Existence of Discrete Eigenvalues for the Dirichlet Laplacian in a Two-Dimensional Twisted Strip

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Abstract
We study the spectrum of the Dirichlet Laplacian operator in twisted strips on ruled surfaces in any space dimension. It is shown that a suitable twisted effect can create discrete eigenvalues for the operator. In particular, we also study the case where the twisted effect “grows” at infinity while the width of the strip goes to zero. In this situation, we find an asymptotic behavior for the eigenvalues.

Keywords
Dirichlet Laplacian · Unbounded strips · Twisted strips · Discrete eigenvalues

Mathematics Subject Classification 81Q10 · 35P15 · 47F05 · 58J50

1 Introduction
Let \( \Omega \) be a strip on a surface in \( \mathbb{R}^d, \ d \geq 2 \), and denote by \( -\Delta^D_\Omega \) the Dirichlet Laplaceian operator in \( \Omega \). A problem extensively studied in the literature is to find spectral information of \( -\Delta^D_\Omega \). If \( \Omega \) is a bounded strip, it is known that the spectrum \( \sigma(-\Delta^D_\Omega) \) is purely discrete. Otherwise, the existence of discrete eigenvalues is a non-trivial property and it depends on the geometry of \( \Omega \) (Borisov et al. 2001, 2002; Chenaud et al. 2005; Clark and Bracken 1996; Duclos and Exner 1995; Duclos et al. 2001; Exner and Kovařík 2005, 2015; Friedlander and Solomyak 2008a,b, 2009; Krejčířík

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At first, consider the following class of planar strips. Let $r : \mathbb{R} \to \mathbb{R}^2$ be a $C^\infty$ curve parameterized by its arc-length $s$ and denote by $k(s)$ its curvature at the point $r(s)$. Consider the case where $\Omega$ is obtained by moving a bounded segment $(c, d)$ along $r$ with respect to its normal vector field. In the pioneering paper (Exner and Šeba 1989), the authors studied the spectral problem of $-\Delta_{\Omega}^D$. In particular, they proved the existence of discrete spectrum for the operator on the conditions that $k(s) \neq 0$, for some $s \in \mathbb{R}$, and that $k(s)$ decays fast enough at infinity. In addition, the authors assumed some regularity for $r$ and that $d - c$ is small enough. In subsequent studies, the results were improved and generalized (Duclos and Exner 1995; Goldstone and Jaffe 1992; Krejčířík and Kříž 2005). In Goldstone and Jaffe (1992), the authors proved the existence of discrete eigenvalues without the restriction on the width of the strip. In Duclos and Exner (1995) the authors estimated the number of discrete eigenvalues. They proved that this number is finite and bounded by a constant that does not depend on the width of $\Omega$. In Krejčířík and Kříž (2005), the authors proved the existence of discrete spectrum on the conditions that $r$ is of class $C^2$, $k(s) \neq 0$, and $k(s) \to 0$, as $|s| \to \infty$; we also emphasize that the authors made an overview of some new and old results on spectral properties of $-\Delta_{\Omega}^D$, including other boundary conditions in $\partial \Omega$.

In Briet et al. (2019), the authors introduced a new, two-dimensional model of strip to study the spectral problem of $-\Delta_{\Omega}^D$. In that paper, $\Omega$ is a strip in $\mathbb{R}^2$ which is built by translating a segment oriented in a constant direction along an unbounded curve in the plane. The spectrum of the operator $-\Delta_{\Omega}^D$ was carefully studied and the model covers different effects: purely essential spectrum, discrete spectrum or a combination of both.

One can consider strips embedded in a Riemannian manifold instead of the Euclidean space. For example, suppose that $\Omega$ is a strip of constant width which is defined as a tubular neighborhood of an infinite curve in a two-dimensional Riemannian manifold. This situation was considered in Krejčířík (2003). In particular, the author proved that the discrete spectrum of $-\Delta_{\Omega}^D$ is nonempty for non-negatively curved strips.

Strips on ruled surfaces are also regions of great interest (Krejčířík and de Aldecoa 2018; Krejčířík and Zahradová 2020; Verri 2019); roughly speaking, a ruled surface is generated by straight lines translating along a curve in the Euclidean space. Consider the case where $\Omega$ is a twisted strip composed of segments translated along the straight line in $\mathbb{R}^3$ with respect to a rotation angle. In Krejčířík and de Aldecoa (2018), assuming that the twisted effect diverges at infinity, the authors studied the spectral problem of $-\Delta_{\Omega}^D$. In particular, they proved that this kind of geometry can create discrete eigenvalues. In a similar situation, in Verri (2019) was proved that the discrete spectrum of $-\Delta_{\Omega}^D$ is nonempty since the twisted effect “grows” at infinity while the width of $\Omega$ goes to zero.

Some results for strips on ruled surfaces can be extended to higher dimensions. In other words, consider $\Omega$ a twisted and bent two-dimensional strip embedded in $\mathbb{R}^d$ with $d \geq 3$. In Krejčířík and Zahradová (2020), the spectrum of the Dirichlet Laplacian $-\Delta_{\Omega}^D$ was carefully studied. The authors also proved that, in the limit when the width of the strip tends to zero, the Dirichlet Laplacian converges in the norm.
resolvent sense to a one-dimensional Schrödinger operator whose potential depends on the deformations of twisting and bending. An interesting point is that the geometric construction of those strips was performed with a \textit{relatively parallel adapted frame} instead of a Frenet frame. In fact, it is known that a Frenet frame of a curve does not need to exist. However, the authors proved that a relatively parallel adapted frame always exists for an arbitrary curve. The goal of this work is to find additional information about the spectrum of the Dirichlet Laplacian in this situation. In the next paragraphs, we present the formal construction of the strip and more details of the problem.

Let $\Gamma : \mathbb{R} \to \mathbb{R}^{n+1}$, $n \geq 1$, be a curve of class $C^{1,1}$ parameterized by its arc-length $s$, i.e., $|\Gamma'(s)| = 1$, for all $s \in \mathbb{R}$. The vector $T(s) := \Gamma'(s)$ denotes its unitary tangent vector at the point $\Gamma(s)$. Note that $T$ is a locally Lipschitz continuous function which is differentiable almost everywhere in $\mathbb{R}$. The number $k(s) := |\Gamma''(s)|$, $s \in \mathbb{R}$, is called the curvature of $\Gamma$ at the position $\Gamma(s)$. In Appendix A of Krejčířik and Zahradová (2020), the authors proved the existence of a relatively parallel adapted frame for the curve $\Gamma$. More exactly, it was shown that there exist $n$ almost-everywhere differentiable normal vector fields $N_1, \ldots, N_n$ so that

\[
\begin{pmatrix}
 T \\
 N_1 \\
 \vdots \\
 N_n
\end{pmatrix}' = \begin{pmatrix}
 0 & k_1 & \cdots & k_n \\
 -k_1 & 0 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots \\
 -k_n & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
 T \\
 N_1 \\
 \vdots \\
 N_n
\end{pmatrix},
\]

where $k_j : \mathbb{R} \to \mathbb{R}$, $j \in \{1, \ldots, n\}$, are locally bounded functions. In particular, the vector $(k_1, \ldots, k_n)$ satisfies $k_1^2 + \cdots + k_n^2 = k^2$.

Now, let $\Theta_j : \mathbb{R} \to \mathbb{R}$, $j \in \{1, \ldots, n\}$, be functions of class $C^{0,1}$ so that

\[
\Theta_1^2 + \cdots + \Theta_n^2 = 1,
\]

and define

\[
N_\Theta := \Theta_1 N_1 + \cdots + \Theta_n N_n.
\]

Consider the region

\[
\Omega_\varepsilon := \{ \Gamma(s) + N_\Theta(s) \varepsilon t : (s, t) \in \mathbb{R} \times (-1, 1) \};
\]

$\Omega_\varepsilon$ is obtained by translating the segment $(-\varepsilon, \varepsilon)$ along $\Gamma$ with respect to a normal field (2). Let $-\Delta^D_{\Omega_\varepsilon}$ be the Dirichlet Laplacian operator in $\Omega_\varepsilon$. More precisely, $-\Delta^D_{\Omega_\varepsilon}$ is defined as the self-adjoint operator associated with the quadratic form

\[
a_\varepsilon(\varphi) = \int_{\Omega_\varepsilon} |\nabla \varphi|^2 dx, \quad \text{dom } a_\varepsilon = H^1_0(\Omega_\varepsilon);
\]

$\nabla \varphi$ denotes the gradient of $\varphi$ correspondent to the metric induced by the immersion defined in (13), Sect. 2. For simplicity, we denote $-\Delta_\varepsilon := -\Delta^D_{\Omega_\varepsilon}$.
Define the function $\Theta : \mathbb{R} \rightarrow \mathbb{R}^n$,

$$\Theta(s) := (\Theta_1(s), \ldots, \Theta_n(s)),$$

and write $|\Theta'(s)| := (\Theta_1^2(s) + \cdots + \Theta_n^2(s))^{1/2}$. Note that $\Theta \in C^{0,1}(\mathbb{R}; \mathbb{R}^n)$. As in Krejčiřík and Zahradová (2020), $\Theta$ is called a twisting vector; if $\Theta' = 0$, $\Omega_\varepsilon$ is called untwisted or purely bent strip; if $k \cdot \Theta := k_1 \Theta_1 + \cdots + k_n \Theta_n = 0$, $\Omega_\varepsilon$ is called unbent or purely twisted strip. Geometrically, interpreting $\Gamma$ as a curve in $\Omega_\varepsilon$, $k \cdot \Theta$ is the geodesic curvature of $\Gamma$: $-|\Theta'|^2/(1 + |\Theta'|^2 \varepsilon^2 t^2)^2$ is the Gauss curvature of $\Omega_\varepsilon$. One can refer to Krejčiřík and Zahradová (2020) for a detailed geometric description of $\Omega_\varepsilon$.

Let $(\pi/2\varepsilon)^2$ be the first eigenvalue of the Dirichlet Laplacian $-\Delta^D_{(-\varepsilon,\varepsilon)}$ in $L^2(-\varepsilon, \varepsilon)$. In Krejčiřík and Zahradová (2020), the authors presented a detailed study of the spectrum of $-\Delta_\varepsilon$. In particular, under the conditions $k \cdot \Theta$, $|\Theta'| \in L^\infty(\mathbb{R})$, $\varepsilon \|k \cdot \Theta\|_{L^\infty(\mathbb{R})} < 1$, and $(k \cdot \Theta)(s) \to 0$, $|\Theta'(s)| \to 0$, as $|s| \to \infty$, they proved that:

(i) $\sigma_{ess}(-\Delta_\varepsilon) = [(\pi/2\varepsilon)^2, \infty]$;
(ii) if $\Theta' = 0$ and $k \cdot \Theta \neq 0$, then the discrete spectrum of $-\Delta_\varepsilon$ is nonempty;
(iii) if $k \cdot \Theta = 0$ and $\Theta' \neq 0$ with $\varepsilon \|\Theta'|_{L^\infty(\mathbb{R})} \leq \sqrt{2}$, then the discrete spectrum of $-\Delta_\varepsilon$ is empty.

The conditions in (i) and (iii) show that a local twisted effect does not create discrete eigenvalues for $-\Delta_\varepsilon$. Then, a natural question is to ask if some appropriated twisted effect can be used to create discrete eigenvalues for the operator. This problem is the subject of this work.

In the first part of this paper, we assume that

$$\Theta \in C^{1,1}(\mathbb{R}; \mathbb{R}^n), \quad (k \cdot \Theta)(s) = 0, \quad \text{and} \quad |\Theta'(s)| = \gamma - \beta(s), \quad \forall s \in \mathbb{R}, \quad (5)$$

where $\gamma$ is a positive number, and $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous, almost-everywhere differentiable function with compact support so that $\beta' \in L^\infty(\mathbb{R})$. Geometrically, the second and third conditions in (5) mean that $\Omega_\varepsilon$ is a purely twisted strip; the properties of the function $\beta(s)$ imply that the twisted effect locally slows down. In this situation, we will find information about the spectrum of $-\Delta_\varepsilon$. In particular, we will show that the conditions in (5) can create discrete eigenvalues for $-\Delta_\varepsilon$. Note that the condition $k \cdot \Theta = 0$ does not necessarily imply that $\Gamma$ is a straight line.

At first, we study the essential spectrum of $-\Delta_\varepsilon$. The strategy is based on a direct integral decomposition of the operator; see Sect. 3. In particular, consider the one-dimensional operator

$$D_\varepsilon(0) := -\frac{\partial^2}{\varepsilon^2} + Y_\varepsilon^0(t), \quad \text{dom } D_\varepsilon(0) = H^1_0(-1, 1) \cap H^2(-1, 1), \quad (6)$$

where $\partial_t := \partial / \partial t$,

$$\quad Y_\varepsilon^0(t) := -\frac{3\gamma^4 \varepsilon^2 t^2}{4h_\varepsilon^4(t)} + \frac{\gamma^2}{2h_\varepsilon^2(t)}, \quad h_\varepsilon(t) := \sqrt{1 + \gamma^2 \varepsilon^2 t^2}. \quad (7)$$
Since \( Y_0^0 \in C^\infty[-1, 1] \), \( D_\varepsilon(0) \) has compact resolvent. Denote by \( \lambda_{\varepsilon, 1}(0) \) its first eigenvalue and by \( u_{\varepsilon, 1}^0 \) the corresponding orthonormal eigenfunction; \( \lambda_{\varepsilon, 1}(0) \) is simple. Take \( \varepsilon_0 > 0 \) so that
\[
Y_0^0 > 0, \quad \forall \varepsilon \in (0, \varepsilon_0).
\]
(8)
Thus, for each \( \varepsilon \in (0, \varepsilon_0) \), \( u_{\varepsilon, 1}^0 \) can be chosen to be real and positive in \((-1, 1)\); see, e.g., Chapter 6 of Evans (2010) for more details. The condition in (8) will be useful in the proof of Proposition 3 in Sect. 3.

Now, we have conditions to state the following result.

**Theorem 1** Assume the conditions (1), (5) and (8). Then,
\[
\sigma_{ess}(-\Delta_\varepsilon) = [\lambda_{\varepsilon, 1}(0), \infty).
\]

Theorem 1 is a consequence of Propositions 2 and 3 of Sect. 3. In that same section, Remark 4 shows an asymptotic behavior for the sequence \{\( \lambda_{\varepsilon, 1}(0) \)\}; in fact, \( \varepsilon^2 \lambda_{\varepsilon, 1}(0) \to (\pi/2)^2 \), as \( \varepsilon \to 0 \).

The next results provide sufficient conditions to the existence of (discrete) eigenvalues for \(-\Delta_\varepsilon \) below \( \lambda_{\varepsilon, 1}(0) \). Let \( s_0 > 0 \) be so that \( \text{supp} \beta \subset [-s_0, s_0] \).

**Theorem 2** Assume the conditions (1), (5) and (8). If \( \int_{-s_0}^{s_0} (|\Theta'(s)|^2 - \gamma^2)ds < 0 \), then there exists \( \varepsilon_1 > 0 \) so that, for each \( \varepsilon \in (0, \varepsilon_1) \),
\[
\inf \sigma(-\Delta_\varepsilon) < \lambda_{\varepsilon, 1}(0),
\]
i.e., \( \sigma_{dis}(-\Delta_\varepsilon) \neq \emptyset \).

The assumption \( \int_{-s_0}^{s_0} (|\Theta'(s)|^2 - \gamma^2)ds < 0 \) in Theorem 2 is not a necessary condition to create discrete eigenvalues for \(-\Delta_\varepsilon \). For example,

**Theorem 3** Assume the conditions (1), (5) and (8). If \( \int_{-s_0}^{s_0} (|\Theta'(s)|^2 - \gamma^2)ds = 0 \), then there exists \( \varepsilon_2 > 0 \) so that, for each \( \varepsilon \in (0, \varepsilon_2) \),
\[
\inf \sigma(-\Delta_\varepsilon) < \lambda_{\varepsilon, 1}(0),
\]
i.e., \( \sigma_{dis}(-\Delta_\varepsilon) \neq \emptyset \).

Theorems 2 and 3 show how an appropriated twisted effect can create discrete eigenvalues for \(-\Delta_\varepsilon \); proofs are presented in Sect. 4.

Now, the goal of the second part of this paper is to discuss a different situation where this phenomenon can also be obtained.

As already commented in the previous paragraphs, in Krejčířík and de Aldecoa (2018); Verri (2019), the authors studied the Dirichlet Laplacian restricted in a two-dimensional twisted (straight) strip in \( \mathbb{R}^3 \). We emphasize the paper (Krejčířík and de Aldecoa 2018) where the authors considered the region
\[
S := \{(s, t \cos \theta(s), t \sin \theta(s)) : (s, t) \in \mathbb{R} \times (a_1, a_2)\},
\]
(9)
where \( \theta : \mathbb{R} \to \mathbb{R} \) is a (locally) Lipschitz continuous function and \( a_1, a_2 \in \mathbb{R} \). One of the results of that work shows that the assumptions

\[
\lim_{|x| \to \infty} |\theta'(s)| = \infty, \quad a_1 a_2 \leq 0, \tag{10}
\]

ensure that the discrete spectrum of the Dirichlet Laplacian in \( S \) is nonempty. As noted by the authors, an interesting geometric fact is that the conditions in (10) show that the two-dimensional strip \( S \) looks at infinity like a three-dimensional tube of annular cross-section \( \{ x \in \mathbb{R}^2 : 0 < |x| < r \} \), where \( r := \max(|a_1|, |a_2|) \). Inspired by Krejčířík and de Aldecoa (2018), in Verri (2019) the author considered a twisted strip similar to that in (9). However, in that work, the twisted effect “grows” at infinity while the width of the strip goes to zero. Then, since the strip is thin enough, it was shown that the discrete spectrum of the Dirichlet Laplacian is nonempty and an asymptotic behavior for the eigenvalues was found. In the next paragraphs, we present an adaptation of the results of Verri (2019) for the model of strips treated in this work.

In this new situation, for \( n \geq 1 \), assume that \( \Theta : \mathbb{R} \to \mathbb{R}^n \) is a \( C^{1,1} \) function, which satisfies (1), and

(I) \( \lim_{|x| \to \infty} |\Theta'(s)| = \infty; \)

(II) \( |\Theta'| \) is decreasing in \((-\infty, 0)\) and increasing in \((0, \infty)\).

Fix a number \( 0 < a < 1/3 \). For each \( \varepsilon > 0 \) small enough, let \( \nu_1(\varepsilon) < 0 \) and \( \nu_2(\varepsilon) > 0 \) so that

\[
|\Theta'(\nu_i(\varepsilon))| = \frac{1}{\varepsilon a}, \quad i \in \{1, 2\}. \tag{11}
\]

Define \( I_{\varepsilon} := (\nu_1(\varepsilon), \nu_2(\varepsilon)) \) and let \( \Theta_{\varepsilon} : \mathbb{R} \to \mathbb{R}^n \) be a function of class \( C^{1,1} \) so that

(III) \( \Theta_{\varepsilon}(s) = \Theta(s), \) for all \( s \in I_{\varepsilon}; \)

(IV) \( |\Theta'_{\varepsilon}(s)| \leq |\Theta'(s)|, \) for all \( s \in \mathbb{R}; \)

(V) \( |\Theta''_{\varepsilon}(s)| \) is non-increasing in \((-\infty, 0)\) and non-decreasing in \((0, +\infty)\).

Now, assume that the sequence \( \{ \Theta_{\varepsilon} \}_{\varepsilon} \) satisfies

(VI) there exists \( K > 0 \) so that

\[
|\Theta'_{\varepsilon}(s)| \leq \frac{K}{\varepsilon a}, \quad |\Theta''_{\varepsilon}(s)| \leq \frac{K}{\varepsilon b}, \quad |\Theta'''_{\varepsilon}(s)| \leq \frac{K}{\varepsilon c}, \quad \forall s \in \mathbb{R},
\]

for all \( \varepsilon > 0 \) small enough, where \( b, c \) are real numbers so that \( b < 1, \ a + c < 2; \)

(VII) \( |\Theta'_{\varepsilon}(s)| \leq |\Theta'_{\varepsilon'}(s)|, \) for all \( s \in \mathbb{R}, \) if \( \varepsilon > \varepsilon'. \)

Finally, we use the notation \( \Theta_{\varepsilon} := (\Theta_{\varepsilon}^1, \ldots, \Theta_{\varepsilon}^n) \), and we assume that

(VIII) \( (\Theta_{\varepsilon}^1)^2 + \cdots + (\Theta_{\varepsilon}^n)^2 = 1. \)

For each \( \varepsilon > 0 \) small enough, let \( \Gamma_{\varepsilon} : \mathbb{R} \to \mathbb{R}^{n+1} \) be a curve of class \( C^{1,1} \) whose curvature \( k_{\varepsilon} \) satisfies

(IX) \( \text{supp} \ k_{\varepsilon} \subset I_{\varepsilon}, \) and \( (k_{\varepsilon} \cdot \Theta_{\varepsilon})(s) = 0, \) for all \( s \in \mathbb{R}. \)

The normal vector fields of \( \Gamma_{\varepsilon} \) are denoted by \( N_{\varepsilon}^1, \ldots, N_{\varepsilon}^n, \) and \( N_{\varepsilon \Theta}^i := \Theta_{\varepsilon}^i N_{\varepsilon}^1 + \cdots + \Theta_{\varepsilon}^n N_{\varepsilon}^n. \)
Consider the strip
\[ \widetilde{\Omega}_\varepsilon := \{ \Gamma_\varepsilon(s) + N_{\widetilde{\Omega}_\varepsilon}(s) \varepsilon t : (s, t) \in \mathbb{R} \times (-1, 1) \}. \]

Geometrically, \( \widetilde{\Omega}_\varepsilon \) is a locally twisting (and locally bending) strip for which the twisted effect “grows” at infinity while its width goes to zero.

Let \(-\Delta_\Omega^D\) be the Dirichlet Laplacian operator in \( \widetilde{\Omega}_\varepsilon \), i.e., the self-adjoint operator associated with the quadratic form
\[ \tilde{a}_\varepsilon(\phi) = \int_{\widetilde{\Omega}_\varepsilon} |\nabla \phi|^2 \, dx, \quad \text{dom} \, \tilde{a}_\varepsilon = H^1_0(\widetilde{\Omega}_\varepsilon). \]  (12)

For simplicity, write \(-\tilde{\Delta}_\varepsilon := -\Delta_\Omega^D\).

**Remark 1** Let \( T \) be a self-adjoint operator that is bounded from below. We denote by \( \{ \lambda_j(T) \}_{j \in \mathbb{N}} \) the non-decreasing sequence of numbers corresponding to the spectral problem of \( T \) according to the Min-Max Principle; see, for example, Theorem XIII.1 in Reed and Simon (1978).

Let \( \Pi(\varepsilon) \) be the infimum of the essential spectrum of \(-\tilde{\Delta}_\varepsilon\). Denote by \( N(\varepsilon) \leq \infty \) the number of eigenvalues \( \lambda_j(-\tilde{\Delta}_\varepsilon) \) of \(-\tilde{\Delta}_\varepsilon\) below \( \Pi(\varepsilon) \). Let \(-\Delta_\mathbb{R}\) be the one-dimensional Laplacian and consider the self-adjoint operator \(-\Delta_\mathbb{R} + (|\Theta'(s)|^2/2)1\) acting in \( L^2(\mathbb{R}) \); \( 1 \) denotes the identity operator in \( L^2(\mathbb{R}) \). Due to the condition (I), this operator has purely discrete spectrum. In Sect. 5 of this work, we prove the following result.

**Theorem 4** Assume the conditions (I)–(IX). For \( \varepsilon > 0 \) small enough, the discrete spectrum \( \sigma_{dis}(-\tilde{\Delta}_\varepsilon) \) is nonempty and \( N(\varepsilon) \to \infty \), as \( \varepsilon \to 0 \). Furthermore, for each \( j \in \mathbb{N} \),
\[ \lim_{\varepsilon \to 0} \left[ \lambda_j(-\tilde{\Delta}_\varepsilon) - \left( \frac{\pi}{2\varepsilon} \right)^2 \right] = \lambda_j \left( -\Delta_\mathbb{R} + \frac{|\Theta'(s)|^2}{2}1 \right). \]

Estimating the number of discrete eigenvalues of the Dirichlet Laplacian operator in unbounded strips is also an interesting problem (Dauge et al. 2012; Dauge and Raymond 2012; Dauge et al. 2015; Verri 2019). In the case of curved strips in \( \mathbb{R}^2 \), in Duclos and Exner (1995), the authors found that this number is finite and bounded by a constant that does not depend on the width of the strip. Now, fix \( \theta \in (0, \pi/2) \) and consider the “broken” strip defined by \( \{(s, t) \in \mathbb{R}^2 : s \tan \theta < |t| < (s + \pi/\sin \theta) \tan \theta \} \). In Dauge et al. (2012), the authors ensured the existence of discrete eigenvalues for the Dirichlet Laplacian operator in that region. They proved that there is a finite number of them and this number can be arbitrarily large if \( \theta \) is taken small enough.

As a consequence of Theorem 4, the number of discrete eigenvalues of \(-\tilde{\Delta}_\varepsilon\) grows when the width of the strip goes to zero. In Verri (2019), a similar result to Theorem 4 was obtained as a consequence of a convergence in the norm resolvent sense of the
operators associated to the problem. In this text, we present a simpler proof where the strategy is based on to find upper and lower bounds for the eigenvalues $λ_j(−Δ_ε)$.

We finish this introduction with some remarks and examples of the model presented.

**Remark 2** Theorem 4 shows that the locally bending effect imposed by (IX) does not affect the final result.

**Example 1** A simple example of a family of curves $\{Γ_ε\}_ε$ satisfying (IX) is the following. Let $Θ$ be a function satisfying (I), and let $Γ : \mathbb{R} → \mathbb{R}^{n+1}$ be a curve of class $C^{2,1}$ whose curvature $k$ has compact support and satisfies $(k \cdot Θ)(s) = 0$, for all $s \in \mathbb{R}$. Define $Γ_ε := Γ$, for all $ε > 0$ small enough. Then, the condition (IX) is satisfied for all $ε > 0$ small enough.

**Remark 3** In the case $n = 2$, conditions (I) and (I) imply that $Θ$ can be written as $Θ = (\cos(ψ), \sin(ψ))$, for some function $ψ ∈ C^{1,1}(\mathbb{R}; \mathbb{R})$ so that $|ψ′(s)| → \infty$, as $|s| → \infty$. In the case $n = 3$, due to the condition (I), $Θ$ can be written as $Θ = (\cos(φ) \cos(ψ), \sin(φ) \cos(ψ), \sin(ψ))$, for functions $φ, ψ ∈ C^{1,1}(\mathbb{R}; \mathbb{R})$. Since $|Θ′| = \sqrt{(φ′)^2 \cos^2(ψ) + (ψ′)^2}$, the condition (I) is satisfied, for example, if $|ψ′(s)| → \infty$, as $|s| → \infty$, or if $ψ(s) ∈ (c_1, c_2) ⊂ (0, π/2)$, for all $s \in \mathbb{R}$, and $|ψ′(s)| → \infty$, as $|s| → \infty$.

**Example 2** Consider $Θ : \mathbb{R} → \mathbb{R}^2$ defined by $Θ(s) = (\cos(s^2), \sin(s^2))$. Some calculations show that $|Θ′(s)| = 2|s|, s \in \mathbb{R}$; $Θ$ satisfies (I)–(II). Fix a number $0 < a < 1/3$. For each $ε > 0$ small enough, define the functions $α_ε : \mathbb{R} → \mathbb{R}$,

$$α_ε(s) := \begin{cases} (-2e^a s - 1)/ε^{2a}, & s \leq -1/ε^a, \\ s^2, & s \in (-1/ε^a, 1/ε^a), \\ (2e^a s - 1)/ε^{2a}, & s \geq 1/ε^a, \end{cases}$$

and $Θ_ε(s) := (\cos(α_ε(s)), \sin(α_ε(s))), s \in \mathbb{R}$. Taking $b = 2a$ and $c = 3a$, the conditions in (VI) are satisfied. In fact, the sequence $\{Θ_ε\}_ε$ satisfies the conditions (III)–(VIII).

**Example 3** Consider the function $Θ : \mathbb{R} → \mathbb{R}^3$ defined by

$$Θ(s) = \cos(s^2) \cos \left( \frac{1}{1+s^2} \right), \sin(s^2) \cos \left( \frac{1}{1+s^2} \right), \sin \left( \frac{1}{1+s^2} \right).$$

Note that the condition (I) is satisfied and the function

$$|Θ′(s)| = \sqrt{4s^2 \cos^2 \left( \frac{1}{1+s^2} \right) + \frac{4s^2}{(1+s^2)^2}}, \quad s \in \mathbb{R},$$

satisfies (I)–(II). Fix a number $0 < a < 1/3$. For each $ε > 0$ small enough, take $ν(ε) > 0$ so that $|Θ′(±ν(ε))| = 1/ε^a$; then, there exists $K > 0$ so that $|ν(ε)| ≤ K/ε^a$, \[ Springer \]
for all $\varepsilon > 0$ small enough. Let $\alpha_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by
\[
\alpha_\varepsilon (s) := \begin{cases} 
-2\nu(\varepsilon) s - \nu(\varepsilon)^2, & s \leq -\nu(\varepsilon), \\
\nu(\varepsilon)^2, & s \in (-\nu(\varepsilon), \nu(\varepsilon)), \\
2\nu(\varepsilon) s - \nu(\varepsilon)^2, & s \geq \nu(\varepsilon);
\end{cases}
\]

note that $\alpha_\varepsilon \in C^{1,1}(\mathbb{R}; \mathbb{R})$. Now, define
\[
\Theta_\varepsilon (s) := \left( \cos(\alpha_\varepsilon(s)) \cos \left( \frac{1}{1+s^2} \right), \sin(\alpha_\varepsilon(s)) \cos \left( \frac{1}{1+s^2} \right), \sin \left( \frac{1}{1+s^2} \right) \right), 
\]

$s \in \mathbb{R}$.

The sequence $\{\Theta_\varepsilon\}_\varepsilon$ satisfies the conditions (III)–(VII), where $b = 2a$ and $c = 3a$.

This paper is organized as follows. In Sect. 2 we present some details of the construction of the region in (3) and we make an usual change of coordinates in the quadratic form in (4). Sections 3 and 4 are dedicated to study the essential and discrete spectrum of $-\Delta_\varepsilon$, respectively. In Sect. 5, we study the spectral problem of $-\tilde{\Delta}_\varepsilon$. In Appendix are presented results that are useful in this text. Along the text, $K$ is used to denote different constants.

### 2 Geometry of the Region and Change of Coordinates

Recall the region $\Omega_\varepsilon$ given by (3) in the Sect. 1 and the straight strip $\Lambda := \mathbb{R} \times (-1, 1)$.

In this section we identify $\Omega_\varepsilon$ with the Riemannian manifold $(\Lambda, G_\varepsilon)$, where $G_\varepsilon$ is given by (14), below. After that, we perform usual changes of coordinates in the quadratic form $a_\varepsilon(\varphi)$.

Consider the map
\[
L_\varepsilon : \mathbb{R}^2 \longrightarrow \mathbb{R}^{n+1}, \quad (s, t) \longmapsto \Gamma(s) + N_\Theta(s)\varepsilon t.
\]

(13)

We have $\Omega_\varepsilon = L_\varepsilon(\Lambda)$. Define the metric $G_\varepsilon := \nabla L_\varepsilon \cdot (\nabla L_\varepsilon)^\perp$. Some calculations show that
\[
G_\varepsilon = \begin{pmatrix} f_\varepsilon^2 & 0 \\
0 & \varepsilon^2 \end{pmatrix}, \quad f_\varepsilon(s, t) := \sqrt{1 + |\Theta'(s)|^2 \varepsilon^2 t^2}.
\]

(14)

Let $J_\varepsilon$ be the Jacobian matrix of $L_\varepsilon$. One has $\det J_\varepsilon = |\det G_\varepsilon|^{1/2} = \varepsilon f_\varepsilon > 0$, for all $(s, t) \in \Lambda$. If $\Gamma$ and $\Theta$ are smooth functions, the map $L_\varepsilon : \Lambda \longrightarrow \Omega_\varepsilon$ is a local smooth diffeomorphism. Then, $\Omega_\varepsilon$ can be identified with the Riemannian manifold $(\Lambda, G_\varepsilon)$. However, as mentioned in the Sect. 1 of this work, the assumptions about $\Gamma$ and $\Theta$ are more general. At first, one has

**Proposition 1** Assume that $\Gamma \in C^{1,1}(\mathbb{R}; \mathbb{R}^{n+1})$ and $\Theta \in C^{0,1}(\mathbb{R}; \mathbb{R}^n)$. Then, the map $L_\varepsilon : \Lambda \longrightarrow \Omega_\varepsilon$ is a local $C^{0,1}$-diffeomorphism.

The proof of this result can be found in Krejčířík and Zahradová (2020). We emphasize that in that work, $k \cdot \Theta = 0$ does not necessarily hold and Proposition 1 is proven under the additional assumptions that $k \cdot \Theta \in L^\infty(\mathbb{R})$ and $\varepsilon \|k \cdot \Theta\|_{L^\infty(\mathbb{R})} < 1$. 

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In particular, Proposition 1 ensures that $\mathcal{L}_\varepsilon$ is a $C^{0,1}$-immersion. In addition, assume that $\mathcal{L}_\varepsilon$ is injective. Thus, the strip $\Omega_\varepsilon$ does not self-intersect and it is interpreted as an immersed submanifold in $\mathbb{R}^{n+1}$. As a consequence, $(\Lambda, \mathcal{G}_\varepsilon)$ is an abstract Riemannian manifold.

Now, we perform a change of coordinates so that the quadratic form $a_\varepsilon(\varphi)$ starts to act in the Hilbert space $L^2(\Omega)$ (with the usual metric of $\mathbb{R}^2$) instead of $L^2(\Omega_\varepsilon)$. At first, consider the unitary operator

$$U_\varepsilon : L^2(\Omega) \longrightarrow L^2(\Lambda, f_\varepsilon \, ds \, dt),$$

and define the quadratic form

$$b_\varepsilon(\psi) := a_\varepsilon \left( U_\varepsilon^{-1} \psi \right) = \int_\Lambda \langle \nabla \psi, \mathcal{G}_\varepsilon^{-1} \nabla \psi \rangle f_\varepsilon \, ds \, dt + \frac{1}{f_\varepsilon^2} \int_\Lambda |\partial_t \psi|^2 f_\varepsilon \, ds \, dt,$$

$$\text{dom } b_\varepsilon := U_\varepsilon \left( H^1_0(\Omega_\varepsilon) \right); \partial_t = \partial/\partial_t \text{ and } \partial_s = \partial/\partial_s. \text{ Then, consider}$$

$$V_\varepsilon : L^2(\Lambda) \longrightarrow L^2(\Lambda, f_\varepsilon \, ds \, dt),$$

$$\psi \mapsto f^{-1/2}_\varepsilon \psi,$$

which is also a unitary operator, and, finally, define

$$c_\varepsilon(\psi) := b_\varepsilon \left( V_\varepsilon \psi \right) = \int_\Lambda \frac{1}{f_\varepsilon^2} \left| \partial_s \psi - \frac{\partial_s f_\varepsilon}{2 f_\varepsilon} \psi \right|^2 ds \, dt + \frac{1}{f_\varepsilon^2} \int_\Lambda |\partial_t \psi|^2 ds \, dt$$

$$+ \int_\Lambda V_\varepsilon |\psi|^2 ds \, dt,$$

where

$$V_\varepsilon(s, t) := -\frac{3|\Theta'(s)|^4 \varepsilon^2 t^2}{4 f_\varepsilon^4(s, t)} + \frac{|\Theta'(s)|^2}{2 f_\varepsilon^2(s, t)}.$$

$\text{dom } c_\varepsilon = V_\varepsilon^{-1}(U_\varepsilon(H^1_0(\Omega_\varepsilon)))$. Due to the conditions in (5), one has $\text{dom } c_\varepsilon = H^1_0(\Lambda)$. Denote by $C_\varepsilon$ the self-adjoint operator associated with the quadratic form $c_\varepsilon(\psi)$.

### 3 Essential Spectrum

This section is dedicated to prove Theorem 1. Recall the functions $h_\varepsilon$ and $Y_\varepsilon^0$ defined by (7) in the Sect. 1. Consider the quadratic form

$$d_\varepsilon(\psi) := \int_\Lambda \frac{|\partial_s \psi|^2}{h_\varepsilon^2} ds \, dt + \frac{1}{\varepsilon^2} \int_\Lambda |\partial_t \psi|^2 ds \, dt + \int_\Lambda Y_\varepsilon^0 |\psi|^2 ds \, dt, \quad \text{dom } d_\varepsilon := H^1_0(\Lambda).$$
Denote by $D_\varepsilon$ the self-adjoint operator associated with $d_\varepsilon(\psi)$. We start with the following result.

**Proposition 2** Assume the conditions in (5). Then, $\sigma_{\text{ess}}(C_\varepsilon) = \sigma_{\text{ess}}(D_\varepsilon)$.

The proof of this result is presented in Appendix. As a consequence, we start to study the essential spectrum of $D_\varepsilon$.

Let $F_s : L^2(\Lambda) \to L^2(\Lambda)$ be the Fourier transform with respect to $s$. $F_s$ is a unitary operator and, for functions in $L^1(\Lambda)$, its action is given by

$$(F_s \psi)(p, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ips} \psi(s, t) ds.$$  

Then, the operator $\hat{D}_\varepsilon := F_s D_\varepsilon F_s^{-1}$ admits the direct integral decomposition

$$\hat{D}_\varepsilon = \int_{\mathbb{R}} D_\varepsilon(p) \, dp,$$

where, for each $p \in \mathbb{R}$, $D_\varepsilon(p)$ is the self-adjoint operator associated with the quadratic form

$$d_\varepsilon(p)(v) := \frac{1}{\varepsilon^2} \int_{-1}^1 |\partial_t v|^2 dt + \int_{-1}^1 Y_\varepsilon^p |v|^2 dt,$$

with $\text{dom} \, d_\varepsilon(p) = H^1_0(-1, 1)$, the case $p = 0$ corresponds to the operator defined by (6) in the Sect. 1. Since $Y_\varepsilon^p \in C^\infty[-1, 1]$, each $D_\varepsilon(p)$ has compact resolvent. Denote by $\{\lambda_{\varepsilon,n}(p)\}_{n \in \mathbb{N}}$ the sequence of eigenvalues of $D_\varepsilon(p)$ and by $\{u_{\varepsilon,n}(p)\}_{n \in \mathbb{N}}$ the sequence of the corresponding normalized eigenfunctions, i.e.,

$$D_\varepsilon(p) u_{\varepsilon,n}(p) = \lambda_{\varepsilon,n}(p) u_{\varepsilon,n}(p), \quad n \in \mathbb{N}, \quad p \in \mathbb{R}.$$  

Due to the decomposition in (15), we have

$$\sigma(D_\varepsilon) = \bigcup_{p \in \mathbb{R}} \sigma(D_\varepsilon(p)) = \bigcup_{n \in \mathbb{N}} \{\lambda_{\varepsilon,n}(p) : p \in \mathbb{R}\}.$$  

In particular, denote $u_{\varepsilon,1}^0 := u_{\varepsilon,1}(0)$. Due to the condition (8) in the Sect. 1, $u_{\varepsilon,1}^0$ can be chosen to be real and positive in $(-1, 1)$. This property will be used in the proof of Proposition 3 below.

**Lemma 1** For each $n \in \mathbb{N}$, $\lambda_{\varepsilon,n}(\cdot)$ is a real analytic function in $p$ and

$$\lim_{p \to \pm \infty} \lambda_{\varepsilon,n}(p) = \infty.$$  

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Proof At first, note that \( \text{dom } D_\varepsilon(p) = \text{dom } D_\varepsilon(0) \), for all \( p \in \mathbb{R} \). One can write \( D_\varepsilon(p) = D_\varepsilon(0) + p^2/\varepsilon^2 \). Since \( h_\varepsilon \geq 1 \), it holds the estimate \( \| (p^2/\varepsilon^2) v \| \leq p^2 \| v \| \), for all \( v \in \text{dom } D_\varepsilon(0) \), for all \( p \in \mathbb{R} \). Then, \( p^2/\varepsilon^2 \) is \( D_\varepsilon(0) \)-bounded with zero relative bound. Consequently, \( \{ D_\varepsilon(p) : p \in \mathbb{R} \} \) is a type A analytic family. By Theorem 3.9 in Kato (1995), \( \lambda_{\varepsilon,n}(\cdot) \) is a real analytic function in \( p \).

Now, for each \( v \in \text{dom } d_\varepsilon(p) \), we have

\[
d_\varepsilon(p)(v) = \frac{1}{\varepsilon^2} \int_{-1}^{1} |\partial_t v|^2 dt + \int_{-1}^{1} Y_\varepsilon p^2 |v|^2 dt \geq \left( \frac{p^2}{1 + \gamma/\varepsilon^2} - \varepsilon^2 \right) \int_{-1}^{1} |v|^2 dt.
\]

As a consequence, for each \( n \in \mathbb{N} \),

\[
\lambda_{\varepsilon,n}(p) = d_\varepsilon(p)(u_{\varepsilon,n}(p)) \geq \frac{p^2}{1 + \gamma/\varepsilon^2} - \varepsilon^2, \quad p \in \mathbb{R}.
\]

Thus, we obtain the punctual limit \( \lambda_{\varepsilon,n}(p) \to \infty \), as \( p \to \pm \infty \). \( \square \)

**Proposition 3** One has \( \sigma(D_\varepsilon) = [\lambda_{\varepsilon,1}(0), \infty) \).

**Proof** Since \( \lambda_{\varepsilon,1}(p) \) is a real analytic function in \( p \), by (16), we have \( [\lambda_{\varepsilon,1}(0), \infty) \subset \sigma(D_\varepsilon) \). Now, we need to show that

\[
(-\infty, \lambda_{\varepsilon,1}(0)) \cap \sigma(D_\varepsilon) = \emptyset. \tag{17}
\]

Take \( \psi \in C_0^\infty(\Lambda) \). Since \( u_{\varepsilon,1}^0 \) is positive, we can write \( \psi(s,t) = \phi(s,t)u_{\varepsilon,1}^0(t) \), with \( \phi \in C_0^\infty(\Lambda) \). Some calculations show that

\[
d_\varepsilon(\psi) - \lambda_{\varepsilon,1}(0) \int_\Lambda |\psi|^2 ds dt
\]

\[
= \int_\Lambda \left( \frac{|\partial_0 \phi|^2}{h_\varepsilon^2} + \frac{|\partial_t \phi|^2}{\varepsilon^2} \right) |u_{\varepsilon,1}^0|^2 ds dt + \frac{2}{\varepsilon^2} \int_\Lambda \frac{\partial_0 \phi}{\partial_0 u_{\varepsilon,1}^0} \partial_0 u_{\varepsilon,1}^0 \partial_1 u_{\varepsilon,1}^0 ds dt
\]

\[
+ \frac{1}{\varepsilon^2} \int_\Lambda |\phi|^2 |\partial_t u_{\varepsilon,1}^0|^2 ds dt + \int_\Lambda |\phi|^2 (Y_\varepsilon^0 u_{\varepsilon,1}^0)^2 - \lambda_{\varepsilon,1}(0)(u_{\varepsilon,1}^0)^2 ds dt.
\]

By integrating by parts, and since \( D_\varepsilon(0)u_{\varepsilon,1}^0 = \lambda_{\varepsilon,1}(0)u_{\varepsilon,1}^0 \), one has

\[
d_\varepsilon(\psi) - \lambda_{\varepsilon,1}(0) \int_\Lambda |\psi|^2 ds dt = \int_\Lambda \left( \frac{|\partial_0 \phi|^2}{h_\varepsilon^2} + \frac{|\partial_t \phi|^2}{\varepsilon^2} \right) |u_{\varepsilon,1}^0|^2 ds dt \geq 0.
\]

Then, we obtain (17). \( \square \)

**Proof of Theorem 1** Apply Propositions 2 and 3.

**Remark 4** For the sequence \( \{\lambda_{\varepsilon,1}(0)\} \), we have the estimate: there exists \( K > 0 \) so that

\[
\int_{-1}^{1} |\partial_t v|^2 dt \leq \varepsilon^2 d_\varepsilon(0)(v) = \int_{-1}^{1} \left( |\partial_t v|^2 + \varepsilon^2 Y_\varepsilon^0 |v|^2 \right) dt
\]
\[ \leq \int_{-1}^{1} |\partial_t v|^2 dt + K \varepsilon^2 \int_{-1}^{1} |v|^2 dt, \]

for all \( v \in H_0^1(-1, 1) \), for all \( \varepsilon > 0 \) small enough. Consequently,

\[ \left( \frac{\pi}{2} \right)^2 \leq \varepsilon^2 \lambda_{\varepsilon,1}(0) \leq \left( \frac{\pi}{2} \right)^2 + O(\varepsilon^2). \]

### 4 Discrete Spectrum

Based on Goldstone and Jaffe (1992), the main idea in the proofs of Theorems 2 and 3 is to find a function \( \psi \in \text{dom } c_\varepsilon \) so that \( (c_\varepsilon(\psi) - \lambda_{\varepsilon,1}(0)) \|\psi\|_2 / \|\psi\|_2 < 0 \).

**Proof of Theorem 2** Take \( \delta > 0 \). Define \( \psi_\delta(s, t) := \phi(s) u_{\varepsilon,1}(t) \), where

\[ \phi(s) := \begin{cases} e^{\delta(s-s_0)}, & s \leq -s_0, \\ 1, & -s_0 \leq s \leq s_0, \\ e^{-\delta(s-s_0)}, & s \geq s_0. \end{cases} \]

Note that \( \psi_\delta \in \text{dom } c_\varepsilon \). Some calculations show that

\[ c_\varepsilon(\psi_\delta) - \lambda_{\varepsilon,1}(0) \int_\Lambda |\psi_\delta|^2 ds dt \]

\[ = \int_\Lambda \left| \frac{\partial_s \psi_\delta}{f_\varepsilon^2} \right|^2 ds dt + \frac{1}{4} \int_\Lambda \left| \frac{\partial_s f_\varepsilon}{f_\varepsilon^4} \psi_\delta \right|^2 ds dt - \mathcal{R} \int_\Lambda \frac{\partial_s f_\varepsilon}{f_\varepsilon^4} \psi_\delta \frac{\partial_s \psi_\delta}{f_\varepsilon^4} ds dt \]

\[ + \frac{1}{\varepsilon^2} \int_\Lambda |\partial_t \psi_\delta|^2 ds dt - \lambda_{\varepsilon,1}(0) \int_\Lambda |\psi_\delta|^2 ds dt + \int_\Lambda V_\varepsilon |\psi_\delta|^2 ds dt \]

\[ = \int_\Lambda \left| \frac{\partial_s \psi_\delta}{f_\varepsilon^2} \right|^2 ds dt + \frac{1}{4} \int_\Lambda \left| \frac{\partial_s f_\varepsilon}{f_\varepsilon^4} \psi_\delta \right|^2 ds dt - \mathcal{R} \int_\Lambda \frac{\partial_s f_\varepsilon}{f_\varepsilon^4} \psi_\delta \frac{\partial_s \psi_\delta}{f_\varepsilon^4} ds dt \]

\[ + \int_\Lambda (V_\varepsilon - Y^0_\varepsilon) |\psi_\delta|^2 ds dt. \]

Now, note that \( f_\varepsilon \to 1, \partial_s f_\varepsilon \to 0, (V_\varepsilon - Y^0_\varepsilon) \to (|\Theta'|^2 - \gamma^2)/2 \), uniformly, as \( \varepsilon \to 0 \). Then,

\[ c_\varepsilon(\psi_\delta) - \lambda_{\varepsilon,1}(0) \|\psi_\delta\|^2 \to \delta + \frac{1}{2} \int_{-s_0}^{s_0} \left( |\Theta'(s)|^2 - \gamma^2 \right) ds, \]

as \( \varepsilon \to 0 \). Since \( \|\psi_\delta\|^2 = 2s_0 + \delta^{-1} \), one has

\[ \frac{c_\varepsilon(\psi_\delta) - \lambda_{\varepsilon,1}(0) \|\psi_\delta\|^2}{\|\psi_\delta\|^2} \to O(\delta^2) + \frac{\delta}{2} \int_{-s_0}^{s_0} \left( |\Theta'(s)|^2 - \gamma^2 \right) ds, \]

as \( \varepsilon \to 0 \).
Since $\int_{s_0}^{s_0} (|\Theta'(s)|^2 - \gamma^2) \, ds < 0$, we can choose $\delta$ small enough so that the limit in (18) is negative. Consequently, there exists $\varepsilon_1 > 0$ so that

$$\frac{c_\varepsilon(\psi_\delta) - \lambda_{\varepsilon,1}(0)\|\psi_\delta\|^2}{\|\psi_\delta\|^2} < 0,$$

for all $\varepsilon \in (0, \varepsilon_1)$. □

**Proof of Theorem 3** Given $\delta > 0$ and $\eta > 0$, define $\psi_{\delta,\eta}(s, t) := \phi_\eta(s) \psi_{\varepsilon,1}(t)$, where

$$\phi_\eta(s) := \begin{cases} e^{\delta(s+s_0)}, & s \leq -s_0, \\
1 + \eta \left( \gamma - |\Theta'(s)| \right), & -s_0 \leq s \leq s_0, \\
e^{-\delta(s-s_0)}, & s \geq s_0.
\end{cases}$$

Note that $\psi_{\delta,\eta} \in \text{dom } c_\varepsilon$. Similarly as in the proof of Theorem 2, we can show that

$$c_\varepsilon(\psi_{\delta,\eta}) - \lambda_{\varepsilon,1}(0)\|\psi_{\delta,\eta}\|^2 \to \delta + O(\eta^2) - \eta \int_{-s_0}^{s_0} \left( |\Theta'(s)| - \gamma \right)^2 \left( |\Theta'(s)| + \gamma \right) \, ds,$$

as $\varepsilon \to 0$. Since $\|\psi_{\delta,\eta}\|^2 = 2s_0 + \delta^{-1} + O(\eta^2)$, one has

$$\frac{c_\varepsilon(\psi_{\delta,\eta}) - \lambda_{\varepsilon,1}(0)\|\psi_{\delta,\eta}\|^2}{\|\psi_{\delta,\eta}\|^2} \to O(\delta^2) + \delta O(\eta^2)$$

$$-\delta \eta \int_{-s_0}^{s_0} \left( |\Theta'(s)| - \gamma \right)^2 \left( |\Theta'(s)| + \gamma \right) \, ds, \hspace{1cm} (19)$$

as $\varepsilon \to 0$. Taking $\eta = \sqrt{\delta}$, again we can choose $\delta$ small enough so that the limit in (19) is negative. Then, there exists $\varepsilon_2 > 0$ so that

$$\frac{c_\varepsilon(\psi_{\delta,\eta}) - \lambda_{\varepsilon,1}(0)\|\psi_{\delta,\eta}\|^2}{\|\psi_{\delta,\eta}\|^2} < 0,$$

for all $\varepsilon \in (0, \varepsilon_2)$. □

### 5 Thin Strips

In this section we present the proof of Theorem 4 stated in the Sect. 1. The strategy will be to establish upper and lower bounds for the eigenvalues $\lambda_j(-\tilde{\Delta}_\varepsilon)$. Recall $-\tilde{\Delta}_\varepsilon$ is the self-adjoint operator associated with the quadratic form $\tilde{a}_\varepsilon(\varphi)$; see (12). Then, the analysis will be based on estimates for $\tilde{a}_\varepsilon(\varphi)$.

Define

$$\tilde{f}_\varepsilon(s, t) := \sqrt{1 + |\Theta_\varepsilon'(s)|^2 \varepsilon^2 t^2}.$$
and consider the Hilbert space $\mathcal{H}_\varepsilon := L^2(\Lambda, f_\varepsilon \, ds \, dt)$; the norm in this space is denoted by $\| \cdot \|_{\mathcal{H}_\varepsilon}$. Performing a change of coordinates similar to that in Sect. 2, $\tilde{a}_\varepsilon(\varphi)$ becomes

$$
\tilde{b}_\varepsilon(\psi) := \int_{\Lambda} \frac{|\delta_\varepsilon \psi|^2}{f_\varepsilon} \, ds \, dt + \frac{1}{\varepsilon^2} \int_{\Lambda} |\partial_t \psi|^2 f_\varepsilon \, ds \, dt,
$$

$\text{dom } \tilde{b}_\varepsilon = H^1_0(\Lambda) \subset \mathcal{H}_\varepsilon$.

**Upper bound.** Denote by $\chi_1(t) := \cos(\pi t / 2)$ the first eigenfunction of the Dirichlet Laplacian $-\Delta^D_{(-1,1)}$ in $L^2(-1,1)$; $(\pi/2)^2$ is the eigenvalue associated with $\chi_1$. Consider the subspace

$$
A_\varepsilon := \{ \varphi_\varepsilon := w(s) \chi_1(t)(\tilde{f}_\varepsilon(s, t))^{-1/2} : w \in H^1(\mathbb{R}) \}
$$

of the Hilbert space $\mathcal{H}_\varepsilon$. The identification $w \mapsto \varphi_w$, $w \in H^1(\mathbb{R})$, motivates the definition of the one-dimensional quadratic form

$$
m_\varepsilon(w) := \tilde{b}_\varepsilon(\varphi_w) - (\pi/2\varepsilon)^2 \|\varphi_w\|_{\mathcal{H}_\varepsilon}^2,
$$

$\text{dom } m_\varepsilon := H^1(\mathbb{R})$. Denote by $M_\varepsilon$ the self-adjoint operator associated with $m_\varepsilon(w)$. In particular, for each $j \in \mathbb{N}$,

$$
\lambda_j(-\tilde{\Delta}_\varepsilon) - \left( \frac{\pi}{2\varepsilon} \right)^2 \leq \lambda_j(M_\varepsilon).
$$

We are going to get upper bounds for the values $\lambda_j(M_\varepsilon)$.

Define the function

$$
W_\varepsilon(s, t) := \frac{-3\left|\Theta'_\varepsilon \cdot \Theta''_\varepsilon(s)\right|^2 \varepsilon^4 t^4}{4\tilde{f}_\varepsilon^6(s, t)} + \frac{(2\left|\Theta''_\varepsilon(s)\right|^2 + 2(\Theta'_\varepsilon \cdot \Theta''_\varepsilon)(s) - 3|\Theta'_\varepsilon(s)|^4)\varepsilon^2 t^2}{4\tilde{f}_\varepsilon^4(s, t)} + \frac{|\Theta'_\varepsilon(s)|^2}{2\tilde{f}_\varepsilon^2(s, t)}.
$$

Recall that we have the condition (V) in the Sect. 1. Then we get the estimates

$$
\|1/\tilde{f}_\varepsilon^2\|_{L^\infty(\Lambda)} \leq \varepsilon^{2-2a},
$$

$$
\|W_\varepsilon - |\Theta'_\varepsilon|^2/2\|_{L^\infty(\Lambda)} \leq K(\varepsilon^{4-2(a+b)} + \varepsilon^{2-2b} + \varepsilon^{2-2(a+c)} + \varepsilon^{2-4a}),
$$

for some $K > 0$, for all $\varepsilon > 0$ small enough. Finally, some calculations show that

$$
m_\varepsilon(w) = \int_{\Lambda} \frac{|w' \chi_1|^2}{\tilde{f}_\varepsilon^2} \, ds \, dt + \int_{\Lambda} W_\varepsilon |w \chi_1|^2 \, ds \, dt
$$

$$
\leq \int_{\mathbb{R}} \left( |w'|^2 + \frac{1}{2} |\Theta'_\varepsilon(s)|^2 |w|^2 \right) \, ds + O(\varepsilon^d) \int_{\mathbb{R}} |w|^2 \, ds.
$$
for all \( w \in H^1(\mathbb{R}) \), for all \( \varepsilon > 0 \) small enough, where \( d = \min\{4 - 2(a + b), 2 - 2b, 2 - (a + c), 2 - 4a\} \). As a consequence, for each \( j \in \mathbb{N} \),

\[
\lambda_j(M_\varepsilon) \leq \lambda_j \left( -\Delta_\mathbb{R} + \frac{|\Theta'_\varepsilon(s)|^2}{2} \right) + O(\varepsilon^d). \tag{21}
\]

By (20) and (21), for each \( j \in \mathbb{N} \),

\[
\lambda_j(-\tilde{\Delta}_\varepsilon) - \left( \frac{\pi}{2\varepsilon} \right)^2 \leq \lambda_j \left( -\Delta_\mathbb{R} + \frac{|\Theta'_\varepsilon(s)|^2}{2} \right) + O(\varepsilon^d). \tag{22}
\]

**Lower bound.** For each \( \varepsilon \geq 0 \) small enough, consider the one-dimensional self-adjoint operator

\[
(S_\varepsilon v)(t) := -v''(t) - \frac{\varepsilon t}{1 + \varepsilon t^2} v'(t), \quad \text{dom } S_\varepsilon = H^1_0((-1,1)),
\]

acting in the Hilbert space \( L^2((-1,1), \sqrt{1 + \varepsilon t^2} dt) \). The particular case \( \varepsilon = 0 \) corresponds to the Dirichlet Laplacian operator \( -\Delta_{(-1,1)}^D \) in \( L^2(-1,1) \).

Denote by \( \Sigma(\varepsilon) \) the first eigenvalue of \( S_\varepsilon \). By the analytic perturbation theory, we can write

\[
\Sigma(\varepsilon) = \left( \frac{\pi}{2} \right)^2 + \delta(\varepsilon)\varepsilon + O(\varepsilon^2),
\]

where

\[
\delta(\varepsilon) := -\int_{-1}^{1} \frac{t}{\sqrt{1 + \varepsilon t^2}} \chi_1'(t) \chi_1(t) dt;
\]

see (Kato 1951) for more details. Consequently, for each \( \psi \in H^1_0(\Lambda) \), we have the estimate

\[
\tilde{b}_\varepsilon(\psi) \geq \int_{\Lambda} \left( \frac{\partial_\varepsilon \psi}{\tilde{f}_\varepsilon} \right)^2 + \Sigma(\varepsilon^2) \frac{|\Theta'_\varepsilon(s)|^2}{\varepsilon^2} \tilde{f}_\varepsilon \psi^2 \right) ds dt.
\]

More exactly,

\[
\tilde{b}_\varepsilon(\psi) - \left( \frac{\pi}{2\varepsilon} \right)^2 \|\psi\|^2_{H_\varepsilon} \geq \int_{\Lambda} \left( \frac{\partial_\varepsilon \psi}{\tilde{f}_\varepsilon} \right)^2 + |\Theta'_\varepsilon(s)|^2 \delta(\varepsilon^2) |\Theta'_\varepsilon(s)|^2 \tilde{f}_\varepsilon \psi^2 \right) ds dt
\]

\[
+ O(\varepsilon^{2 - 4a}) \|\psi\|^2_{H_\varepsilon}.
\]

(23)

Now, define the quadratic form

\[
n_\varepsilon(\psi) := \int_{\Lambda} \left( \frac{\partial_\varepsilon \psi}{\tilde{f}_\varepsilon} + |\Theta'_\varepsilon(s)|^2 \delta(\varepsilon^2) |\Theta'_\varepsilon(s)|^2 \tilde{f}_\varepsilon \psi^2 \right) ds dt.
\]
dom \( n_\varepsilon = H^1_0(\Lambda) \). Denote by \( N_\varepsilon \) the self-adjoint operator associated with \( n_\varepsilon(\psi) \). For each \( j \in \mathbb{N} \), inequality (23) implies

\[
\lambda_j(N_\varepsilon) + O(\varepsilon^{2-4a}) \leq \lambda_j(\tilde{\Delta}_\varepsilon) - \left( \frac{\pi}{2\varepsilon} \right)^2 .
\] (24)

The next step is to find lower bounds for the values \( \lambda_j(N_\varepsilon) \).

**Lemma 2** There exists a number \( K > 0 \) so that

\[
\| (\Theta'_\varepsilon(s))^2 \delta(\varepsilon^2 |\Theta'_\varepsilon(s)|^2) \tilde{f}_\varepsilon - \frac{|\Theta'_\varepsilon(s)|^2}{2} \|_{L^\infty(\Lambda)} \leq K \varepsilon^{1-3a},
\]

for all \( \varepsilon > 0 \) small enough.

**Proof** At first, note that

\[
|\Theta'_\varepsilon(s)|^2 \delta(\varepsilon^2 |\Theta'_\varepsilon(s)|^2) \tilde{f}_\varepsilon - \frac{|\Theta'_\varepsilon(s)|^2}{2} \leq |\Theta'_\varepsilon(s)|^2 \left( |\delta(\varepsilon^2 |\Theta'_\varepsilon(s)|^2)| - \frac{1}{2} |\tilde{f}_\varepsilon| + \frac{1}{2} |\tilde{f}_\varepsilon - 1| \right),
\]

for all \((s, t) \in \Lambda\). Some calculations show that

\[
\delta(\varepsilon^2 |\Theta'_\varepsilon(s)|^2) - \frac{1}{2} = \frac{\pi}{2} \int_{-1}^{1} \left( \frac{1}{\tilde{f}_\varepsilon - 1} \right) t \sin \left( \frac{\pi t}{2} \right) \cos \left( \frac{\pi t}{2} \right) dt .
\]

Since \( \|1/\tilde{f}_\varepsilon - 1\|_{L^\infty(\Lambda)} \leq \varepsilon^{1-a} \), we have the estimate \( \|\delta(\varepsilon^2 |\Theta'_\varepsilon(s)|^2) - 1/2\|_{L^\infty(\Lambda)} \leq \varepsilon^{1-a}/2 \). Thus, along with the condition \(|\Theta'_\varepsilon(s)| \leq K/\varepsilon^a\) and the estimate \( \|\tilde{f}_\varepsilon - 1\|_{L^\infty(\Lambda)} \leq \varepsilon^{1-a} \), we get the result. \( \Box \)

Using Lemma 2 and the estimate \( \|1/\tilde{f}_\varepsilon - 1\|_{L^\infty(\Lambda)} \leq \varepsilon^{1-a} \), one has

\[
n_\varepsilon(\psi) \geq (1 + O(\varepsilon^{1-a})) \int_\Lambda |\partial_s \psi|^2 ds dt
+ \int_\Lambda \frac{|\Theta'_\varepsilon(s)|^2}{2} |\psi|^2 ds dt + O(\varepsilon^{1-3a}) \int_\Lambda |\psi|^2 ds dt .
\]

Since \(|\Theta'_\varepsilon(s)| \leq K/\varepsilon^a\), for all \( s \in \mathbb{R} \), it follows that

\[
n_\varepsilon(\psi) \geq (1 + O(\varepsilon^{1-a})) \int_\Lambda \left( |\partial_s \psi|^2 + \frac{|\Theta'_\varepsilon(s)|^2}{2} |\psi|^2 \right) ds dt ,
\]

for all \( \psi \in H^1_0(\Lambda) \), for all \( \varepsilon > 0 \) small enough.
As a consequence, for each $j \in \mathbb{N}$,
\[
(1 + O(\varepsilon^{1-a}))\lambda_j \left(-\Delta_{\mathbb{R}} + \frac{|\Theta'(s)|^2}{2}\right) \leq \lambda_j(N_{\varepsilon}).
\] (25)

Inequalities (24) and (25) ensure that, for each $j \in \mathbb{N}$,
\[
(1 + O(\varepsilon^{1-a}))\lambda_j \left(-\Delta_{\mathbb{R}} + \frac{|\Theta'(s)|^2}{2}\right) + O(\varepsilon^{2-4a}) \leq \lambda_j(-\Delta_{\varepsilon} - \frac{\pi}{2\varepsilon}^2).
\] (26)

Due to inequalities (22) and (26), we will study the spectral problem of the operator $-\Delta_{\mathbb{R}} + \left(\frac{|\Theta'(s)|^2}{2}\right)1$.

**Proposition 4**  
For each $j \in \mathbb{N}$,
\[
\lim_{\varepsilon \to 0} \lambda_j \left(-\Delta_{\mathbb{R}} + \frac{|\Theta'(s)|^2}{2}\right) = \lambda_j \left(-\Delta_{\mathbb{R}} + \frac{|\Theta'(s)|^2}{2}\right).
\] (27)

**Proof**  
Recall the conditions (I)–(VII) in the Sect. 1. Consider the quadratic forms
\[
y(w) := \int_{\mathbb{R}} \left(|w'|^2 + \frac{|\Theta'(s)|^2}{2}|w|^2\right) \, ds, \quad \text{dom } y = \{w \in H^1(\mathbb{R}) : y(w) < \infty\},
\]
and, for each $\varepsilon > 0$ small enough,
\[
y_\varepsilon(w) := \int_{\mathbb{R}} \left(|w'|^2 + \frac{|\Theta'(s)|^2}{2}|w|^2\right) \, ds, \quad \text{dom } y_\varepsilon = H^1(\mathbb{R});
\]
denote by $Y$ and $Y_\varepsilon$ the self-adjoint operators associated with $y(w)$ and $y_\varepsilon(w)$, respectively. In particular, the condition (III) implies that $|\Theta'(s)| \to |\Theta'(s)|$ pointwise, as $\varepsilon \to 0$. Thus, for each $w \in C^0_0(\mathbb{R})$, $Y_\varepsilon w \to Y w$, as $\varepsilon \to 0$. Since $C^0_0(\mathbb{R})$ is a core of $Y$, given a constant $c > 0$,
\[
(Y_\varepsilon + c\mathbf{1})^{-1} u \to (Y + c\mathbf{1})^{-1} u, \quad \forall u \in L^2(\mathbb{R}).
\] (28)

Now, denote by $\mathcal{C}$ the ideal of all compact operators in the algebra of all bounded operators in $L^2(\mathbb{R})$. Fix $\varepsilon^* > 0$ small enough. By (11) and (V), one has
\[
\text{dist}((Y_{\varepsilon^*} + c\mathbf{1})^{-1}, \mathcal{C}) \leq \left(\liminf_{|s| \to \infty} \frac{|\Theta'(s)|^2}{2}\right)^{-1} \leq 2(\varepsilon^*)^2a.
\] (29)

The condition (VII) implies that
\[
(Y_\varepsilon + c\mathbf{1})^{-1} \leq (Y_{\varepsilon^*} + c\mathbf{1})^{-1},
\] for all $\varepsilon < \varepsilon^*$.
As a consequence of (28), (29) and (30), Proposition 5.3 of Friedlander and Solomyak (2008a) ensures that
\[
\limsup_{\varepsilon \to 0} \| (Y_\varepsilon + c \mathbf{1})^{-1} - (Y + c \mathbf{1})^{-1} \| \leq 2 (\varepsilon^*)^{2\alpha}.
\]
Taking \( \varepsilon^* \to 0 \),
\[
\| (Y_\varepsilon + c \mathbf{1})^{-1} - (Y + c \mathbf{1})^{-1} \| \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]
Then, we obtain (27). \( \square \)

**Proof of Theorem 4** It follows from an application of inequalities (22) and (26), along with Proposition 4. \( \square \)

**Data availability** Not applicable, no new data was generated.

## Appendix

### Stability of the Essential Spectrum

The results of this appendix are simple adaptations of Lemma 4.1 and Proposition 4.2 of Briet et al. (2019), and Lemma 4.2 of Dermenjian et al. (1998).

**Lemma 3** A real number \( \lambda \) belongs to the essential spectrum of \( C_\varepsilon \) if and only if there exists a sequence \( \{\psi_n\}_{n \in \mathbb{N}} \subseteq \text{dom} \ c_\varepsilon \) satisfying the following conditions:

(i) \( \| \psi_n \| = 1 \), for all \( n \in \mathbb{N} \);

(ii) \( (C_\varepsilon - \lambda \mathbf{1}) \psi_n \to 0 \), as \( n \to \infty \), in the norm of the dual space \( (\text{dom} \ c_\varepsilon)^* \);

(iii) \( \text{supp} \ \psi_n \subseteq \Lambda \setminus (-n, n) \times (-1, 1) \), for all \( n \in \mathbb{N} \).

**Proof** It is known that \( \lambda \in \sigma_{\text{ess}}(C_\varepsilon) \) if and only if there exists a sequence \( \{\xi_n\}_{n \in \mathbb{N}} \subseteq \text{dom} \ c_\varepsilon \) satisfying (i), (ii), and (iii'). \( \xi_n \to 0 \) in \( L^2(\Lambda) \), as \( n \to 0 \); see, e.g., Theorem 5 in Krejčiřík and Lu (2014). Let \( \{\psi_n\}_{n \in \mathbb{N}} \) be a sequence satisfying the conditions (i), (ii) and (iii). Consequently, it satisfies (i), (ii) and (iii').

Now, let \( \{\xi_n\}_{n \in \mathbb{N}} \subseteq \text{dom} \ c_\varepsilon \) be a sequence satisfying (i), (ii), and (iii'). Take \( \eta \in C^\infty(\mathbb{R}; \mathbb{R}) \), \( 0 \leq \eta \leq 1 \), \( \eta = 0 \) in \( [-1, 1] \), and \( \eta = 1 \) in \( \mathbb{R}\setminus(-2, 2) \). Define the sequence \( \{\eta_k\}_{k \in \mathbb{N}} \subseteq C^\infty(\Lambda) \), where \( \eta_k(s, t) := \eta(s/k) \). Since \( (1 - \eta_k)(C_\varepsilon + \mathbf{1})^{-1} \) is compact in \( L^2(\Lambda) \), by (iii'), we have \( (1 - \eta_k)(C_\varepsilon + \mathbf{1})^{-1}\xi_n \to 0 \) in \( L^2(\Lambda) \), as \( n \to \infty \), for all \( k \in \mathbb{N} \). Then there exists a subsequence \( \{\xi_{n_k}\}_{k \in \mathbb{N}} \) of \( \{\xi_n\}_{n \in \mathbb{N}} \) so that \( (1 - \eta_k)(C_\varepsilon + \mathbf{1})^{-1}\xi_{n_k} \to 0 \) in \( L^2(\Lambda) \), as \( k \to \infty \). By writing

\[
\xi_{n_k} = (C_\varepsilon + \mathbf{1})^{-1} (C_\varepsilon - \lambda \mathbf{1})\xi_{n_k} + (\lambda + 1)(C_\varepsilon + \mathbf{1})^{-1}\xi_{n_k},
\]

and using (ii), it follows that \( (1 - \eta_k)\xi_{n_k} \to 0 \) in \( L^2(\Lambda) \), as \( k \to \infty \). Thus, we can assume that \( \| \eta_k\xi_{n_k} \| \geq 1/2 \), for all \( k \in \mathbb{N} \). Finally, let us define

\[
\psi_k := \frac{\eta_k \xi_{n_k}}{\| \eta_k\xi_{n_k} \|}, \quad k \in \mathbb{N}.
\]
The sequence \( \{\psi_k\}_{k \in \mathbb{N}} \subset \text{dom } c_\varepsilon \) satisfies the conditions (i) and (iii). It remains to verify (ii), i.e.,

\[
\sup_{\phi \in H^1_0(\Lambda), \phi \neq 0} \frac{|c_\varepsilon(\phi, \psi_k) - \lambda \langle \phi, \psi_k \rangle|}{\|\phi\|_+} \to 0, \quad (31)
\]
as \( k \to \infty \), where \( \|\phi\|_+^2 := c_\varepsilon(\phi) + \|\phi\|^2 \).

Some calculations show that

\[
c_\varepsilon(\phi, \eta_k \xi_{n_k}) - \lambda \langle \eta_k \phi, \xi_{n_k} \rangle = c_\varepsilon(\eta_k \phi, \xi_{n_k}) - \lambda \langle \eta_k \phi, \xi_{n_k} \rangle + \int_{\Lambda} \frac{1}{f_\varepsilon} \phi \partial_s^2 \eta_k \xi_{n_k} dsdt
+ 2 \int_{\Lambda} \frac{1}{f_\varepsilon} \left( \partial_s \phi - \frac{\partial s}{f_\varepsilon} \phi \right) \partial_s \eta_k \xi_{n_k} dsdt.
\]

Since \( \{\xi_{n_k}\}_{k \in \mathbb{N}} \) satisfies (ii), we have

\[
\sup_{\phi \in H^1_0(\Lambda), \phi \neq 0} \frac{|c_\varepsilon(\eta_k \phi, \xi_{n_k}) - \lambda \langle \eta_k \phi, \xi_{n_k} \rangle|}{\|\eta_k \phi\|_+} \to 0,
\]
as \( k \to \infty \). By Hölder’s Inequality and by the estimates \( \|\phi\| \leq \|\phi\|_+ \) and \( c_\varepsilon(\phi) \leq \|\phi\|_+^2 \), we get

\[
\sup_{\phi \in H^1_0(\Lambda), \phi \neq 0} \left\{ \frac{1}{\|\phi\|_+} \int_{\Lambda} \frac{1}{f_\varepsilon} |\phi| |\partial_s^2 \eta_k| |\xi_{n_k}| dsdt \right\}
\leq \|\partial_s^2 \eta_k\|_{L^\infty(\Lambda)} = k^{-2} \|\eta''\|_{L^\infty(\mathbb{R})} \to 0,
\]

and

\[
\sup_{\phi \in H^1_0(\Lambda), \phi \neq 0} \left\{ \frac{1}{\|\phi\|_+} \int_{\Lambda} \frac{1}{f_\varepsilon} \left| \partial_s \phi - \frac{\partial s}{f_\varepsilon} \phi \right| |\partial_s \eta_k| |\xi_{n_k}| dsdt \right\}
\leq \|\partial_s \eta_k\|_{L^\infty(\Lambda)} = k^{-1} \|\eta'\|_{L^\infty(\mathbb{R})} \to 0,
\]
as \( k \to \infty \). Finally, since \( \beta, \beta' \in L^\infty(\mathbb{R}) \), we have

\[
\sup_{\phi \in H^1_0(\Lambda), \phi \neq 0} \left\{ \frac{1}{\|\phi\|_+} \int_{\Lambda} \frac{\partial s f_\varepsilon}{f_\varepsilon^3} \left| \phi \right| |\partial_s \eta_k| |\xi_{n_k}| dsdt \right\} \leq \|\partial_s \eta_k\|_{L^\infty(\Lambda)}
\leq k^{-1} \|\eta'\|_{L^\infty(\mathbb{R})} \to 0.
\]
as \( k \to \infty \). Then, (31) holds true. \( \Box \)

**Remark 5** The same conclusion of Lemma 3 holds true for the operator \( D_\varepsilon \).

**Proof of Proposition 2** Let \( \lambda \in \sigma_{ess}(D_\varepsilon) \), then there exists a sequence \( \{\psi_n\}_{n \in \mathbb{N}} \subset \text{dom } d_\varepsilon \) so that \( \|\psi_n\| = 1, \supp \psi_n \subset \Lambda \setminus (-n, n) \times (-1, 1) \), for all \( n \in \mathbb{N} \), and \( (D_\varepsilon - \lambda \mathbf{1})\psi_n \to 0 \), as \( n \to \infty \), in the norm of the dual space \( (\text{dom } d_\varepsilon)^* \). Recall \( \supp \beta \subset [-s_0, s_0] \), for some \( s_0 > 0 \). Take \( n_0 \in \mathbb{N} \) so that \( n_0 > s_0 \). For each \( \phi \in H^1_0(\Lambda) \), one has

\[
c_\varepsilon(\phi, \psi_n) - \lambda \langle \phi, \psi_n \rangle = d_\varepsilon(\phi, \psi_n) - \lambda \langle \phi, \psi_n \rangle,
\]

for all \( n \geq n_0 \). Consequently,

\[
\sup_{\substack{\phi \in H^1_0(\Lambda) \\ \phi \neq 0}} \frac{|c_\varepsilon(\phi, \psi_n) - \lambda \langle \phi, \psi_n \rangle|}{\|\phi\| + } \longrightarrow 0,
\]

as \( n \to \infty \). The Lemma 3 implies that \( \lambda \in \sigma_{ess}(C_\varepsilon) \). In a similar way, it is possible to show that \( \sigma_{ess}(C_\varepsilon) \subset \sigma_{ess}(D_\varepsilon) \). \( \Box \)

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