Dualities from large $\mathcal{N}$ orbifold equivalence in Chern-Simons-matter theories with flavor

Mitsutoshi Fujita$^{a,b}$

$^a$Department of Physics, University of Washington, Seattle, WA 98195-1560, USA
$^b$Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, Kashiwa, Chiba 277-8583, Japan

E-mail: mf29@uw.edu

Abstract: We study large $N$ orbifold equivalences involving three-dimensional $\mathcal{N}=3$ and $\mathcal{N}=4$ supersymmetric quiver Chern-Simons-matter theories. The gravity dual of the $\mathcal{N}=3$ Chern-Simons-matter theory is described by $AdS_4 \times M_7$ where the tri-Sasaki manifold $M_7$ is known as the Eschenburg space. We find evidence that a large $N$ orbifold equivalence for the $\mathcal{N}=4$ case continues from the M-theory limit to the weak-coupling limit. For the $\mathcal{N}=3$ case, we find a consistent large $N$ equivalences involving a projection changing the nodes of the gauge groups, and also for a projection changing Chern-Simons-levels where for the latter projection, the BPS monopole operators behave as expected in large $N$ equivalence. For both cases we show, using the gravity dual, that the critical temperature of the confinement/deconfinement transition does not change and the entropy behaves as expected under the orbifold equivalence. We show that large $N$ orbifold equivalence changing Chern-Simons-levels can be explained using the planar equivalence in the mirror dual.

Keywords: M-Theory, Chern-Simons Theories, AdS-CFT Correspondence
1 Introduction

A variety of non-perturbative equivalences are known to relate the large $N$ limits of many non-Abelian gauge theories (where $N$ is the rank of the gauge group). These include, for example, equivalences relating $SU(N)$ and $SO(N)$ or $Sp(N)$ theories [1–3], equivalences relating theories with adjoint representation matter fields to theories with bifundamental or other tensor representation matter fields [4–7], and equivalences relating toroidally-compactified theories with differing compactification radii [8]. Underlying these, and other, examples of large $N$ equivalences are a network of orbifold projections. In field theories, an orbifold projection is a mapping which, given an initial ("parent") theory, plus some chosen discrete symmetry of this theory, constructs a new ("daughter") theory by removing from the parent theory all degrees of freedom which are not invariant under the chosen discrete symmetry. In suitable cases, observables in the parent and daughter theories which are invariant under appropriate symmetries coincide in the large $N$ limit [9, 10].

However, not all orbifold projections lead to large $N$ equivalences.\(^1\) Valid large $N$ equivalences appear to be associated with "invertible" orbifold projections, that is, cases where

---

\(^1\)For example, a projection defined by a $\mathbb{Z}_2$ subgroup of the gauge symmetry, combined with $(-1)^F$, maps $U((p+q)N)\, \mathcal{N} = 1$ supersymmetric Yang-Mills theory to an $U(pN) \times U(qN)$ quiver theory with bifundamental fermions. Only in the special case of $p = q$ are the parent and daughter theories related by a large $N$ equivalence.
some orbifold projection maps theory $A$ to theory $B$ while a different projection maps theory $B$ back to theory $A$ (with a smaller gauge group rank). This is an empirical observation based on examining numerous cases, but we are unaware of any general proof. In light of this, it is interesting to consider more diverse examples of orbifold projections and investigate when and if they lead to valid large $N$ equivalences.

In this paper, we discuss several orbifold projections of $d = 3, \mathcal{N} = 6$, ABJM theory, as well as generalizations of ABJM theory which include fundamental representation flavors. The first case we consider is a $\mathbb{Z}_p$ projection which relates the $\mathcal{N} = 6 U(pN)_{kp} \times U(pN)_{-kp}$ ABJM theory to an $\mathcal{N} = 4 [U(N)_{k} \times U(N)_{-k}]^p$ quiver Chern-Simons-matter theory with a $\mathbb{Z}_p$ global symmetry. To see unbroken $\mathbb{Z}_p$ global symmetry in the weak coupling limit, we confirm large $N$ orbifold equivalence using the free theory on $S^1 \times S^2$. We also show large $N$ orbifold equivalence using the AdS/CFT correspondence [11–13] in both the type IIA string theory and the outside of the planar limit of the M-theory where $k$ is fixed and $N$ is taken to be large [14] (see also [15]).

We then add $N_f$ fundamental representation hypermultiplets to these theories. This turns the parent ABJM theory into an $\mathcal{N} = 3 U(pN)_{kp} \times U(pN)_{-kp}$ quiver Chern-Simons-matter theory. Provided $N_f$ is divisible by $p$, $N_f \equiv pn_f$, one may define an orbifold projection mapping theory to an $\mathcal{N} = 3 [U(N)_{k} \times U(N)_{-k}]^p$ quiver Chern-Simons-matter theory with $n_f$ fundamentals at each node and a $\mathbb{Z}_p$ global symmetry. As observables, we perform the projection of the mesonic operators which are neutral sector of large $N$ equivalence. We also show large $N$ orbifold equivalence using the gravity dual in both the type IIA string theory and the outside of the planar limit in the M-theory. The gravity dual of an $\mathcal{N} = 3 [U(N)_{k} \times U(N)_{-k}]^p$ quiver Chern-Simons-matter theory with fundamentals has been studied in [24–27] and for massive flavors in [28, 29]. The gravity dual of Chern-Simons-matter theory with backreacted flavor has been proposed as the $d = 11$ supergravity on $AdS_4 \times \mathcal{M}_7$ where flavor becomes the Kaluza-Klein magnetic monopole as a soliton. Here, tri-Sasaki $\mathcal{M}_7$ is given by the Eschenburg space [34–37] parametrized by three relatively prime numbers $(t_1, t_2, t_3)$ which are read off according to the number and charge of 5-branes in the dual type IIB elliptic brane configuration.

Finally, we consider some level-changing projection which is not understood well. Large $N$ equivalence between ABJM theories in terms of the level-changing projection was analyzed in the paper [16, 17] based on the proof using the AdS/CFT correspondence in the M-theory. Since we need to know the origin of the string theory to understand the level-changing projection, it is interesting to study the level-changing projection furthermore. As a generalization, we apply the level-changing projection for $d = 3 \mathcal{N} = 3$ Chern-Simons-matter

---

Non-equivalence for $p \neq q$ may be easily confirmed by considering the high temperature thermodynamics of these theories.

2There are other $\mathcal{N} \leq 2$ Chern-Simons-matter theories including flavor in which flavor are D6-branes in the type IIA string theory [30–32].

3The cone over tri-Sasaki $\mathcal{M}_7$ is given by a $d = 8$ toric hyperKähler manifold with $Sp(2)$ holonomy and with 3/16 supersymmetry [33].
theories with flavor. We analyze the $AdS_4 \times M_7$ gravity dual in large $N$ orbifold equivalence and show that the curvature radius of the two equivalent theories becomes the same. Since $(t_1, t_2, t_3)$ depend on the discrete number of the orbifold, the equivalence in terms of the level-changing projection is non-trivial. We analyze the BPS monopole operators and Bekenstein-Hawking entropy in large $N$ equivalence. Moreover, we analyze the critical temperature of the confinement/deconfinement transition using the entropy. When the discrete symmetry of the level-changing projection is unbroken, we can observe large $N$ orbifold equivalence in the phase transition. We also show that the planar dominance holds in the very strong coupling limit of the M-theory since the equivalence holds even outside the ’t Hooft limit, as long as there exists a classical gravity dual [19, 20].

The content of this paper is as follows. In section 2, we review the $\mathcal{N}=6$ ABJM theory and $\mathcal{N}=4$ Chern-Simons-matter theories and show the orbifold projection of the $\mathcal{N}=6$ ABJM theory gives a $\mathcal{N}=4$ Chern-Simons-matter theory. After that, we introduce flavor in the Chern-Simons-matter theory and discuss the orbifold equivalence in the presence of flavor. In section 3, we consider the free Chern-Simons-matter theory on $S^1 \times S^2$ at finite temperature and analyze large $N$ orbifold equivalence. In section 4, we analyze large $N$ equivalence in the type IIA gravity dual to the Chern-Simons-matter theory with small amount of flavor by introducing probe D6-branes. In section 5, we analyze the orbifold equivalence between the $\mathcal{N}=3$ Chern-Simons-matter theories using gravity dual with backreacted flavor. As observables invariant under chosen discrete symmetry, we analyze BPS monopole operators in large $N$ orbifold equivalence. We also compare the Bekenstein-Hawking Entropy and the Hawking-Page transition between the parent theory and the daughter theory under the orbifold. In section 6, we explain the orbifold changing Chern-Simons levels by using the mirror symmetry of the type IIB elliptic D3-brane system. It turns out that such orbifold changes the number of nodes and NS5-branes in the mirror theory side.

2 $\mathcal{N}=4$ Chern-Simons-matter theory

![Figure 1](image)

**Figure 1.** A part of quiver diagram of $\mathcal{N}=4$ Chern-Simons-matter theory. $U_I$ represents the $I$-th node. $A_I, B_I$ show chiral multiplets composing $\mathcal{N}=4$ (twisted) hyper multiplet.
In this section, we briefly review $\mathcal{N}=4$ Chern-Simons-matter theory proposed in [38, 43, 44]. The field content in the $d = 3$ $\mathcal{N}=4$ Chern-Simons theory becomes the $\mathcal{N}=4$ vector multiplet $(V_I, \Phi_I)$, $\mathcal{N}=4$ hyper multiplet $(A_I, B_I)$, and $\mathcal{N}=4$ twisted hyper multiplet $(A'_I, B'_I)$. Here, the chiral multiplets $A_I, A'_I$ and $B_I, B'_I$ transform under the gauge symmetry $\prod_{I=1}^{2p} U(N)_I$ with $U(N)_{2p+1} = U(N)_1$ as $(N_I, \overline{N}_{I+1})$. In terms of components, $A_I = (h_I, \psi_{I,\alpha})$, $B_I = (\tilde{h}_I, \tilde{\psi}_{I,\alpha})$, $A'_I = (h'_I, \psi'_{I,\alpha})$, and $B'_I = (\tilde{h}'_I, \tilde{\psi}'_{I,\alpha})$. We consider the quiver diagram with a cyclic configuration in Figure 1. we label hypermultiplets with the associated numbers $n_I$. Numbers $n_I$ are 0 for untwisted hyper multiplets and 1 for twisted hypermultiplets.

The Lagrangian for the hypermultiplets and the Chern-Simons terms is composed from kinetic terms and superpotential as follows:

\[
S_{CS} = \sum_{I=1}^{2p} \frac{ik_I}{2} \left[ \int d^3xd^4\theta \int_0^1 dt \text{tr} \left( V_I D (e^{-2V_I} D e^{2V_I}) \right) - \text{tr} \left( \int d^3 xd^2 \theta \Phi_I^2 + \text{c.c.} \right) \right],
\]

\[
S_{\text{hyper}} = -\sum_{I=1}^{2p} \int d^3xd^4\theta \text{tr} \left( \bar{A}_I e^{2V_I} A_I e^{-2V_{I+1}} + B_I e^{-2V_I} B_I e^{2V_{I+1}} \right)
\]

\[
+ \sum_{I=1}^{2p} \left( \int d^3 xd^2 \theta \sqrt{2} \text{tr} (B_I \Phi_I A_I - B_I A_I \Phi_{I+1} + \text{c.c.}) \right),
\]

where the Chern-Simons level of the $U(N)_I$ gauge group is given by $k_I = k(n_{I+1} - n_I)$. In this paper, we consider the particular case where the hyper multiplets and twisted hyper multiplets are aligned mutually, namely, $k_I = \pm k$ for even(odd) $I$ and the number of the untwisted multiplet is the same as the number of the twisted multiplet [44]. Non-dynamical adjoint fields $\Phi^I$ in F-terms and auxiliary fields in the vector multiplet are integrated out. $SO(4)_R \sim SU(2)_I \times SU(2)_{\text{unt}}$ $R$-symmetry rotates $(h_I, \tilde{h}_I)$ and $(h'_I, \tilde{h}'_I)$, respectively. There are also the baryonic $U(1)_b$ and the diagonal $U(1)_d$ which act on $(h_I, h'_I)$ (and $(\tilde{h}_I, \tilde{h}'_I)$) in two different ways. This symmetry agrees the isometry $(SU(2) \times U(1))^2$ of its moduli space $(\mathbb{C}^2 / \mathbb{Z}_p \times \mathbb{C}^2 / \mathbb{Z}_p) / \mathbb{Z}_k$ [38, 39].

Remember that for two nodes $p = 1$, the $\mathcal{N}=4$ Chern-Simons-matter theory becomes the ABJM theory preserving enhanced $\mathcal{N}=6$ supersymmetry and enlarged $SU(4)_R$ $R$-symmetry for $k > 2$ [18]. In the ABJM theory, two nodes are joined by four links representing chiral multiplets $A_1, A_2, B_1, B_2$. Moreover, the $U(N) \times U(N)$ ABJM theory has enhanced $\mathcal{N}=8$ supersymmetry for $k = 1, 2$ [40, 41] (see also [42]). The amount of supersymmetry of the Chern-Simons-matter theory is also reviewed in the next section in the view of the multiple M2-brane theory on an orbifold of $\mathbb{C}^4$.

2.1 Orbifolds of the $\mathcal{N}=6$ ABJM theory

In this section, we have the review of the orbifold projection of $\mathcal{N}=6 U(nN)_{kn} \times U(nN)_{-kn}$ ABJM action as the multiple M2-brane theory which corresponds to the $p = 1$ case of the
$\mathcal{N}=4$ Chern-Simons-matter theory [21]. Here, we quantized Chern-Simons levels in terms of $n$. In this quantization, 't Hooft coupling $\lambda = N/k$ is independent of $n$. It is known that the ABJM theory is the multiple M2-brane theory placed at the $\mathbb{Z}_{nk}$ orbifold of $\mathbb{C}^4$. Note that this $\mathbb{Z}_{nk}$ orbifold does not correspond to the orbifold projection of the $U(nN)_{kn} \times U(nN)_{-kn}$ ABJM theory.

Here, we focus on a single M2-brane instead of the multiple M2-branes and introduce the 4 complex variables $y^A$ to specify $\mathbb{C}^4$, the isometry of which is $SO(8)$. The orbifold action acts on $y^A$ as $y^A \rightarrow e^{2\pi i/(nk)}y^A$. The orbifold $\mathbb{Z}_{nk}$ action also acts on the spinors of $SO(8)$ as

$$\epsilon = e^{2\pi i(s_1+s_2+s_3+s_4)/(kn)}\epsilon,$$

(2.3)

where $s_i$ ($i = 1, \ldots, 4$) takes the values $\pm 1/2$. The chirality condition of the spinor imposes on $s_i$ the condition that $\sum_{i=1}^4 s_i$ is even giving an 8-dimensional spinor representation. The orbifolded theory then has 6 spinors out of 8 spinors or $d = 3 \mathcal{N} = 6$ supersymmetry for $kn > 2$ ($d = 3 \mathcal{N} = 8$ supersymmetry for $kn = 1, 2$).

The $\mathbb{Z}_n$ orbifold projection preserving $\mathcal{N} = 4$ supersymmetry is given by

$$y^A = e^{2\pi i/nA}y^A, \quad (n_1, n_2, n_3, n_4) = (n, \infty, n, \infty).$$

(2.4)

We can show the supersymmetry by considering the action on the spinor as $\epsilon \rightarrow e^{2\pi i(s_1+s_2)/n}\epsilon$ which leaves 4 spinors.

Secondly, we consider the multiple M2-brane theory. In the non-Abelian gauge theory, we introduce the element of the $\mathbb{Z}_n$ orbifold projection from the element of each gauge group $U(nN) \times U(nN)$ as

$$\Omega = \text{diag}(1_N, \nu_1 N, \nu^2 1_N, \ldots, \nu^{n-1} 1_N),$$

(2.6)

where $1_N$ is the $N \times N$ identity matrix and we have defined the phase $\nu = e^{2\pi i/n}$. We combine chiral multiplets of bi-fundamental matters into the following multiplet transforming under $SU(4)$ enhanced R-symmetry:

$$y^A = (A_1, B_2, \bar{B}_1, \bar{A}_2).$$

(2.7)

The $\mathbb{Z}_n$ action acts on the bifundamental fields $y^A$, $V_I$ and $\Phi_I$ ($I = 1, 2$) as

$$y^1 = A_1 = \nu \Omega A_1 \Omega^{-1}, \quad y^2 = B_2 = \Omega B_2 \Omega^{-1},$$

$$y^3 = B_1 = \nu^{-1} \Omega B_1 \Omega^{-1}, \quad y^4 = A_2 = \Omega A_2 \Omega^{-1},$$

$$V_I = \Omega V_I \Omega^{-1}, \quad \Phi_I = \Omega \Phi_I \Omega^{-1}.$$  

(2.8) \quad (2.9) \quad (2.10)

\begin{footnote}{It is known that changing the basis of the orbifold action, the above orbifold action is equivalent to
\begin{equation}
 y^A = e^{2\pi i/nA}y^A, \quad (n_1, n_2, n_3, n_4) = (n, n, -n, -n).
\end{equation}
(2.5)

The orbifold (2.5) also preserves $\mathcal{N} = 4$ supersymmetry.}

\end{footnote}
We show that the orbifolded theory agrees with the $\mathcal{N} = 4 \ [U(N)_k \times U(N)_{-k}]^n$ Chern-Simons-matter theory with $p = n$ obtained in the previous section. Remember that we quantized the Chern-Simons levels of the mother theory by $n$ and Chern-Simons levels of the daughter theory become $\pm k$ for each node [22]. We observe that the ’t Hooft coupling $\lambda = N/k$ becomes the same between the mother theory and the daughter theory which is the condition of the orbifold equivalence. It is known that the orbifolded theory has the moduli space $\mathbb{C}^2/\mathbb{Z}_{kn} \times \mathbb{Z}_n$.

### 2.2 Adding flavor to the Chern-Simons-matter theory

In this section, we review the $\mathcal{N} = 3$ Chern-Simons-matter theory with flavor which can be constructed from the $d = 3 \mathcal{N} = 4$ Chern-Simons-matter theory by adding flavor fields [27] (see also [24–26]). After that, we discuss the orbifold equivalence between Chern-Simons-matter theories with flavor. We add massless flavor to the $\mathcal{N} = 4$ Chern-Simons-matter theory, namely, $N_F^I = N_0 F$ fundamental hypermultiplets aligned evenly among the different groups $(Q^I_\alpha, \tilde{Q}^I_\alpha)$ ($\alpha = 1, \ldots, N_0, \ I = 1, \ldots, 2p$) transforming under the $I$-th gauge group as $(N, \bar{N})$ with $\sum_I N_F^I = N_F$. Then, we add the D-term to the $\mathcal{N} = 4$ action as follows:

$$S_{\text{flavor} 1} = -\text{Tr} \sum_{\alpha, l} \int d^3 x d^4 \theta \ (\bar{Q}^I_\alpha e^{2V_I} Q^I_\alpha + \tilde{Q}^I_\alpha e^{-2V_I} \tilde{Q}^I_\alpha).$$

The potential term is also added

$$S_{\text{flavor} 2} = -\int d^3 x d^2 \theta \sum_{I, \alpha} i \sqrt{2} \bar{Q}^I_\alpha \Phi_I Q^I_\alpha + \text{c.c.},$$

(2.11)

The non-dynamical field $\Phi_I$ can be integrated out. $R$-symmetry is now broken to $SU(2)_R$ which is the diagonal part of $SU(2)_{\text{unt}} \times SU(2)_t$, while $U(1)_b$ symmetry is unchanged. This theory also has $SU(2)_d$ global symmetry commuting with $SU(2)_R$ [48]. $SU(2)_d$ is also a subgroup of $SU(4)_R$ in the ABJM theory.

For $p = 1$, the mesonic field in this theory is constructed as $\tilde{Q}_1(A_1 B_1) Q_1$ or $\tilde{Q}_1 (A_1 B_1) A_1 Q_2$. If there is one flavor, the former type of the mesonic operator exists and if there are two type of flavors with different gauge groups, both the mesonic operators exist.

We now consider the $\mathbb{Z}_n$ orbifold projection of the Chern-Simons-matter theory with flavor for $p = 1$. The orbifold action for the supersymmetry multiplets except for the fundamental hypermultiplets is given in (2.8). Using the projection matrix (2.6), the $\mathbb{Z}_n$ orbifold action for the fundamental hypermultiplets is given by

$$Q^I_\alpha = \nu^{m^I_\alpha} Q^I_\alpha, \quad \tilde{Q}^I_\alpha = \nu^{-m^I_\alpha} \tilde{Q}^I_\alpha \Omega^{-1},$$

(2.13)

\footnote{After integrating out $\Phi_I$, the superpotential is rewritten as

$$\int d^3 x d^2 \theta \sum_{i=1}^{2p} \frac{i}{\kappa_i} \text{tr}(A_i B_i - B_{i-1} A_{i-1} - Q^I_\alpha \tilde{Q}^I_\alpha)^2. \quad (2.12)$$

The convention of the superpotential is different from that of [27] by the change $A_i \rightarrow B'_i + 1$ and $B_i \rightarrow A'_i + 1$ (See also [25]).}
where $I = 1, 2$ and $m^I_\alpha = 0, 1, \ldots, n - 1$. The factor $m^I_\alpha$ implies the nodes to which the flavor field couples. We do not have $\mathbb{Z}_n$ symmetry in the presence of the flavor in general. If $N_F^0$ is a multiple of $n$, however, one can distribute the flavors evenly among the different groups and recover $\mathbb{Z}_n$ symmetry.\(^6\) Remember that we believe the orbifold equivalence only if $\mathbb{Z}_n$ symmetry is present. Thus, we consider the case that $N_F^0$ is a multiple of $n$ for following section. We show that the orbifolded theory is given by the $\mathcal{N} = 3 \left[ U(N) \times U(N) \right]^n$ Chern-Simons-matter theory with $N_F$ flavors. The orbifold projection of mesonic operators is given by

\begin{equation}
\frac{1}{pN} \tilde{Q}^I_\beta (A_1 B_I) \tilde{Q}^I_\beta \rightarrow \frac{1}{pN} \sum_{I = \text{odd}} \tilde{Q}^I_\alpha (A_1 B_I)^I \tilde{Q}^I_\alpha, \tag{2.14}
\end{equation}

\begin{equation}
\frac{1}{pN} \tilde{Q}^I_\beta (A_1 B_I)^I A_I Q^I_\alpha \rightarrow \frac{1}{pN} \sum_{I = \text{odd}} \tilde{Q}^I_\alpha (A_1 B_I)^I A_I Q^{I+1}_\alpha, \tag{2.15}
\end{equation}

It can be shown that these operators are in the neutral sector of $\mathbb{Z}_p$ symmetry.

### 3 Free Chern-Simons-matter theory on $S^1 \times S^2$

In this section, we analyze the orbifold equivalence for the free Chern-Simons-matter theory on $S^1 \times S^2$ at finite temperature. We first analyze the critical temperature of the phase transition [50, 51]. We consider the large $N$ limit with 't Hooft coupling $\lambda = N/k \ll 1$.

We ignore the contribution of flavor to derive the Hagedorn/deconfinement transition. After taking temporal gauge and integrating out matters, the unitary matrix model appears from compactifying Chern-Simons-matter theory on $S^1 \times S^2$ ($t \sim t + \beta$)

\begin{equation}
Z = \int \prod_{I = 1}^{2p} DU_I \exp \left[ \sum_{I = 1}^{2p} \sum_{n = 1}^{\infty} \frac{1}{n} \left( z_{\text{unt}}^n(x^n) \text{tr}(U_{2i}^n \text{tr}(U_{2i+1}^n) + z^t(x^n) \text{tr}(U_{2i-1}^n \text{tr}(U_{2i}^n) + c.c. \right) + \ldots \right], \tag{3.1}
\end{equation}

where ... shows the contribution of flavor and is ignored at present and $t$, $\text{unt}$ show single-particle partition functions for twisted hyper multiplets, untwisted hypermultiplets as

\begin{equation}
z^t_n = z_{\text{unt}}^n = z_B(x^n) + (-1)^{n+1} z_F(x^n) \equiv z_n, \quad z_B(x) = \frac{2x^4(1 + x)}{(1 - x)^2}, \quad z_F(x) = \frac{4x}{(1 - x)^2}, \tag{3.2}
\end{equation}

where $x = \exp(-\beta)$.

Polyakov loop $U_I = e^{i \beta A_{0,I}}$ satisfies the periodic condition $U_{2p+1} = U_1$ and $U_I^{-1} = U_I^\dagger$. We diagonalize the eigenvalues of holonomy matrix $U_I$ as $U_I = \exp(i \theta_{I,a})$ with $-\pi \leq \theta_{I,a} \leq \pi$ ($a = 1, \ldots, N$) where in the large $N$ limit, each $\theta_I$ is a continuous parameter with a density

\(^6\)For $N_F^0 = n$, we can choose $m^I_\alpha = \alpha - 1$ for $I = 1, 2$. 

---

- 7 –
\( \rho^I(\theta_I) \). The density satisfies \( \int_{-\pi}^{\pi} \rho^I(\theta_I) d\theta_I = 1 \). Using the density, the effective action is written as

\[
N^2 \sum_{1 \leq I,J \leq 2p} \int d\theta_I d\theta'_J \rho^I(\theta_I) \rho^J(\theta_J) \left[ -\delta_{IJ} \log \left| \sin \frac{\theta_I - \theta'_J}{2} \right| + \sum_{n=1}^{\infty} \frac{1}{n} M^{IJ}(z_n) \cos(n(\theta_I - \theta'_J)) \right],
\]

(3.3)

where \( M \) is a \( 2p \times 2p \) matrix as

\[
M = \begin{pmatrix}
0 & -z_n & 0 & \ldots & -z_n \\
-z_n & 0 & -z_n & \ldots & 0 \\
0 & -z_n & 0 & \ldots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
-z_n & 0 & 0 & \ldots & 0
\end{pmatrix}.
\]

(3.4)

The first term in (3.3) is obtained from the change of the measure. Using the Fourier transformation \( \rho_n^I = \int d\theta_I \rho^I(\theta_I) \cos(n\theta_I) \) the effective action is rewritten as

\[
S_{eff} = \sum_n \frac{N^2}{n} \rho_n^I (\delta^{IJ} + M^{IJ}) \rho_n^J,
\]

(3.5)

and \((\delta + M)_{IJ}\) satisfies a circulant determinant formula

\[
\det \begin{pmatrix}
g_1 & g_2 & g_3 & \ldots & g_k \\
g_k & g_1 & g_2 & \ldots & g_{k-1} \\
g_{k-1} & g_k & g_1 & \ldots & g_{k-2} \\
& \ddots & \ddots & \ddots & \ddots \\
g_2 & g_3 & g_4 & \ldots & g_1
\end{pmatrix} = \prod_{I=0}^{k-1} (g_1 + \omega^I g_2 + \omega^{2I} g_3 + \ldots + \omega^{(k-1)I} g_k),
\]

(3.6)

\[
\det(1 + M) = (1 - 2z_n) \prod_{I=1}^{2p-1} (1 - z_n \omega^I - z_n \omega^{(2p-1)I}),
\]

(3.7)

where \( \omega = \exp(\pi i/p) \).

The phase structure can be analyzed using the above effective action. At low temperature, the real eigenvalues of \( M \) are positive and so the trivial saddle point \( \rho_n = 0 \) is dominated when \( 2z_1 < 1 \) as \( z_n \) is monotonically decreasing function of \( n \). The order \( O(N^2) \) contributions are not included in this saddle point and the Casimir energy vanishes for the above \( d = 3 \) Chern-Simons-matter theory. For \( 2z_1 > 1 \), one of the eigenvalues becomes negative and another saddle point where the free energy is of order \( O(N^2) \) is dominated. The confinement/deconfinement phase transition happens at \( 2z_1 = 1 \). Note that the phase transition happens in the same critical temperature \( T_H = 1/\log(17 + 12\sqrt{2}) \) as that of the free ABJM theory analyzed in [49]. It implies that \( \mathbb{Z}_p \) orbifold symmetry is not broken in the free theory...
on $S^1 \times S^2$. We show this unbroken $\mathbb{Z}_p$ symmetry slightly above critical temperature and at high temperature from now on.

Slightly above the critical temperature, the density is non-zero only for $-\theta_{tc} \leq \theta_I \leq \theta_{tc}$. The saddle point of $\theta_I$ is obtained by solving following equation

$$
\int d\theta_I' \rho^I(\theta_I') \cot \left( \frac{\theta_I - \theta_I'}{2} \right) = -2 \sum_J \sum_{n=1}^\infty M_{IJ} \sin(n\theta_I) \rho_{n^J}^I. \tag{3.8}
$$

To obtain the free energy, the approximation $z_n = 0$ for $n > 1$ is used. Namely, only the first winding state in the time direction is excited and this approximation is valid in not high temperature. Reflecting $\mathbb{Z}_p$ symmetry of the Chern-Simons-matter theory, moreover, the densities of eigenvalues should take the same form $\rho^I(\theta) = \rho(\theta)$. Using the above approximation and $\mathbb{Z}_p$ symmetry, the equation (3.8) is rewritten as

$$
\int d\theta_I' \rho(\theta_I') \cot \left( \frac{\theta_I - \theta_I'}{2} \right) = 4z_1 \sin(\theta_I) \rho_1. \tag{3.9}
$$

The solution for (3.9) is given by

$$
\rho(\theta) = \frac{1}{\pi \sin^2 \frac{\theta_c}{2}} \sqrt{\sin^2 \frac{\theta_c}{2} - \sin^2 \frac{\theta}{2} \cos \frac{\theta}{2}} \tag{3.10}
$$

for $-\theta_{tc} \leq \theta \leq \theta_{tc}$. Here, $\theta_c$ satisfies

$$
\sin^2 \frac{\theta_c}{2} = 1 - \sqrt{1 - \frac{1}{2z_1}}, \tag{3.11}
$$

where inside of the root is positive for $2z_1 > 1$. Using $\mathbb{Z}_p$ symmetry of the densities of eigenvalues, the free energy $F = -\log Z/\beta$ is given by

$$
F = -\frac{2pN^2}{\beta} \left( \frac{1}{2 \sin^2 \frac{\theta_c}{2}} + \frac{1}{2} \log \left( \sin^2 \frac{\theta_c}{2} \right) - \frac{1}{2} \right). \tag{3.12}
$$

From the above formula, we can show the relation expected in the orbifold equivalence between the $U(pN) \times U(pN)$ ABJM theory and the $[U(N) \times U(N)]^p$ Chern-Simons-matter theory as

$$
F = \frac{F^{\text{ABJM}}}{p}, \tag{3.13}
$$

where $F^{\text{ABJM}}$ is the free energy of the $U(pN) \times U(pN)$ ABJM theory. We also compare the Polyakov loop vev between the Chern-Simons-matter theories which are equivalent under the orbifold equivalence. Polyakov loop vev normalized by the rank is given by the first moment $\rho_1$ of $\rho(\theta)$ [52]. It can be obtained from (3.10) as

$$
\langle \text{tr}(U_I) \rangle_N = \rho_1 = \begin{cases} 
1 & 2z_1 \geq 1 \\
\frac{1}{4z_1(1 - \sqrt{1 - \frac{1}{2z_1}})} & 0 < 2z_1 \leq 1
\end{cases} \tag{3.14}
$$
Remember that $\rho_1$ becomes $1/2$ at the critical point $2z_1 = 1$. Since $\rho_1$ does not depend on the orbifold action, the above result is consistent with the orbifold equivalence as

$$\langle \text{tr}(U^i) \rangle = \frac{\langle \text{tr}(U_i^{ABJM}) \rangle}{p}. \quad (3.15)$$

Finally, in the high temperature limit, the free energy of the Chern-Simons-matter theory is obtained using the different saddle point $\rho_n = 1$ for all $n$. The result agrees with the relation expected in the orbifold equivalence as

$$F = -28pT^3\zeta(3)N^2, \quad F = \frac{F_{ABJM}^2}{p}, \quad (3.16)$$

where $F_{ABJM}^2$ is the free energy of the ABJM theory.

We can also include flavor for the analysis of the above free theory which changes the order of the confinement/deconfinement transition [53–55]. Including flavor, it is known that large $N$ orbifold equivalence still holds if $Z_p$ symmetry is unbroken.

4 The gravity dual to the $\mathcal{N} = 4$ theory

According to [23], we verify the action of the above orbifold in the dual gravity side and show the orbifold equivalence between the ABJM theory and $\mathcal{N} = 4$ Chern-Simons-matter theory with the equal amount of twisted and untwisted hypermultiplets introduced in section 2 via holography. We consider the dual $AdS_4 \times S^7/Z_{pk} \times Z_p$ geometry of $\mathcal{N} = 4$ SCFT constructed via type IIB $N$ D3, $p$ NS5, and $p$ $(1,k)$ 5-branes. This SCFT corresponds to $\mathcal{N} = 4$ Chern-Simons-matter theory with the equal amount of twisted and untwisted hypermultiplets. It is convenient to represent $S^7/Z_{pk} \times Z_p$ in terms of 4 complex coordinates $X_i$ ($i = 1,2,3,4$) as

$$X_1 = \cos \xi \cos \frac{\theta_1}{2} e^{i\frac{\chi_1+\chi_2}{2}}, \quad X_2 = \cos \xi \sin \frac{\theta_1}{2} e^{i\frac{\chi_1-\chi_2}{2}},$$

$$X_3 = \sin \xi \cos \frac{\theta_2}{2} e^{i\frac{\chi_2+\chi_4}{2}}, \quad X_4 = \sin \xi \sin \frac{\theta_2}{2} e^{i\frac{\chi_2-\chi_4}{2}}, \quad (4.1)$$

where $0 \leq \xi < \pi/2$, $(\chi_1,\chi_2) \sim (\chi_1 + \frac{4\pi}{kp},\chi_2 + \frac{4\pi}{kp}) \sim (\chi_1 + \frac{4\pi}{p},\chi_2)$, $0 \leq \varphi_i < 2\pi$, and $0 \leq \theta_i < \pi$. The $Z_{kp}$ and $Z_p$ orbifold action is written in terms of $X_i$ as

$$\begin{align*}
(X_1, X_2, X_3, X_4) &\sim e^{2\pi i/(kp)}(X_1, X_2, X_3, X_4), \\
(X_1, X_2, X_3, X_4) &\sim (e^{2\pi i/p}X_1, e^{2\pi i/p}X_2, X_3, X_4).
\end{align*} \quad (4.3)$$

According to [26], $X_i$ can be identified with the complex parameters $y^A$ parametrizing $\mathbb{C}^4$. Using this identification, the isometry corresponding to $SU(2) \times SU(2)$ $R$-symmetry of the $\mathcal{N} = 4$ SCFT rotates $(X_1, X_2)$ and $(X_3, X_4)$, respectively. The orbifold action (2.3) and (2.4) for $y^A$ are also consistent with the above identification.
After considering the back-reaction of $N$ M2-branes and the near-horizon limit, the dual geometry of $\mathcal{N} = 4$ Chern-Simons-matter theory is described by

$$
\begin{align*}
    ds^2_{11D} &= \frac{R^2}{4} ds^2_{AdS_4} + R^2 ds^2_7, \\
    ds^2_7 &= d\xi^2 + \frac{1}{4} \cos^2 \xi ((d\chi_1 + \cos \theta_1 d\phi_1)^2 + d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) \\
    &\quad + \frac{1}{4} \sin^2 \xi ((d\chi_2 + \cos \theta_2 d\phi_2)^2 + d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2).
\end{align*}
$$

The isometry of $ds^2_7$ is $(SU(2) \times U(1))^2$ and is interpreted as the global symmetry of $\mathcal{N} = 4$ SCFT. To make the classical M-theory description valid, size of the orbifolded M-circle in terms of $\mathbb{Z}_{pk}$ must be larger than the 11-dimensional Plack length, $R/(l_p kp) \gg 1$ or $pN/(kp)^5 \gg 1$.

According to [27], we also consider the Kaluza-Klein reduction to type IIA string theory setting $\alpha' = 1$ as

$$
\begin{align*}
    ds^2_A &= ds^2_6 + \frac{1}{kp} (dy + A)^2, \\
    e^2 &\equiv \frac{R^3}{kp^3}, \\
    A &= kp \left( \frac{1}{2} \cos^2 \xi (d\psi + \cos \theta_1 d\phi_1) + \frac{1}{2} \sin^2 \xi \cos \theta_2 d\phi_2 \right), \\
    ds^2_6 &= d\xi^2 + \frac{1}{4} \cos^2 \xi \sin^2 \xi ((d\psi + \cos \theta_1 d\phi_1 - \cos \theta_2 d\phi_2)^2 \\
    &\quad + \frac{1}{4} \cos^2 \xi (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) } \\
    &\quad + \frac{1}{4} \sin^2 \xi (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2), \\
    ds^2_{IA} &= l^2 (ds^2_{AdS_4} + 4 ds_6^2), \\
    L^2 &= \frac{R^3}{4kp} = 2^{1/2} \pi \sqrt{\frac{N}{k}}, \\
    0 < y \leq 2\pi, &\quad 0 < \psi \leq \frac{4\pi}{p}, \\
    \chi_1 &= \psi + \frac{2y}{kp}, &\quad \chi_2 &= \frac{2y}{kp}.
\end{align*}
$$

The weakly coupled type IIA theory description is valid when the size of the orbifolded M-circle or the string coupling is small $pN/(kp)^5 \ll 1$. Remember that the metric $ds_6^2$ is equal to the orbifold $\mathbb{C}P^3/\mathbb{Z}_p$. The D2-brane flux and the D6-brane flux $(F_2 = dA)$ are quantized as $N (= \int F_4/(2\pi)^3)$ and $kp (= \int F_2/(2\pi))$, respectively, consistent with the brane configuration since we must have $p (1,k)5$-branes in the type IIA elliptic D3-brane configuration through T-duality. The curvature radius $2^{1/2} \pi \sqrt{N/k}$ of $AdS_4 \times \mathbb{C}P^3/\mathbb{Z}_p$ in the type IIA should be large to make the supergravity description valid. That is, the type IIA description is valid when $pk \ll pN \ll (pk)^5$.

Hereby, we discuss the orbifold equivalence between the $\mathcal{N} = 4 [U(N)k \times U(N)_{-k}]^p$ Chern-Simons-matter theory and the $U(pN)_{pk} \times U(pN)_{-pk}$ ABJM theory. Note that the curvature radius of the above type IIA solution is equal to the curvature radius of the type IIA solution $2^{1/2} \pi \sqrt{N/k}$ dual to the $U(pN)_{pk} \times U(pN)_{-pk}$ ABJM theory. In addition, the D6-brane flux of the latter theory is the same as that of the former theory $pk$. Thus, the type IIA geometry for
$\mathcal{N}=4$ SCFT is related with the type IIA geometry of the ABJM theory by the $\mathbb{Z}_p$ orbifold\footnote{Actually, the region in which both the type IIA theory and the M-theory description are valid is precisely the same form.}. The equivalence should work for any observables that are invariant under the $\mathbb{Z}_p$ projection.

When we include probe D6-branes dual to flavor, we also need to take care of $\mathbb{Z}_p$ discrete symmetry in the presence of flavor. For $\mathcal{N}=4$ SCFT with $2p$ nodes, the D6-brane corresponding to the massless flavor wraps $AdS_4 \times S^3/\mathbb{Z}_{2p}$ inside $AdS_4 \times \mathbb{C}P^3/\mathbb{Z}_p$ without a tadpole problem \cite{27} (see also \cite{25, 26}). In large $N$ orbifold equivalence between the ABJM theory and $\mathcal{N}=4$ SCFT with $2p$ nodes, the D6-brane corresponding to the massless flavor wraps $AdS_4 \times S^3/\mathbb{Z}_{2p}$ inside $AdS_4 \times \mathbb{C}P^3/\mathbb{Z}_p$ without a tadpole problem \cite{27} (see also \cite{25, 26}). In large $N$ orbifold equivalence between the ABJM theory and $\mathcal{N}=4$ SCFT with $2p$ nodes, we should add $2pN_F$ D6-branes to recover $\mathbb{Z}_p$ discrete symmetry of the orbifold. Moreover, we find that the node coupling to fundamentals can be specified using the holonomy of the Wilson line\footnote{We would like to thank A. Karch for pointing out this point.} $\pi_1(S^3/\mathbb{Z}_{2p}) = \mathbb{Z}_{2p}$. We can show that in large $N$ orbifold equivalence, the fluctuations of the D6-brane in the neutral sector in the presence of the $\mathbb{Z}_{2p}$ holonomy coincides between the parent theory and the daughter theory. As an application, the holographic BKT phase transition \cite{28} in terms of the ABJM with flavor can be applied in large $N$ orbifold equivalence since the fluctuation around the massless embedding corresponds to the neutral sector of the orbifold. We leave the general analysis including massive flavor in the future work.

5 The gravity dual to the $\mathcal{N}=3$ Chern-Simons-matter theory with flavor

In this section, we review the $d=8$ transverse geometry of M2-branes describing the $\mathcal{N}=3$ quiver Chern-Simons-matter theory with flavor. This transverse geometry becomes $d=8$ toric hyperKähler manifold, where toric means there is at least two-torus inside it. Using the cone structure of this transverse space, we also observe that the Eschenburg space which is tri-Sasaki manifold gives the gravity dual of the $\mathcal{N}=3$ quiver Chern-Simons-matter theory with flavor.

The metric of $d=8$ toric hyperKähler manifold ($\varphi_i \in (0, 4\pi]$) is given by

$$
\begin{align}
\left\{ & ds^2 = \frac{1}{2} U_{ij} dx_i \cdot dx_j + \frac{1}{2} U^{ij} (d\varphi_i + A_i)(d\varphi_j + A_j) \\
A_i = dx_j \cdot \omega_{ji} = dx_j^a \omega_{ji}^a, & \quad \partial_{x^a_j} \omega_{ki}^b - \partial_{x^a_k} \omega_{ji}^b = \epsilon^{abc} \partial_{x^c_j} U_{ki}, 
\end{align}
$$

(5.1)

where $i, j, k = 1, 2, a, b, c = 1, 2, 3$, and $U^{ij}$ is the inverse matrix of $U_{ij}$.

If we introduce $p$ NS5 and $p$ $(1, k)5$-branes in a type IIB brane setup, we have

$$
U_{ij} = \frac{1}{2} \left( \begin{array}{ccc}
p & kp \\
|\mathbf{x}_1| & \mathbf{x}_1 + k\mathbf{x}_2 & \mathbf{x}_1 + k\mathbf{x}_2 \\
p & kp \\
|\mathbf{x}_1 + k\mathbf{x}_2| & \mathbf{x}_1 + k\mathbf{x}_2 
\end{array} \right)
$$

(5.2)
where $U$ is normalized being consistent with the quantization condition on the flux of $\omega$. We can use the $GL(2)$ transformation to diagonalize the matrix $U$.

\[
(x'_1, x'_2) = (x_1, x_2)G^t = p(x_1, x_1 + kx_2),
\]

\[
(\varphi'_1, \varphi'_2) = (\varphi_1, \varphi_2)G^{-1} = \left( \frac{\varphi_1}{p} - \frac{\varphi_2}{kp}, \frac{\varphi_2}{kp} \right),
\]

\[
G = \begin{pmatrix} p & 0 \\ p & kp \end{pmatrix}, \quad U \to U' = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{1}{|x'_2|} \end{pmatrix}. \tag{5.3}
\]

The orbifold action of $(\varphi'_1, \varphi'_2)$ is given by

\[
(\varphi'_1, \varphi'_2) \sim \left( \varphi'_1 + \frac{4\pi}{p}, \varphi'_2 \right), \quad (\varphi'_1, \varphi'_2) \sim \left( \varphi'_1 - \frac{4\pi}{kp}, \varphi'_2 + \frac{4\pi}{kp} \right). \tag{5.4}
\]

The contribution of $N_F$ flavor is included by adding the following extra $\Delta U$ to $U$:

\[
\Delta U = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \frac{N_F}{|x_2|} \end{pmatrix}. \tag{5.5}
\]

Here, the $GL(2)$ transformation (5.3) acts on $\Delta U$ as

\[
\Delta U \to \Delta U' = \frac{1}{2} \begin{pmatrix} N_F & -N_F \\ kpL & kpL \end{pmatrix}, \quad L = |x'_2 - x'_1|.
\]

Here, we see following symmetry in the above metric. Since $N_F$ is non-zero, there is a common element of $SO(3)$ which rotates $(x'_1, x'_2)$ in order to preserve $L$. Furthermore, we generate $U(1)_b \times U(1)_d$ corresponding to two $U(1)$’s of $(\varphi'_1, \varphi'_2)$ by using gauge transformations of $(A'_1, A'_2)$: $(A'_1, A'_2) = (A'_1 + d\lambda_1, A'_2 + d\lambda_2)$. It is convenient to use the transformation $x'_1 \to -x'_1$, $\varphi'_1 \to -\varphi'_1$ and perform the following reparametrization in terms of three $t$’s:

\[
\Delta U' = \frac{1}{2} \begin{pmatrix} t_2^2 & t_1t_2 \\ t_1t_2 & t_1^2 \end{pmatrix} \begin{pmatrix} t_3|t_1x'_1 + t_2x'_2| & t_3|t_1x'_1 + t_2x'_2| \\ t_3|t_1x'_1 + t_2x'_2| & t_3|t_1x'_1 + t_2x'_2| \end{pmatrix}.
\]

where $(t_1, t_2, t_3) = (N_F, N_F, kp)$ are relatively prime without a divisor as seen in the above metric components [27]. Using $V = U' + \Delta U'$, the metric in (5.1) is rewritten as

\[
ds^2 = \frac{1}{2} V_{ij} dx'_i \cdot dx'_j + \frac{1}{2} V^{ij}(d\varphi'_i + A'_j)(d\varphi'_j + A'_i). \tag{5.6}
\]

It can be shown that $ds^2$ in (5.6) becomes the cone over an $d = 7$ Eschenburg space $\mathcal{M}_7 = S^{(p, kp)}(N_F, N_F, kp)$ with the orbifold action (5.4)\footnote{In the previous paper [27], we did not clarify the inclusion of the orbifold. However, this orbifold does not change the volume of the Eschenburg space.}, namely, $Z_p \times Z_{kp}$. $ds^2$ in (5.6) has the
isometry $SO(3) \times SU(2)_d \times U(1)_b$, where we have $SU(2)_d$ enhanced global symmetry instead of $U(1)_d$ [45] since $t_1 = t_2 \neq t_3$.

Note that by replacing $kp$ with $k'$ in the metric (5.6), we have the cone over an Eschenburg space $\mathcal{M}_7 = S_7^{(p,k')}(N_F, N_F, k')$. Note that this $\mathcal{M}_7$ is the $\mathbb{Z}_p$ orbifold of the $p = 1$ case, $\mathcal{M}_7^{(p=1)} = S_7^{(1,k')}(N_F, N_F, k')$. In other words, operating the $\mathbb{Z}_p$ orbifold on $\mathcal{M}_7^{(p=1)}$, both the metrics become the same. It implies that the $\mathbb{Z}_p$ orbifold equivalence works as seen in next section.

We can also consider the replacement of $N_F$ with $N_F'k$. Then, we have the cone over an Eschenburg space $\mathcal{M}_7 = S_7^{(p,k)}(N_F', N_F', p)$. Note that we have different charges because three charges $(N_F'k, N_F'k, pk)$ are not relatively prime. This $\mathcal{M}_7$ is exactly the $\mathbb{Z}_{kp}$ orbifold of the $k = 1$ case, $\mathcal{M}_7^{(k=1)} = S_7^{(p,p)}(N_F', N_F', p)$ instead of the $\mathbb{Z}_p$ orbifold.

After including the backreaction of $N$ M2-branes on the $d = 8$ transverse space and taking the near horizon limit, we obtain $AdS_4 \times \mathcal{M}_7$,

$$
\begin{align*}
\text{d}s_{11D}^2 &= \frac{R^2}{4} \text{d}s_{AdS_4}^2 + R^2 \text{d}s_7^2, \quad N = \frac{1}{(2\pi \ell_p)^6} \int_{\mathcal{M}_7} \ast F_4, \\
F_4 &= \frac{3}{8} R^3 \text{vol}_{AdS_4}, \quad R^6 \text{vol}(\mathcal{M}_7) = (2\pi \ell_p)^6 N, 
\end{align*}
$$

where the relation $R_{ab}(\mathcal{M}_7) = 6g_{ab}(\mathcal{M}_7)$ is satisfied and $R = 2R_{AdS}$ is the radius of $\mathcal{M}_7$. This background is the gravity dual of the strongly-coupled limit of the $\mathcal{N} = 3$ Chern-Simons-matter theory.

### 5.1 The $\mathbb{Z}_p$ orbifold equivalence

According to [45], we have the following relation of the volume of the Eschenburg space and the radius $R$:

$$\frac{\text{vol}(S_7^7)}{\text{vol}(\mathcal{M}_7)} = \frac{2p(N_F + kp)^2}{(N_F + 2kp)}, \quad R^6 = 2^5 \pi^2 p N_l^6 \cdot \frac{2(N_F + kp)^2}{(N_F + 2kp)},$$

where $\text{vol}(S_7^7) = \pi^4/3$. It is interesting to consider the case of $p = 1$, namely, $\mathcal{M}_7^{(p=1)} = S_7^{(1,k')}(N_F, N_F, k')$. The volume and the radius $R'$ of $\mathcal{M}_7^{(p=1)}$ are given by

$$\frac{\text{vol}(S_7^7)}{\text{vol}(\mathcal{M}_7^{(p=1)})} = \frac{2(N_F + k')^2}{(N_F + 2k')}, \quad R'^6 = 2^5 \pi^2 N_l^6 \cdot \frac{2(N_F + k')^2}{(N_F + 2k')}.$$  

Note that by replacing $kp$ with $k'$ in the metric (5.6) and (5.8), the volume of the Eschenburg space $\text{vol}(\mathcal{M}_7^{(p=1)})$ is found to be the product of $\text{vol}(\mathcal{M}_7)$ and the orbifold factor $p$ as

$$\text{vol}(\mathcal{M}_7)p = \text{vol}(\mathcal{M}_7^{(p=1)}).$$

Using (5.10), we can show that the radius $R$ of $AdS_4 \times \mathcal{M}_7$ is equal to the radius of $AdS_4 \times \mathcal{M}_7^{(p=1)}$ setting $pN$ M2-brane flux of the latter theory. Thus, the formula (5.9) shows that the M2-brane theory with $N$ M2-branes and $\mathcal{M}_7 = S_7^{(p,k')}(N_F, N_F, k')$ are equivalent to that
with $pN$ M2-branes and $\mathcal{M}_7^{(p=1)} = S_7^{(1,k')} (N_F, N_F, k')$ in terms of the orbifold equivalence since the metric of the Eschenburg space becomes the same up to the $\mathbb{Z}_p$ orbifold. Thus, we observe the structure of the orbifold equivalence between the $\mathcal{N} = 3 \ U(pN)_k \times U(pN)_{-k'}$ quiver Chern-Simons-matter theory with $N_F$ flavor and the $\mathcal{N} = 3 \ (U(N)_k \times U(N)_{-k})^p$ quiver Chern-Simons-matter theory with $N_F$ flavor.

5.2 The $\mathbb{Z}_{kp}$ orbifold equivalence

The case $k = 1$ is also interesting, namely, $\mathcal{M}_7^{(k=1)} = S_7^{(p,p)} (N'_F, N'_F, p)$. The volume and the radius $R'$ of $\mathcal{M}_7^{(k=1)}$ are obtained from (5.8) as

$$\frac{\text{vol}(S^7)}{\text{vol}(\mathcal{M}_7^{(k=1)})} = \frac{2p(N'_F + p)^2}{(N'_F + 2p)}, \quad R'^6 = 2\pi^2 pN'^6_f \cdot \frac{2(N'_F + p)^2}{(N'_F + 2p)}. \quad (5.11)$$

Note that by substituting $N_F = kN'_F$ into the metric (5.6) and (5.8), the volume of the Eschenburg space $\text{vol}(\mathcal{M}_7^{(k=1)})$ becomes the product of $\text{vol}(\mathcal{M}_7)$ and the orbifold factor $k$ as

$$\text{vol}(\mathcal{M}_7)k = \text{vol}(\mathcal{M}_7^{(k=1)}). \quad (5.12)$$

Using (5.12), it can be shown that the curvature radius $R$ of $AdS_4 \times \mathcal{M}_7$ is equal to the curvature radius of $AdS_4 \times \mathcal{M}_7^{(k=1)}$ setting $kN$ M2-brane flux of the latter theory. Thus, the formula (5.11) implies the orbifold equivalence between $N$ M2-branes with $S_7^{(p,kp)} (N_F, N_F, kp)$ and $kN$ M2-branes with $S_7^{(p,p)} (N'_F, N'_F, p)$ since the metric of the Eschenburg space becomes the same up to the $\mathbb{Z}_{kp}$ orbifold. Thus, we find that the $\mathcal{N} = 3 \ (U(kN)_1 \times U(kN)_{-1})^p$ quiver Chern-Simons-matter theory with $N'_F$ flavor is equivalent to the $\mathcal{N} = 3 \ (U(N)_k \times U(N)_{-k})^p$ quiver Chern-Simons-matter theory with $N_F = N'_F k$ flavor in view of the orbifold equivalence.

As an example, we consider the BPS observables of Chern-Simons-matter theories including flavor invariant under the $\mathbb{Z}_k$ orbifold projection. We start with $[U(N)_k \times U(N)_{-k}]^p$ and $kN'_F$ fundamentals. There are operators charged under $U(1)_b$ and operators neutral under $U(1)_b$. Hereby, we concentrate on the operators charged under $U(1)_b$. The operators dual to D0-brane is the operator with smallest dimension and charged under $U(1)_b$ [25].

Note that in the Abelian case, the Chern-Simons EOM is satisfied via a constant magnetic flux $m$ on the sphere. Thus, we can construct the operators charged under $U(1)_b$ by introducing diagonal monopole operators. Such a monopole operator is shown to be BPS and is defined as $T^{(m)}$ with the same monopole flux $m(\in \mathbb{Z}_n)$ under all $U(1)^{2p}$ subgroup of $U(N)^{2p}$ gauge groups [56]. In the Chern-Simons-matter theories with $\sum k'_i = 0$, the monopole $T^{(m)}$ has the charges $(mk, -mk, \ldots, -mk)$ under $U(1)^{2p}$ subgroup. Moreover, it is known that the monopole operators $T^{(m)}$ can be charged under any $U(1)$ symmetry via quantum corrections [57, 58]. In the Abelian case, the quantum correction to the R-charge of the monopole operators is [31]

$$\delta R[T^{(m)}] = -\frac{m}{2} \sum_{\psi} R[\psi] = \frac{mkN'_F}{2}, \quad (5.13)$$
where we summed over the R-charge for all fermions. We used the fact that the gaugino has the charge 1 which is cancelled by the R-charges of bifundamentals.

The gauge invariant operators are of the form

$$T^{(m)} \prod_i A_i^{d_i} \quad \text{for } d_i - d_{i+1} = k_i m, \quad (5.14)$$

where $d_i$ is proportional to the D5-brane charges of the $i$-th fivebranes. The above operator has the baryonic charge $\sum_i d_i = mkp$. When $m = 1$, the operator (5.14) is dual to the D0-brane. The conformal dimension of the operators (5.14) is given by

$$\Delta = \frac{1}{2} \left( \sum d_i + mkN_F' \right) = \frac{mk}{2} (p + N'_F). \quad (5.15)$$

Namely, the operators preserving the number $m_1k_1 = m_2k_2$ under the orbifold action are the invariant operators under the $\mathbb{Z}_k$ orbifold projection which does not change the number of nodes. Especially, the operator with flux $m > 1$ in the parent theory can be mapped into the operator dual to the D0-brane in the daughter theory.

5.3 Orbifold equivalence in terms of entropy

In this subsection, we show that the orbifold equivalence works for the Bekenstein-Hawking Entropy. We consider AdS-Schwarzschild black hole metric as

$$ds^2 = \left( \frac{4r^2}{R^2} + 1 - \frac{M}{r} \right) d\tau^2 + \frac{dr^2}{\left( \frac{4r^2}{R^2} + 1 - \frac{M}{r} \right)} + r^2 d\Omega_2^2. \quad (5.16)$$

The above metric describes the finite temperature Chern-Simons-matter theory on $S^1 \times S^2$. The inverse temperature $\beta$ is given by

$$\beta = \frac{\pi R^2 r_0}{3r_0^3 + \frac{R^2}{4}}, \quad (5.17)$$

where $r_0$ describes the horizon radius. Solving (5.17), the horizon radius is represented as

$$r_0 = \frac{\pi R^2}{6\beta} \left( \frac{\pi R^2}{6\beta} \right)^2 \frac{R^2}{12}. \quad (5.18)$$

From (5.18), we find that AdS black holes exist at the temperature larger than $\sqrt{3}/(\pi R)$.

Using the metric (5.7), the Bekenstein-Hawking area law gives the entropy of $N$ M2-branes on the singularity of Cone($\mathcal{M}_7$) per $\text{vol}(S^2)R^2/4$ as (see also [59])

$$S_E = \frac{2^{\frac{3}{2}} \pi^4 N^3 \left( 1 + \sqrt{1 - \frac{3\beta^2}{\pi^2 R^2}} \right)}{3^2 \beta^2 \sqrt{\text{vol} (\mathcal{M}_7)}}. \quad (5.19)$$
We compare (5.19) with the cases for the M-theory on $AdS_4 \times M_7^{(k=1)}(\mathcal{M}_7^{(p=1)})$, where we have $pN(kN)$ numbers of the M2-brane flux. Defining the corresponding entropy $S_{E}^{(p=1)}(S_{E}^{(k=1)})$ for each case, we obtain the following relations as

$$\frac{S_E}{S_{E}^{(p=1)}} = \frac{1}{p} \sqrt{\frac{\text{vol}(\mathcal{M}_7^{(p=1)})}{p \cdot \text{vol}(\mathcal{M}_7)}} = \frac{1}{p}, \quad \frac{S_E}{S_{E}^{(k=1)}} = \frac{1}{k} \sqrt{\frac{\text{vol}(\mathcal{M}_7^{(k=1)})}{k \cdot \text{vol}(\mathcal{M}_7)}} = \frac{1}{k}, \quad (5.20)$$

where we used the relations (5.10) and (5.12) among $\text{vol}(\mathcal{M}_7)$, $\text{vol}(\mathcal{M}_7^{(k=1)})$, and $\text{vol}(\mathcal{M}_7^{(p=1)})$. Usually, the orbifold just affects the geometry through a projection in the internal space, which changes its volume and this will be reflected in the entropy. Thus, the above relations are those expected in the orbifold equivalence. Note that the entropy (5.19) in the M-theory region is consistent with the planar equivalence outside the planar limit [20]. It implies that large $N$ equivalence holds even outside the ’t Hooft limit as long as there is a classical gravity dual.

We can see that the orbifold equivalence also works until the Hawking-Page transition happens. It is known that the Hawking-Page transition [46, 47] between the thermal $AdS$ background and the $AdS$ black hole happens at $\beta_c = \pi R/2$ ($r_0 = R/2$) above the temperature bound $\beta_c < \pi R/\sqrt{3}$ as seen in (5.20). In the field theory side, the Hawking-Page transition can be interpreted as the confinement/deconfinement transition since the free energy is the order parameter of the phase transition which changes from being zero for the thermal $AdS$ in the planar limit into being of order $N^{3/2}$ for the $AdS$ black hole. Note that since the critical temperature does not depend on the orbifold but depends on $R$, the critical temperature is not changed between the mother theory and the daughter theory of the orbifold where we have the same $AdS$ radius. In other words, the $AdS$ part is not affected and this means that the Hawking-Page transition indeed occurs at the same temperature. This result shows that in the field theory side dual to the Eschenburg space, the critical point of the confinement/deconfinement transition does not change under the orbifold action since symmetry of the orbifold is not broken in the deconfinement phase. For the confinement phase corresponding to the thermal $AdS$ case, on the other hand, we can not discuss the equivalence since the free energy vanishes in the planar limit.

### 6 Explanation by using mirror symmetry

Mirror symmetry of the $U(N)_k \times U(N)_{-k}$ ABJM theory is considered in [17, 65] including a step of the mass deformation for fundamentals in the type IIB string theory. In this section, we use mirror symmetry in the type IIB elliptic D3-brane configuration to explain the $\mathbb{Z}_k$ orbifold equivalence of the $\mathcal{N} = 3$ Chern-Simons-matter theories with flavor. Mirror symmetry takes a theory with coupling $g_{YM}^2 = O(1/N)$ to a theory with coupling $g_{YM}^2 \sim 1/g_{YM}^2 = O(N)$. We start with the original theory and after taking a mirror dual, we analyze the orbifold of the mirror theory in the ’t Hooft limit. Remember that when $k$ is small, we should see the
IR fixed points of the original theory of the energy $E/\lambda_{3d} \sim O(1/N^2)^{10}$. The IR fixed point actually describes the region where Yang-Mills terms decouple in the presence of the adjoint mass.

![Wireframe diagrams](s22.png)

**Figure 2.** (a): The $\mathcal{N}=2$ type IIB brane configuration of the original theory for $k = 3$ and $N_F = 12$ (b): The mirror dual of (a)

To obtain the mirror to the $\mathcal{N}=3$ Chern-Simons-matter theories with flavor, we start with the $\mathcal{N}=2$ type IIB brane configuration of the original theory given by $N$ D3-branes, $k$ D5-branes, 2 NS5', and $N_F$ D5-branes with different orientation where we consider $N_F = kN'_F$ numbers of D5 for convenience. They are given in the following table as

|       | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $N$   | $\times$ | $\times$ | $\times$ |$\times$ | | | | | | |
| $k$   | $\times$ | $\times$ | $\times$ | | $\times$ | $\times$ | | | | |
| 2     | $\times$ | $\times$ | $\times$ | | $\times$ | $\times$ | | | | |
| $N_F$ | $\times$ | $\times$ | | | $\times$ | $\times$ | | | | |
| $D5$  | | | | | | | | | | |

Note that since there are 2 transverse directions for both D5-branes and 3 transverse directions for both D5' and NS5', respectively, we can exchange them on the D3-branes without hitting each other [60]. 5-branes are placed on D3-branes and aligned along $x_6$ in the order of 2 NS5', a D5', $N'_F$ D5, a D5', $N'_F$ D5, . . . , and $N'_F$ D5 where D5 and D5' are placed to be symmetric in terms of $\mathbb{Z}_k$ in the absence of 2 NS5'. See Figure 2(a).

We consider the mirror duality exchanging D5-branes with NS5-branes and make D3-branes invariant. The $\mathcal{N}=2$ type IIB brane configuration of the mirror theory is given by $N$ D3-branes, $k$ NS5, 2 D5-branes, and $N_F$ NS5' with different orientation. They are given in the following table as

|       | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $N$   | $\times$ | $\times$ | $\times$ | | | | | | | |
| $k$   | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | | | | | |
| 2     | $\times$ | $\times$ | | $\times$ | $\times$ | | | | | |
| $N_F$ | $\times$ | $\times$ | $\times$ | | | $\times$ | $\times$ | | | |
| $NS5'$| | | | | | | | | | $\times$ |

10When $k \sim N$, there are many massless states in the mirror theory.
5-branes are placed on D3-branes and aligned along $x_6$ in the order of 2 D5, a NS5, $N'_F$, NS5', a NS5, $N'_F$, NS5', ..., and $N'_F$, NS5' where NS5 and NS5' are placed to be rotationally $Z_k$ symmetric in the absence of 2 D5. See Figure 2(b).

The gauge group of the above configuration becomes $[U(N)^2 \times U(N)^{N'_F-1}]^k$. In the aligned NS5 and NS5' configurations, we have the adjoint matter between two NS5'-branes with the same direction and we do not have the adjoint matter between the NS5 and the NS5' [61–63] because we have only $\mathcal{N}=2$ supersymmetry. We consider the cross configuration where each D5-brane is on top of a single NS5-brane. This configuration preserves $d=3$ $\mathcal{N}=2$ supersymmetry and for a single NS5-brane, there are four copies of fundamental chiral multiplets for the two gauge groups via the flavor doubling. The global symmetry associated with these chiral multiplets is enhanced from $U(1)^2$ to $U(1)^4$.

On the other hand, we can take the $Z_k$ orbifold for the mirror theory with $kN$ D3-branes, two D5, one NS5, and $N'_F$ NS5'. We consider the cross configuration where each D5-brane is on top of a NS5. The gauge group of the mother theory is then $\prod_{i=1}^{1+N'_F} U(kN)_i$. The matter content consists of $N'_F - 1$ chiral multiplet in the adjoint $Y_i$ ($i = 3, ..., N'_F + 1$), 1 + $N'_F$ hyper multiplets transforming in the bifundamental representation of $(i, i + 1)$ gauge groups $(A_{i,i+1}, B_{i+1,i})$, 4 chiral multiplets transforming in the fundamental representation under the first and second gauge group $L(a)_1, R(a)_2$, and four chiral multiplets transforming in the anti-fundamental $\tilde{L}(a)_1, \tilde{R}(a)_2$ where $a = 1, 2$. The $\mathcal{N}=2$ superpotential becomes

$$S = \sum_{a=1}^{2} [\tilde{L}(a)_1 A_{12} R(a)_2 - \tilde{R}(a)_2 B_{21} L(a)_1].$$ (6.1)

The $Z_k$ orbifold projection is obtained from the element of each gauge group $\prod_{i=1}^{1+N'_F} U(kN)_i$ and spans a $Z_k$ subgroup [64] as

$$\gamma = \text{diag}(1_N, \omega 1_N, \omega^2 1_N, \ldots, \omega^{k-1} 1_N),$$ (6.2)

where $1_N$ is the $N \times N$ identity matrix and we have defined the phase $\omega = e^{2\pi i/k}$. $k$ should be relatively prime to $N'_F + 1$ since otherwise, the quiver diagram is separated into many parts as also observed in the case of orbifolds of the ABJM theory [22].

The quiver gauge theory is obtained from the $\prod_{i=1}^{1+N'_F} U(kN)_i$ theory by keeping the components that are invariant under the orbifold projection as

$$V_i \rightarrow \gamma V_i \gamma^{-1}, \quad Y_i \rightarrow \gamma Y_i \gamma^{-1},$$ (6.3)

$$A_{i,i+1} \rightarrow \omega \gamma A_{i,i+1} \gamma^{-1}, \quad B_{i+1,i} \rightarrow \omega^{-1} \gamma B_{i+1,i} \gamma^{-1},$$ (6.4)

$$\tilde{L}(a)_1 \rightarrow \tilde{L}(a)_1 \gamma^{-1}, \quad R(a)_2 \rightarrow \omega^{-1} \gamma R(a)_2, \quad (a = 1, 2),$$ (6.5)

$$L(a)_1 \rightarrow \gamma L(a)_1, \quad \tilde{R}(a)_2 \rightarrow \omega \gamma R(a)_2 \gamma^{-1} (a = 1, 2).$$ (6.6)

After the $Z_k$ orbifold, the gauge group becomes $[U(N)^2 \times U(N)^{N'_F-1}]^k$ and realizes the mirror brane configuration. The flavor fields couple to two nodes separated by a NS5-brane.
In addition, the \( \mathbb{Z}_k \) symmetry of the daughter theory is now seen as the symmetry rotating \( 1 + N'_F \) units of the nodes along the quiver diagram in the absence of the flavor. The presence of the flavor breaks \( \mathbb{Z}_k \) symmetry in the UV of the daughter theory. However, according to [65], mirror symmetry implies that in the deep IR of this gauge theory, all 5-branes are gathered in the same position on D3-branes and the global symmetry \( U(1)^4 \) is enhanced to \( U(k) \times U(k) \times U(2) \times U(2) \times U(N_F) \) for both the original theory and the mirror theory. Thus, we seem to recover \( \mathbb{Z}_k \) symmetry in the deep IR.

The flow to the IR fixed point described by the \( \mathcal{N} = 3 \) Chern-Simons-matter theory is not directly given by the above brane configurations. Though the field content is the same, we should give a mass deformation for the fields coming from the D5-branes and D5'-branes in the original \( U(kN)^2 \) theory. In the mirror theory side \( \prod_{i=1}^{1+N'_F} U(kN)_i \), the mass deformation maps to some non-local deformation such as the monopole operator [17]. It implies that the lagrangian description of the mirror dual does not exist. However, it can be shown that large \( N \) orbifold equivalence can be proven by not using the lagrangian description but using the brane configuration non-perturbatively.

7 Discussion

In this paper, we showed two large \( N \) orbifold equivalences between \( d = 3, \mathcal{N} = 3, 4 \) Chern-Simons-matter theories. Here, \( \mathcal{N} = 3 \) Chern-Simons-matter theories include flavor. We first analyzed the \( \mathbb{Z}_p \) orbifold equivalence for the orbifold changing the nodes of the gauge groups. For \( \mathcal{N} = 4 \) case, we found evidence that the \( \mathbb{Z}_p \) orbifold equivalence holds from the M-theory limit to the weak-coupling limit by analyzing the gravity dual and the free theory on \( S^1 \times S^2 \). For the analysis in the free theory, we showed that the free energy, the Polyakov loop vev, and the critical temperature of the phase transition agree with the relation expected in the orbifold equivalence.

For \( \mathcal{N} = 3 \) case with flavor, we can believe the equivalence when flavor is aligned to reflect \( \mathbb{Z}_p \) symmetry in the daughter theory. We showed that the \( \mathbb{Z}_p \) equivalence holds in the M-theory region using the gravity dual. When \( N_F \ll N \), large \( N \) orbifold equivalence using the type IIA string theory could be analyzed by introducing probe D6-branes corresponding to flavor without back reactions of them. Here, the \( \mathbb{Z}_{2p} \) holonomy on the probe brane was used to specify the node coupling to fundamentals. For the case of backreacted flavor, the dilation or the coefficient of the M-circle depends on the internal coordinates [25] under the subtle dimensional reduction to the type IIA superstring theory. We leave large \( N \) equivalence in this type IIA string theory for the future work. It will be interesting to analyze the orbifold equivalence of the Chern-Simons-matter theory with flavor in the weak-coupling limit where we should include finite \( \lambda \) corrections to describe the first-order phase transition instead of the third-order phase transition for zero ’t Hooft coupling.

Secondly, we analyzed the \( \mathbb{Z}_k \) orbifold equivalence changing Chern-Simons-levels in the M-theory region. We confirmed large \( N \) equivalence by computing the BPS monopole operators sensitive to the \( \mathbb{Z}_k \) projection (see also [16]). It will be interesting to apply the
equivalence between the BPS monopole operators for other Chern-Simons-matter theories. We also showed that the critical temperature of Hawking-Page transition does not change since the $AdS$ part is not affected by the orbifold and the Bekenstein-Hawking entropy behaves as expected in large $N$ orbifold equivalence. In the M-theory limit, the entropy (5.19) was also consistent with the planar dominance outside the planar limit [20]. It implies that large $N$ planar equivalence holds even outside the ’t Hooft limit when there exists a classical gravity dual. It is known that however, the large $N$ equivalence is broken when $1/N$ corrections coming from the non-planar diagram are included. In the $Z_k$ equivalence, the change of Chern-Simons-levels can be interpreted as the change of the number of D5-branes in the mirror theory side of the type IIB elliptic D3-brane configuration.

In [16], the $Z_k$ equivalence of Chern-Simons-matter theory was also confirmed from the free energy computed by using the localization method [66–69]. For unquenched flavor case, since the behavior of the free energy [70, 71] was the same as that derived from gravity dual $N^{3/2}/\sqrt{\text{vol}(M_7)}$, it was consistent with the $Z_k$ orbifold equivalence between $U(kN)_1 \times U(kN)_{−1}$ theory with $N_F$ flavor and $U(N)_k \times U(N)_{−k}$ theory with $kN_F$ flavor.

It will also be interesting to analyze large $N$ equivalence between the Kaluza-Klein spectra on the Eschenburg space and that of its orbifold for understanding of the $Z_k$ orbifold projection. Actually, for $(t_1, t_2, t_3) = (1, 1, 1)$, $S_7^{(1,1)}(1,1,1) = N(1,1)$ and its Kaluza-Klein spectra are known in [72–74].

Acknowledgement We would like to thank A. Armoni, S. Cremonesi, S. Kachru, A. Karch, E. Silverstein, S. Sugimoto, T. Takayanagi, and T. Watari for helpful discussions and comments. We would like to thank C. Hoyos and L. Yaffe for collaboration in the initial stage of this project and for helpful and valuable comments. MF is in part supported by JSPS Postdoctoral Fellowship and partly by JSPS Grant-in-Aid for JSPS Fellows No. 25-4348. This work was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

References

[1] C. Lovelace, “Universality At Large N,” Nucl. Phys. B 201, 333 (1982).
[2] A. Cherman, M. Hanada and D. Robles-Lllana, Orbifold equivalence and the sign problem at finite baryon density, Phys. Rev. Lett. 106 (2011) 091603, arXiv:1009.1623 [hep-th].
[3] M. Hanada and N. Yamamoto, Universality of phases in QCD and QCD-like theories, J. High Energy Phys. 1202 (2012) 138, arXiv:1103.5480 [hep-ph].
[4] S. Kachru and E. Silverstein, 4d conformal theories and strings on orbifolds, Phys. Rev. Lett. 80 (1998) 4855, hep-th/9802183.
[5] A. E. Lawrence, N. Nekrasov and C. Vafa, On conformal field theories in four-dimensions, Nucl. Phys. B 533 (1998) 199, hep-th/9803015.
[6] M. Bershadsky, Z. Kakushadze and C. Vafa, String expansion as large N expansion of gauge theories, Nucl. Phys. B 523 (1998) 59, hep-th/9803076.
[7] A. Armoni, M. Chiffon and G. Veneziano, *Exact results in nonsupersymmetric large N orientifold field theories*, Nucl. Phys. B **667** (2003) 170, [hep-th/0302163](https://arxiv.org/abs/hep-th/0302163).

[8] T. Eguchi and H. Kawai, *Reduction of dynamical degrees of freedom in the large N gauge theory*, Phys. Rev. Lett. **48** (1982) 170-172.

[9] P. Kovtun, M. Unsal and L. G. Yaffe, “Necessary and sufficient conditions for non-perturbative equivalences of large N(c) orbifold gauge theories,” JHEP **0507**, 008 (2005) [hep-th/0411177](https://arxiv.org/abs/hep-th/0411177).

[10] P. Kovtun, M. Unsal and L. G. Yaffe, “Can large N(c) equivalence between supersymmetric Yang-Mills theory and its orbifold projections be valid?,” Phys. Rev. D **72**, 105006 (2005) [hep-th/0505075](https://arxiv.org/abs/hep-th/0505075).

[11] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113], [hep-th/9711200](https://arxiv.org/abs/hep-th/9711200).

[12] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from non-critical string theory*, Phys. Lett. B **428** (1998) 105, [hep-th/9802109](https://arxiv.org/abs/hep-th/9802109).

[13] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. **2** (1998) 253, [hep-th/9802150](https://arxiv.org/abs/hep-th/9802150).

[14] E. Kiritsis and V. Niarchos, *Large-N limits of 2d CFTs, quivers and AdS₃ duals*, J. High Energy Phys. **1104** (2011) 113, [arXiv:1011.5900](https://arxiv.org/abs/1011.5900) [hep-th].

[15] A. Armoni and A. Naqvi, “A Non-Supersymmetric Large-N 3D CFT And Its Gravity Dual,” JHEP **0809**, 119 (2008) [arXiv:0806.4068](https://arxiv.org/abs/0806.4068) [hep-th].

[16] M. Hanada, C. Hoyos and H. Shimada, *On a new type of orbifold equivalence and M-theoretic AdS₄/CFT₃ duality*, Phys. Lett. B **707** (2012) 394, [arXiv:1109.6127](https://arxiv.org/abs/1109.6127) [hep-th].

[17] M. Hanada, C. Hoyos and A. Karch, *Generating new dualities through the orbifold equivalence: a demonstration in ABJM and four-dimensional quivers*, J. High Energy Phys. **1201** (2012) 008, [arXiv:1110.3803](https://arxiv.org/abs/1110.3803) [hep-th].

[18] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, *N = 6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, J. High Energy Phys. **0810** (2008) 091, [arXiv:0806.1218](https://arxiv.org/abs/0806.1218) [hep-th].

[19] M. Fujita, M. Hanada and C. Hoyos, *A new large-N limit and the planar equivalence outside the planar limit*, Phys. Rev. D **86** (2012) 026007, [arXiv:1205.0853](https://arxiv.org/abs/1205.0853) [hep-th].

[20] T. Azeyanagi, M. Fujita and M. Hanada, *From the planar limit to M-theory*, [arXiv:1210.3601](https://arxiv.org/abs/1210.3601) [hep-th].

[21] M. Benna, I. Klebanov, T. Klose and M. Smedback, *Superconformal Chern-Simons theories and AdS(4)/CFT(3) correspondence*, J. High Energy Phys. **0809** (2008) 072, [arXiv:0806.1519](https://arxiv.org/abs/0806.1519) [hep-th].

[22] S. Terashima and F. Yagi, *Orbifolding the membrane action*, J. High Energy Phys. **0812** (2008) 041, [arXiv:0807.0368](https://arxiv.org/abs/0807.0368) [hep-th].

[23] Y. Imamura and S. Yokoyama, *N = 4 Chern-Simons theories and wrapped M5-branes in their gravity duals*, [arXiv:0812.1331](https://arxiv.org/abs/0812.1331) [hep-th].

[24] S. Hohenegger and I. Kirsch, *A note on the holography of Chern-Simons matter theories with flavour*, [arXiv:0903.1730](https://arxiv.org/abs/0903.1730) [hep-th].

[25] D. Gaiotto and D. L. Jafferis, *Notes on adding D6 branes wrapping RP₃ in AdS₄ × CP₃*, [arXiv:0903.2175](https://arxiv.org/abs/0903.2175) [hep-th].
[26] Y. Hikida, W. Li and T. Takayanagi, ABJM with flavors and FQHE, arXiv:0903.2194 [hep-th].
[27] M. Fujita and T. -S. Tai, Eschenburg space as gravity dual of flavored $\mathcal{N} = 4$ Chern-Simons-matter theory, J. High Energy Phys. 0909 (2009) 062, arXiv:0906.0253 [hep-th].
[28] K. Jensen, More holographic Berezinskii-Kosterlitz-Thouless transitions, Phys. Rev. D 82 (2010) 046005, arXiv:1006.3066 [hep-th].
[29] G. Zafrir, Embedding massive flavor in ABJM, J. High Energy Phys. 1210 (2012) 056, arXiv:1202.4295 [hep-th].
[30] D. L. Jafferis, Quantum corrections to $\mathcal{N} = 2$ Chern-Simons theories with flavor and their AdS$_4$ duals, arXiv:0911.4324 [hep-th].
[31] F. Benini, C. Closset and S. Cremonesi, Chiral flavors and M2-branes at toric CY$_4$ singularities, J. High Energy Phys. 1002 (2010) 036, arXiv:0911.4127 [hep-th].
[32] E. Conde and A. V. Ramallo, On the gravity dual of Chern-Simons-matter theories with unquenched flavor, J. High Energy Phys. 1107 (2011) 099, arXiv:1105.6045 [hep-th].
[33] J. P. Gauntlett, G. W. Gibbons, G. Papadopoulos and P. K. Townsend, Hyper-Kähler manifolds and multiply intersecting branes, Nucl. Phys. B 500 (1997) 133, hep-th/9702202.
[34] J.H. Eschenburg, New examples of manifolds with strictly positive curvature, Invent. Math. 66 (1982) 469-480. Cohomology of biquotients, Manuscripta Math. 75 (1992), 151-166.
[35] C. P. Boyer and K. Galicki, 3-Sasakian Manifolds, (1999) 123, hep-th/9810250.
[36] C. P. Boyer, K. Galicki, and B. M. Mann, Quaternionic reduction and Einstein manifolds, The geometry and topology of 3-Sasakian manifolds, T. reine angew. Math. 455 (1994), 183-220.
[37] R. Bielawski and A. Dancer, The geometry and the topology of toric hyperkahler manifolds, Comm. Anal. Geom. 8 (2000) 727.
[38] Y. Imamura and K. Kimura, On the moduli space of elliptic Maxwell-Chern-Simons theories, Prog. Theor. Phys. 120 (2008) 509, arXiv:0806.3727 [hep-th].
[39] D. L. Jafferis and A. Tomasiello, A simple class of $\mathcal{N} = 3$ gauge/gravity duals, J. High Energy Phys. 0810 (2008) 101, arXiv:0808.0864 [hep-th].
[40] I. R. Klebanov and G. Torri, M2-branes and AdS/CFT, Int. J. Mod. Phys. A 25 (2010) 332, arXiv:0909.1580 [hep-th].
[41] H. Samtleben and R. Wimmer, $\mathcal{N} = 6$ superspace constraints, SUSY enhancement and monopole operators, J. High Energy Phys. 1010 (2010) 080, arXiv:1008.2739 [hep-th].
[42] A. Gustavsson and S.-J. Rey, Enhanced $\mathcal{N} = 8$ supersymmetry of ABJM theory on $R^8$ and $R^8/Z(2)$, arXiv:0906.3568 [hep-th].
[43] K. Hosomichi, K. M. Lee, S. Lee, S. Lee and J. Park, $\mathcal{N} = 4$ superconformal Chern-Simons theories with hyper and twisted hyper multiplets, J. High Energy Phys. 0807 (2008) 091, arXiv:0805.3662 [hep-th].
[44] Y. Imamura and K. Kimura, $\mathcal{N} = 4$ Chern-Simons theories with auxiliary vector multiplets, J. High Energy Phys. 0810 (2008) 040, arXiv:0807.2144 [hep-th].
[45] K. M. Lee and H. U. Yee, New AdS$_4 \times X_7$ geometries with $CN = 6$ in M theory, J. High Energy Phys. 0703 (2007) 012, hep-th/0605214.
[46] S. W. Hawking and D. N. Page, *Thermodynamics of black holes in anti-de Sitter space*, Commun. Math. Phys. **87** (577) 1983.

[47] E. Witten, *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*, Adv. Theor. Math. Phys. **2** (1998) 505, hep-th/9803131.

[48] M. Ammon, J. Erdmenger, R. Meyer, A. O’Bannon and T. Wrase, *Adding flavor to $AdS_4/CFT_3$*, J. High Energy Phys. **0911** (2009) 125, arXiv:0909.3845 [hep-th].

[49] T. Nishioka and T. Takayanagi, *On Type IIA Penrose Limit and $\mathcal{N} = 6$ Chern-Simons Theories*, J. High Energy Phys. **0808** (2008) 001, arXiv:0806.3391 [hep-th].

[50] B. Sundborg, *The Hagedorn transition, deconfinement and $\mathcal{N} = 4$ SYM theory*, Nucl. Phys. B **573** (2000) 349, hep-th/9908001.

[51] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *The Hagedorn/deconfinement phase transition in weakly coupled large $N$ gauge theories*, Adv. Theor. Math. Phys. **8** (2004) 603, hep-th/0310285.

[52] D. J. Gross and E. Witten, *Possible third order phase transition in the large $N$ lattice gauge theory*, Phys. Rev. D **21** (446) 1980.

[53] H. J. Schnitzer, “Confinement/deconfinement transition of large N gauge theories with N(f) fundamentals: N(f)/N finite,” Nucl. Phys. B **695**, 267 (2004) [hep-th/0402219].

[54] P. Basu and A. Mukherjee, “Dissolved deconfinement: Phase Structure of large N gauge theories with fundamental matter,” Phys. Rev. D **78**, 045012 (2008) [arXiv:0803.1880 [hep-th]].

[55] H. Liu, “Fine structure of Hagedorn transitions,” hep-th/0408001.

[56] M. K. Benna, I. R. Klebanov and T. Klose, “Charges of Monopole Operators in Chern-Simons Yang-Mills Theory,” JHEP **1001**, 110 (2010) [arXiv:0906.3008 [hep-th]].

[57] V. Borokhov, A. Kapustin and X. -k. Wu, “Monopole operators and mirror symmetry in three-dimensions,” JHEP **0212**, 044 (2002) [hep-th/0207074].

[58] V. Borokhov, “Monopole operators in three-dimensional N=4 SYM and mirror symmetry,” JHEP **0403**, 008 (2004) [hep-th/0310254].

[59] S. S. Gubser, I. R. Klebanov and A. W. Peet, *Entropy and temperature of black 3-branes*, Phys. Rev. D **54** (1996) 3915, hep-th/9602135.

[60] A. Hanany and E. Witten, “Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics,” Nucl. Phys. B **492**, 152 (1997) [hep-th/9611230].

[61] S. Elitzur, A. Giveon and D. Kutasov, *Branes and $\mathcal{N} = 1$ duality in string theory*, Phys. Lett. B **400** (1997) 269, hep-th/9702014.

[62] I. Brunner, A. Hanany, A. Karch and D. Lust, *Brane dynamics and chiral nonchiral transitions*, Nucl. Phys. B **528** (1998) 197, hep-th/9801017.

[63] O. Bergman, A. Hanany, A. Karch and B. Kol, *Branes and supersymmetry breaking in three-dimensional gauge theories*, J. High Energy Phys. **9910** (1999) 036, hep-th/9908075.

[64] J. Park, R. Rabadan and A. M. Uranga, $\mathcal{N} = 1$ type IIA brane configurations, chirality and T duality, Nucl. Phys. B **570** (2000) 3, hep-th/9907074.

[65] K. Jensen, A. Karch, *ABJM mirrors and a duality of dualities*, J. High Energy Phys. **0909** (2009) 004, arXiv:0906.3013 [hep-th].

[66] A. Kapustin, B. Willett and I. Yaakov, *Exact results for Wilson loops in superconformal
Chern-Simons theories with matter, *J. High Energy Phys.* **1003** (2010) 089, arXiv:0909.4559 [hep-th].

[67] N. Drukker, M. Marino and P. Putrov, *From weak to strong coupling in ABJM theory*, *Commun. Math. Phys.* **306** (2011) 511, arXiv:1007.3837 [hep-th].

[68] H. Fuji, S. Hirano and S. Moriyama, *Summing up all genus free energy of ABJM matrix model*, *J. High Energy Phys.* **1108** (2011) 001, arXiv:1106.4631 [hep-th].

[69] M. Marino and P. Putrov, *ABJM theory as a Fermi gas*, *J. Stat. Mech.* **1203** (2012) P03001 arXiv:1110.4066 [hep-th].

[70] C. P. Herzog, I. R. Klebanov, S. S. Pufu and T. Tesileanu, *Multi-matrix models and Tri-Sasaki Einstein spaces*, *Phys. Rev. D* **83** (2011) 046001, arXiv:1011.5487 [hep-th].

[71] R. C. Santamaria, M. Marino and P. Putrov, *Unquenched flavor and tropical geometry in strongly coupled Chern-Simons-matter theories*, *J. High Energy Phys.* **1110** (2011) 139, arXiv:1111.6281 [hep-th].

[72] P. Termonia, *The complete $\mathcal{N} = 3$ Kaluza-Klein spectrum of 11D supergravity on $AdS_4 \times N(010)$*, *Nucl. Phys. B* **577** (2000) 341, hep-th/9909137.

[73] P. Fre’, L. Gualtieri and P. Termonia, *The structure of $\mathcal{N} = 3$ multiplets in $AdS_4$ and the complete $Osp(3|4) \times SU(3)$ spectrum of M-theory on $AdS_4 \times N(0,1,0)$*, *Phys. Lett. B* **471** (1999) 27, hep-th/9909188.

[74] M. Billo, D. Fabbri, P. Fre, P. Merlatti and A. Zaffaroni, *Rings of short $\mathcal{N} = 3$ superfields in three dimensions and M-theory on $AdS_4 \times N(0,1,0)$*, *Class. and Quant. Grav.* **18** (2001) 1269, hep-th/0005219.