THE DEGREE OF THE DORMANT OPERATIC LOCUS

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Abstract. Let $X$ be a smooth, projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic $p > 0$. We provide a conjectural formula for the degree of the scheme of dormant $\text{PGL}(r)$-opers on $X$ where $r \geq 2$ (we assume that $p$ is greater than an explicit constant depending on $g, r$). For $r = 2$ a dormant $\text{PGL}(2)$-oper is a dormant indigenous bundle on $X$ in the sense of Shinichi Mochizuki (and his work provides a formula only for $g = 2, r = 2, p \geq 5$, from a different point of view). Recently Yasuhiro Wakabayashi has shown that my conjectural formula holds for $r = 2$ and $p > 2g - 2$.

1. Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$. In his fundamental papers [Mochizuki, 1996, 1999], Shinichi Mochizuki introduced and studied several (log) stacks over the moduli stack, $\mathcal{M}_{g,n}$, of stable log-curves of type $(g, n)$ (by [Kato, 2000] this log stack is isomorphic to the stack of stable, $n$-pointed curves of genus $g$ equipped with the log structure obtained from the divisor at infinity). Amongst them is the stack of $\mathcal{S}_{g,n} \to \mathcal{M}_{g,n}$ of indigenized curves of genus $g$ and $n$ marked points (another name for $\mathcal{S}_{g,n}$ is the stack of Schwarz-torsors). An indigenized curve is a stable log-curve of type $(g, n)$ equipped with an indigenous bundle. A nil-curve of type $(g, n)$ is a stable log curve of type $(g, n)$ which is equipped with an indigenous bundle whose underlying connection is nilpotent with nilpotent residues at marked points. The stack $\mathcal{N}_{g,n} \to \mathcal{M}_{g,n}$ of nil-curves is a closed substack of the stack of indigenized curves. It was shown in [Mochizuki, 1996] that $\mathcal{N}_{g,n} \to \mathcal{M}_{g,n}$ is finite flat of degree $p^{3g-3}$. This is a fundamental result in the subsequent development of Mochizuki’s $p$-adic Teichmüller Theory (see [Mochizuki, 1999]).

One may consider, as I do in the rest of this paper, the restriction of these stacks to $\mathcal{M}_{g,n}$, the moduli stack of smooth, proper curves with $n$-marked points and from now on assume $g \geq 2$ and $n = 0$. I do this, primarily, because the theory of opers over stable log-curves is not yet available, though it is clear to me that the construction of these stacks given in [Mochizuki, 1996], for $r = 2$, extends mutatis mutandi to this case.

Before proceeding further, I recall a few facts about opers (see [Beilinson and Drinfeld, 1997, 2005]; opers in characteristic $p > 0$ are also studied in [Joshi, Ramanan, Xia, and Pauly, 2009]). Let me note that following ideas of [Mochizuki, 1996] one may relativize the [Beilinson and Drinfeld, 1997] construction of the stack of opers on a smooth, proper curve of genus $g$ to construct the stack of $\text{oper}_{\text{ked}}$ over $\mathcal{M}_{g,0}$ (i.e. a curve equipped with an oper on it) or more generally the stack of $\text{oper}_{\text{ked}}$ stable log-curves of type $(g, n)$. But I do not pursue this line of thought here.

Let $X/k$ be a smooth, projective curve of genus $g \geq 2$ on $X$. Let $r \geq 2$ be an integer. In this situation one has at one’s disposal the stack of $\text{PGL}(r)$-opers, denoted $\text{Oper}_r(X)$, on $X$ (this stack is in fact a scheme). For $r = 2$ the stack of $\text{PGL}(2)$-opers is
the stack of indigenous bundles studied by Mochizuki in [Mochizuki, 1996]. Thus opers provide a natural generalization of indigenous bundles. In [Joshi and Pauly, 2009] some of Mochizuki’s results were extended to all ranks, and in particular we defined, what we call the Hitchin-Mochizuki morphism:

\[
\text{Oper}_r(X) \rightarrow \text{Hitchin}_r(X) = \oplus_{i=2}^{r} H^{i}(X, (\Omega^1_X)^{\otimes i})
\]

\[
(V, \nabla, \psi) \rightarrow \psi(V, \nabla)^{1/p}
\]

which maps an oper to the \(p\)-th root of the characteristic polynomial of the \(p\)-curvature of the oper connection. Let \(\mathcal{N}_{g,0}^r(X)\) be the schematic fibre over zero of the Hitchin-Mochizuki morphism. Let me call \(\mathcal{N}_{g,0}^r(X)\) the scheme of nilpotent PGL\((r)\)-opers on \(X\). A result of [Joshi and Pauly, 2009] is that \(\mathcal{N}_{g,0}^r(X)\) is finite (and in we gave a conjectural degree of this map to be \(p^{\dim \text{PGL}(r)(g-1)}\)). For \(r = 2\) this is \([Mochizuki, 1996]\).

The principal object of interest for this note is the stack \(\mathcal{D}_{g,0}^r \rightarrow \mathcal{M}_{g,0}\) of dormant PGL\((r)\)-opers on \(X\) which is a closed substack of \(\mathcal{N}_{g,0}^r\), consisting of nilpotent opers on \(X\) whose underlying indigenous bundle is dormant i.e., the \(p\)-curvature of the connection is zero. This is the dormant operatic locus of the title. The closed subscheme of \(\mathcal{D}_{g,0}^r(X) \subset \mathcal{N}_{g,0}^r(X)\) defined by the equation \(\psi = 0\) (i.e. by the vanishing of the \(p\)-curvature of the oper connection). For brevity I will shorten dormant PGL\((r)\)-opers on \(X\) to dormant opers on \(X\). The dormant locus \(\mathcal{D}_{g,0}^r(X) \subset \mathcal{N}_{g,0}^r(X)\) is also a finite subscheme of Oper\(_r(X)\). For \(r = 2\) this was proved by Mochizuki. Note that my notation for the dormant locus is different from that used in [Mochizuki, 1999] as it is easier to remember: \(\mathcal{N}\) for nilpotent and \(\mathcal{D}\) for dormant.

One is interested in the degree of \(\mathcal{D}_{g,0}^r(X)\). In the present note I describe a conjectural formula for this degree. For \(r = 2, g = 2, p \geq 5\) this was proved by [Mochizuki, 1999], [Osserman, 2007] and [Lange and Pauly, 2008] by completely different methods.

The provenance of this formula is as follows. In a “back-of-the-envelop” calculation done for my NSF grant proposal for 2006 (the proposal was not funded), I had observed that the calculations by [Lange and Pauly, 2008], [Mochizuki, 1999], [Osserman, 2007], for the degree of the dormant (tautologically operatic) locus of rank two bundles for an ordinary (or general) of genus two and calculation of Gromov-Witten invariants of certain Quot-schemes due to [Lange and Newstead, 2003], [Holla, 2004] agreed. In this setup [Lange and Pauly, 2008] had calculated the degree of the dormant locus (as a degree of the Quot-scheme which I have described explicitely in this note) as a certain Chern class; the fact that these two numbers coincide was unequivocally established (in this case). This observed coincidence for \(g = 2, r = 2\) was the basis for Conjecture [Joshi and Pauly, 2009]. Let me note that this conjecture was based on one observed data point. At that time the left hand side of the formula made sense only for \(r = 2\) (by Mochizuki’s work). Subsequent to my work with Christian Pauly ([Joshi and Pauly, 2009]) the left hand side of the conjectural formula now makes sense for all \(p > C(r, g)\) (for a certain explicit constant \(C(r, g)\) which I will explain in the text).

Recently Yasuhiro Wakabayashi has given an ingenious proof of Conjecture [Wakabayashi, 2013] for \(X\) general, \(r = 2\) and \(p > 2g - 2\) in [Wakabayashi, 2013] (part of his PhD Thesis at RIMS). For more comments on Wakabayashi’s work see Section [Joshi and Pauly, 2009].

In Section [Joshi and Pauly, 2009] I show that method of the paper also allows one to obtain a conjectural formula (see Conjecture [Joshi and Pauly, 2009]) for the degree of the rational map \(\mathcal{V} : \mathcal{W}(r, \mathcal{O}_X) \to \mathcal{W}(r, \mathcal{O}_X)\), where \(\mathcal{W}(r, \mathcal{O}_X)\) is the moduli space of semi-stable bundles of trivial determinant, and \(X\) is ordinary curve, and \(\mathcal{V}\) is given by Frobenius pull-back \(V \mapsto F^\ast (V)\).

For several years, since my first observation in 2006, I had lost interest in pursuing or publishing the set of ideas presented here. I am deeply indebted to Shinichi Mochizuki for
suggesting that my conjectural formula be published, also for many conversations around this subject and for his invitation to spend some time at RIMS. I would also like to thank Akio Tamagawa for numerous conversations around many topics of common interest. I am also grateful to RIMS for providing hospitality and an excellent working environment.

2. Notations

Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $S/k$ be a $k$-scheme. Let $X \to S$ be a proper, smooth morphism of relative dimension one over $S$. Let $\sigma : S \to S$ be the absolute Frobenius morphism of $S$. Let $X^{(1)} = X \times_k,^\sigma S$. Then one has a “standard” Frobenius diagram (which commutes!) with $F = F_{X/S}$ the relative Frobenius morphism of $X \to S$:

\[
\begin{array}{ccc}
X & \xrightarrow{F_{\text{abs}}} & X^{(1)} \\
\downarrow f & & \downarrow f^{(1)} \\
S & \xrightarrow{\sigma} & S
\end{array}
\]

(2.1)

Now suppose that $S = \text{Spec}(k)$. Let $J_X$ be the Jacobian of $X$, let $J_X[n]$ be the closed subscheme of $J_X$ which is the kernel of multiplication by $n$. Let $V : J_X \to J_X$ be the verschiebung.

3. Opers

From now on fix a smooth, proper curve $X/k$ of genus $g \geq 2$. Let $\Omega^1_X$ be the sheaf of regular one forms on $X$.

Assume that $p > C(r, g) = r(r - 1)(r - 2)(g - 1)$ (I do this because I will use a crucial result of [Joshi and Pauly, 2009] which requires this assumption about $p$, though most of the general facts about opers which I recall below can be proved without this assumption. Opers were introduced in [Beilinson and Drinfeld, 1997]. Reader will find [Beilinson and Drinfeld, 2005] quite useful. Opers in characteristic $p$ were introduced in [Joshi, Ramanan, Xia, and Yu, 2006] and studied in greater detail in [Joshi and Pauly, 2009]. I refer the reader to these references for basic facts on opers.

An $SL(r)$ oper on $X$ is a triple $(V, V_\bullet, \nabla)$ with $(\det(V), \det(\nabla))$ isomorphic to $(\mathcal{O}_X, \nabla = d)$ and a decreasing flag $V_\bullet$ of subbundles which satisfies Griffiths transversality with respect to $\nabla$ and for all $1 \leq i \leq r$, the $\mathcal{O}_X$-linear morphism induced by $\nabla$:

\[
\text{gr}^i(V) \to \text{gr}^{i-1}(V) \otimes \Omega^1_X
\]

is an isomorphism of line bundles.

A $PGL(r)$-oper is the projectivization $\mathbb{P}(V)$ of an $SL(r)$-oper $(V, \nabla, V_\bullet)$ and together with induced connection on $\mathbb{P}(V)$ and the induced reduction of structure group to a Borel subgroup.

The stack, $\text{Oper}_r(X)$, of $PGL(r)$-opers on $X$ is in fact a scheme, isomorphic to the Hitchin space $\oplus_{i=2}^r H^0(X, (\Omega^1_X)^{\otimes i})$ (see [Beilinson and Drinfeld, 1997, 2005]).

The Hitchin-Mochizuki morphism of $\text{Oper}_r(X)$ is the morphism:

\[
\text{Oper}_r(X) \to \text{Hitchin}_r(X) = \oplus_{i=2}^r H^i(X, (\Omega^1_X)^{\otimes i})
\]

maps an oper to the $p$th root of the $p$-curvature. Let $\mathcal{M}_{g, \theta} \subset \text{Oper}_r(X)$ be the schematic fibre over zero of the Hitchin-mochizuki morphism. Then $\mathcal{M}_{g, \theta}$ is finite (as a scheme) (see [Joshi and Pauly, 2009]) and in fact the Hitchin-Mochizuki morphism is finite. Let
\(\mathcal{D}_{g,0}^r, \mathcal{N}_{g,0}^r\) be the closed subscheme of dormant opers, i.e., the subscheme defined by the vanishing of the \(p\)-curvature.

4. A Quot scheme

Let us fix, for once and for all, a line bundle \(L\) on \(X\) such that

\[
L^\otimes r \otimes (\Omega^1_X)^{\otimes \frac{r(r-1)}{2}} = \mathcal{O}_X.
\]  

Let us note that the choice of Cartier connection on \(V\) always exists. Clearly such an \(L\) is unique up to tensoring by a line bundle of degree \(r\). Thus the scheme of such line bundles is evidently a torsor over \(J_X[r]\).

Next I prove a simple Lemma which will be useful in subsequent constructions. This version of the lemma is taken from [Wakabayashi, 2013] as it is better than my original formulation (in a earlier version of this manuscript).

**Lemma 4.3.** The scheme \(\mathcal{D}_L^{r,0}\) carries a natural action of \(\ker(\mathcal{Y})\) given by \(V \in \mathcal{D}_L^{r,0} \mapsto V \otimes \gamma\) for any \(\gamma \in \mathcal{Y}\).

**Proof.** The lemma is evident from the definition of the action and the fact that if \(V \mapsto F_\ast(L)\) then \(V \otimes \gamma \mapsto F_\ast(L) \otimes \gamma = F_\ast(L \otimes \gamma^p) = F_\ast(L)\). Observe that \(\det(V \otimes \gamma) = \det(V) \otimes \gamma^p\).

Let \(\mathcal{D}_L^{r,0}_{\gamma}\) be the closed subscheme of \(\mathcal{D}_L^{r,0}\) defined by the condition

\[
V \in \mathcal{D}_L^{r,0}_{\gamma} \iff V \in \mathcal{D}_L^{r,0}, \text{ and } \det(V) = \gamma^p.
\]

Then clearly \(\mathcal{D}_L^{r,0}_{\gamma}\) is closed in \(\mathcal{D}_L^{r,0}\) and tensoring with \(\gamma' \in \ker(\mathcal{Y})\) induces an isomorphism \(\mathcal{D}_L^{r,0}_{\gamma} \to \mathcal{D}_L^{r,0}_{\gamma'}\) for any \(\gamma, \gamma' \in \ker(\mathcal{Y})\).

Then, as \(\deg(\ker(\mathcal{Y})) = p^r\), one sees that

\[
\deg(\mathcal{D}_L^{r,0}) = p^r \deg(\mathcal{D}_L^{r,0}_{\gamma})
\]

for any \(\gamma \in \ker(\mathcal{Y})\).

5. The dormant locus of \(SL(r)\)-opers

Let me digress in this section and describe the dormant locus of \(SL(r)\)-opers (defined analogously). In other words one considers vector bundles rather than their projectivizations. So I indicate how to do this in this section. One may consider the dormant locus \(\mathcal{D}_{g,0}^r\), in the stack of \(SL(r)\) opers. For any \(k\)-scheme \(S\), an \(S\)-valued point of this stack is a triple \((V, \nabla, V_\bullet)\) where \(V\) is a vector bundle of rank \(r\) on \(X \times S\); \(\nabla\) is an \(\mathcal{O}_S\)-linear connection on \(V\) which is dormant and \(V_\bullet\) is the oper flag on \(V\), moreover one has an isomorphism \((\det(V), \det(\nabla))\) and \((\mathcal{O}_X, \nabla = d)\). There is a natural action of \(J_X[r]\) on \(\mathcal{D}_{g,0}^r\), which I will now explicate. Let \(V \in \mathcal{D}_{g,0}^r(X)(S)\) and let \(\eta \in J_X[r]\) be a line bundle of order \(r\). Then \(\eta^p\) carries its canonical Cartier connection and the morphism \(\eta \mapsto \eta^p\) is an automorphism of \(J_r[X]\) as \((r, p) = 1\).

Equip \(V \otimes \eta^p\) with the connection from \(V\) and the Cartier connection of \(\eta^p\). Let us note that the choice of Cartier connection on \(\eta^p\) fixes for us a canonical isomorphism on the spaces of connections on \(V\) and \(V \otimes \eta^p\) (the two spaces are \(H^0(X, \Omega^1 \otimes \text{End}(V))\) and \(H^0(X, \Omega^1 \otimes \text{End}(V \otimes \eta^p))\) (respectively). Now equip \(V \otimes \eta^p\) with the filtration
induced from $V$. It is easy to see that this provides the structure of an oper on $V \otimes \eta^p$ and $\det(V \otimes \eta^p) = \mathcal{O}_X$ and the induced connection on the determinant is the trivial connection. Thus one has for any pair $V \in \tilde{\mathcal{D}}_{g,0}^r$ and $\eta \in J_r[X]$ a canonical oper in $\tilde{\mathcal{D}}_{g,0}^r$ and this provides us with an action of $J_r[X]$ on $\tilde{\mathcal{D}}_{g,0}^r$ as claimed.

Further it is evident from our description that the action is also transitive. This action is also compatible with the action of $J_r[X]$ on the quotient line bundle $V \to \text{gr}^r(V)$ (such line bundles $\text{gr}^r(V)$ also form a $J_r[X]$-torsor).

For a line bundle $L$ as in the previous section and a line bundle $\eta \in J_r[X]$, We define a substack of $\tilde{\mathcal{D}}_{g,0}^r$:

\begin{equation}
\tilde{\mathcal{D}}_{g,0}^r(X)_{L,\eta} = \left\{ V \in \tilde{\mathcal{D}}_{g,0}^r(X) | V \to \text{gr}^r(V) = L \otimes \eta^p \right\}.
\end{equation}

This is clearly a substack of $\mathcal{D}_{g,0}^r(X)$. The canonical arrow $\tilde{\mathcal{D}}_{g,0}^r \to \mathcal{D}_{g,0}^r$ given by projectivization of the $\text{SL}(r)$-oper data, makes $\tilde{\mathcal{D}}_{g,0}^r$ into a $J_r[X]$-bundle over $\mathcal{D}_{g,0}^r$ (and induces a non-canonical isomorphism of $\tilde{\mathcal{D}}_{g,0}^r(X)_{L,0}$ and $\mathcal{D}_{g,0}^r$). We will use the notation $\mathcal{D}_{g,0}^r(X)_{L,0}$ for $\mathcal{D}_{g,0}^r$ as a reminder of this non-canonical isomorphism. In particular one sees that

\begin{equation}
\deg(\tilde{\mathcal{D}}_{g,0}^r(X)) = r^{2g} \deg(\mathcal{D}_{g,0}^r(X)_{L,0}) = r^{2g} \deg(\mathcal{D}_{g,0}^r(X)),
\end{equation}

where $0 \in J_X[r]$ is the identity element.

6. A CANONICAL ISOMORPHISM

**Theorem 6.1.** If $p > C(r, g)$ and $g \geq 2$ then the natural morphism

\[ \mathcal{D}_{g,0}^r(X)_{L,0} \to \mathcal{D}_{L,0}^{r,0} \]

\[ V \mapsto V \hookrightarrow F_*(L) \to F_*(L)/V \]

is an isomorphism of scheme.

**Proof.** It will suffice to do this at the level of functor of points. Let $S$ be any connected $k$-scheme. Let $V \in \mathcal{D}_{g,0}^r(X)_{L,0}(S)$. This means one has a vector bundle $V$ on $X \times S$ such that $(F \times \text{id}_S)^*(V)$ is a family of $\text{SL}_r$-opers over $S$. Let me note that $F \times \text{id}_S$ is the relative Frobenius of $X \times S$, by [Joshi and Pauly, 2009] one has $V \hookrightarrow (F \times S)_*(\pi_X^*(L))$ and this gives an $S$-valued point of the quot scheme $\mathcal{Q}_{L,0}^r$. Thus $\mathcal{D}_{g,0}^r(X)_{L,0}(S) \subset \mathcal{Q}_{L,0}^r(S)$. On the other hand if $V \in \mathcal{D}_{L,0}^{r,0}$ then this means one has a quotient

\begin{equation}
0 \to V \to (F \times \text{id}_S)_*(\pi_X^*(L)) \to G \to 0.
\end{equation}

Then adjunction gives $(F \times \text{id}_S)_*(V) \to L$ and a connection on the source and the filtration \cite{Joshi, Ramanan, Xia, and Yu, 2006} gives the oper-flag (by Theorem 5.4.1, Prop. 3.4.2 and Remark 5.4.3 of [Joshi and Pauly, 2009]). This gives an oper and $V$ is a family of stable bundles on $X$, thus $V \in \mathcal{D}_{g,0}^r(X)_{L,0}(S)$. Thus I have proved the assertion. \hfill \Box

**Corollary 6.3.** Let $X$ be a smooth, projective curve of genus $g \geq 2$ over an algebraically closed field of characteristic $p > C(r, g)$. Let $L$ be a line bundle fixed earlier. Then

\begin{equation}
\deg(\mathcal{D}_{g,0}^r(X)_{L,0}) = \frac{1}{p^g} \deg(\mathcal{D}_{L,0}^{r,0})
\end{equation}

and

\begin{equation}
\deg(\tilde{\mathcal{D}}_{g,0}^r(X)_{L,0}) = \frac{r^{2g}}{p^g} \deg(\mathcal{D}_{L,0}^{r,0}).
\end{equation}

**Proof.** The proof is immediate from (6.1), (6.5) and (5.2). \hfill \Box
7. Computing the schematic degree

The preceding discussion reduces the problem of computing the schematic degree of $\mathcal{Q}_{g,0}(X)$ to computing the degree of the quot-scheme $\mathcal{Q}^{r,0}$. This is a special case of a more general sort of problem which is very well-understood over complex numbers under the assumption that the ambient bundles (whose) quot-schemes are being studied are very general and the formula for the degree is a special case of the Vafa-Intrilligator Formula (see [Intriligator, 1991], [Holla, 2004]). This formula can be used, using results of [Joshi and Pauly, 2009] and the results of the preceding sections, to find a conjectural formula for the degree of the dormant operatic locus using the reductions made in the preceding section. I show that my conjecture is true for $g = 2, r = 2$ (where the degree of the dormant locus has been computed by different methods and different authors).

Before I begin let us paraphrase what has been achieved in the preceding sections. The problem of computing the degree has been reduced to calculating the degree of the quot-scheme $\mathcal{Q}^{r,0}$ of quotients of $F^*(L)$ of rank $p - r$ and degree equal to $\deg(F^*(L))$. This is a special case of the following problem, studied by [Popa and Roth, 2003], [Lange and Newstead, 2003], [Holla, 2004] and several others (see references to papers by these authors for a longer list).

Let $E$ be a vector bundle of rank $n$ and degree $d$. Let $1 \leq r \leq n$ be an integer. Let

\begin{equation}
\epsilon_{\max}(E, r) = \max_P(\deg(F))
\end{equation}

where the maximum is taken over all subbundles $F \subset E$ of rank $r$. Let

\begin{equation}
s_r(E) = d \cdot r - n \cdot \epsilon_{\max}(E, r).
\end{equation}

Then by a well-known result of [Mukai and Sakai, 1985], one has

\begin{equation}
s_r(E) \leq r(n - r)g
\end{equation}

and one can give a better estimate by a result of [Hirschowitz, 1988]. Under the assumption that $E$ is very general and very-stable of degree $d$ and rank $n$ one has

\begin{equation}
s_r(E) = r(n - r)(g - 1) + \varepsilon,
\end{equation}

where $\varepsilon$ is the unique integer satisfying the following two conditions

\begin{enumerate}
  \item $0 \leq \varepsilon < n$,
  \item $s_r(E) \equiv rd \mod n$.
\end{enumerate}

The number $\varepsilon$, under these assumptions, is the dimension of every irreducible component of the quot-scheme $\mathcal{Q}^{r,\epsilon_{\max}(E, r)}(E)$ of quotients of $E$ of rank $n - r$ and degree $d - \epsilon_{\max}(E, r)$. By maximality of $E$, every such quotient is locally free and so the associated kernel is always a subbundle. In particular if $s_r(E) = r(n - r)(g - 1)$ then the quot-scheme is zero dimensional and is known to be smooth (this is due to [Popa and Roth, 2003]).

Assume now that $s_r(E) = r(n - r)(g - 1)$. Let

\begin{equation}
N(n, d, r, g) = \deg(\mathcal{Q}^{r,\epsilon_{\max}(E, r)}(E)).
\end{equation}

Then one gets

\begin{align*}
s_r(E) &= r(n - r)(g - 1) \\
\epsilon_{\max}(E, r) &= dr - r(n - r)(g - 1).
\end{align*}

In this setting a formula for the degree of this quot-scheme was described by [Holla, 2004] in terms of Gromov-Witten theory; see [Lange and Newstead, 2003] and its references for the rank two case.
Theorem 7.6 (Holla). Let $k = \mathbb{C}$, let $g \geq 1$, $E$ be a very general and very stable bundle of rank and degree as above. Write $d = ar - b$ with $0 \leq b < r$. Then

$$\deg(\mathcal{Q}^{r,0}(E)) = \frac{(-1)^{(r-1)(b-r)(g-1)^2}}{r!} \sum_{\rho_1, \ldots, \rho_r} \left( \frac{\prod_{i=1}^r \rho_i}{\prod_{i \neq j} (\rho_i - \rho_j)^{g-1}} \right)^{b-r},$$

where $\rho_i^0 = 1$, for $1 \leq i \leq r$ and the sum is over tuples $(\rho_1, \ldots, \rho_r)$ with $\rho_i \neq \rho_j$.

In my setup I want to take $E = F_s(L)$ with

$$\deg(L) = (1 - r)(g - 1).$$

Then using

$$\mu(F_s(L)) = \frac{\deg(L)}{p} + \frac{(p-1)(g-1)}{p},$$

one sees that $d = \deg(E)$ is

$$d = (p - r)(g - 1),$$

and $0 \leq \varepsilon < p$ and

$$s_r(E) \equiv r(p - r)(g - 1) \mod p.$$

Thus one gets $r(p - r)(g - 1) + \varepsilon \equiv r(p - r)(g - 1) \mod p$, as $0 \leq \varepsilon < p$ this means $\varepsilon = 0$ (which is as it should be because in our case one of the main results of [Joshi and Pauly, 2009] says that this quot-scheme is finite) and a similar calculation shows that

$$e_{\max}(E, r) = dr - r(p - r)(g - 1)$$

(7.10)

$$= r(p - r)(g - 1) - r(p - r)(g - 1)$$

(7.11)

By [Joshi and Pauly, 2009] the quot-scheme $\mathcal{Q}^{r,0}_L(E)$ is non-empty and every point of this scheme provides such a subbundle of degree zero.

8. The Degree Conjecture

One can now write down the conjectural formula for the degree of the dormant locus.

Conjecture 8.1. Let $k$ be an algebraically closed field of characteristic $p > C(r, g)$ and $X/k$ be a smooth projective curve over $k$ of genus $g \geq 2$. Let $L$ be a line bundle such that $L^r \otimes (\Omega^1_X)^{r-1} = \mathcal{O}_X$. Then the degree

$$\deg(\mathcal{Q}^{r,0}_g(X)) = \frac{1}{p^g}N(p, (p - r)(g - 1), r, g)$$

(8.2)

$$= \frac{1}{p^g}\deg(\mathcal{Q}^{r,0}_L)$$

$$= \frac{1}{p^g} \sum_{\rho_1, \ldots, \rho_r} \left( \frac{\prod_{i=1}^r \rho_i}{\prod_{i \neq j} (\rho_i - \rho_j)^{g-1}} \right)^{r(g-1)},$$

where $\rho_i^0 = 1$, for $1 \leq i \leq r$ and the sum is over tuples $(\rho_1, \ldots, \rho_r)$ with $\rho_i \neq \rho_j$.

Note that I am conjecturing that the formula [Holla, 2004] and the Vafa-Intrilligator formula [Intriligator, 1991], continues to hold in characteristic $p$ and even holds for the quot-scheme of the specific bundle $F_s(L)$ (which is not general).

To see that the exponents are what I claim they are, one notes that here $n = p$ and on writing $d = (p - r)(g - 1) = pa - b$ with $0 \leq b < p$ gives us

$$d = p(g - 1) - r(g - 1)$$

(8.2)
as \( p > C(r, g) \) with \( b = r(g - 1) \), \( a = (g - 1) \). The power of \((-1)\) in Holla’s formula cancels out with these choice of parameters.

9. **The Formula for Genus Two, Rank Two**

The degree of the quot-scheme in this special case \((g = 2, r = 2 \text{ and } X \text{ is ordinary})\) was calculated by \cite{Lange and Pauly, 2008} by an explicit method (which displays the degree in terms Chern classes of a suitable bundles) and also by Mochizuki in \cite{Mochizuki, 1999}, also see \cite{Osserman, 2007} \((X \text{ general})\) by different methods. In \cite{Holla, 2004} the number of maximal subbundles of a generic bundle was computed \((g = 2, r = 2)\) and this note began with my observation that these two numbers agree. Let me recall these known formulae here.

By \cite{Mochizuki, 1999}, \cite{Lange and Pauly, 2008}, \cite{Osserman, 2007} one has

\[(9.1) \quad \text{deg}(\mathcal{D}_2(X)) = \frac{(p^3 - p)}{24}.\]

On the other hand Holla’s formula \cite{Holla, 2004} gives that

\[(9.2) \quad N(n, d, 2, 2) = \frac{n^3(n^2 - 1)}{24},\]

which gives with \( n = p \) and \( d = p - 2 \), and so conjectural the degree of \( \text{deg}(\mathcal{D}^{r,0}_L) \) is \( \text{deg}(\mathcal{D}^{r,0}_L) = N(p, p - 2, 2, 2) \). This gives with \( g = 2 \) that

\[(9.3) \quad \text{deg}(\mathcal{D}_2(X)) = \frac{1}{p^9} \frac{p^3(p^2 - 1)}{24},\]

which is true by the result of \cite{Mochizuki, 1996}, \cite{Lange and Pauly, 2008} and \cite{Osserman, 2007}.

10. **Additional Comments**

I learnt from Wakabayashi’s preprint that in \cite{Liu and Osserman, 2006} it was shown that for \( r = 2 \), \( \text{deg}(\mathcal{D}^{r,0}_{g,0}) \) is a polynomial in \( p \), of degree \( 3g - 3 \), with rational coefficients. This is based on a different approach (see \cite{Wakabayashi, 2013} for more comments on \cite{Liu and Osserman, 2006}). In \cite{Wakabayashi, 2013} one finds many explicit formulae for \( \text{deg}(\mathcal{D}^{r,0}_{g,0}) \) for small \( g \).

The following remarkable fact is proved \((r = 2)\) in \cite{Wakabayashi, 2013}—the formula in Conjecture 8.1 is equivalent to

\[(10.1) \quad \text{deg}(\mathcal{D}^{r,0}_{g,0}) = 2^{-g} \dim(H^0(\mathcal{M}^{r}_X(2, \mathcal{O}_X), \theta^{p-2}))\]

where \( \mathcal{M}^{r}_X(2, \mathcal{O}_X) \) is the moduli space of semistable bundles of rank \( r = 2 \) with trivial determinant and \( \theta \) is the the theta-line bundle on this moduli space. The space on the right hand side of the above is the space of conformal blocks and its dimension is thus the Verlinde formula! (see \cite{Beauville and Laszlo, 1994} for the terminology).

As I was preparing to speak about this topic and Wakabayashi’s work at a meeting in Nice in June 2013 (hosted by Christian Pauly), it seemed reasonable to me that this should hold for all ranks, and so I suggested at the meeting that Wakabayashi’s observed coincidence should hold for all ranks. More precisely

\[(10.2) \quad \dim(\mathcal{D}^{r,0}_{g,0}) = r^{-g} \dim(H^0(\mathcal{M}^{r}_X(r, \mathcal{O}_X), \theta^{p-r})).\]

Christian Pauly verified immediately after the meeting that the right hand side of Conjecture 8.1 is always the dimension of space of conformal blocks (up to the factor). Thus one has a coincidence of two numbers whose genesis is rather complicated but one has no natural explanation for this numerical coincidence. At any rate one may view the above equality as a subtler form of Conjecture 8.1.
This discussion begs the question: what is the most natural explanation for the co-incidence of these two numbers? What hidden relation exists between dormant opers and spaces of conformal blocks? Further let me note that the spaces $H^0(\mathcal{U}(r)_X, \theta^m)$ of conformal blocks satisfy Fusion rules (which allow one to compute their dimensions combinatorially). Thus it seems reasonable to speculate that same sort of Fusion rules insightfully.

Thus it seems reasonable to speculate that same sort of Fusion rules for computing $D_{g,0}$ still lie hidden. One may, of course, infer these rules from the known rules for the right hand side and the observed coincidence, but this approach is not very insightful.

11. Degree of the Frobenius map

Let $\mathcal{U}_X(r, \mathcal{O}_X)$ be the moduli space of semi-stable bundles of rank $r$ and trivial determinant on $X$. In this section I will explain, that the methods of the preceding sections can be also applied to obtain a conjectural formula for the degree of the generic fibre of the Frobenius map, i.e., the rational map $\mathcal{U}_X(r, \mathcal{O}_X) \to \mathcal{U}_X(r, \mathcal{O}_X)$ given by Frobenius pullback $V \mapsto F^*(V)$ (for $V \in \mathcal{U}_X(r, \mathcal{O}_X)$), in the case when this map is generically finite. Let us begin with this generic finiteness result in the ordinary case. This finiteness result is due to [Mehta and Subramanian, 1995]. I reprove it here for convenience.

**Proposition 11.1.** Let $X/k$ be a smooth, projective curve of genus $g \geq 2$ over an algebraically closed field $k$ of characteristic $p > C(r, g)$. If $X$ is ordinary then the rational map $\mathcal{U}_X(r, \mathcal{O}_X) \to \mathcal{U}_X(r, \mathcal{O}_X)$ given by $V \mapsto F^*(V)$ is generically étale.

**Proof.** Let us write $V = \mathcal{O}_X$ for the trivial bundle of rank $r$. Then $\text{End}(V) = \mathcal{O}_X^2$ and the tangent space at the point $[V] \in \mathcal{U}_X(r, \mathcal{O}_X)$ is given by $H^1(X, \text{End}(V))$ and the map on tangent space is the Frobenius map

$$H^1(X, \text{End}(V)) = \oplus H^1(X, \mathcal{O}_X) \to \oplus H^1(X, \mathcal{O}_X) = H^1(X, \text{End}(V)).$$

As $X$ is ordinary $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is injective and so that $\mathcal{U}$ is injective on the tangent space at $[V] \in \mathcal{U}_X(r, \mathcal{O}_X)$. Thus $\mathcal{U}$ is étale at $[V]$. Thus $\mathcal{U}$ is étale in a neighborhood of $V$. Thus at the generic point of $\mathcal{U}_X(r, \mathcal{O}_X)$ one gets a finite separable extension. Thus this proves the assertion. \qed

For the remainder of this section we assume that $X$ is ordinary.

Let me now provide a conjectural description of the fibre over a general point $V' \in \mathcal{U}_X(r, \mathcal{O}_X)$. Since one is dealing with a general bundle $V'$ one can assume that $V'$ is stable. Further let $F^*(V) = V'$. Thus $V$ is in the fibre over $V'$. By the previous proposition and the fact that $V'$ is general this fibre is finite. One can provide a reasonable description of the fibre over $V'$. Since $F^*(V) = V'$ so by adjunction one has a morphism

$$V \to F_* (V'),$$

and by my assumption that $p > C(r, g)$ I have $V \hookrightarrow F_* (V').$

Thus every point of the fibre $V \in \mathcal{U}^{-1}(V')$ gives an injection

$$0 \to V \to F_* (V') \to G \to 0.$$ 

In other words one has a point of the Quotient scheme

$$\mathcal{Q}_V^{r,0} = \text{Quot}^{r,0}(F_*(V')),$$

of $F_*(V')$ consisting of quotients $F_*(V') \to G \to 0$ such that the kernel $V$ of this map has degree zero and rank $r$. 

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of $F_*(V')$ consisting of quotients $F_*(V') \to G \to 0$ such that the kernel $V$ of this map has degree zero and rank $r$.
Proposition 11.6. In the notation as above, every point $V \hookrightarrow F_s(V') \twoheadrightarrow G$ of $\mathcal{Q}^r_{V'}$ as above has the property that $V$ is a maximal subbundle of $F_s(V')$ and in particular $G$ is locally free.

Proof. Let us recall that $\mu(F_s(V')) = \left(1 - \frac{1}{p}\right)(g - 1)$, and by [Sun, 2008] one knows that $F_s(V')$ is stable as $V'$ is stable. Now define, following [Lange and Newstead, 2003], the degree of $r$-stability of $F_s(V')$:

$$s_r(F_s(V')) = \min_{E}(r \cdot \deg(F_s(V')) - p \cdot r \cdot \deg(E))$$

where the minimum is over all subbundle $E \subset F_s(V')$ of rank $r$ (here $pr = \text{rk}(F_s(V'))$).

One would like to compute this number. We know by [Mukai and Sakai, 1985] that $s_r(F_s(V')) \leq r(pr - r)g$. As $V$ is a subbundle of degree zero,

$$s_r(F_s(V')) \leq r\deg(F_s(V')) = r^2(p - 1)(g - 1).$$

On the other hand, by a result of [Hirschowitz, 1988] one knows that

$$s_r(F_s(V')) = r^2(p - 1)(g - 1) + \varepsilon,$$

where $0 \leq \varepsilon < pr$. Thus one sees that $s_r(F_s(V'))$ satisfies

$$r^2(p - 1)(g - 1) \leq s_r(F_s(V')) \leq r^2(p - 1)(g - 1).$$

Thus one has

$$s_r(F_s(V')) = r^2(p - 1)(g - 1).$$

In particular one sees from this that $V$ is a maximal subbundle of $F_s(V')$.

\[ \square \]

Conjecture 11.12. Suppose $X$ is ordinary. For general stable bundle $V'$ of rank $r$ in $\mathcal{M}_X(r, \mathcal{O}_X)$ the quotient scheme $\mathcal{Q}^r_{V'}$ is finite and reduced.

This conjecture is reasonable and is compatible with the sort of results which are known in characteristic zero.

Let us analyze the quotient scheme $\mathcal{Q}^r_{V'}$ further, so one can extract from it the information about the fibre over $V'$. If $\gamma \in J_X[p]$ is a line bundle of order $p$ and if

$$0 \twoheadrightarrow V \twoheadrightarrow F_s(V')$$

then tensoring this with $\gamma$ one gets

$$0 \twoheadrightarrow V \otimes \gamma \twoheadrightarrow F_s(V') \otimes \gamma = F_s(V' \otimes \gamma^p) = F_s(V').$$

Thus one sees that $V \otimes \gamma$ also corresponds to a point of our quot-scheme and so one sees that $\ker(\gamma)$ acts on $\mathcal{Q}^r_{V'}$.

Let

$$\mathcal{Q}^r_{V', \gamma} = \left\{ V \hookrightarrow F_s(V') \mid \det(V) = \gamma^r, \text{rk}(V) = r \right\}.$$ 

Then $\mathcal{Q}^r_{V', \gamma} \subset \mathcal{Q}^r_{V'}$ is a closed subscheme and one has

$$\mathcal{Q}^r_{V'} = \coprod_{\gamma} \mathcal{Q}^r_{V', \gamma}.$$ 

One has a natural morphism

$$\psi^{-1}(V') \rightarrow \mathcal{Q}^r_{V', 0},$$

given by $V \mapsto [V \hookrightarrow F_s(V')]$.

Conjecture 11.15. Assume $X$ is ordinary and $V'$ is a general stable bundle of rank $r$. Then the morphism constructed above (see 11.14) is an isomorphism of smooth schemes.
Thus one sees that the problem of computing the degree of the Frobenius pull-back reduces to the problem of computing the degree of a certain quot-scheme. This leads one to the following conjectural formula for its degree:

**Conjecture 11.16.** Assume $X$ is ordinary. Let $\{\zeta_i\}$ run over all the complex numbers such that $\zeta_i^{pr} = 1$. The degree of the fibre $\deg(V^{-1}(V'))$ over a general stable bundle $V'$ of degree zero is given by

$$
\deg(Q_{r,0}^0) = \frac{(pr)^{r(g-1)}}{p^g} \sum_{(\zeta_1, \ldots, \zeta_r), \zeta_i \neq \zeta_j} \left( \prod_{i=1}^r \zeta_i \right)^{(r-1)(g-1)} \left( \prod_{i \neq j} (\zeta_i - \zeta_j)^{(g-1)} \right).
$$

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