MOTIVIC DOUBLE ZETA VALUES OF ODD WEIGHT

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Abstract. For odd \( N \geq 5 \), we establish a short exact sequence about motivic double zeta values \( \zeta^m(r, N - r) \) with \( r \geq 3 \) odd, \( N - r \geq 2 \). And from this exact sequence we classify all the relations between motivic double zeta values \( \zeta^m(r, N - r) \) with \( r \geq 3 \) odd, \( N - r \geq 2 \). As corollary, we obtain the rank of a matrix in [10] which was conjectured by Zagier.

1. Introduction

The multiple zeta values are defined by convergent series

\[
\zeta(n_1, \ldots, n_r) = \sum_{0 < k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}, \quad (n_1, \ldots, n_{r-1} > 0, n_r > 1).
\]

We call \( N = n_1 + n_2 + \cdots + n_r \) and \( r \) the weight and depth of \( \zeta(n_1, \ldots, n_r) \) respectively. Let \( \mathcal{Z}_N \) be the \( \mathbb{Q} \)-linear combination of all the weight \( N \) part multiple zeta values. Denote \( \mathcal{Z}_0 = 1 \), then the graded vector space

\[
\mathcal{Z} = \bigoplus_{n \geq 0} \mathcal{Z}_n
\]
is a \( \mathbb{Q} \)-graded algebra. Beware that the notation about multiple zeta values here are different from the notation [8] and [10].

In order to study these numbers, Brown introduced motivic multiple zeta algebra \( \mathcal{H} \) in [3]. The elements in \( \mathcal{H} \) are \( \mathbb{Q} \)-linear combination of motivic multiple zeta values \( \zeta^m(n_1, n_2, \cdots, n_r) \). Define the weight and depth of \( \zeta^m(n_1, n_2, \cdots, n_r) \) similarly. Denote \( \mathcal{D}_r \mathcal{H} \) the elements of \( \mathcal{H} \) of depth \( \leq r \) and \( gr_r \mathcal{H} = \mathcal{D}_r \mathcal{H}/\mathcal{D}_{r-1} \mathcal{H} \). And there is a natural graded \( \mathbb{Q} \)-algebra homomorphism

\[
\eta: \mathcal{H} \to \mathcal{Z}
\]

which satisfy \( \eta(\zeta^m(n_1, n_2, \cdots, n_r)) = \zeta(n_1, \ldots, n_r) \).

For \( N \geq 5 \), odd, denote

\[
\mathfrak{R}_N = \{(c_r)_{3 \leq r \leq N-2, \text{odd}} \mid \sum_{r=1, \text{odd}}^{N-2} c_r \zeta^m(r, N - r) \equiv 0 \text{ mod } \zeta^m(N), c_r \in \mathbb{Q}\}.
\]

And denote \( \mathcal{D}_N^{\text{odd, even}} \) the \( \mathbb{Q} \)-vector sub-quotient space of \( \mathcal{H} \) generated by \( \zeta^m(r, N - r), 3 \leq r \leq N - 2 \), odd modulo the subspace \( \mathbb{Q} \zeta^m(N) \).

For even \( k \), we denote \( W_k^+ \) and \( W_k^- \) as odd and even restricted period polynomial respectively. By using the technique developed by Brown in [2], we obtain

Theorem 1.1. For \( N \geq 5 \), odd, there is an exact sequence

\[
0 \to \mathcal{D}_N^{\text{odd, even}} \overset{\partial}{\to} (gr_1 \mathcal{H}^{\text{odd}} \otimes gr_1 \mathcal{H}^{\text{even}})_N \overset{\nu}{\to} (W_{N-1}^+ \oplus W_{N+1}^-)^\vee \to 0
\]
Where \((W^+_N \oplus W^-_{N+1})^\vee\) means the dual vector space of \(W^+_N \oplus W^-_{N+1}\). \(gr_1 D^e H\) and \(gr_1 D^o H\) are depth-graded 1 part of \(H\) generated by \(\zeta^m(3), \zeta^m(5), \cdots, \zeta^m(2n + 1), \cdots\) and \(\zeta^m(2), \zeta^m(4), \cdots, \zeta^m(2n), \cdots\) respectively. \((gr_1 D^e H \otimes gr_1 D^o H)_N\) means the degree \(N\) part of \((gr_1 D^e H \otimes gr_1 D^o H)\).

The maps \(\partial, v\) will be given in Section 2. The difficult part in the proof of Theorem 1.1 is the exactness \(\text{Im} \partial = \text{Ker} v\). This will follow from some tricky matrix construction and calculation.

Since motivic multiple zeta value also satisfy the double shuffle relation [9], by the main results in Ding Ma [8] there is an injective \(\mathbb{Q}\)-linear map \(\xi: W^+_N \oplus W^-_{N+1} \to \mathcal{R}_N\) for odd \(N\). For \(p \in W^+_N, p(x + y, y) = \sum_{0 \leq r \leq N} \binom{N - 2}{r - 1} b_{N-r,r} x^{N-r} y^{r-2},\)
we have
\(\xi(p) = (b_{N-r,r} - b_{r,N-r})_{3 \leq r \leq N-2, \text{odd}}.\)

For \(p \in W^-_{N+1}, \frac{\partial}{\partial x} p(x + y, y) = \sum_{0 \leq r \leq N} \binom{N - 2}{r - 1} c_{N-r,r} x^{N-r} y^{r-1},\)
we have
\(\xi(p) = (c_{N-r,r} - c_{r,N-r})_{3 \leq r \leq N-2, \text{odd}}.\)

The proof of main theorems of Ding Ma [8] is based on the work of H. Gangl, M. Kaneko, D. Zagier about double shuffle relation between formal symbols. From Theorem 1.1 we can deduce that

**Theorem 1.2.** The map \(\xi: W^+_N \oplus W^-_{N+1} \to \mathcal{R}_N\) is an isomorphism.

Thus for \(N\) odd, we complete the classification of relations between motivic double zeta values \(\zeta^m(r, N - r)\) with \(r\) odd. Recall that for \(N\) even, the classification of relations between motivic double zeta values \(\zeta^m(r, N - r)\) with \(r\) odd is completed by the work of [1].

From Theorem 1.1 we can deduce that the injective map (41) in [10] is also an isomorphism.

## 2. Motivic Galois action

First we will give a very short introduction about mixed Tate motives, the references are [4], [5]. Denote \(MT(\mathbb{Z})\) mixed Tate motives over \(\mathbb{Z}\). The motivic multiple zeta algebra \(H\) is an object in the category of \(MT(\mathbb{Z})\). Denote \(\pi_1(MT(\mathbb{Z}))\) the de-Rham fundamental group of \(MT(\mathbb{Z})\). Then
\[
\pi_1(MT(\mathbb{Z})) = \mathbb{G}_m \ltimes U^{dr}.
\]
Where \(U^{dr}\) is the pro-unipotent algebraic group with free lie algebra generated by \(\sigma_{2n+1}\) of degree \(-(2n+1), n \geq 1\) respect the action of \(\mathbb{G}_m\).

Denote \(\mathfrak{g}\) lie algebra of \(U^{dr}\), then \(\mathfrak{g}\) has a natural action on \(H\). \(\mathfrak{g}\) induced a depth filtration structure from one automorphism subgroup of motivic fundamental group.
of $\mathbb{P} - \{0, 1, \infty\}$. The lie algebra of this automorphism subgroup is $\mathbb{L}(e_0, e_1)$ with Poisson-Ihara bracket $\{ , \}$. 

There is an injective map $i : \mathfrak{g} \rightarrow (\mathbb{L}(e_0, e_1), \{ , \})$ such that $i(\sigma_{2n+1}) = \text{ad}^{2n}(e_0)e_1 + \text{higher depth}$. And the action of $\mathfrak{g}$ on $\mathcal{H}$ factor through $(\mathbb{L}(e_0, e_1), \{ , \})$. So elements of $\mathfrak{g}$ map $gr^D_r \mathcal{H}$ to $gr^D_{r-1} \mathcal{H}$ and all the action factor through $\mathfrak{g}^{ab}$ (see also the analysis in Section 10 of [2]). Denote the action determined by the canonical elements $\tilde{\sigma}_{2n+1}$ (image of $\sigma_{2n+1}$ in $\mathfrak{g}^{ab}$) as 

$$\partial_{2n+1} : gr^D_r \mathcal{H} \rightarrow gr^D_{r-1} \mathcal{H}.$$ 

For odd $N \geq 5$, define 

$$\partial : D_{N, \text{od, ev}} \rightarrow (gr^D_1 \mathcal{H}^{\text{od}} \otimes gr^D_1 \mathcal{H}^{\text{ev}})_N$$ 

as 

$$\partial(\zeta^m(r, N - r)) = \sum_{3 \leq s \leq N - 2, \text{odd}} \zeta^m(s) \otimes \partial_s(\zeta^m(r, N - r)).$$ 

By theorem 3.3 in [3], we know

**Proposition 2.1. For odd $N \geq 5$, $\partial$ is injective.**

Now we want to know the explicit formula of $\partial$. This can be achieved by the motivic method developed in [2]. But here we prefer a more simple and elementary way.

**Proposition 2.2. For $N \geq 5$, odd, $2m + 2n + 1 = N, m, n \geq 1$, we have**

$$\partial(\zeta^m(2m + 1, 2n)) = \sum_{m_1 + n_1 = \frac{1}{2}N - \frac{1}{2}, m_1, n_1 \geq 1} \left[ \delta \left( \frac{m_1}{m, n}, \frac{n_1}{m, n} \right) - \left( \begin{array}{c} 2m_1 \\ m, n \end{array} \right) - \left( \begin{array}{c} 2m_1 \\ n, m - 1 \end{array} \right) \right] \zeta^m(2m_1 + 1) \otimes \zeta^m(2n_1).$$

**Proof:** Since motivic multiple zeta value satisfy the double shuffle relation. And the proof of Proposition 7 in [10] only use double shuffle relation between multiple zeta value. So by Proposition 7 in [10] for $N \geq 5$, odd, $2m + 2n + 1 = N, m, n \geq 1$, we have 

$$\zeta^m(2m + 1, 2n) = \left[ - \left( \frac{N - 1}{2n - 1} \right) - \left( \frac{N - 1}{2m} \right) + 1 \right] \zeta^m(N)$$

$$+ \sum_{m_1 + n_1 = \frac{1}{2}N - \frac{1}{2}, m_1, n_1 \geq 1} \left[ \delta \left( \frac{m_1}{m, n}, \frac{n_1}{m, n} \right) - \left( \begin{array}{c} 2m_1 \\ m, n \end{array} \right) - \left( \begin{array}{c} 2m_1 \\ 2n - 1 \end{array} \right) \right] \zeta^m(2m_1 + 1) \zeta^m(2n_1)$$

Since 

$$\partial(\zeta^m(2m_1 + 1, 2n_1)) = \sum_{1 \leq k \leq \frac{N}{2} - \frac{3}{2}} \zeta^m(2k + 1) \otimes \partial_{2k+1}(\zeta^m(2m_1 + 1) \zeta^m(2n_1))$$

$$= \sum_{1 \leq k \leq \frac{N}{2} - \frac{3}{2}} \zeta^m(2k + 1) \otimes (\partial_{2k+1}(\zeta^m(2m_1 + 1))) \zeta^m(2n_1) + \zeta^m(2m_1 + 1) \partial_{2k+1}(\zeta^m(2n_1)))$$

$$= \zeta^m(2m_1 + 1) \otimes \zeta^m(2n_1)$$
The last identity is due to the fact
\[ \partial_{2k+1}(\zeta^m(2m_1 + 1)) = \delta \left( \frac{2m_1 + 1}{2k + 1} \right), \partial_{2k+1}(\zeta^m(2n_1)) = 0 \]
So we have
\[
\partial(\zeta^m(2m + 1, 2n)) = \sum_{m_1 + n_1 = \frac{1}{2}N - \frac{1}{2}} \left[ \delta \left( \frac{m_1 + n_1}{m, n} \right) - \delta \left( \frac{2m}{2m} \right) - \delta \left( \frac{2n_1}{2n_1 - 1} \right) \right] \zeta^m(2m_1 + 1) \otimes \zeta^m(2n_1).
\]

For even integer \( h > 2 \), if \( P \in \mathbb{Q}[X] \) satisfy
\[
P(X) + X^{h-2}P \left( \frac{-1}{X} \right) = 0,
\]
\[
P(X) + X^{h-2}P(1 - \frac{1}{X}) + (X - 1)^{h-2}P \left( \frac{-1}{X - 1} \right),
\]
then \( P \) is called period polynomial of weight \( h \) over \( \mathbb{Q} \). Denote \( W_h \) the space of period polynomial of weight \( h \) over \( \mathbb{Q} \).
Define
\[
W_h^+ = \{ P \in W_h \mid P(0) = 0, P(X) = X^{h-2}P \left( \frac{1}{X} \right), \}
\]
\[
W_h^- = \{ P \in W_h \mid P(0) = 0, P(X) + X^{h-2}P \left( \frac{1}{X} \right) = 0, \}
\]
then \( W_h = W_h^+ \oplus W_h^- \oplus \{ \mathbb{Q}(X^{h-2} - 1) \} \) by the general theory of period polynomial.
For \( N \geq 5 \), odd, we define
\[
j_1 : W_{N-1}^+ \rightarrow ((g r_1^D \mathcal{H}^{od} \otimes g r_1^D \mathcal{H}^{ev})_N)^\vee
\]
for \( p \in W_{N-1}^+ ;
\[
p = \sum_{r+s=N-3, r,s \geq 1, odd} p_{r,s}X^r, \quad j_1(p)(\zeta^m(2m + 1) \otimes \zeta^m(2n)) = p_{2m-1,2n-1}
\]
and
\[
j_2 : W_{N+1}^- \rightarrow ((g r_1^D \mathcal{H}^{od} \otimes g r_1^D \mathcal{H}^{ev})_N)^\vee
\]
for \( q \in W_{N+1}^- ;
\[
X^{N-2}q' \left( \frac{1}{X} \right) = \sum_{r+s=N-1, r,s \geq 1, odd} t_{r-1,s-1}X^{r-1}, \quad j_2(q)(\zeta^m(2m + 1) \otimes \zeta^m(2n)) = t_{2m,2n}
\]
It is clear that \( j_1, j_2 \) are both injective. We define
\[
j : W_{N-1}^+ \oplus W_{N+1}^- \rightarrow ((g r_1^D \mathcal{H}^{od} \otimes g r_1^D \mathcal{H}^{ev})_N)^\vee
\]
to be the unique map which satisfy \( j \mid_{W_{N-1}^+} = j_1, j \mid_{W_{N+1}^-} = j_2 \). By the symmetry property of \( W_{N-1}^+ \) and anti-symmetry property of \( W_{N+1}^- \) it is easy to show \( j \) is also injective.
We define \( v : (g r_1^D \mathcal{H}^{od} \otimes g r_1^D \mathcal{H}^{ev})_N \rightarrow (W_{N-1}^+ \oplus W_{N+1}^-)^\vee \) to be the dual of \( j \). So we have
Proposition 2.3. For $N \geq 5$, odd, the map

$$v : (\text{gr}^D_1 \mathcal{H}^{\text{od}} \otimes \text{gr}^D_1 \mathcal{H}^{\text{ev}})_N \to (W_{N-1}^+ \oplus W_{N+1}^-)^{\nu}$$

is surjective.

The exactness $\text{Im} \partial = \text{Ker} v$ is more difficult and will be left to section 3. Now we only prove the following

Proposition 2.4. For $N \geq 5$, odd, $\text{Im} \partial \subseteq \text{Ker} v$.

Proof: By the formula of $\partial$ and the definition of $v$, it suffices to prove that

(1) If

$$p(X) = \sum_{m_1 + n_1 = \frac{1}{2}N - \frac{1}{2}} p_{2m_1 - 1, 2n_1 - 1} X^{2m_1 - 1} \in W_{N-1}^+,$$

then

$$\sum_{m_1 + n_1 = \frac{1}{2}N - \frac{1}{2}} p_{2m_1 - 1, 2n_1 - 1} \left[-\delta \left(\frac{m_1}{m}, \frac{n_1}{n}\right) + \left(\frac{2m_1}{2m}ight) + \left(\frac{2m_1}{2n - 1}\right)\right] = 0$$

for all $m + n = \frac{1}{2}N - \frac{1}{2}, m, n \geq 1$.

(2) If $q \in W_{N+1}^-$ and

$$X^{N-2} q'(\frac{1}{X}) = \sum_{m_1 + n_1 = \frac{1}{2}N - \frac{1}{2}} t_{2m_1, 2n_1} X^{2m_1},$$

then

$$\sum_{m_1 + n_1 = \frac{1}{2}N - \frac{1}{2}} t_{2m_1, 2n_1} \left[-\delta \left(\frac{m_1}{m}, \frac{n_1}{n}\right) + \left(\frac{2m_1}{2m}ight) + \left(\frac{2m_1}{2n - 1}\right)\right] = 0$$

for all $m + n = \frac{1}{2}N - \frac{1}{2}, m, n \geq 1$.

Since $p \in W_{N-1}^+$, by definition $p$ satisfy

$$-p(X) + p(1 + X) + X^{N-3} p(1 + \frac{1}{X}) = 0.$$

By comparing the coefficients of $X, X^2, \cdots, X^{N-4}$ in the above formula on both sides, we have

(A) $\sum_{m_1 + n_1 = \frac{1}{2}N - \frac{1}{2}} p_{2m_1 - 1, 2n_1 - 1} \left[-\delta \left(\frac{m_1}{m}, \frac{n_1}{n}\right) + \left(\frac{2m_1 - 1}{2m - 1}\right) + \left(\frac{2m_1 - 1}{2n - 1}\right)\right] = 0$

(B) $\sum_{m_1 + n_1 = \frac{1}{2}N - \frac{1}{2}} p_{2m_1 - 1, 2n_1 - 1} \left[\left(\frac{2m_1 - 1}{2m}\right) + \left(\frac{2m_1 - 1}{2n - 2}\right)\right] = 0$

for all $m + n = \frac{1}{2}N - \frac{1}{2}, m, n \geq 1$. (A) plus (B), then (1) is proved.

For $q \in W_{N+1}^-$, let $C(X) = X^{N-2} q'(\frac{1}{X})$. Statement (2) is equivalent to

$$C(X) - C(1 + X) - X^{N-2} C(1 + \frac{1}{X}) = \text{constant} + \text{odd polynomial in } X.$$
By the anti-symmetry properties \( q(X) = q(-X) = -X^{N-1} q(\frac{1}{X}) \) we have

\[
C(X) - C(1 + X) - X^{N-2} C(1 + \frac{1}{X}) = \frac{(X + 1)^N}{d} \left[ q\left(\frac{-1}{X + 1}\right) - q\left(\frac{X}{X+1}\right) + (X + 1)^{1-N} q(-X) \right] - q'(X)
\]

So statement (2) is proved. \( \square \)

**Remark 2.5.** Proposition [2,4] is essentially the main result of Zagier in Section 6 [10]. And the proof here is nearly the same in [10]. We give the proof here for completeness. So the exact sequence we try to establish provides a more motivic explanation of Zagier’s result.

### 3. Exactness at the middle

In the whole section, fix \( N \) an odd number which \( \geq 5 \). From the formula of \( \partial \) and the definition of \( v \), \( \text{Im} \, \partial = \text{Ker} \, v \) is equivalent to

**Proposition 3.1.** If \( C(X) = \sum_{n=1}^{\frac{1}{2}(N-3)} c_n X^{2n} \) satisfy

\[
C(X) - C(1 + X) - X^{N-2} C\left(1 + \frac{1}{X}\right) = c + \text{odd polynomial in } X,
\]

then \( C(X) = X p(X) + X^{N-2} q'\left(\frac{1}{X}\right) \) for some \( p \in W^{+}_{N-1}, \, q \in W^{-}_{N+1} \).

For convenience denote \( L_{C}(X) = C(X) - C(1 + X) - X^{N-2} C\left(1 + \frac{1}{X}\right) \) for short.

**Lemma 3.2.** If \( C(X) = \sum_{n=1}^{\frac{1}{2}(N-3)} c_{2n} X^{2n} \) satisfy

\[
C(X) - C(1 + X) - X^{N-2} C\left(1 + \frac{1}{X}\right) = c + \text{odd polynomial in } X,
\]

and \( C(X) = X p(X) \) with \( p(X) = X^{N-3} p\left(\frac{1}{X}\right) \), then \( p \in W^{+}_{N-1} \).

**Proof:** Let \( X = 0 \), we know that \( c = -C(1) \). We have

\[
(O) \quad (X + 1)[p(X) - p(1 + X) - X^{N-3} p\left(1 + \frac{1}{X}\right)] = L(X) + p(X) = c + \text{odd in } X.
\]

Replace \( X \) by \( \frac{1}{X} \), we get

\[
\left(\frac{1}{X} + 1\right)[p\left(\frac{1}{X}\right) - p\left(1 + \frac{1}{X}\right) - \frac{1}{X^{N-3}} p(1 + X)] = L\left(\frac{1}{X}\right) + p\left(\frac{1}{X}\right) = c + \text{odd in } \frac{1}{X}.
\]

Multiply by \( X^{N-2} \) and use the fact \( p(X) = X^{N-3} p\left(\frac{1}{X}\right) \), then

\[
(E) \quad (X + 1)[p(X) - p(1 + X) - X^{N-3} p\left(1 + \frac{1}{X}\right)] = cX^{N-2} + \text{even in } X.
\]

By comparing formula \((O)\) and \((E)\), we know

\[
p(X) - p(1 + X) - X^{N-3} p\left(1 + \frac{1}{X}\right) = c\left(1 + X^{N-2}\right) \frac{1}{1 + X}.
\]
Let $X \to -1$ in the above formula, then

$$-p(1) = -C(1) = c(N - 2) = -C(1)(N - 2).$$

So $c = -C(1) = 0$ and $p \in W_{N-1}^+$. □

**Lemma 3.3.** If $C(X) = \sum_{n=1}^{(N-3)} c_n X^{2n}$ satisfy

$$C(X) - C(1 + X) - X^{N-2}C(1 + \frac{1}{X}) = c + \text{odd polynomial in } X,$$

and $C(X) = X^{N-2}q'(\frac{1}{X})$ with $q(X) + X^{N-1}q(\frac{1}{X}) = 0, q(0) = 0$, then $q \in W_{N+1}^-$. 

**Proof:** We have

$$(O') \quad (X + 1)^N \frac{d}{dX} \left[ q(\frac{1}{X + 1}) - q(\frac{X}{X + 1}) + (X + 1)^{-N+1}q(X) \right]$$

$$= L(X) + q'(X) = c + \text{odd in } X$$

Replace $X$ by $\frac{1}{X}$, we get

$$-(\frac{1}{X} + 1)^N X^2 \frac{d}{dX} \left[ q(\frac{X}{X + 1}) - q(\frac{1}{X + 1}) + (\frac{1}{X} + 1)^{-N+1}q(\frac{1}{X}) \right]$$

$$= L(\frac{1}{X}) + q'\left(\frac{1}{X}\right) = c + \text{odd in } \frac{1}{X}$$

Multiply by $X^{N-2}$, then

$$(E') \quad (X + 1)^N \frac{d}{dX} \left[ q(\frac{1}{X + 1}) - q(\frac{X}{X + 1}) + (X + 1)^{-N+1}q(X) \right]$$

$$= cX^{N-2} + \text{even in } X.$$ By comparing formula $(O')$ and $(E')$, we know

$$\frac{d}{dX} \left[ q(\frac{1}{X + 1}) - q(\frac{X}{X + 1}) + (X + 1)^{-N+1}q(X) \right] = c \frac{(X^{N-2} + 1)}{(X + 1)^N}.$$ Integrate on $X$, then

$$q(\frac{1}{X + 1}) - q(\frac{X}{X + 1}) + (X + 1)^{-N+1}q(X) = \frac{c}{N - 1} \frac{X^{N-1} - 1}{(X + 1)^{N-1}} + a$$

Replace $X$ by $\frac{1}{X}$ and use the fact that $q(X) + X^{N-1}q(\frac{1}{X}) = 0$, then

$$-q\left(\frac{1}{X + 1}\right) + q\left(\frac{X}{X + 1}\right) - (X + 1)^{-N+1}q(X) = \frac{c}{N - 1} \frac{X^{N-1} - 1}{(X + 1)^{N-1}} + a.$$ So $a = 0$, the above formula reduce to

$$-q\left(\frac{1}{X + 1}\right) + q\left(\frac{X}{X + 1}\right) - (X + 1)^{-N+1}q(X) = \frac{c}{N - 1} \frac{X^{N-1} - 1}{(X + 1)^{N-1}}.$$ Let $X = 0$, so $c = -(N - 1)q(1) = 0$.

So $q(\frac{1}{X + 1}) - q(\frac{X}{X + 1}) + (X + 1)^{-N+1}q(X) = 0, q \in W_{N+1}^-$. □
Lemma 3.2 and lemma 3.3 say that if $C(X)$ satisfies some kind of symmetry or anti-symmetry condition, then Proposition 3.1 is true. The following lemma shows that there is some kind of symmetrization process.

**Lemma 3.4.** If $C(X) = \frac{1}{2} \sum_{n=1}^{N-3} c_n X^{2n}$ satisfies

$$C(X) - C(1 + X) - X^{N-2}C(1 + \frac{1}{X}) = \text{constant} + \text{odd in } X,$$

let $XP(X) = XC'(X) + X^{N-2}C'(\frac{1}{X})$, then

$$XP(X) - (1 + X)p(1 + X) - X^{N-2}(1 + \frac{1}{X})p(1 + \frac{1}{X}) = \text{constant} + \text{odd in } X.$$

**Proof:** Let $N = 2K + 1$, it suffices to prove that

$$\sum_{j=1}^{K-1} j c_j + (K - j)c_{K-j} \left[ \binom{2j}{2i} + \binom{2j}{2K-2i-1} - \delta\left(\binom{j}{i}\right) \right] = 0$$

for $1 \leq i \leq K - 1$. i.e.

$$\sum_{j=1}^{K-1} j c_j \left[ \binom{2j}{2i} + \binom{2j}{2K-2i-1} - \delta\left(\binom{j}{i}\right) \right] + \sum_{j=1}^{K-1} j c_j \left[ \binom{2K-2j}{2i} + \binom{2K-2j}{2K-2i-1} - \delta\left(\binom{K-j}{i}\right) \right] = 0$$

for $1 \leq i \leq K - 1$.

First we fix some notations. For indeterminant element $T$ and inter $n \geq 1$, denote

$$\binom{T}{n} = \frac{T(T-1)\ldots(T-n+1)}{n!} \in \mathbb{Q}[T].$$

and for $n = 0$, denote $\binom{T}{0} = 1$. For integer $n < 0, m > 0$, define $\binom{m}{n} = 0$.

It is well-known that the polynomials

$$\binom{T}{0}, \binom{T}{1}, \ldots, \binom{T}{n}$$

form a basis of the vector space of polynomial of degree $\leq n$. In fact

$$f(0) \binom{T}{0} + \Delta f(0) \binom{T}{1} + \cdots + \Delta^n f(0) \binom{T}{n} = f(T)$$

where $\Delta^l f(0)$ is the $l$-th difference of $f$ at $T = 0$.

All the differences are defined as

$$\Delta^0 f(T) = f(T)$$

$$\Delta^1 f(T) = f(T + 1) - f(T)$$

$$\Delta^2 f(T) = \Delta^1 f(T + 1) - \Delta^1 f(T) = f(T + 2) - 2f(T + 1) + f(T)$$
Set \( f(T) = \frac{T}{2} \left[ \binom{T}{2i} + \binom{2K-T}{2i} \right] \), then

\[
\Delta^1 f(T) = f(T+1) - f(T) = \frac{T+1}{2} \left[ \binom{T+1}{2i} + \binom{2K-T-1}{2i} \right] - \frac{T}{2} \left[ \binom{T}{2i} + \binom{2K-T}{2i} \right]
\]

\[
= \frac{T}{2} \left( \frac{T}{2i-1} \right) + \frac{1}{2} (T+1) - \frac{T}{2} \left( \frac{2K-T-1}{2i-1} \right) + \frac{1}{2} \left( \frac{2K-T}{2i} \right)
\]

\[
\Delta^2 f(T) = \frac{T+1}{2} \left( \frac{T+1}{2i-1} \right) - \frac{T}{2} \left( \frac{T}{2i-1} \right) + \frac{1}{2} \left[ \left( \frac{T+2}{2i} \right) - \left( \frac{T+1}{2i} \right) \right]
\]

\[
- \left[ \frac{T}{2} \left( \frac{2K-T-2}{2i-1} \right) - \frac{T}{2} \left( \frac{2K-T-1}{2i-1} \right) \right] + \frac{1}{2} \left[ \left( \frac{2K-T-2}{2i} \right) - \left( \frac{2K-T-1}{2i} \right) \right]
\]

\[
= \frac{T}{2} \left( \frac{T}{2i-2} \right) + \frac{2}{2} \left( \frac{T+1}{2i-1} \right) + \frac{T}{2} \left( \frac{2K-T-2}{2i-2} \right) - \frac{T}{2} \left( \frac{2K-T-2}{2i-1} \right).
\]

For \( 0 \leq p \leq 2i+1 \), inductively

\[
\Delta^p f(T) = \frac{T+1}{2} \left( \frac{T+1}{2i-p} \right) + \frac{p}{2} \left( \frac{T+1}{2i-p+1} \right) + (-1)^p \frac{T}{2} \left( \frac{2K-T-p}{2i-p} \right) - (-1)^p \frac{p}{2} \left( \frac{2K-T-p}{2i-p+1} \right).
\]

For \( 0 \leq p \leq 2i+1 \), let

\[
x_p = \Delta^p f(0) = \frac{p}{2} \left[ \left( \frac{1}{2i-p-1} \right) - (-1)^p \left( \frac{2K-p}{2i-p+1} \right) \right].
\]

For \( p > 2i+1 \), the above formula gives \( x_p = 0 \).

\[
(a) \quad \sum_{p=1}^{2i+1} x_p \binom{T}{p} = T \left[ \binom{T}{2i} + \binom{2K-T}{2i} \right]
\]

For \( l \geq 1 \), taking \( l \)-th difference of the above formula on both sides, then

\[
(b) \quad \sum_{p=0}^{2i+1} x_p \binom{T}{p} = \frac{T}{2} \left( \frac{T}{2i-l} \right) + \frac{l}{2} \left( \frac{T+1}{2i-l+1} \right) + (-1)^l \frac{T}{2} \left( \frac{2K-T-l}{2i-l} \right) - (-1)^l \frac{1}{2} \left( \frac{2K-T-l}{2i-l+1} \right)
\]

For \( 1 \leq j \leq K-1 \), let \( T = 2j \) in formula (a) and (b), we have

\[
(c) \quad \sum_{p=1}^{2i+1} x_p \binom{2j}{p} = j \left[ \binom{2j}{2i} + \binom{2K-2j}{2i} \right]
\]
and

\[ (d) \quad \sum_{p=1}^{2l+1} x_p \binom{2j}{2K - p - 1} \]

\[
= \sum_{p=1}^{2l+1} x_p \binom{2j}{2j - 2K + p + 1} \\
= \sum_{p=0}^{2l+1} x_p \binom{2j}{(2K - 2j - 1)p} \\
= j \left( \binom{2j}{2i + 2j - 2K + 1} \right) + \frac{2K - 2j - 1}{2} \left( \binom{2j}{2i + 2j - 2K + 2} \right) \\
- j \left( \binom{2j}{2i} \right) + \frac{2k - 2j - 1}{2} \left( \binom{2j}{2i + 2j - 2K + 2} \right) \\
= j \left( \binom{2j}{2K - 2i - 1} \right) + \frac{2K - 2j - 1}{2} \left( \binom{2j}{2K - 2i - 1} \right) \\
- j \left( \binom{2j}{2i + 2j - 2K + 1} \right) + \frac{2k - 2j - 1}{2} \left( \binom{2j}{2i + 2j - 2K + 2} \right).
\]

By formula (c) and (d),

\[ (e) \quad \sum_{l=1}^{i+1} x_{2l-1} \left[ \left( \binom{2j}{2K - 2l} \right) + \left( \binom{2j}{2l - 1} \right) - \delta \left( \binom{j}{K - l} \right) \right] \\
+ \sum_{l=1}^{i} x_{2l} \left[ \left( \binom{2j}{2l} \right) + \left( \binom{2j}{2K - 2l - 1} \right) - \delta \left( \binom{j}{l} \right) \right] \\
= \sum_{p=1}^{2l+1} x_p \binom{2j}{2K - p - 1} - \sum_{l=1}^{i+1} x_{2l-1} \delta \left( \binom{j}{K - l} \right) - \sum_{l=1}^{i} x_{2l} \delta \left( \binom{j}{l} \right) \\
= j \left[ \binom{2j}{2i} + \binom{2K - 2j}{2i} \right] + j \left( \binom{2j}{2K - 2i - 1} \right) + \frac{2K - 2j - 1}{2} \left( \binom{2j}{2K - 2i - 1} \right) \\
- j \left( \binom{2j}{2i + 2j - 2K + 1} \right) + \frac{2K - 2j - 1}{2} \left( \binom{2j}{2i + 2j - 2K + 2} \right) \\
- \sum_{l=1}^{i+1} x_{2l-1} \delta \left( \binom{j}{K - l} \right) - \sum_{l=1}^{i} x_{2l} \delta \left( \binom{j}{l} \right) \\
= j \left[ \binom{2j}{2i} + \binom{2j}{2K - 2i - 1} \right] - \delta \left( \binom{j}{i} \right) + \binom{2K - 2j}{2i} + \binom{2K - 2j}{2K - 2i - 1} - \delta \left( \binom{K - j}{i} \right).\]

While the last identity is based on checking over all the cases:
Case I, \( i < K - i - 1 \), Subcase 1: \( 1 \leq j \leq i \), Subcase 2: \( i < j < K - i - 1 \),
Subcase 3: \( K - i - 1 \leq j \leq K - 1 \);
Case II, \( i = K - i - 1 \), Subcase 1: \( 1 \leq j < i \), Subcase 2: \( j = i \),
Subcase 3: \( i < j \leq K - 1 \);
Case III, \( i > K - i - 1 \), Subcase 1: \( 1 \leq j < K - i - 1 \), Subcase 2: \( K - i \leq j \leq i \),
Subcase 3: $i < j \leq K - 1$.

From the formula (e),
\[ \sum_{j=1}^{K-1} j c_j \left[ \binom{2j}{2i} + \binom{2K - 2j}{2i} - \delta\left( \frac{j}{i} \right) \right] + \sum_{j=1}^{K-1} j c_j \left[ \binom{2K - 2j}{2i} + \binom{2K - 2j}{2i} - \delta\left( \frac{j}{i} \right) \right] \]
\[ = \sum_{l=1}^{i+1} x_{2l-1} \sum_{j=1}^{K-1} c_j \left[ \binom{2j}{2l} + \binom{2j}{2l} - \delta\left( \frac{j}{l} \right) \right] + \sum_{l=1}^{i} x_{2l} \sum_{j=1}^{K-1} c_j \left[ \binom{2j}{2l} + \binom{2j}{2l} - \delta\left( \frac{j}{l} \right) \right] \]
\[ = 0 \]
So the lemma is proved. 

By Lemma 3.2, Lemma 3.3 and Lemma 3.4, now we prove Proposition 3.1.

**Proof of Proposition 3.1** For $C(X) = \sum_{n=1}^{\frac{(N-3)}{2}} c_{2n} X^{2n}$, let
\[ p(X) = C'(X) + X^{N-3} C'(\frac{1}{X}) \]
and $q(X) = X^{N-1} C(\frac{1}{X}) - C(X)$
It’s easy to check that
\[ p(X) = X^{N-3} p(\frac{1}{X}), \quad q(X) + X^{N-1} q(\frac{1}{X}) = 0, \]
\[ X^{N-2} q'(\frac{1}{X}) = (N - 1)C(X) - X p(X), \quad q(0) = 0. \]
By Lemma 3.4 if $C(X)$ satisfies $L_C(X) =$ constant + odd in $X$, then $L_{p(X)}(X) =$ constant + odd in $X$. By Lemma 3.2 $p \in W_{N-1}^+$. Since $L_C(X) =$ constant + odd in $X$ and $L_{p(X)}(X) =$ constant + odd in $X$, so
\[ L_{X^{N-2} q'(\frac{1}{X})}(X) =$ constant + odd in $X$. 
By Lemma 3.3 $q \in W_{N+1}^-$. 

Now from Proposition 2.1, Proposition 2.3 and Proposition 3.1 Theorem 1.1 is proved. And by counting dimension of the exact sequence, Theorem 1.2 is proved. Thus the classification of all the relations between motivic double zeta values $\zeta^m(r, N - r)$ with $r \geq 3$ odd $N - r \geq 2$ is completed.

**Remark 3.5.** Theorem 1.2 is also true on the formal double zeta values. We omit the proof here. And for more exact sequences about depth-graded motivic multiple zeta values, see [7].
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