Memory effects in adiabatic quantum pumping with parasitic nonlinear dynamics

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The charge current adiabatically pumped through a mesoscopic region coupled to a classical variable obeying a nonlinear dynamics is studied within the scattering matrix approach. Due to the nonlinearity in the dynamics of the variable, an hysteretic behavior of the pumping current can be observed for specific characteristics of the pumping cycle. The steps needed to build a quantum pump working as a memory device are discussed together with a possible experimental implementation.

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I. INTRODUCTION

The quantum pumping proposed by Thouless is a phase coherence effect able to pump dc current by the out-of-phase adiabatic modulation of two system parameters. The phase difference \( \varphi \) of the external signals produces a charge current proportional to \( \omega \sin(\varphi) \), \( \omega \) being the adiabatic pumping frequency. After the pioneering work of Thouless, a scattering matrix approach to the adiabatic quantum pumping has been proposed by Brouwer. Following the Brouwer formulation, the notion of non-interacting quantum pump has been developed in several directions. The most important application is the possibility to achieve a pure spin current injector, which is an important goal in spintronics. Recently the idea of interacting quantum pump, beyond the free electrons model, has been introduced. To get read of non-trivial interaction effects, typically described in the context of the Anderson model for quantum-dot systems, a non-equilibrium Green’s functions (NEGF) theory of the quantum pumping has been developed using several approximations. Alternatively to the NEGF approach, a powerful diagrammatic technique based on a generalized master equation method has been developed. More recently, another interesting study was performed aiming at generalizing Brouwer’s formula for interacting systems to include inelastic scattering events.

From the experimental point of view pumping effects have been analyzed in confined nano-structures, as quantum dots, where the realization of a periodic time-dependent potential can be achieved by modulating gate voltages. Very recently it has been proposed the idea that the presence of a classical state variable inside the scattering region of the system can strongly affect the form of the pumped current. This effect is produced by the linear dynamics of a classical variable which modifies the phase relations introducing a dynamical phase shift. In this work we study the charge current pumped through a mesoscopic region coupled to a parasitic classical variable governed by a weakly nonlinear dynamics supposed adiabatic on the typical time scales of the electron transport.

The organization of the paper is the following: In Sec. II we summarize the scattering matrix approach to the quantum pumping as formulated by M. Moskalets and M. Büttiker in Ref. [4]. In Sec. III we derive a general expression for the current pumped through a mesoscopic region containing a parasitic internal degrees of freedom governed by weakly nonlinear dynamics. In Sec. IV the specific case of the Duffing nonlinear oscillator is discussed, while the conclusions are given in Sec. V.

II. SCATTERING THEORY OF QUANTUM PUMPING

In the standard theory of the quantum pumping (i.e. without internal classical dynamics) a mesoscopic scatterer coupled to two external leads (left/right) taken at the same temperature \( T \) and electrochemical potential \( \mu \approx E_F \), \( E_F \) being the Fermi energy, is considered. When the scatterer is subjected to two controllable parameters \( X_i \), whose adiabatic time dependence is \( X_i(t) = X_i^0 + X_i^\omega \sin(\omega t + \varphi_i) \), the mesoscopic phase coherent sample is characterized by the scattering matrix \( S(X_1(t), X_2(t)) \). Exploiting an instant scattering description and assuming that the pumping amplitudes \( X_i^\omega \) are small quantities compared to \( X_i^0 \), the scattering matrix takes the following form:

\[
S(X_1(t), X_2(t)) \approx S(X_1^0, X_2^0) + s_+ e^{i\omega t} + s_- e^{-i\omega t},
\]

where \( s_\alpha \) are related to the parametric derivatives of the scattering matrix. Under equilibrium condition of the external leads, the electrons with energy \( E \) entering the scattering region (in-states) are described by the Fermi distribution \( f_{\alpha}^\text{in}(E) = f_0(E) = (1 + \exp((E - E_F)/kT))^{-1} \), where \( \alpha \in \{\text{left, right}\} \). On the other hand, the distribution function \( f_{\alpha}^\text{out}(E) \) of the outgoing particles leaving the mesoscopic region and entering the reservoir (namely out-states) is affected by the interaction with the oscillating scatterer allowing the electrons to absorb or emit an energy quantum \( \hbar \omega \). The latter mechanism changes the initial distribution function \( f_0(E) \) and produces a non-equilibrium distribution responsible for the generation of a finite particles current. To compute the non-equilibrium distribution of the outgoing particles \( f_{\alpha}^\text{out}(E) \)
In case of a quantum pump working with two time modulated parameters can be described in terms of the classical Green’s function of the scattering region. The notion of parasitic variable is conceptually similar to the one of parasitic impedance in electrical circuits theory: i.e. a degree of freedom not considered in designing the device whose dynamics may affect the performance of the system in an unexpected (usually unwanted) way. In that context, once the presence of an hidden impedance is recognized, an equivalent circuit model of the whole system can be obtained and a proper control of the device is achieved. In full analogy to this well known situation, we would like to build a scattering theory of the two-parameters Thouless pumping able to take into account the parasitic dynamics. The resulting theory is equally applicable when the internal dynamics is a priori known as is the case for a deformable quantum dot system as considered in Ref. [10]. Thus, when a classical state variable is indirectly excited by the pumping parameters, an internal phase shift may activate additional contributions to the charge current. This mechanism becomes very interesting in the case of a weakly nonlinear dynamics able to produce non-trivial internal phase shifts.

III. QUANTUM PUMPING WITH A PARASITIC NONLINEAR VARIABLE

We now consider the case when the dynamics of the oscillating scatterer is affected by the internal classical dynamics of a parasitic variable $y(t)$. The nature of the variable $y$, if not recognized and included in the pumping model, does not allow to precisely control the system even though the external pumping parameters $X_i(t)$ act as a driving force on the parasitic dynamics. In this way the scattering matrix $S(X_1(t), X_2(t); y(t))$ depends parametrically on the pumping parameters and on the parasitic variable whose dynamics is controlled by the pumping cycle in a way which is a priori unknown. When the equation of motion of $y(t)$ is linear (or when it can be linearized around the working point), the dynamics of the parasitic variable in the presence of external parameters can be described in terms of the classical Green’s function $\chi(t - t')$,

$$y(t) = \int_{-\infty}^{\infty} dt' \chi(t-t') F(X_1(t'), X_2(t')),$$

(7)

$F(X_1, X_2)$ being the forcing term of the differential equation. Once the dynamics has been formally solved the charge current can be computed as done in Eq. (3) or in the multi-parametric case. In the latter case, already discussed in Ref. [10], the dynamical phase shift $\phi_D$ induced by the Fourier transform $|\chi| e^{i\phi_D}$ of $\chi(t - t')$ plays an important role in generating additional contributions to the pumping current. Such additional terms, weighted by $|\chi|$, can give information on the differential operator
related to the parasitic dynamics configuring the quantum pumping as a probe of internal dynamics. When the internal variable follows a nonlinear dynamics all the above considerations are still valid even though new features appear. We consider here the simplest nonlinear dynamics in which the interaction between the electronic degrees of freedom and the parasitic variable is governed by the equation of motion of a forced Duffing oscillator \( y(t) \) in (dimensionless units):

\[
\ddot{y} + \beta \dot{y} + y + \epsilon [-ay + by^3 + \sum_{i=1,2} g_i X_i^r \sin(\tau + \varphi_i)] = 0, \quad (8)
\]

where the \( g_i \) are gain functions, \( a \) is a detuning term able to modify the resonance frequency of the system, while \( b > 0 \) is the nonlinear term. Furthermore, the time \( \tau \) is normalized to the resonance frequency \( \Omega_0 \) of the harmonic oscillator (obtained for \( \epsilon = 0 \)) while the pumping frequency \( \omega \) is tuned on resonance (i.e., \( \omega = \Omega_0 \)). In the weak interaction limit \( \epsilon \ll 1 \) one can measure the linear damping in terms of \( \epsilon \) making the substitution \( \beta \rightarrow \epsilon \beta \). In this way a weakly nonlinear system is obtained which can be treated within the two-timing perturbation theory. Namely two time scales are introduced, one is the fast time scale \( t = \tau \) and the other is the slow time scale \( T = \epsilon \tau \), while the approximate solution of the problem is written as \( y(t, T) \approx y_0(t, T) + \epsilon y_1(t, T) \). The derivative with respect to \( \tau \), i.e., \( d/d\tau \), has to be substituted by the operator \( \partial_t + \epsilon \partial_T \), while \( d^2/d\tau^2 \) (up to the first order in \( \epsilon \)) takes the form \( \partial_T^2 + 2 \epsilon \partial_T \). With the above rules the zero-th order of the Duffing equation reduces to \( \partial_T^2 y_0 + y_0 = 0 \), which is solved by \( y_0(t, T) = R(T) \cos(t + \Phi(T)) \). Using \( y_0(t, T) \) within the first order equation and imposing vanishing secular terms one obtains the following set of equations for \( R(T) \) and \( \Phi(T) \):

\[
\begin{align*}
2R' + \beta R &= f_1 \cos(\Phi) + f_2 \sin(\Phi) \quad (9) \\
2\Phi' R + aR - \frac{3}{4} b R^3 &= -f_1 \sin(\Phi) + f_2 \cos(\Phi),
\end{align*}
\]

where \( f_1 = g_1 X_1^r \cos(\varphi) + g_2 X_2^r \) and \( f_2 = g_1 X_1^r \sin(\varphi) \) (here we fixed \( \varphi_1 = 0 \) and \( \varphi_2 = \varphi \)), while \( R' \) and \( \Phi' \) represent the first derivatives with respect to \( T \). The above equations completely characterize the zero-th order solution of the problem allowing to derive the pumping current produced when a weak nonlinear dynamics is present. When the pumping terms are absent \((f_1 = f_2 = 0)\) the stationary solution is \( y_e = 0 \) and the static scatterer condition is reached. When nonvanishing pumping terms are present the equilibrium condition \( R^2 = 0, \quad \Phi = 0 \) can be written as follows:

\[
R^2 \left\{ \beta^2 + \left[ a - \frac{3}{4} b R^2 \right]^2 \right\} = f_1^2 + f_2^2 \tag{10}
\]

\[
\begin{bmatrix}
\cos(\Phi_e) \\
\sin(\Phi_e)
\end{bmatrix} = M \begin{bmatrix}
\frac{\beta R_e}{\sqrt{f_1^2 + f_2^2}} \\
-a R_e + (3/4) b R^3
\end{bmatrix},
\]

where \( M = (if_1 - i \sigma_y f_2)/\sqrt{f_1^2 + f_2^2} \) is a rotation matrix, being \( \sigma_y \) a Pauli matrix. Once the equilibrium radius \( R_e \) is known solving the first equation in (10) the dynamical phase shift \( \Phi_e \) is determined by the second equation in (10). In this way the asymptotic zero-th order solution is completely determined, \( y_e(\tau) = R_e \cos(\tau + \Phi_e) \).

Using the asymptotic form of the scattering matrix \( S(X_1(\tau), X_2(\tau); y_e(\tau)) \) and proceeding as in the standard case, one obtains the formula of the pumped current:

\[
I_0 = I_{00} + \frac{\epsilon \omega}{2\pi} \sum_{\beta, \beta'} \{ \cos(\Phi_e) \} [R_e X_2^\omega \Lambda_{\alpha \beta}^I \cos(\varphi)] + \sin(\Phi_e) [R_e X_2^\omega \Lambda_{\alpha \beta}^I \sin(\varphi)], \tag{11}
\]

where \( \Lambda_{\alpha \beta}^I = \Im\{\partial_\varphi S_{\alpha \beta} \partial X_j S_{\alpha \beta}^* \} \), while \( I_{00} \) is the standard pumping term as given in Eq. (10). Eq. (11) represents the main result of this work. Its validity goes beyond the Duffing dynamics we are discussing here. Indeed every weakly nonlinear problem of the form

\[
\dot{x} + x + \epsilon h(x, \dot{x}, \tau) = 0 \tag{12}
\]

has a zero-th order asymptotic solution of the form \( x_e(\tau) = R_e \cos(\tau + \Phi_e) \), being \( R_e \) and \( \Phi_e \) determined by the precise form of the nonlinear time-periodic function \( h(x, \dot{x}, \tau) \). Thus Eq. (11) has to be considered valid for the whole class of weakly nonlinear problems. From the experimental point of view, the parasitic nonlinear dynamics of the variable \( y_e(\tau) \) can be probed using Eq. (11) and considering the quantities \( R_e \cos(\Phi_e) \Lambda_{\alpha \beta}^I \) and \( R_e \sin(\Phi_e) \Lambda_{\alpha \beta}^I \) as fitting parameters. For the fitting procedure it can be useful to estimate \( \partial_\varphi S_{\alpha \beta} \) by using the Fisher-Lee relation

\[
S_{\alpha \beta}(E_F, \tau) = \delta_{\alpha \beta} - i \sqrt{\Gamma_\alpha(X_1, X_2) \Gamma_\beta(X_1, X_2) G^\prime(E_F, \tau)}, \tag{13}
\]

where \( G^\prime(E_F, \tau) \) is the instantaneous retarded Green’s function associated to the scattering matrix assumed of resonant-like form

\[
G^\prime(E_F, \tau) = [E_F - \epsilon(y_e(\tau))] + i \sum_s \Gamma_s(X_1(\tau), X_2(\tau))/2 \Big[ E_F - \epsilon(y_e(\tau)) - i \sum_s \Gamma_s(X_1(\tau), X_2(\tau))/2 \Big], \tag{14}
\]

while the tunneling amplitudes \( \Gamma_s \) are related to the Green’s function self-energy. In this way we obtain a model for the parametric derivative of the scattering matrix with respect to the parasitic variable:

\[
(\partial_\varphi S_{\alpha \beta}) = -i \lambda \sqrt{\Gamma_\alpha^0 \Gamma_\beta^0} \exp(-2i\Theta) / (E_F - \epsilon_0)^2 + \left( \sum_s \Gamma_s^0 \right)^2, \tag{15}
\]

where \( \lambda = (\partial_\varphi \epsilon(y_0)) \), \( \Theta = \arctan(\sum_s \Gamma_s^0/|E_F - \epsilon_0|) \), \( \epsilon_0 \) being the energy level of the resonant state inside the scatterer in the absence of pumping terms. When \( \epsilon_0 \approx E_F \) and considering \( \Gamma^0_L \approx \Gamma^0_R = \Gamma_0 \), one obtains the simple relation \( (\partial_\varphi S_{\alpha \beta}) \approx i \lambda \Gamma_0 / \Gamma_0 \) leading to the useful estimation \( \Lambda_{\alpha \beta}^I \approx (\lambda/\Gamma_0) \Re\{(\partial_\varphi S_{\alpha \beta}) \} \). Within
the resonant hypothesis considered above the nonlinear dynamics affects the pumped current mainly via the terms \( \Re \sin(\Phi_e) \) and \( \Re \cos(\Phi_e) \). Thus in case of bistability of the forced nonlinear dynamics an hysteresis of the pumped current induced by \( \Re \) is expected. In the latter case, the quantum pump displays a memory effect controlled by the features of the pumping cycle. To explain this point we now go back to the analysis of the forced Duffing oscillator governing in our specific example the dynamics of \( y \).

IV. RESULTS FOR THE DUFFING OSCILLATOR

The numerical analysis of first equation in (10) is shown in Fig.1 where the equilibrium radius \( \Re \) is reported as a function of the detuning parameter \( a \) and for different pumping phases \( \varphi \). Starting from the upper curve obtained for \( \varphi = 0 \) we observe an hysteretic behavior as a function of \( a \). Increasing the value of the pumping phase the hysteresis disappears when the value \( \varphi \approx \pi/2 \) is reached. The hysteretic behavior of the Duffing oscillator is controlled by the nonlinear term \( b \). In particular for values of \( b \) below the critical value \( b_c \) the curves \( \Re(a) \) are single valued functions, while when \( b_c \) is exceeded (i.e. \( b > b_c \)) a bistable behavior is detected. Since in our case \( b = 3.5 \) is fixed, we have to assume that the pumping phase \( \varphi \) can vary \( b_c \). Very interestingly the above observation implies that the shape of the pumping cycle may affect directly the stability of the nonlinear parasitic variable. In particular the hysteresis indicates that the phase space of the oscillator contains two stable limit cycles characterized by different radius \( \Re^{(1)} \) and \( \Re^{(2)} \), a third unstable limit cycle being present between the stables one. In this way the state of the parasitic variable can be controlled by using the detuning \( a \) (or equivalently the pumping frequency \( \omega \) and the pumping phase \( \varphi \)). Furthermore, the change of state of the parasitic variable produces an observable variation of the pumping current allowing us to modify and detect the state of the parasitic variable \( y \). In this way a quantum-pump-based memory device is obtained. The critical value \( b_c \) of the nonlinear term can be computed exactly differentiating the first relation in (10) with respect to \( a \) and imposing diverging derivative for \( \Re(a) \) (i.e. \( d\Re/da \to \pm \infty \)). After some additional algebra, one obtains the following expression for \( b_c(\varphi) \):

\[
b_c(\varphi) = \frac{32\sqrt{3}\beta^3}{2\pi^2f_1^2 + f_2^2},
\]

where the dependence on the pumping phase \( \varphi \) is hidden in the forcing terms \( f_1 \) and \( f_2 \). The critical curve \( b_c(\varphi) \) is reported in Fig.2 as a function of the pumping phase setting the remaining parameters as done in Fig.1. The dashed line represents the actual value of the nonlinear term \( b = 3.5 \). When the full line lies below the dashed one the condition \( b > b_c \) is fulfilled and an hysteretic behavior is detected in Fig.1. For \( b < b_c \), \( \Re(a) \) is a single valued function. The vertical lines correspond to the pumping phases chosen in Fig.1.
A quantum-pump-based memory device

In this Section we discuss how a memory device based on the quantum pumping works. In order to build a memory device a scattering region containing a nonlinear parasitic dynamics is needed. For instance, a parasitic dynamics can be activated by an unknown capacitive coupling between the scatterer and a charge density located on the mesoscopic substrate which can be usually modeled by a classical RLC circuit. Another more controllable way to produce nonlinear parasitic dynamics is provided by a scatterer able to elastically react in the presence of a charge density on it. Such system is a nanoresonator coupled to external leads whose experimental realization is reported in Ref. [14], while its use as deformable quantum dot in the quantum pumping context has been reported in Ref. [10]. Following a derivation similar to the one given in Ref. [10] and going beyond the linear terms in expanding the charge density, one obtains the center mass dynamics of the scatterer (i.e. the parasitic degree of freedom dynamics) in the form given in Eq. (8). Thus Eq. (8) has to be considered the minimal nonlinear model of the center of mass dynam-}

V. CONCLUSIONS

In conclusion, we have shown an unconventional application of the quantum pumping in which a parasitic nonlinear variable affects the pumping cycle introducing interesting memory effects controlled by the critical parameter $b_c$. The hysteretic threshold $b_c$ can be manipulated acting on the shape of the pumping cycle and the state of the parasitic variable $y$ can be detected measuring the pumping current. The features above are useful in obtaining a memory device based on quantum pumping which could be realized using a nano-resonator similar to the one reported in Ref. [14]. Apart from the technological application of our study, nanoelectromechanical systems are ideal candidates to validate the present theory and explore the parasitic dynamics. Studies of nanoelectromechanical systems are usually performed by reducing their dynamics from that of a continuous model to that of the center of mass dynamics, the so called ”resonator”. However in the real situation the resonator is an extended object whose center of mass motion is affected by different vibrational modes of the structure. Thus, due to the complexity of the problem it is not possible to fix a priori the parameters of the reduced model (e.g. the damping coefficient, the resonance frequency, etc) or recognize the structure of the nonlinear contributions to the damping term. In fact the total energy is differently stored among the vibrational degrees of freedom depending on the working point of the resonator. Thus, even

FIG. 3: Hysteresis loop for the radius $R_e$ as a function of the detuning parameter $a$ by setting the remaining parameters as follows: $\beta = 0.75$, $b = 3.5$, $g_1 X_1^2 = g_2 X_2^2 = 0.35$, $\varphi = 0$. 

controlled by using gate voltages or acting on the frequency of the pumping cycle. Starting from the point $A$ of the loop and decreasing the value of $a$, we move along the lower branch until the point $B$ is reached. As evident, from analyzing the second and third term in Eq. (11), along $AB$ the system presents a continuous variation of the pumping current since $R_e$ and $\Phi_e$ are continuous function of $a$. Further decreasing $a$, moving from the point $B$, produces a jump toward the point $C$ located on the upper branch (i.e. $CD$) of the hysteresis loop. This jump, corresponding to a sizeable change of $R_e$, can be detected by a strong variation in the pumping current if the second and third terms in Eq. (11) are of the same order of magnitude compared to the pumping term. Once the point $C$ is reached, the loop can be closed by increasing $a$ until the state $D$ is achieved. Starting from $D$, a further increasing of the detuning $a$ produces a big jump toward the initial state $A$ of the hysteresis loop. The $DA$ jump (as also the $BC$ jump), detectable in the pumping current, indicates the change of state of the parasitic dynamics (e.g. the center of mass evolution of the nanoresonator). In this way the bistable state of the parasitic variable is recorded by two reference pumping currents taking different values pertaining to the upper or lower branch of the hysteresis loop. The above procedure constitutes the working principle of a memory device based on quantum pumping.
though the resonator dynamics can be taken into account in designing the device, to fix the time evolution of the mechanical degrees of freedom a precise knowledge of the capacitive coupling as a function of the resonator position is needed. Our proposed method can be useful in identifying the model for the capacitive coupling and offer a method to probe the parasitic nonlinear dynamics. A further example of parasitic dynamics is constituted by the capacitive coupling of a classical background charge to the scattering region. Such situation can be easily understood as the capacitive coupling between the scattering region and an effective RLC circuit. However, also in this simple case, the parasitic dynamics can be imagined as the linearization of some nonlinear model around the working point whose dynamical parameters (capacity, differential resistance, etc) can be tested by using our theory.

Finally, a variety of other useful applications such as frequency synchronization, frequency mixing and conversion, and parametric amplification, could be implemented by our proposal playing an important role for nanomechanical devices.

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12 Notice that the choice $b > 0$ does not allow a chaotic behavior of the system.
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14 V. Sazonova et al., Nature (London) 431, 284 (2004); notice the hysteretic behavior reported in Fig. 3b.