Spectral geometry of symplectic spinors

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Abstract
Symplectic spinors form an infinite-rank vector bundle. Dirac operators on this bundle were constructed recently by K. Habermann. Here we study the spectral geometry aspects of these operators. In particular, we define the associated distance function and compute the heat trace asymptotics.

1 Introduction
According to the noncommutative geometry approach, geometry is defined by spectral triples. That is, geometry essentially becomes spectral geometry of natural Dirac type operators. This approach is welcomed by physicists since it bridges up the differences between classical and quantum geometries. This also moves geometry towards traditional areas of Mathematical Physics. Many links with Quantum Field Theory and even particle physics have been established, see [4]. The spectral geometry of Riemannian manifolds has been studied in detail, while the spectral geometry of symplectic manifolds remains a largely uncharted area.

Quantization starts with a Poisson structure, that becomes a symplectic structure in the non-degenerate case. Therefore, from the point of view of Quantum Theory, symplectic manifolds are more important than the Riemannian ones. The purpose of this work is to extend the spectral geometry approach to the symplectic spinors. We address two important aspects, namely the spectral distance function and the heat trace asymptotics.

The symplectic spinors were introduced by B. Kostant [15]. The Dirac operator on symplectic spinors and the corresponding Laplacian were defined by K. Habermann [7,8], who has also studied their basic properties. A nice overview is the monograph [9], whose conventions and notations we mostly follow in this work. The works [7,8] defined two Dirac operators, $D$ and $\tilde{D}$. Since the symplectic spinor bundle has an infinite rank, neither of these two operators is a Dirac operator in the noncommutative geometry sense. Moreover, the relevant Laplacian $P$ appears to be the commutator of $D$ and $\tilde{D}$ (rather than the square of $D$ or $\tilde{D}$). Therefore, the basic notions of spectral geometry cannot be immediately applied.

We shall show that despite the difficulties described above, there exists a modification of the standard distance formula of noncommutative geometry that reproduces the geodesic distance on base manifold $M$ through symplectic Dirac operators. This is probably the most surprising result of this work.

If the base manifold $M$ is almost hermitian, the operator $P$ leaves some subbundles $Q_l$ invariant. These subbundles are of finite rank. Let $P_l$ be a restriction of Laplacian $P$ to $\Gamma(Q_l)$, then $P_l$ is also a Laplace type operator. This will allow us to develop the theory of heat trace asymptotic for $P_l$, identify corresponding invariants and compute a couple of leading terms in the asymptotic expansion. Even more detailed information on the heat trace can be obtained when $\dim M = 2$ and in the particular case $M = CP^1$. We shall consider these cases as examples in the last section of this paper.

2 Preliminaries
In this Section we collect some basic facts that will be useful later. In what refers to symplectic spinors we mostly follow [9].

2.1 Differential geometry of almost hermitian manifolds
Consider a symplectic manifold $(M,\omega)$, $\dim M = 2n$, with $\omega$ being a symplectic form, equipped with an almost complex structure $J$ and a Riemannian metric $g$ related through $g(X,Y) = \omega(X,JY)$ for
The symplectic form \( \omega \) have a canonical form \([14]\). Such frames form a principal \( U \) Almost hermitian manifolds admit unitary tangent frames in that both the metric and the symplectic form have a canonical form \([14]\). Such frames form a principal \( U(n) \) bundle \( U(M) \) over \( M \). We shall follow the notations of \([9]\) and write such frames as \((e_1, \ldots, e_{2n}) = (\hat{e}_1, \ldots, \hat{e}_n, \hat{f}_1, \ldots, \hat{f}_n)\). They satisfy

\[
g(e_i, e_j) = \delta_{ij}, \quad \omega(e_i, \hat{f}_j) = \delta_{ij}, \quad \omega(e_i, \hat{e}_j) = \omega(\hat{f}_i, \hat{f}_j) = 0. \tag{2}\]

The almost complex structure \( J \) acts on the basis as

\[
J \hat{e}_j = \hat{f}_j, \quad J \hat{f}_j = -\hat{e}_j. \tag{3}\]

It is interesting and useful to follow certain analogies with the Yang-Mills theory. By linearity, there is \( u \in \Gamma(T^*M \times \text{End}(U(M))) \) such that for any vector field \( X \)

\[
\nabla_X e_i = (X^\mu) u_{\mu i j} e_j. \tag{4}\]

Whenever it cannot lead to a confusion we shall use the Einstein conventions of summation over repeated indices. The Greek indices \( \mu, \nu, \ldots \) are vector indices corresponding to a local coordinate chart. They are introduced to make connections with the Yang-Mills more explicit. It is easy to check that \( \nabla \) is a symplectic hermitian connection iff \( u \) is antisymmetric and commutes with \( J \), i.e. \( u \) belongs to the \( 2n \)-dimensional real representation of \( u(n) \). Eq. \((4)\) allows to express the Christoffel symbol through the vectors of unitary frame \( e_i \) and the \( U(n) \) connection one-form \( u \). One has the following expression for the torsion:

\[
T(e_i, e_j) = (e_i u) e_j - (e_j u) e_i - [e_i, e_j]. \tag{5}\]

Also the curvature of \( \nabla \) can be expressed through \( u \):

\[
(\mathcal{R}(e_i, e_j) e_k)^\rho = e_i^\rho e_j^\nu F_{\nu \mu k m} e_m^\mu, \tag{6}\]

where

\[
F_{\nu \mu k m} = -\partial_{\nu} u_{\mu k m} + \partial_{\mu} u_{\nu k m} + [u_{\nu}, u_{\mu}]_{k m} \tag{7}\]

is the Yang-Mills type curvature associated to \( u_{\mu} \). We remind that the generators of \( u(n) \) algebra are labeled by the pairs of indices \((k, m)\).

The heat trace asymptotics of Laplace type operators are usually expressed in terms of the Levi-Civita connection \( \nabla^{LC} \) and corresponding curvatures. This connection is related to \( \nabla \) by the text-book formula:

\[
g(\nabla x Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} \left[ g(T(X, Y), Z) - g(T(X, Z), Y) - g(T(Y, Z), X) \right] \tag{8}\]

The following vector, \( \mathcal{T} \), and covector, \( \tau \), fields are associated with the torsion:

\[
\mathcal{T} = \sum_{j=1}^{n} T(\hat{e}_j, \hat{f}_j), \quad \tau(X) = \sum_{k=1}^{2n} g(T(e_k, X), e_k). \tag{9}\]

There is a useful relation which involves the Riemann scalar curvature \( \bar{\rho} \):

\[
\sum_{j=1}^{2n} g(R(e_j, e_k) e_j, e_k) = -\bar{\rho} + 2 \sum_{j=1}^{2n} \nabla^{LC}_{e_j} \tau(e_j) + \sum_{j=1}^{2n} \tau(e_j)^2
\]

\[
-\frac{1}{4} \sum_{j, l}^{2n} \left[ 2g(T(e_j, e_l), e_j) + g(T(e_j, e_l), T(e_j, e_l)) \right]. \tag{10}\]
2.2 Symplectic spinors and relevant operators

Let us remind that the metaplectic group $Mp(2n, \mathbb{R})$ is a two-fold covering of the symplectic group $Sp(2n, \mathbb{R})$. A metaplectic structure is a principal $Mp(2n, \mathbb{R})$ bundle (together with a morphism that ensures consistency with the symplectic structure). The metaplectic group is represented in the space of square integrable functions $L^2(\mathbb{R}^n)$. By using this representation one can define a bundle associated with the metaplectic structure, which is the symplectic spinor bundle $Q$. For the sections of this fiber bundle, $\varphi, \psi \in \Gamma(Q)$, one has a fiber scalar product, $\langle \varphi, \psi \rangle_x$, $x \in M$, and a fiber norm $\|\varphi\|^2_x = \langle \varphi, \varphi \rangle_x$.

By integrating $\langle \varphi, \psi \rangle_x$ over $M$ one obtains a scalar product on $\Gamma(Q)$.

For $X \in \Gamma(TM)$ and $\varphi \in \Gamma(Q)$ one defines the symplectic Clifford multiplication $X \cdot \varphi$ satisfying

$$(X \cdot Y - Y \cdot X) \cdot \varphi = -i\omega(X, Y)\varphi$$

From now on, $1 \equiv \sqrt{-1}$.

The symplectic spinor bundle $Q$ splits into an orthogonal sum of finite rank subbundles $Q^l_j$, $l = 0, 1, 2, \ldots$,

$$\text{rank } Q^l_j = \frac{(l + n - 1)!}{l!(n - 1)!}.$$ (12)

There is an important operator, $H^l_j$, acting on $\varphi \in \Gamma(Q)$

$$H^l_j \varphi = \frac{1}{2} \sum_{j=1}^{2n} e_j \cdot e_j \cdot \varphi.$$ (13)

The operator $H^l_j$ equals to a constant $q_l$ on each of $Q^l_j$, and

$$q_l = -(l - \frac{n}{2}).$$ (14)

In the quantum mechanical language, the Clifford multiplication by $\hat{e}_j$ may be thought of as a canonical coordinate operator, while $\hat{f}_j$ may be thought of as a conjugate momentum operator. Then $H^l_j$ is minus the Hamiltonian of $n$-dimensional harmonic oscillator, while the ladder operators are

$$L^{(\pm)}_j \varphi = (\hat{f}_j \mp \hat{e}_j) \cdot \varphi.$$ (15)

For any $\varphi_0 \in \Gamma(Q^l_j)$

$$L^{(+)} \varphi_0 = 0.$$ (16)

Any symplectic connection $\nabla$ on $M$ induces a covariant derivative on $Q$ that will be denoted by the same letter $\nabla$. We shall need the relation

$$\nabla_X (Y \cdot \varphi) = \nabla_Y X \cdot \varphi + Y \cdot \nabla_X \varphi$$

between the derivative and the Clifford multiplication and the corresponding curvature

$$\nabla^Q(X, Y) \varphi = \nabla_X \nabla_Y \varphi - \nabla_Y \nabla_X \varphi - \nabla_{[X,Y]} \varphi$$

$$= -\frac{i}{2} \sum_{j=1}^{2n} \nabla^Q(X, Y) e_j \cdot J e_j \cdot \varphi.$$ (18)

Given a unitary frame ($e_1, \ldots, e_{2n}$) one defines a pair of Dirac operators

$$D \varphi = -\sum_{j=1}^{2n} J e_j \cdot \nabla_{e_j} \varphi, \quad \bar{D} \varphi = \sum_{j=1}^{2n} e_j \cdot \nabla_{e_j} \varphi.$$ (19)

Note, that the formula for $D$ may be rewritten in the form

$$D \varphi = \sum_{j=1}^{n} (\hat{e}_j \cdot \nabla_{\hat{e}_j} \varphi - \hat{f}_j \cdot \nabla_{\hat{f}_j} \varphi)$$

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Note, that the formula for $D$ may be rewritten in the form

$$D \varphi = \sum_{j=1}^{n} (\hat{e}_j \cdot \nabla_{\hat{e}_j} \varphi - \hat{f}_j \cdot \nabla_{\hat{f}_j} \varphi)$$
which does not use the metric or the almost complex structure.

An associated second order operator is defined through the commutator

\[ \mathcal{P} = i[\tilde{D}, D]. \]  

The principal symbol of \( \mathcal{P} \) is given by the inverse metric. The operator \( \mathcal{P} \) leaves the subbundles \( \mathcal{Q}_l \) invariant, \( \mathcal{P} : \Gamma(\mathcal{Q}_l) \to \Gamma(\mathcal{Q}_l) \), though neither \( \tilde{D} \) nor \( D \) have this property. A restriction of \( \mathcal{P} \) to \( \Gamma(\mathcal{Q}_l) \) will be denoted by \( \mathcal{P}_l \).

### 2.3 Heat trace asymptotics

We shall need some facts from the theory of asymptotic expansion of the heat trace associated with Laplacians [6] (see also [13, 18]).

Let \( V \) be a finite rank vector bundle over a compact Riemannian manifold \( M \) without boundary. Let \( P \) be a Laplace type operator on \( \Gamma(V) \). Then

1. exist a unique endomorphism \( \tilde{E} \) and a unique connection \( \tilde{\nabla} \) of \( V \) such that

\[ P = -(\tilde{\nabla}^2 + \tilde{E}), \]  

where the square in \( \tilde{\nabla}^2 \) is calculated with the Riemannian metric on \( M \) and includes the metric Christoffel symbol.

2. The heat trace

\[ K(P,t) := \text{Tr}(\exp(-tP)), t \in \mathbb{R}_+ \]  

exists and admits a full asymptotic expansion

\[ K(P,t) \sim \sum_{k=0}^{\infty} t^{k-n} a_{2k}(P) \]  

as \( t \to +0 \). Here \( 2n = \text{dim} M \).

3. The heat trace coefficients \( a_{2k} \) are locally computable. Namely, each \( a_{2k} \) is given by the integral over \( M \) of the bundle trace of a local invariant polynomial constructed from the endomorphism \( \tilde{E} \), from the Riemann tensor \( \tilde{R}_{ijkl} \), the curvature \( \tilde{\Omega}_{ij} \) of \( \tilde{\nabla} \), and their derivatives. In particular,

\[ a_0(P) = (4\pi)^{-n} \int_M \text{tr}_V(I), \]  

\[ a_2(P) = (4\pi)^{-n} \frac{1}{2} \int_M \text{tr}_V(6\tilde{E} + \tilde{\rho}), \]  

\[ a_4(P) = (4\pi)^{-n} \frac{1}{24} \int_M \text{tr}_V(60\tilde{\rho}\tilde{E} + 180\tilde{E}^2 + 5\tilde{\rho}^2 - 2\tilde{R}_{ij} \tilde{R}_{ij} + 2\tilde{R}_{ijkl} \tilde{R}_{ijkl} + 30\tilde{\Omega}_{ij} \tilde{\Omega}_{ij}). \]

The Ricci tensor \( \tilde{\text{Ric}}_{ij} = \tilde{R}_{ij} \) and the scalar curvature \( \tilde{\rho} = \tilde{\text{Ric}}_{jj} \) are defined in such a way that \( \tilde{\rho} = 2 \) on the unit \( S^2 \). Summation over the repeated indices is understood.

Note, that the expansion (24) contains even-numbered coefficients and integer powers of \( t \) only. Odd-numbered coefficients appear e.g. on manifolds with boundaries.

Our purpose is to calculate the coefficients \( a_{2k} \) and relate them to geometric invariants of the symplectic spinor bundle.

### 3 The distance

Let us start with basic definitions related to the distance function in noncommutative geometry, see [17] for a brief introduction. Consider a spectral triple \((\mathcal{A}, H, D)\) consisting of an algebra \( \mathcal{A} \), acting on a Hilbert space \( H \) by bounded operators, and of a Dirac operator \( D \). The commutator \([D, a]\) has to be
bounded for all \( a \in \mathcal{A} \) (or at least the set of \( a \) for which \([D,a]\) is bounded has to be dense in \( \mathcal{A} \)). One can define a distance between two states \( x \) and \( y \) on the algebra \( \mathcal{A} \) by the formula \( \|D,a\| \leq 1 \). \hfill (28)

For a commutative spectral triple, \( \mathcal{A} \) is the algebra \( C(M) \) of continuous functions on a compact Riemannian spin\(^*\) manifold \( M \), \( H \) is the space of square-integrable spinors, and \( D \) is the canonical Dirac operator. The pure states \( x \) correspond to points on \( M \) with \( x : a \mapsto a(x) \) being the evaluation map. The distance \( \|\phi\|_x \) forms a distance on \( \gamma \) formed by sections of the symplectic spinor bundle. However, the commutators \([D,pr],a\) practically never bounded, so that Eq. (28) with \( a \) being the \( g^{-1} \) of \( \gamma \) and \( \beta \). Fiberwise, \( \gamma(a) \) is a hermitian matrix. The restriction \( \|\gamma(a)\|_M \leq 1 \) is equivalent to \hfill (29)

\[
\|g^{-1}(da,da)\| \leq 1.
\]

Complexity or reality of the function \( a \) plays no role here. We shall consider real functions in what follows.

In the context of symplectic spinors one may take the same \( \mathcal{A} = C(M) \), the Hilbert space may be formed by sections of the symplectic spinor bundle. However, the commutators \([\mathcal{D},a]\) and \([\mathcal{D},a]\) are practically never bounded, so that Eq. (28) with \( \mathcal{D} \) or \( \mathcal{D} \) instead of \( D \) does not define any interesting distance on \( M \). Therefore, we return again to the usual spin case and replace the condition \( \|\gamma(a)\|_M \leq 1 \) by an equivalent one, which however can be naturally generalized for symplectic spinors. Let us take a section \( \psi_0 \) of spin bundle that has a unit fiber norm at each point of \( M \): \( 1 = \|\psi_0\|_x = \psi_0(x)\psi_0(x) \). Let us take a real \( a = a^* \) in \( \mathcal{A} \) and compute

\[
\|\phi\|_x^2 = \psi_0^\dagger(x)\gamma(a)\gamma(da)\psi(x) = g^{-1}(da,da)\psi_0^\dagger(x)\psi_0(x) = g^{-1}(da,da).
\]

This shows that the condition \( \|\gamma(a)\|_x \leq 1 \) for all points \( x \) on \( M \) may be used instead of the original restriction on the norm \( \|\gamma(a)\|_M \leq 1 \) to define the distance on \( M \).

A similar construction in the symplectic case goes as follows. Take a section \( \varphi_0 \) of \( \mathcal{Q}_0^\dagger \) with a constant fiber norm \( \|\varphi_0\|_x^2 = 2 \) everywhere on \( M \). Then,

\[
[\mathcal{D},a]\varphi_0 = \sum_{j=1}^n \left( (\hat{e}_j a)\hat{e}_j \cdot \varphi_0 + (\hat{f}_j a)\hat{f}_j \cdot \varphi_0 \right) = \sum_{j=1}^n \left( (\hat{e}_j a)(-\frac{1}{2}L_j^{(-)} + (\hat{f}_j a)\frac{1}{2}L_j^{(-)} \varphi_0
\right.
\]

\[
= \sum_{j=1}^n \left(-i(\hat{e}_j a) + (\hat{f}_j a)\right)\frac{1}{2}L_j^{(-)} \varphi_0.
\]

Furthermore,

\[
\frac{1}{2}L_j^{(-)} \varphi_0, \frac{1}{2}L_k^{(-)} \varphi_0_x = \frac{1}{2}(\varphi_0, L_j^{(+)} L_k^{(-)} \varphi_0)_x = \frac{1}{2}(\varphi_0, (L_j^{(-)} L_k^{(+)} + 2)\varphi_0)_x \delta_{jk} = \frac{1}{2}(\varphi_0, \varphi_0)_x \delta_{jk} = \delta_{jk},
\]

where we used Eq. (16). Finally,

\[
\|\mathcal{D},a\varphi_0\|_x^2 = \sum_{j=1}^n (e_j a)^2 = g^{-1}(da,da).
\]

By collecting everything together we arrive at the following

**Proposition 3.1.** Let \( \mathcal{A}, \varphi_0 \) and \( \mathcal{D} \) be as defined above. Then

\[
d(x,y) = \sup_{a \in \mathcal{A}} \{ |a(x) - a(y)| : \|\mathcal{D},a\varphi_0\|_p \leq 1, \forall p \in M \}\.
\]

is the geodesic distance between two points \( x \) and \( y \) on \( M \).
appearing in the canonical form \(\nabla\) of the Laplacian
\[
\nabla^* \nabla \varphi = \nabla^* \nabla \varphi + 1 \sum_{j,k} J e_j \cdot e_k \cdot (R^Q(e_j, e_k) \varphi - \nabla_{T(e_j, e_k)} \varphi) - \nabla_{J^Y} \varphi.
\]

Here \(\nabla^* \Gamma(T^* M \otimes Q) \to \Gamma(Q)\) is the formal adjoint operator of the spinor covariant derivative \(\nabla\). It is easy to see that \(\nabla^* \nabla = -\nabla^2\) in the notations of Eq. (22). After some algebra we obtain the quantities appearing in the canonical form (22) of the Laplacian
\[
\nabla_X \varphi = \nabla_X \varphi + g(X, v) \varphi
\]
\[
v = \frac{1}{2} \sum_{j,k=1}^{2n} T(e_j, e_k) J e_j \cdot e_k + \frac{1}{2} J^Y
\]
\[
E = -\text{div}^LC v - g(v, v) - 1 \sum_{j,k} J e_j \cdot e_k \cdot R^Q(e_j, e_k)
\]

In the last formula the divergence corresponds to the Levi-Civita connection. In local terms, \(\text{div}^LC v = g^{\mu\nu} \nabla^LC_{\mu} v_{\nu}\).

To calculate the traces we have to define the \(u(n)\) representations corresponding to the objects appearing the formulas above. Just looking at the expression (13) for the curvature and at the relations (7) to the Yang-Mills field strength one may conjecture that the combinations \(e_j \cdot J e_k\) are generators of \(u(n)\). Let us show that this is indeed the case.

Let \(A \in u(n)\) be given by a matrix \(A_{jk}\) in the \(2n\)-dimensional real defining representation. This means that \(A_{jk}\) is a real antisymmetric \(2n \times 2n\) matrix which satisfies \(A_{jk} = A_{j+n,k+n}\) and \(A_{j,k+n} = -A_{j+n,k}\) for \(j, k \leq n\). Then
\[
r_Q(A) = \frac{1}{2} \sum_{j,k=1}^{2n} A_{jk} e_k \cdot J e_j.
\]
is a representation of \(A\) on the symplectic spinors. Indeed,
\[
[r_Q(A), r_Q(B)] \varphi = \left(\frac{1}{2}\right)^2 [A_{jk} e_k \cdot J e_j, B_{lm} e_m \cdot J e_l] \cdot \varphi
\]
\[
= \frac{1}{2} [A, B]_{jk} e_k \cdot J e_j \cdot \varphi = r_Q([A, B]) \varphi.
\]

This representation is, of course, reducible. The action is fiberwise, so that it is enough to understand \(r_Q\) on each fiber.

**Lemma 4.1.** On the fibers of \(Q\), the representation \(r_Q\) is equivalent to an \(su(n)\) representation with the Dynkin indices \((l, 0, \ldots, 0)\), while the \(u(1)\) charge is \(q_l\).

**Proof.** In the \(2n\)-dimensional real defining representation non-zero matrix elements of the Cartan generators of \(u(n)\) are \((K_j)_{j,j+n} = -(K_j)_{j+n,j} = 1, j = 1, \ldots, n\). Therefore,
\[
r_Q(K_j) \varphi = \frac{1}{2} (\hat{e}_j \cdot \hat{e}_j + \hat{f}_j \cdot \hat{f}_j) \cdot \varphi
\]
Locally, a fiber of $Q^I_l$ can be viewed as a linear space spanned by products $h_{\alpha_1}(x_1) \cdots h_{\alpha_n}(x_n)$ of the Hermite functions $h_{\alpha}(x)$ with $\alpha_1 + \cdots + \alpha_n = l$ and with Clifford multiplications by $i\partial_j$ and $\partial_j$, respectively [9]. Hence, $r_Q(K_j)h_{\alpha_1}(x_1) = -i(\alpha_1 + \frac{1}{2})h_{\alpha_1}(x_1)$. Let us take the Cartan generators corresponding to ordered positive simple roots of $su(n)$ as $K_1 - K_2$, $K_2 - K_3$, ..., $K_{n-1} - K_n$. By calculating the eigenvalues of this generators on $h_l(x_1)h_{0}(x_2)\cdots h_0(x_n)$, we conclude that this monomial is the highest weight vector of the representation $(l,0,\ldots,0)$. To demonstrate that a fiber of $Q^I_l$ is just this representation and nothing else, it is enough to compare the dimensions (cf [9]),

$$\dim(l,0,\ldots,0) = \frac{(l+n-1)!}{l!(n-1)!} = \text{rank} Q^I_l.$$

(42)

The $u(1)$ charge is, up to the imaginary unit, the eigenvalue of the $u(1)$ generator

$$\sum_{j=1}^{n} r_Q(K_j) = i\mathcal{H}^j,$$

(43)

which is $iq_l$. □

Let us define a projector $\Pi$ on the $u(1)$ generator

$$\Pi r_Q(A) = \frac{i\mathcal{H}^j}{n} \sum_{j=1}^{n} A_{j,j+n}$$

(44)

and a pull-back of $\Pi$ to the $2n$-dimensional real representation, $\Pi r_Q(A) = r_Q(\Pi A)$.

**Corollary 4.2.**

$$\text{tr}_1(r_Q(A)) = \frac{iq_l \text{rank} Q^I_l}{n} \sum_{j=1}^{n} A_{j,j+n}$$

(45)

$$\text{tr}_1(\Pi r_Q(A)\Pi r_Q(B)) = -\frac{q^2 l}{n^2} \text{rank} Q^I_l \sum_{j,k=1}^{n} A_{j,j+n} B_{k,k+n}$$

(46)

$$\text{tr}_1((1-\Pi)r_Q(A)(1-\Pi)r_Q(B)) = \frac{(l+n)!}{4(l-1)!n!} \text{tr} ((1-\Pi)A(1-\Pi)B)$$

(47)

**Proof.** The first two lines above immediately follow from the fact that on each $Q^I_l$ the operator $\mathcal{H}$ is an identity matrix times $q_l$. To get the last line one uses that the trace forms in irreducible representations of a simple Lie algebra are proportional between themselves. I.e., for $A$ and $B$ being matrices in the $2n$ dimensional real representation of $su(n)$

$$\text{tr}_1(r_Q(A)r_Q(B)) = c(n,l) \text{tr} (AB).$$

(48)

Next, we take the quadratic Casimir element $C_2$ and calculate its' trace once by using the relation above, and then by using the fact that $C_2$ in any irreducible representation is a unit matrix times the eigenvalues $C_2(n,l)$,

$$\text{tr}_1 C_2 = \mu_n (n^2 - 1) c(n,l) = \text{rank} Q^I_l C_2(n,l),$$

(49)

where $n^2 - 1 = \dim su(n)$ and $\mu_n$ is a normalization coefficient. Hence,

$$c(n,l) = c(n,1) \frac{\text{rank} Q^I_l C_2(n,l)}{\text{rank} Q^I_l C_2(n,1)}.$$

(50)

The value $c(n,1) = \frac{1}{4}$ can be easily recovered by considering the realization on products of Hermite functions, $C_2(n,l) = l(n+l)(n-1)/n$ is to be found in any textbook, see [1] for instance, and dimensions of relevant representations have been given above. By collecting everything together, one arrives at [17]. □
Explicitly,
\[
\operatorname{tr} \left( (1 - \Pi)A(1 - \Pi)B \right) = \sum_{j,k=1}^{2n} A_{jk} B_{kj} + \frac{2}{n} \sum_{i,l=1}^{n} A_{i,i+n} B_{l,l+n}.
\]

With these formulas, we calculate traces of the terms appearing in Eq. (53) for $E$
\[
\begin{align*}
\operatorname{tr}_l g(v, v) &= \text{rank } \mathbf{Q}^f \left( \frac{1}{4} - \frac{q^2}{n^2} + \frac{(l + n)!}{2n^2(n + 1)} \right) g(\mathbf{\Sigma}, \mathbf{\Sigma}) \\
&\quad - \frac{(l + n)!}{4(l - 1)! (n + 1)!} \sum_{j,k=1}^{2n} g(T(e_j, e_k), T(e_j, e_k)) \\
\text{tr}_l \left( -\sum_{j,k=1}^{2n} J e_j \cdot e_k \cdot R^Q(e_j, e_k) \right) &= \left( -\frac{2q^2}{n^2} \text{rank } \mathbf{Q}^f \right) + \frac{(l + n)!}{n(l - 1)! (n + 1)!} \sum_{i,j=1}^{n} g(R(e_i, e_{i+n}), e_i, e_{i+n}) \\
&\quad - \frac{(l + n)!}{2(l - 1)! (n + 1)!} \sum_{j,k=1}^{2n} g(R(e_j, e_k) e_j, e_k).
\end{align*}
\]

Let us introduce a short hand notation
\[\alpha(n, l) = \frac{l(l + n)}{n(n + 1)}\]

Then the first two heat trace coefficients read
\[
\begin{align*}
a_0(P_l) &= (4\pi)^{-n} \text{rank } \mathbf{Q}^f \text{ vol } (M) \\
a_2(P_l) &= (4\pi)^{-n} \text{rank } \mathbf{Q}^f \int_M \left( \frac{1}{6} + \frac{\alpha(n, l)}{2} - \frac{2q^2}{n^2} \right) \rho + \left( \alpha(n, l) - \frac{2q^2}{n^2} \right) \sum_{i,j=1}^{n} g(R(e_i, e_j), e_i, e_j) \\
&\quad + \alpha(n, l) \sum_{i,j=1}^{2n} \left( \frac{1}{4} g(T(e_i, e_j), e_i, e_j) + \frac{3}{8} g(T(e_i, e_j), e_i, e_j) \right) \\
&\quad + \left( -\frac{1}{4} + \frac{q^2}{n^2} - \frac{\alpha(n, l)}{2n} \right) g(\mathbf{\Sigma}, \mathbf{\Sigma}) - \frac{1}{2} \alpha(n, l) \sum_{j=1}^{2n} \sigma(e_j)^2.
\end{align*}
\]  

**Remarks.** 1. The coefficients (53) and (54) carry no dependence on the metaplectic structure. The structure of invariants (53) - (58) tells us that higher heat kernel coefficients have the same property. In the language of Kac [12] this means that through the heat trace expansion of $P_l$ one can hear the shape of symplectic almost hermitian manifolds, but not of the metaplectic structures.
2. In general, the operator $P$ is not self-adjoint. However, the coefficients (53) and (54) are real. We do not expect this property to hold for higher terms in the heat trace expansion.
3. One may define a family of spectral actions $S_t = \operatorname{Tr} \left( \chi(P_t/A^2) \right)$ similarly to (2) with a cut-off function $\chi$ and a scale parameter $\Lambda$. The large $\Lambda$ expansion of $S_t$ is given by the heat trace asymptotics those structure differs considerably for the standard case of spin Dirac operators with torsion, cf. [10][11].

Note that in four dimensions the spectral action for the spin Dirac operator is restricted by the chiral symmetry [11], that is not present in the symplectic case.

5 **Examples: two dimensions and $CP^1$**

In two dimensions any manifold admits a Kähler structure. To use all advantages of this low-dimensional case, we shall assume that $M$ is a Kähler manifold, so that the torsion vanishes. This reduces considerably
the combinatorial complexity of the heat trace asymptotic expansion and will allow to compute more heat trace coefficients. We also like to mention compact expressions \cite{16} for the heat trace asymptotics of the scalar Laplace operator on Kähler manifolds.

In two dimensions, the expression for curvature simplifies
\[
\mathcal{R}^Q(X, Y)\varphi = -i(R(X, Y)\hat{e}, \hat{f})\mathcal{H}^f \varphi ,
\]
where $\hat{e} \equiv \hat{e}_1$, $\hat{f} \equiv \hat{f}_1$. For vanishing torsion
\[
\mathcal{P}\varphi = \nabla^2 \varphi + 1 \sum_{j,k=1}^{2} J e_j \cdot e_k \cdot \mathcal{R}^Q(e_j, e_k)\varphi = -\nabla^2 \varphi - \rho (\mathcal{H}^f)^2 \varphi
\]
where $\rho = \bar{\rho}$ is the scalar curvature.

Let us consider the operator $\mathcal{P}_l$. We remind that $\mathcal{H}^f|_{\mathcal{Q}^l} = -(l + \frac{1}{2}) \equiv q_l$. The heat trace asymptotics are characterized by the following

**Proposition 5.1.** The operator $\mathcal{P}_l$ has the form \eqref{22} with $\nabla = \nabla$,
\[
E = \rho q_l^2, \quad \Omega_{ij} = \frac{1}{2} q_l \rho \omega_{ij}.
\]

The heat kernel coefficients read
\[
a_0(\mathcal{P}_l) = (4\pi)^{-1} \text{vol } M ,
\]
\[
a_2(\mathcal{P}_l) = (4\pi)^{-1} \frac{1}{6} (1 + 6q_l^2) \int_M \rho
\]
\[
a_4(\mathcal{P}_l) = (4\pi)^{-1} \frac{1}{48 \pi} (2 + 15q_l^2 + 60q_l^4) \int_M \rho^2 .
\]

**Proof.** First, we recall that in two dimensions there is only one independent component of the Riemann tensor, so that if $(R(\hat{e}, \hat{f})\hat{e}, \hat{f}) = r$, then $\nabla \mathcal{R}(\hat{e}, \hat{e}) = \nabla \mathcal{R}(\hat{f}, \hat{f}) = -r$ and $\bar{\rho} = \rho = -2r$. Then, $\nabla = \nabla$ by inspection, so that the corresponding curvature is just $\mathcal{R}^Q$. By Eq. \eqref{55}, we have
\[
\Omega(\hat{e}, \hat{f})\varphi = \frac{1}{2} \rho \mathcal{H}^f (\hat{e}, \hat{f})\varphi ,
\]
that yields the 2nd equation in \eqref{57}. The first equation there follows from \eqref{56}. Substitutions in \eqref{25} – \eqref{27} lead to the desired result. \hfill $\Box$

Let us restrict our attention further by taking $M = \mathbb{C}P^1$. Then the eigenvalues $\lambda_{l,j}$ of $\mathcal{P}_l$ and their degeneracies $m_{l,j}$ read (see \cite[Proposition 6.3.5]{20})
\[
\lambda_{l,j} = 4(l + j + 1)^2 - 3(2l + 1)^2 - 1 , \quad m_{l,j} = 2(l + j + 1), \quad j = 0, 1, 2, \ldots
\]

To calculate the heat trace asymptotics one has to evaluate the asymptotic expansion for
\[
K(\mathcal{P}_l, t) = \sum_{j=0}^{\infty} m_{l,j} e^{-t\lambda_{l,j}} = e^{t(3(2l+1)^2+1)} \sum_{k=1}^{\infty} 2k e^{-4tk^2} .
\]

To this end one may use the Euler-Maclaurin formula
\[
\sum_{m=k}^{q} f(k) = \int_{m}^{q} f(x)dx + \frac{1}{2} (f(q) + f(m)) + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!} (f^{(2i-1)}(q) - f^{(2i-1)}(m))
\]
with $f(x) = 2xe^{-4tk^2}$. Since $f^{(2i-1)} = O(t^{i-1})$, only a finite number of terms on the right hand side of \eqref{63} contribute to any given order of the expansion. Here $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$... are the Bernoulli numbers. We have
\[
\sum_{k=m}^{q} 2k e^{-4tk^2} \simeq e^{-4m^2t} \left( \frac{1}{4t} + (m - \frac{1}{6}) + t\left( \frac{4}{3}m^2 - \frac{1}{15} \right) + O(t^2) \right).
\]
Consequently,

\[ K(P_l, t) \simeq \frac{1}{4t} + \left( \frac{5}{6} - 2m + 2m^2 \right) + t \left( \frac{19}{15} - 6m + 14m^2 - 16m^3 + 8m^4 \right) + O(t^2), \]  

where \( m = l + 1 \). This expansion gives the values of \( a_0, a_2 \) and \( a_4 \) for \( M = CP^1 \).

The \( CP^1 \) with Fubini-Study metric is isometric to \( S^2 \) with the radius \( 1/2 \). Consequently,

\[ \text{vol} CP^1 = \pi, \quad \int_{CP^1} \rho = 8\pi, \quad \int_{CP^1} \rho^2 = 64\pi. \]  

Therefore, (65) - (60) are consistent with (65).

The eigenvalues of \( P_l \) on other complex projective spaces can be found in [19].

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