Pseudoscalar Meson Mixing, the Contribution of the Hadronic Continuum to Deviation from Factorization

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Abstract

The contribution of the hadronic continuum in the QCD sum rule calculation of the parameters entering in pseudoscalar meson mixing is evaluated by making use of simple integration kernels tailored in order to practically eliminate the contribution of the hadronic continuum. This approach avoids the arbitrariness and instability inherent to previous sum rule calculations. An independent evaluation of the mixed quark gluon condensate $\langle QGC \rangle = \langle \bar{g} \sigma_{\mu \nu} G_{\mu \nu} q \rangle$ which enters in the calculation is presented as well as the calculation of the K-meson decay constant $f_K$ to five loops.

1 Introduction

A considerable amount of attention [1] has been devoted to the study of neutral meson mixing. In particular the matrix elements $\langle \bar{K}_0(p') | \Theta_{\Delta S=2} | K_0(p) \rangle$ and $\langle \bar{B}_0(p') | \Theta_{\Delta B=2} | B_0(p) \rangle$ where $\Theta_{\Delta S=2} = (\bar{s}_L \gamma_\mu d_L)(\bar{s}_L \gamma_\mu d_L)$ and $\Theta_{\Delta B=2} = (\bar{b}_L \gamma_\mu d_L)(\bar{b}_L \gamma_\mu d_L)$ which contribute to the mass differences of the neutral mesons and in studies of CP violation have been classic subjects of investigation. The simplest approach (factorization) reduces these matrix elements to the products

$$\langle \bar{K}_0 | \Theta_{\Delta S=2} | K_0 \rangle = \langle \bar{K}_0 | \bar{s}_L \gamma_\mu d_L | 0 \rangle \langle 0 | \bar{s}_L \gamma_\mu d_L | K_0 \rangle$$

$$\langle \bar{B}_0 | \Theta_{\Delta B=2} | B_0 \rangle = \langle \bar{B}_0 | \bar{b}_L \gamma_\mu d_L | 0 \rangle \langle 0 | \bar{b}_L \gamma_\mu d_L | B_0 \rangle$$

deviation from factorization is described by a parameter $B$ which multiplies the above matrix elements. In factorization $B = 1$.

Sophisticated calculations of $B$ appeared in the literature using quark and bag models, lattice calculations and QCD sum rule techniques.

The latter start from a 3-point function involving two pseudoscalar currents in addition to the $\Delta S (\Delta B) = 2$ four quark operator

$$A(p, p')(p, p') = i^2 \int dxdy \epsilon^{ipx-ipy'} \langle 0 | T j_5(x) \Theta_{\Delta S,B=2} (0) j_5(y) | 0 \rangle$$

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Dispersion relations are written for this quantity and intermediate states inserted. The sought for matrix elements are provided by the meson poles. In addition there is a potentially large contribution arising from the pseudoscalar continuum of which not much is known.

The aim of any reliable calculation is to minimize this contribution before neglecting it.

In the case of the K-mesons the contribution of the continuum is damped by use of the Borel (Laplace) transform in which case the damping is provided by an exponential kernel $e^{-tM^2}$. If the parameter $M^2$, the square of the Borel mass, is small the damping is good but the contribution of unknown higher order condensates increases rapidly. If $M^2$ increases the contribution of the higher order condensates decreases but the damping in the resonance region worsens. An intermediate value of $M^2$ has to be chosen. Because $M^2$ is an unphysical parameter the results should not depend on it in a relatively broad interval which is often not the case. The choice of the parameter which signals the onset of perturbative QCD is another source of uncertainty.

In the case of the B-mesons, Koerner et al. [3] use inverse powers of the dispersion variables (moments) which is legitimate because the matrix elements are infrared safe. As usual the potentially large contribution of the hadronic continuum is unknown. They estimate it by using gaps and other ill-known quantities. Recent work on B-meson mixing using Heavy Quark Effective Theory is also available [4].

In this work the aim is to practically eliminate the contribution of the hadronic continua by introducing kernels which vanish at the position of the resonances and are very small in a broad region around them.

The method will also be applied to the evaluation of the quark-gluon mixed condensate $\langle QGC \rangle = m_0^2 \langle 0 | \bar{q}q | 0 \rangle$ which enters in the calculation and to the evaluation of the K-meson decay constant.

2 $\bar{K}^0 - K^0$ Mixing

In the Standard Model (SM) [6] the mixing of the two eigenstates of strangeness is predicted as a higher order process which contributes to the $K_L - K_S$ mass difference through the so called $\Delta S = 2$ box diagram.

The $K_L - K_S$ mass difference $\Delta m$ is a sum of a long distance dispersive contribution $\Delta m_L$ and a short distance one $\Delta m_S$ proportional to the matrix element $\langle \bar{K}^0(p') | \Theta_{\Delta S=2} | K^0(p) \rangle$

With

$$\Theta_{\Delta S=2} = (\tilde{s}\gamma_{\mu}(1 - \gamma_5)d)(\tilde{s}\gamma_{\mu}(1 - \gamma_5)d)$$

Neglecting anomalous dimension factors the parameter $B$ is defined

$$\langle \bar{K}^0(p') | \Theta_{\Delta S=2} | K^0(p) \rangle = \frac{16}{3} B f_K^2 \langle p.p' \rangle$$

$B = 1$ in vacuum saturation and $f_K = .114$ GeV

Sophisticated calculations of $B$ followed using quark and bag models, lattice calculations and QCD sum rules techniques. Unfortunately no single value for $B$ has emerged.

Start with a 3-point function involving two pseudoscalar currents in addition to the $\Delta S = 2$ quark operator

$$A(p,p')(p.p') = i^2 \int \int dxdye^{ipx - ip'y} \langle 0 | Tj_5(x)\Theta_{\Delta S=2}(0)j_5(y) | 0 \rangle$$

where $j_5(x) = \bar{d}(x)\gamma_5s(x)$ is the pseudoscalar current

Dispersion relations for this quantity are written and intermediate states inserted. The $K$-meson poles carry the sought for information in addition there is the contribution of the strange pseudoscalar continuum of which not much is known except that it is dominated by two radial excitations of the $K$, $K(1460)$ and $K(1830)$. In order to damp the unknown contribution of the continuum Borel (Laplace) transforms have been used in
which case the damping is provided by an exponential kernel. As discussed in the introduction I shall proceed otherwise in order to avoid the arbitrariness and instability inherent to this method. In this work I shall use polynomial kernels in order to eliminate the contribution of the unknown continuum. The coefficients of these polynomials are chosen to make the roots coincide with the masses of the radial excitations of the $K$.

The amplitude $A(t=p^2, t'=p'^2, p.p')$ will be studied at fixed $p.p'$ and will be denoted by $A(t,t')$

$A(t,t')$ possesses a double pole, two single poles and cuts on the real $t, t'$ axes extending from $th = (m_K + 2m_π)^2$ to infinity stemming from the strange pseudoscalar intermediate states.

$$A(t,t')(p.p') = 2f_K^2 m_K^4 \langle K^0 |\Theta_{\Delta S=2} | K^0 \rangle + \frac{\Phi(t)}{t - m_K^2} + \frac{\Phi(t')}{t' - m_K^2} + \cdots$$ (7)

Consider now the double integral in the complex $t$ and $t'$ planes

$$\frac{1}{(2\pi i)^2} \int_c \int_{c'} dt dt' P(t) P(t') A(t,t')(p.p')$$ (8)

where $c$ and $c'$ are the contours shown on Fig. 1, $f_K$ is the $K$ decay constant and $P(t)$ is a so far arbitrary entire function.

![Figure 1: The contours of integration c,c'](image)

Because $\Phi(t), \Phi(t')$ have no singularities inside the contours of integration the single poles do not contribute to the double integral and we are left with

$$\frac{2f_K^2 m_K^4}{(m_s + m_d)^2} \langle K^0 |\Theta_{\Delta S=2} | K^0 \rangle P^2(m_K^2) = \frac{1}{(2\pi i)^2} \int_c \int_{c'} dt dt' P(t) P(t') A(t,t')(p.p')$$ (9)

The integrals over the cuts represent the contribution of the pseudoscalar strange continuum. $P(t)$ is now chosen to be a second order polynomial whose roots coincide with the masses squared of the radial excitations of the $K, K(1400)$ and $K(1870)$.

$$P(t) = 1 - a_1 t - a_2 t^2 = 1 - .768 GeV^{-2} t + .14 GeV^{-4} t^2$$ (10)

This choice of $P(t)$ and $P(t')$ which vanishes at the radial excitations of the $K$ and is very small in a very
broad region around them, practically eliminates the contribution of the hadronic continuum and leaves us with the integrals on the circles of large radius \( R \) where \( A(t,t') \) can be replaced by \( A_{QCD}(t,t') \) so that using

\[
\langle \bar{K}_0 | \Theta_{\Delta S=2} | K^0 \rangle = \frac{16}{3} f_K^2 B(p,p')
\]

gives

\[
\frac{32}{3} \frac{f_K^2 m_K^2}{(m_s + m_d)^2} P^2(m_K^2) B = \frac{1}{(2\pi i)^2} \oint dt \oint dt' P(t) P(t') A_{QCD}(t,t')
\]

(11)

\( A_{QCD}(t,t') \) is the sum of a factorizable and a non-factorizable part [2]

\[
A_{QCD} = A_f^{QCD} + A_{nf}^{QCD}
\]

(12)

where

\[
A_f^{QCD} = \frac{8}{3} \Pi_5(t) \Pi_5(t')
\]

(13)

\[
\Pi_5(t) = -\frac{3}{8\pi^2} m_s \ln(-t) + \frac{\langle \bar{d}d + \bar{s}s \rangle}{t} + \frac{m_s}{8t^2} \langle a_s GG \rangle + \cdots
\]

(14)

and

\[
A_{nf}^{QCD} = \frac{2}{3} m_0^2 \langle \bar{q}q \rangle^2 \left( \frac{1}{t^2 p} + \frac{1}{t^2 p'} \right) + \frac{1}{4\pi^2} m_0^2 \langle \bar{q}q \rangle \frac{1}{t^2} \left( \ln(-t) + \ln(-t') \right) - \frac{4\pi^2}{9} \langle \bar{q}q \rangle^2 \langle a_s GG \rangle + \frac{13}{288} m_0^4 \langle \bar{q}q \rangle^2 \frac{1}{t^2 p^2} + \cdots
\]

(15)

Because \( B_f = 1 \), eqs. (11), (13) and (14) yield

\[
\frac{2f_K^2 m_K^2}{m_s + m_d} P(m_K^2) = I
\]

(16)

where

\[
I = \frac{1}{2\pi i} \oint dt P(t) \Pi_5(t)
\]

(17)

\[
= -\frac{3m_s}{8\pi^2} \frac{1}{2\pi i} \oint dt P(t) \ln(-t) + \langle \bar{d}d + \bar{s}s \rangle + \frac{a_1 m_s}{8} \langle a_s GG \rangle
\]

Because \( \ln(-t) \) has a cut on the positive \( t \)-axis which starts at the origin the integral over the circle in the equation above can be transformed into an integral over the real axis so that

\[
I = -\frac{3m_s}{8\pi^2} \int_0^R dt P(t) + \langle \bar{d}d + \bar{s}s \rangle + \frac{a_1 m_s}{8} \langle a_s GG \rangle
\]

(18)

The choice of \( R \) is determined by stability considerations. It should not be too small as this would invalidate the Operator Product Expansion on the circle, nor should it be too large because \( P(t) \) would start enhancing the contribution of the continuum instead of suppressing it. We seek an intermediate range of \( R \) for which the integral in eq. (18) is stable.

The integral \( i(R) = \int_0^R dt P(t) \) is seen to be stable for \( 2GeV^2 \leq R \leq 4GeV^2 \), \( i(R) \approx .83GeV \) as shown in Fig. 2.
Figure 2: The variation of $i(R) = \int_0^R dt P(t)$ as a function of $R$ in Gev

Then

$$I = \langle \bar{d}d + \bar{s}s \rangle + \frac{3}{8\pi^2} m_s i(R) + \frac{a_1 m_s}{8} \langle a_s GG \rangle$$

so that

$$\frac{2 f_K^4}{m_s + m_d} P(m_K^2) = \langle \bar{d}d + \bar{s}s \rangle + \frac{3}{8\pi^2} m_s i(R) + \frac{a_1 m_s}{8} \langle a_s GG \rangle$$

The equation above is dominated by the quark condensate term $\langle \bar{d}d + \bar{s}s \rangle$. Eq. (20) is seen to be a version of the Gell-Mann, Oakes, Renner relation [7] in the strange sector modified by $SU(3) \times SU(3)$ chiral symmetry breaking.

Turn now to the contribution of the non-factorizable part. Similar manipulations lead to

$$\frac{4 f_K^4 m_K^4 P^2(m_K^2) B_{nf}}{(m_s + m_d)^2} = -\frac{1}{2} a_1 m_0^2 \langle \bar{q}q \rangle^2 + \frac{3 m_0^2}{16\pi^2} \langle m_s \bar{q}q \rangle \ln \frac{R}{\mu^2} - \frac{3}{8} \left( \frac{4\pi^2}{9} \langle \bar{q}q \rangle^2 \langle a_s GG \rangle + \frac{13}{288} m_0^4 \langle \bar{q}q \rangle^2 a_1^2 \right)$$

Values of $m_0^2$, which parameterizes the quark-gluon mixed condensate vary over a large range in the literature.

The method presented here offers an independent evaluation of this quantity:

The integral

$$\frac{1}{(2\pi i)^2} \int_c dt P(t)(t - m_K^2) \int_{c'} dt' P(t')(t' - m_K^2) A^{GCD}(t, t') = I'_f + I'_{nf} = 0$$

vanishes because the singularities inside $c$ and $c'$ have been removed.

$$I'_f = \frac{8}{3} I'^2$$

$$I' = \frac{1}{(2\pi i)} \int dt P(t)(t - m_K^2) \left\{ -\frac{3m_K^2}{8\pi^2} \ln(-t) + \frac{\langle \bar{d}d + \bar{s}s \rangle}{t} - m_s \frac{\langle a_s GG \rangle}{8t^2} \right\}$$

$$=- m_K^2 \langle \bar{d}d + \bar{s}s \rangle - \frac{3m_K^2}{8\pi^2} i'(R) - \frac{m_s}{8} \langle a_s GG \rangle (1 + a_1 m_K^2)$$

where $i'(R) = \int_0^R dt P(t)(t - m_K^2)$
The non-factoizable contribution is

\[ I'_{nf} = -\frac{4}{3} m_0^2 \langle \bar{q}q \rangle^2 m_K^2 (1 + a_1 m_K^2) - \frac{m_s}{2 \pi^2} m_0^2 \langle \bar{q}q \rangle m_K^2 i''(R) - m_K^2 \ln \frac{R}{\mu^2} \left[ -\left( \frac{4\pi^2}{9} \langle \bar{q}q \rangle \right) a_{\text{SGL}} + \frac{13}{288} m_0^4 \langle \bar{q}q \rangle^2 \right] (1 + a_1 m_K^2)^2 \]  

(26)

where \( i''(R) = \int_0^R dt [1 + a_1 m_K^2 - (a_1 - a_2 m_K^2) t - a_2 t^2] \).

The condensate \( \langle \bar{d}d + \bar{s}s \rangle \) dominates our equations. It could be obtained from eq. [16], an improved calculation (to five loops) is found in [8], it gives

\[ - (m_s + m_d) \langle \bar{d}d + \bar{s}s \rangle = (0.39 \pm 0.03) \times 10^{-2} \text{ GeV}^4 \]  

(27)

This, with \( (m_s + m_d) = (108 \pm 8) \text{ MeV} \) yields \( m_0^2 \approx 1.0 \text{ GeV}^2 \) which determines \( B_{nf} \)

\[ B_{nf} = -0.09 \text{ or } B \approx 0.91 \]  

(28)

3 \( f_K \) to Five Loops

Theoretical calculations of the weak decay constants \( f_K \) and \( f_\pi \) are of great interest. This has been done recently in the context of the extended Nambu-Jona-Lasinio model [9], using an improved holographic wave function [10] in the light-front quark model [11] or on lattice calculations [12].

I offer here instead a QCD calculation of \( f_K \) to five loops. Start with the correlator

\[ \Pi_{\mu\nu}(t = q^2) = i \int dx e^{iqx} \langle 0 | TA_{\mu}(x) A_{\nu}(0) | 0 \rangle = (q_{\mu} q_{\nu} - g_{\mu\nu} q^2) \Pi^1(t) + q_{\mu} q_{\nu} \Pi^0(t) \]  

(29)

Let \( \Pi(t) = \Pi^{0+1}(t) \) and consider

\[ \int dt P(t) \Pi(t) = f_K^2 P(m_K^2) = \frac{1}{\pi} \int_0^R dt P(t) Im \Pi(t) + \frac{1}{4} \int dt P(t) \Pi^{QC\bar{D}}(t) \]  

(30)

As before the polynomial \( P(t) \) is chosen in order to eliminate the contribution of the integral on the cut. We have now to take into account the axial-vector resonances in addition to the pseudoscalar ones, i.e. \( K_1(1273) \) and \( K_1(1402) \) in addition to \( K(1460) \) and \( K(1830) \).

The choice \( P(t) = 1 - 1.42t^2 + 0.68t^2 - 0.093t^4 \) with the coefficients in appropriate powers of GeV achieves the purpose of eliminating the contribution of the continuum. Here

\[ \Pi^{QC\bar{D}}(t) = \Pi^{pert}(t) + \frac{c_1}{t} + \frac{c_2}{t^2} + \frac{c_3}{t^3} + \cdots \]  

(31)

\[ 4\pi Im \Pi^{pert}(t) = 1 + a_s(r) + a_s^2(r) l_2(t, r) + a_s^3(r) l_3(t, r) + a_s^4(r) l_4(t, r) \]  

(32)

The \( l_i(t, r) \) and the strong coupling constant \( a_s(r) \) are known to 5-loop order [15,16] and the non-perturbative condensates are given in [17].
and light mass eigenstates. The simplest approach (factorization) \([6]\) reduces model dependent and more reliable way.

\[
\begin{align*}
c_1 & = \frac{3m_s^2}{4\pi^2} (1 + \frac{7}{3} a_s) \\
c_2 & = \frac{1}{12} (1 - \frac{11}{18} a_s) \langle a_s GG \rangle + (1 - \frac{a_s}{3}) \langle m_s \bar{s}s \rangle + O(a_s^2) \\
c_3 & = -a_s \frac{32\pi^2}{9} [\langle \bar{q}q \rangle \langle \bar{s}s \rangle + \frac{1}{9} \langle \bar{q}q \rangle^2] + O(a_s^2)
\end{align*}
\]

The integral of \(\Pi_{\text{pert}}(t)\) over the circle is transformed into an integral over the cut once again and finally

\[
2f_K^2 P(m_K^2) = \frac{1}{\pi} \int_0^R dt P(t) Im\Pi_{\text{pert}}(t) + c_1 - a_1c_2 - a_2c_3
\]

With the standard values \(m_s = .10 GeV^4, \langle a_s GG \rangle = .013 GeV^4, \langle \bar{s}s \rangle = .6 \langle \bar{q}q \rangle, \langle \bar{q}q \rangle = .02 GeV^3\), the final result is

\[
f_K = .107 GeV
\]

The pion decay constant \(f_\pi\) could also be studied. In this case data on the continuum is available from \(\tau\) decay and yields \(f_\pi\) \([3]\). The method used above can likewise be applied taking into account the pseudoscalar and axial-vector resonances with the result

\[
f_\pi = .092 GeV
\]

4 \(\bar{B}^0 - B^0\) Mixing

Turn now to \(\bar{B}^0 - B^0\) mixing. Start with the three point correlation function

\[
\Pi(p_1, p_2) = \int \int dx dy e^{i(p_1 \cdot x - p_2 \cdot y)} \langle 0 | T \bar{f}_B(x) O(0) j_B(y) | 0 \rangle
\]

The operator \(j_B = (m_b + m_d) \bar{d}\gamma_5 b = \partial_\mu (\bar{d}_L \gamma_\mu \gamma_5 b_L)\) is the interpolating current for the \(B\) meson and \(\langle 0 | \bar{J}_B(0) | B^0 \rangle = f_B m_B^2\).

The relevant quantity to calculate is the matrix element \(A = \langle \bar{B}^0 | O(\mu) | B^0 \rangle\) where \(O(\mu) = (b_L \gamma_\mu d_L)(\bar{d}_L \gamma_\mu b_L)\) is the local 4-quark operator at the normalization point \(\mu\) which can be used to evaluate the splitting of heavy and light mass eigenstates. The simplest approach (factorization) \([3]\) reduces \(A\) to

\[
A' = \frac{8}{3} \langle \bar{B}^0 | \bar{b}_L \gamma_\mu d_L | 0 \rangle \langle 0 | \bar{b}_L \gamma_\mu d_L | B^0 \rangle = \frac{2}{3} f_B^2 m_B^2
\]

where \(\langle 0 | \bar{b}_L \gamma_\mu d_L | B^0(\mu) \rangle = i f_B p_\mu\)

The deviation from factorization is again parametrized by \(B_B\) defined as \(A = B_B A'\). In factorization \(B_B = 1\).

A nice way to calculate \(B\), using inverse moments, was used in \([3]\). In the present work I shall follow their approach but the contribution of the higher resonances and continuum shall be estimated in a different less model dependent and more reliable way.

A dispersion representation of the correlator reads

\[
\Pi(p_1^2, p_2^2, q^2) = \int dt_1 dt_2 \frac{g(t_1, t_2, q^2)}{(t_1 - p_1^2)(t_2 - p_2^2)}
\]

where \(q = p_1 - p_2\). Consider the moments of the correlation function at \(p_1^2 = p_2^2 = q^2 = 0\)
\[ M_{ij} = \int dt_1 dt_2 \frac{\theta(t_1, t_2, 0)}{t_1^{t_1} t_2^{t_2}} \]

Because the origin is infrared safe \( M_{ij} \) can be computed in QCD [13]. It also has a phenomenological representation

\[ M_{ij}^{\text{ph}} = \frac{8}{3} B_B f_B^2 \frac{m_B^2}{m_B^{2(i+j)}} + \cdots \] (40)

where the ellipses stand for the contribution of the higher resonances and the continuum. Separating the factorizable part we obtain two sum rules

\[ \frac{8}{3} B_B f_B^4 m_B^2 m_B^{2(i+j)} + \Delta_{ij} = M_{ij} \]
\[ \frac{8}{3} f_B^4 m_B^2 m_B^{2(i+j)} + \Delta'_{ij} = M^f_{ij} \] (41)

The l.h.s of the above eqs. represent the phenomenology and the r.h.s the QCD theoretical expressions.

The \( \Delta' \)'s are the contributions of the higher resonances and the continuum. While [3] try to estimate their contribution using gaps and other ill-known parameters, I aim to eliminate it altogether.

For this purpose note that the integrals in eq. (40) are fast convergent so that the bulk of the contribution to the continuum come from the vicinity of the first resonance. If this contribution is eliminated \( \Delta_{ij} \) and \( \Delta'_{ij} \) become negligible and can be discarded. This is done by using instead of eq. (40)

\[ I_{ij} = \int dt_1 dt_2 \frac{\theta(t_1, t_2)}{t_1^{t_1} t_2^{t_2}} (m_1^2 - 1)(m_2^2 - 1) = M_{ij} - m_i^2 M_{i,j+1} + m_i^2 M_{i+1,j} + m_i^4 M_{i+1,j+1} \] (42)

If \( m' \) is close to the mass of the first resonance, the factor \( (m_1^2 - 1) \) annihilates the integrand in its vicinity. Because of the fast convergence due to the denominators the main contribution of the continuum is eliminated and this justifies the neglect of \( \Delta \) and \( \Delta' \).

Taking \( i = j = 2 \) eqs. (42) become

\[ \frac{8}{3} B_B f_B^4 m_B^2 m_B^{2(i+j)} - \frac{2}{m_B^{2(i+j)}} = I_{ij} \]
\[ \frac{8}{3} f_B^4 m_B^2 m_B^{2(i+j)} - \frac{2}{m_B^{2(i+j)}} = I^f_{ij} \] (44)

or

\[ B_B = \frac{M_{22} - 2m^2 M_{23} + m^4 M_{33}}{M^f_{22} - 2m^2 M^f_{23} + m^4 M^f_{33}} \] (45)

The \( M \)'s are the theoretical QCD values and the \( M^f \) their factorizable counterparts.

It was found in [3] that

\[ M_{ij} = \frac{m_b^6 a_{ij}}{m_b^{2(i+j)}} (1 + \frac{a_s}{4} (b_{ij}^f + b_{ij}^n)) + M_{ij}^{\text{non-perturbative}} \] (46)

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Expressions for $M_{ij}^{\text{non-perturbative}}$ are given in [13]. In the present approach their contribution vanishes. The quantities $a_{ij}$, $b_{ij}^{f}$, $b_{ij}^{nf}$ represent the LO, NLO and non-factorizable contributions.

Then, if

$$B_B = \frac{I_{22}}{I_{22}} = 1 + \frac{a - \delta}{4}$$

$$\delta = \frac{a_{22}b_{22}^{nf} - 2(\frac{m'}{m_b})^2a_{23}b_{23}^{nf} + (\frac{m'}{m_b})^4a_{33}b_{33}^{nf}}{a_{22} - 2(\frac{m'}{m_b})^2a_{23} + (\frac{m'}{m_b})^4a_{33}}$$

The corresponding expression in the work of [3] is

$$\delta = b_{22}^{nf} + \delta R + \delta C$$

Where $\delta R$ and $\delta C$ are parameters which account for the resonances and continuum contribution and which they estimate by fitting and using gap parameters. The present approach avoids this arbitrariness.

The $a_{ij}$ are

$$a_{ij} = m_b^{2(i+j)-6}M_{ij}^{LO}$$

$$M_{ij}^{LO} = \int \frac{dt_1dt_2}{t_1t_2} \frac{4}{3}(t_1 + t_2)\varrho(t_1)\varrho(t_2)$$

$$\varrho(t) = \frac{3}{16\pi^2}m_q^2(1 - \frac{m_q^2}{t})$$

which yields

$$a_{22} = \frac{1}{(16\pi^2)^2} \frac{8}{3}, \quad a_{23} = \frac{1}{(16\pi^2)^2} \frac{2}{3}, \quad a_{33} = \frac{1}{(16\pi^2)^2} \frac{1}{6}$$

The $b_{ij}^{nf}$ are given in [3]

$$b_{22}^{nf} = .68, \quad b_{23}^{nf} = 1.22, \quad b_{33}^{nf} = 1.96$$

The Particle Data Group lists two candidates for $m'$, $m'(5.84)$ and $m'(5.97)$. A reasonable choice is then $\frac{(m')^2}{m_b^2} = 2.0$ which yields $\frac{a - \delta}{4} = .006$ or $B_B \simeq 1.0$

5 Discussion

In this work I have studied the contribution of the hadronic continuum to deviations from factorization in neutral K and B meson mixing. In both cases I minimized this contribution before neglecting it by using simple kernels in the dispersion integrals which vanish at the low lying resonances, in the case of the K-meson a polynomial kernel was used, the same method (and kernel ) was also used in the calculation of the quark- gluon mixed condensate as well as the evaluation of the K-meson decay coupling constant $f_K$.

For the B-meson inverse moments were used. These render the integrals fast convergent and concentrate the contribution to the dispersion integral in the vicinity of the first resonance. In this case the kernel used is of the form $(\frac{m'}{t} - 1)$ where $m'$ lies in the vicinity of the first resonance.

This method avoids the arbitrariness and instability inherent to previously used ones.
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