Tunneling radiation of fermions from the non-stationary Kerr black hole

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In this paper, the tunneling radiation of fermions with spin 1/2 and 3/2 from the non-stationary Kerr black hole are investigated. First, according to the Dirac equation for spin 1/2 fermions and Rarita-Schwinger equation for spin 3/2 fermions, we construct the corresponding gamma matrices and the derive Hamilton-Jacobi equation for spin 1/2 and 3/2 fermions. Then, the tunneling behavior of fermions on the event horizon is studied. Finally, we obtain the thermodynamic properties of the non-stationary Kerr black hole. The result shows that the tunneling radiation rate, surface gravity and temperature are all related to $r_H$ and $r'_H$.

I. INTRODUCTION

Based on Einstein’s general relativity, a lot of works have been done for studying the black hole. Those results show that general relativity is an elegant theory. Thus, the enthusiasm of investigating general relativity is aroused as hot topics. In 1974 Hawking discovered the thermodynamic of black hole [1–3]. Considering the quantum effects, Hawking proved that the black holes have the thermal radiation. Hawking’s thermal radiation theory pointed out that the virtual particles inside a black hole via the quantum tunneling effect reach the event horizon and materialize real particles [4–10]. In 2000, Parikh and Wilczek have put forward the tunneling method and investigated the thermal radiation from black holes by using quantum mechanics [11]. In recent years, a lot of significant studies have been done for investigating tunneling radiation of black hole. In Refs. [12–17], Zhang and Zhao have studied the tunneling radiation and provided a explanation for information loss of a black hole. Later, Lin and Yang et al. have further researched on tunneling radiation from non-stationary black holes[18–24]. In Refs. [25, 26], by taking into account the change of background of black holes, the the tunneling radiation needs to be modified. Thus, according to the Klein-Gordon equation in curved spacetimes, researchers in Refs. [27–35] study the tunneling radiation and Hawking temperature of static, steady and dynamic black holes. Those results show that the tunneling radiation and Hawking temperature have significant correction. As we know, the dynamics equation of spin 1/2 and 3/2 fermions in the curved spacetimes are depended on some specific matrices, which leads to complex calculations. For overcome this situation, one can use the Hamilton-Jacobi equation (HJE) to describe the dynamic features of particles in curved spacetimes. This method simplify the process of studying particles dynamic characteristics. Especially the tunneling radiation characteristics of fermions with non-zero spin in curved spacetimes of dynamic black holes.

Despite there are many researchers have been made many achievements in particles in radiation and Hawking temperature, we still need to modified it. In actually, black holes in the universe are dynamic, and it is meaningful to study the quantum tunneling radiation of fermions that spin of non-zero in the dynamic curved spacetime. In this paper, we derive the HJE from Dirac equation and Rarita-Schwinger equation of spin 1/2 and 3/2 fermions. Then using the advanced Eddington coordinate to describe the tunneling radiation and Hawking temperature of axially symmetric dynamic Kerr black hole.

The rest of the paper is organized as follows. In Sec. II, we derived the HJE from the dynamic equation of fermions. In Sec. III, we investigated the tunneling radiation and Hawking temperature in a non-stationary symmetric curved spacetime. The last section is is devoted the discussion and conclusion.

II. THE DYNAMIC EQUATION OF SPIN 1/2 AND 3/2 FERMIONS IN A NON-STATIONARY KERR BLACK HOLE

In the advanced Eddington coordinate, the line element of the non-stationary Kerr black hole can be written as follows [4]:

\[
\begin{align*}
\text{d}s^2 &= - \left( 1 - \frac{2Mr}{\rho^2} \right) \text{d}t^2 + \text{d}x^2 + \text{d}y^2 + \text{d}z^2 - \frac{2Mr\sin^2 \theta}{\rho^2} \text{d}r \text{d}\varphi \\
&\quad + 2a\sin^2 \theta \text{d}r \text{d}\varphi + \rho^2 \text{d}\theta^2 + \left( r^2 + a^2 \right) \left( \frac{2Mr^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta \text{d}\varphi^2, \\
&= \left( \begin{array}{cccc}
0 & 0 & 0 & g_{00} \\
0 & 0 & 0 & g_{10} \\
0 & 0 & 0 & g_{20} \\
0 & 0 & 0 & g_{30}
\end{array} \right),
\end{align*}
\]

(1)

where $\rho^2 = r^2 + a^2 \cos^2 \theta$, $v$ is the standard advanced time. Therefore, the mass of non-stationary Kerr black hole is time dependent, namely, $M = M(v)$. However, the specific angular momentum is a constant [36]. According to Eq. (1), the covariant metric tensor is given by

\[
\left( \begin{array}{cccc}
0 & 0 & 0 & g_{00} \\
0 & 0 & 0 & g_{10} \\
0 & 0 & 0 & g_{20} \\
0 & 0 & 0 & g_{30}
\end{array} \right).
\]

(2)
and the corresponding determinant and the inverse tensors metric are
\[ g = -\rho^4 \sin^2 \theta, \]  
\[ g^{\mu\nu} = (-1)^{\mu+\nu} \frac{\Delta^{\mu\nu}}{g} = \begin{pmatrix} g^{00} & g^{01} & 0 & g^{03} \\ g^{10} & g^{11} & 0 & g^{13} \\ 0 & 0 & g^{22} & 0 \\ g^{00} & g^{01} & 0 & g^{33} \end{pmatrix}, \]
where \( \Delta^{\mu\nu} \) is minors, the non-zero components of the inverse metric tensor are shown as
\[ g^{\mu\nu} = \frac{a^2 \sin^2 \theta}{\rho^2}, \quad g^{01} = \frac{r^2 + a^2}{\rho^2}, \]
\[ g^{03} = \frac{a}{\rho^2}, \quad g^{11} = \frac{r^2 + a^2 - 2Mr}{\rho^2}, \]
\[ g^{13} = \frac{a}{\rho^2}, \quad g^{22} = \frac{1}{\rho^2}, \quad g^{33} = \frac{1}{\rho^2 a^2 \sin^2 \theta}. \]

According to the null-hyper surface condition \( g^{\mu\nu} (\partial_{\nu} F) (\partial_{\mu} F) = 0 \), where \( F \) is the hyper-surface, the position of event horizon is located at \( g^{00} \partial_{\mu} r_H - 2g^{01} \partial_{\mu} t_H + g^{11} + g^{22} \partial_{\mu} r_H = 0 \), with \( \partial_{\mu} r_H = \partial_{\mu} x_H \) and \( \partial_{\mu} t_H = \partial_{\mu} \tilde{r}_H \).

Since we study the tunneling behavior of spin 1/2 and 3/2 fermions from the black hole, it is necessary briefly review the Dirac equation and Rarita-Schwinger equation. It is well known that the kinematic porpoise of spin 1/2 fermions on the event horizon of black holes is describes by Dirac equation
\[ \gamma^\mu D_\mu \Psi + \frac{m}{\hbar} \Psi = 0, \]
where
\[ D_\mu = \partial_\mu + \frac{i}{2} \gamma_\mu \Pi_{\alpha\beta}, \]
\[ \Pi_{\alpha\beta} = \frac{i}{4} [\gamma^\alpha, \gamma^\beta]. \]
Meanwhile, for studying the kinematic porpoise of spin 3/2 fermion on the event of black holes, one needs to employ the Rarita-Schwinger equation, which can be expressed as follow:
\[ \gamma^\mu D_\mu \Psi_\nu + \frac{m}{\hbar} \Psi_\nu = 0, \]
with the supplementary condition
\[ \gamma^\mu \Psi_\nu = 0. \]
Now, considering the spacetimes of non-stationary Kerr black hole, the gamma matrix should satisfy the relation
\[ \{ \gamma^\mu, \gamma^\nu \} = 2 g^{\mu\nu} I. \]

Now, combining Eq. (9) and Eq. (11), the gamma matrix can be constructed as follows:
\[ \gamma^\nu = \left( \frac{a^2 \sin^2 \theta}{\rho^2} \right)^{1/2} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \]
\[ \gamma^\tau = \left[ \frac{r^2 + a^2 - 2Mr}{\rho^2} - \frac{(r^2 + a^2)^2}{\rho^2 a^2 \sin^2 \theta} \right] \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \]
\[ \gamma^\theta = \frac{1}{\rho} \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \]
\[ \gamma^\varphi = \frac{a^2 \sin \theta}{\rho (r^2 + a^2)} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + \frac{1}{\rho^2 (r^2 + a^2)^2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}. \]

where \( \sigma^1, \sigma^2, \sigma^3 \) are the Pauli matrix, and the in the above equation represents the unit matrix. By using the gamma matrix in non-stationary Kerr black hole spacetimes, we will derive the IIJE for fermions with spin 1/2 and 3/2 and study the fermions radiation from the non-stationary Kerr black hole.

### III. THE TUNNELING RADIATION FROM NON-STATIONARY KERR BLACK HOLE OF SPIN 1/2 AND 3/2 FERMIONS

Due to the Dirac equation and the WKB approximation, the wave function of spin 1/2 fermion can be expressed as
\[ \Psi = \xi \exp \left( \frac{i}{\hbar} S \right), \]
where \( S \) is the classical action of fermions, the coefficient term can be expressed as
\[ \xi = \begin{pmatrix} A \\ B \end{pmatrix}, \]
where \( A \) and \( B \) are 1 \times 2 matrices. Substituting Eq. (13) into Eq. (6) and neglecting the higher order terms of \( \hbar \), one yields
\[ \begin{pmatrix} \alpha & \beta \\ \beta & \eta \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \]
The component of \( \alpha, \beta, \eta \) are shown in Eq. (16) are
\[ \alpha = \frac{\sin \theta}{\rho} \frac{\partial S}{\partial \varphi} I + \frac{r^2 + a^2}{\rho \sin \theta} \frac{\partial S}{\partial r} I + \frac{1}{\rho \sin \theta} \frac{\partial S}{\partial \varphi} I + imI, \]
\[ \beta = \begin{pmatrix} \frac{1}{\rho^2} \frac{\partial S}{\partial r} & \frac{1}{\rho (r^2 + a^2 - 2Mr)} \frac{\partial S}{\partial \varphi} \end{pmatrix} \sigma^1 \]
\[ \eta = \frac{\sin \theta}{\rho} \frac{\partial S}{\partial \varphi} I + \frac{r^2 + a^2}{\rho \sin \theta} \frac{\partial S}{\partial r} I + \frac{1}{\rho \sin \theta} \frac{\partial S}{\partial \varphi} I - imI, \]
\[ \beta = \begin{pmatrix} \frac{1}{\rho^2} \frac{\partial S}{\partial r} & \frac{1}{\rho (r^2 + a^2 - 2Mr)} \frac{\partial S}{\partial \varphi} \end{pmatrix} \sigma^1 \]
\[ \alpha = \frac{\sin \theta}{\rho} \frac{\partial S}{\partial \varphi} I + \frac{r^2 + a^2}{\rho \sin \theta} \frac{\partial S}{\partial r} I + \frac{1}{\rho \sin \theta} \frac{\partial S}{\partial \varphi} I + imI, \]
\[
\begin{align*}
\Xi &= \left\{\frac{1}{\rho^2\sin^2\theta} - \Delta - \frac{1}{\rho^2a^2\sin^2\theta}\right\} \frac{\partial S}{\partial \varphi}^3, \\
\Delta &= \frac{a^2\sin^2\theta - a(r^2 + a^2)}{\rho^4} \left( r^2 + a^2 - 2Mr(r^2 - a^2) \right). 
\end{align*}
\]

In order to get non-trivial solution, it requires the determination of Eq. (15) equal to zero, hence, one gets
\[
\det (\alpha \eta - \beta \eta) = 0, 
\]
Combining the Eq. (16) with Eq. (17), which leads
\[
\begin{align*}
&\frac{a^2\sin^2\theta}{\rho^2} \left( \frac{\partial S}{\partial v} \right)^2 + 2r^2 + a^2 \left( \frac{\partial S}{\partial r} \right)^2 \left( \frac{\partial S}{\partial v} \right) \\
+ \frac{2a}{\rho^2} \frac{\partial S}{\partial \varphi} + \frac{r^2 + a^2 - 2Mr}{\rho^2} \left( \frac{\partial S}{\partial r} \right)^2 + 2a \frac{\partial S}{\partial r} \frac{\partial S}{\partial \varphi} \\
+ \frac{1}{\rho^2} \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{\rho^2a^2\sin^2\theta} \left( \frac{\partial S}{\partial \varphi} \right)^2 + m^2 = 0.
\end{align*}
\]

It is observably that Eq. (18) is the HJE, one can get the same result when putting the tensors into the general expressions of HJE, that is \[g^{\mu\nu} \left( \frac{\partial S}{\partial x^\mu} \right) \left( \frac{\partial S}{\partial x^\nu} \right) + m^2 = 0.\] Notably, Eq. (18) is derived from the Dirac equation, hence, it only suitable for describing kinematic porpoise of the spin 1/2 fermions from event horizon of the non-stationary Kerr black hole.

Next, with the same way, one can derive the HJE from Rarita-Schwinger equation. First of all, it is necessary define wave function of Eq. (9) as follows
\[
\Psi_\nu = \begin{pmatrix} A_\nu \\ B_\nu \end{pmatrix} \exp \left( \frac{i}{\hbar} S \right),
\]
In the above expression, we denote the matrices by \[A_\nu = \begin{pmatrix} a_\nu & c_\nu \end{pmatrix}^T, B_\nu = \begin{pmatrix} b_\nu & d_\nu \end{pmatrix}^T,\] and \[a_\nu, b_\nu, c_\nu, d_\nu\] are corresponding matrix. In the semi-classical approximation, we can get
\[
\begin{pmatrix} \alpha & \beta \\ \beta & \eta \end{pmatrix} \begin{pmatrix} A_\nu \\ B_\nu \end{pmatrix} = 0,
\]
where \[\alpha, \beta\] and \[\eta\] are shown as Eq. (16). In order to get non-trivial solution, it requires the determination of Eq. (21) equal to zero, that is
\[
\det (\alpha \beta - \beta \eta) = 0.
\]
The Eq. (21) leads to the following results
\[
\begin{align*}
&\frac{a^2\sin^2\theta}{\rho^2} \left( \frac{\partial S}{\partial v} \right)^2 + 2r^2 + a^2 \left( \frac{\partial S}{\partial r} \right)^2 \left( \frac{\partial S}{\partial v} \right) + 2a \frac{\partial S}{\partial r} \frac{\partial S}{\partial \varphi} \\
+ \frac{r^2 + a^2 - 2Mr}{\rho^2} \left( \frac{\partial S}{\partial r} \right)^2 + 2a \frac{\partial S}{\partial r} \frac{\partial S}{\partial \varphi} + \frac{1}{\rho^2} \left( \frac{\partial S}{\partial \theta} \right)^2 \\
&+ \frac{1}{\rho^2a^2\sin^2\theta} \left( \frac{\partial S}{\partial \varphi} \right)^2 + m^2 = 0,
\end{align*}
\]
which has the same expression as Eq. (18). However, the HJE from Eq. (22) is only for spin 3/2 fermions. Now, we can use the HJE to study the tunneling behavior of spin 1/2 and spin 3/2 fermions from non-stationary Kerr black hole since the their kinematic porpoise can be describe by HJE, which has the same expression. As we know, the event horizon of non-stationary Kerr black hole varies with time \[v.\] Therefore, for calculate the tunneling rate of fermions on the event horizon, one needs to use the general tortoise coordinate transformation \[37:\]
\[
\begin{align*}
&v_* = v - v_0, \\
&\theta_* = \theta - \theta_0,
\end{align*}
\]
where \[v_0\] and \[\theta_0\] are arbitrary constants characterizes the initial state of the hole, respectively. \[\kappa\] denotes an adjustable parameter. The above-mention equations lead to
\[
\begin{align*}
&\frac{\partial}{\partial r} = \frac{2\kappa(r - r_H) + 1}{2\kappa(r - r_H)} \frac{\partial}{\partial r_*}, \\
&\frac{\partial}{\partial \nu} = \frac{\partial}{\partial v_*} - \frac{r_H}{\kappa(r - r_H)} \frac{\partial}{\partial r_*}, \\
&\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta_*} - \frac{r_H}{\kappa(r - r_H)} \frac{\partial}{\partial r_*},
\end{align*}
\]
where \[r_H = \partial_\nu r_H, r'_H = \partial_\theta r_H.\] Substituting Eq. (24) into line element Eq. (18) or Eq. (22), The HJE becomes
\[
\begin{align*}
&a^2\sin^2\theta \left( \frac{\partial S}{\partial \theta_*} \right)^2 + a^2\sin^2\theta \left( \frac{\partial S}{\partial \varphi_*} \right)^2 \\
+ \frac{2a^2\sin^2\theta}{2\kappa(r - r_H)} \left( \frac{\partial S}{\partial \theta_*} \right) \left( \frac{\partial S}{\partial \varphi_*} \right) + \frac{\partial S}{\partial \theta_*} \\
+ \Delta \left[ \frac{2\kappa(r - r_H) + 1}{2\kappa(r - r_H)} \frac{\partial S}{\partial r_*} \right]^2 \\
+ \frac{r_H}{\kappa(r - r_H)} \frac{\partial S}{\partial r_*} - \frac{r_H}{\kappa(r - r_H)} \frac{\partial S}{\partial r_*} + 2a \frac{\partial S}{\partial r} \frac{\partial S}{\partial \varphi} + \frac{1}{\rho^2} \left( \frac{\partial S}{\partial \theta} \right)^2 \\
+ \frac{m^2}{2\kappa(r - r_H)} \left[ \frac{2\kappa(r - r_H) + 1}{2\kappa(r - r_H)} \frac{\partial S}{\partial r_*} \right]^2 = 0.
\end{align*}
\]
Simplifying the Eq. (25), one yields
\[
\frac{B}{C} \left( \frac{\partial S}{\partial r_*} \right)^2 - 2\frac{\partial S}{\partial v_*} \frac{\partial S}{\partial r_*} - 2\frac{\partial S}{\partial \theta_*} \frac{\partial S}{\partial r_*} + 2\kappa(r - r_H) \frac{E}{C} = 0,
\]
with
\[
B = \frac{(a^2\sin^2\theta - 2r^2 + a^2) \frac{\partial S}{\partial r} + (r^2 + a^2 - 2Mr)}{2\kappa(r - r_H)}.
\]
\[ C = a^2 \sin^2 \theta \bar{r}_H - (r^2 + a^2), \]
\[ D = -P_0 \bar{r}'_H - a \bar{r}_H j, \]
\[ E = a^2 \sin^2 \theta \left( \frac{\partial S}{\partial r_*} \right)^2 + P_0 + \frac{j^2}{\sin^2 \theta} - 2aP_0 j \]
\[ + (r^2 + a^2 \cos^2 \theta) m^2, \] (27)

where \( P_0 = \partial_{\theta_0} S, \ j = \partial_{\phi_0} S. \) According to Eq. (26), one obtains an infinite limit of 0/0 type on the event horizon. Therefore, it is necessary to use the L'Hopital law here, which leads to

\[ \lim_{r \to r_H, \theta \to \theta_0} \frac{B}{C} = 1. \] (28)

By solving the Eq. (28) the \( \kappa \) is given by

\[ \kappa = \frac{-2r_H \bar{r}_H + r_H - M}{(1-2\bar{r}_H) \left( r_H^2 + a^2 - a^2 \bar{r}_H \sin^2 \theta_0 \right) + 2r_H^2}. \] (29)

Actually, \( \kappa \) is the surface gravity of non-stationary Kerr black hole. On the event horizon, Eq. (26) can be rewritten as follows:

\[ \left( \frac{\partial S}{\partial r_*} \right)^2 + 2(\omega - \omega_0) \frac{\partial S}{\partial r_*} = 0, \] (30)

where \( \omega \) is the energy of the tunneling particles, and the expression of \( \omega_0 \) is denoted as

\[ \omega_0 = \lim_{r \to r_H} \frac{D}{C} = \frac{-P_0 \bar{r}'_H - a \bar{r}_H j}{a^2 \sin^2 \theta \bar{r}_H - (r_H^2 + a^2)}. \] (31)

Due to the solution of Eq. (30), one gets

\[ \frac{\partial S}{\partial r} = \left[ 1 + \frac{1}{2\kappa (r - r_H)} \right] \frac{\partial S}{\partial r_*} = \frac{2\kappa (r - r_H) + 1}{2\kappa (r - r_H)} \left[ (\omega + \omega_0) \pm (\omega - \omega_0) \right]. \] (32)

After integral the Eq. (32), the result is

\[ S = \int \frac{2\kappa (r - r_H) + 1}{2\kappa (r - r_H)} \left[ (\omega + \omega_0) \pm (\omega - \omega_0) \right] dr \]
\[ = \frac{i\pi}{2\kappa} \left[ (\omega - \omega_0) \pm (\omega - \omega_0) \right], \] (33)

where \( + (-) \) mean the outgoing (incoming) solution. Taking into account both of outgoing and incoming solution, the total tunneling rate of fermions is

\[ \Gamma = \frac{\Gamma_{\text{emission}}}{\Gamma_{\text{absorption}}} = \exp \left[ -\frac{2\pi}{\kappa} (\omega - \omega_0) \right] \]
\[ = \exp \left( -\frac{\omega - \omega_0}{T} \right). \] (34)

Obviously, the expression of \( \kappa \) in Eq. (29) is the surface gravity of non-stationary Kerr black hole on the event horizon. So the Hawking temperature of non-stationary Kerr black hole is

\[ T_H = \frac{\kappa}{2\pi} \frac{(1 - r_H) r_H - M}{(1 - 2r_H) \left( r_H^2 + a^2 - a^2 r_H^2 \sin^2 \theta_0 \right) + 2r_H^2}. \] (35)

Now, we have derived the HJE in the non-stationary Kerr black hole space-time from dynamic equation of spin 1/2 and spin 3/2 fermions. Then discussed the quantum tunneling and Hawking temperature. The Hawking temperature is related to \( r_H \) and \( r'_H \). It should be noted that for the spin of 3/2 fermion, we should start with the matrix Eq. (8) and in the same way to study tunneling rate and Hawking temperature.

**IV. CONCLUSION**

In this paper, we obtain the HJE by describing the Dirac equation of spin 1/2 fermion and Rarita-Schwinger equation of spin 3/2 fermion in the non-stationary curved Kerr black hole spacetime. The conclusion shows that the HJE is a basic condition for the establishment of field equation in a curved spacetimes. Based on the HJE, we have investigated the Hawking temperature and fermions tunneling radiation of axially symmetric dynamic Kerr black hole. The result shows that the tunneling radiation, temperature and surface gravity are all related to \( r_H \) and \( r'_H \). For different spacetimes, there should be different gamma matrices. Therefore, this paper effectively investigated the particles tunneling radiation and Hawking temperature under the axially symmetric dynamic Kerr black hole’s curved spacetime background. We can get when \( a = 0, r'_H = 0 \) the non-stationary Kerr black hole return to the case of Vaidya black hole that also proved the method that we studied is correctness.

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