Perturbation technique for heterogeneously delay-coupled oscillators with a gap

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Heterogeneous delays with a gap are taken into consideration in Kuramoto oscillators. Employing the method of multiple scales, normal forms and detailed dynamical behavior are investigated and a hysteresis loop is found near the subcritically bifurcated coherent state on the Ott-Antonsen’s Manifold. For Gamma distributed delay with fixed variance and mean, we find large gap or large excess kurtosis of delays destroys the loop and significantly increases in the number of coexisted coherent attractors.

Keywords: Kuramoto model; distributed delay with gap; bifurcation; normal form; synchronization

Investigating Kuramoto systems is a fundamental way to understand many nonlinear phenomena involving oscillations, where time lag usually arises during transmitting, responding or both. People usually consider the delays come from certain distribution, such as Gamma distribution on $[0, +\infty)$ for convenience. However, even for fixed expectation and variance, the case sensitively depends on $\tau_0$ when delay distributes on $[\tau_0, +\infty)$. In this paper, on the Ott-Antonsen’s manifold, using the method of multiple scales, a relatively simple approach is given to determine direction and stability of the bifurcated coherent states. With respect to Gamma distributed delay, we verify the results, discuss the effect of the gap $\tau_0$, and conclude that larger gap or larger excess kurtosis of delays not only increases in the number of coherent attractors, but leads to supercritical bifurcation hence avoids the hysteresis loop. We believe these results are of more general to explain some complicated coexistence phenomena.

I. INTRODUCTION

The Kuramoto phase oscillators were used to model diverse situations involving large community of oscillators, where the state of every oscillator is determined by a phase on the unit circle. This model captures essential features of synchronization, observed in many physical models ranging from biology, neural science, lasers, engineering to superconducting Josephson junctions. Kuramoto extended Winfree’s mean-field idea, and confirmed that one population of weakly nearly-identical coupled oscillators could be depicted as a universal model

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, 2, \ldots, N$$

Here the frequencies $\omega_i$ follow some distribution with probability density function (PDF) $g(\omega)$.

In most situations, signal’s transmission and receive both lead to time delays. Thus considering time lag is necessary in many coupled systems. Due to the heterogeneity or the spatio-distribution of oscillators, time delays among oscillators may be heterogenous. They may follow some probability distributions such as Gamma distribution, because the lag can be viewed as a period of awaiting. Lee et al., using the Ott-Antonsen’s manifold reduction method, found that the variation of delay could greatly alter the dynamical behavior. This paper offered a framework for studying the delay heterogeneity, where the results are illustrated with respect to Gamma-distributed time lags in $[0, +\infty)$. After some simulations, both supercritical and subcritical Hopf bifurcations on the mean-field are observed.

On one hand, in mathematical consideration, the mechanism causing the above phenomena is not quite clear yet, which depends on further bifurcation analysis. On the other, sometimes time lag distributes in the interval $[\tau_0, +\infty)$ with a minimal responding time $\tau_0 > 0$. It may be the sum of a Gamma distributed delay $\tau \sim \Gamma(n, \frac{T}{\tau_0})$ and a constant $\tau_0$ (See FIG. I(a)), for example. In this case, by varying $n$, one can still fix the expectation of total delay $\langle \tau \rangle = \langle \tau + \tau_0 \rangle = T + \tau_0$, and its variance $\text{Var}(\tau) = \frac{T^2}{2}$ despite the ratio $\frac{T}{\tau_0}$ varies (See FIG. I(b)). Thus the gap $\tau_0$ may have certain effect on the system dynamics without changing $\langle \tau \rangle$ and $\text{Var}(\tau)$. In this case higher order moment of the data should be considered such as the excess kurtosis, which involves the fourth-order moment and is usually used to measure the “peakedness” of the probability distribution. So far as we know, these two points of view are new and have not been well studied.
Motivated by such two considerations and these pioneer works, we are about to consider a more realistic case: heterogeneous delays with nonnegative minimal delay in a system, or a gap, we say. Now the Kuramoto model reads

\[ \theta_i = \omega_i + \frac{k}{N} \sum_{j=1}^{N} \sin[\theta_j(t - \tau_i) - \theta_i(t)] \]  

(1)

\(i = 1, 2, \ldots, N\), where \(\theta_i(t) \in [0, 2\pi]\) is the phase of the \(i\)th oscillator and \(\omega_i\) is its natural frequency coming from \(\omega, \theta, t\) having PDF \(h(\tau)\) and \(\tau_0\) a constant. Thus \(\tau_i\) can be viewed as following a new distribution with PDF \(h(\tau - \tau_0), \tau \in [\tau_0, +\infty)\).

In this paper, we will give some bifurcation results to system (1). After reducing it onto the Ott-Antonsen’s manifold, a delay differential equation is obtained, in which a Hopf bifurcation at the trivial solution means the coherent state is bifurcating from the incoherent state. In our earlier work, the center manifold reduction method is employed to investigate this bifurcation. However, the calculations depend on a rather complicated decomposition of a Banach space. Here, we make this approach easier, and use the method of multiple scales to give a relatively simple calculation process of the normal forms, by which the direction and the stability of the bifurcated coherent state are determined. For Gamma distributed delay, we calculate all bifurcation points in certain parameter spaces and discuss the effect of the gap. Finally, it is found that, when fixing \(\langle \tau \rangle\) and \(\text{Var}(\tau)\), larger gap (or larger excess kurtosis, equivalently) not only leads to a supercritical bifurcation hence avoids the existence of hysteresis loop, but also significantly increases in the number of coexisted coherent states.

II. REDUCTION

As \(N \to \infty\), the continuity equation of (1) is

\[ \frac{\partial}{\partial t} f + \frac{\partial}{\partial \theta} (\xi f) = 0 \]  

(2)

with a drift term \(\xi(\theta, t) = \omega + \frac{k}{2\pi} (e^{-i\theta}r - e^{-i\theta}r^*)\). The complex-valued “order-parameter” \(r(t)\) is defined by

\[ r(t) = \langle \xi(t - \tau) \rangle = \int_{\tau_0}^{+\infty} \xi(t - \tau) h(\tau - \tau_0) d\tau \]  

(3)

with

\[ \xi(t) = \int_{-\infty}^{+\infty} 2\pi f(\omega, \theta, t)e^{i\theta} d\omega \]  

(4)

The distribution density \(f(\omega, \theta, t)\) characterizes the state of the oscillators’ system at time \(t\) in frequency \(\omega\) and phase \(\theta\).

Now we are about to restate some results about the Ott-Antonsen’s reduction of a system with distributed delay first derived by Lee et al. Rewriting system (2) as

\[ \frac{\partial}{\partial t} f + \frac{\partial}{\partial \theta} \left( \left[ \omega + \frac{k}{2\pi} (e^{-i\theta}r - e^{-i\theta}r^*) \right] f \right) = 0 \]  

(5)

and restricting this partial differential equation on the Ott-Antonsen manifold

\[ \left\{ f : f = \frac{g(\omega)}{2\pi} \left( 1 + \sum_{m=1}^{\infty} \alpha^m(\omega, t)e^{im\theta} + \text{c.c.} \right) \right\} \]

with c.c. the complex conjugate of the formal terms, we substitute the Fourier series of \(f\) into (5). After comparing the coefficient of the same harmonic terms, a reduced equation is obtained

\[ \dot{\alpha}(\omega, t) = -i\omega \alpha(\omega, t) + \frac{k}{2} r^* - \frac{k}{2} r^2 \alpha^2(\omega, t) \]  

(6)

Obviously, from (1) we have \(\xi(t) = \int_{-\infty}^{+\infty} g(\omega)\alpha^*(\omega, t) d\omega\), then Eq. (4) yields

\[ r(t) = \int_{\tau_0}^{+\infty} \int_{-\infty}^{+\infty} g(\omega)\alpha^*(\omega, t - \tau)d\omega h(\tau - \tau_0) d\tau \]  

(7)

For the sake of theoretical analysis, the distribution density \(g(\omega)\) is usually chosen as Lorentzian distribution, that is

\[ g(\omega) = \frac{\Delta}{\pi (\omega - \omega_0)^2 + \Delta^2}, -\infty < \omega < +\infty \]  

(8)

Following Ott and Antonsen’s method, substituting (8) into (7), and using residue theorem, we have

\[ r(t) = \int_{\tau_0}^{+\infty} \alpha^*(\omega_0 - i\Delta, t - \tau) h(\tau - \tau_0) d\tau \]  

(9)
Putting $\omega = \omega_0 - i\Delta$ in Eq. (5) and noticing Eq. (9) yield

$$\dot{\alpha}(t) = -(i\omega_0 + \Delta)\alpha(t) + \frac{k}{k + ev} \int_{\tau_0}^{+\infty} \alpha(t - \tau)h(\tau - \tau_0)d\tau$$

which is a delay differential equation$^{13}$, whose trivial equilibrium stands for the incoherent state. To investigate its stability, we substitute $\alpha = \alpha_0 e^{i\beta t}$ into the linear part of (10) with $\alpha_0 \neq 0$, and obtain

$$\lambda = -(i\omega_0 + \Delta) + \frac{k}{k + ev} \int_{\tau_0}^{+\infty} e^{-\lambda\tau}h(\tau - \tau_0)d\tau$$

If (11) has a root $\lambda = i\beta$, a Hopf bifurcating solution $\alpha(t) = e^{i\beta t}$ into the linear part of (10) with $\alpha_0 \neq 0$, and obtain

$$\lambda = -(i\omega_0 + \Delta) + \frac{k}{k + ev} \int_{\tau_0}^{+\infty} e^{-\lambda\tau}h(\tau - \tau_0)d\tau$$

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In this section, we assume that a Hopf bifurcation occurs in Eq. (11). If Hopf bifurcation occurs at $k = k_0$, two necessary conditions are required: [i] Eq. (11) has a simple root $\lambda = i\beta$ with $\beta \neq 0$ when $k = k_0$; [ii] The so-called transversality condition holds in the sense that $\text{Re}\lambda(k) \neq 0$ at $k = k_0$. To obtain more properties near $k$, we need to calculate the normal form with the aid of multiple scales$^{12}$. (A) Normal forms

Denoting by $k = k_0 + \epsilon v$ with $\epsilon > 0$ and $v$ a detuning parameter which describes the nearness of $k$ to the critical value $k_0$, Eq. (10) can be rewritten into

$$\dot{\alpha}(t) = -(i\omega_0 + \Delta)\alpha(t) + \frac{k}{k + ev} \int_{\tau_0}^{+\infty} \alpha(t - \tau)h(\tau - \tau_0)d\tau$$

For the absence of second order term in (12), the solution of Eq. (12) can be expressed by$^{14}$

$$\alpha(t; \epsilon) := e^{t/2}\alpha_1(T_0, T_1) + \epsilon^{3/2}\alpha_2(T_0, T_1) + \cdots$$

where $T_0 = t$, $T_1 = ct$. The derivative with respect to $t$ is transformed into

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} := D_0 + \epsilon D_1$$

Taylor expanding the term $\alpha(t - \tau) := \alpha(t - \tau; \epsilon)$ gives

$$\alpha(t - \tau; \epsilon) = e^{t/2}\alpha_1(T_0, T_1) + \epsilon^{3/2}\alpha_2(T_0, T_1)$$

Substituting $\alpha(t)$ and $\alpha(t - \tau)$ into (12), and balancing the same order terms of $\epsilon$ in both sides we have

$$D_0\alpha_1(T_0, T_1) = -(i\omega_0 + \Delta)\alpha_1(T_0, T_1) + \frac{k}{k + ev} \int_{\tau_0}^{+\infty} \alpha_1(t - \tau)h(\tau - \tau_0)d\tau$$

and

$$D_0\alpha_2(T_0, T_1) = -(i\omega_0 + \Delta)\alpha_2(T_0, T_1) + \frac{k}{k + ev} \int_{\tau_0}^{+\infty} \alpha_1(t - \tau)h(\tau - \tau_0)d\tau$$

Eq. (13) is a linear equation and has a solution

$$\alpha_1(T_0, T_1) = A(T_1)e^{i\beta T_1}$$

If all roots of Eq. (11) except $i\beta$ have negative real part, $\sigma_m < 0$ for any $m$, then $\alpha_1(T_0, T_1) \rightarrow A(T_1)e^{i\beta T_1}$ as $t \rightarrow +\infty$. We know, the bifurcated solution oscillates in time scale $T_0$, whereas all solutions nearby trend toward it in time scale $T_1$. Thus the dynamics of $A(T_1)$ determines the property of the bifurcation, which will be further calculated in the following.

Substituting $\alpha_1(T_0, T_1)$ into (14) yields

$$D_0\alpha_1(T_0, T_1) = -(i\omega_0 + \Delta)\alpha_1(T_0, T_1) + \frac{k}{k + ev} \int_{\tau_0}^{+\infty} \alpha_1(t - \tau)h(\tau - \tau_0)d\tau$$

Eliminating the terms that lead to secular terms (i.e., the last four terms in the above equation), we have the normal form given by

$$A'(T_1) = \nu \frac{\int_{\tau_0}^{+\infty} e^{-i\beta T}h(\tau - \tau_0)d\tau}{2+k\int_{\tau_0}^{+\infty} \text{e}^{-i\beta T}h(\tau - \tau_0)d\tau} A(T_1)$$

The amplitude equation is

$$|A(T_1)|' = \nu \text{Re} A(T_1) + \text{Re} A(T_1)$$

In fact, regarding $A$ a function of $k$ defined implicitly by (11), we know $a = \lambda'(k)$. According to the fundamental theory about Poincaré–Birkhoff normal form of ODE$^{15}$, the two real parts of $a$ and $b$ determine the direction and stability of the bifurcation.
Precisely, letting $k_0 = \text{inf}\{\bar{k}|\bar{k}\text{ is a Hopf bifurcation value}\}$, we always have $\text{Re}a > 0$. If $\nu > 0$ and $\text{Re}b < 0$, then $A(T_1) \rightarrow \sqrt{\frac{\text{Re}a}{\text{Re}b}}$ as $T_1 \rightarrow \infty$. Thus a branch of stable bifurcations  solutions appears at $k > k_0$ for $\text{Re}b < 0$. Similarly, a branch of unstable solutions appears at $k < k_0$ for $\text{Re}b > 0$. In the latter case, the branch of unstable solutions must go to $k > k_0$, because when $k = 0$ system (10) has a globally stable attractor $\alpha$ and a constant $\tau$ value.

Two kinds of bifurcations are the so-called supercritical and subcritical bifurcations, respectively as shown in FIG. 2. If at some $\bar{k}$, $\text{Re} \ a < 0$, there are two other kinds of bifurcations which can be illustrated by reversing the stability of coherent states.

![FIG. 2. Supercritical bifurcation and subcritical bifurcation (with a hysteresis loop between $k$ and the saddle-node point) near a critical value $k$.](image)

Consider the case time delay comes from an ensemble $\tau$ which is the sum of a Gamma distributed variable $\tilde{\tau}$ and a constant $\tau_0$, $\tilde{\tau} \sim \Gamma(n, \frac{T}{n})$ has PDF

$$h(\tau) = \frac{1}{\Gamma(n)(T/n)^n} \tau^{n-1} e^{-\frac{T}{n} \tau}, \tau > 0$$

where $T$ is the mean value, $\frac{T^2}{2}$ the variance and $\frac{6}{n}$ the excess kurtosis. The PDF of $\tau$, $h(\tau-\tau_0)$ has been shown in FIG. 1(a).

The characteristic equation in this case is given by

$$\lambda = -(i\omega_0 + \Delta) + \frac{k}{2} \int_{\tau_0}^{\infty} e^{-\lambda \tau} h(\tau - \tau_0) d\tau$$

$$= -(i\omega_0 + \Delta) + \frac{k}{2} (1 + \frac{T}{n}) \lambda^{-n} e^{-\lambda \tau_0}$$

Motivated by the above analysis, we are seeking for a root $\lambda = i\beta$ and have

$$i\beta = -(i\omega_0 + \Delta) + \frac{k}{2} (1 + \frac{T}{n}) \beta^{-n} e^{-i\beta \tau_0} \quad (15)$$

Denote by $1 + \frac{T}{n} i\beta = \rho e^{i\theta}$, then

$$\Delta = \frac{k}{2} \rho^{-n} \cos(n\theta + \beta \tau_0)$$

$$\beta + \omega_0 = -\frac{k}{2} \rho^{-n} \sin(n\theta + \beta \tau_0) \quad (16)$$

Obviously, $\theta \in [-\pi/2, \pi/2]$, thus $\theta = \arctan \frac{T \beta}{n}$. Then we have

$$-\frac{\beta + \omega_0}{\Delta} = \tan \left[n \arctan \left(\frac{T \beta}{n}\right) + \beta \tau_0\right]$$

This equation can be solved by a sequence of $\beta's$ with $|\beta| \rightarrow +\infty$, for all parameters fixed. As $|\beta| \rightarrow +\infty$, we have $\rho \rightarrow \infty$, thus $k \rightarrow \infty$ holds from the first equation of (10). Hence we only need to calculate all roots for $\beta \in [-J,J]$ with $J$ relatively large, to obtain the first several bifurcation values of $k$.

If a bifurcation occurs at $\bar{k}$, then employing the normal form theory established above, we have

$$a = \frac{(1 + \frac{i\beta T}{n})^{-n} e^{-i\beta \tau_0}}{2 + \bar{k} \left(\frac{T e^{-i\beta \tau_0}}{1 + \frac{i\beta T}{n}} + \tau_0 e^{-i\beta \tau_0}\right) \left(1 + \frac{i\beta T}{n}\right)^{-n}}$$

$$b = -\frac{\bar{k} e^{i\beta \tau_0} \left(1 - \frac{i\beta T}{n}\right)^{-n}}{2 + \bar{k} \left(\frac{T e^{-i\beta \tau_0}}{1 + \frac{i\beta T}{n}} + \tau_0 e^{-i\beta \tau_0}\right) \left(1 + \frac{i\beta T}{n}\right)^{-n}}$$

We claim that $\text{Re}a > 0$ always holds. In fact, from (15), we have $i\beta + i\omega_0 + \Delta = \frac{k}{2} (1 + \frac{T}{n}) \beta^{-n} e^{-i\beta \tau_0}$, then

$$\text{Sign} \ \text{Re} \ a = \text{Sign} \ \text{Re} \ a^{-1} \quad (16)$$

$$= \text{Sign} \ \text{Re} \ \frac{1 + \left(1 + \frac{T}{n} \tau_0\right) \left(i\beta + i\omega_0 + \Delta\right)}{\frac{T}{n} i\beta + i\omega_0 + \Delta} = \text{Sign} \ \text{Re} \ \frac{1}{\left(i\beta + i\omega_0 + \Delta\right) + \left(1 + \frac{T}{n} \tau_0\right)} > 0$$

Thus only the two kinds of bifurcations shown in FIG. 2 can occur which are distinguished by the sign of $\text{Re}b$. Moreover, using the global Hopf bifurcation theorem, we know all Hopf bifurcation branches are unbounded in the $k$ direction. When $k$ increases, after two, even more Hopf bifurcations occur, together with $\alpha = 0$ is globally stable at $k = 0$, we conclude the number of coexisted coherent states gets larger.

B. The case without gap

When $\tau_0 = 0$, i.e., $\tau$ degenerates into $\tilde{\tau}$ the Gamma distributed variable. This is the case investigated by Lee et al. Hysteresis loop is observed when $T = 3$, and the authors found smaller $n$ led to hysteresis loop while larger one did not. Using the method we established, we can calculate the bifurcation points and the signs of $b$ which are shown in FIG. 3 (a) and (b). One supercritical bifurcation curve intersects with the other subcritical bifurcation curve at a double Hopf bifurcation point HH. With the help of FIG. 2 near the subcritical bifurcation we know there is a stable coherent state coexisting with the incoherent state, i.e., the hysteresis loop. When $T$ decreases, as shown in FIG. 3 (c) and (d), supercritical bifurcation never occurs at the first bifurcation value $k$, which coincides with the previous results. Finally, two remarks should be noticed that [i] in FIG. 3 (b) the saddle node curve is a sketched one as we do not know how to calculate the exact values, theoretically; and [ii] near the double Hopf point HH, the dynamics may be more

![FIG. 3.](image)
complicated such as the quasiperiodic behavior possibly existed on 2-torus, even 3-torus\cite{18}.

![Figure 3](image3.png)

**FIG. 3.** When $\omega_0 = 3$, $\Delta = 1$, $T = 3$, bifurcation curves are shown in (a), with a double Hopf bifurcation labeled by HH. Near HH, number of coherent states is marked in (b). The dotted curve stands for the saddle-node bifurcation curve (sketched) illustrated in FIG. 2. “+/-” denotes sub-critical/supercritical bifurcation. (c) and (d) are bifurcation curves for $T = 1.15$ and $T = 1$.

**C. The case with $\tau_0 > 0$ and effect of excess kurtosis**

In the rest part of this paper, we consider the case with a minimal positive delay, i.e., the case $\tau = \tilde{\tau} + \tau_0$ with $\tilde{\tau}$ a Gamma variable. Using the method above, we can calculate the bifurcation values. In FIG. 4 (a)-(c), we find that increasing $\omega_0$ will delay the occurrence of bifurcation and increase in resonant structure(s) of the dependence of $k$ on $\tau_0$. When $T$, the mean of Gamma distribution $\tilde{\tau}$ is larger, the effect of $\tau_0$ becomes weaker, thus the distributed delay $\tilde{\tau}$ acts predominantly. Increasing the variance of the natural frequency $\Delta$ even further weakens the effect of the gap $\tau_0$.

By fixing $T$, we are about to consider the interactional effect of $\tau_0$ and the variance (or the excess kurtosis) of Gamma distribution characterized by $n$. In FIG. 4 (d), we fix $T = 3$ and investigate the effect of $\tau_0$ and $n$, where we find increasing $n$ will decrease in resonant structure of the dependence of $k$ on $\tau_0$, meanwhile weakens the effect of $\tau_0$.

Now we further discuss how the gap $\tau_0$ has an effect on the system dynamics, when fixing the mean and variance of $\tilde{\tau} + \tau_0$, i.e., the case shown in FIG. 4 (b). Letting $\langle \tilde{\tau} + \tau_0 \rangle = T + \tau_0 = 3$, $\text{Var}(\tilde{\tau} + \tau_0) = \frac{T^2}{n} = 3$, the first Hopf bifurcation value and its direction are drawn in FIG. 5 (a). We find small $T$ (i.e., $\langle \tilde{\tau} \rangle$) makes supercritical bifurcation occur at small $k$. When $T$ is large, the subcritical bifurcation occurs at large $k$, which means that in the case of fixed mean and variance of total delay, large proportion of the gap $\tau_0$ (i.e., small $T$) may destroy the hysteresis loop. In FIG. 5 (b), we draw all bifurcation curves and one can also find that larger $\tau_0$ can significantly increase in the number of coherent attractors. When $T = 0.5$ and $2.8$, respectively, simulations are carried out in FIG. 5 (c) and (d), where we find supercritical bifurcation and subcritical bifurcation in system (1) with

![Figure 4](image4.png)

**FIG. 4.** When $n = 3$, the first bifurcation value $k$ is calculated for $(T, \tau_0) \in [0, 2.8] \times [0, 8]$, (a) $\omega_0 = 3$, $\Delta = 0.3$; (b) $\omega_0 = 3$, $\Delta = 1$; (c) $\omega_0 = 5$, $\Delta = 0.3$. (d) When $\omega_0 = 3$, $\Delta = 0.3$, $T = 3$, the first bifurcation value $k$ is calculated for $(n, \tau_0) \in [0, 2.8] \times [0, 3]$.

![Figure 5](image5.png)

**FIG. 5.** Fixing $\omega_0 = 3$, $\Delta = 1$, $\langle \tilde{\tau} + \tau_0 \rangle = T + \tau_0 = 3$, $\text{Var}(\tilde{\tau} + \tau_0) = \frac{T^2}{n} = 3$, the first bifurcation value and its direction is drawn in (a), and all bifurcation curves are illustrated in (b). The dark region stands for local stable region of incoherent state. Numbers indicates the quantity of coherent attractors. When (c) $T = 0.5(k = 2.6992)$ and (d) $T = 2.8(k = 7.0388)$, $|r|$ is shown in black dot by simulating system (1) with $N = 128$. 
128 oscillators.

It is worthwhile to mention that in this case larger \( \tau_0 \) means smaller \( T \). As we have fixed the variance \( \frac{\tau^2}{n} \), the excess kurtosis \( \frac{\Delta}{n} \) is larger. Hence the sample data of delays are more “concentrated”, which induces a supercritical bifurcation. Thus we conclude that smaller excess kurtosis (decentralized samples) induces subcritical bifurcations and hysteresis loop, while larger one does not.

IV. CONCLUSION

In this paper, we establish a normal form method by extending Nayfeh’s multiple scales to determine the properties of the bifurcated coherent states in a group of Kuramoto oscillators with heterogeneously distributed delays with a gap. We find this a useful way, on the Ott-Antonsen’s manifold, to reveal the detailed dynamics near the critical values such as the direction of bifurcation, stability of bifurcated solutions and certain coexistence phenomena. They can be determined by real part of two variables \( a \) and \( b \). Some numerical results indicate the effect of the gap. As direct applications of our theory, how these parameters affect system dynamics is investigated. For fixed variance and expectation of total delay, of delays are more “concentrated”, which induces a supercritical bifurcation hence avoid the hysteresis loop, and (iii) decrease the bifurcation values, (ii) induce a supercritical bifurcation, and (iii) increase in the number of the coexisted coherent states.

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1Y. Kuramoto, Progr. Theoret. Phys. Suppl. 79, 223 (1984); H. Sakaguchi, and Y. Kuramoto, Progr. Theoret. Phys. 76, 576 (1986); Y. Kuramoto, and I. Nishikawa, J. Statist. Phys. 49, 569 (1987); J.D. Crawford, J. Statist. Phys. 74, 1047 (1994); Y. Kuramoto, in International Symposium on Mathematical Problems in Theoretical Physics, Lecture Notes in Physics, edited by H. Araki Springer-Verlag, Berlin, 1975, Vol. 39; S.H. Strogatz, Nature 410, 268 (2001); Physica D 143, 1 (2000); A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences (Cambridge university press, New York, 2003); B. Ermentrout, and T. Ko, Phil. Trans. Roy. Soc. A 367, 1007 (2009); E.M. Izhikevich, Phys. Rev. E 58, 905 (1998); A. Pikovsky, and M. Rosenblum, Physica D 240, 872 (2011); J.R. Engelbrecht, and R. Mirollo, Chaos 24, 013114 (2014).

2D.C. Michaels, E.P. Matyas, and J. Jaffe, Circulation Res. 61, 704 (1987); C. Liu, D.R. Weaver, S.H. Strogatz, and S.M. Reppert, Cell 91, 855 (1997); Z. Jiang, and M. McCall, J. Opt. Soc. Am. 10, 155 (1993); S.Yu. Kourtchatov, V.V. Likhanskii, A.P. Napartovich, F.T. Arecchi, and A. Lapucci, Phys. Rev. A 52, 4089 (1995); K. Wiesenfeld, P. Colet, and S.H. Strogatz, Phys. Rev. E 57, 1563 (1998).

3Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence (Springer, Berlin, 1984).

4A.T. Winfree, J. Theoret. Biol. 16, 15 (1967).

5M.K. Stephen Yeung, and S.H. Strogatz, Phys. Rev. Lett. 82, 648 (1999).

6S. Kim, S.H. Park, and C.S. Ryu, Phys. Rev. Lett. 79, 2911 (1997); E. Montbrió, D. Pazó, and J. Schmidt, Phys. Rev. E 74, 056201 (2006).

7W. Lee, E. Ott, and T. Antonsen, Phys. Rev. Lett. 103, 044101 (2009).

8E. Ott, and T. Antonsen, Chaos 18, 037113 (2008); Chaos 19, 023117 (2009).

9Y. Yuan, and J. Béclair, SIAM J. Appl. Dyn. Syst. 10, 551 (2011).

10B. Niu, and Y. Guo, Physica D 266, 23 (2014).

11B. Hassard, N.D. Kazarinoff, and Y. Wan, Theory and Applications of Hopf Bifurcation (Cambridge Univ. Press, Cambridge, 1981); T. Faria, and L. Magalhaes, J. Differ. Equations. 122, 181 (1995).

12A.H. Nayfeh, Nonlinear Dynam. 51, 483 (2008); Introduction to Perturbation Techniques (Wiley, New York, 1981); B. Niu, and W. Jiang, Nonlinear Dynam. 70, 43 (2012).

13J. Hale, and S. Lunel, Introduction to Functional Differential Equations (Springer, New York, 1993).

14In fact, the traditional method of multiple scales requires \( a(t; \epsilon) \) \( = \) \( e^{\epsilon^2/2 \alpha_1 (T_0, T_1)} + e^{\epsilon^2 \alpha_2 (T_0, T_1)} + \cdots \) \( \alpha \) 0, \( \alpha \) 0.

15W. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos (Springer, New York, 1980).

16The stability can be obtained easily, by calculating the negative Floquet exponent at \( A(T_1) = \sqrt{\text{Re} \alpha} \).

17J. Wu, Trans. Amer. Math. Soc. 350, 4799 (1998). In this case, with respect to a branch of Hopf bifurcating solutions, either it is unbounded, or it contains \( 2m \) bifurcation points with \( \text{Re} \) \( > \) \( 0 \) at \( m \) points and \( \text{Re} \) \( < \) \( 0 \) at the rest \( m \) points.

18J. Guckenheimer, and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (Springer, New York 1983).