Global Newtonian limit for the Relativistic Boltzmann Equation near Vacuum

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The relativistic Boltzmann Equation

\[ p^\mu \partial_\mu f = C(f, f), \]

The “transport term” is a lorentz inner product with signature \((- + + +)\)

\[ p^\mu \partial_\mu = p_0 \partial_t + p \cdot \nabla_x \]

Here \( p_0 = \sqrt{c^2 + |p|^2} \) is the relativistic energy.

The “collision operator” is \( C(f, h) = C_g(f, h) - C_l(f, h) \).

With Gain term

\[ C_g(f, h) = \frac{c}{2} \int_{\mathbb{R}^3} \frac{dq}{q_0} \int_{\mathbb{R}^3} \frac{dq'}{q_0'} \int_{\mathbb{R}^3} \frac{dp'}{p_0'} W(p, q | p', q') f(p') h(q') \]

And Loss term

\[ C_l(f, h) = \frac{c}{2} \int_{\mathbb{R}^3} \frac{dq}{q_0} \int_{\mathbb{R}^3} \frac{dq'}{q_0'} \int_{\mathbb{R}^3} \frac{dp'}{p_0'} W(p', q' | p, q) f(p) h(q) \]
The kernel \( W(p, q|p', q') \) is called the transition rate:

\[
W(p, q|p', q') = s\sigma(g, \theta)\delta^{(4)}(p^\mu + q^\mu - p'^\mu - q'^\mu),
\]

\( \sigma(g, \theta) \) is the differential cross-section or scattering kernel.

\( p^\mu = (-p_0, p) \) and \( q^\mu = (-q_0, q) \) are relativistic four-vectors: \( p, q \in \mathbb{R}^3 \). The Lorentz inner product is then

\[
p^\mu q_\mu = -p_0 q_0 + p \cdot q.
\]

The energy in the center-of-momentum system is

\[
s = -(p^\mu + q^\mu)(p_\mu + q_\mu) \geq 0
\]

Lastly we define the relative momentum

\[
g^2 = (p^\mu - q^\mu)(p_\mu - q_\mu) \geq 0
\]

And the scattering angle \( \theta \):

\[
\cos \theta = (p^\mu - q^\mu)(p'_\mu - q'_\mu)/g^2
\]
The Collisional Cross Sections can be computed via Quantum Field Theory (QFT), (e.g. Peskin & Schroeder 1995).

They can not be computed from a Scattering Problem because there is no widely accepted theory of relativistic N-Body dynamics (or 2-Body).

Short Range Interactions:

$$\sigma \equiv \text{constant}.$$ 

This “hard-ball” cross section is the relativistic analogue of the hard-sphere kernel in the Newtonian case.
Møller Scattering: electron-electron scattering:

\[\sigma = r_0^2 \frac{1}{u^2(u^2 - 1)^2} \left\{ \frac{(2u^2 - 1)^2}{\sin^4 \theta} - \frac{2u^4 - u^2 - \frac{1}{4}}{\sin^2 \theta} + \frac{1}{4}(u^2 - 1)^2 \right\}.\]

where the magnitude of total four-momentum

\[u = \frac{\sqrt{s}}{2mc}\]

and \(r_0 = \frac{e^2}{4\pi mc^2}\) is the classical electron radius.

Compton Scattering: photon-electron scattering.

\[\sigma = \frac{1}{2} r_0^2 (1 - \xi) \left\{ 1 + \frac{1}{4} \frac{\xi^2(1 - \cos \theta)^2}{1 - \frac{1}{2}\xi(1 - \cos \theta)} + \left( \frac{1 - (1 - \frac{1}{2}\xi)(1 - \cos \theta)}{1 - \frac{1}{2}\xi(1 - \cos \theta)} \right)^2 \right\}\]

where

\[\xi = 1 - \frac{m^2 c^2}{s}.\]
Glassey-Strauss reduction of the collision integrals

- Glassey and Strauss (1993) reduction

\[ C(f, h) = \int_{\mathbb{R}^3 \times S^2} \frac{s\sigma(g, \theta)}{p_0 q_0} B(p, q, \omega)[f(p')h(q') - f(p)h(q)]d\omega dq \]

- with kernel

\[ B(p, q, \omega) \equiv c \frac{(p_0 + q_0)^2 p_0 q_0 \left| \omega \cdot \left( \frac{p}{p_0} - \frac{q}{q_0} \right) \right|}{[(p_0 + q_0)^2 - (\omega \cdot [p + q])^2]^2} \]

- Post-Collisional Momentum:

\[ p' = p + a(p, q, \omega)\omega, \quad q' = q - a(p, q, \omega)\omega, \]

- where: \[ a(p, q, \omega) = \frac{2(p_0 + q_0)p_0 q_0 \left\{ \omega \cdot \left( \frac{q}{q_0} - \frac{p}{p_0} \right) \right\}}{(p_0 + q_0)^2 - \left\{ \omega \cdot (p + q) \right\}^2} \]

- The energies: \[ p'_0 = p_0 + N_0 \text{ and } q'_0 = q_0 - N_0: \]

\[ N_0 \equiv \frac{2\omega \cdot (p + q)\left\{ p_0(\omega \cdot q) - q_0(\omega \cdot p) \right\}}{(p_0 + q_0)^2 - \left\{ \omega \cdot (p + q) \right\}^2} \]
Lorentz Transformations grant another reduction:

\[ C(f, h) = \int_{\mathbb{R}^3 \times S^2} v_c \sigma(g, \theta) [f(p')h(q') - f(p)h(q)] \, d\omega dq. \]

\( v_c = v_c(p, q) \) is the Møller velocity:

\[ v_c(p, q) \equiv \frac{c}{2} \sqrt{\left| \frac{p}{p_0} - \frac{q}{q_0} \right|^2 - \frac{1}{c^2} \left| \frac{p}{p_0} \times \frac{q}{q_0} \right|^2} = \frac{c}{4} \frac{g \sqrt{s}}{p_0 q_0}. \]

The post collisional momentum can be written:

\[ p' = \frac{p + q}{2} + g \left( \omega + (\gamma - 1)(p + q) \frac{(p + q) \cdot \omega}{|p + q|^2} \right), \]

\[ q' = \frac{p + q}{2} - g \left( \omega + (\gamma - 1)(p + q) \frac{(p + q) \cdot \omega}{|p + q|^2} \right), \]

where \( \gamma = (p_0 + q_0)/\sqrt{s} \).
The energies are
\[ p_0' = \frac{p_0 + q_0}{2} + \frac{g}{\sqrt{s}} \omega \cdot (p + q), \]
\[ q_0' = \frac{p_0 + q_0}{2} - \frac{g}{\sqrt{s}} \omega \cdot (p + q). \]

These will be the coordinates we use. As far as we know this is the first time the coordinates are used in a mathematically oriented paper.

More generally can do this reduction with any Lorentz Transformation $\Lambda$ and obtain
\[ P' = \frac{1}{2} \Lambda^{-1} \begin{pmatrix} s \\ g\omega \end{pmatrix}, \quad Q' = \frac{1}{2} \Lambda^{-1} \begin{pmatrix} s \\ -g\omega \end{pmatrix}, \]

Need only
\[ \Lambda(P + Q) = (\sqrt{s}, 0, 0, 0)^t, \]
Lorentz Transformations Mapping Into $p + q = 0$

$$\Lambda(P + Q) = (\sqrt{s}, 0, 0, 0), \ \Lambda(P - Q) = (0, 0, 0, g)$$

$$\Lambda = \begin{pmatrix} \frac{p_0 + q_0}{\sqrt{s}} & \frac{p_1 + q_1}{\sqrt{s}} & \frac{p_2 + q_2}{\sqrt{s}} & \frac{p_3 + q_3}{\sqrt{s}} \\ \Lambda_0^1 & \Lambda_1^1 & \Lambda_2^1 & \Lambda_3^1 \\ \frac{p_0 - q_0}{g} & \frac{p_1 - q_1}{g} & \frac{p_2 - q_2}{g} & \frac{p_3 - q_3}{g} \end{pmatrix}.$$ 

$$\Lambda_0^1 = \frac{2|p \times q|}{\sqrt{s}g} = \frac{|p \times q|}{\sqrt{(p^\mu q_\mu)^2 - c^4}}.$$ 

$$\Lambda_i^1 = \frac{2 \left(p_i \{p_0 - q_0 p^\mu q_\mu\} + q_i \{q_0 - p_0 p^\mu q_\mu\}\right)}{\sqrt{s}g|p \times q|} \quad (i = 1, 2, 3).$$
The collision operator converges (as $c \to \infty$) to

$$Q_\infty(f, g) = \frac{1}{2} \int_{\mathbb{R}^3 \times S^2} |p - q|[f(p')g(q') - f(p)g(q)]d\omega dq.$$ 

This is again when $\sigma = 1$... (other cross sections will give other limits.)

The variables in this $\sigma$-representation are

$$p' = \frac{p + q}{2} + \frac{1}{2}|p - q|\omega, \quad q' = \frac{p + q}{2} - \frac{1}{2}|p - q|\omega,$$

The newtonian limit is again the Classical Boltzmann equation (now in $\sigma$-representation):

$$\partial_t f + p \cdot \nabla_x f = C_\infty(f, f)$$
Previous Results for the relativistic Boltzmann Equation

- **Glassey & Strauss (1991)**
  - compute \( \frac{\partial (p', q')}{\partial (p, q)} = -\frac{p'_0 q'_0}{p_0 q_0} \)

- **Dudyński and Ekiel-Jeżewska (1992)**
  - large data Diperna-Lions renormalized solutions

- **Glassey & Strauss (1993, 1995)**
  - Global Stability of \( e^{-p_0} \) in \( \mathbb{T}_x^3, \mathbb{R}_x^3 \)

- **Andréasson (1996) & Wennberg (1997)**
  - Regularity of the Gain Term

- **Andréasson, Calogero, Illner, (2004)**
  - blowup for gain-term-only,
Previous Results for the relativistic Boltzmann Equation

- Calogero (2004)
  - In $\mathbb{T}_x^3$, Local in Time uniform existence, Newtonian limit,

- Glassey (2006)
  - Global solutions to the Cauchy problem for the relativistic Boltzmann equation with near-vacuum data with $c = 1$,

- Ha, Kim, Lee, Noh (2007)
  - Conditional $L^1$ scattering:
    \[
    \|f^#(t) - f_+(t)\|_{L_{x,p}^1} \to 0, \quad t \to \infty
    \]
    Existence not known in strong enough space for scattering.

- S (2009)
  - Global Existence Near Vacuum in $\mathbb{R}_x^3$ uniform in $c \geq 1$,
  - Validity of Global Newtonian Limit,
  - Solution space sufficient for Scattering
Illner-Shinbrot (1984) method for constructing solutions to the Newtonian System near Vacuum.

**Important Newtonian Symmetry:**

\[ |x + tv|^2 + |x + tu|^2 = |x + tv'|^2 + |x + tu'|^2 \]

Follows from Newtonian conservation of energy:

\[ |v|^2 + |u|^2 = |v'|^2 + |u'|^2 \]

\[ v + u = v' + u' \]

This apparently fails under special relativity:

\[ p_0 + q_0 = p'_0 + q'_0 \]

\[ p + q = p' + q' \]

These relativistic Symmetries make it hard to find a positive dispersive quantity as above.

May not exist. Many have looked for it.
Important Relativistic Invariant:

\[
c^3 \frac{t^2}{p_0} + \frac{p_0}{c} |x|^2 + c^3 \frac{t^2}{q_0} + \frac{q_0}{c} |x + t(\hat{p} - \hat{q})|^2
\]

\[
= c^3 \frac{t^2}{q_0'} + \frac{q_0'}{c} |x + t(\hat{p} - \hat{q}')|^2 + c^3 \frac{t^2}{p_0'} + \frac{p_0'}{c} |x + t(\hat{p} - \hat{p}')|^2.
\]

Difficulty: Temporal components due to coupling of space and time via Lorentz Invariance.

Problem: the transport operator doesn’t appear to produce the kind of time decay rates that seem to be required to exploit this symmetry.

Angular Cut-Off a-la Grad used to handle temporal components of invariant, Cut-Off disappears in Newtonian Limit.
Glassey’s (2006) Theorem

- Glassey gave the first construction of solutions near Vacumm to the relativistic Boltzmann Equation
- Space:
  \[ e^{p_0} (1 + |x \times p|^2)^{1+\delta} f(t, x, p) \leq b_0, \quad 0 < \delta < 1 \]
- Cross Sectional Assumption:
  \[ \sigma(p, q, \omega) \leq \frac{|\omega \cdot (q \times \hat{p})| \tilde{\sigma}(\omega)}{g(1 + g^2)^{\delta+1/2}} \]
  And
  \[ \int_{S^2} d\omega \frac{\tilde{\sigma}(\omega)}{1 + |\omega \cdot z|} \leq c|z|^{-1}. \]
- Glassey’s (2006) Result: Global existence for near Vacuum Data.
Weight:

\[ \rho_c(x, p) = \exp\left(-\alpha p_0 |x|^2 / c\right) J^\beta(q), \quad \alpha, \beta > 0. \]

Above \( J^\beta(q) \) is the relativistic Maxwellian:

\[ J^\beta(q) = (4\pi cK_2(c^2))^{-\beta} e^{-\beta cp_0} \]

Notice that as \( c \to \infty \)

\[ \rho_c(x, p) \to \rho_\infty(x, p) = \exp\left(\alpha |x|^2\right) e^{\beta |p|^2}, \]

which is the Newtonian space for Near Vacuum Solutions.

“Almost invariance” in this space

\[
\begin{align*}
\frac{q'_0}{c} |x + t (\hat{p} - \hat{q}')|^2 + \frac{p'_0}{c} |x + t (\hat{p} - \hat{p}')|^2 & \\
= \frac{p_0}{c} |x|^2 + \frac{q_0}{c} |x + t (\hat{p} - \hat{q})|^2 + \gamma t^2
\end{align*}
\]

Deadly Term: \( \gamma = c^3 \left( \frac{1}{p_0} + \frac{1}{q_0} - \frac{1}{p'_0} - \frac{1}{q'_0} \right) \)
Recall the Bad Term:

$$\gamma = c^3 \left( \frac{1}{p_0} + \frac{1}{q_0} - \frac{1}{p_0'} - \frac{1}{q_0'} \right)$$

If $\gamma \geq 0$, then we would be in business.

In fact we can do better. For $B \geq 0$ and $0 \leq a < 1$ and $t > 0$,

$$h = h(x, p, q, t, c) = \frac{B}{t^2} + a \frac{\beta q_0 |x + t (\hat{p} - \hat{q})|^2}{t^2} / c \geq 0.$$ 

Define the cut-off set

$$\mathcal{B}_c = \{\omega : \gamma \geq -h\}.$$ 

We remark that $h$ can be quite large, and often $\mathcal{B}_c = S^2$.

To handle this term we introduce a cut-off in the cross section

$$\sigma(\omega) = \sigma(\omega) 1_{\mathcal{B}_c}(\omega),$$

Here $1_{\mathcal{B}_c}(\omega)$ is the indicator function of the set $\mathcal{B}_c(\omega)$. 

Robert Strain

Global Newtonian limit for the Relativistic Boltzmann Equation
Bad Term (continued...)

- This is an angular cut-off:

\[ \mathcal{B}(\omega) = \{\omega \in S^2 : \gamma \geq -h}\]

Compare to Grad’s angular cutoff...

- Not at all a limitation in the Newtonian Limit \((c \to \infty)\):

\[ 1_{\mathcal{B}}(\omega) = 1, \quad \forall c \geq c^*(p, q, T) \]

- Otherwise we use a generic collision kernel

\[ \sigma(\omega, p, q) \leq (A_1 + A_2 g^{-\gamma}) \sin^\beta \theta. \]

Above \(A_1\) and \(A_2\) are positive constants. We allow

\[ 0 \leq \gamma < -3, \quad \beta \geq 0. \]
**Local in Time** Existence result. Uniform time of existence $T > 0$ for $c \geq c_0$. Uses Illner - Shinbrot (1984).

Spatially periodic solutions $x \in \mathbb{T}^3$.

Uses existence of solutions to the limiting Hard-Sphere Newtonian Boltzmann equation. (Illner - Shinbrot.)

Establishes (Local) Newtonian Limit as $c \to \infty$.

To our knowledge, all previous results on classical Newtonian limits for Kinetic Equations are **Local in Time**.
e.g. Degond - Palaiseau (1986),
Asano-Ukai (1986),
Schaeffer (1986),
Rendall (1994) Vlasov - Einstein, etc.
Mild Formulation of the Cauchy Problem

- Define the solution along its trajectories

\[ f^\#(t, x, p) = f(t, x + t\hat{p}, p) \]

- The Mild Formulation of the Cauchy Problem

\[ f^\#(t, x, p) = f_0(x, p) + \int_0^t ds \ Q^\#(f, f) \]

- with Collision Operator

\[ Q^\#(f, f) = \int_{\mathbb{R}^N} dq \int_{S^{N-1}} d\omega \ v_c \ \sigma \ f(t, x + t\hat{p}, p')f(t, x + t\hat{p}, q') \]

\[ -\int_{\mathbb{R}^N} dq \int_{S^{N-1}} d\omega \ v_c \ \sigma \ f(t, x + t\hat{p}, q)f(t, x + t\hat{p}, p) \]

- Recall we are in the Center - of - Momentum Variables (seems to be crucial)

- This is the mild form of: \( \partial_t f + \hat{p} \cdot \nabla_x f = Q(f, f) \)
Global Existence Theorem uniform for $c \geq 1$

**Theorem**

Consider initial values $0 \leq f_{0,c}(x, p) \in C^0(\mathbb{R}_x^3 \times \mathbb{R}_p^3)$ and additionally

$$\frac{f_{0,c}(x, p)}{\rho_c(x, p)} \leq b.$$ 

There exists a positive number $b_0$, which is independent of the speed of light $c \geq 1$, with the property that if $b \leq b_0$ then there exists a unique non-negative global solution $f_c(t, x, p)$ to the mild form of the Cauchy problem. This solution satisfies the estimates

$$\|f_c^\#\|_c = \sup_{t,x,p} \frac{f_c^\#(t, x, p)}{\rho_c(x, p)} \leq b_1,$$

The constant $b_1 = b_1(b_0)$ is explicit and does not depend upon $c \geq 1$. 

Robert Strain

Global Newtonian limit for the Relativistic Boltzmann Equation
Theorem (Newtonian Limit)

Suppose that for any \( c_n, c_m \geq c \geq 1 \) and \( \epsilon > 0 \) we have a collection of initial data satisfying the estimates

\[
\| f_{0,c_n} - f_{0,c_m} \|_{L^1_p L^\infty_x} \leq A/c^{1+\epsilon}.
\]

For some uniform constant \( A > 0 \) which is independent of \( c, c_n, c_m \). Further suppose

\[
\| \nabla_x f_{0,c_n} \|_{L^\infty_x L^1_p} + \| \nabla_p f_{0,c_n} \|_{L^\infty_x L^1_p} \leq A_1 < \infty
\]

uniformly in \( c_n \). Then for any fixed \( T > 0 \) (which is allowed to be large) the solution corresponding to these initial data satisfy

\[
\| f_{c_n}(t) - f_{c_m}(t) \|_{L^1_p L^\infty_x} \leq A(T)/c, \quad c \to \infty.
\]

These solutions thereby converge to a solution of the Newtonian Boltzmann equation.
Remarks on the Theorems

We do not assume the existence of any solution to the limit equation, instead we recover global existence in the limit.

Even though convergence is in the weak space $L^{1}_{p}L^{\infty}_{x}$, we still recover

$$f_{cm} \rightarrow f_{\infty}$$

with

$$f_{\infty}(t, x, p) \leq b_{1} e^{-\alpha |x|^{2}} e^{-\beta |p|^{2}}$$

(In the Classical Case you would have $p = v$.)
Relativistic Vlasov-Maxwell-Boltzmann System

- Relativistic Vlasov-Maxwell-Boltzmann System:

\[
\begin{align*}
\partial_t F_+ &+ \frac{p}{p_0} \cdot \nabla_x F_+ + \left( E + \frac{p}{p_0} \times B \right) \cdot \nabla_p F_+ \\
&= C(F_+, F_+) + C(F_+, F_-) \\
\partial_t F_- &+ \frac{p}{p_0} \cdot \nabla_x F_- - \left( E + \frac{p}{p_0} \times B \right) \cdot \nabla_p F_- \\
&= C(F_-, F_-) + C(F_-, F_+) \\
\end{align*}
\]

coupled with Maxwell’s Equations:

\[
\begin{align*}
\partial_t E - \nabla_x \times B &= -\mathcal{J}, \\
\partial_t B + \nabla_x \times E &= 0
\end{align*}
\]

- And constraints: \( \nabla_x \cdot B = 0, \ \nabla_x \cdot E = \rho \)

- Non-Linear coupling

\[
\mathcal{J} = \int_{\mathbb{R}^3} \frac{p}{p_0} (F_+ - F_-) \, dp, \quad \rho = \int_{\mathbb{R}^3} (F_+ - F_-) \, dp
\]
Relativistic Boltzmann Collision Operator

\[ \mathcal{C}(F_+, F_-) = \int_{\mathbb{R}^3 \times S^2} \frac{s}{p_0 q_0} B[F_+(p') F_-(q') - F_+(p) F_-(q)] d\omega dq \]

with kernel

\[ B = B(p, q, \omega) \equiv c \frac{(p_0 + q_0)^2 p_0 q_0 \left| \omega \cdot \left( \frac{p}{p_0} - \frac{q}{q_0} \right) \right|}{[(p_0 + q_0)^2 - (\omega \cdot [p + q])^2]^2}. \]

Post-Collisional Momentum:

\[ p' = p + a(p, q, \omega) \omega, \quad q' = q - a(p, q, \omega) \omega, \]

where:

\[ a(p, q, \omega) = \frac{2(p_0 + q_0)p_0 q_0 \left\{ \omega \cdot \left( \frac{q}{q_0} - \frac{p}{p_0} \right) \right\}}{(p_0 + q_0)^2 - \{\omega \cdot (p + q)\}^2} \]
The conservation of mass, total momentum and total energy for solutions as

\[
\frac{d}{dt} \int_{T^3 \times \mathbb{R}^3} m_+ F_+(t) = \frac{d}{dt} \int_{T^3 \times \mathbb{R}^3} m_- F_-(t) = 0,
\]

\[
\frac{d}{dt} \left\{ \int_{T^3 \times \mathbb{R}^3} p(m_+ F_+(t) + m_- F_-(t)) + \frac{1}{4\pi} \int_{T^3} E(t) \times B(t) \right\} = 0,
\]

\[
\frac{d}{dt} \left\{ \frac{1}{2} \int_{T^3 \times \mathbb{R}^3} (m_+ p_0^+ F_+(t) + m_- p_0^- F_-(t)) + \cdots \right\} = 0.
\]

\[
\cdots + \frac{d}{dt} \left\{ \frac{1}{8\pi} \int_{T^3} |E(t)|^2 + |B(t)|^2 \right\} = 0.
\]

The entropy increasing

\[
- \frac{d}{dt} \int \{ F_+ \ln F_+ + F_- \ln F_- \} \, dx dp \geq 0.
\]

This is Boltzmann’s H-Theorem.
Asymptotic Stability of Relativistic Vlasov-Maxwell-Boltzmann System

Theorem (Guo-S, 2009)

Fix $N \geq 4$, $x \in \mathbb{T}^3$. Let $F_0(x, p) = \mu_{rel} + \sqrt{\mu_{rel}} f_0(x, p) \geq 0$, where $\mu_{rel} = e^{-p_0}$. Assume

Conservation Laws$(F_0, E_0, B_0) = \text{Conservation Laws}(\mu_{rel}, 0, \bar{B})$

Then $\exists C_N > 0$, $\epsilon_N > 0$ small enough such that if

$\mathcal{E}_N(f_0) \leq \epsilon_N$

Then there exists a unique positive global smooth solution with

$\frac{d}{dt} \mathcal{E}_N(t) + \mathcal{D}_N(t) \leq 0$

$\mathcal{D}_N(t)$ measures the dissipation of the linearized collision operator.
**Problem:** derivatives of the collisional map grow

\[ |\nabla_p q'_i| + |\nabla_p p'_i| \leq Cq_0^5 \left( 1 + |p \cdot \omega|^{1/2} 1_{\{|p \cdot \omega| > |p \times \omega|^{3/2}\}} \right). \]

- There is no place to put these moments...
- We develop a long series of non-linear changes of coordinates to facilitate integration by parts and to move these weights onto lower order derivatives.
THANK YOU!