We explore some curiosities in 4d susy RG flows. One issue is that the compelling candidate a-function, from a-maximization with Lagrange multipliers, has a ‘strange branch,” with reversed RG flow properties, monotonically increasing instead of decreasing. The branch flip to the strange branch occurs where a double-trace deformation $\Delta W = O^2$ passes through marginality, reminiscent of the condition for the chiral symmetry breaking, out of the conformal window transition in non-susy gauge theories. The second issue arises from Higgsing vevs for IR-free fields, which sometimes superficially violate the a-theorem. The resolution is that some vevs trigger marginal or irrelevant interactions, leading to $\Delta a = 0$ and decoupled dilaton on a subspace of the moduli space of vacua. This is contrary to classical intuition about Higgsing. This phenomenon often (but not always) correlates with negative R-charge for the Higgsing chiral operator.
1. Introduction

Cardy’s conjecture [1] is that the conformal anomaly $a$ counts the d.o.f. of 4d CFTs, that all 4d RG flow endpoints satisfy $\Delta a \equiv a_{IR} - a_{UV} \leq 0$, and $a > 0$. Intuitively then, $-\Delta a$ gives the net length of the RG flow. There has been recent, renewed interest in the $a$-theorem, following the recent, compelling arguments based on $a$ anomaly matching [3] and associated unitarity constraints on the sign of $\Delta a$ [4,5]. For spontaneous conformal symmetry breaking (or explicit soft breaking, via spurions), $\Delta a$ is related in [4] to the inclusive total scattering cross section $\sigma_{\tau\tau}(s)$ for the dilaton,

$$-\Delta a \equiv a_{UV} - a_{IR} = \frac{f^4}{\pi} \int_{s>0} ds \frac{\sigma_{\tau\tau}(s)}{s^2} \geq 0.$$  

For exactly marginal deformations, there is no RG flow so $\Delta a = 0$. Intuitively, other, interacting deformation will lead to a RG flow and thus have $\sigma_{\tau\tau}(s) \neq 0$, and hence $\Delta a \neq 0$. We will here discuss counter-intuitive examples, of Higgsing flows with $\Delta a = 0$.

The statement that $\Delta a \leq 0$ for endpoints of RG flows is referred to as the “weak version” of the $a$-theorem. There is also the possibility of a stronger version: that there exists a monotonically decreasing $a$-function $\tilde{a}(g)$, that is critical at the ends of RG flows where it reduces to $a$, analogous to the $c$-function in 2d [3]. There is a 4d proposal [9] that was checked in perturbation theory, and explored more recently in e.g. [10,11]. In the susy context, there is another proposal [12,13,14], that we’ll further explore here.

For 4d $\mathcal{N} = 1$ susy theories, results of [15,16] relate $a$ at the endpoints of RG flows to superconformal $U(1)^{R_{\ast}}$ ’t Hooft anomalies. The $U(1)^{R_{\ast}}$ symmetry can often be exactly determined, using $a$-maximization [17] if needed. This “almost proved” the $a$-theorem in the supersymmetric context [17]. The “often” and “almost” qualifiers are needed because of accidental symmetries [18], which have crucial effects [18] if present, see also e.g. [19,20].

As observed in [12], $a$-maximization of $a(R, \lambda)$, with $\lambda$ Lagrange multipliers for the interaction constraints on the superconformal R-charges $R$, gives a compelling candidate...

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1. The fact that $a > 0$ at the endpoints follows from [2], since unitarity implies $c > 0$.

2. There is also the “strongest” conjectured version of the $a$-theorem, that RG flow is gradient flow of $\tilde{a}(g)$ with positive definite metric on the space of deformations. The validity of this conjecture, even in 2d conformal perturbation theory, awaits a better understanding of the appropriate metric on the space of deformations beyond lowest order [17]. In 4d non-susy theories, there are recent works aiming towards perhaps producing counter-examples, e.g. [8] and references therein.
for the stronger versions of the a-theorem: with \( a(\lambda) \) interpreted as an a-function along the RG flow, with \( \lambda \sim g^2 \) and with derivatives related to beta functions \([9,10,11,12]\).

We here explore a few open issues \([14]\). To briefly exhibit them, consider the function

\[
a_1(R) \equiv 3(R - 1)^3 - (R - 1),
\]

(1.2)

which enters in a-maximization \([17]\). As plotted in Fig. 1, \( a_1(R) \) has a local maximum at the free-field value, \( R = \frac{2}{3} \), and a local minimum at \( R = \frac{4}{3} \). Indeed \( a_1(R) = -a_1(2 - R) \), so \( a_1(R = 1) = 0 \), fitting with massive operators contributing \( a = 0 \).

One issue is a puzzling behavior of the conjectured a-function \( a(\lambda) \) in RG flows past \( R(\mathcal{O}) = 1 \), which involves switching branches of a square-root, from one that is normal to one that is strange \([14]\). The flow direction has puzzling reversals on the strange-branch, with \( a(\lambda) \) monotonically increasing rather than decreasing. The endpoints of the flow nevertheless satisfy the weak version of the a-theorem, \( \Delta a < 0 \).

We note that the two branches give operators \( \mathcal{O}_\pm \) with a similar relation as that between an operator and its Legendre-transform source “shadow” operator: the product \( \mathcal{O}_+ \mathcal{O}_- \) is a marginal superpotential term. The branch flip occurs at the dividing line between where the “double-trace” deformation \( \Delta W = \mathcal{O}_c^2 \) is relevant vs irrelevant. In the non-susy context, the lore (see e.g. \([22,23,24]\) and references cited therein) is that having a relevant \( \Delta \mathcal{L} = \mathcal{O}^2 \) triggers the phase change from conformal to chiral symmetry breaking, \( \langle \mathcal{O} \rangle \neq 0 \). The susy case is different, but perhaps the phenomena are somehow related.

![Figure 1: The function \( a_1(R) \).](image-url)
Another curiosity is how Higgsing is compatible with the a-theorem, particularly when the field with $\langle Q \rangle \neq 0$ has $R_{\text{micro}}(Q) \leq 0$. Whenever a theory is Higgsed, the uneaten matter contributions increase $a$, so the a-theorem relies on having a greater contribution from the eaten matter, which moves from point $C$ in Fig. 2 to $R_{\text{IR}}(Q_{\text{eaten}}) = 0$ [14]. This raises the question of what happens if initially point $C$ is at $R_{\text{micro}}(Q) \leq 0$, since then all contributions to $\Delta a$ are apparently contrary to the a-theorem. It is crucial in these cases to account for the accidental symmetries. We explore this issue further here. Needless to say, we do not find a contradiction with the (weak version of the) a-theorem.

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{changea.png}
\caption{The change in $a$ from Higgsing.}
\end{figure}

It is interesting how the contradiction is avoided: the Higgsing can be marginal or trigger an irrelevant interaction, so $\Delta a = 0$ in the end. This does not happen for weakly coupled gauge theories. Intuitively, Higgsing is always relevant, since operator vevs involve boundary conditions at long distance, and results in some fields getting massive, hence $\Delta a < 0$. In SQCD, for example, moving away from the origin of the moduli space generally leads to $\Delta a < 0$, whether the theory is in the conformal window or the IR free-magnetic phase. This fits with the argument of [1] that $-\Delta a$ is related as in (1.1) to the total dilaton scattering cross section. In the examples that we explore, however, the operator Higgsing triggers an irrelevant or marginal interaction. The endpoint of the (Wilsonian) flow has $\Delta a = 0$, meaning that the dilaton is a completely decoupled free field, with $\sigma_{\tau \tau}(s) = 0$.

An illustrative toy model has chiral superfields, $X$ and $\Phi$, with superpotential

$$W = hX\Phi^n.$$  

(1.3)
For $n = 1$, this is a mass term, while for $n \geq 2$ the coupling $h$ is irrelevant in the IR, $h \to 0$, so $X$ and $\Phi$ are IR free massless fields. Now consider deforming this theory by $\langle X \rangle \neq 0$. For $n = 2$, (1.3) changes from being (marginally) irrelevant to relevant, giving $\Phi$ a mass $m_\Phi \sim h \langle X \rangle$, so the IR theory now only has the massless modulus $X$, and $\Delta a < 0$. For $n \geq 3$, on the other hand, (1.3) remains irrelevant even for $\langle X \rangle \neq 0$. Since $\langle X \rangle \neq 0$ triggers an irrelevant interaction, an $\alpha$-function would increase along this flow, with the final result that $\Delta a = 0$ at the endpoint of the flow.

We consider susy gauge theory examples similar to this toy model, where $X$ can be a composite operator made up of charged matter constituents, $X = \prod_i Q_i^{p_i}$. Then $\langle X \rangle \neq 0$ corresponds to $\langle Q_i \rangle = v_i \neq 0$, which Higgses the gauge group. Classically, this gives gauge field and matter masses $\sim v_i$, so one would expect $\Delta a < 0$. Nevertheless, with IR free operators $X$, Higgsing $\langle X \rangle$ can trigger an interaction that can be either relevant or irrelevant, as in (1.3). In the latter case, $\Delta a = 0$. We note a frequent, though not strict, connection between this phenomenon and the sign of $R(Q)$.

**2. Review of susy results**

Supersymmetry relates the dilatation current to a particular $U(1)_R$ current, with

$$\Delta(O_{cp}) = \frac{3}{2} R(O_{cp})$$

for scalar chiral primary operators. This is interpreted as holding along the entire RG flow, even though the associated R-current is conserved only at the SCFT endpoints. Although only gauge invariant operators are observables, it is useful to assign R-charges to the microscopic gauge fields of the underlying lagrangian (if it exists). If the gauge invariant operator is $X = \prod_i Q_i^{p_i}$, then

$$R_{micro}(X) \equiv \sum_i p_i R(Q_i)$$

and $R(X) = R_{micro}(X)$ unless $X$ is an IR-free field:

$$R(X) = \begin{cases} R_{micro}(X) & \text{interacting } X \\ \frac{2}{3} & \text{IR free field } X. \end{cases}$$

In the IR free case, the $U(1)_R$ mixes with an accidental $U(1)_X$ symmetry, which acts only on the IR free field composite $X$. Having $R_{micro}(X) < 2/3$ is a sufficient (by unitarity)
condition that $X$ must be an IR-free field, though not necessary. See e.g. \[20,25,26\] for proposed diagnostics for IR free operators.

The exact superconformal R-symmetry locally maximizes \[17\]

$$a(R) = 3\text{Tr}R^3 - \text{Tr}R, \quad (2.4)$$

with conformal anomaly at the RG fixed point given by $a = a(R_*)$. If there are accidental symmetries associated with IR free field operators $X$, the superconformal R-symmetry maximizes a modified function \[18\]

$$a_{X=\text{free}}(R) = a_{\text{old}}(R) + \text{dim}(X)g(R_X), \quad (2.5)$$

where $\text{dim}(X)$ is the number of $X$ operators, and $g(R)$ is given in terms of (1.2) by

$$g(R) \equiv a_1(2/3) - a_1(R) = \frac{1}{9}(2 - 3R)^2(5 - 3R). \quad (2.6)$$

It was conjectured in \[12\] that $a$-maximization can be extended to the RG flow via

$$a(R, \lambda) = a(R) + \lambda_I \hat{\beta}^I(R) \to a(\lambda) \equiv a(R(\lambda), \lambda), \quad (2.7)$$

with $\lambda_I$ Lagrange multipliers and $\hat{\beta}^I(R)$ linear functions of $R$ that are proportional to the beta functions \[12,13,14,21\]. Here $a(R)$ is given by (2.4), or if there are accidental symmetries by (2.5) (patched to (2.4) where the accidental symmetry arises on the flow). The $\to$ in (2.7) involves determining $R(\lambda)$ by extremizing $a(R, \lambda)$.

The conjectured interpretation is that the $\lambda^I$ are related to the associated coupling constants, $\lambda^I \sim (g^I)^2$. The result for $R(\lambda)$ then gives the exact anomalous dimension $\gamma_i$ of microscopic Lagrangian fields $Q_i$ in terms of their one-loop anomalous dimensions $\gamma^{(1)}(g)$:

$$\gamma_{i \pm} = 3R_{i \pm}(Q_i) - 2 = 1 \mp \sqrt{1 - 2\gamma^{(1)}(g)}, \quad (2.8)$$

with $\gamma^{(1)}(g)$ linear in the $\lambda^I$, and our notation is that $\gamma_{i+}$ is the normal branch, which connects with perturbation theory. The $\gamma_{i-}$ solution in (2.8) is the strange branch, that we’ll discuss further. By construction, $a(\lambda)$ in (2.7) has

$$\frac{\partial a(\lambda)}{\partial \lambda_I} = \hat{\beta}^I = f^I_J(g)\beta^J(g). \quad (2.9)$$

\[3\] Note the normalization $a_{\text{here}} = \frac{32}{3}a_{\text{usual}}$, so $a(\text{free chiral}) = \frac{2}{9}$ and $a(\text{free vector}) = 2$
This suggests the possibility of RG gradient flow \[ 12,14,13 \]
\[
\frac{\partial a}{\partial g^I} = G_{IJ}(g) \beta^J(g), \quad \text{with} \quad G_{IJ}(g) \equiv f^K_J(g) \frac{\partial \lambda^K}{\partial g^I}, \quad (2.10)
\]
if it turns out that \( G_{IJ}(g) \) satisfies the conditions \( G_{IJ}(g) > 0 \) and \( G_{IJ} = G_{JI} \).

Near weak coupling, \( \lambda \ll 1 \) on the normal branch, (2.8) can be matched to perturbation theory, relating \( \lambda_I \) to \( g^I \), and matching (2.8) to perturbation theory \[12,13,14\]; moreover, \( a(\lambda) \) and \( G_{IJ}(g) \) in (2.10) nicely agree \[14\] with Osborn’s perturbative expressions \[27\]. The Lagrange-multiplier method can just as easily be applied to analyze deformations of a strongly coupled or non-Lagrangian initial SCFT, where again \( \lambda \ll 1 \) can be successfully compared with conformal perturbation theory around the UV SCFT \[21\].

3. Cases where \( a(\lambda) \) isn’t monotonically decreasing?

Extremizing the cubic function \( a(R, \lambda) \) in \( R \) in (2.7) has two branches of solutions, \( R_\pm(\lambda) \). The \( R_+(\lambda) \) solution is the normal branch, which connects with perturbation theory around the UV SCFT, while \( R_-(\lambda) \) is the strange branch. \( R_+(\lambda) \) locally maximizes \( a(R, \lambda) \), while \( R_- \) is a local minimum. Some RG flows go from the normal branch in the UV to the strange branch in the IR. A simple example where the flip to the strange branch is needed is the magnetic dual of SQCD, when the dual quarks have \( R(q) < \frac{1}{2} \) \[14\]. The strange branch is generally needed if some field has sufficiently large, positive anomalous dimension. In the simplest examples, there is an operator \( S \) such that the two branches have \( R_+(S) + R_-(S) = 2 \) and the branch flip occurs at \( R_+ = R_- = 1 \).

Recall the notion of “shadow fields” where, for each operator \( \mathcal{O} \) of dimension \( \Delta \), one formally introduces an operator \( \tilde{\mathcal{O}} \) of dimension \( d - \Delta \), whose one-point function can play the role of a source term for \( \mathcal{O} \). This is familiar from AdS/CFT, where the two branches in solving for \( \Delta_\pm(m) \) can be regarded as dimensions of conjugate operators \( \Delta_\pm = \Delta(\mathcal{O}_\pm) \).

With chiral superfields, it is natural to have shadow chiral superfields that are conjugate in terms of superpotential couplings, i.e. \( \mathcal{O}_\pm \) with \( R(\mathcal{O}_+) + R(\mathcal{O}_-) = 2 \). As we discuss, the branch flip to the strange branch corresponds to an exchange in the roles of the operators and their conjugate shadow fields \( \mathcal{O}_\pm \leftrightarrow \mathcal{O}_\mp \), and it occurs at \( \mathcal{O}_+ = \mathcal{O}_- = \mathcal{O} \), where the double trace deformation \( \Delta W = \mathcal{O}^2 \) is marginal.

To illustrate the strange branch phenomenon in a general context, consider deforming an initial SCFT by coupling some of its operators \( \mathcal{O} \) to some additional fields, \( S \):

\[
W = hS\mathcal{O}, \quad (3.1)
\]
with \( h \) the coupling constant. The UV limit of the flow has \( h \rightarrow 0, \ R_{UV}(S) = \frac{2}{3} \), and \( R_{UV}(\mathcal{O}) < 4/3 \) for (3.1) to be relevant in the IR. Generally, the RG flow from (3.1) affects both \( R_S \equiv R(S) \) and \( R_O \equiv R(\mathcal{O}) \), but for simplicity we first consider cases where only \( R_S \) flows, with \( R_O \) fixed. This is the case e.g. in the magnetic dual of SQCD, where \( S \rightarrow M \) and \( \mathcal{O} \rightarrow q\tilde{q}'s \), with \( R_O \) fixed, independent of \( h \), by the symmetry.

In this fixed \( R_O \) case, the Lagrange multiplier description (2.7) gives

\[
a(R_S, \lambda) = a_{UV} + N_S(a_1(R_S) - \frac{2}{9}) + \lambda(R_S + R_O - 2),
\]

(3.2)

where \( a_{UV} \) is the total \( a \) of the UV SCFT (including the UV-free \( S \) contributions), \( N_S \) is the number of \( S \) fields (\( N_s = N_f^2 \) in the SQCD example of [14]), and \( a_1(R_S) \) is the cubic function (1.2). Extremizing the cubic function (3.2) in \( R_S \equiv R(S) \) gives two solutions

\[
R_S(\lambda, \epsilon_S) = 1 - \frac{\epsilon_S}{3} \sqrt{1 - \frac{\lambda}{N_S}},
\]

(3.3)

with \( \epsilon_S = \pm 1 \) labelling the two branches, with \( \epsilon_S = +1 \) the normal branch, and \( \epsilon_S = -1 \) the strange branch. The normal branch has \( \frac{2}{3} \leq R_S \leq 1 \), with \( 0 \leq \lambda \leq \lambda_{max} = N_S \), and includes conformal perturbation theory around the the UV limit of the RG flow. The strange branch is needed for \( R_S > 1 \), and has \( 1 \leq R_S \leq \frac{4}{3} \) (\( R \leq 4/3 \) was also emphasized recently in [28]), corresponding to \( \lambda_{max} \geq \lambda \geq 0 \) [14].

If \( R_O \geq 1 \), then the theory stays on the normal branch, with no puzzling behavior. If, on the other hand, \( R_O < 1 \), then eventually \( R_S \) flows to \( R_S > 1 \), which requires going to the strange branch. In this process, \( \lambda \) initially increases from \( \lambda = 0 \) to \( \lambda = \lambda_{max} \) on the normal branch, and thereafter its flow direction must reverse, while on the strange branch, for \( R_S \) to continue to increase past \( R_S = 1 \). As an extreme example, if \( R_O = 2/3 \), then the theory flows first on the entire normal branch, from \( \lambda = 0 \) in the UV to \( \lambda = \lambda_{max} \) in the middle of the flow, and then the flow continues across the strange branch, back to \( \lambda = 0 \) in the extreme IR. There are two strange aspects of the strange branch: the \( \lambda \) flow direction reverses, and \( a(\lambda) \) increases rather than decreases along this flow.

For fixed \( \lambda \), the normal branch locally maximizes \( a(R, \lambda) \), whereas the strange branch locally minimizes it. Indeed, plugging (3.3) back into (3.2) gives

\[
a(\lambda, \epsilon_S) = a_{UV} + \frac{2}{9} N_S \left[ \epsilon_S \left( 1 - \frac{\lambda}{N_S} \right)^{3/2} - 1 \right] + \lambda(R_O - 1).
\]

(3.4)
So, for fixed $\lambda$, the value of $a$ is lower on the strange branch, $a(\lambda,+) > a(\lambda,-)$. Note that
\[ \frac{da}{d\lambda} = R_S + R_O - 2 = R_O - 1 - \frac{\epsilon_S}{3} \left( 1 - \frac{\lambda}{N_S} \right)^{1/2} = \tilde{\beta}_h \leq 0, \] (3.5)
with a zero only at the endpoint $\lambda = \lambda_*$ of the RG flow, $da/d\lambda|_{\lambda=\lambda_*} = 0$.

The sign (3.5) of the slope remains negative on both branches. On the normal branch, the RG flow has $d\lambda > 0$ and $da < 0$, whereas on the strange branch $d\lambda < 0$ and $da > 0$, so $a(\lambda)$ monotonically decreases as expected on the normal branch, but monotonically increases with the RG flow on the strange branch. Note also that
\[ \frac{d^2a}{d\lambda^2} = \frac{1}{6} \frac{\epsilon_S}{N_S} \left( 1 - \frac{\lambda}{N_S} \right)^{-1/2} \] (3.6)
is everywhere positive on the normal branch and everywhere negative on the strange branch, changing sign through $d^2a/d\lambda^2 = 0$ at $\lambda_{max}$. The branch flip point $\lambda_{max}$ is the global minimum of $a(\lambda,\epsilon_S)$, even though (3.5) is non-vanishing there. On the strange branch, $a(\lambda)$ monotonically increases, up to a local maximum rather than minimum.

Even though $a(\lambda)$ is not monotonically decreasing, the net change $\Delta a = a_{IR} - a_{UV}$ between the flow’s endpoints is consistent with the weak version of the a-theorem, regardless of $\lambda_*$’s branch:
\[ \Delta a = a(\lambda_*,\epsilon_S) - a_{UV} = -N_S g(R_S) < 0. \] (3.7)
Here $g(R_S)$ is given in (2.6), and $g(R_S) > 0$ since $R_S < 4/3$ ($g(R) > 0$ for $R < 5/3$). In particular, at weak coupling $R_S \approx \frac{2}{3} + \frac{1}{3} \gamma_S$, and $g(R_S) \approx \frac{1}{3} \gamma_S^2$.

Since $a(\lambda)$ is not monotonically decreasing, there are two options: (i) $a(\lambda)$ is not a good a-function after all, and needs modification; or (ii) $a(\lambda)$ can be salvaged as an a-function, with some reinterpretation. Regarding possibility (i), note that the $a_{UV}$ and $a_{IR}$ endpoints of $a(\lambda)$ are certainly correct, and we have found no associated contradiction with the weakest version of the a-theorem, $a_{IR} < a_{UV}$ for the RG flow endpoints. So any modification cannot alter the RG flow endpoints. This allows, for example, additive modifications proportional to powers of the beta-functions $a(g) \rightarrow a(g) + f(g) \beta(g)^p$. Because $R(\lambda)$ and $a(\lambda)$ satisfy compelling and nontrivial checks with perturbation theory on the normal branch \([12][13][14]\), we could try to arrange for such modifications to kick in only at high order in perturbation theory, to try to preserve the good aspects on $a(\lambda)$ on the normal branch while fixing the puzzling aspects on the strange branch, outside of the perturbative regime. Such a modification might work, but we have not found any naturally compelling candidate, nor a plausible reinterpretation for option (ii).
A related question is the qualitative shape of the function \( \lambda(h) \), which evidently cannot be one-to-one with the branch flip. For example, \( \lambda = 0 \) gives both \( R_S = 2/3 \), hence \( h = 0 \), on the normal branch, and \( R_S = 4/3 \), hence \( h \neq 0 \) strongly coupled, on the strange branch. The suggestion in [14] is that \( \lambda(h) \) has a “shark-fin” shape, with \( d\lambda(h)/dh^2 > 0 \), in order for the Jacobian factor in the metric \( G_{hh} \) in (2.10) to remain always positive.

The shark fin shape, however, does not really help with the puzzle. It is bad enough that the \( \lambda \) flow direction reverses on the strange branch. If \( d\lambda/dh^2 \) remains positive, then the \( h \) RG flow direction would also need to flip, switching from \( \dot{h} > 0 \) for the flow on the normal branch, to \( \dot{h} < 0 \) on the strange branch. Such a flow direction reversal for a physical coupling would be unphysical: since RG flows are first order, \( \dot{h} = -\beta(h) \), if \( \beta(h) \) is continuous it would have to flow through a zero, \( \dot{h} = \beta(h) = 0 \), where the flow stops.

An alternative possible qualitative shape for \( \lambda(h) \) is to have \( d\lambda/dh^2 < 0 \) after the branch flip, e.g. \( \lambda(h) = \lambda_{\text{max}} \sin(\pi h^2/2h_c^2) \), with branch flip at \( h = h_c \), \( \lambda = \lambda_{\text{max}} \). The advantage is that \( h \) can continue increasing after the branch flip, where \( \lambda \) decreases, consistent with \( \beta(h) \neq 0 \) away from the flow endpoints. The metric \( G_{hh} = f(h)d\lambda/dh \) (2.10) could then only be positive if \( f(h) = \hat{\beta}(R)/\beta(h) \) changes sign at \( h = h_c \). Since \( \hat{\beta}(R) \neq 0 \) until the endpoint, \( f(h_c) = 0 \) requires \( \beta(h_c) = \infty \), (e.g. analogous to the pole of the NSVZ beta function [29] at \( T(G)\alpha/2\pi = 1 \)). This merits further study. But neither shape seems to change the fact that \( a(\lambda) \) is not a monotonically decreasing function on the RG flow: it hits a minimum at the branch flip location and thereafter increases.

The example given in [14] is the magnetic dual of SQCD [30], here with some renaming: \( SU(N_c) \) SQCD with \( N_f^2 \) added singlets, \( S_{ij} \), coupled as in (1.1) to all the mesons \( O_{ij} = M_{ij} = Q_i \tilde{Q}_j \). The symmetries determine \( R(Q) = (N_f - N_c)/N_f \), so \( R(O) = R(M) = 2(N_f - N_c)/N_f \), along the entire RG flow in \( h \). The branch flip to the strange branch is needed if \( R(M) < 1 \), i.e. \( N_f < 2N_c \). The extreme case is the bottom of the conformal window, where \( R(O) \to \frac{2}{3}^+ \), and \( R(S) \to \frac{4}{3}^- \) in the IR, with \( \lambda \to 0 \) in the IR, on the strange branch. Even though \( a(\lambda) \) is not monotonic, the endpoints satisfy \( \Delta a < 0 \) (3.7).

Let us now discuss the branches phenomenon in the completely general formulation of [21] of a-maximization with Lagrange multipliers. The UV limit of the flow is an interacting SCFT \( \mathcal{P} \), deformed by superpotential deformations \( \Delta W = \sum_{\alpha} g^{\alpha} O_{\alpha} \) and/or gauging a subgroup of the global flavor group. Along the resulting flow, the superconformal R-current mixes with the remaining (ungauged) global symmetry currents \( J^\mu_a \):

\[
R = R_\mathcal{P} + d^aq_a, \quad (3.8)
\]
where \( d_a \) are real parameters that are determined by maximizing \( a(\lambda, d) \) w.r.t. \( d^a \):

\[
\frac{\partial a}{\partial d^a} = 0 = -6\tau_{ab} d^b + 9D_{abc} d^b d^c + \Lambda_a. \tag{3.9}
\]

See [21] for details and notation and, for convenience, we here define \( \Lambda_a \equiv \lambda^\alpha \bar{\lambda}^\beta (T_a)^\beta_\alpha - \lambda_G k_a \). The \( \tau_{ab} \) in (3.9) are the coefficients of the global current two-point functions, and \( D_{abc} \) are the ’t Hooft anomaly coefficients, \( D_{abc} = \text{tr} Q_a Q_b Q_c \), both evaluated in the UV SCFT \( \mathcal{P} \). We can choose the sign of the \( Q_a \) charges such that all \( D_{aaa} > 0 \). In conformal perturbation theory near \( \mathcal{P} \), for small \( \Lambda^a \), the solution of (3.9) is

\[
d^a \approx \frac{1}{6}(\tau^{-1})^{ab} \Lambda_b \equiv \frac{1}{6} \Lambda^a,
\]

on the normal-branch solution of (3.9).

The strange branch is needed for sufficiently large \( d^a \). For example, for mixing with a single global current (so we can drop the \( a, b, c \) indices) the solution of (3.9) is

\[
d(\Lambda) = \frac{\tau}{3D} \left( 1 - \epsilon \sqrt{1 - \Lambda D\tau^{-2}} \right), \tag{3.10}
\]

with \( \epsilon = +1 \) the normal branch, and \( \epsilon = -1 \) the strange branch, with the branch flip at

\[
d_{\text{flip}} = d(\Lambda_{\text{max}}) = \frac{\tau}{3D}; \quad \Lambda = \Lambda_{\text{max}} = \tau^2/D. \tag{3.11}
\]

Note also that \( d \) has maximum value, \( d_{\text{max}} = 2d_{\text{flip}} \), which is achieved on the strange branch, \( \epsilon = -1 \), at \( \Lambda = 0 \). In terms of the R-charge (3.8), the branch flip occurs at

\[
R_{\text{flip}} = R_{\mathcal{P}} + \frac{\tau}{3D} Q. \tag{3.12}
\]

Taking \( Q \) to act on the operators as \( Q \mathcal{O}_\alpha = T^\beta_\alpha \mathcal{O}_\beta \), the branch flip happens when the anomalous dimensions of operators change by a sufficiently positive amount, given by

\[
(\gamma_{\text{flip}})^\beta_\alpha = (\gamma_{\mathcal{P}})^\beta_\alpha + \frac{\tau}{D} T^\beta_\alpha. \tag{3.13}
\]

The case (3.3) is mixing with \( U(1)_S \), with \( Q(S) = S \), so \( \tau = D = 1 \), and (3.10) reduces to agree with (3.3) and following expressions.
4. “Double trace” deformations

In the simplest examples (3.1), the branch flip happens at $R(\mathcal{O}) = 1$, which is also where the double-trace operator $\Delta W \sim \mathcal{O}^2$ crosses through marginality. Including the field $S$, the flow from the normal to the strange branch happens when $\Delta W = m_S S^2$ crosses from being relevant to irrelevant. Perhaps this could help resolve the branch flip puzzles.

As we discuss in this section, there is a close connection between the theory (3.1) with added singlets, and the double trace deformations. Indeed, the singlets with coupling (3.1) can emerge dynamically from the double-trace deformations of an initial SCFT by

$$\Delta W = h \mathcal{O}^2. \tag{4.1}$$

We can also consider the product of two different operators,

$$\Delta W = h \mathcal{O}_1 \mathcal{O}_2, \tag{4.2}$$

where the operators $\mathcal{O}_{1,2}$ can even be operators in two initially decoupled theories, $(\text{SCFT})_1$ and $(\text{SCFT})_2$ (see also e.g. [31]). To give a concrete example, we can consider SQCD and (4.1) could be the square of some meson operators, or (4.2) could be meson operators in two different copies of SQCD. If $R(\mathcal{O}) < 1$ (or $R(\mathcal{O}_1) + R(\mathcal{O}_2) < 2$ for (4.2)) in the initial SCFT, then (4.1) is relevant\(^4\) and drives the theory somewhere new.

The correct way to analyze the RG flow is via the usual trick of introducing some new fields $S$, which yields the double-trace deformation (4.1) upon integrating out $S$

$$W = h_S S \mathcal{O} + \frac{1}{2} m_S S^2. \tag{4.3}$$

Likewise, we get (4.2) by adding massive fields $S_1$ and $S_2$, with

$$W = h_1 S_1 \mathcal{O}_1 + h_2 S_2 \mathcal{O}_2 + m_S S_1 S_2 \tag{4.4}$$

The fields $S$ in (4.3) and (4.4) start as free fields, $R_{UV}(S) = 2/3$ for $h = 0$, where the $m_S$ terms are relevant. But one should first analyze the RG flow $h \to h_*$ with $m_S = 0$,

$$W_{\text{int}} = h_S S \mathcal{O}, \tag{4.5}$$

and subsequently include the $m_S \neq 0$ term, as a deformation of that theory. This captures the fact that the $m_S$ terms kick in later in the RG flow to the IR.

\(^4\) If $R(\mathcal{O}) = 1$, one can determine if $\Delta W$ is exactly marginal or marginally irrelevant [32,21].
Let $R_0(O) \equiv R_{UV}(O)$ be the superconformal R-charge at the UV starting point, with all couplings in (4.3) or (4.4) set to zero, and $R_1(O)$ denote the superconformal R-charge at the IR endpoint of (4.3) ($h_S = h_*, m_S = 0$), and $R_2(O)$ denote that at the IR endpoint of (4.3) or (4.4) ($h_\neq 0$, $m_S \neq 0$). Since we suppose (4.1) is relevant, $R_0(O) < 1$, which ensures that the $h$ term in (4.3) is relevant (that only requires $R_0 < 4/3$). The $m_S$ coupling in (4.3) is irrelevant if $R_1(S) > 1$, which means $R_1(O) < 1$, and which is often the case. For example, if $R_0(O) = R_1(O)$, the condition needed for (4.1) to be relevant also implies that the $m_S$ term in (4.3) is irrelevant, $m_S \to 0$.

When the $m_S$ term is irrelevant, the upshot of the double trace deformation (4.1) is an emergent new field, $S$, with superpotential (4.5) so $R(S) = 2 - R(O)$. The $F_S$ equation of (4.5) removes $O$ from the chiral ring. So the upshot is to replace $O \to S$, with $R(O)$ replaced with that of $S$, $R(S) = 2 - R(O)$. This effect has already been discussed in the literature, in the context of many specific examples – see e.g. [33] for SQCD with the meson-squared deformation. In the context of ADS/CFT, the bulk field with mass $m$ maps to operators $O_\pm$ with $\Delta_+ + \Delta_- = 4$, so one is naturally an operator in the CFT and the other is its shadow operator. Then the flow of $hO_-^2$ and associated IR replacement $O_- \to O_+$ is again well known, see e.g. [34].

To connect with the previous section, note that a double trace deformation leads to the emergent $S$ field, with $m_S \to 0$ in the IR, precisely when the flow takes $R(S) > 1$. As in (3.3), this coincides with where the RG flow flips to the $\epsilon_S = -1$ branch.

5. Examples: SQCDM$^2$

In this section we discuss a class of examples that are of interest, both because they illustrate double trace deformations, and for application to later sections, where we consider examples of $\langle Q \rangle \neq 0$ Higgsing of fields with $R_{\text{micro}}(Q) < 0$.

Consider $SU(N_c)$ SQCD with $N_f + N'_f$ fundamental flavors, $N'_f$ of which enter in

$$W_{\text{tree}} = h^{i'i'j'j'}(Q_{i'i'}^{\dagger} \tilde{Q}_{i'i'}) (Q_{j'j'}^{\dagger} \tilde{Q}_{j'j'}).$$

We refer to this theory as SQCDM$^2$. The case where all flavors enter in (5.1), i.e. $N_f = 0$, was considered in earlier works, e.g. [35]. In general, (5.1) is relevant for $N'^{\text{tot}}_f = N_f + N'_f < 2N_c$, in which case (5.1) drives a RG flow to a new IR fixed point. Let us define

$$R(Q) = R(\tilde{Q}) \equiv y, \quad x \equiv N_c/N_f, \quad n \equiv N'_f/N_f.$$

12
Naively, the IR fixed point theory has $R(Q') = R(\tilde{Q}') = \frac{1}{2}$ for the $N'_f$ fields in (5.1), since $R(W) = 2$, and then $U(1)_R$ conservation (anomaly free) gives for the remaining $N_f$ fields

$$y_{\text{naive}} = \frac{n}{2} + 1 - x.$$  \hfill (5.3)

For later use, we note that there is a range of allowed $n$ and $x$ where $y_{\text{naive}} < 0$, suggesting that these theories as candidates for exploring Higgsing by $R < 0$ operators, in this case by $M = Q\tilde{Q}$. However, as we now discuss, the correct treatment of the double trace deformation (5.1) shows that this class of examples actually always has $R(Q) > 0$.

5.1. SQCD coupled to added singlets: SSQCD

The intermediate theory, needed to analyze the double-trace theory (5.1), is the SSQCD theory of [14] (see also e.g. [36]): $SU(N_c)$ SQCD with $N_f + N'_f$ fundamental flavors, coupled to $N'_f^2$ singlets

$$W = h \sum_{i', j'=1}^{N'_f} S^{i'j'} Q'_{i'} \tilde{Q}'_{j'}.$$  \hfill (5.4)

For $N'_f = 0$ it is SQCD and for $N_f = 0$ it is a magnetic dual SQCD. For $N_f$ and $N'_f$ both non-zero, both $R(S)$ and $R(Q')$ vary along the RG flow, and a-maximization is required. Using the definitions (5.2), the R-charge conservation constraints determine

$$R(Q') = R(\tilde{Q}') = 1 + \frac{1 - y - x}{n}, \quad R(S) = 2 \left( \frac{x + y - 1}{n} \right).$$  \hfill (5.5)

The vacuum stability bound condition, needed to avoid $W_{\text{dyn}}$ or a deformed moduli space, is $N_f + N'_f > N_c$, i.e. $x < 1 + n$. a-maximization is then used to determine $y(x, n)$ [14].

For $x \geq x_M(n)$, there is an accidental symmetry associated with $M = Q\tilde{Q}$ being IR free, with $x_M(n)$ determined by the unitarity bound for $M$: $y(x_M(n)) = 1/3$. The SSQCD theory has a $SU(N_f - N_c)$ dual [14], with superpotential (here we add the $m_S$ term)

$$W_{\text{dual}} = SM' + M'q\tilde{q} + Mq'\tilde{q}' + Pq'\tilde{q} + P'q\tilde{q}' + \frac{1}{2}m_S S^2.$$  \hfill (5.6)

Knowledge of the dual shows that there are additional accidental symmetries, that of the IR free magnetic phase, when $x \geq x_{FM} = \frac{2}{3}(1 + n)$. For $n$ large enough, there is a range $x_M(n) < x < x_{FM}$ where $M$ has hit the unitarity bound and decoupled, but the rest of
the theory remains interacting. In this case, the \( M q' q' \) term in (5.6) becomes irrelevant. a-maximization shows that \( y \) remains everywhere positive in any case.

Note that SSQCD has a moduli space where \( \langle S \rangle \neq 0 \). For generic \( \langle S \rangle \) we can integrate out the massive \( Q' \bar{Q}' \) matter fields, and the low-energy theory consists of \( SU(N_c) \) SQCD with \( N_f \) matter fields \( Q, \bar{Q} \). If \( N_f < N_c \), that theory can generate a superpotential,

\[
W_{dyn} = (N_c - N_f) \left( \frac{\det S \Lambda^{3N_c-N_f-N'_f}}{\det M} \right)^{1/(N_c-N_f)}.
\]

(5.7)

5.2. Back to the double-trace theories SQCDM^2

The double trace theory (5.1) is obtained as a \( m_S \) deformed version of the SSQCD theory discussed in [14] and the previous subsection:

\[
W_{elec} = S^{i'j'}Q_{i'}\bar{Q}_{j'} + \frac{1}{2} (m_S)_{i'\bar{i},j'\bar{j}} S^{i\bar{i}} S^{j\bar{j}}, \tag{5.8}
\]

as in (4.3). We can now analyze whether the \( m_S \) term is a relevant or irrelevant deformation of the SSQCD fixed point. If \( m_S \) is relevant, the IR limit of SQCDM^2 has massive \( S^{ij} \) fields, with \( R(S) = R(Q') = R(\bar{Q}') = 1 \), and \( R(Q) \) given by (5.3). If \( m_S \) is irrelevant, the IR limit of SQCDM^2 is the same as that of SSQCD, with emergent \( S^{ij} \) fields.

For the \( N_f = 0 \) case [33,33], the flavor symmetry fixes \( R(Q' \bar{Q}') = 1 - N_c/N'_f \) and then the condition needed for the double-trace term to be relevant, \( N'_f < 2N_c \), always implies that the \( m_S \) term in (5.8) is irrelevant. So this case always has the emergent \( S \) fields (5.8), with \( m_S \to 0 \). In the magnetic dual description, the original quartic superpotential (2.1) is a mass term for the magnetic mesons \( M' \sim Q' \bar{Q}' \), which can be integrated out leading to a quartic superpotential in the dual, \( W_{dual} = \tilde{h}(q\bar{q})^2 \), which is IR irrelevant, \( \tilde{h} \to 0 \). Then \( q\bar{q} \to S \) emerge from the magnetic dual in the IR as new chiral primaries.

For general \( N_f \) and \( N'_f \), the \( m_S \) term in (5.8) is relevant if \( R(S) < 1 \), i.e. if a-maximization in SSQCD yields \( y(x,n) < 1 + \frac{n}{2} - x \), and \( m_S \) is irrelevant otherwise. When \( m_S \) is relevant, its effect is to replace \( y_{SSQCD} \to 1 + \frac{n}{2} - x \), as in (5.3), so in the IR

\[
y_{SQCDM^2}(x,n) = \max(y_{SSQCD}(x,n), 1 + \frac{n}{2} - x). \tag{5.9}
\]

Using a-maximization it is found that \( y_{SSQCD}(x,n) > 0 \) [14], so it follows from (5.9) that \( y_{SQCDM^2}(x,n) > 0 \), unlike (5.3) which can be negative, so \( R_{micro}(M) > 0 \) in all case.

In the later sections we will discuss examples where Higgsing leads to \( \Delta a = 0 \). Neither SSQCD nor SQCDM^2 are like that: in both theories, taking e.g. \( \langle M \rangle \neq 0 \), drives a relevant interaction, with \( \Delta a < 0 \), even in the case where \( M \) is IR free. In the dual (5.6), this is because \( R(q'\bar{q}') < 2 \), so \( \langle M \rangle \) always triggers a relevant deformation, leading to \( \Delta a < 0 \).
6. Higgsing $\langle Q \rangle$: when can it be irrelevant / marginal?

Superperconformal theories generally have a moduli space of vacua, and one can consider deforming the SCFT at the origin by giving expectation values to moduli fields, $\langle X \rangle = v \neq 0$, spontaneously breaking the conformal symmetry, with the modulus $X$ the massless dilaton. The beta functions vanish, but there can be a RG flow in a Wilsonian sense, with $v \to 0$ in the UV limit and the massive modes from $v \neq 0$ integrated out in the IR. The IR limit has a restored, linearly realized conformal symmetry for the remaining massless fields. The “flow” has $a_{UV} = a_{\text{origin}}$ and $a_{IR} = a_{v \neq 0}$, with $-\Delta a \equiv a_{UV} - a_{IR} \geq 0$ related as in (1.1) [4] to the total scattering cross section $\sigma(s)$ for the modulus $X$.

Intuitively, since $\langle X \rangle \neq 0$ changes the field boundary conditions at large distance, it should always be “relevant,” and non-trivially affect the theory in the IR. This classical intuition suggests that vevs $\langle X \rangle \neq 0$ should lead to $\Delta a < 0$, fitting with $\sigma(s) \neq 0$ in (1.1). We will discuss examples where this classical intuition is wrong, and instead $\langle X \rangle$ triggers an irrelevant deformation. Away from the RG flow endpoint, it should effectively lead to an increase of a hypothetical $a$-function. In the deep IR, the irrelevant deformation relaxes away, and the endpoints satisfy $\Delta a = 0$. According to (1.1), $\Delta a = 0$ means that $\sigma_{\tau \tau}(s) = 0$, i.e. the modulus $\tau$ is a completely decoupled free field. We refer to this as irrelevant / marginal Higgsing.

Irrelevant / marginal Higgsing is indeed unusual, e.g. it does not happen in $\mathcal{N} = 4$ or in $\mathcal{N} = 1$ SQCD Higgsing. In those cases, integrating out the massive fields, e.g. the W-bosons on the Coulomb branch of $\mathcal{N} = 4$, leads to dilaton scattering derivative interactions (e.g. superpartners of the $F^4$ terms), and hence $\sigma_{\tau \tau}(s) \neq 0$ and $\Delta a \neq 0$. Likewise, in SQCD, taking $\langle M \rangle \neq 0$ is always a relevant deformation, leading to $\Delta a \neq 0$, even in the free-magnetic range of $N_f$ [17], where the mesons $M$ are IR free. If, for example, the gauge group is completely broken on the moduli space, the low-energy theory in the bulk of the moduli space $\mathcal{M}$ consists of the IR-free moduli fields, so $a_{\text{bulk}} = \frac{3}{2} \dim_C(\mathcal{M})$. The a-theorem implies that the theory at the origin has $a_{\text{origin}} \geq a_{\text{bulk}}$. This is indeed what happens in e.g. SQCD for all $N_f > N_c$.

Again, from the perspective of Fig 2, the fact that $\Delta a \leq 0$ for Higgsing RG flows is non-obvious [14], particularly when the fields $Q$ with $\langle Q \rangle \neq 0$ have $R_{\text{micro}}(Q) < 0$. We will indeed see a frequent correlation between the irrelevant / marginal Higgsing phenomenon associated with $\langle X \rangle$ and negative R-charge, $R_{\text{micro}}(X) \leq 0$. We can also motivate this connection in terms of gauge invariant operators, by considering

$$W = hL(X - X_{\text{micro}}), \quad (6.1)$$
where \( X_{\text{micro}} = \prod_i Q_i^{p_i} \), and \( X \) and \( L \) are added fields, with \( R = 2/3 \) in the UV limit. The \( L X_{\text{micro}} \) term drives a RG flow to where \( R(L) = 2 - R(X_{\text{micro}}) \). The \( L X \) term in (6.1) is relevant if \( R(L) < \frac{4}{3} \), i.e. if \( R_{\text{micro}}(X) > 2/3 \); the added \( L \) and \( X \) fields are then massive and can be integrated out, \( L \to 0 \), \( X \to X_{\text{micro}} \). For \( R(X_{\text{micro}}) \leq 2/3 \), the \( LX \) term in (6.1) is irrelevant, so \( X \) is an emergent, IR-free decoupled field [14].

For \( 0 < R_{\text{micro}}(X) < \frac{2}{3} \) the \( hLX \) interaction is dangerously irrelevant, in that it becomes relevant for \( \langle X \rangle \neq 0 \). This simple argument suggests that \( \langle X \rangle \neq 0 \) will be different if \( R_{\text{micro}}(X) \leq 0 \), since then \( R(L) > 2 \) and \( W = hXL \) remains irrelevant for \( \langle X \rangle \neq 0 \).

We note that irrelevant / marginal \( \langle X \rangle \) indeed often correlates with \( R_{\text{micro}}(X) < 0 \), but we note examples where that is not the case. In one class, operators with \( R_{\text{micro}}(X) < 0 \) have relevant \( \langle X \rangle \). In another class, operators with \( R_{\text{micro}}(X) > 0 \) have irrelevant \( \langle X \rangle \).

Having operators with \( R_{\text{micro}}(X) < 0 \) raises the possibility of a vacuum instability, having conformal symmetry broken by a dynamically generated superpotential, \( W_{\text{dyn}} \). We are here interested in theories with \( W_{\text{dyn}} = 0 \), despite having \( R_{\text{micro}}(X) < 0 \) operators.

7. Some known, IR free examples of Higgsing with \( \Delta a = 0 \) moduli spaces.

The irrelevant / marginal Higgsing phenomenon can be found in a few examples already appearing in the literature (the \( \Delta a = 0 \) curiosity was not noted or emphasized).

7.1. \( R_{\text{micro}}(X) = 0 \) examples with \( \Delta a = 0 \), e.g. SQCD for \( N_f = N_c \)

The matter fields here have \( R_{\text{micro}}(X) = 0 \), compatible with the quantum modified constraint [37], which removes the origin from the moduli space. The low-energy theory on the smooth moduli space is an IR-free theory of the constrained moduli fields \( \mathcal{M} \) (with a WZ term to account for ’t Hooft anomaly matching of the global symmetries [38]), so \( a_{\text{IR}}(\mathcal{M}) = \frac{2}{9} \dim_C \mathcal{M} \). Moving on the moduli space has \( \Delta a = 0 \), so \( \langle X \rangle \) can be regarded as “marginal.” Different points on the moduli space can nevertheless be physically distinguished, by their different massive spectrum and different IR unbroken global symmetries.

This case can be contrasted with SQCD for \( N_f \geq N_c + 1 \), where \( \Delta a < 0 \) and \( R(X) > 0 \). There the moduli space \( \mathcal{M} \) is singular, with additional massless states near the origin, giving \( a_{\text{origin}} > a_{\text{bulk}}(\mathcal{M}) \), where \( a_{\text{bulk}}(\mathcal{M}) = \frac{2}{9} \dim_C \mathcal{M} = \frac{2}{9}(2N_fN_c - (N_c^2 - 1)) \). For example, for \( N_f = N_c + 1 \), \( a_{\text{bulk}}(\mathcal{M}) = \frac{2}{9}N_c^2 \) while the theory at the origin has \( a_{\text{origin}} = \frac{2}{9}(N_c^2 + 2N_f) > a_{\text{bulk}} \), thanks to the additional massless \( B^i \) and \( \tilde{B}^i \) fields at \( M_{ij} = 0 \) [37].
7.2. $R_{\text{micro}}(X) < 0$ affine moduli space examples with $\Delta a = 0$

There are examples where $R_{\text{micro}}(X) < 0$, with $W_{\text{dyn}} = 0$ thanks to a quantum-branch cancellation. The original examples are $SO(N_c)$ with $N_f = N_c - 4$ fundamental flavors $Q_f$ [33]: there are discrete quantum parameters, $\epsilon_{1,2} = \pm 1$, and $W_{\text{dyn}} = 2(\epsilon_1 + \epsilon_2)(\Lambda^{2(N_c - 1)}/\det M)^{1/2}$ vanishes on the $\epsilon_1 = -\epsilon_2$ branches. The $W_{\text{dyn}} = 0$ branch has a smooth moduli space $M$, with no additional massless fields. Indeed, the moduli saturate the ’t Hooft anomalies. Theories of this type, with simple gauge group and $W_{\text{tree}} = 0$, were classified in [40]. The T1-T6 theories there have $\mu_{\text{matter}} < T(G)$, and thus $R_{\text{micro}}(X) < 0$ fields, and are all IR-free, with $a_{IR}(M) = \frac{2}{9} \dim_C M$, constant on all of $M$. In all these theories, the $W_{\text{dyn}} = 0$ branch is lifted (broken susy) upon adding $W_{\text{tree}} \neq 0$.

So Higgsing on $M$ has $\Delta a = 0$: the moduli vevs $\langle M \rangle$ are marginal. According to (1.1), the massless moduli are thus completely decoupled from any other fields, $\sigma(s) = 0$.

7.3. Examples with $\Delta a \neq 0$ Higgsing, fitting with $R_{\text{micro}}(X) > 0$.

Recall the example of $SU(2)$ with matter field $Q \in 4$ [41], with $R_{\text{micro}}(Q) = 3/5$. The $W_{\text{tree}} = 0$ theory has a moduli space $M \cong \mathbb{C}$, with modulus $X = Q^4$. Since $X$, with $R_{\text{micro}}(X) = 12/5$, saturates the $\text{Tr} R$ and $\text{Tr} R^3$ anomaly matching, there are two scenarios [41] for the IR dynamics at $X = 0$: (i): $X$ is a decoupled, IR-free field (then $R_{IR}(X) = 2/3$, via accidental symmetry), or (ii) it is a SCFT, with $R_{SCFT}(X) = 12/5$ (with misleading anomaly matching [42]). Several diagnostics [20,43,25,44] suggest that the correct IR phase is probably (ii), SCFT (so $\Delta W = \lambda X$ is irrelevant, instead of susy-breaking).

The IR free phase scenario (i) would have had $a_{IR,free} = \frac{2}{9}$ everywhere on $M$, so $\Delta a = 0$. The presumably correct SCFT scenario (ii), on the other hand, has $a_{\text{origin}} = 3(\frac{2}{5})^3 - \frac{7}{5}$ (assuming that there are no overlooked accidental symmetries), and $a_{\text{bulk}} = \frac{2}{9}$, so Higgsing by $\langle X \rangle$ has $\Delta a = a_{\text{bulk}} - a_{\text{origin}} < 0$. Assuming scenario (ii) is correct thus exhibits the frequent correlation between $R_{\text{micro}}(X) > 0$ and $\Delta a \neq 0$. More generally, the proposed diagnostic of [21] automatically ensures that Higgsing has $\Delta a$ of correct sign.

Similarly, the $SO(N_c)$ with symmetric tensor examples [42], and the other $\mu_{\text{matter}} > T(G)$ examples classified in [40], all have $R_{\text{micro}}(X) > 0$ moduli, and all lead to SCFTs at the origin, with $a_{SCFT} > a_{IR,free}$. So Higgsing $\langle X \rangle \neq 0$ gives $\Delta a = a_{IR,free} - a_{SCFT} < 0$, non-zero, again correlating $R_{\text{micro}}(X) > 0$ with $\langle X \rangle$ “relevant,” leading to $\Delta a \neq 0$. 

17
7.4. Δa = 0, R_{micro}(X) < 0 examples, with higher power irrelevant interactions

Consider the theory of [13,16,17], SQCD with adjoint X and W_{tree} = TrX^{k+1}, choosing N_f and N_c such that N_c \equiv kN_f - N_c = 1. The magnetic “SU(1)” group is then trivial, corresponding to a smoothly-confining theory, with ’t Hooft anomalies matching between the original electric theory and an IR free dual “confined” theory of generalized mesons and baryons. Consider in particular the k = 2 case, W = TrX^3, with 2N_f - N_c = 1. The dual fields are the generalized mesons and baryons M_1 = (\bar{Q}Q), M_2 = \bar{Q}XQ, B = (Q^{N_f}(XQ)^{N_f-1}), \tilde{B} = (\bar{Q}^{N_f}(X\bar{Q})^{N_f-1}), with [48],

\[ W = \frac{1}{N_f-1} \Lambda^{2N_f-4} (\tilde{B} M_2 \tilde{B} - \det M_2 (M_1 \text{cof} M_2)). \] (7.1)

The mesons have R_{micro}(M_1) = 2 - 4(2N_f - 1)/3N_f < 0 and R_{micro}(M_2) = \frac{8}{3} - 4(2N_f - 1)/3N_f > 0. For large N_f, R_{micro}(M_1) \to -\frac{4}{3} and R_{micro}(M_2) \to 0^+. Of course, since the dual theory is IR free, the physical values are R(M_1) = R(M_2) = R(B) = R(\tilde{B}) = 2/3. The values of R_{micro} remain nevertheless useful to characterizing the irrelevant interactions in (7.1). Clearly, \langle M_2 \rangle is relevant, since it gives a quadratic mass term to B and \tilde{B}; this fits with R_{micro}(M_2) > 0. On the other hand, for \langle M_2 \rangle = 0, it is evident from (7.1) that \langle M_1 \rangle has no effect; this is a marginal subspace of the moduli space, with Δa = 0. This correlates with R_{micro}(M_1) < 0, and is analogous to the toy model \[(1.3)\] for n > 2.

7.5. R_{micro}(X) < 0 counter-examples, with relevant \langle X \rangle leading to Δa ≠ 0.

Recall the case of SO(N_c) with N_f = N_c - 3 \[39\]: the mesons M_{fg} = Q_f \cdot Q_g have R_{micro}(M) < 0, and there is a branch with a quantum moduli space of the unconstrained mesons M_{fg}, with additional massless fields at the origin, W_{low} = M_{ij}q^i q^j. Despite the fact that R_{micro}(M) < 0, and thus R_{micro}(q^2) > 2, taking \langle M \rangle ≠ 0 gives a mass deformation, which is always relevant, leading to Δa < 0. Of course, R(M) = R(q) = 2/3, since the theory is IR free. Even though R_{micro}(M) < 0, the interactions are similar to the n = 2 case of \[(1.3)\] and the above 0 < R_{micro}(X) < 2/3 examples.

7.6. R_{micro}(X) > 0 counter-examples, with irrelevant \langle X \rangle leading to Δa = 0.

Recall the case of Sp(N_c) with N_f = N_c + 2 fundamental matter fields [49]. There is a moduli space given by expectation values of M_{ij} = Q_{ic}Q_{jd}J^{cd}, in the N_f(2N_f - 1) antisymmetric rep of SU(2N_f), with R_{micro}(M) = 2/N_f and superpotential interactions

\[ W = -\frac{\text{Pf } M}{2N_c - 1} \Lambda^{2N_c + 1}, \] (7.2)
and $F_M = 0$ gives the classical moduli space constraint $\frac{1}{2} \text{rank}(M) \leq N_c$. For the case $N_c = 1$, $Sp(1) \cong SU(2)$, the superpotential (7.2) is cubic, and $\langle M \rangle \neq 0$ gives a relevant mass terms for the non-classical, additional $M_{ij}$ fields that are massless at the origin, hence $\Delta a < 0$. This $Sp(1)$ case is in the $SU(N_c)$ with $N_f = N_c + 1$ family, where $\langle M_{ij} \rangle \neq 0$ always gives a mass to some additional baryons, leading always to $\Delta a < 0$.

For higher $Sp(N_c)$, on the other hand, the superpotential (7.2) is order $N_f = N_c + 2$ in the $M$ fields, so $\langle M \rangle$ of low enough rank can trigger an irrelevant interacting, leading to $\Delta a = 0$ in the far IR. This happens on a subspace of the moduli space, where $\langle M \rangle$ has $\frac{1}{2} \text{rank}(\langle M \rangle) \leq N_c - 1$, since then the interaction remaining from (7.2) is cubic or higher order, so is irrelevant, giving $\Delta a = 0$ on this noncompact subspace including the origin:

$$a(\langle M \rangle) = \left\{ \begin{array}{ll}
\frac{2}{9} N_f (2N_f - 1) & \text{for } \frac{1}{2} \text{rank}(M) \leq N_c - 1 \\
\frac{2}{9} N_c (2N_c + 7) & \text{for } \frac{1}{2} \text{rank}(M) = N_c.
\end{array} \right.$$

(7.3)

The $\Delta a = 0$ Higgsing phenomenon occurs on that subspace of the full moduli space where $Sp(N_c)$ is only partially Higgsed, with at least a $Sp(1)$ unbroken, where all $M_{ij}$ fields remain massless and IR free. When $\frac{1}{2} \text{rank}(M) = N_c$, the $Sp(N_c)$ group is fully broken, and the non-classical $M_{ij}$ components have becomes massive; there the the $a$ value in (7.3) comes from simply counting the $Q$ matter fields left uneaten after the Higgs mechanism.

8. Examples of relevant, $\Delta a \neq 0$ Higgsing for $R_{\text{micro}}(X) > 0$ interacting SCFTs

Consider $SU(N_c)$ gauge theory with $N_f$ fundamentals, $Q_f$, $\tilde{Q}_f$ and $N_a$ adjoints, $X_i$, taking $W_{\text{tree}} = 0$. For $N_a = 0$, the theory is SQCD; for $N_a = 1$, the theory flows to the $\hat{A}$ SCFTs (for $1 < N_f < 2N_c$); for $N_a = 2$, the theory flows to the $\hat{O}$ SCFTs (for $N_f < N_c$). See [18,19] for the detailed a-maximization analysis of these latter SCFTs. As there, it is convenient to take the Veneziano limit of large $N_c$ and $N_f$, holding $x \equiv N_c/N_f$ fixed. In this limit, $a_{\text{SCFT}}(N_c, N_f) \rightarrow N_f^2 \tilde{a}_{\text{SCFT}}(x)$.

We now consider the Wilsonian flow associated with $\langle M \rangle$ for $M = Q \tilde{Q}$,

$$\langle Q^{N_f, N_c} \rangle = \langle \tilde{Q}^{N_c, N_f} \rangle = v, \quad \langle X_i \rangle = 0 : \quad (N_c, N_f) \rightarrow (N_c - 1, N_f - 1 + N_a),$$

(8.1)

where one flavor is eaten and $N_a$ additional flavors come from decomposing under $SU(N_c) \rightarrow SU(N_c - 1)$; $X \rightarrow \tilde{X} + Q_X + \tilde{Q}_X + S$, with $S$ a singlet. The Higgsed theory
has, all together, $2N_f - 1 + N_a$ singlets, which are IR-free fields (at point (A) in Fig. 2). The a-theorem for this flow thus states (including $a_{free} = \frac{2}{9}$ for each free singlet)

$$a_{SCFT}(N_c, N_f) \geq a_{SCFT}(N_c - 1, N_f - 1 + N_a) + \frac{2}{9}(2N_f - 1 + N_a).$$

(8.2)

Taking the Veneziano limit, with $x \equiv N_c / N_f$ fixed and $\hat{a} \equiv a / N_f^2$, this gives

$$2(1 - N_a)\hat{a}(x) + ((N_a - 1)x + 1)\frac{d}{dx}a(x) \geq \frac{4}{9}. \quad (8.3)$$

The inequality is satisfied, $\Delta a < 0$, with $\Delta a \neq 0$, so $\langle M \rangle$ is relevant. The fields here have $R_{micro}(Q_f) > 0$ and $R_{micro}(X_i) > 0$, so these examples fit with the frequent correlation between relevant Higgsing and $R_{micro} > 0$.

9. Examples of irrelevant / marginal $\langle M \rangle$, $\Delta a = 0$, with an interacting sector

We now consider the $A_k$ theories [15,16,17]: $SU(N_c)$ SQCD with $N_f$ flavors, adjoint $X$, and $W_{elec} = \text{Tr}X^{k+1}$, which has a $SU(\tilde{N}_c \equiv kN_f - N_c)$ magnetic dual with

$$W_{dual} = -\text{Tr}Y^{k+1} + \sum_{j=1}^{k} M_j \tilde{q}Y^{k-j}q, \quad (9.1)$$

where $M_{j=1...k}$ map to the generalized mesons $M_j \to \tilde{Q}X^{j-1}Q$ on the electric side. A-maximization is used [18] to determine when the $A_k$ theories exist, by analyzing when $W_{elec}$ is a relevant deformation of the $\tilde{A}$ SCFTs. We recall some results, referring the reader to [18] for more details. Again, it simplifies the expressions to take the Veneziano limit of large $N_f$ and $N_c$, holding fixed $x \equiv N_c / N_f$. The electric theory is asymptotically free for $x > x_{FE} = \frac{1}{2}$, and its vacuum stability (avoiding $W_{dyn} \neq 0$) requires $x < k$. The electric description is weak for $x \approx \frac{1}{2}$, and becomes more strongly coupled for larger $x$. The magnetic description is weak for $\tilde{x} \equiv k - x \approx \frac{1}{2}$, and becomes more strongly coupled for larger $\tilde{x}$. In cases where the two descriptions seem to disagree, given that we trust the duality, the more weakly coupled description is more reliable and presumably correct.

The $A_k$ SCFT only exists for $x > x_k$, i.e. where $R(X^{k+1}) \leq 2$ in the $\tilde{A}$ SCFT, to drive relevant $\tilde{A} \to A_k$ RG flow [18]. For the range $x_k < x < k$, the $A_k$ theory has

$$A_k : R_{micro}(X) = \frac{2}{k + 1}, \quad \text{and} \quad R_{micro}(Q) = \frac{k + 1 - 2x}{k + 1}. \quad (9.2)$$
So \( R_{\text{micro}}(Q) < 0 \) for \( x > (k + 1)/2 \), which will be a case of interest here. Using (9.2),

\[
R_{\text{micro}}(M_j) = R_{\text{micro}}(\tilde{Q}X^{j-1}Q) = \frac{2(k + j - 2x)}{k + 1}.
\]  

(9.3)

When \( R_{\text{micro}}(M_j) < \frac{2}{3} \), the unitarity bound requires that \( M_j \) is IR free.

There can be additional accidental symmetries, seen only from the duality. Depending on \( x \), various term in the dual (9.1) are relevant or irrelevant. An extreme case is \( x \geq x_{FM} \equiv k - \frac{1}{2} \), where the entire dual \( SU(\tilde{N}_c) \) theory is IR free. For \( x \) just below \( x_{FM} \), the theory is in a magnetic Banks-Zaks limit, where the \( SU(\tilde{N}_c) \) is barely interacting and \( R \approx 2/3 \) for every field, so every non-cubic term in (9.1) is irrelevant; the meson \( M_k \) in (9.1) has the weak interaction with \( q\bar{q} \) in (9.1), while all mesons \( M_j < k \) are IR free. For the window \( x_k < x < k - \bar{x}_k \) [18], where \( TrX^{k+1} \) and \( TrY^{k+1} \) are both relevant deformations of the electric and magnetic \( \hat{A} \) SCFTs, \( R_{\text{micro}}(M_j) + R(\bar{q}Y^{k-j}q) = 2 \) and the \( M_j \) term in (9.1) is irrelevant precisely when \( R_{\text{micro}}(M_j) \leq \frac{2}{3} \). More generally, a-maximization is required on both sides [18].

We now consider taking \( \langle M_j \rangle = \langle \tilde{Q}X^{j-1}Q \rangle \neq 0 \). The case of \( M_{j=k} = \tilde{Q}X^{k-1}Q \) expectation value, which maps to a \( \bar{q}q \) mass term in the dual (9.1), was considered as a check of the duality in the original works [15,16,17]. It is clear from the dual that \( \langle M_k \rangle \) leads to \( \Delta a \neq 0 \), since it sources a \( \bar{q}q \) mass term, which is always relevant. On the other hand, as seen from the dual (9.1), the other \( \langle M_{j<k} \rangle \) can source irrelevant interactions.

If \( \langle M_j \rangle \) sources an irrelevant interaction, the magnetic description reveals physics that is very different than what would have been expected from the electric description, or from classical intuition about Higgsing removing d.o.f.. The magnetic description shows that in such cases \( \langle M_j \rangle \) is a freely generated subspace of the moduli space, with \( \Delta a = 0 \) everywhere on this space. A necessary condition for this to happen is that \( \langle M_j \rangle \) does not fully break the gauge group, both to avoid classical constraints on \( \langle M_j \rangle \), and also to avoid having \( \langle M_j \rangle \) connect to where the quantum effects can be made small, i.e. fully Higgsing the gauge group with large vevs. In cases where \( \Delta a = 0 \), evidently never-small quantum effects on the \( \langle M_j \rangle \) moduli spaces must eliminate the non-zero \( \Delta a_{\text{classical}} \).

Let us consider the extreme case of \( \langle M_{j=1} \rangle \neq 0 \) in more detail, first in the electric description. As in (8.1), we take \( \langle X \rangle = 0 \) and \( \langle Q_{N_f,N_c} \rangle = \langle \tilde{Q}N_{N_f} \rangle = \nu \), which breaks \( SU(N_c) \rightarrow SU(N_c - 1) \), with \( N_f \rightarrow N_f - 1 + 1 \), with the +1 additional flavor from decomposing the \( SU(N_c) \) adjoint under \( SU(N_c - 1) \), \( X \rightarrow \tilde{X} + QX + \tilde{Q}X + S \), with \( S \) a singlet, with interactions (not bothering with numerical factors)

\[
W_{\text{tree}} = TrX^{k+1} \rightarrow Tr\tilde{X}^{k+1} + \tilde{Q}X \tilde{X}^{k-1}QX + STr\tilde{X}^k + S^{k+1} + \ldots
\]  

(9.4)
For \( x < x_k \) all terms in (9.4) are irrelevant, so we can ignore \( W \) before and after Higgsing,

\[
a_i = a_A(N_c, N_f), \quad a_f = a_A(N_c - 1, N_f) + \frac{2}{9}(2N_f), \quad \text{for} \quad x < x_k, \quad (9.5)
\]
as in (8.2) for \( N_a = 1 \). Here \( a_i \) is computed in the \( SU(N_c) \widehat{A} \) SCFT with \( v = 0 \), using a-maximization as in [18], while \( a_f \) is computed in the Higgsed, \( v \neq 0 \), \( SU(N_c - 1) \) IR theory. For \( x > x_k \) the \( W_{A_k} = \text{Tr}X^{k+1} \) term in (9.4) is relevant for \( a_i \), and the \( W_{A_k,M_k} = \text{Tr}\hat{X}^{k+1} + (M_k)_{N_f,N_f} \), with \((M_k)_{N_f,N_f} = \hat{Q}_XX^{k-1}Q_X \), terms are relevant for \( a_f \), with the other terms irrelevant. The effect of the mesonic \( \Delta W = (M_k)_{N_f,N_f} \) is a RG flow from \( A_k \) to a new IR fixed point, where \( R(Q_X) = 2/(k+1) \), so

\[
a_i = a_{A_k}(N_c, N_f), \quad a_f = a_{A_k,M_k}(N_c - 1, N_f - 1, 1) + \frac{2}{9}(2N_f), \quad \text{for} \quad x > x_k. \quad (9.6)
\]

We can now compute \( \Delta a \equiv a_f - a_i \) for the Higgsing, using (9.3) or (9.6). The result is found to satisfy \( \Delta a \leq 0 \), with \( \Delta a = 0 \) occurring at \( x = (k + 1)/2 \), which is precisely where \( R(Q) = 0 \). Since \( x_k < (k + 1)/2 \) [18], this is in the region of the \( A_k \) SCFT, so (9.6) applies. Indeed, the \( \widehat{A} \) SCFT has \( R(Q) > 0 \), and (9.5) yields \( \Delta a < 0 \).

On the magnetic side, things are much simpler, since the \( \langle M_1 \rangle \) deformation triggers

\[
\Delta W_{\text{dual}} = \langle M_1 \rangle \bar{q}Y^{k-1}q : \quad \text{irrelevant if} \quad R(\bar{q}Y^{k-1}q) > 2, \quad (9.7)
\]
and relevant otherwise. The relevant case certainly leads to \( \Delta a < 0 \), in accord with the a-theorem. The irrelevant case, on the other hand, leads to \( \Delta a \to 0 \) in the IR.

The effect of \( \langle M_1 \rangle \) for \( k = 2 \) was discussed in some detail in [50], where the effect of \( \langle M_1 \rangle \bar{q}Yq \) was analyzed by considering a UV completion, to replace the quartic term in \( W_{\text{dual}} \) with purely cubic terms, with some additional massive flavors integrated-in, such that integrating them back out gives back the quartic terms. Likewise, one could here UV complete \( \Delta W_{\text{IR}} \supset M_j \bar{q}Y^{k-j}q \) to a theory with only cubic interactions, by integrating-in \( k - j \) additional pairs of massive matter fields \( F_i, \tilde{F}_i \) and \( G_j, \tilde{G}_j \), with \( W_{\text{UV}} \supset M_j \bar{q}F_1 + F_1Y\tilde{F}_2 + F_2Y\tilde{F}_3 + m \sum_i F_i\tilde{F}_i \). Then \( M_j \to \langle M_j \rangle \) leads to mass mixing, which can be re-diagonalized, similar to [50]. But for our interests here, such UV completions are not needed, and the IR effect can just as well be obtained directly from (9.1).

To determine \( R(\bar{q}Y^{k-j}q) \), we must account for the relevance or irrelevance of the various terms in (9.1), including \( \text{Tr}Y^{k+1} \) [18]. The a-maximization procedure is best implemented by computer, but we can make some general, qualitative remarks. For \( x \approx x_{FM} = k - \frac{1}{2} \) the theory is in the weakly coupled magnetic Banks-Zaks regime,
$\text{Tr}Y^{k+1}$ is irrelevant (for $k > 2$), and $R(q) = R(\tilde{q}) \approx R(X) \approx \frac{2}{3}$, so $M_k$ is weakly coupled and all $M_{j<k}$ are IR free; then deforming by $\langle M_k \rangle$ is relevant, deforming by $\langle M_{k-1} \rangle$ is weakly relevant, and any other $\langle M_{j<k-1} \rangle$ source irrelevant interactions, leading to $\Delta a = 0$. The extreme case is $\langle M_1 \rangle$, which sources $\tilde{q}Y^{k-1}q$, which for $k > 2$ is certainly irrelevant for sufficiently small $\tilde{x}$. As we increase $\tilde{x}$, it can be seen more generally that $R(\tilde{q}Y^{k-1}q) > R(Y^{k+1})$, since $R(q) > R(Y)$, so $\text{Tr}Y^{k+1}$ becomes relevant before $\langle M_1 \rangle$.

This is nice: the critical $\tilde{x}$, where $\langle M_1 \rangle$ crosses from sourcing an irrelevant to relevant $\Delta W$ in the magnetic description, is where $\text{Tr}Y^{k+1}$ is already relevant, so a-maximization is not needed there. We can simply use $R(Y) = 2/(k+1)$, which determines $R(\tilde{q}Y^{k-1}q) = 2 - R_{\text{micro}}(M_1)$. Therefore, using (9.7), $\langle M_1 \rangle$ becomes irrelevant on the magnetic side precisely when $R_{\text{micro}}(M_1) < 0$, i.e. when the electric theory has $R_{\text{micro}}(Q) \leq 0$.

So the electric and the magnetic dual descriptions both give $\Delta a = 0$ at $x = (k+1)/2$, where $R_{\text{micro}}(Q) = 0$, but they seemingly disagree for $x > (k+1)/2$. The detailed analysis on the electric side (accounting for all visible accidental symmetries) gives $\Delta a \neq 0$ for $x > (k+1)/2$. But the magnetic analysis gives $\Delta a = 0$ for all $x \geq (k+1)/2$, since $\langle M_1 \rangle$ sources $\tilde{q}Y^{k-j}q$, which is there irrelevant. The magnetic result is presumably correct (assuming the duality is correct), since it is more weakly coupled at large $x$. Evidently, the magnetic dual reveals accidental symmetries that were not evident in the electric description, giving $\Delta a = 0$ for $x \geq (k+1)/2$, rather than just at $x = (k+1)/2$.

There are many analogous examples, e.g. using the $D_k$ and $E_k$ theories of [19], which we have checked are quite similar to the $A_k$ case.

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