SOME STRONG LIMIT THEOREMS IN AVERAGING

YURI KIFER

INSTITUTE OF MATHEMATICS
HEBREW UNIVERSITY
JERUSALEM, ISRAEL

Abstract. The paper deals with the fast-slow motions setups in the discrete time $X^\varepsilon((n+1)\varepsilon) = X^\varepsilon(n\varepsilon) + \varepsilon B(X^\varepsilon(n\varepsilon), \xi(n)), \ n = 0, 1, ..., [T/\varepsilon]$ and the continuous time $dX^\varepsilon(t) = B(X^\varepsilon(t), \xi(t/\varepsilon)), t \in [0, T]$ where $B$ is a smooth in the first variable vector function and $\xi$ is a sufficiently fast mixing stationary stochastic process. It is known since [23] that if $\bar{X}$ is the averaged motion then $G^\varepsilon = \varepsilon^{-1/2}(X^\varepsilon - \bar{X})$ weakly converges to a Gaussian process $G$. We will show that for each $\varepsilon$ the processes $\xi$ and $G$ can be redefined on a sufficiently rich probability space without changing their distributions so that $E \sup_{0 \leq t \leq T} |G^\varepsilon(t) - G(t)|^2 M = O(\varepsilon^{\delta}), \ \delta > 0$ which gives also $O(\varepsilon^{\delta/3})$ Prokhorov distance estimate between the distributions of $G^\varepsilon$ and $G$. This provides also convergence estimates in the Kantorovich–Rubinstein (or Wasserstein) metrics. In the product case $B(x, \xi) = \Sigma(x)\xi$ we obtain also almost sure convergence estimates of the form $\sup_{0 \leq t \leq T} |G^\varepsilon(t) - G(t)| = O(\varepsilon^\delta)$ a.s., as well as the Strassen’s form of the law of iterated logarithm for $G^\varepsilon$. We note that our mixing assumptions are adapted to fast motions generated by important classes of dynamical systems.

1. Introduction

Let $X^\varepsilon(t) = X^\varepsilon_x(t)$ be the solution of a system of ordinary differential equations having the form

\begin{equation}
\frac{dX^\varepsilon(t)}{dt} = B(X^\varepsilon(t), \xi(t/\varepsilon)), \ X^\varepsilon(0) = x, \ t \in [0, T]
\end{equation}

where $B(x, \xi(s))$ is a (random) bounded twice differentiable in the first variable vector field on $\mathbb{R}^d$ and $\xi$ is a stationary process on a Polish space $\mathcal{Y}$ which is viewed as a fast motion while $X^\varepsilon$ is considered as a slow motion. Under the ergodicity assumption the limit

\begin{equation}
\bar{B}(x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t B(x, \xi(s))ds
\end{equation}

exists almost surely (a.s.) and it is Lipschitz continuous, as well. Hence, the solution $\bar{X}(t) = \bar{X}_x(t)$ of the equation

\begin{equation}
\frac{d\bar{X}(t)}{dt} = \bar{B}((\bar{X}(t)), \ \bar{X}(0) = x, \ t \in [0, T]
\end{equation}

Date: June 21, 2024.

2000 Mathematics Subject Classification. Primary: 34C29 Secondary: 60F15, 60G40, 91A05.

Key words and phrases. averaging, strong approximations, $\phi$-mixing, stationary process, shifts, dynamical systems.
exists and it is called the averaged motion. It is well known that $X^\varepsilon(t)$ and $\bar{X}_x(t)$ are close uniformly in time on bounded time intervals (cf. [23]). This is a version of the averaging principle which was used in celestial mechanics already in the 18th century though its rigorous justification was obtained only in the middle of the 20th century (see [7]).

The next natural step was to study the error $X^\varepsilon(t) - \bar{X}_x(t)$ of the averaging approximation and it was shown in [23] that the normalized difference $\varepsilon^{-1/2}(X^\varepsilon(t) - \bar{X}_x(t))$ converges weakly as $\varepsilon \to 0$ to a $d$-dimensional Gaussian Markov process $G(t)$ which solves the linear stochastic differential equation

$$dG(t) = \nabla \bar{B}(\bar{X}_x(t))G(t)dt + \sigma(\bar{X}_x(t))dW(t), \ G(0) = 0$$

where $W(t)$ is a standard $d$-dimensional Brownian motion, the diffusion matrix $\sigma$ is obtained via certain limit and for each vector $b(x) = (b_1(x),...,b_d(x))$ we denote by $\nabla b(x)$ the matrix whose $(i,j)$-th element is $\partial b_i(x)/\partial x_j$. In [20] a stronger convergence result was obtained considering in addition to the process $G(t)$ another process $H(t) = H^\varepsilon(t)$ solving the stochastic differential equation

$$dH^\varepsilon(t) = \bar{B}(H^\varepsilon(t))dt + \sqrt{\varepsilon}\sigma(H^\varepsilon(t))dW(t), \ H^\varepsilon_x(0) = H^\varepsilon(0) = x$$

which was suggested by K. Hasselmann (2021 Nobel prize in physics) in the study of climate evolution (see [18]). In some models of weather–climate interactions the former is viewed as a fast chaotic and the latter as the slow motions in the averaging setup above. Supported by heuristic arguments the process $H^\varepsilon$ was introduced in [18] as an approximation for the slow motion $X^\varepsilon$. It was shown in [20] that under sufficiently fast mixing assumptions the process $\xi(t)$ can be redefined on a larger probability space preserving its distribution where there exists a standard Brownian motion $W$ such that for $G(t)$ and $H^\varepsilon(t)$ constructed by (1.4) and (1.5) with such $W$ the $L^1$-norms $E\sup_{0 \leq t \leq T}|X^\varepsilon(t) - \bar{X}_x(t) - \sqrt{\varepsilon} G(t)|^2$ and $E\sup_{0 \leq t \leq T}|X^\varepsilon(t) - H^\varepsilon(t)|^2$ have the order $\varepsilon^{1+\delta}$ with $\delta > 0$ which provides also estimates for the Prokhorov distance between distributions of pairs $X^\varepsilon - \bar{X}$, $\sqrt{\varepsilon}G$ and $X^\varepsilon$, $H^\varepsilon$. It is important to observe that both [23] and [20] assume that the process $\xi(t)$ is sufficiently fast mixing with respect to $\sigma$-algebras generated by itself which greatly restricts applications to the case when $\xi(t)$ is generated by a dynamical system as will be discussed later on. Similar results when $\xi(t)$ is a diffusion depending on the slow motion $X^\varepsilon$ (the fully coupled case) were obtained in [3]. Still, the statistical study of the slow motion driven by a deterministic fast motion fits more the spirit of the chaos theory, and so we will deal in this paper with fast motions $\xi(t)$ which can be generated by important classes of dynamical systems.

In this paper we start with the discrete time setup given by the recurrence relation

$$X^\varepsilon(n+1;\varepsilon) = X^\varepsilon(n;\varepsilon) + \varepsilon B(X^\varepsilon(n;\varepsilon),\xi(n))$$

where $B(x,\xi)$ is a smooth vector field in $x$ on $\mathbb{R}^d$ Borel measurably dependent on $\xi$ and $\xi(n)$ is a vector valued stationary process while sufficiently fast mixing is assumed with respect to a two parameter family of $\sigma$-algebras $\mathcal{F}_{mn}$ as described in the next section. Each random vector $\xi(n)$ is not supposed to be $\mathcal{F}_{mn}$-measurable and, instead, we assume that it is well approximated by its conditional expectations with respect to $\sigma$-algebras $\mathcal{F}_{n-m,n+m}$. As will be explained in more details in the next section, this will enable us to consider processes generated by a large class of
dynamical systems, i.e. when $\xi(\omega, n) = f(F^n\omega)$ for a probability preserving transformation $F$ and a Hölder continuous vector function $f$. The relevant dynamical systems include hyperbolic ones which is the important family of chaotic systems considered sometimes as a model for weather evolution.

For $t \in [0, T]$ we set $X^\varepsilon(t) = X^\varepsilon([t/\varepsilon]\varepsilon)$ and show that the process $\xi(n)$ can be redefined on a richer probability space preserving its distribution where there exists a Brownian motion $W = W_\varepsilon$ such that for $G = G_\varepsilon$ solving (1.4) with such $W = W_\varepsilon$ we have the moment estimates of the form $E \sup_{0 \leq t \leq T} |\varepsilon^{-1/2}(X^\varepsilon(t) - \tilde{X}(t)) - G_\varepsilon(t)|^{2M} = O(\varepsilon^a)$, which gives also the Prokhorov and the Kantorovich–Rubinstein (or Wasserstein) distance estimate of $\varepsilon^{3/2}$ between distributions of $\varepsilon^{-1/2}(X^\varepsilon(t) - \tilde{X}(t))$ and $G$. These results are easily extended to the continuous time case (1.1) if the corresponding process $\xi(t)$ is sufficiently fast mixing. But in applications to dynamical systems the latter is quite restrictive and we consider a more realistic situation where the process $\xi(t)$ is obtained via the so called suspension construction over a sufficiently fast mixing discrete time probability preserving transformation which requires some work and will be done by reducing the problem to the discrete time case. The weak limit theorem and moderate deviations results in this setup were obtained in [25] and similar weak convergence results for some additional classes of dynamical systems were derived in [35] under somewhat different assumptions. Of course, weak convergence results cannot provide any speed estimates while the results of this paper yield explicit moment and Prokhorov distance estimates for each $\varepsilon > 0$. In the product case $B(x, \xi) = \Sigma(x)\xi$ we obtain also almost sure convergence estimates of the form $\sup_{0 \leq t \leq T} |\varepsilon^{-1/2}(X^\varepsilon(t) - \tilde{X}(t)) - G_\varepsilon(t)| = O(\varepsilon^a)$ a.s., as well as the Strassen form of the functional law of iterated logarithm saying that with probability one as $\varepsilon \to 0$ the set of limit points of random functions $(2\varepsilon \log \log(1/\varepsilon))^{-1/2}(X^\varepsilon - \tilde{X})$ coincides with certain compact set in the space of continuous functions on $[0, T]$.

At the end of the introduction we will mention another but quite different averaging problem started from [24] and attracted recently the renewed interest (see [31], [28], [29], [15] and references there). The main difference of the setup considered there with the present paper is that here we are interested in the normalized deviation of $X^\varepsilon$ from the averaged motion $\tilde{X}$ for the time of order $1/\varepsilon$ while in the other setup the averaging motion is zero and we are waiting for the time of order $1/\varepsilon^2$ to see what happens with $X^\varepsilon$. In the first case considered here we end up with a Gaussian process (or a close to it Hasselmann’s diffusion with a small parameter). In the second setup the limiting process there turns out to be a quite general diffusion. Observe also that unlike the present paper, it was possible to obtain strong moment convergence estimates in the second setup only under very special assumptions on coefficients of [28] while without it only weak or almost sure results were obtained and they relied on the rough paths theory technique. The approaches to these two types of averaging problems are mostly different but when dealing with strong limit theorems in [28], [29], [14] and in the present paper the methods have a non empty intersection around preparations to apply the strong approximation theorem from [0].

The end of this paper is the following. In the next section we provide necessary definitions and give precise statements of our results. Section 3 is devoted to necessary estimates both of general nature and more specific to our problem, as well as characteristic functions approximations needed in Section 4 for the strong
approximation theorem. In Section 5 we deal with the continuous time case relying on Sections 3 and 4 after certain discretization. In Section 6 we obtain the a.s. approximation results and the functional law of iterated logarithm in the product case.

2. Preliminaries and main results

2.1. Discrete time case. We start with the discrete time setup which consists of a complete probability space \((\Omega,\mathcal{F},P)\), a stationary sequence of \(d\)-dimensional random vectors \(\xi(n) = (\xi_1(n),\ldots,\xi_d(n))\), \(-\infty < n < \infty\) and a two parameter family of countably generated \(\sigma\)-algebras \(\mathcal{F}_{m,n} \subset \mathcal{F}\), \(-\infty \leq m \leq n \leq \infty\) such that \(\mathcal{F}_{mn} \subset \mathcal{F}_{m,n'} \subset \mathcal{F}\) if \(m' \leq m \leq n \leq n'\) where \(\mathcal{F}_{m\infty} = \bigcup_{n:m \geq m}\mathcal{F}_{mn}\) and \(\mathcal{F}_{-\infty} = \bigcup_{m:m \leq n}\mathcal{F}_{mn}\). It is often convenient to measure the dependence between two sub \(\sigma\)-algebras \(\mathcal{G},\mathcal{H} \subset \mathcal{F}\) via the quantities

\[
\varpi_{b,a}(\mathcal{G},\mathcal{H}) = \sup\{\|E(g\mathcal{G}) - Eg\|_a : g \text{ is } \mathcal{H} - \text{measurable and } \|g\|_b \leq 1\},
\]

where the supremum is taken over real functions and \(\|\cdot\|_c\) is the \(L^c(\Omega,\mathcal{F},P)\)-norm. Then more familiar \(\alpha,\rho,\phi\) and \(\psi\)-mixing (dependence) coefficients can be expressed via the formulas (see [4], Ch. 4)

\[
\alpha(\mathcal{G},\mathcal{H}) = \frac{1}{4}\varpi_{\infty,1}(\mathcal{G},\mathcal{H}), \quad \rho(n) = \varpi_{2,2}(n), \quad \phi(\mathcal{G},\mathcal{H}) = \frac{1}{2}\varpi_{\infty,\infty}(\mathcal{G},\mathcal{H})\quad \text{and} \quad \psi(\mathcal{G},\mathcal{H}) = \varpi_{1,\infty}(\mathcal{G},\mathcal{H}).
\]

We set also

\[
\varpi_{b,a}(n) = \sup_{k \geq 0} \varpi_{b,a}(\mathcal{F}_{-\infty,k},\mathcal{F}_{k+n,\infty})
\]

and accordingly

\[
\alpha(n) = \frac{1}{4}\varpi_{\infty,1}(n), \quad \rho(n) = \varpi_{2,2}(n), \quad \phi(n) = \frac{1}{2}\varpi_{\infty,\infty}(n), \quad \psi(n) = \varpi_{1,\infty}(n).
\]

Furthermore, by the real version of the Riesz–Thorin interpolation theorem or the Riesz convexity theorem (see [10], Section 9.3 and [11], Section VI.10.11) whenever \(\theta \in [0,1]\), \(1 \leq a_0,a_1,b_0,b_1 \leq \infty\) and

\[
\frac{1}{a} = \frac{1 - \theta}{a_0} + \frac{\theta}{a_1}, \quad \frac{1}{b} = \frac{1 - \theta}{b_0} + \frac{\theta}{b_1}
\]

then

\[
\varpi_{b,a}(n) \leq 2(\varpi_{b_0,a_0}(n))^{1-\theta}(\varpi_{b_1,a_1}(n))^\theta.
\]

In particular, using the obvious bound \(\varpi_{b_0,a_0}(n) \leq 2\) valid for any \(b_1 \geq a_1\) we obtain from (2.1) for pairs \((\infty,1), (2,2)\) and \((\infty,\infty)\) that for all \(b \geq a \geq 1\)

\[
\varpi_{b,a}(n) \leq 4(2\alpha(n))^{\frac{1}{2} - \frac{\theta}{2}}, \quad \varpi_{b,a}(n) \leq 2^{1 + \frac{\theta}{2}}(\rho(n))^{\frac{1}{2} + \frac{\theta}{2}}\quad \text{and} \quad \varpi_{b,a}(n) \leq 2^{1 + \frac{\theta}{2}}(\phi(n))^{1 - \frac{\theta}{2}}.
\]

This enables us to replace our assumption below on the decay of the dependence coefficient \(\varpi\) by the corresponding assumptions on the more familiar dependence coefficients mentioned above.

We will assume that \(B : \mathbb{R}^d \times \mathcal{Y} \to \mathbb{R}^d\) is a Borel map with a uniform bound on its \(C^2\) norm in the first argument

\[
\sup_{x \in \mathbb{R}^d} \sup_{y \in \mathcal{Y}} \max_{0 \leq i,j,k \leq d} \max(|B_i(x,y)|, |\frac{\partial B_i(x,y)}{\partial x_j}|, |\frac{\partial^2 B_i(x,y)}{\partial x_j \partial x_k}|) = L < \infty.
\]
Unlike [26], in order to ensure more applicability of our results to dynamical systems, we do not assume that \( \xi(n) \) is \( \mathcal{F}_n \)-measurable and instead we will work with the moment approximation coefficient
\[
(2.4) \quad \rho(a, n) = \sup_{x,m} \max_{i,j,k} \max \left( \|B_i(x, \xi(m))\|_a, \|E(B_i(x, \xi(m)))|\mathcal{F}_{m-n, m+n}\|_a, \|E(\|B_i(x, \xi(m))\|_a|\mathcal{F}_{m-n, m+n})\|_a \right)
\]

Observe that if we assume that \( Y \) is a Banach space with a norm \( |\cdot| \), the functions \( B_i(x, y), \frac{\partial B_i(x, y)}{\partial x_j}, \frac{\partial^2 B_i(x, y)}{\partial x_i \partial x_k} \) are Lipschitz continuous in \( y \) with a constant \( L \) and
\[
(2.5) \quad \sup_m \|\xi(m) - E(\xi(m)|\mathcal{F}_{m-n, m+n})\|_a \leq \frac{1}{2} L^{-1} \rho(a, n)
\]
then (2.4) holds true. In particular, this is satisfied if \( B(x, \xi(m)) = \sigma(x)\xi(m) \) where \( \xi(m) \)'s are random vectors and \( \sigma(x) \) is a matrix function with its \( C^2 \)-norm bounded by \( \ell \). To save notations we will still write \( \mathcal{F}_{mn}, \varpi_{b,a}(n) \) and \( \rho(a, n) \) for \( \mathcal{F}_{m|n}, \varpi_{b,a}([n]) \) and \( \rho(a, [n]) \), respectively, if \( m \) and \( n \) are not integers (or \( \pm \infty \)), where \([\cdot]\) denotes the integral part. We will assume that the coefficients \( \varpi \) and \( \rho \) decay fast enough, namely that for some \( K, M \geq 1 \) large enough,
\[
(2.6) \quad D = \sum_{n=1}^{\infty} n^5 (\varpi_{K,AM}(n) + \rho(K, n)) < \infty.
\]
It turns out that the same proofs work in a seemingly more general setup when we hide the process \( \xi(m) \) and consider instead an invertible probability preserving transformation \( \vartheta : \Omega \to \Omega \), so that \( B(x, \xi(m)) \) is replaced by \( B(x, \vartheta^m \omega) \) where \( B : \mathbb{R}^d \times \Omega \to \mathbb{R}^d \) is measurable and satisfies the regularity conditions (2.3) in the first variable. In fact, if \( \Omega \) is a Lebesgue (standard probability) space it is always possible to pass from this second representation to the first one, so that this becomes just the matter of notations.

Define also \( \hat{B}(x, \xi(m)) = B(x, \xi(m)) - EB(x, \xi(m)) \),
\[
a_{ij}(x, y, m, n) = E(\hat{B}_i(x, \xi(m)) \hat{B}_j(y, \xi(m))) \quad \text{and} \quad a_{ij}(x, m, n) = a_{ij}(x, x, m, n)
\]
where \( \hat{B} = (\hat{B}_1, \ldots, \hat{B}_d) \) and \( \hat{B}_i = B_i - EB_i \) with \( B = (B_1, \ldots, B_d) \). It will be shown in the next section under the conditions of our assertions below that for \( i, j = 1, \ldots, d \) the limits
\[
(2.7) \quad a_{ij}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=m}^{m+n} \sum_{l=m}^{m+n} a_{ij}(x, k, l) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n} \sum_{l=0}^{n} a_{ij}(x, k, l)
\]
exist. We will see that under our conditions the matrix \( A(x) = (a_{jk}(x)) \) is symmetric and twice differentiable in \( x \), and so it has a symmetric Lipschitz continuous in \( x \) square root \( \sigma(x) \), i.e. we have the representation (see [14] and Sections 5.2 and 5.3 in [37]),
\[
(2.8) \quad A(x) = \sigma^2(x),
\]
and both the uniform bound of the norm and the Lipschitz constant of \( \sigma \) will be denoted again by \( L \). In fact, for our purposes it suffices to have the representation \( A(x) = \sigma(x)\sigma^*(x) \) with a Lipschitz continuous matrix \( \sigma \) where \( \sigma^* \) is the conjugate to \( \sigma \). Thus, there exists a unique solution \( H^x \) of the stochastic differential equation
Set also $\hat{G}^\varepsilon(t) = \hat{X}(t) + \sqrt{\varepsilon}G(t)$, $\hat{G}^\varepsilon(0) = \hat{G}^\varepsilon(0) = x$ where $\hat{X}$ and $G$ are given by (1.3) and (1.4), respectively. Our main results in the discrete time case are the following.

2.1. Theorem. Suppose that the conditions (2.3) and (2.6) hold true and a symmetric Lipschitz continuous matrix $\sigma(x)$ satisfying (2.8) is fixed. Then for each $\varepsilon > 0$ the stationary process $\xi(n), 0 \leq n < \infty$ can be redefined preserving its distributions on a richer probability space where there exists a standard Brownian motion $W = W_\varepsilon$ so that the slow motion $X^\varepsilon_\varepsilon$, the diffusion $H^\varepsilon_\varepsilon$ and the Gaussian process $\hat{G}^\varepsilon_\varepsilon$ constructed with these newly defined processes and having the same initial condition $X^\varepsilon(0) = H^\varepsilon(0) = \hat{G}^\varepsilon(0) = x$, satisfy

$$E \sup_{0 \leq t \leq T} |X^\varepsilon(t)|^{2M} \leq C_0(M)\varepsilon^{M+\delta} \quad \text{and}$$

$$E \sup_{0 \leq t \leq T} |X^\varepsilon(t) - \hat{G}^\varepsilon_\varepsilon(t)|^{2M} \leq C_0(M)\varepsilon^{M+\delta},$$

for any integer $M \geq 1$, where we can take $\delta = \frac{1}{2M}$ and $C_0(M) > 0$ does not depend on $\varepsilon$ and can be explicitly estimated from the proof. In particular,

$$E \sup_{0 \leq t \leq T} |\varepsilon^{-1/2}(X^\varepsilon(t) - \hat{X}_\varepsilon(t)) - G(t)|^{2M} \leq C_0(M)\varepsilon^{\delta}.$$ 

Furthermore,

$$E \sup_{0 \leq t \leq T} |\varepsilon^{-1/2}(H^\varepsilon_\varepsilon(t) - \hat{X}_\varepsilon(t)) - G(t)|^{2M} \leq 2^{2M-1}\hat{C}_0(M)\varepsilon^{M},$$

where $\hat{C}_0(M) > 0$ does not depend on $\varepsilon$, provided $G$ and $H_\varepsilon$ are given by (1.3) and (1.4), respectively, with the same Brownian motion $W$.

While (2.12) follows directly from Lemmas 3.6 and 3.7 from Section 3 in order to obtain (2.9) we will rely on the strong approximation theorem from Section 4. Therefore (2.10) will follow from (2.3) and (2.12) with an appropriate $C_0(M) > 0$ which does not depend on $\varepsilon$.

Recall, that the Prokhorov distance $\pi$ between two probability measures $\mu$ and $\nu$ on a metric space $\mathcal{X}$ with a distance function $d$ is defined by

$$\pi(\mu, \nu) = \inf\{\kappa > 0 : \mu(U) \leq \nu(U^\kappa) + \kappa \text{ and } \nu(U) \leq \mu(U^\kappa) + \kappa \text{ for any Borel set } U \text{ on } \mathcal{X}\}$$

where $U^\kappa = \{x \in \mathcal{X} : d(x, y) < \kappa \text{ for some } y \in U \subset \mathcal{X}\}$ is the $\kappa$-neighborhood of $U$. Recall also that the $L^1$ Wasserstein (or Kantorovich–Rubinstein) distance between two probability measures $\mu$ and $\nu$ on $\mathcal{X}$ is defined by

$$w_q(\mu, \nu) = \inf\{(Ed^q(Q, R))^{1/q} : L(Q) = \mu \text{ and } L(R) = \nu\}$$

where the infimum is taken over all random points (variables) $Q$ and $R$ in $\mathcal{X}$ with their distributions $L(Q)$ and $L(R)$ equal $\mu$ and $\nu$ respectively. From Theorem 2.1 we obtain immediately that

$$w_{2M}(L(\sqrt{\varepsilon}X^\varepsilon_\varepsilon), L(G)) \leq C_0^{1/2M}(M)\varepsilon^{\delta/2M}$$

and

$$w_{2M}(L(\sqrt{\varepsilon}H^\varepsilon_\varepsilon), L(G)) \leq 2\hat{C}_0^{1/2M}(M)\varepsilon^{\delta/2M}.$$ 

It is known (see, for instance, Theorem 2 in [17]) that $(\pi(\mu, \nu))^2 \leq w_1(\mu, \nu)$ but since we claim (2.13) only for $M \geq 1$, we will provide below a slightly better estimate.
than what follows from this one. Let $X$ be the metric space of measurable paths $\gamma : [0, T] \to \mathbb{R}^d$ with the uniform metric $d(\gamma, \tilde{\gamma}) = \sup_{0 \leq t \leq T} |\gamma(t) - \tilde{\gamma}(t)|$.

2.2. Corollary. For any $\varepsilon > 0$,

\[(2.14)\]
\[
\pi(\mathcal{L}(\frac{X^\varepsilon - \hat{X}_x}{\sqrt{\varepsilon}}), \mathcal{L}(G)) \leq C_0^{1/3}(M)\varepsilon^{4/3} \quad \text{and} \quad \pi(\mathcal{L}(\frac{H^\varepsilon_x - \hat{X}_x}{\sqrt{\varepsilon}}), \mathcal{L}(G)) \leq \tilde{C}_0^{1/3}(M)\varepsilon^{M/3}
\]

where $G$ is given by (1.4) and, recall, that (2.13) depends only on distributions and not on specific choices of the stationary process $\xi$ and of the Brownian motion $W$ as in Theorem 2.1.

We observe that the only place which forces us to have the estimate with only small fixed powers of $\varepsilon$ in (2.11), (2.12) and (2.13), and not a growing with $M$ power of $\varepsilon$, is the strong approximation estimates of Theorem 2.1 and Lemma 4.4 below which cannot be improved substantially within the current method.

Important classes of processes satisfying our conditions come from dynamical systems. Let $F$ be a $C^2$ Axiom A diffeomorphism (in particular, Anosov) in a neighborhood of an attractor or let $F$ be an expanding $C^2$ endomorphism of a Riemannian manifold $\Omega$ (see [4]), $f$ be either a Hölder continuous vector function or a vector function which is constant on elements of a Markov partition and let $\xi(n) = \xi(n, \omega) = f(F^n(\omega))$. Here the probability space is $(\Omega, \mathcal{B}, P)$ where $P$ is a Gibbs invariant measure corresponding to some Hölder continuous function and $\mathcal{B}$ is the Borel $\sigma$-field. Let $\zeta$ be a finite Markov partition for $F$, then we can take $\mathcal{F}_kl$ to be the finite $\sigma$-algebra generated by the partition $\bigcap_{i=k}^l F^i\zeta$ (or by $\bigcap_{i=k}^l F^{-i}\zeta$ in the non invertible case). In fact, we can take here not only Hölder continuous $f$’s but also indicators of sets from $\mathcal{F}_kl$. The conditions of Theorem 2.1 allow all such functions since the dependence of Hölder continuous functions on $m$-tails, i.e., on events measurable with respect to $\mathcal{F}_{-\infty,-m}$ or $\mathcal{F}_{m,\infty}$, decays exponentially fast in $m$ and the condition (2.6) is even weaker than that. A related class of dynamical systems corresponds to $F$ being a topologically mixing subshift of finite type which means that $F$ is the left shift on a subspace $\Omega$ of the space of one (or two) sided sequences $\omega = (\omega_i, i \geq 0)$, $\omega_i = 1, ..., l_0$ such that $\omega \in \Omega$ if $\pi_{\omega_{i+1} = 1} = 1$ for all $i \geq 0$ where $\Pi = (\pi_{ij})$ is an $l_0 \times l_0$ matrix with 0 and 1 entries and such that $\Pi^n$ for some $n$ is a matrix with positive entries. Again, we have to take in this case $f$ to be a Hölder continuous bounded function on the sequence space above. $P$ to be a Gibbs invariant measure corresponding to some Hölder continuous function and to define $\mathcal{F}_kl$ as the finite $\sigma$-algebra generated by cylinder sets with fixed coordinates having numbers from $k$ to $l$. The exponentially fast $\psi$-mixing, which is the strongest type of mixing among mentioned above, is well known in these cases (see [4]). Among other dynamical systems with exponentially fast $\psi$-mixing we can mention also the Gauss map $Fx = \{1/x\}$ (where $\{x\}$ denotes the fractional part) of the unit interval with respect to the Gauss measure and more general transformations generated by $f$-expansions (see [19]). Gibbs-Markov maps which are known to be exponentially fast $\phi$-mixing (see, for instance, [34]) can be taken as $F$ with $\xi(n) = f(F^n)$ as above. Moreover, (2.2) enables us to apply the results to $\alpha$ (and so also to $\beta$) mixing dynamical systems with a sufficiently fast decay of the $\alpha$ (or $\beta$) dependence coefficient, among them some systems which can be represented via the Young tower construction. Observe that in the above symbolic setups the assumption that $\xi(n, \omega) = f(F^n(\omega))$ is $\mathcal{F}_{nk}$-measurable would mean that $f$ depends only on one
coordinate or is constant on the elements of the basic partition which is, of course, a very restrictive assumption.

2.3. Remark. We believe that a combination of methods from [1], [5] and the present paper will yield a version of Theorem 2.1 for the specific fully coupled averaging setup of the form

$$X^\varepsilon(n+1) = X^\varepsilon(nc) + \varepsilon B(X^\varepsilon(nc), \xi(n)), \quad \xi(n+1) = T_{X^\varepsilon(nc)} \xi(n),$$

$$X^\varepsilon(0) = x, \quad \xi(0) = y$$

where $T_x, x \in \mathbb{R}^d$ is a $C^2$ depending on $x$ family of either $C^2$ expanding transformations or $C^2$ Axiom A diffeomorphisms in a neighborhood of an attractor $\Lambda_x, x \in \mathbb{R}^d$ so that $\Lambda_x$ corresponds to $T_x$. For more details about this setup we refer the reader to [1] and [27]. Still, the treatment of the asymptotical behavior of the slow motion $X^\varepsilon$ in this situation requires substantial machinery from dynamical systems and the theory of perturbations which goes beyond the scope of the present paper, and so this study will be left for another publication. We observe though that it is possible to handle the fully coupled setup only when an appropriate model of slowly changing well mixing fast motions is available which is not the case of a general stationary process $\xi$ considered in the present paper.

2.2. Continuous time case. Here we start with a complete probability space $(\Omega, \mathcal{F}, P)$, a $P$-preserving invertible transformation $\vartheta : \Omega \to \Omega$ and a two parameter family of countably generated $\sigma$-algebras $\mathcal{F}_{m,n} \subset \mathcal{F}, -\infty \leq m \leq n \leq \infty$ such that $\mathcal{F}_{mn} \subset \mathcal{F}_{m',n'} \subset \mathcal{F}$ if $m' \leq m \leq n \leq n'$ where $\mathcal{F}_{m\infty} = \cup_{n \geq m} \mathcal{F}_{mn}$ and $\mathcal{F}_{-\infty n} = \cup_{m \leq n} \mathcal{F}_{mn}$. The setup includes also a (roof or ceiling) function $\tau : \Omega \to (0, \infty)$ such that for some $\bar{L} > 0$,

$$\bar{L}^{-1} \leq \tau \leq \bar{L}.$$

Next, we consider the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ such that $\hat{\Omega} = \{ (\hat{\omega}, t) : \omega \in \Omega, 0 \leq t \leq \tau(\omega), (\omega, \tau(\omega)) = (\vartheta(t[w,0]), t) \}$, $\hat{\mathcal{F}}$ is the restriction to $\hat{\Omega}$ of $\mathcal{F} \times \mathcal{B}[0,\bar{L}]$, where $\mathcal{B}[0,\bar{L}]$ is the Borel $\sigma$-algebra on $[0,\bar{L}]$ completed by the Lebesgue zero sets, and for any $\Gamma \in \hat{\mathcal{F}}$,

$$\hat{P}(\Gamma) = \bar{\tau}^{-1} \int_{\Gamma} \mathbb{I}_\Gamma(\omega, t) dP(\omega) dt$$

where $\bar{\tau} = \int \tau dP = E\tau$,

$E$ denotes the expectation on the space $(\Omega, \mathcal{F}, P)$ and $\hat{E}$ will denote the expectation on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. Finally, we introduce a vector valued stochastic process $\xi(t) = (\xi(t, (\omega, s)), -\infty < t < \infty, 0 \leq s \leq \tau(\omega))$ on $\hat{\Omega}$ satisfying

$$\xi(t, (\omega, s)) = \xi(t + s, (\omega, 0)) = \xi(0, (\omega, t + s))$$

if $0 \leq t + s < \tau(\omega)$ and

$$\xi(t, (\omega, s)) = \xi(0, (\vartheta^k(\omega, u), t))$$

if $t + s = u + \sum_{j=0}^k \tau(\vartheta^j(\omega))$ and $0 \leq u < \tau(\vartheta^k(\omega))$.

This construction is called in dynamical systems a suspension and it is a standard fact that $\xi$ is a stationary process on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and in what follows we will write also $\xi(t, (\omega, 0))$ for $\xi(t, (\omega, 0))$.

We will assume that $X^\varepsilon(t) = X^\varepsilon(t, \omega)$ considered as a process on $(\Omega, \mathcal{F}, P)$ solves the equation (1.1) and the averaged motion $\hat{X}(t)$ is the solution of (1.3) with
\[ \hat{B}(x) = \hat{E}B(x, \xi(0, \omega)). \]

Set
\[
\begin{align*}
b(x, \omega) &= \int_0^\tau B(x, \xi(s, \omega)) ds \\
\rho(a, n) &= \sup_{x, m} \max_{i, j, k} (\|\tau \circ \vartheta^m - E(\tau \circ \vartheta^m | F_{m-n,m+n})\|_a, \|b_i(x, \xi(m)) - E(b_i(x, \xi(m)) | F_{m-n,m+n})\|_a)
\end{align*}
\]

\[
- E(b_i(x, \xi(m)) | F_{m-n,m+n})|_a, \| \frac{\partial b_i(x, \xi(m))}{\partial x_j} - E(\frac{\partial b_i(x, \xi(m))}{\partial x_j} | F_{m-n,m+n})\|_a, \| \frac{\partial^2 b_i(x, \xi(m))}{dx_j dx_k} - E(\frac{\partial^2 b_i(x, \xi(m))}{dx_j dx_k} | F_{m-n,m+n})\|_a).
\]

Since we assume that \( B(x, \zeta) \) is twice differentiable in the first variable the last \( \sup_x \) is still measurable. Observe also that \( b(x, \cdot) \circ \vartheta^k, k \in \mathbb{Z} \) is a stationary sequences of random vectors.

Next, we consider the Gaussian process \( G(t) \) and the diffusion \( H^\varepsilon(t) = H_2^\varepsilon(t) \) given by
\[
dG(t) = \nabla b(\tilde{X}_x(\varepsilon t)) G(t) dt + \sigma(\tilde{X}_x(\varepsilon t)) dW(t), \quad G(0) = 0
\]
and
\[
dH^\varepsilon(t) = \tilde{b}(H^\varepsilon(t)) dt + \sqrt{\sigma} H^\varepsilon(t) dW, \quad H^\varepsilon(0) = H^\varepsilon(0) = x,
\]
respectively, where \( \tilde{b}(x) = E b(x, \cdot) \), \( \sigma^2(x) = A(x) = (a_{ij}(x))_{i,j=1,...,d} \) and
\[
a_{ij}(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 \int_0^t \hat{E} \tilde{b}_i(x, \xi(s, \omega)) \hat{b}_j(x, \xi(s, \omega)) ds du
\]
\[
\lim_{n \to \infty} \frac{1}{\varepsilon} \sum_{k,l=0}^n E (b_i(x, \vartheta^k \omega) - \tau(\vartheta^k \omega) \hat{b}_i(x))(b_j(x, \vartheta^l \omega) - \tau(\vartheta^l \omega) \hat{b}_j(x)),
\]
\[
\hat{B}(x, \xi(s, \omega)) = B(x, \xi(s, \omega)) - E B(x, \xi(s, \omega)) \]
and the limits in (2.18) will be shown to exist in the same way as in (2.7). As before we set also \( G^\varepsilon = \tilde{X} + \varepsilon G \). Our results for this continuous time setup are the following.

2.4. **Theorem.** Suppose that the conditions (2.3), with \( b \) in place of \( B \), and (2.6) hold true with \( \omega \) defined with respect to the \( \sigma \)-algebras \( F_{mn} \). Let a symmetric Lipschitz continuous matrix \( \sigma(x) \) satisfying (2.3) is fixed. Then for each \( \varepsilon > 0 \) the process \( \xi(t), 0 \leq t < \infty \) viewed on the basic probability space \( (\Omega, F, P) \) can be redefined preserving its distributions on a richer probability space where there exists a standard d-dimensional Brownian motion \( W = W_x \) so that the slow motion \( X^\varepsilon \), the diffusion \( H^\varepsilon \), and the Gaussian process \( G^\varepsilon \), constructed with these newly defined processes and having the same initial condition \( X^\varepsilon(0) = H^\varepsilon(0) = G^\varepsilon(0) = x \), satisfy
\[
E \sup_{0 \leq t \leq T} |X^\varepsilon_x(t) - H^\varepsilon_x(t/\varepsilon)|^{2M} \leq C_0(M)(\varepsilon^{M+\delta} + \varepsilon^{(3M-4)/2})
\]
and
\[
E \sup_{0 \leq t \leq T} |X^\varepsilon_x(t) - \hat{G}^\varepsilon_x(t/\varepsilon)|^{2M} \leq C_0(M)(\varepsilon^{M+\delta} + \varepsilon^{(3M-4)/2})
\]
for any integer \( M \geq 1 \) and some \( C_0(M) > 0 \) which do not depend on \( \varepsilon \) while, again, we can take \( \delta = \frac{1}{3M} \). In particular,
\[
E \sup_{0 \leq t \leq T} |\varepsilon^{-1/2}(X^\varepsilon(t) - \tilde{X}_x(t)) - G(t/\varepsilon)|^{2M} \leq C_0(M)(\varepsilon^{\delta} + \varepsilon^{(M-4)/2})
\]
Furthermore,
\[
E \sup_{0 \leq t \leq T} |\varepsilon^{-1/2}(H^\varepsilon_x(t) - \tilde{X}_x(t)) - G(t/\varepsilon)|^{2M} \leq C_0(M)(\varepsilon^{M} + \varepsilon^{(M-4)/2})
\]
where $\hat{C}_0(M)$ does not depend on $\varepsilon$, provided $G$ and $H^x$ are given by (1.4) and (1.2), respectively, with the same Brownian motion $W$. Corollary 2.2 remains true with the same constants in this continuous setup, as well.

The proof of Theorem 2.4 proceeds by reducing the problem to the corresponding limit theorems for certain discrete time processes on the probability space $(\Omega, F, P)$ given by the recurrence relations similar to (1.6) but with certain (random) vector field $b(x, \vartheta^n \omega)$ in place of $B(x, \xi(n, \omega))$ in (1.6). We observe that though the former look slightly more general than the latter, we will still be able to rely on results of Section 3 and 4 since we will use there only the appropriate decay of the approximation coefficient $\rho$ and not any specific properties of the process $\xi$ except for its stationarity which is replaced by the assumption that $\vartheta$ preserves the probability $P$ on $\Omega$. Moreover, if $(\Omega, F, P)$ is a Lebesgue (standard probability) space then we can always represent $b(y, \vartheta^n \omega)$ in the form $b(y, \vartheta^n (n, \omega))$ where $\vartheta(n, \omega) = \xi(\vartheta^n \omega)$ is a real valued stationary process which fits into the discrete time setup of Sections 3 and 4.

The main application to dynamical systems we have here in mind is a $C^2$ Axiom A flow $F^t$ near an attractor which using Markov partitions can be represented as a suspension over an exponentially fast mixing transformation so that we can take $\xi(t) = f \circ F^t$ for a H"older continuous function $f$ and the probability $P$ to be a Gibbs invariant measure constructed by a H"older continuous potential on the base of the Markov partition (see, for instance, [8]). The space $\Omega$ above is the union of $\sigma$-algebras $F_{mn}$ are generated by cylinder sets.

Theorems 2.1 and 2.4 show that both $G^x_\varepsilon$ and $H^x_\varepsilon$ provide the same order of approximation of the slow motion $X^x_\varepsilon$ but depending on circumstances it may be preferable to deal with one or with the other. It is usually easier to study a specific Gaussian process than a general diffusion but, on the other hand, the former can be considered on a differential manifold only in local coordinates while the later can be written there globally in an invariant form. Since the differential equation (1.1) also can be considered on a manifold, we can obtain (2.19) in the form $E \sup_{0 \leq t \leq T} d^2 M(X^x_\varepsilon(t), H^x_\varepsilon(t/\tau)) \leq C_0(M)(\varepsilon^{M+2} + \varepsilon^{(3M-4)/2})$ where $d(\cdot, \cdot)$ is the distance on the manifold.

2.5. Remark. In order to know whether the Gaussian process $G$ and the diffusion $H^x$ are non degenerate, we have to know whether the matrix function $A(x)$ is positively definite for each $x$ which means that the inner product $\langle A(x) \lambda, \lambda \rangle$ is positive for any $x, \lambda \in \mathbb{R}^d$, $\lambda \neq 0$. Set $\gamma_{\lambda, x}(\xi) = \langle \lambda, \hat{B}(x, \xi) \rangle$. Then $\langle A(x) \lambda, \lambda \rangle$ is the limiting variance of the normalized sum $n^{-1/2} \sum_{k=0}^n \gamma_{\lambda, x}(\xi(k))$ of one dimensional random variables. It is known (see, for instance, Chapter 18 in [21]) that this variance is positive if and only if there exist no co-boundary representation $\gamma_{\lambda, x}(\xi(0, \omega)) = f(\vartheta \omega) - f(\omega)$ for some $L^2$ function $f$.

2.3. Almost sure approximation. Here we will restrict ourselves to the product case where $B(x, \xi) = \Sigma(x) \xi$ where $\Sigma(x)$ is a smooth $d \times d$ matrix function and $\xi \in \mathbb{R}^d$. We assume now (2.2), (2.3),

\begin{equation}
(2.23) \quad \sup_x ||\Sigma(x)||_{C^2} \leq L \quad \text{and} \quad \sup_n |\xi(n)| \leq L \text{ a.s.}
\end{equation}
Applying estimates of Lemma 3.5 to our situation we see that the limit
\[
\varsigma_{ij} = \lim_{n \to \infty} \frac{1}{n} \sum_{0 \leq k,l \leq n} E(\xi_i(k)\xi_j(l))
\]
exists and
\[
a_{ij}(x) = \sum_{k,l=1}^{d} \Sigma_{ik}(x)\varsigma_{kl}\Sigma_{lj}(x), \text{ i.e. } A(x) = \Sigma(x)\varsigma^*(x),
\]
and so \(\sigma(x) = \Sigma(x)\varsigma^{1/2}\).

2.6. **Theorem.** The stationary process \(\xi(n), -\infty < n < \infty\) can be redefined preserving its distributions on a richer probability space where there exists a Brownian motion \(W\) with the covariance matrix \(\varsigma\) (at the time 1) such that the slow motion \(X^\varsigma\) solving \(\{L\}\) with \(B(x,\xi) = \Sigma(x)\xi\) and the redefined process \(\xi(n)\) together with the Gaussian process \(G(t) = G_\varsigma(t)\) determined by \(\{L\}\) with \(W(t) = W_\varsigma(t)\) such that \(\varsigma^{1/2}W(t) = \sqrt{\mathcal{W}(t/\epsilon)}\) satisfy
\[
(2.24) \quad \sup_{0 \leq t \leq T} |\epsilon^{-1/2}(X^\varsigma_\epsilon(t) - \bar{X}_\varsigma(t)) - G_\varsigma(t)| = O(\epsilon^\delta) \quad (a.s.)
\]
for some \(\delta > 0\) which can be estimated from the proof.

2.4. **Law of iterated logarithm.** Here we continue working with the product case \(B(x,\xi) = \Sigma(x)\xi\). Let \(\mathcal{C}_d[0,T]\) be the Banach space of continuous vector functions \(\varphi = (\varphi_1,\ldots,\varphi_d)\) on the interval \([0,T]\) with the supremum norm \(\|\varphi\|_{[0,T]} = \max_{1 \leq k \leq d, 0 \leq s \leq T} |\varphi_k(t)|\). It is easy to see that for each \(\varphi \in \mathcal{C}_d[0,T]\) there exists the unique \(\Phi(\varphi) \in \mathcal{C}_d[0,T]\) such that
\[
(2.25) \quad \Phi(\varphi)(t) = \int_0^t \nabla \bar{B}(\bar{X}_\varsigma(s))\Phi(\varphi)(s)ds + \varphi(t)
\]
and the map \(\Phi : \mathcal{C}_d[0,T] \to \mathcal{C}_d[0,T]\) is continuous. Next, introduce another continuous map \(\Psi : \mathcal{C}_d[0,T] \to \mathcal{C}_d[0,T]\) defined by
\[
(2.26) \quad \Psi(\varphi)(t) = \sigma(\bar{X}_\varsigma(t))\varphi(t) - \int_0^t \nabla \sigma(\bar{X}_\varsigma(u))\bar{B}(\bar{X}_\varsigma(u))\varphi(u)du
\]
where for any smooth \(d \times d\) matrix function \(\sigma\) and a vector \(\eta = (\eta_1,\ldots,\eta_d)\) we denote by \(\nabla \sigma(y)\eta\) the \(d \times d\) matrix function with \((\nabla \sigma(y)\eta)_{ij} = \sum_{1 \leq k \leq d} \frac{\partial \sigma_{ik}(y)}{\partial y_k}\eta_k\). Let \(\mathcal{K}\) be the compact set of absolutely continuous vector functions \(\varphi \in \mathcal{C}_d[0,T]\) such that \(\int_0^T (\frac{d\varphi(s)}{ds})^2ds \leq 1\). We will derive the following Strassen's type law of iterated logarithm.

2.7. **Theorem.** Assume that (2.24) holds true and that \(B(x,\xi) = \Sigma(x)\xi\). Then with probability one the set of limit points in the supremum norm as \(\epsilon \to 0\) of random functions
\[
\frac{X^\varsigma_\epsilon(t) - \bar{X}_\varsigma(t)}{\sqrt{2\epsilon \log \log \frac{1}{\epsilon}}} \quad t \in [0,T]
\]
coincides with the compact set \(\Phi\Psi(\mathcal{K})\).
3. Auxiliary estimates

3.1. General lemmas. First, we will formulate three general results which will be used throughout this paper. The following lemma is well known (see, for instance, Lemma 1.3.10 in [20]).

3.1. Lemma. Let $G(x,\omega)$ and $H(x,\omega)$ be a measurable function on the space $(\mathbb{R}^d \times \Omega, \mathcal{B} \times \mathcal{F})$, where $\mathcal{B}$ is the Borel $\sigma$-algebra, such that for each $x \in \mathbb{R}^d$ the function $G(x,\cdot)$ is measurable with respect to a $\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ and $H(x,\cdot)$ is measurable with respect to another $\sigma$-algebra $\mathcal{H} \subset \mathcal{F}$. Then for any $x,y \in \mathbb{R}^d$,

\[(3.1) \quad |E(G(x,\cdot)H(y,\cdot)) - EG(x,\cdot)EH(y,\cdot)| \leq \|G(x,\cdot)\|_r \|H(y,\cdot)\|_q \varpi_{r,p}(\mathcal{H},\mathcal{G})\]

where $1 \leq p,q,r \leq \infty$, $\frac{1}{r} + \frac{1}{s} = 1$ and, as before, $\| \cdot \|_s$ is the $L^s$-norm of random variables.

Proof. By the Hölder inequality,

\[|E(G(x,\cdot)H(y,\cdot)) - EG(x,\cdot)EH(y,\cdot)| = |E((E(G(x,\cdot)|\mathcal{H}) - EG(x,\cdot))H(y,\cdot))|\]
\[\leq \|H(y,\cdot)\|_q \|E(G(x,\cdot)|\mathcal{H}) - EG(x,\cdot)\|_r \leq \|G(x,\cdot)\|_r \|H(y,\cdot)\|_q \varpi_{r,p}(\mathcal{H},\mathcal{G}),\]

as required. \hfill \Box

We will employ several times the following general moment estimate which appeared as Lemma 3.2.5 in [20] for random variables and was extended to random vectors in Lemma 3.4 from [29]. We observe that neither proof of Lemma 3.4 from [29] nor the proof of Lemma 3.2.5 in [20] rely on any weak dependence assumptions, and so the arguments there are still valid no matter whether the dependence conditions are expressed by the coefficient $\vartheta$ as in [20] and [29] or by the coefficient $\varpi$ as here.

3.2. Lemma. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}_j$, $j \geq 1$ be a filtration of $\sigma$-algebras and $\eta_j$, $j \geq 1$ be a sequence of random $d$-dimensional vectors such that $\eta_j$ is $\mathcal{G}_j$-measurable, $j = 1,2,\ldots$. Suppose that for some integer $M \geq 1$,

\[A_{2M} = \sup_{i \geq 1} \sum_{j \geq i} \|E(\eta_j|\mathcal{G}_i)\|_{2M} < \infty\]

where $\|\eta\|_p = (E|\eta|^p)^{1/p}$ and $|\eta|$ is the Euclidean norm of a (random) vector $\eta$. Then for any integer $n \geq 1$,

\[(3.2) \quad E|\sum_{j=1}^n \eta_j|^{2M} \leq 3(2M)!d^M A_{2M}^2 n^M.\]

In order to obtain uniform moment estimates required by Theorem 2.1 we will need the following general estimate which appeared as Lemma 3.7 in [29] where in the last inequality below we use also Lemma 3.2 above. We observe again that the latter lemma was not based on any dependence assumptions, and so its proof is valid in our circumstances, as well.

3.3. Lemma. Let $\eta_1, \eta_2,\ldots, \eta_N$ be random $d$-dimensional vectors and $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \ldots \subset \mathcal{H}_N$ be a filtration of $\sigma$-algebras such that $\eta_m$ is $\mathcal{H}_m$-measurable for each
\( m = 1, 2, ..., N \). Assume also that \( E|\eta_n|^{2M} < \infty \) for some \( M \geq 1 \) and each \( m = 1, ..., N \). Set \( S_m = \sum_{j=1}^{m} \eta_j \). Then

\[
E \max_{1 \leq m \leq N} |S_m|^{2M} \leq 2^{2M-1} (\frac{2M}{2M-1})^{2M} E|H_m|^{2M} + E \max_{1 \leq m \leq N-1} |\sum_{j=m+1}^{N} E(\eta_j|H_m)|^{2M} \leq 2^{2M-1} A_2^{2M} (3(2M)!d^M N^M + N).
\]

We will need also the following moment estimates for sums and iterated sums which appeared as Lemma 3.4 in [15] under the \( \vartheta \)-mixing condition and as Lemma 3.2 in [30] were it was proved under the same general dependence conditions as here.

3.4. Lemma. Let \( \eta(k), \zeta_k(l), k = 0, 1, 2, ..., l = 0, ..., k \) be two sequences of random variables on the probability space \((\Omega, \mathcal{F}, P)\) such that for all \( k, l, n \geq 0 \),

\[
\|\eta(k) - E(\eta(k)|\mathcal{F}_{k-n,k+n})\|_K, \|\zeta_k(l) - E(\zeta_k(l)|\mathcal{F}_{l-n,l+n})\|_K \leq \rho(K, n), \quad \|\eta(k)\|_K, \|\zeta_k(l)\|_K \leq \tilde{\gamma}_K < \infty \quad \text{and} \quad E\eta(k) = E\zeta(k) = 0
\]

where the \( \sigma \)-algebras \( \mathcal{F}_{kl} \) are the same as in Section 2. Then for any \( N, M \geq 1 \),

\[
E \max_{1 \leq n \leq N} \left( \sum_{k=0}^{n} \eta(k) \right)^{2M} \leq C_1^n(M) N^M
\]

and

\[
E \max_{1 \leq n \leq N} \left( \sum_{k=1}^{n} \eta(k) \sum_{l=0}^{k-1} \zeta_k(l) \right)^{2M} \leq C_1^n(\zeta)(M) N^{2M}
\]

where \( C_1(M) > 0 \) depends only on \( \rho, \vartheta, M, K \) and \( \tilde{\gamma}_K \) but it does not depend on \( N \) and on the sequences \( \eta(n), \zeta_k(n), n \geq 1 \) themselves.

3.2. Approximations. The following result shows that the definition (2.7) is legitimate and it estimates also the speed of convergence in this limit which will be needed for comparison of characteristic functions later on.

3.5. Lemma. For each \( x \in \mathbb{R}^d \) the limit (2.7) exists and for all \( m, n \geq 0 \) and \( i, j = 1, ..., d \),

\[
|n a_{ij}(x) - \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} a_{ij}(x, k, l)| \leq \tilde{L} = 4L \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (\rho(K, l/3) + L \varpi_{K,2M}(l/3)).
\]

Moreover, \( a_{ij}(x) \) is twice differentiable for \( i, j = 1, ..., d \) and all \( x \in \mathbb{R}^d \),

\[
\|(a_{ij})|_{C^2} \leq \tilde{L} = 4Ld^2 \sum_{l=0}^{\infty} (\rho(K, l/3) + L \varpi_{K,2M}(l/3)) \quad \text{and} \quad \sup_x |\sigma(x)| \leq \sqrt{Ld}
\]

where \( \sigma(x) \) is the Lipschitz continuous square root of the matrix \( A(x) = (a_{ij}(x)) \) which exists in view of [13] and [37]. Furthermore, for all \( 0 \leq s < t \leq T \) and \( \varepsilon > 0 \),

\[
|\varepsilon^{-1} \int_s^t a_{ij}(\tilde{X}_x(u))du - \sum_{s/\varepsilon \leq k,l \leq t/\varepsilon} a_{ij}(\tilde{X}_x(k\varepsilon), \tilde{X}_x(l\varepsilon), k, l)| \leq C_2(T) \varepsilon^{-2/3}
\]

where \( C_2(T) > 0 \) does not depend on \( \varepsilon \).
Proof. By Lemma 3.1 for any \( m, n, x, y \in \mathbb{R}^d \) and \( i, j = 1, \ldots, d \),
\begin{equation}
|E(\hat{B}_i(x, \xi(m)), \hat{B}_j(y, \xi(n)))| \leq 4L\rho(K, \frac{1}{3}|m - n|)
\end{equation}
\begin{align*}
+ & E(E(\hat{B}_i(x, \xi(m)))|F_{m - \frac{1}{3}|m - n|, m + \frac{1}{3}|m - n|}) \\
\times & E(\hat{B}_j(y, \xi(n)))|F_{n - \frac{1}{3}|m - n|, n + \frac{1}{3}|m - n|})|
\leq 4L(\rho(K, \frac{1}{3}|m - n|) + L\rho_{K, 2M}(\frac{1}{3}|m - n|)).
\end{align*}
Hence, by the stationarity of the process \( \xi \),
\begin{equation}
\lim_{n \to \infty} \sum_{l=0}^{n-1} a_{ij}(x, k, l) = \sum_{l=0}^{\infty} a_{ij}(x, k, l) + \sum_{l=0}^{k} a_{ij}(x, k, l)
\end{equation}
and so
\begin{equation}
a_{ij}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \sum_{l=0}^{\infty} a_{ij}(x, k, l) = \sum_{k=0}^{\infty} a_{ij}(x, k, 0) + \sum_{l=0}^{\infty} a_{ij}(x, 0, l)
\end{equation}
where all limits exist and infinite sums converge in view of (2.6) and (3.9). Thus, by (3.9),
\begin{equation}
|a_{ij}(x) - \frac{1}{n} \sum_{k=0}^{n-1} \sum_{l=0}^{\infty} a_{ij}(x, k, l)| = |\frac{1}{n} \sum_{k=0}^{n-1} \sum_{m=k+1}^{\infty} a_{ij}(x, m, 0)
\end{equation}
\begin{equation}
+ \frac{1}{n} \sum_{k=0}^{n-1} \sum_{l=n-k}^{\infty} a_{ij}(x, 0, l)| \leq L/n^3
\end{equation}
which gives (3.6).
Next, \( |\Delta_k|^{-1}|\hat{B}(x + \Delta_k, \xi) - \hat{B}(x, \xi)| \leq 2L \) which together with the dominated convergence theorem gives that
\begin{equation}
0 = \frac{\partial}{\partial x_k} \hat{B}(x, \xi) = E \frac{\partial}{\partial x_k} \hat{B}(x, \xi).
\end{equation}
Similarly,
\begin{equation}
E \frac{\partial^2}{\partial x_k \partial x_l} \hat{B}(x, \xi) = 0.
\end{equation}
This together with (2.5) enables us to argue in the same way as above to conclude that
\begin{equation}
\frac{\partial a_{ij}(x)}{\partial x_k} = \sum_{m=1}^{\infty} \frac{\partial a_{ij}(x, m, 0)}{\partial x_k} + \sum_{m=0}^{\infty} \frac{\partial a_{ij}(x, 0, m)}{\partial x_k}
\end{equation}
and
\begin{equation}
\frac{\partial^2 a_{ij}(x)}{\partial x_k \partial x_l} = \sum_{m=1}^{\infty} \frac{\partial^2 a_{ij}(x, m, 0)}{\partial x_k \partial x_l} + \sum_{m=0}^{\infty} \frac{\partial^2 a_{ij}(x, 0, m)}{\partial x_k \partial x_l}
\end{equation}
and these series converge absolutely since similarly to (3.6) we see that
\begin{equation}
\max(|\frac{\partial a_{ij}(x, m, 0)}{\partial x_k}|, |\frac{\partial^2 a_{ij}(x, m, 0)}{\partial x_k \partial x_l}|) \leq 4L(\rho(K, m/3) + L\rho_{K, 2M}(m/3)).
\end{equation}
Observe that for any vector \( y \in \mathbb{R}^d \),
\begin{equation}
\tilde{L}d|u|^2 \geq \langle A(x)y, y \rangle = \langle \sigma(x)\sigma(x), y \rangle = |\sigma(x)y|^2,
\end{equation}
and so \( |\sigma(x)| \leq \sqrt{\tilde{L}d} \) completing the proof of (3.7).
Next, by (3.9) and the stationarity of the process $\xi$, for any $0 \leq s < t \leq T$ and an integer $n \geq 1$,

$$\left| \sum_{s/x \leq k \leq t/x} a_{ij}(\hat{X}_x(k\varepsilon), \hat{X}_x(l\varepsilon), k, l) \right| \leq 8L\sum_{m=0}^{\infty} \sum_{k=m}^{\infty} \left( \rho(K, k/3) + L\sigma_{K,2M}(k/3) \right) = 2L\varepsilon.$$

By (3.3) we have also

$$\left| \sum_{s/x \leq k \leq t/x} a_{ij}(\hat{X}_x(k\varepsilon), \hat{X}_x(l\varepsilon), k, l) \right| \leq 2L^3(\frac{t-s}{n})^3 \varepsilon^{-2}.$$

Now, by (3.7),

$$\left| \int_0^t a_{ij}(\hat{X}_x(u))du - \frac{t-s}{n} \sum_{m=0}^{n-1} a_{ij}(\hat{X}_x(s + m\frac{t-s}{n})) \right| \leq 8Ld^2 n(\frac{t-s}{n})^2 \sum_{l=0}^{\infty} \left( \rho(K, l/3) + L\sigma_{K,2M}(l/3) \right).$$

Finally, by (3.6) and the stationarity of the process $\xi$,

$$\left| \varepsilon^{-1}\frac{t-s}{n} a_{ij}(\hat{X}_x(s + m\frac{t-s}{n}) \right| \leq \hat{L}.$$

Combining four last inequalities we derive (3.8) taking $n = [\varepsilon^{-2}/3]$. 

The first step in the proof of estimates of Theorems 2.1 and 2.4 is the following result which is similar to Lemma 3.1 in [20].

3.6. Lemma. For all $\varepsilon, T > 0$, $M \geq 1$ and $x \in \mathbb{R}^d$,

(3.10) \begin{equation}
\sup_{0 \leq t \leq T} |X^\varepsilon_x(t) - H^\varepsilon_x(t)| \leq e^{LT} \left( \sqrt{\varepsilon} \sup_{0 \leq t \leq T} |\varepsilon S^\varepsilon(t) - H^\varepsilon_x(t)| \right) + \int_0^t \sigma(\hat{X}_x(s))dW(s) + \sum_{i=1}^{\frac{5}{T}} \sup_{0 \leq t \leq T} |R^\varepsilon_i(t)|,
\end{equation}

and

(3.11) \begin{equation}
\sup_{0 \leq t \leq T} |H^\varepsilon_x(t) - \hat{H}^\varepsilon_x(t)| \leq e^{LTd} \sum_{i=1}^{\frac{5}{T}} \sup_{0 \leq t \leq T} |R^\varepsilon_i(t)|,
\end{equation}

where $\sigma^2(x) = A(x)$, the stochastic integral is meant in the Itô sense and

$$S^\varepsilon(t) = \sum_{0 \leq k < \lfloor t/\varepsilon \rfloor} \left( B(\hat{X}_x(k\varepsilon), \xi(k)) - \hat{B}(\hat{X}_x(k\varepsilon)) \right),$$

$$R^\varepsilon_i(t) = \int_0^t \left( \nabla B(\hat{X}_x([s/\varepsilon]\varepsilon), \xi([s/\varepsilon])) - \nabla \hat{B}(\hat{X}_x([s/\varepsilon]\varepsilon)) \right) (X^\varepsilon_x(s) - \hat{X}_x([s/\varepsilon]\varepsilon))ds,$$

$$R^\varepsilon_2(t) = \int_0^t \left( C(\hat{X}_x([s/\varepsilon]\varepsilon) + \theta(\hat{X}_x([s/\varepsilon]\varepsilon))X^\varepsilon_x(s) - \hat{X}_x([s/\varepsilon]\varepsilon)) - \hat{X}_x([s/\varepsilon]\varepsilon) \right) ds,$$

$$R^\varepsilon_3(t) = \int_0^t \left( D(\hat{X}_x([s/\varepsilon]\varepsilon) + \theta(\hat{X}_x([s/\varepsilon]\varepsilon))X^\varepsilon_x(s) - \hat{X}_x([s/\varepsilon]\varepsilon)) - \hat{X}_x([s/\varepsilon]\varepsilon) \right) ds,$$

$$R^\varepsilon_4(t) = \sqrt{\varepsilon} \int_0^t \left( \sigma(H^\varepsilon_x(s)) - \sigma(\hat{X}_x(s)) \right)dW(s).$$
\[ R^\varepsilon(t) = \int_0^t (D(\tilde{X}_x(s) + \tilde{\theta}(\varepsilon^x(s) - \tilde{X}_x(s))))(H^\varepsilon_x(s) - \dot{X}_x(s)), (H^\varepsilon_x(s) - \dot{X}_x(s))) ds \]

with \( \nabla B(z, \eta) = (\frac{\partial B(z)}{\partial z_1}, \frac{\partial B(z)}{\partial z_2}) \), \( \langle \cdot, \cdot \rangle \) denoting the inner product in \( \mathbb{R}^d \), \( C(z, \eta) = \nabla^2_B B(z, \eta) = (\nabla^2_B B_i(z, \eta)) \), and \( D(z) = \nabla^2_B B(z) \).

Proof. By the Taylor formula,

\[
X^\varepsilon_x(t) = x + \int_0^{[t/\varepsilon] \varepsilon} B(X^\varepsilon_x(s), \xi([s/\varepsilon])) ds \]

\[ = x + \int_0^{[t/\varepsilon] \varepsilon} B(\tilde{X}_x([s/\varepsilon]) \varepsilon), \xi([s/\varepsilon])) ds \]

\[ + \int_0^{[t/\varepsilon] \varepsilon} \nabla B(\tilde{X}_x([s/\varepsilon]) \varepsilon), \xi([s/\varepsilon]))(X^\varepsilon_x(s) - \tilde{X}_x([s/\varepsilon]) \varepsilon) ds + R^\varepsilon_2([t/\varepsilon] \varepsilon) \]

\[ = x + \int_0^{[t/\varepsilon] \varepsilon} B(\tilde{X}_x([s/\varepsilon]) \varepsilon) ds + \int_0^{[t/\varepsilon] \varepsilon} \nabla B(\tilde{X}_x([s/\varepsilon]) \varepsilon)(X^\varepsilon_x(s) - \tilde{X}_x([s/\varepsilon]) \varepsilon) ds + R^\varepsilon_2([t/\varepsilon] \varepsilon) \]

\[ - \tilde{X}_x([s/\varepsilon]) \varepsilon) ds + \varepsilon S^x(t) + R^\varepsilon_1([t/\varepsilon] \varepsilon) + R^\varepsilon_2([t/\varepsilon] \varepsilon) = x + \int_0^t \dot{B}(X^\varepsilon_x(s)) ds \]

\[ - \int_0^{[t/\varepsilon] \varepsilon} \dot{B}(X^\varepsilon_x(s)) ds + \varepsilon S^x(t) + R^\varepsilon_1([t/\varepsilon] \varepsilon) + R^\varepsilon_2([t/\varepsilon] \varepsilon) - R^\varepsilon_3([t/\varepsilon] \varepsilon) \]

and

\[
H^\varepsilon_x(t) = x + \int_0^t \dot{B}(H^\varepsilon_x(s)) ds + \sqrt{\varepsilon} \int_0^t \sigma(\tilde{X}_x(s)) dW(s) + R^\varepsilon_4(t) = x + \int_0^t \dot{B}(\tilde{X}_x(s)) ds \]

\[ + \int_0^t \nabla \dot{B}(\tilde{X}_x(s))(H^\varepsilon_x(s) - \dot{X}_x(s)) ds + \sqrt{\varepsilon} \int_0^t \sigma(\tilde{X}_x(s)) dW(s) + R^\varepsilon_4(t) + R^\varepsilon_6(t). \]

Hence, by (3.3) and (3.13),

\[
\sup_{0 \leq t \leq s} |X^\varepsilon_x(s) - H^\varepsilon_x(s)| \leq L \int_0^t \sup_{0 \leq u \leq s} |X^\varepsilon_x(u) - H^\varepsilon_x(u)| ds \]

\[ + \sqrt{\varepsilon} \int_0^t |\sqrt{\varepsilon} S^x(t) - \int_0^t \sigma(\tilde{X}_x(s)) dW(s)| + L \varepsilon + \sum_{i=0}^4 \sup_{0 \leq t \leq T} |R^\varepsilon_i(t)|. \]

Employing here the Gronwall inequality, we derive (3.10). Similarly, by (1.4) and the above Taylor representation of \( H^\varepsilon_x(t) \),

\[
\sup_{0 \leq t \leq s} |H^\varepsilon_x(s) - \tilde{X}_x(s) - \sqrt{\varepsilon} G(s)| \]

\[ \leq L d \int_0^t \sup_{0 \leq u \leq s} |H^\varepsilon_x(u) - \tilde{X}_x(u) - \sqrt{\varepsilon} G(u)| ds + \sup_{0 \leq t \leq T} (|R^\varepsilon_4(t)| + |R^\varepsilon_6(t)|). \]

Again, employing the Gronwall inequality we obtain (3.11).

Next, we estimate directly \( R^\varepsilon_i \), \( i = 1, 2, \ldots, 5 \) while the more significant first summand in the right hand side of (3.10) will be treated in the next section by the strong approximations machinery.

### 3.7. Lemma

(i) For any \( T > 0 \) and an integer \( M > 0 \), there exists a constant \( C_T(M) > 0 \) such that for all \( \varepsilon > 0 \),

\[
E \sup_{0 \leq t \leq T} |R^\varepsilon_1(t)|^{2M} \leq C_T(M) \varepsilon^{2M} \]

and

\[
E \sup_{0 \leq t \leq T} |R^\varepsilon_2(t)|^{2M} \leq \tilde{C}_T(M) \varepsilon^{2M} \quad \text{for } i = 2, 3. \]

where \( \tilde{C}_T(M) = C_T(M) L^{2M} d^{3M} T^{3M} e^{2M d T} \).

(ii) Suppose that \( \sigma \) in (1.3) and (1.4) satisfies for all \( x, y \in \mathbb{R}^d \),

\[
|\sigma(x)| \leq C_\sigma \quad \text{and } |\sigma(x) - \sigma(y)| \leq C_\sigma |x - y| \]
where $C_\sigma > 0$ is a constant and $\| \cdot \|$ is the Euclidean norm for vectors or matrices. Then

\begin{align}
\tag{3.16}
E \sup_{0 \leq t \leq T} |R_n^x(t)|^{2M} & \leq 2^{4M} \varepsilon^{2MT} C_\sigma^{4M} T^{2M} M^{6M} (2M - 1)^{-2 \varepsilon^{2M}} \\
\tag{3.17}
\text{and} \\
E \sup_{0 \leq t \leq T} |R_n^x(t)|^{2M} & \leq 4^{5M} L^{2M} C_\sigma^{4M} T^{4M-1} d^{6M} \varepsilon^{4MT} M^{6M} (4M - 1)^{-2 \varepsilon^{2M}}.
\end{align}

\textbf{Proof.} \ (i) Set

\[ \hat{X}_n^x(n\varepsilon) = x + \varepsilon \sum_{k=0}^{n-1} B(\hat{X}_n^x(k\varepsilon)) = x + \int_0^{n\varepsilon} \hat{B}(\hat{X}_n^x([s/\varepsilon]\varepsilon))ds. \]

Then by (2.12), (1.3) and (2.3) for each $n \leq T/\varepsilon$,

\[ |\hat{X}_n^x(n\varepsilon) - \hat{X}_n^x(n\varepsilon)| \leq \int_0^{n\varepsilon} \hat{B}(\hat{X}_n^x(s))ds \leq L \int_0^{n\varepsilon} |\hat{B}(\hat{X}_n^x(s))|ds \leq L^2 T \varepsilon. \]

Again, by (2.3),

\[ |X_n^x(n\varepsilon) - \hat{X}_n^x(n\varepsilon)| \leq \varepsilon \sum_{k=0}^{n-1} |B(\hat{X}_n^x(k\varepsilon)) - B(\hat{X}_n^x(k\varepsilon), \xi(k))| + \varepsilon \sum_{k=0}^{n-1} |B(\hat{X}_n^x(k\varepsilon), \xi(k)) - B(\hat{X}_n^x(k\varepsilon), \xi(k))| + \varepsilon \sum_{k=0}^{n-1} |B(\hat{X}_n^x(k\varepsilon), \xi(k)) - \hat{X}_n^x(k\varepsilon)| + L^2 T \varepsilon \]

since $\hat{B}(x) = EB(x, \xi(k))$.

By the discrete time Gronwall inequality (see [10]) it follows that for all $n \leq T/\varepsilon$,

\[ |X_n^x(n\varepsilon) - \hat{X}_n^x(n\varepsilon)| \leq \varepsilon L^2 T \varepsilon. \]

Next, we apply Lemma B.3 with

\[ \eta_k = B_i(\hat{X}_n^x(k\varepsilon), \xi(k)) - EB_i(\hat{X}_n^x(k\varepsilon), \xi(k)), \quad i = 1, 2, ..., d \]

and $\hat{L} = 2L$ to obtain that

\[ E \max_{0 \leq nT/e} |X_n^x(n\varepsilon) - \hat{X}_n^x(n\varepsilon)|^{2M} \leq 2^{2M-1} \varepsilon^{2MT} d^{2M} (L^6 T^{4M} \varepsilon^{2M} + C_1(M) T^M \varepsilon^M). \]

Combining this inequality with the above estimate for $|\hat{X}_n^x(n\varepsilon) - \hat{X}_n^x(n\varepsilon)|$ and taking into account that by (1.1), (1.3) and (2.3),

\[ \sup_{0 \leq s \leq T} \sup_{0 \leq u \leq \varepsilon} |X_n^x(s + u) - X_n^x(s)| \leq L \varepsilon \text{ and } \sup_{0 \leq s \leq T} \sup_{0 \leq u \leq \varepsilon} |\hat{X}_n^x(s) - \hat{X}_n^x(s)| \leq L \varepsilon \]

we obtain that

\[ \tag{3.18} E \sup_{0 \leq t \leq T} |X_n^x(t) - \hat{X}_n^x(t)|^{2M} \leq \tilde{C}_T(M) \varepsilon^M \]

where $\tilde{C}_T(M) > 0$ does not depend on $\varepsilon$. Applying (3.18) with $2M$ in place of $M$ and taking into account (2.3) we obtain that for $i = 2, 3$,

\[ E \sup_{0 \leq t \leq T} |R_n^x(t)|^{2M} \leq L^{2M} T^{2M} E \max_{0 \leq n \leq T/\varepsilon} |X_n^x(n\varepsilon) - \hat{X}_n^x(n\varepsilon)|^{4M} \leq \tilde{C}_T(2M) \varepsilon^{2M} \]

yielding (3.14).
In order to estimate \( R_1^\varepsilon \) we introduce the process \( Z^\varepsilon \) defined recursively for \( n \geq 1 \)
by
\[
Z^\varepsilon(n\varepsilon) = \varepsilon S^\varepsilon(n\varepsilon) + \varepsilon \sum_{k=0}^{n-1} \nabla B(\tilde{X}_x(k\varepsilon)) Z^\varepsilon(k\varepsilon)
\]
where
\[
Z^\varepsilon(0) = 0,
\]
with \( Z^\varepsilon(0) = 0 \), where \( I \) is the \( d \times d \) identity matrix and \( \nabla B(x) \) is the matrix \( (\partial B_i(x)/\partial x_j) \).

This follows that
\[
Z^\varepsilon(n\varepsilon) = \varepsilon \sum_{k=1}^{n} K^\varepsilon(n,k) (S^\varepsilon(k\varepsilon) - S^\varepsilon((k-1)\varepsilon))
\]
where \( K^\varepsilon(n,n) = I \) and for \( k < n \),
\[
K^\varepsilon(n,k) = \prod_{l=1}^{n-k} (I + \varepsilon \nabla B(\tilde{X}_x((n-l)\varepsilon)))
\]
where we multiply the matrices from the right.

Next, we consider the difference \( U^\varepsilon(n\varepsilon) = X^\varepsilon_x(n\varepsilon) - \tilde{X}_x(n\varepsilon) - Z^\varepsilon(n\varepsilon) \) which by \[(3.12)\text{ and } (3.19)\] satisfies
\[
U^\varepsilon(n\varepsilon) - \varepsilon \sum_{k=0}^{n-1} \nabla B(\tilde{X}_x(k\varepsilon), \xi(k)) U^\varepsilon(k\varepsilon) = V^\varepsilon(n\varepsilon) + R_2^\varepsilon(n\varepsilon)
\]
where
\[
V^\varepsilon(n\varepsilon) = \varepsilon \sum_{k=0}^{n-1} (\nabla B(\tilde{X}_x(k\varepsilon), \xi(k)) - \nabla B(\tilde{X}_x(k\varepsilon))) Z^\varepsilon(k\varepsilon).
\]

By \[(2.3)\] and the discrete time Gronwall inequality (see \[(10)\]),
\[
|U^\varepsilon(n\varepsilon)| \leq e^{LT} (|V^\varepsilon(n\varepsilon)| + |R_2^\varepsilon(n\varepsilon)|).
\]

Hence, we obtain
\[
|R_1^\varepsilon(n\varepsilon)| \leq \varepsilon \sum_{k=0}^{n-1} |\nabla B(\tilde{X}_x(k\varepsilon), \xi(k)) - \nabla B(\tilde{X}_x(k\varepsilon))) (U^\varepsilon(k\varepsilon) + Z^\varepsilon(k\varepsilon))| \leq 2\varepsilon \sum_{k=0}^{n-1} |U^\varepsilon(k\varepsilon)| + \varepsilon \sum_{k=0}^{n-1} |V^\varepsilon(k\varepsilon)|
\]
\[
\leq 2\varepsilon (e^{LT} + 1) \sum_{k=0}^{n-1} |V^\varepsilon(k\varepsilon)| + 2\varepsilon e^{LT} \sum_{k=0}^{n-1} |R_2^\varepsilon(k\varepsilon)|.
\]

It remains to estimate \( |V^\varepsilon(k\varepsilon)| \) since the estimate \[(3.14)\] of \( |R_2^\varepsilon(k\varepsilon)| \) was already established above. Set
\[
\eta(k) = \nabla B(\tilde{X}_x(k\varepsilon), \xi(k)) - \nabla B(\tilde{X}_x(k\varepsilon)), \quad \text{and} \quad \zeta_l(k) = K^\varepsilon(k,l) (B(\tilde{X}_x(l\varepsilon), \xi(l)) - \tilde{B}(X_x(l\varepsilon))).
\]

Then, by \[(3.20)\] and the definition of \( V^\varepsilon \),
\[
V^\varepsilon(n\varepsilon) = \varepsilon^2 J_1(n) + \varepsilon^2 J_2(n) + \varepsilon^2 J_3(n)
\]
where
\[
J_1(n) = (\sum_{k=0}^{n-1} \eta(k))(B(x, \xi(0)) - \tilde{B}(x)), \quad J_2(n) = \sum_{k=0}^{n-1} \eta(k)\sum_{l=0}^{k-1} \zeta_l(k) \text{ and } J_3(n) = \sum_{k=0}^{n-1} \eta(k)\zeta_k(k).
\]

By \[(2.3)\],
\[
|J_1(n)| \leq 4L^2n \leq 4L^2T\varepsilon^{-1}.
\]
Next, we estimate \(|J_2(n)|\) by Lemma 3.4 taking into account that by (2.3) for 
\(k \leq T/\varepsilon\),
\[|K^\varepsilon(k, l)| \leq (1 + Ld\varepsilon)^{k-1} \leq (1 + Ld\varepsilon)^{T/\varepsilon} \leq e^{LTd}\]
where we take the matrix norm \(\|\alpha_{ij}\| = \max_j \sum_i |\alpha_{ij}|\). Hence,
\[|\eta(k)| \leq 2L \text{ and } |\xi_k(l)| \leq 2Le^{LTd}.
\]
Since \(E\eta(k) = E\zeta_k(l) = 0\) and the approximation by conditional expectations condition of Lemma 3.4 is satisfied with the coefficient \(e^{LTd}\rho(K, n)\) we can apply Lemma 3.4 to obtain that
\[E \max_{0 \leq n \leq T/\varepsilon} |J_2(n)|^2M \leq \tilde{C}_1(M)e^{-2M}\]
where \(\tilde{C}_1(M) > 0\) depends only on \(M, L, T, d\) and \(\rho\) but it does not depend on \(\varepsilon\).
Here, we rely also on the assertion of Lemma 3.4 that the constant in (3.5) does not depend on the sequences \(\eta\) and \(\zeta\) themselves since in our case these sequences depend on \(\varepsilon\). Finally, by the above estimates of \(\eta\) and \(\zeta\),
\[|J_3(n)| \leq 4L^2e^{LTd}n \leq 4L^2Te^{LTd} \varepsilon^{-1}.
\]
It follows that
\[E \max_{0 \leq n \leq T/\varepsilon} |V^{\varepsilon(n)}|^2M \leq \varepsilon^{2M}S^{2M-1}(4^{2M}L^{4M}T^{2M}(1 + e^{LTd}) + \tilde{C}_1(M)).\]
This together with (3.14) and (3.21) yields (3.13) completing the proof of (i).

(ii) By (3.15) and the standard uniform martingale moment estimates of stochastic integrals (see, for instance, [22] and [32]),
(3.22)
\[E \sup_{0 \leq t \leq T} |R^\varepsilon_3(t)|^2M \leq \varepsilon^{2M}S^{2M-1}(4^{2M}L^{4M}T^{2M}(1 + e^{LTd}) + \tilde{C}_1(M)).\]
By (1.3), (1.5) and (2.3),
\[|H^\varepsilon_x(s) - \bar{X}_x(s)| \leq LJ_0^n \int_0^s |H^\varepsilon_x(u) - \bar{X}_x(u)| du + L \sqrt{\varepsilon} \sup_{0 \leq u \leq s} \int_0^u \sigma(H^\varepsilon_x(v))dW(v),\]
and so by Gronwall’s inequality,
(3.23)
\[|H^\varepsilon_x(s) - \bar{X}_x(s)| \leq e^{L_0} \sqrt{\varepsilon} \sup_{0 \leq u \leq s} \int_0^u \sigma(H^\varepsilon_x(v))dW(v).\]
Taking \(2M\)-th power of both sides of this inequality, applying the expectation, using the martingale moment inequalities as above and substituting the result into the right hand side of (3.22), we arrive at (3.10).

Finally, by (2.3) and the Cauchy-Schwarz inequality,
\[E \sup_{0 \leq t \leq T} |R^\varepsilon_3(t)|^2M \leq (Ld)^{2M}E(\int_0^T |H^\varepsilon_x(s) - \bar{X}_x(s)|^2 ds)^{2M} \leq (Ld)^{2M}T^{2M-1} \int_0^T E|H^\varepsilon_x(s) - \bar{X}_x(s)|^{4M} ds.
\]
Relying on (3.23) together with the martingale moment estimates of stochastic integrals we derive (3.17) completing the proof of the lemma. □
3.3. Characteristic functions estimates. For any $0 \leq s < t \leq T$ and $\epsilon > 0$ introduce the characteristic function
\[
f^\epsilon_{s,t}(w) = f^\epsilon_{s,t}(x, w) = E \exp(i \langle w, \epsilon^{1/2}(S^x(t) - S^x(s)) \rangle)
= E \exp(i \langle w, \epsilon^{1/2} \sum_{|s/\epsilon| \leq k < t/\epsilon} \hat{B}(\bar{X}_x(k\epsilon), \xi(k)) \rangle), \quad w \in \mathbb{R}^d
\]
where $\langle \cdot, \cdot \rangle$ denotes the inner product and, recall, $\hat{B}(y, \xi(k)) = B(y, \xi(k)) - \hat{B}(y)$. The first step in the strong approximations machinery is an estimate of the corresponding characteristic function which in our situation amounts to the following.

3.8. Lemma. For any $0 \leq s < t \leq T$, $\epsilon > 0$ and $x \in \mathbb{R}^d$,
\[
(3.24) \quad |f^\epsilon_{s,t}(x, w) - \exp(-\frac{1}{2}(\int_s^t A(\bar{X}_x(u))du)w, w))| \leq C_2(T)\epsilon^\varphi
\]
for all $w \in \mathbb{R}^d$ with $|w| \leq \epsilon^{-\varphi/2}$ where we can take $\varphi \leq \frac{1}{30}$ and a constant $C_2(T) > 0$ does not depend on $s, t$ and $\epsilon$.

Proof. Set $n = n_x(s, t) = \lfloor (t-s)\epsilon^{-1} \rfloor$. The left hand side of (3.24) does not exceed 2 and for $n < 16$ we estimate it by $2(16)^{-\nu} n^{-\nu}$ which is at least 2. So, in what follows, we will assume that $n \geq 16$, so that $n^{1/4} \geq 2$. Set $\nu(n) = \lfloor n(n^{1/4} + n^{1/4})^{-1} \rfloor$, $q_k(n) = s/\epsilon + k(n^{1/4} + n^{1/4})$, $r_k(n) = q_{k-1}(n) + n^{1/4}$ for $k = 1, 2, ..., \nu(n)$ with $q_0(n) = s/\epsilon$. Next, we introduce for $k = 1, ..., \nu(n)$,
\[
y_k = y_k(n) = \sum_{q_{k-1}(n) \leq l < r_k(n)} \hat{B}(\bar{X}_x(l\epsilon), \xi(l)), \quad z_k = z_k(n)
= \sum_{r_k(n) \leq l < q_k(n)} \hat{B}(\bar{X}_x(l\epsilon), \xi(l)) \quad \text{and} \quad z_{\nu(n)+1} = \sum_{q_{\nu(n)} \leq l < \epsilon/\epsilon} \hat{B}(\bar{X}_x(l\epsilon), \xi(l)).
\]
Then by Lemma 3.4
\[
(3.25) \quad E|\sum_{1 \leq k \leq \nu(n)+1} z_k|^2 \leq 2\nu(n) \sum_{1 \leq k \leq \nu(n)} E|z_k|^2 + 2E|z_{\nu(n)+1}|^2
\leq 2C_1(1)(\nu(n))^2 n^{1/4} + n^{3/4} + n^{1/4} \leq 6C_1(1)n^{3/4}.
\]
This together with the Cauchy-Schwarz inequality yields,
\[
(3.26) \quad |f^\epsilon_{s,t}(x, w) - E \exp(i \langle w, n^{-1/2} \sum_{1 \leq k \leq \nu(n)} y_k(n) \rangle)|
\leq E|\exp(i \langle w, n^{-1/2} \sum_{1 \leq k \leq \nu(n)+1} z_k \rangle) - 1| \leq n^{-1/2} E|\langle w, \sum_{1 \leq k \leq \nu(n)+1} z_k \rangle|
\leq n^{-1/2} |w| E|\sum_{1 \leq k \leq \nu(n)+1} z_k| \leq \sqrt{6C_1(1)}|w| n^{-1/8}
\]
where we use that for any real $a, b$,
\[
|e^{i(a+b)} - e^{ib}| = |e^{ia} - 1| \leq |a|.
\]
We will obtain (3.24) from (3.26) by estimating
\[
(3.27) \quad |E \exp(i \sum_{1 \leq k \leq \nu(n)} \eta_k) - \exp(-\frac{1}{2}(\int_s^t A(\bar{X}_x(u))du)w, w))| \leq I_1 + I_2
\]
where
\[
\eta_k = \langle w, \sqrt{\epsilon} y_k \rangle, \quad I_1 = |E \exp(i \sum_{1 \leq k \leq \nu(n)} \eta_k) - \prod_{1 \leq k \leq \nu(n)} E e^{i\eta_k}|
\quad \text{and} \quad I_2 = |\prod_{1 \leq k \leq \nu(n)} E e^{i\eta_k} - \exp(-\frac{1}{2}(\int_s^t A(\bar{X}_x(u))du)w, w))|.
\]
First, we write

\[
I_1 \leq \sum_{m=2}^{\nu(n)} \left| \prod_{m+1 \leq k \leq \nu(n)} E e^{i\eta_k} \right| \times |E \exp(i \sum_{1 \leq k \leq m} \eta_k) - E \exp(i \sum_{1 \leq k \leq m-1} \eta_k) e^{i\eta_m}| \leq \sum_{m=2}^{\nu(n)} |E \exp(i \sum_{1 \leq k \leq m} \eta_k) - E \exp(i \sum_{1 \leq k \leq m-1} \eta_k) e^{i\eta_m}| \]

where \( \prod_{\nu(n)+1 \leq k \leq \nu(n)} = 1 \). Next, using the approximation coefficient \( \rho \) and the inequality \(|e^{ia} - e^{ib}| \leq |a - b|\), valid for any real \( a \) and \( b \), we obtain

\[
E|e^{i\eta_m} - \exp(iE(\eta_m|F_{q_{m-1}(n)-n^{1/4}/3,\infty}))| \leq 2\sqrt{\pi n^{3/4}} |w| \rho(K,n^{1/4}/3) \]

and

\[
E|\exp(i \sum_{1 \leq k \leq m-1} \eta_k) - \exp(iE(\sum_{1 \leq k \leq m-1} \eta_k|F_{-\infty,r_{m-1}(n)+n^{1/4}/3})| \leq 2\sqrt{\pi n^{3/4}} |w|(m-1)\rho(K,n^{1/4}/3). \]

Hence, by (3.29), (3.30) and Lemma 3.1

\[
\begin{align*}
I_1 & \leq n^{1/4}(\varpi_{K,2M}(n^{1/4}/3) + 8\sqrt{\pi n}|w| \rho(K,n^{1/4}/3)). \\
I_2 & \leq \sum_{1 \leq k \leq \nu(n)} |E e^{i\eta_k} - \exp(-\frac{1}{2} \langle \int_{q_{k-1}(n)} A(\tilde{X}_x(u)) du, w, w \rangle)| \\
& \leq \frac{\epsilon \eta^2}{2} \sum_{1 \leq k \leq \nu(n)} |E \eta_k^2 - \langle \int_{q_{k-1}(n)} A(\tilde{X}_x(u)) du, w, w \rangle| + \sum_{1 \leq k \leq \nu(n)} (E \eta_k^3 + \frac{\epsilon \eta^3}{4} \langle \int_{q_{k-1}(n)} A(\tilde{X}_x(u)) du, w, w \rangle^2)
\end{align*}
\]

where we use (1.2) and that for any real \( a \),

\[ |e^{ia} - 1 - ia + \frac{a^2}{2} | \leq |a|^3 \text{ and } |e^{-a} - 1 + a| \leq a^2 \text{ if } a \geq 0. \]

Now,

\[
E \eta_k^2 = \epsilon \langle \sum_{i=1}^{d} w_i \sum_{q_{k-1}(n) \leq l < q_k(n)} \beta_i(\tilde{X}_x(l\epsilon), \xi(l)) \rangle^2 \leq \sum_{i,j=1}^{d} w_i w_j \sum_{q_{k-1}(n) \leq l < q_k(n)} \sum_{m < q_k(n)} a_{ij}(\tilde{X}_x(l\epsilon), \tilde{X}_x(m\epsilon), l, m). \]
Hence, by Lemma 3.5

\begin{equation}
|E_{X_k^2} - \langle \left( \int_{q_{k-1}(n)x}^{r_k(n)x} A(\bar{X}_x(u))du \right) w, w \rangle | \leq C_2(T)|w|^2 \varepsilon^{1/3}.
\end{equation}

By Lemma 3.3 with \( \gamma_K = 2L \) and the Hölder inequality,

\begin{equation}
E|\eta|^3 \leq \varepsilon^{3/2}|w|^3 \left( E\left( \sum_{l=q_{k-1}(n)}^{r_k(n)} B(\bar{X}_x(l\varepsilon), \xi(l)) \right) \right)^{3/4} \leq C_1^3/4(2)\varepsilon^{3/2}n^{9/8}|w|^3.
\end{equation}

By the estimate of the norm of the matrix \( A \) in Lemma 3.6

\begin{equation}
|\langle \left( \int_{q_{k-1}(n)x}^{r_k(n)x} A(\bar{X}_x(u))du \right) w, w \rangle | \leq \hat{L}|w|^2n^{3/4}\varepsilon.
\end{equation}

Now, collecting (3.33)–(3.36) we obtain that

\begin{equation}
I_2 \leq C_2(T)|w|^2n^{1/4}\varepsilon^{1/3} + C_1^{3/4}(2)\varepsilon^{3/2}n^{11/8}|w|^3 + \frac{1}{4}\hat{L}^2|w|^4n^{3/2}\varepsilon^2.
\end{equation}

Finally, (3.25)–(3.29), (3.32) and (3.37) yield (3.24) completing the proof. \( \square \)

4. STRONG APPROXIMATIONS

4.1. Strong approximations theorem. We will rely on the following result which is a version of Theorem 1 in [6] with some features taken from Theorem 1 in [13] (see also Theorem 3 in [34]).

4.1. Theorem. Let \( \{V_k, k \geq 1\} \) be a sequence of random vectors with values in \( \mathbb{R}^d \) defined on some probability space \( (\Omega, \mathcal{F}, P) \) such that \( V_k \) is measurable with respect to \( \mathcal{G}_k, k = 1, 2, \ldots \) where \( \mathcal{G}_k, k \geq 1 \) is a filtration of countably generated sub-\( \sigma \)-algebras of \( \mathcal{F} \). Assume that the probability space is rich enough so that there exists on it a sequence of uniformly distributed on \( [0, 1] \) independent random variables \( U_k, k \geq 1 \) independent of \( \bigvee_{k \geq 1} \mathcal{G}_k \). Let \( \mathcal{G} \) be a probability distribution on \( \mathbb{R}^d \) with the characteristic function \( g \). Suppose that for some nonnegative numbers \( \nu_m, \delta_m \) and \( K_m \geq 10^8d, \)

\begin{equation}
E|E(\exp(i\langle w, V_k \rangle)|\mathcal{G}_{k-1}) - g(w)| \leq \nu_k
\end{equation}

for all \( w \in \mathbb{R}^d \) with \( |w| \leq K_k \) and

\begin{equation}
\mathcal{G}\{x : |x| \geq \frac{1}{4}K_k\} < \delta_k.
\end{equation}

Then there exists a sequence \( \{W_k, k \geq 1\} \) of \( \mathbb{R}^d \)-valued random vectors defined on \( (\Omega, \mathcal{F}, P) \) with the properties

(i) \( W_k \) is \( \mathcal{G}_k \vee \sigma\{U_k\} \)-measurable for each \( k \geq 1 \);

(ii) each \( W_k, k \geq 1 \) has the distribution \( \mathcal{G} \) and \( W_k \) is independent of \( \mathcal{G}_{k-1} \) \( \sigma\{U_1, \ldots, U_{k-1}\} \), and so also of \( W_1, \ldots, W_{k-1} \);

(iii) Let \( \varrho_k = 16K_k^{-1}\log K_k + 2\nu_k^{1/2}K_k^2 + 2\delta_k^{1/2} \). Then

\begin{equation}
P(|V_k - W_k| \geq \varrho_k) \leq \varrho_k.
\end{equation}

In particular, the Prokhorov distance between the distributions \( \mathcal{L}(V_k) \) of \( V_k \) and \( \mathcal{L}(W_k) \) of \( W_k \) does not exceed \( \varrho_k \).
4.2. Moment estimates. Here we will consider the block-gap partition similar to Lemma 3.8. Set $N = N_k(T) = \lfloor T/\varepsilon \rfloor$, $q_k = q_k(N) = k\lfloor N^{3/4} + N^{1/4} \rfloor$, $k = 0, 1, \ldots, \nu(\varepsilon)$, where $\nu(\varepsilon) = \left\lfloor \frac{N^{3/4}}{\lfloor N^{3/4} \rfloor} \right\rfloor$, and $r_k = r_k(N) = q_k(N) + \lfloor N^{3/4} \rfloor$ where we consider the intervals $[q_{k-1}(N), r_k(N)]$ as big blocks and the intervals $[r_k(N), q_k(N)]$ as negligible gaps. Now we define the sum

$$Q_k = \sum_{q_{k-1} \leq j < r_k} E\left(\hat{B}(\tilde{X}_x(j\varepsilon), \xi(j))|F_{j-\frac{3}{4}N^{1/4}, j + \frac{3}{4}N^{1/4}}\right)$$

and observe that

$$|\mathbf{S}^\varepsilon(t) - \sum_{1 \leq k \leq \ell_\varepsilon(t)} Q_k| \leq |R^{(1)}(t)| + |R^{(2)}(t)| + |R^{(3)}(t)|$$

where $\ell_\varepsilon(t) = \max\{k : q_k \leq t/\varepsilon\}$,

$$R^{(1)}(t) = \sum_{1 \leq k \leq \ell_\varepsilon(t)} \sum_{q_{k-1} \leq j < r_k} \left(\hat{B}(\tilde{X}_x(j\varepsilon), \xi(j)) - E(\hat{B}(\tilde{X}_x(j\varepsilon), \xi(j))|F_{j-\frac{3}{4}N^{1/4}, j + \frac{3}{4}N^{1/4}})\right),$$

$$R^{(2)}(t) = \sum_{1 \leq k \leq \ell_\varepsilon(t)} \sum_{r_k \leq j < q_k} (\hat{B}(\tilde{X}_x(j\varepsilon), \xi(j)))$$

and

$$R^{(3)}(t) = \sum_{r_{\ell_\varepsilon(t)} \leq j < t/\varepsilon} \hat{B}(\tilde{X}_x(j\varepsilon), \xi(j)).$$

4.2. Lemma. For all $\varepsilon > 0$ and $M \geq 1$,

$$E \sup_{0 \leq t \leq T} \left(|R^{(1)}(t)| + |R^{(2)}(t)| + |R^{(3)}(t)|\right)^2M \leq C_3(M, T)\varepsilon^{-3M/4}$$

where $C_3(M, T) > 0$ does not depend on $\varepsilon$.

Proof. By (4.4),

$$|R^{(1)}(t)| \leq 2T\varepsilon^{-1}\rho(K, \frac{1}{3}T^{1/4}\varepsilon^{-1/4}).$$

Taking into account that the sum in $R^{(2)}(t)$ contains no more than $\sqrt{T}/\varepsilon$ terms we obtain from Lemma 3.4 considered with $\gamma_K = 2L$ that,

$$E \sup_{0 \leq t \leq T} |R^{(2)}(t)|^2M \leq C_1(M)(T/\varepsilon)^M/2.$$ 

Since the sum in $R^{(3)}(t)$ contains no more than $(T/\varepsilon)^{3/4} + (T/\varepsilon)^{1/4}$ terms we obtain again from Lemma 3.4 that

$$E \sup_{0 \leq t \leq T} |R^{(3)}(t)|^2M \leq C_1(M)2^M(T/\varepsilon)^{3M/4}$$

completing the proof. \qed

Next, set $G_k = F_{-\infty, r_k + \frac{3}{4}N^{1/4}}$. The following result is a corollary of Lemmas 3.1 and 3.8.

4.3. Lemma. For any $\varepsilon \geq 0$,

$$E|E(\exp(iw, \varepsilon^{1/2}Q_k)|G_{k-1}) - \exp(-\frac{1}{2}\{(\int_{q_{k-1} \varepsilon}^{r_k \varepsilon} A(\tilde{X}_x(u))du)w, w\})| \leq C_3(T)\varepsilon^p$$

for all $w \in \mathbb{R}^d$ with $|w| \leq \varepsilon^{-p/2}$ where $C_3(T) > 0$ does not depend on $\varepsilon$. 
Proof. By the definition of the coefficient \( \varpi \) and the above notations,
\[
\| E(\exp(i\langle w, \varepsilon^{1/2} Q_k \rangle)|G_{k-1}) - E \exp(i\langle w, \varepsilon^{1/2} Q_k \rangle) \|_{2M} \leq \varpi K, 2M(N^{1/4}/3).
\]
Since \( |e^{i(a+b)} - e^{ib}| \leq |a| \) we obtain from (2.41) that
\[
E \exp(i\langle w, \varepsilon^{1/2} Q_k \rangle) - f^{\varepsilon}_{q_k - \varepsilon r_k}(w) 
\leq \varepsilon^{1/2}|w| \sum_{q_k - \varepsilon < j < r_k} E|\hat{B}(\tilde{X}_x(j\varepsilon)), \xi(j))|f_{j}\varepsilon^{-3/4}N^{1/4}) 
\leq \varepsilon^{1/2}|w|(T/\varepsilon)^{3/4} \rho(K, (T/\varepsilon)^{1/4})
\]
and (4.6) follows from (2.6) and (3.24).

Next, we apply Theorem 4.1 with \( V_k = \varepsilon^{1/2} Q_k, G_k \) the same as in Lemma 4.3 and
\[
g(w) = \exp\left(-\frac{1}{4}\left(\int_{q_k - \varepsilon}^{r_k} A(\tilde{X}_x(u))du\right) w, w\right)
\]
so that \( \Theta \) is the mean zero \( d \)-dimensional Gaussian distribution with the covariance matrix \( f_{q_k - \varepsilon}^{r_k} A(\tilde{X}_x(u))du \). Relying on Lemmas 3.8 and 4.3 we take \( \varphi = \frac{1}{\varpi} \) and apply Theorem 4.1 with
\[
K_k = \varepsilon^{-\varphi/4d} < \varepsilon^{-\varphi/2} \quad \text{and} \quad \nu_k = C_3(T)\varepsilon^{\varphi}.
\]
By the Chebyshev inequality we have also
\[
\Theta\{x : |x| \geq \frac{K_k}{\varphi}\} = P\{|\Psi| \geq \frac{1}{\varphi} \varepsilon^{-\varphi/4d}\} 
\leq 4d(\int_0^{T} \|A(\tilde{X}_x(u))\|\|du\|) \varepsilon^{\varphi/2d} \leq C_4 \varepsilon^{\varphi/2d}
\]
for some \( C_4 > 0 \) which does not depend on \( \varepsilon \), where \( \Psi \) is a random vector with the distribution \( \Theta \).

Now Theorem 4.1 provides us with random vectors \( W_k, k \geq 1 \) satisfying the properties (i)–(iii), in particular, the random vector \( W_k \) has the mean zero Gaussian distribution with the covariance matrix \( (\int_{q_k - \varepsilon}^{r_k} A(\tilde{X}_x(u))du) \), it is independent of \( W_1, ..., W_{k-1} \) and the property (iii) holds true with
\[
\Theta_k = \frac{4}{d} C_4 \varepsilon^{\varphi/4d} \log(1/\varepsilon) + 2 \sqrt{C_3(T)} \varepsilon^{\varphi/4} + 2 C_4^{1/2} \varepsilon^{\varphi/4d}.
\]
As a crucial corollary of Theorem 4.1 we will obtain next a uniform \( L^{2M} \)-bound on the difference between the sums of \( \varepsilon^{-1/2} V_k \)'s and of \( \varepsilon^{-1/2} W_k \)'s. Set
\[
I(n) = I^c(n) = \varepsilon^{-1/2} \sum_{k:r_k \leq n} (V_k - W_k).
\]

4.4. Lemma. For any \( \varepsilon > 0 \) small enough and \( M \geq 1 \),
\[
E \max_{0 \leq n \leq T/\varepsilon} |I(n)|^{2M} \leq C_4(M, T)\varepsilon^{-M+\varphi}/\varphi
\]
where \( \varphi = \frac{1}{\varpi} \) and \( C_4(M) > 0 \) does not depend on \( \varepsilon \).

Proof. The proof of (4.7) will rely on Lemmas 3.2 and 3.3 and so we will have to estimate the conditional expectations appearing there taking into account that \( V_k \) is \( \mathcal{G}_k = \mathcal{F}_{-\infty, r_k + [N^{1/4}]^{-1}} \)-measurable and \( W_k \) is \( \mathcal{G}_k \vee \sigma\{U_1, ..., U_k\} \)-measurable where, recall, \( N = [T/\varepsilon] \). Let \( k > j \geq 1 \). Since \( W_k \) is independent of \( \mathcal{G}_{k-1} \vee \sigma\{U_1, ..., U_{k-1}\} \) we obtain that
\[
E(W_k|\mathcal{G}_j \vee \sigma\{U_1, ..., U_j\}) = EW_k = 0.
\]
Next, since $V_k$ is independent of $\sigma\{U_1, \ldots, U_j\}$ and the latter $\sigma$-algebra is independent of $\mathcal{G}_j$ we obtain that (see, for instance, [9], p. 323),

\begin{equation}
E(V_k | \mathcal{G}_j \vee \sigma\{U_1, \ldots, U_j\}) = E(V_k | \mathcal{G}_j).
\end{equation}

By Lemma 3.3,

\begin{equation}
\|E(V_k | \mathcal{G}_j)\|_{2M} = \sqrt{\mathbb{E}\left[\sum_{i=k^{-1}}^{r_k} E^2\left(E(B(\tilde{X}_k(\varepsilon), j(\varepsilon)) | \mathcal{F}_{i-\frac{1}{2}[N^{1/4}, i+\frac{1}{2}[N^{1/4}]}, \mathcal{F}_{-\infty, r_j+\frac{1}{2}[N^{1/4}]})\right)^2\right]}_{2M} \\
\leq 2L^1 \varepsilon \sum_{i=k^{-1}}^{r_k} \mathbb{E}(K, 2M) (i - r_j - \frac{3}{4}N^{1/4}) \leq \varepsilon \sum_{i=k^{-1}}^{r_k} \mathbb{E}(K, 2M) (i).
\end{equation}

Now, in order to bound $A_{2M}$ from Lemma 3.2 it remains to consider the case $k = n$, i.e. to estimate $\|V_n - W_n\|_{2M}$ and then to combine it with (4.8)-(4.10). By (4.3) and the Cauchy–Schwarz inequality for any $n \geq 1$,

\begin{equation}
E|V_n - W_n|_{2M} = E(|V_n - W_n|_{2M}^2 \mathbb{1}_{|V_n - W_n| \leq \varrho_k}) \\
+ E(|V_n - W_n|_{2M}^2 \mathbb{1}_{|V_n - W_n| > \varrho_k}) \\
\leq \varrho_k^2 + (E|V_n - W_n|_{2M}^4)^{1/2} (P(|V_n - W_n| > \varrho_k)^{1/2} \\
\leq \varrho_k^2 + \varrho_k^4 (E|V_n|_{4M}^4)^{1/2} + (E|W_n|_{4M}^4)^{1/2}).
\end{equation}

By Lemmas 3.1 and 3.4

\begin{equation}
(E|V_n|_{4M}^4)^{1/2} \leq \sqrt{C_1}(2M)^{M/4} (r_k - q_k^{-1})^M \leq \sqrt{C_1}(2M)^{M/4}.
\end{equation}

Since $W_k$ is a mean zero $d$-dimensional Gaussian random vector with the covariance matrix $\int_{q_k-1}^{r_k} A(\tilde{X}_k(u)) du$ having the same distribution as the stochastic integral $\int_{q_k-1}^{r_k} \sigma(\tilde{X}_k(u)) dW(u)$, we obtain that

\begin{equation}
(E|W_k|_{4M}^4)^{1/2} \leq (2M(4M - 1))^M L^{M/4} T^{M/4} \varepsilon^{M/4}.
\end{equation}

Finally, taking into account that the sum of $\mathcal{I}(n)$ contains at most $(T/\varepsilon)^{1/4}$ summands and combining (4.8)-(11.13) with Lemmas 3.2 and 3.3 we derive (1.7) completing the proof (cf. Lemma 4.4 in [29]).

Next, let $W(t), t \geq 0$ be a standard $d$-dimensional Brownian motion and let $\mathcal{E}(t)$ be the Gaussian process given by the stochastic integral

$$\mathcal{E}(t) = \int_{0}^{t} \sigma(\tilde{X}_k(u)) dW(u) \text{ where } \sigma^2(x) = A(x).$$

Then $\mathcal{E}(t) - \mathcal{E}(s)$ has the covariance matrix $\int_{s}^{t} A(\tilde{X}_k(u)) du$, and so the sequences of independent random vectors $\tilde{W}_k = \mathcal{E}(r_k \varepsilon) - \mathcal{E}(q_k^{-1} \varepsilon), k \leq \ell \varepsilon$ and $W_k, k \leq \ell \varepsilon$ have the same distributions. Let $\mathcal{Q}$ and $\mathcal{R}$ be the joint distributions of the sequences of pairs $(V_k, W_k), 1 \leq k \leq \ell \varepsilon$ and of $(\tilde{W}_k, W_k), 1 \leq k \leq \ell \varepsilon$, respectively. Since the marginal of $\mathcal{Q}$ corresponding to the sequence $W_k, 1 \leq k \leq \ell \varepsilon$ coincides with the marginal of $\mathcal{R}$ corresponding to the sequence $\tilde{W}_k, 1 \leq k \leq \ell \varepsilon$ we conclude by Lemma A1 from [6] that we can redefine the process $\mathcal{E}(n), n \in \mathbb{Z}$ preserving its distributions on a richer probability space where there exists a standard $d$-dimensional Brownian motion $W(t), t \in [0, T]$ and a sequence of random vectors $W_k, 1 \leq k \leq \ell \varepsilon$ such that the sequence of pairs $(V_k, W_k), 1 \leq k \leq \ell \varepsilon$ and of $(\tilde{W}_k, W_k), 1 \leq k \leq \ell \varepsilon$ have the joint distributions $\mathcal{Q}$ and $\mathcal{R}$, respectively, where $V_k$’s are constructed by the redefined process $\mathcal{E}$. Now define again $\tilde{W}_k = \int_{q_k^{-1} \varepsilon}^{r_k \varepsilon} \sigma(\tilde{X}_k(u)) dW(u)$ and let $\mathcal{H}_k$ be
the σ-algebra generated by \{W(u), q_k-1\varepsilon \leq u \leq r_k\varepsilon \}. Since \(\hat{W}_k, W\) and \((\hat{W}_k, W)\) have the same joint distributions, we obtain that

\[ E(\hat{W}_k | \mathcal{H}_k) = E(\hat{W}_k | \mathcal{H}_k) = \hat{W}_k \quad \text{a.s.} \]

Then

\[ E|\hat{W}_k - \hat{W}_k|^2 = 2E|\hat{W}_k|^2 - 2E(\hat{W}_k, \hat{W}_k) = 2E|\hat{W}_k|^2 - 2E(\hat{W}_k, E(\hat{W}_k | \mathcal{H}_k)) = 0 \]

where \(\langle \cdot, \cdot \rangle\) is the inner product. Hence,

\[ \hat{W}_k = \hat{W}_k = \Xi(r_k\varepsilon) - \Xi(q_k-1\varepsilon) = \int_{r_k\varepsilon}^{q_k\varepsilon} \sigma(\hat{X}_x(u))dW(u) \quad \text{a.s.} \]

From now on we drop the hat and tilde signs over \(W_k\) and claim in view of the above that \(W_k = \Xi(r_k\varepsilon) - \Xi(q_k-1\varepsilon), 1 \leq k \leq \ell \) satisfy (4.3) and (4.7).

Now, Lemmas 4.2 and 4.4 yield that

(4.14) \( E \sup_{0 \leq t \leq T} |\sqrt{\varepsilon} S^\varepsilon(t) - \int_0^t \sigma(\hat{X}_x(u))dW(u)|^{2M} \leq 4^{2M-1}(C_3(M, T)\varepsilon^{M/4} + C_4(M, T))\varepsilon^{(1+ \frac{d}{2})/2} + E \sup_{0 \leq t \leq T} |J_1(t)|^{2M} + E \sup_{0 \leq t \leq T} |J_2(t)|^{2M} \)

where

\[ J_1(t) = \sum_{1 \leq k \leq \ell \leq t} \int_{r_k\varepsilon}^{q_k\varepsilon} \sigma(\hat{X}_x(u))dW(u) \]

and

\[ J_2(t) = \int_0^t \sigma(\hat{X}_x(u))dW(u) - \sum_{1 \leq k \leq \ell \leq t} \int_{r_k\varepsilon}^{q_k\varepsilon} \sigma(\hat{X}_x(u))dW(u) = \int_0^t \sigma(\hat{X}_x(u))dW(u). \]

By the standard martingale estimates of stochastic integrals (see, for instance, [22] and [22]),

\[ E \sup_{0 \leq t \leq T} |J_1(t)|^{2M} \leq C_5(M, T)\varepsilon^{M/2} \quad \text{and} \quad E \sup_{0 \leq t \leq T} |J_2(t)|^{2M} \leq C_5(M, T)\varepsilon^{M/4} \]

where \(C_5(M, T) > 0\) does not depend on \(\varepsilon\). These together with Lemmas 3.6, 3.7, 4.2 and the estimate (1.14) complete the proof of Theorem 2.1. □

4.3. Proof of Corollary 2.2. For Corollary 2.2 observe that if \(X\) and \(Y\) are two random variables on a metric space \(\mathcal{X}\) with a metric \(d\) then for any \(\gamma > 0\) and a set \(U \subset \mathcal{X}\),

\[ \{X \in U\} \subset \{Y \in U^\gamma\} \cup \{d(X, Y) \geq \gamma\}. \]

Hence, by the Chebyshev inequality

\[ P\{X \in U\} \leq P\{Y \in U^\gamma\} + P\{d(X, Y) \geq \gamma\} \leq P\{Y \in U^\gamma\} + q_{2M}\gamma^{-2M} \]

provided \(E(d(X, Y))^{2M} \leq q_{2M}\). Similarly, \(P\{Y \in U\} \leq P\{X \in U^\gamma\} + q_{2M}\gamma^{-2M}\), and so

\[ \pi(\mathcal{L}(X), \mathcal{L}(Y)) \leq \max(\gamma, q_{2M}\gamma^{-2M}). \]

Taking \(X = \varepsilon^{-1/2}(X^\varepsilon - \bar{X}_x)\) or \(X = \varepsilon^{-1/2}(H^\varepsilon - \bar{X}_x)\) and \(Y = G\) we derive (2.14) from (2.11) and (2.12) by choosing either \(q_{2M} = C_0(M)\varepsilon^\delta, \gamma = C_0^{1/3}(M)\varepsilon^{1/3}, M = 1\) or \(q_{2M} = C_0^{2M+1/3}(M)\varepsilon^{M(2M+1)/3}, \gamma = C_0^{1/3}(M)\varepsilon^{M/3}. \)

□
5. Continuous time case

5.1. Discretization. Introduce the discrete time process $y^\varepsilon$ by the recurrence relation

$$y^\varepsilon_x((k + 1)\varepsilon, \omega) = y^\varepsilon_x(k\varepsilon, \omega) + \varepsilon b(y^\varepsilon_x(k\varepsilon, \omega), \vartheta^k\omega), \quad y^\varepsilon_x(0) = x$$

and $y^\varepsilon_x(t, \omega) = y^\varepsilon_x(k\varepsilon, \omega)$ if $k\varepsilon \leq t < (k + 1)\varepsilon$. The following result can be derived easily from Lemma 3.1 in [25] but we will give its independent proof here for completeness.

5.1. Lemma. For any $\varepsilon > 0$,

$$(5.1) \sup_{x \in \mathbb{R}^d} \sup_{t \in [0, T]} \sup_{\omega \in \Omega} \sup_{0 \leq s \leq \tau(\omega)} |X^\varepsilon_x(t, (\omega, s)) - y^\varepsilon_x(\varepsilon n(t/\varepsilon, \omega), \omega)| \leq \varepsilon LL(2 + eLT + \bar{L}eLT)$$

where for each $t \geq 0$ we set

$$n(t, \omega) = \max\{k \geq 0 : \sum_{j=0}^{k-1} \tau \circ \vartheta^j(\omega) \leq t\}.$$

Proof. First, we write for any $t \leq T/\varepsilon$ and $s \leq \tau(\omega)$ that

$$X^\varepsilon_x(\varepsilon t, (\omega, s)) = x + \varepsilon \int_0^t B(X^\varepsilon_x(\varepsilon u, (\omega, s)), \xi(u, (\omega, s))) du$$

$$= x + \varepsilon \int_0^t B(X^\varepsilon_x(\varepsilon u, (\omega, s)), \xi(u + s, (\omega, 0))) du$$

$$= x + \varepsilon \int_0^t B(X^\varepsilon_x(\varepsilon v - s, (\omega, s)), \xi(v, (\omega, 0))) dv.$$  

Set $X^\varepsilon_{x,s}(\varepsilon v, (\omega, 0)) = X^\varepsilon_x(\varepsilon v - s, (\omega, s))$ for $v \geq s$, so that $X^\varepsilon_{x,s}(\varepsilon s, (\omega, 0)) = x$. Then

$$X^\varepsilon_{x,s}(\varepsilon (t + s), (\omega, 0)) = x + \varepsilon \int_s^{t + s} B(X^\varepsilon_{x,s}(\varepsilon v, (\omega, 0)), \xi(v, (\omega, 0))) dv.$$ 

Hence, by (2.3),

$$(5.2) \quad |X^\varepsilon_x(\varepsilon (t + s), (\omega, 0)) - X^\varepsilon_{x,s}(\varepsilon (t + s), (\omega, 0))|$$

$$\leq |X^\varepsilon_x(\varepsilon s, (\omega, 0)) - x| + \varepsilon L \int_s^{t + s} |X^\varepsilon_{x,s}(\varepsilon v, (\omega, 0)) - X^\varepsilon_{x,s}(\varepsilon (v - s), (\omega, 0))| dv.$$ 

Since by (1.1), (2.3) and (2.15),

$$|X^\varepsilon_x(\varepsilon s, (\omega, 0)) - x| \leq \varepsilon LL \text{ and } |X^\varepsilon_x(\varepsilon (t + s), (\omega, 0)) - X^\varepsilon_x(\varepsilon t, (\omega, 0))| \leq \varepsilon LL,$$

we obtain from (5.2) by the Gronwall inequality that for any $t \leq T/\varepsilon$ and $s \leq \tau(\omega)$,

$$(5.3) \quad |X^\varepsilon_x(\varepsilon t, (\omega, 0)) - X^\varepsilon_x(\varepsilon t, (\omega, s))| \leq |X^\varepsilon_x(\varepsilon t, (\omega, 0)) - X^\varepsilon_x(\varepsilon t, (\omega, 0))|$$

$$+ |X^\varepsilon_{x,s}(\varepsilon (t + s), (\omega, 0)) - X^\varepsilon_{x,s}(\varepsilon (t + s), (\omega, 0))|$$

$$+ |X^\varepsilon_x(\varepsilon (t + s), (\omega, 0)) - X^\varepsilon_x(\varepsilon (t, (\omega, 0))] \leq \varepsilon LL(2 + eLT).$$

It follows that in order to prove (5.1) it suffices to consider there just $X^\varepsilon_x(t, (\omega, 0))$ which we denote by $X^\varepsilon_x(t, \omega)$ and take $t \in [0, T]$. 

Set

$$\Theta_k(\omega) = \sum_{j=0}^{k-1} \tau \circ \vartheta^j(\omega), \quad \Theta_0(\omega) = 0.$$ 

By (2.3) and (2.15) for any $\Theta_n(\omega) \leq t < \Theta_{n+1}(\omega),$

$$(5.4) \quad |X^\varepsilon_x(\varepsilon t, \omega) - X^\varepsilon_x(\varepsilon \Theta_n(\omega), \omega)| \leq \varepsilon \int_{\Theta_n(\omega)}^{t} |B(X^\varepsilon_x(\varepsilon u, \omega), \xi(u, \omega))| du \leq \varepsilon LL.$$
Next, by (2.3), (2.15) and (5.1) for any $n \leq T/\varepsilon$,
\[
|X^e_n(\varepsilon \Theta_n(\omega),\omega) - y^e_n(\varepsilon n,\omega)|
\leq \varepsilon \sum_{k=0}^{n-1} |\int_{\Theta_{k+1}(\omega)}^T \left( X^e_x(\varepsilon u,\omega),\xi(u,\omega) \right) du - b(y^e_x(\varepsilon k,\omega),\theta^k\omega)|
\leq \varepsilon \sum_{k=0}^{n-1} |\int_{\Theta_{k+1}(\omega)}^T \left( X^e_x(\varepsilon \Theta_k(\omega),\omega),\xi(u,\omega) \right) du - b(y^e_x(\varepsilon k,\omega),\theta^k\omega)|
= \varepsilon \sum_{k=0}^{n-1} |b(X^e_x(\varepsilon \Theta_k(\omega),\omega),\theta^k\omega) - b(y^e_x(\varepsilon k,\omega),\theta^k\omega)|
\leq \varepsilon LL \sum_{k=0}^{n-1} |X^e_x(\varepsilon \Theta_k(\omega),\omega) - y^e_x(\varepsilon k,\omega)| + \varepsilon LLT.
\]

By the discrete time Gronwall inequality (see [10]) we obtain from here that for any $n \leq T/\varepsilon$,
\[
|X^e_x(\varepsilon \Theta_n(\omega),\omega) - y^e_x(\varepsilon n,\omega)| \leq \varepsilon LL^2Te^{LLT}
\]
which together with (5.3) and (5.4) yields (5.1). □

Next, set $g(x,\omega) = \tau(\omega)\bar{B}(x)$ and introduce the discrete time process $z^e_x$ by the recurrence relation
\[
z^e_x((k+1)\varepsilon,\omega) = z^e_x(k\varepsilon,\omega) + \varepsilon g(z^e_x(k\varepsilon,\omega),\theta^k\omega), \quad z^e_x(0) = x
\]
and $z^e_t(t,\omega) = z^e_x(k\varepsilon,\omega)$ if $k\varepsilon \leq t < (k+1)\varepsilon$. Observe that $g(x,\omega)$ is obtained in the same way as $b(x,\omega)$ when we replace $B(x,\xi(s,\omega))$ by $\bar{B}(x)$. Hence, we can apply Lemma 5.1 to the pair $\bar{X}_x$ and $z^e_x$ in place of the pair $X^e_x$ and $y^e_x$ and looking carefully at the proof there we see that this lemma can be applied here with the same constants. Thus, for all $\varepsilon > 0$,
\[
(5.5) \sup_{x \in \mathbb{R}^d} \sup_{t \in [0,T]} \sup_{\omega \in \Omega} |\bar{X}_x(t) - z^e_x(\varepsilon n(t/\varepsilon,\omega),\omega)| \leq \varepsilon LL(2 + e^{LT} + Le^{LLT}).
\]

Now observe that
\[
(5.6) \quad \bar{b}(x) = Eb(x,\omega) = \tau \bar{B}(x) = Eg(x,\omega) = \bar{g}(x).
\]

Hence,
\[
\bar{y}_x(t) = x + \int_0^t \bar{b}(\bar{y}_x(s))ds = x + \int_0^t \bar{g}(\bar{y}_x(s))ds = x + \tau \int_0^t \bar{B}(\bar{y}_x(s))ds.
\]

Since
\[
\bar{z}_x(t) = x + \int_0^t \bar{g}(\bar{z}_x(s))ds \quad \text{and} \quad \bar{X}_x(\tau t) = x + \int_0^{\tau t} \bar{B}(\bar{X}_x(s))ds = x + \bar{\tau} \int_0^{\tau t} \bar{B}(\bar{X}_x(\tau u))du,
\]
we conclude by uniqueness of the solutions of the equations above that
\[
(5.7) \quad \bar{y}_x(t) = \bar{z}_x(t) = \bar{X}_x(\tau t) \quad \text{for all} \quad t \in [0,T].
\]

It follows from (5.1), (5.5) and (5.7) that for all $\varepsilon > 0$,
\[
(5.8) \quad \sup_{x \in \mathbb{R}^d} \sup_{t \in [0,T]} \sup_{\omega \in \Omega} \sup_{0 \leq s \leq \tau(t)} |(X^e_x(t,\omega,s) - \bar{X}_x(t)) - (y^e_x(\varepsilon n(t/\varepsilon,\omega),\omega) - \bar{y}_x(\varepsilon n(t/\varepsilon,\omega))) + (z^e_x(\varepsilon n(t/\varepsilon,\omega),\omega) - \bar{z}_x(\varepsilon n(t/\varepsilon,\omega)))| \leq \varepsilon LL(2 + e^{LT} + Le^{LLT}).
\]
5.2. Time change estimates. Next, we compare \( \tilde{y}_x^c(\varepsilon n(t/\varepsilon, \omega), \omega) - \tilde{y}_x(\varepsilon n(t/\varepsilon, \omega)) \) and \( z_x^c(\varepsilon n(t/\varepsilon, \omega), \omega) - \tilde{z}_x(\varepsilon n(t/\varepsilon, \omega)) \) with \( \tilde{y}_x(t/\tau, \omega) - \tilde{y}_x(t/\tilde{\tau}) \) and \( z_x^c(t/\tau, \omega) - \tilde{z}_x(t/\tilde{\tau}) \), respectively.

5.2. Lemma. For all \( \varepsilon > 0 \),

\[
E \sup_{0 \leq t \leq T} |\tilde{y}_x^c(\varepsilon n(t/\varepsilon, \omega), \omega) - \tilde{y}_x(\varepsilon n(t/\varepsilon, \omega))| - (\tilde{y}_x^c(t/\tilde{\tau}, \omega) - \tilde{y}_x(t/\tilde{\tau}))|^{2M} \leq C_0(M)\varepsilon^{(3M-4)/2}
\]

and

\[
E \sup_{0 \leq t \leq T} |\tilde{z}_x^c(\varepsilon n(t/\varepsilon, \omega), \omega) - \tilde{z}_x(\varepsilon n(t/\varepsilon, \omega))| - (\tilde{z}_x^c(t/\tilde{\tau}, \omega) - \tilde{z}_x(t/\tilde{\tau}))|^{2M} \leq C_0(M)\varepsilon^{(3M-4)/2}
\]

where \( C_0(M) > 0 \) does not depend on \( \varepsilon \).

Proof. In the proof we will rely on Lemma 5.1 from [15] saying that for any \( M \geq 1 \) and \( \varepsilon > 0 \),

\[
E|n(t/\varepsilon, \omega) - t/\varepsilon\tilde{T}|^{2M} \leq K(M)(t/\varepsilon\tilde{T})^M
\]

and

\[
E \sup_{0 \leq s \leq t} |n(s/\varepsilon, \omega) - s/\varepsilon\tilde{T}|^{2M} \leq K(M)(t/\varepsilon\tilde{T})^{M+1}
\]

where \( K(M) > 0 \) does not depend on \( \varepsilon \) and \( t \). As at the beginning of the proof of Lemma 5.1 it will be convenient to replace \( \tilde{y}_x \) and \( \tilde{z}_x \) by \( \tilde{y}_x^c \) and \( \tilde{z}_x^c \) given by

\[
\tilde{y}_x^c(n \varepsilon) = x + \varepsilon \int_0^{n \varepsilon} \tilde{b}(\tilde{y}_x(k \varepsilon)) \, ds \\
\tilde{z}_x^c(n \varepsilon) = x + \varepsilon \int_0^{n \varepsilon} \tilde{g}(\tilde{z}_x(k \varepsilon)) \, ds
\]

with \( \tilde{y}_x^c(t) = \tilde{y}_x^c(k \varepsilon) \) and \( \tilde{z}_x^c(t) = \tilde{z}_x^c(k \varepsilon) \) if \( k \varepsilon \leq t < (k+1) \varepsilon \). Then, in the same way as in Lemma 5.1 for all \( n \in N \),

\[
|\tilde{y}_x(n \varepsilon) - \tilde{y}_x^c(n \varepsilon)| \leq L^2 N \varepsilon^2 \quad \text{and} \quad |\tilde{z}_x(n \varepsilon) - \tilde{z}_x^c(n \varepsilon)| \leq L^2 N \varepsilon^2,
\]

and so for all \( 0 \leq t \leq \tilde{T} = \tilde{T} \varepsilon \),

\[
|\tilde{y}_x(t) - \tilde{y}_x^c(t)| \leq L^2 \tilde{T} \varepsilon \quad \text{and} \quad |\tilde{z}_x(t) - \tilde{z}_x^c(t)| \leq L^2 \tilde{T} \varepsilon.
\]

Here we take \( \tilde{T} \) and not just \( T \) since the time \( \varepsilon n(t/\varepsilon, \omega) \) can run, in principle, up to \( \tilde{T} \geq T \). The estimate (5.12) enables us to replace \( \tilde{y}_x \) and \( \tilde{z}_x \) by \( \tilde{y}_x^c \) and \( \tilde{z}_x^c \), respectively, and so we estimate now

\[
E|\tilde{y}_x^c(\varepsilon n(t/\varepsilon, \omega), \omega) - \tilde{y}_x(\varepsilon n(t/\varepsilon, \omega))| - (\tilde{y}_x^c(t/\tilde{\tau}, \omega) - \tilde{y}_x(t/\tilde{\tau}))|^{2M} \leq C_0(M)\varepsilon^{(3M-4)/2}
\]

where by (2.3) and (2.15),

\[
\mathcal{I}_x^c(t, \omega) = \left| \sum_{\min(t/\varepsilon, n(t/\varepsilon, \omega)) \leq k < \max(t/\varepsilon, n(t/\varepsilon, \omega))} \left( b(\tilde{y}_x^c(k \varepsilon, \omega), \tilde{y}_x^c(k \varepsilon, \omega)) \right. \right. - b(\tilde{y}_x(k \varepsilon, \omega), \tilde{y}_x(k \varepsilon, \omega)) \bigg| \leq \tilde{L} \sup_{0 \leq t \leq \tilde{T}} |\tilde{y}_x^c(t, \omega) - \tilde{y}_x(t)| \quad \text{and}
\]

\[
\mathcal{J}_x(t, \omega) = \left| \sum_{\min(t/\varepsilon, n(t/\varepsilon, \omega)) \leq k < \max(t/\varepsilon, n(t/\varepsilon, \omega))} \left( b(\tilde{y}_x^c(k \varepsilon, \omega), \tilde{y}_x^c(k \varepsilon, \omega)) \right. \right. - b(\tilde{y}_x(k \varepsilon, \omega), \tilde{y}_x(k \varepsilon, \omega)) \bigg|, \quad \tilde{L} = L \tilde{T}.
\]

By (3.18) applied to \( \tilde{y}_x^c \) and \( \tilde{y}_x \) in place of \( X_x^c \) and \( X_x \) together with (5.11) (with \( M \) replaced by \( 2M \)) and the Cauchy–Schwarz inequality we obtain

\[
E \sup_{0 \leq t \leq T} |\mathcal{I}_x^c(t, \omega)|^{2M} \leq (\tilde{L} \tilde{T})^{2M}(E \sup_{0 \leq t \leq T} |n(t/\varepsilon, \omega) - t/\varepsilon\tilde{T}|^{4M})^{1/2}
\]

\[
\times (E \sup_{0 \leq t \leq \tilde{T}} |\tilde{y}_x^c(t, \omega) - \tilde{y}_x(t)|^{4M})^{1/2} \leq C_7(M, T)\varepsilon^{1/2}
\]
where \( C_7(M, T) > 0 \) does not depend on \( \varepsilon \).

In order to estimate the second term in the right hand side of (5.13) introduce for \( j = 1, 2, \ldots \) the events

\[
A_j = \{ \omega : (j - 1)\varepsilon^{-1/2} \leq \sup_{0 \leq t \leq T} |n(t/\varepsilon, \omega) - t/\varepsilon T| < j\varepsilon^{-1/2} \}.
\]

In fact, \( A_j \) is empty for any \( j > L T \varepsilon^{-1/2} \) since

\[
\sup_{0 \leq t \leq T} \max \{n(t/\varepsilon, \omega), t/\varepsilon T\} \leq T L \varepsilon^{-1}.
\]

It follows that

\[
(5.16) \quad E \sup_{0 \leq t \leq T} |J_x^\varepsilon(t, \omega)|^{2M} = \sum_{1 \leq j \leq LT \varepsilon^{-1/2}} E \sup_{0 \leq t \leq T} |J_x^\varepsilon(t, \omega)|^{2M}
\]

\[
\leq \sum_{1 \leq j \leq LT \varepsilon^{-1/2}} E (I_{A_j} \max_{0 \leq k \leq LT \varepsilon^{-1}} \max_{1 \leq n < j \varepsilon^{-1/2}} |\sum_{k \leq l \leq k + n} (b(\tilde{g}_x(\varepsilon), \vartheta l, \omega) - Eb(\tilde{g}_x(\varepsilon), \vartheta l, \omega))^2|^{2M})
\]

\[
\leq \sum_{1 \leq j \leq LT \varepsilon^{-1/2}} \sum_{0 \leq k \leq LT \varepsilon^{-1}} R_x^k(j, k, n, M)
\]

where \( I_A \) is the indicator of an event \( A \) and by the Cauchy–Schwarz inequality,

\[
(5.17) \quad R_x^k(j, k, n, M) = E (I_{A_j} \max_{1 \leq n < j \varepsilon^{-1/2}} |\sum_{k \leq l \leq k + n} (b(\tilde{g}_x(\varepsilon), \vartheta l, \omega) - Eb(\tilde{g}_x(\varepsilon), \vartheta l, \omega))^2|^{2M})
\]

\[
\leq (P(A_j))^{1/2} (E (\max_{1 \leq n < j \varepsilon^{-1/2}} |\sum_{k \leq l \leq k + n} (b(\tilde{g}_x(\varepsilon), \vartheta l, \omega) - Eb(\tilde{g}_x(\varepsilon), \vartheta l, \omega))^2|^{4M}))^{1/2}.
\]

For \( j = 1 \) we estimate \( P(A_j) \) just by 1 and for \( j \geq 2 \) we apply (5.11) and the Chebyshev inequality to obtain

\[
(5.18) \quad P(A_j) \leq P\{\sup_{0 \leq t \leq T} |n(t/\varepsilon, \omega) - t/\varepsilon T| \geq (j - 1)\varepsilon^{-1/2}\}
\]

\[
\leq \varepsilon^M (j - 1)^{-2M} E \sup_{0 \leq t \leq T} |n(t/\varepsilon, \omega) - t/\varepsilon T|^{2M}
\]

\[
\leq K(M) \varepsilon^{-(j - 1)^2(M T/\varepsilon)^M + 1}.
\]

The second factor in the right hand side of (5.17) we estimate by Lemma 3.4 with

\[
\eta_j = b(\tilde{g}_x((k + j)\varepsilon), \vartheta^{k+j-1} \omega) - Eb(\tilde{g}_x((k + j)\varepsilon), \vartheta^{k+j} \omega)
\]

(5.19) we derive

\[
E (\max_{1 \leq n < j \varepsilon^{-1/2}} |\sum_{k \leq l \leq k + n} (b(\tilde{g}_x(\varepsilon), \vartheta l, \omega) - Eb(\tilde{g}_x(\varepsilon), \vartheta l, \omega))^2|^{2M}) \leq C_1(M) (j \varepsilon^{-1/2} + 1)^M
\]

where \( C_1(M) > 0 \) does not depend on \( \varepsilon \) and \( j \). Combining (5.16), (5.18), (5.19) we conclude that

\[
(5.20) \quad E \sup_{0 \leq t \leq T} |J_x^\varepsilon(t, \omega)|^{2M} \leq C_b(M) \varepsilon^{-(M + 4)/2}
\]

for some \( C_b(M) > 0 \) which does not depend on \( \varepsilon \). Finally, (5.12), (5.15) and (5.20) yield (5.9) and (5.10) completing the proof of the lemma. \( \square \)

Observe that, as a byproduct, (5.15) together with (5.8)–(5.10) improves the estimate of Theorem A in [12].
5.3. Completing the proof of Theorem 2.4. In the same way as in (3.12), using the Taylor formula we can write
\[ y^\varepsilon_x(n\varepsilon) - \tilde{y}_x(n\varepsilon) = \sum_{0 \leq k < n} \nabla \bar{b}(\tilde{y}_x(k\varepsilon)) (y^\varepsilon_x(k\varepsilon) - \tilde{y}_x(k\varepsilon)) + \varepsilon S^\varepsilon_y(n\varepsilon) + R^\varepsilon_{1,y}(n\varepsilon) + R^\varepsilon_{2,y}(n\varepsilon) \]
and
\[ z^\varepsilon_x(n\varepsilon) - \tilde{z}_x(n\varepsilon) = \sum_{0 \leq k < n} \nabla \bar{g}(\tilde{z}_x(k\varepsilon)) (z^\varepsilon_x(k\varepsilon) - \tilde{z}_x(k\varepsilon)) + \varepsilon S^\varepsilon_z(n\varepsilon) + R^\varepsilon_{1,z}(n\varepsilon) + R^\varepsilon_{2,z}(n\varepsilon) \]
where
\[ S^\varepsilon_y(s) = \sum_{0 \leq k < [s/\varepsilon]} (b(\tilde{y}_x(k\varepsilon), \partial^k \omega) - \bar{b}(\tilde{y}_x(k\varepsilon))) \]
\[ S^\varepsilon_z(s) = \sum_{0 \leq k < [s/\varepsilon]} (g(\tilde{z}_x(k\varepsilon), \partial^k \omega) - \bar{g}(\tilde{z}_x(k\varepsilon))) \]
and by Lemma 3.7
\[ E \left( \sup_{0 \leq t \leq T} |R^\varepsilon_{1,y}(t)|^{2M} + \sup_{0 \leq t \leq T} |R^\varepsilon_{2,y}(t)|^{2M} \right. 
+ \left. \sup_{0 \leq t \leq T} |R^\varepsilon_{1,z}(t)|^{2M} + \sup_{0 \leq t \leq T} |R^\varepsilon_{2,z}(t)|^{2M} \right) \leq C_9(M) \varepsilon^{2M} \]
for some \( C_9(M) > 0 \) which does not depend on \( \varepsilon \).

Next, let \( G \) be the Gaussian process solving the linear equation (2.10). Then by (5.10), (5.17), (5.21) and (5.22),
\[ |y^\varepsilon_x(t) - z^\varepsilon_x(t) - \sqrt{\varepsilon} G(t)| \leq L \bar{L} \int_0^t |y^\varepsilon_x(s) - z^\varepsilon_x(s) - \sqrt{\varepsilon} G(s)| ds 
+ |\varepsilon (S^\varepsilon_y(t) - S^\varepsilon_z(t)) - \sqrt{\varepsilon} \int_0^t \sigma(\bar{X}_x(\tau s)) dW(s)| 
+ |R^\varepsilon_{1,y}(t)| + |R^\varepsilon_{2,y}(t)| + |R^\varepsilon_{1,z}(t)| + |R^\varepsilon_{2,z}(t)| + |\tilde{R}^\varepsilon(t)| \]
where
\[ \tilde{R}^\varepsilon(t) = \bar{\sigma} \int_0^t \nabla B(\bar{X}_x(\tau s)) - \nabla B(\bar{X}_x(\bar{\tau}[s/\varepsilon] \varepsilon)))|G(s)| ds \leq L \bar{L} \varepsilon \int_0^t |G(s)| ds. \]

By the Gronwall inequality,
\[ |\varepsilon^{-1/2}(y^\varepsilon_x(t) - z^\varepsilon_x(t) - G(t))| \leq e^{L \bar{L} T} \left( |\varepsilon (S^\varepsilon_y(t) - S^\varepsilon_z(t)) 
- \int_0^t \sigma(\bar{X}_x(\tau s)) dW(s)| 
+ \varepsilon^{-1/2} |R^\varepsilon_{1,y}(n\varepsilon)| + |R^\varepsilon_{2,y}(n\varepsilon)| + |R^\varepsilon_{1,z}(n\varepsilon)| + |R^\varepsilon_{2,z}(n\varepsilon)| + |\tilde{R}^\varepsilon(t)| \right). \]

In order to estimate \( \tilde{R}^\varepsilon \) we write
\[ E \sup_{0 \leq t \leq T} \left( \int_0^t |G(s)| ds \right)^{2M} = E(\int_0^T |G(s)| ds)^{2M} \leq T^{2M-1} \int_0^T E |G(s)|^{2M} ds. \]
By (2.13), (2.15), (2.16) and the standard moment estimates of stochastic integrals we obtain that for any \( t \in [0, T] \),
\[ E |G(t)|^{2M} \leq 2^{2M-1}(L \bar{L})^{2M} E(\int_0^T |G(s)| ds)^{2M} + 2^{2M-1} E(\int_0^T \sigma(\bar{X}_x(\tau s)) dW(s))^{2M} \]
\[ \leq 2^{2M-1}(L \bar{L})^{2M} T^{2M-1} \int_0^T E |G(s)|^{2M} ds + 2^{2M-1}(M(2M - 1))^M T^M \sup_x |\sigma(x)|^{2M}. \]
Hence, by the Gronwall inequality
\[ E|G(t)|^{2M} \leq \tilde{C}(T,M) \]
\[ = 2^{2M-1}(M(2M-1))^{M}T^{2M} \sup_{x} |\sigma(x)|^{2M} \exp (2^{2M-1}(LL)^{2M}T^{2M-1}), \]
and so by (5.26),
\[ (5.27) \quad E \sup_{0 \leq t \leq T} |\tilde{R}(t)| \leq (LL)^{2M} \varepsilon^{2MT^{2M-1}} \tilde{C}(T,M). \]

Next, in the same way as in Section 4 we construct for each \( \varepsilon > 0 \) the Brownian motion \( W = W_\varepsilon \) such that
\[ (5.28) \quad E \sup_{0 \leq t \leq T} |\sqrt{\varepsilon}(S_\varepsilon^x(t/\varepsilon) - S_\varepsilon^z(t/\varepsilon)) - \int_{0}^{t} \sigma(\tilde{X}(s))dW(s)|^{2M} \leq C_{10}(M)\varepsilon^{\delta} \]
where \( C_{10}(M) > 0 \) does not depend on \( \varepsilon \). This together with (5.8–5.10), (5.23), (5.26) and (5.27) yields (2.20) and (2.21) while (2.19) and (2.22) follow from here and (5.11) taking into account (3.10) and (3.17). The result similar to Corollary 2.2 with the same constants is obtained in the continuous and discrete time cases in the same way. This completes the proof of Theorem 2.4. \( \square \)

6. Almost Sure Approximations and the Law of Iterated Logarithm

6.1. Lemma. For any fixed \( \kappa > 0 \) as \( \varepsilon \downarrow 0 \),
\[ (6.1) \quad \sup_{0 \leq t \leq T} |\tilde{X}_{\varepsilon}(t) - \tilde{X}(t)| = O(\varepsilon^{1-\kappa}) \quad a.s. \]
where \( \tilde{Z} \) was defined by (3.19).

Proof. Though it will suffice for our purposes to have in the right hand side of (6.1) only \( O(\varepsilon^{1+\delta}) \) for an arbitrarily small \( \delta > 0 \), we will prove (6.1) in the stronger form as stated. Using (3.12) we obtain
\[ (6.2) \quad X_{\varepsilon}^{x}(n\varepsilon) - \tilde{X}_{x}(n\varepsilon) = Z_{\varepsilon}^{x}(n\varepsilon) \]
\[ = \varepsilon \sum_{k=0}^{n-1} \nabla B(\tilde{X}(k\varepsilon))(X_{\varepsilon}^{x}(k\varepsilon) - \tilde{X}_{x}(k\varepsilon) - Z_{\varepsilon}^{x}(k\varepsilon)) + R_{1}^{x}(n\varepsilon) + R_{2}^{x}(n\varepsilon), \]
and so by the discrete time Gronwall inequality,
\[ (6.3) \quad \max_{0 \leq n \leq T/\varepsilon} |X_{\varepsilon}^{x}(n\varepsilon) - \tilde{X}_{x}(n\varepsilon) - Z_{\varepsilon}^{x}(n\varepsilon)| \leq e^{\varepsilon T} \sup_{0 \leq t \leq T} (|R_{1}^{x}(t)| + |R_{2}^{x}(t)| + |R_{2}^{x}(t)|). \]
By (3.13) for any \( \lambda > 0 \),
\[ P\{ \sup_{0 \leq t \leq T} |X_{\varepsilon}^{x}(t) - \tilde{X}_{x}(t)| > \varepsilon^{\lambda-\lambda} \} \leq \tilde{C}(T,M)\varepsilon^{2M\lambda}, \]
and so taking \( M \geq \lambda^{-1} \) and \( \varepsilon_{n} = \frac{1}{n} \) we obtain by the Borel–Cantelli lemma that for \( n = 1, 2, \ldots, \)
\[ (6.4) \quad \sup_{0 \leq t \leq T} |X_{\varepsilon}^{x}(t) - \tilde{X}_{x}(t)| = O(\varepsilon^{1-\lambda}) \quad a.s. \]
Now, set \( \Psi_{n}(\varepsilon) = X_{\varepsilon}^{x}(n\varepsilon) \). Then by (1.6),
\[ \Psi_{m+1}(\varepsilon) = x + \varepsilon \sum_{k=0}^{m} B(\Psi_{k}(\varepsilon), \xi(k)), \quad m + 1 \leq T/\varepsilon, \]
and so by (2.3) for \( \varepsilon_n = \frac{1}{n} \leq \varepsilon < \varepsilon_{n-1} = \frac{1}{n-1} \),

\[
|\Psi_m(\varepsilon) - \Psi_m(\varepsilon_n)| \leq \varepsilon L \sum_{k=0}^{m-1} |\Psi_k(\varepsilon) - \Psi_k(\varepsilon_n)| + (\varepsilon - \varepsilon_n)LT\varepsilon^{-1}.
\]

Hence, by the discrete time Gronwall inequality for all \( m \leq T/\varepsilon \),

\[
|\Psi_m(\varepsilon) - \Psi_m(\varepsilon_n)| \leq LT\varepsilon^{LT}(n-1)^{-1}.
\]

This together with (6.3) yields that for all \( \varepsilon > 0 \),

\[
\sup_{0 \leq t \leq T} |X^\varepsilon_x(t) - \bar{X}_x(t)| = O(\varepsilon^{1/2}) \quad \text{a.s.}
\]

Recalling (2.4) and the definition of \( R^2 \) we obtain from (6.7) that for all \( \varepsilon \),

\[
\sup_{0 \leq t \leq T} |R^2(t)| = O(\varepsilon^{1-2\lambda}) \quad \text{a.s.}
\]

Next, by (3.13), similarly to the above, we obtain that for \( \varepsilon_n = \frac{1}{n}, n = 1, 2, ... \) and \( \lambda > 0 \),

\[
\sup_{0 \leq t \leq T} |R^1_n(t)| = O(\varepsilon_n^{1-2\lambda}) \quad \text{a.s.}
\]

In order to derive that as \( \varepsilon \downarrow 0 \),

\[
\sup_{0 \leq t \leq T} |R^1(t)| = O(\varepsilon^{1-2\lambda}) \quad \text{a.s.}
\]

we will show that for all \( t \in [0, T] \) and all \( n > 1 \),

\[
|R^1(t) - R^1_n(t)| \leq C_{11}(n-1)^{-1}
\]

for some \( C_{11} > 0 \) which does not depend on \( n \) and \( \varepsilon \). Indeed, by (2.3) for any \( k \leq T/\varepsilon \),

\[
|\nabla B(\bar{X}_x(k\varepsilon), \xi(k)) - \nabla B(\bar{X}_x(k\varepsilon_n)) - \nabla B(\bar{X}_x(k\varepsilon_n), \xi(k)) - \nabla B(\bar{X}_x(k\varepsilon_n))| \leq 2L^2T(n-1)^{-1}
\]

and by (6.5),

\[
|X^\varepsilon_x(k\varepsilon) - \bar{X}_x(k\varepsilon) - X^\varepsilon_x(k\varepsilon_n) + \bar{X}_x(k\varepsilon_n)| \leq (e^{LT} + 1)LT(n-1)^{-1}
\]

which together with (2.3) and the definition of \( R^1 \) in Lemma 3.6 yields (6.9), and so also (6.8). Now, (6.3), (6.7) and (6.8) yield (6.1) and complete the proof of the lemma.

Extend the definition of \( Z^\varepsilon \) to all \( t \in [0, T] \) by setting \( Z^\varepsilon(t) = Z^\varepsilon(n\varepsilon) \) if \( n\varepsilon \leq t < (n+1)\varepsilon \). The following estimate leads us to the study of a.s. approximations of \( \sqrt{\varepsilon S^\varepsilon} \) with \( S^\varepsilon \) defined in Lemma 3.6.

6.2. Lemma. For all \( \varepsilon > 0 \),

\[
\sup_{0 \leq t \leq T} |Z^\varepsilon(t) - \sqrt{\varepsilon}G(t)| \leq \sqrt{\varepsilon}e^{LT} \left( \sup_{0 \leq t \leq T} |\sqrt{\varepsilon}S^\varepsilon(t)| - \int_0^t \sigma(\bar{X}_x(s))dW(s) + 2\sqrt{\varepsilon}L^2T^2e^{LT} \right)
\]

where \( G \) is given by (1.4).
Proof. First, observe that by (2.3),
\[
|S^\varepsilon(t)| \leq 2LT\varepsilon^{-1}.
\]
Hence, by (2.3) and (3.19),
\[
|Z^\varepsilon(n\varepsilon)| \leq 2LT + \varepsilon L \sum_{k=0}^{n-1} |Z^\varepsilon(k\varepsilon)|,
\]
and so by the discrete time Gronwall inequality
\[
\sup_{0 \leq t \leq T} |Z^\varepsilon(t)| \leq 2LT e^{LT}.
\]
This together with (1.4), (2.3) and (3.19) yields
\[
\sup_{0 \leq t \leq T} |Z^\varepsilon(t)| \leq 2LT e^{LT}.
\]
Applying the Gronwall inequality we arrive at (6.10). □

6.2. A.s. approximations. We will assume now that \( B(x, \xi) = \Sigma(x)\xi \) where \( \Sigma \) is the matrix function satisfying (2.23). This assumption will enable us to rely on the a.s. approximation result for the sequence \( \xi(k), k \in \mathbb{Z} \) itself which is essentially well known (see, for instance, Theorem 2.1 in [30] or Theorem 2.2 in [15] and references there) and it says the following.

6.3. Proposition. The sequence of random vectors \( \xi(n), -\infty < n < \infty \) can be redefined preserving its distributions on a sufficiently rich probability space which contains also a \( d \)-dimensional Brownian motion \( W \) with the covariance matrix \( \varsigma \) so that for any \( N \geq 1 \),
\[
\sup_{0 \leq n \leq N} |S(n) - W(n)| = O(N^{\frac{1}{2}}) \quad \text{a.s.}
\]
for some \( \delta > 0 \) where \( S(n) = \sum_{0 \leq k < n} (\xi(k) - \bar{\xi}) \) and \( \bar{\xi} = E\xi(0) \).

Set also \( S(t) = S(n) \) if \( n \leq t < n + 1 \). Observe that in our circumstances \( \bar{B}(x, \xi(k)) = \Sigma(x)(\xi(k) - \bar{\xi}) \) and \( S^\varepsilon(t) = \sum_{0 \leq k < t/\varepsilon} \Sigma(\bar{X}_x(\varepsilon k))(\xi(k) - \bar{\xi}) \). Next, we set
\[
V^\varepsilon(t) = \sum_{0 \leq k < t/\varepsilon} \Sigma(\bar{X}_x(\varepsilon k))\left( \sum_{t(1-\gamma) \leq k < (t+1)\varepsilon(1-\gamma)} (\xi(k) - \bar{\xi}) \right),
\]
with \( \gamma \in (0, 1) \) to be chosen later on, and estimate
\[
|S^\varepsilon(t) - V^\varepsilon(t)| \leq I^\varepsilon(t)
\]
where
\[
I^\varepsilon(t) = \sum_{0 \leq k < t/\varepsilon} \sum_{t(1-\gamma) \leq k < (t+1)\varepsilon(1-\gamma)} (\Sigma(\bar{X}_x(\varepsilon k)) - \Sigma(\bar{X}_x(\varepsilon^\gamma k)))(\xi(k) - \bar{\xi}).
\]
Since
\[
|\Sigma(\bar{X}_x(\varepsilon k)) - \Sigma(\bar{X}_x(\varepsilon^\gamma k))| \leq L^2 \varepsilon^\gamma
\]
and the constant $C_1(M)$ in the estimate (6.14) of Lemma 3.4 depends only on the $L^\infty$ bound of summands themselves, we apply (3.4) with

$$\eta(k) = \varepsilon^{-\gamma}(\Sigma(\tilde{X}_x(\varepsilon k)) - \Sigma(\tilde{X}_x(\varepsilon l)))(\xi(k) - \xi)$$

to obtain that

$$E \sup_{0 \leq t \leq T} |I^\varepsilon(t)|^{2M} \leq C_{12}(M)\varepsilon^{M(2\gamma - 1)}$$

for some $C_{12}(M) > 0$ which does not depend on $\varepsilon$.

Next, we set

$$\Xi^\varepsilon(t) = \sum_{0 \leq t \leq \varepsilon t} \Sigma(\tilde{X}_x(\varepsilon l))(W((l+1)\varepsilon^{-1} - W(l\varepsilon^{-1})))$$

and

$$J^\varepsilon(t) = \int_0^{t/\varepsilon} \Sigma(\tilde{X}_x(\varepsilon s))dW(s) - \Xi^\varepsilon(t)$$

$$= \sum_{0 \leq t \leq \varepsilon t} \int_{\varepsilon l}^{(l+1)\varepsilon^{-1}} \Sigma(\tilde{X}_x(\varepsilon s)) - \Sigma(\tilde{X}_x(\varepsilon l)))dW(s).$$

By the standard martingale estimates of stochastic integrals (see, for instance, [32]) we obtain that

$$E \sup_{0 \leq t \leq T} |J^\varepsilon(t)|^{2M} \leq C_{13}(M)\varepsilon^{M(2\gamma - 1)}$$

for some $C_{13}(M) > 0$ which does not depend on $\varepsilon$. Now, we employ (6.14) to estimate

$$\sup_{0 \leq t \leq T} |V^\varepsilon(t) - \Xi^\varepsilon(t)| = O(\varepsilon^{-\frac{1}{2} + \delta - \gamma}) \text{ a.s.}$$

Set $\varepsilon_n = \frac{1}{n}$, $n = 1, 2, \ldots$. Then by (6.14),

$$P\left\{ \sqrt{\varepsilon_n} \sup_{0 \leq t \leq T} |I^{\varepsilon_n}(t)| > \varepsilon_n^{\gamma/2} \right\} \leq C_{12}(M)\varepsilon_n^{M\gamma}.$$

Taking here $M \geq 2\gamma - 1$ we obtain by the Borel–Cantelli lemma that

$$\sqrt{\varepsilon_n} \sup_{0 \leq t \leq T} |I^{\varepsilon_n}(t)| = O(\varepsilon_n^{\gamma/2}) \text{ a.s.}$$

Now, let $\varepsilon_{n+1} \leq \varepsilon \leq \varepsilon_n$. By (2.23),

$$|\sqrt{\varepsilon}S^\varepsilon(t) - \sqrt{\varepsilon_n}S^{\varepsilon_n}(t)| \leq 2LT\varepsilon^{-1}(\sqrt{\varepsilon_n} - \sqrt{\varepsilon}) + 2L^3T^2\varepsilon^{-3/2}(\varepsilon_n - \varepsilon)
\leq 2LT(1 + LT)\sqrt{\varepsilon_n}.$$

Again by (2.3),

$$|\sqrt{\varepsilon}V^\varepsilon(t) - \sqrt{\varepsilon_n}V^{\varepsilon_n}(t)| \leq 4LT\varepsilon^{-\gamma}\sqrt{\varepsilon_n} + 2L^3T\varepsilon^{-1}(\varepsilon_n^{\gamma} - \varepsilon)\sqrt{\varepsilon_n} \leq C_{14}\varepsilon_n^{\frac{1}{2} - \gamma}$$

for some $C_{14} > 0$ which does not depend on $\varepsilon$. This together with (6.18) and (6.19) yields that for all $\varepsilon > 0$,

$$\sqrt{\varepsilon} \sup_{0 \leq t \leq T} |I^\varepsilon(t)| = O(\varepsilon^{\gamma/2}) \text{ a.s.}$$

provided $\gamma \leq 1/3$.

Next, by (6.15) similarly to (6.17),

$$P\left\{ \sqrt{\varepsilon_n} \sup_{0 \leq t \leq T} |J^{\varepsilon_n}(t)| > \varepsilon_n^{\gamma/2} \right\} \leq C_{13}(M)\varepsilon_n^{M\gamma}.$$
and by the Borel–Cantelli lemma

\[(6.21) \quad \sqrt{\epsilon_n} \sup_{0 \leq t \leq T} |J^{\gamma_n}(t)| = O(\epsilon_n^{\gamma/2}) \text{ a.s.}\]

Observe that integrating by parts we have

\[(6.22) \int_0^T \left( \Sigma(\bar{X}_x(\epsilon s)) - \Sigma(\bar{X}_x(\epsilon u)) \right) dW(s) = \left( \Sigma(\bar{X}_x(\epsilon v)) - \Sigma(\bar{X}_x(\epsilon u)) \right) W(v) - \epsilon \int_0^T \nabla \Sigma(\bar{X}_x(\epsilon s)) B(\bar{X}_x(\epsilon s)) W(s) ds\]

where (similarly to (2.23)) for a matrix function \(\Sigma(y) = (\Sigma_{ij}(y))\) and a vector \(\eta = (\eta_1, \ldots, \eta_d)\) we denote by \(\nabla \Sigma(y) \eta\) the matrix function such that \((\nabla \Sigma(y) \eta)_{ij} = \sum_{1 \leq k \leq d} \frac{\partial \Sigma_{ij}(y)}{\partial y_k} \eta_k\). This together with (2.23) yields that

\[
\begin{align*}
|J^{\gamma_n}(t) - J^{\gamma}(t)| &\leq \sup_{0 \leq s \leq T/\epsilon} |W(s)| \sum_{0 \leq t \leq \epsilon^{-1} n} \left( |\Sigma(\bar{X}_x(\epsilon^{-1} n l + 1)) - \Sigma(\bar{X}_x(\epsilon^{-1} n l + 1))| + |\Sigma(\bar{X}_x(\epsilon^{-1} n l)) - \Sigma(\bar{X}_x(\epsilon^{-1} n l))| \right) \\
&\quad + \epsilon \left( f(t(l+1)\epsilon^{-1} n) - f(t(l)\epsilon^{-1} n) \right) \left| \nabla \Sigma(\bar{X}_x(\epsilon^{-1} n s)) B(\bar{X}_x(\epsilon^{-1} n s)) \right| ds + \epsilon \left( f(t(l)\epsilon^{-1} n) - f(t(l-1)\epsilon^{-1} n) \right) \left| \nabla \Sigma(\bar{X}_x(\epsilon^{-1} n s)) B(\bar{X}_x(\epsilon^{-1} n s)) \right| ds \\
&\quad + 4LT \max_{0 \leq l \leq \epsilon^{-1} n} |W(l(\epsilon^{-1} n - 1)) - W(l(\epsilon^{-1} n - 1))| \\
&\leq C_{14}(T) \left( \sup_{0 \leq s \leq T/\epsilon} |W(s)| (\epsilon^{-1} n + \epsilon) \right) \\
&\quad \leq C_{14}(T) \left( n^{-1/2} \sup_{0 \leq s \leq T/\epsilon} |W(s)| (\epsilon^{-1} n + \epsilon) \right) \left( m^{-1} \sup_{0 \leq t \leq T/\epsilon} |W(t)| n^{-1} \sup_{0 \leq s \leq T/\epsilon} |W(s)| (\epsilon^{-1} n + \epsilon) \right).
\end{align*}
\]

where \(C_{14}(T) > 0\) does not depend on \(\epsilon\). It follows that

\[
\begin{align*}
\mathcal{J}_n(T) &= \sup_{n+1 \leq l \leq \epsilon n} \sup_{0 \leq t \leq T} |J^{\gamma_n}(t) - J^{\gamma}(t)| \\
&\leq C_{14}(T) \left( n^{-1/2} \sup_{0 \leq s \leq T/\epsilon} |W(s)| (\epsilon^{-1} n + \epsilon) \right).
\end{align*}
\]

The standard martingale uniform estimates for the Brownian motion

\[
E \sup_{0 \leq t \leq (n+1)T} |W(t)|^{2M} \leq \left( \frac{2M}{2M-1} \right)^{2M} n^{2M} (n+2)^M \prod_{k=1}^M (2k-1)
\]

and

\[
E \max_{0 \leq t \leq (n+1)T} \sup_{n+1 \leq l \leq \epsilon n} |W(l(\epsilon^{-1} n - 1)) - W(l(\epsilon^{-1} n - 1))|^{2M} \leq \left( \frac{2M}{2M-1} \right)^{2M} n \sum_{0 \leq l \leq (n+1)T} E |W(l(l+n+1)^{1-\gamma} - n^{1-\gamma})|^{2M}
\]

It follows that

\[
E(\mathcal{J}_n(T))^{2M} \leq C_{15}(T)n^{-2M\gamma}
\]
provided that \( \gamma < \frac{1}{2} \). By the Chebyshev inequality and the Borel–Cantelli lemma we obtain similarly to (6.21) that
\[
\mathcal{J}_n(T) = O(n^{-\gamma/2}) = O(\varepsilon_n^{\gamma/2}) \quad \text{a.s.}
\]
This together with (6.21) yields that for all \( \varepsilon > 0 \),
\[
(6.23) \quad \sqrt{\varepsilon} \sup_{0 \leq t \leq T} |J^\varepsilon(t)| = O(\varepsilon^{\gamma/2}) \quad \text{a.s.}
\]
Introduce \( \hat{W}(t) = \hat{W}_\varepsilon(t) = \sqrt{\varepsilon}W(t/\varepsilon) \) which is another Brownian motion with the covariance matrix \( \varsigma \) at the time 1. Then we can choose a standard \( d \)-dimensional Brownian motion \( W = W_\varepsilon \) such that \( \varsigma^{1/2}W_\varepsilon = \hat{W}_\varepsilon \). Then
\[
(6.24) \quad \sqrt{\varepsilon} \int_0^{t/\varepsilon} \Sigma(\bar{X}_\varepsilon(s))dW(s) = \int_0^t \Sigma(\bar{X}_\varepsilon(u))dW_\varepsilon(u) = \int_0^t \sigma(\bar{X}_\varepsilon(u))dW_\varepsilon(u).
\]
Finally, combining (6.10), (6.12), (6.16), (6.20), (6.23), (6.24) and taking \( \gamma = \delta/2 \) we obtain that
\[
\sup_{0 \leq t \leq T} |Z^\varepsilon(t) - \sqrt{\varepsilon}G(t)| = O(\varepsilon^{(1+\delta)/2}) \quad \text{a.s.}
\]
provided that the Gaussian process \( G \) is given by (1.4) with \( W = W_\varepsilon \) the same as in (6.21). This together with (6.3), (6.4) and (6.8) completes the proof of Theorem 2.6 for the discrete time case while the corresponding assertion in the continuous time case follows as well, in view of the discretization estimates of Section 5. \( \square \)

6.4. Remark. The assumption that \( B(x,\xi) = \Sigma(x)\xi \) enables us to employ the strong approximation theorem to the sums of the sequence \( \xi(n) \), \( -\infty < n < \infty \) itself. Without this assumption we would have to apply the strong approximation theorem to the sums \( S^\varepsilon(t) = \sum_{0 \leq k < [t/\varepsilon]} (B(\bar{X}_\varepsilon(k\varepsilon), \xi(k)) - B(\bar{X}_\varepsilon(k\varepsilon))) \) whose summands change with \( \varepsilon \) and the number of summands depends on \( \varepsilon \), as well, i.e. we have to deal here with arrays. This was possible in Section 3 and 5 when our goal was to obtain estimates in Theorems 2.1 and 2.4 for each fixed \( \varepsilon \) but it is not clear how to apply this machinery to sums with changing summands with the goal to obtain almost sure estimates of the form \( O(\varepsilon^k) \) as \( \varepsilon \to 0 \) and to use this in order to obtain the law of iterated logarithm of Theorem 2.7.

6.3. Law of iterated logarithm. Let \( W \) be the standard \( d \)-dimensional Brownian motion such that \( \varsigma^{1/2}W = W \). Then by (1.4) and (6.24) the Gaussian process \( G = G_\varepsilon \) satisfies
\[
(6.25) \quad G_\varepsilon(t) = \int_0^t \nabla \bar{B}(\bar{X}_\varepsilon(s))G_\varepsilon(s)ds + \sqrt{\varepsilon} \int_0^{t/\varepsilon} \sigma(\bar{X}_\varepsilon(s))dW(s).
\]
Hence,
\[
G_\varepsilon = \Phi(\varphi_\varepsilon) \quad \text{where} \quad \varphi_\varepsilon(t) = \sqrt{\varepsilon} \int_0^{t/\varepsilon} \sigma(\bar{X}_\varepsilon(s))dW(s)
\]
with the map \( \Phi \) defined by (2.26). Integrating by parts we obtain that
\[
\int_0^{t/\varepsilon} \sigma(\bar{X}_\varepsilon(s))dW(s) = \sigma(\bar{X}_\varepsilon(t))W(t/\varepsilon) - \varepsilon \int_0^{t/\varepsilon} \nabla \sigma(\bar{X}_\varepsilon(s))W(s)ds
\]
\[
= \sigma(\bar{X}_\varepsilon(t))W(t/\varepsilon) - \int_0^t \nabla \sigma(\bar{X}_\varepsilon(u))W(u/\varepsilon)du.
\]
Hence,
\[
(6.26) \quad G_\varepsilon = \Phi\Psi(\sqrt{\varepsilon}\zeta_\varepsilon) \quad \text{where} \quad \zeta_\varepsilon(t) = W(t/\varepsilon), \; t \in [0,T].
\]
Observe that (2.25) is a particular case of the second order Volterra equation, and so it has a unique solution which can be seen here directly since if $\Phi_1(\varphi)$ and $\Phi_2(\varphi)$ are two solutions of (2.25), then
\[
|\Phi_1(\varphi)(t) - \Phi_2(\varphi)(t)| \leq L \int_0^t |\Phi_1(\varphi)(s) - \Phi_2(\varphi)(s)|ds
\]
for all $t \in [0, T]$ which by the Gronwall inequality implies that $\Phi_1(\varphi) = \Phi_2(\varphi)$. It follows, in particular, that $\Phi : C_\varepsilon[0, T] \rightarrow C_\varepsilon[0, T]$ is a linear map. In addition, by (2.25) for any two $\varphi, \psi \in C_\varepsilon[0, T]$,
\[
|\Phi(\varphi)(t) - \Phi(\psi)(t)| \leq L \int_0^t |\Phi(\varphi)(s) - \Phi(\psi)(s)|ds + |\varphi(t) - \psi(t)|,
\]
and so by the Gronwall inequality,
\[
\|\Phi(\varphi) - \Phi(\psi)\|_{[0, T]} \leq e^{LT} \|\varphi - \psi\|_{[0, T]},
\]
i.e. $\Phi$ is a continuous map. Concerning $\Psi$ it is clear from (2.26) that it is a linear continuous map of $C_\varepsilon[0, T]$ into itself.

By the Strassen theorem (see [36]) with probability one as $\varepsilon \to 0$ the set of limit points in $C_\varepsilon[0, T]$ of
\[
(\frac{2}{\varepsilon} \log \log \frac{1}{\varepsilon})^{-1/2} \zeta_{\varepsilon} = \frac{\sqrt{2\log \log \frac{1}{\varepsilon}}}{\sqrt{2 \log \log \frac{1}{\varepsilon}}
\]
coincides with the compact set $K \subset C_\varepsilon[0, T]$ defined in Section 2.4. In fact, [36] deals with the limit points of $(\frac{2}{\varepsilon} \log \log \frac{1}{\varepsilon})^{-1/2} \zeta_{\varepsilon}$ as $\varepsilon_n = 1/n \to 0$ but it is not difficult to extend the above assertion to all $\varepsilon \to 0$ and though an even more general result can be found in [2], for readers’ convenience we will provide the direct proof here, as well. Let $\varepsilon_n \leq \varepsilon < \varepsilon_{n-1}$ with $n \geq 4$ and assume that log is the natural logarithm. Then
\[
(6.27) \quad |(\frac{2}{\varepsilon_n} \log \log \frac{1}{\varepsilon_n})^{-1/2} \zeta_{\varepsilon_n}(t) - (\frac{2}{\varepsilon} \log \log \frac{1}{\varepsilon})^{-1/2} \zeta_{\varepsilon}(t)| \leq I_1(n) + I_2(n)
\]
where
\[
I_1(n) = \sup_{\varepsilon_n \leq \varepsilon < \varepsilon_{n-1}} |(\frac{2}{\varepsilon_n} \log \log \frac{1}{\varepsilon_n})^{-1/2} - (\frac{2}{\varepsilon} \log \log \frac{1}{\varepsilon})^{-1/2}| \sup_{0 \leq t \leq T} |\mathbf{W}(nt)|
\]
and
\[
I_2(n) = 3(n-1)^{-1/2} \sup_{0 \leq t \leq T} \sup_{\varepsilon_n \leq \varepsilon < \varepsilon_{n-1}} |\zeta_{\varepsilon_n}(t) - \zeta_{\varepsilon}(t)|
\]
\[
\leq 6(n-1)^{-1/2} \max_{0 \leq k \leq n-1} \sup_{kT \leq t \leq (k+1)T} |\mathbf{W}(t) - \mathbf{W}(kT)|.
\]

Observe that when $3 \leq (n - 1) < u \leq n$, then
\[
|\frac{d}{du} (2u \log \log u)^{-1/2}| \leq (2u)^{-3/2} (\log \log u)^{-1/2} (1 + \log u \log \log u)^{-1} \leq 2(n-1)^{-3/2}
\]
and $|\varepsilon_{n-1} - \varepsilon^{-1}| \leq 1$. This together with the standard moment estimates for the Brownian motion yields that
\[
E(I_1(n))^{2M} \leq C_{15}(M)n^{-2MT^M}
\]
for some $C_{15}(M) > 0$ which does not depend on $n$. By the Chebyshev inequality
\[
P(I_1(n) > n^{-1/2}) \leq C_{15}(M)n^{-M T^M}.
\]
Taking $M = 2$ we obtain from the Borel–Cantelli lemma that with probability one $I_1(n) \leq n^{-1/2}$ for all $n$ large enough, and so
\[
\lim_{n \to \infty} I_1(n) = 0 \quad \text{a.s.}
\]
Again, by the standard moment estimates for the Brownian motion
\[
E(I_2(n))^{2M} \leq 6^{2M}(n - 1)^{-M} \sum_{0 \leq k \leq n - 1} E \sup_{kT \leq t \leq (k+1)T} |W(t) - W(kT)|^{2M}
\]
\[
\leq C_{16}(M)6^{2M}(n - 1)^{-M} nT^M,
\]
where $C_{16}(M) > 0$ which does not depend on $n$, and in the same way as above we conclude that
\[
\lim_{n \to \infty} I_2(n) = 0 \quad \text{a.s.}
\]
This together with (6.27) yields that the sets of limit points of $(\frac{2}{\epsilon_n} \log \log \frac{1}{\epsilon_n})^{-1/2} \zeta_{\epsilon_n}$ as $n \to \infty$ and of $(\frac{2}{\epsilon} \log \log \frac{1}{\epsilon})^{-1/2} \zeta_\epsilon$ as $\epsilon \to 0$ both coincide with $K$.

Finally, since $\Phi$ and $\Psi$ are linear and continuous we obtain from (6.26) that with probability one as $\epsilon \to 0$ the set of limit points of $(2 \log \log \frac{1}{\epsilon})^{-1/2} G_\epsilon$ coincides with the compact set $\Phi \Psi(K)$ which together with (2.36) yields the assertion of Theorem 2.7. □

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Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel

Email address: kifer@math.huji.ac.il