Can parameters of $f_0$-mesons be determined correctly analyzing only $\pi\pi$ scattering?

Yu.S. Surovtsev$^1$, P. Bydžovský$^2$, R. Kamiński$^3$, V.E. Lyubovitskij$^4$, M. Nagy$^5$

$^1$ Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141 980 Dubna, Russia
$^2$ Nuclear Physics Institute, Czech Academy of Sciences, Rež near Prague 25068, Czech Republic
$^3$ Institute of Nuclear Physics, Polish Academy of Sciences, Cracow 31342, Poland
$^4$ Institut für Theoretische Physik, Universität Tübingen, Kepler Center for Astro and Particle Physics, Auf der Morgenstelle 14, D-72076 Tübingen, Germany
$^5$ Institute of Physics, Slovak Academy of Sciences, Bratislava 84511, Slovak Republic

The coupled processes – the $\pi\pi$ scattering and $\pi\pi \to K\bar{K}$ in the $f^G J^{PC} = 0^+0^{++}$ channel – are analyzed (both separately and combined) in a model-independent approach based on analyticity and unitarity and using a uniformization procedure. It is shown: 1) a structure of the Riemann surface of the $S$-matrix for considered coupled processes must be allowed for calculating both amplitudes and resonance parameters, such as the mass and width; 2) the combined analysis of coupled processes is needed as the analysis of only $\pi\pi$ channel does not give correct values of resonance parameters even if the Riemann surface structure is included.

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The study of scalar mesons is very important for such profound topics in particle physics as the QCD vacuum. However, despite of a big effort devoted to studying various aspects of the problem[1] (for recent reviews see [2,3]) a description of this sector is far from being complete. Parameters of the scalar mesons, their nature and status of some of them are still not well settled[1]. E.g., applying our model-independent method in three-channel analyses[6] we have obtained parameters of the $f_0(600)$ and $f_0(1500)$ which differ considerably from results of analyses utilizing other methods (mainly those based on the dispersion relations and Breit–Wigner approaches). Reasons of this difference should be understood because our method is based only on a demand for analyticity and unitarity of the amplitudes using a uniformization procedure. The construction of amplitudes is practically free from any dynamical (model) assumptions using only the mathematical fact that a local behavior of analytic functions determined on the Riemann surface is governed by the nearest singularities on all corresponding sheets. I.e., the obtained parameters of resonances can be considered as free from theoretical prejudice.

First note that in our previous three-channel analyses with the uniformizing variables[6,7] we were enforced to construct a four-sheeted model of the eight-sheeted Riemann surface. This we have achieved neglecting the $\pi\pi$-threshold branch-point which means that we have considered the nearest to the physical region semi-sheets of the initial Riemann surface. This is in the line with our approach of a consistent account of the nearest singularities on all relevant sheets. The two-channel analysis utilizes the full Riemann surface and is, therefore, free of these approximations. To verify a plausibility of our assumptions in the three-channel calculations, we have performed a combined two-channel analysis of data on $\pi\pi \to \pi\pi$, $K\bar{K}$ to check whether the results of our three-channel analyses[6,7] are also obtained in the two-channel consideration. Moreover, to better understand reasons for the above-indicated difference in results, we have performed first the analysis only of the $\pi\pi$ scattering data in the two-channel approach.

The two-channel $S$-matrix is determined on the four-sheeted Riemann surface. The matrix elements $S_{ij}$, where $i,j = 1(\pi\pi)$, $(K\bar{K})$ denote channels, have the right-hand cuts along the real axis of the $s$ complex plane ($s$ is the invariant total energy squared), starting with the channel thresholds $s_i$, and the left-hand cuts related to the crossed channels. The Riemann-surface sheets are numbered according to the signs of analytic continuations of the roots $\sqrt{s} - s_i$ as follows: signs($\text{Im}\sqrt{s} - s_1$, $\text{Im}\sqrt{s} - s_2$) = $++, --, --, ++$ correspond to sheets I, II, III, IV.

The resonance representations on the Riemann surface are obtained from formulas[8] which express analytic continuations of the $S$-matrix elements to unphysical sheets in terms of those on the physical (I) sheet having only resonance zeros (beyond the real axis). Then, starting from the resonance zeros on sheet I, one can obtain an arrangement of poles and zeros of a resonance on the whole Riemann surface (“pole clusters”). In the two-channel case, according to these formulas and a real analyticity amplitudes of all coupled processes have conjugate poles at the same points of complex energy on sheets II, III and IV if $S_{11} = 0$, $S_{11}S_{22} - (S_{12})^2 = 0$ and $S_{22} = 0$, respectively (the superscript I means the matrix elements on sheet I). Therefore three types of pole clusters representing states of different nature arise: (a) when there is a pair of conjugate zeros only in $S_{11} = 0$, (b) only in $S_{22} = 0$, and (c) in both the $S_{11} = 0$ and $S_{22} = 0$. If the coupling of channels is present ($S_{12} \neq 0$) (i.e. a state decays into both channels and/or is exchanged in crossing channels), then this state being of type a is represented in $S_{11}$ by a pair of conjugate poles on sheet II and a pair of conjugate zeros on sheet I and also by a pair of conjugate poles on sheet III and a pair of conjugate zeros on sheet IV at the same complex-energy points, which are to be shifted with respect to the zeros on sheet I. For states of type b, the pair of conjugate poles on sheet III is shifted relative to the pair of poles on sheet IV.
For the states of type $c$, one must consider two pairs of conjugate poles on sheet III. Generally, wide multi-channel states are most adequately represented by pole clusters, because the pole positions are rather stable characteristics for various models, whereas masses and widths are very model-dependent \[8\]. For calculation of the latters one must use the poles on those sheets where they are not shifted (due to the channel couplings) with respect to the zero position on sheet I. In the case of $N$ channels, the poles only on the sheets with the numbers $2^i$ ($i = 1, \cdots, N$ is the number of channel), i.e. II, IV, VIII, \ldots should be used for calculating resonance parameters \[11\].

It is convenient to use the Le Couteur–Newton relations \[10\]. They express the $S$-matrix elements of all coupled processes in terms of the Jost matrix determinant $d(\sqrt{s} - s_1, \cdots, \sqrt{s} - s_N)$ that is a real analytic function with the only square-root branch-points at $\sqrt{s} - s_n = 0$.

A necessary and sufficient condition for existence of the multi-channel resonance is its representation by one of the types of pole clusters. To use this representation of resonances, which is very important for the wide multi-channel states, a uniformizing variable is applied. Analyzing $\pi\pi \rightarrow \pi\pi, K\overline{K}$ we applied the uniformizing variable \[9\] which takes into account, in addition to the $\pi\pi$- and $K\overline{K}$-threshold branch-points, the left-hand branch-point at $s = 0$, related to the $\pi\pi$ crossed channels:

$$v = \frac{m_K \sqrt{s - 4m^2_K \cdot m_\pi \sqrt{s - 4m^2_\pi}}}{\sqrt{s(m^2_K - m^2_\pi)}}. \quad (1)$$

It maps the four-sheeted Riemann surface with two unitary cuts and the left-hand cut onto the $v$-plane. Representation of resonances of various types on the uniformization $v$-plane can be found in \[11\].

On the $v$-plane, $S_{11}(v)$ has no cuts; $S^2_{12}(v)$ and $S^2_{22}(v)$ do have the cuts which arise from the left-hand cut on the $s$-plane, starting at $s = 4(m^2_K - m^2_\pi)$, which is further approximated by a pole

$$d_L = v^{-4}(1 - (p - i \sqrt{1 - p^2})v)^4 (1 + (p + i \sqrt{1 - p^2})v)^4,$$

where $p = 0.903 \pm 0.0004$ from analysis. An explanation of the fourth power of this pole can be found in \[11\].

On the $v$-plane, the function $d(v)$ in the Le Couteur–Newton relations \[9,10\] does not possess any branch points. The main model-independent contribution of resonances, given by the pole clusters, is factorized in the $S$-matrix elements from the background. The possible remaining small (model-dependent) contributions of resonances are supposed to be included in the background. Therefore, $d(v)$ is taken as $d = d_{res}d_Ld_{bg}$ where the resonance part is

$$d_{res} = v^{-M}\prod_{n=1}^{M}(1 - v_n^nv)(1 + v_nv) \quad (2)$$

with $M$ the number of pairs of the conjugate zeros. The background part is

$$d_{bg} = \exp\left[-i \sum_{n=1}^{3}(\sqrt{s - s_n}/2m_n)(\alpha_n + i\beta_n)\right], \quad (3)$$

$$\alpha_n, \beta_n = a_{n1}, b_{n1} + \sum_{k=\eta,\sigma,v} a_{nk}, b_{nk}(s/s_k - 1)\theta(s - s_k)$$

with $s_\eta$ and $s_\sigma$ the $\eta\eta$, $\rho\rho$ and $\omega\omega$ thresholds, respectively, $s_v$ a combined threshold of the $\eta\eta'$, $\rho\rho$ and $\omega\omega$ thresholds; from the analysis: $s_\eta = 1.6558$ GeV$^2$, $s_v = 2.1293$ GeV$^2$.

At present in the scalar sector, there are alternative data for the $\pi\pi$ scattering – \[12\] and \[13\] – which are different considerably in the 0.76 GeV region and especially above 1.45 GeV. Therefore separate analyses using these alternative data are needed. Here we performed the analysis taking for the $\pi\pi$ scattering in interval $0.575 < \sqrt{s} < 1.89$ GeV the data from \[12\] and for $\sqrt{s} < 1$ GeV from many works. References to the latter and to practically all accessible $\pi\pi \rightarrow K\overline{K}$ data used can be found in \[11\]. Analysis using the data \[13\] will be presented in other paper.

First analyzing only the $\pi\pi$ scattering, we supposed an existence of two states (narrow $f_0(1500)$ and wide $f_0'(1500)$) and achieved an excellent description for the phase shift $\delta_1$ and modulus $|S_{11}|$ (the total $\chi^2/NDF \approx 1.07$) with the resonance parameters (Table \[1\]) which largely coincide with estimations of the PDG \[1\] (cf. also \[14\]). The only distinction is observation of the wide $f_0'(1500)$. The fact that this state is not observed in works cited by the PDG is related, it seems, with peculiarities of analysis of data therein.
and scalar particles contribute with the opposite signs due to gauge invariance. $\rho$ left-hand cut from exchanges by the nearest $s$ data. This should not be the case because, in the uniformizing variable, we have allowed for the left-hand branch-point $\pi\pi$ and the very good agreement of obtained $\pi\pi$ of $S$ resonances are described by pole clusters of the same types as in the analysis only of the $\pi\pi$ scattering. In Table II we show the pole positions of resonances on sheets II and IV on the $\sqrt{s}$-plane.

| Sheet | II | IV |
|-------|----|----|
| $f_0(600)$ | $447.5\pm5.9-i(267\pm6.5)$ | |
| $f_0(980)$ | $1001.1\pm3.7-i(20.3\pm2.6)$ | |
| $f_0(1370)$ | $1301.1\pm47.9-i(224\pm49.3)$ | |
| $f_0(1500)$ | $1503.7\pm45.1-i(56.5\pm39.4)$ | $1509.5\pm38.5-i(168\pm40.6)$ |
| $f_0(1710)$ | $1720\pm32.2-i(64.9\pm30.1)$ | |

In the analysis, the $f_0(600)$ and $f_0(980)$ are described by the clusters of type a: $f_0(1370)$, $f_0(1500)$ and $f_0(1710)$, type b: $f_0'(1500)$, type c. The received background parameters are: $a_{11} = -0.0895\pm0.0030$, $a_{10} = 0.04\pm0.03$, $a_{1\sigma} = 0.0\pm0.8$, $a_{1\tau} = 0.0\pm0.7$, $b_{11} = 0.0\pm0.007$, $b_{10} = 0.0\pm0.01$, $b_{1\sigma} = 0.0\pm0.02$, $b_{1\tau} = 0.054\pm0.036$.

In Fig. I we show the fitting only to the $\pi\pi$ scattering data and energy behavior of the phase shift $\phi_{12}$ and modulus of $S_{12}$ calculated using the resonance parameters from this analysis. In spite of the very good description of data and the very good agreement of obtained $\pi\pi$ scattering length $a_0^\pi$ with the experimental results and with the chiral perturbation theory (ChPT) calculations (see Table III), this analysis shows two important flaws: (1) The negative background phase-shift beginning at the $\pi\pi$ threshold ($a_{11} = -0.0895$) is necessary for a successful description of the data. This should not be the case because, in the uniformizing variable, we have allowed for the left-hand branch-point at $s = 0$ which gives a main contribution to the $\pi\pi$ background below 1 GeV. Other possible contributions of the left-hand cut from exchanges by the nearest $\rho$ and $f_0(600)$ mesons practically obliterate each other [9] because vector and scalar particles contribute with the opposite signs due to gauge invariance. (2) Description of the $\pi\pi \rightarrow K\bar{K}$ data, using the same parameters of resonances as in the $\pi\pi$ channel, is satisfactory only for the phase shift $\phi_{12}$ which is due to the approximation of the left-hand cut in $S_{12}$ and $S_{22}$ by the fourth-power pole. The modulus $|S_{12}|$ is described well only from the $K\bar{K}$ threshold to about 1.15 GeV as it should be due to the two-channel unitarity. Above this energy the description fails even qualitatively (Fig. I).

From this we conclude: If the data are consistent, for obtaining correct parameters of wide resonances the combined analysis of coupled processes is needed. Further that analysis of $\pi\pi \rightarrow \pi\pi, K\bar{K}$ is performed successfully. The data for the $\pi\pi$ scattering below 1 GeV admit two solutions for the phase shift – A and B – which differ mainly in the pole position on sheet II of the $f_0(600)$. The total $\chi^2/\text{NDF}$ is 1.53 for the A-solution and 1.44 for B-solution. The resonances are described by pole clusters of the same types as in the analysis only of the $\pi\pi$ scattering. In Table II we show the pole positions of resonances on sheets II and IV on the $\sqrt{s}$-plane.
TABLE II: The pole positions of resonances on sheets II and IV in the $\sqrt{s}$-plane in the combined analysis of the $\pi\pi \to \pi\pi, K\bar{K}$ data. The complete pole-clusters of resonances can be found in [11].

| Sheet  | II               | IV               |
|--------|------------------|------------------|
|        |                  | A-solution       |
| $f_0(600)$ | $517\pm7.8$–i$(393.9\pm6)$ |                  |
| $f_0(980)$ | $1004.6\pm3.9$–i$(25.0\pm2.3)$ |                  |
| $f_0(1370)$ | $1342.9\pm12.2$–i$(221.6\pm30.7)$ |                  |
| $f_0(1500)$ | $1501.1\pm6.4$–i$(56.6\pm6.0)$ |                  |
| $f_0'(1500)$ | $1532.2\pm12.4$–i$(323.2±21)$ | $1519.3\pm18.7$–i$(339.5±42.2)$ |
| $f_0(1710)$ | $1717\pm34.9$–i$(72.9\pm16.2)$ |                  |
|        |                  | B-solution       |
| $f_0(600)$ | $550.6\pm9$–i$(502.1\pm7.2)$ |                  |
| $f_0(980)$ | $1003.2\pm3$–i$(28.9\pm2)$ |                  |
| $f_0(1370)$ | $1336.7\pm14$–i$(251.9\pm27.5)$ |                  |
| $f_0(1500)$ | $1500.3\pm6.3$–i$(57.0\pm6.4)$ |                  |
| $f_0'(1500)$ | $1528.4\pm12.5$–i$(328\pm20.2)$ | $1515.6\pm17$–i$(340.3±34.9)$ |
| $f_0(1710)$ | $1722\pm35.7$–i$(92.3\pm20.3)$ |                  |

The obtained background parameters for the A-solution are: $a_{11} = 0.0 \pm 0.003$, $a_{1\eta} = -1.004 \pm 0.0301$, $a_{1\sigma} = 0.2148 \pm 0.0822$, $a_{1\nu} = 0.0 \pm 0.07$, $b_{1\eta} = b_{1\sigma} = b_{1\nu} = 0$, $b_{1\nu} = 0.012 \pm 0.0287$, $a_{21} = -0.919 \pm 0.107$, $a_{2\eta} = -1.399 \pm 0.348$, $a_{2\nu} = 0.0 \pm 0.7$, $a_{2\nu} = -11.45 \pm 0.75$, $b_{21} = 0.0747 \pm 0.0503$, $b_{2\eta} = b_{2\nu} = 0$, $b_{2\nu} = 4.83 \pm 1.94$; for B-solution: $a_{11} = 0.0 \pm 0.003$, $a_{1\nu} = -0.0913 \pm 0.0327$, $a_{1\nu} = 0.1707 \pm 0.0899$, $a_{1\nu} = 0.0 \pm 0.07$, $b_{1\eta} = b_{1\sigma} = b_{1\nu} = 0$, $b_{1\nu} = 0.006 \pm 0.029$, $a_{21} = -1.338 \pm 0.111$, $a_{2\eta} = -1.119 \pm 0.376$, $a_{2\nu} = 0.0 \pm 0.8$, $a_{2\nu} = -12.13 \pm 0.77$, $b_{21} = 0.018 \pm 0.050$, $b_{2\eta} = b_{2\nu} = 0$, $b_{2\nu} = 4.48 \pm 1.98$.

In the combined analysis both flaws of the only $\pi\pi$-scattering analysis are cured. Now the $\pi\pi$ background below the $K\bar{K}$ threshold is absent ($a_{11} = 0.0$). An arising pseudo-background at the $\eta\eta$ threshold ($a_{1\eta} < 0$) is also clear: this is a direct indication to consider explicitly the $\eta\eta$-threshold branch-point. This was already done in our work [1]. In the combined analysis the $f_0(600)$ parameters are changed considerably receiving new values closer to those obtained in our three-channel analysis [2]. Earlier one noted that wide resonance parameters are largely controlled by the non-resonant background [13]. In part this problem is removed due to allowing for the left-hand branch-point at $s = 0$ in the uniformizing variable.

In the Table III we compare our results for the $\pi\pi$ scattering length $a_0^0$ with results of some other theoretical and experimental works.

TABLE III: The $\pi\pi$ scattering length $a_0^0$.

| $a_0^0$ ($m_{\pi^+}^{-1}$) | Remarks | References |
|--------------------------|---------|------------|
| 0.222 ± 0.008           | Analysis only of $\pi\pi$ scattering | This work |
| 0.230 ± 0.004           | A-solution  | This work |
| 0.282 ± 0.003           | B-solution  | This work |
| 0.26 ± 0.05             | Analysis of the $K \to \pi\pi\nu\nu$ using Roy’s equation | [15] |
| 0.24 ± 0.09             | Analysis of $\pi^- p \to \pi^+\pi^- n$ | [16] |
| 0.2220 ± 0.00128stat    | Experiment on $K_{e4}$ decay | [18] |
| ±0.0050_{sys}±0.0037_{th}|                       |            |
| 0.220 ± 0.005           | ChPT + Roy’s equations | [19, 20] |
| 0.220 ± 0.008           | Dispersion relations and $K_{e4}$ data | [21] |
| 0.26                    | NJL model (I) | [22] |
| 0.28                    | NJL model (II) | [23] |
In the analysis only of $\pi\pi$ scattering and in the A-solution we reproduced with a high accuracy the ChPT results \cite{19, 20} including constraints imposed by the Roy equations. On the other hand, the B-solution is similar to the predictions of the chiral approaches based on the linear realization of chiral symmetry (models of the Nambu–Jona-Lasinio (NJL) type \cite{22, 23}). Taking into account very precise experiments at CERN performed by the NA48/2 \cite{18} and the DIRAC \cite{24} Collaborations, which confirmed the ChPT prediction \cite{19, 20}, one ought to prefer the A-solution.

In summary, a structure of the Riemann surface of the $S$-matrix for coupled processes must be included properly. To calculate resonance parameters, such as masses and widths, one must use poles on those sheets where they are not shifted (due to the channel coupling) in respect of the zeros on sheet I. In the two-channel case, the relevant poles are on sheets II or/and IV depending on the resonance type. Moreover, the combined analysis of coupled processes is needed as the analysis of only $\pi\pi$ channel do not give correct values of resonance parameters even if the Riemann surface structure is included. Finally, in order to be concrete, the main scope of the paper is the scalar mesons, however, the method and conclusions of paper can be applied to other wide resonances, e.g. to vector mesons.

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\begin{thebibliography}{99}
\bibitem{1} J.Beringer \textit{et al.} (PDG), Phys. Rev. D \textbf{86}, 010001 (2012).
\bibitem{2} C. Amsler, N.A. Tornqvist, Phys. Rept. \textbf{389}, 61 (2004).
\bibitem{3} D.V. Bugg, Phys. Rept. \textbf{397}, 257 (2004).
\bibitem{4} F.E. Close, N.A. Tornqvist, J. Phys. G G \textbf{28}, R249 (2002). \textbf{28}, R249 (2002).
\bibitem{5} E. Klempt, A. Zaitsev, Phys. Rept. \textbf{454}, 1 (2007).
\bibitem{6} Yu.S. Surovtsev, P. Bydžovský, R. Kamiński, M. Nagy, Phys. Rev. D \textbf{81}, 016001 (2010).
\bibitem{7} Yu.S. Surovtsev, P. Bydžovský, V.E. Lyubovitskij, Phys. Rev. D \textbf{85}, 036002 (2012).
\bibitem{8} D. Krupa, V.A. Meshcheryakov, Yu.S. Surovtsev, Nuovo Cim. A \textbf{109}, 281 (1996).
\bibitem{9} Yu.S. Surovtsev, D. Krupa, and M. Nagy, Eur. Phys. J. A \textbf{15}, 409 (2002).
\bibitem{10} K.J. Le Couteur, Proc. R. London, Ser. A \textbf{256}, 115 (1960); R.G. Newton, J. Math. Phys. \textbf{2}, 188 (1961); M. Kato, Ann. Phys. \textbf{31}, 130 (1965).
\bibitem{11} Yu. Surovtsev, P. Bydžovský, R. Kamiński, V. Lyubovitskij, M. Nagy, arXiv: 1206.3438 [hep-ph].
\bibitem{12} B. Hyams \textit{et al.}, Nucl. Phys. B \textbf{64}, 134 (1973) [AIP Conf. Proc. \textbf{13}, 206 (1973)]; Nucl. Phys. B \textbf{100}, 205 (1975).
\bibitem{13} R. Kamiński, L. Leśniak, K. Rybicki, Z. Phys. C \textbf{74}, 79 (1997); Eur. Phys. J. direct C \textbf{4}, 4 (2002).
\bibitem{14} R. García-Martín, R. Kamiński, R. Peláez, J. Ruiz de Elvira, Phys. Rev. Lett. \textbf{107}, 072001 (2011).
\bibitem{15} L. Rosselet \textit{et al.}, Phys. Rev. D \textbf{15}, 574 (1977).
\bibitem{16} A.A. Belkov \textit{et al.}, JETP Lett. \textbf{29}, 597 (1979).
\bibitem{17} N.N. Achasov, G.N. Shestakov, Phys. Rev. D \textbf{49}, 5779 (1994).
\bibitem{18} J.R. Batley \textit{et al.}, Eur. Phys. J. C \textbf{70}, 635 (2010).
\bibitem{19} G. Colangelo, J. Gasser and H. Leutwyler, Phys. Lett. B \textbf{488}, 261 (2000).
\bibitem{20} G. Colangelo, J. Gasser, H. Leutwyler, Nucl. Phys. B \textbf{603}, 125 (2001); B. Ananthanarayan, G. Colangelo, J. Gasser, H. Leutwyler, Phys. Rept. \textbf{353}, 207 (2001).
\bibitem{21} R. García-Martín, R. Kamiński, J.R. Peláez, F.J. Ynduráin, Phys. Rev. D \textbf{83}, 074004 (2011).
\bibitem{22} M.K. Volkov, Sov. J. Part. Nucl. \textbf{17}, 186 (1986).
\bibitem{23} A.N. Ivanov, N.I. Troitskaya, Nuovo Cim. A \textbf{108}, 555 (1995).
\bibitem{24} B. Adeva \textit{et al.}, Phys. Lett. B \textbf{704}, 24 (2011).
\end{thebibliography}