On discrete values of bilinear forms

A. Iosevich, O. Roche-Newton and M. Rudnev

Abstract. Let $\omega$ be a nondegenerate skew-symmetric bilinear form in the real plane. We prove that for finite a point set $P \subset \mathbb{R}^2 \setminus \{0\}$, the set $T_\omega(P)$ of nonzero values of $\omega$ on $P \times P$, if nonempty, has cardinality $\Omega(N^{96/137})$.

In the special case when $P = A \times A$, where $A$ is a set of at least two reals, we establish the following sum-product type estimates, corresponding to the symmetric and skew-symmetric form $\omega$:

$$|AA + AA| = \Omega(|A|^{19/12}) \quad \text{and} \quad |AA - AA| = \Omega\left(\frac{|A|^{49/32}}{\log^{3/32}|A|}\right).$$

These estimates improve their basic prototypes $\Omega(N^{2/3})$ and $\Omega(|A|^{3/2})$, which readily follow from the Szemerédi-Trotter theorem.

Bibliography: 28 titles.

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§1. Introduction

Let $\omega$ be a nondegenerate symmetric or skew-symmetric bilinear form on $\mathbb{R}^2$. Consider an $N$-element point set $P \subset \mathbb{R}^2 \setminus \{0\}$, that is $|P| = N$. The question we ask is, what is the minimum cardinality of

$$T_\omega(P) = \{\omega(q, q') : q, q' \in P\} \setminus \{0\}. \quad (1.1)$$

One exceptional case is that the set $T_\omega(P)$ can be empty, given that the form $\omega$ is skew-symmetric and $P$ is supported on a single line through the origin. But what if we exclude this special case?

Throughout the paper, the standard symbols $O$ and $\Omega$ (equivalently $\ll$ and $\gg$) as well as $\Theta$ (which is both $O$ and $\Omega$) are used to imply absolute constants in upper and lower bounds, respectively. The symbols $\lesssim$ and $\gtrsim$ work in the same way as $\ll$ and $\gg$ but also subsume logarithmic factors in the asymptotic parameters $N$ and $|A|$.

The following conjecture appears to be a central open question in discrete projective plane geometry.

Conjecture 1.1. $|T_\omega(P)| \gtrsim N$.

In fact, we have no evidence that the logarithmic factors hidden in the $\gtrsim$ symbol should be there. After establishing partial results towards the resolution of this
problem in the main body of the paper, additional discussion of the conjecture and its relation to other important questions of ‘Hard Erdős’ type is presented in §6.

All the sets we are dealing with are assumed to be finite and have at least two elements. For such a set $A \subset \mathbb{R} \setminus \{0\}$ we use the standard notations for its sum set $A + A = \{a_1 + a_2 : a_1, a_2 \in A\}$, the difference set $A - A$, the product set $AA$, the ratio set $A:A$, proceeding in the same vein with notations for sets arising from other algebraic operations on finite sets.

We write $E(A, B)$ for the additive energy of two sets $A, B \subseteq \mathbb{R}$, that is $E(A, B) = |\{a_1 + b_1 = a_2 + b_2 : a_1, a_2 \in A, b_1, b_2 \in B\}|$, and just $E(A)$ instead of $E(A, A)$.

Define $E(P)$ as the number of quadruples $(q, q', r, r')$ satisfying the equality

$$\omega(q, q') = \omega(r, r') \neq 0, \quad q, q', r, r' \in P. \quad (1.2)$$

The bound $O(N^3 \log N)$ on the number $E(P)$ was erroneously claimed in [9] in the case when $\omega$ is the standard dot or cross product. Similar to the construction in the celebrated paper [5] by Guth and Katz, which settled the plane Erdős distinct distances problem, or in fact, its predecessor [6] by Elekes and Sharir, it was claimed at the outset of [9] that a pair $(q, r) \in \mathbb{R}^4$ defines a line in the standard $\mathbb{R}^4$ embedding of the group $SL_2(\mathbb{R})$, and solutions of (1.2) are in one-to-one correspondence with pairwise intersections of lines. The error consisted in overlooking the fact that, unlike in the set-up of [6] and [5] (see also some generalisation in [20]), the above map from $P \times P$ to lines in $SL_2(\mathbb{R})$ is generally noninjective.

There is a degeneracy inherent in equation (1.2). If a quadruple $(q, q', r, r')$ satisfies this equation then so does $(\lambda q, q', \lambda r, r')$ for any nonzero $\lambda \in \mathbb{R}$. This is clearly not the case if one deals with the corresponding equation

$$\|q - q'\| = \|r - r'\| \neq 0, \quad q, q', r, r' \in P, \quad (1.3)$$

for equal pairs of Euclidean distances. This means that a line in question in $SL_2(\mathbb{R})$ is defined not by the pair $(q, r) \in \mathbb{R}^4$, but by $(q : r) \in \mathbb{R}P^3$, that is as homogeneous coordinates in the projective three-space. This means that lines potentially acquire multiplicities, which significantly worsens the estimates coming from the Guth-Katz theorem.

More recently it was shown by the third author [19], that in the special case when the above multiplicity is uniformly bounded as $O(1)$, there are $O(N^3)$ solutions to (1.2) and hence $|T_\omega(P)| \gg N$ over any field $F$, under the constraint $N \ll p$ if $F$ has positive characteristic $p$. More generally, Theorem 13 in [19] claims the bound $|T_\omega(P)| \gg N^{2/3}$ (if $F$ is the real or complex field this bound follows immediately by an application of the Szemerédi-Trotter theorem). The latter bound, in the general $F$ context, has been slightly improved recently in [14], Theorem 4, to $|T_\omega(P)| \gtrsim N^{10/161}$, for skew-symmetric $\omega$ only. The strategy in [14], Theorem 4, is an elaboration on the proof of Theorem 2.1 in this paper—which first appeared as a preprint in December 2015—and presents the original instance of using the
cross-ratio to break the threshold exponent $2/3$ for the number of distinct values of $\omega$, which is the key novelty in this paper.

Theorem 2.1, our main result, focuses on the real case, where the ideas can be made more clear, and the estimates are somewhat stronger than in the general field context, owing partially to the use of the Szemerédi-Trotter theorem but more to the recent state-of-the-art cross-ratio bound by the third author in [18]. We remark that most of the results in this paper are also valid over the complex field. At least, this is true for Theorem 2.1, where the only modification required is for $\gg$ to be weakened to $\gtrsim$, and also Theorem 2.3. We hesitate to say whether or not this is the case with Theorem 2.2; extending it to the complex field would require combining the arguments we use with the approach set up in [10].

§2. Main results

**Theorem 2.1.** Let $P$ be a set of $N$ points in $\mathbb{R}^2 \setminus \{0\}$. Suppose that the nondegenerate bilinear form $\omega$ is skew-symmetric and $P$ is not supported on a single line through the origin. Then

$$|T_\omega(P)| \gg N^{96/137}.$$

Note that the basic lower bound $|T_\omega(P)| \gg N^{2/3}$ already comes from the (sharp, see §6) bound $O(N^{4/3})$ on the number of appearances of a single nonzero value of $\omega$, given by the Szemerédi-Trotter theorem.

In the special case $P = A \times A$ we establish the following estimates.

**Theorem 2.2.** Let $A \subset \mathbb{R} \setminus \{0\}$ be finite. Then

$$|AA + AA| \gg |A|^{5/4}|A : A|^{1/3}. \quad (2.1)$$

In particular,

$$|AA + AA| \gg |A|^{19/12}. \quad (2.2)$$

This result improves on the known bound

$$|AA \pm AA| \gg |A||A : A|^{1/2}, \quad (2.3)$$

which also follows from a simple application of the Szemerédi-Trotter theorem (see Exercise 8.3.3 in [27]), provided that the ratio set $|A : A|$ is not too large.

To prove Theorem 2.2 we take advantage of the recent sum-product estimates, which are state-of-the-art. These results rely crucially on the Szemerédi-Trotter theorem. Our proof builds on techniques introduced in the recent virtuoso elab-
oration of Solymosi’s approach to sum-products by Konyagin and Shkredov [11] and [12]. See [24] for the original construction and [10] for its adaptation to the complex case. We also remark in this context that Balog [1] was the first to obtain lower bounds for $|AA + AA|$ via this particular geometric approach using Solymosi’s geometric construction.

The proof of Theorem 2.2 does not extend to give a lower bound for $|AA - AA|$. Instead, we combine the Szemerédi-Trotter theorem along with the best known results for the sum-product problem for the case when $|A : A|$ is small (which also rely on the Szemerédi-Trotter theorem) to give the following weaker result.
Theorem 2.3. Let \( A \subset \mathbb{R} \setminus \{0\} \) be finite. Then

\[
|AA - AA| \gg \frac{|A|^{49/32}}{\log^{3/32}|A|}.
\]  

(2.4)

Sections 3–5 present proofs of our main results. They are followed by §6, where the problem on the number of distinct values of symmetric or skew-symmetric forms is put into the context of the Erdős-Szemerédi sum-product conjecture [7].

§3. Preliminary results

We start out by reminding the reader about the Szemerédi-Trotter theorem.

Theorem 3.1. Let \( P \) and \( L \) be finite sets of points and lines respectively in the projective plane \( \mathbb{F}P^2 \), where \( \mathbb{F} \) is a real or complex field. Then

\[
I(P, L) := |\{(p, l) \in P \times L : p \in l\}| \ll |P|^{2/3}|L|^{2/3} + |P| + |L|.
\]

It is convenient to record the following corollary.

Corollary 3.1. For arbitrary finite sets \( A, B, C \) and \( D \) of real or complex numbers, the number of solutions to the equation

\[
a - b = cd, \quad (a, b, c, d) \in A \times B \times C \times D,
\]

(3.1)

is \( O((|A| |B| |C| |D|)^{2/3} + |A| |D| + |B| |C|) \).

Proof. Let \( l_{b,c} \) be the line with equation \( y = cx + b \) and let \( L = \{l_{b,c} : (b, c) \in B \times C\} \). Then, with \( P = D \times A \), note that \( I(P, L) \) is equal to the number of solutions to (3.1). An application of the Szemerédi-Trotter theorem completes the proof.

As a generalisation of Theorem 3.1 we can use its weighted version, which we quote next, even though in the application (4.4), which follows, we essentially use Theorem 3.1 in the worst possible case. This is extremely inefficient, however we do not see how to get a better estimate.

The weighted set-up is a pair \((P, L)\) comprising a finite set of points and a finite set of lines in the projective plane \( \mathbb{F}P^2 \), where \( \mathbb{F} \) is \( \mathbb{R} \) or \( \mathbb{C} \). We have a positive real-valued weight function \( w \) on \( P \cup L \), with the supremum norms \( w_P \) and \( w_L \), and \( L_1 \)-norms \( W_P \) and \( W_L \) on the sets \( P \) and \( L \), respectively. For a pair \((p, l) \in P \times L\) we write \( \delta_{pl} \) as the characteristic function of the event that the point \( p \) is incident to a line \( l \). The number of weighted incidences \( I_w \), defined below, is bounded as follows.

Theorem 3.2. The number of weighted incidences satisfies

\[
I_w := \sum_{p \in P, l \in L} w(p)w(l)\delta_{pl} \ll (w_Pw_L)^{1/3}(W_PW_L)^{2/3} + w_PW_L + w_LW_P.
\]

(3.2)

1The original proof of the Szemerédi-Trotter theorem over the reals appeared in [26]. It was extended to the complex field by Tóth in as early as 2003, but the proof has come out in print only recently, [28]. For the special case of real/complex Cartesian product sets see a very short and elegant argument by Solymosi and Tardos [25].
The statement follows after an easy weight rearrangement argument, followed by an application of the Szemerédi-Trotter theorem (see [8], for example).

**Remark 3.1.** Given information on the weight distribution over \( P \cup L \), the first (main) term in the estimate (3.2) of Theorem 3.2 can easily be improved, using dyadic pigeonholing, to

\[
\sum_{p \in P} w^{3/2}(p)^{2/3} \left( \sum_{l \in L} w^{3/2}(l) \right)^{2/3}.
\]

However, in the forthcoming application (4.4) we are forced to deal with the worst possible scenario of having roughly \( W_P/w_P \) points with the maximum weight \( w_p \) each, and the same concerning the lines. If this is the case, the two estimates meet, and, unfortunately, essentially boil down to fixing two out of six variables and applying the nonweighted Theorem 3.1 in (4.4) below.

The other component of the geometric argument in the proof of Theorem 2.1 consists in the following theorem, which deals with distinct cross-ratios, generated by a finite subset of the real projective line \( \mathbb{RP}^1 \). We recall that, given a quadruple of pairwise distinct points \( a, b, c, d \in \mathbb{RP}^1 \), their cross-ratio is defined as

\[
[a, b, c, d] := \frac{(a - b)(c - d)}{(a - c)(b - d)}.
\]

Let \( A \subset \mathbb{RP}^1 \), with \( |A| \geq 4 \), and set

\[ C(A) := \{ [a, b, c, d] : \forall a, b, c, d \in A \}. \]

**Theorem 3.3.** Let \( A \subset \mathbb{RP}^1 \), with \( |A| \geq 4 \). Then

\[ |C(A)| \gg |A|^{24/11}, \]

in the complex case \( |C(A)| \gtrsim |A|^{24/11} \).

Theorem 3.3 was proved in [18] by the third author, improving the previous bound \( |C(A)| \gg |A|^2 \), implicit in the paper by Solymosi and Tardos [25]. The conjectured bound is \( |C(A)| \gtrsim |A|^3 \). There are \( O(|A|^3) \) distinct cross-ratios if \( A \) is a geometric progression, and we are unaware of any examples that would yield fewer distinct cross-ratios.

**§ 4. Proof of Theorem 2.1**

In two dimensions a nontrivial skew-symmetric form \( \omega \) is given by the matrix

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]

up to a multiplier, so, in what follows, we refer to its values as *areas* and write simply \( T(P) \) for the set of its nonzero values, defined by pairs of noncollinear vectors in \( P \).

Let \( P_1 \subseteq P \) contain all those points of \( P \) that are supported on lines passing through the origin and with no more than \( w_0 \geq 1 \) points and let \( P_2 = P \setminus P_1 \). The parameter \( w_0 \) is to be chosen later. Let \( N_1 \) and \( N_2 \) be the number of elements in
P_1 and P_2, respectively. Let T_1 and T_2 stand for T(P_1) and T(P_2). We will obtain two different lower bounds for the cardinalities of the sets T_1 and T_2.

The first of these improves as w_0 gets smaller, whilst the second bound improves as w_0 increases. Balancing the two lower bounds leads to the desired result.

The first lower bound is the following.

**Lemma 4.1.** Assume that P_1 is not contained in a single line through the origin. Then
\[ |T_1| = \Omega(N_1 w_0^{-1/2}). \] (4.1)

**Proof.** The result follows after an application of Theorem 3.2. For each point in p \in P_1 and each t \in T draw the line \{ q \in \mathbb{R}^2 : \omega(p, q) = t \}. We obtain an arrangement of some weighted set L of lines with the total weight \( W_L = |T_1| N_1 \) and maximum weight \( w_0 < N_1/2 \) (or there is nothing to prove). Consider its incidences with the point set P. Applying Theorem 3.2 with \( w_L = w_0, w_P = 1 \) and \( W_P = N_1 \) we obtain
\[ I_w = O(w_0^{1/3} N_1^{4/3} |T_1|^{2/3}). \]

Note that the last two terms in the estimate (3.2) in Theorem 3.2 are dominated by the first. Otherwise the bound (4.1) holds trivially. On the other hand, we have \( I_w = \Omega(N_1^2) \), since \( I_w \) is equal to the number of pairs of points in P_1 that do not lie on a line through the origin. Estimate (4.1) follows.

Our second bound is as follows.

**Lemma 4.2.** Suppose P_2 lies on at least four distinct lines through the origin. Then
\[ |T_2| = \Omega(N_2^{36/77} w_0^{30/77}). \] (4.2)

**Proof.** Consider the following equation
\[ t_1 t_2 = t_3 t_4 - t_5 t_6, \quad t_1, \ldots, t_6 \in T_2. \] (4.3)

The following lemma is central to the argument.

**Lemma 4.3.** Equation (4.3) has \( \Omega(N_2^{24/11} w_0^{20/11}) \) solutions.

Before proving Lemma 4.3, we will show how (4.2) follows directly from it, and thus so does Theorem 2.1. We apply Corollary 3.1 (or equivalently, we could apply Theorem 3.2) to get an upper bound for the number of solutions of (4.3). Indeed, for any set T, the number of solutions to (4.3) is bounded by
\[ \left| \{(t_1, \ldots, t_6) \in T^6 : t_1 t_2 = t_3 t_4 - t_5 t_6 \} \right| \]
\[ = \sum_{t_3, t_5 \in T} \left| \{(t_1, t_2, t_4, t_6) \in T^6 : t_1 t_2 = t_3 t_4 - t_5 t_6 \} \right| \ll |T|^2 |T|^{8/3} = |T|^{14/3}. \] (4.4)

On the other hand, Lemma 4.3 provides a lower bound for the same quantity. Comparing the two yields (4.2).

This suffices to complete the proof of Theorem 2.1. We balance the estimates (4.1) and (4.2) by setting \( w_0 = N^{82/137} \) and observe that one of \( N_1 \) and \( N_2 \) is at
least \(N/2\), and if the corresponding assumption that \(P_1\) is not supported on a single line through the origin or that \(P_2\) is not supported on fewer than four lines through the origin was false, there would be nothing to prove.

**Proof of Lemma 4.3.** Let \((a, b, c, d)\) be a quadruple of points in \(P_2\), lying in four distinct directions from the origin. Let \(t_{ab} = a_1b_2 - a_2b_1\) be the signed area of a triangle \(Oab\), where \(a = (a_1, a_2)\), \(b = (b_1, b_2)\) and similarly for other pairs of points in the quadruple.

We have the identity

\[
t_{ad}t_{cb} = t_{ab}t_{cd} - t_{ac}t_{bd},
\]

which gives a particular instance of equation (4.3).

There are many ways to verify (4.5): we could do it, for example, by direct calculation using coordinates, trigonometry, or using the cross and scalar product notation from the fact that

\[
(a \times d) \cdot (b \times c) = -c \cdot (b \times (a \times d)) = (a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d);
\]

now replace \(b = (b_1, b_2)\) by \(b^\perp = (-b_2, b_1)\), and make the analogous change for \(c\).

Consider a quadruple of oriented areas \((t_{ab}, t_{cd}, t_{ac}, t_{bd})\). We introduce the equivalence relation \((a, b, c, d) \sim (a', b', c', d')\) if all the corresponding four areas \((t_{ab}, t_{cd}, t_{ac}, t_{bd}) = (t_{a'b'}, t_{c'd'}, t_{a'c'}, t_{b'd'})\) are the same. Note that if \((a, b, c, d)\) and \((a', b', c', d')\) come from different equivalence classes, then the solutions they give to (4.5) are distinct when viewed as solutions to (4.3).

Observe that,

\[
(a, b, c, d) \sim (a', b', c', d') \quad \text{only if} \quad \frac{t_{ab}t_{cd}}{t_{ac}t_{bd}} = \frac{t_{a'b'}t_{c'd'}}{t_{a'c'}t_{b'd'}}.
\]

Here we have the cross-ratio

\[
\frac{t_{ab}t_{cd}}{t_{ac}t_{bd}} = [\delta_a, \delta_b, \delta_c, \delta_d]
\]

of the directions \(\delta_a, \delta_b, \delta_c\) and \(\delta_d\) of \(a, b, c\) and \(d\), respectively, from the origin.

To explain this, we express the points in coordinates like \(a = (a_1, a_2)\) and by choosing the coordinate axes, assume that \(a_1, b_1, c_1\) and \(d_1\) are all nonzero. We then have

\[
\frac{t_{ab}t_{cd}}{t_{ac}t_{bd}} = \frac{(a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1)}{(a_1c_2 - a_2c_1)(b_1d_2 - b_2d_1)}.
\]

Now suppose we set \(\delta_a = a_2/a_1\), say, to be the direction of the line through the origin that passes through \(a\). Dividing both numerator and denominator by \(a_1b_1c_1d_1\) gives

\[
\frac{t_{ab}t_{cd}}{t_{ac}t_{bd}} = \frac{(a_1b_2 - a_2b_1)(c_1d_2 - c_2d_1)}{(a_1c_2 - a_2c_1)(b_1d_2 - b_2d_1)} = \frac{(\delta_b - \delta_a)(\delta_d - \delta_c)}{(\delta_c - \delta_a)(\delta_d - \delta_b)} = [\delta_a, \delta_b, \delta_c, \delta_d].
\]

By Theorem 3.3 there are \(\Omega(\|D\|^{24/11})\) distinct cross-ratios (in the real case) where \(D\) is the set of directions from the origin, defined by \(P_2\). If \(|D| < 4\) but the points are not all collinear there is nothing to prove. In addition, fixing a representative quadruple \((\delta_a, \delta_b, \delta_c, \delta_d)\) of directions with a given cross-ratio, choosing any \(a, b, c\)
and $d$ in $P_2$ in the given directions, modulo the mirror symmetry with respect to the origin, yields different solutions of (4.3). Indeed, the right-hand side of (4.5) will remain the same only if $a$ and $d$ are dilated by the same factor, while $b$ and $c$ are contracted by the same factor. But this clearly changes the left-hand side, unless the dilation factor equals ±1. See [14], Lemma 19, for details of the latter statement.

It remains to show that, basically, we can assume that $|D| = |P_2|/w_0$, for if we have fewer directions things get better. The easiest way of doing this is to use induction in $|D|$ (the more technically involved arguments in [14] cannot not help but run two dyadic pigeonholing procedures). Let $c < 1$ be the universal constant implicit in Theorem 3.3. Take, say $c' = c/10000$. The lemma is trivially true, with the hidden constant $c'$, if $|D| = O(1)$, for we can certainly assume that no line through the origin supports $\Omega(N_2)$ points.

Now partition $P_2$ into $P'_2$ and $P''_2$, where $P'_2$ corresponds to points on lines through the origin supporting between $w_0$ and $2w_0$ points and $P''_2$ is the rest of $P_2$. Then if $P''_2$ contains at least three-quarters of $P_2$, we are done by the induction assumption, for $P_2$ alone will yield at least $c'(3|P_2|/4)^{24/11}(2w_0)^{20/11} \geq c' |P_2|^{24/11}w_0^{20/11}$ solutions to (4.3).

Otherwise $P'_2$ contains at least one-quarter of $P_2$. It determines at least $|P_2|/(8w_0)$ directions. Hence there are at least $c(|P_2|/4)^{24/11}/(8w_0)^{24/11}$ distinct cross-ratios, generated by quadruples of pairwise distinct directions. We fix a representative quadruple $(\delta_a, \delta_b, \delta_c, \delta_d)$ of directions through the origin for each value of the cross-ratio. As we mentioned earlier, any point quadruple $(a, b, c, d)$ lying in $P_2$ in the given directions $(\delta_a, \delta_b, \delta_c, \delta_d)$, modulo the simultaneous change of sign of $a, b, c, d$, yields different solutions of (4.3). Since there are at least $w_0$ independent choices of each variable $a, b, c, d$, we can multiply the latter estimate $c(|P_2|/4)^{24/11}/(8w_0)^{24/11}$ by $w_0^4/2$. This gives a lower bound $c(|P_2|^{24/11}w_0^{20/11})/10000 = c' |P_2|^{24/11}w_0^{20/11}$ for the number of solutions of (4.3).

§ 5. Proof of Theorem 2.2

Let $A$, the finite set of reals, have more than one element. Define

$$d(A) = \min_{C \neq \emptyset} \frac{|AC|^2}{|A||C|}.$$  

To prove Theorem 2.2 we will need the following result, which is Corollary 8 in [11].

**Lemma 5.1.** Let $A$ be a finite sets of reals and suppose that $\alpha_1, \alpha_2$ and $\alpha_3$ are nonzero real numbers. Then the number of solutions to the equation

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0,$$

such that $a_1, a_2, a_3 \in A$, is at most

$$C \cdot d^{1/3}(A) \cdot |A|^{5/3},$$

for some absolute constant $C$.

Lemma 5.1 is a consequence of the Szemerédi–Trotter theorem, building on the work of Li and the second author [13] and Schoen and Shkredov [21].
Proof. Without loss of generality, we may assume that $A$ consists of strictly positive real numbers. Following the notation in [11], for a real nonzero $\lambda$, define

$$\mathcal{A}_\lambda := \left\{ (x, y) \in A \times A : \frac{y}{x} = \lambda \right\},$$

and its projection onto the horizontal axis,

$$A_\lambda := \{ x : (x, y) \in \mathcal{A}_\lambda \}.$$

Note that $|A_\lambda| = |A \cap \lambda A|$ and

$$\sum_\lambda |A_\lambda| = |A|^2. \quad (5.1)$$

For each $\lambda \in A : A$, we identify an arbitrary element from $\mathcal{A}_\lambda$, which we label $(a_\lambda, \lambda a_\lambda)$. Then, fixing distinct two slopes $\lambda_1$ and $\lambda_2$ from $A : A$ and following Balog’s observation [1], we note that at least $|A|^2$ distinct elements of $(AA + AA) \times (AA + AA)$ are obtained by summing pairs of vectors from the two lines through the origin with slope $\lambda_1$ and $\lambda_2$. Indeed,

$$A(a_{\lambda_1}, \lambda_1 a_{\lambda_1}) + A(a_{\lambda_2}, \lambda_2 a_{\lambda_2}) \subset (AA + AA) \times (AA + AA),$$

where for $\lambda \in A : A$

$$A(a_\lambda, \lambda a_\lambda) = \{ (aa_\lambda, \lambda aa_\lambda) : a \in A \}.$$

Note that these $|A|^2$ vector sums have slopes between $\lambda_1$ and $\lambda_2$. This is a consequence of Solymosi’s observation [24] that the sum set of $m$ points on one line through the origin and $n$ points on another line through the origin consists of $mn$ points lying in between the two lines. This fact expresses the linear independence of two vectors in the two given directions, combined with the fact that multiplication by positive numbers preserves the order of reals. Hence the assumption that the points lie inside the positive quadrant of the plane.

Following the strategy in [11], we split the family of $|A : A|$ slopes into clusters of $M$ consecutive slopes, where $2 \leq M \leq |A : A|$ is a parameter to be specified later. The idea is to show that each cluster determines many different elements of $(AA + AA) \times (AA + AA)$. Since the slopes of these elements are between the maximum and minimum values in that cluster, we can then sum over all clusters without overcounting.

If a cluster contains exactly $M$ lines, then it is called a full cluster. Note that there are $\lceil |A : A|/M \rceil \geq |A : A|/(2M)$ full clusters, since we place exactly $M$ lines in each cluster, with the possible exception of the last cluster which contains at most $M$ lines.

Let $U$ be a full cluster, with shallowest slope $\lambda_{\text{min}}$ and steepest slope $\lambda_{\text{max}}$. Let $\mu(U)$ denote the number of elements of $(AA + AA) \times (AA + AA)$ which lie between $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$. Then, by the inclusion-exclusion principle

$$\mu(U) \geq |A|^2 \frac{M(M - 1)}{2} - \sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in U_j : \{\lambda_1, \lambda_2\} \neq \{\lambda_3, \lambda_4\}} E(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \quad (5.2)$$
Choosing the integer-valued parameter

\[ E(\lambda_1, \lambda_2, \lambda_3, \lambda_4) := \left| \left[ [A(a_{\lambda_1}, \lambda_1 a_{\lambda_1}) + A(a_{\lambda_2}, \lambda_2 a_{\lambda_2})] \cap [A(a_{\lambda_3}, \lambda_3 a_{\lambda_3}) + A(a_{\lambda_4}, \lambda_4 a_{\lambda_4})] \right] \right|. \]

The next task is to obtain an upper bound for \( E(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) for an arbitrary quadruple \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) which satisfies the aforementioned conditions.

Suppose that

\[ z = (x, y) \in [A(a_{\lambda_1}, \lambda_1 a_{\lambda_1}) + A(a_{\lambda_2}, \lambda_2 a_{\lambda_2})] \cap [A(a_{\lambda_3}, \lambda_3 a_{\lambda_3}) + A(a_{\lambda_4}, \lambda_4 a_{\lambda_4})], \]

that is,

\[
(x, y) = (aa_{\lambda_1}, a_{\lambda_1} a_{\lambda_1}) + (ba_{\lambda_2}, b_{\lambda_2} a_{\lambda_2}) = (ca_{\lambda_3}, c_{\lambda_3} a_{\lambda_3}) + (da_{\lambda_4}, d_{\lambda_4} a_{\lambda_4})
\]

for some \(a, b, c, d \in A\). Then,

\[ x = aa_{\lambda_1} + ba_{\lambda_2} = ca_{\lambda_3} + da_{\lambda_4} \]

and

\[ y = a_{\lambda_1} a_{\lambda_1} + b_{\lambda_2} a_{\lambda_2} = c_{\lambda_3} a_{\lambda_3} + d_{\lambda_4} a_{\lambda_4}. \]

It follows from the conditions on the quadruple \((\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) that at least one of its members differs from the other three. Without loss of generality take \(\lambda_4 \neq \lambda_1, \lambda_2, \lambda_3\). Then

\[ 0 = a_{\lambda_1} a_{\lambda_1} + b_{\lambda_2} a_{\lambda_2} - c_{\lambda_3} a_{\lambda_3} - d_{\lambda_4} a_{\lambda_4} - \lambda_4(aa_{\lambda_1} + ba_{\lambda_2} - ca_{\lambda_3} - da_{\lambda_4}), \]

and thus

\[ 0 = aa_{\lambda_1}(\lambda_1 - \lambda_4) + ba_{\lambda_2}(\lambda_2 - \lambda_4) + ca_{\lambda_3}(\lambda_4 - \lambda_3). \tag{5.3} \]

Note that the values of \(a_{\lambda_1}(\lambda_1 - \lambda_4)\), \(a_{\lambda_2}(\lambda_2 - \lambda_4)\) and \(a_{\lambda_3}(\lambda_4 - \lambda_3)\) are all nonzero. We have shown that each contribution to \(E(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\) determines a solution to (5.3). Furthermore, the solution \((a, b, c)\) to (5.3) that we obtain via this deduction is unique. That is, if we start out with a different element

\[ z \in [A(a_{\lambda_1}, \lambda_1 a_{\lambda_1}) + A(a_{\lambda_2}, \lambda_2 a_{\lambda_2})] \cap [A(a_{\lambda_3}, \lambda_3 a_{\lambda_3}) + A(a_{\lambda_4}, \lambda_4 a_{\lambda_4})], \]

we obtain a different solution to (5.3). It therefore follows from an application of Lemma 5.1 that

\[ E(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \leq C \cdot d^{1/3}(A)|A|^{5/3}, \]

where \(C\) is an absolute constant. Therefore,

\[ \mu(U) \geq \frac{M(M-1)}{2} |A|^2 - M^4Cd^{1/3}(A)|A|^{5/3} \geq \frac{M^2}{4} |A|^2 - M^4Cd^{1/3}(A)|A|^{5/3}. \tag{5.4} \]

Choosing the integer-valued parameter

\[ M := \left\lfloor \frac{|A|^{1/6}}{\sqrt{8Cd^{1/6}(A)}} \right\rfloor, \tag{5.5} \]

yields

\[ \mu(U) \geq \frac{M^2}{8} |A|^2. \tag{5.6} \]
Recall that, in fact, we need $2 \leq M \leq |A : A|$. It is easy to check that the upper bound for $M$ is satisfied. Further, $M \leq 2$ implies that $d(A) \gg |A|$, in particular $|A : A| \gg |A|^{3/2}$. Then the basic claim (2.3) results in a stronger statement than Theorem 2.2 and we are done.

Thus we proceed under the assumption that $2 \leq M \leq |A : A|$. Summing over the full clusters, of which there are at least $|A : A|/2M$, yields

$$|AA + AA|^2 \geq \frac{|A : A|}{2M} \frac{M^2}{8} |A|^2 \gg \frac{|A : A| |A|^{13/6}}{d^{1/6}(A)}.$$  \hspace{1cm} (5.7)

To complete the proof, we need a suitable upper bound for $d(A)$. We simply use the trivial bound $d(A) \leq |A : A|^2/|A|^2$. Combining this inequality with (5.7) and rearranging, we get

$$|AA + AA| \gg |A : A|^{1/3} |A|^{5/4},$$

which completes the proof of (2.1). To prove (2.2), we simply apply the trivial bound $|A : A| \geq |A|$.

**Proof of Theorem 2.3.** It was established by Shkredov in [23], Theorem 12, that

$$|A : A|^{13} |A - A|^6 \gg \frac{|A|^{23}}{\log^3 |A|}. \hspace{1cm} (5.8)$$

The claim of Theorem 2.3 follows if we plug this into the bound (2.3), rearrange and use the trivial bound $|A - A| \leq |AA - AA|.$

§ 6. Appendix

In this section we give some heuristic arguments, showing the relation of Conjecture 1.1 to other significant topics in geometric and arithmetic combinatorics, first and foremost the Erdős-Szemerédi sum-product problem conjecture [7]. We remind the reader that Erdős and Szemerédi conjectured in 1983 that for a finite integer set $A$,

$$\max(|A + A|, |AA|) \gg |A|^{1+\varepsilon} \text{ for any } \varepsilon < 1. \hspace{1cm} (6.1)$$

Today the question is usually studied in the context of fields rather than rings; current ‘world records’ justify any $\varepsilon < 1/3 + 5/9813$ over $\mathbb{R}$ in [12] and $\varepsilon \leq 1/5$ over a general field $F$ in [17] (with the constraint $|A| \leq p^{5/8}$ if $F$ has positive characteristic $p$).

It was suggested in [4] that the Erdős-Szemerédi conjecture, even in its weak form discussed below, can hardly be expected to get resolved merely by a ‘black box’ application of an incidence theorem. Viewed in this context, the error in [9] was the false presupposition that the main result could be achieved by an adaptation of the proof of the renowned Erdős distance problem, which had been beautifully resolved by Guth and Katz [5]. As the title of [9] indicated, it came in the wake of the celebrated work [5], but as we indicated above, the setup does not work in this case.

Another approach would be to use nontrivial arguments from [22] to prove the bound $d(A) \ll |AA + AA|^2/|A|^3$, which implies that $|AA + AA| \gg |A|^{8/7} |A : A|^{3/7}$. This result is better than Theorem 2.2 when $|A : A|$ is large. However, since this does not result in better unconditional bounds for $|AA + AA|$, we do not pursue the details here.
context. It appears that the Erdős distance problem is very special. For a sharp second moment bound can be established, which is just the upper bound for the number of congruent line segments defined by a point set $P$ in the Euclidean plane after a single application of an incidence theorem in projective three-space $\mathbb{R}P^3$ that Guth and Katz succeeded in proving. (We refer to the union of Theorems 2.10 and 2.11 in [5] as the Guth-Katz theorem.)

The fact that the estimate for distances in the plane in the $L^2$-metric can be converted to a projective estimate for lines intersecting in three dimensions was discovered by Elekes and Sharir [6] by looking at the symmetries in the Euclidean group $SE_2(\mathbb{R})$ that take one point of the plane point set to another. More recently it was shown by Selig and the third author [20] that what matters is not so much the symmetry group action but the dimension and the algebraic structure of the equations stating that two line segments in the plane have the same length. In fact, the space of line segments in the plane is four dimensional, just like the Klein quadric, the space of lines in three-space. So a whole family of two-dimensional metric problems was identified in [20], where the set of pairs of line segments, congruent in the appropriate sense, with endpoints in a $N$-point set $P$ was put into one-to-one correspondence with a set of intersecting pairs of lines for a family of some $N^2$ lines in three dimensions.

The Erdős distance problem is one of these problems; some others are analogous statements about spherical and hyperbolic distances. It was shown in [20] how the Guth-Katz theorem applies to these problems directly, bypassing any symmetry group considerations. Previously, the second and third authors [16] had followed the Elekes-Sharir and Guth-Katz approach in order to show that $N$ points in the plane determine $\Omega(N/\log N)$ distinct Minkowski distances. However, one can see explicitly from [20] that equation (1.2) is not amenable to its general scope.

6.1. Conjecture 1.1 and the Erdős-Szemerédi problem. We take the weak Erdős-Szemerédi conjecture to be the statement that for $A \subset \mathbb{R}$, the assumption $|AA| = O(|A|^{1+\delta})$ implies that $|A+A| = \Omega(|A|^{2-\varepsilon})$ where the small parameters $(\delta, \varepsilon)$ are on a polynomial curve through the origin. The question has been resolved qualitatively in the integer case, and therefore also when $F = \mathbb{Q}$, by Bourgain and Chang [2], though the dependence between $\delta$ and $\varepsilon$ was not made explicit. It is open over other fields.

We shall show that a slightly stronger statement than Conjecture 1.1 implies the weak Erdős-Szemerédi conjecture. Namely, suppose we have the second moment bound $O(|P|^3)$ on the number $E(P)$ of solutions of equation (1.2).

Then, assuming $0 \notin A$, we have:

$$E(A) = |\{(a_1, a_2, a_3, a_4) \in A \times A \times A \times A: a_1 + a_2 = a_3 + a_4\}|$$

$$= |A|^{-4}|\{(a_1, \ldots, a_8) \in A \times \cdots \times A: a_1 a_5/a_5 + a_2 a_6/a_6 = a_3 a_7/a_7 + a_4 a_8/a_8\}|$$

$$\leq |A|^{-4}|\{(p_1, p_2, p_3, p_4, b_1, b_2, b_3, b_4) \in AA \times \cdots \times AA \times A^{-1} \times \cdots \times A^{-1}:$$

$$p_1 b_1 + p_2 b_2 = p_3 b_3 + p_4 b_4\}|$$

$$\ll |AA|^3|A|^{-1},$$

after applying the assumption for the plane point set $AA \times A^{-1}$. 
It follows, by the Cauchy-Schwarz inequality that

\[ |A \pm A| \geqslant \frac{|A|^4}{E(A)} \gg \frac{|A|^5}{|AA|^3}, \]

thus if \(|AA| < |A|^{1+\delta}\), then \(|A \pm A| \gg |A|^{2-3\delta}\).

We would like to point out that this paper only proves lower support bounds: the strongest known \(L^2\)-estimate for the number of solutions of (1.2) is still \(O(N^{10/3})\) for any field \(F\) (this follows from the Szemerédi-Trotter \(L^\infty\)-bound for real and complex numbers and for a general \(F\) was proved in [19]). In the special case \(P = A \times A\) we are not aware of an \(L^2\)-bound that is better than the corresponding \(O(|A|^{13/2})\) (this is better than the estimate for a general point set \(P\), as a line through the origin may not have more than \(|A|\) points of \(A \times A\)).

### 6.2. Example: on popular values of \(\omega\).

We conclude this note by revisiting the well-known Erdős construction showing that there is a plane point set \(P\) of \(\Theta(N)\) points with \(\gg 1\) nonzero values of the form \(\omega\) repeating \(\Omega(N^{4/3})\) times. By the Szemerédi-Trotter theorem a single nonzero value of \(\omega\) can only have \(O(N^{4/3})\) realizations.

Without loss of generality let \(\omega\) be the standard dot product. It can equally well be the cross product. Note that, sadly, the main theorem in this note, Theorem 2.1 only applies to the latter case: we are not aware of a better bound than \(\Omega(N^{2/3})\) for the number of distinct dot products defined by a set of \(N\) points in \(\mathbb{R}^2\).

The equation \(q' \cdot q = 1\) can be viewed as an incidence relation between \(N\) points described by vectors \(q \in P\) and \(N\) lines described by covectors \(q'\). There are well known constructions, similar to the one which follows, where the number of incidences is \(\Omega(N^{4/3})\).

We will also show that the pinned version of equation (1.2)

\[ \omega(q, q') = \omega(q, r') \neq 0, \quad q, q', r' \in P, \quad (6.2) \]

can have as many as \(\Omega(N^{7/3})\) solutions. Therefore, it is impossible to use it to settle Conjecture 1.1 without additional assumptions on \(P\).

To counter this, we do not have evidence that equation (1.2) can possibly have more than \(O(N^3)\) solutions. On the other hand, we cannot prove at the moment an upper bound better than \(O(N^{10/3})\) for the number of solutions of (1.2). The latter bound follows, for example, from the \(O(N^{4/3})\) bound on the number of realizations of a single nonzero value of \(\omega\).

To avoid taking integer parts, take \(N\) to be the 6th power of an even integer. Let \(P\) be the union of the integer grid square \(P_1 = [-\sqrt{N}, \ldots, \sqrt{N}] \times [-\sqrt{N}, \ldots, \sqrt{N}]\) and the set \(P_2\) of approximately \(N\) points, constructed below, with some \(N^{2/3}\) points lying in each of the approximately \(N^{1/3}\) directions from the origin of the set of lines \(L\), defined as follows.

Take all the points in the sub-square \([-\frac{1}{6}\sqrt{N}, \ldots, \frac{1}{6}\sqrt{N}] \times [-\frac{1}{6}\sqrt{N}, \ldots, \frac{1}{6}\sqrt{N}]\) with co-prime coordinates \((a, b)\) and define a family \(L\) of \(\Theta(N^{1/3})\) lines through the origin by the following equations:

\[ bx - ay = 0, \quad |a|, |b| \leqslant \frac{1}{6}\sqrt{N}, \quad \gcd(a, b) = 1. \]

Clearly, \(L = L^\perp\). Each line in \(L\) supports \(\Theta(N^{1/2-1/6}) = \Theta(N^{1/3})\) points of \(P_1\).
Now translate $L$ from the origin to every point in the subset of $P_1$ given by $[-\sqrt{N}/2, \ldots, \sqrt{N}/2] \times [-\sqrt{N}/2, \ldots, \sqrt{N}/2]$. Since each line in $L$ supports $\Theta(N^{1/3})$ points of $P_1$, in the union $\mathcal{L}$ of the translated lines one gets $\Theta(N)$ lines in $\Theta(N^{1/3})$ directions, hence $\Theta(N^{2/3})$ parallel lines in each direction. Each line in $L$ gets translated $\Theta(N)$ times but two translates of $l \in L$ result in the same line in $\mathcal{L}$ if their difference is in $P_1 \cap l$. In particular, since $L = L^\perp$, for every line in $L$ there are $\Theta(N^{2/3})$ lines in $\mathcal{L}$ perpendicular to it, each supporting $\Theta(N^{1/3})$ points of $P_1$.

The equations of the lines in $\mathcal{L} \setminus L$ are

$$bx - ay = bi - aj \neq 0, \quad |a|, |b| \leq \sqrt[3]{N}, \quad \gcd(a, b) = 1, \quad |i|, |j| \leq \frac{\sqrt{N}}{2}.$$

By construction, for every $(a, b)$ there are $\Theta(N^{2/3})$ values of the right-hand side above. Also, there are $\Theta(N)$ distinct points in the set

$$P_2 := \left\{ \left(\frac{b}{bi - aj}, \frac{a}{bi - aj}\right) : |a|, |b| \leq \sqrt[3]{N}, \gcd(a, b) = 1; \ |i|, |j| \leq \frac{\sqrt{N}}{2}, \ bi - aj \neq 0 \right\}.$$

Furthermore, the equation $q \cdot q' = 1$, with $q \in P_1$ and $q' \in P_2$ has $\Omega(N^{4/3})$ solutions. The same can be said about the equation $q \cdot q' = c$, for at least an $O(1)$-set of values of $c$. So there are $\Omega(N^{4/3})$ realizations for each of the $\gg 1$ distinct dot product values on the set $P = P_1 \cup P_2$ of $\Theta(N)$ points.

Observe that the set $P_2$ has the property of being the union of sets of $\Theta(N^{2/3})$ points supported on each of the $\Theta(N^{1/3})$ lines in $L$. In fact, for the purposes of having $\Omega(N^{7/3})$ solutions to equation (6.2) it suffices to take any set $P_2$ with the above property. Indeed, consider (6.2) with $q \in P_2$ and $q', r' \in P_1$. Each nonzero vector $q$ has $\Theta(N^{2/3})$ lines in $\mathcal{L}$ perpendicular to it, each line supporting approximately $N^{1/3}$ points of $P_1$.

Thus the number of solutions of equation (6.2), that is the number of solutions of $q \cdot (q' - r') = 0$ is at least a constant times

$$N \cdot N^{2/3} \cdot (N^{1/3})^2 = N^{7/3}.$$

(6.3)

This example suggests that there is a principal difference between Conjecture 1.1, which has been shown to be related to the Erdős-Szemerédi sum-product conjecture, and the Erdős distance problem. For the analogue of (6.2) set $q = r$ in (1.3). However, an analogue of the above construction which leads to the estimate (6.3) is impossible, for Pach and Tardos [15] proved the bound $O(N^{2.137})$ for the number of isosceles triangles defined by a point set in $\mathbb{R}^2$. Moreover, Erdős distance-type questions would all follow with accuracy up to $o(1)$ in the exponent, from the single distance conjecture, which claims that any distance in the Euclidean plane point set repeats only $O(N^{1+o(1)})$ times (see, for example, the book by Brass, Moser and Pach [3] for details). This is also not the case with single nonzero values of $\omega$, whose set appears to be considerably more volatile.

We finish by observing that an analogue of the construction in this section does not appear to be feasible within a single Cartesian product set $P = A \times A$, where the problem of distinct values of bilinear forms meets the Erdős-Szemerédi conjecture.
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Alexander Iosevich
Department of Mathematics,
University of Rochester, Rochester, NY, USA
E-mail: iosevich@math.rochester.edu

Oliver Roche-Newton
Johannes Kepler University, Linz, Austria
E-mail: o.rochenewton@gmail.com

Misha Rudnev
Department of Mathematics,
University of Bristol, Bristol, UK
E-mail: m.rudnev@bristol.ac.uk, misharudnev@gmail.com

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