HYPERBOLIC VOLUME AND TWISTED ALEXANDER INVARIANTS OF KNOTS AND LINKS

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Abstract. Let $\Delta_{L,\rho_n}(t)$ be the twisted Alexander polynomial with respect to the representation given by the composition of the lift of the holonomy representation of a certain hyperbolic link $L$ and the $n$-dimensional irreducible complex representation of $\text{SL}(2,\mathbb{C})$. We consider a sequence of $\Delta_{L,\rho_n}(t)$ and extract the volume of the complement of $L$ from the asymptotic behaviour of the sequence obtained by evaluating $t = 1$ or $t = -1$.

1. Introduction

One of the known classical knot and link invariants is the Alexander polynomial [1]. This polynomial is derived from the fundamental group of the complement of a knot or link and is determined by its maximal metabelian quotient. This computable invariant plays an important role in the knot theory. In 1990, Lin [15] introduced the twisted Alexander polynomial associated with a knot $K$ and a representation of the knot group $\pi_1(S^3 \setminus K) \to \text{GL}(n, \mathbb{F})$, where $\mathbb{F}$ is a field. Wada [34] generalised this work and showed how to define a twisted polynomial given only a presentation of a group and representations.

Kirk and Livingston [13] gave a construction of the twisted Alexander invariants, which they interpreted in the general framework of Reidemeister torsion, and showed how the work of Lin and Wada could also be interpreted in that framework. We note that these invariants can be calculated using the classical Fox calculus. Since then, various applications of the twisted Alexander invariants have been given. For example, see the survey papers of [8, 21].

A 3-manifold $M$ is called hyperbolic if the interior admits a complete metric of constant curvature $-1$ of finite volume. If $M$ is hyperbolic, it follows from Mostow-Prasad rigidity that the hyperbolic metric is unique up to isometry. Furthermore, its fundamental group $\pi_1(M)$ dominates all geodesic information of $M$. Therefore, the volume of $M$ with respect to the hyperbolic metric is an invariant of $M$ and may be computed theoretically from a presentation of $\pi_1(M)$.

We say that a knot $K$ is hyperbolic if $S^3 \setminus K$ is hyperbolic. In that case, we will denote the volume by $\text{Vol}(K)$. Thurston’s geometrisation theorem states that any knot that is neither a torus knot nor a satellite knot is a hyperbolic knot. The link case is the same.

In this paper, we give a volume formula of hyperbolic knots or links using twisted Alexander invariants. The volume formula in the knot case has already been proven [9].

Müller [23] established a volume formula for a compact hyperbolic 3-manifold using Ray-Singer analytic torsion. By [22] of Müller on the equivalence between Ray-Singer torsion and Reidemeister torsion for unimodular representations, we know that the hyperbolic volume of a compact 3-manifold can be expressed using Reidemeister torsion.

Applying Thurston’s hyperbolic Dehn surgery theorem, Menal-Ferrer and Porti [17] obtained a formula for the volume of a cusped hyperbolic 3-manifold $M$ using the so-called
higher-dimensional Reidemeister torsion invariants, which are associated with representations \( \rho_n : \pi_1(M) \to \text{SL}(n, \mathbb{C}) \) corresponding to the holonomy representation \( \text{Hol}_M : \pi_1(M) \to \text{PSL}(2, \mathbb{C}) \) (see Section 3 for details). In the present paper, we consider the relationship between the higher-dimensional Reidemeister torsion invariants and the twisted Alexander invariants, thereby obtaining a formula for the volume for a hyperbolic link using twisted Alexander invariants.

Let \( L = K_1 \cup \cdots \cup K_b \) be a \( b \)-component link in \( S^3 \). We call \( L \) an algebraically split link if the linking numbers \( \ell_k(K_i, K_j) \) are zero for all \( i, j \) (Definition 4.6). Any knot may be regarded as an algebraically split link. Let \( \Delta_{L, \rho_n}(t) \) be the determinant of the transformation \( L, \rho_n \) (Wada’s notation [34]). For an integer \( k(> 1) \), set \( A_{L, 2k}(t) = \frac{\Delta_{L, \rho_{2k}}(t)}{\Delta_{L, \rho_2}(t)} \) and \( A_{L, 2k+1}(t) = \frac{\Delta_{L, \rho_{2k+1}}(t)}{\Delta_{L, \rho_3}(t)} \). A result of this paper is the following asymptotic formula.

**Theorem 1.1.** Let \( L \) be an algebraically split hyperbolic link in the 3-sphere. Then,

\[
\lim_{k \to \infty} \frac{\log |A_{L, 2k}(1)|}{(2k + 1)^2} = \lim_{k \to \infty} \frac{\log |A_{L, 2k}(1)|}{(2k)^2} = \frac{\text{Vol}(L)}{4\pi}.
\]

The main theorem (Theorem 7.1) provides a generalisation of this formula to a complete, oriented, hyperbolic 3-manifold whose boundary consists of torus cusps. The equation obtained by evaluating \(-1\) instead of 1 also holds (Corollary 7.4).

This paper is organised as follows. In Section 2, we recall the definitions of Reidemeister torsions and twisted Alexander invariants and then fix the notation. Section 3 is devoted to explaining how to obtain the \( n \)-dimensional complex irreducible representation \( \pi_1(M) \to \text{SL}(n, \mathbb{C}) \) via the holonomy representation of \( M \). The important results used to prove the main theorem are those of Menal-Ferrer and Porti [16, 17] and a slight generalisation of a result of Dubois and Yamaguchi [5]. We review the results of Menal-Ferrer and Porti in Section 4 and discuss the generalisation of the results of Dubois and Yamaguchi in Sections 5 and 6. Section 7 is devoted to the proof of the main results. Theorem 1.1 means that we are able to obtain an approximate value of the volume of an algebraically split link via Fox calculus. We give some sample calculations in Section 8.

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2. Preliminaries

In this section, we review the basic notion and results regarding Reidemeister torsion [5, 26, 33].

Let \( C_* = (0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} C_0 \to 0) \) be a chain complex of finite-dimensional vector spaces over a field \( \mathbb{F} \). Choose a basis \( c^i \) of \( C_i \) and a basis \( h^i \) of the \( i \)-th homology group \( H_i(C_*) \) if nonzero. We denote by \( c \) (\( h \), resp.) the collection \( \{c^i\} \) (\( \{h^i\} \), resp.). The torsion of \( C_* \) with this choice of bases is defined as follows.

For each \( i \), let \( b^i \) be a set of vectors in \( C_i \) such that \( d_i(b^i) \) is a basis of \( B_{i-1} = \text{im}(d_{i-1} : C_{i-1} \to C_i) \), and \( \hat{h}^i \) a lift of \( h^i \) in \( Z_i = \ker(d_i : C_i \to C_{i-1}) \). The set of vectors \( d_{i+1}(b^{i+1})\hat{h}^i b^i/c^i \) becomes a basis of \( C_i \). Let \( \det(d_{i+1}(b^{i+1})\hat{h}^i b^i/c^i) \in \mathbb{F}^* \) be the determinant of the transformation matrix between those bases.

**Definition 2.1.** The torsion of the chain complex \( C_* \) with bases \( c \) and \( h \) is

\[
\text{Tor}(C_*, c, h) = \prod_{i=0}^n \det(d_{i+1}(b^{i+1})\hat{h}^i b^i/c^i)^{(-1)^{i+1}} \in \mathbb{F}^*/\{\pm 1\}.
\]
It is known that $\text{Tor}(C_*, c, h)$ is independent of the choice of $b^i$ and the lifts $\tilde{h}^i$. We are interested in the volume, namely, the absolute values of the Reidemeister torsion, so we do not consider its sign in this paper. Thus, the torsion lies in $\mathbb{F}^*/\{\pm 1\}$, but there are ways to avoid the sign indeterminacy.

**Remark 2.2.** In [17], the authors use the power $(-1)^i$ instead of $(-1)^{i+1}$ in Definition 2.1. In that case, the sign of the right-hand side of the equation in Theorem 7.1 in [17] is opposite. See Remark 2.2 and Theorem 4.5 in [26].

Let $W$ be a finite CW-complex and $(V, \rho)$ be a pair of a vector space over $\mathbb{F}$ and a homomorphism of $\pi_1(W)$ into $\text{Aut}(V)$. The vector space $V$ turns into a right $\mathbb{Z}[\pi_1(W)]$-module by using the right action of $\pi_1(W)$ on $V$ given by

$$v \cdot \gamma = \rho^{-1}(\gamma)(v)$$

for $v \in V$ and $\gamma \in \pi_1(W)$. We denote this module by $V_\rho$. The complex of the universal cover of $W$ with integer coefficients $C_*(\widetilde{W}; \mathbb{Z})$ becomes a left $\mathbb{Z}[\pi_1(W)]$-module by the action of $\pi_1(W)$ on $\widetilde{W}$ as the covering transformation group. We define the $V_\rho$-twisted chain complex of $W$ as follows:

$$C_*(W; V_\rho) = V_\rho \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W}; \mathbb{Z}).$$

The boundary operator of the chain complex is defined by linearity and

$$d_i(v \otimes c_i) = (\text{Id} \otimes d_i)(v \otimes c_i) = v \otimes d_i(c_i)$$

for $c_i \in C_i(\widetilde{W}; \mathbb{Z})$. Then, we can define the $V_\rho$-twisted homology of $W$, which we denote by $H_*(W; V_\rho)$.

Let $\{v_1, \ldots, v_n\}$ be a basis of $V$ and let $c_i^1, \ldots, c_i^k_i$ denote the set of $i$-dimensional cells of $W$. We take a lift $\tilde{c}_j^i$ of the cell $c_j^i$ in $\widetilde{W}$. Then, for each $i$, $\tilde{c}_i^j$ is a basis of the $\mathbb{Z}[\pi_1(W)]$-module $C_i(\widetilde{W}; \mathbb{Z})$. Thus, we have the following basis of $C_i(W; V_\rho)$:

$$\tilde{c}_i^j = \{v_1 \otimes \tilde{c}_1^i, \ldots, v_n \otimes \tilde{c}_1^i, \ldots, v_1 \otimes \tilde{c}_k_i^i, \ldots, v_n \otimes \tilde{c}_k_i^i\}.$$

If $H_i(W; V_\rho) \neq 0$, choosing for each $i$ a basis $h^i$ of $H_i(W; V_\rho)$ and set $h = \{h^i\}$, we may define the torsion as

$$\text{Tor}(C_*(W; V_\rho), c, h) \in \mathbb{F}^*/\{\pm 1\}.$$

**Remark 2.3.** The torsion does not depend on the lifts of the cells $\tilde{c}_j^i$ if $\det \rho = 1$. Furthermore, if the Euler characteristic of $W$, say $\chi(W)$, is equal to 0, we can use any basis of $V$ since the torsion is multiplied by the determinant of the base change matrix to the power $\chi(W)$.

Here, we introduce a twisted chain complex with some variables following [5]. This will be done using a $\mathbb{Z}[\pi_1(W)]$-module with variables. We regard $\mathbb{Z}^m$ as the multiplicative group generated by $m$ variables $t_1, \ldots, t_m$, namely,

$$\mathbb{Z}^m = \langle t_1, \ldots, t_m \mid t_i t_j = t_j t_i \forall i, \forall j \rangle$$

and consider a surjective homomorphism $\alpha : \pi_1(W) \to \mathbb{Z}^m$. We abbreviate the $m$ variables $(t_1, \ldots, t_m)$ to $t$ and the rational functions $\mathbb{F}(t_1, \ldots, t_m)$ to $\mathbb{F}(t)$. Furthermore, we write $V(t_1, \ldots, t_m) = V(t)$ for $\mathbb{F}(t) \otimes_{\mathbb{F}} V$ for brevity. The fundamental group $\pi_1(W)$ acts on $V(t)$:

$$\alpha \otimes \rho : \pi_1(W) \to \text{Aut}(V(t)).$$

Thus, $V(t)$ inherits the structure of a right $\mathbb{Z}[\pi_1(W)]$-module as (2.1), so we denote it by $V_\rho(t)$ and consider the associated twisted chain complex $C_*(W; V_\rho(t))$ given by

$$C_*(W; V_\rho(t)) = V_\rho(t) \otimes_{\mathbb{Z}[\pi_1(W)]} C_*(\widetilde{W}; \mathbb{Z})$$
where
\[(2.6) \quad f \otimes v \otimes \gamma \cdot c_s = f \alpha(\gamma) \otimes \rho^{-1}(\gamma)(v) \otimes c_s\]
for any \(f \in \mathbb{F}(t)\), \(v \in V\), \(\gamma \in \pi_1(W)\) and \(c_s \in C_\ast(\tilde{W}; \mathbb{Z})\). We call this chain complex the \(V_\rho(t)\)-twisted chain complex and its homology is denoted by \(H_\ast(W; V_\rho(t))\). Note that the boundary operator is defined by linearity and
\[(2.7) \quad d_i(f \otimes v \otimes c_i) = f \otimes v \otimes d_i(c_i)\]
as in (2.3). If the twisted homology \(H_\ast(W; V_\rho(t)) = 0\), then \(C_\ast(W; V_\rho(t))\) is said to be acyclic.

**Definition 2.4.** If \(C_\ast(W; V_\rho(t))\) is acyclic, then the twisted Alexander invariant of \(W\) is
\[
\Delta_{W,\rho}(t_1, \ldots, t_m) = \Delta_{W,\rho}(t) = \text{Tor}(C_\ast(W; V_\rho(t)), 1 \otimes c, \emptyset) \in \mathbb{F}^*(t_1, \ldots, t_m)/\{\pm 1\}.
\]

Proposition 5.2 asserts that the assumption holds in our setting. Note that the twisted Alexander invariant is determined up to a factor \(\pm t_1^{p_1} \cdots t_m^{p_m}\) like the classical Alexander polynomial (cf. Theorem 2 in [34]).

**Example 2.5.** Let \(K\) be a knot in the 3-sphere \(S^3\) and \(W_K\) be a CW-complex corresponding to \(S^3 - \text{Int}N(K)\). If the representation \(\rho \in \text{Hom}(\pi_1(W_K); \mathbb{C})\) is the trivial homomorphism and \(\alpha\) is the abelianisation of \(\pi_1(W_K)\), that is, \(\alpha : \pi_1(W_K) \rightarrow H_1(W_K; \mathbb{Z}) \cong \langle t \rangle\), then the twisted chain complex \(C_\ast(W_K; V_\rho(t))\) is acyclic and the twisted Alexander invariant \(\Delta_{W,\rho,K}(t)\) is the classical Alexander invariant divided by \((t - 1)\). See Theorem 4 in [18] or Chapter II in [33], for example.

In the present paper, we study a compact 3-manifold \(M\) whose boundary consists of tori. In particular, we discuss a link complement in the 3-sphere. We always suppose a CW-structure \(W\) of \(M\) and use the notation \(C_\ast(W; V_\rho) = C_\ast(W; V_\rho)\). Furthermore, we omit the collection \(c\) of the basis and denote by \(\text{Tor}(C_\ast(M; V_\rho), h)\) under the simple homotopy invariance (see [2] or Theorem 11.32 in [41]). Following these, we denote by \(\Delta_{M,\rho}(t)\) the twisted Alexander invariant for \(M\) and by \(\Delta_{L,\rho}(t)\) for a link \(L\) in \(S^3\).

### 3. Holonomy Representations of the Fundamental Groups of Hyperbolic Links and Their Lifts

Let \(M\) be a connected, oriented, hyperbolic 3-manifold. The hyperbolic structure of \(M\) yields the holonomy representation: \(\text{Hol}_M : \pi_1(M) \rightarrow \text{Isom}^+\mathbb{H}^3\), where \(\text{Isom}^+\mathbb{H}^3\) is the orientation-preserving isometry group of the hyperbolic 3-space \(\mathbb{H}^3\). Using the upper half-space model, \(\text{Isom}^+\mathbb{H}^3\) is identified with \(\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm 1\}\). Thurston proved that \(\text{Hol}_M\) can be lifted to \(\text{SL}(2, \mathbb{C})\) [29].

There were two ways to obtain a linear representation so that we can consider the twisted Alexander invariants: either (i) compose the holonomy representation \(\text{Hol}_M\) with the adjoint representation to obtain \(\pi_1(M) \rightarrow \text{Aut}(\text{sl}_2(\mathbb{C})) \leq \text{SL}(3, \mathbb{C})\) or (ii) lift \(\text{Hol}_M\) to a representation \(\pi_1(M) \rightarrow \text{SL}(2, \mathbb{C})\). The former approach is the focus of [5] and the latter is that of [7]. In particular, the open problem 1 in Section 1.7 of [7] asks whether the twisted Alexander polynomial associated with (i) determines the volume of the complement of a knot. The results in the present paper might be regarded as an answer to that question.

For all \(n > 0\), there exists a unique (up to isomorphism) \(n\)-dimensional complex, irreducible representation of \(\text{SL}(2, \mathbb{C})\), say
\[\sigma_n : \text{SL}(2, \mathbb{C}) \rightarrow \text{SL}(n, \mathbb{C})\].

Our approach in this paper is to use this \(n\)-dimensional representation of \(M\) as the composition of a lift of \(\text{Hol}_M\) with \(\sigma_n\) according to [17]. Thus, some relevant results in the case of \(n = 2\)
or 3 can be found in [5] and [7]. In Section 1.5 of [7], the fact that twisted Alexander invariants associated with representations (i) and (ii) behave differently is deemed very mysterious. However, it might be due to nothing more than the difference between the representations $\sigma_n$ ($n$: even) and $\sigma_n$ ($n$: odd).

Let us give a more precise description of the representations. By Propositions 2.1 and 2.2 in [17], there is a one-to-one correspondence between the lifts of $\text{Hol}_M$ and the spin structures on $M$. Thus, attached to a fixed spin structure $\eta$ on $M$, we have a representation:

\[
\text{Hol}_{(M,\eta)} : \pi_1(M, \eta) \to \text{SL}(2, \mathbb{C}).
\]

The vector space $\mathbb{C}^2$ has the standard action of $\text{SL}(2, \mathbb{C})$. It is known that the pair of the symmetric product $\text{Sym}^{n-1}(\mathbb{C}^2)$ and the induced action by $\text{SL}(2, \mathbb{C})$ gives an $n$-dimensional irreducible representation of $\text{SL}(2, \mathbb{C})$. Concretely, let $V_n$ be the vector space of homogeneous polynomials on $\mathbb{C}^2$ with degree $n - 1$; namely,

\[
V_n = \text{span}_\mathbb{C}(x^{n-1}, x^{n-2}y, \ldots, xy^{n-2}, y^{n-1}).
\]

Then, the symmetric product $\text{Sym}^{n-1}(\mathbb{C}^2)$ can be identified with $V_n$ and the action of $A \in \text{SL}(2, \mathbb{C})$ is expressed as

\[
A \cdot p(x, y) = p(x', y') \quad \text{where} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = A^{-1} \begin{pmatrix} x \\ y \end{pmatrix}
\]

where $p(x, y)$ is a homogeneous polynomial. We denote by $(V_n, \sigma_n)$ the representation given by this action of $\text{SL}(2, \mathbb{C})$, where $\sigma_n$ is the homomorphism from $\text{SL}(2, \mathbb{C})$ to $\text{GL}(V_n)$. It is known that each representation $(V_n, \sigma_n)$ turns into an irreducible $\text{SL}(n, \mathbb{C})$-representation of $\text{SL}(2, \mathbb{C})$ and that every irreducible $n$-dimensional representation of $\text{SL}(2, \mathbb{C})$ is equivalent to $(V_n, \sigma_n)$. Composing $\text{Hol}_{(M,\eta)}$ with $\sigma_n$, we obtain the representation

\[
\rho_n : \pi_1(M, \eta) \to \text{SL}(n, \mathbb{C}).
\]

In the following section, we discuss the twisted Alexander invariants and Reidemeister torsions associated with this representation $\rho_n$.

**Remark 3.1.** There have been several studies recently on Reidemeister torsions and twisted Alexander polynomials associated with an $\text{SL}(n, \mathbb{C})$-representation of fundamental groups. These are called higher-dimensional Reidemeister torsions or twisted Alexander polynomials. See [30, 31, 35, 36].

### 4. Results of Menal-Ferrer and Porti

In this section, we give a short review of the relevant work by Menal-Ferrer and Porti [16, 17].

Suppose $M$ is a complete, oriented, hyperbolic 3-manifold of finite volume. Thus, $M$ is the interior of a compact manifold whose boundary consists of tori $T_1, \ldots, T_b$; that is, $\partial(\text{cl}(M)) = T_1 \cup \cdots \cup T_b$. In what follows, we often abbreviate $\partial(\text{cl}(M))$ to $\partial M$.

Here, we assume the following condition, which is called $(A_M)$ in [5]:

\[
(4.1) \quad \text{the homomorphism } i_* : H_1(\partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z}) \text{ is onto and its restriction } (i|_{T_\ell})_* \text{ to the } \ell\text{-th component of } \partial M \text{ has rank one for all } \ell.
\]

**Remark 4.1.** If $\partial M$ consists of only one torus (i.e., $b = 1$), then $M$ satisfies condition (4.1); see Lemma 6.8 in [10], for example.

Then, we can choose two loops on $T_\ell$, say $\mu_\ell$ and $\lambda_\ell$, such that the pair of the homology classes $[\mu_\ell]$ and $[\lambda_\ell]$ is a basis of $H_1(T_\ell; \mathbb{Z})$ and the image $i_*(\{[\mu_\ell]\})$ generates the subgroup
im(\rho|_{T_\ell})_* and [\lambda_\ell] generates the kernel of (i|_{T_\ell})_* . The loop \mu_\ell is called the meridian and \lambda_\ell is called the longitude.

We say that a spin structure \eta on M is positive on T_\ell if, for all \mu \in \pi_1(T_\ell), we have trace Hol_{M,\eta}(\mu) = +2. Otherwise, we say that \eta is non-positive. A spin structure \eta is acyclic if \eta is non-positive on each T_\ell. Let \gamma be a simple closed curve in \partial M. The holonomy representation can be lifted to SL(2, \mathbb{C}), but we do not know whether a holonomy representation extends to a holonomy representation on M_\eta, where M_\eta is the manifold obtained by the Dehn filling along \gamma. The problem is crucial because a volume formula in [17], which we use and is given if \eta is acyclic, is obtained via Thurston’s hyperbolic Dehn surgery theorem. Corollary 3.4 in [17] gives a sufficient condition to guarantee the existence of an acyclic spin structure. It corresponds to condition (4.1); in other words, it is necessary to prove our volume formula.

Let us consider the homology groups of M twisted by \rho_n that is introduced in Section 3. Furthermore, let V_n be the vector space stated in Section 3 that is, the vector space on which \rho_n(\gamma) acts for \gamma \in \pi_1(M).

**Proposition 4.2** (Corollaries 3.6 and 3.7 in [16]). Suppose \rho_n is associated with an acyclic spin structure.

1. If n is even, then \dim \mathbb{C} H^i(\partial M; V_n) = \dim \mathbb{C} H^i(M; V_n) = 0 for i = 0, 1, 2,
2. If n is odd, then \dim \mathbb{C} H^0(\partial M; V_n) = \dim \mathbb{C} H^2(\partial M; V_n) = B, \dim \mathbb{C} H^1(\partial M; V_n) = 2B, \dim \mathbb{C} H^0(M; V_n) = b for i = 1, 2.

Moreover, Menal-Ferrer and Porti determined a basis of the homology groups. Note that Poincaré duality with coefficients in \rho_n holds (Corollary 3.7 in [17]).

**Proposition 4.3** (Proposition 4.6 in [17]). Suppose n (\geq 3) is odd. Let G_\ell < \pi_1(M) be some fixed realisation of the fundamental group of T_\ell as a subgroup of \pi_1(M). For each T_\ell choose a non-trivial cycle \theta_\ell \in H_1(T_\ell; \mathbb{Z}), and a non-trivial vector v_\ell \in V_n fixed by \rho_n(G_\ell). If i_\ell : T_\ell \rightarrow M denotes the inclusion, then the following holds:

1. A basis for H_1(M; V_n) is given by
   \langle (i_1([v_1 \otimes \theta_1])), \ldots, (i_1([v_b \otimes \theta_b])) \rangle.
2. Let [T_\ell] \in H_2(T_\ell; \mathbb{Z}) be a fundamental class of T_\ell. Then, a basis for H_2(M; V_n) is given by
   \langle (i_1([v_1 \otimes T_1])), \ldots, (i_1([v_b \otimes T_b])) \rangle.

Set \mathbf{h}^1 = \langle (i_1([v_1 \otimes \theta_1])), \ldots, (i_1([v_b \otimes \theta_b])) \rangle, \mathbf{h}^2 = \langle (i_1([v_1 \otimes T_1])), \ldots, (i_1([v_b \otimes T_b])) \rangle, and \mathbf{h} = \{\mathbf{h}^1, \mathbf{h}^2\}. Then, we define the following quotients:

\[ T_{2k+1}(M, \eta) := \frac{\text{Tor}(C_*(M; V_{2k+1}), \mathbf{h}^1)}{\text{Tor}(C_*(M; V_2), \mathbf{h})} \in \mathbb{C}^*/\{\pm1\}, \]

\[ T_{2k}(M, \eta) := \frac{\text{Tor}(C_*(M; V_{2k}), \mathbf{h}^2)}{\text{Tor}(C_*(M; V_2), \mathbf{h})} \in \mathbb{C}^*/\{\pm1\}. \]

It is known that the quantity \mathcal{T}_{2k+1} is independent of the spin structure (Remark 3.1, and hence we shall denote it simply by \mathcal{T}_{2k+1}(M). On the other hand, \mathcal{H}_i(M; V_{2k}) = 0 for i = 1, 2 by Proposition 4.2 then we may define as follows:

**Definition 4.4.**

\[ T_{2k+1}(M) := \frac{\text{Tor}(C_*(M; V_{2k+1}), \mathbf{h}^1)}{\text{Tor}(C_*(M; V_2), \mathbf{h})} \in \mathbb{C}^*/\{\pm1\}, \]

\[ T_{2k}(M, \eta) := \frac{\text{Tor}(C_*(M; V_{2k}))}{\text{Tor}(C_*(M; V_2))} \in \mathbb{C}^*/\{\pm1\}. \]
Note that Proposition 4.2 in [17] proves that $T_{2k+1}(M)$ is independent of the choice of $h$. These invariants are called the higher-dimensional Reidemeister torsion invariants.

Theorem 7.1 in [17] asserts the following.

**Theorem 4.5 (Theorem 7.1 in [17]).** Let $M$ be a connected, complete, hyperbolic 3-manifold of finite volume. Then

$$\lim_{k \to \infty} \frac{\log |T_{2k+1}(M)|}{(2k+1)^2} = \frac{\text{Vol}(M)}{4\pi}.$$  

In addition, if $\eta$ is an acyclic spin structure of $M$, then

$$\lim_{k \to \infty} \frac{\log |T_{2k}(M,\eta)|}{(2k)^2} = \frac{\text{Vol}(M)}{4\pi}.$$  

As noted in Remark 2.2, the sign of the right-hand sides is positive.

Because we focus on knots and links, we define a family of knots and links satisfying condition (4.1). See Chapter 5D in [27].

**Definition 4.6.** Let $L$ be a $b$-component link in $S^3$. One calls $L = K_1 \cup \cdots \cup K_b$ an algebraically split link if the linking numbers $\ell_k(K_i, K_j)$ are zero for all $i, j$.

Because of a Seifert surface for a knot, a knot becomes an algebraically split link. Moreover, the Whitehead link, the Borromean link and a boundary link are examples of algebraically split links. Condition (4.1) is again necessary to prove Proposition 5.2 in Section 5.

5. **Homology with Local Coefficients Corresponding to Twisted Alexander Invariants**

As in the previous section, we let $M$ be a complete, oriented, hyperbolic 3-manifold of finite volume whose boundary consists of tori $T_1 \cup \cdots \cup T_b$ with condition (4.1). In Propositions 4.2 and 4.3 we studied the homology group $H_*(M; V_n)$. In this section, we will investigate $H_*(M; V_n(t))$. According to the description in Section 2, let $\alpha : \pi_1(M) \to \mathbb{Z}^m$ be a surjective homomorphism, and suppose that $\alpha \otimes \rho_n : \pi_1(M) \to \text{Aut}(V_n(t))$ and $C_*(M; V_n(t)) = V_n(t) \otimes_{\mathbb{Z}[\pi_1(M)]} C_*(\tilde{M}; \mathbb{Z})$ are defined as in (2.4) and (2.5).

**Remark 5.1.** Every homomorphism from $\pi_1(M)$ to an Abelian group factors through the abelianisation $H_1(M; \mathbb{Z})$ of $\pi_1(M)$; that is, we have the following commutative diagram:

$$\begin{array}{ccc}
\pi_1(M) & \longrightarrow & H_1(M; \mathbb{Z}) \\
\alpha \downarrow \quad & & \downarrow \alpha_h \\
\mathbb{Z}^m
\end{array}$$

Since we suppose $\alpha$ is a surjective, the induced homomorphism $\alpha_h$ is also surjective. Together with condition (4.1), this includes $m \leq b$.

Let $\mu_\ell$ be the meridian of $T_\ell$. According to Remark 5.1, we assume the following (natural) condition:

$$\alpha(\pi_1(T_\ell)) = \alpha(\mu_\ell) = t_{a(\ell)} \quad (a(\ell) \leq m).$$  

The purpose of this section is to prove the next proposition. Note that the contents in this section are due to Section 3.2.1 in [5] and Sections 2 and 3 in [13].

**Proposition 5.2.** $H_k(M; V_n(t)) = 0$ for any $k$. 
First, we consider the case in which there is one valuable, that is, \( \alpha : \pi_1(M) \to \mathbb{Z} = \langle t \rangle \) such that \( \alpha(\mu_t) = t \) for any \( \mu_t \), where \( \mu_t \) is the meridian of \( T_t \). Condition (4.1) means that \( M \) has the infinite cyclic cover \( \widetilde{M} \) corresponding to \( \alpha \), whose boundary consists of \( b \) component annuli \( A_1 \cup \cdots \cup A_b \). Then, \( \pi_1(\widetilde{M}) = \ker \alpha \) and we have the following short exact sequence
\[
1 \to \pi_1(\widetilde{M}) \to \pi_1(M) \xrightarrow{\alpha} \mathbb{Z} \to 1.
\]
We denote by the same symbol \( \rho_n \) the restriction of the representation \( \rho_n \) of \( \pi_1(M) \) to \( \pi_1(\widetilde{M}) \), and \( M \) is also the universal cover of \( \widetilde{M} \) with covering group \( \pi_1(\widetilde{M}) \). As noted in Section 2, \( C_*(\widetilde{M}; \mathbb{Z}) \) is a left \( \mathbb{Z}[\pi_1(\widetilde{M})] \)-module, by restriction, also a left \( \mathbb{Z}[\pi_1(M)] \)-module on the cells of \( \widetilde{M} \). As in (2.2), we can form the chain complex
\[
C_*(\widetilde{M}; V_n) = V_n \otimes_{\mathbb{Z}[\pi_1(\widetilde{M})]} C_*(\widetilde{M}; \mathbb{Z}).
\]
Furthermore, by the same action as (2.6), we have the chain complex
\[
C_*(M; V_n[t, t^{-1}]) = (\mathbb{C}[t, t^{-1}] \otimes V_n) \otimes_{\mathbb{C}[\pi_1(M)]} C_*(\widetilde{M}; \mathbb{Z}).
\]
It is known that \( C_*(\widetilde{M}; V_n) \cong C_*(M; V_n[t, t^{-1}]) \) (cf. Theorem 2.1 in [13]), and there is the natural inclusion \( C_*(M; V_n[t, t^{-1}]) \subset C_*(M; V_n(t)) \). Moreover, the universal coefficient theorem implies that \( H_*(M; V_n(t)) = V_n(t) \otimes_{\mathbb{C}[t, t^{-1}]} H_*(M; V_n[t, t^{-1}]) \). Thus, we have

**Lemma 5.3.** \( H_*(M; V_n(t)) = 0 \) if and only if \( H_*(\widetilde{M}; V_n) \cong H_*(M; V_n[t, t^{-1}]) \) has no free part.

**Lemma 5.4.** \( H_0(M; V_n(t)) = 0 \).

**Proof.** This lemma is obtained from Proposition 3.5 in [13].

According to [20], the short exact sequence
\[
0 \to C_*(\widetilde{M}; V_n) \xrightarrow{t^{-1}} C_*(\widetilde{M}; V_n) \xrightarrow{t=1} C_*(M; V_n) \to 0
\]
induces the next long exact sequence in the twisted homology, which is called the Wang exact sequence (28, p456).
\[
0 \to H_2(\widetilde{M}; V_n) \xrightarrow{t^{-1}} H_2(\widetilde{M}; V_n) \xrightarrow{t=1} H_2(M; V_n)
\]
\[
\xrightarrow{\delta} H_1(\widetilde{M}; V_n) \xrightarrow{t^{-1}} H_1(\widetilde{M}; V_n) \xrightarrow{t=1} H_1(M; V_n) \to \cdots
\]
Here, \( \delta \) is the so-called connecting map.

**Lemma 5.5.** Suppose \( n \) is even. Then, \( H_k(M; V_n(t)) = 0 \) for any \( k \).

**Proof.** By Proposition 4.2, \( H_k(M; V_n) = 0 \) for \( k = 0, 1, 2 \). From the exact sequence (5.3), we obtain isomorphisms for \( k = 0, 1, 2 \):
\[
0 \to H_k(\widetilde{M}; V_n) \xrightarrow{t^{-1}} H_k(\widetilde{M}; V_n) \xrightarrow{t=1} H_k(M; V_n)
\]
This implies that \( H_k(\widetilde{M}; V_n) \) has no free part, whereupon we have this lemma by Lemma 5.3.

From now on, we consider the \( n \text{: odd} \) case. Applying the short exact sequence (5.2) to the boundaries of \( M \) and \( \widetilde{M} \), namely, \( \{T_t\}_{t=1}^{b} \) and \( \{A_t\}_{t=1}^{b} \), we have
\[
0 \to C_*(A_t; V_n) \xrightarrow{t^{-1}} C_*(A_t; V_n) \xrightarrow{t=1} C_*(T_t; V_n) \to 0.
\]
Furthermore, we have the following commutative diagram since all constructions are natural and $\bigoplus_{t=1}^b H_2(T; V_n) \cong H_2(M; V_n)$ by Propositions 4.2 and 4.3

\begin{equation}
\begin{array}{ccc}
0 & \longrightarrow & \bigoplus_{t=1}^b H_2(T; V_n) \\
\downarrow & & \downarrow \cong \\
\bigoplus_{t=1}^b H_1(A; V_n) & \longrightarrow & \bigoplus_{t=1}^b H_1(A; V_n) \\
\downarrow & & \downarrow \cong \\
\cdots & \longrightarrow & \cdots \\
\end{array}
\end{equation}

(5.5)

Here, $\delta_0$ means the connecting map corresponding to $\delta$ and $i$ is the inclusion of $A_\ell$ into $\overline{M}$.

By Proposition 4.3, we may suppose that $H_2(M; V_n)$ is generated by $[v_\ell \otimes T_\ell]$, where $v_\ell$ is an invariant vector of $\rho_n^{-1}(\tau_1(T_\ell))$. Let us consider the mapping $\delta_0$ in (5.5).

**Lemma 5.6.** $\delta_0([v_\ell \otimes T_\ell]) = [1 \otimes v_\ell \otimes \lambda_\ell]$ where $\lambda_\ell$ is a longitude of $T_\ell$, for all $\ell$.

**Proof.** For ease of notation, denote $C_*(A; V_n)$ by $C_*$ and $C'_*$, and denote $C_*(T; V_n)$ by $C''_*$ in the exact sequence (5.4). Thus, we have a commutative diagram:

\begin{equation}
\begin{array}{ccc}
0 & \longrightarrow & C''_* \\
\downarrow d'_2 & & \downarrow d_2 \\
0 & \longrightarrow & C'_* \\
\end{array}
\end{equation}

(5.6)

Let $z'' = v_\ell \otimes T_\ell \in C''_*$. For $z''$, there is an element in $C_2$, say $c_2$, such that $j_2(c_2) = z''$ since $j_2$ is surjective. By the construction, we may suppose $c_2 = 1 \otimes v_\ell \otimes A_\ell$. Since $[z'']$ is a generator of $H_2(T; V_n)$, $z'' \in \ker d'_2$, then $j_1 \circ d_2(c_2) = d'_2 \circ j_2(c_2) = d'_2(z'') = 0$. Thus, $d'_2(c_2) \in \ker j_1 = \im (t-1)$, so there exists an element $z' \in C'_*$ such that $(t-1)z' = d'_2(c_2)$. Since $d_2(A_\ell) = \mu_\ell \cdot \lambda_\ell - \lambda_\ell$ by condition (4.1), and $v_\ell$ is fixed by $\rho_n^{-1}(\mu_\ell)$, we have

(5.6)

\[ d_2(c_2) = d_2(1 \otimes v_\ell \otimes A_\ell) = \begin{array}{c} 2.7 \\ 4.1 \end{array} \]

\[ = 1 \otimes v_\ell \otimes (\mu_\ell \cdot \lambda_\ell - \lambda_\ell) = \begin{array}{c} 2.6 \\ 5.1 \end{array} \]

\[ = 1 \otimes v_\ell \otimes \lambda_\ell - 1 \otimes v_\ell \otimes \lambda_\ell = (t-1)(1 \otimes v_\ell \otimes \lambda_\ell). \]

Hence, $z' = 1 \otimes v_\ell \otimes \lambda_\ell$. Thus, we have this lemma. (We omit the well-definedness.) \qed

Since $M$ is a 3-manifold with tori boundary, the Euler number $\chi(M) = 0$. Then, the universal coefficient theorem and the Euler-Poincaré theorem imply that $\sum_{d=0}^2 (-1)^d \rk H_d(\overline{M}; V_n) = \chi(M) \times n$. Thus, we have $\rk H_1(\overline{M}; V_n) = \rk H_2(\overline{M}; V_n)$ since $\rk H_0(\overline{M}; V_n) = 0$ (Lemmas 5.3 and 5.4).

The next lemma with Lemma 5.4 concludes the single-valuable case of Proposition 5.2.

**Lemma 5.7.** $H_k(M; V_n(t)) = 0$ for $k = 1, 2$.

**Proof.** Suppose that $\rk H_1(\overline{M}; V_n) = \rk H_2(\overline{M}; V_n) = r(> 0)$. From the exact sequence (5.3), we obtain: $\im (t-1) \cong H_2(\overline{M}; V_n)/\ker (t-1) \cong H_2(\overline{M}; V_n)/\im (t-1) \cong H_2(\overline{M}; V_n)/(t-1) \cong H_2(\overline{M}; V_n)$ $\cong 0$. Therefore, $\ker \delta \cong \im (t-1) \cong 0$. Let $\xi$ be nonzero homology class in $H_2(M; V_n)$ such that $\delta(\xi) = 0 \in H_1(\overline{M}; V_n)$. Then, we can write that $\xi = \sum_{t=1}^b \beta_t [v_t \otimes T_t]$ by Proposition 4.3, and we may regard it as an element in $H_2(T; V_n)$. By Lemma 5.6 we have $\delta_0(\xi) = \delta_0(\sum_{t=1}^b \beta_t [v_t \otimes T_t]) = \sum_{t=1}^b \beta_t [1 \otimes v_t \otimes \lambda_\ell]$. On the other hand, the image of $[1 \otimes v_t \otimes \lambda_\ell]$ by the map $(t = 1)$ is nonzero in $H_1(T; V_n)$ and $H_1(M; V_n)$ by Propositions 4.2.
Therefore, \([1 \otimes v_\ell \otimes \lambda_\ell] \neq 0\) in \(H_1(A_\ell; V_\ell)\) and \(\mu_v [1 \otimes v_\ell \otimes \lambda_\ell] \neq 0\) in \(H_1(M; V_\ell)\) from the commutative diagram (5.5). This contradicts \(\delta(\xi) = 0\), and we have \(\text{rk} H_1(M; V_\ell) = \text{rk} H_2(M; V_\ell) = 0\). Thus, we have this lemma by Lemma 5.3.

**Proof of Proposition 5.2**

The idea presented here is an induction on the number of variables. We consider the fractional field \(\mathbb{C}(t_1, \ldots, t_m)\) instead of \(\mathbb{C}\) in the case of one variable, and \(V_\ell(t) = V_\ell(t_1, \ldots, t_m) = \mathbb{C}(t) \otimes \mathbb{C} V_\ell\) instead of \(V_\ell(t) = \mathbb{C}(t) \otimes \mathbb{C} V_\ell\).

Let \(pr_m : \mathbb{Z}^m \to \mathbb{Z}\) be the projection by substituting \(t_1 = \cdots = t_{m-1} = 1\) and \(pr_{1,\ldots,m-1} : \mathbb{Z}^m \to \mathbb{Z}^{m-1}\) be the projection obtained by substituting \(t_m = 1\). We can have the infinite cyclic cover \(\overline{M}\) of \(M\) corresponding to \(pr_m \circ \alpha\), namely, \(\pi_1(\overline{M}) = \ker(pr_m \circ \alpha)\), and we have the next short exact sequence:

\[1 \to \pi_1(\overline{M}) \to \pi_1(M) \xrightarrow{pr_m \circ \alpha} \mathbb{Z} \to 1.\]

The action of \(\alpha \otimes \rho_n\) on \(\mathbb{C}[t_1, t_\ell^{-1}] \otimes V_\ell(t_1, \ldots, t_m)\) is given by the tensor product of \(pr_m \circ \alpha\) and \((pr_{1,\ldots,m-1}) \circ \alpha \otimes \rho_n\). By Theorem 2.1 in [5], we have:

\[C_\ast(\overline{M}; V_\ell(t_1, \ldots, t_{m-1})) \cong C_\ast(M; \mathbb{C}[t_m, t_\ell^{-1}] \otimes V_\ell(t_1, \ldots, t_{m-1})),\]

that is,

\[H_\ast(\overline{M}; V_\ell(t_1, \ldots, t_{m-1})) \cong H_\ast(M; \mathbb{C}[t_m, t_\ell^{-1}] \otimes V_\ell(t_1, \ldots, t_{m-1})).\]

For the same reason as in (5.2) and (5.3), we have the long exact sequence

\[0 \to H_2(\overline{M}; V_\ell(t_1, \ldots, t_{m-1})) \xrightarrow{t_{m-1}} H_2(M; V_\ell(t_1, \ldots, t_{m-1})) \xrightarrow{t_{m-1}} H_2(M; V_\ell(t_1, \ldots, t_{m-1})) \xrightarrow{t_{m-1}} H_1(M; V_\ell(t_1, \ldots, t_{m-1})) \to\]

Suppose \(m = 2\). Then, the exact sequence \([5.7]\) induces the next isomorphism for \(* = 1\) or 2 by Lemma 5.7.

\[0 \to H_\ast(\overline{M}; V_\ell(t_1)) \xrightarrow{t_{\ast-1}} H_\ast(\overline{M}; V_\ell(t_1)) \to 0.\]

If \(H_\ast(\overline{M}; V_\ell(t_1))\) has a free part, it is a contradiction. Thus, we have \(H_\ast(M; V_\ell(t_1, t_2)) = 0\) for the same reason as Lemma 5.3. Then, we have \(H_\ast(M; V_\ell(t)) = 0\) inductively. This finishes the proof of Proposition 5.2.

### 6. Relationship between Twisted Alexander Invariants and Higher-Dimensional Reidemeister Torsion Invariants

According to Definition 4.4, we define the Reidemeister torsion twisted by \(\rho_n\) stated in Section 3 with preferred bases \(h = \{h^1, h^2\}\) obtained in Proposition 4.3. Let \(M\) be a complete, oriented, hyperbolic 3-manifold whose boundary consists of tori \(T_1 \cup \cdots \cup T_b\) with condition (4.1). This means that one has the meridian \(\mu_\ell\) and a longitude \(\lambda_\ell\) on \(T_\ell\) for each boundary torus \(T_\ell\) as in just after (4.1). Choose the longitude \(\lambda_\ell\) as a non-trivial cycle in \(H_1(T_\ell; \mathbb{Z})\) in Proposition 4.3, namely, let \(h_\ell = \{[v_1 \otimes \lambda_1], \ldots, [v_b \otimes \lambda_b]\}\) and denote \(h_\lambda = \{h^1, h^2\}\). Under this condition, we prove the following theorem in this section.

**Theorem 6.1.** For a positive integer \(k\), one has

1. \(\text{Tor}(C_\ast(M; V_{2k})) = \Delta_{M, \rho_{2k}}(1)\);
2. \(\text{Tor}(C_\ast(M; V_{2k+1}), h_\lambda) = \lim_{t \to 1} \frac{\Delta_{M, \rho_{2k+1}}(t)}{(t - 1)^b}\).
This theorem corresponds to a result of Milnor [18] in the case in which the representation is trivial, Theorem A in [14] in the case of $\rho_2$ and Theorem 5 in [5] in the case of the adjoint representation of $\rho_2$. Actually, (1) is proved from the map at the chain level $C_*(M_L; V_{2k}(t)) \to C_*(M_L; V_{2k}(t))$ induced by evaluation $t = 1$ since the corresponding homologies $H_*(M_L; V_{2k})$ and $H_*(M_L; V_{2k}(t))$ vanish by Proposition 4.2 and Lemma 5.5. (See also Remark 2.11 in [26].)

From now on, we consider the odd dimensional case: $n = 2k + 1 (k = 1, 2, \cdots)$.) The idea of the following arguments is the same as that in Section 6 in [5]. To prove the theorem, we review two lemmas, namely, the base-changing lemma and the multiplicativity lemma. Note that the statements of both hold without the assumption on $h^1$.

Let $C_*$ be a chain complex and $H_i$ its $i$th homology group $H_i(C_*)$. The statement of the base-changing lemma is as follows. See Proposition 0.2 in [25], for example.

**Lemma 6.2.** Suppose $c_0 = \{c^i\}$ and $c' = \{c'^i\}$ are bases of $C_*$, and $h = \{h^i\}$ and $h' = \{h'^i\}$ are bases of $H_i$. Then,

$$\text{Tor}(C_*, c', h') = \prod_{i=0}^{\infty} \left( \frac{[h'^i/c'^i]}{[h^i/c^i]} \right) \text{Tor}(C_*, c, h).$$

In particular, if $C_*$ is acyclic, then

$$\text{Tor}(C_*, c', \emptyset) = \prod_{i=0}^{\infty} \left[ [h'^i/c'^i] \right] \text{Tor}(C_*, c, \emptyset).$$

Let $0 \to V' \xrightarrow{i} V \xrightarrow{j} V'' \to 0$ be a split short exact sequence of finite-dimensional vector spaces and $s$ a section of $j$. Thus, $i \oplus s : V' \oplus V'' \to V$ is an isomorphism. Let $b' = \{b_1', \ldots, b_q'\}$, $b = \{b_1, \ldots, b_q\}$, $b'' = \{b_1'', \ldots, b_q''\}$ be bases of $V'$, $V$, $V''$, respectively, so $p + r = q$. We say that the bases $b'$, $b$ and $b''$ are compatible if the isomorphism $i \oplus s : V' \oplus V'' \to V$ has determinant 1 in the bases $b' \cup b'' = (b_1', \ldots, b_q', b_1'', \ldots, b_q'')$ of $V' \oplus V''$ and $b$ of $V$.

Let us review the multiplicativity property of the Reidemeister torsion. The equation including their signs is known, but we omit the signs since they are not important herein. For the details, see Theorem 3.2 in [19] and Lemma 3.4.2 in [32]. Let $0 \to C'_* \to C_* \to C''_* \to 0$ be an exact sequence of chain complexes. Assume that $C'_*, C_*$ and $C''_*$ are equipped with bases and homology bases. Let $c'^i$, $c^i$ and $c''^i$ denote the preferred bases of $C'_*$, $C_*$ and $C''_*$, respectively, and $h', h, h''$ be their homology bases, respectively. Then, the long exact sequence in homology

$$\cdots \to H_i(C'_*) \to H_i(C_*) \to H_i(C''_*) \to H_{i-1}(C'_*) \to \cdots$$

can be considered as an acyclic chain complex $\mathcal{H}_*$ by setting

$$\mathcal{H}_{3i} = H_i(C''_*), \mathcal{H}_{3i+1} = H_i(C_*), \mathcal{H}_{3i+2} = H_i(C'_*)$$

with preferred bases, respectively.

**Lemma 6.3.** If for all $i$, the bases $c'^i$, $c^i$ and $c''^i$ are compatible, then

$$\text{Tor}(C_*, c, h) = \text{Tor}(C'_*, c', h') \cdot \text{Tor}(C''_*, c'', h'') \cdot \text{Tor}(\mathcal{H}_*, \{h', h, h''\}, \emptyset).$$

In particular, if $C'_*$, $C_*$ and $C''_*$ are acyclic, then

$$\text{Tor}(C_*, c, \emptyset) = \text{Tor}(C'_*, c', \emptyset) \cdot \text{Tor}(C''_*, c'', \emptyset).$$

Let $M$ be a compact hyperbolic 3-manifold as in Sections 4 and 5. Let $C_* = C_*(M; V_{2k+1})$ and $C'_*(t) = C_*(M; V_{2k+1}(t))$ as defined in Section 2, namely, $C_*$ and $C'_*(t)$ are obtained from a CW-complex of $M$; see (2.2) and (2.5). We denote by $c = \{c^i\}$ and $1 \otimes c = \{1 \otimes c^i\}$ the bases of $C_*$ and $C'_*(t)$, respectively. We define $C'_*$ and $C'_*(t)$ as follows: The complex $C'_*$ is the subchain complex of $C_*$, which is a lift of the homology group $H_*(M; V_{2k+1})$. Note that the
boundary operators are all zero by this definition. According to Proposition 4.3 $C'_2$ is spanned by $\{ v_\ell \otimes T_\ell \mid 1 \leq \ell \leq b \}$ over $C$ and $C'_1$ is spanned by $\{ v_\ell \otimes \lambda_\ell \mid 1 \leq \ell \leq b \}$ over $C$, where $\lambda_\ell$ is a longitude on $T_\ell$, associated with the condition (4.1). Set $C'_3 = C'_0 = 0$. Moreover, set $c^2 = \{ v_1 \otimes T_1, \ldots, v_b \otimes T_b \}$ and $c'^1 = \{ v_1 \otimes \lambda_1, \ldots, v_b \otimes \lambda_b \}$, and we say that $C'_s$ is spanned by $c'$. Similarly, we define that $C'_e (t)$ is spanned by $1 \otimes c^2 = \{ 1 \otimes v_1 \otimes T_1, \ldots, 1 \otimes v_b \otimes T_b \}$, $C'_1 (t)$ is spanned by $c'(t) = \{ 1 \otimes v_1 \otimes \lambda_1, \ldots, 1 \otimes v_b \otimes \lambda_b \}$. Set $C'_2(t) = C'_0(t) = 0$ and we say that $C'_e(t)$ is spanned by $1 \otimes c'$. Note that we sometimes abbreviate $1 \otimes c$ to $c(t)$ and $1 \otimes c'$ to $c'(t)$ for simplicity. Similarly, we abbreviate $1 \otimes c'$ ($1 \otimes c''$, resp.) to $c'(t)$ ($c''(t)$, resp.) and $1 \otimes c'^i$ ($1 \otimes c''^i$, resp.) to $c'^i(t)$ ($c''^i(t)$, resp.). From the definition in Section 2 we obtain

$$\lim_{t \to 1} c'(t) = c', \lim_{t \to 1} c''(t) = c'', \text{ and } \lim_{t \to 1} c''^i(t) = c''^i.$$

As in the calculation (5.6), the boundary operator $d'_2$ works as follows:

$$d'_2 : 1 \otimes v_\ell \otimes T_\ell \mapsto (t - 1) \cdot (1 \otimes v_\ell \otimes \lambda_\ell).$$

Thus, we have the subchain complex of $C_s(t)$

$$C'_s(t) : 0 \to C'_3(t) \to C'_2(t) \xrightarrow{d'_2} C'_1(t) \to C'_0(t) \to 0.$$

Next, we define $C'_s (t)$ ($C'_s (t)$, resp.) as the quotient complex $C_s / C'_s (C_s(t) / C'_s(t)$, resp.). We endow it with the basis $c'' (1 \otimes c''$, resp.). Then, we have the following short exact sequence of complexes:

$$0 \to C'_s (t) \to C_s(t) \to C''_s(t) \to 0;$$

$$0 \to C'_s (t) \to C_s(t) \to C''_s(t) \to 0.$$

Lemma 6.4. $H_i (C''_s) = 0$ for $i = 0, 1, 2, 3$.

Proof. By definition, we have $H_i (C''_s) = 0$ for $i = 0, 3$. From the short exact sequence (6.4), we obtain the next long exact sequence

$$\cdots \to H_i (C'_s) \to H_i (C_s) \to H_i (C''_s) \to H_{i-1} (C'_s) \to H_{i-1} (C_s) \to \cdots.$$ 

By the definition of $C'_s$, we have $H_i (C'_s) \cong H_i (C_s)$, which leads to the conclusion that $H_i (C''_s) = 0$ for $i = 1, 2$.

Lemma 6.5. For $i = 0, 1, 2, 3$, the following holds: (1) $H_i (C'_s(t)) = 0$; (2) $H_i (C_s(t)) = 0$; (3) $H_i (C''_s(t)) = 0$.

Proof. (1) Since the map $d'_2$ is invertible, we have $H_s (C'_s (t)) = 0$. (2) This is from Proposition 5.2. (3) This can be proved by the long exact sequence in homology, which is induced by the short exact sequence (6.5).

Using the exact sequence (6.5), we endow $C_s(t)$ with the basis $c'(t) \cup c''(t)$ obtained by lifting and concatenation. Here, we compare the Reidemeister torsions of a hyperbolic 3-manifold $M$ associated with the basis $c'(t) \cup c''(t), c'(t)$ and $c''(t)$. We write $\text{Tor}(C_s(t), c'(t) \cup c''(t), \emptyset)$ ($\text{Tor}(C'_s(t), c'(t), \emptyset)$, $\text{Tor}(C''_s(t), c'(t), \emptyset)$, resp.) the Reidemeister torsion of $C_s(t)$ ($C'_s(t), C''_s(t)$, resp.) computed in the basis $c'(t) \cup c''(t)$ ($c'(t), c''(t)$, resp.). On the other hand, we denote by $\text{Tor}(M; V_{2k+1}(t), c(t), \emptyset)$ the Reidemeister torsion of the chain complex $C_s(t)$ endowed with the basis $c(t)$, which is the twisted Alexander invariant $\Delta_{L,\rho_{2k+1}} (t)$ (see Definition 2.4). Note that $c$ is different from $c' \cup c''$.

Lemma 6.6.

$$\text{Tor}(M; V_{2k+1}(t), c(t), \emptyset) = \text{Tor}(C_s(t), c'(t) \cup c''(t), \emptyset) \cdot \prod_i [c'(t) / c'(t) \cup c''(t)]^{(-1)^i}.$$
Proof. By definition, \( \text{Tor}(M; V_{2k+1}(t), c(t), \emptyset) = \text{Tor}(C_*(t), c(t), \emptyset) \). Hence, we obtain this lemma from Lemma 6.2.

By Lemma 6.7, we have the next lemma. Note that the bases are compatible by definition.

Lemma 6.7.

\[
\text{Tor}(C_*(t), c'(t) \cup c''(t), \emptyset) = \text{Tor}(C'_*(t), c'(t), \emptyset) \cdot \text{Tor}(C''_*(t), c''(t), \emptyset).
\]

Lemma 6.8.

\[
\text{Tor}(C'_*(t), c'(t), \emptyset) = (t - 1)^b.
\]

Proof. From the chain complex (6.3) and (6.2), we obtain: \( \text{Tor}(C'_*(t), c'(t), \emptyset) = \text{det} d'_2 = (t - 1)^b \).

Lemma 6.9.

\[
\text{Tor}(C''_*(t), c''(t), \emptyset) = \det(1 \otimes d_3(b^3)c'^2(t)1 \otimes b^2/c'^2(t)c''^2(t))^{-1} \\
\cdot \det(1 \otimes d_2(b^2)c'^1(t)1 \otimes b^1/c'^1(t)c''^1(t)) \cdot \det(1 \otimes d_1(b^1)/c''^0(t))^{-1}.
\]

Proof. By the definition of the way to choose a set of vectors \( b'^{n+1} \) in \( C''_{i+1} \), \( d_i^{n+1}b'^{n+1} \) is a basis of \( B_i'' = \text{im}(d_i^{n+1} : C''_{i+1} \to C''_i) \). Furthermore, we have

\[
\text{Tor}(C''_*(t), c''(t), \emptyset) = \prod_{l=0}^{2} (1 \otimes d_i^{n+1}b'^{n+1}1 \otimes b'^{n}/c''^n(t))^{-1} \\
\cdot \prod_{l=0}^{2} (1 \otimes d_i^{n+1}b'^{n+1}1 \otimes b'^{n}/c''^n(t))^{-1}.
\]

since \( C''_*(t) \) is acyclic (Lemma 6.5) and \( b'^3 = c''^3 \). For \( i = 1, 2 \), the set of vectors \( 1 \otimes b'^{n} \) in \( C''_i(t) \) generates a subspace on which the boundary operator \( d_i' : C'_i(t) \to C'_{i-1}(t) \) is injective by definition, so \( c'^n(t) \cup d_i+1(1 \otimes b'^{i+1}) \cup 1 \otimes b'^{i} \) may be regarded as a basis of \( C'_*(t) \), where \( 1 \otimes b'^{i} \) is a lift of \( 1 \otimes b'^{n} \) to \( C'_*(t) \). Therefore, we have

\[
\prod_{i=0}^{2} (1 \otimes d_i^{n+1}b'^{n+1}1 \otimes b'^{n}/c''^n(t))^{-1} = \prod_{i=0}^{2} (1 \otimes d_i^{n+1}b'^{n+1}1 \otimes b'^{i}/c'(t)c''^n(t))^{-1}.
\]

This is the desired conclusion up to \( \pm 1 \).

Proof of Theorem 6.1

First, we note that if \( t \) tends to 1, the boundary operator \( d'_2 \) in (6.3) becomes the zero map and the complex \( C'_*(t) \) changes into \( C'_* \). Moreover, if we suppose \( c'^n \) in \( C'_* \) using the exact sequence (6.3), we have the following by Proposition 4.3:

(6.6) \( H_1(M; V_{2k+1}) \) is generated by \( h_1 = \{v_1 \otimes \lambda_1, \ldots, v_b \otimes \lambda_b\} \) and \( \tilde{h}_1 = c'^1 \);

\( H_2(M; V_{2k+1}) \) is generated by \( h_2 = \{v_1 \otimes T_1, \ldots, v_b \otimes T_b\} \) and \( \tilde{h}_2 = c'^2 \).

Here, \( \tilde{h}_1 \) and \( \tilde{h}_2 \) are lifts of \( h_1 \) and \( h_2 \) into \( C'_* \).
Let \( b^{ni+1} \) be a set of vectors in \( C''_{i+1} \) such that \( d_{i+2}''(b^{ni+1}) \) is a basis of \( B_i = \text{im}(d_{i+1}'') : C''_{i+1} \to C''_i \). By Lemmas 6.2 and 6.9 we have

\[
\Tor(C''_s(t), c''_s(t), \emptyset) = \{ [1 \otimes d_3(b^3) c^2(t) 1 \otimes b^2/c^2(t)] \cdot [c^2(t)/c^2(t) c''_s(t)] \}^{-1}
\]

Moreover, by (6.1) and (6.6) together with \( \lim_{t \to 1} \Tor(C''_s(t), c''_s(t), \emptyset) \cdot \lim_{t \to 1} [c^2(t)/c^2(t) c''_s(t)]^{-1} \cdot [1 \otimes d_2(b^2) c^1(t) 1 \otimes b^1/c^1(t)] \cdot [1 \otimes d_1(b^1) / c^0(t)]^{-1} \)

Therefore, by Lemmas 6.6, 6.7 and 6.8 we have

\[
\Tor(M; V_{2k+1}(t), c(t), \emptyset) = (t - 1)^b \cdot \Tor(C''_s(t), c''_s(t), \emptyset) \cdot \prod_i [c^i(t)/c^i(t) c''_s(t)]^{-1}
\]

Moreover, by (6.1) and (6.6) together with \( \lim_{t \to 1} (1 \otimes d_3(b^3)) = d_3b^3 \) and \( \lim_{t \to 1} (1 \otimes b^2) = b^2 \), we obtain

\[
\lim_{t \to 1} [1 \otimes d_3(b^3) c^2(t) 1 \otimes b^2/c^2(t)]^{-1} \cdot [1 \otimes d_2(b^2) c^1(t) 1 \otimes b^1/c^1(t)]
\]

By Definition 2.4 \( \Delta_{L,2k+1}(t) = \Tor(M; V_{2k+1}(t), c(t), \emptyset) \), so we complete the proof of (2) in Theorem 6.1

In the case of the \( m \)-variable twisted Alexander invariant \( \Delta_{L,2k+1}(t_1, \ldots, t_m) \), the right-hand side of Lemma 6.8 is \( \prod_{t=1}^b (t_1(t) - 1) \) by assumption (5.7). Hence, we have

**Corollary 6.10.** For a positive integer \( k \), we have

1. \( \Tor(C_s(M; V_{2k})) = \Delta_{M,2k}(1, \ldots, 1) \)
2. \( \Tor(C_s(M; V_{2k+1}), h_s) = \lim_{t_1, \ldots, t_m \to 1} \frac{\Delta_{M,2k+1}(t_1, \ldots, t_m)}{\prod_{t=1}^b (t_1(t) - 1)} \)

7. **Main results**

Let \( M \) be a complete, oriented, hyperbolic 3-manifold whose boundary consists of torus cusps. We assume that \( M \) satisfies condition (4.1). Set

\[
\mathcal{A}_{M,2k}(t) = \frac{\Delta_{M,2k}(t)}{\Delta_{M,2}(t)}, \quad \mathcal{A}_{M,2k+1}(t) = \frac{\Delta_{M,2k+1}(t)}{\Delta_{M,2}(t)}
\]

Then, we have

**Theorem 7.1.**

\[
\lim_{k \to \infty} \frac{\log |\mathcal{A}_{M,2k}(1)|}{(2k)^2} = \lim_{k \to \infty} \frac{\log |\mathcal{A}_{M,2k+1}(1)|}{(2k + 1)^2} = \frac{\text{Vol}(M)}{4\pi}.
\]
We can obtain a similar volume formula in case of $m$ variables according to Corollary 6.10. If $L$ is an algebraically split link in $S^3$, the complement of $L$ satisfies condition (4.1); see Definition 4.6. Hence, we have Theorem 1.1. Since every knot is algebraically split, we have the following corollary.

**Corollary 7.2** (Theorem 1.1 in [9]). For any hyperbolic knot $K$,

$$\lim_{k \to \infty} \frac{\log |\Delta_M(2k)|(1)}{(2k)^2} = \lim_{k \to \infty} \frac{\log |\Delta_M(2k+1)|(1)}{(2k+1)^2} = \frac{\text{Vol}(K)}{4\pi}. $$

**Remark 7.3.** As in (7.1), $A_{M,\rho_n}(t)$ is defined by dividing the principal part. But it is not essential, especially in the even case. We may describe as follows without a correction term: $\Delta_{M,\rho_2}(t)$ and $\Delta_{M,\rho_3}(t)$.

- $\lim_{k \to \infty} \frac{\log |\Delta_{M,2k}(1)|}{(2k)^2} = \frac{\text{Vol}(M)}{4\pi}$
- $\lim_{k \to \infty} \frac{\log |\Delta_{M,2k+1}(t)|}{(2k+1)^2} = \frac{\text{Vol}(M)}{4\pi}$, where $b$ is the number of the component of $\partial M$.

**Proof of Theorem 7.1.**

Case 1 (even-dimensional representation $\rho_{2k}$ case).

Since $M$ satisfies condition (4.1), all spin structures on $M$ are acyclic by Corollary 3.4 in [17]. Then, we have the conclusion by Theorem 4.5 and (1) in Theorem 6.1.

Case 2 (odd-dimensional representation $\rho_{2k+1}$ case).

Part (2) of Theorem 6.1 implies that $\Delta_{M,\rho_{2k+1}}(t) = (t-1)^b \tilde{\Delta}_{M,\rho_{2k+1}}(t)$ and $\text{Tor}(M;V_{2k+1},h_\lambda) = \tilde{\Delta}_{M,\rho_{2k+1}}(1)$, where $\tilde{\Delta}_{M,\rho_{2k+1}}(t)$ is a rational function. Then,

$$A_{M,2k+1}(1) = \frac{\Delta_{M,\rho_{2k+1}}(1)}{\Delta_{M,\rho_3}(1)} = \frac{\text{Tor}(M;V_{2k+1},h_\lambda)}{\text{Tor}(M;V_3,h_\lambda)} = T_{2k+1}(M)$$

by Definition 4.4. Hence, we have Theorem 7.1 by Theorem 4.5.

**Corollary 7.4.**

$$\lim_{k \to \infty} \frac{\log |A_{M,2k}(-1)|}{(2k)^2} = \lim_{k \to \infty} \frac{\log |A_{M,2k+1}(-1)|}{(2k+1)^2} = \frac{\text{Vol}(M)}{4\pi}.$$  

**Proof.** Let $\hat{M}$ be the 2-fold cyclic covering of $M$, and let $\text{prj}$ be the covering map $\hat{M} \to M$. Then, we have the following exact sequence:

$$1 \to \pi_1(\hat{M}) \xrightarrow{\pi_1} \pi_1(M) \xrightarrow{\otimes} \mathbb{Z}/2\mathbb{Z} \to 1.$$  

Here, we use the pull-back of homomorphisms of $\pi_1(M)$ as homomorphisms of $\pi_1(\hat{M})$. Let $\alpha$ be the surjective homomorphism $\pi_1(M) \to \mathbb{Z} = (t)$ and $\hat{\alpha}$ the pull-back by $\text{prj}$.* We also suppose that $\pi$ factors through $\alpha$, and we use the symbol $\hat{\rho}_n$ for the pull-back of $\rho_n$ by $\text{prj}$. Under this situation, we can apply Corollary 4.4 in [6] where $q = 2$ and $s = t^2$ to our setting. Then, we have

$$\Delta_{\hat{M},\hat{\rho}_n}(s) = \Delta_{\hat{M},\hat{\rho}_n}(t^2) = \Delta_{M,\rho_n}(t) \cdot \Delta_{M,\rho_n}(-t).$$  

Suppose $n$ is odd (i.e., $n = 2k+1$) and we denote by $\Delta_{M,\rho_{2k+1}}(t)$ the fractional function defined in the proof of Theorem 7.1. Since $\Delta_{M,\rho_{2k+1}}(t^2) = (t^2 - 1)^b \Delta_{M,\rho_{2k+1}}(t^2)$ and $\Delta_{M,\rho_{2k+1}}(t) \cdot \Delta_{M,\rho_{2k+1}}(t^2)$...
\[ \Delta_{M,\rho,2k+1}(-t) = (t - 1)^k \Delta_{M,\rho,2k+1}(t) \cdot (-t - 1)^k \Delta_{M,\rho,2k+1}(-t), \]

we have \(|\Delta_{M,\rho,2k+1}(t^2)| = |\Delta_{M,\rho,2k+1}(t)||\Delta_{M,\rho,2k+1}(-t)|\). It is known that \(\text{Vol}(\hat{M}) = 2\text{Vol}(M)\), so we obtain the following from Theorem 7.1:

\[
\frac{2\text{Vol}(M)}{4\pi} = \frac{\text{Vol}(\hat{M})}{4\pi} = \lim_{k \to \infty} \frac{\log |A_{\rho,2k+1}(1)|}{(2k + 1)^2} = \lim_{k \to \infty} \frac{1}{(2k + 1)^2} \log \frac{|\Delta_{M,\rho,2k+1}(1)|}{|\Delta_{M,\rho,3}(1)|}
\]

\[
= \lim_{k \to \infty} \frac{\log |\Delta_{M,\rho,2k+1}(1)|}{(2k + 1)^2} = \lim_{k \to \infty} \frac{\log |\Delta_{M,\rho,2k+1}(1)||\Delta_{M,\rho,2k+1}(-1)|}{(2k + 1)^2}
\]

\[
= \frac{\text{Vol}(M)}{4\pi} + \lim_{k \to \infty} \frac{\log |\Delta_{M,\rho,2k+1}(-1)|}{(2k + 1)^2}.
\]

Therefore, we have the conclusion in the case that \(n\) is odd. In the same way, we can prove it in the case that \(n\) is even. \(\square\)

Taking account of the conjecture on the volume and the coloured Jones polynomial for knots \([12, 24]\), it might be worth considering the evaluation of twisted Alexander invariants at every root of unity.

8. ON CONCRETE CALCULATIONS

In this section, we present some known facts and methods regarding calculating twisted Alexander invariants. We give concrete calculations that provide approximate values of the volumes of the figure eight knot and the Whitehead link complements.

First, we remark on a representation that leads to the hyperbolic volume. Let \(G\) be a finitely generated group and set \(\widetilde{R}(G) = \text{Hom}(G, \text{SL}(2, \mathbb{C}))\). Since \(G\) is finitely generated, \(\widetilde{R}(G)\) may be embedded in a product \(\text{SL}(2, \mathbb{C}) \times \cdots \times \text{SL}(2, \mathbb{C})\) by mapping each representation to the image of a generating set. Thus, \(\widetilde{R}(G)\) is regarded as an affine algebraic set whose defining polynomials are induced by the relators of a presentation of \(G\). It is known that the structure is independent of the choice of the presentation of \(G\) up to isomorphism. For a representation \(\rho \in \widetilde{R}(G)\), its \textit{character} is the map \(\chi_{\rho} : G \to \mathbb{C}\) defined by \(\chi_{\rho}(\gamma) = \text{trace}(\rho(\gamma))\) for \(\gamma \in G\), where trace means the trace of a matrix. Let \(X(G)\) be the set of all characters. For a given \(\gamma \in G\), we define the map \(\tau_{\gamma} : X(G) \to \mathbb{C}\) by \(\tau_{\gamma}(\chi) = \chi(\gamma)\). Then, it is known that \(X(G)\) is an affine algebraic set that embeds in \(\mathbb{C}^n\) with coordinates \((\tau_{\gamma_1}, \ldots, \tau_{\gamma_n})\) for some \(\gamma_1, \ldots, \gamma_n \in G\). This affine algebraic set is called the \textit{character variety} of \(G\). Note that the set \(\{\gamma_1, \ldots, \gamma_n\}\) may be chosen to contain a generating set of \(G\) and the projection \(\widetilde{R}(G) \to X(G)\) given by \(\rho \mapsto \chi_{\rho}\) is surjective.

Let \(M\) be a complete hyperbolic 3-manifold. The character variety \(X(\pi_1(M))\) contains the distinguished component related to the complete hyperbolic structure. The component is defined by containing a lift of the holonomy representation \(\pi_1(M) \to \text{PSL}(2, \mathbb{C})\) determined by the complete hyperbolic structure.

Two representations \(\rho\) and \(\rho'\) are said to be \textit{equivalent} if there is an automorphism \(\psi\) of the representation space such that \(\rho'(\gamma) = \psi \circ \rho(\gamma) \circ \psi^{-1}\) for all \(\gamma \in G\). In general, the equation \(\Delta_{M,\rho}(t) = \Delta_{M,\rho'}(t)\) holds if \(\rho\) and \(\rho'\) are equivalent representations (Section 3 in \([34]\)). If \(\rho_2, \rho'_2 \in \widetilde{R}(\pi_1(M))\) are irreducible representations with \(\chi_{\rho_2} = \chi_{\rho'_2}\), then \(\rho_2\) is equivalent to \(\rho'_2\) (Proposition 1.5.2 in \([4]\)). So, \(\rho_n\) is equivalent to \(\rho'_n\), where \(\rho_n(\rho'_n, \text{resp.})\) is the irreducible
representation to $\text{SL}(n, \mathbb{C})$ stated in Section 3 so that $\Delta_{M, \rho_n}(t) = \Delta_{M, \rho_n'}(t)$. Thus, our theorem holds stated in Section 7 for the representations that are contained in the distinguished component of $X(\pi_1(M))$.

Next, we review spin structures and lifts of $\text{Hol}_M$. As in Propositions 2.1 and 2.2 in [17], there are one-to-one correspondence spin structures of $M$ and lifts of $\text{Hol}_M$. From condition (4.1) and the homology long exact sequence of $(M, \partial M; \mathbb{Z})$ and $(M, \partial M; \mathbb{Z}/2\mathbb{Z})$, we have $b_1(M; \mathbb{Z}) = b_1(M; \mathbb{Z}/2\mathbb{Z}) = b$. Hence, the number of lifts of the holonomy representation of $M$ to $\text{SL}(2, \mathbb{C})$ is equal to $|H^1(M; \mathbb{Z}/2\mathbb{Z})| = 2^b$ by Corollary 2.3 in [17].

In our setting, for each component $K_\ell$ of an algebraically split link $L$, there is the meridian $\mu_\ell$ corresponding to $K_\ell$ and there are two types of lift of the holonomy representation such that trace $\text{Hol}_{(L, \eta)}([\mu_\ell]) = +2$ and trace $\text{Hol}_{(L, \eta)}([\mu_\ell]) = -2$ where $\eta, \eta'$ are corresponding spin structures. More precisely, for a $b$-component hyperbolic link $L = K_1 \cup \cdots \cup K_b$, let $G(L)$ be the fundamental group of the exterior of $L$ in $S^3$. We suppose that $G(L)$ has the following Wirtinger presentation:

$$\langle x_1, x_2, \ldots, x_{1+k_1}, x_2, x_2, \ldots, x_{2+j_2}, \ldots, x_b, x_b, \ldots, x_{b+k_b} \mid r_1, r_2, \ldots, r_s \rangle$$

where the generator $x_{j_k}$ corresponds to the meridian of the $i$th component $K_i$ and $\sum_{k=1}^b j_k = s + 1$. Let $\eta$ be a spin structure of $L$. Then, we have trace $\text{Hol}_{(L, \eta)}(x_{j_k}) = +2$ or trace $\text{Hol}_{(L, \eta)}(x_{j_k}) = -2$ for all $k$. Hence, we have a sequence of signs that consists of $b$-component $(a_1 a_2 \cdots a_b)$ where

$$\begin{cases} a_\ell = + & \text{if trace } \text{Hol}_{(L, \eta)}(x_{j_\ell}) = +2 \\ a_\ell = - & \text{if trace } \text{Hol}_{(L, \eta)}(x_{j_\ell}) = -2 \end{cases}$$

We call this the sign of holonomy lift corresponding to the spin structure $\eta$ and denote by $\rho_2^{a_1 a_2 \cdots a_b}$ the holonomy representation with the spin structure $\text{Hol}_{(L, \eta)}$: $G(L) \rightarrow \text{SL}(2, \mathbb{C})$. Moreover, we obtain the following representation:

$$\rho_n^{a_1 a_2 \cdots a_b} : G(L) \rightarrow \text{SL}(n, \mathbb{C})$$

by composing $\rho_2^{a_1 a_2 \cdots a_b}$ with $\sigma_n$ stated in Section 3. According to (7.1), we set:

$$A_{L, 2k}^{a_1 a_2 \cdots a_b}(t) = \frac{\Delta_{L, \rho_2^{a_1 a_2 \cdots a_b}}(t)}{\Delta_{L, \rho_2^{a_1 a_2 \cdots a_b}}(t)}, \quad A_{M, 2k+1}^{a_1 a_2 \cdots a_b}(t) = \frac{\Delta_{L, \rho_3^{a_1 a_2 \cdots a_b}}(t)}{\Delta_{L, \rho_3^{a_1 a_2 \cdots a_b}}(t)}.

Moreover, the following fact is known.

Remark 8.1. Since an odd-dimensional irreducible complex representation of $\text{SL}(2, \mathbb{C})$ factors through $\text{PSL}(2, \mathbb{C})$, $\Delta_{M, \rho_n}$ and Tor$(M; V_n, h)$ does not change for any sign of the holonomy lift $(a_1 a_2 \cdots a_b)$ if $n$ is odd.

Hence, the sign of the holonomy lift is essential in the case of the even-dimensional representation.

Example 8.2. Let $K$ be the figure eight knot $4_1$. A Wirtinger presentation of $G(K)$ is

$$G(K) = \langle a, b \mid ab^{-1} a^{-1} b a b^{-1} a^{-1} b \rangle.$$

Here, $a$ and $b$ correspond to the meridians of $K$, as shown in Fig[I]. It is known that the holonomy representation of $G(K)$ is given by

$$\text{Hol}_K(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{Hol}_K(b) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix},$$

where $u$ is a complex value satisfying $u^2 + u + 1 = 0$. We use $u = (-1 + \sqrt{-3})/2$ in the following calculations (we have the same results in the case of $u = (-1 - \sqrt{-3})/2$). Then, we
have

$$\rho_+^2 : G(K) \to \text{SL}(2, \mathbb{C}) : \rho_+^2(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho_+^2(b) = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix},$$

$$\rho_-^2 : G(K) \to \text{SL}(2, \mathbb{C}) : \rho_-^2(a) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \quad \rho_-^2(b) = \begin{pmatrix} -1 & 0 \\ u & -1 \end{pmatrix}.$$  

Set $A = \rho_+^2(a)$ and $B = \rho_+^2(b)$. By the definition (Section 3), we have $p(A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}) = p(x - y, y)$, and $(x - y)^2 = x^2 - 2xy + y^2$, $(x - y)y = xy - y^2$. Hence, we have

$$\rho_+^3(a) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$  

By the same calculations, we have

$$\rho_+^3(b) = \begin{pmatrix} 1 & u & u^2 \\ 0 & 1 & 2u \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho_+^4(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}, \quad \rho_+^4(b) = \begin{pmatrix} 1 & u & u^2 & u^3 \\ 0 & 1 & 2u & 3u^2 \\ 0 & 0 & 1 & 3u \\ 0 & 0 & 0 & 1 \end{pmatrix}, \cdots.$$  

Via Fox calculus for $G(K)$, we obtain the denominator of $\Delta_{K,\rho_+^2}(t) = \det(tB - I) = (t - 1)^2$. On the other hand, the numerator of $\Delta_{K,\rho_+^2}(t) = \det((I - t^{-1}AB^{-1})A^{-1} + AB^{-1}A^{-1}B - tB + BAB^{-1}A^{-1}) = \frac{1}{t}(t - 1)^2(t^2 - 4t + 1)$. Thus, we obtain the following rational functions:

$$\Delta_{K,\rho_+^2}(t) = \frac{1}{t^2}(t^2 - 4t + 1), \quad \Delta_{K,\rho_+^2}(t) = -\frac{1}{t^3}(t - 1)(t^2 - 5t + 1),$$  

$$\Delta_{K,\rho_+^2}(t) = \frac{1}{t^4}(t^2 - 4t + 1)^2, \quad \Delta_{K,\rho_+^2}(t) = -\frac{1}{t^5}(t - 1)(t^4 - 9t^3 + 44t^2 - 9t + 1).$$

Using these functions, we have approximations to the volume of $4_1$ as follows:

$$\frac{4\pi \log |A_{K,4}^+(t)|}{4^2} = \frac{\pi \log |t^2 - 4t + 1|}{4} \to_{t = 1} \frac{\pi \log 2}{4} \approx 0.544397 \cdots,$$

$$\frac{4\pi \log |A_{K,5}^+(t)|}{5^2} = \frac{4\pi \log \left| \frac{t^4 - 9t^3 + 44t^2 - 9t + 1}{t^2 - 5t + 1} \right|}{5^2} \to_{t = 1} \frac{4\pi \log \frac{28}{3}}{5^2} \approx 1.12273 \cdots.$$  

The data of $A_{K,n}^+(1)$ for $n \leq 15$ can be found in [9]. We proceeded the calculation so that the following data are obtained.
By similar calculations, we have the following data in the case of \( \rho_2^+ \). Note that the value of \( A_{K,n}(1) \) is the same as \( A_{K,n}(1) \) in the case of \( n: \text{odd} \) as noted in Remark 8.1

| \( n \) | 16 | 20 | 26 | 30 | 36 |
|---|---|---|---|---|---|
| \( \frac{4\pi \log |A_{K,n}(1)|}{n^2} \) | 1.98381 \( \cdots \) | 2.00039 \( \cdots \) | 2.01243 \( \cdots \) | 2.01677 \( \cdots \) | 2.02078 \( \cdots \) |

**Example 8.3.** Let \( L \) be the Whitehead link as illustrated in Fig. [2]. The volume of the complement of \( L \) is equal to \( 4 \times \text{Catalan’s constant} \approx 3.66386 \cdot \cdots \). The link group \( G(L) \) has the following Wirtinger presentation, where \( a \) and \( b \) correspond to the meridians of \( L \) as in Fig. [2]:

\[
G(L) = \pi_1(E(L)) = \langle a, b \mid awa^{-1}w^{-1} \rangle \text{ where } w = bab^{-1}a^{-1}b^{-1}ab.
\]

**Figure 2.**

Hilden et al. [11] showed an explicit description of the character variety of the Whitehead link group. In fact, the component of irreducible characters is given by

\[
X(G(L)) = \{(x, y, v) \in \mathbb{C}^3 \mid p(x, y, v) = 0 \} \setminus R,
\]

where \( p(x, y, v) = xy - (x^2 + y^2 - 2)v + xyv^2 - v^3 \) and \( R = \{x = \pm 2, y = \pm 1\} \cup \{y = \pm 2, v = \pm x\} \); see Section 4.2 in [5]. Let \( \rho : G(L) \to SL(2, \mathbb{C}) \) be an irreducible representation. Then, we can suppose that the pair \( \rho(a) \) and \( \rho(b) \) are expressed as follows after taking conjugation with eigenvectors of \( \rho(a) \) and \( \rho(b) \) if necessary:

\[
\rho(a) = \begin{pmatrix} \alpha & 1 \\ 0 & 1/\alpha \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} \beta & 0 \\ \gamma & 1/\beta \end{pmatrix}.
\]

Here, the local coordinates \( (x, y, v) \) as \( x = \alpha + \alpha^{-1}, y = \beta + \beta^{-1} \) and \( v = \gamma + \alpha\beta + \alpha^{-1}\beta^{-1} \). Hence, the holonomy representation is obtained by \( \alpha + 1/\alpha = \beta + 1/\beta = 2 \) since it is a parabolic one. Therefore, we have four types of representation as follows:

(I) \( \rho_2^{++} : a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 \\ \pm \sqrt{-1} & 1 \end{pmatrix} \); (II) \( \rho_2^{+-} : a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \); (III) \( \rho_2^{-+} : a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \); (IV) \( \rho_2^{-} : a \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \).

We have the following two-variable twisted Alexander invariants using \( \rho_2^{++}, \rho_2^{+-} \), for example:

\[
\Delta_{L,\rho_2^{++}}(t_1, t_2) = \frac{1}{t_2^2} \left( (t_2 - 1)^2 + t_1^2(t_2 - 1)^2 - 2t_1(t_2^2 - (1 \mp \sqrt{-1})t_2 + 1) \right);
\]

\[
\Delta_{L,\rho_2^{-+}}(t_1, t_2) = \frac{1}{t_2^2} \left( (t_2 - 1)(t_2 - 1) - (t_2 - 1)^2 + t_1^2(t_2 - 1)^2 - 2t_1(t_2^2 + (2 \mp 4\sqrt{-1})t_2 + 1) \right).
\]

Then, we have the following data by similar calculations as knot eight.

These calculations were performed using Wolfram Mathematica and MathWorks Matlab.
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
n & 10 & 15 & 20 & 25 & 30 \\
\hline
\frac{4\pi \log |A^+_{L,n}(1)|}{n^2} & 3.43083 \cdots & 3.52207 \cdots & 3.60589 \cdots & 3.61282 \cdots & 3.63810 \cdots \\
\hline
\hline
n & 10 & 16 & 20 & 26 & 30 \\
\hline
\frac{4\pi \log |A^-_{L,n}(1)|}{n^2} & 3.54395 \cdots & 3.61657 \cdots & 3.63358 \cdots & 3.64594 \cdots & 3.65040 \cdots \\
\hline
\frac{4\pi \log |A^{+-}_{L,n}(1)|}{n^2} & 3.54395 \cdots & 3.61657 \cdots & 3.63358 \cdots & 3.64594 \cdots & 3.65040 \cdots \\
\hline
\frac{4\pi \log |A^{--}_{L,n}(1)|}{n^2} & 3.45010 \cdots & 3.58080 \cdots & 3.61071 \cdots & 3.63241 \cdots & 3.64024 \cdots \\
\hline
\end{array}
\]

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