SHEVA–SHEVA–SHEVA: LARGE CREATURES

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Abstract. We develop the theory of the forcing with trees and creatures for an inaccessible \( \lambda \) continuing Roslanowski and Shelah \[14\], \[13\]. To make a real use of these forcing notions (that is to iterate them without collapsing cardinals) we need suitable iteration theorems, and those are proved as well. (In this aspect we continue Roslanowski and Shelah \[15\] and Shelah \[16\], \[17\].)

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0. Introduction

The present paper has two themes.

The first is related to the quest for the right generalization of properness to higher cardinals (that is, for a property of forcing notions that would play in iterations with uncountable supports similar role to that of standard properness in CS iterations). The evidence that there is no straightforward generalization of properness to larger cardinals was given already in Shelah [18] (see [19, Appendix 3.6(2)]). Substantial progress has been achieved in Shelah [16], [17], but the properties there were tailored for generalizing the case no new reals of [19, Ch. V]. Then Rosłanowski and Shelah [15] gave an iterable condition for not collapsing $\lambda^+$ in $\lambda$–support iterations of $\lambda$–complete forcing notions (with possibly adding subsets of $\lambda$). Very recently Eisworth [6] has given another property preserved in $\lambda$–support iterations (and implying that $\lambda^+$ is not collapsed). At the moment it is not clear if the two properties (the one of [15] and that of [6]) are equivalent, though they have similar flavour. However, the existing iterable properties still do not cover many examples of natural forcing notions, specially those which come naturally in the context of $\lambda$–reals. This brings us to the second theme: developing the forcing for $\lambda$–reals.

A number of cardinal characteristics related to the Baire space $\omega^\omega$, the Cantor space $\omega^2$ and/or the combinatorial structure of $[\omega]^{<\omega}$ can be extended to the spaces $\lambda\lambda$, $\lambda^2$ and $[\lambda]^\lambda$ for any infinite cardinal $\lambda$. Following the tradition of Set Theory of the Reals we may call cardinal numbers defined this way for $\lambda\lambda$ (and related spaces) cardinal characteristics of $\lambda$–reals. The menagerie of those characteristics seems to be much larger than the one for the continuum. But to decide if the various definitions lead to different (and interesting) cardinals we need a well developed forcing technology.

There has been a serious interest in cardinal characteristics of the $\lambda$–reals in literature. For example, Cummings and Shelah [5] investigates the natural generalizations $b_\lambda$ of the unbounded number and the dominating number $d_\lambda$, giving simple constraints on the triple of cardinals $(b_\lambda, d_\lambda, 2^{\lambda})$ and proving that any triple of cardinals obeying these constraints can be realized. In a somewhat parallel work [20], Shelah and Spasojević study $b_\lambda$ and the generalization $t_\lambda$ of the tower number. Zapletal [21] investigated the splitting number $\mathfrak{s}_\lambda$ – here the situation is really complicated as the inequality $\mathfrak{s}_\lambda > \lambda^+$ needs large cardinals. One of the sources of interest in characteristics of the $\lambda$–reals is their relevance for our understanding of the club filter on $\lambda$ (or the dual ideal on non-stationary subsets of $\lambda$) – see, e.g., Balcar and Simon [2, §5], Landver [9], Matet and Pawlikowski [10], Matet, Rosłanowski and Shelah [11]. First steps toward developing forcing for $\lambda$–reals has been done long time ago: in 1980 Kanamori [8] presented a systematic treatment of the $\lambda$–perfect–set forcing in products and iterations. Recently, Brown [3, 4] discussed the $\lambda$–superperfect forcing and other tree–like forcing notions.

Our aim in this paper is to provide tools for building forcing notions relevant for $\lambda$–reals (continuing in this Rosłanowski and Shelah [14], [13]) and give suitable iteration theorems (thus continuing Rosłanowski and Shelah [15]). However, we restrict our attention to the case when $\lambda$ is a strongly inaccessible uncountable cardinal (after all, $\aleph_0$ is inaccessible), see 0.3 below.

The structure of the paper is as follows. It is divided into two parts, first one presents iteration theorems, the second one gives examples and applications. In Section A.1 we present some basic notions and methods relevant for iterating
\(\lambda\)–complete forcing notions. The next section, A.2, presents preservation of \(\lambda\)–analogue of the Sacks property (in Theorem A.2.3) as well as preservation of being \(\lambda\lambda\)–bounding (in Theorem A.2.7). Section A.3 introduces fuzzy properness, a more complicated variant of properness over semi-diamonds from [15]. Of course, we prove a suitable iteration theorem (see Theorem A.3.10). Then we give examples for the properties discussed in Part A. We start with showing that a forcing notion useful for uniformization is fuzzy proper (in Section B.4), and then we turn to developing forcing notions built with the use of trees and creatures. In Section B.5 we set the terminology and notation, and in the next section we discuss when the resulting forcing notions have the two bounding properties discussed in §A.2. Section B.7 shows how our methods result in suitably proper forcing notions, and the last section introduces some new characteristics of the \(\lambda\)–reals.

**Notation**

Our notation is rather standard and compatible with that of classical textbooks (like Jech [7]). In forcing we keep the older convention that a stronger condition is the larger one. Our main conventions are listed below.

**Notation 0.1.**

1. For a forcing notion \(\mathbb{P}\), \(\Gamma_\mathbb{P}\) stands for the canonical \(\mathbb{P}\)–name for the generic filter in \(\mathbb{P}\). With this one exception, all \(\mathbb{P}\)–names for objects in the extension via \(\mathbb{P}\) will be denoted with a tilde below (e.g., \(\tau, X\)). The weakest element of \(\mathbb{P}\) will be denoted by \(\emptyset\) and we will always assume that there is one, and that there is no other condition equivalent to it. We will also assume that all forcing notions under considerations are atomless.

2. By “\(\lambda\)–support iterations” we mean iterations in which domains of conditions are of size \(\leq \lambda\). However, we will pretend that conditions in a \(\lambda\)–support iteration \(\bar{Q} = \langle P_\zeta, Q_\zeta : \zeta < \zeta^* \rangle\) are total functions on \(\zeta^*\) and for \(p \in \text{lim}(\bar{Q})\) and \(\alpha \in \zeta^* \setminus \text{Dom}(p)\) we will let \(p(\alpha) = \emptyset_{Q_\alpha}\).

3. For a filter \(D\) on \(\lambda\), the family of all \(D\)–positive subsets of \(\lambda\) is called \(D^+\). (So \(A \in D^+\) if and only if \(A \subseteq \lambda\) and \(A \cap B \neq \emptyset\) for all \(B \in D\).)

The club filter of \(\lambda\) is denoted by \(\mathcal{D}_\lambda\).

4. Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet (\(\alpha, \beta, \gamma, \delta \ldots\)) and also by \(i, j\) (with possible sub- and superscripts).

Cardinal numbers will be called \(\theta, \kappa, \lambda, \mu\) (with possible sub- and superscripts); \(\lambda\) is a fixed inaccessible cardinal (see 0.3).

5. By \(\chi\) we will denote a sufficiently large regular cardinal; \(\mathcal{H}(\chi)\) is the family of all sets hereditarily of size less than \(\chi\). Moreover, we fix a well ordering \(\preceq^*\) of \(\mathcal{H}(\chi)\).

6. A bar above a letter denotes that the object considered is a sequence; usually \(\bar{X}\) will be \(\langle X_i : i < \zeta\rangle\), where \(\zeta\) is the length \(lh(\bar{X})\) of \(\bar{X}\). Sometimes our sequences will be indexed by a set of ordinals, say \(S \subseteq \lambda\), and then \(\bar{X}\) will typically be \(\langle X_\delta : \delta \in S\rangle\).

But also, \(\eta, \nu\) and \(\rho\) (with possible sub- and superscripts) will denote sequences (nodes in quasi trees).

For two sequences \(\eta, \nu\) we write \(\nu \preceq \eta\) whenever \(\nu\) is a proper initial segment of \(\eta\), and \(\nu \preceq \eta\) when either \(\nu \preceq \eta\) or \(\nu = \eta\).

7. We will consider several games of two players. One player will be called Generic or Complete or just I player, and we will refer to this player as “she”. Her opponent will be called Antigeneric or Incomplete or just II player and will be referred to as “he”.

**Notation**

Our notation is rather standard and compatible with that of classical textbooks (like Jech [7]). In forcing we keep the older convention that a stronger condition is the larger one. Our main conventions are listed below.
Definition 0.2. (1) A $\lambda$–quasi tree is a set $T$ of sequences of length $<\lambda$ with the $\prec$–smallest element denoted by root($T$).
(2) A $\lambda$–quasi tree $T$ is a $\lambda$–tree if it is closed under initial segments longer then $\text{lh(root}(T))$.
(3) A $\lambda$–quasi tree $T$ is complete if the union of any $\prec$–increasing sequence of length less than $\lambda$ of members of $T$ is in $T$.
(4) For a $\lambda$–quasi tree $T$ and $\eta \in T$ we define the successors of $\eta$ in $T$, maximal points of $T$, the restriction of $T$ to $\eta$, and the height of $T$ by:
\[
\text{succ}_T(\eta) = \{ \nu \in T : \eta \prec \nu \& \neg(\exists \rho \in T)(\eta \prec \rho \prec \nu) \},
\]
\[
\text{max}(T) = \{ \nu \in T : \text{there is no } \rho \in T \text{ such that } \nu \prec \rho \},
\]
\[
T[\eta] = \{ \nu \in T : \eta \preceq \nu \}, \text{ and } \text{ht}(T) = \sup\{ \text{lh}(\eta) : \eta \in T \}.
\]
We put $\hat{T} = T \setminus \text{max}(T)$.
(5) For $\delta < \lambda$ and a $\lambda$–quasi tree $T$ we let
\[
(T)_\delta = \{ \eta \in T : \text{lh}(\eta) = \delta \} \text{ and } (T)_{<\delta} = \{ \eta \in T : \text{lh}(\eta) < \delta \}.
\]
The set of all limit $\lambda$–branches through $T$ is
\[
\text{lim}_\lambda(T) \overset{\text{def}}{=} \{ \eta : \eta \text{ is a } \lambda\text{–sequence and } (\forall \beta < \lambda)(\exists \alpha > \beta)(\eta[\alpha \in T]) \}.
\]
(6) A subset $F$ of a $\lambda$–quasi tree $T$ is a front of $T$ if no two distinct members of $F$ are $\prec$–comparable and
\[
(\forall \eta \in \text{lim}_\lambda(T) \cup \text{max}(T))(\exists \alpha < \lambda)(\eta[\alpha \in F]).
\]
Note that if $T$ is a complete $\lambda$–quasi tree of height $< \lambda$, then max($T$) is a front of $T$ and every $\prec$–increasing sequence of members of $T$ has a $\prec$–upper bound in max($T$).

In the present paper we assume the following.

Context 0.3. (a) $\lambda$ is a strongly inaccessible cardinal,
(b) $\bar{\lambda} = \langle \lambda_\alpha : \alpha < \lambda \rangle$ is a strictly increasing sequence of uncountable regular cardinals, $\sup_{\alpha < \lambda} \lambda_\alpha = \lambda$,
(c) for each $\alpha < \lambda$,
\[
\prod_{\beta < \alpha} \lambda_\beta < \lambda_\alpha \text{ and } (\forall \xi < \lambda_\alpha)(|\xi|^\alpha < \lambda_\alpha).
\]
A.1. Iterations of Complete Forcing Notions and Trees of Conditions

In this section we recall some basic definitions and facts concerning complete forcing notions and \( \lambda \)-support iterations.

**Definition A.1.1.** Let \( \mathbb{P} \) be a forcing notion.

1. For a condition \( r \in \mathbb{P} \) and a set \( S \subseteq \lambda \), let \( \mathcal{G}_0^\lambda(\mathbb{P}, S, r) \) be the following game of two players, Complete and Incomplete:
   
   - the game lasts \( \lambda \) moves and during a play the players construct a sequence \( \langle (p_i, q_i) : i < \lambda \rangle \) of pairs of conditions from \( \mathbb{P} \) in such a way that \( (\forall j < i < \lambda)(r \leq p_j \leq q_j \leq p_i) \) and at the stage \( i < \lambda \) of the game:
     - if \( i \in S \), then Complete chooses \( p_i \) and Incomplete chooses \( q_i \), and
     - if \( i \not\in S \), then Incomplete chooses \( p_i \) and Complete chooses \( q_i \).
   
   Complete wins if and only if for every \( i < \lambda \) there are legal moves for both players.

2. We say that the forcing notion \( \mathbb{P} \) is \((\lambda, S)\)-strategically complete if Complete has a winning strategy in the game \( \mathcal{G}_0^\lambda(\mathbb{P}, S, r) \) for each condition \( r \in \mathbb{P} \). We say that \( \mathbb{P} \) is strategically \((<\lambda)\)-complete if it is \((\lambda, \emptyset)\)-strategically complete.

3. We say that \( \mathbb{P} \) is \( \lambda \)-complete if every \( \leq \mathbb{P} \)-increasing chain of length less than \( \lambda \) has an upper bound in \( \mathbb{P} \).

4. Let \( N \prec (\mathbb{H}(\chi), \in, ^\prec) \) be a model such that \( ^\prec N \subseteq N \), \( |N| = \lambda \) and \( \mathbb{P} \not\in N \). We say that a condition \( p \in N \) is \((N, \mathbb{P})\)-generic in the standard sense (or just: \((N, \mathbb{P})\)-generic) if for every \( \mathbb{P} \)-name \( \tau \in N \) for an ordinal we have \( p \Vdash \" \tau \in N \" \).

5. \( \mathbb{P} \) is \( \lambda \)-proper in the standard sense (or just: \( \lambda \)-proper) if there is an \( \mathbb{P} \)-name \( \tau \in ^\prec N \) such that for every model \( N \prec (\mathbb{H}(\chi), \in, ^\prec) \) satisfying
   
   \[ ^\prec N \subseteq N, \ |N| = \lambda \quad \text{and} \quad \mathbb{P}, x \in N, \]

   and every condition \( q \in N \cap \mathbb{P} \) there is an \((N, \mathbb{P})\)-generic condition \( p \in \mathbb{P} \) stronger than \( q \).

**Remark A.1.2.**

1. On strategic completeness (and variants) see [16, §A.1]. Plainly, \( \lambda \)-completeness implies strategic \((<\lambda)\)-completeness.

2. Note that if \( \mathbb{P} \) is strategically \((<\lambda)\)-complete and \( D \) is a normal filter on \( \lambda \), then in \( V^\mathbb{P} \) the filter \( D \) generates a proper normal filter on \( \lambda \). [Abusing notation, we may call this filter also by \( D \).]

**Proposition A.1.3.** Suppose that \( \mathbb{P} \) is a \((<\lambda)\)-strategically complete (atomless) forcing notion, \( \alpha^* < \lambda \) and \( q_\alpha \in \mathbb{P} \) (for \( \alpha < \alpha^* \)). Then there are conditions \( p_\alpha \in \mathbb{P} \) (for \( \alpha < \alpha^* \)) such that \( q_\alpha \leq p_\alpha \) and for distinct \( \alpha, \alpha' < \alpha^* \) the conditions \( p_\alpha, p_\alpha' \) are incompatible.
Proof: For \( \alpha < \alpha^* \) let \( \text{st}_\alpha \) be the winning strategy of Complete in the game \( \mathcal{D}_\lambda^\alpha(\mathbb{P}, \emptyset, q_\alpha) \). By induction on \( i < \alpha^* \) we define conditions \( q^i_\alpha, p^i_\alpha \) as follows:

\[
p^0_\alpha = q_\alpha, q^0_\alpha = \text{the answer of Complete to } \langle p^0_\alpha \rangle \text{ according to } \text{st}_\alpha, q^0_\alpha = p^0_\alpha = q_\alpha \text{ for } \alpha > 0.
\]

Suppose that conditions \( p^j_\alpha, q^j_\alpha \) have been defined for \( j < i \), \( \alpha < \alpha^* \) (where \( i < \alpha^* \)) so that

\[
(\alpha) \ (\forall \alpha < \alpha' < i)(q^{\alpha'}_\alpha, q^{\alpha'}_\alpha \text{ are incompatible }),
\]

\[
(\beta) \text{ for each } \alpha < i, (\langle p^j_\alpha, q^j_\alpha \rangle : \alpha \leq j < i) \text{ is a play of } \mathcal{D}_\lambda^\alpha(\mathbb{P}, \emptyset, q_\alpha) \text{ in which Complete uses the strategy } \text{st}_\alpha, \text{ and }
\]

\[
(\gamma) \ p^i_\alpha = q^i_\alpha = q_\alpha \text{ for } \alpha \geq i > j.
\]

For \( \alpha < i \) let \( r_\alpha \) be a condition stronger than all \( q^j_\alpha \) for \( j < i \) (there is one by (\( \beta \))). If every \( r_\alpha \) (for \( \alpha < i \)) is incompatible with \( q_i \), then we let \( p^i_\alpha = r_\alpha \) for \( \alpha < i, p^i_\alpha = q_\alpha \) for \( \alpha \geq i \). Otherwise, let \( \alpha_0 < i \) be the first such that \( r_{\alpha_0}, q_i \) are compatible. Then we may pick two incompatible conditions \( p^i_{\alpha_0}, p^i_\alpha \) above both \( r_{\alpha_0} \) and \( q_i \). Next we let \( p^i_\alpha = r_\alpha \) for \( \alpha < i, \alpha \neq \alpha_0 \) and \( p^i_{\alpha_0} = q_\alpha \) for \( \alpha > i \). Finally, for \( \alpha \leq i, q^i_\alpha \) is defined as the answer of Complete according to \( \text{st}_\alpha \) to \( \langle (p^j_\alpha, q^j_\alpha) : j < i \rangle \sim (p^i_\alpha), \rangle \) and \( q^i_\alpha = q_\alpha \) for \( \alpha > i \).

After the inductive definition is carried out we may pick upper bounds \( p_\alpha \) to \( \langle q^j_\alpha : j < \alpha^* \rangle \) (for \( \alpha < \alpha^* \); exist by (\( \beta \))). The conditions \( p_\alpha \) are pairwise incompatible by (\( \alpha \)), so we are done. \( \square \)

Both completeness and strategic completeness are preserved in iterations:

**Proposition A.1.4.** Suppose that \( \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \zeta^* \rangle \) is a \( \lambda \)-support iteration such that for each \( \alpha < \zeta^* \)

\[
\models_{\mathbb{P}_\alpha} \text{" } \mathbb{Q}_\alpha \text{ is } \lambda \text{-complete."}
\]

Then the forcing \( \mathbb{P}_{\zeta^*} \) is \( \lambda \)-complete.

**Proposition A.1.5.** Suppose \( \mathbb{Q} = \langle \mathbb{P}_i, \mathbb{Q}_i : i < \gamma \rangle \) is a \( \lambda \)-support iteration and for each \( i < \gamma \)

\[
\models_{\mathbb{P}_i} \text{" } \mathbb{Q}_i \text{ is strategically } (<\lambda)\text{-complete }".
\]

Then:

(a) \( \mathbb{P}_\gamma \) is strategically \( (<\lambda)\)-complete.

(b) Moreover, for each \( \varepsilon \leq \gamma \) and \( r \in \mathbb{P}_\varepsilon \) there is a winning strategy \( \text{st}(\varepsilon, r) \) of Complete in the game \( \mathcal{D}_\lambda^\varepsilon(\mathbb{P}_\varepsilon, \emptyset, r) \) such that, whenever \( \varepsilon_0 < \varepsilon_1 \leq \gamma \) and \( r \in \mathbb{P}_{\varepsilon_1} \), we have:

(i) if \( \langle (p_i, q_i) : i < \lambda \rangle \) is a play of \( \mathcal{D}_\lambda^\varepsilon(\mathbb{P}_{\varepsilon_0}, \emptyset, r \varepsilon_0) \) in which Complete follows the strategy \( \text{st}(\varepsilon_0, r \varepsilon_0) \), then \( \langle (p_i \upharpoonright [\varepsilon_0, \varepsilon_1], q_i \upharpoonright [\varepsilon_0, \varepsilon_1]) : i < \lambda \rangle \) is a play of \( \mathcal{D}_\lambda^{\varepsilon_1}(\mathbb{P}_{\varepsilon_1}, \emptyset, r \varepsilon_1) \) in which Complete uses \( \text{st}(\varepsilon_1, r) \);

(ii) if \( \langle (p_i, q_i) : i < \lambda \rangle \) is a play of \( \mathcal{D}_\lambda^\varepsilon(\mathbb{P}_{\varepsilon_1}, \emptyset, r) \) in which Complete plays according to the strategy \( \text{st}(\varepsilon_1, r) \), then \( \langle (p_i \varepsilon_0, q_i \varepsilon_0) : i < \lambda \rangle \) is a play of \( \mathcal{D}_\lambda^\varepsilon(\mathbb{P}_{\varepsilon_0}, \emptyset, r \varepsilon_0) \) in which Complete uses \( \text{st}(\varepsilon_0, r) \);

(iii) if \( \langle (p_i, q_i) : i < i^* \rangle \) is a partial play of \( \mathcal{D}_\lambda^\varepsilon(\mathbb{P}_{i^*}, \emptyset, r) \) in which Complete uses \( \text{st}(\varepsilon_1, r) \) and \( p^i \in \mathbb{P}_{\varepsilon_0} \) is stronger than all \( p_i \varepsilon_0 \) (for \( i < i^* \)), then there is \( p^* \in \mathbb{P}_{\varepsilon_1} \) such that \( p^i = p^* | \varepsilon_0 \) and \( p^* \geq p_i \) for \( i < i^* \).
Definition A.1.6 (Compare [10, A.3.3, A.3.2]).

1. Let $\alpha, \gamma$ be ordinals, $\emptyset \neq w \subseteq \gamma$. A standard $(w, \alpha) \gamma$–tree is a pair $T = (T, \text{rk})$ such that
   - $\text{rk} : T \to w \cup \{\gamma\}$,
   - if $t \in T$ and $\text{rk}(t) = \varepsilon$, then $t$ is a sequence $\langle (t)_{\zeta} : \zeta \in w \cap \varepsilon \rangle$, where each $(t)_{\zeta}$ is a sequence of length $\alpha$,
   - $(T, \prec)$ is a tree with root $\langle \rangle$ and such that every chain in $T$ has a $\prec$–upper bound it $T$.

   We will keep the convention that $T_\gamma^w$ is $(T_\gamma^w, \text{rk}_\gamma^w)$.

2. Suppose that $w_0 \subseteq w_1 \subseteq \gamma$, $\alpha_0 \leq \alpha_1$, and $T_1 = (T_1, \text{rk}_1)$ is a standard $(w_1, \alpha_1) \gamma$–tree. The projection $\text{proj}_{(w_0, \alpha_0)}^\gamma(w_1, \alpha_1)$ of $T_1$ onto $(w_0, \alpha_0)$ is defined as a standard $(w_0, \alpha_0) \gamma$–tree $T_0 = (T_0, \text{rk}_0)$ such that

   $T_0 = \{(t)_\zeta : \zeta \in w_0 \cap \text{rk}_1(t) : t = \langle (t)_{\zeta} : \zeta \in w_1 \cap \text{rk}_1(t) \rangle \in T_1\}$.

   The mapping

   $T_1 \ni \langle (t)_{\zeta} : \zeta \in w_1 \cap \text{rk}_1(t) \rangle \mapsto \langle (t)_{\zeta} : \zeta \in w_0 \cap \text{rk}_1(t) \rangle \in T_0$

   will be denoted $\text{proj}_{(w_0, \alpha_0)}^\gamma(w_1, \alpha_1)$ too.

3. We say that $\bar{T} = \langle T_\alpha : \alpha < \alpha^* \rangle$ is a legal sequence of $\gamma$–trees if for some increasing continuous sequence $\vec{w} = \langle w_\alpha : \alpha < \alpha^* \rangle$ of subsets of $\gamma$ we have
   - (i) $T_\alpha$ is a standard $(w_\alpha, \alpha) \gamma$–tree (for $\alpha < \alpha^*$),
   - (ii) if $\alpha < \beta < \alpha^*$, then $T_\alpha = \text{proj}_{(w_\alpha, \alpha)}^\gamma(T_\beta)$.

4. Suppose that $\bar{T} = \langle T_\alpha : \alpha < \alpha^* \rangle$ is a legal sequence of $\gamma$–trees and $\alpha^*$ is a limit ordinal. Let $w_\alpha \subseteq \gamma$ be such that $T_\alpha$ is a standard $(w_\alpha, \alpha) \gamma$–tree (for $\alpha < \alpha^*$) and let $w = \bigcup_{\alpha < \alpha^*} w_\alpha$. The inverse limit $\text{lim}^{-1}(\bar{T})$ of $\bar{T}$ is a standard $(w, \alpha^*) \gamma$–tree $(T_{\lim}, \text{rk}^\gamma_{\lim})$ such that

   - $(\otimes)$ $T_{\lim}$ consists of all sequences $t$ satisfying
     - (i) $\text{Dom}(t)$ is an initial segment of $w$ (not necessarily proper),
     - (ii) if $\zeta \in \text{Dom}(t)$, then $(t)_{\zeta}$ is a sequence of length $\alpha^*$,
     - (iii) $\langle (t)_{\zeta} : \zeta \in w_\alpha \cap \text{Dom}(t) \rangle \in T_\alpha$ for each $\alpha < \alpha^*$.

5. A legal sequence $\bar{T} = \langle T_\alpha : \alpha < \alpha^* \rangle$ is continuous if for each limit ordinal $\beta < \alpha^*$, $\beta = \lim_{\tau}(\bar{T})$.

6. Let $\bar{Q} = \langle Q_i, \delta_i : i < \gamma \rangle$ be a $\lambda$–support iteration. A standard tree of conditions in $\bar{Q}$ is a system $\bar{p} = \langle p_i : t \in T \rangle$ such that
   - $(T, \text{rk})$ is a standard $(w, \alpha) \gamma$–tree for some $w \subseteq \gamma$ and an ordinal $\alpha$,
   - $p_t \in \mathbb{P}_{\text{rk}(t)}$ for $t \in T$, and
   - if $s, t \in T$, $s \prec t$, then $p_s = p_t | \text{rk}(s)$.

7. Let $\bar{p}^0, \bar{p}^1$ be standard trees of conditions in $\bar{Q}$, $\bar{p}^0 = \langle p_i^0 : t \in T_i \rangle$, where $T_0 = \text{proj}_{(w_0, \alpha_0)}^\gamma(T_1), w_0 \subseteq w_1 \subseteq \gamma, \alpha_0 < \alpha_1$. We will write $\bar{p}^0 \trianglelefteq_{(w_0, \alpha_0)} \bar{p}^1$ (or just $\bar{p}^0 \trianglelefteq \bar{p}^1$) whenever for each $t \in T_1$, letting $t' = \text{proj}_{(w_1, \alpha_1)}^\gamma(t) \in T_0$, we have $p^0_t | \text{rk}(k(t)) \leq p^1_{t'}$.

Remark A.1.7. Concerning Definition A.1.6, note that even though $T_{\lim}$ could be empty, it does satisfy requirements of $\text{lim}^\gamma(\bar{T})$ (so $\text{lim}^{-1}(\bar{T})$ is indeed a standard $(w, \alpha^*) \gamma$–tree). Also, if the sequence $\bar{T}$ is continuous (and $T_\alpha$’s are not empty), then $T_{\lim} \neq \emptyset$. 

Proposition A.1.8. Assume that $\mathcal{Q} = \langle P_i, \mathcal{Q}_i : i < \gamma \rangle$ is a $\lambda$–support iteration such that for all $i < \gamma$ we have

\[ \| P_i \| < \langle \lambda \rangle \] is strategically ($\langle \lambda \rangle$)-complete.

Suppose that $\bar{p} = \langle p_t : t \in T \rangle$ is a standard tree of conditions in $\mathcal{Q}$, $|T| < \lambda$, and $T \subseteq P_\gamma$ is open dense. Then there is a standard tree of conditions $\bar{q} = \langle q_t : t \in T \rangle$ such that $\bar{p} \leq \bar{q}$ and $(\forall t \in T)(rk(t) = \gamma \Rightarrow q_t \in T)$. 

Proof. For $\varepsilon \leq \gamma$ and $r \in \mathbb{P}_\varepsilon$, let $st(\varepsilon, r)$ be a winning strategy of Complete in $\mathcal{Q}^0_\varepsilon(\mathbb{P}_\varepsilon, \emptyset, r)$ as in A.1.3(b). Let

\[ T^{\text{max}} \overset{\text{df}}{=} \{ t \in T : (\exists t' \in T)(t < t') \} = \{ t_\zeta : \zeta < \kappa \} \]

(where $\kappa < \lambda$ is a cardinal). We construct partial plays $\langle p_\zeta^i, q_\zeta^i : i \leq \kappa \rangle$ of $\mathcal{Q}^0_\varepsilon(\mathbb{P}_{rk(p_\zeta)}, \emptyset, p_\zeta)$ (for $\zeta < \kappa$) in which Complete uses strategy $st(rk(p_\zeta), p_\zeta)$ and such that

\[ (\alpha) \text{ if } \zeta < \kappa \text{ and } rk(p_\zeta) = \gamma, \text{ then } p_\zeta^\zeta \in T, \]

\[ (\beta) \text{ if } t < t_\zeta, t < t_\zeta, t_\zeta, \zeta < \kappa, i \leq \kappa, \text{ then } p_\zeta^i | rk(t) = p_\zeta^\zeta | rk(t) \text{ and } q_\zeta^i | rk(t) = q_\zeta^\zeta | rk(t). \]

So suppose we have defined $p_j^\zeta, q_j^\zeta$ for $\zeta < \kappa$, $j < i < \kappa$. First we look at $\langle p_j^\zeta, q_j^\zeta \rangle : j < i$ – it is a play of $\mathcal{Q}^0_\varepsilon(\mathbb{P}_{rk(p_\zeta)}, \emptyset, p_\zeta)$ in which Complete uses $st(rk(p_\zeta), p_\zeta)$, so we may find a condition $p_i^\zeta \in \mathbb{P}_{rk(p_\zeta)}$ stronger than all $p_j^\zeta, q_j^\zeta$ for $j < i$, and such that $rk(p_\zeta) = \gamma \Rightarrow p_i^\zeta \in T$. Next, for $\zeta < \kappa$, $\zeta \neq i$, we define $p_i^\zeta$ as follows. Let $t \in T$ be such that $t < t_\zeta$, $t < t_i$ and $rk(t)$ is the largest possible. We declare that

\[ \text{Dom}(p_i^\zeta) = (\text{Dom}(p_j^\zeta) \cap rk(t)) \cup \bigcup_{j < i} \text{Dom}(q_j^\zeta) \cup \text{Dom}(p_\zeta), \]

and $p_i^\zeta | rk(t) = p_i^\zeta | rk(t)$, and for $\varepsilon \in [rk(t), \gamma]$ we have that $p_i^\zeta(\varepsilon)$ is the $<_{\zeta}^+\text{–first}$ $\mathbb{P}_\varepsilon$-name for a member of $\mathcal{Q}_\varepsilon$ such that

\[ p_i^\zeta | \varepsilon \overset{\text{df}}{=} \text{“ } p_i^\zeta(\varepsilon) \text{ is an upper bound to } \{ p_\zeta(\varepsilon) \} \cup \{ q_j^\zeta(\varepsilon) : j < i \} \text{ ”}. \]

The definition of $p_i^\zeta$’s is correct by A.1.3(b)(iii+ii). Also, by the choice of “the $<_{\zeta}^+\text{–first}$ names” and clause (\beta) at earlier stages we get clause (\beta) for $p_i^\zeta$’s.

Finally we define $q_i^\zeta$ (for $\zeta < \kappa$) as the condition given to Complete by $st(rk(t_\zeta), p_\zeta)$ in answer to $\langle (p_j^\zeta, q_j^\zeta) : j < i \rangle^\zeta$. (Again, one easily verifies (\alpha), (\beta).

The conditions $p_\zeta^\zeta, q_\zeta^\zeta$ are chosen in a similar manner except that we do not have to worry about entering $T$ anymore, so we may take $p_\zeta^\zeta$ to be any bound to the previously defined conditions $p_j^\zeta, q_j^\zeta$, and other $p_\zeta^\zeta, q_\zeta^\zeta$ are defined as earlier.

After the above construction is carried out, for $t \in T$ we let

\[ q_t = p_\zeta^\zeta | rk(t) \text{ for some (equivalently: all) } \zeta < \kappa \text{ such that } t \leq t_\zeta. \]

It should be clear that $\bar{q} = \langle q_t : t \in T \rangle$ is as required. \qed

Let us close this section with recalling an important result on easy ensuring that $\lambda$–support iteration satisfies the $\lambda^{++}$–cc. Its proof is a fairly straightforward modification of the proof of the respective result for CS iterations; see [19], Thm 4.1], Abraham [11, §2] for the CS case, Eisworth [6, §3] for the general case of $\lambda$–support iterations.
**Theorem A.1.9.** Assume $2^\lambda = \lambda^+$, $\lambda^{<\lambda} = \lambda$. Let $\mathcal{Q} = \langle \mathbb{P}_i, \check{Q}_i : i < \lambda^{++} \rangle$ be $\lambda$–support iteration such that for all $i < \lambda^{++}$ we have

- $\mathbb{P}_i$ is $\lambda$–proper,
- $\Vdash_{\mathbb{P}_i}$ “$|\check{Q}_i| \leq \lambda^+$”.

Then the limit $\mathbb{P}_{\lambda^{++}}$ satisfies the $\lambda^{++}$–cc.

### A.2. Bounding properties

The results on preservation in CS iterations of properties like the Sacks property and $\omega$–bounding property were among the earliest in the theory of proper forcing. Here we introduce relatives of these two properties for $\lambda$–reals and we show suitable iteration theorems. For both properties, the properness is “built into the property”.

Recall that $\lambda, \bar{\lambda}$ are assumed to be as specified in Context 1.5.

**Definition A.2.1.** Let $\mathbb{P}$ be a forcing notion.

1. For a condition $p \in \mathbb{P}$ and an ordinal $i_0 < \lambda$ we define a game $\mathcal{G}^{\text{Sacks}}_{\lambda}(i_0, p, \mathbb{P})$ of two players, the Generic player and the Antigeneric player. A play lasts $\lambda$ moves indexed by ordinals from the interval $[i_0, \lambda)$, and during it the players construct a sequence $\langle (s_i, \bar{q}_i, \bar{p}_i) : i_0 \leq i < \lambda \rangle$ as follows. At stage $i$ of the play (where $i_0 \leq i < \lambda$), first Generic chooses $s_i \subseteq \leq^{i+1}\lambda$ and a system $\bar{q}_i^i = \langle q_\eta^i : \eta \in s_i \cap i^{++} \rangle$ such that
   
   (a) $s_i$ is a complete $\lambda$–tree of height $i + 1$, and
   
   $$(\forall \eta \in s_i)(\exists \nu \in s_i)(\nu \subseteq \nu & \text{lh}(\nu) = i + 1),$$
   
   and $\text{lh}(\text{root}(s_i)) = i_0$,

   (b) for all $j$ such that $i_0 \leq j < i$ we have $s_j = s_i \cap \leq^{j+1}\lambda$,

   (c) $q_\eta^i \in \mathbb{P}$ for all $\eta \in s_i \cap i^{++}\lambda$ and

   (d) if $i_0 \leq j < i$, $\nu \in s_i \cap j^{++}\lambda$ and $\nu \subset \eta \in s_i \cap i^{++}\lambda$, then $p_\nu^j \leq q_\eta^i$ and $p \leq q_\eta^i$,

   (e) $|s_i \cap i^{++}\lambda| < \lambda_i$.

Then Antigeneric answers choosing a system $\bar{p}_i^i = \langle p_\eta^i : \eta \in s_i \cap i^{++}\lambda \rangle$ of conditions in $\mathbb{P}$ such that $q_\eta^i \leq p_\eta^i$ for each $\eta \in s_i \cap i^{++}\lambda$.

The Generic player wins a play if she always has legal moves (so the play really lasts $\lambda$ steps) and there are a condition $q \geq p$ and a $\mathbb{P}$–name $\rho$ such that

(\odot) \quad q \Vdash_{\mathbb{P}} “ \rho \in ^\lambda \lambda & (\forall i \in [i_0, \lambda)) (\rho(i + 1) \in s_i & q_\rho^i(i + 1) \in \Gamma_\rho) ”.$

2. We say that $\mathbb{P}$ has the strong $\bar{\lambda}$–Sacks property whenever the Generic player has a winning strategy in the game $\mathcal{G}^{\text{Sacks}}_{\lambda}(i_0, p, \mathbb{P})$ for any $i_0 < \lambda$ and $p \in \mathbb{P}$.

3. We say that $\mathbb{P}$ has the $\bar{\lambda}$–Sacks property if for every $p \in \mathbb{P}$ and a $\mathbb{P}$–name $\tau$ such that $p \Vdash \tau : \lambda \rightarrow V$, there are a condition $q \geq p$ and a sequence $\langle a_\alpha : \alpha < \lambda \rangle$ such that $|a_\alpha| < \lambda_\alpha$ (for $\alpha < \lambda$) and $q \Vdash “ (\forall \alpha < \lambda)(\tau(\alpha) \in a_\alpha) ”.$

**Remark A.2.2.** (1) At a stage $i < \lambda$ of a play of $\mathcal{G}^{\text{Sacks}}_{\lambda}(i_0, p, \mathbb{P})$, the Antigeneric player may play stronger conditions, and using A.1.3 we may require that if $\bar{p}_i = \langle p_\eta^i : \eta \in s_i \cap i^{++}\lambda \rangle$ is his move, then the conditions $p_\eta^i$ are pairwise incompatible. Thus the winning criterion (\odot) could be replaced by

(\odot)^{\cdot} \quad q \Vdash_{\mathbb{P}} “ (\forall i \in [i_0, \lambda)) (\exists \eta \in s_i \cap i^{++}\lambda) (q_\eta^i \in \Gamma_\rho) ”.$
(thus eliminating the use of $\rho$). However, the $\lambda$–branch along which the conditions are from the generic filter will be new (so we cannot replace the name $\rho$ by an object $\rho \in {}^\lambda \lambda$).

(2) Note that if Generic has a winning strategy in $\mathcal{D}_\lambda^{\text{Sacks}}(0, p, \mathbb{P})$, then she has one in $\mathcal{D}_\lambda^{\text{Sacks}}(i_0, p, \mathbb{P})$ for all $i_0 < \lambda$. (Remember: the sequence $\lambda$ is increasing.) The reason why we have $i_0$ as a parameter is a notational convenience.

(3) Plainly, if Generic has a winning strategy in $\mathcal{D}_\lambda^{\text{Sacks}}(i_0, p, \mathbb{P})$, then she has one with the following property:

(\text{nice}) if $s_i, q_i$ are given to Generic as a move at a stage $i \in [i_0, \lambda)$, then for every $\eta \in s_i \cap \lambda$, the set $\{ \alpha < \lambda : \eta^\check{}(\alpha) \in s_i \}$ is an initial segment of $\lambda_i$ and $\eta(j) = 0$ for all $j < i_0$.

Strategies satisfying the condition (\text{nice}) will be called \textit{nice}.

(4) Easily, if $\mathbb{P}$ has the strong $\lambda$–Sacks property, then it is strategically ($<\lambda$–complete and has the $\lambda$–Sacks property.

\textbf{Theorem A.2.3.} Suppose that $\bar{Q} = \langle P_\alpha, Q_\alpha : \alpha < \gamma \rangle$ is a $\lambda$–support iteration such that for all $\alpha < \gamma$:

$$\models_{P_\alpha} " Q_\alpha \text{ has the strong } \lambda \text{–Sacks property } ".$$

Then:

(a) $P_\gamma$ has the $\lambda$–Sacks property.

(b) If $N < (H(\lambda), \in, <^*_\lambda)$, $|N| = \lambda$, $\prec^\lambda N \subseteq N$ and $\lambda, \lambda, p, P, \gamma, \ldots \in N$, $p \in P_\gamma$, then there is an $(N, P_\gamma)$–generic condition $\tau \in P_\gamma$ stronger than $p$.

\textbf{Proof.} (a) First note that each $P_\alpha$ is strategically ($<\lambda$–complete (by \ref{A.1.5} remember \ref{A.2.4})), so our assumptions on $\lambda, \lambda$ hold in intermediate universes $V^{P_\alpha}$.

For $\alpha < \gamma$ and $i_0 < \lambda$ and a $P_\alpha$–name $q$ for a condition in $Q_\alpha$, let $\pi_{\alpha}(i_0, q)$ be the $<^*_\lambda$–first $P_\alpha$–name for a nice (see \ref{A.2.3}) winning strategy of the Generic player in the game $\mathcal{D}_\lambda^{\text{Sacks}}(i_0, q, Q_\alpha)$.

Let $\tau$ be a $P_\gamma$–name for a function from $\lambda$ to $\mathbb{V}$, $p \in P_\gamma$. Pick a model $N < (H(\lambda), \in, <^*_\lambda)$ such that

$$\lambda, \lambda, \tau, p, Q_\gamma, \ldots \in N, \text{ and } |N| = \lambda \text{ and } \prec^\lambda N \subseteq N.$$ 

Note that if $i_0 < \lambda$, $\alpha \in N \cap \gamma$, and $q \in N$ is a $P_\alpha$–name for a member of $Q_\alpha$, then $\pi_{\alpha}(i_0, q) \in N$. Also, as $\bar{Q}$ is a $\lambda$–support iteration of ($<\lambda$–strategically complete forcing notions, we may use $\mathcal{A}_{\lambda, \alpha}$ inside $N$ and for each $\varepsilon \in N \cap \gamma$ and $r \in P_r \cap N$ fix a winning strategy $\text{st}^*(\varepsilon, r) \in N$ of Complete in the game $\mathcal{D}_0^{\lambda}(P_\varepsilon, \emptyset, r)$ so that conditions (i)–(iii) of \ref{A.1.4} hold true.

Fix a list $\bar{I} = (I_\xi : \xi < \lambda)$ of all open dense subsets of $P_\gamma$ from $N$ and a one-to-one mapping $\pi : N \cap \gamma \rightarrow \lambda$. For $i < \lambda$ let $w_i = \pi^{-1}[i]$ (thus $\bar{w} = \langle w_i : i < \lambda \rangle$ is an increasing continuous sequence of subsets of $N \cap \gamma$, each of size $< \lambda$, and $\bigcup_{i<\lambda} w_i = N \cap \gamma$).

By induction on $i < \lambda$ we define sequences $\langle T_i : i < \lambda \rangle$ and $\langle \rho^i, \rho^i_0 : i < \lambda \rangle$ is not a limit ordinal \} such that the following requirements are satisfied.

(\alpha) $\langle T_i : i < \lambda \rangle$ is a continuous legal sequence of $\gamma$–trees; $T_0 \in N$ is a standard $(w_i, i)\gamma$–tree, $|T_i| < \lambda_i$, and for all $t \in T_i$ there is an $t' \in T_i$ such that $t \leq t'$ and $\text{rk}_i(t') = \gamma$.}
(β) For $i < \lambda$ and $t \in T_i$ such that $rk_i(t) < \gamma$ let $\psi_i(t) = \{ (s)_{rk_i(t)} : t < s \in T_i \}$. Then (for each $i, t$ as above) $\emptyset \neq \psi_i(t) \subseteq \prod_{j<i} \lambda_j$ and for each $\eta \in \psi_i(t)$ and

$$i' < \pi(rk_i(t))$$

we have $\eta(i') = 0$.

(γ) If $\xi \in N \cap \gamma$, $\pi(\xi) < i < j < \lambda$, $t \in T_j$, $rk_j(t) = \xi$ and $t' = \text{proj}^{w_{i,j}^{*}}(t) \in T_i$ (so $rk_i(t') = \xi$), then $\psi_i(t') = \{ \eta | i : \eta \in \psi_i(t) \}$.

(δ) $T_0 = \{ \{ \} \}$, $\bar{p}^0 = (p^0_0), \bar{p}^0_0 = p$, and for $i < \lambda$, $\bar{p}^{i+1} = (p^{i+1}_t : t \in T_{i+1})$ and $\bar{p}^{i+1}_s = (p^{i+1}_s : t : t \in T_{i+1})$ are standard trees of conditions in $\mathbb{Q}$, both belonging to $N$ and such that $\bar{p}^{i+1}_s \leq \bar{p}^{i+1}$.

(ε) If $i < j < \lambda$, then $\bar{p}^{i+1} \leq_{w_{i+1,j+1}^{*}} \bar{p}^{j+1}$.

(ζ) If $t_{i+1} \in T_{i+1}$ (for $i < \lambda$) are such that $rk_{i+1}(t_{i+1}) = \gamma$ and $t_{i+1} = \text{proj}^{w_{i+1,j+1}^{*}}(t_{i+1})$ (for $i < j$), then $(\bar{p}^{i+1}_s, t_{i+1}, p^{i+1}_i : i < \lambda)$ is a play of the game $\omega^\lambda_0(\mathbb{P}_\gamma, 0, p)$ in which Complete uses the strategy $st^*(\gamma, p)$.

(η) If $t \in T_{i+1}$, $rk_{i+1}(t) = \gamma$, then $p^{i+1}_t \in \mathbb{I}_\xi$ for all $\xi \leq i$ and $p^{i+1}_t$ forces a value to $\tau(i)$.

(θ) Assume that $\xi \in N \cap \gamma$, $\pi(\xi) = i_0$, and $t \in T_{i+1}$ is such that $rk_{i+1}(t) = \xi$. Let, for $j < i$, $t_j = \text{proj}^{w_{i,j}^{*}, i+1}(t)$ and let $r$ be the $<^\lambda_\gamma$-first $\mathbb{P}_\gamma$-name for a member of $\mathbb{Q}_\xi$ such that

$$\Vdash_{\mathbb{P}_\gamma} \text{"if there is a common upper bound to } \{ p^{j}_t(\xi) : j \leq i_0 \text{ is non-limit } \},$$

then $r$ is such an upper bound, else $r = p(\xi)$.

Furthermore, for $i_0 \leq j < i$ and $\eta \in \psi_{j+1}(t_{j+1})$, fix $s^{j+1}_t \in T_{j+1}$ such that $rk_{j+1}(s^{j+1}_t) > \xi$ and $(s^{j+1}_t) = \eta$, $t_{j+1} < s^{j+1}_t$, and put $p^{j+1}_t = p^{j+1}_{s^{j+1}_t}(\xi)$.

Then the condition $p^{i+1}_t$ forces in $\mathbb{P}_\gamma$ the following:

there is a partial play $(s_j, \bar{q}^j, \bar{r}^j : i_0 \leq j < i)$ of the game $\omega^\lambda_0(\mathbb{P}_\gamma, 0, \mathbb{Q}_\xi)$ in which the Generic player uses the strategy $st^*(i_0, r)$ and, for $i_0 \leq j < i$,

$$(s_j \cap \gamma, \gamma, \lambda = \psi_j(t_{j+1}) \text{ and } \bar{r}^j = (p^{j+1}_t) : \eta \in s_j \cap \gamma, \gamma, \lambda).$$

Let us describe how the construction of $(T_i : i < \lambda)$ and $(\bar{p}^{i+1} : i < \lambda)$ is carried out. We start with letting $T_0 = \{ \{ \} \}$, $p^0_0 = p$ (as in (δ)). Now suppose that we have defined $T_j, \bar{p}^j$ for $j < i < \lambda$ so that clauses (α)–(θ) are satisfied. If $i$ is a limit ordinal, then we let $T_i = \lim(T_j : j < i)$ in $N$ ($\bar{p}^i, \bar{p}_i$ are not defined). It is straightforward to verify conditions (α)–(γ) (use the inductive hypothesis), clauses (δ)–(θ) are not relevant.

So suppose now that $i$ is a successor ordinal, say $i = i_0 + 1$. First we let $T^*$ be a the largest standard $(w_i, i)^{\gamma}$-tree such that $\text{proj}^{w_i, i}(T^*) = T_{i_0}$, and if $t = \langle (t)_{\xi} : \xi \in w_i \cap rk^*(t) \rangle \in T^*$, then $(t)_{\xi}(i_0) \leq \lambda_{i_0}$, and if $\pi(\xi) = i_0$, then $(t)_{\xi}(i_0) \equiv 0$. (Plainly $T^* \subseteq N$ and $|T^*| < \lambda$. Next, for each $t \in T^*$ we define a condition $q_t \in \mathbb{P}_{rk^*(t)} \cap N$ and names $s^o(\xi)$ for ordinals (for $\xi \in w_i \cap rk^*(t)$). For this let us fix $t \in T^*$ and let $t_j = \text{proj}^{w_{i,j}^{*}}(t) \in T_j$ for $j < i$. Put

$$\text{Dom}(q_t) = (w_i \cup \bigcup \{ \text{Dom}(p^{j+1}_t) : j < i \text{ is not a limit } \}) \cap rk^*(t),$$
and for $\zeta \in \text{Dom}(q_t)$ let $q_t(\zeta)$ be a $P_\zeta$-name for a member of $Q_\zeta$ chosen as follows. If $\zeta \in \text{Dom}(q_t) \setminus w_1$, then $q_t(\zeta)$ is the $<^*_\chi$-first $P_\zeta$-name such that

\[ \models_{P_\zeta} " \text{if possible, then } q_t(\zeta) \text{ is an upper bound to } \{ p^j_t(\zeta) : j < i \text{ is non-limit } \} " \]

If $\zeta \in \text{Dom}(q_t) \cap w_1$, then $\alpha^t(\zeta) \in N$ is a $P_\zeta$-name for an element of $\lambda_i$ and $q_t(\zeta)$ is the $<^*_\chi$-first $P_\zeta$-name for a condition in $Q_\zeta$ with the following property.

Let $r$ be the $<^*_\chi$-first $P_\zeta$-name for a member of $Q_\zeta$ such that

\[ \models_{P_\zeta} " \text{if possible, then } r \text{ is an upper bound to } \{ p^j_t(\zeta) : j \leq \pi(\zeta) \text{ is non-limit } \}, \text{ else } r = p(\zeta) " \]

Now, suppose that $G_\zeta \subseteq P_\zeta$ is a generic filter over $V$ and $p^{j+1}_t | \zeta \in G_\zeta$ for all $j < i_0$, and work in $V[G_\zeta]$. Then, by clause $(\theta)$, there is a partial play $(s_j, q^j, \bar{\tau}^j : \pi(\zeta) \leq j < i_0)$ of the game $\mathcal{G}^{\text{Sacks}}(\pi(\zeta), t^{G_\zeta}_\zeta, q^{G_\zeta}_\zeta)$ in which Generic uses $s(t^{G_\zeta}_\zeta)$, and $s_j \cap j+1 = \psi_j(t_j+1) \zeta$ and $\bar{\tau}^j = (\bar{\tau}^j_\eta : \eta \in s_j \cap j+1)$, where $\bar{\tau}^j_\eta = (p^{j+1}_t(\zeta))^{G_\zeta}$ for $s^{j+1}_\eta \in T_{j+1}$ such that $t_{j+1} \zeta < s^{j+1}_\eta$, $(s^{j+1}_\eta) \zeta = \eta$ and $\text{rk}_{j+1}(s^{j+1}) = \gamma$. So we may look at the answer $s_{i_0}$, $\bar{q}^{i_0} = (q^{i_0}_\eta : \eta \in s_{i_0} \cap i_0+1)$ to this play according to the strategy $s(t^{G_\zeta}_\zeta)$. Then, $q_t(\zeta)^{G_\zeta}$ is a condition stronger than all $\bar{r}^j_{(j+1)}$ for $j < i_0$, and such that

\[ \text{if } (t_i)_{i < i_0}, \text{ then } q_t(\zeta)^{G_\zeta} = q^{i_0}_0(\zeta) \]

Also, $\alpha^t(\zeta) = \{ \lambda_i : (t_i)_{i < i_0} \zeta \sim (\alpha) \in s_{i_0} \}$.

If $\pi(\zeta) = i_0$, then we do not have the partial play we started with - the game just begins and we look at the first move of Generic, requiring that $q_t(\zeta)^{G_\zeta}$ is stronger than $r^{G_\zeta}$ and if $(t_i)_{i < i_0} \zeta \zeta$ then $q_t(\zeta)^{G_\zeta} = q^{i_0}_0(\zeta)$.

This finishes the definition of $\bar{q} = (q_t : t \in T^*)$. One easily checks that $\bar{q} \in N$ is a tree of conditions (remember the choice of “the $<^*_\chi$-first names”). Also, by induction on $\zeta \in \text{Dom}(q_t)$, one verifies that $\bar{p}^j \leq w_j, j \bar{q}$ for all non-limit $j < i_0$. (Note that if $\pi(\zeta) = i_0$, $t \in T^*$, and $\text{rk}(t) > \zeta$, then in the inductive process we know that by clause $(\zeta)$

\[ q_t(\zeta) \models_{P_\zeta} " \text{there is a common upper bound to } \{ p^j_t(\zeta) : j \leq \pi(\zeta) \text{ is non-limit } \} " \]

and thus $q_t(\zeta)$ forces that the respective condition $r$ is stronger than all $p^j_t(\zeta)$ (for non-limit $j \leq \pi(\zeta)$).

Next, we use $A_{1,8}$ to pick a standard tree of conditions $\bar{p}^* = (p^*_t : t \in T^*) \in N$ such that $\bar{q} \leq \bar{p}^*$ and for each $t \in T^*$ with $\text{rk}(t) = \gamma$ the condition $p^*_t$ decides the values of all names $\alpha^{t'}(\zeta)$ for $t' \in T^*$, $\zeta \in w_i \cap \text{rk}(t')$ and the value of $\tau(i_0)$ (and let $p^*_t \models " \tau(i_0) = \tau^t_{i_0} "$), and such that $p^*_t \in T_\zeta$ for all $\zeta \leq i_0$. For $t \in T^*$ with $\text{rk}(t) = \gamma$ and for $\zeta \in w_i$ let $\alpha^t(\zeta)$ be the value forced to $\alpha^t(t)$ by $p^*_t$. Since $\alpha^t(\zeta)$ is a $P_\zeta$-name, we have that

\[ t_0 \zeta t_1 \in T^* \& \text{rk}(t_1) = \gamma \& \zeta \in w_i \cap \text{rk}(t_0) \Rightarrow \quad p^*_t \models \alpha^{t_0}(\zeta) = \alpha^{t_1}(\zeta) \]

So we may naturally define $\alpha^t(\zeta)$ also for $t \in T^*$ with $\text{rk}(t) < \gamma$. Now we let

\[ T_i = T_{i+1} = \{ t \in T^* : (\forall \zeta \in w_i \cap \text{rk}(t))(t)_\zeta(\zeta) \in \alpha^t(\zeta) \}

and $p^*_t = p^*_t$ for $t \in T_i$ (thus defining $p^*_s$). Plainly, $T_i \in N$ is a standard $(w_i, i)^{\zeta}$-tree satisfying $(\alpha)\zeta^{\gamma}$, $p^*_s \in N$. Finally, using the properties of the strategies $st^*$
stated in [13.5] b) (and the clause (ζ) from earlier stages) we may pick a standard tree of conditions \( \vec{p} = \langle p_i^t : t \in T_t \rangle \) such that \( \vec{p}^* \leq \vec{p} \) and

if \( t \in T_i, \text{rk}_t(t) = \gamma, t_j = \text{proj}_{w, \iota}^{w, j}(t) \) for non-limit \( j \leq i, \)

then \( \langle p_i^{t+1} : j < i \rangle \) is a sequence of answers of Complete in some partial play of

\[ \mathcal{D}_0^\lambda(\mathbb{P}_\gamma, \emptyset, p) \]

in which she uses the winning strategy \( \text{st}^*(\gamma, p) \).

Now one easily verifies that \( T_t, \bar{p}^t, \bar{p}_* \) satisfy requirements (α)–(θ), thus the construction is complete.

Let \( T_\lambda = \lim (\langle T_j : j < \lambda \rangle) \). We will consider this standard \((N \cap \gamma, \lambda)^\gamma\)-tree in universes \( V^{\mathbb{P}_\xi} \) (for \( \xi \leq \gamma \)), so let us note that forcings \( \mathbb{P}_\xi \) may add new branches in \( T_\lambda \). But if \( (\in V^{\mathbb{P}_\xi}) t \in T_\lambda \) and \( i < \lambda \), then

\[ t|i = \text{proj}_{\mathbb{P}_\gamma}^{N \cap \gamma, \lambda}(t) \in V. \]

Also if \( i < \lambda \) is limit, then the equality \( T_i = \lim (\langle T_j : j < i \rangle) \) holds in \( V^{\mathbb{P}_\xi} \) as well.

We are going to define a condition \( r \in \mathbb{P}_\gamma \) such that \( \text{Dom}(r) = N \cap \gamma \) and the names \( r(\alpha) \) are defined by induction on \( \alpha \in N \cap \gamma \). For \( \alpha \in N \cap \gamma \) we will also choose \( \mathbb{P}_{\alpha + 1} \)-names \( t_\alpha \) for functions in \( \lambda^\lambda \), and we will put \( t^\alpha = \langle t_\beta : \beta < \alpha \land \beta \in N \rangle \). The construction will be carried out so that (for each \( \alpha \in N \cap (\gamma + 1) \)):

1. \( r|\alpha \models_{\mathbb{P}_\alpha} \bar{t}^\alpha \in T_\lambda \),
2. \( r|\alpha \models_{\mathbb{P}_\alpha} \forall i < \lambda \langle \bar{t}^\alpha | \gamma, \alpha \rangle \in \Gamma_{\mathbb{P}_\alpha} \).

Arriving at a limit stage \( \alpha \in N \cap (\gamma + 1) \), we have defined \( r|\alpha \) and \( t^\alpha \), and we should only check that conditions (i)_\alpha, (ii)_\alpha hold (assuming (i)_\beta, (ii)_\beta hold for \( \beta < \alpha, \beta \in N \)).

RE: (i)_\alpha: \( T_\lambda \) is a standard tree, so every chain in \( T_\lambda \) has a \( \prec \)-bound. Now, the first condition follows immediately from the inductive hypothesis.

RE: (ii)_\alpha: Suppose that \( G_\alpha \subseteq \mathbb{P}_\alpha \) is generic over \( V \) and \( r|\alpha \in G_\alpha \). Let \( i < \lambda \) and for \( \beta \leq \alpha \) let \( t_\beta = \langle t^\beta | i \rangle | G_\alpha \cap \mathbb{P}_\beta \in T_i \). Then, by (ii)_\beta, we know that \( \bar{p}^\beta_{t_{i+1}} | \beta \in G_\alpha \cap \mathbb{P}_\beta \) for each \( \beta \in (\alpha \cap N) \). But \( \bar{p}^\beta_{t_{i+1}} | \beta = \bar{p}^\beta_{t_{i+1}} | \beta \) (as \( t^\beta < t_{i+1}^\alpha \)), so remembering that \( \bar{p}^\beta_{t_{i+1}} \in G_\alpha \) we conclude \( \bar{p}^\beta_{t_{i+1}} | \alpha \in G_\alpha \).

Now suppose that we arrived at stage \( \alpha + 1 \in N \cap (\gamma + 1) \) and we have defined \( r|\alpha \), \( t^\alpha \) so that (i)_\alpha + (ii)_\alpha hold. Let \( G_\alpha \subseteq \mathbb{P}_\alpha \) be generic over \( V \), \( r|\alpha \in G_\alpha \). For \( i < \lambda \) let \( t^\alpha_i = \langle \bar{t}^\alpha_i | G_\alpha \rangle \in T_i \) (remember (i)_\alpha). Plainly, \( t^\alpha_j = \text{proj}_{w,j}^{w,j}(t^\alpha_i) \) for \( j < i < \lambda \). By (ii)_\alpha we get \( \bar{p}^\beta_{t_{i+1}} | \alpha \in G_\alpha \) for all \( i < \lambda \).

(∃)_\alpha Let \( i_0 = \pi(\alpha) \) and let \( r \) be the \( <_\lambda \)-first \( \mathbb{P}_\alpha \)-name for an element of \( Q_\alpha \) such that \( r \in V \), of course, and

\[ \models_{\mathbb{P}_\alpha} \text{ "if there is a common upper bound to } \langle p^\alpha_j(\alpha) : j \leq i_0 \text{ is non-limit } \rangle \text{ then } r = \text{p}(\alpha) " \).

(Note: for each \( j^* < \lambda \) the sequence \( \langle t^\alpha_j : j < j^* \rangle \) belongs to the ground model \( V \), and even to \( N \).)

Fix \( j^* < \lambda, j^* > i_0 \) for a moment. In \( V \), for each \( i \in [i_0, j^*] \) and \( \eta \in \psi_{i+1}(t^\alpha_{i+1}) \) let us choose \( s^i_{\eta^*} \in T_{i+1} \) such that \( t^\alpha_{i+1} < s^i_{\eta^*}, (s^i_{\eta^*})_\alpha = \eta \). Now work in \( V[G_\alpha] \).

Since \( \bar{p}^\beta_{t_{i+1}} \in G_\alpha \), we may use clause (θ) of the construction to claim that there is a partial play \( \bar{s}^\alpha = \langle s_i, \bar{q}_i^\beta, \bar{r}^j : i_0 \leq i < j^* \rangle \) of the game \( \mathcal{D}_0^\lambda(S^{\text{Sacks}}_{i_0, \bar{q}_i^\beta, \bar{r}^j}, (Q_\alpha)^{G_\alpha}) \) in
which Generic uses $s \triangleright (i_0, t^\alpha_0)$ and $s_i \triangleright \alpha^1 \lambda = \psi_i^1(t^\alpha_{i+1})$ and $\bar{p}^i = (p^i_{s^1_{n+1}}(\alpha))^{G_\alpha} : \eta \in s_i \triangleright \alpha^1 \lambda$.

It should be clear that (in $V[G_\alpha]$) $\bar{\sigma}^* \triangleright \bar{\sigma}^{**}$ for $i_0 < j^* < j^{**} < \lambda$, so we have a play $\bar{\sigma} = \bigcup_{i_0 < j^* < \lambda} \bar{\sigma}^i = (s_i, q^i, \bar{p}^i : i_0 \leq i < \lambda)$ of the game $S_{\lambda}^{\text{Sacks}}(i_0, t^\alpha_0, (Q_\alpha)^{G_{\alpha}})$ with the respective properties. This play is won by Generic, so there is a condition $q \in (Q_\alpha)^{G_{\alpha}}$ and a $\alpha^1 \lambda$-name $p$ for a member of $\lambda \lambda$ such that $q \supseteq t^\alpha_{i+1}$ and

$$q \vDash \bar{\sigma}^i \triangleq \left( \forall i \in [i_0, \lambda) \left( p^i(i+1) \in \psi_i^1(t^\alpha_{i+1}) \& p^i_{s^1_{n+1}}(\alpha) \in \Gamma(Q_{\alpha}, g_{\alpha}) \right) \right).$$

Let $r(\alpha), \tau_{\alpha}$ be names for the $q, p$ as above (i.e., $r(\alpha)$ is a $\alpha^1 \lambda$-name of a member of $Q_\alpha$ and $\tau_{\alpha}$ is a $\alpha^1 \lambda$-name of a member of $\lambda \lambda$ and $r|\alpha$ forces that they have the property stated in $(\circ)$). It follows from our choices that (i) $\alpha + 1$ hold true, finishing the inductive construction of $r \in V_\gamma$ and $\tau_{\alpha}$.

For $\alpha < \lambda$ let $a_\alpha = \{ \tau_{\alpha}^i : t \in T_{\alpha+1} \& r_{\alpha+1}(t) = \gamma \}$ (remember: $\tau_{\alpha}^i$ is the value forced to $\tau(\alpha)$ by $p^i_{s^1_{n+1}}$). Plainly, $|a_\alpha| < \lambda$, for each $\alpha < \lambda$.

The proof of the iteration theorem will be complete once we show the following claim.

**Claim A.2.3.1.** The condition $r \in V_\gamma$ (defined earlier) is stronger than $p$, it is $(N, P_\gamma)$-generic and $r \vDash \bar{\sigma}^i \triangleq \left( \forall \alpha < \lambda) (\tau(\alpha) \in a_\alpha) \right) \triangleq $.

**Proof of the Claim.** First, by induction on $\alpha \in N \cap (\gamma + 1)$ we are showing that $p|\alpha \leq r|\alpha$. There is nothing to do at limit stages, so let us deal with non-limit ones. Assume we have shown $p|\alpha \leq r|\alpha$.

Suppose that $G_{\alpha} \subseteq P_\alpha$ is generic over $V$, $r|\alpha \in G_{\alpha}$. Let $t^\alpha_0 = (t^\alpha_j) \in T_{\alpha}$, and let $i_0, \tau$ be defined as in $(\circ \circ \circ)$. Since, by (ii), $p^i_\alpha|\alpha \in G_{\alpha}$ (for non-limit $j \leq i_0$) and by the clause $(\zeta)$ of the construction, we get $V[G_{\alpha}] \models \left( \exists \psi \in [i_0, \lambda) \left( p^i|\alpha \neq \psi^i_\alpha \right) \right)$, which is a common upper bound to $p^i_{s^1_{n+1}}(\alpha) : j \leq i_0$ is non-limit $\} \triangleq $, and thus $V[G_{\alpha}] \models \left( \forall j < i_0 \right) (p^i_{s^1_{n+1}}(\alpha))^{G_{\alpha}} \leq t^\alpha_{i+1}$.

By the choice of $r(\alpha)$ we have $r(\alpha)^{G_{\alpha}} \geq t^\alpha_{i+1} \geq p(\alpha)^{G_{\alpha}}$.

Hence $r|\alpha \vDash p(\alpha) \leq r(\alpha)$, as needed.

Now, let $G \subseteq P_\gamma$ be generic over $V$, $r \in G$. For $i < \lambda$ let $t_i = (t^i_j) \in T_i$. By (ii), we know that $p^i_{s^1_{n+1}} \in G$. By clause $(\eta)$ we have $p^i_{s^1_{n+1}} \in I_i$ and (by the definition of $a_i$) $p^i_{s^1_{n+1}} \vDash \tau \in a_i$. The former implies that $G$ intersects $I \cap N$ for each open dense subset of $P_\gamma$ from $N$, the latter gives $G^G(i) \in a_i$.

(b) Included in the proof of (a).

**Definition A.2.4.** Let $P$ be a forcing notion.

1. For a condition $p \in P$ and an ordinal $i_0 < \lambda$ we define a game $\exists^\text{bd} (i_0, p, P)$ like $S_{\lambda}^{\text{Sacks}}(i_0, p, P)$, but demand $A.2.1(\zeta)$ is replaced by

   \[ (\zeta^-) \quad |s_i \cap \alpha^1 \lambda| < \lambda. \]

2. $P$ has the strong $\lambda$-bounding property if Generic has a winning strategy in the game $\exists^\text{bd} (i_0, p, P)$ for every $i_0 < \lambda$, $p \in P$. 


Proof. Then:

1. We say that \( \overline{\text{Definition A.3.2.}} \)

2. Even though we do not prove a real preservation in \( \lambda \)-fuzzy properness in this paper we may do much better:

   a. A number of natural forcing notions were not covered. For the context considered in this section captures the known ZFC limitations a number of \( \lambda \)-properness over semi-diamonds, was introduced in Roslanowski and Shelah \cite{15}. That property worked for any uncountable regular cardinal \( \lambda \geq \tau \). But it was known that for every \( \delta \in \mathcal{N} \) such that \( |N| = \lambda \), \( \mathcal{N} \subseteq N \) and \( \dot{\alpha}, \dot{\alpha} \preceq D \), then there is an \( (N, \mathcal{P}) \)-generic condition \( r \in \mathcal{P} \gamma \) stronger than \( p \).

3. For the context considered in this section we may do much better: fuzzy properness introduced in this section captures more examples. Even though we do not prove a real preservation in \( \lambda \)-support iterations, our iteration theorem A.3.10 is satisfactory for most applications (see sections B.4 and B.8 later).

In this section we fix \( \lambda^\ast, A, W, D \) such that

**A.3. Fuzzy properness over \( \lambda \)**

A properness-type property preserved in \( \lambda \)-support iterations, so called properness over semi-diamonds, was introduced in Roslanowski and Shelah \cite{15}. That property worked for any uncountable regular cardinal \( \lambda \), satisfying \( \lambda^{<\lambda} = \lambda \) (not necessarily strongly inaccessible), so because of the known ZFC limitations a number of natural forcing notions were not covered. For the context considered in this paper we may do much better: fuzzy properness introduced in this section captures more examples. Even though we do not prove a real preservation in \( \lambda \)-support iterations, our iteration theorem \( \text{A.3.10} \) is satisfactory for most applications (see sections B.4 and B.8 later).

In this section we fix \( \lambda^\ast, A, W, D \) such that

**Context A.3.1.**

1. \( \lambda^\ast > \lambda \) is a regular cardinal, \( A \subseteq \mathcal{H}_{<\lambda}(\lambda^\ast), W \subseteq [A]^{\lambda^\ast} \), and if \( a \in W, w \in [a]^{<\lambda^\ast}, f : w \rightarrow a \), then \( f \in a \) (hence also \( 0 \in a \) for \( a \in W \)),

2. for every \( x \in \mathcal{H}(\chi) \) there is a model \( N \prec (\mathcal{H}(\chi), \in, \langle \tau \rangle^\mathcal{H}) \) such that \( |N| = \lambda \), \( \mathcal{N} \subseteq N \), and \( \mathcal{N} \cap A \subseteq W \),

3. \( D \) is a normal filter on \( \lambda \) such that there is a \( D \)-diamond (see A.3.2).

**Definition A.3.2.**

1. We say that \( \tilde{F} = \langle F_\delta : \delta \in S \rangle \) is a \( D \)-pre-diamond sequence if

   a. \( S \in D^+ \) contains all successor ordinals below \( \lambda \), \( \lambda \setminus S \) is unbounded in \( \lambda \), \( 0 \notin S \), and

   b. \( F_\delta \in \delta D \) for all \( \delta \in S \).

2. A convenient \( D \)-diamond is a \( D \)-pre-diamond \( \tilde{F} = \langle F_\delta : \delta \in S \rangle \) such that

   \( (\forall f \in \lambda \lambda)(\{\delta \in S : F_\delta \subseteq f\} \in D^+) \).

**Definition A.3.3.**

Let \( \mathcal{P} \) be a forcing notion. \( \lambda \)-base for \( \mathcal{P} \) over \( W \) is a pair \( (\mathcal{R}, \mathcal{G}) \) such that

1. \( \mathcal{R} \subseteq \mathcal{P} \times \lambda \times A \) is a relation such that

   a. if \( (p, \delta, x) \in \mathcal{R} \) and \( p \leq p' \), then \( (p', \delta, x) \in \mathcal{R} \),
(b) \( \mathcal{Y} = \{ \mathfrak{Y}_{\alpha} : \alpha \in W \} \) where, for each \( \alpha \in W, \mathfrak{Y}_{\alpha} : \lambda \rightarrow [a]^{\lambda} \).

(c) If \( q \in \mathbb{P}, a \in W, \) and \( \delta < \lambda \) is a limit ordinal,
then there are \( p \geq_p q \) and \( x \in \mathfrak{Y}_{\alpha}(\delta) \) such that \( (p, \delta, x) \in \mathfrak{R} \).

If \( \mathfrak{R} \) is understood and \( (p, \delta, x) \in \mathfrak{R} \), then we may say \( p \) obeys \( x \) at \( \delta \).

**Definition A.3.4.** Let \( \mathbb{P} \) be a forcing notion and let \( (\mathfrak{R}, \mathcal{Y}) \) be a \( \lambda \)-base for \( \mathbb{P} \)
over \( W \). Also let a model \( N < (\mathcal{H}(\lambda), \in, <^*_{\lambda}) \) be such that \( |N| = \lambda, \mathcal{P} N \subseteq N, \)
a \( \mathfrak{def} \) \( N \cap A \in W \) and \( \{\lambda, \mathbb{P}, D, \mathfrak{R}\} \in N \). Furthermore, let \( h : \lambda \rightarrow N \) be such that
the range \( \operatorname{Rng}(h) \) of the function \( h \) includes \( \mathbb{P} \cap N \) and let \( F = \{ F_\delta : \delta \in S \} \) be a
\( D \)-pre-diamond sequence.

1. Let \( \mathcal{I} = \langle \mathcal{I}_\alpha : \alpha < \lambda \rangle \subseteq N \) list all open dense subsets of \( \mathbb{P} \) from \( N \).
A sequence \( \bar{p} = \langle p_\alpha : \alpha < \delta \rangle \) of conditions from \( \mathbb{P} \cap N \) of length \( \delta \leq \lambda \) is called
\( \mathcal{I} \)-exact if
\[
(\forall \xi < \delta)(\exists \alpha < \delta)(p_\alpha \in \mathcal{I}_\xi).
\]
2. We say that \( \bar{F} \) is a quasi \( D \)-diamond sequence for \( (N, h, \mathcal{Y}) \) if for some
(equivalently: all) list \( \mathcal{I} = \langle \mathcal{I}_\alpha : \alpha < \lambda \rangle \) of all open dense subsets of \( \mathbb{P} \) from \( N \), for every \( \leq_p \)-increasing sequence \( \bar{p} = \langle p_\alpha : \alpha < \lambda \rangle \subseteq \mathbb{P} \cap N \) such that
\[
E \defeq \{ \delta < \lambda : (p_\alpha : \alpha < \delta) \text{ is } \mathcal{I} \text{-exact} \} \in D
\]
equivalently: \( \bar{p} \) is \( \mathcal{I} \)-exact we have
\[
\{ \delta \in E : (\forall \alpha < \delta)(h \circ F_\delta(\alpha) = p_\alpha) \} \subseteq D^+.
\]
3. For a limit ordinal \( \delta \in S \) we define \( \mathcal{Y}(\delta) = \mathcal{Y}(N, \mathbb{P}, h, \bar{F}, \mathfrak{R}, \mathfrak{Y}, \delta) \) as the set
\[
\{ x \in \mathfrak{Y}_{\alpha}(\delta) : \text{if } (h \circ F_\delta(\alpha) : \alpha < \delta) \text{ is a } \leq_p \text{-increasing sequence of conditions from } \mathbb{P},
\text{then there is a condition } p \in \mathbb{P} \text{ such that }
(\forall \alpha < \delta)(h \circ F_\delta(\alpha) \leq_p p) \text{ and } (p, \delta, x) \in \mathfrak{R} \}
\]
4. Let \( \mathcal{I} = \langle \mathcal{I}_\alpha : \alpha < \lambda \rangle \subseteq N \) list all open dense subsets of \( \mathbb{P} \) from \( N \). A sequence \( \bar{q} = \langle q_{\delta,x} : \delta \in S \text{ limit } \& \ x \in \mathcal{I}_\delta \rangle \subseteq N \cap \mathbb{P} \) is called a weak fuzzy candidate over \( \bar{F} \)
for \( (N, h, \mathbb{P}, \mathfrak{R}, \mathfrak{Y}, \bar{I}) \) whenever \( \emptyset \neq X_\delta \subseteq \mathcal{Y}(\delta) \) (for limit\( \delta \in S \))
and
\( \alpha \) if \( \delta \in S \) and \( X_\delta = \mathcal{Y}(\delta) \) for each limit \( \delta \in S, \) then \( \bar{q} \) is called a fuzzy candidate over \( \bar{F} \)
for \( (N, h, \mathbb{P}, \mathfrak{R}, \mathfrak{Y}, \bar{I}) \).
Omitting \( \bar{I} \) means “for some \( \bar{I} \).”
5. Let \( \bar{q} = \langle q_{\delta,x} : \delta \in S \text{ limit } \& \ x \in \mathcal{I}_\delta \rangle \) be a weak fuzzy candidate \( \bar{F} \)
for \( (N, h, \mathbb{P}, \mathfrak{R}, \mathfrak{Y}, \bar{I}) \), and \( r \in \mathbb{P} \). We define a game \( G^{\text{haz2}}(r, N, \bar{I}, h, \mathbb{P}, \bar{F}, \bar{q}) \) of
two players, the Generic player and the Antigeneric player, as follows. A
play lasts \( \lambda \) moves, in the \( i \)-th move a condition \( r_i \in \mathbb{P} \) and a set \( C_i \in D \)
are chosen such that \( (\forall j < i)(r_j \leq r_i) \), and Generic chooses \( r_i, C_i \)
if \( i \in S = \text{Dom}(\bar{F}) \), and Antigeneric chooses \( r_i, C_i \) if \( i \notin S \). In the end
Generic wins the play if
\( \alpha \) if \( \forall \alpha < \lambda \exists i < \lambda \exists p \in \mathbb{P} \cap N \) \( (p \in \mathcal{I}_\alpha \& p \leq r_i) \), and
Definition A.3.6. Let $A.3.5$ and $A.3.6$.

Let $\overline{\mathcal{T}}$ be a weak fuzzy candidate over $\overline{F}$ for $(N, h, P, \mathfrak{R}, \overline{\mathcal{T}})$. We say that $P$ is a fuzzy candidate over $\overline{\mathcal{T}}$ if $\overline{\mathcal{T}}$ is a fuzzy proper sequence and $(\forall \alpha < \delta)(\exists i < \delta)(h \circ F_i(\alpha) \leq r_i), \overline{\mathcal{T}}$ is $\mathfrak{R}$-increasing.

Then for some $x \in \mathcal{X}_\delta$ we have $q_{\delta,x} \leq r_\delta$.

(6) Let $\overline{q}$ be a weak fuzzy candidate over $\overline{F}$ for $(N, h, P, \mathfrak{R}, \overline{\mathcal{T}})$. We say that a condition $r \in P$ is $(\mathfrak{R}, \overline{\mathcal{T}})$-fuzzy generic for $\overline{q}$ (over $(N, \overline{\mathcal{T}}, h, P, \overline{F})$) if Generic has a winning strategy in the game $\mathcal{D}_\delta^\mathfrak{R}(r, N, \overline{\mathcal{T}}, h, P, \overline{F}, \overline{q})$.

Remark A.3.5. (1) For any two lists $\overline{\mathcal{T}}^1, \overline{\mathcal{T}}^2$ of open dense subsets of $\mathbb{P}$ from $N$, on a club $E$ of $\lambda$ we have

$$\{\overline{\mathcal{T}}^1_\xi : \xi < \delta\} = \{\overline{\mathcal{T}}^2_\xi : \xi < \delta\}$$

for $\delta \in E$. Thus the corresponding notions of exactness agree for $\delta \in E$. As the generic player can choose $C_\delta \subseteq E$, in $\lambda$-compact forcing notions we allow not mention $\overline{\mathcal{T}}$ as a parameter.

(2) Plainly, every fuzzy candidate is a weak fuzzy candidate.

Definition A.3.6. Let $\mathbb{P}$ be a $\lambda$-complete forcing notion.

(1) We say that $\mathbb{P}$ is fuzzy proper over quasi $D$-diamonds for $W$ whenever for some $\lambda$-base $(\mathfrak{R}, \overline{\mathcal{T}})$ for $\mathbb{P}$ over $W$ and for some $c \in \mathcal{H}(\chi)$

$\mathfrak{R}$ if $\mathfrak{R}$ is a fuzzy proper sequence and $\mathfrak{R}$ is $\mathfrak{R}$-increasing.

Then there is $r \in \mathbb{P}$ stronger than $p$ and such that $r$ is $(\mathfrak{R}, \overline{\mathcal{T}})$-fuzzy generic for $\overline{q}$.

We may call $(\mathfrak{R}, \overline{\mathcal{T}})$ and $c$ witnesses for fuzzy properness.

(2) $\mathbb{P}$ is strongly fuzzy proper over quasi $D$-diamonds whenever for some $\lambda$-base $(\mathfrak{R}, \overline{\mathcal{T}})$ for $\mathbb{P}$ over $W$ and for some $c \in \mathcal{H}(\chi)$

$\mathfrak{R}$ if $\mathfrak{R}$ is a fuzzy proper sequence and $\mathfrak{R}$ is $\mathfrak{R}$-increasing.

Then there is a condition $r \in \mathbb{P}$ stronger than $p$ such that $r$ is $(\mathfrak{R}, \overline{\mathcal{T}})$-fuzzy generic for $\overline{q}$.

(3) $\mathbb{P}$ is weakly fuzzy proper over quasi $D$-diamonds whenever for some $\lambda$-base $(\mathfrak{R}, \overline{\mathcal{T}})$ for $\mathbb{P}$ over $W$ and for some $c \in \mathcal{H}(\chi)$

$\mathfrak{R}$ if $\mathfrak{R}$ is a fuzzy proper sequence and $\mathfrak{R}$ is $\mathfrak{R}$-increasing.

Then for some quasi $D$-diamond $\overline{F}$ for $(N, h, P)$ and a weak fuzzy candidate $\overline{q}$ over $\overline{F}$, there is a condition $r \in \mathbb{P}$ stronger than $p$ such that $r$ is $(\mathfrak{R}, \overline{\mathcal{T}})$-fuzzy generic for $\overline{q}$.

(4) $\mathbb{P}$ is fuzzy proper for $W$ if it is fuzzy proper over quasi $D'$-diamonds for every normal filter $D'$ on $\lambda$ (which has diamonds). Similarly for strongly fuzzy and weakly fuzzy proper.
Remark A.3.7. Strong fuzzy properness is very close to properness over semi-diamonds of Røshlanowski and Shelah \[13\] and even closer to properness over diamonds introduced by Eisworth \[9\]. (Note that considering the condition \(A.3.6(\delta)^+\) we may assume that the weak fuzzy candidate \(\bar{q} = \langle q_{\delta,x} : x \in S \rangle\) is such that \(|X_\delta| = 1\) for each relevant \(\delta\), so one may treat it as \(\bar{q} = \langle q_{\delta} : \delta \in S \rangle\) is limit \(\delta\).) Thus fuzzy properness has a flavour of a weaker property. However, the differences in technical details of the conditions introduced in this section and those in \[13\] and/or \[6\] make it unclear if there are any implications between the “properness conditions” in this section and those in the other two papers.

Proposition A.3.8. Let \(N, P, h, \tilde{I}, \mathcal{R}, \mathcal{Q}\) be as in \(A.3.3\). \(\tilde{F} = \langle F_\delta : \delta \in S \rangle\) be a \(D\)-pre-diamond.

1. If the forcing notion \(P\) is \(\lambda\)-complete, then there is a fuzzy candidate \(\bar{q}\) over \(F\) for \((N, h, \mathcal{R}, \mathcal{Q}, \tilde{I})\). In fact we can even demand:
   - (+) for every \(\alpha < \lambda\), for every large enough \(\delta \in S\), \(q_{\delta,x} \in I_\alpha\) for all \(x \in \mathcal{I}(\delta)\).
2. If \(r\) is \(\langle \mathcal{R}, \mathcal{Q}\rangle\)-fuzzy generic for some weak fuzzy candidate \(\bar{q}\), then \(r\) is \((N, P)\)-generic (in the standard sense).
3. Assume that a condition \(r\) is \((N, P)\)-generic (in the standard sense), \(\tilde{F}\) is a quasi \(D\)-diamond and \(\bar{q}\) is a weak fuzzy candidate over \((N, \tilde{I}, h, P, \tilde{F})\). Suppose that Generic has a strategy in the game \(\mathcal{C}_\lambda^{\text{fuzzy}}(r, N, \tilde{I}, h, P, \tilde{F}, \bar{q})\) which guarantees that the result \((r_i, C_i : i < \lambda)\) of the play satisfies \(A.3.4(5))\), then she has a winning strategy in \(\mathcal{C}_\lambda^{\text{fuzzy}}(r, N, \tilde{I}, h, P, \tilde{F}, \bar{q})\) (i.e., one ensuring \((\alpha) + (\beta)\) of \(A.3.4(5))\).
4. If \(P\) is fuzzy proper over quasi \(D\)-diamonds, then it is weakly fuzzy proper over quasi \(D\)-diamonds.
5. If \(P\) is strongly fuzzy proper over quasi \(D\)-diamonds, then it is fuzzy proper over quasi \(D\)-diamonds.

Proof. (1) Immediate (by the \(\lambda\)-completeness of \(P\); remember \(A.3.5(c)\)) and that \(\mathcal{R} \in N\); note that \(\mathcal{Q}_a(\delta) \in N\).

(2) Remember that \(0 \notin S\), so in the game \(\mathcal{C}_\lambda^{\text{fuzzy}}(r, N, \tilde{I}, h, P, \tilde{F}, \bar{q})\) the condition \(r_0\) is chosen by Antigeneric. So if the conclusion fails, then for some \(P\)-name \(\mathcal{V} \in N\) for an ordinal we have \(r \not\Vdash \alpha \in N\). Thus Antigeneric can choose \(r_0 \geq r\) so that \(r_0 \Vdash \alpha = \alpha_0\) for some ordinal \(\alpha_0 \notin N\), what guarantees him to win the play (remember clause (a) of \(A.3.4(5))\).

(3) The Generic player modifies her original strategy as follows. During the play she builds a \(\leq_P\)-increasing sequence of conditions \(\langle p_i : i \in \lambda \setminus S \rangle \subseteq P \cap N\). Arriving to stage \(i + 1\), \(i \in \lambda \setminus S\), she has two sequences: \(\langle r_j, C_j : j \leq i \rangle\) (of the play) and \(\langle p_j : j \in i \setminus S \rangle\) such that \(p_j \leq r_j\). Now Generic picks \(p_i \in P \cap N\) such that

\[
(\forall j \in i \setminus S)(p_j \leq p_i) \quad \text{and} \quad (\forall \xi < i)(p_i \in I_\xi),
\]
and $p_i, r_i$ are compatible (remember: we assumed that $r$ is $(N, P)$–generic). Next she replaces $r_i$ by a common upper bound of $p_i$ and $r_i$, pretending that that was the condition played by her opponent, and then she plays according to her original strategy. One easily verifies that this is a winning strategy for the Generic player.

(4) Straightforward (remember that, by A3.3.3 (3), there is a quasi $D$–diamond and by A3.3.1 (1) there is a fuzzy candidate over it).

(5) Follows from (2).

**Proposition A.3.9.** $\lambda^+$–complete forcing notions are strongly fuzzy proper for $W$. Proof.**

This is essentially a variant of [15, 2.5], but since we did not give the proof there, we will present it fully here.

So suppose that a forcing notion $P$ is $\lambda^+$–complete. Let $\mathfrak{R}^{tr} = \mathfrak{R}^{tr}(P)$ be the trivial relation consisting of all triples $(p, \delta, 0)$ such that $p \in P$ and $\delta < \lambda$ and let $\mathfrak{G}^{tr}$ be such that $\mathfrak{G}^{tr}_{\alpha}(\delta) = \{0\}$ (for each $\delta < \lambda, a \in W$). Assume now that

- $N < (H(\lambda), \in, <^*_\lambda), |N| = \lambda$, $<^*_\lambda N \subseteq N, \lambda, P \in N$, and $a \overset{\text{df}}{=} N \cap A \in W$,
- $p \in P \cap N$, and $h : \lambda \rightarrow N$ satisfies $P \cap N \subseteq \text{Rng}(h)$,
- $F = \langle F_\delta : \delta \in S \rangle$ is a quasi $D$–diamond for $(N, h, P)$ and $\bar{q}$ is a weak fuzzy candidate over $\bar{F}$. Since $\mathfrak{G}^{tr}_{\alpha}(\delta)$ has a one member only we may think of $\bar{q}$ as a sequence $\langle q_\delta : \delta \in S \text{ is limit } \rangle$.

Let $\bar{I} = \langle I_\xi : \xi < \lambda \rangle$ list all open dense subsets of $P$ from $N$.

We are going to build a condition $r \in P$ stronger than $p$ which is $(\mathfrak{R}^{tr}, \mathfrak{G}^{tr})$–generic for $\bar{q}$. For this we inductively build a $\leq_{\mathfrak{G}}$–increasing sequence $\langle r^i : i < \lambda \rangle \subseteq P \cap N$ such that

- $r^0 = p, r^i+1 \in \bigcap_{\xi \leq i} I_\xi$,
- if there is an upper bound to $\{r^j : j < i\} \cup \{q_i\}$, then $r^i$ is such an upper bound.

Then we pick any upper bound $r$ to the sequence $\langle r^i : i < \lambda \rangle$ (remember: $P$ is $\lambda^+$–complete). Now we want to argue that Generic has a winning strategy in the game $\overset{\text{fuzzy}}{G}_\lambda (r, N, \bar{I}, h, P, \bar{F}, \bar{q})$. Since $r$ is $(N, P)$–generic it is enough to give a strategy for the Generic player which ensures that the result of the play satisfies A3.3.5(β) (by A3.3.5(3)). To this end note that there is a club $E_0$ of $\lambda$ such that

- every member of $E_0$ is a limit of ordinals from $\lambda\setminus S$,
- for every $\delta \in E_0$ and $i < \delta$,

$$\{ q \in P : q \geq r^i \text{ or } q, r^i \text{ are incompatible } \} \in \{ I_\xi : \xi < \delta \},$$

Let Generic play so that arriving to a stage $\delta \in S$ of the play she puts the $<^*_\lambda$–first upper bound to the conditions played so far and $E_0$. Why does this strategy work? Let $\langle r_\alpha, C_\alpha : i < \lambda \rangle$ be the result of the play in which Generic plays as described above and let $\delta \in S \cap \bigcap_{i < \delta} C_\alpha$ be a limit ordinal such that $\langle h \circ F_\delta(\alpha) : \alpha < \delta \rangle$ is a $\leq_{\mathfrak{G}}$–increasing $\bar{I}$–exact sequence and

$$(\forall \alpha < \delta)(\exists \bar{I} < \delta)(h \circ F_\delta(\alpha) \leq r_i).$$

Then no $r^i, h \circ F_\delta(\alpha)$ (for $i, \alpha < \delta$) can be incompatible, so (since $\delta \in E_0$ and $\langle h \circ F_\delta(\alpha) : \alpha < \delta \rangle$ is $\bar{I}$–exact) we have also

$$(\forall i < \delta)(\exists \alpha < \delta)(r^i \leq h \circ F_\delta(\alpha)), $$
and hence \( q_δ \) is stronger than all \( r'_i \) (for \( i < δ \)). Therefore \( q_δ \leq r'_δ \leq r_δ \). □

**Theorem A.3.10.** Let \( A, W, D \) be as in **A.3.1** and let \( \bar{Q} = \langle P_α, Q_α : α < ζ^* \rangle \) be a \( λ \)-support iteration of \( λ \)-complete forcing notions, and assume that \( ζ^* \subseteq A \). Suppose also that for each \( ζ < ζ^* \) we have \( \bar{Q}_ζ \) and \( P_ζ \)-names \( \bar{R}_ζ, \bar{G}_ζ \) such that

\[
\Vdash_{P_ζ} \ \text{“} \bar{Q}_ζ \text{ is fuzzy proper over quasi } D \text{-diamonds for } W \\
\text{with witnesses } (\bar{R}_ζ, \bar{G}_ζ) \text{ and } \bar{G}_ζ”.
\]

Then \( P_ζ = \text{lim}(\bar{Q}) \) is weakly fuzzy proper over quasi \( D \)-diamonds.

**Proof.** By **A.3.1** the forcing notion \( P_ζ \) is \( λ \)-complete, so we have to concentrate on showing clause **A.3.6(3)((*)−)** for it. The proof, though not presented as such, is by induction on \( ζ^* \). However, the inductive hypothesis is used only to be able to claim that \( A, W, D \) are as in **A.3.1** when considered in the intermediate universes \( V^{P_ζ} \) (for \( ζ < ζ^* \) — remember **A.3.8(5)**). Thus our assumptions on \( Q_ζ \)'s are meaningful.

Let us fix a convenient \( D \)-diamond sequence \( F^i = \langle F^i_δ : δ \in S \rangle \) (so in particular, \( S \in D^+ \) contains all successors, \( λ \setminus S \) is unbounded in \( λ \) and \( 0 \notin S \)). Put

\[
E_0 = \{ δ < λ : δ \text{ is a limit of points from } λ \setminus S \}, \quad E_1 = (λ \setminus S) \cup E_0.
\]

Plainly, \( E_0, E_1 \) are clubs of \( λ \). Let \( \{ i_α : α < λ \} \) be the increasing enumeration of \( E_1 \) and \( E_2 = E_0 \cap \{ α < λ : i_α = α \} \) (So \( E_2 \) is a club of \( λ \) too).

For each \( α \) in \( W \) fix a one-to-one mapping \( π_α : a \cap ζ^* \rightarrow λ \) such that \( π_α(0) = 0 \) (say, \( π_α \) is the \( <_ζ^* \)-first such function), and for \( α < λ \) let \( w^α_δ = (π_α)^{-1}[i_α] \) (so \( a \cap ζ^* = \bigcup_{α < λ} w^α_δ \)).

For \( ζ ≤ ζ^* \) let \( R^{[κ]}_δ \) consist of all triples \( (p, δ, \bar{x}) \in P_ζ \times λ \times A \) such that for some non-empty \( w ∈ [ζ \cap A]^{< λ} \) we have

\[
\bar{x} = (x_ε : ε ∈ w) \quad \text{and} \quad (\forall ε ∈ w)(p|ε \Vdash_{P_ε} \ “(p(ε), δ, x_ε) ∈ R_ε”).
\]

Next, for \( ζ ≤ ζ^* \), \( a ∈ W \) and \( δ < λ \) we put

\[
\bar{Q}^{[κ]}_δ = \prod \{ \bar{Q}^{[κ]}_ε : ε ∈ w^α_δ \cap ζ \},
\]

thus defining \( \bar{Q}^{[κ]}_δ \) and \( \bar{Q}^{[κ]}_δ = (\bar{Q}^{[κ]}_δ : a ∈ W) \). If \( ζ = ζ^* \) we will omit it (so then we write \( R \) and \( \bar{Q} \)).

**Claim A.3.10.1.** For each \( ζ ≤ ζ^* \), \( (R^{[κ]}, \bar{Q}^{[κ]}) \) is a \( λ \)-base for \( P_ζ \) over \( W \).

**Proof of the Claim.** Immediately by the definition of \( R^{[κ]}, \bar{Q}^{[κ]} \) we see that clauses (a), (b) of **A.3.3** hold (note: \( \bar{Q}^{[κ]}_δ(δ) ⊆ a \) by **A.3.1**). Now, to verify **A.3.3(c)**, suppose \( q ∈ P_ζ \), \( a ∈ W \) and \( δ < λ \) is limit. For each \( ε ∈ w^α_δ \cap ζ \) let \( p(ε), x_ε \) be \( P_ε \)-names such that

\[
q|ε \Vdash_{P_ε} \ “p'(ε) ≥ q(ε) \& x'_ε ∈ Q_ε^{[κ]}(δ) \& (p'(ε), δ, x'_ε) ∈ R_ε”
\]

and for \( ε ∈ \text{Dom}(q) \setminus w^α_δ \) let \( p'(ε) = q(ε) \). This defines a condition \( p' ∈ P_ζ \) stronger than \( q \) (and names \( x'_ε \)). Since \( P_ζ \) is \( λ \)-complete we may find a condition \( p ≥ p' \) and \( x_ε ∈ Q_ε^{[κ]}(δ) \) (for \( ε ∈ w^α_δ \cap ζ \)) such that \( p|ε \Vdash_{P_ε} \ “x'_ε = x_ε” \) (for \( ε ∈ w^α_δ \cap ζ \)). Then, by **A.3.3(a)**, we have \( p|ε \Vdash_{P_ε} \ “(p(ε), δ, x_ε) ∈ R_ε” \) (for each \( ε ∈ w^α_δ \cap ζ \)), and hence \( (p, δ, (x_ε : ε ∈ w^α_δ \cap ζ)) ∈ R^{[κ]} \). □
Our aim now is to show that $P_{\zeta}$ is weakly fuzzy proper with witnesses $(\mathcal{R}, \mathcal{Q})$ and $c = (\zeta \in \zeta^+ \mid (\mathcal{R}, \mathcal{Q}) \in P_{\zeta} \cap \mathcal{S} \mid D, F^*, \mathcal{Q}))$. So suppose that a model

$$N \models (\mathcal{H}(\chi), \varepsilon, \langle \epsilon \rangle^+) \text{ satisfies}$$

$$|N| = \lambda, \quad \langle \mathcal{A} \subseteq N, \quad \lambda, P_{\zeta}, c \in N, \quad a \equiv N \cap A \in W,$$

and $p \in P_{\zeta} \cap N$, and $h : \lambda \rightarrow N$ is such that $P_{\zeta} \cap N \subseteq \text{Rng}(h)$. To simplify the notation later, let $\pi = \pi_\alpha, w_\alpha = w_\alpha^a$ (for $\alpha < \lambda$).

Let us fix a list $\mathcal{I} = \{I_\alpha : \alpha < \lambda \}$ of all open dense subsets of $P_{\zeta}$ from $N$. For $\zeta \in (\zeta^* + 1) \cap N$, let $\mathcal{I}[\zeta] = \{I_\alpha^\zeta : \alpha < \lambda \}$, where $I_\alpha^\zeta = \{p \mid \zeta : p \in I_\alpha\}$. (Note that $\mathcal{I}[\zeta]$ lists all open dense subsets of $P_{\zeta}$ from $N$.) Also for $\zeta \in \zeta^* \cap N$ let $J_\zeta = \{p \in P_{\zeta} : p|\zeta \vdash p(\zeta) \neq \emptyset \}$ (so $J_\zeta$ is an open dense subset of $P_{\zeta}$ from $N$) and let $E_\zeta = \{\alpha \in E_2 : (\forall \zeta \in w_\alpha)(\exists \beta < \alpha)(J_\zeta = I_\beta)\}$. Clearly, $E_\zeta$ is a club of $\lambda$.

Now, using the diamond $\mathcal{F}$ fixed earlier, we are going to define the quasi $D$–diamond sequence $\mathcal{F}$ (and then a weak fuzzy candidate $\bar{q}$ over it) that will be as required by $(\circ)$ of (3.3.4.3). So, for each $\delta \in S$ we let

$$Z(\delta) = \{\zeta \in (\zeta^* + 1) \cap \emptyset : \langle[h \circ F_\delta^\alpha(\alpha) \mid \zeta : \alpha < \delta]\rangle \text{ is a } \leq \text{–increasing } \mathcal{I}[\zeta]\text{–exact sequence of members of } N \cap P_{\zeta}\}$$

and if $Z(\delta) \neq \emptyset$ then we put $\gamma(\delta) = \sup(Z(\delta))$. Note that $Z(\delta) \in N$ and thus $\gamma(\delta) \in N$ (when defined). Now, the pre–diamond $\mathcal{F} = \langle F_\delta : \delta \in S \rangle$ is picked so that for a limit $\delta \in S$:

$$(\circ)_1$$ if $Z(\delta) \neq \emptyset$, then $h \circ F_\delta(\alpha) = (h \circ F_\delta(\alpha)) \upharpoonright \gamma(\delta)$ for all $\alpha < \delta$;

$$(\circ)_2$$ if $Z(\delta) = \emptyset$, then $h \circ F_\delta(\alpha) = \emptyset_{P_{\zeta}}$, for all $\alpha < \delta$.

Then easily $\mathcal{F}$ is a quasi $D$–diamond for $(N, h, P_{\zeta})$ and for each limit $\delta \in S$, $\langle h \circ F_\delta(\alpha) : \alpha < \delta\rangle$ is a $\leq P_{\zeta}$–increasing sequence of conditions from $P_{\zeta} \cap N$.

Just for notational simplicity, we will identify a sequence $\bar{s} = (s_0)$ with its (only) term $s_0$. Thus, below, when we talk about a standard $(w, 1)^{\zeta^*}$–tree $T$, we think of $T$ as a set of sequences $t = \langle(t)\zeta : \zeta \in w \cap \text{rk}(t)\rangle$ where $(t)\zeta$’s do not have to be sequences.

Now we are going to define sequences $\bar{p} = \langle p_i : i < \lambda \rangle \subseteq P_{\zeta} \cap N$. $\langle T_\delta : \delta \in S \rangle$ is limit ), and $\langle q_{\delta, t} : \delta \in S \text{ is limit } \& t \in T_\delta \rangle \subseteq P_{\zeta} \cap N$ such that for a limit ordinal $\delta \in S$:

(i) $T_\delta = (T_\delta, \text{rk}_\delta)$ is a standard $(w_\delta, 1)^{\zeta^*}$–tree, and (under the identification mentioned earlier) $\{t \in T_\delta : \text{rk}_\delta(t) = \zeta \in \mathcal{Z}_\alpha^\zeta(\delta)\}$ for $\zeta \in w_\delta \cup \{\zeta^*\}$,

(ii) $\langle q_{\delta, t} : t \in T_\delta \rangle$ is a standard tree of conditions in $\mathcal{Q}$,

(iii) $p_i \leq p_j \leq p_j$ for $i < j < \lambda$,

(iv) if $j < \lambda$, then $w_j \subseteq \text{Dom}(p_j)$ and $(\forall \varepsilon \in w_j)(\forall t' > j)(p_j(\varepsilon) = p_{j'}(\varepsilon))$,

(v) if $t \in T_\delta$, $\text{rk}_\delta(t) = \zeta$, then $q_{\delta, t} \in P_{\zeta} \cap N$ is such that

(a) $\bigcup_{\alpha < \delta} \text{Dom}(h \circ F_\delta(\alpha)) \cup \bigcup_{\alpha \leq \delta} \text{Dom}(p_i) \cap \zeta \subseteq \text{Dom}(q_{\delta, t})$,

(b) $(\forall \alpha < \delta)(h \circ F_\delta(\alpha) \mid \zeta \leq q_{\delta, t})$, and

(c) if $\varepsilon \in \text{Dom}(q_{\delta, t}) \setminus w_\delta$, then

$q_{\delta, t} \mid \varepsilon \upharpoonright \langle t \rangle \text{ if the set } \{p_i(\varepsilon) : i < \delta\} \cup \{(h \circ F_\delta(\alpha))(\varepsilon) : \alpha < \delta\}$

has an upper bound in $\mathcal{Q}_\zeta$,

then $q_{\delta, t}(\varepsilon)$ is such an upper bound ”,
(d) if $\varepsilon \in \text{Dom}(q_{\delta,t}) \cap w_\delta$, then

$q_{\delta,t} \models \varepsilon \vdash \langle \text{if the set } \{p_i(\varepsilon) : i < \delta\} \cup \{(h \circ F_\delta(\alpha))(\varepsilon) : \alpha < \delta\} \rangle$

has an upper bound which obeys $(t)_\varepsilon$ at $\delta$, then $q_{\delta,t}(\varepsilon)$ is such an upper bound,

else $q_{\delta,t}(\varepsilon)$ is an upper bound of $\{(h \circ F_\delta(\alpha))(\varepsilon) : \alpha < \delta\}$ which obeys $(t)_\varepsilon$ at $\delta$ ",

(e) $q_{\delta,t} \in \bigcap_{\xi<\delta} T_\xi^{[\delta]}$,

(vi) if $t \in T_\delta$, $\zeta = \text{rk}_\delta(t) < \zeta^*$, $\zeta' \in w_\delta \cup \{\zeta^*\}$ and $t', t'' \in T_\delta$ are such that $\text{rk}_\delta(t) = \text{rk}_\delta(t'') = \zeta'$, $t < t'$, $t < t''$ and $t' \neq t''$, then

$q_{\delta,t} \models P_\zeta \langle \text{the conditions } q_{\delta,t'}(\zeta) \text{ and } q_{\delta,t''}(\zeta) \text{ are incompatible } \rangle$,

(vii) if $t \in T_\delta$ and $\varepsilon \in \text{Dom}(q_{\delta,t}) \setminus w_\delta$, then $\varepsilon \in \text{Dom}(p_\delta)$ and

$p_\delta \models \varepsilon \vdash \langle \text{if } q_{\delta,t} \vDash \varepsilon \in \Gamma_\varepsilon \text{ and } \{p_i(\varepsilon) : i < \delta\} \cup \{q_{\delta,t}(\varepsilon)\} \text{ has an upper bound in } Q_\varepsilon, \text{ then } p_\delta(\varepsilon) \text{ is such an upper bound } \rangle$,

(viii) if $t \in T_\delta$, $\text{rk}_\delta(t) = \zeta < \zeta^*$, $x \in \mathcal{P}_\delta(\delta)$ and

$q_{\delta,t} \not\models P_\zeta \langle \text{there is no condition stronger than all } (h \circ F_\delta(\alpha))(\zeta) \text{ for } \alpha < \delta \text{ which obeys } x \text{ at } \delta \rangle$

then there is $t' \in T_\delta$ such that $t < t'$ and $(t')_\zeta = x$.

Assume $\delta < \lambda$ and we have defined $p_i, T_i, q_{i,t}$ for relevant $i < \delta$ and $t$. If $\delta$ is not a limit ordinal from $S$, then only $p_\delta \in P_\zeta \cap \mathcal{N}$ needs to be defined, and clauses (iii), (iv) can be easily taken care of. So suppose that $\delta \in S$ is limit.

First we let $T'_\delta$ be a standard $(w_\delta,1)^{\zeta^*}$–tree such that $\{t \in T'_\delta : \text{rk}_\delta(t) = \zeta\} = \mathcal{P}_\delta^{[\delta]}(\delta)$ (for $\zeta \in w_\delta \cup \{\zeta^*\}$). For $t \in T'_\delta$ we define a condition $r_t \in \mathcal{P}_\delta(\zeta)$ so that

$$\text{Dom}(r_t) = \left( \bigcup_{\alpha<\delta} \text{Dom}(h \circ F_\delta(\alpha)) \cup \bigcup_{i<\delta} \text{Dom}(p_i) \cup w_\delta \right) \cap \text{rk}_\delta(t),$$

and for each $\zeta \in \text{Dom}(r_t)$, $r_t(\zeta)$ is the $<_{\lambda}^*$–first $P_\zeta$–name for a condition in $Q_\zeta$ such that:

if $\zeta \in w_\delta$, then

$r_t[\zeta \vdash P_\zeta \langle \text{if the family } \{p_i(\zeta) : i < \delta\} \cup \{(h \circ F_\delta(\alpha))(\zeta) : \alpha < \delta\} \text{ has an upper bound which obeys } (t)_\zeta \text{ at } \delta, \text{ then } r_t(\zeta) \text{ is such an upper bound, if the previous is impossible, but there is an upper bound of } (h \circ F_\delta(\alpha))(\zeta) : \alpha < \delta \text{ which obeys } (t)_\zeta \text{ at } \delta \text{ then } r_t(\zeta) \text{ is such an upper bound, if neither from the previous two possibilities holds, then } r_t(\zeta) \text{ is an upper bound of } (h \circ F_\delta(\alpha))(\zeta) : \alpha < \delta \rangle \rangle$,

and if $\zeta \notin w_\delta$, then

$r_t[\zeta \vdash P_\zeta \langle \text{if the family } \{p_i(\zeta) : i < \delta\} \cup \{(h \circ F_\delta(\alpha))(\zeta) : \alpha < \delta\} \text{ has an upper bound, then } r_t(\zeta) \text{ is such an upper bound, if this is not possible, then } r_t(\zeta) \text{ is just an upper bound of } (h \circ F_\delta(\alpha))(\zeta) : \alpha < \delta \rangle \rangle$. 

Plainly, $|T'_\theta| < \lambda$ and $\bar{r} = \langle r_t : t \in T'_\theta \rangle$ is a standard tree of conditions, and it belongs to $N$ (remember: $\lambda < \lambda' \subseteq N$). So using $\textbf{1.1.8}$ in $N$ we may pick a standard tree of conditions $r^* = \langle r'_t : t \in T''_\theta \rangle \in N$ such that $\bar{r} \leq r^*$ and for each $t \in T''_\theta$ and $\zeta \in w_3 \cap r_k(t)$ the condition $r'_t|\zeta$ decides the truth value of the sentence

\[ r'_t(\zeta) \text{ obeys } (t)_\zeta \text{ at } \delta \text{ (with respect to } \mathfrak{U}_{\zeta}) \]

(remember the choice of $r'_t(\zeta)$ for $\zeta \in w_3$ and $\textbf{A.3.3}((a)$)). Put

$T_\delta = \{ t \in T''_\theta : \text{ for each } \zeta \in w_3 \cap r_k(t), r'_t|\zeta \text{ obeys } (t)_\zeta \text{ at } \delta \}$,

and notice that $T_\delta \in N$ is a standard $(w_3,1)^{\ast}$--tree. Note also that for each $t \in T_\delta$ there is $t' \in T_3$ such that $t \leq t'$ and $r_k(t') = \zeta^\ast$. Finally, using $\textbf{1.1.8}$ and next $\textbf{A.1.8}$ in $N$ we may choose a standard tree of conditions $\langle q_{\delta,t} : t \in T_3 \rangle \in N$ which satisfies clauses (vi) and (v)(e) and such that $r'_t \leq q_{\delta,t}$ (for $t \in T_3$). It should be clear that then $T_\delta$ and $\langle q_{\delta,t} : t \in T_3 \rangle$ satisfy all the relevant demands from our list ((i)-(viii)). Now finding a condition $p_\delta \in \mathbb{P}_{\zeta} \cap N$ which satisfies (iii)+(iv)+(vii) is straightforward. (Note that, by (vi), if $t', t'' \in T_3$, $\varepsilon \in \text{Dom}(q_{\delta,t'}) \cap \text{Dom}(q_{\delta,t''})$, $\varepsilon \notin w_3$ and the conditions $q_{\delta,t'} | \varepsilon, q_{\delta,t''} | \varepsilon$ are compatible, then $q_{\delta,t'} | \{ (\varepsilon+1) = q_{\delta,t''} | (\varepsilon+1) \}$.)

For a limit ordinal $\delta \in S$ and $\zeta \in N \cap (\zeta^\ast + 1)$ we let

- $\lambda^{[\zeta]}_\delta = \{ t|\zeta : t \in T_\delta \text{ & } r_k(t) = \zeta^\ast \};$
- if $s \in \lambda^{[\zeta]}_\delta$, then $q_{s,t}^{[\zeta]} = q_{s,t}|\zeta$ for some (equivalently: all) $t \in T_3$ such that $s \leq t$ and $r_k(t) = \zeta^\ast$;
- $q^{[\zeta]}_\delta = \langle q^{[\zeta]}_s : \delta \in S \text{ is limit & } s \in \lambda^{[\zeta]}_\delta \rangle$;
- $h^{[\zeta]} : \lambda \rightarrow N$ is such that $h^{[\zeta]}(\gamma) = (h(\gamma))|\zeta$ provided $h(\gamma)$ is a function, and $h^{[\zeta]}(\gamma) = 0_{\mathfrak{F}_\zeta}$ otherwise.

Plainly, $\emptyset \neq \lambda^{[\zeta]}_\delta \subseteq 2^{\mathfrak{U}}_\delta(\delta)$ (remember (i)) and $h^{[\zeta]} : \lambda \rightarrow N$ is such that $\mathbb{P}_{\zeta} \cap N \subseteq \text{Rng}(h^{[\zeta]}).$ Moreover, one easily verifies the following claim.

**Claim A.3.10.2.** Let $\zeta \in N \cap (\zeta^\ast + 1)$. Then $\bar{F}$ is a quasi $D$--diamond sequence for $(N, h^{[\zeta]}, \mathbb{P}_{\zeta})$ and $\bar{F}^{[\zeta]}$ is a weak fuzzy candidate over $\bar{F}$ for $(N, h^{[\zeta]}, \mathbb{P}_{\zeta}, \mathfrak{U}_{\zeta}, 2^{\mathfrak{U}}_\delta(\delta), \tilde{\mathfrak{U}}^{[\zeta]}_\delta)$.

We may write $\bar{q}, q_{\delta,t}, X_3$ for $q^{[\zeta]}_\delta, q^{[\zeta]}_{s,t}, \lambda^{[\zeta]}_3$ respectively. Also note that, in the context of our definitions, the functions $h$ and $h^{[\zeta]}$ behave the same, so we may identify them. Of course, we are going to define an $(\mathfrak{F}, \mathfrak{U})$--fuzzy generic condition $r \in \mathbb{P}_{\zeta}$ for $\bar{q}$ over $\bar{F}$, but before that we have to introduce more notation used later and prove some important facts.

For $\zeta \in \zeta^\ast \cap N$ we define a function $h^{(\zeta)}$ and $\mathbb{P}_{\zeta}$--names $\bar{S}^{(\zeta)}, \bar{F}^{(\zeta)}, \bar{\mathfrak{U}}^{(\zeta)}_{\delta}, \bar{X}^{(\zeta)}_3, \bar{F}^{(\zeta)}$ and $\bar{q}^{(\zeta)}$ so that:

- $h^{(\zeta)} : \lambda \rightarrow N$ is such that if $h(\gamma)$ is a function, $\zeta \in \text{Dom}(h(\gamma))$ and $(h(\gamma)\langle \zeta \rangle)\langle \zeta \rangle$ is a $\mathbb{P}_{\zeta}$--name then $h^{(\zeta)}(\gamma) = ((h(\gamma))\langle \zeta \rangle)\langle \zeta \rangle$, otherwise $h^{(\zeta)}(\gamma) = 0_{\mathfrak{F}_\zeta}$;
- $\text{if } \mathbb{P}_{\zeta} \text{--names } \bar{S}^{(\zeta)} = \{ \delta \in S : \text{ if } \delta \text{ is limit then } \delta > \pi(\zeta) \text{ & } (\exists s \in \lambda^{(\zeta)}_\delta)(q_s^{[\zeta]} \in \Gamma_{\mathfrak{F}_{\zeta}})^\prime\}$;
- $\text{if } \mathbb{P}_{\zeta} \text{--names } \bar{q}^{(\zeta)} = \{ q^{(\zeta)}_\delta : \delta \in S^{(\zeta)} \text{ is limit & } x \in X^{(\zeta)}_3 \}$, where:
- $\text{if } \mathbb{P}_{\zeta} \text{--names } \bar{q}^{(\zeta)} = \{ q^{(\zeta)}_\delta : \delta \in S^{(\zeta)} \text{ is limit & } x \in X^{(\zeta)}_3 \}$, where:
- $\text{if } \mathbb{P}_{\zeta} \text{--names } \bar{q}^{(\zeta)} = \{ q^{(\zeta)}_\delta : \delta \in S^{(\zeta)} \text{ is limit & } x \in X^{(\zeta)}_3 \}$, where:
- $\text{if } \mathbb{P}_{\zeta} \text{--names } \bar{q}^{(\zeta)} = \{ q^{(\zeta)}_\delta : \delta \in S^{(\zeta)} \text{ is limit & } x \in X^{(\zeta)}_3 \}$, where:
Assume that this fails. Then we can find a condition $p \in S(N)$. Naturally, we treat

Thus we take $\langle \cdot \rangle$ as a $P$-name for a function from $\lambda$ to $N[\Gamma_{\xi}]$. Observe that $p \in S(N)$. Naturally, we treat

\[ p \in \Gamma_{\xi} \] \[ \text{as a limit of elements of } \triangle \]

\[ \{ p : p \in T_{\alpha} \wedge p \models \text{a } \xi \} \]

Naturally, we treat $h(\xi)$ as a $P$-name for a function from $\lambda$ to $N[\Gamma_{\xi}]$. Observe that $p \in S(N)$. Naturally, we treat

Together with choosing $\xi < \delta$, we may find an ordinal $\xi \in I_{\xi}$ for elements of $\triangle$ (for $\xi \leq j$) such that

\[ \xi \in I_{\xi} \] \[ \text{where:} \]

\[ p \models \xi \in I_{\xi} \] \[ \text{if } j' < j \text{ from } \lambda \setminus S, \text{ then } p_j \models \xi \in I_{\xi} \] \[ \text{and } \]

\[ p_j \models \xi \in I_{\xi} \] \[ \text{if } \delta < j, \delta \in \bigcap \xi \in I_{\xi} \text{ then } p_j \models \xi \in I_{\xi} \] \[ \text{for all } \xi < \delta. \]

\[ \langle r, q \rangle : j < \lambda \] \[ \text{of game } \angle \text{fuzzy}(r, N, \lambda, h, p, F, q) \]

\[ \text{in which Generic uses her winning strategy and Antigeneric plays as follows.} \]

\[ \text{Together with choosing } r_j \text{ (for } j \in \lambda \setminus S), \text{ the Antigeneric player chooses also} \]

\[ \text{side conditions } p_j \models \xi \in I_{\xi} \text{ for elements of } \triangle \text{ (for } \xi \leq j) \text{ such that} \]

\[ p_j \models \xi \in I_{\xi} \] \[ \text{if } \delta < j, \delta \in \bigcap \xi \in I_{\xi} \text{ then } p_j \models \xi \in I_{\xi} \] \[ \text{for all } \xi < \delta. \]

\[ (1) \text{ Will follow from } (2). \]

\[ (2) \text{ Assume that this fails. Then we can find a condition } r^* \in P, \text{ a } P \text{-name } \bar{q} = \langle q'_\alpha : \alpha < \lambda \rangle \text{ for an increasing sequence of conditions from } \triangle \text{ and } P \text{-names } A_\xi, B_\xi \text{ for members of } D \cap \bigcap \xi \in I_{\xi} \text{ such that } r^* \models P \]

\[ r^* \models \langle q'_\alpha : \alpha < \lambda \rangle \] \[ \text{is } \bar{q} \text{-exact } \]

\[ (\forall \xi \in \bigcap \xi \in I_{\xi} \text{ (by } \langle r_j \rangle \text{ for } j \in \lambda \setminus S) \text{ the Antigeneric player completes } p_j \models \xi \in I_{\xi} \text{ for elements of } \triangle \text{ such that} \]

\[ p_j \models \xi \in I_{\xi} \] \[ \text{if } \delta < j, \delta \in \bigcap \xi \in I_{\xi} \text{ then } p_j \models \xi \in I_{\xi} \] \[ \text{for all } \xi < \delta. \]

\[ \langle r, q \rangle : j < \lambda \] \[ \text{of game } \angle \text{fuzzy}(r, N, \lambda, h, p, F, q) \]

\[ \text{in which Generic uses her winning strategy and Antigeneric plays as follows.} \]

\[ \text{Together with choosing } r_j \text{ (for } j \in \lambda \setminus S), \text{ the Antigeneric player chooses also} \]

\[ \text{side conditions } p_j \models \xi \in I_{\xi} \text{ for elements of } \triangle \text{ (for } \xi \leq j) \text{ such that} \]

\[ p_j \models \xi \in I_{\xi} \] \[ \text{if } \delta < j, \delta \in \bigcap \xi \in I_{\xi} \text{ then } p_j \models \xi \in I_{\xi} \] \[ \text{for all } \xi < \delta. \]
\( \langle p_j \& (q_i^j) : j < \delta \rangle \). By the choice of \( \bar{F} \) we know that \( h(F_\delta(j)) \) is a condition in \( \bar{P}_\xi \)
(so a function) and hence \( h^{(C)}(F_\delta(j)) = g_\sigma^j \) for all \( j < \delta \). Also
\[
(\forall i < \delta)(\exists j \in \delta \setminus S)(h^{[k]} \circ F_\delta(i) \leq_{\bar{P}_\xi} h^{[k]} \circ F_\delta(j) = p_j \leq_{\bar{P}_\xi} r_j).
\]
Since the play is won by Generic, for some \( s \in \mathcal{X}^{[k]} \) we have \( q_{\delta,s}^{[k]} \leq_{\bar{P}_\xi} r_\delta \). But then
\[
r_\delta \models \\delta \in S^{(C)} \cap \bigcup_{\xi < \lambda} B_\xi \quad \& \quad (h^{(C)} \circ F_\delta(\alpha) : \alpha < \delta) = \bar{q}^*|\delta ^* ,
\]
a contradiction.

(3) Straightforward (remember the choice of \( q_{\delta,i} \)'s, specifically clauses (v)(b,d,e) and (vii)).

Now we are going to define an \( (\mathcal{R}, \bar{q}) \)-fuzzy generic condition \( r \in \bar{P} \) for \( \bar{q} \) over 
\( (N, \bar{T}, h, \bar{P}_\xi, \bar{F}) \) in the most natural way. Its domain is \( \text{Dom}(r) = \zeta^* \cap N \) and for each \( \zeta \in \zeta^* \cap N \)
\[
r \upharpoonright \zeta \models a \ r(\zeta) \geq p_r(\zeta) + 1(\zeta) \text{ is an } (\mathcal{R}_{\zeta}, \bar{q}^{\zeta}) \text{-fuzzy generic condition for } \bar{q}^{\zeta}
\]
over \( (N[\bar{F}_\zeta], \bar{T}^{\zeta}, h^{\zeta}, \bar{Q}_{\zeta}, \bar{F}^{(\zeta)}) \).”

[So \( r \geq p_i \) for all \( i < \lambda \).]

**Claim A.3.10.4.** For every \( \zeta \in (\zeta^* + 1) \cap N \), the Generic player has a winning strategy in the game \( C^\text{fuzzy}(r \upharpoonright \zeta, N, \bar{T}^{\zeta}, h^{\zeta}, \bar{P}_\zeta, \bar{F}, \bar{q}^{\zeta}) \).

**Proof of the Claim.** We will prove the claim by induction on \( \zeta \in (\zeta^* + 1) \cap N \). After we are done with stage \( \zeta \in (\zeta^* + 1) \cap \lambda \), we know that \( r \upharpoonright \zeta \) is \( (\mathcal{R}^{\zeta}, \bar{q}^{\zeta}) \)-fuzzy generic for \( \bar{q}^{\zeta} \) over \( (N[\bar{F}_\zeta], \bar{T}^{\zeta}, h^{\zeta}, \bar{Q}_{\zeta}, \bar{F}^{(\zeta)}) \) for \( \zeta, \lambda, N \) this implies that \( r(\zeta) \) is well-defined (remember A.3.10.3). Of course for \( \zeta = \zeta^* \) we finish the proof of the theorem.

Suppose that \( \zeta \in (\zeta^* + 1) \cap N \) and we know that \( r \upharpoonright \zeta' \) is \( (\mathcal{R}^{\zeta'}, \bar{q}^{\zeta'}) \)-fuzzy generic for \( \bar{q}^{\zeta'} \) over \( (N[\bar{F}_\zeta], \bar{T}^{\zeta'}, h^{\zeta'}, \bar{P}_\zeta, \bar{F}) \) for all \( \zeta' \in N \cap \zeta \). We are going to define a winning strategy for Generic in the game \( C^\text{fuzzy}(r \upharpoonright \zeta, N, \bar{T}^{\zeta}, h^{\zeta}, \bar{P}_\zeta, \bar{F}, \bar{q}^{\zeta}) \).

First, for \( \varepsilon \in \zeta \cap N \) fix a \( \bar{P}_\varepsilon \)-name \( \bar{s}_t_{\varepsilon} \) such
\[
r \upharpoonright \varepsilon \models \quad \text{st}_t_{\varepsilon} \text{ is a winning strategy of the Generic player in the game } C^\text{fuzzy}(r(\varepsilon), N[\bar{F}_\varepsilon], \bar{T}(\varepsilon), h(\varepsilon), \bar{Q}_\varepsilon, \bar{F}(\varepsilon), \bar{q}(\varepsilon)) \).
\]
We will think of \( \text{st}_t_{\varepsilon} \) as a name for a function from \( \lambda \)-sequences of members of \( \bar{Q}_\varepsilon \times D \) (thought of as pairs of sequences of the same length \( \lambda \)) to \( \bar{Q}_\varepsilon \times D \) such that if \( \langle \bar{s}, \bar{C} \rangle \in \text{Dom}(\text{st}_t_{\varepsilon}) \) and \( \bar{s} \) has an upper bound, then the first coordinate of \( \text{st}_t_{\varepsilon}(\bar{s}, \bar{C}) \) is such an upper bound (and, of course, any play according to \( \text{st}_t_{\varepsilon} \) is won by Generic). (In a play of \( C^\text{fuzzy}(r(\varepsilon), N[\bar{F}_\varepsilon], \bar{T}(\varepsilon), h(\varepsilon), \bar{Q}_\varepsilon, \bar{F}(\varepsilon), \bar{q}(\varepsilon)) \) only the values of \( \text{st}_t_{\varepsilon} \) at “legal partial plays according to \( \text{st}_t_{\varepsilon} \)” matter, but it is notionally convenient to have \( \text{st}_t_{\varepsilon} \) giving values for all sequences of elements of \( \bar{Q}_\varepsilon \times D \), even if first coordinates are not increasing, as well as for sequences after which Antigeneric should play.)

We will describe the strategy \( \text{st} \) by giving the answers of Generic on intervals \( S \cap [i_\alpha, i_{\alpha+1}) \) for \( \alpha < \lambda \), where, remember, \( (i_\alpha : \alpha < \lambda) \) is the increasing enumeration of \( E_1 \). Aside the Generic player will construct sequences \( \langle r'_j(\varepsilon) : j' < \lambda, \varepsilon \in \zeta \cap N \rangle \) and \( \langle C^{\delta}_j(\varepsilon) : j' \delta \varepsilon < \lambda, \varepsilon \in \zeta \cap N \rangle \) so that
Then for $j$ applying successively $\hat{st}_\xi$ of the play so far. Now Generic looks at ordinals $\varepsilon$ determined as follows. If $r \mathrel{\lhd} r_\alpha^{\varepsilon} \in \mathcal{P}_\varepsilon$ is stronger than all $r_j \mathrel{\lhd} \varepsilon$ for $j \in (i_\alpha + 1) \setminus S$ (the conditions played in the game), then

$$ p^* \models P \quad " \forall j \in (i_\alpha + 1) \setminus S) (r_j \mathrel{\lhd} r_\alpha^{\varepsilon}) \quad ", $$

and

$$ (*)_4 \quad \text{if } \alpha < \lambda, \varepsilon \in w_{\alpha+1} \cap \zeta \text{ and } r_{i_\alpha + 1} \text{ is the condition played by Generic at stage } i_\alpha + 1 \in S, \text{ then} $$

$$ r_{i_\alpha + 1} \mathrel{\lhd} \varepsilon \models P \quad " \forall j' < i_\alpha + 1 \{} (r_j' \mathrel{\lhd} r_{i_\alpha + 1}^{\varepsilon}) \quad ", $$

and

$$ (*)_5 \quad \text{for each } \varepsilon \in N \cap \zeta,$noline

$$ r \mathrel{\lhd} \varepsilon \models P \quad " \langle r_j' \mathrel{\lhd} \varepsilon, \Delta \mathcal{C}_j^{\varepsilon} : j < \lambda \rangle \text{ is a play of the game} $$

$$ \mathcal{C}_\lambda^{\text{fuzzy}}(r(\varepsilon), N[\Gamma_{\varepsilon}], \bar{\varepsilon}(\varepsilon), h(\varepsilon), \bar{\varepsilon}_{\alpha}(\varepsilon), \bar{\varepsilon}_{\alpha}^{\text{fuzzy}}(\varepsilon)) \text{ in which Generic uses } \hat{st}_\varepsilon. $$

So suppose that $\alpha < \lambda, \delta = \min (S \cap [i_\alpha, i_{\alpha+1}])$ and $\langle r_j, C_j : j < \delta \rangle$ is the result of the play so far. Now Generic looks at ordinals $\varepsilon \in w_{\alpha+1} \cap \zeta$. She lets the $\mathcal{P}_\varepsilon$-names $r_j' \mathrel{\lhd} \varepsilon, C_j^{\varepsilon}$ be such that $\langle r_j' \mathrel{\lhd} \varepsilon, \Delta \mathcal{C}_j^{\varepsilon} : j' < i_\alpha + 1 \rangle$ is forced by $r \mathrel{\lhd} \varepsilon$ to be a play of $\mathcal{C}_\lambda^{\text{fuzzy}}(r(\varepsilon), N[\Gamma_{\varepsilon}], \bar{\varepsilon}(\varepsilon), h(\varepsilon), \bar{\varepsilon}_{\alpha}(\varepsilon), \bar{\varepsilon}_{\alpha}^{\text{fuzzy}}(\varepsilon))$ in which the moves are determined as follows. If $\varepsilon \in w_{\alpha+1}$, then we have already the play below $i_\alpha$ and the names $r_j^{\varepsilon} \mathrel{\lhd} i_\alpha, C_j^{\varepsilon} \mathrel{\lhd} i_\alpha \epsilon$ are such that

- if $i_\alpha = \delta$ (i.e., $i_\alpha \in S$ and $r_i, C_i$ have been chosen for $i < i_\alpha$ only), then

$$ r \mathrel{\lhd} \varepsilon \models P \quad " \langle r_j^{\varepsilon} \mathrel{\lhd} i_\alpha, \Delta \mathcal{C}_j^{\varepsilon} \mathrel{\lhd} i_\alpha \rangle \text{ is the value of } \hat{st}_\varepsilon $$

applied to $\langle r_j' \mathrel{\lhd} \varepsilon, \Delta \mathcal{C}_j^{\varepsilon} : j < i_\alpha \rangle, "$

- if $i_\alpha < \delta$ (i.e., $i_\alpha \notin S$ so $r_i, C_i$ are already chosen for $i \leq i_\alpha$), then

$$ r \mathrel{\lhd} \varepsilon \models P \quad " \langle r_j^{\varepsilon} \mathrel{\lhd} i_\alpha, \Delta \mathcal{C}_j^{\varepsilon} \mathrel{\lhd} i_\alpha \rangle \text{ if } (\forall j < i_\alpha) (r_j^{\varepsilon} \mathrel{\lhd} r_{i_\alpha}^{\varepsilon}(\varepsilon)) \text{ then } r_j^{\varepsilon} \mathrel{\lhd} \varepsilon \text{ otherwise } r_j^{\varepsilon} \mathrel{\lhd} \varepsilon \text{ is the first coordinate of } \hat{st}_\varepsilon \text{ applied to the play so far, and } \mathcal{C}_j^{\varepsilon} \mathrel{\lhd} i_\alpha \epsilon = \bigcap_{j \leq i_\alpha} C_j \text{ for all } \varepsilon < \lambda \quad ". $$

Then for $j \in (i_\alpha, i_{\alpha+1})$ (and $\varepsilon \in w_{\alpha} \cap \zeta$) the names $r_j^{\varepsilon} \mathrel{\lhd} \varepsilon, C_j^{\varepsilon} \mathrel{\lhd} \varepsilon$ are determined by applying successively $\hat{st}_\varepsilon$, that is

$$ r \mathrel{\lhd} \varepsilon \models P \quad " \langle r_j^{\varepsilon} \mathrel{\lhd} \varepsilon, \Delta \mathcal{C}_j^{\varepsilon} \mathrel{\lhd} \varepsilon \rangle \text{ is the value of } \hat{st}_\varepsilon $$

applied to $\langle r_j^{\varepsilon} \mathrel{\lhd} \varepsilon, \Delta \mathcal{C}_j^{\varepsilon} : j' < j \rangle, "$

If $\varepsilon \in (w_{\alpha+1} \setminus w_{\alpha}) \cap \zeta$, then the Generic player defines the names $r_j^{\varepsilon} \mathrel{\lhd} \varepsilon, C_j^{\varepsilon} \mathrel{\lhd} \varepsilon$ somewhat like above, but starting with subscript $j = 0$. Thus
Suppose that the sequence determined by applying successively \( \gamma \), \( \delta > \iota \) is the first coordinate of the value of \( \delta \) at \( \langle r_j(\varepsilon), \bigcap_{i<\iota_a} C_i \rangle \) for all \( \xi < \lambda \)

\[
C_0^\xi(\varepsilon) = \bigcap_{i<\iota_a} C_i
\]

\[\text{and } C_0^\xi(\varepsilon) = \bigcap_{i<\iota_a} C_i \text{ for all } \xi < \lambda \]

• if \( \iota_a = \delta \), then

\[
r|\varepsilon| \models_P \psi \quad \text{"} r_0(\varepsilon) = \text{the first coordinate of the value of } \delta \text{ at } \langle r_j(\varepsilon), \bigcap_{i<\iota_a} C_i \rangle \quad \text{"}
\]

and \( C_0^\xi(\varepsilon) = \bigcap_{i<\iota_a} C_i \) for all \( \xi < \lambda \)

• if \( \iota_a < \delta \), then

\[
r|\varepsilon| \models_P \psi \quad \text{"} \text{if } (\forall j < \iota_a)(r_j(\varepsilon) \leq r_{\iota_a}(\varepsilon)) \text{ then } r_0(\varepsilon) = r_{\iota_a}(\varepsilon) \quad \text{"}
\]

otherwise \( r_0(\varepsilon) \) is the first coordinate of the value of \( \delta \) at \( \langle r_j(\varepsilon), \bigcap_{i<\iota_a} C_i \rangle \) for all \( \xi < \lambda \)

Lastly, for \( 0 < j < \iota_{a+1} \) (and \( \varepsilon \in (w_{\iota_a+1} \setminus w_\alpha) \cap \zeta \)) the names \( r_j^*(\varepsilon), C_j^*(\varepsilon) \) are determined by applying successively \( \delta \) like earlier.

This defines the names \( r_j^*(\varepsilon), C_j^*(\varepsilon) \) for \( j < \iota_{a+1}, \xi < \lambda \) and \( \varepsilon \in w_{\iota_a+1} \cap \zeta \). It is straightforward to check that the requirements \((*)_1-(*_3)\) and \((*)_5\) restricted to \( \varepsilon \in w_{\iota_a+1} \cap \zeta \) (and with “\( j < \iota_{a+1} \)” replaced by “\( j < \iota_{a+1} \)” are satisfied.

Next, using the fact that \( P_\zeta \) is \( \lambda \)-closed and \((*)_3\) of the choice above, Generic picks a condition \( r^* \in P_\zeta \) such that

\[
(*)_6 \quad r^* \geq r_j \quad \text{for every } j < \delta,
\]

\[
(*)_7 \quad r^* \models \psi \quad \text{"} r_j^*(\varepsilon) \leq r^*(\varepsilon) \quad \text{"} \quad \text{for every } j < \iota_{a+1} \text{ and } \varepsilon \in w_{\iota_a+1} \cap \zeta,
\]

\[
(*)_8 \quad r^* \models \bigcap \mathcal{T}_\xi, \quad \text{and}
\]

\[
(*)_9 \quad \text{for every } j', \xi < \iota_{a+1} \text{ and } \varepsilon \in w_{\iota_a+1} \cap \zeta, \quad \text{the condition } r^* \models \psi \quad \text{decides the value of } C_j^*(\varepsilon), \quad \text{say } r^* \models \psi \quad \text{"} C_j^*(\varepsilon) = C_j^*(\varepsilon) \quad \text{"}, \quad \text{where } C_j^*(\varepsilon) \in D \cap V.
\]

If \( \iota_a \in S \) (so \( \delta = \iota_a \) is a limit ordinal), then Generic picks a condition \( r^+ \in P_\zeta \) stronger than \( r^* \) and such that for every \( t \in \mathcal{A}_0^\zeta \) and \( \varepsilon \in (w_{\delta} \cap \zeta) \cup \{ \zeta \} \) we have:

\[
(*)_{10} \quad \text{either the conditions } r^+ \models \psi \quad \text{and } q_{\delta}^l(\varepsilon) \models \psi \quad \text{are incompatible, or } q_{\delta}^l(\varepsilon) \models \psi \models \psi, \quad \text{or}
\]

\[
(*)_{11} \quad \text{if } \varepsilon \models w_{\delta} \cap \zeta \text{ and } q_{\delta}^l(\varepsilon) \models \psi, \text{ and } s \in \mathcal{A}_0^\zeta \text{ is such that } t|\varepsilon = s|\varepsilon, \quad \text{then}
\]

\[
either r^+ \models \psi \quad \text{"} q_{\delta}^l(\varepsilon), r^+(\varepsilon) \quad \text{are incompatible } \text{" or } r^+ \models \psi \quad \text{"} q_{\delta}^l(\varepsilon) \models \psi \quad \text{"}. \]

If \( \delta > \iota_a \) (i.e., \( \iota_a \notin S \)) then Generic lets \( r^+ = r^* \). Finally, for every \( j \in [\iota_a, \lambda_{a+1}) \cap S \) she plays

\[
r_j = r^+ \quad \text{and} \quad C_j = E_3 \cap \bigcup\{ C_j^\xi(\varepsilon) : j', \xi < \iota_{a+1}, \varepsilon \in w_{\iota_a+1} \cap \zeta \}.
\]

Plainly, \( r_{\iota_a+1} \) satisfies clause \((*)_4\).

Why does the strategy described above work?

Suppose that \( (r_j, C_j : j < \lambda) \) is a play of the game \( \mathcal{G}(r \upharpoonright \zeta, N, \mathcal{F}[\alpha], P_\zeta, F, \mathcal{F}) \) in which the Generic player used this strategy and let \( \langle r_j^*(\varepsilon) : j' < \lambda, \varepsilon \in \zeta \cap N \rangle \) and \( \langle C_j^*(\varepsilon) : j', \xi < \lambda, \varepsilon \in \zeta \cap N \rangle \) be the sequences she constructed aside.

First let us argue that condition \( 2^{3.4.5}(\beta) \) holds. We will show slightly more than actually needed to help later with clause \( (\alpha) \). Below remember that ordinals \( \gamma(\delta) \) were defined when we picked our quasi-\( D \)-diamond \( \mathcal{F} \), and if \( \varepsilon < \gamma(\delta) \) then the sequence \( \langle h^{(\varepsilon+1)} \circ F_\beta(\alpha) : \alpha < \delta \rangle \) is \( \mathcal{F}^{(\varepsilon+1)} \)-exact. Now, suppose that a limit ordinal \( \delta \in S \cap \bigcup_{j<\delta} C_j \) (so in particular \( \delta \in E_3 \)) is such that
(ⅳ) \( w_{\delta} \cap \zeta \subseteq \gamma(\delta) \) and \((\forall \alpha < \delta)(\exists j < \delta)(h^{[\zeta]} \circ F_3^{\delta}(\alpha) \leq r_j)\).

(So then \((\forall \alpha < \delta)(h^{[\zeta]} \circ F_3^{\delta}(\alpha) \leq r_{\delta})\). Note also that by the choice of \(E_3\), if \(h^{[\zeta]} \circ F_3^{\delta}(\alpha) : \alpha < \delta \) is \(\bar{T}^{[\zeta]}\)-exact, then \(w_{\delta} \cap \zeta \subseteq \gamma(\delta)\).)

We are going to choose \(t \in X^{[\zeta]}_{\delta} \) and show that \(q_{\delta,t}^{\zeta} \leq r_{\delta}\). We do this by induction on \(\varepsilon \in (\zeta + 1) \cap N\), defining \(t[\varepsilon] \in T_{\delta}\) and showing that \(q_{\delta,t[\varepsilon]}^{\zeta}[\varepsilon] \leq r_{\delta}[\varepsilon]\) (and for \(\varepsilon = \zeta\) we get the desired conclusion). Limit stages and the initial stage \(\varepsilon = 0\) are trivial, so assume that we have defined \(t[\varepsilon]\) and have shown \(q_{\delta,t[\varepsilon]}^{\zeta}[\varepsilon] \leq r_{\delta}[\varepsilon]\) (where \(\varepsilon \in \zeta \cap N\)), and let us consider the restrictions to \(\varepsilon + 1\).

If \(\varepsilon \notin w_{\delta}\) then \(t[(\varepsilon + 1)] = t[\varepsilon]\) (so it has been already defined). Suppose also that \(\varepsilon \in \text{Dom}(q_{\delta,t[\varepsilon]}^{\zeta})\) (otherwise there is nothing to do). Look at the clause (v) of the choice of \(q_{\delta,t[\varepsilon]}\) at the beginning: \(r_{\delta} \geq r \geq p_{\delta}\) (and (ⅳ)) implies that

\[
\text{then } \forall \alpha < \delta \quad q_{\delta,t[\varepsilon]}^{\zeta}[\varepsilon] \leq p_\alpha[\varepsilon] \leq r_{\delta}[\varepsilon] \]

But then also by the clause (vii) there, \(r_{\delta}[\varepsilon] \models \langle q_{\delta,t[\varepsilon]}^{\zeta}[\varepsilon] \leq p_\alpha[\varepsilon] \leq r_{\delta}[\varepsilon] \rangle\), so we are done.

Suppose now that \(\varepsilon \in w_{\delta} \cap \zeta\) (and thus \(\varepsilon < \gamma(\delta)\)). Since \(\delta \in E_3\), we know that arriving to stage \(\delta\) of the game, Generic has already defined \(r_j^{\zeta}(\varepsilon), C_j^{\zeta}(\varepsilon)\) for \(j < \delta\) and \(\xi < \lambda\) (remember \((*)_2\)). Moreover, the condition \(r_{\delta}[\varepsilon]\) forces that (remember: \(h^{[\zeta]} \circ F_3^{\delta}(\alpha) : \alpha < \delta\) is \(\bar{T}^{[\zeta]}\)-exact):

- the sequence \(\langle h^{(\varepsilon)} \circ F_3^{\delta}(\alpha) : \alpha < \delta \rangle\) is \(\leq \bar{Q}_\varepsilon\)-increasing \(\bar{T}^{(\varepsilon)}\)-exact, and
- \(\langle r_j^{\zeta}(\varepsilon), \Delta C_j^{\zeta}(\varepsilon) : j < \delta \rangle\) is a play according to \(\text{st}_{\varepsilon}\) (by \((*)_9\)), and
- \(\delta \in \bar{S}^{(\varepsilon)} \cap \bigcap_{j, \xi < \delta} \Delta C_j^{\zeta}(\varepsilon)\) (remember \((*)_9\) and the choice of \(C_{\alpha + 1}\) for \(\alpha < \delta\)),
  and hence also \(\delta \in \bar{S}^{(\varepsilon)} \cap \bigcap_{j, \xi < \delta} \Delta C_j^{\zeta}(\varepsilon)\),
- \((\forall j < \delta)(\exists j' < \delta)(r_j^{\zeta}(\varepsilon) \leq r_{j'}^{\zeta}(\varepsilon))\) and \((\forall j < \delta)(\exists j' < \delta)(r_j^{\zeta}(\varepsilon) \leq r_{j'}^{\zeta}(\varepsilon))\) (by \((*)_3 + (**)_4\)), so also
- \((\forall \alpha < \delta)(\exists j < \delta)(h^{(\zeta)} \circ F_3^{\delta}(\alpha) \leq \bar{Q}_\varepsilon r_j^{\zeta}(\varepsilon))\).

Since \(\text{st}_{\varepsilon}\) is a name for a winning strategy, we may conclude that (by \((**)_7\))

\[
\forall \alpha < \delta \quad \exists x \in X^{[\zeta]}_{\delta}(\varepsilon) \quad (h^{(\varepsilon)} \circ F_3^{\delta}(\alpha) \leq \bar{Q}_\varepsilon r_j^{\zeta}(\varepsilon) \leq r_{\delta}(\varepsilon))\]

Now look at \((**)_{11}\) remembering clause (vi) of the choice of \(\bar{q}\): by them there is a unique \(x \in \bar{P}_{\delta}(\zeta)\) such that letting \((t)[\varepsilon] = x\) we get \(t[(\varepsilon + 1)] \in T_{\delta}\) satisfying \(q_{\delta,t[\varepsilon + 1]}^{\zeta}[\varepsilon] \leq r_{\delta}[\varepsilon] \).)

This completes the inductive proof of \(\text{A.3.4.5}(\beta)\).

Why does \(\text{A.3.4.5}(\alpha)\) hold? To show this condition, it is enough to prove that (ⅴ) holds for unboundedly many \(\delta \in \bar{S} \cap \Delta C_j\) (remember clause (v) of the choice of \(q_{\delta,t[\varepsilon]}^{\zeta}\) and what we have already shown). We do this considering various characters of \(\zeta\).
Clearly, if $\beta < \alpha < \lambda$, then $s_\beta[(\varepsilon_\alpha \cap \varepsilon_\beta)] \leq s_\alpha[(\varepsilon_\alpha \cap \varepsilon_\beta)]$,

$$(d)_\alpha \ s_\alpha \in \bigcap_{\gamma < \alpha} I_{\gamma}^{[\varepsilon_\alpha]}.$$ 

So suppose that we have defined $s_\beta$’s for $\beta < \alpha$. For $\beta < \alpha$ let

$I_{\alpha, \beta} = \{s \in P_{\varepsilon_\alpha} : \text{ either } s_\beta[\varepsilon_\alpha \leq s,} \n
or s_\beta[\varepsilon_\alpha, s \text{ are incompatible }\}.$

Clearly $I_{\alpha, \beta} \subseteq N$ is an open dense subset of $P_{\varepsilon_\alpha}$. Since the condition $r[\varepsilon_\alpha]$ is $(N, P_{\varepsilon_\alpha})$-generic and the increasing sequence $(r_j[\varepsilon_\alpha] : j < \lambda)$ enters all open dense subsets of $P_{\varepsilon_\alpha}$ from $N$ (by $(*)_\lambda$), we may find $s_\alpha \in \bigcap_{\beta < \alpha} I_{\alpha, \beta} \cap \bigcap_{j < \alpha} I_{j}^{[\varepsilon_\alpha]}$ such that $s_\alpha \leq r_j[\varepsilon_\alpha]$ for all large enough $j < \lambda$. By $(a)_\alpha$ (for $\beta < \alpha$) we conclude that $s_\alpha$ and $s_\beta[\varepsilon_\alpha]$ cannot be incompatible, and hence clauses $(a)_\alpha$–$(d)_\alpha$ are satisfied.

Now, let conditions $s'_\alpha \in P_{\varepsilon} \cap N$ (for $\alpha < \lambda$) be such that $\text{Dom}(s'_\alpha) = \bigcup_{\beta \leq \alpha} \text{Dom}(s_\beta)$ and $s'_\alpha(\varepsilon)$ (for $\varepsilon \in \text{Dom}(s'_\alpha)$) is the $<^*_{\alpha}$-first $P_{\varepsilon}$-name for a condition in $\mathbb{Q}$ satisfying

$s'_\alpha(\varepsilon) \vDash \text{ "}(\forall \beta < \alpha)(s'_\beta(\varepsilon) \leq s'_\alpha(\varepsilon))\text{"}$

and if there is a $\gamma \in [\alpha, \lambda)$ such that $(\forall \beta < \alpha)(s'_\beta(\varepsilon) \leq s_\gamma(\varepsilon))$

then $s'_\alpha(\varepsilon) = s_\alpha(\varepsilon)$ for the first such $\gamma$.

Then the sequence $\langle s'_\alpha : \alpha < \lambda \rangle$ is $\leq_{P_{\varepsilon}}$-increasing and

$(\forall \alpha < \lambda)(\forall \varepsilon < \varepsilon_\alpha)(s_\alpha(\varepsilon) \vDash \text{ "} s'_\alpha(\varepsilon) \vDash \text{"}).$

So it follows from $(d)_\alpha$ that for each $\varepsilon \in u$ there is a club $C' \subseteq \lambda$ such that $\langle s'_\alpha \varepsilon : \alpha < \delta \rangle$ is $\mathbb{D}[\varepsilon]$-exact for all $\delta \in C'$. Also, as clearly $\langle \forall \alpha < \lambda)(\exists j < \lambda)(s'_\alpha \leq r_j)$, we may pick a club $C^*$ of $\lambda$ such that

$(\forall \delta \in C^*)(\forall \alpha < \delta)(\exists \delta < \delta)(s'_\alpha \leq r_j).$

Now, as $\mathbb{F}'$ is a $D$–diamond, for unboundedly many $\delta \in S \cap \Delta C' \cap \bigcap_{j < \lambda} C' \cap C^*$ we have $\langle s'_\alpha : \alpha < \delta \rangle = \langle h \circ F'_\delta(\alpha) : \alpha < \delta \rangle$. Plainly, defining $F'_\delta$ for those $\delta$ we had clause $(\circ)_\lambda$ with $\gamma(\delta) \geq \zeta$ and hence $\langle s'_\alpha : \alpha < \delta \rangle = \langle h[\varepsilon] \circ F_{\delta}(\alpha) : \alpha < \delta \rangle$. Therefore $(\exists)_\beta$ holds for those $\delta$ (remember the choice of $C^*$).

$\zeta$ is a limit ordinal of cofinality $\geq \lambda$.

Let $\langle \varepsilon_\alpha : \alpha < \lambda \rangle \subseteq \zeta \cap N$ be an increasing continuous sequence cofinal with $\zeta \cap N$, $\varepsilon_0 = 0$. By induction on $\alpha < \lambda$ choose conditions $s_\alpha$ such that

$(a)_\alpha \ (\exists j < \lambda)(s_\alpha \leq r_j),$

$(b)_\alpha \ s_\alpha \in P_{\varepsilon_\alpha} \cap N,$

$(c)_\alpha \text{ if } \beta < \alpha < \lambda, \text{ then } s_\beta \leq s_\alpha,$

$(d)_\alpha \ s_\alpha \in \bigcap_{\gamma < \alpha} I_{\gamma}^{[\varepsilon_\alpha]}.$

[Possible as $r[\varepsilon_\alpha]$ is $(N, P_{\varepsilon_\alpha})$-generic and by $(*)_\lambda$. For each $\alpha < \lambda$, for some club $C'_\alpha$ of $\lambda$ we have

$(\forall \delta \in C'_\alpha)(\langle s_\gamma \varepsilon : \gamma < \delta \rangle$ is $\mathbb{D}[\varepsilon_\alpha]$–exact).]

Also for a club $C^*$ of $\lambda$ we have $(\forall \delta \in C^*)(w_\delta \cap \zeta \subseteq \varepsilon_\delta)$. Like before, as $\mathbb{F}'$ is a $D$–diamond, for unboundedly many $\delta \in S \cap \Delta C' \cap \bigcap_{j < \lambda} C' \cap C^*$ we have $\langle s_\alpha : \alpha < \delta \rangle = \langle h \circ F'_\delta(\alpha) : \alpha < \delta \rangle$. Plainly, for those $\delta$ we have $\gamma(\delta) = \varepsilon_\delta$ and also $\langle s_\alpha : \alpha < \delta \rangle = \langle h[\varepsilon] \circ F_{\delta}(\alpha) : \alpha < \delta \rangle$, and thus $(\exists)_\beta$ holds (remember clause $(a)_\alpha$).
$\zeta$ is a successor ordinal.
Like before (remember that, letting $\zeta = \zeta' + 1$, the condition $r\upharpoonright \zeta'$ is $(N, \mathbb{P}_\zeta)$–generic and it forces that $r(\zeta')$ is $(N[\mathbb{G}_\zeta], \mathbb{Q}_\zeta)$–generic).

Remark A.3.11. In A.3.1 we may have $\bar{S} = \langle S_a : a \in W \rangle$ and $\bar{D} = \langle D_a : a \in W \rangle$ be such that each $D_a$ is a normal filter on $\lambda$, $S_a \in D_a^+$ satisfies the relevant demands of A.3.2(1), and require that there is a $D_a$ diamond $\langle F_\delta^a : \delta \in S_a \rangle$. Then in all definitions and results we may replace $D, S$ by $D_a, S_a$, where $a = N \cap A$. In particular, this way we get the notions of fuzzy properness over quasi $\bar{D}$–diamonds which behave nicely in iterations.
Part B

Building suitably proper forcing notions

B.4. A CREATURE–FREE EXAMPLE

In this section we show that a natural forcing notion uniformizing colourings on ladder systems is fuzzy proper. (This forcing is a relative of $Q^*$ from [15, 4.6–4.8].)

Here we assume that:

**Context B.4.1.**

1. $\lambda^+ > \lambda$ is a regular cardinal, $A = H_{<\lambda}(\lambda^+)$ and $W \subseteq [A]^\lambda$ is as in [A.3.1] and $\lambda \subseteq a$ for each $a \in W$.
2. $\xi^* < \lambda$, $S^* \subseteq S_{\lambda^+}^\delta \overset{\text{def}}{=} \{\delta < \lambda^+: \text{cf}(\delta) = \lambda\}$ and for $\beta \in S^*$:
   - (a) $B_\beta \subseteq \beta$ is a club of $\beta$ of order type $\text{otp}(B_\beta) = \lambda$,
   - (b) $h_\beta : B_\beta \rightarrow \xi^*$.

Let $\bar{B} = \langle B_\beta : \beta \in S^* \rangle$, $\bar{h} = \langle h_\beta : \beta \in S^* \rangle$.

The forcing notion $Q^* = Q^*(S^*, \bar{B}, \bar{h})$ is defined as follows:

**a condition in $Q^*$** is a tuple $p = (u^p, v^p, \bar{e}^p, h^p)$ such that

1. $u^p \in [\lambda^+]^{<\lambda}$, $v^p \in [S^*]^{<\lambda}$,
2. $\bar{e}^p = \langle e^p_\beta : \beta \in v^p \rangle$, where each $e^p_\beta$ is a closed bounded subset of $B_\beta$, and $e^p_\beta \subseteq u^p$, and
3. $\sup(e^p_\beta) = \sup(u^p \cap \beta)$ (for $\beta \in v^p$), and if $\beta_1 < \beta_2$ are from $v^p$, then $\sup(e^p_{\beta_2}) > \beta_1$ and $\sup(e^p_{\beta_1}) > \sup(B_{\beta_2} \cap \beta_1)$,
4. $h^p : u^p \rightarrow \xi^*$ is such that $$(\forall \alpha \in v^p)(\forall e^p_\beta)(h^p(\alpha) = h_\beta(\alpha));$$

the order $\leq$ of $Q^*$ is such that $p \leq q$ if and only if $u^p \subseteq u^q$, $h^p \subseteq h^q$, $v^p \subseteq v^q$, and for each $\beta \in v^p$ the set $e^q_\beta$ is an end-extension of $e^p_\beta$.

A tuple $p = (u^p, v^p, \bar{e}^p, h^p)$ satisfying clauses (a), (b) and (d) above will be called a pre-condition. Note that every pre-condition can be extended to a condition in $Q^*$.

**Proposition B.4.2.**

1. The forcing notion $Q^*$ is $\lambda^+$–complete, it satisfies the $\lambda^+$–chain condition and $|Q^*| = \lambda^+$.
2. If $p \in Q^*$, $\alpha < \lambda^+$, $\beta \in S^*$ and $\delta < \lambda$, then there is a condition $q \geq p$ such that $\alpha \in u^q$, $\beta \in v^q$ and $(\forall \beta' \in v^p)(\text{otp}(e^q_{\beta'}) > \delta)$.

**Proof.** (1) Verification of the chain condition is a straightforward application of the $\Delta$–lemma. To check that $Q^*$ is $\lambda^+$–complete suppose that $\langle p_i : i < j \rangle$ is a $\leq_{Q^*}$–increasing sequence of conditions from $Q^*$, $j < \lambda$. Let $r = (u^r, v^r, \bar{e}^r, h^r)$ be such that

\[
\begin{align*}
u^r &= \bigcup_{i<j} v^{p_i}, \quad \text{and for } \beta \in v^r \\
e^r_\beta &= \bigcup_{i<j} \{e^p_\beta : \beta \in v^{p_i} \land i < j \} \cup \{\sup(\bigcup_{i<j} \{e^p_\beta : \beta \in v^{p_i} \land i < j \})\} \\
u^r &= \bigcup_{i<j} v^{p_i} \cup \bigcup_{i<j} \{e^r_\beta : \beta \in v^r\} \\
h^r &\supseteq \bigcup_{i<j} h^{p_i},
\end{align*}
\]

(2) If $p \in Q^*$, $\alpha < \lambda^+$, $\beta \in S^*$ and $\delta < \lambda$, then there is a condition $q \geq p$ such that $\alpha \in u^q$, $\beta \in v^q$ and $(\forall \beta' \in v^q)(\text{otp}(e^q_{\beta'}) > \delta)$.
and if $\alpha \in e_0^* \setminus \bigcup \{ e_{ij}^* : \beta \in v^p, i < j \}$, then $h^*(\alpha) = h_\beta(\alpha)$. Using clause (c) for $p_1$'s one easily sees that $r$ is a pre-condition. Extend it to a condition $q \in \mathcal{Q}^*$. 

(2) Should be clear. □

**Proposition B.4.3.** $\mathcal{Q}^*$ is fuzzy proper for $W$.

*Proof.* Suppose that $D$ is a normal filter on $\lambda$ such that there is a $D$–diamond. We will show that $\mathcal{Q}^*$ is fuzzy proper over quasi $D$–diamonds. First we define a $\lambda$–base $(\mathcal{R}^*, \mathcal{Q}^*)$ for $\mathcal{Q}^*$ over $W$. Let $\mathcal{R}^*$ be the set of all triples $(p, \delta, x)$ such that $p \in \mathcal{Q}^*$, $\delta \in \lambda$ and $x$ is a function with $\text{Dom}(x) \subseteq u^p$ and $(\forall \alpha \in \text{Dom}(x))(h^p(\alpha) = x(\alpha))$.

Now suppose that $a \in W$ and let $\pi_a$ be the $<_\chi$–first one-to-one mapping from $a \cap \lambda^+$ to $\lambda$. For a limit ordinal $\delta < \lambda$ we put

$$x_0^\delta = (\pi_a)^{-1}[\delta] \cup \{ \alpha < \lambda^+ : \alpha = \sup(\alpha \cap (\pi_a)^{-1}[\delta]) \},$$

and then

$$\mathcal{Y}^*_a(\delta) = \{ x : x \text{ a function from } x_0^\delta \cap a \text{ to } \xi^* \}. $$

This defines $\mathcal{Y}^*_a$ and $\mathcal{Q}^*_a = \{ \mathcal{Y}^*_a : a \in W \}$. It is easy to check that $(\mathcal{R}^*, \mathcal{Q}^*)$ is a $\lambda$–base for $\mathcal{Q}^*$ (for [3.3.3 c] use repeatedly [3.4.2]). Assume now that

- $N < (\mathcal{H}(\chi), e, <_\chi)$, $|N| = \lambda$, $<_\lambda N \subseteq N$, $\lambda, \mathcal{Q}^*, B, h, S^*, \mathcal{R}^* \in N$, and
- $a \overset{\text{def}}{=} N \cap A \in W$, $p \in \mathcal{Q}^* \cap N$,
- $\mathcal{I} = (\mathcal{I}_\xi : \xi < \lambda)$ lists all open dense subsets of $\mathcal{Q}^*$ from $N$,
- $h : \lambda \rightarrow N$ satisfies $\mathcal{Q}^* \cap N \subseteq \text{Rng}(h)$, and
- $\mathcal{F} = (F_\delta : \delta \in S)$ is a quasi $D$–diamond for $(N, h, \mathcal{Q}^*)$ and $\bar{q}$ is a fuzzy candidate over $\mathcal{F}$.

For limit $\delta \in S$ let $\mathcal{Y}(\delta) = \mathcal{Y}(N, \mathcal{Q}^*, h, \bar{F}, \mathcal{R}^*, \mathcal{Q}^*, \delta)$ be as defined in [3.3.4 (3)] (and thus $\bar{q} = \langle q_{\delta, x} : \delta \in S \text{ is limit } \& x \in \mathcal{Y}(\delta) \rangle$). Also let $E_0$ be the set of all $\delta < \lambda$ which are limits of members of $\lambda \setminus S$ (so it is a club of $\lambda$).

We are going to show that the condition $p$ is $(\mathcal{R}^*, \mathcal{Q}^*)$–fuzzy generic for $\bar{q}$. Note that, as $\mathcal{Q}^*$ satisfies the $\lambda^+$–cc, the condition $p$ is $(N, \mathcal{Q}^*)$–generic (in the standard sense). So, by [3.3.5 (3)], it is enough to give a strategy of the Generic player in the game $\mathcal{C}^{\text{fuzzy}}(p, N, \mathcal{I}, h, \mathcal{Q}^*, \bar{F}, \bar{q})$ which guarantees that the result $\langle r_i, C_i : i < \lambda \rangle$ of the play satisfies [3.3.5 (3)]($\beta$).

Suppose that we arrive to a stage $\delta \in S$ and $\langle r_i, C_i : i < \delta \rangle$ is the sequence played so far. First, Generic picks the $<_\chi$–first condition $r_\delta'$ stronger than all $r_i$'s played so far and such that

- if $\delta$ is limit and $(\exists x \in \mathcal{Y}(\delta))(\exists r \in \mathcal{Q}^*)(q_{\delta, x} \leq r \& (\forall i < \delta)(r_i \leq r))$,

then $q_{\delta, x} \leq r_\delta'$ for some $x \in \mathcal{Y}(\delta)$.

Then she plays the $<_\chi$–first condition $r_\delta$ above $r_\delta'$ such that

- if $\beta \in v^r$, then $\text{otp}(e_{n}^r) > \delta$, and

- $(*)_2 (\pi_a)^{-1}[\delta] \subseteq v^r $ and $(\pi_a)^{-1}[\delta] \cap S^* \subseteq v^r$. 

The set $C_\delta$ played at this stage is $(\alpha, \lambda) \cap E_0$, where $\alpha$ is the first ordinal such that

- $(*)_3 \pi_a[u^r \cap N] \subseteq \alpha$, and the set

$$\{ q \in \mathcal{Q}^* : (\pi_a)^{-1}[\delta] \subseteq u^q \& (\pi_a)^{-1}[\delta] \cap S^* \subseteq v^q \& (\forall \beta \in v^q)(\text{otp}(e_{\beta}^q) > \delta) \}$$

(which is an open dense subset of $\mathcal{Q}^*$ from $N$; remember [3.3.2] is in $\{ \mathcal{I}_\xi : \xi < \alpha \}$,

- $(*)_4 \text{ otp}(B_\beta \cap (\sup(e_{\beta}^q) + 1)) < \alpha$ for all $\beta \in v^r$,
One easily verifies that the above conditions indeed define a pre-condition (remem-
ber (**5)) if \( \beta \in v^* \) and \( a \cap \beta \setminus (\sup(e_\beta^\alpha) + 1) \neq \emptyset \), then there is \( \gamma \in a \cap \beta \setminus (\sup(e_\beta^\alpha) + 1) \) with \( \pi_\gamma(\gamma) < \alpha \).

Why does this strategy work (i.e., why does it ensure (**5))? Let \( (r_i, C_i : i < \lambda) \) be a play according to this strategy, and suppose that \( \delta \in S \cap \Delta \cap C_i \) is a limit ordinal such that \( (h \circ F_\delta(\alpha) : \alpha < \delta) \) is a \( \leq_{\text{Q}} \) -increasing \( \mathcal{I} \)-exact sequence of conditions from \( \mathbb{Q}^* \cap N \) such that \((\forall \alpha < \delta)(\exists i < \delta)(h \circ F_\delta(\alpha) \leq r_i)\).

Note that then

\[
(*)_6 \text{ if } \beta \in \bigcup_{i < \delta} v^{r_i}, \text{ then } \mathrm{otp}(\bigcup_{i < \delta} e_\beta^{r_i}) = \delta \text{ and } \bigcup_{i < \delta} e_\beta^{r_i} \text{ is an unbounded subset of } \{ \varepsilon \in B_\beta : \mathrm{otp}(\varepsilon \cap B_\beta) < \delta \}, \text{ and }
\]

\[
(*)_7 \text{ if } \bigcup_{i < \delta} v^{r_i} \cap N = (\pi_\delta)^{-1}[\delta] = \bigcup_{\alpha < \delta} u^{h \circ F_\delta(\alpha)} \text{ and } \bigcup_{i < \delta} v^{r_i} \cap N = (\pi_\delta)^{-1}[\delta] \cap S^* = \bigcup_{\alpha < \delta} v^{h \circ F_\delta(\alpha)}.
\]

We want to show that

\[
\square \text{ (for some } x \in \mathcal{Y}(\delta), \text{ there is a common upper bound to } \{ r_i : i < \delta \} \cup \{ q_\delta, x \} \text{ (which, by the definition of our strategy, will finish the proof). For } \beta \in S^* \text{ let } \gamma_\beta \in B_\beta \text{ be such that } \mathrm{otp}(B_\beta \cap \gamma_\beta) = \delta. \text{ Now, let a pre-condition } r'' = (u^{r''}, v^{r''}, e^{r''}, h^{r''}) \text{ be such that }
\]

- \( v^{r''} = \bigcup_{i < \delta} v^{r_i}, u^{r''} = \bigcup_{i < \delta} u^{r_i} \cup \{ \gamma_\beta : \beta \in v^{r''} \} \)
- \( e^{r''}_\beta = \bigcup_{\beta \in v^{r''}} e^{r_i}_\beta \cup \{ \gamma_\beta \} \) (for \( \beta \in v^{r''} \)), and
- \( h^{r''} : \bigcup_{i < \delta} h^{r_i} \subseteq h^{r''} \) and \( h^{r''}(\gamma_\beta) = h_\beta(\gamma_\beta) \).

One easily verifies that the above conditions indeed define a pre-condition (remember (**6)). Also, note that if \( \beta \in v^{r''} \), then \( \gamma_\beta \in e^{r''}_\beta \setminus \bigcup_{i < \delta} e^{r_i}_\beta \) and each \( e^{r''}_\beta \) is a proper subset of \( \bigcup_{i < \delta} e^{r_i}_\beta \). Moreover, if \( \beta \in v^{r''} \) and \( \gamma_\beta \in N \), then \( \gamma_\beta = \sup(u^{r''} \cap N \cap \gamma_\beta) = \sup((\pi_\delta)^{-1}[\delta] \cap \gamma_\beta) \) (by (**5), (**7); remember also (**5)). Now, extend \( r'' \) to a pre-condition \( r''' \) such that \( u^{r'''} = u^{r''} \cup x_0 \), \( v^{r'''} = v^{r''} \) and \( e^{r''}_\beta = e^{r_i}_\beta \) for \( \beta \in v^{r'''} \) (clearly possible). Let \( x = h^{r'''}[x_0] \) (note that \( x_0 \subseteq a \)). Since \( r''' \) is stronger than all \( h \circ F_\delta(\alpha) \) (for \( \alpha < \delta \)), any condition stronger than \( r''' \) witnesses that \( x \in \mathcal{Y}(\delta) \). Now we put

- \( u^* = u^{q_\delta, x} \cup u^{r''}, v^* = v^{q_\delta, x} \cup v^{r''}, h^* = h^{q_\delta, x} \cup h^{r''} \)
- if \( \beta \in v^{q_\delta, x} \), then \( e^*_\beta = e^{q_\delta, x}_\beta \), and if \( \beta \in v^{r''} \setminus N \), then \( e^*_\beta = e^{r''}_\beta \).

Note that \( h^* \) is a function from \( u^* \) to \( \xi^* \) by (**7) (remember the choice of \( x \)) and that \( q_\delta, x \in N \) is stronger than all \( h \circ F_\delta(\alpha) \)'s. Also, if \( \beta \in v^{r''} \setminus N \) then \( (\beta \in v^{q_\delta, x} \) and \( e^{q_\delta, x} \) is an end-extension of \( e^{r''}_\beta \) (remember (**5), (**8)). Hence \((u^*, v^*, e^*, h^*)\) is a pre-condition stronger than both \( q_\delta, x \) and \( r''' \). Extending it to a condition in \( \mathbb{Q}^* \) we conclude (\square), thus completing the proof of (**3).

**Corollary B.4.4.** Assume that \( \lambda \) is a strongly inaccessible cardinal, \( 2^\lambda = \lambda^+ \), \( 2^{\lambda^+} = \lambda^{++} \) and \( D \) is a normal filter on \( \lambda \) such that there is a \( D \)-diamond. Then there is a forcing notion \( \mathbb{P} \) such that:
• $\mathbb{P}$ is $\lambda$–complete weakly fuzzy proper over quasi $D$–diamonds for $W$ and it satisfies the $\lambda^{++}$–cc.

• in $V^\mathbb{P}$, $2^\lambda = 2^{\lambda^+}$ and for every $\xi^*, S^\ast, \bar{B}, \bar{h}$ as in $[4.4,1](2)$ there is $h : \lambda^{++} \rightarrow \xi^*$ such that for every $\beta \in S^\ast$ the set

$$\{\alpha \in B_\beta : h_\beta(\alpha) = h(\alpha)\}$$

contains a club.

**Proof.** The forcing notion $\mathbb{P}$ is the limit $\mathbb{P}_\lambda^{++}$ of a $\lambda$–support iteration $\dot{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \dot{Q}_\alpha : \alpha < \lambda^{++} \rangle$, where each $\dot{Q}_\alpha$ is forced to be $\mathbb{Q}^*(S_\lambda^{\alpha^+}, \bar{B}_\alpha, \bar{h}_\alpha)$ for some $\bar{B}_\alpha, \bar{h}_\alpha$. Then, by [4.1.9] [4.3.10] and [4.4.3] we are sure that $\mathbb{P}$ satisfies the $\lambda^{++}$–cc, it is weakly fuzzy proper over quasi $D$–diamonds for $W$ and it has a dense subset of size $\lambda^{++}$. Consequently we may arrange suitable bookkeeping to take care of all $\mathbb{P}_{\xi^+}$–names $\bar{B}, \bar{h}$ for objects as in $[4.4,1](2)$ – the details and the rest should be clear. □

### B.5. Trees and Creatures

Let us introduce the notation used in the forcing notions we want to build. The terminology here is somewhat parallel to that of [44] §1.2, 1.3, but there are some differences as the context is different. We start with the tree case.

**Definition B.5.1.** Let $H : \lambda \rightarrow \mathcal{H}(\lambda^+)$. 

1. A $\lambda$–tree creature for $H$ is a tuple

$$t = (\eta, \text{dis}, \text{pos}, \text{nor}) = (\eta[t], \text{dis}[t], \text{pos}[t], \text{nor}[t])$$

such that $\text{dis} \in \mathcal{H}(\lambda^+)$, $\text{nor} \in \lambda + 1$,

$$\eta \in \bigcup_{\alpha < \lambda} \prod_{\beta < \alpha} H(\beta), \quad \text{and} \quad \emptyset \neq \text{pos} \subseteq \{\nu \in \bigcup_{\alpha < \lambda} \prod_{\beta < \alpha} H(\beta) : \eta < \nu\}.$$ 

TCR$^\lambda[H]$ is the family of all $\lambda$–tree creatures for $H$.

For $\eta \in \bigcup_{\alpha < \lambda} \prod_{\beta < \alpha} H(\beta)$ we let $\text{TCR}^\lambda_{\eta}[H] = \{t \in \text{TCR}^\lambda[H] : \eta[t] = \eta\}$.

2. Let $K \subseteq \text{TCR}^\lambda[H]$. A tree–composition operation on $K$ is a mapping $\Sigma$ with values in $\mathcal{P}(K)$ and the domain consisting of systems $(t_\nu : \nu \in \hat{T})$ such that

- $T$ is a complete $\lambda$–quasi tree of height $\text{ht}(T) < \lambda$,
- for each $\nu \in \hat{T}$, $t_\nu \in K$ satisfies $\text{succ}(\nu) = \text{pos}[t_\nu]$,
- and
- if $t \in \Sigma(t_\nu : \nu \in \hat{T})$, then $\eta[t] = \text{root}(T)$ and $\text{pos}[t] \subseteq \text{max}(T)$,
- if $t \in \Sigma(t_\nu : \nu \in \hat{T})$ and $t_\nu \in \Sigma(s_\rho^\nu : \rho \in \hat{T}_\nu)$ (for $\nu \in \hat{T}$), then $t \in \Sigma(s_\rho^\nu : \rho \in \bigcup_{\nu \in \hat{T}} \hat{T}_\nu)$, and
- for each $t \in K$ we have $\langle t \rangle \in \text{dom}(\Sigma)$ and $t \in \Sigma(t)$.

Then $(K, \Sigma)$ is called a $\lambda$–tree creating pair (for $H$).

3. A $\lambda$–tree creating pair $(K, \Sigma)$ is local if

- $(t_\nu : \nu \in T) \in \text{Dom}(\Sigma)$ implies $\text{ht}(T) = \text{lh}(\text{root}(T)) + 1$ (and so $T = \{\text{root}(T)\} \cup \text{pos}[\text{root}(T)]$), and
- $t' \in \Sigma(t)$ implies $\text{nor}[t'] \leq \text{nor}[t]$.
We say that $(K, \Sigma)$ is very local if, additionally, for every $\nu \in \bigcup_{\alpha < \lambda} \prod_{\beta < \lambda} H(\beta)$ such that $K \cap TCR_p^{\Sigma} H \neq \emptyset$ there is $t^*_\nu \in K \cap TCR_p^{\Sigma} H$ satisfying $(\forall t \in K \cap TCR_p^{\Sigma} H)(t \in \Sigma ylabel t^*_\nu)$. The tree creature $t^*_\nu$ may be called the minimal creature at $\nu$.

(4) If $(K, \Sigma)$ is a very local $\lambda$–tree creating pair, then the minimal tree $T^*$ for $(K, \Sigma)$ and the minimal condition $p^*$ for $(K, \Sigma)$ are defined by

$$T^* = T^*(K, \Sigma) = \{ \eta \in \bigcup_{\alpha < \lambda} \prod_{\beta < \lambda} H(\beta) : (\forall \alpha < \text{lh}(\eta))(\eta[(\alpha + 1) \in \text{pos}[t^*_\eta]]) \}$$

$$p^* = p^*(K, \Sigma) = \{ t^*_\nu : \nu \in T^* \}.$$ 

[Note that, in the general case, $T^*$ could be empty, but in real applications this can be easily avoided.]

**Definition B.5.2.** Let $(K, \Sigma)$ be a $\lambda$–tree creating pair for $H$.

(1) We define the forcing notion $\mathcal{Q}^{\text{tree}}_1(K, \Sigma)$ by:

- **conditions** are systems $p = \langle t_\eta : \eta \in T \rangle$ such that

  a. $\emptyset \neq T \subseteq \bigcup_{\alpha < \lambda} \prod_{\beta < \alpha} H(\beta)$ is a complete $\lambda$–quasi tree with $\text{max}(T) = \emptyset$;

  b. $t_\eta \in TCR_p^{\Sigma} \cap K$ and $\text{pos}[t_\eta] = \text{succr}(\eta)$;

  c. for every $\eta \in \text{lim}_T(\lambda)$, $\lim(\lambda)_{t_\eta : \alpha < \lambda} : \eta \alpha T \alpha = \lambda$;

- **the order** be given by:

  $$\langle t^1_\eta : \eta \in T^1 \rangle \leq \langle t^2_\eta : \eta \in T^2 \rangle$$ if and only if

  $T^2 \subseteq T^1$ and for each $\eta \in T^2$ there is a complete $\lambda$–quasi tree $T_{0, \eta} \subseteq (T^1)^{[\eta]}$ of height $\text{ht}(\text{T}_{0, \eta}) < \lambda$ such that $t^2_\eta \in \Sigma(t^1_\eta : \nu \in T_{0, \eta})$.

  If $p = \langle t_\eta : \eta \in T \rangle$ then we write $\text{root}(p) = \text{root}(T)$, $T^p = T$, $t^p_\eta = t_\eta$ etc.

(2) Let $D^*$ be a filter on $\lambda$. The forcing notion $\mathcal{Q}^{\text{tree}}_D(K, \Sigma)$ is defined similarly, replacing the condition (c)_1 by

- (c)_D:

  for some set $Y = Y^p \in D^*$ we have

  $$(\forall \delta \in Y)(\forall \eta \in (T)_\delta)(\eta \text{nor}[t^*_\eta] \geq |\delta|).$$

  [The set $Y^p$ above may be called a witness for $p \in \mathcal{Q}^{\text{tree}}_D(K, \Sigma)$]

(3) The forcing notion $\mathcal{Q}^{\text{tree}}_c(K, \Sigma)$ is defined by replacing the condition (c)_1 by

- (c)_c:

  a. $(\forall \eta \in T)(\exists \nu \in T)(\eta \delta \nu \in T)(\eta \delta \nu \in T)(\eta \delta \nu \text{ nor}[t^*_\eta] \geq |\delta|)$, and

  b. $(\forall \eta \in T)(\eta \text{ nor}[t^*_\eta] = 0$ or $\eta \text{ nor}[t^*_\eta] \geq |\delta|)$$

(4) If $p \in \mathcal{Q}^{\text{tree}}_c(K, \Sigma)$ and $\eta \in T^p$, then we let $p^{[\eta]} = \langle t^p_\nu : \nu \in (T^p)^{[\eta]} \rangle$.

(5) For the sake of notational convenience we define partial order $\mathcal{Q}^{\text{tree}}_D(K, \Sigma)$ in the same manner as $\mathcal{Q}^{\text{tree}}_c(K, \Sigma)$ above but we omit the requirement (c)_c.

**Definition B.5.3.** Let $(K, \Sigma)$ be a $\lambda$–tree creating pair for $H$, $t \in K$. We define a relation $\preceq^{\Sigma}_t$ on $\Sigma(t)$ by

$$t^t \preceq^{\Sigma}_t t''$$ if and only if

$t^t, t'' \in \Sigma(t)$ and

$$t'' \in \Sigma(t').$$

If $(K, \Sigma)$ is very local, $t^*_\nu$ is the minimal creature at $\nu$, then $\preceq^{\Sigma}_t$ is also denoted by $\preceq^{\Sigma}_{t^*_\nu}$.

**Remark B.5.4.** (1) Note that the relation $\preceq^{\Sigma}_t$ is transitive and reflexive.
(2) If \((K, \Sigma)\) is local and \(p \in \bigcup_{\delta} \text{Tree}(K, \Sigma)\), then \(T^p\) is a complete \(\lambda\)-tree.

Now we are going to describe the non-tree case of forcing with creatures. For sake of simplicity we restrict ourselves to what corresponds to forgetful creatures of \([1, 12.5]\).

**Definition B.5.5.** Let \(H : \lambda \rightarrow \mathcal{H}(\lambda^+)\).

1. A **forgetful \(\lambda\)-creature** for \(H\) is a tuple
   \[ t = (\alpha_{dn}, \alpha_{up}, \text{dis}, \text{val}, \text{nor}) = (\alpha_{dn}[t], \alpha_{up}[t], \text{dis}[t], \text{val}[t], \text{nor}[t]) \]
   such that \(\text{dis} \in \mathcal{H}(\lambda^+), \text{nor} \in \lambda + 1, \alpha_{dn} < \alpha_{up} < \lambda\) and \(\emptyset \neq \text{val} \subseteq \prod_{\alpha_{dn} \leq \beta < \alpha_{up}} \mathcal{H}(\beta)\).
   
   \(\text{CR}^\lambda[H] \) is the family of all forgetful \(\lambda\)-creatures for \(H\).

2. Let \(K \subseteq \text{CR}^\lambda[H]\). A **composition operation on** \(K\) is a mapping \(\Sigma\) with values in \(\mathcal{P}(K)\) and the domain consisting of systems \(\langle t_i : i < j \rangle \subseteq K\) such that \(j < \lambda\) and
   
   \[ \alpha_{up}[t_i] = \alpha_{dn}[t_{i+1}] \text{ for } i < j + 1 < j, \text{ and } \sup_{i < j} \alpha_{up}[t_i]: i' < i = \alpha_{dn}[t_i] \text{ for limit } i < j, \]
   
   and if \(t \in \Sigma(t_i : i < j)\), then
   
   - \(\alpha^- = \alpha_{dn}[t] = \alpha_{dn}[t_0], \alpha^+ = \alpha_{up}[t] = \sup_{i < j} \alpha_{up}[t_i] \text{ for } i < j, \text{ and } \)
   
   - \(\text{val}[t] \subseteq \{ \nu \in \prod_{\alpha^- \leq \beta < \alpha^+} \mathcal{H}(\beta) : (\forall i < j) (\nu[\alpha_{dn}[t_i], \alpha_{up}[t_i]] \in \text{val}[t_i]) \}\),
   
   and
   
   - if \(t_i \in \Sigma(s_i^\lambda : \zeta < \zeta_i)\) for \(i < j\) and \(t \in \Sigma(t_i : i < j)\),
     
     then \(t \in \Sigma(s_i^\lambda : \zeta < \zeta_i, i < j)\), and
   
   - for each \(t \in K\) we have \(\langle t \rangle \in \text{Dom}(\Sigma)\) and \(t \in \Sigma(t)\).

   Then \((K, \Sigma)\) is called a **\(\lambda\)-creating pair** (for \(H\)).

3. We say that \((K, \Sigma)\) is **local** if for each \(t \in K\)
   
   - \(\alpha_{up}[t] = \alpha_{dn}[t] + 1\), and
   
   - \(t' \in \Sigma(t)\) implies \(\text{nor}[t'] \leq \text{nor}[t]\).

   It is **very local** if, additionally, for each \(\alpha < \lambda\) there is \(t^*_\alpha \in K\) such that \(\alpha_{dn}[t^*_\alpha] = \alpha\) and for every \(t \in K\) with \(\alpha_{dn}[t] = \alpha\) we have \(t \in \Sigma(t^*_\alpha)\). The creature \(t^*_\alpha\) will be called the **minimal creature** \(t^*_\alpha\) of \(\alpha\).

4. For \(j < \lambda\), a **\(j\)-approximation** for \((K, \Sigma)\) is a pair \((w, \langle t_i : i < j \rangle)\) such that \(t_i \in K\),
   
   \[ \alpha_{up}[t_i] = \alpha_{dn}[t_{i+1}] \text{ for } i < j + 1 < j, \text{ and } \sup_{i < j} \alpha_{up}[t_i]: i' < i = \alpha_{dn}[t_i] \text{ for limit } i < j, \]
   
   and \(w \in \prod_{\alpha < \alpha_{dn}[t_0]} \mathcal{H}(\alpha)\).

5. For a **\(j\)-approximation** \((w, \langle t_i : i < j \rangle)\) for \((K, \Sigma)\) we let
   
   \[ \text{pos}(w, \langle t_i : i < j \rangle) = \{ v \in \prod_{\alpha < \alpha^+} \mathcal{H}(\alpha) : w \triangleright v \text{ and for all } i < j v[\alpha_{dn}[t_i], \alpha_{up}[t_i]] \in \text{val}[t_i] \}, \]
   
   where \(\alpha^+ = \sup_{i < j} \alpha_{up}[t_i] : i < j\).

**Definition B.5.6.** Let \((K, \Sigma)\) be a \(\lambda\)-creating pair for \(H\).
We define the forcing notion $Q^*(K, \Sigma)$:

**Conditions** are pairs $p = (\bar{w}, \bar{t})$ such that

(a) $\bar{t} = (t_i : i < \lambda)$ is a sequence of $\lambda$-creatures from $K$ satisfying

\[
\alpha_{\text{up}}[t_i] = \alpha_{\text{dn}}[t_{i+1}] \quad \text{for } i < i + 1 < \lambda, \text{ and}
\]

\[
\sup\{\alpha_{\text{up}}[t_i] : i' < i\} = \alpha_{\text{dn}}[t_i] \quad \text{for limit } i < \lambda,
\]

(b) $\bar{w} \in \prod_{\alpha < \alpha_{\text{dn}}[t_0]} H(\alpha)$

(c) $\lim (\text{nor}[t_i] : i < \lambda) = \lambda$

**The order** is given by:

\[
(w, (t_i^1 : i < \lambda)) \leq (w', (t_i^2 : i < \lambda)) \text{ if and only if } \forall i < \lambda (t_i^1 \leq t_i^2)
\]

for some continuous strictly increasing sequence $\langle i_\zeta : \zeta < \lambda \rangle$ we have

\[
w^2 \in \text{pos}(w, (t_i^1 : i < i_0)) \quad \text{and} \quad (\forall \zeta < \lambda) (t_i^2 \in \Sigma(t_i^1 : i_\zeta < i < i_\zeta)).
\]

If $p = (w, (t_i : i < \lambda))$, then we write $w^p = w$, $t_i^p = t_i$ (for $i < \lambda$).

(2) Let $D^*$ be a filter on $\lambda$. The forcing notion $Q^*_D(K, \Sigma)$ is defined similarly, replacing the condition (c) by

(c)$_D^*$ for some set $Y = Y^p \in D^*$ we have

\[
(\forall i \in Y)(\text{nor}[t_i] \geq |\alpha_{\text{dn}}[t_i]|).
\]

[The set $Y^p$ above may be called a witness for $p \in Q^*_D(K, \Sigma)$.]

(3) For the sake of notational convenience we define partial order $Q^*_\emptyset(K, \Sigma)$ in the same manner as $Q^*_\emptyset(K, \Sigma)$ above but we omit the requirement (c)$_\emptyset$. If $(K, \Sigma)$ is very local, then the minimal condition $p^*$ for $(K, \Sigma)$ is

\[
p^* = p^*(K, \Sigma) = (\langle \rangle, (t_\alpha^* : \alpha < \lambda)) \in Q^*_\emptyset(K, \Sigma),
\]

where $t_\alpha^*$ is the minimal creature at $\alpha$.

(4) The relations $\preceq_\Sigma^\lambda$ and $\preceq_\Sigma^\lambda = \preceq_\Sigma^\lambda$ are defined in a way parallel to $\preceq_\Sigma^\lambda$.

B.6. GETTING COMPLETENESS AND BOUNDING PROPERTIES

In this section we introduce properties of $\lambda$–tree creating pairs ensuring that the resulting forcing notions are complete or strategically complete. Next we show that adding bounds on the size of $H(\alpha)$ guarantees strong bounding properties from section A.2. Finally we will introduce parallel completeness conditions for the case of $\lambda$–creating pairs.

**Definition B.6.1.** Let $(K, \Sigma)$ be a $\lambda$–tree creating pair for $H$, $\kappa$ be a cardinal (and $\lambda, \bar{\lambda}$ be as in [3.3]).

(1) We say that a $\lambda$–tree creature $t \in K$ is $\kappa$-complete (for $(K, \Sigma)$) if

(a) for every $\preceq_\Sigma^\lambda$–increasing chain $\langle t_\alpha : \alpha < \delta \rangle \subseteq \Sigma(t)$ with $\delta < \kappa$ and $\text{nor}[t_\alpha] > 0$, there is $t_\delta \in \Sigma(t)$ such that $t_\delta \preceq_\Sigma^\lambda t_\alpha$ and $\text{nor}[t_\delta] \geq \min\{\text{nor}[t_\alpha] : \alpha < \delta\}$,

(b) if $t' \in \Sigma(t)$, $\text{nor}[t'] = 0$, then $|\text{pos}[t']| = 1$ and $\Sigma(t') = \{t'\}$,

(c) if $\nu \in \text{pos}[t]$, then there is $t' \in \Sigma(t)$ such that $\text{pos}[t'] = \{\nu\}$ and $\text{nor}[t'] = 0$.

(2) $t \in K$ is said to be exactly $\kappa$–complete if it is $\kappa$–complete and

(iii) if $\bar{t} = (t_\alpha : \alpha < \kappa) \subseteq \Sigma(t)$ is a strictly $\preceq_\Sigma^\lambda$–increasing chain, then $\bar{t}$ has no $\preceq_\Sigma^\lambda$–upper bound in $\Sigma(t)$, but $\bigcap_{\alpha < \kappa} \text{pos}[t_\alpha] \neq \emptyset$.

(3) We say that $(K, \Sigma)$ is $\bar{\lambda}$–complete (exactly $\bar{\lambda}$–complete, respectively) if
of the game, after a sequence $\langle q_i : i < \delta \rangle$ (for $\delta < \lambda$).

We say that $(K, \Sigma)$ is just $\lambda$–complete if it satisfies (a) above and (b)$^*$ each minimal creature $t^*_p$ is $\lambda$–complete.

**Proposition B.6.2.** Assume that $(K, \Sigma)$ is a very local $\lambda$–tree creating pair for $H$, $D^*$ is a $\langle \lambda \rangle$–complete filter on $\lambda$. Let $P$ be one of the forcing notions $Q^{\text{tree}}(K, \Sigma)$, $Q^{\text{tree}}_D(K, \Sigma)$, or $Q^{\text{tree}}_c(K, \Sigma)$.

1. If $(K, \Sigma)$ is $\check{\lambda}$–complete, then $P$ is strategically $(\langle \lambda \rangle)–complete$.
2. If $(K, \Sigma)$ is just $\lambda$–complete, then $P$ is $\lambda$–complete.
3. If $(K, \Sigma)$ is exactly $\lambda$–complete, then $P$ is $\lambda$–complete.

**Proof.** (1) Let $P \in \{Q^{\text{tree}}(K, \Sigma), Q^{\text{tree}}_D(K, \Sigma), Q^{\text{tree}}_c(K, \Sigma)\}$ and let $r \in P$. Consider the following strategy $st$ of Complete in the game $\sigma^0_3(P, \emptyset, r)$. At stage $j < \lambda$ of the game, after a sequence $\langle (p_i, q_i) : i < j \rangle$ has been constructed (so $p_j$ is the move of Incomplete), she plays the $\langle \lambda \rangle$–first condition $q_j \in P$ stronger than $p_j$ and such that $\text{lh}(\text{root}(q_j)) > j + \omega$.

Why is this a winning strategy? Suppose that the players have arrived at a limit stage $\delta < \lambda$ of the game. Complete has used $st$ and $\langle (p_i, q_i) : i < \delta \rangle$ is the result of the game so far. Our aim is to show that the (increasing) sequence $\langle q_i : i < \delta \rangle$ has an upper bound in $P$. To this end we are going to define a condition $q = \langle t_\eta : \eta \in T \rangle \in P$ inductively defining $(T)_\alpha$ and $t_\eta$ for $\alpha < \lambda, \eta \in (T)_\alpha$. First we declare $\text{root}(T) = \bigcup_{i<\delta} \text{root}(q_i)$ and we note that

$$\text{root}(T) \in \bigcap_{i<\delta} T^{q_i} \text{ and } \delta \leq \text{lh}(\text{root}(T)) < \lambda^+_{\text{lh}(\text{root}(T))}.$$ 

Now we may choose $t^*_{\text{root}(T)} \in \text{TCR}^{\lambda}_{\text{root}(T)}[H]$ so that $t^*_{\text{root}(T)} \in \Sigma(t^*_P_{\text{root}(T)})$ (for all $i < \delta$) and $\text{nor}[t^*_P_{\text{root}(T)}] \geq \min \{\text{nor}[t^*_P_{\text{root}(T)}] : i < \delta \}$, and we declare $\text{pos}[t^*_P_{\text{root}(T)}] \subseteq T$ (thus defining $(T)_{\text{lh}(\text{root}(T)) + 1}$). Next we proceed inductively in a similar manner: suppose that $(T)_\alpha$ has been already defined and it is included in $\bigcap_{i<\delta} T^{q_i}$. For each $\eta \in (T)_\alpha$ choose $t_\eta$ such that

$$\langle \forall i < \delta \rangle(t_\eta \in \Sigma(t^*_P_{\text{root}(T)})) \text{ and } \text{nor}[t_\eta] \geq \min \{\text{nor}[t^*_P_{\text{root}(T)}] : i < \delta \},$$

and declare $\text{pos}[t_\eta] \subseteq T$. (So after this step $(T)_{\alpha+1}$ is defined.) If $\alpha < \lambda$ is limit and $(T)_\beta$ has been defined for $\beta < \alpha$, then we let $(T)_\alpha$ consist of all $\eta$ such that $\eta|\beta \in (T)_\beta$ whenever $\text{lh}(\text{root}(T)) \leq \beta < \alpha$, and then we choose $t_\eta$ (for $\eta \in (T)_\alpha$ like above).

This way we build a condition $\langle t_\eta : \eta \in T \rangle \in Q^{\text{tree}}(K, \Sigma)$, and it is very straightforward to verify that this condition is actually in $P$, and it is stronger than all $q_i$ (for $i < \delta$).

(2), (3) Similar. 

The exact $\check{\lambda}$–completeness may seem to be very strange and/or strong. But as a matter of fact it is easy to modify any $\check{\lambda}$–complete $\lambda$–tree creating pair to one that is exactly complete (and the respective forcing notions are very close).

**Definition B.6.3.** Let $(K, \Sigma)$ be a very local $\check{\lambda}$–complete $\lambda$–tree creating pair for $H$. We define the $\check{\lambda}$–exactivity $(K^{\text{ex}(\check{\lambda})}, \Sigma^{\text{ex}(\check{\lambda})})$ of $(K, \Sigma)$ as follows.
Let \( \eta \in \bigcup_{\alpha < \lambda} \bigcap_{\beta < \alpha} \mathbf{H}(\beta) \). We let \( K^{\text{ex}(\lambda)} \cap \text{TCR}^\lambda \eta[\mathbf{H}] \) consist of all \( \lambda \)-tree creatures \( t \) such that

- \( \eta[t] = \eta \),
- \( \text{dis}[t] = \{ \xi : \xi \leq \delta \} \), where \( t_0 = t^*_\eta \) is the minimal creature at \( \eta \) for \( (K, \Sigma) \), \( \delta < \lambda \text{lh}(\eta) \), and \( \xi < \zeta \leq \delta \Rightarrow t_\xi \subseteq_\Sigma t_\zeta \not= t_\xi \),
- \( \text{pos}[t] = \text{pos}[t^*_\delta] \),
- \( \text{nor}[t] = \min\{ \text{nor}[t_\xi] : \xi \leq \delta \} \).

Then, for \( t', t \in K^{\text{ex}(\lambda)} \cap \text{TCR}^\lambda \eta[\mathbf{H}] \) we let \( t' \in \Sigma^{\text{ex}(\lambda)}(t) \) if and only if \( \text{dis}[t] \subseteq \text{dis}[t'] \).

**Proposition B.6.4.** Assume \((K, \Sigma)\) is a very local \( \lambda \)-complete \( \lambda \)-tree creating pair. Then \((K^{\text{ex}(\lambda)}, \Sigma^{\text{ex}(\lambda)})\) is a very locally exactly \( \lambda \)-complete \( \lambda \)-tree creating pair. The minimal creature for it at \( \eta \) is \( t^{\ast*}_\eta \) such that \( \text{dis}[t^{\ast*}_\eta] = \{ t^*_\eta \} \).

**Proof.** Easy. \( \square \)

**Theorem B.6.5.** Suppose that

- (a) \( \forall \alpha < \lambda \)(\( |\mathbf{H}(\alpha)| < \lambda_\alpha \)), and
- (b) \( (K, \Sigma) \) is a \( \bar{\mu} \)-complete very local \( \lambda \)-tree creating pair for \( \mathbf{H} \) for some strictly increasing sequence \( \bar{\mu} = \{ \mu_\alpha : \alpha < \lambda \} \) of regular cardinals such that \( \mu_\alpha < \lambda \) for \( \alpha < \lambda \), and
- (c) \( D^* \) is a normal filter on \( \lambda \).

Then the forcing notion \( \mathcal{Q}^{\text{ex}(\lambda)}_{\mathcal{D}^*}(K, \Sigma) \) has the strong \( \bar{\lambda} \)-Sacks property

**Proof.** Let \( i_0 < \lambda \) and \( p \in \mathcal{Q}^{\text{ex}(\lambda)}_{\mathcal{D}^*}(K, \Sigma) \). Just for notational simplicity we assume that \( \mathbf{H}(\alpha) \in \lambda_\alpha \) for all \( \alpha < \lambda \) and \( \text{lh}(\text{root}(p)) \leq i_0 \). We are going to describe a strategy for the Generic player in the game \( \mathfrak{S}_{\bar{\mu}}^\lambda(i_0, p, \mathcal{Q}^{\text{ex}(\lambda)}_{\mathcal{D}^*}(K, \Sigma)) \). In the course of the play she will also choose sets \( Y_{i+1} \in D^* \) and \( \lambda \)-tree creatures \( t_\nu \).

First Generic picks \( \eta \in T^p \) such that \( \text{lh}(\eta) > i_0 \) and she starts the game with playing \( s_{i_0} = \{ \eta[i_0 + 1] \} \) and \( q^{i_0}_{\eta[i_0 + 1]} = p^{[\eta]} \). She also picks \( t_\eta[i_0] \in \Sigma(t^p_{\eta[i_0 + 1]}) \) such that \( \text{pos}[t_{\eta[i_0 + 1]}] = \{ \eta[i_0 + 1] \} \) (remember B.6.1(1)(a)).

Arriving at a successor stage \( j = i + 1 \) of the play the players have determined \( s_i, q^i, \bar{p} \) and \( Y_i \) so that, in addition to the demands of the game, for each \( \eta \in s_i \cap i+1 \lambda \) we have \( \eta \subseteq \text{root}(q^i_{\eta}) \). Now for each \( \eta \in s_i \cap i+1 \lambda \) Generic picks \( \nu_\eta \in T^{\bar{p}_i} \) strictly extending \( \eta \) and she plays

\[
\begin{align*}
s_{i+1} = s_i \cup \{ \nu_\eta[i + 2] : \eta \in s_i \}, & \quad q^{i+1}_{\nu_\eta[i + 2]} = (p^i_{\eta})^{[\nu_\eta]} \quad \text{for } \eta \in s_i \cap i+1 \lambda. \\
\end{align*}
\]

She also fixes a set \( Y_{i+1} \in D^* \) of limit ordinals included in \( \bigcap_{\eta \in s_i} \mathcal{H}^{\eta[i + 2]} \) (recall B.5.2(2)) and for \( \eta \in s_i \cap i+1 \lambda \) she lets \( t_\eta \in \Sigma(t^p_\eta) \) be such that \( \text{pos}[t_\eta] = \{ \nu_\eta \} \).

Now suppose that the players have arrived to a limit stage \( \delta < \lambda \) of the game, and assume that \( \delta \not\in \bigcap_{i < \delta} Y_{i+1} \). Generic lets \( s^*_\delta \) consist all sequences \( \eta \) of length \( \delta \) such that \( \eta[i + 1] \in s_i \) whenever \( i_0 \leq i < \delta \). For each \( \eta \in s^*_\delta \) she first picks a condition \( r_\eta \) stronger than all \( p^i_{\eta[i + 1]} \) for \( i_0 \leq i < \delta \) (there is one by arguments as in the proof of B.6.2(1)) and then she chooses \( \nu_\eta \in T^{r_\eta} \) strictly extending \( \eta \). Then she plays

\[
\begin{align*}
s_\delta = s^*_\delta \cup \{ \nu_\eta[\delta + 1] : \eta \in s^*_\delta \}, & \quad q^{\delta}_{\nu_\eta[\delta + 1]} = (r_\eta)^{[\nu_\eta]} \quad \text{for } \eta \in s^*_\delta. \\
\end{align*}
\]

The \( \lambda \)-tree creatures \( t_\eta \) (for \( \eta \in s^*_\delta \)) are chosen as above: \( t_\eta \in \Sigma(t^p_\eta), \text{pos}[t_\eta] = \{ \nu_\eta \} \).
Finally suppose that we are at a limit stage \( \delta < \lambda \) of the game and \( \delta \in \bigcap_{i<\delta} Y_i \). Let \( s_\delta^* \) be defined as above and for each \( \eta \in s_\delta^* \) let \( r_\eta \) be a condition stronger than all \( p_\eta^{\langle i+1 \rangle} \) for \( i_0 \leq i < \delta \) and such that \( \text{root}(r_\eta) = \eta \) and \( \text{nor}(t_\eta^{r_\eta}) \geq |\delta| \) (there is one by arguments as in the proof of Proposition B.6.2(1) and the choice of the \( Y_i \)'s). Then she plays

\[
s_\delta = s_\delta^* \cup \{ \text{pos}[t_\eta^{r_\eta}] : \eta \in s_\delta^* \}, \quad q_\nu^\delta = (r_\eta)^{[\nu]} \quad \text{for } \nu \in \text{pos}[t_\eta^{r_\eta}], \, \eta \in s_\delta^*.
\]

She also lets \( t_\eta = t_\eta^{r_\eta} \in \Sigma(t_\eta^{r_\eta}) \) (for \( \eta \in s_\delta^* \)).

It should be clear that the strategy described above always tells \( \text{Generic} \) to play legal moves (remember Proposition B.6.3(c)). It should also be clear that if \( \langle (s_i, q_i^i, p_i^i) : i_0 \leq i < \lambda \rangle \) is the result of a play of \( \mathcal{C}^{\text{sacks}} \langle i_0, p, \mathbb{Q}^\text{tree}_{D^*}(K, \Sigma) \rangle \) in which \( \text{Generic} \) uses that strategy, then letting \( T = \bigcup \{ s_i : i_0 \leq i < \lambda \} \) and \( q = (t_\eta : \eta \in T) \) we get a condition in \( \mathbb{Q}^\text{tree}_{D^*}(K, \Sigma) \) (as witnessed by \( \Delta Y_{i+1} \)) stronger than \( p \) and forcing that

\[
"(\exists \rho \in {}^\lambda \alpha)(\forall i \in [i_0, \lambda))(\rho(i+1) \in s_i \& q^i_{\rho(i+1)} \in \Gamma_{\mathbb{Q}^\text{tree}_{D^*}(K, \Sigma)}").
\]

\[\square\]

**Theorem B.6.6.** Assume that \( (\forall \alpha < \lambda)(|H(\alpha)| < \lambda) \), and \( (K, \Sigma), D^* \) satisfy (b), (c) of Proposition B.6.3. Then the forcing notion \( \mathbb{Q}^\text{tree}_{D^*}(K, \Sigma) \) has the strong \( \lambda \)-bounding property.

**Proof.** Similar to Proposition B.6.5 \( \square \)

The above two theorems are applicable to forcing notions of the type \( \mathbb{Q}^\text{tree}_{D^*}(K, \Sigma) \) as they may be treated as a special case (under the assumptions as there):

**Proposition B.6.7.** Assume that \( (\forall \alpha < \lambda)(|H(\alpha)| < \lambda) \) and \( (K, \Sigma) \) is a \( \check{\mu} \)-complete very local \( \lambda \)-tree creating pair for \( H \) (for some \( \check{\mu} \)). Then the forcing notions \( \mathbb{Q}^\text{tree}_{D^*}(K, \Sigma) \) and \( \mathbb{Q}^\text{tree}_{\mathbb{D}_*}(K, \Sigma) \) are equivalent.

Turning to the case of \( \lambda \)-creating pairs (and forcing notions of the form \( \mathbb{Q}^\ast_{D^*}(K, \Sigma) \)), we do easy ways to ensure they are suitably complete (parallel to Proposition B.6.1 and B.6.2). However at present we do not know how to get the strong bounding properties of section A.2 (which were obviously tailored for trees).

**Definition B.6.8.**

1. For a \( \lambda \)-creating pair \( (K, \Sigma) \) and \( t \in K \) we define when \( t \) is \( \kappa \)-complete and exactly \( \kappa \)-complete like in Proposition B.6.1(2) (but with \text{val} replacing \text{pos}).

2. If \( (K, \Sigma) \) is very local, then we say that it is \( \lambda \)-complete (exactly \( \lambda \)-complete, resp.) if each minimal creature \( t_\alpha^* \) is \( \lambda^+ \)-complete (exactly \( \lambda^+ \)-complete, resp.).

**Proposition B.6.9.** Assume that \( (K, \Sigma) \) is a very local \( \lambda \)-creating pair for \( H \), \( D^* \) is a normal filter on \( \lambda \). Let \( \mathbb{P} \) be either the forcing notion \( \mathbb{Q}^\ast_{\mathbb{D}_*}(K, \Sigma) \) or \( \mathbb{Q}^\text{tree}_{\mathbb{D}_*}(K, \Sigma) \).

1. If \( (K, \Sigma) \) is \( \check{\lambda} \)-complete, then \( \mathbb{P} \) is strategically \( (\langle \lambda \rangle \) \)-complete.

2. If \( (K, \Sigma) \) is exactly \( \check{\lambda} \)-complete, then \( \mathbb{P} \) is \( \lambda \)-complete.
B.7. Getting fuzzy properness

In this section we show that the forcing notions with trees and creatures may fit the fuzzy proper framework. Note that even though the forcing notions covered by Theorems B.6.6 and B.7.2 below are also covered by Theorem B.6.6 the results here still have value if we want to iterate that forcing notions with ones which do not have the strong $\lambda$-bounding property.

**Theorem B.7.1.** Let $A, W, D$ be as in A.3.1 and let $D^*$ be a normal filter on $\lambda$ such that for some $S_0 \in D^*$ we have $\lambda \setminus S_0 \in D$. Let $\bar{\mu}$ be an increasing sequence of regular cardinals cofinal in $\lambda$. Assume that $(K, \Sigma)$ is an exactly $\bar{\mu}$–complete very local $\lambda$–tree creating pair for $H$, and $|H(\alpha)| < \lambda$ for each $\alpha < \lambda$. Then the forcing notion $Q_{D^*}^{\tr}(K, \Sigma)$ is strongly fuzzy proper over quasi $D$–diamonds for $W$.

**Proof.** By B.6.2 we know that $Q_{D^*}^{\tr}(K, \Sigma)$ is $\lambda$–complete.

Let $\bar{\mathfrak{N}}^{\tr}$, $\bar{\mathfrak{Q}}^{\tr}$ be the trivial $\lambda$–base defined as in the proof A.3.3 (but for $\bar{\mathfrak{P}} = Q_{D^*}^{\tr}(K, \Sigma)$). We are going to show that for this $\lambda$–base and for $c = (\lambda, K, \Sigma)$ the condition A.3.6 (2) holds. So assume that $N, h, \bar{F} = (F_\delta : \delta \in S)$ and $\bar{\eta} = \langle q_{\delta, x} : \delta \in S \text{ limit and } x \in X_\delta \rangle$ are as there and $p \in Q_{D^*}^{\tr}(K, \Sigma) \cap N$. Note that $X_\delta = \{0\}$ (for all relevant $\delta$) and thus we may think that $\bar{\eta} = q_{\delta} : \delta \in S \text{ limit}$.

Let $\bar{I} = \langle I_\xi : \xi < \lambda \rangle$ list all open dense subsets of $Q_{D^*}^{\tr}(K, \Sigma)$ from $N$. For $i < \lambda$ let $\xi_i$ be such that $I_{\xi_i}$ consist of conditions $p \in Q_{D^*}^{\tr}(K, \Sigma)$ with $\text{lh}(\text{root}(p)) > i$, and let $E$ be a club of $\lambda$ such that

\[(\forall \delta \in E)(\forall i < \delta)(\forall \alpha < \bar{\eta})\text{ is limit and } \xi_i < \delta).\]

By induction on $\alpha < \lambda$ choose conditions $p_\alpha \in Q_{D^*}^{\tr}(K, \Sigma) \cap N$ and sets $Y_\alpha \in D^*$ such that

1. $p_0 = p$, $\text{root}(p_\alpha) = \text{root}(p)$, and $p_\alpha \leq p_\beta$ and $Y_\beta \subseteq Y_\alpha \subseteq S_0$ for $\alpha < \beta < \lambda$,
2. $Y_\alpha$ witnesses $p_\alpha \in Q_{D^*}^{\tr}(K, \Sigma)$ (see B.6.2 (2)),
3. for every $\alpha < \beta < \lambda$ and $\nu \in (T^{p_\alpha})_\alpha$ we have $\nu \in T^{p_\beta}$ and $t^{p_\beta}_\nu = t^{p_\alpha}_\nu$,
4. if $\alpha < \lambda$ is a successor, $\xi < \alpha$ and $\eta \in (T^{p_\alpha})_\alpha$, then for some $\nu \in (T^{p_\alpha})^{[\eta]}$ we have: $(p_\beta)^{[\nu]} = I_{\xi}$ and $(\forall \nu \in T^{p_\alpha})(\eta \leq \rho < \nu \Rightarrow \text{nor}[t^{p_\alpha}_\rho] = 0)$,
5. if $\delta \in \bigcap_{\alpha < \delta} Y_\alpha$ is a limit ordinal, then $\delta \in Y_\beta$ for every $\beta \geq \delta$,
6. if $\delta \in S \cap E \setminus S_0$ and $(h \circ F_\delta(i) : i < \delta)$ is an increasing $\bar{I}$–exact sequence of members of $N \cap Q_{D^*}^{\tr}(K, \Sigma)$ such that

\[(\forall \alpha < \delta)(\exists i < \delta)(p_\alpha \leq h \circ F_\delta(i)),\]

and $\eta \in (T^{p_\delta})_\delta$ is such that every $h \circ F_\delta(i)$ is compatible with $(p_\delta)^{[\eta]}$, then $(p_\delta)^{[\eta]} \leq q_\delta = (p_\delta)^{[\text{root}(q_\delta)]}$ and

\[(\forall \rho \in T^{p_\delta})(\eta \leq \rho < \text{root}(q_\delta) \Rightarrow \text{nor}[t^{p_\delta}_\rho] = 0).\]

[Note that there is at most one $\eta$ as above; remember the choice of $E$.]

It should be clear that the inductive construction of the $p_\alpha$’s and $Y_\alpha$’s is possible (for (v) remember $\delta < \lambda$; note also that there is no collision between (v) and (vi) because $Y_\alpha \subseteq S_0$). Now letting $\text{root}(r) = \text{root}(p)$, $T^r = \bigcup_{\alpha < \lambda} (T^{p_\alpha})_\alpha$, $t^r_{\alpha} = t^{p_\alpha}_{\text{root}(r)+1}$ we get a condition $r \in Q_{D^*}^{\tr}(K, \Sigma)$ (as witnessed by $\Delta Y_\alpha$). Also note that $r$ is stronger than all $p_\alpha$’s.

**Claim B.7.1.1.** The condition $r$ is $(\bar{\mathfrak{N}}^{\tr}, \bar{\mathfrak{Q}}^{\tr})$–fuzzy generic for $\bar{\eta}$. 
Proof of the Claim. First note that the condition \( r \) is \((N, \mathbb{Q}^\text{tree}_{D^*}(K, \Sigma))\)-generic by clause (iv) above. Therefore we may use \ref{A.3.4}(3), and it is enough that we show that the Generic player has a strategy in the game \( \mathbb{Q}_{\lambda}^{\text{fuzzy}}(r, N, \check{I}, h, \mathbb{Q}^\text{tree}_{D^*}(K, \Sigma), \check{F}, \check{q}) \) which guarantees that the result \( \langle r_i, C_i : i < \lambda \rangle \) of the play satisfies \ref{A.3.4}(5)(\( \beta \)). Let us describe such a strategy.

First, for \( \alpha < \lambda \) let \( \zeta_\alpha < \lambda \) be such that
\[
(\forall q \in \mathcal{Z}_\alpha)( \text{ either } p_\alpha \leq q \text{ or } p_\alpha, q \text{ are incompatible } ),
\]
and let \( E' = \{ \delta \in E : (\forall \alpha < \delta)(\zeta_\alpha < \delta) \} \) (it is a club of \( \lambda \)).

Now, suppose that during a play of \( \mathbb{Q}_{\lambda}^{\text{fuzzy}}(r, N, \check{I}, h, \mathbb{Q}^\text{tree}_{D^*}(K, \Sigma), \check{F}, \check{q}) \) the players have arrived at stage \( i \in S \) having constructed a sequence \( \langle r_j, C_j : j < i \rangle \).

If either \( i \) is a successor ordinal or \( i \notin \bigcap_{j < i} C_j \), then the Generic player plays
\[
(\forall j < i)(\forall \alpha < i)(\forall \eta)(\exists q)(\forall \delta)(\zeta_\alpha < \eta < \delta) \text{ is incompatible with some } q)
\]
for each \( j < i \) and \( \alpha < i \).

If \( i \in \bigcap_{j < i} C_j \) is a limit ordinal (so also \( i \in E' \setminus S_0 \)), then Generic asks
\[
(\forall j < i)(\exists \alpha < i)(\exists q)(\forall \delta)(\zeta_\alpha < \eta < \delta) \text{ is incompatible with some } q)
\]
for each \( j < i \) and \( \alpha < i \).

If the answer to (\( \ast \)) is “yes”, then Generic looks at clause (vi) (of the choice of \( p_\alpha \)’s) and \( \eta = \bigcup_{j < i} \text{root}(r_j) \) (note that \( \text{lh}(\eta) = i \)). If \( (p_i)_{i=0}^{\eta} \) is incompatible with some \( h \circ F_i(\alpha), \alpha < i \), then she plays \( C_i, r_i \) as in the successor case.

[Note that if the answer to (\( \ast \)) is “no” and \( h \circ F_i(\alpha), \alpha < i \) is increasing \( \check{I} \)-exact, then for some \( j < i \) and \( \alpha < i \) the conditions \( p_j \) and \( h \circ F_j(\alpha) \) are incompatible, and hence \( r_i \) and \( h \circ F_i(\alpha) \) are incompatible.]

If the answer to (\( \ast \)) is “no”, then she plays like at the successor stage.

Now, suppose that during a play of \( \mathbb{Q}_{\lambda}^{\text{fuzzy}}(r, N, \check{I}, h, \mathbb{Q}^\text{tree}_{D^*}(K, \Sigma), \check{F}, \check{q}) \) the players have arrived at stage \( i \in S \).

If the answer to (\( \ast \)) is “yes”, then Generic plays the \( r_i \) and \( h \circ F_i(\alpha), \alpha < i \) is increasing \( \check{I} \)-exact, then for some \( j < i \) and \( \alpha < i \) the conditions \( p_j \) and \( h \circ F_j(\alpha) \) are incompatible, and hence \( r_i \) and \( h \circ F_i(\alpha) \) are incompatible.]

If the answer to (\( \ast \)) is “yes”, then Generic looks at clause (vi) (of the choice of \( p_\alpha \)’s) and \( \eta = \bigcup_{j < i} \text{root}(r_j) \) (note that \( \text{lh}(\eta) = i \)). If \( (p_i)_{i=0}^{\eta} \) is incompatible with some \( h \circ F_i(\alpha), \alpha < i \), then she plays \( C_i, r_i \) as in the successor case.

[Note that then \( r_i, h \circ F_i(\alpha) \) are incompatible.]

Otherwise \( \eta \subseteq \text{root}(q_i) \in T_{p_i}, q_i = \langle p_i \rangle_{\text{root}(q_i)} \) and \( \langle q \in T_{p_i} : (\forall \rho \in T_{p_i})((\exists \eta)(\exists \rho < \text{root}(q_i)) \Rightarrow |\text{pos}(\rho^1) = 1) \rangle \). Therefore, \( \text{root}(q_i), T_{p_i} \subseteq T_{r_j} \) and \( q_i \subseteq \langle r_j \rangle_{\text{root}(q_j)} \) (for each \( j < i \)). So the Generic player can play \( C_i = E' \setminus i \) and the \( <^*_\lambda \)-first condition \( r_i \) stronger than all \( r_j \) (for \( j < i \) and \( q_i \)).

It follows immediately from the comments stated during the description of the strategy that every play according to it satisfies \ref{A.3.4}(5)(\( \beta \)), finishing the proof of the claim and that of the theorem.

\( \square \)

**Theorem B.7.2.** Let \( A, W, D \) be as in \ref{A.3.1} and let \( D^* \) be a normal filter on \( \lambda \) such that for some \( S_0 \in D^* \) we have \( \lambda \setminus S_0 \in D \). Assume that \( (K, \Sigma) \) is an exactly \( \check{\lambda} \)-complete very local \( \check{\lambda} \)-creating pair for \( H, |H(\alpha)| < \lambda \) for each \( \alpha < \lambda \). Then the forcing notion \( \mathbb{Q}^n_{\lambda} (K, \Sigma) \) is strongly fuzzy proper over quasi \( D \)-diamonds for \( W \).

**Proof.** Like \ref{B.7.1}. \( \square \)

**Theorem B.7.3.** Let \( A = \mathcal{H}_{<\lambda}(\check{\lambda}^*) \), \( \lambda^* > \lambda \) and \( W \subseteq [A]^{<\lambda} \) be as in \ref{A.3.1}. Let \( \check{\mu} \) be an increasing sequence of regular cardinals cofinal in \( \lambda \). Suppose that \( (K, \Sigma) \) is an exactly \( \check{\mu} \)-complete very local \( \check{\lambda} \)-tree creating pair for \( H, (\forall \alpha < \lambda)(|H(\alpha)| < \lambda) \), and \( D^* \) is a normal filter on \( \lambda \). Then the forcing notion \( \mathbb{Q}^n_{\lambda} (K, \Sigma) \) is fuzzy proper for \( W \).

**Proof.** Like \ref{B.7.1}. \( \square \)
Proof. The proof closely follows the lines of that of [B.7.1]. Let $D$ be a normal filter on $\lambda$ such that there is a $D$–diamond.

Just only to simplify somewhat the definition of a $\lambda$–base which we will use, let us assume that $\bigcup \prod_{\alpha<\lambda} H(\alpha) \subseteq a$ for every $a \in W$. Now we let $R = R(K,\Sigma)$ consist of all triples $(p,\delta,\eta)$ such that $\delta < \lambda$, $\eta \in \prod_{\alpha<\delta} H(\alpha)$ and $p \in Q^\text{tree}_{D^*(K,\Sigma)}$ satisfies $\eta < \text{root}(p)$. Next, for $a \in W$ let $\mathcal{Q}_a = \mathcal{Q}_a(K,\Sigma) : \lambda \to [a]^{<\lambda}$ be given by $\mathcal{Q}_a(\delta) = \prod_{\alpha<\delta} H(\alpha) \subseteq a$ (for $\delta < \lambda$). It should be clear that $(R,\mathcal{Q})$ is a $\lambda$–base for $Q^\text{tree}_{D^*(K,\Sigma)}$ over $W$.

We claim that $(R,\mathcal{Q})$ and $c = (\bar{\lambda}, H, K, \Sigma)$ witness the condition $(\bar{\alpha})$ of [A.3.6(1)]. To this end, let $N, h, \bar{F} = (F_\delta : \delta \in S)$ and $\bar{q} = (q_{\delta,x} : \delta \in S \text{ limit } & x \in X_\delta)$ be as in [A.3.6(1)](\bar{\alpha}), $p \in Q^\text{tree}_{D^*(K,\Sigma)} \cap N$. Let $\mathcal{I} = (I_\xi : \xi < \lambda)$ list all open dense subsets of $Q^\text{tree}_{D^*(K,\Sigma)}$ from $N$. For $i < \lambda$ let $\xi_i$ be such that $I_{\xi_i}$ consist of conditions $p \in Q^\text{tree}_{D^*(K,\Sigma)}$ with $\text{lh}(\text{root}(p)) > i$, and let $E$ be a club of $\lambda$ such that
\[(\forall \delta \in E)(\forall i < \delta)(\delta \text{ is limit and } \xi_i < \delta).\]

By induction on $\alpha < \lambda$, like in [B.7.1] (but note the change in (vi) below!), we choose conditions $p_\alpha \in Q^\text{tree}_{D^*}(K,\Sigma) \cap N$ and sets $Y_\alpha \subset D^*$ such that

(i) $p_0 = p$, $\text{root}(p_\alpha) = \text{root}(p)$, and $p_\alpha \leq p_\beta$ and $Y_\beta \subseteq Y_\alpha$ for $\alpha < \beta < \lambda$,

(ii) $Y_\alpha$ witnesses $p_\alpha \in Q^\text{tree}_{D^*}(K,\Sigma)$,

(iii) for every $\alpha < \beta < \lambda$ and $\nu \in (T^{p_\beta})_\alpha$ we have $\nu \in T^{p_\beta}$ and $\nu^{\bar{D}} = \nu'_{\bar{\beta}}$,

(iv) if $\alpha < \lambda$ is a successor ordinal and $\eta \in (T^{p_\alpha})_\alpha$, then for some $\nu \in (T^{p_\alpha})_\alpha$ we have: $(p_\alpha)^{[\nu]} \in \bigcap_{\xi < \alpha} I_{\xi}$ and $(\forall \rho \in T^{p_\alpha})(\eta \leq \rho < \nu \Rightarrow \text{nor}^{[p_\rho]} = 0),$

(v) if $\delta \in \bigcap_{\alpha<\delta} Y_\alpha$ is a limit ordinal, then $\delta \in Y_\delta$ for every $\beta \geq \delta$,

(vi) if $\delta \in S \cap E$, $(h \circ F_\delta(i) : i < \delta)$ is increasing $\tilde{F}$–exact, $\eta = \bigcup_{i<\delta} \text{root}(h \circ F_\delta(i))$ and $\text{lh}(\eta) = \delta$, and $(\forall \alpha < \delta)(\exists i < \delta)(p_\alpha \leq h \circ F_\delta(i))$, then $(\eta \in T^{p_\beta}$ and) for every $\nu \in \text{pos}^{[p_\beta]} \cap \bigcap_{i<\delta} \text{pos}^{[h \circ F_\delta(i)]}$ we have $\nu^{[\bar{D}]} \leq q_{\delta,\nu} = (p_\delta)^{[\text{root}(q_{\delta,\nu})]}$ and $(\forall \rho \in T^{p_\beta})(\nu \leq \rho < \text{root}(q_{\delta,\nu}) \Rightarrow \text{nor}^{[p_\rho]} = 0).

[Note that, in the situation as in (vi), $X_\delta = \bigcap_{i<\delta} \text{pos}^{[h \circ F_\delta(i)]}$.]

Plainly, the inductive construction of the $p_\alpha$’s and $Y_\alpha$’s is possible (for (v) remember $\delta < \lambda_\delta$). Now letting root$(r) = \text{root}(p)$, $T^* = \bigcup_{\alpha<\lambda} (T^{p_\alpha})_\alpha$, $t^*_\nu = t^{p_{h^{(\nu)+1}}}_{\nu}$ we get a condition $r \in Q^\text{tree}_{D^*}(K,\Sigma)$ stronger than all $p_\alpha$’s.

**Claim B.7.3.1.** The condition $r$ is $(R,\mathcal{Q})$–fuzzy generic for $\bar{q}$.

**Proof of the Claim.** It is very much like [B.7.1.1] We note that $r$ is $(N, Q^\text{tree}_{D^*}(K,\Sigma))$–generic (by clause (iv)), and therefore it is enough to show that the Generic player has a strategy in the game $\mathcal{O}^\text{fuzzy}_{\lambda}(r, N, \bar{F}, \bar{\mathcal{I}}, \bar{q})$ which guarantees that the result $(r_1, C_1 : i < \lambda)$ of the play satisfies $\text{A.3.3(5)}(\beta)$ (remember $\text{A.3.8(3)}$). Let us describe such a strategy. First, for $\alpha < \lambda$ let $\zeta_\alpha < \lambda$ be such that
\[(\forall q \in I_{\zeta_\alpha})(\text{either } p_\alpha \leq q \text{ or } p_\alpha, q \text{ are incompatible }),\]
and let $E' = \{ \delta \in E : (\forall \alpha < \delta)(\zeta_\alpha < \delta) \}$ (it is a club of $\lambda$).
Now, suppose that during a play of $\mathcal{C}_{\lambda}^{\text{fuzzy}}(r, N, \tilde{I}, h, Q_{D}^{\text{free}}(K, \Sigma), \bar{F}, \bar{q})$ the players have arrived at stage $i \in S$ having constructed a sequence $\langle r_{j}, C_{j} : j < i \rangle$.

If either $i$ is a successor ordinal or $i \notin \bigcap_{j<i} C_{j}$, then Generic plays the $<\chi$–first condition $r_{i} \in Q_{D}^{\text{free}}(K, \Sigma)$ such that $(\forall j < i)(r_{j} \leq r_{i})$ and $\text{lh}(\text{root}(r_{i})) > i$ and $C_{i} = E' \setminus \text{lh}(\text{root}(r_{i}))$.

If $i \in \bigcap_{j<i} C_{j} \subseteq E'$ is a limit ordinal, then Generic asks

\[ (*) \text{ is } (h \circ F_{i}(\alpha) : \alpha < i) \text{ an increasing } \tilde{I} \text{–exact sequence such that } \langle \forall \alpha < i \rangle(\exists j < i)(h \circ F_{i}(\alpha) \leq r_{j})? \]

If the answer to $(*)$ is “no”, then she plays like at the successor stage.

If the answer to $(*)$ is “yes”, then Generic takes $\eta = \bigcup_{j<i} \text{root}(r_{j})$ and she notes that $\text{lh}(\eta) = \delta$ (by the choice of $C_{j}$’s at successor stages) and $\eta = \bigcup_{i<\delta} \text{root}(h \circ F_{i}(\delta))$.

Also, by the exactness and the choice of $E'$, we have

\[ (\forall j < i)(\exists \alpha < i)(p_{j} \leq h \circ F_{i}(\alpha)). \]

So now Generic looks at clause (vi) of the choice of $p_{\alpha}$’s. She picks (say, the $<\chi$–first) $\nu \in \bigcap_{j<i} \text{pos}[p_{i}'_{j}] \subseteq \bigcap_{j<i} \text{pos}[\text{ho}_F(T_{\eta}(\alpha))]$ and notices that (by (vi)) $\nu \preceq \text{root}(q_{i,\nu}) \in T_{\eta}$, $q_{i,\nu} = (p_{i})_{\text{root}(q_{i,\nu})}$ and $(\forall \rho \in T_{\eta})((\nu \preceq \rho \prec \text{root}(q_{i,\nu}) \Rightarrow |\text{pos}[p_{\rho}'_{i}]| = 1)$.

Therefore, $\text{root}(q_{i,\nu}) \in T^{r_{j}}$ and $q_{i,\nu} \leq (T^{r_{j}})[\text{root}(q_{i,\nu})]$ (for each $j < i$). So the Generic player can play $C_{i} = E' \setminus i$ and the $<\chi$–first condition $r_{j}$ stronger than all $r_{j}$ (for $j < i$) and $q_{i,\nu}$.

Easily, the strategy described above has the required property, and the proof is completed.

\[ \Box \]

\textbf{Problem B.7.4.} Unlike that was in the case of \[ \text{B.7.3,} \] it is not clear how the proof of \[ \text{B.7.3} \] can be modified to get the parallel result for non-tree case. So, assuming that $A, W, H$ and $D^{*}$ are as in \[ \text{B.7.3} \] and $(K, \Sigma)$ is an exactly $\lambda$–complete very local $\lambda$–creating pair for $H$, is the forcing notion $Q_{D}^{\text{free}}(K, \Sigma)$ fuzzy proper for $W$?

\textbf{B.8. More examples and applications}

Here we are going to present some direct applications of the methods developed in this paper. Though we do keep our basic assumptions from \[ \text{B.3} \] we are going to introduce more parameters, so let us fully state the context we are working in now.

\textit{Context B.8.1.} (a) $\lambda$ is a strongly inaccessible cardinal, $2^{\lambda} = \lambda^{+}$, and $2^{\lambda^{+}} = \lambda^{++}$, and

(b) $\bar{\mu} = (\mu_{\alpha} : \alpha < \lambda)$, $\bar{\lambda} = (\lambda_{\alpha} : \alpha < \lambda)$ and $\bar{\kappa} = (\kappa_{\alpha} : \alpha < \lambda)$ are strictly increasing sequences of uncountable regular cardinals, each cofinal in $\lambda$,

(c) for each $\alpha < \lambda$,

\begin{itemize}
  \item $\mu_{\alpha}^{+} < \lambda_{\alpha} < \kappa_{\alpha}$,
  \item $\prod_{\beta < \alpha} \lambda_{\beta} < \lambda_{\alpha}$ and $(\forall \xi < \lambda_{\alpha})(|\xi|^\alpha < \lambda_{\alpha})$,
\end{itemize}

(d) $A = H_{<\alpha}(\lambda^{+})$, $\lambda^{*} > \lambda$ and $W \subseteq [A]^{\lambda}$ are as in \[ \text{\underline{A.3.3}} \]

(e) $D$ is a normal filter on $\lambda$ such that there is a $D$–diamond.
Let us recall some notions related to cardinal characteristics of $\lambda$-reals.

**Definition B.8.2.**

1. Let $S_\mu$ be the family of all sequences $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ such that $a_\alpha \in [\lambda]^{<\mu_\alpha}$ (for all $\alpha < \lambda$). We define

   \[
   c(\mu) = \min \{ |\mathcal{Y}| : \mathcal{Y} \subseteq S_\mu \land (\forall f \in \lambda \lambda)(\exists \bar{a} \in \mathcal{Y})(\forall \alpha < \lambda)(f(\alpha) \in a_\alpha) \},
   \]

   \[
   c_\text{cl}(\mu) = \min \{ |\mathcal{Y}| : \mathcal{Y} \subseteq S_\mu \land (\forall f \in \lambda \lambda)(\exists \bar{a} \in \mathcal{Y})(\{\alpha < \lambda : f(\alpha) \in a_\alpha\} \subseteq (\mathcal{D}_\lambda)^+) \},
   \]

   and also

   \[
   c_\text{li}(\mu) = \min \{ |\mathcal{G}| : \mathcal{G} \subseteq \prod_{\alpha < \lambda} \mu_\alpha \text{ and } (\forall f \in \lambda \lambda)(\exists g \in \mathcal{G})(\{\alpha < \lambda : f(\alpha) \neq g(\alpha)\} \subseteq \mathcal{D}_\lambda) \}
   \]

2. For an ideal $\mathcal{I}$ of subsets of a set $\mathcal{X}$, the covering number $\text{cov}(\mathcal{I})$ of $\mathcal{I}$ is

   \[
   \text{cov}(\mathcal{I}) = \min \{ |\mathcal{Y}| : \mathcal{Y} \subseteq \mathcal{I} \land \bigcup \mathcal{Y} = \mathcal{X} \}.
   \]

**Proposition B.8.3.** It is consistent that $c(\bar{\lambda}) < e_{\text{cl}}(\bar{\mu})$.

**Proof.** Let $H_0(\alpha) = \mu_\alpha$ (for $\alpha < \lambda$) and let $K_0$ consist of all $\lambda$–tree creatures $t \in \text{TCR}^{\lambda \lambda}[H_0]$ such that:

- if $\text{dis}[t] = \mu_{\text{th}(\eta[t])}$, then $\text{pos}[t] = \{ \eta[t]^{-}(\xi) : \xi < \mu_{\text{th}(\eta[t])} \}$ and $\text{nor}[t] = \mu_{\text{th}(\eta[t])}$,
- if $\text{dis}[t] < \mu_{\text{th}(\eta[t])}$, then $\text{pos}[t] = \{ \eta[t]^{-}(\text{dis}[t]) \}$ and $\text{nor}[t] = 0$.

Let $\Sigma_0$ be a local tree-composition operation on $K_0$ (so its domain consists of singletons only) such that

- if $\text{dis}[t] < \mu_{\text{th}(\eta[t])}$, then $\Sigma_0(t) = \{ t \}$,
- if $\text{dis}[t] = \mu_{\text{th}(\eta[t])}$, then $\Sigma_0(t) = \{ t' \in K_0 : \eta[t'] = \eta[t] \}$.

It should be clear that $(K_0, \Sigma_0)$ is a very local exactly $\bar{\lambda}$–complete tree creating pair. The forcing notion $Q^{\text{tree}}_{\lambda \lambda}(K_0, \Sigma_0)$ has the strong $\lambda$–Sacks property (by B.6.5).

Let $W$ be the canonical $Q^{\text{tree}}_{\lambda \lambda}(K_0, \Sigma_0)$–name for the generic function in $\prod_{\alpha < \lambda} \mu_\alpha$, so

\[
\forall \preceq_{\lambda_{\alpha}}^\text{tree}_{\lambda \lambda}(K_0, \Sigma_0) \text{ "root}(p) \triangleleft W".
\]

Then we have

\[
\forall \preceq_{\lambda_{\alpha}}^\text{tree}_{\lambda \lambda}(K_0, \Sigma_0) \text{ "}(\forall f \in \prod_{\alpha < \lambda} \mu_\alpha \cap V)(\{\alpha < \lambda : W(\alpha) = f(\alpha)\} \subseteq (\mathcal{D}_\lambda)^+)".
\]

Now let $\mathbb{P}$ be the limit of a $\lambda$–support iteration, $\lambda^{++}$ in length, of the forcing notions $Q^{\text{tree}}_{\lambda \lambda}(K_0, \Sigma_0)$. Then, by [B.2.3] $\Delta_{1.3} + \Delta_{1.3} + \Delta_{3.10}$

- $\mathbb{P}$ is $\lambda$–complete, $\lambda$–proper and satisfies the $\lambda^{++}$–cc, and it has a dense subset of size $\lambda^{++}$, thus forcing with $\mathbb{P}$ does not collapse cardinals,
- $\mathbb{P}$ has the $\lambda$–Sacks property, it is weakly fuzzy proper for $W$,
- $\mathbb{P}$ has the $\lambda$–Sacks property, it is weakly fuzzy proper for $W$,
- $\mathbb{P}$ is $\lambda$–complete, satisfies the $\lambda^{++}$–cc, and $c(\bar{\lambda}) = \lambda^{++}$

$\square$

**Remark B.8.4.** The forcing $Q^{\text{tree}}_{\lambda \lambda}(K_0, \Sigma_0)$ is a “bounded relative” of $\mathcal{D}_\lambda$ from [B.14.10] (remember B.6.7). It is also a generalization of the forcing notions $\mathcal{D}_\lambda$ from [B.12].
Proposition B.8.5. It is consistent that $c(\lambda) < c_\omega(\bar{\mu}^+) = c(\bar{\mu}^+)$, where $\bar{\mu}^+ = \langle \mu_\alpha^+ : \alpha < \lambda \rangle$.

Proof. Let $H_1(\alpha) = \mu_\alpha^+$ (for $\alpha < \lambda$) and let $K'_I$ consist of all $\lambda$–tree creatures $t \in TCR^\lambda[H_1]$ such that:

- $\text{dis}[t] \subseteq \mu_{\text{lh}(\eta[t])}$, either $|\text{dis}[t]| = 1$ or $\text{dis}[t]$ is a club of $\mu_{\text{lh}(\eta[t])}$;
- $\text{pos}[t] = \{ \eta[t] \smallsetminus (\xi : \xi \in \text{dis}[t]) \}$;
- if $|\text{dis}[t]| = 1$ then $\text{nor}[t] = 0$, if $|\text{dis}[t]| > 1$ then $\text{nor}[t] = \mu_{\text{lh}(\eta[t])}$.

Let $\Sigma'_I$ be a local tree-composition operation on $K'_I$ such that

$$\Sigma'_I(t) = \{ t' \in K'_I : \eta[t'] = \eta[t] \& \text{dis}[t'] \subseteq \text{dis}[t] \}.$$  

Then $(K'_I, \Sigma'_I)$ is a very local $\bar{\mu}$–complete $\lambda$–tree creating pair. Let $(K_1, \Sigma_1)$ be the $\bar{\mu}$–exactivity of $(K'_I, \Sigma'_I)$ (see B.8.5), thus $(K_1, \Sigma_1)$ is a very local exactly $\bar{\mu}$–complete $\lambda$–tree creating pair. The forcing notion $Q_{D_\lambda}^{\text{tree}}(K_1, \Sigma_1)$ is $\lambda$–complete fuzzy proper for $W$ and it has the strong $\bar{\lambda}$–Sacks property. Also, letting $W$ be the canonical name for the generic function in $\prod_{\alpha < \lambda}^\mu^+$ (i.e., \( p \Vdash_{Q_{D_\lambda}^{\text{tree}}(K_1, \Sigma_1)} \text{root}(p) < W^+ \)), we have

$$\Vdash Q_{D_\lambda}^{\text{tree}}(K_1, \Sigma_1) \quad " \left( \forall \bar{a} \in S_{\mu^+} \cap V \right) \left( \{ \alpha < \lambda : W(\alpha) \notin a_\alpha \} \in D_\lambda \right) \cdot"$$

Let $\mathbb{P}$ be the limit of a $\lambda$–support iteration, $\lambda^++$ in length, of the forcing notions $Q_{D_\lambda}^{\text{tree}}(K_1, \Sigma_1)$. Then (by A.2.3 + A.1.9 + A.1.4 + A.3.10) we have:

- $\mathbb{P}$ is $\lambda$–complete, $\lambda$–proper and satisfies the $\lambda^{++}$–cc, and it has a dense subset of size $\lambda^{++}$, thus forcing with $\mathbb{P}$ does not collapse cardinals,
- $\mathbb{P}$ has the $\bar{\lambda}$–Sacks property, it is weakly fuzzy proper for $W$,
- $\Vdash \mathbb{P} \upharpoonright 2^\lambda = 2^{\lambda^+} = \lambda^{++} = c_\omega^{\bar{\mu}^+}$ and $c(\bar{\lambda}) = \lambda^+$

\[ \square \]

Remark B.8.6. The result in [B.8.5] is of interest as it shows that the $\lambda$–versions of cardinal characteristics of the reals may behave totally differently from their “ancestors”. Recall that if for an increasing function $f \in \omega\omega$ we let $S_f$ consist of all sequences $\bar{a} = \langle a_n : n < \omega \rangle$ with $a_n \in [\omega]^{|f(n)| + 1}$ (for $n < \omega$), then

$$\min \{ |Y| : Y \subseteq S_f \& (\forall h \in \omega^n)(\exists \bar{a} \in Y)(\forall n < \omega)(h(n) \in a_n) \} = \min \{ |Y| : Y \subseteq S_f \& (\forall h \in \omega^n)(\exists \bar{a} \in Y)(\forall n < \omega)(h(n) \in a_n) \}$$

for any increasing $f, g \in \omega\omega$.

The $\lambda$–tree creating pair $(K_1, \Sigma_1)$ may be treated (in some sense) as a special case of the $\lambda$–tree creating pairs $(K(\bar{A}), \Sigma(\bar{A}))$ from B.8.10 below.

Definition B.8.7. Let $\mathcal{A}$ be a family of subsets of $\kappa$ such that $\kappa \in \mathcal{A}$.

1. A game $\mathcal{O}^*(\mathcal{A}, \mu)$ of two players, I and II, is defined as follows. A play lasts $\mu$ moves, in the $\alpha$th move a set $A_\alpha \in \mathcal{A}$ is chosen, and player I chooses $A_\alpha$ for every $\alpha$. In the end player II wins if $\bigcap_{\alpha < \mu} A_\alpha \neq \emptyset$.

2. The family $\mathcal{A}$ is a $\mu$–category prebase on $\kappa$ if player II has a winning strategy in the game $\mathcal{O}^*(\mathcal{A}, \mu)$ and $(\forall A \in \mathcal{A})(\forall \xi < \kappa)(\exists B \in \mathcal{A})(B \subseteq A \setminus \{ \xi \})$.

3. A set $X \subseteq \kappa$ is $\mathcal{A}$–presmall if

$$(\forall A \in \mathcal{A})(\exists B \in \mathcal{A})(B \subseteq A \setminus X).$$

Of course, every $\mu^+$–complete uniform filter $D^*$ on $\kappa$ is a $\mu$–category base on $\kappa$ and then a set is $D$–presmall if and only if its complement is in $D^*$.
Definition B.8.8. (1) A $\lambda$–smallness base on $\kappa$ is a sequence $\langle A_\alpha : \alpha < \lambda \rangle$ such that each $A_\alpha$ is a $\lambda_\alpha$–category prebase on $\kappa_\alpha$.

Let $\mathcal{A}$ be a $\lambda$–smallness base on $\kappa$.

(2) Let $T \subseteq \bigcup_{\alpha < \lambda} \prod_{\beta < \lambda} \kappa_\alpha$ be a complete $\lambda$–tree with $\text{max}(T) = \emptyset$ and $D^\ast$ be a filter on $\lambda$. We say that

- $T$ is $\mathcal{A}$–small if for every $\eta \in (T)_\alpha$, $\alpha < \lambda$, the set $\{ \xi < \kappa_\alpha : \eta ^\prec (\xi) \in T \}$

is $A_\alpha$–presmall;
- $T$ is $(D^\ast, \mathcal{A})$–small if

\[
\{ \alpha < \lambda : \text{for every } \eta \in (T)_\alpha \text{ the set } \{ \xi < \kappa_\alpha : \eta ^\prec (\xi) \in T \} \text{ is } A_\alpha \text{–presmall} \} \in D^\ast.
\]

(3) Let $\mathcal{J}_\kappa(\mathcal{A})$ consist of all subsets $X$ of $\prod_{\alpha < \lambda} \kappa_\alpha$ such that $X \subseteq \bigcup_{\varepsilon < \lambda} \lim_\varepsilon(T_\varepsilon)$ for some $\mathcal{A}$–small trees $T_\varepsilon \subseteq \bigcup_{\alpha < \lambda} \prod_{\beta < \lambda} \kappa_\alpha$ (for $\varepsilon < \lambda$).

$\mathcal{J}_\kappa(D^\ast, \mathcal{A})$ is defined similarly, replacing “$\mathcal{A}$–small” by “$(D^\ast, \mathcal{A})$–small”.

Proposition B.8.9. Let $\mathcal{A}$ be a $\lambda$–smallness base on $\kappa$. Then both $\mathcal{J}_\kappa(\mathcal{A})$ and $\mathcal{J}_\kappa(D^\ast, \mathcal{A})$ are proper $\lambda^+$–complete ideals of subsets of $\prod_{\alpha < \lambda} \kappa_\alpha$, $\mathcal{J}_\kappa(\mathcal{A}) \subseteq \mathcal{J}_\kappa(D^\ast, \mathcal{A})$.

They contain singletons and $\lambda < \text{cov}(\mathcal{J}_\kappa(D^\ast, \mathcal{A})) \leq \text{cov}(\mathcal{J}_\kappa(\mathcal{A}))$.

Proposition B.8.10. Let $\mathcal{A}$ be a $\lambda$–smallness base on $\kappa$ and $D^\ast$ be a normal filter on $\lambda$. It is consistent that $\text{cov}(\mathcal{J}_\kappa(D^\ast, \mathcal{A})) > \lambda^+$.

Proof. First we define a $\lambda$–tree creating pair $(K(\mathcal{A}), \Sigma(\mathcal{A})) = (K, \Sigma)$. For $\alpha < \lambda$ let $H(\alpha) = \kappa_\alpha$ and let $\text{st}_\alpha$ be a winning strategy of player II in the game $\Sigma^\ast(\mathcal{A}_\alpha, \kappa_\alpha)$.

Let $K$ consist of all $\lambda$–tree creatures $t \in \text{TCR}_\lambda^\lambda[H]$ such that letting $\alpha = \text{lh}(\eta [t])$:

- either $\text{dis}[t] = (\delta, \langle A^t_i : i < \delta \rangle)$, where $\delta < \lambda_\alpha$ and $\langle A^t_i : i < \delta \rangle$ is (an initial segment of) a play of $\Sigma^\ast(\mathcal{A}_{\alpha}, \kappa_\alpha)$ in which player II uses strategy $\text{st}_\alpha$,
- or $\text{dis}[t] = (\xi)$ for some $\xi < \kappa_\alpha$;
- if $\text{dis}[t] = (\xi)$, then $\text{pos}[t] = \{ \eta [t] ^\prec (\xi) \}$ and $\text{nor}[t] = 0$;
- if $\text{dis}[t] = (\delta, \langle A^t_i : i < \delta \rangle)$, then $\text{pos}[t] = \{ \eta [t] ^\prec (\xi) : \xi \in \bigcap_{i < \delta} A_i \}$ and $\text{nor}[t] = \alpha + 1$. (If $\delta = 0$ then we stipulate $\text{pos}[t] = \{ \eta [t] ^\prec (\xi) : \xi < \kappa_\alpha \}$.)

The domain of the tree composition operation $\Sigma$ consists of singletons only, and if $\text{nor}[t] = 0$ then $\Sigma(t) = \{ t \}$, if $\text{nor}[t] > 0$, $\alpha = \text{lh}(\eta [t])$ and $\text{dis}[t] = (\delta, \langle A^t_i : i < \delta \rangle)$, then $\Sigma(t)$ consists of those $t' \in K \cap \text{TCR}_\lambda^\lambda[H]$ for which:

- either $\text{nor}[t'] = 0$ and $\text{pos}[t'] \subseteq \text{pos}[t]$,
- or $\text{nor}[t'] > 0$, $\text{dis}[t'] = (\delta', \langle A^{t'}_i : i < \delta' \rangle)$ and $\langle A^{t'}_i : i < \delta' \rangle \subseteq \langle A^t_i : i < \delta \rangle$.

Claim B.8.10.1. $(K, \Sigma)$ is an exactly $\lambda$–complete very local tree creating pair for $H$. Hence the forcing notion $\mathbb{Q}^\text{tree}_D(K, \Sigma)$ is fuzzy proper for $\mathbb{W}$.

Proof of the Claim. Should be clear. \qed

We finish the proof of the proposition in a standard way: we force with $\lambda$–support iteration, $\lambda^{++}$ in length, of the forcing notion $\mathbb{Q}^\text{tree}_D(K(\mathcal{A}), \Sigma(\mathcal{A}))$. \qed
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