LUBIN-TATE THEORY AND OVERCONVERGENT HILBERT MODULAR FORMS OF LOW WEIGHT

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Abstract. Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $\Gamma$ be the Galois group of the cyclotomic extension of $K$. Fontaine’s theory gives a classification of $p$-adic representations of $\text{Gal} (\overline{K}/K)$ in terms of $(\varphi, \Gamma)$-modules. A useful aspect of this classification is Berger’s dictionary which expresses invariants coming from $p$-adic Hodge theory in terms of these $(\varphi, \Gamma)$-modules.

In this article, we use the theory of locally analytic vectors to generalize this dictionary to the setting where $\Gamma$ is the Galois group of a Lubin-Tate extension of $K$. As an application, we show that if $F$ is a totally real number field and $v$ is a place of $F$ lying above $p$, then the $p$-adic representation of $\text{Gal} (F_v/F_v)$ associated to a finite slope overconvergent Hilbert eigenform which is $F_v$-analytic up to a twist is Lubin-Tate trianguline. Furthermore, we determine a triangulation in terms of a Hecke eigenvalue at $v$. This generalizes results in the case $F = \mathbb{Q}$ obtained previously by Chenevier, Colmez and Kisin.

Contents

1. Introduction 2
1.1. Structure of the article 5
1.2. Notations and conventions 5
1.3. Acknowledgments 5
2. Locally $K$-analytic and pro $K$-analytic vectors 5
2.1. Locally analytic and pro analytic vectors 5
2.2. Locally analytic vectors in $\hat{K}_\infty$-semilinear representations 6
2.3. Pro analytic vectors in $B_{\text{dR}}$ 8
3. Lubin-Tate $p$-adic Hodge theory 8
3.1. The modules $D_{\text{Sen}, K}$ and $D_{\text{dif},K}$ 8
3.2. The modules $D_{\text{HT}, K}$ and $D_{\text{dR}, K}$ 10
3.3. The modules $D_{\text{cris}, K}$ and $D_{\text{st}, K}$ 11
4. Big period rings 12
4.1. The rings $\tilde{B}_{\text{rig}}^\dagger$ and $\tilde{B}_{\text{log}}^\dagger$ 12
4.2. Pro $K$-analytic vectors 14
5. Lubin-Tate $(\varphi_q, \Gamma_K)$-modules 16
5.1. $K$-analytic $(\varphi_q, \Gamma_K)$-modules 16
5.2. The modules $D_{\ast, K}$ and the extended dictionary 17
6. Lubin-Tate trianguline representations of dimension 2 19
6.1. Characters of the Weil group 20
6.2. Extensions 20
6.3. Spaces of Lubin-Tate trianguline $(\varphi_q, \Gamma_K)$-modules of dimension 2 20
1. Introduction

Let $p$ be a prime number. Kisin showed in [Ki03] that the $p$-adic representation $\rho_f$ of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ attached to a finite slope $p$-adic eigenform $f$ has a very special property: its restriction to $\text{Gal}((\mathbb{Q}_p/\mathbb{Q}_p)$ always has a crystalline period. Even better, this period is an eigenvector for crystalline Frobenius, with eigenvalue coinciding with that arising from the Hecke action of $U_p$ on $f$. Consequently, Kisin was able to verify the Fontaine-Mazur conjecture for these $p$-adic representations. In a subsequent work [Co08], Colmez gave a reinterpretation of Kisin’s result to the effect that the $(\varphi, \Gamma)$-module attached to $\rho_f|_{\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)}$ is an extension of $(\varphi, \Gamma)$-modules of rank 1, where $\Gamma \cong \mathbb{Z}_p^\times$ is the Galois group of the cyclotomic extension of $\mathbb{Q}_p$. Colmez coined the term “trianguline” for the $p$-adic representations satisfying this property, and studied them in detail in dimension 2. Then in [Co10] he attached to any 2-dimensional trianguline representation of $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$ a unitary Banach representation of $\text{GL}_2(\mathbb{Q}_p)$. By a suitable continuity argument he was able to extend this procedure to any 2-dimensional $p$-adic representation of $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$, thereby constructing the $p$-adic Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$. This circle of ideas came to a satisfying conclusion when Emerton used this correspondence in [Em11] to show that the trianguline property at $\text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$ characterizes these 2-dimensional representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which are attached to finite slope $p$-adic eigenforms.

In this article, we are concerned with performing the reinterpretation step of Colmez in an analogous story when $\mathbb{Q}$ is replaced with a totally real number field $F$. Namely, the $p$-adic representation $\rho_f$ of $\text{Gal}(\mathbb{Q}_v/\mathbb{Q}_p)$ is attached to a finite slope $p$-adic Hilbert eigenform $f$, and we would like to show $\rho_f$ is trianguline at a place $v | p$. However, when $F_v \neq \mathbb{Q}_p$, there is more than one meaning one can attach to the phrase “$\rho_f$ is trianguline at a place $v | p$”. On the one hand, there are cyclotomic trianguline $\text{Gal}(\overline{F}_v/F_v)$-representations. These are the trianguline representations in the sense of Nakamura in [Na09]; in that setting, $\Gamma$ is the Galois group of the cyclotomic extension of $F_v$ and is isomorphic to an open subgroup of $\mathbb{Z}_p^\times$. On the other hand, there are Lubin-Tate trianguline $\text{Gal}(\overline{F}_v/F_v)$-representations in the sense of Fourquaux and Xie in [FX13], where $\Gamma = \Gamma_{F_v}$ is the Galois group of a Lubin-Tate extension and is isomorphic to $\mathcal{O}_{F_v}^\times$. The representation $\rho_f|_{\text{Gal}(\overline{F}_v/F_v)}$ has been known to be cyclotomic trianguline for a while by the global triangulation theory of Kedlaya-Pottharst-Xiao in [KPX14] and Liu in [Li12]. Although these results have found many applications, it seems like this notion of cyclotomic triangulinity may not be the most suitable for applications to a hypothetical $p$-adic Langlands correspondence for $\text{GL}_2(F_v)$. Rather, the replacement of $\mathbb{Z}_p^\times$ by $\mathcal{O}_{F_v}^\times$ seems more natural, which leads us in this work to focus on the notion of Lubin-Tate triangulinity of Fourquaux and Xie.
Let us explain what the difficulties are in proving such a result when $F_v \neq \mathbb{Q}_p$. The methods of [KPX14] and [Li12] can still be used to show the existence of a crystalline period which is a Frobenius eigenvector. However, Colmez’s reformulation of this condition in terms of $(\varphi, \Gamma)$-module for $F_v = \mathbb{Q}_p$ relies on Berger’s dictionary, which expresses invariants coming from $p$-adic Hodge theory in terms of these $(\varphi, \Gamma)$-modules. This dictionary is only available in the cyclotomic setting. Indeed, the proof of this dictionary ultimately relies on Sen theory and the Cherbonnier-Colmez theorem. Unfortunately, a direct attempt to use these methods breaks down whenever $F_v \neq \mathbb{Q}_p$, because of the failure of the Tate-Sen axioms for Lubin-Tate extensions.

Now let $K$ be a finite extension of $\mathbb{Q}_p$. Recall that a representation $V$ of $\text{Gal}(\overline{K}/K)$ with coefficients in $K$ is called $K$-analytic if $C_p \otimes_K^\tau V$ is trivial for each nontrivial embedding $\tau : K \to \overline{K}$. In [BC16], Berger and Colmez were able to find a certain generalization of Sen theory for Lubin-Tate extensions of $K$ using ideas coming from the theory of $K$-locally analytic vectors. Berger then used this theory in [Be16] to prove that $K$-analytic representations are overconvergent, so that we can associate to $V$ a Lubin-Tate $(\varphi_q, \Gamma_K)$-module $D^\dagger_{\text{rig},K}(V)$ over the (Lubin-Tate) Robba ring $B^\dagger_{\text{rig},K}$ (see §5). By adapting the original techniques of [Be02] to the setting of $K$-locally analytic vectors, we are able to extend Berger’s dictionary to Lubin-Tate extensions of $K$. Our first main result is the following (see Theorem 5.4).

**Theorem A.** Let $V$ be a $K$-analytic representation of $G_K$. For $* \in \{\text{Sen, dif, } dR, \text{ cris, st}\}$, there is a natural isomorphism

$$D_{*,K}(V) \cong D_{*,K} \left( D^\dagger_{\text{rig},K}(V) \right).$$

For the definition of the functors $D_{*,K}$ we refer to §3. When $K = \mathbb{Q}_p$ they coincide with the usual definitions and the theorem is already known, but note that in contrast, it is not a-priori clear how to even define $D_{\text{Sen},K}$ and $D_{\text{dif},K}$ in the general case.

When $V$ is 2-dimensional, one can deduce from Theorem A that the Lubin-Tate triangulinity of $V$ can be detected from the existence of a crystalline period which is a Frobenius eigenvector. On the other hand, techniques going back to the original paper of Colmez show that the cyclotomic triangulinity of $V$ can also be detected in a similar way. From this we deduce that the two notions of triangulinity actually coincide for $K$-analytic representations of dimension 2 (see Theorem 6.8 for a more precise version).

**Theorem B.** Let $V$ be a 2-dimensional $K$-analytic representation of $G_K$. Then $V$ is Lubin-Tate trianguline if and only if $V$ is cyclotomic trianguline.

As mentioned above, it is known that the local Galois representations associated to finite slope overconvergent Hilbert eigenforms are cyclotomic trianguline. It is then natural to use Theorem B to translate this into a Lubin-Tate triangulinity result, provided that this local representation is analytic. Furthermore, it is possible to explicitly determine this triangulation, generalizing previous work of Chenevier and Colmez (see Theorem 7.4 for a more precise statement). To state the result, let $\rho_f$ be the $p$-adic representation of $\text{Gal}(\overline{F}/F)$ associated
to a finite slope overconvergent Hilbert eigenform of weights \((k, 1, \ldots, 1)\) at \(v\) and determinant \(\eta_{\text{cyc}}^{w-1}\) for some potentially unramified character \(\eta\). For the sake of simplifying the introduction, assume here that the restriction of \(\rho_f\) to a decomposition group \(G_{F_v} = \Gal(F_v/F_v)\) has coefficients in \(F_v\) and that \(k, w \in \mathbb{Z}\). Removing these assumptions only requires introducing the following notation. To state the result, choose a uniformizer \(\pi_v\) of \(F_v\), write \(\chi_{\pi_v}\) for the corresponding Lubin-Tate character and let \(a_v \in F_v^\times\) be such that \(U_v = a_v f\). If \(y \in F_v^\times\), write \(\mu_y : F_v^\times \to F_v^\times\) for the character defined by \(\mu_y(z) = y^{\text{val}(z)}\). Let \(\chi \in \text{val}(z)\) be the character \(x(z) = z\) and \(x_0 : F_v^\times \to F_v^\times\) be the character \(x_0(z) = x/\pi_v\). We say that \(f\) is \(\eta\)-analytic up to a twist if the same holds for \(\rho_f|_{G_{F_v}}\).

**Theorem C.** Suppose that \(f\) is \(F_v\)-analytic up to a twist. Then it is Lubin-Tate trianguline. If \(D_{\text{rig}, F_v}^\dagger (\rho_f|_{G_{F_v}})\) is the \((\varphi, \Gamma_{F_v})\)-module over \(B_{\text{rig}, F_v}^\dagger\) associated to \(\rho_f|_{G_{F_v}}\), a triangulation is given by the short exact sequence

\[
0 \to B_{\text{rig}, F_v}^\dagger (\delta_1) \to D_{\text{rig}, K}^\dagger (\rho_f|_{G_{F_v}}) \to B_{\text{rig}, F_v}^\dagger (\delta_2) \to 0,
\]

where \(\delta_2 = \delta_1^{-1}\text{det}(V)\) and \(\delta_1 : F_v^\times \to F_v^\times\) is a character. Here and \(\delta_1\) and \(\rho_f|_{G_{F_v}}\) satisfy the following.

1. If \(k \notin \mathbb{Z}_{\geq 1}\) then \(\delta_1 = \mu_{a_v} x_0^{\frac{k-1}{2}} \left( N_{F_v/Q_p} \circ x_0 \right)^{\frac{1-w}{2}}\) and \(\rho_f|_{G_{F_v}}\) is irreducible and not Hodge-Tate.

2. If \(k \in \mathbb{Z}_{\geq 1}\) and \(\text{val}(a_v) < \frac{k-1}{2} + \frac{w-1}{2} [F_v : \mathbb{Q}_p]\), then \(\delta_1 = \mu_{a_v} x_0^{\frac{k-1}{2}} \left( N_{F_v/Q_p} \circ x_0 \right)^{\frac{1-w}{2}}\) and \(\rho_f|_{G_{F_v}}\) is irreducible and potentially semistable.

3. If \(k \in \mathbb{Z}_{> 1}\) and \(\text{val}(a_v) = \frac{k-1}{2} + \frac{w-1}{2} [F_v : \mathbb{Q}_p]\), then either

   a. \(\delta_1 = \mu_{a_v} x_0^{\frac{k-1}{2}} \left( N_{F_v/Q_p} \circ x_0 \right)^{\frac{1-w}{2}}\) and \(\rho_f|_{G_{F_v}}\) is reducible, nonsplit and potentially crystalline.

   b. \(\delta_1 = x_0^{1-k} \mu_{a_v} x_0^{\frac{k-1}{2}} \left( N_{F_v/Q_p} \circ x_0 \right)^{\frac{1-w}{2}}\) and \(\rho_f|_{G_{F_v}}\) is a sum of two characters and potentially crystalline.

4. If \(k \in \mathbb{Z}_{\geq 1}\) and \(\text{val}(a_v) > \frac{k-1}{2} + \frac{w-1}{2} [F_v : \mathbb{Q}_p]\), then \(\delta_1 = x_0^{1-k} \mu_{a_v} x_0^{\frac{k-1}{2}} \left( N_{F_v/Q_p} \circ x_0 \right)^{\frac{1-w}{2}}\) and \(\rho_f|_{G_{F_v}}\) is irreducible, Hodge-Tate and not potentially semistable.

The condition on the weights at \(v\) to be of the form \((k, 1, \ldots, 1)\) is necessary but not sufficient for \(f\) to be \(F_v\)-analytic up to a character twist. In fact, the computations of [Na09, Proposition 2.10] and [FX13, Theorem 0.3] suggest that this stronger condition of \(F_v\)-analyticity cuts out a locus of codimension \([F_v : \mathbb{Q}_p] - 1\) inside the locus of weights \((k, 1, \ldots, 1)\) at \(v\) of the Hilbert eigenvariety. However, under suitable local-global compatibility conjectures, it contains all classical points of weights \((k, 1, \ldots, 1)\). We believe it might be possible to obtain a version of Theorem C for arbitrary \(f\) if one works with \((\varphi, \Gamma_{F_v})\)-modules over multivariable Robba rings as in [Be13].

Finally, we make some further speculations. For simplicity, assume that \(p\) is inert in \(F\). The small slope condition \(\text{val}(a_p) < \frac{k-1}{2} + \frac{w-1}{2} [F_v : \mathbb{Q}_p]\) in Theorem C agrees with the optimal bound in partial weight 1 conjectured in an unpublished note of Breuil (see Proposition 4.3 of [Br10]). This suggests that \(F_v\)-analytic finite slope \(p\)-adic Hilbert eigenforms of weights \((k, 1, \ldots, 1)\) and \(\text{val}(a_p) < \frac{k-1}{2} + \frac{w-1}{2} [F_v : \mathbb{Q}_p]\) should be classical. If such a classicality criterion were known, an argument as in Theorem 6.6 of [Ki03] using our Theorem 7.4 would verify...
the Fontaine-Mazur conjecture for representations which arise from $F_{\pi}$-analytic finite slope $p$-adic Hilbert eigenforms. Conversely, if the Fontaine-Mazur conjecture were known in our context then Theorem 7.4 would imply such a classicality criterion. See §7.2 for a more precise discussion.

In another direction, suppose again that $p$ is inert in $F$ and that $V$ is a $p$-adic representation of $\text{Gal}(\overline{F}/F)$ which is irreducible, totally odd, unramified at almost all primes and Lubin-Tate trianguline at $p$. Recall again the theorem of Emerton (Theorem 1.2.4 of [Em11]) which asserts that if $F = \mathbb{Q}$ and $V$ satisfies certain technical conditions then $V$ is the character twist of the Galois representation attached to an (elliptic) overconvergent $p$-adic eigenform of finite slope. In light of Theorem 7.4, we ask the following.

**Question 1.1.** Is $V$ necessarily the character twist of a representation attached to an $F_{\pi}$-analytic overconvergent $p$-adic Hilbert eigenform of finite slope?

### 1.1. Structure of the article

§2 contains preliminaries regarding locally $K$-analytic vectors. In §3 we define the functors $D_{\ast, K}$ for $\ast \in \{\text{Sen}, \text{dif}, \text{dR}, \text{cris}, \text{st}\}$ and prove their basic properties. In §4 we study some big period rings and their $K$-locally analytic vectors. Theorem A is proved in §5. This proof involves reinterpreting several constructions in $p$-adic Hodge theory in terms of $K$-locally analytic vectors, as well as some computations with rather large rings of periods, and so involves most of the work done in §§2-5. In §6 we relate this to Lubin-Tate triangulinity and prove Theorem B. Finally, in §7 we prove Theorem C, concluding with an example.

### 1.2. Notations and conventions

The field $K$ denotes a finite extension of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}_K$, uniformizer $\pi$, and residue field $k$. The field $K_0 = W(k)[1/p]$ is the maximal unramified subextension of $K$. Let $q = p^f$ be the cardinality of the residue field and $e$ the absolute ramification index of $K$, so that $[K : \mathbb{Q}_p] = ef$. We let $\Sigma_K$ denote the set of embeddings of $K$ into $\overline{\mathbb{Q}}_p$.

Denote by $G_K$ the absolute Galois group of $K$. If $\mathcal{F}$ is a formal Lubin-Tate group associated to $\pi$, then $K_n = K(\mathcal{F}[\pi^n])$ and $K_\infty = \cup_{n \geq 1} K_n$ are abelian extensions of $K$ which depend only on $\pi$. The Lubin-Tate character $\chi_\pi : G_K \to \mathcal{O}_K^\times$ is the character given by the action of $G_K$ on $\mathcal{F}[\pi_\infty]$. It induces an isomorphism of $\Gamma_K = \text{Gal}(K_\infty/K)$ with $\mathcal{O}_K^\times$. Its kernel is $H_K = \text{Gal}(\overline{K}/K_\infty)$, and $G_K/H_K = \Gamma_K$. The cyclotomic character $\chi_{\text{cyc}}$ of $G_K$ satisfies the relation $N_{K/k_p} \circ \chi_\pi = \chi_{\text{cyc}} \eta$ for an unramified character $\eta$.

A $K$-linear representation $V$ of $\text{Gal}(\overline{K}/K)$ is called $K$-analytic if $\mathbb{C}_p \otimes_{\mathbb{Q}_p} V$ is trivial for each $\tau \in \Sigma_K \setminus \{\text{Id}\}$.

Finally, all characters and representations appearing in this article are assumed to be continuous. We normalize the $p$-adic valuation and $p$-adic logarithm so that $\text{val}_p(p) = 1$ and $\log(p) = 0$. The Hodge-Tate weight of $\chi_{\text{cyc}}$ is 1.

### 1.3. Acknowledgments

2. Locally $K$-analytic and pro $K$-analytic vectors

#### 2.1. Locally analytic and pro analytic vectors

We briefly recall the treatment given in §2 of [Be16] and in §2 of [BC16].
Let $W$ be a Banach $\mathbb{Q}_p$-linear representation of $\Gamma_K$. Given an open subgroup $H$ of $\Gamma_K$ with coordinates $c_1, ..., c_d : H \tilde{\to} \mathbb{Z}_p^d$, we have the subspace $W^{H-\text{an}}$ of $H$-analytic vectors in $W$. These are the elements $w \in W$ for which there exists a sequence of vectors $\{w_k\}_{k \in \mathbb{N}^d}$ with $w_k \to 0$ and $g(w) = \sum_{k \in \mathbb{N}^d} a_k(g)^k w_k$ for all $g \in H$. Write $W^\text{la} = \cup_H W^{H-\text{an}}$ for the subspace of locally analytic vectors of $W$. If $W$ is a Fréchet space whose topology is defined by a countable sequence of seminorms, let $W_i$ be the Hausdorff completion of $W$ for the $i$'th norm, so that $W = \operatorname{lim}_{\leftarrow} W_i$ is a projective limit of Banach spaces. We write $W^\text{pa} = \operatorname{lim}_{\leftarrow} W^\text{la}_i$ for the subspace of pro analytic vectors.

For $n \gg 0$, we have an isomorphism $l : \Gamma_{K_n} \to \pi^n \mathcal{O}_K$, given by $g \mapsto \log(\chi_\pi(g))$. We have the subspace $W^{\Gamma_{K_n}-\text{an}, \text{K-\text{la}}}$ of vectors which are $K$-analytic on $\Gamma_{K_n}$, i.e. such that there exists a sequence $\{w_k\}_{k \in \mathbb{N}}$ with $\pi^{nk} w_k \to 0$ and $g(w) = \sum_{k \in \mathbb{N}} l(g)^k w_k$ for all $g \in \Gamma_{K_n}$. We write $W^K-\text{la} = \cup_{n \geqslant 0} W^{\Gamma_{K_n}-\text{an}, \text{K-\text{la}}}$ for the subspace of $K$-locally analytic vectors of $W$. If $W = \operatorname{lim}_{\leftarrow} W_i$ is a Fréchet space as above, we write $W^K-\text{pa} = \operatorname{lim}_{\leftarrow} W^K_{i-\text{la}}$ for the subspace of pro $K$-analytic vectors. We extend the definitions of locally $\widehat{K}$-analytic vectors and pro $K$-analytic vectors to LB and LF spaces (i.e. filtered colimits of Banach spaces and Fréchet spaces) in the obvious way.

For each $\tau \in \Sigma_K$, there is a differential operator $\nabla_\tau \in K^{\text{Gal}} \otimes \mathcal{O}_p \operatorname{Lie}(\Gamma_K)$ (see §2 of [Be16]) where $K^{\text{Gal}}$ is the Galois closure of $K$. It is defined in such a way that if $W$ is $K^{\text{Gal}}$-linear, then for $w \in W^\text{la}$ and $g \in \Gamma_{K_n}$ with $n \gg 0$ we have $g(w) = \sum_{k \in \mathbb{N} \cap \mathbb{K}} l(g)^k w_k$, where $l(g)^k = \prod_{\tau \in \Sigma_K} \tau(l(g))^{k_\tau}$ and $\nabla_\tau(w) = \prod_{\tau \in \Sigma_K} \nabla_\tau(w)$. In other words, we can think of $\tau \circ l$ as giving coordinates for $\Gamma_K$ and $\nabla^k(w)$ as being an iterated directional derivative of $w$. In particular, $W^K-\text{la}$ is the subspace of $W^\text{la}$ where $\nabla_\tau = 0$ for each $\tau \in \Sigma_K \setminus \{\text{Id}\}$. On $W^K-\text{la}$ (or on $W^K-\text{pa}$ if $W$ is Fréchet) we write $\nabla = \nabla_\text{Id}$ when there is no danger of confusion; it is given by the formula

$$\nabla(w) = \lim_{g \to 1} \frac{g(w) - w}{\log(\chi_\pi(g))},$$

and we have $\nabla(w) = \frac{\log(\chi_\pi(w))}{\log(\chi_\pi(g))}$ when $g$ is sufficiently close to 1.

The next lemma is proved in the same way as Proposition 2.2 and Proposition 2.4 of [Be16].

**Lemma 2.1.** Let $B$ be a Banach (resp. Fréchet) $\Gamma_K$-ring and let $W$ be a free $B$-module of finite rank, equipped with a compatible action of $\Gamma_K$. If the $B$ module has a basis $w_1, ..., w_d$ in which the function $\Gamma_K \to \text{GL}_d(B) \subset M_d(B), g \mapsto \text{Mat}(g)$ is locally $K$-analytic (resp. pro $K$-analytic), then $W^K-\text{la} = \oplus_{j=1}^d B^{K-\text{la}} w_j$ (resp. $W^K-\text{pa} = \oplus_{j=1}^d B^{K-\text{pa}} w_j$).

### 2.2. Locally analytic vectors in $\widehat{K}\infty$-semilinear representations.

Let $L$ be a finite extension of $K$, and write $\Gamma_L = \text{Gal}(L_\infty/L)$ where $L_\infty = L\widehat{K}\infty$. Recall that following result (Proposition 2.10 of [Be16]):

**Proposition 2.2.** $\widehat{L}_\infty^{-\text{la}} = L_\infty$.

The purpose of this subsection is to prove a similar descent result for representations.

**Theorem 2.3.** Let $W$ be a finite dimensional $\widehat{L}_\infty$-semilinear representation of $\Gamma_L$. Then the natural map $\widehat{L}_\infty \otimes_{L_\infty} W^{K-\text{la}} \to W$ is an isomorphism.
This is proved in §4 of [BC16] under the assumption that \( K \) is Galois over \( \mathbb{Q}_p \). Here we shall adapt the methods of ibid. to get rid of this assumption.

First, we reduce to the case where \( L \) is Galois over \( \mathbb{Q}_p \).

**Lemma 2.4.** Suppose that Theorem 2.3 holds for \( M = L^{\text{Gal}} \), the Galois closure of \( L \) over \( \mathbb{Q}_p \). Then Theorem 2.3 holds for \( L \).

**Proof.** Let \( L \) be any finite extension of \( K \) and let \( W \) be a finite dimensional \( \hat{L}_\infty \)-semilinear representation of \( \Gamma_K \). Write \( W_M = \hat{M}_\infty \otimes \hat{L}_\infty W \), so that \( W_M \) is a finite dimensional \( \hat{M}_\infty \)-semilinear representation of \( \Gamma_M \). Note that \( W_M \) is actually endowed with a semilinear \( \text{Gal}(M_\infty/L) \)-action, which restricts to a \( \Gamma_M \) action. By the assumption, we have \( \hat{M}_\infty \otimes M_\infty W_M^{K^{\text{la}}} \cong W_M \). On the other hand, the extension \( \text{Gal}(M_\infty/L_\infty) \) is finite, so we are in the setting for completed Galois descent (see §2.2 of [BC09]). We have

\[
W^{K^{\text{la}}} = W^{K^{\text{la}}} \cap W = (W^{K^{\text{la}}}_M)^{\text{Gal}(M_\infty/L_\infty)}
\]

which implies that \( \hat{M}_\infty \otimes M_\infty W^{K^{\text{la}}}_M \cong \hat{M}_\infty \otimes L_\infty W^{K^{\text{la}}} \). We then have the following chain of natural isomorphisms

\[
\hat{L}_\infty \otimes L_\infty W^{K^{\text{la}}} \cong \left( \hat{M}_\infty \otimes L_\infty W^{K^{\text{la}}}_M \right)^{\text{Gal}(M_\infty/L_\infty)} \\
\cong \left( \hat{M}_\infty \otimes M_\infty W^{K^{\text{la}}}_M \right)^{\text{Gal}(M_\infty/L_\infty)} \\
\cong (W_M)^{\text{Gal}(M_\infty/L_\infty)} \\
\cong W
\]

whose composition is the natural map \( \hat{L}_\infty \otimes L_\infty W^{K^{\text{la}}} \to W \), which proves the claim. \( \square \)

**Proposition 2.5.** If \( \tau \in \Sigma_K \setminus \{ \text{Id} \} \) and \( K^{\text{Gal}} \subset L \), there exists an element \( x_\tau \in \hat{L}_\infty \) such that \( g(x_\tau) = x_\tau + \tau((g)) \) for \( g \in G_{K^{\text{Gal}}} \). In particular, \( \nabla_\tau(x_\tau) = 1 \) and \( \nabla_\sigma(x_\tau) = 0 \) for \( \sigma \neq \tau \).

**Proof.** By §3.2 of [Fo09], there exists an element \( \xi_\tau \in \mathbb{C}_p^\times \) such that \( \xi_\tau g(\xi_\tau) = \tau(\chi_\pi(g)) \) for \( g \in G_{K^{\text{Gal}}} \). This equation makes it clear that \( \xi_\tau \) lies in the completion of \( K^{\text{Gal}}K_\infty \), which is contained in \( \hat{L}_\infty \). Now take \( x_\tau = -\log \xi_\tau \). \( \square \)

For each \( n \geq 1 \) and for each \( \tau \neq \text{Id} \), choose \( x_\tau \) as in Proposition 2.5 and let \( x_{n,\tau} \in L_\infty \) be such that \( ||x_\tau - x_{n,\tau}|| \leq p^{-n} \). For \( k \in \mathbb{N}^{\Sigma_K \setminus \{ \text{Id} \}} \) we write \( (x-x_n)^k = \prod_{\tau \in \Sigma_K \setminus \{ \text{Id} \}} (x_\tau - x_{n,\tau})^{k_\tau} \).

**Proof of Theorem 2.3.** By Lemma 2.4, we may assume \( L \) is Galois over \( \mathbb{Q}_p \). Recall that by Theorem 1.7 of [BC16], the natural map \( \hat{L}_\infty \otimes L_\infty W^{\text{la}} \to W^{\text{la}} \) is an isomorphism. Therefore, it is enough to prove that the natural map \( \hat{L}_\infty \otimes L_\infty W^{K^{\text{la}}} \to W^{\text{la}} \) is an isomorphism. To prove injectivity, take \( \sum_{i=1}^n \alpha_i \otimes x_i \in \hat{L}_\infty \otimes L_\infty W^{K^{\text{la}}} \) of minimal length such that \( \sum_{i=1}^n \alpha_i x_i = 0 \). We may assume that \( \alpha_1 = 1 \). For each \( \tau \neq \text{Id} \), we have \( \nabla_\tau (\sum_{i=1}^n \alpha_i x_i) = \sum_{i=2}^n \nabla_\tau (\alpha_i) x_i \), so by minimality \( \nabla_\tau (\alpha_i) = 0 \) for all \( i \). This means that each \( \alpha_i \in \hat{L}_\infty \), so \( \sum_{i=1}^n \alpha_i \otimes x_i = 0 \).

To prove surjectivity, we give a sketch, omitting all details of convergence; these can be provided in exactly the same way as in §4 of [BC16]. For each \( z \in W^{\text{la}} \), and for each
\[ i \in \mathbb{N}_{\Sigma_K \setminus \{\text{Id}\}}, \]

let

\[ y_i = \sum_{k \in \mathbb{N}_{\Sigma_K \setminus \{\text{Id}\}}} (-1)^{|k|} (x - x_n)^k \nabla^{k+i}(z) \begin{pmatrix} k + i \end{pmatrix}. \]

One can show that \( y_i \in W^{Ia} \). By Proposition 2.5, for each \( \tau \in \Sigma_K \setminus \{\text{Id}\} \) we have \( \nabla_{\tau} (x - x_n)^k = k_{\tau} (x - x_n)^{k-1} \), where 1 is the tuple \((k_\sigma) \in \mathbb{N}_{\Sigma_K} \) with \( k_{\tau} = 1 \) and \( k_{\sigma} = 0 \) for \( \sigma \neq \tau \). By a direct calculation this implies that \( \nabla_{\tau} (y_i) = 0 \), so that \( y_i \in W^{K-Ia} \). Finally, the identity

\[ z = \sum_{i \in \mathbb{N}_{\Sigma_K \setminus \{\text{Id}\}}} y_i (x - x_n)^i \]

shows that \( z \in \widehat{L}_\infty \otimes \widehat{\mathbb{L}}_{\infty} W^{Ia} \).

\[ \square \]

2.3. **Pro analytic vectors in** \( B_{dR}^{+} \). The ring \( B_{dR}^{+} \) contains an element \( t_K \) for which each \( g \in G_K \) acts by \( g(t_K) = \chi_{\pi}(g)t_K \). It differs from the usual \( t \) by a unit, but it has the advantage that it carries an action of \( \Gamma_K \), which is moreover \( K \)-analytic. As \( B_{dR}^{+}/t_K \cong \mathbb{C}_p \), the quotients \((B_{dR}^{+}/t_K)^{H_K} \) for \( l \geq 1 \) are Banach \( \Gamma_K \)-rings. The ring and \((B_{dR}^{+})^{H_K} \) is a Fréchet \( \Gamma_K \)-ring and \((B_{dR})^{H_K} \) is an LF \( \Gamma_K \)-ring.

**Proposition 2.6.**

1. \((B_{dR}^{+}/t_K)^{H_K,K-Ia} = K_\infty[t_K]/t_K^{l} \).

2. \((B_{dR}^{+})^{H_K,K-pa} = K_\infty[[t_K]] \).

3. \((B_{dR})^{H_K,K-pa} = K_\infty ((t_K)) \).

**Proof.** (3) follows from (2) and (2) follows from (1). To prove (1), we argue by induction. For \( l = 1 \), this is Proposition 2.2. For \( l \geq 2 \), we have a short exact sequence

\[ 0 \to \mathbb{C}_p(l-1) \to B_{dR}^{+}/t_K^{l} \to B_{dR}^{+}/t_K^{l-1} \to 0. \]

Taking \( H_K \) invariants and \( K \)-locally analytic vectors is left exact, so we have

\[ 0 \to K_\infty(l-1) \to (B_{dR}^{+}/t_K^{l})^{H_K,K-Ia} \to (B_{dR}^{+}/t_K^{l-1})^{H_K,K-Ia} = K_\infty[t_K]/t_K^{l-1}. \]

This shows that \( \dim K_\infty (B_{dR}^{+}/t_K^{l})^{H_K,K-Ia} \leq l \), so the containment \( K_\infty[t_K]/t_K^{l} \subset (B_{dR}^{+}/t_K^{l})^{H_K,K-Ia} \) has to be an equality, concluding the proof.

\[ \square \]

3. **Lubin-Tate \( p \)-adic Hodge theory**

The goal of this section is to provide constructions and properties of several of Fontaine’s functors on \( p \)-adic representations where \( \mathbb{Q}_p \)-coefficients are systematically replaced by \( K \)-coefficients. Throughout, we fix a \( K \)-linear \( G_K \)-representation \( V \) of dimension \( d \).

3.1. **The modules** \( D_{Sen,K} \) and \( D_{\text{diff},K} \). When \( K = \mathbb{Q}_p \), the modules \( D_{Sen,K} \) and \( D_{\text{diff},K} \) can be defined using the method of Sen (see §4 of [BC08]). It is unavailable for \( K \neq \mathbb{Q}_p \), so we make use of locally analytic and pro analytic vectors instead.

We set \( W_{+,l} = (B_{dR}^{+}/t_K^{l})^{H_K} \) for \( l \geq 1 \), \( W_{+} = (B_{dR}^{+} \otimes K V)^{H_K} \) and \( W = (B_{dR} \otimes K V)^{H_K} \).

By Proposition 2.6, we have \( K_\infty[t_K]/t_K^{l} \)-submodules \( D_{\text{diff},K}^{+,l}(V) = W_{+,l}^{K-Ia} \) for \( l \geq 1 \), a \( K_\infty[[t_K]] \)-submodule \( D_{\text{diff},K}^{+}(V) = W_{+}^{K-pa} \) and a \( K_\infty ((t_K)) \)-vector space \( D_{\text{diff},K}(V) = W^{K-pa} \). The subspace \( D_{\text{diff},K}^{+,l}(V) \) is also called \( D_{Sen,K}(V) \), and was already constructed in [BC16].
Lemma 3.1. The natural map $B^+_\text{dr}/t'_K \otimes_{K_{\infty}[t_K]/t'_K} W_{+l} \to B^+_\text{dr}/t'_K \otimes_K V$ is an isomorphism.

Proof. It suffices to prove that $H^1(H_K, \text{GL}_d(B^+_\text{dr}/t'_K)) = 1$. When $l = 1$ this is true by almost étale descent. For $l \geq 2$, we have a short exact sequence

$$1 \to I + t^{-1}_K M_d (B^+_\text{dr}/t'_K) \to \text{GL}_d(B^+_\text{dr}/t'_K) \to \text{GL}_d(B^+_\text{dr}/t'_{K-1}) \to 1.$$  

As $I + t^{-1}_K M_d (B^+_\text{dr}/t'_K) \cong M_d(\mathbb{C}_p(l-1))$, the group $H^1(H_K, I + t^{-1}_K M_d (B^+_\text{dr}/t'_K))$ is trivial, and we conclude by induction.

We will also need the following.

Lemma 3.2. Let $w \in W_{+l}$ and suppose that $t_K \cdot w \in D^+_{\text{diff}, K}(V)$. Then $w \in D^+_{\text{diff}, K}(V)$.

Proof. Since $\nabla(t_K w) = t_K(w + \nabla(w))$, we have that $\frac{\nabla(t_K w)}{t^{k}_K}$ is divisible by $t_K$ for $k \geq 1$. If $t_K \cdot w \in D^+_{\text{diff}, K}(V)$, there exists an $n \gg 0$ such that for $g \in \Gamma_n$, we have $g(t_K w) = \sum_{k \geq 0} l(g)^k t_K w_k$, where $w_k = t^{-1}_K \frac{\nabla(t_K w)}{t^{k}_K}$. Therefore, $g(w) = \chi_p(g^{-1}) \sum_{k \geq 0} l(g)^k w_k$, so $w$ is locally $K$-analytic.

Proposition 3.3. 1. The natural map $B^+_\text{dr}/t'_K \otimes_{K_{\infty}[t_K]/t'_K} D^+_{\text{diff}, K}(V) \to B^+_\text{dr}/t'_K \otimes_K V$ is an isomorphism, and $D^+_{\text{diff}, K}(V)$ is a free $K_{\infty}[t_K]/t'_K$-module of rank $d$.

2. The natural map $B^+_\text{dr} \otimes_{K_{\infty}[t_K]} D^+_{\text{diff}, K}(V) \to B^+_\text{dr} \otimes_K V$ is an isomorphism, and $D^+_{\text{diff}, K}(V)$ is a free $K_{\infty}[t_K]$-module of rank $d$.

3. The natural map $B^+_\text{dr} \otimes_{K_{\infty}[t_K]} D^+_{\text{diff}, K}(V) \to B^+_\text{dr} \otimes_K V$ is an isomorphism, and $D^+_{\text{diff}, K}(V)$ is a $K_{\infty}((t_K))$-vector space of dimension $d$.

Proof. Recall that $D^+_{\text{diff}, K}(V) = W_{+l}^{K-\text{la}}$. By Lemma 3.1, proving (1) reduces to showing that the natural map $\tilde{\mathcal{F}}_{\infty} \otimes_{K_{\infty}} W_{+l}^{K-\text{la}} \to W_{+l}$ is an isomorphism and that $W_{+l}^{K-\text{la}}$ is a free $K_{\infty}[t_K]/t'_K$-module of rank $d$. By Theorem 2.3, this is true if $l = 1$. For $l \geq 2$, we have a short exact sequence

$$0 \to W_{+l}^{K-\text{la}}(l-1) \to W_{+l}^{K-\text{la}} \to W_{+l-1}^{K-\text{la}}.$$  

By the case $l = 1$, we know that $W_{+l}^{K-\text{la}}(l-1)$ contains linearly independent elements $e_1, \ldots, e_d$ which are all divisible by $t_{K-1}^l$. Writing $f_i = t_{K-1}^l e_i$ for $1 \leq i \leq d$, the elements $f_1, \ldots, f_d$ span a free submodule $W'_{+l}$ of $W_{+l}$ which surjects onto $W_{+l-1}$ which contains $W_{+l}$; so $W'_{+l} = W_{+l}$. It now suffices to show that the $f_i$ are locally $K$-analytic, and this follows from Lemma 3.2. This concludes the proof of (1).

As each $W_{+l}^{K-\text{la}}$ is a free $K_{\infty}[t_K]/t'_K$-module of rank $d$, we have that $W_{+l}^{K-\text{pa}} = \lim_{\leftarrow} W_{+l}^{K-\text{la}}$ is a free $K_{\infty}[t_K]$-module of rank $d$, and the chain of isomorphisms

$$B^+_\text{dr} \otimes_{K_{\infty}[t_K]} D^+_{\text{diff}, K}(V) \cong B^+_\text{dr} \otimes_{K_{\infty}[t_K]} W_{+l}^{K-\text{pa}} \cong \lim_{\leftarrow} \left( B^+_\text{dr}/t'_K \otimes_{K_{\infty}[t_K]/t'_K} W_{+l}^{K-\text{la}} \right) \cong \lim_{\leftarrow} \left( B^+_\text{dr}/t'_K \otimes_K V \right) \cong B^+_\text{dr} \otimes_K V,$$
whose composition is the natural map $B_{\text{dr}}^+ \otimes_{K[[t_K]]} D_{\text{dif},K}^+(V) \to B_{\text{dr}}^+ \otimes_K V$. This proves (2), and (3) follows immediately since $D_{\text{dif},K}(V) = \colim_i D_{\text{dif},K}(V(i))$.

Recall from $\S 2$ that the modules $D_{\text{Sen},K}$ and $D_{\text{dif},K}$ are both endowed with a canonical differential operator. We write $\Theta_{\text{Sen},K}, \nabla_{\text{dif},K}$ respectively for the operators acting on $D_{\text{Sen},K}, D_{\text{dif},K}$. The operator $\Theta_{\text{Sen},K}$ is $K_\infty$-linear, while $\nabla_{\text{dif},K}$ is a derivation over $\nabla_{K_\infty((t_K))} = t_K \frac{\partial}{\partial t_K}$.

The following result serves to complete the analogy with the usual $D_{\text{Sen}}$.

**Proposition 3.4.** The following are equivalent.
1. $\mathbb{C}_p \otimes_K V \cong \bigoplus_{i=1}^d \mathbb{C}_p(\chi_{\pi}^{n_i})$, where the $n_i \in \mathbb{Z}$.
2. $D_{\text{Sen},K}(V) \cong \bigoplus_{i=1}^d K_\infty(\chi_{\pi}^{n_i})$, where the $n_i \in \mathbb{Z}$.
3. $\Theta_{\text{Sen},K}$ is semisimple with integer eigenvalues $\{n_i\}_{i=1}^d$.

**Proof.** Suppose $v_1, \ldots, v_d$ is a basis of $\mathbb{C}_p \otimes_K V$ for which $g(v_i) = \chi_{\pi}^{n_i}(g)v_i$ for $g \in G_K$. The action of $G_K$ on each $v_i$ factors through $\Gamma_K$ and is locally $K$-analytic, so $v_i \in D_{\text{Sen},K}(V)$, which shows (1) implies (2). Next, (2) implies (3) because the action of $\Theta_{\text{Sen},K}$ on $\chi_{\pi}^{n_i}$ is given by multiplication with $n_i$. Finally, suppose that (3) holds, and let $v_1, \ldots, v_d$ be a basis of $D_{\text{Sen},K}(V)$ for which $\Theta_{\text{Sen},K}(v_i) = n_i v_i$. By integrating the action of $\Gamma_K$, we see that $g \in \Gamma_K$ acts by $g(v_i) = \eta_i(g) \chi_{\pi}^{n_i}(g)v_i$, where $\eta_i$ is a finite order character of $\Gamma_K$. Then $\mathbb{C}_p(\eta_i \chi_{\pi}^{n_i}) \cong \mathbb{C}_p(\chi_{\pi}^{n_i})$, so

$$\mathbb{C}_p \otimes_K V \cong \mathbb{C}_p \otimes_{K_\infty} D_{\text{Sen},K}(V)$$
$$\cong \bigoplus_{i=1}^d \mathbb{C}_p(\eta_i \chi_{\pi}^{n_i})$$
$$\cong \bigoplus_{i=1}^d \mathbb{C}_p(\chi_{\pi}^{n_i}).$$

If the conditions of Proposition 3.4 hold for some $n_i \in \mathbb{Z}$, the $n_i$ are called the $K$-Hodge-Tate weights of $V$.

### 3.2. The modules $D_{\text{HT},K}$ and $D_{\text{dr},K}$

Let $B_{\text{HT},K}, B_{\text{dr},K}, B_{\text{dr}}$ respectively be the rings $\mathbb{C}_p[t_K, t_K^{-1}], B_{\text{dr}}^+, B_{\text{dr}}$. We set $D_{\text{HT},K}(V) = (B_{\text{HT},K} \otimes_K V)^{G_K}$, $D_{\text{dr},K}(V) = (B_{\text{dr},K} \otimes_K V)^{G_K}$ and $D_{\text{dr},K}(V) = (B_{\text{dr}} \otimes_K V)^{G_K}$. We say that $V$ is $K$-Hodge-Tate (resp. positive $K$-de Rham, resp. $K$-de Rham) if $\dim_K D_{\text{HT},K}(V) = d$ (resp. $\dim_K D_{\text{dr},K}(V) = d$, resp. $\dim_K D_{\text{dr},K}(V) = d$).

**Lemma 3.5.** $(\mathbb{C}_p \otimes_K V)^{G_K} = (D_{\text{Sen},K}(V))^{\Gamma_K}$.  

**Proof.** By Proposition 3.3, we have

$$(\mathbb{C}_p \otimes_K V)^{G_K} = (\mathbb{C}_p \otimes D_{\text{Sen},K}(V))^{G_K} = \left(\hat{K}_\infty \otimes_{K_\infty} D_{\text{Sen},K}(V)\right)^{G_K}.$$

As $\left(\hat{K}_\infty \otimes_{K_\infty} D_{\text{Sen},K}(V)\right)^{G_K}$ is fixed by the action of $\Gamma_K$, it is also locally $K$-analytic on $\Gamma_K$, so it is contained in $\left(\hat{K}_\infty \otimes_{K_\infty} D_{\text{Sen},K}(V)\right)^{K-\text{la}}$. But according to Lemma 2.1,

$$\left(\hat{K}_\infty \otimes_{K_\infty} D_{\text{Sen},K}(V)\right)^{K-\text{la}} = D_{\text{Sen},K}(V).$$

$\square$
Proposition 3.6. 1. $D_{\text{HT}, K}(V) = \bigoplus_{t \in \mathbb{Z}} (D_{\text{Sen}, K}(V)^{G_K})_t$.

2. The natural map $B_{\text{HT}, K} \otimes_K D_{\text{HT}, K}(V) \to B_{\text{HT}, K} \otimes_K V$ is an isomorphism if $V$ is $K$-Hodge-Tate.

3. $V$ is $\mathbb{C}_p$-admissible if and only if $\Theta_{\text{Sen}, K} = 0$ on $D_{\text{Sen}, K}(V)$.

Proof. We have $D_{\text{HT}, K}(V) = \bigoplus_{t \in \mathbb{Z}} (\mathbb{C}_p t^K \otimes_K V)^{G_K}$, so (upon twisting $V$ with an appropriate power of $\chi$) the equality $D_{\text{HT}, K}(V) = \bigoplus_{t \in \mathbb{Z}} (D_{\text{Sen}, K}(V)^{G_K})_t$ reduces to the verification that $(\mathbb{C}_p \otimes_K V)^{G_K} = (D_{\text{Sen}, K}(V))^{G_K}$, which was done in the previous lemma. This proves (1). To prove (2), suppose $V$ is $K$-Hodge-Tate. Then by (1) and Proposition 3.4 we have $D_{\text{Sen}, K}(V) \cong \bigoplus_{t \in \mathbb{Z}} K_{\infty}(\chi^u)$, which gives the second isomorphism in

$$B_{\text{HT}, K} \otimes_K D_{\text{HT}, K}(V) \cong B_{\text{HT}, K} \otimes_K \left( \bigoplus_{t \in \mathbb{Z}} (D_{\text{Sen}, K}(V)^{G_K})_t \right)$$

$$\cong B_{\text{HT}, K} \otimes_{K_{\infty}} D_{\text{Sen}, K}(V)$$

$$\cong B_{\text{HT}, K} \otimes_K V.$$

Finally, (3) follows from Proposition 3.4.

By the same logic one obtains similar results for $D_{\text{dR}, K}$.

Proposition 3.7. 1. $D_{\text{dR}, K}(V) = D_{\text{dR}, K}(V)^{G_K}$.

2. The natural map $B_{\text{dR}, K} \otimes_K D_{\text{dR}, K}(V) \to B_{\text{dR}, K} \otimes_K V$ is an isomorphism if $V$ is $K$-de Rham.

3. $V$ is $K$-de Rham if and only if $\nabla_{\text{dR}, K}$ has a full set of sections on $D_{\text{dR}, K}(V)$.

3.3. The modules $D_{\text{cris}, K}$ and $D_{\text{st}, K}$. Recall that $B_{\text{max}}^+$ is a period ring similar to Fontaine’s $B_{\text{cris}}^+$. The element $t^K$ introduced in §2.3 actually lies in $B_{\text{max}}^+ \otimes_K K$. Denote by $B_{\text{max}, K}^+$, $B_{\text{st}, K}$, $B_{\text{max}, K}^+$, $B_{\text{st}, K}^+$, $B_{\text{max}, K}$, $B_{\text{st}, K}$ respectively the rings $B_{\text{max}}^+ \otimes_K K$, $B_{\text{st}}^+ \otimes_K K$, $B_{\text{max}, K}^+ \left[ \frac{1}{t^K} \right]$ and $B_{\text{st}, K}^+ \left[ \frac{1}{t^K} \right]$. These rings carry a $\varphi_q = \varphi^f$-action, and the usual monodromy operator $N$ of $B_{\text{st}}^+$ extends to $B_{\text{st}, K}$ with $B_{\text{st}, K}^{N=0} = B_{\text{max}, K}$. If $L$ is a finite extension of $K$, we set $D_{\text{cris}, K}^+(V) = D_{\text{dR}, K}^+(V)^{G_L}$, $D_{\text{st}, K}^+(V) = B_{\text{st}, K}^+ \otimes_K V)^{G_L}$.

To a filtered $(\varphi_q, N)$-module $D$ over $L_0 \otimes_{K_0} K$ one can associate two polygons. The Hodge polygon $P_H(D)$, whose slopes have lengths according to the jumps in the filtration; and the Newton polygon $P_N(D)$, whose slopes match the slopes of $\varphi_q$ with respect to the valuation $\text{val}_p$. We say $D$ is admissible if the endpoints of $P_H(D)$ and $P_N(D)$ are the same and if $P_H(D_0)$ lies below $P_N(D_0)$ for every subobject $D_0$ of $D$.

If $D$ is a filtered $(\varphi_q, N)$-module over $L_0 \otimes_{K_0} K$, then

$$I^K_{\varphi_q} (D) := \left( L_0 \otimes_{\varphi_q} K \right) \left[ \varphi^f \otimes (L_0 \otimes_{K_0} K)[\varphi_q] \right] D$$

is a filtered $(\varphi, N)$-module over $L_0 \otimes_{\varphi_q} K$. The following is proved in §3 of [KR09] under the assumption that $N = 0$, but the proof in the general case is the same. Note that in loc. cit. this statement is actually proved for $(B_{\text{max}}^+ \otimes_K V)^{G_L}$ instead of $D_{\text{cris}, K}^+(V) = \left( B_{\text{max}, K}^+ \left[ \frac{1}{t^K} \right] \otimes_K V \right)^{G_L}$, but these coincide for $K$-analytic representations.

\footnote{It is $\log_{f} u$ for the element $u$ introduced in §4.1 below.}
**Proposition 3.8.** Let $L$ be a finite extension of $K$, and suppose that $V \in \text{Rep}_K(G_L)$ is $K$-analytic. Then

$$I^K_{\Theta_p}(D_{\text{st},K}(V)) = D_{\text{st},\Theta_p}(V).$$

Furthermore, $D_{\text{st},K}(V)$ is admissible if and only if $D_{\text{st},\Theta_p}(V)$ is admissible.

We say that $V$ is $K$-potentially semistable if for some finite extension $L$ of $K$ and for some $n \in \mathbb{Z}$ we have $\text{rank}_{L_0 \otimes_{K_0} K} D_{\text{st},K}(V) = d$.

**Corollary 3.9.** Suppose that $V \in \text{Rep}_E(G_K)$ is $K$-analytic for $K \subset E$. Then the following are equivalent.

1. $V$ is de Rham.
2. $V$ is $K$-de Rham.
3. $V$ is potentially semistable.
4. $V$ is $K$-potentially semistable.

**Proof.** The equivalence between (1) and (3) is the $p$-adic monodromy theorem, which is Theorem 0.7 of [Be02]. The equivalence between (3) and (4) follows from Proposition 3.8. It remains to prove that (1) and (2) are equivalent. Indeed, we have $D_{\text{dR}}(V) \cong \oplus_{\tau \in \Sigma_K} D_{\text{dR},K}(V^\tau)$, with $V^\tau$ being the $\tau$-twist of $V$. Since $V$ is assumed $K$-analytic, the representation $V^\tau$ is $\mathbb{C}_p$-admissible for $\tau \neq \text{Id}$, so it also $K$-de Rham. This implies that $\dim_K D_{\text{dR},K}(V^\tau) = d$, so $\dim_K D_{\text{dR}}(V) = \dim_{\Theta_p} V$ if and only if $\dim_K D_{\text{dR},K}(V) = \dim_K V$, as required.

We conclude with a lemma that will be used in the proof of Theorem 6.8.

**Lemma 3.10.** Suppose that $V \in \text{Rep}_E(G_K)$ is $K$-analytic for $K \subset E$ and that $L = K$, and let $\alpha \in E^\times$. Then $I^K_{\Theta_p}(D_{\text{cris},\Theta_p}(V))^{\varphi_q = \alpha} = (K_0 \otimes_{\mathbb{Q}_p} E)D_{\text{cris},K}(V))^{\varphi_q = \alpha}$.

**Proof.** In general, if $D$ is a filtered $\varphi_q$-module over $E$, then $I^K_{\Theta_p}(D)^{\varphi_q = \alpha} = (K_0 \otimes_{\mathbb{Q}_p} E)^{\varphi_q = \alpha}$ because the $\varphi_q$ action is $K_0 \otimes_{\mathbb{Q}_p} E$-linear. The lemma now follows from Proposition 3.8.

### 4. Big period rings

If $\mathcal{F}$ is the formal $\mathcal{O}_K$-module associated to $\pi$ as in §1.2, we choose a coordinate $T$ for $\mathcal{F}$ so that for $a \in \mathcal{O}_K$ we have a power series $[a] = [a](T)$ corresponding to the action of $a$ on $\mathcal{F}$. For $n \geq 0$ we choose elements $u_n \in \mathcal{O}_{\mathcal{C}_p}$ such that $u_0 = 0$, $u_1 \neq 0$ and $[\pi](u_n) = u_{n-1}$.

### 4.1. The rings $\tilde{\mathcal{B}}^+_{\text{rig}}$ and $\tilde{\mathcal{B}}^+_{\text{log}}$

This subsection provides ramified counterparts for the constructions given in §2 of [Be02] in the case $K = K_0$. Recall the notations from §1.2 and set $\mathcal{O}_{\mathcal{C}_p} = \lim\left(\mathcal{O}_{\mathcal{C}_p}/\pi^{n \to \infty} \mathcal{O}_{\mathcal{C}_p}/\pi^{n \to \infty} \ldots \right)$. We equip $\mathcal{O}_{\mathcal{C}_p}$ with the valuation $|(\pi_n)_{n \geq 0}| = \lim_{n \to \infty}|x_n|^q_n$ where $x_n \in \mathcal{O}_{\mathcal{C}_p}$ is a lift of $\pi_n$. Denote by $\tilde{\mathcal{A}}^+_0$, $\tilde{\mathcal{A}}^+$ respectively the rings $W\left(\mathcal{O}_{\mathcal{C}_p}\right)$ and $\tilde{\mathcal{A}}^+_0 \otimes \mathcal{O}_{K_0} \mathcal{O}_K$. Then $\overline{\mathcal{u}} = (\overline{\pi_n})_{n \geq 0}$ lies in $\mathcal{O}_{\mathcal{C}_p}$, and by §8 of [Co02] there exists an element $u \in \tilde{\mathcal{A}}^+$ which lifts $\overline{\mathcal{u}}$ and which satisfies $\varphi_q(u) = [\pi](u)$ and $g(u) = [\chi_{\pi}(g)](u)$ for $g \in \Gamma_K$. 

Let \( \varpi \in \mathcal{O}_{\mathbb{C}_p} \) be any element with \( |\varpi| = p^{-p/p-1} \). Given \( r, s \in \mathbb{Z}_{\geq 0}[1/p] \) with \( r \leq s \), and given \( \Lambda \in \{ \hat{A}_0, \hat{A} \} \), we set
\[
\Lambda^{[r,s]} = \Lambda^+ \left\langle \frac{p}{[\varpi]^r}, \left[ \frac{[\varpi]^s}{p} \right] \right\rangle,
\]
the completion of \( \Lambda^+ \left\langle \frac{p}{[\varpi]^r}, \left[ \frac{[\varpi]^s}{p} \right] \right\rangle \) with respect to the \((p,[\varpi])\)-adic topology.\(^2\) We write \( \tilde{B}_0^{[r,s]} = \hat{A}_0^{[r,s]}[1/p] \) and \( \tilde{B}^{[r,s]} = \hat{A}^{[r,s]}[1/\pi] \).

**Lemma 4.1.** 1. \( \tilde{A}_0^{[r,s]} \otimes_{\mathcal{O}_K \hat{O}_K} \mathcal{O}_K = \tilde{A}^{[r,s]} \).
2. \( \tilde{B}_0^{[r,s]} \otimes_{K_0} K = \tilde{B}^{[r,s]} \).

**Proof.** (2) follows from (1). To prove (1), we write
\[
\tilde{A}^+ \left\langle \frac{p}{[\varpi]^r}, \left[ \frac{[\varpi]^s}{p} \right] \right\rangle = \bigoplus_{i=0}^{r-1} \pi^i \hat{A}_0^+ \left\langle \frac{p}{[\varpi]^r}, \left[ \frac{[\varpi]^s}{p} \right] \right\rangle
\]
as \( \hat{A}_0^+ \)-modules. Now take the \((p,[\varpi])\)-adic completion of both sides. \( \square \)

We denote by \( \tilde{B}_{\text{rig},0}^{t,r}, \tilde{B}_{\text{rig},0}^{t} \) and \( \tilde{B}_{\text{rig}}^{t} \) respectively the rings \( \cap_{r \leq s} \tilde{B}_0^{[r,s]}, \cap_{r \leq s} \tilde{B}^{[r,s]}, \cup_{r > 0} \tilde{B}_{\text{rig},0}^{t,r} \) and \( \cup_{r > 0} \tilde{B}_{\text{rig}}^{t} \). The \( \varphi \) and \( G_K \) actions on \( \hat{A}_0^+ \) (resp. the \( \varphi_q \) and \( G_K \) actions on \( \tilde{A}^+ \)) extend to \( \tilde{B}_{\text{rig},0}^{t} \) (resp. to \( \tilde{B}_{\text{rig}}^{t} \)). The following is proved in Proposition 2.23 of [Be02] in the case \( K = \mathbb{Q}_p \), but the same proof works in the general case.

**Proposition 4.2.** There exists a unique map \( \log : \hat{A}^+ \to \tilde{B}_{\text{rig}}^t[X] \) satisfying \( \log(\pi) = 0 \), \( \log([\overline{u}]) = X \), \( \log([x]) = 0 \) for \( x \in \mathbb{F}_q \) and \( \log(xy) = \log(x) + \log(y) \), such that if \( [x] - 1 \) is sufficiently close to 1, we have
\[
\log [x] = \sum_{n \geq 1} (-1)^{n-1} \left( \frac{[x] - 1}{n} \right).
\]

Moreover, if \( x \in \mathcal{O}^\times_{\mathbb{C}_p} \) then \( \log [x] \in \tilde{B}_{\text{rig}}^t \).

Write \( p^\flat \) and \( \pi^\flat \) for the elements \( \left( \frac{p}{p^{1/q}}, \ldots \right) \) and \( \left( \frac{\pi}{\pi^{1/q}}, \ldots \right) \) of \( \mathcal{O}_{\mathbb{C}_p} \). We set \( \tilde{B}_{\log,0}^t = \tilde{B}_{\text{rig},0}^{t} \left[ \log [p^\flat] \right] \) and \( \tilde{B}_{\log}^t = \tilde{B}_{\text{rig}}^{t} \left[ \log [\pi^\flat] \right] \). Since \( u/\overline{u} \equiv 1 \mod \pi \), we have \( \log (u/\overline{u}) \in \tilde{B}_{\log}^t \); on the other hand, \( \overline{u}^{q-1}/(\pi^\flat)^q \) is a unit of \( \mathcal{O}^\times_{\mathbb{C}_p} \), so \( \log (\overline{u})^{q-1}/(\pi^\flat)^q \in \tilde{B}_{\log}^t \) as well. Combining these two observations, we see that in \( \tilde{B}_{\log}^t \) we have
\[
q \log [\pi^\flat] = (q-1) \log u - (q-1) \log (u/\overline{u}) - \log (\overline{u})^{q-1}/(\pi^\flat)^q
\]
\( \equiv (q-1) \log u \mod \tilde{B}_{\log}^{t} \),
so we also have \( \tilde{B}_{\log}^t = \tilde{B}_{\text{rig}}^t \left[ \log u \right] \).

The \( \varphi \) (resp. \( \varphi_q \)) action on \( \tilde{B}_{\text{rig},0}^{t} \) (resp. on \( \tilde{B}_{\text{rig}}^{t} \)) extends to \( \tilde{B}_{\log,0}^{t} \) (resp. to \( \tilde{B}_{\log}^{t} \)) by setting \( \varphi (\log [p^\flat]) = p \log [p^\flat] \) and \( g(\log [p^\flat]) = \log [g(p^\flat)] \) (resp. \( \varphi_q (\log [\pi^\flat]) = q \log [\pi^\flat] \) and \( g(\log [\pi^\flat]) = \log [g(\pi^\flat)] \)).

\(^2\)In some references the completion is taken with respect to the \( p \)-adic topology, but this makes no difference because \( p \) divides a power of \( [\varpi] \).
Proposition 4.3. 1. \( \tilde{B}_{\text{rig},0}^t \otimes K_0 K = \tilde{B}_{\text{rig}}^t \). 2. \( \tilde{B}_{\log,0}^t \otimes K_0 K = \tilde{B}_{\log}^t \).

Proof. For \( r \leq s \) we have by Lemma 4.1 that \( \tilde{B}_0^{[r,s]} \otimes K_0 K = \tilde{B}^{[r,s]} \). As \( K \) is finite free over \( K_0 \), this implies

\[
\tilde{B}_{\text{rig},0}^t \otimes K_0 K = \left( \cap_{r \leq s} \tilde{B}_0^{[r,s]} \right) \otimes K_0 K
\]

\[
= \cap_{r \leq s} \left( \tilde{B}_0^{[r,s]} \otimes K_0 K \right)
\]

\[
= \cap_{r \leq s} \tilde{B}^{[r,s]}
\]

\[
= \tilde{B}_{\text{rig}}^t.
\]

For (2), we write \( \pi^n = pv \) with \( v \in \mathcal{O}_K^\times \). We can find \( v^\flat = \left( \pi, v^{1/q}, ... \right) \in \mathcal{O}_{K_p} \) such that \([\pi^n] = [p^s] \left[ v^n \right] \), so \( e \log [\pi^n] \equiv \log [p^s] \mod \tilde{B}_{\text{rig}}^t \), and

\[
\tilde{B}_{\log}^t = \tilde{B}_{\text{rig}}^t \left[ \log \left[ p^s \right] \right]
\]

\[
= \left( \tilde{B}_{\text{rig},0}^t \otimes K_0 K \right) \left[ \log \left[ p^s \right] \right]
\]

\[
= \tilde{B}_{\log,0}^t \otimes K_0 K.
\]

\[ \square \]

4.2. Pro K-analytic vectors. Let \( \tilde{B}_{\text{rig},K}^t \) be the Robba ring, i.e. the ring of power series \( f(T) = \sum_{k \in \mathbb{Z}} a_k T^k \) with \( a_k \in K \) and such that \( f(T) \) converges on some nonempty annulus \( r < |T| < 1 \). The ring \( \tilde{B}_{\text{rig},K}^t \) can be viewed as a subring of \( \tilde{B}_{\text{rig}}^t = \left( \tilde{B}_{\text{rig}}^t \right)^{H_K} \) by identifying \( T \) with the element \( u \) of \( \S 4.1 \). It has induced \( \varphi_q \) and \( \Gamma_K \) actions.

Recall the following result (Theorem B of [Be16]), which determines the ring of pro K-analytic vectors in \( \tilde{B}_{\text{rig},K}^t \).

**Theorem 4.4.** \( \left( \tilde{B}_{\text{rig},K}^t \right)^{K-\text{pa}} = \cup_{n \geq 0} \varphi_q^{-n} \left( \tilde{B}_{\text{rig},K}^t \right) \).

On the other hand, we can also write \( \tilde{B}_{\log, K}^t = \left( \tilde{B}_{\log}^t \right)^{H_K} \). The goal of this subsection is to obtain an analogous result for \( \tilde{B}_{\log, K}^t \).

**Proposition 4.5.** We have \( \log u \in \left( \tilde{B}_{\log, K}^t \right)^{K-\text{pa}} \).

Before we give a proof of Proposition 4.5, we record the following consequence. Let \( \tilde{B}_{\log, K}^t = \tilde{B}_{\text{rig}, K}^t \left[ \log T \right] \), thought of as a subring of \( \tilde{B}_{\text{rig}, K}^t \). The \( \varphi_q \) action on \( \log T \) is given by \( \varphi_q (\log T) = q \log T + \log \left( \left[ \pi \right] (T)/T^q \right) \), where \( \log \left( \left[ \pi \right] (T)/T^q \right) \in \tilde{B}_{\text{rig}, K}^t \).
Theorem 4.6. \( \left( \bigoplus_{i=0}^{d} \tilde{B}_{\text{rig},K}^i \cdot (\log u)^i \right)^{K}_{\text{pa}} = \bigcup_{n \geq 0} \varphi_q^n \left( B_{\log,K}^i \right). \)

Proof. Fix \( d \geq 0. \) As \( g(\log u) = \log u + \log \frac{g(u)}{u} \) for \( g \in \Gamma_K, \) the submodule \( \bigoplus_{i=0}^{d} \tilde{B}_{\text{rig},K}^i \cdot (\log u)^i \) is closed under the \( \Gamma_K \)-action. By Proposition 4.5, the elements \( 1, log u, \ldots, (\log u)^d \) from a \( \tilde{B}_{\text{rig},K}^i \)-basis of this submodule for which the action is pro-\( K \)-analytic. Combining Lemma 4.7 and Theorem 4.4, we obtain

\[
\left( \bigoplus_{i=0}^{d} \tilde{B}_{\text{rig},K}^i \cdot (\log u)^i \right)^{K}_{\text{pa}} = \bigoplus_{i=0}^{d} \left( \bigcup_{n \geq 0} \varphi_q^n \left( B_{\text{rig},K}^i \right) \right) (\log u)^i.
\]

Taking the colimit as \( d \to \infty \) shows that \( \left( \tilde{B}_{\log,K}^i \right)^{K}_{\text{pa}} = \left( \bigcup_{n \geq 0} \varphi_q^{-n} \left( B_{\text{rig},K}^i \right) \right) [\log u]. \) It remains to show that \( \bigcup_{n \geq 0} \varphi_q^{-n} \left( B_{\log,K}^i \right) \) is contained \( \left( B_{\log,K}^i \right) \) as the inclusion in the other direction is obvious. Assume the opposite and let \( f = \sum_{i=0}^{d} a_i (\log u)^i \) be an element of \( \varphi_q^{-n} \left( B_{\log,K}^i \right) \) with \( a_i \in \tilde{B}_{\text{rig},K}^i \) and \( d \) minimal such that \( f \) is not contained in \( \left( \bigcup_{n \geq 0} \varphi_q^{-n} \left( B_{\text{rig},K}^i \right) \right) \) \( \log u \). As \( \varphi_q^{-n} \cdot B_{\log,K}^i \) and \( \varphi_q(\log u) \equiv q \log u \mod B_{\text{rig},K}^i \), examining the coefficient of \( (\log u)^d \) reveals that \( \varphi_q^{-n} (a_d) \in B_{\text{rig},K}^i \), providing a contradiction. \( \square \)

We now proceed to prove Proposition 4.5 (compare with §4 of [Be16]). We do so in several steps. If \( t \geq 1 \), we denote by \( L_{A_t}(\mathcal{O}_K) \) the space functions of \( \mathcal{O}_K \) which are analytic on closed discs of radius \( |\pi|^t \). For \( a \in \mathcal{O}_K \), write \( [a](T) = \sum_{n \geq 1} c_n(a) T^n \). Each \( c_n(a) \) is a polynomial of degree at most \( n \) in \( a \), and \( c_n(\mathcal{O}_K) \subset \mathcal{O}_K \).

Lemma 4.7. \( ||c_n||_{L_{A_t}} \leq |\pi|^{-\frac{n}{q^t(q-1)}}. \)

Proof. Recall that de Shalit has constructed in [dS16] a Mahler basis \( \{ g_n(T) \}_{n \geq 0} \) such that \( g_n(T) \) is a polynomial in \( \overline{K}[T] \) of degree \( n \) and such that \( ||w_n,T g_n||_{L_{A_t}} = 1 \), where \( w_n,T \) is an integer satisfying \( w_n,T \leq \frac{n}{q^t(q-1)} \). As \( c_n \) has degree at most \( n \), we can write \( c_n = \sum_{i=0}^{n} b_{n,i}g_i \) for some \( b_{n,i} \in \mathcal{O}_K \), and so \( ||c_n||_{L_{A_t}} \leq \sum_{1 \leq i \leq n} ||g_i|| \leq |\pi|^{-\frac{n}{q^t(q-1)}}. \) \( \square \)

Recall that \( g(\log u) = \log u + \log \left( \frac{[a(\log u)]}{u} \right) \), where \( a = \chi_{\pi}(g) \). We write

\[
\log \left( \frac{[a(\log u)]}{u} \right) = \sum_{n=1}^{\infty} d_n(a) u^n.
\]

Lemma 4.8. \( ||d_n||_{L_{A_t}} \leq |\pi|^{-\frac{2n}{q^t(q-1)} + o(n)}. \)

Proof. Write \( \frac{[a(\log u)]}{u} - 1 = \sum_{n \geq 0} e_n(a) u^n \), where \( e_0(a) = a - 1 \) and \( e_n(a) = c_{n+1}(a) \) for \( n \geq 1 \). Then \( d_n \) is a sum of functions of the form

\[
\frac{(-1)^m}{m} \sum_{(k_1, \ldots, k_m) \in \mathbb{Z}_{\geq 0}^m \atop k_1 + \ldots + k_m = n} \prod_{i=1}^{m} e_{k_i}.
\]

LUBIN-TATE THEORY AND OVERCONVERGENT HILBERT MODULAR FORMS OF LOW WEIGHT 15
and it suffices to bound each such function by $|\pi|^{-\frac{2n}{q(q-1)}}$, where $o(n)$ does not depend on $m$.

Fix $(k_1, \ldots, k_m) \in \mathbb{Z}_{\geq 0}^m$ with $k_1 + \ldots + k_m = n$. Let $h$ be the number of $1 \leq i \leq m$ such that $k_i \geq 1$. Then by Lemma 4.7 we have

$$\left\| \prod_{i:1 \leq k_i} e_{k_i} \right\|_{LA_t} \leq |\pi|^{-\frac{2n}{q(q-1)}} |\pi|^{-\frac{2n}{q(q-1)}}.$$

On the other hand, $|e_0|_{LA_t} \leq |\pi|^t$, so

$$\left\| \frac{1}{m} \prod_{i:k_i=0} e_{k_i} \right\|_{LA_t} \leq \frac{1}{m} |\pi|^t |m-h| \leq p^{\text{top}(m)-t/\max\{0,m-n\}} = |\pi|^{o(n)}.$$

Combining the two inequalities we obtain the claim. \hfill \Box

Proof of Proposition 4.5. Write $r_n = p^{n-1}(p-1)$ and let $s \leq t$. It is enough to show that $\log u$ is $K$-analytic on $\Gamma_K$ as a vector of $\widetilde{\mathbb{B}}^{[r_n,r_t]} = \left( \mathbb{B}^{[r_n,r_t]} \right)^{H_K}$. Since $g(\log u) = \log u + \log \left( \frac{|a(u)|}{u} \right)$ for $a = \chi_n(g)$, we need to verify that $||d_n||_{LA_{t+1}} |u^n|_{[r_n,r_t]} \to 0$ as $n \to 0$. By the maximum principle, we have $||u||_{[r_n,r_t]} = |u||_{[r_t]} = |\pi|^{1/q-1(q-1)}$, and so by Lemma 4.7

$$\left\| d_n \right\|_{LA_{t+1}} |u^n|_{[r_n,r_t]} \leq |\pi|^n \left[ \frac{2n}{q-1(q-1)} - \frac{n}{q-1(q-1)} + o(1) \right],$$

which approaches $0$, as required. \hfill \Box

5. LUBIN-TATE $(\varphi, \Gamma_K)$-MODULES

In this section we recall how to attach a $(\varphi, \Gamma_K)$-module over $B_{rig,K}^\dagger$ to a $K$-analytic $p$-adic representation of $G_K$, and we express the invariants of §3 in terms of these $(\varphi, \Gamma_K)$-modules.

5.1. $K$-analytic $(\varphi, \Gamma_K)$-modules. Let $A_K$ be the ring of power series $f(T) = \sum_{k \in \mathbb{Z}} a_k T^k$ with $a_k \in \mathcal{O}_K$ such that $\text{val}_p(a_k) \to 0$ as $k \to -\infty$, and let $B_K = A_K[1/\pi]$. These rings have a $(\varphi, \Gamma_K)$-action given by $\varphi_T = \left[ \pi \right](T)$ and a $\Gamma_K$-action given by $g(T) = \left[ \chi_T(g) \right](T)$ for $g \in \Gamma_K$. A $(\varphi, \Gamma_K)$-module is then a finite-dimensional $B_K$ vector space $D_K$ which has commuting semilinear $\varphi$ and $\Gamma_K$ actions. We say it is étale if there exists a basis of $D_K$ for which $\text{Mat}((\varphi)) \in \text{GL}_{d}(A_K)$.

Kisin and Ren have shown in §1 of [KR09] how to associate to any $V \in \text{Rep}_E(G_K)$ an étale $(\varphi, \Gamma_K)$-module $B_K$ which we denote $D_K(V)$. Furthermore, one has the following result.

Theorem 5.1. The functor $V \mapsto D_K(V)$ induces an equivalence of categories

$$\{E\text{-representations of } G_K \} \leftrightarrow \{\text{étale } (\varphi, \Gamma_K)\text{-modules over } B_K \otimes_K E \}.$$

Now let $B_{rig,K}^\dagger$ be the subring of $B_K$ which consists of power series which converge on some nonempty annulus $r \leq |T| < 1$. We say that a $(\varphi, \Gamma_K)$-module over $B_K \otimes_K E$ is overconvergent if $D_K = D_{rig,K}^\dagger \otimes_{B_K} B_K$ where $D_{rig,K}^\dagger$ is a $(\varphi, \Gamma_K)$-module over $B_{rig,K} \otimes_K E$, and a representation $V \in \text{Rep}_E(G_K)$ is said to be overconvergent if $D_K(V)$ is. As $B_{rig,K}^\dagger$ is a subring of $B_{rig,K}^\dagger$, for such a $(\varphi, \Gamma_K)$-module we can form $D_{rig,K}^\dagger = D_K \otimes_{B_K} B_{rig,K}^\dagger$ and $D_{log,K}^\dagger = D_K \otimes_{B_K} B_{log,K}^\dagger$. 
In the case $K = \mathbb{Q}_p$, Cherbonnier and Colmez have proven in [CC98] that $D_K(V)$ is always overconvergent. Unfortunately, this is no longer true whenever $K \neq \mathbb{Q}_p$ (see Theorem 0.6 of [FX13]). However, the analogue of the Cherbonnier-Colmez theorem does hold for $K$-analytic representations, and, even better, we can characterize the $(\varphi_q, \Gamma_K)$-modules which arise in this way. More precisely, a $(\varphi_q, \Gamma_K)$-module $D_{\text{rig}, K}$ over $B_{\text{rig}, K}$ is called $K$-analytic if $D_{\text{rig}, K} = \left(D_{\text{rig}, K}^+\right)^{K-\text{pa}}$. Then one has the following result (Theorems C and D of [Be16]).

**Theorem 5.2.** 1. If $V \in \text{Rep}_E(G_K)$ is $K$-analytic, then $D_K(V)$ is overconvergent.

2. The functor $V \mapsto D_{\text{rig}, K}(V)$ gives an equivalence of categories

$$\left\{\text{analytic } E\text{-linear representations of } G_K\right\}
\longleftrightarrow
\left\{\text{étale } K\text{-analytic } (\varphi_q, \Gamma_K)\text{-modules over } B_{\text{rig}, K}^+ \otimes_K E\right\}$$

3. If $V \in \text{Rep}_E(G_K)$ is $K$-analytic, then there exists a natural $G_K$-equivariant isomorphism $\widetilde{B}_{\text{rig}}^+ \otimes_K V \cong \widetilde{B}_{\text{rig}}^+ \otimes_{B_{\text{rig}, K}}^+ D_{\text{rig}, K}^+(V)$.

All characters are overconvergent, so split 2-dimensional representation are always overconvergent. For non-split representations, Theorem 5.2 implies the following.

**Corollary 5.3.** Let $V \in \text{Rep}_E(G_K)$ be a nonsplit 2-dimensional representation. The following are equivalent.

1. $V$ is overconvergent.

2. Either $V$ is $K$-analytic up to a character twist or $V$ is an extension of the trivial representation by itself.

**Proof.** If $V(\delta)$ is $K$-analytic then it is overconvergent by Theorem 5.2. In addition, Theorem 0.3 of [FX13] shows that every extension of the trivial representation by itself is overconvergent, so (2) implies (1). In the converse direction, let $V$ be an overconvergent representation.

**Case 1:** $V$ is absolutely irreducible. Then Corollary 4.3 of [Be13] implies that $V(\delta)$ is $K$-analytic for some character $\delta$. Corollary 4.3 of [Be13] is proved there in the setting where $K$ is an unramified extension of $\mathbb{Q}_p$; it is a consequence of Theorem 4.2 of ibid. This assumption can be removed, because Theorem 4.2 of ibid is reproven in [Be16] without assuming $K$ is unramified.

**Case 2:** After extending scalars, which does not matter for the question of overconvergence, we may assume $V$ is reducible and non-split, and after performing a character twist we may further assume it is an extension of $1$ by $E(\delta)$ with $\det(V) = \delta$. If $\delta = 1$, we are done. Otherwise, by Theorem 0.4 of [FX13] a nontrivial extension of $1$ by $E(\delta)$ can only occur if $\delta$ is $K$-analytic, and since $\delta \neq 1$ this implies that $V$ is also $K$-analytic by Theorem 0.3 of [FX13].

### 5.2. The modules $D_{\ast, K}$ and the extended dictionary.

Recall that for $n \geq 0$ we set $r_n = p^{n-1}(p-1)$. For $r > 0$ we let $n(r)$ be the minimal $n$ such that $r_n \geq r$. If $I$ is a closed interval and $r_0 = \frac{r-1}{p} \in I$, then for $\overline{B}^+ \in \mathfrak{S}$ the completion map $\overline{A}^+ \to \overline{B}_{\text{dr}}^+$ extends to a map $t_0 : \overline{B}^+ \to \overline{B}_{\text{dr}}^+$. More generally if $r_n \in I$ then one has the map $t_0 \circ \varphi_q^n : \overline{B}^+ \to \overline{B}_{\text{dr}}^+$. Now let $B_{\text{rig}, K}^+ = \overline{B}_{\text{rig}, K}^+ \cap B_{\text{rig}, K}^+$, then for $n \geq n(r)$ the map
above restricts to give \( \iota_0 \circ \varphi_q^{-n} : B_{\text{rig},K}^+ \to K_n ((t_K)) \subset B_{\text{dR}} \). As \( B_{\text{rig},K}^+ = \cup_{r>0} B_{\text{rig},K}^{+,r} \), each \((\varphi_q, \Gamma_K)\)-module \( D_{\text{rig},K}^{+,r} \) over \( B_{\text{rig},K}^{+,r} \) descends to \( D_{\text{rig},K}^{+,r} \) over \( B_{\text{rig},K}^+ \) for some \( r > 0 \). Finally, let \( t_K = \log_T(T) \in B_{\text{rig},K}^+ \); it belongs to \( B_{\text{rig},K}^{+,\varphi_q} \) and \( \iota_0(t_K) \) coincides with the usual \( t_K \) of \( B_{\text{dR}}^+ \) as in \( \S 2 \) and \( \S 3 \). We set

\[
D_{\text{Sen},K} \left( D_{\text{rig},K}^{\dagger} \right) = \left( D_{\text{rig},K}^{\dagger,r} \otimes_{\theta \circ \varphi_q^{-n}} K_n ((t_K)) \right) \otimes_{K_n} K_{\infty},
\]

\[
D_{\text{dif},K} \left( D_{\text{rig},K}^{\dagger} \right) = \left( D_{\text{rig},K}^{\dagger,r} \otimes_{\theta \circ \varphi_q^{-n}} K_n ((t_K)) \right) \otimes_{K_n} K_{\infty} ((t_K)),
\]

\[
D_{\text{dR},K} \left( D_{\text{rig},K}^{\dagger} \right) = D_{\text{dif},K} \left( D_{\text{rig},K}^{\dagger} \right) \Gamma_K,
\]

\[
D_{\text{cris},K} \left( D_{\text{rig},K}^{\dagger} \right) = \left( D_{\text{rig},K}^{\dagger} [1/t_K] \right) \Gamma_K,
\]

\[
D_{\text{st},K} \left( D_{\text{rig},K}^{\dagger} \right) = \left( D_{\text{log},K}^{\dagger} [1/t_K] \right) \Gamma_K.
\]

One verifies that \( D_{\text{Sen},K} \left( D_{\text{rig},K}^{\dagger} \right) \) and \( D_{\text{dif},K} \left( D_{\text{rig},K}^{\dagger} \right) \) are independent of the choice of \( n \). The main theorem of this section is the following.

**Theorem 5.4.** Let \( V \) be \( K \)-analytic representation of \( G_K \). For \( * \in \{ \text{Sen, dif, dR, cris, st} \} \), we have a natural isomorphism

\[
D_{*,K}(V) \cong D_{*,K} \left( D_{\text{rig},K}^{\dagger}(V) \right).
\]

**Proof.** Set \( D_{\text{rig},K}^+ = D_{\text{rig},K}^{\dagger}(V) \). For \( r, n \gg 0 \), we have a natural map \( D_{\text{rig},K}^{\dagger,r} \xrightarrow{\theta \circ \varphi_q^{-n}} W \), where \( W = (\mathbb{C}_p \otimes_K V)^{H_K} \). The image of \( \theta \circ \varphi_q^{-n} \) is by definition \( D_{\text{Sen},K} \left( D_{\text{rig},K}^{\dagger} \right) \), which is a \( K_{\infty} \)-submodule of rank \( d = \dim_K V \). As \( \theta \circ \varphi_q^{-n} \) is \( \Gamma_K \) equivariant, it maps \( \text{pro} \ K \)-analytic vectors to locally \( K \)-analytic vectors, so the image lands in \( W^{K_{\infty} \text{la}} = D_{\text{Sen},K}(V) \). Comparing ranks we get the desired isomorphism for \( * = \text{Sen} \). Replacing \( \theta \circ \varphi_q^{-n} \) by \( \iota_0 \circ \varphi_q^{-n} \) we similarly get a map \( D_{\text{dif},K}^+(V) \to D_{\text{dif},K} \left( D_{\text{rig},K}^{\dagger,r} \right) \) of two \( K_{\infty} \) \((t_K)\)-modules of rank \( d \), whose reduction mod \( t_K \) is the isomorphism \( D_{\text{Sen},K}(V) \xhookrightarrow{} D_{\text{Sen},K} \left( D_{\text{rig},K}^{\dagger} \right) \). Thus by Nakayama’s lemma we have \( D_{\text{dif},K}^+(V) \cong D_{\text{dif},K} \left( D_{\text{rig},K}^{\dagger} \right) \). As \( D_{\text{cris},K} = D_{\text{st},K}^{N=0} \), it remains to prove the comparison for \( * = \text{st} \). Twisting \( V \) by an appropriate power of \( \chi_x \), we further reduce to proving that \( D_{\text{st},K}^+(V) = \left( D_{\text{log},K}^{\dagger} \right) \Gamma_K \). By Lemma 5.5 below, we have

\[
D_{\text{st},K}^+(V) = \left( \overline{B}_{\log,K}^+ \otimes_K V \right)^{G_K}
\]

\[
= \left( \overline{B}_{\log,K}^+ \otimes_{B_{\log,K}} D_{\log,K}(V) \right)^{G_K}
\]

\[
= \left( \overline{B}_{\log,K}^+ \otimes_{B_{\log,K}} D_{\log,K}(V) \right)^{\Gamma_K}.
\]
On the one hand, this implies that \( D^+_{s, K}(V) \subset (D^+_{	ext{log}, K})_{\Gamma_K} \). On the other hand, vectors which are fixed by \( \Gamma_K \) are also pro \( K \)-analytic on \( \Gamma_K \), so

\[
D^+_{s, K}(V) = \left( \tilde{B}^+_{\text{log}, K} \otimes B^+_{\text{log}, K} D^+_{\text{log}, K}(V) \right)_{\Gamma_K, K-\text{pa}}
\]

\[
= \left( \left( \tilde{B}^+_{\text{log}, K} \otimes B^+_{\text{log}, K} D^+_{\text{log}, K}(V) \right)_{K-\text{pa}} \right)_{\Gamma_K}.
\]

Since \( V \) is \( K \)-analytic, Theorem 5.2 implies that \( D^+_{\text{log}, K}(V) \) is pro \( K \)-analytic, and so by Lemma 2.1 we have

\[
\left( \tilde{B}^+_{\text{log}, K} \otimes B^+_{\text{log}, K} D^+_{\text{log}, K}(V) \right)_{K-\text{pa}} = \left( \tilde{B}^+_{\text{log}, K} \otimes B^+_{\text{log}, K} D^+_{\text{log}, K}(V) \right)_{K-\text{pa}}.
\]

Applying Theorem 4.6, we deduce

\[
D^+_{s, K}(V) \subset \left( \bigcup_{n \geq 0} \varphi^{-n}_q \left( B^+_{\text{log}, K} \otimes B^+_{\text{log}, K} D^+_{\text{log}, K}(V) \right) \right)_{\Gamma_K}.
\]

Thus \( \varphi^{-n}_q(D^+_{s, K}(V)) \subset \left( D^+_{\text{log}, K} \right)_{\Gamma_K} \) for some \( n \gg 0 \). If \( e_1, \ldots, e_l \) is a basis of \( D^+_{s, K}(V) \) then \( \varphi^{-n}_q(e_1), \ldots, \varphi^{-n}_q(e_l) \) gives another basis of \( D^+_{s, K}(V) \) which lies in \( \left( D^+_{\text{log}, K} \right)_{\Gamma_K} \). This concludes the proof.

The following lemma was used in the proof of Theorem 5.4.

**Lemma 5.5.** Let \( V \in \text{Rep}_E(G_K) \). Then

\[
D^+_{s, K}(V) = \left( \tilde{B}^+_{\text{log}} \otimes K V \right)^{G_K}.
\]

**Proof.** Given an automorphism \( \sigma : K_0 \to K_0 \), let \( V^\sigma \) be the \( \sigma \)-twist of \( V \). Then we have \( G_K \)-compatible identifications \( B^+_{s, \mathbb{Q}_p} V^\sigma \cong \oplus \sigma B^+_{s, K_0} V^\sigma \) and \( \tilde{B}^+_{\text{log, } 0} \otimes_{\mathbb{Q}_p} V^\sigma \cong \oplus \sigma \tilde{B}^+_{\text{log, } 0} \otimes_{K_0} V^\sigma \).

Now by Proposition 3.4 of [Be02] we have \( (B^+_{s, \mathbb{Q}_p} V)^{G_K} = (\tilde{B}^+_{\text{log, } 0} \otimes_{\mathbb{Q}_p} V)^{G_K} \) and hence by projecting to the \( \sigma = \text{Id} \) component

\[
D^+_{s, K}(V) = (B^+_{s, K_0} V)^{G_K} = \left( \tilde{B}^+_{\text{log, } 0} \otimes_{K_0} V \right)^{G_K}.
\]

Finally, by Proposition 4.3 we have \( \tilde{B}^+_{\text{log}} = \tilde{B}^+_{\text{log, } 0} \otimes_{K_0} K \) so \( \left( \tilde{B}^+_{\text{log, } 0} \otimes_{K_0} V \right)^{G_K} = \left( \tilde{B}^+_{\text{log}} \otimes K V \right)^{G_K} \). \( \square \)

**Remark 5.6.** The definitions given in this section for \( D_{\text{Sen}, K}, D_{\text{dif}, K}, D_{\text{dR}, K}, D_{\text{cris}, K}, D_{s, K} \) make sense for non étale \( K \)-analytic \( (\varphi_q, \Gamma_K) \)-modules. The properties of these modules which were proved in §3 carry over with no difficulty to this more general case.

6. **Lubin-Tate trianguline representations of dimension 2**

We continue with the convention that \( E \) is a finite extension of \( \mathbb{Q}_p \) which contains \( K \).
Definition 6.1. 1. A $(\varphi_q, \Gamma_K)$-module over $B_{\text{rig},K}^\dagger \otimes_K E$ is called Lubin-Tate trianguline if it can be written as a successive extension of $(\varphi_q, \Gamma_K)$-modules of rank 1.

2. An $E$-linear $K$-analytic representation $V$ is called Lubin-Tate trianguline if $D_{\text{rig},K}^\dagger(V)$ is Lubin-Tate trianguline.

In the case $K = \mathbb{Q}_p$, trianguline $(\varphi_q, \Gamma_K)$-modules of dimension 2 were first studied by Colmez in [Co08]. We shall be concerned with Lubin-Tate trianguline $(\varphi_q, \Gamma_K)$-modules of dimension 2 which were studied by Fourquaux and Xie in [FX13].

6.1. Characters of the Weil group. Recall that if $W_K$ is the Weil group of $K$, local class field theory gives a natural isomorphism $W_K^{ab} \cong K^\times$. This allows us to identify characters $\delta : K^\times \to E^\times$ with characters $W_K^{ab} \to E^\times$, the identification given by

$$\delta(\text{Frob}_p^n g) = \delta(p^n)$$

for $g \in \text{Gal}(K^{ab}/K^\text{an})$ and $n \in \mathbb{Z}$. To such characters $\delta$ we associate the $(\varphi_q, \Gamma_K)$-module $\left(B_{\text{rig},K}^\dagger \otimes_K E\right)(\delta)$. It is a $(\varphi_q, \Gamma_K)$-module of rank 1 with a basis $e_\delta$, where $\varphi_q(e_\delta) = \delta(\pi)e_\delta$ and $e_\delta = \delta(\chi_\pi(g))$ for $g \in \Gamma_K$. Note that this $(\varphi_q, \Gamma_K)$-module is étale exactly when $\delta$ is unitary; in this case, if $\delta$ is locally $K$-analytic, the module $\left(B_{\text{rig},K}^\dagger \otimes_K E\right)(\delta)$ corresponds under the equivalence of categories in §5.1 to the extension of $\delta$ to $\text{Gal}(\overline{K}/K)$. Proposition 1.9 of [FX13] shows that all $K$-analytic $(\varphi_q, \Gamma_K)$-modules of rank 1 over $B_{\text{rig},K}^\dagger \otimes_K E$ are obtained in this way. We write $\mathcal{I}_{\text{an}} = \mathcal{I}_{\text{an}}(E)$ for the set of locally $K$-analytic Weil characters. There are two characters in $\mathcal{I}_{\text{an}}$ of particular interest: the inclusion character $x : K^\times \to E^\times$ and the character $\mu_\lambda(z) = \lambda^{\text{val}_e(z)}$. To $\delta \in \mathcal{I}_{\text{an}}$ we associate to the weight $w(\delta) = \frac{\log_p \delta(u)}{\log_p u}$ where $u \in \mathcal{O}_K^\times$ is any element with $\log_p u \neq 0$, and then $w(\delta)$ does not depend on $u$. If $\delta$ is unitary and $w(\delta) \in \mathbb{Z}$ then $w(\delta)$ is the $K$-Hodge-Tate weight of the associated character of $\text{Gal}(\overline{K}/K)$ (see §3).

6.2. Extensions. Given $\delta_1, \delta_2 \in \mathcal{I}_{\text{an}}$ we consider the set of extensions

$$0 \to \left(B_{\text{rig},K}^\dagger \otimes_K E\right)(\delta_1) \to D_{\text{rig},K}^\dagger \to \left(B_{\text{rig},K}^\dagger \otimes_K E\right)(\delta_2) \to 0$$

in the category of $K$-analytic $(\varphi_q, \Gamma_K)$-modules. These extensions are classified by a finite-dimensional $E$-vector space $H_{\text{an}}^1(\delta_1 \delta_2^{-1})$, whose dimension is determined in Theorem 0.3 of [FX13] as follows.

Theorem 6.2. $\dim_E H_{\text{an}}^1(\delta_1 \delta_2^{-1}) = 2$ if $\delta_1 \delta_2^{-1} = x^{-i}$ for $i \in \mathbb{Z}_{\geq 1}$ or if $\delta_1 \delta_2^{-1} = \mu_q \cdot x^i$ for $i \in \mathbb{Z}_{\geq 0}$. Otherwise, $\dim_E H_{\text{an}}^1(\delta_1 \delta_2^{-1}) = 1$.

6.3. Spaces of Lubin-Tate trianguline $(\varphi_q, \Gamma_K)$-modules of dimension 2. There is an action of $G_m(E) = E^\times$ on $H_{\text{an}}^1(\delta_1 \delta_2^{-1})$, and extensions which lie in the same orbit of this action give rise to isomorphic $(\varphi_q, \Gamma_K)$-modules. Following §6 of [FX13] we write

$$\mathcal{I}_{\text{an}}^\#(E) = \{ s = (\delta_1, \delta_2, \mathcal{L}) : \delta_1, \delta_2 \in \mathcal{I}_{\text{an}}(E), \mathcal{L} \in H_{\text{an}}^1(\delta_1 \delta_2^{-1}) \setminus \{0\} \} / G_m(E).$$

By Theorem 6.2, each pair of characters $\delta_1, \delta_2 \in \mathcal{I}_{\text{an}}$ give rise either to a unique point $(\delta_1, \delta_2, \infty)$ of $\mathcal{I}_{\text{an}}^\#$ in the generic case or a $\mathbb{P}^1(E)$-family of points of $\mathcal{I}_{\text{an}}^\#$ in the non generic
case. To each such \( s \) we associate the corresponding \((\varphi, \Gamma_K)\)-module \( D_{\text{rig}, K}^+(s) \) which is an extension of \( \left( B_{\text{rig}, K}^+ \otimes_K E \right) (\delta_1) \) by \( \left( B_{\text{rig}, K}^+ \otimes_K E \right) (\delta_2) \).

Inside \( \mathcal{S}^\text{an} \), we consider the subset \( \mathcal{S}^\text{an}_+ \) of real interest given by these \( s \in \mathcal{S}^\text{an} \) such that
\[
\text{val}_s(\delta_1 \delta_2) = 0 \quad \text{and} \quad \text{val}_s(\delta_1(\pi)) \geq 0.
\]
For example, all étale \( K \)-analytic Lubin-Tate trianguline \((\varphi, \Gamma_K)\)-modules appear in \( \mathcal{S}^\text{an}_+ \).

For an element \( s \in \mathcal{S}^\text{an}_+ \) we associate two invariants: the slope \( u(s) = \text{val}_s(\delta_1(\pi)) \) and the weight \( w(s) = w(\delta_1 \delta_2^{-1}) = w(\delta_1) - w(\delta_2) \). We then have the following partition
\[
\mathcal{S}^\text{an}_+ = \mathcal{S}^\text{ng}_+ \mathcal{S}^\text{cris}_+ \mathcal{S}^\text{st}_+ \mathcal{S}^\text{ord}_+ \mathcal{S}^\text{ncl}_+,
\]
where
\[
\mathcal{S}^\text{ng}_+ = \{ s \in \mathcal{S}^\text{an}_+ : w(s) \notin \mathbb{Z}_{\geq 1} \},
\mathcal{S}^\text{cris}_+ = \{ s \in \mathcal{S}^\text{an}_+ : w(s) \in \mathbb{Z}_{\geq 1}, u(s) < w(s), \mathcal{L} = \infty \},
\mathcal{S}^\text{st}_+ = \{ s \in \mathcal{S}^\text{an}_+ : w(s) \in \mathbb{Z}_{\geq 1}, u(s) < w(s), \mathcal{L} \neq \infty \},
\mathcal{S}^\text{ord}_+ = \{ s \in \mathcal{S}^\text{an}_+ : w(s) \in \mathbb{Z}_{\geq 1}, u(s) = w(s) \},
\mathcal{S}^\text{ncl}_+ = \{ s \in \mathcal{S}^\text{an}_+ : w(s) \in \mathbb{Z}_{\geq 1}, u(s) > w(s) \}.
\]
(Here \( \text{ng} \) and \( \text{ncl} \) are abbreviations for “non-geometric” and “non-classical”). We also write \( \mathcal{S}^{\text{an}}_0 = \{ s \in \mathcal{S}^\text{an}_+ : u(s) = 0 \} \) and \( \mathcal{S}^{\text{st}}_+ = \mathcal{S}^{\text{cris}}_+ \cap \mathcal{S}^{\text{ncl}}_+ \). Each subset \( \mathcal{S}^{\text{st}}_+ \) above is named according to the behaviour that the \( s \in \mathcal{S}^{\text{st}}_+ \) exhibit. For example, \( s \in \mathcal{S}^\text{an}_+ \) is étale if and only if \( s \notin \mathcal{S}^\text{ncl}_+ \), and in that case if \( s \in \mathcal{S}^\text{cris}_+ \) (resp. \( s \in \mathcal{S}^{\text{st}}_+ \), \( s \in \mathcal{S}^{\text{ord}}_+ \)) then \( D_{\text{rig}, K}^+(s) \) comes from a potentially crystalline (resp. semistable but non-crystalline, potentially ordinary) \( E \)-representation up to a twist. See §6 of [FX13] for more details.

**Lemma 6.3.** Let \( D_{\text{rig}, K}^+ \) be a \( K \)-analytic \((\varphi, \Gamma_K)\)-module over \( B_{\text{rig}, K}^+ \otimes_K E \). Then
\[
\nabla \left( t_K D_{\text{dif}, K}^+ \left( D_{\text{rig}, K}^+ \right) \right) \subset t_K D_{\text{dif}, K}^+ \left( D_{\text{rig}, K}^+ \right).
\]

**Proof.** Use the identity \( \nabla(t_k x) = t_k x + t_k \nabla(x) = t_k(x + \nabla(x)) \).

The following shows that if \( s \in \mathcal{S}^{\text{an}}_+ \setminus (\mathcal{S}^{\text{ng}}_+ \cup \mathcal{S}^{\text{ncl}}_+) \) then \( D_{\text{rig}, K}^+(s) \) is comes from a de Rham \( E \)-representation up to a twist (see Corollary 3.9).

**Corollary 6.4.** Let \( s = (\delta_1, \delta_2, \mathcal{L}) \in \mathcal{S}^{\text{an}}_+ \) and suppose that \( w(\delta_1), w(\delta_2) \in \mathbb{Z} \) with \( w(\delta_1) > w(\delta_2) \). Then \( D_{\text{rig}, K}^+(s) \) is \( K \)-de Rham.

**Proof.** We may assume that \( \delta_1 = 1 \). Write \( \delta = \delta_2 \) so that and \( w(\delta) < 0 \). Then \( D_{\text{dif}, K}^+ = D_{\text{dif}, K}^+ \left( D_{\text{rig}, K}^+ \right) \) is an extension of the form
\[
0 \to K_{\infty}[[t_K]] \otimes_K E \to D_{\text{dif}, K}^+ \to D_{\text{dif}, K}^+ \left( \left( B_{\text{rig}, K}^+ \otimes_K E \right) (\delta) \right) \to 0.
\]
Take \( e_\delta \in D_{\text{dif}, K}^+ \left( \left( B_{\text{rig}, K}^+ \otimes_K E \right) (\delta) \right) = D_{\text{dif}, K}^+ \left( \left( B_{\text{rig}, K}^+ \otimes_K E \right) (\delta) \right)^{\Gamma_K=1} \) and lift it to \( D_{\text{dif}, K}^+ \).
If we take \( e = 1 \in K_{\infty}[[t_K]] \otimes_K E \) then \( e, e_\delta \) is a basis of \( D_{\text{dif}, K}^+ \), and the action of \( \nabla_{\text{dif}, K} \) on
\( \mathbf{D}_{\text{dif}, K}^+ \) in the basis \( e, e_\delta \) is given by

\[
\text{Mat}(\nabla_{\text{dif}, K}) = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix},
\]

where \( f \in K_\infty[[t_K]] \otimes_K E \). As \( w(\delta) < 0 \), we have that \( e_\delta \) is divisible by \( t_K \) so by Lemma 6.3 we have that \( f \) is divisible by \( t_K \). As \( \nabla_{\text{dif}, K} = t_K \partial_{t_K} \) on \( K_\infty[[t_K]] \) we may find an \( h \in K_\infty[[t_K]] \otimes_K E \) with \( \nabla_{\text{dif}, K}(h) = f \). Then \( e \) and \( e_\delta - he \) give a full set of sections for \( \nabla_{\text{dif}, K} \) on \( \mathbf{D}_{\text{dif}, K}^+ \). By Proposition 3.7 (which applies according to Remark 5.6) we are done.

**Remark 6.5.** The argument of Corollary 6.4 can be generalized to show that if \( \mathbf{D}_{\text{rig}, K}^+ \) is a \( K \)-analytic Lubin-Tate trianguline \((\varphi_q, \Gamma_K)\)-module which is a successive extension of \((\mathbf{B}_{\text{rig}, K} \otimes_K E)\) \((\delta_i)\) for \( i = 1, \ldots, d \) with \( w(\delta_i) \) integers satisfying \( w(\delta_1) > \ldots > w(\delta_d) \), then \( \mathbf{D}_{\text{rig}, K}^+ \) is \text{K-de Rham}.

### 6.4. Lubin-Tate triangulation of a \( p \)-adic representation of dimension 2

The following generalizes Proposition 2.4.2 of [BC08].

**Proposition 6.6.** Let \( \mathbf{D}_{\text{rig}, K}^+ \) be a \( K \)-analytic \((\varphi_q, \Gamma_K)\)-module over \( \mathbf{B}_{\text{rig}, K}^+ \otimes_K E \). Then for \( \alpha \in E^\times \) and \( i \in \mathbb{Z} \) we have

\[
\text{Fil}^i \left( \mathbf{D}_{\text{cris}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) \right) \cap \mathbf{D}_{\text{cris}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) = \{ t_K \mathbf{D}_{\text{rig}, K}^+ \cap \mathbf{D}_{\text{cris}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) \}^\alpha.
\]

**Proof.** We may reduce to the case \( i = 0 \) by twisting. Since \( \mathbf{D}_{\text{cris}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) = \left( \mathbf{D}_{\text{rig}, K}^+ \right)^{\Gamma_K} \) and \( \mathbf{D}_{\text{dif}, K}^+ = \mathbf{D}_{\text{rig}, K}^+ \cap \mathbf{D}_{\text{cris}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) \rangle \), for \( n \gg 0 \) we have

\[
\varphi_q^n \left( \mathbf{D}_{\text{cris}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) \right) = (t_0 \circ \varphi_q^{-n})^{-1} \left( \mathbf{D}_{\text{dif}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) \right).
\]

If \( x \in \mathbf{D}_{\text{cris}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) \) satisfies \( \varphi_q(x) = x \alpha \), then \( x = \alpha^{-n} \varphi_q^n(x) \), so \( (t_0 \circ \varphi_q^{-n})(x) \) lies in \( \mathbf{D}_{\text{dif}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) \) for \( n \gg 0 \), whence \( x \in \mathbf{D}_{\text{rig}, K}^+ \). Conversely, if \( x \in \mathbf{D}_{\text{rig}, K}^+ \cap \mathbf{D}_{\text{cris}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) \varphi_q^n = \alpha \)

we have \((t_0 \circ \varphi_q^{-n})(x) \in \mathbf{D}_{\text{dif}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right)\) for \( n \gg 0 \), so \( x \in \varphi_q^n \left( \mathbf{D}_{\text{cris}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) \right) \) and the relation \( x = \alpha^n \varphi_q^{-n}(x) \) implies \( x \in \mathbf{D}_{\text{cris}, K}^+ \left( \mathbf{D}_{\text{rig}, K}^+ \right) \).

Following §3 of [Ch08], we compute the triangulation of a representation of dimension 2 in terms of a crystalline period.

**Proposition 6.7.** Let \( V \) be a 2-dimensional \( E \)-linear \( K \)-analytic representation of \( G_K \). Then \( V \) is Lubin-Tate trianguline if and only if there exists a \( K \)-analytic character \( \eta : G_K \rightarrow \mathcal{O}_E^\times \) and \( \alpha \in E^\times \) such that \( \mathbf{D}_{\text{cris}, K}(V(\eta))^{\varphi_q = \alpha} \neq 0 \). Moreover, if \( i \) is the largest integer such that \( \text{Fil}^i \mathbf{D}_{\text{cris}, K}(V(\eta))^{\varphi_q = \alpha} \not\subseteq \text{Fil}^{i+1} \mathbf{D}_{\text{cris}, K}(V(\eta))^{\varphi_q = \alpha} \), then \( \mathbf{D}_{\text{rig}, K}(V) \) is an extension of \( \left( \mathbf{B}_{\text{rig}, K} \otimes_K E \right) \left( \delta_1 \right) \) by \( \left( \mathbf{B}_{\text{rig}, K} \otimes_K E \right) \left( \delta_2 \right) \) where \( \delta_1 = \eta^{-1} \mu_\alpha x^{-i} \) and \( \delta_2 = \eta \mu_\alpha^{-1} x^i \text{det}(V) \).
Proof. If $V$ is Lubin-Tate trianguline, then $\text{D}^+_{\text{rig},K}(V)$ contains a submodule of rank 1 isomorphic to $\left( B_{\text{rig},K}^+ \otimes_K E \right)(\delta)$ for some $\delta \in \mathcal{J}_{\text{an}}$. Taking $\eta : G_K \to \mathcal{O}_E^*$ defined by $\eta(g) = \delta^{-1}(\chi_\pi(g))$ we have $\text{D}^+_{\text{cris},L}(V(\eta))^{\varphi_q = \delta(\pi)} = \text{D}^+_{\text{rig},K}(V(\eta))^{\Gamma_L \varphi_q = \delta(\pi)} \neq 0$. Conversely, suppose that such an $\alpha$ and $\eta$ exist. We shall show $V$ is Lubin-Tate trianguline with the described triangulation. Twisting by a power of $\chi_\pi$, we may assume that $i = 0$ and that $\text{D}^+_{\text{cris},K}(V(\eta))^{\varphi_q = \alpha}$ contains an element $f \notin \text{Fil}^1\text{D}_{\text{cris},K}(V(\eta))^{\varphi_q = \alpha}$. By what we have proven in §5, we have

$$\text{D}^+_{\text{cris},K}(V(\eta))^{\varphi_q = \alpha} = \text{D}^+_{\text{rig},K}(V(\eta))^{\Gamma_K = 1, \varphi_q = \alpha},$$

so $f \in \text{D}^+_{\text{rig},K}(V(\eta))^{\Gamma_K = 1, \varphi_q = \alpha}$. By taking its span and twisting by $\eta^{-1}$ we get a rank 1 sub $(\varphi_q, \Gamma_K)$-module of $\text{D}^+_{\text{rig},K}(V)$. The ideal $I$ generated by the coefficients of $f$ in a basis of in $\text{D}^+_{\text{rig},K}(V(\eta))$ is stable under the actions of $\varphi_q$ and $\Gamma_K$. As $B_{\text{rig},K}^+ \otimes_K E$ is a Bézout domain and $I$ is finitely generated, it is principal, and we conclude from Lemma 1.1 of [FX13] that $I = (\mu_n)$ for $n \in \mathbb{Z}_{\geq 0}$. Proposition 6.6 shows that $n = 0$, and this means that

$$\left( B_{\text{rig},K}^+ \otimes_K E \right) \cdot f(\eta^{-1}) \cong \left( B_{\text{rig},K}^+ \otimes_K E \right)(\eta^{-1} \mu_0)$$

is a rank 1 saturated submodule of $\text{D}^+_{\text{rig},K}(V)$. We then have

$$\text{D}^+_{\text{rig},K}(V) / \left( B_{\text{rig},K}^+ \otimes_K E \right)(\eta^{-1} \mu_0) \cong \left( B_{\text{rig},K}^+ \otimes_K E \right)(\eta \mu_0 \cdot \det(V))$$

by the classification of $(\varphi_q, \Gamma_K)$-modules of rank 1. 

Finally, we conclude with the proof of Theorem B from the introduction. To do so, we first recall what are cyclotomic trianguline representations. Let $K_{\text{cyc}}^\infty = K(\mu_{p^\infty})$ be the cyclotomic extension of $K$ and let $K'_0$ be the maximal unramified extension of $K_0$ in $K_{\text{cyc}}^\infty$. The ring $B_{\text{rig},K}^{\text{cyc}}$ is the ring of power series $\sum_{n \in \mathbb{Z}} a_n T^n$ with $a_n \in K'_0$ and such that $f(T)$ converges on some nonempty annulus $r < |T| < 1$. The ring is endowed with a Frobenius-semilinear $\varphi$ action and a semilinear $\Gamma_{\text{cyc}}^K$-action. If $K = K_0$ then $\varphi(T) = (1 + T)^p - 1$ and $\gamma(T) = (1 + T)^{\gamma_{\text{cyc}}(\gamma)} - 1$, but in general the action has to do with the theory of lifting the field of norms and is more complicated.

We can then define a notion of a $(\varphi, \Gamma_{\text{cyc}}^K)$-module over $B_{\text{rig},K}^{\text{cyc}} \otimes_{\mathbb{Q}_p} E$ analogous to the notion of a $(\varphi, \Gamma_K)$-module over $B_{\text{rig},K}^+ \otimes_{\mathbb{Q}_p} E$. If $V$ is an $E$-linear representation of $G_K$, one can associate to $V$ a $(\varphi, \Gamma_{\text{cyc}}^K)$-module $\text{D}^{\text{cyc}}_{\text{rig},K}(V)$ over $B_{\text{rig},K}^{\text{cyc}} \otimes_{\mathbb{Q}_p} E$. Now let $\delta : K^\times \to E^\times$ a continuous character; we can define a $(\varphi, \Gamma_{\text{cyc}}^K)$-module $\left( B_{\text{rig},K}^{\text{cyc}} \otimes_{\mathbb{Q}_p} E \right) \left( \delta \right)$ in the following way. If $\delta$ is unitary, then it corresponds to a character $\delta : G_K \to E^\times$, and we set

$$\left( B_{\text{rig},K}^{\text{cyc}} \otimes_{\mathbb{Q}_p} E \right) \left( \delta \right) = \text{D}^{\text{cyc}}_{\text{rig},K}(V)(\delta(\Delta)).$$

If $\delta |_{\mathcal{O}_K^\times} = 1$, set

$$\left( B_{\text{rig},K}^{\text{cyc}} \otimes_{\mathbb{Q}_p} E \right) \left( \delta \right) = \left( B_{\text{rig},K}^{\text{cyc}} \otimes_{\mathbb{Q}_p} E \right)[\varphi] \otimes \left( B_{\text{rig},K}^{\text{cyc}} \otimes_{\mathbb{Q}_p} E \right)[\varphi] E e_{\delta},$$

where $\varphi_\delta(e_\delta) = \delta(\pi)e_\delta$. For general $\delta$, write $\delta = \delta_1 \delta_2$ where $\delta_1$ is unitary and $\delta_2 |_{\mathcal{O}_K^\times}$ and set

$$\left( B_{\text{rig},K}^{\text{cyc}} \otimes_{\mathbb{Q}_p} E \right) \left( \delta \right) = \left( B_{\text{rig},K}^{\text{cyc}} \otimes_{\mathbb{Q}_p} E \right) \left( \delta_1 \right) \otimes \left( B_{\text{rig},K}^{\text{cyc}} \otimes_{\mathbb{Q}_p} E \right) \left( \delta_2 \right).$$
An $E$-linear representation $V$ of $G_K$ is said to be cyclotomic trianguline if $D^\dagger_{\text{rig,K}}(V)$ is a successive extension $(\varphi, \Gamma^\cyc_K)$-modules of the form $\left(B^\dagger_{\text{rig,K}} \otimes_{\mathbb{Q}_p} E\right)(\delta)$. This is the same notion of triangulinity which appears in [Na09, KPX14, Li12], but we give it a different name here to distinguish it from Lubin-Tate triangulinity.

**Theorem 6.8.** Let $V$ be a 2-dimensional $E$-linear $K$-analytic representation of $G_K$. The following are equivalent.

1. $V$ is cyclotomic trianguline.
2. There exists a $K$-analytic character $\eta : \mathcal{O}_K^\times \to E^\times$ and $\alpha \in E^\times$ such that $D_{\text{cris},\mathbb{Q}_p}(V(\eta))^{\varphi_\eta = \alpha}$ is nonzero.
3. There exists a $K$-analytic character $\eta : \mathcal{O}_K^\times \to E^\times$ and $\alpha \in E^\times$ such that $D_{\text{cris},K}(V(\eta))^{\varphi_\eta = \alpha}$ is nonzero.
4. $V$ is Lubin-Tate trianguline.

**Proof.** The equivalence between 3 and 4 was proven in Proposition 6.7, while the equivalence between 2 and 3 follows from Lemma 3.10. It remains to prove the equivalence of 1 and 2. This equivalence seems to be well known but due to a lack of suitable reference when $K \neq \mathbb{Q}_p$ we give a proof here.

**Proof.** If $V$ is cyclotomic trianguline, then $D^\dagger_{\text{rig,K}}(V)$ can be written as an extension

$$0 \to \left(B^\dagger_{\text{rig,K}} \otimes_{\mathbb{Q}_p} E\right)(\delta_1) \to D^\dagger_{\text{rig,K}}(V) \to \left(B^\dagger_{\text{rig,K}} \otimes_{\mathbb{Q}_p} E\right)(\delta_2) \to 0.$$

Since $V$ is $K$-analytic, $\delta_1$ is also $K$-analytic. Twisting by $\delta_1|_{\mathcal{O}_K^\times}^{-1}$, we may assume $\delta_1|_{\mathcal{O}_K^\times} = 1$. It then follows from [KPX14, Example 6.2.6] that

$$D_{\text{cris}}\left(\left(B^\dagger_{\text{rig,K}} \otimes_{\mathbb{Q}_p} E\right)(\delta_1)\right) = I^K_{\mathbb{Q}_p}(Ee_{\delta_1}),$$

where $\varphi_\eta(e_{\delta_1}) = \delta_1(\pi)e_{\delta_1}$. It follows that $D_{\text{cris},\mathbb{Q}_p}(V)^{\varphi_\eta = \delta_1(\pi)} \neq 0$. \hfill $\square$

Conversely, suppose that 2 holds. By replacing $V$ with a $K$-analytic twist, we may assume that $D^\dagger_{\text{cris},\mathbb{Q}_p}(V)^{\varphi_\eta = \alpha} = D_{\text{cris},\mathbb{Q}_p}(V)^{\varphi_\eta = \alpha} \neq 0$. It follows from Berger’s dictionary that

$$D^\dagger_{\text{rig,K}}(V)^{\Gamma^\cyc_K,\varphi_\eta = \alpha} = D^\dagger_{\text{cris},\mathbb{Q}_p}(V)^{\varphi_\eta \neq 0},$$

so that $D^\dagger_{\text{rig,K}}(V)^{\Gamma^\cyc_K,\varphi_\eta = \alpha}$ contains a $(\varphi_\eta, \Gamma^\cyc_K)$ invariant $E$-line, and hence $\left(B^\dagger_{\text{rig,K}} \otimes_{\mathbb{Q}_p} E\right)(\delta)$ where $\delta|_{\mathcal{O}_K^\times} = 1$ and $\delta(\pi_K) = \alpha$. This sub $B^\dagger_{\text{rig,K}} \otimes_{\mathbb{Q}_p} E$-module may not be saturated, but it follows from [KPX14, Corollary 6.2.9] that $D^\dagger_{\text{rig,K}}(V)$ contains a saturated module of the form $\left(B^\dagger_{\text{rig,K}} \otimes_{\mathbb{Q}_p} E\right)(\delta')$. In particular, $D^\dagger_{\text{rig,K}}(V)$ is an extension of two rank 1 $(\varphi_\eta, \Gamma^\cyc_K)$-modules, so $V$ is cyclotomic trianguline. \hfill $\square$

7. **Overconvergent Hilbert modular forms**

7.1. **Overconvergent Hilbert eigenforms.** We briefly recall what we need about the cuspidal Hilbert eigenvariety of Andreatta, Iovita and Pilloni (see [AIP16]).
Let $F$ be a totally real number field, $\Sigma$ the set of embeddings of $F$ in $\overline{\mathbb{Q}}$ and $N \in \mathbb{Z}_{\geq 4}$. A choice of an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ determines a decomposition $\Sigma = \Pi_{v|p} \Sigma_{F_v}$ where each $v$ is a place of $F$ lying over $p$. Let $L$ be a finite extension of $\mathbb{Q}_p$ which contains $F^{\text{Gal}}$. The weight space for the algebraic group $\mathrm{GL}_2$ is $W = \text{Spf} \left( \mathcal{O}_L \left[ \left( \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p \right)^{\times} \times \mathbb{Z}_p \right] \right)^{\text{rig}}$. If $f$ is a classical Hilbert eigenform on $F$ of tame level $N$, its weight is a tuple $\text{wt}(f) = \left( \left(k_\tau\right)_{\tau \in \Sigma}, w \right) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ satisfying $k_\tau \equiv w \mod 2$ for each $\tau \in \Sigma$. It is then identified with the point in $W$ corresponding to the character $(z_1, z_2) \mapsto \left( \prod_{\tau \in \Sigma} \tau(z_1)^{k_\tau}\right) z_2^w$. The cuspidal Hilbert eigenvariety of tame level $N$ is a certain rigid analytic space $\mathcal{E}$ which gives a $p$-adic interpolation of classical Hilbert eigenforms. More precisely, it is a rigid analytic space together with a weight map $\text{wt} : \mathcal{E} \to W$ whose points parametrize overconvergent Hilbert modular forms of finite slope together with a choice of Hecke eigenvalues at places $v|p$. We summarize its properties below (see §5 of [AIP16]).

**Theorem 7.1.** 1. The map $\text{wt} : \mathcal{E} \to W$ is, locally on $\mathcal{E}$ and $W$, finite and surjective.

2. For each $\kappa \in W(\mathbb{C}_p)$, the fiber $\text{wt}^{-1}(\kappa)$ is in bijection with finite slope Hecke eigenvalues appearing in the space of overconvergent cusp forms of weight $\kappa$, level $N$ and coefficients in $\mathbb{C}_p$.

3. There exists a universal Hecke character $\lambda : \mathcal{H}^{NP} \otimes \mathcal{U}_p \to \mathcal{O}_\mathcal{E}$. Here, $\mathcal{H}^{NP}$ is the abstract Hecke algebra away from $Np$, and $\mathcal{U}_p$ is the $\mathbb{Q}_p$-algebra generated by the $U_v$-operators for $v|p$.

4. There is a universal pseudo-character $T : \text{Gal}(\overline{F}/F) \to \mathcal{O}_\mathcal{E}$ which is unramified for $1 \nmid Np$ such that $T(Frob_1) = \lambda(T_1)$ for the arithmetic Frobenius $Frob_1$.

5. For each $\kappa \in W$ there exists a semisimple Galois representation $\rho \kappa : \text{Gal}(\overline{F}/F) \to \text{GL}_2(\overline{K}(x))$ which is unramified for $1 \nmid Np$ and which is characterized by $T(\rho \kappa) = T_\kappa$ and $\det(\rho \kappa) = N\text{m}_F/\mathbb{Q}(1) \lambda_\kappa(S_1)$.

6. The generalized Hodge-Tate weights of $\rho \kappa |_{G_{F_v}}$ are $\left\{ \frac{w-k_\tau}{2}, \frac{w+k_\tau-2}{2} \right\}_{\tau \in \Sigma_{F_v}}$.

We fix a place $v|p$ in $F$ and place ourselves in the setting of §1.2 with $K = F_v$, $\pi = \pi_v$ a uniformizer of $F_v$, etc. We extend scalars if necessary so that $\rho \kappa |_{G_{F_v}}$ is $F_v^{\text{Gal}}$-linear.

**Proposition 7.2.** For $x \in \mathcal{E}$, we have

$$D^+_{\text{cris}, F_v} \left( \rho \kappa |_{G_{F_v}} \left( \prod_{\tau \in \Sigma_{F_v}} \left( \tau \circ \chi_{\kappa_v} \right)^{\frac{w-k_\tau}{2}} \right) \right) \neq 0.$$

**Proof.** For classical Hilbert modular forms of cohomological weights this is known by Saito’s local-global compatibility results in [Sa09]. The regular classical points are Zariski dense in $\mathcal{E}$ by the classicality criterion in [Bi16], so the claim follows from the global triangulation results in Theorem 6.3.13 of [KPx14] or in Theorem 4.4.2 of [Li12].

**7.2. Lubin-Tate triangulation.** Let $x \in \mathcal{E}$ and consider $\rho \kappa |_{G_{F_v}}$ as an $E$-linear representation for some finite extension $\mathbb{Q}_p \subset E$ which contains $F_v^{\text{Gal}}$ and $\overline{K}(x)$. In this section we shall assume $\rho \kappa |_{G_{F_v}}$ is nonsplit. The split case is less interesting and can be easily dealt with. By Corollary 5.3 we get the following.

**Proposition 7.3.** $\rho \kappa |_{G_{F_v}}$ is overconvergent if and only if it is $F_v$-analytic up to a twist.
Let us assume then that \( \rho_s|_{G_{F_v}} \) is \( s \) \( F_v \)-analytic up to a twist, so that that the weights at \( \Sigma_{F_v} \) are \((k, 1, ..., 1)\) where \( k = k_d \). Let \( a_v \) be eigenvalue of \( U_v \) for the corresponding Hecke operator of \( v \). Then \( \alpha_v = \pi e^{-\frac{k-1}{2} \left( N_{F_v}/Q_p(\pi) \right)} \) \((1-w)/2 \) \( a_v \) interpolates to a function on \( E \) (see Remark 4.7 of [AIP16]). Upon writing

\[
V = \rho_s^v|_{G_{F_v}} \left( \chi_{\pi e^{-\frac{k}{2} \left( N_{F_v}/Q_p(\pi) \circ \chi_{\pi} \right)}} \right),
\]

Proposition 7.2 becomes the statement \( D^+_{\text{cris},F_v}(V)^{\varphi_v = \alpha_v} \neq 0 \). The representations \( \rho_s|_{G_{F_v}} \) and \( V \) differ by a dual and a character twist, so according to Corollary 5.3 their overconvergence and Lubin-Tate triangulinity are equivalent. However, \( V \) is \( F_v \)-analytic with \( F_v \)-Hodge-Tate weights 0 and \( k-1 \) which makes it nicer to work with.

The following is a generalization of Proposition 5.2 of [Ch08].

**Theorem 7.4.** The representation \( V \) is Lubin-Tate trianguline. We have \( D^+_{\text{rig},F_v}(s) = D^+_{\text{rig},F_v}(s) \) for \( s = (\delta_1, \delta_1^{-1} \det(V), \mathcal{L}) \in \mathcal{S}_+^{\text{an}} \), where

1. If \( k \notin \mathbb{Z}_{\geq 1} \) then \( \delta_1 = \mu_{\alpha_v}, \mathcal{L} = \infty \) and \( s \in \mathcal{S}_+^{\text{rig}} \).
2. If \( k \in \mathbb{Z}_{\geq 1} \) and \( \text{val}_{\pi_v}(\alpha_v) < k-1 \) then \( \delta_1 = \mu_{\alpha_v} \) and either
   a. \( \mathcal{L} = \infty \), in which case \( s \in \mathcal{S}_+^{\text{cris}} \).
   b. \( \mathcal{L} \neq \infty \), in which case \( s \in \mathcal{S}_+^{\text{rig}} \). This is only possible if \( 2 \text{val}_{\pi_v}(\alpha_v) + [F_v : Q_p] = k-1 \).
3. If \( k \in \mathbb{Z}_{\geq 1} \) and \( \text{val}_{\pi_v}(\alpha_v) = k-1 \), then \( \delta_1 = \mu_{\alpha_v}, \mathcal{L} = \infty \) and \( s \in \mathcal{S}_+^{\text{ord}} \).
4. If \( k \in \mathbb{Z}_{\geq 1} \) and \( \text{val}_{\pi_v}(\alpha_v) > k-1 \) then \( \delta_1 = x^{1-k} \mu_{\alpha_v}, \mathcal{L} = \infty \) and \( s \in \mathcal{S}_+^{\text{rig}} \).

**Proof.** By Proposition 6.7, we know \( V \) is Lubin-Tate trianguline and a triangulation is determined by the by largest \( i \in \mathbb{Z} \) with \( \text{Fil}^i D_{\text{cris},F_v}(V)^{\varphi_v = \alpha_v} \notin \text{Fil}^{i+1} D_{\text{cris},F_v}(V)^{\varphi_v = \alpha_v} \). It remains to determine \( i \) in each case; it is a nonnegative \( F_v \)-Hodge-Tate weight of \( V \). If \( k \notin \mathbb{Z}_{\geq 1} \) then \( i = 0 \), so (1) is settled and we may assume \( k \in \mathbb{Z}_{\geq 1} \).

Assume that \( \text{val}_{\pi_v}(\alpha_v) < k-1 \) and suppose by contradiction that \( i = k-1 \). Then \( D^+_{\text{rig},F_v}(V) \) has \( \left( B^+_{\text{rig},F_v} \otimes_{F_v} E \right) \left( x^{1-k} \mu_{\alpha_v} \right) \) as a subobject, and the latter has slope \( \text{val}_{\pi_v}(\alpha_v) - (k-1) < 0 \) which contradicts Kedlaya’s slope filtration theorem. Thus \( i = 0 \). For the equality in part (b) of (2), observe that \( \mathcal{L} \neq \infty \) can only occur if \( \dim E H^i_{\text{an}}(\delta_1 \delta_2^{-1}) > 1 \), which by Theorem 6.2 implies \( \delta_1 \delta_2^{-1} = \mu_{q^{-1}} x^{k-1} \). This proves (2).

For (3), suppose by contradiction that \( i = k-1 \). Then \( \delta_1 = x^{1-k} \mu_{\alpha_v} \) and \( s \in \mathcal{S}_+^{\text{cris}} \mathcal{S}_+^{\text{rig}} \), so by Corollary 6.4 we have that \( V \) is de Rham. A similar argument to Lemma 6.7 of [Ki03] shows that \( V \) must be split, contradicting our assumption that \( \rho_s|_{G_{F_v}} \) is nonsplit.

Finally, suppose that \( \text{val}_{\pi_v}(\alpha_v) > k-1 \) and suppose by contradiction that \( i = 0 \). Then \( D^+_{\text{rig},F_v}(V) \) is an extension of \( \left( B^+_{\text{rig},F_v} \otimes_{F_v} E \right) \left( \delta_1 \right) \) by \( \left( B^+_{\text{rig},F_v} \otimes_{F_v} E \right) \left( \delta_2 \right) \) with \( w(\delta_1) = 0 \) and \( w(\delta_2) = 1 - k \). This implies by Corollary 6.4 that \( V \) is \( F_v \)-de Rham, and hence also \( F_v \)-potentially semistable by Corollary 3.9. But this contradicts admissibility because \( \text{val}_{\pi_v}(\alpha_v) > k-1 \).

**Remark 7.5.**
(1) If $k, w \in \mathbb{Z}$ then $\text{val}_{\pi_v}(\alpha_v) = \frac{k-1}{2} + \frac{w-1}{2} [F_v : \mathbb{Q}_p] + \text{val}_{\pi_v}(a_v)$. The small slope condition $0 \leq \text{val}_{\pi_v}(\alpha_v) \leq k - 1$ can then be rewritten as
\[
\frac{1-k}{2} + \frac{w-1}{2} [F_v : \mathbb{Q}_p] \leq \text{val}_{\pi_v}(a_v) \leq \frac{k-1}{2} + \frac{w-1}{2} [F_v : \mathbb{Q}_p].
\]
(2) The parameter $\mathcal{L} \neq 0$ appearing in the case 2(b) is described in the work of Ding (Corollary 2.3 of [Di17]) in the following way. Upon considering these points of $\mathcal{E}$ with weights $(\kappa, 1, \ldots, 1)$ in a small affinoid neighborhood of $x$, one has
\[
\mathcal{L}(x) = -2 \frac{d \log \alpha_v}{d\kappa} |_{\kappa = k}.
\]
(3) When $F = \mathbb{Q}$ and $k \geq 2$, Coleman’s classicality theorem ([Co97]) says that $f$ is classical if and only if $\text{val}_v(a_p) < k - 1$ or $\text{val}_v(a_p) = k - 1$ and $f$ is not in the image of $\Theta^{k-1}$, where $\Theta$ is the operator which acts on $q$-expansions by $q^{\frac{d}{\text{val}_v}}$. Analogously, we can give a prediction in general when $p$ is an inert prime in $F$. We expect that an $F_p$-analytic form $f$ is classical if and only if $\text{val}_v(a_p) < k - 1$ or $\text{val}_v(a_p) = k - 1$ and $f$ is not in the image of $\Theta_{\text{Id}}^{k-1}$. Here $\Theta_{\text{Id}}$ is the Theta operator in the direction of the identity embedding, as constructed in §15 of [AG05]. If such a classicality statement were known, one could argue as in §6 of [Ki03] and deduce the Fontaine-Mazur conjecture for the representations attached to $F_p$-analytic finite slope Hilbert eigenforms.

(4) If we allow $\rho_f|_{G_{F_v}}$ to be split, it is also possible that $k \in \mathbb{Z}_{\geq 1}$, $\text{val}_{\pi_v}(\alpha_v) = k - 1$ and $\text{Fil}^{k-1}D_{\text{cris},(F_v)(V)^{\alpha_v = \alpha_v}} = 0$. Our expectation is that if $f$ itself is not classical then $f = \Theta_{\text{Id}}^{k-1}g$ for some eigenform $g$, so that this is the only case where $\rho_f|_{G_{F_v}}$ can be de Rham without $f$ itself being classical. In the case of $F = \mathbb{Q}$, this is known by §6 of [Ki03].

7.3. Example: the eigenform of Moy and Spencer. In this section we shall test our results for a classical Hilbert eigenform. It is not too easy to find explicit classical Hilbert eigenforms for which Theorem 7.4 gives any new information beyond that which already exists in the literature. The case where $v$ splits in $F$ is well understood, and for CM Hilbert eigenforms of $F$ the local representation at $F_v$ splits so Theorem 7.4 is rather trivial. That’s why we shall consider in this subsection the non-CM Hilbert eigenform of partial weight 1 found by Moy and Spencer in [MS15]. To the best of the author’s knowledge, it is the only example in the literature of a non-CM classical Hilbert eigenform of partial weight 1.

Recall that if $f$ is a classical Hilbert eigenform of level $\Gamma_1(N)$ and nebentypus $\varepsilon$, the Hecke polynomial $P_v(X)$ at a place $v$ with $a_v \neq 0$ is given by
\[
P_v(X) = \begin{cases} X - a_v & \text{if } v \mid N \\ X^2 - c(v, f)X + \varepsilon(v)N_{F_v/\mathbb{Q}}^{w-1} & \text{if } v \nmid N \end{cases},
\]
where $c(v, f)$ is the $T_v$-eigenvalue. When $v \nmid N$, raising the level of $f$ gives two eigenforms $f_1, f_2$ whose attached $p$-adic representations $\rho_f$ coincide and such that $\{a_v(f_1), a_v(f_2)\}$ are the two roots of $P_v(X)$. Using Theorem 7.4, this gives rise to two different triangulations of $V = \rho_f^\vee|_{G_{F_v}} \left( \frac{1}{\chi_{\pi_v}} (N_{F_v/\mathbb{Q}} \chi_{\pi_v})^{w-1} \right)$. Whenever local-global compatibility holds, the Hecke
polynomial is equal to the characteristic polynomial of the action of $\varphi_q$ on $D^+_{\text{cris}} \left( \rho_f|_{G_{F_v}} \right)$. Thus the valuation of $c(v, f)$ determines the valuations of the eigenvalues of $\varphi_q$ by the method of the Newton polygon. This observation is used in the computations below.

Next recall that the main theorem of [MS15] finds for $F = \mathbb{Q}(\sqrt{5})$ a non CM cuspidal Hilbert eigenform $f$ of weights $(t_1, k_2, w) = (5, 1, 5)$, level $\Gamma_1(14)$, nebentypus $\varepsilon$ with conductor $7(\infty_1)(\infty_2)$. For the following examples, we let $p$ be a prime in the range $[2, 11]$, $v$ a place of $F$ lying over that prime and $\rho_f$ the associated $p$-adic Galois representation of $f$. We set

$$V = \rho_f^\vee|_{G_{F_v}} \left( \chi_{\pi_v}^{-2} (N_{F_v/\mathbb{Q}_p} \circ \chi_{\pi_v})^2 \right),$$

which differs from $\rho_f|_{G_{F_v}}$ only by a dual and a crystalline twist. We shall examine the behaviour of $V$ for different $v$. When $v \neq (2)$ local-global compatibility holds by Remark 1.5 of [Ne15], while in $v = (2)$ we shall assume it holds, though it seems to be still conjectural in this case. Local-global compatibility implies that $\rho_f|_{G_{F_v}}$ is de-Rham, and since its Hodge-Tate weights at each nontrivial embedding of $F_v$ are $\{0, 0\}$, it is also $F_v$-analytic. Given an eigenvalue $a_v$ of $U_v$, Theorem 7.4 produces a point $s \in \mathcal{S}^\text{an}$. The table in §3 of [MS15] computes the values of $a_v$ for such $v$. It has to lie in the range given by Remark 7.5(1).

**Examples.**

1. The place $v = (2)$ lies over the inert prime $p = 2$ and the valuation bound is $\text{val}_v(a_v) \in [2, 6]$. Since the character has conductor prime to $2$ and the level at $2$ is $\Gamma_0(2)$, the local component $\pi_2(f)$ is Steinberg (up to an unramified quadratic twist). A suitable local-global compatibility theorem predicts that $V$ is semistable noncrystalline and $s \in \mathcal{S}^\text{semistable}_+$. In particular, the condition of case 2(b) of Theorem 7.4 predicts that $\text{val}_2(a_2) = 3$, which is confirmed by §3 of [MS15].

2. The place $v = (3)$ lies over the inert prime $p = 3$ and the valuation bound is $\text{val}_v(a_v) \in [2, 6]$. The place $v$ is coprime to the level, so the local component $\pi_3(f)$ is unramified principal series. By local-global compatibility, $V$ is crystalline. By §3 of [MS15], we have $\text{val}_3(c(3, f)) = 2$, so that the two $U_v$-eigenvalues have valuations 2 and 4. Then $V$ has two triangulations, giving rise to $s_1 \in \mathcal{S}^\text{cryst}_0$ and $s_2 \in \mathcal{S}^\text{crys}_+$. This is similar to the case of $v = (3)$.

3. The place $v = (\sqrt{5})$ lies over the ramified prime $p = 5$ and the valuation bound is $\text{val}_v(a_v) \in [2, 6]$. By §3 of [MS15], we have $\text{val}_v(c(v, f)) = 2$, and the triangulations in this case behave similar to the case of $v = (3)$.

4. The place $v = (7)$ lies over the inert prime $p = 7$ and the valuation bound is $\text{val}_v(a_v) \in [2, 6]$. The character has conductor divisible by $7$ and the level at $7$ is $\Gamma_0(7)$, so the local component $\pi_7(f)$ is ramified principal series. After an abelian extension it becomes unramified principal series, so by local-global compatibility $V$ is crystalline. By §3 of [MS15], we have $\text{val}_7(a_7) = 3$, so $V$ gives rise to $s \in \mathcal{S}^\text{cryst}_0 \setminus \mathcal{S}^\text{crys}_0$.

5. The place $v = \left( \frac{7+\sqrt{5}}{2} \right)$ lies over the split prime $p = 11$ and the valuation bound is $\text{val}_v(a_v) \in [0, 4]$. By §3 of [MS15], we have $\text{val}_v(c(v, f)) = 0$, and the triangulations in this case behave similar to the case of $v = (3)$ and $v = (\sqrt{5})$. 


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