Exact Methods for Self Interacting Neutrinos

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Abstract. The effective many-body Hamiltonian which describes vacuum oscillations and self interactions of neutrinos in a two flavor mixing scheme under the single angle approximation has the same dynamical symmetries as the well known BCS pairing Hamiltonian. These dynamical symmetries manifest themselves in terms of a set of constants of motion and can be useful in formulating the collective oscillation modes in an intuitive way. In particular, we show that a neutrino spectral split can be simply viewed as an avoided level crossing between the eigenstates of a mean field Hamiltonian which includes a Lagrange multiplier in order to fix the value of an exact many-body constant of motion. We show that the same dynamical symmetries also exist in the three neutrino mixing scheme by explicitly writing down the corresponding constants of motion.

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Neutrinos are produced in copious amounts by various astrophysical sources. For example, a core collapse supernova releases %99 of the gravitational binding energy of the pre-supernova in the form of neutrinos [see Refs. 1–4, for review]. These neutrinos are excellent probes into the physics of the supernova and believed to play a role in the supernova dynamics as well as the subsequent r-process nucleosynthesis. Black hole accretion disks are also likely to be major sources of neutrinos [5, 6]. In the Early Universe neutrinos were produced abundantly and influenced the Big Bang nucleosynthesis [see Refs. 7, 8, for review].

Determining the impact neutrinos requires a careful study of their energy distribution. Although the initial energy distribution can be given with a specific model, it is subsequently modified as the neutrinos undergo flavor evolution subject to the refractive effects of the background matter. The refraction of neutrinos due to the other particles in the background such as protons, neutrons and electrons, is proportional to the forward scattering amplitude since it is only in the forward direction that the scattering amplitudes add up coherently. This gives rise to the well known MSW effect in the Sun. But if the neutrino density is sufficiently high, then the self refraction of neutrinos can also make a significant contribution to the flavor evolution as is the case for the core collapse supernovae [9], the Early Universe [10] and in black hole accretion disks [11]. When neutrinos scatter off each other, those diagrams in which neutrinos completely exchange their momenta also add up coherently in addition to the forward scattering diagrams [12]. The exchange diagrams couple neutrinos with different energies and turn flavor evolution of the system into a nonlinear many-body phenomenon. A rich set of flavor oscillation modes arises due to this nonlinearity which is a subject of intense study in the recent years. In particular, collective flavor oscillation modes in which neutrinos of different energies oscillate with the same frequency were identified and thoroughly studied in terms of their astrophysical implications [13–22].

Salient features of collective flavor oscillations can be captured by a simplified model in which neutrinos undergo vacuum oscillations and self interactions in the absence of a net leptonic background under the so-called single angle approximation. It was pointed out in Ref. [23] that, in the case of a two neutrino mixing scheme, this model has the same dynamical symmetries as the reduced pairing Hamiltonian which is used in the context of the BCS model of superconductivity to describe the electron pairs in the conduction band of a metal and also in the context of the nuclear shell model to describe the nucleon pairs occupying the valance shell of a nucleus. These dynamical symmetries guarantee that the model is exactly solvable in both its original many-body form and in the framework of the commonly used mean field approximation.

The goal of this contribution is to emphasize that a thorough examination of the symmetries and exact solutions of the self interacting neutrinos would be helpful in developing a deeper insight into the nature of the collective flavor oscillations. Such a study is naturally complementary to the numerical techniques that have been developed and successfully applied to the problem so far. As an example, we will consider the adiabatic flavor evolution of
neutrinos as they radiate from a source and undergo spectral splits by exchanging parts of their spectra. This behavior was first observed in the numerical simulations of the system under the mean field approximation [see Ref. 24, for a review]. They were analytically explained in terms of the adiabatic time evolution of the instantaneously stable mean field configurations viewed from a rotating frame of reference in the neutrino isospin space [22]. Here we offer an alternative view of a spectral split as an avoided level crossing between the eigenstates of a mean field Hamiltonian which includes a Lagrange multiplier in order to fix the value of an exact many-body constant of motion which cannot be otherwise fixed in the mean field approximation scheme [23].

Note that the dynamical symmetries and the corresponding constants of motion of the self interacting neutrinos were so far examined only for two flavor mixing. It is natural to ask if similar symmetries also exist for three flavor mixing. We show that the answer is positive by explicitly writing down the constants of motion of the exact many-body Hamiltonian describing the vacuum oscillations and self interactions of three neutrino flavors in the single angle approximation. The implications of these symmetries, including their roles in the multiple spectral splits are subject to further study and will be reported elsewhere.

We start by formulating the problem for two mixing flavors. We take them to be $\nu_e$ and an orthogonal flavor that we denote by $\nu_x$ which can be either $\nu_\mu$ or $\nu_\tau$ or a normalized combination of them. The particle operator for a neutrino of flavor $\alpha = e,x$ with momentum $p$ is denoted by $a_\alpha (p)$. Typically, additional quantum numbers besides the momentum are needed to distinguish the neutrinos but we choose to keep our formulas simple by not explicitly displaying them in our notation. Instead, $p$ can be viewed as a multiple index like $(p,s_1,s_2,\ldots)$.

It is useful to introduce the isospin operator $J_p = (J_p^+, J_p^-, J_p^3)$ whose components are given by

$$J_p^+ = a_e^*(p) a_e(p), \quad J_p^- = a_x^*(p) a_x(p), \quad J_p^3 = \frac{1}{2} (a_e^*(p) a_e(p) - a_x^*(p) a_x(p)) .$$

(1)

Note that we use boldface letters to indicate vectors in momentum space (e.g. $p$) and arrows to indicate vectors in flavor space (e.g. $J$). The components of the isospin operator obey the $SU(2)$ commutation relations

$$[J_p^+ , J_q^-] = 2 \delta_{pq} J_p^3 , \quad [J_p^3 , J_q^\pm] = \pm \delta_{pq} J_q^\pm$$

(2)

such that one has as many mutually commuting $SU(2)$ isospin algebras as the number of neutrinos. It follows from Eq. (1) that each isospin algebra is realized in the spin-1/2 representation and that the electron neutrino is isospin up.

The particle operators in the mass basis are denoted by $a_i (p)$ where $i = 1, 2$ indicates the eigenstate with mass $m_i$. The transformation from the flavor basis into mass basis is a global rotation in the sense that it is the same for neutrinos of all energies. One can also express the isospin operator in terms of its components in the mass basis, i.e.,

$$\bar{J} = (\bar{J}^+ , \bar{J}^- , \bar{J}^3)$$

where

$$\bar{J}_p^+ = a_1^*(p) a_2(p) , \quad \bar{J}_p^- = a_2^*(p) a_1(p) , \quad \bar{J}_p^3 = \frac{1}{2} (a_1^*(p) a_1(p) - a_2^*(p) a_2(p)) .$$

(3)

In order to avoid confusion, we use curly letters $\mathcal{J}^a$ to denote the components of isospin operator in the mass basis. They satisfy the same commutation relations as those given in Eq. (2).

At this point we introduce our summation convention for all isospin operators as follows:

$$\bar{J}_p = \sum_{|p|=p} \bar{J}_p \quad \text{and} \quad \bar{J} = \sum_p \bar{J}_p .$$

(4)

Here, $\bar{J}_p$ represents the total isospin operator of all neutrinos with the same energy $p$ and $\bar{J}$ represents the total isospin operator of all neutrinos. Since the operators $\bar{J}_p$ and $\bar{J}$ are sums of individual $SU(2)$ operators, their components also obey the $SU(2)$ commutation relations.

The Hamiltonian describing the vacuum oscillations of neutrinos can be written as

$$H_\nu = \sum_p \left( \frac{m_e^2}{2p} a_e^*(p) a_1(p) + \frac{m_x^2}{2p} a_x^*(p) a_2(p) \right) = \sum_p \omega_p \bar{B} \cdot \bar{J}_p .$$

(5)

Here $\omega_p = (m_2^2 - m_1^2)/2p$ is the vacuum oscillation frequency of the neutrino with energy $p$. Note that the summation convention introduced in Eq. (4) is used for neutrinos with the same energy. In Eq. (5), $\bar{B}$ is a vector which points in the negative direction along the third axis in the mass basis. Its components are given by

$$\bar{B} = (0,0,-1)_{\text{max}} = (\sin 2\theta, 0, -\cos 2\theta)_{\text{max}}$$

(6)
in mass and flavor bases, respectively. We also note that the equality in Eq. (5) is correct up to a term proportional to identity which can always be subtracted from the Hamiltonian.

The effect of scattering on the neutrino flavor oscillations in matter can be described by an effective Hamiltonian which takes into account only those terms which add up coherently over the scatterers. In the case of the neutrino-neutrino scattering, only forward scattering diagrams in which there is no momentum transfer between the particles, and exchange diagrams in which particles swap their momenta add up coherently. As a result, the effective Hamiltonian which describes self interactions of neutrinos have the following form \([25, 26]\):

\[
H_{\nu\nu} = \frac{\mu}{2} \sum_{p,q} \left[ a_\nu^\dagger(p) a_\nu(p) a_\nu^\dagger(q) a_\nu(q) + a_\nu^\dagger(p) a_\nu(p) a_\nu(q) a_\nu^\dagger(q) + a_\nu^\dagger(p) a_\nu(p) a_\nu^\dagger(q) a_\nu(q) + a_\nu^\dagger(p) a_\nu(p) a_\nu(q) a_\nu^\dagger(q) \right].
\] (7)

Here \(\mu = \sqrt{2} G_F / V\) where \(V\) denotes the quantization volume. Note that the Hamiltonian in Eq. (7) is valid within the so called single angle approximation in which the dependence of the scattering amplitudes on the angle between the propagation direction of neutrinos is ignored. It can be shown that the Hamiltonian in Eq. (7) is equal to

\[
H_{\nu\nu} = \mu \vec{J} \cdot \vec{J}
\]

up to some terms which are proportional to identity. The Hamiltonian in Eq. (8) has the same form in both the mass and flavor bases because these bases are related by a global rotation which leaves all scalar products invariant.

The total Hamiltonian describing the self interactions of neutrinos together with vacuum oscillations is given by the sum of the terms in Eqs. (5) and (8), i.e.,

\[
H = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu \vec{J} \cdot \vec{J}.
\] (9)

This Hamiltonian belongs to one of the three classes of exactly solvable Hamiltonians which were first systematically studied by Gaudin in 1976 [27, 28]. Gaudin was interested in finding integrable Hamiltonians which describe spin systems in an external magnetic field with long range spin-spin interactions. He classified those integrable Hamiltonians that he identified into three classes which are known as the rational, trigonometric and elliptic models. The Hamiltonian in Eq. (9) belongs to the class of rational models. As will be discussed in more detail below, it has as many constants of motion (or invariants) as the number of energy modes in the system. These constants of motion were identified by Gaudin and are known as the “rational Gaudin magnet Hamiltonians.” It should be noted that although Gaudin was interested in real spins as opposed to isospins that appear in Eq. (9), this distinction is not important as long as the integrability and the exact solutions are concerned because both the real spins and the isospins obey the same algebra given in Eq. (2).

In the language of neutrino isospin, the invariants identified by Gaudin are given by

\[
h_p = \vec{B} \cdot \vec{J}_p + \frac{2\mu}{\omega_p - \omega_q} \frac{\vec{J}_p \cdot \vec{J}_q}{\omega_p - \omega_q}.
\] (10)

It is straightforward to show that these operators commute with one another and with the neutrino Hamiltonian:

\[
[h_p, h_q] = 0 \quad \text{and} \quad [H, h_p] = 0 \quad \text{for all } p \text{ and } q.
\] (11)

It is possible to express the invariants of the model in a number of alternative ways using linear or nonlinear combinations of those given in Eq. (10). In particular, their sum is simply equal to

\[
\sum_p h_p = -\vec{J}^3 = \frac{N_2 - N_1}{2}
\] (12)

where \(N_1\) is the total number of neutrinos in the \(p\)th mass eigenstate. Note that the total number of neutrinos, \(N_1 + N_2\), is also constant because we consider only the vacuum oscillations and scatterings of the neutrinos. Therefore Eq. (12) tells us that \(N_1\) and \(N_2\) are individually conserved.

Another case where the constants of motion take a simple form is the \(\mu \to 0\) limit where all neutrino-neutrino interactions cease. In this limit, we have

\[
\lim_{\mu \to 0} h_p = -\vec{J}_p^3 = \frac{n_2(p) - n_1(p)}{2}
\] (13)
where \( n_i(p) \) is the total number of neutrinos in the \( i \)th mass eigenstate with energy \( p \). Note that the total number of neutrinos in a given energy mode, \( n_1(p) + n_2(p) \), is conserved for any value of \( \mu \) because neutrinos either keep their momenta or exchange it in the current model. Therefore, Eq. (13) simply expresses the fact that \( n_1(p) \) and \( n_2(p) \) are individually conserved in \( \mu \to 0 \) limit. However, away from the \( \mu \to 0 \) limit and except for the combination in Eq. (12), the constants of motion in Eq. (10) are nontrivial and cannot be expressed in terms of the neutrino number operators.

It should also be noted that the operators in Eq. (10) are invariant only under the ideal conditions, i.e., when the single angle approximation is adopted, there is no net leptonic background, and the volume occupied by the neutrinos is fixed (\( \mu = \text{constant} \)). However, the constants of motion may still be useful away from these ideal conditions. For example, one can decompose the Hamiltonian into ideal and non-ideal parts as

\[
H = H_{\text{ideal}} + H_{\text{non-ideal}}.
\]

In this case, the time evolution of the “constants of motion” will only be due to the non-ideal part, i.e.,

\[
\frac{d}{dt} h_p = -i[h_p, H_{\text{non-ideal}}].
\]

because they commute with the ideal part. Therefore the invariants can provide a convenient set of variables subject to a simpler time evolution.

It is worth mentioning that the Hamiltonian in Eq. (9) was studied and its integrability was already known before Gaudin’s work. In fact it was first introduced in 1957 by Bardeen, Cooper and Schrieffer in order to describe the pairing of valance electrons in a superconductor [29]. In the context of electron pairs, the role of the Gaudin’s spins or the neutrino isospin is played by pair quasi-spin operator. The (reduced) BCS pairing Hamiltonian is given by

\[
H_{\text{BCS}} = \sum_k 2\epsilon_k t_k^3 - GT^+T^-.
\]

It describes a set of spin up (\( c_{k\uparrow} \)) and spin-down (\( c_{k\downarrow} \)) electrons (Cooper pairs) which can occupy a set of single particle energy levels denoted by \( \epsilon_k \). The components of the quasi-spin operator \( \vec{t}_k \) are given by

\[
t_k^+ = c_{k\uparrow}^\dagger c_{k\downarrow}, \quad t_k^- = c_{k\downarrow} c_{k\uparrow} \quad \text{and} \quad t_k^3 = \frac{1}{2} \left( c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow} - 1 \right)
\]

and they obey the same \( SU(2) \) commutation relations as those given in Eq. (2). In the quasi-spin scheme, a single particle level \( \epsilon_k \) has quasi-spin up if it is occupied by a pair and quasi-spin down if it is not. \( \vec{T} = \sum \vec{t}_k \) denotes the total quasi-spin of all levels and \( G > 0 \) is the pairing strength. The BCS pairing Hamiltonian is also used in nuclear shell model to describe pairing between the nucleons in the valance shell.

It is easy to see that in the mass basis where \( \vec{B} = (0, 0, -1) \), the neutrino Hamiltonian in Eq. (9) has the same form as the BCS pairing Hamiltonian in Eq. (16) up to an overall minus sign and a term proportional to \( \mu J_3^3 (J_3^3 - 1) \). The overall minus sign may have dynamical consequences on the stability of some solutions but it is irrelevant for a discussion of the symmetries and the resulting exact solvability of both models. The term \( \mu J_3^3 (J_3^3 - 1) \) is also unimportant in this context because \( J_3^3 \) is itself a constant of motion as was shown in Eq. (12).

The exact solvability of the pairing model was first shown by Richardson in 1963 [30] who found its exact eigenstates and eigenvalues using the method of Bethe ansatz [31]. This method gives analytical expressions for the eigenstates and yields corresponding eigenvalues in terms of the roots of some algebraic equations which are known as the Bethe ansatz equations. Although these equations still call for a numerical approach in generic cases, the resulting problem is significantly less challenging than a brute force diagonalization of the Hamiltonian. In fact many numerical and analytical techniques were developed to solve them, especially in the limit of a large number of particles [32, 33]. In recent years, the problem of numerically solving the equations of Bethe ansatz received renewed interest [34], particularly in connection with the quench dynamics of superconductors away from the stability [35]. Pairing models and the solutions of the related Bethe ansatz equations also receive attention in recent years due to their connections with the conformal field theories and the matrix models [36, 37]. Reviews can be found in Refs. [38, 39].

A mean field type approximation is usually employed in the case of both self interacting neutrinos and the BCS model. The exact solvability extends to the mean field case as was shown by Yuzbashyan et al who derived formal solutions of the resulting mean field equations in the context of the BCS model [40]. Collective modes of behavior for Cooper pairs were also analyzed in the same reference. Note that two of these collective modes were already known in the context of neutrinos as synchronized and bipolar oscillations. More recently, all of these modes were identified and classified for neutrinos in an independent study [41].
In the mean field approximation, neutrino-neutrino interactions are represented by an effective one-body scheme in which each neutrino interacts with an average potential created by all other neutrinos. One way to implement this approximation is to employ the operator product linearization through which the quadratic operator \( \vec{J}_p \cdot \vec{J}_q \) is approximated as

\[
\vec{J}_p \cdot \vec{J}_q \sim \langle \vec{J}_p \rangle \cdot \langle \vec{J}_q \rangle + \langle \vec{J}_p \rangle \cdot \langle \vec{J}_q \rangle - \langle \vec{J}_p \rangle \cdot \langle \vec{J}_q \rangle.
\]

Linearization of the neutrino evolution equations is also discussed in Ref. [42]. The expectation values in the above equations should be calculated with respect to a state which satisfies the condition \( \langle \vec{J}_p \rangle \cdot \langle \vec{J}_q \rangle = \langle \vec{J}_p \rangle \cdot \langle \vec{J}_q \rangle \). This amounts to truncating the Hilbert Space of the problem by excluding the entangled states because this condition is satisfied only by the non-entangled states. These states are also the coherent states of the orthogonal SU(2) algebras presented in Eq. (1) [43]. The expectation values of the isospin operators, i.e., \( \vec{P}_p = 2 \langle \vec{J}_p \rangle \), are called the polarization vectors where the factor of 2 is included for convenience. The polarization vectors are also subject to the summation rule introduced in Eq. (4). Application of the mean field approximation to the neutrino Hamiltonian given in Eq. (9) yields

\[
H \sim H^{\text{MF}} = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu \vec{P} \cdot \vec{J}
\]

where \( \vec{P} \) is the total polarization vector which is the total potential that each neutrino interacts with. This approximation is consistent only if the mean field evolves in line with the evolution of the particles which collectively create it. In the Heisenberg picture, this can be formulated by first calculating the quantum mechanical equation of motion of the isospin operator from the mean field Hamiltonian, i.e., \( d\vec{J}_p / dt = -i[\vec{J}_p, H^{\text{MF}}] \) and then taking the expectation values of both sides [23]. This yields the mean field consistency equations

\[
\frac{d}{dt} \vec{P}_p = (\omega_p \vec{B} + \mu \vec{P}) \times \vec{P}_p
\]

which should be satisfied for every momentum mode \( p \).

A straightforward calculation shows that the mean field Hamiltonian given in Eq. (19) does not commute with the exact many-body constants of motion given in Eq. (10). However, their expectation values

\[
I_p = 2 \langle \vec{J}_p \rangle = \vec{B} \cdot \vec{P}_p + \mu \sum_{q \neq p} \frac{\vec{P}_q \cdot \vec{P}_p}{\omega_p - \omega_q}
\]

are still independent of time as can be directly verified from mean field consistency equations. Non-conservation of the exact many-body constants of motion in the mean field picture is particularly evident in the case of the invariant given in Eq. (12) because the occupation numbers of the first and second mass eigenstates clearly change as the neutrinos interact with the mean field through the term \( \mathcal{P}^+ \vec{J}^- + \mathcal{P}^- \vec{J}^+ \) in Eq. (19). However, the average occupation numbers \( \langle N_1 \rangle \) and \( \langle N_2 \rangle \) continue to be invariant since \( \langle N^3 \rangle = 2 \langle \vec{J}^3 \rangle \) is conserved.

The invariants of the neutrino Hamiltonian represent the dynamical symmetries of the system which should be carefully studied in order to understand the collective modes of behavior that the neutrinos display. Here, we would like to consider a particular example concerning the adiabatic evolution of neutrinos from a region of high neutrino density (like the surface of a proto-neutron star) into the vacuum. Numerical simulations of the mean field equations under the relevant conditions showed that neutrinos completely exchange parts of their spectra above or below a critical energy by the time they reach the vacuum. This behavior is known as a spectral split and was analytically explained in terms of the instantaneously stable solutions of the mean field consistency equations viewed from a rotating reference frame in the isospin space in such a way that the rotation frequency of the frame yields the split frequency [22, 44]. Here we consider this phenomenon from a different perspective, namely as an avoided level crossing between the energy eigenvalues of the mean field Hamiltonian which includes a Lagrange multiplier to fix the occupation numbers \( N_1 \) and \( N_2 \) to chosen initial values. In this scheme, the value of the Lagrange multiplier yields the split frequency. In what follows, we closely follow Ref. [23] where this approach was originally developed.

According to the adiabatic theorem, if a Hamiltonian varies slowly enough, then a system which initially occupies one of its eigenstates evolves in such a way that it continues to occupy the same instantaneous eigenstate, as long as there is an energy gap between this particular eigenstate and the others. One can easily find the eigenstates of the mean field Hamiltonian given in Eq. (19) but since the Hamiltonian involves the mean field \( \vec{P} \), its eigenstates necessarily involve \( \vec{P} \) as a parameter as well. On the other hand, these eigenstates should also satisfy the mean field consistency condition \( \vec{P} = 2 \langle \vec{J} \rangle \) and it is easy to see that this condition restricts the value of the mean field \( \vec{P} \). Since
the component of $\vec{P}$ along $\vec{B}$ is equal to $\langle N_1 \rangle - \langle N_2 \rangle$, this tells us that, although the average total occupancies of mass eigenstates are conserved, not all possible values are allowed for them in a steady state solution due to the self consistency requirement of the mean field approximation. This is in contrast with the exact many-body picture where the many-body Hamiltonian can be diagonalized simultaneously with the number operators $N_1$ and $N_2$ and therefore there is a steady state solution (i.e., an eigenstate of the many-body Hamiltonian) for all possible values of these total occupancies.

One way to accommodate any possible set of occupation numbers in the mean field picture is to fix them by introducing a Lagrange multiplier $\omega$, before adopting the mean field approximation as follows:

$$ (H + \omega \mathcal{F}_p^3)^{\text{MF}} = - \sum_p (\omega_p - \omega_c) \mathcal{F}_p^3 + \mu \vec{B} \cdot \mathcal{F}. $$

(22)

This is equivalent to viewing the problem from a rotating reference frame in the isospin space as pointed out in Refs. [22, 44]. In either case, the extra degree of freedom $\omega$ can be used to set the desired occupation numbers.

The instantaneous eigenstates of the Hamiltonian in Eq. (22) can be found with the following transformation:

$$ \begin{pmatrix} \alpha_1(p) \\ \alpha_2(p) \end{pmatrix} = \begin{pmatrix} \cos \theta_p & e^{i\delta} \sin \theta_p \\ -e^{-i\delta} \sin \theta_p & \cos \theta_p \end{pmatrix} \begin{pmatrix} \alpha_1(p) \\ \alpha_2(p) \end{pmatrix}. $$

(23)

Here $\theta_p$ and $\delta$ are given by

$$ \sin 2\theta_p = \sqrt{1 - \frac{(\omega_c - \omega_p + \mu \mathcal{F}^3)^2}{(\omega_c - \omega_p + \mu \mathcal{F}^3)^2 + \mu^2 \mathcal{F}^+ \mathcal{F}^-}} \quad \text{and} \quad e^{i\delta} = \frac{\mathcal{F}^+}{|\mathcal{F}|}. $$

(24)

Note that here $\theta_p$ is not summed over all directions as defined in Eq. (4). It is indexed with $p$ because it does not depend on the direction of the momentum. $\alpha_1(p)$ and $\alpha_2(p)$ are the instantaneous non-interacting degrees of freedom of the mean field Hamiltonian. In other words, when the Hamiltonian in Eq. (22) is expressed in terms of them, it has the form of a free Hamiltonian:

$$ (H + \omega \mathcal{F}_p^3)^{\text{MF}} = \sum_p \lambda_p \left( \alpha_1^\dagger(p) \alpha_1(p) - \alpha_2^\dagger(p) \alpha_2(p) \right). $$

(25)

Here $\lambda_p$ is given by

$$ \lambda_p = \frac{1}{2} \sqrt{(\omega_c - \omega_p + \mu \mathcal{F}^3)^2 + \mu^2 \mathcal{F}^+ \mathcal{F}^-}. $$

(26)

At any given moment, instantaneous eigenstates of the Hamiltonian in Eq. (22) can be written in terms of $\alpha_1(p)$ and $\alpha_2(p)$. For example,

$$ \prod_p |\alpha_1^\dagger(p) \rangle \langle 0| $$

(27)

is a particular instantaneous eigenstate. The other eigenstates can be written similarly in terms of $\alpha_1$’s and $\alpha_2$’s.

It is easy to show that the new basis given in Eq. (23) coincides with the flavor basis when neutrinos occupy a very small volume ($V \to 0$, $\mu \to 0$) and with the mass basis when they occupy a very large volume ($V \to \infty$, $\mu \to 0$). If initially there is more $\nu_e$ than $\nu_x$ in the system, then Eqs. (23) and (24) can be used to show that

$$ \lim_{\mu \to 0} \alpha_1(p) = a_c(p) \quad \text{and} \quad \lim_{\mu \to 0} \alpha_2(p) = a_x(p). $$

(28)

If the opposite is true, i.e., initially there is more $\nu_x$ then $\nu_e$, then $a_e$ and $a_x$ should be exchanged in the above equation. In the limit where neutrino density approaches to zero, Eqs. (23) and (24) give

$$ \lim_{\mu \to 0} \alpha_1(p) = \begin{cases} \alpha_1(p) & \omega_p < \omega_c \\ \alpha_2(p) & \omega_p > \omega_c \end{cases} \quad \text{and} \quad \lim_{\mu \to 0} \alpha_2(p) = \begin{cases} \alpha_1(p) & \omega_p > \omega_c \\ \alpha_2(p) & \omega_p < \omega_c \end{cases}. $$

(29)

Eq. (28) tells us that when neutrinos are released from the neutrinosphere in a supernova, the system occupies one of the eigenstates of the Hamiltonian in Eq. (22) because the neutrino density is initially very high and all neutrinos emerge in flavor eigenstates. Under the adiabatic evolution conditions, the system stays in the same eigenstate but $\alpha_1$
and $\alpha_2$ slowly evolve from flavor to the mass basis. According to Eq. (29), all neutrinos are converted from flavor basis into mass basis by the time the neutrino density drops to zero.

As an example, let us consider an initial state which simply consists of electron neutrinos with a box spectrum as shown on the left hand side of Fig. (1). According to Eq. (28), this distribution corresponds to the eigenstate given in Eq. (27) in the limit where $\mu \to \infty$. In the opposite limit where $\mu \to 0$, Eq. (29) tells us that the same eigenstate corresponds to the distribution seen on the right hand side of Fig. (1). Therefore those neutrinos which oscillate faster than a critical frequency evolve into the second mass eigenstate whereas others evolve into the first mass eigenstate.

At this point, we would like to leave the mean field approximation scheme and return to the exact many-body Hamiltonian given in Eq. (9) and its dynamical symmetries represented by the constants of motion given in Eq. (10). Although we have worked with two mixing flavors so far, it is easy to show that similar symmetries exist for three mixing flavors as well. In order to describe three neutrino flavors, one should first generalize the concept of neutrino isospin. For this purpose, we introduce the operators

$$[\mathcal{T}_{ij}(p), \mathcal{T}_{kl}(q)] = \delta^3(p - q) \left( \delta_{ik} \mathcal{T}_{jl}(p) - \delta_{il} \mathcal{T}_{kj}(p) \right)$$

for $i, j, k = 1, 2, 3$. These operators generalize the mass isospin defined in Eq. (3). We use curly letters to denote the components of the operators in mass basis to comply with our earlier convention. Note that we also generalize the summation convention introduced in Eq. (4) to the three flavor case. For example, $\mathcal{T}_{ij}$ represents the $ij$ component of the generalization of the total isospin of all neutrinos. These operators now obey the $SU(3)$ commutation relations:

$$[\mathcal{T}_{ij}(p), \mathcal{T}_{kl}(q)] = \delta^3(p - q) \left( \delta_{ik} \mathcal{T}_{jl}(p) - \delta_{il} \mathcal{T}_{kj}(p) \right)$$

The Hamiltonian which describes the self interactions and the vacuum oscillations of neutrinos is given by

$$H = -\sum_p \left( \frac{\delta m_{11}^2 + \delta m_{13}^2}{6p} \mathcal{T}_{11}(p) + \frac{\delta m_{12}^2 + \delta m_{32}^2}{6p} \mathcal{T}_{22}(p) + \frac{\delta m_{13}^2 + \delta m_{32}^2}{6p} \mathcal{T}_{33}(p) \right) + \frac{\mu}{2} \sum_{i,j=1}^{3} \mathcal{T}_{ij} \mathcal{T}_{ji}$$

where $\delta m_{ij}^2 = m_i^2 - m_j^2$ and the single angle approximation is again adopted in describing the neutrino-neutrino interactions. This Hamiltonian reduces to the one given in Eq. (9) up to a trace term if it is restricted to a two flavor subspace. It is a simple exercise in algebra to show that the operators

$$h_p = -\left( \frac{\delta m_{11}^2 + \delta m_{31}^2}{6p} \mathcal{T}_{11}(p) + \frac{\delta m_{12}^2 + \delta m_{32}^2}{6p} \mathcal{T}_{22}(p) + \frac{\delta m_{13}^2 + \delta m_{32}^2}{6p} \mathcal{T}_{33}(p) \right) + \mu \sum_{q \neq p} \frac{3}{2} \sum_{i,j=1}^{3} \mathcal{T}_{ij}(p) \mathcal{T}_{ij}(q)$$

are constants of motion of the Hamiltonian given in Eq. (32), i.e., they obey

$$[h_p, h_q] = 0 \quad \text{and} \quad [H, h_p] = 0 \quad \text{for all } p.$$  

If restricted to a two flavor subspace, these constants of motion are equivalent to the ones given in Eq. (10) up to a trace term and an overall multiplicative constant.

We emphasize that a study of self interacting neutrinos as a many-body system with its underlying symmetries can lead us to a simple understanding of its collective modes of behavior. To this end, we presented the invariants representing the dynamical symmetries in the exact many-body formalism. In the two flavor mixing scheme, we
used an example in which a single spectral split can be simply viewed as an avoided level crossing of the mean field Hamiltonian if the value of a many-body constant of motion is fixed with a Lagrange multiplier. We showed that the many-body Hamiltonian describing three mixing flavors with self interactions also has the same dynamical symmetries. Whether these symmetries can be used to explain multiple spectral splits of neutrinos for two or three mixing flavors and whether they lead to other, possibly more interesting collective behavior modes are the subjects of our ongoing research and will be discussed elsewhere.

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