Hamiltonian description of vortex systems

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Abstract

In the framework of 2D ideal Hydrodynamics a vortex system is defined as a smooth vorticity function having few positive local maxima and negative local minima separated by curves of zero vorticity. Invariants of such structures are discussed following from the vorticity conservation law and invertibility of Lagrangian motion. Hamiltonian formalism for vortex systems is developed by introducing new functional variables diagonalizing the original non-canonical Poisson bracket.

Keywords: Vortex, continuum Hamiltonian system, Poisson bracket, vorticity, 2D hydrodynamics

1. Introduction

Fundamentals of Hamiltonian formalism in application to hydrodynamics systems were developed in [1,2] and proved to be an efficient tool in handling a variety of fluid mechanics problems. Here that approach is applied to a relatively simple object: a vortex system in the framework of ideal 2D hydrodynamics. Our starting point is the vorticity conservation equation

$$\frac{\partial \Omega}{\partial t} + J(\psi, \Omega) = 0 (1)$$

written in the Hamiltonian form as, [3]

$$\frac{\partial \Omega}{\partial t} = \{H, \Omega\},$$

where the non-canonical Poisson bracket is expressible as

$$\{F, G\}_\Omega = \int \Omega(r) J_{x,y} \left( \frac{\delta F}{\delta \Omega(r)}, \frac{\delta G}{\delta \Omega(r)} \right) dr. \tag{2}$$

Here $F = F(\Omega), G = G(\Omega)$ are smooth functionals, $r = (x, y)$, and $J_{x,y}(f, g) = f_x g_y - f_y g_x$ the Jacobian. Henceforth the integration region is the whole plane $R^2$ unless other indicated. The Hamiltonian is given by

$$H = \frac{1}{2} \int |\nabla \psi|^2 dr = -\frac{1}{2} \int \psi \Omega dr \tag{3}$$

that implies

$$\frac{\delta H}{\delta \Omega} = -\psi. \tag{4}$$

Combining (2) and (3) we extend (1) to any functional

$$\frac{\partial F(\Omega)}{\partial t} = \int \frac{\delta F}{\delta \Omega(r)} J_{x,y}(\psi, \Omega(r)) dr. \tag{5}$$
Next we assume that $\Omega(r)$ decays fast enough as $|r| \to \infty$ and, hence, the stream function is expressed in terms of vorticity as

$$\psi(r) = \frac{1}{2\pi} \int \ln(r - r')\Omega(r')dr'. \quad (6)$$

From (4) and (6)

$$H = \frac{1}{4\pi} \int \int \ln(r_1 - r_2)\Omega(r_1)\Omega(r_2)dr_1dr_2. \quad (7)$$

The main objectives of this work are:

- Formulate and prove rigorously conservation laws for the topography of vorticity $\Omega$ such as the number of critical points (points where $\nabla\Omega = 0$), the vorticity values at the critical points, and the number of distinct level curves defined by $\Omega(r) = w$, corresponding to a fixed value $w$, that also will be called contour lines.

- Derive translation equations for the critical points

- Derive and discuss equations for contour lines in both a Hamiltonian form and in form of closed integro-differential equations. In this regard our efforts can be viewed as an extension of the contour dynamics, [4, 5], to smooth vorticity functions

- Demonstrate how some well-known models such as point vortex systems and FAVOR, [6], could be derived from the underlying vorticity class.

Now we specify the class of functions $\Omega(r)$ called the $N$-vortex system. Let

$$H_{\Omega}(r) = \begin{pmatrix} \Omega_{xx} & \Omega_{xy} \\ \Omega_{xy} & \Omega_{yy} \end{pmatrix}$$

be the Hessian of vorticity. Assume that

(i) $\Omega$ has exactly $N$ extrema (maxima or minima) at points $z_k = (\xi_k, \eta_k)$, $k = 1, ..., N$, i.e.

$$\nabla\Omega(z_k) = 0, \quad \det(H_{\Omega}(z_k)) > 0.$$

(ii) The set $\Gamma_0 = \{r \in R^2 \mid \Omega(r) = 0\}$ is either empty or divides the plane in $N \geq 2$ distinct regions $G_k$, $k = 1, ..., N$ such that each $G_k$ contains exactly one extremum and the vorticity has the same sign for all points in $G_k$. Thus, summarizing

$$R^2 = \bigcup_k G_k, \quad G_k \cap G_j = \emptyset, \quad \Omega(\partial G_k) = 0, \quad z_k \in G_k.$$

(iii) For any two adjacent regions $G_k$ and $G_j$ the signs in $G_k$ and $G_j$ are opposite.

The last condition can be ensured by assumption that the set $\Gamma_0$ of zero vorticity lines is an Euler graph [7]. For such graphs the corresponding regions can be painted by two colors only in a way that any two adjacent regions have opposite colors. Our ‘colors’ mean positive and negative vorticity. Examples of some important vortex systems satisfying to (i-iii) are shown in Fig.1.

A remarkable fact proven in the text is that the structure (i-iii) and, in particular, the value of $N$ is conserved by equation (1). In other words critical points neither are being born nor dying during the system evolution. Moreover, the vorticity values at the critical points $\omega_k = \Omega(z_k)$ are conserved as well.

We do not address specifically saddles of $\Omega(r)$ (hyperbolic critical points) as not important in context of our goals. However, it is worth noting that their number and vorticity values are also conserved. We can say even more assuming that the Euler graph representing $\Gamma_0$ has exactly four edges incident to each vertex like in two last panels of Fig.1. Namely, in this case each vertex (an intersection of two zero vorticity lines) is a saddle and vice versa: each saddle is a vertex of the graph. Thereby the vorticity
at each saddle is zero. There is one more important invariant that is a consequence of two fundamental laws: vorticity conservation in Lagrangian particles and incompressibility. For a fixed \( w > 0 \) let \( n(w) \) be the number of disjoint connected regions where \( \Omega > w \) and the number of disjoint connected regions where \( \Omega < w \) if \( w < 0 \). We will show that \( n(w) \) is also conserved by (1). In other words, no merging of vorticity patches is possible in ideal 2D hydrodynamics. We take an opportunity to clearly spell out that obvious claim because in last years the vortex merger problem was addressed in many works, i.e. [8], but not always initial equations were formulated explicitly.

An important role of the extreme points is that they serve as natural poles for local polar coordinates in parametrization of vorticity lines at each particular region \( G_k \). Let \( r = \rho_k(\varphi, w) \) be the polar equation of the single contour line corresponding to the vorticity level \( w \) at a certain region

\[
\Gamma_{w,k} = \{ r \in G_k \mid \Omega(r) = w \}. \tag{8}
\]

We derive closed evolution equations for \( \rho_k, \; k = 1, ..., N \) that explicitly show interacting between vorticity contours corresponding to different regions \( G_k \) and different vorticity levels \( w \). An essential drawback of polar parametrization is that the very assumption that curve can be represented in polar form with a single valued \( \rho(\varphi) \) is not conserved by the system. Thus such a consideration is valid for small integration times only or if one considers a small vicinity of an extremum point or, equivalently, small values of \( \rho_k \) where curve (8) is well approximated by an ellipse. This is why we first address a general parametrization (still pinned to \( G_k \)), \( r = \hat{r}(p, w) \) of \( \Gamma_{w,k} \) where \( p \) is a positive parameter, say the length of arc (natural parametrization). Such a parametrization is free from the above drawback.

In both cases, polar and natural parametrization, the resulting equations are too sophisticated (non-linear, integro-differential, non-decoupling) to efficiently work on them. Exceptions are monopoles and dipoles ( \( N = 1, 2 \) respectively). Thus, for now the equations describing evolution of contours (8) for \( N > 2 \) are of purely theoretical interest only: our point is to clarify conservation laws concerning with vorticity topography and to shed light on the underlying Hamiltonian structure rewritten in new phase variables \( \{ \rho_k(\varphi, w) \} \) or \( \{ \hat{r}_k(p, w) \}, \; k = 1, ..., N \).

Worth noting that condition (iii) is not necessary for deriving the mentioned equations. Its intention is to exclude unstable vortex structures once and forever. For the same reason we do not address bifurcation points where the determinant of Hessian is zero.

The paper is organized as follows. In Section 1 we state and give a sketch of proof of most important conservation laws for the considered vortex systems. In Section 2 the main result on the Poisson bracket diagonalization is proven. As a consequence, equations for vorticity lines and extrema translation are derived for a general parametrization. These equations are specified and discussed for the polar parametrization in Section 3. Examples of application of the equations to monopoles and dipoles
are given in Section 4. Section 5 contains a short discussion and conclusions. Finally, some details are brought to Appendix.

1. Invariants

PROPOSITION 1. The number of critical points and their type is conserved by equation (1).

PROOF. Let \( r(t, r_0) = r \) be the position of a Lagrangian particle at moment \( t \) starting from \( r_0 \) and \( J(r, r_0) = \partial(r)/\partial(r_0) \) be the Jacobi matrix of the diffeomorphism \( T : r_0 \to r \). Direct computations based on the vorticity conservation equation (1) give \( \nabla \Omega(r) = J \nabla \Omega(r_0) \) and \( H_\Omega(r) = J H_\Omega(r_0) J^* \) if \( \nabla \Omega(r_0) = 0 \), the star means transposition. Now the statement follows from \( \det(J) = 1 \) that is a consequence of incompressibility, [9].

Due to assumptions (i-ii) for any \( w \neq 0 \) the set \( \Gamma_w = \{ r | \Omega(r) = w \} \) consists of a finite number of closed curves and the number \( n(w) \) of such curves obviously does not exceed \( N \). The next statement shows that \( n(w) \) is conserved as well.

PROPOSITION 2 Assume for definiteness that \( w > 0 \) and at the initial moment region \( D = \{ r | \Omega(r) > w \} = D_1 \cup D_2 \) consists of two disjoint regions. Then \( D \) remains so for all moments.

PROOF. Assume that at some moment \( t \) the regions have merged. Let \( r_1 \in D_1 \) and \( r_2 \in D_2 \). Consider a continuous curve \( C \) joining \( T(r_1) \) and \( T(r_2) \) completely belonging to the merger, i.e. \( C \in T(D_1) \cup T(D_2) \). That means the vorticity at each point of \( C \) is greater than \( w \), \( \Omega(C) > w \). Thus \( \Omega(T^{-1}(C)) > w \) as well, but the curve \( T^{-1}(C) \) for sure lies partially outside both \( D_1 \) and \( D_2 \) and hence at some point of the curve \( \Omega < w \) that is a contradiction. By invertibility of \( T \) a connected region cannot be broken down in two distinct ones either.

PROPOSITION 3 The local extrema are conserved, more exactly the equations

\[
\frac{\partial \xi_k}{\partial t} = -\psi_y(\xi_k, \eta_k), \quad \frac{\partial \eta_k}{\partial t} = \psi_x(\xi_k, \eta_k)
\]

hold true for any \( k = 1, ..., N \).

PROOF. Let us fix a certain \( G_k \) and drop sub \( k \) in the following below computations for simplicity. With the assumption that there is a single critical point of \( \Omega \) in \( G \) we can represent its coordinates as the following functionals of \( \Omega \), [10]

\[
\xi(\Omega) = \int_G x \delta(\Omega_x) \delta(\Omega_y) S(\Omega) dr, \quad \eta(\Omega) = \int_G y \delta(\Omega_x) \delta(\Omega_y) S(\Omega) dr,
\]

where \( S(\Omega) = \Omega_{xx} \Omega_{yy} - \Omega_{xy}^2 \) is the determinant of the Hessian. The idea behind such a representation is that a unique solution of

\[
f(r) = 0, \quad r \in R^n, \quad f : R^n \to R^n
\]

is expressible as

\[
r_0 = \int_{R^n} r \delta(f(r)) \left| \det \left( \frac{\partial f}{\partial r} \right) \right| dr,
\]

where \( \partial f/\partial r \) is the Jacobi matrix of map (10).

Taking variational derivatives in (9) obtain, [10]

\[
\frac{\delta \xi}{\delta \Omega(x,y)} = \delta'(\Omega_x) \delta(\Omega_y) S(\Omega), \quad \frac{\delta \eta}{\delta \Omega(x,y)} = \delta'(\Omega_y) \delta(\Omega_x) S(\Omega),
\]
and plug each expression in (5). After changing the variables in the integrals \( u = \Omega(x, y) \) and \( v = \Omega_y(x, y) \) arrive at (9).

2. Poisson bracket diagonalization. Evolution and translation equations for vorticity lines

Assume that vorticity values at the extrema are ordered \( \omega_1 > \omega_2 > ... > \omega_N \). Let

\[
x = \xi_k + \hat{x}_k(p, w), \quad y = \eta_k + \hat{y}_k(p, w), \quad (p, w) \in D_k = \{0 \leq p < L_k(w), \ 0 < w < \omega_k\}
\]

be an arbitrary parametrization of the vorticity line \( \Gamma_{w,k} \) (see (8)), corresponding to level \( w \) in region \( G_k \); \( p \) is a positive parameter with the upper limit \( L_k(w) \) depending in general on \( w \). For example, in the case of natural parametrization it is the length of a particular contour. According to (11), the origin of local rectangular coordinate system \((\hat{x}, \hat{y})\) is placed at \( z_k = (\xi_k, \eta_k) \).

**ASSUMPTION** The map \((p, w) \to (x, y)\) defined by (11) is one-to-one map of \( D_k \) onto \( G_k \) for a fixed \((\xi_k, \eta_k)\).

The assumption implies that the inversion of (11) can be written as

\[
p = p_k(x, y), \quad w = \Omega(x, y),
\]

where \( p_k(x, y) \) is a function determined by a specific parametrization and it depends on region \( G_k \).

Introduce \( e = (1, 1) \),

\[
\mathbf{V}(r) = \frac{1}{S(\Omega)} \begin{pmatrix} -\Omega_{xy} & \Omega_{xx} \\ \Omega_{yy} & -\Omega_{yx} \end{pmatrix}, \quad \hat{r}_k(p, w) = (\hat{x}_k(p, w), \hat{y}_k(p, w)), \quad \nabla \delta(r) = (\delta'(x)\delta(y), \delta(x)\delta'(y))^t.
\]

The following statement proven in Appendix plays a key role in further computations.

**LEMMA.** Consider the vector function \( \hat{r}(p, w) \) of local coordinates on \( \Gamma_{k,w} \) in \( G = G_k \) as a functional of \( \Omega(r) \), then its variational derivative is given by

\[
\frac{\delta \hat{r}(p, w)}{\delta \Omega(r)} = \frac{\hat{r}_w(p, w)}{g(p, w)} \left( \delta(\Omega(r) - w)\delta(p(r) - p) - e \frac{\partial}{\partial p}(\hat{r}(p, w)\mathbf{V}(z)\nabla \delta(r - z)) \right). \tag{12}
\]

where \( g(p, w) = \hat{x}_wp - \hat{y}_p\hat{x}_w \)

For brevity we dropped subscripts \( k \) in (12). Set

\[
\zeta_k(p, w) = \int_{\omega_k}^{\omega_p} g_k(p, u) du \tag{13}
\]

and let \( F = F(\zeta_1, ..., \zeta_N) \) be a smooth functional of new variables. Obviously, it is also a functional of \( \Omega \) that will be denoted by the same symbol \( F(\Omega) \). Next, introduce the range \( R_k \) of values of \( \Omega(r) \), \( r \in G_k \) that is either interval \((0, \omega_k)\) or \((\omega_k, 0)\) depending on the sign of \( \omega_k \). Finally, let

\( S(w) = \{ k \in \{1, 2, ..., N\} \mid w \in R_k \} \) be the list of all regions containing a piece of \( \Gamma_w = \{ r \mid \Omega(r) = w \} \).

**PROPOSITION 4** Assume that \( \delta F/\delta \Omega(r) \) is a smooth function of \( r \), then

\[
\frac{\delta F}{\delta \zeta_k(p, w)} = \left. \frac{\delta F}{\delta \Omega(r)} \right|_{r = \zeta_k + \hat{r}_k(p, w)}
\]
for \( w \neq \omega_k \) and

\[
\{ \zeta_k(p, w), F \} = L_k(F),
\]

where

\[
L_k(F) = \frac{\partial}{\partial p} \left\{ \sum_{j \in S(w)} \left( \frac{\delta F}{\delta \zeta_j(q, w)} \right)_{q=p_j(\Delta z_{kj}+\hat{r}_k(p, w))} - \nabla \frac{\delta F}{\delta \Omega(r)} \right|_{r=\hat{z}_k} \cdot \hat{r}_k(p, w) \right\}, \quad \Delta z_{kj} = \hat{z}_k - \hat{z}_j.
\]

Notice that the Poisson bracket \( \{ \zeta_k(p, w), \zeta_j(q, u) \} \) is not defined for \( k = j \).

Main steps in derivation of (14) are as follows. First, using (13), the chain rule, and

\[
\frac{\delta \zeta(p, w)}{\delta \Omega(r)} = \delta(p-q)\delta(w-u)\hat{r}_p(q, u) - \frac{\partial}{\partial p} (\hat{r}(p, w) \nabla \delta(r - z)),
\]

that leads to the first statement. Then plug the obtained expression in (2) and break down the integration over \( R^2 \) in integration over \( G_j, j = 1, \ldots, n(w) \). Finally we proceed to local coordinates \((\hat{x}_j, \hat{y}_j)\).

Thus, setting in (14) \( F = H \) we get Hamiltonian equations for the new variables

\[
\frac{\partial \zeta_k(p, w)}{\partial t} = L_k(H).
\]

To get a closed system in terms of variables \( \zeta_k \) we again change the integration over the whole plane in (7) with integration over distinct \( G_k, k = 1, 2, \ldots, N \). The result is \( H = \sum_{k,j=1}^{N} H_{kj} \) where

\[
H_{kj} = \frac{s_k s_j}{4\pi} \int_0^{\omega_k} \int_0^{\omega_j} \int_0^{L_k(w_1)} \int_0^{L_j(w_2)} w_1 w_2 \zeta_k(p_1, w_1) \zeta_j(p_2, w_2) \ln D_{kj}(p_1, w_1, p_2, w_2) dp_1 dp_2 dw_1 dw_2,
\]

where

\[
s_k = \text{sgn}(\omega_k), \quad D_{kj} = \sqrt{(\Delta \xi_{kj} + \hat{x}_k(p_1, w_1) - \hat{x}_j(p_2, w_2))^2 + (\Delta \eta_{kj} + \hat{y}_k(p_1, w_1) - \hat{y}_j(p_2, w_2))^2},
\]

\[
\Delta \xi_{kj} = \xi_k - \xi_j, \quad \Delta \eta_{kj} = \eta_k - \eta_j.
\]

A cumbersome expression for Hamiltonian is a trade-off for a diagonal Poisson bracket. Finally, \( \hat{x}, \hat{y} \) should be expressed in terms of \( \zeta \). That can be easily done in the case of a polar parametrization we consider below.

3. Polar parametrization

Assume now \( p = \varphi \) is a polar angle and for each \( G_k \) introduce local polar coordinates

\[
x = \xi_k + \rho_k(\varphi, w) \cos \varphi, \quad y = \eta_k + \rho_k(\varphi, w) \sin \varphi, \quad (\varphi, w) \in D_k = [0, 2\pi] \times [0, \omega_k], \quad (x, y) \in G_k,
\]

where \( \rho_k(\varphi, w) \) is the distance from \( z_k \) to the point on the contour in direction \( \varphi \). In other words, the closed curve \( \Gamma_{w, k} \) is covered by equation \( r = \rho_k(\varphi, w) \). Easy to see that the new phase variable introduced in (13) now becomes \( \zeta_k = \rho_k^2(\varphi, w)/2 \) and (15) implies
POPOSITION 5

\[
\frac{1}{4} \frac{\partial \rho_k^2(\varphi, w)}{\partial t} = \frac{\partial}{\partial \varphi} \left\{ \frac{\delta H}{\delta \rho_k^2(\varphi, w)} + \sum_{j \neq k} \frac{\rho_j(\theta_{kj}, w)}{\rho_k^2(\varphi, w)} \left. \frac{\delta H}{\delta \rho_j^2(\theta, w)} \right|_{\theta = \theta_{kj}(\varphi)} - \rho_k(\varphi, w) \left( D_\varphi \frac{\delta H}{\delta \rho_k} \right)_{w = \omega_k} \right\}, \quad (17)
\]

where \(D_\varphi\) is the derivative in direction given by \(\varphi\) and \(\theta = \theta_{kj}(\varphi)\) is the solution of

\[
\frac{\Delta \eta_{kj} + \rho_j(\theta, w) \sin \theta}{\Delta \xi_{kj} + \rho_j(\theta, w) \cos \theta} = \tan \varphi.
\]

The last term in the braces reflects an effect on the shape of \(k\)-th vorticity line due to the vortex peak motion.

Notice that for the polar coordinates (16) turns into

\[
H_{kj} = \frac{s_k s_j}{16\pi} \int_0^{\omega_k} \int_0^{\omega_k} \int_0^{2\pi} \int_0^{2\pi} w_1 w_2 (\rho_k^2 w_1 (\rho_j^2) w_2) \ln D_{kj}(\varphi_1, \varphi_2, \varphi_3) d\varphi_1 d\varphi_2 d\varphi_3 d\varphi_4,
\]

where

\[
\rho_k = \rho_k(\varphi_1, w_1), \quad \rho_j = \rho_j(\varphi_2, w_2), \quad D_{kj} = \sqrt{(\Delta \xi_{kj} + \rho_k \cos \varphi_1 - \rho_j \cos \varphi_2)^2 + (\Delta \eta_{kj} + \rho_k \sin \varphi_1 - \rho_j \sin \varphi_2)^2}
\]

To get a closed system for \(\rho_k\) one should rewrite (17) in terms of stream function

\[
\psi_j(\theta, w) = \psi(\xi_j + \rho_j(\theta, w) \cos \theta, \eta_j + \rho_j(\theta, w) \sin \theta).
\]

The result is

\[
\frac{1}{2} \frac{\partial \rho_k(\varphi, w)}{\partial t} = -\frac{1}{\rho_k(\varphi, w)} \frac{\partial}{\partial \varphi} \left\{ \psi_k(\varphi, w) + \sum_{j \neq k} \psi_j(\theta_{kj}, w) - \rho_k(\varphi, w) D_\varphi \psi(\varphi, \omega_k) \right\},
\]

and then substitute for \(\psi\) the following expression

\[
\psi_j(\theta, w) = -\frac{1}{2\pi} \sum_{\alpha = 1}^N s_\alpha \int_0^{2\pi} \int_0^{\omega_\alpha} u_\alpha(\alpha, \vartheta)(\partial \rho_\alpha(\vartheta, \varphi))/\partial \vartheta) d\vartheta \ln D_{j\alpha}(\theta, \varphi, u) d\vartheta \varphi , \quad (18)
\]

derived from (8).

Translation equations for the critical points we give in terms of complex variables \(z_k = \xi_k + i\eta_k\) by plugging (18) in (9)

\[
\frac{\partial z_k}{\partial t} = \frac{s_k}{2\pi i} \int_0^{\omega_k} \int_0^{\omega_k} \rho_k(\varphi, w) e^{-i\varphi} d\varphi dw + \sum_{j \neq k} \frac{s_j}{2\pi i} \int_0^{\omega_j} \int_0^{\omega_j} w_j(\varphi, w) (\partial \rho_j(\varphi, w)/\partial \varphi) d\varphi dw.
\]

Finally, let us express other well known invariants of (1) in terms of the new variables. First, notice Casimir functionals \(K(\Omega) = \int K(\Omega(r)) d\varphi\), where \(K(\cdot)\) is an arbitrary smooth function. Breaking the integral over \(R^2\) in regions \(G_j, \ j = 1, ..., N\) and proceeding to new variables \((\xi_j, \eta_j, \rho_j)\) one gets

\[
K = -\sum_j \frac{s_j}{2} \int_0^{\omega_j} K(w) \frac{\partial}{\partial w} \left( \int_0^{2\pi} \rho_j^2(\varphi, w) d\varphi \right) dw.
\]
Notice that the inner integral is simply the doubled area of the region bounded by $\Gamma_{w,j}$.

Then, in the same manner the vorticity first momentum

$$c = \int (x + iy)\Omega(r)\,dr$$

can be obtained

$$c = \sum_j s_j \int_0^{\omega_j} \int_0^{2\pi} \left( \frac{1}{2} z_j \rho_j^2(\varphi, w) + \frac{1}{3} \rho_j^3(\varphi, w)e^{i\varphi} \right) d\varphi dw.$$

4. Monopole and Dipole

A monopole is defined by conditions $N = 1$, $\Omega(r) > 0$, $\mathbf{r} \in \mathbb{R}^2$. Denote $M = \omega_1 > 0$ the maximum value of the vorticity and introduce local polar coordinates with pole at the point of maximum $z = (\xi, \eta)$.

In other words, the closed curve $\Gamma_w$ is covered by equation $r = \rho(\varphi, w)$ in local polar coordinates. As it was already noticed $\rho(\varphi, w)$ does not remain a single valued function in the process of evolution except trivial cases such as circular contours for all $w$. Thus, if we interpret $\rho(\varphi, w)$ as a distance, then all the following equations hold true only during finite time (probably small) of integration. However, if we treat $\rho(\varphi, w)$ as a generalized distance (a pseudo inverse of $\Omega$ with respect to radial variable $r$) i.e.

$$\rho(\varphi, w) = \int_0^\infty I_{(0,\infty)}(\Omega(\xi + r \cos \varphi, \eta + r \sin \varphi) - w) \, dw,$$

then the following below evolution equations hold true for all $t$. This is because their derivation is based on the variational derivative of $\rho$ in $\Omega$ obtained from (19) rather than on the distance interpretation where it is assumed that the ray from $z$ in direction $\varphi$ intersects $\Gamma_w$ once. If it intersects the contour few times, then $\rho$ given by (19) is the sum of the distances to all the intersection points.

In the considered case (17) becomes

$$\frac{1}{4} \frac{\partial^2 \rho(\varphi, w)}{\partial t} = \frac{\partial}{\partial \varphi} \left\{ \delta H \left( \frac{\delta H}{\delta \rho^2(\varphi, w)} \right) - \rho(\varphi, w) \left( D \frac{\delta H}{\delta \rho^2(\varphi, w)} \right) \bigg|_{w=M} \right\},$$

where $\rho = \rho(\varphi, w)$ and

$$H = \frac{1}{16\pi} \int_0^M \int_0^M \int_0^{2\pi} \int_0^{2\pi} w_1 w_2 (\rho_1^2)_{w_1} (\rho_2^2)_{w_2} \ln D(\varphi_1, w_1, \varphi_2, w_2) d\varphi_1 d\varphi_2 dw_1 dw_2$$

with

$$\rho_1 = \rho(\varphi_1, w_1), \quad \rho_2 = \rho(\varphi_2, w_2), \quad D = \sqrt{(\rho_1 \cos \varphi_1 - \rho_2 \cos \varphi_2)^2 + (\rho_1 \sin \varphi_1 - \rho_2 \sin \varphi_2)^2}.$$

Equation (20) was first appeared in [10] where the singular term, describing effects of the vortex motion on its shape, was missing. Yet, here the expression for Hamiltonian is significantly simplified.

The translation equation in terms of the complex coordinate $z = \xi + i\eta$ becomes

$$\frac{\partial z^*}{\partial t} = \frac{1}{2\pi i} \int_0^M \int_0^{2\pi} \rho(\varphi, w)e^{-i\varphi} d\varphi dw.$$
Notice a similarity with the contour dynamics, [4], where the object of study was a vorticity patch of value \( \Omega(r) = M \) bounded by a closed curve \( r = \rho(\varphi) \) with zero vorticity outside. Not difficult to show that in this case (1) again leads to a Hamiltonian system resulting in the following evolution equation

\[
\frac{1}{2} \frac{\partial}{\partial t} \rho^2(\varphi) = \frac{2}{M} \frac{\partial}{\partial \varphi} \frac{\delta H}{\delta \rho^2} = - \frac{\partial}{\partial \varphi} \psi(\varphi),
\]  

(22)

where \( \psi(\varphi) = \psi(\varphi, r)|_{r = \rho(\varphi)} \) and \( \psi(\varphi, r) = \psi(\xi + r \cos \varphi, \eta + r \sin \varphi) \). In this case the pole \((\xi, \eta)\) is usually placed at the patch centroid. It is now easy to get a closed equation for \( \rho(\varphi) \) from (22) by using

\[
\psi(\varphi, r) = -\frac{M}{4\pi} \int_0^{2\pi} \left[ \rho^2(\theta) - r (\rho(\theta) \sin(\theta - \varphi)) \right] \ln \left( r^2 + \rho^2(\theta) - 2r \rho(\theta) \cos(\theta - \varphi) \right) d\theta.
\]  

(23)

Strange enough, we could not find in the literature an absolutely correct closed equation obtained from (22) after differentiating in the right hand side. For example, in [4] and [12] both, the expression (23) and the equation (22), involving stream function, were correct, but a mistake was made when differentiating stream function in \( \varphi \).

Returning to the case of smooth vorticity notice that the stream function expression in coordinates \((\varphi, w)\)

\[
\psi(\varphi, w) = -\frac{1}{4\pi} \int_0^{2\pi} \int_0^M u \rho(\theta, u) \rho_w(\theta, u) \ln \left( \rho^2(\varphi, w) + \rho^2(\theta, u) - 2\rho(\theta, u) \rho(\varphi, w) \cos(\theta - \varphi) \right) dud\theta
\]

is somewhat simpler than (23) because \( \varphi \) shows up only under the logarithm. Another advantage of a continuous monopole compared to a patch is that the maximum (the vortex head) moves along stream lines while the centroid of patch certainly does not.

Summing up, a closed equation for \( \rho(\varphi, w) = \rho(t, \varphi, w) \) can be written in form

\[
\frac{\partial}{\partial t} \rho^2 = \frac{\partial}{\partial \varphi} N(\rho^2), \quad \rho^2|_{t=0} = p(\varphi, w),
\]

(24)

where \( p(\cdot, \cdot) \) is an initial condition and a non-linear integro-differential operator is

\[
N(\rho^2)(\varphi, w) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^M \left( u \rho^2(\theta, u) \right) \ln \left( D + 4\rho(\theta, u) \rho(\varphi, w) \cos(\theta - \varphi) \right) dud\theta
\]

with \( D = \rho^2(\varphi, w) + \rho^2(\theta, u) - 2\rho(\theta, u) \rho(\varphi, w) \cos(\theta - \varphi) \).

Obviously, in general the equation (24) is not simpler than the original equation (1). However, in one particular case, we are about to discuss, making use of (24) is more efficient than that of (1). Namely, we suggest a natural asymptotic procedure bridging the contour dynamics and the smooth vorticity case.

For that let us assume that the initial vorticity for set up (1) is represented as \( \Omega(r, \varphi) = MS \left( \frac{r}{R(\varphi)} \right) \)

where dimensionless function \( S(x) \) defined on \([0, \infty)\) satisfies : \( S(0) = 1, \quad S'(x) < 0, \quad S(\infty) = 0 \) and \( R(\varphi) \) is a certain space scale depending on the direction. Introduce the following scaling

\[
\Omega_\epsilon(r, \varphi) = MS \left( \left[ \frac{r}{R(\varphi)} \right]^{1/\epsilon} \right),
\]

(25)

that for small \( \epsilon \) converts a continuously distributed vorticity to a patch

\[
\lim_{\epsilon \to 0} \Omega_\epsilon(r, \varphi) = \begin{cases} M, & r < R(\varphi) \\ 0, & r > R(\varphi) \end{cases}
\]
Assume that the equation $r = R(\varphi)$ represents an ellipse, i.e, the solution of (24) with the initial elliptic patch $\Omega_0$ is the well-known Kirchhoff vortex, [12]. Any attempt to correct that solution for small $\epsilon$ fails because the derivative of $\Omega_\epsilon(r, \varphi)$ at $\epsilon = 0$ is infinite. However, proceeding to $\rho_\epsilon(\varphi, w) = R(\varphi) \left[ S^{-1} \left( \frac{w}{M} \right) \right]$, obtained from (25) we arrive at an analytic function of $\epsilon$. That allows for application of a standard perturbation approach with details given in Appendix.

We show results in Fig. 2 from which one can see that the vorticity lines are losing elliptic shape, nevertheless preserving the central symmetry. The latter obviously follows from the original equation (1). Thus, the integral on the right-hand side of (21) is zero and the vortex center does not move.

The only purpose of the example was to illustrate a well-posedness of the suggested perturbation procedure.

Now we consider a dipole determined by $N = 2, \quad \omega_1 = M > 0, \omega_2 = m < 0$. From (17) and the general translation equation (next to (19)) one gets

**PROPOSITION 6.**

$$
\frac{1}{4} \frac{\partial \rho_1^2(\varphi, w)}{\partial t} = \frac{\partial}{\partial \varphi} \left\{ \frac{\delta H}{\delta \rho_1^2(\varphi, w)} - \rho_1(\varphi, w) \left( D_\varphi \frac{\delta H}{\delta \rho_1^2(\varphi, w)} \right) \right\} w=M,
$$

$$
\frac{1}{4} \frac{\partial \rho_2^2(\varphi, w)}{\partial t} = \frac{\partial}{\partial \varphi} \left\{ \frac{\delta H}{\delta \rho_2^2(\varphi, w)} - \rho_2(\varphi, w) \left( D_\varphi \frac{\delta H}{\delta \rho_2^2(\varphi, w)} \right) \right\} w=m,
$$

$$
\frac{\partial z_1^*}{\partial t} = \frac{1}{2\pi i} \int_0^M \int_0^{2\pi} \rho_1(\varphi, w)e^{-i\varphi} d\varphi dw + \frac{1}{2\pi i} \int_0^M \int_0^{2\pi} \frac{w\rho_2(\varphi, w)(\partial \rho_2(\varphi, w)/\partial w) dwd\varphi}{z_1 - z_2 - \rho_2(\varphi, w)e^{i\varphi}},
$$

$$
\frac{\partial z_2^*}{\partial t} = \frac{1}{2\pi i} \int_m^0 \int_0^{2\pi} \rho_2(\varphi, w)e^{-i\varphi} d\varphi dw + \frac{1}{2\pi i} \int_0^M \int_0^{2\pi} \frac{w\rho_1(\varphi, w)(\partial \rho_1(\varphi, w)/\partial w) dwd\varphi}{z_2 - z_1 - \rho_1(\varphi, w)e^{i\varphi}},
$$

where an expression for Hamiltonian in terms of $\rho_1, \rho_2, \xi_1, \eta_1, \xi_2, \eta_2$ can be obtained from Proposition 5.
Evolution and translation equations from Proposition 6 were first announced in [13]. Here they are essentially simplified and corrected.

Notice that for vortices of identical shape $\rho_1 \equiv \rho_2$ the distance between poles and the angle between the axis through poles and the $x$-axis are conserved $\partial (z_1 + z_2)/\partial t = 0$, the fact well known in contour dynamics.

To illustrate the use of (27) let us consider a strong positive point vortex with maximum $M$ centered at the origin and a weak negative satellite with the minimum $m$, $|m| \ll M$, initially axisymmetric and centered at $(0, R)$. We assume that the stream function $\psi_1(r)$ of the positive vortex is not affected by the satellite. In addition we neglect the influence of the velocity field generated by the the satellite on itself. In other words, it is considered as a passive scalar driven by the velocity field of the strong vortex. Hence, the first equation in (28) turns to $\rho_1(t, \varphi, w) = \rho_1(0, \varphi, w)$ and the second one turns to a closed equation

$$\frac{1}{2} \frac{\partial \rho_2^2(\varphi, w)}{\partial t} = -\frac{\partial}{\partial \varphi} \psi_1 (\rho_2(\varphi, w)^2 + R^2 - 2R\rho_2(\varphi, w) \cos \varphi).$$

Notice that $w$ is included in the equation simply as a parameter. Proceeding to the polar coordinates $(r, \theta)$ with a pole at the origin $(0, 0)$, i.e. setting $r^2 = \rho_2(\varphi, w)^2 + R^2 - 2R\rho_2(\varphi, w) \cos \varphi$, $\sin \theta = \rho_2 \sin \varphi/r$ obtain

$$\frac{\partial r}{\partial t} + \frac{1}{r} \frac{\partial \psi_1(r)}{\partial r} \frac{\partial r}{\partial \theta} = 0.$$

The equation is integrable for any $\psi_1(r)$, but it makes a physical sense only if the background vortex is a point vortex. i.e. $\psi_1 = k \ln r$, where $k$ is its intensity. The solution is given by

$$r^2 - 2rR \cos \left( \theta - \frac{kt}{r^2} \right) = C(w),$$

where constant $C(w)$ is defined by the vorticity level and shape of the initial satellite.

It can be shown that the vorticity line of level $w$ spirals into limit cycle $r = R - r_0(w)$ where $r_0(w)$ is the radius of the $w$-vorticity line for the initial satellite and the number of cycles in spiral $n \approx c(w)\omega_0 t$, $\omega_0 = k/R^2$ angular velocity, $t$ time, $c$ constant depending on vorticity level.

Notice that the integral (28) follows directly from the original equation (1) after linearization, [14]. The mentioned work addressed physical aspects of the solution and validation of the considered approximation. Moreover, a general case of a distributed background vortex was also discussed in [14].

5. Discussion and Conclusions

We have introduced a class of vorticities extending the contour dynamics[4,5] to the case of continuously distributed vorticity and developed a Hamiltonian formalism for that class. The relation with
contour dynamics was revealed and discussed in the text. In that regard the suggested approach could be called "continuum contour dynamics".

In this section we, first, present explicit scalings transforming the suggested class to two well known models, the point vortex system, [11], and FAVOR, [6].

As for the former, let \( \Omega_k(r) = \Omega(r)I_{G_k}(r) = \tilde{\Omega}_k(r - z_k) \) be the \( k \)-th vortex written in the local coordinate system with origin at \( z_k \), then

\[
\Omega_\epsilon(r) = \frac{1}{\epsilon^2} \sum_k \tilde{\Omega}_k \left( \frac{r - z_k}{\epsilon} \right) \rightarrow \sum_k \tilde{\omega}_k \delta(r - z_k),
\]
as \( \epsilon \to 0 \), where

\[
\tilde{\omega}_k = \frac{1}{2} \int_0^{2\pi} \int_0^{\omega_j} \rho_k^2(\varphi, w) dw d\varphi.
\]

For the latter, let

\[
R^2_k(\varphi) = -|\omega_k| \frac{\partial^2}{\partial w} \bigg|_{w=\omega_k}
\]
be a characteristic space scale near the \( k \)-th vortex peak. Obviously \( r = R_k(\varphi) \) is an ellipse and vorticity lines \( \Omega_k = w \) with \( w \) close to \( \omega_k \) are well approximated by \( r = c(w)R_k(\varphi) \) with a constant depending on the vorticity level. Introduce dimensionless distance \( \hat{r} = r/R_k(\varphi) \) and define \( \Omega_k(\varphi, \hat{r}) = \hat{\Omega}_k(\varphi, r) \) where \( (\varphi, r) \) are local polar coordinates. Introduce

\[
\Omega_\epsilon(r) = \sum_k \hat{\Omega}_k \left( \varphi, \hat{r}^{1/\epsilon} \right),
\]
then

\[
\Omega_\epsilon(r) \rightarrow \sum_k \tilde{\omega}_k I_{E_k}(r)
\]
as \( \epsilon \to 0 \), where

\[
E_k = \{ (\varphi, r) | r < R_k(\varphi) \}
\]
is a Kirchhoff elliptic patch, [12].

The described limiting procedures lead to the well known Hamiltonian formulations of the point vortex system, [2], and FAVOR, [15]. However, details of a transition from Proposition 5 to equations presented in in the mentioned papers are out of the scope of our present work.

Then, notice that most of the above results can be extended to a similar class of vortices on an arbitrary two-dimensional Riemann manifold \( \mathbb{M} \) with metric \( dS = s(x, y) dxdy, (x, y) \in D \), where \( D \) is a region in the \( (x, y) \)-plane, \( s = s(x, y) \) is the metric density. This is possible because the vorticity conservation equation on \( \mathbb{M} \) similar to (1)

\[
\frac{\partial \Omega}{\partial t} + s^{-1} J(\psi, \Omega) = 0
\]
can be written in the Hamiltonian form as well

\[
\frac{\partial q}{\partial t} = \{ q, H \},
\]
where \( q = s\Omega \) is the phase variable and the non-canonical Poisson bracket is expressible similarly to (2) as

\[
\{ F, G \} = \int_D \Omega(r) J_{x,y} \left( \frac{\delta F}{\delta q(r)}, \frac{\delta G}{\delta q(r)} \right) dr,
\]
with Hamiltonian \( H = -\frac{1}{2} \int_D q \psi d\mathbf{r} \). The ideal hydrodynamics in the plane is given by \( s \equiv 1, \ D = R^2 \). For a sphere of unit radius \( x = \lambda \) is the longitude, \( y = \theta \) is the latitude and \( D = [0, 2\pi] \times [0, \pi] \), and, finally, for periodic boundary conditions on a rectangle \([0, a] \times [0, b] \) in a torus, \( s \equiv 1/(ab), \ D = [0, 2\pi] \times [0, 2\pi] \)

Dipoles on a sphere were discussed in [13].

Finally, summing up the results presented we conclude that, from the application point of view, there is no convincing evidence yet that the equations in terms of vorticity lines and coordinates of extrema could be more efficient in analytical/numerical studies of vortex systems than traditional approaches. However, certain similarities with the contour dynamics, a theory proved to be useful, give hopes for a better future.

Theoretically, the suggested approach gives a useful insight in conservation laws concerning with the vorticity topography. Yet, a traditionally interesting problem of a Poisson bracket diagonalization is solved for the suggested class of vorticities. However, we admit that the diagonalization here was a goal itself unlike a similar procedure for Hasegawa-Mima equation, [15,16] that has led to canonical variables and ultimately to advances in the weak turbulence theory. Technically, canonical variables could be introduced for the Hamiltonian system discussed here as well, but they would hardly make a clear physical sense.

6. Appendix

6.1 Proof of Lemma

Identity \( p(\dot{x}(p, w), \dot{y}(p, w)) = p \) after differentiation in \( p \) and \( w \) gives \( p_x = \dot{y}_w/g, \ p_y = -\dot{x}_w/g \). The same identity implies

\[
p_x \frac{\delta \dot{x}}{\delta \Omega(\mathbf{r})} + p_y \frac{\delta \dot{y}}{\delta \Omega(\mathbf{r})} = 0 \quad \text{or} \quad \dot{y}_w \frac{\delta \dot{x}}{\delta \Omega(\mathbf{r})} - \dot{x}_w \frac{\delta \dot{y}}{\delta \Omega(\mathbf{r})} = 0. \tag{A1}
\]

Taking variational derivatives in identity \( \Omega(\xi + (\dot{x}(p, w), \eta + (\dot{y}(p, w)) = w \) obtain

\[
\frac{\delta \Omega(\mathbf{r})}{\delta x} + \Omega_x(\delta \xi + \delta \dot{x}) + \Omega_y(\delta \eta + \delta \dot{y}) = 0. \tag{A2}
\]

From \( \Omega(x, y) = w \) it follows that \( \Omega_x = -\dot{y}/g, \ \Omega_x = \dot{x}/g \). Plug these expressions in (A2) and solve (A1-A2) for \( \delta \dot{x}/\delta \Omega(\mathbf{r}) \) and \( \delta \dot{y}/\delta \Omega(\mathbf{r}) \). The result is

\[
\frac{\delta \dot{x}(p, w)}{\delta \Omega(\mathbf{r})} = \frac{\dot{x}_w(p, w)}{g(p, w)} \left( \delta(\Omega(x, y) - w)\delta(p(x, y) - p) - \dot{y}_w \frac{\delta \xi}{\delta \Omega(\mathbf{r})} - \dot{x}_w \frac{\delta \eta}{\delta \Omega(\mathbf{r})} \right), \quad \frac{\delta \dot{y}(p, w)}{\delta \Omega(\mathbf{r})} = \frac{\delta \dot{x}}{\delta \Omega(\mathbf{r})} \frac{\dot{y}_w}{\dot{x}_w}.
\]

Substituting expressions for the variational derivatives of \( \xi \) and \( \eta \) on page 4, converting them to the delta functions in \( x \) and \( y \), arrive at (12).

6.2 Perturbation procedure

Assume for the initial condition in (24)

\[
p(\varphi, w) = p(\varphi, w; \epsilon) = p_0(\varphi, w) + \epsilon p_1(\varphi, w) + ...
\]

and represent the solution in a similar way

\[
p^2(\varphi, w) = p^2_0(\varphi, w) + \epsilon p^2_1(\varphi, w) + ...
\]

and get

\[
\frac{\partial}{\partial t} p^2_n = L_{n-1}(p^2_n), \quad p^2_n|_{t=0} = p_n(\varphi, w), \tag{A3}
\]

13
where
\[ L_{n-1}(\rho^2) = \frac{\delta N(\rho^2)}{\delta \rho^2} \bigg|_{\rho=\rho_{n-1}} \]
is the linearization of \( N \) at the previous correction.

Take the initial condition in form (26) and denote
\[ f(w) = \ln S^{-1} \left( \frac{w}{M} \right). \]
Thus, in the first order of \( \epsilon \)
\[ p(\varphi, w) = R^2(\varphi) + 2\epsilon R^2(\varphi)f(w) \]
Because \( p_0 \) does not depend on \( w \), the zero approximation \( \rho_0^2 = \rho_0^2(t, \varphi) \) does not depend on \( w \) as well and in fact is nothing but the solution of the contour dynamics equation corresponding to the initial condition at \( \epsilon = 0 \). In addition, assume that \( R(\varphi) \) is symmetric about the pole thereby the singular term in (24) vanishes \( (\psi_\xi = \psi_\eta = 0) \), i.e. the vortex center does not move. That implies essential simplifications in the linearized equation (A3) for \( n = 1 \)
\[ \frac{\partial}{\partial t} \rho_1^2(\varphi, w) = -\int_0^{2\pi} K(\varphi, \theta) \int_0^M \rho_1^2(\theta, u)du d\theta, \quad \rho_1^2|_{t=0} = 2R^2(\varphi)f(w), \quad (A4) \]
where the kernel is
\[ K(\varphi, \theta) = \frac{1}{2\pi} \frac{\partial}{\partial \varphi} \ln \left( \rho_0^2(\varphi) + \rho_0^2(\theta) - 2\rho_0(\theta)\rho_0(\varphi)\cos(\theta - \varphi) \right). \]
Integrate both parts of (A4) in \( w \) and get
\[ \frac{\partial}{\partial t} z(\varphi) = -M \int_0^{2\pi} K(\varphi, \theta)z(\theta)d\theta, \quad z|_{t=0} = 2\bar{f} R^2(\varphi), \]
where
\[ z(\varphi) = \frac{1}{M} \int_0^M \rho_1^2(\varphi, w)dw, \quad \bar{f} = \frac{1}{M} \int_0^M f(w)dw. \]
This equation is easy to solve numerically and then retrieve \( \rho_1(t, \varphi, w) \) itself using the initial condition given in (A4). The result is
\[ \rho_1^2(t, \varphi, w) = z(t, \varphi) + 2R^2(\varphi)(t, \varphi) \left( f(w) - \bar{f} \right). \]
Converting
\[ \rho^2(t, \varphi, w) = \rho_0^2(t, \varphi) + \epsilon \rho_1^2(t, \varphi, w) \]
with respect to \( w \), we obtain the first order approximation for the vorticity itself
\[ \Omega_\epsilon(\xi + r \cos \varphi, \eta + r \sin \varphi) = MS \left( \exp \left\{ \frac{r^2 - \rho_0^2(t, \varphi) - \epsilon (z(t, \varphi) - z(0, \varphi))}{2\epsilon R^2(\varphi)} \right\} \right). \]

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