POTENTIAL THEORY AND A CHARACTERIZATION OF POLYNOMIALS IN COMPLEX DYNAMICS

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Abstract. We obtain a measure theoretical characterization of polynomials among rational functions on \( \mathbb{P}^1 \), which generalizes a theorem of Lopes. Our proof applies both classical and dynamically weighted potential theory.

1. Introduction

We are interested in a measure theoretical characterization of polynomials among rational functions \( f \) of degree \( d > 1 \) on \( \mathbb{P}^1 \). Recall that the Fatou set \( F(f) \) of \( f \) is the region of normality of iterates \( \{ f^k; k \in \mathbb{N} \} \) in \( \mathbb{P}^1 \), which by the definition is open. The Julia set \( J(f) \) is the complement of \( F(f) \) and it is known to be non-empty. Both \( F(f) \) and \( J(f) \) are \( f \)-invariant. The characterization that we have in mind is provided by the following theorem:

Theorem 1. Let \( f \) be a rational function on \( \mathbb{P}^1 \) of degree \( d > 1 \). Suppose that the point \( \infty \) belongs to a Fatou component \( D_\infty \) of \( f \), and that \( f(D_\infty) = D_\infty \). Then the following are equivalent:

(i) \( f \) is a polynomial.

(ii) The balanced measure \( \mu_f \) of \( f \) coincides with the harmonic measure \( \nu \) of \( D_\infty \) with pole \( \infty \).

The probability measure \( \mu_f \) in Theorem 1 is also known to be the (unique) maximal entropy measure of \( f \), which was constructed by Lyubich \[8\] and by Freire, Lopes and Mañé \[9\] (see the next paragraph for more historical remarks). Under the additional assumption that \( f(\infty) = \infty \), Theorem 1 was proved by Lopes \[9\]. The Lopes’s theorem was stated earlier in Oba and Pitcher \[12, Theorem 6\], but proved only partially. Lalley gave a probabilistic proof of Lopes’s theorem (\[7, \S 6\]). In all those proofs, a key role is played by the same equality, which is a consequence of a pullback formula for the logarithmic potential of balanced measure. We will give a simple and conceptually new proof of both this formula and the key equality in an improved form (Lemma 3.3 and Claim 1), which will enable us to prove Theorem 1. Mañé and da Rocha \[10\] also studied Lopes’s theorem in relation to calculations of the entropy of invariant measures on the Julia set.

Date: December 13, 2010.

2010 Mathematics Subject Classification. Primary 37F10; Secondary 31A05.
Key words and phrases. balanced measure, harmonic measure, complex dynamics, Lopes’s theorem, Brolin’s theorem, weighted potential theory.

The first author is partially supported by JSPS Grant-in-Aid for Young Scientists (B), 21740096.
Brolin’s theorem [3, Theorem 16.1] says that the pullbacks $(f^k)^*\delta_a/d^k$ converge weakly to the harmonic measure $\nu = \nu_\infty$ of $D_\infty$ with pole at $\infty$ when $f$ is a polynomial and $\delta_a$ is the Dirac measure at a non-exceptional point $a \in \mathbb{C}$ of $f$. As a generalization of Brolin’s theorem, the balanced measure $\mu_f$ was first obtained as the weak limit of pullbacks $(f^k)^*\delta_a/d^k$ for a rational function $f$ and a non-exceptional point $a \in \mathbb{P}^1$ (by Lyubich [5] and by Freire, Lopes and Mañé [6] independently). However, in this article, we will not use these equidistribution results.

In the next section we recall the definition of measures $\mu_f$ and $\nu$. Our proof of Theorem 1 is an application of both classical and dynamically weighted potential theory. For related results on (generalized) polynomial-like maps, see [13, 16].

Notation 1.1. We denote the origin of $\mathbb{C}^2$ by 0. Let $\pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1$ be the canonical projection so that $\pi(z_0, z_1) = z_1/z_0$ if $z_0 \neq 0$ and $\pi(z_0, z_1) = \infty$ if $z_0 = 0$. Let $\|\cdot\|$ be the Euclidean norm on $\mathbb{C}^2$, and put $(z_0, z_1) \wedge (w_0, w_1) := z_0w_1 - z_1w_0$ on $\mathbb{C}^2 \times \mathbb{C}^2$. A function on $\mathbb{P}^1$ is said to be $\delta$-subharmonic (DSH) if it is locally the difference of two subharmonic functions. We normalize $d^c$ so that $dd^c = (i/\pi)\partial\overline{\partial}$. An important example of a value of the $dd^c$-operator is the generalized Laplacian of the $\delta$-subharmonic function $\log |\cdot - w|$ ($w \in \mathbb{C}$) on $\mathbb{P}^1$, which equals $dd^c \log |\cdot - w| = \delta_w - \delta_\infty$, where $\delta_w$ denotes the Dirac measure at $w \in \mathbb{P}^1$.

2. Rational functions and probability measures on $\mathbb{P}^1$

Balanced measure $\mu_f$. For more details, see [6, §4], [15, §1] and [2, Chapitre VIII].

Let $f$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$. A lift

$$F(z_0, z_1) = (F_0(z_0, z_1), F_1(z_0, z_1))$$

of $f$ is a non-degenerate homogeneous polynomial endomorphism of algebraic degree $d$ on $\mathbb{C}^2$ in that $\pi \circ F = f \circ \pi$ and $F^{-1}(0) = \{0\}$, and is uniquely determined up to multiplication by a constant in $\mathbb{C}^*$.

The dynamical Green function of $F$ is

$$G^F := \lim_{k \to \infty} \frac{1}{d^k} \log \|F^k\| : \mathbb{C}^2 \to \mathbb{R} \cup \{-\infty\}.$$ 

This convergence is uniform on $\mathbb{C}^2 \setminus \{0\}$, so $G^F$ is continuous there and plurisubharmonic on $\mathbb{C}^2$. It follows from the definition of $G^F$ that

$$d \cdot G^F = G^F \circ F,$$

and from homogeneity of $F$ that for every $p \in \mathbb{C}^2$ and every $c \in \mathbb{C}^*$,

$$G^F(c \cdot p) = G^F(p) + \log |c|,$$

$$G^F(p) = G^F(0) + \frac{1}{d - 1} \log |c|.$$ 

The function $G^F(1, \cdot)$ is continuous on $\mathbb{C}$ and $\delta$-subharmonic on $\mathbb{P}^1$, and the balanced measure $\mu_f$ is defined by the unique probability measure on $\mathbb{P}^1$ satisfying

$$dd^c G^F(1, \cdot) = \mu_f - \delta_\infty.$$

It follows from (2.3) that the left-hand side of (2.4) is independent of the choice of $F$, hence the measure $\mu_f$ is well-defined. It is also known that $\text{supp } \mu_f = J(f)$ and $J(f)$ is perfect.
From (2.1), the $\mu_f$ is balanced and invariant under $f$: namely,

$$(2.5) \quad f^*\mu_f = \mu_f = f_*\mu_f.$$ 

We recall that the pullback $f^*\phi$ of continuous function $\phi$ on $\mathbb{P}^1$ is defined by $\phi \circ f$, and the push-forward $f_*\phi$ is

$$f_*\phi(z) := \frac{1}{d} \sum_{w \in f^{-1}(z)} \phi(w),$$

where the sum takes into account of the multiplicity of $f$ at each $w$. Both $f^*\phi$ and $f_*\phi$ are continuous on $\mathbb{P}^1$, and we may define respectively the push-forward $f^*\mu$ and the pullback $f_*\mu$ of (Radon) measure $\mu$ by duality.

**Harmonic measure $\nu = \nu_\infty$.** For a finite Borel measure $\nu$ on $\mathbb{C}$ with compact support, its logarithmic potential on $\mathbb{C}$ is

$$(2.6) \quad p_\nu(z) := \int_{\mathbb{C}} \log |z - w| d\nu(w) = \nu(\mathbb{C}) \log |z| + O(|z|^{-1})$$

as $z \to \infty$, and the logarithmic energy of $\nu$ is $I_\nu := \int_{\mathbb{C}} p_\nu d\nu$. A compact set $K$ in $\mathbb{C}$ is said to be polar if

$$\sup \{ I_\nu; \text{supp } \nu \subset K, \nu(\mathbb{C}) = 1 \} = -\infty.$$ 

If $K$ is non-polar, then by Frostman’s theorem, there is the unique probability measure $\nu = \nu_K$ (the equilibrium measure of $K$) which attains the supremum in the above. The measure $\nu$ has the support on the exterior boundary $\partial_e K$ of $K$, and satisfies that $p_\nu \equiv I_\nu$ on $K \setminus E$, where $E$ is a (possibly empty) $F$ polar subset of $\partial_e K$. Moreover, $p_\nu > I_\nu$ on $D_\infty$ by the minimum principle.

For a domain $D$ in $\mathbb{P}^1$ which contains $\infty$ and whose complement $\mathbb{C} \setminus D$ is non-polar, the harmonic measure $\nu = \nu_\infty$ of $D$ with pole $\infty$ is determined by $\nu_{\mathbb{C} \setminus D}$ (cf. [4, Theorem 4.3.14]). Under the situation in Theorem 1, we will compute $I_{\mu_f}$ and see directly that $I_{\mu_f} > -\infty$, so $\mathbb{C} \setminus D_{\infty}$ is non-polar (see Lemma 3.2 below). Hence $D_\infty$ admits the harmonic measure $\nu = \nu_\infty (= \nu_{\mathbb{C} \setminus D_\infty})$ with pole at $\infty$.

**Dynamically weighted potential theory.** A function

$$\mathbb{C}^2 \times \mathbb{C}^2 \ni (p, q) \mapsto \log |p \wedge q| - G^F(p) - G^F(q) \in \mathbb{R} \cup \{-\infty\}$$

descends to a weighted kernel $\Phi_F(z, w)$ ($p \in \pi^{-1}(z), q \in \pi^{-1}(w)$) on $\mathbb{P}^1$. For a Radon measure $\mu$, its $F$-potential is a $\delta$-subharmonic function

$$U_{F, \mu}(z) := \int_{\mathbb{P}^1} \Phi_F(z, w) d\mu(w).$$

It can be computed directly that $d\delta d\delta U_{F, \mu} = \mu - \mu(\mathbb{P}^1) \mu_f$, so the potential $U_{F, \mu_f}$ of $\mu = \mu_f$ is harmonic on $\mathbb{P}^1$, and hence constant, say, $U_{F, \mu_f} \equiv V_F$. The constant $V_F$ has been computed as

$$(2.7) \quad V_F = -\frac{1}{d(d-1)} \log |\text{Res } F|$$

in [4, Theorem 1.5]. We will compute it in a different way in Appendix.
3. A proof of Theorem [1]

Let $f$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$, and $F = (F_0, F_1)$ a lift of $f$. We note that

$$f(z) = F_1(1, z)/F_0(1, z).$$

Put $d_0 := \deg F_0(1, z)$ and $d_1 := \deg F_1(1, z)$, and let $a_F, b_F$ be the coefficients of the maximal degree term of $F_0(1, z), F_1(1, z)$, respectively.

Suppose that a Fatou component $D_\infty$ of $f$ contains $\infty$, and that $f(D_\infty) = D_\infty$.

Lemma 3.1. For every $z \in \mathbb{C}$,

$$p_{\mu_f}(z) = G^F(1, z) - G^F(0, 1).$$

In particular, $p_{\mu_f}$ is continuous on $\mathbb{C}$.

Proof. Recall that $U_{F, \mu_f}^c = V_F$ on $\mathbb{P}^1$. Hence for every $z \in \mathbb{C}$,

$$p_{\mu_f}(z) = \int_{\mathbb{C}} \log |z - w| d\mu_f(w)$$

$$= U_{F, \mu_f}(z) + G^F(1, z) + \int_{\mathbb{C}} G^F(1, w) d\mu_f(w)$$

$$= G^F(1, z) + C_F,$$

where we put $C_F := V_F + \int_{\mathbb{C}} G^F(1, w) d\mu_f(w)$. Hence from (2.2) and (2.6),

$$0 = \lim_{z \to \infty} \left( p_{\mu_f}(z) - \log |z| \right) = \lim_{z \to \infty} G^F(1/z, 1) + C_F,$$

so that $C_F = -G^F(0, 1)$. \hfill \Box

The following computation of $I_{\mu_f}$ may be of independent interest, and was proved in [12 Theorem 4] under the restrictions $f(\infty) = \infty$ and $d_0 < d - 1$, with no reference to $G^F$.

Lemma 3.2. The complement $\mathbb{C} \setminus D_\infty$ of $D_\infty$ is non-polar, and $D_\infty$ admits the harmonic measure $\nu = \nu_\infty$ with pole $\infty$. The energy $I_{\mu_f}$ of $\mu_f$ is computed as

$$e^{I_{\mu_f}} = e^{-2G^F(0, 1)} |\text{Res } F|^{d/(d - 1)}.$$

Here $\text{Res } F := a_F^{d-d_1} b_F^{d-d_0} R(F_0(1, z), F_1(1, z))$ is the homogeneous resultant of $F$, where $R(P(z), Q(z))$ is the resultant of two polynomials $P(z)$ and $Q(z)$.

Proof. Integrating the equality in Lemma 3.1 in $d\mu_f(z)$, we get

$$I_{\mu_f} = \int_{\mathbb{C}} p_{\mu_f} d\mu_f = \int_{\mathbb{C}} G^F(1, \cdot) d\mu_f - G^F(0, 1)(> -\infty),$$

which with $\text{supp } \mu_f \subset J(f) \subset \mathbb{C} \setminus D_\infty$ implies that $\mathbb{C} \setminus D_\infty$ is non-polar. It also follows from the proof of the previous lemma that

$$-G^F(0, 1) = C_F = V_F + \int_{\mathbb{C}} G^F(1, \cdot) d\mu_f,$$

where $C_F$ has been introduced in the proof of the previous lemma. The value of $V_F$ has already been computed in (2.7). Hence

$$I_{\mu_f} = \frac{1}{d(d - 1)} \log |\text{Res } F| - 2G^F(0, 1).$$

\hfill \Box
The following was proved in [12, p307] and [9, p398] under the assumption
\( f(\infty) = \infty \) and \( z \in J(f) \) (again with no reference to \( G^F \)).

**Lemma 3.3.** For every \( z \in \mathbb{C} \setminus f^{-1}(\infty) \),
\[
p_{\mu_f}(f(z)) = d \cdot p_{\mu_f}(z) - \log |F_0(1, z)| + (d - 1)G^F(0, 1).
\]

**Proof.** For every \( z \in \mathbb{C} \setminus f^{-1}(\infty) \), from (2.1),
\[
G^F(1, f(z)) = G^F(F(1, z)) - \log |F_0(1, z)| = d \cdot G^F(1, z) - \log |F_0(1, z)|.
\]
Now Lemma 3.1 completes the proof. \( \square \)

We now proceed with the proof of our theorem. Let us denote the harmonic
measure of \( D_\infty \) with pole at \( \infty \) by \( \nu = \nu_\infty \).

**Claim 1.** On \( \mathbb{C} \setminus f^{-1}(D_\infty) \),
\[
|F_0(1, \cdot)| = e^{(d-1)(I_{\mu_f} + G^F(0, 1))}.
\]

**Proof.** The assumption \( f(D_\infty) = D_\infty \) implies \( D_\infty \subset f^{-1}(D_\infty) \). Hence \( p_{\mu_f} \circ f = p_{\mu_f} \equiv I_{\mu_f} \) on \( \mathbb{C} \setminus f^{-1}(D_\infty) \). Now Lemma 3.3 completes the proof. \( \square \)

Suppose that \( F_0(1, \cdot) \) is non-constant, that is, \( d_0 > 0 \). Our refinement of the key
equality make the following reduction possible.

**Claim 2.** \( F(f) = D_\infty \).

**Proof.** If \( F(f) \neq f^{-1}(D_\infty) \), then \( F(f) \setminus f^{-1}(D_\infty) \) is a non-empty open subset
of \( \mathbb{C} \setminus f^{-1}(D_\infty) \). By the identity theorem, Claim 1 proves \( F_0(1, \cdot) \) must be constant, which
contradicts the assumption \( d_0 > 0 \). Hence \( F(f) = f^{-1}(D_\infty) \), which
implies that \( F(f) = f(F(f)) = D_\infty \). \( \square \)

By Claim 1 \( J(f) \) is contained in the lemniscate
\[
L := \{ z \in \mathbb{C} : |F_0(1, z)| = e^{(d-1)(I_{\mu_f} + G^F(0, 1))} \}.
\]
We note that each component of \( L \) is a (possibly non-simple) closed curve in
\( \mathbb{C} \setminus f^{-1}(\infty) \), which is real-analytic except for finitely many singularities; more
precisely, for each \( z_0 \in L \), there are an \( n \in \mathbb{N} \), a M"obius transformation \( K \) and
a local holomorphic coordinate \( h \) around \( z_0 \) such that \( h(z_0) = 0 \), \( K(F_0(1, z_0)) = 0 \),
\( \text{Im}K(F_0(1, \cdot)) \equiv 0 \) on \( L \) and \( K(F_0(1, h^{-1}(w))) = w^n \) around \( 0 \). In particular, \( L \)
around \( z_0 \) is the image of \( \bigcup_{-n \leq j < n} \{ w : \arg w = j\pi/n \} \cup \{ 0 \} \) under \( h^{-1} \), and \( n \geq 2 \)
if and only if \( z_0 \) is a critical point of \( F_0(1, \cdot) \).

Fix a component \( l \) of \( L \) intersecting \( J(f) \).

**Claim 3.** For every \( k \in \mathbb{N} \), \( f^k(l) \subset L \).

**Proof.** Fix \( z_0 \in l \cap J(f) \). Since \( L \) has at most \( d_0 \) components, there exists \( \delta > 0 \)
such that \( \{ |z - z_0| < \delta \} \cap J(f) \subset l \). Hence from the perfectness of \( J(f) \), \( z_0 \) is a
non-isolated point of \( l \cap J(f) \).

For every \( k \in \mathbb{N} \), let \( L_k \) be the component of \( L \) containing \( f^k(z_0) \). By the same
argument as the above, there is \( \delta_k > 0 \) such that \( \{ |z - f^k(z_0)| < \delta_k \} \cap J(f) \subset L_k \).
Hence if $\delta > 0$ is small enough, then $f^k(\{|z - z_0| < \delta\} \cap l \cap J(f)) \subset L_k$. Then by an argument involving the identity theorem \[\text{\ref{IdentityTheorem}}\], $f^k(l) \subset L_k(\subset L)$.

\begin{claim}
\begin{enumerate}[(i)]
\item $l \subset J(f)$.
\end{enumerate}
\end{claim}

\begin{proof}
Suppose that $l \cap F(f) \neq \emptyset$. Then there exists $z_0 \in l \cap D_\infty$ by Claim \[\text{\ref{Claim2}}\] and hence $p_{\mu_f}(z_0) = \nu(z_0) > I_f = I_{\mu_f}$. By Claim \[\text{\ref{Claim3}}\] for every $k \in \mathbb{N}$, we have $f^{k-1}(z_0) \subset L \subset \mathbb{C} \setminus f^{-1}(\infty)$, which with Lemma \[\text{\ref{Lemma2}}\] (and the definition of $L$) implies that $p_{\mu_f}(f^k(z_0)) - I_{\mu_f} = d \cdot (p_{\mu_f}(f^{k-1}(z_0)) - I_{\mu_f})$. Hence
\[
p_{\mu_f}(f^k(z_0)) - I_{\mu_f} = d^k \cdot (p_{\mu_f}(z_0) - I_{\mu_f}) > 0,
\]
so $\lim_{k \to \infty} p_{\mu_f}(f^k(z_0)) = \infty$. On the other hand, by Claim \[\text{\ref{Claim3}}\] we have $(f^k(z_0)) \subset L$, and since $p_{\mu_f}$ is upper semicontinuous, we have $\sup_{k \in \mathbb{N}} p_{\mu_f}(f^k(z_0)) \leq \sup_L p_{\mu_f} < \infty$. This is a contradiction.  
\end{proof}

Let $U$ be a component of $\mathbb{P}^1 \setminus l$ not containing $\infty$. Then $\partial U \subset J(f)$. By the maximum modulus principle, it follows that $U \subset \mathbb{C} \setminus L$, and from $J(f) \subset L$, we have $U \subset F(f)$. Hence $U$ is a Fatou component of $f$, and by Claim \[\text{\ref{Claim2}}\] we must have $U = F(f) = D_\infty$. This contradicts our assumption $\infty \not\in U$.

Now the proof of \[\text{\ref{Theorem1}}\] is complete.

\[\text{\ref{Theorem1}} \Rightarrow \text{\ref{Theorem2}}\]. Here we will give a proof of the assertion $\mu_f = \nu$ without the equidistribution results mentioned in Section \[\text{\ref{Theorem1}}\] using only computation of capacity.

Suppose that $f$ is a polynomial, or equivalently, that $F_0(1, z) = a_F$ on $\mathbb{C}$. By a direct computation, Lemma \[\text{\ref{Lemma1}}\] implies that
\[
eq \exp \left( \frac{1}{\log |b_F|} \cdot (|a_F|^{1/d} - 1) \right) = |a_F|^{1/d} e^{1/d}.
\]
Brolin’s theorem which we mentioned in Section \[\text{\ref{Theorem1}}\] was based on the computation of $e^{\nu_f}$ (Lemma \[\text{\ref{Lemma2}}\] as $e^{\nu} = |b_F|^{1/d} e^{1/d}$. Hence $I_{\mu_f} = I_{\nu}$, and from the uniqueness of $\nu = \nu_{\mathbb{C} \setminus D_\infty}$, we have $\mu_f = \nu$.

Now the proof of Theorem \[\text{\ref{Theorem2}}\] is complete.

\begin{acknowledgment}
We thank David Drasin for his comments on an earlier version of this paper, which helped us improve the presentation of our results.
\end{acknowledgment}

4. Appendix: A Computation of $V_F$

Let $f$ be a rational function of degree $d > 1$, and $F = (F_0, F_1)$ be a lift of $f$. For completeness, we give a direct computation \[\text{\ref{Computation1}}\] of $V_F$, again without using the equidistribution theorem. Indeed, we can give a proof of the equidistribution theorem based on \[\text{\ref{Computation1}}\]. For the original computation of $V_F$, which uses the equidistribution theorem, see DeMarco \[\text{\ref{DeMarco2}}\] Theorem 1.5. After having written this appendix, we learned that similar computations and formulas appeared in Appendix A in \[\text{\ref{DeMarco2}}\].

We continue to use the notation $d_0, d_1, a_F, b_F$ as in \[\text{\ref{Computation1}}\]. Let us write as $F_0(1, z) = a_F \prod_{j=1}^{d_0} (z - w_j)$. Then since $F : \mathbb{C}^2 \to \mathbb{C}^2$ is homogeneous and of topological degree
Integrating this in $\mathbb{C}^2$,
$$|F(p) \wedge (0, 1)| = |F_0(p)| = |a_F||p \wedge (0, 1)|^{d-d_0} \prod_{j} |p \wedge (1, w_j)|$$
and the leading coefficient is computed as
$$= (|a_F|(|F_0(0, 1)|)^{1/d})^{d-d_0} \prod_{j} |F_1(1, w_j)|^{1/d} \prod_{q \in F^{-1}(0, 1)} |p \wedge q|^{1/d},$$
and we also obtain an important formula
$$= (|a_F|^d b_F)^{d-d_0} \cdot |a_F|^{-d_1} |R(F_0(1, z), F_1(1, z))|^{1/d} = |\text{Res } F|^{1/d}.$$ Hence on $\mathbb{C}^2$,
$$|F(p) \wedge (0, 1)| = |\text{Res } F|^{1/d} \prod_{q \in F^{-1}(0, 1)} |p \wedge q|^{1/d}. \quad (4.1)$$
From (2.7), $G^F(F(p)) = d \cdot G^F(p)$ and $G^F(q) = G^F(0, 1)/d$ ($q \in F^{-1}(0, 1)$). Hence the log of (4.1) descends to $\mathbb{P}^1$ as
$$\Phi_F(f(z), \infty) = \frac{1}{d} \log |\text{Res } F| + \int_{\mathbb{P}^1} \Phi_F(z, w) d(f^* \delta_\infty)(w).$$
Integrating this in $d\mu_f(z)$,
$$\int_{\mathbb{P}^1} \Phi_F(f(z), \infty) d\mu_f(z) = \frac{1}{d} \log |\text{Res } F| + \int_{\mathbb{P}^1} U_{F, \mu_f}(w) d(f^* \delta_\infty)(w),$$
and from $f_* \mu_f = \mu_f$ and $U_{F, \mu_f} \equiv V_F$,
$$V_F = \int_{\mathbb{P}^1} \Phi_F(\infty, \cdot) d\mu_f = \int_{\mathbb{P}^1} \Phi_F(\infty, f(z)) d\mu_f(z) = \frac{1}{d} \log |\text{Res } F| + \int_{\mathbb{P}^1} V_F d(f^* \delta_\infty)(w) = \frac{1}{d} \log |\text{Res } F| + d \cdot V_F.$$ Now the proof of (2.7) is completed. \qed

**Remark 4.2.** From (4.1),
$$1 = \prod_{p \in F^{-1}(1, 0)} |F(p) \wedge (0, 1)| = |\text{Res } F|^{(1/d) d^2} \prod_{p \in F^{-1}(1, 0), q \in F^{-1}(0, 1)} |p \wedge q|^{1/d},$$
and we also obtain an important formula
$$|\text{Res } F| = \prod_{p \in F^{-1}(1, 0), q \in F^{-1}(0, 1)} |p \wedge q|^{-1/d^2}.$$ **REFERENCES**

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