Conformally covariant boundary correlation functions with a quantum group

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Abstract. Particular boundary correlation functions of conformal field theory are needed to answer some questions related to random conformally invariant curves known as Schramm-Loewner evolutions (SLE). In this article, we introduce a correspondence and establish its fundamental properties, which are used in the companion articles [JJK13, KP14] for explicitly solving two such problems. The correspondence associates Coulomb gas type integrals to vectors in a tensor product representation of a quantum group, a $q$-deformation of the Lie algebra $\mathfrak{sl}(2; \mathbb{C})$. We show that desired properties of the functions are guaranteed by natural representation theoretical properties of the vectors.

1. Introduction

Boundary correlation functions in conformal field theories in general, and, more particularly, questions about random conformally invariant curves, frequently lead to quite similar systems of partial differential equations whose boundary conditions are specified in terms of the asymptotic behavior of solutions. The main result of this article provides a systematic method to construct explicit solutions. The method is a form of the so called “hidden quantum group symmetry of conformal field theories” [MR89, GS90, RRRA91, FW91, FFK89, BMP90, Var92] (see also the textbooks [GRAS96, Fuc92, Var95]). In this article, we establish properties which are directly relevant for solving the PDE problems arising in the theory of Schramm-Loewner evolutions (SLE). In two companion articles, the results are applied to find explicit answers to the questions of boundary visit probabilities of chordal SLEs [JJK13] and the pure geometries of multiple SLEs [KP14].

The method we introduce relies on the representation theory of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ in the generic, semisimple case (for the precise definitions, see Section 2). Informally, $\mathcal{U}_q(\mathfrak{sl}_2)$ is a deformation of the Lie algebra $\mathfrak{sl}_2$ of traceless complex $2 \times 2$-matrices, with a complex deformation parameter $q$ that we assume not to be a root of unity or zero. As an algebra, $\mathcal{U}_q(\mathfrak{sl}_2)$ is generated by an invertible Cartan element $K$, and raising and lowering operators $E$ and $F$, which shift the eigenvalues of $K$ by multiplicative factors $q^2$ and $q^{-2}$, respectively. The algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ has, for all positive integers $d$,
an irreducible representation \( M_d \) of dimension \( d \), which \( q \)-deforms the corresponding irreducible of \( \mathfrak{sl}_2 \). Tensor products of these representations are defined by equipping \( \mathcal{U}_q(\mathfrak{sl}_2) \) with a Hopf algebra structure, and they decompose into direct sums of irreducible subrepresentations. We call any direct sum of copies of the one-dimensional representation \( M_1 \) a trivial representation.

In the applications of our method, the conformally covariant boundary correlation functions are defined on the chamber

\[
\mathfrak{X}_n = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \left| x_1 < \cdots < x_n \right. \right\}.
\]

We form a tensor product of \( n \) irreducible representations of the quantum group, and to its vectors we associate certain functions. In general terms, we want to show that desired properties of the functions follow from representation theoretical properties of the vectors. Most importantly, we will show the following. Functions \( \mathcal{F}[v] \) associated to highest weight vectors \( v \) are well defined on the above chamber domain \( \mathfrak{X}_n \), and they satisfy a system of partial differential equations arising in conformal field theory. Asymptotics of the functions can be read off from projections to subrepresentations. Homogeneity degree of the function is related to the eigenvalue of the Cartan element \( K \), and for vectors in the trivial subrepresentation, the associated function is covariant under all Möbius transformations.

1.1. Informal statement of the main result. We now outline more concretely the main result of this article, whose precise statement will be given in Theorems 4.16 and 4.17 in Section 4.8 once all relevant notation and conventions have been introduced. Examples of its applications are discussed in Section 1.2.

At intermediate steps, we need an auxiliary anchor point \( x_0 \), and we use the restricted chamber

\[
\mathfrak{X}_n^{(x_0)} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \left| x_0 < x_1 < \cdots < x_n \right. \right\}.
\]

Fix \( n \) irreducible representations \( M_{d_1}, \ldots, M_{d_n} \). To all vectors

\[
v \in \bigotimes_{i=1}^n M_{d_i} = M_{d_n} \otimes \cdots \otimes M_{d_1}\]

we linearly associate functions

\[
\mathcal{F}^{(x_0)}[v] : \mathfrak{X}^{(x_0)}_n \to \mathbb{C}
\]

defined by certain integrals, by a rule that will be detailed in Section 4.1. When the raising operator annihilates \( v \), i.e. \( E.v = 0 \), we call \( v \) a highest weight vector. For such vectors, the function \( \mathcal{F}^{(x_0)}[v] \) will be shown to be independent of \( x_0 \), and it thus gives rise to a well defined function on the chamber \( \mathfrak{X}_n \), denoted simply by \( \mathcal{F}[v] \).

We refer to the above mappings \( \mathcal{F}^{(x_0)} \) as “the spin chain - Coulomb gas correspondence”, since they map the state spaces \( \bigotimes_{i=1}^n M_{d_i} \) of finite quantum spin chains to spaces of screened correlation functions in the Coulomb gas formalism of conformal field theory. Such screened correlation functions have been studied since [DF84].

The content of our main theorem of this article — stated informally below — is that under the spin chain - Coulomb gas correspondence, natural representation theoretical properties of the vectors ensure desired properties of the functions.

**Theorem** (Theorems 4.16 and 4.17 in Section 4). For the mappings \( v \mapsto \mathcal{F}^{(x_0)}[v] \) defined in Section 4.7 the properties of the vector \( v \in \bigotimes_{i=1}^n M_{d_i} \) ensure properties of the function \( \mathcal{F}^{(x_0)}[v] \) as follows:

(PDE): If \( v \) is a highest weight vector, then the function \( \mathcal{F}[v] : \mathfrak{X}_n \to \mathbb{C} \) satisfies \( n \) linear homogeneous partial differential equations of orders given by the dimensions \( d_1, \ldots, d_n \) of the factors of the tensor product.

(COV): The functions have the following covariance properties:

- The function \( \mathcal{F}[v] \) (resp. \( \mathcal{F}^{(x_0)}[v] \)) is translation invariant.
We make a few observations about the interpretation of the above result, and comparisons to related research.

- The functions \( \mathcal{F}(x_0)[v] \) are given by integrals over auxiliary screening variables (see Sections 3.3 and 4.1), and slightly informally, the quantum group can be thought of as acting on the integration surfaces: the generator \( F \) increases the dimension of the integration surface (i.e., the number of screening variables), and the generator \( E \) decreases it.
- A precise version of the quantum group action on integration surfaces has been given by Felder and Wieczerkowski, who define an action of \( U_q(\mathfrak{sl}_2) \) on a suitable homology theory \([FW91]\). Our representation of the quantum group could be obtained from this homology via a degenerate evaluation. The evaluation in particular renders the infinite dimensional Verma modules in the work of Felder and Wieczerkowski into just the finite dimensional irreducible representations \( M_d \).
- The special properties of the highest weight vectors can be seen to arise from the closedness of the associated integration surface — a version of Stokes formula can be used in these cases without boundary terms (Lemma 4.7 and Corollary 4.8).
- The most common way to obtain Möbius covariance in the Coulomb gas formalism of conformal field theory is to ensure a charge neutrality, which takes into account a background charge. The integrals associated to vectors in the trivial subrepresentation do not satisfy this simplest charge neutrality, but rather fall short of it precisely by the amount of the background charge. Our full Möbius covariance statement for these vectors requires a little more work (Proposition 4.15).
- In the presentation of this article, we have opted for straightforwardness and self-containedness.

We illustrate the use of the correspondence and our main result by briefly describing two applications, which both arise from the theory of SLEs.

SLEs are random curves in planar domains that were introduced by Schramm \([Sch00]\) as candidates of scaling limits of interfaces in statistical mechanics models at criticality. SLEs are constructed by a growth process of the curve, encoded in a Loewner chain, in such a way as to ensure the fundamental properties of conformal invariance (of the law associated to different domains) and domain Markov property (which describes the conditional law of the continuation of the curve, given a segment of it). In all SLE variants, a single parameter \( \kappa > 0 \) captures some of the most important properties of the curve — in physical terms \( \kappa \) determines the universality class of the underlying statistical mechanics.
model. In our applications, $\kappa$ determines the deformation parameter of the quantum group according to $q = e^{i\pi/\kappa}$.

The simplest setup of SLEs concerns curves living in a simply connected domain, starting from one marked boundary point and ending at another. A classification result sometimes referred to as “Schramm’s principle” states that such random non-self-traversing curves with domain Markov property and conformally invariant laws are uniquely characterized by the parameter $\kappa$. The term chordal SLE$_\kappa$ is used for these random curves. Chordal SLE$_\kappa$, with particular values of $\kappa$, are known to be the scaling limits of interfaces in the presence of the simplest (Dobrushin) boundary conditions in various critical models of statistical mechanics, see e.g. [Smi01, CN07, LSW04, Zha08, CDCH+13].

1.2.1. Application to multiple SLEs. Multiple SLEs arise from trying to generalize Schramm’s principle to cases where several interfaces are present: in a simply connected domain with $2N$ marked boundary points we may have $N$ curves connecting pairwise the marked points. Such processes have been studied in [Dub07, BBK05, Gra07, KL07], and in some cases they are known to be the scaling limits of lattice model interfaces in the presence of alternating boundary conditions [CS12, Izy11].

Multiple random curves with conformally invariant laws and domain Markov properties are no longer specified by the parameter $\kappa$ alone, but for a fixed $\kappa < 8$, a finite dimensional convex set of possible laws exists. The extremal points of the set of multiple SLE laws were called pure geometries in [BBK05]. Coulomb gas integrals for this problem were considered in [Dub06, Kyt07]. The problem is very closely related to crossing probabilities, for which a solution using Coulomb gas integrals has been recently obtained in the series of articles [FK12, FK13, FK14a, FK14b]. In comparison with the approaches of [Dub06, FK12], the Coulomb gas integrals in our approach have the advantage of treating all the obtained in the series of articles [FK12, FK13, FK14a, FK14b].

An explicit Loewner chain construction of a multiple SLE uses a partition function [Dub07, BBK05], which is most convenient to write down for the reference domain of the upper half plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \Im(z) > 0 \}$ as

$$Z(x_1, \ldots, x_{2N})$$

where the marked points on the real line $x_i \in \mathbb{R} = \partial \mathbb{H}$, $i = 1, \ldots, 2N$, are ordered as

$$x_1 < \cdots < x_{2N}.$$  

A stochastic reparametrization invariance of the random curve requires the PDEs

$$\frac{\kappa}{2} \frac{\partial^2}{\partial x_i^2} + \sum_{j \neq i} \left( \frac{2}{x_j - x_i} \frac{\partial}{\partial x_j} - \frac{2h}{(x_j - x_i)^2} \right) Z(x_1, \ldots, x_{2N}) = 0 \quad \text{for all } i = 1, \ldots, 2N,$$

where $h = \frac{\kappa - 2}{\kappa}$, see [Dub07]. Moreover, conformal invariance of the law of the random curves requires the following Möbius covariance:

$$Z(x_1, \ldots, x_{2N}) = \prod_{i=1}^{2N} \mu'(x_i)^h \times Z(\mu(x_1), \ldots, \mu(x_{2N}))$$

for any conformal map $\mu$ of $\mathbb{H}$ onto itself, which preserves the order of the marked points.

The pure geometries of multiple SLEs are labeled by the planar pair partitions $\alpha$ of $2N$ points, and the problem is to find the corresponding partition functions $Z_\alpha$. The requirements of covariance and PDEs are the same for all pure geometries $\alpha$, but the boundary conditions depend on $\alpha$. By comparison with Bessel processes, the asymptotic behavior of $Z_\alpha$ as $|x_{j+1} - x_j| \to 0$ should depend on whether the points indexed $j$ and $j + 1$ form a pair of $\alpha$ or not, see [BBK05]. More precisely, when $x_{j-1} < \xi < x_{j+2}$,

$$\lim_{x_j, x_{j+1} \to \xi} \frac{Z_\alpha(x_1, \ldots, x_{2N})}{|x_{j+1} - x_j|^\Delta} = \begin{cases} 0 & \text{if } [j, j + 1] \not\in \alpha \\ Z_\alpha(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N}) & \text{if } [j, j + 1] \in \alpha \end{cases}$$
where $\Delta = -2h = \frac{z-6}{\kappa}$, and $\hat{\alpha}$ is the planar pair partition of $2N-2$ points obtained from $\alpha$ by removing the pair $[j, j + 1]$.

To apply the correspondence of the present article to multiple SLEs, one forms the $2N$-fold tensor product of two-dimensional irreducibles $M_2$ of the quantum group. The trivial subrepresentation in this tensor product, consisting of vectors $v \in M_2^{\otimes 2N}$ such that $E.v = 0$ and $K.v = v$, is of dimension equal to the Catalan number $C_N = \frac{1}{N+1} \binom{2N}{N}$, which coincides with the number of planar pair partitions of $2N$ points. One then wants to judiciously choose in this subrepresentation

$$v_\alpha \in M_2^{\otimes 2N},$$

indexed by the the planar pair partitions $\alpha$, so that $Z_\alpha = F[v_\alpha]$ will be the desired multiple SLE partition functions of the pure geometry $\alpha$.

For vectors $v_\alpha \in M_2^{\otimes 2N}$, the $2N$ differential equations of order two, guaranteed by the (PDE)-part of our main theorem, turn out to be exactly the equations (1.3) needed for the reparametrization invariance of the multiple SLE. Moreover, the full Möbius covariance guaranteed by the (COV)-part turns out to be exactly (1.4), as we wanted. The main task is then to choose $v_\alpha$ in such a way that the boundary conditions (1.5) are satisfied.

The asymptotics property (ASY) refers to the decomposition of a tensor product of two representations into irreducible subrepresentations, which in this case simply reads

$$M_2 \otimes M_2 \cong M_1 \oplus M_3.$$

For this case, the possible exponent values appearing in the statement (ASY) are $\Delta_{1,2} = -2h = \frac{z-6}{\kappa}$ and $\Delta_{3,3} = \frac{2}{\kappa} > \Delta_{1,2}$. Letting $\pi_{j,j+1}$ denote the projection determined by picking the one-dimensional irreducible in the direct sum decomposition of the tensor product of the $j$:th and $j+1$:st factors, Equation (1.5) will be guaranteed if we have

$$\pi_{j,j+1}(v_\alpha) = \begin{cases} 0 & \text{if } [j, j + 1] \notin \alpha \\ v_\alpha & \text{if } [j, j + 1] \in \alpha \end{cases},$$

where $v_\alpha \in M_2^{\otimes (2N-2)}$ is the vector corresponding to the planar pair partition $\hat{\alpha}$ of $2N-2$ points obtained from $\alpha$ by removing the pair $[j, j + 1]$.

In KPT1, this problem will be analyzed in detail, and in particular, it will be shown that there is a unique collection of vectors $v_\alpha$ satisfying the above requirements, up to an overall normalization. The construction of the multiple SLE partition functions for the pure geometries thus crucially relies on the results of the present article.

1.2.2. Boundary visit probabilities for chordal SLE. For concrete probabilistic information about the SLE random curves, it is natural to study the probabilities for SLE curves to visit small neighborhoods of given points. In fact, probabilities to visit infinitesimal neighborhoods can be appropriately renormalized to obtain finite amplitudes known as SLE Green’s functions, see LST1, AKLT2, LZ13, LW13. A second application of our method, considered in JK13, concerns the order-refined multi-point boundary Green’s function of the chordal SLE, i.e. the probability amplitude for visits to several boundary points in a prescribed order.

For simplicity, we may fix the domain to be the upper half plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \Im(z) > 0 \}$, the starting point $x \in \mathbb{R}$ and the end point at $\infty$. We order the points to be visited on the real line $\mathbb{R} = \partial \mathbb{H}$, and label them by a superscript $-$ or $+$, according to whether they are on the left or right of the starting point $x$, and thus denote

$$y_L < \cdots < y_{2-} < y_1^- < x < y_1^+ < y_2^+ < \cdots < y_R^+,$$
where $L$ and $R$ are the number of points to be visited on the left and right, respectively. With a given order $\omega$ of visits, a suitably renormalized probability of visits

$$\zeta_\omega(y_L, \ldots, y_1^\pm, x, y_1^\mp, \ldots, y_R^\pm)$$

can then be defined, see \cite{JJK13} for details. This function is by construction translation invariant and homogeneous. It can moreover be argued to satisfy linear homogeneous partial differential equations: a second order equation, and $L + R$ third order equations. These PDEs themselves do not depend on the order $\omega$ of visits, but the boundary conditions do. When two of the arguments are close to each other, the probabilities of visits are asymptotic to similar ones with one point removed from the list of visits. Here we content ourselves to noting that these conditions amount to specifying the asymptotic behavior of $\zeta_\omega$ on a codimension one boundary of its domain of definition: they concern the cases when either $|y^\pm_{i+1} - y_i^\pm| \to 0$ or $|y_i^\pm - x| \to 0$. For the full list of equations and boundary conditions for this chordal SLE boundary visit question, we refer to \cite{JJK13}.

To apply the correspondence of the present article to these boundary visit probabilities of chordal SLE, one considers the tensor product representation

$$M_3^\otimes R \otimes M_2 \otimes M_3^\otimes L.$$ 

Again the task is to judiciously choose vectors $v_\omega$ in it, such that the multi-point boundary Green’s function corresponding to the visit order $\omega$ can be obtained in the form $\zeta_\omega = F[v_\omega]$. The highest weight vector condition $E.v_\omega = 0$ guarantees by the property (PDE) the desired second and third order partial differential equations, and the Cartan generator eigenvalue equation $K.v_\omega = q.v_\omega$ guarantees by the property (COV) the correct homogeneity degree in addition to translation invariance. The most nontrivial part is to satisfy the boundary conditions when either $|y^\pm_{i+1} - y_i^\pm| \to 0$ or $|y_i^\pm - x| \to 0$. The property (ASY) is again suitable for this purpose: the decomposition $M_3 \otimes M_3 \cong M_1 \oplus M_3 \oplus M_3$ applies to the case $|y^\pm_{i+1} - y_i^\pm| \to 0$, and the decompositions $M_3 \otimes M_2 \cong M_2 \oplus M_4$ and $M_2 \otimes M_3 \cong M_2 \oplus M_4$ apply respectively to the cases $|y_1^\pm - x| \to 0$ and $|x - y_1^\mp| \to 0$. SLE boundary visits are treated in more detail in \cite{JJK13}, where it is in particular observed that the requirements for the vectors $v_\omega$ uniquely specify them. Again, the main results of the present article are instrumental for finding the explicit formulas for these order-refined multi-point boundary visit probabilities of the chordal SLE.

### 1.3. Organization of this article.
In Section 2, we introduce notation, fix conventions, and prove auxiliary results about $q$-combinatorics, the quantum group $U_q(\mathfrak{sl}_2)$, its representations, and their tensor products. The main part of this article is Sections 3 and 4. We have divided the material as follows. Section 3 contains definitions and properties of functions that are used in defining and studying the correspondence. This part does not use the quantum group in any way — only some $q$-combinatorial lemmas are used. Section 4 starts with the definition of the correspondence, and proceeds with the proofs of the properties stated in the main theorems in the order that we have found the most straightforward. It concludes with the precise statement of the main result, Theorems 4.16 and 4.17. Finally, in Section 5 we treat two further properties: a generalization of the asymptotics statement and a hidden manifestation of the periodicity of the domain boundary.

**Acknowledgments.** We thank Denis Bernard, Steven Flores, Philippe Di Francesco, Christian Hagendorf, Clément Hongler, Kostya Izyurov, Niko Jokela, Matti Järvinen, Peter Kleban, and Jake Simmons for useful comments and interesting discussions.

This work was supported by the Academy of Finland grant “Conformally invariant random geometry and representations of infinite dimensional Lie algebras”. E.P. is supported by the Finnish National Doctoral Programme in Mathematics and its Applications.
The main purpose of this section is to fix notation and conventions about the quantum group $U_q(sl_2)$. We also include auxiliary results of $q$-combinatorial flavor, which are needed later on in the article.

### 2.1. Q-numbers and some combinatorial formulas

Let $q \in \mathbb{C} \setminus \{0\}$, and assume further that $q$ is not a root of unity, i.e. $q^m \neq 1$ for all $m \in \mathbb{Z} \setminus \{0\}$. Define, for $m \in \mathbb{Z}$ and for $n, k \in \mathbb{N}$, $0 \leq k \leq n$, the $q$-integers as

\[
[m] = \frac{q^m - q^{-m}}{q - q^{-1}} = q^{m-1} + q^{m-3} + \ldots + q^{3-m} + q^{1-m}
\]

the $q$-factorials as

\[
[n]! = \prod_{m=1}^{n} [m],
\]

and the $q$-binomial coefficients as

\[
\begin{bmatrix} n \atop k \end{bmatrix} = \frac{[n]!}{[k]! [n-k]!}.
\]

The following "$q$-combinatorial formulas" will be used in later calculations.

**Lemma 2.1.**

(a): We have

\[
[q^{d-\ell}] [\ell] = \frac{1}{q - q^{-1}} \sum_{u=0}^{\ell-1} (q^{d-2u} - q^{-d+2u})
\]

(b): For a permutation $\sigma \in S_n$ of $\{1, \ldots, n\}$, denote the set of inversions of $\sigma$ by

\[
\text{inv}(\sigma) = \{(i, j) \mid i < j \text{ and } \sigma(i) > \sigma(j)\}.
\]

Then we have

\[
\sum_{\sigma \in S_n} q^{-2\#\text{inv}(\sigma)} = q^{-\binom{n}{2} [n]!}.
\]

(c): We have

\[
\sum_{1 \leq r_1 < r_2 < \ldots < r_k \leq n} q^{-2 \sum_{j=1}^{r_j} (r_j - j)} = q^{-k(n-k)} \begin{bmatrix} n \atop k \end{bmatrix}.
\]

(d): We have, for any $\beta$ and $n \in \mathbb{N}$

\[
\sum_{m=0}^{n} \begin{bmatrix} n \atop m \end{bmatrix} (-1)^m q^{m\beta} = q^{\frac{1}{2} n \beta} \prod_{s=0}^{n-1} \left( q^{\frac{1}{2} (n-1-\beta) - s} - q^{\frac{1}{2} (\beta+1-n) + s} \right).
\]

**Proof.** For part (a), use the definition $[d-\ell] = \frac{q^{d-\ell} - q^{d-\ell+2}}{q - q^{-1}}$ and the finite geometric series $[\ell] = q^{\ell-1} + q^{\ell-3} + \ldots + q^{-\ell+1}$ to get the asserted formula

\[
(q - q^{-1}) [\ell] [d-\ell] = \frac{q^{d-\ell} - q^{d-\ell+2}}{q - q^{-1}} \left( q^{\ell-1} + q^{\ell-3} + \ldots + q^{-\ell+1} \right)
\]

\[
= \left( q^{\ell-1} + q^{\ell-3} + \ldots + q^{2\ell+1} \right) - \left( q^{2\ell-1-d} + q^{2\ell-3+d} + \ldots + q^{-d+1} \right).
\]
For part (b), use the same finite geometric series to rewrite the right hand side as

\[ q^{-\binom{n}{2}} [n!] = \prod_{k=1}^{n} \left( \sum_{l=0}^{k-1} q^{-2l} \right). \]

The inversions \((i, j) \in \text{inv}(\sigma)\) of \(\sigma \in S_n\) can be grouped according to the smaller index \(i\): we have

\[ \#\text{inv}(\sigma) = \sum_{i=1}^{n} \#\text{inv}_i(\sigma), \quad \text{where} \]

\[ \text{inv}_i(\sigma) = \{ j \mid j > i \text{ and } \sigma(j) < \sigma(i) \}. \]

In fact, permutations \(\sigma\) are in bijection with the sequences \((\#\text{inv}_i(\sigma))_{i=1}^{n}\). Then, in the expansion of the product \(\prod_{k=1}^{n} (\sum_{l=0}^{k-1} q^{-2l})\) the choice of the term in the \(k\)-th factor can be thought of as corresponding to the choice of \(\#\text{inv}_{n+1-k}(\sigma)\). The product is thus expanded as a sum over permutations \(\sigma\), with coefficients \(\prod_{i=1}^{n} q^{-2\#\text{inv}_i(\sigma)} = q^{-2\#\text{inv}(\sigma)}\).

The proofs of (c) and (d) are both based on \(q\)-Pascal triangles. Let \(L_k^{(n)}\) and \(R_k^{(n)}\) denote the left hand side and right hand side of assertion (c), respectively. Splitting the sum defining \(L_k^{(n)}\) according to whether \(r_k = n\) or \(r_k < n\), we obtain a Pascal triangle type recursion

\[ L_k^{(n)} = q^{-2(n-k)} L_{k-1}^{(n-1)} + L_k^{(n-1)}. \]

A straightforward calculation using the definition \((2.1)\) of \(q\)-integers shows that also \(R_k^{(n)} = q^{-2(n-k)} R_{k-1}^{(n-1)} + R_k^{(n-1)}\). The equality \(L_k^{(n)} = R_k^{(n)}\) follows from this recursion, together with the initial observation \(L_k^{(1)} = R_k^{(1)}\).

For the proof of (d), proceed by induction on \(n\). For \(n = 0\) both sides are equal to 1. Let \(R^{(n)}(\beta)\) denote the right hand side, and note that we can write

\[ R^{(n+1)}(\beta) = (1 - q^{n+\beta}) \times R^{(n)}(\beta - 1). \]

Assuming the asserted formula for \(R^{(n)}(\beta - 1)\), we expand in powers of \(q^{\beta}\) as follows

\[ R^{(n+1)}(\beta) = \sum_{m} (-1)^{m} q^{m\beta} \left( q^{-m} \left[ \begin{array}{c} n \\ m \end{array} \right] + q^{1+n-m} \left[ \begin{array}{c} n \\ m-1 \end{array} \right] \right). \]

It now suffices to apply the following simple identity

\[ q^{-m} \left[ \begin{array}{c} n \\ m \end{array} \right] + q^{1+n-m} \left[ \begin{array}{c} n \\ m-1 \end{array} \right] = \left[ \begin{array}{c} n+1 \\ m \end{array} \right]. \]

\[ \square \]

2.2. The quantum group and its representations. We now give a definition of the quantum group \(U_q(\mathfrak{sl}_2)\) by generators and relations. We also concretely describe the irreducible representations \(M_d\), and record needed results about the decompositions of their tensor products.

2.2.1. Definition of the quantum group. The quantum group \(U_q(\mathfrak{sl}_2)\) is the (associative unital) algebra over \(\mathbb{C}\) generated by \(E, F, K, K^{-1}\) subject to the relations

\[ KK^{-1} = 1 = K^{-1}K, \quad KE = q^2EK, \quadKF = q^{-2}FK, \quad EF - FE = \frac{1}{q - q^{-1}} (K - K^{-1}). \]

There is a unique Hopf algebra structure on \(U_q(\mathfrak{sl}_2)\) with the coproduct, an algebra homomorphism

\[ \Delta: U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2), \]
given on the generators by the expressions

\[(2.6)\]
\[
\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F.
\]

With the coproduct, we make the tensor product of two representations \(M'\) and \(M''\) again a representation. The action of \(U_q(\mathfrak{sl}_2)\) on \(M' \otimes M''\) is defined so that if

\[
\Delta(X) = \sum_i X'_i \otimes X''_i \in U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)
\]

and \(v' \in M', v'' \in M''\), then

\[
X.(v' \otimes v'') = \sum_i (X'_i.v') \otimes (X''_i.v'') \in M' \otimes M''.
\]

Note that we generally cannot canonically identify \(M' \otimes M''\) with \(M'' \otimes M'\) as representations, because the coproduct \(\Delta\) is not cocommutative. However, the coproduct is coassociative, that is \((\text{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}) \circ \Delta\), and therefore the canonical identification \((M' \otimes M'') \otimes M''' \cong M' \otimes (M'' \otimes M''')\) is an isomorphism of representations. More generally, we may talk about multiple tensor products without specifying the positions of parentheses. For calculations with \(n\)-fold tensor products one needs the \((n-1)\)-fold coproduct \(\Delta^{(n)}: U_q(\mathfrak{sl}_2) \rightarrow \left(U_q(\mathfrak{sl}_2)\right)^{\otimes n}\)

\[
\Delta^{(n)} = (\Delta \otimes \text{id}^{\otimes (n-2)}) \circ (\Delta \otimes \text{id}^{\otimes (n-3)}) \circ \cdots \circ (\Delta \otimes \text{id}) \circ \Delta.
\]

We record for later use the following easily verified expressions for the \((n-1)\)-fold coproducts of the generators.

**Lemma 2.2.** We have

\[
\Delta^{(n)}(K) = K^{\otimes n},
\]

\[
\Delta^{(n)}(E) = \sum_{i=1}^{n} 1^{\otimes (i-1)} \otimes E \otimes K^{\otimes (n-i)},
\]

\[
\Delta^{(n)}(F) = \sum_{i=1}^{n} (K^{-1})^{\otimes (i-1)} \otimes F \otimes 1^{\otimes (n-i)}.
\]

### 2.2.2. Irreducible representations of the quantum group.

We will use representations which can be thought of as \(q\)-deformations of the irreducible representations of the semisimple Lie algebra \(\mathfrak{sl}_2(\mathbb{C})\). For the statement of the Lemma below, recall the definition of \(q\)-integers \([m]\) from Equation (2.1).

**Lemma 2.3.** For every positive integer \(d\), there is an irreducible representation \(M_d\) of \(U_q(\mathfrak{sl}_2)\) with a basis \(e^{(d)}_0, e^{(d)}_1, \ldots, e^{(d)}_{d-1}\) and the action of the generators defined by

\[
K.e^{(d)}_j = q^{d-1-2j} e^{(d)}_j
\]

\[
F.e^{(d)}_j = \begin{cases} 
    e^{(d)}_{j+1} & \text{if } j \neq d - 1 \\
    0 & \text{if } j = d - 1
\end{cases}
\]

\[
E.e^{(d)}_j = \begin{cases} 
    [j][d-j] e^{(d)}_{j-1} & \text{if } j \neq 0 \\
    0 & \text{if } j = 0
\end{cases}
\]

Any \(d\)-dimensional irreducible representation of \(U_q(\mathfrak{sl}_2)\), where the \(K\)-eigenvalues are integer powers of \(q\), is isomorphic to \(M_d\).
\textbf{Proof.} It is easy to check that the formulas defining the action respect the relations (2.3). Moreover, since the $q$-integers $[m]_q$ are non-vanishing for $m \neq 0$, $M_0$ is clearly irreducible. If $V$ is an irreducible representation, then $K$ is diagonalizable on $V$, because the sum of $K$-eigenspaces is a subrepresentation. If in a finite dimensional irreducible representation $V$ the $K$-eigenvalues are integer powers of $q$, then an eigenvector $v_0$ of $K$ of eigenvalue $q^{m_0}$ with $m_0 \in \mathbb{Z}$ maximal must be annihilated by $E$. By a standard calculation one then shows that the linear span of the $m_0 + 1$ vectors $v_0, E.v_0, E^2.v_0, \ldots, E^{m_0}.v_0$ forms a subrepresentation isomorphic to $M_{m_0+1}$. The last assertion follows from this. 

\begin{align*}
\text{2.2.3. Tensor products of the irreducible representations.} \text{ Tensor products of the representations defined in Section 2.2.2 decompose to direct sums of irreducible subrepresentations of the same type. We need concrete descriptions of such decompositions, as given in the following Gordan formulas. Recall that the action of $U_q(\mathfrak{sl}_2)$ on tensor products is defined using the coproduct (2.6), and recall also the definition of $q$-factorials $[n]_q$ from Equation (2.3).}

\text{Lemma 2.4. Consider the tensor product representation $M_{d_2} \otimes M_{d_1}$. For any}
\end{align*}

\[ m \in \{0, 1, \ldots, \min(d_1, d_2) - 1\}, \]

\begin{equation}
\text{denote } d = d_1 + d_2 - 1 - 2m, \text{ and define}
\end{equation}

\[ T^{l_1, l_2}_{0, m} (d_1, d_2) = \delta_{l_1 + l_2, m} \times (-1)^{l_1} \frac{[d_1 - 1 - l_1]! [d_2 - 1 - l_2]!}{[l_1]! [d_1 - 1]! [l_2]! [d_2 - 1]!} \frac{q^{l_1(d_1 - l_1)}}{(q - q^{-1})^m}. \]

\begin{equation}
\text{Then the vector}
\end{equation}

\[ \tau_0 = \tau_0^{(d_1, d_2)} = \sum_{l_1, l_2} T^{l_1, l_2}_{0, m} (d_1, d_2) \times (e_{l_2} \otimes e_{l_1}) \]

\text{satisfies $E.\tau_0 = 0$ and $K.\tau_0 = q^{d-1} \tau_0$ (i.e., $\tau_0$ is a highest weight vector). The subrepresentation of $M_{d_2} \otimes M_{d_1}$ generated by $\tau_0$ is isomorphic to $M_d$.}

\begin{equation}
\text{The tensor product representation decomposes to a direct sum of irreducibles}
\end{equation}

\[ M_{d_2} \otimes M_{d_1} \cong M_{d_1+d_2-1} \oplus M_{d_1+d_2-3} \oplus \cdots \oplus M_{d_1-d_2+1}. \]

\text{Proof. The coefficients $T^{l_1, l_2}_{0, m}$ above are non-zero only for $l_1 + l_2 = m$, so we can write $\tau_0 = \sum_k T^k_{0, m} e_{-m-k} \otimes e_k$. Since $\Delta(K) = K \otimes K$, we have}

\[ K(e_{-m-k} \otimes e_k) = (q^{d-1-2(m-k)} e_{-m-k}) \otimes (q^{d_1-1-2k} e_k) = q^{d-1} (e_{-m-k} \otimes e_k) \]

\text{with $d = d_1 + d_2 - 1 - 2m$, and therefore obviously $K.\tau_0 = q^{d-1} \tau_0$. Using $\Delta(E) = E \otimes K + 1 \otimes E$ we get}

\[ E(e_{-m-k} \otimes e_k) = [m-k][d_2-m+k] q^{d_1-1-2k} (e_{-m-k-1} \otimes e_k) + [k][d_1-k] (e_{-m-k} \otimes e_{k-1}), \]

\text{and thus}

\[ E.\tau_0 = \sum_k \left( [m-k][d_2-m+k] q^{d_1-1-2k} T^k_{0, m} \right) \times (e_{-m-k-1} \otimes e_k). \]

\text{The coefficients satisfy $T^k_{0, m} = -T^k_{0, m} \times [(m-k)[d_2-m+k]]/[d_1-k-1][d_1-k-1] q^{d_1-1-2k}$, and hence we get $E.\tau_0 = 0$.}

\text{From Lemma 2.3 and the properties $E.\tau_0 = 0$ and $K.\tau_0 = q^{d-1} \tau_0$ it is clear that the vector $\tau_0$ generates a subrepresentation of $M_{d_2} \otimes M_{d_1}$ isomorphic to $M_d$. The dimension $d_1d_2$ of the representation $M_{d_2} \otimes M_{d_1}$ equals the sum of dimensions $d = d_1 + d_2 - 1 - 2m$ over the allowed values of $m$. Thus, the entire representation is a direct sum of these subrepresentations.} \qed
Remark 2.5. In view of Equation (2.9), we may freely interpret \( M_d \) as a subrepresentation of \( M_{d_2} \otimes M_{d_1} \), with the embedding normalized so as to map the basis vectors \( e_i^{(d)} \) of Lemma 2.3 to the vectors

\[
\tau_i^{(d_1,d_2)} := F_i^{ d_1,d_2}_0
\]

with \( \tau_0^{(d_1,d_2)} \) given by the formulas (2.8) and (2.7). We denote the coefficients of these vectors in the tensor product basis by \( T_{1:m}^{d_1,d_2}(d_1,d_2) \), so that

\[
(2.10) \quad \tau_i^{(d_1,d_2)} = \sum_{l_1,l_2} T_{1:m}^{d_1,d_2}(d_1,d_2) \times (e_{l_2} \otimes e_{l_1}).
\]

The vectors \( \tau_i^{(d_1,d_2)} \) also form a basis of \( M_{d_2} \otimes M_{d_1} \).

3. Various forms of the integral functions

In this section, we introduce the functions in terms of which the spin chain - Coulomb gas correspondence is defined and studied. The functions

\[
\rho_{m_1,...,m_n}, \rho_{m_1,...,m_n}, \varphi_{l_1,...,l_n}, \Omega_{l_1,...,l_{n-1},l,m:l_{n+1},...,l_n}
\]

will all be defined by integrals of essentially the same multivalued integrand — the differences lie in the choice of the integration surface and the choices of branch and rephasing of the integrand, which are often easiest to indicate by figures. The definition of the correspondence will use \( \varphi_{l_1,...,l_n}^{(x_0)} \) as basis functions, and the other functions are used for proving properties of the correspondence. For this purpose, various properties of the functions and relations among them are proven in this section.

A parameter \( \kappa \in (0,\infty) \setminus \mathbb{Q} \) is fixed throughout, and the deformation parameter \( q = e^{i\pi\ell/\kappa} \) is chosen. We also fix the number \( n \in \mathbb{N} \) of variables, and real parameters \( d_1, \ldots, d_n \), which in the end will be taken to be dimensions of the representations.

We use the shorthand notation \( x = (x_1, \ldots, x_n) \) for the arguments of the functions. The domain of definition will be either the chamber \( X_n \) or the restricted chamber \( X_n^{(x_0)} \), given by (1.1) or (1.2), respectively, so that we always assume the variables ordered according to

\[
x_0 < x_1 < \cdots < x_n.
\]

For fixed \( x \) and \( x_0 \), the value of the function will be written as an integral of Dotsenko-Fateev type [DPS], as in the Coulomb gas formalism of conformal field theory. The integrand is a branch of the following multivalued function, a product of powers of differences,

\[
f_1^{(\ell)}(x; \omega) = f_1^{(\ell)}(x_1, \ldots, x_n; w_1, \ldots, w_\ell)
\]

\[
(3.1)
\prod_{1 \leq i < j \leq n} (x_j - x_i)^{2(d_i-1)(d_j-1)} \prod_{1 \leq i \leq n} (w_r - x_i)^{-\frac{d_i}{2}} \prod_{1 \leq r < s \leq \ell} (w_s - w_r)^{-\frac{d_i}{2}},
\]

and the auxiliary variables \( w_1, \ldots, w_\ell \) are to be integrated over. More precisely, the integrand will be defined on some simply connected subset of

\[
\mathcal{M}^{(\ell)} = \mathcal{M}^{(\ell)}_{x_1,...,x_n} := \{ (w_1, \ldots, w_\ell) \in (\mathbb{C} \setminus \{ x_1, \ldots, x_n \})^\ell \mid w_r \neq w_s \text{ for all } r \neq s \}.
\]

Remark 3.1. The logarithmic differential of the multivalued function \( f_1^{(\ell)} \) is the single-valued one-form

\[
d \left( \log \left( f_1^{(\ell)}(x; \omega) \right) \right) = \sum_{r=1}^{\ell} \left( \sum_{i=1}^{n} \frac{4(1-d_i)/\kappa}{w_r - x_i} + \sum_{s\neq r} \frac{8/\kappa}{w_r - w_s} \right) dw_r.
\]
Thus, to define a branch of the integrand on a simply connected subset of \( \mathcal{M}(\ell) \), it is sufficient to give its value at some point \( w' \), and then set

\[
f_{\text{branch}}(x; w) = f_{\text{branch}}(x; w') \times \exp \left( \int_{w'}^w \, d \left( \log f^{(i)}(x; \cdot) \right) \right),
\]

where the path of integration from \( w' \) to \( w \) stays in the simply connected subset.

We will frequently partition the variables \( w_1, \ldots, w_\ell \) to \( n \) subsets of sizes \( m_1, \ldots, m_n \), in which case we use the notation

\[
I^{(i)} = I^{(i)}_{m_1, \ldots, m_n} = \left\{ r \in \mathbb{Z} \left| \sum_{j=1}^i m_j \geq r > \sum_{j=1}^{i-1} m_j \right. \right\} \quad (i = 1, \ldots, n)
\]

for the partition of the indices.

### 3.1. Real integral functions as integrals over a product of simplices

The integrand [3.2] has a constant phase on the following simply connected real subset of \( \mathcal{M}(\ell) \), a product of simplices of dimensions \( m_1, \ldots, m_n \) with \( \sum_{i=1}^n m_i = \ell \),

\[
\mathcal{R}_{m_1, \ldots, m_n} = \left\{ (w_1, \ldots, w_\ell) \in \mathbb{R}^\ell \left| \begin{array}{l}
x_0 < w_1 < w_2 < w_3 < \cdots < w_{m_1} < x_1, \\
x_1 < w_{m_1+1} < \cdots < w_{m_1+m_2} < x_2, \\
\vdots \\
x_{n-1} < w_{m_1+\cdots+m_{n-1}+1} < \cdots < w_{m_1+\cdots+m_n} < x_n 
\end{array} \right. \right\}.
\]

We define the real-valued functions \( \rho^{(x_0)}_{m_1, \ldots, m_n} : \mathbb{R}^\ell \rightarrow \mathbb{R} \) as integrals over this set

\[
\rho^{(x_0)}_{m_1, \ldots, m_n}(x) := \prod_{1 \leq i < j \leq n} |x_j - x_i|^{\frac{1}{2}(d_i-1)(d_j-1)} 
\times \prod_{\mathcal{R}_{m_1, \ldots, m_n}} \prod_{1 \leq r < s \leq \ell} |w_r - w_s|^{\frac{1}{2}(d_r-1)} \prod_{1 \leq r \leq s \leq \ell} |w_s - w_r|^{\frac{1}{2}} \, dw_1 \cdots dw_\ell.
\]

In applications, it is usually desirable to write the final results in terms of these functions, because of their transparent definition and real-valuedness.

The integrals \( \rho^{(x_0)}_{m_1, \ldots, m_n}(x) \) are convergent for large enough \( \kappa \) — the precise condition is

\[
\kappa > 4 \times \left( \max_{1 \leq i \leq n} d_i - 1 \right),
\]

and this will often be implicitly assumed. Nevertheless, our main results are valid for all irrational positive \( \kappa \) — they are obtained by meromorphic analytic continuation in \( \kappa \). Indeed, the analytic continuation of the real integrals \( \rho^{(x_0)}_{m_1, \ldots, m_n}(x) \) can be done by regularizing the divergent integrals, as discussed in [JJK13]. If such a regularization is performed by the method of counterterms, one can see, in principle explicitly, that for \( \Re(\kappa) > 0 \) the only singularities are isolated poles at some rational values of \( \kappa \).

### 3.2. Integrals over a product of deformed hypercubes

It is natural to extend the integrand above from the real submanifold \( \mathcal{R}_{m_1, \ldots, m_n} \subset \mathcal{M}(\ell) \) to an open subset. A convenient choice for intermediate manipulations is the simply connected subset

\[
\mathcal{M}_{m_1, \ldots, m_n} := \left\{ w \in \mathcal{M}(\ell) \left| \forall i \, \forall r \in I^{(i)} : \Re(x_{i-1}) < \Re(w_r) < \Re(x_i), \\
\forall i \, \forall r, s \in I^{(i)}, r < s : w_s - w_r \in \mathbb{C} \setminus i\mathbb{R}_+ \right. \right\}.
\]
On $\tilde{\mathfrak{M}}_{m_1,\ldots,m_n}$, we choose a branch of the multivalued function $f^{(i)}(x;\cdot)$ of (3.1), and rephase it so that it becomes real and positive on $\mathcal{R}_{m_1,\ldots,m_n} \subset \tilde{\mathfrak{M}}_{m_1,\ldots,m_n}$. This function

$$f^{\approx}_{m_1,\ldots,m_n}(x;\cdot) : \tilde{\mathfrak{M}}_{m_1,\ldots,m_n} \to \mathbb{C}$$

can be defined for example using Remark 3.1, i.e., by fixing a point $w' \in \mathcal{R}_{m_1,\ldots,m_n}$, setting the value at that point equal to the absolute value of $f^{(i)}(x;w')$, and analytically continuing by integrating the single valued logarithmic differential.

In particular, we can express the real integral function $\tilde{\rho}^{(x_0)}_{m_1,\ldots,m_n}(x)$ as an integral of $f^{\approx}(x;\cdot)$

$$\tilde{\rho}^{(x_0)}_{m_1,\ldots,m_n}(x) = \int_{\mathcal{R}_{m_1,\ldots,m_n}} f^{\approx}_{m_1,\ldots,m_n}(x;w_1,\ldots,w_{\ell}) \, dw_1 \cdots dw_{\ell}.$$

We next define the closely related function

$$\tilde{\rho}'^{(x_0)}_{m_1,\ldots,m_n}(x) := \int_{\mathcal{R}_{m_1,\ldots,m_n}} f^{\approx}_{m_1,\ldots,m_n}(x;w_1,\ldots,w_{\ell}) \, dw_1 \cdots dw_{\ell},$$

where the integration surface $\tilde{\mathcal{R}}_{m_1,\ldots,m_n} \subset \tilde{\mathfrak{M}}_{m_1,\ldots,m_n}$ is such that for any $r \in I^{(i)}$, the variable $w_r$ is integrated from $x_{r-1}$ to $x_r$. In view of the definition of the simply connected set $\tilde{\mathfrak{M}}_{m_1,\ldots,m_n}$, this unambiguously determines the homotopy type of the integration surface $\tilde{\mathcal{R}}_{m_1,\ldots,m_n}$, and consequently the function $\tilde{\rho}'^{(x_0)}_{m_1,\ldots,m_n}(x)$. Figure 3.1 illustrates how the variables turn around each other in this integration, and indicates the choice of a point where the integrand is rephased to be positive.

**Lemma 3.2.** The function

$$\tilde{\rho}'^{(x_0)}_{m_1,\ldots,m_n} : \mathfrak{X}_{n}^{(x_0)} \to \mathbb{C}$$

is related to the real integral function (3.3) by

$$\tilde{\rho}'^{(x_0)}_{m_1,\ldots,m_n}(x) = \left( \prod_{i=1}^{n} q^{-\left(\frac{\kappa}{n}\right)} \left[ m_i! \right] \right) \times \rho^{(x_0)}_{m_1,\ldots,m_n}(x), \quad \text{for } x \in \mathfrak{X}_{n}^{(x_0)}.$$

**Proof.** Note that $\rho$ is obtained by integration over the set $\mathcal{R}_{m_1,\ldots,m_n}$ which is a product of simplices, whereas $\tilde{\rho}$ is obtained by integration over the set $\tilde{\mathcal{R}}_{m_1,\ldots,m_n}$ which is a product of slightly deformed hypercubes. We split each of the $m_i$-dimensional hypercubes to $m_i!$ simplices, and thus express $\tilde{\rho}'^{(x_0)}_{m_1,\ldots,m_n}$ as a sum of $\prod_{i=1}^{n} (m_i!)$ terms, each of which is a phase factor times $\rho^{(x_0)}_{m_1,\ldots,m_n}$.

The variables $w_r$, $r \in I^{(i)}$, are integrated over one of the slightly deformed hypercubes. We select the deformed hypercube integration contour so that the variables are on the real axis, except when the distance between some of the variables becomes smaller than a chosen $\varepsilon > 0$. In view of the integrand, proportional to (3.1), the contribution from cases with some $|w_r - x_i| < \varepsilon$ is $O(\varepsilon^{1-\frac{1}{2}\max(d_i-1)})$, and
the further contribution from cases with some $|w_s - w_r| < \varepsilon$ is $O(\varepsilon^{1+\frac{2}{n}})$. We may thus neglect these contributions, which tend to zero as $\varepsilon \searrow 0$, and only consider cases with variables $w_r$ on the real line in some definite order.

We encode the possible orderings of the variables $w_r$ by $n$-tuples $(\sigma^{(1)}, \ldots, \sigma^{(n)})$, where $\sigma^{(i)}$ is a permutation of $I^{(i)}$: the associated order of the variables is

\[
w_{\sigma^{(i)}(r)} < w_{\sigma^{(i)}(r')}, \quad \text{for all } r, r' \in I^{(i)} \text{ such that } r < r',
\]

\[
w_{\sigma^{(i)}(r)} < w_{\sigma^{(i)}(r')}, \quad \text{for all } r \in I^{(i)}, r' \in I^{(j)} \text{ such that } i < j.
\]

By our definition, the phase of $f_{m_1, \ldots, m_n}$ is positive when the ordering of the variables is the one corresponding to all identity permutations, $\sigma^{(i)} = \text{id}_{I^{(i)}}$ for $i = 1, \ldots, n$. In the limit $\varepsilon \searrow 0$, the integration over the set where the variables respect this standard ordering thus simply reproduces $\rho_{m_1, \ldots, m_n}^{(x)}$. If the ordering among the variables $w_r$, $r \in I^{(i)}$, is given by some other permutation $\sigma^{(i)}$, then the phase factors accumulated from exchanging the orders of these variables is $q^{-2 \times \#\text{inv}(\sigma^{(i)})}$, where

\[
\text{inv}(\sigma^{(i)}) = \left\{ r, s \in I^{(i)} \mid r < s \text{ and } \sigma^{(i)}(r) > \sigma^{(i)}(s) \right\}
\]

denotes the set of inversions of $\sigma^{(i)}$. Apart from these phase factors, the contribution of the integral from the set corresponding to the ordering $(\sigma^{(1)}, \ldots, \sigma^{(n)})$ coincides with the integral over the standard ordered part. In conclusion, we have

\[
\tilde{\rho}^{(x)}_{m_1, \ldots, m_n} = \sum_{\sigma^{(1)}, \ldots, \sigma^{(n)}} \left( \prod_{i=1}^{n} q^{-2 \times \#\text{inv}(\sigma^{(i)})} \right) \times \rho_{m_1, \ldots, m_n}^{(x)}.
\]

By Lemma 2.1(b) we simplify the prefactor to the asserted form.

\[\square\]

3.3. Basis functions as integrals over families of non-intersecting loops. Let us now define the integrals $\varphi_{l_1, \ldots, l_n}^{(x)}$, which will serve as our basis functions in defining the spin chain - Coulomb gas correspondence. The integration surfaces are certain families of non-intersecting loops used also in [FW91]. In contrast to the earlier integrals $\rho_{m_1, \ldots, m_n}^{(x)}$ and $\tilde{\rho}^{(x)}_{m_1, \ldots, m_n}$, for the convergence of these integrals it suffices that $\Re(\kappa) > 0$.

Let

\[
l_1, \ldots, l_n \in \mathbb{Z}_{\geq 0}, \quad \ell = \sum_{i=1}^{n} l_i,
\]

and define the partition $\left( I^{(i)}_{l_1, \ldots, l_n} \right)_{i=1}^{n}$ of the indices of the $w$-variables as in (3.2). The integration surface $\Sigma_{l_1, \ldots, l_n}^{[\rho]}$ — a family of non-intersecting loops illustrated in Figure 3.2 — and the associated integrand $\int_{I_{l_1, \ldots, l_n}^{[\rho]}}$, are defined as follows:

- Each of the variables $w_r$, for $r \in I^{(i)}$, makes a simple loop in $\mathbb{C} \setminus \{x_1, \ldots, x_n\}$ starting and ending at the anchor point $x_0$, and encircling the point $x_i$ once in the positive direction. The loop of $w_r$, for $r \in I^{(i)}$, must never cross the lines $x_j + i\mathbb{R}_+$ for $j \neq i$, nor the lines $w_s + i\mathbb{R}_+$ for $s \in I^{(i)}$ with $j < i$.
- The $l_i$ loops around $x_i$, are nested in such a way that if $r, s \in I^{(i)}$ and $r < s$, then the loop of $w_s$ encircles the loop of $w_r$.
- There is a point $w' = (w'_1, \ldots, w'_n) \in \Sigma_{l_1, \ldots, l_n}^{[\rho]}$, also illustrated in Figure 3.2, such that

\[
x_1 < w'_1 < w'_2 < w'_3 < \cdots < w'_{l_1} < x_2,
\]

\[
x_2 < w'_{l_1+1} < w'_{l_1+2} < \cdots < w'_{l_1+l_2} < x_3,
\]

\[
\vdots
\]

\[
x_n < w'_{l_1+\ldots+l_{n-1}+1} < \cdots < w'_{l_1+\ldots+l_n},
\]
Figure 3.2. The integration surface $\mathcal{L}^\oplus_{l_1,\ldots,l_n}$. The point where the integrand $f_{l_1,\ldots,l_n}(x;\cdot)$ is rephased to be positive is marked by red circles.

As the integrand, we choose a branch and rephasing of the function $f^{(\ell)}(x;\cdot)$ of (3.1), so as to make it positive at $w'$. Such an integrand

$$f_{l_1,\ldots,l_n}(x;\cdot) : \mathcal{L}^\oplus_{l_1,\ldots,l_n} \to \mathbb{C}$$

can again be defined using the single valued logarithmic differential as explained in Remark 3.1.

The integrals of $f^\oplus$ over $\mathcal{L}^\oplus$,

$$\varphi^{(x_0)}_{l_1,\ldots,l_n}(x) = \int_{\mathcal{L}^\oplus_{l_1,\ldots,l_n}} f_{l_1,\ldots,l_n}(x;w_1,\ldots,w_{\ell}) \, dw_1 \cdots dw_{\ell},$$

define functions

$$\varphi^{(x_0)}_{l_1,\ldots,l_n} : \mathcal{X}^{(x_0)} \to \mathbb{C}.$$

The transformation rules of these functions under translation and scaling are the following.

**Lemma 3.3.** For any $\xi \in \mathbb{R}$ we have

$$\varphi^{(x_0+\xi)}_{l_1,\ldots,l_n}(x_1+\xi,\ldots,x_n+\xi) = \varphi^{(x_0)}_{l_1,\ldots,l_n}(x_1,\ldots,x_n),$$

and for any $\lambda > 0$ we have

$$\varphi^{(\lambda x_0)}_{l_1,\ldots,l_n}(\lambda x_1,\ldots,\lambda x_n) = \lambda^{\Delta^{d_1,\ldots,d_n}(\ell)} \times \varphi^{(x_0)}_{l_1,\ldots,l_n}(x_1,\ldots,x_n),$$

where $\ell = \sum_{i=1}^n l_i$ and

$$\Delta^{d_1,\ldots,d_n}(\ell) = \frac{2}{\kappa} \sum_{i<j} (d_i - 1)(d_j - 1) - \frac{4}{\kappa} \sum_i (d_i - 1) + \frac{8}{\kappa} \frac{\ell(\ell-1)}{2} + \ell.$$

**Proof.** For the first statement, make the changes of variables $w_i' = w_i + \xi$. For the second, make the change of variables $w_i' = \lambda w_i$, and notice that the integrand has the scaling

$$f_{l_1,\ldots,l_n}^{(\lambda x;\lambda w)} = \lambda^{\frac{\lambda}{\kappa} \sum_{i<j} (d_i - 1)(d_j - 1) - \frac{4}{\kappa} \sum_i (d_i - 1) + \frac{8}{\kappa} \frac{\ell(\ell-1)}{2} + \ell} f_{l_1,\ldots,l_n}^{(x;w)},$$

and that the Jacobian of the change of variables in the $\ell$-dimensional integral is $\lambda^\ell$. □

As another obvious remark, we record the fact that when some tensorand $M_{d_i}$ is the trivial representation, the function is actually independent of the corresponding variable.

**Lemma 3.4.** If $d_i = 1$ for some $i \in \{1,\ldots,n\}$, then for any fixed values of the other variables $x_j$, $j \neq i$, the function $\varphi^{(x_0)}_{l_1,\ldots,l_n}$ is constant as a function of the variable $x_i$. 

Lemma 3.5. Consider the case \( n \) point but different phase on these two line segments. Indeed, on the first line segment posed to linear combinations of real integrals 3.3.1. For two values of the variable \( d \) for \( x \) and \( x' \) to \( x \) constant as a function of \( x \), because all the exponents of differences in which \( x \) appears are proportional to \( d - 1 \). Therefore, for two values of the variable \( x \), the values of the function \( \varphi(x) \) are in fact given by the same integral. \( \square \)

3.3.1. **The basis functions in the case of one variable.** The basis functions \( \varphi^{(x_0)}_\ell \) can be decomposed to linear combinations of real integrals \( \rho^{(x_0)}_{m_1, \ldots, m_n} \). We start from the simplest case of one point, \( n = 1 \). Even this simple case will have an important consequence, Lemma 3.8.

**Lemma 3.5.** Consider the case \( n = 1 \), and denote \( d_1 = d \in \mathbb{Z}_{>0} \), \( l_1 = \ell \), \( x_1 = x \). Then we have

\[
\varphi^{(x_0)}_\ell (x) = \left( [\ell]! \prod_{m=1}^{\ell} (q^{d-m} - q^{m-d}) \right) \times \rho^{(x_0)}_{\ell} (x).
\]

In particular, \( \varphi^{(x_0)}_\ell \) is identically zero if \( \ell \geq d \).

**Proof.** We will prove that \( \varphi^{(x_0)}_\ell (x) = q^{\ell} \left( \prod_{m=1}^{\ell} (q^{d-m} - q^{m-d}) \right) \times \rho^{(x_0)}_{\ell} (x) \) and the asserted equality will then follow from Lemma 3.2. We achieve this by decomposing each of the loops based at \( x_0 \) encircling \( x \) to two pieces: one from \( x_0 \) to \( x \) and the other from \( x \) back to \( x_0 \). We start from the innermost loop and inductively proceed outwards. The procedure is illustrated in Figure 3.3.

Let \( C_1, \ldots, C_\ell \) be the contours of integration of the variables \( w_1, \ldots, w_\ell \), so that \( C_1 \) is a loop encircling \( x \), and for \( r > 1 \) the loop \( C_r \) encircles the entire loop \( C_{r-1} \). The basis function then reads

\[
\varphi^{(x_0)}_\ell (x) = \int_{C_1} dw_1 \cdots \int_{C_\ell} dw_\ell \int_{C_{\ell}} dw_1 f^{\mathbb{R}}_\ell (x; w_1, \ldots, w_\ell),
\]

see also Figure 3.3(a). Recall that the branch and phase of the integrand are chosen so that

\[
f^{\mathbb{R}}_\ell (x; w_1, \ldots, w_\ell) = \prod_{r=1}^{\ell} (w_r - x)^{\frac{1}{2} (1-d)} \times \prod_{1 \leq r < s \leq \ell} (w_s - w_r)^{\frac{1}{2}}
\]
is positive at the “midpoint” \( w' \) of the loops (illustrated by the red dots in the figure). We may thus fix the branches of each of the factors above so that they are positive when \( x < w_1 < w_2 < \cdots < w_\ell \).

Choose a small \( \varepsilon > 0 \). To decompose the first loop, that is, the innermost integral

\[
\int_{C_1} f^{\mathbb{R}}_\ell dw_1,
\]
deform it to a line segment along the real axis from \( x_0 \) to \( x - \varepsilon \), a circle of radius \( \varepsilon \) around \( x \), and a line segment along the real axis from \( x - \varepsilon \) to \( x_0 \). The innermost integral then becomes essentially

\[
\int_{x_0}^{x-\varepsilon} f^{\mathbb{R}}_\ell dw_1 + \oint_{\partial B_\varepsilon (x)} f^{\mathbb{R}}_\ell dw_1 + \int_{x-\varepsilon}^{x_0} f^{\mathbb{R}}_\ell dw_1,
\]

but in this expression we have abused notation and hidden the important phase factors. In particular, the first and last terms do not cancel — the factor \( (x - w_1)^{\frac{1}{2} (1-d)} \) in the integrand \( f^{\mathbb{R}}_\ell \) has a constant but different phase on these two line segments. Indeed, on the first line segment \( (x - w_1)^{\frac{1}{2} (1-d)} = q^{d-1} |x - w_1|^{\frac{1}{2} (1-d)} \) and on the second \( (x - w_1)^{\frac{1}{2} (1-d)} = q^{1-d} |x - w_1|^{\frac{1}{2} (1-d)} \), because from the reference point \( w' \) we must take \( w_1 \) half a turn around \( x \) in the negative or positive direction, respectively. The
(a) The proof of Lemma 3.5 consists of manipulating the integral of $\varphi^{(x_0)}(x)$, which has $\ell$ non-intersecting positively oriented loops around $x$, anchored at $x_0$, as illustrated in this figure.

(b) First, the integration over the innermost loop $C_1$ — corresponding to the variable $w_1$ — is rewritten according to the illustrations in this figure. The small circular arc can be neglected. Two pieces between $x_0$ and $x$ remain, and their contributions are the same up to orientation and phase factors.

(c) Once the integration over $w_1$ has been rewritten as an integral from $x_0$ to $x$, we start manipulating the integration over the next loop $C_2$ — corresponding to the variable $w_2$. The rewriting is illustrated in this figure. The result consists of two pieces, and upon relabeling the dummy integration variables $w_1$ and $w_2$ in one of them, the contributions are seen to be the same up to phase factors and signs.

Figure 3.3. Illustrations for the proof of Lemma 3.5
Proof.
We use the method of the proof of Lemma 3.5 to decompose the main theorems.

We proceed similarly with the integration contours $C_2, \ldots, C_\ell$, cutting each of them into two pieces between $x_0$ and $x$, and a negligible loop around $x$. The example of the second loop $C_2$ is illustrated in Figure 3.3(b). On each piece of the $r$-th loop $C_r$, we rephase the factors of the integrand that contain $w_r$, in order to finally compare with the integral $\tilde{\rho}$. The first piece of the integration contour $C_r$ is a path from $x_0$ to $x$ below the variables $w_s$, for $s < r$, and we extract a phase factor $q^{d-1}$ resulting from taking $w_r$ half a turn around $x$ in the negative direction. The second piece of the integration contour $C_r$ is a path from $x$ to $x_0$ above the variables $w_s$, for $s < r$, and we not only take $w_r$ in the positive direction around $x$, but we also take it positively around all $w_s$, $s < r$, in order to reach a position where the corresponding piece of the contour remains in the subset $\tilde{\mathcal{W}}$. The phase factor accumulated in this case is $q^{1-d+2(r-1)}$. After extracting all these phase factors, the remaining integral is, up to an orientation, equal to the integral $\tilde{\rho}_\ell(x_0)(x)$. The final result is

$$\varphi^{(x_0)}_\ell(x) = \prod_{r=1}^{\ell} (q^{d-1} - q^{1-d+2(r-1)}) \times \int_{C_\ell} f_{x_0}^\infty(x; w_1, \ldots, w_\ell),$$

and we finish the proof by simplifying the prefactor

$$\prod_{r=1}^{\ell} (q^{d-1} - q^{1-d+2(r-1)}) = q^\binom{\ell}{2} \prod_{m=1}^{\ell} (q^{d-m} - q^{m-d})$$

and using Lemma 3.2 If $\ell \geq d$, the product contains a factor which vanishes, and then $\varphi^{(x_0)}_\ell(x) = 0$. □

3.3.2. The basis functions in the case of two variables. In the next Lemma, we write, in the case $n = 2$, the basis functions $\varphi^{(x_0)}$ in terms of the integrals $\tilde{\rho}^{(x_0)}$, which in turn could be written in terms of the real integrals $\rho^{(x_0)}$ by Lemma 3.2 This result is used several times in the course of proving the main theorems.

Lemma 3.6. Let $d_1, d_2 \in \mathbb{Z}_{>0}$ and $0 \leq l_1 < d_1, 0 \leq l_2 < d_2$. Then we have

$$\varphi^{(x_0)}_{l_1,l_2}(x_1, x_2) = q^{\binom{l_1}{2} + \binom{l_2}{2}} (q - q^{1-l_1})^{\frac{l_1-1}{2}} \frac{[d_1-1]!}{[d_1-l_1-1]!} \frac{[d_2-1]!}{[d_2-l_2-1]!} \times \sum_{m=0}^{l_2} q^{m(l_2-l_1-1)} \binom{l_2}{m} \tilde{\rho}^{(x_0)}_{l_1+m,l_2-m}(x_1, x_2).$$

Proof. We use the method of the proof of Lemma 3.5 to decompose the $l_1$ loops around $x_1$ and the $l_2$ loops around $x_2$ to paths from $x_0$ to $x_1$ and $x_2$, respectively. We obtain

$$\varphi^{(x_0)}_{l_1,l_2}(x_1, x_2) = \prod_{i=1,2} \left( q^{\binom{l_i}{2}} \prod_{t=1}^{l_i} (q^{d_i-t} - q^{t-d_i}) \right) \times \tilde{c}^{(x_0)}_{l_1,l_2}(x_1, x_2),$$

where $\tilde{c}^{(x_0)}_{l_1,l_2}$ is the function defined by an integral as in Figure 3.4. The variables $w_1, \ldots, w_{l_1}$ are integrated from $x_0$ to $x_1$ and the variables $w_{l_1+1}, \ldots, w_{l_1+l_2}$ from $x_0$ to $x_2$ in such a way that for all
We finally write
\[ q^{(t_l-1)m} \times \sum_{1 \leq s_1 < s_2 < \cdots < s_m \leq l_2} q^{-2\sum_p (s_p - p)} \times \hat{\rho}_{l_1+m,l_2-m}^{(x_o)}. \]
By Lemma 2.1(c) we simplify the sum of the prefactors to the form \[ q^{-m(l_2-m)} \left[ \frac{l_2}{m} \right], \]
and obtain
\[ \hat{\omega}_{l_1,l_2}^{(x_o)} = \sum_{m=0}^{l_2} q^{(d_{l_1+1}+m-l_2)m} \left[ \frac{l_2}{m} \right] \times \hat{\rho}_{l_1+m,l_2-m}^{(x_o)}. \]
We finally write
\[ q^{(t_l)} \prod_{l=1}^{l_1} (q^{d_{l_1}} - q^{t-l_{l_1}}) = q^{(t_l)} (q - q^{-1})^l_{l_1} \frac{[d_{l_1} - 1]!}{[d_{l_1} - 1 - l_1]!}, \]
and the expression for \( \varphi_{l_1,l_2}^{(x_o)} \) then takes the form
\[ \varphi_{l_1,l_2}^{(x_o)}(x_1,x_2) = q^{(t_l)} + (q - q^{-1})^{l_1+l_2} \frac{[d_{l_1} - 1]! [d_{l_2} - 1]!}{[d_{l_1} - 1 - l_1]! [d_{l_2} - 2]!}, \]
\[ \times \sum_{m=0}^{l_2} q^{(d_{l_1+1}+m-l_2)m} \left[ \frac{l_2}{m} \right] \hat{\omega}_{l_1+m,l_2-m}^{(x_o)}(x_1,x_2). \]
This finishes the proof. \( \square \)
3.3.3. The general basis function reduction. In the case of general \( n \), the basis functions \( \varphi^{(x_0)} \) can still be written, in principle explicitly, in terms of the real integrals \( \rho^{(x_0)} \). The proof of our main results, however, do not rely on this explicit formula.

**Lemma 3.7.** We have

\[
\varphi_{l_1, \ldots, l_n}^{(x_0)}(x) = \sum_{m_1, \ldots, m_n} C_{l_1, \ldots, l_n}^{m_1, \ldots, m_n} \times \tilde{\rho}_{m_1, \ldots, m_n}(x),
\]

with some coefficients \( C_{l_1, \ldots, l_n}^{m_1, \ldots, m_n} \), which are zero unless \( \sum_i l_i = \sum_i m_i \) and \( \sum_i l_i \leq \sum_i m_i \) for all \( j = 1, \ldots, n \).

**Proof.** We only give a rough outline, and leave the details to the reader. By a generalization of the method used in the proof of Lemma 3.6 one can show that

\[
\varphi_{l_1, \ldots, l_n}^{(x_0)} = \prod_{i=1}^n q^{l_i} \prod_{t=1}^{l_i} (q^{d_i-t} - q^{-d_i}) \sum_{(k_j^{(i)})\leq i \leq n; 1 \leq j \leq i} C_{l_1, \ldots, l_n}^{(k_j^{(i)})} \times \tilde{\rho}_{(k_j^{(i)})}^{(x_0)},
\]

where

\[
C_{l_1, \ldots, l_n}^{(k_j^{(i)})} \equiv \prod_{i} \left[ k_i; k_2; \ldots; k_i \right] = \frac{[\ell_i]}{\prod_{j=1}^i [k_j]!}.
\]

The desired coefficients of \( \tilde{\rho}_{m_1, \ldots, m_n} \) are then expressible as sums of the coefficients above,

\[
C_{l_1, \ldots, l_n}^{m_1, \ldots, m_n} = \prod_{i=1}^n q^{l_i} \prod_{t=1}^{l_i} (q^{d_i-t} - q^{-d_i}) \sum_{(k_j^{(i)})} C_{l_1, \ldots, l_n}^{(k_j^{(i)})} \times \tilde{\rho}_{(k_j^{(i)})}^{(x_0)},
\]

and where we have used the following \( q \)-multinomial coefficients

\[
[k_i; k_2; \ldots; k_i] = \frac{[\ell_i]}{\prod_{j=1}^i [k_j]!}.
\]

The following important particular case already follows from the simple calculations performed in the proof of Lemma 3.5.

**Lemma 3.8.** Whenever \( l_i \geq d_i \) for some \( i = 1, \ldots, n \), we have

\[
\varphi_{l_1, \ldots, l_n}^{(x_0)}(x_1, \ldots, x_n) \equiv 0.
\]

**Proof.** We may rearrange the integrations over those \( l_i \) \( w \)-variables which encircle \( x_i \) similarly as in the case of Lemma 3.5. After this rearrangement, the result of the whole integral \( \varphi_{l_1, \ldots, l_n}^{(x_0)}(x) \) is the factor \( [\ell_i]! \prod_{m=1}^{l_i} (q^{d_i-m} - q^{m-d_i}) \) times an integral which is convergent for large \( \kappa \). The result thus again vanishes if \( l_i \geq d_i \) and \( \kappa \) is large, and by analyticity in \( \kappa \) the same conclusion is valid for all values \( \kappa > 0 \). \( \square \)
3.4. Mixed integral functions for asymptotics. To extract the asymptotic behavior of our functions as \(|x_{j+1} - x_j| \to 0\), we rewrite them in yet another way. We still keep \(d_1, \ldots, d_n\) fixed, and for any
\[
l_1, \ldots, l_{j-1}, l_{j+2}, \ldots, l_n \in \mathbb{Z}_{\geq 0}, \quad l \in \mathbb{Z}_{\geq 0}, \quad m \in \mathbb{Z}_{\geq 0}
\]
we define an integration surface
\[
\mathcal{M}_{l_1, \ldots, l_{j-1}; l, m; l_{j+2}, \ldots, l_n}^\circ
\]
rephased integrand
\[
\int_{l_1, \ldots, l_{j-1}; l, m; l_{j+2}, \ldots, l_n}^\circ (x; w)
\]
and an integral function
\[
\alpha_{l_1, \ldots, l_{j-1}; l, m; l_{j+2}, \ldots, l_n}(x)
\]
as follows. The surface \(\mathcal{M}^\circ\) is a mixture between a family of non-intersecting loops (see Section 3.3) and a deformed hypercube (see Section 3.2) as illustrated in Figure 3.5. We content ourselves to give the following slightly informal descriptions:

- There are \(m\) variables integrated from \(x_j\) to \(x_{j+1}\), and they turn around each other like in the deformed hypercube integrals of Section 3.2.
- For \(i \notin \{j, j+1\}\) there are \(l_i\) loops around \(x_i\) starting and ending at the anchor point \(x_0\), and they are nested and turning around each other like in the earlier families of non-intersecting loops of Section 3.3.
- There are \(l\) loops around the entire paths of the deformed hypercube integrals from \(x_j\) to \(x_{j+1}\), and they are nested and turning around each other like in the families of non-intersecting loops, as if the entire deformed hypercube would be a single point.
- The rephasing and branch choice of the integrand is such that \(f^\circ\) is positive at a point \(w'\) illustrated by the red circles in Figure 3.5.

The integral function \(\alpha^{(x_0)}\) is then defined as
\[
\alpha_{l_1, \ldots, l_{j-1}; l, m; l_{j+2}, \ldots, l_n}^{(x_0)}(x) := q^{(\circ)} \frac{1}{[m]!} \times \int_{\mathcal{M}_{l_1, \ldots, l_{j-1}; l, m; l_{j+2}, \ldots, l_n}^\circ} \int_{l_1, \ldots, l_{j-1}; l, m; l_{j+2}, \ldots, l_n}^\circ (x; w) \, dw.
\]
These, just like the integrals $\bar{\rho}$, are convergent for large enough $\kappa$, namely $\kappa > 4 \times (\max(d_j, d_{j+1}) - 1)$. They can also be analytically continued in $\kappa$. The prefactor is included to make the integrals more closely related to the integrals over a real simplex, see Lemma 3.2.

In order to state the results about the asymptotics, we use the exponents

$$\Delta_{d}^{d',d''} := \frac{2(1 + d^2 - (d')^2 - (d'')^2) + \kappa(d' + d'' - d - 1)}{2\kappa},$$

and define the multiplicative constants $B_{d}^{d',d''}$ for $\kappa > 4 \times (\max\{d', d''\} - 1)$ by the convergent integral over an $m$-dimensional simplex, with $m = \frac{1}{2} (d' + d'' - 1 - d)$,

$$B_{d}^{d',d''}(x_1, \ldots, x_n) := \int_{0}^{1} \int_{0}^{1} \cdots \int_{0}^{1} dw_1 \cdots dw_n \prod_{r=1}^{m} w_r^{-\frac{1}{4}(d' - 1)} \prod_{r=1}^{m} (1 - w_r)^{-\frac{1}{4}(d'' - 1)} \prod_{1 \leq r < s \leq m} (w_s - w_r)^{\frac{1}{2} \kappa}.$$  

For general values of $\kappa$, the multiplicative constants $B_{d}^{d',d''}$ are defined by the analytic continuation of this generalized beta-function. The analytic continuation can be done by a method of counterterms as in [JJK13], and one sees that for $\Re(\kappa) > 0$ the only singularities are isolated poles at some rational values of $\kappa$, in analogy with the usual beta-integral.

Remark 3.9. In the case $d = d' + d'' - 3$ (that is $m = 1$), the multiplicative constants are ordinary beta-functions, namely we have

$$B_{d}^{d',d''} = \int_{0}^{1} dw (1 - w)^{-\frac{1}{4}(d' - 1)} (1 - \bar{w})^{-\frac{1}{4}(d'' - 1)} = \frac{\Gamma(1 - \frac{1}{2}(d' - 1)) \Gamma(1 - \frac{1}{2}(d'' - 1))}{\Gamma(2 - \frac{1}{2}(d' + d'' - 2))}.$$  

Remark 3.10. Introducing the Kac labeled conformal weights

$$h_{1,d} := \frac{(d - 1)(2(d + 1) - \kappa)}{2\kappa},$$

one may express the overall homogeneity degree of $\varphi^{(x_0)}_{l_1, \ldots, l_n}$, given in Lemma 3.3, as

$$\Delta_{d_1, \ldots, d_n}(\ell) = h_{1,\sum_{i=1}^{n} d_i - n + 1 - 2\ell} - \sum_{i=1}^{n} h_{1,d_i},$$

where $\sum_{i=1}^{n} d_i - n + 1 - 2\ell$ should be understood as one of the dimensions of the irreducible subrepresentations of the tensor product $\otimes_{i} M_d_i$. We furthermore note that the exponents (3.3) are of this form,

$$\Delta_{d_1,d_2}(\ell) = \Delta_{d}^{d_1,d_2} \quad \text{with} \quad d = d_1 + d_2 - 1 - 2\ell.$$  

We now show an asymptotics property, which is particularly simple for the functions $\alpha^{(x_0)}$. This will be used later, in Proposition 4.4, to establish the general asymptotics statement.

Lemma 3.11. As $x_j$ and $x_{j+1}$ tend to a common limit $\xi$ (with $x_{j-1} < \xi < x_{j+2}$), we have

$$\frac{\alpha^{(x_0)}_{l_1, \ldots, l_{j-1}, j, l_{j+2}, \ldots, l_n}(x_1, \ldots, x_n)}{(x_{j+1} - x_j)^{\Delta}} \rightarrow B \times \varphi^{(x_0)}_{l_1, \ldots, l_{j-1}, j, l_{j+2}, \ldots, l_n}(x_1, \ldots, x_{j-1}, \xi, x_{j+2}, \ldots, x_n),$$

where $d = d_j + d_{j+1} - 1 - 2m$, the exponent $\Delta$ and the constant $B$ are given by

$$\Delta = \Delta_{d}^{d_1,d_{j+1}} \quad \text{and} \quad B = B_{d}^{d_1,d_{j+1}},$$

and $\varphi^{(x_0)}$ on the right hand side is the basis function of $n-1$ points associated to the choice of dimensions $d_1, \ldots, d_{j-1}, d, d_{j+2}, \ldots, d_n$. 

Proof. As a warm-up, note that by a simple scaling (compare with Lemma 3.3), the integrals \( \rho_{0,m}(x_1, x_2) \) over deformed hypercubes have the behavior
\[
q^{(m)} \frac{1}{m!} \times \rho_{0,m}(x_1, x_2) = \rho_{0,m}(x_1, x_2) = (x_2 - x_1)^{\Delta_{d_1,d_2}} B_{d_1,d_2}^d,
\]
for \( \kappa \) large enough so that both sides are given by convergent integrals. In the general case, the same idea is used with the dominated convergence theorem applied to a subset of the integration variables. Consider the factors in the integrand \( \int_{\mathcal{I}^0_{1,\ldots,i_m,\ldots,i_n}} (x; w) \) that involve \( x_j \) or \( x_{j+1} \), or any of the \( m \) variables integrated over the deformed hypercube in \( \mathcal{I}^0_{1,\ldots,i_m,\ldots,i_n} \). The integral over these \( m \) variables, divided by \(|x_{j+1} - x_j|^\Delta\), tends to the integrand of the function \( \varphi_{1,\ldots,i_{j-1},i_{j+1},\ldots,i_n}(x_1, \ldots, x_{j-1}, \xi, x_{j+2}, \ldots, x_n) \), times \( B \). Notice that the other integration contours remain bounded away from these points, so dominated convergence theorem gives the asserted limit.

To conclude, note that both sides of the asserted formula are analytic in \( \kappa \). \( \square \)

4. The spin chain - Coulomb gas correspondence

In this section, we define the spin chain - Coulomb gas correspondence, and show how the representation theoretical properties are translated to properties of the functions. By a succession of small steps which combine formulas for the representations of the quantum group given in Section 2 with properties of the functions established in Section 3 in the end of this section we will have proven the main result, whose precise formulation is given in Theorems 4.16 and 4.17.

From Section 2, we use the \( d \)-dimensional irreducible representation \( M_d \) of the quantum group \( \mathcal{U}_q(sl_2) \), defined in Lemma 2.2.2. Its basis \( \{e_i^{(d)}\}_{i=0,1,\ldots,d-1} \), introduced in the same lemma, is also used below. For simplicity of notation we often omit the superscript reference to the dimension \( d \).

Informally, the role of the quantum group is to act on the functions of Section 3 by modifying the integration surfaces and branch choices of the integrand.

4.1. Definition of the correspondence. Fix \( d_1, \ldots, d_n \in \mathbb{Z}_{>0} \). Denote by \( M_{d_i} \) the \( d_i \)-dimensional irreducible representation of \( \mathcal{U}_q(sl_2) \) defined in Section 2.2.2. Consider the tensor product representation
\[
\bigotimes_{i=1}^n M_{d_i} = M_{d_n} \otimes M_{d_{n-1}} \otimes \cdots \otimes M_{d_2} \otimes M_{d_1}.
\]
The order of tensorands will always be as shown on the right hand side, but for brevity we usually use the notation on the left hand side, and the above order is implicitly understood.

We will define a mapping
\[
\mathcal{F}^{(x_0)}_{d_1,\ldots,d_n} : \bigotimes_{i=1}^n M_{d_i} \to C^\infty(\mathcal{X}^{(x_0)}_n)
\]
from the tensor product (4.1) to smooth functions on the restricted chamber \( \mathcal{X}_n \). To simplify the notation, when the dimensions \( d_1, \ldots, d_n \) are clear from the context, we usually omit the subscripts and write simply \( \mathcal{F}^{(x_0)}_n = \mathcal{F}^{(x_0)}_{d_1,\ldots,d_n} \). In the representation \( M_{d_i} \), let \( e_i \) denote the basis vector obtained by applying the generator \( F \) repeatedly \( l_i \) times to the highest weight vector \( e_0 \), as in Section 2.2.2. The mapping (4.2) is defined by setting the images of the tensor product basis vectors to be the functions defined in Section 3.3
\[
\left( \mathcal{F}^{(x_0)}_n[e_{i_1} \otimes \cdots \otimes e_{i_l}] \right)(x_1, \ldots, x_n) = \varphi_{l_1,\ldots,l_n}(x_1, \ldots, x_n),
\]
and extending linearly.
4.2. **Auxiliary formula for two points.** The following result will be used both for the proofs of asymptotics in Section 4.3 and for the anchor point independence, Proposition 4.5. In fact, this result is literally a special case of the latter.

**Lemma 4.1.** If \( v \in M_{d_2} \otimes M_{d_1} \) is such that \( E.v = 0 \), then we have

\[
\left( \mathcal{F}(x_0)[v] \right)(x_1, x_2) = \sum_m c_m \times \tilde{\rho}_{0,m}(x_1, x_2)
\]

for some coefficients \( c_m \in \mathbb{C} \). In particular \( \left( \mathcal{F}(x_0)[v] \right)(x_1, x_2) \) is then independent of \( x_0 \).

**Proof.** By Lemma 2.4 and Remark 2.5, the vectors \( v \in M_{d_2} \otimes M_{d_1} \) that satisfy \( E.v = 0 \) are linear combinations of the vectors \( \tilde{\psi}_0 \) given by (2.7), with \( d = d_1 + d_2 - 1 - 2m \). It thus suffices to prove the statement for

\[
v = \tilde{\psi}_0 = \sum_{k=0}^m \tilde{\rho}_{0,m} \times (e_{m-k} \otimes e_k),
\]

and correspondingly

\[
\left( \mathcal{F}(x_0)[\tilde{\psi}_0] \right)(x_1, x_2) = \sum_{k=0}^m \tilde{\varphi}_{k,m} \times \rho_{k,m-k}(x_1, x_2),
\]

with \( \tilde{\varphi}_{k,m-k} \) given by (2.7). We use Lemma 3.6 to rewrite \( \tilde{\varphi}_{k,m-k} \) as a linear combination of \( \tilde{\rho}_{k+t,m-k-t} \) for \( t = 0, 1, \ldots, m - k \). After straightforward simplifications and a change of the summation index \( k \) to \( u = k + t \), we obtain

\[
\mathcal{F}(x_0)[\tilde{\psi}_0] = \sum_{u=0}^{m} \tilde{\rho}_{u,m-u} \times \left( (-1)^u \frac{m!}{(m-u)!} \sum_{t=0}^{u} (-1)^t \frac{q^t(u-1)}{[t]!} \frac{1}{[u]!} \right).
\]

The above sum over \( t \) simplifies by Lemma 2.1 (d) to

\[
\frac{1}{[u]!} \sum_{t=0}^{u} (-1)^t q^t(u-1) \left[ \begin{array}{c} u \\ t \end{array} \right] = \frac{1}{[u]!} q^u(u-1) \prod_{s=0}^{u-1} (q^s - q^s) = \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{if } u > 0. \end{cases}
\]

Thus, finally, when \( d = d_1 + d_2 - 1 - 2m \), we obtain the expression

\[
\left( \mathcal{F}(x_0)[\tilde{\psi}_0] \right)(x_1, x_2) = \frac{1}{[m]!} \tilde{\rho}_{0,m}(x_1, x_2).
\]

\[\square\]

4.3. **Asymptotics via projections to subrepresentations.** The asymptotics are easiest for the functions \( \alpha^{(t)}_{l_1,\ldots,l_{j-1};0,m;l_{j+1},\ldots,l_n} \) of Section 3.4. Here it is therefore desirable to write the function \( \mathcal{F}[v](x) \) in terms of these functions. We start from the following particular case.

**Lemma 4.2.** Let \( \tau_{0}^{(d,d+1)} \) be as in Lemma 2.4, with \( d = d_j + d_{j+1} - 1 - 2m \), and suppose that

\[
v = e_{i_1} \otimes \cdots \otimes e_{i_{j+2}} \otimes \tau_{0}^{(d,d_j+1)} \otimes e_{i_{j-1}} \otimes \cdots \otimes e_{i_1}.
\]

Then we have

\[
\mathcal{F}(x_0)[v](x) = \alpha^{(x_0)}_{l_1,\ldots,l_{j-1};0,m;l_{j+1},\ldots,l_n}(x).
\]
Proof. Notice first that for \( n = 2 \) we have \( q_{0,m}^{(x_0)}(x_1, x_2) = q^{(x_0)}_{1/m_1} \times 1^{(x_0)}_{0,m}(x_1, x_2) \), so the assertion is the same as Equation (4.4). The general case is similar. Indeed, the vector \( v \in \bigotimes_{i=1}^{n} M_{d_i} \) can be expanded in the standard tensor product basis as

\[
v = \sum_{k=0}^{m} T^{k,m-k}_{0;m}(d_j, d_{j+1}) \times (e^{(d_0)}_{i_0} \otimes \cdots \otimes e_{l_{j+2}}^{(d_{j+2})} \otimes e_{m-k}^{(d_{j+1})} \otimes e_{k}^{(d_{j})} \otimes e_{l_{j-1}}^{(d_{j-1})} \otimes \cdots \otimes e_{l_1}^{(d_{1})}),
\]

according to Lemma 2.4 Equations (2.8) and (2.7). Correspondingly, we obtain

\[
\mathcal{F}^{(x_0)}[v](x) = \sum_{k=0}^{m} T^{k,m-k}_{0;m}(d_j, d_{j+1}) \times \varphi^{(x_0)}_{l_1,\ldots,l_{j-1},k,m-k,l_{j+2},\ldots,l_n}(x).
\]

The terms in this sum are all integrals of the same integrand (up to phase factors). As for the integration contours, the number of loops around any \( x_i \) is fixed, with the exceptions of \( i = j \) and \( i = j + 1 \). The loops around \( x_j \) and \( x_{j+1} \) can be combined by essentially the same calculation that was done in Lemmas 3.6 and 4.1 with the asserted result. \( \square \)

With the above particular case established, we now proceed to the following generalization.

Lemma 4.3. If

\[
v = e_{l_{n}} \otimes \cdots \otimes e_{l_{j+2}} \otimes \tau^{(d_{d_j}, d_{j+1})}_{l_k} \otimes e_{l_{j-1}} \otimes \cdots \otimes e_{l_1},
\]

then (with \( d = d_{j} + d_{j+1} - 2m \))

\[
\mathcal{F}^{(x_0)}[v](x) = \alpha^{(x_0)}_{l_1,\ldots,l_{j-1};l,m;l_{j+2},\ldots,l_n}(x).
\]

Proof. By definition of the correspondence, we have

\[
\mathcal{F}^{(x_0)}[v](x) = \sum_{l_{j},l_{j+1}} T^{l_{j},l_{j+1}}_{l,m}(d_j, d_{j+1}) \times \varphi^{(x_0)}_{l_1,\ldots,l_n}(x),
\]

with the coefficients \( T^{l_{j},l_{j+1}}_{l,m}(d_j, d_{j+1}) \) given by Equation (2.10) for \( t_{l_k}^{(d_{d_j}, d_{j+1})} = F_{l_k}^{(d_{d_j}, d_{j+1})} \). Therefore, our goal is to rewrite, for any \( l \geq 0 \),

\[
\alpha^{(x_0)}_{l_1,\ldots,l_{j-1};l,m;l_{j+2},\ldots,l_n}(x) = \sum_{l_{j},l_{j+1}} T^{l_{j},l_{j+1}}_{l,m}(d_j, d_{j+1}) \times \varphi^{(x_0)}_{l_1,\ldots,l_n}(x),
\]

with the same coefficients. To achieve this, we will proceed by recursion on \( l \). The base case \( l = 0 \) was the content of Lemma 4.2. Using the formula (2.6) for the coproduct \( \Delta(F) \), we see that the coefficients (2.10) satisfy the recursion

\[
T^{l_{j},l_{j+1}}_{l,m} = q^{1-d_{j+1}+2l_{j+1}} T^{l_{j-1},l_{j+1}}_{l-1,m} + T^{l_{j},l_{j+1}-1}_{l-1,m}.
\]

Hence it suffices to show the same recursion for the coefficients appearing in Equation (4.5). For that, note that the difference between the integrations defining \( \alpha^{(x_0)}_{\ldots,l-1,m;\ldots}(x) \) and \( \alpha^{(x_0)}_{\ldots,l,m;\ldots}(x) \) is that the latter has one extra integration variable, integrated along a loop that surrounds both \( x_j \) and \( x_{j+1} \) and their associated \( u \)-variables. Assume (4.5) for the former, and decompose the extra loop of the latter to two pieces, loops anchored at \( x_0 \) that surround \( x_j \) and \( x_{j+1} \) separately. The desired recursion follows by comparing the phase factors after the decomposition. \( \square \)
Fix an index \( j \in \{1, \ldots, n-1 \} \). Recall Lemma 2.4 especially the decomposition (2.9) of a tensor product to subrepresentations \( M_{d,j+1} \otimes M_d \cong \bigotimes_{i=1}^{d} M_{d,i} \). For any \( d \) appearing in this sum, the linear map
\[
\ell_{j,j+1}^{(d)} : \bigotimes_{i=j+2}^{n} M_{d,i} \otimes M_{d} \otimes \bigotimes_{i=1}^{j-1} M_{d,i} \to \bigotimes_{i=1}^{n} M_{d,i},
\]
where the exponent (4.6)
\[
\begin{align*}
\ell_{j,j+1}^{(d)} (e_l \otimes \cdots \otimes e_{l_{j+2}} \otimes e_l \otimes e_{l_{j-1}} \otimes \cdots \otimes e_1) \\
= e_l \otimes \cdots \otimes e_{l_{j+2}} \otimes \tau_{l}^{(d,d_{j+1})} \otimes e_{l_{j-1}} \otimes \cdots \otimes e_1
\end{align*}
\]
is an embedding that respects the action of \( \mathcal{U}_q(\mathfrak{sl}_2) \). Hence we may interpret the shorter tensor product as a subrepresentation,
\[
(4.7) \quad \bigotimes_{i=j+2}^{n} M_{d,i} \otimes M_{d} \otimes \bigotimes_{i=1}^{j-1} M_{d,i} \subseteq \bigotimes_{i=1}^{n} M_{d,i},
\]
We denote the projection to this subrepresentation by
\[
\pi_{j,j+1}^{(d)} : \bigotimes_{i=1}^{n} M_{d,i} \to \bigotimes_{i=1}^{n} M_{d,i},
\]
and we denote by
\[
\tilde{\pi}_{j,j+1}^{(d)} : \bigotimes_{i=1}^{n} M_{d,i} \to \bigotimes_{i=1}^{n} M_{d,i} \otimes M_{d} \otimes \bigotimes_{i=1}^{j-1} M_{d,i},
\]
the projection combined with the identification (4.7), so that \( \pi_{j,j+1}^{(d)} = \tilde{\pi}_{j,j+1}^{(d)} \). A vector \( v \in \bigotimes_{i=1}^{n} M_{d,i} \) lies in this subrepresentation if and only if \( \pi_{j,j+1}^{(d)} (v) = v \), and in this case we typically denote \( \hat{v} = \pi_{j,j+1}^{(d)} (v) \).

We are now ready to write down the asympotics of the functions as two consecutive variables tend to a common limit. By the following proposition, the asympotics are determined by the submodule decompositions above.

**Proposition 4.4.** If \( v \in M_{d_1} \otimes \cdots \otimes M_{d_n} \) satisfies \( \pi_{j,j+1}^{(d)} (v) = v \), and we denote
\[
\hat{v} = \pi_{j,j+1}^{(d)} (v) \in \bigotimes_{i=j+2}^{n} M_{d,i} \otimes M_{d} \otimes \bigotimes_{i=1}^{j-1} M_{d,i},
\]
then we have the asymptotics
\[
\lim_{x_j,x_{j+1} \to \xi} (x_{j+1} - x_j)^{-\Delta_{d_1,d_{j+1}}^{d_2,d_{j+1}}} \times \mathcal{F}_{d_1,d_2,d_{j+1}}^{(x_0)} ([\hat{v}]) (x_1, \ldots, x_n),
\]
where the exponent \( \Delta_{d_1,d_{j+1}}^{d_2,d_{j+1}} \) is given by (3.5) and the multiplicative constant \( B_{d_1,d_{j+1}}^{d_2,d_{j+1}} \) by (3.6).

**Proof.** The vector \( v \) can be expressed as a linear combination of \( e_{l_n} \otimes \cdots \otimes e_{l_{j+2}} \otimes \tau_{l}^{(d_{j+1},d_{j+1})} \otimes e_{l_{j-1}} \otimes \cdots \otimes e_{l_1} \). For these vectors, the assertion follows by combining Lemmas 4.3 and 3.11. \( \square \)

### 4.4. Anchor point independence

Next we show that for highest weight vectors, the corresponding functions become well defined on the chamber \( X_n \) of (1.1).

**Proposition 4.5.** If \( v \in M_{d_1} \otimes \cdots \otimes M_{d_n} \) is such that \( E.v = 0 \), then we have
\[
\left( \mathcal{F}^{(x_0)} (v) \right) (x) = \sum_{m_2,m_3,\ldots,m_n} c_{m_2,\ldots,m_n} \times \rho_{0,m_2,\ldots,m_n} (x)
\]
for some coefficients \( c_{m_2,\ldots,m_n} \in \mathbb{C} \). In particular \( \left( \mathcal{F}^{(x_0)} (v) \right) (x) \) is independent of \( x_0 \).
Proof of Proposition 3.5. As a warm up, observe that in the case \(n = 1\) the statement is immediate: by Lemma 3.5, the basis vectors \(e_{i_1} \in M_{d_1}\) are mapped to \(\mathcal{F}(x_0)[e_{i_1}] \propto \rho_{i_1}^{(x_0)}\) and we have \(E.e_{i_1} = 0\) only if \(l_1 = 0\). For the case \(n = 2\) the statement was already shown in Lemma 4.1.

We proceed by induction on \(n\). Let \(n > 2\) and write \(v\) in the basis \(e_{i_1} \otimes \cdots \otimes e_{i_3} \otimes \tau^{(d)}_i\), with \(l_1, l_3, \ldots, l_n\) non-negative integers and \(d = d_1 + d_2 - 2m\) for some \(m\) as in Lemma 2.4,

\[
v = \sum_{l_1, l_3, \ldots, l_n} b_{l_1 l_3 \ldots l_n} \times e_{i_1} \otimes \cdots \otimes e_{i_3} \otimes \tau^{(d)}_i.
\]

Separate the parts corresponding to fixed values of \(d\), and denote (no summation over \(d\) here)

\[
v^{(d)} = \sum_{l_1, l_3, \ldots, l_n} b_{l_1 l_3 \ldots l_n} \times e_{i_1} \otimes \cdots \otimes e_{i_3} \otimes \tau^{(d)}_i \in \bigotimes_{i=1}^n M_{d_i}
\]

and

\[
\pi^{(d)}_{1,2}(v^{(d)}) = \tilde{v}^{(d)} = \sum_{l_1, l_3, \ldots, l_n} b_{l_1 l_3 \ldots l_n} \times e_{i_1} \otimes e_{i_3} \otimes e_{i_2} \in \left(\bigotimes_{i=1}^n M_{d_i}\right) \otimes M_2.
\]

Now \(\tilde{v}^{(d)}\) also satisfies \(E.\tilde{v}^{(d)} = 0\), and the tensor product only has \(n - 1\) tensorands. By the induction hypothesis,

\[
\left(\mathcal{F}(x_0)[\tilde{v}^{(d)}]\right)(\xi, x_3, \ldots, x_n) = \sum_{m_3 \ldots m_n} c_{m_3 \ldots m_n}^{(d)}(x_1, x_2, x_3, \ldots, x_n),
\]

where the implicit dimension parameters are \(d, d_3, d_4, \ldots, d_n\) now. In view of Lemma 4.3 and the rearrangement procedure of integration contours similar to Lemmas 3.7 and 3.6, one sees that also

\[
\left(\mathcal{F}(x_0)[v^{(d)}]\right)(x_1, x_2, x_3, \ldots, x_n) = \sum_{m_3 \ldots m_n} c_{m_3 \ldots m_n}^{(d)}(x_1, x_2, x_3, \ldots, x_n).
\]

The conclusion is finally obtained by summing over \(d\), since \(v = \sum_d v^{(d)}\). □

Remark 4.6. When \(\kappa\) is large enough, (3.4), the integrals \(\tilde{\rho}_{m_1 \ldots m_n}\) are convergent. When \(E.v = 0\), the above proposition then shows that \(\mathcal{F}(x_0)[v](x_1, \ldots, x_n)\) is independent of \(x_0\). By analyticity in \(\kappa\), this independence of \(x_0\) holds for all \(\kappa\), and we get a well-defined function \(\mathcal{F}[v]: X_n \to \mathbb{C}\) by

\[
\mathcal{F}[v](x_1, \ldots, x_n) = \mathcal{F}(x_0)[v](x_1, \ldots, x_n)
\]

for any \(x_0 < x_1\).

4.5. Integration by parts formula. According to the definition given in Section 1.1, the vectors \(v \in M_{d_1} \otimes \cdots \otimes M_{d_l}\) determine linear combinations of integration surfaces. With a suitable interpretation, there is a homology theory for these, in which the boundary operator corresponds to the action of the quantum group generator \(E\), see [FW91]. We will only make use of a version of Stokes formula, i.e., integration by parts, which we state and prove next.

Lemma 4.7. Let \(l_1, \ldots, l_n \in \mathbb{Z}_{\geq 0}\) and \(\ell = \sum_j l_j\). Suppose that \(g(w_\ell; x; w_1, \ldots, w_{\ell-1})\) is a holomorphic function of the \(\ell\) variables \(w_1, \ldots, w_{\ell-1}\) defined on \(\mathbb{H}^\ell\), which is symmetric in the \(\ell - 1\) variables \(w_1, \ldots, w_{\ell-1}\). Then we have

\[
\int_{\mathbb{H}^\ell} \sum_{r=1}^\ell \frac{\partial}{\partial w_r} \left( g(w_\ell; x; w_1, \ldots, w_{r-1}, w_{r+1}, \ldots, w_{\ell}) f^{\mathbb{H}}_{l_1 \ldots l_n}(x; w) \right) dw_1 \cdots dw_\ell
\]

\[
= \sum_{j=1}^n \left\{ (q^{-1} - q) [l_j] [d_j - l_j] q^{\sum_{i<j}(d_i - 1 - 2l_i)} \right\}
\times \left( \prod_{r=1}^\ell \left( \frac{\gamma(x; w_1, \ldots, w_{\ell-1}) f^{\mathbb{H}}_{l_1 \ldots l_j - 1 \ldots l_n}(x; w_1, \ldots, w_{\ell-1})}{\prod_{r=1}^\ell (x; w_1, \ldots, w_{\ell-1})} \right) dw_1 \cdots dw_{\ell-1} \right\},
\]
where
\[
\gamma(x; w_1, \ldots, w_{\ell-1}) = \prod_{i=1}^{n} |x_i - x_i|^{-\frac{2}{d_i}} \prod_{s \neq r} |x_s - w_r|^{\frac{n}{d}} g(x_0; x; w_1, \ldots, w_{\ell-1}).
\]

**Proof.** Let us first perform an integration by parts in a single term in (4.8). Fix $r$, and let $x_j$ be the point encircled by the loop of $w_r$ in $\Omega_{t_1, \ldots, t_n}$, and denote by $u$ the number of $s < r$ such that $w_s$ also encircles the same point $x_j$, that is, $u = r - 1 - \sum_{i<j} l_i$. We then perform integration by parts in the integral over $w_r$, and notice that the boundary terms from the beginning and end points of the loop only differ by a phase (formally $w_r = x_0$ both at the beginning and the end, but on different sheets of a Riemann surface). After this integration by parts, the contribution of the $r$:th term becomes
\[
g \sum_{i<j} (d_i-1-2l_i) (q^{1-d_j+2u} - q^{-1+d_j-2u}) \times \int_{\Omega_{t_1, \ldots, t_j-1, \ldots, t_n}} \left( \gamma(x; w_1, \ldots, w_{\ell-1}) f^{\oplus}_{t_1, \ldots, t_j-1, \ldots, t_n}(x; w_1, \ldots, w_{\ell-1}) \right) dw_1 \cdots dw_{\ell-1},
\]
where we relabeled the other integration variables and used the assumption of symmetric dependence of $g$ on them (and a similar property of $f$). We collect the terms corresponding to the same $j$, and use Lemma 2.1(a) in the form
\[
\sum_{j=0}^{l_j-1} (q^{d_j-1-2u} - q^{-d_j+1+2u}) = (q^{-1} - q) [l_j] [d_j - l_j]
\]
to simplify the sum of these terms. This concludes the proof. \hfill \square

**Corollary 4.8.** Let $\ell \in \mathbb{Z}_{\geq 0}$. Suppose that $g(w_{\ell}; x; w_1, \ldots, w_{\ell-1})$ is a holomorphic function of the $\ell$ variables $w_1, \ldots, w_{\ell}$ defined on $2\mathbb{H}^{(\ell)}$, which is symmetric in the $\ell - 1$ variables $w_1, \ldots, w_{\ell-1}$. If
\[
v = \sum_{l_1, \ldots, l_n \geq 0 \atop l_1 + \cdots + l_n = \ell} t_{l_1, \ldots, l_n} e_{l_1} \otimes \cdots \otimes e_{l_n}
\]
satisfies $E.v = 0$, then
\[
\sum_{l_1, \ldots, l_n} t_{l_1, \ldots, l_n} \int_{2\mathbb{H}^{(\ell)}} \sum_{r=1}^{\ell} \frac{\partial}{\partial w_r} \left( g(w_{\ell}; x; w_1, \ldots, w_{\ell-1}, w_{r+1}, \ldots, w_{\ell}) f^{\oplus}_{l_1, \ldots, l_n}(x; w) \right) dw_1 \cdots dw_{\ell} = 0.
\]

**Proof.** On the left hand side of the asserted formula, we use the previous lemma in each term, and write the left hand side as
\[
(q^{-1} - q) \sum_j \sum_{l_1, \ldots, l_n} [l_j] [d_j - l_j] q \sum_{i<j} (d_i-1-2l_i) t_{l_1, \ldots, l_n} \left( \int_{\Omega_{t_1, \ldots, t_j-1, \ldots, t_n}} \gamma f^{\oplus}_{l_1, \ldots, t_j-1, \ldots, t_n} \right).
\]
The requirement $E.v = 0$ amounts to the following equations
\[
\sum_{l_1, \ldots, l_n} [l_j] [d_j - l_j] q \sum_{i<j} (d_i-1-2l_i) t_{l_1, \ldots, l_n} = 0 \quad \text{for all } j.
\]
This shows that for each $j$ the coefficient in (4.9) vanishes. \hfill \square
4.6. Partial differential equations. Now we turn to partial differential equations satisfied by the functions associated to highest weight vectors.

Define, for any \( j = 1, \ldots, n \) and \( p \in \mathbb{Z} \), the first order partial differential operators

\[
\mathcal{L}_p^{(j)} := - \sum_{l \neq j} \left( (x_i - x_j)^{1+p} \frac{\partial}{\partial x_i} + (1 + p) h_{1,d_i} (x_i - x_j)^p \right),
\]

where \( h_{1,d} \) is given by Equation (3.7). Define also, for any \( j = 1, \ldots, n \), a partial differential operator of order \( d_j \) by the Benoit & Saint-Aubin formula [BSA88]

\[
\mathcal{D}_{d_j}^{(j)} := \sum_{k=1}^{d_j} \prod_{n_1 + \ldots + n_k = d_j} \frac{\prod_{j=1}^{k-1}(\sum_{i=1}^{n_j}(\sum_{i=j+1}^{n_j} n_j))}{(\kappa/4)^{d_i-k} (d_j - 1)!} L_{-n_1} \cdots L_{-n_k}.
\]

Lemma 4.9. The function \( f^{(0)} \) satisfies \( \mathcal{D}_{d_j}^{(j)} f^{(0)} = 0 \), for any \( j = 1, \ldots, n \).

Proof. This explicit differential equation for the product of powers of differences

\[
f^{(0)}(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i)^{\mathbf{x}_j - \mathbf{x}_i} \times f^{(0)}(x; w)
\]

follows for instance from properties of vertex operators, see e.g. [Fel89].

Remark 4.10. For any concrete case with definite value of \( d_j \), the above lemma always amounts to an equality of rational functions of fixed degree, which could also be checked in a straightforward (if tedious) manner — for the case \( d_j = 2 \), see e.g. [Kyt07].

Corollary 4.11. The function \( f^{(\ell)} \) satisfies, for any \( j = 1, \ldots, n \),

\[
\left( \mathcal{D}_{d_j}^{(j)} f^{(\ell)} \right)(x; w) = \sum_{r=1}^{\ell} \frac{\partial}{\partial w_r} \left( g(w_r; x; w_1, \ldots, w_{r-1}, w_{r+1}, \ldots, w_n) \times f^{(\ell)}(x; w) \right),
\]

where \( g \) is a rational function which is symmetric in the last \( \ell - 1 \) variables, and whose only poles are where some of its arguments coincide.

Proof. Denote \( \hat{n} = n + \ell \) and

\[
\hat{x} = (x_1, \ldots, x_n, w_1, \ldots, w_\ell)
\]

and define the function, a product of powers of differences of variables,

\[
\hat{f}^{(0)}(\hat{x}) = f^{(\ell)}(x; w).
\]

The conformal weights \( h \) associated to the variables \( w_r \) are \( h = h_{1,-1} = 1 \). We will apply Lemma 4.9 to the function \( \hat{f}^{(0)} \) of \( \hat{n} \) variables. We keep the notation \( \mathcal{L}_p^{(j)} \) and \( \mathcal{D}_{d_j}^{(j)} \) for the differential operators in the \( n \) variables \( x_1, \ldots, x_n \), and use the notation \( \hat{\mathcal{L}}_p^{(j)} \) and \( \hat{\mathcal{D}}_{d_j}^{(j)} \) for the differential operators in \( \hat{n} = n + \ell \) variables, that are appropriate for the application of the previous lemma. Explicitly, we have

\[
\hat{\mathcal{L}}_p^{(j)} = \mathcal{L}_p^{(j)} - \sum_{r=1}^{\ell} D_{w_r} M_{(w_r-x_j)}^{1+p},
\]

where \( D_{w_r} \) is the differential operator \( \frac{\partial}{\partial w_r} \), and \( M_{(w_r-x_j)}^{1+p} \) is the multiplication operator by the function \( (w_r - x_j)^{1+p} \), and we used the fact that, since \( h_{1,-1} = 1 \),

\[
D_{w_r} M_{(w_r-x_j)}^{1+p} = (w_r - x_j)^{1+p} \frac{\partial}{\partial w_r} + (1 + p) h_{1,-1} (w_r - x_j)^p.
\]
The operator $\mathcal{D}_{d_i}^{(j)}$ is a linear combination of terms, which we expand by the binomial formula
\[
\mathcal{L}_{-n_1}^{(j)} \cdots \mathcal{L}_{-n_k}^{(j)} = \sum_{A \subseteq \{1, \ldots, k\}} O_A^1 \cdots O_A^k,
\]
where $O_A^a = \begin{cases} \mathcal{L}_{n_a}^{(j)} & \text{if } a \notin A \\ \sum_r D_{w_r} M_{(w_r-x_j)^{i-n_a}} & \text{if } a \in A \end{cases}$.

The conclusion of Lemma 4.9 reads
\[
0 = \left( \mathcal{D}_{d_i}^{(j)} f^{(0)} \right)(\mathbf{x}) = \left( \mathcal{D}_{d_i}^{(j)} f^{(\ell)} \right)(\mathbf{x}; \mathbf{w}).
\]

Expanding by the binomial formula, we observe that the terms with $A = \emptyset$ give precisely the LHS of the assertion, namely $\mathcal{D}_{d_i}^{(j)} f^{(\ell)}$. When $A \neq \emptyset$, choose the minimal $a \in A$, and write the term in the form
\[
\sum_r D_{w_r} \mathcal{L}_{-n_1}^{(j)} \cdots \mathcal{L}_{-n_{a-1}}^{(j)} M_{(w_r-x_j)^{i-n_a}} O_{a+1}^A \cdots O_k^A f^{(\ell)}
\]
by noticing that $\mathcal{L}_{-n_1}^{(j)} \cdots \mathcal{L}_{-n_{a-1}}^{(j)}$ does not contain $w_r$ and can be moved inside the differentiation $D_{w_r}$.

These remaining terms put together constitute the RHS of the assertion
\[
\sum_{r=1}^\ell \frac{\partial}{\partial w_r} \left( g(w_r; \mathbf{x}; w_1, \ldots, w_{r-1}, w_{r+1}, \ldots, w_k) \times f^{(\ell)}(\mathbf{x}; \mathbf{w}) \right).
\]

It is easy to see that $g$ has no poles where the variables do not coincide, and is symmetric in its last $\ell - 1$ variables.

We now conclude by the important property that the functions which correspond to highest weight vectors satisfy the Benoît & Saint-Aubin partial differential equations.

**Proposition 4.12.** If $E.v = 0$, then the function $\mathcal{F}[v]$ satisfies $\mathcal{D}_{d_i}^{(j)} \mathcal{F}[v] = 0$.

**Proof.** By linearity, we may assume that $v$ is an eigenvector of $K$, i.e. that the number of integration variables $\ell$ is fixed. Dominated convergence ensures that we can take the differential operator $\mathcal{D}_{d_i}^{(j)}$ inside the integral, and thus let it act directly to the integrand $f^{(\ell)}$. Corollary 4.11 then implies that $\mathcal{D}_{d_i}^{(j)} \mathcal{F}[v](\mathbf{x})$ is a linear combination of terms of the form studied in the integration by parts formula, Lemma 4.7. Assuming $E.v = 0$ it follows from Corollary 4.8 that this linear combination vanishes.

**Remark 4.13.** If one of the tensorands is a trivial representation, $d_i = 1$, the $i$:th PDE is merely
\[
\frac{\partial}{\partial x_i} \mathcal{F}[v](x_1, \ldots, x_i, \ldots, x_n) = 0,
\]
which is also a consequence of the statement of Lemma 3.4. If $d_i = 2$, the $i$:th PDE is similar to (1.3), and it can always be interpreted as a local martingale property of the function for a chordal SLE$_\kappa$.

#### 4.7. Möbius covariance.

Translation invariance and scaling covariance were shown for all basis functions $\varphi^{(x_0)}$ in Lemma 3.3. We will now show that if a vector $v$ is in a trivial (one-dimensional) subrepresentation of the entire tensor product, then the corresponding function $\mathcal{F}[v]$ transforms covariantly under all Möbius transformations. The covariance weights are $h_{1,d_i}$, given by (3.7).

We first record a property of the integrand, which will be used in the proof of Möbius covariance.

**Lemma 4.14.** If $\ell = \frac{1}{2} \sum_{i=1}^n (d_i - 1)$, then the function $f^{(\ell)}$ satisfies the partial differential equation
\[
\left( \sum_{i=1}^n (x_i^2 \frac{\partial}{\partial x_i} + 2h_{1,d_i} x_i) \right) f^{(\ell)}(\mathbf{x}; \mathbf{w}) = \sum_{r=1}^\ell \frac{\partial}{\partial w_r} \left( g(w_r; \mathbf{x}; w_1, \ldots, w_{r-1}, w_{r+1}, \ldots, w_k) \times f^{(\ell)}(\mathbf{x}; \mathbf{w}) \right),
\]
where $g$ is a rational function which is symmetric in the last $\ell - 1$ variables, and whose only poles are where some of its arguments coincide.
Proof. We will do an explicit calculation that shows the claimed identity, with

\[
g(w; x; w_1, \ldots, w_r, w_{r+1}, \ldots, w_{\ell}) = -w_r + \prod_{i=1}^{n} (w_r - x_i)^{d_i-1} \prod_{s \neq r} (w_r - w_s)^{-2}.
\]

One can start by observing, by a direct calculation that uses the assumption \(\ell = \frac{1}{2} \sum_{i=1}^{n} (d_i - 1)\), that

\[
\left(\sum_{i=1}^{n} \left( x_i^2 \frac{\partial}{\partial x_i} + 2h_{1,d_i} x_i \right) \right) f^{(\ell)}(x; w) = - \sum_{r=1}^{\ell} \frac{\partial}{\partial w_r} \left( w_r^2 f^{(\ell)}(x; w) \right) + \left( 1 - \frac{4}{n} \right) \left( \sum_{i=1}^{n} (1 - d_i) x_i + 2 \sum_{r=1}^{\ell} w_r \right) f^{(\ell)}(x; w).
\]

Comparing with the explicit \(g\) in Equation (4.10), the claim now reduces to the vanishing of (4.11)

\[
\sum_{r=1}^{\ell} \prod_{i=1}^{n} (w_r - x_i)^{d_i-1} \prod_{s \neq r} (w_r - w_s)^{-2} \left( \sum_{j=1}^{n} d_j - 1 \right) \frac{1}{w_r - x_j} - 2 \sum_{u \neq r} \frac{1}{w_r - w_u} - 2 \sum_{r=1}^{\ell} w_r + \sum_{i=1}^{n} (d_i - 1) x_i,
\]

which is a rational function in the variables \(w_r, r = 1, \ldots, \ell\). Note that there are no poles of (4.11) except possibly poles of degree at most three at \(w_r = w_s\) for some \(r \neq s\). To show that these points are in fact not poles, it is by symmetry sufficient to consider the Laurent series expansion in, for example, the difference \(\epsilon = w_2 - w_1\). This can be done in a straightforward manner. In addition, one verifies that with fixed \(w_2, \ldots, w_{\ell}\), as \(w_1 \to \infty\), the function tends to zero. Thus the expression (4.11) is an entire function of \(w_1\) tending to zero at infinity, and as such vanishes identically. This concludes the proof.

Proposition 4.15. If \(E.v = 0\) and \(K.v = v\), then for any Möbius transformation \(\mu: \mathbb{H} \to \mathbb{H}\) such that \(\mu(x_1) < \mu(x_2) < \cdots < \mu(x_n)\), the function \(F[v]\) satisfies

\[
\prod_{i=1}^{n} \mu'(x_i)^{h_1,d_i} \times F[v](\mu(x_1), \ldots, \mu(x_n)) = F[v](x_1, \ldots, x_n).
\]

Proof. Any Möbius transformation \(\mu: \mathbb{H} \to \mathbb{H}\) can be written as a composition of a translation \(z \mapsto z + \xi\) (for some \(\xi \in \mathbb{R}\), a scaling \(z \mapsto \lambda z\) (for some \(\lambda > 0\), and a special conformal transformation \(z \mapsto \frac{z - 1}{az - 1}\) (for some \(a \in \mathbb{R}\)). Lemma 3.3 shows the assertion for translations and scalings. It suffices to prove the statement for special conformal transformations.

For the special conformal transformation, we may assume that \(x_1 < 0\) and \(x_n > 0\), by precomposing with a translation if necessary. Then, the special conformal transformation \(\mu_a(z) = \frac{z}{1 + az}\) respects the order of the boundary points \(x_1, \ldots, x_n\) if \(a \in (\frac{1}{x_n}, \frac{1}{x_1})\). To obtain the general case we will integrate an infinitesimal form of the formula starting from the trivial case of \(a = 0\).

Consider a term

\[
\prod_{i=1}^{n} \mu'(x_i)^{h_1,d_i} \times \int_{L_{11}^{\infty}, \ldots, L_{1n}^{\infty}} f^{\infty}_{L_{11}, \ldots, L_{1n}}(\mu_a(x_1), \ldots, \mu_a(x_n); w_1, \ldots, w_\ell) \, dw_1 \cdots dw_\ell
\]

on the left hand side of the asserted equation. Using \(\frac{d}{da} \mu_a(z) = -\mu_a(z)^2\) and \(\frac{d}{da}(\mu_a(z)^h) = -2h \mu_a(z) \times \mu_a'(z)^h\) we compute its derivative with respect to \(a\),

\[
\frac{d}{da} \left( \prod_{i=1}^{n} \mu'(x_i)^{h_1,d_i} \times \int_{L_{11}^{\infty}, \ldots, L_{1n}^{\infty}} f^{\infty}_{L_{11}, \ldots, L_{1n}}(\mu_a(x_1), \ldots, \mu_a(x_n); w) \, dw_1 \cdots dw_\ell \right)
\]

\[=
\left( \prod_{i=1}^{n} \mu'(x_i)^{h_1,d_i} \right) \times \int_{L_{11}^{\infty}, \ldots, L_{1n}^{\infty}} (L_{11} f^{\infty}_{L_{11}, \ldots, L_{1n}})(\mu_a(x_1), \ldots, \mu_a(x_n); w) \, dw_1 \cdots dw_\ell,
\]
Recall the following notation, to be used in the statement below. The exponents show how the properties of the vector results about the spin chain - Coulomb gas correspondence, and finish their proofs. These results differential operators (Main theorems about the correspondence).

4.8. Thus, by Lemma 4.14, we have

\[ \mathcal{L}_1 = -\sum_{i=1}^{n} \left( x_i^2 \frac{\partial}{\partial x_i} + 2h_{1,d,x_i} \right). \]

The assumption \( K.v = v \) guarantees that in these terms \( \ell = \sum l_i \) takes the value \( \ell = \frac{1}{2} \sum_{i=1}^{n} (d_i - 1) \). Thus, by Lemma 4.14 we have

\[ (\mathcal{L}_1 f_{l_1,\ldots,l_n}^\circ(x;w)) = \sum_{r=1}^{\ell} \frac{\partial}{\partial w_r} \left( g(w_r; x; w_1, \ldots, w_{r-1}, w_{r+1}, \ldots, w_\ell) \times f_{l_1,\ldots,l_n}^\circ(x;w) \right), \]

where \( g \) single valued and symmetric with respect to the last \( \ell - 1 \) variables. Since \( E.v = 0 \) we can apply Corollary 4.8 to the \( a \)-derivative of the left hand side of the asserted formula, and get

\[ \frac{d}{da} \left( \prod_{i=1}^{n} \mu_i^{h_{1,d,i}} \times \mathcal{F}[v](\mu(x_1), \ldots, \mu(x_n)) \right) = 0. \]

It now follows that also the left hand side of the asserted formula is constant in \( a \) for \( a \in (\frac{1}{x_n}, \frac{1}{x_1}) \). At \( a = 0 \) we have \( \mu_n = \text{id}_{\mathcal{H}} \), so this constant equals \( \mathcal{F}[v](x_1, \ldots, x_n) \).

4.8. Main theorems about the correspondence. We now give the precise statements of the main results about the spin chain - Coulomb gas correspondence, and finish their proofs. These results show how the properties of the vector \( v \in M_{d_n} \otimes \cdots \otimes M_{d_1} \) translate to properties of the function \( (x_0, x_1, \ldots, x_n) \mapsto \mathcal{F}(x_0)[v](x_1, \ldots, x_n) \). Three types of properties of the functions are considered: partial differential equations (PDE), covariance properties (COV) and asymptotics (ASY).

Recall the following notation, to be used in the statement below. The exponents \( h_{1,d} \) and \( \Delta_{d_1,\ldots,d_n}^{d_1,\ldots,d_n} \) are given by

\[ h_{1,d} = \frac{(d - 1)(2(d + 1) - \kappa)}{2\kappa} \quad \text{and} \quad \Delta_{d_1,\ldots,d_n}^{d_1,\ldots,d_n} = h_{1,d} - \sum_{i=1}^{n} h_{1,d,i}. \]

The multiplicative constants \( B_{d_1}^{d',d''} \) are defined for \( \kappa > 4 \times (\max \{ d', d'' \} - 1) \) by the convergent integral over an \( m \)-dimensional simplex, with \( m = \frac{1}{2} ((d' + d'' - 1) - d) \),

\[ B_{d_1}^{d',d''} = \int_0^1 dw_1 \int_0^1 dw_2 \cdots \int_0^1 dw_m \prod_{r=1}^{m} w_r^{\frac{\kappa}{2}(d' - 1)} \prod_{1 \leq r < s \leq m} (w_s - w_r)^\frac{\kappa}{2} \prod_{r=1}^{m} \left( 1 - w_r \right)^\frac{\kappa}{2} \prod_{r=1}^{m} \left( w_s - w_r \right)^\frac{\kappa}{2}, \]

and for general values of \( \kappa \) by the analytic continuation of this generalized beta-function. The partial differential operators \( \mathcal{D}_{d_1}^{(j)} \) (which depend also on \( d_1, \ldots, d_n \)) are defined in terms of

\[ \mathcal{L}_p^{(j)} = -\sum_{i \neq j} \left( (x_i - x_j)^{1+p} \frac{\partial}{\partial x_i} + (1 + p) h_{1,d,i} (x_i - x_j)^p \right), \]

by the Benoit & Saint-Aubin formula

\[ \mathcal{D}_{d_1}^{(j)} = \sum_{k=1}^{d_1} \sum_{n_1, \ldots, n_k \geq 1 \atop n_1 + \cdots + n_k = d_j} \frac{(\kappa/4)^{d_j - k} (d_j - 1)^{12}}{\prod_{j=1}^{k-1} \sum_{i=1}^{j} n_i (\sum_{i=j+1}^{k} n_i)} \times \mathcal{L}_{-n_1}^{(j)} \cdots \mathcal{L}_{-n_k}^{(j)}. \]

The most important cases concern highest weight vectors, i.e. vectors \( v \) satisfying \( E.v = 0 \). We first state, however, the properties that do not depend on this assumption.

**Theorem 4.16.** Let \( v \in M_{d_n} \otimes \cdots \otimes M_{d_1} \). The function \( (x_0; x_1, \ldots, x_n) \mapsto \mathcal{F}(x_0)[v](x_1, \ldots, x_n) \) satisfies the following properties:
Let \( v \in M_d \otimes \cdots \otimes M_d \) satisfy \( E.v = 0 \). Then \( F(x_0)[v](x_1, \ldots, x_n) \) is independent of \( x_0 \), and thus defines a function \( F[v] \) on \( \mathbb{H}_n \). This function \( (x_1, \ldots, x_n) \mapsto F[v](x_1, \ldots, x_n) \) satisfies the following properties:

\( \text{COV)}: \text{For any } \xi \in \mathbb{R}, \text{ we have the translation invariance} \)

\[
F(x_0 + \xi)[v](x_1 + \xi, \ldots, x_n + \xi) = F(x_0)[v](x_1, \ldots, x_n).
\]

If furthermore \( K.v = q^{d-1}v \), then for any \( \lambda > 0 \) we have the scaling covariance

\[
F(\lambda x_0)[v](\lambda x_1, \ldots, \lambda x_n) = \lambda \Delta^{n-\Delta_j} \times F(x_0)[v](x_1, \ldots, x_n).
\]

\( \text{PDE): With } D^{(j)}_d \text{ the differential operator } E[v] = 0 \quad \text{for } j = 1, \ldots, n. \)

\( \text{COV): For any } \xi \in \mathbb{R}, \text{ we have the translation invariance} \)

\[
F[v](x_1 + \xi, \ldots, x_n + \xi) = F[v](x_1, \ldots, x_n).
\]

If furthermore \( K.v = q^{d-1}v \), then for any \( \lambda > 0 \) we have the scaling covariance

\[
F[v](\lambda x_1, \ldots, \lambda x_n) = \lambda \Delta^{n-\Delta_j} \times F[v](x_1, \ldots, x_n).
\]

If furthermore \( K.v = v \), then we have full Möbius covariance

\[
\prod_{j=1}^n \mu'(x_j^h d_j) \times F[v](\mu(x_1), \ldots, \mu(x_n)) = F[v](x_1, \ldots, x_n)
\]

for any Möbius transformation \( \mu: \mathbb{H} \to \mathbb{H} \) such that \( \mu(x_1) < \mu(x_2) < \cdots < \mu(x_n) \).

\( \text{ASY): If } v = \pi^{(d)}_{j,j+1}(v) \text{ and we denote} \)

\[
\hat{v} = \pi^{(d)}_{j,j+1}(v) \in \left( \bigotimes_{i=j+2}^n M_{d_i} \right) \otimes M_d \otimes \left( \bigotimes_{i=1}^{j-1} M_{d_i} \right)
\]

then also \( E.\hat{v} = 0 \) and we have

\[
\lim_{x_j, x_{j+1} \to \xi} \left( (x_{j+1} - x_j)^{\Delta^{d_j}} \times F[v](x_1, \ldots, x_n) \right) = B^{-\Delta^{d_j}}_d \times F[v](x_1, \ldots, x_{j-1}, \xi, x_{j+2}, \ldots, x_n).
\]
Proof of Theorem 4.17. That \( F^{(x_0)}[v] \) does not depend on \( x_0 \) follows from Proposition 4.15 and Remark 4.6 after it — we may define

\[
F[v](x_1, \ldots, x_n) = F^{(x_0)}[v](x_1, \ldots, x_n)
\]

for any \( x_0 < x_1 \).

The statement (PDE) was shown in Proposition 4.12.

Of the statements (COV), the first two are direct consequences of the corresponding statements in Theorem 4.16 for \( F^{(x_0)}[v] \). The third statement was shown in Proposition 4.15.

The statement (ASY) is a direct consequence of the corresponding statement in Theorem 4.16 for \( F^{(x_0)}[v] \). \( \square \)

5. Further properties

In this section, we establish two more properties of the correspondence. First, in Section 5.1, we treat a generalization of the asymptotics property (ASY), to the case where more than two of the variables tend to a common limit. Then, in Section 5.2, we apply the general asymptotics statement to consider a generalization of the asymptotics property (ASY), to the case where more than two of the variables tend to a common limit. Then, in Section 5.2, we apply the general asymptotics statement to consider a generalization of the asymptotics property (ASY), to the case where more than two of the variables tend to a common limit.

5.1. Asymptotics as several variables tend to a common limit. Suppose that \( 1 \leq j < k \leq n \).

We first address what happens to the function

\[
F^{(x_0)}[v](x_1, \ldots, x_j, \ldots, x_k, \ldots, x_n)
\]

in the limit

\[
x_j, x_{j+1}, \ldots, x_{k-1}, x_k \to \xi \quad \text{taken in such a way that} \quad \frac{x_i - x_j}{x_k - x_j} \to \eta_i \quad \text{for} \quad i \in \{j, j + 1, \ldots, k - 1, k\}.
\]

We assume that \( x_{j+1} - \xi < x_{k+1} - x_k \) and \( 0 = \eta_j < \eta_{j+1} < \cdots < \eta_{k-1} < \eta_k = 1 \).

The case \( k = j + 1 \), when only two points come together, was the content of part (ASY) of Theorems 4.17 and 4.16. It was based on the decomposition

\[
M_{d_{j+1}} \otimes M_{d_j} \cong \bigoplus_d M_d.
\]

An important difference in the present case will be that in the decomposition

\[
M_{d_k} \otimes \cdots \otimes M_{d_{j+1}} \cong \bigoplus_d m_d M_d
\]

some of the multiplicities \( m_d \) may be greater than one. Note, however, that any vector in \( m_d M_d \subset M_{d_k} \otimes \cdots \otimes M_{d_{j+1}} \) may be written as a linear combination of vectors of the form \( F^l \tau_0 \), where \( 0 \leq l < d \) and \( \tau_0 \) is some highest weight vector of an irreducible subrepresentation of dimension \( d \) (there are \( m_d \) linearly independent such highest weight vectors). It is therefore possible to deduce the behavior of the functions in general from the following result.

Proposition 5.1. Suppose that \( \tau_0 \in M_{d_k} \otimes \cdots \otimes M_{d_{j+1}} \) satisfies \( E.\tau_0 = 0 \) and \( K.\tau_0 = q^{d-1}v \). Let

\[
v = e_{t_n} \otimes \cdots \otimes e_{t_{k+1}} \otimes F^l \tau_0 \otimes e_{t_{j-1}} \otimes \cdots \otimes e_{t_i} \in M_{d_k} \otimes \cdots \otimes M_{d_i}
\]

\[
\hat{v} = e_{t_n} \otimes \cdots \otimes e_{t_{k+1}} \otimes e_{t_{j-1}} \otimes \cdots \otimes e_{t_1} \in M_{d_k} \otimes \cdots \otimes M_{d_{k+1}} \otimes M_d \otimes M_{d_{k-1}} \otimes \cdots \otimes M_{d_i},
\]

and let \( \Delta = \Delta_d \cdots \Delta_k \) as in Section 4.8. Then in the limit (5.1) we have

\[
\frac{F^{(x_0)}[v](x_1, \ldots, x_n)}{|x_k - x_j|^\Delta} \to F[\tau_0](\eta_j, \ldots, \eta_k) \times F^{(x_0)}[\hat{v}](x_1, \ldots, x_{j-1}, \xi, x_{k+1}, \ldots, x_n).
\]
We have preferred to formulate the above proposition for a vector $v$ of specific form. The reason is that the limit function’s dependence on the ratios $\eta_j, \ldots, \eta_k$ depends on the $d$-dimensional irreducible subrepresentation whose highest weight vector is $\tau_0$. Note, however, that the exponent $\Delta = \Delta_d^{j\ldots k}$ in the asymptotics is the same for all $d$-dimensional subrepresentations of $M_{j\ldots k}$, and therefore a similar limit exists also more generally, as stated in the following.

**Corollary 5.2.** Suppose that $v \in \bigotimes_{i=1}^n M_{d_i}$ belongs to the subrepresentation

$$\left( \bigotimes_{i=k+1}^n M_{d_i} \right) \otimes (m_d M_d) \otimes \left( \bigotimes_{i=1}^{j-1} M_{d_i} \right).$$

Then the expression $\mathcal{F}^{(x_0)}[v](x_1, \ldots, x_n) \propto |x_k - x_j|^{-\Delta}$, with $\Delta = \Delta_d^{j\ldots k}$, has a limit [5.1].

**Proof of Proposition [5.1]** The proof follows a strategy parallel to the simpler asymptotics properties shown before. The crucial step is to rearrange the integrations to a form where there are deformed hypercube type contours between those variables which tend to a common limit, and all other contours are loops which either encircle one other variable, or encircle all the points with a common limit together. Once rearranged this way, dominated convergence theorem may be applied to complete the proof. The rearrangement itself is done in two steps, analogous to the two lemmas in Section 4.3. We only sketch the proofs, as the omitted details are reasonably straightforward modifications of the case already considered.

Analogously to Lemma [4.2] we first handle the case $l = 0$ in which $\tau_0$ is a highest weight vector. We use the fact that the highest weight vector $\tau_0$ corresponds to a closed integration contour. This is expressed precisely in Proposition [4.5] which allows us to rearrange the integrals in $\mathcal{F}^{(x_0)}[\tau_0](x, \ldots, x)$ so that no integration contour starts from $x_0$:

$$\left( \mathcal{F}^{(x_0)}[\tau_0] \right)(x_j, \ldots, x_k) = \sum c_{m_{j+1}, \ldots, m_k}^{(x_0)}(x_{j+1}, \ldots, x_k).$$

The sum $m_{j+1} + \cdots + m_k$ of the indices is fixed, and it equals $\frac{1}{2}(\sum_{i=j}^{k} d_i - k + j - d)$. For the vector $v = e_{l_0} \otimes \cdots \otimes e_{l_{k+1}} \otimes \tau_0 \otimes e_{l_{j-1}} \otimes \cdots \otimes e_{l_1},$ by the same rearrangements of integrals, we obtain the expression

$$\left( \mathcal{F}^{(x_0)}[v] \right)(x_1, \ldots, x_n) = \sum c^{(x_0)}_{m_{j+1}, \ldots, m_k} \alpha_{l_0, \ldots, l_{j-1}, \{m_{j+1}, \ldots, m_k\}; i_{k+1}, \ldots, i_n}(x_1, \ldots, x_n),$$

where $\alpha_{l_0, \ldots, l_{j-1}, \{m_{j+1}, \ldots, m_k\}; i_{k+1}, \ldots, i_n}(x_1, \ldots, x_n)$ are generalizations of the mixed integrals $\alpha_{l_0, \ldots, l_{j-1}; \{m_{j+1}, \ldots, m_k\}; i_{k+1}, \ldots, i_n}(x_1, \ldots, x_n)$ defined in Section 3.4. The integration contours to variables $x_j, \ldots, x_k$ are as in $p_{l_0, \ldots, l_{j-1}; \{m_{j+1}, \ldots, m_k\}; i_{k+1}, \ldots, i_n}$ and all other contours $x_i$ are encircled by $l_i$ non-intersecting nested loops based at the anchor point $x_0$.

The next step is to consider the case of general $l$, in a manner analogous to Lemma [4.3]. By comparison with the $(k - j)$-fold coproduct formula for the quantum group generator $F$ given in Lemma [2.2], one shows recursively in $l$ that for vectors

$$v = e_{l_0} \otimes \cdots \otimes e_{l_{k+1}} \otimes F^l \tau_0 \otimes e_{l_{j-1}} \otimes \cdots \otimes e_{l_1},$$

the integrals can be rearranged to

$$\left( \mathcal{F}^{(x_0)}[v] \right)(x_1, \ldots, x_n) = \sum c^{(x_0)}_{m_{j+1}, \ldots, m_k} \alpha_{l_0, \ldots, l_{j-1}; i, \{m_{j+1}, \ldots, m_k\}; i_{k+1}, \ldots, i_n}(x_1, \ldots, x_n),$$

where $\alpha_{l_0, \ldots, l_{j-1}; i, \{m_{j+1}, \ldots, m_k\}; i_{k+1}, \ldots, i_n}(x_1, \ldots, x_n)$ are generalizations of the mixed integrals $\alpha_{l_0, \ldots, l_{j-1}; i, \{m_{j+1}, \ldots, m_k\}; i_{k+1}, \ldots, i_n}(x_1, \ldots, x_n)$. Compared to the case $l = 0$, the new feature is that the variables $x_j, \ldots, x_k$ together with all integration contours connected to them are encircled by $l$ non-intersecting nested loops based at the anchor point $x_0$.

To reach the conclusion, we need to perform the limit [5.1] of $\alpha_{l_0, \ldots, l_{j-1}; i, \{m_{j+1}, \ldots, m_k\}; i_{k+1}, \ldots, i_n}$. Once we divide by $|x_k - x_j|^\Delta$, dominated convergence theorem can be applied to the integration over all variables whose contour is a loop, since these contours remain bounded away from the points $x_j, \ldots, x_k$ and any hypercube type integration contours between them. The loop type integration contours are
the same as for \( F^{(x_0)}[v](x_1, \ldots, x_n) \). The integral over the hypercube type contour
divided by \(|x_k - x_j|^\Delta\) tends to the integrand of \( F^{(x_0)}[v] \) multiplied by \( F[\tau_0](\eta_j, \ldots, \eta_k) \). The asserted result follows. \(\square\)

\section{Moving one point to infinity.}

In the Möbius covariant case, we will now consider what happens to the function
\[ F[v](x_1, \ldots, x_n) \]
as \( x_n \to +\infty \). For this, we will need to be able to move from the trivial subrepresentation of \( \bigotimes_{j=1}^n M_{d_j} \) to the sum of copies of \( d_n \)-dimensional irreducible subrepresentations of \( \bigotimes_{j=1}^{n-1} M_{d_j} \).

Symmetrically, we consider what happens to the above function as \( x_1 \to -\infty \), in which case we will need to be able to move from the trivial subrepresentation of \( \bigotimes_{j=1}^n M_{d_j} \) to the sum of copies of \( d_1 \)-dimensional irreducible subrepresentations of \( \bigotimes_{j=2}^n M_{d_j} \). The following lemma provides the needed mappings in the two cases.

\begin{lemma}
Let \( H_1 \subset \bigotimes_{j=1}^n M_{d_j} \) denote the trivial subrepresentation
\[ H_1 = \left\{ v \in \bigotimes_{j=1}^n M_{d_j} \mid E.v = 0, K.v = v \right\} . \]

(a) Any vector \( v \in H_1 \) can be written uniquely in the form
\[ v = \sum_{l_n=0}^{d_n-1} A^+_n (e_n \otimes (F^{d_n-1-l_n} \tau_0^+)), \quad \text{with } A^+_n = (-1)^{d_n-1-l_n} q^{(l_n+1)(d_n-1-l_n)}, \]
where \( \tau_0^+ \in \bigotimes_{j=1}^{n-1} M_{d_j} \) satisfies \( E.\tau_0^+ = 0 \) and \( K.\tau_0^+ = q^{d_n-1}\tau_0^+ \). The mapping \( v \mapsto \tau_0^+ = R_+(v) \)
defines a linear isomorphism \( R_+: H_1 \to H^+_{d_n} \) to the space
\[ H^+_{d_n} = \left\{ \tilde{v} \in \bigotimes_{j=1}^{n-1} M_{d_j} \mid E.\tilde{v} = 0, K.\tilde{v} = q^{d_n-1}\tilde{v} \right\} \]
of highest weight vectors of irreducible subrepresentations of dimension \( d_n \) in \( \bigotimes_{j=1}^{n-1} M_{d_j} \).

(b) Any vector \( v \in H_1 \) can be written uniquely in the form
\[ v = \sum_{l_1=0}^{d_1-1} A^-_1 (\tilde{F}^{d_1-1-l_1} \tau_0^-) \otimes e_1), \quad \text{with } A^-_1 = (-1)^{d_1-1-l_1} q^{(l_1-1)(d_1-1-l_1)}, \]
where \( \tau_0^- \in \bigotimes_{j=2}^n M_{d_j} \) satisfies \( E.\tau_0^- = 0 \) and \( K.\tau_0^- = q^{d_1-1}\tau_0^- \). The mapping \( v \mapsto \tau_0^- = R_-(v) \)
defines a linear isomorphism \( R_-: H_1 \to H^-_{d_1} \) to the space
\[ H^-_{d_1} = \left\{ \tilde{v} \in \bigotimes_{j=2}^n M_{d_j} \mid E.\tilde{v} = 0, K.\tilde{v} = q^{d_1-1}\tilde{v} \right\} \]
of highest weight vectors of irreducible subrepresentations of dimension \( d_1 \) in \( \bigotimes_{j=2}^n M_{d_j} \).
\end{lemma}

\textbf{Proof.} The two parts are similar, so we only give the details for part (a). Any vector \( v \) in the tensor product \( \bigotimes_{j=1}^n M_{d_j} \) can be written in the form \( v = \sum_{l_n} e_{l_n} \otimes u_{l_n} \) with unique vectors \( u_{l_n} \in \bigotimes_{j=1}^{n-1} M_{d_j} \).
From the eigenvalue property \( K.e_{l_n} = q^{d_n-1-2l_n} e_{l_n} \) and the coproduct formula \( \Delta(K) = K \otimes K \), it follows that
\[ K.v = \sum_{l_n} q^{d_n-1-2l_n} e_{l_n} \otimes (K.u_{l_n}). \]
The assumption $K.v = v$ thus implies $K.\varepsilon_{l_n} = q^{1-d_n+2l_n} \varepsilon_{l_n}$. Similarly, from the property $E.\varepsilon_{l_n} = [l_n][d_n-l_n] \varepsilon_{l_n-1}$, and the coproduct formula $\Delta(E) = E \otimes K + 1 \otimes E$, and the already established $K$-eigenvalue of $\varepsilon_{l_n}$, it follows that

$$E.v = \sum_{l_n} e_{l_n} \otimes \left( E.\varepsilon_{l_n} + [l_n+1][d_n-l_n-1]q^{3-d_n+2l_n} \varepsilon_{l_n+1} \right).$$

The assumption $E.v = 0$ thus implies $E.\varepsilon_{l_n} = -[l_n+1][d_n-l_n-1]q^{3-d_n+2l_n} \varepsilon_{l_n+1}$. In particular, $\varepsilon_{l_n-1}$ is a highest weight vector of an irreducible submodule of dimension $d_n$. We denote this vector by $\varepsilon_{l_n-1} = \tau_0^+$. We furthermore claim that this vector determines the other $\varepsilon_{l_n}$ uniquely as $\varepsilon_{l_n} = A_1^+ F^{d_n-l_n-1, \tau_0^+}$. To see this, use the coproduct formula $\Delta(F) = F \otimes 1 + K^{-1} \otimes F$, to get

$$F.v = \sum_{l_n} e_{l_n} \otimes \left( \varepsilon_{l_n-1} + q^{1-d_n+2l_n} F.\varepsilon_{l_n} \right).$$

Note that this expression for $F.v$ must vanish, since $v$ is in the trivial subrepresentation. Therefore the other $\varepsilon_{l_n}$ are obtained recursively from $\varepsilon_{l_n-1} = \tau_0^+$, by $\varepsilon_{l_n-1} = -q^{1-d_n+2l_n} F.\varepsilon_{l_n}$. The solution of this recursion is the asserted formula $\varepsilon_{l_n} = A_1^+ F^{d_n-l_n-1, \tau_0^+}$, and since this formula indeed satisfies $E.v = 0$ and $K.v = v$, we get that the linear mapping $R_+$ defined by $R_+(v) = \varepsilon_{l_n-1}$ is bijective.

We now show that the behavior of Möbius covariant functions as $x_n \to +\infty$ (resp. $x_1 \to -\infty$) can be expressed in terms of the identification $R_+$ (resp. $R_-$) defined in Lemma 5.3.

**Proposition 5.4.** Let $v \in H_1 \subset \bigotimes_{j=1}^{n-1} M_d$, and use the mappings defined in Lemma 5.3 to construct $R_+(v) \in H_{d_1}^+ \subset \bigotimes_{j=1}^{n-1} M_d$ and $R_-(v) \in H_{d_1}^- \subset \bigotimes_{j=2}^{n} M_d$. Then we have

$$\lim_{s \to +\infty} \left( \frac{s^{2h_1.d_n} \times F[v](x_1, \ldots, x_{n-1}, s)}{s} \right) = C_+ \times F[R_+(v)](x_1, \ldots, x_{n-1}),$$

with $C_+ = (q-q^{-1})^{-1} [d_n-1]^2 \times B_1^{d_n}$ and

$$\lim_{s \to -\infty} \left( \frac{s^{2h_1.d_n} \times F[v](s, x_2, \ldots, x_n)}{s} \right) = C_- \times F[R_-(v)](x_2, \ldots, x_n),$$

with $C_- = (q^{-2}-1)^{-1} [d_1-1]^2 \times B_1^{d_1}$.

**Proof.** Again, the two cases are similar, so we only give the details about the first. We write the vector $v$ in the form given by Lemma 5.3 (a), as $v = \sum \varepsilon_{l_n} \otimes F^{d_n-l_n, \tau_0^+}$, where $\tau_0^+ = R_+(v)$ is a highest weight vector of a $d_n$-dimensional subrepresentation of $\bigotimes_{j=1}^{n-1} M_d$. We then write, using homogeneity,

$$s^{2h_1.d_n} \times F[v](x_1, \ldots, x_{n-1}, s) = s^{h_1.d_n-\sum_{j=1}^{n-1} h_1.d_j} \times F[v](\frac{x_1}{s}, \ldots, \frac{x_{n-1}}{s}, 1),$$

We apply Proposition 5.1 to this. More precisely, for all terms $\varepsilon_{l_n} \otimes F^{d_n-l_n, \tau_0^+}$ we have

$$\lim_{s \to +\infty} \left( \frac{s^{2h_1.d_n} \times F[v](\varepsilon_{l_n} \otimes F^{d_n-l_n, \tau_0^+})(x_1, \ldots, x_{n-1}, s)}{s} \right) = \lim_{s \to +\infty} \left( \frac{s^{h_1.d_n-\sum_{j=1}^{n-1} h_1.d_j} \times F[v](\varepsilon_{l_n} \otimes F^{d_n-l_n, \tau_0^+})(\frac{x_1}{s}, \ldots, \frac{x_{n-1}}{s}, 1)}{s} \right) = F[\tau_0^+](x_1, \ldots, x_{n-1}) \times F[\varepsilon_{l_n} \otimes F^{d_n-l_n, \tau_0^+}](0, 1).$$

Now the vector $v$ reads $v = \sum \varepsilon_{l_n} \otimes F^{d_n-l_n, \tau_0^+}$, so by linearity we have

$$\lim_{s \to +\infty} \left( \frac{s^{2h_1.d_n} \times F[v](x_1, \ldots, x_{n-1}, s)}{s} \right) = F[\tau_0^+](x_1, \ldots, x_{n-1}) \times F[v](\sum \varepsilon_{l_n} \otimes e_{d_n-1-l_n})(0, 1) = (q-q^{-1})^{-1} [d_n-1]^2 \times B_1^{d_n} \times F[\tau_0^+](x_1, \ldots, x_{n-1}),$$

where in the last step we used the facts that $\sum \varepsilon_{l_n} \otimes e_{d_n-1-l_n} = (q-q^{-1})^{-1} [d_n-1]^2 \times \tau_0^{1(d_n,d_n)}$ in the notation of Equation (2.8), and $F[x_0](\tau_0^{1(d_n,d_n)})(0, 1) = B_1^{d_n} d_n$ by Proposition 4.4. 

□
Remark 5.5. Proposition [5.4] can be seen in two ways: it allows us to trade Möbius covariance to dependence on one variable less. Directly by the statement, any Möbius covariant function of \( n \) variables may be viewed as a function of \( n - 1 \) variables — we can either get rid of the variable \( x_1 \) or \( x_n \). Conversely, if we assume that \( v \in \bigotimes_{j=1}^{n} M_{d_j} \) lies in a \( d \)-dimensional irreducible subrepresentation, then we can view it as a Möbius covariant function of \( n + 1 \) variables — either associated to the vector \( R_+^{-1}(v) \in M_d \otimes \bigotimes_{j=1}^{n} M_{d_j} \) or to the vector \( R_-^{-1}(v) \in \bigotimes_{j=1}^{n} M_{d_j} \otimes M_d \), and with the additional variable respectively to the right or to the left of all other variables.

5.3. Cyclic permutations of variables. Remark 5.5 suggests yet another interesting way to interpret the operations \( R_+ \) and \( R_- \) in Lemma 5.3. Namely, in view of Proposition 5.4, the composition \( S = R_-^{-1} \circ R_+ \) gives rise to cyclic permutations of variables in our functions.

Consider the Möbius covariant case, and denote by \( H_1(V) \) the maximal trivial subrepresentation of a representation \( V \). Let \( v \in V \) be in the trivial subrepresentation \( H_1(M_{d_n} \otimes M_{d_{n-1}} \otimes \cdots \otimes M_{d_1}) \). Then the vector \( S(v) \) is in the trivial subrepresentation \( H_1(M_{d_{n-1}} \otimes \cdots \otimes M_{d_1} \otimes M_{d_n}) \), and the function \( \mathcal{F}[S(v)] \) could be thought of as corresponding to the original function \( \mathcal{F}[v] \) when the variables are ordered as \( x_n < x_1 < x_2 < \cdots < x_{n-1} \).

When the number of variables is \( n \), and the operation of moving the rightmost variable to the left of all others is repeated \( n \) times, one expects to recover the original function. Above we have defined this operation by its action on vectors in the trivial subrepresentation of the \( n \)-fold tensor product as

\[
S^{(d_n)}: H_1(M_{d_n} \otimes M_{d_{n-1}} \otimes \cdots \otimes M_{d_1}) \rightarrow H_1(M_{d_{n-1}} \otimes \cdots \otimes M_{d_1} \otimes M_{d_n}),
\]

where we now emphasize that the definition of the operation \( S^{(d_n)} = R_-^{-1} \circ R_+ \) depends on the dimension \( d_n \). The correct \( n \):th iterate is thus

\[
S^{(d_1)} \circ S^{(d_2)} \cdots \circ S^{(d_{n-1})} \circ S^{(d_n)}: H_1(M_j M_{d_j}) \rightarrow H_1(M_j M_{d_j}),
\]

This mapping turns out to be not exactly the identity, but rather a constant multiple of the identity.

To see that \( S^{(d_n)} \circ \cdots \circ S^{(d_1)} \) is a constant multiple of the identity on \( H_1(\bigotimes_{j=1}^{n} M_{d_j}) \), it is convenient to characterize the vectors in \( H_1(\bigotimes_{j=1}^{n} M_{d_j}) \) by their projections to different subrepresentations. One then needs two basic commutative diagrams, which express how to exchange the order of the projections and the operations \( S^{(d_j)} \).

Let

\[
\hat{\pi}^{(6)}_{j,j+1}: \bigotimes_{i=1}^{n} M_{d_j} \rightarrow \left( \bigotimes_{i=j+2}^{n} M_{d_j} \right) \otimes M_{d_{j+1}} \otimes \left( \bigotimes_{i=1}^{j-1} M_{d_j} \right)
\]

denote the projection \( M_{d_{j+1}} \otimes M_{d_j} \rightarrow M_{d_{j+1}} \) acting in the \( j \):th and \( j + 1 \):st tensorands, as in Section 4.3. If \( j < n - 1 \), then the projection obviously commutes with the operation \( S^{(d_n)} \), according to the following square diagram

\[
\begin{array}{ccc}
H_1(M_{d_n} \otimes M_{d_{n-1}} \otimes \cdots \otimes M_{d_1}) & \xrightarrow{S^{(d_n)}} & H_1(M_{d_n} \otimes M_{d_{n-1}} \otimes \cdots \otimes M_{d_{j+1}} \otimes \cdots \otimes M_{d_1}) \\
S^{(d_n)} & \xrightarrow{S^{(d_n)}} & H_1(M_{d_{n-1}} \otimes \cdots \otimes M_{d_1} \otimes M_{d_n}) & \xrightarrow{S^{(d_n)}} & H_1(M_{d_{n-1}} \otimes \cdots \otimes M_{d_{j+1}} \otimes \cdots \otimes M_{d_1} \otimes M_{d_n}).
\end{array}
\]
If \( j = n - 1 \), then a diagram of the above type doesn’t make sense — instead, we apply two \( S \)-operations before the projection. The following pentagon diagram commutes up to a multiplicative constant

\[
H_1(M_{d_n} \otimes M_{d_{n-1}} \otimes M_{d_{n-2}} \otimes \cdots \otimes M_{d_1}) \xrightarrow{\hat{\pi}_{1,2}^{(n-1)}_n} H_1(M_\delta \otimes M_{d_{n-2}} \otimes \cdots \otimes M_{d_1})
\]

\[
H_1(M_{d_{n-1}} \otimes M_{d_{n-2}} \otimes \cdots \otimes M_{d_1} \otimes M_{d_n}) \xrightarrow{\hat{\pi}_{1,2}^{(n-2)}} H_1(M_{d_{n-2}} \otimes \cdots \otimes M_{d_1} \otimes M_\delta)
\]

\[
H_1(M_{d_{n-2}} \otimes \cdots \otimes M_{d_1} \otimes M_{d_n} \otimes M_{d_{n-1}}) = \cdots
\]

Using the above square and pentagon diagrams [5.2] and [5.3], one can finally show that also the larger diagram [5.4] below commutes up to constants. We choose a sequence of intermediate dimensions \( \delta_2, \delta_3, \ldots, \delta_{n-2}, \delta_{n-1} = d_n \), to specify an element of the dual of \( H_1(\bigotimes_{i=1}^n M_{d_i}) \) through a sequence of projections. In this diagram, the top left \( H_1 \) stands for \( H_1(\bigotimes_{i=1}^n M_{d_i}) \), and moving downwards amounts to a cyclic permutation of the tensorands by an \( S \)-operation, and moving to the right reduces the number of tensorands by a projection. In the rightmost column, only two tensorands remain and the spaces \( H_1 \) stand for \( H_1(M_{d_1} \otimes M_{d_n}) \) — note that \( \delta_{n-1} = d_n \), as is needed for \( M_{d_n} \otimes M_{\delta_{n-1}} \) to contain a trivial subrepresentation (see Lemma 2.4). The diagram

\[
H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_1)}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} \cdots \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1 = \cdots
\]

\[
H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_1)}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} \cdots \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1
\]

\[
H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} \cdots \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1
\]

\[
H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} \cdots \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1
\]

\[
H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} \cdots \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1
\]

\[
H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} \cdots \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1
\]

\[
H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} \cdots \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1
\]

\[
H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} \cdots \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1
\]

\[
H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} \cdots \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1
\]

\[
H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_2)}} \cdots \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1 \xrightarrow{\hat{\pi}_{1,2}^{(d_{n-1})}} H_1
\]

commutes up to constants, and we then deduce that

\[
\hat{\pi}_{1,2}^{(d_{n-1})} \circ \cdots \circ \hat{\pi}_{1,2}^{(d_2)} = \text{const.} \times \hat{\pi}_{1,2}^{(d_{n-1})} \circ \cdots \circ \hat{\pi}_{1,2}^{(d_2)} \circ S^{(d_1)} \circ \cdots \circ S^{(d_n)}.
\]

The constant above is independent of the sequence of dimensions \( \delta_2, \ldots, \delta_{n-1} \). Such projections span the dual of \( H_1(\bigotimes_{i=1}^n M_{d_i}) \), which allows us to conclude that

\[
S^{(d_1)} \circ \cdots \circ S^{(d_n)} = \text{const.} \times \text{id}_{H_1(\bigotimes_{i=1}^n M_{d_i})}.
\]
Thus, the $S$-operations give rise to a projective action of cyclic permutations on our Möbius covariant functions $F[v] : X_n \to \mathbb{C}$.

6. Conclusions and outlook

We have defined “the spin chain - Coulomb gas correspondence”, which associates screened Coulomb gas correlation functions to vectors in a tensor product of representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. Natural representation theoretical properties of the vectors have been shown to imply properties of the corresponding functions.

The results presented here are used in [JJK13, KP14] to explicitly solve two interesting problems about SLEs. Other conformally covariant boundary correlation functions could be treated similarly with the techniques of the present article.

The results of this article apply in the generic, semisimple case, in which the deformation parameter $q$ of the quantum group is not a root of unity. If $q$ were a root of unity — that is $\kappa \in \mathbb{Q}$ — the representation theory of the quantum group would become non-semisimple and the corresponding functions would have degeneracies (exceptional linear dependencies or poles as a function of $\kappa$). Extending a version of “the spin chain - Coulomb gas correspondence” to these degenerate cases is a natural topic of future research — some of the non-semisimple representation theory has been analyzed in, e.g., [BFGT09], and examples of degeneracies of the functions have been resolved in, e.g., [FK13].

The boundary correlation functions obtained in this correspondence have conformal weights labeled by the first row of the Kac table. This is sufficient for those applications that served as our primary motivation, but the correspondence could possibly be generalized by considering another type of screening charges, and an appropriate “two-screening quantum group” [DFS].

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