Abstract

We present the concept of a feedback-based topological acoustic metamaterial as a tool for realizing autonomous and active guiding of sound beams along arbitrary curved paths in free two-dimensional space. The metamaterial building blocks are acoustic transducers, embedded in a slab waveguide. The transducers generate a desired dispersion profile in closed-loop by processing real-time pressure field measurements through preprogrammed controllers. In particular, the metamaterial can be programmed to exhibit analogies of quantum topological wave phenomena, which enables unconventional and exceptionally robust sound beam guiding. As an example, we realize the quantum valley Hall effect by creating, in closed-loop, an alternating acoustic impedance pattern across the waveguide, traversed by artificial trajectories of different shapes, which are reconfigurable in real-time. Due to topological protection, the sound waves between the plates remain localized on the trajectories, and do not back-scatter by the sharp corners or imperfections in the design. The feedback-based design can be used to realize arbitrary physical interactions in the metamaterial, including non-local, nonlinear, time-dependent, or non-reciprocal couplings, paving the way to new unconventional acoustic wave guiding on the same reprogrammable platform.

1 Introduction

Controlling wave propagation in acoustic systems is an essential requirement in advanced engineering applications, such as acoustic imaging, acoustic signature cloaking, noise cancellation, vibration suppression, and more. The idea to control sound waves by artificially designing the medium in which they propagate received a considerable interest over the years, and has recently manifested itself through the emergent concept of metamaterials.

Metamaterials are artificially designed structures, usually of periodic nature, composed of sub-components denoted by unit cells. For sufficiently large wavelengths, much larger than the lattice features, metamaterials effectively act as a continuous material, whose properties are determined by the collective dynamic behavior of their unit cells. As such, metamaterials can exhibit properties that are unavailable in natural materials. This capability has drawn immense attention of the scientific and engineering communities. The use of metamaterials in the control of wave propagation began with photonic [1] crystals, demonstrating negative refraction [2,3,4,5,6], superlensing [7], electro-magnetic cloaking [8,9,10,11], left-handed transmission-line waveguides [12,13], and more [14,15]. At a later stage the metamaterial concept was extended to acoustic and elastic systems, resulting in artificial structures with unusual effective acoustic properties that are capable to control the propagation of sound in new ways [16,17,18,19,20]. Notable applications are acoustic cloaking [21,22,23], metamaterials with a negative acoustic refractive index that can bend, focus and shape sound fields in unconventional fashions [24,25,26,27,28], acoustic leaky wave antennas [29], Dirac-cone-like dispersion [30], subwavelength imaging [31,32], and many more, as summarized in [33]. Similarly, elastic metamaterials are used to control vibrations and waves in solid materials [34,35,36,37,38,39,40].

A special class of systems that has emerged in the last few years is topological metamaterials, which draws inspiration from the condensed matter branch of quantum physics [41,42,43,44,45]. In quantum systems, the topological properties of the electronic band-structure of solids can be exploited to achieve unique and exciting functionalities. One such functionality, known as...
topological wave phenomena, is electrical insulation in the solid interior, while conduction of current is supported only along edges, interfaces or boundaries. Remarkably, these edge waves are immune to backscattering in the presence of a broad class of imperfections and impurities, including localized defects and sharp corners. The role of topology manifests itself in the ability to predict the boundary properties of finite sized materials only by knowing the bulk properties of infinite sized materials [46]. The robustness of the boundary wave properties, captured by topological protection, and the exceptional immunity of the waves to back-scattering has recently inspired the search for analogies in classical systems, substituting the electronic band-structure with acoustic or photonic dispersion relations. Generating topological waves in acoustics or elasticity is particularly advantageous, due to the ability to shape these waves beam-like narrow, which is obviously uncommon for sound or vibration.

As a result, metamaterials supporting topologically protected wave propagation have been realized in diverse fields, including photonics [47,48,49,50], optomechanics [51,52], acoustics [53,54,55,56], elasticity [57,58,59,60], and more. While there are several classes of quantum topological effects, each having a different underlying physical mechanism [61], the common requirement in their realization is breaking a certain form of symmetry of the system. One class employs breaking time reversal symmetry and results in uni-directional edge waves [42,62,63,64,65]. Another class is achieved by breaking spatial symmetry in a periodic lattice (while preserving time reversal symmetry), which supports bi-directional edge waves [43,44]. The Quantum Valley Hall Effect (QVHE) [66], which is at the center of our work, belongs to this class, and can be realized in a structure as simple as a bipartite lattice with a single degree of freedom per site. Attaching two such lattices with flipped partitions will support an exceptionally robust wave propagation along the interface. Examples of mechanical and acoustic topological metamaterials that invoke spatial symmetry breaking, include altering the spacing between scatterers [55] or bottle-like Helmholtz resonators [56] in acoustic waveguides, shifting elastic resonators on plates [60], modifying spring constants in mass-spring lattices [67], or designing arrays of pendula with intricate couplings [68]. Particularly, the QVHE was demonstrated in a vibrating plate with elastic resonators of two different masses [59], in an acoustic lattice with scatterers of two different refractive indices [69], or in a flexible membrane sprayed by rigid particles of two different radii [70].

To date, most of the metamaterial design is based on fixed elements, where the unit cells have given shape and dimensions. Such designs result in fixed dynamic properties, including effective constitutive parameters, interactions between sites, dispersion relation, etc., which are also limited to a particular operating frequency. For a topological metamaterial, such a design would result, for example, in a single quantum effect being mimicked, with a single waveguiding trajectory at a fixed frequency range. These limitations inspired recent attempts to construct topological metamaterials with tunable properties [71,72,73,74,75,76,77,78,79,80,81], where active elements were incorporated in artificial mass-spring lattices, elastic sheets and electric circuits.

In this work we present the feedback-based design method, which converts a bare structure into an autonomous topological acoustic waveguide. The topological properties are created exclusively by real-time feedback operation of embedded acoustic transducers, and can be tuned and reconfigured by changing a control program alone, without any structural modifications. As a result, back-scattering-immune sound beams can be guided along artificial trajectories of any desired shape, and in different frequency ranges, on the same physical platform.

Here we employ the embedded control mechanism to generate the acoustic analogy of the QVHE. In Sec. 2 we describe in detail the acoustic platform on which we propose to realize the feedback-based design concept. We then discuss the conditions for a classical analogy of the QVHE to be met on this platform. In Sec. 3 we present the target closed-loop metamaterial, and analyze its dispersion properties and the corresponding topological characteristics. In Sec. 3.3 we demonstrate, using dynamical simulations, the guiding of robust and back-scattering-immune acoustic beams in real-time along reprogrammable curved trajectories. In Sec. 4.1 we derive the mathematical model of the target system, and in Sec. 4.2 the control algorithm that generates this system. The work is discussed and summarized in Sec. 5.

2 Feedback-based acoustic metamaterial setup

In Sec. 2.1 we introduce the concept of a feedback-based topological metamaterial, and its potential realization on an acoustic platform. In Sec. 2.2 we present the target system, which supports sound propagation according to the acoustic analogy of the QVHE, and describe in detail how it is obtained by our feedback-based design.

2.1 Feedback-based design mechanism

The general principle of a feedback-based metamaterial design is that the couplings between the metamaterial sites, and its consequent dynamical properties, are determined by a reprogrammable electronic feedback controller. The underlying mechanism includes application of external inputs to a host structure. The inputs depend on measured responses in selected locations, which are processed and fed back in real-time according to targeted closed-loop schemes. The control actuators, which are transducers embedded in a base waveguide, constitute the metamaterial unit cells. The particular transducers depend on the implementation platform. In this work we focus on an acoustic platform.

We consider a model for an acoustic waveguide, consisting of two rigid parallel plates separated by a small air gap, as illustrated in Fig. 1(a). The waveguide supports a continuous two-dimensional sound wave propagation between the plates. The
upper plate is hollowed in a periodic pattern, embedding an array of identical acoustic actuators (loudspeakers), facing the air gap through the holes. The pattern defines an artificial discrete lattice on top of the continuous acoustic field, with the actuators constituting the lattice sites. The choice of the holes spacing determines the lattice constant, and sets a limit for the actuators external diameter. The space between the plates along the metamaterial edges can be either sealed or left open, depending on the desired boundary conditions. The actuators are used as acoustic velocity source generators, the role of which is creating a desired acoustic pressure field between the plates. The pressure field is measured by acoustic sensors (microphones) that are embedded in the waveguide along the same pattern as the actuators. The sensors can be either attached to the actuators themselves or mounted in mirror positions on the opposite plate, facing inwards, and assumed small enough to not significantly disturb the measured field. The measured signals are processed by synchronized micro-processors (not shown) according to a pre-programmed algorithm, and fed back to the actuators.

This real-time closed-loop operation constitutes the underlying mechanism of our feedback-based acoustic metamaterial. The processing algorithm, as well as the exact mapping from the sensors to the actuators, depend on the particular couplings that need to be created, and is exclusively defined by the control program. In this work we develop a program that drives the sound propagation between the plates to replicate the QVHE.

### 2.2 Target acoustic metamaterial supporting the QVHE

As was discussed in Sec. 1, the QVHE belongs to a family of effects, originating from quantum physics, that support topologically protected wave propagation. Here we outline the structural conditions for a classical analogy of this effect to take place. The most basic structure that is capable to support the QVHE is a uniform honeycomb lattice, as illustrated by the gold circles in Fig. 1(b), which is turned into a lattice with a certain property alternating between the sites, as captured by the gold and black circles in Fig. 1(c). This transition involves breaking space inversion symmetry within a two-site unit cell, as outlined by the black parallelogram. In a quantum system the lattice sites represent vacancies of alternating potential for hopping electrons [61]. The simplest classical mechanical analogy is a lattice of alternating masses connected by springs [59]. In a passive acoustic metamaterial the system in Fig. 1(c) could be achieved e.g. by attaching Helmholtz resonators [82] (cavities on necks) of different geometries at sites $A$ and $B$ to the upper plate of the waveguide. This would result in alternating discrete changes of acoustic impedance $Z$ at these locations, which is the ratio between the sound pressure field $p$ and the flow velocity $v$ at the resonator entrance [82].

In our system, however, no passive elements are included. Any changes of spatial symmetry will be achieved only via active feedback control of the pressure field. We therefore span the upper waveguide plate by identical acoustic transducers in the uniform honeycomb pattern of Fig. 1(b). These transducers create, using real-time feedback control, discrete changes of impedance, denoted by $Z_A$ and $Z_B$, at sites $A$ and $B$ of each unit cell in Fig. 1(b), imitating Helmholtz resonators at these locations. The control action thereby converts the uniform pattern of Fig. 1(b) to the target closed-loop metamaterial, which has...
In order to design a control algorithm that generates such a partition and in order to calculate the dispersion relation of the target metamaterial, we first need to develop a theoretical model for it. In Sec. 4.1 we mathematically formulate the coupling of the pressure $p$ at any continuous position $r$ between the plates to impedance changes at arbitrary discrete locations across the waveguide upper plate. By this model, each of the unit cells of the target metamaterial is governed by

$$c^2 \nabla^2 p(r,s) = s^2 p(r,s) + \rho c^2 \eta \frac{\delta}{Z_A(s)} p(R_A; s) \delta(r - R_A) + \rho c^2 \eta \frac{\delta}{Z_B(s)} p(R_B; s) \delta(r - R_B),$$

(1)

where $c = 340 \text{ m/s}$ is the speed of sound in air, $\rho = 1.21 \text{ kg/m}^3$ is the mass density of air, $\delta(r - R_j)$ [1/m$^2$], $j = A, B$, is the Dirac delta function indicating the location $R_j$ within a unit-cell, and $\eta$ [m] (defined below (8)) indicates the effect of the distance $d$ between the plates. The complex variable $s$ in Laplace domain is commonly used in control design, and is related to the frequency domain as $s = i\omega$. When control is turned off and the loudspeakers in Fig. 1(a) are assumed ideal, i.e., their surface is rigid when inactive, the target system comprises fully sealed cavities, retaining the slab waveguide. We then obtain $Z_j(s) \rightarrow \infty$, $j = A, B$, and (1) retrieves the classical two-dimensional wave equation, which in time domain reads $c^2 \nabla^2 p(r,t) = \rho \partial^2 p/r,t)$.

A Helmholtz resonator affects the acoustic impedance at its vicinity, imitating such a resonator by our control system implies controlling the acoustic impedance at each loudspeaker location. This can be achieved by measuring the pressure $p$ with a microphone at that location, processing it through a controller, and generating the required flow velocity $v$ with the corresponding loudspeaker, as illustrated in Fig. 1(d). Although the generated velocity is perpendicular to the propagation field in the waveguide, it is similarly coupled to the horizontal velocity field as with passive upper plate resonators. Since the impedances $Z_A(s)$ and $Z_B(s)$ are determined within the control program, their particular expressions can be arbitrary, up to hardware limits and causality. Here we use expressions corresponding to Helmholtz resonators. In a one-dimensional setting, as depicted in Fig. 1(e), a Helmholtz resonator is a cavity of volume $\text{Vol}$ on a neck of length $L_n$ and area $A_n$, attached at location $x_0$ to a tube of cross-section area $A_c$. This acoustic resonator is analogous to a mechanical mass-spring resonator (Fig. 1(f)-left) of mass $M_b$ and spring constant $K_h$, with the impedance relating the force $p_1$ applied to the mass and its resulting velocity $v_h$. Another analogy, with which the notion of impedance is naturally associated, is an electrical LC circuit (Fig. 1(f)-right), of inductance $M_b$ and capacitance $1/K_h$, with the impedance relating the voltage $p_1$ and the current $v_h$. The acoustic resonator the impedance $Z_h$ relates the pressure $p_1$ and the flow velocity $v_h$ at the resonator entrance, and is given by

$$Z_h(s) = \frac{p_1(x_0; s)}{v_h(s)} = M_h s + D_h + \frac{K_h}{s}. \quad (2)$$

$D_h$ represents dissipation, whereas $M_h$ and $K_h$ are the equivalent acoustic mass and spring constant, respectively determined by the neck and cavity parameters, as $M_h = \rho L_n$ [kg/m$^2$] and $K_h = A_n \rho c^2/\text{Vol}$ [N/m$^3$]. Following (2), the impedances of the target resonators read

$$Z_A(s) = M_A s + D_h + \frac{K_A}{s}, \quad Z_B(s) = M_B s + D_h + \frac{K_B}{s}. \quad (3)$$

Equations (1) and (3) represent the target closed-loop acoustic metamaterial, which we create using the feedback-based design. A detailed derivation of the model (1) and of the corresponding control scheme is presented in Sec. 4, but before that, in Sec. 3 we calculate the dispersion relation of (8) to establish and analyze its topological properties.

### 3 Results - The target closed-loop system

In this section we calculate the frequency dispersion of the target closed-loop system, captured by the model in (1) and (3), and schematically illustrated in Fig. 1(c),(e). We show that the dispersion relation of this system exhibits the properties required for the topologically protected wave propagation according to the QVHE acoustic analogy.

#### 3.1 Dispersion characteristics of the bulk metamaterial

Since the model in (1) is a hybridization of a continuous acoustic pressure field and discretely spaced impedance changes, the frequency dispersion with wavevector, $\omega(k)$, cannot be calculated directly using a traveling harmonic wave solution $e^{i(k \cdot r - \omega t)}$. We therefore invoke a semi-analytical approach, denoted by the Plane Wave Expansion method [60,83,84], to calculate the frequency dispersion of the infinite periodic metamaterial of Figs. 1(b,c). The unit cell is depicted in Fig. 2(a)-left, where the gold circles indicate the locations of the target resonators, given by $R_A = R_A d_1 + R_A d_2$ and $R_B = -R_A$ in the real lattice space. The wavevector $k = k_1 b_1 + k_2 b_2$ is evaluated in the reciprocal lattice space along a one-dimensional path connecting the high symmetry points $M - \Gamma - K - M$, as illustrated in Fig. 2(a)-right, enclosing the Irreducible Brillouin Zone (the shaded
Fig. 2. Dispersion relation of an infinite hybrid continuous-discrete target metamaterial. (a) Left - unit cell. Right - the first Brillouin zone. (b) Dispersion relation of the metamaterial when control is off (the slab waveguide). (c) Dispersion relation and (g) eigenmode at the $K$ point of the metamaterial when control is turned on and creates a hexagonal pattern of identical impedances. (d) Zoom-in at the low frequency bands in (c), which are decoupled from the higher bands due to control. (e) Low frequency dispersion and (h) eigenmodes at the $K$ point with $M_A = 1.1M_h$, $M_B = 0.9M_h$. A gap is opened between the bands. (f) Low frequency dispersion and (i) eigenmodes at the $K$ point with $M_A = 0.9M_h$, $M_B = 1.1M_h$. The dispersion profile is identical to (e), but the modes are flipped, indicating a topological transition.

The calculation details of frequency evolution with respect to the wavevector as the solutions of an augmented eigenvalue problem appear in Appendix A.

First, we consider the metamaterial in Fig. 1(a) for the trivial open-loop (uncontrolled) case, which indicates a bare waveguide without any changes of impedance, obtained for $Z_A(s) = Z_B(s) \to \infty$ in (1). This is a fully continuous system with a standard dispersion of the two-dimensional wave equation, which is indeed retrieved by our calculation, as appears in Fig. 2(b). As expected for a uniform impedance system, the dispersion curve, folded here into the Irreducible Brillouin Zone, is gapless (truncated at $10^6 [kHz]$ in the figure), i.e. traveling waves are supported at any temporal frequency.

Next we consider a waveguide with identical impedance changes $Z_A(s) = Z_B(s) = Z_h(s)$ in every unit cell of the honeycomb pattern in Fig. 1(b) with $a = 0.05 [m]$, imitating identical target resonators of impedance $Z_h(s)$ (defined in (2)) for $Vol = 4.917 \cdot 10^{-6} [m^3]$, $A_n = 4.524 \cdot 10^{-6} [m^2]$ and $L_n = 10^{-3} [m]$. The distance between plates in the calculation of $\eta$ in (1), as defined below (8), was $d = 5 \cdot 10^{-3} [m]$. The system is now a continuous-discrete hybridization. Remarkably, introducing the periodically spanned change of impedance, even when space inversion symmetry within the unit cell is preserved, decouples the lower frequency bands from the higher bands, as illustrated in Fig. 2(c). This allows for an isolated working regime, which is encircled in red in Fig. 2(c) and enlarged in Fig. 2(d). The low frequency regime consists of two bands connected at a single (Dirac) point $K$, marked by a black dot in the figure. This two-band wave dispersion diagram resembles the dispersion of a purely discrete bipartite lattice with a single degree of freedom per site, for which the two bands constitute the entire spectrum, such as in the electronic band-structure of graphene [86]. The corresponding eigenmode at the $K$ point is depicted in Fig. 2(g), indicating the pressure field distribution in the unit cell. Since the target resonators in this case are identical, the pressure distribution is equal at both resonators locations $R_A$ and $R_B$.

Our ultimate goal, however, is target resonators of different impedance $Z_A(s) \neq Z_B(s)$, implying space inversion symmetry breaking in the unit cell, which we achieve here by the embedded feedback control operation. For Helmholtz resonators, the impedance of which is of the form given in (3), the difference can be obtained either by imitating $K_A \neq K_B$, or $M_A \neq M_B$, or both. Assuming, for example, $M_A \neq M_B$ and $K_A = K_B$, we set $M_A = M_h(1 + \varepsilon)$ and $M_B = M_h(1 - \varepsilon)$ and a free design area) [85].


parameter $\varepsilon \in (-1, 1)$. We performed the calculation for two values, $\varepsilon = 0.1$ and $\varepsilon = -0.1$. The resulting dispersion relations are shown in Fig. 2(e) and (f), respectively. In both cases a gap is opened between the two bands at the $K$ point, indicating that for frequencies within the gap, wave propagation is not supported in the metamaterial bulk.

The appearance of a gap due to space inversion symmetry breaking is a central feature in topological systems, as discussed next. Since the transition between the gapped states occurs only through a gap closing, together with a nonzero topological invariant carried by each band (see Appendix B), the systems corresponding to $\varepsilon > 0$ and to $\varepsilon < 0$ are topologically different. Although the gapped dispersion profile is identical for both values of $\varepsilon$ (since for an infinite system flipping the masses is essentially equivalent to translation), the transition between the two systems is still captured in the dispersion data. Specifically, it is captured by the eigenmodes corresponding to the two frequencies at the $K$ point, which are labeled by the red and blue dots in Figs. 2(e) and (f). When the value of $\varepsilon$ is flipped, the eigenmodes, which are respectively depicted in Fig. 2(h) and (i), are flipped as well. Topological wave propagation, which is the goal of our metamaterial design, is obtained on an interface of the topologically different $\varepsilon > 0$ and $\varepsilon < 0$ systems. To obtain the interface states in the dispersion relation, we consider a periodic metamaterial with an extended unit cell, which contains the interface, as outlined in Sec. 3.2.

### 3.2 Dispersion characteristics of a finite-sized metamaterial

As was discussed in Sec. 3.1 and captured by Fig. 2(e) and (f), when the impedance at sites $A$ and $B$ is different, a gap opens in the frequency dispersion of the infinite periodic system. As a result, harmonic waves of frequencies that lie within the gap cannot propagate in the bulk, implying that at these frequencies the system behaves as an insulator for acoustic waves. In this section we investigate the effect of an interface of identical impedance at sites $A$ and $B$ on the frequency dispersion. We consider a metamaterial that is infinite along the $x$ axis and finite along the $y$ axis, spanned by a periodic pattern of target closed-loop resonators, as illustrated in Fig. 3(a). This pattern is periodic in the $x$ axis with each vertical strip constituting an extended unit cell, or a super-cell. The black and white circles respectively indicate impedances $Z_A(s)$ and $Z_B(s)$, as defined in (3), created in real-time by the embedded control system. Each super-cell contains an interface of identical impedance, here, for example, $Z_A(s)$, encircled in red in Fig. 3(a).

Calculating the dispersion relation of this system by the Plane Wave Expansion method, as we did for the fully infinite system in Appendix A, now becomes more involved, as we need to distinguish between spatial derivatives $p_{xx}$ and $p_{yy}$ in (1). Since the $y$ coordinate is now finite, $p_{yy}$ will not be eliminated, but will yield a polynomial eigenvalue problem, which is also differential. The common resort in calculating dispersion of semi-infinite systems is the Finite Element method [60]. We take a different approach by developing an equivalent tractable model of a purely discrete system, the sites of which coincide with the pattern in Fig. 3(a), thus preserving the insight of the analytical treatment. As detailed in Appendix C, we adjust the parameters of the equivalent system until we obtain an exceptional fitting of the dispersion relation with the original hybrid continuous-discrete system in the fully infinite configuration. We therefore regard all calculations performed on the equivalent model as absolute representatives of the original model. The resulting frequency dispersion for the super-cell of thirty two sites (comprising sixteen primitive two-site cells) is presented in Fig. 3(b). The bulk states are plotted in black.

We observe that a band gap is formed exactly at the same frequency region as for the fully infinite configuration (Fig. 2(e) and (f)). In addition, a new state, which is plotted in red, emerges inside the gap. This state corresponds to the interface of identical impedances. The dispersion plot for a $Z_A(s)$ interface case is quite similar to Fig. 3(b), yet not identical, as such system cannot be converted into the one with a $Z_B(s)$ interface by a simple translation. Since the dispersion of the bulk metamaterial is

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**Fig. 3.** Band-structure of an equivalent discrete target metamaterial, infinite-periodic in the $x$ direction and finite in the $y$ direction. (a) Lattice schematic, comprising $Z_A$ (black circles) and $Z_B$ (white circles) impedances, with an interface (red). (b) The band-structure, calculated for a 16 two-site cells strip, which includes bulk states (black) and an interface state (red). (c) The corresponding eigenmodes at the $K$ point.
gapped at the region of the interface state, waves of frequencies at that region can propagate only along the interface. Due to the topological property of the dispersion relation (evaluated for the infinite configuration in Appendix B), these waves are expected to be strictly confined to the interface, and to remain localized on it in the presence of sharp turns and corners. This expectation is validated by the corresponding eigenmode (calculated for $\omega = 1.38 \, [kHz]$), depicted in Fig. 3(c), which is clearly localized on the interface. The resulting time domain wave propagation is demonstrated in Sec. 3.3.

3.3 Dynamical simulations of the feedback-based metamaterial demonstrating topologically protected wave propagation

In this section we demonstrate that the feedback-based design of Sec. 2 indeed converts a slab waveguide into an acoustic metamaterial that supports topologically protected wave propagation according to the QVHE. We perform dynamical simulations of the equivalent discrete system of Sec. 3.2, whose dispersion model was fitted in Appendix C to the original continuous-discrete metamaterial. The simulated system corresponds to the two-dimensional waveguide in Fig. 1(a). It is finite-sized in both $x$ and $y$ axes, and contains $8 \times 16$ honeycomb cells. To demonstrate the versatility of the underlying control mechanism, we program the metamaterial to generate topological interfaces of identical adjacent impedance, of two different shapes. The first interface is created by control program 1, and is $Z$-shaped, as illustrated in Fig. 4(a). Each circle corresponds to an acoustic transducer that is driven in a real-time closed loop. The black and white fillings respectively indicate the controllers $H_A(s)$ and $H_B(s)$ in (12), which create the closed-loop impedances $Z_A(s)$ and $Z_B(s)$. We stress that there is no physical interface between the waveguide plates, and all the actuators are identical. The values used for the simulation are those that were used in Sec. 3.2, and are given in Sec. 3.1 with the addition of a small damping $D_h = 0.01$ to the target impedance.

We perform three different simulations, each of them for a different combination of the source input location and frequency. The time duration of all the simulations is $t_f = 0.5 \, [sec]$. The first simulation comprises a source at a location indicated by the

![Fig. 4. Dynamical simulations of the feedback-based acoustic metamaterial. (a) The mounting pattern of the acoustic transducers, which are all identical and operate in a real-time closed-loop according to control laws $H_A(s)$ (black circles) and $H_B(s)$ (white circles), respectively creating $Z_A(s)$ and $Z_B(s)$ impedances. An artificial $Z$-shaped interface (red) is defined in the loop by the control action alone. The system is excited three different times, at locations indicated by the blue, red and green arrows (arrow direction has no meaning). (b) Time response to excitation in the bulk (blue arrow) at a bulk band-gap frequency $f_1 = 1.38kHz$. (c) Time response to excitation on the interface (red arrow) at a bulk band-gap frequency $f_2 = 1.38kHz$. (d) Time response to excitation in the bulk (green arrow) at a bulk state frequency $f_3 = 1kHz$. (e) The same platform as in (a) with control laws $H_A(s)$ (black circles) and $H_B(s)$ (white circles), creating a straight line artificial interface (red). (f) Time response to excitation on the interface in (e) (red arrow) at a bulk band-gap frequency $f_4 = 1.38kHz$.](image-url)
blue arrow, operated at the frequency $f_1 = 1.38 \text{ kHz}$. The resulting time response is plotted in Fig. 4(b). Since the frequency falls within the bulk band gap, and the source is located at the bulk far from the interface, no wave propagation takes place. The second simulation comprises a source of the same frequency $f_2 = 1.38 \text{ kHz}$, but is located on the interface, as indicated by the red arrow. The resulting time response is plotted in Fig. 4(c). Since the interface state in Fig. 3(b) lies within the bulk band gap, waves propagate along the interface only. Due to topological protection, implied by the topological invariant that is calculated in Appendix B, the waves are completely immune to back-scattering from the sharp corners of the $Z$ shaped interface, and remain localized on the interface while smoothly traversing the corners. Reflection from the metamaterial boundaries at the $Z$ shape ends does take place, though, as these are not accounted in the semi-infinite dispersion in Fig. 3(b). As for the control effort required for the closed-loop operation, the highest control inputs amplitude of the acoustic actuators was recorded on the interface and at its primary vicinity, reaching eight times the source amplitude. The control effort can be reduced by reducing the band gap, which, in turn, will reduce the wave decay length outside the interface. In the third simulation the source loudspeaker is located at the same position as in the first case, as indicated by the green arrow, and operates at the frequency $f_3 = 1 \text{ kHz}$. The resulting time response is plotted in Fig. 4(d). Since $f_3$ falls within the bulk states of the dispersion relation, as shown in Fig. 3(b), waves are propagating everywhere along the metamaterial, and are reflecting from its boundaries back and forth (which is true regardless of the source location). The interface shape and orientation are determined exclusively by the control program, and can be rearranged at will. In Fig. 3(e) the same physical platform is programmed to create an artificial interface in a form of a straight line, and denoted by control program 2. We perform a fourth simulation, in which the system is excited on this new interface (red arrow in (e)) at frequency $f_4 = 1.38 \text{ kHz}$. The resulting time response is depicted in Fig. 3(f), where a topologically protected sound beam is guided along the straight line interface.

We therefore demonstrated that the feedback-based metamaterial designed in Sec. 2 indeed supports the acoustic analogy of the QVHE. All the required conditions, including space inversion symmetry breaking within the bulk unit cells and an interface between two values with flipped topological characteristics, were created by the control program, and can be reprogrammed upon the user’s request.

4 Methods

4.1 Theoretical model of the target closed-loop

In this section we derive the target metamaterial model (1). We begin the analysis from the one-dimensional tube-like waveguide of Fig. 1(e), and then generalize it to the actual two-dimensional waveguide. We consider the effect of a side branch resonator of neck area $A_n$, located at $x_0$, on the sound field in the tube, which was originated e.g. by a source at the left end $x = 0$. Although such system is discussed in most acoustic textbooks, the prime focus is usually on the transmission and reflection of sound through the impedance change introduced by the resonator. Our goal is deriving the governing partial differential equations for the pressure and velocity fields, accounting explicitly for a resonator that can be mounted at any location along the tube. For a wavelength large enough compared to the tube cross-section $A$, the sound propagation may be considered in the $x$ direction only. It is determined by the continuity of pressure and continuity of normal flow velocity equations (the acoustic analogue of Kirchhoff voltage and current laws, respectively), which read

$$p_1(x_0;s) = p_2(x_0;s) = p_0(x_0;s), \quad (4a)$$

$$A_sv_1(x_0;s) - A_pv_h(s) = A_vv_2(x_0;s). \quad (4b)$$

The downstream velocities $v_1, v_2$ and the pressure at the resonator entrance $p_1, p_2, p_0$ are marked on Fig. 1(e). We now employ the acoustic constitutive equations [28,39,40,82], which relate the time evolution of the pressure to the velocity at each location along the tube. In Laplace domain these equations have the form

$$sp(x;s) = -\rho c^2 v_x(x;s), \quad (5a)$$

$$p_x(x;s) = -\rho sv(x;s), \quad (5b)$$

where multiplication by $s$ stems from differentiation with respect to time. By (5b) and (4b) we obtain $v_1(x_0;s) \propto p_x(x_0;s)$ and $v_2(x_0;s) \propto p_x(x_0 + dx;s)$, where $x_0$ and $x_0 + dx$ represent the locations slightly before and slightly after the resonator opening, respectively. Combining it with (2), gives

$$A_v \frac{1}{\rho s} [p_x(x_0 + dx;s) - p_x(x_0;s)] = A_h \frac{1}{Z_0(s)} p(x_0;s). \quad (6)$$
Dividing (6) by dx, taking the limit \( dx \to 0 \) and using (5a), we obtain

\[
c^2 p_{xx}(x,s) = s^2 p(x,s) + \rho c^2 \eta_0 \frac{s}{Z_0(s)} p(x_0,s) \delta(x-x_0),
\]

(7)

where \( \eta_0 = A_n/A_t \) is the cross-sections ratio, and \( \delta(x-x_0) \) has the units of one over length. Equation (7) is our theoretical model for the acoustic pressure field in a one-dimensional waveguide with a side branch resonator. Note that despite the fact that both pressure and velocity were involved in the derivation leading to (7), this final equation does not have velocity in it. This is important for our ability to generalize this one-dimensional model also to our two-dimensional waveguide, since in two dimensions the velocity is a vector, and it is easier to work with the pressure, which is a scalar. Our generalization to a two-dimensional waveguide with multiple resonators of different impedances \( Z_j(s) \) at arbitrary locations \( R_j \), is thus of the form

\[
c^2 \nabla^2 p(r,s) = s^2 p(r,s) + \rho c^2 \eta \sum_{R_j} p(R_j,s) \delta(r-R_j),
\]

(8)

where the length scale \( \eta = A_n/d \) captures the effect of the distance \( d \) between the plates, whereas \( \delta(r-R_j) \) obtains the units of one over length squared. The model in (1) is a particular case of (8).

### 4.2 Controller design

In this section we derive the core of our feedback-based design approach, which is the underlying real-time control mechanism of the acoustic metamaterial in Sec. 2. The first step in a control scheme design is determining the available inputs for accessing the open-loop plant, and the available outputs for measurement. While the common output in acoustics is the pressure field, measured by microphones, the acoustic input representation is not unique, but depends on the actuator (loudspeaker) model. In particular, a loudspeaker can be regarded either as a source of flow velocity or of pressure \([82,87,88]\). In this work we regard the measured pressure is processed through the controller that both pressure and velocity were involved in the derivation leading to (7), this final equation does not have velocity in it. Since the control goal is creating a desired acoustic impedance \( Z_j(s) \) at the corresponding location \( R_j \), we set the control law at \( R_j \) to a collocated pressure feedback

\[
v_j(s) = -H_j(s)p(R_j;s),
\]

(10)

The measured pressure is processed through the controller

\[
H_j(s) = \frac{1}{Z_j(s)}.
\]

(11)

The control algorithm in (10)-(11) realizes the target metamaterial in (8) in real-time. The resulting closed-loop system is stable as long as the target resonators include damping. The advantage of this control scheme is that it can generate a metamaterial with any desired couplings (within the hardware limits) between the sites, including non-collocated interactions. The collocation of the feedback in (10) was only due to the type of coupling required for the QVHE, as was discussed in Sec. 4.1. For the particular target resonators at sites \( A \) and \( B \) in (1), the impedance of which is given in (3), the controller in (11) takes the leading phase forms

\[
H_A(s) = \frac{s}{M_A s^2 + D_h s + K_A}, \quad H_B(s) = \frac{s}{M_B s^2 + D_h s + K_B}.
\]

(12)

### 5 Conclusion

We presented a method to design acoustic metamaterials supporting propagation of sound beams of arbitrary reconfigurable shapes, in two-dimensional free space. The underlying platform is a slab waveguide with an embedded feedback control mechanism, which enables shaping the sound pressure field between the plates in real-time, in a way that mimics quantum topological
wave phenomena. Specifically, the model includes identical acoustic actuators mounted in a periodic pattern in one of the plates, and operated according to measurements from a mirror pattern of acoustic sensors, which are processed through autonomous electronic controllers. As an example, we programmed the metamaterial to mimic the QVHE, using a theoretical model that we developed for the target closed-loop system. The required spatial symmetry breaking was created by a collocated pressure feedback at each lattice site, augmenting the continuous pressure field by a discrete pattern of alternating acoustic impedances. We then demonstrated that the closed-loop system obtained a topological dispersion profile corresponding to the QVHE.

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A Infinite system dispersion calculation using the Plane Wave Expansion method

Here we present a detailed frequency dispersion calculation of the system in (1), which models the acoustic metamaterial in Fig. 1(b),(c), when operated in closed-loop. For the sake of the calculation according to Bloch theory of periodic systems [90], in this section the metamaterial is assumed of infinite extension. The calculation can be therefore folded into a single unit cell. Following the unit cell geometry in Fig. 2(b)-left, one obtains \( b = a/\sqrt{3} \) (where \( a \) is the lattice constant and \( b \) is the distance between the \( A \) and \( B \) sites), as well as \( R_{A1} = R_{A2} = b \sin 30^\circ / \sin 120^\circ / 2 \), leading to \( R_{A1} = R_{A2} = a/6 \). Since this system is a hybridization of the continuous pressure field \( p(\mathbf{r}; s) \) and the discretely located transducers, we can calculate its dispersion relation using, for example, the Plane Wave Expansion method [60,83,84]. This method assumes a series solution of traveling harmonic waves, \( p(\mathbf{r}, t) = e^{i\omega t} P(\mathbf{r}) \), where

\[
P(\mathbf{r}) = \sum_{m,n=-M}^{M} p_G e^{-i(k+G) \cdot \mathbf{r}}.
\]

(A.1)

Here \( \mathbf{r} = r_1 \mathbf{d}_1 + r_2 \mathbf{d}_2 \) is the position vector in the real lattice space, and \( \mathbf{k} = k_1 \mathbf{b}_1 + k_2 \mathbf{b}_2 \) is the base wavevector in the reciprocal lattice space, defined by

\[
\mathbf{d}_1 = (a,0), \quad \mathbf{d}_2 = (a/2, \sqrt{3}a/2), \quad \mathbf{b}_1 = 2\pi/a \left( 1, -1/\sqrt{3} \right), \quad \mathbf{b}_2 = 2\pi/a \left( 0, 2/\sqrt{3} \right).
\]

(A.2)

\( G = m \mathbf{b}_1 + n \mathbf{b}_2 \) is its expansion, where \( m \) and \( n \) are integer indices, which span across \(-M : M\), and \( M \) is the series truncation order. Since \( G \) covers a two-dimensional grid, the total number of its entries is \( N^2 \), where \( N = 2M + 1 \). Since \( p(\mathbf{r}; s) \) transforms to frequency domain when \( s = i\omega \), we substitute (A.1) in (1), which reads

\[
\frac{c^2}{\omega^2} \sum_{n,m=-M}^{M} |\mathbf{k} + G|^2 e^{-i(k+G) \cdot \mathbf{r}} p_G = -\sum_{n,m=-M}^{M} \left\{ e^{-i(k+G) \cdot \mathbf{r}} + \rho c^2 \eta \sum_{j=A,B} e^{-i(k+G) \cdot \mathbf{R}_j / \tilde{Z}_j(\omega)} \delta(\mathbf{r} - \mathbf{R}_j) \right\} p_G,
\]

where \( \tilde{Z}_{A,B}(\omega) = i\omega Z_{A,B}(i\omega) = -M_{A,B} \omega^2 + K_{A,B} \) (for the sake of the calculation we set \( D_h \rightarrow 0 \)). Multiplying (A.3) by \( e^{i(k+\tilde{G}) \cdot \mathbf{r}} \), where \( \tilde{G} = \tilde{m} \mathbf{b}_1 + \tilde{n} \mathbf{b}_2 \) for some specific values \( \tilde{m} \) and \( \tilde{n} \), we obtain

\[
\frac{c^2}{\omega^2} \sum_{n,m=-M}^{M} |\mathbf{k} + \tilde{G}|^2 e^{-i(G-\tilde{G}) \cdot \mathbf{r}} p_G = -\sum_{n,m=-M}^{M} \left\{ e^{-i(G-\tilde{G}) \cdot \mathbf{r}} + \rho c^2 \eta e^{i(k+\tilde{G}) \cdot \mathbf{r}} \sum_{j=A,B} e^{-i(k+G) \cdot \mathbf{R}_j / \tilde{Z}_j(\omega)} \delta(\mathbf{r} - \mathbf{R}_j) \right\} p_G.
\]

(A.4)
Due to the orthogonality property of the Fourier series, we have

\[ \iint_{A_c} e^{-i(G \cdot \hat{G}) r} \, \text{d}A_c = \begin{cases} A_c, & G = \hat{G} \\ 0, & G \neq \hat{G} \end{cases}, \quad \iint_{A_c} f(r) \delta(r - R_a) \, \text{d}A_c = f(R_a), \]  
(A.5)

where \( A_c = a^2/\sqrt{3} \) is the unit cell area (Fig. 1(b)), and \( a \) is the lattice constant. Integrating (A.4) over a unit cell then gives

\[ \frac{e^2}{\omega^2} |k + \hat{G}|^2 A_c p_G = -A_c p_G - \rho c^2 \eta \sum_{m,n= -M}^{M} \sum_{j=\alpha,B} e^{-i(G + \hat{G}) R_j} Z_j(\omega) p_G. \]  
(A.6)

Using matrix formulation, we define \( \sum_{m,n=-M}^{M} e^{-i(G + \hat{G}) R_j} p_G = E_j p_G \), where

\[ E_j = e^{i \left[ G_1 G_2 \cdots G_{N^2} \right]^T R_j \cdot \delta R_j \left[ G_1 G_2 \cdots G_{N^2} \right]}. \]  
(A.7)

Following the target resonators coordinates that are given in (A.2), we obtain \( G \cdot R_A = 2\pi a(m + n)/6 \), and \( G \cdot R_B = -2\pi a(m + n)/6 \). Substituting (A.7) into (A.6) and using the explicit form of \( Z_j(\omega) \), (A.6) takes the form

\[ [M_A M_B \omega^4 - (M_A K_B + M_B K_A) \omega^2 + K_A K_B] \left( c^2 |k + \hat{G}|^2 A_c - \omega^2 \right) p_G = -\rho c^2 \eta \omega^2 \left[ (M_B \omega^2 + K_B) E_A + (M_A \omega^2 + K_A) E_B \right] p_G, \]  
(A.8)

where \( E_j \) is defined in (A.7). Rearranging (A.8) results in the following polynomial eigenvalue problem,

\[ (A_3 \lambda^3 + A_2 \lambda^2 + A_1 \lambda + A_0) p_G = 0, \quad \lambda = \omega^2, \]  
(A.9)

where

\[ \begin{align*}
A_3 &= M_A M_B A, & A_2 &= M_A M_B c^2 B - (M_A K_B + M_B K_A) A - \rho c^2 \eta C_M, \\
A_1 &= -(M_A K_B + M_B K_A) c^2 B + K_A K_B A + \rho c^2 \eta C_K, & A_0 &= K_A K_B c^2 B,
\end{align*} \]  
(A.10)

and

\[ A = A_c I_{N^2}, \quad B = -A_c \begin{pmatrix} |k + \hat{G}_1|^2 & |k + \hat{G}_2|^2 & \cdots & |k + \hat{G}_{N^2}|^2 \end{pmatrix}, \quad C_M = M_A E_A + M_B E_B, \quad C_K = K_A E_A + K_B E_B. \]  
(A.11)

We then rewrite (A.9)-(A.11) in a companion form to obtain an augmented linear eigenvalue problem

\[ \lambda \mathbf{P} \mathbf{v} = \mathbf{Q} \mathbf{v}, \]  
(A.12)

where

\[ \mathbf{P} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & A_3 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ -A_0 & -A_1 & A_2 \end{pmatrix}. \]  
(A.13)

and \( \mathbf{v} \) is the augmented eigenvector of length \( 3N^2 \). The dispersion relations of the infinite closed-loop system that are shown in Fig. 2 are the first two solutions of (A.12)-(A.13) for \( N = 4 \).
The topological characteristic of systems supporting the QVHE (and its classical analogies) manifests itself through the valley Chern number \[61\], evaluated for the gapped bands of the infinite system dispersion profile. In our system the relevant bands are the two low frequency range solutions of the augmented eigenvalue problem in (A.12)-(A.13) for \( \varepsilon \neq 0 \), depicted in Fig. 2(e),(f). For each band, a Chern number is given by the formula \[61\]

\[ C = \frac{1}{2\pi} \int_{BZ} \Omega_v(k) \, dk^2, \quad \Omega_v(k) = \nabla_k \times (-i v(k) | \nabla_k v(k) |), \]  

(B.1)

where \( \Omega_v(k) \) is the Berry curvature \[61\], \( v(k) \) is the corresponding eigenstate and integration is performed over the entire Brillouin zone. As a consequence of time reversal symmetry, the Chern number in our system is zero, but a different topological index, the valley Chern number, results in a finite quantized value. The valley Chern number is defined as

\[ C_V = C_K - C_{K'} \]

where \( C_K \) and \( C_{K'} \) are computed by integrating the Berry curvature over the infinite wavevector space of the linearized low frequency model around the high symmetry points \( K \) and \( K' \), as illustrated in Fig. B.1(a).

Employing the numerically optimized algorithm \[91\], we obtain a Berry curvature comprised of alternating vortexes at the \( K \) and \( K' \) points, as illustrated in Fig. B.1(a). We integrate the Berry curvature over a small region of radius \( R_k \) around \( K \) or \( K' \), and find that the result (divided by \( 2\pi \)) converges to \( \pm 1/2 \) as the band-gap between the two lowest bands, \( \Delta(\varepsilon) \), gets smaller, and as \( R_k \) gets larger. \( \Delta(\varepsilon) \) increases with \( |\varepsilon| \), and their exact functional dependence can be found by linearizing the eigenvalue problem in (A.12)-(A.13) around \( K \) or \( K' \), where the gap size is controlled by \( \varepsilon \), and deriving an effective low frequency model, where \( \Delta \) directly describes the gap. Here we look at a limit of a small \( \varepsilon \), where we can assume \( \Delta(\varepsilon) \propto \varepsilon \), and in Fig. B.1(b) we plot the Berry phase around \( K \) as a function of \( \varepsilon \), finding a linear dependence consistent with

\[ \frac{1}{2\pi} \int_{R_k} \Omega_v(k) \, dk^2 = \pm \frac{1}{2} \left( 1 - \alpha \frac{\Delta(\varepsilon)}{R_k} \right). \]  

(B.2)

The pre-factor \( \alpha \) results from the linearization and accounts for units, as \( \Delta(\varepsilon) \) is an effective gap term in units of frequency, \( R_k \) is in units of \( 1/a \), and the Berry phase is unit-less. While equation (B.2) was numerically deduced from the model, where \( R_k \) can be increased up to a finite limit for the integration to encompass solely a single Dirac point, it also describes the linearized low energy model around the \( K \) and \( K' \) points. For the linearized models, the limit \( R_k \to \infty \) leads to, as prescribed by equation (B.1), \( C_K, C_{K'} \to \pm 1/2 \), resulting in a non-vanishing valley Chern number and an associated topologically non-trivial phase, analogous to the QVHE. We note that when \( \varepsilon \) flips sign, the values of \( C_K \) and \( C_{K'} \) flip sign as well, corresponding to the topological phase transition discussed in Sec. 3.1.

Fig. B.1. (a) Contour plot of the Berry curvature in the first Brillouin zone, for \( \varepsilon = 0.1 \). The non-zero curvature values are centered around the \( K \) and \( K' \) high symmetry points of Fig. 2(a)-right. (b) The Chern number \(|C_K|\) is linearly approaching \( 1/2 \) with \( \varepsilon \to 0 \), as expected from equation (B.2).
The design parameter is given by
\[ a = \frac{\omega}{2c} \]
where \( a \) is the target resonator opening area (as before) and \( A \) is the target resonator opening area (as before) and \( b \) is the artificial tube length. As a result, we obtain a periodic hexagonal net with two sites per unit cell, labeled by \( A \) and \( B \), as captured within the black parallelogram. The second stage is approximating the continuous pressure field in those artificial tubes by equivalent lumped acoustic springs. The governing target closed-loop equations of the \((m,n)\) unit cell for each of these sites, then respectively take the form
\[
B_0 \left( p_{m,n+1}^B + p_{m,n}^B + p_{m+1,n}^B - 3p_{m,n}^A \right) = M_0 s^2 p_{m,n}^A + M_0 B_0 \eta \frac{s}{Z_A(s)} p_{m,n}^A, \tag{C.1a}
\]
\[
B_0 \left( p_{m,n-1}^A + p_{m,n}^A + p_{m-1,n}^A - 3p_{m,n}^B \right) = M_0 s^2 p_{m,n}^B + M_0 B_0 \eta \frac{s}{Z_B(s)} p_{m,n}^B, \tag{C.1b}
\]
where \( B_0 = \rho c^2 / b \, [kg/\text{s}^2/\text{m}^2] \) and \( M_0 = \rho b \, [kg/m^2] \) are the equivalent acoustic spring and mass of each artificial tube, respectively. The design parameter is given by \( \eta = A_n / A_T \) and is dimensionless, as expected for a one-dimensional waveguide, where \( A_n \) is the target resonator opening area (as before) and \( A_T \) is the cross-section area of the artificial tube.

In order to fit the approximated model in (C.1) to the exact one in (8), we first calculate the dispersion relation of the open system in Sec. 3.2. The only part of (1) to be discretized is the continuous two-dimensional wave equation on the left hand side. We thus completely preserve the coupling of the target resonators on the right hand side. We perform the discretization in two stages. First, we limit the sound pressure wave propagation to direct routes between the target resonators locations, through, for example, artificial one-dimensional tubes, as illustrated in Fig. C.1(a). Here \( a \) is the same lattice constant as in the continuous case, and \( b \) is the artificial tube length. As a result, we obtain a periodic hexagonal net with two sites per unit cell, labeled by \( A \) and \( B \), as captured within the black parallelogram. The second stage is approximating the continuous pressure field in those artificial tubes by equivalent lumped acoustic springs. The governing target closed-loop equations of the \((m,n)\) unit cell for each of these sites, then respectively take the form
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