ON SIMILARITY OF PERTURBED MULTIPLICATION OPERATORS

R. O. HRYNIV AND YA. V. MYKYTYUK

Abstract. Let $S$ be the multiplication operator by an independent variable $x$ in $L_2(0, 1)$ and $V$ be an integral operator of Volterra type. We find conditions for $T := S + V$ to be similar to $S$ and discuss some generalisations of the results obtained to an abstract setting.

1. Introduction

In the Hilbert space $\mathcal{H} = L_2(0, 1)$, consider the operator $S$ of multiplication by an independent variable, $(Sf)(x) = xf(x)$, and its perturbation $T := S + V$, where

$$(Vf)(x) := \int_0^x v(x, t)f(t) \, dt.$$  

(1)

We assume throughout the paper that the kernel $v$ is Lebesgue measurable on $[0, 1] \times [0, 1]$, $v(x, t) = 0$ if $x < t$, and that the induced integral operator $V$ is bounded in $\mathcal{H}$.

Operators of similar type appear, e.g., in the so-called Friedrichs model [1, 2] or in polymerisation chemistry, where $T$ describes the evolution of a polymer system near dynamical equilibrium [3]. In both examples the asymptotic behaviour of the group $e^{itT}$ (in particular, uniform boundedness, or Lyapunov stability, of $e^{itT}$) is of much importance, which poses a problem of similarity of $T$ to a selfadjoint operator. For the case when $v(x, t) = \phi(x)\psi(t)$ for $0 \leq t \leq x \leq 1$ and $\phi, \psi \in \mathcal{H}$ this problem was studied in detail in the paper [4], where the following result was established.

Theorem A. Let $\phi \psi \equiv 0$ and suppose that there exist moduli of continuity $\omega_1, \omega_2$ such that $\phi \in \text{Lip} (\omega_1)$, $\psi \in \text{Lip} (\omega_2)$, and

$$\int_0^\infty \frac{\omega_1(\tau)\omega_2(\tau)}{\tau} < \infty.$$  

(2)

Date: December 20, 2021.

1991 Mathematics Subject Classification. Primary 47A05; Secondary 47G10, 47B38.

Key words and phrases. Multiplication operators, integral operators, similarity.
Then the operator $T$ is similar to a selfadjoint one.

Moreover, a sharp analysis of behaviour of the operator $T$ resolvent near the real axis shows the necessity of condition (2) in the sense that if it does not hold, then the operator $T$ need not be similar to a selfadjoint one; in [4] the corresponding examples are constructed for $\omega_1(\tau) = |\ln \tau|^{-\delta_1}$ and $\omega_2(\tau) = |\ln \tau|^{-\delta_2}$ with $\delta_1 + \delta_2 < 1$.

The main aim of this paper is to find conditions under which the perturbed operator $T$ is similar to the unperturbed operator $S$. We consider the perturbations $V$ of the general form (1) and follow a line of attack due to Friedrichs [1, Ch. II.6]. Namely, we find sufficient conditions for existence of a bounded operator $K$ with spectral radius $r(K)$ smaller than $1/2$ such that

$$T(I + K) = (I + K)S. \quad (3)$$

Our main results are as follows.

**Theorem 1.** Suppose that the kernel

$$w(x, t) := \frac{|v(x, t)|}{x - t}, \quad x, t \in [0, 1],$$

generates an integral operator $W$ that is bounded in $H$ and has spectral radius $r(W)$ less than $1/2$. Then the operators $T$ and $S$ are similar.

**Corollary 2.** Suppose that there exists a function $q \in L_1(0, 1)$ such that $w(x, t) \leq q(x - t)$ for all $x, t$, $0 \leq t < x \leq 1$. Then the operators $T$ and $S$ are similar.

Note that under the assumptions of Theorem A we have

$$|v(x, t)| \leq |\phi(x)\psi(t)| + |\phi(t)\psi(x)| = (|\phi(x)| - |\phi(t)|)(|\psi(t)| - |\psi(x)|) \leq \omega_1(x - t)\omega_2(x - t),$$

and therefore Corollary 2 applies with $q(\tau) = \omega_1(\tau)\omega_2(\tau)/\tau$ and proves the claim of Theorem A. Corollary 2 also admits kernels $v$ for which the norm

$$\|v\|_\alpha := \text{ess sup}_{0 \leq t < x \leq 1} (x - t)^{1-\alpha}|v(x, t)|$$

is finite for some $\alpha > 0$; see [4, 5] and references therein for related details on similarity of Volterra operators to fractional integration operators. We remark that the results of [4] imply that the condition $q \in L_1(0, 1)$ cannot be weakened.

In fact, under our approach $V$ need not be an integral operator of the form (1). We allow $V$ from the algebra $A$ of operators leaving invariant functions with support in $[a, 1]$, for any $a \in [0, 1)$, that are majorised in a certain sense. The corresponding concepts are based on the theory
of nonnegative operators in a Banach space with a positive cone and are developed in Section 2. Abstract results on similarity of the operators \( T \) and \( S \) are established in Section 3, and then used to prove Theorem 1 and Corollary 2 in Section 4. Finally, in the last section we comment on some straightforward generalisations of the main results.

Throughout the paper we shall denote by \( r(T) \) the spectral radius of a bounded operator \( T \); recall that \( r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} \).

2. Nonnegative operators and some auxiliary results

We start by recalling some concepts of linear spaces with a positive cone (see, e.g., [7]). Denote by \( H^+ = \{ f \in H \mid f(x) \geq 0 \text{ a.e. in } [0,1] \} \) and \( \mathcal{B}_+(H) = \{ A \in \mathcal{B}(H) \mid A H_+ \subset H_+ \} \) the cones of nonnegative elements in \( H \) and nonnegative operators in \( \mathcal{H} \), respectively. As usual, for any \( f, g \in H \) and \( A, B \in \mathcal{B}(H) \) we write \( f \geq g \) and \( A \geq B \) if \( f - g \in H^+ \) and \( A - B \in \mathcal{B}_+(H) \), respectively. Recall that \( H^+ \) is a generating cone and hence for any \( f \in H \) the absolute value \( |f| \) exists as an element of \( H^+ \); in the present context \( |f| \) is the function defined by \( |f|(x) = |f(x)| \). The cone \( \mathcal{B}_+(H) \), on the contrary, is not generating and hence the absolute value \( |A| \) cannot be defined for all \( A \in \mathcal{B}(H) \). We shall point out the class of operators, for which the absolute value is well defined.

An operator \( B \) is said to majorise \( A \) (written \( B \succ A \) or \( A \preceq B \)) if \( |Af| \leq |Bf| \) for all \( f \in H \). Evidently, \( A \preceq B \) implies that \( B \) is nonnegative, \( B \geq \lambda A \) for all \( \lambda \) with \( |\lambda| \leq 1 \), and that

\[
||(Af, g)|| \leq (|Af|, |g|) \leq (|Bf|, |g|) \leq \|B\|\|f\|\|g\|
\]

for all \( f, g \in H \), i.e., \( \|A\| \leq \|B\| \). Moreover, if \( B \) majorises \( A \), then the inequality

\[
|A^n f| \leq B|A^{n-1} f| \leq \cdots \leq B^{n-1}|Af| \leq B^n |f| \quad \forall f \in H
\]

shows that \( A^n \propto B^n \) for any \( n \in \mathbb{N} \) and hence \( r(A) \leq r(B) \).

Put

\[
\mathcal{M}(A) := \{ B \in \mathcal{B}_+(H) \mid B \succ A \}
\]

and

\[
\mathcal{B}_M(H) := \{ A \in \mathcal{B}(H) \mid \mathcal{M}(A) \neq \emptyset \}.
\]

Observe that \( \mathcal{B}_M(H) \) is a multiplicative cone in \( \mathcal{B}(H) \), i.e., \( AB \in \mathcal{B}_M(H) \) whenever \( A, B \in \mathcal{B}_M(H) \).
Example 3. Let $A$ and $B$ be integral operators in $\mathcal{H}$ with continuous kernels $a$ and $b$ respectively. Then $B \in \mathcal{M}(A)$ if and only if $|a(x,t)| \leq b(x,t)$ everywhere in $[0,1] \times [0,1]$ and $A \in \mathcal{B}_M(\mathcal{H})$ if and only if the kernel $|a|$ induces a bounded integral operator in $\mathcal{H}$.

To some extent, the above example is generic as any bounded operator in $\mathcal{H}$ is a strong limit of integral operators with continuous kernels. In fact, let $r$ be a smooth nonnegative function such that $\text{supp } r \subset [0,1]$ and $\int r = 1$. Denote by $R_\varepsilon$ the integral operator 

$$(R_\varepsilon f)(x) = \int_0^1 r_\varepsilon(x-t) f(t) \, dt, \quad r_\varepsilon(x) = \varepsilon^{-1} r(x/\varepsilon),$$

and for any $A \in \mathcal{B}(\mathcal{H})$ and $\varepsilon > 0$ put

$$A_\varepsilon := R_\varepsilon AR_\varepsilon.$$ 

Note that $R_\varepsilon \to I$ as $\varepsilon \to 0$ strongly in $\mathcal{H}$ [8, Ch. III.2], whence

$$s\lim_{\varepsilon \to 0} A_\varepsilon = A.$$ 

Moreover, since $R_\varepsilon$ is a Hilbert-Schmidt operator, $A_\varepsilon$ is an integral operator with some kernel $a_\varepsilon$ (see details in [4]). Denoting by $\delta_\varepsilon(u) := \delta(u - \xi)$ the delta function centered at the point $\xi$, we see that

$$a_\varepsilon(x,t) = (A_\varepsilon \delta_t, \delta_x) = (AR_\varepsilon \delta_t, R_\varepsilon^* \delta_x) = (Ar_\varepsilon, \hat{r}_\varepsilon, x),$$

where $r_\varepsilon(t) := r_\varepsilon(u-t)$ and $\hat{r}_\varepsilon(x) := r_\varepsilon(x-t)$ are continuous functions in $t$ and $x$ respectively with values in $\mathcal{H}$, whence the kernel $a_\varepsilon$ is continuous.

This observation is heavily used to justify the following statement.

Lemma 4. For any $A \in \mathcal{B}_M(\mathcal{H})$, the set $\mathcal{M}(A)$ contains the minimal element $|A|$ called the absolute value of $A$. In other words, $|A| \in \mathcal{M}(A)$ and $|A| \leq B$ for any $B \in \mathcal{M}(A)$.

Proof. Suppose that $A \in \mathcal{B}_M(\mathcal{H})$ and $B \in \mathcal{M}(A)$ and put $A_\varepsilon = R_\varepsilon AR_\varepsilon$, $B_\varepsilon := R_\varepsilon BR_\varepsilon$ with the above constructed $R_\varepsilon$. Then

$$|(A_\varepsilon f, g)| = |(AR_\varepsilon f, R_\varepsilon^* g)| \leq (B |R_\varepsilon f|, |R_\varepsilon^* g|) \leq (BR_\varepsilon |f|, R_\varepsilon^* |g|) = (B_\varepsilon |f|, |g|)$$

for any $f, g \in \mathcal{H}$, which implies the inequality

$$|a_\varepsilon(x,t)| \leq b_\varepsilon(x,t)$$

(4)

for the corresponding kernels.

Denote by $A_\varepsilon^+$ the integral operator induced by the kernel $|a_\varepsilon(x,t)|$. Due to inequality (4) the operator $A_\varepsilon^+$ is bounded, $A_\varepsilon^+ \leq B_\varepsilon$, and $\|A_\varepsilon^+\| \leq \|B_\varepsilon\|$. Moreover, $A_\varepsilon^+ \geq A_\varepsilon$ for any $\varepsilon > 0$ by construction.
$B_\varepsilon \to B$ strongly as $\varepsilon \to 0$, the norms $\|A_\varepsilon^+\|$ are bounded uniformly in $\varepsilon > 0$ and the set $\{A_\varepsilon^+\}_{\varepsilon>0}$ is weakly compact in $B(H)$. Consequently there exists a sequence $\varepsilon_n \to 0$ and an operator $A_0^+$ such that $A_\varepsilon^+ \to A_0^+$ weakly as $n \to \infty$. Passing to the limit $\varepsilon \to 0$ in $A_\varepsilon^+ \leq B_\varepsilon$ and $A_\varepsilon^+ \ni A_\varepsilon$, we find that $A_0^+ \leq B$ and $A_0^+ \ni A$. Since $B \in \mathcal{M}(A)$ was arbitrary, these inequalities prove that $A_0^+$ is the weak limit of $A_\varepsilon^+$ as $\varepsilon \to 0$ and that $A_0^+$ is the desired absolute value $|A|$ of $A$. \hfill \Box

Let $\chi_a$ denote the multiplication operator by the characteristic function of the interval $[a, 1]$ and

$$\mathcal{A} := \{A \in B(H) \mid A\chi_a = \chi_a A\chi_a, \quad \forall a \in [0, 1]\}.$$ 

$\mathcal{A}$ is clearly a weakly closed subalgebra of $B(H)$. Put

$$A_+ := \mathcal{A} \cap B_+(H)$$

and note that $|A| \in A_+$ for any $A \in B_M(H) \cap \mathcal{A}$.

**Remark 5.** Suppose that $A \in \mathcal{A}_+$ and $A_\varepsilon = R_\varepsilon A R_\varepsilon$, $\varepsilon > 0$. Then $A_\varepsilon$ is an integral operator with a continuous kernel $a_\varepsilon(x,t)$ such that $a_\varepsilon(x,t) = 0$ for $t > x$. Therefore $A_\varepsilon$ is a Volterra operator (see, e.g., [10, Sect. 68] and $A$ is a strong limit of a sequence of Volterra operators.

The above remark implies the following result.

**Lemma 6.** Suppose that $A \in \mathcal{A}_+$; then $[S,A] := SA - AS$ belongs to $\mathcal{A}_+$.

**Proof.** It suffices to notice that $[S,A_\varepsilon]$ is an integral operator with the nonnegative kernel $(x-t)a_\varepsilon(x,t)$ and hence belongs to $\mathcal{A}_+$. \hfill \Box

### 3. Similarity of the operators $S$ and $T$

In this section, we shall study the question of existence of an operator $K \in B(H)$ satisfying equation (3). Denoting $[S,K] := SK - KS$, we rewrite (3) as the equation

$$[S,K] + VK + V = 0$$

and apply a modification of Friedrichs successive approximation method [1, Ch. II.6] to solve the latter.

**Lemma 7.** Suppose that $V \in B_M(H) \cap \mathcal{A}$ and $[S,W] \in \mathcal{M}(V)$ for some $W \in \mathcal{A}_+$. Then the equation

$$[S,K] = V$$

has a solution $K \in B_M(H) \cap \mathcal{A}$ such that $K \preccurlyeq W$. 


Proof. Since \([S, W] \succ V\), we have \(R_\varepsilon [S, W] R_\varepsilon \succ R_\varepsilon VR_\varepsilon =: V_\varepsilon\). By Lemma 6, \(R_\varepsilon S \leq SR_\varepsilon\), whence
\[ R_\varepsilon [S, W] R_\varepsilon = R_\varepsilon SWR_\varepsilon - R_\varepsilon WSR_\varepsilon \leq SW_\varepsilon - W_\varepsilon S = [S, W_\varepsilon] \]
and \([S, W_\varepsilon] \succ V_\varepsilon\), where \(W_\varepsilon = R_\varepsilon WR_\varepsilon\). It follows that the kernels \(v_\varepsilon\) and \(w_\varepsilon\) of the operators \(V_\varepsilon\) and \(W_\varepsilon\) satisfy the inequality
\[ \left| \frac{v_\varepsilon(x, t)}{x - t} \right| \leq w_\varepsilon(x, t); \]
in particular, \(V_\varepsilon \in A\). Denote by \(K^{(\varepsilon)}\) the integral operator induced by the kernel \(v_\varepsilon(x, t)/(x - t)\). By the above inequality \(K^{(\varepsilon)}\) is a bounded operator belonging to the algebra \(A\), \(K^{(\varepsilon)} \succ W_\varepsilon\), and \([S, K^{(\varepsilon)}] = V_\varepsilon\).
Therefore the norms \(\|K^{(\varepsilon)}\|\) are bounded uniformly in \(\varepsilon > 0\) and there exist a bounded operator \(K\) and a sequence \(\varepsilon_n \to 0\) such that \(K^{(\varepsilon_n)}\) converge weakly to \(K\) as \(n \to \infty\). Passing to the limit \(\varepsilon_n \to 0\) in the relations
\[ SK^{(\varepsilon)} - K^{(\varepsilon)}S = V_\varepsilon, \quad K^{(\varepsilon)} \succ W_\varepsilon \]
we find that \(K\) solves the equation \([S, K] = V\) and satisfies \(K \succ W\).
The proof is complete. \(\square\)

Corollary 8. Suppose that \(U, V \in \mathcal{B}_M(\mathcal{H}) \cap A\) and \([S, W] \in \mathcal{M}(V)\) for some \(W \in A_+\). Then the equation
\[ [S, K] = VU \]
has a solution \(K \in \mathcal{B}_M(\mathcal{H}) \cap A\) such that \(K \succ W|U|\).

Proof. Since \([S, W|U|] = [S, W]|U| + W[S, |U|] \geq [S, W]|U| \succ VU\) by Lemma 6 and the assumptions of the corollary, Lemma 7 applies with \(W|U|\) and \(VU\) instead of \(W\) and \(V\), and the claim follows. \(\square\)

Theorem 9. Suppose that \(V \in A\) and that there exists an operator \(W \in A_+\) such that \(r(W) < 1\) and \([S, W] \succ V\). Then the equation
\[ [S, K] + VK + V = 0 \] (5)
has a solution \(K \in \mathcal{B}_M(\mathcal{H}) \cap A\).

Proof. We shall seek for \(K\) of the form
\[ K = \sum_{k=1}^{\infty} K_n, \]
where \(K_n\) are found recursively from the relations
\[ [S, K_n] = -VK_{n-1}, \quad n \in \mathbb{N}, \]
with $K_0 := I$. In virtue of Lemma 7 and Corollary 8 we find successively $K_n \in A$, $n = 1, 2, \ldots$, such that $K_n \preceq W[K_{n-1}] \preceq W^n$. Therefore $\|K_n\| \leq \|W^n\|$, which shows that the series $\sum_{n=1}^{\infty} K_n$ converges absolutely in the uniform operator topology and its sum $K$ satisfies the equality

$$[S, K] = \sum_{n=1}^{\infty} [S, K_n] = -\sum_{n=1}^{\infty} VK_{n-1} = -VK - V.$$

The theorem is proved.

**Corollary 10.** If under the assumptions of Theorem 9 $r(W) < 1/2$, then equation (5) has a solution $K \in B_M(H) \cap A$ with $r(K) < 1$.

**Proof.** Observe that the solution $K$ constructed in the proof of Theorem 9 satisfies the inequality $K \preceq W(I - W)^{-1}$. Therefore $r(K) \leq r(W(I - W)^{-1})$, and it suffices to note that $r(W(I - W)^{-1}) < 1$ if $r(W) < 1/2$.

4. **Proof of the main results**

**Proof of Theorem 1.** It suffices to notice that the assumptions of Theorem 9 and Corollary 8 are satisfied for the integral operator $W$ with the kernel $w(x, t) := \frac{|v(x, t)|}{x - t}$.

**Theorem 11.** Suppose that $w$ is a positive kernel on $[0, 1] \times [0, 1]$ such that $w(x, t) = 0$ if $t > x$ and $w(x, t) \leq q(x - t)$ if $x \geq t$, where $q \in L_1(0, 1)$. Then the induced integral operator $W$ is a Volterra operator in $\mathcal{H}$ and $\|W\| \leq \|q\|_1$.

**Proof.** We prove first that $W$ is bounded in $\mathcal{H}$. In fact,

$$\int_0^x w(x, t) \, dt \leq \int_0^x q(x - t) \, dt = \int_0^x q(t) \, dt \leq \|q\|_1,$$

$$\int_t^1 w(x, t) \, dx \leq \int_t^1 q(x - t) \, dx = \int_0^{1-t} q(x) \, dx \leq \|q\|_1,$$

whence $\|W\| \leq \|q\|_1$ by the Schur test [3, Theorem 5.2]. Next, put

$$w^{(m)} = \begin{cases} w(x, t) & \text{if } q(x - t) \leq m, \\ 0 & \text{if } q(x - t) > m. \end{cases}$$
Then the induced integral operator $W^{(m)}$ is a Volterra operator in $\mathcal{H}$ (see, e.g., [10, Sect. 68]) and

$$\|W - W^{(m)}\| \leq \int_{q>m} q(t) \, dt \to 0$$

as $m \to \infty$ in view of $q \in L_1(0,1)$ and the above arguments. Therefore $W$ is a Volterra operator as well and the lemma is proved. \qed

Corollary 2 now easily follows from Theorems 1 and 11.

5. SOME GENERALISATIONS

In this section, we comment on some straightforward generalisations of the main results. Observe first that the arguments of Sections 2 and 3 work for the Banach space $L_p(a,b)$ with $-\infty \leq a < b \leq \infty$ arbitrary and any $p \in [1,\infty)$. Next, $S$ can be replaced by the multiplication operator by any increasing function $\phi(x)$. The analogue of Theorem 1 reads as follows.

**Theorem 1′.** Suppose that $\phi$ strictly increases on $(a,b)$ and that the kernel

$$w(x,t) := \left| \frac{v(x,t)}{\phi(x) - \phi(t)} \right|$$

induces a bounded integral operator in $L_p(a,b)$ of spectral radius less than $1/2$. Then the operators $S$ of multiplication by $\phi$ and $T := S + V$, where

$$(V f)(x) := \int_a^x v(x,t) f(t) \, dt,$$

are similar in $L_p(a,b)$. In particular, $S$ and $T$ are similar if $w(x,t) \leq q(x - t)$ for some $q \in L_1(a,b)$.

Observe also that most results of the paper hold in an arbitrary Banach lattice $\mathcal{X}$ of functions over $(a,b)$ provided the identity operator in $\mathcal{X}$ is the strong limit of integral operators with continuous kernels.

**Remark.** After this paper was finished, M. M. Malamud drew our attention to his note [12], where the statements of Theorem 1 (under the assumption $r(W) = 0$) and Corollary 2 were announced without proof.

**References**

[1] Friedrichs, K. O. *Perturbation of Spectra in Hilbert Space*. Amer. Math. Soc., Providence, RI, 1965.

[2] Faddeev, L. D. On Friedrichs model in the perturbation theory of continuous spectrum// *Trudy MIAN*. 1964. V. 73. P. 292-313 (Russian).
[3] Kokholm, N. J. Spectral analysis of perturbed multiplication operators occurring in polymerisation chemistry// Proc. Roy. Soc. Edinburgh Sect. A. 1989. V. 113. No.1–2. P. 119–148.

[4] Naboko S. N. and Tretter C. Lyapunov stability of a perturbed multiplication operator// Contributions to operator theory in spaces with an indefinite metric. The Heinz Langer anniversary volume on the occasion of his 60th birthday. Basel: Birkhäuser, 1998. Oper. Theory Adv. Appl. V. 106. P. 309–326.

[5] Freeman, J. M. Volterra operators similar to $J : f(x) \mapsto \int_0^x f(y) dy$// Trans. Amer. Math. Soc. 1965. V. 116. P. 181–192.

[6] Malamud, M. M. Similarity of Volterra operators and related problems in the theory of differential equations of fractional orders// Trudy Moscov. Mat. Obshch. 1994. V. 55. P. 73–148 (Russian); transl. in Trans. Moscow Math. Soc. 1994. V. 55. P. 57-122.

[7] Krein M. G., Rutman M. A. Linear operators leaving invariant a cone in a Banach space// Uspekhi Mat. Nauk. 1948. V. 3. No. 1. P. 3–95 (Russian); transl. in Functional Analysis and Measure Theory, Transl. Amer. Math. Soc. 1962. V. 10. P. 5–105.

[8] Stein E. M. Singular Integrals and Differentiability Properties of Functions. Princeton Univ. Press, Princeton, N. J., 1970.

[9] Halmos, P. R., Sunder V. S. Bounded Integral Operators on $L^2$ Spaces. Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[10] Riesz F. and Sz.-Nagy B. Leçons D’analyse Fonctionnelle. Akadémiai Kiadó, Budapest, 1972.

[11] Schaefer H. H. Banach Lattices and Positive Operators. Springer-Verlag, Berlin, 1974.

[12] Malamud M. M.// Uspekhi Mat. Nauk. 1977. V. 32. No. 5. P. 167. (Russian)

Institute for Applied Problems of Mechanics and Mathematics,
3b Naukova str., 79601 Lviv, Ukraine
E-mail address: hryniv@mebm.lviv.ua

Lviv National University, 1 Universytetska str., 79602 Lviv, Ukraine
E-mail address: yamykytyuk@yahoo.com