Bowen’s equations for upper metric mean dimension with potential

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Abstract

Firstly, we introduce a new notion called induced upper metric mean dimension with potential, which naturally generalizes the definition of upper metric mean dimension with potential given by Tsukamoto to more general cases, then we establish variational principles for it in terms of upper and lower rate distortion dimensions and show there exists a Bowen’s equation between induced upper metric mean dimension with potential and upper metric mean dimension with potential.

Secondly, we continue to introduce two new notions, called BS metric mean dimension and Packing BS metric mean dimension on arbitrary subsets, to establish Bowen’s equations for Bowen upper metric mean dimension and Packing upper metric mean dimension with potential on subsets. Besides, we also obtain two variational principles for BS metric mean dimension and Packing BS metric mean dimension on subsets.

Finally, the special interest about the Bowen upper metric mean dimension of the set of generic points of ergodic measures are also involved.

1 Introduction

Mean topological dimension introduced by Gromov [Gro99] is a new topological invariant in topological dynamical systems. Later, Lindenstrauss and Weiss [LW00] introduced the notion called metric mean dimension to capture the complexity of infinite topological entropy systems and revealed the well-known fact that metric mean dimension is an upper bound of mean topological dimension. Therefore, metric mean dimension plays a vital role in dimension
theory and deserves some special attentions. Very recently, Lindenstrauss and Tsukamoto’s pioneering work [LT18] showed a first important relationship between mean dimension theory and ergodic theory, which is an analogue of classical variational principle for topological entropy. More discussions associated with this result can be found in [GS21, CDZ22]. From that time on, Lindenstrauss and Tsukamoto’s work inspired more and more researchers to inject ergodic theoretic ideas into mean dimension theory by constructing some new variational principles, and we refer to [VV17, LT19, T20, GS21, S21, CLS21, W21] for more details. Before stating our main results, we list some basic notions and recall some necessary backgrounds.

By a pair \((X, f)\) we mean a topological dynamical system (TDS for short), where \(X\) is a compact metrizable topological space and \(f\) is a continuous self-map on \(X\). The set of metrics on \(X\) compatible with the topology is denoted by \(\mathcal{D}(X)\). We denote by \(C(X, \mathbb{R})\) the set of all real-valued continuous functions of \(X\) equipped with the supremum norm. By \(M(X), M(X, f), E(X, f)\) we denote the sets of all Borel probability measures on \(X\), all \(f\)-invariant Borel probability measures on \(X\), all ergodic measures on \(X\), respectively.

In the setting of quasi-circles, Bowen [B79] firstly found the Hausdorff dimension of certain compact set is exactly the unique root of the equation defined by the topological pressure of geometric potential function, which was later known as Bowen’s equation. In 2000, Barreira and Schmeling [BS00] introduced the notion of BS dimension (or called \(u\)-dimension in that paper) on subsets and proved that BS dimension is the unique root of the equation defined by topological pressure of additive potential function. The non-additive setting and non-uniform setting about Bowen’s equation can be found in [B96] and [C11], respectively. Later, Xing and Chen [XC15] extended the work of Jaerisch et al. [JKL14] to general topological dynamical systems and introduced a notion called induced topological pressure that specializes the BS dimension, and they revealed an important link between the induced topological pressure and the classical topological pressure is Bowen’s equation. Based on these work, the first purpose of this paper is to establish Bowen’s equation for upper metric mean dimension with potential on the whole phase space.

One says that a topological dynamical system \((X, f)\) admits marker property if for any \(N > 0\), there exists an open set \(U \subset X\) with property that

\[
U \cap f^n U = \emptyset, \quad 1 \leq n \leq N, \quad \text{and} \quad X = \bigcup_{n \in \mathbb{Z}} f^n U.
\]

The symbolic version of marker property was first introduced by Krieger in [K82] and then the simplified version of Krieger’s marker lemma was given in [B83, Lemma 2.2], and the non-symbolic version of marker property was defined in [D06, Definition 2] based on the Krieger’s marker lemma [K82, B83]. For example, free (no aperiodic points) minimal systems and their extensions [L99, Lemma 3.3], an aperiodic finite-dimensional TDS [G15, Theorem 6.1] and an extension of an aperiodic TDS which has a countable number of minimal
subsystems [G17, Theorem 3.5] have marker property. In general, the marker property implies the aperiodicity and whether the aperiodicity implies the marker property or not is still an open problem posed by Gutman in [G15, Problem 5.4] and [G17, Problem 3.4]. As an application, marker property have been extensively used to deal with the embedding problems, readers can turn to [L99, G15, GLT16, G17, LT19, T20, GT20] for more details of this aspect.

Tsukamoto [T20] introduced a notion called upper metric mean dimension with potential and proved the following

**Theorem A.** Let $(X, f)$ be a TDS admitting the marker property. Then for all $d \in \mathcal{D}(X)$,

\[
\overline{\text{mdim}}_M(X, f, d, \varphi) = \sup_{\mu \in \mathcal{M}(X, f)} \left( \overline{\text{rdim}}(X, f, d, \mu) + \int_X \varphi d\mu \right),
\]

where $\mathcal{D}(X) = \{ d \in \mathcal{D}(X) : \text{mdim}(X, f, d, \varphi) = \overline{\text{mdim}}_M(X, f, d, \varphi) \}$, $\text{mdim}(X, f, \varphi)$ denotes mean dimension with potential $\varphi$, see [T20, Subsection 1.2] for its explicit definition. $\overline{\text{mdim}}_M(X, f, d, \varphi)$ is upper metric mean dimension with potential $\varphi$ given in subsection 2.1. $\overline{\text{rdim}}(X, f, d, \mu)$ and $\overline{\text{rdim}}(X, f, d, \mu)$ respectively denote lower and upper rate distortion dimensions, see [T20, Section 2] for their precise definitions.

We remark that Theorem A can be directly deduced from [T20, Corollary 1.7, Theorem 1.8]. Here, we borrow some ideas from [JKL14, XC15] to define induced upper metric mean dimension with potential and establish a Bowen’s equation for upper metric mean dimension with potential on the whole phase space.

**Theorem 1.1.** Let $(X, f)$ be a TDS with a metric $d \in \mathcal{D}(X)$ and $\varphi, \psi \in C(X, \mathbb{R})$ with $\psi > 0$. Suppose that $\overline{\text{mdim}}_M(X, f, d, \varphi) < \infty$. Then $\overline{\text{mdim}}_{M, \psi}(X, f, d, \varphi)$ is the unique root of the equation $\overline{\text{mdim}}_M(X, f, d, \varphi - \beta \psi) = 0$, where $\overline{\text{mdim}}_{M, \psi}(X, f, d, \varphi)$ is called $\psi$-induced upper metric mean dimension with potential $\varphi$ defined in Subsection 2.1.

**Theorem 1.2.** Let $(X, f)$ be a TDS admitting marker property and $\varphi, \psi \in C(X, \mathbb{R})$ with $\psi > 0$. Then for all $d \in \mathcal{D}(X)$,

\[
\overline{\text{mdim}}_{M, \psi}(X, f, d, \varphi) = \sup_{\mu \in \mathcal{M}(X, f)} \left\{ \frac{\overline{\text{rdim}}(X, f, d, \mu)}{\int \psi d\mu} + \frac{\int \varphi d\mu}{\int \psi d\mu} \right\},
\]

We would like to emphasize that only Theorem 1.2 and subsequent Corollary 3.20 need the assumption of marker property and hold for some "nice" metrics. It is not clear if we can
remove the assumption of marker property in Theorem 1.2. More precisely, it is unclear if for any dynamical system \((X, f)\), there exists a metric \(d \in \mathcal{D}(X)\) such that \(mdim(X, f, \phi) = mdim_M(X, f, d, \phi)\). This open problem was also mentioned in [GLT16, LT19, T20].

In 1973, Bowen [B73] introduced Bowen topological entropy resembling the definition of Hausdorff dimension for any Borel subset \(Z\) of \(X\). In that paper, he proved the following three important results.

(i) When \(Z = X\), Bowen topological entropy \(h_{top}(f, X)\) coincides with the classical topological entropy.

(ii) If \(\mu \in M(X, f)\) and \(Y \subset X\) with \(\mu(Y) = 1\), then the measure-theoretic entropy denoted by \(h_\mu(f)\) is less than or equal to the Bowen topological entropy \(h_{top}(f, Y)\).

(iii) If \(\mu \in E(X, f)\), then the measure-theoretic entropy \(h_\mu(f)\) is equal to \(h_{top}(f, G_\mu)\), where the set \(G_\mu = \{x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x)) = \int \phi d\mu\}\) for any \(\phi \in C(X, \mathbb{R})\) denotes the set of generic points of \(\mu\).

In 2012, Feng and Huang [FH12] introduced measure-theoretical upper and lower Brin-Katok local entropies for Borel probability measures and obtained variational principles for Bowen topological entropy and Packing topological entropy on subsets. Wang and Chen [WC12] showed the variational principles still holds for BS dimension and Packing BS dimension on subsets. Following the idea of the definition of Hausdorff dimension, Lindenstrauss and Tuskamoto [LT19] introduced mean Hausdorff dimension, which is proved to be an upper bound of mean dimension. The version of mean Hausdorff dimension with potential can be found in [T20]. Later, Wang [W21] introduced Bowen upper metric mean dimension on subsets and established an analogous variational principle for Bowen upper metric mean dimension on subsets. After that, Cheng et al. [CLS21] introduced several types of upper metric mean dimensions with potential on arbitrary subsets through Carathéodory-Pesin structures, which is an analogue of the theory of topological pressure of non-compact, and they also established a variational principle for Bowen upper metric mean dimension with potential on subsets under some conditions. Inspired by the ideas used in [BS00, WC12], in this paper we introduce the notions of BS metric mean dimension and Packing BS metric mean dimension on subsets, which allows us to establish Bowen’s equations for Bowen upper mean dimension and Packing upper metric mean dimension with potential on subsets. Moreover, two variational principles for BS metric mean dimension and Packing BS metric mean dimension on subsets are also obtained analogous to [FH12, WC12, W21]. Finally, we extend Bowen’s three important results to the framework of Bowen upper metric mean dimension.

**Theorem 1.3.** Let \((X, f)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) and \(Z\) be a non-empty subset of \(X\). Suppose that \(\phi \in C(X, \mathbb{R})\) with \(\phi > 0\). Then
(i) if \( \text{mdim}_M(f, X, d) < \infty \), then \( \text{BSmdim}_M, Z, f(\varphi) \) is the unique root of the equation \( \text{mdim}_M, Z, f(-t\varphi) = 0 \); (ii) if \( \text{Pmdim}_M(f, X, d) < \infty \), then \( \text{BSpmdim}_M, Z, f(\varphi) \) is the unique root of the equation \( \text{Pmdim}_M, Z, f(-t\varphi) = 0 \),

where \( \text{mdim}_M, Z, f(-t\varphi) \) denote Bowen upper metric mean dimension with potential \(-t\varphi\) on \( Z \) and Packing upper metric mean dimension with potential \(-t\varphi\) on \( Z \), respectively. \( \text{BSmdim}_M, Z, f(\varphi), \text{BSpmdim}_M, Z, f(\varphi) \) are respectively called BS metric mean dimension on \( Z \) with respect to \( \varphi \) and Packing BS metric mean dimension on \( Z \) with respect to \( \varphi \).

**Theorem 1.4.** Let \((X, f)\) be a TDS and \( K \) be a non-empty compact subset of \( X \). Suppose that \( \varphi \in C(X, \mathbb{R}) \) with \( \varphi > 0 \). Then for all \( d \in \mathcal{D}(X) \),

\[
\text{BSmdim}_{M, K, f}(\varphi, d) = \limsup_{\varepsilon \to 0} \frac{\sup \{ h_{\varphi, \mu}(f, \varepsilon) : \mu \in M(X), \mu(K) = 1 \}}{\log \frac{1}{\varepsilon}},
\]

\[
\text{BSpmdim}_{M, K, f}(\varphi, d) = \limsup_{\varepsilon \to 0} \frac{\sup \{ \overline{h}_{\varphi, \mu}(f, \varepsilon) : \mu \in M(X), \mu(K) = 1 \}}{\log \frac{1}{\varepsilon}},
\]

where \( h_{\varphi, \mu}(f, \varepsilon) \) and \( \overline{h}_{\varphi, \mu}(f, \varepsilon) \) are two notions related to the measure-theoretical lower and upper BS entropies of \( \mu \), see definition 3.13 for their precise definitions.

**Theorem 1.5.** Let \((X, f)\) be a TDS with a metric \( d \in \mathcal{D}(X) \), then the following statements hold.

(i) Suppose that \( \mu \in M(X, f) \). If \( Y \subset X \) and \( \mu(Y) = 1 \), then

\[
\limsup_{\varepsilon \to 0} \frac{h^{BK}_{\mu}(f, d, \varepsilon)}{\log \frac{1}{\varepsilon}} \leq \text{mdim}_M(f, Y, d).
\]

(ii) Suppose that \( \mu \in E(X, f) \). If \( \limsup_{\varepsilon \to 0} \frac{h^{BK}_{\mu}(f, d, \varepsilon)}{\log \frac{1}{\varepsilon}} = \limsup_{\varepsilon \to 0} \frac{h^{BK}_{\mu}(f, d, \varepsilon)}{\log \frac{1}{\varepsilon}} \), then

\[
\text{mdim}_M(f, G_{\mu}, d) = \limsup_{\varepsilon \to 0} \frac{PS(f, d, \mu, \varepsilon)}{\log \frac{1}{\varepsilon}} = \limsup_{\varepsilon \to 0} \frac{h^{K}_{\mu}(f, d, \varepsilon)}{\log \frac{1}{\varepsilon}} = \limsup_{\varepsilon \to 0} \frac{h^{BK}_{\mu}(f, d, \varepsilon)}{\log \frac{1}{\varepsilon}} = \limsup_{\varepsilon \to 0} \frac{h^{BK}_{\mu}(f, d, \varepsilon)}{\log \frac{1}{\varepsilon}} = \text{rdim}_{L, \infty}(X, f, d, \mu).
\]
All notions mentioned in Theorem 1.5 are explicated in Subsection 3.4.

The rest of this paper is organized as follows. In section 2, we introduce the notion of induced metric mean dimension with potential in subsection 2.1, and we prove Theorem 1.1 and Theorem 1.2 in subsection 2.2. The section 3 is divided into four parts. In subsection 3.1, we recall some basic definitions of upper metric mean dimension with potential on subsets and collect some standard facts. Theorem 1.3 and Theorem 1.4 are proved in subsection 3.2 and subsection 3.3, respectively. We give the proof of Theorem 1.5 in subsection 3.4.

2 The upper metric mean dimension with potential on the whole phase space

In section 2, we focus on the upper metric mean dimension with potential on the whole space. We introduce induced upper metric mean dimension with potential on the whole phase space in subsection 2.1, and we major the Bowen’s equation for upper metric mean dimension with potential on the whole space in subsection 2.2.

2.1 Induced upper metric mean dimension with potential

In this subsection, we present some useful notions associated with upper metric mean dimension with potential and then introduce the notion of induced upper metric mean dimension with potential.

Let $n \in \mathbb{N}$. For $x, y \in X$, we define the $n$-th Bowen metric $d_n$ on $X$ as

$$d_n(x, y) = \max_{0 \leq j \leq n-1} d(f^j(x), f^j(y)).$$

For each $\epsilon > 0$, the Bowen open ball and closed ball of radius $\epsilon$ and order $n$ in the metric $d_n$ around $x$ are respectively given by

$$B_n(x, \epsilon) = \{ y \in X : d_n(x, y) < \epsilon \},$$

$$\overline{B}_n(x, \epsilon) = \{ y \in X : d_n(x, y) \leq \epsilon \}.$$  

For a non-empty subset $Z \subset X$, one says that a set $E$ is an $(n, \epsilon)$-spanning set of $Z$ if for any $x \in Z$, there exists $y \in E$ such that $d_n(x, y) < \epsilon$. The smallest cardinality of $(n, \epsilon)$-spanning set of $Z$ is denoted by $r_n(f, d, \epsilon, Z)$. One says that a set $F \subset Z$ is an $(n, \epsilon)$-separated set of $Z$ if $d_n(x, y) \geq \epsilon$ for any $x, y \in F$ with $x \neq y$. The maximal cardinality of $(n, \epsilon)$-separated set of $Z$ is denoted by $s_n(f, d, \epsilon, Z)$.

Let $\varphi, \psi \in C(X, \mathbb{R})$ with $\psi > 0$. For all $n \geq 1, x \in X$, we set $S_n \varphi(x) := \sum_{i=0}^{n-1} \varphi(f^i(x))$ and $m := \min_{x \in X} \psi(x) > 0$. We now recall that the equivalent definition of upper metric mean dimension with potential defined by separated set in [T20].
Let $0 < \epsilon < 1$, $d \in \mathcal{D}(X)$, and $\varphi \in C(X, \mathbb{R})$. Set

$$\#_{\text{sep}}(X, d_n, S_n \varphi, \epsilon) = \sup \{ \sum_{x \in F_n} (1/\epsilon)^{S_n \varphi(x)} : F_n \text{ is an } (n, \epsilon)\text{-separated set of } X \},$$

and

$$P(X, f, d, \varphi, \epsilon) = \lim_{n \to \infty} \frac{\log \#_{\text{sep}}(X, d_n, S_n \varphi, \epsilon)}{n}.$$ 

**Upper metric mean dimension with potential $\varphi$** is given by

$$\overline{\text{mdim}}_M(X, f, d, \varphi) = \lim_{\epsilon \to 0} \frac{P(X, f, d, \varphi, \epsilon)}{\log \frac{1}{\epsilon}}.$$ 

Specially, $\overline{\text{mdim}}_M(X, f, d) = \overline{\text{mdim}}_M(X, f, d, 0)$ recovers the definition of the upper metric mean dimension of $X$ given by Lindenstrauss and Weiss in [LW00].

**Definition 2.1.** Let $(X, f)$ be a TDS with a metric $d \in \mathcal{D}(X)$ and $\varphi, \psi \in C(X, \mathbb{R})$ with $\psi > 0$. For $T > 0$, set

$$S_T := \{ n \in \mathbb{N} : \exists x \in X \text{ such that } S_n \psi(x) \leq T \text{ and } S_{n+1} \psi(x) > T \}.$$ 

For each $n \in S_T$ and $\epsilon > 0$, put

$$X_n = \{ x \in X : S_n \psi(x) \leq T \text{ and } S_{n+1} \psi(x) > T \},$$

$$P_{\psi, T}(X, f, d, \varphi, \epsilon) = \sup \left\{ \sum_{n \in S_T} \sum_{x \in F_n} (1/\epsilon)^{S_n \varphi(x)} : F_n \text{ is an } (n, \epsilon)\text{-separated set of } X_n, n \in S_T \right\}.$$ 

We define the $\psi$-induced upper metric mean dimension with potential $\varphi$ as

$$\overline{\text{mdim}}_{M, \psi}(X, f, d, \varphi) = \lim_{\epsilon \to 0} \lim_{T \to \infty} \frac{1}{T \log \epsilon} \log P_{\psi, T}(X, f, d, \varphi, \epsilon).$$

**Remark 2.2.**

(i) If $S_T \neq \emptyset$, then for each $n \in S_T$, we have $n \leq \lfloor \frac{T}{m} \rfloor + 1$, where $\lfloor \frac{T}{m} \rfloor$ denotes the integer part of $\frac{T}{m}$ and $m = \min_{x \in X} \psi(x)$. In other words, $S_T$ is a finite set.

(ii) Take $\psi = 1$, then the $\psi$-induced upper metric mean dimension with potential $\varphi$ is reduced to the upper metric mean dimension with potential $\varphi$, that is, $\overline{\text{mdim}}_{M, 1}(X, f, d, \varphi) = \overline{\text{mdim}}_M(X, f, d, \varphi)$.

(iii) $\overline{\text{mdim}}_{M, \psi}(X, f, d, \varphi) > -\infty$.

In fact, analogous to the definition of the classical topological pressure, the $\psi$-induced upper metric mean dimension with potential $\varphi$ can be also given by spanning set.
Proposition 2.3. Let \((X, f)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) and \(\varphi, \psi \in C(X, \mathbb{R})\) with \(\psi > 0\). Set

\[
Q_{\psi, T}(X, f, d, \varphi, \epsilon) = \inf \left\{ \sum_{n, x \in E_n} \frac{1}{\epsilon} S_n \varphi(x) : E_n \text{ is an } (n, \epsilon)\text{-spanning set of } X_n, n \in S_T \right\}.
\]

Then

\[
\overline{\text{mdim}}_{M, \psi}(X, f, d, \varphi) = \lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T \log \frac{1}{\epsilon}} \log Q_{\psi, T}(X, f, d, \varphi, \epsilon).
\]

Proof. Let \(0 < \epsilon < 1, n \in S_T\). Note that a maximal \((n, \epsilon)\)-separated set \(F_n\) of \(X_n\) is also an \((n, \epsilon)\)-spanning set of \(X_n\). Then

\[
Q_{\psi, T}(X, f, d, \varphi, \epsilon) \leq P_{\psi, T}(X, f, d, \varphi, \epsilon).
\]

Therefore,

\[
\lim_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T \log \frac{1}{\epsilon}} \log Q_{\psi, T}(X, f, d, \varphi, \epsilon) \leq \overline{\text{mdim}}_{M, \psi}(X, f, d, \varphi).
\]

On the other hand, let \(0 < \epsilon < 1\) and \(\gamma(\epsilon) := \sup \{|\varphi(x) - \varphi(y)| : d(x, y) < \epsilon\}\). For \(n \in S_T\), let \(E_n\) be an \((n, \frac{\epsilon}{2})\)-spanning set of \(X_n\) and \(F_n\) be an \((n, \epsilon)\)-separated set of \(X_n\). Consider a map \(\Phi : F_n \to E_n\) by assigning each \(x \in F_n\) to \(\Phi(x) \in E_n\) satisfying \(d_n(x, \Phi(x)) < \frac{\epsilon}{2}\). Then \(\Phi\) is injective.

Thus

\[
\geq (2/\epsilon)\gamma(\frac{\epsilon}{2m+1}) \sum_{n \in S_T} \sum_{x \in F_n} \frac{1}{\epsilon} S_n \varphi(x) + S_n \varphi(x) \\
\geq (2/\epsilon)^{-\gamma(\frac{\epsilon}{2m+1})} \sum_{n \in S_T} \sum_{x \in F_n} \frac{1}{\epsilon} S_n \varphi(x).
\]

It follows that

\[
\limsup_{T \to \infty} \frac{1}{T} \log Q_{\psi, T}(X, f, d, \varphi, \epsilon) \geq - \frac{1}{m} \gamma(\epsilon) \log \frac{2}{\epsilon} - \frac{1}{m} \log 2 + \limsup_{T \to \infty} \frac{1}{T} \log Q_{\psi, T}(X, f, d, \varphi, \epsilon).
\]

Since \(\epsilon \to 0, \gamma(\epsilon) \to 0\), we finally deduce that

\[
\limsup_{\epsilon \to 0} \limsup_{T \to \infty} \frac{1}{T \log \frac{1}{\epsilon}} \log Q_{\psi, T}(X, f, d, \varphi, \epsilon) \geq \overline{\text{mdim}}_{M, \psi}(X, f, d, \varphi).
\]
2.2 Bowen’s equation for upper metric mean dimension with potential on the whole phase space

In this subsection, we prove Theorem 1.1 and Theorem 1.2. To this end, we need to examine the relationship between $\text{mdim}_M(X, f, d, \varphi)$ and $\text{mdim}_{M, \psi}(X, f, d, \varphi)$, which will be useful for the forthcoming proof.

**Theorem 2.4.** Let $(X, f)$ be a TDS with a metric $d \in \mathcal{D}(X)$ and $\varphi, \psi \in C(X, \mathbb{R})$ with $\psi > 0$. For $T > 0$, define

$$G_T := \{ n \in \mathbb{N} : \exists x \in X \text{ such that } S_n \psi(x) > T \}. $$

For each $n \in G_T$ and $\epsilon > 0$, define

$$Y_n = \{ x \in X : S_n \psi(x) > T \},$$

$$R_{\psi, T}(X, f, d, \varphi, \epsilon) = \sup \left\{ \sum_{n \in G_T} \sum_{x \in F'_n} (1/\epsilon)^{S_n \varphi(x)} : F'_n \text{ is an } (n, \epsilon)\text{-separated set of } Y_n, n \in G_T \right\}. $$

Then

$$\text{mdim}_{M, \psi}(X, f, d, \varphi) = \inf \{ \beta \in \mathbb{R} : \limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi, T}(X, f, d, \varphi - \beta \psi, \epsilon) < \infty \}. \quad (2.1)$$

We use the convention that $\inf \emptyset = \infty$.

**Proof.** For $n \in \mathbb{N}, x \in X$, we define $m_n(x)$ as the unique positive integer satisfying that

$$(m_n(x) - 1)||\psi|| < S_n \psi(x) \leq m_n(x)||\psi||. \quad (2.2)$$

For any $x \in X$, we have

$$(1/\epsilon)^{-\beta||\psi||m_n(x)} (1/\epsilon)^{-\beta||\psi||} \leq (1/\epsilon)^{-\beta S_n \psi(x)} \leq (1/\epsilon)^{-\beta||\psi||m_n(x)} (1/\epsilon)^{\beta||\psi||} \quad (2.3)$$

for all $\beta \in \mathbb{R}$.

Fix $0 < \epsilon < 1$. Define

$$R^{(1)}_{\psi, T}(X, f, d, \varphi, \{ \beta||\psi||m_n + ||\beta||\psi|| \}_{n \in G_T}, \epsilon) =$$

$$\sup \left\{ \sum_{n \in G_T} \sum_{x \in F'_n} (1/\epsilon)^{S_n \varphi(x) - \beta||\psi||m_n(x) - ||\beta||\psi||} : F'_n \text{ is an } (n, \epsilon)\text{-separated set of } Y_n, n \in G_T \right\},$$

$$R^{(2)}_{\psi, T}(X, f, d, \varphi, \{ \beta||\psi||m_n - ||\beta||\psi|| \}_{n \in G_T}, \epsilon) =$$

$$\sup \left\{ \sum_{n \in G_T} \sum_{x \in F'_n} (1/\epsilon)^{S_n \varphi(x) - \beta||\psi||m_n(x) + ||\beta||\psi||} : F'_n \text{ is an } (n, \epsilon)\text{-separated set of } Y_n, n \in G_T \right\}.$$
Set
\[ A = \inf \{ \beta \in \mathbb{R} : \limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi,T}^{(1)}(X, f, d, \varphi, \{ \beta \| \psi \| m_n + \| \beta \| \| \psi \| \} )_{n \in G_T, \epsilon} < \infty \}, \]
\[ B = \inf \{ \beta \in \mathbb{R} : \limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi,T}^{(1)}(X, f, d, \varphi - \beta \psi, \epsilon) < \infty \}, \]
\[ C = \inf \{ \beta \in \mathbb{R} : \limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi,T}^{(2)}(X, f, d, \varphi, \{ \beta \| \psi \| m_n - \| \beta \| \| \psi \| \} )_{n \in G_T, \epsilon} < \infty \}. \]

By (2.3), we have \( A \leq B \leq C \). To get (2.1), it suffices to show
\[ \text{mdim}_{M, \psi}(X, f, d, \varphi) \leq A, \quad \text{and} \quad C \leq \text{mdim}_{M, \psi}(X, f, d, \varphi). \]

Firstly, we show that \( \text{mdim}_{M, \psi}(X, f, d, \varphi) \leq A \). Let \( \beta < \text{mdim}_{M, \psi}(X, f, d, \varphi) \), we can choose a positive number \( \delta > 0 \) and a sequence of positive number \( 0 < \epsilon_k < 1 \) such that \( \beta + \delta < \text{mdim}_{M, \psi}(X, f, d, \varphi) \), and
\[ \text{mdim}_{M, \psi}(X, f, d, \varphi) = \lim_{k \to \infty} \limsup_{T \to \infty} \frac{1}{\log(1/\epsilon_k) T} \log P_{\psi,T}(X, f, d, \varphi, \epsilon_k). \]

Hence, there exists \( K_0 \in \mathbb{N} \) such that for any \( k > K_0 \), we can choose a subsequence \( \{ T_{j_k} \} \in \mathbb{N} \) converges to \( \infty \) as \( j \to \infty \) satisfying that
\[ (1/\epsilon_k)^{T_{j_k}(\beta + \frac{\delta}{2})} < P_{\psi,T_{j_k}}(X, f, d, \varphi, \epsilon_k). \]

For every \( j \in \mathbb{N} \) and \( n \in S_{T_{j_k}} \) there is an \( (n, \epsilon) \)-separated set \( F_n \) of \( X_n \) so that
\[ (1/\epsilon_k)^{T_{j_k}(\beta + \frac{\delta}{2})} < \sum_{n \in S_{T_{j_k}}} \sum_{x \in F_n} (1/\epsilon_k)^{S_n \varphi(x)}. \] (2.4)

Claim 1: Let \( T > 0 \). If \( S_T \neq \emptyset \), then for each \( n \in S_T \), we have
\[ \frac{T}{\| \psi \|} - 1 < n \leq \left\lfloor \frac{T}{m} \right\rfloor + 1, \]
where \( m = \min_{x \in X} \psi(x) > 0 \).

Proof of the Claim 1: Let \( n \in S_T \), then there exists a point \( x \in X \) such that \( S_n \psi(x) \leq T \) and \( S_{n+1} \psi(x) > T \). It follows that \( T - \| \psi \| < S_n \psi(x) \leq T \), which implies the desired claim.

Taking \( T_{j_k} \) arbitrarily, note that \( T_{j_k} \to \infty \), then we can choose \( T_{j_{k+1}} \) such that
\[ \left\lfloor \frac{T_{j_k}}{m} \right\rfloor + 1 < \frac{T_{j_{k+1}}}{\| \psi \|} - 1. \]

Repeating this process, we can choose a subsequence \( T_{j_k} \) of \( T_j \) that converges to \( \infty \) as \( k \to \infty \). Without loss of generality, we still denote the subsequence \( T_{j_k} \) by \( T_j \).

Claim 2: \( S_{T_i} \cap S_{T_j} = \emptyset \) with \( i \neq j \).
Proof of the Claim 2: If $S_{T_i} \cap S_{T_j} \neq \emptyset$, we assume that $i < j$ and let $n \in S_{T_i} \cap S_{T_j}$, then there exists $x_1, x_2 \in X$ such that $S_n \psi(x_1) \leq T_i$ and $S_{n+1} \psi(x_1) > T_i$, $S_n \psi(x_2) \leq T_j$ and $S_{n+1} \psi(x_2) > T_j$. By Claim 1, we have

$$n \leq \left[ \frac{T_i}{m} \right] + 1 < \frac{T_j}{||\psi||} - 1 < n,$$

so we get a contradiction.

Note that for each $j \in \mathbb{N}$ and $n \in S_{T_j}$, if $x \in F_n$, then we have $T_j - ||\psi|| < S_n \psi(x) \leq T_j$. Together with inequality (2.2), we get

$$||\psi||m_n(x) - T_j | < 2||\psi||. \quad (2.5)$$

Observed that $-\beta||\psi||m_n(x) \geq -\beta T_j - 2||\psi||$. This gives us

$$R_{\psi,T}^{(1)}(X, f, d, \varphi, \{\beta||\psi||m_n + |\beta||\psi||\}_{n \in G_T}, \epsilon) \geq \sum_{j \in \mathbb{N}, T_j - ||\psi|| > T} \sum_{n \in S_{T_j}} \sum_{x \in F_n} (1/\epsilon_k)^{S_n \varphi(x) - \beta T_j} \geq (1/\epsilon_k)^{-3||\psi||} \sum_{j \in \mathbb{N}, T_j - ||\psi|| > T} \sum_{n \in S_{T_j}} \sum_{x \in F_n} (1/\epsilon_k)^{S_n \varphi(x) - \beta T_j} \geq (1/\epsilon_k)^{-3||\psi||} \left( \frac{1}{\epsilon_k} \right)^{1/2} T_j \quad \text{by (2.4)}$$

$$= \infty.$$ 

It follows that for any $\beta < \overline{m\text{dim}}_{M,\psi}(X, f, d, \varphi)$, we have

$$\limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi,T}^{(1)}(X, f, d, \varphi, \{\beta||\psi||m_n + |\beta||\psi||\}_{n \in G_T}, \epsilon) = \infty. \quad (2.6)$$

Therefore, we obtain $\overline{m\text{dim}}_{M,\psi}(X, f, d, \varphi) \leq A$.

If $\overline{m\text{dim}}_{M,\psi}(X, f, d, \varphi) = \infty$, let $P < \overline{m\text{dim}}_{M,\psi}(X, f, d, \varphi)$, by slightly modifying the above proof, one can show for any $\beta < P$,

$$\limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi,T}^{(1)}(X, f, d, \varphi, \{\beta||\psi||m_n + |\beta||\psi||\}_{n \in G_T}, \epsilon) = \infty.$$ 

Since $P$ is arbitrary, and using the convention, we know that $A = \inf \emptyset = \infty$. Hence

$$\overline{m\text{dim}}_{M,\psi}(X, f, d, \varphi) = A = \infty.$$

Now, we turn to show $C \leq \overline{m\text{dim}}_{M,\psi}(X, f, d, \varphi)$. We assume that $\overline{m\text{dim}}_{M,\psi}(X, f, d, \varphi) < \infty$, otherwise there is nothing to prove. Let $\delta > 0$, using the definition of $\overline{m\text{dim}}_{M,\psi}(X, f, d, \varphi)$, there is a $0 < \epsilon_0 < 1$ so that for every $0 < \epsilon < \epsilon_0$,

$$\limsup_{T \to \infty} \frac{1}{\log(1/\epsilon)T} \log P_{\psi,T}(X, f, d, \varphi, \epsilon) < \overline{m\text{dim}}_{M,\psi}(X, f, d, \varphi) + \frac{\delta}{2}.$$
Hence, we can choose an \( l_0 \in \mathbb{N} \) such that for any \( l \geq l_0 \),
\[
P_{\psi,lm}(X, f, d, \varphi, \epsilon) < (1/\epsilon)^{l_0 m} \min\{\text{dim}_{M,\psi}(X, f, d, \varphi) + 2\}
\]
\[
\frac{\delta}{3} l_0 m - \Delta - 1 > 0,
\]
where \( \Delta = 3\|\psi\|(\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta) \) and \( m = \min_{x \in X} \psi(x) \).

For \( n \in S_{lm} \), let \( F_n \) be an \((n, \epsilon)\)-separated set of \( X_n \). Then for each \( x \in F_n \), we have
\[
\|\|\|\psi\|\|_{m_n}(x) - lm| < 2\|\|\|\psi\|\|.
\]

Therefore, we obtain that
\[
- (\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta)\|\psi\|_{m_n}(x) \\
\leq -lm(\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta) + 2\|\|\|\psi\|\|(\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta)).
\]

For sufficiently large \( T \), \( n \in G_T \), let \( F_n' \) be an \((n, \epsilon)\)-separated set of \( Y_n \). Then for each \( x \in F_n' \), there exists a unique \( l \geq l_0 \) such that \((l - 1)m < S_n \psi(x) \leq lm \). So \( S_{n+1} \psi(x) = S_n \psi(x) + \psi(f^n x) > lm \). It follows that
\[
R_{\psi,T}^{(2)}(X, f, d, \varphi, \{\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta\}\|\psi\|_{m_n} - \|\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta\|\psi\|)
\]
\[
\leq \sum_{l \geq l_0} \sup \left\{ \sum_{n \in S_{lm}} \left( 1/\epsilon \right)^{S_{n} \varphi(x) - \text{dim}_{M,\psi}(X, f, d, \varphi) + \delta} \|\psi\|_{m_n} - \|\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta\|\psi\| \right\},
\]
\[
F_n \text{ is an } (n, \epsilon)-\text{separated set of } X_n, n \in S_{lm}
\]
\[
\leq (1/\epsilon)^{3\|\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta\|\psi\|} \sum_{l \geq l_0} \sup \left\{ \sum_{n \in S_{lm}} \sum_{x \in F_n} \left( 1/\epsilon \right)^{S_{n} \varphi(x) - \text{dim}_{M,\psi}(X, f, d, \varphi) + \delta} \right\},
\]
\[
F_n \text{ is an } (n, \epsilon)-\text{separated set of } X_n, n \in S_{lm} \text{ by (2.7)}
\]
\[
\leq (1/\epsilon)^{3\|\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta\|\psi\|} \sum_{l \geq l_0} \left( 1/\epsilon \right)^{\Delta} \frac{4}{3} l_0 m - \frac{\epsilon}{1 - \epsilon^m} < \frac{1}{1 - \epsilon^m} \text{ by (2.8)}.
\]

Therefore, we get
\[
\lim_{\epsilon \to 0} \lim_{T \to \infty} \left\{ (\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta)\|\psi\|_{m_n} - (\text{dim}_{M,\psi}(X, f, d, \varphi) + \delta)\|\psi\| \right\} \leq 1.
\]

That is to say, \( C \leq \text{dim}_{M,\psi}(X, f, d, \varphi) + \delta \), and hence we obtain \( C \leq \text{dim}_{M,\psi}(X, f, d, \varphi) \) by letting \( \delta \to 0 \).
Corollary 2.5. Let \((X, f)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) and \(\varphi, \psi \in C(X, \mathbb{R})\) with \(\psi > 0\). Then
\[
\overline{\text{mdim}}_{M, \psi}(X, f, d, \varphi) = \inf\{\beta \in \mathbb{R} : \text{mdim}_M(X, f, d, \varphi - \beta \psi) \leq 0\}.
\]

Proof. Let \(\beta \in \{\beta \in \mathbb{R} : \limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi, T}(X, f, d, \varphi - \beta \psi, \epsilon) < \infty\}\), and let \(M := \limsup_{\epsilon \to 0} \limsup_{T \to \infty} R_{\psi, T}(X, f, d, \varphi - \beta \psi)\). Then we can find \(0 < \epsilon_0 < 1\) such that for any \(0 < \epsilon < \epsilon_0\), there is a \(T_0 \in \mathbb{N}\) so that for all \(T \geq T_0\), we have
\[
R_{\psi, T}(X, f, d, \varphi - \beta \psi, \epsilon) < M + 1.
\]

There exists a subsequence \(n_j\) that converges to \(\infty\) as \(j \to \infty\) such that
\[
P(X, f, d, \varphi - \beta \psi, \epsilon) = \limsup_{n \to \infty} \frac{\log \#\text{sep}(X, d_{n_j}, S_{n_j}(\varphi - \beta \psi), \epsilon)}{n_j}
\]
\[
= \lim_{j \to \infty} \frac{\log \#\text{sep}(X, d_{n_j}, S_{n_j}(\varphi - \beta \psi), \epsilon)}{n_j}.
\]

Therefore, for each \(T \geq T_0\), there exists sufficiently large positive number \(n_j > T\) such that \(S_{n_j}(\psi(x)) > T\) for all \(x \in X\). Hence, \(n_j \in G_T\). Let \(F_{n_j}\) be an \((n_j, \epsilon)\)-separated set of \(X\). Then
\[
\sum_{x \in F_{n_j}} (1/\epsilon)^{S_{n_j}(\psi(x)-\beta \psi(x))} < M + 1,
\]
which shows that \(P(X, f, d, \varphi - \beta \psi, \epsilon) \leq 0\). This yields that \(\overline{\text{mdim}}_M(X, f, d, \varphi - \beta \psi) \leq 0\). By Theorem 2.4, we deduce that
\[
\overline{\text{mdim}}_{M, \psi}(X, f, d, \varphi) \geq \inf\{\beta \in \mathbb{R} : \text{mdim}_M(X, f, d, \varphi - \beta \psi) \leq 0\}.
\]

The following proposition describes some properties of the function \(\overline{\text{mdim}}_M(X, f, d, \varphi - \beta \psi)\) with respect to \(\beta\), which is useful for establishing the Bowen’s equation for upper metric mean dimension with potential on the whole phase space.

Proposition 2.6. Let \((X, f)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) and \(\varphi, \psi \in C(X, \mathbb{R})\) with \(\psi > 0\). Consider the map \(\beta \in \mathbb{R} \mapsto \overline{\text{mdim}}_M(X, f, d, \varphi - \beta \psi)\). The following statements hold.

(i) If \(\overline{\text{mdim}}_M(X, f, d, \varphi - \beta_0 \psi) = \infty\) for some \(\beta_0 \in \mathbb{R}\), then the map \(\overline{\text{mdim}}_M(X, f, d, \varphi - \psi)\) is infinite.

(ii) If \(\overline{\text{mdim}}_M(X, f, d, \varphi - \beta_0 \psi) < \infty\) for some \(\beta_0 \in \mathbb{R}\), then the map \(\overline{\text{mdim}}_M(X, f, d, \varphi - \psi)\) is finite, strictly decreasing and continuous on \(\mathbb{R}\). Moreover, the equation \(\overline{\text{mdim}}_M(X, f, d, \varphi - \beta \psi) = 0\) has unique (finite) root.
Proof. Given $0 < \epsilon < 1$. For $\beta_1, \beta_2 \in \mathbb{R}$ and each $n \in \mathbb{N}$,

$$
\sum_{x \in E} \left( \frac{1}{\epsilon} S_n \varphi(x) - \beta_2 S_n \psi(x) - n|\beta_1 - \beta_2| ||\psi|| \right)
\leq \sum_{x \in E} \left( \frac{1}{\epsilon} S_n \varphi(x) - \beta_1 S_n \psi(x) \right)
\leq \sum_{x \in E} \left( \frac{1}{\epsilon} S_n \varphi(x) - \beta_2 S_n \psi(x) + n|\beta_1 - \beta_2| ||\psi|| \right),
$$

where $E$ is an $(n, \epsilon)$-separated set of $X$.

Therefore,

$$
\overline{m\dim}_M (X, f, d, \varphi - \beta_2 \psi) - ||\beta_1 - \beta_2| ||\psi|| \leq \overline{m\dim}_M (X, f, d, \varphi - \beta_1 \psi)
\leq \overline{m\dim}_M (X, f, d, \varphi - \beta_2 \psi) + ||\beta_1 - \beta_2| ||\psi||.
$$

This yields that $\overline{m\dim}_M (X, f, d, \varphi - \beta_1 \psi) < \infty$ if and only if $\overline{m\dim}_M (X, f, d, \varphi - \beta_2 \psi) < \infty$, which confirms our corresponding statements.

Under the assumption of (ii), we prove the remaining statements.

It follows from the inequality (2.10) that

$$
|\overline{m\dim}_M (X, f, d, \varphi - \beta_1 \psi) - \overline{m\dim}_M (X, f, d, \varphi - \beta_2 \psi)| \leq ||\psi|| ||\beta_1 - \beta_2||.
$$

This tells us the map $\overline{m\dim}_M (X, f, d, \varphi - \cdot \psi)$ is continuous on $\mathbb{R}$.

Let $\beta_1, \beta_2 \in \mathbb{R}$ with $\beta_1 < \beta_2$, and fix $0 < \epsilon < 1$. Let $F_n$ be an $(n, \epsilon)$-separated set of $X$, we have

$$
\sum_{x \in F_n} \left( \frac{1}{\epsilon} S_n \varphi(x) - \beta_2 S_n \psi(x) \right)
= \sum_{x \in F_n} \left( \frac{1}{\epsilon} S_n \varphi(x) - \beta_1 S_n \psi(x) + (\beta_1 - \beta_2) S_n \psi(x) \right)
\leq \sum_{x \in F_n} \left( \frac{1}{\epsilon} S_n \varphi(x) - \beta_1 S_n \psi(x) + (\beta_1 - \beta_2) nm \right),
$$

where $m = \min_{x \in X} \psi(x) > 0$.

Then we obtain that

$$
\overline{m\dim}_M (X, f, d, \varphi - \beta_2 \psi) \leq \overline{m\dim}_M (X, f, d, \varphi - \beta_1 \psi) - (\beta_2 - \beta_1)m,
$$

which implies that the map $\overline{m\dim}_M (X, f, d, \varphi - \cdot \psi)$ is strictly decreasing.

If $\overline{m\dim}_M (X, f, d, \varphi) = 0$, then $0$ is the unique root of the equation $\overline{m\dim}_M (X, f, d, \varphi - \beta \psi) = 0$. 
If $\text{mdim}_M(X, f, d, \varphi) \neq 0$, we assume that $\text{mdim}_M(X, f, d, \varphi) > 0$, taking $\beta_1 = 0$ and $\beta_2 = h > 0$ in (2.11), then

$$\text{mdim}_M(X, f, d, \varphi - h\psi) \leq \text{mdim}_M(X, f, d, \varphi) - hm.$$ 

Hence, the unique root $\beta$ of the equation $\text{mdim}_M(X, f, d, \varphi - \beta\psi) = 0$ satisfies $0 < \beta \leq \text{mdim}_M(X, f, d, \varphi)/m$.

For the case $\text{mdim}_M(X, f, d, \varphi) < 0$, taking $\beta_1 = h < 0$ and $\beta_2 = 0$ in (2.11) again,

$$\text{mdim}_M(X, f, d, \varphi - h\psi) \leq \text{mdim}_M(X, f, d, \varphi) - hm.$$ 

Then the unique root $\beta$ of the equation $\text{mdim}_M(X, f, d, \varphi - \beta\psi) = 0$ satisfies $\text{mdim}_M(X, f, d, \varphi)/m \leq \beta < 0$.

Corollary 2.7. Let $(X, f)$ be a TDS with a metric $d \in \mathcal{D}(X)$ and $\varphi, \psi \in C(X, \mathbb{R})$ with $\psi > 0$. Then

$$\text{mdim}_{M, \psi}(X, f, d, \varphi) = \inf\{\beta \in \mathbb{R} : \text{mdim}_M(X, f, d, \varphi - \beta\psi) \leq 0\}$$ 

$$= \sup\{\beta \in \mathbb{R} : \text{mdim}_M(X, f, d, \varphi - \beta\psi) \geq 0\}.$$ 

Proof. If there exists $\beta_0 \in \mathbb{R}$ such that $\text{mdim}_M(X, f, d, \varphi - \beta_0\psi) = \infty$, then by Proposition 2.6, $\text{mdim}_M(X, f, d, \varphi - \beta\psi) = \infty$ for all $\beta \in \mathbb{R}$. Using Corollary 2.5, we obtain that

$$\text{mdim}_{M, \psi}(X, f, d, \varphi) = \sup\{\beta \in \mathbb{R} : \text{mdim}_M(X, f, d, \varphi - \beta\psi) \geq 0\}$$ 

$$= \inf\{\beta \in \mathbb{R} : \text{mdim}_M(X, f, d, \varphi - \beta\psi) \leq 0\} = \inf \emptyset = \infty.$$ 

Now, we can assume that $\text{mdim}_M(X, f, d, \varphi - \beta\psi) \in \mathbb{R}$ for any $\beta \in \mathbb{R}$.

Next, we show that

$$\text{mdim}_{M, \psi}(X, f, d, \varphi) \leq \inf\{\beta \in \mathbb{R} : \text{mdim}_M(X, f, d, \varphi - \beta\psi) < 0\}.$$ 

(2.12)

Let $\beta \in \mathbb{R}$ with $\text{mdim}_M(X, f, d, \varphi - \beta\psi) = 2a < 0$. Then there exists $0 < \epsilon_0 < 1$ such that for any $0 < \epsilon < \epsilon_0$, we can choose $N_0$ such that for $n \geq N_0$, one has

$$\sup\left\{ \sum_{x \in F_n} (1/\epsilon)^{S_n(\varphi(x) - \beta\psi(x))} : F_n \text{ is an } (n, \epsilon)\text{-separated set of } X \right\} < (1/\epsilon)^{an}.$$ 

This implies that for sufficiently large $T$, we have

$$R_{\psi, T}(X, f, d, \varphi - \beta\psi, \epsilon) \leq \sum_{n \geq N_0} \sup_{F_n} \sum_{x \in F_n} (1/\epsilon)^{S_n(\varphi(x) - \beta\psi(x))}$$ 

$$\leq \sum_{n \geq N_0} (1/\epsilon)^{an}$$ 

$$< \frac{1}{1 - \epsilon^{-a}}.$$
We finally obtain that \( \lim_{\epsilon \to 0} \sup_{T \to \infty} R_{\psi,T}(X, f, d, \varphi - \beta \psi, \epsilon) \leq 1 \). It follows from Theorem 2.4 that

\[
\inf \{ \beta \in \mathbb{R} : \overline{\text{mdim}}_M(X, f, d, \varphi - \beta \psi) < 0 \} \\
\geq \inf \{ \beta \in \mathbb{R} : \limsup_{\epsilon \to 0} \sup_{T \to \infty} R_{\psi,T}(X, f, d, \varphi - \beta \psi, \epsilon) < \infty \} \\
= \overline{\text{mdim}}_{M,\psi}(X, f, d, \varphi).
\]

By virtue of Proposition 2.6, we know that

\[
\inf \{ \beta \in \mathbb{R} : \overline{\text{mdim}}_M(X, f, d, \varphi - \beta \psi) < 0 \} \\
= \inf \{ \beta \in \mathbb{R} : \overline{\text{mdim}}_M(X, f, d, \varphi - \beta \psi) \leq 0 \} \\
= \sup \{ \beta \in \mathbb{R} : \overline{\text{mdim}}_M(X, f, d, \varphi - \beta \psi) \geq 0 \}.
\] (2.13)

Combining the facts (2.12), (2.13) and Corollary 2.5, we finish the proof.

Now, we are ready to prove the Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** By Proposition 2.6, the equation \( \overline{\text{mdim}}_M(X, f, d, \varphi - \beta \psi) = 0 \) has unique root \( \beta \). By Corollary 2.7, we know the root \( \beta \) is exactly equal to \( \overline{\text{mdim}}_{M,\psi}(X, f, d, \varphi) \).

**Proof of Theorem 1.2.** It suffices to show

\[
\overline{\text{mdim}}_{M,\psi}(X, f, d, \varphi) = \sup_{\mu \in M(X,f)} \left\{ \int \varphi d\mu - \beta \int \psi d\mu \right\},
\]

and the remaining equality can be obtained similarly.

Firstly, we show \( LHS \geq RHS \). Let \( \beta > \overline{\text{mdim}}_{M,\psi}(X, f, d, \varphi) \). By Corollary 2.7, we have

\[
0 \geq \text{mdim}(X, f, d, \varphi - \beta \psi) \\
= \sup_{\mu \in M(X,f)} \left\{ \int \varphi d\mu - \beta \int \psi d\mu \right\} \\
= \sup_{\mu \in M(X,f)} \left\{ \int \psi d\mu \left( \frac{\text{rdim}(X, f, d, \mu)}{\int \psi d\mu} + \frac{\varphi d\mu}{\int \psi d\mu} - \beta \right) \right\},
\]

which implies that \( \overline{\text{rdim}}_{(X,f,d,\mu)} + \frac{\varphi d\mu}{\int \psi d\mu} \leq \beta \) for all \( \mu \in M(X,f) \). This shows the inequality \( LHS \geq RHS \).
Next, we prove the converse inequality $LHS \leq RHS$ by using the same method. Let $\beta < \overline{mdim}_{M,\varphi}(X, f, d, \varphi)$. By Corollary 2.7, we have

$$\overline{mdim}_M(X, f, d, \varphi - \psi) = \sup_{\mu \in M(X, f)} \left\{ \frac{rdim(X, f, d, \mu)}{\int \psi d\mu} + \int \varphi d\mu - \beta \int \psi d\mu \right\}$$

using Theorem A

$$= \sup_{\mu \in M(X, f)} \left\{ \int \psi d\mu \left( \frac{rdim(X, f, d, \mu)}{\int \psi d\mu} + \int \varphi d\mu - \beta \int \psi d\mu \right) \right\} \geq 0,$$

which yields that $\frac{rdim(X, f, d, \mu)}{\int \psi d\mu} + \int \varphi d\mu \geq \beta$ for some $\mu \in M(X, f)$. This shows the inequality $LHS \leq RHS$. This completes the proof.

\[\square\]

3 The metric mean dimension with potential on subsets

The section 3 is divided into four parts. In subsection 3.1, we recall some basic definitions of upper metric mean dimension with potential on subsets and collect some standard facts. The subsection 3.2 is devoted to establishing the Bowen’s equations for upper metric mean dimension with potential on subsets. The subsection 3.3 is designed to obtain variational principles for BS metric mean dimension and Packing BS metric mean dimension, and in subsection 3.4 we focus on the upper metric mean dimension of generic points of ergodic measures.

3.1 Several types of upper metric mean dimension with potential

We first recall the definitions of the upper metric mean dimension of arbitrary subset of $X$ defined by Carathéodory structures using covering method introduced by Wang [W21] and Cheng et al. [CLS21]. Besides, we also apply the Packing method used in fractal geometry to define the Packing upper metric mean dimension with potential on subsets. Furthermore, some basic properties related by these quantities are derived.

**Definition 3.1.** Let $0 < \epsilon < 1$ and $\lambda \in \mathbb{R}$. For $Z \subset X$, $\varphi \in C(X, \mathbb{R})$ and $d \in \mathcal{D}(X)$, we define

$$M(f, d, Z, \varphi, \lambda, N, \epsilon) = \inf \left\{ \sum_{i \in I} e^{-n_i \lambda + \log \frac{1}{\epsilon} \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i} \varphi(y)} \right\},$$

where the infimum is taken over all finite or countable covers $\{B_{n_i}(x_i, \epsilon)\}_{i \in I}$ of $Z$ with $n_i \geq N$.

$$\overline{m}(f, d, Z, \varphi, \lambda, N, \epsilon) = \inf \left\{ \sum_{i \in I} e^{-N \lambda + \log \frac{1}{\epsilon} \sup_{y \in B_{N}(x_i, \epsilon)} S_{N} \varphi(y)} \right\},$$

where the infimum is taken over all finite or countable covers $\{B_{n_i}(x_i, \epsilon)\}_{i \in I}$ of $Z$ with $n_i = N$. 
Let
\[ M(f, d, Z, \varphi, \lambda, \epsilon) = \lim_{N \to \infty} M(f, d, Z, \varphi, \lambda, N, \epsilon), \]
\[ \overline{m}(f, d, Z, \varphi, \lambda, \epsilon) = \limsup_{N \to \infty} \overline{m}(f, d, Z, \varphi, \lambda, N, \epsilon). \]

It is readily to check that \( M(f, d, Z, \varphi, \lambda, \epsilon) \), \( \overline{m}(f, d, Z, \varphi, \lambda, \epsilon) \) have a critical value of parameter \( \lambda \) jumping from \( \infty \) to \( 0 \). We respectively denote their critical values as
\[ \text{mdim}_{M,Z,f}(\varphi,d) := \inf \left\{ \lambda : M(f,d,Z,\varphi,\lambda,\epsilon) = 0 \right\} = \sup \left\{ \lambda : M(f,d,Z,\varphi,\lambda,\epsilon) = \infty \right\}, \]
\[ \text{upmdim}_{M,Z,f}(\varphi,d) := \inf \left\{ \lambda : \overline{m}(f,d,Z,\varphi,\lambda,\epsilon) = 0 \right\} = \sup \left\{ \lambda : \overline{m}(f,d,Z,\varphi,\lambda,\epsilon) = \infty \right\}. \]

Put
\[ \text{mdim}_{M,Z,f}(\varphi,d) = \lim_{\epsilon \to 0} \frac{\text{mdim}_{M,Z,f}(\varphi,d,\epsilon)}{\log \frac{1}{\epsilon}}, \]
\[ \text{upmdim}_{M,Z,f}(\varphi,d) = \lim_{\epsilon \to 0} \frac{\text{upmdim}_{M,Z,f}(\varphi,d,\epsilon)}{\log \frac{1}{\epsilon}}. \]

The quantities \( \text{mdim}_{M,Z,f}(\varphi,d), \text{upmdim}_{M,Z,f}(\varphi,d) \) are called Bowen upper metric mean dimension with potential \( \varphi \), \( u \)-upper metric mean dimension with potential \( \varphi \) on the set \( Z \), respectively. Furthermore, we sometimes omit \( d \) in these quantities when \( d \) is clear. Specially, \( \text{mdim}_M(f,Z,d) := \text{mdim}_{M,f,Z}(0,d) \) is called the Bowen upper metric mean dimension on \( Z \).

**Remark 3.2.** Let \( Z \subset X \). Define
\[ \text{mdim}_M(Z, f, d, \varphi) := \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n \log \frac{1}{\epsilon}} \log \inf_{E_n} \left\{ \sum_{x \in E_n} e^{\log \frac{1}{\epsilon} S_n \varphi(x)} \right\}, \]
where the infimum \( E_n \) ranges over all \((n, \epsilon)\)-spanning sets of \( Z \).

By a standard method, one can check
\[ \text{mdim}_M(Z, f, d, \varphi) = \limsup_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n \log \frac{1}{\epsilon}} \log \sup_{F_n} \left\{ \sum_{x \in F_n} e^{\log \frac{1}{\epsilon} S_n \varphi(x)} \right\} \]
\[ = \text{upmdim}_{M,Z,f}(\varphi,d), \quad (3.1) \]
where the supremum \( F_n \) ranges over all \((n, \epsilon)\)-separated sets of \( Z \).

Using the fact [CLS21, Proposition 2.2] that if \( Z \) is a \( f \)-invariant compact subset, then
\[ \text{mdim}_{M,Z,f}(\varphi) = \text{upmdim}_{M,Z,f}(\varphi). \]
Hence
\[ \text{mdim}_M(Z, f, d, \varphi) = \text{mdim}_{M,X,f}(\varphi,d). \]
Definition 3.3. Let $0 < \epsilon < 1$ and $\lambda \in \mathbb{R}$. For $Z \subset X$, $\varphi \in C(X, \mathbb{R})$ and $d \in \mathcal{D}(X)$, we define

$$P(f, d, Z, \varphi, \lambda, N, \epsilon) = \sup \left\{ \sum_{i \in I} e^{-n_i \lambda + \log \frac{1}{\epsilon} \cdot \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i} \varphi(y)} \right\},$$

where the supremum is taken over all finite or countable pairwise disjoint closed families $\{B_{n_i}(x_i, \epsilon)\}_{i \in I}$ of $Z$ with $n_i \geq N, x_i \in Z$.

The quantity $P(f, d, Z, \varphi, \lambda, N, \epsilon)$ is non-increasing as $N$ increases, so we define

$$P(f, d, Z, \varphi, \lambda, \epsilon) = \lim_{N \to \infty} P(f, d, Z, \varphi, \lambda, N, \epsilon).$$

Set

$$\mathcal{P}(f, d, Z, \varphi, \lambda, \epsilon) = \inf \left\{ \sum_{i=1}^{\infty} P(f, d, Z_i, \varphi, \lambda, \epsilon) : \bigcup_{i \geq 1} Z_i \supseteq Z \right\}.$$

It is readily to check that the quantity $\mathcal{P}(f, d, Z, \varphi, \lambda, \epsilon)$ has a critical value of parameter $\lambda$ jumping from $\infty$ to 0. We define the critical value as

$$\underbar{\text{Pdim}}_{M,Z,f}(\varphi,d,\epsilon) := \inf \{ \lambda : \mathcal{P}(f, d, Z, \varphi, \lambda, \epsilon) = 0 \} = \sup \{ \lambda : \mathcal{P}(f, d, Z, \varphi, \lambda, \epsilon) = \infty \}.$$

Let $\overline{\text{Pdim}}_{M,Z,f}(\varphi,d) = \limsup_{\epsilon \to 0} \frac{\underbar{\text{Pdim}}_{M,Z,f}(\varphi,d,\epsilon)}{\log \frac{1}{\epsilon}}$. We call the quantities $\underbar{\text{Pdim}}_{M,Z,f}(\varphi,d)$, $\overline{\text{Pdim}}_{M,Z,f}(\varphi,d)$ Packing upper metric mean dimension with potential $\varphi$ on the set $Z$, Packing upper metric mean dimension on the set $Z$, respectively. We sometimes omit the metric $d$ in above quantities when $d$ is clear.

The following proposition presents some basic properties related by these quantities.

Proposition 3.4. Let $(X, f)$ be a TDS with a metric $d \in \mathcal{D}(X)$ and $\varphi \in C(X, \mathbb{R})$.

(i) If $Z_1 \subset Z_2 \subset X$, then $\text{mdim}_{M,Z_1,f}(\varphi) \leq \text{mdim}_{M,Z_2,f}(\varphi)$, $\text{upmdim}_{M,Z_1,f}(\varphi) \leq \text{upmdim}_{M,Z_2,f}(\varphi)$, $\overline{\text{Pdim}}_{M,Z_1,f}(\varphi,d) \leq \overline{\text{Pdim}}_{M,Z_2,f}(\varphi,d)$.

(ii) If $Z$ is a finite union of some $Z_i$, that is, $Z=\bigcup_{i=1}^{N} Z_i$, then $\text{mdim}_{M,Z,f}(\varphi) = \max_{1 \leq i \leq N} \text{mdim}_{M,Z_i,f}(\varphi)$, $\text{upmdim}_{M,Z,f}(\varphi) = \max_{1 \leq i \leq N} \text{upmdim}_{M,Z_i,f}(\varphi)$, $\overline{\text{Pdim}}_{M,Z,f}(\varphi) = \max_{1 \leq i \leq N} \overline{\text{Pdim}}_{M,Z_i,f}(\varphi)$.

(iii) For any non-empty subset $Z \subset X$,

$$\text{mdim}_{M,Z,f}(\varphi) \leq \overline{\text{Pdim}}_{M,Z,f}(\varphi) \leq \text{upmdim}_{M,Z,f}(\varphi).$$

Further, if $Z$ is compact and $f$-invariant, then

$$\text{mdim}_{M,Z,f}(\varphi) = \overline{\text{Pdim}}_{M,Z,f}(\varphi) = \text{upmdim}_{M,Z,f}(\varphi).$$
Proof. (i) and (ii) follow directly from the Definitions 3.1 and 3.3.

(iii) Let $0 < \epsilon < 1$ and $\gamma(4\epsilon) = \sup\{|\varphi(x) - \varphi(y)| : \delta(x, y) \leq 4\epsilon\}$, and let $n \in \mathbb{N}$ and $A \subset X$. Let $R$ be the largest cardinality such that there exists a pairwise disjoint family \{\(B_n(x_i, \epsilon)\}\}_{i=1}^R with $x_i \in A$. Then

$$\bigcup_{i=1}^R B_n(x_i, 3\epsilon) \supseteq A.$$ 

Let $\lambda \in \mathbb{R}$, then

$$M(f, d, A, \varphi, \lambda, n, 3\epsilon) \leq \sum_{i=1}^R e^{-n\lambda + \log \frac{1}{3\epsilon} \sup_{y \in B_n(x_i, 3\epsilon)} S_n \varphi(y)} \leq \sum_{i=1}^R e^{-n\lambda + \log \frac{1}{3\epsilon} \sup_{y \in B_n(x_i, \epsilon)} S_n \varphi(y) + \log \frac{1}{\epsilon} n \gamma(4\epsilon)} \leq \sum_{i=1}^R e^{-n\lambda + \log \frac{1}{3\epsilon} \sup_{y \in B_n(x_i, \epsilon)} S_n \varphi(y) - \log \frac{1}{\epsilon} n ||\varphi|| + \log \frac{1}{\epsilon} n \gamma(4\epsilon)} \leq P(f, d, A, \varphi, \lambda - \log 3 ||\varphi|| - \log \frac{1}{3\epsilon} \cdot \gamma(4\epsilon), n, \epsilon).$$

Hence for any $\bigcup_{i \geq 1} Z_i \supseteq Z$, we have

$$M(f, d, Z, \varphi, \lambda, 3\epsilon) \leq \sum_{i \geq 1} M(f, d, Z_i, \varphi, \lambda, 3\epsilon) \leq \sum_{i \geq 1} P(f, d, Z_i, \varphi, \lambda - \log 3 ||\varphi|| - \log \frac{1}{3\epsilon} \cdot \gamma(4\epsilon), \epsilon).$$

This implies that

$$\text{mdim}_{M,f}(\varphi, d, 3\epsilon) \leq P\text{mdim}_{M,f}(\varphi, d, \epsilon) + \log 3 ||\varphi|| + \gamma(4\epsilon) \log \frac{1}{3\epsilon}.$$

Therefore, we finally obtain $\text{mdim}_{M,Z,f}(\varphi) \leq P\text{mdim}_{M,Z,f}(\varphi)$.

We continue to verify that $P\text{mdim}_{M,Z,f}(\varphi) \leq \text{upmdim}_{M,Z,f}(\varphi)$. We may assume that $P\text{mdim}_{M,Z,f}(\varphi) > -\infty$, otherwise there is nothing left to prove. Let $-\infty < t < s < P\text{mdim}_{M,Z,f}(\varphi)$. Then we can choose a subsequence $0 < \epsilon_k < 1$ that converges to 0 as $k \to \infty$ such that

$$P\text{mdim}_{M,Z,f}(\varphi, d) = \lim_{k \to \infty} \frac{P\text{mdim}_{M,Z,f}(\varphi, d, \epsilon_k)}{\log \frac{1}{\epsilon_k}} > s.$$

Therefore, there is $K_0 \in \mathbb{N}$ satisfying for any $k > K_0$,

$$P\text{mdim}_{M,Z,f}(\varphi, d, \epsilon_k) > s \log \frac{1}{\epsilon_k}.$$ 

This means that $P(f, d, Z, \varphi, s \log \frac{1}{\epsilon_k}, \epsilon_k) \geq P(f, d, Z, \varphi, s \log \frac{1}{\epsilon_k}, \epsilon_k) = \infty.$
Fix such a $k > K_0$. For any $N \in \mathbb{N}$, we can find a countable pairwise disjoint family $\{B_{n_i}(x_i, \epsilon_k)\}_{i \in I}$ with $x_i \in Z$ and $n_i \geq N$ such that

$$
\sum_{i \in I} e^{-n_i \cdot \log \frac{1}{\epsilon_k} + \log \frac{1}{\epsilon_k} \sup_{y \in B_{n_i}(x_i, \epsilon_k)} S_{n_i} \varphi(y)} > 1.
$$

For any $l \geq N$, we set $E_l = \{x_{n_i} : n_i = l, i \in I\}$. So

$$
\sum_{l \geq N} \sum_{x \in E_l} e^{-l \cdot \log \frac{1}{\epsilon_k} + \log \frac{1}{\epsilon_k} \sup_{y \in B_{n_i}(x, \epsilon_k)} S_l \varphi(y)} > 1,
$$

where $\gamma(\epsilon) := \sup\{|\varphi(x) - \varphi(y)| : d(x, y) \leq \epsilon\}$.

There must exist an $l_N \geq N$ such that

$$
\sum_{x \in E_{l_N}} e^{-l_N \cdot (s - \gamma(\epsilon_k)) \log \frac{1}{\epsilon_k} + \log \frac{1}{\epsilon_k} S_{l_N} \varphi(x)} > (1 - e^{(t-s) \log \frac{1}{\epsilon_k}}) e^{(t-s) l_N \log \frac{1}{\epsilon_k}}.
$$

Namely, we get $\sum_{x \in E_{l_N}} (1/\epsilon_k) S_{l_N} \varphi(x) > (1 - e^{(t-s) \log \frac{1}{\epsilon_k}})(1/\epsilon_k)^{(t-\gamma(\epsilon_k)) l_N}$, where $E_{l_N}$ is an $(l_N, \epsilon_k)$-separated set of $Z$. This gives us that

$$
\limsup_{N \to \infty} \frac{1}{N \log \frac{1}{\epsilon_k}} \log \sup_{E_N} \left\{ \sum_{x \in E_N} (1/\epsilon_k) S_N \varphi(x) \right\} \geq t - \gamma(\epsilon_k),
$$

where the supremum ranges over all $(N, \epsilon_k)$-separated sets of $Z$.

Note that $\gamma(\epsilon_k) \to 0$ as $k \to \infty$, combining the fact mentioned in remark 3.2, we finally deduce that $\text{upmdim}_{M,Z,f}(\varphi, d) \geq t$. Letting $t \to \text{Pmdim}_{M,Z,f}(\varphi, d)$, we get the desired result.

The last statement follows from (iii) and the fact [CLS21, Proposition 2.2] stating that if $Z$ is a $f$-invariant compact subset, then $\text{mdim}_{M,Z,f}(\varphi) = \text{upmdim}_{M,Z,f}(\varphi)$.

\[\Box\]

### 3.2 Bowen’s equation for upper metric mean dimension with potential on subsets

We begin this subsection with studying some basic properties of the functions defined by the Bowen upper metric mean dimension with potential and Packing upper metric mean dimension with potential on a subset of $X$. Then we define BS metric mean dimension and Packing BS metric mean dimension and show that they are exactly the unique root of the corresponding Bowen’s equations.
Given a non-empty subset \( Z \subset X \) that does not need to be invariant or compact, and let \( \varphi \in C(X, \mathbb{R}) \). Consider the following functions

\[
\phi(t) = \overline{\dim}_{M,Z,f}(t\varphi, d), \\
\Phi(t) = \underline{\dim}_{M,Z,f}(t\varphi, d).
\]

**Proposition 3.5.** Let \((X, f)\) be a TDS and \( Z \subset X \) be a non-empty subset. Suppose that \( \varphi \in C(X, \mathbb{R}) \) with \( \varphi < 0 \). Then for all \( t \in \mathbb{R} \), one has \( \overline{\dim}_{M,Z,f}(t\varphi) > -\infty \), and \( \overline{\dim}_{M,Z,f}(t\varphi) < \infty \) if and only if \( \overline{\dim}_{M}(f, Z) < \infty \).

**Proof.** Set \( m = \min_{x \in X} \varphi(x) \). Let \( 0 < \epsilon < 1 \) and \( t \geq 0 \). Then for each \( N \), we have

\[
M(f, d, Z, t\varphi, tm \log \frac{1}{\epsilon}, N, \epsilon) = \inf \left\{ \sum_{i \in I} e^{-n_i tm \log \frac{1}{\epsilon} + t \log \frac{1}{\epsilon} \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i} \varphi(y)} \right\}
\]

\[
\geq \inf \left\{ \sum_{i \in I} e^{-n_i tm \log \frac{1}{\epsilon} + n_i tm \log \frac{1}{\epsilon}} \right\} \geq 1,
\]

where the infimum ranges over all finite or countable covers \( \{B_{n_i}(x_i, \epsilon)\}_{i \in I} \) of \( Z \) with \( n_i \geq N \).

Therefore, \( \overline{\dim}_{M,Z,f}(t\varphi) \geq tm > -\infty \). Now, fix a \( t_0 > 0 \). Then for all \( t < 0 \), we have \( \overline{\dim}_{M,Z,f}(t\varphi) \geq \overline{\dim}_{M,Z,f}(t_0 \varphi) > -\infty \) by the monotonicity of \( M(f, d, Z, \varphi, \lambda, N, \epsilon) \) with respect to \( \varphi \).

For the second statement,

\[
\inf \left\{ \sum_{i \in I} e^{-n_i \lambda - |t| n_i \|\varphi\| \log \frac{1}{\epsilon}} \right\} \leq \inf \left\{ \sum_{i \in I} e^{-n_i \lambda + t \log \frac{1}{\epsilon} \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i} \varphi(y)} \right\}
\]

\[
\leq \inf \left\{ \sum_{i \in I} e^{-n_i \lambda + |t| n_i \|\varphi\| \log \frac{1}{\epsilon}} \right\},
\]

where the infimum ranges over all finite or countable covers \( \{B_{n_i}(x_i, \epsilon)\}_{i \in I} \) of \( Z \) with \( n_i \geq N \).

Note that \( \overline{\dim}_{M}(f, Z) = \overline{\dim}_{M,Z,f}(0) \), this implies that

\[
\overline{\dim}_{M}(f, Z) - |t| \|\varphi\| \leq \overline{\dim}_{M,Z,f}(t\varphi) \leq \overline{\dim}_{M}(f, Z) + |t| \|\varphi\|.
\]

Hence for all \( t \in \mathbb{R} \), \( \overline{\dim}_{M,Z,f}(t\varphi) < \infty \) if and only if \( \overline{\dim}_{M}(f, Z) < \infty \). \( \square \)

**Proposition 3.6.** Let \( \varphi \) be a negative and continuous function on \( X \). Suppose that \( \overline{\dim}_{M}(f, X) < \infty \). Then the function \( \phi(t) \) is strictly decreasing and Lipschitz, the equation \( \phi(t) = 0 \) has unique (finite) root \( s \) and \( -\frac{1}{m} \overline{\dim}_{M}(f, Z) \leq s \leq -\frac{1}{M} \overline{\dim}_{M}(f, Z) \), where \( m = \min_{x \in X} \varphi(x) \) and \( M = \max_{x \in X} \varphi(x) \).
Proof. Let \( t_1, t_2 \in \mathbb{R} \) with \( t_1 > t_2 \). Let \( 0 < \epsilon < 1 \) and \( N \in \mathbb{N} \). Given a cover \( \{ B_{n_i}(x_i, \epsilon) \}_{i \in I} \) of \( Z \) with \( n_i \geq N \), then we have

\[
\sum_{i \in I} e^{-n_i \lambda + t_1 \log \frac{1}{\epsilon}} \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i} \varphi(y) \leq \sum_{i \in I} e^{-n_i \lambda + t_2 \log \frac{1}{\epsilon}} \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i} \varphi(y) + (t_1 - t_2)n_i M \log \frac{1}{\epsilon}.
\]

From this relation, we deduce that

\[
\overline{\text{mdim}}_{M,Z,f}(t_1 \varphi) \leq \overline{\text{mdim}}_{M,Z,f}(t_2 \varphi) + (t_1 - t_2)M,
\]

which implies that \( \phi(t) \) is strictly decreasing with respect to \( t \) on \( \mathbb{R} \).

Similarly,

\[
\overline{\text{mdim}}_{M,Z,f}(t_2 \varphi) + (t_1 - t_2)m \leq \overline{\text{mdim}}_{M,Z,f}(t_1 \varphi).
\]

Taking Lipschitz constant \( L := -m \), we see that

\[
|\overline{\text{mdim}}_{M,Z,f}(t_1 \varphi) - \overline{\text{mdim}}_{M,Z,f}(t_2 \varphi)| \leq L|t_1 - t_2|.
\]

Letting \( t_1 = h > 0, t_2 = 0 \) in (3.3), then

\[
\overline{\text{mdim}}_{M,Z,f}(h \varphi) \leq \overline{\text{mdim}}_{M}(f, Z) - h(-M).
\]

Therefore, \( \overline{\text{mdim}}_{M,Z,f}(\frac{1}{-m} \overline{\text{mdim}}_{M}(f, Z) \cdot \varphi) \leq 0 \). Again, letting \( t_1 = h > 0, t_2 = 0 \) in (3.4), then

\[
\overline{\text{mdim}}_{M,Z,f}(h \varphi) \geq \overline{\text{mdim}}_{M}(f, Z) - h(-m).
\]

This gives us that \( \overline{\text{mdim}}_{M,Z,f}(\frac{1}{-m} \overline{\text{mdim}}_{M}(f, Z) \cdot \varphi) \geq 0 \).

Using the intermediate value theorem of continuous function, we know that the equation \( \phi(t) = 0 \) has unique non-negative root \( s \) and

\[
-s \leq \frac{\overline{\text{mdim}}_{M}(f, Z)}{-m} < \infty.
\]

By slightly modifying the method used in Proposition 3.6, we have the following

**Proposition 3.7.** Let \( \varphi \) be a negative and continuous function on \( X \). Suppose that \( \overline{\text{Pdim}}_{M}(f, X) < \infty \). Then the function \( \Phi(t) \) is strictly decreasing and Lipschitz. Moreover, the equation \( \Phi(t) = 0 \) has unique root.

Analogous to the setting of BS dimension \([\text{BS00}]\) and Packing BS dimension \([\text{WC12}]\) on arbitrary subset defined by Carathéodory structures, we define two new notions called BS metric mean dimension and Packing BS metric mean dimension on subsets.
Definition 3.8. For $0 < \epsilon < 1, N \in \mathbb{N}, \lambda \in \mathbb{R}, Z \subset X$ and $\varphi \in C(X, \mathbb{R})$ with $\varphi > 0, d \in \mathcal{D}(X)$, we define

$$R(f, d, \varphi, \lambda, Z, N, \epsilon) = \inf \left\{ \sum_{i \in I} e^{-\lambda \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i} \varphi(y)} \right\},$$

where the infimum is taken over all finite or countable covers $\{B_{n_i}(x_i, \epsilon)\}_{i \in I}$ of $Z$ with $n_i \geq N$.

Since $R(f, d, \varphi, \lambda, Z, N, \epsilon)$ is non-decreasing as $N$ increases, we define

$$R(f, d, \varphi, \lambda, Z, \epsilon) = \lim_{N \to \infty} R(f, d, \varphi, \lambda, Z, N, \epsilon).$$

There is a critical value of the parameter $\lambda$ that jumps from $\infty$ to 0. We define such critical value $R(X, f, d, \varphi, Z, \epsilon)$ as

$$R(f, d, \varphi, Z, \epsilon) = \inf \{ \lambda : R(f, d, \varphi, \lambda, Z, \epsilon) = 0 \},$$

$$= \sup \{ \lambda : R(f, d, \varphi, \lambda, Z, \epsilon) = \infty \}.$$

Let

$$\overline{BSdim}_{M, f}(\varphi, d) = \limsup_{\epsilon \to 0} \frac{R(f, d, \varphi, Z, \epsilon)}{\log \frac{1}{\epsilon}}.$$

The quantity $\overline{BSdim}_{M, f}(\varphi, d)$ is said to be BS metric mean dimension on the set $Z$ with respect to $\varphi$ (or simply BS metric mean dimension). We sometimes omit $d$ and write $\overline{BSdim}_{M, f}(\varphi)$ instead of $\overline{BSdim}_{M, f}(\varphi, d)$ when $d$ is clear.

Definition 3.9. For $0 < \epsilon < 1, N \in \mathbb{N}, \lambda \in \mathbb{R}, Z \subset X$ and $\varphi \in C(X, \mathbb{R})$ with $\varphi > 0, d \in \mathcal{D}(X)$, we define

$$P_p(f, d, \varphi, Z, \lambda, N, \epsilon) = \sup \left\{ \sum_{i \in I} e^{-\lambda \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i} \varphi(y)} \right\},$$

where the supremum is taken over all finite or countable pairwise disjoint closed families $\{B_{n_i}(x_i, \epsilon)\}_{i \in I}$ of $Z$ with $n_i \geq N, x_i \in Z$.

The quantity $P_p(f, d, \varphi, Z, \lambda, N, \epsilon)$ is non-increasing as $N$ increases, so we define

$$P_p(f, d, \varphi, Z, \lambda, \epsilon) = \lim_{N \to \infty} P_p(f, d, \varphi, Z, \lambda, N, \epsilon).$$

Define

$$\mathcal{P}_p(f, d, \varphi, Z, \lambda, \epsilon) = \inf \left\{ \sum_{i=1}^{\infty} P_p(f, d, \varphi, Z_i, \lambda, \epsilon) : \cup_{i \geq 1} Z_i \supseteq Z \right\}.$$

The quantity $\mathcal{P}_p(f, d, \varphi, Z, \lambda, \epsilon)$ has a critical value of parameter $\lambda$ jumping from $\infty$ to 0. We define such critical value as

$$\overline{BSPdim}_{M, f}(\varphi, d, \epsilon) : = \inf \{ \lambda : \mathcal{P}_p(f, d, \varphi, Z, \lambda, \epsilon) = 0 \},$$

$$= \sup \{ \lambda : \mathcal{P}_p(f, d, \varphi, Z, \lambda, \epsilon) = +\infty \}.$$
Let

\[ {\text{BSPmdim}}_{M,Z,f}(\varphi, d) = \limsup_{\epsilon \to 0} \frac{{\text{BSPmdim}}_{M,Z,f}(\varphi, d, \epsilon)}{\log \frac{1}{\epsilon}}. \]

\( {\text{BSPmdim}}_{M,Z,f}(\varphi, d) \) is called Packing BS metric mean dimension on the set \( Z \) with respect to \( \varphi \) (or simply Packing BS metric mean dimension), and we sometimes omit \( d \) and write \( {\text{BSPmdim}}_{M,Z,f}(\varphi) \) instead of \( {\text{BSPmdim}}_{M,Z,f}(\varphi, d) \) when \( d \) is clear.

**Remark 3.10.**

(i) For any \( Z \subset X, \) \( 0 \leq {\text{BSmdim}}_{M,Z,f}(\varphi) \leq {\text{BSPmdim}}_{M,Z,f}(\varphi). \)

(ii) \( {\text{BSmdim}}_{M,Z,f}(1) = mdim_M(f, Z), \) \( {\text{BSPmdim}}_{M,Z,f}(1) = Pmdim_M(f, Z). \)

We now are ready to verify the Theorem 1.3.

**Proof of Theorem 1.3.** Let \( 0 < \epsilon < 1. \) Note that for each \( N, \)

\[ M(f, d, Z, -\frac{\lambda \varphi}{\log \frac{1}{\epsilon}}, 0, N, \epsilon) = R(f, d, \varphi, Z, \lambda, N, \epsilon). \]

Let \( s > {\text{BSmdim}}_{M,Z,f}(\varphi). \) Then \( R(f, d, \varphi, Z, s \log \frac{1}{\epsilon}, \epsilon) < 1 \) for sufficiently small \( \epsilon > 0. \) Hence \( M(f, d, Z, -s \varphi, 0, \epsilon) < 1, \) which implies that \( \overline{mdim}_{M,Z,f}(-s \varphi) \leq 0. \) Using the continuity of \( \phi \) obtained in Proposition 3.6, we obtain

\[ \overline{mdim}_{M,Z,f}(-{\text{BSmdim}}_{M,Z,f}(\varphi) \cdot \varphi) \leq 0 \]

after letting \( s \to \overline{BSmdim}_{M,Z,f}(\varphi). \)

Let \( s < \overline{BSmdim}_{M,Z,f}(\varphi). \) There exists a subsequence \( 0 < \epsilon_k < 1 \) that converges to \( 0 \) as \( k \to \infty \) such that

\[ \overline{BSmdim}_{M,Z,f}(\varphi, d) = \lim_{k \to \infty} \frac{\overline{BSmdim}_{M,Z,f}(\varphi, d, \epsilon_k)}{\log \frac{1}{\epsilon_k}}. \]

It follows that

\[ R(f, d, \varphi, Z, s \log \frac{1}{\epsilon_k}, \epsilon_k) > 1 \]

for all sufficiently large \( k. \) This shows \( M(f, d, Z, -s \varphi, 0, \epsilon_k) > 1. \) Similarly, we can deduce that

\[ \overline{mdim}_{M,Z,f}(-{\text{BSmdim}}_{M,Z,f}(\varphi) \cdot \varphi) \geq 0. \]

Hence, by Proposition 3.6, \( \overline{BSmdim}_{M,Z,f}(\varphi) \) is the unique root of the equation \( \overline{mdim}_{M,Z,f}(-t \varphi) = 0. \)

Using the relation \( \mathcal{P}(f, d, Z, -\frac{\lambda \varphi}{\log \frac{1}{\epsilon}}, 0, \epsilon) = \mathcal{P}_p(f, d, \varphi, Z, \lambda, \epsilon), \) one can similarly deduce that \( \overline{BSPmdim}_{M,Z,f}(\varphi) \) is the unique root of the equation \( \overline{Pmdim}_{M,Z,f}(-t \varphi) = 0. \)
The following corollary shows that the BS metric mean dimension is a special case of $\psi$-induced upper metric mean dimension with potential $0$.

**Corollary 3.11.** Let $(X, f)$ be a TDS with a metric $d \in \mathcal{D}(X)$ and $\psi \in C(X, \mathbb{R})$ with $\psi > 0$. Then

$$\overline{mdim}_{M, \psi}(X, f, d, 0) = \overline{BSmdim}_{M, X, f}(\psi).$$

**Proof.** If $\overline{mdim}_M(f, X, d) = \infty$, by Remark 3.2, then

$$\overline{mdim}_M(f, X, d) = \overline{mdim}_{M, X, f}(0, d) = \overline{mdim}_{M, X, f}(0, -\psi, d) = \overline{mdim}_M(X, f, d, 0, -\psi, d) = \infty.$$

Taking $\varphi = 0$ in Proposition 2.6, we get

$$\overline{mdim}_M(X, f, d, -\beta \psi, d) = \infty$$

for all $\beta \in \mathbb{R}$. By Corollary 2.7, we have

$$\overline{mdim}_{M, \psi}(X, f, d, 0) = \inf\{\beta \in \mathbb{R} : \overline{mdim}_M(X, f, d, -\beta \psi) \leq 0\} = \inf \emptyset = \infty.$$

Set $M := \max_{x \in X} \psi(x) > 0$ and $\lambda \geq 0$. For each $0 < \epsilon < 1$ and $N \in \mathbb{N}$,

$$R(f, d, \psi, \lambda, X, N, \epsilon) = \inf \left\{ \sum_{i \in I} e^{-\lambda \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i} \psi(y)} \right\} \geq \inf \left\{ \sum_{i \in I} e^{-\lambda M_{n_i}} \right\} = M(f, d, X, 0, M\lambda, N, \epsilon),$$

where the infimum is taken over all finite or countable covers $\{B_{n_i}(x_i, \epsilon)\}_{i \in I}$ of $X$ with $n_i \geq N$.

From this relation, we finally get that

$$\infty = \frac{\overline{mdim}_M(f, X, d)}{M} \leq \overline{BSmdim}_{M, X, f}(\psi, d).$$

Therefore,

$$\overline{mdim}_{M, \psi}(X, f, d, 0) = \overline{BSmdim}_{M, X, f}(\psi, d) = \infty.$$

For the case $\overline{mdim}_M(f, X, d) < \infty$, by remark 3.2, we have $\overline{mdim}_M(f, X, d) = \overline{mdim}_M(X, f, d) < \infty$. By Theorem 1.1, we have

$$\overline{mdim}_{M, X, f}(-\overline{mdim}_{M, \psi}(X, f, d, 0) \cdot \psi, d) = \overline{mdim}_{M, X, f}(X, f, d, -\overline{mdim}_{M, \psi}(X, f, d, 0) \cdot \psi, d) = 0.$$

Combing with Theorem 1.3, we obtain

$$\overline{mdim}_{M, \psi}(X, f, d, 0) = \overline{BSmdim}_{M, X, f}(\psi, d)$$

by the uniqueness of the root of the equation. \qed
As a direct consequence of Theorem 1.2 and Corollary 3.11, we have established a variational principle for BS metric mean dimension as follows.

**Corollary 3.12.** Let \((X, f)\) be a TDS admitting marker property and \(\psi \in C(X, \mathbb{R})\) with \(\psi > 0\). Then for all \(d \in D'(X)\), one has
\[
\text{BSdim}_{M, X, f}(\psi, d) = \sup_{\mu \in M(X, f)} \left\{ \frac{\text{rdim}(X, f, d, \mu)}{\int \psi d\mu} \right\}
\]

\[
= \sup_{\mu \in M(X, f)} \left\{ \frac{\text{rdim}(X, f, d, \mu)}{\int \psi d\mu} \right\}.
\]

### 3.3 Variational principles for BS and Packing BS metric mean dimension on subsets

In Corollary 3.12, we have established a variational principle for BS metric mean dimension on the whole phase space in terms of rate distortion dimensions over invariant measures. In this subsection, we proceed to establish the variational principles for BS metric mean dimension and Packing BS metric mean dimension on subsets. The following abundant critical ingredients are due to [WC12].

**Definition 3.13.** [WC12, Definition 3.8] Let \(\mu \in M(X), \varphi \in C(X, \mathbb{R})\) with \(\varphi > 0\), we define
\[
h_{\varphi, \mu}(f, \epsilon) = \int \liminf_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{S_n \varphi(x)} d\mu,
\]
\[
\overline{h}_{\varphi, \mu}(f, \epsilon) = \int \limsup_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{S_n \varphi(x)} d\mu.
\]

Let \(h_{\varphi, \mu}(f) = \lim_{\epsilon \to 0} h_{\varphi, \mu}(f, \epsilon), \overline{h}_{\varphi, \mu}(f) = \lim_{\epsilon \to 0} \overline{h}_{\varphi, \mu}(f, \epsilon)\). We call the quantities \(h_{\varphi, \mu}(f), \overline{h}_{\varphi, \mu}(f)\) the measure-theoretical lower and upper BS entropies of \(\mu\), respectively.

**Remark 3.14.** If \(\mu \in E(X, f)\), by Birkhoff ergodic theorem and Brin-Katok formula, then \(h_{\varphi, \mu}(f) = \overline{h}_{\varphi, \mu}(f, \epsilon), \overline{h}_{\varphi, \mu}(f) = \overline{h}_{\varphi, \mu}(f, \epsilon)\). When \(\varphi = 1\), the measure-theoretical lower and upper BS entropies of \(\mu\) is reduced to the classical Brin-Katok entropy formula [BK83].

**Lemma 3.15.** [M95, Theorem 2.1] Let \((X, d)\) be a compact metric space. Suppose that \(B = \{B(x_i, r_i)\}_{i \in I}\) is a family of open (or closed) balls in \(X\). Then there exists a finite or countable subfamily \(B' = \{B(x_i, r_i)\}_{i \in I'}\) of pairwise disjoint balls in \(B\) such that
\[
\bigcup_{B \in B} B \subseteq \bigcup_{i \in I'} B(x_i, 5r_i).
\]

**Definition 3.16.** Let \(\varphi \in C(X, \mathbb{R})\) with \(\varphi > 0\) and \(\psi\) be a non-negative bound function on \(X\), and let \(\lambda \in \mathbb{R}\) and \(N \in \mathbb{N}, \epsilon > 0\). Define
\[
W(f, d, \varphi, \psi, \lambda, N, \epsilon) = \inf \left\{ \sum_{i \in I} c_i e^{-\lambda \sup_{y \in B_{n_i}(x_i, \epsilon)} S_{n_i} \varphi(x)} \right\},
\]
where the infimum ranges over all finite or countable families \( \{(B_{n_i}(x_i, \epsilon), c_i)\}_{i \in I} \) satisfying \( 0 < c_i < \infty, x_i \in X, n_i \geq N, \) and

\[
\sum_{i \in I} c_i \chi_{B_{n_i}(x_i, \epsilon)} \geq \psi,
\]

where \( \chi_E \) denotes the characteristic function of \( E. \)

For \( Z \subset X, \) set \( W(f, d, \varphi, Z, \lambda, N, \epsilon) := W(f, d, \varphi, x, Z, \lambda, N, \epsilon). \) Since the quantity \( W(f, d, \varphi, Z, \lambda, N, \epsilon) \) is non-decreasing as \( N \) increases, so we define

\[
W(f, d, \varphi, Z, \lambda, \epsilon) = \lim_{N \to \infty} W(f, d, \varphi, Z, \lambda, N, \epsilon).
\]

There is a critical value of \( \lambda \) so that \( W(f, d, \varphi, Z, \lambda, \epsilon) \) jumps from \( \infty \) to \( 0. \) We define such critical value as

\[
\overline{\text{Wmdim}}_{M, Z, f}(\varphi, d, \epsilon) := \inf \{ \lambda : W(f, d, \varphi, Z, \lambda, \epsilon) = 0 \},
\]

\[
= \sup \{ \lambda : W(f, d, \varphi, Z, \lambda, \epsilon) = \infty \}.
\]

Let \( \overline{\text{Wmdim}}_{M, Z, f}(\varphi, d) = \limsup_{\epsilon \to 0} \frac{\overline{\text{Wmdim}}_{M, Z, f}(\varphi, d, \epsilon)}{\log \frac{1}{\epsilon}}, \) and we call the quantity \( \overline{\text{Wmdim}}_{M, Z, f}(\varphi, d) \) the weighted BS metric mean dimension on the set \( Z \) with respect to \( \varphi. \)

Wang and Chen [WC12, Lemma 5.1] proved the following proposition.

**Proposition 3.17.** Let \( (X, f) \) be a TDS with a metric \( d \in \mathcal{D}(X) \) and \( \varphi \in C(X, \mathbb{R}) \) with \( \varphi > 0, \) and let \( 0 < \epsilon < 1 \) and \( Z \subset X. \) Then

\[
R(f, d, \varphi, Z, \lambda + \delta, N, 6\epsilon) \leq W(f, d, \varphi, Z, \lambda, N, \epsilon) \leq R(f, d, \varphi, Z, \lambda, N, \epsilon)
\]

holds for all \( \lambda > 0, \delta > 0. \) Consequently, \( \overline{\text{BSmdim}}_{M, Z, f}(\varphi, d) = \overline{\text{Wmdim}}_{M, Z, f}(\varphi, d). \)

**Lemma 3.18 (BS Frostman’s lemma).** [WC12, Lemma 6.1] Let \( K \) be a non-empty compact subset of \( X \) and \( \lambda \geq 0, \epsilon > 0, N \in \mathbb{N}, \varphi \in C(X, \mathbb{R}) \) with \( \varphi > 0. \) Suppose that \( c := W(f, d, \varphi, K, \lambda, N, \epsilon) > 0. \) Then there exists a Borel probability measure \( \mu \in M(X) \) such that \( \mu(K) = 1 \) and

\[
\mu(B_n(x, \epsilon)) \leq \frac{1}{c} e^{-\lambda S_n \varphi(x)}
\]

holds for all \( x \in K, n \geq N. \)

The following proposition can be proved by following the line of the first part of the proof given in [WC12, Theorem 7.2].

**Proposition 3.19.** Let \( (X, f) \) be a TDS with a metric \( d \in \mathcal{D}(X) \) and \( \varphi \in C(X, \mathbb{R}) \) with \( \varphi > 0. \) Let \( K \) be a non-empty compact subset of \( X. \) Then for any \( 0 < \epsilon < 1 \) and \( \mu \in M(X) \) with \( \mu(K) = 1, \) we have \( (1 - \frac{\gamma(2\epsilon)}{m}) \underline{L}_{\varphi, \mu}(f, \epsilon) \leq R(f, d, \varphi, K, \frac{\epsilon}{2}), \) where \( m = \min_{x \in X} \varphi(x) > 0. \)
Next, we give the proof of Theorem 1.4.

**Proof of Theorem 1.4.** Firstly, we show

\[ \text{BSdim}_{M,K,f}(\varphi, d) = \limsup_{\epsilon \to 0} \sup \left\{ \frac{h_{\varphi,\mu}(f, \epsilon), \mu \in M(X), \mu(K) = 1}{\log \frac{1}{\epsilon}} \right\}. \]

It is clear that \( \text{LHS} \geq \text{RHS} \) follows from the Proposition 3.19. On the other hand, we assume that \( \text{BSdim}_{M,K,f}(\varphi, d) > 0 \). By Proposition 3.17, we know that \( \text{BSdim}_{M,K,f}(\varphi, d) = \text{Wdim}_{M,K,f}(\varphi, d) \). Let \( 0 < \lambda < \text{Wdim}_{M,K,f}(\varphi, d) \). Then we can find a sequence \( 0 < \epsilon_k < 1 \) that converges to 0 as \( k \to \infty \) so that

\[ \text{Wdim}_{M,K,f}(\varphi, d) = \lim_{k \to \infty} \frac{\text{Wdim}_{M,K,f}(\varphi, d, \epsilon_k)}{\log \frac{1}{\epsilon_k}} > \lambda. \]

Hence, for all sufficiently large \( k \), there is \( N_0 \in \mathbb{N} \) such that \( c := W(f, d, \varphi, \lambda \log \frac{1}{\epsilon_k}, Z, N_0, \epsilon_k) > 0. \) By virtue of Lemma 3.18, there exists a Borel probability measure \( \mu \in M(X) \) such that \( \mu(K) = 1 \) and

\[ \mu(B_n(x, \epsilon_k)) \leq \frac{1}{c} e^{-\lambda \log \frac{1}{\epsilon_k} S_n \varphi(x)} \]

holds for all \( x \in X, n \geq N_0. \)

This gives us that

\[ \sup_{\epsilon \to 0} \left\{ \frac{h_{\varphi,\mu}(f, \epsilon), \mu \in M(X), \mu(K) = 1}{\log \frac{1}{\epsilon}} \right\} \geq \frac{h_{\varphi,\mu}(f, \epsilon_k)}{\log \frac{1}{\epsilon_k}} \geq \lambda. \]

for all sufficiently large \( k \), which implies that \( \text{LHS} \leq \text{RHS} \).

Next, we prove that

\[ \text{BSPmdim}_{M,K,f}(\varphi, d) = \limsup_{\epsilon \to 0} \sup \left\{ \frac{h_{\varphi,\mu}(f, \epsilon), \mu \in M(X), \mu(K) = 1}{\log \frac{1}{\epsilon}} \right\}. \]

Fix a sufficiently small \( \epsilon \) with \( 0 < \epsilon < 1. \) We may assume that \( \text{BSPmdim}_{M,K,f}(\varphi, d, \epsilon) > 0 \).

Let \( 0 < s < \text{BSPmdim}_{M,K,f}(\varphi, d, \epsilon) \). By [WC12, Theorem 3.12, Part 2], there is a \( \mu \in M(X) \) with \( \mu(K) = 1 \) such that for any \( x \in K \), there exists a subsequence \( n_i := n_i(x) \) so that

\[ \mu(B_{n_i}(x, \epsilon)) \leq C \cdot e^{-s \cdot S_{n_i} \varphi(x)}, \]

where \( C \) is a constant that does not depend on the points of \( K. \)

It follows that \( \overline{h}_{\varphi,\mu}(f, \epsilon) \geq s \), and we obtain that

\[ \overline{h}_{\varphi,\mu}(f, \epsilon) \geq \text{BSPmdim}_{M,K,f}(\varphi, d, \epsilon), \]

after letting \( s \to \text{BSPmdim}_{M,K,f}(\varphi, d, \epsilon) \), which yields that \( \text{RHS} \geq \text{LHS} \).
Let \( \mu \in M(X) \) with \( \mu(K) = 1 \). We assume that \( \overline{h}_{\varphi, \mu}(f, 2\epsilon) > 0 \). Let \( 0 < s < \overline{h}_{\varphi, \mu}(f, 2\epsilon) \). We can choose \( \delta > 0 \) and a Borel set \( A \subset K \) with \( \mu(A) > 0 \) such that

\[
\limsup_{n \to \infty} \frac{-\log \mu(B_n(x, 2\epsilon))}{S_n \varphi(x)} > s + \delta
\]

for all \( x \in A \).

Next, we show \( P_p(f, d, K, \varphi, s(1 - \frac{\gamma(\epsilon)}{m}), \frac{\epsilon}{5}) = \infty \), where \( m = \min_{x \in X} \varphi(x) > 0 \) and \( \gamma(\epsilon) = \sup \{ |\varphi(x) - \varphi(y)| : d(x, y) \leq \epsilon \} \). To this end, it suffices to show for any \( E \subset A \) with \( \mu(E) > 0 \), we have \( P_p(f, d, \varphi, E, s(1 - \frac{\gamma(\epsilon)}{m}), \frac{\epsilon}{5}) = \infty \). Fix such a set \( E \), define

\[
E_n := \{ x \in E : \mu(B_n(x, 2\epsilon)) < e^{-(s+\delta)S_n \varphi(x)} \}.
\]

Then we have \( E = \cup_{n \geq N} E_n \) for any \( N \in \mathbb{N} \). Fix such a \( N \), by \( \mu(E) = \mu(\cup_{n \geq N} E_n) \), then there is a \( n \geq N \) so that

\[
\mu(E_n) \geq \frac{1}{n(n+1)} \mu(E).
\]

Fix such \( n \), consider a family of closed cover \( \{ B_n(x, \frac{\epsilon}{5}) : x \in E_n \} \) of \( E_n \). By Lemma 3.15 (replacing \( d \) with the Bowen metric \( d_n \)), then there exists a finite pairwise disjoint subfamily \( \{ B_n(x_i, \frac{\epsilon}{5}) : x_i \in E_n \}_{i \in I} \), where \( I \) is a finite index set, such that

\[
\cup_{i \in I} B_n(x_i, \epsilon) \supseteq \cup_{x \in E_n} B_n(x, \frac{\epsilon}{5}) \supseteq E_n.
\]

For each \( i \in I \), we have

\[
\sup_{y \in B_n(x_i, \frac{\epsilon}{5})} S_n \varphi(y) \leq S_n \varphi(x_i) + n \gamma(\epsilon)
\]

\[
\leq S_n \varphi(x_i) + \frac{\sup_{y \in B_n(x_i, \frac{\epsilon}{5})} S_n \varphi(y)}{m} \gamma(\epsilon).
\]

Hence,

\[
P_p(f, d, \varphi, E, s(1 - \frac{\gamma(\epsilon)}{m}), N, \frac{\epsilon}{5}) \geq P_p(f, d, \varphi, E_n, s(1 - \frac{\gamma(\epsilon)}{m}), N, \frac{\epsilon}{5})
\]

\[
\geq \sum_{i \in I} e^{-s(1 - \frac{\gamma(\epsilon)}{m}) \sup_{y \in B_n(x_i, \frac{\epsilon}{5})} S_n \varphi(y)}
\]

\[
\geq \sum_{i \in I} e^{-sS_n \varphi(x_i)}
\]

\[
= \sum_{i \in I} e^{-(s+\delta)S_n \varphi(x_i)} e^{\delta S_n \varphi(x_i)}
\]

\[
\geq e^{nm\delta} \sum_{i \in I} \mu(B_n(x_i, \epsilon))
\]

\[
\geq e^{nm\delta} \mu(E_n)
\]

\[
\geq e^{nm\delta} \frac{\mu(E)}{n(n+1)}.
\]
Letting $N \to \infty$, we obtain that $P_p(f, d, E, s(1 - \frac{\gamma(\epsilon)}{m}), \frac{\epsilon}{5}) = \infty$. This gives us that

$$\overline{BSP}^\text{mdim}_{M,K,f}(\varphi, d, \frac{\epsilon}{5}) \geq s(1 - \frac{\gamma(\epsilon)}{m}).$$

Letting $s \to h_{\varphi,\mu}(f, 2\epsilon), \mu \in M(X)$ with $\mu(K) = 1$. This implies that

$$(1 - \frac{\gamma(\epsilon)}{m}) \sup \{ h_{\varphi,\mu}(f, 2\epsilon) : \mu \in M(X), \mu(K) = 1 \} \leq \overline{BSP}^\text{mdim}_{M,K,f}(\varphi, d, \frac{\epsilon}{5}),$$

which yields that $\text{LHS} \geq \text{RHS}$.

\[\square\]

### 3.4 Bowen upper metric mean dimension of the set of generic points

We first collect several types of measure-theoretical entropies defined by invariant measures (or ergodic measures) as candidates to characterize the Bowen upper metric mean dimension of the sets of generic points of ergodic measures.

Let $(X, f)$ be a TDS with a metric $d \in \mathfrak{D}(X)$. Given $\mu \in M(X, f)$, by $h_{\mu}(f)$ we denote the measure-theoretical entropy of $\mu$.

(i) Measure-theoretical entropy given from the viewpoint of the local perspective. Put

$$h^{BK}_\mu(f, d, \epsilon) = \int \liminf_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{n} d\mu,$$

$$\overline{h}^{BK}_\mu(f, d, \epsilon) = \int \limsup_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{n} d\mu.$$

Brin and Katok [BK83] showed that $h_{\mu}(f) = \lim_{\epsilon \to 0} h^{BK}_\mu(f, d, \epsilon) = \lim_{\epsilon \to 0} \overline{h}^{BK}_\mu(f, d, \epsilon)$ for all $\mu \in M(X, f)$. If $\mu$ is an ergodic measure, they also showed that for each fixed $\epsilon > 0$,

$$\liminf_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{n}, \limsup_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{n}$$

are both constants for $\mu$-a.e $x \in X$. In this case, we still denote

$$h^{BK}_\mu(f, d, \epsilon) = \liminf_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{n},$$

$$\overline{h}^{BK}_\mu(f, d, \epsilon) = \limsup_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{n}.$$

(ii) Measure-theoretical entropy defined by separated set and spanning set.

Put

$$PS(f, d, \mu, \epsilon) = \inf_{F, \mu} \limsup_{n \to \infty} \frac{1}{n} \log s_n(f, d, \epsilon, X_n, F),$$

where

$$s_n(f, d, \epsilon, X_n, F) = \inf_{F, \mu} \limsup_{n \to \infty} \frac{1}{n} \log s_n(f, d, \epsilon, X_n, F),$$

and

$$s_n(f, d, \epsilon, X_n, F) = \inf_{F, \mu} \limsup_{n \to \infty} \frac{1}{n} \log s_n(f, d, \epsilon, X_n, F).$$
where the infimum runs over all neighborhoods of $\mu$ in $M(X)$ and $X_{n,F} = \{ x \in X : \frac{1}{n} \sum_{j=1}^{n} \delta f_j(x) \in F \}$.

If $\mu \in E(X, f)$, Pfister and Sullivan [PS07] proved that $h_{\mu}(f) = \lim_{\epsilon \to 0} PS(f, d, \mu, \epsilon)$.

Let $\delta \in (0, 1)$. Put

$$h^K_{\mu}(f, d, \epsilon, \delta) = \limsup_{n \to \infty} \frac{1}{n} \log r_n(\mu; d, \epsilon, \delta),$$

$$h^K_{\mu}(f, d, \epsilon) = \lim_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r_n(\mu; d, \epsilon, \delta),$$

where $r_n(\mu; d, \epsilon, \delta) = \min\{ \# F : \mu(\cup_{x \in F} B_n(x, \epsilon)) > 1 - \delta, F \subset X \}$.

For $\mu \in E(X, f)$, Katok [K80] showed that $h_{\mu}(f) = \lim_{\epsilon \to 0} h^K_{\mu}(f, d, \epsilon, \delta)$ for any $\delta \in (0, 1)$.

(iii) The last candidate comes from information theory. Recall that upper and lower rate distortion dimensions are respectively given by

$$\overline{r\dim}(X, f, d, \mu) = \limsup_{\epsilon \to 0} \frac{R(d, \mu, \epsilon)}{\log \frac{1}{\epsilon}},$$

$$\underline{r\dim}(X, f, d, \mu) = \liminf_{\epsilon \to 0} \frac{R(d, \mu, \epsilon)}{\log \frac{1}{\epsilon}},$$

where $R(d, \mu, \epsilon)$ denotes the rate distortion function.

Replacing $R(d, \mu, \epsilon)$ by $R_{L\infty}(d, \mu, \epsilon)$, one can similarly define upper $L^\infty$-rate distortion dimension $\overline{r\dim}_{L\infty}(X, f, d, \mu)$ and lower $L^\infty$-rate distortion dimension $\underline{r\dim}_{L\infty}(X, f, d, \mu)$, where $R_{L\infty}(d, \mu, \epsilon)$ denotes $L^\infty$-rate distortion function. Due to the forthcoming proof does not refer to the definitions of $R(d, \mu, \epsilon)$ and $R_{L\infty}(d, \mu, \epsilon)$, we omit their precise definitions and refer readers to [CT06, LT18, LT19] for more details.

Inspired by the method used in [ZC18, Theorem 1.2], we proceed to prove Theorem 1.5, (i).

Proof of Theorem 1.5, (i). Let $\mu \in M(X, f)$ with $\mu(Y) = 1$. There exists an increasing sequence $Y_n$ of compact subsets of $Y$ satisfying $\mu(Y_n) > 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

Therefore,

$$\overline{mdim}_{M,f,Y}(0, d, \epsilon) \geq \overline{mdim}_{M,f,Y_n}(0, d, \epsilon) = \lim_{n \to \infty} \overline{mdim}_{M,f,Y_n}(0, d, \epsilon).$$

Put $\mu_n := \mu|Y_n$, that is, for any Borel set $A \in \mathcal{B}(X)$, $\mu_n(A) = \frac{\mu(A \cap Y_n)}{\mu(Y_n)}$. Take $\varphi = 1$ in Proposition 3.19, note that $h^K_{\mu_n}(f, d, \epsilon) = h^K_{\mu}(f, \epsilon)$ and $\overline{mdim}_{M,f,Y_n}(0, d, \frac{\epsilon}{2}) = R(f, d, Y_n, \frac{\epsilon}{2})$. Then

$$(1 - \gamma(2\epsilon))h^K_{\mu_n}(f, d, \epsilon) \leq \overline{mdim}_{M,f,Y_n}(0, d, \frac{\epsilon}{2}).$$
Hence
\[
\frac{h_{BK}^{M}}{m}(f, d, \epsilon) = \int_{Y_n} \liminf_{m \to \infty} \frac{1}{m} \log \mu_n(B_m(x, \epsilon)) d\mu_n
\]
\[
= \frac{1}{\mu(Y_n)} \int_{Y_n} \liminf_{m \to \infty} \frac{1}{m} \log \frac{\mu(B_m(x, \epsilon) \cap Y_n)}{\mu(Y_n)} d\mu
\]
\[
\geq \frac{1}{\mu(Y_n)} \int_{Y_n} \liminf_{m \to \infty} \frac{1}{m} \log \mu(B_m(x, \epsilon)) d\mu.
\]

Letting \( n \to \infty \), we have
\[
\liminf_{n \to \infty} \frac{h_{BK}^{M,f,Y}(0, d, \epsilon/2)}{2} \geq \lim_{n \to \infty} \frac{h_{BK}^{M,f,Y}(0, d, \epsilon/2)}{2} \geq \lim_{n \to \infty} (1 - \gamma(2\epsilon)) h_{BK}^{M,f}(f, d, \epsilon) \geq (1 - \gamma(2\epsilon)) h_{BK}^{M}(f, d, \epsilon).
\]

This implies that \( \limsup_{\epsilon \to 0} \frac{h_{BK}^{M,f}(f, d, \epsilon)}{\log \frac{1}{\epsilon}} \leq \tilde{mdim}_M(f, Y, d) \).

\[\square\]

**Corollary 3.20.** Let \((X, f)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) and \(\mu \in E(X, f)\). Then

\[
\limsup_{\epsilon \to 0} \frac{h_{BK}^{M,f}(f, d, \epsilon)}{\log \frac{1}{\epsilon}} \leq \tilde{mdim}_{G_M\mu}(f, d).
\]

**Proposition 3.21.** Let \((X, f)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) and \(\mu \in E(X, f)\). Then for each \(\epsilon > 0\),
\[
\lim_{n \to \infty} \frac{h_{BK}^{K}(f, d, 2\epsilon)}{2} \leq \lim_{n \to \infty} \frac{h_{BK}^{K}(f, d, \epsilon)}{2}.
\]

**Proof.** Fix \(\epsilon > 0\) and let \(F(x, \epsilon) = \limsup_{n \to \infty} -\frac{\log \mu(B_n(x, \epsilon))}{n}\). Since \(f(B_{n+1}(x, \epsilon)) \subset B_n(f(x, \epsilon))\) for all \(x \in X\) and \(n \in \mathbb{N}\) and \(\mu \in M(X, f)\), we have
\[
\mu(B_{n+1}(x, \epsilon)) \leq \mu(f^{-1}f(B_{n+1}(x, \epsilon))) = \mu(B_{n+1}(x, \epsilon)) \leq \mu(B_n(f(x, \epsilon))).
\]

This yields that \(F(f(x, \epsilon), \epsilon) \leq F(x, \epsilon)\). It follows from \(\mu \in E(X, f)\) that \(F(x, \epsilon)\) is a constant \(\mu\)-a.e \(x \in X\). Let \(s > \frac{h_{BK}^{K}(f, d, \epsilon)}{2}\). Fix \(\delta \in (0, 1)\) and set
\[
X_N := \{x \in X : \mu(B_n(x, \epsilon)) > e^{-ns}, \forall n \geq N\}.
\]

Then we have \(\mu(\cup_{N \geq 1} X_N) = 1\). There exists \(N_0\) such that for any \(N \geq N_0\), \(\mu(X_N) > 1 - \delta\). For each \(N \geq N_0\), let \(E_N\) be the \((N, 2\epsilon)\)-separated set of \(X_N\) with maximal cardinality. Note that the Bowen balls \(B_N(x, \epsilon), x \in E_N\) are pairwise disjoint, hence we have
\[
1 \geq \mu(\cup_{x \in E_N} B_N(x, \epsilon)) = \sum_{x \in E_N} \mu(B_N(x, \epsilon)) > \sum_{x \in E_N} e^{-Ns}.
\]

Then \(r_N(\mu; d, 2\epsilon, \delta) \leq \#E_N \leq e^{Ns}\) for all \(N \geq N_0\), which implies \(\bar{h}_{BK}^{K}(f, d, 2\epsilon) \leq \bar{h}_{BK}^{K}(f, d, \epsilon)\). \[\square\]
Proposition 3.22. [W21, Proposition 4.3, Proposition 6.1] Let \((X, f)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) and \(\mu \in E(X, f)\). Then for each \(\epsilon > 0\),

\[
R_{L^\infty}(d, \mu, \epsilon) \leq \overline{h}_\mu^K(f, d, \epsilon) \leq PS(f, d, \mu, \epsilon) \leq R_{L^\infty}(d, \mu, \frac{1}{6} \epsilon)
\]

and \(
\underbar{mdim}_{M,G_\mu}(0, d, \epsilon) \leq PS(f, d, \mu, \epsilon).
\)

Finally, we give the proof of Theorem 1.5, (ii).

Proof of Theorem 1.5, (ii). By virtue of Proposition 3.22 and Proposition 3.23, we have

\[
\overline{mdim}_M(f, G_\mu, d, 12\epsilon) = \overline{mdim}_{M,f,G_\mu}(0, d, 12\epsilon) \\
\leq PS(f, d, \mu, 12\epsilon) \\
\leq R_{L^\infty}(d, \mu, 2\epsilon) \\
\leq \overline{h}_\mu^K(f, d, 2\epsilon) \\
\leq \overline{h}_\mu^{BK}(f, d, \epsilon).
\]

Combining the assumption \(\lim \sup_{\epsilon \to 0} \frac{h_{BK}(f, d, \epsilon)}{\log \frac{1}{\epsilon}} = \lim \sup_{\epsilon \to 0} \frac{\overline{h}_\mu^{BK}(f, d, \epsilon)}{\log \frac{1}{\epsilon}}\) and Corollary 3.21, this completes the proof.

One says that a compact metric space \((X, d)\) admits tame growth of covering numbers if for each \(\theta > 0\),

\[
\lim_{\epsilon \to 0} \epsilon^\theta \log r_1(f, d, \epsilon, X) = 0.
\]

This condition was introduced by Lindenstrauss and Tsukamoto [LT18] to show the metric mean dimensions defined by Bowen metric and average metric coincide, see [LT18, Lemma 26], which was proved as a fairly mild condition [LT18, Lemma 4]. Namely, every compact metrizable space admits a metric with the property of tame growth of covering numbers, and some examples satisfying such condition can be found in [LT19, Example 3.9].

Under the assumption of tame growth of covering numbers, Wang [W21, Theorem 1.7] also showed if \(\mu \in E(X, f)\), then

\[
\overline{rdim}(X, f, d, \mu) = \overline{rdim}_{L^\infty}(X, f, d, \mu).
\]

Together with Theorem 1.5, we immediately deduce the following.

Corollary 3.23. Let \((X, f)\) be a TDS with a metric \(d \in \mathcal{D}(X)\) admitting tame growth of covering numbers, and suppose that \(\mu \in E(X, f)\) satisfying \(\lim \sup_{\epsilon \to 0} \frac{h_{BK}(f, d, \epsilon)}{\log \frac{1}{\epsilon}} = \lim \sup_{\epsilon \to 0} \frac{\overline{h}_\mu^{BK}(f, d, \epsilon)}{\log \frac{1}{\epsilon}}\). Then

\[
\overline{mdim}_M(f, G_\mu, d) = \overline{rdim}(X, f, d, \mu) = \overline{rdim}_{L^\infty}(X, f, d, \mu).
\]
Example 3.24. Let $\sigma : [0, 1]^Z \to [0, 1]^Z$ be the shift on alphabet $[0, 1]$, where $[0, 1]$ is the unit interval with the standard metric. Equipped $[0, 1]^Z$ with a metric given by

$$d(x, y) = \sum_{n \in Z} 2^{-|n|} |x_n - y_n|.$$ 

Then $([0, 1]^Z, d)$ has the tame growth of covering numbers, see [LT19, Example 3.9]. Let $\mu = \mathcal{L}^Z$, where $\mathcal{L}$ is the Lebesgue measure on $[0, 1]$. For each $\varepsilon > 0$, $x \in [0, 1]^Z$. Let $r = \lceil \log_2 4 \varepsilon \rceil + 1$. Then $\sum_{|n| > r} 2^{-|n|} < \frac{\varepsilon}{2}$. Put

$$I_n(x, \varepsilon) := \{ y \in [0, 1]^Z : |x_i - y_i| < \frac{\varepsilon}{6}, \forall -r \leq i \leq n + r \},$$

$$J_n(x, \varepsilon) := \{ y \in [0, 1]^Z : |x_i - y_i| < \varepsilon, \forall 0 \leq i \leq n \}.$$ 

One can check that $I_n(x, \varepsilon) \subset B_n(x, \varepsilon) \subset J_n(x, \varepsilon)$.

It is clear that $\mu(I_n(x, \varepsilon)) \geq \frac{(\frac{\varepsilon}{6})^{n+2r}}{4}, \mu(J_n(x, \varepsilon)) \leq (4\varepsilon)^n$. This implies that

$$\log \frac{1}{4\varepsilon} \leq h_{BK}^{\mu}(\sigma, d, \varepsilon) \leq \log \frac{6}{\varepsilon},$$

which tells us that $\limsup_{\varepsilon \to 0} \frac{\log h_{BK}^{\mu}(\sigma, d, \varepsilon)}{\log \frac{1}{\varepsilon}} = \limsup_{\varepsilon \to 0} \frac{\log h_{BK}^{\mu}(\sigma, d, \varepsilon)}{\log \frac{1}{\varepsilon}} = 1.$

It is well-known that $mdim_M(\sigma, [0, 1]^Z, d) = 1$, see [LT18, Section II, E. Example]. By Corollary 3.21, $1 \leq mdim_M(\sigma, G_{\mu}, d) \leq mdim_M(\sigma, [0, 1]^Z, d) = 1$. So $mdim_M(\sigma, G_{\mu}, d) = 1$. By [LT18, Example 22], we know $rdim([0, 1]^Z, \sigma, d, \mu) = rdim_{L^\infty}([0, 1]^Z, \sigma, d, \mu) = 1$. Finally, $mdim_M(\sigma, G_{\mu}, d) = rdim([0, 1]^Z, \sigma, d, \mu) = rdim_{L^\infty}([0, 1]^Z, \sigma, d, \mu) = 1$.

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