Minimal Length Uncertainty Relation and Ultraviolet Regularisation

Achim Kempf

Department of Applied Mathematics & Theoretical Physics
University of Cambridge
Cambridge CB3 9EW, U.K.

Gianpiero Mangano

INFN, Sezione di Napoli, and Dipartimento di Scienze Fisiche,
Università di Napoli Federico II,
I-80125 Napoli, Italy

Abstract

Studies in string theory and quantum gravity suggest the existence of a finite lower limit $\Delta x_0$ to the possible resolution of distances, at the latest on the scale of the Planck length of $10^{-35}$ m. Within the framework of the euclidean path integral we explicitly show ultraviolet regularisation in field theory through this short distance structure. Both rotation and translation invariance can be preserved. An example geometry is studied in detail.
1 Introduction

As has been known for long, the combination of relativistic and quantum effects implies that the conventional notion of distance breaks down the latest at the Planck scale, which is about $10^{-35}m$. The basic argument is that the resolution of small distances requires test particles of short wavelength and thus of high energy. At sufficiently small scale, i.e. close to the Planck scale, the gravitational effect of the test particle’s energy significantly disturbs the space-time structure which was tried. Studies on gedanken experiments therefore suggest the existence of a finite limit $\Delta x_0$ to the possible resolution of distances. String theory, as a theory of quantum gravity, should allow a deeper understanding of what could happen at such extreme scales. Indeed, several studies in string theory yielded a certain type of correction to the uncertainty relation

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + ...), \quad \beta > 0 \quad (1)$$

which, as is easily verified, implies a finite minimal uncertainty $\Delta x_0 = \sqrt{\hbar \beta}$. Therefore, $\Delta x_0 > 0$ can be viewed as a fuzzyness of space, or also as a consequence of the nonpointlikeness of the fundamental particles. It seems that, in string theory, intuitively, the input of more energy does eventually no longer allow to improve the spatial resolution, as this energy starts to enlarge the probed string. References are e.g. [1]-[6], and recently [7]. For recent reviews, see e.g. [8, 9].

Using the usual definition of uncertainties, ($|\psi\rangle$ normalised)

$$(\Delta x)_{\psi} = \langle \psi | (x - \langle x|\psi\rangle)^2|\psi\rangle^{1/2} \quad (2)$$

the uncertainty relation Eq.1 implies a small correction term to the commutation relation in the associative Heisenberg algebra:

$$[x, p] = i\hbar(1 + \beta p^2 + ...) \quad (3)$$

For studies on the technical and conceptual implications of these and more general types of correction terms see [10]-[13]. We remark that those studies arose from work (e.g. [14]) in the seemingly unrelated field of quantum groups, in which this type of commutation and uncertainty relations had appeared independently (first in [15]). A standard reference on quantum groups is [16].

For the general case of $n$-dimensions it appears that no consensus has been reached in the literature on which generalisation of Eq.3 i.e. which particular correction terms to the uncertainty relations could arise as a gravity effect in the ultraviolet, or as a string effect. Let us therefore here consider small correction terms of a general form ($x_i^\dagger = x_i$, $p_i^\dagger = p_i$)

$$[x_i, p_j] = i\hbar(\delta_{ij} + \beta_{ijkl}p_kp_l + ...) \quad (4)$$
with the coefficients $\beta_{ijkl}$ (and also possible terms of higher power in the $p_i$) chosen such that the corresponding uncertainty relations imply a finite minimal uncertainty $\Delta x_0 > 0$. We will for simplicity normally assume $[p_i, p_j] = 0$, but we allow $[x_i, x_j] \neq 0$. We will generally refer to such modified commutation relations as describing a 'noncommutative geometry'. Let us keep in mind however that it is the correction terms to the $x, p$ commutation relations which induce $\Delta x_0 > 0$. A noncommutativity of the $x_i$ will not be necessary for the appearance of a finite minimal uncertainty $\Delta x_0$.

In short, the key mechanism which leads to ultraviolet regularisation in the presence of a minimal uncertainty $\Delta x_0$ is the following:

On ordinary geometry, i.e. with the ordinary commutation relations underlying, the states of maximal localisation are position eigenstates $|x\rangle$, for which the uncertainty in position vanishes. Crucially, these maximal localisation states are nonnormalisable. Therefore, their scalar product is not a function but the Dirac $\delta$-distribution $\langle x|x'\rangle = \delta(x - x')$. As is well known, (for a recent reference see [19]), in the formulation of local interaction in field theory it is the ill-definedness of the product of these and related distributions which give rise to ultraviolet divergencies.

A finite minimal uncertainty $\Delta x_0$ will yield normalisable maximal localisation states, and thereby regularise the ultraviolet. More precisely, as we will see, there exist geometries, i.e. generalised commutation relations of the type of Eqs.4 such that there exists a minimal uncertainty $\Delta x_0 > 0$, with the vectors of maximal localisation $|x^{ml}\rangle$ obeying:

$$\langle x^{ml}|x^{ml}\rangle = 1, \quad \Delta_{x|x^{ml}} = \Delta x_0, \quad \langle x^{ml}|x|x^{ml}\rangle = x \quad \text{with} \quad x \in \mathbb{R} \quad (5)$$

It follows that due to their normalisability, the scalar product

$$\tilde{\delta}(x, y) := \langle x^{ml}|y^{ml}\rangle \quad (6)$$

is a function rather than a distribution.

A simple example is the one-dimensional case of Eq.3 with $\beta > 0$ and no higher order corrections. For this case the scalar product of the maximal localisation states has been calculated in [12]:

$$\tilde{\delta}(x, y) = \frac{1}{\pi} \left( \frac{x - y}{2\hbar \sqrt{\beta}} - \left( \frac{x - y}{2\hbar \sqrt{\beta}} \right)^3 \right)^{-1} \sin \left( \frac{x - y}{2\hbar \sqrt{\beta}} \right) \quad (7)$$

Note that the poles of the first factor are cancelled by zeros of the sine function, so that $\tilde{\delta}$ is a regular function. For a graph see Fig.3 in [12]. The analogous result for the case with also a finite minimal uncertainty in momentum has been worked out in [14].

We consider it to be an attractive feature of this short distance structure that it
will not require the breaking of translation and rotation invariance, while also being compatible with possible (e.g. quantum group-) generalisations of these symmetries. Also, this regularisation will not require to cut momentum space.

A general approach for the formulation of quantum field theory on such geometries has been developed in [11], with a general result on infrared regularisation in [13], and preliminary results on ultraviolet regularisation in [13]. Our aim here is to show the general mechanism, both abstractly and explicitly, by which a minimal uncertainty in position regularises the ultraviolet, i.e. we show how a \( \Delta x_0 \) could indeed provide a natural ultraviolet cutoff in quantum field theory.

While we will focus here on commutation relations which induce a finite \( \Delta x_0 > 0 \), the general framework does allow for generic commutation relations. Let us therefore also mention some of those studies which suggest such more general commutation relations.

For example, the approach by Doplicher et al. [20] suggests the existence of specific corrections to the \( x, x \) commutation and uncertainty relations. One of the arguments there is that the improvement of a position measurement in one direction ultimately requires a delocalisation in orthogonal directions, in order to reduce the gravitationally disturbing energy density of the probing particle. A possible noncommutativity of the position operators was probably first discussed in [21], developing a line of thought which has been followed since, mainly by Russian schools, see e.g. [22]. In the context of noncommuting position operators, see also [23]. Other studies, e.g. [24], suggest a length dependence of the minimal uncertainty in length measurements. Correction terms specifically to the \( p, p \) commutation relations have been discussed e.g. in [1, 25]. The approach of ‘generalised quantum dynamics’ by S. Adler [26] allows for generic commutation relations and a possible generalisation of the underlying Hilbert space to a quaternionic or octonic space. In this approach the ordinary canonical commutation relations have been derived as a first order approximation in a statistical averaging process [27].

Further, it should also be of interest to apply the noncommutative geometric concepts developed in [28], in particular to study the modifications to the differential and integral calculus over such generalised quantum phase spaces.

We note that, technically, the appearance of correction terms to the canonical commutation relations can generally also be viewed as a nontrivial and nonunique change of generators from the \( x_0, p_0 \) which obey \( [x_0, p_0] = i\hbar \) to new sets of generators. Examples for such algebra homomorphisms \( \rho \) for the case of Eq.3 are for example \( \rho : x_0 \to x = x_0 + \beta p_0 x_0 p_0, \ p_0 \to p_0 \), or also \( \rho : x_0 \to x = x_0, \ p_0 \to p = \beta^{-1/2} \tan(p_0^{1/2}) \).

The reason why a slight change in the commutation relations is able to introduce a drastically new short distance structure is not only that expectation values of a function of operators generally do not equal the function of the expectation values. Technically, the reason is of course that algebra homomorphisms \( \rho \) which change the commutation relations of the generators are necessarily noncanonical transformations,
i.e. unlike symmetries, the $\rho$ cannot be implemented as unitary (nor as antilinear antiunitary) transformations. Unitaries $U$ generally preserve any chosen commutation relations, say $h(x,p) = 0$, since $h(x,p) = 0 \Rightarrow h(x',p') = h(UxU^\dagger,UpU^\dagger) = Uh(x,p)U^\dagger = 0$. Thus, any change in the commutation relations introduces new features into the theory, such as the appearance of a $\Delta x_0 > 0$, which we will here focus on.

2 General framework

2.1 Partition Function

Let us consider the example of euclidean charged scalar $\phi^4$-theory, in its formulation on position space

$$Z[J] := N \int D\phi \; e^{-\frac{1}{\hbar^2} \int d^4x \left[ \phi^4 - \partial_i \partial^i + m^2 c^2 \right] \phi + \frac{1}{2} \left( \phi \phi^* - \phi^* \phi \right) J - J^* \phi}$$  \hspace{1cm} (8)$$

with $N$ a normalisation factor. Fourier transformation allows to express the action functional in momentum space, which is of course to choose the plane waves as a Hilbert basis in the space of fields which is formally being summed over. Equivalently, the action functional can be expressed in any arbitrary other Hilbert basis, such as e.g. a Hilbert basis of Hermite functions. In fact, it is not necessary to specify any choice of basis. Fields can be identified as vectors in the representation space $F$ of the associative Heisenberg algebra $A$ with the canonical commutation relations:

$$[x_i,p_j] = i\hbar \delta_{ij} \quad i,j = 1,...,4$$  \hspace{1cm} (9)$$

Since the functional analytic structure is analogous to the situation in quantum mechanics, we formally extend the Dirac notation for states to fields, i.e. $\phi(x) = \langle x | \phi \rangle$ and $\phi(p) = \langle p | \phi \rangle$. We recall that, via $\langle x | p \rangle = (2\pi\hbar)^{-2} \exp(ixp/\hbar)$, the $\hbar$ which appears in the Fourier factor $e^{ixp/\hbar}$ of the transformation from position to momentum space stems from the $\hbar$ of Eq.9. Of course, the simple quantum mechanical interpretation of fields $|\phi\rangle$ and in particular of the position and momentum operators of Eq.9 does not simply extend, due to the relativistically necessary existence of anti-particles, see [29]. However, this formulation clarifies the functional analytic structure of the action functional [11, 13]:

$$Z[J] = N \int D\phi \; e^{-\frac{\alpha}{\hbar^2} \left( \phi^4 + m^2 c^2 \right) \phi - \frac{1}{4} \left( \phi^4 + \phi^4 \right) + \langle \phi | J \rangle + \langle J | \phi \rangle}$$  \hspace{1cm} (10)$$

The pointwise multiplication $\ast$ of fields is crucial for the description of local interaction. It maps two fields onto one field, i.e. $\ast : F \otimes F \rightarrow F$ and it normally reads:

$$\ast = \int d^4x \; |x\rangle \otimes \langle x| \otimes \langle x|$$  \hspace{1cm} (11)$$
so that, in our notation:

\[(\phi_1 \ast \phi_2)(y) = \langle y|\phi_1 \ast \phi_2 \rangle = \int d^4x \langle y|x \rangle \langle x|\phi_1 \rangle \langle x|\phi_2 \rangle = \phi_1(y)\phi_2(y)\]  

(12)

On general geometries we read Eq.11 with the \(|x⟩\) denoting the vectors of maximal localisation i.e. we are integrating over the position expectation values of the maximal localisation vectors:

\[\ast = \int d^4x |x^{ml}⟩ \otimes ⟨x^{ml}| \otimes ⟨x^{ml}|\]  

(13)

In Eq.10, in order to make the units more transparent, we introduced an arbitrary unit length \(l\), so that the fields \(|\phi⟩\) become unitless. \(l\) could trivially also be reabsorbed in the definition of the fields. As is easily seen, on ordinary geometry the vectors \(|x⟩\) have units \(\text{length}^{-2}\), so that \(|\phi_1 \ast \phi_2⟩\) has units \(\text{length}^{-2}\), implying that the coupling constant \(\lambda\) (of the unregularised theory) is unitless. As is to be expected in a regularised situation, this changes on general geometries with normalisable maximal localisation vectors. Due to \(⟨x^{ml}|x^{ml}⟩ = 1\) the \(|x^{ml}⟩\) do not carry units, so that the coupling \(\lambda\) is no longer unitless.

We recall that on ordinary geometry the position eigenvectors are the maximal localisation vectors, implying that the application of the definition Eq.11 for \(\ast\) in the partition function describes the maximally local interaction. The apparent ‘nonlocality’ introduced in Eq.13 is of the size of the minimal position uncertainty in the underlying geometry. Within the framework, physical processes, including measurement processes, obey the uncertainty relations. We therefore conclude that the so-defined interactions are observationally strictly local since the apparent nonlocality could not be observed - due to the fuzzyness \(\Delta x_0\) of the underlying geometry.

In our formulation of field theories on noncommutative geometries we will stick to the abstract form of the action functional and the partition function, as e.g. given in Eq.10, i.e. we will not introduce any changes "by hand" into the form of the action functional. The switching on of corrections to the underlying geometry will automatically manifest itself in the explicit form of the resulting Feynman rules. The correction terms to the commutation relations induce modifications to the action of the operator \((p^2 + m^2)\), and to the properties of the maximally localised fields \(|x^{ml}⟩\), which will both crucially enter into the Feynman rules.

We remark that, as a new feature, some generalised commutation relations will have nontrivial unitarily nonequivalent representations, as the well-known theorem by v. Neumann no longer applies. It has been suggested that such cases could correspond to a noncommutative geometric analog of geometries with horizons or nontrivial topology [25].

2.2 Feynman rules

For explicitness, let us specify some arbitrary Hilbert basis \(|n⟩\)ₙ in the space \(F\) of fields on which the generalised commutation relations are represented. While this
basis can be continuous, discrete, or generally a mixture of both, we here use the convenient notation for \( n \) discrete. We recall that the discreteness or continuousness of the choice of basis is unrelated to the issue of regularisation. \( F \) is separable even in the case of the ordinary commutation relations, i.e. discrete Hilbert bases (such as the Fock basis) also exist on ordinary geometry. We remark that on ordinary geometry, and e.g. ‘on position space’, the situation is slightly subtle since the propagator and the vertex are then distributions. The situation will become simpler for \( \Delta x_0 > 0 \), as the distributions will turn into regular functions.

Fields, operators and \( * \) are expanded in the \( \{|n\} \) basis as

\[
\phi_n = \langle n | \phi \rangle \quad \text{and} \quad (p^2 + m^2c^2)_{nm} = \langle n | p^2 + m^2c^2 | m \rangle
\]

and

\[
* = \sum_{n_i} L_{n_1,n_2,n_3} |n_1\rangle \otimes \langle n_2 | \otimes \langle n_3 | \quad (15)
\]

Thus

\[
|\phi \star \phi'\rangle = \sum_{n,m,r} L_{nmr} \langle m | \phi \rangle \langle r | \phi' \rangle |n\rangle \quad (16)
\]

i.e.

\[
(\phi \star \phi')_n = L_{nrs} \phi_r \phi'_s \quad (17)
\]

In this Hilbert basis the partition function Eq.10 thus reads, summing over repeated indices:

\[
Z[J] = N \int_F D\phi \ e^{-\frac{\bar{h}^2}{2l^2} \phi^*_n (p^2 + m^2c^2)_{n_1n_2} \phi_{n_2} - \frac{\hbar^4}{4l^8} L_{n_1n_2n_3}^* L_{n_1n_2n_3} L_{n_4n_5n_3} \phi^*_n \phi_{n_2} \phi_{n_3} \phi_{n_4} \phi_{n_5} + \phi_n^* J_n + J_n^* \phi_n} \quad (18)
\]

Pulling the interaction term in front of the path integral, completing the squares, and carrying out the gaussian integrals yields

\[
Z[J] = N' e^{-\frac{\lambda l^4}{4 l^8} L_{n_1n_2n_3}^* L_{n_1n_2n_3} L_{n_4n_5n_3} \phi^*_n \phi_{n_2} \phi_{n_3} \phi_{n_4} \phi_{n_5}} \ e^{-\frac{\hbar^2}{12 l^8} J_n^* (p^2 + m^2c^2)^{-1} J_m} \quad (19)
\]

We can therefore read off the Feynman rules for the propagator and the vertex

\[
G_{nm} = \left( \frac{\hbar^2/l^2}{p^2 + m^2c^2} \right)_{nm}, \quad \Gamma_{rstu} = -\frac{\lambda l^4}{4!} L_{nrs}^* L_{ntu} \quad (20)
\]

Note that the earlier arbitrarily introduced constant \( l \) drops out of the Feynman rules since each vertex attaches to four propagators.

Explicitly, Eq.13 yields the structure constants:

\[
L_{n_1,n_2,n_3} = \int d^4x \ \langle n_1 | x^m | n_2 \rangle \langle x^m | n_2 \rangle \langle x^m | n_3 \rangle \quad (21)
\]
On ordinary geometry, we recover with \(|x^{\mu l}\rangle = |x\rangle\), and e.g. choosing the position representation \(|n\rangle = |x\rangle:\)

\[
L^{(\Delta x_0=0)}_{x,x',x''} = \delta^4(x-x')\delta^4(x-x'')
\]  

(22)

In the general case with \(\Delta x_0 > 0\), as we said, the coupling constant picks up units. We can however still define a unitless \(\lambda\) by splitting off suitable factors of \(l\). Let us also choose \(l = \Delta x_0\). Any other choice for \(l\) would amount to a redefinition of the coupling constant \(\lambda\).

As abstract operators, i.e. without specifying a Hilbert basis in the space of fields, the free propagator and the lowest order vertex then read, using the definition Eq.6:

\[
G = \frac{\hbar^2}{(\Delta x_0)^2(p^2 + m^2c^2)}
\]  

(23)

\[
\Gamma = -\frac{\lambda}{4!} \int d^4x\, d^4y\, \delta^4(x^{\mu l}, y^{\mu l}) \left| y^{\mu l} \right\otimes \left| y^{\mu l} \right\otimes \left\langle x^{\mu l} \right| \otimes \left\langle x^{\mu l} \right|
\]  

(24)

We can now use the Feynman rules Eqs.23,24 to explicitly check for UV regularisation on noncommutative geometries \(\mathcal{A}\) with \(\Delta x_0 > 0\).

### 2.3 Regularisation

Let us first consider the tadpole graph (see Fig.1).

![Fig1: The tadpole graph. The notation is meant to indicate \(\Delta x_0 > 0\), i.e. the fuzzyness of space-time, or the particles’ nonpointlikeness.](image)

Using Eqs.19,20,21, or directly Eqs.23,24, yields its expression as an operator:

\[
\frac{2\lambda\hbar^2}{(4!)^2(\Delta x_0)^2} \int d^4x\, d^4y\, \delta^4(x, y) \left\langle x^{\mu l} \right| \frac{1}{p^2 + m^2c^2} \left| y^{\mu l} \right\otimes \left| y^{\mu l} \right\otimes \left\langle x^{\mu l} \right|
\]  

(25)

As is well known, ordinarily this graph is quadratically divergent for large momenta. On position space the divergence, or rather the ill-definedness of this graph, arises not through the large scale integrals, but instead at short distances, i.e. as \(x \to y\).

On our noncommutative geometries this graph is however well defined: Due to the
normalisability of the maximal localisation vectors, their scalar product $\tilde{\delta}^4$ is a function bounded by 1, rather than a distribution. In the second factor, which consists of matrix elements of the propagator, the operator $(p^2 + m^2 c^2)^{-1}$ is bounded. Therefore, again due to the normalisability of the $|x^{ml}\rangle$ also these matrix elements are bounded functions of $x$ and $y$. Thus the short-distance divergence is indeed removed on the noncommutative geometry.

In the case $m = 0$ the operator $1/p^2$ is unbounded, which, as is well known, can lead to infrared divergencies at large distances. A relevant question in this context is of course whether on a geometry with a finite minimal uncertainty in momentum this infrared problem could be avoided. Indeed, as has been shown in [13], the existence of a finite $\Delta p_0 > 0$ implies that the operator $1/p^2$ is as well behaved as if it contained a mass term, i.e. it is a bounded self-adjoint operator. Since we are here primarily interested in the ultraviolet behaviour, let us in the following assume the infrared to be regularised either through $m > 0$, or e.g. through $\Delta p_0 > 0$ (examples of noncommutative geometries with both, finite minimal uncertainties in position $\Delta x_0$ and in momentum $\Delta p_0$ are known, see [10]).

The tadpole graph could of course have been avoided by normal ordering the interaction lagrangian. Let us therefore consider the further example of the normally logarithmically divergent ‘fish’ graph (see Fig2).

\[ \text{Fig2: The fish graph. The thick lines for the propagator and the vertex are meant to indicate the presence of a finite minimal position uncertainty } \Delta x_0. \]

It requires two vertices and two propagators:

\[
\frac{2\lambda^2 \hbar^4}{(4!)^2(\Delta x_0)^4} \int \frac{d^4x_1 \, d^4x_2 \, d^4x_3 \, d^4x_4}{(\Delta x_0)^{16}} \langle x_2^{ml} | \frac{1}{p^2 + m^2 c^2} | x_3^{ml} \rangle^2 \times \\
\times \tilde{\delta}^4(x_1, x_2) \tilde{\delta}^4(x_3, x_4) | x_1^{ml} \rangle \otimes | x_1^{ml} \rangle \otimes \langle x_4^{ml} \rangle \otimes \langle x_4^{ml} \rangle \]  

Ordinarily, in position space, the propagator $\langle x_2 | (p^2 + m^2 c^2)^{-1} | x_3 \rangle$ is divergent for $x_2 \to x_3$. Nevertheless, it is well defined as a distribution. However, its square

\\

9
\[ \langle x_2 | (p^2 + m^2c^2)^{-1} | x_3 \rangle^2 \text{ is not } 1. \]

In contrast, since on the noncommutative geometry the matrix elements of the propagator \( \langle x_2^{ml} | (p^2 + m^2c^2)^{-1} | x_3^{ml} \rangle \) are bounded, also for \( x_2 \rightarrow x_3 \), arbitrary high powers \( \langle x_2^{ml} | (p^2 + m^2c^2)^{-1} | x_3^{ml} \rangle^r, r \in \mathbb{N} \), are also well defined functions of \( x_2 \) and \( x_3 \). Again, the short distance structure is found to be regularised.

In fact, it is obvious that the short distance structure of all graphs is regularised, since in arbitrary graphs at most finite powers of matrix elements of the propagator, and powers of \( \tilde{\delta} \) can appear, which both are now bounded regular functions.

We should note, however, that although we have seen that the ultraviolet divergencies are absent, we cannot generally exclude that a noncommutative geometry could introduce new types of divergencies. This will have to be investigated case by case.

### 2.4 External Symmetry

The one-dimensional uncertainty relation Eq.\[\text{[1]}\] has no unique \( n \)-dimensional generalisation. Therefore, any particular choice for the corrections to the commutation relations in \( n \) dimensions will require motivation from string theory or quantum gravity. There is also the possibility of generalised external and internal symmetry groups (e.g. quantum groups) at the Planck scale, see e.g. \[10, 30, 31, 32\]. We will here not attempt to develop such arguments any further. Let us here instead consider the constraints which can be posed by requiring conventional translation and rotation invariance of the commutation relations.

We start with a general ansatz for \( x, p \) commutation relations in \( n \) dimensions

\[ [x_i, p_j] = i\hbar \Theta_{ij}(p) \tag{27} \]

where we require that only the ultraviolet is affected, i.e. \( \Theta_{ij}(p) \) shall be allowed to significantly differ from \( \delta_{ij} \) only for large momenta.

As we said, we assume \( [p_i, p_j] = 0 \). (We remark that it has been argued that if the final theory of quantum theory on curved space does contain momentum operators, these should be generators of a generalised definition of translation on curved space, in which case \( [p_i, p_j] = 0 \) would express the absence of curvature on position space \[23\].)

The remaining commutation relations among the \( x_i \) are then determined through the Jacobi identities, yielding \[24\]:

\[ [x_i, x_j] = i\hbar \left\{ x_a, \Theta^{-1}_{ar} \Theta_{s|j} \Theta_{j|s} \right\} \tag{28} \]

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\(^1\)We remark that the ansatz of Differential Renormalisation, see e.g. \[19\], starts here by replacing the ill defined square of the propagator (nonuniquely) by the derivative of a well defined distribution, thereby introducing a length scale.
For simplicity we adopted the geometric notation, with \{,\} and \[,] standing for (anti-)commutators and with \( Q_{s} = \partial/\partial p_{s} Q \).

We observe that the \( x, p \) commutation relations Eqs.\( \ref{27} \) are translation invariant in the sense that they are preserved under the transformations

\[
x_{i} \rightarrow x_{i} + d_{i}, \quad p_{i} \rightarrow p_{i}, \quad d_{i} \in \mathbb{R}, \quad i = 1, ... n
\]  

On the other hand, for generic \( \Theta \), the commutation relations Eqs.\( \ref{28} \) are not invariant under translations, i.e. the generators obtained through the transformations Eqs.\( \ref{29} \) do not obey Eq.\( \ref{28} \). We can, however, enforce translation invariance by requiring \( \Theta \) to yield \([x_{i}, x_{j}] = 0\). We read off from Eq.\( \ref{28} \) that a sufficient and necessary condition for this to hold is, (summing over \( i \))

\[
\Theta_{ia} \partial_{p_{i}} \Theta_{bc} = \Theta_{ib} \partial_{p_{i}} \Theta_{ac}
\]  

which may be viewed as expressing the absence of curvature on momentum space, by the same arguments as above. Of course, central correction terms may still be added on the RHS of the \( x, x \) commutation relations, without spoiling translation invariance, e.g. terms of the form suggested in \( \\ref{20} \).

The requirement of rotation invariance further imposes

\[
\Theta_{ij}(p) = f(p^2) \delta_{ij} + g(p^2) \, p_{i} p_{j}
\]  

so that Eq.\( \ref{30} \) takes the form \( \\ref{33} \):

\[
g = \frac{2 ff'}{f - 2p^2 f'}
\]  

where the prime denotes \( d/dp^2 \). Under these conditions translations and rotations do respect the commutation relations, i.e. they are quantum canonical transformations, and can indeed be implemented as unitary transformations. The translations are given by

\[
U(d) := e^{d \cdot T}
\]  

with \([T_{i}, x_{j}] = \delta_{ij}\), and where we denoted the scalar product \( \sum_{i=1}^{n} d_{i} T_{i} \) by \( d \cdot T \). Since in the ‘naive’ definition of translations in Eqs.\( \ref{29} \) there is no explicitly built-in ‘knowledge’ of the new short distance structure, the anti-hermitean generators \( T_{i} \) are not given by the \( -p_{i}/i\hbar \) directly. Instead, they are

\[
T_{i} = \frac{p_{i}}{-i\hbar f(p^2)}
\]  

as is not difficult to verify. As a consequence of the new short distance structure the translators \( T_{i} \) will be found to be bounded operators, technically as we will see (Eq.\( \ref{42} \)), because \( f \) eventually goes linearly with \( p \) for large \( p \).
Analogously, rotations

\[ U(\Theta) = e^{\Theta_{ij} M_{ij}} \]  

are generated by the operators

\[ M_{ij} = \frac{1}{-i\hbar f(p^2)} (p_i x_j - p_j x_i) \]  

which obey

\[ [p_i, M_{jk}] = \delta_{ik} p_j - \delta_{ij} p_k \]  
\[ [x_i, M_{jk}] = \delta_{ik} x_j - \delta_{ij} x_k \]  
\[ [M_{ij}, M_{kl}] = \delta_{ik} M_{jl} - \delta_{il} M_{jk} + \delta_{jl} M_{ik} - \delta_{jk} M_{il} \]  
as usually.

3 Explicit example

In the following we will illustrate the formalism with an explicit example of a non-commutative geometry.

3.1 Choice of commutation relations

If we require our example geometry to obey translation and rotation invariance, there still appears to be considerable freedom in choosing the functions \( f \) and, through Eq.32, the function \( g \). Many choices may not lead to geometries with a minimal uncertainty \( \Delta x_0 > 0 \). In particular, Eq.32 indicates that \( g \) can develop singularities. A detailed investigation into the various possibilities is in progress [34]. Here, in order to obtain a well behaved example geometry we simply force there not to appear a singularity by imposing, as the simplest choice (\( \beta > 0 \)):

\[ g = \beta \]  

(40)

Thus, Eq.32 then reads

\[ f' = \frac{\beta f}{2(f + \beta p^2)} \]  

(41)

which is solved by

\[ f = \frac{\beta p^2}{\sqrt{1 + 2\beta p^2} - 1} \]  

(42)

The Taylor expansion around the origin is well behaved:

\[ f = 1 + \frac{\beta}{2} p^2 + O((\beta p^2)^2) \]  

(43)
so that, if we choose $\beta$ e.g. at around the Planck scale $\beta^{-1/2} \approx p_{\text{pl}}$ then $f$ significantly deviates from the identity only for large momenta of that scale.

We therefore obtain the commutation relations:

$$[x_i, p_j] = i\hbar \left( \frac{\beta p^2}{(1 + 2\beta p^2)^{1/2}} \delta_{ij} + \beta p_i p_j \right)$$  \hspace{1cm} (44)

$$[x_i, x_j] = 0$$  \hspace{1cm} (45)

$$[p_i, p_j] = 0$$  \hspace{1cm} (46)

We remark that, assuming translation and rotation invariance, the correction terms to the commutation relations are in fact unique to first order in $\beta$: Eq.32 yields $f = 1 + \beta / 2 p^2 + O(\beta^2)$ and $g = \beta + O(\beta^2)$, so that

$$[x_i, p_j] = i\hbar \left( (1 + \beta / 2 p^2) \delta_{ij} + \beta p_i p_j + O(\beta^2) \right)$$  \hspace{1cm} (47)

and $[x_i, x_j] = 0 + O(\beta^2)$, $[p_i, p_j] = 0$, which of course coincides with what we obtain from Eqs.44 to first order in $\beta$.

We remark that concerning the possible choices of commutation relations it should generally be interesting to investigate the interplay of the technical constraints with the input and physical intuition from string theory and quantum gravity. In particular, as follows from the relation between the translators and the momenta, Eq.34, the rule for the addition of extremely large momenta is modified through $(p+k)_i = p_if^{-1}(p^2) + k_if^{-1}(k^2)$. There should exist an interpretation in terms of the effects of gravity at the Planck scale, similar to the well known effect of momentum nonconservation through gravity on large scales ($T^{\mu\nu;\nu} = 0$ rather than $T^{\mu\nu,\nu} = 0$). This may e.g. be related to the old idea of possible curvature in momentum space, in which a generalised parallelogram rule for the addition of momenta has been discussed, see [21], and more recently [22]. It has of course long been suggested that, more drastically, both rotation and translation invariance may be generalised or broken at the Planck scale. Any physical intuition for this could and should then also provide guidance for the generalisation of Eq.13 to account for the then position (and possibly orientation) dependence of the short distance structure of the geometry. This will at first require a case by case study.

3.2 Hilbert space representations

The commutation relations Eqs.44,45,46 still find a Hilbert space representation in the spectral representation of the momenta $p_i$ (since momentum space is still commutative and there is no finite minimal uncertainty in momentum, $\Delta p_0 = 0$)

$$x_i \psi(p) = i\hbar \left( (f' + p^2g' + \frac{n+1}{2}g)p_i + f \partial_{p_i} + g p_ip_j \partial_{p_j} \right) \psi(p)$$  \hspace{1cm} (48)

$$p_i \psi(p) = p_i \psi(p)$$  \hspace{1cm} (49)

$$\langle \psi_1 | \psi_2 \rangle = \int d^n p \psi^*_1(p) \psi_2(p)$$  \hspace{1cm} (50)
where $\psi(p) = \langle p|\psi \rangle$ and $\langle p|p' \rangle = \delta(p - p')$. $x_i$ and $p_i$ are symmetric operators on the dense domain $D := S_\infty$. This representation holds for any choice of $f$ and $g$, as can be checked directly. The case of commutation relations with general $\Theta$ is covered in [25].

A further representation of the commutation relations Eqs.[44,45,46], which will prove convenient for practical calculations, is obtained by using that the translators $T_i$ are anti-hermitean and have a spectral representation on the Hilbert basis $\{|\rho\rangle \, |\rho \in I_n\}$ of vectors obeying $T_i.|\rho\rangle = \rho_i/\hbar|\rho\rangle$ with

$$I_n = \{\rho \in \mathbb{R}^n \, | \, \rho^2 < 2/\beta\}$$

(51)

i.e. the $T_i$ are bounded operators. The unitary transformation which maps from momentum space to the spectral representation of the $T_i$ has the matrix elements:

$$\langle \rho|p \rangle = (1 - \beta \rho^2/2)^{-n/2}(1 + \beta \rho^2/2)^{1/2} \frac{\rho_i}{1 - \beta \rho^2/2}$$

(52)

The operator representations and the scalar product then read in $\rho$-space:

$$x_i.\psi(\rho) = \frac{\hbar}{2} \partial_\rho \psi(\rho)$$

(53)

$$p_i.\psi(\rho) = \frac{\rho_i}{1 - \beta \rho^2/2} \psi(\rho)$$

(54)

$$\langle \psi_1|\psi_2 \rangle = \int_{I_n} d^n \rho \, \psi^*_1(\rho) \psi_2(\rho)$$

(55)

where $\psi(\rho) = \langle \rho|\psi \rangle$ and $\langle \rho|\rho' \rangle = \delta^m(\rho_i - \rho'_i)$. Note that, as is easy to see in this representation, the momentum operators $p_i$ are still unbounded.

We still have to prove that the geometry defined through the commutation relations Eqs.[44,45,46] does in fact imply a finite minimal uncertainty $\Delta x_0 > 0$, rather than e.g. a discretisation of position space. Before we do this in the next section, let us note an important representation theoretic consequence of the existence of a minimal uncertainty $\Delta x_0 > 0$:

A general argument shows that commutation relations which imply a finite minimal uncertainty in position cannot find a Hilbert space representation on a spectral representation of the the position operators: The uncertainty relations hold in all $*$-representations of the commutation relations. On the other hand, as is easily seen, e.g. in the example of Eq.[4], an eigenvector to an observable necessarily has vanishing uncertainty in this observable. Thus, if the uncertainty relations imply a finite uncertainty in positions, they exclude the existence of any position eigenvectors in any physical domain, i.e. on any domain on which the commutation relations are represented. In particular, in cases where $\Delta x_0 > 0$ and $\Delta p_0 > 0$ both position and momentum representations are ruled out and one has to resort to other Hilbert bases, as e.g. in [14, 13].
To be precise, let us assume that the commutation relations are represented on some dense domain $D \subset H$ in a Hilbert space $H$. Ordinarily, there would exist sequences $\{|\psi_n\rangle \in D\}$ with position uncertainties decreasing to zero (e.g. Gaussian approximations to the position eigenvectors). In the presence of a finite $\Delta x_0 > 0$, however, there exists a minimal uncertainty ‘gap’, i.e. there are no vectors $|\psi\rangle \in D$ which would have an uncertainty in positions in the interval $[0, \Delta x_0]$, so that now

$$\exists \neg \{|\psi_n\rangle \in D\} : \lim_{n \to \infty} (\Delta x_0)_{|\psi_n\rangle} = 0$$

(56)

Technically, the position operators are merely symmetric on representations $D$ of the commutation relations. Their deficiency indices are nonvanishing and equal, implying the existence of a family of self-adjoint extensions in $H$, though, crucially of course, not in $D$. This functional analytic structure was first found in [10].

As is easily seen, there do exist formal position eigenvectors in $H$:

$$\psi_\xi(\rho) = \left[ \left(\frac{\beta}{2\pi}\right)^{\frac{n}{2}} \frac{\Gamma(n/2)}{2} \right]^{\frac{1}{2}} e^{-i\xi \cdot \rho / \hbar}$$

(57)

Concerning the normalisation, recall that the surface of the $(n-1)$-dimensional unit sphere reads $S_n = \int d\Omega_n = 2\pi^{n/2} / \Gamma(n/2)$. The scalar product can be calculated to be

$$\langle \psi_\xi | \psi_\eta \rangle = \left(\frac{\beta}{2\pi}\right)^{\frac{n}{2}} \frac{\Gamma(n/2)}{2} \int_{I_n} d\rho \ e^{-i(\eta-\xi) \cdot \rho / \hbar}$$

$$= \left(\frac{\sqrt{2}h^{1/2} \beta}{|\xi - \eta|}\right)^{\frac{n}{2}} \Gamma \left(\frac{n}{2} + 1\right) J_{n/2} \left(\frac{\sqrt{2}|\xi - \eta|}{h^{1/2} \beta}\right)$$

(58)

where $J_{n/2}$ is the Bessel function of the first kind of order $n/2$. The zeros of the scalar product determine the self-adjoint extensions of the on $D$ densely defined $x_i$ (for any chosen $\xi$, all $\eta$’s such that $|\xi - \eta|$ is a zero of $J_{n/2}$ correspond to the eigenvectors of one self-adjoint extension). However, as is readily verified, none of these vectors is in the domain of the $p_i$. Thus, as is to be expected when $\Delta x_0 > 0$, none of the family of self-adjoint extensions of the $x_i$ is in the domain of the representation of the commutation relations. In the one-dimensional case $n = 1$ we recover the results obtained in [12], in particular the scalar product of the ‘formal position eigenvectors’ (technically of eigenvectors of the adjoints $x_i^*$, which are not self-adjoint, nor symmetric):

$$\langle \psi_\xi | \psi_\eta \rangle = \frac{\hbar^{1/2} \beta \sin \left(\frac{\sqrt{2}|\xi - \eta|}{h^{1/2} \beta}\right)}{\sqrt{2}|\xi - \eta|}$$

(59)

There is, however, a natural generalisation of the position space representation. To this end we define a Hilbert space representation of the commutation relations on ‘quasi-position space’, see [12, 14]:

$$\psi(x) := \langle x^m | \psi \rangle$$

(60)
These quasi-position functions $\psi(x)$ are obtained by projecting the fields $|\psi\rangle$ onto the fields of maximal localisation $|x^{ml}\rangle$ and they do of course turn into the ordinary position space representation for $\Delta x_0 \to 0$.

### 3.3 Maximally localised fields

Let us now prove that $\Delta x_0 > 0$ by explicitly calculating the maximally localised fields.

As is well known the $\Delta x_i \Delta p_i$ uncertainty relations are derived from the positivity of the norm:

$$||((x_i - \langle x_i \rangle) + i k (p_i - \langle p_i \rangle))|\psi\rangle|| \geq 0$$  \hspace{1cm} (61)

Thus, the vectors on the boundary of the region allowed by the uncertainty relations obey the squeezed state equation:

$$(x_i - \langle x_i \rangle) + i k (p_i - \langle p_i \rangle)|\psi\rangle = 0$$  \hspace{1cm} (62)

Due to the symmetry of the underlying geometry we do not loose generality by calculating the maximally localised field $|x^{ml}\rangle$, around the origin, i.e. with $\langle 0^{ml}|x_i|0^{ml}\rangle = 0$ and $\langle 0^{ml}|p_i|0^{ml}\rangle = 0$. In $\rho$-space Eq.62 reads

$$\left( i \hbar \partial_{\rho^i} + i k \frac{\rho^i}{1 - \beta \rho^2/2} \right) \psi_k(\rho) = 0$$  \hspace{1cm} (63)

Due to rotational symmetry, $|0^{ml}\rangle$ can only depend on $p^2$, so that Eq.63 becomes

$$\partial_{\rho^2} \psi_k(\rho^2) = - \frac{k}{2 \hbar (1 - \beta \rho^2/2)} \psi_k(\rho^2)$$  \hspace{1cm} (64)

whose normalised solutions read ($k > 0$):

$$\psi_k(\rho^2) = \left[ \left( \frac{\beta}{2\pi} \right)^{\frac{k}{2}} \frac{\Gamma \left( \frac{2k}{\hbar \beta} + \frac{n}{2} + 1 \right)}{\Gamma \left( \frac{2k}{\hbar \beta} + 1 \right)} \right]^{\frac{1}{2}} \left( 1 - \beta \rho^2/2 \right)^{\frac{k}{n}}$$  \hspace{1cm} (65)

We can now calculate the squared uncertainty in position as a function of $k$:

$$(\Delta x)^2_{\psi_k} = \frac{hk}{4} \frac{4k + n\hbar \beta}{2k + \hbar \beta}$$  \hspace{1cm} (66)

The minimum is reached for

$$k_0 = \frac{\hbar \beta}{2} \left( 1 + \sqrt{1 + \frac{n}{2}} \right)$$  \hspace{1cm} (67)
We therefore find the finite minimal uncertainty $\Delta x_0$:

$$
(\Delta x_0)^2 = (\Delta x)^2_{\psi_k_0} = \frac{\hbar^2 \beta}{4} \sqrt{1 + \frac{n}{2}} \left( \sqrt{1 + \frac{n}{2}} + 1 \right)^2
$$

(68)

The field $|0^{ml}\rangle = |\psi_{k_0}\rangle$ of maximal localisation around the origin therefore reads, in the $\rho$-representation:

$$
\langle \rho | 0^{ml} \rangle = \psi_{k_0} (\rho^2) = N^{1/2}(n) \left( \beta / 2 \pi \right)^{n/4} \left( 1 - \beta \rho^2 / 2 \right)^{1/2 + \sqrt{1/4 + n/8}}
$$

(69)

where we defined:

$$
N(n) := \frac{\Gamma \left( 2 + n/2 + \sqrt{1 + n/2} \right)}{\Gamma \left( 2 + \sqrt{1 + n/2} \right)}
$$

(70)

The $\rho$-space representation of the fields of maximal localisation around arbitrary position expectation values $\xi$ now follow by translation:

$$
\langle \rho | x^{ml} \rangle = \langle \rho | e^{x T} | 0^{ml} \rangle = \langle \rho | 0^{ml} \rangle e^{-ix \cdot \rho/\hbar}
$$

(71)

Using Eq. 52, we eventually obtain the maximally localised fields in momentum space,

$$
\langle p | x^{ml} \rangle = (\beta / 2 \pi)^{n/4} \frac{\beta p^2}{1 + 2 \beta p^2 - \sqrt{1 + 2 \beta p^2}}
$$

(72)

$$
\times \left( \frac{\sqrt{1 + 2 \beta p^2} - 1}{\beta p^2} \right)^{1 + n/2 + \sqrt{1 + 2 \beta p^2}} \exp \left( -i \frac{x \cdot p}{\hbar} \frac{\sqrt{1 + 2 \beta p^2} - 1}{\beta p^2} \right)
$$

This expression is of course also the quasi-position representation of the plane wave with momentum $p$. We observe that in quasi-position space the fields can now not have arbitrarily fine ripples. Indeed, from the argument in the exponent in Eq. 72 we read off that for increasing momentum the wavelength in quasi-position space only tends towards a finite minimal wavelength

$$
\lambda_0 = \pi \hbar \sqrt{2 \beta}
$$

(73)

which is reached as the momentum $p_i$ tends to infinity. The situation is perhaps comparable to the speed of light as a fundamental limit, in which case also the kinematics lets the energy diverge as the fundamental limit is approached. In the Appendix we give the unitary transformation from the $\rho$-representation to the quasi-position representation and back, and we prove the completeness of the set of maximally localised fields.
3.4 Feynman rules

As we saw in Sec. 2.2, the Feynman rules are composed of two basic functions, related to the vertex and to the propagator respectively:

\[ \tilde{\delta}(x^{ml}, y^{ml}) := \langle x^{ml} | y^{ml} \rangle \quad \text{and} \quad G(x, y) := \frac{\hbar^2}{(\Delta x_0)^2} (p^2 + m^2 c^2)^{-1} |y^{ml}\rangle \]  

(74)

The calculation of \( \tilde{\delta} \), i.e. of the scalar product of maximally localised fields is straightforward, in particular in the \( \rho \)-representation. The result is of course independent of the choice of Hilbert basis. Choosing spherical coordinates:

\[ \tilde{\delta}(x - y) = \langle x^{ml} | y^{ml} \rangle \]  

(75)

\[ \text{for } \beta > 0 \]

\[ = \left( \frac{\beta}{2\pi} \right)^{n/4} N^{1/2}(n) \int d^n \rho \left( 1 - \beta \rho^2 / 2 \right)^{1+\sqrt{1+n/2}} e^{i(x-y) \cdot \rho / \hbar} \]

\[ = \left( \frac{\hbar \sqrt{2\beta}}{|x - y|} \right)^{1+\frac{n}{2}+\sqrt{1+n/2}} \Gamma \left( 2 + \frac{n}{2} + \sqrt{1+n/2} \right) J_{1+\frac{n}{2}+\sqrt{1+n/2}} \left( \frac{\sqrt{2}|x - y|}{\hbar \sqrt{\beta}} \right) \]

\[ \text{Fig.3: Plot of the scalar product of maximally localised fields } \tilde{\delta}(x - y) \text{ versus } |x - y|/\hbar \sqrt{\beta}. \]

\( \tilde{\delta} \) generalises the Dirac \( \delta \)-distribution for \( \Delta x_0 > 0 \).

We note that \( \tilde{\delta}(x - y) \) is also the quasi-position wave function of the around \( y \) maximally localised field \( |y^{ml}\rangle \). Because of the finite norm of the maximally localised fields, \( \tilde{\delta}(x - y) \) had of course to come out as a regular function. Its graph is plotted in Fig.3.

Recall that on ordinary geometry, i.e. when \( \Delta x_0 = 0 \), the propagator in position
space \( G(x - y) \) can only be defined as a distribution, and that it is the ill-definedness of its square (as well as of higher powers) which gives rise to ultraviolet divergencies, see e.g. [19]. We know from Sec.2.3 that \( G(x - y) \) must now be a well-defined function without singularities. Explicitly, let us consider the free propagator matrix elements:

\[
G(x - y) = \frac{\hbar^2}{(\Delta x_0)^2} \langle x^m | (p^2 + m^2c^2)^{-1} | y^m \rangle
\]

\[
= \frac{N(n)\hbar \beta}{(\Delta x_0)^2} \left( \frac{1}{2\pi} \right)^{n/2} \int_{I_n} d^n \rho \frac{(1 - \beta \rho^2/2)^{3+\sqrt{1+n/2}}}{\rho^2 + m^2c^2(1 - \beta \rho^2/2)^2} e^{i(x-y) \cdot \rho/\hbar} \quad (76)
\]

The massive propagator cannot be simply expressed in terms of elementary or special functions. However, for arbitrary nonvanishing mass, \( G(x - y) \) can be uniformly bounded by

\[
|G(x - y)| \leq \frac{\hbar^2}{m^2c^2(\Delta x_0)^2} \quad (77)
\]

which shows that the propagator is well behaved for all distances, in particular also for \( |x - y| \to 0 \).

Since the small distance behaviour is independent of the mass, let us also consider the simpler massless propagator. Using spherical coordinates and introducing the dimensionless variables

\[
t = \rho \sqrt{\beta/2}, \quad s = \cos \theta
\]

we obtain

\[
G(x - y) = \frac{N(n)\hbar \beta}{2(\Delta x_0)^2} \left( \frac{1}{\pi} \right)^{n/2} S_{n-1} \int_0^1 \int_0^\pi d\theta (\sin \theta)^{n/2} t^n (1 - t^2)^{3+\sqrt{1+n/2}} e^{i\theta \cos \theta}
\]

\[
= \frac{N(n)\hbar \beta}{2(\Delta x_0)^2} \left( \frac{1}{\pi} \right)^{n/2} S_{n-1} \int_0^1 \int_{-1}^1 ds (1 - s^2)^{(n-3)/2} t^{n-3}
\]

\[\times (1 - t^2)^{3+\sqrt{1+n/2}} e^{i\theta s} \quad (79)\]

where \( d := \sqrt{2|x-y|/(\hbar\beta)} \). Performing the integration over \( s \) and then over \( t \), and after some simplification, one finally obtains for \( n > 2 \):

\[
G(x - y) = 2^n \frac{(3 + \sqrt{1+n/2})(2 + \sqrt{1+n/2})}{(n-2)\sqrt{1+n/2}(1 + \sqrt{1+n/2})^2 (2 + n/2 + \sqrt{1+n/2})} \times \Gamma_n \left( -1 + n/2; \frac{3 + n/2 + \sqrt{1+n/2}}{n/2}; -|x-y|^2/(2\hbar^2\beta) \right)
\]

\[
= \frac{\hbar^2}{m^2c^2(\Delta x_0)^2} \frac{(3 + \sqrt{1+n/2})(2 + \sqrt{1+n/2})}{(n-2)\sqrt{1+n/2}(1 + \sqrt{1+n/2})^2 (2 + n/2 + \sqrt{1+n/2})} \times \Gamma_n \left( -1 + n/2; \frac{3 + n/2 + \sqrt{1+n/2}}{n/2}; -|x-y|^2/(2\hbar^2\beta) \right)
\]

\[
\Gamma_n \left( -1 + n/2; \frac{3 + n/2 + \sqrt{1+n/2}}{n/2}; -|x-y|^2/(2\hbar^2\beta) \right)
\]

\[
\text{where we have used the explicit expressions for } (\Delta x_0)^2 \text{ and } N(n) \text{ given in Eqs. 68, 70.}
\]
In the particular case of four euclidean dimensions the last expression can be cast in a much simpler form. For $n = 4$, Eqs. [68, 70] for $(\Delta x_0)^2$, $N(n)$ and the definition of $\tilde{\delta}(x - y)$ yield

$$G(x - y) = \frac{36 + 20\sqrt{3}}{6 + 4\sqrt{3}} \frac{h^2 \beta}{|x - y|^2} \left(1 - \tilde{\delta}(x - y)\right)$$

(81)

Therefore, the propagator can be expressed as the product of the usual zero mass propagator and a smooth cut-off function, which has the following behaviour for large and short distances

$$h^2 \beta \left(1 - \tilde{\delta}(x - y)\right) \sim \frac{1}{8 + 2\sqrt{3}} |x - y|^2 \left(1 + \mathcal{O}\left(\left(\frac{\sqrt{2}|x - y|}{h\sqrt{\beta}}\right)^2\right)\right),$$

for $|x - y| << h\sqrt{\beta}$

(82)

$$h^2 \beta \left(1 - \tilde{\delta}(x - y)\right) \sim h^2 \beta \left(1 + \mathcal{O}\left(\left(\frac{\sqrt{2}|x - y|}{h\sqrt{\beta}}\right)^{-7/2 - \sqrt{3}}\right)\right),$$

for $|x - y| >> h\sqrt{\beta}$

In particular $G(x - y)$ is a well behaved function in the short distance regime and tends to a finite limit for $|x - y| = 0$, while for distances larger than $h\sqrt{\beta}$ it rapidly approaches the well known $|x - y|^{-2}$ behaviour of the free massless propagator on ordinary geometry.

## 4 Summary and Outlook

Studies in string theory and quantum gravity provide theoretical evidence for various types of correction terms to the canonical commutation relations. Measurable effects could in principle appear anywhere between the presently resolvable scale of $10^{-18} m$ and the Planck scale of $10^{-35} m$, although expected close to the Planck scale.

Our aim here was to show that the existence of even an at present unmeasurably small $\Delta x_0$, for example at about the Planck length, could have a drastic effect in field theory, namely by rendering the theory ultraviolet finite. We note that this new short distance behaviour would truly be a quantum structure, in the sense that it has no classical analog. Further, the presence of a $\Delta x_0 > 0$ is compatible with both, generalised symmetries as well as with conventional rotation and translation symmetry.

The existence of this short distance structure would raise a number of conceptual issues, in particular, it would lead to a generalised notion of local interaction. Strictly speaking, the maximally local interaction term on generalised geometries is neither
local nor nonlocal in the conventional sense. Instead, it is ‘observationally’ local in the sense that it is local as far as distances can be resolved on the given geometry. Similarly, questions such as whether the unitarity of time evolution is broken or conserved, or whether local gauge invariance is broken or conserved also seem not to be applicable in the usual sense. Instead, on generalised geometries the notions of ‘time evolution’ or ‘local’ gauging may need to be redefined, analogously to how local interaction is generalised into ‘observationally’ local interaction. This is under investigation.

There exist a number of immediate technical issues which need to be addressed, for example the significance of Eq.32 and its pole structure, and Wick rotation. It should of course also be worth exploring the possible usefulness of the approach as a mere regularisation method.

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Appendix

We prove the completeness of the set of maximally localised fields and we give the unitary transformation which connects the $\rho$-representation with the quasi-position representation for the example geometry considered in Section 3.

In order to see that the set of maximally localised fields $\{|x_{ml}\rangle\}$ is complete, we use that a set of vectors $|\psi_{\lambda}\rangle$ in a Hilbert space is complete iff from $\langle \phi | \psi_{\lambda} \rangle = 0$, $\forall |\psi_{\lambda}\rangle$ follows that $\langle \phi | = 0$. Consider now $\langle \phi | x_{ml} \rangle$:

$$\langle \phi | x_{ml} \rangle = N^{1/2}(n)(\beta/2\pi)^{n/4} \int_{I_n} d^n \rho \ (1 - \beta \rho^2/2)^{1/2+\sqrt{1/4+n/8}} e^{ix \cdot \rho/\hbar} \phi(\rho)$$  \hfill (83)

Using the mean value theorem

$$\langle \phi | x_{ml} \rangle = N^{1/2}(n)(\beta/2\pi)^{n/4}(1 - \beta \rho^2/2)^{1/2+\sqrt{1/4+n/8}} \int_{I_n} d^n \rho \ e^{ix \cdot \rho/\hbar} \phi(\rho)$$

$$= N^{1/2}(n) \frac{2}{n\Gamma(n/2)} (1 - \beta \rho^2/2)^{1/2+\sqrt{1/4+n/8}} \langle \phi | \psi_x \rangle$$  \hfill (84)

where the $|\psi_x\rangle$ are eigenvectors to the (nonhermitean) operators $x_i^*$ (see Eq.57) and $\rho$ is some value in the open interval $]0, \sqrt{2}/\beta]$. The set of vectors $|\psi_x\rangle$ is the collection of all eigenbases to the self-adjoint extensions of the $x_i$, and is therefore overcomplete (for the details of the functional analysis see [10, 12, 14, 17]). Further, the factor in front of the scalar product in Eq.84 never vanishes nor diverges for any $\rho$. Thus, if the right hand side of Eq.84 vanishes for all $x$ this implies $\langle \phi | = 0$, which had to be shown.

The completeness of the set of maximally localised fields means that we obtain the full information on any $|\psi\rangle$ when collecting its projections $\langle x_{ml} | \psi \rangle$ on the $|x_{ml}\rangle$, i.e. the quasi-position representation truly represents the fields. Indeed, the mapping e.g. from $\rho$-space to quasi-position space is invertible:

Using the explicit expressions Eq.69,71 for the maximally localised fields in the $\rho$-representation we obtain for arbitrary $|\psi\rangle$ the quasi-position wave function $\psi(x)$ expressed in terms of its $\rho$-representation $\psi(\rho)$ as

$$\psi(x) = \langle x_{ml} | \psi \rangle = N^{1/2}(n)(\beta/2\pi)^{n/4} \int_{I_n} d^n \rho \ (1 - \beta \rho^2/2)^{1/2+\sqrt{1/4+n/8}} e^{ix \cdot \rho/\hbar} \psi(\rho)$$  \hfill (85)

with the inverse:

$$\psi(\rho) = N^{-1/2}(n)(\beta/2\pi)^{-n/4}(2\pi \hbar)^{-n}(1 - \beta \rho^2/2)^{-1/2-\sqrt{1/4+n/8}} \int d^n x \ e^{-ix \cdot \rho/\hbar} \psi(x)$$  \hfill (86)

Let us denote the mapping from $\rho$-space to quasi-position space (Eq.85) by $U$. The identity $U^{-1}U = 1$ is easily verified by inserting Eq.85 into Eq.86.
In the quasi-position representation the scalar product and the action of the position and momentum operators then read:

$$
\langle \psi | \phi \rangle = N^{-1}(n)(\beta / 2\pi)^{-n/2}(2\pi \hbar)^{-n} \int d^n \rho \ (1 - \beta \rho^2 / 2)^{-1 + \sqrt{1 + n/2}}
$$

$$
\int \int d^nx \ d^ny \ e^{i(x-y) \cdot \rho / \hbar} \psi^*(x) \phi(y)
$$

$$
p^i \psi(x) = -i \hbar \sum_{r=0}^\infty (\hbar^2 \beta \Delta / 2)^r \frac{\partial}{\partial x_i} \psi(x)
$$

$$
x^i \psi(x) = \left( x^i + i \frac{\hbar \beta}{2} \left( 1 + \sqrt{1 + n/2} \right) p^i \right) \psi(x)
$$

where $\Delta = \sum_i \frac{\partial^2}{\partial x_i^2}$. Note that the action of $p_i$, given in Eq.88 (and also used in Eq.89) is well defined on quasi-position wave functions $\psi(x)$, since they Fourier decompose into wavelengths not smaller than the finite minimal wavelength $\lambda_0$ (Eq.73). In this context see also [12] where the concept of quasi-position representation was first introduced.