Hardy–Stein identity for pure-jump Dirichlet forms

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Abstract

We prove the \( L^p \) variant of the Hardy–Stein identity for Sobolev–Bregman forms associated with pure-jump Dirichlet forms, under rather mild assumptions. Along the way, we obtain a general result in terms of the \( p \)-form defined in a more abstract way.

1 Introduction

In this article we prove the Hardy–Stein identity for a rather general regular conservative pure-jump Dirichlet form

\[
\mathcal{E}(u, v) = \frac{1}{2} \iint_{E \times E \setminus \text{diag}} (u(y) - u(x))(v(y) - v(x)) J(dx, dy)
\]

on a locally compact separable metric space \( E \), with Borel \( \sigma \)-algebra \( \mathcal{B} \) and a positive Radon measure \( m \) such that \( \text{supp}(m) = E \). Here \( \text{diag} \) is the diagonal of the cartesian product \( E \times E \) and \( J \) is the symmetric jumping measure on \( E \times E \setminus \text{diag} \). Let \( (P_t)_{t \geq 0} \) be a strongly continuous semigroup of contractions associated to the Dirichlet form \( \mathcal{E} \).

The following is the main result of this paper.

2020 Mathematics Subject Classification: Primary 31C25; Secondary 42B25, 60J35, 60G51.

Key words and phrases: Hardy–Stein identity, nonlocal operator, Littlewood–Paley theory.
**Theorem 1.1.** Let \( p \in (1, \infty) \). Let \( \mathcal{E} \) be a regular conservative pure-jump Dirichlet form given by (1.1). Assume that:

(i) For every \( t > 0 \) and \( f \in L^p(m) \) we have \( P_t f \in C(E) \), that is, \( P_t f \) is continuous on \( E \).

(ii) The semigroup \( (P_t)_{t \geq 0} \) is strongly stable on \( L^p(m) \), i.e., for every \( f \in L^p(m) \),

\[
\| P_T f \|_p \to 0 \quad \text{when } T \to \infty.
\]

If \( f \in L^p(m) \), then

\[
\int_E |f|^p \, dm = \int_0^\infty \int E \times E \setminus \text{diag} F_p(P_t f(x), P_t f(y)) J(dx, dy) dt.
\]

Here \( F_p(a, b) := |b|^p - |a|^p - pa^{p-1}(b - a) \) is the Bregman divergence.

In general, it is known that for every regular Dirichlet form \( \mathcal{E} \), there exists (unique in a certain sense) \( m \)-symmetric Hunt process whose Dirichlet form is \( \mathcal{E} \). For more details see [FOT11, Chapter 7]. On the other hand, there is a one-to-one correspondence between the family of closed symmetric forms \( \mathcal{E} \) on \( L^2(m) \) and the family of non-positive definite self-adjoint operators \( L \) on \( L^2(m) \) (see [FOT11, Theorem 1.3.1.]). We use these facts to present some examples of Dirichlet forms within the present settings.

**Example 1.2.** The classical example of a pure-jump Dirichlet form that fulfills our assumptions is the Dirichlet form associated with the fractional Laplacian. More precisely, let \( 0 < \alpha < 2 \) and denote by \( L = \mathcal{L} \) the fractional Laplacian on \( E = \mathbb{R}^d \) equipped with the Lebesgue measure. This operator induces the Dirichlet form given by (1.1), where the jumping measure satisfies \( J(dx, dy) = \nu(y - x) dx \, dy \) and

\[
\nu(y) := \frac{A_{d,\alpha}}{|y|^{d+\alpha}}.
\]

In the above formula the constant \( A_{d,\alpha} \) is given by

\[
A_{d,\alpha} := \frac{2^\alpha \Gamma \left( \frac{d+\alpha}{2} \right)}{\pi \Gamma \left( -\frac{\alpha}{2} \right)}.
\]

It is well-known that assumptions (i) and (ii) hold in this case.
Example 1.3. More generally, Theorem 1.1 applies to Dirichlet forms associated with pure-jump symmetric Lévy processes on $E = \mathbb{R}^d$ with Lévy measure $\nu$ fulfilling the Hartman–Wintner condition, i.e.,

$$\lim_{|\xi| \to \infty} \frac{\psi(\xi)}{\log(|\xi|)} = \infty,$$

where $\psi$ is the characteristic exponent of Lévy process given by $\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot x)) \nu(dx)$. Then the Dirichlet form is given by (1.1), where the jumping measure satisfies $J(dx,dy) = \nu(dy - x)dx$. It is known that assumptions (i) and (ii) hold in this case. For more details, we refer the reader to [BBL16] and [BGP], where the Hardy–Stein identity in this context was already considered.

The following example show that the main result of this paper allows us to formulate the Hardy–Stein in more general cases. It concerns more general spaces than the Euclidean space $\mathbb{R}^d$.

Example 1.4. Let $n \in \mathbb{N}_+$ and let $d$ be any real number from $(0, n]$. Let $E$ be a closed $d$-set in $\mathbb{R}^n$ with measure $m$. That is, there are constants $C_1, C_2 > 0$ such that

$$(1.4) \quad C_1 r^d \leq m(B(x,r)) \leq C_2 r^d, \quad \text{for all } x \in E, r \in (0, 1].$$

Here $B(x,r)$ is the intersection of the set $E$ and a ball in $\mathbb{R}^n$ of radius $r$ centered at $x$. One may show, that $m$ is equivalent to the $d$-dimensional Hausdorff measure restricted to $E$. Typical examples of such sets are self-similar sets and Riemannian manifolds embedded into a Euclidean space $\mathbb{R}^n$. In addition to (1.4), we assume that

$$(1.5) \quad m(B(x,r)) \leq C_2 r^d, \quad \text{for every } x \in E, r > 0.$$

For example, assumption (1.5) is fulfilled for $E$ bounded in $\mathbb{R}^n$.

Fix $0 < \alpha < 2$. Let $\phi : E \times E \to (0, \infty)$ be a symmetric function such that there are some constants $C_3, C_4 > 0$ such that

$$C_3 \leq \phi(x,y) \leq C_4 \quad \text{for m-a.e. } x, y \in E.$$

Consider the Dirichlet form (1.1) with jumping measure

$$J(dx,dy) = \frac{\phi(x,y)}{|y-x|^{d+\alpha}} m(dx) m(dy).$$

The process associated with the Dirichlet form within the present settings is called the stable-like process.
For basic properties of such Dirichlet form, we refer the reader to \[\text{CK03}\]. One may show that this Dirichlet form is conservative. One can show that assumption (i) holds. For the reader’s convenience, we sketch the proof of this claim in Appendix B. Condition (ii) is satisfied whenever \(m(E) = \infty\). If \(m(E) < \infty\), then (ii) fails. In this case, however, modified Hardy–Stein identity holds; see Appendix A.

Along the way of proving the main result, in Theorem 3.1 we give a more general variant of the Hardy–Stein identity:

\[
\int_E |f|^p \, dm = p \int_0^\infty \mathcal{E}_p[P_t f] \, dt, \quad f \in L^p(m),
\]

where \(\mathcal{E}_p\) is the \(p\)-form defined as the limit of appropriate approximate forms \(\mathcal{E}^{(t)}(u, u^{(p-1)}) = \langle u - P_t u, u^{(p-1)} \rangle / t\). The \(p\)-form may be treated as an extension of the classical quadratic Dirichlet form.

In the case of pure-jump Dirichlet forms, in Theorem 4.3 we derive the explicit formula for the \(p\)-form

\[
\mathcal{E}_p[u] = \frac{1}{p} \iint_{E \times E \setminus \text{diag}} F_p(u(x), u(y)) \, J(dx, dy)
\]

for continuous functions \(u\) from the domain \(\mathcal{D}(\mathcal{E}_p)\) of \(p\)-form.

The Hardy–Stein identity was originally proved for the classical Laplace operator as a consequence of Green’s theorem and the chain rule \(\Delta u^p = p(p-1)u^{p-2}|\nabla u|^2 + pu^{p-1}\Delta u\). We refer to Lemma 1 and Lemma 2 in Stein \[\text{Ste70a, p. 86–88}\]. The identity in the non-local case can be shown using analytic methods as in \[\text{BBL16}\] or in \[\text{BGP}\]. On the other hand, in \[\text{BK19}\] the identity is derived from Itô’s lemma.

The Hardy–Stein identity has found applications in the Littlewood–Paley theory, especially to the proof of \(L^p\)-boundedness of the square function and Fourier multipliers. For more details, we refer to \[\text{BBL16}\] and \[\text{BK19}\]. The Hardy–Stein identity also gives a characterization of Hardy spaces \[\text{BDL14}\] and it is used to prove Douglas-type identities \[\text{Bog+20}\].

The goal of this paper is to extend the previous Hardy–Stein type identities to more general Dirichlet forms which correspond to pure-jump regular Dirichlet form under certain mild assumptions. Our main tool is the theory of Dirichlet forms and \(p\)-forms. To prove Theorem 3.1 we adopt the approach from \[\text{BGP, Theorem 15}\].

The \(p\)-form (or Sobolev–Bregman form) corresponding to the fractional Laplacian (defined as a double integral of the Bregman divergence) was
studied in [KL21], [Bog+21], and for more general Lévy operators in [BGP], but similar expression appeared already in [Bak+95, Lemma 7.2]. A similar object was frequently used also in [Bog+20].

In this work, we begin with a non-standard definition of the $p$-form – as a limit of appropriate approximating forms $\mathcal{E}^{(t)}(u, u^{(p-1)})$. This gives access to an extended class of operators. The same definition was used in the case of the Brownian motion in [BGP, Section 8]. At this moment it is known that this definition is equivalent to the commonly used definition in terms of the Bregman divergence for pure-jump Lévy operators [Bog+21, Lemma 7], [BGP, Proposition 13]. In Theorem 4.3 we show that for more general pure-jump Dirichlet form the equivalence remains true for continuous functions. Equivalence of the two definitions for discontinuous functions in the domain of the $p$-form remains an open problem.

Independently from the main subject of this article, the Hardy–Stein identity, we are interested in the relationship between the domains of the Dirichlet form and the $p$-form. In [Bog+21, Lemma 7] and [BGP, Proposition 13] it was shown that for a pure-jump Lévy operators, with certain assumptions about the Lévy measure, the domain $\mathcal{D}(\mathcal{E}_p)$ of the $p$-form consists of functions of the form $u^{(p/2)}$, where $u$ is a function in the domain $\mathcal{D}(\mathcal{E})$ of the Dirichlet form. In Theorem 4.3 we show the inclusion $\mathcal{D}(\mathcal{E}_p) \subseteq \mathcal{D}(\mathcal{E})^{(p/2)}$ in the general pure-jump case, and we conjecture that, in fact, equality holds.

The structure of the article is as follows. In Section 2 we introduce the notions of a Dirichlet form, its semigroup, the definition of the corresponding $p$-form, and derivatives of $L^p$-valued functions and we discuss their basic properties. The Hardy–Stein identity in the general case is proved in Section 3. In Section 4 we consider pure-jump Dirichlet forms and we show that the corresponding $p$-form $\mathcal{E}_p$ for continuous functions is given by a double integral. This proves the Hardy–Stein identity for such Dirichlet forms (Theorem 4.1). In Appendix A we discuss the sufficient condition for assumption (iii) and the situation when this assumption is not necessarily true. In Appendix B we prove assumption (i) in case of Example 1.4.

## 2 Preliminaries

We consider a regular conservative Dirichlet form $\mathcal{E}$ on $L^2(E, \mathcal{B}, m)$ where $(E, \mathcal{B}, m)$ is a measure space. For simplicity we will write $L^p(m) := L^p(E, \mathcal{B}, m)$ for any $p \in [1, \infty]$. We assume that $E$ is a locally compact
separable metric space and $m$ is a positive Radon measure on $E$ such that $	ext{supp}(m) = E$, defined on the $\sigma$-algebra $\mathcal{B}$ of all Borel sets in $E$. By $C(E)$ we denote the class of continuous functions on $E$. Let $(P_t)_{t \geq 0}$ be the strongly continuous semigroup of contractions associated with the Dirichlet form $\mathcal{E}$. Recall that for every $p \in [1, \infty)$ $(P_t)_{t \geq 0}$ is a strongly continuous semigroup of contractions on $L^p(m)$. For a function $u \in L^p(m)$ we have

$$P_t u(x) = \int_E u(y) P_t(x, dy), \quad t \geq 0, x \in E.$$ 

Here $P_t(x, dy)$ is the probability kernel associated with the operator $P_t$. To emphasize symmetry, we write $P_t(dx, dy) := P_t(x, dy)m(dx)$; then $P_t(dx, dy) = P_t(dy, dx)$.

We use the notation

$$a^{(\kappa)} := |a|^\kappa \text{sgn } a,$$

whenever above expression makes sense. Note that

$$(|x|^{\kappa})' = \kappa x^{(\kappa - 1)}, \quad \text{if } x \in \mathbb{R}, \ \kappa > 1 \text{ or } x \in \mathbb{R}\{0\},$$

and

$$(x^{(\kappa)})' = \kappa |x|^{\kappa - 1}, \quad \text{if } x \in \mathbb{R}, \ \kappa \geq 1 \text{ or } x \in \mathbb{R}\{0\}.$$

Let $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$. For $u \in L^p(m), v \in L^q(m)$ we use the notation

$$\langle u, v \rangle := \int_E u(x)v(x) m(dx).$$

For $t > 0$ and $u \in L^p(m), v \in L^q(m)$ we define

$$(2.1) \quad \mathcal{E}^{(t)}(u, v) := \frac{1}{t} \langle u - P_t u, v \rangle.$$ 

Let $u \in L^p(m)$. We define the nonlinear functional

$$\mathcal{E}_p[u] := \lim_{t \to 0^+} \mathcal{E}^{(t)}(u, u^{(p-1)})$$

with its natural domain

$$\mathcal{D}(\mathcal{E}_p) := \left\{ u \in L^p(m) : \text{finite } \lim_{t \to 0^+} \mathcal{E}^{(t)}(u, u^{(p-1)}) \text{ exists} \right\}.$$ 

We call $\mathcal{E}_p$ the $p$-form corresponding to the Dirichlet form $\mathcal{E}$. 
When $p = 2$, then $\mathcal{E}_2$ is just the usual Dirichlet form $\mathcal{E}(u, u)$ with domain $\mathcal{D}(\mathcal{E}_2) = \mathcal{D}(\mathcal{E})$.

It is well-known that if $u \in L^2(m)$, then $\mathcal{E}^{(t)}(u, u)$ is non-increasing as a function of $t$ [FOT11, Lemma 1.3.4].

We consider the infinitesimal generator $L_p$ of the semigroup $(P_t)_{t \geq 0}$ on $L^p(m)$:

$$L_p u := \lim_{t \to 0^+} \frac{1}{t} (P_t u - u) \quad \text{in } L^p(m)$$  \hspace{1cm} (2.2)

with the natural domain

$$\mathcal{D}(L_p) := \left\{ u \in L^p(m) : \lim_{t \to 0^+} \frac{1}{t} (P_t u - u) \text{ exists in } L^p(m) \right\}.$$

Of course, $\mathcal{D}(L_p) \subseteq \mathcal{D}(\mathcal{E}_p)$ and

$$\mathcal{E}_p[u] = -\langle L_p u, u^{(p-1)} \rangle, \quad u \in \mathcal{D}(L_p).$$

We use the Bregman divergence: a function $F_p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, where $p > 1$, defined by

$$F_p(a, b) := |b|^p - |a|^p - pa^{(p-1)}(b - a).$$

We also use symmetrized Bregman divergence

$$H_p(a, b) := \frac{1}{2} (F_p(a, b) + F_p(b, a)) = \frac{p}{2} (b - a) \left( b^{(p-1)} - a^{(p-1)} \right).$$

Note that $F_p(a, b)$ is the second-order Taylor remainder of the convex map $\mathbb{R} \ni a \mapsto |a|^p \in \mathbb{R}$, we have $F_p \geq 0$ and also $H_p \geq 0$. Furthermore, $F_2(a, b) = H_2(a, b) = (b - a)^2$.

The following estimate was proved in [LS93, Lemma 1] for $H_p$ in place of $F_p$.

**Lemma 2.1.** Let $p \in (1, \infty)$. There exist constants $c_p, C_p > 0$ such that

$$c_p (b^{(p/2)} - a^{(p/2)})^2 \leq F_p(a, b) \leq C_p (b^{(p/2)} - a^{(p/2)})^2$$  \hspace{1cm} (2.3)

for all $a, b \in \mathbb{R}$.

**Proof.** If $a = 0$, we have $F_p(a, b) = |b|^p$ and the statement is obvious. If $a \neq 0$, then we let $x := b/a$ and we arrive at the following equivalent formulation of (2.3):

$$c_p (x^{(p/2)} - 1)^2 \leq F_p(x, 1) \leq C_p (x^{(p/2)} - 1)^2,$$  \hspace{1cm} (2.4)
where $F_p(x, 1) = |x|^p - p(x - 1)$. The above expressions define continuous and positive functions of $x \neq 1$. Therefore to prove (2.4) it is enough to notice that

$$\lim_{x \to \pm \infty} \frac{F_p(x, 1)}{(x^{p/2} - 1)^2} = 1$$

and, by L'Hôpital's rule,

$$\lim_{x \to 1} \frac{F_p(x, 1)}{(x^{p/2} - 1)^2} = \lim_{x \to 1} \frac{x^{p-1} - 1}{(x^{p/2} - 1) x^{(p-2)/2}} = \frac{2(p - 1)}{p}.$$

The above limits are finite and positive. This completes the proof. \(\square\)

For every $u \in L^1(m)$, by symmetry of $P_t$, we have

$$\int_E P_t u(x) \, m(dx) = \int_E \left( \int_E u(y) \, P_t(x, dy) \right) \, m(dx)$$

$$= \int_E u(y) \left( \int_E P_t(y, dx) \right) \, m(dy) = \int_E u(x) \, m(dx).$$

Let $u \in L^p(m)$. Using (2.5) for $|u|^p \in L^1(m)$ we can write

$$\mathcal{E}^{(t)}(u, u^{(p-1)}) = \frac{1}{t} \langle u - P_t u, u^{(p-1)} \rangle$$

$$= -\frac{1}{t} \iint_{E \times E} u^{(p-1)}(x)(u(y) - u(x)) \, P_t(dx, dy)$$

$$= \frac{1}{pt} \int_E P_t(|u|^p)(x) \, m(dx) - \frac{1}{pt} \int_E |u|^p(x) \, m(dx)$$

$$- \frac{1}{t} \iint_{E \times E} u^{(p-1)}(x)(u(y) - u(x)) \, P_t(dx, dy)$$

$$= \frac{1}{pt} \iint_{E \times E} F_p(u(x), u(y)) \, P_t(dx, dy).$$

In particular, we see that $\mathcal{E}^{(t)}(u, u^{(p-1)}) \geq 0$, and so $\mathcal{E}_p[u] \geq 0$ whenever $u \in D(\mathcal{E}_p)$.

By symmetry of $P_t$ we can also write

$$\mathcal{E}^{(t)}(u, u^{(p-1)}) = \frac{1}{pt} \iint_{E \times E} H_p(u(x), u(y)) \, P_t(dx, dy).$$
Let $p \in [1, \infty)$ and let $I \subseteq [0, \infty)$ be an interval. For a mapping $I \ni t \mapsto u(t) \in L^p(m)$ we denote
\[
\Delta_h u(t) := u(t + h) - u(t) \quad \text{if } t, t + h \in I.
\]

We say that $u$ is \textit{continuous} on $I$ with values in $L^p(m)$ if $\Delta_h u(t) \to 0$ in $L^p(m)$ as $h \to 0$ for every $t \in I$, and we say that $u$ is \textit{continuously differentiable} (or shortly $C^1$) on $I$ with values in $L^p(m)$ if $u'(t) := \lim_{h \to 0} \frac{1}{h} \Delta_h u(t)$ exists in $L^p(m)$ for every $t \in I$ and the mapping $I \ni t \mapsto u'(t) \in L^p(m)$ is continuous.

The following two elementary results are proved rigorously in \cite[Lemmas 15, 16]{Bog22}.

\textbf{Lemma 2.2} (\cite{Bog22}). Let $p \in (1, \infty)$. If $I \ni t \mapsto u(t)$ is $C^1$ on $I$ with values in $L^p(m)$, then $|u|^p$ is $C^1$ on $I$ with values in $L^1(m)$ and $(|u|^p)' = pu^{(p-1)}u'$.

Let $f \in L^p(m)$ and let $u(t) := P_t f \in L^p(m)$. If $f \in \mathcal{D}(L_p)$, then $u'(t) = L_p P_t f = P_t L_p f = L_p u(t)$. We know that if $p > 1$, then $(P_t)_{t \geq 0}$ is an analytic semigroup on $L^p(m)$ \cite[p. 67]{Ste70b}. In particular, for every $t > 0$ and $f \in L^p(m)$ the derivative $\frac{d}{dt} P_t f = u'(t)$ exists in $L^p(m)$. Hence $P_t f \in \mathcal{D}(L_p)$ and $u'(t) = L_p P_t f = L_p u(t)$.

\textbf{Corollary 2.3} (\cite{Bog22}). Let $f \in \mathcal{D}(L_p)$ and $u(t) := P_t f$. Then $|u(t)|^p$ is $C^1$ on $[0, \infty)$ with values in $L^1(m)$, with derivative
\[
(2.8) \quad (|u(t)|^p)' = pu(t)^{(p-1)}u'(t) = pu(t)^{(p-1)}L_p u(t), \quad t \geq 0.
\]

\section{Hardy–Stein identity in the general case}

In this section we prove the Hardy–Stein identity for arbitrary regular Dirichlet form. The explicit form of the right-hand side of following identity depends on the specific Dirichlet form. In particular, in Section \ref{section:general} we will give an explicit expression for pure-jump Dirichlet forms.

\textbf{Theorem 3.1}. Let $p \in (1, \infty)$. Assume that condition (ii) from Theorem \ref{Theorem:Stein} holds. For every $f \in L^p(m)$,
\[
(3.1) \quad \int_E |f|^p \, dm = p \int_0^\infty \mathcal{E}_p[P_t f] \, dt.
\]
Proof. Consider first \( f \in \mathcal{D}(L_p) \) and fix \( T > 0 \). Let \( u(t) := P_t f \). By Corollary 2.3, \( |u|^p \) is \( C^1 \) on \([0, T]\) with values in \( L^1(m) \) with derivative \((|u(t)|^p)' = pu(t)^{(p-1)} L_p u(t)\). The integral is a continuous linear functional on \( L^1(m) \), hence \([0, T] \ni t \mapsto \int_E |u(t)|^p dm \) is \( C^1 \) and

\[
\frac{d}{dt} \int_E |u(t)|^p dm = \int_E \frac{d}{dt} |u(t)|^p dm = \int_E pu(t)^{(p-1)} L_p u(t) dm
= p(L_p u(t), u(t)^{(p-1)}) = -p\mathcal{E}_p[u(t)].
\]

Therefore, we can write

\[
\int_E |f|^p dm - \int_E |P_T f|^p dm = - \left( \int_E |u(T)|^p dm - \int_E |u(0)|^p dm \right)
= - \int_0^T \frac{d}{dt} \int_E |u(t)|^p dm dt
= p \int_0^T \mathcal{E}_p[u(t)] dt.
\]

From the strong stability of the semigroup (assumption (ii)), we have \( \int_E |P_T f|^p dm \to 0 \) when \( T \to \infty \). Since \( \mathcal{E}_p[u(t)] \geq 0 \), the right-hand side tends to \( p \int_0^\infty \mathcal{E}_p[u(t)] dt \) as \( T \to \infty \). Therefore

\[
\int_E |f|^p dm = p \int_0^\infty \mathcal{E}_p[P_t f] dt.
\]

Next, we relax the assumption that \( f \in \mathcal{D}(L_p) \). Let \( f \) be an arbitrary function in \( L^p(m) \) and let \( s > 0 \). Recall that \( P_s f \in \mathcal{D}(L_p) \). Thus, (3.1) holds for \( P_s f \):

\[
\int_E |P_s f|^p dm = p \int_s^\infty \mathcal{E}_p[P_t f] dt.
\]

Since \( (P_t)_{t \geq 0} \) is a strongly continuous semigroup on \( L^p(m) \) and \( f \mapsto \int_E |f|^p dm \) is a continuous functional on \( L^p(m) \), the left-hand side tends to \( \int_E |f|^p dm \) when \( s \to 0^+ \). Since \( \mathcal{E}_p[P_t f] \geq 0 \), the right-hand side tends to \( p \int_0^\infty \mathcal{E}_p[u(t)] dt \) by the monotone convergence theorem. \( \square \)

Remark 3.2. Notice that without assumption (ii) the identity takes form

\[
\int_E |f|^p dm - \lim_{T \to \infty} \int_E |P_T f|^p dm = p \int_0^\infty \mathcal{E}_p[P_t f] dt,
\]
and in particular, the limit on the left-hand side exists. This situation is discussed in more detail in Appendix A.

4 Pure-jump Dirichlet forms

In this section we consider a pure-jump regular Dirichlet form: we assume that the Dirichlet form \( E \) is given by (1.1). Here and below, \( \text{diag} := \{(x, y) \in E \times E : x = y\} \). Measure \( J \), the so-called jumping measure, is a symmetric positive Radon measure on the \( E \times E \setminus \text{diag} \).

Our goal is to propose explicit form of \( p \)-form for such Dirichlet form.

**Lemma 4.1.** We have

\[
\lim_{t \to 0^+} \frac{1}{t} \int_{E \times E} f(x, y) P_t(dx, dy) \to J(dx, dy) \quad \text{vaguely on } E \times E \setminus \text{diag} \quad \text{when } t \to 0^+.
\]

An analogous result for the resolvent rather than the semigroup \((P_t)\) was showed in [FOT11, (3.2.7)]. The proof is similar and we omit it.

We consider the class \( U \) of non-negative functions \( f \) on \( E \times E \) such that

\[
\lim_{t \to 0^+} \frac{1}{t} \int_{E \times E} f(x, y) P_t(dx, dy) = \int_{E \times E \setminus \text{diag}} f(x, y) J(dx, dy) < \infty.
\]

We know that if \( u \in D(E) \) and \( f(x, y) = (u(y) - u(x))^2 \), then \( f \in U \) because

\[
\lim_{t \to 0^+} \frac{1}{t} \int_{E \times E} (u(y) - u(x))^2 P_t(dx, dy) = \lim_{t \to 0^+} 2E^{(t)}(u, u) = 2E[u] = \int_{E \times E \setminus \text{diag}} (u(y) - u(x))^2 J(dx, dy).
\]

Here we used (2.7).

**Lemma 4.2.** Suppose that \( 0 \leq f \leq g \), \( f = g = 0 \) on \( \text{diag} \), \( f, g \in C(E \times E) \) and \( g \in U \). Then \( f \in U \).

**Proof.** Fix \( \varepsilon > 0 \). Since \( g \in U \), we have

\[
\int_{E \times E \setminus \text{diag}} g(x, y) J(dx, dy) < \infty.
\]

Therefore,

\[
\int_{E \times E \setminus \text{diag}} f(x, y) J(dx, dy) \leq \int_{E \times E \setminus \text{diag}} g(x, y) J(dx, dy) < \infty
\]
and there is a compact subset $K \subseteq E \times E \setminus \text{diag}$ such that

\[(4.2) \quad \int_{K^c} g(x, y) \, J(dx, dy) < \varepsilon.\]

Let $\varphi \in C_c(E \times E \setminus \text{diag})$ be such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ on $K$. Since $f$ is continuous, we have $\varphi \cdot f \in C_c(E \times E \setminus \text{diag})$, hence from (4.1) we obtain

\[(4.3) \quad \lim_{t \to 0^+} \frac{1}{t} \int_{E \times E \setminus \text{diag}} \varphi(x, y) f(x, y) \, P_t(dx, dy) = \int_{E \times E \setminus \text{diag}} \varphi(x, y) f(x, y) \, J(dx, dy).\]

Using (4.2) we can write

\[(4.4) \quad \int_{E \times E \setminus \text{diag}} (1 - \varphi(x, y)) f(x, y) \, J(dx, dy) \leq \int_{E \times E \setminus \text{diag}} (1 - \varphi(x, y)) g(x, y) \, J(dx, dy) \leq \int_{K^c} g(x, y) \, J(dx, dy) < \varepsilon.\]

We also have

\[
\frac{1}{t} \int_{E \times E \setminus \text{diag}} (1 - \varphi(x, y)) f(x, y) \, P_t(dx, dy) \leq \frac{1}{t} \int_{E \times E \setminus \text{diag}} (1 - \varphi(x, y)) g(x, y) \, P_t(dx, dy) \\
= \frac{1}{t} \int_{E \times E \setminus \text{diag}} g(x, y) \, P_t(dx, dy) \\
- \frac{1}{t} \int_{E \times E \setminus \text{diag}} \varphi(x, y) g(x, y) \, P_t(dx, dy).
\]

Since $g \in \mathcal{U}$ and $g$ is continuous, we have $\varphi \cdot g \in C_c(E \times E \setminus \text{diag})$, and thus, using (4.1), we find that as $t \to 0^+$, the right-hand side converges to

\[
\int_{E \times E \setminus \text{diag}} g(x, y) \, J(dx, dy) - \int_{E \times E \setminus \text{diag}} \varphi(x, y) g(x, y) \, J(dx, dy) \\
= \int_{E \times E \setminus \text{diag}} (1 - \varphi(x, y)) g(x, y) \, J(dx, dy) \leq \int_{K^c} g(x, y) \, J(dx, dy) < \varepsilon.
\]

Here we used (4.2). Therefore,

\[(4.5) \quad \limsup_{t \to 0^+} \frac{1}{t} \int_{E \times E \setminus \text{diag}} (1 - \varphi(x, y)) f(x, y) \, P_t(dx, dy) < \varepsilon.\]
Finally, from (4.3), (4.4) and (4.5) we get

\[
\lim_{t \to 0^+} \sup \left| \frac{1}{t} \int \int_{E \times E \setminus \text{diag}} f(x, y) P_t(dx, dy) - \int \int_{E \times E \setminus \text{diag}} f(x, y) J(dx, dy) \right| \\
\leq \lim_{t \to 0^+} \sup \left| \frac{1}{t} \int \int_{E \times E \setminus \text{diag}} \varphi(x, y) f(x, y) P_t(dx, dy) - \int \int_{E \times E \setminus \text{diag}} \varphi(x, y) f(x, y) J(dx, dy) \right| \\
+ \lim_{t \to 0^+} \frac{1}{t} \int \int_{E \times E \setminus \text{diag}} (1 - \varphi(x, y)) f(x, y) P_t(dx, dy) \\
+ \int \int_{E \times E \setminus \text{diag}} (1 - \varphi(x, y)) f(x, y) J(dx, dy) < 0 + \epsilon + \epsilon = 2\epsilon.
\]

Since \(\epsilon > 0\) is arbitrary,

\[
\lim_{t \to 0^+} \frac{1}{t} \int \int_{E \times E \setminus \text{diag}} f(x, y) P_t(dx, dy) = \int \int_{E \times E \setminus \text{diag}} f(x, y) J(dx, dy).
\]

\[\square\]

**Theorem 4.3.** Let \(u \in \mathcal{D}(\mathcal{E}_p)\). Then \(u^{(p/2)} \in \mathcal{D}(\mathcal{E})\). Moreover, if \(u \in C(E)\), then

\[
(4.6) \quad \mathcal{E}_p[u] = \frac{1}{p} \int \int_{E \times E \setminus \text{diag}} F_p(u(x), u(y)) J(dx, dy).
\]

**Proof.** Using (2.6), the fact that \(F_2(a, b) = (b - a)^2\) and Lemma 2.1, we find that

\[
\mathcal{E}^{(t)} \left( u^{(p/2)}, u^{(p/2)} \right) = \frac{1}{2t} \int \int_{E \times E} \left( (u(y))^{(p/2)} - (u(x))^{(p/2)} \right)^2 P_t(dx, dy) \\
\leq c_p^{-1} \frac{1}{2t} \int \int_{E \times E} F_p(u(x), u(y)) P_t(dx, dy) \\
= \frac{p}{2c_p} \mathcal{E}^{(t)} \left( u, u^{(p-1)} \right).
\]

Since \(u \in \mathcal{D}(\mathcal{E}_p)\), the right-hand side converges to a finite limit \(\frac{p}{2c_p} \mathcal{E}_p[u]\) as \(t \to 0^+\), and since the left-hand side is non-increasing as a function of \(t\), a finite limit \(\lim_{t \to 0^+} \mathcal{E}^{(t)} \left( u^{(p/2)}, u^{(p/2)} \right)\) exists, i.e., \(u^{(p/2)} \in \mathcal{D}(\mathcal{E})\).

Let us additionally assume that \(u \in C(E)\). Denote \(f(x, y) := F_p(u(x), u(y))\), and \(g(x, y) := C_p \left( (u(y))^{(p/2)} - (u(x))^{(p/2)} \right)^2\), where \(C_p\) is as
in (2.3). Since \( u^{p/2} \in \mathcal{D}(\mathcal{E}) \), we have \( g \in \mathcal{U} \), and by (2.3) we have \( f \leq g \). Moreover, \( f = g = 0 \) on \( \text{diag} \) and \( f, g \in C(E \times E) \) because \( u \in C(E) \). Therefore, we can use Lemma 4.2 and conclude that \( f \in \mathcal{U} \). This means that

\[
\mathcal{E}_p[u] = \lim_{t \to 0^+} \mathcal{E}^{(t)}(u, u^{p-1}) = \lim_{t \to 0^+} \frac{1}{pt} \iint_{E \times E} F_p(u(x), u(y)) P_t(dx, dy)
= \frac{1}{p} \iint_{E \times E \setminus \text{diag}} F_p(u(x), u(y)) J(dx, dy).
\]

Now, we are ready to prove the main result of the article.

**Proof of Theorem 1.1.** It is enough to note that, by assumption, for every \( t > 0 \) we have \( P_t f \in C(E) \cap \mathcal{D}(\mathcal{E}_p) \), and so we can rewrite formula (3.1) using (4.6) with \( u = P_t f \).

\( \square \)

### A Strong stability of the heat semigroup

In this section we discuss a sufficient condition for assumption (ii) as well as the question of how the main result of this paper changes without this assumption.

In general, when assumption (ii) is not necessarily true, the Hardy-Stein identity takes the following form

\[
\int_E |f|^p \, dm - \lim_{T \to \infty} \left\Vert P_T f \right\Vert_p^p = \int_0^\infty \iint_{E \times E \setminus \text{diag}} F_p(P_T f(x), P_T f(y)) J(dx, dy) \, dt.
\]

The question is, under what assumptions the term \( \lim_{T \to \infty} \left\Vert P_T f \right\Vert_p^p \) is equal to zero or can be written in a simpler form.

For \( p \in [1, \infty] \) we define the operator \( P_\infty : L^p(m) \to L^p(m) \) by

\[
P_\infty f := \lim_{T \to \infty} P_T f, \quad \text{in } L^p(m).
\]

First of all, the following fact holds.

**Fact A.1.** For \( p = 2 \), the operator \( P_\infty \) is the orthogonal projection onto the kernel of the generator \( L_2 \) (given by (2.2)).
Proof. Using spectral theorem (in the multiplication version) for the semi-
group \((P_t)_{t \geq 0}\) on \(L^2(m)\) there is a measure space \((X, \Sigma, \mu)\), function \(\lambda \in L^\infty(\mu)\) (here, for any \(p \in [1, \infty]\) we denote \(L^p(\mu) := L^p(X, \Sigma, \mu)\)) and a
unitary operator \(U : L^2(m) \to L^2(\mu)\) such that
\begin{equation}
U^* P_t U = M_t
\end{equation}
where, for every \(t \geq 0\)
\[ M_t g(x) := e^{-t \lambda(x)} g(x), \quad g \in L^2(\mu) \]
is the multiplication operator. Moreover, the following equality holds
\begin{equation}
U^* (-L_2) U = M,
\end{equation}
where \(Mg(x) := \lambda(x) g(x)\). The operator \(-L_2\) is a non-negative definite
self-adjoint operator on \(L^2(m)\), therefore \(\lambda \geq 0 \mu\)-almost everywhere.

It is enough to note that for any \(g \in L^2(\mu)\)
\[ M_\infty g(x) := \lim_{T \to \infty} M_T g(x) = 1_{(\lambda=0)}(x) g(x) \quad \text{for } \mu\text{-almost every } x, \]
i.e., \(M_\infty\) is the orthogonal projection onto the kernel of the operator \(M\). By \((A.2)\) and \((A.3)\), \(P_\infty = UM_\infty U^*\) is the orthogonal projection onto the
kernel of \(-L_2 = UMU^*\).

Using the above fact, we can show that when the jumping measure \(J\) is
irreducible in an appropriate sense, then for every \(f \in L^p(m)\) the function \(P_\infty f\) is constant a.e.

**Fact A.2.** Let \(p \in (1, \infty)\). Assume that for every \(A \in \mathcal{B}\)
\begin{equation}
m(A) > 0, m(A^c) > 0 \implies J(A \times A^c) > 0.
\end{equation}
Then for every \(f \in L^p(m)\)
\[ P_\infty f = \bar{f} \quad \text{m-a.e.,} \]
where
(a) \(\bar{f} \equiv 0\) when \(m(E) = \infty\);
(b) \(\bar{f}\) is equal to the mean value of the function \(f\), i.e.,
\[ \bar{f} \equiv \frac{1}{m(E)} \int_E f \, dm \]
when \(m(E) < \infty\).
Remark A.3. One can prove similar results when the space $E$ can be divided into mutually non-accessible sets.

Proof of the fact. Let us first assume that $p = 2$. By Fact A.1 the following equality holds

$$
\mathcal{E}(P_{\infty}f, P_{\infty}f) = -\langle L_2 P_{\infty}f, P_{\infty}f \rangle = 0.
$$

Therefore, by (1.1) and assumption (A.4) we obtain that $P_{\infty}f$ is constant $m$-a.e. Let $c$ be this constant.

When $m(E) = \infty$, every $m$-a.e. constant function on $L^2(m)$ must be equal to zero $m$-a.e. Otherwise, when $m(E) < \infty$, we have $f, P_{\infty}f \in L^2(m) \subseteq L^1(m)$ and we can write

$$
\int_E f \, dm = \lim_{T \to \infty} \int_E f \, dm = \lim_{T \to \infty} \int_E P_T f \, dm = \lim_{T \to \infty} \langle P_T f, 1 \rangle = \langle P_{\infty} f, 1 \rangle
$$

and therefore $c = \bar{f}$. Here we used (2.5).

Now, we relax the assumption about $p$. Let $p \neq 2$ and fix $\varepsilon > 0$. By density, there is a function $g \in L^1(m) \cap L^\infty(m)$ such that

(A.5) \[ \|f - g\|_p < \varepsilon. \]

Of course, by the contraction property, we have also

(A.6) \[ \|P_T f - P_T g\|_p < \varepsilon. \]

By an interpolation argument we have $g \in L^2(m) \cap L^p(m)$ and therefore $P_{\infty} g = \bar{g}$. Moreover, by conservativeness, we have $P_t \bar{g} = \bar{g}$.

For $p > 2$, using log-convexity of the $L^p$-norm with respect to $p$ and the contraction property, we get

$$
\|P_T g - \bar{g}\|_p^p \leq \|P_T g - \bar{g}\|_2^2 \|P_T g - \bar{g}\|_{\infty}^{p-2} \leq \|P_T g - \bar{g}\|_2^2 \|g - \bar{g}\|_{\infty}^{p-2} \to 0
$$

when $T \to \infty$. Similarly, for $p < 2$

$$
\|P_T g - \bar{g}\|_p^p \leq \|P_T g - \bar{g}\|_2^{2p-2} \|P_T g - \bar{g}\|_1^{2-p} \leq \|P_T g - \bar{g}\|_2^{2p-2} \|g - \bar{g}\|_1^{2-p} \to 0
$$

when $T \to \infty$.

We claim that in the case $m(E) < \infty$

(A.7) \[ \|\bar{g} - \bar{f}\|_p < m(E)^{1/p-1} \varepsilon. \]
Indeed, by Jensen’s inequality

$$|\bar{g} - \bar{f}|^p = m(E)^{-p} \left| \int_E (g - f) \, dm \right|^p \leq m(E)^{-p} \int_E |g - f|^p \, dm < m(E)^{-p} \varepsilon^p.$$ 

This implies (A.7). When $m(E) = \infty$, then, by definition $\bar{f}, \bar{g} \equiv 0$ hence of course $\|\bar{g} - \bar{f}\|_p = 0$.

Finally, in the case $m(E) < \infty$, using (A.5), (A.6) and (A.7) we obtain

$$\|P_T f - \bar{f}\|_p \leq \|P_T f - P_T g\|_p + \|P_T g - \bar{g}\|_p + \|\bar{g} - \bar{f}\|_p < \varepsilon (1 + m(E)^{1/p - 1}) + \|P_T g - \bar{g}\|_p$$

and therefore

$$\limsup_{T \to \infty} \|P_T f - \bar{f}\|_p \leq \varepsilon (1 + m(E)^{1/p - 1}).$$

Since $\varepsilon > 0$ was arbitrary, $\lim_{T \to \infty} \|P_T f - \bar{f}\|_p = 0$.

Similarly $\lim_{T \to \infty} \|P_T f - \bar{f}\|_p = \lim_{T \to \infty} \|P_T f\|_p = 0$ in the case $m(E) = \infty$. 

**Corollary A.4.** Let $p \in (1, \infty)$. Assume that (A.4) holds. If $m(E) = \infty$, then assumption (ii) holds.

**Corollary A.5.** Let $p \in (1, \infty)$. Assume that (A.4) holds. If $m(E) < \infty$, then identity (A.1) can be rewritten in the following form

(A.8)

$$\int_E |f|^p \, dm - \frac{1}{m(E)^{p-1}} \left| \int_E f \, dm \right|^p = \int_0^\infty \int_{E \times E \setminus \text{diag}} F_p(P_t f(x), P_t f(y)) \, J(dx, dy) \, dt.$$ 

**Proof.** It is enough to utilize (A.1), Fact A.2 and notice that $\lim_{T \to \infty} \|P_T f\|_p^p = \|\bar{f}\|_p^p = m(E)^{1-p} \left| \int_E f \, dm \right|^p$. 

**B Continuity of $P_t f$ for Dirichlet forms on $d$-sets**

In this section, we sketch the proof of condition (iii) within the context of Example 1.4.

When assumption (1.5) holds, the semigroup $(P_t)_{t \geq 0}$ has Hölder continuous kernel $p_t(x, y)$ such that for some constants $c_1, c_2 > 0$

(B.1) 

$$c_1 \min \left\{ \frac{1}{t^{d/\alpha}}, \frac{t}{|x - y|^{d+\alpha}} \right\} \leq p_t(x, y) \leq c_2 \min \left\{ \frac{1}{t^{d/\alpha}}, \frac{t}{|x - y|^{d+\alpha}} \right\}$$
for all $x, y \in E$ and $t \in (0,1]$; see Theorem 1.1. and Theorem 4.14. in [CK03]. This means that $P_t f(t > 0)$ is given by

$$P_t f(x) = \int_E f(y) p_t(x, y) \, m(dy), \quad x \in E,$$

for every $f \in L^p(m), p \in [1,\infty]$. 

To prove assumption (i) we will use the upper bound from (B.1) and Hölder continuity of $p_t$. 

Let us first consider $f \in L^1(m)$ and $t \in (0,1)$.

Then there are the constants $c(t), \beta > 0$ such that $|p_t(x_1, y) - p_t(x_2, y)| \leq c(t)|x_1 - x_2|^{\beta}$ for all $x_1, x_2, y \in E$. By a simple calculation

$$|P_t f(x_1) - P_t f(x_2)| \leq c(t)|x_1 - x_2|^{\beta} \|f\|_1,$$

hence $P_t f \in C(E)$. 

Now, let $f \in L^\infty(m)$ and $t \in (0,1)$. Let $f_k := f 1_{B(x_0, k)}$ for some arbitrary $x_0 \in E$. Let $K \subseteq E$ be any compact set. Then for all $x \in K$, by (B.1) and standard calculations, we can show that for sufficiently large $k$

$$|P_t f(x) - P_t f_k(x)| \leq c_2\|f\|_\infty \int_{E \setminus B(x_0, k)} \frac{m(dy)}{|y - x_0|^{d+\alpha}}.$$

Therefore, $P_t f_k$ converges uniformly to $P_t f$ on $K$ when $k \to \infty$ for every compact set $K$. Since $f_k \in L^1(m), P_t f_k \in C(E)$, hence also $P_t f \in C(E)$.

Finally, for $f \in L^p(m)$ with any $p \in (1, \infty)$ it is enough to use decomposition $f = f 1_{\{|f| \geq 1\}} + f 1_{\{|f| < 1\}} \in L^1(m) + L^\infty(m)$.

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