The nerve of a crossed module

Ivan Yudin
Departamento de Matematica
Universidade de Coimbra
Apartado 3008
3001-454 Coimbra
Portugal
yudin@mat.uc.pt *

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Abstract
We give an explicit description for the nerve of crossed module of categories.

1 Introduction

Let $T$ be a topological space. It is said that $T$ has a type $k$ if all the homotopy groups $\pi_n(T)$ are zero for $n > k$. It is known that the categories of groups and of 1-types are equivalent. In [EM45] Eilenberg and Maclane constructed for every group $G$ a simplicial set $BG$ such that the topological realization $|BG|$ of $BG$ is the corresponding 1-type. In fact they gave three different description for $BG$ called homogeneous, non-homogeneous and matrix description. They used these descriptions to get the explicit chain complex that computes the cohomology groups of $|BG|$. This was the born of the homology theory for algebraic objects. It turns out that non-homogeneous description of $BG$ is the most useful one. This description was used by Hochschild in [Hoc46] to define the Hochschild complex for an arbitrary associative algebra $A$ that coincides with the complex constructed by Eilenberg and Maclane when $A$ is a group algebra. It also inspired the definition of the nerve of small category and definition of Barr cohomology. In fact, it is difficult to image the modern mathematics without non-homogeneous description of $BG$.

In [Whi49] Whitehead showed that 2-types can be described by crossed modules of groups. Blakers constructed in [Blad48] for every crossed module of groups

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(A, G) the complex \(N^B(A, G)\) whose geometrical realization is the 2-type corresponding to \((A, G)\). In fact he has done this for arbitrary crossed complexes of groups that describe \(k\)-type for any \(k \in \mathbb{N}\). In the case of \(k = 1\) his description coincides with the matrix description of Eilenberg-Maclane for \(BG\).

In this article we give an explicit description of a simplicial set \(N(A, C)\) for an arbitrary crossed monoid \((A, C)\) in terms of certain matrices. This simplicial set is isomorphic to the one constructed by Blakers in case \((A, C)\) is a crossed module of groups \((A, G)\). The difference is that the elements of \(N_k(A, C)\) are described as collections of elements in \(A\) and \(G\) without any relations between them, however the elements of \(N^B_k(A, G)\) are described as collections of elements in \(A\) and \(G\) that should satisfy certain conditions between them.

The paper is organized as follows. In Section 2 we recall the definition of simplicial set and their elementary properties. Section 3 contains the main result of the paper. Namely, we describe the simplicial set \(N(A, C)\) for an arbitrary crossed monoid \((A, C)\). In Theorem 3.2 we prove that \(N(A, C)\) is indeed a simplicial set.

In Section 4 we prove that \(N(A, C)\) is 4-coskeletal. Moreover, in case \((A, C)\) is a crossed module of groups we also check that the homotopy groups of \((A, C)\) and \(N(A, C)\) are isomorphic in this case.

In the next version of this paper we shall give a comparison between our construction and the construction of Blakers [Bla48] and the construction of Moerdijk and Svensson [MS93].

2 Simplicial set

For the purpose of this paper a simplicial set is a sequence of sets \(X_n, n \geq 0\) with maps \(d_j : X_n \to X_{n+1}\) and \(s_j : X_n \to X_{n-1}\), \(0 \leq j \leq n\) such that for \(i < j\):

\[
\begin{align*}
    d_j d_k &= d_{k-1} d_j \\ 
    d_j s_k &= s_{k-1} d_j \\ 
    d_j s_j &= \text{id} \\ 
    d_{j+1} s_j &= \text{id} \\ 
    d_k s_j &= s_j d_{k-1} \\ 
    s_j s_{k-1} &= s_k s_j.
\end{align*}
\]

The \(n\)-truncated simplicial set is defined as a sequence of sets \(X_0, \ldots, X_n\) with the maps \(d_j : X_k \to X_{k-1}\), \(s_j : X_k \to X_{k+1}\) for all \(k\) and \(j\) they have sense, that satisfy the same identities as same-named maps for a simplicial set.

We denote the category of simplicial sets by \(\Delta^{op}Sets\) and the category of \(n\)-truncated simplicial sets by \(\Delta^{op}_nSets\). Then we have an obvious forgetful functor \(tr^n : \Delta^{op}Sets \to \Delta^{op}_nSets\). This functor has a right adjoint \(cosk^n : \Delta^{op}_nSets \to \Delta^{op}Sets\). The composition functor \(cosk^n tr^n\) will be denoted by \(\text{Cosk}^n\). Thus
**Proof.** Note that for every\( X \) in \[\text{Dus02}\] it is shown that these maps coincide with \( b_N \) \( N > n \) if for every \( \text{Proposition 2.1}. \) Let \( X \) be an isomorphism. Thus if \( X \) is an isomorphism. But now the set on the right hand side of (7) is \( (\text{Cosk}^n X)_N \) for a Kan complexes. \( \text{As n-th component of } \eta_X \text{ decomposes into the product of } \tau \text{ and } b_n \) we get that it is an isomorphism.

Define the set \( \bigwedge^n X \) of \( l \)-horns in dimension \( n \) to be the collection of \( n \)-tuples \((x_0, \ldots, x_l, \ldots, x_n)\) of elements in \( X_{n-1} \) such that \( d_j x_k = d_{k-1} x_j \) for all \( 0 \leq j < k \leq n - 1 \) different from \( l \). There are the natural maps

\[
b^n_l \colon X_n \longrightarrow \bigwedge_l^n X \quad x \mapsto (d_0 (x), \ldots, d_l (x), \ldots, d_n (x)).
\]

A complex \( X \) is said to be a Kan complex if the maps \( b^n_l \) are surjective for all \( 0 \leq l \leq n \). We define now based homotopy groups \( \pi_n(X, x) \) for a Kan complexes \( X \). We follow to the exposition of \[\text{Smil01}\] on the pages 27-28. Let \( x \in X_0 \). Then all the degenerations \( s_{i_1} \ldots s_{i_l} (x) \) of \( x \) in degree \( n \) are mutually equal and will be denoted by the same letter \( x \). We define \( \pi_n(X, x) \) to be the set

\[
\{ y \in X_n | b^n (y) = (x, \ldots, x) \}
\]
factorized by the equivalence relation

\[ y \sim z \iff \exists w \in X_{n+1}: b_n^{n+1}(w) = (x, \ldots, x, y, z). \]

That \( \sim \) is indeed an equivalence relations for a Kan set is shown at the end of page 27 of [Smi01]. Now we define a multiplication on \( \pi_n(X, x) \) as follows. Let \([y], [z] \in \pi_n(X, x)\) be equivalence classes containing \(y\) and \(z\), respectively. Then the tuple

\[ (x, \ldots, x, y, \varnothing, z) \]

is an element of \( \bigwedge^{n+1} \). Therefore there is an element \( w \in X_{n+1} \) such that \( b_n^{n+1}(w) = (x, \ldots, x, y, \varnothing, z) \). We define \([y][z] = [d_n(w)]\). Again it is shown in [Smi01], that this product is well defined and associative, \([x]\) is the neutral element, and if \( n \geq 2 \) the product is commutative.

There is a connection between coskeletal and Kan conditions for a simplicial set. To see this we start with

**Proposition 2.2.** Let \((x_0, \ldots, x_l, \ldots, x_n) \in \bigwedge^n X\). Then

\[(y_0, \ldots, y_{n-1}) = (d_{l-1}x_0, \ldots, d_{l-1}x_{l-1}, d_lx_{l+1}, \ldots, d_lx_n) \in \bigwedge^{n-1} X. \quad (8)\]

**Proof.** Suppose \(0 \leq j < k \leq l-1\). Then

\[ d_ky_j = d_k(d_{l-1}x_j) = d_kd_{l-1}x_j = d_{l-2}d_kx_j = d_{l-2}d_jx_{k+1} = d_j(d_{l-1}x_{k+1}) = d_jy_{k+1}. \]

For \(0 \leq j \leq l-1 < k \leq n-1\) we get

\[ d_ky_j = d_kd_{l-1}x_j = d_{l-1}d_kx_{k+1} = d_{l-1}d_jx_{k+2} = d_jd_{l-2}x_{k+2} = d_jy_{k+1}. \]

Finally for \( l-1 < j < k \leq n-1\) we have

\[ d_ky_j = d_kd_{l-1}x_{j+1} = d_{l-1}d_{k+1}x_{j+1} = d_{l-1}d_{j+1}x_{k+2} = d_jd_{l-2}x_{k+2} = d_jy_k. \]

Thus we have a well defined map \( \beta^n_l : \bigwedge^n X \to \bigwedge^{n-1} X \) given by (8).

As a simple corollary of Proposition 2.2 we get

**Corollary 2.3.** Suppose \( b_n \) and \( b_{n-1} \) are surjections. Then for every \(0 \leq l \leq n\) the maps \( \beta^n_l \) are surjections.

**Proof.** Let \( x = (x_0, \ldots, x_l, \ldots, x_n) \in \bigwedge^n X\). Then by Proposition 2.2

\[ \beta^n_l x = (d_{l-1}x_0, \ldots, d_{l-1}x_{l-1}, d_lx_{l+1}, \ldots, d_lx_n) \in \bigwedge^{n-1} X. \]

Since \( b_{n-1} \) is surjective there is \( x_l \in X_{n-1} \) such that \( d_jx_l = d_{l-1}x_j \) for \( 0 \leq j \leq l-1 \) and \( d_jx_l = d_lx_{j+1} \) for \( l \leq j \leq n-1 \). Therefore \((x_0, \ldots, x_n) \in \bigwedge^n X\) and since \( b_n \) is surjective there is \( z \in X_n \) such that \( d_jz = x_j, 0 \leq j \leq n \).
3 Category crossed monoids

Let C be a small category. We denote by C₀ the set of objects and by C₁ the set of morphisms of C. We will write s(α) for the source and t(α) for the target of the morphism α ∈ C₁. If F: C → Mon is a contravariant functor from C to the category of monoids, for α ∈ C(s, t) and m ∈ F(t) we write m^α for the result of applying F(α) to m.

A crossed monoid over C is a contravariant functor A: C → Mon together with a collection of functions ∂ₜ: A(t) → C(t, t), t ∈ C₀, such that

\[∂ₜ(a) = t\] (9)
\[α∂ₜ(α) = ∂ₜ(α)\] (10)
\[ab = ba^{∂ₜ b}\] (11)

for all s, t ∈ C₀, α ∈ C(s, t), a, b ∈ A(t). We will write eₓ for the unit of A(x), x ∈ C₀.

A morphism from a crossed module (A, C) to a crossed module (B, C̃) is a pair (f, F), where F: C → C̃ is a functor and f is a collection of homomorphisms fₓ: A(x) → B(F(x)) of monoids such that

\[fₓ(αᵃ) = fₓ(α)^F(x)\] (12)
\[F(∂ₜ(a)) = ∂F(∂ₜ(a))\] (13)

for all s, t ∈ C₀, α ∈ C(s, t), a ∈ A(t). We denote the category of crossed monoids over small categories by XMon. Note that XMon contains a full subcategory XMod of crossed modules whose objects (A, C) are such that C is a groupoid and A(t) is a group for every t ∈ C₀.

Now we describe the nerve functor N: XMon → △^opSets into the category of simplicial sets. Define N₀(A, C) = C₀. For n ≥ 1 we define Nₙ(A, G) to be the set of n × k upper triangular matrices M = (mᵢⱼ)ᵢ≤ⱼ such that there is a sequence x(M) = (x₀(M), . . . , xₙ(M)) of objects in C such that

- mᵢⱼ ∈ C(xᵢ, xⱼ), 1 ≤ j ≤ n;
- mᵢⱼ ∈ A(xᵢ) for 1 ≤ i < j ≤ n.

We will identify N₁(A, C) with C₁. We extend function x on N₀(A, cat) = C₀ by x(p) := (p).

Below we will sometimes indicate the empty places with the sign ∅.

Define s₀: N₀(A, C) by s₀(p) = 1_p, p ∈ C₀. For n ≥ 1 and 0 ≤ j ≤ n the matrix M ∈ Nₙ₊₁(A, C) will be constructed from M ∈ Nₙ(A, C) as follows

1. first insert eₓᵢ(M) at the (j + 1)-st place of every row i above the j + 1-st row;

1Upper triangular means that the places in the matrix under the diagonal are empty.
2. insert \((\emptyset, \ldots, \emptyset, 1_{x_j(M)}, e_{x_j(M)}, \ldots, e_{x_j(M)})\) as the \(j + 1\)-st row, where 
\(1_{x_{j+1}(M)}\) stay on the \((j + 1)\)-st place.

3. shift all elements below \((j + 1)\)-st row one position to the right.

**Example 3.1.** For \(M \in N_3(A, C)\), \(j = 1\), and \((x_0, x_1, x_2, x_3, x_4) = x(M)\) we get

\[
\begin{pmatrix}
m_{11} & m_{12} & m_{13} \\
m_{22} & m_{23} & m_{33}
\end{pmatrix}
\]  

\rightarrow

\[
\begin{pmatrix}
e_{x_1} & m_{12} & m_{13} \\
l_{x_1} & e_{x_1} & e_{x_1} \\
m_{22} & m_{23} & m_{33}
\end{pmatrix}
\]

\rightarrow

\[
\begin{pmatrix}
m_{11} & e_{x_1} & m_{12} & m_{13} \\
e_{x_1} & l_{x_1} & e_{x_1} & e_{x_1} \\
m_{22} & m_{23} & m_{33}
\end{pmatrix}
\].

Note that in the case \(j = 0\) the first step is skipped and in the case \(j = n\) the last step is skipped.

Now define \(d_0 : N_1(A, C) \to N_0(A, C)\) to be \(s : C_1 \to C_0\), and \(d_1 : N_1(A, C) \to N_0(A, C)\) to be \(t : C_1 \to C_0\). Let \(n \geq 2\) and \(M \in N_n(A, C)\). We construct the matrix \(d_j(M) \in N_{n-1}(A, C)\) as follows:

1. if \(j = 0\) we just delete the first row;
2. if \(j = n\) delete the last column;
3. if \(1 \leq j \leq n - 1\)
   
   (a) at every row above the \(j\)-th row we multiply elements at \(j\)-th and \((j + 1)\)-st places;
   
   (b) shift all the elements at \(j\)-th row and below one position to the left;
   
   (c) replace \(j\)-th and \((j + 1)\)-st rows with the row:

\[
(\emptyset, \ldots, \emptyset, m_{jj}\partial(m_{j,j+1})m_{j+1,j+1}, m_{j,j+2}^\eta_{j+1,j+1}m_{j+1,j+2}, \ldots, m_{j,n-1}^\eta_{j+1,n-1}m_{j+1,n}),
\]

where

\[
\eta_{jk} = m_{j+1,j+k}\partial(m_{j+1,j+k+2} \ldots m_{j+1,k}). \tag{14}
\]
For example

\[
\begin{pmatrix}
    m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\
    m_{22} & m_{23} & m_{24} & m_{25} \\
    m_{33} & m_{34} & m_{35} \\
    m_{44} & m_{45} \\
    m_{55}
\end{pmatrix}
\]

Steps (a) and (b)

\[
\begin{pmatrix}
    m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\
    m_{22} & m_{23} & m_{24} & m_{25} \\
    m_{33} & m_{34} & m_{35} \\
    m_{44} & m_{45} \\
    m_{55}
\end{pmatrix}
\]

Step (c)

\[
\begin{pmatrix}
    m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\
    m_{22} & m_{23} & m_{24} & m_{25} \\
    m_{33} & m_{34} & m_{35} \\
    m_{44} & m_{45} \\
    m_{55}
\end{pmatrix}
\]

**Theorem 3.2.** Let \((A, C)\) be a crossed monoid. The sequence of sets \(N_n (A, C)\) with the maps \(s_j, d_j\) defined above is a simplicial set.

**Proof.** We have to check that the maps \(d_j\) and \(s_j\) satisfy the simplicial identities. For a convenience we divide them into two groups. Let \(M \in N_n (A, C)\). In the first group we put the identities

\[
\begin{align*}
    d_j d_{j+1} (M) &= d_j^2 (M) \quad & d_j s_j (M) &= M \\
    s_{j+1} s_j (M) &= s_j^2 (M) \quad & d_{j+1} s_j (M) &= M \quad & d_j s_{j+1} (M) &= s_j d_j (M).
\end{align*}
\]

The rest of the identities

\[
\begin{align*}
    d_j d_k (M) &= d_{k-1} d_j (M) \quad & d_j s_k (M) &= s_{k-1} d_j (M) \\
    s_j s_{k-1} (M) &= s_{k-1} s_j (M) \quad & d_k s_j (M) &= s_j d_{k-1} (M),
\end{align*}
\]

where \(j < k - 1\), will be in the second group.

Note that the effect of action of all above maps on the \(i\)-th row of the matrix \(M\) for \(i < j\) is the same as the effect of action of the same named maps on the nerve of \(A (x_i (M))\). Therefore the equality of the matrices above the \(j\)-th row follows from the standard description of the nerve of monoid.
Now the matrices \( s_{j+1}s_j(M) = s_j^2(M) \) are equal strictly under the \((j+1)-\)st as this part is obtained by shifting the part of \( M \) under the \((j-1)-\)st row two positions in the south-east direction in both of them. Let \( x = x_j(M) \). The \( j \)-th row of \( s_{j+1}s_j(M) \) is obtained from the sequence \((\emptyset, \ldots, \emptyset, 1_x, e_x, \ldots, e_x)\) by inserting \( e_x \) after \( 1_x \) and thus coincides with the \( j \)-th row of \( s_j^2(M) \). Since \( x_{j+1}(s_j(M) = x) \) the \((j+1)-\)st row of \( s_{j+1}s_j(M) \) is the sequence \((\emptyset, \ldots, \emptyset, 1_x, e_x, \ldots, e_x)\) of the appropriate length. The \((j+1)-\)st row of \( s_j s_j(M) \) is equal to the \( j \)-th row of \( s_j(M) \) and thus is the same sequence. This shows that \( s_{j+1}s_j = s_j^2(M) \).

Now for the rest of matrices in the first group the part strictly below the \( j \)-th row is obtained by shifting the elements of \( M \) back and forth. It is not difficult to see that these shifts bring the same-named elements to the same positions in all four pairs of matrices.

Similarly the parts strictly below the \( j \)-th row in matrices of second group are obtained by applying the map with greater index and moving elements around. Again the same elements will be in the same places.

Thus we have only to check that the \( j \)-th rows are equal in every pair of matrices.

We start with the matrices of the second group. Thus from now on \( k - 1 > j \). In this case the \( j \)-th row of \( d_j d_k(M) \) is calculated from \( j \)-th and \((j+1)-\)st rows of \( d_k(M) \):

\[
\begin{align*}
    m_{j,j} & \quad m_{j,j+1} & \cdots & m_{j,km_{j,k+1}} & \cdots & m_{j,n} \\
    m_{j+1,j+1} & \cdots & m_{j+1,km_{j+1,k+1}} & \cdots & m_{j+1,n}.
\end{align*}
\]

Now the sequence of \( \eta \)'s defined by \((14)\) for the \((j+1)-\)st row of \( d_k(M) \) is

\[
(\eta_{j+1,j+1}, \ldots, \hat{\eta}_{j+1,k}, \ldots, \eta_{j+1,n}).
\]

Therefore the \( j \)-th row of \( d_j d_k(M) \) is

\[
(\emptyset, \ldots, \emptyset, m_{j,j}\partial(m_{j,j+1})m_{j+1,j+1}, m_{j,j+2}m_{j+1,j+2}, \ldots, m_{j,k}m_{j,k+1})^{\eta_j+1}_{j+1,k} m_{j+1,km_{j+1,k+1}, \ldots, m_{j,n}^{\eta_j+1,n-1}m_{j+1,n}}.
\]

Now the \( j \)-th row of \( d_{k-1}d_j(M) \) is obtained from the \( j \)-th row of \( d_j(M) \) by multiplying elements in the \((k-1)-\)st and \( k \)-th columns:

\[
\begin{align*}
    (m_{j,j}\partial(m_{j,j+1})m_{j+1,j+1}, m_{j,j+1}m_{j,j+2}m_{j+1,j+2}, \ldots, m_{j,k}m_{j,k+1})^{\eta_j+1}_{j+1,k-1} m_{j+1,km_{j+1,k+1}, \ldots, m_{j,n}^{\eta_j+1,n-1}m_{j+1,n}}.
    \end{align*}
\]

Thus the \( j \)-th rows of \( d_j d_k(M) \) and \( d_{k-1}d_j(M) \) are the same outside the \((k-1)-\)th column, where the most complicated looking elements are. By \((15)\) we get

\[
\begin{align*}
    m_{j,k}^{\eta_j+1,k-1} m_{j+1,km_{j+k+1}^{\eta_j+1,k-1}} m_{j+1,km_{j+1,k+1}^{\eta_j+1,k-1}} &= \\
    = m_{j,k}^{\eta_j+1,k-1} m_{j+1,km_{j+k+1}^{\eta_j+1,k-1}} m_{j+1,km_{j+1,k+1}^{\eta_j+1,k-1}} (15) \\
    = m_{j,k}^{\eta_j+1,k-1} m_{j+1,km_{j+1,k}^{\eta_j+1,k-1}} m_{j+1,km_{j+1,k}^{\eta_j+1,k-1}}.
\end{align*}
\]
This shows that \( d_j d_k (M) = d_{k-1} d_j (M) \).

Now we consider the pair of matrices \( s_j s_k (M) \) and \( s_k s_j (M) \). Denote \( x_j (M) \) by \( x \). The \( j \)-th row of \( s_j s_k (M) \) is the sequence \((1_x, e_x, \ldots, e_x)\) of the appropriate length. Now the \( j \)-th row of \( s_k s_j (M) \) is obtained from the similar sequence, which is shorter by one element, by inserting this missing element. Thus \( s_j s_k -1 (M) = s_k s_j (M) \).

The \( j \)-th row of \( d_j s_k (M) \) is obtained from \( j \)-th and \((j+1)\)-st rows of \( s_k (M) \):

\[
\begin{pmatrix}
\emptyset & \ldots & \emptyset & m_{j,j} & m_{j,j+1} & m_{j,j+2} & \ldots & e_x_j & \ldots & m_{j,n}
\end{pmatrix},
\]

where \( e \)'s are in the \((k+1)\)-st column. Since \( \partial (e_{x,j+1}) = 1_{x_{j+1}} \) it is immediate that the corresponding sequence of \( \eta \)'s has the form

\[
\eta_{j+1,j+1}, \ldots, \eta_{j+1,k-1}, \eta_{j+1,k}, \eta_{j+1,k+1}, \ldots, \eta_{j+1,n},
\]

that is it is obtained from the sequence of \( \eta \)'s for \( M \) by duplicating \( \eta_{j+1,k} \). Since \( e_{x,j}^{\prime} = e_{x,j} \) we see that the \( j \)-th row of \( d_j s_k (M) \) can be obtained from the \( j \)-th row of \( d_j (M) \) by inserting \( e_{x,j} \) at place \( k \). Thus the \( j \)-th row of \( d_j s_k (M) \) is equal to the \( j \)-th row of \( s_{k-1} d_j (M) \).

Further the \( j \)-row of \( s_j d_{k-1} (M) \) is a sequence of appropriate length

\[
(\emptyset, \ldots, \emptyset, 1_x, e_x, \ldots, e_x),
\]

where \( x = x_j (M) \). The \( j \)-th row of \( d_k s_j (M) \) is obtained from the one element longer sequence by multiplying two neighboring \( e_x \). As \( e_x^2 = e_x \) we get that \( d_k s_j (M) = s_j d_{k-1} (M) \).

It is left to consider the equalities in the first group. First we will show that the \( j \)-th rows of \( d_j d_{j+1} (M) \) and \( d_j d_j (M) \) are the same. First we consider the most left elements of these rows. For \( d_j d_{j+1} (M) \) it is equal to

\[
m_{j,j} \partial (m_{j,j+1} m_{j,j+2}) (m_{j+1,j+1} \partial (m_{j+1,j+2}) m_{j+2,j+2})
\]

and for \( d_j^2 (M) \):

\[
(m_{j,j} \partial (m_{j,j+1}) m_{j+1,j+1}) \partial (m_{j,j+2} m_{j+1,j+2}) m_{j+2,j+2}.
\]

These two elements are equal since

\[
\partial (m_{j,j+2}) m_{j+1,j+1} = m_{j+1,j+1} \partial (m_{j,j+2}^\prime)
\]

by (10). Now let \( l > j \). We will compute the element at the place \((j,l)\) in \( d_j d_{j+1} (M) \) and \( d_j^2 (M) \). First note that the sequence of \( \eta \)'s for the \((j+1)\)-st row of \( d_j (M) \) coincide with the sequence of \( \eta \)'s of the \((j+2)\)-nd row of \( M \).

Taking to the account shift of columns on two positions to the left the element of \( d_j^2 (M) \) at the place \((j,l)\) is

\[
\left( m_{j,j+2} m_{j+1,l+2} \right)^{\eta_{j+2,l+1}} m_{j+2,l+2}.
\]
To compute the corresponding element in $d_j d_{j+1} (M)$ we have to find
\[
\eta_{j+1,t} (d_{j+1} (M)) = d_{j+1} (M)_{j+1,j+1} \partial (d_{j+1} (M)_{j+1,j+2} \cdots d_{j+1} (M)_{j+1,l}) \\
= m_{j+1,j+1} \partial (m_{j+1,j+2} m_{j+2,j+2} \\
\times \partial (m_{j+2,j+3} \cdots m_{j+2,l+1})).
\]
Now iterating (15) we can write the product under the $\partial$ as
\[
m_{j+2,j+2} m_{j+2,j+3} \cdots m_{j+2,l+1} = (m_{j+1,j+3} \cdots m_{j+1,l+1}) m_{j+2,j+2} m_{j+2,j+3} \cdots m_{j+2,l+1}.
\]
Since by (10)
\[
m_{j+2,j+2} \partial ((m_{j+1,j+3} \cdots m_{j+1,l+1}) m_{j+2,j+2}) = \\
\partial (m_{j+1,j+3} \cdots m_{j+1,l+1}) m_{j+2,j+2}
\]
we get
\[
\eta_{j+1,t} (d_{j+1} (M)) = m_{j+1,j+1} \partial (m_{j+1,j+2} m_{j+1,j+3} \cdots m_{j+1,l+1}) \\
\times m_{j+2,j+2} \partial (m_{j+2,j+3} \cdots m_{j+2,l+1}) \\
= \eta_{j+1,t+1} \eta_{j+2,t+1}.
\]
Therefore the $(j, l)$-th element of $d_j d_{j+1} (M)$ is
\[
d_{j+1} (M)_{j+1,l+1}^{\eta_{j+1,t+1} \eta_{j+2,t+2}} d_{j+1} (M)_{j+1,l+1} = m_{j,l+1}^{\eta_{j+1,t+1} \eta_{j+2,t+2}} m_{j,l+2}^{\eta_{j+2,t+2}} m_{j+2,l+2};
\]
which is equal to (16). Therefore $d_j d_{j+1} (M) = d_j^2 (M)$.

Now the $j$-th row of $d_j s_j (M)$ is obtained from the $j$-th and $(j + 1)$-st rows of $s_j (M)$:
\[
\begin{pmatrix}
\varnothing & \cdots & \varnothing & m_{j,j} & e_x & m_{j,j+1} & \cdots & m_{j,n} \\
\varnothing & \cdots & \varnothing & 1_x & e_x & \cdots & e_x
\end{pmatrix},
\]
where $x = x_j (M)$. We see that the corresponding sequence of $\eta$’s consist from $1_x$ repeated the required number of times. Now
\[
m_{j,j} \partial (e_x) 1_x = m_{j,j} \\
m_{j,j}^1 e_x = m_{j,l} \\
l \geq j + 1.
\]
Therefore $d_j s_j (M) = M$.

The $j$-th row of $d_j s_{j-1} (M)$ is obtained from the $j$-th and $(j + 1)$-st rows of $s_{j-1} (M)$:
\[
\begin{pmatrix}
\varnothing & \cdots & \varnothing & 1_x & e_x & \cdots & e_x \\
\varnothing & \cdots & \varnothing & m_{j,j} & m_{j,j+1} & \cdots & m_{j,n}
\end{pmatrix},
\]
where \( x = x_{j-1}(M) \). The required sequence of \( \eta \)'s is the \( j \)-th sequence of \( \eta \)'s for \( M \). Now

\[
1_x \partial (e_x) m_{j,j} = m_{j,j} \\
e_x m_{j,l} = e_x = m_{j,l} \quad \text{for } l > j.
\]

Therefore \( d_j s_{j-1}(M) = M \).

Finally we consider the \( j \)-th row of \( d_j s_{j+1}(M) \) and \( s_j d_j(M) \). The \( j \)-th row of the second matrix is obtained from the \( j \)-th row of \( d_j(M) \) by inserting \( e_x, x = x_j(d_j(M)) = x_{j+1}(M) \), at the place \( j + 1 \). The \( j \)-th row of \( d_j s_{j+1}(M) \) is obtained from the \( j \)-th and \((j+1)\)-st rows of \( s_{j+1}(M) \):

\[
\left( \emptyset \ldots \emptyset \ m_{j,j} \ m_{j,j+1} \ e_x \ m_{j,j+2} \ldots m_{j,n} \\
\emptyset \ldots \emptyset \emptyset \ m_{j+1,j+1} \ e_{x_{j+1}} \ m_{j+1,j+2} \ldots m_{j+1,n} \right)
\]

We see that the corresponding sequence of \( \eta \)'s is obtained from the \((j+1)\)-st sequence of \( \eta \)'s for \( M \) by repeating \( \eta_{j+1,j+1} \) twice. It is straightforward not that the \( j \)-th row of \( d_j s_{j+1}(M) \) is obtained from the \( j \)-th row of \( d_j(M) \) by inserting \( e_{x_{j+1}} \) at the place \( j + 1 \). Thus \( d_j s_{j+1}(M) = s_j d_j(M) \). \(\square\)

4 Coskeletal property

In this section we investigate coskeletality of \( N(A,C) \) for a given crossed monoid \((A,C)\). For every \( n \geq 2 \) we denote by \( \tilde{N}_n(A,C) \) the set of triples \((M^0, M^n, m)\), where \( M^0, M^n \in N_{n-1}(A,C), m \in A(s(m^0_{1,1})) \) are such that \( d_{n-1}M^0 = d_0M^n \).

We have an obvious map

\[
\lambda_n : N_n(A,C) \to \tilde{N}_n(A,C) \\
M \mapsto (d_0M, d_n M, m_{1,1}).
\]

The map is a bijection and we will denote the inverse of \( \lambda_n \) by \( \mu_n \). The following picture explains how to construct \( \mu_n(M^0, M^n, m) \in N_n(A,C) \) for \((M^0, M^n, m) \in \tilde{N}_n(A,C)\):

\[\mu(M^0, M^n, m) = \]

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Now we investigate the effect of applying $d_j$ to $\mu_n (M^0, M^n, m)$. For $j = 0$ and $j = n$ we have by definition
\[
d_0 \mu_n (M^0, M^n, m) = M^0 \\
d_n \mu_n (M^0, M^n, m) = M^n.
\]
Now for $1 \leq j \leq n - 1$:
\[
d_0 d_j \mu_n (M^0, M^n, m) = d_{j-1} d_0 \mu (M^0, M^n, m) = d_{j-1} M^0 \\
d_{n-1} d_j \mu_n (M^0, M^n, m) = d_j d_{n-1} \mu (M^0, M^n, m) = d_j M^n.
\]
Therefore $d_j \mu_n (M^0, M^n, m) = \mu_{n-1} (d_{j-1} M^0, d_j M^n, m')$, where $m'$ is the element at the north-east corner of $d_j (M^0, M^n, m)$. If $2 \leq j \leq n - 2$, then $m' = m$. For $j = 1, n - 1$ it looks more complicated. Namely, for $j = 1$ we get
\[
m' = m_{1,i}^o \beta (m_{1,2}^o \ldots m_{1,n-2}^o) m_{1,n-1}^0
\]
and for $j = n - 1$
\[
m' = m_{1,n-1}^n m,
\]
where $m_{1,j}^o$ and $m_{1,j}^n$ are the entries of $M^0$ and $M^n$, respectively.

**Theorem 4.1.** Let $(A, C)$ be a crossed monoid. Then the simplicial set $N(A, C)$ is 4-coskeletal.

**Proof.** Let $n \geq 5$. We have to check that $b^n : N_n (A, C) \rightarrow \bigwedge^n N(A, C)$ is a bijection. Define the map $\nu_n : \bigwedge^n N(A, C) \rightarrow \tilde{N}_n (A, C)$ by
\[
\nu_n : (M^0, \ldots, M^n) \mapsto (M^0, M^n, m_{1,n-1}^2),
\]
where $m_{1,n-1}^2$ is the element of $M^2$ at the upper-right corner. We get a commutative triangle
\[
\begin{array}{ccc}
N_n (A, C) & \xrightarrow{b_n} & \bigwedge^n N(A, C) \\
\downarrow{\lambda_n} & & \downarrow{\nu_n} \\
\tilde{N}_n (A, C). & & \\
\end{array}
\]
Since $\lambda_n$ is a bijection it follows that $b_n$ is injective. Now for $(M^0, \ldots, M^n) \in \bigwedge^n N(A, C)$ we define $M = \mu_n \nu_n (M^0, \ldots, M^n)$. We claim that $b_n (M) = (M^0, \ldots, M^n)$. In fact
\[
d_0 M = d_0 \mu_n (M^0, M^n, m_{1,n-1}^2) = M^0 \\
d_n M = d_n \mu_n (M^0, M^n, m_{1,n-1}^2) = M^n \\
d_j M = d_j \mu_n (M^0, M^n, m_{1,n-1}^2) = \mu_{n-1} (d_{j-1} M^0, d_j M^n, m') \\
= \mu_{n-1} (d_0 M^j, d_{n-1} M^j, m').
\]
If $2 \leq j \leq n - 2$, then $m' = m^2_{1, n-1}$. If $j = 2$ then
\[ d_2 M = \mu_{n-1} (d_0 M^2, d_{n-1} M^2, m^2_{1, n-1}) = \mu_{n-1} \lambda_{n-1} M^2 = M^2. \]

If $3 \leq j \leq n - 2$, then the element of $d_{j-1} M^2$ at the right-upper corner is the same as for $M^2$, and similarly for $d_2 M^j$ and $M^{j-1}$. As $d_j M^2 = d_2 M^{j-1}$ we get $m^2_{1, n-1} = m^j_{1, n-1}$ and therefore
\[ d_j (M) = \mu_{n-1} \left( d_0 M^j, d_{n-1} M^j, m^j_{1, n-1} \right) = M^j. \]

For $j = n - 1$ the element $m'$ in (20) is $m^n_{1, n-1} m^2_{1, n-1}$ by (18). Now
\[ m^n_{1, n-1} m^2_{1, n-1} = (d_2 M^n)_{1, n-2} m^2_{1, n-1} \]
\[ = (d_{n-1} M^2)_{1, n-2} m^2_{1, n-1} \]
\[ = m^2_{1, n-2} m^2_{1, n-1} \]
\[ = (d_{n-2} M^2)_{1, n-2} = (d_2 M^{n-1})_{1, n-2} = m^{n-1}_{1, n-1}. \]

Note that in the first step we used $n - 1 \geq 4$ which is equivalent to our assumption $n \geq 5$. Combining with (20) we get
\[ d_{n-1} M = \mu_{n-1} (d_0 M^{n-1}, d_{n-1} M^{n-1}, m^{n-1}_{1, n-1}) = M^{n-1}. \]

For $j = 1$ the element $m'$ in (20) is given by (17):
\[ m' = (m^2_{1, n-1})^0_{1, i-1} \partial (m^2_{1, n-1})^0_{1, i-1} m^0_{1, n-1}. \quad (21) \]

We have to show that this product is equal to $m^1_{1, n-1}$. We have
\[ m^1_{1, n-1} \xrightarrow{n-1 \geq 4} (d_2 M^1)_{1, n-2} = (d_1 M^3)_{1, n-2} \]
\[ = (m^3_{1, n-1})^0_{1, i-1} \partial (m^3_{1, n-1})^0_{1, i-1} m^3_{2, n-1}. \]

This formula already looks similar to (21). It is only left to identify the elements in both formulas. For $2 \leq i \leq n - 1$ we have
\[ m^3_{2, i} = (d_0 M^3)_{1, i-1} = (d_2 M^0)_{1, i-1} = \begin{cases} m^0_{11}, & i = 2 \\ m^0_{12} m^0_{13}, & i = 3 \\ m^0_{1i}, & i \geq 4. \end{cases} \]

In particular
\[ m^3_{22} \partial (m^3_{23} \ldots m^3_{2, n-2}) = m^0_{11} \partial (m^0_{12} \ldots m^0_{1, n-2}). \]

Thus it is left to show $m^3_{1, n-1} = m^2_{1, n-1}$. This follows from
\[ m^3_{1, n-1} = (d_2 M^3)_{1, n-2} = (d_2 M^2)_{1, n-2} = m^2_{1, n-1}. \]

Finally $d_{n-1} M = \mu_{n-1} (d_0 M^{n-1}, d_{n-1} M^{n-1}, m^{n-1}_{1, n-1}) = M^{n-1}$. □
Theorem 4.2. Let \((A, C)\) be a crossed monoid such that

- for every object \(t \in C\) the monoid \(A(t)\) has left and right cancellation properties;
- for every morphism \(\gamma \in C\) the map \(a \rightarrow a^\gamma\) from \(A(t(\gamma))\) to \(A(s(\gamma))\) is injective.

Then \(N(A, C)\) is 3-coskeletal

Remark 4.3. Note that crossed modules satisfy the conditions of the theorem.

Proof. We already saw in Theorem 4.1 that \(N(A, C)\) is 4-coskeletal. Therefore it is enough to show that \(b_4 : N(A, C) \rightarrow \bigwedge^4 N(A, C)\) is a bijection. We define the map \(\nu_4 : \bigwedge^4 N(A, C) \rightarrow \tilde{N}_4(A, C)\) by (19). Then \(\lambda_4 = \nu_4 b_4\) is a bijection. Therefore \(b_4\) is injective. For \((M^0, \ldots, M^4)\) we define \(M = \mu_4 \nu_4 (M^0, \ldots, M^4)\).

In the same way as in the proof of Theorem 4.1 we get \(d_0 M = M^0, d_4 M = M^4\) and \(d_2 M^2 = M^2\). Now by (20) and (18) we get

\[
d_3 M = \mu_3 (d_0 M^3, d_1 M^3, m^4_{13} m^2_{13}).
\]

To get \(d_3 M = M^3\) we have to show that \(m^4_{13} m^2_{13} = m^3_{13}\). We have the following equalities

\[
m^3_{12} m^3_{13} = (d_2 M^3)_{12} = (d_2 M^2)_{12} = m^2_{12} m^2_{13} \quad (22)
\]
\[
m^3_{12} = (d_3 M^3)_{12} = (d_3 M^4)_{12} = m^4_{12} \quad (23)
\]
\[
m^2_{12} = (d_3 M^2)_{12} = (d_2 M^4)_{12} = m^4_{12} m^4_{13}. \quad (24)
\]

Therefore

\[
m^3_{12} m^3_{13} \overset{22}{=} m^2_{12} m^2_{13} \overset{23}{=} m^4_{12} m^4_{13} \overset{23}{=} m^3_{12} m^4_{13} m^2_{13}
\]

and by left cancellation for \(A(x_1 (M))\) we get \(m^4_{13} = m^4_{13} m^2_{13}\) as required.

Now by (20) and (17) we get

\[
d_1 M = \mu_{n-1} \left( d_0 M^1, d_3 M^1, (m^2_{13})^{m^0_{12} \circ (m^0_{12})} m^0_{13} \right).
\]

To prove \(d_1 M = M^1\) it is enough to check that \(m^1_{13} = (m^2_{13})^{m^0_{12} \circ (m^0_{12})} m^0_{13}\). We have the equalities

\[
(m^1_{13})^{m^2_{12} m^3_{23}} = (d_1 M^1)_{12} = (d_1 M^2) = (m^2_{13})^{m^2_2 m^2_{23}} \quad (25)
\]
\[
m^2_{23} = (d_0 M^2)_{12} = (d_1 M^0)_{12} = (m^0_{13})^{m^0_0 m^0_{23}} \quad (26)
\]
\[
m^1_{23} = (d_0 M^1)_{12} = (d_0 M^0)_{12} = m^0_{23} \quad (27)
\]
\[
m^2_{22} = (d_0 M^2)_{11} = (d_1 M^0)_{11} = m^0_{11} \circ (m^0_{12}) m^0_{22} \quad (28)
\]
\[
m^1_{22} = (d_0 M^1)_{11} = (d_0 M^0)_{11} = m^0_{22}. \quad (29)
\]
We get
\[
\left( m^{1}_{13} \right)^{m_{0}^{22}} m_{0}^{13} \quad \left( m^{2}_{13} \right)^{m_{2}^{22}} m_{2}^{13} = \left( m^{1}_{13} \right)^{m_{0}^{22}} m_{0}^{13}
\]
Now from the right cancellation property for \( A(x_{1}(M)) \) and injectivity of the action of \( C \) we obtain
\[
m_{13}^{1} = \left( m^{2}_{13} \right)^{m_{0}^{12}} m_{0}^{13}
\]
as required. \( \square \)

5 Kan property

Recall that the nerve \( N(C) \) of a category \( C \) is a Kan simplicial set if and only if \( C \) is a groupoid. In this section we prove that the nerve \( N(A,C) \) of a crossed monoid \( (A,C) \) is a Kan complex if and only if \( (A,C) \) is a crossed module.

Suppose \( (A,C) \) is crossed module. Then by Proposition 4.2 the set \( N(A,C) \) is 3-coskeletal. Therefore for \( n \geq 5 \) by Corollary 2.3 the maps \( b_{n}^{j} \) are surjective. Now \( N(C) \) can be embedded into \( N(A,C) \) by putting appropriate units over the diagonal. At levels 0 and 1 this embedding is a bijection. As \( N(C) \) is a Kan complex we get that the Kan condition holds for \( N(A,C) \) at degrees 0 and 1. Moreover, \( N(C) \to N(A,C) \) induces the isomorphisms between sets \( \Lambda^{2}_{N}(C) \) and \( \Lambda^{2}_{N}(A,C) \), \( 0 \leq j \leq 2 \). As \( N_{2}(C) \) is a subset of \( N_{2}(A,C) \) and the restriction of \( b_{j}^{2}: N_{2}(A,C) \to \Lambda^{2}_{N}(A,C) \) to \( N_{2}(C) \) coincide with \( b_{j}^{2}: N_{2}(C) \to \Lambda^{2}_{N}(C) \) we get the Kan property at level 2. Thus only the surjectivity of maps \( b_{j}^{3}: N_{3}(A,C) \to \Lambda^{3}_{N}(A,C) \), \( 0 \leq j \leq 3 \), and \( b_{j}^{4}: N_{4}(A,C) \to \Lambda^{4}_{N}(A,C) \), \( 0 \leq j \leq 4 \), should be checked in order to show that \( N(A,C) \) is a Kan simplicial set.

To deal with this problem we start by description of the image of \( b_{3}: N_{3}(A,C) \to \Lambda^{3}_{N}(A,C) \).

**Proposition 5.1.** Let \( (A,C) \) be a crossed module. Then \( (M^{0}, M^{1}, M^{2}, M^{3}) \) from \( \Lambda^{3}_{N}(A,C) \) lies in the image of \( b^{3} \) if and only if
\[
\left( m^{3}_{12} \right)^{m_{2}^{22}} m_{12}^{1} = \left( m_{12}^{2} \right)^{m_{0}^{22}} m_{12}^{0}.
\]

**Proof.** The only if part is true for an arbitrary crossed monoid \( (A,C) \). In fact,
let \( M \in N_3(A, C) \), then

\[
b_3(M) = \left( \begin{array}{cc}
m_{22} & m_{23} \\
m_{33} & \end{array} \right), \left( \begin{array}{cc}
m_{11}\partial(m_{12})m_{22} & (m_{13})m_{22}m_{23} \\
m_{33} & \end{array} \right),
\left( \begin{array}{cc}
m_{11}m_{12}m_{13} & m_{22}\partial(m_{23})m_{33} \\
m_{33} & \end{array} \right), \left( \begin{array}{cc}
m_{11} & m_{12} \\
m_{22} & \end{array} \right)
\]

Therefore

\[
((d_3M)_{12})^{(d_3M)_{22}}(d_1M)_{12} = (m_{12})^{m_{22}}(m_{13})^{(m_{22})m_{23}}
\]

\[
= (m_{12}m_{13})^{m_{22}m_{23}} = ((d_2M)_{12})^{(d_3M)_{22}}(d_0M)_{12}.
\]

Now suppose that \((A, C, d)\) is a crossed module and \((M^0, \ldots, M^3) \in \Lambda^3 N(A, C)\) satisfies (30). We define \(M := \mu_3\left(M^0, M^3, (m_{12}^3)^{-1}m_{12}^2\right)\). Then \(d_0M = M^0\) and \(d_3M = M^3\). Moreover, by (20) and (17)

\[
d_1M = \mu_2\left(d_0M^1, d_2M^1, (m_{12}^3)^{-1}m_{12}^2\right)^{m_{11}^0}m_{12}^0
\]

Since \(m_{11}^0 = d_2M^0 = d_0M^3 = m_{22}^3\) we get

\[
\left((m_{12}^3)^{-1}m_{12}^2\right)^{m_{11}^0}m_{12}^0 = \left((m_{12}^3)^{-1}m_{12}^2\right)^{m_{22}^0}m_{12}^0 = m_{11}^0
\]

and therefore \(d_1M = \mu_2\left(d_0M^1, d_2M^1, m_{12}^1\right) = M^1\). Now by (20) and (18)

\[
d_2M = \mu_2\left(d_0M^2, d_2M^2, m_{12}^3\left(m_{12}^3\right)^{-1}m_{12}^2\right) = \mu_2\left(d_0M^2, d_2M^2, m_{12}^2\right) = M^2.
\]

Now we can handle Kan property at level 3 of \(N(A, C)\).

**Proposition 5.2.** Let \((A, C)\) be a crossed module. Then for \(0 \leq j \leq 3\) the maps \(b^3_j: N(A, C) \to \Lambda^3 N(A, C)\) are surjective.

**Proof.** For every \(0 \leq j \leq 3\) and \(M^* \in \Lambda^3 N(A, C)\) we will construct \(M^j \in N_2(A, C)\) that extends \(M^*\) to \(\left(M^0, M^1, M^2, M^3\right) \in \text{Im}(b^3)\). The diagonal elements of \(M^j\) are determined from the equalities

\[
m^j_{11} = d_2M^j = \begin{cases}
d_jM^3 & j < 3 \\
d_2M^2 & j = 3
\end{cases}
\]

\[
= \begin{cases}
m_{22}^3 & j = 0 \\
(m_{12}^3)\partial(m_{12}^3)m_{22}^3 & j = 1 \\
m_{11}^3 & j = 2 \\
m_{11}^3 & j = 3
\end{cases}
\]

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\[ m_{22}^j = d_0 M^j = \begin{cases} d_0 M^1 & j = 0 \\ d_{j-1} M^0 & j > 0 \end{cases} = \begin{cases} m_{22}^1 & j = 0 \\ m_{22}^0 & j = 1 \\ m_{11}^0 \partial (m_{12}^0) m_{22}^0 & j = 2 \\ m_{11}^0 & j = 3. \end{cases} \]

The element at the right upper corner of \( M^j \) is uniquely determined from (30). The care should be taken for \( j = 3 \): in this case we replace \( m_{22}^3 \) in (30) by \( m_{11}^0 \). We automatically get that

\[ d_0 M^j = \begin{cases} d_0 M^1 & j = 0 \\ d_{j-1} M^0 & j > 0 \end{cases} \quad d_2 M^j = \begin{cases} d_2 M^3 & j < 3 \\ d_2 M^2 & j = 3. \end{cases} \]

Thus we have only to check that

\[ d_1 M^j = \begin{cases} d_1 M^2 & j \leq 1 \\ d_{j-1} M^1 & j \geq 2. \end{cases} \]

Below is the required computation. For \( j = 0 \) we have:

\[ d_1 M^0 = m_{11}^0 \partial (m_{12}^0) m_{22}^0 = m_{22}^3 \partial \left( \left( (m_{12}^0)^{-1} m_{12}^3 \right)^{m_{22}^3} m_{12}^1 \right) m_{22}^1 \]

\[ = \partial \left( (m_{12}^0)^{-1} \right) \partial (m_{12}^1) m_{22}^3 \partial (m_{12}^1) m_{22}^1 \]

\[ = \partial \left( (m_{12}^0)^{-1} \right) \left( m_{11}^0 \right)^{-1} (d_1 M^3) \partial (m_{12}^1) m_{22}^1 \]

\[ = \partial \left( (m_{12}^0)^{-1} \right) \left( d_2 M^2 \right)^{-1} m_{11}^0 \partial (m_{12}^1) m_{22}^1 \]

\[ = \partial \left( (m_{12}^0)^{-1} \right) (m_{11}^0)^{-1} (d_1 M^1) = \partial \left( (m_{12}^0)^{-1} \right) (m_{11}^0)^{-1} (d_1 M^2) \]

\[ = m_{22}^2 = d_0 M^2. \]

For \( j = 1 \) we get

\[ d_1 M^1 = m_{11}^1 \partial (m_{12}^1) m_{22}^1 = m_{11}^1 \partial (m_{12}^3) m_{22}^3 \partial \left( \left( (m_{12}^0)^{-1} m_{12}^3 \right)^{m_{22}^3} m_{12}^0 \right) m_{22}^0 \]

\[ = m_{11}^3 \partial \left( m_{12}^0 \right)^{-1} m_{12}^1 m_{22}^3 \partial (m_{12}^0) m_{22}^0 \]

\[ = m_{11}^0 \partial (m_{12}^0) m_{22}^0 \partial (m_{12}^0) m_{22}^0 = m_{11}^2 \partial (m_{12}^2) (d_1 M^0) \]

\[ = m_{11}^2 \partial (m_{12}^2) (d_0 M^2) = m_{11}^2 \partial (m_{12}^2) m_{22}^2 = d_1 M^2. \]
For \( j = 2 \) we replace \( m_{32}^3 \) in (30) by \( m_{11}^0 \) and obtain
\[
d_1 M^2 = m_{11}^2 \partial (m_{12}^2) m_{22}^2 \\
= m_{11}^3 \partial \left( m_{12}^3 \left( m_{12}^0 (m_{12}^0)^{-1} \right)^{m_{11}^0} \right) m_{11}^0 \partial (m_{12}^0) m_{22}^0 \\
= m_{11}^3 \partial (m_{12}^3) m_{11}^0 \partial \left( m_{12}^0 (m_{12}^0)^{-1} \right) \partial (m_{12}^0) m_{22}^0 \\
= m_{11}^3 \partial (m_{12}^3) m_{11}^0 \partial (m_{12}^0) m_{02} = (d_1 M^3) \partial (m_{12}^0) m_{22}^0 \\
= (d_0 M^1) \partial (m_{12}^1) m_{22}^1 = m_{11}^1 \partial (m_{12}^1) m_{22}^1 = d_1 M^1.
\]

Finally for \( j = 3 \) we get
\[
d_1 M^3 = m_{11}^3 \partial (m_{12}^3) m_{22}^3 = m_{11}^2 \partial \left( m_{12}^2 \left( m_{12}^0 (m_{12}^0)^{-1} \right)^{m_{11}^0} \right) m_{11}^0 \\
= m_{11}^2 \partial (m_{12}^2) m_{11}^0 \partial \left( m_{12}^0 (m_{12}^0)^{-1} \right) \\
= (d_1 M^2) \left( m_{22}^2 \right)^{-1} (d_1 M^0) \left( m_{22}^0 \right)^{-1} \partial \left( m_{12}^0 \right)^{-1} \\
= (d_1 M^1) \left( d_0 M^2 \right)^{-1} (d_0 M^0) \left( d_0 M^0 \right)^{-1} \partial \left( m_{12}^0 \right)^{-1} \\
= m_{11}^0 \partial (m_{12}^1) m_{22}^1 \left( m_{22}^1 \right)^{-1} \partial \left( m_{12}^1 \right)^{-1} = m_{11}^2 = d_2 M^1.
\]

\( \square \)

Now we check the Kan condition at the level 4.

**Proposition 5.3.** Let \((A, C)\) be a crossed module. Then for all \( 0 \leq j \leq 4 \) the map \( b_j^4 : N_4 (A, C) \to \Lambda_j^4 N (A, C) \) is surjective.

**Proof.** We know by Proposition 4.2 that \( b_4 : N_4 (A, C) \to \Lambda_4^4 N (A, C) \) is surjective. Thus if we show that for every \( 0 \leq j \leq 4 \) any \( M^* \in \Lambda_j^4 N (A, C) \) can be extended by \( M^j \in N_4 (A, C) \) to an element of \( \Lambda_j^4 N (A, C) \), the proposition will be proved. The existence of such \( M^j \) is equivalent to \( \beta_j^4 M^* \in \text{Im} \left( b^4 \right) \), where \( \beta_j^4 : \Lambda_j^4 N (A, C) \to \Lambda_j^4 N (A, C) \) is defined on page 41. Therefore by Proposition 5.4 we have to check that (30) holds for \( \beta_j^4 M^* \).

Before doing this let us introduce some notation. We define the elements
\[
m_{22} := m_{11}^0 = m_{32}^3 = m_{22}^4 \\
m_{23} := m_{12}^0 = m_{43}^4 \\
m_{33} := m_{12}^0 = m_{33}^4.
\]

This definitions should be understand in a way that left hand side element is defined to be any of available element in \( M^* \) on the right hand side. Moreover, if more then one element on the right hand side is available then all choices give the same result. The last assertion follows from \( M^* \in \Lambda_4^4 N (A, C) \).
Now we define the matrix $W = (w_{st})_{s,t=0}^4$ to be

\[
\begin{pmatrix}
\emptyset & m_{23}^1 & m_{23}^2 & m_{23}^3 & m_{23}^4 \\
m_{23}^0 & \emptyset & m_{13}^2 & m_{23}^2 & m_{23}^3 \\
(\beta^1_{13})_{m_{13}^0} & m_{13}^0 & m_{13}^2 & \emptyset & m_{13}^3 \\
m_{12}^0 m_{13}^0 & m_{12}^1 m_{13}^1 & m_{12}^2 m_{13}^3 & \emptyset & m_{12}^4 \\
m_{12}^0 & m_{12}^1 & m_{12}^2 & m_{12}^3 & \emptyset \\
\end{pmatrix}
\]

or in other terms

\[
w_{st} = \begin{cases} (d_{s-1}M^t)_{12} & s < t \\
(d_{s}M^t)_{12} & t > s. \end{cases}
\]

For a given $j$ and $M^* \in \bigwedge^j N(A, C)$ only the elements outside of $j$-th column of $W$ are defined. Moreover, the $j$-th row of $W$ gives the upper-corner elements of the matrices $\beta^j_j M^*$ and the relation between them equivalent to (30) can be read off from the $j$-th column. It follows from $M^* \in \bigwedge^j N(A, C)$ that if we remove $j$-th column and $j$-th row from $W$ then the resulting matrix is symmetric. We will use this fact in the computations below.

Now for $j = 0$ we have to check that $w_{02}w_{01}^{-1} = (w_{04}^{-1} w_{03})^{m_{33}}$. We have

\[
w_{01} = \left( w_{31}^{-1} w_{41} \right)^{m_{33}} w_{21} \quad w_{03} = \left( w_{23}^{-1} w_{43} \right)^{m_{22}} w_{13} \\
w_{02} = \left( w_{32}^{-1} w_{42} \right)^{m_{22} \beta(m_{23}) m_{33}} w_{12} \quad w_{04} = \left( w_{24}^{-1} w_{34} \right)^{m_{22}} w_{14}.
\]

Moreover $m_{23} = m_{43}^4 = w_{04}$ and for any $a \in A(t)$ holds $a^{\beta(w_{04})} = w_{04}^{-1} a w_{04}$. Since $W$ is symmetric we get

\[
w_{04}^{-1} w_{03} = w_{14}^{-1} \left( w_{34}^{-1} w_{24} w_{13}^{-1} w_{33} \right)^{m_{22}} w_{13} \\
w_{03} w_{01}^{-1} = \left( w_{14}^{-1} \right)^{m_{33}} w_{21} = \left( w_{34}^{-1} w_{24} w_{13}^{-1} w_{33} \right)^{m_{33}} w_{21} = \left( w_{14}^{-1} \right)^{m_{33}} w_{34} w_{24} w_{13}^{-1} w_{33} = \left( w_{14}^{-1} \right)^{m_{33}}.
\]

For $j = 1$ we have to check that $w_{12}w_{10}^{-1} = (w_{14}^{-1} w_{13})^{m_{33}}$. We have

\[
w_{10} = \left( w_{30}^{-1} w_{40} \right)^{m_{33}} w_{20} \quad w_{13} = \left( w_{43}^{-1} w_{23} \right)^{m_{22}} w_{03} \\
w_{12} = \left( w_{42}^{-1} w_{32} \right)^{m_{22} \beta(m_{23}) m_{33}} w_{02} \quad w_{14} = \left( w_{34}^{-1} w_{24} \right)^{m_{22}} w_{04}.
\]

Therefore taking into account that $m_{23} = m_{04}$ we get

\[
w_{14}^{-1} w_{13} = w_{04}^{-1} \left( w_{24}^{-1} w_{34} w_{43}^{-1} w_{23} \right)^{m_{22}} w_{03} = w_{04}^{-1} \left( w_{24}^{-1} w_{23} \right)^{m_{22}} w_{03} \\
w_{12} w_{10}^{-1} = \left( w_{04}^{-1} \right)^{m_{33}} w_{02} w_{01}^{-1} \left( w_{40}^{-1} w_{30} \right)^{m_{33}} w_{01}^{-1} \left( w_{40}^{-1} w_{30} \right)^{m_{33}} = \left( w_{14}^{-1} \right)^{m_{33}}.
\]
For $j = 2$ we have to check that $w_{21}w_{20}^{-1} = (w_{24}^{-1}w_{23})^{m_{22}\partial(m_{23})m_{33}}$. We have

\[
\begin{align*}
w_{20} &= (w_{40}^{-1}w_{30})^{m_{33}}w_{10} \\
w_{21} &= (w_{41}^{-1}w_{31})^{m_{33}}w_{01}
\end{align*}
\]

Therefore

\[
(w_{24}^{-1}w_{23})^{m_{22}\partial(m_{23})} = w_{04}^{-1}w_{14}w_{14}^{-1}(w_{34}^{-1}w_{43})^{m_{22}w_{13}^{-1}w_{03}^{-1}w_{04}} \\
+ w_{14}^{-1}w_{13}w_{03}^{-1}w_{04} \\
w_{21}w_{20}^{-1} = (w_{41}^{-1}w_{31})^{m_{33}}w_{01}w_{10}^{-1}(w_{30}^{-1}w_{40})^{m_{33}} \\
= (w_{41}^{-1}w_{31}w_{03}^{-1}w_{04})^{m_{33}} = (w_{24}^{-1}w_{23})^{m_{22}\partial(m_{23})m_{33}}.
\]

For $j = 3$ we have to check that $w_{31}w_{30}^{-1} = (w_{34}^{-1}w_{32})^{m_{22}}$. We have

\[
\begin{align*}
w_{30} &= w_{30}^{-1}w_{20}w_{10}^{-1} \\
w_{31} &= w_{31}^{-1}w_{21}w_{10}^{-1} \\
w_{32} &= w_{32}^{-1}w_{22}w_{12}^{-1} \\
w_{33} &= w_{33}^{-1}w_{23}w_{13}^{-1}.
\end{align*}
\]

Therefore

\[
(w_{31}w_{30}^{-1})^{\partial(w_{30})m_{33}} = (w_{04}^{-1}w_{41})^{m_{33}}w_{21}w_{01}^{-1}w_{10}w_{20}^{-1}(w_{40}^{-1}w_{04})^{m_{33}} \\
= (w_{04}^{-1}w_{41})^{m_{33}}w_{21}w_{20}^{-1} \\
(w_{34}^{-1}w_{32})^{m_{22}\partial(w_{30})m_{33}} = (w_{04}^{-1}w_{14}w_{04}^{-1}w_{24}^{-1}w_{04})^{m_{33}w_{12}w_{02}^{-1}}(w_{04}^{-1}w_{24}^{-1}w_{04})^{m_{33}w_{12}w_{02}^{-1}} \\
= (w_{04}^{-1}w_{14})^{m_{33}}w_{12}w_{02}^{-1} = (w_{31}w_{30})^{\partial(w_{30})m_{33}}.
\]

And now the required equality follows from the invertibility of action of $C$ on $A$.

For $j = 4$ we have to check that $w_{41}w_{40}^{-1} = (w_{43}^{-1}w_{42})^{m_{22}}$. We have

\[
\begin{align*}
w_{40} &= w_{40}^{-1}w_{30}w_{10}w_{20}^{-1} \\
w_{41} &= w_{31}^{-1}w_{21}w_{21}^{-1} \\
w_{42} &= w_{32}^{-1}w_{22}w_{12}^{-1} \\
w_{43} &= w_{33}^{-1}w_{23}w_{13}^{-1}.
\end{align*}
\]

Note that this time we can not use $m_{23} = w_{04}$, instead we will use $m_{23} = w_{40}$. We get

\[
\begin{align*}
(w_{41}w_{40}^{-1})^{\partial(w_{40})m_{33}} &= (w_{40}^{-1}w_{41})^{m_{33}}w_{20}w_{10}^{-1}(w_{50}^{-1}w_{31})^{m_{33}w_{01}w_{21}^{-1}} \\
(w_{43}^{-1}w_{42})^{m_{22}\partial(w_{40})m_{33}} &= (w_{40}^{-1}w_{13}w_{03}^{-1}(w_{23}^{-1})^{m_{22}w_{10}^{-1}w_{40}})^{m_{33}} \\
&\times (w_{40}^{-1}w_{32}^{-1}w_{40})^{m_{33}w_{02}w_{12}^{-1}} \\
&= (w_{40}^{-1})^{m_{33}}w_{13}w_{03}^{-1}w_{40}^{m_{33}}w_{02}w_{12}^{-1} \\
&= w_{20}w_{10}^{-1}(w_{30}^{-1}w_{13}w_{03}^{-1}w_{30})^{m_{33}w_{10}w_{20}^{-1}w_{02}w_{12}^{-1}} \\
&= w_{20}w_{10}^{-1}(w_{30}^{-1}w_{13})^{m_{33}w_{10}w_{12}^{-1}}
\end{align*}
\]

and the required equality follows from the invertibility of action of $C$ on $A$. □
Now we can compute homotopy groups of \( N(A, C) \) for a crossed module \((A, C)\). Let \( t \in C_0 \). Then \( \pi_2(N(A, C), t) \) is given by the classes \([g]\) of elements \( g \in C_1 \) such that \( s(g) = t(g) = t \), that is \( g \in C(t, t) \). Two elements \( g_1, g_2 \in C(t, t) \) belong to the same class if and only if there is an element \( M \in N_2(A, C) \) such that \( b^2(M) = (1_t, g_1, g_2) \). This implies \( m_{11} = 1_t \) and \( m_{22} = g_2 \). Therefore \( g_1 = \partial(m_{12}) \) and \( m_{22} = \partial(m_{12}) g_2 \). As the element \( m_{12} \in A(t) \) can be chosen arbitrary we see that \( g_1 \) and \( g_2 \) are in the same class if and only if \( g_1 \text{Im} \partial_2 = g_2 \text{Im} \partial_1 \). Note that \( \text{Im} \partial_2 \) is a normal subgroup of \( C(t, t) \) as for all \( g \in C(t, t) \) and \( a \in A(t) \) we have \( g^{-1} \partial(a) g = \partial(a'^2) \). Thus \( \pi_1 \) can be identified with the quotient group \( C(t, t)/\partial(A(t)) \) as a set. Now we show that the composition law on \( \pi_1(N(A, C), t) \) coincides with the composition law of \( C(t, t)/\partial(A(t)) \). Let \([g_1], [g_2] \in \pi_1(N(A, C), t)\). Then

\[
M := \begin{pmatrix} g_1 & e_t \\ g_2 & \end{pmatrix} \in N_2(A, C)
\]

is the preimage of \([g_1, \varnothing, g_2] \in \prod^2_1 N(A, C) \) under \( b^2_2 \). Therefore \([g_1][g_2] = [d_2(M)] = [g_1 g_2]\).

Now we compute \( \pi_2 := \pi_2(N(A, C), t) \). The elements of \( \pi_2 \) are the classes \([M] \) of elements \( M \in N_2(A, C) \) such that \( d_j M = 1_t, 0 \leq j \leq 2 \). This implies \( m_{11} = m_{22} = 1_t \) and \( m_{12} \in Ker(\partial_2) \). Two elements \( M^1, M^2 \in N_2(A, C) \) belong to the same class if and only if exists \( M \in N_2(A, C) \) such that \( b^3 M = (s_0(1_t), s_0(1_t), M^1, M^2) \). Such \( M \) necessarily has the form

\[
\mu_3(s_0(1_t), M^2, e_t) = \begin{pmatrix} 1_t & m_{12}^2 & e_t \\ 1_t & e_t & 1_t \\ 1_t & \end{pmatrix}
\]

and therefore \( m_{12}^2 = m_{12}^1 \). Thus we see that \( \pi_2 = Ker(\partial_2) \) as a set. Let \( M^1, M^2 \in \pi_2 \). Then

\[
M := \mu_3(s_0(1_t), M^2, m_{12}^1) = \begin{pmatrix} 1_t & m_{12}^2 & m_{12}^1 \\ 1_t & m_{12}^1 & 1_t \\ 1_t & \end{pmatrix}
\]

is the preimage of \([s_0(1_t), M^1, \varnothing, M^2] \) under \( b^3_2 \). Therefore

\[
M^1 M^2 = d_2 M = \begin{pmatrix} 1_t & m_{12}^2 m_{12}^1 \\ 1_t & \end{pmatrix}
\]

As \( Ker(\partial_2) \) is a commutative group we see that \( \pi_2 \) and \( Ker(\partial_2) \) are isomorphic as groups.

Since \( N(A, C) \) is a 3-coskeletal set all other homotopy groups of \( N(A, C) \) are trivial. Thus \( N(A, C) \) is a 2-type.

**Proposition 5.4.** Let \((A, C)\) be a crossed module of monoids such that \( N(A, C) \) is a Kan set. Then \((A, C)\) is a crossed module.
Proof. We have to show that $C$ is a groupoid and that for every $t \in C_0$ the monoid $A(t)$ is a group. Let $g \in C(s,t)$. Then $(g, 1_s, \emptyset)$ and $(\emptyset, 1_t, g)$ are elements of $\bigwedge^2_2 N(A,C)$ and $\bigwedge^2_0 N(A,C)$ respectively. Since $N(A,C)$ is a Kan simplicial set there exist their preimages $M^1$ and $M^2$ in $N_2(A,C)$ under $b_2^1$ and $b_2^0$ respectively. Then $1_s = d_2 M^1 = g \partial (m_{12}^1) m_{22}^1$ and $1_t = d_2 M^2 = m_{11}^2 \partial t (m_{22}^2 g)$, which shows that $g$ has left and right inverse. By the usual trick they are equal to each other.

Now let $a \in A(t)$. We consider

$$
\begin{pmatrix}
1_t & e_t & a \\
1_t & e_t & 0 \\
1_t & 0 & 1_t
\end{pmatrix},
\begin{pmatrix}
1_t & e_t & e_t \\
1_t & e_t & 0 \\
1_t & 0 & 1_t
\end{pmatrix},
\begin{pmatrix}
1_t & e_t & e_t \\
1_t & e_t & 0 \\
0 & 1_t & 1_t
\end{pmatrix},
\begin{pmatrix}
1_t & e_t & e_t \\
1_t & e_t & 0 \\
0 & 1_t & 1_t
\end{pmatrix},
\begin{pmatrix}
1_t & e_t & e_t \\
1_t & e_t & 0 \\
0 & 1_t & 1_t
\end{pmatrix},
\begin{pmatrix}
1_t & e_t & a \\
1_t & e_t & 0 \\
1_t & 0 & 1_t
\end{pmatrix},
$$

which is an element of $\bigwedge^5_2 N(A,C)$. Since $N(A,C)$ is a Kan simplicial set there exists $M$ in the preimage of this element under $b_2^5$. This matrix necessarily has the form

$$
\begin{pmatrix}
1_t & e_t & a & m_{15} \\
1_t & e_t & a & 0 \\
1_t & 0 & 1_t & 1_t
\end{pmatrix}.
$$

From the explicit form for $d_1 M$ and $d_3 M$ we see that $m_{15}$ is the inverse element to $a$ in $A(t)$. \hfill \Box

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