Einstein–Yang–Mills strings

Dmitri V. Gal’tsov and Evgeny A. Davydov
Department of Theoretical Physics, Moscow State University, 119899, Moscow, RUSSIA

Mikhail S. Volkov
Laboratoire de Mathématiques et Physique Théorique,
CNRS-UMR 6083, Université de Tours,
Parc de Grandmont, 37200 Tours, FRANCE

We present globally regular vortex-type solutions for a pure SU(2) Yang-Mills field coupled to gravity in 3+1 dimensions. These gravitating vortices are static, cylindrically symmetric and purely magnetic, and they support a non-zero chromo-magnetic flux through their cross section. In addition, they carry a constant non-Abelian current, and so in some sense they are analogs of the superconducting cosmic strings. They have a compact central core dominated by a longitudinal magnetic field and endowed with an approximately Melvin geometry. This magnetic field component gets color screened in the exterior part of the core, outside of which the fields approach exponentially fast those of the electrovacuum Bonnor solutions with a circular magnetic field. In the far field zone the solutions are not asymptotically flat but tend to vacuum Kasner metrics.

PACS numbers: 04.20.-q, 04.25.Dm, 11.15.-q, 11.27.+d

Classical solutions in non-Abelian gauge models coupled to gravity in four space-time dimensions attracted a lot of attention after the discovery by Bartnik and McKinnon of particle-like solutions in the Einstein-Yang-Mills (EYM) theory and after the construction of the corresponding black holes (see for a review). Since then a variety of static and stationary axisymmetric EYM solutions have been considered, but static, cylindrically symmetric EYM configurations have remained unexplored for some reason. To fill this gap, we consider in this letter static, cylindrically symmetric, purely magnetic, globally regular four-dimensional EYM solutions of cosmic string type.

Surprisingly, we find strings endowed with a constant current along their symmetry axis,
and so in some sense they can be considered as analogs of Witten’s superconducting cosmic strings. In the latter case the superconductivity arises due to the presence of two complex scalars in the theory. In our case their role is played by two color components of the self-interacting Yang-Mills field, which behave exactly in the same way as the scalars in Witten’s model. Specifically, one of them develops a non-zero condensate value in the vortex core and vanishes at infinity, while the other one behaves the other way round. Since the gauge symmetry of our theory is not broken, the gauge field coupled to the current is long-range. The slow fall-off of the energy density in the direction orthogonal to the string then does not let the solutions to be asymptotically flat, although they are asymptotically Ricci flat. This is the principal difference of the EYM strings as compared to the previously studied currentless Abelian self-gravitating cosmic strings of the Nielsen-Olesen type.

In what follows we present a numerical evidence for the existence of a family of globally regular EYM string solutions labeled by an integer winding number \( \nu \) and by a real parameter \( p \) determining the value of the magnetic field at the symmetry axis. These EYM strings are filled with a self-interacting chromo-magnetic field giving rise to a constant current along the string symmetry axis and also to a chromo-magnetic flux through the string cross section. The flux and current can be defined in a gauge-invariant way within the approach of asymptotic symmetries. The geometry of the solutions interpolates between flat Minkowskian geometry at the symmetry axis and vacuum Kasner geometry in the far field zone. Interestingly, these solutions survive even if the coupling to gravity is off and the metric is flat everywhere. They describe then ‘pure Yang-Mills superconducting strings’ with a regular central core outside of which there remains only a U(1) component of the gauge field, which, being coupled to the current, diverges logarithmically for large \( r \). As a result, the total energy per unit length for the flat space solutions diverges at large \( r \). Getting back to curved space, this divergence is cured by gravity.

We consider the SU(2) Einstein-Yang-Mills (EYM) theory

\[
S = \int d^4x \sqrt{-g} \left( -\frac{R}{16\pi G} - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} \right),
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \varepsilon_{abc} A_\mu^b A_\nu^c \) and \( e \) is the gauge coupling constant. With \( A = \frac{1}{2} \tau_a A_\mu^a dx^\mu \) where \( \tau_a \) are the Pauli matrices, the local SU(2) gauge symmetry of the action is expressed by \( A \rightarrow U(A + id)U^{-1} \) where \( U(x) \in SU(2) \). We shall be interested in static, cylindrically symmetric systems invariant under the action of the three commuting Killing
vectors $\partial_t, \partial_\varphi, \partial_z$. The spacetime metric then can be parameterized as

$$ds^2 = l^2 \{ N(r)^2 dt^2 - H(r)^2 dr^2 - L(r)^2 d\varphi^2 - K(r)^2 dz^2 \},$$

(2)

where $l$ is the length scale and all the other quantities are dimensionless. For the gauge field we make the following purely magnetic ansatz,

$$A = \tau_2 R(r) \, dz + \tau_3 P(r) \, d\varphi.$$

(3)

Using Eqs.(2), (3) we obtain

$$S = \frac{1}{\kappa e} \int d^4x \mathcal{L} \text{ where } \kappa = \frac{8\pi G}{(el)^2}$$

and

$$H L N K = N' K' \frac{N^2}{NK} + \frac{N'L'}{NL} + L' K' \frac{L^2}{LK} - \frac{\kappa R^2}{2K^2} - \frac{\kappa P^2}{2L^2} - \frac{\kappa H^2 R^2 P^2}{2K^2 L^2}.$$  

(4)

We notice that the reduced Lagrangian $\mathcal{L}$ is invariant under $P \rightarrow \alpha P, L \rightarrow \alpha L, R \rightarrow \beta R, K \rightarrow \beta K, N \rightarrow N, H \rightarrow H$, with constant $\alpha, \beta$. It also admits a discrete symmetry $P \leftrightarrow R, L \leftrightarrow K$, which is equivalent to $z \leftrightarrow \varphi$. Varying $\mathcal{L}$ gives the field equations, which can be represented in the form

$$N'' = \left( \frac{N'^2}{N^2} + \frac{L'S'}{LS} - \frac{L'^2}{L^2} + \kappa \frac{P^2 R^2}{S^2} \right) N,$$

$$L'' = \left( \frac{N'S'}{NS} - \kappa \frac{P'^2}{L^2} \right) L,$$

$$S'' = -\kappa \frac{N^2 R^2 P^2}{S},$$

$$\left( \frac{L^2}{S} R' \right)' = \frac{N^2 P^2}{S} R,$$

$$\left( \frac{S}{L^2} P' \right)' = \frac{N^2 R^2}{S} P,$$

(5)

plus a first order constraint equation

$$\frac{N'S'}{NS} + \frac{L'S'}{LS} - \frac{L'^2}{L^2} = \frac{\kappa}{2} \left( \frac{P'^2}{L^2} + \frac{L^2 R'^2}{S^2} - \frac{N^2 P^2 R^2}{S^2} \right).$$  

(6)

Here $S = LK$ and, after varying, we have imposed the coordinate condition $H = N$. The energy density is

$$T^0_0 = \frac{P^2}{2N^2 L^2} + \frac{R'^2}{2N^2 K^2} + \frac{P^2 R^2}{2S^2},$$

(7)

while the other non-trivial components of $T^\mu_\nu$ are obtained by choosing the signs in front of the three terms in this formula as $T^r_r \sim (-, -, +)$, $T^z_z \sim (+, -, -)$, $T^\varphi_\varphi \sim (-, +, -)$, such
that $T^\mu_\mu = 0$. Associated with the two continuous global symmetries of $\mathcal{L}$ there are two Noether charges

$$
Q_1 = S \left( \frac{N'}{N} - \frac{L'}{L} - \kappa \frac{PP'}{L^2} \right), \\
Q_2 = S \left( \frac{N'}{N} - \frac{S'}{S} + \frac{L'}{L} - \kappa \frac{L^2 RR'}{S^2} \right),
$$

(8)

whose conservation can be checked straightforwardly.

We want the fields to be regular at the symmetry axis $r = 0$. This implies that one should have $P(0) = \nu \in \mathbb{Z}$, since in this case, with $U = e^{-\frac{\mu}{2} r^3 \varphi}$, one can pass to the gauge where the gauge field will be regular at the axis: $2A = U r^2 U^{-1} R dz + \tau_3 (P - \nu) d\varphi$. The metric should become Minkowskian for $r \to 0$. The most general local power series solution of Eqs.(5) with such boundary conditions reads

$$
P = \nu - pr^2 + O(r^4), \quad R = qr^\nu + O(r^{\nu+2}), \quad N = 1 + O(r^2), \quad S = r + O(r^3), \quad L = r + O(r^3),
$$

(9)

where $p, q$ are two integration constants. This solution fulfills also the constraint (6). Inserting (9) to (8) fixes the values of the Noether charges: $Q_1 = 2\nu \kappa p - 1$ and $Q_2 = 0$. We can now integrate the equations starting from the origin on, with the boundary condition (9), and this reveals the following picture. Fixing a value of $p$ and varying $q$, the numerical solution generically terminates at a finite point $r = r_\ast(q)$ where $P'$ becomes large and either positive or negative, depending on $q$. Adjusting properly the value of $q$, we can postpone further and further the moment $r_\ast(q)$ where the numerical procedure crashes. We thus extend the solution from the axis farther and farther to the asymptotic region, and we observe then that the amplitude $P$ approaches zero very quickly with growing $r$. As a result, we conclude that at large $r$ the globally regular solutions approach configurations with $P = 0$.

Setting $P = 0$ in Eqs.(5), the SU(2) gauge field configuration becomes U(1), in which case the general solution of Eqs.(5), let us call it $N_0, L_0, S_0, R_0$, reads

$$
L_0 = a_1 x^{1+m} + a_2 x^{1-m}, \quad N_0 = a_3 x^{m^2-1} L_0, \\
S_0 = a_4 x, \quad R_0 = a_5 + a_6 \frac{x^{1-m}}{L_0}, \quad x = r - r_0,
$$

(10)

where $m, a_1, \ldots, a_6, r_0$ are 8 integration constants. Imposing the constraint (6) would imply also that $a_6 \sqrt{\kappa a_1} = \pm a_4 \sqrt{2a_2}$, but we do not demand this at the moment. We conclude that
FIG. 1: The globally regular solution of Eqs.(5) with $p = 0.8, \nu = 1$. The central core extends up to $\xi \approx 1.2$, after which the configuration practically coincides with the Abelian solution (10) of Bonnor with $m = -1.94$. For $\xi \geq 3$ the gauge field dies out and the solution becomes Kasner.

at large $r$ our solution is given by

\[
L = L_0 + \delta L, \quad N = N_0 + \delta N, \quad S = S_0 + \delta S,
\]

\[
R = R_0 + \delta R, \quad P = \delta P,
\]

(11)

where the deviations $\delta L, \ldots, \delta P$ vanish as $r \to \infty$. To determine them we insert (11) to Eqs.(5) and linearize with respect to the deviations. This gives

\[
\delta P = a_7 \frac{L_0}{\sqrt{S_0}} \exp\left(-\frac{L_0 N_0 R_0}{S_0} x\right) (1 + \ldots),
\]

(12)

where $a_7$ is a new integration constant and the dots stand for the subleading terms. It follows also that $\delta N, \delta L, \delta S,$ and $\delta R$ are all of the order $(\delta P)^2$ and no new integration constants appear in these variations, since the background solution $N_0, L_0, S_0, R_0$ already contains the maximal number of integration constants.

Summarizing, the local solution at large $r$ contains 9 free parameters and is given by Eqs.(10)-(12), while that at small $r$ contains 2 integration constants in Eq.(9). We then integrate numerically Eqs.(5) to extend these local solutions to the intermediate region where we match them. This gives us global solutions in the interval $r \in [0, \infty)$. To fulfill the 10 matching conditions for the 5 functions in Eqs.(3) and for their first derivatives, we have in our disposal $2+9=11$ free parameters, and so there is one free parameter left after the matching. This parameter, we choose it to be $p$ in Eq.(9), labels the global solutions.

No special care is to be taken about the constraint (6), since being imposed at $r = 0$ it ‘propagates’ to all values of $r$. As a result, the values of the asymptotic parameters of
FIG. 2: The parameter $m(\nu, p)$, the Tolman mass $M(\nu, p)$ and the current $I(\nu, p) \equiv I_2/(2\pi)$ for the globally regular solutions with $\nu = 1, 2$.

the global solutions found by the matching will automatically satisfy $a_6 \sqrt{\kappa a_1} = \pm a_4 \sqrt{2a_2}$. Similarly, the axis values of the Noether charges $Q_1 = 2\nu \kappa p - 1$, $Q_2 = 0$ propagate to the large $r$ region and impose the asymptotic conditions

$$a_4(m^2 - 1) = Q_1, \quad (2 + m)a_4 + 2\sqrt{2\kappa a_1 a_2}a_5 = 0. \tag{13}$$

We thus obtain a family of globally regular solutions of Eqs. labeled by a real parameter $p > 0$ and by an integer $\nu$, and we use the length scale for which $\kappa = 2$. The typical solution is shown in Fig.1. These EYM strings have a compact central core filled with a chromo-magnetic field, outside of which $P \approx 0$ and the remaining gauge field becomes effectively Abelian. The interior par of the core, where $R \approx 0$, is also effectively Abelian and can be well described by the Melvin solution. The latter can be obtained from the general solution by applying the $z \leftrightarrow \varphi$ symmetry and adjusting the constants such that the boundary conditions at $r = 0$ are fulfilled:

$$N = K = 1 + \frac{\kappa}{2} p^2 r^2, \quad L = \frac{r}{N}, \quad P = \nu - \frac{m^2}{N}. \tag{14}$$

Since this inner solution has a magnetic field along the $z$ axis, we call it z-string. The metric in this limit acquires an additional symmetry with respect to Lorentz boosts in the $(0, z)$ plane, and the energy-momentum tensor in this limit is such that $T_0^0 = T_z^z = -T_r^r = -T_\varphi^\varphi$.

The outer part of the central core is the transition region where the non-Abelian effects become essential, since the $R$ field starts growing there, playing a role of the effective mass term for the $P$ field, and so the latter starts falling down to zero exponentially fast. As a result, outside of the central core one has $P \to 0$ and the whole configuration approaches
FIG. 3: The Euclidean 2-geometry of the \((r, \varphi)\) string orthogonal section can be realized on a 2-surface of revolution in three dimensional Euclidean space obtained by rotating the shown planar curve around the Z-axis. The curve shown corresponds to the string with \(\nu = 1, p = 0.75\).

exponentially fast one of the Abelian Bonnor solutions \(\text{[10]}\). These, in their turn, contain an intermediate region filled with a circular magnetic field, which we call \(\phi\)-string and which has \(T^0_0 = T^\varphi_\varphi = -T^r_r = -T^z_z\). This magnetic field falls off polynomially for large \(r\), and in the far field zone the solutions approach asymptotically vacuum Kasner metrics. Summarizing, the whole solution can be viewed as a non-linear superposition of a \(z\)-string placed at small \(r\) and a \(\phi\)-string located at larger \(r\), which is demonstrated by the profile of the radial energy density, \(SN^2T^0_0\), clearly showing two peaks (see Fig.1).

It is natural to wonder if the solutions can be asymptotically flat, as in the case of gravitating Nielsen-Olesen strings \([4]\). However, taking a linear combination of the first equation in \([3]\) with the constraint \([6]\) and integrating gives

\[
\lim_{r \to \infty} S \frac{N'}{2N} = \int_0^\infty SN^2T^0_0 dr.
\]

(15)

The integral on the right here is manifestly positive and equals the Tolman mass, \(M\), while the left hand side is equal to \((m^2 + |m|)a_4\). As a result, one always has \(m \neq 0\) and so solutions with \(T^0_0 \neq 0\) cannot approach flat metric for large \(r\).

Apart from the mass \(M\), another important parameter of the EYM strings is the current flowing along them. Although there is no Higgs field in the theory, it is still possible to introduce conserved and gauge invariant currents within the asymptotic symmetries approach of Abbott and Deser \([5]\). Specifically, since our solutions are asymptotically vacuum, one has \(A = A^\infty + a\), where \(A^\infty = iU_\infty dU_\infty^{-1}\) and \(a \to 0\) as \(r \to \infty\), but otherwise \(a\) need not to
be small. Defining \( D_\mu a_\nu = \nabla_\mu a_\nu - i [A_\mu, a_\nu] \) one can introduce

\[
f^{(\infty) a}_{\mu \nu} = \text{tr} \left[ \left( D_\mu a_\nu - D_\nu a_\mu \right) U_\infty^a U_\infty^{-1} \right]
\]

and then the conserved and gauge invariant currents are \( J^a_\mu = \nabla_\sigma f^{(\infty) a}_{\sigma \mu} \). The total current through the string cross section is then given by

\[
\frac{I_a}{2\pi} = \int_0^\infty J^a_z \sqrt{-g} dr = \delta_a^2 \lim_{r \to \infty} \frac{L^2}{S} R' = -2m \sqrt{\frac{2}{\kappa}} a_1 a_2 \delta_a^2.
\]

Our strings thus carry a persistent current, and for this reason they can be called ‘superconducting’.

At the axis we have similarly \( A = A^0 + a \), where \( A^0 = i U_0 dU_0^{-1} \) and \( a \to 0 \) for \( r \to 0 \). Defining then \( f^{(0) a}_{\mu \nu} \) as in (16) but with ‘\( \infty \)’ replaced by ‘\( 0 \)’, we can calculate its flux through the string cross section, \( \Sigma \),

\[
\int_{\Sigma} f^{(0) a}_{\mu \nu} dx^\mu \wedge dx^\nu = -\nu \delta_a^2,
\]

and this is ‘quantized’ as for the Nielsen-Olesen vortex.

The mass, current and the parameter \( m \) as functions of \( p \) are shown in Fig.2 for solutions with \( \nu = 1, 2 \). We see that \( m \) is always negative and tends to zero for \( p \to 0 \), in which limit the gauge field becomes pure gauge with \( R(r) = 0 \) and \( P(r) = \nu \), so its energy vanishes and the metric becomes flat. For \( p = 1/(2\kappa \nu) \), when \( Q_1 = 0 \), Eq. (13) implies that \( m = -1 \), in which case \( g_{00} \) becomes proportional to \( g_{\varphi \varphi} \) for large \( r \), and so the metric acquires the asymptotical boost symmetry in the \( (t, \varphi) \) plane.

Another interesting limit is \( \kappa \to 0 \) with fixed \( p \), in which case the gravity switches off and the geometry becomes flat. The gauge field, however, remains non-trivial. Such Minkowski space solutions also show a regular central core where \( R \approx 0 \) and \( P \neq 0 \), outside of which \( P \) is exponentially small, but the large \( r \) behavior of \( R \) is now different. Specifically, for small \( \kappa \) one has \( m = -\sqrt{\kappa/2 I} \), \( a_1 - 1/2 = O(\sqrt{\kappa}) \), \( a_2 \to 1 - a_1 \), using which in Eq. (10) and taking the limit gives \( R = C_0 + I \ln(r) \) for large \( r \). Of course, the same results are obtained by solving the two Yang-Mills equations in (5) with \( N = 1, L = S = r \). The amplitude \( R \) is thus now logarithmically divergent at infinity, as it should be, since we effectively have here a U(1) gauge field coupled to the conserved current. We thus obtain ‘superconducting strings’ made of pure Yang-Mills field in flat space, and these, to the best of our knowledge, have also never been described in the literature. Getting back to the self-gravitating case,
one can say that the gravity provides a ‘dynamical cut-off’ by rendering $R$ finite at large $r$, which makes the total energy finite.

It is interesting to visualize the two-dimensional geometry $dl^2 = N^2 dr^2 + L^2 d\varphi^2$ of the $(r, \varphi)$ plane orthogonal to the string in terms of embeddings. The same geometry can be realized on a 2-surface of revolution in three dimensional Euclidean space with coordinates $X, Y, Z$ obtained by rotating around the $Z$-axis the planar curve defined parametrically by relations $X = \pm L(r), Z = Z(r)$ with $Z^2 = N^2 - L^2$. This defines a ‘flowerpot’ surface shown in Fig.3. We observe that the circumference of a circle centered at the string axis in the plane orthogonal to the string grows up slower than the radius of the circle.

Summarizing, we have presented superconducting cosmic string solutions for a pure self-gravitating Yang-Mills field. We have not studied their stability yet. As a matter of fact, many solutions in the EYM theory are unstable. However, since we now have the conserved current, there is a chance that it may stabilize the solutions. Other interesting problems to study would be to consider the black hole generalizations for our solutions or to include other fields. We also notice that replacing in all above formulas $t \leftrightarrow z$ gives solutions with an electric field. They do not have a current, but possess instead a chromo-electric charge.

The work of D.V.G. and E.A.D. was supported in part by the RFBR grant 02-04-16949.

[1] R. Bartnik and J. Mckinnon, Phys. Rev. Lett. 61, 141 (1988).
[2] M. S. Volkov and D. V. Gal’tsov, Phys. Rept. 319, 1 (1999).
[3] E. Witten, Nucl. Phys. B249, 557 (1985).
[4] D. Garfinkle, Phys. Rev. D32, 1323 (1985); M. Christensen, A.L. Larsen and Y. Verbin, Phys. Rev. D60, 125012 (1999).
[5] L.F. Abbott and S. Deser, Phys.Lett. B116, 259 (1982).
[6] W.B. Bonnor, Proc.Phys.Soc.Lond. A66, 145 (1953).