Fox pairings of Poincaré duality groups

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Abstract

This paper develops the study of Fox pairings of a group $G$ from the viewpoint of group cohomology. We compute some cohomology groups of Fox pairings of $G$, where $G$ admits a Poincaré duality group pair. We also suggest fundamental Fox pairings and higher Fox pairings.

Keywords

Fox pairing, group cohomology, Poincaré duality, derivations

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1 Introduction

Let $\mathbb{K}[G]$ be the group ring of a group $G$ over a commutative ring $\mathbb{K}$. Let $M$ be a $\mathbb{K}[G]$-bimodule and $\text{aug} : \mathbb{K}[G] \to \mathbb{K}$ be the augmentation map. A Fox pairing (of $G$) [MT1, MT2, Tur1, Tur2] is defined to be a $\mathbb{K}$-bilinear map $\eta : \mathbb{K}[G] \times \mathbb{K}[G] \to M$ satisfying

$$\eta(a_1a_2, b) = \eta(a_1, b)\text{aug}(a_2) + a_1\eta(a_2, b), \text{ for any } a_1, a_2, b \in \mathbb{K}[G],$$

(1)
\[ \eta(a, b_1 b_2) = \eta(a, b_1) b_2 + \text{aug}(b_1) \eta(a, b_2), \quad \text{for any } a, b_1, b_2 \in \mathbb{K}[G]. \] (2)

See [MT2, Sections 6–8], Example 2.2, and Appendix A for examples. There have been studies of Fox pairings and their applications in the case that G is a surface group and \( M = \mathbb{K}[G] \); for example, they give a generalization of (logarithms of) Dehn twists (see [MT1]) and suggest a 3-dimensional description of the Goldman-Turaev Lie bialgebra (see [Mas]). See also [Tur2] for Fox pairings from knots in the 3-sphere.

This paper develops the study of Fox pairings from the viewpoint of the group cohomology of Poincaré duality pairs. We fix subgroups \( S_1, \ldots, S_m \subset G \) and denote the union \( \bigcup_{k=1}^m S_k \) by \( S \).

Roughly speaking, if the pair \((G, S)\) satisfies a Poincaré-Lefschetz duality over \( \mathbb{K} \), it is called a \( \mathbb{K}\)-Poincaré duality pair (see §3 for the definition). In Section 2, we review group cohomology and show (Proposition 2.3) that the set of Fox pairings is almost in a 1-1 correspondence with the set of the double cocycles, i.e., \( Z^1(G, S; Z^1(G; \mathbb{M})) \). Accordingly, in subsequent sections, we study the cohomology \( H^1(G, S; H^1(G; \mathbb{M})) \).

In Section 3, we first show (Proposition 3.2) that every Fox pairing of an \( n \)-dimensional Poincaré duality pair is null-cohomologous if \( n > 2 \). Thus, we concentrate on 2-dimensional Poincaré duality pairs and compute the cohomology. The following is a typical example.

**Theorem 1.1** (see Theorem 4.1). *If the pair \((G, S)\) is a 2-dimensional Poincaré duality pair over \( \mathbb{K} \) and \( S \neq \emptyset \), then the cohomology \( H^1(G, S; H^1(G; \mathbb{K}[G])) \) is isomorphic to \( \mathbb{K} \).*

We call a generator of the cohomology the fundamental Fox pairing of \((G, S)\) (Definition 4.2). Furthermore, we examine the uniqueness of the fundamental Fox pairing from a certain condition; see Theorem 4.3 and Corollary 4.4. As examples, if \( G \) is a surface group, then the fundamental Fox pairing is equal to the Fox pairing, as defined by Turaev [Tur1] (see Proposition 6.1).

We also explore Fox pairings of other groups. We first observe the fundamental Fox pairings of some orbifolds of dimension two (see Proposition 6.4). Next, in Section 7 examines the conditions of constructing Fox pairings from 3-manifold groups and shows that some homotopy groups are obstacles to the construction; see Proposition 7.1. Furthermore, Section 8 examines the commutator subgroups of knot groups and presents a \( \mathbb{Q}\)-duality with a comparison to the work of Turaev [Tur1, Tur2]. In addition, as a generalization of Fox pairings, Section 9 introduces higher Fox pairings of the \( \mathbb{K}\)-Poincaré duality pair of higher cohomological dimension, where we will show (Theorem 9.3) that the set of higher Fox pairings is closely related to the relative homology \( H_2(G, S; \mathbb{K}) \). For example, if \( G \) is a knot group in the 3-sphere, we point out a higher Fox pairing in the form of a Gysin map that is obtained from a Seifert surface; see Example 9.6. Furthermore, we also discuss the existence of higher Fox pairings for aspherical closed \( n \)-manifolds and the case \( S = \emptyset \); see Proposition 9.7 and Example 9.8.

To conclude, the above computations of the cohomology \( H^n(G, S; H^n(G; \mathbb{K}[G])) \) predict non-trivial (higher) Fox pairings, although it remains a problem for the future to describe these non-trivial Fox pairings concretely and to give their applications.
Conventional notation. We always write $G$ for a group and $\mathbb{K}$ for a commutative ring. By $\mathbb{K}[G]$, we mean the group ring of $G$. Furthermore, we fix a finite family of subgroups $S_1, \ldots, S_m \subset G$ such that $S_i \cap S_j = \{1\}$ for any $i \neq j$ (possibly, every $S_i$ is empty). We denote the union $\bigcup_{k=1}^{m} S_m$ by $\mathcal{S}$.

2 Fox pairings from the viewpoint of group cohomology

The purpose of this section is to suggest an approach to Fox pairings in terms of group (co)-homology.

We will begin by reviewing the (relative) group (co)-homology. Denote the group ring of a group $G$ over $\mathbb{K}$ by $\mathbb{K}[G]$. For $n \geq 1$, let $F_n(G)$ be the free $\mathbb{K}[G]$-module with basis $\{[g_1] \cdots [g_n]; g_i \in G\}$. Given a left $\mathbb{K}[G]$-module $A$, we define $C_n(G; A)$ to be $A \otimes_{\mathbb{K}[G]} F_n(G)$ and the differential operator $\partial_n$ by

$$(g_1^{-1}a) \otimes [g_2] \cdots [g_n] + \sum_{i=1}^{n} (-1)^i a \otimes [g_1] \cdots [g_{i-1}] [g_i g_{i+1}] [g_{i+2}] \cdots [g_n] + (-1)^n a \otimes [g_1] \cdots [g_{n-1}].$$

Then, we define the group homology $H_n(G; A)$ from this complex. The relative homology $H_n(G, \mathcal{S}; A)$ is defined to be the homology of the quotient complex $C_n(G; A)/ \sum_{k=1}^{m} C_n(S_k; A)$. Dually, given a right $\mathbb{K}[G]$-module $B$, we define $C_n^*(G; B)$ to be $F_n(G) \otimes_{\mathbb{K}[G]} B$ and another differential operator $\partial_n^*$ by

$$[g_2] \cdots [g_n] \otimes b + \sum_{i=1}^{n} (-1)^i [g_1] \cdots [g_i g_{i+1}] \cdots [g_n] \otimes b + (-1)^n [g_1] \cdots [g_{n-1}] \otimes bg^{-1}.$$

We can define the homology groups $H_n(G; B)$ and $H_n(G, \mathcal{S}; B)$ in the same fashion.

Dually, let $C_n^*(G; A)$ be the module $\text{Hom}(F_n(G), A)$ consisting of left $\mathbb{K}[G]$-module homomorphisms $f$. Furthermore, we define $C_n^*(G, \mathcal{S}; A)$ to be the submodule consisting of such $f$'s satisfying $f(l_{i,1}, \ldots, l_{i,n}) = 0$ for any $i \leq m$ and any $l_{i,1}, \ldots, l_{i,n} \in S_i$. For $f \in C^n(G; A)$, define $\delta_n^*(f) \in C^{n+1}(G; A)$ to be $f \circ \partial_{n+1}$. We denote by $Z_n^*(G, \mathcal{S}; A)$ the submodule of $C_n^*(G, \mathcal{S}; A)$ consisting of $n$-cocycles.

Example 2.1. Consider the case $n = 1$. A left (resp. a right) derivation $\partial : \mathbb{K}[G] \to A$ is a $\mathbb{K}$-homomorphism satisfying $\partial(ab) = \partial(a)\text{aug}(b) + a\partial(b)$ (resp. $\partial(ab) = \partial(a)b + \text{aug}(a)\partial(b)$).

Then, by the definition of $\delta_1^*$, $Z_1^*(G, A)$ is identified with the set of left derivations.

In addition, we can define the cap-product $\cap$; see, e.g., [BE] for the definition.

Next, we give some examples of Fox pairings:

Example 2.2. Let $M$ be a $\mathbb{K}[G]$-bimodule. If there are a $\mathbb{K}[G]$-bimodule homomorphism $\nu : M \otimes_{\mathbb{K}[G]} M \to M$ and a left derivation $D_l$ and a right one $D_r$ over $\mathbb{K}[G]$, the map $\mathbb{K}[G] \times \mathbb{K}[G] \to M$ which takes $a \otimes b$ to $\nu(D_l(a), D_r(b))$ is a Fox pairing. We denote the
pairing by $\eta_{D_i \otimes D_i}$. For example, when $G$ is a free group $\pi$ and $M = K[\pi]$, any Fox pairing is a sum of such Fox pairings arising from derivations (see [MT1, Section 2.5]).

For $c \in M$, the map that sends $(a, b)$ to $(\text{aug}(a) - a)(\text{aug}(b) - b)$ is a Fox pairing, where $a, b \in K[G]$. Such a Fox pairing is said to be inner. Furthermore, given a Fox pairing $\eta$ with $M = K[G]$, the mapping $(g, h) \mapsto g\eta(h^{-1}, g^{-1})h$ is also a Fox pairing, where the overline means the involution of $K[G]$ defined by $\bar{g} = g^{-1}$. We write $\eta^t$ for the mapping and call it the transpose of $\eta$.

Now we will give a characterization of every Fox pairing in terms of group cocycles (Proposition 2.3 below). Let $M$ be a $K[G]$-bimodule and $\mathcal{S}, \mathcal{S}'$ be collections of subgroups. Let us denote the set of Fox pairings by $\mathcal{FP}(G, M)$, which is canonically turned into a $K$-module. Define a submodule of $\mathcal{FP}(G; M)$ by setting

$$\{\eta : K[G] \otimes K[G] \rightarrow M : \text{Fox pairing} \mid \eta(l \otimes g) = \eta(g \otimes l') = 0 \text{ for any } l \in \mathcal{S}, l' \in \mathcal{S}', g \in G \},$$

and denote it by $\mathcal{FP}(G, \mathcal{S}, \mathcal{S}'; M)$. Take two copies, $G_1, G_2$, of $G$. Regarding $M$ as a right $K[G_1]$-module, we can define the complex $C_r^1(G_1; M)$. Since the left action of $G$ on $M$ gives rise to an action of $G$ on $C_r^1(G_1; M)$, we can consider another complex $C^*_l(G_2; Z^1_r(G_1; M))$. For a 1-cocycle $f \in Z^1_r(G_2; Z^1_r(G_1; M))$, we define a map $\eta_f : G_2 \times G_1 \rightarrow M$ by $\eta_f(g, h) := (f(g))(h)$, which bilinearly extends to $\mathbb{K}[G] \times \mathbb{K}[G] \rightarrow M$ as a Fox pairing.

**Proposition 2.3.** Let $M$ be a $K[G]$-bimodule. The correspondence $f \mapsto \eta_f$ gives rise to a $K$-module isomorphism,

$$Z^1_l(G_2, \mathcal{S}; Z^1_r(G_1, \mathcal{S}'; M)) \cong \mathcal{FP}(G, \mathcal{S}, \mathcal{S}'; M).$$

Furthermore, if $\eta_f$ is inner, then $f$ lies in $B^1_l(G_2, \mathcal{S}; Z^1_r(G_1, \mathcal{S}'; M))$.

**Proof.** First, consider the case $\mathcal{S} = \mathcal{S}' = \emptyset$. By linearity, we have a bijection,

$$\mathcal{FP}(G, M) \leftrightarrow \{\eta : G \times G \rightarrow M \mid \eta \text{ satisfies } (1) \text{ and } (2)\}.$$  \hspace{1cm} (3)

On the other hand, if we regard $C^n(G; M)$ as the set $\{G^n \rightarrow M\}$, we have canonical bijections,$

$$C^1_l(G_2; C^1_r(G_1; M)) \leftrightarrow \text{Map}(G_2, \text{Map}(G_1, M)) \leftrightarrow \text{Map}(G_2 \times G_1, M).$$

We can readily see that the restriction on $Z^1_l(G_2; Z^1_r(G_1; M))$ is onto the right-hand side of (3) and that the composite of the bijections is equal to the required correspondence.

For the cases $\mathcal{S} \neq \emptyset$ and $\mathcal{S}' \neq \emptyset$, we can easily check that the restriction on $\mathcal{FP}(G, \mathcal{S}, \mathcal{S}'; M)$ of the bijection above is the required isomorphism.

The final claim can be readily shown by referring to the definitions of inner Fox pairings and the isomorphisms. \hfill \box

### 3 Fox pairings of duality groups of higher dimension

As a consequence of Proposition 2.3, it is reasonable to discuss the cohomology $H(G_2, \mathcal{S}; H^1(G_1; M))$. In particular, we should focus on the case where $H^1(G_1; M)$ does not vanish. For applications, we will hereafter mainly consider the case $M = K[G]$.  

\hspace{1cm} 4
Now let us discuss such (non)-vanishing cases with $M = \mathbb{K}[G]$. As the Shapiro lemma indicates (see, e.g., (6.4) in [Bro, Section III. 6]), if $G$ is finite, $H^1(G; \mathbb{K}[G])$ vanishes. So, we should consider groups of infinite order. As an example, consider duality groups, where a group $L$ is a duality group of dimension $n$, i.e., there are a $\mathbb{K}[G]$-module $D$ a homology $n$-class $\mu \in H_n(L; D)$ such that the cap-product with $\mu$ gives an isomorphism $H^*(L; M) \cong H_{n-1}(L; D \otimes \mathbb{K})$ for any coefficient $M$. Moreover, $G$ is a virtual duality group of dimension $n$, if there is a subgroup $L$ of $G$ of infinite index that is a duality group of dimension $n$. Sections VIII. 8–10 in [Bro] give examples.

**Proposition 3.1.** If $G$ is a virtually duality group of dimension $n \neq 1$, then $H^1(G; \mathbb{K}[G])$ vanishes as well.

**Proof.** Notice that $H^*(G; \mathbb{K}[G]) \cong H^*(L; \mathbb{K}[L])$, by (6.4) in [Bro, Section III. 6], and $H_*(L; \mathbb{K}[L]) \cong H_*(\text{pt}; \mathbb{K})$, by the Shapiro lemma; thus, the duality implies $H^1(G; \mathbb{K}[G]) \cong H_{n-1}(L; (D \otimes \mathbb{K})[L]) \cong H_{n-1}(\text{pt}; D \otimes \mathbb{K}) = 0$. \(\Box\)

As a result of Proposition 3.1, we should focus on duality groups of dimension one or Poincaré duality groups of low dimension in order to find non-trivial Fox pairings from duality groups.

At this point, we should review $\mathbb{K}$-Poincaré duality pairs in the sense of [BE, §6]. The pair $(G, S)$ is a $\mathbb{K}$-Poincaré duality pair of dimension $n$ ($\mathbb{K}$-PD$_n$ pair, for short) if there is a homology class $e \in H_n(G, S; \mathbb{K})$ with trivial coefficients such that the cap products

$$e \cap \bullet : H^i(G, S; M) \to H_{n-i}(G, M), \quad H^i(G; M) \to H_{n-i}(G, S; M)$$

are isomorphisms for any $\mathbb{K}[G]$-module $M$. For example, for any orientable aspherical compact manifold $X$, if the inclusion $\partial X \hookrightarrow X$ induces an injection $\pi_1(\partial X) \hookrightarrow \pi_1(X)$, then the pair $(\pi_1(X), \pi_1(\partial X))$ is a $\mathbb{K}$-PD$_n$ group for any ring $\mathbb{K}$; see [BE, Theorem 6.3]. Similarly to Proposition 3.1, we compute the first cohomology with $M = \mathbb{K}[G]$ as follows:

**Proposition 3.2.** Let the pair $(G, S)$ be a $\mathbb{K}$-PD$_n$ pair.

1. If $n \geq 2$, then the relative $H^1(G, S; \mathbb{K}[G])$ vanishes.

2. If $n \geq 3$, then the non-relative $H^1(G; \mathbb{K}[G])$ vanishes. If $n = 2$ and $S = \emptyset$, then $H^1(G; \mathbb{K}[G])$ also vanishes.

3. If $n = 2$ and $S \neq \emptyset$, then there is an isomorphism,

$$H^1(G; \mathbb{K}[G]) \cong \text{Ker}(\bigoplus_{k: 1 \leq k \leq m} \text{aug} : \bigoplus_{k: 1 \leq k \leq m} \mathbb{K}[G/S_k] \to \mathbb{K}). \quad (4)$$

Here, if $H^1(G; \mathbb{K}[G])$ is the cohomology of $\mathbb{Z}_r(G; \mathbb{K}[G])$, which can be regarded as a left $\mathbb{K}[G]$-module, the isomorphism (4) holds in the sense of left $\mathbb{K}[G]$-modules.

**Proof.** By duality and the Shapiro lemma, $H^1(G, S; \mathbb{K}[G]) \cong H_{n-1}(G; \mathbb{K}[G]) \cong H_{n-1}(\text{pt}; \mathbb{K}) = 0$ for $n \geq 2$. Next, $H^1(G; \mathbb{K}[G]) \cong H_{n-1}(G, S; \mathbb{K}[G])$ is isomorphic to $H_{n-2}(S; \mathbb{K}[G])$ by the
homology long exact sequence and \( H_*(G, \mathbb{K}[G]) = 0 \) for \( * \geq 1 \) and \( n \geq 3 \). By the Shapiro lemma again, the homology \( H_{n-2}(S; \mathbb{K}[G]) \) is zero; hence, we have proven the second claim. A similar discussion holds for the case of \( n = 2 \) and \( S = \emptyset \).

Finally, to prove (4), consider the homology long exact sequence

\[
0 = H_1(G; \mathbb{K}[G]) \to H_1(G, S; \mathbb{K}[G]) \to H_0(S; \mathbb{K}[G]) \xrightarrow{\text{inc}_*} H_0(G; \mathbb{K}[G]) \to 0.
\]

By the Shapiro lemma again, the third and fourth terms are computed as

\[
H_0(S; \mathbb{K}[G]) \cong \bigoplus_{k:1 \leq k \leq m} \mathbb{K}[G/S_k], \quad H_0(G; \mathbb{K}[G]) \cong H_0(\text{pt}; \mathbb{K}) \cong \mathbb{K},
\]

and the map \( \text{inc}_* \) coincides with the augmentation map. Therefore, by duality again, the cohomology \( H^1(G; \mathbb{K}[G]) \cong H_1(G, S; \mathbb{K}[G]) \) is isomorphic to the kernel \( \text{Ker}(\text{aug}) \), as required. Furthermore, if \( M \) is also regarded as a left \( \mathbb{K}[G] \)-module, it is not hard to check that the above isomorphisms are left \( \mathbb{K}[G] \)-module homomorphisms.

Consequently, we shall focus on the case \( n = 2 \) and the non-relative cohomology \( H^1(G; \mathbb{K}[G]) \).

4 Fox pairings of Poincaré duality pairs of dimension two

In this section, we will focus on Fox pairings of Poincaré duality pairs of dimension two. The cohomology consisting of Fox pairings is computed as follows (see §5 for the proof):

**Theorem 4.1.** Let the pair \((G, S)\) be a \( \mathbb{K} \)-PD\(_2\) pair with \( S \neq \emptyset \). Then, there is an isomorphism,

\[
H^1(G_2, S; H^1(G_1; \mathbb{K}[G])) \cong \mathbb{K}. \tag{5}
\]

Meanwhile, concerning the non-relative cohomology, there is an exact sequence,

\[
0 \to \mathbb{K} \to H^1(G_2; H^1(G_1; \mathbb{K}[G])) \to \frac{\text{Ker}(\text{aug} : \bigoplus_{k=1}^m \mathbb{K}[G/S_k] \to \mathbb{K})}{\{a \cdot \zeta \cdot a\}_{\zeta \in S}} \to 0. \tag{6}
\]

Theorem 4.1 implies that, while there are infinitely non-nullcohomologous Fox pairings, there is a unique Fox pairing as a basis of \( H^1(G_2, S; H^1(G_1; \mathbb{K}[G])) \cong \mathbb{K} \) in (5). Thus, we define

**Definition 4.2.** The fundamental Fox pairing is a Fox pairing \( G^2 \to \mathbb{K}[G] \), which represents a basis of \( H^1(G_2, S; H^1(G_1; \mathbb{K}[G])) \cong \mathbb{K} \) in (5).

Furthermore, we will discuss uniqueness of Fox pairings under some conditions.

**Theorem 4.3** (See Section 5 for the proof). Let the pair \((G, S)\) be a \( \mathbb{K} \)-PD\(_2\) pair. Then, a Fox pairing \( \eta : G^2 \to \mathbb{K}[G] \) and \( a_s \in \mathbb{K}[G] \) uniquely associated with \( s \in S \) exist such that

\[
\eta(s, g) = a_s(1 - g) \quad \text{for any } g \in G, \tag{7}
\]

and \( \eta \) represents a basis of \( H^1(G_2, S; H^1(G_1; \mathbb{K}[G])) \cong \mathbb{K} \) in (5).
Furthermore, if $S \cong \mathbb{Z}$ and $K$ is a field, we can choose a Fox pairing $\eta$ satisfying

$$\eta(\zeta, g) = 1 - g \text{ for any } g \in G,$$

where $\zeta$ is a generator of $S_1 \cong \mathbb{Z}$.

**Corollary 4.4.** Let $(G, S)$ be a $K$-PD$_2$ pair satisfying $S \cong \mathbb{Z}$, and $K$ be a field. Let $\eta$ be the Fox pairing satisfying (8). Then, any group isomorphism $f : G \to G$ satisfying $f(S) = S$ preserves $\eta$. More precisely, $\eta(f(g), f(h)) = f(\eta(g, h))$ for any $g, h \in G$.

**Proof.** We can easily check that the map $\eta' : G^2 \to K[G]$ which sends $(g, h)$ to $f^{-1}(\eta(f(g), f(h)))$ is a Fox pairing and satisfies (8). Thus, $\eta = \eta'$ by uniqueness. Namely, $f$ preserves $\eta$. \(\square\)

In addition, we will discuss representatives of the cohomology classes.

**Theorem 4.5.** Let $(G, S)$ be a $K$-PD$_2$ pair with $S \neq \emptyset$. Then, every cohomology class of the cohomology groups in (5) and (6) is represented by a Fox pairing. Moreover, if $K$ is a PID, then every cohomology class of these cohomologies is represented by a sum of the Fox pairings like in Example 2.2.

The proof is in Appendix A. Here, we should note that the fundamental Fox pairing is not always represented by a cocycle in the relative $Z^1(G_2; S; Z^1(G_1; M))$, but is always represented by a cocycle in the non-relative $Z^1(G_2; Z^1(G_1; M))$; see Proposition 6.1 for examples.

## 5 Proofs of Theorems 4.1 and 4.3

A hasty reader may skip this section.

**Proof of Theorem 4.1.** Let the coefficient $M$ be the kernel in (4), i.e., $M = \text{Ker}(\text{aug} : \bigoplus_{k=1}^m K[G/S_k] \to K)$. Consider the long exact sequence,

$$H_2(G; K) \to H_1(G; M) \to \bigoplus_{k=1}^m H_1(G; K[G/S_k]) \to H_1(G; K) \to H_0(G; M) \to \bigoplus_{k=1}^m H_0(G; K[G/S_k]) \to H_0(G; K) \to 0 \quad \text{(exact)}.$$

Notice that $H_i(G; K[G/S_k]) \cong H_i(S_k; K)$, by the Shapiro lemma. Thus, the long sequence reduces to the exact sequence,

$$0 \to H_1(G; M) \to \bigoplus_{k=1}^m H_1(S_k; K) \to H_0(G; M) \to K^m \to K \to 0, \quad (9)$$

and the map $\text{aug}_*$ is equal to the induced map $H_1(S; K) \to H_1(G; K)$ with trivial coefficients from the inclusion $S_k \hookrightarrow G$. Also notice that, by the homology long exact sequence,

$$H_2(G; K) = 0 \to H_2(G, S; K) \to \bigoplus_{k=1}^m H_1(S_k; K) \to H_1(G; K),$$

the kernel of the map $\text{aug}_*$ is isomorphic to $H_2(G, S; K) \cong H^0(G; K) \cong K$. In summary, since $H^1(G, S; M) \cong H_1(G; M)$, we can readily obtain (5) from (9).
To prove the sequence (10) in the latter claim, let us write \( B \) for \( H^1(G_1; \mathbb{K}[G]) \) for short. Consider the homology long exact sequence,

\[ \oplus_{k=1}^{m} H_1(S_k; B) \rightarrow H_1(G_2; B) \rightarrow H_1(G_2, S; B) \rightarrow \oplus_{k=1}^{m} H_0(S_k; B) \rightarrow H_0(G_2; B). \tag{10} \]

We will observe each term. Since each \( S_i \) is a \( \mathbb{K}\)-PD_1 group (see [BE, Theorem 4.2]), the first term \( H_1(S_k; B) \) is isomorphic to the invariant part \( H^0(S_k; B) = B^{S_k} \), which is zero. The last term can be shown to be zero by checking the (co)-invariant part of \( B \). We can invoke the claim above to show that the second term is \( \mathbb{K} \), and the fourth term is the coinvariant of \( B = \text{Ker}(\text{aug}: \oplus_{k=1}^{m} \mathbb{K}[G/S_k] \rightarrow \mathbb{K}) \). Since the third term in (10) is \( H^1(G_2; H^1(G_1; \mathbb{K}[G])) \) by duality, the exact sequence (10) turns out to be the required one (6).

Next, we will prove Theorem 4.3. For this, we will need a lemma.

**Lemma 5.1.** There is no non-trivial derivation \( D : G \rightarrow \mathbb{K}[G] \) such that \( D(\zeta) = 0 \) for any \( \zeta \in S \).

**Proof.** This follows straightforwardly from \( H^1(G, S; \mathbb{K}[G]) = 0 \) by Proposition 3.2 and \( B^1(G, S; \mathbb{K}[G]) = 0 \) by definition.

**Proof of Theorem 4.3.** Let \( \eta \) be the fundamental Fox pairing, which lies in \( Z^1(G, Z^1(G; \mathbb{K}[G])) \) by Theorem 4.5. From the definition of \( H^1(G, S; H^1(G; \mathbb{K}[G])) \), for any \( s \in S \), \( \eta(s, \bullet) \) is null-cohomologous in \( Z^1_r(G_1; \mathbb{K}[G]) \); thus, there is \( a_s \in \mathbb{K}[G] \) such that \( \eta(s, h) = a_s(1 - h^{-1}) \) as required.

Next, we will show uniqueness. Suppose there is another such Fox pairing \( \eta' \). Then, \( \eta - \eta'(\zeta, a) = 0 \) for any \( a \in G \) and \( \zeta \in S \). Thus, the transpose of \( \eta - \eta' \) lies in \( Z^1(G; Z^1(G, S; \mathbb{K}[G])) = 0 \) by Lemma 5.1. That is, \( \eta = \eta' \), as desired.

We will prove the final statement. Suppose \( m = 1 \), and \( S_1 \cong \mathbb{Z} \). By Lemma 5.2 below, there is a left derivation \( D : G \rightarrow \mathbb{K}[G] \) satisfying \( D(\zeta) = a_\zeta - \text{aug}(a_\zeta) + (1 - \zeta)b_\zeta \) for some \( b_\zeta \in \mathbb{K}[G] \). Let us define \( k_\eta \in \mathbb{K} \) to be \( \text{aug}(a_\zeta) \) and a Fox pairing \( \eta'' \) by setting

\[ \eta''(g, h) = \eta(g, h) - D(g)(1 - h) - (1 - g)b_\zeta(1 - h). \]

Then, if we can show that \( k_\eta \neq 0 \), then \( k_\eta^{-1}\eta'' \) implies the existence of the Fox pairing, since \( \mathbb{K} \) is a field by assumption.

Thus, let us show that \( k_\eta \neq 0 \). Suppose \( k_\eta = 0 \). Then, \( \eta \in Z^1(G, S; Z^1(G; \mathbb{K}[G])) \) by Proposition 2.3. Then, the transposed \( '^t \eta \) lies in \( Z^1(G; Z^1(G, S; \mathbb{K}[G])) \), which is zero by Lemma 5.1. Thus, \( \eta \) is zero and does not represent a generator of \( H^1(G, S; H^1(G; \mathbb{K}[G])) \), which is a contradiction.

**Lemma 5.2.** Suppose \( m = 1 \) and \( S_1 \cong \mathbb{Z} \). Then, for any non-zero \( b \in \text{Ker}(\text{aug}) \subset \mathbb{K}[G] \), a left derivation \( D : G \rightarrow \mathbb{K}[G] \) and \( c \in \mathbb{K}[G] \) exist such that \( D(\zeta) = b - (1 - \zeta)c \).
Proof. Consider the (co)-homology long exact sequences:

$$
\begin{align*}
H^1(G, S; \mathbb{K}[G]) &\xrightarrow{\cong} H^1(G; \mathbb{K}[G]) \xrightarrow{\cong} H^1(S; \mathbb{K}[G]) \\
H_1(G; \mathbb{K}[G]) = 0 &\xrightarrow{\cong} H_1(G, S; \mathbb{K}[G]) \xrightarrow{(\text{inc})} H_0(S; \mathbb{K}[G]) \xrightarrow{(\text{inc})} H_0(G; \mathbb{K}[G]) \cong \mathbb{K} \rightarrow 0.
\end{align*}
$$

Here, the vertical maps are the cap products in duality. By reconsidering the proof of the isomorphism $H_0(S; \mathbb{K}[G]) \cong \mathbb{K}[G]/(1 - \zeta)$ in the Shapiro lemma, the map $(\text{inc})_\ast$ coincides with the augmentation map. Hence, diagram chasing leads to the conclusion that, for any derivation $D' : L \to \mathbb{K}[G]$ such that $D'(\zeta) = b$, there is an extension $D$ of $D'$ such that $D(\zeta) = b$ up to coboundary. Namely $D(\zeta) = b - (1 - \zeta)c$ for some $c \in \mathbb{K}[G]$. 

\section{Fox pairings of surface groups and 2-dimensional orbifolds}

We will examine surface groups as a typical example of Poincaré pairs of dimension two and discuss a theorem in \cite{Tur1}. Take a connected compact surface $\Sigma$ with boundaries. Let $G$ be $\pi_1(\Sigma)$ and $S$ be $\pi_1(\partial \Sigma)$. If $\Sigma$ is orientable, we let $\mathbb{K}$ be $\mathbb{Z}$; otherwise, we let $\mathbb{K}$ be $\mathbb{Z}/2$. Then, by Poincaré-Lefschetz duality, the pair $(G, S)$ is a $\mathbb{K}$-PD$_2$ pair.

Following \cite{Tur1}, §§1.4–1.6 (see also Section 3.1 in \cite{Mas}), we can define a Fox pairing $\eta_\Sigma : G^2 \to \mathbb{K}[G]$, as follows. Fix a base point $\ast \in \partial \Sigma$, and $\bullet, \dagger \in \partial \Sigma$ be additional points such that $\bullet < * < \dagger$ in the same component of $\partial \Sigma$. Let $I$ be the interval $[0, 1]$, and $\nu_{\bullet \ast}$ the path from $\bullet$ to $\ast$, and $\nu_{\ast \dagger}$ the path from $\ast$ to $\dagger$. Let $\overline{\nu}_{\bullet \ast}$ and $\overline{\nu}_{\ast \dagger}$ be the respective paths with opposite orientations. For $a, b \in \pi_1(\Sigma, \ast)$, we choose a loop $\alpha$ based at $\bullet$ such that $\overline{\nu}_{\bullet \ast} \alpha \overline{\nu}_{\bullet \ast}$ represents $a$ and a loop $\beta$ based at $\dagger$ such that $\overline{\nu}_{\ast \dagger} \beta \overline{\nu}_{\ast \dagger}$ represents $b$. Here, we may assume that $\alpha$ and $\beta$ are generic immersions and that they cross transversally. Then,

$$
\eta_\Sigma^{\text{pre}}(a, b) := \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta) \overline{\nu}_{\bullet \ast} \alpha_p \beta_p \overline{\nu}_{\ast \dagger} \in \mathbb{K}[\pi_1(\Sigma)].
$$

Here, the sign $\varepsilon_p(\alpha, \beta) \in \{\pm 1\}$ is defined to be $+1$ if and only if a unit tangent vector of $\alpha$ at $p$ followed by a unit tangent vector of $\beta$ at $p$ gives a positively-oriented frame of $\Sigma$. If $\Sigma$ is not orientable, $\varepsilon_p(\alpha, \beta)$ is always one. The bilinear extension of $\eta_\Sigma^{\text{pre}}$ is written as $\eta_\Sigma : \mathbb{K}[G]^2 \to \mathbb{K}[G]$ and is known to be a Fox pairing. Regarding the generator $\nu \in S \cong \mathbb{Z}$, it is known that $\eta_\Sigma(\nu, G) \subset (\nu - 1)\mathbb{Z}[G]$ (see Page 232 in \cite{Tur1}). Thus, $\eta_\Sigma$ can be regarded as a 1-cocycle in $Z^1(G_2, S; H^1(G_1; \mathbb{K}[G])) \cong \mathbb{K}$ as in Proposition 6.3.

**Proposition 6.1.** The Fox pairing $\eta_\Sigma$ represents the fundamental Fox pairing of $\pi_1(\Sigma)$.

**Proof.** The augmentation $\text{aug} : \mathbb{K}[G] \to \mathbb{K}$ as the transformation coefficient induces $\mathbb{K} \cong H^1(G_2, S; H^1(G_1; \mathbb{K}[G])) \to H^1(G_2, S; H^1(G_1; \mathbb{K})) \cong H^1(G_2, S; \mathbb{K}) \otimes H^1(G_1; \mathbb{K})$. Notice that $\text{aug}(\eta_\Sigma^{\text{pre}}(a, b)) \in \mathbb{K}$ is equal to the intersection number of $a$ and $b$ by definition. Therefore, $\text{aug}_\ast(\eta_\Sigma)$ is equal to the intersection form $I_\Sigma$ on $H^1(\Sigma; \mathbb{K})$. Since all of the entries of $I_\Sigma$ lie in $\{-1, 0, 1\}$, $\eta_\Sigma$ must be a generator of $H^1(G_2, S; H^1(G_1; \mathbb{K}[G])) \cong \mathbb{K}$, as required. 

\section*{References}

1. Tur1. Turaev, V.G., 
2. Mas. Massey, W.S., 
3. Proposition 2.3.

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Furthermore, let us show that uniqueness immediately follows from Theorem 4.3 and make a comparison with [Tur1].

**Corollary 6.2.** Let the pair \((G, S)\) be \((\pi_1(\Sigma), \pi_1(\partial \Sigma))\), as above. Let \(\zeta_1, \ldots, \zeta_m \in \pi_1(\partial \Sigma) \cong \square^m \mathbb{Z}\) be generators. If a Fox pairing \(\eta : G^2 \to \mathbb{K}[G]\) satisfies

\[
\eta(\zeta_k, g) = \eta_\Sigma(\zeta_k, g) \quad \text{for any } g \in G, \ k \leq m,
\]

then the pairing \(\eta\) is equal to \(\eta_\Sigma\). Such a pairing is a representative of a generator of \((5)\).

A stronger version of this corollary is given in [Tur1, Theorem I and Corollary] for the case that \(\Sigma\) is orientable.

Next, we present a procedure for expressing the Fox pairings of a two-dimensional orbifold under some conditions. Let \(\Sigma_{orb}\) be a compact 2-dimensional orbifold with circle boundaries. Suppose that a finite group \(\Gamma\) and a surjective homomorphism \(f : \pi_1(\Sigma_{orb}) \to \Gamma\) satisfying

\[
f(\pi_1(\partial \Sigma_{orb})) = \{1\}
\]

exist such that the associated covering space is an oriented surface \(\Sigma\) and admits a group extension,

\[
0 \to \pi_1(\Sigma) \xrightarrow{i} \pi_1(\Sigma_{orb}) \xrightarrow{f} \Gamma \to 0.
\]

Then, by following the discussion in [BE, Theorem 7.6] or [Fow, Proposition 5.3], we can easily establish the following proposition.

**Proposition 6.3.** Under the above assumption, \((\pi_1(\Sigma_{orb}), \pi_1(\partial \Sigma_{orb}))\) is a \(\mathbb{Q}\)-PD\(_2\) pair.

Moreover, as in [Bro, Section III.6], the Shapiro lemma yields an isomorphism,

\[
\mathcal{I} : H^1(\pi_1(\Sigma_{orb}); \mathbb{Q}[\pi_1(\Sigma_{orb})]) \cong H^1(\pi_1(\Sigma); \mathbb{Q}[\pi_1(\Sigma)]).
\]

Thus, the restriction map \(\pi_1(\Sigma) \hookrightarrow \pi_1(\Sigma_{orb})\) yields a homomorphism,

\[
\mathcal{I} \otimes \text{res}_*: H^1(\pi_1(\Sigma_{orb}), \pi_1(\partial \Sigma_{orb}); H^1(\pi_1(\Sigma_{orb}); \mathbb{Q}[\pi_1(\Sigma_{orb})])) \to H^1(\Sigma, \partial \Sigma; H^1(\Sigma; \mathbb{Q}[\pi_1(\Sigma)]))
\]

[Bro, Proposition III.10.4] uses transfer maps to show that \(\mathcal{I} \otimes \text{res}_*\) is injective; since the image and domain of \(\mathcal{I} \otimes \text{res}_*\) are computed as \(\mathbb{Q}\) by Theorem 4.1, \(\mathcal{I} \otimes \text{res}_*\) is an isomorphism. As is known [Bro], the inverse map is constructed by a transfer map \(\text{Tr}\) with respect to \(\text{res}_*\). Hence, if we explicitly describe the transfer map on a chain level as in [Bro, Section III.9], we can express \(\text{Tr}_*(\eta_\Sigma)\) as well. To summarize:

**Proposition 6.4.** Under the above assumption, the fundamental Fox pairing of the orbifold group \(\pi_1(\Sigma_{orb})\) over \(\mathbb{Q}\) is represented by \(c_\Sigma \text{Tr}_*(\eta_\Sigma)\) for some \(c_\Sigma \in \mathbb{Q}^\times\).

We conclude this section by mentioning (co)-bracket structures. According to the discussion in [MT1, Remark 7.4], such a Fox pairing defines a bracket \([,] : \mathbb{K}[G/\text{conj}] \otimes \mathbb{K}[G/\text{conj}] \to \mathbb{K}[G/\text{conj}]\), where \(G = \pi_1(\Sigma_{orb})\), and \(G/\text{conj}\) is the set of conjugacy classes of \(G\). It might be an interesting to see whether the bracket defines a Lie algebra structure on \(\mathbb{K}[G/\text{conj}]\). Incidentally, Appendix B discusses the existence of the cobracket on \(\mathbb{K}[G/\text{conj}]\).
7 Fox pairings of 3-manifold groups

In this section, we focus on Fox pairings derived from 3-manifold groups. Let $X$ be a connected orientable compact 3-manifold, where the boundary $\partial X$ is non-empty. Let $\iota$ be the inclusion $\iota : \partial X \hookrightarrow X$. By Proposition 3.2 if $X$ is aspherical and $\iota_* : \pi_1(\partial X) \to \pi_1(X)$ is injective, then every Fox pairing is null-cohomologous. Thus, we shall consider non-aspherical 3-manifolds and show (Propositions 7.1 and 7.2) that non-trivial Fox pairings of $\pi_1(X)$ are derived from the homotopy groups $\pi_*(X)$ and the (non)-injectivity of $\iota_*$. We will begin by discussing the first cohomology groups of $\pi_1(X)$.

**Proposition 7.1** (cf. Proposition 3.2). Let $G$ be $\pi_1(X)$, and $\mathcal{S}$ be $\pi_1(\partial X)$ as above. Assume that $\text{Im}(\iota_*)$ is not zero in $\pi_1(X)$. Then, the relative $H^1(X, \partial X; \mathbb{K}[G])$ is isomorphic to $\pi_2(X) \otimes \mathbb{K}$; in particular, the following isomorphism holds:

$$H^1(\pi_1(X); H^1(\partial X; \mathbb{K}[G])) \cong H^1(\pi_1(X); \pi_2(X) \otimes \mathbb{Z} \mathbb{K}).$$

(13)

Furthermore, there is an exact sequence,

$$0 \rightarrow \pi_2(X) \otimes \mathbb{Z} \mathbb{K} \rightarrow H^1(X; \mathbb{K}[G]) \rightarrow H_1(\partial X; \mathbb{K}[G]) \rightarrow 0.$$

**Proof.** Let $\tilde{X}$ be the universal cover of $X$. Then, we immediately prove the former claim by

$$H^1(X, \partial X; \mathbb{K}[G]) \cong H_2(X; \mathbb{K}[G]) \cong H_2(\tilde{X}; \mathbb{K}) \cong \pi_2(X) \otimes \mathbb{K}.$$

Here, the three isomorphisms are obtained by invoking duality, Shapiro lemma, and Hurewicz theorem, respectively. Next, consider the homology long exact sequence,

$$H_2(\partial X; \mathbb{K}[G]) \rightarrow H_2(X; \mathbb{K}[G]) \rightarrow H_2(X, \partial X; \mathbb{K}[G]) \rightarrow H_1(\partial X; \mathbb{K}[G]) \rightarrow H_1(X; \mathbb{K}[G]).$$

(14)

By duality, the first term is $H^0(\partial X; \mathbb{K}[G])$, which is zero by assumption, and the last term vanishes because of $H_1(\tilde{X}; \mathbb{K}) = 0$. Hence, the required sequence is nothing but the dual of (14).

Concerning Fox pairings, we shall focus on the cohomology in (13). Let $P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 X$ be the Postnikov tower of $X$ (see, e.g., [McC] Section 4.3 for the definition). The cohomology is estimated as follows:

**Proposition 7.2.** In the above notation, there is an exact sequence,

$$0 \rightarrow H^1(\pi_1(X); \pi_2(X) \otimes \mathbb{K}) \rightarrow H^4(\pi_1(X); \mathbb{K}) \rightarrow H^4(P_2(X); \mathbb{K}),$$

(15)

and there is a surjection from the invariant part $(\pi_3(X) \otimes \mathbb{K})^{\pi_1(X)}$ to the last term $H^4(P_2(X); \mathbb{K})$. 


Proof. By the Leary-Serre cohomology spectral sequence of the Postnikov tower (see [McC, Lemma 8bis.27] for the details of the homological one), we have the exact sequences,

\[ H^3(P_2(X); \mathbb{K}) \to H^1(\pi_1(X); \pi_2(X) \otimes \mathbb{K}) \to H^4(\pi_1(X); \mathbb{K}) \to H^4(P_2(X); \mathbb{K}), \quad (16) \]

\[ 0 \to H^n(P_{n-1}(X); \mathbb{K}) \to H^n(\pi_1(X); \mathbb{K}) \to (\pi_n(X) \otimes \mathbb{K})^\pi_1(X), \quad (17) \]

\[ (\pi_3(X) \otimes \mathbb{K})^\pi_1(X) \to H^4(P_2(X); \mathbb{K}) \to H^4(P_3(X); \mathbb{K}). \quad (18) \]

Then, the desired (15) is readily due to (16) and (17). Furthermore, the required surjection arises from (18) and (17) with \( n = 3 \). \( \square \)

To summarize, to compute the cohomology \( H^1(X, \partial X; H^1(X; \mathbb{K}[G])) \), it is important to compute \( H^4(\pi_1(X); \mathbb{K}) \) and the homotopy groups \( \pi_*(X) \). Similarly, the other cohomology groups,

\[ H^1(\pi_1(X), \pi_1(\partial X); H^1(X; \mathbb{K}[G])), \quad \text{and} \quad H^1(\pi_1(X); H^1(X; \mathbb{K}[G])) \]

can be estimated from spectral sequences. However, by the loop theorem and the sphere theorem, most 3-manifolds with infinite fundamental groups are aspherical and satisfy the injectivity of \( \iota_* : \pi_1(\partial X) \to \pi_1(X) \). Furthermore, we should mention that in cases where \( \pi_2(X) \) is infinitely generated, it might be hard to find non-trivial Fox pairings using the above propositions.

8 Duality of commutator subgroups of knot groups

Some Fox pairings of the commutator subgroups of knot groups are discussed in [Tur2, Theorem E] and [Tur1, Appendix 3]. Here, we make a conjecture (Problem 8.1) and state and prove Proposition 8.2 that relates to this conjecture.

Let \( K \) be a knot in an integral homology 3-sphere \( \Sigma \). Since \( H_1(\Sigma \setminus K; \mathbb{Z}) \cong \mathbb{Z} \), we have an infinite cyclic cover, \( \tilde{E}_K \), of \( \Sigma \setminus K \). If \( \Sigma \setminus K \) is irreducible, the knot group pair \( (\pi_1(\Sigma \setminus K), \mathbb{Z} \times \mathbb{Z}) \) is known to be a PD\(_3\)-pair; thus, by Proposition 3.2, the Fox pairings of \( \pi_1(\Sigma \setminus K) \) are trivial. In contrast, we will focus on the commutator subgroup \([\pi_1(\Sigma \setminus K), \pi_1(\Sigma \setminus K)] = \pi_1(\tilde{E}_K)\) and pose a question.

**Problem 8.1.** Let \( G = \pi_1(\tilde{E}_K) \), and \( S \) be the subgroup \( \pi_1(\partial \tilde{E}_K) \). Then, is the pair \((G, S)\) a \( \mathbb{Q}\)-PD\(_2\) pair?

The duality theorem only with trivial coefficients \( \mathbb{Q} \) is known to the Milnor pairing [Mil]. However, if the problem is positively solved, we can discuss a generalization of [Tur1, Theorem 1] and [Tur2, Theorem E] as follows.
Furthermore, if \( \pi \) define a right evaluation \( \eta \) if the right evaluation \( \eta \) is residually nilpotent, then

\[
\eta(g,h) + \eta^t(g,h) = (1 - g)(1 - h) \quad \text{for any } g, h \in G.
\]  

We give a comparison with existing results. If \( M \) is replaced by a completed module of \( \mathbb{K}[G] \), the same statement in the knot case is also shown in \cite[Theorem E]{Tur2}, and is not used by homology algebra. Here, the point is that our statement discusses before completing the module \( \mathbb{K}[G] \).

Finally, we will give the proof of Proposition \ref{prop:higherFoxPairing}. For a group \( G \), let \( I \subset \mathbb{K}[G] \) be the augmentation ideal. Consider the inverse limit \( \hat{\mathbb{K}}[G] := \lim_n \mathbb{K}[G]/I^n \). As is classically known \cite[Appendix A.]{Qui} this \( \hat{\mathbb{K}}[G] \) is isomorphic to the group ring \( \hat{\mathbb{K}}[G] \), where \( \hat{\mathbb{K}} \) the Malcev completion of \( G \), that is, \( \hat{G} = \lim G/G_n \otimes \mathbb{Q} \), where \( G \supset G_2 \supset G_3 \supset \cdots \) is the lower central series of \( G \), and \( G/G_n \otimes \mathbb{Q} \) is the rationalization of \( G/G_n \). Note that the canonical map \( G \to \hat{G} \) is injective if and only if \( G \) is torsion free and residually nilpotent. If so, the induced map \( \mathbb{Q}[G] \to \mathbb{Q}[\hat{G}] \) is injective. In addition, Turaev shows \cite[Theorem E]{Tur2} that, if \( G = \pi_1(\tilde{E}_K) \), there is a unique Fox pairing \( \hat{\eta} : \mathbb{Q}[G] \times \mathbb{Q}[G] \to \hat{\mathbb{Q}}[\hat{G}] \) satisfying

\[
\hat{\eta}(\zeta, g) = 1 - g, \quad \hat{\eta}(g,h) + \hat{\eta}^t(g,h) = (1 - g)(1 - h) \quad \text{for any } g, h \in G.
\]  

**Proof of Proposition \ref{prop:higherFoxPairing}.** The former statement is immediately obtained from Theorems \ref{thm:poincareDuality} and \ref{thm:higherFoxPairing}. To prove the latter one, notice that the knot group \( \pi_1(\Sigma \setminus K) \) is torsion free. Thus, so is \( G \). Therefore, we have injections \( \mathbb{Q}[G] \hookrightarrow \mathbb{Q}[\hat{G}] \hookrightarrow \hat{\mathbb{Q}}[\hat{G}] \). Hence, \eqref{eq:higherFoxPairing} implies \eqref{eq:higherFoxPairing2}. \( \square \)

### 9 Higher Fox pairings of Poincaré duality pairs

Now let us introduce higher Fox pairings and study the cohomology of higher Fox pairings of Poincaré duality pairs. We fix integers \( n, m \in \mathbb{N} \) and a \( \mathbb{K}[G] \)-bimodule \( M \).

For a \( \mathbb{K} \)-bilinear map \( \eta : \mathbb{K}[G]^{\otimes m} \times \mathbb{K}[G]^{\otimes n} \to M \) and \( a \in \mathbb{K}[G]^{\otimes m}, b \in \mathbb{K}[G]^{\otimes n} \), we can define a right evaluation \( \eta_b : \mathbb{K}[G]^{\otimes m} \to M \) that sends \( c \) to \( \eta(c,b) \) and a left evaluation \( \eta^a : \mathbb{K}[G]^{\otimes n} \to M \) that sends \( d \) to \( \eta(a,d) \). The maps \( a \eta \) and \( \eta_b \) can be regarded as maps \( C^m \to M \) and \( G^m \to M \), respectively.

**Definition 9.1.** Let \( S \) and \( S' \) be finite families of subgroups of a group \( G \).

A \( \mathbb{K} \)-bilinear map \( \eta : \mathbb{K}[G]^{\otimes m} \times \mathbb{K}[G]^{\otimes n} \to M \) is a **(higher) Fox pairing (of type \((m, n)\))** if the right evaluation \( \eta_b : G^m \to M \) is an \( m \)-cocycle in \( C^m_i(G, S; M) \) and the left evaluation \( a \eta \) is an \( n \)-cocycle in \( C^n_i(G, S'; M) \), for any \( a \in \mathbb{K}[G]^{\otimes m}, b \in \mathbb{K}[G]^{\otimes n} \).

We denote the module of all such higher Fox pairings by \( \mathcal{HFP}_{m,n}(G, S, S'; M) \).
If \( m = n = 1 \), this definition is obviously the (original) Fox pairing. Take two copies, \( G_1, G_2, \) of \( G \). For a cocycle \( f \in \mathbb{Z}^n(G_2; Z^n(G_1; M)) \), we define a map \( \eta_f : (G_2)^m \times (G_1)^n \to M \) by \( \eta_f(g, h) := (f(g))(h) \), which bilinearly extends to \( \mathbb{K}[G]^m \times \mathbb{K}[G]^n \to M \) as a higher Fox pairing. In the same way as Proposition 2.3, we can easily prove the following.

**Proposition 9.2.** Let \( M \) be a \( \mathbb{K}[G] \)-bimodule. The correspondence \( f \mapsto \eta_f \) gives rise to a \( \mathbb{K} \)-module isomorphism,

\[
Z^m_1(G_2, S; Z^n_1(G_1, S'; M)) \cong HFP_{m,n}(G, S, S'; M).
\]

Furthermore, as in the proof of Theorem 4.1, we can easily make a generalization as follows:

**Theorem 9.3.** Let \((G, S)\) be a \( \mathbb{K} \)-PD\( _n \) pair with \( S \neq \emptyset \). Then, there is an isomorphism,

\[
H^{n-1}(G_2, S; H^{n-1}(G_1; \mathbb{K}[G])) \cong H_2(G, S; \mathbb{K}).
\]

Furthermore, we define the Gysin maps of Fox pairings as follows. Let \((G, S)\) be a \( \mathbb{K} \)-PD\( _n \) pair and \((G', S')\) be a \( \mathbb{K} \)-PD\( _n \) pair. Take a group homomorphism \( f : (G, S) \to (G', S) \). Then, we define a Gysin map using the composite maps,

\[
f_! : H^{n-1}(G, S; H^{n-1}(G; \mathbb{K}[G])) \cong H_1(G, S; H_1(G, S; \mathbb{K}[G])) \to H_1(G', S'; H_1(G', S'; \mathbb{K}[G'])) \cong H^{\ell-1}(G', S', H^{\ell-1}(G', \mathbb{K}[G'])).
\]

Here, the first and last isomorphisms are due to the duality. Similarly, we can define the Gysin map starting from the relative cohomology:

\[
f_! : H^{n-1}(G, S; H^{n-1}(G; \mathbb{K}[G])) \cong H_1(G; H_1(G, S; \mathbb{K}[G])) \to H_1(G', S'; \mathbb{K}[G']) \cong H^{\ell-1}(G', S', H^{\ell-1}(G', \mathbb{K}[G'])).
\]

Concerning the latter Gysin map, as in the functorial discussion in the proof of Theorem 4.1, we can show a functorial result for Theorem 9.3.

**Proposition 9.4.** The Gysin map \( f_! \) in (22) is equal to the canonical pushforward,

\[
f_* : H_2(G, S; \mathbb{K}) \to H_2(G', S'; \mathbb{K}).
\]

**Example 9.5.** If \( n = \ell = 2 \), then the homology \( H_2(G, S; \mathbb{K}) \) is isomorphic to \( H^0(G; \mathbb{K}) \cong \mathbb{K} \). Therefore, the Gysin map \( f_! \) in (22) is always an isomorphism. Namely, for any fundamental Fox pairing \( \eta \) of \((G, S)\), \( f_!(\eta) \) is also a fundamental Fox pairing of \((G', S')\).

**Example 9.6.** As in Section 8, let us consider a link \( L : \sqcup_{k=1}^q S^1 \to \Sigma \) with \( q \) components, where \( \Sigma \) is an integral homology 3-sphere and \( q \in \mathbb{N} \). In many cases (e.g., when the complement \( \Sigma \setminus L \) is irreducible), the complement \( \Sigma \setminus L \) is aspherical, and the inclusion \( \iota : \partial(\Sigma \setminus L) \to \Sigma \setminus L \) induces an injection \( \iota_* : \pi_1(\partial(\Sigma \setminus L)) \to \pi_1(\Sigma \setminus L) \). If so, \( \Sigma \setminus L \) is aspherical; hence,
the pair \((G, \mathcal{S}) = (\pi_1(\Sigma \setminus L), \pi_1(\partial(\Sigma \setminus L)))\) is a \(\mathbb{K}\text{-PD}_3\) pair for any ring \(\mathbb{K}\). By Alexander duality, \(H_2(G, \mathcal{S}; \mathbb{K}) \cong \mathbb{K}^q\). Hence, Theorem 9.3 implies that there are \(q\) non-trivial higher Fox pairings from \((G, \mathcal{S})\). An open problem is to find a way to explicitly express Fox pairings as representatives of the homology classes of \(H_2(G, \mathcal{S}; \mathbb{K}) \cong \mathbb{K}^q\).

Let \(F \subset \Sigma \setminus L\) be a Seifert surface whose boundary is the \(k\)-th component of \(L\). Consider the inclusion \(\iota : (\pi_1(F), \pi_1(\partial F)) \to (G, \mathcal{S})\), and let \(\eta_F\) be the fundamental Fox pairing of \(\pi_1(F)\). Then, the fundamental Fox pairing of \((G, \mathcal{S})\) as the \(k\)-th basis of \(H_2(G, \mathcal{S}; \mathbb{K}) \cong \mathbb{K}^q\) is equal to \(\iota(\eta_F)\) by definitions.

Finally, let us briefly discuss the case \(\mathcal{S} = \emptyset\) and higher Fox pairings for aspherical closed \(n\)-manifolds. Here, we will prove an easy proposition:

**Proposition 9.7.** Let the pair \((G, \mathcal{S})\) be a \(\mathbb{K}\text{-PD}_n\) pair with \(\mathcal{S} = \emptyset\).

Then, \(H^s(G; \mathbb{K}[G])\) is zero if \(s \neq n\), and \(\mathbb{K}\) otherwise. Furthermore, \(H^i(G; H^n(G; \mathbb{K}[G]))\) is isomorphic to the ordinary cohomology \(H^i(G; \mathbb{K})\). In particular, if \(t = n\), then the \(n\)-th cohomology \(H^n(G; H^n(G; \mathbb{K}[G]))\) is isomorphic to \(\mathbb{K}\).

**Proof.** By duality and the Shapiro lemma, \(H^s(G; \mathbb{K}[G]) \cong H_{n-s}(G; \mathbb{K}[G]) \cong H_{n-s}(pt; \mathbb{K})\), which proves the former claim. Since \(G\) acts trivially on \(H_{n-k}(pt; \mathbb{K})\), we have \(H^i(G; H^n(G; \mathbb{K}[G])) \cong H^i(G; \mathbb{K})\), as claimed. \(\square\)

As an example, suppose that \(X\) is an aspherical closed \(n\)-manifold with orientation and \(n \geq 2\), and \(G = \pi_1(X)\); then, \(H^n(G; H^n(G; \mathbb{K}[G])) \cong \mathbb{K}\). This means that such a manifold admits no original Fox pairing, but does admit a non-trivial higher Fox pairing. An open problem is to find a way to concretely express higher Fox pairings as a representative n-cocycle. We conclude this paper by giving a higher Fox pairing of \(\mathbb{Z}^n\).

**Example 9.8.** Let \(G = \mathbb{Z}^n\) and \(\mathbb{K}\) be a PID. Then, \(G\) is a \(\mathbb{K}\text{-PD}_n\) pair, since \(K(G, 1)\) is a torus \((S^1)^n\). We fix a basis \(t_1, \ldots, t_n \in G\), and regard \(\mathbb{K}[G]\) as the Laurent polynomial ring \(\mathbb{K}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]\). Notice that, if \(n = 1\), then a basis of \(H^1(\mathbb{Z}; \mathbb{K}[\mathbb{Z}])\) is represented by the derivation \(D_i\), which takes \(t_1^\ell\) to \(\delta_1 \sum_{k=0}^{\ell-1} t_i^k\) if \(\ell \geq 0\), and to \(\delta_{ij} \sum_{k=0}^{\ell-1} t_i^{-k}\) if \(\ell < 0\). By the Künneth theorem, \(H^n(G; \mathbb{K}[G]) \cong \mathbb{K}\) is represented by the cross product \(D_1 \times D_2 \times \cdots \times D_n\). In particular, the higher Fox pairing \(\eta\) of a basis of \(H^n(G; H^n(G; \mathbb{K}[G])) \cong \mathbb{K}\) is described as \((D_1 \times \cdots \times D_n) \otimes (D_1 \times \cdots \times D_n)\). More precisely, by the definition of the cross product, the map \(\eta : \mathbb{K}[G]^n \times \mathbb{K}[G]^n \to \mathbb{K}[G]\) is given by

\[
\eta((t_1^{e_1,1} \cdots t_n^{e_1,n}, \ldots, t_1^{e_n,1} \cdots t_n^{e_n,n}), (t_1^{f_1,1} \cdots t_n^{f_1,n}, \ldots, t_1^{f_n,1} \cdots t_n^{f_n,n})) = \prod_{k=1}^n (D_k(t_k^{e_{k,1},k} - \sum_{\ell=k+1}^{n} t_\ell^{e_{k,\ell},k})) \prod_{k=1}^n (D_k(t_k^{f_{k,1},k} - \sum_{\ell=k+1}^{n} t_\ell^{f_{k,\ell},k})).
\]

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A Fox pairings from seven-term exact sequences

In this appendix, we observe Fox pairings from seven-term exact sequences and give the proof of Theorem 4.5.

Let us assume $G$ to be a group and $M$ to be a $\mathbb{K}[G]$-bimodule. By Proposition 2.3, it is reasonable to discuss the cohomology $H^1(G_1; H^1(G_2; M))$. For this, let us consider the Lyndon-Hochschild-Serre spectral sequence associated with the extension,

$$0 \longrightarrow G_1 \overset{i}{\longrightarrow} G_2 \times G_1 \longrightarrow G_2 \longrightarrow 0,$$

where $i(g) = (1, g)$ for any $g \in G_1$. Then, as is known [DHW], the tail of the spectral sequence reduces to an exact sequence of seven terms:

$$0 \rightarrow H^1(G_2; M^{G_1}) \rightarrow H^1(G_2 \times G_1; M) \rightarrow H^1(G_1; M)^{G_2} \rightarrow H^2(G_2; M^{G_1}) \rightarrow \rightarrow H^2(G_2 \times G_1; M) \overset{j}{\rightarrow} H^1(G_2; H^1(G_1; M)) \rightarrow H^3(G_2; M^{G_1}).$$

Here, $M^{G_1}$ is the invariant part $\{ m \in M \mid g \cdot m = m \text{ for any } g \in G_1 \}$, and the fifth term $H^2(G_2 \times G_1; A)_1$ is the kernel of the restriction of $i^*$, i.e.,

$$H^2(G_2 \times G_1; M)_1 = \ker(i^* : H^2(G_2 \times G_1; M) \rightarrow H^2(G_1; M)).$$

In particular, if $M^{G_1} = 0$ or $H^*(G_2; M^{G_1}) = 0$, then $\rho$ is an isomorphism. For example, we can easily verify that if $G$ is of infinite order and $M = \mathbb{Z}[G]$, then $M^{G_1} = 0$. Therefore, it is sensible to focus on the map $\rho$ and the second cohomology $H^2(G_2 \times G_1; M)$.

Let us review the construction of the map $\rho$ from [DHW]. Take the maps

$$f : (G_2 \times G_1) \times (G_2 \times G_1) \rightarrow M, \quad k : G_2 \rightarrow M,$

such that $f$ lies in $Z^2(G_2 \times G_1; M)$ and $\delta^1(k)(a, b) = f((1, a), (1, b))$ for any $a, b \in G_2$. Now, let us define $\rho_f : G_2 \times G_1 \rightarrow M$ by

$$\rho_f(a, b) = f((a, 1), (1, b)) - f((1, b), (a, 1)) + q(a)(1 - b^{-1}) + (1 - a)q(b)$$

for $(a, b) \in G_2 \times G_1$. Then, we can easily verify that $\rho_f$ lies in $Z_1^1(G_2; Z_1^1(G_1; M))$. According to Sections 6 and 10.3 in [DHW], the correspondence $f \mapsto \rho_f$ is equal to the map $\rho$.

Now we will give an example and make a comparison of the cross product and Example 2.2.

**Example A.1.** Suppose that $\mathbb{K}$ is a PID, and $M$ is $\mathbb{K}[G] \otimes_{\mathbb{K}} \mathbb{K}[G]$ and that $G$ is of infinite order. Here, $M$ is acted on by

$$g(a \otimes b) = ga \otimes b, \quad (a \otimes b) \cdot h = a \otimes bh^{-1} \quad \text{for} \quad a, b \in G, g \in G_2, h \in G_1.$$  

Then, $H^0(G_1; \mathbb{K}[G]) = \mathbb{K}[G]^{G_1} = 0$. Thus, the Künneth theorem in local coefficients implies that the cross product

$$\times : H^1(G_2; \mathbb{K}[G]) \otimes H^1(G_1; \mathbb{K}[G]) \rightarrow H^2(G_2 \times G_1; \mathbb{K}[G] \otimes_{\mathbb{K}} \mathbb{K}[G]),$$
is an isomorphism. Furthermore, since we can easily verify that $M^{G_1} = 0$, it follows from \[24\] that the composite
\[
\rho \circ \times : H^1(G_2; \mathbb{K}[G]) \otimes H^1(G_1; \mathbb{K}[G]) \rightarrow H^1(G_2; H^1(G_1; M))
\]
is an isomorphism. For two derivations $D_l \in Z_1^1(G_2; \mathbb{K}[G]), D_r \in Z_1^1(G_1; \mathbb{K}[G])$, we can verify by construction that $\rho \circ (D_l \times D_r)$ is equal to $\rho_{D_l \otimes D_r}$ in Example \[2.2\]. To summarize, in the above situation with $M = \mathbb{K}[G] \otimes \mathbb{K}[G]$, every Fox pairing is a sum of some Fox pairings in Example \[2.2\] up to coboundary.

Next, for practice, let us examine the case of $M = \mathbb{K}[G]$. Let $\mu : \mathbb{K}[G] \otimes \mathbb{K}[G] \rightarrow \mathbb{K}[G]$ be the $\mathbb{K}$-bilinear map defined by $g \otimes h \mapsto gh^{-1}$. Then, we have an exact sequence of $\mathbb{K}[G_1 \times G_2]$-modules,
\[
0 \rightarrow \text{Ker}(\mu) \rightarrow \mathbb{K}[G] \otimes \mathbb{K}[G] \rightarrow \mathbb{K}[G] \rightarrow 0 \quad \text{(exact)}.
\]
Moreover, we have the exact sequence,
\[
H^2(G_2 \times G_1; \mathbb{K}[G] \otimes \mathbb{K}[G]) \overset{\mu_\ast}{\rightarrow} H^2(G_2 \times G_1; \mathbb{K}[G]) \overset{\delta}{\rightarrow} H^3(G_2 \times G_1; \text{Ker}(\mu)). \tag{27}
\]
If $M^{G_1} = \mathbb{K}[G]^{G_1} = 0$, the map $\rho$ in \[24\] is an isomorphism. Notice that, from \[26\], any Fox pairing from the image $\text{Im}(\mu_\ast)$ is the Fox pairing $\eta_{D_l \otimes D_r}$ in Example \[2.2\]. Thus, if the connecting homomorphism $\delta$ is not zero, there are other examples of Fox pairings, which do not arise from Example \[2.2\].

**Proof of Theorem 4.2** Let us prove the first claim. Let $(G, S)$ be a PD$_2$-pair by assumption. By the injection in \[6\], we may consider only cohomology classes of the non-relative $H^1(G_2; H^1(G_1; \mathbb{K}[G]))$. Consider the commutative diagrams,
\[
\begin{array}{ccccccccc}
0 & \rightarrow & B^1(G; Z^1(G; \mathbb{K}[G])) & \rightarrow & Z^1(G; Z^1(G; \mathbb{K}[G])) & \rightarrow & H^1(G; Z^1(G; \mathbb{K}[G])) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & B^1(G; H^1(G; \mathbb{K}[G])) & \rightarrow & Z^1(G; H^1(G; \mathbb{K}[G])) & \rightarrow & H^1(G; H^1(G; \mathbb{K}[G])) & \rightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & & & & \delta & & \\
& & & & & & & & \rightarrow & & \\
& & & & & & & & H^2(G; B^1(G; \mathbb{K}[G])).
\end{array}
\]

Here, the horizontal arrows are canonical exact sequences, and the right vertical map is derived from the $\delta$-functor associated with
\[
0 \rightarrow B^1(G; \mathbb{K}[G]) \rightarrow Z^1(G; \mathbb{K}[G]) \rightarrow H^1(G; \mathbb{K}[G]) \rightarrow 0.
\]
The bottom-right term $H^2(G; B^1(G; \mathbb{K}[G])) \cong H_0(G, S; B^1(G; \mathbb{K}[G]))$ is zero. Therefore, any element of $H^1(G; H^1(G; \mathbb{K}[G]))$ is represented by an element of $Z^1(G; Z^1(G; \mathbb{K}[G]))$, i.e., a Fox pairing, as required.

Next, to prove the second claim, we assume that $\mathbb{K}$ is a PID. By duality, $H^m(G; \text{Ker}(\mu))$ is zero if $m \geq 2$. Therefore, the $E_2$-term $H^p(G_2; H^q(G_1; \text{Ker}(\mu)))$ with $p > 1, q > 1$ of the
LHS-spectral sequence from \([23]\) is zero. In particular, \(E_3^3 = H^3(G_2 \times G_1; \operatorname{Ker}(\mu))\) vanishes; the map \(\mu_*\) in \([27]\) is surjective.

Since any PD\(_2\)-pair \((G, S)\) is of infinite order, \(\mathbb{K}[G]^{G_1} = 0\). By assumption, the relative cohomology \(H^2(G_2 \times G_1, G_1; \mathbb{K}[G])\) is isomorphic to \(H^2(G_2 \times G_1; \mathbb{K}[G])\) because of \(H^2(G; \mathbb{K}[G]) = 0\). Thus, every 2-cocycle \(f\) in \(Z^2(G_2 \times G_1; M)\) may satisfy that the restriction of \(f\) on \(\{1\} \times G_1\) is zero. Thus, it follows from \([24]\) that every Fox pairing \([\eta]\) in the cohomology \(H^1(G_2; H^1(G_1; \mathbb{K}[G]))\) is represented by a 2-cocycle in \(H^2(G_2 \times G_1; \mathbb{K}[G])\), or in \(H^2(G_2 \times G_1; \mathbb{K}[G] \otimes \mathbb{K}[G])\) by the surjectivity of \(\mu_*\). Since any such 2-cocycle is a sum of cross products as shown in Example \([A.1]\) \(\eta\) is represented by a sum of Fox pairings in Example \([2.2]\) as required.

\[\Box\]

**B On quasi-derivations and cobrackets**

Here, we discuss quasi-derivations, as presented in Section 2.3 of \([\text{Mas}]\), and prove some lemmas. A **quasi-derivation ruled by a Fox pairing** \(\eta : \mathbb{K}[G] \times \mathbb{K}[G] \to M\) is a \(\mathbb{K}\)-linear map \(q : \mathbb{K}[G] \to M\) satisfying

\[
q(ab) = q(a)b + aq(b) + \eta(a, b), \quad \text{for any } a, b \in \mathbb{K}[G].
\]

**Lemma B.1.** Let \(\eta\) be a Fox pairing, and \(A = \mathbb{K}[G]\). Furthermore, let \(M_{\text{conj}}\) be \(M\) with the action of \(G\) defined by setting \(a \mapsto gag^{-1}\), where \(a \in M, g \in G\).

The map \(\kappa_\eta : G^2 \to M\) that \((g, h)\) sends to \(\eta(g, h)h^{-1}g^{-1}\) is a 2-cocycle of \(G\) in the coefficients \(M\). Furthermore, there is a quasi-derivation ruled by \(\rho\) if and only if the cohomology class of \(\kappa_\eta\) vanishes in \(H^2(G; M_{\text{conj}})\).

**Proof.** From \([1]\) and \([2]\), we can easily check that \(\delta^2(\kappa_\eta) = 0\). For a map \(q : A \to M\), we define \(q' : A \to M\) by \(q'(a) := q(a)a^{-1}\). Since equation \([28]\) is equivalent to \(\delta^1(q')(a, b) = \eta(a, b)b^{-1}a^{-1}\) for any \(a, b \in G\), the existence of \(q\) is equivalent to \([\kappa_\eta] = 0 \in H^2(G; M_{\text{conj}})\). \(\Box\)

Since \(H^2(G; \mathbb{K}[G]_{\text{conj}})\) is isomorphic to the Hochschild cohomology \(HH^2(\mathbb{K}[G])\) (see, e.g., \([\text{Wei}, \text{Corollary 9.7.5}]\)), sometimes it can be computed by using the techniques of Hochschild cohomology. Concerning Poincaré duality groups, we can immediately show the following.

**Lemma B.2.** If \((G, S)\) is a PD\(_n\)-pair with \(n \geq 2\) and \(S \neq \emptyset\), then \(H^2(G; \mathbb{K}[G]_{\text{conj}})\) vanishes. In particular, any Fox pairing \(\eta\) of \(G\) admits a quasi-derivation ruled by \(\eta\).

According to \([\text{Mas}, \text{Lemma 2.6}]\), if a Fox pairing \(\eta\) satisfies \((\eta + \eta')(g, h) = k(1-g)(1-h)\) for some \(k \in \mathbb{K}\) and there is a quasi-derivation ruled by \(\eta\), we can define a cobracket \(\mathbb{K}[G/\text{conj}] \to \mathbb{K}[G/\text{conj}] \otimes \mathbb{K}[G/\text{conj}]\). It would be interesting to construct such cobrackets starting from a Poincaré duality pair \((G, S)\), e.g., when \(G\) arises from an orbifold as in Section \([6]\).

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