Small-amplitude nonlinear waves on a black hole background

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Abstract

Let $G(x)$ be a $C^0$ function such that $|G(x)| \leq K|x|^p$ for $|x| \leq c$, for constants $K, c > 0$. We consider spherically symmetric solutions of $\Box_g \phi = G(\phi)$ where $g$ is a Schwarzschild or more generally a Reissner-Nordström metric, and such that $\phi$ and $\nabla \phi$ are compactly supported on a complete Cauchy surface. It is proven that for $p > 4$, such solutions do not blow up in the domain of outer communications, provided the initial data are small. Moreover, $|\phi| \leq C(max\{v, 1\})^{-1}$, where $v$ denotes an Eddington-Finkelstein advanced time coordinate.

The interaction between the geometry of black hole backgrounds and the behaviour (in the large) of linear and non-linear waves plays a fundamental role in general relativity. The very stability of the simplest black hole solutions like Schwarzschild and Kerr could depend on subtle features of this interaction. Moreover, fine aspects of the behaviour of these waves, such as the phenomenon of decaying tails, have physical importance, both from the point of view of far away observers, and for those brave enough to cross the “event horizon”.

In [8], we studied the problem of collapse of a spherically symmetric self-gravitating scalar field, i.e. solutions of the system

\begin{align*}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= 2T_{\mu\nu} \\
\Box_g \phi &= 0 \\
T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}^2\phi_{,\lambda},
\end{align*}

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arising from spherically symmetric initial data, in the case where it is known \textit{a priori} that a black hole forms.\footnote{For the definition of the notion of black hole, see \cite{6}. Black holes form for generic large initial data in view of Christodoulou’s proof of weak cosmic censorship \cite{5, 6} for spherically symmetric solutions to (1)–(3). In the presence of an additional non-trivial Maxwell field, for which the results of \cite{8} also apply, the existence of a black hole follows from \cite{6}.} The main result of \cite{8} was a proof of Price’s \textit{law}, heuristically derived in \cite{20, 9}. The mathematical content of this “law” is a set of decay rates for the scalar field in appropriately defined null coordinates on the so-called \textit{domain of outer communications} or \textit{exterior} of the black hole. In particular, we showed decay $|\phi| \leq v^{-3+\varepsilon}$ in an Eddington-Finkelstein-like advanced time $v$-coordinate along the event horizon, and decay $|r \phi| \leq u^{-2}$ in retarded time $u$ along null infinity.

The problem of the self-gravitating scalar field studied in \cite{8} is non-linear. In particular, the black hole geometry depends on the solution, and even properly identifying the black hole exterior is a non-trivial part of the problem. The results of \cite{8} can be specialized, however, to the easier problem of the study of spherically symmetric solutions to the equation

$$\Box_g \phi = 0$$

on a fixed \textit{Reissner-Nordström} background spacetime $(\mathcal{M}, g)$. These spacetimes constitute a two-parameter family of solutions to the Einstein-Maxwell equations, with parameters mass $M$ and charge $e$. The one-parameter subfamily of vacuum solutions defined by $e = 0$ is known as the \textit{Schwarzschild} family. For parameter values $0 \leq |e| < M$, these spacetimes contain black and white holes, and the so-called exterior region can be covered by coordinates $(r, t)$ ranging in $(M + \sqrt{M^2 - e^2}, \infty) \times (-\infty, \infty)$ such that the metric takes the form:

$$g = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 d\sigma_{S^2},$$

where $d\sigma_{S^2}$ denotes the standard metric on the unit two-sphere. In \cite{8}, decay rates were obtained both in the (static) $t$ coordinate, as well as in Eddington-Finkelstein retarded and advanced time coordinates $u$ and $v$. (The latter are indispensable for understanding the decay of the radiation tail along null infinity and the behaviour along the event horizon.) Previously, decay rates had been shown only in the static $t$ coordinate \cite{15, 17} under the additional restrictive assumption that the solution is supported away from the event horizon’s sphere of bifurcation.\footnote{We also note earlier results showing boundedness \cite{16} and decay without a rate \cite{22}.}

In contrast to the situation in Minkowski space, there is no decay rate in $t$ which is uniform in $r$, in view of the geometry of the horizon.

In this paper, we shall explore the implications of the decay proven in \cite{8} for the behaviour of small data solutions to the semi-linear wave equation

$$\Box_g \phi = G(\phi)$$

on a fixed Reissner-Nordström background $(\mathcal{M}, g)$, where the nonlinearity is assumed to decay sufficiently fast at 0:

$$|G(x)| \leq K |x|^p \text{ for } |x| \leq c$$

(5)
for some constants $K, c > 0$. The study of small-data solutions to (4) under assumption (5) is often referred to as the “John problem”. The point of (5) is that no special structure is assumed of the non-linearity; in particular, it is not assumed that $G(x) = V'(x)$ for a nonnegative potential $V \geq 0$. Our main result (Theorem 1) states that for $p > 4$, small-data spherically symmetric solutions of (4) remain regular on the closure of the exterior region $J^- (I^+) \cap J^+ (I^-)$ of Schwarzschild or Reissner Nordström. Moreover, $|\phi| \leq C v^{-1}$ in advanced Eddington-Finkelstein time $v$, and, for $r \geq r_0 > M + \sqrt{M^2 - \epsilon^2}$, $|\phi(r, t)| \leq C(r_0)t^{-1}$, but with $C \to \infty$ as $r_0 \to M + \sqrt{M^2 - \epsilon^2}$! Note that our assumptions on the initial data in Theorem 1 are the geometrically appropriate ones; no special conditions are imposed at the sphere of bifurcation $H_A^+ \cap H_B^+$ of the event horizon.

The global study of the equation (4) on a Schwarzschild background was initiated in [1, 18], in the case where the non-linearity $G$ is given precisely by $G(x) = |x||x|^2$. Although this is a special case of (13), the phenomenology of this particular equation is completely different from the general case, as this non-linearity is of the form $V'(x)$ for a potential $V \geq 0$. In view of the nonnegativity of $V$, the specific form of $G$, and the Sobolev embedding $H^1 \hookrightarrow L^6$, global regularity for large data solutions can be proven using only the energy estimate arising from contraction of the energy momentum tensor of $\phi$ with the Killing vector $\partial_t$. In particular, no appeal need be made to decay for the linearized problem. Under spherical symmetry, similar results can in fact be proven in the self-gravitating case, i.e. for the system

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2T_{\mu\nu}$$

$$\Box g\phi = V'(\phi)$$

$$T_{\mu\nu} = \phi,\mu\phi,\nu - \frac{1}{2}g_{\mu\nu}\phi,\lambda\phi,\lambda - g_{\mu\nu}V(\phi),$$

for a wide class of potentials. Specifically, if the potential is bounded below, i.e. $V \geq -C > -\infty$, then a weak form of stability has been proven in [7] for arbitrary spherically symmetric solutions of (6)–(8), in the case where it was assumed a priori that the past of infinity has a nontrivial complement. If $V \geq 0$, the results of [7] imply in particular that null infinity is complete. These

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3In stating results, we apply here standard notations from global Lorentzian geometry. See Hawking and Ellis [11]. The reader unfamiliar with these global concepts should note that our result implies in particular “global existence” with respect to Schwarzschild or Reissner-Nordström time $t$.

4In particular, the sufficiently low power

5These methods have recently been extended [19] to the case of a Kerr background, where $\partial_t$ is no longer everywhere timelike in the domain of outer communications. In this case, global regularity is proven even though the solution is allowed to grow.

6i.e., that the solution does not blow up in the past of infinity.

7In the spherically symmetric self-gravitating case, an energy estimate is provided via the coupling by considering evolution equations for the Hawking mass $m$. 

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questions received attention recently in the string theory community in the context of possible cosmic censorship violation.

In contrast to the above, for the case of the “John problem” considered here, where we do not assume a special structure for $G$, results are only expected for small data solutions, and for $p$ sufficiently large. We recall the situation in Minkowski space: The study of small amplitude solutions of the equation

$$\Box \phi = |\phi|^p$$

(9)
on $\mathbb{R}^{3+1}$ was initiated by F. John, who showed that for $p > 1 + \sqrt{2}$, solutions to (9) with sufficiently small, compactly supported initial data remain regular for all time, and moreover, decay to 0 along all inextendible causal geodesics. On the other hand, for $1 < p < 1 + \sqrt{2}$, F. John established the existence of small data spherically symmetric solutions which blow up in finite time. This blow-up result was later extended by J. Schaeffer to $p = 1 + \sqrt{2}$.

Recently, it was proven in [3] that for equation (9) on a Schwarzschild background for $1 < p < 1 + \sqrt{2}$, there exist solutions arising from arbitrarily small compactly supported data which blow up on the exterior region $J^- (I^+) \cap J^+ (I^-)$. In view of the restriction $p > 4$ in our Theorem 1, it would be interesting to understand the behaviour in the interval $1 + \sqrt{2} \leq p \leq 4$, in particular, to determine the critical power of $p$ separating stability and blow up.

Besides addressing what is by now a classical problem in non-linear wave equations, our motivation for studying this “John problem” in the black hole context arises as follows: Experience from non-linear stability problems for the Einstein vacuum equations without symmetry strongly suggests that using decay rates of linearized fields will be an essential part of any argument. In the black hole context, implementing this approach introduces a number of technical challenges, as the notion of “time” with respect to which decay is to be measured must necessarily be substantially altered. In this context, the present problem appears to concern the simplest geometrically well-defined non-linear equation where “global regularity” is indeed shown via decay for the linearized problem. Moreover, we believe that the estimates proven here clarify the role of the event horizon–and the celebrated red-shift effect–for ensuring stability. Indeed, it is not clear whether the results shown here could be obtained using only the (non-uniform) $t$-decay shown in [6, 17], which do not precisely express behaviour up to the horizon. Further clarification of these issues may shed light on the problem of the non-linear stability of the Kerr vacuum solution.

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8Consideration of the general $n$-dimensional problem leads to the critical power $p_c$ defined as the positive root of the quadratic equation $(n - 1)p_c^2 - (n + 1)p_c - 2 = 0$; this power first arose in work of W. Strauss. For analogues of F. John’s result for $\mathbb{R}^{n+1}$, see [10], and references therein.

9The spherically symmetric geometric wave operator on Schwarzschild can be reinterpreted, after multiplication of $\phi$ by $r$, as the $\mathbb{R}^{1+1}$ wave operator, with an additional positive potential, and this is the approach taken in [3]. For extensions of the John problem to large classes of linear potentials, not necessarily positive, see [15], and references therein.

10For simplicity, we appeal in this paper only to the uniform $v$-decay estimate—not the complete set of decay rates—proven in [3]. We expect that a more detailed study should lower the constraint $p > 4$. It is not clear, however, whether the estimates of [3] alone will be able to retrieve the result for $p > 1 + \sqrt{2}$, for comparison with the case of $\mathbb{R}^{4+1}$, see [13].
1 Schwarzschild and Reissner-Nordström

Let \((M, g)\) denote a non-extremal Reissner-Nordström solution corresponding to parameters \(M, e\), with \(0 \leq |e| < M\). (We refer the reader to standard references, for instance [11].) By convention here, the term Reissner-Nordström solution will always refer to a globally hyperbolic spacetime which is the Cauchy development of an asymptotically flat hypersurface Σ with two ends\(^{11}\). Considering the quotient

\[ \mathcal{Q} = M/\text{SO}(3), \]

and defining the area-radius function \( r : \mathcal{Q} \to \mathbb{R} \)

\[ r(q) = \sqrt{\text{Area}(q)/4\pi}, \quad (10) \]

we have that

\[ g = \bar{g} + r^2 d\sigma_{S^2}, \]

where \( d\sigma_{S^2} \) denotes the standard metric on the unit sphere, and where \( \bar{g} \) defines a 1 + 1-dimensional Lorentzian metric on \( \mathcal{Q} \). The Lorentzian manifold \((\mathcal{Q}, \bar{g})\) can be covered by a bounded global null coordinate system \((\bar{u}, \bar{v})\). The range of such a coordinate system is depicted, as a subset of 2-dimensional Minkowski space, in the Schwarzschild \((M > 0, e = 0)\) case:

\(^{11}\text{i.e., in the } e \neq 0 \text{ case, we do not mean an analytic (or otherwise) extension thereof} \)
and the charged Reissner-Nordström (\( M > |e| > 0 \)) case:

These depictions are commonly known as *Penrose diagrams*. The curve \( S \) depicts the projection to \( Q \) of a particular choice of complete Cauchy surface \( \Sigma \subset M \).

We define the *Hawking mass* function \( m : Q \to \mathbb{R} \) by

\[
m = \frac{r}{2} (1 - \bar{g}(\nabla r, \nabla r)),
\]

and the *mass ratio*

\[
\mu = \frac{2m}{r}.
\]

The mass \( m \) is related to the parameter \( M \) by

\[
M = m + \frac{e^2}{2r}.
\]

The sets \( J^- (\mathcal{I}_B^+) \cap J^+ (\mathcal{I}_A^-) \) and \( J^- (\mathcal{I}_B^+) \cap J^+ (\mathcal{I}_B^-) \), the two so-called *domains of outer communications*, can each be covered by a coordinate system \((r, t)\) with coordinate range \((r_+, \infty) \times (-\infty, \infty)\), where \( r \) is defined by \( \text{[10]} \), and in which the metric \( \bar{g} \) of \( Q \) takes the form

\[
\bar{g} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1}dr^2.
\]

Here \( r_+ \) is defined by

\[
r_+ = M + \sqrt{M^2 - e^2}.
\]

One immediately sees that the vector field \( \partial_t \) is timelike Killing in these domains.
To define a null coordinate system, we first introduce the so-called Regge-Wheeler coordinate
\[ r_\ast = r + r_+^2 (r_+ - r_-)^{-1} \log |r - r_+| - r_-^2 (r_+ - r_-)^{-1} \log |r - r_-|, \tag{12} \]
where
\[ r_- = M^2 - \sqrt{M^2 - e^2}. \]
This coordinate satisfies
\[ \frac{dr_\ast}{dr} = (1 - \mu)^{-1} \]
We now set
\[ u = t - r_\ast, \quad v = t + r_\ast. \]
With respect to these coordinates, the metric \( \tilde{g} \) takes the form
\[ \tilde{g} = -(1 - \mu) du dv = - \left( 1 - \frac{2M}{r} + \frac{e^2}{r^2} \right) du dv \]
in \( J^- (I_+^+ \cup I^-) \) and \( J^- (I_+^+ \cup I^-) \); i.e. \( (u, v) \) are indeed null. Individually, the coordinates \( u \) and \( v \) are each known as Eddington-Finkelstein retarded and advanced time coordinates, respectively.\(^\text{12}\) Each of the domains of outer communications is covered by the range \( (-\infty, \infty) \times (-\infty, \infty) \).

The notion of asymptotic flatness is captured by the fact that \( 1 - \mu \rightarrow 1 \) as \( r \rightarrow \infty \). To examine the geometry of the horizon, first, we note that the timelike Killing vector field \( \partial_t \) extends to a null tangent Killing vector field on the event horizon \( \mathcal{H} = \mathcal{H}_A^+ \cup \mathcal{H}_B^+ \), vanishing at the sphere of bifurcation \( \mathcal{H}_A^+ \cap \mathcal{H}_B^+ \). Thus \( 1 - \mu = 0 \) along \( \mathcal{H} \), and moreover \( m \) and \( r \) are constant. In terms of the parameters, we have
\[ m = 2r = 2r_+ = 2(M + \sqrt{M^2 - e^2}). \]

This vanishing of \( 1 - \mu \) relative to the Regge-Wheeler \( (t, r_\ast) \) and null \( (u, v) \) coordinates is captured by:
\[ \hat{C}_1 e^{\frac{(r_+ - r_-)}{2r_+^2}(v-u)} = \hat{C}_1 e^{\frac{(r_+ - r_-)}{r_+^2}r_\ast} \leq C_1 (r - r_+) \leq 1 - \mu \leq C_2 (r - r_+) \leq \hat{C}_2 e^{\frac{(r_+ - r_-)}{2r_+^2}(v-u)} = \hat{C}_2 e^{\frac{(r_+ - r_-)}{2r_+^2}(v-u)} \tag{13} \]
for small enough \(-\infty < r_\ast^* \), and for some positive constants, \( C_1, \hat{C}_1, C_2, \) and \( \hat{C}_2 \), depending only on \( M \) and \( e \).

\[^{12}\text{The term Eddington-Finkelstein coordinate system, however, typically refers to the hybrids \( (u, r) \) and \( (v, r) \).}\]
2 Local existence

Let $G$ be a bounded $C^\infty$ function of its argument, let $(\mathcal{M}, g)$ be as in the previous section, and let $\Sigma$ be a Cauchy surface. We have the following

**Proposition 1.** Let $\phi_0$ and $\phi_1$ be $C^\infty$ functions on $\Sigma$. There exists a unique globally hyperbolic subset $D \subset \mathcal{M}$, and a $C^\infty$ function $\phi : D \to \mathbb{R}$ such that

1. $\Sigma \subset D$ is a Cauchy surface, and $\phi|_\Sigma = \phi_0$, $N^\alpha \phi_\alpha = \phi_1$, where $N$ is the future-directed unit normal to $\Sigma$.
2. $\Box_g \phi = G(\phi)$
3. The set $D$ is maximal, i.e. if $D \subset \tilde{D}$ is globally hyperbolic, with $N_\tilde{\phi} : \tilde{D} \to \mathbb{R}$ satisfying the above, then $\tilde{D} = D$.

We shall call $D$ the maximal domain of existence of the solution. Moreover, if $\Sigma$ is spherically symmetric, and $\phi_0$ and $\phi_1$ are constant on the $SO(3)$ orbits, then $D$ is spherically symmetric and $\phi$ is constant on the $SO(3)$ orbits, and thus descends to a function on $D_\mathbb{Q} = D/\text{SO}(3)$.

In the spherically symmetric case, the regularity assumption in the statement can be lowered to $G \in C^0_{\text{loc}}$, $\phi_0 \in C^1_{\text{loc}}$, $\phi_1 \in C^0_{\text{loc}}$. The resulting solution $\phi$ is then $C^1$. In a slight abuse of notation, we will denote $D_\mathbb{Q}$ again as $D$. It turns out that blowup can be easily characterized in terms of the $L^\infty$ norm of $\phi$, i.e. we have

**Proposition 2.** Let $q \in \mathcal{Q} \cap \overline{D}$. If $J^-(q) \cap D \neq \emptyset$, and

$$\sup_{x \in J^-(q) \cap D} |\phi(x)| < \infty,$$

then $q \in D$.

A similar statement can be made about the past. The proofs of the above proposition are completely standard and are thus omitted.

3 The main theorem

The main theorem of this paper is the following:

**Theorem 1.** Let $\Sigma$ be a spherically symmetric Cauchy surface in a Reissner-Nordström spacetime $(\mathcal{M}, g)$, with parameters $0 \leq e < M$, and let $\phi_0$, $\phi_1$ be $C^1$, $C^0$ functions, respectively, on $\Sigma$, constant on the $SO(3)$ orbits, and supported in $\Sigma \cap \{ r \leq R \}$, for some $R < \infty$. Let $D$ denote the maximal domain of existence for the initial value problem defined by Proposition 1, where $G$ is assumed to satisfy \[ with $p > 4$. Then there exists an $\epsilon$, depending only on $M$, $e$, $\Sigma$, and $R$, such that for $13$

\[ |\phi_0|_{C^1} < \epsilon, \]

\[13\text{The norm above refers to the induced Riemannian metric on } \Sigma.\]
\[ |\phi_1|_{C^0} < \epsilon, \]

\( \mathcal{D} \) contains the closures of the two domains of outer communications,

\[ (J^-(I_A^+) \cap J^+(I_A^-)) \cup (J^-(I_B^+) \cap J^+(I_B^-)) \subset \mathcal{D}. \]

Moreover,

\[ |\phi(u, v)| \leq C(\max\{v, 1\})^{-1}, \]

and, for each fixed \( r_0 > r_+ \),

\[ |\phi(r, t)| \leq C(r_0)(|t| + 1)^{-1} \quad r \geq r_0, \]

where \( v \) and \( t \) denote an Eddington-Finkelstein advanced time coordinate and a static time coordinate, respectively, on either of the two domains of outer communications.

### 4 Proof of Theorem 1: The bootstrap

In view of simple arguments of Cauchy stability, and the fact that the data are compactly supported on \( \Sigma \), it follows, that given any \( t_0 \), say \( t = 1 \), then for \( \epsilon \) sufficiently small, we have

\[ \{ t = 1 \} \subset \mathcal{D} \]

and

\[ |\phi|_{t=1}|_{C^1} \leq \tilde{\epsilon}, \quad (14) \]

\[ |N\phi|_{t=1}|_{C^0} \leq \tilde{\epsilon}. \quad (15) \]

Here again, the norms refer to the induced Riemannian metric of \( \{ t = 1 \} \), and \( N \) refers to a unit future directed normal. The relation of \( \mathcal{S} = \Sigma/\text{SO}(3) \) and \( \{ t = 1 \} \) could be given, for instance, as follows:

Note \( \phi|_{t=1} \) will not in general be compactly supported in \( (r_+, \infty) \). It will be supported in \( (r_+, \bar{R}) \) for some \( \bar{R} < \infty \). Moreover, we have \( \tilde{\epsilon} = \tilde{\epsilon}(\epsilon, \Sigma, \bar{R}, M, e) \to 0 \), as \( \epsilon \to 0 \).
In proving our theorem, it clearly suffices to show that
\[ D \supset J^+(\Sigma) \cap J^-(I_A^+) \cap J^+(I_A^-). \]

Define the set \( B \subset D \cap J^-(I_A^+) \cap J^+(I_A^-) \) by
\[ B = \left\{ q \in D \cap J^-(I_A^+) \cap J^+(I_A^-) : \sup_{x \in J^-(q) \cap \{ t \geq 1 \} \cap D} v_+(x) |\phi(x)| \leq B \right\} \tag{16} \]
where \( v_+ \) denotes \( \max\{v, 1\} \), and for a constant \( B \) to be determined later.

This subset is clearly non-empty and open for \( B > \bar{v}_+ (R, 1) \). In view of Proposition 2, proving Theorem 1 reduces to showing that \( B \) is closed. This would follow immediately from

**Theorem 2.** For \( q \in B \), we have
\[ \sup_{x \in J^-(q) \cap \{ t \geq 1 \} \cap D} v_+(x) |\phi(x)| \leq \frac{B}{2}. \]

To prove this, we shall view our non-linear problem in \( B \) as a solution of
\[ \Box_y \psi = F, \tag{17} \]
where \( F \) is to satisfy the estimates arising from (16) if \( F \) is set equal to the non-linearity. We thus turn in the next section to the study of this linear inhomogeneous equation. The proof of Theorem 2 will then follow easily in Section 6.

5 The linear problem

First some formulas: Written with respect to Regge-Wheeler coordinates, equation (17) takes the form
\[ (1 - \mu)^{-1} \left( \partial_t^2 \psi - r^{-2} \partial_r (r^2 \partial_r \psi) \right) = F. \tag{18} \]

Alternatively, relative to the null coordinates \((u = t - r_*, v = t + r_*)\) equation (17) can be written as a first order system
\begin{align*}
\partial_v (r \partial_u \psi) &= \frac{(1 - \mu)}{2r} r \partial_u \psi + \frac{(1 - \mu)}{4} r F, \tag{19} \\
\partial_u (r \partial_v \psi) &= \frac{(1 - \mu)}{2r} r \partial_u \psi + \frac{(1 - \mu)}{4} r F \tag{20}
\end{align*}
for the pair of functions \( \theta = r \partial_u \psi \) and \( \zeta = r \partial_v \psi \), or by a single equation
\[ \partial_v \partial_u (r \psi) = -\left( M - \frac{\mu^2}{r} \right) \left( 1 - \mu \right) \frac{1}{2r^2} \psi + \frac{(1 - \mu)}{4} r F. \tag{21} \]
We also record the following form of equation (19), crucial to understanding the so-called red-shift effect

\[
\frac{\partial}{\partial v} \left( \frac{r \partial_u \psi}{1 - \mu} \right) + \frac{1}{r} \left( \frac{M - 2e^2}{r} \right) \frac{r \partial_u \psi}{1 - \mu} = \frac{r \partial_u \psi}{2r} + \frac{1}{4} rF.
\]

(22)

This effect has been discussed in [8].

From (18) and (19)–(20) we can derive the following energy estimates. Given \((u_1, v_1)\), for \(t \leq \frac{(v_1 + u_1)}{2}\), define

\[
E_t(u_1, v_1)[\psi] = \int_{-t + v_1}^t \left( |\partial_t \psi(t, r_*)|^2 + |\partial_r \psi(t, r_*)|^2 \right) dr_*
\]

\[
= 2 \int_{-t - u_1}^{-t + v_1} \left( |r \partial_u \psi(t, r_*)|^2 + |r \partial_v \psi(t, r_*)|^2 \right) dr_*.
\]

For \(t_1 = \frac{(v_1 + u_1)}{2} \geq t_2 \geq t_3\), we have the following energy identity:

\[
E_{t_2}^{(u_1, v_1)}[\psi] + \int_{2t_2 - u_1}^{2t_2 - v_1} |r \partial_u \psi(u_1, v)|^2 dv + \int_{2t_2}^{2t_3 - v_1} |r \partial_u \psi(u, v_1)|^2 du
\]

\[
= E_{t_3}^{(u_1, v_1)}[\psi] + \frac{1}{2} \int_{t_3}^{t_2} \int_{-t - u_1}^{-t + v_1} (1 - \mu) F(t, r_*) \partial_\psi(t, r_*) r^2 dr_* dt.
\]

The estimate refers thus to the region depicted below:

We also have the following characteristic energy identity in the characteristic rectangle \([u_2, u_1] \times [v_2, v_1]\):

\[
\int_{u_2}^{u_1} |r \partial_u \psi(u, v_1)|^2 du + \int_{v_2}^{v_1} |r \partial_v \psi(u_1, v)|^2 dv = \int_{u_2}^{u_1} |r \partial_u \psi(u, v_0)|^2 du + \int_{v_2}^{v_1} |r \partial_v \psi(u_2, v)|^2 dv
\]

\[
+ \frac{1}{4} \int_{u_2}^{u_1} \int_{v_2}^{v_1} (1 - \mu) F(u, v)(\partial_u \psi(u, v) - \partial_v \psi(u, v)) r^2 dv du.
\]

The reader can refer to the diagram of Theorem [H]

5.1 The homogeneous problem

The linear homogeneous problem, i.e. the case \(F = 0\), was studied in [8], where pointwise decay estimates were proven. In this section, we will present these
estimates in a form suitable for understanding the inhomogeneous problem via Duhamel’s principle.

First we have the following preliminary result:

**Theorem 3.** Fix $r_0 > r_+$, and let $(u_1, v_1)$ be an arbitrary point. Let $t_0 < (u_1 + v_1)/2$, and consider the lighter-shaded region

\[ E = J^- (r_1, t_1) \cap J^+ (\{ t = t_0 \} \cap \{ r \geq r_0 \} ) \]

depicted below:\(^{14}\):

\[ (u_3, \tilde{v}) \]

\[ (u_2, v_2) \]

\[ (u_1, v_1) \]

\[ (u_3, v_3) \]

\[ (2t_0 - v_1, v_1) \]

\[ t = t_0 \]

Let $\psi$ be a solution to

\[ \Box \psi = 0 \]

on $J^- (u_1, v_1) \cap J^+ (\{ t = t_0 \})$. Let $(u_2, v_2) \in E$. Then there exists a constant $A$, depending only on $r_0$, $e$, and $M$, such that

\[ r |\psi(\tilde{u}, v_2)|^2 \leq A \left( r |\psi(u_2, v_2)|^2 + E^{(\tilde{u}, v_2)}_{t_0} [\psi] \right) \]

(23)

\[ r^3 |\partial_u \psi(\tilde{u}, v_2)|^2 \leq A \left( r^3 |\partial_u \psi(u_2, v_2)|^2 + E^{(\tilde{u}, v_2)}_{t_0} [\psi] \right) , \]

(24)

for any point $\tilde{u} \geq u_2$ with $(\tilde{u}, v_2) \in E$.

On the other hand, if $(u_3, v_3)$ is any point in $J^- (u_1, v_1) \cap J^+ (\{ t = t_0 \})$, then

\[ \left( \frac{r |\partial_u \psi(u_3, \tilde{v})|}{1 - \mu} \right)^2 \leq A \left( e^{-\frac{r - (u_3, v_3)}{1 - \mu}} \left( \frac{r |\partial_u \psi(u_3, v_3)|}{1 - \mu} \right)^2 + E^{(u_3, \tilde{v})}_{t_0} [\psi] \right) \]

(25)

for any $\tilde{v} \geq v_3$ such that $(u_3, \tilde{v}) \in J^- (r_1, t_1) \cap J^+ (\{ t = t_0 \})$.

Finally, if

\[ C = \sup_{E \cap \{ r \geq r_0 \}} \{ |r \psi|(r, t_0), |r^2 \partial_u (r \psi)(r, t_0)| \} \]

then

\[ |r \psi(r, t)|^2 + |r^2 \partial_u \psi(r, t)|^2 + |r^2 \partial_u (r \psi)(r, t)|^2 \leq A \left( C^2 + E^{(u_1, v_1)}_{t_0} [\psi] \right) , \]

(26)

throughout $J^- (r_1, t_1) \cap \{ r \geq r_0 \} \cap \{ t \geq t_0 \}$.

\(^{14}\)Note that despite the diagram, we are not necessarily assuming that $r(u_1, v_1) \geq r_0$. 

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Proof. Equations (23), (24), and (26) can be easily deduced from the equations (19)–(21) and the energy estimates of the previous section. For the estimate (25), we write the homogeneous wave equation in the red-shift from given by (22):
\[
\partial_v r \partial_u \psi + \frac{1}{r^2} \left( M - \frac{e^2}{r} \right) r \partial_u \psi = \frac{1}{2r} r \partial_v \psi.
\]
Introducing the integrating factor \( e^{\int_{v_3}^v \frac{1}{r} (M - \frac{e^2}{r^2}) dv} \) and integrating in \( v \) we obtain
\[
\frac{r \partial_u \psi(u_3, \tilde{v})}{1 - \mu} = e^{-\int_{v_3}^v \frac{1}{r} (M - \frac{e^2}{r^2}) dv} \frac{r \partial_u \psi(u_3, v_3)}{1 - \mu} + \int_{v_3}^v e^{-\int_{v_3}^{v'} \frac{1}{r} (M - \frac{e^2}{r^2}) dv'} \frac{1}{2r} r \partial_v \psi dv'.
\]
Since
\[
M - \frac{e^2}{r} \geq M - \frac{e^2}{r_o} = \frac{r_+ - r_-}{2} > 0,
\]
and \( r \) is increasing in the \( v \)-direction, we conclude
\[
\frac{r |\partial_u \psi(u_3, \tilde{v})|}{1 - \mu} \leq e^{-\int_{v_3}^v \frac{1}{r} (v - v_3) dv} \frac{r |\partial_u \psi(u_3, v_3)|}{1 - \mu} + \int_{v_3}^v e^{-\frac{1}{2r} (v - v_3)} \frac{1}{2r} r |\partial_v \psi(u_3, v)| dv.
\]
Applying Cauchy-Schwarz gives
\[
\frac{r |\partial_u \psi(u_3, \tilde{v})|}{1 - \mu} \leq e^{-\frac{1}{2r} (v - v_3)} \int_{v_3}^v \frac{r |\partial_u \psi(u_3, v_3)|}{1 - \mu} + A \left( \int_{v_3}^v r |\partial_u \psi(u_3, v)|^2 dv \right)^{\frac{1}{2}}.
\]

We now turn to decay estimates. The full results of [8] are complicated to state, as different regions must be treated separately. In this paper we shall only use the following:

**Theorem 4.** Let \((u_1, v_1), (u_2, v_2)\) be points with \((u_1, v_1) \in J^+(u_2, v_2)\), i.e. with \(u_1 \geq u_2, v_1 \geq v_2\). Consider the characteristic rectangle
\[
\mathcal{R} = J^+(u_2, v_2) \cap J^-(u_1, v_1)
\]
depicted below:

![Diagram](image-url)
Let \( \psi \) be a solution to \( \Box_g \psi = 0 \) on \( \mathcal{R} \). Let
\[
C = \max \left\{ \sup_{u_2 \leq u \leq u_1} |r \partial_v \psi(u_2, v)| + |r^2 \partial_v(r \psi)(u_2, v)|, \sup_{u_2 \leq u \leq u_1} |(1 - \mu)^{-1} \partial_u \psi(u, v_2)| \right\}.
\]
Then there exists a constant \( A \) depending only on \( r_0 = r(u_2, v_2), e \) and \( M \) such that
\[
\psi(u_1, v_1) \leq AC(v_1 - v_2)^{-1}.
\]
The above theorem concerns a characteristic initial value problem. We can combine Theorems 3 and 4 to derive decay estimates for solutions of the homogeneous Cauchy problem.

**Theorem 5.** Let \((u_1, v_1)\) be an arbitrary point, and let \(t_0 < (v_1 + u_1)/2\). Let \( \psi \) be a solution to
\[
\Box_g \psi = 0
\]
on \( J^-(u_1, v_1) \cap J^+(\{t = t_0\}) \). Let
\[
C = \sup_{t = t_0} \{(1 - \mu)^{-\frac{1}{2}} r^2 |\partial_t \psi|, (1 - \mu)^{\frac{1}{2}} r^2 |\partial_r \psi|, r^2 |\partial_r (r \psi)|, r |\psi| \}.
\]
(28)
Pick a point \((u_2, v_2)\) \(\in J^-(u_1, v_1) \cap \{t = t_0\}\):

It follows that
\[
|\psi(u_1, v_1)| \leq AC(v_1 - v_2)^{-1},
\]
where \( A \) depends only on \( r_0 = r(u_2, v_2), e \) and \( M \).

**Proof.** We first show that \( E_{t_0}^{(u_1, v_1)}[\psi] \leq AC^2 \). Note that \( \partial_{r_+} \psi = (1 - \mu) \partial_r \psi \).

\[
\left( E_{t_0}^{(u_1, v_1)}[\psi] \right)^2 = \int_{t_0 - u_1}^{t_0 + v_1} r^2 (|\partial_t \psi|^2 + |\partial_{r_+} \psi|^2) \, dr,
\]
\[
= \int_{r(r_+ = t_0 - u_1)}^{r(r_+ = t_0 + v_1)} (1 - \mu)^{-1} r^2 (|\partial_t \psi|^2 + |\partial_{r_+} \psi|^2) \, dr
\]
\[
\leq C^2 \int_{r_+}^{\infty} r^{-2} \, dr \leq C^2 r_+^{-1}.
\]

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We apply now Theorem 3. From (25) and (28), the estimate
\[
| (1 - \mu)^{-1} \partial_u \psi(u, v_2) | \leq e^{-(r_0^*-r_0^*)(v_2-v_0(u))}(1 - \mu)^{-1} \partial_u \psi(u, v_0(u)) | + AC \\
\leq e^{-(r_0^*-r_0^*)(v_2-v_0(u))}(1 - \mu)^{-\frac{1}{2}C + AC} + AC
\]
(30)
follows for \( u_2 \leq u \leq u_1 \), where \( v_0(u) \) denotes the \( v \)-coordinate of the intersection of the outgoing constant-\( u \) ray with \( t = t_0 \). Since,
\[
v_2 - v_0 = 2(r_0^* - r_0^*(u, v_2)),
\]
we have by (13), for sufficiently small \( r_0^* = r_0^*(u, v_2) \),
\[
(1 - \mu(r_0^*))^{-\frac{1}{2}e} e^{-(r_0^*-r_0^*)(v_2-v_0)} \leq A(1 - \mu(r_0^*))^{-\frac{1}{2}e} e^{-(r_0^*-r_0^*)r_0^*} \\
\leq Ae^{-\frac{(r_0^*-r_0^*)(v_2-v_0)}{2r_0^*}} \\
= Ae^{-r_0^*(v_2-v_0)} \\
\leq A.
\]
(31)
(Note that by convention here, \( A \) does not always represent the same constant.)
Since the above estimate is also clearly true for large \( r \), putting together (30) and (31) yields
\[
\sup_{u_2 \leq u \leq u_1} (1 - \mu)^{-1} | \partial_u \psi(u, v_2) | \leq AC.
\]
(32)
On the other hand, from (26), we have
\[
| \partial_v \psi(u_2, v) | \leq ACr^{-1}, \\
| \partial_v(r\psi)(u_2, v) | \leq ACr^{-2},
\]
for \( v_1 \leq v \leq v_2 \), in view of the fact that \( r \geq r_0 \) for such \( (u_0, v) \). Our result now follows immediately from Theorem 4.

Remark 1. Observe that the constant \( C \) in the above theorem is dominated by the (Riemannian) \( C^1 \) norm of \( r^3 \psi \),
\[
C \leq | r^3 \psi |_{t=t_0} C^1.
\]

Remark 2. The quantity \( \partial_u \psi/\partial_u r \) is clearly independent of the choice of retarded time \( \bar{u} \) and coincides with \( -(1 - \mu)^{-1} \partial_u \psi \) for Eddington-Finkelstein \( u \). As one can choose a global regular null coordinate system, such that \( \partial_u r \leq -1 \) along the ray \( u = u_2 \) in a neighborhood of \( \mathcal{H}^+_+ \cap \{ u = u_2 \} \), one could have alternatively derived (32) by appealing to Cauchy stability.
5.2 The inhomogeneous problem

To prove Theorem 2, we must derive estimates for solutions of the equation

$$\Box_g \Psi = F,$$

with vanishing initial data at \( t = 1 \). Let \( \psi(r, t; s) \) denote the solution of \( \Box_g \psi = 0 \) evaluated at \( (r, t) \) with initial condition

$$\psi(\cdot, s) = 0, \quad \partial_t \psi(\cdot, s) = (1 - \mu)F(\cdot, s).$$

Using representation (17) we can then write the solution of (33) as

$$\Psi(r, t) = \int_1^t \psi(r, t; s) ds.$$

We prove the following theorem:

**Theorem 6.** Let \((u_1, v_1)\) be given, let \( \Psi \) satisfy (33) on \( J^-(u_1, v_1) \cap \{ t \geq 1 \} \), with \( \Psi = 0, \partial_t \Psi = 0 \) on \( J^-(u_1, v_1) \cap \{ t = 1 \} \), and let \( \alpha > 4 \). Then there exists a constant \( A \), depending only on \( M, e, \) and \( \alpha \), such that

$$|\Psi(u_1, v_1)| \leq A(v_1)^{-1} \sup_{J^-(r, t) \cap \{ t \geq 1 \}} v_+^\alpha |F(u, v)|,$$

where \( v_+ = \max(v, 1) \).

**Proof.** Let \( r_1 = r(u_1, v_1), t_1 = (v_1 + u_1)/2 \). We will fix some \( r_0 > r_+ \), to be determined later, and distinguish the cases \( r_1 \geq r_0 \) and \( r_1 < r_0 \).

5.2.1 Estimates in the region \( r \geq r_0 \)

Consider first the case \( r_1 \geq r_0 \). We have \( v_1 = t_1 + r_1^\omega \geq t_1 + r_0^\omega \).

Let us take \( s > 2v_1^\omega \), for \( 0 \leq \omega < 1 \). To compute \( \psi(r_1, t_1; s) \), we decompose

$$J^-(r_1, t_1) \cap \{ t = s \} = \{ (u, v) : u = 2s - v, v_1 - 2(t_1 - s) \leq v \leq v_1 \}$$

as depicted:
i.e. we have
\[ I_1 = \{(u, v) : u = 2s - v, v_1 - 2(t_1 - s) \leq v \leq v_1^\omega\}, \]
\[ I_2 = \{(u, v) : u = 2s - v, v_1^\omega \leq v \leq v_1\}. \]

Note that on the set \( I_1 \),
\[ 2r_* = v - u = 2(v - s) \leq -2v_1^\omega \]
and therefore, by (13), we have
\[ 1 - \mu(r, s) \leq \hat{C}_2 e^{-\frac{(r + r_0)}{v_1^\omega}}, \quad (r, s) \in I_1. \]

By the above estimate, (23) and (34), we have
\[
\left| \psi(r_1, t_1; s) \right|^2 \leq A^2 (E_s[\psi(\cdot, \cdot; s)])^2 \\
= A^2 \int_{I_1 \cup I_2} (1 - \mu)|F(r, s)|^2 r^2 \, dr \\
\leq A^2 \left( e^{-\frac{(r + r_0)}{v_1^\omega}} \sup_{I_1} |F(r, s)|^2 + v_1^\omega \sup_{I_2} |F(r, s)|^2 \right), \quad (37)
\]
where in the last inequality we used that \( r \leq C' v_1 \) on \( I_2 \).

On the other hand, for \( s \leq 2v_1^\omega \) we can apply Theorem 5 for the points \( (r_1, t_1) = (u_1, v_1) \) and \( (r_0, s) = (u_2, v_2) \), where
\[ v_2 = s + r_0^* \leq 2v_1^\omega. \]
(In the last inequality, we assumed that \( r_0 \) is chosen in particular so that \( r_0^* \leq 0 \).) Then,
\[
\left| \psi(r_1, t_1; s) \right| \leq Av_1^{-1} \sup_r (1 - \mu)^\frac{1}{2} r^2 |F(r, s)| \\
\leq Av_1^{-1} \sup_{r^* \leq -\frac{s}{2}} (1 - \mu)^\frac{1}{2} r^2 |F(r, s)| + Av_1^{-1} \sup_{r^* \geq -\frac{s}{2}} (1 - \mu)^\frac{1}{2} r^2 |F(r, s)|.
\]

For points \( r^* \leq -\frac{s}{2} \), estimate (13) gives
\[ 1 - \mu \leq \hat{C}_2 e^{-\frac{(r + r_0)}{v_1^\omega}}, \]
while for \( r^* \geq -\frac{s}{2} \), we clearly have
\[ v = r_* + s \geq \frac{s}{2}. \]
Thus, for $\delta > 0$,

$$
|\psi(r_1, t_1; s)| \leq A v_1^{-1} e^{\frac{(r_+ - r_0)}{2r_+}} \sup_{r \leq -\frac{\delta}{2}} r^3 |F(r, s)| + A v_1^{-1} s^{-1-\delta} \sup_{r \geq -\frac{\delta}{2}} |v^{1+\delta} r^3 F(r, s)|.
$$

(38)

Applying (35), and the estimates (37) and (38), we obtain

$$
|\Psi(r_1, t_1)| \leq t_1 A e^{\frac{(r_+ - r_0)}{2r_+}} \sup_{J^-(r_1, t_1)} |F(r, s)| + A t_1 v_1^{\frac{\alpha}{2}} \sup_{J^-(r_1, t_1)} |F(u, v)| + A v_1^{-1} \sup_{J^-(r_1, t_1)} v_+^{4+\delta} |F(r, s)|
$$

$$
\leq (v_1)^{\frac{\alpha}{2}} \sup_{J^-(r_1, t_1)} |F(u, v)| + (v_1)^{\frac{\alpha}{2}} \sup_{J^-(r_1, t_1)} v_+^{4+\delta} |F(r, s)|,
$$

where we have used the fact that for $r_1 \geq r_0 > r_+$, we have $(v_1)_+ \geq \sim t_1$, with constant depending on $r_0$.

Clearly, there exists a constant $H$, depending only on $M$ and $e$, such that for $r \geq H$, we have $v = t + r_+ \geq r$ in the region $t \geq 1$. It follows immediately that for all $r$, we have $r \leq H + |\nu|$ in the region $t \geq 1$. We obtain, for any $\alpha \geq 0$,

$$
|\Psi(r_1, t_1)| \leq A (v_1)^{\frac{\alpha}{2}} \sup_{J^-(r_1, t_1)} v_+^{\alpha} |F(u, v)|
$$

$$
+ A (v_1)^{-1} \sup_{J^-(r_1, t_1)} v_+^{4+\delta} |F(u, v)|
$$

(39)

Choosing

$$
\alpha = 4 + \delta, \quad \omega = \frac{7}{2(4 + \delta)}
$$

we obtain

$$
|\Psi(r_1, t_1)| \leq A (v_1)^{-1} \sup_{J^-(r_1, t_1)} v_+^{\alpha} |F(u, v)|.
$$

(40)

5.2.2 Estimates in the region $r < r_0$.

Consider now the case $r_+ < r_1 < r_0$. In this region we shall use the red-shift effect manifest\(^\text{15}\) in the equation (22):

$$
\partial_r \left( \frac{r \partial_r \Psi}{1 - \mu} + \frac{1}{r^2} \left( M - \frac{\nu^2}{r} \right) \right) = \frac{r \partial_r \Psi}{1 - \mu} + \frac{1}{4} r F.
$$

\(^\text{15}\)See [8]. The point, as we shall see, is that we can extract decay in $v$ for $\partial_u \Psi$, even though we are integrating in $v$ towards the future.
For fixed $u$, let $v_*(u)$ be defined by $t(u, v_*(u)) = 1$. Using the integrating factor $e^{\int_{v_*(u)}^v r^{-2}(M-e^2r^{-1})} dv$ and the vanishing of the initial data on $\{t = 1\}$, we obtain

$$\frac{r \partial_u \Psi(u, v)}{1 - \mu} = \frac{1}{4} \int_{v_*(u)}^v e^{-\int_{v_*(u)}^w r^{-2}(M-e^2r^{-1})} \left(2 \partial_v \Psi(u, v') + rF(u, v')\right) dv'.$$

Integrating by parts we derive

$$\frac{r \partial_u \Psi(u, v)}{1 - \mu} = \frac{1}{2} \Psi(u, v) + \int_{v_*(u)}^v e^{-\int_{v_*(u)}^w r^{-2}(M-e^2r^{-1})} \left(-2r^{-2}(M-e^2r^{-1})\Psi(u, v') + rF(u, v')\right) dv'.$$

Since in the region under consideration, we have the bound $r_+ < r \leq r_0$, and also,

$$M - \frac{e^2}{r} > M - \frac{e^2}{M + \sqrt{M^2 - e^2}} = \sqrt{M^2 - e^2} > 0,$$

it follows that

$$\frac{\partial_u \Psi(u, v)}{1 - \mu} \leq A|\Psi(u, v)| + A \int_{v_*(u)}^v e^{-c(v-v')} \left(|\Psi(u, v')| + |F(u, v')|\right) dv'.$$

Given $v$, let $u_*(v)$ be defined by $r(u_*(v), v) = r_0$. Then

$$|\Psi(u, v)| \leq |\Psi(u_*(v), v)| + A \int_{u_*(v)}^u (1 - \mu(u', v))|\Psi(u', v)| du'$$

$$+ A \int_{u_*(v)}^u \int_{v_*(u')} e^{-c(v-v')} \left(|\Psi(u', v')| + |F(u', v')|\right) dv'(1 - \mu(u', v)) du'.$$

We now observe that in the region $r_+ < r \leq r_0$, we can estimate $|1 - \mu| < \delta$ for some $\delta$ that can be made arbitrarily small provided that $r_0$ is sufficiently close to $r_+$. Moreover,

$$\int_{u_*(v)}^u |1 - \mu(u', v)| du' = r(u, v) - r_0 < \delta.$$

Therefore, for any $u \geq u_*(v)$,

$$|\Psi(u, v)| \leq |\Psi(u_*(v), v)| + A\delta \sup_{u' \in [u_*(v), u]} \Psi(u', v)$$

$$+ A\delta \sup_{u' \in [u_*(v), u]} \int_{v_*(u')} e^{-c(v-v')} \left(|\Psi(u', v')| + |F(u', v')|\right) dv'.$$
Split
\[
\int_{v_\star(u')}^v e^{-c(v-v')} \left( |\psi(u', v')| + |F(u', v')| \right) dv' = \\
\int_{v_\star(u')}^{v/2} e^{-c(v-v')} \left( |\psi(u', v')| + |F(u', v')| \right) dv' + \\
\int_{v/2}^v e^{-c(v-v')} \left( |\psi(u', v')| + |F(u', v')| \right) dv' 
\]
and estimate
\[
\int_{v_\star(u')}^{v/2} e^{-c(v-v')} \left( |\Psi(u', v')| + |F(u', v')| \right) dv' 
\leq A e^{-cv/2} \sup_{v' \in [v_\star(u'), v]} \left( |\Psi(u', v')| + |F(u', v')| \right), \\
\int_{v/2}^v e^{-c(v-v')} \left( |\Psi(u', v')| + |F(u', v')| \right) dv' 
\leq A \sup_{v' \in [v/2, v]} \left( |\Psi(u', v')| + |F(u', v')| \right). 
\]
Returning to (41), we derive
\[
|\Psi(u, v)| \leq |\Psi(u_\star(v), v)| + A \delta \sup_{u' \in [u_\star(v), u]} \Psi(u', v) \\
+ A \delta \sup_{u' \in [u_\star(v), u]} \sup_{v' \in [v/2, v]} \left( |\Psi(u', v')| + |F(u', v')| \right) \\
+ A \delta e^{-cv} \sup_{u' \in [u_\star(v), u]} \sup_{v' \in [v_\star(u'), v]} \left( |\Psi(u', v')| + |F(u', v')| \right). 
\]
Thus,

\[
|v_+^\beta \Psi(u, v)| \leq v_+^\beta |\Psi(u_*(v), v)| + A \delta v_+^\beta \sup_{u' \in [u_*(v), u]} \Psi(u', v) \\
+ v_+^\beta A \delta \sup_{u' \in [u_*(v), u]} \sup_{v' \in [v, v/2, v]} |\Psi(u', v')| \\
+ A \delta v_+^\beta e^{-c v} \sup_{u' \in [u_*(v), u]} \sup_{v' \in [v, (u')]} |\Psi(u', v')| \\
+ |v_+^\beta \Psi(u_*(v), v)| + A \delta v_+^\beta \sup_{u' \in [u_*(v), u]} |\Psi(u', v)| \\
+ A \delta \sup_{u' \in [u_*(v), u]} \sup_{v' \in [v/2, v]} |(v_+')^\beta \Psi(u', v')| \\
+ A \delta \sup_{u' \in [u_*(v), u]} \sup_{v' \in [v, (u')] \neq 0} |\Psi(u', v')| \\
+ A \delta \sup_{u' \in [u_*(v), u]} \sup_{v' \in [v, (u')] \neq 0} |F(u', v')| \\
\leq v_+^\beta |\Psi(u_*(v), v)| + A \delta v_+^\beta \sup_{u' \in [u_*(v), u]} |\Psi(u', v)| \\
+ 2 A \delta \sup_{u' \in [u_*(v), u]} \sup_{v' \in [v, (u')] \neq 0} |(v_+')^\beta \Psi(u', v')| \\
+ 2 A \delta \sup_{u' \in [u_*(v), u]} \sup_{v' \in [v, (u')] \neq 0} |(v_+')^\beta |F(u', v')|).
\]

The constant \( A \) is independent of the choice of \( r_0 \) and thus of \( \delta \). A fortiori, the above estimate applies also when \( |v_+^\beta \Psi(u, v)| \) on the left hand side is replaced by \( \sup_{u' \in [u_*(v), u]} \sup_{v' \in [v, (u')] \neq 0} |(v_+')^\beta \Psi(u', v')| \). Thus, if \( r_0 \) is chosen close enough to \( r_+ \), so that \( \delta \) is sufficiently small, we have at our point \((r_1, t_1)\) the estimate

\[
|\Psi(r_1, t_1)| \leq 2(v_1)_+^{\beta} \left( \sup_{v \in [v', v_1]} v_+^\beta |\Psi(u_*(v), v)| \right) \\
+ A \delta v_+^\beta \sup_{u \in [u_*(v_1), u_1]} \sup_{v' \in [v, (u_1)]} |F(u, v)|, \tag{42}
\]

for any \( \beta \geq 0 \). Here \( v' \) denotes the \( v \)-coordinate of the intersection of the curve \( r = r_0 \) and \( \{t = 1\} \). Observe that the first term on the right hand side of the above inequality can be rewritten as \( \sup_{1 \leq t \leq t_1} v_+^\beta |\Psi(r_0, t)| \); the supremum is thus taken over \( \Psi \) evaluated at points in a subset of the region \( r \geq r_0 \). We choose

\[
\beta = 1.
\]
From (40) and (42), we infer now that

$$|\Psi(r_1, t_1)| \leq A(v_1)_+^{-1} \max_{J^{-}(r_1, t_1)} v_+^n |F(u, v)|,$$  \hspace{1cm} (43)

for all \((r_1, t_1).\)

\[\Box\]

6 Proof of Theorem 2

Recall the set \(B\) and the constant \(B\) defined by (16). Let \((r_1, t_1) = (u_1, v_1) \in B.\) Let \(F = G(\phi)\) and consider \(\Psi\) a solution of \(33\) in \(J^{-}(u_1, v_1)\) with vanishing data at \(t = 1,\) and let \(\psi\) be a solution of the homogeneous equation with data

$$\psi|_{J^{-}(u_1, v_1) \cap \{t = 1\}} = \phi|_{J^- (u_1, v_1) \cap \{t = 1\}},$$

$$\partial_t \psi|_{J^{-}(u_1, v_1) \cap \{t = 1\}} = \partial_t \phi|_{J^- (u_1, v_1) \cap \{t = 1\}}.$$

We have that

$$\phi = \psi + \Psi$$

in \(J^{-}(u_1, v_1) \cap J^+(\{t = 1\}).\)

By Theorems 3 and 4, it follows that

$$|\psi(u_1, v_1)| \leq E(v_1)_+^{-1},$$  \hspace{1cm} (45)

where \(E = E(\epsilon, M, e, \Sigma, R) \to 0\) as \(\epsilon \to 0.\) If \(0 \leq B < c,\) then by assumption 5,

$$|G(\phi(u, v))| \leq K|\phi(u, v)|^p, \forall (u, v) \in B$$

Thus, by the definition 16 of the set \(B,\) we have

$$|F(u, v)| \leq KB^p v_+^{-p}.$$  \hspace{1cm} (46)

Therefore for \(p > 4,\) choosing \(\alpha = p\) and applying Theorem 6 we have

$$|\Psi(u_1, v_1)| \leq A(v_1)_+^{-1} KB^p$$

and thus, by (46) and (47),

$$|\psi(u_1, v_1)| \leq A(v_1)_+^{-1} KB^p + E(v_1)_+^{-1}.$$  \hspace{1cm} (47)

Let us define \(B = 4E,\) and let us require \(E\) (and thus \(\epsilon\)) to be sufficiently small so that \(AKB^p < \frac{B}{4}.\) Inequality 16 then yields

$$|\psi(u_1, v_1)| \leq \frac{B}{2}(v_1)_+^{-1},$$

as required.
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