HOMOTOPY METHOD FOR A CLASS OF MULTIOBJECTIVE OPTIMIZATION PROBLEMS WITH EQUILIBRIUM CONSTRAINTS

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Abstract. In this paper, we present a combined homotopy interior point method for solving multiobjective programs with equilibrium constraints. Under suitable conditions, we prove the existence and convergence of a smooth homotopy path from almost any interior point to a solution of the K-K-T system. Numerical results are presented to show the effectiveness of this algorithm.

1. Introduction. Given functions $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^p$, $g : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^s$, $G : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^l$, and $F : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$. In this paper, we are devoted to the study of the multiobjective optimization problems with equilibrium constraints (MOPECs) in finite dimensional spaces defined as follows:

$$
\min \ f(x, y)
$$

s.t. $Z \subseteq \mathbb{R}^{n+m}$,

$$
y \in S(x),
$$

where $Z = \{(x, y) \in \mathbb{R}^{n+m} : g(x, y) \leq 0\}$ is a nonempty closed convex set. For $x \in X$, $S(x)$ is the solution set of a parametric variational inequality problem (PVI)

$$
y \in S(x) \iff F(x, y)^T(v - y) \geq 0, \forall v \in C(x),
$$

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where \( C(x) = \{ y \in \mathbb{R}^m : G(x, y) \leq 0 \} \), \( X = \{ x \in \mathbb{R}^n : (x, y) \in Z, \text{ for some } y \in \mathbb{R}^m \} \).

Throughout this paper, we suppose that \( f, g \) and \( F \) are triply continuously differentiable and that \( G \) is sufficiently smooth. The feasible set of the MOPECs is denoted by \( E \), which is nonempty in this paper.

In practice, many problems can be formulated as MOPECs \([1]\), see reference \([27]\). It has extensive applications in practical areas such as traffic control, economic modeling, engineering design, and so on. When the number of the objects reduces to one, this kind of problem is the mathematical programs with equilibrium constraints (MPEC). It is well known that the constraints of MPEC destroy the convexity, connectedness and closeness of the feasible region \([17]\). Meanwhile, MOPECs has the complexity caused by multiple objects. Then, MOPECs is generally difficult to deal with.

During the past two decades, many researchers have paid attentions to the mathematical programs with equilibrium constraints. There were many literatures about basic theory, various effective algorithms and applications of the MPEC \([4, 12, 15, 20]\). However, the papers about MOPECs were relatively few. They mainly studied the necessary optimization of the MOPECs and the constraints qualification of problem were abstract functions. In 2003, Ye and Zhu derived Fritz John type necessary optimality conditions and lead to Kuhn-Tucker type necessary optimality conditions under various constraint qualifications \([27]\). In 2007, Bao et al. used modern tools of variational analysis and generalized differentiation to gain the verified necessary optimality conditions for general problems \([2]\). In 2009, Boris S. Mordukhovich revealed a new Fredholm constraint qualification and established new qualified necessary optimality conditions for broad classes of the MOPECs in finite-dimensional and infinite-dimensional spaces \([21]\).

In this paper, we solve the multiobjective optimization problems with equilibrium constraints by combined homotopy method. The homotopy method established by Kellogg et al. \([11]\), Smale \([23]\) and Chow et al. \([3]\) is a powerful tool for solving non-linear problems, fixed point problems and complementarity problems (see \([14, 16, 26]\) and references therein). The advantage of this method is global convergence under certain weak assumptions and it was widely applied for solving other mathematic problems. In \([24, 29]\), the authors respectively solved multiobjective programming problem by homotopy method. Otherwise, the homotopy method was also used to deal with variational inequality problems \([5, 25, 30]\). In 2007, Li considered to solve MPEC by homotopy method in \([13]\). In this paper, we solve the multiobjective problem with general equilibrium constraints. Under certain assumptions, we get the equivalence formulation of the MOPECs \([1]\). Then, under some weak conditions, the existence and convergence of a smooth homotopy path from almost any initial interior point to a solution of the KKT system of equivalence problem.

The rest of this paper is organized as follows. In Section 2, we reformulate MOPECs as a general multiobjective programming under suitable assumptions. In Section 3, we recall some definitions and properties. In Section 4, we construct a new combined homotopy mapping and prove the existence and the convergence of a smooth homotopy path from almost any interior initial point to a K-K-T point of MOPECs under some assumptions. In Section 5, we give two numerical results to show the effectiveness of this algorithm. In Section 6, the conclusions are given.
2. Preparation. In what follows we consider the follower’s parametric variational problem:

\[ y \in S(x) \Leftrightarrow F(x, y)^T(v - y) \geq 0, \forall v \in C(x). \]

**Assumption 1.**

(A1) For each \( x \in X \) and \( i = 1, \cdots, l \), \( G_i(x, \cdot) \) is a convex function in the second argument.

(A2) For \((x, y) \in \Omega, \nabla_y F(x, y) + \sum_{i=1}^{l} (\nabla^2_{yy} G_i(x, y) + \nabla_y G_i(x, y) \nabla_y G_i(x, y)^T) \) is positive definite, where \( \Omega = \{(x, y) \in \mathbb{R}^{n+m} : g(x, y) \leq 0, G(x, y) \leq 0\} \).

**Lemma 2.1.** Let \( F \) be a twice continuously differentiable, \( G_i(i \in \{i, \cdots, l\}) \) be triply continuously differentiable. Suppose that each \( G_i(i \in \{1, \cdots, l\}) \) is convex function in the second argument. Then \( y \in S(x) \) if and only if there exists a unique \( u \in \mathbb{R}^l_+ \) such that

\[
\begin{cases}
F(x, y) + \nabla_y G(x, y)u = 0, \\
u \geq 0, G(x, y) \leq 0, UG(x, y) = 0,
\end{cases}
\]

where \( U = \text{diag}(u) \). \( \Box \) is called the K-K-T system of problem \( \Box \) (see \[10\]).

We denote the multipliers \( u \in \mathbb{R}^l_+ \) satisfying \( \Box \) by \( M(x, y) \). Let us write an explicit formulation for the set:

\[
M(x, y) = \{u \in \mathbb{R}^l_+ : F(x, y) + \sum_{i \in I(x, y)} u_i \nabla_y G_i(x, y) = 0, u_i = 0, \forall i \notin I(x, y)\},
\]

where \( I(x, y) = \{i : G_i(x, y) \neq 0\} \). It is easy to show that multiplier map \( M \) is closed by applying to a limit argument to the K-K-T system \( \Box \).

**Lemma 2.2.** \[18\] Let \( q_i(x)(i = 1, \cdots, l) \) be convex functions, if \( \Omega^0_q \) is nonempty, then for all \( x \in \partial \Omega_q, \{\nabla q_i(x), i \in I_q(x)\} \) is positive independent, i.e.,

\[
\sum_{i \in I_q(x)} u_i \nabla q_i(x) = 0, \quad u_i \geq 0, \forall i \notin I_q(x) \Rightarrow u_i = 0, (i \in I_q(x)).
\]

where \( \Omega_q = \{x|q_i(x) \leq 0, \ i = 1, \cdots, l\}, \Omega^0_q = \{x|q_i(x) < 0, \ i = 1, \cdots, l\}, \partial \Omega_q = \Omega_q \setminus \Omega^0_q, I_q(x) = \{i \in \{1, \cdots, l\}|q_i(x) = 0\} \).

In order to introduce the following lemmas, we need to introduce the sequentially bounded constraint qualification (SBCQ) and the Mangasarian-Fromovitz constraint qualification (MFCQ) for the MOPECs \( \Box \), are:

\( \text{SBCQ} \): For any convergent sequence \((x^{(k)}, y^{(k)}) \subset E\), there exists for each \( k \) multiplier vector \( u^{(k)} \in M(x^{(k)}, y^{(k)}) \) and \( u^{(k)} \) is bounded;

\( \text{MFCQ} \): The MFCQ is said to hold at a vector \( y \in C(x) \) if there exists a vector \( v \in \mathbb{R}^l \) such that

\[
v^T \nabla g_i(x, y) < 0, \ \text{for all} \ i \in I(x, y).
\]
Proposition 1. [17] Let \( F, g \) be twice continuously differentiable and each \( G_i(i \in \{1, \cdots, l\}) \) be triply continuously differentiable; let \( Z \) be closed. If the MFCQ holds at all pairs \((x, y) \in E\), then the SBCQ holds on \( E \).

Lemma 2.3. [17] Let \( F, g \) be twice continuously differentiable and each \( G_i(i \in \{1, \cdots, l\}) \) be triply continuously differentiable; let \( Z \) be closed. Suppose that each \( G_i(x, \cdot) \) is convex for all \( x \in X \). Assume that the SBCQ holds on \( E \) for the set-valued map \( M \) defined above, then the problem (1) is equivalent to the following minimization problem in the variables \((x, y, u)\)

\[
\begin{align*}
\min & \quad f(x, y) \\
\text{s.t.} & \quad g(x, y) \leq 0, \\
& \quad F(x, y) + \nabla_y G(x, y) u = 0, \\
& \quad u \geq 0, G(x, y) \leq 0, UG(x, y) = 0.
\end{align*}
\]

Lemma 2.4. [19] Let \( G_i(i \in \{1, \cdots, l\}) \) be triply continuously differentiable. Then, the MFCQ holds at the pair \((x, y) \in X \times C(x)\) if and only if

\[
\sum_{i \in I(x, y)} u_i \nabla_y G_i(x, y) = 0, u_i \geq 0, i \in I(x, y) \Rightarrow u_i = 0, \forall i \in I(x, y).
\]

With this setup, MOPECs (1) is equivalent to the problem (5). For solving optimization problem (5), we construct the following homotopy equation:

\[
h(\theta, t) = \begin{pmatrix} F(x, y) + \nabla_y G(x, y) u \\ UG(x, y) + te \end{pmatrix} = 0,
\]

where \( \theta = (x, y, u)^T \), \( e = (1, 1, \cdots, 1)^T \in \mathbb{R}^l \), \( t \in (0, 1] \).

Assumption 2.

(A3) for \( t \in (0, 1], \theta \in \Omega_2(t), \{\nabla g_i(\theta), i \in I(\theta), \nabla h(\theta, t)\} \) is full column rank.

Lemma 2.5. [6] Suppose that Assumptions A1-A3 hold, the set

\[
\Omega^0 = \{(x, y) \in \mathbb{R}^{n+m} : g(x, y) < 0, G(x, y) < 0\}
\]

is nonempty. when \( t \to 0 \), for all given point \((x, y) \in \Omega^0\), the solution curve \(\{y(t), u(t), t \in (0, 1]\\} \) of the homotopy equation (6) is continuous, bounded, unique, and as \( t \to 0 \), \( y(t) \) tends to the solution of the equation (6).

Let \( f(\theta) = f(x, y, u), g(\theta) = g(x, y, u) \), summarizing above, the MOPECs (1) is given by

\[
\begin{align*}
\min & \quad f(\theta) \\
\text{s.t.} & \quad g(\theta) \leq 0 \\
& \quad h(\theta, t) = 0
\end{align*}
\]

when \( t \to 0 \), then problem (7) is equal to MOPECs (1). In the following, we need only solving the MOPECs (7).
3. Some definitions and properties. For convenience, let
\[ \Omega_1(t) = \{ \theta \in \mathbb{R}^{n+m} \times \mathbb{R}_+^l : g(\theta) < 0, \ h(\theta, t) = 0 \}, \]
\[ \Omega_2(t) = \{ \theta \in \mathbb{R}^{n+m} \times \mathbb{R}_+^l : g(\theta) \leq 0, \ h(\theta, t) = 0 \}, \]
\[ \Omega(t) = \Omega_1(t) \times \Lambda^+ \times \mathbb{R}_+^l \times \mathbb{R}^{l+m}, \partial \Omega_1(t) = \{ \theta \in \Omega_2(t) : \prod_{i=1}^s g_i(\theta) = 0 \}, \]
\[ I(\theta) = \{ i \in \{ 1, \cdots, s \} : g_i(\theta) = 0 \}, \Lambda^+ = \{ \lambda \in \mathbb{R}_+^p : \sum_{i=1}^p \lambda_i = 1 \}, \]
\[ \Lambda^{++} = \{ \lambda \in \mathbb{R}_+^p : \sum_{i=1}^p \lambda_i = 1 \}. \]

For solving the MOPECs \([7]\), except Assumptions A1-A3, we make the following assumptions hold, which will be used throughout this paper.

**Assumption 3.**

(A4) \( \forall t \in [0, 1], \ \Omega_1(t) \) is nonempty, bounded, connected,

(A5) \( \forall \theta \in \Omega_2(t), t \in [0, 1] \)
\[ \{ \theta + \sum_{i \in I(\theta)} \alpha_i \nabla g_i(\theta) + \nabla h_\theta(\theta, 1) \beta : \alpha_i \geq 0, i \in I(\theta), \beta \in \mathbb{R}^{m+l} \} \cap \Omega_2(1) = \{ \theta \}, \]

**Remark 1.** The Assumption A5 is called the normal cone condition.

**Definition 3.1.** A point \((\bar{x}, \bar{y}) \in E\) is said to be an efficient solution to (MOPECs), if there is no \((x, y) \in E\) such that \(f(x, y) \leq f(\bar{x}, \bar{y})\).

**Definition 3.2.** Let \(U \subset \mathbb{R}^n\) be an open set, and let \(\varphi : U \rightarrow \mathbb{R}^p\) be a smooth mapping. If \(\text{Range }[\partial \varphi(x)/\partial x] = \mathbb{R}^p\) for all \(x \in \varphi^{-1}(y)\), then \(y \in \mathbb{R}^p\) is a regular value and \(x \in \mathbb{R}^n\) is a regular point.

**Lemma 3.3.** (Parametric form of the Sard theorem on a smooth manifold; see [28]). Let \(Q, N, P\) be smooth manifolds of dimensions \(q, m, p\) respectively, let \(\varphi : Q \times N \rightarrow P\) be a \(C^r\) map, where \(r \geq \max\{0, m-p\}\). If \(0 \in P\) is a regular value of \(\varphi\), then for almost all \(\alpha \in Q, 0\) is a regular value of \(\varphi(\alpha, \cdot)\).

**Lemma 3.4.** (Inverse image theorem; see [22]). If \(0\) is a regular value of the mapping \(\varphi_\alpha(\cdot) \triangleq \varphi(\alpha, \cdot)\), then \(\varphi^{-1}_\alpha(0)\) consists of some smooth manifolds.

**Lemma 3.5.** (Classification theorem of one-dimensional manifold; see [22]). A one-dimensional smooth manifold is diffeomorphic to a unit circle or a unit interval.

4. Main results. Let \(\theta \in \Omega_2(t)\). We say that \(\theta\) is a K-K-T point of MEOPCs \([7]\) if there exists \((\lambda, \alpha, \beta) \in \mathbb{R}_+^p \times \mathbb{R}_+^s \times \mathbb{R}^{m+l}\), such that
\[ \nabla f(\theta) \lambda + \nabla g(\theta) \alpha + \nabla h(\theta, t) \beta = 0, \]
\[ h(\theta, t) = 0, \]
\[ g(\theta) \leq 0, \ Ag(\theta) = 0, \]
\[ 1 - \sum_{i=1}^p \lambda_i = 0, \] (8)

where \(\nabla f(\theta) = (\nabla f_1(\theta), \nabla f_2(\theta), \cdots, \nabla f_p(\theta)) \in \mathbb{R}^{(n+m+l) \times p}\), \(\nabla g(\theta) = (\nabla g_1(\theta), \nabla g_2(\theta), \cdots, \nabla g_s(\theta)) \in \mathbb{R}^{(n+m+l) \times s}\), \(\nabla h(\theta, t) = (\nabla h_1(\theta, t), \nabla h_2(\theta, t), \cdots, \nabla h_{m+l}(\theta, t)) \in \mathbb{R}^{(n+m+l) \times (m+l)}\), \(A = \text{diag}(\alpha)\).

To solve the K-K-T system, we construct a homotopy equation as follows:
\[ H(w, w^{(0)}, t) = \begin{pmatrix}
(1 - t)(\nabla f(\theta) + \nabla g(\theta)\alpha + \nabla h(\theta, t)\beta + t(\theta - \theta^{(0)})
\end{pmatrix}_{h(\theta, t)}
\begin{pmatrix}
Ag(\theta) - tA^{(0)}g(\theta^{(0)})
(1 - t)(1 - \sum_{i=1}^{p} \lambda_i)e + t(\lambda^2 - (\lambda^{(0)})^2)
\end{pmatrix} = 0,
\]

where \( w^{(0)} = (\theta^{(0)}, \lambda^{(0)}, \alpha^{(0)}, \beta^{(0)})^T \in \Omega_1(1) \times \Lambda^+ \times \mathbb{R}^n_+ \times \{0\}, \ w = (\theta, \lambda, \alpha, \beta)^T \in \mathbb{R}^{n+m+n+1} \times \mathbb{R}^n_+ \times \mathbb{R}^{m+n+1}, \lambda^2 = (\lambda^2_1, \lambda^2_2, \ldots, \lambda^2_p)^T, \ t \in (0, 1]. \]

When \( t = 1 \), the homotopy equation \( H(\theta) \) becomes
\[ \nabla \theta h(\theta, t) + \theta - \theta^{(0)} = 0, \]
\[ h(\theta, 1) = 0 \]
\[ Ag(\theta) - A^{(0)}g(\theta^{(0)}) = 0, \]
\[ \lambda^2 - (\lambda^{(0)})^2 = 0. \]

If \( \beta \neq 0, \) \( (10a) \) contradicts to Assumption A5. Then, we get \( \beta = 0, \) \( \theta = \theta^{(0)}. \)

Since \( g(\theta^{(0)}) < 0, \) \( (10c) \) implies that \( \alpha = \alpha^{(0)}. \) Equation \( (10d) \) show that \( \lambda = \lambda^{(0)}. \)
That is, when \( t = 1, \) the equation \( H(w, w^{(0)}, t) = 0 \) has only one solution \( w^{(0)} = (\theta^{(0)}, \lambda^{(0)}, \alpha^{(0)}, 0)^T. \)

When \( t = 0, \) the homotopy equation \( H(w, w^{(0)}, t) = 0 \) turns to the K-K-T system \( \{ \}. \)

When \( w^{(0)} \) is given, we rewrite \( H(w, w^{(0)}, t) \) as \( H_{w^{(0)}}(w, t). \) Let \( H_{w^{(0)}}^{-1}(0) = \{(w, t) \in \Omega(t) \times (0, 1] : H_{w^{(0)}}(w, t) = 0\}. \)

**Theorem 4.1.** Suppose that \( H_{w^{(0)}}(w, t) \) is defined as in \( \{ \} \) and let \( f, g, \) and \( h \) be triply continuously differentiable functions. Assumptions A1-A5 hold. Then, for almost all initial points \( w^{(0)} = (\theta^{(0)}, \lambda^{(0)}, \alpha^{(0)}, \beta^{(0)})^T \in \Omega_1(1) \times \Lambda^+ \times \mathbb{R}^n_+ \times \{0\}, \) 0 is a regular value of \( H \) and \( H_{w^{(0)}}^{-1}(0) \) consists of some smooth curves. In addition, there is a smooth curve noted by \( \Gamma_{w^{(0)}} \), which is starting from \( (w^{(0)}, 1). \)

**Proof.** To prove the conclusion, we only need to consider the sub-matrix \( \frac{\partial H(w, w^{(0)}, t)}{\partial (\theta, \theta^{(0)}, \lambda^{(0)}, \alpha^{(0)})} \) of the Jacobi matrix of the \( H(w, w^{(0)}, t). \) \( \forall (w, t) = \Omega(t) \times (0, 1], \)
\[ \frac{\partial H(w, w^{(0)}, t)}{\partial (\theta, \theta^{(0)}, \lambda^{(0)}, \alpha^{(0)})} = \begin{pmatrix}
Q & -tI_{m+n+1} & 0 & 0 \\
\nabla \theta h(\theta, t)^T & 0 & 0 & 0 \\
A\nabla g(\theta)^T & -tA^{(0)}\nabla g(\theta^{(0)})^T & 0 & -\frac{q}{4}t(\lambda^{(0)})^2 I_p \\
0 & 0 & 0 & 0
\end{pmatrix}, \]
where \( Q = (1 - t) \left( \sum_{i=1}^{p} \lambda_i \nabla^2 f_i(\theta) + \sum_{j=1}^{n} \alpha_j \nabla^2 g_j(\theta) \right) + \sum_{k=1}^{m+n+1} \beta_k \nabla^2 h_k(\theta, t) + tI_{m+n+1}. \)

Because \( \theta^{(0)} \in \Omega_1(1), g(\theta^{(0)}) < 0, \) and \( \nabla \theta h(\theta, t) \) is full of column, then \( \frac{\partial H(w, w^{(0)}, t)}{\partial (\theta, \theta^{(0)}, \lambda^{(0)}, \alpha^{(0)})} \) is full of row. We can known that \( \frac{\partial H(w, w^{(0)}, t)}{\partial (w, w^{(0)}, t)} \) is full of row, and 0 is the regular value of mapping \( H. \) That is, 0 is a regular value of \( H_{w^{(0)}}(w, t). \)
By the parametric form of the Sard theorem, for almost all \((\theta(0), \lambda(0), \alpha(0), \beta(0)) \in \Omega_1(1) \times \Lambda^{++} \times R^*_+ \times \{0\}\), 0 is a regular value of \(H_{w(0)}\). By the inverse image theorem, \(H^{-1}_{w(0)}(0)\) consists of some smooth curves. Since \(H^{-1}_{w(0)}(w(0), 1) = 0\), there must be a smooth curve, denoted by \(\Gamma_{w(0)}\), starting from \((w(0), 1)\).

**Lemma 4.2.** Suppose that \(H\) is defined as in \([4]\) and Assumptions A3, A4 hold. Then for almost all initial points \(w(0) = (\theta(0), \lambda(0), \alpha(0), \beta(0)) \in \Omega_1(1) \times \Lambda^{++} \times R^*_+ \times \{0\}\), if \(0\) is regular value of \(H_{w(0)}(w, t)\), then the projection of the smooth curve \(\Gamma_{w(0)}\) on the component \(\lambda\) is bounded.

**Theorem 4.3.** Let \(f, g, h\) be triply continuously differentiable functions, and Assumptions A1-A5 hold. For almost all the initial point \(w(0) = (\theta(0), \lambda(0), \alpha(0), \beta(0)) \in \Omega_1(1) \times \Lambda^{++} \times R^*_+ \times \{0\}\), \(0\) is a regular value of \(H\), then the curve \(\Gamma_{w(0)} \subset \Omega(t) \times \{0, 1\}\) is bounded.

**Proof.** By Theorem 4.1, from almost all \(w(0) = (\theta(0), \lambda(0), \alpha(0), \beta(0)) \in \Omega_1(1) \times \Lambda^{++} \times R^*_+ \times \{0\}\), \(0\) is a regular value of \(H\) and \(H^{-1}_{w(0)}(0)\) contains a smooth curve \(\Gamma_{w(0)}\) starting from \((w(0), 1)\). Suppose \(\Gamma_{w(0)} \subset \Omega(t) \times \{0, 1\}\) is unbounded, by lemma 4.1 and lemma 4.2, we know that there exists a subsequence denoted also by \(\{w(k), t_k\} \subset \Gamma_{w(0)}\) such that \(x(k) \to x^*, y(k) \to y^*, u(k) \to u^* t_k \to t^*, \lambda(k) \to \lambda^*\), \(||(\alpha(k), \beta(k))|| \to \infty\), as \(k \to \infty\).

Let \(I = \{i \in \{1, \ldots, s\} : \alpha^*_i(k) \to \infty\}\), \(J = \{j \in \{1, \ldots, m + l\} : \beta^*_j(k) \to \infty\}\).

By \(H(w(k), w(0), t_k) = 0\), \((w(k), t_k)\) satisfy the following equations:

\[(1 - t_k)(\nabla f(\theta(k)) \lambda(k) + \nabla g(\theta(k)) \alpha(k) + \nabla h(\theta(k), t_k) \beta(k) + t_k(\theta(k) - \theta(0))) = 0, \quad (11a)\]
\[h(\theta(k), t_k) = 0, \quad (11b)\]
\[A^{(k)} g(\theta(k)) - t A^{(0)} g(\theta(0)) = 0, \quad (11c)\]
\[\lambda(k) = \sum_{i=1}^{p} \lambda_i(k) e - t_k((\lambda(k))^{(0)} - (\lambda(0))^{(0)}) = 0. \quad (11d)\]

(i) When \(t^* \in [0, 1]\), from \((11a)\), we have

\[-(1 - t_k)(\nabla f(\theta(k)) \lambda(k) + \sum_{i \in I} \alpha^*_i(k) \nabla g_i(\theta(k)) + \sum_{j \in J} \beta^*_j(k) \nabla h_j(\theta(k), t_k) =
\]
\[-(1 - t_k)(\nabla f(\theta(k)) \lambda(k) + \sum_{i \in I} \alpha^*_i(k) \nabla g_i(\theta(k)) + \sum_{j \in J} \beta^*_j(k) \nabla h_j(\theta(k)) - t_k(\theta(k) - \theta(0)).\]

Suppose

\[\alpha^*_i = \lim_{k \to \infty} \frac{\alpha^*_i(k)}{\max_{i \in I, j \in J} \{|\alpha^*_i(k)|, |\beta^*_j(k)|\}}, \quad \beta^*_j = \lim_{k \to \infty} \frac{\beta^*_j(k)}{\max_{i \in I, j \in J} \{|\alpha^*_i(k)|, |\beta^*_j(k)|\}}\]

when \(k \to \infty\),

\[-(1 - t_k)(\nabla f(\theta(k)) \lambda(k) + \sum_{i \in I} \alpha^*_i(k) \nabla g_i(\theta(k)) + \sum_{j \in J} \beta^*_j(k) \nabla h_j(\theta(k), t_k) - t_k(\theta(k) - \theta(0)))/ \max_{i \in I, j \in J} \{|\alpha^*_i(k)|, |\beta^*_j(k)|\} \to 0,\]
this contradicts to Assumption A3.

(ii) When $t^* = 1$, there three cases are possible. (a) $I = \emptyset$, $J \neq \emptyset$. (b) $I \neq \emptyset$, $J = \emptyset$. (c) $I \neq \emptyset$, $J \neq \emptyset$.

Case (a). Because $I = \emptyset$ and $|\beta_j^{(k)}| \to \infty$, then $\beta_j^{(k)} = \frac{\beta_j^{(k)}}{|\beta_j^{(k)}|}$ is bounded. When $k \to \infty$, let $\bar{\beta}(k) \to \beta^*$, by the equation (11a) we have $\nabla g_h(\theta^*,1)\beta^* = 0$. This contradicts to Assumption A3.

Case(b). Because $I \neq \emptyset$, $J = \emptyset$, we have

$$\lim_{k \to \infty} \left| \sum_{i \in I} (1 - t_k)\alpha_i^{(k)} \nabla g_i(\theta^{(k)}) + \sum_{j=1}^{m+l} \beta_j^{(k)} \nabla g_h(\theta^{(k)},t_k) \right| = 0,$$

then, $\lim_{k \to \infty} \left| \sum_{i \in I} (1 - t_k)\alpha_i^{(k)} \nabla g_i(\theta^{(k)}) + \sum_{j=1}^{m+l} \beta_j^{(k)} \nabla g_h(\theta^{(k)},t_k) \right|$ exists.

Let

$$(1 - t_k)\alpha_i^{(k)} \to \alpha_i^*, \; i \in I,$$

$$\beta_j^{(k)} \to \beta_j^*, \; j = 1, \ldots, m + l,$$

when $k \to \infty$, $\theta^* + \sum_{i \in I} \alpha_i^* \nabla g_i(\theta^*) + \sum_{j=1}^{m+l} \beta_j^* \nabla g_h(\theta^*,1) = \theta^{(0)}$, this contradicts to Assumption A5.

Case(c). We rewrite (11a) as

$$(1 - t_k)\nabla f(\theta^{(k)}) + ((1 - t_k)\nabla g(\theta^{(k)}), \nabla g_h(\theta^{(k)},t_k)) \left( \frac{\alpha^{(k)}}{\beta^{(k)}} \right) + t_k (\theta^{(k)} - \theta^{(0)}) = 0.$$

Because $\| (\alpha^{(k)}, \beta^{(k)}) \| \to \infty$, $(\alpha^{(k)}, \beta^{(k)}) = \frac{(\alpha^{(k)}, \beta^{(k)})}{\| (\alpha^{(k)}, \beta^{(k)}) \|}$ is bounded, there exist a finite subsequence denoted by itself. When $k \to \infty$, we suppose $(\alpha^{(k)}, \beta^{(k)}) \to (\alpha^*, \beta^*)$. It is obvious that $(\alpha^*, \beta^*) \neq 0$, then $\nabla g_h(\theta^{(k)})\alpha^* + \nabla g_h(\theta^{(k)},1)\beta^* = 0$, this contradicts to Assumption A3. So $\{(\alpha^{(k)}, \beta^{(k)})\}$ is bounded. \\

\begin{theorem}
Suppose $f$, $g$, $F$ are triply continuously differentiable and $G$ is sufficiently smooth, Assumptions A1-A5 hold. Then, when $t \to 0$, the solution of K-K-T system (8) exists and for almost all the initial point $w^{(0)} = (\theta^{(0)}, \lambda^{(0)}, \alpha^{(0)}, \beta^{(0)})^T \in \Omega_1(1) \times \Lambda^+ \times \mathbb{R}_+^m \times \{0\}$, $H^{-1}_{w^{(0)}}(0)$ contains a curve $\Gamma_{w^{(0)}}$ starting from $(w^{(0)}, 1)$. When $t \to 0$, the limit set $\Gamma \times \{0\} \subseteq \Omega_2(0) \times \Lambda^+ \times \mathbb{R}_+^m \times \{0\}$ of $\Gamma_{w^{(0)}}$ is nonempty and if $(w^*, 0)$ is the limit point of $\Gamma_{w^{(0)}}$, $w^*$ is the solution of the K-K-T system (8).
\end{theorem}

\begin{proof}
By Theorem 4.1, for almost all points $w^{(0)} = (\theta^{(0)}, \lambda^{(0)}, \alpha^{(0)}, \beta^{(0)})^T \in \Omega_1(1) \times \Lambda^+ \times \mathbb{R}_+^m \times \{0\}$, $0$ is a regular value of $H$ and $H^{-1}_{w^{(0)}}(0)$ consists of some smooth curves. Among them, there is a smooth curve noted by $\Gamma_{w^{(0)}}$ starting from $(w^{(0)}, 1)$.

By the classification theorem of one dimensional smooth manifolds, $\Gamma_{w^{(0)}}$ is diffeomorphic to a unit circle or unit interval. Because
The predictor-corrector algorithm and numerical experiments.

5. Algorithm 5.1 (Euler-Newton method).

Step 1. Compute an initial point.

(a) given $u = u_0$, compute function $h(\theta) = 0$, then we get some initial points

$$(w^{(0)}, 1) \in \Omega(1) \times \mathbb{A}^+ \times \mathbb{R}^*_{++} \times \{0\} \times \{1\};$$

(b) choose an initial step length $h_0 > 0$, and three small positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Let $k = 0$.

Step 2. Compute the direction $\zeta(k)$ of the predictor step:

(a) compute a unit tangent vector $\xi(k) \in\mathbb{R}^{2(m+1)+n+p+s+1}$ of $\Gamma_{w^{(0)}}$ at $(w^{(0)}, t_k)$;

(b) determine the direction $\zeta(k)$ of the predictor step.

If the sign of the determinant

$$DH_{w^{(0)}}(w^{(k)}, t_k) \xi(k)^T$$

is $(-1)^{m+1+s+n+p+1}$, take $\zeta(k) = \xi(k)$, or, take $\zeta(k) = -\xi(k)$, where $DH(w, w^{(0)}, t)$ is the Jacobi matrix of the $H(w, w^{(0)}, t)$.

Step 3. Compute a corrector point $(\tilde{w}^{(k+1)}, t_k)$:

$$(\tilde{w}^{(k)}, \tilde{t}_k) = (w^{(k)}, t_k) + h_k \zeta^{(k)},$$

$$(w^{(k+1)}, t_{k+1}) = (\tilde{w}^{(k)}, \tilde{t}_k) - DH_{w^{(0)}}(\tilde{w}^{(k)}, \tilde{t}_k)^* H_{w^{(0)}}((\tilde{w}^{(k)}, \tilde{t}_k))$$

$g(\theta^{(0)}) < 0$, it is easy to know that $\partial H_{w^{(0)}}(w, 1)/\partial w$ is nonsingular. Then, $\Gamma_{w^{(0)}}$ is diffeomorphic to a unit interval.

Let $(w^*, t^*)$ be the limit of $\Gamma_{w^{(0)}}$, the following three cases are possible:

(i) $(w^*, t^*) \in \Omega(t^*) \times \{1\}$;

(ii) $(w^*, t^*) \in \partial \Omega(t^*) \times (0, 1]$;

(iii) $(w^*, t^*) \in \partial \Omega(t^*) \times \{0\}$.

We know that the function $H_{w^{(0)}}(w, 1) = 0$ has a unique solution $(w^{(0)}, 1)$ in the set $\Omega(1) \times \{1\}$, then, case (i) is not possible. Obviously, by Theorem 4.3, case (ii) will not happen. Then, case (iii) is the unique possible case and $w^*$ is the solution of the K-K-T system \[8\].

5. The predictor-corrector algorithm and numerical experiments. In this section, in a framework, a predictor corrector procedure (c.f., e.g., [1]) for numerically tracing the homotopy path which is defined by the homotopy equation $H(w, w^{(0)}, t) = 0$.

**Algorithm 5.1 (Euler-Newton method).**

Step 1. Compute an initial point.

(a) given $u = u_0$, compute function $h(\theta) = 0$, then we get some initial points

$$(w^{(0)}, 1) \in \Omega(1) \times \mathbb{A}^+ \times \mathbb{R}^*_{++} \times \{0\} \times \{1\};$$

(b) choose an initial step length $h_0 > 0$, and three small positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Let $k = 0$.

Step 2. Compute the direction $\zeta^{(k)}$ of the predictor step:

(a) compute a unit tangent vector $\xi^{(k)} \in\mathbb{R}^{2(m+1)+n+p+s+1}$ of $\Gamma_{w^{(0)}}$ at $(w^{(0)}, t_k)$;

(b) determine the direction $\zeta^{(k)}$ of the predictor step.

If the sign of the determinant

$$DH_{w^{(0)}}(w^{(k)}, t_k) \xi^{(k)}^T$$

is $(-1)^{m+1+s+n+p+1}$, take $\zeta^{(k)} = \xi^{(k)}$, or, take $\zeta^{(k)} = -\xi^{(k)}$, where $DH(w, w^{(0)}, t)$ is the Jacobi matrix of the $H(w, w^{(0)}, t)$.

Step 3. Compute a corrector point $(w^{(k+1)}, t_k)$:

$$(\tilde{w}^{(k)}, \tilde{t}_k) = (w^{(k)}, t_k) + h_k \zeta^{(k)},$$

$$(w^{(k+1)}, t_{k+1}) = (\tilde{w}^{(k)}, \tilde{t}_k) - DH_{w^{(0)}}(\tilde{w}^{(k)}, \tilde{t}_k)^* H_{w^{(0)}}((\tilde{w}^{(k)}, \tilde{t}_k))$$

$g(\theta^{(0)}) < 0$, it is easy to know that $\partial H_{w^{(0)}}(w, 1)/\partial w$ is nonsingular. Then, $\Gamma_{w^{(0)}}$ is diffeomorphic to a unit interval.
where \( DH_{w^{(0)}}(w, t)^+ = DH_{w^{(0)}}^T(DH_{w^{(0)}}(w, t)DH_{w^{(0)}}(w, t)^T)^{-1} \) is the Moore-Penrose inverse of \( DH_{w^{(0)}}(w, t) \).

If \( t > 1 \), let \( h_k = h_k \frac{t_k}{t_{k+1}} \), go to step3.

If \( \|H_{w^{(0)}}(w^{(k+1)}, t_{k+1})\| \leq \varepsilon_1 \), let \( h_{k+1} = min\{h_0, 2h_k\} \), go to step4.

If \( \|H_{w^{(0)}}(w^{(k+1)}, t_{k+1})\| \in (\varepsilon_1, \varepsilon_2) \), let \( h_{k+1} = h_k \), go to step 4.

If \( \|H_{w^{(0)}}(w^{(k+1)}, t_{k+1})\| \geq \varepsilon_2 \), let \( h_{k+1} = max\{\frac{1}{2}h_0, \frac{1}{2}h_k\} \), go to step3.

**Step 4.** If \( w^{(k+1)} \in \Omega(t) \times \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^{m+l} \) and \( t_{k+1} > \varepsilon_3 \), let \( k = k+1 \) and go to step2.

If \( w^{(k+1)} \in \Omega(t) \times \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^{m+l} \) and \( t_{k+1} < 0 \), let \( h_k = h_k \frac{t_k}{t_{k+1}} \), go to step3.

If \( w^{(k+1)} \not\in \Omega(t) \times \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^{m+l} \), let \( h_k = \frac{h_k}{t_{k+1}} \), go to step3.

If \( w^{(k+1)} \in \Omega(t) \times \mathbb{R}^p_+ \times \mathbb{R}^m_+ \times \mathbb{R}^{m+l} \) and \( |t_{k+1}| \leq \varepsilon_3 \), then stop.

**Remark 2.** In the paper, we determine the predictor direction by computing the tangent vector at a point \( (w, t) \) on \( \Gamma_{w^{(0)}} \). For tracing to the K-K-T point, we must go along the positive direction. The sign of the tangent vector is determined by \( \frac{DH_{w^{(0)}}(w^{(k)}, t_k)}{\xi^{(k)^T}} \).

**Numerical experiments 5.2.**

To show the efficiency of the homotopy method for solving MOPECs, we implement the Algorithm 5.1 for two test problems. All numerical experiments are done by running MATLAB on a PC with CPU of 2.0 GHZ and RAM 1.99 GB. Consider the MOPECs (1), f, g, G are given as follows. Because of the lack of the test problem, We didn’t compare to other algorithms. For every test problem, we get four initial point \( w^{(0)} = (x^{(0)}, y^{(0)}, \lambda^{(0)}, \alpha^{(0)}, \beta^{(0)}) \in \Omega_1(1) \times \Lambda^{++} \times \mathbb{R}^p_+ \times \{0\} \) tracing the limiting point.

**Example 5.1** We have

\[
\begin{align*}
    f(x, y) &= \left( \frac{1}{2}(x_1 - y_1)^2 + (x_2 - y_2)^2 \right) \\
    g_1(x, y) &= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 1000 \leq 0, \\
    g_2(x, y) &= (x_1 - 1)^2 + (x_2 - 1)^2 + y_1^2 + y_2^2 - 5000 \leq 0, \\
    F(x, y) &= \left( -4 + 10y_2 + 2x_1 \right), \\
    G_1(x, y) &= x_2 + y_1 - 4 \leq 0, \\
    G_2(x, y) &= x_1 + y_2 - 4 \leq 0.
\end{align*}
\]

The results are shown in Table 1.

**Example 5.2** We have

\[
\begin{align*}
    f(x, y) &= \left( \frac{1}{2}(x_1 - y_1)^2 + (x_2 - y_2)^2 \right) \\
    g(x, y) &= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 1000 \leq 0, \\
    F(x, y) &= \left( -4 + 8y_1 + 2y_2 \right), \\
    G(x, y) &= x_1^2 + x_2^2 + y_1^2 + y_2^2 - 20 \leq 0.
\end{align*}
\]

The results are shown in Table 2.
6. Conclusion. In this paper, based on the theory and algorithm of the multi-objective optimization and mathematical programs with variational inequality, we solve MOPECs by homotopy method. Under some assumptions which have been introduced, the original problem is reformed as a problem which can be solved by homotopy method and existence and convergence of a smooth homotopy method is proved. The numerical results show the proposed method is feasible and efficient.

In the future, we attempt to find new and weaker assumptions. It is desirable that those assumptions be easily verifiable and possibly be associated from a practical point of view with convergence analysis of nonlinear optimization algorithms.

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