Numerical Simulations of Electrohydrodynamics
flow model based on Burgers’ Equation with
Transport of Bubbles

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Abstract. In this paper we present numerical models for electrodynamical
flows with time-dependent electrical fields with transport of bubbles.
Such models are applied in e-jet printing, e.g., additive manufacturing
(AM), and convective cooling, electrostatic precipitation, plasma assisted
combustion.
We present a coupled model, consisting of two models. The first model is
the underlying hydrodynamical simulation model, which is based on the
Burgers’ equation. The second model is a transport model based on level-
set equations, which transport the bubbles in the flow field. We derive the
modeling of the two coupled modeling equations and discuss decoupled
and coupled versions for solving such delicate problems. In the numerical
analysis, we discuss the solvers methods, which are based on splitting
approaches and level-set techniques. In the first numerical results, we
present decoupled and coupled model simulations of the bubbles in the
liquid flow.

Keywords: electro-hydrodynamics model, hydrodynamical model, coupling
methods, splitting approach

AMS subject classifications. 35K25, 35K20, 74S10, 70G65.

1 Introduction

We are motivated to simulate bubble transport and their formation in liquid,
based on E-fields and different flow-fields. Such formation and transport of gas-
bubbles in liquid are important for the controlled production in additive man-
facturing (AM), convective cooling, electrostatic precipitation, plasma assisted
combustion, see [7], [9], [14] and [8].

For simulating such processes, we are motivated to extend an underlying
hydrodynamics model, which is based on the Burgers’ equation with an elect-
rodynamical equation and additional a two-phase equation, which models the
bubble transport and modifications, see [2].
We consider the single models and coupled them into a multi-physics model based on the different modeled processes, i.e., flow-field model, E-field model and phase field model. Here, the numerical solvers are important, while we deal with different scale-dependent models, see [3] and [4].

We present splitting methods, which separate each modeling equation and solve them with the optimal solver for each equations, see [6].

We apply a modified model of the surface tension, see [12], which is important for the formation and transport of the bubbles in liquids, see [11] and [5]. We present first numerical results with decoupled and coupled flow-fields.

The paper is organized as following: The modeling problems and their solvers are presented in Section 2. The solver of the models are discussed in Section 3. The numerical experiments are presented in Section 4. In the contents, that are given in Section 5, we summarize our results.

## 2 Mathematical Model

The modeling is based on an electrohydrodynamics model coupled with phase field models.

We have the following coupled modeling equations:

- Electrical field equation
- Phase-field equation
- Fluid-flow equation

### 2.1 Modeling: Electrical field equations

We deal with the electrohydordynamics. We have given the Poisson equation as:

$$\nabla \cdot (\varepsilon_0 \varepsilon_r \nabla \phi) = -\rho_e,$$

(1)

$\varepsilon_0$ is the permittivity of vacuum, $\varepsilon_r$ the dielectric constant, $\phi$ the potential and $\rho_e$ the free charge density.

The free charge conservation is given as:

$$\frac{\partial \rho_e}{\partial t} + \mathbf{u} \cdot \nabla \rho_e = -\nabla \cdot (\sigma \nabla \rho_e),$$

(2)

where $\sigma$ is the electrical conductivity, $\mathbf{u}$ the velocity field and $\mathbf{E} = -\nabla \phi$ is the electrical field. While the second term on the left hand side presents the convection of the free charges and the right-hand side term presents the transport by electromigration.

### 2.2 Modeling: Phase field equations

We deal with the volume-fraction $c$, which satisfies the advection equation:

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = 0,$$

(3)

where $\mathbf{u}$ is the velocity field and $c = 1$ is the bubble/drop phase, while $c = 0$ is the bulk fluid phase.
Remark 1. The full phase field equation is given by the Cahn-Hilliard equation, while the parameter \( c \) is the phase parameter.

### 2.3 Modeling: Fluid-flow equation

We deal with the fluid-flow equation, which satisfy the Navier-Stokes equation:

\[
\frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \nabla) \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g} + \mathbf{f}_{sf} + \mathbf{f}_{ef},
\]

\( \nabla \cdot \mathbf{u} = 0, \)

where \( \mathbf{u} \) is the velocity field and \( p \) is the pressure, \( \mu \) is the dynamic viscosity, \( \rho \) is the fluid density, \( \mathbf{f}_{sf} \) is the surface tension, \( \mathbf{f}_{ef} \) is the electrostatic force, \( \mathbf{g} \) is the gravitational acceleration.

The forces are given as:

\[
\mathbf{f}_{sf} = \sigma \kappa (\nabla c),
\]

\[
\mathbf{f}_{ef} = \nabla \tau_e = \rho_e \mathbf{E} - \frac{1}{2} |\mathbf{E}|^2 \nabla \epsilon,
\]

where \( \sigma \) is the surface tension, \( \kappa \) is the interface curvature with \( \kappa (\nabla c) = \nabla \cdot \left( \frac{\nabla c}{|\nabla c|} \right) \) and \( \tau_e \) is the Maxwell stress tensor.

### 3 Solvers and splitting approaches

We deal with the modeling and solver-ideas, as presented in Fig. 1.

**Fig. 1.** Numerical cycle of the near-far-field coupling.

#### 3.1 Reinitialization of the Level-Set method

Based on the problem of the conservation of the volume of the level-set method, we deal with the improvement of the reinitialisation, see [?].

We have the following algorithm of the reinitialization:
1. We initialize with $t = 0$ and $c(0) = c_{\text{init}}$. we apply $n = 1$ and we have the indicator function of the density $\rho(t_0) = \rho_{\text{init}} = 1.0$ and the volume of the bubble with $V(t_0) = V_{\text{init}}$.
2. We compute with the next level-set step with time-step $\Delta t = t_n - t_{n-1}$. We have the old concentration at time $t_{n-1}$ with $c(t_{n-1})$ and we obtain the new concentration $c(t_n)$ at $t_n$.
3. We compute the volume of the bubble $V(t_n)$ in the new time-point $t_n$.
4. We apply the correction, which is based on the factor:
   \[ \rho(t_n) = \rho(t_{n-1}) \frac{V(t_{n-1})}{V(t_n)}, \quad (8) \]
5. We apply the reinitialization of the concentration, which is given as:
   \[ c_{\text{reinit}}(t_n) = \rho(t_n)c(t_n). \quad (9) \]
6. We set $c(t_n) = c_{\text{reinit}}(t_n)$ and apply $n = n + 1$, if $n + 1 = N + 1$ we are done, else we go to Step 2.

The idea of the algorithm is given in the Figure 2.

**Fig. 2.** Reinitialization in the level set step.

### 3.2 Solver methods for the Burgers’ and Navier-Stokes equation

In the following, we discuss the discretization of the Burger’s and NS equation.

**3.2.1 Discretization of the Burgers’ equation** We apply the following splitting approach to the 2D Burgers’ equation, which is given as:

\[ \partial_t u = -uu_x - v u_y + D (u_{xx} + u_{yy}) + f_{cl,x}(t), \quad (10) \]
\[ \partial_t v = -u v_x - v v_y + D (v_{xx} + v_{yy}) + f_{cl,y}(t), \quad (11) \]
with initial and boundary conditions.

We rewrite into the operator-notation as:

\[ \partial_t U = A(U) + BU + F_{el}, \]

where \( A \) is the nonlinear term (flow-term), \( B \) is the linear viscosity-term and \( F_{el} \) is the right-hand side electrical field term.

We apply the following splitting approach:

– **A-Step (Burgers’ Step)** with explicit time-discretization:
  We apply the finite difference or finite volume scheme for the viscous Burgers’ equation and deal with the following discretized differential equations:

\[ U_{1}^{n+1} = U_{1}^{n} - (U_{1}^{n} \nabla)U_{1}^{n}, \quad U_{1}(t^n) = U(t^n), \]

\[ U_{2}^{n+1} = U_{2}^{n} + BU_{2}^{n+1}, \quad U_{2}(t^n) = U_{1}(t^{n+1}), \]

where \( B \) is the semi-discretized operator of the diffusion operator.

– **B-Step (Diffusion Step)** with implicit time-discretization:
  We apply the finite difference or finite volume scheme for the Diffusion equation and deal with the following discretized differential equations:

\[ U_{2}^{n+1} = (I - B)^{-1}U_{2}^{n}, \quad U_{2}(t^n) = U_{1}(t^{n+1}), \]

where we discretize the spatial operator with upwinding methods.

– **C-Step (RHS-Step)**
  We apply the RHS-operator

\[ U_{3}^{n+1} = U_{3}^{n} + \Delta t F_{n+1}^{n+1}, \quad U_{3}(t^n) = U_{2}(t^{n+1}), \]

where we apply the rhs with \( F_{n+1} = (f_{el,x}(t^{n+1}), f_{el,y}(t^{n+1}))^{T} \). We obtain the solution of the Burgers’ equation \( U(t^{n+1}) = U_{3}^{n+1} \). Then, we could go on to the first step.

### 3.2.2 Discretization of the NS equation

We apply the following splitting approach to the Navier-Stokes equation.

We deal with the 2D NS equation, which is given as:

\[ \partial_t u + p_x = -(u^2)_x - (uv)_y + \frac{1}{Re}(u_{xx} + u_{yy}), \]

\[ \partial_t v + p_y = -(uv)_x - (v^2)_y + \frac{1}{Re}(v_{xx} + v_{yy}), \]

\[ u_x + v_y = 0, \]
with initial and boundary conditions.

We rewrite into the operator-notation as:

\[ \partial_t U = A(U) + BU + CP, \]
\[ \nabla \cdot U = 0, \]  
(21)  
(22)

where \( A \) is the nonlinear term (flow-term), \( B \) is the linear viscosity-term and \( P \) is the pressure-term.

We apply the following splitting approach:

- A-Step (Burgers’ Step):
  
  We apply the finite difference or finite volume scheme for the viscous Burgers’ equation and deal with the following semi-discretized differential equations:

  \[ \partial_t U_1 = A(U_1) + BU_1, \quad U_1(t^n) = U(t^n), \]  
  \[ \quad \text{(23)} \]

  where \( A \) is the semi-discretized operator of the nonlinear convection operator of the NS equation and \( B \) is the semi-discretized operator of the diffusion operator of the NS equation.

- B-Step (Correction-Step)
  
  We apply the pressure term in the NS equation, which is solved as following:

  We apply an implicit formulation as:

  \[ \frac{1}{\Delta t}(U_2^{n+1} - U_2^n) = -\nabla P^{n+1}, \quad U_2(t^n) = U_1(t^{n+1}), \]  
  \[ \quad \text{(25)} \]

  where we apply the divergence of the solution and we have \( \nabla \cdot U_2^{n+1} = 0 \).

  Then, we reformulate with an additional derivation into the following Poisson equation:

  \[ -\Delta P^{n+1} = \frac{1}{\Delta t} (\nabla \cdot U_2^n). \]  
  \[ \quad \text{(26)} \]

  Therefore we solve:

  \[ F^n = \nabla \cdot U_2^n, \quad \text{Computation of the right hand side}, \]  
  \[ -\Delta P^{n+1} = F^n, \quad \text{Solving the Poisson’s equation}, \]  
  \[ G^{n+1} = \nabla P^{n+1}, \quad \text{Computation of the pressure-gradient}, \]  
  \[ U_2^{n+1} = U_2^n - \Delta t \cdot G^{n+1}, \quad \text{Correction of the velocity field}, \]  
  \[ \quad \text{(27)-(30)} \]

  where we obtain the solution of the NS equation \( U(t^{n+1}) = U_2^{n+1} \). Then, we could go on to the first step.
4 Numerical Experiments

In the next subsections, we deal with the different test-experiments based on the coupled equations.

We apply the following test- and Simulation-Software, that allows a flexibilisation of the software with respect to the EHD-model:

![Software-flexibilisation diagram]

Fig. 3. Software-flexibilisation.

4.1 Test example 1: Slow and fast velocity fields

In the following, we deal with the modeling-equation:

The modeling equations of the gas-bubble in the liquid is given by a transport equation, which is based on the idea of the volume of fluid and level set methods.

We assume the indicator functions \( u_i(x, t) \) with \( i = 1, \ldots, M \), while \( M \) is the number of bubble-sources.

We define the values of the function as:

\[
\begin{align*}
\forall i = 1, \ldots, M, \\
u_i(x, t) = 1, & \quad \text{gas-phase,} \\
u_i(x, t) = 0, & \quad \text{liquid-phase,} \\
0 < u_i(x, t) < 1, & \quad \text{interface.}
\end{align*}
\]  

Furthermore, we assume the absence of phase changes and then, we deal with the transport equation of the gas-phase as:

\[
\begin{align*}
\frac{\partial u_i}{\partial t} + \nabla (v u_i) &= 0, \quad (x, y, t) \in \Omega \times [0, T], \quad i = 1, \ldots, M, \\
u_i(x, 0) &= u_{i,0}(x), \\
i = 1, \ldots, M,
\end{align*}
\]

where \( v \) is the velocity of the two-phase flow equation, and \( c_i \) are the indicator functions of the gas-phases and \( u_{i,0}(x) \) are the initial conditions.
**Assumption 1** We deal with the following assumption to reduce the computational amount of the far-field model:

- We can assume a constant velocity in the reactor.
- We assume that the water and the gas are not interacting or change the constant velocity.
- We have such small velocities, such that the transport equations of the gas-bubbles are sufficient, see [13].
- We deal with a space and time-dependent velocity: \( \mathbf{v} = \mathbf{v}(x, t) \) in the transport equations (32)-(34), further, we also fulfill the CFL-condition of the explicit scheme, with
  \[
  \Delta t \leq \frac{\Delta x}{\max_{(x, t) \in \Omega \times [0, T]} \{||\mathbf{v}(x, t)||\}}.
  \]
- The concentration of the bubbles are given as \( c_i(x, t) = c_{i,0}(x) u_i(x, t) \), while \( c_{i,0}(x) \) is the initial concentration of the bubble.

We study 2 different examples:

1. Example 1: Two layer problem:
   We have the following velocity field for the bubble-transport, see Figure 4.
   For the low velocity-field we have \( \mathbf{v} = (1, 0)^t \) for the high velocity-field, we have \( \mathbf{v} = (2, 0)^t \), as shown in Fig. 4.

   ![Velocity-field](image)

   **Fig. 4.** Velocity-field to produce thin and large bubbles.

   The results are computed with a MATLAB-code and are given in see Figure 5.

2. Example 2: Multi-layer problem:
   In the following, we deal with five different layers, while we could also extend the number of layers. We have the following velocity field for the bubble-transport, see Figure 6. For the lowest velocity-field we have \( v = (1, 0)^t \), for the low velocity-field, we have \( v = (2, 0)^t \) and for the high velocity-field, we have \( v = (3, 0)^t \).

   The results are computed with the MATLAB-code and are given in see Figure 7.
Fig. 5. Transport of 2 bubbles in a velocity field with different layers (low velocity-field with \( v = 1.0 \) and high velocity field with \( v = 2 \)), the level-set function (left figures) and the contour functions (right figures) are given at at time \( t = 10 \), \( t = 30 \) and \( t = 45 \) (from top to bottom).

### 4.2 Test example 2: Decoupled EHD-model with Burgers’ equation

We apply the following decoupled model, which is applied with \( p = 0 \):

\[
\frac{\partial u}{\partial t} + \rho (u \nabla) u = \mu \nabla^2 u + f_{ef}, \quad x \in [0, 100] \times [0, 400], \quad t \in [0, T_{end}], \tag{35}
\]

\[
u(x, 0) = \begin{cases} 
1.5, & 12.5 \leq x \leq 56.25 \text{ and } 150.0 \leq x \leq 225.0, \\
1.0, & \text{else} \end{cases} \tag{36}
\]

where \( u \) is the 2D-velocity field and \( p \) is the pressure, \( \mu \) is the dynamic viscosity, \( \rho \) is the fluid density, \( f_{ef} \) is the electrostatic force. We apply \( \rho = 1.0 \) and \( \mu = 0.01 \) and \( T_{end} = 40 \).

The forces are given as:

\[
f_{ef} = \nabla \tau_e = \rho_e E_e, \tag{37}
\]

where we have \( E = E_1 + E_2 \),

\[
E_1(x, y) = \begin{cases} 
(sin(4\pi t/T), sin(4\pi t/T))^t, & \text{for } (x, y) \in (0.45, 0.55) \times (0.45, 0.55) \\
0, & \text{else}
\end{cases} \text{ and}
\]

\[
E_2(x, y) = \begin{cases} 
-(sin(4\pi t/T), sin(4\pi t/T))^t, & \text{for } (x, y) \in (0.75, 0.85) \times (0.75, 0.85) \\
0, & \text{else}
\end{cases}
\]
The velocity field is applied in the phase field model, which is given as:

\[
\frac{\partial \tilde{c}_i}{\partial t} + \nabla (u \tilde{c}_i) = 0, \quad (x, y, t) \in \Omega \times [0, T], \quad i = 1, \ldots, M, \tag{38}
\]

\[
\tilde{c}_i(x, 0) = \tilde{c}_{i,0}(x), \quad i = 1, \ldots, M, \tag{39}
\]

where \(u\) is the velocity of the two-phase flow equation, and \(\tilde{c}_i\) are the indicator functions of the gas-phases and \(\tilde{c}_{i,0}(x)\) are the initial conditions. The indicator functions \(\tilde{c}_i(x, t)\) is given as:

\[
\begin{cases}
\tilde{c}_i(x, t) = 1, & \text{gas-phase,} \\
\tilde{c}_i(x, t) = 0, & \text{liquid-phase,} \\
0 < \tilde{c}_i(x, t) < 1, & \text{interface,}
\end{cases} \tag{41}
\]

with \(i = 1, \ldots, M\), while \(M\) is the number of bubble-sources.

Further, we have the CFL-condition of the Burgers’ Equation and the level-set equation for the time-interval \([t^n, t^{n+1}]\) is given as:

- CFL-Condition for the Burgers’ equation:

\[
\Delta t \leq \frac{1}{\max\{|u^n|\} + \frac{2D}{\Delta x}}, \tag{42}
\]

where \(\Delta t\) is the time-step, \(\Delta x = \Delta y\) is the spatial step and \(\max\{|u^n|\}\) is the absolute value of the velocity-field in time-step \(t^n\) and \(D\) is the diffusion parameter.
Fig. 7. Transport of 2 bubbles in a velocity field with 5 different layers (low $v = 1.0$, medium $v = 2$ and high velocity field with $v = 3$), the level-set function (left figures) and the contour functions (right figures) are given at at time $t = 10$, $t = 30$ and $t = 40$ (from top to bottom).

- CFL-Condition for the Level-set equation:

$$\Delta t \leq \frac{\Delta x}{\max \{u^n\}},$$

where $\Delta t$ is the time-step, $\Delta x = \Delta y$ is the spatial step and $\max \{u^n\}$ is the absolute value of the velocity-field in time-step $t^n$.

- EHD-model without E-field, we have $\rho_e = 0.0$:

The velocity field is computed by the Burgers’ equation in advance at the beginning, see the Figures 8 (here without the E-field). Further, we apply the computed velocity field based on the decoupled Burgers’ equation (without the E-field) into the phase field model, which is computed with modified level-set method, see the Figures 8.

- EHD-model with E-field, we have $\rho_e = 0.01$

The velocity field is computed by the Burgers’ equation and is computed in previous at the beginning, while we also apply the maximum amplitude of the E-field, see the Figures 9 (here with the E-field). Further, we apply the computed velocity field based on the decoupled Burgers’ equation (with the E-field) into the phase field model, which is computed with modified level-set method, see the Figures 9.
Fig. 8. Upper figures: Velocity-field of the Burgers’ equation without E-field in contour visualization (upper left figure) and velocity-field of the Burgers’ equation without E-field in vectorial-visualization (upper right figure). Lower Figures: Transport of 2 bubbles in a velocity field with decoupled velocity field computed by the Burgers’ equation without an E-field, the level-set function is given in the lower left figure and the contour functions is given in the lower right figure at time $t = 50$.

4.3 Test example 3: Coupled EHD-model with Burgers’ equation

We apply the following coupled model, which is applied with $p = 0$ and we assume, that the velocity of the Burgers’ equation are in the scales of the Level-set equation, means we also have to apply Burgers’ equation simultaneously to the Level-set equation, means we have $\Delta t_{LS} \approx \Delta_{Burgers}$.

Here, we have a fast process that influences the bubbles, while the velocity field influences the speed of the bubbles in the field, we have to deal with the repeatedly update, see Figure 10.

We have the Burgers’ equation, which is given as:

$$\rho \frac{\partial u}{\partial t} + \rho (u\nabla) u = \mu \nabla^2 u + f_{ef},\, x \in [0,100] \times [0,400],\, t \in [0,T_{end}],$$  \hspace{1cm} (44)

$$u(x,0) = \begin{cases} 1.5, & 12.5 \leq x \leq 56.25 \text{ and } 150.0 \leq x \leq 225.0, \\ 1.0, & \text{else} \end{cases}$$  \hspace{1cm} (45)

where $u$ is the 2D-velocity field and $p$ is the pressure, $\mu$ is the dynamic viscosity, $\rho$ is the fluid density, $f_{ef}$ is the electrostatic force. We apply $\rho = 1.0$ and $\mu = 0.01$ and $T_{end} = 40$.

The forces are given as:

$$f_{ef} = \nabla \tau_e = \rho_e E,$$  \hspace{1cm} (46)
Fig. 9. Upper figures: Decoupled velocity-field of the Burgers’ equation with weak E-field, which is computed in a maximum of the amplitudes (contour visualization, left figure), right figure: Velocity-field of the Burgers’ equation with weak E-field (vectorial-visualization, right figure). Lower Figures: Transport of 2 bubbles in a velocity field with decoupled velocity field computed by the Burgers’ equation with an E-field, the level-set function is given in the lower left figure and the contour functions is given in the lower right figure at time $t = 50$.

where we have $E = E_1 + E_2$

$E_1(x, y) = \begin{cases} 
(|\sin(2\pi t)|, |\sin(2\pi t)|)^t, & \text{for } (x, y) \in (0.45, 0.55) \times (0.45, 0.55) \\
0 & \text{else}
\end{cases}$

$E_2(x, y) = \begin{cases} 
(|\sin(2\pi t)|, |\sin(2\pi t)|)^t, & \text{for } (x, y) \in (0.75, 0.85) \times (0.75, 0.85) \\
0 & \text{else}
\end{cases}$

we assume only positive E-fields.

Further, we have the CFL-condition of the Burgers’ Equation and the level-set equation for the time-interval $[t^n, t^{n+1}]$ is given as:

- CFL-Condition for the Burgers’ equation:

$$\Delta t \leq \frac{1}{\frac{\max\{u^n\}}{\Delta x} + \frac{2D}{\Delta x}}, \quad (47)$$

where $\Delta t$ is the time-step, $\Delta x = \Delta y$ is the spatial step and $\max\{u^n\}$ is the absolute value of the velocity-field in time-step $t^n$ and $D$ is the diffusion parameter.

- CFL-Condition for the Burgers’ equation:

$$\Delta t \leq \frac{1}{\frac{\max\{u^n\}}{\Delta x} + \frac{2D}{\Delta x}}, \quad (48)$$
where $\Delta t$ is the time-step, $\Delta x = \Delta y$ is the spatial step and $\max\{u^n\}$ is the absolute value of the velocity-field in time-step $t^n$ and $D$ is the diffusion parameter.

The velocity field is applied in the phase field model, which is given as:

$$\frac{\partial \tilde{c}_i}{\partial t} + \nabla (u \tilde{c}_i) = 0, \ (x, y, t) \in \Omega \times [0, T], \ i = 1, \ldots, M, \ (49)$$
$$\tilde{c}_i(x, 0) = \tilde{c}_{i,0}(x), \ (50)$$
$$i = 1, \ldots, M, \ (51)$$

where $u$ is the velocity of the two-phase flow equation, and $\tilde{c}_i$ are the indicator functions of the gas-phases and $\tilde{c}_{i,0}(x)$ are the initial conditions. The indicator functions $\tilde{c}_i(x, t)$ is given as:

$$\begin{cases} 
\tilde{c}_i(x, t) = 1, & \text{gas-phase,} \\
\tilde{c}_i(x, t) = 0, & \text{liquid-phase,} \\
0 < \tilde{c}_i(x, t) < 1, & \text{interface,}
\end{cases} \ (52)$$

with $i = 1, \ldots, M$, while $M$ is the number of bubble-sources.

Further, we have the CFL-condition of the Burgers’ Equation and the level-set equation for the time-interval $[t^n, t^{n+1}]$ is given as:

- CFL-Condition for the Burgers’ equation:

$$\Delta t \leq \frac{1}{\max\{u^n\}} + \frac{2D}{\Delta x}, \ (53)$$

where $\Delta t$ is the time-step, $\Delta x = \Delta y$ is the spatial step and $\max\{u^n\}$ is the absolute value of the velocity-field in time-step $t^n$ and $D$ is the diffusion parameter.
CFL-Condition for the Level-set equation:

\[ \Delta t \leq \frac{\Delta x}{\max \{u^n\}} \]  \hspace{1em} (54)

where \( \Delta t \) is the time-step, \( \Delta x = \Delta y \) is the spatial step and \( \max \{u^n\} \) is the absolute value of the velocity-field in time-step \( t^n \).

EHD-model without E-field, we have \( \rho_e = 0.0 \):

The velocity field is computed by the Burgers’ equation, see the Figures 11 (here without the E-field). Further, we apply the computed velocity field based on the Burgers’ equation (without the E-field) into the phase field model, which is computed with modified level-set method, see the Figures 11.

Fig. 11. Upper figures: Coupled velocity-field of the Burgers’ equation without E-field in contour visualization (upper left figure) and velocity-field of the Burgers’ equation without E-field in vectorial-visualization (upper right figure). Lower Figures: Transport of 2 bubbles in a velocity field with coupled velocity field computed by the Burgers’ equation without an E-field, the level-set function is given in the left lower figure and the contour functions is given in the lower right figure at time \( t = 50 \).

EHD-model with E-field, we have \( \rho_e = 0.01 \):

The velocity field is computed by the Burgers’ equation, see the Figures 12 (here with the E-field). Further, we apply the computed velocity field based on the Burgers’ equation (with the E-field) into the phase field model, which is computed with modified level-set method, see the Figures 12.
Fig. 12. Upper figures: Coupled velocity-field of the Burgers’ equation with weak E-field, which is computed by each time-step, in contour visualization (upper left figure) and velocity-field of the Burgers’ equation with weak E-field in vectorial-visualization (upper right figure). Lower Figures: Transport of 2 bubbles in a velocity field with coupled velocity-field computed by the Burgers’ equation with an E-field, the level-set function is given in the lower left figure and the contour functions is given in the lower right figure at time $t = 50$.

5 Conclusion

In this paper, we discussed coupled models, which are based on electro-hydrodynamics equations and transport models, which are based on phase-field equations. The models are coupled via splitting approaches and we could present decoupled and coupled versions of the electrohydrodynamical model with the phase field model. We obtained numerical results, while we have to be careful with the different scales of the bubble-oscillations in the E-field and the flow-scale if the hydrodynamical equation. We solved such problems with time- and space-step controls. First numerical results are presented with the coupled EHD and phase-field models, here we could see the influence of the different and changing velocities, due to the E-Field on the bubbles.

References

1. T. Adams, S. Giani, and W.M. Coombs. A high-order elliptic PDE based level set reinitialisation method using a discontinuous Galerkin discretisation. Journal of Computational Physics, 379:373-391, 2019.

2. N. Behera, S. Mandal and S. Chakraborty. Electrohydrodynamic settling of drop in uniform electric field: beyond Stokes flow regime. Journal of Fluid Mechanics, Cambridge University Press, 881:498-523, 2019.
3. J. Geiser. *Iterative Splitting Methods for Differential Equations*. Numerical Analysis and Scientific Computing Series, Taylor & Francis Group, Boca Raton, London, New York, 2011.

4. J. Geiser. *Multicomponent and Multiscale Systems: Theory, Methods, and Applications in Engineering*. Springer, Cham, Heidelberg, New York, Dordrecht, London, 2016.

5. J. Geiser and P. Mertin. *Modelling approach of a near-far-field model for bubble formation and transport*. arXiv:1809.00899 (Preprint arXiv:1809.00899), September 2018.

6. J. Geiser. *Multiscale Modelling and Splitting Approaches for Fluids composed of Coulomb-interacting Particles*. Special-Issue: Problems with Multiple Time-scales Mathematical, Journal: Mathematical and Computer Modelling of Dynamical Systems, edited by J.Geiser, Taylor and Francis, Abingdon, UK, accepted January 2018.

7. H. Gu *, M.H.G. Duits and F. Mugele. *Droplets Formation and Merging in Two-Phase Flow Microfluidics*. Int. J. Mol. Sci., 12:2572-2597, 2011.

8. T. Hayashi, S. Uehara, H. Takana and H. Nishiyama. *Gas-liquid two-phase chemical reaction model of reactive plasma inside a bubble for water treatment*. Proceeding of the 22nd International Symposium on Plasma Chemistry July 5-10, Antwerp, Belgium, 2015.

9. R. Kumar and N.R. Kuloor. *The formation of bubbles and drops*. Advances in Chemical Engineering, 8:255-368, 1970.

10. A. Obeidat and S.P.A. Bordas. *An implicit boundary approach for viscous compressible high Reynolds flows using a hybrid remeshed particle hydrodynamics method*. Journal of Computational Physics, 386: 1-17, 2019.

11. J.A. Simmons and J.E. Sprittles and Y.D. Shikhmurzaev. *The formation of a bubble from a submerged orifice*. European Journal of Mechanics - B/Fluids, 53, Supplement C, 24-36, 2015.

12. M. Sussman and M. Ohta. *A Stable and Efficient Method for Treating Surface Tension in Incompressible Two-Phase Flow*. SIAM J. Sci. Comput., 31(4), 2447-2471, 2009.

13. Y.-Y. Tsui, C.-Y. Liu and S.-W. Lin. *Coupled level-set and volume-of-fluid method for two-phase flow calculations*. Numerical Heat Transfer, Part B: Fundamentals, 71(2): 173-185, 2017.

14. G.Q. Yang, B. Du, and L.S. Fan. *Bubble formation and dynamics in gas-liquid-solid fluidization - A review*. Chemical Engineering Science 62, 2-27, 2007.