Non-minimally coupled dark fluid in Schwarzschild spacetime

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Abstract

If one assumes a particular form of non-minimal coupling, called the conformal coupling, of a perfect fluid with gravity in the fluid-gravity Lagrangian then one gets modified Einstein field equation. In the modified Einstein equation, the effect of the non-minimal coupling does not vanish if one works with spacetimes for which the Ricci scalar vanishes. In the present work we use the Schwarzschild metric in the modified Einstein equation, in presence of non-minimal coupling with a fluid, and find out the energy-density and pressure of the fluid. In the present case the perfect fluid is part of the solution of the modified Einstein equation. We also solve the modified Einstein equation, using the flat spacetime metric and show that in presence of non-minimal coupling one can accommodate a perfect fluid of uniform energy-density and pressure in the flat spacetime. In both the cases the fluid pressure turns out to be negative. Except these non-trivial solutions it must be noted that the vacuum solutions also remain as trivial valid solutions of the modified Einstein equation in presence of non-minimal coupling.

1 Introduction

General relativity (GR) is now accepted as the standard theory of gravity whose application involves a wide variety of scales. From the cosmic scale to the scale of neutron stars GR works efficiently. Except the vacuum solutions all the other scenarios in GR requires the presence of a gravitating fluid whose energy density and pressure curves space and time. The dynamics of our universe is often described by considering it as a perfect fluid. A perfect fluid is one that has isotropic pressure, no viscosity and no heat conduction.

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However, the validity of the perfect fluid description obviously depends on the length scale used by the observer. We can imagine the constituent bodies to be analogous to those in a fluid if and only if the mean free path between them is much smaller than the length scale used. If the mean free path of the constituent particles of the system is small compared to the curvature scale, we can consider the fluid to be a perfect fluid (in the simplest approximation) in a locally flat spacetime and can apply minimal coupling (MC) prescription to write down the energy momentum tensor of the fluid in more longer length scales where curvature effects become prominent. The root of the procedure lies in the Einstein’s equivalence principle (EEP), which states that in small enough regions of spacetime the laws of physics reduce to those of special relativity. To get the equations in curved spacetime, we simply need to change the flat Lorentzian metric by the actual metric for the curved spacetime and also replace the partial derivatives by the covariant derivatives. This includes the assumption that in the matter Lagrangian, there is no coupling between the dynamical variables describing the fluid and the curvature tensor \[1\].

In presence of non-minimal coupling (NMC), which may arise if the mean free path of the matter distribution is comparable or larger than the curvature scales, we will see that EEP does not hold as new effects become apparent in the locally flat spacetime limit\[^1\].

Recent observations of the Universe \[^3,4\], using information from the CMBR and type I supernovae data, have revealed that only 4% of the total energy density is in the form of baryonic matter. Rest of the 96% consists of dark energy (73%) and dark matter (23%), none of which has yet been experimentally detected. In an earlier work \[^5\] the authors have suggested that the interesting behavior of the dark sector of the universe may have a connection with fluids non-minimally coupled to gravity. The authors of Ref. \[^5\] have argued that dark matter system, being weakly interacting, may have very long mean free path (~ 1000 Gpc); so at the length scale at which these kind of matter can be regarded as a fluid, spacetime no longer remains flat and the effect of curvature starts affecting the fluid variables directly. In that case, the system can be described as a perfect fluid in a curved spacetime which is non-minimally coupled to gravity. As soon as we consider NMC of matter with gravity, the assumption that dynamical variables do not couple directly with the curvature tensor breaks down and the EEP is also violated\[^2\].

To include the effects of NMC of fluids with gravity one has to use the action of perfect fluids in presence of gravity. The concept of relativistic perfect fluids and their properties are well described in Ref. \[^9\] and \[^10\], but in these references the authors do not use the action principle. The action principle for relativistic fluids in general and for MC fluid in a gravitational field is well studied \[^11\], \[^12\], \[^13\], \[^14\]. The authors of Ref. \[^5\] expanded the terms in the fluid action by introducing new NMC terms which they call the conformal coupling, where the fluid variables couple with the Ricci scalar. There can be another kind of a NMC where the fluid variables couple to the contracted form of the Ricci tensor with the fluid 4-velocities, called the disformal coupling. In the present work we will only focus on the conformal coupling case. The generalization of the results for the disformally coupled case can be done in a separate work. An interesting outcome of the NMC of a perfect fluid to curvature is that the gravitational field equation gets modified. In absence of NMC one obtains the conventional Einstein equation for the metric in presence of the perfect fluid whereas in presence of NMC the Einstein equation gets modified. The NMC

\[^1\]One can see how NMC of gravity and Electromagnetic fields affect the equivalence principle in Ref.\[^2\]

\[^2\]The idea of dark matter fluid non-minimally coupled to gravity is also addressed in several other works as in Refs. \[^6\], \[^7\] and \[^8\].
alters the Planck mass and the energy momentum tensor of the fluid which now becomes dependent on the curvature scale. In such a scenario one can analyze the modification of the cosmological equations and get interesting outcomes. One naturally expects that the cosmological solutions will be modified as the fluid content now becomes non-minimally coupled to curvature [5].

In the present work we have tried to address a different problem which may arise in theories where a perfect fluid is non-minimally coupled to gravity. We know that in Einstein gravity the Schwarzschild solution is a vacuum, Ricci-flat solution which is asymptotically flat. Ricci flat solutions in general do not admit presence of matter and are vacuum solutions of Einstein’s equation in GR. We pose the question, can a Ricci-flat metric admit of matter in the presence of NMC terms? In the present article we discuss about two simple Ricci-flat cases, the Minkowski spacetime and that of the Schwarzschild spacetime. The result we get is interesting as because we see that except the standard Schwarzschild solution in vacuum the altered Einstein equation, which includes the effect of the non-minimal coupling, also allows presence of matter in Schwarzschild spacetime. This result opens up another question regarding the asymptotic flatness of the Schwarzschild spacetime. The Schwarzschild spacetime is known to be asymptotically flat and consequently the Schwarzschild solution in presence of a perfect fluid must also be asymptotically flat. In GR the flat spacetime solution is devoid of matter but can one obtain a solution of the altered Einstein equation where a flat spacetime can also accommodate a perfect fluid? If the answer is yes then the Schwarzschild solution in presence of a perfect fluid can match the asymptotically flat spacetime in presence of the same perfect fluid. It turns out that when one includes the NMC term in the basic gravity-fluid action one can obtain flat spacetime solution with a fluid with uniform energy density. The conventional vacuum solution remains a trivial alternative. In presence of NMC of fluid with curvature we will see that if one demands that the number density of particles in the fluid and energy density of the fluid be positive definite the pressure of the resultant fluid has to be negative. Consequently we propose an interesting solution of the altered Einstein equation in presence of NMC where the Schwarzschild solution can accommodate a perfect fluid with negative pressure. It is a natural question that how such a fluid can exist without a backreaction which modifies the form of the metric? The answer lies in the NMC term, in the fluid-gravity Lagrangian, which absorbs all the back reaction which the fluid may produce. As because the pressure of the perfect fluid in the Schwarzschild spacetime turns out to be negative we assume it to be a dark fluid. In this scenario blackholes may be immersed in a dark fluid which fills up all the space outside the blackhole with a varying energy density and pressure.

Before ending the introduction we must point out that in GR one can always accommodate a perfect fluid in the physically relevant spatial regions in the Schwarzschild spacetime[15, 16, 17, 18] where the Einstein equation $R_{\mu\nu} = 0$ holds true. As Schwarzschild metric is a vacuum solution in GR, consequently the perfect fluid in such a spacetime must not be backreacting. Alternatively, the metric must affect the fluid properties via the minimal prescription procedure but the fluid cannot affect the Schwarzschild metric properties in GR. One can apply the conservation of the energy-momentum tensor of the perfect fluid in such a case and find out the energy density profile of the test fluid [18]. In the case of non-minimal coupling one can always accommodate a perfect fluid in the Schwarzschild spacetime where the energy-momentum tensor of the perfect fluid naturally conserved as it is a part of the altered Einstein equation. Once NMC effects
are taken into account the fluid present in the Schwarzschild metric need not be a test fluid. In section 4 we will present some properties of a minimally coupled perfect fluid in the presence of Schwarzschild spacetime. The result of that section will be used to make a comparison between the properties of a non-minimally coupled fluid and a minimally coupled fluid in the Schwarzschild spacetime.

The materials in this paper are presented in the following way. In the next section we specify the formulation of the problem which we will like to address in the present paper. The notations and conventions will also be defined in the next section. The solution of the modified Einstein equation in presence of NMC, for the simplest cases where \( R = 0 \), will be presented in section \( \text{B} \). The first subsection of section \( \text{B} \) will focus on the fluid solutions for the case of flat spacetime. The second subsection of section \( \text{B} \) has the main solution of the modified Einstein equation in Schwarzschild spacetime. In both of the cases in section \( \text{B} \) we will find out the properties of the fluids assuming the background metric to be known. The case of a perfect fluid minimally coupled to gravity, in the Schwarzschild spacetime, is presented in section \( \text{H} \). The penultimate section presents a brief overview on the stability of fluid perturbations in the fixed Schwarzschild metric. The last section concludes the article and contains the summary of the main results obtained in this paper.

2 Formulation of the problem

In this section we formulate the problem after describing the basic equations and the conventions and notations we will use extensively in this paper. We start with the definition of stress-energy tensor \( T_{\mu \nu} \) for a perfect fluid in terms of its energy density \( \rho \) and pressure \( p \):

\[
T_{\mu \nu} = (\rho + p) u_{\mu} u_{\nu} + p g_{\mu \nu}.
\]

Here \( u_{\mu} \) is the fluid 4-velocity which is a timelike 4-vector normalized as \( u^{\mu} u_{\mu} = -1 \) and \( g_{\mu \nu} \) is the metric tensor. For a minimally coupled perfect fluid, the action can be written as:

\[
S_{\text{fluid}} = \int d^4 x \sqrt{-g} F(n, s) + J^\mu (\nabla_\mu \phi + s \nabla_\mu \theta + \beta_A \nabla_\mu \alpha^A).
\]  

Here, \( J^\mu = \sqrt{-g} n u^\mu \) is a vector density field, \( n \) is the proper number density and \( s \) is the entropy per particle. \( F(n, s) \) is a scalar function later revealed to be the negative of the energy density \( \rho \). The other variables \( \phi \) and \( \theta \) are spacetime scalars related to chemical free energy, local temperature of the fluid\(^{[12]}\). Except these variables one also has the Lagrangian coordinates of the fluid element \( \alpha^A \) where \( A \) runs from 1 to 3. The \( \beta_A \) are space time scalars which represents Lagrange multipliers responsible for constraining the fluid 4-velocity. In our notation \( \nabla_\mu \) stands for the covariant derivative and in our convention,

\[
\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu \alpha} A^\alpha,
\]

where \( A^\nu \) is an arbitrary 4-vector and

\[
\Gamma^\nu_{\mu \alpha} = \frac{1}{2} g^{\nu \kappa} (\partial_\mu g_{\kappa \alpha} + \partial_\kappa g_{\mu \alpha} - \partial_\alpha g_{\mu \kappa}),
\]

stands for the affine connection coefficient. Except the first term \( \sqrt{-g} F(n, s) \) all the other terms in the action integral imposes some constraints on the fluid flow and do not appear
explicitly in the fluid SET. One can connect the terms in the action with the SET by using the standard relation

\[ T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}}. \]

If one associates the energy density and the pressure appearing in the SET in the following way,

\[ \rho = -F, \quad \text{and} \quad p = -n \frac{\partial F}{\partial n} + F, \tag{2} \]

then the action principle can correctly reproduce the terms in the SET of a perfect fluid.

If one wants to accommodate non-minimal conformal coupling in the gravity-fluid action then the resultant action can be written as:

\[ S = \frac{M_P^2}{2} \int d^4x \sqrt{-g} \left[ 1 + \alpha_c F_c(n, s) \right] R + S_{\text{fluid}} \tag{3} \]

Where \( S_{\text{fluid}} \) corresponds to the minimally coupled fluid action as given in Eq. (1) and \( M_P = 1/\sqrt{8\pi G} \) gives the reduced Planck mass, \( G \) being the universal gravitational constant. Here \( F_c(n, s) \) is a function of the fluid variables \( n \) and \( s \).

In the Eq. (3) the Ricci scalar is represented by \( R \). By varying the metric in the action given in Eq. (3) one obtains the modified Einstein equation:

\[ M_\ast^2 G_{\mu\nu} = T_{\mu\nu}^{\text{eff}}, \tag{4} \]

where the effective Planck mass is given by \( M_\ast^2 = M_P^2(1 + \alpha_c F_c) \) and the effective SET has the form

\[ T_{\mu\nu}^{\text{eff}} = g_{\mu\nu} F - h_{\mu\nu} n \left( \frac{\partial F}{\partial n} + \alpha_c M_P^2 \frac{R F_c}{\partial n} \right) - \alpha_c M_P^2 \left( g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \nabla_\beta F_c - \nabla_\mu \nabla_\nu F_c \right). \tag{5} \]

Here \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) is the transverse projection tensor. In the present case the modified Euler equation take the following form:

\[ n \left( \frac{\partial F}{\partial n} + \alpha_c M_P^2 \frac{R F_c}{\partial n} \right) u^\sigma - h^\sigma^\alpha \nabla_\alpha \left[ F - n \frac{\partial F}{\partial n} - \alpha_c M_P^2 \left( n \frac{\partial F_c}{\partial n} - F_c \right) R \right] \]

\[ + \alpha_c M_P^2 F_c h^\sigma^\alpha \nabla_\alpha R = 0. \tag{6} \]

For a minimally coupled fluid, the Euler equation is given as:

\[ (\rho + p) u^\mu \nabla_\mu u^\sigma + h^{\sigma\nu} \nabla_\nu p = 0. \tag{7} \]

If one defines

\[ \rho_{\text{tot}} = - \left( F + \alpha_c M_P^2 R F_c \right), \quad p_{\text{tot}} = n \frac{\partial \rho_{\text{tot}}}{\partial n} - \rho_{\text{tot}}, \tag{8} \]

then Eq. (6) can also be written as

\[ (\rho_{\text{tot}} + p_{\text{tot}}) u^\mu \nabla_\mu u^\sigma + h^{\sigma\nu} \nabla_\nu p_{\text{tot}} - \alpha_c M_P^2 F_c h^{\sigma\alpha} \nabla_\alpha R = 0, \tag{9} \]
which serves as the modified Euler equation in presence of non-minimal conformal fluid-gravity coupling.

Comparing Eq. (7) and Eq. (9), we see that NMC introduces an extra force term which depends on the gradient of the curvature. Also, total energy density and total pressure becomes directly dependent on curvature which is a signature of NMC. In [5], the authors have shown that the effect of NMC remains even in the weak field limit as the effective Poisson equation (for the Newtonian potential, $\phi_N$) in such a limit becomes

$$\nabla^2 \phi_N = 4\pi G (\rho - \alpha_c \nabla^2 F'_c),$$

where $F'_c = F_c/(4\pi G)$. This effect shows that EEP is violated in presence of NMC.

The above description of the basic theory of NMC of a fluid with spacetime curvature opens up new questions regarding the solution of the effective Einstein equation as given in Eq. (4). In Eq. (4), the left hand side contains the Einstein tensor $G_{\mu\nu}$ and the right hand side contains the effective SET. One can assume some appropriate forms of $F$ and $F_c$ and solve the effective Einstein equation for the metric. On the other hand one may also use the equation the opposite way, where one feeds in a known spacetime metric to see what are the resultant forms of $F$ and $F_c$ which arises in such a spacetime. Out of various spacetimes one important class of spacetimes is the one for which the tensor $G_{\mu\nu} = 0$, which constitutes the class of vacuum solutions in GR. Does a vacuum solution in GR remain a vacuum solution of the effective Einstein equation in presence of NMC? To answer the question one must choose some $g_{\mu\nu}$ for which $G_{\mu\nu} = 0$ and Eq. (4) becomes $T_{\mu\nu}^{\text{eff}} = 0$, which can be written explicitly as:

$$g_{\mu\nu} F - h_{\mu\nu} n \frac{\partial F}{\partial n} - \alpha_c M_P^2 \left( g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \nabla_\beta F_c - \nabla_\mu \nabla_\nu F'_c \right) = 0,$$

and solve it to see whether one gets some reasonable forms of $F$ and $F_c$. In the above equation we have used the fact that $R = 0$ for all those solutions which have $G_{\mu\nu} = 0$. In general when one works with non-minimally coupled fluid one assumes $F$ and $F_c$ to be some known functions of $n$ and $s$. In the present case we do not know the exact $n$ and $s$ dependence of $F$ or $F_c$ and consequently one has to find out how $F_c$ depends on $n$ and $s$. The discussions on the next two sections show that in these cases one can uniquely find out $F$ once one knows the coordinate dependence of $F_c$. In general $F_c$ will be found out by solving a differential equation, an equation which can be treated as a dynamical equation for $F_c$. As a consequence in the present article the status of $F_c$ is more like a dynamical scalar field whose solution gives us the form of $F$. To treat $F_c$ as a dynamical scalar field one also has to take a variation of $F_c$ in the action integral in Eq. (3). The resulting field equation one gets by varying $F_c$ comes out to be $R = 0$, the subsector in which we are working. So our way of treating $F_c$ as an independent field works perfectly for all those spacetimes where one has $R = 0$. In the present article we will assume the fluid to be non-thermal and assume $F$ and $F_c$ are independent of $s$. The modified Einstein equation, as given in Eq. (10) will yield $F_c$ and $F$ as functions of spacetime coordinates.

\[\text{3In Ref. [5] the authors miss a negative sign in the expression of pressure in Eq. (2.4) which has been corrected in the present work. We would like to point out that the definition of } \rho_{\text{tot}} \text{ used in Eq. (8) differs from its form in Ref. [5] where the authors use } \rho_{\text{tot}} = F + \frac{\alpha_c}{c^2} M_P^2 R F_c \text{ instead of the form given in Eq. (8). The expression of the energy density used here is consistent as in the absence of NMC we do obtain } \rho_{\text{tot}} = -F \text{ as one expects for a minimally coupled perfect fluid. The sign convention adopted in the present paper is consistent with a metric signature } \{-,+,+,+\} \]

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Using the forms of these functions one can always express \( n \) as a function of the spacetime coordinates. Once the spacetime dependence of \( n \) is known one can invert the relationship and express both \( F \) and \( F_c \) as functions of \( n \). For the case of a non-minimally coupled fluid in flat spacetime one gets \( F_c \) to be a function of \( n \) and position coordinates, in such a way that \( F_c \) becomes zero when \( n \) vanishes.

If the differential equation, Eq. (10), is satisfied for a non-zero \( F \) and non-zero \( F_c \) then one can unambiguously say that the vacuum solution in GR can accommodate a perfect fluid in presence of NMC. In this article we will specifically deal with Schwarzschild solution in GR, which is a known vacuum solution in GR, and try to see whether a spacetime defined by the Schwarzschild solution can accommodate a perfect fluid when one uses the Schwarzschild metric in the modified Einstein equation. As the Schwarzschild solution is asymptotically flat one has to also investigate whether the Minkowski spacetime also admits some matter in the presence of non-minimal coupling. In the next section we will start our analysis with the flat Minkowski solution and then move on to the Schwarzschild solution.

### 3 Effect of non-minimal coupling for the simplest cases where the Ricci scalar is zero

In this section we will describe how a NMC affects the modified Einstein equation for those spacetimes which has \( R = 0 \). We will start with the simplest case, the flat spacetime, and then move on to the next case of Schwarzschild spacetime. As the Schwarzschild spacetime is asymptotically flat so the effect of NMC in flat spacetime becomes important.

#### 3.1 Non-minimally coupled fluid in flat spacetime

The line element for flat spacetime is given by:

\[
\text{d}s^2 = -dt^2 + dx^2 + dy^2 + dz^2.
\]  

(11)

As \( R_{\mu\nu} = 0 \) and \( R = 0 \), we have \( G_{\mu\nu} = 0 \). Before proceeding further, we will make minor changes to Eq. (10) in terms of the symbols used. We will like to rewrite Eq. (10) as

\[
g_{\mu\nu} \tilde{F} - h_{\mu\nu} n \frac{\partial \tilde{F}}{\partial n} - \alpha_c M_p^2 \left( g_{\mu\nu} \Box \tilde{F}_c - \nabla_{\mu} \nabla_{\nu} \tilde{F}_c \right) = 0,
\]  

(12)

where the previous symbols (corresponding to \( F \) and \( F_c \)) are only represented in different notations without any new physical input. Here and henceforth \( \Box = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \). This change in notation will help us to write down the equations in a more compact way. Now we introduce the variables \( F \) and \( F_c \) as

\[
F = \frac{\tilde{F}}{M_p^2}, \quad F_c = \alpha_c \tilde{F}_c.
\]  

(13)

In terms of the new variables as defined above the modified Einstein equation in flat spacetime becomes,

\[
g_{\mu\nu} F - (u_\mu u_\nu + g_{\mu\nu}) n \frac{\partial F}{\partial n} - (g_{\mu\nu} \partial_\alpha \partial^\alpha F_c - \partial_\mu \partial_\nu F_c) = 0,
\]  

(14)
where the metric is as given in Eq. (11) and the fluid 4-velocity is assumed to be \( u_\mu = (-1, 0, 0, 0) \). The perfect fluid is assumed to be at rest in the flat four dimensional manifold. Assuming \( F \) and \( F_c \) to be independent of time, the above equation yields

\[
F - \partial_i \partial_i F_c = 0,
\]

and

\[
F - n \frac{\partial F}{\partial n} - \left( \partial_i \partial_i F_c - \frac{\partial^2 F_c}{\partial x^2} \right) = 0,
\]

\[
F - n \frac{\partial F}{\partial n} - \left( \partial_i \partial_i F_c - \frac{\partial^2 F_c}{\partial y^2} \right) = 0,
\]

\[
F - n \frac{\partial F}{\partial n} - \left( \partial_i \partial_i F_c - \frac{\partial^2 F_c}{\partial z^2} \right) = 0,
\]

which are the \( 0-0 \) and \( i-i \) components of Eq. (14). Here \( i \) runs from 1 to 3. The last three equations imply

\[
\frac{\partial^2 F_c}{\partial x^2} = \frac{\partial^2 F_c}{\partial y^2} = \frac{\partial^2 F_c}{\partial z^2},
\]

which immediately gives

\[
F_c = C(x^2 + y^2 + z^2) = Cr^2,
\]

where \( C \) is a constant. From Eq. (15) one can get the form of \( F \) as

\[
F = 6C.
\]

One can now use these forms of \( F_c \) and \( F \) in the above equations and find out

\[
n \frac{\partial F}{\partial n} = 2C,
\]

which shows that \( \partial F/\partial n \) is non-zero if \( n \) does not vanish identically. In order to satisfy the fact that \( F \) is a constant with \( r \), but \( \partial F/\partial n \) is non zero, one must have

\[
F(n) = \alpha n^{\frac{1}{3}},
\]

which yields,

\[
n = \left( \frac{F(n)}{\alpha} \right)^3 = \left( \frac{6C}{\alpha} \right)^3,
\]

if Eq. (22) is satisfied. In the above equations \( \alpha \) is another numerical constant. In flat spacetime one can indeed have a fluid, which is non-minimally coupled to curvature, only if the density of the fluid particles remains a constant throughout spacetime. In the present case the existence of this energy-density does not curve spacetime as because the effect of it is completely cancelled by the existence of non-zero \( F_c \).

In the present case we see that

\[
F_c(n, r) = \frac{\alpha n^{\frac{1}{3}} r^2}{6},
\]
which is zero if \( n \) vanishes, but it also depends upon the radial coordinate \( r \). The above equation shows that \( F_c \) increases with distance, at infinite distance this coupling blows up but the number density \( n \) remains always a constant. Also from Eq. (8) we can write the energy density and pressure of the fluid to be,

\[
\rho_{\text{tot}} = -6C, \quad p_{\text{tot}} = 4C, \quad (26)
\]

The above relations show that we have to take \( C < 0 \) so that \( \rho \) remains positive definite. To make the number density \( n \) to be positive definite we also require to have \( \alpha < 0 \). Consequently one can write the expressions of the energy density and pressure of the fluid also as

\[
\rho_{\text{tot}} = -\alpha n^{\frac{1}{3}}, \quad p_{\text{tot}} = \frac{2\alpha n^{\frac{1}{3}}}{3}, \quad \alpha < 0, \quad n > 0, \quad (27)
\]

predicting

\[
p_{\text{tot}} = -\frac{2}{3}\rho_{\text{tot}}. \quad (28)
\]

The fluid which satisfies the modified Einstein equation in flat spacetime must have negative pressure. These expressions for the flat spacetime will be used to verify the asymptotic limit of Schwarzschild solution in presence of NMC.

The modified Einstein equation, in flat spacetime, in presence of NMC can certainly have the above non-trivial and interesting solution which to our understanding was not presented before. On the other hand one can get the standard vacuum solution, as one gets in GR, by choosing the constant \( C = 0 \) in the above analysis. In that case the energy density, pressure of the fluid and the number density of the fluid constituents vanish identically.

### 3.2 Non-minimally coupled fluid in the Schwarzschild spacetime

In this section we analyze the solution of the modified Einstein equation in presence of NMC using the Schwarzschild metric. In GR we know the Schwarzschild metric is a solution of the Einstein equation in vacuum. We want to first verify whether the modified Einstein equation allow the Schwarzschild spacetime to have matter. Next we will try to figure out the energy density and pressure of the fluid in the Schwarzschild spacetime.

We write the line element for Schwarzschild spacetime as

\[
ds^2 = -(1 - \frac{2m}{\tilde{r}})dt^2 + \frac{d\tilde{r}^2}{1 - \frac{2m}{\tilde{r}}} + \tilde{r}^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (29)
\]

The usage of \( \tilde{r} \) instead of \( r \) is deliberate and the reason for choosing such a convention will become clearer very soon. As like the flat spacetime analysis, we will write the basic equation, Eq. (10) as,

\[
g_{\mu\nu} \tilde{F} - h_{\mu\nu} \frac{\partial \tilde{F}}{\partial n} - \alpha_c M_p^2 \left( g_{\mu\nu} \Box \tilde{F}_c - \nabla_\mu \nabla_\nu \tilde{F}_c \right) = 0, \quad (30)
\]

where the tildes over the quantities \( F \) and \( F_c \) are there for purely technical reasons and shortly we will redefine the quantities as \( F \) and \( F_c \) in appropriate ways. We will work
in the comoving frame and hence $u^i = 0$, $i = 1, 2, 3$ and $u^0 u^0 = -1/g_{00}$. Now we can redefine various quantities as,

$$r = \tilde{r}/m, \quad F = \alpha_c \tilde{F}_c, \quad F = \tilde{F} m^2/M_p^2,$$

(31)

using which we will write the following equations in this section. In this discussion we will assume that $n$, $s$, $F$ and $F_c$ are time-independent quantities. Using the above notation the 0–0 component of the modified Einstein equation becomes

$$F - \Box F_c - \frac{1}{(1 - \frac{2}{r})} \nabla_0 \nabla_0 F_c = 0,$$

(32)

where the notation is defined as

$$\nabla_t \nabla_t F_c = - \frac{1}{m^2 r^2} \left( 1 - \frac{2}{r} \right) \frac{\partial F_c}{\partial r} = \frac{1}{m^2} \nabla_0 \nabla_0 F_c.$$

In the above case we use the coordinate $\tilde{r}$ to calculate $\nabla_t \nabla_t F_c$ and then express the result in terms of $r$, as a result of which a factor of $1/m^2$ comes out. The double covariant derivatives in terms of the dimensionless $r$ modulo the $1/m^2$ factor is defined as $\nabla_0 \nabla_0 F_c$.

Suitably manipulating the other components of the modified Einstein equation, in the Schwarzschild spacetime, one gets:

$$F - n \frac{\partial F}{\partial n} - \Box F_c + \left( 1 - \frac{2}{r} \right) \nabla_1 \nabla_1 F_c = 0,$$

(33)

$$r^2 \left( 1 - \frac{2}{r} \right) \nabla_1 \nabla_1 F_c - \nabla_2 \nabla_2 F_c = 0,$$

(34)

$$\nabla_3 \nabla_3 F_c - \sin^2 \theta \nabla_2 \nabla_2 F_c = 0,$$

(35)

where the double covariant derivative $\nabla_\tilde{r} \nabla_\tilde{r} F_c$ using coordinate $\tilde{r}$ is related to the quantity $\nabla_1 \nabla_1 F_c$ via the following relation,

$$\nabla_\tilde{r} \nabla_\tilde{r} F_c = \frac{1}{m^2} \left[ \frac{\partial^2 F_c}{\partial t^2} + \frac{1}{r^2(1 - \frac{2}{r})} \frac{\partial F_c}{\partial r} \right] = \frac{1}{m^2} \nabla_1 \nabla_1 F_c.$$

The other double covariant derivatives appearing in the above equations are given as,

$$\nabla_\theta \nabla_\theta F_c = \nabla_2 \nabla_2 F_c, \quad \nabla_\phi \nabla_\phi F_c = \nabla_3 \nabla_3 F_c,$$

where one calculates the quantities $\nabla_\theta \nabla_\theta F_c$ and $\nabla_\phi \nabla_\phi F_c$ using $\tilde{r}$ and re-express the result in terms of $r$ in $\nabla_2 \nabla_2 F_c$ and $\nabla_3 \nabla_3 F_c$. In these cases one does not get factors of $1/m^2$ due to the change of variables.

Solving the modified Einstein equations, in the Schwarzschild spacetime, we will get the functional forms of $F$, $F_c$ and $n$. If these quantities turns out to be non-vanishing then there can be a perfect fluid in the Schwarzschild spacetime in such a way that the modified Einstein equation is satisfied. In the present case one starts with Eq. (35) which can be written as

$$\frac{\partial^2 F_c}{\partial \phi^2} - \sin^2 \theta \frac{\partial^2 F_c}{\partial \theta^2} + \sin \theta \cos \theta \frac{\partial F_c}{\partial \theta} = 0.$$

(36)
There can be various kind of solutions of the above equation. By using the method of separation of variables one gets the following type of solutions:

\[
F_c(r, \theta, \phi) = \begin{cases} 
    f(r), \\
    f(r) \cos \theta, \\
    f(r) \sin \theta \cos \phi, \\
    f(r) \sin \theta \sin \phi, \\
    f(r) \sin \theta \cos \phi \log(\tan(\frac{\theta}{2})),
\end{cases}
\]

(37)

where the first solution is an obvious one. Moreover the first solution can be uniquely chosen keeping spherical symmetry in mind. Consequently we discard those solutions where \( F_c \) explicitly depend on \( \theta \) and \( \phi \). The only possible spherically symmetric solution of \( F_c \) must be a function of \( r \) alone.

To find out the form of \( f(r) \) we will use Eq. (34) which can be written explicitly as:

\[
 r(r - 2) \frac{d^2 f}{dr^2} + (3 - r) \frac{df}{dr} = 0,
\]

(38)

which can be transformed into a first order differential equation by using a new variable \( g(r) \) defined as \( g(r) = df/dr \). The resulting first order equation is

\[
\frac{dg}{g} + \left[ -\frac{3}{2r} + \frac{1}{2(r-2)} \right] dr = 0,
\]

(39)

which after integration yields the result,

\[
\frac{dF_c}{dr} = A \sqrt{\frac{r^3}{r-2}},
\]

(40)

where \( A \) is an integration constant. One can integrate the above expression and finally obtain

\[
F_c(r) = \begin{cases} 
    \tilde{B} + \tilde{A} \left[ 3 \sin^{-1} \sqrt{\frac{r}{2}} - \frac{3}{2} \sqrt{r(2-r)} \\
    -\frac{1}{2} \sqrt{r^3(2-r)} \right]; \quad r < 2, \\
    B + A \left[ \frac{1}{2} \sqrt{r^3(r-2)} + \frac{3}{2} \sqrt{r(r-2)} \\
    + 3 \log \left( \sqrt{\frac{r}{2}} + \sqrt{\frac{r-2}{2}} \right) \right]; \quad r > 2,
\end{cases}
\]

(41)

where the solutions for the regions \( r > 2 \) and \( r < 2 \) are separately specified. Since \( r < 2 \) is not physically accessible we will always take \( r > 2 \) and hence the physically relevant solution is

\[
F_c = B + A \left[ \frac{1}{2} \sqrt{r^3(r-2)} + \frac{3}{2} \sqrt{r(r-2)} + 3 \log \left( \sqrt{\frac{r}{2}} + \sqrt{\frac{r-2}{2}} \right) \right], \quad r > 2.
\]

(42)

From Eq. (32) one can now find out the expression of \( F \) as

\[
F = \frac{3(r - 2)}{r^2} \frac{dF_c}{dr} \]

(43)

\[
= 3A \sqrt{\frac{r - 2}{r}},
\]

(44)
showing that $F$ tends to zero as $r$ approaches the value 2, near the event horizon. From the last equation one can easily calculate

$$\frac{dF}{dr} = \frac{3}{r^2} \frac{dF_c}{dr},$$  \hspace{1cm} (45)

a result which will be useful for the calculation of $n$ as a function of $r$. Using the above results in Eq. (33) one gets

$$n \frac{\partial F}{\partial n} = \left( r - 3 \right) \frac{dF_c}{dr}.$$  \hspace{1cm} (46)

Writing the last equation as,

$$n \frac{(dF/dr)}{(dn/dr)} = \left( r - 3 \right) \frac{dF_c}{dr},$$  \hspace{1cm} (47)

and using the result of Eq. (45) one gets the functional form of $n$ as

$$n(r) = D \frac{|r - 3|}{r},$$  \hspace{1cm} (48)

where $D$ is a numerical constant. This shows that $n(r)$ is not a constant in the finite $r$ region of the Schwarzschild spacetime. More over $n(r)$ vanishes at $r = 3$ and remains finite at $r = 2$.

We can now calculate all the properties of the perfect fluid in the Schwarzschild spacetime. In the present case if we want to make the energy-density of the fluid to be positive then the constant $A$ has to be smaller than zero. Using Eq. (8) we get

$$\rho_{tot} = -3A \sqrt{\frac{r - 2}{r}} , \quad p_{tot} = A \frac{(2r - 3)}{r} \sqrt{\frac{r}{r - 2}} , \quad A < 0,$$  \hspace{1cm} (49)

which yields

$$\frac{p_{tot}}{\rho_{tot}} = - \left( \frac{2r - 3}{3r - 6} \right),$$  \hspace{1cm} (50)

which reproduces the result in flat spacetime, given in Eq. (28), in the limit of large $r$. In this case also we see that for positive definite energy density one must have a negative pressure for finite values of $r$. Consequently the perfect fluid which is non-minimally coupled to gravity cannot have a positive pressure and acts like an exotic dark fluid. From the expressions of the fluid variables one sees that $\rho_{tot} \to 0$ as $r \to 2$ from the right hand side. On the other hand $p_{tot} \to -\infty$ as $r \to 2$ from the right hand side. Consequently one finds that

$$-\infty < \left( \frac{p_{tot}}{\rho_{tot}} \right) < -\frac{2}{3},$$  \hspace{1cm} (51)

where the limiting values, of the equation of state of the fluid, are attained at $r = 2$ and $r \to \infty$. The equation of state of the fluid is a function of $r$ and it increases as $r$ increases. At $r = 3$ the equation of state becomes $(p_{tot}/\rho_{tot}) = -1$ and in between $r = 2$ and $r = 3$ the equation of state decreases from $-1$ to $-\infty$. The phantom divide \cite{19,20} happens at $r = 3$. The plots of $\rho_{tot}$ and $p_{tot}$ with respect to $r$ are presented in Fig. 1 and Fig. 2.
value of $A$ has been taken as minus one. The specific choice of $A$ is justified in Eq. (59). The value of $A$ is related to the large $r$ asymptotic values of $\rho_{\text{tot}}$ and $p_{\text{tot}}$.

One can express all the physical fluid parameters in terms of $n$. As there is only one fluid in question and we do not take into account thermal equilibrium, entropy density $s$ does not play a part in this discussion. From Eq. (48) one sees that

$$r = \frac{3D}{D - n}, \quad r > 3,$$

which directly shows that the constant $D$ should be such that $D > n$ for $r > 3$. One can now express the energy-density in terms of $n$ as

$$\rho_{\text{tot}}(n) = -3A \left[ \left( \frac{2}{3D} \right) n + \frac{1}{3} \right]^{1/2}, \quad r > 3. \quad (53)$$

Working similarly one can get,

$$\rho_{\text{tot}}(n) = -3A \left[ \frac{1}{3} - \left( \frac{2}{3D} \right) n \right]^{1/2}, \quad 2 < r < 3, \quad (54)$$

which again restrains $D$ in this region as here one requires $D > 2n$ for a real value of $\rho_{\text{tot}}(n)$. The above results show that although the radial derivative of $n(r)$ is a discontinuous function of $r$ at $r = 3$, where $n = 0$, $\rho_{\text{tot}}(n)$ remains a continuous function of $n$ in
the whole region of physical relevance. One can also express pressure as a function of $n$ in the various regions as:

$$p_{tot}(n) = A \left( 1 + \frac{n}{D} \right) \left[ \frac{3}{1 + \left( \frac{2}{D} \right)n} \right]^{1/2}, \quad r > 3; \quad (55)$$

and

$$p_{tot}(n) = A \left( 1 - \frac{n}{D} \right) \left[ \frac{3}{1 - \left( \frac{2}{D} \right)n} \right]^{1/2}, \quad 2 < r < 3; \quad (56)$$

which again shows that pressure varies continuously with $n$. From the above expressions one can immediately notice that $\rho_{tot} + p_{tot} > 0$ for $r > 3$ and $\rho_{tot} + p_{tot} < 0$ for $2 < r < 3$. From the form of Eq. $(22)$ and Eq. $(48)$ one can identify that

$$D = \left( \frac{6C^2}{\alpha} \right)^{3}, \quad (57)$$

as the Schwarzschild result tends to the flat spacetime result in the large $r$ limit. On the other hand if one matches the value of the energy density in Schwarzschild spacetime for large $r$ values to that of the flat spacetime result in Eq. $(26)$ one gets $C = A/2$. Using this result in the last equation one gets a relationship between the various constants as

$$D = 3A \left( \frac{36C^2}{\alpha^3} \right); \quad (58)$$
The above equation connects the Schwarzschild spacetime results with the flat spacetime results. As such the above equation does not specify any particular set of values of the constants and may be valid for a range of values for the constants, giving various fluid profiles. In this article we will choose a specific set of values of the constants as

$$A = -1, \quad D = 1, \quad C = -\frac{1}{2}, \quad \alpha = -3, \quad (59)$$

for which Eq. (58) holds. One can always assume $B = 0$, in the expression of $F_c$ in Eq. (42), without any loss of generality. For finite values of $r$ one can always express $F_c(n)$, in Eq. (12), as a function of $n$ using the functional relationship between $n$ and $r$. For very large values of $r$ something interesting happens. For large enough $r$, one gets $F_c \sim \frac{A}{2} r^2$ from Eq. (12). In this large $r$ limit one cannot simply express $F_c$ as purely a function of $n$ as $n$ becomes constant for large $r$. Remembering that $C = A/2$, one sees that in the large $r$ limit one gets $F_c \sim Cr^2$ which is exactly similar to the relation in Eq. (20), a result familiar from our flat spacetime analysis.

The plots of $\rho_{\text{tot}}$ and $p_{\text{tot}}$ with respect to $n$ are presented in Fig. 3 and Fig. 4. Looking at the plots one may like to think that both $\rho_{\text{tot}}$ and $p_{\text{tot}}$ are double valued functions of $n$ for some values of $n$, but that is not the case here. In the figures we have specified the various radial regions for the different branches. The number density vanishes at $r = 3$, where both the branches meet continuously. The parameters required to draw the plots are taken from Eq. (59).
4 Minimally coupled fluid in the Schwarzschild spacetime

To understand the particular properties of non-minimally coupled fluid in Schwarzschild spacetime it may be helpful to contrast the results with those of the minimally coupled fluid in the Schwarzschild spacetime. In this case the fluid is not backreacting on the spacetime and consequently the Schwarzschild spacetime remains unaltered in the presence of the fluid. For the case of minimal coupling one has the SET of the fluid as given in Eq. (1) in section 2. To maintain energy-momentum conservation one has to assume
\[ \nabla \mu T^{\mu \nu} = 0. \] (60)

To compare the two cases, in this case also we assume that \( u_\mu \) is the fluid 4-velocity which is a timelike 4-vector is normalized as \( u^\mu u_\mu = -1 \) and \( g_{\mu \nu} \) is the Schwarzschild spacetime metric as given in Eq. (29). Assuming time-independent \( \rho \) and \( p \) we see that \( \nabla_\mu T^{\mu \theta} = 0 \) is an identity and reveals no information regarding the fluid variables. The two equations, \( \nabla_\mu T^{\mu \theta} = 0 \) and \( \nabla_\mu T^{\mu \phi} = 0 \) yield the important relations
\[ \left( \frac{\partial p}{\partial \theta} \right) = 0, \quad \left( \frac{\partial p}{\partial \phi} \right) = 0, \] (61)
which says that for the perfect fluid at rest in presence of the Schwarzschild spacetime, the pressure must be purely a function of \( r \). From the equation \( \nabla_\mu T^{\mu \phi} = 0 \) one gets
\[ \frac{dp}{dr} + \frac{\rho + p}{r^2(1 - \frac{2}{r})} = 0, \] (62)
where \( r = \tilde{r}/m \). The above result shows that one cannot accommodate a pressure-less, static perfect fluid, minimally coupled to gravity, in the Schwarzschild spacetime. If one assumes that the equation of state of the perfect fluid, minimally coupled to gravity, is simply given by \( p = \omega \rho \) then the solution of the above differential equation is

\[
\rho(r) = \rho_0 \left( 1 - \frac{2}{r} \right)^{-\frac{(1+\omega)}{2}},
\]

(63)

where \( \rho_0 \) is a constant energy-density at very large \( r \). In this simple case one sees that if \( \omega = 0 \) then \( \rho(r) = 0 \) for \( r > 2 \), a fact we already know. On the other hand if \( \omega = -1 \) then \( \rho(r) \) is constant throughout the region \( r > 2 \), which is the familiar property of vacuum energy. If \( \omega = 1/3 \) then the energy-density (as well as pressure) is maximum near the event horizon and the energy-density decreases as one moves away to large values of \( r \). If \( \omega \ll -1 \) then the energy-density is again maximum near the event horizon but the pressure is negative and dips to negative infinity near the event horizon. Only if \( -1 < \omega < 0 \) then \( \rho(r) \) tends to zero as \( r \) nears the event horizon and pressure remains negative always.

From the above discussion we see that the difference between non-minimally coupled fluid and minimally coupled fluid in Schwarzschild spacetime is remarkable. The most important difference is related to the status of the equations which describe the nature of these fluids. In the case of minimally coupled fluid, the SET was assumed to be present and one had to invoke the energy-momentum conservation equation for the fluid to mathematically formulate the properties of such a fluid. For the case of NMC fluid, the equations governing the nature of the fluid directly came from the modified Einstein equation. The SET automatically satisfied energy-momentum conservation condition as it was a part of the modified Einstein equation. The non-minimally coupled fluid has a specific energy-density and pressure in the Schwarzschild spacetime where as the minimally coupled fluid only gives a relationship between \( \rho \) and \( p \) and nothing more. If one does not assume a barotropic fluid then one has to find out the pressure of the fluid by assuming various forms of \( \rho(r) \). All the analysis in this paper is based on the assumption of a static fluid and we hope there will be more interesting properties of these fluids if they have non-zero 3-velocity components. In the next section we will briefly opine on the effect of fluid velocity perturbations on the fluid properties when the fluid is non-minimally coupled to gravity.

5 A brief note on the effect of perturbations of the fluid velocity

One can study the stability of the solution so obtained by perturbative methods. There can be two kinds of perturbations involved in the present case. In the first kind, we perturb the fluid 4-velocity keeping the background metric fixed. In the second kind, one can perturb the metric itself and study the properties of non-minimally coupled fluids in perturbed Schwarzschild spacetime. The second case leads to a new problem where one has to solve for the fluid properties in an altered metric. In this section we will opine on perturbations of the first kind where the fluid velocity is perturbed and set up the perturbation equations up to the first order keeping the background metric fixed.
In subsection 3.2 we have the solutions of the field equation, Eq. (30) in the static case, where

\[ u^0 \neq 0, \quad u^i = 0. \]

In the case of the static fluid in the Schwarzschild spacetime we had \( u^0 u^0 = -1/g_{00} \). Now let there be a finite, but small perturbation in the velocity of the fluid element

\[ U^0 = u^0(r) + \delta u^0(t, x), \quad U^i = \delta u^i(t, x), \]

where \( U^\mu(t, x) \) are the new perturbed 4-velocity components. From the normalization of \( U^\mu \) one can check that \( \delta u^0(t, x) = 0 \) up to first order in the perturbation. Due to the perturbation in the fluid velocity all of the terms in Eq. (30) now becomes time-dependent. One must first note that the fluid velocity enters the dynamical equations directly via \( h_{\mu\nu} \).

The diagonal terms in \( h_{\mu\nu} \) does not change up to first order of perturbation in the fluid 4-velocity, and consequently the equations obtained from the diagonal terms in Eq. (30) must remain the same. Consequently the perturbations on \( F(t, x), F_c(t, x) \) and \( n(t, x) \) must also satisfy the old equations:

\[ \delta F - \Box \delta F_c - \frac{1}{(1 - \frac{2}{r})} \nabla_0 \nabla_0 \delta F_c = 0, \]

\[ \delta F - \delta \left( n \frac{\partial F}{\partial n} \right) - \Box \delta F_c + \left( 1 - \frac{2}{r} \right) \nabla_1 \nabla_1 \delta F_c = 0, \]

\[ r^2 \left( 1 - \frac{2}{r} \right) \nabla_1 \nabla_1 \delta F_c - \nabla_2 \nabla_2 \delta F_c = 0, \]

\[ \nabla_3 \nabla_3 \delta F_c - \sin^2 \theta \nabla_2 \nabla_2 \delta F_c = 0. \]

In our notation

\[ F(t, x) = F_0(r) + \delta F(t, x), \quad F_c(t, x) = F_{c0}(r) + \delta F_c(t, x), \quad n \frac{\partial F}{\partial n} = \left( n \frac{\partial F}{\partial n} \right)_0 + \delta \left( n \frac{\partial F}{\partial n} \right), \]

where the subscript zero stands for the unperturbed values of the quantities.

The above equations do not show any dependence of the perturbed quantities on the 3-velocity perturbations. To get the dependence of the perturbed energy density, non-minimal coupling and the number density one has to write the non-diagonal terms in Eq. (30). Up to first order in the perturbation of the fluid velocity one must note that

\[ \delta h_{ij} = 0, \quad \delta h_{0i} = u_0 \delta u_i, \]

where \( i, j \) runs from one to three. Consequently the non-diagonal terms of Eq. (30) gives

\[ u_0 \delta u_i \left( n \frac{\partial F}{\partial n} \right)_0 - \nabla_0 \nabla_i \delta F_c = 0. \]

The interesting thing to observe is that up to first order of perturbations in the fluid velocity the diagonal equations keeps the same form as those of the unperturbed solutions, although the perturbations are now functions of space and time. The non-diagonal equations specify the velocity perturbations in terms of the non-minimal coupling. This observation is only true to the first order of perturbations in the fluid velocity, in the second order of perturbations the diagonal equations will be directly affected by the velocity perturbations.
The way to solve the above set of equations for the perturbations is a complicated matter as these equations can have non-trivial time dependence. In this article we will not discuss the details about the general method of solving such equations, it is beyond the scope of the present paper. We will end this brief note on the fluid perturbations in the fixed background with an interesting observation regarding the non-diagonal perturbation equations. They reveal an interesting feature of the present problem. From the form of Eq. (69) one can see that each of the equations contain one time derivative. As a result both $\delta u_i$ and $\delta F_c$ cannot have time dependent parts as $e^{i\lambda t}$ for a real $\lambda$. This shows that the perturbations in the fluid velocity cannot be oscillating in time as $e^{i\lambda t}$ and consequently the perturbations must be either growing or decaying with time or in other words the perturbations are unstable. This fact may not be a completely unexpected as we have seen that the background solution predicts a fluid with negative pressure throughout space and it is well known that perturbations in a negative pressure fluid are prone to be unstable as pressure forces cannot withstand the force of fluid compression. The unstable perturbations in the present system can be of various kinds, some of which does not affect the energy density while others can affect energy density and pressure. Perturbations growing in time may ultimately destroy the background set up where as perturbations decaying in time will be unable to destabilize the background fluid properties.

6 Conclusion

In this article we have applied the modified Einstein equations, arising out of non-minimal coupling of a perfect fluid with gravity, in the particular case of the Schwarzschild spacetime. Knowing the metric solution we derive the properties of the fluid in such a spacetime. Although Schwarzschild spacetime has $R = 0$, but the effects of the NMC, introduced in the fluid-gravity action, remains in the modified Einstein equation. The very fact that a non-minimal coupling term in the gravity-fluid Lagrangian can affect the solutions of the modified Einstein equation for the Schwarzschild spacetime, for which $R = 0$, is interesting. In this article we have also presented the solution of the modified Einstein equation for the flat spacetime. There also we find an interesting non-trivial solution which predicts that in presence of NMC there can be a fluid in flat spacetime, whose energy-density and pressure are constants. The important point about the solution is that the pressure of such a fluid must be negative. The flat spacetime solution is important for the present work as Schwarzschild spacetime is asymptotically flat, and consequently all the large $r$ behavior of the fluid present in the Schwarzschild spacetime must merge with the flat spacetime results.

In this article we have assumed $F_c$ to be like an independent scalar field, which depends on the fluid parameters, and plays the most important role for non-minimal coupling of the fluid with gravity. Our method always works when one chooses those spacetimes for which $R = 0$. At the end we want to stress that although we gave a slightly different interpretation of $F_c$, it still retains all its basic properties. For the time-independent and radially symmetric solutions, one can easily check from the $1 - 1$ component of the modified Einstein equation in the Schwarzschild spacetime that if there is no ambient fluid, that is if $F = 0$ (and consequently $n = 0$), one gets $F_c$ to be a constant. From the form of the fluid-gravity action, as given in Eq. (3), it can be checked that in such a case one gets back the Einstein equation of GR in vacuum and the effect of NMC gets
completely wiped out as expected. This implies that non-trivial effects of NMC arises, through the effect of $F_c$, only when there is an ambient fluid in the curved spacetime. In the finite $r$ regions of the Schwarzschild spacetime one can write $F_c(n)$ by using the form of $n(r)$ in Eq. (48). The fact that $F_c$ depends on the radial coordinate, for the case of flat spacetimes, is directly related to the fact that in flat spacetime $n$ is a constant. If $F_c$ remains purely a function of $n$ and hence becomes a constant when $F \neq 0$, then one gets an inconsistent Einstein equation from the form of the action in Eq. (3). For a relevant form of $F_c$ in flat spacetime it must depend upon some coordinates, and in a spherically symmetric situation it is natural that it depends upon the radial coordinate $r$. Moreover as we were working with a static fluid in a time-independent set up, one can trivially show that the functional form of $n(r)$ satisfies the constraint $\nabla_\mu (nu^\mu) = 0$. So one can indeed interpret $n(r)$ as a number density specifying the fluid.

In the Schwarzschild spacetime it is seen that one can always accommodate a perfect fluid whose nature changes radially. In the region $2 \leq r < 3$ the fluid is a phantom fluid and it becomes a dark fluid with $-1 \leq (p_{\text{tot}}/\rho_{\text{tot}}) \leq -(2/3)$ in the region $r \geq 3$. One can keep the energy-density of the perfect fluid to be positive in the physically accessible regions of spacetime. The interesting thing to note is that the perfect fluid, non-minimally coupled to gravity in the Schwarzschild spacetime, must have negative pressure. One cannot have non-minimally coupled positive pressure fluid in the Schwarzschild spacetime. This is very different from the case of minimally coupled perfect fluid in the Schwarzschild spacetime, where one can have all sorts of perfect fluids which have zero, positive or negative pressure. In this article we have analyzed the region of spacetime for which $r > 2$, outside the event horizon as in the Schwarzschild spacetime that is the region accessible to an observer at infinity.

In the case of non-minimally coupled fluid one sees that the fluid has negative pressure throughout but in regions $r > 3$, the quantity $\rho_{\text{tot}} + p_{\text{tot}} > 0$ and only in the region $r < 3$ one has $\rho_{\text{tot}} + p_{\text{tot}} < 0$ and the pressure diverges in the negative direction where as the energy-density tends to zero near the event horizon. The expression of the energy-density and pressure of the fluid shows that even if $n = 0$ the energy-density and pressure does not turn out to be zero. This is a very interesting property of the non-minimally coupled fluid in the Schwarzschild spacetime. The $n$ independent part of the fluid parameters specify the dark energy like contributions. In the region $r > 3$, decrease in $n$ is accompanied by decrease in $\rho_{\text{tot}}$, but once in the region $2 < r < 3$, $\rho_{\text{tot}}$ decreases as number density increases. In the region $2 < r < 3$ the energy-density $\rho_{\text{tot}}(n)$ is a decreasing function of $n$ because in this region $\rho_{\text{tot}} + p_{\text{tot}} < 0$. From the expressions of energy-density and pressure in Eq. (3) one can verify that when $\rho_{\text{tot}} + p_{\text{tot}} < 0$ one must have $(\partial \rho_{\text{tot}}/\partial n) < 0$ as number density $n(r)$ cannot be negative. This interesting behavior of energy-density can also be obtained in minimally coupled fluids in GR when the weak energy condition is violated, as can be checked from the expression of pressure in Eq. (2). In our particular case as $\rho_{\text{tot}} + p_{\text{tot}}$ is positive for $r > 3$ and negative for $r < 3$ attaining zero value at $r = 3$ where we have $n(\partial \rho_{\text{tot}}/\partial n) = 0$. At $r = 3$ it is seen that $(\partial \rho_{\text{tot}}/\partial n)$ is discontinuous but not singular and so $n$ becomes zero at $r = 3$. We have seen that for the background solution the pressure diverges at the horizon. One must note that pressure is a scalar function and consequently a divergence in its value shows a physical effect independent of any coordinate choice. The divergence of the pressure function at the horizon therefore is a physical effect and not due to some wrong choice of coordinates.

The present observations regarding the existence of non-minimally coupled dark fluid
in presence of the Schwarzschild spacetime turns out to be interesting as one can pinpoint most of the properties of the fluid from the modified Einstein equation. In this paper we assumed that the fluid is at rest in the Schwarzschild coordinates. One can also get back the vacuum solutions in the present case by suitably choosing the constants appearing in the analysis. The present paper shows that simplest of the blackholes can be surrounded by an exotic perfect fluid whose property changes with radial distance. More over this fluid does not vanish at large distance from the event horizon, the energy-density and the pressure of the fluid becomes constant in the large \( r \) limit. As because the system is made up of negative pressure fluid the fluid perturbations on the system turn out to be unstable. Some perturbations may lead to distabilization of the background solutions. In the case of minimally coupled fluid, one has to always assume that the fluid is not backreacting on the spacetime, an assumption which is at best an approximation because any fluid which has high enough energy-density or pressure may actually modify the original spacetime itself. Unlike the minimally coupled case, the fluid properties in the case of non-minimal coupling, are obtained using the proper modified Einstein equation and consequently the solutions are always valid and one may not treat the fluid as a test probe in the Schwarzschild spacetime.

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