SOME GRADIENT ESTIMATES FOR THE HEAT EQUATION ON DOMAINS AND FOR AN EQUATION BY PERELMAN

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ABSTRACT. In the first part, we derive a sharp gradient estimate for the log of Dirichlet heat kernel and Poisson heat kernel on domains, and a sharpened local Li-Yau gradient estimate.

In the second part, without explicit curvature assumptions, we prove a global upper bound for the fundamental solution of an equation introduced by G. Perelman, i.e. the heat equation of the conformal Laplacian under backward Ricci flow. Further, under nonnegative Ricci curvature assumption, we prove a qualitatively sharp, global Gaussian upper bound. The idea is to combine the Nash and Davies heat kernel estimate with a Sobolev imbedding by Hebey, together with a Hamilton type gradient estimate.

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1. INTRODUCTION

The goal of the paper is to establish certain new point-wise or gradient estimates for the heat equation in both the fixed metric and the Ricci flow case. Gradient estimates for the heat equation are important tools in geometric analysis as pointed out in the papers [LY], [H], [CH], [ACDH] and others. In this paper, for the fixed metric case, two gradient estimates are proven. One is a sharp gradient estimate for the log of Dirichlet heat kernel and Poisson heat kernel on domains. This can be viewed as a boundary version of the well known Li-Yau gradient estimate (see Section 2 for a restatement). It can also be viewed as another step in the long running process of heat kernel or Poisson kernel estimate starting with the Gaussian formula and Poisson formula. As far as boundary gradient estimate is concerned, only the Neumann boundary case was treated in [LY] and [Wa]. The Dirichlet case is different in that solutions vanish on the boundary. Therefore, the estimate is different from both the Li-Yau theorem and its generalization in [Wa]. The result seems to be new even for Euclidean domains. The other result is a sharpened local Li-Yau gradient estimate that matches the global one. As well known, even for manifolds

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with nonnegative Ricci curvature, the local Li-Yau estimate differs with the global one by a nontrivial factor. Here we show that this factor can be chosen as one. Hence the global and local Li-Yau estimate are identical. We expect the result to have applications in heat kernel estimate on manifolds. These two results are presented in section 2.

In sections 3 and later we turn to the case when the metric evolves by the Ricci flow. In the fundamental paper [P], Perelman discovered a monotonicity formula for equation (4.0) below, which can be regarded as the heat equation of the conformal Laplacian under backward Ricci flow. Perelman’s formula can be thought of as a gradient estimate. Using this estimate together with the reduced distance function, he then applies the maximum principle to prove a lower bound for the fundamental solution of (4.0), whenever it exists. The reduced distance incorporates the scalar curvature as an integral part. However it does not seem that the maximum principle alone will yield a two sided bound. Here we will use the Nash method to show that certain long time upper bound also holds. The bounds involve more classical geometric quantities such as the best constants in Sobolev embedding, which depend only on Ricci curvature lower bound and injectivity radius. Under no explicit curvature assumption, we prove a global on-diagonal upper bound for the fundamental solution on (4.0). The bound is good in the sense that it matches the on diagonal bound in the fixed metric case. However, we are not able to prove a good off-diagonal bound without further assumptions on curvature. Nevertheless, the on-diagonal upper bound does not have the usual, trouble making exponentially growing term even when the Ricci curvature changes sign. When the Ricci curvature is nonnegative, we obtain a qualitatively sharp Gaussian upper bound. This is presented in Section 5.

We will use the following notations throughout the paper. $\mathbf{M}$ denotes a compact Riemannian manifold without boundary, unless stated otherwise; $g, R_{ij}$ will be the metric and Ricci curvature; $\nabla, \Delta$ the corresponding gradient and Laplace-Beltrami operator; $c$ with or without index denote generic positive constant that may change from line to line. In case the metric $g(t)$ evolves with time, then $d(x,y,t)$ will denote the corresponding distance function; $d\mu(x,t)$ denotes the volume element under $g(t)$; We will still use $\nabla, \Delta$ the corresponding gradient and Laplace-Beltrami operator, when no confusion arises.

To close the introduction, we point out that all results in the paper are stated for compact manifolds or bounded domains. However similar results can be proven for noncompact manifolds under appropriate assumptions near infinity.

2. LOG DERIVATIVE ESTIMATES FOR DIRICHLET HEAT KERNEL AND POISSON HEAT KERNEL ON DOMAINS.

There have been several log gradient estimates available for the heat kernel on complete manifolds, compact manifolds without boundary and for the Neumann heat kernel. In the introduction, we mentioned the papers [LY] and [Wa]. For compact manifolds without boundary, we refer the reader to the papers [Sh], [H] (Corollary 1.3), [No], [MS], [Hs] and [ST]. However, an estimate for the Dirichlet heat kernel is clearly missing. This is done in the next theorem. The estimate is sharp in general as can be seen from the heat kernel formula for the Euclidean half space. Let us mention that for bounded domains, the large time behavior of heat kernels is determined by the first eigenvalue and eigenfunction. So we will only deal with the most interesting, small time case.
Theorem 2.1. Let $D$ be a bounded $C^2$ domain in a Riemannian manifold and $G = G(x,t;y,0)$ and $P = P(x,t;y,0)$ be the Dirichlet heat kernel and Poisson heat kernel respectively. Also let $\rho(x) = \text{dist}(x,\partial D)$ and $d(x,y)$ be the Riemannian distance. Given $T > 0$, there exists a constant $C$ depending on $T$ and $D$ such that

\begin{equation}
|\nabla_x \log G(x,t;y,0)| \leq \begin{cases} 
\frac{C}{\rho(x)}, & \rho(x) \leq \sqrt{t}; \\
\frac{C}{\sqrt{t}} \left[ 1 + \frac{d(x,y)}{\sqrt{t}} \right], & \rho(x) > \sqrt{t}; 
\end{cases}
\end{equation}

for all $x,y \in D$ and $0 < t < T$; and for all $x \in D$, $y \in \partial D$ and $0 < t < T$,

\begin{equation}
|\nabla_x \log P(x,t;y,0)| \leq \begin{cases} 
\frac{C}{\rho(x)}, & \rho(x) \leq \sqrt{t}; \\
\frac{C}{\sqrt{t}} \left[ 1 + \frac{d(x,y)}{\sqrt{t}} \right], & \rho(x) > \sqrt{t}; 
\end{cases}
\end{equation}

Proof.

Let us prove (2.1) first. The proof of (2.2) is similar and will be sketched later. As explained earlier, the most interesting case for the derivative estimate is for small time. Hence we can take $T$ to be sufficiently small. Here we will take $T$ so small that the boundary Harnack principle of [FGS] holds when $\rho(x) \leq 2T$. Here we notice that even though the boundary Harnack principle was proven in the Euclidean case in that paper, it is still valid in the current case. This is so because we can cover the boundary of $D$ by a finite number of metric balls with radius less than the injectivity radius. And then we can convert the Laplace-Beltrami operator into an elliptic equation with smooth coefficients in $\mathbb{R}^n$.

For a fixed $t_0 \in (0, T)$ and $y \in D$, we write

$f(x,t) = G(x,t;y,-t_0), \quad x \in D, t > 0; \quad (2.3)$

$\Omega_{t_0} = \{(z,\tau) \mid x \in D, 0 < \tau \leq t_0, \rho(z) \geq \sqrt{\tau}\}.$

Fixing $(x,t) \in D \times [0, t_0] - \Omega_{t_0}$, we can apply the gradient estimate in Theorem 1.1 of [SZ] on the cube

$Q_{x,t} = B(x,\rho(x)) \times [t - \frac{\rho^2(x)}{2}, t] \subset D \times [-t_0, t_0].$

This gives us

\begin{equation}
\frac{\nabla f(x,t)}{f(x,t)} \leq \frac{C}{\rho(x)} (1 + \log \frac{A}{f(x,t)}).
\end{equation}

Here $A = \sup_{Q_{x,t}} f$. For a proof of (2.4) and that of Theorem 1.1 in [SZ], please go to Theorem 3.1 in the next section, which contains Theorem 1.1 in [SZ] as a special case.

Now we apply the standard Harnack inequality of [LY] on manifold to reach

$A = \sup_{Q_{x,t}} f \leq c_1 f(x,t + \rho(x)^2).$

Then the boundary Harnack inequality of [FGS] gives us

$f(x,t + \rho(x)^2) \leq c_2 f(x,t)$

since $f$ vanishes on $\partial D \times (-t_0, t_0)$. Therefore

\begin{equation}
A = \sup_{Q_{x,t}} f \leq c_3 f(x,t) \quad (2.5)
\end{equation}
Substituting (2.5) to (2.4), we deduce, for \((x,t) \in D \times [0, t_0] - \Omega_t\),

\[
(2.6) \quad \frac{|\nabla f(x,t)|}{f(x,t)} \leq \frac{C}{\rho(x)}
\]

This proves the first part of (2.1).

Next we work in \(\Omega_t\). Let us observe that on the sides of \(\partial \Omega_t\), there holds \(\rho(x) = \sqrt{t}\). Hence for such \(x\), inequality (2.6) becomes

\[
\frac{|\nabla f(x,t)|}{f(x,t)} \leq \frac{C}{\sqrt{t}}
\]

i.e.

\[
(2.7) \quad \frac{|\nabla f(x,t)|^2}{f(x,t)} \leq \frac{C}{t} f(x,t), \quad \rho(x) = \sqrt{t}.
\]

Let \(m = \sup_{\Omega_t} f\), then for any \(b > 0\), we have

\[
f \log \frac{bm}{f} \geq f \log b.
\]

Now we use the calculation in the proof of Theorem 1.1 in [H] (p115) to reach

\[
\Delta (f \log \frac{bm}{f}) - \partial_t (f \log \frac{bm}{f})
\]

\[
= (\Delta f - \partial_t f) \log b + (\Delta - \partial_t)(f \log \frac{m}{f})
\]

\[
= -\frac{|\nabla f|^2}{f}.
\]

Also

\[
(\Delta - \partial_t)(\frac{|\nabla f|^2}{f}) = \frac{2}{f} \left| \partial_i f \partial_j f - \frac{\partial_i f \partial_j f}{f} \right|^2 + 2R_{ij} \frac{\partial_i f \partial_j f}{f}
\]

\[
\geq -2K \frac{|\nabla f|^2}{f}.
\]

Here \(-K\) is the lower bound of the Ricci curvature. Therefore, for

\[
h = \frac{t}{1 + 2Kt} \frac{|\nabla f|^2}{f} - f \log \frac{bm}{f},
\]

we have

\[
\Delta h(x,t) - \partial_t h(x,t) \geq 0, \quad (x,t) \in \Omega_{x_0}.
\]

We \(t = 0\), it is clear that \(h \leq 0\). On the sides of \(\partial \Omega_{t_0}\), i.e., when \(\rho(x) = \sqrt{t}\), one can choose \(b\) sufficiently large so that

\[
f \log \frac{bm}{f} \geq C f \geq t \frac{|\nabla f|^2}{f}.
\]

Here we just used (2.7) and the constant \(C\) is from there too. Therefore \(h \leq 0\) on the sides of \(\partial \Omega_{t_0}\). When \(T\) is sufficiently small, we know that \(\partial \Omega_{t_0}\) is connected and we can
By the lower bound estimate in [Z], there holds when \( \rho \) when

\[
G(x, 2t_0; y, 0) = G(x, 2t_0; y, 0),
\]

that

\[
\frac{\| \nabla_x G(x, 2t_0; y, 0) \|}{G(x, 2t_0; y, 0)} \leq C(T, K) \sqrt{t_0} \left[ 1 + \sqrt{\log \frac{m}{G(x, 2t_0; y, 0)}} \right]
\]

when \( \rho(x) \geq \sqrt{2t_0} \). Here

\[
m = \sup_{\Omega_t} f = \sup_{\rho(z) \geq \sqrt{2T} \rho(z) \leq t_0} G(z, \tau; y, -t_0).
\]

Making a change of variables \( t_0 \to t_0/2 \), we have

\[
\frac{\| \nabla_x G(x, t_0; y, 0) \|}{G(x, t_0; y, 0)} \leq C(T, K) \sqrt{t_0} \left[ 1 + \sqrt{\log \frac{m}{G(x, t_0; y, 0)}} \right]
\]

when \( \rho(x) \geq \sqrt{t_0} \).

By the Dirichlet heat kernel upper bound in Davies [Da], we know that

\[
m \leq C \left( \frac{\rho(y)}{\sqrt{t_0}} \wedge 1 \right) \frac{1}{|B(y, \sqrt{t_0})|}.
\]

By the lower bound estimate in [Z], there holds

\[
G(x, t_0; y, 0) \geq C \left( \frac{\rho(y)}{\sqrt{t_0}} \wedge 1 \right) \frac{1}{|B(y, \sqrt{t_0})|} \frac{1}{e^{-cd(x,y)^2/t_0}}.
\]

Here we note that the lower bound was proven under the assumption that the Ricci curvature is nonnegative. However for short time behavior this assumption is not necessary. Substituting (2.13) and (2.14) to (2.12), we obtain

\[
\frac{\| \nabla_x G(x, t_0; y, 0) \|}{G(x, t_0; y, 0)} \leq C(T, K) \sqrt{t_0} \left[ 1 + \frac{d(x, y)}{\sqrt{t_0}} \right]
\]

when \( \rho(x) \geq \sqrt{t_0} \). Now (2.1) follows from (2.7) and (2.15).

To prove (2.2), let us recall the results in [Da] (upper bound) and [Z] (lower bound): there exists \( c_1 \) and \( c_2 \) such that

\[
\frac{1}{c_1} \left( \frac{\rho(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\rho(y)}{\sqrt{t}} \wedge 1 \right) \frac{1}{|B(y, \sqrt{t})|} e^{-d(x,y)^2/(c_2 t)} \leq
\]

\[
G(x, t; y, 0) \leq c_1 \left( \frac{\rho(x)}{\sqrt{t}} \wedge 1 \right) \left( \frac{\rho(y)}{\sqrt{t}} \wedge 1 \right) \frac{1}{|B(y, \sqrt{t})|} e^{-c_2 d(x,y)^2/t}.
\]

for all \( x, y \in D \) and \( 0 < t \leq T \).

Given \( y \in \partial D \), the Poisson heat kernel is defined as

\[
P(x, t; y, s) = -\frac{\partial}{\partial_n y} G(x, t; y, 0).
\]
Therefore one has the two-sided bound
\[ \frac{1}{c_1} \left( \frac{\rho(x)}{\sqrt{t}} \wedge 1 \right) \frac{1}{|B(y, \sqrt{t})|} e^{-d(x,y)^2/(ct^2)} \leq P(x, t; y, 0) \leq c_1 \left( \frac{\rho(x)}{\sqrt{t}} \wedge 1 \right) \frac{1}{|B(y, \sqrt{t})|} e^{-c_2 d(x,y)^2/t}. \]

The rest of the proof for (2.2) is identical to that of (2.1). □

Our next theorem provides a sharpened local Li-Yau estimate. In 1986 Li and Yau proved the following famous estimate.

**Theorem (Li-Yau [LY]).** Let \( M \) be a complete manifold with dimension \( n \geq 2 \), \( \text{Ricci}(M) \geq -K, K \geq 0 \). Suppose \( u \) is any positive solution to the heat equation in \( B(x_0, R) \times [t_0 - T, t_0] \subset M \times [t_0 - T, t_0] \). Then, for any \( \alpha \in (0, 1) \), there exists a constant \( c = c(n, \alpha) \) such that

\[ \alpha \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{c}{R^2} + \frac{c}{T} + cK, \]

in \( B(x_0, R/2) \times [t_0 - T/2, t_0] \).

Moreover, if \( M \) has nonnegative Ricci curvature and \( R = \infty \), i.e. \( B(x_0, R) = M \), then

\[ \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{c_n}{T}. \]

Let us observe that, even in the case of nonnegative Ricci curvature, the first local estimate does not match the second global estimate completely, due to the presence of the parameter \( \alpha < 1 \). Here we show that \( \alpha \) can be taken as 1 modulo a lower order term. We mention that our estimate in the next theorem is new only in the local sense. The global estimate was already proven in [Y] by using a more involved quantity. The very short proof, simpler than previous ones, is based on a modification of an idea in [H] and the cut-off method in [LY].

**Theorem 2.2.** Let \( B(x_0, R) \) be a geodesic ball in a Riemannian manifold \( M \) with dimension \( n \geq 2 \) such that \( \text{Ricci}|_{B(x_0, R)} \geq -K, K \geq 0 \). Suppose \( u \) is any positive solution to the heat equation in \( B(x_0, R) \times [t_0 - T, t_0] \). Then

\[ \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{c_n}{R^2} + \frac{c_n}{T} + c_nK + c_n\sqrt{K} \sup_u \frac{|\nabla u|}{u} + \frac{c_n}{R} \sup_u \frac{|\nabla u|}{u}, \]

in \( B(x_0, R/2) \times [t_0 - T/2, t_0] \). Here \( c_n \) depends only on the dimension \( n \).

**Proof.**

By direct computation (see [H]), we have

\[ (\Delta - \partial_t)(\frac{|\nabla u|^2}{u}) = \frac{2}{u} \left| \partial_i \partial_j u - \frac{\partial_i u \partial_j u}{u} \right|^2 + 2R_{ij} \frac{\partial_i u \partial_j u}{u}. \]

In view of the estimate

\[ \left| \partial_i \partial_j u - \frac{\partial_i u \partial_j u}{u} \right|^2 \geq \frac{1}{n} \left( \Delta u - \frac{|\nabla u|^2}{u} \right)^2, \]

the above implies

\[ (\Delta - \partial_t)(\frac{|\nabla u|^2}{u}) \geq \frac{2}{nu} \left( \Delta u - \frac{|\nabla u|^2}{u} \right)^2 + 2R_{ij} \frac{\partial_i u \partial_j u}{u}. \]
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Since $\Delta u$ is also a solution to the heat equation, it follows that

$$(\Delta - \partial_t)(-\Delta u + \frac{|\nabla u|^2}{u}) \geq \frac{2}{nu} \left(\Delta u - \frac{|\nabla u|^2}{u}\right)^2 - 2K\frac{|\nabla u|^2}{u}.$$ 

Let us write

$$q = -\Delta u + \frac{|\nabla u|^2}{u} = \frac{|\nabla u|^2}{u} - u_t.$$ 

Then $q$ satisfies

$$(\Delta - \partial_t)q \geq \frac{2}{nu}q^2 - 2K\frac{|\nabla u|^2}{u}.$$ 

Define

$$H = q/u.$$ 

Then $H$ satisfies

$$(\Delta - \partial_t)H \geq \frac{2}{n}H^2 - 2K\frac{|\nabla u|^2}{u^2} - 2\nabla H \nabla \ln u.$$ 

Now we can use the Li-Yau idea of cut-off functions to derive the desired bound. The only place that may cause difficulty is that $H$ may change sign. However it turns out that it does not hurt. Here is the detail. Let $\psi(x,t)$ be a smooth cut-off function supported in $Q_{R,T} = B(x_0, R) \times [t_0 - T, t_0]$, satisfying the following properties

1. $\psi = \psi(d(x, x_0), t) \equiv \psi(r, t) = 1$ in $Q_{R/2, T/4}$, $0 \leq \psi \leq 1$.

2. $\psi$ is decreasing as a radial function in the spatial variables.

3. $|\partial_r \psi| \leq \frac{C_1}{R}$, $|\partial^2_r \psi| \leq \frac{C_2}{R^2}$ when $0 < a < 1$.

4. $|\frac{\partial \psi}{\psi}| \leq \frac{C}{T}$.

Then, from (2.16) and a straightforward calculation, one has

$$(2.17)$$

$$\Delta(\psi H) - (\psi H)_t - 2\frac{\nabla \psi}{\psi} \cdot \nabla(\psi H) + 2\psi K\frac{|\nabla u|^2}{u^2} + 2\nabla(\psi H) \nabla \ln u$$

$$\geq \frac{2}{n}\psi H^2 + (\Delta \psi) H - 2\frac{|\nabla \psi|^2}{\psi} - H - \psi_t H + 2H \nabla \psi \nabla \ln u$$

$$= \frac{2}{n}\psi H^2 - 2\frac{|\nabla \psi|^2}{\psi} H + (\partial_r^2 \psi + (n-1)\frac{\partial_r \psi}{r} + \partial_r \psi \partial_r \log \sqrt{g})H - \psi_t H + 2H \nabla \psi \nabla \ln u.$$ 

Suppose that at $(y, s)$, the function $\psi H$ reaches a maximum. If the value is non-positive, there is nothing to prove. So we assume the maximum value is positive. Then (2.17) shows

$$2\psi K\frac{|\nabla u|^2}{u^2} + 2\frac{|\nabla \psi|^2}{\psi} H \geq \frac{2}{n}\psi H^2 + (\partial_r^2 \psi + (n-1)\frac{\partial_r \psi}{r} + \partial_r \psi \partial_r \log \sqrt{g})H - \psi_t H + 2H \nabla \psi \nabla \ln u.$$ 

In the above, the only term we need extra care of is

$$\partial_r \psi \partial_r \log \sqrt{g} H.$$
Note that $-C/R \leq \partial_r \psi / \psi a \leq 0$, $\partial_r \log \sqrt{g} \leq \sqrt{K}$ and $H(y, s) > 0$. Therefore
\[
2\psi K \frac{\nabla u^2}{u^2} + 2 \frac{\nabla \psi}{\psi} H + 2 \frac{\psi H \nabla \psi}{\sqrt{g}} \nabla \ln u \\
\geq \frac{2}{nu} \psi H^2 + (\partial_r^2 \psi + (n - 1) \frac{\partial_r \psi}{r}) H - C \sqrt{K} \psi^a H/R - \psi_t H.
\]
This shows that
\[
\psi H^2 = \psi \left( \frac{\nabla u^2}{u^2} - \frac{u_t}{u} \right)^2 \leq \left( \frac{c_n}{R^4} + \frac{c_n}{T^2} + c_n K^2 \right) + c_n K \frac{\nabla u^2}{u} + \left( \frac{c_n}{R} \nabla \ln u \right)^2.
\]
Hence
\[
\frac{\nabla u^2}{u^2} - \frac{u_t}{u} \leq \frac{c_n}{R^2} + \frac{c_n}{T} + c_n K + c_n \sqrt{K} \frac{\nabla u}{u} + \frac{c_n}{R} \frac{\nabla u}{u}.
\]
in the half parabolic cube. □

3. GRADIENT ESTIMATES ON THE LOG TEMPERATURE UNDER BACKWARD AND FORWARD RICCI FLOW

In this section we will prove certain localized or global gradient bound on the heat equation under backward and forward Ricci flow, i.e. equations (3.1) and (3.2) below. This estimate is a generalization of the results in [H] and [SZ], where the heat equation under a fixed metric is studied. Similar estimates for the conjugate heat equation (i.e. when $\Delta$ is replaced by $\Delta - R$ in (3.1) or (3.2)) were proven in [Ni3], [CKNT] and [CCGGIKLLN] Chapter 8. This estimate then also relies on the derivative of the scalar curvature $R$.

The current estimate under the forward Ricci flow ((3.2)) will be useful for Section 5, where we will prove a global Gaussian upper estimate for Perelman’s equation under nonnegative Ricci curvature assumption.

Recall that the heat equation under backward and forward Ricci flow are given by

(3.1) \[
\begin{cases}
\Delta u - \partial_t u = 0, \\
\frac{d}{dt} g_{ij} = 2R_{ij}
\end{cases}
\]

and

(3.2) \[
\begin{cases}
\Delta u - \partial_t u = 0, \\
\frac{d}{dt} g_{ij} = -2R_{ij}.
\end{cases}
\]

For (3.1) we have the following:

**Theorem 3.1.** Let $M$ be a compact Riemannian manifold equipped with a family of Riemannian metric evolving under the backward Ricci flow in (3.1).

(a) (local estimate). Suppose $u$ is any positive solution to (3.1) in $Q_{R,T} = \{(x, t) \mid x \in M, d(x, x_0, t) < R, t \in [t_0 - T, t_0]\}$ such that the Ricci $\geq -k$ throughout. Suppose also $u \leq M$ in $Q_{R,T}$. Then there exists a dimensional constant $c$ such that
\[
\frac{|\nabla u(x, t)|}{u(x, t)} \leq c \left( \frac{1}{R} + \frac{1}{T^{1/2}} + \sqrt{k} \right) \left( 1 + \log \frac{M}{u(x, t)} \right)
\]
in $Q_{R/2, T/2}$.
(b). (global estimate) Suppose $u$ is any positive solution to (3.1) in $M \times [0,T]$. Under the assumption that $\text{Ricci} \geq 0$, it holds

$$\frac{\|\nabla u(x,t)\|}{u(x,t)} \leq \frac{1}{t^{1/2}} \sqrt{\log \frac{M}{u(x,t)}}$$

for $M = \sup_{M \times [0,T]} u$ and $(x,t) \in M \times [0,T]$.

**Remark.** As pointed out in [SZ], the local and global estimate can not replace each other. Also note that there is no other curvature assumption in part (b), nor any constants.

**Proof of Theorem 3.1 (a).**

We will use the idea in [SZ] with certain modifications to handle the changing nature of the metric. Suppose $u$ is a solution to the heat equation in the statement of the theorem in the parabolic cube $Q_{R,T}$. It is clear that the gradient estimate in Theorem 3.1 is invariant under the scaling $u \rightarrow u/M$. Therefore, we can and do assume that $0 < u \leq 1$.

Write

$$f = \log u, \quad w \equiv |\nabla \log(1-f)|^2 = \frac{|
abla f|^2}{(1-f)^2}.$$

Since $u$ is a solution to the heat equation, simple calculation shows that

$$\Delta f + |\nabla f|^2 - f_t = 0.$$

We will derive an equation for $w$. First notice that

$$w_t = \frac{2\nabla f(\nabla f)_t}{(1-f)^2} + \frac{2|\nabla f|^2}{(1-f)^3} + \frac{2\text{Ric}}{(1-f)^2}\nabla f, \nabla f$$

$$= \frac{2\nabla f\nabla(\Delta f + |\nabla f|^2)}{(1-f)^2} + \frac{2|\nabla f|^2(\Delta f + |\nabla f|^2)}{(1-f)^3} - \frac{2\text{Ric}}{(1-f)^2}\nabla f, \nabla f$$

In local orthonormal system, this can be written as

(3.3) $$w_t = \frac{2f_i f_{ij} + 4f_i f_j f_{ij}}{(1-f)^2} + \frac{2f_i^2 f_{jj}}{(1-f)^3} - \frac{2R_{ij} f_i f_j}{(1-f)^2}.$$

Here and below, we have adopted the convention $f_i^2 = |\nabla f|^2$ and $f_{ii} = \Delta f$.

Next

(3.4) $$\nabla w = \left(\frac{f_i^2}{(1-f)^2}\right)_j = \frac{2f_i f_{ij}}{(1-f)^2} + 2\frac{f_i^2 f_j}{(1-f)^3}.$$

It follows that

(3.5) $$\Delta w = \left(\frac{f_i^2}{(1-f)^2}\right)_{jj}$$

$$= \frac{2f_i^2}{(1-f)^2} + \frac{2f_i f_{ij}}{(1-f)^2} + \frac{4f_i f_{ij} f_j}{(1-f)^3}$$

$$+ \frac{4f_i f_{ij} f_j}{(1-f)^3} + 2\frac{f_i^2 f_{jj}}{(1-f)^3} + 6\frac{f_i^2 f_j^2}{(1-f)^4}.$$
By (3.5) and (3.3),
\[
\Delta w - w_t = \frac{2f_i^2}{(1-f)^2} + \frac{2f_i f_{ijj} - f_{j,iij}}{(1-f)^2} \\
+ 6\frac{\nabla f^4}{(1-f)^4} + 8\frac{f_i f_{ij} f_j}{(1-f)^3} + 2\frac{f_i^2 f_{jj}}{(1-f)^3} \\
- 4\frac{f_i f_{ij} f_j}{(1-f)^2} - 2\frac{f_i^2 f_{jj}}{(1-f)^3} - 2\frac{\nabla f^4}{(1-f)^3} + 2R_{ij} f_i f_j.
\]

The 5th and 7th terms on the righthand side of this identity cancel each other. Also, by Bochner’s identity
\[
f_i f_{ijj} - f_{j,iij} = f_j (f_{j,iii} - f_{iiij}) = R_{ij} f_i f_j.
\]
So the second term doubles with the last term. Therefore
\[
\Delta w - w_t = \frac{2f_i^2}{(1-f)^2} + 6\frac{\nabla f^4}{(1-f)^4} + 8\frac{f_i f_{ij} f_j}{(1-f)^3} - 4\frac{f_i f_{ij} f_j}{(1-f)^2} - 2\frac{\nabla f^4}{(1-f)^3} + 4R_{ij} f_i f_j.
\]

Notice from (3.4) that
\[
\nabla f \nabla w = \frac{2f_i f_{ij} f_j}{(1-f)^2} + \frac{f_i^2 f_{jj}}{(1-f)^3}.
\]
Hence
\[
0 = 4\frac{f_i f_{ij} f_j}{(1-f)^2} - 2\nabla f \nabla w + 4\frac{\nabla f^4}{(1-f)^3}.
\]
\[
0 = -4\frac{f_i f_{ij} f_j}{(1-f)^3} + [2\nabla f \nabla w - 4\frac{\nabla f^4}{(1-f)^3}] \frac{1}{1-f}.
\]
Adding (3.6) with (3.7) and (3.8), we deduce
\[
\Delta w - w_t = \frac{2f_i^2}{(1-f)^2} + 2\frac{\nabla f^4}{(1-f)^4} + 4\frac{f_i f_{ij} f_j}{(1-f)^3} \\
+ \frac{2}{1-f} \nabla f \nabla w - 2\nabla f \nabla w + 2\frac{\nabla f^4}{(1-f)^3} + 4R_{ij} f_i f_j.
\]
Since
\[
\frac{2f_i^2}{(1-f)^2} + 2\frac{\nabla f^4}{(1-f)^4} + 4\frac{f_i f_{ij} f_j}{(1-f)^3} \geq 0,
\]
we have
\[
\Delta w - w_t \geq \frac{2f}{1-f} \nabla f \nabla w + 2\frac{\nabla f^4}{(1-f)^3} - 4kw.
\]
Since \(f \leq 0\), it follows that
\[
\Delta w - w_t \geq \frac{2f}{1-f} \nabla f \nabla w + 2(1-f)\frac{\nabla f^4}{(1-f)^3} - 4kw.
\]
GRADIENT ESTIMATE

\begin{equation}
\Delta w - w_t \geq \frac{2f}{1-f} \nabla f \nabla w + 2(1-f)w^2 - 4kw.
\end{equation}

From here, we will use a cut-off function to derive the desired bounds. Let \( \psi = \psi(x,t) \) be a smooth cut-off function supported in \( Q_{R,T} \), satisfying the following properties

1. \( \psi = \psi(d(x,x_0,t),t) \equiv \psi(r,t); \psi(x,t) = 1 \) in \( Q_{R/2,T/4} \), \( 0 \leq \psi \leq 1 \).
2. \( \psi \) is decreasing as a radial function in the spatial variables.
3. \( \frac{\partial \psi}{\psi} \leq \frac{C}{R} \), \( \frac{\partial^2 \psi}{\psi^2} \leq \frac{C}{R^2} \) when \( 0 < a < 1 \).
4. \( \frac{\partial \psi}{\psi^{1/2}} \leq \frac{C}{\sqrt{T}} \).

Then, from (3.9) and a straightforward calculation, one has

\begin{equation}
\Delta (\psi w) + b \cdot \nabla (\psi w) - 2\frac{\nabla \psi}{\psi} \cdot \nabla (\psi w) - (\psi w)_t
\end{equation}

\begin{equation}
\geq 2\psi(1-f)w^2 + (b \cdot \nabla \psi)w - 2\frac{|\nabla \psi|^2}{\psi}w + (\Delta \psi)w - \psi_tw - 4kw\psi,
\end{equation}

where we have written \( b = -\frac{2f}{1-f} \nabla f \).

Comparing with the heat equation under a fixed metric, the last term \( -\psi_tw \) is more complicated. It is given by

\begin{equation}
-\psi_tw = -\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial r} \frac{\partial (d(x,x_0,t))}{\partial t} w.
\end{equation}

By our assumption that \( \text{Ricci} \geq -k \) and that \( -c/R \leq \frac{\partial \psi}{\partial r} \leq 0 \), we have

\begin{equation}
-\psi_tw \geq -\frac{\partial \psi}{\partial t} w - ckw\psi^{1/2}.
\end{equation}

Here \( \frac{\partial \psi}{\partial r} \equiv \frac{\partial \psi(r,t)}{\partial t} \).

Therefore

\begin{equation}
\Delta (\psi w) + b \cdot \nabla (\psi w) - 2\frac{\nabla \psi}{\psi} \cdot \nabla (\psi w) - (\psi w)_t
\end{equation}

\begin{equation}
\geq 2\psi(1-f)w^2 + (b \cdot \nabla \psi)w - 2\frac{|\nabla \psi|^2}{\psi}w + (\Delta \psi)w - \frac{\partial \psi}{\partial t} w - ckw\psi^{1/2}.
\end{equation}

Suppose the maximum of \( \psi w \) is reached at \( (x_1,t_1) \). By [LY], we can assume, without loss of generality that \( x_1 \) is not in the cut-locus of \( M \). Then at this point, one has, \( \Delta (\psi w) \leq 0 \), \( (\psi w)_t \geq 0 \) and \( \nabla (\psi w) = 0 \). Therefore

\begin{equation}
2\psi(1-f)w^2(x_1,t_1) \leq -[ (b \cdot \nabla \psi)w - 2\frac{|\nabla \psi|^2}{\psi}w + (\Delta \psi)w - \frac{\partial \psi}{\partial t} w ](x_1,t_1) + ckw\psi^{1/2}.
\end{equation}

We need to find an upper bound for each term of the right-hand side of (3.11).
\[(b \cdot \nabla \psi)w \leq \frac{2|f|}{1-f} |\nabla f| |\nabla \psi| \leq 2w^{3/2} |f| |\nabla \psi|\]

\[= 2[\psi(1 - f)w^2]^{3/4} \frac{f |\nabla \psi|}{[\psi(1 - f)]^{3/4}}\]

\[\leq \psi(1 - f)w^2 + c \frac{(f |\nabla \psi|)^4}{[\psi(1 - f)]^3}.
\]

This implies

\[(3.12) \quad |(b \cdot \nabla \psi)w| \leq (1 - f)\psi w^2 + c \frac{f^4}{R^4(1 - f)^3}.
\]

For the second term on the righthand side of (3.11), we proceed as follows

\[\frac{|\nabla \psi|^2}{\psi}w = \psi^{1/2} w \left| \frac{\nabla \psi}{\psi^{1/2}} \right|^2 \leq \frac{1}{8} \psi w^2 + c \frac{1}{R^4}.
\]

Furthermore, by the properties of \(\psi\) and the assumption of on the Ricci curvature, one has

\[-(\Delta \psi)w = -(\partial^2_r \psi + (n - 1) \frac{\partial_r \psi}{R} + \partial_r \psi \partial_r \log \sqrt{g})w \leq \left( |\partial^2_r \psi| + 2(n - 1) \frac{|\partial_r \psi|}{R} \right)w + c \frac{\sqrt{k} w \sqrt{\psi}}{R} \]

\[\leq \psi^{1/2} w \frac{|\partial^2_r \psi|}{\psi^{1/2}} + \psi^{1/2} w 2(n - 1) \frac{|\partial_r \psi|}{R \psi^{1/2}} + c \frac{\sqrt{k} w \sqrt{\psi}}{R} \]

\[\leq \frac{1}{8} \psi w^2 + c \left( \frac{|\partial^2_r \psi|}{\psi^{1/2}} \right)^2 + \frac{c k}{R^2}.
\]

Therefore

\[(3.14) \quad -(\Delta \psi)w \leq \frac{1}{8} \psi w^2 + c \frac{1}{R^4}.
\]

Now we estimate \(\left| \frac{\partial \psi}{\partial t} \right| w\).

\[\left| \frac{\partial \psi}{\partial t} \right| w = \psi^{1/2} w \left| \frac{\partial \psi}{\partial t} \right| \psi^{1/2} \]

\[\leq \frac{1}{8} \left( \psi^{1/2} w \right)^2 + c \left( \frac{|\partial \psi|}{\psi^{1/2}} \right)^2.
\]

This shows

\[(3.15) \quad \left| \frac{\partial \psi}{\partial t} \right| w \leq \frac{1}{8} \psi w^2 + c \frac{1}{T^2}.
\]

Substituting (3.12)-(3.15) to the righthand side of (3.11), we deduce,

\[2(1 - f)\psi w^2 \leq (1 - f)\psi w^2 + c \frac{f^4}{R^4(1 - f)^3} + \frac{1}{2} \psi w^2 + c \frac{1}{R^4} + \frac{c}{T^2} + \frac{c k}{R^2} + k w \sqrt{\psi}.
\]
Recall that $f \leq 0$, therefore the above implies
\[ \psi w^2(x_1, t_1) \leq c \frac{f^4}{R^4(1 - f)^4} + \frac{1}{2} \psi w^2(x_1, t_1) + \frac{c}{R^4} + \frac{c}{T^2} + ck^2. \]
Since $\frac{f^4}{(1 - f)^4} \leq 1$, the above shows, for all $(x, t) \in Q_{R,T}$,
\[ \psi^2(x, t)w^2(x, t) \leq \psi^2(x_1, t_1)w^2(x_1, t_1) \]
\[ \leq \psi(x_1, t_1)w^2(x_1, t_1) \]
\[ \leq c \frac{c}{R^4} + \frac{c}{T^2} + ck^2. \]
Notice that $\psi(x, t) = 1$ in $Q_{R/2, T/4}$ and $w = |\nabla f|^2/(1 - f)^2$. We finally have
\[ \frac{|\nabla f(x, t)|}{1 - f(x, t)} \leq \frac{c}{R} + \frac{c}{\sqrt{T}} + c\sqrt{k}. \]
We have completed the proof of Theorem 3.1 (a) since $f = \log(u/M)$ with $M$ scaled to 1.

**Proof of Theorem 3.1 (b).**

The proof is almost identical to that of Theorem 1.1 in [H] except for an additional curvature term. By direct computation, we have
\[ (\Delta - \partial_t)\left(\frac{|\nabla u|^2}{u}\right) \geq \frac{2}{u} \left| \partial_i \partial_j u - \frac{\partial_i u \partial_j u}{u} \right|^2. \]
In the above, comparing with the fixed curvature case (2.10), there is no more term containing the Ricci curvature. By (2.9), it holds
\[ \Delta(u \log \frac{M}{u}) - \partial_t (u \log \frac{M}{u}) = -\frac{|\nabla u|^2}{u}. \]
Since
\[ (\Delta - \partial_t)(t \frac{|\nabla u|^2}{u}) \geq -\frac{|\nabla u|^2}{u} \]
the maximum principle implies that
\[ \frac{|\nabla u|^2}{u^2} \leq \frac{1}{t} u \log \frac{M}{u}. \]

The remainder of the section deals with (3.2). For (3.2), we no longer have the nice cancelation effect that associated with (3.1). So we only obtain the following global gradient estimate under curvature assumptions.

**Theorem 3.2.** Let $M$ be a complete Riemannian manifold equipped with a family of Riemannian metric evolving under the forward Ricci flow in (3.2) with $t \in [0, T]$. Suppose $u$ is any positive solution to (3.2) in $M \times [0, T]$. Then, it holds
\[ \frac{|\nabla u(x, t)|}{u(x, t)} \leq \sqrt{\frac{1}{t} \log \frac{M}{u(x, t)}} \]
for $M = \sup_{M \times [0, T]} u$ and $(x, t) \in M \times [0, T]$. 

Moreover, the following interpolation inequality holds for any \( \delta > 0 \), \( x, y \in \mathcal{M} \) and \( 0 < t \leq T \):

\[
u(y, t) \leq c_1 u(x, t)^{1/(1+\delta)} M^{\delta/(1+\delta)} e^{c_2 d(x, y, \delta)^2/t}.
\]

Here \( c_1, c_2 \) are positive constants depending only on \( \delta \).

**Proof of Theorem 3.2.**

This again is almost the same as that of Theorem 1.1 in [H]. By direct calculation

\[
\Delta(\log \frac{M}{u}) - \partial_t (\log \frac{M}{u}) = -\frac{\nabla^2 u}{u},
\]

\[
(\Delta - \partial_t)(\frac{\nabla^2 u}{u}) = 2 \frac{u}{u} \left| \partial_i \partial_j u - \frac{\partial_i u \partial_j u}{u} \right|^2 \geq 0.
\]

The first inequality follows immediately from the maximum principle since

\[
t \frac{\nabla^2 u}{u} - u \log \frac{M}{u}
\]

is a sub-solution of the heat equation.

To prove the second inequality, we set

\[
l(x, t) = \log(\frac{M}{u(x, t)}).
\]

Then the first inequality implies

\[
|\nabla \sqrt{l(x, t)}| \leq \frac{1}{\sqrt{t}}.
\]

Fixing two points \( x \) and \( y \), we can integrate along a geodesic to reach

\[
\sqrt{\log(\frac{M}{u(x, t)})} \leq \sqrt{\log(\frac{M}{u(y, t)})} + \frac{d(x, y, t)}{\sqrt{t}}.
\]

The result follows by squaring both sides. \( \square \)

4. **Pointwise and gradient estimate for the fundamental solution to an equation of Perelman’s**

In the paper [P] Perelman introduced an equation which after time reversal becomes

\[
\begin{aligned}
  \Delta u - Ru - \partial_t u &= 0, \\
  \frac{d}{dt} g_{ij} &= 2 R_{ij}.
\end{aligned}
\]

Here as before \( \Delta \) is the Laplace-Beltrami operator with respect to the metric \( g_{ij} \) evolving by the backward Ricci flow. \( R \) is the scalar curvature. This equation and the associated monotonicity formula have proven to be of fundamental importance. Using the maximum principle and reduced distance, Perelman proved a lower bound for the fundamental solution to (4.0). An outstanding feature of the estimate is that it does need any explicit curvature assumption. The information on curvature is encoded in the reduced distance. From the analysis point of view, it would be desirable to establish an upper bound for the fundamental solution too. Here we first prove an upper bound under no explicit curvature assumptions. The bound is in terms of more traditional geometric quantities, i.e. the best constant in Sobolev imbedding or Yamabe constant, which are controlled by the lower bound of the Ricci curvature and injectivity radius. Under more restrictive curvature assumptions, we are able to prove a Gaussian like upper bound. Let us mention the method
by maximum principle alone does not seem to yield the upper bound. Our method is based on the one by J. Nash. For related results on local lower and upper bounds for fundamental solutions of (3.2) and for a global lower bound for the conjugate of (3.2) in the spirit of Perelman, please see the interesting papers [G], [Ni1] and [Ni2].

In order to state our theorem, we need to recall two concepts. One is the Yamabe constant and the other is the best constant in the Sobolev imbedding.

Given a Riemannian metric \( g(t) \) the Yamabe constant is

\[
Y(t) \equiv \inf \int \frac{||\nabla \phi||^2 + \frac{n-2}{4(n-1)} R \phi^2}{\int \phi^{2n/(n-2)} d\mu(x,t)} \, d\mu(x,t)^{(n-2)/n}.
\]

The other is a Sobolev imbedding theorem due to E. Hebey [Heb] which is a refined form (on the controlling constants) of the result by T. Aubin [ACDH]:

**Theorem S.** Let \( M \) be a complete (compact or noncompact) Riemannian \( n \)-manifold. Suppose the Ricci curvature is bounded below by \( k \) and the injectivity radius is bounded below by \( i > 0 \). For any \( \epsilon > 0 \), there exists \( B(g) = B(\epsilon, n, k, i) \) such that for any \( \phi \in W^{1,2}(M) \),

\[
\left( \int_M |u|^{2n/(n-2)} d\mu(g) \right)^{(n-2)/n} \leq (K(n)^2 + \epsilon) \int_M |\nabla u|^2 d\mu(g) + B(g) \int_M u^2 d\mu(g).
\]

Here \( K(n) \) is the best constant in the Sobolev imbedding in \( \mathbb{R}^n \).

We also need to mention the result by Hebey and Vaugon [HV] where Theorem S is proven with \( \epsilon = 0 \). However, then the constant \( B \) may depend on the derivative of the curvature tensor which is harder to control. For our purpose, it suffices to fix the \( \epsilon \) as any positive constant, say \( 1 \).

The following theorem is the main result of the section. It contains three statements. The first one is an upper bound controlled by the Yamabe constants, the second is an upper bound controlled by the constant \( B(g) \) in the Sobolev imbedding Theorem S. They may seem technical at the first glance. However, the third statement of the theorem provides a clarification. It shows that these upper-bounds are the proper extension of on-diagonal upper bound for the heat kernel in the fixed metric case. Recall that for a compact Riemannian manifold \( M \) without boundary, the heat kernel \( G \) satisfies the following on-diagonal upper bound:

\[
G(x, t; y, s) \leq c_1 \max\left\{ \frac{1}{(t-s)^{n/2}}, 1 \right\}
\]

for some constant \( c_1, c_2 > 0 \) and for all \( t > s \) and \( x, y \in M \).

Here are some additional notations for the theorem. We will write

\[
R^- = -\min\{R(x, t), 0\}
\]

where \( R(x, t) \) is the scalar curvature under the metric \( g(t) \). When the scalar curvature changes sign, the theorem will also involve the expression

\[
\frac{1}{(\max R^-(\cdot, t))^{-1} + (t-s)}.
\]

This quantity is regarded as 0 when \( R(\cdot, t) \geq 0 \).
Remark 4.1. In statements (b) and (c) of Theorem 4.1 below, the controlling constants depend only on the dimension, the lower bound of Ricci curvature and the lower bound of injectivity radii. By the result of Cheeger [Che], if one assumes that the sectional curvatures are bounded between two constants and the volume of geodesic balls of radius 1 is bounded below by a positive constant, then the injectivity radii are bounded from below by a positive constant. Therefore, the controlling constants in (b) and (c) depend only on the bound of sectional curvature, the lower bound of volume of balls of radius 1 and dimension. The same can be said for Theorem 5.1 below. The upshot is that the length of time and the incompatibility of metric at different time do not destroy the bound.

**Theorem 4.1.** Suppose equation (4.0) has a smooth solution in the time interval \([s,t]\) and let \(G\) be the fundamental solution of (4.0). Then the following statements hold.

(a). Suppose the Yamabe invariant \(Y(g(\tau)) > 0\) for \(\tau \in [s,t]\), then

\[
G(x,t;y,s) \leq \frac{c_n}{\left( \int_{s}^{(t+s)/2} e^{2c_n a(\tau)/n} Y(\tau)d\tau \int_{t+s/2}^{t} e^{2c_n a(\tau)/n} \left[1 + (t - \tau) \max R^-(\cdot,t)\right]^{-4/n} Y(\tau)d\tau \right)^{n/4}}.
\]

Here \(a(\tau) = \int_{\tau}^{s} \frac{1}{(\max R^-(\cdot,t))^{-1} + (t-l)}dl\).

(b). Let \(B(g(\tau))\) be the best constant in the Sobolev imbedding Theorem S. Then

\[
G(x,t;y,s) \leq \frac{c_n}{\left( \int_{s}^{(t+s)/2} e^{2H(\tau)c_n/n} d\tau \int_{t+s/2}^{t} \left[1 + (t - \tau) \max R^-(\cdot,t)\right]^{-4/n} e^{-2H(\tau)c_n/n} d\tau \right)^{n/4}}
\]

with

\[H(\tau) = \int_{\tau}^{s} \left[B(g(l)) + \frac{1}{\max R^-(\cdot,t)} - 1 + (t-l)\right]dl\].

(c). In the special case that \(R^-(\cdot,t) \geq 0\) and \(\text{Ric}(g(\tau)) \geq k\) and the injectivity radius is bounded below by \(i > 0\), for all \(\tau \in [s,t]\), then

\[
G(x,t;y,s) \leq C(n,B) \max\left\{1, \frac{1}{(t-s)^{n/2}}\right\}.\]

Here \(B\) only depends on \(n\), \(k\) and \(i\). Moreover

\[
\frac{|\nabla_y G(x,t;y,s)|^2}{G(x,t;y,s)^2} \leq C(n,B) \frac{1}{(t-s)} \log \frac{\max\left\{1, \frac{1}{(t-s)^{n/2}}\right\}}{G(x,t;y,s)}.
\]

**Remark 4.2.** Recently, in a paper [CL], Chang and Lu, proved a derivative estimate for the Yamabe constant under the Ricci flow. It can be coupled with this theorem to obtain better upper bound on \(G\).

Professor Lei Ni also informs us that he also knows a result on upper bound in the case of certain Sobolev inequality.

**Proof of part (a).** Without loss of generality, we take \(s = 0\) here and later. Let \(G\) be the fundamental solution to (4.0). By the reproducing property

\[
G(x,t;0) = \int G(x,t;z,t/2)G(z,t/2;y,0)d\mu(z,t/2),
\]
there holds
\begin{equation}
G(x, t; y, 0) \leq \left[ \int G^2(x, t; z, t/2) d\mu(z, t/2) \right]^{1/2} \left[ \int G^2(z, t/2; y, 0) d\mu(z, t/2) \right]^{1/2}.
\end{equation}

Therefore an upper bound follows from pointwise estimate on the two quantities
\begin{equation}
p(t) = \int G^2(x, t; y, s) d\mu(x, t),
\end{equation}
\begin{equation}
q(s) = \int G^2(x, t; y, s) d\mu(y, s).
\end{equation}

Let us estimate \( p(t) \) in (4.2) first.

It is clear that
\begin{equation}
\frac{d}{dt} p(t) = 2 \int G[\Delta G - RG] d\mu(x, t) + \int G^2 R d\mu(x, t).
\end{equation}

Here and later we omit the arguments on \( G \) and differential operators when no confusions appear. Therefore
\begin{equation}
\frac{d}{dt} p(t) \leq -\int \|\nabla G\|^2 + RG^2 d\mu(x, t) - \frac{3n - 2}{4(n - 1)} \int RG^2 d\mu(x, t).
\end{equation}

Let \( Y(t) \) be the Yamabe constant with respect to \( g(t) \), i.e.
\begin{equation}
Y(t) = \inf \frac{\int [\|\nabla \phi\|^2 + \frac{n - 2}{4(n - 1)} R\phi^2] d\mu(x, t)}{\left( \int \phi^{2n/(n - 2)} d\mu(x, t) \right)^{(n - 2)/n}}.
\end{equation}

By Hölder's inequality
\begin{equation}
\int G^2 d\mu(x, t) \leq [ \int G^{2n/(n - 2)} d\mu(x, t) ]^{(n - 2)/(n + 2)} [ \int G d\mu(x, t) ]^{4/(n + 2)},
\end{equation}
we arrive at the 'conformal' Nash inequality
\begin{equation}
\int G^2 d\mu(x, t) \leq c_n Y(t)^{-n/(n + 2)} [ \int [\|\nabla \phi\|^2 + \frac{n - 2}{4(n - 1)} R\phi^2] d\mu(x, t) ]^{n/(n + 2)} [ \int G d\mu(x, t) ]^{4/(n + 2)}.
\end{equation}

It is easy to check that
\begin{equation}
\int G(x, t; y, s) d\mu(x, t) = 1.
\end{equation}

Therefore (4.5) becomes
\begin{equation}
\int G^2 d\mu(x, t) c_n \leq c_n Y(t)^{-n/(n + 2)} [ \int [\|\nabla \phi\|^2 + \frac{n - 2}{4(n - 1)} R\phi^2] d\mu(x, t) ]^{n/(n + 2)}.
\end{equation}

Substituting this to (4.4), we deduce
\begin{equation}
p'(t) \leq -c_n p(t)^{(n+2)/n} Y(t) - \frac{3n - 2}{4(n - 1)} \int RG^2 d\mu(x, t).
\end{equation}
It well-known (see [CK] e.g.) that the scalar curvature \( R \) satisfies the inequality
\[
\frac{dR}{dt} + \Delta R + c_n R^2 \leq 0.
\]
This implies
\[
R(y, \tau) \geq -\frac{1}{(\max R^-(\cdot, \tau))^{-1} + c_n (t - \tau)}, \quad \tau < t.
\]
Here and later, if \( R(\cdot, t) \geq 0 \), then the above fraction is regarded as zero. Hence, for \( \tau \in (s, t) \),
\[
p'(\tau) \leq -c_n p(\tau)^{(n+2)/n} Y(\tau) + \frac{c_n}{(\max R^-(\cdot, \tau))^{-1} + (t - \tau)} p(\tau).
\]
Let
\[
a(\tau) = \int_s^\tau \frac{1}{(\max R^-(\cdot, l))^{-1} + (t - l)} dl.
\]
Then the above ordinary differential inequality becomes
\[
(e^{-c_n a(\tau)})' p(\tau) \leq c_n (e^{-c_n a(\tau)})^{(n+2)/n} e^{2c_n a(\tau)/n} Y(\tau).
\]
Integrating from \( s \) to \( t \), we deduce
\[
e^{-c_n a(t)} p(t) \leq \frac{c_n}{\int_s^t e^{2c_n a(\tau)/n} Y(\tau) d\tau}^{n/2}.
\]
This immediately shows that
\[
\int G^2(x, t; y, s) d\mu(x, t) = p(t) \leq \frac{c_n e^{c_n a(t)}}{\int_s^t e^{2c_n a(\tau)/n} Y(\tau) d\tau}^{n/2}.
\]
(4.6)

Next we estimate \( q(s) \) in (4.3). Due to the asymmetry of the equation, the computation is different. Notice that the second entries of \( G \) satisfies the backward heat equation. i.e.
\[
\Delta_y G(x, t; y, s) + \partial_s G(x, t; y, s) = 0.
\]
This gives
\[
q'(s) = -2 \int G \Delta G d\mu(y, s) + \int R G^2 d\mu(y, s).
\]
Hence
\[
q'(s) \geq \int [\|\nabla G\|^2 + R G^2] d\mu(y, s).
\]
(4.7)

By the same argument as before we arrive at the Nash inequality
\[
\int G^2 d\mu(y, s) \leq c_n Y(s)^{-n/(n+2)} \left[ \int [\|\nabla \phi\|^2 + \frac{n-2}{4(n-1)} R \phi^2] d\mu(y, s) \right]^{n/(n+2)} \left[ \int G d\mu(y, s) \right]^{4/(n+2)}.
\]
(4.8)

This time we have to compute the quantity
\[
I(s) \equiv \int G(x, t; y, s) d\mu(y, s).
\]
It is clear that
\[ I'(s) = \int G(x, t; y, s) R(y, s) d\mu(y, s). \] (4.9)

Recall that
\[ R(y, \tau) \geq -\frac{1}{(\max R^-(\cdot, t))^{-1} + c_n(t - \tau)}, \quad \tau < t. \]

Combining this with (4.9) we deduce
\[ I'(\tau) \geq -\frac{1}{(\max R^-(\cdot, t))^{-1} + c_n(t - \tau)} I(\tau). \]

Integrating from \( s \) to \( t \) and noting that \( I(t) = 1 \), we obtain
\[ I(s) \leq 1 + c_n(t - s) \max R^-(\cdot, t). \] (4.10)

Substituting (4.10) to (4.8), we deduce
\[ \int [||\nabla G||^2 + RG^2] d\mu(y, s) \]
\[ = \int [||\nabla G||^2 + \frac{n - 2}{4(n - 1)} RG^2] d\mu(y, s) + \frac{3n - 2}{4(n - 1)} \int RG^2 d\mu(y, s) \]
\[ \geq [q(s)]^{(n+2)/n} \left[ 1 + c_n(t - s) \max R^-(\cdot, t) \right]^{-4/n} Y(s) - \frac{c_n q(s)}{(\max R^-(\cdot, t))^{-1} + (t - s)}. \]

Here, again we used the lower bound on the scalar curvature, given just below (4.9).

This and (4.7) together imply that, for \( \tau \in (s, t) \),
\[ q'(\tau) \geq c_n q(\tau) [1 + (t - \tau) \max R^-(\cdot, t)]^{-4/n} Y(\tau) - \frac{c_n q(\tau)}{(\max R^-(\cdot, t))^{-1} + (t - \tau) q(\tau)}. \]

Let again
\[ a(\tau) = \int_s^\tau \frac{1}{(\max R^-(\cdot, t))^{-1} + (t - \tau)} d\tau. \]

Then
\[ q(\tau) e^{c_n a(\tau)}' \geq c_n q(\tau) e^{c_n a(\tau)} \left[ 1 + (t - \tau) \max R^-(\cdot, t) \right]^{-4/n} Y(\tau). \]

Integrating from \( s \) to \( t \), we obtain
\[ q(s) = \int G^2(x, t; y, s) d\mu(y, s) \]
\[ \leq \frac{c_n e^{-c_n a(s)}}{\left( \int_s^t e^{-2c_n a(\tau)/n} \left[ 1 + (t - \tau) \max R^-(\cdot, t) \right]^{-4/n} Y(\tau) d\tau \right)^{n/2}}. \] (4.12)

Now (4.6) and (4.12) respectively imply that
\[ \int G^2(z, t/2; y, 0) d\mu(z, t/2) \leq \frac{c_n e^{c_n a(t/2)}}{\left( \int_0^{t/2} e^{2c_n a(\tau)/n} Y(\tau) d\tau \right)^{n/2}}. \] (4.13)
\[
\int G^2(x, t; z, t/2) d\mu(z, t/2) \\
\leq \frac{c_n e^{-c_n a(t/2)}}{\left( \int_{t/2}^t e^{-2c_n a(\tau)/n} \left[ 1 + (t - \tau) \max R^{-}(\cdot, t) \right]^{-4/n} Y(\tau) d\tau \right)^{n/2}}.
\]

By (4.1), (4.13) and (4.14), we arrive at the following upper bound
\[
G(x, t; y, 0) \\
\leq \frac{c_n}{\left( \int_0^{t/2} e^{2c_n a(\tau)/n} Y(\tau) d\tau \int_{t/2}^t e^{-2c_n a(\tau)/n} \left[ 1 + (t - \tau) \max R^{-}(\cdot, t) \right]^{-4/n} Y(\tau) d\tau \right)^{n/4}}.
\]

This proves part (a).

**Proof of Part (b).**

We generally follow the previous arguments between (4.1) and (4.14) to derive an upper bound. The difference is that we will use the Sobolev inequality (Theorem S) instead of the Yamabe constant.

As before, by Hölder’s inequality and Theorem S, we arrive at the Nash type inequality
\[
\int G^2 d\mu(x, t) \\
\leq \left[ \int [c_n |\nabla G|^2 d\mu(x, t) + B(g(t)) \int_M G^2 d\mu(x, t)]^{n/(n+2)} \left[ \int G d\mu(x, t) \right]^{4/(n+2)} \right].
\]

Here and later \(c_n\) is a dimensional constant that may change from line and to line. Since, again,
\[
\int G d\mu(x, t) = 1,
\]
we have
\[
\int |\nabla G|^2 d\mu(x, t) \geq c_n \left[ \int_M G^2 d\mu(x, t) \right]^{(n+2)/n} - c_n B(g(t)) \int G^2 d\mu(x, t).
\]

Combing (4.16) with (4.4) under again the notation (4.2), we obtain
\[
p'(t) \leq -c_n p(t)^{(n+2)/n} + c_n B(g(t))p(t) - \int_M R G^2 d\mu(x, t).
\]

Fixing \(s\) and \(t\), for any \(\tau \in (s, t)\), we still have the lower bound for the scalar curvature (just after (4.9))
\[
R(\cdot, \tau) \geq -\frac{1}{(\max R^{-}(\cdot, t))^{-1} + c_n(t - \tau)} , \quad \tau < t.
\]

Therefore
\[
p'(\tau) \leq -c_n p(\tau)^{(n+2)/n} + c_n h(\tau)p(\tau),
\]
with
\[
h(\tau) = B(g(\tau)) + \frac{1}{(\max R^{-}(\cdot, t))^{-1} + (t - \tau)}.
\]
Let $H(\tau)$ be the anti-derivative of $h(\tau)$ such that $H(s) = 0$. Then
\[
\left( e^{-c_n H(\tau)} p(\tau) \right)' \leq -c_n \left( e^{-c_n H(\tau)} p(\tau) \right)^{(n+2)/n} e^{2c_n H(\tau)/n}.
\]
Integrating from $s$ to $t$, we arrive at
\[
(4.18) \quad p(t) \leq \frac{c_n e^{c_n H(t)}}{\left( \int_s^t e^{2H(\tau)c_n/nd\tau} \right)^{n/2}}.
\]
Our next task is to bound
\[
q(s) = \int G^2(x, t; y, s) d\mu(y, s).
\]
Clearly the counter-parts of (4.7), (4.10) and (4.15) still hold, i.e.
\[
q'(s) \geq \int (|\nabla G|^2 + RG^2) d\mu(y, s).
\]
\[
I(s) = \int G d\mu(y, s) \leq 1 + c_n (t-s) \max R^-(\cdot, t).
\]
\[
\int G^2 d\mu(y, s) \leq \left[ \int c_n |\nabla G|^2 d\mu(y, s) + B(g(s)) \int G^2 d\mu(x, t) \right]^{n/(n+2)} \left[ \int G d\mu(y, s) \right]^{4/(n+2)}.
\]
Also
\[
R(y, s) \geq -\frac{1}{\max R^-(\cdot, t)^{-1} + c_n (t-s)}.
\]
These four inequalities imply that
\[
q'(s) \geq c_n q(s)^{(n+2)/n} \left[ 1 + (t-s) \max R^-(\cdot, t)^{-1/n} - c_n h(s) q(s) \right].
\]
Here $h(s)$ is given by the expression just below (4.17) with $\tau$ replaced by $s$. Now, for fixed $s$ and $t$ and any $\tau \in (s, t)$, the above differential inequality on $q'(s)$ is still valid for $q'(\tau)$ when $s$ is replaced by $\tau$. Let $H(\tau)$ be the antiderivative of $h(\tau)$ with $H(s) = 0$. Then it is clear that
\[
\left( e^{c_n H(\tau)} q(\tau) \right)' \geq c_n \left( e^{c_n H(\tau)} q(\tau) \right)^{(n+2)/n} \left[ 1 + (t-\tau) \max R^-(\cdot, t)^{-1/n} e^{-2H(\tau)c_n/n} \right].
\]
Integrating from $s$ to $t$, we arrive at
\[
(4.19) \quad q(s) \leq \frac{c_n e^{-c_n H(s)}}{\left( \int_s^t [1 + (t-\tau) \max R^-(\cdot, t)^{-1/n} e^{-2H(\tau)c_n/nd\tau}] \right)^{n/2}}.
\]
By (4.18), we have
\[
(4.20) \quad p(t/2) = \int G^2(z, t/2; y, 0) d\mu(z, t/2) \leq \frac{c_n e^{c_n H(t/2)}}{\left( \int_0^{t/2} e^{2H(\tau)c_n/nd\tau} \right)^{n/2}}.
\]
Also, (4.19) shows

\[ q(t/2) = \int G^2(x, t; z, t/2) \, d\mu(z, t/2) \leq \frac{c_n e^{-H(t/2)}}{\left( \int_{t/2}^t [1 + (t - \tau) \max R^{-\cdot}(\cdot, t)]^{-4/n} e^{-2H(\tau)/n} \, d\tau \right)^{n/2}}. \]

Here

\[ H(t/2) = \int_{t/2}^t [B(g(\tau)) + \frac{1}{(\max R^{-\cdot}(\cdot, t))^{-1} + (t - \tau)}] \, d\tau. \]

Multiplying (4.20) and (4.21), and using (4.1), we have proven the on-diagonal upper bound

\[ G(x, t; y, 0)^2 \leq c_n \left( \frac{\int_{t/2}^t e^{2H(\tau)c_n/n} \, d\tau}{\left( \int_{t/2}^t [1 + (t - \tau) \max R^{-\cdot}(\cdot, t)]^{-4/n} e^{-2H(\tau)c_n/n} \, d\tau \right)^{n/2}} \right)^{n/2}. \]

This gives part (b).

**Proof of part (c)**.

In the special case that \( R(\cdot, t) \geq 0 \) and \( Ricc(g(\tau)) \geq k \) uniformly and the injectivity radius is uniformly bounded below by \( i \), then

\[ H(\tau) = \int_0^\tau [B(g(l)) + \frac{1}{(\max R^{-\cdot}(\cdot, t))^{-1} + (t - l)}] \, dl = c_n B(n, k, i) \tau \]

and

\[ \frac{1}{(\max R^{-\cdot}(\cdot, t))^{-1} + (t - \tau)} = 0. \]

Hence the above immediately shows

\[ G(x, t; y, 0) \leq C(c_n, B) \max\left\{ \frac{1}{\left( \frac{t}{n/2} \right)^n}, 1 \right\}. \]

The gradient estimate follows from Hamilton’s argument in Theorem 1.1 [H], which can be easily generalized to the present case. For \( z \in M \) and \( \tau \in [0, t/2] \), let

\[ v(z, \tau) = G(x, t; z, \tau). \]

Then \( v \) is a solution to the backward heat equation \( \Delta v + v_\tau = 0 \). By direct computation

\[ (\Delta + \partial_\tau) \frac{|\nabla v|^2}{v} = \frac{2}{v} \left| \partial_i \partial_j v - \frac{\partial_i v \partial_j v}{v} \right|^2 \geq 0. \]

Let \( A \) be the maximum of \( v \) in the time interval \([0, t/2]\). By the above estimate

\[ A \leq C(c_n, B) \max\left\{ \frac{1}{\left( \frac{t}{n/2} \right)^n}, 1 \right\}. \]
Direct computation shows
\[
\Delta (v \log \frac{A}{v}) + \partial_\tau (v \log \frac{A}{v})
= (\Delta v + \partial_\tau v) \log A + (\Delta + \partial_\tau)(v \log \frac{A}{v})
= -\frac{\|\nabla v\|^2}{v}.
\]
Let \( \phi = (t/2) - \tau \), then it is clear that, for
\[
h = \phi \frac{\|\nabla v\|^2}{v} - v \log \frac{A}{v},
\]
there holds
\[
\Delta h + \partial_\tau h \geq 0.
\]
By the maximum principle, applied backward in time, we have
\[
\frac{\|\nabla_y v\|^2}{v^2} \leq C \frac{1}{r} \log \frac{1}{\tau/m, 1}
\]
for \( \tau \in [0, t/4] \).

5. THE CASE OF NONNEGATIVE RICCI CURVATURE

In this section, we specialize to the case of nonnegative Ricci curvature. We establish certain Gaussian type upper bound for the fundamental solution of (4.0). We will begin with the traditional method of establishing a mean value inequality via Moser’s iteration and a weighted estimate in the spirit of Davies [Da]. However, there is some difficulty in applying this method directly due to the lack of control of the time derivative of the distance function. The new idea to overcome this difficulty is to use the interpolation result of Theorem 3.2 and the bound in Theorem 4.1 (c).

The following are some additional notations for this section. We will use \( B(x, r; t) \) to denote the geodesic ball centered at \( x \) with radius \( r \) under the metric \( g(t) \); \( |B(x, r; t)|_s \) to denote the volume of \( B(x, r; t) \) under the metric \( g(s) \).

The main result of this section is Theorem 5.1 below. Note that the theorem is qualitatively sharp in general since it matched the well-known Gaussian upper bound for the fixed metric case. Also there is no assumption on the comparability of metrics at different times. In this theorem, we assume the manifold is compact. This accounts for the extra 1 on the Gaussian upper bound. Even in the case of fixed metric, the heat kernel converges to a positive constant for large time. The theorem still holds for certain noncompact manifolds under suitable assumptions. In this case the extra 1 in the upper bound should be replaced by 0.

Remark 5.1. As mentioned in section 4 (Remark 4.1), the controlling constants in the theorem below can be made to depend only on the bound of sectional curvature, the lower bound of volume of balls of radius 1 and the dimension.

In the case \( \text{Ricci} \geq -k \) with \( k > 0 \), then certain integral Gaussian bound similar to the one below (5.13) can still be proven by the same method. However, so far we are not able to derive a pointwise Gaussian upper bound without an exponentially growing term
This is due to a lack of an efficient mean value inequality for the second entries of the fundamental solution, which satisfies (3.2) after a time reversal.

**Theorem 5.1.** Assume that equation (4.0) has a smooth solution in the time interval $[0, T]$ and let $G$ be the fundamental solution of (4.0). Suppose that $\text{Ricci} \geq 0$ and that the injectivity radius is bounded from below by a positive constant $i$ throughout. Then the following statement holds.

For any $s, t \in (0, T)$ and $x, y \in M$, there exist a dimensional constant $c_n$, a dimension less constant $c$ and a constant $A$ depending only on $i$ such that

$$G(x, t; y, s) \leq c_n A \left(1 + \frac{1}{(t - s)^{n/2}} + \frac{1}{|B(x, \sqrt{t - s}, t)|_s}\right) e^{-cd(x,y,s)^2/(t-s)}.$$

**Proof**

It is obvious that we only have to deal with the case that $B(x, 2\sqrt{t - s}, s)$ is a proper sub-domain of $M$. Otherwise, $\sqrt{t - s}/2 \geq d(x, y, s)$ for any $x, y \in M$. So the exponential term is mute and the result is already proven by Theorem 4.1 (c).

First we use Moser’s iteration to prove a mean value inequality. The only new factor is a cancelation effect induced by the backward Ricci flow. So we will be brief in the presentation at this part of the proof.

Let $u$ be a positive solution to (4.0) in the region

$$Q_{\sigma r}(x, t) \equiv \{(y, s) \mid z \in M, t - (\sigma r)^2 \leq s \leq t, d(y, x, s) \leq \sigma r\}.$$

Here $r > 0, 2 \geq \sigma \geq 1$. Given any $p \geq 1$, it is clear that

$$\Delta u^p - p Ru^p - \partial_t u^p \geq 0.

(5.1)$$

Let $\phi : [0, \infty) \to [0, 1]$ be a smooth function such that $|\phi'| \leq 2/((\sigma - 1)r)$, $\phi' \leq 0$, $\phi \geq 0$, $\phi(\rho) = 1$ when $0 \leq \rho \leq r$, $\phi(\rho) = 0$ when $\rho \geq \sigma r$. Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function such that $|\eta'| \leq 2/((\sigma - 1)r)^2$, $\eta' \geq 0$, $\eta \geq 0$, $\phi(s) = 1$ when $t - r^2 \leq s \leq t$, $\phi(s) = 0$ when $s \leq t - (\sigma r)^2$.

Writing $w = u^p$ and using $w\psi^2$ as a test function on (5.2), we deduce

$$\int \nabla(w\psi^2)\nabla w d\mu(y, s)ds + p \int R w^2 \psi^2 d\mu(y, s)ds \leq - \int (\partial_s w) w \psi^2 d\mu(y, s)ds.

(5.3)$$

By direct calculation

$$\int \nabla(w\psi^2)\nabla w d\mu(y, s)ds = \int |\nabla(w\psi)|^2 d\mu(y, s)ds - \int |\nabla\psi|^2 w^2 d\mu(y, s)ds.

(5.4)$$

Next we estimate the righthand side of (5.3). Here we will use the backward Ricci flow.

$$- \int (\partial_s w) w \psi^2 d\mu(y, s)ds = \int w^2 \psi \partial_s \psi d\mu(y, s)ds + \frac{1}{2} \int (w\psi)^2 Rd\mu(y, s)ds - \frac{1}{2} \int (w\psi)^2 d\mu(y, t).

$$

Observe that

$$\partial_s \psi = \eta(s) \phi'(d(y, x, s)) \partial_s d(y, x, s) + \phi(d(y, x, s))\eta'(s) \leq \phi(d(y, x, s))\eta'(s).$$
This is so because $\phi' \leq 0$ and $\partial_s d(y, x, s) \geq 0$ under the backward Ricci flow with nonnegative Ricci curvature. Hence

$$-
\int (\partial_s w)w\varepsilon^2 d\mu(y, s)ds
\leq \int w^2 \psi(d(y, x, s))\eta(s)d\mu(y, s)ds + \frac{1}{2} \int (w\psi)^2 Rd\mu(y, s)ds - \frac{1}{2} \int (w\psi)^2 d\mu(y, t).$$

Combing (5.3) to (5.5), we obtain, in view of $p \geq 1$ and $R \geq 0$,

$$\int |\nabla(w\psi)|^2 d\mu(y, s)ds + \frac{1}{2} \int (w\psi)^2 d\mu(y, t) \leq \frac{c}{(\sigma - 1)^2 r^2} \int_{Q_{\sigma r}(x, t)} w^2 d\mu(y, s)ds.$$  

By Hölder’s inequality

$$\int (\psi w)^{2(1+(2/n))} d\mu(y, s) \leq \left( \int (\psi w)^{2n/(n-2)} d\mu(y, s) \right)^{(n-2)/n} \left( \int (\psi w)^2 d\mu(y, s) \right)^{2/n}.$$  

Let us assume that $B(x, \sigma r, s)$ is a proper sub-domain of $M$. In this case, for manifolds with nonnegative Ricci curvature, it is well-known that the following Sobolev imbedding holds (see [Sa] e.g.)

$$\left( \int (\psi w)^{2n/(n-2)} d\mu(y, s) \right)^{(n-2)/n} \leq \frac{c_n \sigma^2 r^2}{|B(x, \sigma r, s)|^{2/n}_{2(\sigma r)^2}} \int [\nabla(\psi w)]^2 + r^{-2}(\psi w)^2] d\mu(y, s).$$

For $s \in [t - (\sigma r)^2, t]$, by the assumption that the Ricci curvature is nonnegative, it holds

$$B(x, \sigma r, s) \supseteq B(x, \sigma r, t); \quad |B(x, \sigma r, s)| \geq |B(x, \sigma r, t)|_{t - (\sigma r)^2}.$$  

Therefore we have

$$\left( \int (\psi w)^{2n/(n-2)} d\mu(y, s) \right)^{(n-2)/n} \leq \frac{c_n \sigma^2 r^2}{|B(x, \sigma r, t)|^{2/n}_{2(\sigma r)^2}} \int [\nabla(\psi w)]^2 + r^{-2}(\psi w)^2] d\mu(y, s).$$

For $s \in [t - (\sigma r)^2, t]$. Substituting (5.7) and (5.8) to (5.6), we arrive at the estimate

$$\int_{Q_{r}(x, t)} w^{2\theta} d\mu(y, s)ds \leq c_n \frac{r^2}{|B(x, \sigma r, t)|^{2/n}_{t - (\sigma r)^2}} \left( \frac{1}{(\sigma - 1)^2 r^2} \int_{Q_{\sigma r}(x, t)} w^2 d\mu(y, s)ds \right)^\theta,$$

with $\theta = 1 + (2/n)$. Now we apply the above inequality with the parameters $\sigma_0 = 1, \sigma_i = 1 - \sum_{j=1}^i 2^{-j-1}$ and $p = \theta^i$. This shows a $L^2$ mean value inequality

$$\sup_{Q_{r/2}(x, t)} u^2 \leq \frac{c_n}{r^2 |B(x, r, t)|^{1/2}_{t-r^2}} \int_{Q_{r}(x, t)} u^2 d\mu(y, s)ds.$$  

From here, by a generic trick of Li and Schoen [LS], applicable here since it uses only the doubling property of the metric balls, we arrive at the $L^1$ mean value inequality

$$\sup_{Q_{r/2}(x, t)} u \leq \frac{c_n}{r^2 |B(x, r, t)|^{1/2}_{t-r^2}} \int_{Q_r(x, t)} u d\mu(z, \tau)d\tau.$$
Fixing $y \in M$ and $s < t$, we apply (5.10) on $u = G(\cdot, \cdot; y, s)$ with $r = \sqrt{t-s}/2$. Note that $\int_M u(z, \tau) d\mu(z, \tau) = 1$. The doubling property of the geodesic balls show that

$$G(x, t; y, s) \leq \frac{c_n}{|B(x, \sqrt{t-s})|}$$

when $|B(x, \sqrt{t-s})|$ is a proper subdomain of $M$.

Without loss of generality, we take $s = 0$. We begin by using a modified version of the exponential weight method due to Davies [Da]. Pick a point $x_0 \in M$, a number $\lambda < 0$ and a function $f \in L^2(M, g(0))$. Consider the functions $F$ and $u$ defined by

$$F(x, t) \equiv e^{\lambda d(x, x_0, t)} u(x, t) \equiv e^{\lambda d(x, x_0, t)} \int G(x, t; y, 0) e^{-\lambda d(y, x_0)} f(y) d\mu(y, 0).$$

It is clear that $u$ is a solution of (4.0). By direct computation, we have

$$\partial_t \int F^2(x, t) d\mu(x, t) = \partial_t \int e^{2\lambda d(x, x_0, t)} u^2(x, t) d\mu(x, t)$$

$$= 2\lambda \int e^{2\lambda d(x, x_0, t)} \partial_t d(x, x_0, t) u^2(x, t) d\mu(x, t) + \int e^{2\lambda d(x, x_0, t)} u^2(x, t) R(x, t) d\mu(x, t)$$

$$+ 2 \int e^{2\lambda d(x, x_0, t)} [\Delta u - R(x, t) u(x, t)] u(x, t) d\mu(x, t).$$

By the assumption that $\text{Ricci} \geq 0$ and $\lambda < 0$, the above shows

$$\partial_t \int F^2(x, t) d\mu(x, t) \leq 2 \int e^{2\lambda d(x, x_0, t)} u \Delta u d\mu(x, t).$$

Using integration by parts, we turn the above inequality into

$$\partial_t \int F^2(x, t) d\mu(x, t)$$

$$\leq -4\lambda \int e^{2\lambda d(x, x_0, t)} u \nabla d(x, x_0, t) \nabla u d\mu(x, t) - 2 \int e^{2\lambda d(x, x_0, t)} |\nabla u|^2 d\mu(x, t).$$

Observe also

$$\int |\nabla F(x, t)|^2 d\mu(x, t) = \int |\nabla (e^{\lambda d(x, x_0, t)} u(x, t))|^2 d\mu(x, t)$$

$$= \int e^{2\lambda d(x, x_0, t)} |\nabla u|^2 d\mu(x, t) + 2\lambda \int e^{2\lambda d(x, x_0, t)} u \nabla d(x, x_0, t) \nabla u d\mu(x, t)$$

$$+ \lambda^2 \int e^{2\lambda d(x, x_0, t)} |\nabla d|^2 u^2 d\mu(x, t).$$

Combining the last two expressions, we deduce

$$\partial_t \int F^2(x, t) d\mu(x, t) \leq -2 \int |\nabla F(x, t)|^2 d\mu(x, t) + \lambda^2 \int e^{2\lambda d(x, x_0, t)} |\nabla d|^2 u^2 d\mu(x, t).$$

By the definition of $F$ and $u$, this shows

$$\partial_t \int F^2(x, t) d\mu(x, t) \leq \lambda^2 \int F(x, t)^2 d\mu(x, t).$$
Upon integration, we derive the following $L^2$ estimate
\begin{equation}
(5.11) \quad \int F^2(x,t) d\mu(x,t) \leq e^{\lambda^2 t} \int F^2(x,0) d\mu(x,0) = e^{\lambda^2 t} \int f(x)^2 d\mu(x,0).
\end{equation}

Recall that $u$ is a solution to (4.0). Therefore, by the mean value inequality (5.9), the following holds
\[ u(x,t)^2 \leq \frac{c_n}{t|B(x, \sqrt{t/2}, t)|_{t/2}} \int_{t/2}^{t/2} \int_{B(x, \sqrt{t/2}, \tau)} u^2(z, \tau) d\mu(z, \tau) d\tau. \]

i.e. By the definition of $F$ and $u$, it follows that
\[ u(x,t)^2 \leq \frac{c_n e^{-2\lambda\sqrt{t/2}}}{t|B(x, \sqrt{t/2}, t)|_{t/2}} \int_{t/2}^{t/2} \int_{B(x, \sqrt{t/2}, \tau)} e^{-2\lambda d(z,x_0,\tau)} F^2(z, \tau) d\mu(z, \tau) d\tau. \]

In particular, this holds for $x = x_0$. In this case, for $z \in B(x_0, \sqrt{t/2}, \tau)$, there holds $d(z,x_0,\tau) \leq \sqrt{t/2}$. Therefore, by the assumption that $\lambda < 0$,
\[ u(x_0,t)^2 \leq \frac{c_n e^{-2\lambda\sqrt{t/2}}}{t|B(x_0, \sqrt{t/2}, t)|_{t/2}} \int_{t/2}^{t/2} \int_{B(x_0, \sqrt{t/2}, \tau)} F^2(z, \tau) d\mu(z, \tau) d\tau. \]

This combined with (5.11) shows that
\[ u(x_0,t)^2 \leq \frac{c_n e^{\lambda^2 t - \lambda \sqrt{t/2}}}{|B(x_0, \sqrt{t/2}, t)|_{t/2}} \int f(y)^2 d\mu(y,0). \]

i.e.
\begin{equation}
(5.12) \quad \left( \int G(x_0,t; z,0) e^{-\lambda d(z,x_0,0)} f(z) d\mu(z,0) \right)^2 \leq \frac{c_n e^{\lambda^2 t - \lambda \sqrt{t/2}}}{|B(x_0, \sqrt{t/2}, t)|_{t/2}} \int f(y)^2 d\mu(y,0).
\end{equation}

Now, we fix $y_0$ such that $d(y_0,x_0,0)^2 \geq 4a^2 t$ with $a > 1$ to be chosen later. Then it is clear that, by $\lambda < 0$ and the triangle inequality,
\[ -\lambda d(z,x_0,0) \geq -a \lambda d(x_0,y_0,0) \]
when $d(z,y_0,0) \leq \sqrt{t}$. In this case, (5.12) implies
\begin{equation}
(5.13) \quad \left( \int_{B(y_0, \sqrt{t}, 0)} G(x_0,t; z,0) f(z) d\mu(z,0) \right)^2 \leq \frac{c_n e^{2a \lambda d(x_0,y_0,0) + \lambda^2 t - \lambda \sqrt{t/2}}}{|B(x_0, \sqrt{t/2}, t)|_{t/2}} \int f(y)^2 d\mu(y,0).
\end{equation}

Now we take
\[ \lambda = -\frac{d(x_0,y_0,0)}{bt}. \]

Take $b > 0$ and $a > 0$ sufficiently large. Then (5.13) shows, for some $c > 0$,
\[ \int_{B(y_0, \sqrt{t}, 0)} G^2(x_0,t; z,0) d\mu(z,0) \leq \frac{c_n e^{-cd(x_0,y_0,0)^2/t}}{|B(x_0, \sqrt{t/2}, t)|_{t/2}}. \]

Hence, there exists $z_0 \in B(y_0, \sqrt{t}, 0)$ such that
\[ G^2(x_0,t; z_0,0) \leq \frac{c_n e^{-cd(x_0,y_0,0)^2/t}}{|B(x_0, \sqrt{t/2}, t)|_{t/2}|B(x_0, \sqrt{t}, 0)|_0}. \]
By the doubling property of the geodesic balls, it implies
\begin{equation}
G^2(x_0; t; z_0, 0) \leq \frac{c_n e^{-cd(x_0,y_0,0)^2/t}}{|B(x_0, \sqrt{t}, t)|_0||B(x_0, \sqrt{t}, 0)|_0}.
\end{equation}

Finally, let us remind ourself that $G(x_0, t; \cdot, \cdot)$ is a solution to the conjugate equation of (4.0), i.e.
\[ \Delta_z G(x, t; z; \tau) + \partial_{\tau} G(x, t; z, \tau) = 0.\]

Therefore Theorem 3.2 can be applied to it after a reversal in time. Consequently, for $\delta > 0, C > 0$,
\begin{equation}
G(x_0, t; y_0, 0) \leq CG^{1/(1+\delta)}(x_0, t, z_0, 0)M^{\delta/(1+\delta)},
\end{equation}
where $M = \sup_{M \times [0,t/2]} G(x_0, t, \cdot, \cdot)$. By Theorem 4.1, part (c), there exists a constant $A > 0$, depending only on the lower bound of the injectivity radius such that
\[ M \leq A \max\{1, t/n/2, 1\}.\]

This, (5.14) and (5.15) show, with $\delta = 1$, that
\[ G(x_0, t; y_0, 0)^2 \leq \max\{c_n, 1\} \frac{A e^{-cd(x_0,y_0,0)^2/t}}{|B(x_0, \sqrt{t}, t)|_0||B(x_0, \sqrt{t}, 0)|_0}.\]

By the assumption that the Ricci curvature is nonnegative, we have
\[ |B(x_0, \sqrt{t}, t)|_0 \leq |B(x_0, \sqrt{t}, 0)|_0.\]

Therefore
\[ G^2(x_0, t; y_0, 0) \leq \max\{c_n, 1\} \frac{A e^{-cd(x_0,y_0,0)^2/t}}{|B(x_0, \sqrt{t}, t)|_0}.\]

Consequently
\[ G(x_0, t; y_0, 0) \leq c_n A \left(1 + \frac{1}{t/n/2} + \frac{1}{|B(x_0, \sqrt{t}, t)|_0}\right) e^{-cd(x_0,y_0,0)^2/t}.\]

Since $x_0$ and $y_0$ are arbitrary, the proof is done.

\begin{flushright}
\Box
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**References**

[Au] Aubin, Thierry, *Problèmes isopérimétriques et espaces de Sobolev*. (French) J. Differential Geometry 11 (1976), no. 4, 573–598.

[ACDH] Auscher, Pascal; Coulhon, Thierry; Duong, Xuan Thinh; Hofmann, Steve, *Riesz transforms on manifolds and heat kernel regularity*. Ann. Sci. cole Norm. Sup. (4) 37 (2004), no. 6, 911–957.

[Che] Cheeger, Jeff, *Finiteness theorems for Riemannian manifolds*. Am. J. Math. 92(1970), 61-75.

[CH] B. Chow and R. Hamilton, *Constrained and linear Harnack inequalities for parabolic equations*, Invent. Math. 129 (1997), 213–238.

[CK] Chow, Bennett; Knopf, Dan, *The Ricci flow: an introduction*. Mathematical Surveys and Monographs, 110. American Mathematical Society, Providence, RI, 2004
[CGGGIIKLLN] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, Jim Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, Lei Ni. The Ricci flow: Techniques and Applications. In preparation.

[CKNT] K. Ecker, D. Knopf, L. Ni and P. Topping. Heat balls, monotone quantities and local mean value formulae on evolving Riemannian manifolds, preprint.

[CL] Shu-Cheng Chang and Peng Lu. Evolution of Yamabe constants under Ricci flow, preprint 2006.

[Da] Davies, E. B. Heat kernels and spectral theory. Cambridge Tracts in Mathematics, 92. Cambridge University Press, Cambridge, 1990.

[FGS] Fabes, Eugene B.; Garofalo, Nicola; Salsa, Sandro. A backward Harnack inequality and Fatou theorem for nonnegative solutions of parabolic equations. Illinois J. Math. 30 (1986), no. 4, 536–565.

[G] Guenther, C. The fundamental solution on manifolds with time-dependent metrics, J. Geom. Anal. 12 (2002), 425–436.

[H] Hamilton, Richard S. A matrix Harnack estimate for the heat equation. Comm. Anal. Geom. 1 (1993), no. 1, 113–126.

[Heb] Hebey, Emmanuel. Optimal Sobolev inequalities on complete Riemannian manifolds with Ricci curvature bounded below and positive injectivity radius. Amer. J. Math. 118 (1996), no. 2, 291–300.

[Hs] Hsu, E. P. Estimates of derivatives of the heat kernel on a compact Riemannian manifold. Proc. Amer. Math. Soc. 127 (1999), no. 12, 3739–3744.

[HV] Hebey, Emmanuel and Vaugon, Michel. Meilleures constantes dans le thôrème d’inclusion de Sobolev. (French) Ann. Inst. H. Poincaré Anal. Non Linéaire 13 (1996), no. 1, 57–93.

[LS] Li, Peter; Schoen, Richard. $L^p$ and mean value properties of subharmonic functions on Riemannian manifolds. Acta Math. 153 (1984), no. 3-4, 279–301.

[LY] Li, P.; Yau, S.T., On the parabolic kernel of the Schrödinger operator, Acta Math., 156 (1986) 153-201.

[MS] Malliavin, P. and Stroock, D. W., Short time behavior of the heat kernel and its logarithmic derivatives, J. of Diff. Geom., 44, No. 3 (1996), 550-570.

[Ni1] Ni, Lei. Ricci flow and nonnegativity of sectional curvature. Math. Res. Lett. 11 (2004), no. 5-6, 883–904.

[Ni2] Ni, Lei. A matrix Li-Yau-Hamilton inequality for Kaehler-Ricci flow, J. Differential Geom. to appear.

[Ni3] Ni, Lei. A note on Perelman’s LYH inequality, Comm. Analysis and Geometry, to appear.

[No] Norris, J. R., Path integral formulae for heat kernels and their derivatives, Prob. Theory and Related Fields, 94 (1993), 525-541.

[P] Perelman, Grisha. The entropy formula for the Ricci flow and its geometric applications, Math. ArXiv, math.DG/0211159

[Sa] Saloff-Coste, Laurent, Uniformly elliptic operators on Riemannian manifolds. J. Differential Geom. 36 (1992), no. 2, 417–450.

[Sh] Sheu, S. J., Some estimates of the transition density of a nondegenerate diffusion Markov process. Ann. Probab. 19 (1991), no. 2, 538–561.

[ST] Stroock, D. W.; Turetsky, J., Upper bounds on derivatives of the logarithm of the heat kernel. Comm. Anal. Geom. 6 (1998), no. 4, 669–685.

[SZ] P. Souplet and Qi S. Zhang, sharp gradient estimate and Yau’s Liouville theorem for the heat equation on noncompact manifolds, Bulletin LMS, to appear.

[Wa] Wang, Jiaping. Global heat kernel estimates. Pacific J. Math. 178 (1997), no. 2, 377–398.

[Y] Yau, S.T., On the Harnack inequalities for partial differential equations, Comm. Analysis and Geometry, Vol.2, No. 3, (1994), 431–450.

[Z] Zhang, Qi S. The global behavior of heat kernels in exterior domains. J. Funct. Anal. 200 (2003), no. 1, 160–176.

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