Mirror Map as Generating Function of Intersection Numbers: Toric Manifolds with Two Kähler Forms

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Abstract: In this paper, we extend our geometrical derivation of the expansion coefficients of mirror maps by localization computation to the case of toric manifolds with two Kähler forms. In particular, we consider Hirzebruch surfaces $F_0$, $F_3$ and Calabi-Yau hypersurface in weighted projective space $\mathbb{P}(1, 1, 2, 2, 2)$ as examples. We expect that our results can be easily generalized to arbitrary toric manifolds.

1. Introduction

In the study of mirror symmetry, the gauged linear sigma model is expected to play an important role [19]. It has been considered to be slightly different from the topological (non-linear) sigma model, whose correlation function is nothing but the Gromov-Witten invariant. Let us restrict our attention to the genus 0 Gromov-Witten invariants of toric manifolds. The moduli space used in the topological (non-linear) sigma model is the moduli space of stable maps, which is a compactification of the moduli space of holomorphic maps from $\mathbb{C}P^1$ to a toric manifold by using stable maps. On the other hand, the moduli space used in the gauged sigma model is another compactification (toric compactification) of the moduli space of holomorphic maps from $\mathbb{C}P^1$ to a toric manifold. In this case, we use “quasi-maps” from $\mathbb{C}P^1$ to toric manifolds to compactify the moduli space. In [17] and [18], Losev, Nekrasov, and Shatashvili called them “freckled instantons” and asserted that they play the role of the origin of the mirror transformation in the mirror computation of the Gromov-Witten invariants of the Calabi-Yau manifold. A quasi map $f: \mathbb{C}P^1 \rightarrow X$ is a map which allows some points in $\mathbb{C}P^1$ whose images are undefined by $f$ (in [17] and [18], these points are called “freckles”). Therefore, a quasi map is not an actual map in some cases. The merit of using toric compactification is that the boundary structure of toric compactification is simpler than the one of stable map compactification. Since the moduli space is different, the correlation functions of the gauged linear sigma model do not always coincide with the corresponding Gromov-Witten invariants. In [19], Morrison and Plesser computed the generating functions of the three
point functions of the two gauged linear sigma models that correspond to Calabi-Yau 3-folds in \( CP^4 \) and \( \mathbb{P}(1, 1, 2, 2, 2) \) and observed that they coincide with the B-model Yukawa-couplings used in the mirror computation of the Gromov-Witten invariants of these Calabi-Yau 3-folds. Therefore, the generating function of correlation functions of the gauged linear sigma model is expected to reproduce the object of the B-model in the mirror computation. In this paper, we construct “the moduli space of quasi maps with two marked points”. We first consider the moduli space of quasi-maps (freckled instantons) of degree \( d \) from \( CP^1 \) to a toric manifold \( X \) with well-defined images at two marked points: \((1 : 0), (0 : 1) \in CP^1 \). Then we divide this space by the \( C^\times \) action induced from the \( C^\times \) action on \( CP^1 \) that keeps the two marked points fixed. We call this space \( M_{P0,2}(X, d) \). \( M_{P0,2}(X, d) \) turns out to be non-compact because the quasi maps whose image of either \((1 : 0)\) or \((0 : 1)\) is undefined are excluded. Therefore, we compactify it by adding boundary parts. These boundary parts consist of chains of quasi maps (with well defined images at \((1 : 0), (0 : 1)\)) connected at the images of the two marked points. We denote the resulting space by \( \hat{M}_{P0,2}(X, d) \). From this construction, we can regard \( \hat{M}_{P0,2}(X, d) \) as the moduli space that has moderate characteristics between the moduli of stable maps and the moduli space of the gauged linear sigma model. It is well known that the boundary structure of the moduli space of genus 0 stable maps is described by tree graphs. In contrast, the boundary structure of \( \hat{M}_{P0,2}(X, d) \) is described only by line graphs because it includes quasi maps. Therefore, application of localization theorem to \( M_{P0,2}(X, d) \) is easier than the moduli space of stable maps.

With this set-up, our general conjecture is the following.

**General Conjecture.** The 2-point correlation functions computed by using \( \hat{M}_{P0,2}(X, d) \) give us the information of the crucial process used in the mirror computation of the Gromov-Witten invariants. More specifically, some of these 2-point correlation functions give us the expansion coefficients of the mirror map used in the mirror computation. The other 2-point functions are translated into the 2-point Gromov-Witten invariants via the (generalized) mirror transformation caused by the mirror map.

Of course, the above conjecture is a little bit abstract. For example, we have to define the 2-point correlation function computed by using \( M_{P0,2}(X, d) \). We will give more explicit details in the following part of this section. Before we turn to details, we remark here that this paper is a continuation of our previous work [12], which is our first paper aiming at establishing the above conjecture when the toric manifold is \( CP^{N-1} \).

In [12], we proposed a residue integral representation of the virtual structure constant \( \hat{L}_{n,k,d} \), which is a B-model analogue of genus 0 Gromov-Witten invariants of the degree \( k \) hypersurface in \( CP^{N-1} \) (we denote this hypersurface by \( M_k \)). \( \hat{L}_{n,k,d} \) is our candidate of the above 2-point correlation function. The virtual structure constant \( \hat{L}_{n,k,d} \) is a rational number which is non-zero if and only if \( 0 \leq n \leq N - 1 - (N - k)d \). It is defined by the initial condition

\[
\sum_{n=0}^{k-1} \hat{L}_{n,k,1} w^n = k \prod_{j=1}^{k-1} ((k - j) + jw), \tag{1.1}
\]

and the recursive formulas that represent \( \hat{L}_{n,k,d} \) as a weighted homogeneous polynomial in \( \hat{L}_{m,k',d'} \) \((d' \leq d)\). We will show the explicit form of the recursive formulas in Sect. 2. Let us first review the main results on the virtual structure constants presented in [14, 15]. For this purpose, we introduce the genus 0 degree \( d \) two-point Gromov-Witten