Dependency Stochastic Boolean Satisfiability: 
A Logical Formalism for NEXPTIME Decision Problems with Uncertainty

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Abstract

Stochastic Boolean Satisfiability (SSAT) is a logical formalism to model decision problems with uncertainty, such as Partially Observable Markov Decision Process (POMDP). SSAT, however, is limited by its descriptive power within the PSPACE-complexity class. More complex problems, such as the NEXPTIME-complete Decentralized POMDP (Dec-POMDP), cannot be succinctly encoded with SSAT. To provide a logical formalism of such problems, we generalize Dependency Quantified Boolean Formula (DQBF), a representative problem in the NEXPTIME-complete class, to its stochastic variant, named Dependency SSAT (DSSAT), and show that DSSAT is also NEXPTIME-complete. To demonstrate the descriptive power of DSSAT, we further establish a polynomial-time reduction from Dec-POMDP to DSSAT. Our results may encourage DSSAT solver development to enable potential broad applications.

1 Introduction

The success of satisfiability (SAT) solvers (Biere, Heule, and van Maaren 2009) in numerous applications including artificial intelligence (Nilsson 2014; Russell and Norvig 2016), electronic design automation (Marques-Silva and Sakallah 2000), software verification (Bérard et al. 2013; Jhala and Majumdar 2009), etc., has encouraged the development of more advanced decision procedures for satisfiability with respect to more complex logics beyond pure propositional. For example, solvers of the satisfiability modulo theories (SMT) (De Moura and Bjørner 2011; Barrett and Tinelli 2018) accommodate first order logic fragments; quantified Boolean formula (QBF) (Giunchiglia et al. 2005; Büning and Bubeck 2009) allows both existential and universal quantifiers; stochastic Boolean satisfiability (SSAT) (Littman, Majercik, and Pitassi 2001; Majercik 2009) models uncertainty by using random quantification; and dependency QBF (DQBF) (Balabanov, Chiang, and Jiang 2014; Scholl and Wimmer 2018) equips Henkin quantifiers capable of describing multi-player games with partial information. Due to their simplicity and generality, the satisfiability formulations of various logic constraints are under active investigation.

Among the quantified decision procedures, QBF and SSAT are closely related and share large commonality. While SSAT generalizes QBF to allow random quantifiers to model decision under uncertainty, they both have the same PSPACE-complete complexity (Stockmeyer and Meyer 1973). A number of SSAT solvers have been developed and applied in probabilistic planning, formal verification of probabilistic design, partially observable Markov Decision Process (POMDP), and analysis of software security. For example, solver MAXPLAN in (Majercik and Littman 1998) encodes a conformant planning problem as an exist-random quantified SSAT formula; solver ZANDER in (Majercik and Littman 2003) deals with partially observable probabilistic planning by formulating the problem as a general SSAT formula; solver DC-SSAT in (Majercik and Boots 2005) relies on a divide-and-conquer approach to speedup the solving of a general SSAT formula. Authors in (Lee, Wang, and Jiang 2017; Lee, Wang, and Jiang 2018) developed SSAT solvers ressat and erssat, respectively, for random-exist and exist-random quantified SSAT formulas, and show that they can be applied for the formal verification of probabilistic design formulated in (Lee and Jiang 2018). Partially observable Markov Decision Process (POMDP) has also been studied under the formalism of SSAT in (Majercik and Littman 2003; Salmon and Poupart 2019). Recently, authors in (Salmon and Poupart 2019) established bi-directional polynomial-time reductions between SSAT and POMDP. The analysis of software security is investigated as an exist-random quantified SSAT formula in (Fremont, Rabe, and Seshia 2017).

In view of the close relation between QBF and SSAT, we raise the question what would be the formalism that extends DQBF to the stochastic domain. The main results of this work include 1) formulating the new logic of dependency SSAT (DSSAT) in Section 3, 2) characterizing the NEXPTIME-complete complexity of DSSAT in Section 4, and 3) showing its application in modeling decentralized POMDP (Dec-POMDP) in Section 5. Our results may shed
light on the development of DSSAT solvers to enable potential broad applications.

2 Preliminaries

In this section, we provide background knowledge to facilitate the subsequent discussion. In particular, we will introduce Stochastic Boolean Satisfiability (SSAT), Dependency Quantified Boolean Formula (DQBF), and Decentralized POMDP (Dec-POMDP).

In the following, Boolean values TRUE and FALSE are represented by symbols \( \top \) and \( \bot \), respectively; They are also treated as 1 and 0, respectively, in arithmetic computation. Boolean connectives \( \neg, \vee, \land, \Rightarrow, \Leftrightarrow \) are interpreted in their conventional semantics. Given a variable set \( V \), an assignment \( \alpha \) is a mapping from each variable \( x \in V \) to \( B = \{ \top, \bot \} \), and we denote the set of all assignments over \( V \) by \( \mathcal{A}(V) \). An assignment \( \alpha \) satisfies a Boolean formula \( \phi \) over a variable set \( V \) if \( \phi \) yields \( \top \) after substituting all occurrences of every variable \( x \in V \) with its assigned value \( \alpha(x) \) and simplifying \( \phi \) under the semantics of Boolean connectives.

A Boolean formula \( \phi \) over a variable set \( V \) is a tautology if every assignment \( \alpha \in \mathcal{A}(V) \) satisfies \( \phi \).

2.1 Stochastic Boolean Satisfiability

SSAT is first proposed in (Papadimitriou 1985) as games against nature. An SSAT formula \( \Phi \) over a variable set \( V = \{ x_1, \ldots, x_n \} \) is of the form \( Q_1 x_1 \ldots Q_n x_n \phi \), where each \( Q_i \in \{ \exists, \forall \} \) and Boolean formula \( \phi \) over \( V \) is quantifier-free. Symbol \( \exists \) denotes an existential quantifier, and \( \forall \) denotes a randomized quantifier, which requires the probability that the quantified variable equals \( \top \) to be \( p \in [0, 1] \). Given an SSAT formula \( \Phi \), the quantification structure \( Q_1 x_1 \ldots Q_n x_n \) is called the prefix, and the quantifier-free Boolean formula \( \phi \) is called the matrix.

Let \( x \) be the outermost variable in the prefix of an SSAT formula \( \Phi \). The satisfying probability of \( \Phi \), denoted by \( \Pr[\Phi] \), is defined recursively by the following four rules:

\begin{align*}
\text{a) } & \Pr[\top] = 1, \\
\text{b) } & \Pr[\bot] = 0, \\
\text{c) } & \Pr[\Phi] = \max\{\Pr[\Phi_{\neg x}], \Pr[\Phi_{x}]\}, \text{ if } x \text{ is existentially quantified}, \\
\text{d) } & \Pr[\Phi] = (1 - p) \Pr[\Phi_{\neg x}] + p \Pr[\Phi_{x}], \text{ if } x \text{ is randomly quantified by } \forall \).
\end{align*}

where \( \Phi_{\neg x} \) and \( \Phi_{x} \) denote the SSAT formulas obtained by eliminating the outermost quantifier of \( x \) via substituting the value of \( x \) in the matrix with \( \top \) and \( \bot \), respectively.

The decision version of SSAT is stated as follows. Given an SSAT formula \( \Phi \) and a threshold \( \theta \in [0, 1] \), decide whether \( \Pr[\Phi] \geq \theta \). On the other hand, the optimization version asks to compute \( \Pr[\Phi] \). The decision version of SSAT is shown to be PSPACE-complete in (Papadimitriou 1985).

2.2 Dependency Quantified Boolean Formula

The concept of DQBF is pioneered in (Peterson and Reif 1979) as multiple-person alternation. In contrast to the linearly ordered prefix used in QBF, i.e., an existentially quantified variable will depend on all of its outer universally quantified variables, the quantification structure is extended with an arbitrary dependency in DQBF.

A DQBF \( \Phi \) over a set \( V = \{ x_1, \ldots, x_n, y_1, \ldots, y_m \} \) is of the form \( \forall x_1 \ldots \forall x_n \exists y_1(D_{y_1}) \ldots \exists y_m(D_{y_m}) \phi \), where each \( D_{y_j} \subseteq \{ x_1, \ldots, x_n \} \) denotes the set of variables that \( y_j \) can depend on, and Boolean formula \( \phi \) over \( V \) is quantifier-free. We denote the set \( \{ x_1, \ldots, x_n \} \) (resp. \( \{ y_1, \ldots, y_m \} \) of universally (resp. existentially) quantified variables of \( \Phi \) by \( V_s^\forall \) (resp. \( V_s^\exists \)).

Given a DQBF \( \Phi \), it is satisfied if for each variable \( y_j \), there exists a function \( f_j : A(D_{y_j}) \rightarrow B \), such that after eliminating variables in \( V_s^\exists \) by substituting them with their corresponding functions respectively, matrix \( \phi \) becomes a tautology over variables in \( V_s^\forall \). The set of functions \( F = \{ f_1, \ldots, f_m \} \) is called a set of Skolem functions for \( \Phi \). In other words, \( \Phi \) is satisfied by \( F \) if

\[ \min_{\beta \in \mathcal{A}(V_s^\forall)} \mathbb{I}_{\phi|\beta}(\beta) = 1, \]

where \( \mathbb{I}_{\phi|\beta}(\cdot) \) is the indicator function to indicate whether an assignment over variables in \( V_s^\forall \) belongs to the set of satisfying assignments of matrix \( \phi \), when variables in \( V_s^\exists \) are substituted by their Skolem functions in \( F \). That is, \( \phi|\beta = \{ \beta_1, \ldots, \beta_{|F|}, f_1[\beta], \ldots, f_m[\beta] \equiv \top \} \), where \( f_j[\beta] \) is the logical value derived by substituting every \( x_i \in D_{y_j} \) with \( \beta(x_i) \) in function \( f_j \).

The satisfiability problem of DQBF is NEXPTIME-complete (Peterson, Reif, and Azhar 2001).

2.3 Decentralized POMDP

Dec-POMDP is a formalism to model multiagent systems under uncertainty and with partial information. Its computational complexity was shown to be NEXPTIME-complete in (Bernstein et al. 2002). In the following, we briefly review the definition, optimality criteria, and value function of Dec-POMDP (Oliehoek, Amato, and others 2016).

A Dec-POMDP is formally specified by a tuple \( M = (I, S, \{ A_i \}, T, \rho, \{ O_i \}, \Omega, \Delta_0, h) \), where \( I = \{ 1, \ldots, n \} \) is a finite set of agents, \( S \) is a finite set of states, \( A_i \) is a finite set of actions of agent \( i, T : S \times (A_1 \times \cdots \times A_n) \rightarrow [0, 1] \) is a transition distribution function with \( T(s, a) = \Pr[s'|s, a] \), the probability to transit to state \( s' \) from state \( s \) after taking actions \( a \), \( \rho : S \times (A_1 \times \cdots \times A_n) \rightarrow \mathbb{R} \) is a reward function with \( \rho(s, a) \) giving the reward for being in state \( s \) and taking actions \( a \), \( O_i \) is a finite set of observations for agent \( i, \Omega : S \times (A_1 \times \cdots \times A_n) \times (O_1 \times \cdots \times O_n) \rightarrow [0, 1] \) is an observation distribution function with \( \Omega(s', a, o) = \Pr[o|s', a] \), the probability to receive observation \( o \) after taking \( a \) and transiting to state \( s', \Delta_0 : S \rightarrow [0, 1] \) is an initial state distribution function with \( \Delta_0(s) = \Pr[s^0 \equiv s] \), the probability for the initial state \( s^0 \) being state \( s \), and \( h \) is a planning horizon, which we assume finite in this work.

Given a Dec-POMDP \( M \), we aim at maximizing the expected total reward \( \mathbb{E} \left[ \sum_{t=0}^{h-1} \rho(s^t, a^t) \right] \) through searching an optimal joint policy for the team of agents. Specifically, a policy \( \pi_i \) of agent \( i \) is a mapping from the agent’s observation history, i.e., a sequence of observations \( o_i^t = o_i^0, \ldots, o_i^t \).
received by agent $i$, to an action $a_t^{i+1} \in A_i$. A joint policy for the team of agents $\pi = (\pi_1, \ldots, \pi_n)$ maps the agents’ joint observation history $\vec{O} = (o_1^{\mathcal{O}_1}, \ldots, o_n^{\mathcal{O}_n})$ to actions $a_t^{i+1} = (a_1^{i+1}, \ldots, a_n^{i+1})$. We shall focus on deterministic policies only, as it was shown that every Dec-POMDP with a finite planning horizon has a deterministic optimal joint policy (Olteanu, Spaan, and Vlassis 2008).

To assess the quality of a joint policy $\vec{\pi}$, its value is defined to be $E[\sum_{t=0}^{n-1} \rho(s_t, a_t)|\Delta_0, \vec{\pi}]$. The value function $V(\vec{\pi})$ can be computed in a recursive manner, where for $t = h - 1$, $V^\pi(s^{h-1}, \vec{O}^{h-2}) = \rho(s^{h-1}, \vec{\pi}(s^{h-2}))+\sum_{s^{h+1} \in S} \sum_{\sigma \in \Delta} \Pr[s^{h+1}, \sigma|s^{h}, \vec{\sigma}^{h}] |V^\pi(s^{h+1}, \vec{O})$ (1)

where

$\Pr[s^{h+1}, \sigma|s^{h}, \vec{\sigma}^{h}] = T(s^{h}, \vec{\pi}(\vec{O}^{h}), s^{h+1}, \vec{\sigma}^{h}, \vec{\sigma})$.

The recursive computation of the value functions in Eq. (1) is called the Bellman Equation for Dec-POMDP. Finally, the value of a joint policy $V(\vec{\pi}) = \sum_{s \in S} \Delta_0(s) \sum_{\vec{O} \in \Delta} \rho(s, \vec{\pi}(\vec{O}))$.

We use the symbol $\vec{O}^{h-1}$ to denote the observation history at the first stage, i.e., $t = 0$, which contains no observations.

3 Dependency Stochastic Boolean Satisfiability

In this section, we generalize DQBF to its stochastic variant, named Dependancy Stochastic Boolean Satisfiability (DSSAT).

A DSSAT formula $\Phi$ over $V = \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ is of the form $\mathcal{Y}^p_1 x_1 \ldots \mathcal{Y}^p_n x_n \mathcal{E}^q_1 y_1(D_{y_1}) \ldots \mathcal{E}^q_m y_m(D_{y_m}) \phi$, where each $D_{y_j} \subseteq \{x_1, \ldots, x_n\}$ denotes the set of variables that variable $y_j$ can depend on, and Boolean formula $\phi$ over $V$ is quantifier-free. We denote the set $\{x_1, \ldots, x_n\}$ (resp. $\{y_1, \ldots, y_m\}$) of randomly (resp. existentially) quantified variables by $\mathcal{V}_\Phi^D$ (resp. $\mathcal{V}_\Phi^E$).

Given a DSSAT formula $\Phi$ and a set of Skolem functions $\mathcal{F} = \{f_j : A(D_{y_j}) \rightarrow \mathbb{B}| j = 1, \ldots, m\}$, the satisfying probability $\Pr[\Phi|\mathcal{F}]$ of $\Phi$ with respect to $\mathcal{F}$ is defined by the following equation:

$\Pr[\Phi|\mathcal{F}] = \sum_{\alpha \in A(V^D_\mathcal{F})} \mathbb{1}_{\phi|\mathcal{F}}(\alpha) \times w(\alpha),$

where $\mathbb{1}_{\phi|\mathcal{F}}(\cdot)$ is the indicator function defined in Section 2.2 and $w(\alpha) = \prod_{j=1}^m (1 - p_j)^{1 - \mathbb{1}_{\phi|\mathcal{F}}(\alpha_s)}$. In other words, the satisfying probability is the summation of weights of satisfying assignments over $\mathcal{V}_\Phi^D$. Note that the weight of an assignment can be understood as its occurring probability in the space of $A(V^D_\mathcal{F})$.

The decision version of DSSAT is stated as follows. Given a DSSAT formula $\Phi$ and a threshold $\theta \in [0, 1]$, decide whether there exists a set of Skolem functions $\mathcal{F}$ such that $\Pr[\Phi|\mathcal{F}] \geq \theta$. On the other hand, the optimization version asks to find a set of Skolem functions to maximize the satisfying probability of $\Phi$.

We remark that, the formulation of DSSAT could be extended by incorporating universal quantities, which has also been done for SSAT, resulting in a unified framework named extended SSAT (Majercik 2009), consisting of both QBF and SSAT. In an analogous manner, an extended DSSAT formula $\Phi$ over variables $\{x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_l\}$ is of the form $\mathcal{Y}^p_1 z_1 \ldots \mathcal{Y}^p_l z_l |Vx_1 \ldots \forall x \exists y_1(D_{y_1}) \ldots \exists y_m(D_{y_m}) \phi, and each $D_{y_j} \subseteq \{x_1, \ldots, x_n, z_1, \ldots, z_l\}$ denotes the set of randomly and universally quantified variables which variable $y_j$ can depend on. The satisfying probability of $\Phi$ with respect to a set of Skolem functions $\mathcal{F} = \{f_j : A(D_{y_j}) \rightarrow \mathbb{B}| j = 1, \ldots, m\}$ is defined by the following equation:

$\Pr[\Phi|\mathcal{F}] = \sum_{\alpha \in A(V^D_\mathcal{F})} \min_{\beta \in A(V^E_\mathcal{F})} \mathbb{1}_{\phi|\mathcal{F}}((\alpha, \beta)) \times w(\alpha)$.

Note that although the extended formulation could increase the “practical” expressiveness, the computational complexity will not be changed.

4 DSSAT Complexity

In the following, we show that the decision version of DSSAT is NEXPTIME-complete.

**Theorem 1. DSSAT is NEXPTIME-complete.**

**Proof.** To show that DSSAT is NEXPTIME-complete, we have to show that it belongs to the NEXPTIME complexity class and that it is NEXPTIME-hard.

First, to see why DSSAT belongs to the NEXPTIME complexity class, observe that the Skolem function for an existentially quantified variable can be guessed and constructed as a truth table in nondeterministic exponential time. Given the guessed Skolem functions, the evaluation of the matrix, summation of weights of satisfying assignments, and comparison against the threshold $\theta$ can also be performed in exponential time. Overall, the whole procedure is done in nondeterministic exponential time, and hence DSSAT belongs to the NEXPTIME complexity class.

Second, to see why DSSAT is NEXPTIME-hard, we reduce NEXPTIME-complete problem DQBF to DSSAT as follows. Given a DQBF $\Phi_Q = \forall x_1 \ldots \forall x_n \exists y_1(D_{y_1}) \ldots \exists y_m(D_{y_m}) \phi$, we construct a DSSAT formula $\Phi_S = \mathcal{Y}^{0.5}_1 x_1 \ldots \mathcal{Y}^{0.5}_n x_n \mathcal{E}^q_1 y_1(D_{y_1}) \ldots \mathcal{E}^q_m y_m(D_{y_m}) \phi$ by changing every universal quantifier to a randomized quantifier with probability 0.5. Note that the reduction can be done in polynomial time with respect to the size of $\Phi_Q$. We would like to show that $\Phi_Q$ is satisfiable if and only if there exists a set of Skolem functions $\mathcal{F}$ such that $\Pr[\Phi_S|\mathcal{F}] \geq 1$.

The “only if” direction: As $\Phi_Q$ is satisfiable, there exists a set of Skolem functions $\mathcal{F}$ such that after substituting variables $y_1, \ldots, y_m$ with their corresponding Skolem
functions, matrix \( \phi \) becomes a tautology over variables \( x_1, \ldots, x_n \), i.e., every assignment \( \alpha \in \mathcal{A}(\{x_1, \ldots, x_n\}) \) satisfies \( \phi \). Therefore, \( \Pr[\Phi_S]\) \( = 1 \geq 1 \).

The “if” direction: As there exists a set of Skolem functions \( \mathcal{F} \) such that \( \Pr[\Phi_S]\) \( \geq 1 \), after substituting variables \( y_1, \ldots, y_m \) with their corresponding Skolem functions, every assignment \( \alpha \in \mathcal{A}(\{x_1, \ldots, x_n\}) \) must satisfy \( \phi \). Otherwise, the satisfying probability \( \Pr[\Phi_S]\) \( \geq 1 \) as the weight of some unsatisfying assignment is missing from the summation. Therefore, \( \Phi_S \) is satisfiable due to the existence of \( \mathcal{F} \).

When DSSAT is extended with universal quantifiers, its complexity remains in NEXPTIME. Therefore the following corollary is immediate.

**Corollary 1.** The DSSAT extended with universal quantifiers is NEXPTIME-complete.

## 5 Reducing Dec-POMDP to DSSAT

After defining DSSAT and proving that it is NEXPTIME-complete, in this section we demonstrate its descriptive power to model NEXPTIME-complete problems by constructing a polynomial-time reduction from Dec-POMDP to DSSAT. Our reduction is an extension of that from POMDP to SSAT proposed in [Salmon and Poupart 2019].

In essence, given a Dec-POMDP \( \mathcal{M} \), we will construct in polynomial time a DSSAT formula \( \Phi \) such that there is a joint policy \( \pi \) for \( \mathcal{M} \) with value \( V(\pi) \) if and only if there is a set of Skolem functions \( \mathcal{F} \) for \( \Phi \) with satisfying probability \( \Pr[\Phi_S]\) \( \geq 1 \), such that \( V(\pi) = \Pr[\Phi_S]\).

First we introduce the variables used in construction of the DSSAT formula and their domains. To improve readability, as in [Salmon and Poupart 2019] we allow a variable \( x \) to take values from a finite set \( \mathcal{X} = \{x_1, \ldots, x_K\} \). Under this setting, a randomized quantifier \( \exists \) over variable \( x \) specifies a distribution \( \Pr[x = x_i]\) for each \( x_i \in \mathcal{X} \). We also define a scaled reward function:

\[
r(s, \bar{a}) = \frac{\rho(s, \bar{a}) - \min_{s', \bar{a}'} \rho(s', \bar{a}')} {\sum_{s', \bar{a}'} \rho(s', \bar{a}') - \min_{s', \bar{a}'} \rho(s', \bar{a}')} \]

such that \( r(s, \bar{a}) \) forms a distribution over all pairs of \( s \) and \( \bar{a} \), i.e., \( \forall s, \bar{a}, r(s, \bar{a}) \geq 0 \) and \( \sum_s r(s, \bar{a}) = 1 \). We will use the following variables:

- \( x^s_i \in S \): the state at stage \( t \),
- \( x^a_i \in A_i \): the action taken by agent \( i \) at stage \( t \),
- \( x^o_i \in O_i \): the observation received by agent \( i \) at stage \( t \),
- \( x^r_t \in S \times (A_1 \times \ldots \times A_n) \): the reward earned at stage \( t \),
- \( x^T_t \in S \): transition distribution at stage \( t \),
- \( x^O_t \in O_1 \times \ldots \times O_n \): observation distribution at stage \( t \),
- \( x^p_t \in \mathcal{B} \): used to sum up rewards across stages.

We represent elements in the sets \( S, A_i, \) and \( O_i \) by integers, i.e., \( S = \{0, 1, \ldots, |S| - 1\} \), etc., and use indices \( s, a_i, \) and \( o_i \) to iterate through them, respectively. On the other hand, a special treatment is required for variables \( x^s_i \) and \( x^o_i \), as they range over Cartesian products of several sets. We will give a unique number to an element in a product set as follows. Consider \( \bar{Q} = Q_1 \times \ldots \times Q_n \), where each \( Q_i \) is a finite set. An element \( \bar{q} = (q_1, \ldots, q_n) \in \bar{Q} \) is numbered by \( N(q_1, \ldots, q_n) = \sum_{i=1} |q_i| (|Q_{i-1}| - 1) \).

The DSSAT formula below encodes the above equation:

\[
\{x^s_0 \geq 0 \land \sum x^s_0 (D_{x^s_0}) \ldots \sum x^s_0 (D_{x^{n-1}_0}) \phi, \}
\]

where in the prefix the distribution of \( x^0 \) follows \( \Pr[x^0 = s] = \Delta_0(s) \), the distribution of \( x^0 \) follows \( \Pr[x^0 = N_r(s, \bar{a})] = r(s, \bar{a}) \), each \( D_{x^s_0} = \emptyset \), and the matrix

\[
\phi = \bigwedge_{s \in S} \bigwedge_{\bar{a} \in \mathcal{A}} [x^0_s = s \land x^0_\bar{a} = \bar{a} \rightarrow x^0_s = N_r(s, \bar{a})].
\]

As the existentially quantified variables have no dependency on randomized quantified variable, the DSSAT formula is effectively an exist-random quantified SSAT formula.

For a general Dec-POMDP with \( h > 1 \), we follow the two steps proposed in [Salmon and Poupart 2019], namely policy selection and policy evaluation, and adapt the policy selection step for the multiagent setting in Dec-POMDP.

Agent \( i \)'s policy is selected by the prefix of the SSAT formula proposed in [Salmon and Poupart 2019]:

\[
\exists x^s_0 \land \forall x^o_0 \land \exists x^i_1 \land \exists x^o_1 \land \ldots \land \exists x^o_{h-1} \land x^{s-h} \land x^{h-1} \land x^p_{h-1}.
\]

In the above quantification, variable \( x^p_t \) is introduced to sum up rewards earned at different stages. It takes values from \( \mathcal{B} \), and follows a uniform distribution, i.e.,

\[
\Pr[x^p_t = \top] = \Pr[x^p_t = \bot] = 0.5.
\]

As proposed in [Salmon and Poupart 2019], when \( x^p_t = \bot \), the process is stopped and the reward at stage \( t \) is earned; when \( x^p_t = \top \), the process is continued to stage \( t+1 \).

Note that variables \( \{x^p_t\} \) are shared across all agents. With the help of variable \( x^p_t \), rewards earned at different stages are summed up with an equal weight \( 2^{-h} \). Variable \( x^s_i \) also follows a uniform distribution \( \Pr[x^s_i = \alpha_i] = \frac{1}{|O_i|} \), which scales the satisfying probability by \( |O_i|^{-1} \) at each stage. Therefore, we need to re-scale the satisfying probability accordingly in order to obtain the correct satisfying probability corresponding to the value of a joint policy. The scaling factor will be derived in the proof of Theorem 2.

As there are \( n \) agents, and each agent's actions can only depend on its own observations, for the selection of a joint policy it is not obvious how to combine the quantification, i.e., the selection of a policy, of each agent into a linearly
ordered prefix required by SSAT, without suffering an exponential translation cost. On the other hand, DSSAT allows to specify the dependency of an existentially quantified variable freely and is suitable to encode the selection of a joint policy. In the prefix of the DSSAT formula, variable $x_i \triangleq$ depends on $D_{x_i} = \{x_{i,0}, \ldots, x_{i,t}, x_{o,0}, \ldots, x_{o,t-1}\}$.

Next, the policy evaluation step is exactly the same as in [Salmon and Poupart 2019]. The following quantification computes the value of a joint policy:

$$
\begin{align*}
\forall x_i \forall x_o \exists x_t, t = 0, \ldots, h - 1.t = 0, \ldots, h - 2
\end{align*}
$$

Variables $x_i$ follow a uniform distribution $Pr[x_i = s] = |S|^{-1}$ except for variable $x_i^0$, which follows the initial distribution specified by $Pr[x_i^0 = s] = \Delta_0(s)$; variables $x_o$ follow the distribution of the reward function $Pr[x_o = N_r(s, \bar{a})] = r(s, \bar{a})$; variables $x_t$ follow the state transition distribution $Pr[x_{t, s, \bar{a}} = s'] = T(s, \bar{a}, s')$; variables $x_{o,t}$ follow the observation distribution $Pr[x_{o, t, \bar{a}} = \Omega(t, \bar{a})] = \Omega(s', \bar{a}, \bar{d})$. Note that these variables encode the random mechanism of a Dec-POMDP and are hidden from agents. That is, variables $x_i$ do not depend on the above variables.

The CNF formulas to encode $M$ are as follows. Formula 2 encodes that when $x_i^0 \equiv \bot$, the process is stopped, the observation $x_o^t$ and next state $x_o^{t+1}$ are set to a preserved value $0$, and $x_o^{t+1} \equiv \bot$. Formula 3 ensures the process is stopped at the last stage. Formula 4 ensures the reward at the first stage is earned when the process is stopped, i.e., $x_i^0 \equiv \bot$. Formula 5 requires the reward at stage $t > 0$ is earned when $x_o^{t-1} \equiv \top$ and $x_i^t \equiv \bot$. Formula 6 encodes the transition distribution from state $s$ to state $s'$ given actions $\bar{a}$ are taken. Formula 7 encodes the observation distribution to receive observation $\bar{d}$ under the situation that state $s'$ is reached after actions $\bar{a}$ are taken.

The correctness of the proposed reduction from Dec-POMDP to DSSAT is stated in the following theorem.

**Theorem 2.** In the above reduction from a Dec-POMDP $M$ to a DSSAT formula $\Phi$, there exists a joint policy $\pi$ for $M$ with value $V(\pi)$ if and only if there is a set of Skolem functions $F$ for $\Phi$ with satisfying probability $Pr[\Phi|F]$, such that $V(\pi) = Pr[\Phi|F]$.

**Proof.** We prove this statement by induction over the planning horizon $h$.

First, consider an arbitrary Dec-POMDP $M$ with $h = 1$. For the “only if” direction, consider a joint policy $\pi$ for $M$ which specifies $\bar{a} = (a_1, \ldots, a_n)$ where agent $i$ will take action $a_i$. For this joint policy, the value is computed as $V(\pi) = \sum_{s \in S} \Delta_0(s)r(s, \bar{a})$. Based on $\pi$, we derive a set of Skolem functions $F$ where $x_i^0 = a_i$ for each $i \in I$. To compute $Pr[\Phi|F]$, we cofactor the matrix with $F$ and arrive at the following CNF formula:

$$
\forall s \in S \{x_i^0 \neq s \lor x_o^0 \equiv N_r(s, \bar{a})\},
$$

and the satisfying probability of $\Phi$ with respect to $F$ is

$$
Pr[\Phi|F] = \sum_{s \in S} Pr[x_i^0 = s] Pr[x_o^0 = N_r(s, \bar{a})]
= \sum_{s \in S} \Delta_0(s)r(s, \bar{a}) = V(\pi).
$$

Note in the above argument, only equalities are involved, and hence can be reversed to prove the “if” direction.

For the induction step, assume that the statement holds for any Dec-POMDP with a planning horizon of $h$. For a Dec-POMDP with a planning horizon of $h + 1$, consider a joint policy $\pi_{h+1}$ with value $V(\pi_{h+1})$. Note that as a joint policy is a mapping from observation histories to actions, we can build a corresponding set of Skolem functions $F_{h+1}$ to simulate joint policy $\pi_{h+1}$ for the DSSAT formula. The derivation of satisfying probability with respect to $F_{h+1}$ is shown on the next page. Note that to obtain the correct value of the joint policy, we need to re-scale the satisfying probability by a scaling factor $k^{-1} = 2^h(|\mathcal{O}| |S| h^{-1})$. As $Pr[\Phi|F_{h+1}] = V(\pi_{h+1})$, the theorem is proved according to the principle of mathematical induction.

**5.1 Discussion**

Below we count the numbers of variables and clauses in the resulting DSSAT formula with respect to the input size of the given Dec-POMDP. For one stage, there are $3 + 2(|I| + |S||\bar{A}|)$ variables, and therefore in total the number of variables is $O(h(|I| + |S||\bar{A}|))$ asymptotically. On the other hand, the number of clauses per stage is $2 + |I| + |S||\bar{A}| + |S|^2|\bar{A}| + |S||\bar{A}||\bar{O}|$, and hence the total number of clauses is $O(h(|I| + |S||\bar{A}||\mathcal{S}| + |\bar{O}|))$. Overall, we show that the proposed reduction is polynomial-time with respect to the input size of the Dec-POMDP.

**5.2 Example**

Here we use a Dec-POMDP with two agents and planning horizon $h = 2$ as an example to illustrate how the constructed DSSAT formula encodes the derivation of the value of a joint policy. Given a joint policy $(\pi_1, \pi_2)$ for agent 1 and 2, let the actions taken at $t = 0$ be $\bar{a}^0 = (a_1^0, a_2^0)$ and the actions taken at $t = 1$ under certain observations $\bar{d}^0 = (a_1^1, a_2^1)$ be $\bar{a}^1 = (a_1^0, a_2^1)$. The value of this joint policy is computed by the Bellman equation in Eq. 1 as

$$
V(\pi) = \sum_{s \in S} \Delta_0(s^0)[r(s^0, \bar{a}^0) + \sum_{s' \in S} T(s^0, \bar{a}^0, s^1) \Omega(s^1, \bar{a}^0, \bar{d}^0)r(s^1, \bar{a}^1)].
$$

The decision tree to solve the converted DSSAT formula is shown in Figure 1. At $t = 0$, after taking actions $\bar{a}^0$, variable $x_o^0$ splits into two cases: when $x_o^0 \equiv \bot$, i.e., the left branch, the expected reward $\Delta_0(s^0)r(s^0, \bar{a}^0)$ to start from state $s^0$ and take actions $\bar{a}^0$ will be earned for $t = 0$; on the other hand, when $x_o^0 \equiv \top$, observation $\bar{d}^0$ is received, based on which the agents will select their actions $\bar{a}^1$ at
\[
\begin{align*}
\bigwedge_{0 \leq t \leq h-2} [x_p^t &\equiv \top \rightarrow \bigwedge_{i \in I} x_i^{t+1} \equiv 0 \land x_p^{t+1} \equiv \bot] \quad (2) \\
\bigwedge_{s \in S} \bigwedge_{\bar{a} \in \bar{A}} [x_p^0 &\equiv \top \land x_s^0 \equiv s \land \bigwedge_{i \in I} x_i^0,0 \equiv a_i \rightarrow x_i^0 \equiv N_r(s, \bar{a})] \quad (3) \\
\bigwedge_{1 \leq t \leq h-1} \bigwedge_{s \in S} \bigwedge_{\bar{a} \in \bar{A}} [x_p^{t-1} \equiv \top \land x_s^t \equiv s \land \bigwedge_{i \in I} x_i^{t},t \equiv a_i \rightarrow x_i^t \equiv N_r(s, \bar{a})] \quad (4) \\
\bigwedge_{0 \leq t \leq h-2} \bigwedge_{s \in S} \bigwedge_{\bar{a} \in \bar{A}} [x_p^t \equiv \top \land x_s^{t+1} \equiv s' \land \bigwedge_{i \in I} x_i^{t},t \equiv a_i \land \bigwedge_{i \in I} x_i^{t+1} \equiv o_i \rightarrow x_i^{t+1} \equiv N_\Omega(\bar{a})] \quad (5) \\
\bigwedge_{0 \leq t \leq h-2} \bigwedge_{s \in S} \bigwedge_{\bar{a} \in \bar{A}} \bigwedge_{s' \in S} [x_p^t \equiv \top \land x_s^{t+1} \equiv s' \land \bigwedge_{i \in I} x_i^{t},t \equiv a_i \land \bigwedge_{i \in I} x_i^{t+1} \equiv o_i \rightarrow x_i^{t+1} \equiv N_\Omega(\bar{a})] \quad (6) \\
\bigwedge_{0 \leq t \leq h-2} [x_p^t \equiv \top \land x_s^{t+1} \equiv s' \land \bigwedge_{i \in I} x_i^{t},t \equiv a_i \land \bigwedge_{i \in I} x_i^{t+1} \equiv o_i \rightarrow x_i^{t+1} \equiv N_\Omega(\bar{a})] \quad (7)
\end{align*}
\]

\[
\Pr[\Phi | x_{h+1}] = \sum_{v^0, \ldots, v^h} \sum_{a^0, \ldots, a^h} \sum_{s^0, \ldots, s^h} \prod_{t=0}^{h-1} \Pr[x_p^t \equiv v^t, x_s^t \equiv s^t, x_o^t \equiv o^t, x_r^t] \prod_{t=0}^{h-1} \Pr[x_T^t, x_\Omega^t | x_p^t \equiv v^t, x_s^t \equiv s^t, x_o^t \equiv o^t] \\
= 2^{-(h+1)} \sum_{t=0}^{h+1} \sum_{s^0, \ldots, s^h} \prod_{i=0}^{h-1} \Pr[x_i^t \equiv s^t, x_o^t \equiv o^t, x_r^t] \prod_{t=0}^{h-2} \Pr[x_T^t, x_\Omega^t | v^t, s^t, o^t] \\
= 2^{-(h+1)} |\bar{O}|^{-h} \sum_{t=0}^{h+1} \sum_{s^0, \ldots, s^h} \prod_{i=0}^{h-1} \Pr[x_i^t \equiv s^t, x_o^t \equiv o^t, x_r^t] \prod_{t=0}^{h-2} \Pr[x_T^t, x_\Omega^t | v^t, s^t, o^t] \\
= 2^{-(h+1)} (|\bar{O}| \cdot |S|)^{-h} \sum_{t=0}^{h+1} \sum_{s^0, \ldots, s^h} \prod_{i=0}^{h-1} \Pr[x_i^t \equiv s^0] \prod_{t=0}^{h-2} \Pr[x_T^t, x_\Omega^t | v^t, s^t, o^t] \\
= 2^{-(h+1)} (|\bar{O}| \cdot |S|)^{-h} \sum_{t=0}^{h+1} \sum_{s^0, \ldots, s^h} \prod_{i=0}^{h-1} \Pr[x_i^t \equiv s^0] \prod_{t=0}^{h-2} \Pr[x_T^t, x_\Omega^t | v^t, s^t, o^t] \\
= \kappa \sum_{t=0}^{h+1} \sum_{s^0, \ldots, s^h} \sum_{t=0}^{h-1} \Delta_0(s^0) r(s^{t-1}, a^{t-1}) \prod_{t=0}^{h-2} T(s^t, a^t, s^{t+1}) \Omega(s^{t+1}, a^t, o^t) \\
= \kappa \sum_{s^0 \in S} \Delta_0(s^0) r(s^{t-1}, a^0) + \sum_{s^1 \in S} \sum_{s^0 \in S} T(s^0, a^0, s^1) \Omega(s^1, a^0, o^0) \Pr[\Phi | x_s] \\
= \kappa \sum_{s^0 \in S} \Delta_0(s^0) r(s^{t-1}, a^0) + \sum_{s^1 \in S} \sum_{s^0 \in S} T(s^0, a^0, s^1) \Omega(s^1, a^0, o^0) V(\tilde{\pi}_h)(by \ induction \ hypothesis) \\
= \kappa V(\tilde{\pi}_{h+1})(by \ Eq. 1)
\]
Again, variable $x^1_p$ will split into two cases, but this time $x^1_p$ is forced to be $\perp$ as it is the last stage. The expected reward $\Delta_0(s^0) T(s^0, \vec{a}^0, s^1) \Omega(s^1, \vec{a}^1, s^2) r(s^2, \vec{a}^2)$ to start from state $s^0$, transit to $s^1$ with actions $\vec{a}^0$ and receive observations $\vec{o}^0$, and take actions $\vec{a}^1$ in state $s^1$ will be earned under the branch of $x^0_p \equiv \perp$ for $t = 1$. Note that the randomized quantifiers over variables $x^0_p$, $x^1_p$, and $x^2_p$ will scale the satisfying probability by the factors labelled on the edges, respectively. Therefore, to obtain the correct value of the joint policy we have to re-scale the satisfying probability by $2^{|S||O_1 \times O_2|}$.

### 6 Conclusions

In this paper, we extended DQBF to its stochastic variant DSSAT; and proved that DSSAT is also NEXPTIME-complete. Compared to SSAT, DSSAT has the potential to succinctly model NEXPTIME-complete decision problems with uncertainty. As a concrete example, we showed a polynomial-time reduction from the NEXPTIME-complete Dec-POMDP to DSSAT. As DSSAT serves as a logical formalism for NEXPTIME-complete decision problems, we envisage its potential broad applications. For future work, we plan to develop a DSSAT solver.

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Figure 1: The decision tree of a Dec-POMDP example with two agents and $h = 2$.