Leibniz Homology, Characteristic Classes and K-theory

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1 Introduction

In this paper we identify many striking elements in Leibniz (co)homology which arise from characteristic classes and K-theory. For a group $G$ and a field $k$ containing $\mathbb{Q}$, it is shown that all primary characteristic classes, i.e. $H^*(BG; k)$, naturally inject into certain Leibniz cohomology groups via an explicit chain map. Moreover, if $f : A \to B$ is a homomorphism of algebras or rings, the relative Leibniz homology groups $HL_* (f)$ are defined, and if in addition $f$ is surjective with nilpotent kernel, $A$ and $B$ algebras over $\mathbb{Q}$, then there is a natural surjection

$$HL_{*+1}(gl(f)) \to HC_*(f),$$

where $HC_*(f)$ denotes relative cyclic homology, and $gl(f) : gl(A) \to gl(B)$ is the induced map on matrices. Here again, the above surjection is realized via an explicit chain map, and offers a relation between Leibniz homology and K-theory, since by work of T. Goodwillie \[3\], there is an isomorphism

$$K_{*+1}(f) \otimes \mathbb{Q} \to HC_*(f)$$

between the relative theories when $f$ satisfies the above hypotheses.

Both explicit chain maps mentioned above involve an initial homomorphism

$$\varphi_* : HL_{*+1}(A) \to HH_*(A)$$

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from Leibniz to Hochschild homology, which on the chain level is simply a
version of the antisymmetrization map. In many cases, $\varphi_*$ is seen to be surjec-
tive, and from $HH_*(A)$ there are known maps to other types of homologies,
such as cyclic homology, or when $A = k[G]$, $HH_*(A)$ maps to $H_*(BG)$. The
various homologies can be assembled into a curious commutative diagram:

$$\begin{align*}
HL_{s+1}(A) & \longrightarrow HH_*(A) \\
\downarrow & \quad \downarrow \\
H_{s+1}^{\text{Lie}}(A) & \longrightarrow HC_*(A),
\end{align*}$$

where $H_{s}^{\text{Lie}}(A)$ denotes Lie-algebra homology.

The map from Leibniz to Hochschild homology is studied in §2, while the
map to $H_*(BG)$ appears in §3, and the map to cyclic homology in the relative
case is in the final paragraph. The appendix offers an alternative calculation
of the primitives of $HL_*(gl(A))$, and fills a gap in a proof in [8]. Combined
with the author’s previous results on foliations, Leibniz cohomology contains
primary characteristic classes (this paper), secondary characteristic classes [9],
and variations of secondary classes [10]. In this way, the $HL_*$ theory can
be viewed as a “proto-homology.”

## 2 From Leibniz to Hochschild Homology

Recall that J.-L. Loday has defined Leibniz homology for the category of Lie
algebras, and more generally for the category of Leibniz algebras [7]. In this
paper we begin with an associative algebra $A$ over a commutative ring $k$, and
consider $A$ as a Lie algebra via its algebra structure:

$$[a, b] = ab - ba, \quad a, b \in A.$$ 

In most cases $A$ will be unital, although that assumption is not necessary for
the definition of $HL_*(A)$, the Leibniz homology of $A$. The latter is defined
as the homology of the chain complex $CL_*(A)$:

$$k \leftarrow^0 A \leftarrow^{[\cdot, \cdot]} A^\otimes 2 \leftarrow \ldots \leftarrow A^\otimes n \leftarrow^d A^\otimes (n+1) \leftarrow \ldots ,$$

where

$$d(a_0 \otimes a_1 \otimes \ldots \otimes a_n) =
\sum_{0 \leq i < j \leq n} (-1)^{j+1} (a_0, a_1, \ldots, a_{i-1}, [a_i, a_j], a_{i+1}, \ldots, \hat{a}_j, \ldots, a_n). \quad (2.1)$$
The projection from the tensor powers of $A$ to the exterior powers

$$A^\otimes n \to A^\wedge n$$

induces a natural map

$$HL_s(A) \to H^\text{Lie}_s(A)$$

from Leibniz homology to Lie-algebra homology, where again the algebra $A$ is viewed as a Lie algebra. We now define a natural map

$$HL_{s+1}(A) \to HH_s(A)$$

to Hochschild homology, $HH_s(A)$, which when $k$ is a characteristic zero field, yields a commutative diagram:

$$\begin{array}{ccc}
HL_{s+1}(A) & \longrightarrow & HH_s(A) \\
\downarrow & & \downarrow \\
H^\text{Lie}_{s+1}(A) & \longrightarrow & HC_s(A),
\end{array}$$

(2.2)

with $HC_s(A)$ denoting cyclic homology.

Recall that $HH_s(A)$ is the homology of the chain complex $CHH_s(A)$:

$$A \leftarrow^b A^\otimes 2 \leftarrow \ldots \leftarrow A^\otimes n \leftarrow^b A^\otimes (n+1) \leftarrow \ldots,$$

where $b : A^\otimes (n+1) \to A^n$ is defined using the face maps of the cyclic bar construction [6, 1.1.1]

$$b = \sum_{i=0}^{n} (-1)^i d_i$$

$$d_i(a_0, a_1, \ldots, a_n) = (a_0, a_1, \ldots, a_i a_{i+1}, \ldots, a_n), \quad 0 \leq i < n$$

$$d_n(a_0, a_1, \ldots, a_n) = (a_n a_0, a_1, a_2, \ldots, a_{n-1}).$$

(2.3)

Note that $b : A^\otimes 2 \to A$ is simply the bracket $[\ , \ ] : A^\otimes 2 \to A$, since

$$b(a_0, a_1) = a_0 a_1 - a_1 a_0,$$

from which follows

$$HL_1(A) \simeq HH_0(A) \simeq HC_0(A) \simeq H^\text{Lie}_1(A).$$
In fact, every arrow in diagram (2.2) is an isomorphism in the special case \( \ast = 0 \).

Consider now the map of chain complexes

\[
\varphi : CL_{*+1}(A) \to CHH_*(A)
\]

\[
\varphi_n : A^{\otimes(n+1)} \to A^{\otimes(n+1)}
\]

given by \( \varphi_0 = 1, \varphi_1 = 1, \) and for \( n \geq 2 \),

\[
\varphi_n(a_0, a_1, \ldots, a_n) = \sum_{\sigma \in S_n} (\text{sgn} \sigma) \left( a_0, a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(n)} \right),
\]

(2.5)

where \( S_n \) is the symmetric group on \( n \) letters. Of course, formula (2.5) could be used to define \( \varphi_1 \), since \( S_1 \) is the trivial group. Also, the summation remains invariant if \( \sigma^{-1} \) is replaced with \( \sigma \) in all subscripts, which reconciles various descriptions of antisymmetrization maps in the literature.

\textbf{Lemma 2.1.} The \( k \)-module homomorphism

\[
\varphi : CL_{*+1}(A) \to CHH_*(A)
\]

is a map of chain complexes.

\textit{Proof.} It follows at once that \( b \varphi_1 = \varphi_0 d \). For \( n \geq 2 \),

\[
b \circ \varphi_n(a_0, a_1, a_2, \ldots, a_n) = \sum_{\sigma \in S_n} (\text{sgn} \sigma) \left( a_0a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots a_{\sigma^{-1}(n)} \right)
\]

\[
- \left( a_0, a_{\sigma^{-1}(1)}a_{\sigma^{-1}(2)}, a_{\sigma^{-1}(3)}, \ldots, a_{\sigma^{-1}(n)} \right)
\]

\[
+ \cdots + (-1)^{i-1}(a_0, \ldots, a_{\sigma^{-1}(i)}a_{\sigma^{-1}(i+1)}, \ldots, a_{\sigma^{-1}(n)})
\]

\[
+ \cdots + (-1)^n(a_{\sigma^{-1}(n)}a_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n-1)})
\]

(2.6)

On the other hand,

\[
\varphi_{n-1} \circ d(a_0, a_1, \ldots, a_n) =
\]

\[
\varphi_{n-1} \left( \sum_{j=1}^{n} (-1)^{j+1} \left( [a_0, a_j], a_1, \ldots, \hat{a}_j, \ldots, a_n \right) \right)
\]

\[
+ \varphi_{n-1} \left( \sum_{1 \leq i < j \leq n} (-1)^{j+1} (a_0, a_1, \ldots, [a_i, a_j], \ldots, \hat{a}_j, \ldots, a_n) \right).
\]

(2.7)
Note that the terms
\[ \sum_{\sigma \in S_n} (\text{sgn} \sigma) \left( (a_0 a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(n)}) + (-1)^n (a_{\sigma^{-1}(n)} a_0, a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n-1)}) \right) \] (2.8)
match the terms
\[ \phi_{n-1} \left( \sum_{j=1}^{n} (-1)^{j+1} ([a_0, a_j], a_1, \ldots, \hat{a}_j, \ldots, a_n) \right), \]
and the remaining terms in the sums for \( b \circ \phi_n \) and \( \phi_{n-1} \circ d \) also match. Thus,
\[ b \circ \phi_n = \phi_{n-1} \circ d. \]

\( \square \)

**Corollary 2.2.** There is a natural induced homomorphism
\[ \phi_* : HL_{s+1}(A) \rightarrow HH_s(A). \]

The chain map \( \phi \) belongs to the genre of constructions known as antisymmetrization maps [6, 1.3.4]. In fact, it follows from [6, 1.3.5] that \( \phi \) descends to the exterior powers on the domain
\[ \epsilon_n : A \otimes \Lambda^n(A) \rightarrow A^{\otimes(n+1)}, \]
and there is an induced map
\[ \epsilon_* : H_{s}^{\text{Lie}}(A; A) \rightarrow HH_s(A) \] (2.9)
where \( H_{s}^{\text{Lie}}(A; A) \) denotes Lie-algebra homology with coefficients in the adjoint representation:
\[ \text{ad}(a)(b) = [a, b], \quad a, \ b \in A. \]

For completeness, recall that \( H_{s}^{\text{Lie}}(A; A) \) is the homology of the complex
\[
A \leftarrow A \otimes A \leftarrow A \otimes A^{\wedge 2} \leftarrow \ldots \leftarrow A \otimes A^{\wedge (n-1)} \leftarrow d A \otimes A^{\wedge n} \leftarrow \ldots ;
\]
\[ d(a_0 \otimes a_1 \wedge a_2 \wedge \ldots a_n) = \]
\[
\sum_{1 \leq i < j \leq n} (-1)^{j+1} (a_0 \otimes a_1 \wedge \ldots a_{i-1} \wedge [a_i, a_j] \wedge a_{i+1} \wedge \ldots \hat{a}_j \ldots \wedge a_n) + \sum_{j=1}^{n} (-1)^{j+1} (\text{ad}(a_0)(a_j) \otimes a_1 \wedge a_2 \wedge \ldots \hat{a}_j \ldots \wedge a_n). \]
The projection $A^\otimes(n+1) \to A \otimes A^\wedge n$ induces a map on homology
\[ HL_{*+1}(A) \to H^\text{Lie}_*(A; A). \] (2.10)

**Lemma 2.3.** There is a commutative diagram
\[
\begin{array}{ccc}
HL_{*+1}(A) & \xrightarrow{\varphi_*} & H^\text{Lie}_*(A; A) \\
\downarrow & & \downarrow_{\epsilon_*} \\
H_{*}^\text{Lie}(A; A) & \xrightarrow{\theta_*} & HH_{*}(A)
\end{array}
\]

*Proof.* This follows from corollary (2.2) and maps (2.9) and (2.10) \(\square\)

When $k$ is a characteristic zero field, the cyclic homology of $A$ may be computed from the complex $C^\Lambda_*(A)$ [6, 2.1.4]:
\[
A \leftarrow A^\otimes 2/(1-t) \leftarrow \ldots A^\otimes n/(1-t) \leftarrow A^\otimes (n+1)/(1-t) \leftarrow \ldots,
\]
where $\mathbb{Z}/(n+1)$ acts on $A^\otimes (n+1)$ via
\[
t(a_0, a_1, \ldots, a_n) = (-1)^n(a_n, a_0, a_1, \ldots, a_{n-1}).
\]

There is a chain map [3, 10.2.3]
\[
\begin{align*}
\theta : & \Lambda^{*+1}_*(A) \to C^\Lambda_*(A) \\
\theta : & \Lambda^{n+1}_*(A) \to A^\otimes(n+1)/(1-t) \\
\theta(a_0 \wedge a_1 \wedge \ldots \wedge a_n) = & \sum_{\sigma \in S_n} (\text{sgn} \, \sigma)(a_0, a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)} \ldots, a_{\sigma^{-1}(n)}),
\end{align*}
\] (2.11)
which induces a homomorphism
\[
\theta_* : H^\text{Lie}_* (A) \to HC_*(A).
\]

Recall that there is a natural map $I : HH_*(A) \to HC_*(A)$ [3, 2.2.1], which in the characteristic zero case is induced by the projection
\[
\begin{align*}
CHH_*(A) & \to C^\Lambda_*(A) \\
A^\otimes(n+1) & \to A^\otimes(n+1)/(1-t).
\end{align*}
\] (2.12)
Lemma 2.4. When $k$ is a characteristic zero field, there is a commutative diagram

\[
\begin{array}{ccc}
HL_{*+1}(A) & \xrightarrow{\varphi_*} & HH_*(A) \\
\downarrow \pi & & \downarrow I \\
HL_{*+1}(A) & \xrightarrow{\theta_*} & HC_*(A).
\end{array}
\]

Proof. This follows immediately, since both $\varphi_*$ and $\theta_*$ are induced by antisymmetrization maps, and both $\pi$ and $I$ are induced by projections. \qed

Of course, when $* = 0$, every arrow in lemma (2.4) is an isomorphism.

The map $\varphi_* : HL_{*+1}(A) \to HH_*(A)$ has a nice interpretation when $A$ is a smooth algebra over a Noetherian ring $k$. By definition of smooth, $A$ is assumed to be commutative and unital over $k$ (See [6, 3.4.1] for a discussion of smooth algebras.) Let $\Omega^n_{A|k}$ be the $A$-module of differential forms, where

\[
\Omega^0_{A|k} = A, \quad \Omega^n_{A|k} = \Lambda^n_A(\Omega^1_{A|k}),
\]

and $\Omega^1_{A|k}$ form the Kähler differentials, generated as an $A$-module by the symbols $da, a \in A$, subject to the relations

\[
\begin{align*}
d(\lambda a + \mu b) &= \lambda da + \mu db, \quad \lambda, \mu \in k, \quad a, b \in A \\
d(ab) &= a(db) + b(da), \quad a, b \in A
\end{align*}
\] (2.13)

By a theorem of Hochschild, Kostant, and Rosenberg [3], when $A$ is smooth over $k$, the antisymmetrization

\[
\epsilon_* : \Omega^*_{A|k} \to HH_*(A)
\] (2.14)

is an isomorphism of graded algebras. Of course, for any commutative algebra $A$, $[a, b] = 0$ for all $a, b \in A$, and

\[
HL_*(A) \simeq T(A),
\] (2.15)

where $T(A) = \sum_{k \geq 0} A^{\otimes k}$ denotes the tensor algebra on $A$.

Lemma 2.5. If $A$ is smooth over $k$, then the natural map

\[
\varphi_* : HL_{*+1}(A) \to HH_*(A)
\]

is surjective.
Proof. There is a surjective $k$-module homomorphism

$$HL_{n+1}(A) \simeq A^{(n+1)} \to \Omega^n_{A|k}$$

given on homogeneous elements by

$$p(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0 \, da_1 \wedge da_2 \wedge \ldots \wedge da_n.$$ 

The result now follows from the commutativity of the diagram

$$\begin{array}{ccc}
HL_{*+1}(A) & \xrightarrow{p} & \Omega^n_{A|k} \\
\downarrow \phi_* & \swarrow & \downarrow e_* \\
\phi_* & \to & HH_*(A).
\end{array}$$

When $A$ is the smooth algebra $C^\infty(M)$ of real-valued differentiable functions on a differentiable manifold $M$, the reader is invited to identify the various homology groups in lemma (2.4) using the calculations of $HC_*(C^\infty(M))$ in terms of de Rham cohomology [6, 3.4.12].

In general, the map $\phi_* : HL_{*+1}(A) \to HH_*(A)$ is not surjective, even in the commutative case, since from the definition of $\phi$

$$\text{Im}[\phi_*(HL_{n+1}(A))] \subset HH_n^{(n)}(A),$$

where $HH_n^{(n)}(A)$ denotes the $n$-th summand in the $\lambda$-decomposition of Hochschild homology over $Q$. See [8, 4.5] for details about this decomposition. Of course, $HH_n^{(n)}(A)$ is not necessarily isomorphic to $HH_n(A)$. In the next section we investigate another important case in which $\phi_*$ is surjective.

3 Relation to Characteristic Classes

In this paragraph we prove that the natural map

$$\phi_* : HL_{*+1}(gl(A)) \to HH_*(gl(A)) \simeq HH_*(A)$$

is onto for a unital algebra over a characteristic zero field $k$. Furthermore, when $A$ is the group ring $k[G]$, there is a surjective homomorphism

$$HH_*(k[G]) \to H_*(k[G]) := H_*(BG; k),$$
and, in fact, $H_*(BG; k)$ is a direct summand of $HH_*(k[G])$. Thus, on cohomology

$$H^*(BG; k) \rightarrow HL^{*+1}(gl(A))$$

is injective, and $HL^{*+1}(gl(A))$ contains all characteristic classes in $H^*(BG; k)$.

Specifically, let $gl(A) = \lim\limits_{\rightarrow} gl_n(A)$ be the Lie algebra of infinite matrices over $A$ with finitely many nonzero entries. Note that the Lie algebra structure on $gl(A)$ is actually induced from the ring structure of $M(A) = \lim\limits_{\rightarrow} M_n(A)$, where $M_n(A) = gl_n(A)$ is the ring of all $n \times n$ matrices over $A$. Recall that \cite{2, 3, 10.6.5}

$$HL_*(gl(A)) \simeq T(HH_*(A)[1]),$$

where $T$ denotes the tensor algebra. See also \cite{8}. To understand the map $\varphi_*$ on homology, the isomorphism in equation (3.1) must be understood on the chain level. The chain complex $CL_*(gl(A))$ is quasi-isomorphic to $L_*(A)$ \cite{3, 10.6.7}:

$$k[S_1] \otimes_k A \leftarrow k[S_2] \otimes_k A^\otimes 2 \leftarrow \ldots \leftarrow k[S_n] \otimes_k A^\otimes n \leftarrow \ldots,$$

where $S_n$ denotes the symmetric group on $n$ letters. There is an explicit chain map

$$CL_*(gl(A)) \rightarrow L_*(A)$$

$$[gl(A)]^\otimes n \rightarrow ([gl(A)]^\otimes n)_{gl(k)} \overset{\Theta}{\rightarrow} k[S_n] \otimes A^\otimes n$$

with $[gl(A)]^\otimes n \rightarrow ([gl(A)]^\otimes n)_{gl(k)}$ being the projection onto the quotient by the adjoint action. For $\sigma \in S_n$ and $E_{ij}^a$ the elementary matrix with only one possible nonzero entry $a \in A$ in the $ij$ position,

$$\Theta(E_{1\sigma(1)}^a \otimes E_{2\sigma(2)}^a \otimes \ldots \otimes E_{n\sigma(n)}^a) = \sigma \otimes (a_1, a_2, \ldots, a_n)$$
For more details, see [8, 10.2.11].

Now, \( HH_*(A)[1] \) is the direct summand of \( HL_*(gl(A)) \) that arises from the homology of the complex \( P_*(A) \):

\[
k[U_1] \otimes A \leftarrow k[U_2] \otimes A^{\otimes 2} \leftarrow \ldots \leftarrow k[U_n] \otimes A^{\otimes n} \leftarrow \ldots,
\]

where \( U_n \) is the conjugacy class of the cyclic shift in \( S_n \). The complex \( P_*(A) \) is a summand of \( L_*(A) \) [8], and an alternative calculation of \( H_*(P_*(A)) \) appears in the appendix. Also needed for the identification of \( \text{Im}(\varphi_*) \) is an explicit description of the trace map isomorphism from Morita invariance

\[
\text{tr}_*: HH_*(gl(A)) \to HH_*(A).
\]

For \( pM = [p m_{ij}] \in gl(A) \),

\[
\text{tr}(0M \otimes 1M \otimes \ldots \otimes nM) = \sum 0m_{i1}i2 \otimes 1m_{i2}i3 \otimes 2m_{i3}i4 \otimes \ldots \otimes nm_{in+1}i1,
\]

where the sum is over all indices \( i_1, i_2, \ldots, i_{n+1} \).

**Lemma 3.1.** Let \( A \) be a unital algebra over a characteristic zero field \( k \). Then the composition

\[
\text{tr}_* \circ \varphi_*: HL_{*+1}(gl(A)) \to HH_*(A)
\]

is surjective.

**Proof.** The summand of \( HL_{*+1}(gl(A)) \) isomorphic to \( HH_*(A)[1] \) can be represented via chains which are \( k \)-linear combinations of terms:

\[
E^{a_1}_{1\tau(1)} \otimes E^{a_2}_{2\tau(2)} \otimes \ldots \otimes E^{a_n}_{n\tau(n)},
\]

where \( \tau \) is the cyclic shift given by the cycle \( (1, 2, 3, \ldots, n) \) [8]. It is enough to compute

\[
\text{tr} \circ \varphi(E^{a_1}_{1\tau(1)} \otimes E^{a_2}_{2\tau(2)} \otimes \ldots \otimes E^{a_n}_{n\tau(n)}) = a_1 \otimes a_2 \otimes \ldots \otimes a_n.
\]

Thus, any element in \( HH_*(A) \) can be represented as a chain in \( CL_{*+1}(gl(A)) \).
Consider the case of a group ring $A = k[G]$, and the associated bar construction $\{G^n\}_{n \geq 0}$. The face maps $d_i : G^n \to G^{n-1}$ are given by

$$d_i(g_1, g_2, \ldots, g_n) = \begin{cases} (g_2, g_3, \ldots, g_n) & \text{if } i = 0, \\ (g_1, g_2, g_{i+1}, \ldots, g_n) & \text{if } 1 \leq i \leq n - 1, \\ (g_1, g_2, \ldots, g_{n-1}) & \text{if } i = n, \end{cases}$$

with $H_\ast(BG)$ denoting the homology of the complex $B_\ast(G) = k[G^n] \simeq (k[G])^\otimes n$, $n \geq 0$.

There are natural simplicial maps

$$\pi : CHH_\ast(k[G]) \to B_\ast(G)$$
$$\pi : k[G^{n+1}] \to k[G^n]$$
$$\pi(g_0, g_1, g_2, \ldots, g_n) = (g_1, g_2, \ldots, g_n)$$
$$\iota : B_\ast(G) \to CHH_\ast(k[G])$$
$$\iota : k[G^n] \to k[G^{n+1}]$$
$$\iota(g_1, g_2, \ldots, g_n) = ((g_1 g_2 \cdots g_n)^{-1}, g_1, g_2, \ldots, g_n),$$

and the composition $\pi \circ \iota$ is the identity on $B_\ast(G)$. Thus follows the known lemma \[\text{[1]}\]

**Lemma 3.2.** \[\text{[1]}\] The graded group $H_\ast(BG)$ is a direct summand of $HH_\ast(k[G])$

for any coefficient ring $k$.

By composing the chain maps $\pi$, $\text{tr}$, and $\varphi$, we have

**Theorem 3.3.** Let $k$ be a characteristic zero field and $A = k[G]$. Then

$$(\pi \circ \text{tr} \circ \varphi)_* : HL_{\ast+1}(gl(A)) \to H_\ast(BG; k)$$

is surjective, and

$$(\pi \circ \text{tr} \circ \varphi)^* : H^\ast(BG; k) \to HL^{\ast+1}(gl(A))$$

in injective.

Thus, all characteristic classes over $k$ naturally inject into Leibniz cohomology.
4 Relation to K-theory

A fundamental theorem of T. Goodwillie relates relative algebraic K-theory, $K_*(f)$, to relative cyclic homology, $HC_*(f)$. In particular, if $f : R \to S$ is a homomorphism of simplicial rings such that the induced map $\pi_0(R) \to \pi_0(S)$ is surjective with nilpotent kernel, then

$$K_n(f) \otimes \mathbb{Q} \simeq HC_{n-1}(f) \otimes \mathbb{Q}. \quad (4.1)$$

Although for Leibniz homology we are working in the category of discrete rings, the following results may be extended to the simplicial case. For a homomorphism $f : A \to B$ of discrete rings, we define the relative Leibniz homology groups, $HL_*(f)$, we then consider the map on matrices $gl(f) : gl(A) \to gl(B)$, and show that over a characteristic zero field, the composition

$$(\text{tr} \circ \varphi)_* : HL_{*+1}(gl(f)) \to HH_*(f)$$

is onto. If furthermore, $f : A \to B$ is surjective with nilpotent kernel, then

$$I : HH_*(f) \to HC_*(f)$$

is also surjective, as well as the composition

$$I \circ (\text{tr} \circ \varphi)_* : HL_{*+1}(gl(f)) \to HC_*(f). \quad (4.2)$$

Above, all relative homology groups are taken with coefficients in a characteristic zero field.

Recall briefly the construction of relative homology in a general setting [11, p. 46–47]. Let $f : C_n \to C'_n$, $n \geq 0$, be a homomorphism between any two chain complexes $C_*$ and $C'_*$, and define the mapping cone of $f$ as

$$M_n(f) = C_{n-1} \oplus C'_n, \quad n \geq 1,$$

with boundary map

$$\partial(c, c') = (-\partial c, \partial c' + f c)$$

By definition, $H_*(f)$ are the homology groups of $M_*[-1] = M_{*+1}$, which fit into a long exact sequence

$$\ldots \to H_n(C_*) \xrightarrow{f_*} H_n(C'_*) \xrightarrow{\alpha_*} H_n(f) \xrightarrow{p_*} H_{n-1}(C_*) \to \ldots$$

$$\ldots \to H_1(f) \xrightarrow{p_*} H_0(C) \xrightarrow{f_*} H_0(C'_*) \xrightarrow{\alpha_*} H_0(f) \to 0, \quad (4.3)$$
where \( p : M_*(f) \to C_{* - 1} \) and \( \alpha : C'_* \to M_*(f) \) are given by
\[
p((c, c')) = c, \quad \alpha(c') = (0, c').
\]

If \( f : A \to B \) is a homomorphism of discrete rings, then the above construction yields \( HL_*(f) \) by considering the chain map \( f : CL_*(A) \to CL_*(B) \). Moreover, the relative homology group construction is functorial, and there are natural maps
\[
\varphi_* : HL_{*+1}(f) \to HH_*(f) \\
\text{tr}_* : HH_*(gl(f)) \to HH_*(f) \\
I : HH_*(f) \to HC_*(f).
\]

**Theorem 4.1.** Suppose that \( A \) and \( B \) are unital algebras over a characteristic zero field \( k \), and \( f : A \to B \) is an algebra homomorphism. For the map \( gl(f) : gl(A) \to gl(B) \), the composition
\[
(\text{tr} \circ \varphi)_* : HL_{*+1}(gl(f)) \to HH_*(gl(f)) \to HH_*(f)
\]
is surjective.

**Proof.** From equation (3.1)
\[
HL_*(gl(A)) \simeq T(HH_*(A)[1]) \\
HL_*(gl(B)) \simeq T(HH_*(B)[1])
\]
From equation (3.3) the summand of \( HL_*(gl(A)) \) isomorphic to \( HH_*(A)[1] \) can be computed from the complex \( P_*(A) \), and similarly for \( gl(B) \). Consider the \( k \)-linear homomorphism
\[
P(f) : P_*(A) \to P_*(B) \\
P(f) : k[U_n] \otimes A^\otimes n \to k[U_n] \otimes B^\otimes n \\
P(f)(\sigma \otimes (a_1, a_2, \ldots, a_n)) = \sigma \otimes (f(a_1), f(a_2), \ldots, f(a_n)).
\]
We have the following commutative diagram with exact rows:
\[
\begin{array}{c}
\longrightarrow H_{*+1}(P_*(A)) \longrightarrow H_{*+1}(P_*(B)) \longrightarrow H_{*+1}(P(f)) \longrightarrow \\
\downarrow \quad \downarrow \quad \downarrow \\
\longrightarrow HH_*(A) \longrightarrow HH_*(B) \longrightarrow HH_*(f) \longrightarrow
\end{array}
\]
where the dimensions in the top row are inherited from the chain complex for Leibniz homology. By the 5-lemma, \( H_*(P(f)) \simeq HH_*(f)[1] \). Let \( z \) denote an element in the mapping cone

\[
CL_*(gl(A)) \to CL_*(gl(B))
\]

de the form

\[
(E_{1}\tau(1) \otimes E_{2}\tau(2) \otimes \ldots \otimes E_{n}\tau(n), E_{1}^{b}\tau'(1) \otimes E_{2}^{b}\tau'(2) \otimes \ldots \otimes E_{n+1}^{b}\tau'(n+1)),
\]

where \( \tau \) is the cyclic shift in \( S_n \) and \( \tau' \) is the cyclic shift in \( S_{n+1} \). The chain map \( \Theta \) in equation \((3.4)\) may be defined on the respective mapping cones, and

\[
\Theta(z) = (\tau \otimes (a_1, a_2, \ldots, a_n), \tau' \otimes (b_1, b_2, \ldots, b_{n+1})),
\]

where the latter is in fact an element of \( M(P(f)) \). Also, at the level of mapping cones, we have

\[
(tr \circ \varphi)(z) = ((a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_{n+1})).
\]

The theorem follows, since the isomorphism

\[
H_*(P(f)) \simeq HH_*(f)[1]
\]

is realized by sending the class of \( \Theta(z) \) to the class of \( (tr \circ \varphi)(z) \).

For relative cyclic homology, there is a long exact sequence

\[
\ldots \to HH_*(f) \xrightarrow{I} HC_*(f) \xrightarrow{S} HC_{n-2}(f) \xrightarrow{B} HH_{n-1}(f) \xrightarrow{I} \ldots.
\]

By work of Goodwillie, we have

**Lemma 4.2.** If \( f : A \to B \) is a homomorphism of unital algebras over a characteristic zero field with nilpotent kernel, then the map

\[
I : HH_n(f) \to HC_n(f)
\]

is surjective.

**Proof.** From \([3, p. 399]\), the map \( S : HC_n(f) \to HC_{n-2}(f) \) is zero. \( \square \)

**Corollary 4.3.** Under the hypotheses of lemma \((4.2)\), the natural map

\[
I \circ (tr \circ \varphi)_* : HL_{n+1}(gl(f)) \to HC_*(f)
\]

is surjective.
5 Appendix: The Homology of $P_\ast(A)$

The results of this paper concerning the surjectivity of the homomorphism

$$HL_{s+1}(gl(A)) \to HH_s(A)$$

rely heavily on the calculation of the homology of the complex $P_\ast(A)$:

$$k[U_1] \otimes A \leftarrow k[U_2] \otimes A^\otimes 2 \leftarrow \ldots \leftarrow k[U_{n+1}] \otimes A^\otimes (n+1) \leftarrow \ldots ,$$

which form the primitive elements of $HL_\ast(gl(A))$. In this appendix we offer a calculation of $H_\ast(P_\ast(A))$ which fills a gap in a previous proof, namely lemma (2.6) of [8]. We prove that if $A$ is a unital $k$-algebra, then

$$H_\ast(P_\ast(A)) \simeq HH_\ast(A),$$

where now $P_n(A) = k[U_{n+1}] \otimes A^\otimes(n+1)$.

Recall that $U_n$ is the conjugacy class of the cyclic shift in the symmetric group $S_n$. The collection $\{U_{n+1}\}_{n \geq 0}$ form a presimplicial set with face maps $d_i$, but lack degeneracies. Let $N^{cy}_\ast(A)$ be the cyclic bar construction on $A$ with faces given in equation (2.3). Let

$$k[U_{s+1}] \otimes N^{cy}_\ast(A)$$

be the presimplicial $k$-module with

$$\{ k[U_{s+1}] \otimes N^{cy}_\ast(A) \}_n = k[U_{n+1}] \otimes A^\otimes(n+1),$$

and face maps

$$d_i(\sigma \otimes \bar{a}) = d_i(\sigma) \otimes d_i(\bar{a}).$$

The complex $P_\ast(A)$ is simply $k[U_{s+1}] \otimes N^{cy}_\ast(A)$ together with its boundary map constructed as the alternating sum of the face maps. In [8] it is shown that the complex $k[U_{s+1}]$ is acyclic, but without degeneracies, the Eilenberg-Zilber and Künneth theorems cannot be applied to calculate $H_\ast(P_\ast(A))$. To remedy this, we invoke the Dold-Kan functor [5], which to any chain complex $K_\ast$ (over $k$), associates the simplicial $k$-module

$$D_n(K_\ast) = \text{Hom}(C(\Delta[n]), K_\ast),$$

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where Hom denotes chain maps over \( k \). Also, \( \Delta[n] \) is the simplicial model for the \( n \)-simplex, \( C'(\Delta[n]) \) is the free \( k \)-module on the elements of \( \Delta[n] \), and

\[
C(\Delta[n]) = \frac{C''(\Delta[n])}{\epsilon C(\Delta[n])},
\]

with \( \epsilon C(\Delta[n]) \) denoting the submodule of \( C''(\Delta[n]) \) generated by the degenerate elements. See \([5]\) for further details.

Using properties of the Dold-Kan functor, the complexes

\[
k[U_{*+1}] \otimes N^{cy}(A), \quad D_*(k[U_{*+1}]) \otimes N^{cy}(A)
\]

are quasi-isomorphic, as well as the complexes

\[
k[U_{*+1}] \otimes N^{cy}(A), \quad D_*(k[U_{*+1}] \otimes N^{cy}(A)).
\]

Define a chain map

\[
\psi : D_*(k[U_{*+1}]) \otimes N^{cy}(A) \to D_*(k[U_{*+1}] \otimes N^{cy}(A))
\]

\[
\psi : \text{Hom}(C(\Delta[n]), k[U_{*+1}]) \otimes A^{n+1} \to \text{Hom}(C(\Delta[n]), k[U_{*+1}] \otimes A^{n+1})
\]

as follows. Recall that \( \delta^i : [n-1] \to [n], i = 0, 1, 2, \ldots, n \) is given by

\[
\delta^i(j) = \begin{cases} 
  j, & j < i \\
  j+1, & j \geq i.
\end{cases}
\]

Let \( \alpha \in (\Delta[n])_q \), \( \alpha : [q] \to [n], q \leq n \). Then

\[
\alpha = \delta^{n-1} \circ \ldots \circ \delta^2 \circ \delta^1
\]

\[
[q] \xrightarrow{\delta^1} [q+1] \xrightarrow{\delta^2} [q+2] \xrightarrow{\delta^3} \ldots \xrightarrow{\delta^{n-q}} [n-1] \xrightarrow{\delta^{n-q}} [n],
\]

and the factorization of \( \alpha \) is well-defined up to cosimplicial identities. Suppose that \( \vec{a} \in A^{n+1} \) and

\[
g : C(\Delta[n]) \to k[U_{*+1}]
\]

is a chain map. We define

\[
\psi(g \otimes \vec{a})(\alpha) = g(\alpha) \otimes (d_{i_1} \circ d_{i_2} \circ \ldots \circ d_{i_{n-q}}(\vec{a})).
\]

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Now, \( H_*(D_*(k[u_{*+1}]) \otimes N^{cy}(A)) \simeq HH_*(A) \), since \( D_*(k[U_{*+1}]) \) is an acyclic simplicial \( k \)-module. Also,

\[
H_*(D_*(k[u_{*+1}] \otimes N^{cy}(A))) \simeq H_*(k[U_{*+1}] \otimes N^{cy}(A)).
\]

Define an inclusion of chain complexes

\[
N^{cy}(A) \rightarrow k[U_{*+1}] \otimes N^{cy}(A)
\]

\[
A^{\otimes(n+1)} \rightarrow k[U_{n+1}] \otimes A^{\otimes(n+1)}
\]

\[
\vec{a} \mapsto \tau_{n+1} \otimes \vec{a},
\]

where \( \tau_{n+1} \) is the cyclic shift in \( U_{n+1} \). We then have a commutative diagram of chain complexes, where the diagonal and vertical arrows are inclusions:

\[
\begin{array}{ccc}
N^{cy}(A) & \rightarrow & k[U_{*+1}] \otimes N^{cy}(A) \\
\downarrow & & \downarrow \\
D_*(k[U_{*+1}]) \otimes N^{cy}(A) & \rightarrow & D_*(k[U_{*+1}] \otimes N^{cy}(A)).
\end{array}
\]

On homology

\[
\begin{array}{ccc}
HH_*(A) & \rightarrow & H_*(k[U_{*+1}] \otimes N^{cy}(A)) \\
\downarrow & & \downarrow \\
H_*(D_*(k[U_{*+1}]) \otimes N^{cy}(A)) & \rightarrow & H_*(D_*(k[U_{*+1}] \otimes N^{cy}(A))).
\end{array}
\]

It follows that

\[
HH_*(A) \rightarrow H_*(k[U_{*+1}] \otimes N^{cy}(A))
\]

is an isomorphism, and \( HH_*(A) \simeq H_*(P_*(A)) \). In fact, every arrow in diagram (5.1) is an isomorphism.

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