Abstract—We address the problem of finding the minimum decomposition of a permutation in terms of transpositions with non-uniform cost. For arbitrary non-negative cost functions, we describe polynomial-time, constant-approximation decomposition algorithms. For metric-path costs, we describe exact polynomial-time decomposition algorithms. Our algorithms represent a combination of Viterbi-type algorithms and graph-search techniques for minimizing the cost of individual transpositions, and dynamic programming algorithms for finding minimum cost cycle decompositions. The presented algorithms have applications in information theory, bioinformatics, and algebra.

I. INTRODUCTION

Permutations are ubiquitous combinatorial objects encountered in areas as diverse as mathematics, computer science, communication theory, and bioinformatics. The set of all permutations of \( n \) elements – the symmetric group of order \( n! \), \( S_n \) – plays an important role in algebra, representation theory, and analysis of algorithms [1]–[4]. As a consequence, the properties of permutations and the symmetric group have been studied extensively.

One of the simplest ways to generate an arbitrary permutation is to apply a sequence of transpositions - swaps of two elements - on a given permutation, usually the identity permutation. The sequence of swaps can be reversed in order to recover the identity permutation from the original permutation. This process is referred to as sorting by transpositions.

A simple result, established by Cayley in the 1860’s, asserts that the minimum number of transpositions needed to sort a permutation is \( n - 1 \) if \( n \) is odd, and \( n \) if \( n \) is even. Cayley’s result is based on a simple constructive argument, which reduces to a linear-complexity procedure for breaking cycles into sub-cycles. Sorting a permutation is equivalent to finding the transposition distance between the permutation and the identity permutation. Since permutations form a group, the transposition distance between two arbitrary permutations equals the transposition distance between the identity permutation and the composition of the inverse of one permutation and the other permutation.

We address the substantially more challenging question: assuming that each transposition has a non-negative, but otherwise arbitrary cost, is it possible to find the minimum sorting cost and the sequence of transpositions used for this sorting in polynomial time? In other words, can one compute the cost-constrained transposition distance between two permutations in polynomial time? Although at this point it is not known if the problem is NP hard, at first glance, it appears to be computationally difficult, due to the fact that it is related to finding minimum generators of groups and the subset-sum problem [5]. Nevertheless, we show that large families of cost functions – such as costs based on metric-paths – have exact polynomial-time decomposition algorithms. Furthermore, we devise algorithms for approximating the minimum sorting cost for any non-negative cost function, with an approximation constant that does not exceed four.

Our investigation is motivated by three different applications.

The first application pertains to sorting of genomic sequences, while the second application is related to a generalization of the notion of a chemical channel (also known as trapdoor channel [6]). The third application is in the area of coding for storage devices.

Genomic sequences – such as DNA sequences – evolved from one common ancestor, and therefore frequently contain conserved subsequences. During evolution or during the onset of a genomic disease, these subsequences are subject to mutations, and they may exchange their locations. As an example, genomes of cancer cells tend to contain the same sequence of blocks as normal cells, but in a reshuffled (permuted) order. This finding motivated a large body of work on developing efficient algorithms for reverse-engineering the sequence of shuffling steps performed on conserved subsequences. With a few exceptions, most of the methods for sorting use reversals rather than transpositions, they follow the uniform cost model (each change in the ordering of the blocks is equally likely) and the most parsimonious sorting scenario (the sorting scenario with smallest number of changes is the most likely explanation for the observed order). Several approaches that do not fit into this framework were described in [7], [8]. Sorting by cost-constrained transpositions can be seen as a special instance of the general subsequence sorting problem, where the sequence is allowed to break at three or four points. Unfortunately, the case of two sequence breakpoints, corresponding to so called reversals, cannot be treated within this framework.

The second application arises in the study of chemical channels. The chemical channel is a channel model in which symbols are used to describe molecules, and where the channel permutes the molecules in a queue using adjacent transpositions [4]. In information theory, the standard chemical channel model assumes that there are only two molecules, and that the channel has only two states - hence the use of adjacent transpositions. If all the molecules are different, and the channel is 1Usually, the channel is initialized by a molecule that may appear in the queue as well.
allowed to output molecules with time-varying probabilities, one arrives at a channel model for which the output is a cost-constrained permutation of the input. Finding the minimum cost sequence decomposition therefore represents an important step in the maximum likelihood decoding algorithm for the channel.

The third application is concerned with flash memories and rank permutation coding (see [9] and [10]). In this case, one is also interested in sorting permutations using adjacent transpositions and computing the Kendall distance between permutations [11]. If one considers more precise charge leakage models for memory cells, the costs of adjacent transpositions become non-uniform. This can easily be captured by a transposition cost model in which non-adjacent transpositions have unbounded cost, while the costs of adjacent transpositions are unrestricted. Hence, the proposed decomposition algorithms can be used as part of general soft-information rank modulation decoders.

Our findings are organized as follows. Section II introduces the notation followed in the remainder of the paper, as well as relevant definitions. Sections III and IV contain the main results of our study: a three-stage polynomial-time algorithm for general cost-constrained sorting of permutations, an exact polynomial-time algorithm for sorting with metric-path costs, as well as a complexity analysis of the described techniques. Section V contains the concluding remarks.

II. NOTATION AND DEFINITIONS

A permutation \( \pi \) of \( \{1,2,\cdots,n\} \) is a bijection from \( \{1,2,\cdots,n\} \) to itself. The set of permutations of \( \{1,2,\cdots,n\} \) is denoted by \( S_n \), and is called the symmetric group on \( \{1,2,\cdots,n\} \). A permutation can be represented in several ways. In the two-line notation, the domain is written on top, and its image below. The one-line representation is the second row of the two-line representation. A permutation may also be represented as the set of elements and their images.

For example, one can write a permutation \( \pi \) as \( \pi(1) = 3, \pi(2) = 1, \pi(3) = 2, \pi(4) = 5, \pi(5) = 4 \), or more succinctly as \( \pi = 31254 \), or in the two-line notation as

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 1 & 2 & 5 & 4 \\
\end{pmatrix}
\]

Yet another way of writing a permutation is via a set of mappings, for example \( \pi = \{1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2, 4 \rightarrow 5, 5 \rightarrow 4\} \). It will be helpful to think of a permutation as a mapping from positions to objects. For example, \( \pi(1) = 3 \) means object 3 occupies position 1. Alternatively, we can also say that element 1 is a predecessor of element 3. If not otherwise stated, the word predecessor will be henceforth used in this context.

The product \( \pi_2\pi_1 \) of two permutations \( \pi_1 \) and \( \pi_2 \) is the permutation obtained by first applying \( \pi_1 \) and then \( \pi_2 \) to \( \{1,2,\cdots,n\} \), i.e., the product represents the composition of \( \pi_1 \) and \( \pi_2 \).

The functional digraph of a function \( f : \{1,2,\cdots,n\} \rightarrow \{1,2,\cdots,n\} \), denoted by \( G(f) \), is a directed graph with vertex set \( \{1,2,\cdots,n\} \) and an edge from \( i \) to \( f(i) \) for each \( i \in \{1,2,\cdots,n\} \). We use the words vertex and element interchangeably. For a permutation \( \pi \) of \( \{1,2,\cdots,n\} \), \( G(\pi) \) is a collection of disjoint cycles, since the in-degree and out-degree of each vertex is exactly one. Each cycle can be written as a \( k \)-tuple \( \sigma = (a_1a_2\cdots a_k) \), where \( k \) is the length of the cycle and \( a_{k+1} = a_1 \). For each cycle of length \( k \), the indices are evaluated modulo \( k \), so that \( a_{k+1} \) equals \( a_1 \). A planar embedding of \( G(\pi) \) can be obtained by placing vertices of each of the disjoint cycles on disjoint circles. We hence reserve the symbol \( \sigma \) for single cycles, and \( \pi \) for multiple cycle permutations.

We use \( G(\pi) \) to refer to the planar embedding of the functional digraph of \( \pi \) on circles, as well as the functional digraph of \( \pi \). As a convention, we do not explicitly indicate the direction of edges on the circle. Instead, we assume a clockwise direction and treat \( G(\pi) \) as a non-directional graph, unless otherwise stated.

A cycle of length two is called a transposition. A transposition decomposition \( \tau \) (or simply a decomposition) of a permutation \( \pi \) is a sequence \( t_{i_1}\cdots t_{i_l} \) of transpositions \( t_i \) whose product is \( \pi \). Note that the transpositions are applied from right to left. A sorting \( s \) of a permutation \( \pi \) is a sequence of transpositions that transform \( \pi \) into \( \nu \), where \( \nu \) denotes the identity element of \( S_n \). In other words, \( \sigma\pi = \nu \). Note that a decomposition \( \tau \) in reverse order equals a sorting \( s \) of the same permutation.

The cycle representation of a permutation is the list of its cycles. For example, the cycle representation of 31254 is (132)(45). Cycles of length one are usually omitted. The product of non-disjoint cycles is interpreted as a product of permutations. As an illustration, \((124)(213) = ((124)(3))(213)(4)) = (2)(134)\).

A permutation \( \pi \) is said to be odd (even) if the number of pairs \( a,b \in \{1,2,\cdots,n\} \) such that \( a < b \) and \( \pi(a) > \pi(b) \) is odd (even). If a permutation is odd (even), then the number of transpositions in any of its decompositions is also odd (even).

The following definitions regarding graphs \( G = (V,E) \) will be used throughout the paper. An edge with endpoints \( u \) and \( v \) is denoted by \((uv) \in E\). A graph is said to be planar if it can be embedded in the plane without intersecting edges. The subgraph of \( G \) induced by the vertices in the set \( S \subseteq V \) is denoted by \( G[S] \). The degree of a vertex \( v \) in \( G \) is denoted by \( \deg_G(v) \) or, if there is no ambiguity, by \( \deg(v) \). Deletion of an edge \( e \) from a graph \( G \) is denoted by \( G - e \) and deletion of a vertex \( v \) and its adjacent edges from \( G \) is denoted by \( G - v \). The same notions can be defined for multigraphs - graphs in which there may exist multiple edges between two vertices.

We say that an edge \( e \) in \( G \) is a cut edge for two vertices \( a \) and \( b \), denoted by \( a,b \)-cut edge, if in \( G - e \) there exists no path between \( a \) and \( b \). The well known Menger’s theorem [12] asserts that the minimum number of edges one needs to delete from \( G \) to disconnect \( a \) from \( b \) is also the maximum number of pairwise edge-disjoint paths between \( a \) and \( b \). This theorem holds for multigraphs as well.

Let \( T(\tau) \) be a (multi)graph with vertex set \( \{1,2,\cdots,n\} \) and edges \( (a_i,b_i) \) for each transposition \( t_i = (a_i,b_i) \) of \( \tau \). We use the words transposition and edge interchangeably. The
embedding of $T(\pi)$ with vertex set $\{1,2,\ldots,n\}$ into $G(\pi)$ is also denoted by $T(\pi)$. In the derivations to follow, we make frequent use of the spanning trees of the (multi)graphs $T(\pi), G(\pi)$ and $G(\pi) \cup T(\pi)$. A spanning tree is a standard notion in graph theory: it is a tree that contains all vertices of the underlying (multi)graph.

We are concerned with the following problem: given a non-negative cost function $\varphi$ on the set of transpositions, the cost of a transposition decomposition is defined as the sum of costs of its transpositions. The task is to find an efficient algorithm for generating the Minimum Cost Transposition Decomposition (MCD) of a permutation $\pi \in S_n$. The cost of the MCD of a permutation $\pi$ under cost function $\varphi$ is denoted by $M_\varphi(\pi)$.

For a non-negative cost function $\varphi$, let $K(\varphi)$ be the undirected complete graph in which the cost of each edge $(ab)$ equals $\varphi(a,b)$. The cost of a graph $G \subseteq K(\varphi)$ is the sum of the costs of its edges,

$$\text{cost}(G) = \sum_{(ab) \in G} \varphi(a,b).$$

The shortest path, i.e., the path with minimum cost, between $i$ and $j$ in $K(\varphi)$ is denoted by $\varphi^*(i,j)$.

The following definitions pertaining to cost functions are useful in our analysis. A cost function $\varphi$ is a metric if for $a, b, c \in \{1, 2, \ldots, n\}$

$$\varphi(a,c) \leq \varphi(a,b) + \varphi(b,c).$$

A cost function $\varphi$ is a metric-path cost if it is defined in terms of a weighted path, denoted by $\varphi_s$. The weights of edges $(uv)$ in $\varphi_s$ are equal to $\varphi(u,v)$, and the cost of any transposition $(ij)$ equals

$$\varphi(i,j) = \sum_{t=1}^{T} \varphi(c_t, c_{t+1}),$$

where $c_1 \cdots c_{T+1}, c_1 = i, c_{T+1} = j$, represents the unique path between $i$ and $j$ in $\varphi_s$. The path $\varphi_s$ is called the defining path of $\varphi$. A cost function $\varphi$ is an extended-metric-path cost function if for a defining path $\varphi_s$, $\varphi(i,j)$ is finite only for the edges $(ij)$ of the defining path, and unbounded otherwise.

Applying a transposition $(ab)$ to a permutation $\pi$ is equivalent to exchanging the predecessors of $a$ and $b$ in $G(\pi)$. We define a generalization of the notion of a transposition, termed $h$-transposition, where the predecessor of $a$ can be changed independently of the predecessor of $b$. For example, let $a, b, c, d \in \{1, 2, \ldots, n\}$ and let $\pi(c) = a$ and $\pi(d) = b$. Let $\pi' = (c, (a \rightarrow b)) \pi$, where we used $(c, (a \rightarrow b))$ to denote an $h$-transposition. This $h$-transposition takes $c$, the predecessor of $a$, to $b$, without modifying the predecessor of $b$. That is, we have a mapping in which $\pi'(c) = \pi'(d) = b$, and $a$ has no predecessor. Note that $\pi'$ is no longer a bijection, and several elements may be mapped to one element. A transposition represents the product of a pair of $h$-transpositions, as in

$$(ab)\pi = (\pi^{-1}(a), (a \rightarrow b)) (\pi^{-1}(b), (b \rightarrow a)) \pi.$$  

An $h$-decomposition $h$ of a permutation $\pi$ is a sequence of $h$-transpositions such that $hx = \pi$. Similar to transpositions, a cost $\psi(a,b) \geq 0$ can be assigned to $h$-transpositions $(c, (a \rightarrow b))$, where $c$ is the predecessor of $a$. Note that the cost $\psi$ is not dependent on $c$. We say that the transposition cost $\varphi$ and the $h$-transposition cost $\psi$ are consistent if for all transpositions $(ab)$ it holds that $\varphi(a,b) = \psi(a,b) + \psi(b,a)$.

For a permutation $\pi$ and a transposition $(ab)$, it can be easily verified that $(ab)\pi$ is no less than the number of cycles in $\pi$. The minimum cost of an MLD of $\pi$, with respect to cost function $\varphi$, is denoted by $L(\pi)$. For example, $(132)(45) = (45)(23)(12)$ is decomposed into three transpositions. In particular, if $\pi$ is a single cycle, then the MLD of the cycle has length $n - 1$. A cycle of length $k$ has $k^k - 2$ MLDs \[13\]. An MCD is not necessarily an MLD, as illustrated by the following example.

**Example 1.** Consider the cycle $\sigma = (1 \cdots 5)$ with $\varphi(i,i+1) = 3$ and $\varphi(i,i+2) = 1$. It is easy to verify that $(14)(13)(52)(14)(13)$ is an MCD of $\sigma$ with cost six, i.e., $M(\sigma) = 6$. However, as we shall see later, the cost of a minimum cost MLD is eight, i.e., $L(\sigma) = 8$. One such MLD is $(14)(23)(13)(45)$ \[14\].

Our approach to finding the minimum cost decomposition of a permutation consists of three stages:

1. First, we find the minimum cost decomposition for each individual transposition. In particular, we show that the minimum cost decomposition of a transposition can be obtained by recursively substituting transpositions with triples of transpositions. This step is superfluous for the case when the cost function is a metric.
2. In the second step, we consider cycles only and assume that each transposition cost is optimized. Cycles have the simplest structure among all permutations, and furthermore, any permutation is a collection of cycles. Hence, several approximation algorithms operate on individual cycles and combine their decompositions. As part of this line of results, we describe how to find the minimum cost MLD and show that its cost is not more than a constant factor higher than that of the corresponding MCD. We also present a particularly simple-to-implement class of decompositions whose costs lie between the cost of a minimum MLD and a constant multiple of the cost of an MCD.
3. We generalize the results obtained for single cycles to permutations with multiple cycles.

**III. Optimizing Individual Transposition**

Let $\tau$ be a transposition decomposition and let $(ab)$ be a transposition in $\tau$. Since a transposition is an odd permutation, it may only be written as the composition of an odd number of transpositions. For example,

$$(ab) = (ac)(bc)(ac),$$

where $c \in \{1, 2, \cdots, n\}$ and $c \neq a, b$. It is straightforward to see that any decomposition of a transposition of length three must be of the form \[1\], with a possible reversal of the roles of the elements $a$ and $b$. 


If $\varphi(ab) > 2\varphi(ac) + \varphi(bc)$, then replacing $(ab)$ by $(ac)(bc)(ac)$ reduces the overall cost of $(ab)$. Thus, the first step of our decomposition algorithm is to find the optimal cost of each transposition. As will be shown, it is straightforward to develop an algorithm for finding minimum cost decompositions of transpositions of the form $i$. One such algorithm – Alg. I – performs a simple search on the ordered set of transpositions in order to check if their product, of the form of $i$, yields a decomposition of lower cost for some transposition. It then updates the costs of transpositions and performs a new search for decompositions of length three that may reduce some transposition cost.

The optimized costs produced by the algorithm are denoted by $\varphi^*$. Note that $\varphi^*(a,b) \leq 2\varphi^*(a,x) + \varphi^*(b,x)$, for any $x \neq a, b$. Although an optimal decomposition of the form produced by Alg. I is not guaranteed to produce the overall minimum cost decomposition of any transposition, we show that this is indeed the case after the expositions associated with Alg. I.

Observe that if the cost function is such that

$$\varphi(b,c) + 2\varphi(a,c) \geq \varphi(a,b), \quad a,b,c \in \{1,2,\ldots,n\},$$

as in Example 1, Alg. I is redundant and can be omitted when computing the MCD. In particular, if the cost function is a metric, then Alg. I is not needed.

The input to the algorithm Alg. I is an ordered list $\Omega$ of transpositions and their costs. Each row of $\Omega$ corresponds to one transposition and is of the form $[\varphi(a,b)]$. Sorting of $\Omega$ means reordering its rows so that transpositions are sorted in increasing order of their costs. The output of the algorithm is a list with the same format, but with minimized costs for each transposition.

**Algorithm 1: Optimize-Transposition-Costs($\Omega$)**

1: Input: $\Omega$ (the list of transpositions and their cost)
2: Sort $\Omega$
3: for $i \leftarrow 2 : |\Omega|$ do
4:   $(a_1b_1) \leftarrow \Omega(i)$
5:   $\varphi_1 \leftarrow \varphi(a_1,b_1)$
6:   for $j \leftarrow 1 : i - 1$ do
7:     $(a_2b_2) \leftarrow \Omega(j)$
8:     $\varphi_2 \leftarrow \varphi(a_2,b_2)$
9:     if $\{a_1b_1\} \cap \{a_2b_2\} \neq \emptyset$ then
10:        $a_{com} \leftarrow \{a_1,b_1\} \cap \{a_2,b_2\}$
11:        $\{a_3,b_3\} \leftarrow \{a_1,a_2,b_1,b_2\} - \{a_{com}\}$
12:        if $\varphi_1 + 2\varphi_2 < \varphi(a_3,b_3)$ then
13:           update $\varphi(a_3,b_3)$ in $\Omega$
14:   end if
15: end for
16: Sort $\Omega$

Lemma 2. Alg. I optimizes the costs $\varphi$ of all transpositions with respect to the triple transposition decomposition.

*Proof:* Let $\Omega_i$ be the list $\Omega$ at the beginning of iteration $i$, obtained immediately before executing line 4 of Alg. I. We prove, by induction, that transpositions in $\Omega_i(1:i)$ have minimum triple decomposition costs that do not change in subsequent iterations of the algorithm, and that the transpositions in $\Omega_i(i+1:|\Omega|)$ cannot be written as a product of transpositions exclusively in $\Omega_i(1:i-1)$ that have smaller overall cost.

The claim is obviously true for $i = 2$.

Assume that the claim holds for $i$. Let $t_1 = \Omega_i(i)$ and consider $s \in \Omega_i(i+1:|\Omega|)$. By the induction assumption, $s$ cannot be written as a product of transpositions exclusively in $\Omega_i(1:i-1)$ having smaller overall cost. Thus, the cost of $s$ may be reduced only if one can write $s$ as $t_2t_1t_2$, where $t_2 \in \Omega(1:i-1)$. The list $\Omega_{i+1}$ is obtained after considering all such transpositions, updating $\varphi$ and sorting $\Omega_i$. The transposition of minimum cost in $\Omega_{i+1}(i+1:|\Omega|)$ is $\Omega_{i+1}(i+1)$. Now $\Omega_{i+1}(i+1:|\Omega|)$ cannot be written in terms of transpositions in $\Omega_{i+1}(1:i)$ only, and hence the cost of any transposition in $\Omega_{i+1}(i+1:|\Omega|)$ cannot be reduced below the cost of $\Omega_{i+1}(i+1)$. Hence, the cost of $\Omega_{i+1}(i+1)$ is minimized.

**Example 3.** The left-most list in (3) represents the input $\Omega$ to the algorithm, with transpositions in increasing order of their costs. The two lists that follow represent updates of $\Omega$ produced by Alg. I. In the first step, the algorithm considers the transposition $(13)$, for $i = 2$, and the transposition $(34)$, for $j = 1$. Using these transpositions we may write $(34)(13)(34) = (14)$. The initial cost of $(14)$ is 12 which exceeds $2\varphi(3,4) + \varphi(1,3) = 8$. Hence, the list representing $\Omega$ is updated to form the second list in (3). Next, for $i = 3$ and $j = 1$, the algorithm considers $(24)$ and $(34)$. Since $(34)(24)(34) = (23)$, we update the cost of $(23)$ from 23 to 11 as shown in the third list in (3). Additional iterations of the algorithm introduce no further changes in the costs.

\[
\begin{pmatrix}
(34) & 2 \\
(13) & 4 \\
(24) & 7 \\
(14) & 12 \\
(12) & 15 \\
(23) & 23 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
(34) & 2 \\
(13) & 4 \\
(24) & 7 \\
(14) & 8 \\
(12) & 15 \\
(23) & 23 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
(34) & 2 \\
(13) & 4 \\
(24) & 7 \\
(14) & 8 \\
(12) & 15 \\
(23) & 23 \\
\end{pmatrix}
\]

Upon executing the algorithm, the cost of each transposition is set to its minimal value. Only after the last stage of the MCD approximation algorithm is completed will each transposition be replaced by its minimal cost decomposition. For each index $i$ the number of operations performed in the algorithm is $O(|\Omega|)$. Thus, the total complexity of the algorithm is $O(|\Omega|^2)$. Since $|\Omega|$ is at most equal to the number of transpositions, we have $|\Omega| = \binom{n}{2}$. Hence, the complexity of Alg. I equals $O(n^4)$.

In the analysis that follows, denote the optimized transposition costs by the superscript *, as in $\varphi^*$.

Since the transposition costs are arbitrary non-negative values, it is not clear that the minimum cost decomposition of a transposition is necessarily of the form generated by Alg. I. This algorithm only guarantees that one can identify the optimal sequence of consecutive replacements of transpositions by triples of transpositions. Hence, the minimum cost of a transposition $(ab)$ may be smaller than $\varphi^*(a,b)$, i.e. there may
be decompositions of length five, seven, or longer, which allow for an even smaller decomposition cost of a transposition.

Fortunately, this is not the case: we first prove this claim for decompositions of length five via exhaustive enumeration and then proceed to prove the general case via the use of Menger’s theorem for multigraphs \[1\]. We choose to provide the example of length-five decompositions since it illustrates the difficulty of proving statements about non-minimal decompositions of permutations using exhaustive enumeration techniques. Graphical representations, on the other hand, allow for much more general and simpler proofs pertaining to non-minimal decompositions of transpositions.

We start by considering all possible transposition decompositions of length five, for which the transposition costs are first optimized via Alg. \[1\]. In other words, we investigate if there exist decompositions of \((ab)\) of length five that have cost smaller than \(\varphi^*(a,b)\). Once again, observe that the costs of all transpositions used in such decompositions are first optimized via a sequence of triple-transposition decompositions. To reduce the number of cases, we present the following lemma restricting the possible configurations in a multigraph corresponding to the decomposition of a transposition \((ab)\).

**Lemma 4.** Let \(\tau\) be a decomposition of a transposition \((ab)\). The multigraph \(\mathcal{M} = \mathcal{T}(\tau)\), where \(\tau\) does not contain \((ab)\), has the following properties:

1. Both \(a\) and \(b\) have degree at least one.
2. The degree of at least one of the vertices \(a\) and \(b\) is at least two.
3. Every vertex of \(\mathcal{M} = \mathcal{T}(\tau)\), other than \(a\) and \(b\), appears in a closed path (cycle) with no repeated edges in \(\mathcal{M}\).

**Proof:** The proof follows from the simple observations that:

1. In order to swap \(a\) and \(b\), both \(a\) and \(b\) must be moved.
2. If both vertices \(a\) and \(b\) have degree one, then \(a\) and \(b\) are moved exactly once. This is only possible only if \((ab)\) \(\in\) \(\tau\).
3. Let \(\tau = t_{m} \cdots t_2 t_1\). Let \(t_i\) be the transposition with the smallest index \(i\) that includes \(x \in \mathcal{V}(\mathcal{M})\). In the permutation \(\tau_1 = t_1 \cdots t_2 t_1\), \(x\) is not in its original location but rather occupies the position of another element, say, \(y\). As we shall see in the proof of Lemma \[9\] and Example \[10\], this means that there is a path from \(x\) to \(y\) in \(\mathcal{T}(\tau_1)\). Similarly, there must exist a path from \(y\) to \(x\) in \(\mathcal{T}(t_m \cdots t_i t_{i+1} t_i + 1)\). Thus there is a closed path with no repeated edges from \(x\) to itself in \(\mathcal{M}\).

Let \(x_1, x_2, \cdots, x_N\) be vertices included in the decomposition \(\tau\) other than \(a\) and \(b\). If \(E(\mathcal{M})\) denotes the number of edges in the multigraph \(\mathcal{M} = \mathcal{T}(\tau)\), then

\[
2 |E(\mathcal{M})| \geq 2N + 3,
\]

and, from part [3] it holds that \(\deg(x_i) \geq 2\). Hence,

\[
N \leq \frac{2 |E(\mathcal{M})| - 3}{2} = |E(\mathcal{M})| - 2. \tag{4}
\]

Suppose that \(\tau = t_5 t_4 t_3 t_2 t_1\) is the minimum cost decomposition of \((ab)\) with cost \(\phi\), and that the cost of the optimal decomposition produced by Alg. \[1\] exceeds \(\phi\). Then there is no vertex \(x\) such that

\[
G_1 = \{(ax), (ax), (bx)\}
\]
is a subset of edges in the multigraph \(\mathcal{M}\) since, in that case,

\[
\varphi^*(a,b) \leq 2\varphi^*(a,x) + \varphi^*(b,x) \leq \phi.
\]

Also, there exists no pair of vertices \(x, y\) such that

\[
G_2 = \{(ax), (bx), (ay), (by)\}
\]
is a subset of edges in the multigraph \(\mathcal{M}\). To prove this claim, suppose that \(G_2 \subseteq E(\mathcal{M})\). Without loss of generality, assume that

\[
\varphi^*(a,x) + \varphi^*(b,x) \leq \varphi^*(a,y) + \varphi^*(b,y).
\]

Then,

\[
\varphi^*(a,b) \leq 2\varphi^*(a,x) + \varphi^*(b,x) \leq 2\varphi^*(a,x) + 2\varphi^*(b,x) \leq \text{cost } (G_2) \leq \phi.
\]

Hence, any decomposition of length five that contains \(G_2\) must have cost at least \(\varphi^*(a,b)\).

For any five-decomposition \(\tau\), we have \(|E(\mathcal{M})| = 5\) and, thus, \(N \leq 3\). We consider all five-decompositions of \((ab)\) such that \(\mathcal{M}\) is \(G_1\)-free and \(G_2\)-free, and which contain at most five vertices in \(\mathcal{M}\). Assume that the three extra vertices, in addition to \(a\) and \(b\), are \(c, d\), and \(e\). We now show that for each decomposition of length five, there exists a decomposition obtained via Alg. \[1\] with cost at most \(\phi\), denoted by either \(\mu\) or \(\mu'\). The following scenarios are possible.

1. Suppose that \(\deg(a) = 2\) and \(\deg(b) = 1\). Furthermore, suppose that there exist a vertex that is adjacent to both \(a\) and \(b\) in \(\mathcal{M}\). Without loss of generality, assume that \((ad) \in \mathcal{M}, (bd) \in \mathcal{M}\) (Figure \[1\]). We consider two cases, depending on the existence of the edge \((cd)\) in \(\mathcal{M}\).

First, assume that \((cd) \in \mathcal{M}\) (Figure \[1\]). If \(\varphi^*(a,c) + \varphi^*(c,d) \leq \varphi^*(a,d)\), then the decomposition

\[
\mu = (ac)(cd)(bd)(ac)
\]
has cost at most \(\phi\). Note that \(\mu\) can be obtained from Alg. \[1\] since

\[
\mu = (ac)(bc)(ac) = (ab).
\]

On the other hand, if \(\varphi^*(a,c) + \varphi^*(c,d) > \varphi^*(a,d)\), then the decomposition \(\mu' = (ad)(bd)(ad)\) has cost at most \(\phi\).

Next assume that \((cd) \notin \mathcal{M}\). Since both \(c\) and \(d\) each must lie on a cycle, the only possible decompositions
Figure 1: Possible $G_1$— and $G_2$—free configurations for $\mathcal{M} = T(\tau)$ when $\tau$ is a five-decomposition of $(ab)$. Note that other configurations can be obtained from these by relabeling $a$ and $b$, and relabeling $c$, $d$, and $e$ since they are symmetric.

of $(ab)$ are shown in Figure 1c. Now, if $\varphi^*(ad) \leq \varphi^*(d,e) + \varphi^*(e,c) + \varphi^*(a,c)$, then the decomposition

$$
\mu = (ad)(bd)(ad)
$$

has cost at most $\phi$. On the other hand, if $\varphi^*(ad) > \varphi^*(d,e) + \varphi^*(e,c) + \varphi^*(a,c)$, then the decomposition

$$
\mu' = (ac)(ec)(ed)(bd)(ed)(ce)(ac)
$$

has cost at most $\phi$. Note that $\mu'$ can be obtained from Alg. 1 since

$$
\mu' = (ac)(ec)(be)(ec)(ac)
$$

$$
= (ac)(cb)(ac)
$$

$$
= (ab).
$$

2) Suppose that $\deg(a) = 2$ and $\deg(b) = 1$, but that there is no vertex adjacent to both $a$ and $b$. Without loss of generality, assume $c$ and $d$ are adjacent to $a$ and $e$ is adjacent to $b$ (Figure 1d). Since $c$, $d$, and $e$ each must lie on a cycle, one must include two more edges in the graph, as shown in Figure 1e. Since $d$ and $c$ have a symmetric role in the decomposition, we may without loss of generality, assume that $\varphi^*(a,c) + \varphi^*(c,e) \leq \varphi^*(a,d) + \varphi^*(d,e)$. Let $\mu$ be equal to

$$
\mu = (ac)(ce)(eb)(ec)(ac).
$$

Similarly to (5), it is easy to see that the cost of $\mu$ is at most $\phi$ and that it can be obtained from Alg. 1.

3) Assume that $\deg(a) = \deg(b) = 2$ (Figure 1f). Since $e$ and $c$ must lie on a cycle, the fifth transposition in the decomposition must be $(ec)$ (Figure 1g). If $\varphi^*(a,d) + \varphi^*(b,d) \leq \varphi^*(b,e) + \varphi^*(e,c) + \varphi^*(c,a)$, then the decomposition

$$
\mu = (ad)(bd)(ad)
$$

has cost at most $\phi$. Otherwise, if $\varphi^*(a,d) + \varphi^*(b,d) > \varphi^*(b,e) + \varphi^*(e,c) + \varphi^*(c,a)$, the decomposition

$$
\mu' = (ac)(ec)(be)(ce)(ac)
$$

has cost at most $\phi$. Note that both $\mu$ and $\mu'$ represent decompositions of a form optimized over by Alg. 1.

4) Suppose that $\deg(a) = 3$, $\deg(b) = 1$, and that all edges adjacent to $a$ and $b$ are simple (not repeated). Without loss of generality, assume that $e$ is adjacent to both $a$ and $b$ (Figure 1h). One edge must complete cycles that include $c$, $d$, and $e$. Since creating such cycles with one edge is impossible, this configuration is impossible.

5) Suppose that $\deg(a) = 3$, $\deg(b) = 1$, one edge adjacent to $a$ appears twice, and there is a vertex adjacent to both $a$ and $b$. Without loss of generality, assume that this vertex is $d$ (Figure 1i). Since $d$ must be in a cycle, it must be adjacent to the “last edge”, i.e., the fifth transposition. If the last edge is $(ed)$, then one more edge is needed to create a cycle passing through $c$. Thus, the last edge cannot be $(ed)$. The only other choice is $(cd)$ (Figure 1j). Now, if $\varphi^*(a,d) \geq \varphi^*(c,d)$, then the decomposition

$$
\mu = (ac)(cd)(bd)(cd)(ac)
$$

has cost at most $\phi$. Otherwise, if $\varphi^*(a,d) < \varphi^*(c,d)$, the decomposition

$$
\mu' = (ad)(bd)(ad)
$$

has cost at most $\phi$.

6) Suppose that $\deg(a) = 3$, $\deg(b) = 1$, and no vertex is adjacent to both $a$ and $b$. The two possible cases are shown in Figures 1k and 1l. Since one edge cannot create all the necessary cycles, both configurations are impossible.

Next, we state a general theorem pertaining to the optimality of Alg. 1.

Theorem 5. The minimum cost decompositions of all transpositions are generated by Alg. 1.

Proof: The proof proceeds in two steps. First, we show that the multigraph $\mathcal{M}$ for a transposition $(ab)$ cannot have more than one $a,b$-cut edge. If $\mathcal{M}$ has no $a,b$-cut edge, then there exist at least two edge-disjoint paths between $a$ and $b$ in
$\mathcal{M}$. This claim follows by invoking Menger’s theorem. The costs of the paths can be combined, leading to a cost of the form induced by a transposition decomposition optimized via $H$. If the multigraph has exactly one $a,b$-cut edge, this case can be reduced to the case of no $a,b$-cut edge. This completes the proof.

Before proving the impossibility of the existence of more than one $a,b$-cut edge, we explain how a $a,b$-cut edge imposes a certain structure in the decomposition of $(ab)$. Consider the decomposition $t_m t_{m-1} \cdots t_i \cdots t_1$ of $(ab)$ and suppose that $t_i = (x_1 y_1)$ is an $a,b$-cut edge, as shown in Figure 2a. Let $\pi_j = t_j \cdots t_1$. Since there exists a path between $a$ to $b$, there also exists a path between $a$ and $x_1$ that does not use the edge $(x_1 y_1)$. Thus, in $\mathcal{M} - (x_1 y_1)$, $a$ and $x_1$ are in the same “component”. Denote this component by $B_1$. Similarly, a component, denoted by $B_2$, must contain both the vertices $b$ and $y_1$. Since there is no transposition in $\pi_{i-1}$ with endpoints in both $B_1$ and $B_2$, there is no element $z \in B_1$ such that $\pi_{i-1} (z) \in B_2$. Similarly, there is no element $z \in B_2$ such that $\pi_{i-1} (z) \in B_1$. This implies that $\pi_{i-1} (a) \in B_1$ and $\pi_{i-1} (b) \in B_2$. Since $(x_1 y_1)$ is the only edge connecting $B_1$ and $B_2$, we must have

$$\pi_{i-1}^{-1} (x_1) = a,$$

$$\pi_{i-1}^{-1} (y_1) = b,$$

and

$$\pi_{i-1}^{-1} (x_2) = b,$$

$$\pi_{i-1}^{-1} (y_2) = a.$$

Now suppose there are at least two $a,b$-cut edges in $T$ as shown in Figure 2b. Let the decomposition of $(ab)$ be $t_m \cdots t_i \cdots t_2 \cdots t_1$, where $t_i = (x_2 y_2)$ and $t_i = (x_1 y_1)$, for some $i < l$. Define $B_1$, $B_2$, and $B_3$ to be the components containing $a$, $y_1$, and $y_2$, respectively, in $\mathcal{M} - (x_1 y_1) - (x_2 y_2)$. By the same reasoning as above we must have

$$\pi_{i-1}^{-1} (x_2) = a,$$

$$\pi_{i-1}^{-1} (y_2) = b.$$

However, this cannot be true: after applying $(x_1 y_1)$, the successor of $b$ belongs to $B_1$, and there are no other edges connecting the two components of the multigraph. Hence, the successor of $b$ before transposing $(x_2 y_2)$ (that is, the successor of $b$ in $\pi_{i-1}$) cannot be $y_2$.

Since $\mathcal{M}$ cannot contain more than one $a,b$-cut edge, it must contain either one $a,b$-cut edge or it must contain no $a,b$-cut edges.

Consider next the case when there is no $a,b$-cut edge in $\mathcal{M}$. In this case, based on Menger’s theorem, there must exist at least two pairwise edge disjoint paths between $a$ and $b$. The cost of one of these paths has to be less than or equal to the cost of the other path. Refer to this path as the minimum path. Clearly, the cost of the decomposition of $(ab)$ described by $\mathcal{M}$ is greater than or equal to twice the cost of the minimum path.

Let the edges of the minimum path be $(a z_1) (z_1 z_2) (z_m \cdots z_m) (z_m b)$, for some integer $m$. The cost of $(ab)$ is greater than or equal to

$$2\varphi^*(a, z_1) + 2\varphi^*(z_1, z_2) + \cdots + 2\varphi^*(z_m \cdots z_m) + 2\varphi^*(z_m, b) \geq \varphi^*(a, z_1) + \varphi^*(z_1, z_2) + \cdots + \varphi^*(z_m \cdots z_m) + \varphi^*(z_m, b) \geq \varphi^*(a, z_2) + \varphi^*(z_2, z_3) + \cdots + \varphi^*(z_m, b) \geq \cdots \geq \varphi^*(a, z_m) + \varphi^*(z_m, b) \geq \varphi^*(a, b),$$

and the cost of the decomposition associated with $\mathcal{M}$ cannot be smaller than the cost of the optimal decomposition produced by Alg. 1.

Next, consider the case when there is one $a,b$-cut edge in $\mathcal{M}$. In this case, we distinguish two scenarios: when $x_1 = a$, and when $x_1 \neq a$.

In the former case, the transposition $(a y_1)$ plays the role of the transposition $(a z_1)$ and the remaining transpositions used in the decomposition lie in the graph $\mathcal{M} - (a y_1)$. Since $\mathcal{M} - (a y_1)$ has no $a,b$-cut edge, continuing with line two of (6) proves that the cost of the decomposition associated with $\mathcal{M}$ cannot be smaller than $\varphi^*(a, b)$.

In the later case, the procedure we described for the case $x_1 = a$ is first applied to the multigraph containing the edge $(x_1 y_1)$ and the sub-multigraph containing the vertex $a$. As a result, the edge $(x_1 y_1)$ is replaced by $(a y_1)$, with cost greater than or equal to $\varphi^*(a, y_1)$. Applying the same procedure again, now for the case $x_1 = a$, proves the claimed result.

As an illustration, one can see in Figures 1a-11 that the multigraphs corresponding to decompositions of length five have no more than one $a,b$-cut edge.

A quick inspection of Alg. 1 reveals that it has the structure of a Viterbi-type search for finding a minimum cost path in a transposition graph. An equivalent search procedure can be devised to operate on the graph $K(\varphi)$, rather than on a trellis. The underlying search algorithm is described in the Appendix, and is based on a modification of the well known Bellman-Ford procedure [15].

**Definition 6.** For an arbitrary path $p = c_1 c_2 \cdots c_m$ in $K(\varphi)$,
the transposition path cost is defined as

$$\tilde{\varphi}(p) = 2 \sum_{i=1}^{m} \varphi(c_i, c_{i+1}) - \max_i \varphi(c_i, c_{i+1}).$$

Let $\hat{p}(a, b)$ be a path with minimum transposition path cost among paths between $a$ and $b$. That is,

$$\tilde{\varphi}(\hat{p}(a, b)) = \min_p \tilde{\varphi}(p),$$

where the minimum is taken over all paths $p$ in $K(\varphi)$ between $a$ and $b$. Furthermore, let $p^*(a, b)$ be the standard shortest path between $a$ and $b$ in the cost graph $K(\varphi)$.

Lemma 7. The minimum cost of a transpositions $(a, b)$ is at most $\varphi(\hat{p}(a, b))$.

Proof: Suppose that $\hat{p} = c_0c_1 \cdots c_mc_m+1$ where $a = c_0$ and $b = c_m+1$. Note that, for any $0 \leq i \leq m$,

$$(ab) = (c_0c_1 \cdots c_{i-1}c_i)(c_{i+1}c_{i+2} \cdots c_mc_m+1)(c_{m+1} \cdots c_{i}c_0), \quad (7)$$

Choose $i = \arg\max_j \varphi(c_j, c_{j+1})$ so that $(c_{i+1}c_i)$ is the most costly edge in $\hat{p}(a, b)$.

Each of the cycles in (7) can be decomposed using the edges of $p$ as

$$(c_0c_1 \cdots c_{i-1}c_i) = (c_{i-1}c_i) \cdots (c_{2}c_1)(c_0c_1),$$

$$(c_{i+1}c_{i+2} \cdots c_{m}c_{m+1})(c_{m+1} \cdots c_{i}c_0) = (c_{i+1}c_{i+2} \cdots (c_{m-1}c_m)(c_{m+1}c_m+1). \quad (8)$$

Thus, the minimum cost of $(ab)$ does not exceed

$$2 \sum_{j=0}^{m} \varphi(c_j, c_{j+1}) - \varphi(c_i, c_{i+1}).$$

Lemma 8. The minimum cost of a transposition $(a, b)$ equals the minimum transposition path cost $\varphi(\hat{p}(a, b))$. That is,

$$\varphi^*(a, b) = \varphi(\hat{p}(a, b)).$$

Proof: Suppose $\tau$ is the minimum cost decomposition of $(ab)$. Let $M = T(\tau)$, and note that cost $(M) = \varphi^*(a, b)$.

In the proof of Theorem 5 we showed that $M$ has at most one $a, b$-cut edge. Suppose that $M$ has no $a, b$-cut edge. Then there are two edge-disjoint paths between $a$ and $b$ in $M$. Define the minimum path as in Theorem 5. Suppose the minimum path is $p = c_0c_1 \cdots c_mc_m+1$. It is easy to see that

$$\varphi(\hat{p}(a, b)) \leq \varphi(p) \leq \text{cost } (M) = \varphi^*(a, b).$$

From Lemma 7, we have $\varphi^*(a, b) \leq \varphi(\hat{p}(a, b))$. Hence, in this case, we conclude that $\varphi^*(a, b) = \varphi(\hat{p}(a, b))$.

Next, suppose that $M$ has one $a, b$-cut edge, as shown in Figure 2a. Menger’s theorem implies that there are two edge-disjoint paths between $a$ and $x_1$ and two edge-disjoint paths between $b$ and $y_1$. Let $p_1$ be the path with smaller cost among the pair of paths between $a$ and $x_1$, and similarly, let $p_2$ be the path with smaller cost between the pair of paths between $b$ and $y_1$. Let $p$ be the path obtained by concatenating $p_1$, the edge $(x_1y_1)$, and $p_2$. Note that $\varphi(p(a, b)) \leq \varphi(p) \leq \text{cost } (M) \leq \varphi^*(a, b).$ Since $\varphi^*(a, b) \leq \varphi(p(a, b))$, we have $\varphi^*(a, b) = \varphi(p(a, b))$. ■

It is easy to see that $\varphi^*(a, b) \leq 2 \text{cost } (p^*(a, b))$ since we have

$$\varphi^*(a, b) = \varphi(\hat{p}(a, b)) \leq \varphi(p^*(a, b)) \leq 2 \text{cost } (p^*(a, b)). \quad (9)$$

Note that the Bellman-Ford Alg. presented in the Appendix, finds the paths $\hat{p}$ in $K(\varphi)$ between a given vertex $s$ and all other vertices in the graph.

Lemma 8 provides an easy method for computing $\varphi^*(i, j)$ when there is only one path with finite cost between $i$ and $j$ in $K(\varphi)$. For example, for an extended-metric cost function $\varphi$, we have

$$\varphi^*(i, j) = 2 \sum_{i=1}^{l} \varphi(c_i, c_{i+1}) - \max_{1 \leq j \leq k} \varphi(c_i, c_{i+1}), \quad (10)$$

where $\hat{p}(i, j) = c_1 \cdots c_{i+1}, c_1 = i, c_{i+1} = j$, is the unique path between $i$ and $j$ in $T_s$.

IV. Optimizing Individual Cycles

We consider next the cost optimization problem over single cycles. First, we find the minimum cost MLD via a dynamic programming algorithm. The minimum cost MLD is obtained with respect to the optimized cost function $\varphi^*$ of the previous section. For simplicity, we henceforth omit the superscript in the cost whenever there is no ambiguity in terms which cost function is used.

We also present a second algorithm to find decompositions whose cost, along with the cost of the minimum cost MLD, is not more than a constant factor higher than the cost of the MCD. Both algorithms are presented for completeness.

The results in this section apply to any cycle $\sigma$. However, for clarity of presentation, and without loss of generality, we consider the cycle $\sigma = (12 \cdots k)$.

A. Minimum Cost, Minimum Length Transposition Decomposition

Recall that the vertices of $G(\sigma)$ are placed on a circle. For an MLD $\tau$ of a permutation $\pi$ with $\ell$ cycles, $T(\tau)$ is a forest with $\ell$ components; each tree in the forest is the decomposition of one cycle of $\pi$. This can be easily seen by observing that each cycle corresponds to a tree. The following lemma provides a rigorous proof for this statement.

Lemma 9. The graph $T(\tau)$ of an MLD $\tau$ of a cycle $\sigma$ is a tree.

Proof: First, we show that $T(\tau)$ is connected. The decomposition $\tau$ transform the identity permutation $\pi$ to $\sigma$ by
transposing pairs of elements. Note that every transposition
exchanges the predecessors of two elements. In $\in$, each element
$i$ is a fixed point (i.e., it is its own predecessor) and in $\sigma$, $i$
is the predecessor of $\sigma(i)$. Thus there exists a path, formed by
a sequence of transpositions, between $i$ and $\sigma(i)$. An instance
of such a path is described in Example 10.

To complete the proof, observe that $T(\tau)$ has $k$ vertices and
$k - 1$ edges, since an MLD of a cycle of length $k$ contains
$k - 1$ transpositions. Hence, $T(\tau)$ is a tree.

As already pointed out, we provide an example that
illustrates the existence of a path from $i$ to $\sigma(i)$ in the
decomposition $\tau$ of $\sigma$, for the special case when $\sigma$ is a cycle of
length two.

Example 10. Consider the cycle $\sigma = (14)$. It is easy to
see that $\tau = (34)(12)(23)(34)(12)$ is a decomposition of
$\sigma$. Figure 3 illustrates a path from vertex 1 to vertex 4 in
$T(\tau)$. For instance, the transposition (12) in $\tau$ corresponds to
the edge (12) in $T(\tau)$, as shown in Figure 3b and the transposition (23) corresponds to the edge (23) etc. The cycle $\sigma$ is a cycle in $\pi_e$ in Figure 4a. The path from 1 to 4 in $T(\tau)$ is $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$.

For related ideas regarding permutation decompositions and
graphical structures, the interested reader is referred to [16].

The following definitions will be used in the proof of a
lemma which states that $G(\sigma) \cup T(\tau)$ is planar, provided that
$\tau$ is an MLD of $\sigma$.

Let $R$ be the region enclosed by edges of $G(\sigma)$. Let $T$ be
a tree with vertex set $\{1, 2, \ldots, k\}$, such that $G(\sigma) \cup T$
is planar. Since $T$ is a tree with edges contained in $R$, the edges of
$T$ partition $R$ into smaller regions; each of these parts is the
enclosure of a subset of edges of $G(\sigma) \cup T$ and includes
the vertices of these edges. These vertices can be divided into
corner vertices, lying at the intersection of at least two regions, and
inner vertices, belonging only to one region. In Figure 4a
$G(\sigma)$ with vertices $V = \{1, 2, 3, 4, 5, 6\}$ is partitioned into
four regions, $R_1, R_2, R_3$ and $R_4$. In $R_2$, vertices 1 and 3 are
corner vertices, while vertex 2 is an inner vertex.

Lemma 11. For an MLD $\tau = t_1 \cdots t_k - 1$ of $\sigma$, $T(\tau) \cup G(\sigma)$
is planar. That is, for $t_i = (a_i a_{i+1})$, where $a_1 < a_2$, and $t_j = (b_1 b_2)$, where $b_1 < b_2$, if $a_1 < b_1 < a_2$, then $a_1 < b_2 < a_2$.

Proof: Note that $\tau^{-1} \sigma = i$. Let $\tau_i = t_{i-1} \cdots t_1$. Since
$\tau$ is an MLD of $\sigma$, $\tau_i \sigma$ has $i$ cycles. The proof proceeds by
showing that for all $1 \leq i \leq k$, the following two claims are true:
(I) $G(\sigma) \cup T(\tau_i)$ is planar.

(II) Each cycle of $\tau_i$ corresponds to a subregion $R$ of $G(\sigma) \cup T(\tau_i)$. The cycle corresponding to $R$ contains all of its inner vertices and some of its corner vertices but no other vertex.

Both claims (I) and (II) are obvious for $i = 1$. We show that if (I) and (II) are true for $\tau_i$, then they are also true for $\tau_{i+1}$.

Let $t_i = (ab)$. Clearly, $\tau_{i+1} = t_i \tau_i$ has one more cycle
than $\tau_i \sigma$, and by assumption, $G(\sigma) \cup T(\tau_i)$ is planar and
partitioned into a set of subregions. Note that $a$ and $b$ are in
the same cycle, and thus are inner or corner vertices of some
subregion $R^*$ of $G(\sigma) \cup T(\tau_i)$. The edge $(ab)$ divides $R^*$
into two subregions, $R_a$ and $R_b$ (without crossing any edge
in $G(\sigma) \cup T(\tau_i)$). This proves (I). Let the cycle corresponding to $R^*$ be

$$
\mu = (aa_1 \cdots a_l bb_1 \cdots b_l)
$$

as seen in Figure 4b. Then,

$$(ab) \mu = (aa_1 \cdots a_l) (bb_1 \cdots b_l).$$

Now the cycles $(aa_1 \cdots a_l)$ and $(bb_1 \cdots b_l)$ in $\tau_{i+1}$ correspond to subregions $R_a$ and $R_b$, respectively, as see in Figure 4c. This proves claim (II) since the cycle corresponding to each subregion contains all of its inner vertices and some of its corner vertices but no other vertex.

The following lemma establishes a partial converse to the
previous lemma.
Lemma 12. For a cycle \( \sigma \) and a spanning tree \( T \) over the vertices \( \{1, 2, \ldots, k\} \), for which \( G(\sigma) \cup T \) is planar, there exists at least one MLD \( \tau \) of \( \sigma \) such that \( T = T(\tau) \).

**Proof:** We prove the lemma by recursively constructing an MLD corresponding to \( T \). If \( k = 2 \), then \( T \) has exactly one edge and the MLD is the transposition corresponding to that edge. For \( k > 2 \), some vertex has degree larger than one. Without loss of generality, assume that \( \deg(1) > 1 \). Let
\[
r = \max \{u : |u| \in E(T)\}.
\]
Since \( T \) is a tree, \( T - (1r) \) has two components. These two components have vertex sets \( \{1, \ldots, s\} \) and \( \{s + 1, \ldots, k\} \), for some \( s \). It is easy to see that
\[
(1 \cdots k) = (s + 1 \cdots k 1)(1 \cdots s).
\]
Let
\[
T' = T[\{1, \ldots, s\}],
\]
\[
T'' = T[\{s + 1, \ldots, k, 1\}].
\]
Note that \( T' \) and \( T'' \) have fewer than \( k \) vertices. Furthermore, \( T' \cup G((1 \cdots s)) \) and \( T'' \cup G((s + 1 \cdots k 1)) \) are planar. Thus, from the induction hypothesis, \( (1 \ldots s) \) and \( (s + 1 \ldots k 1) \) have decompositions \( r' \) and \( r'' \) of length \( s - 1 \) and \( k - s \), respectively. By (11), \( r'' r' \) is an MLD for \( \sigma \).

**Example 13.** In Figure 5, we have \( r = 10 \) and \( s = 8 \). The cycle \( (1 \cdots 12) \) can be decomposed into two cycles,
\[
(1 \cdots 12) = (9 10 11 12 1)(1 2 \cdots 8).
\]
Now, each of these cycles is decomposed into smaller cycles, for example
\[
(9 10 11 12 1) = (9 10)(10 11 12 1),
\]
\[
(1 \cdots 8) = (8 1)(1 \cdots 7).
\]
The same type of decomposition can be performed on the cycles \( (9 10), \ldots, (17) \).

Since any MLD of a cycle can be represented by a tree that is planar on the circle, the search for an MLD of minimum cost only needs to be performed over the set of planar trees. This search can be performed using a dynamic program, outlined in Alg. 2. Lemma 14 establishes that Alg. 2 produces a minimum cost MLD.

**Algorithm 2 MIN-COST-MLD**

1: Input: Optimized transposition cost function \( \Phi^* \) where \( \Phi^*_{ij} = \Phi^*(i, j) \) (Output of Alg. 1)
2: \( C(i, j) \leftarrow r \) for \( i, j \in [k] \)
3: \( C(i, i) \leftarrow 0 \) for \( i \in [k] \)
4: \( C(i, i + 1) \leftarrow \Phi^*(i, i + 1) \) for \( i \in [k] \)
5: for \( l = 2 \cdots k - 1 \) do
6: \( \text{for } i = 1 \cdots k - l \text{ do} \)
7: \( j \leftarrow i + l \)
8: \( \text{for } i \leq s < r \leq j \text{ do} \)
9: \( A \leftarrow C(i, s) + C(s + 1, r) + C(r, j) + \Phi^*(i, r) \)
10: \( \text{if } A < C(i, j) \text{ then} \)
11: \( C(i, j) \leftarrow A \)

**Lemma 14.** The output cost of Alg. 2, \( C(1, k) \), equals \( L(\sigma) \).

**Proof:** The algorithm finds the minimum cost MLD of \((1 \cdots k)\) by first finding the minimum cost of MLDs of shorter cycles of the form \((i \cdots j)\), where \( 1 \leq i < j \leq k \). We look at the computations performed in the algorithm from a top-down point of view.

Let \( C_T(i, j) \) be the cost of the decomposition of the cycle \( \sigma^{i:j} = (i \cdots j) \), using edges of \( T[\{i, \ldots, j\}] \), where \( T \) is an arbitrary planar spanning tree over the vertices \( \{1, \ldots, k\} \) arranged on a circle. For a fixed \( T \), let \( r \) and \( s \) be defined as in the proof of Lemma 12. We may write
\[
(i \cdots j) = (s + 1 \cdots r)(ir)(r \cdots j)(i \cdots s)
\]
where \( i \leq s < r \leq j \). Thus
\[
C_T(i, j) = C_T(s + 1, r) + \Phi^*(i, r) + C_T(r, j) + C_T(i, s).
\]
Define \( C(i, j) = C_T^*(i, j) \), where
\[
T^* = \operatorname{arg min}_T C_T(i, j)
\]
denotes a tree that minimizes the cost of the decomposition of \((i \cdots j)\). Then, we have
\[
C(i, j) = C(s + 1, r^*) + \Phi^*(i, r^*) + C(r^*, j) + C(i, s^*),
\]
where \( s^* \) and \( r^* \) are the values that minimize the right-hand-side of (13) under the constraint \( 1 \leq i \leq s < r \leq j \). Since the cost of each cycle can be computed from the cost of shorter cycles, \( C(i, j) \) can be obtained recursively, with initialization
\[
C(i, i + 1) = \Phi^*(i, i + 1).
\]
The algorithm searches over \( s \) and \( r \) and computes \( C(1, k) \) using (14) and (15).

Although these formulas are written in a recursive form, Alg. 2 is written as a dynamic program. The algorithm first computes \( C(i, j) \) for small values of \( i \) and \( j \), and then finds the cost of longer cycles. That is, for each \( 2 \leq l \leq k - 1 \)
in increasing order, \( C(i, i + l) \) is computed by choosing its optimal decomposition in terms of costs of smaller cycles.

**Example 15.** As an example, let us find the minimum cost decomposition of the cycle \( \sigma = (1234) \) using the above algorithm. Let \( \Phi \) be the matrix of transposition costs, with \( \Phi_{ij} = \varphi(i, j) \):

\[
\Phi = \begin{bmatrix}
0 & 5 & 10 & 3 \\
-0 & 2 & 3 & 0 \\
-0 & 0 & 9 & 0 \\
-0 & -0 & 0 & 0
\end{bmatrix}, \quad (16)
\]

After optimizing the transposition costs in \( \Phi \) via Alg. 1, we obtain \( \Phi^* \), shown beneath \( \Phi \). From Alg. 2 we obtain

\[
C(1, 3) = C(2, 3) + \varphi^*(1, 2) = 7, (s, r) = (1, 2),
\]

\[
C(2, 4) = C(2, 3) + \varphi^*(2, 4) = 5, (s, r) = (3, 4).
\]

Consider the cycle \((1234)\), where \( i = 1 \) and \( j = 4 \). The algorithm compares \( \frac{1}{4} = 6 \) ways to represent the cost of this cycle using the cost of shorter cycles. The minimum cost is obtained by choosing \( s = 2 \) and \( r = 4 \), so that

\[
C(1, 4) = C(2, 4) + \varphi^*(1, 4) = 8.
\]

Writing \( C \) as a matrix, where \( C(i, j) = C_{ij} \), we have:

\[
C = \begin{bmatrix}
0 & 5 & 7 & 8 \\
-0 & 2 & 5 & 0 \\
-0 & 0 & 7 & 0 \\
-0 & -0 & 0 & 0
\end{bmatrix}
\]

Note that we can modify the above algorithm to also find the underlying MLD by using (12) to write the decomposition of every cycle with respect to \( r \) and \( s \) that minimize the cost of the cycle. For example, from (12), by substituting the appropriate values of \( r \) and \( s \), we obtain

\[
(1234) = (234)(14) = (34)(24)(14).
\]

The initialization steps are performed in \( O(k) \) time. The algorithm performs a constant number of steps for each \( i, j, r \), and \( s \) such that \( 1 \leq i \leq s < r \leq j \leq k \). Hence, the computational cost of the algorithm is \( O(k^4) \).

Note that Alg. 2 operates on the optimized cost function \( \varphi^* \), obtained as the output of Alg. 1. Figure 6 illustrates the importance of first reducing individual transposition costs using Alg. 1 before applying the dynamic program. Since the dynamic program can only use \( k - 1 \) transpositions of minimum cost, it cannot optimize the individual costs of transpositions and strongly relies on the reduction of Alg. 1 for producing low cost solutions. In Figure 6 the transposition costs were chosen independently from a uniform distribution over \([0, 1]\).

**Figure 6:** The average minimum MLD cost vs the length of the cycle. Transposition costs are chosen independently and uniformly in \([0, 1]\).

**B. Constant-factor approximation for cost of MCD**

For the cycle \( \sigma = (12\cdots k) \) and \( 1 \leq j \leq k \), consider the decomposition

\[
(j + 1 \ j + 2) \ (j + 2 \ j + 3) \cdots \ (k - 1 \ k) \ (k1) \ (12) \ (23) \cdots (j - 1 \ j).
\]

The cost of this decomposition equals

\[
\sum_{i \in \sigma} \varphi^*(i, \sigma(i)) - \varphi^*(j, \sigma(j)).
\]

To minimize the cost of the decomposition, we choose \( j \) such that the transpositions \((j \ j + 1)\) has maximum cost. This choice leads to the decomposition

\[
(j^* + 1 \ j^* + 2) \ (j^* + 2 \ j^* + 3) \cdots \ (k - 1 \ k) \ (k1) \ (12) \ (23) \cdots (j^* - 1 \ j^*). \quad (17)
\]

where

\[
j^* = \arg\max_{j \in \sigma} \varphi^*(j, \sigma(j)).
\]

The decomposition in (17) is termed the Simple Transposition Decomposition (STD) of \( \sigma \). The cost of the STD of \( \sigma \), denoted by \( S(\sigma) \), equals

\[
S(\sigma) = \sum_{i \in \sigma} \varphi^*(i, \sigma(i)) - \varphi^*(j^*, \sigma(j^*)).
\]

**Theorem 16.** For a cycle \( \sigma \), \( M(\sigma) \leq L(\sigma) \leq S(\sigma) \leq 4M(\sigma) \).

**Proof:** Clearly, \( M(\sigma) \leq L(\sigma) \). It is easy to see that the STD is itself an MLD and, thus, \( L(\sigma) \leq S(\sigma) \). For \( S(\sigma) \),
we have
\[ S(\sigma) = \sum_{i \in \sigma} \varphi^* (i, \sigma (i)) - \varphi^* (j^*, \sigma (j^*)) \]
\[ \leq \sum_{i \in \sigma} \varphi^* (i, \sigma (i)) \]
\[ \leq 2 \sum_{i \in \sigma} \text{cost} (p^* (i, \sigma (i))) \]  
(18)

where the last inequality follows from \([9]\). To complete the proof, we need to show that $M(\sigma) \geq \frac{1}{2} \sum_i \text{cost} (p^* (i, \sigma (i)))$. Since this result is of independent importance in our subsequent derivations, we state it and prove it in Lemma 19.

In order to prove Lemma 19 we first prove Lemma 17 and a corollary.

Consider a transposition cost function $\varphi$ and a h-transposition cost function $\psi$. Recall from Section 11 that
\[ (\sigma^{-1} (a), (a \rightarrow b)) (\sigma^{-1} (b), (b \rightarrow a)) \sigma = (ab) \sigma. \]

**Lemma 17.** The minimum cost of an h-decomposition of $\sigma$ is upper-bounded by the cost of the MCD of $\sigma$, provided that $\varphi$ and $\psi$ are consistent.

**Proof:** We prove the lemma by showing that there exists an h-decomposition of $\sigma$ with cost $M(\sigma)$. Suppose that the MCD of $\sigma$ is $\tau = t_m t_{m-1} \cdots t_1$, where $t_i = (a_i b_i)$, $1 \leq i \leq m$, and where $m$ is the length of the MCD. Let the permutation $t_i t_{i-1} \cdots t_1$ be denoted by $\sigma_i$. The cost of the MCD is
\[ M_{\varphi} (\sigma) = \sum_{i=1}^m \varphi (a_i, b_i). \]

By replacing each transposition $t_i = (a_i b_i)$ in $\tau$ by a corresponding pair of h-transpositions $\left(\sigma_{i-1}^{-1} (a_i), (a_i \rightarrow b_i)\right) \left(\sigma_{i-1}^{-1} (b_i), (b_i \rightarrow a_i)\right)$, one can see that
\[ M(\sigma) = \sum_{i=1}^m \left(\psi (a_i, b_i) + \psi (b_i, a_i)\right), \]

since $\psi$ and $\varphi$ are consistent. Hence, the h-decomposition
\[ \left(\sigma_{m-1}^{-1} (b_m), (b_m \rightarrow a_m)\right) \left(\sigma_{m-1}^{-1} (a_m), (a_m \rightarrow b_m)\right) \cdots \left(\sigma_0^{-1} (b_1), (b_1 \rightarrow a_1)\right) \left(\sigma_0^{-1} (a_1), (a_1 \rightarrow b_1)\right) \]

has cost $M(\sigma)$. In other words, decomposing each transposition in an MCD into h-transpositions establishes the claimed result.

**Corollary 18.** For a fixed $\varphi$ and a cycle $\sigma$, one has
\[ M(\sigma) \geq \max_{H} \min_{\psi} C_{\psi} (H) \]

where the maximum is taken over all h-transposition costs $\psi$ consistent with $\varphi$, and the minimum is taken over all h-decompositions $H$ of $\sigma$ with cost $C_{\psi} (H)$.

**Lemma 19.** It holds that $M(\sigma) \geq \frac{1}{2} \sum_i \text{cost} (p^* (i, \sigma (i)))$.

**Proof:** Define $\psi_{1/2}$ as
\[ \psi_{1/2} (a, b) = \psi_{1/2} (b, a) = \varphi (a, b) / 2. \]

It is clear that $\psi_{1/2}$ is consistent with $\varphi$. Hence, we have
\[ M(\sigma) \geq \max_{\psi} \min_{H} C_{\psi} (H) \]
\[ \geq \min_{H} C_{\psi_{1/2}} (H) \]
\[ = \sum_{i} \sum_{(ab) \in p^* (i, \sigma (i))} \psi_{1/2} (a, b) \]
\[ = \frac{1}{2} \sum_{i} \text{cost} (p^* (i, \sigma (i))) \]

where $(*)$ follows from the fact that the minimum cost h-decomposition uses the shortest path $p^* (i, \sigma (i))$ between $i$ and $\sigma (i)$. In this case, $i$ becomes the predecessor of $\sigma (i)$ through the following sequence of h-transpositions:
\[ (i, (v_m \rightarrow \sigma (i))) \cdots (i, (v_1 \rightarrow v_2)) (i, (i \rightarrow v_1)) \]

where $p^* (i, \sigma (i)) = iv_1v_2\cdots v_m\sigma (i)$ is the shortest path between $i$ and $\sigma (i)$.

Observe that Theorem 16 asserts that a minimum cost MLD never exceeds the cost of the corresponding MCD by more than a factor of four. Hence, a minimum cost MLD represents a good approximation for an MCD, independent of the choice of the cost function. On the other hand, STDs and their corresponding path search algorithms are attractive alternatives to MLDs and dynamic programs, due to the fact that they are particularly simple to implement.

**Example 20.** Consider the cycle $\sigma = (12345)$ and the cost function $\varphi$, with $\varphi (2, 4) = \varphi (2, 5) = \varphi (3, 5) = 1$, and $\varphi (i, j) = 100$ for all remaining transposition. First, observe that the costs are not reduced according to Alg. 1. Nevertheless, one can use the upper-bound for the transposition cost in terms of the shortest paths defined in the proof of Lemma 19. In this case, one obtains
\[ M(\sigma) \geq \frac{1}{2} (100 + 2 + 3 + 2 + 100) = 103.5. \]

For example, the second term in the sum corresponds to a path going from 2 to 5 and then from 5 to 3. The cost of this path is two.

Since $M(\sigma)$ has to be an integer, it follows that $M(\sigma) \geq 104$.

The optimized cost function, $\varphi^*$, obtained from Alg. 1 gives
\[ \varphi^*(i, j) = \begin{cases} 1, & (ij) \in \{(25), (35), (24)\} \\ 3, & (ij) \in \{(23), (45)\} \\ 5, & (ij) = (34) \\ 100, & \text{otherwise} \end{cases} \]

A minimum cost MLD can be computed using the dynamic program of Alg. 2. One minimum cost MLD equals $\tau_L = (45) (35) (12) (25)$, and has cost $L(\sigma) = 105$. By substituting each of the transposition in $\tau_L$ with their minimum cost transposition decomposition, we obtain $(24) (25) (24) (35) (12) (25)$.

It is easy to see that $\tau_n = (12) (23) (34) (45)$ is the STD of $\sigma$ with cost $S(\sigma) = 100 + 3 + 5 + 3 = 111$. 

Hence, the inequality $M(\sigma) \leq L(\sigma) \leq S(\sigma) \leq 4M(\sigma)$ holds. Furthermore, note that $\sigma$ is an even cycle, and hence must have an even number of transpositions in any of its decompositions. This shows that $M(\sigma) = L(\sigma) = 105$.

C. Metric-Path and Extended-Metric-Path Cost Functions

We show next that for two non-trivial families of cost functions, one can improve upon the bounds of Theorem 16. For metric-path cost functions, a minimum cost MLD is actually an MCD, i.e., $L(\sigma) = M(\sigma)$. For extended-metric-path costs, it holds that $L(\sigma) \leq 2M(\sigma)$.

Note that metric-path costs are not the only cost functions which admit MCDs of the form of MLDs – another example includes star transposition costs. For such costs, one has $\varphi(i, j) = \infty$ for all $i, j$ except for one index $i$. The remaining costs are arbitrary, but non-negative. The proof for this special case is straightforward and hence omitted.

**Lemma 21.** For a cycle $\sigma$ and a metric-path cost function $\varphi$, $L(\sigma) \leq \frac{1}{2} \sum_i \varphi(i, \sigma(i)) = \frac{1}{2} \sum_i \text{cost}(p^*(i, \sigma(i)))$.

**Proof:** The equality in the lemma follows from the definition of metric-path cost functions.

We recursively construct a spanning tree $T(\sigma)$ of cost $B(\sigma) = \frac{1}{2} \sum_i \varphi(i, \sigma(i))$, such that $G(\sigma) \cup T(\sigma)$ is planar. Since $T(\sigma)$ corresponds to an MLD, $L(\sigma) \leq B(\sigma)$.

The validity of the recursive construction can be proved by induction. For $k = 2$, $T(\sigma)$ is the edge $(12)$. Assume next that the cost of $T(\sigma)$ for any cycle of length $\leq k - 1$ equals $B(\sigma)$.

For a cycle of length $k$, without loss of generality, assume that the vertex labeled 1 is a leaf in $\Theta_s$, the defining path of $\varphi$, and that $t$ is its parent. We construct $T(\sigma)$ from smaller trees by letting

$$T(\sigma) = (1t) \cup T((2 \cdots t)) \cup T((t \cdots k)).$$

See Figure 7 for an illustration. The cost of $T(\sigma)$ is equal to $B((2 \cdots t)) + B((t \cdots k)) + \varphi(1, t)$. Note that we can write

$$B((2 \cdots t)) = \frac{1}{2} \sum_{i=2}^{t-1} \varphi(i, \sigma(i)) + \frac{1}{2} \varphi(2, t)$$

$$= \frac{1}{2} \sum_{i=1}^{t-1} \varphi(i, \sigma(i)) + \frac{1}{2} \varphi(2, t) - \frac{1}{2} \varphi(1, 2),$$

$$B((t \cdots k)) = \frac{1}{2} \sum_{i=t}^{k-1} \varphi(i, \sigma(i)) + \frac{1}{2} \varphi(t, k)$$

$$= \frac{1}{2} \sum_{i=t}^{k-1} \varphi(i, \sigma(i)) + \frac{1}{2} \varphi(t, k) - \frac{1}{2} \varphi(1, k).$$

Since $\varphi(1, 2) = \varphi(1, t) + \varphi(t, 2)$ and $\varphi(1, k) = \varphi(1, t) + \varphi(t, k)$, it follows that

$$B((2 \cdots t)) + B((t \cdots k)) = B((1 \cdots k)) - \varphi(1, t).$$

This completes the proof of the Lemma.

**Theorem 22.** For a cycle $\sigma$ and a metric-path cost function, one has

$$L(\sigma) = M(\sigma) = \frac{1}{2} \sum_i \varphi(i, \sigma(i)).$$

**Proof:** Since $L(\sigma) \geq M(\sigma)$, it suffices to show that $L(\sigma) \leq \frac{1}{2} \sum_i \varphi(i, \sigma(i))$ and $M(\sigma) \geq \frac{1}{2} \sum_i \varphi(i, \sigma(i))$.

**Lemma 21** establishes that $L(\sigma) \leq \frac{1}{2} \sum_i \varphi(i, \sigma(i))$. From **Lemma 19** it also follows that

$$M(\sigma) \geq \frac{1}{2} \sum_i \text{cost}(p^*(i, \sigma(i))).$$

Since $\varphi$ is a metric-path cost function, we have $\varphi(i, \sigma(i)) = \text{cost}(p^*(i, \sigma(i)))$. This proves the claimed result.

**Theorem 23.** For extended-metric-path cost functions $\varphi$, $\mathcal{L}_{\varphi_e}(\sigma) \leq 2M_{\varphi_e}(\sigma)$.

**Proof:** We prove the theorem by establishing that

$$\mathcal{L}_{\varphi_e}(\sigma) \leq \frac{1}{2} \sum_i \text{cost}(p^*(i, \sigma(i))) = 2M_{\varphi_e}(\sigma),$$

where $p^*(i, \sigma(i))$ is the shortest path between $i$ and $\sigma(i)$ in $\mathcal{K}(\varphi_e)$ and is calculated with respect to the cost function $\varphi_e$.

Let $\Theta_s$ be the defining path of an extended-metric-path cost $\varphi_e$. Consider the metric-path cost function, $\varphi_m$, with defining path $\Theta_s$, and with costs of all edges $(i, j) \in \Theta_s$ doubled. If the edge $(i, j) \notin \Theta_s$, and if $c_1c_2 \cdots c_{t+1}$ is the unique path from $c_1 = i$ to $c_{t+1} = j$ in $\Theta_s$, then

$$\varphi_m(i, j) = \sum_{l=1}^{t} \varphi_m(c_l, c_{l+1}) = 2 \sum_{l=1}^{t} \varphi_e(c_l, c_{l+1}).$$

By (10), $\varphi_e(i, j) \leq \varphi_m(i, j)$, for all $i, j$. Hence, $\mathcal{L}_{\varphi_e}(\sigma) \leq \mathcal{L}_{\varphi_m}(\sigma)$. Now, following along the same lines of the proof of
Figure 8: MLD (a) and MCD (b) for \( \sigma = (12345) \). Edge labels denote the order in which transpositions are applied.

Lemma 21 it can be shown that

\[
L_{\varphi_{\ast}}(\sigma) \leq L_{\varphi_{\ast}}(\sigma) = \frac{1}{2} \sum_i \varphi_{\ast}(i, \sigma(i)) = \sum_i \text{cost}(p^{\ast}(i, \sigma(i))),
\]

which proves (a).

Note that Lemma 19 holds for all non-negative cost functions, including extended-metric-path cost functions. Thus,

\[
M_{\varphi_{\ast}}(\sigma) \geq \frac{1}{2} \sum_i \text{cost}(p^{\ast}(i, \sigma(i))),
\]

which proves (b).

Example 24. Consider the cycle \( \sigma = (12345) \) and the extended-metric-path cost function of Example 1.

By inspection, one can see that an MCD of \( \sigma \) is \((14) (13) (35) (24) (14) (13)\), with cost \( M(\sigma) = 6 \). A minimum cost MLD of \( \sigma \) is \((14) (23) (13) (45)\), with cost \( L(\sigma) = 8 \). The STD is \((12) (23) (34) (45)\), with cost \( S(\sigma) = 12 \). Thus, we observe that the inequality \( L(\sigma) \leq S(\sigma) \leq 2M(\sigma) \) is satisfied.

V. OPTIMIZING PERMUTATIONS WITH MULTIPLE CYCLES

Most of the results in the previous section generalize to permutations with multiple cycles without much difficulty. We present next the generalization of those results.

Let \( \pi \) be a permutation in \( S_n \), with cycle decomposition \( \sigma_1\sigma_2\cdots\sigma_\ell \). A decomposition of \( \pi \) with minimum number of transpositions is the product of MLDs of individual cycles \( \sigma_i \). Thus, the minimum cost MLD of \( \pi \) equals

\[
L(\pi) = \sum_{i=1}^\ell L(\sigma_i).
\]

The STD of \( \pi \) is the product of the STDs of individual cycles \( \sigma_i \).

The following theorem generalizes the results presented for single cycle permutations to permutations with multiple cycles.

Theorem 25. Consider a permutation \( \pi \) with cycle decomposition \( \sigma_1\sigma_2\cdots\sigma_\ell \), and cost function \( \varphi \). The following claims hold.

1) \( S(\pi) \leq 2 \sum_i \text{cost}(p^{\ast}(i, \pi(i))) \).
2) \( M(\pi) \geq \frac{1}{2} \sum_i \text{cost}(p^{\ast}(i, \pi(i))) \).
3) \( L(\pi) \leq S(\pi) \leq 4M(\pi) \).
4) If \( \varphi \) is a metric-path cost function, then

\[
M(\pi) = L(\pi).
\]
5) If \( \varphi \) is an extended-metric-path cost function, then

\[
L(\pi) \leq 2M(\pi).
\]

Proof:

1) For each cycle it holds that

\[
S(\pi) \leq 2 \sum_{i \in \pi_i} \text{cost}(p^{\ast}(i, \pi(i))),
\]

which can be seen by referring to (15) in the proof of Theorem 16. Thus,

\[
S(\pi) = \sum_{i=1}^\ell S(\pi_i) \leq 2 \sum_{i=1}^\ell \text{cost}(p^{\ast}(i, \pi(i))) = 2 \sum_{i=1}^n \text{cost}(p^{\ast}(i, \pi(i))).
\]

2) The same argument as in Lemma 19 applies without modifications.

3) For each \( 1 \leq t \leq \ell \), from the proof of Theorem 16, we have \( L(\pi_i) \leq S(\pi_i) \). Consequently,

\[
L(\pi) = \sum_{i=1}^\ell L(\pi_i) \leq \sum_{i=1}^\ell S(\pi_i) = S(\pi).
\]

Furthermore, from parts 1 and 2 of this theorem, it follows that \( S(\pi) \leq 4M(\pi) \). Therefore \( L(\pi) \leq 4M(\pi) \).

4) From Lemma 21 for each \( \pi_i \), it holds that

\[
L(\pi_i) \leq \frac{1}{2} \sum_{i \in \pi_i} \text{cost}(p^{\ast}(i, \pi(i))) .
\]

By summing over all cycles, we obtain

\[
L(\pi) \leq \frac{1}{2} \sum_{i=1}^n \text{cost}(p^{\ast}(i, \pi(i))) .
\]

The claimed result follows from part 2 and the fact that \( M(\pi) \leq L(\pi) \).

5) From the proof of Theorem 23, we have \( L(\pi_i) \leq \sum_{i \in \pi_i} \text{cost}(p^{\ast}(i, \pi(i))) \). By summing over all cycles, we obtain

\[
L(\pi) \leq \sum_{i=1}^n \text{cost}(p^{\ast}(i, \pi(i))) \leq 2M(\pi),
\]

where the last inequality follows from part 2 of this theorem.
A. Merging cycles

In Section 4, we demonstrated that the minimum cost of an MLD for an arbitrary permutation represents a constant approximation for an MCD. The MLD of a permutation represents the product of the MLDs of individual cycles of the permutation. Clearly, optimization of individual cycle costs may not lead to the minimum cost decomposition of a permutation. For example, it may happen that the cost of transpositions within a cycle are much higher than the costs of transpositions between elements in different cycles. It is therefore useful to analyze how merging of cycles may affect the overall cost of a decomposition.

We propose a simple merging method that consists of two steps:

1) Find a sequence of transpositions

\[ \tau' = t_{k-1} \cdots t_1 \]

so that \( \sigma' = \tau' \pi \) is a single cycle. Ideally, this sequence should have minimum cost, although this is not required in the proofs to follow.

2) Find the minimum cost MLD \( \tau \) of \( \sigma' \).

The resulting decomposition is of the form \( \tau^{(1-r)} \).

Suppose that \( \pi \) has \( k \) cycles. Joining \( k \) cycles requires \( k - 1 \) transpositions. Hence, each \( t_i \) is a transposition joining two cycles of \( \pi \). The cost of \( \tau' \) equals \( \sum_{i=1}^{k-1} \varphi(a_i, b_i) \), where \( t_i = (a_i b_i) \). The cost of the resulting decomposition using \( \tau' \) and the single cycle MLD equals

\[ C = \sum_{i=1}^{k-1} \varphi(a_i, b_i) + M(\sigma'). \quad (20) \]

Since \( \pi = t_1 \cdots t_k \sigma' \), we also have

\[ L(\sigma') \leq 4M(\sigma') \leq 4 \left( \sum_{i=1}^{k-1} \varphi(a_i, b_i) + M(\pi) \right). \quad (21) \]

Hence, from (20) and (21), \( C \) is upper bounded by

\[ C \leq \sum_{i=1}^{k-1} \varphi(a_i, b_i) + 4 \left( \sum_{i=1}^{k-1} \varphi(a_i, b_i) + M(\pi) \right) \leq 5k \varphi_{\max} + 4M(\pi), \quad (22) \]

where \( \varphi_{\max} \) is the highest cost in \( \varphi \). The approximation ratio, defined as \( C/M(\pi) \), is upper bounded by

\[ \alpha \leq 4 + \frac{5k}{n-k} \varphi_{\min} = 4 + \frac{5k/n \varphi_{\max}}{1 - k/n \varphi_{\min}}, \quad (23) \]

which follows from the fact \( M(\pi) \geq (n-k) \varphi_{\min} \), where \( \varphi_{\min} \) is the smallest cost in \( \varphi \), assumed to be nonzero.

Although \( \alpha \) is bounded by a value strictly larger than four, according to the expression above, this does not necessarily imply that merging cycles is sub-optimal compared to running the MLD algorithm on individual cycles. Furthermore, if the MCDs of single cycles can be computed correctly, one can show that

\[ C = \sum_{i=1}^{k-1} \varphi(a_i, b_i) + M(\sigma'). \quad (24) \]

\[ \leq 2 \sum_{i=1}^{k-1} \varphi(a_i, b_i) + M(\pi) \]

\[ \leq 2k \varphi_{\max} + M(\pi) \quad (26) \]

The approximation ratio in this case is upper bounded by

\[ \alpha \leq 1 + \frac{2k}{n-k} \varphi_{\max} = 1 + \frac{2k/n \varphi_{\max}}{1 - k/n \varphi_{\min}}. \]

Lemma 26. Let \( \pi \) be a randomly chosen permutation from \( S_n \). Given that the MCDs of single cycles can be computed correctly, and provided that \( \varphi_{\max} = o(n/\log n) \), \( \alpha \) goes to one in probability as \( n \to \infty \).

Proof: Let \( X_n \) be the random variable denoting the number of cycles in a random permutation \( \pi_n \in S_n \). It is well known that \( EX_n = \sum_{j=1}^n \frac{1}{j} = H(\pi) \) and that \( EX_n (X_n - 1) = (EX_n)^2 - \sum_{j=1}^n \frac{1}{j^2} \) [17]. Here, \( H(n) \) denotes the \( n \)th Harmonic number. Thus

\[ EX^2_n = O \left( (\ln n)^2 \right). \quad (27) \]

which shows that \( X_n/n \to 0 \) in quadratic mean as \( n \to \infty \). Hence \( X_n/n \to 0 \) in probability. By Slutsky’s theorem [18], \( \alpha \to 1 \) in probability as \( n \to \infty \).

In the following example, all operations are performed modulo 10, with zero replaced by 10.

Example 27. Consider the permutation \( \pi = \sigma_1 \sigma_2 \), where \( \sigma_1 = (1 \ 7 \ 3 \ 9 \ 5) \) and \( \sigma_2 = (2 \ 8 \ 4 \ 10 \ 6) \), and the cost function \( \varphi \),

\[ \varphi(i, j) = \begin{cases} 1, & d(i, j) = 1 \\ \infty, & \text{otherwise} \end{cases} \]

where \( d(i, j) = \min \{|i-j|, 10-|i-j|\} \). Note that \( \text{cost}(p^*(i, j)) = d(i, j) \).

where \( p^* \) is the shortest path from \( i \) to \( j \). We make the following observations regarding the decompositions of \( \pi \).

1) MCD: We cannot find the MCD of \( \pi \), but we can easily obtain the following bound:

\[ M(\pi) \geq \left| \frac{1}{2} \sum_{i=1}^{10} \text{cost}(p^*(i, \pi(i))) \right| = \frac{2 \cdot 5 \cdot 4}{2} = 20. \]

2) MLD: As before, let the output of Alg. 1 be denoted by \( \varphi^* \). We have \( \varphi^*(i, j) = 2d(i, j) - 1 \). The minimum cost MLDs for the cycles are

\[ (1 \ 7 \ 3 \ 9 \ 5) = (1 \ 9 \ 3 \ 7 \ 13 \ 95), \]
\[ (2 \ 8 \ 4 \ 10 \ 6) = (2 \ 10 \ 48 \ 24 \ 610), \]

each of cost 20. The MLD of \( \pi \) is the concatenation of the MLDs of \( \sigma_1 \) and \( \sigma_2 \):

\[ \pi = (2 \ 10 \ 48 \ 24 \ 610 \ 19 \ 37 \ 13 \ 95), \]

with overall cost equal to 40.
3) STD: It can be shown that the STD of $\sigma_1$ is

$$\sigma_1 = (1 \ 7 \ 3 \ 9 \ 5),$$

and the transposition path cost,

$$\bar{\varphi} (p) = 2 \text{cost} (p) - \max_i \varphi (v_i, v_{i+1}) .$$

The goal is to find the path that minimizes the transposition path cost in (29).

Before describing our algorithm, we briefly review the standard Single-Source Bellman-Ford shortest path algorithm, and its relaxation techniques.

Given a fixed source $s$, for each vertex $v \neq s$, the algorithm maintains an upper bound on the distance between $s$ and $v$, denoted by $D (v)$. Initially, for each vertex $v$, we have $D (v) = \varphi (s, v)$.

"Relaxing" an edge $(uv)$ means testing that the upper-bounds $D (u)$ and $D (v)$ satisfy the conditions,

$$D (u) \leq D (v) + w, \quad D (v) \leq D (u) + w,$$

where $w$ denotes the cost of the edge $(uv)$. If the above conditions are not satisfied, then one of the two upper-bounds can be improved, since one can reach $u$ by passing through $v$, and vice versa.

In our algorithm, we maintain the upper-bound for two types of costs. The source $s$ is an arbitrary vertex in $K(\varphi)$. For a path between $s$ and a vertex $v$, we use $D_1 (v)$ to denote the bound on the minimum transposition path cost, and we use $D_2 (v)$ to denote the bound on twice the minimum cost of the path. From the definitions of these costs, it is clear that

$$D_2 (u) \leq 2w + D_2 (v),$$

where $w$ denotes the cost of the edge $(uv)$. If the above conditions are not satisfied, then one of the two upper-bounds can be improved, since one can reach $u$ by passing through $v$, and vice versa.

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the relaxation of edge \((cd)\) reduces the cost of \(D_1(d)\) from 12 to 10. Continuing with the algorithm, we obtain the final result in Figure 9f. Note that in this example, the result obtained after the first pass is the final result. In general, however, the final costs may be obtained only after all \(n - 1\) passes are performed.

**Lemma 28.** Given \(n\), a cost function \(\varphi\), and a source \(s\), after the execution of Alg. 4, one has \(D_1(u) = \varphi^*(s, u)\) and \(D_2(u) = 2\text{cost}(\varphi^*(s, u))\).

**Proof:** Let \(\hat{p}(s, u) = v_1v_2\cdots v_{m+1}\) be the path that minimizes \(\varphi(p)\) among all paths \(p\) between \(v_1 = s\) and \(v_{m+1} = u\). Since any path \(p\) has at most \(n\) vertices, we have \(m \leq n - 1\). The algorithm makes \(n - 1\) passes and in each pass relaxes all edges of the graph. Thus, there exist a subsequence of relaxations that relax \((v_1v_2), (v_2v_3), \ldots, (v_mv_{m+1})\), in that order. The proof for the claim regarding \(D_1(u)\) follows by invoking the path-relaxation property and the fact that \(\varphi^*(s, u) = \varphi(\hat{p}(s, u))\). The proof for the claim regarding \(D_2(u)\) is similar.

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Figure 9: SINGLE-PAIR BELLMAN-FORD on a 6-vertex graph. The costs $(D_1(u), D_2(u))$, are shown inside each vertex. Edges that are not drawn have weight $\infty$. 

(a) Initialization

(b) Relaxation of $(bd)$

(c) Relaxation of $(cd)$

(d) Relaxation of $(be)$ and $(cf)$

(e) Relaxation of $(de)$ and $(df)$

(f) Relaxation of $(fe)$. 