Parameter-dependent linear ordinary differential equations and topology of domains

Vyacheslav M. Boyko†, Michael Kunzinger‡ and Roman O. Popovych†‡§

† Institute of Mathematics of National Academy of Sciences of Ukraine, 3 Tereshchenkivska Str., Kyiv-4, 01601 Ukraine
‡ Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria
§ Mathematical Institute, Silesian University in Opava, Na Rybníčku 1, 746 01 Opava, Czech Republic

E-mail: †boyko@imath.kiev.ua, ‡michael.kunzinger@univie.ac.at, §rop@imath.kiev.ua

The well-known solution theory for (systems of) linear ordinary differential equations undergoes significant changes when introducing an additional real parameter. Properties like the existence of fundamental sets of solutions or characterizations of such systems via nonvanishing Wronskians are sensitive to the topological properties of the underlying domain of the independent variable and the parameter. In this work we give a complete characterization of the solvability of such parameter-dependent systems in terms of topological properties of the domain. In addition, we also investigate this problem in the setting of Schwartz distributions.

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1 Introduction

The solution theory of p-th order linear ordinary differential equations (ODEs)

$$\sum_{i=0}^{p} g^i(x) \frac{d^i u}{dx^i} = f(x),$$

where $u$ is the unknown function, the independent variable $x$ varies in some interval $(a,b)$, $g^0, \ldots, g^p, f \in C((a,b))$ and $g^p(x) \neq 0$ for all $x \in (a,b)$, is a classical subject that is a part of most textbooks in the field (e.g., [1, 3, 15]). The set of all (classical) solutions to the homogeneous equation ($f = 0$) forms a $p$-dimensional vector subspace $V$ of $C^p((a,b))$, while the solution space of (1) is an affine subspace obtained by translating $V$ by any particular solution of (1). Any tuple of $p$ linearly independent solutions $(\varphi^1, \ldots, \varphi^p)$ of the homogeneous equation is called a fundamental set of solutions, and setting $\varphi_{s-1} := d^{s-1} \varphi/dx^{s-1}$, $s = 1, \ldots, p$, we write

$$W(\varphi^1, \ldots, \varphi^p) := \det \begin{pmatrix} \varphi^1 & \cdots & \varphi^p \\ \varphi_1 & \cdots & \varphi_1 \\ \vdots & \ddots & \vdots \\ \varphi_{p-1} & \cdots & \varphi_{p-1} \end{pmatrix}$$

for the corresponding Wronskian. Solutions $\varphi^1, \ldots, \varphi^p$ of the homogeneous equation form a fundamental set if and only if the Wronskian $W(\varphi^1, \ldots, \varphi^p)$ does not vanish at $x_0 \in (a,b)$, and therefore it vanishes nowhere on $(a,b)$ in view of the Liouville–Ostrogradski formula,

$$W(\varphi^1, \ldots, \varphi^p)(x) = W(\varphi^1, \ldots, \varphi^p)(x_0) \exp \left( - \int_{x_0}^{x} \frac{g^{p-1}(x')}{g^p(x')} dx' \right).$$
In this case, \((\varphi^1, \ldots, \varphi^p)\) is a basis of the vector space of solutions to the homogeneous equation. See, e.g., [9] Section IV.8.iii or [15] Section 19.II. A particular solution to the inhomogeneous equation \((6)\) is given by (cf. [1] Proposition (14.3))

\[
u(x) = \sum_{s=1}^{p} (-1)^{p-s} \varphi^s(x) \int_{x_0}^{x} \psi^s(x') \, dx'
\]

with \(\psi^s := \frac{f}{g^p} \frac{W(\varphi^1, \ldots, \varphi^s, \ldots, \varphi^p)}{W(\varphi^1, \ldots, \varphi^p)}\). \hspace{1cm} (3)

Finally, we recall the well-known fact that any \(p\)th order equation of the form \((6)\) can be rewritten as a linear system of first-order ordinary differential equations in the normal Cauchy form, so that the solution theory of (scalar) equations of the form \((6)\) can be reduced to that of such systems. Concretely, setting

\[v^s := u_{s-1}, \quad s = 1, \ldots, p,\]

the equation \((6)\) is equivalent to the system

\[v_1 = A(x) v + F(x)\]

for \(v = (v^1, \ldots, v^p)^T\), where

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
-g^2 \frac{\partial^2}{g^p} - g^2 \frac{\partial^2}{g^p} - g^2 \frac{\partial^2}{g^p} \cdots - g^{p-2} \frac{\partial^2}{g^p} - g^{p-1} \frac{\partial^2}{g^p}
\end{pmatrix}, \quad F = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\frac{f}{g^p}
\end{pmatrix}.
\hspace{1cm} (5)

In this paper we address the question of how this solution theory changes if the coefficient functions and the right hand side in \((6)\) are allowed to additionally depend on a real parameter \(t\). Thus we shall be investigating ODEs of the form

\[Pu = \sum_{i=0}^{p} g^i(t, x) u_i = f(t, x),\]

where, analogously to the above, \(u_i := \partial^i u / \partial x^i\), \(i = 0, \ldots, p\), as well as systems of the form

\[v_1 = A(t, x) v + F(t, x),\]

for \((t, x)\) varying in some open subset \(\Omega\) of \(\mathbb{R}^2\). Here \(g^0, \ldots, g^p, f \in C(\Omega, \mathbb{R}^p), g^p(t, x) \neq 0 \) for all \((t, x) \in \Omega, A \in C(\Omega, M_p(\mathbb{R}))\) and \(F \in C(\Omega, \mathbb{R}^p)\). We are, in particular, interested in determining the influence of the topology of \(\Omega\) on the structure of the solution spaces of \((6)\) and of \((7)\). Basic examples show that solvability may completely break down already for very simple sets \(\Omega\) (e.g. for \(\Omega\) the punctured plane, cf. Example 4.1 below). Conversely, for nice enough domains, e.g. for rectangles, the parameter-dependent theory is practically the same as in the single-variable case. We want to find out which properties of the domain determine the solvability of linear parameter-dependent ODEs. Indeed, we will completely characterize the solvability of \((6)\) and of \((7)\) in terms of a topological property of \(\Omega\), namely the so-called \(x\)-simplicity of \(\Omega\), a notion well-known from elementary integration theory (cf., e.g., [9]). In addition, we characterize the existence of fundamental sets of solutions of \((6)\) or of fundamental matrices of \((7)\), and the nonvanishing of the corresponding Wronskians, again in terms of the \(x\)-simplicity of \(\Omega\) (or its connected components).

As \((6)\) may also be viewed as a specific kind of linear partial differential equation, the question of existence of solutions in terms of properties of the underlying domain bears some resemblance
to notions like Hörmander’s concept of \( P \)-convexity \([4, 9]\). In particular, it is of interest to address the problems stated above also within the framework of Schwartz distributions.

In the remainder of this introduction we fix some notations and outline the content of the sections to follow. Let us briefly comment on our choice of notation and style of presentation. Our original motivation for studying parameter-dependent linear ODEs derives from our desire to develop a more rigorous theory of Darboux transformations for \((1+1)\)-dimensional linear evolution equations than the existing ones; cf. \([10, 11, 12]\). This explains why we primarily focus on scalar equations and just outline the corresponding results for systems (contrary to the standard approach in the ODE literature). It also justifies the notation of variables, \( x \) for the independent variable and \( t \) for the parameter, as well as their order.

**Notation.** Given a subset \( U \) of the \((t, x)\)-plane, the projection \( U \) to the \( t \)-axis is denoted by \( \text{pr}_t U \), where \( \text{pr}_t : \mathbb{R}^2 \rightarrow \mathbb{R} \) is the projection \( (t, x) \mapsto t \), and for each \( t_0 \in \text{pr}_t U \) the set \( U_{t_0} \) is the projection of the section of \( U \) by the line \( t = t_0 \) to the \( x \)-axis,

\[
\text{pr}_t U := \{ t \in \mathbb{R} \mid \exists x \in \mathbb{R} : (t, x) \in U \}, \quad \text{pr}_t U \ni t \mapsto U_t := \{ x \in \mathbb{R} \mid (t, x) \in U \}.
\]

For a function \( g : U \rightarrow \mathbb{R} \), the expression “\( g \neq 0 \) on \( U \)” means that \( g(z) \neq 0 \) for any \( z \in U \).

For an open set \( \Omega \) of the \((t, x)\)-plane, \( C^\omega(\Omega) \) is the space of real analytic functions on \( \Omega \). \( C_p^{\omega}(\Omega) \) with \( p \in \mathbb{N} \) denotes the subspace of functions from \( C(\Omega) \) that admit derivatives with respect to \( x \) up to order \( p \), and these derivatives are continuous on \( \Omega \). Analogously, \( C_p^\infty(\Omega) \) denotes the subspace of functions from \( C(\Omega) \) that are real analytic with respect to \( x \) and whose derivatives with respect to \( x \) are continuous on \( \Omega \). \( DO(\Omega) \), \( DO^\infty(\Omega) \), \( DO^\omega(\Omega) \) and \( DO^\infty(\Omega) \) denote the sets of linear differential operators in \( x \) (hence of the form \([4]\)) with coefficients from \( C(\Omega) \), \( C^\infty(\Omega) \), \( C^\omega(\Omega) \) and \( C^\infty(\Omega) \), respectively, and whose leading coefficients do not vanish on the entire (open) set \( \Omega \). \( DO_1(\Omega) \), \( DO_p^\omega(\Omega) \), \( DO_p^\infty(\Omega) \), \( DO_p^\infty(\Omega) \) and \( DO_p^\infty(\Omega) \) are, respectively, the subsets of operators from \( DO(\Omega) \), \( DO^\infty(\Omega) \), \( DO_p^\omega(\Omega) \) and \( DO_p^\infty(\Omega) \) whose leading coefficients are equal to one. The notation \( \text{ord} \) \( P \) and \( \text{lcoef} \) \( P \) is used for the order and the leading coefficient of the operator \( P \in DO(\Omega) \), respectively. If \( \text{ord} P = p \), then we view the operator \( P \) as a map from \( C_p^\omega(\Omega) \) to \( C(\Omega) \), and thus (classical) solutions of the equation \( Pu = 0 \) belong to \( C_p^{\omega}(\Omega) \).

Expressions of the form \( \zeta \psi \) for some functions \( \zeta \in C(\text{pr}_t \Omega) \) and \( \psi \in C(\Omega) \) should always be interpreted as the product of \( \psi \) by the pullback of \( \zeta \) to \( \Omega \) with respect to the map \( \text{pr}_t |_{\Omega} \).

As already noted above, for a function \( u \) of \((t, x)\) we set \( u_i := \partial^i u / \partial x^i \), \( i \in \mathbb{N} \), \( u_0 := u \), and \( \partial_x = \partial / \partial x \). We also employ, depending upon convenience or necessity, the notation \( u_x = u_1 \).

By \( W(\varphi^1, \ldots, \varphi^p) \) we denote the Wronskian of functions \( \varphi^1, \ldots, \varphi^p \in C^\omega_x(\Omega) \) in the variable \( x \), i.e., \( W(\varphi^1, \ldots, \varphi^p) = \det(\partial_x^{s-1} \varphi^s)_{s,s' = 1, \ldots, p} \).

The indices \( s, s' \) and \( s'' \) run from 1 to \( p \), and summation with respect to repeated indices is always understood.

The plan of the paper is as follows: In Section 2 we introduce sets simple with respect to a variable (in our case, \( x \)), and prove some basic topological properties of such sets. In Section 3 we provide an appropriate notion of fundamental sets of solutions to homogeneous linear parameter-dependent ODEs and relate it to the nonvanishing of the corresponding Wronskian. We also characterize both concepts in terms of \( x \)-simplicity of (pieces of) the underlying domain \( \Omega \). The inhomogeneous setting is studied in Section 4 where we derive necessary and sufficient conditions for solvability in terms of \( x \)-simplicity and also quantify the ‘degree of non-solvability’ in case some connected component of \( \Omega \) fails to be \( x \)-simple. In Section 5 we then turn to the distributional setting, singling out the relevant case of \( C^0 \)-semiregular distributions. The final Section 6 is devoted to the study of systems of parameter-dependent linear ODEs. In Appendix A we prove a structure theorem for distributions with vanishing partial derivatives on domains that are simple with respect to a variable. This is required for deriving the general form of distributional solutions to parameter-dependent (systems of) linear ODEs on such domains in Sections 5 and 6.


## 2 Sets simple with respect to a variable

Given a subset $U$ of the $(t,x)$-plane, by $lb_U$ and $ub_U$ we denote the lower and upper bounds of $U$ in $x$, which are functions from $pr_I U$ to $\mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$lb_U(t) := \inf U_t, \quad ub_U(t) := \sup U_t, \quad t \in pr_I U.$$  

In view of the inequality $lb_U(t) \leq x \leq ub_U(t)$ for any $t \in pr_I U$ and any $x \in U_t$, the functions $lb_U$ and $ub_U$ may attain values only from $\mathbb{R} \cup \{-\infty\}$ and $\mathbb{R} \cup \{+\infty\}$, respectively. It is obvious that $U \subseteq \{(t,x) \in \mathbb{R}^2 \mid t \in pr_I U, lb_U(t) \leq x \leq ub_U(t)\}$. See Figure 1 that illustrates some objects related to $x$-simplicity.

**Lemma 2.1.** If a subset $\Omega$ of the $(t,x)$-plane is open, then its projection $I := pr_t \Omega$ is an open subset of $\mathbb{R}$ and the functions $a := lb_{\Omega}$ and $b := ub_{\Omega}$ are upper and lower semi-continuous, respectively.

**Proof.** Since for any ball contained in $\Omega$ its projection to the $t$-axis is an open interval contained in $I$, it is obvious that $I$ is an open set. Fix an arbitrary $t_0 \in I$. If $a(t_0) \in \mathbb{R}$, then for any $\varepsilon > 0$ with $a(t_0) + \varepsilon < b(t_0)$, the point $z_0 = (t_0, a(t_0) + \varepsilon)$ belongs to $\Omega$ and thus there exists a $\delta > 0$ such that the ball $B_{\delta}(z_0)$ is contained in $\Omega$. Therefore, for any $t \in (t_0 - \delta, t_0 + \delta)$ we have $t \in I$ and $a(t) < a(t_0) + \varepsilon$. Analogously, if $a(t_0) = -\infty$, then for an arbitrary $N > 0$ with $-N < b(t_0)$, the point $z_0 = (t_0, -N)$ belongs to $\Omega$ and again there exists a $\delta > 0$ such that the ball $B_{\delta}(z_0)$ is contained in $\Omega$. Hence for any $t \in (t_0 - \delta, t_0 + \delta)$ we have $t \in I$ and $a(t) < -N$. In total, this means that the function $a$ is upper semi-continuous on $I$. The lower semi-continuity of $b$ is proved in a similar way. \(\square\)

In the above notation, we have $\Omega \subseteq \{(t,x) \mid t \in I, a(t) < x < b(t)\}$ if the set $\Omega$ is open.

**Definition 2.2.** We call a subset $U$ of the $(t,x)$-plane an $x$-simple set if the intersection of $U$ by any line $t = t_0 \in \mathbb{R}$ is an open interval within this line or the empty set.

Equivalently, a subset $U$ of the $(t,x)$-plane is called an $x$-simple set if there exist a subset $I$ of the $t$-axis and functions $a,b : I \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ with $a(t) < b(t)$ for any $t \in I$ such that

$$U = \{(t,x) \mid t \in I, a(t) < x < b(t)\}.$$  

(8)

Then $I = pr_I U$, $a = lb_U$ and $b = ub_U$. Note that this definition of $x$-simple set is similar to but in fact different from and, in certain sense, more general than the one used in elementary calculus (cf., e.g., [9, p. 341]).

**Lemma 2.3.** An $x$-simple subset $\Omega$ of the $(t,x)$-plane is open if and only if its projection $I := pr_t \Omega$ is an open subset of $\mathbb{R}$ and its lower and upper bounds, $a := lb_{\Omega}$ and $b := ub_{\Omega}$, are upper and lower semi-continuous functions on $I$, respectively. Moreover, in this case there exists a smooth function $\theta : I \rightarrow \mathbb{R}$ such that $a(t) < \theta(t) < b(t)$ for any $t \in I$.

**Proof.** The necessity of the first claim follows from Lemma 2.1. Let us prove its sufficiency. Suppose that the set $I$ is open and the functions $a$ and $b$ are upper and lower semi-continuous, respectively. Fix an arbitrary $(t_0, x_0) \in \Omega$. Then $a(t_0) < x_0 < b(t_0)$ and thus $\varepsilon := \frac{1}{2} \min \{x_0 - a(t_0), b(t_0) - x_0, 1\} > 0$. Hence there exists a neighborhood $V$ of $t_0$ in $I$ such that for all $t \in V$, $a(t) < x_0 - \varepsilon$ and $b(t) > x_0 + \varepsilon$, i.e., the neighborhood $V \times (x_0 - \varepsilon, x_0 + \varepsilon)$ of $t_0$ is contained in $\Omega$. Therefore, the set $\Omega$ is open.

Supposing now that $\Omega$ is $x$-simple, we can cover the open set $I$ by a family of open subsets $(V_j)_{j \in J}$, where $J$ is some index set, such that for any $j \in J$ there exists $c_j \in \mathbb{R}$ with $a(t) < c_j < b(t)$ for any $t \in V_j$. Let $(\chi^j)_{j \in J}$ be a $C^\infty$-partition of unity that is subordinate to $(V_j)_{j \in J}$,
i.e., \((\supp \chi^j)_{j \in J}\) is locally finite and \(\supp \chi^j \subseteq V_j\) for any \(j \in J\) [7, Theorem A.1]. Then the function \(\theta := \sum_{j \in J} \chi^j c_j\) belongs to \(C^\infty(I)\), and for any \(t \in I\) we obtain

\[
a(t) = \sum_{j \in J} \chi^j(t) a(t) < \sum_{j \in J} \chi^j(t) c_j = \theta(t) < \sum_{j \in J} \chi^j(t) b(t) = b(t).
\]

(Here non-strict inequalities are clear for any \(j \in J\), but also for any \(t \in I\) there exists \(j_0 \in J\) with \(\chi^{j_0}(t) > 0\), and for this term we have \(\chi^{j_0}(t) a(t) < \chi^{j_0}(t) c_j < \chi^{j_0}(t) b(t)\), which implies that strict inequalities hold for the entire sums.)

**Corollary 2.4.** Any \(x\)-simple open connected set is simply connected.

**Proof.** Suppose that the set \(\Omega \subseteq \mathbb{R}^2\) is \(x\)-simple, open and connected. In view of Lemma 2.3, there exists a smooth function \(\theta \in C^\infty(\text{pr}_1 \Omega)\) whose graph is contained in \(\Omega\). (In fact, the continuity of \(\theta\) is sufficient for the further proof.) Fix a point \(t_0 \in \text{pr}_1 \Omega\) and consider an arbitrary continuous path \(\gamma : S^1 \to \Omega, S^1 \ni \tau \mapsto (\gamma^1(\tau), \gamma^2(\tau)) \in \Omega\), where \(S^1\) is the unit circle. The path \(\gamma\) can be shrunk to the point \((t_0, \theta(t_0))\) within \(\Omega\) using the map from the unit disk to \(\Omega\) that is defined by

\[
(\rho, \tau) \mapsto \begin{cases} 
(\gamma^1(\tau) + (1 - 2\rho)(t_0 - \gamma^1(\tau)), \theta(\gamma^1(\tau) + (1 - 2\rho)(t_0 - \gamma^1(\tau)))) & , \rho \in [0, \frac{1}{2}], \\
(\gamma^1(\tau), \gamma^2(\tau) + 2(1 - \rho)(\theta(\gamma^1(\tau)) - \gamma^2(\tau))) & , \rho \in [\frac{1}{2}, 1],
\end{cases}
\]

where \((\rho, \tau)\) are the ‘polar’ coordinates on the disk, \(\rho \in [0, 1]\) and \(\tau \in S^1\). (Roughly speaking, we first shrink the path \(\gamma\) along the \(x\)-direction to the arc \(\{(t, \theta(t)) \mid t \in \text{pr}_1 \gamma(S^1)\}\) of the graph of the function \(\theta\) and then shrink this arc along itself to the point \((t_0, \theta(t_0))\)).

Note that an open simply connected set is not in general \(x\)-simple. An example of such a set is \(\mathbb{R}^2 \setminus ([0, +\infty) \times \{0\})\).

The following lemma introduces an essential technical tool for our further investigation: it identifies, within any non-\(x\)-simple set, a certain configuration that will allow us to construct differential operators on \(\Omega\) with ‘problematic’ behavior.
Lemma 2.5. For any open connected non-x-simple subset $\Omega$ of the $(t, x)$-plane, there exist $\tilde{t}_0, \varepsilon, \tilde{x}_1, \tilde{x}_2 \in \mathbb{R}$ with $\varepsilon > 0$, and $\tilde{x}_1 \leq \tilde{x}_2$ such that, up to reflections in $t$, the set $\Omega$ does not intersect a closed subset $\Upsilon$ of the line segment $\{\tilde{t}_0\} \times [\tilde{x}_1, \tilde{x}_2]$ with $(\tilde{t}_0, \tilde{x}_1), (\tilde{t}_0, \tilde{x}_2) \in \Upsilon$ and contains the subset

$$[\tilde{t}_0 - \varepsilon, \tilde{t}_0] \times [\tilde{x}_1 - \varepsilon, \tilde{x}_2 + \varepsilon] \setminus \Upsilon.$$  \hspace{1cm} (9)

Proof. Since the set $\Omega$ is not $x$-simple, there exists $t_0 \in \text{pr}_t \Omega$ such that $\Omega_{t_0}$ is not connected, i.e., for some $x_0, x_1, x_2, x_3 \in \mathbb{R}$ with $x_0 < x_1 \leq x_2 < x_3$ we have $[x_0, x_1), (x_2, x_3] \subset \Omega_{t_0}$ and $x_1, x_2 \not\in \Omega_{t_0}$; see Figure 2. Since the set $\Omega$ is connected, there exists a (continuous) path $\gamma: [0, 1] \rightarrow \Omega$ with $\gamma(0) = (t_0, x_0)$ and $\gamma(1) = (t_0, x_3)$. Without loss of generality, we can assume the map $\gamma$ injective.\footnote{Using the openness of $\Omega$, we can additionally assume that the image of $\gamma$ is a polygonal line, but this is not essential for the present proof.} Let

$$\tau_0 = \sup \{ \tau \in [0, 1] \mid \gamma(\tau) \in (t_0) \times [x_0, x_1) \}, \quad \tau_1 = \inf \{ \tau \in [0, 1] \mid \gamma(\tau) \in (t_0) \times (x_1, x_3]\}.$$  

Replacing $\gamma$ by its subpath $\gamma|_{[\tau_0, \tau_1]}$, $(t_0, x_0)$ by $\gamma(\tau_0)$, $(t_0, x_3)$ by $\gamma(\tau_1)$ and $x_2$ by the supremum of the relative complement of $\Omega_{t_0}$ in the new interval $(x_0, x_3)$, we can also assume that $(t_0, x_0)$ and $(t_0, x_3)$ are the only common points of $\gamma([0, 1])$ with $(t_0) \times [x_0, x_3]$. We complete $\gamma([0, 1])$ by $(t_0) \times (x_0, x_3)$ to a simple closed curve, which we denote by $C$. According to the Jordan curve theorem, this curve divides the $(t, x)$-plane into the (bounded) interior $\Gamma$ and the (unbounded) exterior $\tilde{\Gamma}$. Up to reflections in $t$, we can assume that there exists a neighborhood $U$ of the point $(t_0, x_1)$ such that $U \cap \Gamma$ and $U \cap \tilde{\Gamma}$ contain only points with negative and positive values of $t - t_0$, respectively. Set

$$\tilde{t}_0 = \inf \{ t \in \mathbb{R} \mid \exists x \in \mathbb{R}: (t, x) \in (C \cup \Gamma) \setminus \Omega \}.$$  

If $\tilde{t}_0 = t_0$, then we set $\tilde{x}_1 = x_1$, $\tilde{x}_2 = x_2$ and $\varepsilon = \frac{1}{2} \text{dist} \left( \{\tilde{t}_0\} \times [\tilde{x}_1, \tilde{x}_2], \gamma([0, 1]) \right).$ (The distance is measured between disjoint compact sets and thus $\varepsilon > 0$.)
Otherwise, the compactness of \((C \cup \Gamma) \setminus \Omega\) implies that there exists \(\tilde{x}_0 \in \mathbb{R}\) with \((\tilde{t}_0, \tilde{x}_0) \in \Gamma \setminus \Omega\). Since the set \(\Gamma\) is bounded, the line \(t = \tilde{t}_0\) intersects the curve \(C\) in at least one point with \(x\)-coordinate less than \(\tilde{x}_0\) and in at least one point with \(x\)-coordinate greater than \(\tilde{x}_0\). Therefore the values

\[
\hat{x}_1 = \sup \{ x \in \mathbb{R} \mid (\tilde{t}_0, x) \in \gamma([0,1]), x < \tilde{x}_0 \}, \quad \tilde{x}_1 = \inf \{ x \in \mathbb{R} \mid (\tilde{t}_0, x) \notin \Omega, x > \hat{x}_1 \},
\]

\[
\hat{x}_2 = \inf \{ x \in \mathbb{R} \mid (\tilde{t}_0, x) \in \gamma([0,1]), x > \tilde{x}_0 \}, \quad \tilde{x}_2 = \sup \{ x \in \mathbb{R} \mid (\tilde{t}_0, x) \notin \Omega, x < \hat{x}_2 \}
\]

are well defined as the supremum (resp. infimum) of a nonempty set that is bounded from above (resp. below), and \(\tilde{x}_1 \leq \tilde{x}_2\). The line segment \(\{(\tilde{t}_0) \times (\hat{x}_1, \hat{x}_2)\}\) does not intersect the curve \(C\) and contains the point \((\tilde{t}_0, \tilde{x}_0)\), which belongs to the interior \(\Gamma\). Therefore, this segment is contained in \(\Gamma\). The value of \(\varepsilon\) is defined as in the previous case.

The chosen values of \(\tilde{t}_0, \varepsilon, \hat{x}_1\) and \(\tilde{x}_2\) then satisfy the claimed properties. \(\square\)

**Definition 2.6.** If there exists an open interval \(I\) of the \(t\)-axis such that the intersection of a subset \(U\) of the \((t,x)\)-plane by the strip \(I \times \mathbb{R}\) has an \(x\)-simple connected component, then we call this component an \(x\)-simple piece of \(U\).

**Remark 2.7.** Suppose that for an open set \(\Omega\) of the \((t,x)\)-plane the subset \(J\) of \(t\)'s from \(\text{pr}_t \Omega\) with connected \(\Omega_t\)'s is dense in \(\text{pr}_t \Omega\). Then the set \(\Omega\) contains no \(x\)-simple pieces if and only if the complement of \(J\) in \(\text{pr}_t \Omega\) is also dense in \(\text{pr}_t \Omega\).

**Example 2.8.** The set \(\Omega := \{(0,1) \times (0,1)\} \setminus \{(2^{-k}l, 1 - 2^{-k}), l = 1, \ldots, 2^k - 1, k \in \mathbb{N}\}\) is open, connected, and contains no \(x\)-simple pieces. See Figure 3.

Open \(x\)-simple regions naturally arise in the context of fundamental sets of solutions of linear ordinary differential equations depending on a parameter. Moreover, several properties of such equations depend on whether the underlying domain \(\Omega\) of the independent variable \(x\) and the parameter \(t\) is \(x\)-simple and how the \(x\)-simplicity is combined with the connectedness, in particular, whether all the connected components of \(\Omega\) or at least some of them are connected or whether the domain \(\Omega\) has \(x\)-simple pieces. These properties include

- the existence of fundamental sets of solutions and of sets of solutions with nonvanishing Wronskians,
Figure 4. Variants of combining $x$-simplicity, connectedness and simple connectedness: (a) connected, $x$-simple and thus simply connected set; (b) connected, non-$x$-simple and thus multiply connected set; (c) non-$x$-simple, simply connected set; (d) disconnected $x$-simple set (each of its connected components is necessarily $x$-simple and thus simply connected); (e) disconnected non-$x$-simple set whose connected components are $x$-simple and thus simply connected; (f) disconnected non-$x$-simple set having a non-$x$-simple connected component.

- the relation between these two kinds of solution sets,
- the existence of solutions that are not identically zero for such homogeneous equations and
- the general existence of solutions for such inhomogeneous equations.

See Figure 4 for some variants of combining $x$-simplicity, connectedness and simple connectedness.

### 3 Fundamental sets of solutions of homogeneous linear ordinary differential equations depending on a parameter

Given a homogeneous linear $p$th order ordinary differential equation with the independent variable $x$ and the parameter $t$ and with continuous coefficients defined on an open set $\Omega \subseteq \mathbb{R}^2$ of $(t,x)$, the question is whether there exist $p$ continuous solutions of this equation with non-vanishing Wronskian on $\Omega$. In general, the answer is negative, as is illustrated by the following example.

**Example 3.1.** Consider the linear homogeneous first-order ordinary differential equation

$$P: \ u_x = \frac{u}{x^2 + t^2} \quad \text{on} \quad \Omega = \mathbb{R}^2 \setminus \{(0,0)\}.$$

For each fixed $t$, its general solution is

$$u = C \exp \left( \frac{1}{t} \arctan \frac{x}{t} \right) \quad \text{if} \quad t \neq 0, \quad u = C \exp \left( -\frac{1}{x} \right) \quad \text{if} \quad t = 0,$$

---

$^2$By this we mean solutions from $C^p_0(\Omega)$, cf. the notations agreed upon in Section 1.
where $C$ is an arbitrary constant. This solution is well defined on the entire $\Omega = \mathbb{R}$ if $t \neq 0$, and should be considered separately on each $x$-semiaxis, $\mathbb{R}_+$ and $\mathbb{R}_-$, if $t = 0$. The functions

$$u = \zeta^+(t) \exp \left( \frac{1}{t} \arctan \frac{x}{t} \right), \quad t > 0, \ x \in \mathbb{R},$$

$$u = \zeta^-(t) \exp \left( \frac{1}{t} \arctan \frac{x}{t} \right), \quad t < 0, \ x \in \mathbb{R},$$

where the parameter function $\zeta^+$ (resp. $\zeta^-$) runs through $C(\mathbb{R}_+)$ (resp. $C(\mathbb{R}_-)$), represent the general solutions of the equation $\mathcal{P}$ on the domains $\mathbb{R}_+ \times \mathbb{R}$ and $\mathbb{R}_- \times \mathbb{R}$, respectively. The question is whether there exists a solution of $\mathcal{P}$ that is continuous and nonvanishing on the entire $\Omega$. Suppose that this is the case, and that $u = \varphi(t,x)$ is such a solution. Define the function $\zeta(t) := \varphi(t,-1), \ t \in \mathbb{R}$. We have $\zeta \in C(\mathbb{R})$ and

$$\varphi(t,x) = \begin{cases} 
\zeta(t) \exp \left( \frac{1}{t} \arctan \frac{x}{t} + \frac{\pi}{2|t|} - \arctan \frac{t}{x} \right) & \text{if } t \neq 0, \ x \in \mathbb{R}, \\
\zeta(0) \exp \left( -\frac{1}{x} - 1 \right) & \text{if } t = 0, \ x \in \mathbb{R}_-. 
\end{cases}$$

Here we use the equality

$$\frac{1}{t} \arctan \frac{x}{t} - \frac{\pi}{2|t|} \sgn x = -\frac{1}{t} \arctan \frac{t}{x} \quad \text{if } t \neq 0, \ x \neq 0. \quad (10)$$

The right hand side function is continuous on $\mathbb{R}^2 \setminus \{(0) \times [0, +\infty)\}$ but cannot be continuously extended to $\Omega$ if $\zeta(0) \neq 0$ since for $x > 0$ and $t \to 0$ we obtain

$$\zeta(t) \exp \left( \frac{1}{t} \arctan \frac{x}{t} + \frac{\pi}{2|t|} - \arctan \frac{t}{x} \right) = \zeta(t) \exp \left( -\frac{1}{t} \arctan \frac{t}{x} + \frac{\pi}{|t|} - \arctan \frac{t}{x} \right) \to \infty,$$

where the sign of infinity coincides with the sign of $\zeta(0)$. In other words, the equation $\mathcal{P}$ has no (continuous) solution that is nonzero on the entire domain $\Omega$. Moreover, any solution of this equation on $\Omega$ vanishes on the half-axis $\{0\} \times (-\infty, 0)$, and the corresponding function $\zeta$ is $O(e^{-\pi/|t|})$ as $t \to 0$. Consider the solution

$$\varphi^1(t,x) = \begin{cases} 
\exp \left( \frac{1}{t} \arctan \frac{x}{t} - \frac{\pi}{2|t|} + \arctan \frac{t}{x} \right) & \text{if } t \neq 0, \ x \in \mathbb{R}, \\
\exp \left( \frac{1}{x} - 1 \right) & \text{if } t = 0, \ x \in \mathbb{R}_+, \\
0 & \text{if } t = 0, \ x \in \mathbb{R}_-. 
\end{cases}$$

Since $\varphi^1(t,1) = 1$, any solution $\varphi$ of the equation $\mathcal{P}$ on $\Omega$ can be represented as $\varphi = \tilde{\varphi} \varphi^1$, where $\tilde{\varphi} := \varphi(t,1) \in C(\mathbb{R})$. In this sense the function $\varphi^1$ constitutes a fundamental set of solutions of this equation on $\Omega$.

**Definition 3.2.** Given a linear ordinary differential equation $\mathcal{P}$: $Pu = 0$ on an open subset $\Omega$ of the $(t,x)$-plane, where $P \in DO(\Omega)$ with $\text{ord } P = p$ and $t$ plays the role of a parameter, we say that functions $\varphi^s \in C^p(\Omega)$, $s = 1, \ldots, p$, satisfying this equation constitute

- a **fundamental set of solutions** of $\mathcal{P}$ on $\Omega$ if any solution $u$ of $\mathcal{P}$ can uniquely be represented in the form $u = \zeta^s \varphi^s$ for certain functions $\zeta^s \in C(\text{pr}_t \Omega)$;
- a **locally fundamental set of solutions** of $\mathcal{P}$ on $\Omega$ if there exists an open cover of the set $\Omega$ (by which we shall always understand a cover consisting of open subsets of $\Omega$) such that the restriction of any solution $u$ of $\mathcal{P}$ to any element $U$ of the cover, $u|_U$, can uniquely be represented in the form $u|_U = \zeta^s \varphi^s|_U$ for certain functions $\zeta^s \in C(\text{pr}_t U)$.
Lemma 3.3. Any solutions $\varphi^s \in C^p_x(\Omega)$, $s = 1, \ldots, p$, of an equation $Pu = 0$ with $P \in DO(\Omega)$ and $\text{ord } P = p$ that satisfy the condition $W(\varphi^1, \ldots, \varphi^p) \neq 0$ on $\Omega$ constitute a locally fundamental set of solutions of this equation.

Proof. It suffices to consider a covering of $\Omega$ by balls $U_j := B_{\varepsilon_j}(z_j) \subseteq \Omega$, $j \in J$, where $J$ is some index set, and $z_j = (t_j, x_j) \in \Omega$. For an arbitrary solution $u$ of the equation $Pu = 0$ and for each of these balls, we have the representation $u|_{U_j} = \zeta^j u^j$ where the functions $\zeta^j \in C^2((t_j - \varepsilon_j, t_j + \varepsilon_j))$ are defined, for each $t \in (t_j - \varepsilon_j, t_j + \varepsilon_j)$, as solutions of the system $\zeta^j(t) \varphi^s - \varphi^j = u^j - u^s$. Let $s' = 1, \ldots, p$.

Theorem 3.4. Given an open subset $\Omega$ of the $(t, x)$-plane, the following are equivalent:

(i) Any homogeneous linear ordinary differential equation $Pu = 0$ with $P \in DO(\Omega)$ admits a fundamental set of solutions on $\Omega$ with nonvanishing Wronskian.

(ii) $\Omega$ is an $x$-simple region.

Proof. (ii) $\Rightarrow$ (i): Consider an arbitrary $P \in DO(\Omega)$. In view of Lemma 2.3, there exists a function $\theta \in C^\infty(I)$ with $I = \text{pr}_t \Omega$ such that its graph is contained in $\Omega$. For each $t \in I$ and $s \in \{1, \ldots, p\}$, we consider the initial value problem for the equation $Pu = 0$ on $\Omega$ with the initial conditions $u_{s' - 1} = \delta_{ss'}$, $s' = 1, \ldots, p$, at $x = \theta(t)$ and then vary $t$ through $I$. Here $\delta_{ss'}$ is the Kronecker delta. The collection of the solutions $\varphi^s: \Omega \to \mathbb{R}$ of the above problems then satisfies the required properties.

(i) $\Rightarrow$ (ii): Supposing that the open set $\Omega$ is not $x$-simple, we distinguish two cases.

First, we assume that each connected component of $\Omega$ is an $x$-simple set but the entire $\Omega$ is not. This means that there are connected components $U_1$ and $U_2$ of $\Omega$ with overlapping projections $\text{pr}_t U_1$ and $\text{pr}_t U_2$ to the $t$-axis. Suppose that for some $P \in DO(\Omega)$, of some order $p$ the equation $Pu = 0$ possesses a fundamental set of solutions $\varphi^1, \ldots, \varphi^p$ on $\Omega$. In view of the previous part of the proof, this equation possesses sets of $p$ solutions with nonzero Wronskians on each connected component of $\Omega$ and hence it does on the entire $\Omega$. Therefore the Wronskian of any fundamental set of solutions of $P$ does not vanish on $\Omega$. Using Lemma 2.3, we fix a function $\theta \in C^\infty(\text{pr}_t U_1)$ whose graph is contained in $U_1$. There is a solution $\psi$ of $P\psi = 0$ on $\Omega$ and $\psi = 0$ on $U_2$. By assumption, $\psi = \zeta^s \varphi^s$ for some functions $\zeta^s \in C(\text{pr}_t \Omega)$. These functions vanish on $\text{pr}_t U_2$ since $\psi \equiv 0$ on $U_2$ and thus the solution $\psi$ vanishes on the intersection of the strip $\{(t, x) | t \in \text{pr}_t U_2, x \in \mathbb{R}^2\}$ with $\Omega$. But this contradicts the fact that $\psi(t, \theta(t)) = 1$ for $t \in \text{pr}_t U_1 \cap \text{pr}_t U_2$. Therefore, for any $P \in DO(\Omega)$ with such an $\Omega$, the equation $Pu = 0$ possesses no (global) fundamental set of solutions on $\Omega$.

Henceforth we may therefore assume that some connected component of $\Omega$ is not an $x$-simple region. Applying Lemma 2.3 to this component, we get that up to reflections in $t$, the set $\Omega$ contains, for some $\bar{t}_0, \varepsilon, \bar{x}_1, \bar{x}_2 \in \mathbb{R}$ with $\varepsilon > 0$, $\bar{x}_1 < \bar{x}_2$ and for some closed subset $\bar{Y}$ of the line segment $\bar{t}_0 \times [\bar{x}_1, \bar{x}_2]$, the subset $\bar{Y} \setminus \bar{Y}$ and does not intersect $\Omega$. Consider any $P \in DO(\Omega)$ with $G(t) \to -\infty$ as $t \to \bar{t}_0^-$, where

$$G(t) := \int_{\bar{x}_1 - \varepsilon}^{\bar{x}_2 + \varepsilon} \frac{g^{p-1}(t, x)}{g^p(t, x)} \, dx, \quad t \in [\bar{t}_0 - \varepsilon, \bar{t}_0),$$

$p = \text{ord } P$, and $g^p$ and $g^{p-1}$ denote the leading and subleading coefficients of $P$, respectively. An example of such coefficients is given by $g^p(t, x) = 1$ and $g^{p-1}(t, x) = -c((x - \bar{x}_1)^2 + (t - \bar{t}_0)^2)^{-1}$ with $c > 0$ for $(t, x) \in \Omega$, cf. Example 3.4. Then $W(\varphi^1, \ldots, \varphi^p) = 0$ on $\{\bar{t}_0\} \times [\bar{x}_1 - \varepsilon, \bar{x}_1]$ for any solutions $\varphi^1, \ldots, \varphi^p$ of the equation $Pu = 0$. Indeed, it suffices to prove this claim only for the point $(\bar{t}_0, \bar{x}_1 - \varepsilon)$. Supposing that it is not the case, by the Liouville–Ostrogradski formula we obtain $W(\varphi^1, \ldots, \varphi^p)(t, \bar{x}_2 + \varepsilon) = W(\varphi^1, \ldots, \varphi^p)(t, \bar{x}_1 - \varepsilon)e^{-G(t)} \to \infty$, $t \to \bar{t}_0^-$, which contradicts the continuity of $\varphi^1, \ldots, \varphi^p$ at $(\bar{t}_0, \bar{x}_2 + \varepsilon)$.

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3 If $G(t) \to +\infty$ as $t \to \bar{t}_0^-$, it is necessary to carry out a reflection in $x$ permuting the points $\bar{x}_1 - \varepsilon$ and $\bar{x}_2 + \varepsilon$. 

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Corollary 3.5. If a connected component of an open set of \( \Omega \) is not an \( x \)-simple region, then for each \( p \in \mathbb{N} \) there exists an infinite-parameter family of equations of the form \( Pu = 0 \) with \( P \in DO_1^p(\Omega) \) of order \( p \) such that the Wronskian of any \( p \) solutions of any of them vanishes on the same line segment \( \{t_0\} \times [x_1, x_2] \) contained in \( \Omega \).

**Proof.** We follow the proof of Theorem 3.4 and consider an operator \( P \in DO_1^p(\Omega) \) of the form \( P = \sum_{q=0}^{p} q^0 \partial_x^q \), where \( g^p(t, x) = 1 \), \( g^{p-1}(t, x) = -f(t, x)((x-\bar{x})^2 + (t-\bar{t})^2)^{-1} \) for \( (t, x) \in \Omega \), \( g^q, q = 0, \ldots, p - 2 \), are arbitrary elements of \( C^\infty(\Omega) \), and \( f \) is an arbitrary positive function in \( C^\infty(\Omega) \) that is separated from zero on the intersection of a neighborhood of \( (\bar{t}, \bar{x}) \) with \( \Omega \). The coefficients of the Taylor expansions of the functions \( f \) and \( g^q \), \( q = 0, \ldots, p - 2 \), can serve as parameters of the family of equations \( Pu = 0 \), which obviously has the required properties. 

Corollary 3.6. If each connected component of an open non-\( x \)-simple set \( \Omega \) is \( x \)-simple, then any equation \( Pu = 0 \) with \( P \in DO(\Omega) \) admits sets of ord \( P \) solutions with Wronskians vanishing on \( \Omega \) and no fundamental set of solutions on \( \Omega \).

Corollary 3.7. Given an open \( x \)-simple subset \( \Omega \) of the \((t, x)\)-plane, a solution set \( \{\varphi^1, \ldots, \varphi^P\} \) of a \( p \)th order linear ordinary differential equation \( \mathcal{P}: Pu = 0 \) with \( P \in DO(\Omega) \) is fundamental on \( \Omega \) if and only if the Wronskian of these solutions does not vanish on \( \Omega \).

Corollary 3.8. 1. If an open set \( \Omega \) has an \( x \)-simple piece, then any differential equation \( Pu = 0 \) with \( P \in DO(\Omega) \), possesses a solution that is not identically zero on \( \Omega \).

2. If there are \( x \)-simple pieces of \( \Omega \) with overlapping projections to the \( t \)-axis, then any equation of the above form admits no fundamental set of solutions on \( \Omega \).

**Proof.** Fix \( P \in DO(\Omega) \) with ord \( P = p \) and let \( U \) be an \( x \)-simple piece of \( \Omega \). In view of Lemma 2.8, there exists a function \( \theta \in C^\infty(\text{pr}_t U) \) whose graph is contained in \( U \). For each \( t \in \text{pr}_t U \), we consider the initial value problem for the equation \( \mathcal{P}: Pu = 0 \) on \( U_t \) with the initial conditions \( u_{s-1} = \chi^s(t), s = 1, \ldots, p, \) at \( x = \theta(t) \) and then vary \( t \) through \( \text{pr}_t U \). Here \( \chi^1, \ldots, \chi^P \) are (smooth) bump functions on \( \text{pr}_t U \). The continuation of the solution of this problem by zero to \( \Omega \) gives a solution of \( \mathcal{P} \) on \( \Omega \) as required.

Suppose that there is another \( x \)-simple piece \( \tilde{U} \) of \( \Omega \) such that \( \text{pr}_t U \cap \text{pr}_t \tilde{U} \neq \emptyset \). We choose a value \( t_0 \in \text{pr}_t U \cap \text{pr}_t \tilde{U} \) and additionally set the condition \( \chi^1(t_0) \neq 0 \) in the above construction, which results in a solution \( \psi \in C^\infty(\Omega) \) of \( \mathcal{P} \) with \( \text{supp} \psi \subset U \) and \( \psi(t_0, \theta(t_0)) \neq 0 \). If the equation \( \mathcal{P} \) admitted a fundamental set \( \{\varphi^1, \ldots, \varphi^P\} \) of solutions on \( \Omega \), where \( p = \text{ord} P \), then the restrictions of these solutions to \( U \) and to \( \tilde{U} \) would form fundamental sets of solutions of \( \mathcal{P} \) on \( U \) and on \( \tilde{U} \), respectively. Thus, Corollary 3.7 would imply that the Wronskian of these solutions does not vanish on \( U \cup \tilde{U} \). Let us analyze the expansion \( \psi = \zeta^s \varphi^s \), where \( \zeta^s \in C(\text{pr}_t \Omega) \). Since \( \psi = 0 \) on \( \tilde{U} \), the functions \( \zeta^s \) would vanish on \( \text{pr}_t \tilde{U} \), which contradicts the condition \( \psi(t_0, \theta(t_0)) \neq 0 \).

If an open set \( \Omega \) contains no \( x \)-simple pieces, then there may exist a differential equation \( Pu = 0 \) with \( P \in DO(\Omega) \), possessing only the zero solution on \( \Omega \).

**Example 3.9.** On the “infinitely punctured open square” \( \Omega \) presented in Example 2.8, we consider the equation \( u_x = H(t, x)u \), where

\[
H(t, x) := \sum_{k=1}^{\infty} \sum_{l=1}^{2^{k-1}} \frac{4^{-k} c_{kl}}{(x - 1 + 2^{-k} l)^2 + (t - 2^{-k} l)^2}, \quad (t, x) \in \Omega,
\]

with positive constants \( c_{kl} \) such that \( \{c_{kl}, k, i \in \mathbb{N}\} \) is bounded above by a (positive) constant \( C \). Note that \( H \in C^\infty(\Omega) \), being a locally uniformly convergent sum of real analytic functions on \( \Omega \). Indeed, take an arbitrary point \( z_0 = (t_0, x_0) \in \Omega \) and fix \( \delta > 0 \) such that the ball \( B_{2\delta}(z_0) \)
is contained in $\Omega$. Then the series for $H$ is dominated on $B_\delta(z_0)$ by the convergent series $\sum_{k=1}^{\infty} \sum_{l=1}^{2^k-1} \delta^{-2} C 4^{-k}$.

Let $\psi$ be a solution of this equation on $\Omega$. According to the proof of Theorem 3.4, the function $\psi$ vanishes on the set $\Omega \cap (\bigcup_{k=1}^{\infty} \bigcup_{l=1}^{2^k-1} \{2^{-k}l\} \times (0, 1 - 2^{-k}))$, which is dense in $\Omega$. Hence this function vanishes on the entire $\Omega$.

Moreover, the above consideration allows us to conclude by induction that for any $p \in \mathbb{N}$ the equation $(\partial_x - H)^p u = 0$ admits only the zero solution on $\Omega$.

Example 3.9 can be generalized to the following assertion.

**Theorem 3.10.** If an open set $\Omega$ contains no $x$-simple pieces, and the subset $J$ of $t$’s from $\text{pr}_I \Omega$ with connected $\Omega_I$’s is dense in $\text{pr}_I \Omega$, then for each $p \in \mathbb{N}$ there exists an infinite-parameter family of equations of the form $Pu = 0$ with $P \in \text{DO}^2_\omega(\Omega)$ of order $p$ that possess only the zero solution on $\Omega$.

**Proof.** Given a set $\Omega$ with the prescribed properties and $I := \text{pr}_I \Omega$, we can consider each connected component of $\Omega$ separately, and thus we can assume that $\Omega$ is connected. We define the set $\Theta := \{(t, x) \in \mathbb{R}^2 \mid t \in I, \text{lb}_\Omega(t) < x < \text{ub}_\Omega(t)\}$, which is open and $x$-simple with $\text{pr}_I \Theta = I$. (Moreover, it is the minimal $x$-simple set that contains $\Omega$.) We fix a function $\theta \in C^\infty(I)$ with graph contained in $\Theta$, which exists in view of Lemma 2.3.

We choose a countable subset $\{t_k, k \in \mathbb{N}\}$ in $J$ that is dense in $J$ and thus in $I$. For each $k$, we have $\Omega_{t_k} = (a_k, b_k)$, where $a_k := \text{lb}_\Omega(t_k) \in \mathbb{R} \cup \{-\infty\}$ and $b_k := \text{ub}_\Omega(t_k) \in \mathbb{R} \cup \{+\infty\}$; see Figure 5. Since the set $\Omega$ is open, there exists a sequence of rectangles $[(t_k - \delta_{ki}, t_k + \delta_{ki}) \times [a_{ki}, b_{ki}]]$, $i \in \mathbb{N}$, where $\delta_{ki} \downarrow 0$, $a_{ki} \downarrow a_k$ and $b_{ki} \uparrow b_k$ strictly monotonically as $i \to \infty$, and $a_{ki} < \theta(t) < b_{ki}$ for $t \in [t_k - \delta_{ki}, t_k + \delta_{ki}]$, and that are contained in $\Omega$. There exists a sequence of points $(t_{ki}, x_{ki}) \in \Theta \setminus \Omega, i \in \mathbb{N}$, such that $t_{ki} \in (t_k - \delta_{ki}, t_k + \delta_{ki})$ and hence either $x_{ki} < a_{ki}$ or $x_{ki} > b_{ki}$ for each $i \in \mathbb{N}$. Indeed, if this was not the case for some $i$, then the set $\Omega$ would possess the $x$-simple
piece \( \Omega \cap (t_k - \delta_{k_i}, t_0 + \delta_{k_i}) \times \mathbb{R} \), which contradicts the assumption of the lemma. Therefore, the sequence \((x_{k_i}, i \in \mathbb{N})\) has limit points that are less than \(a_k\) or greater than \(b_k\). Define\(^4\)

\[
K_+ := \left\{ k \in \mathbb{N} \mid \lim_{i \to \infty} x_{k_i} \geq b_k \right\}, \quad K_- := \left\{ k \in \mathbb{N} \setminus K_+ \mid \lim_{i \to \infty} x_{k_i} \leq a_k \right\},
\]

endowing these sets with the natural order inherited from \(\mathbb{N}\). Only one of them may be empty. If \(K_+ \neq \emptyset\), then for each \(k \in K_+\), we can assume without loss of generality (by selecting a subsequence) that \(x_{k_i} > b_k\) for any \(i \in \mathbb{N}\). Define \(x'_{k_i} := \inf\{x \in (b_{k_i}, x_{k_i}] \mid x \notin \Omega_{k_i}\}\). As a result, we construct the countable tuple \((t_{k_i}, x'_{k_i}), k \in K_+, i \in \mathbb{N})\]. In a similar way, if \(K_- \neq \emptyset\), then for each \(k \in K_-\), we can assume without loss of generality that \(x_{k_i} < a_k\) for any \(i \in \mathbb{N}\). Then set \(x'_{k_i} := \sup\{x \in [x_{k_i}, a_k) \mid x \notin \Omega_{k_i}\}\). This gives the countable tuple \((t_{k_i}, x'_{k_i}), k \in K_-, i \in \mathbb{N})\).

We define the function

\[
H(t, x) := \sum_{k \in K_+} \sum_{i=1}^{\infty} \frac{2^{-k-i}c_{k_i}}{(x - x'_{k_i})^2 + (t - t_{k_i})^2} - \sum_{k \in K_-} \sum_{i=1}^{\infty} \frac{2^{-k-i}c_{k_i}}{(x - x'_{k_i})^2 + (t - t_{k_i})^2}, \quad (t, x) \in \Omega,
\]

where the \(c_{k_i}\) are positive constants\(^5\) such that \(c_{k_i}, k, i \in \mathbb{N}\) is bounded above. (These \(c_{k_i}\) can serve as a family of infinitely many parameters, cf. the formulation of the proposition.) The function \(H\) is real analytic on \(\Omega\), which is shown similarly to Example 3.9. Let us prove that the equation \(u_x = H(t, x)u\) possesses only the zero solution on \(\Omega\).

Any solution \(\psi \in \mathcal{C}_c^\infty(\Omega)\) of this equation vanishes on all the line segments \({t_{k_i}} \times (a_{k_i}, x'_{k_i})\), \(k \in K_+, i \in \mathbb{N}\), and \({t_{k_i}} \times (x'_{k_i}, b_{k_i})\), \(k \in K_-, i \in \mathbb{N}\). We will show this for arbitrary fixed \(k \in K_+\) and \(i \in \mathbb{N}\). (The proof for \(k \in K_-\) is similar.) It suffices to prove that \(\psi(t_{k_i}, b_{k_i}) = 0\). There exists \(b'_{k_i} \in \Omega_{k_i}\) that is greater than \(x'_{k_i}\). Since the set \(\Omega\) is open, there exists \(\delta > 0\) such that both the balls \(B_{2\delta}(t_{k_i}, b_{k_i})\) and \(B_{2\delta}(t_{k_i}, b'_{k_i})\) are contained in \(\Omega\). Then also \([b_{k_i}, b'_{k_i}] \subseteq \Omega, \) for any \(t \in J \cap [t_{k_i} - \delta, t_{k_i} + \delta]\). Similarly to the proof of Theorem 3.4, the assumption \(\psi(t_{k_i}, b_{k_i}) \neq 0\) implies that

\[
\psi(t, b'_{k_i}) = \psi(t, b_{k_i}) \exp \left( \int_{b_{k_i}}^{b'_{k_i}} H(t, x) \, dx \right) \to 0 \quad \text{as} \quad t \to t_{k_i} \quad \text{within} \quad J \cap [t_{k_i} - \delta, t_{k_i} + \delta],
\]

which contradicts the continuity of \(\psi\) at the point \((t_{k_i}, b'_{k_i})\).

As a result, we have \(0 = \psi(t_{k_i}, \theta(t_{k_i})) \to \psi(t_k, \theta(t_k))\) as \(i \to \infty\), and hence \(\psi(t_k, \theta(t_k)) = 0\). Therefore, \(\psi = 0\) on the union \(\bigcup_{k=1}^{\infty} \{t_k\} \times \Omega_{k}\), which is dense in \(\Omega\). This finally implies that \(\psi = 0\) on \(\Omega\).

It is easy to prove by induction using the above claim on the equation \(u_x = H(t, x)u\) as both the base case and a base for proving the inductive step that for any \(p \in \mathbb{N}\) the equation \((\partial_x - H)^p u = 0\) admits only the zero solution on \(\Omega\).

The following is an analogue of Theorem 3.10 for an arbitrary open subset of the \((t, x)\)-plane without \(x\)-simple pieces only for equations with coefficients in \(C^\infty_\omega(\Omega)\).

**Theorem 3.11.** An open set \(\Omega\) contains no \(x\)-simple pieces if and only if for each \(p \in \mathbb{N}\) there exists an infinite-parameter family of equations of the form \(Pu = 0\) with \(P \in \mathcal{D}^{\omega}_{x, 1}(\Omega)\) of order \(p\) that possess only the zero solution on \(\Omega\).

**Proof.** We prove the sufficiency of the absence of \(x\)-simple pieces for existence of equations with only the zero solution since the necessity follows from point 1 of Corollary 3.3. Thus, suppose that an open set \(\Omega\) contains no \(x\)-simple pieces.

\(^4\) It suffices for each \(k\) to belong to a single set, either \(K_+\) or \(K_-\).

\(^5\) In general, there may be repeated points, but this is not essential for the further construction.

\(^6\) These constants can be replaced by functions from \(C^\infty(\Omega)\) each of which is positive on \(\Omega\), bounded above by the same constant \(C\) on \(\Omega\) and separated from zero on the intersection of a neighborhood of the corresponding point \((t_{k_i}, x'_{k_i})\) with \(\Omega\).
Choose a countable dense subset \( \{ (t_k^+, x_k^+) \}, k \in \mathbb{N} \) of \( \Omega \). We consider nested open subsets \( \Omega_k \), \( k \in \mathbb{N} \), of \( \Omega \), \( \Omega_1 := \Omega \supset \Omega_2 \supset \Omega_3 \supset \cdots \), and points \( (t_k, x_k) \in \Omega_k \) with \( (t_1, x_1) := (t_1^+, x_1^+) \) and \( (t_k - t_k^+)^2 + (x_k - x_k^+)^2 < k^{-2} \) for \( k > 1 \). The subsets \( \Omega_k \) with \( k > 1 \) will be defined recursively later.

For each \( k \in \mathbb{N} \), we implement the following procedure.

There exists \( \delta_k > 0 \) such that \( I_k := \{ (t_k - \delta_k, t_k + \delta_k) \} \). Define the functions \( a^k : I_k \to \mathbb{R} \cup \{ -\infty \} \) and \( b^k : I_k \to \mathbb{R} \cup \{ +\infty \} \) by \( a^k(t) := -\inf \{ x \in \mathbb{R} | [x, x_k] \subset \Omega \} \) and \( b^k(t) := -\sup \{ x \in \mathbb{R} | [x, x_k] \subset \Omega \} \). These functions are upper and lower semi-continuous on \( I_k \), respectively; cf. the proof of Lemma 2.4. Indeed, fix an arbitrary \( t \in I_k \). If \( a^k(t) \in \mathbb{R} \), then for any \( \epsilon > 0 \) with \( a^k(t) + \epsilon < x_k \), the interval \( \{ a^k(t) + \epsilon, x_k \} \) is contained in \( \Omega \) and thus there exists a \( \delta > 0 \) such that \( (t - \delta, t + \delta) \subset I_k \) and \( (t - \delta, t + \delta) \times [a^k(t) + \epsilon, x_k] \subset \Omega \). Therefore, for any \( t' \in (t - \delta, t + \delta) \) we have \( a^k(t') < a^k(t) + \epsilon \). Analogously, if \( a^k(t) = -\infty \), then for an arbitrary \( N > 0 \) with \( -N < x_k \), the interval \( \{ -N, x_k \} \) is contained in \( \Omega_k \) and again there exists a \( \delta > 0 \) such that \( (t - \delta, t + \delta) \subset I_k \) and \( (t - \delta, t + \delta) \times [-N, x_k] \subset \Omega \). Hence for any \( t' \in (t - \delta, t + \delta) \) we have \( a^k(t') < -N \). In total, this means that the function \( a^k \) is upper semi-continuous on \( I_k \).

The lower semi-continuity of \( b^k \) is proved in a similar way.

We distinguish four possible cases. For each of Cases 2–4, we assume that the conditions of the previous cases do not hold.

1. There exists a sequence \( (t_{k,m})_{m \in \mathbb{N}} \) contained in \( I_k \) and strictly monotonically converging to \( t_k \) such that \( \beta_k := \limsup_{m \to \infty} b^k(t_{k,m}) > b^k(t_k) \) and \( \beta_k \subset \Omega_{t_k} \neq \emptyset \). Set \( \Lambda_k := \{ 0 \} \) and \( t_{k,0} := t_k \).

2. There exists a sequence \( (t_{k,m})_{m \in \mathbb{N}} \) contained in \( I_k \) and strictly monotonically converging to \( t_k \) such that \( \alpha_k := \liminf_{m \to \infty} a^k(t_{k,m}) < a^k(t_k) \) and \( \alpha_k, \beta_k \subset \Omega_{t_k} \neq \emptyset \). Set \( \Lambda_k := \{ 0 \} \) and \( t_{k,0} := t_k \).

3. There exists a sequence \( (t_{k,l})_{l \in \mathbb{N}} \) contained in \( I_k \) and strictly monotonically converging to \( t_k \) such that for each \( l \in \mathbb{N} \) there exists a sequence \( (t_{k,l,m})_{m \in \mathbb{N}} \) contained in \( I_k \) and strictly monotonically converging to \( t_{k,l} \) with \( \beta_{k,l} := \limsup_{m \to \infty} b^k(t_{k,l,m}) > b^k(t_{k,l}) \) and \( (a_{k,l}, b^k(t_{k,l})) \subset \Omega_{t_{k,l}} \neq \emptyset \). Set \( \Lambda_k := \mathbb{N} \).

4. There exists a sequence \( (t_{k,l})_{l \in \mathbb{N}} \) contained in \( I_k \) and strictly monotonically converging to \( t_k \) such that for each \( l \in \mathbb{N} \) there exists a sequence \( (t_{k,l,m})_{m \in \mathbb{N}} \) contained in \( I_k \) and strictly monotonically converging to \( t_{k,l} \) with \( \alpha_{k,l} := \liminf_{m \to \infty} a^k(t_{k,l,m}) < a^k(t_{k,l}) \) and \( (a_{k,l}, b^k(t_{k,l})) \subset \Omega_{t_{k,l}} \neq \emptyset \). Set \( \Lambda_k := \mathbb{N} \).

Let us show that one of the above cases necessarily holds. Indeed, otherwise there exists \( \delta_k' \) with \( 0 < \delta_k' < \delta_k \) such that the restrictions of \( a^k \) and \( b^k \) on the interval \( I_k' := (t_k - \delta_k', t_k + \delta_k') \) have none of the properties associated with these cases. Consider the intersection \( \Upsilon \) of \( \Omega \) with the strip \( I_k' \times \mathbb{R} \) and partition it into three parts,

\[
\Upsilon^- := \{ (t, x) \in \Omega \mid t \in I_k', x \leq a_k(t) \}, \\
\Upsilon_0 := \{ (t, x) \in \Omega \mid t \in I_k', a_k(t) < x < b_k(t) \}, \\
\Upsilon^+ := \{ (t, x) \in \Omega \mid t \in I_k', x \geq b_k(t) \};
\]

see Figure 6. In fact, \( \Upsilon_0 = \{ (t, x) \in \mathbb{R}^2 \mid t \in I_k', a_k(t) < x < b_k(t) \} \). From this it is obvious that \( \Upsilon_0 \) is a subset of \( \Omega \) that is \( x \)-simple and connected. Since the lower and upper bounds of \( \Upsilon_0 \) in \( x, a_k|_{I_k'} \) and \( b_k|_{I_k'} \), are upper and lower semi-continuous, respectively, then Lemma 2.4 implies that \( \Upsilon_0 \) is an open set. Hence \( \Upsilon_0 \) is not a connected component of \( \Upsilon \); otherwise \( \Upsilon_0 \) would be an \( x \)-simple piece of \( \Omega \). This implies that \( \Upsilon^- \cup \Upsilon^+ \neq \emptyset \) and there exists a continuous path \( \gamma = (\gamma^1, \gamma^2) : [0, 1] \to \Upsilon \) such that \( \gamma(0) \in \Upsilon_0 \) and \( \gamma(1) \in \Upsilon^- \cup \Upsilon^+ \). Define \( \tau_0 := \sup \{ \tau \in [0, 1] \mid \gamma([0, \tau]) \subset \Upsilon_0 \} \). Since the set \( \Upsilon_0 \) is open, the point \( \gamma(\tau_0) \) does not belong to it and thus it belongs to \( \Upsilon^- \cup \Upsilon^+ \), say to \( \Upsilon^+ \). It is obvious that \( b_k(\gamma^2(\tau)) > \gamma^2(\tau) \) for \( \tau \in [0, \tau_0) \), and \( b_k(\gamma^1(\tau_0)) < \gamma^2(\tau_0) \). Therefore, for \( \hat{\tau} := \gamma^1(\tau_0) \) we have \( \hat{\tau} \in I_k', \hat{\beta} := \limsup_{\tau \to \hat{\tau}} b_k(t) > b_k(\hat{\tau}) \) and \( (b_k(\hat{\tau}), \hat{\beta}) \subset \Omega_k \neq \emptyset \), which contradicts the conditions for \( I_k' \).
Denote the set of \(k\)'s related to Cases 1 and 3 by \(K_+\) and the set of \(k\)'s related to Cases 2 and 4 by \(K_-\). Thus, \(K_+ \cup K_- = \mathbb{N}\) and \(K_+ \cap K_- = \emptyset\). We define

\[
\Omega_{k+1} = \Omega_k \setminus \{\{t_k, t_{kl}, t_{klm}, l \in \Lambda_k, m \in \mathbb{N}\} \times \mathbb{R}\},
\]

i.e., the set \(\Omega_{k+1}\) is obtained from \(\Omega_k\) by excluding all lines with fixed values of \(t\) that are involved in the \(k\)th step. Clearly there exists a point \((t_{k+1}, x_{k+1}) \in \Omega_{k+1}\) with and \((t_{k+1} - t_k)^2 + (x_{k+1} - x_k)^2 < (k+1)^{-2}\). Hence the above recursion procedure is well defined.

We define the function

\[
H(t, x) := \sum_{k \in K_+} \sum_{l \in \Lambda_k} 2^{-k-l} c_{kl} \chi^k(t) \frac{2^{-k-l} c_{kl} \chi^k(t)}{(x - b^k(t_{kl}))^2 + (t - t_{kl})^2} - \sum_{k \in K_-} \sum_{l \in \Lambda_k} \frac{2^{-k-l} c_{kl} \chi^k(t)}{(x - a^k(t_{kl}))^2 + (t - t_{kl})^2},
\]

for \((t, x) \in \Omega\), where \(c_{kl}\) are positive constants\(^\text{4}\) such that \(\{c_{kl}, k \in \mathbb{N}, l \in \Lambda_k\}\) is bounded above by a (positive) constant \(C\), and the functions \(\chi^k \in C^\infty(\text{pr}_1 \Omega)\) satisfy the properties

\[
\chi^k(t_{kl}) = 1, \quad \chi^k(t) \geq 0 \quad \forall t \in \text{pr}_1 \Omega, \quad \text{supp} \chi^k \subseteq I_k.
\]

(again, these \(c_{kl}\) can serve as a family of infinitely many parameters, cf. the formulation of the theorem.) The function \(H\) belongs to the space \(C^2_\infty(\Omega)\) being a locally uniformly convergent sum of functions in \(C^\infty(\Omega)\). Indeed, take an arbitrary point \(z_0 = (t_0, x_0) \in \Omega\) and fix \(\delta > 0\) such that the ball \(B_\delta(z_0)\) is contained in \(\Omega\). Then the series for \(H\) is dominated on \(B_\delta(z_0)\) by the convergent series \(\sum_{k=1}^\infty \sum_{l \in \Lambda_k} C\delta^{-2}2^{-k-l}\).

Let us prove that the equation \(u_x = H(t, x)u\) possesses only the zero solution on \(\Omega\). Any solution \(\psi \in C^1_\infty(\Omega)\) of this equation vanishes on all the line segments \(\{t_{kl}\} \times (a^k(t_{kl}), b^k(t_{kl}))\), \(k \in \mathbb{N}, l \in \Lambda_k\), and hence on all the line segments \(\{t_k\} \times (a^k(t_k), b^k(t_k))\), \(k \in \mathbb{N}\). We will show this for arbitrary fixed \(k \in K_+\) and \(l \in \Lambda_k\). The proof for \(k \in K_-\) is similar. Since the union of

\[\text{These constants can be replaced by functions from } C^\infty_\infty(\Omega) \text{ each of which is positive on } \Omega, \text{ bounded above by the same constant } C \text{ on } \Omega \text{ and separated from zero on the intersection of } \Omega \text{ by a neighborhood of the corresponding point, } (t_{kl}, a^k(t_{kl})) \text{ if } k \in K_- \text{ or } (t_{kl}, b^k(t_{kl})) \text{ if } k \in K_+.
\]

\[\text{For each fixed } q \in \mathbb{N}, \text{ we can obtain } q \text{ times continuously differentiable of } H \text{ with respect to } t \text{ by setting more restrictive conditions on the parameters } c_{kl}. \text{ More precisely, denote by } C^{q, q}_\infty(\Omega) \text{ the subspace of functions in } C^\infty(\Omega) \text{ that are continuously differentiable with respect to } t \text{ } q \text{ times, with each of these derivatives belonging to } C^\infty(\Omega). \text{ Then the above function } H \text{ belongs to } C^{q, q}_\infty(\Omega) \text{ if additionally } c_{kl} < C/ \max\{1, |(\partial^{q'} \chi^k/\partial t^{q'})| (t), t \in \text{pr}_1 \Omega, q' = 1, \ldots, q\} \text{ for all } k \in \mathbb{N} \text{ and all } l \in \Lambda_k.\]
the line segments \( \{t_k\} \times (a^k(t_k), b^k(t_k)) \), \( k \in \mathbb{N} \), is dense in \( \Omega \), this will imply that the function \( \psi \) vanishes identically on \( \Omega \).

We fix a \( y_1 \in (a^k(t_k), b^k(t_k)) \) and a \( y_2 \in (b^k(t_k), \beta_{kl}) \). Selecting a subsequence if necessary, we can assume without loss of generality that the sequence \( (b^k(t_{klm}))_{m \in \mathbb{N}} \) converges to \( \beta_{kl} \). Since the function \( a^k \) is upper semi-continuous on \( I_k \) and \( b^k(t_{klm}) \to \beta_{kl} > b^k(t_k) \) as \( m \to \infty \), there exists \( N_1 \in \mathbb{N} \) such that \( [y_1, y_2] \subset (a^k(t_{klm}), b^k(t_{klm})) \subset \Omega_{t_{klm}} \) for any \( m > N_1 \). Since \( \chi^k(t_{kl}) = 1 \) and \( t_{klm} \to t_{kl} \) as \( m \to \infty \), there exists \( N_2 \in \mathbb{N} \) such that \( \chi^k(t_{klm}) > 1/2 \) for any \( m > N_1 \). Further we consider only values of \( m \) greater than \( N := \max(N_1, N_2) \). We have \( \chi^k(t_{klm}) = 0 \) for any \( k > k \) and any \( m \in \mathbb{N} \) and thus

\[
H(t_{klm}, x) \geq \frac{2^{-k-l-1}c_{kl}}{(x - b^k(t_k))^2 + (t - t_{kl})^2} - C\delta_{kl}^{-2},
\]

where \( \delta_{kl} := \text{dist}(\mathbb{R} \setminus I_k, \{t_{kl}, t_{klm}, m \in \mathbb{N}\}) \). Consequently, the assumption \( \psi(t_{kl}, y_1) \neq 0 \) implies that

\[
\psi(t_{klm}, y_2) = \psi(t_{klm}, y_1) \exp \left( \int_{y_1}^{y_2} H(t_{klm}, x) \, dx \right) \to \infty \quad \text{as} \quad m \to \infty,
\]

which contradicts the continuity of \( \psi \) at the point \( (t_{kl}, b_{kl}) \). Therefore, the function \( \psi \) vanishes at \( (t_{kl}, y_1) \) and thus it vanishes on the entire line segment \( \{t_{kl}\} \times (a^k(t_k), b^k(t_k)) \).

Similarly to the proof of Theorem 3.10 we use the above claim on the equation \( u_x = H(t, x)u \) as both the base case and a base for proving the inductive step and derive that for any \( p \in \mathbb{N} \) the equation \( (\partial_x - H)^pu = 0 \) admits only the zero solution on \( \Omega \).

4 Existence of solutions of inhomogeneous linear ordinary differential equations with parameter

As illustrated by the following example, an inhomogeneous linear \( p \)th order ordinary differential equation with independent variable \( x \) and parameter \( t \) and with real analytic coefficients and right hand side defined on an open set \( \Omega \subseteq \mathbb{R}^2 \) of \( (t, x) \) may possess no continuous solutions on \( \Omega \) at all.

**Example 4.1.** Similarly to Example 3.1, consider the elementary linear inhomogeneous first-order ordinary differential equation

\[
\mathcal{P}: \quad u_x = -\frac{1}{x^2 + t^2} \quad \text{on} \quad \Omega = \mathbb{R}^2 \setminus \{(0, 0)\},
\]

which corresponds to the operator \( P := \partial_x \in \mathcal{D}(\mathcal{O}^2(\Omega)) \). For each fixed \( t \), its general solution is

\[
u = \frac{1}{t} \arctan \frac{x}{t} + C \quad \text{if} \quad t \neq 0, \quad u = \frac{1}{x} + C \quad \text{if} \quad t = 0,
\]

where \( C \) is an arbitrary constant. This solution is well defined on the entire \( \Omega_t = \mathbb{R} \) if \( t \neq 0 \), and should be separately considered on each \( x \)-semiaxis, \( \mathbb{R}_+ \) and \( \mathbb{R}_- \), if \( t = 0 \). The functions

\[
u = \frac{1}{t} \arctan \frac{x}{t} + \zeta^+(t), \quad t > 0, \quad x \in \mathbb{R},
\]

\[
u = \frac{1}{t} \arctan \frac{x}{t} + \zeta^-(t), \quad t < 0, \quad x \in \mathbb{R},
\]

where the parameter function \( \zeta^+ \) (resp. \( \zeta^- \)) runs through \( C(\mathbb{R}_+) \) (resp. \( C(\mathbb{R}_-) \)), represent the general solutions of the equation \( \mathcal{P} \) on the domains \( \mathbb{R}_+ \times \mathbb{R} \) and \( \mathbb{R}_- \times \mathbb{R} \), respectively. The question is whether there exists a solution of \( \mathcal{P} \) that is continuous on the entire \( \Omega \). Suppose that
this is the case, and that \( u = \varphi(t, x) \) is such a solution. Define the function \( \zeta(t) := \varphi(t, -1) \), \( t \in \mathbb{R} \). We have \( \zeta \in C(\mathbb{R}) \) and

\[
\varphi(t, x) = \begin{cases} 
\frac{1}{t} \arctan \frac{x}{t} + \frac{\pi}{2|t|} - \frac{\arctan t}{t} + \zeta(t) & \text{if } t \neq 0, \ x \in \mathbb{R}, \\
-\frac{1}{x} - 1 + \zeta(0) & \text{if } t = 0, \ x \in \mathbb{R}.,
\end{cases}
\]

where we use the equality (10). Here the right hand side is continuous on \( \mathbb{R}^2 \setminus \{(0) \times [0, +\infty)\) but cannot be continuously extended to \( \Omega \) since for \( x > 0 \) and \( t \to 0 \) we obtain

\[
\frac{1}{t} \arctan \frac{x}{t} + \frac{\pi}{2|t|} - \frac{\arctan t}{t} + \zeta(t) = \frac{1}{t} \arctan \frac{t}{x} + \frac{\pi}{|t|} - \frac{\arctan t}{t} + \zeta(t) \to +\infty.
\]

In other words, the equation \( P \) has no (continuous) solution on the entire domain \( \Omega \).

**Example 4.2.** Generalizing Example [4.1] consider the family of elementary linear inhomogeneous first-order ordinary differential equations

\[
P_f: \ (x^2 + t^2)u_x = f(t, x) \quad \text{on} \quad \Omega = \mathbb{R}^2 \setminus \{(0, 0)\},
\]

with the operator \( P := (x^2 + t^2)\partial_x \in \text{DO}^\omega(\Omega) \), where the parameter function \( f \) runs through the subset \( F \) of functions from \( C(\Omega) \) whose values at certain upper or lower half-neighborhoods of \( (0, 0) \) are separated from zero, i.e., for each element \( f \) of \( F \) there exist \( \delta, \varepsilon > 0 \) such that, up to reflections in \( t \) and function values, \( f(t, x) \geq \delta \) for \( (t, x) \in (0, \varepsilon) \times [-\varepsilon, \varepsilon] \). Supposing that the equation \( P_f \) admits a solution \( \varphi \in C^1(\Omega) \), we obtain

\[
\varphi(t, \varepsilon) = \varphi(t, -\varepsilon) + \int_{-\varepsilon}^{\varepsilon} \frac{f(t, x)}{x^2 + t^2} \, dx \geq \varphi(t, -\varepsilon) + \int_{-\varepsilon}^{\varepsilon} \frac{\delta \, dx}{x^2 + t^2},
\]

which contradicts the continuity of \( \varphi \) at \( (0, \varepsilon) \). In other words, for any \( f \in F \) the equation \( P_f \) has no (continuous) solution on the entire domain \( \Omega \). Therefore, the quotient space \( C(\Omega)/\text{im} \ P \) is infinite dimensional. Additionally assuming \( f \in C^\infty(\Omega) \) or \( f \in C^\omega(\Omega) \), we also conclude that the quotient spaces \( C^\infty(\Omega)/P(C^\infty(\Omega)) \) and \( C^\omega(\Omega)/P(C^\omega(\Omega)) \) are infinite dimensional.

**Theorem 4.3.** Given an open subset \( \Omega \) of the \((t, x)\)-plane, any inhomogeneous linear ordinary differential equation \( Pu = f \) with \( P \in \text{DO}(\Omega) \) and \( f \in C(\Omega) \) admits solutions on the entire \( \Omega \) if and only if each connected component of \( \Omega \) is an \( x \)-simple set.

**Proof.** Without loss of generality, we may assume that the set \( \Omega \) itself is connected.

Suppose that the set \( \Omega \) is \( x \)-simple. In view of Lemma 2.3 there exists a function \( \theta \in C^\infty(I) \) with \( I = \text{pr}_t \Omega \) such that its graph is contained in \( \Omega \). Consider an arbitrary \( P \in \text{DO}(\Omega) \) with \( p = \text{ord} \ P \). Theorem 3.3 implies that the equation \( Pu = 0 \) admits a fundamental set of solutions on \( \Omega \) with Wronskian nonvanishing on the entire \( \Omega \), \( \{\varphi^s, s = 1, \ldots, p\} \). Using the Lagrange method of variation of constants, for any \( f \in C(\Omega) \) the general solution of the equation \( Pu = f \) can be represented in the form \( u = \psi + \sum_{s=1}^p \zeta^s \varphi^s \). Here the tuple \( (\zeta^1, \ldots, \zeta^p) \) runs through \( C(I, \mathbb{R}^p) \) and \( \psi \in C^p(\Omega) \) is a particular solution of this equation that is defined by (cf. 3)

\[
\psi(t, x) = \sum_{s=1}^p \varphi^s(t, x) \int_{\theta(t)}^x \psi^s(t, x') \, dx', \quad (t, x) \in \Omega
\]

with

\[
\psi^s := (-1)^{p-s} \frac{f}{\text{lcoef } P} \frac{W(\varphi^1, \ldots, \varphi^s, \ldots, \varphi^p)}{W(\varphi^1, \ldots, \varphi^p)}, \quad s = 1, \ldots, p.
\]
For proving \( \psi \in C^0_c(\Omega) \) it suffices to switch from the equation (6) to the equivalent linear system of first-order ordinary differential equations in the normal form (7) with \( A \) and \( F \) defined by (5). In other words, the equation \( Pu = f \) possesses a family of solutions that are continuous on the entire \( \Omega \) and parameterized by \( p \) arbitrary continuous functions of \( t \).

Conversely, let \( \Omega \) be an open set that is not \( x \)-simple. In view of Lemma 2.5, there exist \( \tilde{t}_0, \varepsilon, \tilde{x}_1, \tilde{x}_2 \in \mathbb{R} \) with \( \varepsilon > 0 \) and \( \tilde{x}_1 \leq \tilde{x}_2 \) such that up to reflections in \( t \), the set \( \Omega \) contains a subset of the form \( [\tilde{t}_0 - \varepsilon, \tilde{t}_0] \times [\tilde{x}_1 - \varepsilon, \tilde{x}_2 + \varepsilon] \setminus \Upsilon \), where \( \Upsilon \) is a closed subset of \( [\tilde{t}_0] \times [\tilde{x}_1, \tilde{x}_2] \) that is disjoint from \( \Omega \) and contains the points \((\tilde{t}_0, \tilde{x}_1)\) and \((\tilde{t}_0, \tilde{x}_2)\). The equation \( u_x = ((x - \tilde{x}_1)^2 + (t - \tilde{t}_0)^2)^{-1} \) has no (continuous) solution on the entire domain \( \Omega \); cf. Example 4.2.

If a connected component of an open set \( \Omega \) is not \( x \)-simple, then in fact we can show much more than just the existence of an inhomogeneous linear ordinary differential equation \( Pu = f \) with \( P \in DO(\Omega) \) and \( f \in C(\Omega) \) that possesses no continuous solutions on the entire \( \Omega \).

**Theorem 4.4.** If a connected component of an open set \( \Omega \) is not \( x \)-simple, then for each \( P \in DO(\Omega) \) the quotient space \( C(\Omega)/\ker P \) is infinite dimensional.

**Proof.** We again apply Lemma 2.5 to obtain the existence, up to reflections in \( t \), of a subset of the “rectangular” shape in \( \Omega \). Below we continue to use the notation of this lemma. There exist \( \delta_k \in \mathbb{R}_+ \), \( k \in \mathbb{N} \), with \( \delta_k \geq \delta_{k+1} \) for any \( k \in \mathbb{N} \) such that

\[
\Xi := ([\tilde{t}_0 - \varepsilon - \delta_1, \tilde{t}_0] \times (\tilde{x}_1 - \varepsilon - \delta_1, \tilde{x}_2 + \varepsilon + \delta_1) \\
\cup [\tilde{t}_0, \tilde{t}_0 + \delta_1] \times (\tilde{x}_1 - \varepsilon - \delta_1, \tilde{x}_1 - \varepsilon/2) \cup \bigcup_{k=2}^{\infty} [\tilde{t}_0, \tilde{t}_0 + \delta_k] \times [\tilde{x}_1 - \varepsilon/k, \tilde{x}_1 - \varepsilon/(k + 1)]
\]

is a subset of \( \Omega \); see Figure 7. By construction, the set \( \Xi \) is \( x \)-simple and open. In the capacity of a smooth function \( \theta \) related to \( \Xi \) according to Lemma 2.3, we can choose the constant function \( \theta(t) = \tilde{x}_1 - \varepsilon \), \( t \in ([\tilde{t}_0 - \varepsilon - \delta_1, \tilde{t}_0 + \delta_1]) \).

For an arbitrary operator \( P \in DO(\Omega) \), we consider the corresponding inhomogeneous linear differential equations, \( P_f : Pu = f \) with \( f \in C(\Omega) \). We will prove that there exist an infinite number of linearly independent continuous right hand sides \( f \) such that for any solution \( \psi \) of \( P_f \) on \( \Xi \) we have a sequence \( (\tilde{t}_k^*, x_k^*) \), \( k \in \mathbb{N} \) of points in \( \Xi \) with \( \tilde{t}_k^* \rightarrow \tilde{t}_0 \), \( x_k^* \rightarrow \tilde{x}_3 \in (\tilde{x}_2, \tilde{x}_2 + \varepsilon] \) and \( \psi(\tilde{t}_k^*, x_k^*) \rightarrow \infty \) as \( k \rightarrow \infty \). (We will choose \( \tilde{t}_k^* = \tilde{t}_0 - \varepsilon/k \).) This means that such solutions...
cannot be extended to (continuous) solutions of $\mathcal{P}_f$ on the entire $\Omega$. In other words, this implies that the equation $\mathcal{P}_f$ admits no smooth solutions on $\Omega$.

We elucidate the basic ideas of the proof by first treating the particular case of first-order differential operators. Thus, we consider an arbitrary operator $P$ of the form $P := h^1(t, x)\partial_x + h^0(t, x)$ with $h^0, h^1 \in C(\Omega)$ and $h^1 \neq 0$ on $\Omega$. The function $\varphi \in C^1(\Xi)$ defined by

$$\varphi(t, x) := \exp \int_{x_1 - \varepsilon}^{x} \frac{h^0(t, x')}{h^1(t, x')} \, dx', \quad (t, x) \in \Xi,$$

constitutes a fundamental set of solutions of the equation $\mathcal{P}_0$ on $\Xi$ and satisfies the initial condition $\varphi = 1$ at $x = \tilde{x}_1 - \varepsilon$ for $t \in (t_0 - \varepsilon - \delta_1, \tilde{t}_0 + \delta_1)$. Note that this solution is positive on $\Xi$, and this specific feature of first-order operators from $DO(\Omega)$ has no counterpart in higher orders. We set $t_{0k} := t_0 - \varepsilon - \delta_1$ and $t^*_{k} := t_k - \varepsilon/k, k \in \mathbb{N}$. For each $k \in \mathbb{N}$, there exist $b_k > 0$ and $\varepsilon_k$ with $0 < \varepsilon_k \leq \varepsilon$ such that $|h^1(t^*_{k}, x)|/|\varphi(t^*_{k}, x) - k|b_k$ for $x \in B_{\varepsilon_k}(\tilde{x}_1) = \{x \in \mathbb{R} \mid |x - \tilde{x}_1| \leq \varepsilon_k\}$. We take a continuous function $f^k$ of $x \in \mathbb{R}$ and a continuous function $\chi^k$ of $t \in \mathbb{R}$, respectively, satisfying the properties

$$\supp f^k \subseteq B_{\varepsilon_k}(\tilde{x}_1), \quad f^k(x) > 0 \quad \forall x \in \mathbb{R}, \quad \int_{-\infty}^{+\infty} f^k(x) \, dx \geq k b_k \left(1 + \frac{1}{\varphi(t^*_{k}, \tilde{x}_2 + \varepsilon)}\right),$$

$$\chi^k(t^*_{k}) = 1, \quad \chi^k(t) \geq 0 \quad \forall t \in \mathbb{R}, \quad \supp \chi^k \subseteq [t_{k-1}, t_{k+1}]$$

and construct the function $f := (\sum_{k=1}^{\infty} f^k \chi^k) |_{\Omega}$, which is continuous on $\Omega$. Any solution $\psi$ of $\mathcal{P}_f$ on $\Xi$ can be represented in the form

$$\psi(t, x) = \varphi(t, x) \left(\psi(t, \tilde{x}_1 - \varepsilon) + \int_{x_1 - \varepsilon}^{x} \frac{f(t, x')}{h^1(t, x') \varphi(t, x')} \, dx'\right).$$

Since $\psi \in C^1(\Xi)$, it is bounded on the line segment $[t_0 - \varepsilon, t_0] \times \{\tilde{x}_1 - \varepsilon\}$, i.e., there exists a constant $C > 0$ such that $|\psi(t, \tilde{x}_1 - \varepsilon)| \leq C$ for any $t \in [t_0 - \varepsilon, t_0]$. Estimating the value of $\psi$ at $(t^*_{k}, \tilde{x}_2 + \varepsilon)$ for $k > C$, we obtain

$$|\psi(t^*_{k}, \tilde{x}_2 + \varepsilon)| \geq \varphi(t^*_{k}, \tilde{x}_2 + \varepsilon) \left(1 - \frac{b_k}{\varphi(t^*_{k}, \tilde{x}_2 + \varepsilon)}\right) > k,$$

which completes the proof for $\text{ord} = 1$. Here $x^*_{k} := \tilde{x}_2 + \varepsilon$ for any $k \in \mathbb{N}$.

Now we consider the general case of order $\text{ord} = p$. Following the proof of Theorem 3.4, we choose the fundamental set of solutions $\{\varphi^1, s = 1, \ldots, p\}$ of the homogeneous equation $\mathcal{P}_0$ on $\Xi$ that satisfy the initial conditions $\varphi^s_{s-1} = \delta_{ss'}, s' = 1, \ldots, p$, at $x = \tilde{x}_1 - \varepsilon$ with $t$ varying through $(t_0 - \varepsilon - \delta_1, \tilde{t}_0 + \delta_1)$. Recall that $\delta_{ss'}$ denotes the Kronecker delta. The Wronskian $W := W(\varphi^1, \ldots, \varphi^p)$ does not vanish on $\Xi$, and thus it does not vanish at the points $(t^*_{k}, \tilde{x}^1)$, $k \in \mathbb{N}$, where again $t^*_{0} := \tilde{t}_0 - \varepsilon - \delta_1$ and $t^*_{k} := \tilde{t}_0 - \varepsilon/k, k \in \mathbb{N}$.

We fix $k \in \mathbb{N}$ and set $\varphi^{k_1s} := \varphi^{s}(t^*_{k}, \cdot) \in C^p((\tilde{x}_1 - \varepsilon - \delta_1, \tilde{x}_2 + \varepsilon + \delta_1), s = 1, \ldots, p$. Now choose $s_1 \in \{1, \ldots, p\}$ such that $|\varphi^{k_1s_1}(\tilde{x}_1)| = \max_s |\varphi^{k_1s}(\tilde{x}_1)|$. This absolute value is greater than zero since $W(t^*_{k}, \tilde{x}^1) \neq 0$. Set

$$\varphi^{k_2s_1} := \varphi^{k_1s_1}, \quad \varphi^{k_2s} := \varphi^{k_1s} - \frac{\varphi^{k_1s}(\tilde{x}_1)}{\varphi^{k_1s_1}(\tilde{x}_1)} \varphi^{k_1s_1}, \quad s \neq s_1.$$

For the transition matrix from $\varphi^{k_1s}$ to $\varphi^{k_2s}$, its determinant equals one, the absolute value of each of its entries is not greater than one, and its inverse has the same properties. Therefore, the Wronskian of $\varphi^{k_2s}$ coincides with $W(t^*_{k}, \cdot)$. Then we recursively iterate the above procedure, repeating it for ascending orders of derivatives. More specifically, on the $s'$th step, where $s' \in \{1, \ldots, p - 1\}$, we choose $s' = N_{s'-1} := \{1, \ldots, p\} \setminus \{s_1, \ldots, s'_{s'-1}\}$ such that

$$|\varphi^{k_2s'}(\tilde{x}_1)| = \max_s |\varphi^{k_2s'_{s'}}(\tilde{x}_1) | s \in N_{s'-1}.$$
Recall that a subscript of a function denotes the corresponding number of differentiations with respect to $x$, $f_s := \partial^s_x f$. The above maximal absolute value is greater than zero since the Wronskian of $(\varphi^{k,s})_s$ coincides with $W(t_k, \cdot)$ and hence it does not vanish at $\bar{x}$. We define

$$\varphi^{k,s+1,s} := \varphi^{k,s,s}, \quad i = 1, \ldots, s,$$

$$\varphi^{k,s+1,s} := \varphi^{k,s,s} - \frac{\varphi^{k,s,s}(\bar{x}_1)}{\varphi^{k,s,s,s}(\bar{x}_1)} \varphi^{k,s+1,s}, \quad s \in N^+.$$

For the transition matrix from $(\varphi^{k,s})_s$ to $(\varphi^{k,s+1,s})_s$, again its determinant equals one, the absolute value of each of its entries is not greater than one, and its inverse has the same properties.

The above procedure results in the functions $\varphi^{kps} \in C^p((\bar{x}_1 - \varepsilon - \delta_1, \bar{x}_2 + \varepsilon + \delta_1), s = 1, \ldots, p$ with Wronskian coinciding with $W(t_k, \cdot)$. Since the Wronskian $W$ does not vanish on $\Xi$, there exists $x_k^* \in [\bar{x}_2 + \varepsilon/2, \bar{x}_2 + \varepsilon]$ such that $\varphi^{kps}(x_k^*) \neq 0$. We also have $\varphi^{kps}_1(\bar{x}_1) = 0, 1 \leq j < i \leq p$. Consequently, the $(p - 1)$th order sub-Wronskians $W^{ks}_{s_1} := W(\varphi^{kps})_{s \neq s_1}$ satisfy the conditions $W^{ks}_{s_1}(\bar{x}_1) = 0, i = 1, \ldots, p - 1$, and $W^{ks}_{s_1}(\bar{x}_1) \neq 0$, and hence there exist $b_k > 0$ and $\varepsilon_k$ with $0 < \varepsilon_k \leq \varepsilon$ such that

$$\begin{vmatrix} W^{kps}(x) \\ (W \cdot lcof P)(t_k^*, x) \end{vmatrix} \geq b_k, \quad \begin{vmatrix} W^{ks_i}(x) \\ (W \cdot lcof P)(t_k^*, x) \end{vmatrix} \leq \frac{b_k |\varphi^{kps}(x_k^*)|}{4p \max_s |\varphi^{kps}(x_k^*)|}, \quad i = 1, \ldots, p - 1,$$

for any $x \in \overline{B}_{\varepsilon_k}(\bar{x}_1)$. We pick a continuous function $f^k$ of $x \in \mathbb{R}$ and a continuous function $\chi^k$ of $t \in \mathbb{R}$, respectively, satisfying the properties

$$\supp f^k \subseteq \overline{B}_{\varepsilon_k}(\bar{x}_1), \quad f^k(x) \geq 0 \quad \forall x \in \mathbb{R},$$

$$\frac{2k}{b_k |\varphi^{kps}(x_k^*)|} \left(1 + \sum_{s=1}^p |\varphi^{kps}(x_k^*)| \right) \leq \int_{-\infty}^{+\infty} f^k(x) \, dx \leq \frac{4k}{b_k |\varphi^{kps}(x_k^*)|} \left(1 + \sum_{s=1}^p |\varphi^{kps}(x_k^*)| \right),$$

$$\chi^k(t_k^*) = 1, \quad \chi^k(t) \geq 0 \quad \forall t \in \mathbb{R}, \quad \supp \chi^k \subseteq [t_k^- - 1, t_k^+ + 1]$$

and construct the function $f := (\sum_{k=1}^\infty f^k \chi^k) |_{\Omega}$. Any solution $\psi$ of $P f$ on $\Xi$ can be represented at $t = t_k^*$ in the form

$$\psi(t_k^*, x) = \sum_{s=1}^p \varphi^{kps}(x) \left( \sum_{s'=1}^p d_{kss'} \psi_{s'-1}(t_k^*, \bar{x}_1 - \varepsilon) + \int_{\bar{x}_1 - \varepsilon}^x (-1)^{p-s} f(t, x') W^{ks}(x') \, dx' \right).$$

Here the matrix $(d_{kss'})_{s, s'=1,\ldots, p}$ is the inverse of the Wronskian matrix $(\varphi^{k1s})_{s, s'=1,\ldots, p}$ at $\bar{x}_1 - \varepsilon$, which coincides with the transition matrix from $(\varphi^{k1s})_s$ to $(\varphi^{kps})_s$ in view of $\varphi^{k1s}(\bar{x}_1 - \varepsilon) = \delta_{s' s}$. Hence the set of the matrices $(d_{kss'})_{s, s'=1,\ldots, p}$ with $k$ running through $\mathbb{N}$ is bounded. Since $\psi \in C^\infty(\Xi)$, the function $\psi$ and each of its derivatives with respect to $x$ are bounded on the line segment $[t_0 - \varepsilon, t_0] \times \{\bar{x}_1 - \varepsilon\}$. As a result, there exists a constant $C > 0$ such that $|\sum_{s'=1}^p d_{kss'} \psi_{s'-1}(t, \bar{x}_1 - \varepsilon)| \leq C$ for any $t \in [t_0 - \varepsilon, t_0]$ and for any $s \in \{1, \ldots, p\}$.

Minorizing the value of $|\psi|$ at $(t_k^*, x_k^*)$ for $k > C$, we use the above representation for $\psi(t_k^*, x)$, compute a lower bound of the absolute value of the summand

$$\varphi^{kps}(x_k^*) \int_{\bar{x}_1 - \varepsilon}^{x_k^*} \frac{(-1)^{p-s} f(t, x') W^{ks}(x')}{(W lcof P)(t_k^*, x')} \, dx'$$

and subtract upper bounds of the absolute values of the other summand from this lower bound.
We arrive at
\[
\left| \psi(t_k^+, x_k^+) \right| \geq \left| \varphi^{kps}(x_k^+) \right| b_k \left( 1 + \sum_{s=1}^{p} |\varphi^{kps}(x_k^+)| \right) \\
- \sum_{s \neq p} \left| \varphi^{kps}(x_k^+) \right| b_k \left( \frac{4k}{4p \max_{s'} |\varphi^{kps'}(x_k^+)|} \right) \\
- C \sum_{s=1}^{p} \left| \varphi^{kps}(x_k^+) \right| \\
= k + (k - C) \sum_{s=1}^{p} \left| \varphi^{kps}(x_k^+) \right| > k
\]
if \( k > C \). Therefore, \( |\psi(t_k^+, x_k^+)| \to +\infty \) as \( k \to \infty \). There is a convergent subsequence of the sequence \( (x_k^+)_{k \in \mathbb{N}} \), and the limit of this subsequence belongs to the interval \( [\tilde{x}_2 + \varepsilon/2, \tilde{x}_2 + \varepsilon] \). \( \square \)

An inspection of the above proof also shows the following:

**Corollary 4.5.** If a connected component of an open set of \( \Omega \) is not \( x \)-simple, then for each \( P \in \text{DO}^\infty(\Omega) \) the quotient space \( C^\infty(\Omega)/P(C^\infty(\Omega)) \) is infinite dimensional.

## 5 Distributional solutions of linear ordinary differential equations with parameter

In contrast to usual ordinary differential operators, an operator \( P \) from \( \text{DO}^\infty(\Omega) \), where \( \Omega \) is an \( x \)-simple open subset of \( \mathbb{R}^2 \), is never hypoelliptic. At the same time, for any \( f \in C^\infty(\Omega) \) we can represent the general distributional solution of the equation \( Pu = f \) on \( \Omega \) in terms of a fundamental set of smooth solutions of this equation.

**Proposition 5.1.** Given an \( x \)-simple open subset \( \Omega \) of \( \mathbb{R}^2 \), an arbitrary \( P \in \text{DO}^\infty(\Omega) \) of order \( p \in \mathbb{N} \) and an arbitrary \( f \in C^\infty(\Omega) \), the general solution of the equation \( Pu = f \) in \( \mathcal{D}'(\Omega) \) can be represented in the form \( u = T_\psi + \varphi^s \cdot (\zeta^s \otimes T_1_\varepsilon) \)|\( \Omega \), where \( T_\psi \) and \( T_1_\varepsilon \) are the regular distributions associated with a particular solution \( \psi \in C^\infty(\Omega) \) of this equation and with the indicator function \( 1_\varepsilon \) of \( \mathbb{R} \), respectively, \( \{ \varphi^s, s = 1, \ldots, p \} \) is a fundamental set of smooth solutions of this equation on \( \Omega \) and each \( \zeta^s \) runs though \( \mathcal{D}'(pr_\varepsilon \Omega) \).

The proof of this proposition follows from Proposition 6.13 below, using the equivalence of (6) and (7) via (5). Proposition 5.1 can be generalized to right hand sides of lower regularity. Thus, for \( f \in C(\Omega) \) or \( f \in \mathcal{D}'(\Omega) \) it suffices to replace the condition \( \psi \in C^\infty(\Omega) \) by the condition \( \psi \in C^p(\Omega) \) or \( T_\psi \) by \( \psi \in \mathcal{D}'(\Omega) \), respectively. For \( f \in C^0(\mathcal{D}'(\Omega)) \) we should substitute the condition \( \psi \in C^0(\mathcal{D}'(\Omega)) \), which leads to a result in the spirit of [5, Theorem 4.4.8]. Here \( C^0(\mathcal{D}'(\Omega)) \) is the space of distributions on \( \Omega \) that are \( C^0 \)-semiregular in \( x \). We call an element \( u \) of \( \mathcal{D}'(\Omega) \) \( C^0 \)-semiregular in \( x \) if for any open rectangle \( I \times J \subset \Omega \) we have that \( u \) restricted to \( I \times J \) is in \( C(J, \mathcal{D}'(I)) \), cf. [5, 13]. The space \( C^0(\mathcal{D}'(\Omega)) \) is defined analogously.

**Example 5.2.** Although the equation \( u_x = H(t, x^2)u \) from Example 3.9 has no nonzero smooth solutions on \( \Omega \), it admits nonzero distributional solutions on this set, for example \( u = \delta_{t_0} \otimes T_\eta \), where \( \delta_{t_0} \) is the Dirac delta at \( t_0 \in (0, 1) \) \( \{ 2^{-k}l, l = 1, \ldots, 2^k - 1, k \in \mathbb{N} \} \), and \( T_\eta \) is the regular distribution associated with the smooth function \( \eta \in C^\infty((0, 1)) \), \( \eta(x) := \exp \left( \int_{t_0}^{t_{1/4}} H(t_0, x') dx' \right) \), \( x \in (0, 1) \). It is obvious that the Dirac delta can be replaced by an arbitrary linear combination of its derivatives. The equation constructed in the proof of Theorem 3.10 also possesses similar nonzero distributional solutions.
As the previous example suggests, $\mathcal{D}'(\Omega)$ is not necessarily the best choice for seeking solutions of equations of the form $Pu = f$ with $P \in \mathcal{D}^\infty(\Omega)$ and $f \in C^\infty(\Omega)$ since for this space one can in fact solve such equations on each slice $\Omega_t$, $t \in \mathcal{P}_t\Omega$ separately, and slice solutions do not affect each other. It is more natural to look for solutions in the space $C^p(\Omega)\mathcal{D}'(\Omega)$ of distributions on $\Omega$ that are $C^p$-semiregular. Modifying the definition of the semiregularity in $x$ by permuting $t$ and $x$, we call an element $u$ of $\mathcal{D}'(\Omega)$ $C^p$-semiregular in $t$ if for any open rectangle $I \times J \subset \Omega$ we have that $u$ restricted to $I \times J$ is in $C(I, \mathcal{D}'(J))$. Analogously, we can also define distributions on $\Omega$ that are $C^q$-semiregular in $t$ with $q \in \mathbb{N}$ or $C^\infty$-semiregular in $t$.

**Proposition 5.3.** Given an $x$-simple open subset $\Omega$ of $\mathbb{R}^2$, an arbitrary $P \in \mathcal{D}^\infty(\Omega)$ and an arbitrary $f \in C(\Omega)$ (resp. $f \in C^\infty(\Omega)$), any solution of the equation $Pu = f$ in $C^0_t(\mathcal{D}'_x(\Omega))$ (resp. $C^p(\mathcal{D}'_x(\Omega))$) is a regular distribution associated with a classical (resp. smooth) solution of this equation.

Again, it is better to carry out the proof for the linear system of first-order ordinary differential equations that is equivalent to the equation $Pu = f$ via 5, see Proposition 6.14 below.

### 6 Parameter-dependent linear systems of ordinary differential equations

The results of the previous sections on single parameter-dependent linear ordinary differential equations can easily be extended to parameter-dependent linear systems of ordinary differential equations in the canonical form. Any such system is equivalent to a linear system of first-order ordinary differential equations in the normal Cauchy form (cf. Section 4). It is then evident that the results of Sections 3, 4 and 5 have direct analogues for the system case. Except for the case of distributional solutions we shall therefore omit the proofs and confine ourselves to stating the results below.

Let $M_p(\mathbb{R})$ denote the set of $p \times p$ matrices with real coefficients. Consider a system of linear ordinary differential equations $\mathcal{P}: v_x = A(t,x)v$ on an open subset $\Omega$ of the $(t,x)$-plane, where $A \in C(\Omega, M_p(\mathbb{R}))$, $v = (v^1, \ldots, v^p)^\top$ is the unknown vector function of $(t,x)$, $x$ is the independent variable and $t$ plays the role of a parameter. This system can be interpreted as a vector differential equation. Its matrix counterpart, $M_x = A(t,x)M$ with $M \in M_p(\mathbb{R})$, is denoted by $\mathcal{P}_m$. We assume that (classical) solutions of the system $\mathcal{P}$ and the matrix equation $\mathcal{P}_m$ belong to $C^1(\Omega, \mathbb{R}^p)$ and $C^1(\Omega, M_p(\mathbb{R}))$, respectively.

**Definition 6.1.** We say that a matrix-valued function $\Phi \in C^1(\Omega, M_p(\mathbb{R}))$ satisfying the equation $\mathcal{P}_m$, $\Phi_x = A(t,x)\Phi$, is

- a **fundamental matrix** of $\mathcal{P}$ on $\Omega$ if any solution $v$ of $\mathcal{P}$ can uniquely be represented in the form $v = \Phi \zeta$ for some function $\zeta \in C(\mathcal{P}_t\Omega, \mathbb{R}^p)$;
- a **locally fundamental matrix** of $\mathcal{P}$ on $\Omega$ if there exists an open cover of the set $\Omega$ such that the restriction of any solution $v$ of $\mathcal{P}$ to any element $U$ of the cover, $v|_U$, can uniquely be represented in the form $v|_U = \Phi|_U \zeta$ for some function $\zeta \in C(\mathcal{P}_tU, \mathbb{R}^p)$.

**Lemma 6.2.** Any solution $\Phi$ of the matrix equation $\mathcal{P}_m$ with determinant nonvanishing on $\Omega$ is a locally fundamental matrix of the system $\mathcal{P}$.

**Theorem 6.3.** Given an open subset $\Omega$ of the $(t,x)$-plane, the following are equivalent:

1. Any homogeneous linear system of first-order ordinary differential equations $v_x = A(t,x)v$ with $A \in C(\Omega, M_p(\mathbb{R}))$, where $t$ plays the role of a parameter, admits a fundamental matrix on $\Omega$ with determinant nonvanishing on the entire $\Omega$.
2. $\Omega$ is an $x$-simple region.
Corollary 6.4. If a connected component of an open set of $\Omega$ is not an $x$-simple region, then for each $p \in \mathbb{N}$ there exists an infinite-parameter family of $p \times p$ matrix equations of the form $M_x = A(t,x)M$ with $A \in C^\infty(\Omega,M_p(\mathbb{R}))$ such that the determinant of any solution of each of them vanishes on the same line segment $\{t_0\} \times [x_1,x_2]$ contained in $\Omega$.

Corollary 6.5. If each connected component of an open non-$x$-simple set $\Omega$ is $x$-simple, then any $p$-vector equation of the form $v_x = A(t,x)v$ with $A \in C(\Omega,M_p(\mathbb{R}))$ admits no fundamental matrix on $\Omega$ although the associated matrix equation has solutions with determinants nonvanishing on $\Omega$.

Corollary 6.6. Given an open $x$-simple subset $\Omega$ of the $(t,x)$-plane, a solution $\Phi$ of a $p \times p$ matrix equations of the form $M_x = A(t,x)M$ with $A \in C(\Omega,M_p(\mathbb{R}))$ is a fundamental matrix on $\Omega$ for the associated vector equation $v_x = A(t,x)v$ if and only if the determinant of $\Phi$ does not vanish on $\Omega$.

Corollary 6.7. 1. If an open set $\Omega$ has an $x$-simple piece, then any system of differential equations $v_x = A(t,x)v$ with $A \in C(\Omega,M_p(\mathbb{R}))$, possesses a solution that is not identically zero on $\Omega$.

2. If there are $x$-simple pieces of $\Omega$ with overlapping projections to the $t$-axis, then any system of the above form admits no fundamental matrix on $\Omega$.

Proposition 6.8. If an open set $\Omega$ contains no $x$-simple pieces, and the subset $J$ of $t$’s from $pr_t \Omega$ with connected $\Omega_t$’s is dense in $pr_t \Omega$, then for each $p \in \mathbb{N}$ there exists an infinite-parameter family of $p$-vector equations of the form $v_x = Av$ with $A \in C^\infty(\Omega,M_p(\mathbb{R}))$ that possess only the zero solution on $\Omega$.

Theorem 6.9. An open set $\Omega$ contains no $x$-simple pieces if and only if for each $p \in \mathbb{N}$ there exists an infinite-parameter family of $p$-vector equations of the form $v_x = Av$ with $A \in C(\Omega,M_p(\mathbb{R}))$ that possess only the zero solution on $\Omega$.

Theorem 6.10. Given an open subset $\Omega$ of the $(t,x)$-plane, any inhomogeneous linear system of first-order ordinary differential equations $v_x = Av + F$ with $A \in C(\Omega,M_p(\mathbb{R}))$ and $F \in C(\Omega,\mathbb{R}^p)$, where $t$ plays the role of a parameter, admits continuous solutions on the entire $\Omega$ if and only if each connected component of $\Omega$ is an $x$-simple set.

Theorem 6.11. If a connected component of an open set of $\Omega$ is not $x$-simple, then for each $A \in C(\Omega,M_p(\mathbb{R}))$ the quotient space $C(\Omega,\mathbb{R}^p)/(\partial_x - A)(C^1(\Omega,\mathbb{R}^p))$ is infinite dimensional.

Corollary 6.12. If a connected component of an open set of $\Omega$ is not $x$-simple, then for each $A \in C^\infty(\Omega,M_p(\mathbb{R}))$ the quotient space $C^\infty(\Omega,\mathbb{R}^p)/(\partial_x - A)(C^\infty(\Omega,\mathbb{R}^p))$ is infinite dimensional.

Finally, we turn to the case of distributional solutions of parameter-dependent systems:

Proposition 6.13. Given an $x$-simple open subset $\Omega$ of $\mathbb{R}^2$, an arbitrary $A \in C^\infty(\Omega,M_p(\mathbb{R}))$ and an arbitrary $F \in C^\infty(\Omega,\mathbb{R}^p)$, the general solution of the system $v_x = Av + F$ in $D'(\Omega,\mathbb{R}^p)$ can be represented in the form $v = T_\psi + \Phi \cdot (\zeta \otimes T_{1_\mathbb{R}})|_\Omega$, where the tensor product is understood componentwise, $T_\psi$ and $T_{1_\mathbb{R}}$ are the regular distributions associated with a particular solution $\psi \in C^\infty(\Omega,\mathbb{R}^p)$ of this system and with the indicator function $1_\mathbb{R}$ of $\mathbb{R}$, respectively, $\Phi$ is a smooth fundamental matrix of this system on $\Omega$, and $\zeta$ runs through $D'(pr_t \Omega,\mathbb{R}^p)$.

Proof. We fix a particular solution $\psi \in C^\infty(\Omega,\mathbb{R}^p)$ of the system $v_x = Av + F$ and a smooth fundamental matrix $\Phi$ of this system on $\Omega$ (cf. Theorem 6.9). If a distribution $v \in D'(\Omega,\mathbb{R}^p)$ satisfies the system $v_x = Av + F$, then the distribution $\tilde{v} := \Phi^{-1}(v - T_\psi)$ satisfies the system $\tilde{v}_x = 0$. By Theorem 6.2 the general distributional solution of the latter system is $\tilde{v} = (\zeta \otimes T_{1_\mathbb{R}})|_\Omega$, where $\zeta$ runs though $D'(pr_t \Omega,\mathbb{R}^p)$.

\[\square\]
Similarly to Proposition 5.1, Proposition 6.13 can be extended to right hand sides of lower regularity. Thus, for \( F \in C(\Omega, \mathbb{R}^p) \) or \( F \in D'(\Omega, \mathbb{R}^p) \) it suffices to replace the condition \( \psi \in C^\infty(\Omega, \mathbb{R}^p) \) by the condition \( \psi \in C^1_{\text{c}}(\Omega, \mathbb{R}^p) \) or \( \psi_T \) by \( \psi \in D'(\Omega, \mathbb{R}^p) \), respectively.

**Proposition 6.14.** Given an \( x \)-simple open subset \( \Omega \) of \( \mathbb{R}^2 \), an arbitrary \( A \in C^\infty(\Omega, M_p(\mathbb{R})) \) and an arbitrary \( F \in C(\Omega, \mathbb{R}^p) \) (resp. \( F \in C^\infty(\Omega, \mathbb{R}^p) \)), any solution of the system \( v_x = Av + F \) in \( C^1(D^*_x)(\Omega, \mathbb{R}^p) \) (resp. \( C^\infty(D^*_x)(\Omega, \mathbb{R}^p) \)) is a regular distribution associated with a classical (resp. smooth) solution of this system.

**Proof.** In the notation of the proof of Proposition 6.13, if \( \tilde{v} \) is \( C^0 \)-semiregular in \( t \) (resp. \( C^\infty \)-semiregular in \( t \)), then \( \tilde{v} \) is of the same semiregularity in \( t \) and hence the corresponding tuple \( \zeta \) belongs to \( C(\text{pr}_t \Omega, \mathbb{R}^p) \) (resp. \( C^\infty(\text{pr}_t \Omega, \mathbb{R}^p) \)). \( \square \)

**Appendix A  Distributions with vanishing partial derivatives**

In this appendix we collect some results required in Sections 5 and 6 for deriving the general form of distributional solutions to linear (systems of) ODEs.

For a distribution \( u \in \mathcal{D}'(\mathbb{R}^{n+1}) \) it is well known (cf. [14 Chapitre IV, § 5], [2 Theorem 4.3.4]) that \( \partial_{x_{n+1}}u = 0 \) if and only if \( u \) is of the form \( v \otimes T_{1^y} \) for some \( v \in \mathcal{D}'(\mathbb{R}^n) \). For \( u \in \mathcal{D}'(\Omega) \) with \( \Omega \) an arbitrary open subset of \( \mathbb{R}^n \) such a result cannot be expected. Nevertheless, we shall show that if \( \Omega \) is \( x_{n+1} \)-simple, then a suitable generalization does indeed hold.

We first note that [2 Theorem 4.3.4] remains true for more general products, and we include a proof for the sake of completeness:

**Theorem A.1.** Let \( X \subset \mathbb{R}^n \) be open, \( Y = (a, b), -\infty \leq a < b \leq \infty, n \in \mathbb{N}_0 \) (setting \( X \times Y := Y \) in case \( n = 0 \)), and let \( u \in \mathcal{D}'(X \times Y) \). Then

\[
\partial_y u = 0 \iff \exists v \in \mathcal{D}'(X): u(x, y) = v \otimes T_{1^y}.
\]

**Proof.** (\( \Rightarrow \)): \( \partial_y(v \otimes T_{1^y}) = v \otimes \partial_y T_{1^y} = v \otimes 0 = 0 \).

(\( \Rightarrow \)): Pick some \( \chi \in \mathcal{D}(Y) \) with \( \int_Y \chi(y) \, dy = 1 \) and define \( v: \mathcal{D}(X) \rightarrow \mathbb{R} \) by

\[
\langle v, \psi \rangle := \langle u(x, y), \psi(x) \otimes \chi(y) \rangle \quad \text{for all} \quad \psi \in \mathcal{D}(X).
\]

Then \( v \in \mathcal{D}'(X) \) and for any \( \varphi \in \mathcal{D}(X \times Y) \) we have

\[
\langle v \otimes T_{1^y}, \varphi \rangle = \langle v(x), (T_{1^y}, \varphi(x, y)) \rangle = \langle v(x), \int_Y \varphi(x, y) \, dy \rangle = \langle u(x, y), \int_Y \varphi(x, y) \, dy' \otimes \chi(y) \rangle.
\]

Hence \( \langle u - v \otimes T_{1^y}, \varphi \rangle = \langle u, \varphi \rangle \), where \( \varphi(x, y) = \varphi(x, y) - \int_Y \varphi(x, y') \, dy' \otimes \chi(y) \). Now for any \( x \in X \) we have \( \int_Y \varphi(x, y) \, dy = 0 \), so that \( \psi(x, y) := \int_y^x \varphi(x, y') \, dy' \) defines an element of \( \mathcal{D}(X \times Y) \) and satisfies \( \partial_y \psi = \varphi \). Consequently,

\[
\langle u - v \otimes T_{1^y}, \varphi \rangle = \langle u, \varphi \rangle = \langle u, \partial_y \psi \rangle = -\langle \partial_y u, \psi \rangle = 0,
\]

which completes the proof. \( \square \)

In analogy to Definition 5.2 we say that an open subset \( \Omega \) of \( \mathbb{R}^n_x \times \mathbb{R}^{}_y \) (where the subscripts refer to the names of the corresponding variables) is \( y \)-**simple** if, for all \( x \in \mathbb{R}^n \), the intersection \( \Omega_x := (\{x\} \times \mathbb{R}) \cap \Omega \) is connected or is the empty set. Using this terminology, we have:

**Theorem A.2.** Let \( \Omega \subset \mathbb{R}^n_x \times \mathbb{R}^{}_y \) be open and \( y \)-simple \((n \in \mathbb{N}_0)\), and let \( u \in \mathcal{D}'(\Omega) \). Then

\[
\partial_y u = 0 \iff \exists v \in \mathcal{D}'(\text{pr}_X(\Omega)) : u(x, y) = (v \otimes T_{1^y})|_\Omega.
\]

(Here \( \text{pr}_X \) denotes the projection into \( \mathbb{R}^n_x \).)
Proof. Only the direction \((\Rightarrow)\) requires a proof. Thus let \(\{X_i \times Y_i \mid i \in J\}\) be an open cover of \(\Omega\) with \(X_i\) open in \(\mathbb{R}^n\) and \(Y_i\) open intervals in \(\mathbb{R}_y\). Then \(\partial_y(u|_{X_i \times Y_i}) = 0\) for all \(i \in J\), and so by Theorem \([A.1]\) there exist \(v^i \in D'(X_i)\) such that \(u|_{X_i \times Y_i} = v^i \otimes T_{1_{Y_i}}\).

We now show that \(\{v^i \mid i \in J\}\) forms a coherent family of distributions associated to the covering \(\{X_i \mid i \in J\}\) of \(\text{pr}_X \Omega\). To see this, suppose that \(X_i \cap X_j \neq \emptyset\). We have to show that then \(v^i|_{X_i \cap X_j} = v^j|_{X_i \cap X_j}\). We distinguish two cases:

First, if \(Y_i \cap Y_j \neq \emptyset\), then \(U := (X_i \times Y_i) \cap (X_j \times Y_j) \neq \emptyset\), and therefore

\[
v^i \otimes T_{1_{Y_i}}|_U = u|_U = v^j \otimes T_{1_{Y_j}}|_U,
\]

so that indeed \(v^i|_{X_i \cap X_j} = v^j|_{X_i \cap X_j}\).

Second, suppose that \(Y_i \cap Y_j = \emptyset\) and fix arbitrary \(x \in X_i \cap X_j\) and \(y_i \in Y_i\), \(y_j \in Y_j\), assuming without loss of generality that \(y_i < y_j\). Since \(\Omega\) is \(y\)-simple, we may pick a finite subset \(J'\) of \(J\) such that \(\{X_k \times Y_k \mid k \in J'\}\) is a minimal covering of \(\{x\} \times [y_i, y_j]\). Let \(J' = \{k_1, \ldots, k_p\}\), \(Y_{k_i} = (a_{k_i}, b_{k_i})\), \(l = 1, \ldots, p\), and assume without loss of generality that \(a_{k_1} < a_{k_2} < \cdots < a_{k_p}\). Then

\[
U := \bigcap_{k \in J'} X_k
\]

is an open neighborhood of \(x\), and by the previous case we have

\[
v^i|_U = v^{k_1}|_U = \cdots = v^{k_p}|_U = v^j|_U,
\]

which allows us to conclude that \(v^i|_{X_i \cap X_j} = v^j|_{X_i \cap X_j}\) also in this case.

By the sheaf property of distributions \([14, \text{Chapitre I, § 3, Théorème IV}]\), it follows that there exists a unique \(v \in D'(\text{pr}_X(\Omega))\) such that \(v|_{X_i} = v^i\) for all \(i \in J\). Thus

\[
u|_{X_i \times Y_i} = v^i \otimes T_{1_{Y_i}} = v|_{X_i} \otimes T_{1_{Y_i}} = (v \otimes T_{1_{\Omega_y}})|_{X_i \times Y_i}.
\]

Again by the sheaf property of distributions, \(u = (v \otimes T_{1_{\Omega_y}})|_{\Omega}\). \(\square\)

Remark A.3. The condition on the simplicity of the domain with respect to the variable involved in the distributional derivative is essential in Theorem \([A.2]\). Indeed, given a nonempty open set \(\Omega \subset \mathbb{R}^n_x \times \mathbb{R}_y\) \((n \in \mathbb{N}_0)\) that is not \(y\)-simple, we may take some \(x_0 \in \mathbb{R}^n\) such that the intersection \(\Omega_{x_0} := (\{x_0\} \times \mathbb{R}) \cap \Omega\) is non-empty and not connected. For \(-\infty \leq y_0 < y_1 \leq y_2 < y_3 \leq +\infty\) such that \(\{x_0\} \times (y_0, y_1)\) and \(\{x_0\} \times (y_2, y_3)\) are connected components of \(\Omega_{x_0}\) and for any different and nonzero \(c_1, c_2 \in \mathbb{R}\), consider the distribution

\[
u := \left(\delta_{x_0} \otimes (c_1 T_{1_{(y_0, y_1)}} + c_2 T_{1_{(y_2, y_3)}})\right)|_{\Omega} \in D'(\Omega).
\]

Then \(u\) is not of the form given in Theorem \([A.2]\) although \(\partial_x u = 0\).

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