Geodesic path for the minimal energy cost in shortcuts to isothermality

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Shortcut to isothermality is a driving strategy to steer the system to its equilibrium states within finite time, and enables evaluating the impact of a control promptly. Finding optimal scheme to minimize the energy cost is of critical importance in applications of this strategy in pharmaceutical drug test, biological selection, and quantum computation. We prove the equivalence between designing the optimal scheme and finding the geodesic path in the space of control parameters. Such equivalence allows a systematic and universal approach to find the optimal control to reduce the energy cost. We demonstrate the current method with examples of a Brownian particle trapped in controllable harmonic potentials.

Introduction.— Boosting system to its steady state is critical to promptly evaluate the impact of a control [1–8]. In biological systems, the quest to timely evaluate the impact of therapy or genotypes posts a requirement to steer the system to reach its steady state with a considerable tunable rate [1–5]. In adiabatic quantum computation, the task of solving the optimization problem is converted to the problem of driving systems from a trivial ground state to another nontrivial ground state. The speedup of the computational process needs to steer the system to the target ground state in finite time [6–8]. These quests to tune the system within finite time while keep it in equilibrium is eagerly needed.

Shortcut to isothermality was proposed as a finite-time driving strategy to steer the system evolving along the path of instantaneous equilibrium states [9]. The strategy has been applied in reducing transition time between equilibrium states [10–12], improving the efficiency of free-energy estimation [13], constructing finite-time heat engines [14–16], and controlling biological evolutions [4, 5]. The cost of the finite-time operation is the additional energy cost due to irreversibility posted by the fundamental thermodynamic law. Minimizing such cost is in turn relevant to optimize the heat engine [17–19] and reconstruct the energy landscape of biological macromolecules [20–22]. A question arises naturally, how to find the optimal control protocol to minimize the irreversible energy cost in shortcuts to isothermality.

In this Letter, we present a systematic approach for finding the optimal protocol to minimize the energy cost. In Fig. 1, we show the equivalence of designing the optimal control to finding the geodesic path on a Riemannian manifold, spanned by the control parameters [23–27]. In turn, the powerful tools developed in geometry are adapted for solving the optimal control protocol. Our scheme is exemplified with a single Brownian particle in the harmonic potential with controllable stiffness and central position.

Geometric approach — The system is described by the Hamiltonian $H_o(\vec{x}, \vec{p}, \hat{\lambda}) = \sum_i p_i^2/2 + U_o(\vec{x}, \vec{p}, \hat{\lambda})$ with the coordinate $\vec{x} \equiv (x_1, x_2, \cdots, x_N)$ and the momentum $\vec{p} \equiv (p_1, p_2, \cdots, p_N)$. It is immersed in a thermal reservoir with a constant temperature $T$. $\hat{\lambda}(t) \equiv (\lambda_1, \lambda_2, \cdots, \lambda_M)$ are time-dependent control parameters. For simplicity, we have set the mass of the system as a unit. In the shortcut scheme, an auxiliary Hamiltonian $H_a(\vec{x}, \vec{p}, t)$ is added to steer the system evolving along the instantaneous equilibrium state of the original Hamiltonian $H_o$ in the finite-time interval $\tau$ with boundary conditions $H_a(0) = H_a(\tau) = 0$. The dynamical evolution under the total Hamiltonian $H = H_o + H_a$ is described by the Langevin equation as

$$\dot{x_i} = \frac{\partial H}{\partial p_i},$$
$$\dot{p_i} = -\frac{\partial H}{\partial x_i} - \gamma \dot{x_i} + \xi_i(t), \quad (1)$$
where $\gamma$ is the dissipation rate and $\xi \equiv (\xi_1, \xi_2, \ldots, \xi_N)$ are random variables of the Gaussian white noise. The evolution equation of the system distribution $\rho(\mathbf{x}, \mathbf{p}, t) = \delta(\mathbf{x} - \mathbf{x}(t))\delta(\mathbf{p} - \mathbf{p}(t))$ for a trajectory $[\mathbf{x}(t), \mathbf{p}(t)]$ is described by the Liouville equation as $\partial_t \rho = -\sum_i[\partial_{x_i}(\mathbf{x}, \mathbf{p}, t) + \partial_{p_i}(\mathbf{x}, \mathbf{p}, t)]$. By averaging over different noise realizations $[\xi(t)]$, we obtain the evolution of the observable probability distribution $P(\mathbf{x}, \mathbf{p}, t) \equiv \langle \rho(\mathbf{x}, \mathbf{p}, t) \rangle = \int D[\mathbf{x}(t)]D[\mathbf{p}(t)] \mathcal{F}[\mathbf{x}(t), \mathbf{p}(t)]\delta(\mathbf{x} - \mathbf{x}(t))\delta(\mathbf{p} - \mathbf{p}(t))$ as [28]
\[
\frac{\partial P}{\partial t} = \sum_i \left[ -\frac{\partial}{\partial x_i} \left( \frac{\partial H}{\partial p_i} P \right) + \frac{\partial}{\partial p_i} \left( \frac{\partial H}{\partial x_i} P + \gamma \frac{\partial^2 P}{\partial p_i^2} \right) + \frac{\beta}{2} \frac{\partial^2 P}{\partial p_i^2} \right],
\]
where $\beta \equiv 1/(k_B T)$ is the inverse temperature with the Boltzmann constant $k_B$. Here $\mathcal{F}[\mathbf{x}(t), \mathbf{p}(t)]$ is the probability of the trajectory $[\mathbf{x}(t), \mathbf{p}(t)]$ associated with a noise realization $[\xi(t)]$ [29]. To ensure the instantaneous equilibrium distribution
\[
P(\mathbf{x}, \mathbf{p}, t) = P_{eq}(\mathbf{x}, \mathbf{p}, \tilde{\lambda}) = e^{\mathcal{F}(\mathbf{x}, \mathbf{p}, \tilde{\lambda}) - H_{eq}(\mathbf{x}, \mathbf{p}, \tilde{\lambda})},
\]
the auxiliary Hamiltonian is proved [9] to have the form $H_{eq}(\mathbf{x}, \mathbf{p}, \tilde{\lambda}) = \tilde{\lambda} \cdot \mathbf{f}(\mathbf{x}, \mathbf{p})$ with $\tilde{\lambda} \cdot \mathbf{f}(\mathbf{x}, \mathbf{p})$ satisfying
\[
\sum_i \left[ \gamma \frac{\partial^2 f_i}{\partial p_i^2} - \gamma p_i \frac{\partial f_i}{\partial p_i} + \frac{\partial f_i}{\partial x_i} - \frac{p_i}{\partial x_i} \right] = \frac{dF}{dt} - \frac{\partial U}{\partial \lambda_\mu} - \frac{\partial U}{\partial \lambda_\mu},
\]
where $F \equiv -\beta^{-1} \ln[\int d\mathbf{x}d\mathbf{p} \exp(-\beta H_0)]$ is the free energy. The boundary conditions are presented explicitly as $\tilde{\lambda}(0) = \tilde{\lambda}(\tau) = 0$.

The cost of the energy in the shortcut scheme is evaluated by the average work $W \equiv \langle \int_0^\tau dt \partial_\mu H \rangle_\xi$ [30–33], explicitly as
\[
W = \Delta F + \gamma \sum_i \int_0^\tau dt \int d\mathbf{x}d\mathbf{p} \left( \frac{\partial H_{eq}}{\partial p_i} \right)^2 P_{eq},
\]
where $\Delta F = F(\tilde{\lambda}(\tau)) - F(\tilde{\lambda}(0))$ is the free energy difference. Detailed derivation of Eq. (5) is presented in the supplementary materials [28]. To consider the finite-time effect, we define the irreversible work $W_{irr} \equiv W - \Delta F$, which follows
\[
W_{irr} = \gamma \sum_{\mu > 0} \int_0^\tau dt \lambda_\mu \lambda_\nu \left( \frac{\partial f_\mu}{\partial p_i} \frac{\partial f_\nu}{\partial p_i} \right)_ {eq},
\]
with $\langle \cdot \rangle_ {eq} = \int d\mathbf{x}d\mathbf{p} \cdot P_{eq}$. It follows from Eq. (6) that the integrand scales as $t^{-2}$ reducing the time $s \equiv t/\tau$, which results in the $1/\tau$ scaling [23] of the irreversible work, i.e., $W_{irr} \propto 1/\tau$. Such a $1/\tau$ scaling, predicted in various finite-time studies [18, 34–43], was recently verified for the ideal gas system [44] at the long-time limit. It is worth noting that in the shortcut scheme the current scaling is valid for any duration time $\tau$ with no requirement of the long-time limit [23, 25, 41, 45, 46].

In the space of the control parameters $\tilde{\lambda}$, we define a positive semi-definite metric
\[
g_{\mu \nu} = \gamma \sum_i \left( \frac{\partial f_\mu}{\partial p_i} \frac{\partial f_\nu}{\partial p_i} \right)_ {eq},
\]
whose positive semi-definiteness is proved in the supplementary materials [28]. With this metric, the length of a curve in the current geometric space is characterized via the thermodynamic length [23, 25–27, 45] as
\[
L = \int_0^\tau dt \sum_{\mu \nu} \sqrt{\lambda_\mu \lambda_\nu g_{\mu \nu}},
\]
which provides a lower bound of the irreversible work $W_{irr}$ as
\[
W_{irr} \geq \frac{L^2}{\tau}.
\]
The lower bound is reached with the optimal control scheme $\tilde{\lambda}(t)$ ($0 < t < \tau$), determined by the geodesic equation
\[
\ddot{\lambda}_\mu + \sum_\nu \Gamma^\mu_{\kappa \nu} \dot{\lambda}_\kappa \dot{\lambda}_\nu = 0,
\]
with the given boundary conditions $\tilde{\lambda}(0)$ and $\tilde{\lambda}(\tau)$. Here the Christoffel symbol is defined as $\Gamma^\mu_{\kappa \nu} \equiv \frac{1}{2} \sum_\gamma (g^{-1})_{\mu \gamma} (\partial_{\lambda_\kappa} g_{\nu \gamma} + \partial_{\lambda_\nu} g_{\kappa \gamma} - \partial_{\lambda_\gamma} g_{\kappa \nu})$. For the case with the single control parameter $\lambda(t)$, the analytical solution [26] for Eq. (9) is obtained as $\lambda(t) = (\lambda(\tau) - \lambda(0))/g(\lambda(t)^{-1})/\int_0^\tau dt' g(\lambda(t')^{-1})$, with $g = \gamma \lambda^2$. For the case with multiple parameters, the shooting method is an available option which treats the two-point boundary-value problem as an initial-value problem [47]. See the supplementary materials for details about the shooting method to our problems [28].

The strategy of current formalism is shown in Fig. 1. Firstly, we obtain the control operators $\tilde{\lambda}(\mathbf{x}, \mathbf{p})$ in Fig. 1(a) by solving Eq. (4). Secondly, the metric $g_{\mu \nu}$ in Fig. 1(b) for the parametric space is calculated via Eq. (7). Finally, the optimal control is obtained by solving the geodesic equation in Eq. (9). The current strategy provides an effective approach to find the optimal control to minimize the energy cost, i.e., the total work done during the shortcut-to-isothermal process. The strategy is illustrated through two examples with one or two control parameters as follows.

**Brownian motion in the harmonic potential**– The Brownian particle is trapped by the one-dimensional breathing harmonic potential with tunable stiffness $\lambda(t)$ under the Hamiltonian $H_0(x, p, \lambda) = p^2/2 + \lambda(t)x^2/2$. Its auxiliary Hamiltonian was derived in Ref. [9] as $H_0(x, p, \lambda) = \tilde{\lambda}f(x, p, \lambda)$ with $f = 1/(4\gamma \lambda)(p - \gamma x)^2 + \lambda x^2$. The metric in Eq. (7) in this case reduces to [28]
\[
g = \frac{\lambda + \gamma^2}{4\gamma^2 \lambda^2}.
\]
And the lower bound of the irreversible work is reached by the protocol satisfying the geodesic equation $\lambda + \gamma^2 \lambda^2 = \tau\gamma$. 

The geodesic equation follows

\[ \ddot{\lambda}_1 - \frac{\dot{\lambda}_1^2(2\gamma^2 + 2\lambda_1)}{2\lambda_1(\gamma^2 + \lambda_1)} = 0, \]

\[ \ddot{\lambda}_2 - \frac{2\lambda_2\dot{\lambda}_2}{\lambda_1} + \frac{\dot{\lambda}_1^2\lambda_2(\gamma^2 + 2\lambda_1)}{2\lambda_1^2(\gamma^2 + \lambda_1)} = 0, \]  

with the boundary conditions \( \tilde{\lambda}(0) \) and \( \tilde{\lambda}(\tau) \).

The optimal scheme can be obtained by solving equations above using a general numerical method, i.e., the shooting method [47]. Here we firstly solve these equations numerically to provide a general perspective on our scheme. With the initial point \( \tilde{\lambda}(0) \), we choose an initial rate \( \dot{\tilde{\lambda}}(0) \) and solve the geodesic equation with the Euler algorithm to obtain a trial solution \( \tilde{\lambda}^{(0)}(\tau) \). Newton’s method is utilized for updating the rate \( \dot{\tilde{\lambda}}(0+) \) to reduce the distance between the trial solution \( \tilde{\lambda}^{(0)}(\tau) \) and the target point \( \tilde{\lambda}(\tau) \). In the simulation, we have chosen the parameters \( \tilde{\lambda}(0) = (1, 1), \tilde{\lambda}(\tau) = (16, 2), k_B T = 1, \) and \( \gamma = 1 \). The geodesic path for the optimal control is illustrated as \( \tilde{\lambda}^{(gp)}(t) \) (triangles) in Fig. 2.

Fortunately, the analytical geodesic protocol for Eq. (14) can be obtained as

\[ \dot{\lambda}_1 = \frac{w_b}{\tau} \sqrt{\frac{\lambda_1^2}{\lambda_1 + \gamma^2}}, \]
\[ \dot{\lambda}_2 = m_b \tau + n_b, \]  

where \( w_b = -[2\sqrt{1 + \gamma^2/\lambda_1} + \ln(\sqrt{1 + \gamma^2/\lambda_1} - 1) - \ln(\sqrt{1 + \gamma^2/\lambda_1} + 1)]\lambda_1(0), m_b = (\lambda_2(\tau) - \lambda_1(0))/\lambda_1(0), \) and \( n_b = \lambda_2(0)/\lambda_1(0) \) are constants. In Fig. 2, we show the match between the optimal control obtained from the numerical calculation \( \tilde{\lambda}^{(gp)}(t) \) (triangles) and the analytical solution \( \tilde{\lambda}^{(gp)}(t) \) (solid lines). For the comparison, we also show the protocol of the simple linear control \( \tilde{\lambda}^\text{lin}(t) = (\tilde{\lambda}(\tau) - \tilde{\lambda}(0))t/\tau + \tilde{\lambda}(0) \).

To validate our results of optimization, we calculate the irreversible work for the single Brownian particle in the controllable harmonic potential with two control parameters by solving the Langevin equation (1) through the Euler algorithm [28, 48]. The average work is calculated by the ensemble average of the stochastic work over \( 10^5 \) stochastic trajectories. Details of the simulation are presented in the supplementary material [28]. In Fig. 3, we plot the irreversible work \( W_{\text{irr}} \) as a function of duration \( \tau \in \{0.1, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0\} \) for both the geodesic path (red circles) and the simple linear control (blue squares). The geodesic protocol results in a lower irreversible work than that from the linear protocol. The black line shows the analytical results \( W_{\text{irr}} = \mathcal{L}^2/\tau \), where the thermodynamic length \( \mathcal{L} \) is calculated as

\[ \mathcal{L} = \int_0^\tau dt \sum_{\mu\nu} \sqrt{\lambda_\mu \lambda_\nu g_{\mu\nu}} = \sqrt{w_b^2/(4\beta \gamma)} + \gamma m_b^2. \]
Our findings simplify the procedure of finding the optimal control parameters to find the optimal control scheme by applying the tools of Riemannian geometry.

\[ g_{\mu \nu} \] in Eq. (7) without the need to treat the system on a case-by-case basis. An intuitive determination of the performance of the controls is allowed with the proportional relation in Eq. (8) between the minimal energy cost and the square of the length of the geodesic path.

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**Conclusions.**—In summary, we have provided a geometric approach to find the optimal control scheme to steer the evolution of the system along the path of instantaneous equilibrium states to reduce the energy cost. The proven equivalence between designing the optimal control and finding the geodesic path in the parametric space allows the application of the methods developed in Riemannian geometry to solve the optimization problem in thermodynamics. We have applied our approach into the Brownian particle system tuned by both one and two control parameters to find the optimal control for reducing energy cost. Analytical and numerical results have verified that the geodesic protocol can largely reduce the irreversible work in the shortcut scheme. Our strategy shall provide an effective tool to design the optimal finite-time control with the lowest energy cost.

Our results demonstrate that the optimal control with the minimal energy cost to transfer the system between equilibrium states is to steer the system evolving along the geodesic path. Once the initial and final equilibrium states are given, the geodesic path is determined by the geodesic equation (9) for the given system. The dynamics of the system is covered by the metric \( g_{\mu \nu} \) in Eq. (7) without the need to treat the system on a case-by-case basis. An intuitive determination of the performance of the controls is allowed with the proportional relation in Eq. (8) between the minimal energy cost and the square of the length of the geodesic path.

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Supplementary materials: Geodesic path for the minimal energy cost in shortcuts to isothermality

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The supplementary materials are devoted to provide detailed derivations in the main context.

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I. THE MODIFIED KRAMERS EQUATION

The Kramers equation [1] was developed for describing systems with the form of Hamiltonian $H_o = \frac{\vec{p}^2}{2} + U_o(\vec{x}, \vec{\lambda})$. In the main text, we consider systems controlled by shortcuts to isothermality with the form of Hamiltonian $H = H_o + H_a$, with boundary conditions $H_a(0) = H_a(\tau) = 0$ at the initial time $t = 0$ and the final time $t = \tau$. In this section, we derive a modified Kramers equation [2] for the total Hamiltonian $H$.

The evolution equation of the system probability distribution $\rho(\vec{x}, \vec{p}, t) = \delta(\vec{x} - \vec{x}(t))\delta(\vec{p} - \vec{p}(t))$ for a trajectory $[\vec{x}(t), \vec{p}(t)]$ is governed by the Liouville equation as

$$\frac{\partial \rho}{\partial t} = -\sum_i \left[ \frac{\partial}{\partial x_i} (\dot{x}_i \rho) + \frac{\partial}{\partial p_i} (\dot{p}_i \rho) \right],$$

(1)

where the evolution of variables is described by the Langevin equation as

$$\dot{x}_i = \frac{\partial H}{\partial p_i},$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} - \gamma \dot{x}_i + \xi_i(t).$$

(2)

With Eqs. (1) and (2), we obtain

$$\frac{\partial \rho}{\partial t} = \sum_i \left[ -\frac{\partial}{\partial x_i} (\frac{\partial H}{\partial p_i} \rho) + \frac{\partial}{\partial p_i} (\frac{\partial H}{\partial x_i} \rho + \gamma \frac{\partial H}{\partial p_i} \rho) - \frac{\partial}{\partial p_i} (\xi_i \rho) \right].$$

(3)
An observable probability to describe the average effect of the probability distribution over different realizations of $[\xi(t)]$ can be defined as

$$P(\vec{x}, \vec{p}, t) \equiv \langle \rho(\vec{x}, \vec{p}, t) \rangle_{\xi}$$

$$= \int \int D[\vec{x}(t)]D[\vec{p}(t)] T[\vec{x}(t), \vec{p}(t)] \rho(\vec{x}, \vec{p}, t)$$

$$= \int \int D[\vec{x}(t)]D[\vec{p}(t)] T[\vec{x}(t), \vec{p}(t)] \delta(\vec{x} - \vec{x}(t)) \delta(\vec{p} - \vec{p}(t)),$$

where $T[\vec{x}(t), \vec{p}(t)]$ is the probability of the trajectory $[\vec{x}(t), \vec{p}(t)]$ associated with a noise realization $[\xi(t)]$ [1]. In the following, we derive the evolution equation for the observable probability $P(\vec{x}, \vec{p}, t)$.

Equation (3) can be rewritten as

$$\frac{\partial \rho}{\partial t} = -\hat{L}_d \rho - \hat{L}_s \rho,$$

with the deterministic operator

$$\hat{L}_d(t) \equiv \sum_{i} \left[ \frac{\partial}{\partial x_i} (\frac{\partial H}{\partial p_i}) - \frac{\partial}{\partial p_i} (\frac{\partial H}{\partial x_i} + \gamma \frac{\partial H}{\partial p_i}) \right],$$

and the stochastic operator

$$\hat{L}_s(t) \equiv \sum_{i} \xi_i \frac{\partial}{\partial p_i}.$$

Introduce a new probability $\phi(\vec{x}, \vec{p}, t)$ that satisfies

$$\rho(\vec{x}, \vec{p}, t) = \mathcal{T} e^{-\int_0^t \hat{L}_d(t') \, dt'} \phi(\vec{x}, \vec{p}, t) = \hat{R}(t) \phi(\vec{x}, \vec{p}, t),$$

where $\hat{R}(t) \equiv \mathcal{T} e^{-\int_0^t \hat{L}_d(t') \, dt'}$ with $\mathcal{T}$ representing the time-order operator. Substituting Eq. (6) into Eq. (5), we obtain

$$\frac{\partial \phi}{\partial t} = -\hat{O} \phi,$$

where $\hat{O}(t) \equiv \hat{R}^{-1}(t) \hat{L}_s(t) \hat{R}(t)$ with $\hat{R}^{-1}(t) \equiv (\mathcal{T} e^{-\int_0^t \hat{L}_d(t') \, dt'})^{-1}$. Equation (7) has a formal solution as

$$\phi(\vec{x}, \vec{p}, t) = \mathcal{T} e^{-\int_0^t \hat{O}(t') \, dt'} \phi(\vec{x}, \vec{p}, 0)$$

$$= \sum_{n} (-1)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \hat{O}(t_n) \hat{O}(t_{n-1}) \cdots \hat{O}(t_1) \phi(\vec{x}, \vec{p}, 0).$$

Averaging it over different realizations of $[\xi]$, we derive that

$$\langle \phi(\vec{x}, \vec{p}, t) \rangle_{\xi} = \sum_{n} (-1)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \hat{O}(t_n) \hat{O}(t_{n-1}) \cdots \hat{O}(t_1) \langle \xi \rangle_{\xi} \phi(\vec{x}, \vec{p}, 0)$$

$$= \sum_{n} (-1)^{-2n} \int_0^t dt_2n \int_0^{t_{2n}} dt_{2n-1} \cdots \int_0^{t_2} dt_1 \hat{O}(t_{2n}) \hat{O}(t_{2n-1}) \cdots \hat{O}(t_1) \langle \xi \rangle_{\xi} \phi(\vec{x}, \vec{p}, 0).$$

In the second step, we have considered the fact that the noise $\dot{\xi}$ is Gaussian satisfying $\langle \xi_i(t) \rangle_{\xi} = 0$ and $\langle \xi_i(t) \xi_j(t') \rangle_{\xi} = 2\gamma k_B T \delta_{ij} \delta(t - t')$. The higher-order moments that contain odd number of $\xi_i$ are zero. The remaining terms containing even number of $\xi_i$ can be decomposed into a sum of products of the second order moment $\langle \xi_i(t) \xi_j(t') \rangle_{\xi}$. For example, the fourth order moment follows

$$\langle \xi_i(t_1) \xi_j(t_2) \xi_k(t_3) \xi_l(t_4) \rangle_{\xi} = \langle \xi_i(t_1) \xi_j(t_2) \rangle_{\xi} \langle \xi_k(t_3) \xi_l(t_4) \rangle_{\xi} + \langle \xi_i(t_1) \xi_k(t_3) \rangle_{\xi} \langle \xi_j(t_2) \xi_l(t_4) \rangle_{\xi} + \langle \xi_i(t_1) \xi_l(t_4) \rangle_{\xi} \langle \xi_j(t_2) \xi_k(t_3) \rangle_{\xi}.$$
In the right hand side of Eq. (10), the second and third terms vanish because of the time order $t \geq t_{2n} \cdots \geq t_{1}$. Therefore, Eq. (9) reduces to the form

$$
\langle \phi(\vec{x}, \vec{p}, t) \rangle_\xi = \sum_n \left( \int_0^t dt_2 \int_0^{t_{2n}} dt_2 \langle \hat{O}(t_{2n}) \hat{O}(t_{2n-1}) \rangle_\xi \left( \int_0^{t_{2n-2}} dt_2 \int_0^{t_{2n-3}} \langle \hat{O}(t_{2n-3}) \hat{O}(t_{2n-2}) \rangle_\xi \right) \right) 
\times \cdots \left( \int_0^{t_3} dt_2 \int_0^{t_2} dt_3 \langle \hat{O}(t_2) \hat{O}(t_1) \rangle \right) \phi(\vec{x}, \vec{p}, 0).
$$

One of the integral in Eq. (11) is calculated as

$$
\int_0^{t_3} dt_2 \int_0^{t_2} dt_3 \langle \hat{O}(t_2) \hat{O}(t_1) \rangle
= \sum_{ij} \int_0^{t_2} dt_2 \int_0^{t_1} dt_3 \langle \xi_i(t_2) \xi_j(t_1) \rangle \hat{R}^{-1}(t_2) \partial \hat{R}(t_2) \hat{R}^{-1}(t_1) \partial \hat{R}(t_1)
= 2\gamma k_B T \sum_{ij} \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 \delta(t_2-t_1) \hat{R}^{-1}(t_2) \partial \hat{R}(t_2) \hat{R}^{-1}(t_1) \partial \hat{R}(t_1)
= \gamma k_B T \sum_i \int_0^{t_3} dt_2 \hat{R}^{-1}(t_2) \partial^2 \hat{R}(t_2).
$$

Then Eq. (11) proceeds as

$$
\langle \phi(\vec{x}, \vec{p}, t) \rangle_\xi = \left[ \sum_n (\gamma k_B T)^n \sum_i \int_0^t dt_2 \hat{R}^{-1}(t_2) \partial^2 \hat{R}(t_2) \right] \left( \sum_i \int_0^{t_{2n}} dt_2 \hat{R}^{-1}(t_{2n}) \partial^2 \hat{R}(t_{2n-2}) \right)
\times \cdots \left( \sum_i \int_0^{t_4} dt_2 \hat{R}^{-1}(t_2) \partial^2 \hat{R}(t_2) \right) \phi(\vec{x}, \vec{p}, 0)
= \left[ \sum_n (\gamma k_B T)^n \sum_i \int_0^t dt_2 \hat{R}^{-1}(t_2) \partial^2 \hat{R}(t_2) \right] \left( \sum_i \int_0^{t_{2n-1}} dt_2 \hat{R}^{-1}(t_{2n-2}) \partial^2 \hat{R}(t_{2n-3}) \right)
\times \cdots \left( \sum_i \int_0^{t_3} dt_2 \hat{R}^{-1}(t_2) \partial^2 \hat{R}(t_2) \right) \phi(\vec{x}, \vec{p}, 0).
$$

Taking derivative of Eq. (13) over time $t$, we obtain

$$
\frac{\partial}{\partial t} \langle \phi(\vec{x}, \vec{p}, t) \rangle_\xi = \left[ \sum_{n=0}^\infty (\gamma k_B T)^n \sum_i \hat{R}^{-1}(t_2) \partial^2 \hat{R}(t_2) \right] \left( \sum_i \int_0^t dt_2 \hat{R}^{-1}(t_{2n-1}) \partial^2 \hat{R}(t_{2n-1}) \right)
\times \cdots \left( \sum_i \int_0^{t\infty} dt_2 \hat{R}^{-1}(t_2) \partial^2 \hat{R}(t_2) \right) \phi(\vec{x}, \vec{p}, 0)
= \gamma k_B T \left( \sum_i \hat{R}^{-1}(t_2) \partial^2 \hat{R}(t_2) \right) \sum_{n=0}^\infty (\gamma k_B T)^n \left( \sum_i \int_0^t dt_2 \hat{R}^{-1}(t_{2n-1}) \partial^2 \hat{R}(t_{2n-1}) \right)
\times \cdots \left( \sum_i \int_0^{t\infty} dt_2 \hat{R}^{-1}(t_2) \partial^2 \hat{R}(t_2) \right) \phi(\vec{x}, \vec{p}, 0)
= \gamma k_B T \left( \sum_i \hat{R}^{-1}(t_2) \partial^2 \hat{R}(t_2) \right) \langle \phi(\vec{x}, \vec{p}, t) \rangle_\xi.
$$

Therefore, the observable probability $P(\vec{x}, \vec{p}, t) = \langle \rho(\vec{x}, \vec{p}, t) \rangle_\xi = \hat{R}(t) \langle \phi(\vec{x}, \vec{p}, t) \rangle_\xi$ follows the evolution equation

$$
\frac{\partial P}{\partial t} = -\hat{L}_d \hat{R}(t) \langle \phi(\vec{x}, \vec{p}, t) \rangle_\xi + \hat{R}(t) \frac{\partial}{\partial t} \langle \phi(\vec{x}, \vec{p}, t) \rangle_\xi
= (-\hat{L}_d + \gamma k_B T \sum_i \partial^2 \hat{R}(t) \langle \phi(\vec{x}, \vec{p}, t) \rangle_\xi
= \sum_i \left[ -\frac{\partial}{\partial \vec{x}_i} \left( \frac{\partial H}{\partial \vec{p}_i} P \right) + \frac{\partial}{\partial \vec{p}_i} \left( \frac{\partial H}{\partial \vec{x}_i} P + \gamma \frac{\partial H}{\partial \vec{p}_i} P \right) + \gamma k_B T \frac{\partial^2 P}{\partial \vec{p}_i^2} \right],
$$

(15)
II. SHORTCUTS TO ISOTHERMALITY WITH MULTIPLE CONTROL PARAMETERS

The framework of the shortcut scheme was originally developed for systems with single control parameter [2–4]. To establish a general formalism, we extend this framework for systems with multiple control parameters.

In the shortcut scheme, the system evolves according to Eq. (15) with $H = H_0 + H_a$. Substituting the instantaneous equilibrium distribution

$$P_{eq}(\vec{x}, \vec{p}, \vec{\lambda}) = e^{\beta[F(\vec{\lambda}) - H_0(\vec{x}, \vec{p}, \vec{\lambda})]}$$

(16)

into Eq. (15), we obtain the requirement for the auxiliary Hamiltonian $H_a$ as

$$\sum_{\mu} \left( \frac{dF}{d\lambda_{\mu}} - \frac{\partial U_0}{\partial \lambda_{\mu}} \right) \dot{\lambda}_{\mu} = \sum_i \left( \frac{\gamma}{\beta} \frac{\partial^2 H_a}{\partial p_i^2} - \gamma p_i \frac{\partial H_a}{\partial p_i} + \frac{\partial H_a}{\partial x_i} \frac{\partial U_0}{\partial x_i} - p_i \frac{\partial H_a}{\partial x_i} \right).$$

(17)

The solution for the auxiliary Hamiltonian is $H_a(\vec{x}, \vec{p}, t) = \sum_{\mu} \dot{\lambda}_{\mu} f_{\mu}(\vec{x}, \vec{p}, \vec{\lambda})$ with $f_{\mu}(\vec{x}, \vec{p}, \vec{\lambda})$ satisfying

$$\frac{dF}{d\lambda_{\mu}} - \frac{\partial U_0}{\partial \lambda_{\mu}} = \sum_i \left( \frac{\gamma}{\beta} \frac{\partial^2 f_{\mu}}{\partial p_i^2} - \gamma p_i \frac{\partial f_{\mu}}{\partial p_i} + \frac{\partial f_{\mu}}{\partial x_i} \frac{\partial U_0}{\partial x_i} - p_i \frac{\partial f_{\mu}}{\partial x_i} \right).$$

(18)

Once the form of the original potential $U_0$ is given, we can solve Eq. (18) for the function $f_{\mu}(\vec{x}, \vec{p}, \vec{\lambda})$. With the boundary condition

$$\dot{\lambda}(0) = \dot{\lambda}(\tau) = 0,$$

(19)

we can realize the shortcut scheme for systems with multi-parameters.

III. GENERAL FORMALISM: THE MEAN WORK DONE IN THE PROCESS DRIVEN BY SHORTCUTS TO ISOTHERMALITY

In this section, we will derive the work done in the shortcut scheme, and show its geometric expression.

A. Geometric approach to the irreversible work

The work done in an individual stochastic trajectory reads [2]

$$w[\vec{x}(t), \vec{p}(t)] = \int_0^\tau dt \frac{\partial H_0(\vec{x}(t), \vec{p}(t), \vec{\lambda})}{\partial t} + \int_0^\tau dt \frac{\partial H_a(\vec{x}(t), \vec{p}(t), t)}{\partial t}$$

$$= \int_0^\tau dt \frac{\partial H_0(\vec{x}(t), \vec{p}(t), \vec{\lambda})}{\partial t} + \sum_i \int_0^\tau dt \left( \frac{dH_a}{dt} - \dot{x}_i(t) \frac{\partial H_a}{\partial x_i} - \dot{p}_i(t) \frac{\partial H_a}{\partial p_i} \right)$$

$$= \int_0^\tau dt \frac{\partial H_0(\vec{x}(t), \vec{p}(t), \vec{\lambda})}{\partial t} - \sum_i \int_0^\tau dt \left( \dot{x}_i(t) \frac{\partial H_a(\vec{x}(t), \vec{p}(t), t)}{\partial x_i} + \dot{p}_i(t) \frac{\partial H_a(\vec{x}(t), \vec{p}(t), t)}{\partial p_i} \right).$$

(20)
In the above derivations, we have used integration by part and considered the boundary conditions in Eq. (19). Taking an ensemble average over the trajectory work \( w \) in Eq. (20), we obtain the mean work as

\[
W = \langle w \rangle_{\xi} = \iint D[\vec{x}(t)|D[\vec{p}(t)]|\mathcal{T}[\vec{x}(t), \vec{p}(t)]w[\vec{x}(t), \vec{p}(t)]
\]

\[
= \iint D[\vec{x}(t)|D[\vec{p}(t)]|\mathcal{T}[\vec{x}(t), \vec{p}(t)] \int_{0}^{\tau} dt \iint d\vec{x}d\vec{p}\delta(\vec{x} - \bar{x}(t))\delta(\vec{p} - \bar{p}(t))
\times [\frac{\partial H_{o}(\bar{x}(t), \bar{p}(t), \bar{\lambda})}{\partial t} - \sum_{i}(\dot{x}_i(t)\frac{\partial H_{o}(\bar{x}(t), \bar{p}(t), t)}{\partial x_i} + \dot{p}_i(t)\frac{\partial H_{o}(\bar{x}(t), \bar{p}(t), t)}{\partial p_i})]
\]

\[
= \int_{0}^{\tau} dt \iint d\vec{x}d\vec{p}\frac{\partial H_{o}(\bar{x}(t), \bar{p}(t), \bar{\lambda})}{\partial t} \langle \delta(\vec{x} - \bar{x}(t))\delta(\vec{p} - \bar{p}(t)) \rangle_{\xi}
\]

\[
- \sum_{i}(\frac{\partial H_{o}(\bar{x}, \bar{p}, t)}{\partial x_i} \langle \dot{x}_i(t)\delta(\vec{x} - \bar{x}(t))\delta(\vec{p} - \bar{p}(t)) \rangle_{\xi} + \frac{\partial H_{o}(\bar{x}, \bar{p}, t)}{\partial p_i} \langle \dot{p}_i(t)\delta(\vec{x} - \bar{x}(t))\delta(\vec{p} - \bar{p}(t)) \rangle_{\xi})
\]

\[
= \int_{0}^{\tau} dt \iint d\vec{x}d\vec{p}\frac{\partial H_{o}(\bar{x}(t), \bar{p}(t), \bar{\lambda})}{\partial t} P(\bar{x}, \bar{p}, t)
\]

\[
- \sum_{i}(\frac{\partial H_{o}(\bar{x}, \bar{p}, t)}{\partial x_i} \langle \dot{x}_i(t)\rho(\bar{x}, \bar{p}, t) \rangle_{\xi} + \frac{\partial H_{o}(\bar{x}, \bar{p}, t)}{\partial p_i} \langle \dot{p}_i(t)\rho(\bar{x}, \bar{p}, t) \rangle_{\xi})]
\]

(21)

In the shortcut scheme, the system keeps in the instantaneous equilibrium state, \( P = P_{eq} = \exp[\beta(F - H_{o})] \), which leads to \( 2 \Delta F = \int_{0}^{\tau} dt \iint d\vec{x}d\vec{p}P_{eq}\partial_{t}H_{o} \). The irreversible work \( W_{irr} = W - \Delta F \) then follows as

\[
W_{irr} = W - \iint d\vec{x}d\vec{p}\sum_{i} \int_{0}^{\tau} dt(\frac{\partial H_{o}(\bar{x}, \bar{p}, \bar{\lambda})}{\partial t} \langle \dot{x}_i(t)\rho(\bar{x}, \bar{p}, t) \rangle_{\xi} + \frac{\partial H_{o}(\bar{x}, \bar{p}, t)}{\partial p_i} \langle \dot{p}_i(t)\rho(\bar{x}, \bar{p}, t) \rangle_{\xi})
\]

(22)

With Eq. (2), we can calculate \( \langle \dot{x}_i(t)\rho(\bar{x}, \bar{p}, t) \rangle_{\xi} \) and \( \langle \dot{p}_i(t)\rho(\bar{x}, \bar{p}, t) \rangle_{\xi} \) as

\[
\langle \dot{x}_i(t)\rho(\bar{x}, \bar{p}, t) \rangle_{\xi} = \iint D[\vec{x}(t)|D[\vec{p}(t)]|\mathcal{T}[\bar{x}(t), \bar{p}(t)]\dot{x}_i(t)\delta(\vec{x} - \bar{x}(t))\delta(\vec{p} - \bar{p}(t))
\]

\[
= \iint D[\vec{x}(t)|D[\vec{p}(t)]|\mathcal{T}[\bar{x}(t), \bar{p}(t)]\delta(\vec{x} - \bar{x}(t))\delta(\vec{p} - \bar{p}(t)) \frac{\partial H(\vec{x}(t), \vec{p}(t), t)}{\partial p_i}
\]

\[
= \frac{\partial H(\bar{x}, \bar{p}, t)}{\partial p_i} \iint D[\vec{x}(t)|D[\vec{p}(t)]|\mathcal{T}[\bar{x}(t), \bar{p}(t)]\delta(\vec{x} - \bar{x}(t))\delta(\vec{p} - \bar{p}(t))
\]

\[
= \frac{\partial H(\bar{x}, \bar{p}, t)}{\partial p_i} P(\bar{x}, \bar{p}, t),
\]

(23)

and

\[
\langle \dot{p}_i(t)\rho(\bar{x}, \bar{p}, t) \rangle_{\xi} = \iint D[\vec{x}(t)|D[\vec{p}(t)]|\mathcal{T}[\bar{x}(t), \bar{p}(t)]\dot{p}_i(t)\delta(\vec{x} - \bar{x}(t))\delta(\vec{p} - \bar{p}(t))
\]

\[
= \iint D[\vec{x}(t)|D[\vec{p}(t)]|\mathcal{T}[\bar{x}(t), \bar{p}(t)]\delta(\vec{x} - \bar{x}(t))\delta(\vec{p} - \bar{p}(t)) (-\frac{\partial H(\vec{x}(t), \vec{p}(t), t)}{\partial x_i}) - \gamma \dot{x}_i(t) + \xi_i(t)
\]

\[
= -\gamma \frac{\partial H(\bar{x}, \bar{p}, t)}{\partial x_i} \rho(\bar{x}, \bar{p}, t) + \langle \xi_i(t)\rho(\bar{x}, \bar{p}, t) \rangle_{\xi}
\]

(24)
Since the stochastic force $\xi(t)$ commutes with the deterministic operator $\hat{L}_d(t)$, we have
\[
\langle \xi_i(t) \rho(\vec{x}, \vec{p}, t) \rangle_{\xi} = \langle \xi_i(t) \hat{R}(t) \phi(\vec{x}, \vec{p}, 0) \rangle_{\xi}
\]
\[
= \hat{R}(t) \langle \xi_i(t) \phi(\vec{x}, \vec{p}, t) \rangle_{\xi}
\]
\[
= \hat{R}(t) \langle \xi_i(t) \sum_{n=0}^{\infty} (-1)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \hat{O}(t_n) \hat{O}(t_{n-1}) \cdots \hat{O}(t_1) \rangle_{\xi} \phi(\vec{x}, \vec{p}, 0)
\]
\[
= \hat{R}(t) \langle \sum_{n=0}^{\infty} (-1)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 \hat{O}(t_n) \hat{O}(t_{n-1}) \cdots \hat{O}(t_1) \rangle_{\xi} \phi(\vec{x}, \vec{p}, 0)
\]
\[
= -\hat{R}(t) \gamma (k_B T) \hat{R}^{-1}(t) \frac{\partial}{\partial \rho_i} \hat{R}(t) \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^t dt_{n-1} \cdots \int_0^{t_2} dt_1 \hat{O}(t_n) \hat{O}(t_{n-1}) \cdots \hat{O}(t_1) \rangle_{\xi} \phi(\vec{x}, \vec{p}, 0)
\]
\[
= -\gamma k_B T \frac{\partial}{\partial \rho_i} \hat{R}(t) \langle \phi(\vec{x}, \vec{p}, t) \rangle_{\xi}
\]
\[
= -\gamma k_B T \frac{\partial}{\partial \rho_i} P(\vec{x}, \vec{p}, t).
\]
Combining Eqs. (23), (24), and (25), we obtain
\[
W_{irr} = -\sum_i \int_0^t dt \int d\vec{x} d\vec{p} \left[ \frac{\partial H_a}{\partial \rho_i} \frac{\partial H}{\partial \rho_i} P_{eq} - \frac{\partial H_a}{\partial \rho_i} \frac{\partial H}{\partial \rho_i} P_{eq} + \gamma \frac{\partial H}{\partial \rho_i} P_{eq} + \gamma k_B T \frac{\partial P_{eq}}{\partial \rho_i} \right]
\]
\[
= -\sum_i \int_0^t dt \int d\vec{x} d\vec{p} \left[ \frac{1}{\beta} \frac{\partial H_a}{\partial \rho_i} P_{eq} + \frac{1}{\beta} \frac{\partial H_a}{\partial \rho_i} P_{eq} - \gamma \frac{\partial H_a}{\partial \rho_i} P_{eq} \right]
\]
\[
= -\sum_i \int_0^t dt \int d\vec{x} d\vec{p} \left[ \frac{1}{\beta} \frac{\partial H_a}{\partial \rho_i} P_{eq} + \frac{1}{\beta} \frac{\partial H_a}{\partial \rho_i} P_{eq} - \gamma \frac{\partial H_a}{\partial \rho_i} P_{eq} \right]
\]
\[
= \gamma \sum_{\mu\nu} \int_0^t dt \lambda_{\mu} \lambda_{\nu} \int d\vec{x} d\vec{p} \frac{\partial f_{\mu}}{\partial \rho_i} \frac{\partial f_{\nu}}{\partial \rho_i} P_{eq}
\]
\[
= \sum_{\mu\nu} \int_0^t dt \lambda_{\mu} \lambda_{\nu} g_{\mu\nu},
\]
with the metric
\[
g_{\mu\nu} = \gamma \sum_i \int d\vec{x} d\vec{p} \frac{\partial f_{\mu}}{\partial \rho_i} \frac{\partial f_{\nu}}{\partial \rho_i} P_{eq} = \gamma \sum_i \left\langle \frac{\partial f_{\mu}}{\partial \rho_i} \frac{\partial f_{\nu}}{\partial \rho_i} \right\rangle_{eq}.
\]
In the derivations of Eq. (26), we have used integration by part and assumed that the boundary term
\[
\sum_i \left. \frac{\partial H_a}{\partial x_i} P_{eq} \right|_{x_i=\pm\infty} = \sum_i \left. \frac{\partial H_a}{\partial x_i} P_{eq} \right|_{x_i=\pm\infty} = 0,
\]
and
\[
\sum_i \left. \frac{\partial H_a}{\partial \rho_i} P_{eq} \right|_{\rho_i=\pm\infty} = \sum_i \left. \frac{\partial H_a}{\partial \rho_i} P_{eq} \right|_{\rho_i=\pm\infty} = 0.
\]
Equation (26) is equivalent to Eq. (7) in the main text.
B. Positive semi-definiteness of the metric $g_{\mu\nu}$

The positive semi-definiteness of the metric in Eq. (27) are guaranteed by the structure, $g_{\mu\nu} = \gamma \sum_i (\partial p_i f_{\mu} \partial p_i f_{\nu})_{\text{eq}}$. For any vector $\vec{v} \equiv (v_1, v_2, \cdots, v_M)$, we have

\[
\vec{v}^T g \vec{v} = \gamma \sum_{\mu\nu} v_\mu v_\nu \sum_i \left( \frac{\partial f_{\mu}}{\partial p_i} \frac{\partial f_{\nu}}{\partial p_i} \right)_{\text{eq}} \\
= \gamma \sum_i \left( (\sum_\mu v_\mu \frac{\partial f_{\mu}}{\partial p_i})(\sum_\nu v_\nu \frac{\partial f_{\nu}}{\partial p_i}) \right)_{\text{eq}} \\
= \gamma \sum_i \left( \sum_\mu v_\mu \frac{\partial f_{\mu}}{\partial p_i} \right)^2_{\text{eq}} \\
= \gamma \sum_i \int d\vec{x} d\vec{p} \sum_\mu v_\mu \frac{\partial f_{\mu}}{\partial p_i}^2.
\]

(28)

The integrand in Eq. (28) is non-negative, which ensures the non-negativity of $\vec{v}^T g \vec{v}$. Therefore, we prove that the metric in Eq. (27) is positive semi-definite.

C. The $1/\tau$ scaling of the irreversible work

It is natural to choose the function form of the protocol $\tilde{\lambda}(t) = \tilde{\Lambda}(t/\tau)$. After a change of variable $s \equiv t/\tau$, the control protocol $\tilde{\Lambda}(s)$ is independent of the protocol duration $\tau$. The irreversible work in Eq. (26) follows

\[
W_{\text{irr}} = \gamma \sum_{\mu\nu} \int_0^\tau dt \dot{\lambda}_\mu(t) \dot{\lambda}_\nu(t) \int d\vec{x} d\vec{p} \frac{\partial f_{\mu}}{\partial p_i} \frac{\partial f_{\nu}}{\partial p_i} P_{\text{eq}}(\vec{x}, \vec{p}, \tilde{\lambda}(t)) \\
= \gamma \sum_{\mu\nu} \int_0^1 ds \Lambda'_\mu(s) \Lambda'_\nu(s) \int d\vec{x} d\vec{p} \frac{\partial f_{\mu}}{\partial p_i} \frac{\partial f_{\nu}}{\partial p_i} P_{\text{eq}}(\vec{x}, \vec{p}, \tilde{\Lambda}(s), \tilde{\lambda}(s)),
\]

where the prime in $\Lambda'_\mu(s)$ represents the derivative of $\Lambda_\mu(s)$ with respect to $s$. The irreversible work $W_{\text{irr}}$ is inversely proportional to the protocol duration $\tau$.

IV. GENERAL FORMALISM: SOLVING THE GEODESIC EQUATION WITH SHOOTING METHOD

According to Eq. (26), the task of designing the optimal protocol in the shortcut scheme is converted to finding the geodesic path in the parameter space. The geodesic path is obtained by solving the geodesic equation

\[
\ddot{\lambda}_\mu + \frac{1}{2} \sum_{\nu \kappa} (g^{-1})_{\mu\nu}(g_{\nu\kappa} \frac{\partial g_{\kappa}}{\partial \lambda_\nu} + g_{\nu\kappa} \frac{\partial g_{\kappa}}{\partial \lambda_\nu} - g_{\nu\kappa} \frac{\partial g_{\kappa}}{\partial \lambda_\nu} \frac{\partial g_{\kappa}}{\partial \lambda_\nu}) \dot{\lambda}_\mu \dot{\lambda}_\nu = 0,
\]

(29)

with boundary conditions $\tilde{\lambda}(0) = \tilde{\lambda}_0$, $\tilde{\lambda}(\tau) = \tilde{\lambda}_\tau$, and $\tilde{\lambda}(0) = \dot{\tilde{\lambda}}(0) = 0$. Shooting method is one of the popular tools that treats the above two-point boundary values problem as an initial value problem [5]. Specifically, the shooting method solves the initial value problem

\[
\dot{\lambda}_\mu = y_\mu(t, \vec{x}, \vec{p}, \dot{\lambda}) \equiv \frac{1}{2} \sum_{\nu \kappa} (g^{-1})_{\mu\nu}(\frac{\partial g_{\nu\kappa}}{\partial \lambda_\nu} - \frac{\partial g_{\nu\kappa}}{\partial \lambda_\nu} - \frac{\partial g_{\nu\kappa}}{\partial \lambda_\nu}) \dot{\lambda}_\nu \dot{\lambda}_\kappa,
\]

(30)

with the initial conditions $\dot{\lambda}(0) = \vec{d}$ and $\dot{\lambda}(0+) = \vec{d}$. We remark here that the first order derivation $\dot{\lambda}(0)$ is not continuous at $t = 0$, noticing $\dot{\lambda}(0) = 0$. The initial rate $\vec{d}$ is updated until the solution of Eq. (30) satisfies the boundary condition $\tilde{\lambda}(\tau) = \tilde{\lambda}_\tau$. The shooting method can be realized by using the Euler algorithm to solve Eq. (30) and Newton’s method [6] to approach the final condition $\dot{\lambda}(\tau) = \dot{\lambda}_\tau$. To update the initial rate $\vec{d}$, we treat the protocol
as a function of the initial rate, i.e., $\tilde{X}(t, \tilde{d})$, and define $z_{\mu\nu}(t, \tilde{d}) \equiv \partial_{d\nu}\lambda_{\mu}(t, \tilde{d})$. At the final time $\tau$, it follows that in Newton’s method, the solution of the equation $\tilde{X} = \tilde{X}(\tau, \tilde{d})$ is approximated as the solution of the equation

$$\tilde{X} \approx \tilde{X}(\tau, \tilde{d}^{(k)}) + z(\tau, \tilde{d}^{(k)})(\tilde{d}^{(k+1)} - \tilde{d}^{(k)}),$$

(31)

where $\tilde{d}^{(k)}$ represents the current iteration and $\tilde{d}^{(k+1)}$ represents the next iteration. Rearranging Eq. (31) yields

$$\tilde{d}^{(k+1)} = \tilde{d}^{(k)} + z^{-1}(\tau, \tilde{d}^{(k)})(\tilde{X} - \tilde{X}(\tau, \tilde{d}^{(k)})),$$

(32)

which gives the process to obtain each new iteration $\tilde{d}^{(k+1)}$ from the previous iteration $\tilde{d}^{(k)}$. Here, $z^{-1}(\tau, \tilde{d}^{(k)})$ in Eq. (32) is obtained by solving the evolution equation as

$$\tilde{z}_{\mu\nu}(t, \tilde{d}) = \frac{\partial \tilde{X}_{\mu}}{\partial d_{\nu}} = \frac{\partial y_{\mu}}{\partial d_{\nu}}$$

$$= \sum \kappa \frac{\partial y_{\mu}}{\partial \lambda_{\kappa}} \frac{\partial \lambda_{\kappa}}{\partial d_{\nu}} + \frac{\partial y_{\mu}}{\partial \lambda_{\kappa}} \frac{\partial \lambda_{\kappa}}{\partial d_{\nu}}$$

$$= \sum \kappa \frac{\partial y_{\mu}}{\partial \lambda_{\kappa}} \tilde{z}_{\kappa\nu} + \frac{\partial y_{\mu}}{\partial \lambda_{\kappa}} \tilde{z}_{\kappa\nu},$$

(33)

which is derived by taking derivative of Eq. (30) over $\tilde{d}$. The accompanied initial conditions follow as $z_{\mu\nu}(0, \tilde{d}) = 0$ and $\tilde{z}_{\mu\nu}(0+, \tilde{d}) = \delta_{\mu\nu}$. The shooting method to solve the geodesic equation (29) is summarized as follows. Firstly, choosing a proper initial rate $\tilde{d}^{(1)}$, we solve Eqs. (30) and (33) to obtain the first iteration $\tilde{X}(\tau, \tilde{d}^{(1)})$ and $z(\tau, \tilde{d}^{(1)})$. Secondly, we get the updated rate $\tilde{d}^{(2)}$ by using Eq. (32) and repeat the first step to solve for the next iteration $\tilde{X}(\tau, \tilde{d}^{(2)})$ and $z(\tau, \tilde{d}^{(2)})$. The iterator finally stops in the $k$th iteration when $|\tilde{X} - \tilde{X}(\tau, \tilde{d}^{(k)})| < \epsilon$ with $\epsilon$ representing the termination precision. Then, the solution of the geodesic equation (29) is $\tilde{X}(t, \tilde{d}^{(k)})$.

V. EXAMPLE: UNDERDAMPED BROWNIAN MOTION

A. The auxiliary Hamiltonian for an one-dimensional system

We consider an underdamped Brownian particle system in an one-dimensional harmonic potential with the Hamiltonian

$$H_o(x, p, \tilde{X}) = \frac{p^2}{2} + U_o(x, \tilde{X}),$$

(34)

where $U_o(x, \tilde{X}) = \lambda_1(t)x^2/2 - \lambda_2(t)x$ is a controllable potential. The auxiliary Hamiltonian $H_a = H_a(x, p, t)$ follows

$$\left( \frac{dF}{d\lambda_1} - \frac{\partial U_o}{\partial \lambda_1} \right) \dot{\lambda}_1 + \left( \frac{dF}{d\lambda_2} - \frac{\partial U_o}{\partial \lambda_2} \right) \dot{\lambda}_2 = \frac{\gamma}{\beta} \frac{\partial^2 H_a}{\partial p^2} - \frac{\gamma p}{\beta} \frac{\partial H_a}{\partial p} + \frac{\partial U_o}{\partial p} \frac{\partial H_a}{\partial x} - \frac{p}{\beta} \frac{\partial H_a}{\partial x}.$$  

(35)

The function $f_1$ and $f_2$ in the auxiliary Hamiltonian $U_a(x, p, t) = \dot{\lambda}_1 f_1(x, p, \lambda_1, \lambda_2) + \dot{\lambda}_2 f_2(x, p, \lambda_1, \lambda_2)$ satisfy the following equations

$$\frac{\gamma}{\beta} \frac{\partial^2 f_1}{\partial p^2} - \frac{\gamma p}{\beta} \frac{\partial f_1}{\partial p} + \frac{\partial U_o}{\partial x} \frac{\partial f_1}{\partial p} - \frac{\partial f_1}{\partial x} = \frac{dF}{d\lambda_1} - \frac{\partial U_o}{\partial \lambda_1},$$

(36)

and

$$\frac{\gamma}{\beta} \frac{\partial^2 f_2}{\partial p^2} - \frac{\gamma p}{\beta} \frac{\partial f_2}{\partial p} + \frac{\partial U_o}{\partial x} \frac{\partial f_2}{\partial p} - \frac{\partial f_2}{\partial x} = \frac{dF}{d\lambda_2} - \frac{\partial U_o}{\partial \lambda_2}.$$  

(37)

By assuming that $f_1 = a_1(t)p^2 + a_2(t)xp + a_3(t)p + a_4(t)x^2 + a_5(t)x$, we can exactly derive the form

$$f_1(x, p, \lambda_1, \lambda_2) = \frac{1}{4\gamma \lambda_1}[(p - \gamma x)^2 + \lambda_1 x^2] - \frac{\lambda_2 p}{2\lambda_1} + \left( \frac{\gamma \lambda_2}{2\lambda_1} - \frac{\lambda_2}{2\gamma \lambda_1} \right)x.$$  

(38)
With similar derivations, we can obtain
\[ f_2(x, p, \lambda_1, \lambda_2) = \frac{p}{\lambda_1} - \frac{\gamma x}{\lambda_1}. \]  
(39)

Therefore, the auxiliary Hamiltonian takes the form
\[ H_a(x, p, t) = \lambda_1 \{ \frac{1}{4\gamma \lambda_1} [(p - \gamma x)^2 + \lambda_1 x^2] - \frac{\lambda_2 p}{2\lambda_1^2} + \left( \frac{\gamma \lambda_2}{2\lambda_1} - \frac{\lambda_2}{2\gamma \lambda_1} \right)x \} + \dot{\lambda}_2 \left( \frac{p}{\lambda_1} - \frac{\gamma x}{\lambda_1} \right). \]  
(40)

**B. The geodesic protocol for an underdamped Brownian particle system**

The metric in Eq. (27) for the current underdamped Brownian motion takes the form
\[ g = \left( \begin{array}{ccc} \frac{1}{4\beta\gamma \lambda_1^2} & + & \frac{\gamma \lambda_2^2}{\lambda_1^2} & - & \frac{2\lambda_2}{\lambda_1^2} \\ -\frac{\gamma \lambda_2^2}{\lambda_1^2} & + & \frac{\gamma \lambda_2^2}{\lambda_1^2} & - & \frac{2\lambda_2}{\lambda_1^2} \\ \end{array} \right), \]
which results in the geodesic equations for the minimal work as
\[ \ddot{\lambda}_1 - \frac{\lambda_1^2}{\lambda_1} \frac{(3\gamma^2 + 2\lambda_1)}{2\lambda_1(\gamma^2 + \lambda_1)} = 0, \]
\[ \ddot{\lambda}_2 - \frac{2\lambda_1 \dot{\lambda}_2}{\lambda_1} - \frac{\lambda_2^2}{\lambda_1} \frac{2(2\lambda_1 + \gamma^2)}{2\lambda_1^2(\gamma^2 + \lambda_1)} = 0. \]  
(41)

Two boundary conditions \( \dot{\lambda}(0) = \ddot{\lambda}(0) = \dddot{\lambda}(\tau) = \dddot{\lambda}(\tau) \) are accompanied with the geodesic equation. We first solve Eq. (41) using the shooting method mentioned above. The geodesic equation (41) can be rewritten as
\[ \dot{\lambda}_1 = y_1(t, \ddot{\lambda}, \lambda) \equiv \frac{\lambda_1^2(2\lambda_1 + 3\gamma^2)}{2\lambda_1(\gamma^2 + \lambda_1)}, \]
\[ \dot{\lambda}_2 = y_2(t, \ddot{\lambda}, \lambda) \equiv \frac{2\lambda_1 \dot{\lambda}_2}{\lambda_1} - \frac{\lambda_2^2}{\lambda_1} \frac{2(2\lambda_1 + \gamma^2)}{2\lambda_1^2(\gamma^2 + \lambda_1)}. \]  
(42)

As shown in Sec. IV, the algorithm to solve Eq. (42) proceeds as follows:

In the simulation, we set the parameters as \( \lambda(0) = (1, 1), \lambda(\tau) = (16, 2), k_B T = 1, \) and \( \gamma = 1. \) The initial rate is chosen as \( \dot{\lambda} = (1, 1) \) , the operation time is \( \tau = 1 \) with the time step \( \delta t = 10^{-3} \) and the termination precision is set as \( \epsilon = 10^{-4} \). The simulation results are presented in Fig. 2 of the main text.

Fortunately, the geodesic equation (41) can also be solved analytically. Substituting the auxiliary Hamiltonian in Eq. (40) into the irreversible work in Eq. (26), we obtain
\[ W_{\text{irr}} = \frac{\lambda_1^2}{4\beta\gamma \lambda_1^2} + \frac{\gamma \lambda_2^2}{4\beta \lambda_1^2} + \frac{\gamma \lambda_2^2}{2\lambda_1^2} \frac{\lambda_1^2}{\lambda_1^2} - \frac{2\gamma \lambda_1 \lambda_2 \lambda_2}{\lambda_1^2} + \frac{2\gamma \lambda_2^2}{\lambda_1^2} \]
\[ = \frac{\lambda_1^2}{4\beta \lambda_1^2} \left( \lambda_1 + \gamma^2 \right) + \gamma \left( \frac{\lambda_1 \lambda_2}{\lambda_1^2} - \frac{\lambda_2 \lambda_1^2}{\lambda_1^2} \right)^2 \]
\[ = \frac{\lambda_2^2}{4\beta \lambda_1^2} \left( \lambda_1 + \gamma^2 \right) + \gamma \frac{d}{dt} \left( \frac{\lambda_2}{\lambda_1} \right)^2. \]  
(43)

We can simplify the expression in Eq. (43) with a new set of parameters,
\[ \dot{\gamma} \equiv \dot{\lambda}_1 \sqrt{\frac{\lambda_1 + \gamma^2}{4\beta \lambda_1^2}}, \dot{\lambda}_2 \equiv d \frac{dt}{d t} \left( \sqrt{\frac{\lambda_2}{\lambda_1}} \right), \]  
(44)
and the irreversible work follows \( W_{\text{irr}} = \dot{\gamma}^2 + \dot{\lambda}_2^2 \), indicating a flatten manifold in the geometric space. The corresponding geodesic equation follows \( \dot{\gamma} = 0 \) and \( \dot{\lambda}_2 = 0 \) which gives
\[ \dot{\lambda}_1 = \frac{w_b}{\tau} \sqrt{\frac{\lambda_1^3}{\lambda_1 + \gamma^2}}, \]
\[ \frac{\lambda_2}{\lambda_1} = \frac{m_b t}{\tau} + n_b, \]  
(45)
Algorithm 1: Shooting method

Data: Choose $\tilde{\lambda}(0^+)=\tilde{d}(k)$, with $k=0$; Choose $\delta t$ such that $N \delta t = \tau$ where $N$ is the number of steps.

Result: Optimal control protocol $\lambda(\tau)$;

1. while $(|\tilde{\lambda}' - \tilde{\lambda}(\tau)| > \epsilon)$ do
2. \hspace{1em} $k = k + 1$;
3. \hspace{1em} $\lambda(0) = \tilde{\lambda}^{0}$, $\dot{\tilde{\lambda}}(0) = \tilde{d}(k)$, $z(0) = 0$, $\dot{z}_{11}(0) = 0$, $\dot{z}_{22}(0) = 0$;
4. \hspace{1em} for $m = 0, 1, 2, \ldots, N - 1$ do
5. \hspace{2em} for $\mu = 1, 2$ do
6. \hspace{3em} $\lambda_{\mu}(m+1)\delta t = \lambda_{\mu}(m\delta t) + \dot{\lambda}_{\mu}(m\delta t)\delta t$;
7. \hspace{2em} end
8. \hspace{2em} end
9. \hspace{2em} $z_{\mu}(m+1)\delta t = z_{\mu}(m\delta t) + \dot{z}_{\mu}(m\delta t)\delta t$;
10. \hspace{1em} end
11. $d_{1}(k+1) = d_{1}(k) + \frac{\lambda_{1}'(\tau)z_{22} - (\lambda_{1}'(\tau)z_{22})_{12}}{s_{1,1}^{2} + s_{2,1}^{2}}$;
12. $d_{2}(k+1) = d_{2}(k) + \frac{\lambda_{2}'(\tau)z_{22} - (\lambda_{2}'(\tau)z_{22})_{12}}{s_{2,1}^{2} + s_{1,1}^{2}}$;
13. end

where the parameters $w_b = -[2\sqrt{1 + \gamma^2/\lambda} + \ln(\sqrt{1 + \gamma^2/\lambda} - 1) - \ln(\sqrt{1 + \gamma^2/\lambda} + 1)]\lambda_{1}(\tau)$, $m_b = (\lambda_{2}(\tau)\lambda_{1}(0) - \lambda_{2}(0)\lambda_{1}(\tau))/(\lambda_{1}(\tau)\lambda_{1}(0))$, and $n_b = \lambda_{2}(0)/\lambda_{1}(0)$. The final geodesic protocol is shown as Fig. 2 in the main text.

C. The stochastic simulations

The motion of the Brownian particle is governed by the Langevin equation as
\[
\dot{x} = \frac{\partial H_0}{\partial p} + \frac{\partial H_a}{\partial x}, \quad \dot{p} = -\frac{\partial H_0}{\partial x} - \frac{\partial H_a}{\partial p} - \gamma \dot{x} + \xi(t),
\] (46)
where $\xi$ represents the standard Gaussian white noise satisfying $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2\gamma k_b T \delta(t-t')$. We introduce the characteristic length $l_c \equiv (k_b T \lambda(1))^{1/2}$, the characteristic times $\tau_1 = m/\gamma$ and $\tau_2 = \tau/\lambda(1)$ to define the dimensionless coordinate $\tilde{x} \equiv x/l_c$, momentum $\tilde{p} \equiv p r/(m l_c)$, time $s \equiv t/\tau$, and the control protocol $\tilde{\lambda} \equiv \lambda/(k l_c^2)$. The dimensionless Langevin equation follows
\[
\dot{\tilde{x}}' = \tilde{p} + \alpha \tau^2 \frac{\partial H_a}{\partial \tilde{p}},
\]
\[
\dot{\tilde{p}}' = -\alpha \tau^2 \frac{\partial H_0}{\partial \tilde{x}} - \alpha \tau^2 \frac{\partial H_a}{\partial x} - \tilde{\tau} \tilde{x} + \tilde{\tau} \sqrt{2\alpha \tau \tilde{\zeta}(s)},
\] (47)
where $\tilde{\tau} \equiv \tau/\tau_1$ and $\alpha \equiv \tau_1/\tau_2$. The prime represents the derivative with respect to dimensionless time $s$. $\tilde{\zeta}(s)$ is a Gaussian white noise satisfying $\langle \tilde{\zeta}(s) \rangle = 0$ and $\langle \tilde{\zeta}(s_1)\tilde{\zeta}(s_2) \rangle = \delta(s_1 - s_2)$. The Hamiltonian $H_0$ and $H_a$ are rewritten with the dimensionless parameters as
\[
\tilde{H}_0(\tilde{x}, s) = \frac{1}{\alpha \tau^2} \frac{\tilde{p}^2}{2} + \frac{1}{2} \tilde{\lambda}_1 \tilde{x}^2 - \tilde{\lambda}_2 \tilde{x}
\] (48)
and
\[
\tilde{H}_a(\tilde{x}, \tilde{p}, s) = \frac{\tilde{\lambda}_1'}{4 \tilde{\tau} \tilde{\lambda}_1} \left[ \frac{1}{\alpha \tau^2} (\tilde{p} - \tilde{\tau} \tilde{x})^2 + \tilde{\lambda}_1 \tilde{x}^2 \right]
\]
\[
- \frac{\tilde{\lambda}_1 \tilde{\lambda}_2}{2 \alpha \tau^2 \tilde{\lambda}_1^2} (\tilde{p} + \tilde{\tau} \tilde{x} - \alpha \tilde{\tau} \tilde{\lambda}_1 \tilde{x}) + \tilde{\lambda}_2' \left( \frac{\tilde{p}}{\alpha \tau^2 \tilde{\lambda}_1} - \frac{\tilde{x}}{\alpha \tilde{\tau} \tilde{\lambda}_1} \right).
\] (49)
We solve the Langevin equation (47) by using the Euler algorithm as
\[
\tilde{x}(s + \delta s) = \tilde{x}(s) + \tilde{p}\delta s + \alpha\tau^2 \frac{\partial \tilde{H}_a}{\partial \tilde{p}} \delta s,
\]
\[
\tilde{p}(s + \delta s) = \tilde{p}(s) - \alpha\tau^2 \frac{\partial \tilde{H}_a}{\partial \tilde{x}} \delta s - \alpha\tau^2 \frac{\partial \tilde{H}_b}{\partial \tilde{x}} \delta s - \tilde{\tau}(\tilde{p} + \alpha\tau^2 \frac{\partial \tilde{H}_a}{\partial \tilde{p}}) \delta s + \tilde{\tau}\sqrt{2\alpha\tau\delta s}\theta(s),
\]
where \(\delta s\) is the time step and \(\theta(s)\) is a random number sampled from Gaussian distribution with zero mean and unit variance. The trajectory work of the system takes
\[
\tilde{w} \equiv \frac{w}{k_B T} = \int_0^1 \left( \frac{\partial \tilde{H}_o}{\partial s} + \frac{\partial \tilde{H}_a}{\partial s} \right) ds \\
\approx \sum \left( \frac{\partial \tilde{H}_o}{\partial s} + \frac{\partial \tilde{H}_a}{\partial s} \right) \delta s.
\]
In the simulation, we have chosen the parameters \(\tilde{\lambda}(0) = (1,1), \tilde{\lambda}(\tau) = (16,2), k_B T = 1, \gamma = 1, \) and \(m = 1\). The mean work is obtained as the ensemble average over the trajectory work of \(10^5\) stochastic trajectories. We perform the simulation for different duration \(\tau \in \{0.1, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}\).

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