AMENABILITY, TUBULARITY, AND EMBEDDINGS INTO $\mathcal{R}^\omega$

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Dedicated to Ed Effros on the occasion of his 70th birthday

ABSTRACT. Suppose $M$ is a tracial von Neumann algebra embeddable into $\mathcal{R}^\omega$ (the ultraproduct of the hyperfinite $\Pi_1$-factor) and $X$ is an $n$-tuple of selfadjoint generators for $M$. Denote by $\Gamma(X; m, k, \gamma)$ the microstate space of $X$ of order $(m, k, \gamma)$. We say that $X$ is tubular if for any $\epsilon > 0$ there exist $m \in \mathbb{N}$ and $\gamma > 0$ such that if $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \Gamma(X; m, k, \gamma)$, then there exists a $k \times k$ unitary $u$ satisfying $|u x_i u^* - y_i|_2 < \epsilon$ for each $1 \leq i \leq n$. We show that the following conditions are equivalent:

- $M$ is amenable (i.e., injective).
- $X$ is tubular.
- Any two embeddings of $M$ into $\mathcal{R}^\omega$ are conjugate by a unitary $u \in \mathcal{R}^\omega$.

1. INTRODUCTION

One version of Voiculescu’s free entropy involves matricial microstates. Given an $n$-tuple of selfadjoint elements $X = \{x_1, \ldots, x_n\}$ in a von Neumann algebra $M$ with trace $\varphi$, denote by $\Gamma(X; m, k, \gamma)$ the set of all $n$-tuples of selfadjoint $k \times k$ matrices $(a_1, \ldots, a_n)$ such that for any $1 \leq p \leq m$ and $1 \leq i_1, \ldots, i_p \leq n$,

$$|tr_k(a_{i_1} \cdots a_{i_p}) - \varphi(x_{i_1} \cdots x_{i_p})| < \gamma$$

where $tr_k$ denotes the normalized trace on the $k \times k$ matrices. Such an $n$-tuple $(a_1, \ldots, a_n)$ is called a microstate for $X$ and the sets $\Gamma(X; m, k, \gamma)$ are called the microstate spaces of $X$.

One can use microstates and random matrices to obtain nonisomorphism results (for an overview see [9]). At some point all of these kinds of arguments rely on the fact that amenability (=hyperfiniteness=injectivity by [1]; see also [2]-[6] and [8]) forces an extremely rigid condition on the microstate spaces. When $X''$ is amenable then the following is true: given $\epsilon > 0$ there exist $m \in \mathbb{N}$ and $\gamma > 0$ such that for any two elements $\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_n) \in \Gamma(X; m, k, \gamma)$ there exists a $k \times k$ unitary $u$ such that for each $1 \leq i \leq n$, $|u \xi_i u^* - \eta_i|_2 < \epsilon$. Roughly speaking then, a microstate space of $X$ is the neighborhood of the unitary orbit of a single microstate of the space.

In this note we observe the converse. To be more exact, suppose $X''$ embeds into $\mathcal{R}^\omega$ and satisfies the aforementioned geometric property: for any $\epsilon > 0$ there exist $m \in \mathbb{N}$ and $\gamma > 0$ such that if $\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_n) \in \Gamma(X; m, k, \gamma)$, then there exists a $k \times k$ unitary $u$ satisfying $|u \xi_i u^* - \eta_i|_2 < \epsilon, 1 \leq i \leq n$. We show that $X''$ is semidiscrete in the sense of [5] and thus, by [2] amenable.

We also notice that this characterization is equivalent to the condition that any two embeddings $\sigma, \pi$ of $X''$ into $\mathcal{R}^\omega$ are conjugate by a unitary $u \in \mathcal{R}^\omega$. In the course of the proof we will introduce seemingly weaker notions of tubularity: finite tubularity, quasitubularity, and finite quasitubularity. All of these notions will be shown to coincide with amenability as well.

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2. Some Other Characterizations of Amenability

Throughout $M$ will be a von Neumann algebra with tracial, faithful, normal state $\varphi$ and $X = \{x_1, \ldots, x_n\}$ is a finite generating set of selfadjoints of $M$. Fix a nontrivial ultrafilter $\omega$ of $\mathbb{N}$; there exists an obvious net $i : \Lambda \to \mathbb{N}$ such that for any bounded sequence of complex numbers, $\langle c_n \rangle_{n=1}^{\infty}$, $\omega(\langle c_n \rangle_{n=1}^{\infty}) = \lim_{n \in \Lambda} c_n(\lambda)$. $\mathcal{R}$ will be a fixed copy of the hyperfinite $II_1$-factor, and $\mathcal{R}^\omega$ will be the associated ultraproduct construction with respect to $\omega$ and $\mathcal{R}$. $Q : \ell^\infty(\mathcal{R}) \to \mathcal{R}^\omega$ is the obvious quotient map. By "embedding" we mean a normal, injective $*$-homomorphism which preserves units and the traces.

We will be constantly working with finite tuples of operators and for this reason introduce the following, somewhat abusive notation. For an ordered $n$-tuple $\xi = \{\xi_1, \ldots, \xi_n\}$ in a von Neumann algebra $\mathcal{N}$ and a unitary $u \in \mathcal{N}$ $u \xi u^* = \{u \xi_1 u^*, \ldots, u \xi_n u^*\}$. If $F$ is a map from $\mathcal{N}$ into some set $S$, then we carelessly write $F(\xi)$ for $\{F(\xi_1), \ldots, F(\xi_n)\}$. If $\eta = \{\eta_1, \ldots, \eta_n\} \subset \mathcal{N}$, then $\xi - \eta = \{\xi_1 - \eta_1, \ldots, \xi_n - \eta_n\}$ and $|\xi|_2 = (\sum_{i=1}^{n} \tau(\xi_i^2))^{1/2}$ where $\tau$ is a tracial state on $\mathcal{N}$. We will assume throughout that $M$ embeds into $\mathcal{R}^\omega$. For $k \in \mathbb{N}$, $M_k(\mathbb{C})$ denotes the $k \times k$ matrices and $U_k$ is the unitary group of $M_k(\mathbb{C})$.

**Definition 2.1.** $X$ is $N$-tubular ($N \in \mathbb{N}$) if for any $\epsilon > 0$ there exist $m \in \mathbb{N}$ and $\gamma > 0$, such that if $\xi_1, \ldots, \xi_{N+1} \in \Gamma(X; m, k, \gamma)$, then there exists a $u \in U_k$ satisfying $|u \xi_i u^* - \xi_j|_2 < \epsilon$ for some $1 \leq i < j \leq N + 1$. $X$ is finitely tubular if $X$ is $N$-tubular for some $N \in \mathbb{N}$. $X$ is simply tubular if $X$ is 1-tubular.

Thus, $X$ is finitely tubular if it can be encapsulated in the unitary orbits of no more than $N$ of its microstates. Clearly, tubularity coincides with the definition given in the introduction. We have a similar notion for quasitubularity:

**Definition 2.2.** $X$ is $N$-quasitubular ($N \in \mathbb{N}$) if for any $\epsilon > 0$ there exist $m \in \mathbb{N}$ and $\gamma > 0$, such that for any $\xi_1, \ldots, \xi_{N+1} \in \Gamma(X; m, k, \gamma)$ there exists a $p$ (dependent on $\xi_1, \ldots, \xi_{N+1}$) and a unitary $u$ in $M_k(\mathbb{C}) \otimes M_p(\mathbb{C})$ satisfying $|u(\xi_i \otimes I_p)u^* - \xi_j \otimes I_p|_2 < \epsilon$ for some $1 \leq i < j \leq N + 1$. $X$ is finitely quasitubular if $X$ is $N$-quasitubular for some $N$. $X$ is simply quasitubular if $X$ is 1-quasitubular.

**Remark 2.3.** Obviously if $X$ is $N$-tubular, then $X$ is $N$-quasitubular.

**Lemma 2.4.** If $X$ is tubular, then any two embeddings $\sigma, \pi$ of $X'$ into $\mathcal{R}^\omega$ are conjugate by a unitary in $\mathcal{R}^\omega$.

**Proof.** Suppose $X$ is tubular and $\sigma, \pi : X' \to \mathcal{R}^\omega$ are two embeddings. Find algebras $A_k \subset \mathcal{R}$ such that for each $k$, $A_k \simeq M_{r(k)}(\mathbb{C})$ for some $r(k) \in \mathbb{N}$ and such that for each $1 \leq j \leq n$ there exist sequences $y_j = \langle y_{jk} \rangle_{k=1}^{\infty}, z_j = \langle z_{jk} \rangle_{k=1}^{\infty} \in \ell^\infty(\mathcal{R})$ satisfying $y_{jk}, z_{jk} \in A_k$ for each $k$, $\pi(y_j) = Q(y_j)$, and $\sigma(x_j) = Q(z_j)$.

For each $p \in \mathbb{N}$ there exists by tubularity a corresponding $m(p) \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, if $\xi, \eta \in \Gamma(F; m(p), k, m(p)^{-1})$, then there exists a $u \in U_k$ satisfying $|u \xi u^* - \eta|_2 < p^{-1}$. For each $k$ pick a unitary $w_k \in A_k \simeq M_{r(k)}(\mathbb{C})$ satisfying

$$\max_{1 \leq j \leq n} |w_k y_{jk} w_k^* - z_{jk}|_2 = \inf_{w \in U(A_k)} \left( \max_{1 \leq j \leq n} |wy_{jk} w^* - z_{jk}|_2 \right)$$

where $U(A_k)$ ($\simeq U_k$) is the unitary group of $A_k$. Define $w = \langle w_{jk} \rangle_{k=1}^{\infty}, w \in \ell^\infty(M)$. Given an $N \in \mathbb{N}$, for $\lambda$ sufficiently large $\langle y_{i(\lambda)}, \ldots, y_{m(\lambda)} \rangle, \langle z_{i(\lambda)}, \ldots, z_{m(\lambda)} \rangle \in \Gamma(F; m(N), i(\lambda), m(N)^{-1})$ which in turn implies that for such $\lambda$ and all $1 \leq j \leq n$

$$|w_{i(\lambda)} y_{(\lambda)j} w_{i(\lambda)}^* - z_{i(\lambda)j}|_2 < N^{-1}.$$ Set $u = Q(w) \in \mathcal{R}^\omega$. We have that for all $1 \leq j \leq n$, $\sigma(x_j) = u \pi(x_j) u^*$ which completes the proof. \qed
Lemma 2.5. If any two embeddings \( \sigma \) and \( \pi \) of \( \mathcal{M} \) into \( \mathcal{R}' \) are conjugate by a unitary in \( \mathcal{R}' \), then \( X \) is quasitubular.

Proof. Suppose by contradiction that \( X \) is not quasitubular. For some \( \epsilon_0 > 0 \) and any \( m \in \mathbb{N} \) and \( \gamma > 0 \) there exists an \( N \in \mathbb{N} \) and \( \xi, \eta \in \Gamma(X; m, N, \gamma) \) such that for all \( p \in \mathbb{N} \),

\[
\inf_{u \in U_{N_p}} |u(\xi \otimes I_p)u^* - \eta \otimes I_p|_2 > \epsilon_0.
\]

Thus, for each \( m \in \mathbb{N} \) we can find a corresponding \( N_m \in \mathbb{N} \) and \( \xi_m, \eta_m \in \Gamma(X; m, N_m, m^{-1}) \) such that for any \( k \in \mathbb{N} \) \( \inf_{u \in U_{N_k}} |u(\xi_m \otimes I_k)u^* - \eta_m \otimes I_k|_2 > \epsilon_0 \). Without loss of generality we may assume that the operator norms of any of the elements in \( \xi_m \) or \( \eta_m \) are strictly less than \( C = \max_{x \in X} \|x\| + 1 \).

For each \( m, \mathcal{R} = M_{N_m}(\mathbb{C}) \otimes \mathcal{R}_m \) where \( \mathcal{R}_m \simeq \mathcal{R} \); define \( x_m = \xi_m \otimes I \) and \( y_m = \eta_m \otimes I \) with respect to this decomposition of \( \mathcal{R} \) and set \( x = \langle x_m \rangle_{m=1}^\infty \) and \( y = \langle y_m \rangle_{m=1}^\infty \). It is not too hard to see that we can find two embeddings \( \pi, \sigma : M \to \mathcal{R}' \) satisfying \( \pi(X) = Q(x) \) and \( \sigma(X) = Q(y) \).

By hypothesis there exists a unitary \( u \in \mathcal{R}' \) satisfying \( \sigma(x) = u\pi(x)u^* \) for all \( x \in M \). We can find some \( w = \langle w_m \rangle_{m=1}^\infty \in l^\infty(\mathcal{R}) \) such that \( Q(w) = u \). Because \( u \) is a unitary we can assume that for each \( m, w_m \in \mathcal{R} \) is a unitary. The condition \( \sigma(x) = u\pi(x)u^* \) implies that there exists a \( \lambda \in \Lambda \) such that

\[
|w_{i(\lambda)}(\xi_{i(\lambda)} \otimes I)w_{i(\lambda)}^* - \eta_{i(\lambda)} \otimes I|_2 = |w_{i(\lambda)}x_{i(\lambda)}w_{i(\lambda)}^* - y_{i(\lambda)}|_2 < \epsilon_0/3C.
\]

Now, \( w_{i(\lambda)} \in \mathcal{R} = M_{N_{i(\lambda)}}(\mathbb{C}) \otimes \mathcal{R}_{i(\lambda)} \) and by standard approximations we can find some \( p_0 \in \mathbb{N} \), a unital \(*\)-algebra \( A_\lambda \simeq M_{p_0}(\mathbb{C}) \) with \( A_\lambda \subset \mathcal{R}_{i(\lambda)} \), and a unitary \( u \in M_{N_{i(\lambda)}} \otimes A_\lambda \subset \mathcal{R} \) satisfying \( |u - w_{i(\lambda)}|_2 < \epsilon_0/3C \). It is then clear that

\[
\inf_{v \in U_{i(\lambda)p_0}} |v(\xi_{i(\lambda)} \otimes I_{p_0})v^* - \eta_{i(\lambda)} \otimes I_{p_0}|_2 \leq |u(\xi_{i(\lambda)} \otimes I_{p_0})u^* - \eta_{i(\lambda)} \otimes I_{p_0}|_2 < \epsilon_0.
\]

which contradicts the initial hypothesis. \( \square \)

We now present a lemma which is undoubtedly known but which we will prove for completeness. Recall that a von Neumann algebra \( N \) is semidiscrete if there exist nets \( \langle \phi_j \rangle_{j \in \Omega} \) and \( \langle \psi_j \rangle_{j \in \Omega} \) of unital completely positive maps \( \phi_j : N \to M_{n_j}(\mathbb{C}) \) and \( \psi_j : M_{n_j}(\mathbb{C}) \to N, n_j \in \mathbb{N} \) such that for any \( x \in N, \langle \phi_j \circ \psi_j \rangle(x) \to x \) \( \sigma \)-weakly. This is not the original definition of semidiscreteness found in [5] (which demands that the maps only have finite rank, but it is equivalent to that definition by [3] and [4]).

Semidiscreteness, which was introduced in [5], is yet another characterization of amenability ([1] again).

Lemma 2.6. Suppose \( A \) is a tracial von Neumann algebra. Assume that for some finite set of generators \( F \) of \( A \) and any \( \epsilon > 0 \) there exist an embedding \( \pi_\epsilon \) of \( A \) into a tracial von Neumann algebra \( A_\epsilon \) and a unital, finite dimensional algebra \( B_\epsilon \subset A_\epsilon \) with the property that every element of \( \pi_\epsilon(F) \) is contained in the \( \epsilon \)-neighborhood of \( B_\epsilon \) with respect to the \( | \cdot |_2 \)-norm of \( A_\epsilon \). Then \( A \) is semidiscrete, and thus amenable.

Proof. The hypothesis implies that for any finite set \( S \) of \( A \) and any \( \epsilon > 0 \) there exists an embedding \( \pi_\epsilon \) of \( A \) into a tracial von Neumann algebra \( A_\epsilon \) and a finite dimensional unital subalgebra \( B_\epsilon \) of \( A_\epsilon \) such that every element of \( \pi_\epsilon(S) \) is contained in the \( \epsilon \)-neighborhood of \( B_\epsilon \) with respect to \( | \cdot |_2 \). This is because the elements of \( \pi_\epsilon(F) \) can be approximated in \( | \cdot |_2 \)-norm by elements in \( B_\epsilon \) with operator norms no bigger than the maximum of the operator norms of elements in \( F \) (one uses conditional expectations as we do below) and because multiplication is \( | \cdot |_2 \)-continuous on operator norm bounded sets. This gives the implication for \( S \) consisting of polynomials of elements from \( F \) and the general case follows immediately.
To complete the proof it suffices to show that for any finite $S \subset A$ we can construct sequences of u.c.p. maps $\phi_n : A \to M_{k_n} (\mathbb{C})$ and $\psi_n : M_{k_n} (\mathbb{C}) \to A$, $k_n \in \mathbb{N}$, such that for any normal linear functional $f$ on $N$, $f((\phi_n \circ \psi_n)(x)) \to f(x)$ as $n \to \infty$. Thus, let the finite subset $S$ of $A$ be given. By the first paragraph for each $n \in \mathbb{N}$ we can find a tracial von Neumann algebra $A_n$, an embedding $\pi_n : A \to A_n$ and a finite dimensional, unital subalgebra $B_n$ of $A_n$ such that every element of $S$ is contained in the $n^{-1}$-neighborhood of $B_n$ with respect to the $\cdot \nu$-norm of $A_n$. Define $\phi_n : A \to B_n$ to be the composition of $\pi_n$ with the conditional expectation $E_n$ of $A_n$ onto $B_n$. Define $F_n$ to be the conditional expectation of $A_n$ onto $\pi_n (A)$ restricted to $B_n$; so $F_n : B_n \to \pi_n (A)$. Define $\psi_n : B_n \to A$ to be the composition of $F_n$ with $\pi_n^{-1}$. Obviously for any $x \in S$

$$\|(\psi_n \circ \phi_n)(x) - x\| = |F_n (E_n (\pi_n (x))) - \pi_n (x)| \leq |E_n (\pi_n (x)) - \pi_n (x)| < \frac{1}{n}.$$  

Now $(\psi_n \circ \phi_n)(x)_{n=1}^{\infty}$ is a sequence in $A$ uniformly bounded in the operator norm and $(\psi_n \circ \phi_n)(x) \to x$ in the $\cdot \nu$-norm. This implies $(\psi_n \circ \phi_n)(x) \to x$ $\sigma$-weakly for every $x \in S$. \hfill \square

**Remark 2.7.** Suppose $N$ is a von Neumann algebra, and $A \subset N$ is a von Neumann subalgebra. By the above lemma if for some finite set of generators $F$ of $A$ and any $\epsilon > 0$ there exists a finite dimensional algebra $B \subset N$ such that every element of $F$ is contained in the $\epsilon$-neighborhood of $B$ with respect to $\cdot \nu$, then $A$ is amenable.

**Lemma 2.8.** If $X$ is quasitubular, then $M$ is amenable.

**Proof.** For each $m \in \mathbb{N}$ there exist a $k(m) \in \mathbb{N}$ and a $\xi_m = (\xi_{1m}, \ldots, \xi_{nm}) \in \Gamma (X; m, k(m), m^{-1})$. $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \mathcal{R}_3$ where $\mathcal{R}_i \cong \mathcal{R}$, $i = 1, 2, 3$. Define for $1 \leq j \leq n$, $y_j = (I \otimes \xi_{jm} \otimes I)_{m=1}^{\infty} \in \ell^\infty (\mathcal{R})$ where $I \otimes \xi_{jm} \otimes I \in \mathcal{R}_1 \otimes \mathcal{B}_m \otimes \mathcal{R}_3 \subset \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \mathcal{R}_3 = \mathcal{R}$ and $M_{k(m)} (\mathbb{C}) \cong \mathcal{B}_m \subset \mathcal{R}_2$. It is clear that there exists a (trace preserving) embedding $\pi : X'' \to \mathcal{R}^\omega$ satisfying $\pi (x_j) = Q (y_j)$. Given $\epsilon > 0$ we can find by quasitubularitity an $m_0 \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ and $\xi, \eta \in \Gamma (X; m_0, k, m_0^{-1})$ there exists a corresponding $p(k) \in \mathbb{N}$ and unitary $u$ of $M_k (\mathbb{C}) \otimes M_{p(k)} (\mathbb{C})$ satisfying $|u (\xi \otimes I_{p(k)}) u^\ast - \eta \otimes I_{p(k)}| \leq \epsilon$. Fix once and for all, an $N \in \mathbb{N}$ such that there exists a $\zeta \in \Gamma (X; m_0, N, m_0^{-1})$. We can regard $\zeta$ as a subset of $\mathcal{R}_1$ by finding a copy $A$ of the $N \times N$ matrices in $\mathcal{R}_1$ and for each $m$ consider $\zeta \otimes I \otimes I, I \otimes \zeta \otimes I \subset (A \otimes \mathcal{B}_m) \otimes \mathcal{R}_3 \subset \mathcal{R}$. For each $m$ we can find an algebra $D_m$ isomorphic to a full matrix algebra, and a unitary $u_m$ of $A \otimes \mathcal{B}_m \otimes D_m$ such that $$|u_m (\zeta \otimes I \otimes I) u_m^\ast - I \otimes \xi \otimes I|_2 < \epsilon.$$  

Define $\rho : A \to \mathcal{R}^\omega$ by $\rho (x) = Q (u_m (x \otimes I \otimes I) u_m^\ast)_{m=1}^{\infty}$. We have now shown that for $\epsilon > 0$, every element of $\pi (X)$ is within the $\cdot \nu$-ball of the finite dimensional full matrix algebra $\rho (A)$.

By the remark this implies that the von Neumann algebra generated by $\pi (X)$ is semidiscrete, and hence, amenable. Since $\pi$ is an isomorphism, $X'' = M$ is amenable. \hfill $\square$

At this point we have already demonstrated the equivalence claimed in the introduction. We will now prove $\epsilon$ more. Notice that the content of the observation below is the implication that finite quasitubularity implies amenability. This will be very much like the proof above modulo some technicalities. We could have gone straight to the following more technical argument without proving Lemma 2.8, but the proof of Lemma 2.8 has the advantage of being more lucid.

**Lemma 2.9.** The following are equivalent:

1. $M$ is amenable.
2. $X$ is tubular.
3. $X$ is finitely tubular.
4. $X$ is quasitubular.
5. $X$ is finitely quasitubular.
(6) Any two embeddings of $M$ into $\mathcal{R}^\omega$ are conjugate by a unitary in $\mathcal{R}^\omega$.

Proof. By [2] $M$ is amenable iff $M$ is hyperfinite and thus by [7], $X$ must be tubular. Hence, (1) $\Rightarrow$ (2). By Lemma 2.4, (2) $\Rightarrow$ (6) and by Lemma 2.5 (6) $\Rightarrow$ (4). Clearly (4) $\Rightarrow$ (5).

I will now show (5) $\Rightarrow$ (1). Suppose $X$ is finitely quasitubular. Find the smallest $N \in \mathbb{N}$ for which $X$ is $N$-quasitubular. By Lemma 2.6 we can assume $N > 1$. $X$ is not $(N - 1)$-quasitubular, which implies that we can find some $\epsilon_1 > 0$, such that for any $m \in \mathbb{N}$ there exists some $k \in \mathbb{N}$, and $\xi_1, \ldots, \xi_N \in \Gamma(X; m, k, m^{-1})$ such that for any $p \in \mathbb{N}$ and unitary $u$ of $M_k(\mathbb{C}) \otimes M_p(\mathbb{C})$,

$$\min_{1 \leq i \neq j \leq N} |u(\xi_i \otimes I_p)u^* - \xi_j \otimes I_p|_2 > \epsilon_1.$$ 

Let $\epsilon_1 > \epsilon > 0$ be given. There exists an $m_0 \in \mathbb{N}$ such that if $\xi_1, \ldots, \xi_{N+1} \in \Gamma(X; m_0, k, m_0^{-1})$ then there exists some $p \in \mathbb{N}$ and unitary $u$ in $M_{k_0}(\mathbb{C}) \otimes M_p(\mathbb{C})$ ($u$ and $p$ dependent on the $\xi_i$) satisfying $|u(\xi_i \otimes I_p)u^* - \eta_2|_2 < \epsilon$ for some $1 \leq i < j \leq N + 1$. On the other hand, by our initial remark, there is some $k(0) \in \mathbb{N}$, and $\eta_1, \ldots, \eta_N \in \Gamma(X; m_0, k(0), m_0^{-1})$ such that for any $p \in \mathbb{N}$ and unitary $u$ of $M_{k(0)}(\mathbb{C}) \otimes M_p(\mathbb{C})$,

$$\min_{1 \leq i < j \leq N} |u(\eta_i \otimes I_p)u^* - \eta_j \otimes I_p|_2 > \epsilon > \epsilon.$$ 

Now there exists a sequence $\langle \zeta_m \rangle_{m=1}^\infty$ such that for each $m$ there exists a $k(m) \in \mathbb{N}$ with $\zeta_m \in \Gamma(X; m, k(m), m^{-1})$. Identifying $M_{k(m)}(\mathbb{C})$ with $M_{k(0)}(\mathbb{C}) \otimes M_{k(m)}(\mathbb{C})$ for all $m \geq m_0$ we have that $\xi_1 \otimes I_{k(0)}, \ldots, \xi_N \otimes I_{k(0)}, I_{k(0)} \otimes \zeta_m \in \Gamma(X; m_0, k(0), m_0^{-1})$. (1) and the $N$-quasitubularity of $X$ implies that there must exist an $1 \leq i_m \leq N$, a $p_m \in \mathbb{N}$, and a unitary $u_m$ of $M_{k(0)}(\mathbb{C}) \otimes M_{p_m}(\mathbb{C})$ satisfying

$$|u_m(\xi_{i_m} \otimes I_{p_m})u_m^* - \zeta_m \otimes I_{p_m}|_2 < \epsilon.$$ 

$\langle \zeta_m \rangle_{m=1}^\infty$ is a sequence taking integral values between 1 and $N$ and thus $i_m = i$ for some $1 \leq i \leq N$ and infinitely many $m$. Without loss of generality assume $i_m = 1$ for some increasing sequence $\langle m_q \rangle_{q=1}^\infty$ of $\mathbb{N}$.

For each $q \in \mathbb{N}$ set $\theta_q = \zeta_{m_q} \in \Gamma(X; m_q, k(m_q), m_q^{-1})$. $\mathcal{R} = \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \mathcal{R}_3$ where $\mathcal{R}_i \simeq \mathcal{R}$, $i = 1, 2, 3$. Define $Y = \langle I \otimes \theta_q \otimes I \rangle_{q=1}^\infty \in (\ell^\infty(\mathcal{R}))^n$ where $I \otimes \theta_q \otimes I \in R_1 \otimes B_q \otimes R_3 \subset \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes \mathcal{R}_3 = \mathcal{R}$ and $M_{k(m_q)}(\mathbb{C}) \simeq B_q \subset R_2$. Because $\lim_{q \to \infty} m_q = \infty$, it is clear that there exists a (trace preserving) embedding $\pi : X^\omega \to \mathcal{R}^\omega$ satisfying $\pi(X) = Q(Y)$. We can regard $\xi_1$ as a subset of $\mathcal{R}_1$ by fixing a copy $A$ of the $k(0) \times k(0)$ matrices in $\mathcal{R}_1$ and for each $q$ consider $\xi_1 \otimes I \otimes I, I \otimes \theta_q \otimes I \subset (A \otimes B_q) \otimes \mathcal{R}_3 \subset \mathcal{R}$. For each $q$ (2) provides an algebra $D_q$ isomorphic to a full matrix algebra, and a unitary $v_q$ of $A \otimes B_q \otimes D_q$ such that

$$|v_q(\xi_1 \otimes I \otimes I)v_q^* - I \otimes \theta_q \otimes I|_2 < \epsilon.$$ 

Define $\rho : A \to \mathcal{R}^\omega$ by $\rho(x) = Q(v_q(x \otimes I \otimes I)v_q^\infty)_{q=1}^\infty$. We have shown that every element of $\pi(X)$ is within the $\|\cdot\|_2 \epsilon$-ball of the finite dimensional full matrix algebra $\rho(A)$. Thus, for every $\epsilon > 0$ there exists an isomorphic copy of $M$ in $\mathcal{R}^\omega$ such that $X$ identified in $\mathcal{R}^\omega$ with respect to this embedding is within the $\|\cdot\|_2 \epsilon$-ball of a finite dimensional subalgebra of $\mathcal{R}^\omega$. It follows that $M$ is amenable by Lemma 2.6. (1) follows.

We now have the equivalence of conditions (1), (2), (4), (5) and (6). To conclude, (3) $\Rightarrow$ (5) = (2) $\Rightarrow$ (3) so condition (3) is equivalent to all the other conditions as well.

Remark 2.10. It should be fairly obvious to the reader by now that conditions (1)-(6) of Lemma 2.9 are equivalent to the condition that there exist finitely many embeddings $\pi_1, \ldots, \pi_n$ of $M$ into $\mathcal{R}^\omega$.
such that for any other embedding $\sigma$ of $M$ into $R^\omega$, there exists a $1 \leq i \leq n$ and a unitary $u$ in $R^\omega$ such that $\sigma$ is conjugate to $\pi_i$ via $u$.

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