ROST NILPOTENCE AND ÉTALE MOTIVIC COHOMOLOGY
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Abstract. A smooth projective scheme $X$ over a field $k$ is said to satisfy the Rost nilpotence principle if any endomorphism of $X$ in the category of Chow motives that vanishes on an extension of the base field $k$ is nilpotent. We show that an étale motivic analogue of the Rost nilpotence principle holds for all smooth projective schemes over a perfect field. This provides a new approach to the question of Rost nilpotence and allows us to obtain an elegant proof of Rost nilpotence for surfaces, as well as for birationally ruled threefolds over a field of characteristic 0.

1. Introduction

Let $X$ be a smooth projective scheme over a field $k$, and let $\text{Chow}(k)$ denote the category of Chow motives over $k$, see [22], [26], for instance. We say that the Rost nilpotence principle holds for $X$ if for any field extension $E/k$, the kernel of the homomorphism $\text{End}_{\text{Chow}(k)}(X) \to \text{End}_{\text{Chow}(E)}(X_E)$ consists of nilpotent elements. This was first proved by Rost for any smooth projective quadric over a field [24]; it follows from this result that there is a decomposition of the Chow motive of a quadric into simpler motives, which is an essential tool in Voevodsky’s proof of the Milnor conjecture [29]. Chernousov, Gille and Merkurjev [7] proved that the Rost nilpotence principle holds for isotropic projective homogeneous varieties for a semisimple algebraic group. Later, Gille showed that the Rost nilpotence principle holds for geometrically rational surfaces (in arbitrary characteristic) [18], [19], and for smooth, projective, geometrically integral surfaces (in characteristic 0) [19]. The Rost nilpotence principle has proved to be very useful in the study of motivic decompositions and is expected to hold for all smooth projective schemes.

In order to prove that Rost nilpotence holds for a smooth projective scheme $X$ over a field $k$ of characteristic 0, it suffices to show that for a finite Galois field extension $E/k$ the kernel of the restriction map $\text{End}_{\text{Chow}(k)}(X) \to \text{End}_{\text{Chow}(E)}(X_E)$ consists of nilpotent elements. The approach by Gille to prove this statement for surfaces uses two nontrivial results. The first input is a result of Rost [24, Proposition 1], originally proved using Rost’s fibration spectral sequence for cycle modules (see also [6] for a purely intersection-theoretic proof). The second input is a Galois cohomological description of codimension 2 cycles on a scheme that are annihilated after base change to $E$, obtained by Colliot-Thélène and Raskind.

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building on ideas of Bloch used in the study of codimension 2 cycles on rational surfaces [3]. More precisely, the essential tool is the following vanishing result of Colliot-Thélène [9, Theorem 1, Remark 5.2] and Suslin [27, Theorem 5.8]: $H^1(\text{Gal}(E/k), K_2E(X)/K_2E) = 0$, where $X$ is a geometrically irreducible variety with a $k$-rational point. We remark that the analogue of this vanishing result for higher Galois cohomology groups does not hold, thus Gille’s proof does not generalize to higher dimensions.

We consider the étale motivic or Lichtenbaum cohomology groups defined as the hypercohomology groups of Bloch’s cycles complex $Z(n)_{\text{ét}}$, considered as a complex of étale sheaves; for details, see Section 3. A correspondence $\alpha$ in $\text{End}_{\text{Chow}}(X)$ acts on the Lichtenbaum cohomology groups of $X \times_k X$, after inverting the exponential characteristic of the base field. Thus an evident étale motivic analogue of the Rost nilpotence principle is to ask whether this action in the étale setting is nilpotent. We show that Rost nilpotence holds in this setting in arbitrary dimension in the following sense:

**Theorem 1.1.** Let $X$ be a smooth projective scheme over a perfect field $k$. Let $\alpha \in \text{End}_{\text{Chow}}(X)$ be a correspondence such that for a Galois field extension $E$ of $k$ the image $\alpha_E \in \text{End}_{\text{Chow}}(E)(X_E)$ is trivial. Then the action of $\alpha$ on the Lichtenbaum cohomology groups of $X \times_k X$ is nilpotent, after inverting the exponential characteristic.

We remark that analogous to the category Chow$(k)$ of Chow motives over a field $k$, one can construct a category Chow$_L(k)$ of étale motivic or Lichtenbaum Chow motives. In particular, the proof of Theorem 1.1 shows that the analogue of Rost nilpotence in the étale motivic or Lichtenbaum setting holds for schemes of arbitrary dimension, provided the underlying field $k$ is perfect.

There is canonical map from Chow groups to Lichtenbaum Chow groups, which allows us to compare Rost nilpotence in the usual sense with the étale motivic variant proved in Theorem 1.1. Using the Bloch-Kato conjecture, proved by Rost-Voevodsky [29], [30], the kernel of this comparison map can be identified with the quotient of a group, which can be computed via a spectral sequence in terms of cohomology groups of certain well-studied sheaves. We analyze the action of a correspondence which is annihilated by a Galois field extension of the base field on the kernel of the comparison map. In case of dimension $\leq 2$ over a field of characteristic 0, our approach yields an elegant proof of the Rost nilpotence principle, see Theorem 4.1. We note that even for surfaces this generalizes the result of Gille [19], since we do not have to impose the condition of geometric integrality. Moreover, we can improve the bound on the nilpotence exponent obtained by Gille, see Remark 4.2. We also show that the Rost nilpotence principle holds for birationally ruled threefolds over a field of characteristic 0. We note that in this case the restriction on the characteristic is also needed because of use of the weak factorization theorem [1] in the proof.
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Notation. Let $X$ be a scheme over a field $k$; we assume $X$ to be separated, of finite type and equidimensional. By a surface we mean a scheme of dimension $2$ and by a threefold we mean a scheme of dimension $3$. We write $X^{(i)}$ for the points of codimension $i$, and $\text{CH}_i(X)$ (resp. $\text{CH}^i(X)$) for the Chow group of algebraic cycles of dimension $i$ (resp. codimension $i$) on $X$ modulo rational equivalence [15]. In particular, if the $k$-scheme $X$ is of dimension $d$ over $k$, we have $\text{CH}_i(X) = \text{CH}^{d-i}(X)$.

If $E/k$ is a field extension, we set $X_E = X \times_k \text{Spec} E$; if $\overline{k}$ is an algebraic closure of $k$, then $\overline{X} = X \times_k \text{Spec} \overline{k}$. If $f : X \to Y$ is a morphism of schemes over $k$, its pullback along $\text{Spec} E \to \text{Spec} k$ will be denoted by $f_E : X_E \to Y_E$. For an integral $k$-scheme $X$, we write $k(X)$ for its function field and $E(X)$ for the function field of $X_E$. If $x$ is a point of $X$, then $k(x)$ will denote its residue field.

2. Correspondences, Chow motives and Rost nilpotence

In this section, we set up the notation for the article and give a precise statement of the Rost nilpotence principle. For more details on the basic properties of Chow motives and their relationship with motivic cohomology, we refer the reader to [22], [26], [31, Chapter 5]. For cycle modules and their basic properties, we refer the reader to [25] (see also [13]).

Let $\text{SmProj}/k$ be the category of smooth projective schemes over a field $k$. The category $\text{Corr}_0^0(k)$ of correspondences of degree $0$ over $k$ has the same objects as $\text{SmProj}/k$ and as morphisms

$$\text{Hom}_{\text{Corr}_0^0(k)}(X, Y) = \bigoplus_{j=1}^r \text{CH}^{\dim Y}(X_j \times Y),$$

where $X_1, \ldots, X_r$ are the irreducible components of $X$. If $f \in \text{Corr}_0^0(X, Y)$ and $g \in \text{Corr}_0^0(Y, Z)$, then their composition $g \circ f \in \text{Corr}_0^0(X, Z)$ is defined by the formula

$$g \circ f = p_{XZ}(p_{XY}^*(f) \cdot p_{YZ}^*(g)),$$

where $p_{XY}$, $p_{YZ}$ and $p_{XZ}$ are the projection maps from $X \times_k Y \times_k Z$ to $X \times_k Y$, $Y \times_k Z$ and $X \times_k Z$, respectively, and $\cdot$ is the intersection product. The category $\text{Chow}^\text{eff}(k)$ of effective Chow motives is the idempotent completion of $\text{Corr}_0^0(k)$. We will denote the category of Chow motives by $\text{Chow}(k)$ and by $h : \text{SmProj}/k \to \text{Chow}(k)$ the canonical functor that associates with a smooth scheme its Chow
motive. For an object $M$ of Chow($k$) and $i \in \mathbb{Z}$, we will denote by $M(i)$ its $i$th Tate twist. To simplify notation, we will henceforth write

$$\Hom_k(X, Y) = \Hom_{\text{Chow}(k)}(h(X), h(Y)),$$

$$\text{End}_k(X) = \text{End}_{\text{Chow}(k)}(h(X)).$$

We will be interested in the action of $\text{End}_k(X)$ (correspondences of degree 0 from $X$ to itself) on the Chow groups of the self-product $X \times_k X$; this action is simply given by composition of correspondences

$$(2.1) \quad \text{End}_k(X) \times \text{CH}^n(X \times_k X) \to \text{CH}^n(X \times_k X); (\alpha, \beta) \mapsto \alpha \circ \beta.$$  

Let $E/k$ be a field extension. Then $X \mapsto X_E$ induces a restriction functor $\text{res}_{E/k}: \text{Chow}(k) \to \text{Chow}(E)$. If $M$ is an object (resp. $f$ is a morphism) in Chow($k$), we write $M_E$ (resp. $f_E$) for its image under the restriction map $\text{res}_{E/k}(M)$. With this setup, we can state the Rost Nilpotence Principle:

**Rost Nilpotence Principle.** Let $X$ be a smooth projective scheme over a field $k$. Then the Rost nilpotence principle holds for $X$ if for every $\alpha \in \text{End}_k(X)$ such that $\alpha_E = 0$ for some field extension $E/k$, there exists an integer $N$ (possibly depending on $\alpha$) such that $\alpha^N = 0$, i.e. $\alpha$ is nilpotent as a correspondence.

3. Étale motivic cohomology and actions of correspondences

In this section, we recall the definition and basic properties of étale motivic or Lichtenbaum cohomology and give the proof of Theorem 1.1. Throughout this section, we will denote by $p$ the exponential characteristic of $k$.

Let $X$ be a smooth scheme over $k$ and let $z^n(X, \bullet)$ be the cycle complex defined by Bloch [4] whose homology groups define the higher Chow groups

$$\text{CH}^n(X, m) = H_m(z^n(X, \bullet)).$$

The presheaf $z^n(\_, \bullet): U \mapsto z^n(U, \bullet)$ is a sheaf on the (small) étale site (see [17, Section 2.2]), and therefore defines a complex of sheaves in the étale and Zariski topology. It is shown in [4] that the cycle complex is covariantly functorial for proper maps and contravariantly functorial for arbitrary maps of smooth schemes over a field (the latter assertion requires a moving lemma, which is proved in [5]).

If $A$ is an abelian group, we have the complex $A_X(n) = (z^n(\_, \bullet) \otimes A)[-2n]$ of Zariski sheaves (on $X_{\text{Zar}}$) and the analogous complex of $A_X(n)_{\text{ét}}$ of étale sheaves (on $X_{\text{ét}}$). The motivic and étale motivic or Lichtenbaum cohomology groups with coefficients in $A$ are defined as the hypercohomology groups of these complexes

$$H^m_M(X, A(n)) = \Pi^m_{\text{Zar}}(X, A_X(n)),$$

$$H^m_\text{ét}(X, A(n)) = \Pi^m_{\text{ét}}(X, A_X(n)_{\text{ét}}).$$

With this definition, one has $H^m_M(X, \mathbb{Z}(n)) = \text{CH}^n(X, 2n - m)$ for all $m, n$; in particular, if $m = 2n$, then $H^{2n}_M(X, \mathbb{Z}(n)) = \text{CH}^n(X)$ is the usual Chow group.
Analogously, one defines for \( m = 2n \) the Lichtenbaum Chow groups by
\[
\text{CH}^n_L(X) = H^{2n}_L(X, \mathbb{Z}(n)),
\]
and more generally for \( m \geq 0 \), the higher Lichtenbaum Chow groups by
\[
\text{CH}^n_L(X, m) = H^{2n-m}_L(X, \mathbb{Z}(n)).
\]
Note that \( \text{CH}^n_L(X, m) = 0 \) for \( n < 0 \), because \( \mathbb{Z}(n)_{\text{\acute{e}t}} \) is trivial for \( n < 0 \). If \( \pi : X_{\text{\acute{e}t}} \to X_{\text{Zar}} \) denotes the canonical morphism of sites, then the associated adjunction \( \mathbb{Z}_X(n) \to R\pi_*\pi^*\mathbb{Z}_X(n) = R\pi_*\mathbb{Z}_X(n)_{\text{\acute{e}t}} \) induces comparison (or cycle class) maps
\[
(3.1) \quad \text{CH}^n(X, m)^\gamma \to \text{CH}^n_L(X, m),
\]
for all \( m, n \). With rational coefficients, the adjunction \( \mathbb{Q}_X(n) \to R\pi_*\pi^*\mathbb{Q}_X(n) \) is an isomorphism (see [21, Théorème 2.6], for example). Thus, rationally, we have
\[
(3.2) \quad \text{CH}^n(X, m) \otimes \mathbb{Q} \cong \text{CH}^n_L(X, m) \otimes \mathbb{Q},
\]
for all \( m, n \). Geisser-Levine have shown in [16, Theorem 8.5] and [17, Theorem 1.5] that if \( \ell \) is a prime and \( r \) is a positive integer, one has on \( X_{\text{\acute{e}t}} \) the quasi-isomorphisms
\[
(Z/\ell^r\mathbb{Z})_X(n) \xrightarrow{\sim} \begin{cases} 
\mu_{\ell^r}^{\otimes n}, & \text{if } \ell \neq \text{char}(k); \\
\nu_r(n)[-n], & \text{if } \ell = \text{char}(k),
\end{cases}
\]
where \( \nu_r(n) \) is the \( n \)-th logarithmic de Rham-Witt sheaf [23], [20]. Let
\[
(Q/\mathbb{Z})_X(n) = \bigoplus_{\ell} Q_{\ell}/\mathbb{Z}_{\ell}(n),
\]
where \( \ell \) runs through all primes, and where
\[
Q_{\ell}/\mathbb{Z}_{\ell}(n) = \begin{cases} 
\lim_{r \to \infty} \mu_{\ell^r}^{\otimes n}, & \text{if } \ell \neq \text{char}(k); \\
\lim_{r \to \infty} \nu_r(n)[-n], & \text{if } \ell = \text{char}(k).
\end{cases}
\]
Therefore, with divisible coefficients Lichtenbaum and étale cohomology coincide
\[
(3.3) \quad H^m_L(X, \mathbb{Q}/\mathbb{Z}(n)) = H^m_{\text{\acute{e}t}}(X, (Q/\mathbb{Z})_X(n)).
\]
There are product maps on motivic cohomology (which are induced from the usual external product of cycles at the level of cycle complexes followed by pullback along the diagonal)
\[
H^m_M(X, R(n)) \otimes H^{m'}_M(X, R(n')) \to H^{m+m'}_M(X, R(n + n')),
\]
and similar product maps for the Lichtenbaum cohomology groups with coefficients in any commutative ring \( R \). Both motivic and Lichtenbaum cohomology groups are contravariantly functorial for arbitrary morphisms between smooth schemes. In order to get an action of correspondences on Lichtenbaum cohomology groups by a formula analogous to (2.1), we need appropriate covariant functoriality of Lichtenbaum cohomology groups, which we briefly describe below,
using comparison with extension groups in the triangulated category $\text{DM}_{\text{ét}}(k, R)$ of étale motives (see [31, Chapter 5] or [8]). We will use the notation and terminology of [8]. It has been shown in [8, Section 7.1] that there is a canonical map

$$\rho_{X}^{m,n} : H^m_{\text{ét}}(X, \mathbb{Z}(n)) \to H^m_{\text{ét}}(X, \mathbb{Z}(n)),$$

where $H^m_{\text{ét}}(X, \mathbb{Z}(n))$ is defined to be the group $\text{Hom}_{\text{DM}_{\text{ét}}(k, R)}(\mathbb{Z}(X), \mathbb{Z}(n)[m])$. By [8, Theorem 7.1.2], $\rho_{X}^{m,n}$ becomes an isomorphism after tensoring with $\mathbb{Z}[1/p]$. By [12, Corollary 6.2.4], any projective morphism $f : X \to Y$ of relative dimension $r$ between smooth schemes induces Gysin/pushforward morphisms

$$f_{*} : H^m_{\text{ét}}(X, R(n)) \to H^{m-2r}_{\text{ét}}(Y, R(n-r))$$

satisfying the projection formula. Moreover, the cycle class map

$$\sigma : H^m_{\text{ét}}(X, R(n)) \xrightarrow{\sim} H^m_{\text{ét}}(X, R(n)) \xrightarrow{\rho_{X}^{m,n}} H^m_{\text{ét}}(X, R(n))$$

is compatible with pushforwards with respect to projective maps between regular schemes, where on the left-hand side one considers the usual pushforwards on Chow groups and on the right-hand side the Gysin morphisms of [12] (see [8, Remark 7.1.12]). One therefore gets an action of $\text{End}_{k}(X)$ by the formula analogous to (2.1) on the groups $H^m_{\text{ét}}(X \times_k X, R(n))$:

$$\text{End}_{k}(X) \times H^m_{\text{ét}}(X \times_k X, R(n)) \to H^m_{\text{ét}}(X \times_k X, R(n));$$

$$(\alpha, \beta) \mapsto p_{13*}(p_{12}^{*}\sigma(\alpha) \cdot p_{23}^{*}\beta),$$

where $p_{ij}$ denotes the projection map $X \times_k X \times_k X \to X \times_k X$ on the $i, j$th components. Consequently, one gets an action of $\text{End}_{k}(X)$ on the Lichtenbaum cohomology groups $H^m_{\text{ét}}(X \times_k X, R(n))$ after inverting the exponential characteristic (that is, after tensoring with $\mathbb{Z}[1/p]$).

**Remark 3.1.** In fact, with the terminology of [8], the category $\text{DM}_{\text{ét}}(k, R)$ satisifies the Grothendieck six functor formalism along with absolute purity and duality properties (see [8, Corollary 5.5.5, Theorem 5.6.2 and Theorem 6.2.17]).

The main ingredient in the proof of Theorem 1.1 is the Hochschild-Serre spectral sequence

$$E_2^{r,s} = H^r(G, H^s_{L}(Y_E; \mathbb{Z}(n))) \Rightarrow H^{r+s}_{L}(Y; \mathbb{Z}(n))$$

for étale motivic cohomology (see, for example, [10, page 31]), where $G$ denotes the Galois group of the Galois field extension $E/k$. For the convenience of the reader, we briefly recall its construction and justify the convergence. Let $\text{Shv}(Y_{\text{ét}})$ denote the category of étale sheaves on $Y$ and let $G\text{-Mod}$ denote the category of the category of $\mathbb{Z}[G]$-modules. Given a cochain complex $C^{\bullet}$ of étale sheaves on $Y$, one has the hypercohomological spectral sequence [32, 5.7.9] associated with the functor $\text{Shv}(Y_{\text{ét}}) \to G\text{-Mod}$ defined by $\mathcal{F} \mapsto \mathcal{F}(Y_{E})^{G}$:

$$E_2^{r,s} = H^r(G, \mathbb{H}^s_{\text{ét}}(C^{\bullet}(Y_E))) \Rightarrow \mathbb{H}^{r+s}_{\text{ét}}(C^{\bullet}(Y)).$$
which converges if the complex $C^\bullet$ is bounded or cohomologically bounded (see [21, 2C] and the references cited there). The spectral sequence (3.5) can be seen as the hypercohomology spectral sequence of the complex $Z_Y(n)_{\text{et}}$. Note that étale hypercohomology of the complex $Z_Y(n)_{\text{et}}$ is Zariski hypercohomology of the complex $R\pi_*Z_Y(n)_{\text{et}}$ of Zariski sheaves on $Y$. Consider the exact triangle

$$R\pi_*Z_Y(n)_{\text{et}} \rightarrow R\pi_*\mathbb{Q}_Y(n)_{\text{et}} \rightarrow R\pi_*(\mathbb{Q}/\mathbb{Z})_Y(n)_{\text{et}} \rightarrow R\pi_*Z_Y(n)_{\text{et}}[1].$$

The hypercohomology spectral sequence for $R\pi_*\mathbb{Q}_Y(n)_{\text{et}}$ converges since it is cohomologically bounded (since $\mathbb{Q}_Y(n) \simeq R\pi_*\mathbb{Q}_Y(n) = R\pi_*\mathbb{Q}_Y(n)_{\text{et}}$ and since $Y$ has finite Zariski cohomological dimension). The hypercohomology spectral sequence for $(R\pi_*\mathbb{Q}/\mathbb{Z})_Y(n)_{\text{et}}$ converges, the complex being bounded. Consequently, the spectral sequence (3.5) converges.

We are now set to give a proof of Theorem 1.1. We will treat the cases when the base field $k$ is of characteristic 0 and positive characteristic separately.

**Proof of Theorem 1.1.**

*Proof in the case $\text{char } (k) = 0.$*

We may assume that $E$ is a finite Galois field extension of $k$ with $G = \text{Gal}(E/k)$. Consider the Hochschild-Serre spectral sequence

$$E_2^{r,s} = H^r(G, H^s_L(Y_E, \mathbb{Z}(n))) \Rightarrow H^{r+s}_L(Y, \mathbb{Z}(n)).$$

Clearly, we have $E_2^{r,s} = 0$, if $r < 0$. From (3.3), it follows that $H^s_L(Y_E, \mathbb{Q}/\mathbb{Z}(n)) = H^s_{\text{et}}(Y_E, (\mathbb{Q}/\mathbb{Z})_Y(n)) = 0$ for $s < 0$. Hence, the long exact sequence of hypercohomology associated to the exact triangle

$$Z_Y(n)_{\text{et}} \rightarrow \mathbb{Q}_Y(n)_{\text{et}} \rightarrow (\mathbb{Q}/\mathbb{Z})_Y(n)_{\text{et}} \rightarrow Z_Y(n)_{\text{et}}[1]$$

shows that the group $H^s_L(Y_E, \mathbb{Z}(n))$ is isomorphic to $H^s_L(Y_E, \mathbb{Q}(n))$ if $s < 0$, and consequently a $\mathbb{Q}$-vector space in this case. Therefore, we have $E_2^{r,s} = H^r(G, H^s_L(Y_E, \mathbb{Z}(n))) = 0$, if $s < 0$ and $r > 0$. This implies that the filtration on $H^r_L(Y, \mathbb{Z}(n))$ induced by the spectral sequence (3.6) is always finite.

Since the spectral sequence (3.6) is functorial and compatible with products, the action of an element of $\text{End}_k(X)$ respects the filtration induced by the Hochschild-Serre spectral sequence. If $\alpha \in \text{End}_k(X)$ is such that $\alpha_E = 0$, then $\alpha_E$ acts by the zero map on each of the motivic cohomology groups $H^s_L(Y_E, \mathbb{Z}(n))$ and hence, on every term $E_2^{r,s}$ of (3.6). Consequently, the action of $\alpha$ on the Lichtenbaum cohomology groups $H^s_L(Z_Y(n))$ is nilpotent. \hfill \Box

*Proof in the case $\text{char } (k) = p > 0.$*

We have an exact triangle

$$\mathbb{Z}[1/p]_Y(n)_{\text{et}} \rightarrow \mathbb{Q}_Y(n)_{\text{et}} \rightarrow (\mathbb{Q}/\mathbb{Z})_Y(n)_{\text{et}} \rightarrow \mathbb{Z}[1/p]_Y(n)_{\text{et}}[1]$$

in the derived category of étale sheaves on $Y$, where $(\mathbb{Q}/\mathbb{Z})_Y(n)_{\text{et}} = \bigoplus_{\ell \neq p} \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)$, where $\ell$ runs through all the primes except $p$. An argument analogous to the one
used in the characteristic 0 case shows that we have a convergent Hochschild-Serre spectral sequence after inverting the exponential characteristic

\[(4.3) \quad CH^r Y, (Z/p^n(n)) \Rightarrow H^r_{\text{et}}(Y, \mathbb{Z}/p^n(n)).\]

for a Galois extension \(E/k\) with Galois group \(G\). One can now follow the proof in the characteristic 0 case step-by-step to show that the action of \(\alpha \in \text{End}_k(X)\) with \(\alpha_E = 0\) is nilpotent on the groups \(H^m_{\text{et}}(Y, \mathbb{Z}[1/p](n))\), for every \(m, n \in \mathbb{Z}\).

Remark 3.2. The proof shows that for every such correspondence \(\alpha\) with \(\alpha_E = 0\) for a Galois extension \(E\) of \(k\), the index of nilpotence \(N\) of the action of \(\alpha\) on \(H^m_{\text{et}}(Y, \mathbb{Z}(n))\) can be taken to be \(\min\{\text{cd}(k), m\} + 1\), where \(\text{cd}(k)\) is the cohomological dimension of the field \(k\). In particular, the index of nilpotence does not depend on \(\alpha\).

4. Applications to Rost nilpotence

In this section, we use Theorem 1.1 and the Bloch-Kato conjecture (proved by Rost-Voevodsky [30]) to obtain a reformulation of the Rost nilpotence principle. We then use this to study Rost nilpotence for schemes of dimension \(\leq 3\).

Let \(Y\) be a smooth projective scheme over a field \(k\) of characteristic 0. If \(F\) is a sheaf on \(Y_{\text{et}}\), we let \(\mathcal{H}^p_{\text{et}}(F)_Y\) the Zariski sheaf associated with the presheaf \(U \mapsto H^p_{\text{et}}(U, F)\) on the Zariski site \(Y_{\text{Zar}}\). We will abuse notation and write \(\mathcal{H}^p_{\text{et}}(F)\) for \(\mathcal{H}^p_{\text{et}}(F)_Y\), whenever there is no confusion.

Let \(\pi : Y_{\text{et}} \to Y_{\text{Zar}}\) be the canonical morphism of sites. In the derived category of complexes of Zariski sheaves on \(Y\), we have \(\mathbb{Z}_Y(d) \xrightarrow{\sim} \tau_{\geq d+1}R\pi_*\mathbb{Z}_Y(d)_{\text{et}}\) ([28], [29, Theorem 6.6], [17]; also see [21, 2D]), and hence the distinguished triangle

\[(4.1) \quad \mathbb{Z}_Y(d) \to R\pi_*\mathbb{Z}_Y(d)_{\text{et}} \to \tau_{\geq d+2}R\pi_*\mathbb{Z}_Y(d)_{\text{et}} \to \mathbb{Z}_Y(d)[1].\]

The associated long exact sequence of Zariski hypercohomology groups yields the exact sequence

\[(4.2) \quad CH^r Y, 1 \to H^{2d-1}(Y, \tau_{\geq d+2}R\pi_*\mathbb{Z}_Y(d)_{\text{et}}) \to CH^d Y \to CH^r Y.\]

The group \(H^{2d-1}(Y, \tau_{\geq d+2}R\pi_*\mathbb{Z}_Y(d)_{\text{et}})\) is the abutment of the hypercohomology spectral sequence

\[(4.3) \quad E^p_{2q} = H^p Y, \tau_{\geq d+2}R\pi_*\mathbb{Z}_Y(d)_{\text{et}} \Rightarrow H^{p+q}(Y, \tau_{\geq d+2}R\pi_*\mathbb{Z}_Y(d)_{\text{et}}).\]

It follows from the quasi-isomorphism \(\mathbb{Z}_Y(d) \xrightarrow{\sim} \tau_{\leq d+1}R\pi_*\mathbb{Z}_Y(d)_{\text{et}}\) together with [21, Corollaire 2.8], that

\[(4.4) \quad R^q\tau_{\geq d+2}R\pi_*\mathbb{Z}_Y(d)_{\text{et}} = \begin{cases} 0, & \text{if } q \leq d + 1; \\ \mathcal{H}^{q-1}_{\text{et}}(\mathbb{Q}/\mathbb{Z}(d)), & \text{if } q \geq d + 2. \end{cases}\]

In particular, the \(E_2\)-terms of (4.3) are either trivial, or can be identified with the cohomology groups \(H^p_{\text{Zar}} Y, \mathcal{H}^{q-1}_{\text{et}}(\mathbb{Q}/\mathbb{Z}(d))\). This facilitates the study of schemes of dimension \(\leq 3\), as far as Rost nilpotence is concerned. We begin by observing
that this yields a simple proof of Rost nilpotence for surfaces in characteristic 0, generalizing [19, Theorem 9].

**Theorem 4.1.** Let $X$ be a smooth, projective scheme of dimension $\leq 2$ over a field $k$ of characteristic 0. Suppose that $\alpha \in \text{End}_k(X)$ is such that $\alpha_E = 0$ for a Galois extension $E/k$. Then $\alpha^{\circ N} = 0$, for some positive integer $N$, that is, $\alpha$ is nilpotent as a correspondence.

**Proof.** Set $d = \dim X$ and $Y = X \times_k X$. We first assume that $E/k$ is a Galois extension. The case $d = 0$ is trivial, and the case $d = 1$ follows from Theorem 1.1, since $\text{CH}^1(Y) \cong \text{CH}_1^L(Y)$, because of the quasi-isomorphism $\mathbb{Z}(1)_{\text{ét}} \cong \mathbb{G}_m[-1]$. If $d = 2$, then we have from (4.3) and (4.4), the vanishing $H^3(Y, \tau \geq 4 R\pi_* Z_Y(2)_{\text{ét}}) = 0$.

Hence, the canonical comparison map $\text{CH}^2(Y) \to \text{CH}^2_L(Y)$ is injective and the claim now follows from Theorem 1.1.

By a standard argument, it suffices to consider this case. Explicitly, if $E/k$ is an arbitrary field extension, we can find a tower of field extensions $k \subset F \subset E$, where $F/k$ is purely transcendental and $E/F$ is algebraic. Since $F/k$ is purely transcendental, the restriction map $\text{res}_{F/k} : \text{End}_k(X) \to \text{End}_F(X_F)$ is an isomorphism (see [14, Proposition 2.1.8], for example). Thus, in order to show that $\alpha$ is nilpotent, it suffices to show that $\alpha_F \in \text{End}_F(X_F)$ is nilpotent. Since $\text{char}(k) = 0$, we can find a tower of fields $F \subset E \subset E'$ with $E'/F$ Galois. Since $\alpha_E = 0$, we have $\alpha_{E'} = 0$, hence $\alpha$ is nilpotent. \qed

**Remark 4.2.** The proof Theorem 1.1 shows that the index of nilpotence $N$ in Theorem 4.1 can be taken to be $\min\{cd(k), 2 \dim_k(X)\} + 1$. This improves the bounds on the nilpotence exponent obtained by Gille [19, 2.5, Corollary and Remark 11].

We next consider the case of smooth projective threefolds over a field $k$ with $\text{char}(k) = 0$. Since $X$ is a threefold, we have the identification

$$H^3(Y, \tau \geq 5 R\pi_* Z_Y(3)_{\text{ét}}) = H^0_{\text{zar}}(Y, H^4_{\text{ét}}(\mathbb{Q}/\mathbb{Z}(3))) = H^4_{\text{nr}}(Y, \mathbb{Q}/\mathbb{Z}(3)), $$

where $H^4_{\text{nr}}(Y, \mathbb{Q}/\mathbb{Z}(3))$ is the unramified cohomology group of $Y = X \times_k X$. Thus by (4.2), we have an exact sequence

$$\text{CH}^3(Y, 1) \to H^5_L(Y, \mathbb{Z}(3)) \to H^4_{\text{nr}}(Y, \mathbb{Q}/\mathbb{Z}(3)) \to \text{CH}^3(Y) \to \text{CH}^3_L(Y).$$

We wish to study the action of $\text{End}_k(X)$ on the exact sequence (4.5). The motivic cohomology groups can be identified as

$$H^m_M(Y, \mathbb{Z}(n)) = \text{Hom}_{DM^M(k, \mathbb{Z})}(\mathbb{Z}(X), \mathbb{Z}(n)[m]).$$
where $DM^{\text{eff}}(k, \mathbb{Z})$ is the triangulated category of effective motives in the sense of [31]. Note that we have a compatible action of any $\alpha \in \text{End}_k(X)$ on the complexes $Z(Y)$ and $R\pi_*\pi^*Z(Y)$ defined by the same formula as in (3.4) as the maps involved are all defined at the level of complexes. We therefore have an induced compatible action of $\alpha$ on the cone of the comparison map $Z(Y) \to R\pi_*\pi^*Z(Y)$. The long exact sequence (4.2) is obtained by applying $\text{Hom}_{DM^{\text{eff}}(k, \mathbb{Z})}(\cdot, Z(n)[m])$ to the exact triangle in $DM^{\text{eff}}(k, \mathbb{Z})$ arising from the above comparison map. Consequently, we get a compatible $\text{End}_k(X)$-action on (4.5). Along with the exact sequence (4.5) and Theorem 1.1, this immediately implies the following criterion.

**Lemma 4.3.** Let $X$ be a smooth projective integral threefold over a field $k$ with $\text{char}(k) = 0$ and set $Y = X \times_k X$. Assume $\alpha \in \text{End}_k(X)$ is such that $\alpha_E = 0$ for a Galois extension $E/k$. Then the action of $\alpha$ is nilpotent on $H^4_{\text{nr}}(Y, \mathbb{Q}/\mathbb{Z}(3))$ if and only if it is nilpotent on $\text{CH}^3(Y)$.

Thus, if $\text{char}(k) = 0$, then the argument in the proof of Theorem 4.1 shows that Rost nilpotence holds for $X$ if and only if the criterion in Lemma 4.3 is satisfied.

**Remark 4.4.** It is easy to see that the criterion from Lemma 4.3 applies to rational threefolds. Indeed, we may assume that $E/k$ is finite. Since $X$ is rational, so is $Y = X \times_k X$, and $H^4_{\text{nr}}(Y, \mathbb{Q}/\mathbb{Z}(3)) \simeq H^4_{\text{et}}(k, \mathbb{Q}/\mathbb{Z}(3))$. Since $\alpha_E = 0$, the action of $\alpha$ on $H^q_{\text{et}}(E, \mathbb{Q}/\mathbb{Z}(3)) \simeq H^0_{\text{zar}}(Y_E, \mathcal{H}^q_{\text{et}}(\mathbb{Q}/\mathbb{Z}(3)))$ is trivial, and the claim follows now from the Hochschild-Serre spectral sequence $H^p(\text{Gal}(E/k), H^q_{\text{et}}(E, \mathbb{Q}/\mathbb{Z}(3))) \Rightarrow H^{p+q}_{\text{et}}(k, \mathbb{Q}/\mathbb{Z}(3))$.

In particular, if $\text{char}(k) = 0$, then Rost nilpotence holds for $X$.

We now show how Lemma 4.3 can be used to prove Rost nilpotence for birationally ruled threefolds over a field of characteristic 0. Recall that a threefold is said to be birationally ruled if it is birational to $S \times \mathbb{P}^1$, where $S$ is a surface. To this end, we show first that the Rost nilpotence principle is a birational invariant property of threefolds.

**Lemma 4.5.** Let $X$ be a smooth projective threefold over an arbitrary field $k$ and let $\phi: \tilde{X} \to X$ be the blow-up of $X$ with smooth center $C$. Then $X$ satisfies the Rost nilpotence principle if and only if $\tilde{X}$ does.

**Proof.** If $C$ has pure codimension $r$ in $X$, the motive of $\tilde{X}$ is given by the formula

$$h(\tilde{X}) \simeq h(X) \oplus \left( \bigoplus_{i=1}^{r-1} h(C)(i) \right),$$

see [22, Section 9]. Since $\dim X = 3$, we only have to consider the cases when $\dim C$ is 0 or 1. When $\dim C \leq 1$, it is easy to see that $\text{End}_k(h(C))$ injects into
Remark 4.7. The discussion preceding Theorem 4.1 suggests an approach to prove Rost nilpotence for schemes $X$ of dimension $d \geq 3$, which we briefly outline. Assume $\alpha \in \text{End}_k(X)$ is such that $\alpha_E = 0$ for a Galois extension $E/k$. One may attempt to prove Rost nilpotence for $X$ by showing that for all $d + 1 \leq p \leq 2d - 2$ the action of the correspondence $\alpha$ is nilpotent on the cohomology groups $H^{p-2d+2}_\text{zar}(X \times_k E, H^d_{\text{et}}(\mathbb{Q}/\mathbb{Z}(d)))$ to obtain by (4.3) a nilpotent action of $\alpha$ on the hypercohomology group $\mathbb{H}^{2d-1}(X \times_k E, \tau_{\geq d+2}R\pi_*\mathbb{Z}(d)_{\text{et}})$, compatible with the action of $\alpha$ on $\text{CH}^d_1(X \times_k E, 1)$ and $\text{CH}^d(X \times_k E)$. In view of Theorem 1.1, this would imply that the action of $\alpha$ on $\text{CH}^d(X \times_k E)$ is nilpotent.
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