Heat transfer in the boundary layer in an incompressible fluid in terms of waveguide turbulence model

V A Zharov, R S Selim
Moscow Institute of Physics and Technology (State University) Moscow region, Zhukovsky, St. Gagarin, 16, FALT, MIPT.

E-mail: selim.rs@phystech.edu

Abstract. The possibility of the description of the heat transfer processes in the developed turbulent boundary layer on the flat plate at zero angle attack with zero longitudinal pressure gradient in incompressible fluid is considered as a consequence of the waveguide theory of turbulent boundary layer. It is shown that the logarithmic distribution of temperature along normal coordinates arises in a natural way. It is pointed out the possibility of the intensification of the heat transfer due to generation of own modes of the temperature pulsations. The problem of heat transfer in a boundary layer on a plate flow round an incompressible fluid using the collocation method with Chebyshev polynomial is considered.

The dependence of the eigenfrequency of the least damped mode on the wave number is obtained. The amplitude of the excited mode obeys the direct resonance equation if external excitation is applied to the system.

Keywords: turbulent boundary layer – incompressible fluid – equation of temperature fluctuations – spectral problem – Chebyshev polynomials – resonance.

1. Introduction
In monographs [1], [2] and [3] the theory of turbulent boundary layer and heat transfer in a developed turbulent boundary layer is presented. In these monographs, the missing information, in addition to the laws of conservation, is derived from the experiment and, from a practical point of view, provides comprehensive information about the processes of momentum, energy and heat transfer. On the basis of qualitative analysis of the physics of the processes presented in the monographs [1], [3], which allowed to determine the scale of the corresponding quantities, in [4], while making a number of additional assumptions consistent with the theory of similarity and dimension, an asymptotic model of heat transfer was built, which revealed the characteristic features of the boundary layer at large Reynolds numbers.

In [5], [6] the waveguide model of a developed turbulent boundary layer on a plate at zero angle of attack with zero longitudinal pressure gradient is proposed. This model allows us to obtain results consistent with experimental data, as well as to predict some new phenomena. For example, the waveguide model predicts the possibility of resonance amplification of the amplitude of temperature fluctuations in the presence of some external periodic effect on the boundary layer. This study uses a collocation method with Chebyshev polynomials as basic functions for an approximate solution of the spectral problem for the temperature fluctuation equation. This solution makes it possible to
demonstrate the effect of resonance effects on the turbulent boundary layer. During the last ten years a number of spectral methods have been proposed for the numerical solution of the Navier-Stokes equations for incompressible fluids. Most of them concern the equations in velocity-pressure variables, with various ways of handling the incompressibility condition [7, 8, 9].

The aim of this paper is to solve the spectral problem numerically for the equation of temperature fluctuations in a turbulent boundary layer on a plate with a zero longitudinal pressure gradient by collocation with Chebyshev polynomials as basic functions. The dependence of the eigen frequency of the least damping model on the wave number, which shows, that the amplification of the fluctuation amplitude is possible. The Amplitude of the excited mode obeys the equation of direct resonance, if the system is attached to an external excitation.

2. Problem statement

Next, we consider the consequences of the waveguide model [5], [6] of the developed turbulent boundary layer, in which asymptotic features arise naturally from physical ideas about the nature of the pulsations of the developed turbulent boundary layer. As a characteristic example, the problem of heat transfer in the boundary layer on a plate, streamlined incompressible fluid is considered.

Next, we take the characteristic scale for the mean values $(U_0, V_0, T)$ of $-xL, -y\delta, -z\delta, T_\infty$.

\[ U_0 - U_\infty, \quad \frac{t - \delta^\ast}{U_\infty}, \quad \frac{V_0 - (\delta^\ast / L)U_\infty}{U_\infty}, \quad \frac{\Re_{\delta} - U_\infty \delta^\ast \rho / L}{U_\infty}, \quad \text{and for pulsation values, derivatives of} \]

\[ x, y, z \to \delta^\ast, \quad (u', v', w') = \sqrt{\delta^\ast / L} [u, v, w], \quad T - T_\infty, \quad T' = -\frac{\delta^\ast / LT_\infty}{U_\infty}, \quad \delta^\ast / L = \epsilon^2 \] [6].

From the estimates of the characteristic values it follows that the problem for average temperatures becomes similar to the problem of the average speed – before the members of the thermal conductivity is the value $1/\{\epsilon^2 \Re_{\delta}\}$, with respect to scales in the formulation of the developed turbulent boundary layer, it is assumed that this value is small together with the value $1/\Re_{\delta}$, i.e. $\Re_{\delta} >> 1, \epsilon^2 \Re_{\delta} >> 1$. If you follow the above estimates, the equation for temperature fluctuations in the first place it is necessary to leave the members

\[ \frac{\partial T'}{\partial t} + U_0 \frac{\partial T'}{\partial x} + v \frac{\partial T'}{\partial y} = \frac{1}{\Pr \epsilon^2 \Re_{\delta}} \nabla^2 T'. \] (1)

In this form, the equation resembles the Squire equation. It also has induced mode Collina Schlichting members. The difference is only in the term proportional to the second derivative of $y$. From this equation there are fashions that should be reviewed on a direct resonance with the mode Collina - Schlichting. This resonance can occur due to a suitable choice of Prandtl number.

The induced member $v \cdot \frac{\partial}{\partial y} T_0$ creates a turbulent heat transfer with some turbulent thermal conductivity. The waveguide model makes it possible to determine the coefficient of turbulent thermal conductivity. There are no derivatives of temperature in $z$, since the homogeneity of the mean values in this direction is assumed, then we should proceed with the spectral representation of pulsations [8].

Averaging this expression over $t_0$ and over $t_1$ yields an expression for the turbulent thermal conductivity coefficient in the discrete representation of the coherent structure:

\[ \lambda_i = -\frac{1}{(2\pi)^3} \left< v_{\alpha_i k_i} v_{\alpha_i k_i} e^{i(k_i x - \omega_i t_0 - (k_i \cdot k) \xi)} \right> / \omega_i, \] where values $v_{\alpha_i k_i}$ determined by the eigenfunctions of the Orr-Sommerfeld equation (Fig.1, and Fig.2).
The properties of the eigenfunctions of the main mode of the Orr-Sommerfeld equation for the Reynolds number $Re_{\delta^*}=10000$ and the dimensionless wave number are used for the estimation $\alpha=1$. By entering the scale $\Delta = \frac{1}{Pr \varepsilon^2 Re_{\delta^*}} \ll 1$, here $Pr$ - Prandtl molecular number, 1 and $1/ \varepsilon Re_{\delta^*}$.

![Fig.1. The Eigen function of the Orr-Sommerfeld equation with $\alpha=1, \beta=0, Re_{\delta^*}=10000$ (the least decaying mode)](image1)

![Fig.2. Fragment of the square of the modulus of the eigen function near zero (dotted line: $16y^4$, breakpoint $6.018+16.44y$)](image2)

It is possible to construct a composite temperature distribution formula using a three-layer scheme of the structure of the developed turbulent boundary layer [6], as displayed in Fig. 3:

$$
\theta(y) = cA + cB \ln \left(1 + e^{-\frac{c\theta}{\varepsilon^*}} \right) - \left(cA + cB \ln y \right),
$$

$$
\theta(y) = \frac{T(y) - T_w}{T_\infty - T_w}.
$$

$A = 8.772284365113652$, $B = 0.030143990383676093$, $c = 0.111$.

![Fig.3. Temperature distribution in the boundary layer](image3)
In connection with the waveguide interpretation of the developed turbulent boundary layer there is a simple interpretation of the possibility of strengthening the turbulent heat transfer.

In the equation for temperature fluctuations can be identified operator

\[
(-i \omega + i \alpha U(y))T'_{\omega \alpha} - \frac{1}{\Pr \varepsilon^2 \Re_{\alpha}} \left( \frac{\partial^2}{\partial y^2} - \alpha^2 \right) T'_{\omega \alpha} = 0
\]  

(3)

Which at zero values of temperature fluctuations on the outer boundary and on the wall has a countable set of eigen modes. Next, we consider the time formulation of the problem when the wave number \(\alpha\) assumed to be given.

The frequency \(\omega\), or alternatively \(c = \frac{\omega}{\alpha}\), appears as an eigenvalue in the equation for temperature fluctuations, and together with eigen functions is usually complex. The imaginary part \(\omega\) will indicate whether a particular wave is stable or not: unstable waves are characterized by a positive imaginary part. This work is aimed at solving the basic differential equations of temperature fluctuations using the collocation method. It is assumed that the solution of the problem will be the decomposition of the solution by a Chebyshev polynomial \(T_n(y)\) of the first kind with some unknown constants. For solving the equations of temperature fluctuations first, we introduce the following polynomial decomposition Chebyshev

\[
T'_{\omega \alpha}(y) = \sum_{n=0}^{N} a_n T_n(y)
\]

(4)

Where \(T_n(y)\) are polynomials of Chebyshev order \(n\). Equation (4) is then substituted into the equation (3). Postulating that equation (3) is satisfied at zeros of Chebyshev polynomial, and taking into account boundary conditions, this method converts temperature pulsations and given boundary conditions into a matrix equation. By solving a system of algebraic equations [10] one can compute approximate eigenvalues and eigen functions for different mean flow models. Resonance is a phenomenon that occurs when the frequency at which a force is applied periodically is equal to or nearly equal to one of the natural frequencies of the system it acts on. This causes the system to oscillate with greater amplitude than when applying force at other frequencies. The latter phenomenon can be considered if we add external excitation to the temperature fluctuation equation:

\[
\frac{\partial T'(t, y, x)}{\partial t} + U(y) \frac{\partial T'(t, y, x)}{\partial x} - \frac{1}{\Pr \varepsilon^2 \Re_{\alpha}} \nabla^2 T'(t, y, x) = h(y) \exp(-i \omega t + i \alpha x)
\]

(5)

Here \(h(y)\) is one of the eigenfunctions of the operator on the right side of the equation (5), \(i^2 = -1\), we look for a solution for \(T'(t, y, x)\) as \(T = r(t) h(y) \exp(-i \omega t + i \alpha x)\). As shown by simple calculations, the amplitude of the excited mode satisfies the equation

\[
r'(t) + i(\omega - \omega_r) r(t) = 1
\]

(6)

(Here \(\omega = \omega_r + i \omega_i, \omega_r = \Re(\omega)\).) If the flow is subject to periodic influences with a frequency coinciding with the real part of the mode frequency, which has the lowest decrement, it is possible to increase the temperature pulsation, if the absolute value of the imaginary part of the frequency of this mode is less than 1.
Fig. 4. shows the dependence of the real and imaginary part of the frequency on the wave number, when the set of parameters equal, $\text{Re}_{\delta^*} = 10000$, $\varepsilon^2 \left( \text{Re}_{\delta^*} \right) = 0.005$, $\text{Pr} = 1$, where $\text{Pr}$ is the molecular Prandtl number. It can be seen that at different values of the wave number, the imaginary part of the eigen frequency of the least damped mode, or the absolute value is less than one, i.e., an increase in temperature pulsations is in principle possible, as displayed below in Fig. 4.

$$m = 50, \text{Pr} = 1, \text{Re}_{\delta^*} = 10000$$

- $\text{Re} \left( \omega \right)$
- $\text{Im} \left( \omega \right)$

3. Conclusions

In this study, the solution for the equation of the temperature fluctuations is proposed based on the spectral Chebyshev collocation method. Collocation points (zeros of Chebyshev polynomials) are used in the differential equation and boundary conditions to derive a matrix of unknown constants. The dependence of the eigen frequency of the least damped mode on the wave number is obtained, which shows that amplification of the pulsation amplitude is possible. The amplitude of the excited mode obeys the equation if an external force is applied to the system. It is shown that the logarithmic dependence of the temperature pulsation on the normal coordinates of the body surface takes place.

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