Exact uniform approximation and Dirichlet spectrum in dimension at least two

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Abstract
For $m \geq 2$, we determine the Dirichlet spectrum in $\mathbb{R}^m$ with respect to simultaneous approximation and the maximum norm as the entire interval $[0, 1]$. This complements previous work of several authors, especially Akhunzhanov and Moshchevitin, who considered $m = 2$ and Euclidean norm. We construct explicit examples of real Liouville vectors realizing any value in the unit interval. In particular, for positive values, they are neither badly approximable nor singular. Thereby we obtain a constructive proof of the main claim in a recent paper by Beresnevich, Guan, Marnat, Ramírez and Velani, who proved existence of such vectors but without being able to provide any concrete value in the Dirichlet spectrum. Our constructive proof is considerably shorter and less involved than previous work on the topic. Moreover, it is flexible enough to show that the according set of vectors with prescribed Dirichlet constant has large packing dimension and rather large Hausdorff dimension as well, thereby contributing to the metrical problem raised in the aforementioned paper by Beresnevich et alia. We further establish a more general result on exact uniform approximation, applicable to a wide class of approximating functions. Moreover, minor twists in the proof yield similar, slightly weaker results when restricting to a certain class of classical fractals or considering other norms on $\mathbb{R}^m$. In an Appendix we address the situation of a linear form.

Keywords Dirichlet spectrum · Cantor set · Hausdorff dimension

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1 Dirichlet spectrum

Let $\|x\|$ be the distance of $x \in \mathbb{R}$ to the nearest integer and for $x \in \mathbb{R}^m$ let $\|x\| = \max\{\|x_1\|, \ldots, \|x_m\|\}$. Given $\xi \in \mathbb{R}^m$, define the non-increasing, piecewise constant, right-continuous function

$$
\psi_\xi(Q) = \min_{1 \leq q \leq Q} \|q\xi\|,
$$

where $q$ ranges over the positive integers up to $Q$. Let us then call

$$
\Theta(\xi) := \limsup_{Q \to \infty} Q^{1/m} \psi_\xi(Q),
$$

the Dirichlet constant of $\xi$, which is thereby considered with respect to simultaneous approximation and the maximum norm. Define the Dirichlet spectrum $\mathbb{D}_m$ as the set of all values that the Dirchlet constant takes, i.e.

$$
\mathbb{D}_m = \{\Theta(\xi) : \xi \in \mathbb{R}^m\}.
$$

(Note: Occasionally, as in [2, 3], the $m$-th power of $\Theta(\xi)$ is taken, leading to an accordingly altered Dirichlet spectrum $\mathbb{D}_m^m$). For the accordingly defined Lagrange spectrum when considering the lower limit in (1) instead, see [1] for a very general result. The set $\mathbb{D}_m$ is contained in the interval $[0, 1]$ by Dirichlet’s Theorem. It was proved in [10] that $\Theta(\xi) = 1$ for Lebesgue almost all $\xi \in \mathbb{R}^m$, see also the very recent paper by Kleinbock, Strömbergsson, Yu [23] for a considerably refined result and further references. For $m = 1$, the Dirichlet spectrum is a rather complicated, well-studied object, see [2] for a wealth of references. In particular, it is known that $\mathbb{D}_1$ is not an interval, and contained in $[0] \cup [1/2, 1]$ by a result of Khintchine [19]. See further for example [13, 15, 16, 24] for refined metrical claims when restricting to $\xi \in \mathbb{R}$ with $\Theta(\xi) = 1$ when $m = 1$. It is worth mentioning that for $m = 1$, Davenport and Schmidt [11] showed that, besides rational numbers, precisely numbers with bounded partial quotients induce $\Theta(\xi) < 1$. These coincide with the set of badly approximable numbers for which $\liminf_{Q \to \infty} Q \psi_\xi(Q) > 0$. The claim is no longer true for any $m > 1$ and the accordingly defined set of badly approximable vectors in $\mathbb{R}^m$ inducing $\liminf_{Q \to \infty} Q^{1/m} \psi_\xi(Q) > 0$. However, the implication that any badly approximable vector is Dirichlet improvable holds in any dimension, again a result due to Davenport and Schmidt [10, Theorem 2].

For $m \geq 2$, the set of vectors that satisfy $\Theta(\xi) = 0$, commonly referred to as singular vectors, has Hausdorff and packing dimension $m^2/(m + 1)$, see [7] for Hausdorff dimension and [8, 9] for packing dimension. Moreover, it is easy to see that the $(m - 1)$-dimensional set of vectors that are $Q$-linearly dependent together with $\{1\}$ shares this property $\Theta(\xi) = 0$. Hence $\{0, 1\} \subseteq \mathbb{D}_m$ for any $m \geq 2$. For $m = 2$ and with respect to the Euclidean norm, results on the Dirichlet spectrum were obtained by Akhunzhanov and Shatskov [3] and Akhunzhanov and Moshchevitin [2]. In [3] it is shown that this Dirichlet spectrum is an interval which in some natural sense is as large as it can be. For arbitrary norms, very recently structural results for $\mathbb{D}_2$ were obtained...
by Kleinbock and Rao [21], see also [22]. For \( m \geq 2 \) and in the dual setting of a linear form in \( m \) variables, some results on the Dirichlet spectrum are immediate from [4, 25], see Theorem 11.2 in the Appendix. Also from [7] some metrical information can be inferred. None of these results however implies the existence of any non-empty interval where the Dirichlet spectrum with respect to some norm is dense, when \( m \geq 3 \).

In Corollary 2 we provide an interval contained in the Dirichlet spectrum, for a wide class of norms on \( \mathbb{R}^m \).

# 2 Determination of Dirichlet spectrum in \( \mathbb{R}^m \)

We show that if \( m \geq 2 \), there exist (Liouville) vectors with any prescribed Dirichlet constant in \([0, 1]\). In fact this set is rather large in some metrical sense.

**Theorem 2.1** Let \( m \geq 2 \). For any \( c \in [0, 1] \), there exists a set \( \mathcal{A}_{m,c} \subseteq \mathbb{R}^m \) of packing dimension \( m-1 \) consisting of \( \xi \in \mathbb{R}^m \) satisfying

\[
\Theta(\xi) = c
\]  

(2)

and for every \( N \) we have

\[
\liminf_{Q \to \infty} Q^N \psi_{\xi}(Q) = 0.
\]  

(3)

In particular \( \mathbb{D}_m = [0, 1] \).

The claim is very much in line with the result for \( m = 2 \) and the Euclidean norm quoted in Sect. 1. As noticed in Sect. 1 the set of singular vectors in \( \mathbb{R}^m \) has packing dimension \( m-1 + 1/(m+1) \), hence we should expect an estimate

\[
\lim_{c \to 0^+} \text{dim}_P(\{\xi \in \mathbb{R}^m : \Theta(\xi) = c\}) \leq m - 1 + \frac{1}{m+1},
\]

possibly with equality. Consequently for small \( c \) our metrical claim appears to be very close to optimal. Moreover, results by Cheung and Chevallier [7] imply the according result for Hausdorff in place of packing dimension. Note that on the other hand that property (3) forces the Hausdorff dimension of \( \mathcal{A}_{m,c} \) to be 0 by Jarník-Besicovich Theorem [17]. See however Theorems 3.1, 3.2 below for non-trivial Hausdorff dimension results when we drop hypothesis (3). Theorem 2.1 is an immediate corollary of the more general Theorem 2.2 below for a much larger class of uniform approximating functions.

**Definition 1** For \( m \geq 2 \) a fixed integer and \( \Phi : \mathbb{N} \to (0, 1) \) any function, we define decay properties \((d1), (d2), (d3)\) and for \( \gamma > 0 \) the property \((d4(\gamma))\) as follows.

\[(d1)\] Assume

\[
\Phi(t) < t^{-1/m}, \quad t \geq t_0.
\]
(d2) Assume 
\[ \lim_{t \to \infty} t^{\frac{1}{m-1}} \Phi(t) = \infty. \]

(d3) Assume 
\[ \liminf_{\alpha \to 1^+} \liminf_{t \to \infty} \frac{\Phi(\alpha t)}{\Phi(t)} \geq 1, \quad \text{if } m \geq 3, \]
and 
\[ \liminf_{\alpha \to 1^-} \liminf_{t \to \infty} \frac{\Phi(\alpha t)}{\Phi(t)} \geq 1, \quad \text{if } m = 2, \tag{4} \]

where \( t \) and \( \alpha t \) are considered integers so that the expression is well-defined. If \( m = 2 \), additionally assume that for any \( \epsilon > 0 \) and all \( t \geq t_0(\epsilon) \) the estimate 
\[ \min\{\Phi(x) : 1 \leq x < (1 - \epsilon)t, \ x \in \mathbb{N}\} / \Phi(t) > 1 \tag{5} \]
holds

(d4(\( \gamma \))) Assume for given \( \gamma > 0 \) and some \( \eta > 0 \), we have 
\[ \Phi(t) > \eta t^{-\gamma}, \quad t \geq t_0. \]

An alternative formulation of (d3) is that for every \( \epsilon_0 > 0 \) there is \( \epsilon_1 > 0, t_0 > 0 \) such that for any \( \alpha \in (1, 1 + \epsilon_1) \) (resp. \( \alpha \in (1 - \epsilon_1, 1) \) when \( m = 2 \)) and \( t \geq t_0 \) we have 
\[ \frac{\Phi(\alpha t)}{\Phi(t)} \geq 1 - \epsilon_0. \]

The conditions (4), (5) for \( m = 2 \) in (d3) are implied if \( \Phi \) is decreasing, but slightly relax this property. For \( m \geq 3 \), alternatively to (d3) the conditions (4), (5) are sufficient as well, see Remark 5 below. Hence we may just assume that \( \Phi \) is decreasing in place of (d3) in this case as well. Our more general result reads as follows.

**Theorem 2.2** Let \( m \geq 2 \) an integer and \( \Phi \) satisfy (d1), (d2), (d3). Then there exist uncountably many \( \xi \in \mathbb{R}^m \) for which the claims (C1), (C2), (C3) below hold:

(C1) We have 
\[ \psi_{\xi}(Q) < \Phi(Q), \quad Q \geq Q_0. \]

(C2) For any \( \epsilon > 0 \), we have 
\[ \psi_{\xi}(Q) > (1 - \epsilon)\Phi(Q) \]
for certain arbitrarily large \( Q \).

(C3) Property (3) holds for any given \( N \).

If for some \( \gamma > 0 \) the function \( \Phi \) satisfies \((d4(\gamma))\), then the packing dimension of the set of \( \xi \) as above is at least \( m(1 - \gamma) \).

We can always choose \( \gamma = 1/(m - 1) \) by \((d2)\), however we require \( \gamma \geq 1/m \) by \((d1)\), so the bound always lies in the short interval \([m - 1 - \frac{1}{m - 1}, m - 1]\). It is rather satisfactory and seems to exhaust the method, it coincides with the estimate in [32, Theorem 2.1] for the sets of points singular of order at least \( \gamma \). If we assume that \( \Phi \) is decreasing, then we may relax \((d4(\gamma))\) by requiring its inequality only for some unbounded sequence of values for \( t \). Theorem 2.1 represents the special case \( \Phi(t) = ct^{-1/m} \) if \( c \in (0, 1) \), which clearly satisfies \((d1), (d2), (d3), (d4(1/m))\), and slightly altered functions in the special cases \( c \in \{0, 1\} \). If we admit a factor \( 1 + \epsilon \) in the right hand side of \((C1)\) and are given an explicit rate of divergence in \((d2)\), from our proof we may give a rate for \( Q \) in terms of \( \epsilon \) for which \((C1), (C2)\) hold.

We further remark that claim \((C2)\) and property \((d2)\) imply that the coordinates of \( \xi \) in Theorem 2.2 together with \( \{1\} \) are linearly independent over \( \mathbb{Q} \), in other words \( \bar{\xi} \) is totally irrational. We go on to comment on potential relaxations/removal of the conditions \((d2), (d3)\) in Sect. 6.4.

Denote by \( \text{Bad}_m \) the set of badly approximable vectors in \( \mathbb{R}^m \) as introduced in Sect. 1. Our claim \((3)\) means that \( \bar{\xi} \) in Theorem 2.2 are Liouville vectors and hence clearly not badly approximable. Write

\[
\text{Di}_m(c) = \{ \xi \in \mathbb{R}^m : \psi_\xi(Q) \leq cQ^{-1/m}, \ Q \geq Q_0 \} \subseteq \{ \xi \in \mathbb{R}^m : \Theta(\xi) \leq c \},
\]

so that \( \text{Di}_m = \bigcup_{c \leq 1} \text{Di}_m(c) \) is the set of \( m \)-dimensional Dirichlet improvable vectors. Further denote by \( \text{Sing}_m \) the set of singular vectors in \( \mathbb{R}^m \), defined via the property \( \lim_{Q \to \infty} Q^{1/m} \psi_\xi(Q) = 0 \), or equivalently \( \cap_{c > 0} \text{Di}_m(c) \). The next corollary of Theorem 2.2 slightly refines Theorem 2.1.

**Corollary 1** Let \( m \geq 2 \) be an integer. For any \( c \in (0, 1] \), the set

\[
\text{Di}_m(c) \setminus \left( \cup_{\epsilon > 0} \text{Di}_m(c - \epsilon) \cup \text{Bad}_m \right)
\]
has packing dimension at least $m - 1$. In particular, the same applies to the set

$$FS_m := \text{Di}_m \setminus (\text{Bad}_m \cup \text{Sing}_m).$$

The latter claim extends the main result from [4] in two directions. Firstly, we also give a metrical result instead of only proving uncountability, thereby contributing towards the metrical problem of determining the Hausdorff dimension of the “folklore set” $FS_m$ formulated in [4, § 3.4]. As indicated below Theorem 2.1, asymptotically as $c \to 0$, our metrical bound is probably sharp up to an additive error $O(m^{-1})$. When considering Hausdorff dimension instead, results by Cheung and Chevallier [7] actually imply the upper bound $m - 1 + \frac{1}{m+1} + o(1)$ as $c \to 0$ for the set $\text{Di}_m(c) \setminus \text{Di}_m(\delta)$, with positive error term for any $c > 0$ and some explicitly computable $\delta = \delta(c) \in (0, c)$ and $c$ small enough. However the results from [7] do not allow for taking $\delta$ arbitrarily close to $c$, nor do they restrict to $\text{Bad}_m^c$. Note also that as follows from Kleinbock and Mirzadeh [20, Theorem 1.5], the set $\text{Di}_m(c)$ has Hausdorff dimension less than $m$ for any $c < 1$. On the other hand, it is conjectured in [4, Problem 3.1] that $FS_m$ has full Hausdorff dimension.

Secondly, we emphasize that by the first claim we can also prescribe an exact Dirichlet constant (in the simultaneous approximation setting). Note that in [4, 25] the deep, unconstructive result of Roy [26] on parametric geometry of numbers was used. As a consequence of this setup, from [4, Theorem 1.5] and [25], one can only provide a countable partition of $[0, 1]$ into intervals with each having non-empty intersection with $\mathbb{D}_m$, see the “Appendix”. We should however remark that additional specifications on various exponents of approximation within these sets $FS_m$ can be made according to [4, 25]. We should also note that the dual setting of a linear form in $m$ variables is treated in [4]. On the other hand, our constructive proof of Theorem 2.2 is elementary and rather short, based on ideas from the proof of [27, Theorem 2.5], with some twists.

### 3 On Hausdorff dimension

We further provide a considerably weaker bound regarding Hausdorff dimension, which we will denote by $\dim_H$, however still of order $\gg m$. Note that for non-trivial results we can no longer impose (3), as the set of Liouville numbers has Hausdorff dimension 0. We consider slightly larger sets than in Corollary 1, namely for $m \geq 2$ an integer and $c \in [0, 1]$, our focus is now on the sets

$$F_{m,c} := \bigcap_{\epsilon > 0} (\text{Di}_m(c + \epsilon) \setminus \text{Di}_m(c - \epsilon)) \setminus \text{Bad}_m$$

$$= \{ \xi \in \mathbb{R}^m : \Theta(\xi) = c \} \setminus \text{Bad}_m \subseteq FS_m.$$

We show
Theorem 3.1 For any \( m \geq 2 \) and \( c \in [0, 1] \) we have

\[
\dim_H(\mathcal{F}_{m,c}) \geq \frac{\sqrt{m(m^2 - m + 1)}}{(m + \sqrt{m(m^2 - m + 1)})^2} > 0.
\]

(6)

Asymptotically as \( m \to \infty \), uniformly in \( c \in [0, 1] \) we have the stronger lower bound

\[
\dim_H(\mathcal{F}_{m,c}) \geq \frac{3}{8} m - o(m).
\]

(7)

The estimate (6) is only of order \( 1 - o(1) \) as \( m \to \infty \), so the latter asymptotical bound (7) is indeed significantly stronger. It is natural to expect that \( c \mapsto \dim_H(\mathcal{F}_{m,c}) \) decays, which is not reflected in our result. In contrast to Theorem 2.2, our estimates are probably far from the true value no matter how small \( c > 0 \) is chosen. Indeed, it is reasonable to conjecture \( \dim_H(\mathcal{F}_{m,c}) = \dim_H(\text{Dim}(c)) \) for any \( c \in (0, 1) \), in particular we expect

\[
\dim_H(\mathcal{F}_{m,c}) = m - 1 + \frac{1}{m + 1} + o(1), \quad \text{as } c \to 0^+,
\]

and

\[
\dim_H(\mathcal{F}_{m,c}) = m - o(1), \quad \text{as } c \to 1^-,
\]

see the comments below Corollary 1. See further [32, Theorem 2.2] for a stronger bound of order \( m - 4 + O(m^{-1}) \) for the Hausdorff dimension of the larger set of (inhomogeneously) singular vectors obtained from essentially the same method in a simplified setting. Recall there was no such discrepancy to [32] for the packing dimension result, as remarked below Theorem 2.2. Theorem 3.1 can be generalized to the situation of \( \Phi(t) \) satisfying \((d1), (d2), (d3), (d4)(\gamma)) \) for some \( \gamma > 0 \) as in Theorem 2.2, we do not explicitly state it.

We next derive a theorem that contains information on ordinary approximation as well, very much in the spirit of [4, 25]. Let \( \lambda(\xi) \) denote the ordinary exponent of simultaneous rational approximation to \( \xi \in \mathbb{R}^m \), defined as the supremum of \( \lambda > 0 \) such that

\[
\liminf_{Q \to \infty} Q^{\lambda} \psi_{\xi}(Q) = \liminf_{q \to \infty} q^\lambda \|q\xi\| < \infty.
\]

Then \( \lambda(\xi) \in [1/m, \infty) \) for any \( \xi \in \mathbb{R}^m \) by Dirichlet’s Theorem. Denote by

\[
\mathcal{W}_m(\lambda) = \{\xi \in \mathbb{R}^m : \lambda(\xi) = \lambda\} \subseteq \mathbb{R}^m
\]

\( \mathcal{W}_m(\lambda) \) is the set of \( \xi \) such that the ordinary exponent of simultaneous rational approximation to \( \xi \) is equal to \( \lambda \). Theorem 3.1 states that

\[
\dim_H(\mathcal{F}_{m,c}) \geq \frac{3}{8} m - o(m).
\]

(7)
the pairwise disjoint levelsets of vectors with precise ordinary exponent $\lambda \in [1/m, \infty]$. Note that $\text{Bad}_m \subseteq \mathcal{W}_m(\frac{1}{m})$. Let

$$\beta = \frac{1 + \sqrt{5}}{2}$$

be the golden ratio.

**Theorem 3.2** Let $m \geq 2$ be an integer, and $c \in (0, 1]$ and $\lambda \in (\beta, \infty)$. Then the Hausdorff dimension of the set

$$\mathcal{W}_m(\lambda) \cap F_{m,c} \subseteq \mathcal{W}_m(\lambda) \cap FS_m$$

is positive and a lower bound independent of $c$ is explicitly computable. Asymptotically as $\lambda \to \infty$, i.e. for $\lambda \geq \lambda_0(m)$, it is of order

$$\dim_H(\mathcal{W}_m(\lambda) \cap F_{m,c}) \geq \frac{m}{2\lambda} - O(\lambda^{-1}),$$

where the implied constant is effectively computable and does not depend on $m, c, \lambda$.

By Jarník-Besicovich Theorem [17], we have $\dim_H(\mathcal{W}_m(\lambda)) = (m + 1)/(\lambda + 1)$, so for large $m$ our asymptotical bound is basically sharp up to a factor 2.

We may again extend the claim to a setup involving in place of $F_{m,c}$ sets derived from more general uniform approximation functions $\Phi$ via imposing $(C1), (C2)$. Besides, with small modifications in the proof below, we can prescribe the order of ordinary approximation more exactly up to an asymptotical factor $1 + o(1)$ as $Q \to \infty$. More precisely, take any function $\Psi : \mathbb{N} \to (0, 1)$ of decay $o(t^{-\beta-\epsilon})$ as $t \to \infty$ for some $\epsilon > 0$, and conversely satisfying $(d4(\gamma))$ for some $\gamma > 0$. Derive $\mathcal{W}_m(\Psi)$ the set of $\zeta \in \mathbb{R}^m$ satisfying

$$\liminf_{Q \to \infty} \frac{\psi_\zeta(Q)}{\Psi(Q)} = \liminf_{Q \to \infty} \frac{\|q \zeta\|}{\Psi(q)} = 1.$$

Then $\mathcal{W}_m(\Psi) \cap F_{m,c}$ has positive Hausdorff dimension, effective lower bounds can be given subject to the decay rate of $\Psi$. The special case $\Psi(t) = \Psi_{a,\lambda}(t) := at^{-\lambda}$ for $\lambda > \beta$ and $a > 0$ parameters turns out to result in the same bounds as Theorem 3.2 (see Sect. 9.4), thereby refining it since $\mathcal{W}_m(\Psi_{a,\lambda}) \subseteq \mathcal{W}_m(\lambda)$. We sketch the proof of this generalized claim in Sect. 9.4, but want to compare this version of Theorem 3.2 to [25]. There it was shown that there is some explicitly computable $\kappa_m \in (0, 1)$, so that for any $\lambda > 1/m$ and fixed $c \in (0, 1]$, the set of $\zeta \in \mathbb{R}^m$ inducing simultaneously

$$\liminf_{Q \to \infty} \frac{\psi_\zeta(Q)}{\Psi_{a,\lambda}(Q)} \in [\kappa_m, 1], \quad \limsup_{Q \to \infty} \frac{\psi_\zeta(Q)}{\Psi_{a,\lambda}(Q)} \in [\kappa_m, 1],$$

is uncountable. Our claim allows for prescribing the order of both ordinary and uniform approximation considerably more precisely, provides a metric claim, and permits more...
flexibility in the choice of functions $\Psi$ locally, for the cost of requiring a faster decay rate for $\Psi$.

The lower bound $\beta$ for $\lambda$ can in fact be improved to $\lambda > 1$ with a rather technical argument, we prefer to only sketch the proof in Sect. 9.4 below. However, that seems to be the limit of the method. On the other hand, we strongly expect the Hausdorff dimension of the sets in (8) to decay as a function of $\lambda \geq 1/m$ (and to increase in $c \in (0, 1)$). The above remarks on more general $\Psi$ still apply for any $\Psi(t) = o(t^{-1-\epsilon})$.

4 Cantor sets and other norms

4.1 Other norms

Consider any norm $|.|$ on $\mathbb{R}^m$ and let $\mathbb{D}_m = \{ \Theta^{|.|}(\xi) : \xi \in \mathbb{R}^m \}$ be the Dirichlet spectrum with respect to $|.|$ derived from the Dirichlet constant

$$
\Theta^{|.|}(\xi) := \limsup_{Q \to \infty} Q^{1/m} \min_{1 \leq q \leq Q, (p, q) \in \mathbb{Z}^{m+1}} |q_\xi - p| = \limsup_{Q \to \infty} Q^{1/m} \min_{1 \leq q \leq Q, q \in \mathbb{Z}} |q_\xi|
$$

with $q_\xi := (\pm ||q_\xi_1||, \ldots, \pm ||q_\xi_m||) \in \mathbb{R}^m$, for some choice of signs. The latter identity holds as we can exclude the case $|q_\xi_i - p_i| > 1/2$ for some $i$ and large $Q$, as then the limit expression would tend to infinity like $Q^{1/m}$ independent of the chosen norm. Let us call a norm $|.|$ expanding if $|x| \geq |\pi_j(x)|$ for all $x \in \mathbb{R}^m$ and $1 \leq j \leq m$, where $\pi_j : \mathbb{R}^m \to \mathbb{R}^m$ are the orthogonal projections to the coordinate axes. Then

**Corollary 2** Let $m \geq 2$ and $|.|$ be any expanding norm on $\mathbb{R}^m$. Let

$$e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0), \quad 1 \leq i \leq m,$

be the canonical base vectors and let $\Omega := \min_{1 \leq i \leq m} |e_i| > 0$. Then $[0, \Omega] \subseteq \mathbb{D}_m$. The set of $\xi$ realizing any given value $c \in [0, \Omega]$ has packing dimension at least $m - 1$.

See also Remark 2 below on Hausdorff dimension estimates. The result in particular applies to any $p$-norm $|x|_p = (\sum |x_i|^p)^{1/p}$, $p \geq 1$ (it turns out we may take $L_p$-spaces for $1/2 < p < 1$ as well), and shows that the interval $[0, 1]$ is contained in the according Dirichlet spectrum. This may be compared with the results for $m = 2$ and Euclidean norm $p = 2$ by Akhunzhanov, Shatskov [3] and Akhunzhanov, Moshchevitin [2] recalled in Sect. 1. However, it is not hard to construct norms that are not expanding, for example if its unit ball $\{x \in \mathbb{R}^m : |x| \leq 1\}$ is defined by a generic rotation of an ellipsoid (not a ball) in standard position. The deduction of Corollary 2 relies on the fact that for the real vectors $\xi$ constructed in Sect. 6.2, for any $Q$ inducing large values of $Q^{1/m} \psi_\xi(Q)$ we may choose $q < Q$ so that some component $\|q_\xi_i\|$ with fixed $i$ of our choice induces $\psi_\xi(Q) = \|q_\xi\| = \|q_\xi_i\|$ and significantly exceeds all the other $\|q_\xi_j\|, j \neq i$. This will give us any desired upper bound in $[0, \Omega]$ for the Dirichlet constant. The reverse lower bound will be immediate from the assumption
of the norm being expanding and our result \((C2)\) for the maximum norm. We provide more details in Sect. 6.3.

### 4.2 Dirichlet spectrum for Cantor sets

Fix \(b \geq 2\) an integer. Define the Cantor set \(C_b\) as the set of all real numbers that admit a base \(b\) representation only using digits 0 and 1, i.e. of the form

\[
\sum_{j=1}^{\infty} w_j b^{-j}, \quad w_j \in \{0, 1\}.
\]

For \(b = 3\), the set \(C_b\) is quite similar to the Cantor middle third set, where however the triadic digits 0, 2 are considered. We want to consider the \(m\)-fold Cartesian product set

\[
K = K_{m,b} = C_b^m = C_b \times \cdots \times C_b.
\]

We refer to the arXiv version [33] of this article for similar results in a slightly more general setting, and to [31] for a subsequent further generalisation. We claim that in the assertions of Sect. 2, we may restrict to \(\xi \in K\) upon taking a smaller, fixed multiplicative constant \(R < 1\) in \((C2)\) and adjusting the metrical claims. Assuming some Diophantine condition, we even get a precise analogue of Theorem 2.2. Let us first define some more properties.

For \(b \geq 2\) an integer and \(R \in (0, 1)\) parameters, we define a modified decay property \((d^3'(b, R))\) and a Diophantine property \((D(b))\) on functions \(\Phi : \mathbb{N} \to (0, 1)\).

**Definition 2** Let \(b \geq 2\) an integer and \(R \in (0, 1)\). We define

\((d^3'(b, R))\) For some \(\epsilon > 0\) we have

\[
\min\{\Phi(x) : 1 \leq x \leq (b + \epsilon)t, \ x \in \mathbb{N}\} > R\Phi(t), \quad t \geq t_0.
\]

\((D(b))\) For any fixed \(\epsilon > 0\), the inequalities

\[
(1 - \epsilon)\Phi(b^B) < b^{-A} < \Phi(b^B)
\]

hold for certain arbitrarily large pairs of positive integers \(A, B\).

Property \((d^3'(b, R))\) relaxes \((d3)\). The Diophantine property \((D(b))\) is rather mild and applies to most reasonable functions, however with the unfortunate exception of functions \(\Phi(t) = ct^{-r/s}\) for a rational number \(r/s\) and \(c > 0\).

Denote by \(\dim(K) = m \log 2 / \log b\) the Hausdorff (or packing) dimension of \(K\) above. Our result reads as follows.
Theorem 4.1 Let \( m \geq 2 \) and \( K \) be as above. Assume \( \Phi \) satisfies \((d1), (d2)\).

(i) Assume \( \Phi \) also satisfies \((d3')(b, R)\) for some given \( R \in (0, 1) \). Then there exist uncountably many \( \xi \in K \) for which we have \((C1), (C3)\) and

\[
(C2') \quad \text{We have} \quad \psi_{\xi}(Q) > R\Phi(Q)
\]

for certain arbitrarily large \( Q \).

(ii) Assume \( \Phi \) satisfies \((D(b))\) as well. Then there exist uncountably many \( \xi \in K \) satisfying \((C1), (C2)\) and \((C3)\).

If for some \( \gamma > 0 \) the function \( \Phi \) satisfies \((d4(\gamma))\), then the packing dimension of the set of \( \xi \) in (i), (ii) is at least \( \dim(K)(1-\gamma) = m(1-\gamma) \log 2/\log b \). Moreover, the set of vectors satisfying \((C1)\) and \((C2')\) but not \((C3)\) in (i) has positive Hausdorff dimension.

A Hausdorff dimension result in (ii) above would require some density assumption on the integers \( A, B \) in \((D(b))\). We will show that when \( \Phi(q) = cq^{-1/m} \), property \((d3'(b, R))\) holds for \( R = b^{-1/m} \). Hence we get the following variant of Corollary 1.

Corollary 3 Let \( m \geq 2 \) and \( K \) as above. Then for any \( c \in (0, 1] \) the set

\[
K \cap (D_m(c) \setminus (D_m(b^{-1/m}c) \cup \text{Bad}_m)) \subseteq K \cap FS_m
\]

has packing dimension at least \( \dim(K)(1-1/m) = (m-1) \log 2/\log b \) and positive Hausdorff dimension. In particular the same holds for \( K \cap FS_m \).

We may calculate an effective positive lower bound for the Hausdorff dimension as well with the method of Sect. 9.2, however the expression becomes rather cumbersome. Moreover, a variant of Theorem 3.2 can be derived, we do not state it. Corollary 3 induces a countable partition of \([0, 1]\) into intervals that each have non-empty intersection with

\[
\mathbb{D}_{m,K} := \left\{ \limsup_{Q \to \infty} Q^{1/m} \psi_{\xi}(Q) : \xi \in K \right\}.
\]

This is similar to the situation in [4, 25] (or [7]), see Theorem 11.2 in the Appendix of the paper, but restricting to fractal sets. We present a class of functions that satisfy \((D(b))\) for any \( b \geq 2 \) and thus the stronger claim of Theorem 4.1.

Corollary 4 Assume \( K \) is as above. Let \( \Phi(q) = cq^{-\tau} \) for irrational \( \tau \in \left(\frac{1}{m}, \frac{1}{m-1}\right) \) and any \( c > 0 \). Then \( \Phi \) satisfies \((d1), (d2), (D(b))\) and thus the set of vectors \( \xi \in K \) satisfying \((C1), (C2)\) and \((C3)\) has packing dimension at least \( \dim(K)(1-\tau) = m(1-\tau) \log 2/\log b \), and if we drop \((C3)\) also positive Hausdorff dimension.

Unfortunately, the function \( \Phi(t) = ct^{-1/m} \) does not satisfy hypothesis \((D(b))\) for any \( b \geq 2 \), therefore we cannot conclude that \( \mathbb{D}_{m,K} = [0, 1] \). We want to remark that...
for ordinary simultaneous approximation and fast enough decaying $\Phi$, similar results on exact approximation to successive powers of elements of Cantor sets $C_{b,w}$, thereby restricting to the Veronese curve, have been obtained in [28, §2.2].

The sets $K$ in this section can be obtained as the attractor of an iterated function system (IFS) consisting of a finite set of contracting maps $f_j(x) = x/b + u_j/b$ on $\mathbb{R}^m$ with integer vectors $u_j \in \mathbb{Z}^m$. We ask whether our results extend to more general situations.

**Proposition 1** Let $K \subseteq \mathbb{R}^m$ be any uncountable attractor of an IFS. Is the set $K \cap \text{FS}_m$ non-empty? Is it at least true for any IFS consisting of contracting functions of the form $f_j(x) = A_jx + b_j$ with $A_j \in \mathbb{Q}^{m \times m}$, $b_j \in \mathbb{Q}^m$?

See the preprint [31] succeeding the first version of this paper for partial affirmative results.

5 Some remarks and forthcoming work

Our results can be interpreted as prescribing extremal values (primarily maxima, but also minima in a logarithmic setting in Theorem 3.2) of the first successive minimum of some classical parametric lattice point problem, see for example [34], in particular [34, Theorem 1.4]. Our method relies heavily on ideas from the proof of [27, Theorem 2.5], which contains a considerably good description of higher successive minima functions as well. Thus it seems plausible that similar results on extremal values of the according parametric higher successive minima functions can be obtained when combining our proofs below with some more ingredients from the proof of [27, Theorem 2.5].

By transference inequalities, it is possible to derive from our Theorem 2.1 some information on the Dirichlet spectrum of a linear form in $m$ variables as investigated in [4, 25]. After the first version of the present paper, in [30] the analogous result in the linear form setting has been proved. Nevertheless, here we provide weaker results based on transference in the Appendix.

Analogous results of Theorems 2.1, 2.2 for (a) $p$-adic approximation, (b) weighted approximation and/or (c) systems of linear forms, seem in reach by refinements and generalizations of the method presented below. Verification of (a), (b), (c) seems increasingly challenging, in particular the impression of the author is that some fundamentally new concepts need to be introduced for (c) in full generality. The question of a weighted version for linear forms, i.e. solving (b) and (c) at once, was raised in [4, Problem 4.1]. The author plans subsequent work on these topics.

6 Proof of Theorem 2.2 for $m \geq 3$: existence claim

6.1 Construction of suitable vectors $\xi$

Define an increasing sequence of positive integers recursively as follows: For the initial terms, observe that by (d2), for any large enough positive integer $Q$ we have $\Phi(Q) > Q^{-1/(m-1)}$. Let $H > 1$ be large enough, depending on the chosen function
Φ, to be specified below. Define the initial $m$ terms by $a_j = H^j$ for $1 \leq j \leq m$. For $n \geq 1$, having constructed the first $mn$ terms $a_1, \ldots, a_{mn}$, let the next $m$ terms be given by the recursion

$$a_{mn+1} = a_{mn}^M_n$$  \hspace{1cm} (9)

and

$$a_{mn+2} = a_{mn+1}^2, \quad a_{mn+3} = a_{mn+1}^3, \quad \ldots, \quad a_{mn+m-1} = a_{mn+1}^{m-1},$$  \hspace{1cm} (10)

and finally

$$a_{mn+m} = a_{m(n+1)} = L_n \cdot a_{mn+m-1} \in (a_{mn+m-1}, a_{mn+1}^m]$$

with integers $M_n \to \infty$ that tend to infinity fast enough, to be made precise below, and the integer $L_n$ defined as

$$L_n = \max\{z \in \mathbb{N} : a_{mn+1}^{-1} < \Phi(Q), \ 1 \leq Q \leq za_{mn+m-1}\}.$$ 

Important properties of $L_n$ induced by $(d1), (d2)$ are captured in the following proposition.

**Proposition 6.1** For $H$ chosen large enough, the positive integers $L_n$ are well-defined and satisfy

$$L_n \leq a_{mn+1}, \quad n \geq 1,$$  \hspace{1cm} (11)

and

$$\lim_{n \to \infty} L_n = \infty.$$  \hspace{1cm} (12)

**Proof** Let $n \geq 1$. Having chosen $H$, we may assume $M_1, \ldots, M_n$ and thus $a_1, \ldots, a_{mn+m-1}$ chosen as well via (9), (10). A problem may occur when determining $a_{mn+m}$ via $L_n$. Notice first that by $(d1)$ we have $\Phi(t) \to 0$ as $t \to \infty$, and hence $L_n$ is not infinite. In fact $(d1)$ directly implies $a_{mn+m} \leq a_{mn+1}^m$ for $n \geq 1$ upon choosing $H$ large enough (since in $(d1)$ the restriction $t \geq t_0$ is given), hence (11) holds. Conversely, by assumption $(d2)$ we have the estimate $\Phi(t) > t^{-1/(m-1)}$ for all large integers $t \geq t_1 = t_1(\Phi)$. For $z = 1$, by (10) if $H = a_1 \geq t_1$ (and thus $a_{mn+1} > t_1$) we have

$$a_{mn+1}^{-1} = (a_{mn+1}^{m-1} - t_1^{-1/(m-1)} < \Phi(Q), \quad t_1 \leq Q \leq za_{mn+1}^m = za_{mn+m-1}.$$ 

Since there are only finitely many positive integers smaller than $t_1$, for any large $t \geq t_2(\Phi)$ obviously we have

$$\min\{\Phi(1), \Phi(2), \ldots, \Phi(t_1)\} > t^{-1/(m-1)}.$$
We may assume $H$ was chosen large enough that $H \geq t_2^{1/(m-1)}$ as well. Then

$$\min\{\Phi(1), \Phi(2), \ldots, \Phi(t_1)\} > t_2^{-1/(m-1)} \geq H^{-1} = a_1^{-1} > a_{mn+1}^{-1}.$$ 

Thus we have $a_{mn+1}^{-1} < \Phi(Q)$ for $1 \leq Q \leq 1 \cdot a_{mn+1}$. Hence $L_n \geq 1$ is well-defined. A similar argument shows (12). Let $C > 1$ be arbitrary. Then by (d2), for large $t \geq t_3(\Phi, C)$ we have $\Phi(t) > C t^{-1/(m-1)}$. For large values $t_3 \leq Q \leq C a_{mn+1} = C a_{mn+m-1}$ we get

$$a_{mn+1}^{-1} = (a_{mn+1}^{-1})^{1/(m-1)} \leq C^{1/(m-1)} Q^{-1/(m-1)} < C Q^{-1/(m-1)} < \Phi(Q).$$

On the other hand, again for large enough $t \geq t_4(\Phi, C)$ we have

$$\min\{\Phi(1), \Phi(2), \ldots, \Phi(t_3)\} > t_4^{-1/(m-1)}.$$

As $a_n \to \infty$, for $n \geq n_0(t_4) = n_0(\Phi, C)$ large enough we have $a_{mn+1} > t_4^{1/(m-1)}$. Then

$$\min\{\Phi(1), \Phi(2), \ldots, \Phi(t_3)\} > t_4^{-1/(m-1)} > a_{mn+1}^{-1}.$$ 

Hence $a_{mn+1}^{-1} < \Phi(Q)$ for all integers $1 \leq Q \leq C a_{mn+m-1}$ as soon as $n \geq n_0(\Phi, C)$ is large enough. Thus, as $C$ was arbitrary, (12) follows.

In particular indeed $(a_j)_{j \geq 1}$ is strictly increasing. We observe further that

$$a_j | a_{j+1}, \quad j \geq 1. \quad (13)$$

Define the components $\xi_i$ of $\xi$ via

$$\xi_i = \sum_{n=0}^{\infty} \frac{1}{a_{mn+i}}, \quad 1 \leq i \leq m, \quad (14)$$

that is we sum the reciprocals over the indices congruent to $i$ modulo $m$. We claim that all assertions $(C1), (C2), (C3)$ of the theorem hold for $\xi = (\xi_1, \ldots, \xi_m)$ if we choose $M_n$ suitably large in each step. We remark that taking a power of $a_{mn}$ in (9) is just for convenience, it suffices to let

$$a_{mn+1} = M_n' a_{mn} \quad (15)$$

for large enough $M_n'$. We will assume the latter in place of (9) occasionally. We conclude this section with a very elementary observation to be applied below.
Proposition 6.2 Let \( u/v \in \mathbb{Q} \) be reduced. Then for any integers \( m \geq 2, L \geq 1, M \geq 2, \) the fraction

\[
\frac{u}{v} + \frac{1}{(v^{m-1}L)^M} = \frac{uL^{M}v^{(m-1)M-1} + 1}{L^{M}v^{(m-1)M}}
\]

is reduced as well.

Obviously the numerator is congruent to 1 modulo any prime divisor of \( Lv, \) and the claim follows.

6.2 Proof of (C1), (C2), (C3)

Proof of (C3): Take \( Q = q = a_{mn} \) for \( n \) large. By (13) all \( qa_j^{-1} \) for \( j \leq mn \) are integers. Hence \( \|q\xi\| = \|q\xi_1\| = a_{mn}(a_{mn+1}^{-1} + a_{m(n+1)+1}^{-1} + \cdots). \) The main contribution clearly comes from the first term \( a_{mn}/a_{mn+1}, \) indeed we may estimate

\[
\|q\xi\| = \|q\xi_1\| \leq 2 \frac{a_{mn}}{a_{mn+1}} = 2a_{mn}^{-M_{n+1}} = 2Q^{-M_{n+1}}. \tag{16}
\]

Since we assume \( M_n \to \infty \) it suffices to take \( n \) large enough for given \( N. \)

Proof of (C1): For simplicity write

\[
d_n = a_{mn+1}, \quad n \geq 1.
\]

Let \( Q > 1 \) be an arbitrary large number and \( k \) be the index with \( a_k \leq Q < a_{k+1}. \) Let \( g \in \{0, 1, 2, \ldots, m-1\} \) be the residue class of \( k \) modulo \( m \) so that we may write \( k = mf + g \) for another integer \( f \geq 0. \) We consider three cases.

Case 1: Assume \( g = 0. \) This means \( a_{mf} \leq Q < a_{mf+1}. \) Let \( q = a_k = a_{mf} \leq Q. \) Then very similar as in (16) we see

\[
\|q\xi\| = \|q\xi_1\| \leq 2 \frac{a_{mf}}{a_{mf+1}} = 2a_{mf}^{-M_f+1} = 2a_{mf+1}^{-M_{f+1}} < 2Q^{-M_f} = 2Q^{-M_{f+1}}.
\]

Now since \( m \geq 3 \) and we can assume \( M_f \geq 3, \) from (d2) we easily see that the right hand side is less than \( \Phi(Q) \) for sufficiently large \( Q \geq Q_0 \) or equivalently \( f \geq f_0. \) Thus \( \psi_\xi(Q) < \Phi(Q) \) for \( Q \) in these intervals.

Case 2: Assume \( 1 \leq g \leq m-2. \) Then \( Q < a_{k+1} \leq a_{mf+1} = d_{f}^{m-1} \) and \( a_{k+1}/a_k = d_{f}. \) Let \( q = a_{mf+1} = d_{f} \leq Q. \) Isolating the first term \( a_k/a_{k+1} = d_{f}^{-1} \) in \( q\xi_2, \) we easily verify

\[
\|q\xi\| = \|q\xi_2\| \leq \frac{1}{d_{f}} + O(d_{f}^{-1}) \leq 2Q^{-1/(m-1)} < \Phi(Q) \tag{17}
\]

\( \psi \) Birkhäuser
for $Q \geq Q_0$ or equivalently $f$ large enough by property $(d2)$. This again implies $\psi_\xi(Q) < \Phi(Q)$ for the $Q$ in question.

**Case 3:** Now assume $g = m - 1$. By construction of $L_f$ and since $a_{k+1} = a_{m(f+1)} = L_f a_{mf+m-1}$, we have

$$d_f^{-1} < \Phi(Q), \quad 1 \leq Q \leq a_{k+1}.$$ 

With $q = d_f \leq Q$ again, as in (17) we have

$$\|q_\xi\| = \|q_\xi\| \leq \frac{1}{d_f} + O(d_f^{-1}). \quad (18)$$

Combining, we derive the estimate

$$\|q_\xi\| = \|q_\xi\| < \Phi(Q)$$

as well if we choose $M_{f+1}$ and hence $d_{f+1} = a_{m(f+1)+1}$ in the next step large enough that the error term in (18) is small enough. Again $\psi_\xi(Q) < \Phi(Q)$ for $Q$ in these intervals follows. Since we covered all large numbers with our cases, $(C1)$ follows.

**Remark 1** Note that Case 3 is the critical point where we needed the assumption $m > 2$. Indeed, for $m = 2$ we would not have $a_{2(f+1)} = a_{2f+2} = a_{2f+1}^2$ but rather $a_{2f+2} = L_f a_{2f+1}$, and for $\Phi$ slowly decaying, like $\Phi(t) = c t^{-1/2}$, the outcome $\|q_\xi\| = \|q_\xi\| \leq 1/L_f + O(d_f^{-1})$ would be larger than the bound in (18). Returning to $m \geq 3$, we further remark that in Case 2 we could take $q = a_k$ instead and obtain $\|q_\xi\| = \|q_\xi\| \leq d_f^{-1} + O(d_f^{-1})$.

**Proof of (C2):** For some large integer $f$ let

$$Q = Q_f = a_{m(f+1)} - 1. \quad (19)$$

Let $1 \leq q \leq Q$ be any integer and $s = s(q)$ be the maximum index with $a_s$ divides $q$, and take $s = 0$ and let $a_0 = 1$ if no such $s$ exists. Recall $d_f = a_{mf+1}$. We claim that

$$\|q_\xi\| \geq \frac{1}{d_f} + O(Qd_f^{-1}). \quad (20)$$

We again distinguish two cases.

**Case I:** We have $s \leq mf$, or equivalently $a_s \leq a_m f$. Then we claim

$$\|q_\xi\| \geq \|q_\xi\| \geq d_f^{-1} + O(Qd_f^{-1}). \quad (21)$$
Write $e_s = e_s(q) = a_{s+1}/a_s \in \mathbb{Z}$ for simplicity, which is an integer by (13). By assumption we have $q = B \cdot a_s$ with some integer $B = B_q$ with $e_s \nmid B$ and $B = q/a_s \leq Q/a_s < a_{m(f+1)}/a_s$. We split $q \xi_1 = U_q + V_q$ with

$$V_q = B a_s \sum_{j=1}^{f} a_{jm+1}^{-1} = B a_s \frac{D_f}{a_{fm+1}}, \quad U_q = B a_s \sum_{j=f+1}^{\infty} a_{jm+1}^{-1},$$

where

$$D_f = a_{fm+1} \sum_{j=1}^{f} a_{jm+1}^{-1} \in \mathbb{Z}$$

is an integer by (13), and independent of $q$. By Proposition 6.2 applied to $u/v = \sum_{j=1}^{f-1} a_{mj+1}^{-1}$, $M = M_{f-1}$, $L = L_{f-1}$,

and an inductive argument, we see that $(D_f, a_{mf+1}) = 1$. Note hereby that in the base case $f = 1$ of the induction, the hypothesis is easily checked. On the other hand, since $e_s \nmid B$ we have $B/e_s = B a_s / a_{s+1}$ is not an integer. Thus also $B a_s / a_{mf+1} = (B/e_s) \cdot (a_{s+1}/a_{mf+1}) \notin \mathbb{Z}$ since by assumption of Case I we have $mf + 1 \geq s + 1$ and thus $a_{s+1} | a_{mf+1}$ by (13). Combining, we see that $V_q \notin \mathbb{Z}$. Since the denominator in reduced form divides $a_{mf+1}$, it has distance at least $a_{mf+1}^{-1} = d_f^{-1}$ from any integer. Finally we can estimate

$$|U_q| \leq B a_s \cdot 2 a_{m(f+1)+1}^{-1} = 2 q a_{f+1}^{-1} \leq 2 Q d_{f+1}^{-1},$$

and the claim (21) and thus (20) follows.

Case II: Assume $s > mf$. Then by (19) we may write $s = mf + h - 1$ with an integer $h \in \{2, 3, \ldots, m\}$.

Case IIa: First assume $h \neq m$. Write again $e_s = a_{s+1}/a_s \in \mathbb{Z}$ and $q = B \cdot a_s$ with some integer $B$ with $e_s \nmid B$ and $B = q/a_s \leq Q/a_s < a_{m(f+1)}/a_s$. Now observe that by assumption $h \neq m$ from (10) we infer

$$e_s = d_f. \quad \text{(22)}$$

Hence if we split $q \xi_h = Y_q + Z_q$ with

$$Y_q = B a_s \sum_{j=nm+h, j \leq s} a_j^{-1} = B \sum_{j=nm+h, j \leq s} \frac{a_j}{a_j} \in \mathbb{Z}, \quad Z_q = B a_s \sum_{j=nm+h, j > s} a_j^{-1}$$


then \( Y_q \in \mathbb{Z} \) by (13), and separating the first term \( Ba_s/a_{s+1} \) from the sum of \( Z_q \) we may write

\[
Z_q = q \xi_h - Y_q = \frac{Ba_s}{a_{s+1}} + Ba_s \cdot O(a_{s+1+m}^{-1}) = \frac{B}{e_s} + O(Qd_{f+1}^{-1}).
\]

Now again since \( e_s \nmid B \), the term \( B/e_s \) is not an integer and thus has distance at least \( 1/e_s \) from any integer. So indeed by (22) we see

\[
\|q \xi\| \geq \|q \xi_h\| \geq e_s^{-1} - O(Qd_{f+1}^{-1}) = d_f^{-1} - O(Qd_{f+1}^{-1}).
\]

Case IIb: Finally assume \( h = m \), or equivalently \( s = k \). Then note that (11) implies

\[
e_s = a_{s+1} = \frac{am_f + m}{am_f + m - 1} = L_f \leq amf Reg + \frac{m - M_f + 1}{amf + m - 1}.
\]

Choosing \( M_f + 1 \) sufficiently large in the next step, this will be arbitrarily small. On the other hand, by construction of \( L_f \) and (19), for some integer \( L_f amf + m - 1 + 1 \leq Q' \leq (L_f + 1)amf + m - 1 \) we have

\[
d_f^{-1} \geq Q' \Phi(Q') = \Phi(\alpha_f \cdot Q), \quad \alpha_f = \frac{Q'}{Q} \frac{Q'}{am(f+1) - 1} = \frac{Q'}{L_f amf + m - 1 - 1}.
\]

It is obvious that \( Q' > Q \) and thus \( \alpha_f \) is a constant. Conversely, we have

\[
\alpha_f \leq \frac{(L_f + 1)amf + m - 1}{L_f amf + m - 1 - 1} = \frac{L_f + 1}{L_f} \cdot \frac{L_f amf + m}{L_f amf + m - 1 - 1}
\]

hence clearly \( \alpha_f = 1 + o(1) \) as \( f \to \infty \) by (12). Thus we have \( \alpha_f \to 1^+ \) as \( f \to \infty \). Now in view of (24), (25) and (20), for any given \( \epsilon_1 > 0 \) a suitably large choice of \( M_f + 1 \) in the next step will ensure that

\[
\|q \xi\| > (1 - \epsilon_1) \Phi(Q'),
\]

Choosing \( M_f + 1 \) sufficiently large in the next step, this will be arbitrarily small. On the other hand, by construction of \( L_f \) and (19), for some integer \( L_f amf + m - 1 + 1 \leq Q' \leq (L_f + 1)amf + m - 1 \) we have

\[
d_f^{-1} \geq Q' \Phi(Q') = \Phi(\alpha_f \cdot Q), \quad \alpha_f = \frac{Q'}{Q} \frac{Q'}{am(f+1) - 1} = \frac{Q'}{L_f amf + m - 1 - 1}.
\]

It is obvious that \( Q' > Q \) and thus \( \alpha_f \) is a constant. Conversely, we have

\[
\alpha_f \leq \frac{(L_f + 1)amf + m - 1}{L_f amf + m - 1 - 1} = \frac{L_f + 1}{L_f} \cdot \frac{L_f amf + m}{L_f amf + m - 1 - 1}
\]

hence clearly \( \alpha_f = 1 + o(1) \) as \( f \to \infty \) by (12). Thus we have \( \alpha_f \to 1^+ \) as \( f \to \infty \). Now in view of (24), (25) and (20), for any given \( \epsilon_1 > 0 \) a suitably large choice of \( M_f + 1 \) in the next step will ensure that

\[
\|q \xi\| > (1 - \epsilon_1) \Phi(Q'),
\]

Choosing \( M_f + 1 \) sufficiently large in the next step, this will be arbitrarily small. On the other hand, by construction of \( L_f \) and (19), for some integer \( L_f amf + m - 1 + 1 \leq Q' \leq (L_f + 1)amf + m - 1 \) we have

\[
d_f^{-1} \geq Q' \Phi(Q') = \Phi(\alpha_f \cdot Q), \quad \alpha_f = \frac{Q'}{Q} \frac{Q'}{am(f+1) - 1} = \frac{Q'}{L_f amf + m - 1 - 1}.
\]

It is obvious that \( Q' > Q \) and thus \( \alpha_f \) is a constant. Conversely, we have

\[
\alpha_f \leq \frac{(L_f + 1)amf + m - 1}{L_f amf + m - 1 - 1} = \frac{L_f + 1}{L_f} \cdot \frac{L_f amf + m}{L_f amf + m - 1 - 1}
\]

hence clearly \( \alpha_f = 1 + o(1) \) as \( f \to \infty \) by (12). Thus we have \( \alpha_f \to 1^+ \) as \( f \to \infty \). Now in view of (24), (25) and (20), for any given \( \epsilon_1 > 0 \) a suitably large choice of \( M_f + 1 \) in the next step will ensure that

\[
\|q \xi\| > (1 - \epsilon_1) \Phi(Q'),
\]
uniformly in $1 \leq q \leq Q$. From this by property (d3), for $\epsilon_2 > 0$ and $f \geq f_0(\epsilon_2)$ we infer

$$\|q^{\xi}\| > (1 - \epsilon_2) \cdot (1 - \epsilon_1) \Phi(Q) = (1 - \epsilon_3) \Phi(Q)$$

where

$$\epsilon_3 = 1 - (1 - \epsilon_1)(1 - \epsilon_2)$$

is small. Since $q \leq Q$ was arbitrary, this means $\psi_{q}(Q) > (1 - \epsilon_3) \Phi(Q)$. We may make $\epsilon_3$ arbitrarily small by choosing $f$ large enough and consequently $\epsilon_1, \epsilon_2$ small enough. Now we may let the according $\epsilon_3 = \epsilon_3(n)$ of the $n$-th step of the construction tend to 0 as $n \to \infty$, and claim (C2) follows for the induced $\xi$.

Since we can choose infinitely many distinct $M_n$ and thus $a_{mn+1}$ in each step of the construction in Sect. 6.1, and it can easily be arranged for many of them to induce pairwise distinct $\xi$, our method gives rise to a continuum of $\xi$ with the properties of the theorem. We prove the stronger metrical assertion in Sect. 7 below.

6.3 Proof of Corollary 2

Choose $\xi$ as in Sect. 6.1 for $\Phi(t) = ct^{-1/m}$, $c \in (0,1)$. Recall $\Omega := \min_{1 \leq i \leq m} |e_i|$ and let $\overline{\Omega} := \max_{1 \leq i \leq m} |e_i|$. By relabelling indices if necessary, we may assume $|e_2| = \Omega$. We show that $\xi$ has Dirichlet constant $\Theta^{1/|\xi|}(\xi) = c\Omega$ with respect to $|\cdot|$. For simplicity write $\eta_i = \eta_i(q) := |q^{\xi_i}| > 0$, $1 \leq i \leq m$, so that $|q^{\xi_{\frac{k}{2}}}| = \sum |\eta_i e_i|$ for $q^{\frac{k}{2}} \in \mathbb{R}^m$ as defined in Sect. 4.1 and the same sign choices. The proof of (C1) in Case 1 with $q = a_{mf} \leq Q < a_{mf+1}$ yields again negligibly small values by

$$|\eta_i e_i| = \eta_i \cdot |e_i| \leq \eta_i \cdot \overline{\Omega} < q^{1+\frac{1}{mf}} \overline{\Omega} = o(Q^{-1/m}), \quad 1 \leq i \leq m,$$

as $Q \to \infty$, independent of the norm. In Cases 2 and 3, for $q = d_f = a_{mf+1} < Q < a_{m(f+1)} < d_f$ we see that

$$|\eta_2 e_2| = (d_f^{-1} + o(d_f^{-1}))|e_2| \leq cQ^{-1/m}(1 + o(1)) \cdot \Omega$$

(27)

as $Q \to \infty$ whereas for $i \neq 2$ the construction of the $\xi_i$ easily implies

$$|\eta_i e_i| \leq d_f^2(1 + o(1))|e_i| \leq d_f^2(1 + o(1))\overline{\Omega} = o(d_f^{-1}) = o(Q^{-1/m}).$$

(28)

Since $|q^{\xi_{\frac{k}{2}}}| \leq \sum |\eta_i e_i|$ by triangle inequality, reviewing all cases we get $\Theta^{1/|\xi|}(\xi) \leq c\Omega$. The reverse inequality $\Theta^{1/|\xi|}(\xi) \geq c\Omega$ follows from

$$|q^{\xi_{\frac{k}{2}}}| \geq \max_{1 \leq i \leq m} \eta_i |e_i| \geq \max_{1 \leq i \leq m} \eta_i \cdot \min_{1 \leq i \leq m} |e_i|$$

$$> c(1 - \epsilon)Q^{-1/m} \min_{1 \leq i \leq m} |e_i| = c(1 - \epsilon)\Omega Q^{-1/m}$$
for certain large $Q \geq Q_0(\varepsilon)$ and any integer $1 \leq q < Q$, where we used that $|.|$ is expanding and property (C2). A very similar argument above applies in fact to any $\xi \in S$ as defined in Sect. 7.1 below, so the final metrical claim is inherited from Theorem 2.1.

**Remark 2** The results on Hausdorff dimension from Sect. 3 can be transferred as well by a minor modification of the sets $Q_i$ from the proofs in Sect. 9.1 below. More concretely, we have to impose some restrictions on the digit choices near the left endpoints of the intervals of type (ii) of Lemma 9.1 to guarantee (27), (28). This can be done without affecting the Hausdorff dimension of $\prod Q_i$. We omit details.

### 6.4 On relaxing conditions (d2), (d3)

We believe that for $m \geq 3$, the order in (d2) can be significantly relaxed. Our proof above followed the main outline from the proof of [27, Theorem 2.5] with the special choice $\eta_1 = \eta_2 = \cdots = \eta_m = 1/m$ (letter $k$ was used in place of $m$ in [27]). Choosing $\eta_i$ differently, which essentially means altering (10), depending on a rough given decay rate of $\Phi$, a similar approach may ideally allow for replacing (d2) by the weaker, natural condition $t\Phi(t) \to \infty$ (which coincides with the exact condition (d2) when $m = 2$ and is necessary for any $m \geq 2$ when $\xi$ is totally irrational [18, Satz 9, p. 199]). However, some technical obstacles have to be mastered. Recall that for any $\xi \in \mathbb{R}^m \setminus Q^m$ we have $\lim sup_{Q \to \infty} Q\psi_\xi(Q) \geq 1/2$, as mentioned in Sect. 1.

Similarly, we believe that (d3) can be dropped. Notice that we have a free choice of $M_f$ in every step, so it suffices to find some $M_f$ for which for given $\epsilon > 0$ the induced $L_f$ satisfies $\Phi((L_f + 1)a_{mf+1}) > (1 - \epsilon)\Phi(L_f a_{mf+1})$. Now given $\epsilon > 0$, for small enough $\delta = \delta(m, \epsilon) > 1$ there are arbitrarily large $T$ so that $\Phi(\delta T) > (1 - \epsilon)\Phi(T)$, otherwise it is easy to see that (d2) cannot hold. It remains however unclear to us if we can choose $T$ of the given form $L_f a_{mf+1}$. In this matter it may be helpful that, as remarked in Sect. 6.1, we can relax (9) by asking $a_{mf+1} = M_n' a_{mf}$ for large enough $M_n'$.

### 7 Proof of Theorem 2.2 for $m \geq 3$: metrical claim

#### 7.1 Special case $a_n = 2^{c_n}$

Assume for simplicity first the $a_n$ constructed in Sect. 6.1 are of the form $a_n = 2^{c_n}$ for every $n$ with an increasing integer sequence $c_n$, so that the binary expansions of the $\xi_i$ become

$$\xi_i = \sum_{n=0}^{\infty} 2^{-c_{mn+i}}, \quad 1 \leq i \leq m.$$  

The proof is done in two steps. The first key observation is that for given $\Phi$, we have some freedom in the construction of $\xi$ in Sect. 6.1. We will find a Cantor type
set consisting of $ζ ∈ ℝ^m$ sharing the same properties (C1), (C2), (C3). Then in the second step we use a very similar strategy as in the proof of [32, Theorem 2.1] based on a result of Tricot [35] to find the claimed lower bound for the packing dimension of this set.

Step 1: It is clear that the binary digits of all $ξ_i$ as above at positions $c_{mn}+1$, $c_{mn}+2$, ..., $c_{mn+1}−1$ are 0 (whereas $ξ_m$ resp. $ξ_1$ has digit 1 at $c_{mn}$ resp. $c_{mn+1}$). Let small $ε > 0$ be given and $γ ∈ (0, 1)$ as in the theorem. For $1 ≤ i ≤ m$, let $S_i = S_i(γ )$ be the set of $ξ_i ∈ ℝ$ with binary expansions $ξ_i = ∑_{j≥1} g_{i,j} 2^{-j}$ derived from the binary expansion of $ξ_i$ from Sect. 6.1 by altering its digit from 0 to an arbitrary digit $g_{i,j} ∈ \{0, 1\}$ at places $j$ in the intervals

$$I_n = \{[c_{mn+1}(γ + ε)] + 1, [c_{mn+1}(γ + ε)] + 2, \ldots, c_{mn+1}−1\}, \quad n ≥ 0.$$  

Let $S = \prod_{i=1}^m S_i(γ ) ⊆ [0, 1)^m$ be the set of arising real vectors $ξ = (ξ_1, \ldots, ξ_n)$. Note that still $g_{i,j} = 0$ at places $j ∈ [c_{mn}+1, [c_{mn+1}(γ + ε)]] \cap ℤ$ for $1 ≤ i ≤ m$, $n ≥ 1$.

We verify (C1), (C2), (C3) for any $ξ ∈ S$. We start with the proof of (C2) that is analogous to the proof for $ξ$ from Sect. 6.1, upon some twists that we explain now. Since we include certain terms $2^{-j}$ for $j ∈ I_n$ to the partial sums defining the $ξ_i ∈ S_i$, we need to slightly redefine the integer $D_f = D_f(ξ )$ in Case I resp. $Y_q = Y_q(ξ )$ and $Z = Z_q(ξ )$ in Case II, depending on the choice of $ζ ∈ S$. The coprimality condition $(D_f, a_{mf+1}) = 1$ in Case I is further guaranteed since at position $j = c_{mf+1}$ the binary digit of any $ξ_i ∈ S_i$ still equals $g_{i,c_{mf+1}} = 1$ as $c_{mf+1} \notin I_n$, implying that any derived $D_f(ξ )$ is odd, but $a_{mf+1} = 2^m$ is a power of 2. Similarly, the first, main term of $Z_q$ obtained from truncating the binary expansion of any $ξ_h ∈ S_h$ after position $j = c_h+1$ is still bounded from below by $d_f^{-1}$. Indeed, it may be written

$q r_s = B a_s \cdot r_s$ with the rational numbers $r_s ∈ S_h(γ _h) := ∑_{j=c_h}^{c_h+1} g_{h,j} (γ _h) 2^{-j}$ where $g_{h,j} (γ _h)$ denotes the binary digit of $ξ_h ∈ S_h$ at position $j$, depending on the choice of $ζ ∈ S$. We have $g_{h,c_{h+1}} (γ _h) = g_{h,c_{mf+1}} (γ _h) = 1$ for any $ξ ∈ S$, since by $c_{mf+1} \notin I_n$ we have not changed the digit of $ξ_h$ there. So $r_s = v_s 2^{-c_{h+1}} = v_s a_{s+1}$ with some odd integer numerator $v_s$. Finally, since $v_s$ is odd and $B / (a_{s+1}/a_s)$ by assumption, $q r_s = B v_s a_s a_{s+1}$ is not an integer and thus has distance at least $2^{−(c_{s+1}−c_s)} = d_s a_{s+1} ≥ d_f^{-1}$ from any integer, as before.

Now let us turn to (C1), (C3). We have chosen the left interval endpoints of the $I_n$ large enough that we also satisfy (C3) and Case 1 of (C1) with the same choice $q = a_{mf} = 2^m$, for any $ξ ∈ S$. We only show the latter. We may assume $c_{mn+1}/c_{mn} → ∞$ as $n → ∞$, which corresponds to $M_n → ∞$ in Sect. 6.1 (we assume this in this section; we may also use the other assumption involving $M'_n$ a large enough power of 2 from (15)). We estimate

$$∥q ξ∥ ≤ 2^{−(γ + ε)c_{mf+1}+c_{mf}} < θ \cdot (2^{c_{mf}})^{−γ} = θ \cdot a_{mf+1}^{−γ} < θ \cdot Q^{−γ} < θ \cdot η^{−1} Φ(Q), \quad Q < a_{mf+1},$$

for arbitrarily small $θ = θ(f) > 0$ as $f → ∞$, where we also used property $(d4(γ ))$. Then choosing $f$ large enough and thus $θ$ small enough, indeed we again infer $∥q ξ∥ < £$
\(\Phi(Q)\). The proofs of the remaining cases of (C1) for any \(\zeta \in S\) are again essentially unaffected. Hence indeed all \(\zeta \in S\) satisfy the claims of Theorem 2.2.

**Step 2:** Now we show that \(S = S(\gamma)\) has packing dimension at least \(m(1 - \gamma)\). This works similar as the proof of [32, Theorem 2.1]. Let \(\mu = \gamma^{-1} - 1\). First we claim that we can write any given real vector \(y \in \mathbb{R}^m\) as the sum of an element of \(S\) and a vector with ordinary exponent of binary approximation (to be defined below) at least \(\mu - \vep\), for small \(\vep > 0\) that tends to 0 as \(\vep\) does. More precisely, if we let

\[ \mathcal{V}_m^{(2)}(\lambda) = \{ x \in \mathbb{R}^m : \liminf_{t \to \infty} (2^t)^{\lambda} \psi_x(2^t) < \infty \} \subseteq \bigcup_{\tau \geq \lambda} \mathcal{W}_m(\tau), \quad \lambda > 0, \]

then we claim that for some small \(\vep > 0\) (that tends to 0 as \(\vep\) above does) we have the identity of sets

\[ \mathcal{V}_m^{(2)}(\mu - \vep) + S = \mathbb{R}^m. \]  

(29)

Given \(y \in \mathbb{R}^m\), we construct a representation \(x + \zeta = y\) for \(x \in \mathcal{V}_m^{(2)}(\mu - \vep), \zeta \in S\). We take any \(\zeta \in S\) whose coordinates \(\zeta_i, 1 \leq i \leq m\), have the same binary digit as the corresponding component \(y_i\) of \(y\) within the intervals \(I_n\). This is possible by the construction of \(S\). Then we let \(x := y - \zeta\). Notice that for \(q_n = 2^{|e_{mn+1}|(\gamma + \vep)}\), with exponent equal to the left interval endpoint of \(I_n\), and some integer vector \(N = N_n\), the vector \(N/q_n\) is a good rational approximation to \(x\). More precisely, by our choice of \(\mu\) and the construction of \(I_n\), for small \(\vep > 0\) depending on \(\vep\) above, we have

\[ \|q_n x\| \leq 2 \cdot 2^{-|I_n|} \ll 2^{-(1-\gamma-\vep)c_{mn+1}} \ll q_n^{-(\mu-\vep)}, \quad n \geq 1. \]

Thus indeed \(x \in \mathcal{V}_m^{(2)}(\mu - \vep)\) for small \(\vep > 0\). The construction is complete and the claim (29) is proved.

Write \(\dim_H\) and \(\dim_P\) for Hausdorff and packing dimension, respectively. For simplicity set \(V = \mathcal{V}_m^{(2)}(\mu - \vep)\). Next we claim that

\[ \dim_H(V) \leq \frac{m}{\mu - \vep + 1} = m\gamma + O(m\vep), \]

(30)

where the implied constant depends on \(\gamma\) only. This can be done by a standard covering argument and was already observed in a more general form in [32, Lemma 5.6]. See next paragraph for an alternative proof using Lemma 7.1. Combining (29), (30) with a result by Tricot [35] and the well-known property that the Hausdorff dimension of a set does not increase under a Lipschitz map, we conclude

\[ \dim_P(S) \geq \dim_H(S \times V) - \dim_H(V) \geq \dim_H(S + V) - \dim_H(V) \geq m - m\gamma - O(m\vep), \]

and since \(\vep\) and thus \(\vep\) can be arbitrarily small, the claim of Theorem 2.2.
Remark 3 In the light of the short proof of Corollary 3 in Sect. 10.2 below, the above argument already implies the weaker claim that $D_{m}(c) \setminus (D_{m}(2^{-1/m}c) \cup \text{Bad}_{m}) \subseteq FS_{m}$ has packing dimension at least $m - m\gamma$ for any $c \in (0, 1]$. More generally, for $\Phi$ as in Theorem 2.2, analogous claims hold upon replacing the factor $(1 - \epsilon)$ in $(C2)$ by $2^{-1/m}$.

7.2 General case

Unfortunately, as alluded in Remark 3, we cannot guarantee that $a_n$ are integral powers of 2 in general, without losing information on the exact Dirichlet constant. Here we explain how to alter the construction to the general case, however omit a few technical details. Given any integer sequence $k_j$ satisfying (13), i.e., $k_j|k_{j+1}$, it is not hard to verify that any number in $[0, 1)$ can be expressed as $\sum_{j \geq 1} g_j/k_j$ with integers $g_j \in \{0, 1, 2, \ldots, k_j+1/k_j - 1\}$, via some kind of greedy expansion. Call $k_j$ bases and $g_j$ digits. If $k_{j+1}/k_j = b$ for $j \geq 1$ then this becomes just the usual $b$-ary expansion.

Now we choose $a_{nn+1} = 2^Z a_{mn}$ for some integer $Z = Z_n$ so that $M_n' = 2^{Z_n}$ in the notation of Sect. 6.1 where we assume $(15)$ in place of $(9)$, and apply the above construction to the digits $g_{i,j}$ of $\xi_i$ in this setting, in place of the binary digits. Concretely, we choose any $a_j$ as a base integer and call these main bases. At every main base integer $k_j = a_u$, for $i \equiv u \mod m$ with $i \in \{1, 2, \ldots, m\}$ we choose the digit of $\xi_i$ as $g_{i,j} = 1$, and put $g_{i,j} = 0$ for the other $i \not\equiv u \mod m$, very similar to the binary digit construction above. We further define additional intermediate bases within intervals $(a_{mn}, a_{mn+1})$ as follows. Let $Y = Y_n < Z_n$ be an integer so that $2^Y a_{mn} = a_{mn+1}^\gamma + \epsilon$ with small $\epsilon > 0$, which is clearly possible since the approximation can be made precise up to a factor 2. Now within any interval $(a_{mn}, a_{mn+1})$, we choose the intermediate bases $2^Y a_{mn}, 2^{Y+1} a_{mn}, 2^{Y+2} a_{mn}, \ldots, 2^{Z-1} a_{mn} = a_{mn+1}/2$. We define our sequence of bases $(k_j)_{j \geq 1}$ as the increasingly ordered union of all main and intermediate bases. By a similar argument as in the binary construction Sect. 7.1, again we can choose the digits $g_{i,j} \in \{0, 1\}$ of any component $\xi_i$ with respect to the intermediate bases $2^Y a_{mn}, 2^{Y+1} a_{mn}, \ldots, 2^{Z-1} a_{mn}$ freely without violating (C1), (C2), (C3). Call $S^* \subseteq \mathbb{R}^m$ the according set of real vectors $\xi$ induced by the above digit restrictions. Since the good approximations are not powers of 2, we need a slightly different argument than in Sect. 7.1 for the optimal result.

Lemma 7.1 Let $\epsilon = (e_j)_{j \geq 1}$ be any strictly increasing sequence of positive integers (that thus tends to infinity) and $\tau > 0$. Then the set $V_{m,\epsilon}(\tau)$ of vectors $\chi \in \mathbb{R}^m$ satisfying

$$\|e_j\chi\| \leq e_j^{-\tau}, \quad j \geq 1,$$

(31)

has Hausdorff dimension at most $m/(\tau + 1)$.

Proof The set $V_{m,\epsilon}(\tau)$ is clearly contained in the set of $\chi$ for which estimate (31) has infinitely many solutions for a given subsequence $(e_{j_k})_{k \geq 1}$ of the $e_j$. By the convergence case of Jarník-Besicovich Theorem [17] in a setup involving arbitrary approximation functions (no monotonicity is required since we only care for the convergence part, moreover $m > 1$), choosing the approximation function $\Psi$ with support
only on the $e_{jk}$ and there equal to $\Psi(e_{jk}) = e_{jk}^{-\tau}$, the latter set has Hausdorff $v$-measure 0 if the sum of $e_{jk}^{m-v(\tau+1)}$ over $k \geq 1$ converges. As soon as $v > m/(\tau + 1)$, by choosing a sparse enough subsequence $e_{jk}$ of the $e_j$, the criterion obviously holds. This means the Hausdorff dimension the latter set is at most $m/(\tau + 1)$, thus the same applies to our smaller original set $V_{m,e}(\tau)$ as well.

By a similar argument as in Sect. 7.1, we can write given $y \in \mathbb{R}^m$ as a sum $x + \zeta$ where $\zeta \in S^*$ and $x$ has the property that $\|q_n x\| \ll q_n^{-(\mu - \varepsilon)}$ at the places $q_n = 2^{\tau_n} a_{mn}$ for $n \geq 1$. So we may apply Lemma 7.1 to $e_j = 2^{\varepsilon_j} a_{mj}$ and $\tau = \mu - \varepsilon$, which yields that the according set $V_{m,e}(\tau)$ of $x$ has Hausdorff dimension at most $m/(\mu - \varepsilon + 1)$ again. The estimate for the packing dimension of $S^*$ follows now analogously to Sect. 7.1 from Tricot’s result.

**Remark 4** It can be shown with aid of [12, Example 4.6, 4.7] that Lemma 7.1 states the precise Hausdorff dimension of $V_{m,e}(\tau)$, hence we cannot hope for an improvement by some refined treatment of this set. The proofs of Theorems 3.1, 3.2 below are based on this strategy.

### 8 Proof of Theorem 2.2 for $m = 2$

For $m = 2$, we have to alter our sequence $(a_n)_{n \geq 1}$. Take arbitrary $a_1 > 1$ large enough depending on $\Phi$ and let $a_2 = a_1^2$, and for $n \geq 1$ recursively we set

$$a_{2n+1} = a_{2n}^{M_n}, \quad a_{2n+2} = \tilde{L}_n a_{2n+1}.$$  

Here again $M_n$ is a fast growing sequence of integers, but now

$$\tilde{L}_n = \min\{z \in \mathbb{N} : z^{-1} < \Phi(Q) : 1 \leq Q \leq za_{2n+1}\}.$$  

By a similar argument as in Proposition 6.1, if $a_1$ was chosen large enough, condition (d2) guarantees that this is a well-defined, finite number. We omit details. Note that this slightly differs from $L_n$ in Sect. 6.1. Indeed, now the reverse inequality $\tilde{L}_n > d_n := a_{2n+1}$ holds by (d1) for all $n \geq 1$ given that $a_1$ was chosen large enough, and clearly $\tilde{L}_n \to \infty$ follows. Then again define $\xi_i$ for $i \in \{1, 2\}$ as in (14).

The proof of (C3) is identical to Sect. 6.1 by considering $Q = q = a_{2n}$ for large $n$. For (C1), we again let $Q$ be arbitrary, large and let $k$ be the index with $a_k \leq Q < a_{k+1}$ and write $k = 2f + g$ with $g \in \{0, 1\}$. We consider the same cases again. Case 1 is inferred very similarly from (d2) with $q = a_k$, and again leads to $\|q \xi_i\| = \|q \xi_1\| < \Phi(Q)$. There is a small, notable twist though when $r \Phi(t)$ increases to infinity rather slowly. Notice that by (d2), for any $C > 1$ and large enough $t_0 = t_0(\Phi, C)$ we will have $\Phi(t) > C t_0^{-1}$ for every $1 \leq t \leq t_0$ (by a similar argument as in the proof of Proposition 6.1). Hence having constructed $a_{mf}$, we take $C = C_f = 3a_{mf}$ and then choose $M_f$ large enough so that this applies for
\( t_0 = a_{mf+1} = a_{mf} \). Then again we see

\[
\|q_\xi\| = \|q_\xi_1\| \leq 2 \frac{a_{mf}}{a_{mf+1}} = 2a_{mf}a_{mf+1}^{-1} < Ca_{mf+1}^{-1} < \Phi(Q).
\]

Case 2 of (C1) is empty. In Case 3 of (C1), with \( q = d_f = a_{2f+1} \) we now instead of (17) get a bound

\[
\|q_\xi\| = \|q_\xi_2\| \leq \frac{1}{L_f} + O(Qd_{f+1}^{-1}).
\]  

(32)

By definition of \( \tilde{L}_f \) and since we can choose \( M_{f+1} \) in the next step arbitrarily large, we again see \( \|q_\xi\| = \|q_\xi_2\| < \Phi(Q) \) for any \( Q \leq a_{2n+2} \).

We follow the proof of (C2) as in Sect. 6.2. Again we let \( Q = a_{2f+2} - 1 \) for large \( f \) and consider the same cases I, II. In Case I, we get the same estimate (21) by the same argument. However, since now \( \tilde{L}_f > d_f \) we have \( d_f^{-1} > \tilde{L}_f^{-1} \). In Case IIa we again get a lower bound \( \|q_\xi\| \geq e_5^{-1} - O(Qd_{f+1}^{-1}) \) but now \( e_5 = \tilde{L}_f \). Case IIb also gives the same bound \( e_5^{-1} - O(Qd_{f+1}^{-1}) \). Since we noticed \( 1/\tilde{L}_f < 1/d_f \), in any case we get

\[
\|q_\xi\| \geq \frac{1}{L_f} + O(Qd_{f+1}^{-1}).
\]  

(33)

The error term can be made \( o(\tilde{L}_f^{-1}) \) if we choose \( M_{f+1} \) large enough in every step. Moreover, \( (\tilde{L}_f - 1)^{-1} \geq \Phi(Q') \) for some \( Q' \in [1, (\tilde{L}_f - 1)a_{2f+1}] \) by definition of \( \tilde{L}_f \). Assume for the moment that \( \Phi \) is decreasing. Then we may a fortiori assume \( Q' = (\tilde{L}_f - 1)a_{2f+1} \) and (33) implies

\[
\|q_\xi\| \geq (1 - \epsilon_1)\Phi(Q') = (1 - \epsilon_1)\Phi(\tilde{a}_f(a_{2f+2} - 1)),
\]

\[
\tilde{a}_f = \frac{Q'}{a_{2f+2} - 1} = \frac{(\tilde{L}_f - 1)a_{2f+1}}{L_f a_{2f+1} - 1}.
\]

Now obviously the right expression implies \( \tilde{a}_f < 1 \), hence our observation \( \tilde{L}_n \to \infty \) implies \( \tilde{a}_f \to 1^- \). Therefore we can use the first condition for \( m = 2 \) in (d3) to conclude very similarly as in Sect. 6.2 that

\[
\|q_\xi\| \geq (1 - \epsilon_1)(1 - \epsilon_2)\Phi(a_{2f+2} - 1) = (1 - \epsilon_3)\Phi(Q)
\]

where \( \epsilon_3 = 1 - (1 - \epsilon_1)(1 - \epsilon_2) \) is small (in fact we may let \( \epsilon_2 = 0 \) when \( \Phi \) is decreasing, but not in the general case). Now drop the assumption that \( \Phi \) is decreasing. By the latter condition (5) of (d3) for \( m = 2 \), it still follows that we may choose \( Q' \geq (1 - \epsilon_5)(\tilde{L}_f - 1)a_{2f+1} \) with \( \epsilon_5 = \epsilon_5(f) \to 0 \) as \( f \to \infty \). Then \( \tilde{a}_f \) as above still satisfies \( \tilde{a}_f \to 1^- \) as \( f \to \infty \) and thus \( \epsilon_5 \to 0 \), and an analogous argument as in
the case of decreasing $\Phi$ applies upon carrying the additional factor $(1 - \epsilon_5)$ throughout the argument. Finally the metrical claim also follows analogously to Sect. 7. We omit the details.

**Remark 5** For general $m \geq 2$, define again $a_{mn+1} = a_{mn}^{M_n}$ and constant quotients $a_{mn+j+1} / a_{mn+j} = \tilde{L}_n$ for $1 \leq j \leq m - 1$ with

$$\tilde{L}_n = \min\{z \in \mathbb{N} : z^{-1} < \Phi(Q) : 1 \leq Q \leq z^{m-1}a_{mn+1}\}.$$ 

Then analogous arguments would lead to an alternative, slightly shorter proof of Theorem 2.2 upon assuming the conditions (4), (5) instead in (d3). We preferred to include the longer proof for $m \geq 3$ since when adapting the alternative above construction to Cantor sets as in Sect. 4.2, without additional argument the bound in Theorem 4.1 would become weaker.

### 9 Proof of Hausdorff dimension estimates

We again emphasize that $\xi$ will not be Liouville vectors in this section, which is reflected in bounding the growth of the $M_n$ (in fact rather $M'_n$) in the construction of Sect. 6.1.

#### 9.1 Metric preliminaries

To prove Theorem 3.1 resp. Theorem 3.2, similar as in Sect. 7, we construct a Cantor type subset of the corresponding set $F_{m,c}$ resp. $F_{m,c} \cap \mathcal{W}_m(\lambda)$ whose Hausdorff dimension can be estimated/evaluated. The fractal set will be as in the following lemma, for optimized parameters $\gamma_1, \gamma_2$ under certain side conditions and $H_n = a_{mn}$. The notation $A \asymp B$ means $A \ll B \ll A$ in the sequel.

**Lemma 9.1** Let $m \geq 2$ an integer and $\gamma_2 \geq \gamma_1 > 1$ be real numbers. Assume $(c_n)_{n \geq 1}$ is an increasing sequence of positive integers and let $h_n = c_{mn}$ and $H_n = 2^{h_n}$ for $n \geq 1$. Assume

$$H_n \asymp H_n^{\gamma_2 m}, \quad n \geq 2.$$  \hfill (34)

Let a sequence $(\delta_n)_{n \geq 1}$ satisfy $\delta_n > \gamma_1$ for $n \geq 1$ and

$$\delta_n = \gamma_2 + o(1), \quad n \to \infty.$$ \hfill (35)

For $1 \leq i \leq m$, let $Q_i \subseteq [0, 1)$ be the set of real numbers $\xi_i$ whose binary expansion $\xi_i = \sum_{j \geq 1} g_{i,j} 2^{-j}$ has an arbitrary digit $g_{i,j} \in \{0, 1\}$ at places of the form

(i) For $1 \leq i \leq m$ in intervals $j \in [\gamma_1 h_n, \delta_n h_n - 1] \cap \mathbb{Z}$

(ii) For $3 \leq i \leq m - 1$, in intervals $j \in [2\delta_n h_n + 1, i\delta_n h_n - 1] \cap \mathbb{Z}$

(iii) For $i = m$, in intervals $j \in [2\delta_n h_n + 1, h_n + 1 - 1] \cap \mathbb{Z}$
for all \( n \geq 1 \), and a prescribed digit \( g_{i,j} \in \{0, 1\} \) elsewhere. Then their Cartesian product \( Q := \prod_{i=1}^{m} Q_i \) has Hausdorff dimension at least

\[
\dim_H(Q) \geq 2 \frac{\gamma_2 - \gamma_1}{\gamma_1(\gamma_2 m - 1)} + \sum_{i=3}^{m} \min \left\{ \frac{m(\gamma_2 - \gamma_1) + i - 2}{2(m\gamma_2 - 1)}, \frac{(i - 1)\gamma_2 - \gamma_1}{\gamma_1(m\gamma_2 - 1)} \right\},
\]

(36)

and alternatively

\[
\dim_H(Q) \geq m \cdot \frac{\gamma_2 - \gamma_1}{\gamma_1(\gamma_2 m - 1)}.
\]

(37)

We prove Lemma 9.1. Our sets \( Q_i \) can be interpreted as a special case of a construction from Falconer’s book [12, Example 4.6].

**Proposition 9.2** (Falconer) Let \([0, 1] = E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots \) be a decreasing sequence of sets, with each \( E_n \) a union of a finite number of disjoint closed intervals (called \( n \)-th level basic intervals), with each interval of \( E_{n-1} \) containing \( P_n \geq 2 \) intervals of \( E_n \), which are separated by gaps of length at least \( \epsilon_n \), with \( 0 < \epsilon_{n+1} < \epsilon_n \) for each \( n \), which tend to 0 as \( n \to \infty \). Then the set

\[
F = \bigcap_{i \geq 1} E_i
\]

satisfies

\[
\dim_H(F) \geq \liminf_{n \to \infty} \frac{\log(P_1 P_2 \cdots P_{n-1})}{-\log(P_n \epsilon_n)}.
\]

From our setup it can be seen that \( F = Q_i \) for \( 1 \leq i \leq m \) from Lemma 9.1 meet the requirements of Proposition 9.2 with parameters

\[
P_n \asymp H_n^{\delta_n - \gamma_1} = H_n^{\gamma_2 - \gamma_1 - o(1)}, \quad \epsilon_n \asymp H_n^{-\delta_n} = H_n^{-\gamma_2 + o(1)}
\]

(38)

as \( n \to \infty \) for all \( 1 \leq i \leq m \), and alternatively with

\[
P_{2n} \asymp H_n^{\gamma_2 - \gamma_1 + o(1)}, \quad \epsilon_{2n} \asymp H_n^{-\gamma_2 + o(1)}
\]

\[
P_{2n+1} \asymp H_n^{(i-2)\gamma_2 + o(1)}, \quad \epsilon_{2n+1} \asymp H_n^{-i\gamma_2 + o(1)}
\]

(39)

for \( 3 \leq i \leq m \). We provide more details. Assume the interval construction is done up to level \( 2n-1 \) which prescribes digits up to position \( \lfloor \gamma_1 h_n \rfloor - 1 \). Then the free binary digit choice within \( j \in \lfloor \gamma_1 h_n, \delta_n h_n \rfloor = \lfloor \gamma_1 h_n, (\gamma_2 + o(1)) h_n \rfloor \) means that we split each interval given after step \( 2n-1 \) in the next step into \( 2^{\delta_n h_n - \lfloor \gamma_1 h_n \rfloor} = 2^{(\gamma_2 - \gamma_1 + o(1)) h_n} = H_n^{\gamma_2 - \gamma_1 + o(1)} \) subintervals, each of length \( \asymp 2^{-\gamma_1 h_{n+1}} = H_{n+1}^{-\gamma_1} \) due to the subsequent digits vanishing until position \( j = \lfloor \gamma_1 h_{n+1} \rfloor \), and two neighboring intervals roughly.
at distance $\epsilon_{2n} \times 2^{-\delta_n} h_n - 2^{-\gamma_1} h_{n+1} = H_n^{-\gamma_2 + o(1)} - H_{n+1}^{-\gamma_1} = H_n^{-\gamma_2 + o(1)}$ apart. A very similar idea applies in the next step to estimate $P_{2n+1}, \epsilon_{2n+1}$, where we distinguish between various $i$ and for $i > 2$ use that for $i = m$ we also have $H_{n+1} = H_i^{-\gamma_2(1+o(1))}$ by (34). Inserting (39), (34) in Proposition 9.2 we may omit lower order terms and obtain for $3 \leq i \leq m$ that

\[
\dim_H(\mathcal{Q}_i) \geq \liminf_{n \to \infty} \frac{\log(P_1 P_2 \ldots P_{n-1}) - \log(P_n \epsilon_n)}{2 \gamma_2 \log H_n}
\]

\[
= \min \left\{ \liminf_{n \to \infty} \frac{\log(P_1 P_2 \ldots P_{2n})}{\log(P_{2n+1} \epsilon_{2n+1})}, \liminf_{n \to \infty} \frac{\log(P_1 P_2 \ldots P_{2n-1})}{\log(P_{2n} \epsilon_{2n})} \right\}
\]

\[
= \min \left\{ \frac{\gamma_2 - \gamma_1 + \frac{(i-2)\gamma_2}{\gamma_2 m} \sum_{j=0}^{\infty} (\gamma_2 m)^{-j} \log H_n}{2 \gamma_2 \log H_n}, \frac{(i-1)\gamma_2 - \gamma_1}{\gamma_1 (\gamma_2 m - 1)} \right\},
\]

where the last identity requires a short computation involving the geometric sum formula. Similarly, for any $1 \leq i \leq m$ from (38), (34) and Proposition 9.2 we get

\[
\dim_H(\mathcal{Q}_i) \geq \frac{(\gamma_2 - \gamma_1) \sum_{j=1}^{\infty} (\gamma_2 m)^{-j} \log H_n}{\gamma_1 \log H_n} = \frac{(\gamma_2 - \gamma_1) \frac{1}{1-\frac{1}{\gamma_2 m}}}{m \gamma_1 \gamma_2} = \frac{\gamma_2 - \gamma_1}{\gamma_1 (\gamma_2 m - 1)}.
\]

Combining the respective estimates with the general fact $\dim_H(\prod A_i) \geq \sum \dim_H(A_i)$ for any $A_1, \ldots, A_m \subseteq \mathbb{R}^m$, see [12], with $A_i = \mathcal{Q}_i$ proves the claims of the lemma.

### 9.2 Proof of Theorem 3.1

We assume $m \geq 3$ here, for $m = 2$ the proof works very similarly. We show how to derive Theorem 3.1 from Lemma 9.1. Assume the real parameters $\gamma_1, \gamma_2$ satisfy the stronger hypothesis

\[
m(\gamma_1 - 1) > \gamma_2 \geq \gamma_1 > 1 + \frac{1}{m}.
\]

In fact the setup (40) automatically requires $\gamma_1 > 1 + \frac{1}{m-1}$. We assume (15) now in place of (9) and take $(a_n)_{n \geq 1}$ the sequence constructed in Sect. 6.1 with the specialization $\Phi(t) = ct^{-1/m}$, where $c \in (0, 1)$ is fixed. Moreover assume for the moment $a_n = 2^{c_n}$ for integers $c_n$. Take $M'_n \asymp a_{mn}^{\gamma_2 - 1}$ the integer power of 2 closest to $a_{mn}^{\gamma_2 - 1}$ for $n \geq 1$, so that we have

\[
a_{mn+1} = M'_n a_{mn} \asymp a_{mn}^{\gamma_2} = 2^{c_{mn} \gamma_2}, \quad n \geq 1.
\]
and let for \( n \geq 1 \) further

\[ H_n = a_{mn}, \quad h_n = c_{mn}. \]

Note that our particular case \( \Phi(t) = ct^{-1/m} \) implies (34). Indeed then \( L_n \asymp a_{mn+1} \) and

\[ H_{n+1} = a_{m(n+1)} = L_n a_{mn+1}^{-1} \times a_{mn+1} \times a_{mn} = H_n^{2m}. \]

(For general \( \Phi(t) \) under \((d^4(\gamma))\) we would get \( H_n^{2m} \gg H_n \gg H_n^{2/\gamma} \).) Let \( \delta_n > 1 \) be defined by

\[ a_{mn+1} = a_{mn}^{\delta_n}, \quad n \geq 1, \quad (42) \]

which satisfies (35) as required in Lemma 9.1 by (41), more precisely \( H_n^{\delta_n} \asymp H_n^{2/\gamma} \).

(The definition of \( \delta_n \) formally agrees with \( M_n \) in Sect. 6.1, however since we do not assume it is an integer here as we instead impose (15), we prefer to change notation for clarity.)

Consider the real numbers \( \xi_i \) as in Sect. 6.1 when \( a_n = 2^{c_n} \) as above, i.e. \( \xi_i = \sum_{n \geq 0} 2^{-c_{mn+i}} \). Then in the binary expansion the digit of \( \xi_i \) is 1 at places \( c_{mn+i}, n \geq 0, \) and 0 elsewhere. From these we form the sets \( Q_i = Q_i(\gamma_1, \gamma_2) \subseteq [0, 1), 1 \leq i \leq m, \) as in Lemma 9.1 consisting of the real numbers \( \zeta_i = \sum g_{i,j} 2^{-j} \), where \( g_{i,j} \in \{0, 1\} \) depends on \( \zeta_i \), obtained from the \( \xi_i \) when we change the binary digit \( g_{i,j} \) of \( \xi_i \) from 0 to an arbitrary digit in \( \{0, 1\} \) in certain intervals accordingly to (i), (ii), (iii). Then, any \( \zeta_i = \sum_{j \geq 1} g_{i,j} 2^{-j} \in Q_i \) still has binary digit \( g_{i,j} = 0 \) at places \( j \) within the following intervals for all \( n \geq 1 \):

(i*) For \( 1 \leq i \leq m - 1, \) within intervals

\[ j \in [\max\{i, 2\} \delta_{n-1} c_{m(n-1)} + 1, \gamma_1 c_{mn}) \cap \mathbb{Z} \]

\[ = [\max\{i, 2\} \delta_{n-1} h_{n-1} + 1, \gamma_1 h_n) \cap \mathbb{Z}, \]

which contains \([h_n + 1, \gamma_1 h_n) \cap \mathbb{Z}.

(ii*) For \( i = m, \) within intervals

\[ j \in [c_{mn} + 1, \gamma_1 c_{mn}) \cap \mathbb{Z} = [h_n + 1, \gamma_1 h_n) \cap \mathbb{Z}. \]

(iii*) For \( i = 2, \) within intervals

\[ j \in [c_{mn+1} + 1, 2c_{mn+1} - 1] \cap \mathbb{Z} = [\delta_n h_n + 1, 2\delta_n h_n - 1) \cap \mathbb{Z}. \]

(iv*) For \( i = 1 \) and \( 3 \leq i \leq m, \) within intervals

\[ j \in [c_{mn+1} + 1, 2c_{mn+1} - 1] \cap \mathbb{Z} = [\delta_n h_n + 1, 2\delta_n h_n] \cap \mathbb{Z}. \]
Note that \( i \delta_n h_n = i c_{mn+1} = c_{mn+i} \) for \( 1 \leq i \leq m-1 \) and \( n \geq 1 \) by (10), (42). We claim

**Lemma 9.3** Any \( \zeta \in Q = \prod Q_i \) as above satisfies

\[
\limsup_{Q \to \infty} Q^{1/m} \psi_{\zeta}(Q) = c,
\]

i.e. \( C1' \), \( C2 \) for \( \Phi(t) = ct^{-1/m} \), where \( C1' \) is \( C1 \) up to admitting a factor \( 1 + \varepsilon(Q) \) with \( \varepsilon(Q) \to 0 \) as \( Q \to \infty \) in the right hand side.

**Proof of Lemma 9.3** Take \( \zeta \in Q \). We first verify \( C1' \). Assume we are in Cases 2, 3 of the proof of \( C1 \) in Sect. 6.2. We again consider integers \( q = a_{mf+1} = 2^{c_{mf+1}} = H_f^i \approx H_f \gamma_2 \) that satisfy \( q < Q < a_{mf+1} \leq a_{mf+1}^{c_{mf+1}} \). By \((ii^*)\), we have essentially the same estimates for \( \|q \xi_2\| \) as in Sect. 6.2 and by \((i^*\star)\) the other \( \|q \xi_i\|, i \neq 2, \) take smaller values, so

\[
\|q \xi\| = \|q \xi_2\| \leq a_{mf+1}^{-1} + O(a_{mf+1}^{-1} a_{mf+1}^{-1}), \quad \xi \in Q.
\]

As in the proof in Sect. 6.2, by construction \( a_{mf+1}^{-1} = d_f^{-1} < \Phi(Q) = cQ^{-1/m} \) for any \( Q < a_{mf+1} \), Hence, upon admitting a factor \( 1 + \varepsilon \) coming from the lower order error term (which however we cannot control as freely here in view of (41)), we satisfy \( C1 \).

In Case 1 of the proof of \( C1 \) in Sect. 6.2, we use \((i^*\star), (ii^*)\) and \( \gamma_1 - 1 > \gamma_2/m \) from (40) when \( \Phi(t) = ct^{-1/m} \) (under \( (d4') \)) in general \( \gamma_1 - 1 > \gamma_2 \gamma_1 \) to guarantee the same estimate. Indeed, for \( q = a_{mf} = 2^{h_f} = H_f \) and any \( Q < a_{mf+1} = H_f^{b_0} \leq H_f^{\gamma_2+o(1)} \) we calculate

\[
\|q \xi\| \ll 2^{-\frac{h_f}{h_f} + \gamma_f} \approx a_{mf+1}^{-1} + o(a_{mf+1}^{-1}), \quad \text{as } f \to \infty, \quad \zeta \in Q.
\]

Hence \( \|q \xi\| < cQ^{-1/m} = \Phi(Q) \) for \( f \geq f_0 \). The proof of \( (C2) \) is almost analogous to the classical case in Sect. 6.2, with two minor changes. Firstly, we again use \( \gamma_1 > 1 + 1/m \) from (40) and \((i^*), (ii^*)\) for the error term \( O(Q d_f^{-1}) = O(a_{mf+1} a_{mf+1}) \) from (20) to be negligible. Indeed, by \( \gamma_1 > 1 + 1/m \) from (40) and \((i^*), (ii^*)\), this error term can be estimated \( \ll a_{mf+1}^{-1} \gamma_1 \),

while by construction the main term \( d_f^{-1} \) is just slightly smaller than \( \Phi(a_{mf+1}) = c a_{mf+1}^{-1/m} \), so it is \( a_{mf+1}^{-1/m} \). (In general under condition \( (d4') \) on \( \Phi \), we require \( \gamma_1 > 1 + \gamma \) for the same conclusion.) Secondly our digital variations from \((i)\) resp. \((ii)\) induce slightly different integers \( D_f = D_f(\zeta) \) in Case I resp. \( Y_q = Y_q(\zeta) \) and \( Z_q = Z_q(\zeta) \) in Case II, now depending on the choice of \( \zeta \in Q \). However, the according crucial properties hold again for similar reasons as in Sect. 7.1. Note that the remainder terms are essentially unaffected in view of \((i^*), (ii^*)\). Hence our claim is proved. \( \square \)
Notice further that any \( \xi \in \prod Q_i \) is not badly approximable as the stronger claim \( \lambda(\xi) \geq 1 > 1/m \) is easily verified via \((iii^*), (iv^*)\) by considering \( \|q\xi\| \) for integers of the form \( q = a_{mn+1} = 2b_nh_n \). See also Sect. 9.3 below. Combining this with Lemma 9.3, we see that the set \( F_{m,c} \) contains \( Q \). Thus we may apply Lemma 9.1 to bound its Hausdorff dimension from below as in (36), i.e. we obtain

\[
\dim_H(F_{m,c}) \geq 2 \frac{\gamma_2 - \gamma_1}{\gamma_1(\gamma_2 m - 1)} + \sum_{i=3}^{m} \min \left\{ \frac{m(\gamma_2 - \gamma_1) + i - 2}{2(m\gamma_2 - 1)}, \frac{(i-1)\gamma_2 - \gamma_1}{\gamma_1(m\gamma_2 - 1)} \right\}.
\]

(43)

Note that for any pair \( \gamma_1, \gamma_2 \) with \( \gamma_1 > \gamma_2/m + 1 \) and \( \gamma_2 > 1 + \frac{1}{m-1} \) the assumption (40) holds. We optimize the parameters \( \gamma_1, \gamma_2 \) under this restriction. The bound (43) clearly decays in \( \gamma_1 \), so we choose some \( \gamma_1 < 2\gamma_2/m + 1 \). To give an asymptotical estimate, let \( \gamma_2 = m^\sigma \) for fixed \( \sigma \in (0,1) \). Then \( 1 < \gamma_1 < 1 + 2m^{\sigma-1} \) is just slightly larger than 1, so it is asymptotically negligible in (43). Thus, for large \( m \) we then check that the left expression in the minimum of (43) is of order \( 1/2 + o(1) \) while the right is of order \( i/m + o(m) \). Consequently, the minimum in (43) equals \( i/m + o(1) \) for \( 3 \leq i \leq \lfloor m/2 \rfloor \) and \( 1/2 + o(1) \) for \( \lfloor m/2 \rfloor + 1 \leq i \leq m \). Thus, omitting the small positive first expression \( 2(\gamma_2 - \gamma_1)/(\gamma_1(m\gamma_2 - 1)) \), as \( m \to \infty \) we bound the right hand side in (43) from below by

\[
\left( \sum_{i=3}^{\lfloor m/2 \rfloor} \frac{i}{m} - o(m) \right) + \left( \sum_{i=\lfloor m/2 \rfloor + 1}^{m} \frac{1}{2} - o(m) \right) = \frac{m}{8} + \frac{m}{4} - o(m) = \frac{3}{8}m - o(m),
\]

the claimed asymptotical estimate (7).

Alternatively, by Lemma 9.1 and the inclusion \( F_{m,c} \supseteq Q \), the Hausdorff dimension of \( F_{m,c} \) can be estimated from below by (37). We now prove (6) by optimizing the parameters \( \gamma_1, \gamma_2 \). We already noticed that the expression in (37) decreases in \( \gamma_1 \). Hence in view of (40) we again take parameters related by the identity \( \gamma_1 = \frac{\gamma_2}{m} + 1 + \epsilon \) with small \( \epsilon > 0 \) and bare in mind that we require \( \gamma_2 > 1 + \frac{1}{m-1} \) for the conditions (40) to be satisfied. Since \( \epsilon > 0 \) can be arbitrarily small, we want to maximize the function

\[
\gamma_2 \mapsto m \frac{m-1}{m} \gamma_2 - 1 = \frac{(m-1)\gamma_2 - m}{\gamma_2^2 + (m - \frac{1}{m})\gamma_2 - 1}
\]

over \( \gamma_2 > 1 + \frac{1}{m-1} \). By differentiation we verify that the maximum is attained at

\[
\gamma_2 = \frac{m + \sqrt{m(m^2 - m + 1)}}{m - 1} > 1 + \frac{1}{m - 1},
\]

representing the positive solution of \( (m - 1)x^2 - 2mx + m(1 - m) = 0 \). Inserting in the function gives the desired estimate (6) after a short simplification.
Finally, we may generalize Lemma 9.1 to the situation where $H_n$ are not powers of 2, and consequently drop the assumption $a_n = 2^{c_n}$, essentially by the argument explained in detail in Sect. 7.2. We omit recalling the strategy.

9.3 Proof of Theorem 3.2

Assume $\gamma_1, \gamma_2$ satisfy besides (40) also

$$(\gamma_1 - 1)^2 > \gamma_2. \quad (44)$$

We again assume for simplicity that $a_n = 2^{c_n}$ for integers $c_n$, the general case can be obtained as in Sect. 7.2. Recall the sets $Q_i$ induced by $\gamma_1, \gamma_2$, constructed in Sect. 9.1 from freely altering binary digits of $\xi$ from Sect. 6.1 in intervals of type (i), (ii), (iii). We slightly alter $Q_1$ by imposing the additional new condition

(iv) any $\zeta_1 \in Q_1$ has binary digit $g_{1,j} = 1$ at position $j = [\gamma_1 h_n]$, for $n \geq 1$.

Denote this set by $Q_1^* \subseteq Q_1$ and let

$$Q^* := Q_1^* \times \prod_{i=2}^m Q_i \subseteq \mathbb{R}^m$$

for simplicity. We claim

**Lemma 9.4** For any $\zeta \in Q^*$ as above we have

$$\lambda(\zeta) = \gamma_1 - 1.$$  

We remark that $\gamma_1 - 1 > \beta > 1$ by assumptions (40) and (44). We believe that in fact $\lambda(\zeta) = \max\{\gamma_1 - 1, 1\}$ for generic $\zeta \in \prod_{i=1}^m Q_i$ whenever $\gamma_1, \gamma_2$ are related by (40), however we are unable to prove it. The lower bound 1 hereby comes from (iii*), (iv*). The proof of Lemma 9.4 relies on the following standard result on rational approximation to a single real number.

**Proposition 9.5** Let $x \in \mathbb{R}$. Assume for a reduced fraction $p/q \in \mathbb{Q}$ and $\tau > 2$ we have

$$|x - \frac{p}{q}| = q^{-\tau}.$$  

Then for any rational $\tilde{p}/\tilde{q} \neq p/q$ with $q \leq \tilde{q} \ll q^{\tau-1}$ for a sufficiently small absolute implied constant, we have $|x - \tilde{p}/\tilde{q}| \geq \tilde{q}^{\tau-2}/2$ or equivalently $\|\tilde{q}x\| \geq \tilde{q}^{-1}/2$.

Proposition 9.5 follows from Legendre Theorem stating that $|r/s - x| < s^{-2}/2$ implies that $r/s$ (after reduction) must be a convergent of the continued fraction expansion of $x$, and the relation $s_{k+1} \asymp |s_k x - r_k|^{-1}$ between two consecutive convergents $r_k/s_k$ and $r_{k+1}/s_{k+1}$. See [29, Proposition 4.2] for a short proof of the latter fact. Alternatively, Minkowski’s Second Convex Body Theorem directly implies Proposition 9.5, see also [19].
Proof of Lemma 9.4 Let \( \zeta \in Q^* \) be given. By properties \((i^*)\), \((ii^*)\) from Sect. 9.2, the integers \( H_n = 2^{\delta n} = a_{mn} \) induce an estimate
\[
\| H_n \zeta \| \asymp \| H_n \zeta_1 \| \asymp 2^{-h_n \gamma_1 + h_n} = H_n^{-(\gamma_1 - 1)}, \quad n \geq 1.
\] (45)
For the lower bound we have used the non-zero digit assumption \((iv)\). The lower estimate \( \lambda(\zeta) \geq \gamma_1 - 1 \) follows directly from (45). Assume now conversely to the claim of the lemma that we have strict inequality \( \lambda(\zeta) > \gamma_1 - 1 \). Then for some \( \varepsilon > 0 \), there are arbitrarily large integers \( q > 0 \) with the property
\[
\| q \zeta \| < q^{-(\gamma_1 - 1) - \varepsilon}.
\] (46)
Let \( q \) be such an integer and let \( f \) be the index with \( H_f \leq q < H_{f+1} \). Recall that \( H_f = a_{mf} \) and \( H_{f+1} = a_{m(f+1)} \) and the notation \( d_n = a_{mn+1} = a_{mn} = H_n^{\delta_n} \) for any \( n \geq 1 \) and (34). The proof of \((C2)\) in Sect. 6.2 (or Sect. 9.2) shows that for any \( q < H_{f+1} \) we have
\[
\| q \zeta \| \geq d_f^{-1} + O(H_{f+1}d_f^{-1}) = H_f^{-d_f} + O(H_{f+1}d_f^{-1}).
\] (47)
We verify that the error term is of smaller order than the main term. Indeed, (35) and (34) imply
\[
H_{f+1}d_f^{-1} = H_f^{1-d_f+1} = H_f^{1-\gamma_2+o(1)} = H_f^{-(m\gamma_2^2 - m\gamma_2) + o(m)},
\]
so again using (35) it boils down to showing that \( \gamma_2 < m\gamma_2^2 - m\gamma_2 \) or equivalently \( 1 + \frac{1}{m} < \gamma_2 \) which is clear by (40). Hence, combining (47) with (46) and using again (35) and (44), we conclude
\[
q < H_f^{\frac{\gamma_2^2}{\gamma_2} + o(1)} < H_f^{\gamma_1 - 1}, \quad f \geq f_0.
\]
On the other hand, by (45) for \( n = f \) and Proposition 9.5, we get the contradictory claim \( q \gg H_f^{\gamma_1 - 1 + \varepsilon} \gg H_f^{\gamma_1 - 1} \), unless \( q = PH_f, P \in \mathbb{Z} \setminus \{0\} \) is a multiple of \( H_f = a_{mf} \). In fact, in the latter case if \( \| H_f \zeta_1 \| = \| H_f \zeta_1 - p_1 \| \) and \( \| q \zeta \| = |q \zeta_1 - r_1| \) for integers \( p_1, r_1 \), then we must have \( q/H_f = r_1/p_1 = P \), i.e. \( P(H_f, p_1) = (q, r_1) \). But then from (45) we get
\[
\| q \zeta \| \geq \| q \zeta_1 \| = P\| H_f \zeta_1 \| \geq \| H_f \zeta_1 \| \gg H_f^{-(\gamma_1 - 1)} \geq q^{-(\gamma_1 - 1)},
\] again contradicting (46) for large \( f \) (or equivalently \( q \)). \( \square \)
Recall \( \beta = \frac{1 + \sqrt{5}}{2} \) and let \( \lambda \in (\beta, \infty) \) be given. Let
\[
\gamma_1 = \lambda + 1, \quad \gamma_2 \in (\lambda + 1, \min\{m\lambda, \lambda^2\}) \neq \emptyset.
\] (48)
Then (40), (44) hold. Consider the derived set \( Q^* \). By Lemma 9.4 we have \( Q^* \subseteq \mathcal{W}_m(\lambda) \), and by Lemma 9.3 and since \( Q^* \subseteq Q \), we infer \( Q^* \subseteq F_{m,c} \) as well, hence

\[
Q^* \subseteq \mathcal{W}_m(\lambda) \cap F_{m,c}.
\]

It is further clear from the proof of Theorem 3.1 that for the Hausdorff dimension of \( Q^* \) the lower bounds (36), (37) still apply, as condition (iv) is metrically negligible. Hence, we get a positive Hausdorff dimension of our set (8). For the asymptotical estimate as \( \lambda \to \infty \), we may assume \( \lambda^2 > m\lambda \) so by (48) we may take \( \gamma_2 \) just slightly smaller than \( m\lambda \), let us denote this by \( \gamma_2 = m\lambda - o(1) \). Inserting in (36) we observe that for large \( \lambda \) the right term in the minimum is smaller, and identifying main terms gives \( i/(\lambda m) - m^{-1}\lambda^{-1}(1 + o(1)) \) as a lower estimate for \( 3 \leq i \leq m \), which sums up over \( i \) to \( m/(2\lambda) - O(\lambda^{-1}) \), the claimed asymptotic bound. Theorem 3.2 is proved.

### 9.4 Generalizations

We sketch how to modify the construction of Sect. 9.3 to get the refined claims on the exact order of ordinary approximation indicated below Theorem 3.2. Let \( \Psi(t) \) be any approximation function of decay \( o(t^{-\beta-\epsilon}) \) as \( t \to \infty \). Keep the notation \( H_n = 2^{h_n} = 2^{c_{mn}} = a_{mn} \) and for simplicity let \( \nu_n := -[\log \Psi(H_n)/\log 2] \). Note first that the binary digits of \( \Psi(H_n) \in (0,1) \) at positions 1, 2, \ldots, \( \nu - 1 \) are 0, while the digit is non-zero at position \( \nu \). Now we alter (i) and (iv) from Sects. 9.1 and 9.3 by suitably prescribing at step \( n \) simultaneously the binary digits \( g_{i,j} = g_j \) of all \( \xi_i \), \( 1 \leq i \leq m \), at positions

\[
\{ -\nu_n + h_n, -\nu_n + h_n + 1, \ldots, -\nu_n + h_n + n \}.
\]

Concretely, in \( J_n \) we choose the binary digits \( g_{i,j} = z_j \) the same as for \( \Psi(H_n) = \sum_{u \geq 1} z_u 2^{-u} \), \( z_u \in \{0,1\} \). We emphasize that in (i) we do no longer impose free digit choices in the subintervals \( J_n \). Then we again choose arbitrary binary digits for \( \xi_i \), \( 1 \leq i \leq m \), in the interval \( [-\nu_n + n + 1, c_{mn+1} - 1] \cap \mathbb{Z} \), corresponding to \( [\gamma_1 h_n, \gamma_2 h_n] \cap \mathbb{Z} \) essentially. From this construction, and since multiplication by \( H_n = 2^{h_n} \) shifts the comma by \( h_n = \log H_n/\log 2 \) units to the right, it is easily checked that \( \|H_n\xi\|/\Psi(H_n) = 1 + O(2^{-n}) \). So as \( n \to \infty \) indeed we get a factor \( 1 + o(1) \). In the remaining intervals of the form \( [c_{mn+1}, h_{n+1}] \), we copy the digit conditions (ii), (iii) from Sect. 9.1. If we assume the parametric constants defined as

\[
\gamma_1(n) = -\frac{\log \Psi(H_n)}{\log H_n} + 1, \quad \gamma_2(n) = (\gamma_1(n), \min(m(\gamma_1(n) - 1), (\gamma_1(n) - 1)^2)) \neq \emptyset.
\]

to have the according properties (40), (44), the same arguments as in Sect. 9.3 show that the arising set of real vectors \( \hat{Q}(\Psi) \) is contained in \( \mathcal{W}_m(\Psi) \cap F_{m,c} \). Moreover, the intervals \( J_n \) are short enough not to affect the asymptotics (38), (39) for every \( n \), with \( \gamma_i = \gamma_i(n) \). Given explicit lower and upper bounds for \( -\log \Psi(t)/\log t \), we obtain intervals for \( \gamma_i(n) \) uniformly for \( n \geq 1 \), and may again infer metrical claims with some “dynamical variant” of Lemma 9.1. For example, assuming \( \Psi(t) = t^{-\lambda+o(1)} \),
for some $\lambda \in (\beta, \infty)$ as $t \to \infty$, the exact same estimates (36), (37) can be deduced
and we choose $\gamma_1 = \lambda + 1$, $\gamma_2 = m\gamma_1 - o(1)$ again for optimization. We leave the
details to the reader.

We keep similar variants of (ii), (iii) from Sect. 9.3 with $\gamma_i = \gamma_i(n)$, in particular we want the $\zeta_i$
to have digit 0 at positions $\log H_n / \log 2 + 1$, $\log H_n / \log 2 + 2$, $\ldots$, $−\lfloor \log \Psi(H_n) / \log 2 \rfloor$ after the comma. Thereby we obtain essentially a subset $	ilde{Q}$ of $Q$ from Sect. 9.1 again, but with parameters $\gamma_i$ depending on $n$ given as

$$
\gamma_1(n) = -\frac{\log \Psi(H_n)}{\log H_n} + 1, \quad \gamma_2(n) \in (\gamma_1(n), \min \{m(\gamma_1(n) - 1), (\gamma_1(n) - 1)^2\}) \neq \emptyset.
$$

Multiplication by $H_n = 2^{h_n}$ shifts the comma by $h_n = \log H_n / \log 2$ units to the right. So by our variants of (ii), (iii) the binary expansions of the $m + 1$ numbers $H_n\zeta_i$, $1 \leq i \leq m$, and $H_N\Psi(H_n)$ start after the comma with $−\lfloor \log \Psi(H_n) / \log 2 \rfloor − \log H_n / \log 2$ many 0 digits, followed by $|J_n| = n$ identical digits the first of which is non-zero. Therefore $\|H_n\zeta_i / \Psi(H_n)\| = 1 + O(2^{-n})$, so as $n \to \infty$ indeed we get a factor $1 + o(1)$. The arguments from proof of Lemma 9.4 further show that these values $Q = H_n$ essentially induce the local mimima of $\psi_\zeta(Q) / \Psi(Q)$ if $\gamma_i(n)$ satisfy (40), (44) for every large $n$. Thus $\tilde{Q}$ is contained in $\mathcal{W}_m(\Psi) \cap F_{m,c}$.

Finally we sketch how to argue when $1 < \lambda \leq \beta$. We let $\gamma_1 = \lambda + 1 > 2$ again
and $\gamma_2 > \gamma_1$ sufficiently close to $\gamma_1$. Then we fix the last coordinate

$$
\zeta_m = \sum_{n=1}^{\infty} 2^{-h_n} + \sum_{n=1}^{\infty} 2^{-\lfloor \gamma_1 h_n \rfloor},
$$

and take the other $\zeta_i \in Q_i = Q_i(\gamma_1, \gamma_2)$, $1 \leq i \leq m - 1$, as defined above with binary
digit 0 in according intervals of types (i*), (ii*), (iii*), (iv*). Note that $q = H_n = 2^{h_n}$
and $q = 2^{\lfloor \gamma_1 h_n \rfloor} \sim H_n^{\gamma_1}$ induce small values $\|q\zeta_m\|$. Conversely, it can be deduced
from the “Folding Lemma” by the same line of arguments as in [5] that we can only have $\|q\zeta_m\| < q^{−(\gamma_1 − 1)−e}$ and hence (46), if $q$ is an integral multiple of integers of
either of these two forms. To exclude these cases, we can use a similar strategy as in
the proof of (C2) in Sect. 6.2 involving some case distinctions, assuming $\gamma_2 > \gamma_1$ was
chosen sufficiently close to $\gamma_1 = \lambda + 1$. For the Hausdorff dimensions of our Cantor
type sets of $\zeta \in F_{m,c} \cap \mathcal{W}_m(\lambda)$ we get a lower bound $\dim_H(\prod_{i=1}^{m-1} Q_i \times \{\zeta_m\}) = \dim_H(\prod_{i=1}^{m-1} Q_i) \geq \sum_{i=1}^{m-1} \dim_H(Q_i) > 0$. We omit the technical details again.

10 Proof of Cantor set results

10.1 Proof of Theorem 4.1

We restrict to $m \geq 3$, for $m = 2$ we alter accordingly to Sect. 8. Since $w_j \in \{0, 1\}$,
very similarly as in Sect. 7.1 we can take $a_j = b^{c_j}$ for an increasing sequence of integers $c_j$
in the construction in Sect. 6.1, up to redefining $L_n = b^{\ell_n}$ as the smallest
integral power of $b$ so that $a_{mn+1}^{−1} < \Phi(Q')$ for any $Q' \leq b^{\ell_n}a_{mn+1}$. The analogue of
Proposition 6.2 is easily checked as well in our setting. The proof of (C3), (C1) works identically as for Theorem 2.2. The proof of claim (20) further works analogously as Theorem 2.2 up to (25). Below, now we only have $Q' \leq bLfam_{f+m-1}$ in place of $Q' \leq (Lf + 1)a_{m_{f+m-1}}$. Hence $1 \leq \alpha_{f} \leq ba_{m_{f+m-1}} - 1 = b + o(1)$ as $f \to \infty$. Then for large $f$ the decay condition $(d3'_{f}(b, R))$ yields a factor $R$ in place of $1 - \epsilon_{2}$ and (i) follows.

For (ii), first recall that we may relax assumption (9) to $a_{mn+1}$ being just a large enough multiple, in place of power, of $a_{mn}$. That is to use $M'_{n}$ in place of $M_{n}$. This corresponds to $c_{mn} < c_{mn+1}$ in place of $c_{mn}|c_{mn+1}$, so that we can choose $A, B$ in $(D(b))$ freely. As remarked above, an according variant of Proposition 6.2 holds in our setting $a_{j} = b^{\epsilon_{j}}$ as well. Then with large $A, B$ as in $(D(b))$ we may choose $c_{mn+1} = A_{n} = A$ and $c_{mn+m-1} + \ell_{n} = B_{n} = B$, i.e. $\ell_{n} = B_{n} - A_{n}$, in step $n$ so that the quotient $\Phi(b^{B})/b^{-A} > 1$ is arbitrarily close to 1 again. Hence we may choose $\alpha_{f} = Q'/Q_{f}a_{mn-1} - 1 > 1$ arbitrarily close to 1 again, and consequently get the same result as in Theorem 2.2.

The bound on the packing dimension for $K = C_{b,[0,1]}^{m}$ follows similarly as for $\mathbb{R}^{m}$. We now instead have that the sumset $(K \cap \mathcal{V}) + (K \cap \mathcal{S})$ contains $K$, with $\mathcal{V} = \gamma_{b}(\mu_{-\epsilon})$ as in Sect. 7.1 but for general $b \geq 2$, and conclude with Tricot's estimate again. Here we use the more general [32, Lemma 5.6] to bound the Hausdorff dimension of $K \cap \mathcal{V}$ from above. See also the very similar proof of [32, Theorem 4.1].

Regarding Hausdorff dimension in (i) upon dropping (C3), a positive lower bound follows from very similar arguments as in Sect. 9.1. In (38) resp. (39), we need to replace the estimate for $P_{n}$ by $P_{n} \asymp H_{n}((\gamma_{2} - \gamma_{1}) log 2/\log b)$ resp. $P_{2n} \asymp H_{n}((\gamma_{2} - \gamma_{1}) log 2/\log b)$ and $P_{2n+1} \asymp H_{n}(\gamma_{2} - \gamma_{1}) log 2/\log b$ and keep $\epsilon_{n}$ unchanged. We omit the cumbersome calculation but again refer to the proof of [32, Theorem 4.2] which uses the same strategy.

10.2 Proof of Corollary 3 and Corollary 4

We verify property $(d3'(b, R))$ for $R = b^{-1/m}$ for the Cantor set setting.

**Lemma 10.1** Let $m \geq 2, b \geq 2$ be integers and $c \in (0, 1)$). Then for any $\epsilon > 0$ there exist arbitrarily large integers $A, B$ with

$$cb^{-B/m}b^{-1/m} \leq b^{-A} < cb^{-B/m}.$$  

**Proof** It suffices to take $B$ the largest integer with $b^{-A} < cb^{-B/m}$. By maximality of $B$ the other inequality holds as well.

The lemma states that $\Phi(q) = cq^{-1/m}$ satisfies $(d3'(b, R))$ for any pair $(b, R)$ with an integer $b \geq 2$ and $R = b^{-1/m}$, and Corollary 3 follows from part (i) of Theorem 4.1.

**Lemma 10.2** Let $m \geq 2, b \geq 2$ be integers, $c \in (0, 1)$ and $\tau > 0$ irrational. Then for any $\epsilon > 0$ there exist arbitrarily large integers $A, B$ with

$$(1 - \epsilon)c b^{-B\tau} < b^{-A} \leq cb^{-B\tau}.$$  

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Proof Taking logarithms the claim becomes

\[ 0 < (A - B\tau) \log b - \log c < -\log(1 - \epsilon). \]

Since \( \tau \) is irrational, the set of values of \( A - \tau B \), and thus also of \( (A - B\tau) \log b - \log c \) when taking all integer pairs \( A, B \), is dense in \( \mathbb{R} \). The claim follows. \( \Box \)

From the lemma we see that \( \Phi(q) = cq^{-\tau} \) for \( \tau \) irrational satisfies \( (D(b)) \) for any integer \( b \geq 2 \), and part (ii) of Theorem 4.1 implies Corollary 4.

We finish this section with a remark. Assume we are given an effective upper bound for the irrationality exponent of \( \tau \). Then, using a result on uniform inhomogeneous approximation due to Bugeaud and Laurent [6], it is possible to state upper bounds for the smallest \( A, B \) satisfying the hypothesis of Lemma 10.2 in terms of \( \epsilon \), independent from \( c \). Similar to the remark below Theorem 2.2, this in turn implies an effective rate at which we can let \( \epsilon \to 0 \) in terms of \( Q \) in Corollary 4, if we admit a factor \( 1 + \epsilon \) in condition (C1). We do not make this explicit here.

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11 Appendix: The linear form problem

Let \( \langle \cdot, \cdot \rangle \) be the standard scalar product on \( \mathbb{R}^m \) and for \( y = (y_1, \ldots, y_m) \in \mathbb{R}^m \) denote by \( |y|_\infty = \max_{1 \leq i \leq m} |y_i| \) the maximum norm. Recall further the notation \( \| \cdot \| \) introduced in Sect. 1. For \( c^* \in (0, 1] \), let \( \text{Di}_m^*(c^*) \) be the set of \( \xi \in \mathbb{R}^m \) for which the system

\[ 0 < |y|_\infty \leq Q^*, \quad \| \langle y, \xi \rangle \| \leq c^* Q^{* - m} \tag{49} \]

has a solution in an integer vector \( y \), for all large parameters \( Q^* \). By a variant of Dirichlet’s Theorem we have \( \text{Di}_m^*(1) = \mathbb{R}^m \) for any \( m \geq 1 \). Let again \( \text{Di}_m^* = \bigcup_{c < 1} \text{Di}_m^*(c) \). Let \( \text{Bad}_m^* \) be the set of badly approximable linear forms, its defining property being that for some \( c^* > 0 \) and all \( Q^* \) there is no integer vector solution to (49). For completeness further define accordingly \( \text{Sing}_m^* = \bigcap_{c > 0} \text{Di}_m^*(c) \) and \( \text{FS}_m^* = \text{Di}_m^* \setminus (\text{Bad}_m^* \cup \text{Sing}_m^*) \). We point out the well-known identities

\[ \text{Di}_m = \text{Di}_m^*, \quad \text{Sing}_m = \text{Sing}_m^*, \quad \text{Bad}_m = \text{Bad}_m^*, \quad \text{FS}_m = \text{FS}_m^*. \]

Note however that the sets \( \text{Di}_m(c) \) and \( \text{Di}_m^*(c) \) do not coincide for the same parameter \( c < 1 \). Nevertheless, it has been shown in [30] appearing after the first version of the present paper that the according Dirichlet spectrum \( \mathbb{D}_m^* \) with respect to one linear form equals \( [0, 1] \) as well. Similar metrical results as in Sect. 2, Sect. 3 can be shown for the linear form setting as well, however details are not explicitly carried out in [30]. From Corollary 1 and a transference result phrased below, we obtain the following more modest claim that preceded [30].
**Theorem 11.1** For $m \geq 2$ an integer and any $c^* \in (0, 1]$, if we let

$$
\omega = \omega(m, c^*) := (m + 1)^{-m^2 - m} (c^*)^{m^2} \in (0, c^*),
$$

then the set

$$
\text{Dim}^*(c^*) \setminus (\text{Dim}^*(\omega) \cup \text{Bad}^*) \subseteq FS^*_m
$$

has packing dimension at least $m - 1$ and Hausdorff dimension at least as in Theorem 3.1.

As $m \to \infty$, the value $\omega(m, c^*)$ asymptotically satisfies

$$
\omega(m, c^*) > (c^*)^{m^2} e^{-m^2 \log m - o(m^2 \log m)}.
$$

Our result should be compared with the following partial claim of [4, Theorem 1.5] (see also [25]) obtained from a very different, unconstructive method.

**Theorem 11.2** (Beresnevich, Guan, Marnat, Ramírez, Velani) Let $m \geq 2$ an integer and

$$
\kappa_m = e^{-20(m+1)^3(m+10)}.
$$

Then the set

$$
\text{Dim}^*(c^*) \setminus (\text{Dim}^*(\kappa_m c^*) \cup \text{Bad}_m) \subseteq FS^*_m
$$

is uncountable.

We should remark that in the statement we suppressed some more information on other exponents of approximation given in [4, Theorem 1.5]. Moreover, the exact shape of the polynomial in the exponent of $\kappa_m$, in particular the leading coefficient 20, can be readily optimized with sharper estimates at certain places in [4]. Note that in contrast to our result, the value $\kappa_m$ in Theorem 11.2 is independent from $c^*$. We see that for large $m$ and large enough $c^* \in (0, 1]$, roughly as soon as $c^* > e^{-m^2}$, our bound from Theorem 11.1 is stronger. For $c^*$ very close to 1, we basically can reduce the quartic polynomial in $m$ within the exponent in $\kappa_m$ to a quadratic polynomial.

For the deduction of Theorem 11.1 it is convenient to apply a transference result by German [14] based on geometry of numbers, which improves on previous results by Mahler. Concretely, we use the following special cases of [14, Theorem 7], where we implicitly include the upper estimate $\Delta_d^{-1} \leq d^{1/2}$ from [14, § 2] for the quantity $\Delta_d$ defined there, where $d = m + 1$ in our situation.

**Theorem 11.3** (German) Let $m \geq 1$ and $\xi \in \mathbb{R}^m$. Let $X, U$ be positive parameters.
(i) Let $x \in \mathbb{Z}$ and assume 

$$0 < |x| \leq X, \quad \| \xi x \| \leq U.$$ 

Then there exists $y^* \in \mathbb{Z}^m$ so that 

$$0 < |y^*|_\infty \leq Y, \quad \| (y^*, \xi) \| \leq V,$$

where 

$$Y = (m + 1)^{1/(2m)} X^{1/m}, \quad V = (m + 1)^{1/(2m)} X^{1/m-1} U.$$

(ii) Let $y \in \mathbb{Z}^m$ and assume 

$$0 < |y|_\infty \leq X, \quad \| (y, \xi) \| \leq U.$$ 

Then there exists $x' \in \mathbb{Z}$ so that 

$$0 < |x'| \leq Y', \quad \| x' \xi \| \leq V'$$

where 

$$Y' = (m + 1)^{1/(2m)} XU^{1/m-1}, \quad V' = (m + 1)^{1/(2m)} U^{1/m}.$$ 

We now prove our claim.

**Proof of Theorem 11.1** Let $c \in (0, 1]$ to be chosen later. By Corollary 1 and Theorem 3.1, the set $\text{Di}_m(c) \setminus (\cup_{\epsilon > 0} \text{Di}_m(c - \epsilon) \cup \text{Bad}_m) = \text{Di}_m(c) \setminus (\cup_{\epsilon > 0} \text{Di}_m(c - \epsilon) \cup \text{Bad}_m^*)$ has the stated metrical properties. Take any $\xi$ in this set. Take arbitrary, large $Y$ and put $X = (Y(m + 1)^{-1/(2m)})^m$, which is also large. Now the hypothesis of (i) from Theorem 11.3 holds when we take 

$$U = cX^{-1/m}.$$ 

From the conclusion we get $y^* \in \mathbb{Z}^m$ that satisfies 

$$0 < |y^*|_\infty \leq Y = (m + 1)^{1/(2m)} X^{1/m}$$

and 

$$\| (y^*, \xi) \| \leq V = (m + 1)^{1/(2m)} X^{1/m-1} U = c(m + 1)^{1/(2m)} X^{-1} = c^* Y^{-m},$$

where 

$$c^* = c(m + 1)^{1/2+1/(2m)}.$$ (50)
Observe this holds for all large $Y$, so $\xi \in \text{Di}_m^*(c^*)$. 

Now assume that for some $\tilde{c} \in (0, 1)$ we have $\xi \in \text{Di}_m^*(\tilde{c})$. That means for all large $X$ we may take 

$$ U = \tilde{c}X^{-m} $$

and the hypothesis of (ii) from Theorem 11.3 is satisfied for some $y \in \mathbb{Z}^m$. From the conclusion we get the existence of a positive integer $x'$ satisfying 

$$ 0 < |x'| \leq Y' = (m + 1)^{1/(2m)}XU^{1/m-1} = (m + 1)^{1/(2m)}\tilde{c}^{1/m-1}X^m $$

and 

$$ \|x'\xi\| \leq V' = (m + 1)^{1/(2m)}U^{1/m} = (m + 1)^{1/(2m)}\tilde{c}^{1/m}X^{-1} = \tilde{C}Y'^{-1/m}, $$

where 

$$ \tilde{C} = (m + 1)^{1/2+1/(2m)}\tilde{c}^{1/m^2}. \tag{51} $$

Now, since $Y'$ can be any large number by choosing $X$ suitably, we infer $\xi \in \text{Di}_m(\tilde{C})$. If $\tilde{C} < c$ and assuming $c \leq 1$, this contradicts our choice of $\xi$. By (50), the latter condition $c \leq 1$ clearly holds as soon as $c^* \leq 1$, so we require $\tilde{C} \geq c$, which by (51) and (50) leads to 

$$ \tilde{c} \geq ((m + 1)^{-1/2-1/(2m)}c)^{m^2} = (m + 1)^{-m^2/2-m/2}c^{m^2} = (m + 1)^{-m^2-m}c^{*m^2}. $$

Combining our results we see that $\xi \in \text{Di}_m^*(c^*) \setminus (\text{Di}_m^*((m + 1)^{-m^2-m}c^{*m^2}) \cup \text{Bad}_m^*)$. \hfill \Box

It may be possible to derive similar, possibly stronger, effective results when combining Corollary 1 with the essential method of Davenport and Schmidt [10], however in reverse direction (we need the conclusion from simultaneous approximation to linear form instead of the other way round), instead of [14]. We also want to refer to [4, § 4], in particular [4, Lemma 4.9], in this context.

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