Triharmonic Curves in \(f\)-Kenmotsu Manifolds

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Abstract

The aim of this paper is to study triharmonic curves in three dimensional \(f\)-Kenmotsu manifolds. We investigate necessary and sufficient conditions for Frenet curves, and specifically for slant and Legendre curves to be triharmonic. Then we prove that triharmonic Frenet curves with constant curvature are Frenet helices in three dimensional \(f\)-Kenmotsu manifolds. Next, we give a nonexistence theorem that there is no triharmonic Legendre curve in three dimensional \(f\)-Kenmotsu manifolds.

Keywords: \(f\)-Kenmotsu manifold, \(k\)-harmonic curve, triharmonic curve, slant curve, Legendre curve

MSC: 53C25, 53C43, 58E20.

1 Introduction

In 1964, Eells and Sampson [4] introduced \(k\)-harmonic maps (polyharmonic maps) as the critical points of the \(k\)-energy functional defined by

\[
E_k^{ES}(\psi) = \frac{1}{2} \int_M |(d + d^*)^{r}\psi|^2 v_g, \quad r \geq 1
\]

for \(\psi \in C^\infty(M, \bar{M})\) where \(M, \bar{M}\) are Riemannian manifolds. Here, when \(\gamma : I \subset \mathbb{R} \rightarrow M\) is an arc-length parametrized curve, substituting \(\gamma' = T\) where \(T\) is the unit tangent vector field of \(\gamma\), the Euler-Lagrange equation reduces to

\[
\tau_k(\gamma) = \nabla^2_{T} T + \sum_{l=0}^{k-2} (-1)^{l} R^M(\nabla^2_{T}^{3-l} T, \nabla^{l}_{T} T) T = 0, \quad k \geq 1.
\]

Solutions of equation (2) are called as \(k\)-harmonic curves. Particularly, for \(k \geq 1\), any harmonic curve is a polyharmonic curve. Then \(k\)-harmonic curves called as proper \(k\)-harmonic if they are not harmonic, [10].

Besides, Eells and Lemaire, studied in detail \(k\)-energy functionals and \(k\)-harmonic maps, in [5] and Maeta investigated \(k\)-harmonic curves into Riemannian manifolds with constant sectional curvature, in [8].
The theory of harmonic maps has been applied to a wide variety of fields, primarily to differential and Riemannian geometry. Harmonic maps between two Riemannian manifolds are critical points of the energy functional

$$E(\psi) = \frac{1}{2} \int_M |d\psi|^2 v_g.$$  

Namely, $$\psi: (M, g) \rightarrow (\bar{M}, \bar{g})$$ is called as harmonic map if

$$\tau(\psi) = -d^* d\psi = \text{trace} \nabla d\psi = 0$$

where $$M, \bar{M}$$ are Riemannian manifolds. Here $$\tau(\psi)$$, which is the tension field of $$\psi$$, is the Euler-Lagrange equation of the energy functional $$E(\psi)$$. $$d$$ is the exterior differentiation, $$d^*$$ is the codifferentiation, $$\nabla$$ is the connection induced from the Levi-Civita connection $$\nabla^M$$ of $$M$$ and the pull-back connection $$\nabla^\psi$$, [3, 8].

On the other hand, Jiang studied first and second variation formulas of the bienergy functional $$E_2(\psi)$$ whose critical points are called as biharmonic maps, [6]. There have been a rich literature on biharmonic maps like as harmonic maps.

In this study, we focus on arc-length parametrized triharmonic Frenet curves, which are the solutions of (2) for $$k = 3$$, given as the following equality

$$\tau_3(\gamma) = \nabla^5 T + R^M(\nabla^3 T, T)T - R^M(\nabla^2 T, \nabla T)T = 0.$$  

(3)

It should be noted that every harmonic curve is also a triharmonic curve. Nevertheless, biharmonic curves are not necessarily triharmonic curves and vice versa triharmonic curves not need to be biharmonic. Thus, as Montaldo and Pampano pointed out in [10], that the study of triharmonic curves could be a completely different problem than biharmonic curves in general.

Although specific studies on harmonic and biharmonic curves are quite rich, triharmonic curves are generally briefly examined in studies under the title of $$k$$-harmonic curves. However, it has become more interesting exclusively with this recent work by Montaldo and Pampano in [10] which they investigated the possibility of constructing triharmonic curves in a Riemannian manifold with nonconstant curvature. For these reasons, the purpose of this paper is to study triharmonic curves with constant and nonconstant curvature in $$f$$-Kenmotsu manifold in order to add a new perspective to study of triharmonic curves and to motivate readers work on this new topic.

Our study consists of the following sections. Section 2 is reserved for an introduction to $$f$$-Kenmotsu manifolds, slant curves and Legendre curves. In Section 3, triharmonicity conditions of a Frenet curve are obtained and these conditions are studied under various cases. Then, detailed interpretations are given about the triharmonicity conditions of slant and Legendre curves.

### 2 $$f$$-Kenmotsu manifolds

A differentiable manifold $$M^{2n+1}$$ is called almost contact metric manifold with the almost contact metric structure $$(\varphi, \xi, \eta, g)$$, if it admits a tensor field $$\varphi$$
of type $(1, 1)$, vector field $\xi$, 1-form $\eta$ and Riemannian metric tensor field $g$ satisfying the following conditions;

\begin{align*}
\varphi^2 &= -I + \eta \otimes \xi, \\
\eta(\xi) &= 1, \quad \varphi \xi = 0, \quad \eta(X) = g(X, \xi) \quad (4) \\
g(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y)
\end{align*}

where $I$ denotes the identity transformation and $X, Y \in \Gamma(TM)$, \cite{1}.

An almost contact metric manifold is said be $f$-Kenmotsu manifold if the Levi-Civita connection $\nabla$ of $g$ satisfies

\begin{align*}
(\nabla_X \varphi) Y &= f(g(X, Y) \xi - \eta(Y)\varphi X), \\
\nabla_X \xi &= f(X - \eta(X)\xi) \quad (5) \\
\nabla_X \eta &= -f^2 \eta(\xi)
\end{align*}

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$ and $X, Y \in \Gamma(TM)$, \cite{7}.

It is well known that on a 3-dimensional Riemannian manifold, curvature tensor field $R$ given as below;

\begin{align*}
R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + Ric(Y, Z)X \\
&- Ric(X, Z)Y - \frac{r}{2} \{g(Y, Z)X - g(X, Z)Y\}
\end{align*}

where $X, Y, Z \in \Gamma(TM)$, $Ric$ is the Ricci tensor, $Q$ is the Ricci operator and $r$ is the scalar curvature of $M$, \cite{9}.

Thus for a 3-dimensional $f$-Kenmotsu manifold, the curvature tensor field becomes;

\begin{align*}
R(X, Y)Z &= (\frac{r}{2} + 2(f^2 + f'))(g(Y, Z)X - g(X, Z)Y) \\
&- (\frac{r}{2} + 3(f^2 + f'))(g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi\eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X)
\end{align*}

\quad (7)

Now let us consider an arc-length parametrized curve $\gamma : I \subset \mathbb{R} \to M$ in a $n$-dimensional Riemannian manifold $(M, g)$. If $E_1, E_2, ..., E_r$ orthonormal vector fields satisfying

\begin{align*}
E_1 &= \gamma' \\
\nabla_T E_1 &= k_1 E_2, \\
\nabla_T E_2 &= -k_1 E_1 + k_2 E_3, \\
&\vdots \\
\nabla_T E_r &= -k_{r-1} E_{r-1}
\end{align*}

along $\gamma$, then $\gamma$ is called a Frenet curve of osculating order $r$. Here $k_1, ..., k_{r-1}$ are positive functions on $I$ and $1 \leq r \leq n$. With this for $k_1, ..., k_{r-1}$ are non-zero positive constants; a Frenet curve of osculating order 1 is called geodesic, of osculating order 2 is called circle if $k_1$ = constant $> 0$, and of osculating order $r \geq 3$ is called helix of order $r$, \cite{10}.

Now let remind the definitions and properties of slant and Legendre curves.
Definition 1. \( \gamma(s) : I \subset \mathbb{R} \rightarrow M \) is called a slant curve if the contact angle \( \theta : I \rightarrow [0, 2\pi) \) of given by \( \cos \theta(s) = g(T(s), \xi) \) is a constant function.

In particular, if \( \theta = \frac{\pi}{2}(\text{or} \frac{3\pi}{2}) \) then \( \gamma(s) \) is called a Legendre curve, \( [2] \).

Remark 1. For a slant curve in \( f \)-Kenmotsu manifolds, we have \( [2] \)

\[ \eta(N) = -\frac{f}{k_1} (\sin \theta)^2, \]

where \( |\sin \theta| \leq \min \frac{k_1}{f} \) and

\[ \eta(B) = \frac{|\sin \theta|}{k_2} \sqrt{k_1^2 - f^2 (\sin \theta)^2}. \]

Remark 2. For a Legendre curve in \( f \)-Kenmotsu manifolds, we have \( [2, 11] \)

\[ N = -\xi, \quad k_1 = f |\gamma|, \quad k_2 = 0, \]

and

\[ \eta(T) = 0, \quad \eta(N) = -1, \quad \eta(B) = 0. \]

3 Triharmonic curves in \( f \)-Kenmotsu manifolds

Let \((M, \varphi, \xi, \eta, g)\) be a 3-dimensional \( f \)-Kenmotsu manifold and \( \gamma(s) : I \subset \mathbb{R} \rightarrow M \) be a non-geodesic Frenet curve parametrized by arc length \( s \). The Serret-Frenet frame \( \{T = \gamma'(s), N, B\} \) along \( \gamma \), respectively; the tangent, the principal normal and the binormal vector fields. Then the Serret-Frenet formulas are given as;

\[
\begin{align*}
\nabla_T T &= k_1 N \\
\nabla_T N &= -k_1 T + k_2 B \\
\nabla_T B &= -k_2 N
\end{align*}
\]

(8)

where \( k_1 \) and \( k_2 \) are the curvature and the torsion of the curve, respectively.

By using Serret-Frenet formulas given in \( [5] \) and applying \( \nabla_T T = k_1 N \) as many times as needed, we obtained

\[
\begin{align*}
\nabla_T^2 T &= -k_1^2 T + k_1' N + k_1 k_2 B, \quad (9) \\
\nabla_T^2 T &= -3k_1 k_1'T + (-k_1'' - k_1 k_2')N + (2k_1' k_2 + k_1 k_2')B, \quad (10)
\end{align*}
\]

\[
\begin{align*}
\nabla_T^3 T &= (k_1^3 + k_1^2 k_2 - 4k_1' k_1 - 3k_1^2)T \\
&- (6k_2' k_1 + 3k_2 k_1^2 + 3k_1 k_2 k_2' - k_1')N \\
&+ (3k_1' k_2 + k_1 k_2'' + 3k_1' k_2 - k_2 k_1^3 - k_1^3 k_2)B \quad (11)
\end{align*}
\]
and finally
\[
\nabla_T^3 T = (10k_1^3k_1 + 5k_1k_1^2k_2 + 5k_1k_1^3 - 10k_1^2k_1^2 + 5k_2k_2^3k_1^3)T \\
+ (k_1^3 + 2k_1^3k_2 - 10k_1^2k_1^3 + 15k_1k_1k_1^3 - 12k_2k_2k_1^3)T \\
- 6k_2^3k_1^2 + 3k_1k_2k_2^2 + k_1k_2^3 + k_1^4(\eta + N)N \\
+ (-9k_1^3k_2k_1 - 4k_1k_2k_1 - 6k_1k_2^2k_2 + 4k_1k_2k_2 + 6k_1k_2) \\
+ 4k_1k_2k_1 + k_1k_2^3 + k_2k_1^3)B.
\]

On the other hand, by substituting (9), (10) and \( \nabla_T T = k_1 N \) into the curvature tensor formula (10), we get

\[
R^M(\nabla_T^3 T, T) = \left((k_1^3 - k_1^2 - k_1k_2^3)(\frac{r}{2} + 3(f^2 + f'))\eta(N)\eta(T) \right)T \\
+ (2k_1k_1^2 + k_2k_2^2)(\frac{r}{2} + 3(f^2 + f'))\eta(B)\eta(T) \right)T \\
+ \left ((k_1^3 - k_1^2 - k_1k_2^3)(\frac{r}{2} + 2(f^2 + f')) \right)N \\
- (2k_1k_2 + k_1k_2)(\frac{r}{2} + 3(f^2 + f'))\eta(T)^2 \right)B \\
- \left ((k_1^3 - k_1^2 - k_1k_2^3)(\frac{r}{2} + 3(f^2 + f'))\eta(N) \right) \\
+ (2k_1k_2 + k_1k_2)(\frac{r}{2} + 3(f^2 + f'))\eta(B) \right)\xi
\]

and

\[
R^M(\nabla_T^3 T, \nabla_T T) = \left((k_1^3 + 3(f^2 + f'))\eta(N)\eta(T) \right)T \\
+ \left (k_1^3 + 2(f^2 + f') - k_1^3 + 3(f^2 + f'))\eta(T)^2 \right)N \\
+ k_1k_2k_2(\frac{r}{2} + 3(f^2 + f'))\eta(B)\eta(T) \right)N \\
- \left (k_1^3 + 3(f^2 + f'))\eta(N)\eta(T) \right)B \\
- \left (k_1^3 + 3(f^2 + f'))\eta(N) \right)\xi.
\]
After these calculations, by substituting these equations to the following formula
\[ \tau_3(\gamma) = \nabla_T^3 T + R^M(\nabla_T^3 T, T)T - R^M(\nabla_T^2 T, \nabla_T T)T = 0, \]
we obtained the triharmonicity condition of a Frenet curve in a 3-dimensional \( f \)-Kenmotsu manifold as follows:

\[
\tau_3(\gamma) = \left[ 10k_1^3k_1' + 5k_1k_1'k_2^2 - 5k_1k_1'' - 10k_1'k_1 + 5k_2k_2'k_1^2 \\
+ (k_1'' - 2k_1^3 - k_1k_2^2)(\frac{r}{2} + 3(f^2 + f'))\eta(N)\eta(T) \\
+ (2k_1'k_2 + k_1k_2')(\frac{r}{2} + 3(f^2 + f'))\eta(B)\eta(T) \right] T \\
+ \left[ k_1^2 + 2k_1^3k_2^2 - 10k_1^2k_1'' - 15k_1(k_1')^2 - 12k_2k_2'k_1 - 6k_2k_1'' \\
- 3k_1(k_2)^2 - 4k_1k_2k_2'' + k_1k_2 + k_1^{(4)} \\
+ (k_1'' - 2k_1^3 - k_1k_2^2)((\frac{r}{2} + 2(f^2 + f')) - (\frac{r}{2} + 3(f^2 + f'))\eta(T)^2) \\
- k_1^2k_2(\frac{r}{2} + 3(f^2 + f'))\eta(B)\eta(T) \right] N \\
+ \left[ - 9k_1^2k_2k_1' - 4k_1^2k_1 - 6k_1k_2k_2' + 4k_1''k_2 + 6k_1''k_2' + 4k_1'k_2'' + k_1k_2'' \\
- k_1^2k_1 + (2k_1'k_2 + k_1k_2')(\frac{r}{2} + 2(f^2 + f')) - (\frac{r}{2} + 3(f^2 + f'))\eta(T)^2) \\
+ k_1^2k_2(\frac{r}{2} + 3(f^2 + f'))\eta(N)\eta(T) \right] B \\
- \left[ (k_1'' - 2k_1^3 - k_1k_2^2)((\frac{r}{2} + 3(f^2 + f'))\eta(N) \\
+ (2k_1'k_2 + k_1k_2')(\frac{r}{2} + 3(f^2 + f'))\eta(B) \right] \xi \\
= 0
\]

(15)

As a result, we can state the following theorem:

**Theorem 3.** Let \( \gamma(s) : I \to M \) be a non-geodesic, arc-length parametrized Frenet curve in a 3-dimensional \( f \)-Kenmotsu manifold \((M, \varphi, \xi, \eta, g)\). Then
\( \gamma(s) \) is a triharmonic Frenet curve if and only if the following equations hold,

\[
\begin{cases}
k_1k_1''' + 2k_1k_1'' - 2k_1^3k_1' - k_2^3k_2' - k_1^2k_2k_2' = 0, \\
k_1'' - k_2^3k_2' - 6k_3^2k_1' - 15k_1(k_1')^2 - 12k_2k_2k_1' - 3k_1(k_2')^2 + k_1^3 + k_1k_2^2 \\
+ 2k_1k_3^2 + (k_1'' - k_2^3k_1')(\frac{r}{2} + 2(f^2 + f') - (\frac{r}{2} + 3(f^2 + f'))(\eta(T)^2 + \eta(N)^2)) \\
-(k_1^3k_2\eta(T) + (2k_2k_1 + k_1k_2)\eta(N))(\frac{r}{2} + 3(f^2 + f'))\eta(B) = 0,
\end{cases}
\]

where \( k_1 = k_1(s) \) and \( k_2 = k_2(s) \) are the curvature and the torsion of \( \gamma(s) \), respectively.

Proof. By taking the inner product of (15), respectively with \( T, N, B \) vector fields, we get a system of differential equations given in (16).

Now, we investigate results of Theorem 3 in four cases.

**Case I:** If \( k_1 = constant > 0 \) then (16) reduces to;

\[
\begin{cases}
k_1^2k_2k_2' = 0, \\
4k_1k_2k_2'' + 3k_1(k_2')^2 - k_1k_2' + 2k_1^3k_2 \\
+(2k_1^2 + k_1k_2')(\frac{r}{2} + 2(f^2 + f') - (\frac{r}{2} + 3(f^2 + f'))(\eta(T)^2 + \eta(N)^2)) \\
-(k_1^2k_2\eta(T) + k_1k_2'\eta(N))(\frac{r}{2} + 3(f^2 + f'))\eta(B) = 0,
\end{cases}
\]

Hence, we give the following theorem;

**Theorem 4.** Let \( \gamma(s) \) be a non-geodesic proper triharmonic curve with constant curvature \( k_1 \) and nonzero torsion \( k_2 \) in a 3-dimensional \( f \)-Kenmotsu manifold. Then \( \gamma(s) \) is a Frenet helix. Further, \( k_1 \) and \( k_2 \) satisfy the differential equation system:

\[
\begin{cases}
(k_1^3 + k_2^2)^2 = (2k_1^2 + k_2^2)(\frac{r}{2} + 2(f^2 + f') - (\frac{r}{2} + 3(f^2 + f'))(\eta(T)^2 + \eta(N)^2)) \\
+ k_1k_2\eta(T)\eta(B)(\frac{r}{2} + 3(f^2 + f')),
\end{cases}
\]

(18)
Proof. Since $\gamma(s)$ is a proper triharmonic curve and from first equation of (17); if curvature $k_1$ is a constant, then the torsion $k_2$ is necessarily constant, namely the curve is a Frenet helix. Thereafter, assuming that the curvature $k_1$ and the torsion $k_2$ are constant, second and third equations of (17) become, the first and the second equation of (18), respectively.

Case II: If $k_1 = \text{constant} > 0$ and $k_2 = 0$, then (16) reduces to:

$$\begin{cases} k_1^5 - 2k_1^3 \left[ \frac{r}{2} + 2(f^2 + f') - \left( \frac{r}{2} + 3(f^2 + f') \right) \eta(T)^2 + \eta(N)^2 \right] = 0, \\ 2k_1^3 \left( \frac{r}{2} + 3(f^2 + f') \right) \eta(N) \eta(B) = 0. \end{cases}$$ (19)

In the second equation of (19) since $k_1 > 0; \left( \frac{r}{2} + 3(f^2 + f') \right)$ or $\eta(N)$ or $\eta(B)$ can be equal to zero. However if $\eta(N) = 0$, we obtain that $\gamma$ is a Legendre curve. But as we mentioned in Remark 2 for a Legendre curve in $f$-Kenmotsu manifold $\eta(N) = -1$. This is a contradiction with our assumption $\eta(N) = 0$. Therefore, this subcase was not taken into consideration. For this reason we examine Case II in two subcases.

Subcase II-1: If $\left( \frac{r}{2} + 3(f^2 + f') \right) = 0$, then (19) reduces to

$$k_1^5 + 2(f^2 + f') = 0.$$

Hence, we have the following theorem.

**Theorem 5.** Let $\gamma(s)$ is a non-geodesic Frenet curve with constant curvature $k_1$ and zero torsion $k_2$ in a 3-dimensional $f$-Kenmotsu manifold of constant scalar curvature $r = 3k_1^2$. Then $\gamma(s)$ is a proper triharmonic curve for $f^2 + f' < 0$ if and only if

$$f(s) = \frac{k_1}{\sqrt{2}} \tan \left( \frac{1}{2} (\sqrt{2}k_1 c_1 - \sqrt{2}k_1 s) \right)$$

where $c_1 > s$ is an integration constant.

Subcase II-2: If $\eta(B) = 0$, then (19) reduces to

$$k_1^5 - 2k_1^3 \left( \frac{r}{2} + 2(f^2 + f') - \left( \frac{r}{2} + 3(f^2 + f') \right) \eta(T)^2 + \eta(N)^2 \right) = 0.$$

Then, we get;

**Theorem 6.** Let $\gamma(s)$ is a non-geodesic Frenet curve with constant curvature $k_1$ and zero torsion $k_2$ in a 3-dimensional $f$-Kenmotsu manifold and $\eta(B) = 0$. Then $\gamma(s)$ is a proper triharmonic curve if and only if

$$k_1 = \sqrt{2} \left( \frac{r}{2} + 2(f^2 + f') - \left( \frac{r}{2} + 3(f^2 + f') \right) \eta(T)^2 + \eta(N)^2 \right).$$
Case IV: If $k_1 \neq \text{constant}$ and $k_2 = \text{constant} \neq 0$, then (16) reduces to:

$$
\begin{aligned}
&k_1k_1'''' + 2k_1'k_1'' - 2k_1^3k_1' - k_2^2k_1 = 0,
&\left\{
\begin{array}{l}
\begin{aligned}
&k_1^{(4)} - 10k_1^2k_1'' - 6k_1^2k_1' - 15k_1(k_1')^2 + k_1^3 + k_1k_1'' + 2k_1^3k_2^2 \\
&+(k_1'' - 2k_1^3k_2)(r + 2(f^2 + f') - \left(\frac{r}{2} + 3(f^2 + f')(\eta(T)^2 + \eta(N)^2)\right)
\end{aligned}
\end{array}
\right.
\end{aligned}
$$

(20)

So we have the following theorem.

**Theorem 7.** Let $\gamma(s)$ be a non-geodesic Frenet curve in a 3-dimensional $f$-Kenmotsu manifold. Then $\gamma(s)$ is a proper triharmonic curve with nonconstant curvature $k_1$ and nonzero constant torsion $k_2$ if and only if the following equations are provided,

$$
\begin{aligned}
&k_1k_1'''' + 2k_1'k_1'' - 2k_1^3k_1' - k_2^2k_1 = 0,
&\left\{
\begin{array}{l}
\begin{aligned}
&k_1^{(4)} - 10k_1^2k_1'' - 6k_1^2k_1' - 15k_1(k_1')^2 + k_1^3 + k_1k_1'' + 2k_1^3k_2^2 \\
&+(k_1'' - 2k_1^3k_2)(r + 2(f^2 + f') - \left(\frac{r}{2} + 3(f^2 + f')(\eta(T)^2 + \eta(N)^2)\right)
\end{aligned}
\end{array}
\right.
\end{aligned}
$$

(21)

Case IV: If $k_1 \neq \text{constant}$ and $k_2 = 0$, then (16) reduces to:

$$
\begin{aligned}
&k_1k_1'''' + 2k_1'k_1'' - 2k_1^3k_1' = 0,
&\left\{
\begin{array}{l}
\begin{aligned}
&k_1^{(4)} - 10k_1^2k_1'' - 15k_1(k_1')^2 + k_1^3 \\
&+(k_1' - 2k_1^3)(r + 2(f^2 + f') - \left(\frac{r}{2} + 3(f^2 + f')(\eta(T)^2 + \eta(N)^2)\right) = 0,
\end{aligned}
\end{array}
\right.
\end{aligned}
$$

(22)

Since $\left(\frac{r}{2} + 3(f^2 + f')\right)$ or $\eta(B)$ can be equal to zero in the third equation of (22) we examine Case IV in two subcases.
Subcase IV-1: If \((\frac{r}{2} + 3(f^2 + f')) = 0\), \(k_1 \neq \text{constant}\) and \(k_2 = 0\) then (22) reduces to
\[
\begin{align*}
&k_1 k_1''' + 2k_1' k_1'' - 2k_1^3 k_1' = 0, \\
&k_1^{(4)} - 10k_1^2 k_1'' - 15k_1 (k_1')^2 + k_5^5 + (k_1'' - 2k_1^3)(\frac{r}{2} + 2(f^2 + f')) = 0
\end{align*}
\]
(23)

Thus we obtain the following theorem.

**Theorem 8.** Let \(\gamma(s)\) be a non-geodesic Frenet curve with nonconstant curvature \(k_1\) and zero torsion \(k_2\) in a 3-dimensional \(f\)-Kenmotsu manifold. Then \(\gamma(s)\) is a proper triharmonic curve if and only if
\[
k_1(s) = \frac{\sqrt{5}}{s},
\]
\[
f(s) = \frac{9}{2s}
\]
and
\[
r(s) = -\frac{189}{2s^2}.
\]

**Proof.** We use the methods in reference [10] to find the solutions system of differential equations (23). Initially, the first equation of (23) is multiplied by \(5k_1\) and integrated, then we have
\[
5k_1^2 k_1'' - 2k_1^5 = c_1
\]
(24)

where \(c_1\) a real constant. Similarly, when (24) is multiplied by \(2k_1^{-2}k_1'\) and then integrated, we get
\[
5(k_1')^2 = k_1^4 - 2k_1^{-3}c_1 + c_2,
\]
(25)

for another real constants \(c_1\) and \(c_2\). For an explicit solution, by considering \(c_1 = c_2 = 0\); \(k_1(s) = \frac{\sqrt{5}}{s}\) is obtained as a solution of (25).

For \(k_1'' \neq 2k_1^3\), by substituting \(k_1(s) = \frac{\sqrt{5}}{s}\) and its derivatives into the second equation of (23), we get \(f(s) = \frac{9}{2s}\) and \(r(s) = -\frac{189}{2s^2}\).

Subcase IV-2: If \(\eta(B) = 0\), \(k_1 \neq \text{constant}\) and \(k_2 = 0\) then (22) reduces to
\[
\begin{align*}
&k_1 k_1''' + 2k_1' k_1'' - 2k_1^3 k_1' = 0, \\
&k_1^{(4)} - 10k_1^2 k_1'' - 15k_1 (k_1')^2 + \frac{r}{2} + 2(f^2 + f') - (\frac{r}{2} + 3(f^2 + f'))(\eta(T)^2 + \eta(N)^2) = 0.
\end{align*}
\]
(26)

Then, we conclude the following theorem.
Theorem 9. Let $\gamma(s)$ be a non-geodesic Frenet curve with nonconstant curvature $k_1$, zero torsion $k_2$ and $\eta(B) = 0$ in a 3-dimensional $f$-Kenmotsu manifold. Then $\gamma(s)$ is a proper triharmonic curve if and only if following system of differential equations are satisfied;

$$
\begin{align*}
&k_1k_1''' + 2k_1k_1'' - 2k_1^3k_1' = 0, \\
&k_1^{(4)} - 10k_1^2k_1'' - 15k_1(k_1')^2 + k_1^5 \\
&+(k_1'' - 2k_1^3)(\frac{r}{2} + 2(f^2 + f') - (\frac{r}{2} + 3(f^2 + f'))(\eta(T)^2 + \eta(N)^2)) = 0.
\end{align*}
$$

(27)

3.1 Triharmonic slant curves in $f$-Kenmotsu manifolds

In the following subsection we derive the triharmonicity conditions for slant curves in $f$-Kenmotsu manifolds and discuss the particular cases of $k_1$ and $k_2$.

Theorem 10. Let $\gamma(s)$ be a non-geodesic arc-length parametrized slant curve in a 3-dimensional $f$-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$. Then $\gamma(s)$ is a triharmonic curve if an only if

$$
\begin{align*}
&k_1k_1''' + 2k_1k_1'' - 2k_1^3k_1' - k_1^2k_2k_1' - k_1^2k_2 = 0, \\
&k_1^{(4)} - 10k_1^2k_1'' - 6k_1^3k_1'' - 4k_1k_2k_1'' - 15k_1(k_1')^2 - 12k_1k_2k_1' - 3k_2(k_2')^2 \\
&+k_1^2 + k_1k_2 + 2k_1^2k_2 + (k_1'' - 2k_1 - k_1k_2')(\frac{r}{2} + 2(f^2 + f')) \\
&-(\frac{r}{2} + 3(f^2 + f'))((\cos\theta)^2 + \frac{f^2}{k_1^2}((\sin\theta)^2)) \\
&-(k_1^2k_2\cos\theta - (k_1k_2 + k_1k_2')\frac{f}{k_1}((\sin\theta)^2)(\frac{r}{2} + 3(f^2 + f'))\sqrt{\frac{\sin\theta}{k_1} + \sqrt{k_1^2 - f^2(\sin\theta)^2}} = 0,
\end{align*}
$$

$$
\begin{align*}
&4k_2k_1'' + 6k_1^2k_2' - 4k_1^2k_2 - 9k_2^2k_2 - 4k_2^2k_1 - 6k_1k_2k_2 - k_1^2k_2 + k_1k_2'' \\
&(2k_1k_2 + k_1k_2')(\frac{r}{2} + 2(f^2 + f')) \\
&+(\frac{r}{2} + 3(f^2 + f'))((\cos\theta)^2 + \frac{\sin\theta}{k_1^2}(k_1^2 - f^2(\sin\theta)^2)) \\
&-(k_1^2k_2\cos\theta - (k_1 - 2k_1k_2)\frac{f}{k_1}((\sin\theta)^2)(\frac{r}{2} + 3(f^2 + f'))\sqrt{\frac{\sin\theta}{k_1} + \sqrt{k_1^2 - f^2(\sin\theta)^2}} = 0,
\end{align*}
$$

(28)

where $k_1 = k_1(s)$ and $k_2 = k_2(s)$ are the curvature and the torsion of $\gamma(s)$, respectively.

Here we examine some cases for the triharmonic slant curves in 3-dimensional $f$-Kenmotsu manifold.
Case I: If \( k_1 = \text{constant} > 0 \) and \( k_2 \neq 0 \), then (28) reduces to:

\[
\begin{align*}
{k_1^2 k_2 k_2'} & = 0, \\
\{-4k_1 k_2 k_2'' - 3k_1 (k_2')^2 + k_1 k_1' + 2k_1 k_2^2 - (2k_1^2 + k_1 k_2') (\frac{r}{2} + 2(f^2 + f')) \} \\
+ (2k_1^2 + k_1 k_2') (\frac{r}{2} + 3(f^2 + f')) ((\cos \theta)^2 + \frac{f^2}{k_1^2} (\sin \theta)^4) \\
- (k_1^2 k_2 \cos \theta - (k_1 k_2') \frac{f}{k_1} (\sin \theta)^2) (\frac{r}{2} + 3(f^2 + f')) (\frac{\sin \theta}{k_1} \sqrt{k_1^2 - f^2 (\sin \theta)^2}) & = 0, \\
\end{align*}
\]

Then from the first equation of (29), we obtain that \( k_2(s) = \text{constant} \) so we have the following theorem from Case I:

**Theorem 11.** Let \( \gamma(s) \) be a non-geodesic proper triharmonic slant curve in a 3-dimensional \( f \)-Kenmotsu manifold with constant curvature \( k_1 \) and nonzero torsion \( k_2 \). Then \( \gamma(s) \) is a slant helix. Besides, \( k_1 \) and \( k_2 \) satisfy the following system of differential equations:

\[
\begin{align*}
\begin{cases}
(k_1^2 + k_2^2)'' & = (2k_1^2 + k_2^2) (\frac{r}{2} + 2(f^2 + f')) - \left( 2k_1^2 + k_2^2 \right) ((\cos \theta)^2 + \frac{f^2}{k_1^2} (\sin \theta)^4) \\
- k_1 k_2 \cos \theta \frac{\sin \theta}{k_1} \sqrt{k_1^2 - f^2 (\sin \theta)^2} (\frac{r}{2} + 3(f^2 + f')) \\
(k_1 k_2 \cos \theta + (2k_1^2 + k_2^2) \frac{\sin \theta}{k_1} \sqrt{k_1^2 - f^2 (\sin \theta)^2} (\frac{r}{2} + 3(f^2 + f')) \frac{f}{k_1} (\sin \theta)^2 & = 0. \\
\end{cases}
\end{align*}
\]

Case II: If \( k_1 = \text{cons.} > 0 \) and \( k_2 = 0 \), then (28) reduces to:

\[
\begin{align*}
\begin{cases}
k_1^3 - 2k_1^3 (\frac{r}{2} + 2(f^2 + f')) + 2k_1^3 (\frac{r}{2} + 3(f^2 + f')) ((\cos \theta)^2 + \frac{f^2}{k_1^2} (\sin \theta)^4) & = 0, \\
2k_1^3 \frac{\sin \theta}{k_1} \sqrt{k_1^2 - f^2 (\sin \theta)^2} (\frac{r}{2} + 3(f^2 + f')) \frac{f}{k_1} (\sin \theta)^2 & = 0. \\
\end{cases}
\end{align*}
\]

Then, we obtain the following theorem.

**Theorem 12.** Let \( (M, \varphi, \xi, \eta, g) \) be a 3-dimensional \( f \)-Kenmotsu manifold and \( \gamma : I \to M \) be a non-geodesic slant curve. Then for \( k_1 = \text{cons.} > 0 \) and \( k_2 = 0 \), \( \gamma \) is a proper triharmonic curve if and only if \( M \) is of constant scalar curvature \( r = -3k_1^2 \) and

\[
f(s) = \frac{k_1}{\sqrt{2}} \tan \left( \frac{1}{2} (\sqrt{2}k_1 c_1 - \sqrt{2}k_1 s) \right).
\]
3.2 Triharmonic Legendre curves in $f$-Kenmotsu manifolds

In this subsection we discuss the triharmonicity condition for Legendre curves in $f$-Kenmotsu manifolds.

**Theorem 13.** There is no triharmonic Legendre curve in a 3-dimensional $f$-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$.

**Proof.** When Theorem 3 is rewritten according to the properties of the Legendre curve given in Remark 2, following differential equations are obtained:

\[
\begin{align*}
\left\{ \begin{array}{l}
k_1^3 k_1''' + 2k_1^2 k_1' - 2k_1 k_1'' = 0, \\
k_1^{(4)} - 10k_1^3 k_1'' - 15k_1 (k_1')^2 + k_1^5 - (k_1'' - 2k_1^3)(f^2 + f') = 0. \\
\end{array} \right. \\
\end{align*}
\]  

(32)

From the first equation of (32), via differential equation solving methods (such as in Theorem 3), $k_1(s) = \frac{\sqrt{5}}{s}$ obtained, [10]. By using $k_1(s) = \frac{\sqrt{5}}{s}$ and its derivatives in the second equation of (32), we have $f(s) = \frac{9}{2s}$, but this is a contradiction with the definition of Legendre curves in 3-dimensional $f$-Kenmotsu manifold. 

4 Conclusion

There are limited number of studies investigating triharmonic curves as opposed to harmonic and biharmonic curves. In this study, triharmonic curves are studied in three dimensional $f$-Kenmotsu manifolds and original theorems related to slant and Legendre curves are obtained. For this reason, we think that our study will be a guide not only for the author who studies $k$-harmonic curves, but also for the authors who study slant and Legendre curves.

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