A theorem of Mislin for cohomology of fusion systems and applications to block algebras of finite groups

Constantin-Cosmin Todea

Department of Mathematics, Technical University of Cluj-Napoca, Str. G. Baritiu 25, Cluj-Napoca 400027, Romania

Abstract

We give an algebraic proof for a theorem of Mislin in the case of cohomology of saturated fusion systems defined on $p$-groups when $p$ is odd. Some applications of this theorem to block algebras of finite groups are also given.

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1. Introduction

A saturated fusion system $\mathcal{F}$ on a finite $p$-group $P$ is a category whose objects are the subgroups of $P$ and whose morphisms satisfy certain axioms mimicking the behavior of a finite group $G$ having $P$ as a Sylow subgroup. The axioms of saturated fusion systems were invented by Puig in early 1990’s. We follow [2] for basic facts regarding saturated fusion systems. Let $k$ be an algebraically closed field of characteristic $p$. We denote by $H^\ast(G,k)$ the cohomology algebra of a group $G$ with trivial coefficients. We denote by $H^\ast(\mathcal{F})$ the subalgebra of $\mathcal{F}$-stable elements in $H^\ast(P,k)$, i.e. the cohomology algebra of the saturated fusion system $\mathcal{F}$, which is the subalgebra of $H^\ast(P,k)$ consisting of elements $\zeta \in H^\ast(P,k)$ such that

$$\text{res}_Q^P(\zeta) = \text{res}_\phi(\zeta),$$

for any $\phi \in \text{Hom}_\mathcal{F}(Q,P)$ and any subgroup $Q$ of $P$.

A celebrated theorem of Mislin in [15] regarding the control of fusion in group cohomology has now a new short algebraic proof for $p$ odd thanks to Benson, Grodal and Henke, see [4]. Other algebraic proofs are now completed in [7], [17] and [20]. Also, in [8, Remark 5.8] Linckelmann suggests a topological proof for Mislin’s theorem in the case...
of block algebras of finite groups, more precisely for cohomology of fusion systems associated to blocks. We prove this theorem of Mislin in the general context of cohomology of saturated fusion systems for $p$ odd, by completing all the missing pieces.

$H^*(\mathcal{F})$ is a graded-commutative finitely generated $k$-algebra, hence we can associate the spectrum of maximal ideals, i.e. the variety denoted $V_\mathcal{F}$. Let $\mathcal{G}$ be a saturated fusion subsystem of $\mathcal{F}$ defined on the same finite $p$-group $P$. We have an inclusion map

$$i : H^*(\mathcal{F}) \rightarrow H^*(\mathcal{G}),$$

which induces a map on varieties

$$i^* : V_\mathcal{G} \rightarrow V_\mathcal{F}.$$ 

The main result of this paper is the following theorem which contains Mislin’s theorem for saturated fusion systems as a special case, when $p$ is odd.

**Theorem 1.1.** Let $\mathcal{G}$ be a saturated fusion subsystem of $\mathcal{F}$ defined on the same finite $p$-group $P$ and $p$ an odd prime. If for each $\zeta \in H^*(\mathcal{G})$ we have $\zeta^{p^r} \in \text{Im}(i)$ for some $r \geq 0$, then $\mathcal{G} = \mathcal{F}$. Moreover we have $H^*(\mathcal{F}) = H^*(\mathcal{G})$ if and only if $\mathcal{G} = \mathcal{F}$.

We will deduce Theorem 1.1 from a theorem given in terms of saturated fusion systems ([4, Theorem B]). Let $\mathcal{G}$ be a saturated fusion subsystem of $\mathcal{F}$ defined on the same finite $p$-group $P$. For shortness, we will say that $\mathcal{G}$ controls $p$-fusion in $\mathcal{F}$ on elementary abelian $p$-subgroups if $\text{Hom}_{\mathcal{G}}(E_1, E_2) = \text{Hom}_{\mathcal{F}}(E_1, E_2)$ for all $E_1, E_2 \leq P$, where $E_1, E_2$ runs over the set of elementary abelian $p$-subgroups of $P$. Another ingredient for proving Theorem 1.1 is the following theorem which says that controlling $p$-fusion on elementary abelian subgroups happens if and only if $i^*$ is a bijective map. [1, Theorem 2] and [14, Proposition 10.9] are similar statements for group cohomology.

**Theorem 1.2.** Let $\mathcal{G}$ be a saturated fusion subsystem of $\mathcal{F}$ defined on the same finite $p$-group $P$. Then $i^*$ is surjective. Moreover we have that $i^*$ is an injective map if and only if $\mathcal{G}$ controls $p$-fusion in $\mathcal{F}$ on elementary abelian $p$-subgroups.

To prove Theorem 1.2 we will use Quillen stratification for cohomology of saturated fusion systems given by Markus Linckelmann in [13], for which the proof is the same as for block algebras [11], with some minor adjustments. Since it is not appeared in this form in the literature, for completeness, we state it in the next lines. For any subgroup $Q$ of $P$ denote by $V_Q$ the maximal ideal spectrum of $H^*(Q, k)$, and set $V^+_Q = V_Q \setminus \bigcup_{R \leq Q} \text{res}^Q_R V_R$.

Denote by $V_{\mathcal{F}, Q}$ and $V^+_{\mathcal{F}, Q}$ the images of $V_Q$ and $V^+_Q$ in $V_\mathcal{F}$ under the map $r^*_{\mathcal{F}, Q}$ induced by the algebra homomorphism $r_{\mathcal{F}, Q} : H^*(\mathcal{F}) \rightarrow H^*(Q, k)$ given by composing the inclusion $H^*(\mathcal{F}) \subseteq H^*(P, k)$ with the restriction $\text{res}^P_Q : H^*(P, k) \rightarrow H^*(Q, k)$. 

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Theorem 1.3. ([13, Theorem 1]) With the notation above, the following hold.

(i) The variety $V_{\mathcal{F}}$ is the disjoint union of the locally closed subvarieties $V^+_{\mathcal{F},E}$, where $E$ runs over a set of representatives of the $\mathcal{F}$-isomorphism classes of elementary abelian subgroups of $P$.

(ii) Let $E$ be an elementary abelian subgroup of $P$. The group $W_{\mathcal{F}}(E) = \text{Aut}_{\mathcal{F}}(E)$ acts on $V^+_E$ and the restriction map $\text{res}^P_E$ induces an inseparable isogeny $V^+_E / W_{\mathcal{F}}(E) \to V^+_F$.

Remark 1.1. From [18] we know that any saturated fusion system $\mathcal{F}$ can be identified with $\mathcal{F}_P(G)$ for some finite group $G$ where in the last category the morphisms are given by conjugation with elements from $G$. It follows that for any elementary abelian $p$-subgroup $E$ of $P$ we can identify $\text{Aut}_{\mathcal{F}}(E)$ with $N_G(E)/C_G(E)$. Now, similar arguments to [6, Theorem 9.1.7] assure us that $\text{Aut}_{\mathcal{F}}(E)$ acts free on $V^+_E$.

In Section 2 we will prove Theorem 1.1 and 1.2. In Section 3 we give applications of Mislin’s Theorem to the case of block algebras of finite groups. It is known that if two blocks are basic Morita equivalent then there is an isomorphism between the defect groups which induces an equivalence between their local categories [16, 7.6.6]. We will prove that in some cases Mislin’s theorem and an inclusion between some subalgebras of stable elements in Hochschild cohomology algebras of group algebras over the defect groups (with respect to the source algebras of the blocks; see [8]) can give us an alternative method to obtain an equality between the associated fusion systems (Brauer categories). Mislin’s Theorem and cohomological techniques have implications for other results involving block algebras of finite groups. For example, under some assumptions (which are requested to apply Mislin’s theorem as above) we will show that two block algebras, which are splendid stable equivalent in the sense of Linckelmann [10], have the same fusion systems (Theorem 3.2).

2. Proofs of theorems from Section 1

Proof of Theorem 1.2. Clearly $\text{Ker}(i)$ is a nilpotent ideal and $H^*(\mathcal{G})$ is finitely generated as $H^*(\mathcal{F})$-module. This yields the surjectivity of $i^*$.

Suppose that $\mathcal{G}$ controls $p$-fusion in $\mathcal{F}$ on elementary abelian $p$-subgroups. Let $m_1, m_2 \in V_\mathcal{G}$ such that $i^*(m_1) = i^*(m_2)$. By Quillen stratification for $V_\mathcal{G}$ (Theorem 1.3), there are two elementary abelian $p$-subgroups $E_1, E_2 \leq P$ uniquely up to $\mathcal{G}$-isomorphism and $\gamma_1 \in V^+_{E_1}, \gamma_2 \in V^+_{E_2}$ such that

$$m_1 = r_{\mathcal{G}, E_1}(\gamma_1), m_2 = r_{\mathcal{G}, E_2}(\gamma_2)$$
and let

\[ (i^* \circ r^*_{\mathcal{F}, E_1})(\gamma_1) = (i^* \circ r^*_{\mathcal{F}, E_2})(\gamma_2), \]

and since \( r^*_{\mathcal{F}, E_1} \circ i = r^*_{\mathcal{F}, E_2} \), we obtain that \( r^*_{\mathcal{F}, E_1}(\gamma_1) = r^*_{\mathcal{F}, E_2}(\gamma_2) \). Quillen stratification for \( V_{\mathcal{F}} \) (Theorem 1.3) gives us that \( E_1 \) is \( \mathcal{F} \)-isomorphic to \( E_2 \) hence \( E_1 \) is \( \mathcal{G} \)-isomorphic to \( E_2 \). This allow us to choose \( E_1 = E_2 = E \) such that \( r^*_{\mathcal{F}, E}(\gamma_1) = r^*_{\mathcal{F}, E}(\gamma_2) \). The inseparable isogeny from Theorem 1.3 (ii) is given by \( r^*_{\mathcal{F}, E} \) so \( \gamma_1, \gamma_2 \) are in the same orbit of the action of \( W_{\mathcal{G}}(E) \) on \( V^+_E \). The control of fusion on elementary abelian \( p \)-subgroups assure us that \( \gamma_1, \gamma_2 \) are in the same orbit of the action of \( W_{\mathcal{G}}(E) \) on \( V^+_E \). In conclusion

\[ m_1 = r^*_{\mathcal{G}, E}(\gamma_1) = r^*_{\mathcal{G}, E}(\gamma_2) = m_2. \]

Conversely suppose that \( i^* \) is injective. First we prove that if \( E_1, E_2 \) are \( \mathcal{F} \)-isomorphic elementary abelian \( p \)-subgroups then they are also \( \mathcal{G} \)-isomorphic. So let \( E_1, E_2 \) be \( \mathcal{F} \)-isomorphic. Then \( r^*_{\mathcal{F}, E_1}(V^+_E) = r^*_{\mathcal{F}, E_2}(V^+_E) \), hence

\[ i^*(r^*_{\mathcal{F}, E_1}(V^+_E)) = i^*(r^*_{\mathcal{F}, E_2}(V^+_E)). \]

Since \( i^* \) is injective we obtain that \( V^+_E = V^+_E \), hence \( E_1, E_2 \) are \( \mathcal{G} \)-isomorphic.

Secondly we prove that \( \text{Aut}_{\mathcal{F}}(E) = \text{Aut}_{\mathcal{G}}(E) \) for any elementary abelian \( p \)-subgroup \( E \) of \( P \). Since \( V^+_E = i^*(V^+_E) \) we obtain that \( i^* \) induces a bijection between \( V^+_E \) and \( V^+_E \). This bijection, the inclusion \( \text{Aut}_{\mathcal{G}}(E) \subseteq \text{Aut}_{\mathcal{F}}(E) \), Remark 1.1 and the similar statements for \( V^+_E \) give the desired equality; the definitions of the bijection from \( V^+_E \) to \( V^+_E / \text{Aut}_{\mathcal{E}}(E) \) (Theorem 1.3) and of \( i^* \) are important for showing this equality.

Let \( \varphi \in \text{Hom}_{\mathcal{F}}(E_1, E_2) \) which give us the decomposition

\[ E_1 \xrightarrow{\varphi_1} \varphi(E_1) \xrightarrow{\varphi(E_1)} E_2, \]

where \( \varphi_1 : E_1 \rightarrow \varphi(E_1) \) is an isomorphism in \( \mathcal{F} \) hence, from the above, we have that \( E_1 \) is \( \mathcal{G} \)-isomorphic to \( \varphi(E_1) \). It follows that there is \( \alpha : E_1 \rightarrow \varphi(E_1) \) an isomorphism in \( \mathcal{G} \) such that \( \alpha^{-1} \circ \varphi_1 \in \text{Aut}_{\mathcal{F}}(E_1) \), that is \( \alpha^{-1} \circ \varphi_1 \in \text{Aut}_{\mathcal{G}}(E_1) \). It is easy to see now that \( \varphi \in \text{Hom}_{\mathcal{G}}(E_1, E_2) \).

Proof of Theorem 1.1 From the hypothesis we obtain that \( i^* \) is an \( F \)-isomorphism. In particular \( i^* \) is bijective. By Theorem 1.2 we get that \( \mathcal{G} \) controls \( p \)-fusion in \( \mathcal{F} \) on elementary abelian \( p \)-subgroups, hence [4, Theorem B] assure us that \( \mathcal{G} = \mathcal{F} \).

3. Applications of Mislin’s theorem to block algebras of finite groups

Let \( H, G \) be two finite groups with a common \( p \)-subgroup \( P \). Let \( b \) be a block of \( kG \) and let \( c \) be a block of \( kH \) with the same defect group \( P \). Let \( P_i, P_j \) be defect pointed
groups of $G_{(b)}$, respectively $H_{(c)}$ and $i \in \gamma, j \in \delta$ some source idempotents. Let $(P, eP)$ be a maximal $(G, b)$-Brauer pair associated to $P_G$ and $(P, fP)$ be a maximal $(H, c)$-Brauer pair associated to $P_S$. For any subgroup $R$ of $P$, there is a unique block $e_R$ of $C_G(R)$ such that $\text{Br}_R(i)e_R \neq 0$. Then $(R, e_R)$ is a $(G, b)$-Brauer pair and $e_R$ is also the unique block of $C_G(R)$ such that $(R, e_R) \leq (P, e_P)$. We define $\mathcal{T}(P, e_P)(G, b)$ as the category which has as objects the set of subgroups of $P$; for any two subgroups $R, S$ of $P$ the set of morphisms from $R$ to $S$ in $\mathcal{T}(P, e_P)(G, b)$ is the set of (necessarily injective) group homomorphisms $\varphi : R \to S$ for which there is an element $x \in G$ satisfying $\varphi(u) = xux^{-1}$ for all $u \in R$ and satisfying $x(R, e_R) \leq (S, e_S)$. The category $\mathcal{T}(P, e_P)(G, b)$ is sometimes called the Brauer category of $b$ with respect to the choice of $(P, e_P)$ and is a saturated fusion system. The analogous definitions give $\mathcal{T}(P, f_P)(H, c)$.

For the rest of this section we assume that $\mathcal{T}(P, f_P)(H, c)$ is a subsystem of $\mathcal{T}(P, e_P)(G, b)$ and $p$ is an odd prime.

We refer the reader to [8] for results regarding transfer maps and stable elements in Hochschild cohomology algebras. Recall that if $A, B$ are two symmetric $k$-algebras and $X$ is a bounded complex of $A - B$-bimodules whose components are projective as left and right modules, there is a graded $k$-linear map $t_X : \text{HH}^* (B) \to \text{HH}^* (A)$ called the transfer map associated to $X$ [8, Definition 2.9]. If $\pi_X = t_X^0 (1_B) \in Z(A)$ (the relatively $X$-projective element) is invertible then $T_X = \pi_X^{-1} t_X$ is called the normalized transfer map associated to $X$. If both $\pi_X, \pi_X^*$ are invertible then $T_X$ induces a graded $k$-algebra isomorphism $T_X : \text{HH}^*_X (B) \to \text{HH}^*_X (A)$. Here $\text{HH}^*_X (A)$ is the graded subalgebra of $X$-stable elements in $\text{HH}^* (A)$; more precisely $\zeta \in \text{HH}^*_X (A)$ if there is $\theta \in \text{HH}^* (B)$ such that $\zeta \otimes \text{Id}_X = \text{Id}_X \otimes \theta$ in $\text{Ext}_A \otimes B^p (X, X)$, see [19, Definition 2]. For example, we denote by $\text{HH}^*_{ikGi}(kP)$ (respectively $\text{HH}^*_{jkHj}(kP)$) the subalgebra of $ikGi$-stable ($jkHj$-stable) elements in the Hochschild cohomology algebra of $kP$. $ikGi$ (respectively $jkHj$) is called the source algebra of $b$ (respectively $c$) and we obtain decompositions as $kP - kP$-bimodules

$$
\text{ikGi} \cong \bigoplus_{g \in \gamma\gamma_{G,b}} k[P_gP], \quad \text{jkHj} \cong \bigoplus_{h \in \gamma\gamma_{H,c}} k[P_hP]
$$

where $\gamma_{G,b} \subseteq [P \setminus G/P]$ and $\gamma_{H,c} \subseteq [P \setminus H/P]$, see [21, Theorem 44.3].

Next proposition may be regarded as a tool for checking when two blocks of two finite groups have the same local structure and is a first application of Mislin’s theorem (Theorem 11) to block algebras. Recall that $\delta_P : \text{H}^*(P, k) \to \text{HH}^*(kP)$ is the embedding defined in [8, Proposition 4.5].

**Proposition 3.1.** With the above assumptions and notations if

$$
\text{HH}^*_{jkHj}(kP) \cap \delta_P(\text{H}^*(P, k)) \subseteq \text{HH}^*_{ikGi}(kP)
$$
then \( \mathcal{F}_{(P,fp)}(H, c) = \mathcal{F}_{(P,ep)}(G, b) \). In particular if \( ikGi \) is isomorphic with a direct summand of \( jkH j \) as \( kP - kP \)-bimodules then \( \mathcal{F}_{(P,fp)}(H, c) = \mathcal{F}_{(P,ep)}(G, b) \).

**Proof.** The second statement follows from the first if we use [8, Proposition 3.5, (iv)]. For the first statement, we know from [8, Proposition 5.4] that

\[
\delta_P(H^*(\mathcal{F}_{G,b})) \subseteq \text{HH}^*_{ikGi}(kP).
\]

Moreover [19, Theorem 1] is a nice result which tell us more precisely that

\[
\delta_P(H^*(\mathcal{F}_{G,b})) = \text{HH}^*_{ikGi}(kP) \cap \delta_P(H^*(P,k)),
\]

\[
\delta_P(H^*(\mathcal{F}_{H,c})) = \text{HH}^*_{jkH j}(kP) \cap \delta_P(H^*(P,k)).
\]

It follows that \( \delta_P(H^*(\mathcal{F}_{(P,ep)}(G,b))) = \delta_P(\mathcal{F}_{(P,fp)}(H,c)) \), hence

\[
H^*(\mathcal{F}_{(P,ep)}(G,b)) = H^*(\mathcal{F}_{(P,fp)}(H,c)).
\]

Now, Theorem 1.1 gives us our desired conclusion.

**Remark 3.1.** There exists blocks for which we can apply Proposition 3.1. For example let \( kGb \) and \( kHc \) be two block algebras for which there is \( M \) an indecomposable direct summand of the \( kGb - kHc \)-bimodule \( kGi \otimes_{kP} jkH j \) which induces a Morita equivalence (i.e. there is a Morita equivalence between the blocks induced by a \( p \)-permutation bimodule), see [10, Theorem 4.1] or [16, 7.5.1]. In this case we actually have an isomorphism of \( kP - kP \)-bimodules between the source algebras. Moreover if \( H \leq G \) such that \( Y_{G,b} \subseteq Y_{H,c} \), or \( P \subseteq G \) such that we can choose \( [N_{G}(P_{\gamma})/P_{C}(P)] \subseteq [N_{H}(P_{\delta})/P_{C}(P)] \) then we can apply Proposition 3.1.

It is well known that if \( kHc \) and \( kGb \) are splendid stable equivalent then their cohomology algebras are isomorphic \( H^*(\mathcal{F}_{(P,ep)}(G,b)) \cong H^*(\mathcal{F}_{(P,fp)}(H,c)) \). With our assumptions Mislin’s theorem applied to block algebras gives a new method to prove that if two block algebras are splendid stable equivalent then the associated fusion systems are the same.

**Theorem 3.2.** Let \( b,c \) be two blocks with the above assumptions. Let \( X \) be a bounded complex of \( kHc - kGb \)-bimodules whose components are isomorphic with direct sums of direct summands of \( kH j \otimes_{kQ} ikG \), where \( Q \) runs over the set of subgroups of \( P \). If \( X \) induces a splendid stable equivalence (i.e. \( X \otimes_{kGb} X^* \cong kHc \oplus U_c, X^* \otimes_{kHc} X \cong kGb \oplus U_b \), where \( U_c \) is a bounded complex of projective \( kHc \)-bimodules and \( U_b \) is a bounded complex of projective \( kGb - kGb \)-bimodules) then \( \mathcal{F}_{(P,fp)}(H,c) = \mathcal{F}_{(P,ep)}(G,b) \).
Proof. We mimic the proof of [9, Theorem 5.5., (ii)] to obtain the following commutative diagram

\[
\begin{CD}
H^*(\mathcal{F}_{(P,eP)}(G,b)) @>{T_{kGi}\circ \delta_P}>> HH_{X^*}(kGb).
\end{CD}
\]

where \(T_X, T_{kGi}, T_{kHj}\) are the normalized transfer maps. Since the projective elements \(\pi_X, \pi_{X^*}\) are invertible ([9, Theorem 5.5, (i)]) the map \(T_X\) is an isomorphism. For completeness we repeat some of the arguments of that proof. By [8, 3.2.3] we may choose symmetrizing forms such that \(\pi_{kGi} = 1_{kGb}, \pi_X = 1_{kHc}\), or equivalently \(T_{kGi} = t_{kGi}, T_X = t_X\). The relatively projective elements \(\pi_{kHj}, \pi_{jkH}\) are still invertible, [8, Proposition 5.4, (iv)]. In order to show the commutativity of the above diagram we need to show that \(T_{jkH} \circ T_X \circ T_{kGi} \circ \delta_P = \delta_P \circ i\). This is equivalent to showing that

\[
\pi_{jkH}^{-1} \circ t_{jkH} \circ t_X \circ t_{kGi} \circ \delta_P = \delta_P \circ i,
\]

hence equivalent to

\[
\pi_{jkH}^{-1} \circ t_{jX,i} \circ \delta_P = \delta_P \circ i.
\]

With our notations by [8, Proposition 5.7, (iv)] we know that the map \(t_{jX,i}\) acts as multiplication by \(\pi_{jX,i} \) on \(\delta_P(H^*(\mathcal{F}_{(P,eP)}(G,b)))\), so all we need to show is that \(\pi_{jX,i} = \pi_{jkH}\). But this is true since by [8, 3.2] we have

\[
\pi_{jX,i} = t_{jX,i}^0(1_{kP}) = (t_{jkH}^0 \circ t_X^0 \circ t_{kGi}^0)(1_{kP}) = t_{jkH}^0(t_X^0(1_{kGb})) = t_{jkH}^0(1_{kHc}) = \pi_{jkH}.
\]

We denote by \(\tau_b\) the injective graded \(k\)-algebra homomorphism \(T_{kGi} \circ \delta_P\). Similarly \(\tau_c\) is \(T_{kHj} \circ \delta_P\). By [12, Theorem 1.1] \(\tau_b, \tau_c\) induce isomorphisms of varieties

\[
\tau_b^*: X_{kGb} \to V_{\mathcal{F}_{(P,eP)}(G,b)}
\]

\[
\tau_c^*: X_{kHc} \to V_{\mathcal{F}_{(P,fP)}(H,c)}
\]

hence \(\tau_b^*, \tau_c^*\) are bijective maps; where \(X_{kGb}, X_{kHc}\) are the varieties of the Hochschild cohomology algebras \(HH^*(kGb), HH^*(kHc)\). The first diagram yields

\[
T_X \circ \tau_b = \tau_c \circ i
\]

hence

\[
\tau_b^* \circ T_X^* = i^* \circ \tau_c^*.
\]

Since \(T_X\) is an isomorphism it follows that \(i^*\) is injective. Now Theorem [1,1] give us the conclusion. \(\square\)
In the last remark of this paper we give a new consequence of Mislin’s theorem. We obtain a generalization of a theorem of Watanabe [22, Theorem 2] to the class of all finite groups, not just for \( p \)-solvable groups.

**Remark 3.2.** Let \( R \) be a normal subgroup of \( P \) such that \( N_G(P) \leq N_G(R) \). We denote by \( d \) the Brauer correspondent of \( b \) in \( N_G(R) \), that is \( d \) is the unique block of \( N_G(R) \) such that \( Br_P(d) = Br_P(b) \), where \( Br_P \) is the Brauer homomorphism from \( (kG)^P \) to \( kC_G(P) \). Set \( N = N_G(R) \). Then \( b = d^G \) and \((P, e_P)\) is a maximal \((N, d)\)-Brauer pair. Moreover \( \mathcal{F}_{(P, e_P)}(N, d) \) is a subsystem of \( \mathcal{F}_{(P, e_P)}(G, b) \). Now from Theorem \[ \text{Theorem 1.1} \] we have that \( H^*(\mathcal{F}_{(P, e_P)}(N, d)) = H^*(\mathcal{F}_{(P, e_P)}(G, b)) \) if and only if \( \mathcal{F}_{(P, e_P)}(N, d) = \mathcal{F}_{(P, e_P)}(G, b) \).

**References**

[1] J. L. Alperin, On a theorem of Mislin, J. Purre Appl. Alg. 206 (2006), 55-58.

[2] M. Aschbacher, R. Kessar, B. Oliver, Fusion systems in algebra and in topology, Cambridge University Press, Cambridge, 2011.

[3] D. J. Benson, Representations and cohomology II: Cohomology of groups and modules, Cambridge University Press, Cambridge, 1991.

[4] D. J. Benson, J. Grodal, E. Henke, Group cohomology and control of \( p \)-fusion, Inv. Math. (2013), to appear.

[5] C. Broto, R. Levi, B. Oliver, The homotopy theory of fusion systems, J. Amer. Math. Soc. 16 (2003), 779-856.

[6] L. Evens, The Cohomology of Groups, Clarendon Press, Oxford, New York, Tokyo, 1991.

[7] A. Hida, Control of fusion and cohomology of trivial source modules, J. Alg. 317 (2007), 462-470.

[8] M. Linckelmann, Transfer in Hochschild cohomology of blocks of finite groups, Alg. Repres. Theory 2 (1999), 107-135.

[9] M. Linckelmann, Varieties in block theory, J. Alg. 215 (1999), 460-480.

[10] M. Linckelmann, On splendid derived and stable equivalences between blocks of finite groups, J. Alg 242 (2001), 819-843.

[11] M. Linckelmann, Quillen stratification for block varieties, J. Pure Appl. Algebra 172 (2002), 257270.
[12] M. Linckelmann, The Hochschild and block cohomology varieties are isomorphic, J. London Math Soc 81 (2010), 381-411.

[13] M. Linckelmann, Quillen stratification for fusion systems, preprint-private communication.

[14] D. Quillen, The spectrum of an equivariant cohomology ring II, Ann. Math. 94 (1971), 573-602.

[15] G. Mislin, On group homomorphisms inducing mod-$p$ isomorphisms, Comment. Math. Helvetici 65 (1990), 454-461.

[16] L. Puig, On the Local Structure of Morita and Rickard Equivalences between Brauer Blocks, Progress in Mathematics (Boston, Mass.), Vol. 178, Birkhäuser, Basel, 1999.

[17] T. Okuyama, On a theorem of Mislin on cohomology isomorphism and control of fusion, Cohomology Theory of Finite Gourups, RIMS Kokyuroky, Kyoto University, vol. 1466, 2006, pp.86-92.

[18] S. Park, Realizing a fusion system by a single finite group, Arch. Math. 94 (2010), 405-410.

[19] H. Sasaki, Cohomology of block ideals of finite group algebras and stable elements, Alg. Repres. Theory 16 (2013), 1039-1049.

[20] P. Symonds, On cohomology isomorphisms of groups, J. Alg 313 (2007), 802-810.

[21] J. Thévenaz, G-Algebras and Modular Representation Theory. Clarendon Press, Oxford (1995)

[22] A. Watanabe, Note on blocks of $p$-solvable groups with same Brauer category (Cohomology Theory of Finite Groups and Related Topics), http://hdl.handle.net/2433/48047 1466 (2006), 115-118.