List-Decodable Subspace Recovery via Sum-of-Squares

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Abstract

We give the first efficient algorithm for the problem of list-decodable subspace recovery. Our algorithm takes input \( n \) samples \( \alpha n (\alpha \ll 1/2) \) are generated i.i.d. from Gaussian distribution \( \mathcal{N}(0, \Sigma) \) on \( \mathbb{R}^d \) with covariance \( \Sigma \), of rank \( r \) and the rest are arbitrary, potentially adversarial outliers. It outputs a list of \( \mathcal{O}(1/\alpha) \) projection matrices guaranteed to contain a projection matrix \( \Pi \) such that \( \|\Pi - \Pi^*\|_F^2 = \kappa^4 \log(r)\tilde{O}(1/\alpha^2) \). Here, \( \Pi^* \) is the projection matrix to the range space of \( \Sigma \). The algorithm needs \( n = d^{\log(r)\tilde{O}(1/\alpha^2)} \) samples and runs in time \( n^{\log(r)\tilde{O}(1/\alpha^4)} \) time where \( \kappa \) is the ratio of the largest to smallest non-zero eigenvalues of \( \Sigma \).

Our algorithm builds on the recently developed framework for list-decodable learning via the sum-of-squares (SoS) method \([KKK19, RY20]\) with some key technical and conceptual advancements. Our key conceptual contribution involves showing a (SoS “certified”) lower bound on the eigenvalues of covariances of arbitrary small subsamples of an i.i.d. sample of a certifiably anti-concentrated distribution. One of our key technical contributions gives a new method that allows error reduction “within SoS” with only a logarithmic cost in the exponent in the running time (in contrast to polynomial cost in \([KKK19, RY20]\)).

In a concurrent and independent work, Raghavendra and Yau proved related results for list-decodable subspace recovery \([RY20]\).

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\(^1\)All \( \tilde{O} \) hide polylogarithmic factors in \( 1/\alpha \)
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1 Introduction

An influential recent line of work [KLS09, ABL13, DKK+16, LRV16, CSV17, KS17a, KS17b, HL17, DKK+18, DKS17a, KKM18] has focused on designing algorithms for basic statistical estimation tasks in the presence of adversarial outliers. This has led to a body of work on outlier-robust estimation of basic parameters of distributions such as mean, covariance [DKK+16, DKS17b, CDG19, DKK+17, DKK+18, CDGW19] and moment tensors [KS17b] along with applications to "downstream" learning tasks such as linear and polynomial regression [DKS17c, KKM18, DKK+19, PSBR18]. The upshot of this line of work is a detailed understanding of efficient robust estimation when the fraction of inliers (≫ 1/2), but a fixed fraction of arbitrary adversarial outliers in the input data.

In this work, we focus on the harsher list-decodable estimation model where the fraction of inliers $\alpha$ is $\ll 1/2$ - i.e., a majority of the input sample are outliers. First considered in [BBV08] in the context of clustering, this was proposed as a model for untrusted data in a recent influential recent work of Charikar, Steinhardt and Valiant [CSV17]. Since unique recovery is information-theoretically impossible in this setting, the goal is to recover a small (ideally $O(1/\alpha)$) size list of parameters one of which is guaranteed to be close to those of the inlier distribution. A recent series of works have resulted in a high-level blueprint based on the sum-of-squares method for list-decodable estimation yielding algorithms for list-decodable mean estimation [DKS18, KS17a] and linear regression [KKK19, RY20].

We extend this line of work by giving the first efficient algorithm for list-decodable subspace recovery. In this setting, we are given data with $\alpha$ fraction inliers generated i.i.d. according to $\mathcal{N}(0, \Sigma,)^2$ on $\mathbb{R}^d$ with a (possibly low-rank, say $r < d$) covariance matrix $\Sigma$, and rest being arbitrary outliers. We give an algorithm that succeeds in returning a list of size $O(1/\alpha)$ that contains a $\hat{\Pi}$ satisfying $\|\hat{\Pi} - \Pi^*\|_F \leq \log(r\kappa)\tilde{O}(\kappa^4/\alpha^2)$ where $\Pi^*$ is the projector to the range space of $\Sigma$, and $\kappa$ is the ratio of the largest to smallest non-zero eigenvalues of $\Sigma$. Our Frobenius norm recovery guarantees are the strongest possible and imply guarantees in other well-studied norms such as spectral norm or principle angle distance between subspaces. Our algorithm runs in time $n^{\log(r\kappa)\tilde{O}(1/\alpha^4)}$ and requires $n = d^{\log(r\kappa)\tilde{O}(1/\alpha^2)}$ samples.

Our results work more generally for any distribution $D$ that satisfies certifiable anti-concentration and mild concentration properties (concentration of PSD forms). Certifiable anti-concentration was first defined and studied in recent works on list-decodable regression [KKK19, RY20]. Gaussian distribution and uniform distribution on sphere (restricted to a subspace) are natural examples of distributions satisfying this property. We note that Karmalkar et. al. [KKK19] proved that anti-concentration of $D$ is necessary for list-decodable regression (and thus also subspace recovery) to be information theoretically possible.

Why List-Decodable Estimation? List-decodable estimation is a strict generalization of related and well-studied clustering problems (for e.g., list-decodable mean estimation generalizes clustering

\footnote{Our techniques naturally extend to distributions with non-zero means but we will omit this generalization to not complicate the notation.}
spherical mixture models, list-decodable regression generalizes mixed linear regression). In our case, list-decodable subspace recovery generalizes the well-studied problem of subspace clustering where given a mixture of $k$ distributions with covariances non-zero in different subspaces, the goal is to recover the underlying $k$ subspaces [AGGR98, CFZ99, GNC99, PJAM02, AY00]. Algorithms in this model thus naturally yield robust algorithms for the related clustering formulations. In contrast to known results, such algorithms allow “partial recovery” (e.g. for example recovery $k-1$ or fewer clusters) even in the presence of outliers that garble up one or more clusters completely.

Another important implication of list-decodable estimation is algorithms for unique recovery that work all the way down to the information-theoretic threshold (i.e. fraction of inliers $\alpha > 1/2$). Thus, specifically in our case, we obtain an algorithm for (uniquely) estimating the subspace spanned by the inlier distribution $D$ whenever the fraction of inliers satisfy $\alpha > 1/2$ - the information theoretically minimum possible value. We note that such a result will follow from outlier-robust covariance estimation algorithms [DKK+16, LRV16, CDGW19] whenever $\alpha$ is sufficiently close to 1. While prior works do not specify precise constants, all known works appear to require $\alpha$ at least $> 0.75$.

1.1 Our Results

We are ready to formally state our results. Our results apply to input samples generated according to the following model:

**Model 1.1** (Robust Subspace Recovery with Large Outliers). For $0 \leq \alpha < 1$ and $r < d$, let $\mu \in \mathbb{R}^d$, $\Sigma_* \in \mathbb{R}^{d \times d}$ be a rank $r$ PSD matrix and let $D$ be a distribution on $\mathbb{R}^d$ with mean $\mu_*$ and covariance $\Sigma_*$. Let $\text{Sub}_D(\alpha, \Sigma_*)$ denote the following probabilistic process to generate $n$ samples, $x_1, x_2 \ldots x_n$ with $\alpha n$ inliers $I$ and $(1-\alpha)n$ outliers $O$:

1. Construct $I$ by choosing $\alpha n$ i.i.d. samples from $D$.

2. Construct $O$ by choosing the remaining $(1-\alpha)n$ points arbitrarily and potentially adversarially w.r.t. the inliers.

**Remark 1.2.** We will mainly focus on the case when $\mu_* = 0$. The case of non-zero $\mu_*$ can be easily reduced to the case of $\mu_* = 0$ by modifying samples by randomly pairing them up and subtracting off samples in each pair (this changes the fraction of inliers from $\alpha$ to $\alpha^2$).

**Remark 1.3.** Our results naturally extend to the harsher strong contamination model (where one first chooses an i.i.d. sample from $D$ and then corrupts an arbitrary $(1-\alpha)$ fraction of them) with no change in the algorithm.

An $\eta$-approximate list-decodable subspace recovery algorithm takes input a sample $S$ drawn according to $\text{Sub}_D(\alpha, \Sigma_*)$ and outputs a list $L$ of absolute constant (depending only on $\alpha$) such that there exists a $\Pi \in L$ satisfying $\|\Pi - \Pi_*\|_F^2 \leq \eta$, where $\Pi_*$ is the projector to the range space of $\Sigma_*$. Before stating our results we observe that since list-decodable subspace recovery strictly generalizes list-decodable regression (by viewing samples as $d+1$ dimensional points with a rank $d$ covariance), we can import the result of Karamalkar, Klivans and Kothari [KKK19] that shows the information-theoretic necessity of anti-concentration of the distribution $D$. 


Theorem 1.6 (Large-List Subspace Recovery). There exists a distribution $\mathcal{D}$ that $(\alpha + \varepsilon)$-anti-concentrated for every $\varepsilon > 0$ but there is no algorithm for $\alpha/2$-approximate list-decodable subspace recovery for $\text{Sub}_{\mathcal{D}}(\alpha, \Sigma_\alpha)$ that outputs a list of size $< d$.

The distribution $\mathcal{D}$ is simply the uniform distribution on an affine subcube of dimension $n - 1$ of $\{0, 1\}^n$ (and more generally, $q$-ary discrete cube).

Our first main result shows that given any arbitrarily small $\eta > 0$, we can recover a polynomial (in the rank $r$) size list of subspaces that contains a $\hat{\Pi}$ satisfying $\|\hat{\Pi} - \Pi\|_F^2 \leq \eta$. The surprising aspect of this result is that we can get an error that can be made arbitrarily small (independent of the rank $r$ or the dimension $d$) at the cost of increasing the list size from a fixed constant to polynomially large in the rank $r$ of $\Sigma_\alpha$. This result crucially relies on our new exponential error reduction method (see Lemma 4.7).

Theorem 1.5 (Large-List Subspace Recovery). Let $\text{Sub}_{\mathcal{D}}(\alpha, \Sigma_\alpha)$ be such that $\Sigma_\alpha$ has rank $r$ and condition number $\kappa$, and $\mathcal{D}$ is $\kappa$-certifiably $(\varepsilon, \alpha/2)$-anti-concentrated. For any $\eta > 0$, there exists an algorithm that takes input $n \geq n_0 = (kd \log(d))/\mathcal{O}(\kappa)$ samples from $\text{Sub}_{\mathcal{D}}(\alpha, \Sigma_\alpha)$ and outputs a list $\mathcal{L}$ of size $O(1/\alpha^k \log(r\kappa/\eta))$ of projection matrices such that with probability at least 0.99 over the draw of the sample and the randomness of the algorithm, there is a $\hat{\Pi} \in \mathcal{L}$ satisfying $\|\hat{\Pi} - \Pi\|_F^2 \leq \eta$. The algorithm has time complexity at most $n^{\mathcal{O}(k^2 \log(r\kappa/\eta))}$.

We use a new pruning procedure to get the optimal list size of $O(1/\alpha)$ at the cost of increasing the Frobenius error to $\tilde{O}(k^4 \log(r)/\alpha^2)$.

Theorem 1.6 (List-Decodable Subspace Recovery). Let $\text{Sub}_{\mathcal{D}}(\alpha, \Sigma_\alpha)$ be such that $\Sigma_\alpha$ has rank $r$ and condition number $\kappa$, and $\mathcal{D}$ is $\mathcal{N}(0, \Sigma_\alpha)$. Then, there exists an algorithm that takes as input $n = n_0 \geq (d \log(d)/\alpha^2)^{\tilde{O}(1/\alpha^2)}$ samples from $\text{Sub}_{\mathcal{D}}(\alpha, \Sigma_\alpha)$ and outputs a list $\mathcal{L}$ of $O(1/\alpha)$ projection matrices such that with probability at least 0.99 over the draw of the sample and the randomness of the algorithm, there is a $\hat{\Pi} \in \mathcal{L}$ satisfying $\|\hat{\Pi} - \Pi\|_F^2 \leq \tilde{O}(k^4 \log(r)/\alpha^2)$. The algorithm has time complexity at most $n^{\tilde{O}(\log(r\kappa)/\alpha^4)}$.

As discussed above, our results immediately extends by means of a simple reduction to the case when $\mu$ is non-zero.

Corollary 1.7 (Large-List Affine Recovery). Let $\text{Sub}_{\mathcal{D}}(\alpha, \Sigma_\alpha)$ be such that $\Sigma_\alpha$ has rank $r$ and condition number $\kappa$, and $\mathcal{D}$ is $\mathcal{N}(\mu, \Sigma_\alpha)$. Then, there exists an algorithm that takes as input $n = n_0 \geq (d \log(d)/\alpha^4)^{\tilde{O}(1/\alpha^2)}$ samples from $\text{Sub}_{\mathcal{D}}(\alpha, \Sigma_\alpha)$ and outputs a list $\mathcal{L}$ of $O(1/\alpha^2)$ projection matrices such that with probability at least 0.99 over the draw of the sample and the randomness of the algorithm, there is a $\hat{\Pi} \in \mathcal{L}$ satisfying $\|\hat{\Pi} - \Pi\|_F^2 \leq \tilde{O}(k^4 \log(r)/\alpha^4)$. The algorithm has time complexity at most $n^{\tilde{O}(\log(r\kappa)/\alpha^8)}$.

1.2 Related Work

Subspace Clustering. Prior work on subspace recovery focused on the closely related problem of subspace clustering in high dimension, where the goal is to partition a set of points into $k$-clusters according to their underlying subspaces. Subspace clustering methods have found numerous applications in computer vision tasks such as image compression [HWHM06], motion segmentation...
etc. Algorithms for subspace clustering include iterative methods, algebraic and statistical methods and spectral techniques. We refer the readers to the following surveys for a comprehensive overview [EV13, PHL04]. Elhamifar and Vidal [EV13] also introduced sparse subspace clustering, building on the compressed sensing and matrix completion literature. Soltanolkotabi et. al. [SEC+14] extend sparse subspace clustering to work in the presence of noise and provide rigorous algorithmic guarantees. They assume the outliers contribute a small fraction of the input and are distributed uniformly distributed of the unit sphere.

**Robust Subspace Recovery.** A recent line of work on robust subspace recovery has focused on projection pursuit techniques, $\ell_1$-PCA (robust PCA), exhaustive subspace search and robust covariance estimation. Here, the goal is to recover a set of inliers that span a single low-dimensional space. Projection pursuit algorithms iteratively find directions that maximize a scale function. The scale function often accounts on outliers and thus may be non-convex. McCoy and Tropp [MT+11] consider one such function and develop a rounding which approximates the global optimizer. The $\ell_1$ or Robust PCA objective replaces the Frobenius norm objective with a sum of absolute values objective, since it is less sensitive to outliers. While this formulation is non-convex and NP-hard in general, many special cases are tractable, as discussed here [VN18]. Hardt and Moitra [HM13] provide a worst-case exhaustive search algorithm, where both the inliers and outliers are required to be in general position and the inliers are generated deterministically. For a more comprehensive treatment of robust subspace recovery we refer the reader to [LM18].

In a concurrent and independent work, Raghavendra and Yau proved related results for list-decodable subspace recovery [RY20].

## 2 Technical Overview

In this section, we give a high level overview of our algorithm and the new ideas that go into making it work. At a high level, our algorithm generalizes the framework for list-decodable estimation recently used to obtain an efficient algorithm for list-decodable regression in the recent work of [KKK19].

In the list-decodable subspace recovery problem, our input is a collection of samples $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$, an $an$ of which are drawn i.i.d. from distribution $D$ with mean $0$ and unknown covariance $\Sigma_\star$ of rank $r$. For the purpose of this overview, we will think of $\Sigma_\star$ itself being a projection matrix $\Pi_\star$. Our algorithm starts from a polynomial feasibility program that simply tries to find a subset of sample that contains at least an $an$ points such that all of these points lie in a subspace of dimension $r \leq d$. We can encode these two requirements as the following system $A_{11}$ of polynomial constraints as follows:
we can make sure that there are solutions with a non-trivial intersection with the inliers by finding a distribution $\mathcal{A}$ whenever $\mu$ can all be far from the ground truth subspace.

A feasible solution $w$ whenever $w_i(1 - \Pi)x_i = 0$ forces $x_i$ to be in the subspace projected to by $\Pi$ whenever $w_i = 1$.

### 2.1 Designing an Inefficient Algorithm

A feasible solution $(w, \Pi)$ to the aforementioned constraint system (ignoring for now, the issue of efficiency), results in a subset of $\alpha n$ samples that span a subspace of dimension $r$. However, there can be multiple $r$ dimensional subspaces that satisfy this requirement for various $\alpha n$ subsets chosen entirely out of the outliers\(^3\). Thus, even if we were to find a solution to this program, it’s not clear how to recover a subspace close to the one spanned by the inliers.

**High-Entropy Distributions.** In order to force our solution to (2.1) to give us information about the true inliers, it seems beneficial to try to find not one but multiple solution pairs $(w^i, \Pi_i)$ for $i = 1, 2, ...$ so that at least one of the $w^i$ indicates a subset that has a substantial intersection with the true inliers. An important conceptual insight in (see Overview section in [KKK19] for a longer discussion) is to thus ask for a probability distribution (which, at this point can be thought of as a method to ask for multiple solutions) $\mu$ over solutions $(w, \Pi)$ satisfying (2.1). It turns out that we can make sure that there are solutions $(w, \Pi)$ in the support of $\mu$ where $w$ indicates a subset with a non-trivial intersection with the inliers by finding a distribution $\mu$ so that $\|\sum_{i=1}^n \mathbb{E} w_i\|_2^2$ is **minimized**. This constraint serves as a proxy for **high entropy distributions**. Formally, we can conclude the following useful result that shows that the expected (over $\mu$) intersection of a subset indicated by $w$ and the inliers is at least $\alpha$ fraction of the inliers.

**Proposition 2.1.** Let $\mu$ be a distribution on $(w, \Pi)$ satisfying $\mathcal{A}_{w,\Pi}$. Then, $\mathbb{E} \sum_{i \in I} w_i \geq \alpha |I|$.

This result follows by a simple "weight-shifting" argument (if the distribution is over $w$ that do not intersect enough with the inliers, we can shift probability mass on the inliers and decrease $\|\sum_{i=1}^n \mathbb{E} w_i\|_2^2$).

\(^3\)See Section 3 in [KKK19] for examples showing how outliers can generate $\exp(O(d))$ many possible subspaces that can all be far from the ground truth subspace.
Anti-Concentration. Our distribution over \( \mu \) is guaranteed to contain \( w \) with at least \( \alpha \) fraction of the points of \( I \) in the intersection. Our hopes of finding information about the true subspace are pinned on such “good” \((w, \Pi)\) at this point. We would like that for such \( w, \Pi \) matches the ground truth subspace projected to by \( \Pi \). Let \( S \) be the ”intersection indices”, i.e., the set of indices of samples in \( I \) for which \( w_i = 1 \). Why should this be true? Since we have no control over \( S \), it could, a priori, consist of the points in \( I \) that span only a proper subspace, say \( V \) of the ground truth subspace. In this case, \( \Pi \) may not equal \( \Pi_* \).

The key observation is that in this “bad” case, there is a vector \( v \) that is in the orthogonal complement of \( \Pi V \) inside \( \Pi \), such that \( \langle x_i, v \rangle = 0 \) for every \( i \in S \). That is, there’s a direction that inliers have a zero projection in \( \alpha \) fraction of the times. Such an eventuality is ruled out if we force \( D \), the distribution of the inliers to be \textit{anti-concentrated}.

**Definition 2.2 (Anti-Concentration).** A \( \mathbb{R}^d \)-valued random variable \( Y \) with mean 0 and covariance \( \Sigma \) is \( \delta \)-anti-concentrated if for all \( v \) satisfying \( v^\top \Sigma v > 0 \), \( \mathbb{P}[\langle Y, v \rangle = 0] < \delta \). A set \( T \subseteq \mathbb{R}^d \) is \( \delta \)-anti-concentrated if the uniform distribution on \( T \) is \( \delta \)-anti-concentrated.

The following proposition is now a simple corollary:

**Proposition 2.3 (High Intersection Implies Same Subspace (TV Distance to Parameter Distance)).** Let \( S \) be a sample of size \( n \) from \( \text{Sub}_D(\alpha, \Sigma^*, r) \) for a projection matrix \( \Sigma_* = \Pi_* \) of rank \( r \) such that the inliers \( I \) are \( \alpha \)-anti-concentrated. Let \( T \subseteq S \) be a subset of size \( \alpha \) such that \( \Pi x = x \) for every \( x \in T \) for some projection matrix \( \Pi \) of rank \( r \). Suppose \( |T| \geq \alpha |S| \). Then, \( \Pi = \Pi_* \).

**Proof.** Let \( I - \Pi = \sum_{i=1}^{d-r} v_i v_i^\top \) for an orthonormal set of vectors \( v_i \)'s. Since \( \Pi x = x \) for every \( x \in T \), \( \langle x, v_i \rangle = 0 \) for every \( x \in T \). Thus, \( \mathbb{P}_{x \sim I}[\langle x, v_i \rangle = 0] \geq |T \cap I|/|I| \geq \alpha \). Since \( I \) is \( \alpha \)-anti-concentrated, this must mean that \( v_i^\top \Pi v_i = 0 \).

Thus, \( \sum_i v_i^\top \Pi^* v_i = \text{tr}(\Pi^* \sum_{i=1}^{d-r} v_i v_i^\top) = \text{tr}(\Pi^* (I - \Pi)) = 0 \). Or \( \text{tr}(\Pi^*) = \text{tr}(\Pi \cdot \Pi^*) \). On the other hand, by Cauchy-Schwarz inequality, \( \text{tr}(\Pi \cdot \Pi^*) \leq \sqrt{\text{tr}(\Pi^2) \text{tr}(\Pi^*^2)} = \text{tr}(\Pi) \) with equality iff \( \Pi = \Pi^* \). Here, we used the facts that \( \Pi = \Pi^2, (\Pi^*)^2 = \Pi^* \) and that \( \text{tr}(\Pi) = \text{tr}(\Pi^*) = r \). Thus, \( \Pi = \Pi^* \). \( \square \)

**Inefficient Algorithm for Anti-Concentrated Distributions.** We can use the lemma above to give an \textit{inefficient} algorithm for list-decodable subspace recovery.

**Lemma 2.4 (Identifiability for Anti-Concentrated inliers).** Let \( S \) be a sample drawn according to \( \text{Sub}_D(\alpha, \Sigma^*, r) \) such that the inliers \( I \) are \( \delta \)-anti-concentrated for \( \delta \leq \alpha \). Then, there is an (inefficient) randomized algorithm that finds a list \( L \) of projectors of rank \( r \) of size \( 20/(\alpha - \delta) \) such that \( \Pi_* \in L \) with probability at least 0.99.

**Proof.** Let \( \mu \) be any maximally uniform distribution over soluble subset-projection pairs \((w, \Pi)\) where \( w \) indicates a set \( S \) of size at least \( \alpha n \). For \( k = 20/(\alpha - \delta) \), let \((S_1, \Pi_1), (S_2, \Pi_2), \ldots, (S_k, \Pi_k)\) be i.i.d. samples from \( \mu \). Output \( \Pi_1, \Pi_2, \ldots, \Pi_k \). To finish the proof, we will show that there is an \( i \) such that \( |S_i \cap I| \geq \frac{\alpha + \delta}{2} |I| > \delta |I| \). Then, we can then apply Proposition 2.3 to conclude that \( \Pi_i = \Sigma_* \).

By Proposition 2.1, \( \mathbb{E}_{S \sim \mu} |S \cap I| \geq \alpha |I| \). Thus, by averaging, \( \mathbb{P}_{S \sim \mu}[|S \cap I| \geq \frac{\alpha + \delta}{2} |I|] \geq \frac{\alpha}{2} |I| \). Thus, the probability that at least one of \( S_1, S_2, \ldots, S_k \) satisfy \( |S_i \cap I| \geq \frac{\alpha + \delta}{2} |I| \) is at least \( 1 - (1 - \frac{\alpha}{2})^k \geq 0.99 \). \( \square \)
2.2 Efficient Algorithm

Our key technical contributions are in making the above inefficient algorithm into an efficient algorithm using the sum-of-squares method. As in prior works, it is natural at this point to consider the algorithm that finds a pseudo-distribution minimizing $\sum_{i \leq n} w_i^2$ and satisfying $A_{w, \Pi}$. This is indeed our starting point.

A precise discussion of pseudo-distributions and sum-of-squares proofs appears in Section 3. At this point, we can simply think of pseudo-distributions as objects similar to the distribution $\mu$ that appeared above for all "properties" that have a low-degree sum-of-squares proofs. Sum-of-squares proofs are a system of reasoning about polynomial inequalities under polynomial inequality constraints. It turns out that the analog of Proposition 2.1 can be proven easily even for pseudo-distributions.

In the following we point out three novel technical contributions that go into making the inefficient algorithm discussed above into an efficient one.

Unconstrained Formalization of Certifiable Anti-Concentration The key technical step is to find a sum-of-squares proof of the "high-intersection implies same subspace" property. This is a bit tricky because it relies on the anti-concentration property of $D$ which does not have natural formalization as a polynomial inequality. Thankfully, recent works [KKK19, RY20] formalized this property within the SoS system in slightly different ways.

Our proofs are more attuned to the formalization in [KKK19]. But for technical reasons the precise formulation proposed in [KKK19] is not directly useful for us. Briefly and somewhat imprecisely put, anti-concentration formalizations posit that there be a low-degree SoS proof (in the variable $v$) for polynomial inequalities of the form $\mathbb{E}_D p^2((x, v)) \leq \delta$ for a univariate polynomial $p$ that approximates a Dirac Delta function at 0. In the prior works, this requirement was formulated in a constrained manner ($\|v\|^2_2 \leq 1$ implies $\mathbb{E}_D p^2((x, v)) \leq \delta$). For the application to subspace recovery, natural arguments require unconstrained versions of the above inequality (i.e. that hold without the norm bound constraint on $v$). Definition 5.1 formulates this condition precisely. One can then modify the constructions of polynomials used in [KKK19] and show that this notion of anti-concentration holds for natural distribution families such as Gaussians.

Spectral Bound on Subsamples Given our modified formalization of anti-concentration, we give a sum-of-squares proof of the analog of Proposition 2.3. This statement (see Lemma 4.5) is a key technical contribution of our work and we expect will find applications in future works.

It can be seen as a SoS version of results that relate total variation distance (this corresponds to the $1 - \alpha$ where $\alpha$ is the normalized intersection size) between two certifiably anti-concentrated distributions to the Frobenius norm distance between their covariances.

Exponential Error Reduction and Large List Rounding The proof of Lemma 4.5 involves a new technical powering step that allows exponential error reduction. This step allows exponentially reducing the error guarantee of list-decoding at the cost of blowing up the list-size by applying
a natural extension of the rounding “by votes” method introduced in [KKK19]. Our powering technique is quite general and will likely find new uses in list-decodable estimation.

**Pruning Lists** In order to get optimal list size bounds, the last step in our algorithm introduces a "pruning method" on the list obtained by rounding pseudo-distributions. It involves a simple test based on new fresh sample that uses \( O(1/\alpha) \) additional fresh samples, say \( x_1, x_2, \ldots, x_q \) and selects a member \( \Pi \) of the large list such that \( \|\Pi x\|^2_2 \) is a large enough fraction of \( \|x\|^2_2 \).

### 3 Preliminaries

Throughout this paper, for a vector \( v \), we use \( \|v\|_2 \) to denote the Euclidean norm of \( v \). For a \( n \times m \) matrix \( M \), we use \( \|M\|_2 = \max_{\|x\|_2=1} \|Mx\|_2 \) to denote the spectral norm of \( M \) and \( \|M\|_F = \sqrt{\sum_{i,j} M_{i,j}^2} \) to denote the Frobenius norm of \( M \). For symmetric matrices we use \( \geq \) to denote the PSD/Loewner ordering over eigenvalues of \( M \). For a \( n \times n \), rank-\( r \) symmetric matrix \( M \), we use \( U \Lambda U^T \) to denote the Eigenvalue Decomposition, where \( U \) is a \( n \times r \) matrix with orthonormal columns and \( \Lambda \) is a \( r \times r \) diagonal matrix denoting the eigenvalues. We use \( M^\dagger = U \Lambda^\dagger U^T \) to denote the Moore-Penrose Pseudoinverse, where \( \Lambda^\dagger \) inverts the non-zero eigenvalues of \( M \). If \( M \geq 0 \), we use \( M^{1/2} = U \Lambda^{1/2} U^T \) to denote taking the square-root of the non-zero eigenvalues. We use \( \Pi = UU^T \) to denote the Projection matrix corresponding to the column/row span of \( M \). Since \( \Pi = \Pi^2 \), the pseudo-inverse of \( \Pi \) itself, i.e. \( \Pi^\dagger = \Pi \).

In the following, we define pseudo-distributions and sum-of-squares proofs. Detailed exposition of the sum-of-squares method and its usage in average-case algorithm design can be found in [FKP19] and the lecture notes [BS16].

#### 3.1 Pseudo-distributions

Let \( x = (x_1, x_2, \ldots, x_n) \) be a tuple of \( n \) indeterminates and let \( \mathbb{R}[x] \) be the set of polynomials with real coefficients and indeterminates \( x_1, \ldots, x_n \). We say that a polynomial \( p \in \mathbb{R}[x] \) is a sum-of-squares (sos) if there are polynomials \( q_1, \ldots, q_r \) such that \( p = q_1^2 + \cdots + q_r^2 \).

Pseudo-distributions are generalizations of probability distributions. We can represent a discrete (i.e., finitely supported) probability distribution over \( \mathbb{R}^n \) by its probability mass function \( D : \mathbb{R}^n \to \mathbb{R} \) such that \( D \geq 0 \) and \( \sum_{x \in \text{supp}(D)} D(x) = 1 \). Similarly, we can describe a pseudo-distribution by its mass function by relaxing the constraint \( D \geq 0 \) to passing certain low-degree non-negativity tests.

Concretely, a level-\( \ell \) pseudo-distribution is a finitely-supported function \( D : \mathbb{R}^n \to \mathbb{R} \) such that \( \sum_x D(x) = 1 \) and \( \sum_x D(x) f(x)^2 \geq 0 \) for every polynomial \( f \) of degree at most \( \ell/2 \). (Here, the summations are over the support of \( D \).) A straightforward polynomial-interpolation argument shows that every level-infinite pseudo-distribution satisfies \( D \geq 0 \) and is thus an actual probability distribution. We define the pseudo-expectation of a function \( f \) on \( \mathbb{R}^d \) with respect to a pseudo-
distribution $D$, denoted $\mathbb{E}_{D(x)} f(x)$, as
\[
\mathbb{E}_{D(x)} f(x) = \sum_x D(x) f(x) .
\] (3.1)

The degree-$\ell$ moment tensor of a pseudo-distribution $D$ is the tensor $\mathbb{E}_{D(x)}(1, x_1, x_2, \ldots, x_n)^{\otimes \ell}$. In particular, the moment tensor has an entry corresponding to the pseudo-expectation of all monomials of degree at most $\ell$ in $x$. The set of all degree-$\ell$ moment tensors of probability distributions is a convex set. Similarly, the set of all degree-$\ell$ moment tensors of degree $d$ pseudo-distributions is also convex. Unlike moments of distributions, there’s an efficient separation oracle for moment tensors of pseudo-distributions.

**Fact 3.1 ([Sho87, Par00, Nes00, Las01]).** For any $n, \ell \in \mathbb{N}$, the following set has a $n^{O(\ell)}$-time weak separation oracle (in the sense of [GLS81]):
\[
\left\{ \mathbb{E}_{D(x)}(1, x_1, x_2, \ldots, x_n)^{\otimes d} \mid \text{degree-$d$ pseudo-distribution $D$ over } \mathbb{R}^n \right\} .
\] (3.2)

This fact, together with the equivalence of weak separation and optimization [GLS81] allows us to efficiently optimize over pseudo-distributions (approximately)—this algorithm is referred to as the sum-of-squares algorithm. The level-$\ell$ sum-of-squares algorithm optimizes over the space of all level-$\ell$ pseudo-distributions that satisfy a given set of polynomial constraints (defined below).

**Definition 3.2** (Constrained pseudo-distributions). Let $D$ be a level-$\ell$ pseudo-distribution over $\mathbb{R}^n$. Let $\mathcal{A} = \{f_1 \geq 0, f_2 \geq 0, \ldots, f_m \geq 0\}$ be a system of $m$ polynomial inequality constraints. We say that $D$ satisfies the system of constraints $\mathcal{A}$ at degree $r$, denoted $D \models_r \mathcal{A}$, if for every $S \subseteq [m]$ and every sum-of-squares polynomial $h$ with $\deg h + \sum_{i \in S} \max\{\deg f_i, r\}$, $\mathbb{E}_{D} h \cdot \prod_{i \in S} f_i \geq 0$.

We write $D \models \mathcal{A}$ (without specifying the degree) if $D \models_r \mathcal{A}$ holds. Furthermore, we say that $D \models \mathcal{A}$ holds approximately if the above inequalities are satisfied up to an error of $2^{-n^\epsilon} \cdot \|h\| \cdot \|\prod_{i \in S} f_i\|$, where $\|\cdot\|$ denotes the Euclidean norm\(^4\) of the coefficients of a polynomial in the monomial basis.

We remark that if $D$ is an actual (discrete) probability distribution, then we have $D \models \mathcal{A}$ if and only if $D$ is supported on solutions to the constraints $\mathcal{A}$. We say that a system $\mathcal{A}$ of polynomial constraints is explicitly bounded if it contains a constraint of the form $\{\|x\|^2 \leq M\}$. The following fact is a consequence of Fact 3.1 and [GLS81].

**Fact 3.3** (Efficient Optimization over Pseudo-distributions). There exists an $(n + m)^{O(\ell)}$-time algorithm that, given any explicitly bounded and satisfiable system\(^5\) $\mathcal{A}$ of $m$ polynomial constraints in $n$ variables, outputs a level-$\ell$ pseudo-distribution that satisfies $\mathcal{A}$ approximately.

### 3.2 Sum-of-squares proofs

Let $f_1, f_2, \ldots, f_r$ and $g$ be multivariate polynomials in $x$. A sum-of-squares proof that the constraints $\{f_1 \geq 0, \ldots, f_m \geq 0\}$ imply the constraint $\{g \geq 0\}$ consists of polynomials $(p_S)_{S \subseteq [m]}$ such that
\[
g = \sum_{S \subseteq [m]} p_S \cdot \prod_{i \in S} f_i .
\] (3.3)

\(^4\)The choice of norm is not important here because the factor $2^{-n^\epsilon}$ swamps the effects of choosing another norm.

\(^5\)Here, we assume that the bitcomplexity of the constraints in $\mathcal{A}$ is $(n + m)^{O(1)}$. 
We say that this proof has degree \( \ell \) if for every set \( S \subseteq [m] \), the polynomial \( p_\ell \Pi_{i \in S} f_i \) has degree at most \( \ell \). If there is a degree \( \ell \) SoS proof that \( \{ f_i \geq 0 \mid i \leq r \} \) implies \( \{ g \geq 0 \} \), we write:

\[
\{ f_i \geq 0 \mid i \leq r \} \models_{\ell} \{ g \geq 0 \}.
\] (3.4)

For all polynomials \( f, g : \mathbb{R}^n \to \mathbb{R} \) and for all functions \( F : \mathbb{R}^n \to \mathbb{R}^m \), \( G : \mathbb{R}^n \to \mathbb{R}^k \), \( H : \mathbb{R}^p \to \mathbb{R}^n \) such that each of the coordinates of the outputs are polynomials of the inputs, we have the following inference rules.

The first one derives new inequalities by addition/multiplication:

\[
\frac{\mathcal{A} \models_{\ell} \{ f \geq 0, g \geq 0 \}}{\mathcal{A} \models_{\ell} \{ f + g \geq 0 \}}, \quad \frac{\mathcal{A} \models_{\ell} \{ f \geq 0 \}}{\mathcal{A} \models_{\ell+\ell'} \{ f \cdot g \geq 0 \}}.
\] (Addition/Multiplication Rules)

The next one derives new inequalities by transitivity:

\[
\frac{\mathcal{A} \models_{\ell} B, \mathcal{B} \models_{r'} C}{\mathcal{A} \models_{\ell+r'} C}.
\] (Transitivity Rule)

Finally, the last rule derives new inequalities via substitution:

\[
\frac{\{ F \geq 0 \} \models_{\ell} \{ G \geq 0 \}}{\{ F(H) \geq 0 \} \models_{\ell, \deg(H)} \{ G(H) \geq 0 \}}.
\] (Substitution Rule)

Low-degree sum-of-squares proofs are sound and complete if we take low-level pseudo-distributions as models. Concretely, sum-of-squares proofs allow us to deduce properties of pseudo-distributions that satisfy some constraints.

**Fact 3.4 (Soundness).** If \( D \models_{\ell} \mathcal{A} \) for a level-\( \ell \) pseudo-distribution \( D \) and there exists a sum-of-squares proof \( \mathcal{A} \models_{r'} \mathcal{B} \), then \( D \models_{r'+r} \mathcal{B} \).

If the pseudo-distribution \( D \) satisfies \( \mathcal{A} \) only approximately, soundness continues to hold if we require an upper bound on the bit-complexity of the sum-of-squares \( \mathcal{A} \models_{r'} \mathcal{B} \) (number of bits required to write down the proof). In our applications, the bit complexity of all sum of squares proofs will be \( n^{O(\ell)} \) (assuming that all numbers in the input have bit complexity \( n^{O(1)} \)). This bound suffices in order to argue about pseudo-distributions that satisfy polynomial constraints approximately.

The following fact shows that every property of low-level pseudo-distributions can be derived by low-degree sum-of-squares proofs.

**Fact 3.5 (Completeness).** Suppose \( d \geq r' \geq r \) and \( \mathcal{A} \) is a collection of polynomial constraints with degree at most \( r \), and \( \mathcal{A} + \{ \Sigma_{i=1}^n x_i^2 \leq B \} \) for some finite \( B \).

Let \( \{ g \geq 0 \} \) be a polynomial constraint. If every degree-\( d \) pseudo-distribution that satisfies \( D \models_{\ell} \mathcal{A} \) also satisfies \( D \models_{r'} \{ g \geq 0 \} \), then for every \( \varepsilon > 0 \), there is a sum-of-squares proof \( \mathcal{A} \models_{\ell} \{ g \geq -\varepsilon \} \).

We will use the following Cauchy-Schwarz inequality for pseudo-distributions:
Fact 3.6 (Cauchy-Schwarz for Pseudo-distributions). Let $f, g$ be polynomials of degree at most $d$ in indeterminate $x \in \mathbb{R}^d$. Then, for any degree $d$ pseudo-distribution $\tilde{\mu}$, $\tilde{\mathbb{E}}_{\tilde{\mu}}[fg] \leq \sqrt{\tilde{\mathbb{E}}_{\tilde{\mu}}[f^2] \tilde{\mathbb{E}}_{\tilde{\mu}}[g^2]}$.

Fact 3.7 (Hölder’s Inequality for Pseudo-Distributions). Let $f, g$ be polynomials of degree at most $d$ in indeterminate $x \in \mathbb{R}^d$. Fix $t \in \mathbb{N}$. Then, for any degree $dt$ pseudo-distribution $\tilde{\mu}$, $\tilde{\mathbb{E}}_{\tilde{\mu}}[f^{t-1}g] \leq \left(\tilde{\mathbb{E}}_{\tilde{\mu}}[f^t]\right)^{\frac{1}{t}} \left(\tilde{\mathbb{E}}_{\tilde{\mu}}[g^t]\right)^{\frac{1}{t}}$.

The following fact is a simple corollary of the fundamental theorem of algebra:

Fact 3.8. For any univariate degree $d$ polynomial $p(x) \geq 0$ for all $x \in \mathbb{R}$, $\frac{1}{d!} \{ p(x) \geq 0 \}$.

This can be extended to univariate polynomial inequalities over intervals of $\mathbb{R}$.

Fact 3.9 (Fekete and Markov-Lukacs, see [Lau09]). For any univariate degree $d$ polynomial $p(x) \geq 0$ for $x \in [a, b]$, $\{ x \geq a, x \leq b \} \frac{1}{d!} \{ p(x) \geq 0 \}$.

Fact 3.10. Let $A \succeq 0$ be a $d \times d$ matrix. Then,

$$\left\{ \mathbf{v}^\top A \mathbf{v} \succeq 0 \right\}.$$  

Reweightings Pseudo-distributions. The following fact is easy to verify and has been used in several works (see [BKS17] for example).

Fact 3.11 (Reweighting). Let $\tilde{\mu}$ be a pseudo-distribution of degree $k$ satisfying a set of polynomial constraints $\mathcal{A}$ in variable $x$. Let $p$ be a sum-of-squares polynomial of degree $t$ such that $\tilde{\mathbb{E}}[p(x)] \neq 0$. Let $\tilde{\mu}'$ be the pseudo-distribution defined so that for any polynomial $f$, $\tilde{\mathbb{E}}_{\tilde{\mu}'}[f(x)] = \tilde{\mathbb{E}}_{\tilde{\mu}}[f(x)p(x)]/\tilde{\mathbb{E}}_{\tilde{\mu}}[p(x)]$. Then, $\tilde{\mu}'$ is a pseudo-distribution of degree $k - t$ satisfying $\mathcal{A}$.

4 Algorithm

In this section, we describe an efficient algorithm for list-decodable subspace recovery. Let $\mathcal{A}_{w, \Pi}$ be the following system of polynomial inequality constraints in indeterminates $w, \Pi$.

$$\mathcal{A}_{w, \Pi} : \begin{cases} \sum_{i\in[n]} w_i = \alpha n \\ \forall i \in [n]. \quad w_i (\mathbb{I} - \Pi) x_i = 0 \\ \forall i \in [n]. \quad w_i^2 = w_i \\ \Pi^2 = \Pi \\ \text{Tr}(\Pi) = r \end{cases}$$

(4.1)

Our algorithm finds a pseudo-distribution consistent with $\mathcal{A}_{w, \Pi}$. It then uses the large-list rounding algorithm as a first step to get a polynomial (in $d$) size list that contains a subspace that is $\eta$-close in Frobenius norm to the range space of $\Sigma_\cdot$. Finally, we apply a pruning procedure to obtain a $O(1/\alpha)$ size from the large list procedure.
Algorithm 4.1. List-Decodable Subspace Recovery

**Given:** Sample $S = \{x_1, x_2, \ldots, x_n\} = I \cup O$ of size $n$ drawn according to $\text{Sub}_D(\alpha, \Sigma)$ such that the $D$ is $k$-certifiably $(c, \delta)$-anti-concentrated, has mean 0 and the condition number of $\Sigma$, is $\kappa$.

**Operation:**
1. Let $t = \Delta \cdot \left(\frac{\log^{5}(1/\alpha) \log(r \kappa)}{\alpha^2}\right)$ for a large enough constant $\Delta > 0$.
2. Compute a $(t + 2k)$-degree pseudo-distribution $\tilde{\mu}$ satisfying $\mathcal{A}_{w,\Pi}$ that minimizes $\left\| \sum_{i=1}^{n} \tilde{E}[w_i] \right\|_2^2$.
3. Run Large-List Rounding with $\eta = 0.1$ (Algorithm 4.2) to output a $O(1/\alpha^t)$ sized list $L'$.
4. Run pruning (Algorithm 4.3) on $L'$ and output the resulting list $L$.

**Output:** A list $L$ of $O(1/\alpha)$ projection matrices containing a $\tilde{\Pi} \in L$ satisfying $\|\tilde{\Pi} - \Pi_*\|_F^2 \leq \tilde{O}(\kappa^4 \log(r)/\alpha^2)$.

Algorithm 4.2. Large List Rounding

**Given:** A pseudo-distribution $\tilde{\mu}$ of degree $t + 2k$ satisfying $\mathcal{A}_{w,\Pi}$ and minimizing $\left\| \sum_{i \in n} \tilde{E}[w_i] \right\|_2^2$ such that $t = \Delta \cdot \left(\frac{\log^{5}(1/\alpha) \log(r \kappa/\eta)}{\alpha^2}\right)$, for a large constant $\Delta$, accuracy parameter $\eta > 0$.

**Operation:** Repeat $\ell = O(1/\alpha^t)$ times:
1. Let $S \subset [n]$ such that $|S| = an$. Draw $S$ with probability proportional to $\binom{n}{S} \tilde{E}_{\tilde{\mu}}[w_S]$.
2. Let $\tilde{\Pi} = \frac{\tilde{E}_{\tilde{\mu}}[w_S]}{\tilde{E}_{\tilde{\mu}}[w_S]}$ be the corresponding matrix. Compute the Eigenvalue Decomposition of $\tilde{\Pi} = \tilde{U} \tilde{\Lambda} \tilde{U}^\top$ and let $\hat{\Pi} = \tilde{U}_r \tilde{U}_r^\top$, where $\tilde{U}_r$ are the eigenvectors corresponding to the top-$r$ eigenvalues of $\tilde{\Pi}$.
3. Add $\hat{\Pi}$ to the list $L'$.

**Output:** A list $L' \subseteq \mathbb{R}^d$ of size $O(1/\alpha^t)$ containing a Projection matrix $\hat{\Pi} \in L'$ satisfying $\|\hat{\Pi} - \Pi_*\|_F^2 < \eta$. 

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Algorithm 4.3. Pruning Lists

**Given:** A list $L'$ of $d \times d$ projection matrices, a threshold $\tau = \tilde{O}(\kappa^4 \log(r)/\alpha^2)$, $O(1/\alpha)$ fresh samples $S$, drawn according to $\text{Sub}_D(\alpha, \Sigma_i)$.

**Operation:** For $i = 1, 2, \ldots, |S|$:

1. Compute the subset $L_i \subseteq L$ of matrices $\hat{\Pi}$ such that $\| (\mathbb{I} - \hat{\Pi}) x_i \|_2^2 \leq \tau$.
2. If $L_i'$ is non-empty, pick an arbitrary matrix $\hat{\Pi}$ from this set and add it to $L$.

**Output:** A $L \subseteq \mathbb{R}^d$ of size $O(1/\alpha)$ such that there exists a Projection matrix $\hat{\Pi} \in L$ satisfying $\| \hat{\Pi} - \Pi \|_F^2 \leq \tau$.

4.1 Analysis of Algorithm 4.1.

The following theorem captures the guarantees we prove on Algorithm 4.1.

**Theorem 4.4** (List-Decoradable Subspace Recovery, restated). Let $\text{Sub}_D(\alpha, \Sigma_i)$ be such that $\Sigma_\alpha$ has rank $r$ and condition number $\kappa$, and $D$ is $\mathcal{N}(0, \Sigma_i)$. Then, Algorithm 4.1 takes as input $n = n_0 \geq (d \log(d)/\alpha^2)^{\tilde{O}(1/\alpha^2)}$ samples from $\text{Sub}_D(\alpha, \Sigma_i)$ and outputs a list $L$ of $O(1/\alpha)$ projection matrices such that with probability at least 0.99 over the draw of the sample and the randomness of the algorithm, there is a $\hat{\Pi} \in L$ satisfying $\|\hat{\Pi} - \Pi\|_F^2 \leq \tilde{O}(\kappa^4 \log(r)/\alpha^2)$. Further, Algorithm 4.1 has time complexity at most $n^{\tilde{O}(\log(r\kappa)/\alpha^4)}$.

Our proof of Theorem 4.4 is based on the following four pieces. The key technical piece is the following consequence of the constraint system $\mathcal{A}_{w, \Pi}$ in the low-degree SoS proof system.

**Lemma 4.5.** Given $\delta > 0$ and any $t \in \mathbb{N}$, and an instance of $\text{Sub}_D(\alpha, \Sigma_i)$, such that the inlier distribution $D$ has mean 0 and is $k$-certifiably $(C, \delta)$-anti-concentrated,

$$
\mathcal{A}_{w, \Pi} \left[ \prod_{i \in I} \left( \frac{1}{|I|} \sum_{i \in I} w_i \right)^t \| \Pi - \Pi_\alpha \|_F^t = \left( \frac{1}{|I|} \sum_{i \in I} w_i \right)^t 2^{k/2} \text{Tr}(M \Pi M)^{k/2} \leq (2r \kappa)^{k/2} \delta \right].
$$

where $\kappa$ is the condition number of $\Sigma_\alpha$ and $\Pi_\alpha$ is the corresponding rank-$r$ Projection matrix.

Next, we show that “high-entropy” pseudo-distributions must place a large enough weight on the inliers. This is similar to the usage of high-entropy pseudo-distributions in [KKK19].

**Lemma 4.6** (Large weight on inliers from high-entropy constraints). Let $\bar{\mu}$ pseudo-distribution of degree $\geq t$ that satisfies $\mathcal{A}_{w, \Pi}$ and minimizes $\| \hat{\Pi}_{\bar{\mu}} \sum_{i \in [n]} w_i \|_2$. Then, $\frac{1}{|I|} \hat{\mathbb{E}} \left[ (\sum_{i \in I} w_i)^t \right] \geq \alpha^t$.

The above two lemmas allow us to argue that our large-list rounding algorithm (Algorithm 4.2) succeeds.
Lemma 4.7 (Large-List Subspace Recovery, Theorem 1.5 restated). Let Sub$_D$(α, Σ$_r$) be such that Σ$_r$ has rank $r$ and condition number $κ$, and $D$ is $k$-certifiably $(c, α/2)$-anti-concentrated. For any $η > 0$, there exists an algorithm that takes input $n ≥ n_0 = (kd \log(d))^{O(k)}$ samples from Sub$_D$(α, Σ$_r$) and outputs a list $L$ of size $O(1/α^k \log(rκ/η))$ of projection matrices such that with probability at least 0.99 over the draw of the sample and the randomness of the algorithm, there is a $\tilde{Π} \in L$ satisfying $\|\tilde{Π} − Π_r\|_F^2 ≤ η$. The algorithm has time complexity at most $n^{O(k^2 \log(rκ/η))}$.

Finally, we show that we can prune the list output by Algorithm 4.2 to a list of size $O(1/α)$ such that it still contains a Projection matrix close to $Π_r$. Formally,

Lemma 4.8 (Pruning Algorithm). Let $L'$ be the list output by Algorithm 4.2. Given $O(1/α)$ fresh samples from Sub$_D$(α, Σ$_r$), Algorithm 4.3 outputs a list $L$ of size $O(1/α)$ such that with probability at least $99/100$, there exists a projection matrix $\tilde{Π} \in L$ satisfying $\|\tilde{Π} − Π_r\|_F^2 ≤ \tilde{O}(κ^4 \log(r)/α^2)$.

Theorem 4.4 follows easily by combining the above claims:

Proof of Theorem 4.4. It follows from Theorem 5.2 that $N(0, Σ_r)$ is $O(\log^5(1/δ)/δ^2)$-certifiably $(c, δ)$-anti-concentrated. Since the inliers are drawn from $N(0, Σ_r)$ it suffices to set $δ = α/2$. By Lemma 5.8 that the uniform distribution on $I$ is also $O(\log^5(1/α)/α^2)$-certifiably $(c, α)$-anti-concentrated if the number of samples are at least $n_0 = (d \log(d)/α^2)^{O(1/α^2)}$. Since the hypothesis of Lemma 4.7 is now satisfies with $k = \log^5(1/α)/α^2$ and $η = 0.1$, Algorithm 4.2 runs in time $n^{\tilde{O}(\log(r)/α^2)}$ and outputs a list $L'$ of size $(1/α)^{O(\log(r)/α^2)}$ such that with probability at least $99/100$, it contains a projector $\tilde{Π}$ satisfying $\|\tilde{Π} − Π_r\|_F^2 ≤ 0.1$. Recall, $Π_r$ is the projector corresponding to $Σ_r$.

Since we now have a list satisfying the hypothesis for Lemma 4.8 and access to $O(1/α)$ fresh samples we can conclude that Algorithm 4.3 outputs a list of size $O(1/α)$ which contains a projection matrix $\tilde{Π}$ satisfying $\|\tilde{Π} − Π_r\|_F^2 ≤ \tilde{O}(κ^4 \log(r)/α^2)$, as desired. The overall running time is dominated by Algorithm 4.2, which completes the proof.

4.2 Analyzing $A_w,Π$: Proof of Lemma 4.5

We first show that covariance of all large enough subsamples of certifiably anti-concentrated samples have lower-bounded eigenvalues. Recall, for a PSD matrix $Σ_r$, $UU^T$ denotes the Eigenvalue Decomposition and $Π_r = UU^T$ denotes the corresponding rank-$r$ Projection matrix.

Lemma 4.9 (Covariance of Subsets of Certifiably Anti-Concentrated Distributions). Let $S = \{x_1, x_2, \ldots, x_n\} ⊆ R^d$ be $k$-certifiably $(C, δ)$-anti-concentrated with $\frac{1}{n} \sum_{x \in S} xx^T = Σ$. Then,

\[
\{ w_i^2 = w_i \mid i \} \left[\frac{w_i}{2k} \left\{ \frac{1}{n} \sum_{i=1}^n \|v\|_2^{k-2} \left( w_i (Σ^{1/2}x_i, v)^2 \right) \right\} \right], \tag{4.2}
\]

Proof. Let $p$ be the degree $k$ polynomial provided by Definition 5.1 applied to $S$. Thus, for each $1 ≤ i ≤ n$, we must have:

\[
\frac{1}{2k} \left\{ \|v\|_2^{k-2} (Σ^{1/2}x_i, v)^2 + δ^2 p^2((Σ^{1/2}x_i, v)) ≥ δ^2 \|v\|_2^{k} \right\} .
\]
Observe that
\[
\{ w_i^2 = w_i \} \bigg\vert \frac{w_i}{2} \{ w_i \geq 0 \}.
\]

Using the above along with (Addition/Multiplication Rules) for manipulating SoS proofs, we must have:
\[
\left\{ w_i^2 = w_i \mid \forall i \right\} \bigg\vert \frac{w_i}{2} \left\{ \frac{1}{n} \sum_{i=1}^{n} ||v||_{2}^{k-2} w_i (\Sigma^{t/2} x_i, v)^2 + \delta^2 \frac{1}{n} \sum_{i=1}^{n} w_i p^2 (\Sigma^{t/2} x_i, v) \right\} \geq \delta^2 \frac{1}{n} \sum_{i=1}^{n} w_i ||v||_2^k.
\]

Rearranging yields:
\[
\left\{ w_i^2 = w_i \mid \forall i \right\} \bigg\vert \frac{w_i}{2k} \left\{ \frac{1}{n} \sum_{i=1}^{n} ||v||_{2}^{k-2} w_i (\Sigma^{t/2} x_i, v)^2 + \delta^2 \frac{1}{n} \sum_{i=1}^{n} w_i ||v||_2^k \right\} \geq \delta^2 \frac{1}{n} \sum_{i=1}^{n} w_i ||v||_2^k.
\]

Next, observe that \( \left\{ w_i^2 = w_i \right\} \bigg\vert \frac{w_i}{2} \left\{ (1 - w_i)^2 = (1 - w_i)^2 \geq 0 \right\} \). Thus, \( \left\{ w_i^2 = w_i \right\} \bigg\vert \frac{w_i}{2} \left\{ w_i \leq 1 \right\} \). As a consequence, \( \left\{ w_i^2 = w_i \right\} \bigg\vert \frac{w_i}{2} \left\{ w_i p^2 (\Sigma^{t/2} x_i, v) \leq p^2 (\Sigma^{t/2} x_i, v) \right\} \). Summing up over \( 1 \leq i \leq n \) yields:
\[
\left\{ w_i^2 = w_i \mid \forall i \right\} \bigg\vert \frac{w_i}{2k} \left\{ \frac{1}{n} \sum_{i=1}^{n} w_i p^2 (\Sigma^{t/2} x_i, v) \right\} \leq \frac{1}{n} \sum_{i=1}^{n} p^2 (\Sigma^{t/2} x_i, v) \leq C \delta ||v||_2^k,
\]
where in the final inequality on the RHS above, we used the second condition from Definition 5.1 satisfied by \( S \). Plugging this back in (4.3), we thus have:
\[
\left\{ w_i^2 = w_i \mid \forall i \right\} \bigg\vert \frac{w_i}{2k} \left\{ \frac{1}{n} \sum_{i=1}^{n} w_i (\Sigma^{t/2} x_i, v)^2 \right\} \geq \delta^2 \left( \frac{1}{n} \sum_{i=1}^{n} w_i - C \delta \right) ||v||_2^k,
\]
as desired.

\[\square\]

**Lemma 4.10** (Technical SoS fact about Powering). For indeterminates \( a, b, Z \) and any \( t \in \mathbb{N} \),
\[
\left\{ a \geq 0, b \geq 0, (a - b)Z \leq 0 \right\} \bigg\vert \frac{a^t}{t} \left\{ (a^t - b^t)Z \leq 0 \right\}
\]
\[
(4.5)
\]

**Proof.** We have:
\[
\left\{ a \geq 0, b \geq 0 \right\} \bigg\vert \frac{a^t}{t} \left\{ \sum_{i=0}^{t-1} a^{t-i} b^i \geq 0 \right\}.
\]

Using the above identity with (Addition/Multiplication Rules) yields:
\[
\left\{ a \geq 0, b \geq 0, (a - b)Z \leq 0 \right\} \bigg\vert \frac{a^t}{t} \left\{ (a - b) \left( \sum_{i=0}^{t-1} a^{t-i} b^i \right) Z \leq 0 \right\}.
\]

Using the identity: \( (a^2 - \delta) \left( \sum_{i=0}^{t-1} a^{t-1-i} b^i \right) = a^t - b^t \), we finally obtain:
\[
\left\{ a \geq 0, b \geq 0, (a - b)Z \leq 0 \right\} \bigg\vert \frac{a^t}{t} \left\{ (a^t - b^t)Z \leq 0 \right\}.
\]

\[\square\]
Proof of Lemma 4.5. We begin by applying Lemma 4.9 to the set $I$. Observe, the uniform distribution on $I$ is $k$-certifiably $(C, \delta)$-anti-concentrated. Thus,

$$\{ w_i^2 = w_i \mid \forall i \} \left\{ \frac{1}{|I|} \sum_{i \in I} w_i (\Sigma_{i}^{t/2} x_i, v)^2 \|v\|_2^{-2} \geq \delta^2 \left( \frac{\sum_{i \in I} w_i}{|I|} - C \delta \right) \|v\|_2^{-k} \right\}$$

(4.6)

Let $M = I - \Pi$. Since $x_i = \Sigma_{i}^{1/2} \Sigma_{i}^{t/2} x_i$, we have the following polynomial identity (in indeterminates $\Pi, v$) for any $i$:

$$\langle \Sigma_{i}^{-t/2} x_i, \Sigma_{i}^{1/2} M v \rangle = \langle M x_i, v \rangle .$$

By using the (Substitution Rule) for manipulating SoS proofs and substituting $v$ with the polynomial $\Sigma_{i}^{1/2} M v$, we thus obtain:

$$\{ \forall i \in [n] w_i^2 = w_i \} \left\{ \frac{w_i}{|I|} \sum_{i \in I} w_i (M x_i, v)^2 \|\Sigma_{i}^{t/2} M v\|^{-2} \geq \delta^2 \left( \frac{\sum_{i \in I} w_i}{|I|} - C \delta \right) \|\Sigma_{i}^{t/2} M v\|_2^{-k} \right\}$$

(4.7)

Next, observe that $\mathcal{A}_{w, \Pi} \left\{ \{w_i M x_i = 0 \forall i\} \right\}$ and thus,

$$\mathcal{A}_{w, \Pi} \left\{ \langle w_i M x_i, v \rangle^2 = w_i \langle M x_i, v \rangle^2 = 0 \forall i \right\} .$$

Combining this with (4.7), we thus have:

$$\mathcal{A}_{w, \Pi} \left\{ 0 \geq \delta^2 \left( \frac{1}{|I|} \sum_{i \in I} w_i - C \delta \right) \|\Sigma_{i}^{t/2} M v\|_2^{-k} \right\}$$

(4.8)

Using (Addition/Multiplication Rules) to multiply throughout by the constant $1/\delta^2$ yields:

$$\mathcal{A}_{w, \Pi} \left\{ 0 \geq \left( \frac{1}{|I|} \sum_{i \in I} w_i - C \delta \right) \|\Sigma_{i}^{t/2} M v\|_2^{-k} \right\}$$

(4.9)

Applying Lemma 4.10 with $a = \frac{1}{|I|} \sum_{i \in I} w_i$, $b = C \delta$ and $Z = \|\Sigma_{i}^{t/2} M v\|_2^{-k}$, we obtain:

$$\mathcal{A}_{w, \Pi} \left\{ 0 \geq \left( \left( \frac{1}{|I|} \sum_{i \in I} w_i \right)^t - (C \delta)^t \right) \|\Sigma_{i}^{t/2} M v\|_2^{-k} \right\}$$

(4.10)

Let $\lambda_{max}$ be the largest eigenvalue of $\Sigma_{+}$. By applying (Addition/Multiplication Rules) and multiplying by $1/\lambda_{max}$ throughout, we can work with $1/\lambda_{max} \Sigma_{+}$, and thus assume that $\lambda_{max} = 1$. Let $\lambda_{min}$ be the smallest non-zero eigenvalue of $\Sigma_{+}$. Then, $\lambda_{min} = \frac{1}{k}$.

Recall, $\Sigma_{+} = U \Lambda U^T$ and $\Pi_{+} = U \Pi U^T$. Then, from the above, $\Sigma_{+} - \lambda_{min} \Pi_{+} \geq 0$ and thus, using (3.10), we have:

$$\mathcal{A}_{w, \Pi} \left\{ \lambda_{min} v^T M \Pi M v \leq v^T M \Sigma_{+} M v \right\} .$$
Using the (Addition/Multiplication Rules) repeatedly we thus obtain:

\[ \frac{1}{k} \left\{ \lambda_{\min}^{k/2} (v^\top M\Pi, Mv)^{k/2} \leq (v^\top M\Sigma, Mv)^{k/2} \right\} \].  \hspace{1cm} (4.11) 

Since \( \lambda_{\max} = 1 \) and \( M^2 = M \), we have:

\[ \mathcal{A}_{w,\Pi} \left[ \frac{v,\Pi}{4} \right] \left\{ v^\top M\Sigma, Mv \leq ||Mv||_2^2 = v^\top Mv = v^\top (I - \Pi)v = ||v||_2^2 - ||\Pi v||_2^2 \leq ||v||_2^2 \right\} \]

Using the (Addition/Multiplication Rules) repeatedly again, we obtain:

\[ \mathcal{A}_{w,\Pi} \left[ \frac{v,\Pi}{4} \right] \left\{ (v^\top M\Sigma, Mv)^{k/2} \leq ||v||_2^k \right\} \]  \hspace{1cm} (4.12)

Using (4.11) and (4.12) with (4.10), we thus have:

\[ \mathcal{A}_{w,\Pi} \left[ \frac{v,\Pi}{2k} \right] \left\{ \left( \frac{1}{|I|} \sum_{i \in I} w_i \right) ^t (v^\top M\Pi, Mv)^{k/2} \leq \frac{1}{\lambda_{\min}^{k/2}} \left( \frac{1}{|I|} \sum_{i \in I} w_i \right) ^t (v^\top M\Sigma, Mv)^{k/2} \leq \frac{1}{\lambda_{\min}^{k/2}} \delta^t ||v||_2^k \right\} \].  \hspace{1cm} (4.13)

Let \( g \sim \mathcal{N}(0, I) \). Then, using the above with the substitution \( v = g \), we have:

\[ \mathcal{A}_{w,\Pi} \left[ \frac{v,\Pi}{2k} \right] \left\{ \left( \frac{1}{|I|} \sum_{i \in I} w_i \right) ^t \text{Tr}(M\Pi, M)^{k/2} = \left( \frac{1}{|I|} \sum_{i \in I} w_i \right) ^t (\mathbb{E} g^\top M\Pi, Mg)^{k/2} \leq \left( \frac{1}{|I|} \sum_{i \in I} w_i \right) ^t \mathbb{E}(g^\top M\Pi, Mg)^{k/2} \leq \frac{1}{\lambda_{\min}^{k/2}} \delta^t ||g||_2^k = \frac{r^{k/2}}{\lambda_{\min}^{k/2}} \delta^t \right\}, \]  \hspace{1cm} (4.14)

where the inequality follows from the SoS Hölder’s inequality.

Next,

\[ \{ \Pi^2 = \Pi \} \left[ \frac{\Pi}{2} \right] \left\{ ||\Pi||_F^2 = \text{Tr}(\Pi^2) = \text{Tr}(\Pi) = r \right\}. \]

And also,

\[ \{ \Pi^2 = \Pi \} \left[ \frac{\Pi}{2} \right] \left\{ M^2 = (I - \Pi)^2 = I - 2\Pi + \Pi^2 = I - \Pi = M \right\}. \]

Thus,

\[ \mathcal{A}_{w,\Pi} \left[ \frac{\Pi}{2} \right] \left\{ ||\Pi - \Pi||_F^2 = ||\Pi||_F^2 + ||\Pi||_F^2 - 2 \text{Tr}(\Pi\Pi) = 2r - 2 \text{Tr}(\Pi\Pi) \right. \]

\[ = 2 \text{Tr}((I - \Pi)\Pi) = 2 \text{Tr}(M\Pi, M) = 2 \text{Tr}(M^2\Pi, M) = 2 \text{Tr}(M\Pi, M) \} \].

\[ \mathcal{A}_{w,\Pi} \left[ \frac{\Pi}{2} \right] \left\{ ||\Pi - \Pi||_F^2 = ||\Pi||_F^2 + ||\Pi||_F^2 - 2 \text{Tr}(\Pi\Pi) = 2r - 2 \text{Tr}(\Pi\Pi) \right. \]

\[ = 2 \text{Tr}((I - \Pi)\Pi) = 2 \text{Tr}(M\Pi, M) = 2 \text{Tr}(M^2\Pi, M) = 2 \text{Tr}(M\Pi, M) \} \].  \hspace{1cm} (4.15)
Using (Addition/Multiplication Rules) and combining with (4.14), we thus obtain:

\[
\mathcal{A}_{\omega,\Pi} \left[ \frac{1}{|I|} \sum_{i \in I} w_i \right]^t \left\| \Pi - \Pi_\omega \right\|_F^k = \left( \frac{1}{|I|} \sum_{i \in I} w_i \right)^t 2^{k/2} \text{Tr}(M \Pi M^k) 2^{k/2} \leq (2r / \lambda_{\text{min}})^{k/2} \delta^t.
\]

Noting that \( \lambda_{\text{min}} = 1/\kappa \) completes the proof.

\[\square\]

4.3 High-Entropy Pseudo-distributions: Proof of Lemma 4.6

Fact 4.11 (Similar to the proof of Lemma 4.3 in [KKK19]). Let \( \tilde{\mu} \) be a pseudo-distribution of degree at least 2 on \( w_1, w_2, \ldots, w_n \) that satisfies \( \{ w_i^2 = w_i \forall i \} \cup \{ \sum_{i=1}^n w_i = \alpha n \} \) and minimizes \( \left\| \sum_{i=1}^n \tilde{\mathbf{E}}[w_i] \right\|_2^2 \). Then, \( \frac{1}{|I|} \sum_{i \in I} \tilde{\mathbf{E}}[w_i] \geq \alpha \).

We defer the proof of this Fact to Appendix A.1.

Proof of Lemma 4.6. From Fact 4.11, we have that \( \frac{1}{|I|} \sum_{i \in I} \tilde{\mathbf{E}}[w_i] \geq \alpha \). Applying Hölder’s inequality for pseudo-distributions with \( f = 1 \) and \( g = \frac{1}{|I|} \sum_{i \in I} w_i \) gives:

\[
\frac{1}{|I|^t} \tilde{\mathbf{E}} \left( \sum_{i \in I} w_i \right)^t \geq \frac{1}{|I|^t} \left( \sum_{i \in I} \tilde{\mathbf{E}} w_i \right)^t \geq \alpha^t.
\]

\[\square\]

4.4 Rounding Pseudo-distributions to a Large List: Proof of Lemma 4.7

In this subsection, we analyze Algorithm 4.2 and show that it returns a list \( \mathcal{L}' \) that contains a projection matrix \( \hat{\Pi} \) close to \( \Pi \). The key step in our proof is the following lemma:

Lemma 4.12. Given \( t \in \mathbb{N} \), and an instance of \( \text{Sub}_{c_2}(\alpha, \Sigma) \) such that \( I \) is \( k \)-certifiably \((C, \delta)\)-anti-concentrated, let \( \tilde{\mu} \) be a degree-(\( 2k + t \)) pseudo-distribution satisfying \( \mathcal{A}_{\omega,\Pi} \) and minimizing \( \left\| \tilde{\mathbf{E}}[w] \right\|_2 \). Then,

\[
\frac{1}{\tilde{\mathbf{E}}[\sum_{i \in I} w_i]^t} \sum_{S \subseteq I, |S| \leq t} \left( \frac{I}{S} \right) \tilde{\mathbf{E}} \left[ w_S \left\| \Pi - \Pi_\omega \right\|_F^k \right] \leq \left( \frac{8\delta}{\alpha} \right)^t (2r \kappa)^{k/2}.
\]

where \( \binom{n}{k} \) is the coefficient of the monomial indexed by \( S \).

Proof. From Lemma 4.5, we have for every \( t, \ell \in \mathbb{N} \),

\[
\mathcal{A}_{\omega,\Pi} \left[ \frac{1}{|I|^t} \left( \sum_{i \in I} w_i \right)^t \left\| \Pi - \Pi_\omega \right\|_F^k \right] \leq \frac{1}{|I|^t} \left( \sum_{i \in I} w_i \right)^t \left\| \Pi - \Pi_\omega \right\|_F^k \leq (2r \kappa)^{k/2} \delta^t.
\]

Since \( \tilde{\mu} \) satisfies \( \mathcal{A}_{\omega,\Pi} \) and has degree at least \( t + k \), taking pseudo-expectation yields:

\[
\tilde{\mathbf{E}} \left[ \frac{1}{|I|^t} \left( \sum_{i \in I} w_i \right)^t \left\| \Pi - \Pi_\omega \right\|_F^k \right] \leq (2r \kappa)^{k/2} \delta^t.
\]
yields: \( \frac{1}{|I'|} \mathbb{E}_\tilde{\mu} \left[ (\sum_{i \in I} w_i)^t \right] \leq \frac{1}{\alpha} \). Multiplying both sides by \( \frac{1}{\mathbb{E}_\tilde{\mu} (\sum_{i \in I} w_i)^t} \), we obtain:

\[
\frac{1}{\mathbb{E}_\tilde{\mu} \left[ (\sum_{i \in I} w_i)^t \right]} \mathbb{E}_\tilde{\mu} \left[ \left( \sum_{i \in I} w_i \right)^t \| \Pi - \Pi_* \|_F^k \right] \leq \left( \frac{8\delta}{\alpha} \right)^t r(2r\kappa)^{k/2}.
\]

For any monomial \( w_s \), let \( w_{s'} \) be its multilinearization. Then, observe that:

\[
\{ w_i^2 = w_i \; \forall i \} = \frac{w}{t} \{ w_S = w_{s'} \}.
\]

Therefore, we have

\[
\mathcal{A}_{w,\Pi} \left[ \frac{w}{t} \right] \left( \left( \sum_{i \in I} w_i \right)^t \| \Pi - \Pi_* \|_F^k = \sum_{S \subseteq I, |S| \leq t} \frac{(I_{\omega})}{I_{\omega}} w_S \| \Pi - \Pi_* \|_F^k \right).
\]

Combining equations 4.16 and 4.17, we have

\[
\frac{1}{\mathbb{E}_\tilde{\mu} \left[ (\sum_{i \in I} w_i)^t \right]} \sum_{S \subseteq I, |S| \leq t} \frac{(I_{\omega})}{I_{\omega}} \mathbb{E}_\tilde{\mu} \left[ w_S \| \Pi - \Pi_* \|_F^k \right] \leq \left( \frac{8\delta}{\alpha} \right)^t r(2r\kappa)^{k/2}.
\]

which concludes the proof.

Next, we show that sampling a subset of size \( t \) indicated by thee \( w \)'s proportional to the marginal pseudo-distribution on this set results in an empirical estimator that is close to \( \Pi_* \) with constant probability.

**Lemma 4.13.** Let \( \tilde{\mu} \) be a pseudo-distribution of degree at least \( t + 2k \) satisfying \(\mathcal{A}_{w,\Pi} \) and minimizing \( \| \mathbb{E}_\tilde{\mu} [w] \|_2 \). Let \( S \subseteq I, |S| \leq t \) be chosen randomly with probability proportional to \( \left( \frac{I_{\omega}}{I_{\omega}} \right) \mathbb{E}_\tilde{\mu} [w_S] \). Let \( \tilde{\mu}_S \) be the pseudo-distribution obtained by reweighting \( \tilde{\mu} \) by the SoS polynomial \( w_S^2 \). Then, with probability at least 9/10 over the draw of \( S, \| \mathbb{E}_{\tilde{\mu}_S} [\Pi] - \Pi_* \|_F^k \leq 10r(2r\kappa)^{k/2}(8\delta)^t \alpha^{-t}. \)

**Proof.** Rewriting the conclusion of Lemma 4.12, we have:

\[
\frac{1}{\mathbb{E}_\tilde{\mu} \left[ (\sum_{i \in I} w_i)^t \right]} \sum_{S \subseteq I, |S| \leq t} \frac{(I_{\omega})}{I_{\omega}} \mathbb{E}_\tilde{\mu} \left[ w_S \| \Pi - \Pi_* \|_F^k \right] \leq \left( \frac{8\delta}{\alpha} \right)^t r(2r\kappa)^{k/2}.
\]

Further, \( \sum_{S \subseteq I, |S| \leq t} \frac{(I_{\omega})}{I_{\omega}} \mathbb{E}_\tilde{\mu} [w_S] = \mathbb{E}_\tilde{\mu} \left( \sum_{i \in I} w_i \right)^t \). Thus, \( \frac{(I_{\omega})}{I_{\omega}} \mathbb{E}_\tilde{\mu} [w_S] \) is a probability distribution, \( \zeta \), over \( S \subseteq I, |S| \leq t \). Thus, we can rewrite (4.19) as simply:

\[
\mathbb{E}_{S \sim \zeta} \left[ \frac{\mathbb{E}_\tilde{\mu} [w_S \| \Pi - \Pi_* \|_F^k]}{\mathbb{E}_\tilde{\mu} [w_S]} \right] \leq \left( \frac{8\delta}{\alpha} \right)^t r(2r\kappa)^{k/2}.
\]
By Markov’s inequality, a $S \sim \zeta$ satisfies

$$\frac{\mathbb{E}[w_S^2 \| \Pi - \Pi_t \|^2_F]}{\mathbb{E}[w_S]} \leq 10r(2r\kappa)^{k/2}(8\delta)^t \alpha^{-t}$$

with probability at least $9/10$. Finally, observe that by Fact 3.11, $\mathbb{E}_\beta \| \Pi - \Pi_t \|^2_F = \frac{\mathbb{E}_\beta[w_S(\Pi - \Pi_t)]^2}{\mathbb{E}[w_S]}$. Thus, with probability at least $9/10$ over the choice of $S \sim \zeta$, $\mathbb{E}_\beta \left[ \| \Pi - \Pi_t \|^k_F \right] \leq 10r(2r\kappa)^{k/2}(8\delta)^t \alpha^{-t}$. By Cauchy-Schwarz inequality applied with $f = 1$ and $g = \| \Pi - \Pi_t \|^k_F$, we have: $\| \mathbb{E}_\beta[(\Pi - \Pi_t)]\|^k_F \leq \mathbb{E}_\beta \left[ \| \Pi - \Pi_t \|^k_F \right]$. Thus, $\| \mathbb{E}_\beta[\Pi] - \Pi_t \|^k_F \leq 10r(2r\kappa)^{k/2}(8\delta)^t \alpha^{-t}$. This completes the proof.

**Proof of Lemma 4.7.** We note that since $\mathcal{D}$ is $k$-certifiably $(c, \delta)$-anti-concentrated, sampling $n_0 = (kd \log(d))^k$ suffices for the uniform distribution over $\mathcal{I}$ to be $k$-certifiably $(c, 2\delta)$-anti-concentrated (this follows from Lemma 5.8). We then observe that by Lemma 4.11,

$$\frac{\mathbb{E}_\beta[(\Sigma_{i \in I} w_i)^2]}{\|I\|^2} = \frac{\mathbb{E}_\beta[(\Sigma_{i \in I} w_i)^2]}{\mathbb{E}_\beta[(\Sigma_{i \in I} w_i)]^2} \geq \alpha^t.$$  

Therefore, with probability at least $9\alpha^t/10$, $w_S \subset I$ and the conclusion of Lemma 4.13 holds. However, the resulting matrix $\overline{\Pi} = \mathbb{E}_\beta[\Pi]$ need not be a Projection matrix.

From Lemma 4.13, we can now conclude $\| \overline{\Pi} - \Pi_t \|^2_F \leq (10r(2r\kappa)^{k/2}(8\delta)^t \alpha^{-t})^{2/k} \leq c r^2 \kappa (\delta/\alpha)^{2t/k}$. Setting $t = \Theta\left(\frac{k \log(kr/\eta')}{\log(\delta/\alpha)}\right)$ in Lemma 4.13 suffices to bound $\| \overline{\Pi} - \Pi_t \|^2_F \leq \eta'$. It follows that for all $i \in [d]$, with probability at least $9\alpha^t/10$,

$$\lambda_i^2(\overline{\Pi}) = \lambda_i^2(\Pi_t) \pm \left(10r(2r\kappa)^{k/2}(8\delta)^t \alpha^{-t}\right) = \lambda_i^2(\Pi_t) \pm \eta'.$$

Since $\Pi_t$ is an actual rank-$r$ Projection matrix, for $i \in [r]$,

$$\lambda_i^2(\overline{\Pi}) \in [1 - \eta', 1 + \eta']$$

and for $i \in [r + 1, d]$, $\lambda_i^2(\overline{\Pi}) \in [-\eta', \eta']$. Further, since $\Sigma_{i=1}^r \lambda_i^2(\overline{\Pi}) \geq (1 - \eta')r$ and trace is exactly $r$, $\|U_t \Lambda_t \|_F^2 = \Sigma_{i=r+1}^n \lambda_i^2(\overline{\Pi}) \leq r \eta'$.

Now recall, $\overline{\Pi} = U_t U_t^\top$ is the corresponding Projection matrix we obtain in Algorithm 4.2, where $U_t$ are the eigenvectors corresponding to the top-$r$ eigenvalues of $\overline{\Pi}$. Therefore,

$$\| \overline{\Pi} - \Pi_t \|^2_F = \| \overline{\Pi} - \Pi_t + \Pi - \Pi_t \|^2_F \leq 2 \left( \| \overline{\Pi} - \Pi_t \|^2_F + \| U_t U_t^\top - U \Lambda U^\top \|^2_F \right) \leq 4 \left( \eta' + \| U_t (\Lambda_t - \Pi_t) U_t^\top \|^2_F + \| U_t \Lambda_t U_t^\top \|^2_F \right) \leq 4 \left( \eta' + \| (\Lambda_t - \Pi_t) \|^2_F + \| \Pi_t \|^2_F + r \eta' \right) \leq 6r \eta'$$

Setting $\eta' = \eta/6r$, we get $t = \Delta\left(\frac{k \log(kr/\eta')}{\log(\delta/\alpha)}\right)$, for a sufficiently large constant $\Delta$. Repeating $O(1/\alpha^t)$ times, with probability $99/100$, the resulting list contains a Projection matrix $\hat{\Pi}$ such that $\| \hat{\Pi} - \Pi_t \|^2_F \leq \eta$. The claim follows by choosing $\delta = \alpha/2$. The running time is dominated by computing a $(t + 2k)$-degree pseudo-distribution which requires $n^{O(k^2 \log(kr/\eta))}$ time.
4.5 Pruning the List: Proof of Lemma 4.8

Fact 4.14 (Concentration of Quadratic Forms of Subgaussians, Theorem 2.1 [HKZ12]). Let \( x \) be a \( \sigma \)-subgaussian random variable on \( \mathbb{R}^d \), i.e., \( \mathbb{E} \exp(\sigma^T(x - \mu)) \leq \exp(||v||^2\sigma^2/2) \) for all \( v \in \mathbb{R}^d \). Then, for any a matrix \( A \) and for any \( t > 0 \), we have

\[
\mathbb{P} \left[ ||Ax||_2^2 - \mathbb{E} ||Ax||_2^2 > t \right] \leq 2 \exp \left( -\frac{t^2}{2||A^TA||_F^2} \right)
\]

Fact 4.15 (Subspace Distance). Let \( \Pi_1, \Pi_2 \) be rank-\( r \) Projection matrices. Then, \( ||(I - \Pi_2)\Pi_1||_F^2 = \frac{1}{2} \||\Pi_1 - \Pi_2||^2_F \).

Proof. Using \( ||M||_F^2 - \text{Tr}(M^TM) \), we have

\[
||\Pi - \Pi_2||_F^2 = \text{Tr}[(\Pi - \Pi_2)^T(\Pi - \Pi_2)\Pi_1] = \text{Tr}[\Pi(\Pi - \Pi_2)(\Pi - \Pi_2)\Pi_1] \\
= \frac{1}{2} \left( \text{Tr}[\Pi_1^2] + \text{Tr}[\Pi_2^2] - 2 \text{Tr}[\Pi_1\Pi_2] \right)
\]

(4.21)

where we repeatedly use \( \Pi_1 = \Pi_1^2, \Pi_2 = \Pi_2^2 \) and the cyclic property of the trace.

Lemma 4.16 (Testing Distinct Subspaces with One Sample). Let \( \Sigma_1 \) be any rank-\( r \) Covariance matrix. Let \( \Pi_1 \) be the corresponding rank-\( r \) Projection matrix and \( \Pi_2 \) be any fixed rank \( r \) Projection matrix. Then, for any \( 0 < \zeta < 1 \),

\[
\mathbb{P}_{x \sim \mathcal{N}(0, \Sigma_1)} \left[ \|(I - \Pi_2)x\|_2^2 \geq \frac{(1 - \zeta)\lambda_{\text{min}}}{2} \||\Pi_1 - \Pi_2||^2_F \right] \geq 1 - \exp \left( -c \min \left( \frac{\zeta^2}{\kappa^4}, \frac{\zeta}{\kappa^2} \right) \left( \||\Pi_1 - \Pi_2||^2_F \right) \right)
\]

for a fixed constant \( c \).

Proof. Observe,

\[
\mathbb{E}_{x \sim \mathcal{N}(0, \Sigma_1)} \left[ \|(I - \Pi_2)x\|_2^2 \right] = \mathbb{E}_{g \sim \mathcal{N}(0, I)} \left[ \left\| \left( (I - \Pi_2)^T\Sigma_1^{-1/2}g \right) \right\|_2^2 \right] \geq \mathbb{E}_{g \sim \mathcal{N}(0, I)} \left[ \lambda_{\text{min}} \left\| (I - \Pi_2)\Pi_1 g \right\|_2^2 \right] \\
= \frac{\lambda_{\text{min}}}{2} \||\Pi_1 - \Pi_2||_F^2
\]

(4.22)

where the last equality follows from Fact 4.15. Similarly, \( \mathbb{E}_{x \sim \mathcal{N}(0, \Sigma_1)} \left[ \|(I - \Pi_2)x\|_2^2 \right] \leq \frac{\lambda_{\text{max}}}{2} \||\Pi_1 - \Pi_2||_F^2 \). Since \((I - \Pi_2)\) is a projector, \( \|(I - \Pi_2)\Sigma_1\|_2 \leq \lambda_{\text{max}} \). Applying Fact 4.14 with \( A = ((I - \Pi_2)\Sigma_1)^T(I - \Pi_2)\Sigma_1, \|A\|_F^2 = \|(I - \Pi_2)\Sigma_1\|_2^2 \leq \lambda_{\text{max}}^2 \|(I - \Pi_2)\Sigma_1\|_2^2, \|A^TA\|_F = \|(I - \Pi_2)^T(I - \Pi_2)\Sigma_1\|_F \leq \lambda_2^2 \cdot \|(I - \Pi_2)\Pi_1\|_F \) and \( t = \zeta \lambda_{\text{min}} \||\Pi_1 - \Pi_2||_F^2 / 2 \):

\[
\mathbb{P} \left[ \left\| (I - \Pi_2)x \right\|_2^2 - \frac{\lambda_{\text{min}}}{2} \||\Pi_1 - \Pi_2||_F^2 > \frac{\zeta \lambda_{\text{min}}}{2} \||\Pi_1 - \Pi_2||_F^2 \right] \leq 2 \exp \left( -c \min \left( \frac{\zeta^2}{\kappa^4}, \frac{\zeta}{\kappa^2} \right) \left( \||\Pi_1 - \Pi_2||_F^2 \right) \right)
\]

(4.23)
Rearranging the terms yields the claim. □

We are now ready to prove Lemma 4.8:

**Proof of Lemma 4.8.** Let \( t = \tilde{O}(1/\alpha^2) \) and let \( \tau = c \kappa^4 t \log(1/\alpha) \). Observe, by Markov, taking 100/\( \alpha \) fresh samples, with probability 99/100, there is at least 1 sample from the inlier set \( I \). For the samples that are not inliers, we have no guarantees on the projector we add to our list \( L \). Let the \( i \)-th iteration of Algorithm 4.3 correspond to \( x_i \sim \mathcal{N}(0, \Sigma_i) \), i.e. \( x_i \in I \). For a fixed projector \( \hat{\Pi} \in L' \) such that \( \|\hat{\Pi} - \Pi_i\|_F^2 = \Omega(\kappa^4 t \log(1/\alpha)) \), it follows from Lemma 4.16, that with probability at least \( 1 - \Omega(1/\alpha^2) \),

\[
\| (I - \Pi) x_i \|_2^2 > \frac{\lambda_{\min}}{4} \| \Pi_1 - \Pi_2 \|_F^2 \geq \lambda_{\min} \kappa^4 t \log(1/\alpha)
\]

Since our list size is at most \( O(1/\alpha^4) \), with probability at least 99/100, simultaneously for all projectors \( \hat{\Pi} \in L', \|\hat{\Pi} - \Pi_i\|_F^2 = \Omega(\kappa^4 t \log(1/\alpha)) \),

\[
\| (I - \hat{\Pi}) x_i \|_2^2 > \frac{\lambda_{\min}}{2} \| \hat{\Pi} - \Pi_i \|_F^2 > \lambda_{\min} \kappa^4 t \log(1/\alpha)
\]

(4.24)

Recall, if \( x \sim \mathcal{N}(0, \Sigma_i) \), \( E_x \sim \mathcal{N}(0, \Sigma_i) \|\| x \|_2^2 \| = \text{Tr}[\Sigma_i] \). By Markov, with probability at least 99/100,

\[
\| x \|_2^2 = O(\text{Tr}[\Sigma_i]).
\]

Dividing out (4.24) by \( \| x \|_2^2 \), with probability at least 99/100,

\[
\| (I - \hat{\Pi}) x_i \|_2^2 > \frac{\lambda_{\min} \kappa^4 t \log(1/\alpha)}{\text{Tr}[\Sigma_i]} = \Omega(\kappa^4 t \log(1/\alpha))
\]

where the last inequality follows from \( \lambda_{\min}/\text{Tr}[\Sigma_i] \geq 1 \). Therefore, the set of projectors in the sub-list \( L'_i \) in Algorithm 4.3 only contains projectors \( \hat{\Pi} \) such that \( \|\hat{\Pi} - \Pi_i\|_F^2 \leq \kappa^4 t \log(1/\alpha) \leq \tau \). By Lemma 4.13, \( L' \) is guaranteed to have a projector \( \hat{\Pi} \) such that \( \|\hat{\Pi} - \Pi_i\|_F^2 \leq 10/\Delta \) for a large constant \( \Delta \). Observe,

\[
\frac{\| (I - \hat{\Pi}) x \|_2^2}{\| x \|_2^2} = \| (I - \hat{\Pi}) x/\| x \|_2 \|_2^2 = \| (I - \hat{\Pi}) \Pi_i g/\| g \|_2 \|_2^2 \leq \| (I - \hat{\Pi}) \Pi_i \|_F^2 = \| \hat{\Pi} - \Pi_i \|_F^2 \leq \tau
\]

Therefore, \( L'_i \) is non-empty. Algorithm 4.3 selects one projector from \( L'_i \) arbitrarily and the claim follows. □

## 5 Certifiable Anti-Concentration

In this section, prove basic facts about certifiable anti-concentration. We start by recalling the definition again.

**Definition 5.1** (Certifiable Anti-Concentration). A zero-mean distribution \( D \) with covariance \( \Sigma \) is \( 2k \)-certifiably \((\delta, C\delta)\)-anti-concentrated if there exists a univariate polynomial \( p \) of degree \( d \) such that:
will rely on the following construction of a low-degree polynomial with certain important properties:

Thus, in particular, $d / 2$.

Theorem 5.2. (Certifiable Anti-Concentration of Gaussians.) Given $0 < \delta < 1 / 2$, there exists $k = O \left( \frac{\log^{2}(1 / \delta)}{\delta} \right)$ such that $N(0, \Sigma)$ is $k$-certifiably $(C, \delta)$-anti-concentrated.

Our proof of Theorem 5.2 will rely on the following construction of a low-degree polynomial with certain important properties:

Lemma 5.3 (Core Indicator for Strictly Sub-Exponential Tails). Given a univariate distribution $D$ with mean $0$ and variance $\sigma \leq 1$ such that

1. Anti-Concentration: for all $\eta > 0$, $P_{X \sim D}[|x| \leq \eta \sigma] \leq c_{1} \eta$,
2. Strictly Sub-Exponential Tail: for all $k_{1} < 2$, $P_{X \sim D}[|x| \geq t \sigma] \leq \exp(-t^{2}/k_{1}) / c_{2}$,

for some fixed $c_{1}, c_{2} > 1$. Then, for any $\delta > 0$, there exists a degree $d = O \left( \frac{\log^{2}(1 / \delta)}{\delta^{2}(1 / \delta)} \right)$ even polynomial $q$ satisfying:

1. $|x| \leq \delta$, $q(x) = 1 \pm \delta$, and,
2. $\sigma^{2} E_{X \sim D}[q^{2}(x)] \leq 10c_{1}c_{2} \delta$.

We will also use the following basic fact about even polynomials.

Lemma 5.4 (Structure of Even Polynomials). For any even univariate polynomial $q(t)$ of degree $d$, $\|v\|_{2}^{2d} q^{2}(\langle x, v \rangle / \|v\|_{2})$ is a polynomial in vector-valued indeterminate $v$ and further,

\[
\frac{\|v\|_{2}^{2d}}{\|v\|_{2}^{2} q^{2}(\langle x, v \rangle / \|v\|_{2})} > 0.
\]

Proof: The conclusion requires us to prove that $\|v\|_{2}^{2d} q^{2}(\langle x, v \rangle / \|v\|_{2})$ is a sum-of-squares polynomial in vector-valued variable $v$. Let $q(t) = \sum_{i \in d} c_{i} t^{i}$. Since $q(t)$ is even,

\[
q(t) = \frac{1}{2}(q(t) + q(-t)) = \frac{1}{2} \left( \sum_{i \in d} c_{i} t^{i} + c_{i}(-t)^{i} \right) = \sum_{1 \leq i \leq d / 2} c_{2i} t^{2i}.
\]

Thus, in particular, $d$ is even and $q(t) = r(t^{2})$ for some polynomial $r$ of degree $d / 2$. Substituting $t = \langle x, v \rangle / \|v\|_{2}$, we have:

$\|v\|_{2}^{2d} q^{2}(\langle x, v \rangle / \|v\|_{2}) = \|v\|_{2}^{2d} \left( \sum_{i \in d / 2} c_{2i} \|v\|_{2}^{2i} \langle x, v \rangle^{2i} \right)^{2} = \left( \sum_{i \leq d / 2} c_{2i} \|v\|_{2}^{2d-2i} \langle x, v \rangle^{2i} \right)^{2}$ which is a sum-of-squares polynomial in $v$. \qed
Now, we are ready to prove that Gaussians are certifiably anti-concentrated under our new definition:

**Proof of Theorem 5.2.** Let \( x \sim \mathcal{N}(0, \Sigma) \). We begin with the following polynomial:

\[
p(v) = \|v\|^2 q(\Sigma^t x, v) / \|v\|^2
\]

where \( q \) is the degree \( d = \Theta\left(\frac{\log(1/\delta)}{\delta^2}\right) \) polynomial from Lemma 5.3. By Fact 5.4, \( p \) is indeed a univariate polynomial in \( v \). We will prove that \( \mathcal{N}(0, \Sigma) \) is \( 2d \)-certifiably \((C, \delta)\)-subgaussian for some absolute constant \( C > 0 \) using the polynomial \( p \).

Consider the polynomial \( g(x) = x^2 + \delta^2 q^2(x) - \delta^2/4 \). If \( |x| > \delta \) then, \( g(x) \geq 3\delta^2/4 \geq 0 \). On the other hand, if \( |x| \leq \delta \), using that \( q^2(x) = (1 + \delta)^2 \geq \frac{1}{4} \) for every \( \delta \leq 1/2 \), \( g(x) \geq 0 \). Thus, \( g \) is a univariate, non-negative polynomial. Using Fact 3.8 we thus obtain:

\[
\left\{ \frac{v}{\|v\|_2} \right\} \{ x^2 + \delta^2 q^2(x) \geq \delta^2/4 \},
\]
or, equivalently, \( x^2 + \delta^2 q^2(x) - \delta^2/4 = s(x) \) for a SoS polynomial \( s \) of degree at most \( 2d \). Since \( q \) is even, the LHS is invariant under the transformation \( x \to -x \). Thus, \( s \) is an even polynomial.

Substituting \( x = \frac{(\Sigma^t x, v)}{\|v\|_2} \), we thus have:

\[
\frac{(\Sigma^t x, v)^2}{\|v\|^2} + \delta^2 q^2 \left( \frac{(\Sigma^t x, v)}{\|v\|_2} \right) - \frac{\delta^2}{2} = \left( \frac{(\Sigma^t x, v)}{\|v\|^2} \right)
\]

Multiplying out by \( \|v\|_2^{2d} \) and using the definition of \( p \) gives us the polynomial (in \( v \)) identity:

\[
\|v\|_2^{2d-2} (\Sigma^t x, v)^2 + \delta^2 p^2 \left( \frac{(\Sigma^t x, v)}{\|v\|_2} \right) - \frac{\delta^2}{4} \|v\|_2^{2d} = \|v\|_2^{2d} s \left( \frac{(\Sigma^t x, v)}{\|v\|_2} \right)
\]

Since \( s \) is an even polynomial, it follows from Fact 5.4, \( \|v\|_2^{2d} s \left( \frac{(\Sigma^t x, v)}{\|v\|_2} \right) \) is a sum-of-squares in \( v \). Thus, we can conclude:

\[
\left\{ \frac{v}{\|v\|_2} \right\} \{ \|v\|_2^{2d-2} (\Sigma^t x, v)^2 + \delta^2 p^2 \left( \frac{(\Sigma^t x, v)}{\|v\|_2} \right) \geq \frac{\delta^2}{4} \|v\|_2^{2d} \}
\]

which completes the proof of the first inequality in Definition 5.1. By rotational invariance of Gaussians, \( \mathbb{E}_{x \sim \mathcal{N}(0, I)} [q(x, v)] \) is just a function of \( \|v\|_2^2 q^2 \). Thus \( \|v\|_2^2 \mathbb{E}_{x \sim \mathcal{N}(0, \Sigma)} [q^2 \left( \frac{(\Sigma^t x, v)}{\|v\|_2} \right)] \) is a polynomial in \( \|v\|_2^2 \). Since \( \Sigma^t x \) has variance 1, it follows from the definition of \( p \) and \( q \) that \( \mathbb{E}_{x \sim \mathcal{D}} [p^2(\Sigma^t x, v)] \leq C \delta \|v\|_2^{2d} \), for \( C = 10c_1c_2 \). Therefore, applying Fact 3.8

\[
\left\{ \|v\|_2^2 \mathbb{E}_{x \sim \mathcal{D}} [p^2(\Sigma^t x, v)] \leq C \delta \|v\|_2^{2d} \}
\]

\( \square \)

The proof above naturally extends to the uniform distribution on the unit sphere.
Theorem 5.5. (Certifiable Anti-Concentration of Gaussians.) Given $0 < \delta \leq 1/2$, there exists $k = O\left(\frac{\log^5(1/\delta)}{\delta}\right)$ such that the uniform distribution on the unit sphere is $k$-certifiably $(C, \delta)$-anti-concentrated.

Next, we observe that our definition of certifiable anti-concentration is invariant under linear transformations:

Lemma 5.6. (Affine Invariance.) Let $x \sim D$ such that $D$ is $k$-certifiably $(C, \delta)$-anti-concentrated distribution. Then, for any $A \in \mathbb{R}^{m \times d}$, the random variable $Ax$ has a $k$-certifiably $(C, \delta)$-anti-concentrated distribution.

In particular, this yields that certifiable-anti-concentration is preserved under taking linear projections of a distribution.

Corollary 5.7. (Closure under taking projections) Let $x \sim D$ such that $D$ is $k$-certifiably $(C, \delta)$-anti-concentrated distribution on $\mathbb{R}^d$. Let $V$ be any subspace of $\mathbb{R}^d$ and let $\Pi_V$ be the associated projection matrix. Then, the random variable $\Pi_V x$ has a $k$-certifiably $(C, \delta)$-anti-concentrated distribution.

Next, we show that anti-concentration is preserved under sampling, i.e. if $D$ is anti-concentrated, then the uniform distribution over $n$ samples from $D$ is also anti-concentrated.

Lemma 5.8. (Certifiable Anti-Concentration under Sampling.) Let $D$ be $k$-certifiably $(c, \delta)$-anti-concentrated Sub-Exponential distribution. Let $S$ be a set of $n = \Omega((kd \log(d))^k)$ i.i.d. samples from $D$. Then, with probability at least $1 - 1/d$, the uniform distribution on $S$ is $k$-certifiably $(2c, \delta)$-anti-concentrated.

Proof. Let $p$ be a degree-$k$ that witnesses anti-concentration of $D$. We show that $p$ also witnesses anti-concentration of the uniform distribution on $S$, denoted by $D'$. It suffices to show that $\mathbb{E}_{x \sim D'}[p^2(\langle \Sigma^{t/2} x, v \rangle)] = \Theta(\mathbb{E}_{x \sim D'}[p^2(\langle \Sigma^{t/2} x, v \rangle)])$. We can represent $p^2(\langle \Sigma^{t/2} x, v \rangle)$ in the monomial basis as $\langle c(\Sigma^{t/2} x), c(\Sigma^{t/2} x) \rangle^T$, where $c(\Sigma^{t/2} x)$ are the coefficients of $p(\langle \Sigma^{t/2} x, v \rangle)$ and $(1, v)_{\Theta d}$ are all monomials of degree at most $d$. Using concentration of polynomials of Sub-exponential random variables, for all $i, j \in [d^k],$

$$\mathbb{P}_{x \sim D'}\left(\mathbb{E}_{x \sim D'}\left[c(\Sigma^{t/2} x)_i c(\Sigma^{t/2} x)_j\right] - \mathbb{E}_{x \sim D}\left[c(\Sigma^{t/2} x)_i c(\Sigma^{t/2} x)_j\right]\right)^2 > \varepsilon^2 \leq \exp\left(-\left(\frac{\varepsilon n}{\mathbb{E}_{x \sim D}[c(\Sigma^{t/2} x)_i c(\Sigma^{t/2} x)_j]^2}\right)^{1/2}\right)$$

Setting $\varepsilon = \mathbb{E}_{x \sim D}[c(\Sigma^{t/2} x)_i c(\Sigma^{t/2} x)_j]$ and union bounding over all $i$ and $j,$

$$\mathbb{P}\left(\sum_{i,j \in [d^k]} \left(\mathbb{E}_{x \sim D'}[c(\Sigma^{t/2} x)_i c(\Sigma^{t/2} x)_j] - \mathbb{E}_{x \sim D}[c(\Sigma^{t/2} x)_i c(\Sigma^{t/2} x)_j]\right)^2 > \frac{\mathbb{E}[c(\Sigma^{t/2} x)_i c(\Sigma^{t/2} x)_j]^2}{4}\right) \leq d^{2k} \exp\left(-\left(\frac{n}{d^{O(k)}}\right)^{1/2}\right)$$

Setting $n = \Omega((kd \log(d))^k)$ suffices to bound the above probability by $1/d.$
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### A Appendix

We begin with showing that a $d$-dimension Gaussian vector that spans an $r \leq d$ subspace is $\delta$-anti-concentrated in the subspace, for any $\delta > 0$.

**Proposition A.1 (Anti-Concentration).** For all $\delta > 0$, $\Pr_{x \sim \mathcal{N}(0, \Sigma)}[|\langle x, v \rangle| \leq \delta \sqrt{v^\top \Sigma v}] \leq \delta$ whenever $v^\top \Sigma v > 0$.

*Proof.* Let $\Sigma$ be a rank-$r$ covariance matrix and $\mathcal{N}(0, \Sigma)$ be the corresponding Gaussian distribution over vectors in $\mathbb{R}^d$. Let $\Pi$ be the corresponding rank-$r$ projection matrix. We first observe that only the subspace of $\mathbb{R}^d$ spanned by $\Sigma$ has non-zero measure. Restricted to this subspace, we show that $x \sim \mathcal{N}(0, \Sigma)$ is $\delta$-anti-concentrated for all $\delta > 0$. Note, this is equivalent to considering vectors of the form $\Pi v$ for any $v \in \mathbb{R}^d$. Recall, the PDF of a multivariate Gaussian denoted by $\mathcal{N}(0, \Sigma)$ is given by

$$p(x, \Sigma) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp \left( -\frac{1}{2} x^\top \Sigma^+ x \right)$$

where $(\Sigma)^+$ inverts the non-zero eigenvalues of $\Sigma$ and $\det^+$ is the pseudo-determinant. Now, we observe that for any non-zero $v \in \mathbb{R}^d$ and $x \sim \mathcal{N}(0, \Sigma)$, $\{(\Pi v, x) = 0\}$ defines a rank-$(k-1)$ subspace. It is well known that the Gaussian measure on a lower dimensional subspace of span($\Sigma$) is 0. Formally,

$$\int_{(\Pi v, x) = 0} dp(x, \Sigma) = 0 \quad (A.1)$$

Therefore, for all $v \in \mathbb{R}^d$, $\Pr_{x \sim \mathcal{N}(0, \Sigma)}[\langle x, \Pi v \rangle = 0] = 0$. For all $v$ in the kernel of $\Sigma$, $\nu^\top \Sigma v = 0$.

For any $v$ such that the quadratic form in non-zero, from stability of Gaussians, it follows that $\langle x, v \rangle \sim \mathcal{N}(0, \nu^\top \Sigma \nu)$. Recall, the PDF of a univariate Gaussian denoted by $\mathcal{N}(0, \nu^\top \Sigma \nu)$ is given by

$$p(x) = \frac{1}{\sqrt{2\pi\nu^\top \Sigma \nu}} \exp \left( -\frac{x^2}{\nu^\top \Sigma \nu} \right)$$

Then,

$$\Pr \left[ |x| \leq \delta \sqrt{\nu^\top \Sigma \nu} \right] = \int_{-\delta \sqrt{\nu^\top \Sigma \nu}}^{\delta \sqrt{\nu^\top \Sigma \nu}} \frac{1}{\sqrt{2\pi\nu^\top \Sigma \nu}} \exp \left( -\frac{x^2}{\nu^\top \Sigma \nu} \right) dx \leq \int_{-\delta \sqrt{\nu^\top \Sigma \nu}}^{\delta \sqrt{\nu^\top \Sigma \nu}} \frac{1}{\sqrt{2\pi\nu^\top \Sigma \nu}} dx \leq \delta$$

\[ \square \]

Using standard concentration arguments, we can derive a robust version of anti-concentration on a set of samples drawn from

**Proposition A.2 (Anti-Concentration of Gaussian Samples).** Fix any $\delta > 0$ and let $\{x_1, x_2, \ldots, x_n\} \sim \mathcal{N}(0, \Sigma)$. Then, whenever $n \geq n_0$ for some $n_0 = \Omega(d/\delta^2)$, with probability at least $1 - 1/e^d$ over the draw of $x_i$s, for every $v$ such that $\nu^\top \Sigma \nu > 0$, $\frac{1}{n} \sum_{i=1}^{n} 1 \left( |\langle x_i, v \rangle| < 2\delta \sqrt{\nu^\top \Sigma \nu} \right) \leq 2\delta$. 30
**Proof.** By Proposition A.1, for each $i \in [n]$, for all $v$, $\mathbb{P}[|\langle x_i, v \rangle| \leq \delta \sqrt{\Sigma_v}] \leq \delta$. Therefore,

$$
\mathbb{E} \left[ \frac{1}{n} \sum_{i \in [n]} 1 \left( |\langle x_i, v \rangle| < \delta \sqrt{\Sigma_v} \right) \right] = \frac{1}{n} \sum_{i \in [n]} \mathbb{P} \left[ |\langle x_i, v \rangle| < \delta \sqrt{\Sigma_v} \right] \leq \delta
$$

By Chernoff, for any $v$,

$$
\mathbb{P} \left[ \frac{1}{n} \sum_{i \in [n]} 1 \left( |\langle x_i, v \rangle| < \delta \sqrt{\Sigma_v} \right) \geq 2 \right] \leq \exp \left( -\frac{4\delta n}{3} \right) \tag{A.2}
$$

Next, we construct a $\delta / \sqrt{d}$ net in $\mathbb{R}^d$, denoted by $\mathcal{T}$, such that for any $v$, there exists $v' \in \mathcal{T}$ in the net and $\|v - v'|_2 \leq \delta / \sqrt{d}$. By standard constructions, $|\mathcal{T}| \leq (\sqrt{d} / \delta)^d$. Then, by setting $n = \Omega(d \log(d / \delta))$, with probability at least $1 - 1/e^d$, for all $v' \in \mathcal{T}$,

$$
\frac{1}{n} \sum_{i \in [n]} 1 \left( |\langle x_i, v' \rangle| < \delta \sqrt{\Sigma_v} \right) \leq 2 \delta
$$

By construction, for all $v \notin \mathcal{T}$, $|\langle x_i, v - v' \rangle| \leq \|x_i\|_2 \delta / \sqrt{d} \leq 2 \delta$ and the claim follows. $\square$

## A.1 Proof of Fact 4.11

For completeness, we provide a proof of Fact 4.11. The proof strategy is similar to the proof of Lemma 4.3 in [KKK19].

**Fact 5.3 (High-Entropy Psuedo-Distribution Restated.)** Let $\tilde{\mu}$ be a pseudo-distribution of degree at least 2 on $w_1, w_2, \ldots, w_n$ that satisfies $\{w_i^2 = w_i \forall i\} \cup \{\Sigma_{i=1}^n w_i = an\}$ and minimizes $\|\Sigma_{i=1}^n \tilde{E}[w_i]\|_2^2$. Then, $\frac{1}{|I|} \sum_{i \in I} \tilde{E}[w_i] \geq \alpha$.

**Proof.** Let $u = \frac{1}{an} \tilde{E}[w]$ be a non-negative vector with entries summing to 1. Let $u_I = \sum_{i \in I} u_i$ denote the fraction of of mass on the inliers and $u_O = 1 - u_I$. Let $\tilde{\mu}$ be the minimal pseudo-distribution. For sake of contradiction, assume $u_I < \alpha$. We can then exhibit a pseudo-distribution $\tilde{\mu}'$ that satisfies $\mathcal{A}$ and $\|\Sigma_{i=1}^n \tilde{E}[w_i]\|_2^2 < \|\Sigma_{i=1}^n \tilde{E}[w_i]\|_2^2$, contradicting minimality. Consider the real distribution $\tilde{\mu}'$ that is supported on the inliers and $\Pi = \Pi_I$. This distribution clearly satisfies $\mathcal{A}_{w,\Pi}$ and thus any convex combination of $\tilde{\mu}'$ and $\tilde{\mu}$ also satisfies $\mathcal{A}_{w,\Pi}$. For some $\lambda > 0$, let $\bar{\mu}_\lambda = \lambda \tilde{\mu}' + (1 - \lambda) \tilde{\mu}$ be the corresponding mixed distribution.

We begin with lower bounding $\|u\|_2^2$ in terms of $u_I$ and $u_O$. It is easy to see that the minimum norm is obtained by spreading the mass $u_I$ uniformly over the inliers and $u_O$ uniformly over the outliers. Therefore,

$$
\|u\|_2^2 \geq \left( \frac{u_I}{an} \right)^2 \cdot \alpha n + \left( \frac{u_O}{(1 - \alpha) n} \right)^2 \cdot (1 - \alpha) n = \frac{1}{an} \left( u_I^2 + \frac{u_O^2 \cdot \alpha}{1 - \alpha} \right)
$$

Now, consider $u_\lambda = \frac{1}{an} \tilde{E}_{\lambda \mu} w$. Then,

$$
\|u_\lambda\|_2^2 = (1 - \lambda)^2 \|u\|_2^2 + \frac{\lambda^2}{an} + 2\lambda(1 - \lambda) \frac{u_I}{an}
$$
Therefore,
\[
\|u_\lambda\|^2 - \|u_\lambda\|^2 \geq \frac{\lambda}{\alpha n} (2 - \lambda) \left( u_I^2 + u_0^2 \cdot \frac{\alpha}{1 - \alpha} \right) - \lambda - 2(1 - \lambda) \frac{u_0}{\alpha n} \\
\geq \frac{\lambda(2 - \lambda)}{\alpha n} \left( u_I^2 + u_0^2 \cdot \frac{\alpha}{1 - \alpha} - u_I \right)
\]
By assumption, \(u_I < \alpha\) and thus
\[
u^2 + (1 - u_I)^2 \cdot \frac{\alpha}{1 - \alpha} - u_I = \frac{(1 - \alpha)u_I(u_I - 1) + \alpha(1 - u_I)^2}{1 - \alpha} = \frac{(1 - u_I)(\alpha(1 - u_I) - (1 - \alpha)u_I)}{1 - \alpha}
\]
\[> 0\]
Therefore, picking \(\lambda\) such that \((A.3)\) is strictly greater than 0 suffices.

### A.2 Proof of Lemma 5.3

In this Subsection, we describe our construction of the core indicator polynomial. Our construction is derived from the polynomial approximation to the sign function in [DGJ+09] with a key difference. We do not require an upper envelope to the sign function, and thus obtain a simpler polynomial, which is even.

**Lemma 5.3 (Core Indicator Restated.)** Given a univariate distribution \(\mathcal{D}\) with mean 0 and variance \(\sigma \leq 1\) such that

1. **Anti-Concentration:** for all \(\eta > 0\), \(\mathbb{P}_{x \sim \mathcal{D}}(|x| \leq \eta \sigma) \leq c_1 \eta\),
2. **Sub-Exponential Tail:** for all \(k < 2\), \(\mathbb{P}_{x \sim \mathcal{D}}(|x| \geq \sigma) \leq \exp(-t^{2/k}/c_2)\),

for some fixed \(c_1, c_2 > 1\). Then, for any \(\delta > 0\), there exists a degree \(d = O\left(\frac{\log^{(4k)/(2-k)}(1/\delta)}{\delta^{(2-k)}}\right)\) even polynomial \(q\) such that for all \(|x| \leq \delta\), \(q(x) = 1 \pm \delta\) and \(\sigma^2 \mathbb{E}_{x \sim \mathcal{D}}[q^2(x)] \leq 10c_1c_2\delta\).

We start with recalling the following basic fact about growth of polynomials.

**Fact A.3.** (Growth of Polynomials [Riv74].) Let \(a(x)\) be a polynomial of degree at most \(d\) such that \(|a(x)| \leq b\) for all \(x \in [-1, 1]\). Then, \(|a(x)| \leq b|2x|^d\) for all \(|x| > 1\).

We first show the existence of a low-degree indicator approximator polynomial that is even. We use an approximation to the sign function from [DGJ+09]:

**Lemma A.4.** (Sign Polynomial.) Let \(a = \Theta(\varepsilon^2/\log(1/\varepsilon))\). There exists a degree-\(O(1/a)\) polynomial \(\ell(x)\) such that:

1. for all \(|x| \in [a, 1]\), \(\ell(x) \in [\text{sign}(x) - \varepsilon^2, \text{sign}(x) + \varepsilon^2]\)
2. for all \(x \in [-a, a]\), \(\ell(x) \in [1 - \varepsilon^2, 1 + \varepsilon^2]\)
3. \(\ell\) is monotonically increasing in \((-\infty, -1] \cup [1, \infty)\)
4. $\ell$ is an odd polynomial.

5. $|\ell(x)| \leq (1 + \varepsilon^2)(2|x|)^d$ for all $|x| > 1$

**Proof.** The first three properties follow from the construction in Theorem 4.5 [DGJ⁺09]. The fourth property follows from observing this polynomial has the form $\ell(x) = x r(x^2)$. From Fact A.3, we can conclude that $|\ell(x)| \leq (1 + \varepsilon^2)(2|x|)^d$ for all $|x| > 1$. \hfill $\Box$

**Lemma A.5. (Indicator Polynomial.)** Given $\delta > 0$ and $L \geq 1$, let $\varepsilon^2 = \delta / L$. Then, there exists a polynomial $q$ of degree $d = O(L \log(L/\delta) / \delta)$ such that $q(0) = 1$ and

1. $q$ is an even polynomial.

2. $q(x) \in [-3\varepsilon^2, 3\varepsilon^2]$ for all $x \in [2\delta, L] \cup [-L, -2\delta]$.

3. $q(x) \in [-1 - \varepsilon^2, 1 + \varepsilon^2]$ for all $x \in [\delta, 2\delta] \cup [-2\delta, -\delta]$.

4. $q(x) \in [1 - 3\varepsilon^2, 1 + 3\varepsilon^2]$ for all $x \in [-\delta, \delta]$.

5. $q(x) < 4(|4x|)^d$ for all $|x| > L$.

**Proof.** Let $\ell$ be the polynomial from Lemma A.4. We then define

$$q(x) = \frac{\ell\left(\frac{x+\delta}{L} + a\right) - \ell\left(\frac{x-\delta}{L} - a\right)}{2\ell(\delta/L + a)}$$

It is easy to see $q(0) = 1$, since $\ell$ is an odd polynomial. Next, we observe that $q$ is an even polynomial:

$$q(-x) = \frac{p\left(\frac{-x+\delta}{L} + a\right) - p\left(\frac{-x-\delta}{L} - a\right)}{2p(\delta/L + a)} = q(x)$$

Now, for all $x \in [\delta + 2al, L]$, $\ell\left(\frac{x+\delta}{L} + a\right) = \text{sign}\left(\frac{x+\delta}{L} + a\right) \pm \varepsilon^2 = 1 \pm \varepsilon^2$ and $\ell\left(\frac{x-\delta}{L} - a\right) = 1 \pm \varepsilon^2$ and thus assuming $\delta > a$, $q(x) = (\pm 4\varepsilon^2)/2(1 \pm \varepsilon^2) = \pm 3\varepsilon^2$. A similar argument holds for $x \in [-L, -\delta - aL]$. Now, we show that $q(x)$ is close to 1 for $x \in [-\delta, \delta]$. Here, $\ell\left(\frac{x+\delta}{L} + a\right) = 1 \pm \varepsilon^2$ and $\ell\left(\frac{x-\delta}{L} - a\right) = -1 \pm \varepsilon^2$. Therefore, $q(x) = \frac{-3\varepsilon^2}{2 \pm \varepsilon^2} = 1 \pm 3\varepsilon^2$. Setting $aL = \delta$ suffices, therefore $q$ has degree at most $O(L \log(L/\delta) / \delta)$. Further, for all $|x| \in [\delta, \delta + aL]$, $q(x) = \pm (1 \pm \varepsilon^2)$. Finally, for $|x| > L$, $q(x) \leq 4(|4x|)^d$. \hfill $\Box$

We can now blackbox the proof of Lemma A.1 from [KKK19] since the aforementioned Lemma constructs an appropriate polynomial to approximate the indicator function. Additionally, the polynomial we obtain is even and suffices for Lemma 5.3.