Knight move in chromatic cohomology

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Abstract

In this paper we prove the knight move theorem for the chromatic graph cohomologies with rational coefficients introduced by L. Helme-Guizon and Y. Rong. Namely, for a connected graph $\Gamma$ with $n$ vertices the only non-trivial cohomology groups $H^{i,n-i}(\Gamma)$, $H^{i,n-i-1}(\Gamma)$ come in isomorphic pairs: $H^{i,n-i}(\Gamma) \cong H^{i+1,n-i-2}(\Gamma)$ for $i \geq 0$ if $\Gamma$ is non-bipartite, and for $i > 0$ if $\Gamma$ is bipartite. As a corollary, the ranks of the cohomology groups are determined by the chromatic polynomial. At the end, we give an explicit formula for the Poincaré polynomial in terms of the chromatic polynomial and a deletion-contraction formula for the Poincaré polynomial.

Introduction

Recently, motivated by the Khovanov cohomology in knot theory [8], Laure Helme-Guizon and Yongwu Rong [5] developed a bigraded cohomology theory for graphs. Its main property is that the Euler characteristic with respect to one grading and the Poincaré polynomial with respect to the other grading give the chromatic polynomial of the graph.

There is a long exact sequence relating the cohomology of a graph with the cohomologies of graphs obtained from it by contraction and deletion of an edge. It generalizes the classical contraction-deletion rule for the chromatic polynomial. This sequence is an important tool in proving various properties of the cohomology (see [5]). In particular, for a connected graph $\Gamma$ with $n$ vertices, it allows to prove that the cohomologies are concentrated on two diagonals $H^{i,n-i}(\Gamma)$ and $H^{i,n-i-1}(\Gamma)$. We prove that $H^{i,n-i}(\Gamma)$ is isomorphic to $H^{i+1,n-i-2}(\Gamma)$ ("knight move") for all $i$ with the exception of $i = 0$ for a bipartite graph $\Gamma$. An analogous theorem for Khovanov cohomology in knot theory was proved in [9]. Our proof follows the same idea of considering an additional differential $\Phi$ on the chromatic cochain complex of the graph $\Gamma$. This differential anticommutes with the original differential $d$. The associated spectral sequence collapses at the term $E_2$, and so this term is given by the cohomologies of $\Gamma$ with respect to the differential $\Phi + d$. They turn out to be trivial with a small exception (see Theorem 3.4 for the precise statement). On the other hand, the term $E_1$ of the spectral sequence is represented by the cohomology of our graph. The differential in the term $E_1$ induced by the map $\Phi$ gives the desired isomorphism. Its existence implies that the chromatic polynomial of a graph determines its cohomology groups with coefficients in a field of characteristic 0. For simplicity we work here with rational coefficients. The construction of the cohomology is based on the algebra $\mathbb{Q}[x]/(x^2)$ and reflects its properties. In general, for chromatic cohomologies based on other algebras $\mathcal{A}$, the cohomology groups are supported on more than two diagonals, and we do not expect them to be determined by the chromatic polynomial.

It would be interesting to adapt the recent techniques of spanning trees [12],[3] and the Karoubi envelopes [2] from the Khovanov cohomology theory to
the chromatic cohomology. This might lead to another proof of our key Theorem 3.4 and to a deeper understanding of combinatorics.

This work was motivated by computer calculations of the chromatic homology by M. Chmutov, which revealed certain patterns in the Betti numbers. Part of this work has been completed during the Summer’05 VIGRE working group “Knot Theory and Combinatorics” at the Ohio State University funded by NSF, grant DMS-0135308. Y. Rong was partially supported by the NSF grant DMS-0513918. The authors would like to thank L. Helme-Guizon and J. Przytycki for numerous discussions, S. Duzhin and anonymous referees for valuable comments.

1 Definitions and preliminary results

For a graph $\Gamma$ with ordered edges, a state $s$ is a spanning subgraph of $\Gamma$, that is a subgraph of $\Gamma$ containing all the vertices of $\Gamma$ and a subset of the edges. The number of edges in a state is called its dimension. An enhanced state $S$ is a state whose connected components are colored in two colors: $x$ and 1. The number of connected components colored in $x$ is called the degree of the enhanced state. The cochain group $C^{i,j}$ is defined to be the real vector space spanned by all enhanced states of dimension $i$ and degree $j$. These notions are illustrated on Figure 1 similar to Bar-Natan’s [1]. Here every square box represents a vector space spanned by all enhanced states with the indicated underlying state. The direct sum of these vector spaces located in the $i$-th column gives the cochain group $C^i = \bigoplus_j C^{i,j}$. The boxes are labeled by strings of 0’s and 1’s which encode the edges participating in the corresponding states. To turn the cochain groups into a cochain complex we define a differential $d^{i,j} : C^{i,j} \rightarrow C^{i+1,j}$. On a vector space corresponding to a given state (box) the differential can be defined as adding an edge to the corresponding state in all possible ways, and then coloring the connected components of the obtained state according to the following rule. Suppose we have an enhanced state $S$ with an underlying state $s$ and we are adding an edge $e$. Then, if the number of connected components is not changed, we preserve the same coloring of connected components of the new state $s \cup \{e\}$. If $e$ connects two different connected components of $s$, then the color of the new component of $s \cup \{e\}$ is defined by the multiplication

$$1 \times 1 := 1, \quad 1 \times x := x, \quad x \times 1 := x, \quad x \times x := 0.$$  

In the last case, the enhanced state is mapped to zero. In cases where the number of edges of $s$ whose order is less than that of $e$ is odd, we take the target enhanced state $s \cup \{e\}$ with the coefficient $-1$. These are shown in the picture above by arrows with little circles at their tails. See [5] for more details as well as for a proof of the main property $d^{i+1,j} \circ d^{i,j} = 0$ converting our cochain groups into a bigraded cochain complex $C^{*,*}(\Gamma)$. We call its cohomology groups
the chromatic cohomology of the graph $\Gamma$:

$$H^{i,j}(\Gamma) := \frac{\text{Ker} \left( d : C^{i,j}(\Gamma) \to C^{i+1,j}(\Gamma) \right)}{\text{Im} \left( d : C^{i-1,j}(\Gamma) \to C^{i,j}(\Gamma) \right)}.$$}

The vector space corresponding to a single vertex without edges is isomorphic to the algebra of truncated polynomials $\mathcal{A} := \mathbb{Q}[x]/(x^2)$. Using this algebra we can think about a box space of an arbitrary graph $\Gamma$ as a tensor power of the algebra $\mathcal{A}$ whose tensor factors are in one-to-one correspondence with the connected components of the state. Then our multiplication rule for the differential turns out to be the multiplication operation $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ in the algebra $\mathcal{A}$. This approach allows to generalize the definition of the chromatic cohomology to an arbitrary algebra $\mathcal{A}$ (see \[6, 11\] for further development of this approach). In particular, the cohomology of $\Gamma$ with respect to the differential $\Phi + d$ that we will study later can be understood as the cohomology associated with the algebra $\mathcal{C} = \mathbb{Q}[x]/(x^2 - 1)$.

The following facts are known (see \[5, 7\])
If $P_{\Gamma}(\lambda)$ denotes the chromatic polynomial of $\Gamma$ then
\[ P_{\Gamma}(1 + q) = \sum_{i,j} (-1)^i q^j \dim(H^{i,j}(\Gamma)) . \]

For a graph $\Gamma$, having an edge $e$, let $\Gamma - e$ and $\Gamma/e$ denote the graphs obtained from $\Gamma$ by deletion and contraction of $e$, respectively. Then, there exists a long exact sequence (for any $j$)
\[ 0 \rightarrow H^{0,j}(\Gamma) \rightarrow H^{0,j}(\Gamma - e) \rightarrow H^{0,j}(\Gamma/e) \rightarrow H^{1,j}(\Gamma) \rightarrow \ldots \]

For a graph $\Gamma$ with $n \geq 2$ vertices,
\[ H^{i,j}(\Gamma) = 0 \text{ if } i > n - 2. \]

For a graph $\Gamma$ with $n$ vertices and $k$ connected components the cohomologies are concentrated on $k + 1$ diagonals:
\[ H^{i,j}(\Gamma) = 0 \text{ unless } n - k \leq i + j \leq n. \]

For a loopless connected graph $\Gamma$ with $n$ vertices only the following 0-cohomologies are nontrivial:
\[ H^{0,n}(\Gamma) \cong H^{0,n-1}(\Gamma) \cong \mathbb{Q} \text{ for bipartite graphs, and } H^{0,n}(\Gamma) \cong \mathbb{Q} \text{ for non-bipartite graphs.} \]

2 The differentials $\Phi$ and $\Phi + d$

Definition 2.1. Define a map $\Phi : C^i(\Gamma) \rightarrow C^{i+1}(\Gamma)$ in the same way as $d$ except using the algebra $B$:
\[ 1 \times \Phi 1 = 0, 1 \times \Phi x = 0, x \times \Phi 1 = 0, x \times \Phi x = 1. \]

Note that the algebra $B$ is not unital.

Proposition 2.2. The map $\Phi$ is a differential, i.e. $\Phi^2 = 0$.

This is a consequence of the algebra structure on $B$. See [8,11] for details.

Proposition 2.3. The cochain bicomplex $C^{*,*}(\Gamma)$ with differentials $d$ and $\Phi$ is independent of the ordering of edges.

Proof. The proof is the same as that of [5, Theorem 14], where an isomorphism $f$ between chain complexes with different edge orderings was constructed. The isomorphism $f$ also commutes with $\Phi$. \qed

Proposition 2.4. $\Phi \circ d + d \circ \Phi = 0$.

Proof. The map $\Phi + d : C^i(\Gamma) \rightarrow C^{i+1}(\Gamma)$ can be described in the same way as $d$ except using the algebra $C = \mathbb{Q}[x]/(x^2 - 1)$:
\[ 1 \times_{\Phi + d} 1 = 1, \quad 1 \times_{\Phi + d} x = x, \quad x \times_{\Phi + d} 1 = x, \quad x \times_{\Phi + d} x = 1. \]

Of course, $C$ is isomorphic to $\mathbb{Q}^2$ as a vector space. Therefore $\Phi + d$ is a differential. But then
\[ 0 = (\Phi + d)^2 = \Phi^2 + (\Phi \circ d + d \circ \Phi) + d^2, \]
and so $\Phi \circ d + d \circ \Phi = 0$. \qed


3 Cohomology of \( \Phi + d \)

Consider the cohomologies \( H^i_{\Phi + d} \) with respect to the differential \( \Phi + d \). These cohomologies are not graded spaces anymore, since \( \Phi + d \) does not preserve the grading. Instead, they have a natural filtration which is preserved by \( \Phi + d \). We will discuss this in more detail at the end of the section.

**Remark 3.1.** The map \( \Phi + d \) commutes with the maps from the short exact sequence of complexes associated with \( \Gamma/e, \Gamma, \) and \( \Gamma - e \), in the same way as \( d \) does. Therefore is a long exact sequence of the new cohomology groups:

\[
0 \to H^0_{\Phi + d}(\Gamma) \to H^0_{\Phi + d}(\Gamma/e) \to H^1_{\Phi + d}(\Gamma/e) \to H^1_{\Phi + d}(\Gamma) \to H^1_{\Phi + d}(\Gamma/e) \to \cdots
\]

As in [5], all cohomology groups of a graph containing a loop are trivial and that multiple edges of a graph can be replaced by single ones without altering the cohomology groups.

**Remark 3.2.** Let \( a_0 = \frac{1}{2}(x + 1), a_1 = \frac{1}{2}(x - 1) \). Then \( \{a_0, a_1\} \) forms a new basis for the algebra \( \mathcal{C} \) with

\[
a_0 \times_{\Phi + d} a_0 = a_0, \quad a_1 \times_{\Phi + d} a_1 = -a_1, \quad a_0 \times_{\Phi + d} a_1 = a_1 \times_{\Phi + d} a_0 = 0.
\]

Under the new basis, the enhanced states \( S \) are now spanning graphs \( s \) whose components are colored by \( a_0 \) and \( a_1 \). If \( \Gamma \) is bipartite, we can split its vertex set \( V(\Gamma) = V_0 \cup V_1 \) into the two parts \( V_0 \) and \( V_1 \). This gives two particular enhanced states in dimension 0: \( S_0 \) (resp. \( S_1 \)) is the coloring on \( V(\Gamma) \) with each vertex in \( V_i \) (resp. \( V_{1-i} \)) colored by \( a_i \). The property that \( a_0 \times_{\Phi + d} a_1 = a_1 \times_{\Phi + d} a_0 = 0 \) immediately implies

**Lemma 3.3.** \( (\Phi + d)(S_0) = (\Phi + d)(S_1) = 0 \), in other words, \( S_0 \) and \( S_1 \) are both cocycles in \( C^0_{\Phi + d}(\Gamma) \).

In fact, \( \{S_0, S_1\} \) forms a basis of \( H^0_{\Phi + d}(\Gamma) \) as the next theorem states.

**Theorem 3.4.** Let \( \Gamma \) be a connected graph.

1. If \( \Gamma \) is not bipartite, then \( H^i_{\Phi + d}(\Gamma) = 0 \) for all \( i \).

2. If \( \Gamma \) is bipartite, then \( H^i_{\Phi + d}(\Gamma) = 0 \) for all \( i > 0 \), \( H^0_{\Phi + d}(\Gamma) \cong \mathbb{Q}^2 \) with basis \( \{S_0, S_1\} \) described above.

**Proof.** We induct on \( m \), the number of edges in \( \Gamma \).

If \( m = 0 \), \( \Gamma \) consists of one vertex and no edges. We have \( H^0(\Gamma) \cong \mathcal{C} \cong \mathbb{Q}^2 \), and \( H^i(\Gamma) = 0 \) for all \( i > 0 \).

Suppose that the theorem is true for all connected graphs with less than \( m \) edges. Let \( \Gamma \) be a connected graph with \( m \) edges. We consider two cases.

Case A. \( \Gamma \) is a tree. Let \( e \) be a pendant edge. By [6 Proposition 3.4], \( H^i_{\Phi + d}(\Gamma) \cong H^i_{\Phi + d}(\Gamma/e) \otimes \mathcal{A}' \) where \( \mathcal{A}' = \langle x \rangle \cong \mathbb{Q} \) is a subspace in \( \mathcal{C} \) spanned by \( x \). The graph \( \Gamma/e \) is a tree with one less edge and therefore we can apply induction. It follows that \( H^0_{\Phi + d}(\Gamma) \cong \mathcal{C} \cong \mathbb{Q}^2 \), and \( H^i_{\Phi + d}(\Gamma) = 0 \) for all \( i > 0 \).
Case B. \( \Gamma \) is not a tree. It must contain an edge \( e \) that is not a bridge. Consider the long exact sequence

\[
0 \rightarrow H_{\Phi^+d}(\Gamma) \rightarrow H_{\Phi^+d}(\Gamma - e) \xrightarrow{\gamma} H_{\Phi^+d}(\Gamma/e) \rightarrow H_{\Phi^+d}(\Gamma) \rightarrow H_{\Phi^+d}(\Gamma - e) = 0
\]

Here the last term \( H_{\Phi^+d}(\Gamma - e) = 0 \) since \( \Gamma - e \) satisfies the induction hypothesis. By Lemma 3.5 below, we have three subcases.

**Subcase 1.** All three graphs \( \Gamma, \Gamma - e, \Gamma/e \) are non-bipartite. In this case, we have \( H^0_{\Phi^+d}(\Gamma - e) = H^0_{\Phi^+d}(\Gamma/e) = 0 \) by the induction hypothesis. Hence \( H^0_{\Phi^+d}(\Gamma) = H^1_{\Phi^+d}(\Gamma) = 0 \).

**Subcase 2.** The graphs \( \Gamma \) and \( \Gamma - e \) are bipartite, while \( \Gamma/e \) is not. In this case, we have \( H^0_{\Phi^+d}(\Gamma/e) = 0, H^0_{\Phi^+d}(\Gamma - e) \cong \mathbb{Q}^2 \). It follows that \( H^0_{\Phi^+d}(\Gamma) \cong \mathbb{Q}^2, H^1_{\Phi^+d}(\Gamma) = 0 \) for all \( i > 0 \).

**Subcase 3.** The graphs \( \Gamma - e \) and \( \Gamma/e \) are bipartite, while \( \Gamma \) is not. In this case, we have \( H^0_{\Phi^+d}(\Gamma - e) \cong H^0_{\Phi^+d}(\Gamma/e) \cong \mathbb{Q}^2 \). The spaces have bases \( \{ S_0(\Gamma - e), S_1(\Gamma - e) \} \) and \( \{ S_0(\Gamma/e), S_1(\Gamma/e) \} \), respectively. The two endpoints of \( e \) in \( \Gamma - e \) must be labeled by the same color, for otherwise \( \Gamma/e \) would not be bipartite. The property \( a_i \times_{\Phi^+d} a_i = \pm a_i \) then implies that the connecting map \( \gamma \) sends \( S_i(\Gamma - e) \) to \( \pm S_i(\Gamma/e) \) for \( i = 0, 1 \). Indeed, \( \gamma \) acts on an enhanced state \( S \) by inserting the edge \( e \) and adjusting the coloring according to the multiplication rule. Therefore \( \gamma \) is an isomorphism. It follows that \( H^0_{\Phi^+d}(\Gamma) = H^1_{\Phi^+d}(\Gamma) = 0 \).

**Lemma 3.5.** Let \( \Gamma \) be a connected graph and let \( e \in E(\Gamma) \) be an edge that is not a bridge. Then the possible bipartiteness of the triple \( \Gamma, \Gamma - e, \Gamma/e \) is shown in the following table (✓ stands for bipartite graphs, while – stands for non-bipartite ones).

| Case | \( \Gamma \) | \( \Gamma - e \) | \( \Gamma/e \) |
|------|---------------|----------------|--------------|
| 1    | –             | –              | –            |
| 2    | ✓             | ✓              | –            |
| 3    | –             | ✓              | ✓            |

**Proof.** By our assumption, the three graphs \( \Gamma, \Gamma - e, \Gamma/e \) are all connected. Thus by definition, \( \Gamma \) (resp. \( \Gamma - e, \Gamma/e \)) is bipartite if and only if \( P_{\Gamma}(2) = 2 \) (resp. \( P_{\Gamma - e}(2) = 2, P_{\Gamma/e}(2) = 2 \)). On the other hand, the contraction-deletion rule says \( P_{\Gamma - e}(2) = P_{\Gamma}(2) + P_{\Gamma/e}(2) \).

If \( \Gamma - e \) is not bipartite, then \( P_{\Gamma - e}(2) = 0 \) which implies that \( P_{\Gamma}(2) = P_{\Gamma/e}(2) = 0 \) and therefore \( \Gamma \) and \( \Gamma/e \) are both non-bipartite. However, this is exactly the first case.

If \( \Gamma - e \) is bipartite, then \( P_{\Gamma - e}(2) = 2 \) which implies that either \( P_{\Gamma}(2) = 2 \) and \( P_{\Gamma/e}(2) = 0 \), or \( P_{\Gamma}(2) = 0 \) and \( P_{\Gamma/e}(2) = 2 \). The first possibility yields case 2 while second possibility yields case 3. \( \blacksquare \)
Filtered Cohomology

The cochain bicomplex $C^* \langle \Gamma \rangle$ has a natural filtration $C^i, \leq j(\Gamma) := \bigoplus_{k=0}^{j} C^i, k(\Gamma)$;

$$C^i, \leq 0(\Gamma) \subseteq C^i, \leq 1(\Gamma) \subseteq C^i, \leq 2(\Gamma) \subseteq \cdots \subseteq C^i, \leq n(\Gamma) = C^i(\Gamma).$$

The differential $\Phi + d$ preserves this filtration, because $d$ has bidegree $(1, 0)$ and $\Phi$ has bidegree $(1, -2)$. So one can talk about the cohomology groups $H_{\Phi+d}(C^* \langle \Gamma \rangle)$. Since for every $j$ there is an embedding of complexes $C^i, \leq j(\Gamma) \subseteq C^i(\Gamma)$, we have the corresponding homomorphism of cohomology groups $H_{\Phi+d}(C^i, \leq j(\Gamma)) \to H_{\Phi+d}(C^i(\Gamma))$. We denote the image of this homomorphism by $H^i, \leq j_{\Phi+d}(\Gamma)$. Thus we have a filtration

$$H^i, \leq 0_{\Phi+d}(\Gamma) \subseteq H^i, \leq 1_{\Phi+d}(\Gamma) \subseteq H^i, \leq 2_{\Phi+d}(\Gamma) \subseteq \cdots \subseteq H^i, \leq n_{\Phi+d}(\Gamma) = H^i_{\Phi+d}(\Gamma).$$

It is easy to see that the basic cocycles $S_0, S_1$ constructed in Remark 3.2 both have degree $n$. So they belong to $C^0, \leq n_{\Phi+d}(\Gamma)$, however their difference $S_0 - S_1$ belongs to $C^0, \leq n-1_{\Phi+d}(\Gamma)$. As a direct consequence of this and Theorem 3.4 we have the following description of the filtered cohomologies.

**Corollary 3.6.** For a connected graph $\Gamma$ with $n$ vertices:

1. If $\Gamma$ is not bipartite, then $H^i, \leq j_{\Phi+d}(\Gamma) = 0$, for all $i$ and $j$;
2. If $\Gamma$ is bipartite, then $H^i, \leq j_{\Phi+d}(\Gamma) = 0$, for $i \geq 1$, and

$$H^0, \leq n_{\Phi+d}(\Gamma) = H^0_{\Phi+d}(\Gamma) \cong \mathbb{Q}^2, \quad H^0, \leq n-1_{\Phi+d}(\Gamma) \cong \mathbb{Q}, \quad H^0, \leq n-2_{\Phi+d}(\Gamma) = 0.$$

4 Knight Move

Since $\Phi$ anticommutes with $d$, it acts (“knight move”) on the $d$-cohomology groups $\Phi: H^{i, j}(\Gamma) \to H^{i+1, j-2}(\Gamma)$.
The following theorem is a version of a standard theorem for spectral sequences in homological algebra.

**Theorem 4.1 (Knight Move).** Let $\Gamma$ be a connected graph with $n$ vertices. Then there is an isomorphism $\varphi$ of the following quotient spaces

$$\varphi : H^{i \leq j}_+ \Phi(\Gamma)/H^{i \leq j-1}_+ \Phi(\Gamma) \cong \frac{\text{Ker} \left( \Phi : H^{i,j}(\Gamma) \to H^{i+1,j-2}(\Gamma) \right)}{\text{Im} \left( \Phi : H^{i-1,j+2}(\Gamma) \to H^{i,j}(\Gamma) \right)}.$$ 

**Proof.** For every bicomplex $\{C_{\ast \ast}(\Gamma), \Phi\}$ there is a standard way to associate a spectral sequence (see [10, p.47], except our indices $i, j$ are different). Our original complex $\{C_{\ast \ast}(\Gamma), \Phi\}$ is the $E_0$ term of the spectral sequence. Our cohomology groups $\{H_{\ast \ast}(\Gamma), \Phi\}$ together with the differential $\Phi : H^{i,j}(\Gamma) \to H^{i+1,j-2}(\Gamma)$ form the term $E_1$. Its cohomology groups, which are on the right-hand side of our isomorphism $\varphi$:

$$E_2^{i,j}(\Gamma) = \frac{\text{Ker} \left( \Phi : H^{i,j}(\Gamma) \to H^{i+1,j-2}(\Gamma) \right)}{\text{Im} \left( \Phi : H^{i-1,j+2}(\Gamma) \to H^{i,j}(\Gamma) \right)}$$

form the term $E_2$. Its differential has bidegree $(1,-4)$. When the cohomology groups are concentrated on two diagonals it is too “long”, so it is zero. Therefore the spectral sequence collapses at the term $E_2$. In other words, $E_{\infty} = E_2$. The standard theorem ([10, Theorem 2.15]) claims that the spectral sequence converges to the bigraded vector space associated with the filtration

$$H^{i \leq 0}_\Phi(\Gamma) \subseteq H^{i \leq 1}_\Phi(\Gamma) \subseteq H^{i \leq 2}_\Phi(\Gamma) \subseteq \cdots \subseteq H^{i \leq n-i}_\Phi(\Gamma) = H^i_\Phi(\Gamma).$$

That is, the spaces $E_2^{i,j} = E_\infty^{i,j}$ are isomorphic to the corresponding quotient spaces of the filtration $H^{i \leq j}_\Phi(\Gamma)/H^{i \leq j-1}_\Phi(\Gamma)$. So our theorem is a direct consequence of this general theorem for spectral sequences.

**Corollary 4.2.** For a connected graph $\Gamma$ with $n$ vertices:

1. If $\Gamma$ is not bipartite, $\Phi : H^{i,n-i}(\Gamma) \to H^{i+1,n-i-2}(\Gamma)$ is an isomorphism for all $i$.
2. If $\Gamma$ is bipartite, $\Phi : H^{i,n-i}(\Gamma) \to H^{i+1,n-i-2}(\Gamma)$ is an isomorphism for all $i \geq 1$; the map $\Phi : H^0,n(\Gamma) \to H^{1,n-2}(\Gamma)$ has one-dimensional kernel.

This is a direct consequence of Corollary \ref{Knight_move} and Theorem \ref{spectral_sequence}.

## 5 Applications

In this section we give three applications of the knight move theorem. The first one is a computation of the 1-dimensional homologies of a connected graph. The second is an expression for the Poincaré Polynomial in terms of the chromatic polynomial. This result shows that the ranks of the homologies carry no new information as compared to the chromatic polynomial. The last theorem shows
that the long exact sequence of homologies splits into a collection of short exact sequences starting with the term $H^1(\Gamma/e)$.

The key observation here is that for a connected graph $\Gamma$ with $n$ vertices,

$$H^{i-\leq j}(\Gamma)/H^{i-\leq j-1}(\Gamma)$$

is almost always 0. In fact, it is only non-zero for bipartite $\Gamma$, for $i = 0$, and for $j = n$ or $n - 1$ (Corollary 3.9). In this particular case, $H^{i-\leq n}_{\phi+d}(\Gamma) \cong \mathbb{Q}^2$, while $H^{0,\leq n-1}_{\phi+d}(\Gamma) \cong \mathbb{Q}$, and $H^{0,\leq n-1}_{\phi+d}(\Gamma)/H^{0,\leq n-1}_{\phi+d}(\Gamma) \cong \mathbb{Q}$. So, according to the knight move theorem, there is an isomorphism almost everywhere between the two diagonals.

5.1 Homologies of dimension 1

Recall that a simple graph is a graph that does not have more than one edge between any two vertices and no edge starts and ends at the same vertex.

Theorem 5.1. Let $\Gamma$ be a connected and simple graph with $n$ vertices and $m$ edges. Then

$$H^{1,n-1}(\Gamma) = \mathbb{Q}^{m-n+1}, \quad H^{1,n-2}(\Gamma) = 0, \quad H^{1,n-2}(\Gamma) \cong \mathbb{Q}, \quad \text{otherwise}$$

Proof. Since the cohomologies are non-zero only on the two diagonals, the only degrees we need to worry about are $n - 1$ and $n - 2$. First, let us find the cohomologies of degree $n - 1$ (which does not require the knight move theorem).

Notice that $d(C^{1,n-1}(\Gamma)) = 0$ since $C^{2,n-1}(\Gamma) = 0$ for a simple graph. Hence

$$\dim(H^{1,n-1}(\Gamma)) = \dim(C^{1,n-1}(\Gamma)) - \dim(d(C^{0,n-1}(\Gamma))) = \dim(C^{1,n-1}(\Gamma)) - (\dim(C^{0,n-1}(\Gamma)) - \dim(H^{0,n-1}(\Gamma))) = \dim(C^{1,n-1}(\Gamma)) - (n - \dim(H^{0,n-1}(\Gamma))) = m - n + \dim(H^{0,n-1}(\Gamma)).$$

According to [4] Theorem 39,

$$H^{0,j}(\Gamma) \cong \mathbb{Q} \quad \text{for} \quad \begin{cases} j = n & \text{if } \Gamma \text{ is bipartite} \\ j = n - 1 & \text{otherwise} \end{cases}$$

Hence, if $\Gamma$ is bipartite, $\dim(H^{1,n-1}(\Gamma)) = m - n + 1$ and if not, then $\dim(H^{1,n-1}(\Gamma)) = m - n$.

Now let us calculate the cohomologies of degree $n - 2$. If $\Gamma$ is not bipartite, then, by the knight move theorem, $\Phi : H^{0,n}(\Gamma) \rightarrow H^{1,n-2}(\Gamma)$ is an isomorphism. But, again using [4] Theorem 39, $H^{0,n}(\Gamma) \cong \mathbb{Q}$. Therefore $H^{1,n-2}(\Gamma) \cong \mathbb{Q}$.

Next consider the case when $\Gamma$ is bipartite. From the knight move theorem, we know that $\ker(\Phi : H^{0,n}(\Gamma) \rightarrow H^{1,n-2}(\Gamma)) \cong \mathbb{Q}$. However, $H^{0,n}(\Gamma)$ is itself isomorphic to $\mathbb{Q}$. So, $\im(\Phi : H^{0,n}(\Gamma) \rightarrow H^{1,n-2}(\Gamma)) = 0$. Applying the knight move to the next step gives us

$$0 = \frac{\ker(\Phi : H^{1,n-2}(\Gamma) \rightarrow H^{2,n-4}(\Gamma))}{\im(\Phi : H^{0,n}(\Gamma) \rightarrow H^{1,n-2}(\Gamma))} \cong H^{1,n-2}(\Gamma),$$

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since \( H^{2,n-4}(\Gamma) \) lies off the two diagonals. So \( \Gamma \) has no 1-dimensional cohomologies in degree \( n-2 \).

5.2 The Poincaré polynomial and the chromatic polynomial

The Poincaré polynomial \( R_\Gamma(t, q) := \sum_{i,j} t^i q^j \dim(H^{i,j}(\Gamma)) \) of a connected graph \( \Gamma \) splits into two homogeneous parts, \( R^n_\Gamma(t, q) \) and \( R^{n-1}_\Gamma(t, q) \), since all cohomologies are concentrated on two diagonals.

**Theorem 5.2.** For a connected graph \( \Gamma \)

\[
R^n_\Gamma = \begin{cases} 
q^2 t R^{n-1}_\Gamma + q^{n-1} t, & \text{if } \Gamma \text{ is bipartite} \\
q^2 R^{n-1}_\Gamma, & \text{otherwise}
\end{cases}
\]

**Proof.** Suppose \( \Gamma \) is not bipartite. Then \( H^{0,n-1}(\Gamma) = 0 \). So the conclusion is just a consequence of Corollary 4.2.

Now suppose \( \Gamma \) is bipartite. Then the situation is similar to the previous case, except

1. The polynomial for the lower diagonal gains a term \( q^{n-1} \) coming from \( H^{0,n-1} \), and
2. The polynomial for the upper diagonal gains a previously unaccounted term \( q^{n} \) which has been mapped to 0 by \( \Phi \). These two complications account for the two additional terms in the expression for \( R^n_\Gamma(t, q) \).

**Remark 5.3.** We know that \( R_\Gamma(t, q) = R^n_\Gamma(t, q) + R^{n-1}_\Gamma(t, q) \).

If \( \Gamma \) is not bipartite, this gives \( R_\Gamma(t, q) = \left(1 + \frac{q^2}{t} \right) R^{n-1}_\Gamma(t, q) \). Plugging in \( t = -1 \) gives \( P_\Gamma(1 + q) = (1 - q^2) R^{n-1}_\Gamma(-1, q) \), or \( \frac{P_\Gamma(1+q)}{(1-q^2)} = R^{n-1}_\Gamma(-1, q) \). However, since each term in \( R^{n-1}_\Gamma(t, q) \) has degree \( n-1 \), knowledge of \( R^{n-1}_\Gamma(-1, q) \) is sufficient to fully determine \( R^{n-1}_\Gamma(t, q) \). Therefore \( R_\Gamma(t, q) \) is determined by \( P_\Gamma(1+q) \).

If \( \Gamma \) is bipartite, this gives

\[
\frac{P_\Gamma(1+q) - q^n - q^{n+1}}{(1-q^2)} = R^{n-1}_\Gamma(-1, q).
\]

This also means that \( R_\Gamma(t, q) \) is determined by \( P_\Gamma(1+q) \).

**Corollary 5.4.** For a connected graph \( \Gamma \)

\[
R_\Gamma(t, q) = \begin{cases} 
(-1)^{n-1} t^n + q^2 \frac{t + q^2}{t^2 - q^2} P_\Gamma \left( \frac{t - q}{t} \right) + q^n (t+1) \frac{t}{t+q}, & \text{if } \Gamma \text{ is bipartite} \\
(-1)^{n-1} t^n + q^2 \frac{t + q^2}{t^2 - q^2} P_\Gamma \left( \frac{t - q}{t} \right), & \text{otherwise}
\end{cases}
\]

**Proof.** This is a direct result of the calculation described in the remark.
5.3 Deletion-contraction formula for the Poincaré polynomial

Using the bipartiteness table of the triple $\Gamma, \Gamma - e, \Gamma/e$ from Lemma 5.5 we get the following theorem.

**Theorem 5.5.** Let $\Gamma$ be a simple, connected graph with $n$ vertices, and let $e$ be an edge that is not a bridge. Then in cases 1 and 2 (when $\Gamma/e$ is not bipartite)

$$R_{\Gamma}(t, q) = R_{\Gamma - e}(t, q) + tR_{\Gamma/e}(t, q),$$

while in case 3 (when $\Gamma$ is not bipartite but $\Gamma - e$ and $\Gamma/e$ are)

$$R_{\Gamma}(t, q) = R_{\Gamma - e}(t, q) + tR_{\Gamma/e}(t, q) - q^{n-1}(t+1).$$

The theorem follows from Corollary 5.4 and the deletion-contraction formula for the chromatic polynomial $P_{\Gamma}(\lambda)$.

It implies the following relation between the dimensions of the homology spaces

$$\dim(H^{i,j}(\Gamma)) = \dim(H^{i,j}(\Gamma - e)) + \dim(H^{i-1,j}(\Gamma/e))$$

for all $i$ and $j$ in cases 1 and 2, and for $(i, j) \neq (0, n - 1)$ or $(1, n - 1)$ in case 3. For the exceptional values of $(i, j)$ in case 3 we have

| $(i, j)$ | $\dim(H^{i,j}(\Gamma))$ | $\dim(H^{i,j}(\Gamma - e))$ | $\dim(H^{i-1,j}(\Gamma/e))$ |
|----------|-------------------------|-----------------------------|-----------------------------|
| $(0, n - 1)$ | 0                      | 1                           | 0                           |
| $(1, n - 1)$ | $m - n$                | $m - n$                     | 1                           |

where $m$ is the number of edges of $\Gamma$.

This relation between the dimensions gives the following splitting of the long exact sequence into short ones.

**Proposition 5.6.** Let $\Gamma$ be a simple, connected graph with $n$ vertices, and let $e$ be an edge that is not a bridge. Then the connecting homomorphisms $\varphi$ in the long exact sequence

$$0 \to H^{0,j}(\Gamma) \to H^{0,j}(\Gamma - e) \xrightarrow{\varphi} H^{0,j}(\Gamma/e) \to H^{1,j}(\Gamma) \to H^{1,j}(\Gamma - e) \xrightarrow{\varphi} H^{1,j}(\Gamma/e) \to \cdots$$

are identically 0 unless $i = 0, j = n - 1$, $\Gamma$ is not bipartite, while $\Gamma - e$ and $\Gamma/e$ are bipartite.

In the exceptional case where $\Gamma$ is not bipartite, while $\Gamma - e$ and $\Gamma/e$ are bipartite the connection map $\varphi: H^{0,n-1}(\Gamma - e) \to H^{0,n-1}(\Gamma/e)$ is an isomorphism of one-dimensional vector spaces.
References

[1] D. Bar-Natan, On Khovanov’s categorification of the Jones polynomial, Algebraic and Geometric Topology 2 (2002) 337-370. http://www.maths.warwick.ac.uk/agt/AGTVol2/agt-2-16.abs.html

[2] D. Bar-Natan, S. Morrison, The Karoubi Envelope and Lee’s Degeneration of Khovanov Homology. Preprint math.GT/0606542

[3] A. Champanerkar, I. Kofman, Spanning trees and Khovanov homology. Preprint math.GT/0607510

[4] L. Helme-Guizon, A Categorification for the Chromatic Polynomial. Dissertation, George Washington University.

[5] L. Helme-Guizon, Y. Rong, A categorification for the chromatic polynomial, Algebraic and Geometric Topology 5, 1365-1388 (2005). http://www.maths.warwick.ac.uk/agt/AGTVol5/agt-5-53.abs.html

[6] L. Helme-Guizon, Y. Rong, Graph Cohomologies from Arbitrary Algebras. Preprint math.QA/0506023

[7] L. Helme-Guizon, J. Przytycki, Y. Rong, Torsion in Graph Homology. Preprint math.GT/0507245

[8] M. Khovanov, A categorification of the Jones polynomial, Duke Mathematical Journal 101 (2000) 359–426. Preprint math.QA/9908171

[9] Eun Soo Lee, An endomorphism of the Khovanov invariant. Preprint math.GT/0210213

[10] J. McCleary, A User’s Guide to Spectral Sequences, 2-d edition, Cambridge University Press, 2001.

[11] J. Przytycki, When the theories meet: Khovanov homology as Hochschild homology of links. Preprint math.GT/0509334

[12] S. Wehrli, A spanning tree model for Khovanov homology. Preprint math.GT/0409328

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