Long-living Bloch oscillations of matter waves in optical lattices

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It is shown that by properly designing the spatial dependence of the nonlinearity it is possible to induce long-living Bloch oscillations of a localized wavepacket in a periodic potential. The results are supported both by analytical and numerical investigations and are interpreted in terms of matter wave dynamics displaying dozens of oscillation periods without any visible distortion of the wave packet.

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The phenomenon of Bloch oscillations (BO), predicted by Bloch in 1928 in his celebrated paper on the dynamics of a band electron in a steady electrical field [1], represents a problem of non exhausted interest. Besides solid state physics, where the phenomenon has been observed only in recent times after the development of the superlattice technology [2], it is now possible to observe BO also in other fields such as nonlinear optics, using light beams in arrays of waveguides [3] or in photorefractive crystals [4], and atomic physics, using Bose-Einstein condensates (BEC) loaded in optical lattices (OLs) [5, 6, 7]. Apart its fundamental significance, the interest in BO arises mainly from perspectives of their practical applications. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions. In this context we mention the use of BO for metrological tasks, including relatively precise definitions.
the linear regime it is appropriate to start our consider-
ations from the underlining linear eigenvalue problem
\(-d^2 \psi_q/dx^2 + U(x) \psi_q = \varepsilon(q) \psi_q\), where \(\psi_q\) is the stan-
dard Bloch function corresponding to the wave vector \(q\). The periodicity of the linear OL introduces a band
structure in the spectrum e.g. the existence of chemical
potential functions \(\varepsilon_n(q)\) which are periodic in reciprocal
space \(q \in [-1,1]\), with \(n\) denoting the band index. In
the following we shall consider only the lowest chemical
potential band, a situation typical for most BEC applica-
tions, and omit the band index. In presence of nonlin-
erarity, stationary localized states (gap-solitons) can exist
only if chemical potentials are inside gaps (see e.g. [18]).
This is not the case of a nonstationary (moving) soli-
ton whose velocity, in the leading order, coincides with
the group velocity of the carrier Bloch wave and mathe-

dematically can be described by a multiple-scale expansion
provided the linear force is weak enough \(\gamma^{1/2} \ll 1\). To
this regard we search for localized solutions of Eq.(1) in
the form \(\psi = \gamma^{1/2} \psi_1 + \gamma \psi_2 + \cdots\), where at each order \(\psi_j\)
denotes a function of a set of scaled temporal \(t_n = \gamma^{n/2} t\)
and spatial \(x_n = \gamma^{n/2} x\) \((n = 1, 2, ...),\) variables. Since for
BO the solution, thought as a wavepacket of Bloch states
\(\tilde{\varphi}_q\) with the carrier wave-vector depending on the slow
time \(q = q(\tau)\), must scan the whole band, the ansatz for
the first order \(\psi_1\) is chosen, in analogy with the Houston
functions [19] of the underlying linear theory, of the form
\[
\psi_1 = A(\tau, \xi) e^{i \mathcal{E}(\tau)} \varphi_{q(\tau)}(x) \tag{2}
\]
where the phase \(\mathcal{E}(t)\) is determined by the equation
d\mathcal{E}/dt = \varepsilon(q(\gamma t))\) and we have introduced specific
notations for the slow variables \(\tau = \gamma t\) and \(\xi = \sqrt{\gamma} (x - v(\tau) t)\).
This peculiarity of the asymptotic expansion will man-
ifest at the third order of the small parameter, i.e.
\(O(\gamma^{3/2})\), lower orders of the expansion being obtained
in standard manner (see e.g. [13]). Since the temporal
dependence of the wavevector is not specified, we impose
the constraint that \(q\) must follow the well known semi-
classical equation for the linear BO which in our scaled
variables acquires the simplest form \(q = \tau [21]\). Then,
in the order \(\gamma^{3/2}\) we arrive at the NLS equation for the
slowly varying amplitude
\[
iA_\tau + \frac{1}{2M(q)} A_{\xi \xi} - \chi(q) |A|^2 A = 0. \tag{3}
\]
Here we defined the group velocity \(v(q) = d\varepsilon(q)/dq\),
the effective mass \(M(q) = (2d^2 \varepsilon(q)/dq^2)^{-1}\) and the effective
nonlinearity \(\chi(q) = \int_0^\infty G(x)|\varphi_{q(\gamma t)}(x)|^4 dx\). We
emphasize that although we have indicated \(q\) as an argu-
ment in the above definitions, the group velocity, the
effective mass and the effective nonlinearity are func-
tions of the slow time \(\tau\). Introducing the\( +\) and \( -\)
subindexes to denote properties at top and bottom lim-
its of the band, we have that \(\varphi^{(+)} = \varphi_{q=0}, \varphi^{(-)} = \varphi_{q=1},\)
\(\chi^{(+)} = \int_0^\infty G(x)|\varphi^{(+)}(x)|^2 dx, \) and \(\mp M^{(\pm)} > 0\). As it is
clear, if \(G(x) = \text{const}, \) then \(\chi^{(+)} > 0\) and from Eq.(3)
it follows that while envelope solitons are available at one
edge of the band (the edge for which \(M \chi < 0\), they can-
not exist at the other edge. Since for BO the soliton
must travel along the whole band, it will necessarily reach the edge
where it undergoes strong dispersion (due to defocu-
sing action of nonlinearity \(M \chi > 0\), this leading to
destruction of BO [17].

From this analysis it is clear that an essential control
parameter of the problem is \(\sigma(q) = -M(q) \chi(q)\) and
in order to have long-living BO we must require that
\(\sigma(q) > 0\) for all \(q\), this assuring a focusing nonlinearity
for the soliton dynamics along the whole band. The
above condition can be achieved by means of a proper
design of the nonlinear lattice and will be optimized if
the condition \(\sigma = \text{const} > 0\) is satisfied. Analytically it
is easy to consider the general case in which \(\sigma(\tau)\) is a
slowly varying function \(|d\sigma/dq| \ll \sigma\). Indeed, introd-
cing the new temporal variable \(\tau = \int_0^\tau d\tau/M(q(\tau))\) and
the function \(A = A/\sqrt{\sigma}\) we obtain the NLS equation
with a dissipative term \(i A_\tau + \frac{1}{2M(q)} A_{\xi \xi} + |A|^2 A = i \frac{d}{dq} \frac{\sigma}{M(q)} A\)
whose approximate solution can be found by means of
soliton perturbation theory [22]. For the partic-
ular case of a static soliton in the frame moving with the
velocity \(v(\tau)\), the solution reads \(A = A_0 \exp \left( \Gamma(\tau) + \frac{i}{2} \frac{A_0^2}{\int_0^\tau \sqrt{\sigma(\tau)} d\tau} \right) \) \(\text{sech}(A_0 |\sigma(\tau)|^{1/4} \xi)\) with
\(A_0\) the constant amplitude of the soliton.

In order to check the above predictions we have per-
formed direct numerical simulations of Eq. with
\(U(x) = -V \cos(2x)\) and \(G(x) = G_0 + G_1 \cos(2x)\), where
\(V\) and \(G_1\) denote the amplitudes of the linear and non-
linear lattices, respectively, \(G_0\) is the "average" nonlin-
erarity, which in all numerical simulations reported be-
low is chosen to be \(G_0 = -0.777\). We denote by \(G_1^{(\pm)}\)
the value of \(G_1\) at which effective nonlinearity \(\chi^{(\pm)} \) be-
comes zero. Respectively if \(G_1 \geq G_1^{(\pm)}\) we have that
\(\chi^{(\pm)} \geq 0\) and the condition for the existence of enve-
lope solitons (alternatively, the instability of the Bloch
waves) at the both band edges \(M^{(\pm)} \chi^{(\pm)} < 0\) is met
when \(G_1^{(+) < G_1 < G_1^{(-)}\). This corresponds to the
domain of parameters between the lines in Fig.1 where
\(G_1^{(\pm)}\) is vs \(V\) are depicted (e.g. point B in Fig.1). Conse-
quently, domains of parameters \(G_1 < G_1^{(+)\) (e.g. point A in Fig.1]\) and
\(G_1 > G_1^{(-)}\) (e.g. point C in Fig.1) allow for the exist-
ence of small amplitude envelope solitons only at lower
or upper band edge, correspondingly. Thus long-lived
BOs can be expected in the domain of parameters be-
tween the two curves \(G_1^{(+)}(V)\) in Fig.1.

Below we concentrate on the choice of the parameters
corresponding to the specific points A, B, and C, in
Fig.1. These are situations deviating form the opti-

mally designed lattices, as is clearly seen in panels (b)
and (c) of Fig.1. Indeed, while the points \(q_0, \) and \(q_0\) are
relatively close to each other, they do not exactly coincide,
what results in the singularity of $\sigma(q)$ at the point $q_M$ and in the existence of the interval $q_M < q < q_C$ where envelope solitons do not exist (however outside this interval $\sigma$ is a relatively slow function of the wavevector).

In Fig. 2a we present the time evolution of a small amplitude envelope soliton obtained by direct numerical integration of Eq. (1). As initial condition we used the stationary envelope soliton of Eq. (1) with $\gamma = 0$, whose chemical potential belongs to the semi-infinite gap very close to the lowest allowed band. From this figure the existence of long-living BO with the period $T_s \approx 2 \cdot 10^3$ and spatial amplitude $X_s \approx 32.5 \tau$, perfectly matching the semiclassical estimates $2/\gamma$ and $(e^{+\gamma} - e^{-\gamma})/(2\gamma)$, respectively, is quite evident. Notice that the turning points of the spin dynamics correspond to values of $e$ close to the bottom of the band.

In Fig. 2b we have compared the shape of the soliton at the fifth right turning points of the BO with the profile of the stationary state at the top of the band, from which we see that they practically coincide (the small difference is ascribed to the radiative effects of the BO dynamics). Remarkably accurate prediction of the theory is verified by comparing the dynamical profiles of the soliton at the turning points of the BO. The profiles at the times $t = 10^3; 3 \cdot 10^3; 5 \cdot 10^3; 7 \cdot 10^3$ were practically indistinguishable from the one depicted for $t = 9 \cdot 10^3$ in Fig. 2b: Numerical simulations performed on longer time scales (up to $t = 2 \cdot 10^4$) showed no appreciable decay of the BO and perfect recovering of the soliton shape at the turning points.

To appreciate the importance of the parameter design implied by our theory, we depict in Fig. 2b the dynamics of a soliton corresponding to point A in Fig. 1. We see that in this case the BO undergo fast decay due to spreading of the wave packet. The difference in the two types of BO is clearly seen also from the dependence of the chemical potential on time, computed as $\varepsilon(t) = \frac{1}{N} \int_{-\infty}^{\infty} \left( |\psi(x)|^2 + \mathcal{U}(x) |\psi|^2 + \mathcal{G}(x) |\psi|^4 \right) dx$, with $N = \int_{-\infty}^{\infty} |\psi|^2 dx$ denoting the number of atoms. This is shown in Fig 2a, from which we see that stable BO correspond to perfectly periodic trajectory (solid line) while decaying BO correspond to decays of the oscillations of the chemical potential (dashed line). Similar phenomena exist also if the BO is started from stationary states at the top, rather than at the bottom, of the band for the same parameter values of points $B, C$, in Fig 1b), as one can see from Fig. 3

In closing this letter we wish to discuss possible experimental settings for observing long-living BO in BECs trapped in uniformly accelerated linear and nonlinear OLS (with acceleration $2\gamma$): $\mathcal{U}(X - \gamma T^2)$ and $\mathcal{G}(X - \gamma T^2)$, respectively. The corresponding model equation in di-
dimensionless variables take the form

$$i \psi_T = -\psi_{XX} + U(X - \gamma T^2)\psi + G(X - \gamma T^2)|\psi|^2\psi. \quad (4)$$

After substituting \( \psi = e^{-i[(X-\gamma T^2)\gamma T + \gamma T^2/3]} \psi \) and introducing new independent variables \( x = X - \gamma T^2 \) and \( t = T \), Eq. (4) takes the form of a NLS equation with external linear force (1). In this normalization the energy is measured in units of the recoil energy \( E_r = \hbar^2 \pi^2/(2md^2) \), where \( d \) is the lattice spacing and \( m \) is the mass of bosons, and the spatial and temporal variables are measured in units of \( d/\pi \) and \( h/E_r \), respectively. To check the experimental feasibility of the proposed setting, we consider a \(^{87}\text{Rb} \) condensate in a trap with a transverse radial size \( a = 2 \mu \text{m} \) and with a period of linear and nonlinear lattices \( d = 1 \mu \text{m} \). Then dimensionless parameters of long-living BO, depicted in Fig.2a and 2b will correspond to soliton, containing \( N \approx 3800 \) atoms, driven by external force \( 1.17 \cdot 10^{-27} \text{N} \) which is obtained, when the acceleration of linear and nonlinear lattices is of order of \( \sim 8 \text{mm/s} \). At the same time the above parameters of the nonlinear lattice could be created by the spatial variation (obtained by the optically induced Feshbach resonance) of the bosonic s-wave scattering length \( a_s(x) = a_s^{(0)} + a_s^{(1)} \cos(2\pi x/d) \) with the “average” value \( a_s^{(0)} = -1.554 \text{nm} \) and the amplitude \( a_s^{(1)} = 2 \text{nm} \). For the parameters of Fig.1 the long-living BO will occur when the amplitude of the spatial variation of the scattering length is in the range \( 1.948 \text{nm} < a_s^{(1)} < 2.058 \text{nm} \). This show that the long lived BO reported in this Letter can indeed be observed in the experimental settings available today.

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