On the core-halo distribution of dark matter in galaxies

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\begin{abstract}
We investigate the distribution of dark matter in galaxies by solving the equations of equilibrium of a self-gravitating system of massive fermions (‘inos’) at selected temperatures and degeneracy parameters within general relativity. Our most general solutions show, as a function of the radius, a segregation of three physical regimes: 1) an inner core of almost constant density governed by degenerate quantum statistics; 2) an intermediate region with a sharply decreasing density distribution followed by an extended plateau, implying quantum corrections; 3) an asymptotic, $\rho \propto r^{-2}$ classical Boltzmann regime fulfilling, as an eigenvalue problem, a fixed value of the flat rotation curves. This eigenvalue problem determines, for each value of the central degeneracy parameter, the mass of the ino as well as the radius and mass of the inner quantum core. Consequences of this alternative approach to the central and halo regions of galaxies, ranging from dwarf to big spirals, for SgrA*, as well as for the existing estimates of the ino mass, are outlined.

\textbf{Keywords:} Dark matter, Self-gravitating fermions, Galaxies: halos, Galaxies: nuclei
\end{abstract}

1. Introduction

The problem of identifying the masses and the fundamental interactions of the dark matter particles is currently one of the most fundamental issues in physics and astrophysics. The first astrophysical and cosmological constraints on the mass of the dark matter particle appeared in \cite{20, 21, 22, 11, 23, 24}. As we will show, some inferences on the dark matter particle mass can be derived from general considerations based solely on quantum statistics and gravitational interactions on galaxy scales.

An important open issue in astrophysics is the description of the dark matter in terms of collisionless massive particles. Attempts have been presented to put constraints on its phase-space density by knowing its evolution from the cosmological decoupling until the approximate time of virialization of a dark matter halo. Phenomenological attempts have been proposed in the past in terms of Maxwellian-like, Fermi-Dirac-like or Bose-Einstein-like distribution functions. Since the 80’s all the way up to the present, the problem of modeling the distribution of dark matter in terms of self-gravitating quantum particles has been extensively studied and contrasted against galactic observables. In \cite{6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22}, and references therein, this problem was studied by considering Fermi-Dirac statistics in different regimes, from the fully degenerate to the dilute one, and for different fermion masses going from few eV to keV. Instead, in \cite{20, 21, 22, 11, 23, 24} the same problem was analyzed in terms of Bose-Einstein condensates with particle masses from $10^{-25}$ eV up to few eV.

It is our opinion that in the fermionic case, a clear differentiation of a quantum degenerate core and an almost classical halo, has never been properly implemented. In particular it has been neglected the crucial role of comparing and contrasting different configurations, for fixed halo boundary conditions. As we will show, this leads to a very specific eigenvalue problem for the mass of the inos.

In this Letter we formulate the general problem of the dark matter distribution in galaxies based in the following assumptions: 1) that the dark matter phase-space density is described Fermi-Dirac quantum statistics; 2) that the equilibrium equations for the configurations be solved, for completeness, within a general relativistic treatment; 3) we set the boundary condition for all dark matter profiles associated with a specific galaxy type (dwarfs, spirals, and big spirals), to have, in each case, the same value of the flat rotation curve. Having established this procedure in section 2, we evidence in section 3: i) the new core-halo distribution of dark matter density, which is composed by a dense compact core governed by almost degenerate quantum statistics; ii) a semi-degenerate transition, followed by a dilute halo governed by Boltzmann classic statistics; iii) for each central degeneracy parameter we determine as an eigenvalue problem, the mass and radius of the inner quantum core, as well as the corresponding ino mass; and iii) we model a theoretical correlation between the inner quantum core mass and the halo mass, for galaxy types from dwarf up to big spirals. From these considerations clearly follows that the determination of the ino mass is uniquely

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established by the properties of the inner quantum core and the asymptotic boundary conditions, and it cannot be determined in a dark matter distribution governed only by a Boltzmannian distribution, which is independent of the mass of the ino. In section 4 we summarize and discuss our results.

2. Equilibrium equations and boundary conditions

Following [23, 15], we here consider a system of general relativistic self-gravitating bare massive fermions in thermodynamic equilibrium. No additional interactions are initially assumed for the fermions besides their fulfillment of quantum statistics and the relativistic gravitational equations. In particular, we do not assume weakly interacting particles as in [5]. We refer to this bare particles more generally as inos, leaving the possibility of additional fundamental interactions to be determined by further requirements to be fulfilled by the model. Already this treatment of bare fermions leads to a new class of equilibrium configurations and, correspondingly, to new limits to the ino mass. This is a necessary first step in view of a final treatment involving additional interactions to be treated self-consistently, as we will soon indicate.

The density and pressure of the fermion system are given by

$$\rho = \frac{m}{3\pi^2} \int f(p) \left[1 + \frac{\epsilon(p)}{mc^2}\right] d^3p, \quad (1)$$

$$P = \frac{1}{3\pi^2} \int f(p) \left[1 + \frac{\epsilon(p)}{mc^2}\right]^{-1} \left[1 + \frac{\epsilon(p)}{2mc^2}\right] \epsilon d^3p, \quad (2)$$

where the integration is over all the momentum space, $f_p = (\exp[(\epsilon - \mu)/(kT)] + 1)^{-1}$ is the distribution function, $\epsilon = \sqrt{\epsilon^2p^2 + m^2c^4} - mc^2$ is the particle kinetic energy, $\mu$ is the chemical potential, and $k$ is the Boltzmann constant. $h$ is Planck’s constant, $c$ is the speed of light, and $m$ is the ino particle mass. We do not include the presence of anti-fermions, i.e. we consider temperatures $T < mc^2/k$.

The Einstein equations for the spherically symmetric metric $g_{\mu\nu} = \text{diag}(\epsilon^v, -\epsilon^\lambda, -r^2, -r^2\sin^2\theta)$, where $\nu$ and $\lambda$ depend only on the radial coordinate $r$, together with the thermodynamic equilibrium conditions of Tolman [20], $\epsilon^{\nu/2}T = \text{constant}$, and Klein [27], $\epsilon^{\nu/2}(\mu + mc^2) = \text{constant}$, can be written as

$$\frac{d\dot{M}}{dr} = 4\pi r^2 \dot{\rho}, \quad (3)$$

$$\frac{d\dot{\theta}}{dr} = -\frac{1 - \beta_0(\theta - \theta_0)}{\beta_0} \dot{M} + 4\pi \dot{P}^{\beta_0^3} \frac{1}{r^2(1 - 2\dot{M}/\dot{r})}, \quad (4)$$

$$\frac{dv}{dr} = \frac{2(M + 4\pi \dot{P}^{\beta_0^3})}{r^2(1 - 2\dot{M}/\dot{r})}, \quad (5)$$

$$\dot{\beta}_0 = \beta(r) e^{-\nu(r)/\mu(r)}. \quad (6)$$

The following dimensionless quantities were introduced: $\dot{r} = r/\chi, M = GM/(c^2\chi), \dot{\rho} = G\chi^2p/c^2, \dot{P} = G\chi^2P/c^4$, where $\chi = 2\pi^{3/2}(h/m\epsilon)(m_p/m)$, with $m_p = \sqrt{\hbar/c}$ the Planck mass, and the temperature and degeneracy parameters, $\beta = kT/(mc^2)$ and $\theta = \mu/(kT)$, respectively. The constants of the Tolman and Klein conditions are evaluated at the center $r = 0$, indicated with a subscript ‘0’.

The system variables are $[M(r), \theta(r), \beta(r), v(r)]$. We integrate Eqs. (3–6) for given initial conditions at the center, $r = 0$, in order to be consistent with the observed dark matter halo mass $M(r = r_h) = M_h$ and radius $r_h$, defined in our model at the onset of the flat rotation curves. The circular velocity is

$$v(r) = \sqrt{\frac{GM(r)}{r - 2GM(r)/c^2}}, \quad (7)$$

which at $r = r_h$, is $v(r = r_h) = v_h$.

It is interesting that a very similar set of equations have been re-derived in 2002 in [4] apparently disregarding the theoretical approach already implemented in 1990 in [23]. They integrated the Einstein equations fixing a fiducial mass of the ino of $m = 15 \text{ keV}/c^2$, and they derived a family of density profiles for different values of the central degeneracy parameter at a fixed temperature consistent with an asymptotic circular velocity $v_\infty = 220 \text{ km/s}$. They conclude that a self-gravitating system of such inos could offer an alternative to the interpretation of the massive black hole in the core of SgrA* [28]. Although this result was possible at that time, it has been superseded by new constraints imposed by further observational limits on the trajectory of S-stars such as S1 and S2 [28, 29].

In this Letter we give special attention to the halo boundary conditions determined through the flat rotation curves. We integrate our system of equations using different boundary conditions to the ones imposed in [4] and reaching different conclusions. We first apply this model to typical spiral galaxies, similar to our own galaxy, adopting dark matter halo parameters [30, 31].

$$r_h = 25 \text{ kpc}, \quad v_h = 168 \text{ km/s}, \quad M_h = 1.6 \times 10^{11} M_\odot. \quad (8)$$

The initial conditions are $M(0) = 0, v(0) = 0, \theta(0) = \theta_0$ and $\beta(0) = \beta_0$. We integrate Eqs. (3–6) for selected values of $\theta_0$ and $m$, corresponding to different degenerate states of the gas at the center of the configuration. The value of $\beta_0$ is actually an eigenvalue which is found by a trial and error procedure until the observed values of $v_h$ and $M_h$ at $r_h$ are obtained. We show in Fig. 1 the density profiles and the rotation curves as a function of the distance for a wide range of parameters $(\theta_0, m)$, for which the boundary conditions in (8) are exactly fulfilled.

3. Dark matter profiles: from dwarf to big spiral galaxies

The phase-space distribution encompasses both the classical and quantum regimes. Correspondingly, the integration of the equilibrium equations leads to three marked
different regimes (see Fig. 1): a) the first consisting in a core of quantum degenerate fermions. These cores are characterized by having $\theta(r) > 0$. The core radius $r_c$ is defined by the first maximum of the velocity curve. A necessary condition for the validity of this quantum treatment of the core is that the interparticle mean-distance, $\bar{r}$, be smaller or of the same order, of the thermal de Broglie wavelength of the inos, $\lambda_B = h/\sqrt{2\pi m kT}$. As we show below (see Fig. 2), this indeed is fulfilled in all the cases here studied. b) A second regime where $\theta(r)$ goes from positive to negative values for $r > r_c$, all the way up to the so called classical domain where the quantum corrections become negligible. This transition region consists in a sharply decreasing density followed by an extended plateau. c) The classical regime described by Boltzmann statistics and corresponding with $\theta(r) \ll -1$ (for $r \gtrsim r_h$), in which the solution tends to the Newtonian isothermal sphere with $\rho \sim r^{-2}$, where the flat rotation curve sets in.

We define the core mass, the circular velocity at $r_c$, and the core degeneracy as $M_c = M(r_c)$, $v_c = v(r_c)$ and $\theta_c = \theta(r_c)$, respectively. In Table 1 we show the core properties of the equilibrium configurations in spiral galaxies, for a wide range of $(\theta_0, m)$. For any selected value of $\theta_0$ we obtain the correspondent ino mass $m$ to fulfill the halo properties [6], after the above eigenvalue problem of $\beta_0$ is solved.

It is clear from Table 1 and Fig. 1 that the mass of the core $M_c$ is strongly dependent on the ino mass, and that the maximum space-density in the core is considerably larger than the maximum value considered in [5] for a Maxwellian distribution. Interestingly, as can be seen from Fig. 1 the less degenerate quantum cores in agreement with the halo observables [6], are the ones with the largest sizes, of the order of halo-distance-scales. In this limit, the fermion mass acquires a sub-keV minimum value which is larger, but comparable, than the corresponding sub-keV bound in [6], for the same halo observables. Indeed, their formula gives a lower limit $m \approx 0.05\text{ keV/}c^2$ using the proper value for the King radius, $r_K \approx 8.5\text{ kpc}$, for $\sigma = \sqrt{2/5} \nu_h$ and $\rho_0 = 2.5 \times 10^{-2} \rho_0/\text{pc}^3$, associated to the Boltzmann density profile in Fig. 1.

In the case of a typical spiral galaxy, for an ino mass of $m \approx 10\text{ keV/}c^2$, and a temperature parameter $\beta_0 \approx 10^{-7}$, obtained from the observed halo rotation velocity $v_0$, the de Broglie wavelength $\lambda_B$ is higher than the interparticle mean-distance in the core $\bar{r}$, see Fig. 2, safely justifying the quantum-statistical treatment applied here.
neutron stars, where nuclear fermion interactions strongly influence the mass-radius relation (see, e.g., [32]). This may well make the mass and radius of this dark matter quantum core to fulfill the observational constraints imposed by the S2 star [33].

We further compare and contrast in Fig. 3 our theoretical curves in Fig. 1 with observationally inferred ones. It is interesting that the quantum and relativistic treatment of the configurations considered here are characterized by the presence of central cored structures unlike the typical cuspy configurations obtained from a classic non-relativistic approximation such as the ones of numerical N-body simulations in [34]. This naturally leads to a solution to the well-known core-cusp discrepancy [35]. Such a difference between the ino’s core and the cuspy NFW profile, as well as the possible black hole nature of the compact source in SgrA*, will certainly reactivate the development of observational campaigns in the near future. There the interesting possibility, in view of the BlackHole-Cam Project based on the largest Very Long Baseline Interferometry (VLBI) array [4] to verify the general relativistic effects expected in the surroundings of the central compact source in SgrA*. Such effects depend on whether the source is modeled in terms of the RAR model presented here (with the possible inclusion of fermion interactions [32]), or as a black hole. To compare and contrast these two alternatives is an observational challenge now clearly open.

Following the analysis developed here for a typical spiral, we have also considered two new different sets of physical dark matter halos: \( r_h = 0.6 \) kpc; \( v_h = 13 \) km/s; \( M_h = 2 \times 10^7 M_\odot \) for typical dwarf spheroidal galaxies, e.g. [39]; and \( r_h = 75 \) kpc; \( v_h = 345 \) km/s; \( M_h = 2 \times 10^{12} M_\odot \) for big spiral galaxies, as analyzed in [40]. For big spirals, \( \lambda_B/l_c = 5.3 \), while for typical dwarfs galaxies \( \lambda_B/l_c = 4.1 \), justifying the quantum treatment in both cases.

A remarkable outcome of the application of our model to this type is that for the same ino mass, \( m \sim 10 \) keV/\( c^2 \), we obtain respectively core masses \( M_c \sim 10^4 M_\odot \) and radii \( r_c \sim 10^{-1} \)pc for dwarf galaxies, and core masses \( M_c \sim 10^7 M_\odot \) and radii \( r_c \sim 10^{-2} \)pc for big spirals. This leads to a possible alternative to intermediate \( (\sim 10^4 M_\odot) \) and more massive \( (\sim 10^7 M_\odot) \) black halos. This is thus a possible alternative to intermediate \( (\sim 10^4 M_\odot) \) and more massive \( (\sim 10^7 M_\odot) \) black halos.
holes, thought to be hosted at the center of the galaxies. We have obtained, out of first principles, a possible universal relation between the dark matter halos and the super massive dark central objects. For a fixed ino mass \( m = 10 \, \text{keV/c}^2 \), we found the \( M_c-M_h \) correlation law

\[
\frac{M_c}{10^6 M_\odot} = 2.35 \left( \frac{M_h}{10^7 M_\odot} \right)^{0.52},
\]

valid for core masses \( \sim [10^4, 10^7] M_\odot \) (corresponding to dark matter halo masses \( \sim [10^7, 10^{12}] M_\odot \)). Regarding the observational relation between massive dark compact objects and bulge dispersion velocities in galaxies (the \( M_c-\sigma \) correlation [41]), it can be combined with two observationally inferred relations such as the \( \sigma-V_c \) and the \( V_c-M_h \) correlations, where \( V_c \) is the observed halo circular velocity and \( M_h \) a typical halo mass. This was done in [12] to find, by transitivity, a new correlation between central mass concentrations and halo dark masses (\( M_c-M_h \)). Interestingly, such a correlation matches the one found above in Eq. (9) in the range \( M_c = [10^6, 10^7] M_\odot \), without assuming the black hole hypothesis. The appearance of a core surrounded by a non-relativistic halo, is a key feature of the configurations presented in this Letter. It cannot however be extended to quantum cores with masses of \( \sim 10^6 M_\odot \). Such core masses, observed in Active Galactic Nuclei (AGN), overcome the critical mass value for gravitational collapse \( M_{cr} \sim M_{pl}^3/m^2 \) for keV-fermions, and therefore these cores have to be necessarily black holes [18]. The characteristic signatures of such supermassive black-holes, including jets and X-ray emissions, are indeed missing from the observations of the much quiet SgrA* source, or the centers of dwarf galaxies.

4. Conclusions

A consistent treatment of self-gravitating fermions within general relativity has been here introduced and solved with standard boundary conditions appropriate to flat rotation curves observed in galactic halos of spiral and dwarf galaxies. A new structure has been identified: 1) a core governed by quantum statistics; 2) a velocity of rotation at the surface of this core which is bounded independently of the mass of the particle and remarkably close to the asymptotic rotation curve; 3) a semi-degenerate region leading to an asymptotic regime described by a pure Boltzmann distribution, consistent with the flat rotation curves observed in galaxies.

For \( m \sim 10 \, \text{keV/c}^2 \) a universal relation between the mass of the core \( M_c \) and the mass of the halo \( M_h \) has been found. This universal relation applies in a vast region of galactic systems, ranging from dwarf to big spiral galaxies with core masses \( \sim [10^4, 10^7] M_\odot \) (corresponding to dark matter halo masses \( \sim [10^7, 10^{12}] M_\odot \)).

Starting from the basic treatment here introduced, of bare self-gravitating fermions, we are currently examining the possibility to introduce new types of interactions among the inos, considering, for example, right-handed sterile neutrinos in the minimal standard model extension (see e.g. [13]), as a viable candidate for the ino particles in our new scenario. The relevance of self-interactions in ultracold atomic collisions has been already shown in laboratory, for example, for (effective) Fermi gases, e.g. \( ^6\text{Li} \), at temperatures of fractions of the Fermi energy. These systems can be studied in terms of a grand-canonical many-body Hamiltonian in second quantization, with a term accounting for fermion-fermion interaction [14], similarly as done in [53]. This is expected to verify the possibility of the radius of the quantum core to become consistent with the observations of SgrA* [28, 29], and so open the way to identify additional fundamental interactions in the ino physics.

After this generalized treatment, we will further address the issue of the implications of these kev-fermions in cosmology.

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