SU($N$) ANTIFERROMAGNETS AND THE PHASE STRUCTURE OF QED IN THE STRONG COUPLING LIMIT

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Abstract

We examine the strong coupling limit of both compact and non-compact quantum electrodynamics (QED) on a lattice with staggered Fermions. We show that every SU($N_L$) quantum antiferromagnet with spins in a particular fundamental representation of the SU($N_L$) Lie algebra and with nearest neighbor couplings on a bipartite lattice is exactly equivalent to the infinite coupling limit of lattice QED with the number of flavors of electrons related to $N_L$ and the dimension of spacetime, $D + 1$. There are $N_L$ 2-component Fermions in $D = 1$, $2N_L$ 2-component Fermions in $D = 2$ and $2N_L$ 4-component Fermions in $D = 3$. We find that, for both compact and non-compact QED, when $N_L$ is odd the ground state of the strong coupling limit breaks chiral symmetry in any dimensions and for any $N_L$ and the condensate is an isoscalar mass operator. When $N_L$ is even, chiral symmetry is broken if $D \geq 2$ and if $N_L$ is small enough and the order parameter is an isovector mass operator. We also find the exact ground state of the lattice Coulomb gas as well as a variety of related lattice statistical systems with long-ranged interactions.
1 Introduction

It is reasonably well established that, as the bare coupling constant of massless quantum electrodynamics (QED) is increased, there is a phase transition which breaks chiral symmetry and generates an electron mass. The mechanism is similar to that in the Nambu-Jona-Lasinio model [1] where chiral symmetry breaking occurs when the four–Fermion interaction is sufficiently strong and attractive that a bound state and the resulting symmetry breaking condensate forms. In the case of QED it is the Coulomb attraction of electrons and positrons which, as the electric charge is increased, gets strong enough to form a condensate. In QED, this phase transition has been seen in both three and four spacetime dimensions using numerical as well as approximate analytic techniques. In four dimensions, numerical simulations of compact [2] and noncompact [3] lattice QED with dynamical Fermions indicate presence of a phase transition as the bare electric coupling is increased. For compact QED the transition is first order and for non–compact QED it appears to be of second order.

For non-compact QED this phase transition is also found in the continuum using approximate analytical techniques such as the solution of Schwinger-Dyson equations in the quenched ladder approximation [4, 5, 6, 7, 8, 9]. The critical behavior has the additional interesting feature that certain perturbatively non-renormalizable operators such as four–Fermion operators can become relevant there [8, 9].

In three spacetime dimensions, a similar behavior is found in the large $N$ expansion [10, 11] where the inverse of the number of Fermion flavors, $1/N$, plays the role of coupling constant. Both analytical techniques [10, 11] and numerical simulations [12] find a critical value of $N$, above which QED is chirally symmetric and below which the chiral symmetry is broken.

Furthermore, there are some recent analytic proofs that the strong coupling limit of QED breaks chiral symmetry. Salmhofer and Seiler [13] showed using a Euclidean spacetime approach and staggered Fermions that in four dimensions, four-flavor QED (as well as some other $U(N_C)$ gauge theories with $N_C \leq 4$) has a chiral symmetry breaking ground state when the electric charge is infinite. Subsequently, using staggered Fermions in the Hamiltonian approach it was shown [14, 15, 16] that the strong coupling limit of QED has a chiral symmetry breaking ground state in any spacetime dimension greater than two and when there are a specific number of Fermion flavors (two four-
component in 2+1 and four four-component Fermions in 3+1 dimensions, also for other \(U(N_C)\) gauge theories with any \(N_C\). This is an elaboration of previous arguments for chiral symmetry breaking \([18]\) using strong coupling Hamiltonian methods (for a review, see \([19]\)). Also, it is partially motivated by an interesting previous work of Smit \([20]\) where he uses naive and Wilson Fermions (and also finds a particular kind of mapping to an \(SU(N)\) antiferromagent) to analyze chiral symmetry breaking in QED.

If the chiral symmetry breaking phase transition in four dimensions is of second order, as it seems to be in non–compact QED \([3]\), it provides a nonperturbative zero of the beta-function for the renormalized coupling constant. Furthermore, if the critical behavior there differs from mean field theory, the resulting ultraviolet fixed point would allow QED, in the limit of infinite cutoff, to avoid the Landau pole, or Moscow zero \([21, 22, 23]\) which otherwise renders it trivial. It is an interesting and nontrivial question whether, via this mechanism, QED could be an example of a nontrivial field theory which exists in four dimensions.

The existence of the strong-coupling phase transition has long been advocated by Miransky \([5, 6]\). He visualized the mechanism for chiral symmetry breaking as a “collapse” of the electron-positron wave–function, similar to the behavior of the supercritical hydrogen atom with bare proton charge \(Z > 137\). He argued that there are two ways to screen supercritical charges. In the case of a supercritical nucleus, the high electric field produces electron-positron pairs, ejects the positrons and absorbs the electrons to screen its charge. In the case of supercritical electron and positron charge, he argued that the pair production is suppressed by Fermion mass, so a system can stabilize itself by increasing the electron and positron masses; thus the tendency to break chiral symmetry. Miransky studied continuum QED in the quenched approximation, using the ladder Schwinger-Dyson equations to sum the planar photon-exchange graphs. In this approximation, there is a line of critical points, beginning at bare coupling \(\epsilon^2 = 0\) and ending at \(\epsilon_c^2 = \pi/3\) where the theory breaks chiral symmetry dynamically and has an interesting continuum limit. A behavior very similar to this was found in lattice simulations using quenched Fermions \([24, 25]\). At least some of its qualitative features are expected to survive the presence of Fermion loops in realistic QED.

In this Paper we shall examine Miransky’s collapse phenomenon in a more physical context by studying a lattice version of QED which is similar to a condensed matter system. We do this by arguing that the lattice ap-
approximation to QED with staggered Fermions in any number of dimensions, \( D > 1 \), resembles a condensed matter system of lattice electrons in an external magnetic field and with a half-filled band (this argument was also given in \([14, 15, 16]\)). In that picture, the breaking of chiral symmetry and the analog of the collapse phenomenon is the formation of either charge or isospin density waves and the resulting reduction of the lattice translation symmetry from translations by one site to translations by two sites. It occurs when the exchange interaction of the electrons, which is attractive, dominates the tendency of the direct Coulomb interaction and the kinetic energy to delocalize charge, giving an instability to the formation of commensurate charge density waves. This forms a gap in the Fermion spectrum and a particular mass operator obtains a vacuum expectation value. This gives an intuitive picture of how strong attractive interactions in a field theory can form a coherent structure. Here, the commensurate density waves in the condensed matter system correspond to a modulation of the vacuum charge or isospin density at the ultraviolet cutoff wavelength in the field theory.

We are interested in massless quantum electrodynamics with action

\[
S = \int d^{D+1}x \left( -\frac{\Lambda^{3-D}}{4e^2} F_{\mu\nu} F^{\mu\nu} + \sum_{a=1}^{N_F} \bar{\psi}^a \gamma^\mu (i\partial_\mu + A_\mu) \psi^a \right)
\]  

(1)

where \( \Lambda \) is the ultraviolet cutoff, \( e \) is the dimensionless electric charge and there are \( N_F \) flavors of \( 2^{[(D+1)/2]} \)-dimensional Dirac spinors. (Here, \( [(D+1)/2] \) is the largest integer less than or equal to \( (D+1)/2 \).) In even dimensions (when \( D+1 \) is even), the flavor symmetry is \( SU_R(N_F) \times SU_L(N_F) \). In odd dimensions there is no chirality and the flavor symmetry is \( SU(N_F) \). What is usually referred to as chiral symmetry there is a subgroup of the flavor group. We shall use a lattice regularization of (1) and study the limit as \( e^2 \to \infty \). We shall use \( N_L \) flavors of staggered fermions. In the naive continuum limit, this gives \( N_F = N_L \) 2-component fermions in \( \text{D}=1 \), \( N_F = 2N_L \) 2-component fermions in \( \text{D}=2 \) and \( N_F = 2N_L \) 4-component fermions in \( \text{D}=3 \). Though the lattice theory reduces to (1) in the naive continuum limit, the lattice regularization breaks part of the flavor symmetry. All of the operators which are not symmetric are irrelevant and vanish as the lattice spacing is taken to zero. In \( \text{D}=3 \), the \( SU_R(N_F) \times SU_L(N_F) \) symmetry is reduced to \( SU(N_F/2) \) (in \( \text{D}=3 \), \( N_L = N_F/2 \)) and translation by one site in the 3 lattice directions. These discrete transformations correspond to
discrete chiral transformations in the continuum theory. In D=2, the lattice
has $SU(N_F/2)$ (in D=2, $N_L = N_F/2$) symmetry and two discrete (“chiral”) symmetries. In D=1, the $SU_L(N_F) \times SU_R(N_F)$ symmetry is reduced to $SU(N_F)$ (in D=1, $N_L = N_F$) and a discrete chiral transformation. In each case, the discrete chiral symmetry is enough to forbid Fermion mass and it is the spontaneous breaking of this symmetry which we shall examine and which we shall call “chiral symmetry breaking”. We shall also consider the possibility of spontaneous breaking of the $SU(N_F)$ flavor symmetry. In the continuum the $SU(N_F)$ corresponds to a vector–like symmetry subgroup of the full flavor group. Since the lattice and continuum theories have different symmetries, the spectrum of Goldstone bosons, etc. would be different in the two cases. In the following, we shall not address the source of these differences but will define QED by its lattice regularization and discuss the realization of the symmetries of that theory only.

In this Paper we shall prove that, in the strong coupling limit, lattice QED with $N_L$ lattice flavors of staggered Fermions is **exactly equivalent** to an $SU(N_L)$ quantum antiferromagnet where the spins are in a particular fundamental representation of the $SU(N_L)$ Lie algebra. Furthermore, mass operators of QED correspond to staggered charge and isospin density operators in the antiferromagnet. Thus, the formation of charge-density waves corresponds to chiral symmetry breaking and the dynamical generation of an iso–scalar Fermion mass whereas Néel order of the antiferromagnet corresponds to dynamical generation of Fermion mass with an iso–vector condensate. As a result of this correspondence we can use some of the properties of the quantum antiferromagnets to deduce features of strongly coupled QED.

We find that, in the infinite coupling limit, the properties of the electronic ground state of compact and non-compact QED are identical. The fact that compact QED confines and non-compact QED (at least in high enough dimensions) does not confine affects only the properties of the gauge field wavefunctions and the elementary excitations.

We find that the case of even $N_L$ and odd $N_L$ are very different:

When $N_L$ is odd, the vacuum energy in the strong coupling limit is proportional to $e^2$, the square of the electromagnetic coupling constant. We also find that chiral symmetry is broken in the strong coupling limit for all odd $N_L \geq 1$ and for all spacetime dimensions $D + 1 \geq 2$. The mass operator
which obtains a nonzero expectation value is a Lorentz and flavor Lie algebra scalar. There are also mass operators which are Lorentz scalars and which transform nontrivially under the flavor group which could get an expectation value and break the flavor symmetry spontaneously if $N_L$ is small enough. We expect (but do not prove) that when $N_L$ increases to some critical value there is a phase transition to a disordered phase.

In contrast, when $N_L$ is even we find that the vacuum energy in the strong coupling limit is of order 1. We find that chiral symmetry is broken when the spacetime dimension $D + 1 \geq 3$ and when $N_L$ is small enough. The mass operator which gets an expectation value is a Lorentz scalar and transforms nontrivially under the flavor $SU(N_L)$. Thus, flavor symmetry is spontaneously broken. As in the case of odd $N_L$, we expect that there is an upper critical $N_L$ where there is a transition to a disordered phase.

In the course of our analysis, we find the exact ground state of the generalized classical Coulomb gas model in $D$–dimensions with Hamiltonian

$$H_{coul} = \frac{e^2}{2} \sum_{x,y} \rho(x) g(x - y) \rho(y)$$

(2)

where the variable $\rho(x)$ lives at the sites of a square lattice with spacing one (with coordinates $(x_1, \ldots, x_D)$ and $x_i$ are integers) and takes on values

$$-\frac{N_L}{2}, -\frac{N_L}{2} + 1, \ldots, \frac{N_L}{2} + 1, \frac{N_L}{2}, N_L \text{ an odd integer}$$

(3)

and the interaction is long-ranged

$$g(x - y) \sim |x - y|^{2-D}, \text{ as } |x - y| \to \infty$$

(4)

In any space dimension, $D \geq 1$ we prove that there are two degenerate ground states which have the Wigner lattice configurations

$$\rho(x) = \pm \frac{1}{2} (-1)^{\sum_{i=1}^{D} x_i}$$

(5)

When $N_L = 1$ this is a long-ranged Ising model which is known to have antiferromagnetic order, even in one dimension. It is also in agreement with a known result about the Ising model in two dimensions [26].

$^4$1+1 dimensions is a special case which we will discuss later (Section 4).
The difference between even and odd $N_L$ can be seen to arise from a certain frustration encountered when, with odd $N_L$, one simultaneously imposes the conditions on lattice QED which should lead to charge conjugation invariance, translation invariance and Lorentz invariance of the continuum limit. This frustration is absent when $N_L$ is even. This is not an anomaly in the conventional sense of the axial anomaly or a discrete anomaly encountered in the quantization of gauge theories, as no symmetries are incompatible, but it is nevertheless an interesting analog of the anomaly phenomenon.

Note that in 3 spacetime dimensions the difference between even and odd $N_L$ is the difference between an even and odd number of 4-component Fermions. Our result that for an odd number of 4-component Fermions the chiral symmetry is always broken for large coupling seems to contradict the continuum analysis in [10, 11]. We do not fully understand the reason for this, but speculate that it is related to the lattice regularization. Their model is very similar to ours in that they effectively work in the strong coupling limit when they replace the ultraviolet regularization which comes from having a Maxwell term in the QED action by a large momentum cutoff. In our case we have a lattice cut-off and the result should be very similar. Note that we agree with them when $N_L$ is even and there is an even number of 4-component Fermions. In that case, we are also in qualitative agreement with recent numerical work [12] which, since it uses Euclidean staggered Fermions and there is a further Fermion doubling due to discretization of time, can only consider the case where there is an even number of 4-component Fermions. The anomalous behavior that we find with odd numbers of 4-component Fermions exposes a difficulty with treating the flavor number $N$ as a continuous parameter.

A hint as to why 2+1-dimensional Fermions should come in four-component units appears when we formulate compact QED in the continuum using the SO(3) Georgi-Glashow model with spontaneous breaking of the global symmetry, $SO(3) \to U(1)$ and the limit of large Higgs mass [27]. We begin with the model

$$\mathcal{L} = \frac{1}{4e^2} \sum_{a=1}^{3} (F_{a\mu})^2 + \frac{1}{2} \sum_{a=1}^{3} (D\phi)^a \cdot (D\phi)^a + \frac{\lambda}{4} (\sum_{a=1}^{3} \phi^a \phi^a - v^2)^2$$  \hspace{1cm} (6)

The spectrum contains a massless photon which in the $\lambda \to \infty$ and $v^2 \to \infty$ is the only light excitation. Since the U(1) gauge group is a subgroup of
SO(3), it is compact. We wish to make 2+1-dimensional electrodynamics by coupling this model to Fermions. It is known that, if the resulting theory is to preserve parity and gauge invariance simultaneously, we must use an even number of two-component Fermions which, in the minimal case are also SU(2) doublets [28]

$$L_F = \bar{\psi} \left( i \gamma \cdot \nabla + \gamma \cdot A + \gamma^5 (m + g \phi \cdot \sigma) \right) \psi$$

(7)

Here, the Fermions are four-component and have a parity invariant mass and Higgs coupling. Each SU(2) doublet contains two electrons (which can be defined so that both components of the doublet have the same sign of electric charge), thus the basic Fermion for compact QED has eight components. The maximal chiral symmetry is obtained in the massless case with vanishing Higgs coupling. It is possible, by suitable choice of mass and Higgs coupling i.e. $m = \pm g \langle \phi \rangle$, (and also reduction of the chiral symmetry) to make four of the Fermions heavy, leaving four massless components. This is consistent with parity and gauge invariance. It is also interesting to note that staggered Fermions on a Euclidean lattice, where time is also discretized, produce eight component continuum Fermions.

In the lattice gauge theory, the coupling of gauge fields to the electron field does not distinguish between compact and non-compact QED. Therefore, the above considerations for the Fermion content should apply to both cases.

Similarly, in four dimensions it is known that the Fermion multiplicity obtained by lattice regularization is the one which is compatible with the axial anomaly [29, 30]. An alternative way to see this constraint is to again form compact QED using the Georgi-Glashow model. Then, because of Witten’s SU(2) anomaly [31], we are required to use at least two (and generally an even number of) 2-component Weyl Fermions which are at least SU(2) doublets. This seems to indicate that the minimal Fermion would have eight components. This is the minimal number that we find using staggered Fermions in the Hamiltonian approach (see Appendix B). It is interesting that in Euclidean staggered Fermions produce sixteen components, and therefore in our terminology always lead to the case with even $N_L$.

A different context where a correspondence between spin systems and gauge theory appears is in the study of strongly correlated electron systems in condensed matter physics. An important issue there is the correspondence of spin systems such as the quantum Heisenberg antiferromagnet (which is
equivalent to the strong coupling limit of the Hubbard model at 1/2 filling) and certain limits of various kinds of lattice gauge theories. Gauge theory-like states can be obtained as an approximate low energy theory in the mean field approximation. An example is the “flux phases” suggested in \[32, 33, 34\] and also various dimerized phases \[35\] which are disordered and under some conditions compete with the Néel phase, particularly in 2+1 dimensions (for a review, see \[36\]). Affleck and Marston \[34\] showed how to get the flux phase from mean field theory. It is a locally stable but not global minimum of the free energy of an antiferromagnetic spin system and could presumably be stabilized by adding certain operators to the Hamiltonian. The low-energy limit in this phase resembles strongly coupled 2+1-dimensional lattice QED with four species of massless 2-component electrons (because of the two spin states of the lattice electrons, this is the case \(N_L = 2\) in 2+1-dimensions).

The picture that we shall advocate in this paper is that there should be a phase transition between the flux phase and the Néel ordered phase of the antiferromagnet. This phase transition is governed by the strength of the effective electromagnetic coupling constant. The Heisenberg antiferromagnet, which is known to have a Néel ordered ground state, is obtained in the limit of infinite electric coupling constant. For weaker couplings, the system can be in the flux phase where the electrons are massless. As the coupling is increased, the Néel state is recovered by formation of a commensurate spin density wave\[5\] which corresponds to spontaneous chiral symmetry breaking and the generation of iso-vector Fermion mass in the effective QED. (Spin in the condensed matter system corresponds to iso-spin in effective QED.)

In Section II we discuss the formulation of QED on a lattice. We give a detailed discussion of discrete symmetries and also of gauge fixing which is necessary to make non-compact QED well-defined. In Section III we discuss the strong coupling limit and show how quantum antiferromagnets are obtained in the strong coupling limit for both compact and non-compact QED. We discuss the properties of the electron ground state for both cases of \(N_L\) even and odd. We also discuss the implications of the mapping between antiferromagnets and strong coupling gauge theories for the symmetries of the ground state. Section IV is devoted to concluding remarks.

We review some of our notation in Appendix A and the essential features

\[5\] This is a commensurate charge density wave for each spin state whose phase is such that the condensate has zero electric charge.
of the staggered Fermion formalism, with emphasis on those aspects which are important for our arguments, in Appendix B. Appendix C is devoted to a review of the Fermion formulation of spin systems.
2 QED on a Lattice

In this Section we shall set the Hamiltonian formalism of Abelian gauge fields on a lattice. For the most part, this formalism can be found in some of the classic reviews of lattice gauge theory, [19] for example. A novel feature of the present Section is a careful treatment of the normal ordering of the charge operator and a discussion of the ensuing discrete symmetries. This normal ordering turns out to be important if the continuum limit is to have the correct behavior under $C$, $P$ and $T$ transformations. It will also be important in our later solution of the strong coupling limit.

2.1 Hamiltonian and Gauge Constraint

We shall discretize space as a cubic lattice and, in order to use the Hamiltonian formalism, time is left continuous. We use units in which the lattice spacing, the speed of light and Planck’s constant are all equal to one. (See Appendix A for a summary of our notation and Appendix B for a review of staggered Fermions.) Lattice gauge fields are introduced through the link operators

$$U_i(x) \equiv e^{i A_i(x)}$$

(8)

which correspond to the link $[x, i]$ of the lattice. Electric fields propagate on links of the lattice and the electric field operator $E_i(x)$ associated with the link $[x, i]$ is the canonical conjugate of the gauge field

$$[A_i(x), E_j(y)] = i \delta_{ij} \delta(x - y)$$

(9)

The gauge field and electric field operators obey the relations

$$A_{-i}(x) = -A_i(x - \hat{i}) , \quad E_{-i}(x) = -E_i(x - \hat{i})$$

(10)

The Hamiltonian (of non-compact QED) is

$$H_{NC} = \sum_{[x,i]} \frac{e^2}{2} E_i^2(x) + \sum_{[x,i,j]} \frac{1}{2e^2} B^2[x, i, j]$$

$$+ \sum_{[x,i]} \left( \bar{\psi}^{\alpha\dagger}(x + i) e^{i A_i(x)} \psi^\alpha(x) + \text{h.c.} \right)$$

(11)
where the second term contains a sum over plaquettes and the magnetic field is defined as the curvature of the gauge field at the plaquette $[x, i, j],$

$$B[x, i, j] = A_i(x) + A_j(x + \hat{i}) + A_{-i}(x + \hat{i} + \hat{j}) + A_{-j}(x + \hat{j}) = A_i(x) - A_j(x) + A_j(x + \hat{i}) - A_i(x + \hat{j}) \quad (12)$$

As is discussed in Appendix B, the hopping parameter $t_{[x,i]}$ contains phases which produce a background magnetic flux $\pi$ per plaquette. In the weak coupling continuum limit, the magnitude of $t_{[x,i]}$ is one, $|t|^2 \equiv |t_{[x,i]}|^2 = 1$ in order that the speed of the free photon and free electron fields are equal, i.e. so that the low frequency dispersion relations for both the photon and electron have the same speed of light. However, in order to obtain a relativistic continuum limit in general it is necessary to make $|t|$ a function of $e^2$. We shall find that in the limit where $e^2$ is large, the speed of light is proportional to $|t|/e$ and it is necessary that $|t| \sim e$ as $e^2 \rightarrow \infty$.

The Hamiltonian we have written in (11) is appropriate to non-compact QED. If we wish to study compact QED we must make the Hamiltonian symmetric under the field translation

$$A_i(x) \rightarrow A_i(x) + 2\pi \quad (13)$$

for any $[x, i]$. This is accomplished by replacing the second term in (11) by a periodic function of the magnetic flux, so that (for compact QED)

$$H_C = \sum_{[x,i]} \frac{e^2}{2} E_i(x)^2 + \sum_{[x,i,j]} \frac{2}{e^2} \sin^2 \left( B[x, i, j]/2 \right)$$

$$+ \sum_{[x,i]} \left( t_{[x,i]} \psi^a \dagger (x + i)e^{iA_i(x)} \psi^a (x) + h.c. \right) \quad (14)$$

Both (11) and (14) reduce to the standard Hamiltonian of QED in the naive weak coupling continuum limit. Away from that limit the behavior of the dynamical systems described by the two Hamiltonians can differ significantly. For example, in the strong coupling limit compact QED is a confining theory whereas non-compact QED is not confining. Also, the phase transition seen in numerical simulation of the two theories differs. In the compact case the phase transition associated with chiral symmetry breaking is generally of first order whereas it is second order for the non-compact case. The source of some of these differences generally have to do with the
symmetry (13). In (11) and (14) we have introduced $N_L$ flavors of lattice Fermions labelled by the index $a = 1, \ldots, N_L$.

In both compact and non-compact QED, the Hamiltonian is supplemented with the constraint of gauge invariance. The gauge transformations of the dynamical variables,

$$
\Lambda : \quad A_i(x) \to A_i(x) + \nabla_i \Lambda(x)
$$
$$
\Lambda : \quad E_i(x) \to E_i(x)
$$
$$
\Lambda : \quad \psi^a(x) \to e^{i\Lambda(x)} \psi^a(x)
$$
$$
\Lambda : \quad \psi^{\alpha\dagger}(x) \to \psi^{\alpha\dagger}(x)e^{-i\Lambda(x)}
$$

are generated by the operator

$$
G_{\Lambda} \equiv \sum_x \left( -\nabla_i \Lambda(x) E_i(x) + \Lambda(x) \left( \psi^{\alpha\dagger}(x) \psi^a(x) - N_L/2 \right) \right)
$$

The local generator of gauge transformations where $\Lambda$ has compact support is

$$
\frac{\partial G_{\Lambda}}{\partial \Lambda(x)} \equiv G(x) = \nabla \cdot E(x) + \psi^{\alpha\dagger}(x) \psi^a(x) - N_L/2
$$

Both (16) and (17) commute with the Hamiltonians in (11) and (14).

In (16) and (17) we have subtracted the constant $N_L/2$ from the charge density operator in order to make the gauge generator odd under the usual charge conjugation transformation

$$
\xi : \quad A_i(x) \to -A_i(x)
$$
$$
\xi : \quad E_i(x) \to -E_i(x)
$$
$$
\xi : \quad \psi^a(x) \to (-1)^\sum_{k=1}^D x_k \psi^{\alpha\dagger}(x)
$$
$$
\xi : \quad \psi^{\alpha\dagger}(x) \to (-1)^\sum_{k=1}^D x_k \psi^a(x)
$$

In fact, the Fermionic charge term in (16) can be put in the manifestly odd form $\frac{1}{2}[\psi^{\alpha\dagger}(x), \psi^a(x)]$. Of course, charge conjugation symmetry of the lattice theory is necessary to obtain charge conjugation of the continuum theory. We shall see later that, particularly at strong coupling, the subtraction term in (16) plays an important role. It seems to have been ignored in previous literature (for example, see [18, 19]). Its presence is particularly important
when $N_L$ is odd since the charge operator has no zero eigenvalues in that case (the eigenvalues of $\psi^\dagger \psi$ are integers).

Chiral symmetry is related to translation invariance by one site. The Hamiltonians (11) and (14) are invariant under the transformations

\[
\chi_j : \quad A_i(x) \rightarrow A_i(x + \hat{j}) \\
\chi_j : \quad E_i(x) \rightarrow E_i(x + \hat{j}) \\
\chi_j : \quad \psi^a(x) \rightarrow (-1)^{\sum_{k=j+1}^{D} x_k} \psi^a(x + \hat{j}) \\
\chi_j : \quad \psi^{a\dagger}(x) \rightarrow (-1)^{\sum_{k=j+1}^{D} x_k} \psi^{a\dagger}(x + \hat{j})
\]

for $j = 1, \cdots, D$.

In the following we shall use the charge conjugation symmetry which is a combination of these two transformations:

\[
C \equiv \xi \chi_1
\]

This is necessary if the mass operators which we define in Appendix B is to be invariant under charge conjugation symmetry. Also, we shall see that the strong coupling ground state is invariant under $C$ but not under either $\xi$ or $\chi_j$ by themselves.

The dynamical problem of Hamiltonian lattice gauge theory is to find the eigenstates of the Hamiltonian operator (11) or (14) and out of those eigenstates to find the ones which are gauge invariant, i.e. which obey the physical state condition (or, the “Gauss’ law” constraint)

\[
G(x)|\Psi_{\text{phys.}}\rangle = 0
\]

Note that the gauge constraint and physical state condition are the same for both compact and non-compact QED. In the case of compact QED there is the additional symmetry (13) which, being a large gauge symmetry, can be represented projectively. In fact, when we later work in the Schrödinger picture we shall require that the quantum states which are functions of a configuration of the gauge field transform as

\[
|A_i(x) + 2\pi n\rangle = \exp (in\theta[x, i]) |A_i(x)\rangle
\]

There is a separate parameter $\theta[x, i]$ for each link of the lattice. These parameters originate in a way similar to the theta–angle in ordinary QCD. The
symmetry (22) together with the commutator (9) imply that the spectrum of the electric field operator has eigenvalues which are separated by integers and offset by $\theta$:

\[ \text{spectrum}[E_i(x)] = \{\theta[x, i] + \text{integers} \} \quad (23) \]

The Hamiltonian and gauge constraints can be obtained from the gauge invariant Lagrangian

\[
L = \sum_x \psi^\dagger(x) \left( i \partial_t - A_0(x) \right) \psi(x) + \sum_{[x, i]} E_i(x) \dot{A}_i(x) \\
+ \sum_{[x, i]} E_i(x) \nabla_i A_0(x) + \sum_x A_0(x) N_L/2 - H \quad (24)
\]

where the temporal component of the gauge field has been introduced to enforce the gauge constraint and the time derivative terms give the correct symplectic structure. Note that, in order to get Lorentz invariance of the Fermion spectrum in the weak coupling (naive) continuum limit, we require half-filling of the Fermionic states, i.e. that the total charge defined by

\[
\sum_x \left( \psi^\dagger(x) \psi(x) - N_L/2 \right)
\]

has zero vacuum expectation value.

Here we have considered massless QED. As well as the gauge invariance and charge conjugation invariance discussed above, the Hamiltonian is symmetric under the discrete chiral transformations (19) which on the lattice corresponds to symmetry under translation by one site. In later Sections, we shall consider the possibility of spontaneous breaking of this symmetry.

### 2.2 Gauge fixing and quantization

We shall quantize the gauge fields in the Schrödinger picture. The quantum states are functions of the link operators which are represented by the classical variables $A_i(x)$ and the electric field operators are derivatives

\[
E_i(x) \equiv -i \frac{\partial}{\partial A_i(x)} \quad (25)
\]
We shall also consider the usual Fock representation of the Fermion anticommutator. The empty vacuum is the cyclic vector

$$\psi^a(x)|0> = 0 \quad \forall a, x$$

and Fermions occupying lattice sites are created by $\psi^a(x)$.

### 2.2.1 Compact QED

In compact QED the spectrum of the gauge generator is discrete and a state which obeys the physical state condition can be normalized, thus implying that there is no need for additional gauge conditions. The basis wavefunctions for compact QED (in the basis where the Fermions density and electric field operators are diagonal) are $\Psi[n(x)]\Phi[A]$ with the Fermion states

$$\Psi[n(x)] = \prod_x \prod_{a=1}^{N_L} (\psi^a(x))^{n_a(x)}|0>$$

labeled by vectors $n(x) = (n_1(x), \cdots, n_{N_L}(x))$ with $n_a(x) = 0$ or 1, and the photon states

$$\Psi[A] = \exp \left(i \sum e_i(x)A_i(x) \right)$$

where the eigenvalues $e_i(x)$ of the electric field operator are in spectrum $[E_i(x)]$ (23). Furthermore the states of the photon field are normalized using the inner product

$$< \Phi_1[A], \Phi_2[A] > = \prod_{[x,i]} \int_0^{2\pi} \frac{dA_i(x)}{2\pi} \Phi_1^\dagger[A]\Phi_2[A]$$

and the Fermion states have conventional inner product given by $< 0|0 > = 1$ and the canonical anticommutator relations of the fermion field operators. The physical state condition (21) gives the additional restriction that

$$\hat{\nabla}_i e_i(x) = -\rho(x) = -\sum_x \sum_a (n_a(x) - 1/2)$$

where $\rho(x)$ is the charge density (i.e. the eigenvalue of $\psi^a(x)\psi^a(x) - N_L/2$). Pictorially, we can think of this as containing lines of electric flux joining sites.
whose charges are non-zero and also closed loops of electric flux. In the strong coupling limit, the Hamiltonian is diagonal in the basis \([28]\). This gives a natural explanation of confinement in compact QED in the strong coupling region. If we add a particle–antiparticle pair to a state in \([28]\) it must be accompanied by at least a single line of electric flux. The energy of such a line of flux is proportional to its length. Therefore the electron-positron interaction grows linearly with distance and is confining. This is in contrast to the situation in non-compact QED where the electric flux is not quantized. In that case, a particle–antiparticle pair can have many lines with arbitrarily small flux. The energy of the field is minimized by the usual Coulomb dipole configuration. In high enough dimensions this is not a confining interaction.

2.2.2 Non–Compact QED

In contrast to the case of compact QED, in non–compact QED the generator \(G(x)\) \([17]\) of gauge transformations has a continuous spectrum. In order to obtain a normalizable ground state it is therefore necessary to fix the additional gauge freedom.

In order to separate the gauge orbits of the photon field, we shall need to define the transverse and longitudinal components of the gauge fields. The transverse projection operator is

\[
T_{ij} = \delta_{ij} - \frac{\nabla_i \hat{\nabla} j}{\nabla \cdot \hat{\nabla}}
\]

and the longitudinal projection operator is

\[
L_{ij} = \frac{\nabla_i \hat{\nabla} j}{\nabla \cdot \hat{\nabla}}
\]

They have the usual property of projection operators,

\[
T^2 = T \quad L^2 = L \quad TL = 0 = LT
\]

and also,

\[
1 = T + L
\]

The transverse and longitudinal parts of the electric and gauge fields are obtained by

\[
A^T_i(x) \equiv T_{ij} A_j(x) \quad A^L_i(x) \equiv L_{ij} A_j(x)
\]
\[ E^T_i(x) \equiv T_{ij}E_j(x) \quad E^L_i(x) \equiv L_{ij}E_j(x) \] (36)

Note that there is an ambiguity in the precise definition of the projection operators \( T \) and \( L \) on a finite lattice due to the zero mode of the lattice laplacian. We have fixed this so that constant fields, \( A_i(x) = a_i = const., \) are purely transverse, \( a_i = a^T_i \) (see Appendix A).

In order to quantize, it is necessary to solve the gauge constraint

\[ \mathcal{G}(x) = \hat{\nabla} \cdot E(x) + \rho(x) \sim 0 \] (37)

with

\[ \rho(x) \equiv \psi^a \dagger(x) \psi^a(x) - N_L/2 \] (38)

This is most easily accomplished by gauge fixing. The procedure \([37]\) is to find a gauge fixing condition which has a nonzero commutator with the gauge constraint. An example is the Coulomb gauge condition

\[ \chi(x) = \hat{\nabla} \cdot A(x) \sim 0 \] (39)

The commutator of the gauge condition with the gauge generator is

\[ [\chi(x), \mathcal{G}(y)] = -i \nabla \cdot \hat{\nabla} \delta(x - y) \] (40)

which is a non-degenerate matrix. We can then solve the two constraints by eliminating the longitudinal parts of the gauge and electric fields,

\[ A^L_i(x) = 0 \] (41)

\[ E^L_i(x) = -\nabla_i \frac{1}{-\nabla \cdot \hat{\nabla}} \rho(x) \equiv -\sum_y \nabla_i(x) \frac{1}{-\nabla \cdot \hat{\nabla}} |y \rho(y) \] (42)

The remaining degrees of freedom obey canonical commutation relations which are derived from Dirac brackets. These brackets project the canonical Poisson brackets onto the constrained phase space. Given a set of constraints, \( \xi_A \) with a non-degenerate Poisson bracket, \( \det \{ \xi_A, \xi_B \}_PB \neq 0, \) Dirac brackets for dynamical variables are obtained from Poisson brackets as

\[ \{ P, Q \}_DB = \{ P, Q \}_PB - \{ P, \xi_A \}_PB \{ \xi_A, \xi_B \}_PB^{-1} \{ \xi_B, Q \}_PB \] (43)
In the present case, the brackets of the remaining variables are not modified. The commutator for the transverse photon and electric fields is

\[ [A^T_i(x), E^T_j(y)] = iT_{ij}\delta(x - y) \]  \hspace{1cm} (44)

The Hamiltonian depends only on the transverse photon and electric field and on the charged Fermion fields,

\[ H = \sum_{[x,i]} \left( \frac{e^2}{2} (E^T_i(x))^2 + \sum_{[x,i]} \frac{1}{2e^2} A^T_i(x)(-\nabla \cdot \hat{\nabla})A^T_i(x) \right) \]

\[ + \sum_{x} \frac{e^2}{2} \rho(x) - \nabla \cdot \hat{\nabla} \rho(x) \]

\[ + \sum_{[x,i]} t_{[x,i]} \left( \psi_{a\dagger}(x + i)e^{iA^T_i(x)}\psi_{a}(x) + \text{h.c.} \right) \]  \hspace{1cm} (45)

The Coulomb interaction has appeared as a result of the solution of Gauss’ law.

This gauge fixing, which we have done following Dirac’s procedure [37], can always be implemented by a canonical transformation. Gauss’ law is solved by taking the ansatz for the physical states

\[ |\text{phys}>= \exp \left( i \sum_{[x,i]} A_i(x) \cdot \nabla \frac{1}{-\nabla \cdot \hat{\nabla}} \rho(x) \right) |A^T> \equiv U|A^T> \]  \hspace{1cm} (46)

The exponential operator in (46) generates the unitary transformation

\[ \tilde{A}_i(x) = UA_i(x)U^\dagger = A_i(x) \]

\[ \tilde{E}_i(x) = UE_i(x)U^\dagger = E_i(x) - \nabla \frac{1}{-\nabla \cdot \hat{\nabla}} \rho(x) \]

\[ \tilde{\psi}(x) = U\psi(x)U^\dagger = \exp \left( \frac{i}{\nabla \cdot \hat{\nabla}} \hat{\nabla} \cdot A(x) \right) \psi(x) \]

\[ \tilde{\psi}^\dagger(x) = U\psi^\dagger(x)U^\dagger = \psi^\dagger(x) \exp \left( -\frac{i}{\nabla \cdot \hat{\nabla}} \hat{\nabla} \cdot A(x) \right) \]  \hspace{1cm} (47)

Note that the transformation of \( \psi \) and \( \psi^\dagger \) removes the longitudinal part of the gauge field from the covariant hopping term,

\[ \tilde{\psi}^\dagger(x + i)e^{i\tilde{A}_i(x)}\tilde{\psi}(x) = \psi^\dagger(x + i)e^{iA^T_i(x)}\psi(x) \]  \hspace{1cm} (48)
Substituting the canonically transformed fields into the Hamiltonian yields the Hamiltonian \[ \mathcal{H} \] which is decoupled from \[ A^l_i(x) \] together with the canonically transformed Gauss’ law which now states that \[ E^l = 0 \]. The longitudinal parts \[ E^l \] and \[ A^l \] can now be dropped from the phase space and the resulting quantum theory is the same as that obtained from Dirac’s procedure for solving the constraints.

For \( D = 1 \) (1+1 dimensional spacetime), \( \mathcal{H} \) is the Hamiltonian of the lattice Schwinger model. It is worth pointing out that in that case, \( A^T \) and \( E^T \) just comprise one quantum mechanical degree of freedom,

\[
A^T = \sum_x A(x), \quad E^T = \sum_x E(x)
\]

This corresponds to the fact that the only physical degree of freedom of the photon field on a 1 dimensional compact space is the Wilson loop (U(1) holonomy) \( \exp(i \sum_x A(x)) \).

### 3 Strong Coupling Expansion

Although the results are very similar, the implementation of the strong coupling expansion is somewhat different in the two cases of compact and non-compact QED. We shall treat the two cases separately.

#### 3.1 Non–Compact QED

The conventional strong coupling, \( e^2 \to \infty \) limit is difficult to implement for noncompact QED since the leading terms in \( e^2 \) in the Hamiltonian have a continuum spectrum. The alternative, but related procedure is the hopping parameter expansion, i.e. an expansion in the parameter \(|t|\) in equation (45).

The terms in this expansion are very similar to a strong coupling expansion in that they contain inverse powers of \( e^2 \).

We begin by separating the Hamiltonian into two parts, a leading order part

\[
H_0 = \sum_{[x,i]} \frac{e^2}{2} (E^T_i(x))^2 + \sum_{[x,i]} \frac{1}{2e^2} A^T_i(x)(-\nabla \cdot \hat{\nabla})A^T_i(x) + \sum_{x,y} \frac{e^2}{2} \rho(x)(x|\frac{1}{-\nabla \cdot \hat{\nabla}}|y)\rho(y)
\]

\[ (49) \]
whose ground state we shall attempt to find exactly and a next-to-leading order part

\[ H_1 = \sum_{[x,i]} t_{[x,i]} \left( \psi^a_\dagger(x + i) e^{i A^T(x)} \psi^a(x) + \text{h.c.} \right) \]  

(50)

which we treat as a perturbation.

We first examine the structure of the ground state of \( H_0 \). First of all, it is a direct sum of the free transverse photon Hamiltonian and the Coulomb interaction which depends only on the Fermion operators. The wave-function therefore factorizes into a wavefunction for the free photon ground state and a wavefunction for the ground state of the four-Fermion operator in \( \text{(49)} \). In the Schrödinger picture, the photon ground state is the Gaussian

\[ \Phi_{\text{photon}}[A] = \frac{1}{C} \exp \left\{ -\frac{1}{2e^2} \sum_x A^T_i(x) \sqrt{-\nabla \cdot \hat{\nabla}} A^T_i(x) \right\} \]  

(51)

\( C \) the normalization constant) and the photon contribution to the ground state energy is just the ground state energy of the free photon theory and is of order zero in \( e^2 \) as well as \( |t| \).

The nature of the ground state of the four-Fermion part depends on the number of Fermion flavors, \( N_L \). In particular it is quite different when \( N_L \) is even or odd and we shall treat these two cases separately.

### 3.1.1 \( N_L \) Even

If \( N_L \) is an even number, the ground state of the operator

\[ H_{\text{coul}} = \sum_{x,y} \frac{e^2}{2} \rho(x)(x) \frac{1}{-\nabla \cdot \hat{\nabla}} |y\rangle \rho(y) \]  

(52)

is the state \(|g.s.>\rangle\) where

\[ \sum_{a=1}^{N_L} \psi^a_\dagger(x) \psi^a(x) \ |g.s.> = \frac{N_L}{2} \ |g.s.> \]  

(53)

i.e. with every site of the lattice half-occupied. It is easy to see that this is the case by noting that \( H_{\text{coul}} \) is a non-negative operator and that the states with zero charge density are zero eigenvalues of \( H_{\text{coul}} \).
This ground state is degenerate. At each site the quantum state is given by

\[ \prod_{i=1}^{N_L/2} \psi_{a_i\dagger} |0> \]  

(54)

Since this quantity is antisymmetric in the indices \(a_1, \ldots, a_{N_L/2}\) it takes on any orientation of the vector in the representation of the flavor symmetry group SU\((N_L)\) with Young Tableau made of one column with \(N_L/2\) boxes \((m = N_L/2\) in Fig. 1).

**Figure 1:** The representation of SU\((N_L)\) at each site when \(N_L\) is even.

As in ref. [14, 15, 16] we observe that the degeneracy must be resolved by diagonalizing perturbations in the hopping parameter expansion. The first order perturbations vanish. The first non–trivial order is second order,

\[ \delta_2 = - < g.s. | H_1 \frac{1}{H_0 - E_0} H_1 | g.s. > \]  

(55)
This matrix element can be evaluated by noting that $H_1$ creates an eigenstate of $H_0$ different from the ground states with additional energy

$$\Delta E = \frac{e^2}{2} + \frac{e^2}{2} (D) \nabla_1(x) \frac{1}{-\nabla \cdot \nabla} |x\rangle = e^2$$

Diagonalizing the matrix of second order perturbations is equivalent to finding the spectrum of the effective Hamiltonian

$$H_{\text{eff}} = \frac{2|t|^2}{e^2} \sum_{[x,i]} \psi^{\dagger b}(x+\hat{i}) \psi^b(x) \psi^{\dagger a}(x) \psi^a(x+\hat{i})$$

$$= \frac{2|t|^2}{e^2} \sum_{[x,i]} J_{ab}(x) J_{ba}(x+\hat{i})$$

(57)

where the operators $J_{ab}(x) = \psi^{\dagger a}(x) \psi^b(x) - \frac{1}{2} \delta_{ab}$, are the generators of the $U(N_L)$ given in equation (C3) of Appendix C and obeying the Lie algebra in equation (C2).

The constraint (53) on the total occupation number of each site,

$$\rho(x) = \sum_{a=1}^{N_L} J_{aa}(x) \sim 0$$

reduces to $SU(N_L)$ (see Appendix C) and projects onto the irreducible representation given by the Young Tableau in Fig. 1. (This is one of the fundamental representations of $SU(N_L)$.) Furthermore, (57) is just the Hamiltonian of the $SU(N_L)$ antiferromagnet in that representation.

It is straightforward to see that the higher orders in the hopping parameter expansion also have higher orders of $1/e^2$. In fact, if we consider the following limit,

$$e^2 \to \infty, \quad |t|^2 \to \infty$$

$$|t|^2/e^2 = \text{constant}$$

(58)

all higher order perturbative contributions to both the wavefunction and the energy vanish. Thus, in this limit, QED is exactly equivalent to an $SU(N_L)$ antiferromagnet. That (58) is the correct limit to take can be seen from the fact that, if the antiferromagnet in (57) is in an ordered state, the speed of
the spin-waves, which are the gapless low-energy excitations is proportional to $|t|/e$. They have linear dispersion relation $\omega(k) \sim |k|$ and play the role of massless goldstone bosons for broken flavor symmetry. Their speed should be equal to the speed of light, which is one in our units. This implies that $|t|/e$ should be adjusted so that the spin-wave spectrum is relativistic, $\omega(k) = |k|$. Hence the limit in (58).

When $N_L = 2$, this model is the quantum Heisenberg antiferromagnet in the $j = 1/2$ representation. It is known to have a Néel ordered ground state in $D \geq 3$ [38] and there is good numerical evidence that it has Néel order in $D=2$. The antiferromagnetic order parameter is the mass operator

$$ \tilde{\Sigma} = \sum_x (-1)^{\sum_{i=1}^D x_i} \psi(x) \bar{\psi}(x) $$

which obtains a vacuum expectation value in the infinite volume limit. Thus, when $N_L = 2$ the strong coupling limit breaks chiral symmetry and generates electron mass. It is interesting that in this case there is an iso–vector condensate. In the strong coupling limit this seems unavoidable. The only way to get an iso–scalar condensate is with a charge density wave. However such a configuration always has infinite coulomb energy compared to an electric charge neutral but isospin carrying condensate.

The low energy excitations of this systems (with energies of order $|t|^2/e^2$ are spin waves. All other excitations have energies which go to infinity in the limit (58). The spin waves are the pions which are the scalar Goldstone bosons arising from spontaneous breaking of the vector flavor symmetry $SU(2) \rightarrow U(1)$.

For large $N_L$ there is some evidence that the SU($N_L$) antiferromagnet in these specific representations has a disordered ground state [39]. Particularly in 2+1-dimensions it is known that for infinite $N_L$ the ground state is disordered [34] Although it is beyond the scope of this paper, it would be interesting to investigate the $N_L$ dependence of the ground state further. We shall comment on this in Section IV.

3.1.2 $N_L$ Odd

When $N_L$ is odd the charge density operator $\rho(x)$ (38) which enters the Coulomb Hamiltonian (52) has no zero eigenvalues. Therefore the Coulomb energy of the ground state is necessarily of order $e^2$ for large $e^2$. 

Since the Coulomb Hamiltonian commutes with the charge density operator, \( \rho(x) \), they can be diagonalized simultaneously. The ground state of the Coulomb Hamiltonian should therefore also be an eigenstate of \( \rho(x) \). Therefore, to find the spectrum of (52) we consider all states which are also eigenfunctions of the local density, i.e. where at a given site \( x \),

\[
\rho(x) = -N_L/2, -N_L/2 + 1, \ldots, N_L/2
\]

(60)

with the constraint of global neutrality

\[
\sum_x \rho(x) = 0
\]

For convenience, we consider the system on a finite spatial lattice \( V = V_R \) with periodic boundary conditions (see Appendix A). Then the momenta \( k \) appearing in the Fourier transform are discrete. We consider the Coulomb Hamiltonian in momentum space

\[
H_{\text{coul}} \equiv \frac{e^2}{2} \sum_{x,y} \rho(x)\frac{1}{\nabla \cdot \nabla} \rho(x) = \frac{e^2}{2} \sum_k \frac{1}{|V|} \sum_{i=1}^D \sin^2(k_i/2) |\tilde{\rho}(k)|^2
\]

(61)

where \( |V| \) is the total number of spatial lattice sites and \( \tilde{\rho}(k) \) is the fourier transform of the charge operator \( \rho(x) \). Since \( 0 \leq \sin^2(k_i/2) \leq 1 \) we can derive the lower bound for the Coulomb energy as

\[
H_{\text{coul}} \geq \frac{e^2}{2 |V|} \sum_k \frac{1}{4D} |\tilde{\rho}(k)|^2 = \frac{e^2}{8D} \sum_x \rho(x)^2 \geq \frac{|V|e^2}{32D}
\]

(the latter estimate follows from \( \rho(x)^2 \geq 1/4 \)) This bound is saturated by the charge distributions

\[
\tilde{\rho}_0(k) = \pm \frac{|V|}{2} \delta_{k,\bar{\pi}}
\]

where \( \bar{\pi} = (\pi, \ldots, \pi) \) is the vector for which \( \sum_{i=1}^D \sin^2(k_i/2) \) in the denominator of (61) takes its maximum value. These are allowed configurations of the charge density,

\[
\rho_0(x) = \pm \frac{1}{2} (-1)^{\sum_{i=1}^D x_i}
\]
which give the ground state configurations of the Coulomb system. The electric field in these ground states is easily deduced from Gauss’ law,

$$E_i^0(x) = \pm \frac{1}{4D} (-1)^{\sum_{i=1}^D x_i}$$

These configurations break chiral symmetry in that they are not invariant under the transformation $\chi_1$ in (19) but they are symmetric under charge conjugation $C$ defined in (20).

The ground state energy per lattice site is

$$E_{0\text{(coul)}}/|V| = \frac{e^2}{32D}$$

Note that it is of order $e^2$. This is in contrast to the ground state energy when the number of lattice Fermion flavors is even, which is of order $|t|^2/e^2 \sim 1$.

The ground states that we have found are highly degenerate in that only the number of Fermions at each site is fixed. Their quantum state can still take up any orientation in the vector space which carries the representation of the flavor $SU(N_L)$ given by the Young Tableaux in Fig. 2.
Figure 2: Representation of $SU(N_L)$ on each site of the even sublattice $A$ and the odd sublattice $B$ when $N_L$ is odd.

We have divided the lattice into two sublattices: $A$ is all points where $\sum_i x_i$ is even and $B$ where $\sum_i x_i$ is odd. Then, the differing occupation numbers on sites on each sublattice yield different representations of $SU(N_L)$.

Again, this degeneracy must be resolved by diagonalizing the perturbations, which are non-zero in second order and the problem is equivalent to diagonalizing the antiferromagnet Hamiltonian (57), this time with the representations depicted in Fig. 2. Also, in the limit (58) this correspondence is exact.

The strong coupling ground states that we find when $N_L$ is odd contains a charge density wave. The staggered charge density operator has expectation values

$$\frac{1}{|V|} < \sum_x (-1)^{\sum_{k=1}^{D} x_k} \psi^\dagger(x) \psi(x) > = \pm 1/2$$

This condensate is an isoscalar and we have shown that it must always occur in all dimensions. When $N_L > 1$ mass operators with certain generators of $SU(N_L)$ could have expectation values if the ground state has antiferromagnetic order. However, unlike the case of even $N_L$, the antiferromagnetic order is not required in order to have chiral symmetry breaking.

The ground state we find breaks chiral symmetry. This is a true dynamical symmetry breaking since, in infinite volume, the ground states which are related by a chiral transformation are never mixed in any order of strong coupling perturbation theory. Furthermore, there are no local operators which couple them.

We conclude that the strong coupling ground state breaks chiral symmetry for any odd $N_L$ and in any dimensions. As in the case of even $N_L$ there is also the possibility (and for small $N_L$ the likelihood) that the $SU(N_L)$ antiferromagnet we obtain here is in a Néel state and the flavor symmetry is also broken. We shall not pursue this possibility here but refer the reader to the literature [39].

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3.2 Compact QED

The difference between compact and non-compact QED resides in the quantization of the gauge fields. In all cases the Fermionic state is identical in the two cases. In compact QED the eigenstates of the electric field operator are normalizable and can be used for the ground state. In this case we separate the Hamiltonian into three terms,

\[ H_0 = \sum_{[x,i]} \frac{e^2}{2} E_i^2(x) \]
\[ H_1 = \sum_{[x,i]} t_{[x,i]} \left( \psi_{x}^{\dagger}(x)e^{iA_i(x)}\psi(x) + h.c. \right) \]
\[ H_2 = \sum_{[x,i,j]} \frac{2}{e^2} \sin^2 \left( B[x,j]/2 \right) \]

In the strong coupling limit it is necessary to solve Gauss’ law (21) for the electric fields and the charge distribution in such a way as to minimize \( H_0 \).

When \( N_L \) is even, the charge operator has zero eigenvalues and Gauss’ law has the solution where \( E_i(x) = 0 \), which is an obvious minimum of the \( H_0 \), and there are \( N_L/2 \) fermions on each site. This is similar to the situation in non–compact QED when \( N_L \) is even. Also, the degeneracy of this state must be resolved in the same way, resulting in the effective Hamiltonian (57) which describes the SU(\( N_L \)) antiferromagnet in the representation with Young Tableau having one column with \( N_L/2 \) boxes shown in fig. 1. Again, we expect that this system has Néel order in \( D \geq 2 \) if \( N_L \) is small enough and the chiral symmetry of electrodynamics is broken, along with the SU(\( N_L \)) flavor symmetry.

When \( N_L \) is odd, since the charge density operator has no non-zero eigenvalues, it is impossible to find a zero eigenstate of the Gauss’ law constraint operator without some electric field. The problem which we must solve is to minimize the energy functional \( \sum E^2 \) subject to the constraint \( \nabla \cdot E = -\rho \) where at each site \( \rho \) has one of the values

\[-\frac{N_L}{2}, -\frac{N_L}{2} + 1, \ldots, \frac{N_L}{2} \quad N_L \text{ an odd integer}\]

It is straightforward to show that the charge distribution and electric field which one obtains is identical to those in the case of non–compact QED with
odd \(N_L\),

\[
\rho_0(x) = \frac{1}{2}(-1)^{\sum_{i=1}^{D} x_i} \quad E_i(x) = \frac{1}{4D}(-1)^{\sum_{i=1}^{D} x_i}
\]

The ground state degeneracy is again resolved by diagonalizing perturbations and, again the true ground state of the strong coupling limit is the ground state of the effective Hamiltonian (57) when the SU(\(N_L\)) spins take on the configurations in Fig.2.

Notice that in the ground state, the electric fields are not integers, but on each link, the spectrum of the electric field operator is \(1/4D+\) integers. The “theta angles” \(1/4D\) survive all orders in strong coupling perturbation theory.

### 4 Remarks

In this paper we have analyzed the possibility of chiral symmetry breaking in the strong coupling limit of quantum electrodynamics using the Hamiltonian picture and a lattice regularization. We chose to use staggered Fermions because they give the closest analog to interesting condensed matter physics systems. Also, unlike Wilson Fermions which, in the Hamiltonian picture, have no chiral symmetry at all, they have a discrete chiral invariance which forbids Fermion mass and it is sensible to ask questions about dynamical mass generation.

In 1+1 dimensions, staggered Fermions give \(N_L\) species of 2-component Dirac Fermions. When \(N_L = 1\) we obtain the Schwinger model with a lattice regularization. Also, in this case, we have found that the chiral symmetry is broken dynamically. Of course, due to the staggered Fermion regularization there is no continuous chiral symmetry, which is as it should be since it should be impossible to regularize the Schwinger model so that there is simultaneous continuous chiral and gauge symmetry. However, to match the solution of the continuum Schwinger model, the Fermion should obtain mass. This indeed happens in our strong coupling limit by spontaneous symmetry breaking. (Although we disagree with some aspects of the formalism, we agree with the results of reference [18] on this point.)

This result should not be confined to strong coupling, but should persist for all coupling, i.e. the critical coupling in \(D = 1\) should be at \(e^2 = 0\). We conjecture that this sort of symmetry breaking for small \(e^2\) is a manifestation
of the Peierls instability — the tendency of a one dimensional Fermi gas to form a gap at the Fermi surface. This happens with any infinitesimal interaction.

In fact, this must also happen for the case where $N_L$ is even. Then, there cannot be any spin order in 1 dimension. However, anomalies break the isoscalar chiral symmetry in the continuum theory and should also do so here. This means that there should be a dynamical generation of charge density wave which would be driven by the Peierls instability. It also implies that for $N_L = 2$ for example, the ground state in the strong coupling limit would not be a Heisenberg antiferromagnet, but would be alternating empty site and site with two electrons in a spin singlet state. This state, even though it has large coulomb energy, avoids the infrared divergences of gapless Fermions.

In higher dimensions, $D \geq 2$, it would be interesting to explore the possibility of phase transitions between different symmetry breaking patterns for the $SU(N_L)$ flavor symmetry as one varies $N_L$. There is already some work on this subject in the condensed matter physics literature on $SU(N_L)$ antiferromagnets \cite{39}. They analyze the $SU(N_L)$ antiferromagnet which is similar to the strong coupling limit of an $U(N_C)$ gauge theory (see \cite{15, 16} for details) which is in the representation corresponding to a rectangular Young Tableau with $N_L$ rows and $N_C$ columns. They work in the large $N_C$ limit and show that there is a phase transition from the spin ordered Néel phase to a disordered phase when $N_L \sim N_C$. In our case $N_C = 1$ so their analysis is not accurate. Nevertheless, we expect that there should be a phase transition to a disordered phase as $N_C$ is increased. For odd $N_L$ the chiral symmetry is always broken and the question we are asking is whether the flavor symmetry is also broken. For even $N_L$ possible phase transition is relevant to both chiral and flavor symmetry breaking.

First of all, when $N_C = 2$, we have the $j = 1/2$ Heisenberg antiferromagnet which is known to have an ordered ground state in $D \geq 3$ and is also very likely to have an ordered ground state in $D = 2$. Furthermore, when $N_L \rightarrow \infty$, the ground state is known to be disordered in $D = 2$ \cite{34} and is likely also the case in $D = 3$. In between there $N_L = 2$ and $N_L = \infty$ there should be a phase transition. It is interesting to speculate that the order–disorder phase transition which occurs as one increases $N_L$ in the $SU(N_L)$ antiferromagnet is the same one that appears in the study of chiral symmetry breaking in 2+1–dimensional QED in the continuum \cite{10, 11} where they find that chiral symmetry is broken only if the number of flavors is less
than a certain critical value. As we noted in the introduction, their work is effectively in the strong coupling (large $e^2$) limit.

Our results also indicate that, besides the critical $N_L$, for a fixed $N_L$ which is small enough, there should also be a critical coupling constant $e^2$ and, in fact, a critical line in the $N_L-e^2$ plane where there is a second order phase transition between a spin ordered chiral symmetry breaking phase and a disordered (and possibly chirally symmetric phase). We speculate that in 3+1–dimensions a similar situation could occur.

A Notation

In this paper we consider the lattice regularization most suited to the Hamiltonian formalism where time is continuous and space is a square lattice with lattice spacing one. We use a finite spatial lattice $V_R$ with lattice sites

$$x \equiv (x_1, \ldots, x_D), \quad -R \leq x_i < R \quad (A1)$$

were $R$ is a positive integer and $|V_R| = (2R)^D$ is the total number of lattice sites. In the thermodynamic limit, $R \to \infty$.

The lattice sites are connected by unit vectors

$$\hat{1} = (1, 0, \ldots) \quad \hat{2} = (0, 1, \ldots) \quad \ldots \quad (A2)$$

and the oriented link between the lattice site $x$ and $x + \hat{i}$ is denoted $[x, i]$. The link oriented in the opposite direction is denoted $-[x, \hat{i}]$. On the finite lattice $V_R$ we identify lattice sites $x$ and $x + 2R\hat{i}$. Then links obey the identity

$$[x, -i] = -[x - \hat{i}, \hat{i}] \quad (A3)$$

The boundary of the $[x, \hat{i}]$ are the two points

$$\delta[x, \hat{i}] = (x + \hat{i}) - x \quad (A4)$$

Also, a plaquette with corners $x, x + \hat{i}, x + \hat{i} + \hat{j}, x + \hat{j}$ and with sides $[x, \hat{i}], [x + \hat{i}, \hat{j}], [x + \hat{i} + \hat{j}, -i], [x + \hat{j}, -j]$ is denoted as $[x, i, j]$ and has the boundary

$$\delta[x, i, j] = [x, i] - [x, j] + [x + \hat{i}, j] - [x + \hat{j}, i] \quad (A5)$$
It also obeys the identities

\[ [x, j, i] = -[x, i, j] \quad [x, -i, j] = -[x - i, i, j] \] (A6)

It is also possible to introduce higher dimensional structures, elementary cubes, etc.

We shall also introduce lattice derivative operators, the forward difference operator

\[ \nabla_i f(x) = f(x + \hat{i}) - f(x) \] (A7)

and the backward difference operator

\[ \hat{\nabla}_i f(x) = f(x) - f(x - \hat{i}) \] (A8)

The lattice Laplacian is

\[ \nabla \cdot \hat{\nabla} f(x) \equiv \sum_{i=1}^{D} \left( f(x + \hat{i}) - 2f(x) + f(x - \hat{i}) \right) \] (A9)

The functions which we shall consider are functions from either the lattice sites, links or plaquettes to the real numbers. The Fourier transform of a function on lattice sites is given by

\[ \tilde{f}(k) = \sum_{x \in \mathbb{V}_R} e^{ik \cdot x} f(x) \] (A10)

with \( \tilde{f} \) a function on the reciprocal lattice (momentum space) \( \tilde{V}_R \) with lattice sites

\[ k = (k_1, \ldots, k_D), \quad k_i = \frac{2\pi}{2R} \times \text{integers}, \quad -\pi < k_i \leq \pi \] (A11)

The inverse Fourier transform is

\[ f(x) = \frac{1}{|\mathbb{V}_R|} \sum_{k} e^{-ik \cdot x} \tilde{f}(k) \] (A12)

(note the the number of sites of the lattice \( \mathbb{V}_R \) and its reciprocal lattice \( \tilde{V}_R \) are equal). The periodic delta functions on \( \mathbb{V}_R \) and \( \tilde{V}_R \) are given by

\[ \delta(x) = \frac{1}{|\mathbb{V}_R|} \sum_{x} e^{-ik \cdot x} \] (A13)
and
\[ \delta(k) = \frac{1}{|V_R|} \sum_{k \in V_R} e^{ik \cdot x} \] (A14)

and the Parseval relation is
\[ \sum_{x \in V_R} f^*(x) g(x) = \frac{1}{|V_R|} \sum_{k \in V_R} \tilde{f}^*(-k) g(k) \] (A15)

In momentum space, the lattice Laplacian is diagonal,
\[ (\nabla \cdot \nabla \tilde{f})(k) = -4 \sum_{i=1}^{D} \sin^2(k_i/2) \tilde{f}(k) \] (A16)

From this it follows that its inverse \( \frac{1}{\nabla \cdot \nabla} \) is unambiguously defined only on functions \( f \) with \( \tilde{f}(0) = \sum_{x \in V_R} f(x) = 0 \). We can extend its definition to all functions \( f \) by setting
\[ \frac{1}{\nabla \cdot \nabla} \tilde{f}(0) = 0 \] (A17)

In position space, the integral kernel of \( \frac{1}{\nabla \cdot \nabla} \) is just the Green function for the Laplacian,
\[ (x| \frac{1}{-\nabla \cdot \nabla} |y) = \sum_{k \in V_R \notin \{0\}} e^{ik \cdot (x-y)} \frac{1}{4 \sum_{i=1}^{D} \sin^2(k_i/2)} \] (A18)

For the infinite lattice \( V_\infty \), the momentum space is no longer a lattice but the Brillouin zone
\[ \tilde{V}_\infty = \Omega_B \equiv \{ k = (k_1, \ldots, k_D) | k_i \text{ real}, -\pi < k_i \leq \pi \} \] (A19)

The inverse of the Fourier transform (A10) is then
\[ f(x) = \int_{\Omega_B} \frac{d^D k}{(2\pi)^D} e^{-ik \cdot x} \tilde{f}(k) \] (A20)

corresponding to the delta function on the infinite lattice,
\[ \delta(x) = \int_{\Omega_B} \frac{d^D k}{(2\pi)^D} e^{ik \cdot x} \] (A21)
The Green function for the Laplacian on $V_\infty$ is given by

\[
(x| \frac{1}{-\nabla \cdot \nabla} | y) = \int_{\Omega_B} \frac{d^D k}{(2\pi)^D} e^{i k \cdot (x-y)} \frac{1}{4\sum_i \sin^2 k_i / 2}
\]  

(A22)

and is well defined in dimensions $D > 2$. In two dimensions it is defined by one additional subtraction which removes the logarithmic divergence in the integration. In one dimension it must be defined by solving the Laplace equation with a source explicitly.

In the main text we abbreviate $\sum_{x \in V_R}$ as $\sum_x$ and similarly in momentum space.

\section*{B Lattice Fermions}

Throughout this paper we shall use the staggered Fermion formalism which was originally developed by Kogut and Susskind. This formalism is well known and the details can be found in the papers of Susskind and collaborators [40, 17, 18, 19] and Kluberg-Stern et.al. [41]. Here we shall review the basic features and make some observations which are necessary for our present discussion. Some of these observations have already been made in [14, 15, 16].

We shall use the staggered Fermion formalism since we believe that it gives the closest possible analog to the lattice Fermions encountered in condensed matter physics. As a regularization of Fermions in relativistic quantum field theory, this formalism has the disadvantage that chiral symmetries are discrete, rather than continuous. The method should be regarded as adding some formally irrelevant operators to the Hamiltonian. These operators make the Hamiltonian local but break the continuous chiral symmetry down to a discrete subgroup. (Actually, there is a non-local chiral symmetry. However, being non-local it is not a useful symmetry in that, for example, it does not imply the existence of Goldstone Bosons in the phase where it is broken.)

Thus, we can really only address questions about discrete chiral symmetry breaking. This should be enough to tell us whether mass generation, and in fact what sort of mass generation, is possible.
B.1 Review of Staggered Fermions

The purpose of the staggered Fermion method is to minimize Fermion doubling which always accompanies lattice Fermions. Generally, staggered Fermions are obtained by the spin-diagonalization method. To implement this method, we begin with the naively latticized Dirac Hamiltonian,

\[ H_F = \frac{1}{2} \sum_{[x,j]} \left( \psi^\dagger (x) i \alpha^j \nabla_j \psi(x) - (\nabla_j \psi^\dagger)(x) i \alpha^j \psi(x) \right) \]  

(B1)

where \( \alpha^j \) are the \( 2^{[(D+1)/2]} \)-dimensional Dirac \( \alpha \)-matrices. (Here \( [(D + 1)/2] \) is the integer part of \( (D + 1)/2 \).) They are Hermitean, \( \alpha^{j\dagger} = \alpha^j \) and obey the Clifford algebra

\[ \{ \alpha^i, \alpha^j \} = 2\delta^{ij} \]  

(B2)

They are therefore unitary matrices, \( \alpha^{i\dagger} \alpha^i = 1 \). Using the properties of the difference operator, (B1) can be presented in the form

\[ H_F = -\frac{i}{2} \sum_{[x,j]} \left( \psi^\dagger (x + \hat{j}) \alpha^j \psi(x) - \psi^\dagger (x) \alpha^j \psi(x + \hat{j}) \right) \]  

(B3)

Since the Dirac matrices are unitary, the naive lattice Fermion Hamiltonian in (B3) resembles a condensed matter Fermion hopping problem with a background \( U(2^{[(D+1)/2]}) \) gauge field given by the \( \alpha \)-matrices. In any plaquette of the lattice, \([x,i,j]\), this background field has curvature

\[ \alpha^i \alpha^j \alpha^{i\dagger} \alpha^{j\dagger} = -1 \]  

(B4)

The curvature resides in the U(1) subgroup of \( U(2^{[(D+1)/2]}) \) and has exactly half of a U(1) flux quantum per plaquette. This is true in any dimensions. We observe that either 1/2 or zero flux quanta are the only ones allowed by translation invariance and parity and time reversal symmetries of the Hamiltonian.

Since the curvature if the \( \alpha \)-matrices is U(1)-valued, we should be able to do a gauge transform which presents the matrices themselves as U(1) valued gauge fields (i.e. diagonal). A specific example of such a gauge transform due to Kluberg-Stern et. al. [41] is

\[ \psi(x) \rightarrow (\alpha^1)^{x_1}(\alpha^2)^{x_2} \cdots (\alpha^D)^{x_D} \psi(x) \]  

(B5)
Then

\[ \psi^\dagger(x + \hat{j}) \alpha^j \psi(x) \rightarrow (-1) \sum_{k=1}^{D-1} x_k \psi^\dagger(x + \hat{j}) \psi(x) \]  

(B6)

The resulting Hamiltonian is

\[ H_F = -\frac{i}{2} \sum_{[x,j]} (-1)^{\sum_{k=1}^{D-1} x_k} \left( \psi^\dagger(x + \hat{j}) \psi(x) - \psi^\dagger(x) \psi(x + \hat{j}) \right) \]  

(B7)

This describes \(2^{[(D+1)/2]}\) identical copies of Fermions with the same Hamiltonian which must all give Fermions with the same spectrum as the original Hamiltonian in (B1). Staggered Fermions are obtained by choosing one of these copies. This reduces the Fermion doubling by a factor of the dimension of the Dirac matrices, \(2^{[(D+1)/2]}\).

In the staggered Fermion method, we treat the components of the original lattice Dirac Hamiltonian as flavors, rather than components of the relativistic spinor necessary for Lorentz invariance. The spinor components now reside on adjacent lattice sites. In this method, the continuous chiral symmetry of the massless Hamiltonian, under the transformation \(\psi \rightarrow e^{i\gamma^5 \theta} \psi\), is lost. There is a discrete chiral symmetry, corresponding to translations by one lattice site in any direction. Explicitly,

\[ \psi(x) \rightarrow (-1)^{\sum_{k=1}^{D} x_k} \psi(x + \hat{j}) \]  

(B8)

is a symmetry of the Hamiltonian \(B7\) and corresponds to a discrete chiral transformation.

Mass operators correspond to staggered charge densities. The operator

\[ \Sigma = \sum_x (-1)^{\sum_{k=1}^{D} x_k} \psi^\dagger(x) \psi(x) \]  

(B9)

changes sign under the chiral transformations \(B8\) and corresponds to a certain Dirac mass.

With staggered Fermions there is still a certain amount of Fermion doubling. The doubling can be counted by noting that the staggered Fermion Hamiltonian \(B7\) is invariant under translations by two lattice sites. Therefore, a unit cell is a unit hypercube of the lattice, containing \(2^D\) sites and staggered Fermions correspond to a \(2^D\) component spinor. The dimension
of the Dirac matrices is \( d^{(D+1)/2} \). Therefore the number of Dirac spinors we obtain is \( 2^D/2^{(D+1)/2} \). For lower dimensions the minimum number of continuum flavors can be tabulated as

| \( d \)   | dim. of Dirac matrices | No. of flavors |
|---------|------------------------|---------------|
| 1 + 1   | 2                      | 1             |
| 2 + 1   | 2                      | 2             |
| 3 + 1   | 4                      | 2             |

Only in 1+1-dimensions do we get a single species of Dirac Fermion.

### B.2 Explicit Example in 3+1 Dimensions

For simplicity in notation, the formulas here and in the following subsection are given for an infinite spatial lattice \( V_\infty \).

To see how to take the continuum limit explicitly, consider the case of \( d = 3 + 1 \). There, we divide the lattice into eight sublattices and label the spinor components as

\[
\psi(\text{even, even, even}) \equiv \psi_1 \quad \psi(\text{odd, even, odd}) \equiv \psi_7
\]

\[
\psi(\text{even, odd, even}) \equiv \psi_6 \quad \psi(\text{even, even, odd}) \equiv \psi_5
\]

\[
\psi(\text{odd, odd, even}) \equiv \psi_4 \quad \psi(\text{odd, even, odd}) \equiv \psi_3
\]

\[
\psi(\text{even, odd, odd}) \equiv \psi_2 \quad \psi(\text{odd, odd, odd}) \equiv \psi_8
\]

In terms of these spinors, the Hamiltonian (B7) can be written as the matrix operator

\[
H = \int_{\Omega_B} d^3k \psi^\dagger(k) A^i \sin k_i \psi(k)
\]

where

\[
\tilde{\Omega}_B = \{ k_i : -\pi/2 < k_i \leq \pi/2 \}
\]
is the Brillouin zone of the (even,even,even) sublattice,

\[ A^i = \begin{pmatrix} 0 & \alpha^i \\ \alpha^i & 0 \end{pmatrix} \]  

(B16)

and

\[ \alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \]  

(B17)

are a particular representation of the Dirac matrices.

In this representation the mass operator is

\[ \sum_x (-1)^{\sum_{x_i}^{D} x_i} \psi^\dagger(x) \psi(x) = \int_{\Omega_B} \frac{d^3k}{(2\pi)^3} \psi^\dagger(k) B \psi(k) \]  

(B18)

where

\[ B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  

(B19)

The Fermion spectrum is

\[ \omega(k) = \sqrt{\sum_{i=1}^{3} \sin^2 k_i + m^2} \]  

(B20)

and only the region \( k_i \sim 0 \) is relevant to the continuum limit. We have normalized \( \psi(k) \) so that

\[ \{ \psi(x), \psi^\dagger(y) \} = \delta(x - y), \quad \{ \psi(k), \psi^\dagger(l) \} = \delta(k - l) \]  

(B21)

If we define

\[ \beta = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \]  

(B22)

and the unitary matrix

\[ M = \frac{1}{2} \begin{pmatrix} 1 - \beta & 1 + \beta \\ 1 + \beta & 1 - \beta \end{pmatrix} \]  

(B23)

and

\[ \psi = M \psi' \]  

(B24)
with
\[ \psi' = (\psi_a, \psi_b) \] (B25)

the Hamiltonian is
\[ H_f = \int_{\Omega_B} d^3k \begin{pmatrix} \psi_a^\dagger & \psi_b^\dagger \end{pmatrix} \begin{pmatrix} \alpha^i \sin k_i - \beta m & 0 \\ 0 & \alpha^i \sin k_i + \beta m \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \] (B26)

In the low momentum limit, \( \sin k_i \sim k_i \), with Fermion density 1/2 per site so that the Fermi level is at the intersection point of the positive and negative energy bands, we obtain 2 continuum Dirac Fermions.

This describes two flavors of 4-component Dirac Fermions and the Dirac masses for each component given by the staggered charge density have opposite signs. Thus the charge density breaks the discrete chiral symmetry. It also breaks a flavor symmetry which, in the absence of mass, mixes the two continuum Fermions.

### B.3 General Continuum Limit

We shall now consider the continuum limit in a general number of dimensions. A formalism much like (but not exactly the same as) the present one can be found in [41].

We shall begin with the Hamiltonian (B7),
\[ H_f = -\frac{i}{2} \sum_{[x,j]} (-1)^{\sum_{k=1}^{D} x_k} \left( \psi^\dagger(x + \hat{j}) \psi(x) - \psi^\dagger(x) \psi(x + \hat{j}) \right) \]

We consider an elementary hypercube of the lattice with sides of length 1 and \( 2^D \) sites generated by taking a site all of whose coordinates are even and adding to it the vectors
\[ \vec{\alpha} = (\alpha_1, \ldots, \alpha_D) \quad \alpha_i = 0 \text{ OR } 1 \] (B27)

We also decompose the lattice into \( 2^D \) sublattices generated by taking a site of the elementary hypercube, (even, even, ...) + \( \vec{\alpha} \) for some \( \vec{\alpha} \) and translating it by all even multiples of lattice unit vectors, \( \hat{i} \). We label the Fermions which
reside on the sublattice of each of the corners of the elementary hypercube as \( \psi_{\alpha_1, \ldots, \alpha_D}(x) \). In momentum space the Hamiltonian is

\[
H_f = \int_{\tilde{\Omega}_B} d\mathbf{k} \sum_{i=1}^D \psi^\dagger_{\alpha_1, \ldots, \alpha_D}(k) \Gamma_i^i \alpha_1, \ldots, \alpha_D \beta_1, \ldots, \beta_D \sin k_i \psi_{\beta_1, \ldots, \beta_D}(k)
\]

(B28)

where \( \tilde{\Omega}_B = \{ k_i : -\pi/2 < k_i \leq \pi/2 \} \) is the Brillouin zone of the (even,even,\ldots) sublattice, the momentum space Fermions have the anticommutator

\[
\{ \psi_{\alpha_1, \ldots, \alpha_D}(k), \psi^\dagger_{\beta_1, \ldots, \beta_D}(k') \} = \left( \prod_{i=1}^D \delta_{\alpha_i \beta_i} \right) \delta(k - k')
\]

(B29)

and the Dirac tensors are

\[
\Gamma_i^i \alpha_1, \ldots, \alpha_D \beta_1, \ldots, \beta_D = \delta_{\alpha_1, \alpha_i} \ldots \delta_{\alpha_{i-1}, \alpha_{i-1}} \sigma_{\alpha_i \beta_i} \delta_{\alpha_{i+1}, \beta_{i+1}} \ldots \delta_{\alpha_D, \beta_D} (-1)^{\sum_{k=1}^{i-1} \alpha_k}
\]

(B30)

They obey the Clifford algebra

\[
\Gamma^i \Gamma^j + \Gamma^j \Gamma^i = 2\delta^{ij}
\]

(B31)

The spectrum of the Dirac operator is \( \omega(k) = \pm \sqrt{\sum_{i=1}^D \sin^2 k_i} \). Note that, to set the Fermi level of the Fermions at the degeneracy point where the two branches of the spectrum meet, it is necessary that the Fermion states are exactly half-filled. This is also required for charge–conjugation invariance, or particle–hole symmetry of the vacuum state.

The staggered charge density operator \( \Sigma \) is equivalent to a mass operator where

\[
\Sigma = \int_{\tilde{\Omega}_B} d^D \mathbf{k} \psi^\dagger_{\alpha_1, \ldots, \alpha_D}(k) \Gamma_0^0 \alpha_1, \ldots, \alpha_D \beta_1, \ldots, \beta_D \psi_{\beta_1, \ldots, \beta_D}(k)
\]

(B32)

where

\[
\Gamma_0^0 \alpha_1, \ldots, \alpha_D \beta_1, \ldots, \beta_D = \delta_{\alpha_1, \beta_1} \ldots \delta_{\alpha_D, \beta_D} (-1)^{\sum_{k=1}^D \alpha_k}
\]

(B33)

Here, \( \Gamma^0 \) satisfies the algebra

\[
\Gamma^0 \Gamma^i + \Gamma^i \Gamma^0 = 0
\]

(B34)

and

\[
\Gamma^0 \Gamma^0 = 1
\]

(B35)
Thus the spectrum of the operator $H_f + m \sum$ is

$$\omega(k) = \pm \sqrt{\sum_{i=1}^{D} \sin^2 k_i + m^2}$$

(B36)

which is the spectrum of a relativistic Fermion in the limit $k \sim 0$.

Here, we count the number of flavors of Fermions obtained in the continuum limit by noting that (B28) describes a 2$^D$-component Fermion. In $D$ dimensions the Dirac matrices are $[(D + 1)/2]$ dimensional, therefore the continuum limit of (B28) describes 2$^D / 2^{[(D+1)/2]}$ species of Dirac Fermions. These are tabulated up to dimension 4 in (B.1).

C Fermion Representation of SU($N$) Quantum Antiferromagnet

The Hamiltonian for an U($N$) quantum antiferromagnet is

$$H_{\text{AFM}} = \frac{g^2}{2} \sum_{<x,y>} J_{ab}(x) J_{ba}(y)$$

(C1)

where $J_{ab}(x)$, $a, b = 1, \cdots, N_L$, obey current algebra relations associated with the Lie algebra of U($N$),

$$[J_{ab}(x), J_{cd}(y)] = (J_{ad}(x) \delta_{bc} - J_{cb}(x) \delta_{ad}) \delta(x - y)$$

(C2)

and where $<x, y>$ denotes the link connecting sites $x$ and $y$ on a bipartite lattice. For simplicity, we shall take the lattice to be cubic. Here, we have used a particular basis for the SU($N$) algebra which can be conveniently represented by Fermion bilinear operators,

$$J_{ab}(x) = \bar{\psi}^a(x) \psi^b(x) - \delta^{ab}/2$$

(C3)

The representation of the algebra on each site $x$ is fixed by specifying the Fermion number of the states,

$$\rho(x) = \sum_a J_{aa}(x)$$

(C4)
For example, the Fermion vacuum state $|0\rangle$ such that

$$\psi^a(x)|0\rangle = 0, \quad \forall a, x$$  \hspace{1cm} (C5)

is the singlet state, the states with $m \leq N$ Fermions per site,

$$\prod_x \psi^{a_1\dagger}(x)\psi^{a_2\dagger}(x)\ldots\psi^{a_m\dagger}(x)|0\rangle$$

corresponds to $\rho(x) = m - N/2$ for all $x$ and the irreducible representation with the Young Tableau

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{When there are $m$ Fermions per site, the representation of $SU(N_L)$ has Young Tableau with one column of $m$ boxes.}
\end{figure}

For each site $x$, $\rho(x)$ is the generator of the U(1) subgroup of U($N$). Using a basis $T^i = (T^i)^\ast$, $i = 1, \ldots, N^2 - 1$, of the Lie algebra of SU($N$) in the fundamental representation normalized so that $\text{tr}(T^iT^j) = T_{ab}^i T_{ba}^j = \delta^{ij}/2$, 

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and using

\[ T_{ab}^i T_{cd}^i = \frac{1}{2} \delta_{ad} \delta_{bc} - \frac{1}{2N} \delta_{ab} \delta_{cd} \]  

(C6)

it is convenient to introduce

\[ J^i(x) = \psi^\dagger(x) T_{ab}^i \psi^a(x) \]  

(C7)

obeying current algebra of the Lie algebra of SU(N), and to write the Hamiltonian (C1) as

\[ H_{AFM} = \frac{g^2}{N} \sum_{<x,y>} \rho(x) \rho(y) + H_{SU(N)} \]  

(C8)

with

\[ H_{SU(N)} = g^2 \sum_{<x,y>} J^i(x) J^i(y) \]  

(C9)

is the Hamiltonian of an SU(N) antiferromagnet. From this it is obvious that by fixing the \( \rho(x) \), \( H_{AFM} \) is reduced to an SU(N) antiferromagnet.

For example, the familiar \( j = 1/2 \) SU(2) Heisenberg antiferromagnet is obtained from (C1), \( N = 2 \), by using the identity

\[ \frac{\vec{\sigma}_{ab}}{2} \cdot \frac{\vec{\sigma}_{cd}}{2} = \frac{1}{2} \delta_{ad} \delta_{bc} - \frac{1}{2} \delta_{ab} \delta_{cd} \]  

(C10)

corresponding to (C6) for \( N = 2 \).

Generally, when \( N \) is even we will consider the representations where \( m = N/2 \), so that the Fermion occupation of each site is \( N/2 \) and \( \rho(x) = 0 \). When \( N \) is odd we divide the lattice into two sublattices such that the nearest neighbors of all sites of one sublattice are in the other sublattice (when this is possible the lattice is said to be bipartite). When \( N \) is odd, the representation of SU(N) has \( (N+1)/2 \) Fermions, i.e. \( \rho(x) = 1/2 \), on the sites of one sublattice and \( (N-1)/2 \) Fermions, i.e. \( \rho(x) = -1/2 \) on the sites of the other sublattice.
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