'$ and $o''$ be two consecutive intersections of its edge and $Equ$ (resp., $\overline{Equ}$). By Lemma 3, removing superfluous loops, if necessary, either there are no intersections with $Z$ between $o'$ and $o''$, or there are intersections with the edges of two-sided $N_2$-dipoles between $o'$ and $o''$ and $Z$ intersects $Equ$ (resp., $\overline{Equ}$) between $o'$ and $o''$, gets into the $R_2$-annulus, envelops a vertex of at least one of these two-sided $N_2$-dipoles, turns back to $Equ$ (resp., $\overline{Equ}$) and returns to the $R_1$-annulus.

**Lemma 4.** There exists a finite succession of admissible moves that disposes all two-sided dipoles of $P$ in the order.

**Proof of Lemma 4.** By $m'$ denote the number of inversions on $Equ$, by $m''$ the number of inversions on $\overline{Equ}$. If $m' + m'' = 0$, there is nothing to prove. Let $m' + m'' > 0$.

If $m' > 0$, at first consider $Equ$. Let $D_1$ and $D_2$ be two neighboring two-sided $N_1$- and $N_2$-dipoles forming the inversion on $Equ$ such that there are no other dipoles between them. We can assume that $D_2$ is not enveloped by $Z$, otherwise move $Z$ over $D_2$. The permutation of $D_1$ and $D_2$ decreases $m'$ by 1. Induction on $m'$ gives that all two-sided dipoles form the order on $Equ$.

If $m'' > 0$, apply the same procedure to $\overline{Equ}$. ■

**Lemma 5.** There exists a finite succession of admissible moves that disposes all edges of two-sided $N_2$-dipoles in the $R_2$-annulus on the left side of two-sided $R_1$-pieces of $Z$. 
The proof of Lemma 5 is similar to the proof of Lemma 3.

So all two-sided dipoles of $P$ form the order both on $\overline{Equ}$ and on $\overline{Equ}$. In addition two-sided $N_1$-dipoles (resp., $N_2$-dipoles) are near by to each other and intersect the equators side by side. Replace all these two-sided $N_1$-dipoles (resp., $N_2$-dipoles) by one two-sided dipole $\Delta_1$ (resp., $\Delta_2$) with the edge’s label $\nu_1$ (resp., $\nu_2$) equal to the product of the labels of all these two-sided $N_1$-dipoles (resp., $N_2$-dipoles), i.e., $\nu_1$ (resp., $\nu_2$) belongs to $N_1$ (resp., $N_2$). If $\nu_1$ (resp., $\nu_2$) is equal to the identity in $F$, remove the dipole $\Delta_1$ (resp., $\Delta_2$).

**STEP 6.** **Finale.**

The picture $P$ can contain at most one edge-circle $Z$, at most one two-sided $N_1$-dipole $\Delta_1$ (with the label $\nu_1$), at most one two-sided $N_2$-dipole $\Delta_2$ (with the label $\nu_2$) and two conjugated poles $p_u$ and $p_v$. By $\alpha$ (resp., $\beta$) denote the label of the edge conjugating the pole $p_u$ (resp., $p_v$). Below $P$ will be transformed by isotopy, by moving $Z$ over $\Delta_1$ and $\Delta_2$, by conjugation of poles. For simplicity of notation the labels of $\Delta_1$ and $\Delta_2$, the labels of edges conjugating $p_u$ and $p_v$ will be again denoted by $\nu_1$, $\nu_2$, $\alpha$, $\beta$.

There are three possibility:

**Case A.** **There is no $Z$ in $P$.

We have Case (1) of Assertion 1, i.e. $Lab_{p_u}^+(\overline{Equ}) = \alpha(\nu_1\nu_2)\alpha^{-1}$, $Lab_{p_v}^+(\overline{Equ}) = \beta(\nu_1\nu_2)\beta^{-1}$.

**Case B.** **There is $Z$ in $P$ and $Z$ is homotopic to $Conj$.

By isotopy, moving $Z$ over $\Delta_1$, $\Delta_2$ and the conjugated poles, dispose $Z$ near by $Conj$ in a parallel way to $Conj$ so that $Z$ intersects each of the equators exactly at one point. Thus we have Case (1) of Assertion 1, i.e., $Lab_{p_u}^+(\overline{Equ}) = \alpha(\omega^k\nu_1\nu_2)\alpha^{-1}$, $Lab_{p_v}^+(\overline{Equ}) = \beta(\omega^k\nu_1\nu_2)\beta^{-1}$, where $\omega$ is the label of $Z$.

**Case C.** **There is the edge-circle $Z$ in $P$ and $Z$ is homotopic to a simple closed path circuiting $Conj$ $|k|$ times and the equators $|l|$ times, where $l,k \in \mathbb{Z}$, $|l| \geq 1$.

If there is no dipole $\Delta_2$ in $P$, by isotopy and moving $Z$ over $\Delta_1$ and the conjugated poles, dispose $Z$ in such a way that the $|k|$ circuits of $Z$ along $Conj$ are near by $Conj$ and the $|l|$ circuits of $Z$ along the equators are in the $R_1$-annulus. Thus we have Case (1) of Assertion 1, i.e., $Lab_{p_u}^+(\overline{Equ}) = \alpha(\omega^k\nu_1)\alpha^{-1}$, $Lab_{p_v}^+(\overline{Equ}) = \beta(\omega^k\nu_1)\beta^{-1}$, where $\omega$ is the label of $Z$.

It remains to consider the case when there exists $\Delta_2$ in $P$. By isotopy and moving $Z$ over $\Delta_1$, $\Delta_2$ and the conjugated poles, dispose $Z$ in such a way that the $|k|$ circuits of $Z$ along $Conj$ are near by $Conj$ and the $|l|$ circuits of $Z$ along the equators start in the $R_1$-annulus, go in a
parallel way to each other to the edge of $\Delta_2$, envelope its vertex after intersecting $\overline{E_{\text{equ}}}$ and return to the $R_1$-annulus. Thus we have Case (2) of Assertion 1: $\text{Lab}^+_{p_u}(\overline{E_{\text{equ}}}) = \alpha(\omega^k \nu_1 \omega^{-l} \nu_2 \omega') \alpha^{-1}$, $\text{Lab}^+_{p_v}(\overline{E_{\text{equ}}}) = \beta(\omega^k \nu_1 \nu_2 \omega') \beta^{-1}$, where $\omega$ is the label of $Z$. ■

**Remark 1.** It follows from the proof of Assertion 1 that the integer $L = L(u,v,R_1,R_2)$ such that $|\alpha|, |\beta|, |\omega|, |l| \leq L$, can be chosen as $90(|h_1| + |h_2|)(1 + 2l_R + 2l_R^2 + \ldots + 2l_R^{m_V/2-1})$, where $l_R$ is the length of the longest word of $R_1 \cup R_2$, $m_V$ is the number of vertices in the initial picture $P$.

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