Strong convergence theorems for the general split variational inclusion problem in Hilbert spaces

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Abstract
The purpose of this paper is to introduce and study a general split variational inclusion problem in the setting of infinite-dimensional Hilbert spaces. Under suitable conditions, we prove that the sequence generated by the proposed new algorithm converges strongly to a solution of the general split variational inclusion problem. As a particular case, we consider the algorithms for a split feasibility problem and a split optimization problem and give some strong convergence theorems for these problems in Hilbert spaces.

Keywords: general split variational inclusion problem; split feasibility problem; split optimization problem; quasi-nonexpansive mapping; zero point; resolvent mapping

1 Introduction
Let C and Q be nonempty closed convex subsets of real Hilbert spaces $H_1$ and $H_2$, respectively. The split feasibility problem (SFP) is formulated as

$$P_C(I - \gamma A^*(I - P_Q)A)x^*,$$

where $A : H_1 \to H_2$ is a bounded linear operator. In 1994, Censor and Elfving [1] first introduced the SFP in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. It has been found that the SFP can also be used in various disciplines such as image restoration, computer tomography and radiation therapy treatment planning [3–5]. The SFP in an infinite-dimensional real Hilbert space can be found in [2, 4, 6–10]. For comprehensive literature, bibliography and a survey on SFP, we refer to [11].

Assuming that the SFP is consistent, it is not hard to see that $x^* \in C$ solves SFP if and only if it solves the fixed point equation

$$x^* = P_C(I - \gamma A^*(I - P_Q)A)x^*,$$

where $P_C$ and $P_Q$ are the metric projection from $H_1$ onto $C$ and from $H_2$ onto $Q$, respectively, $\gamma > 0$ is a positive constant, and $A^*$ is the adjoint of $A$. 

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A popular algorithm to be used to solve the SFP (1.1) is due to Byrne's CQ-algorithm [2]:
\[ x_{k+1} = P_C \left( (I - \gamma_k A^*(I - P_Q)A)x_k \right), \quad k \geq 1, \]
where \( \gamma_k \in (0, 2/\lambda) \) with \( \lambda \) being the spectral radius of the operator \( A^*A \).

On the other hand, let \( H \) be a real Hilbert space, and \( B \) be a set-valued mapping with \( \text{domain } D(B) := \{ x \in H : B(x) \neq \emptyset \} \). Recall that \( B \) is called monotone, if \( \langle u - v, x, x - y \rangle \geq 0 \) for any \( u \in Bx \) and \( v \in By; B \) is maximal monotone, if its graph \( \{(x, y) : x \in D(B), y \in Bx\} \) is not properly contained in the graph of any other monotone mapping. An important problem for set-valued monotone mappings is to find \( x^* \in H \) such that \( 0 \in B(x^*) \). Here, \( x^* \) is called a zero point of \( B \). A well-known method for approximating a zero point of a maximal monotone mapping defined in a real Hilbert space \( H \) is the proximal point algorithm first introduced by Martinet [12] and generated by Rockafellar [13]. This is an iterative procedure, which generates \( \{x_n\} \) by \( x_1 = x \in H \) and
\[ x_{n+1} = J_B^{\beta_n} x_n, \quad n \geq 1, \quad \text{(1.2)} \]
where \( \{\beta_n\} \subset (0, \infty) \), \( B \) is a maximal monotone mapping in a real Hilbert space, and \( J_B^r \) is the resolvent mapping of \( B \) defined by \( J_B^r = (I + rB)^{-1} \) for each \( r > 0 \). Rockafellar [13] proved that if the solution set \( B^{-1}(0) \) is nonempty and \( \lim \inf_{n \to \infty} \beta_n > 0 \), then the sequence \( \{x_n\} \) in (1.2) converges weakly to an element of \( B^{-1}(0) \). In particular, if \( B \) is the sub-differential \( \partial f \) of a proper convex and lower semicontinuous function \( f : H \to \mathbb{R} \), then (1.2) is reduced to
\[ x_{n+1} = \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2\beta_n} \|y - x_n\|^2 \right\}, \quad \forall n \geq 1. \quad \text{(1.3)} \]
In this case, \( \{x_n\} \) converges weakly to a minimizer of \( f \). Later, many researchers have studied the convergence problems of the proximal point algorithm in Hilbert spaces (see [14–21] and the references therein).

Motivated by the works in [14–17] and related literature, the purpose of this paper is to introduce and consider the following general split variational inclusion problem.

Let \( H_1 \) and \( H_2 \) be two real Hilbert spaces, \( B_i : H_1 \to H_1 \) and \( K_i : H_2 \to H_2, i = 1, 2, \ldots \) be two families of set-valued maximal monotone mappings, \( A : H_1 \to H_2 \) be a linear and bounded operator, and \( A^* \) be the adjoint of \( A \). The so-called general split variational inclusion problem is
\[ \text{to find } x^* \in H_1 \text{ such that } 0 \in \bigcap_{i=1}^{\infty} B_i(x^*) \text{ and } 0 \in \bigcap_{i=1}^{\infty} K_i(Ax^*). \quad \text{(1.4)} \]

The following examples are special cases of (GSVIP) (1.4).

**Classical split variational inclusion problem.** Let \( B : H_1 \to H_1 \) and \( K : H_2 \to H_2 \) be set-valued maximal monotone mappings. The so-called classical split variational inclusion problem (CSVIP) is
\[ \text{to find } x^* \in H_1 \text{ such that } 0 \in B(x^*) \text{ and } 0 \in K(Ax^*), \quad \text{(1.5)} \]
which was introduced by Moudafi [17]. It is obvious that problem (1.5) is a special case of (GSVIP) (1.4). In [17], Moudafi proved that the iteration process

\[ x_{n+1} = f^B_n(x_n + \gamma A^* f^K_n - I)Ax_n \]

converges weakly to a solution of problem (1.5), where \( \lambda \) and \( \gamma \) are given positive numbers.

**Split optimization problem.** Let \( f : H_1 \to \mathbb{R}, g : H_2 \to \mathbb{R} \) be two proper convex and lower semicontinuous functions. The so-called split optimization problem (SOP) is

to find \( x^* \in H_1 \) such that \( f(x^*) = \min_{y \in H_1} f(y) \) and \( g(Ax^*) = \min_{z \in H_2} g(z) \). (1.6)

Denote by \( B = \partial (f) \) and \( K = \partial (g) \), then \( B \) and \( K \) both are maximal monotone mappings, and problem (1.6) is equivalent to the following classical split variational inclusion problem, i.e.

\[ \text{to find } x^* \in H_1 \text{ such that } 0 \in \partial (f(x^*)) \text{ and } 0 \in \partial (g(Ax^*)). \] (1.7)

**Split feasibility problem.** As in (1.1), let \( C \) and \( Q \) be two nonempty closed convex subsets of real Hilbert spaces \( H_1 \) and \( H_2 \), respectively and \( A \) be the same as above. The split feasibility problem is

\[ \text{to find } x^* \in C \text{ such that } Ax^* \in Q. \] (1.8)

It is well known that this kind of problems was first introduced by Censor and Elfving [1] for modeling inverse problems arising from phase retrievals and in medical image reconstruction [2]. Also it can be used in various disciplines such as image restoration, computer tomography and radiation therapy treatment planning.

Let \( i_C, i_Q \) be the indicator function of \( C, Q \), i.e.,

\[ i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C; \end{cases} \quad i_Q(x) = \begin{cases} 0, & \text{if } x \in Q, \\ +\infty, & \text{if } x \notin Q. \end{cases} \] (1.9)

Then \( i_C \) and \( i_Q \) both are proper convex and lower semicontinuous functions, and its subdifferentials \( \partial i_C \) and \( \partial i_Q \) are maximal monotone operators. Consequently problem (1.8) is equivalent to the following ‘split optimization problem’ and ‘Moudafi’s classical split variational inclusion problem,’ i.e.,

\[ \text{to find } x^* \in H_1 \text{ such that } i_C(x^*) = \min_{y \in H_1} i_C(y) \text{ and } i_Q(Ax^*) = \min_{z \in H_2} i_Q(z) \]

\[ \Leftrightarrow \text{ to find } x^* \in H_1 \text{ such that } 0 \in \partial (i_C(x^*)) \text{ and } 0 \in \partial (i_Q(Ax^*)). \] (1.10)

For solving (GSVIP) (1.4), in our paper we propose the following iterative algorithms:

\[ x_{n+1} = \alpha_n x_n + \xi_n f(x_n) + \sum_{i=1}^{\infty} \gamma_n f^K_n \left[ x_n - \lambda_n A^* (I - f^K_n) A x_n \right], \quad \forall n \geq 0, \] (1.11)
where $f : H_1 \to H_1$ is a contraction mapping with a contractive constant $k \in (0, 1)$, $(\alpha_n)$, $(\xi_n)$, and $(\gamma_n)$ are sequence in $[0, 1]$ satisfying some conditions. Under suitable conditions, some strong convergence theorems for the sequence proposed by (1.11) to a solution for (GSVIP) (1.4) in Hilbert spaces are proved. As a particular case, consider the algorithms for a split feasibility problem and a split optimization problem and give some strong convergence theorems for these problems in Hilbert spaces. Our results extend and improve the related results of Censor and Elfving [1], Byrne [2], Censor et al. [3–5], Rockafellar [13], Moudafi [14, 17], Eslamian and Latif [15], Eslamian [21], and Chuang [22].

2 Preliminaries

Throughout the paper, we denote by $H$ a real Hilbert space, $C$ be a nonempty closed and convex subset of $H$ and $F(T)$ denote by the set of fixed points of a mapping $T$. Let $(x_n)$ be a sequence in $H$ and $x \in H$. Strong convergence of $(x_n)$ to $x$ is denoted by $x_n \to x$, and weak convergence of $(x_n)$ to $x$ is denoted by $x_n \rightharpoonup x$. For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_Cx$. This point satisfies

$$
\|x - P_Cx\| \leq \|x - y\|, \quad \forall y \in C.
$$

The operator $P_C$ is called the metric projection. The metric projection $P_C$ is characterized by the fact that $P_Cx \in C$ and

$$
\langle x - P_Cx, P_Cx - y \rangle \geq 0, \quad \forall x \in H, y \in C.
$$

Recall that a mapping $T : C \to H$ is said to be nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. $T$ is said to be quasi-nonexpansive, if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$ for every $x \in C$ and $p \in F(T)$. It is easy to see that $F(T)$ is a closed convex subset of $C$ if $T$ is a quasi-nonexpansive mapping. Besides, $T$ is said to be a firmly nonexpansive, if

$$
\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle \quad \forall x, y \in C;
$$

$$
\iff \|Tx - Ty\|^2 \leq \|x - y\|^2 - \|I - T\| x - (I - T)y\|^2 \quad \forall x, y \in C.
$$

Lemma 2.1 (demi-closed principle) Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to H$ be a nonexpansive mapping, and let $(x_n)$ be a sequence in $C$. If $x_n \rightharpoonup w$ and $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$, then $Tw = w$.

Lemma 2.2 [23] Let $H$ be a (real) Hilbert space. Then for all $x, y \in H$,

$$
\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle. \tag{2.1}
$$

Lemma 2.3 [24] Let $H$ be a Hilbert space and let $(x_n)$ be a sequence in $H$. Then, for any given sequence $(\lambda_n) \subset (0, 1)$ with $\sum_{n=1}^{\infty} \lambda_n = 1$ and for any positive integers $i, j$ with $i < j$,

$$
\left\| \sum_{n=1}^{\infty} \lambda_n x_n \right\|^2 \leq \sum_{n=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j \|x_i - x_j\|^2. \tag{2.2}
$$
Lemma 2.4  Let \( \{a_n\} \) be a sequence of nonnegative real numbers, \( \{b_n\} \) be a sequence of real numbers in \((0,1)\) with \( \sum_{n=1}^{\infty} b_n = \infty \), \( \{u_n\} \) be a sequence of nonnegative real numbers with \( \sum_{n=1}^{\infty} u_n < \infty \), \( \{t_n\} \) be a real numbers with \( \limsup_{n \to \infty} t_n \leq 0 \). If
\[
a_{n+1} \leq (1 - b_n)a_n + b_n t_n + u_n, \quad \text{for each } n \geq 1,
\]
then \( \lim_{n \to \infty} a_n = 0 \).

Lemma 2.5  [25]  Let \( \{a_n\} \) be a sequence of real numbers such that there exists a subsequence \( \{n_i\} \) of \( \{n\} \) such that \( a_{n_i} < a_{n_{i+1}} \) for all \( i \in \mathbb{N} \). Then there exists a nondecreasing sequence \( \{m_k\} \subset \mathbb{N} \) such that \( m_k \to \infty \), \( a_{m_k} \leq a_{m_k+1} \) and \( a_k \leq a_{m_k+1} \) are satisfied by all (sufficiently large) numbers \( k \in \mathbb{N} \). In fact, \( m_k = \max\{j \leq k : a_j < a_{j+1}\} \).

Lemma 2.6  [22]  Let \( H \) be a real Hilbert space, \( B : H \to 2^H \) be a set-valued maximal monotone mapping, \( \beta > 0 \), and let \( J_\beta^B \) be the resolvent mapping of \( B \).

(i) For each \( \beta > 0 \), \( J_\beta^B \) is a single-valued and firmly nonexpansive mapping;
(ii) \( D(J_\beta^B) = H \) and \( F(J_\beta^B) = B^{-1}(0) := \{x \in D(B) : 0 \in Bx\} \);
(iii) \( (I - J_\beta^B) \) is a firmly nonexpansive mapping for each \( \beta > 0 \);
(iv) suppose that \( B^{-1}(0) \neq \emptyset \), then for each \( x \in H \), each \( x^* \in B^{-1}(0) \) and each \( \beta > 0 \)
\[
\|x - J_\beta^B x\|^2 + \|J_\beta^B x - x^*\|^2 \leq \|x - x^*\|^2;
\]
(v) suppose that \( B^{-1}(0) \neq \emptyset \). Then \( \langle x - J_\beta^B x, J_\beta^B x - w \rangle \geq 0 \) for each \( x \in H \) and each \( w \in B^{-1}(0) \), and each \( \beta > 0 \).

Lemma 2.7  Let \( H_1, H_2 \) be two real Hilbert spaces, \( A : H_1 \to H_2 \) be a linear bounded operator and \( A^* \) be the adjoint of \( A \). Let \( B : H_2 \to 2^{H_2} \) be a set-valued maximal monotone mapping, \( \beta > 0 \), and let \( J_\beta^B \) be the resolvent mapping of \( B \), then

(i) \( \| (I - J_\beta^B)Ax - (I - J_\beta^B)Ay \|^2 \leq \| (I - J_\beta^B)Ax - (I - J_\beta^B)Ay \|^2 \);
(ii) \( \| A^*(I - J_\beta^B)Ax - A^*(I - J_\beta^B)Ay \|^2 \leq \| A^2(I - J_\beta^B)Ax - (I - J_\beta^B)Ay, Ax - Ay \|^2 \);
(iii) if \( \rho \in (0, \frac{2}{\|A\|^2}), \) then \( (I - \rho A^*(I - J_\beta^B)A) \) is a nonexpansive mapping.

Proof  By Lemma 2.6(iii), the mapping \( (I - J_\beta^B) \) is firmly nonexpansive, hence the conclusions (i) and (ii) are obvious.

Now we prove the conclusion (iii).

In fact, for any \( x, y \in H_1 \), it follows from the conclusions (i) and (ii) that
\[
\| (I - \rho A^*(I - J_\beta^B)A)x - (I - \rho A^*(I - J_\beta^B)A)y \|^2
= \|x - y\|^2 - 2\rho \langle x - y, A^*(I - J_\beta^B)Ax - A^*(I - J_\beta^B)Ay \rangle
+ \rho^2 \| A^*(I - J_\beta^B)Ax - A^*(I - J_\beta^B)Ay \|^2
\leq \|x - y\|^2 - 2\rho \langle Ax - Ay, (I - J_\beta^B)Ax - (I - J_\beta^B)Ay \rangle
+ \rho^2 \| A\|^2 \langle I - J_\beta^B)Ax - (I - J_\beta^B)Ay \|^2
\leq \|x - y\|^2 - \rho (2 - \rho \| A \|^2) \langle I - J_\beta^B)Ax - (I - J_\beta^B)Ay \|^2
\leq \|x - y\|^2 \quad \text{(since } \rho (2 - \rho \| A \|^2) \geq 0 \).
\]
This completes the proof of Lemma 2.7.
3 Main results

The following lemma will be used in proving our main results.

Lemma 3.1 Let $H_1$ and $H_2$ be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear and bounded operator, and $A^*$ be the adjoint of $A$. Let $B_i : H_1 \to 2^{H_1}$, and $K_i : H_2 \to 2^{H_2}$, $i = 1, 2, \ldots$, be two families of set-valued maximal monotone mappings, and let $\beta > 0$ and $\gamma > 0$. If $\Omega \neq \emptyset$ (the solution set of (GSVIP) (1.4)), then $x^* \in H_1$ is a solution of (GSVIP) (1.4) if and only if for each $i \geq 1$, for each $\gamma > 0$ and for each $\beta > 0$

$$x^* = J_{\beta}^{B_i} (x^* - \gamma A^* (I - J_{\beta}^{K_i}) A x^*).$$  \hfill (3.1)

Proof Indeed, if $x^*$ is a solution of (GSVIP) (1.4), then for each $i \geq 1$, $\gamma > 0$ and $\beta > 0$, $x^* \in B_i^{-1}(0)$ and $Ax^* \in K_i^{-1}(0)$, i.e., $x^* = J_{\beta}^{B_i} x^*$ and $Ax^* = J_{\beta}^{K_i} Ax^*$.

This implies that $x^* = J_{\beta}^{B_i} (x^* - \gamma A x^* (I - J_{\beta}^{K_i}) A x^*)$.

Conversely, if $x^*$ solves (3.1), by Lemma 2.6(v), we have

$$\langle x^* - (x^* - \gamma A x^* (I - J_{\beta}^{K_i}) A x^*), y - x^* \rangle \geq 0, \quad \forall y \in B_i^{-1}(0).$$

Hence we have

$$\langle (I - J_{\beta}^{K_i}) A x^*, Ay - Ax^* \rangle \geq 0, \quad \forall y \in B_i^{-1}(0).$$ \hfill (3.2)

On the other hand, by Lemma 2.6(v) again

$$\langle (Ax^* - J_{\beta}^{K_i} Ax^*), J_{\beta}^{K_i} Ax^* - v \rangle \geq 0, \quad \forall v \in K_i^{-1}(0).$$ \hfill (3.3)

Adding up (3.2) and (3.3), we have

$$\langle Ax^* - J_{\beta}^{K_i} Ax^*, J_{\beta}^{K_i} Ax^* + Ay - Ax^* - v \rangle \geq 0, \quad \forall y \in B_i^{-1}(0), \text{ and } v \in K_i^{-1}(0).$$

Simplifying it, we have

$$\|Ax^* - J_{\beta}^{K_i} Ax^*\|^2 \leq \langle Ax^* - J_{\beta}^{K_i} Ax^*, Ay - v \rangle \geq 0, \quad \forall y \in B_i^{-1}(0), \text{ and } v \in K_i^{-1}(0).$$ \hfill (3.4)

By the assumption that $\Omega \neq \emptyset$. Taking $w \in \Omega$, hence for each $i \geq 1$ $w \in B_i^{-1}(0)$ and $Aw \in K_i^{-1}(0)$. In (3.4), taking $y = w$ and $v = Aw$, then we have

$$\|Ax^* - J_{\beta}^{K_i} Ax^*\|^2 = 0.$$  

This implies that $Ax^* = J_{\beta}^{K_i} Ax^*$, and so $Ax^* \in K_i^{-1}(0)$ for each $i \geq 1$. Hence from (3.1), $x^* = J_{\beta}^{B_i} x^*$, i.e., $x^* \in B_i^{-1}(0)$. Hence $x^*$ is a solution of (GSVIP) (1.4).

This completes the proof of Lemma 3.1. \qed
We are now in a position to prove the following main result.

**Theorem 3.2** Let $H_1, H_2, A, A^*, \{B_j\}, \{K_i\}, \Omega$ be the same as in Lemma 3.1. Let $f : H_1 \to H_1$ be a contractive mapping with contractive constant $k \in (0, 1)$. Let $\{\alpha_n\}, \{\xi_n\}, \{\gamma_{n,i}\}$ be the sequences in $(0, 1)$ with $\alpha_n + \xi_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$, for each $n \geq 0$. Let $\{\beta_i\}$ be a sequence in $(0, \frac{2}{1-k})$, and $\{\lambda_{n,i}\}$ be a sequence in $(0, \frac{\alpha}{1-k})$. Let $\{x_n\}$ be the sequence defined by (1.1). If $\Omega \neq \emptyset$ and $\forall 0$ the following conditions are satisfied:

(i) $\lim_{n \to \infty} \xi_n = 0$, and $\sum_{n=0}^{\infty} \xi_n = \infty$;

(ii) $\lim \inf_{n \to \infty} \alpha_n \gamma_{n,i} > 0$ for each $i \geq 1$;

(iii) $0 < \lim \inf_{n \to \infty} \lambda_{n,i} \leq \lim \sup_{n \to \infty} \lambda_{n,i} < \frac{2}{1-k}$,

then $x_n \to x^* \in \Omega$ where $x^* = P_\Omega f(x^*)$, where $P_\Omega$ is the metric projection from $H_1$ onto $\Omega$.

**Proof** (I) First we prove that $\{x_n\}$ is bounded.

In fact, letting $z \in \Omega$, by Lemma 3.1, for each $i \geq 1$,

$$z = f_{\beta_i}^i [z - \lambda_{n,i} A^* (I - f_{\beta_i}^i) A z].$$

Hence it follows from Lemma 2.7(iii) that for each $i \geq 1$ and each $n \geq 1$ we have

$$\|x_{n+1} - z\| = \left\| \alpha_n x_n + \xi_n f(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i} f_{\beta_i}^i [x_n - \lambda_{n,i} A^* (I - f_{\beta_i}^i) A x_n] - z \right\|$$

$$\leq \alpha_n \|x_n - z\| + \xi_n \|f(x_n) - z\| + \sum_{i=1}^{\infty} \gamma_{n,i} \|f_{\beta_i}^i [x_n - \lambda_{n,i} A^* (I - f_{\beta_i}^i) A x_n] - z\|$$

$$\leq \alpha_n \|x_n - z\| + \xi_n \|f(x_n) - z\| + \sum_{i=1}^{\infty} \gamma_{n,i} \|f_{\beta_i}^i [x_n - \lambda_{n,i} A^* (I - f_{\beta_i}^i) A x_n] - z\|$$

$$\leq \alpha_n \|x_n - z\| + \xi_n \|f(x_n) - z\| + \sum_{i=1}^{\infty} \gamma_{n,i} \|x_n - z\|$$

$$= (1 - \xi_n) \|x_n - z\| + \xi_n \|f(x_n) - z\|$$

$$\leq (1 - \xi_n) \|x_n - z\| + \xi_n \|f(x_n) - f(z)\| + \xi_n \|f(z) - z\|$$

$$\leq (1 - \xi_n (1 - k)) \|x_n - z\| + \frac{\xi_n (1 - k)}{1 - k} \|f(z) - z\|$$

$$\leq \max \left\{ \|x_n - z\|, \frac{1}{1 - k} \|f(z) - z\| \right\}.$$

By induction, we can prove that

$$\|x_n - z\| \leq \max \left\{ \|x_0 - z\|, \frac{1}{1 - k} \|f(z) - z\| \right\}, \quad \forall n \geq 0. \quad (3.5)$$

This implies that $\{x_n\}$ is bounded, so is $\{f(x_n)\}$.

(II) Now we prove that for each $j \geq 1$

$$\alpha_n \gamma_{n,j} \|x_n - f_{\beta_i}^i [x_n - \lambda_{n,i} A^* (I - f_{\beta_i}^i) A x_n]\|^2$$

$$\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \xi_n \|f(x_n) - z\|^2, \quad \text{for each } i \geq 1. \quad (3.6)$$
Indeed, it follows from Lemma 2.3 that for any positive $j \geq 1$

$$\|x_{n+1} - z\|^2 = \|\alpha_n x_n + \xi_n f(x_n) + \sum_{i=1}^{\infty} \gamma_n f_{\beta_i} [x_n - \lambda_{n,i} A^* (I - f_{\beta_i}^{K_i}) A x_n] - z\|^2$$

$$\leq \alpha_n \|x_n - z\|^2 + \xi_n \|f(x_n) - z\|^2$$

$$+ \sum_{i=1}^{\infty} \gamma_n \|f_{\beta_i} [x_n - \lambda_{n,i} A^* (I - f_{\beta_i}^{K_i}) A x_n] - z\|^2$$

$$- \alpha_n \gamma_n \|x_n - f_{\beta_i}^{K_i} [x_n - \lambda_{n,i} A^* (I - f_{\beta_i}^{K_i}) A x_n]\|^2$$

$$\leq (1 - \xi_n) \|x_n - z\|^2 + \xi_n \|f(x_n) - z\|^2$$

$$- \alpha_n \gamma_n \|x_n - f_{\beta_i}^{K_i} [x_n - \lambda_{n,i} A^* (I - f_{\beta_i}^{K_i}) A x_n]\|^2.$$

Simplifying it, (3.6) is proved.

By the assumption that $\Omega \neq \emptyset$, and it is easy to prove that $\Omega$ is closed and convex. This implies that $P_\Omega$ is well defined. Again since $P_\Omega f : H_1 \to \Omega$ is a contraction mapping with contractive constant $k \in (0,1)$, there exists a unique $x^* \in \Omega$ such that $x^* = P_\Omega f x^*$. Since $x^* \in \Omega$, it solves (GSVIP) (1.4). By Lemma 3.1,

$$x^* = f_{\beta_i}^{K_i} (x^* - \lambda_{n,i} A^* (I - f_{\beta_i}^{K_i}) A x^*), \quad \forall j \geq 1, n \geq 0. \quad (3.7)$$

(III) Now we prove that $x_n \to x^*$.

In order to prove that $x_n \to x^*$ (as $n \to \infty$), we consider two cases.

Case 1. Assume that $\{\|x_n - x^*\|\}$ is a monotone sequence. In other words, for $n_0$ large enough, $\{\|x_n - x^*\|\}_{n \geq n_0}$ is either nondecreasing or non-increasing. Since $\{\|x_n - x^*\|\}$ is bounded, $\{\|x_n - x^*\|\}$ is convergence. Again since $\lim_{n \to \infty} \xi_n = 0$, and $\{f(x_n)\}$ is bounded, from (3.6) we get

$$\lim_{n \to \infty} \alpha_n \gamma_n \|x_n - f_{\beta_i}^{K_i} [x_n - \lambda_{n,i} A^* (I - f_{\beta_i}^{K_i}) A x_n]\|^2 = 0.$$

By condition (ii), we obtain

$$\lim_{n \to \infty} \|x_n - f_{\beta_i}^{K_i} [x_n - \lambda_{n,i} A^* (I - f_{\beta_i}^{K_i}) A x_n]\| = 0. \quad (3.8)$$

Now we prove that

$$\limsup_{n \to \infty} |f(x^*) - x^*, x_n - x^*| \leq 0. \quad (3.9)$$

To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to w, \lambda_{n_k,i} \to \lambda_i \in (0, \frac{2}{\|A\|^2})$ for each $i \geq 1$, and

$$\limsup_{n \to \infty} |f(x^*) - x^*, x_{n_k} - x^*| = \lim_{n_k \to \infty} |f(x^*) - x^*, x_{n_k} - x^*|. \quad (3.10)$$
It follows from (3.8) that

\[
\|J_{\bar{B}_i}[x_n - \lambda_iA^*(I - J_{\bar{B}_i})A x_n] - x_n\|
\]

\[
\leq \|J_{\bar{B}_i}[x_n - \lambda_iA^*(I - J_{\bar{B}_i})A x_n] - J_{\bar{B}_i}[x_n - \lambda_n A^*(I - J_{\bar{B}_i})A x_n]\|
\]

\[
+ \|J_{\bar{B}_i}[x_n - \lambda_n A^*(I - J_{\bar{B}_i})A x_n] - x_n\|
\]

\[
\leq \|\big[ x_n - \lambda_iA^*(I - J_{\bar{B}_i})A x_n\big] - \big[ x_n - \lambda_n A^*(I - J_{\bar{B}_i})A x_n\big]\|
\]

\[
+ \|J_{\bar{B}_i}[x_n - \lambda_n A^*(I - J_{\bar{B}_i})A x_n] - x_n\|
\]

\[
\leq |\lambda_i - \lambda_n| \|A^*(I - J_{\bar{B}_i})A x_n\|
\]

\[
+ \|J_{\bar{B}_i}[x_n - \lambda_n A^*(I - J_{\bar{B}_i})A x_n] - x_n\| \to 0 \quad \text{(as } n \to \infty). \]

For each \(i \geq 1, J_{\bar{B}_i}[I - \lambda_i A^*(I - J_{\bar{B}_i})A]\) is a nonexpansive mapping. Thus from Lemma 2.1, \(w = J_{\bar{B}_i}[I - \lambda_i A^*(I - J_{\bar{B}_i})A]w\). By Lemma 3.1 \(w \in \Omega\), i.e., \(w\) is a solution of (GSVIP) (1.4). Consequently we have

\[
\limsup_{n \to \infty} f(x^*) = \lim_{n \to \infty} f(x^*, x_n - x^*)
\]

\[
= \{f(x^*) - x^*, w - x^*\} \leq 0.
\]

(IV) Finally, we prove that \(x_n \to P_{\Omega}f(x^*)\).

In fact, from Lemma 2.2 we have

\[
\|x_{n+1} - x^*\|^2
\]

\[
\leq \|x_n - x^*\|^2 + \sum_{i=1}^{\infty} \gamma_{n_i} J_{\bar{B}_i}[x_n - \lambda_n A^*(I - J_{\bar{B}_i})A x_n] - x^*\|^2
\]

\[
+ 2\xi _n f(x_n) - x^*, x_{n+1} - x^*\]

\[
\leq (1 - \xi _n)^2 \|x_n - x^*\|^2 + 2\xi _n f(x_n) - f(x^*), x_{n+1} - x^*\]

\[
\leq (1 - \xi _n)^2 \|x_n - x^*\|^2 + 2\xi _n \|x_n - x^*\| \|x_{n+1} - x^*\|^2 + 2\xi _n f(x^*) - x^*, x_{n+1} - x^*\]

\[
\leq (1 - \xi _n)^2 \|x_n - x^*\|^2 + \xi _n \|x_{n+1} - x^*\|^2 + \|x_n - x^*\|^2
\]

\[
+ 2\xi _n f(x^*) - x^*, x_{n+1} - x^*\].

Simplifying it, we have

\[
\|x_{n+1} - x^*\|^2 \leq (1 - \xi _n)^2 + \xi _n \|x_n - x^*\|^2 + 2\xi _n f(x^*) - x^*, x_{n+1} - x^*\]

\[
\leq 1 - 2\xi _n + \xi _n \|x_n - x^*\|^2 + \frac{\xi _n^2}{1 - \xi _n k} \|x_n - x^*\|^2
\]

\[
+ 2\xi _n f(x^*) - x^*, x_{n+1} - x^*\]

\[
\leq (1 - \eta_0) \|x_n - x^*\|^2 + \eta_0 \delta_n, \quad \forall n \geq 0,
\]
where \( \delta_n = \frac{\xi_n M}{2(1-k)} + \frac{1}{1-k} (f(x^*) - x_n, x_{n+1} - x^*) \), \( M = \sup_{n \geq 0} \|x_n - x^*\|^2 \), and \( \eta_n = \frac{2(1-k)\xi_n}{1-k} \). It is easy to see that \( \eta_n \rightarrow 0 \), \( \sum_{n=1}^{\infty} \eta_n = \infty \), and \( \limsup_{n \rightarrow \infty} \delta_n \leq 0 \). Hence by Lemma 2.4, the sequence \( \{x_n\} \) converges strongly to \( x^* = P_{Qf}(x^*) \).

Case 2. Assume that \( \{\|x_n - x^*\|\} \) is not a monotone sequence. Then, by Lemma 2.3, we can define a sequence of positive integers: \( \{\tau(n)\} \), \( n \geq n_0 \) (where \( n_0 \) large enough) by

\[
\tau(n) = \max \left\{ k \leq n : \|x_k - x^*\| \leq \|x_{k+1} - x^*\| \right\}.
\]

Clearly \( \{\tau(n)\} \) is a nondecreasing sequence such that \( \tau(n) \rightarrow \infty \) as \( n \rightarrow \infty \), and for all \( n \geq n_0 \)

\[
\|x_{\tau(n)} - x^*\| \leq \|x_{\tau(n)+1} - x^*\|.
\]

Therefore \( \{\|x_{\tau(n)} - x^*\|\} \) is a nondecreasing sequence. According to Case (1), \( \lim_{n \rightarrow \infty} \|x_{\tau(n)} - x^*\| = 0 \) and \( \lim_{n \rightarrow \infty} \|x_{\tau(n)+1} - x^*\| = 0 \). Hence we have

\[
0 \leq \|x_n - x^*\| \leq \max \{\|x_n - x^*\|, \|x_{\tau(n)} - x^*\|\} \leq \|x_{\tau(n)+1} - x^*\| \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

This implies that \( x_n \rightarrow x^* \) and \( x^* = P_{Qf}(x^*) \) is a solution of (GSVIP) (1.4).

This completes the proof of Theorem 3.2. \( \square \)

In Theorem 3.2, if \( B_i \) and \( K_i \) are two proper, convex and lower semicontinuous functions, and \( A : H_1 \rightarrow H_2 \) be a linear and bounded operators. The so-called **split optimization problem** (SOP) is

\[
\min_{y \in H_1} h(y) \text{ and } \min_{z \in H_2} g(Az).
\]
Denote by \( \partial h = B \) and \( \partial g = K \). It is known that \( B : H_1 \rightarrow 2^{H_1} \) (resp. \( K : H_2 \rightarrow 2^{H_2} \)) is a maximal monotone mapping, so we can define the resolvent \( f^B_\beta = (I + \beta B)^{-1} \) and \( f^K_\beta = (I + \beta K)^{-1} \), where \( \beta > 0 \). Since \( x^* \) and \( Ax^* \) is a minimum of \( h \) on \( H_1 \) and \( g \) on \( H_2 \), respectively, for any given \( \beta > 0 \), we have

\[
x^* \in B^{-1}(0) = F(f^B_\beta), \quad \text{and} \quad Ax^* \in K^{-1}(0) = F(f^K_\beta).
\]

(4.2)

This implies that the (SOP) (4.1) is equivalent to the split variational inclusion problem (SVIP) (4.2). From Theorem 3.3 we have the following.

**Theorem 4.1** Let \( H_1, H_2, A, B, K, h, g \) be the same as above. Let \( f, \{\alpha_n\}, \{\xi_n\}, \{\gamma_n\} \) be the same as in Theorem 3.3. Let \( \beta > 0 \) be any given positive number, and \( \{\lambda_n\} \) be a sequence in \((0, \frac{2}{\|A\|^2})\). Let \( \{x_n\} \) be a sequence generated by \( x_0 \in H_1 \)

\[
\begin{align*}
  y_n &= \text{argmin}_{z \in H_2} \{g(z) + \frac{1}{2\beta} \|z - Ax_n\|^2 \}, \\
  z_n &= x_n - \lambda_n A^* (Ax_n - y_n), \\
  w_n &= \text{argmin}_{y \in H_1} \{h(y) + \frac{1}{2\beta} \|y - z_n\|^2 \}, \\
  x_{n+1} &= \alpha_n x_n + \xi_n f(x_n) + \gamma_n w_n, \quad n \geq 0.
\end{align*}
\]

(4.3)

If \( \Omega \neq \emptyset \), the solution set of the split optimization problem (4.1), and the following conditions are satisfied:

(i) \( \lim_{n \to \infty} \xi_n = 0 \), and \( \sum_{n=0}^{\infty} \xi_n = \infty \);

(ii) \( \lim \inf_{n \to \infty} \alpha_n \gamma_n > 0 \);

(iii) \( 0 < \lim \inf_{n \to \infty} \lambda_n \leq \lim \sup_{n \to \infty} \lambda_n < \frac{2}{\|A\|^2} \),

then \( x_n \to x^* \in \Omega \) where \( x^* = P_\Omega f(x^*) \).

**Proof** Since \( \partial h = B \), \( \partial g := K \), and \( y_n = \text{argmin}_{z \in H_2} \{g(z) + \frac{1}{2\beta} \|z - Ax_n\|^2 \} \), we have

\[
0 \in \left[K(z) + \frac{1}{\beta} (z - Ax_n)\right]_{z=y_n}, \quad \text{i.e.,} \quad Ax_n \in (\beta K + I)(y_n).
\]

This implies that

\[
y_n = f^K_\beta (Ax_n).
\]

(4.4)

Similarly, from (4.3), we have

\[
w_n = f^B_\beta (z_n).
\]

(4.5)

From (4.3)-(4.5), we have

\[
w_n = f^K_\beta (x_n - \lambda_n A^* (I - f^K_\beta) Ax_n).
\]

(4.6)

Therefore (4.3) can be rewritten as

\[
x_{n+1} = \alpha_n x_n + \xi_n f(x_n) + \gamma_n f^K_\beta (x_n - \lambda_n A^* (I - f^K_\beta) Ax_n), \quad n \geq 0.
\]

(4.7)

The conclusion of Theorem 4.1 can be obtained from Theorem 3.3 immediately. □
4.2 Application to split feasibility problem

Let \( C \subset H_1 \) and \( Q \subset H_2 \) be two nonempty closed convex subsets and \( A : H_1 \to H_2 \) be a bounded linear operator. Now we consider the following split feasibility problem, i.e.: to find

\[
x^* \in C \text{ such that } Ax^* \in Q.
\] (4.8)

Let \( i_C \) and \( i_Q \) be the indicator functions of \( C \) and \( Q \) defined by (1.9). Let \( N_C(u) \) be the normal cone at \( u \in H_1 \) defined by

\[
N_C(u) = \{ z \in H_1 : \langle z, v - u \rangle \leq 0, \forall v \in C \}.
\]

Since \( i_C \) and \( i_Q \) both are proper convex and lower semicontinuous functions on \( H_1 \) and \( H_2 \), respectively, and the subdifferential \( \partial i_C \) of \( i_C \) (resp. \( \partial i_Q \) of \( i_Q \)) is a maximal monotone operator, we can define the resolvents \( J^{\beta i_C}_\partial \) of \( \partial i_C \) and \( J^{\beta i_Q}_\partial \) of \( \partial i_Q \) by

\[
J^{\beta i_C}_\partial(x) = (I + \beta \partial i_C)^{-1}(x), \quad \forall x \in H_1,
\]

\[
J^{\beta i_Q}_\partial(x) = (I + \beta \partial i_Q)^{-1}(x), \quad \forall x \in H_2,
\]

where \( \beta > 0 \). By definition, we know that

\[
\partial i_C(x) = \{ z \in H_1 : i_C(x) + \langle z, y - x \rangle \leq i_C(y), \forall y \in H_1 \}
\]

\[
= \{ z \in H_1 : \langle z, y - x \rangle \leq 0, \forall y \in C \} = N_C(x), \quad x \in C.
\]

Hence, for each \( \beta > 0 \), we have

\[
u = J^{\beta i_C}_\partial(x) \iff x - u \in \beta N_C(u)
\]

\[
\iff \langle x - u, y - u \rangle \leq 0, \quad \forall y \in C \iff u = P_C(x).
\]

This implies that \( J^{\beta i_C}_\partial = P_C \). Similarly \( J^{\beta i_Q}_\partial = P_Q \). Taking \( h(x) = i_C(x) \) and \( g(x) = i_Q(x) \) in (4.1), then the (SFP) (4.8) is equivalent to the following split optimization problem:

to find \( x^* \in H_1 \) such that \( i_C(x^*) = \min_{y \in H_1} i_C(y) \) and \( i_Q(Ax^*) = \min_{z \in H_2} i_Q(z) \). (4.9)

Hence, the following result can be obtained from Theorem 4.1 immediately.

**Theorem 4.2** Let \( H_1, H_2, A, A^*, i_C, i_Q \) be the same as above. Let \( f, \{\alpha_n\}, \{\xi_n\}, \{\gamma_n\} \) be the same as in Theorem 4.1. Let \( \{\lambda_n\} \) be a sequence in \((0, \frac{2}{\|A\|^2})\). Let \( \{x_n\} \) be the sequence defined by

\[
x_{n+1} = \alpha_n x_n + \xi_n f(x_n) + \gamma_n P_C \left[ x_n - \lambda_n A^*(I - P_Q) A x_n \right], \quad \forall n \geq 0.
\] (4.10)

If the solution set of the split optimization problem (4.4) \( \Omega_2 \neq \emptyset \), and the following conditions are satisfied:
Theorem 4.3 Theorem 4.2 extends and improves the main results in Censor and Elfving [1] and Byrne [2].

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors read and approved the final manuscript.

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