Statistics of the Radiating Relativistic Electrons

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Outline

- Preliminaries
- Statistical properties of recoils
- Straggling function
- Comparative analysis
- Verifications and illustrations
- Summary and outlook
Processes and Model
‘All models are wrong but some are useful!’ (George Box)

quantum processes

- radiation in periodic fields
- ionization losses of relativistic electrons
- ionization losses of heavy particles
- ...

model

- Independent identically distributed (i.i.d.) recoils
- Compact support of the recoil spectrum $w(\omega)$, $0 < \omega_{\text{min}} \leq \omega \leq \omega_{\text{max}} < \infty$
- Normalised spectrum: $\int_{-\infty}^{\infty} w(\omega) \, d\omega = 1$
- Small magnitude of a recoil, $\omega_{\text{max}} \ll \gamma/e/p$

Key model suggestion: the recoil spectrum independent of the particle energy
Scheme of the Process. Straggling function
Mathematical formulation “Subordinate to compound Poisson Process”

Non-recoiled particles $= \exp(-x)$ ($x$ is the average number of recoils per particle)
Rigorous Solutions for Arbitrary Recoil Spectrum
Characteristic functions, Bulyak, Shul’ga (2016, 2017)

**evolution of spectrum**

\[ \hat{f} = \hat{f}_0 \exp[x(\tilde{\omega} - 1)] \]

**straggling function**

\[ \hat{S}_x = \hat{\omega} e^{x(\tilde{\omega} - 1)} \]

**moments of spectrum**

\[ \bar{\gamma}(x) = \bar{\gamma}_0 - x \bar{\omega} \]
\[ \text{Var}[\gamma](x) = (\bar{\gamma} - \bar{\gamma})^2 = \text{Var}[\gamma_0] + x \bar{\omega}^2 \]
\[ \ldots \]

**SF moments**

\[ \bar{\epsilon} = (1 + x) \bar{\omega} \]
\[ \text{Var}[\epsilon] = (1 + x) \bar{\omega}^2 - \bar{\omega}^2 \]
\[ \ldots \]

\( \hat{g} \) is the Fourier transform, \( \check{g} \) the inverse Fourier transform.

**Density of energy losses distribution defined by**

- recoil spectrum \( w(\omega) \)
- average number of recoils \( x = \int_0^Z P(z') \, dz' / \bar{\omega} \)
Mathematically the characteristic function is the solution for SF
Practically, SF may be restored at limits
- ‘Poisson’, $x \lesssim 1$, a few self states may be derived
- ‘Fokker–Planck’, $x \to \infty$, a few first moments sufficient

How does SF evolve in between the limits?
Analysis: BS $\Leftrightarrow \alpha$-stable

Goals: stability parameter, $\alpha$, and scale $c$ (half width at 1/e height)

### CF of $\alpha$-stable distribution

$$\hat{\phi}(s) = \exp \{i s \mu - |c s|^\alpha [1 - i \beta \text{sgn}(s) \Phi]\}$$

$$\Phi = \begin{cases} \tan \left( \frac{\pi \alpha}{2} \right), & \alpha \neq 1, \\ -\frac{2}{\pi} \log |s|, & \alpha = 1, \end{cases}$$

- $\alpha \in (0, 2]$ stability parameter
- $\beta \in [-1, 1]$ skewness parameter
- $c \in (0, \infty)$ scale parameter
- $\mu \in (-\infty, \infty)$ location parameter

### CF of BS distribution

$$\hat{S}_x(s) = \hat{\nu}(s) \exp [x(\hat{\nu}(s) - 1)]$$

Physical reasons

- $\alpha \in (1, 2] = 2$ Gaussian $x \to \infty$
- $c \sim \sqrt{x \omega^2}$ FP $x \to \infty$
- $\mu = x \overline{\omega}$ energy conservation
Analysis of Straggling Function

Central Limit Theorem: if $x \to +\infty$ distribution $\to$ Gaussian for i.i.d.

Gauss $\alpha = 2$, Landau $\alpha = 1$, $\beta = 1$ distributions

$\mu = 2$, $c = 1$

SF in helical undulator, $K = 1$
BS has nonlinear logarithmic dependence of $s$. “Linearising” it at $s_*$, the root of

$$\Re[x (1 - \hat{w}(s_*))] = 1 = |\pi cs_*|^\alpha ,$$

yields

$$\alpha(x) = \frac{sD_s\Re[\hat{w}]}{1 - \Re[\hat{w}]} \bigg|_{s=s_*} ,$$

where $D_s = \partial \cdot / \partial s$

Scaling parameter

$$c(x) = [xm_\alpha[w]]^{1/\alpha} = \frac{1}{\pi s_*}$$

with $\alpha$–moment of the recoil

$$m_\alpha[w] = \int \omega^\alpha w(\omega) \, d\omega , \quad \omega > 0$$
Validation of BS Distribution at Limits

Rigorous: if $x \to \infty$ then $\alpha \to 2$ Central Limit Theorem

Landau distribution (L.D. Landau, 1944)

- Bounded Pareto distribution
  $\to$ Rutherford cross section
  when $a \to 0$, $b \to \infty$
- $\bar{\omega} \to 0$ such that $P = x \bar{\omega}$ finite
- $\lim_{a \to 0, b \to \infty} \alpha = 1$
No Contradiction Between CLT and Landau Distribution

Practical case: \( a > 0, \ b < \infty \ \rightarrow \ \alpha > 1 \)

1 GeV electrons traversing tungsten target

Within model limits \( \alpha - 1 \ll 1 \)
Helical Undulator Stability Parameter

\( K = 0.01, 0.3, 1 \)

When \( K \) increases

- later the Gaussian distribution attracted
- thicker the tail
- faster the scale parameter increases
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Summary

General properties of straggling spectrum

\[
\hat{S}_{BS} = \hat{w} \exp [x(\hat{w} - 1)]
\]

\[
\alpha(x) = \frac{sD_s \Re[\hat{w}]}{1 - \Re[\hat{w}]} \bigg|_{s=s_*}
\]

\[
c(x) = \{x \, m_\alpha[w]\}^{1/\alpha}
\]

when number of recoils increases

- (variable) stability parameter increases: \(1 < \alpha(x) \leq 2\)
- type of diffusion varies from ballistic to normal
- width of spectrum increases as \(x^{1/\alpha}\), maximum rate at the beginning

Efficiency of a source of hard EM radiation – spectral brightness – inverse to the width of spectrum

Losses of particles-radiators occur due to tail(s) of the spectrum