INTEGRATION OF D-DIMENSIONAL 2-FACTOR SPACES
COSMOLOGICAL MODELS BY REDUCING TO THE GENERALIZED
EMDEN-FOWLER EQUATION

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Abstract

The $D$-dimensional cosmological model on the manifold $M = R \times M_1 \times M_2$ describing the evolution of 2
Einsteinian factor spaces, $M_1$ and $M_2$, in the presence of multicomponent perfect fluid source is considered.
The barotropic equation of state for mass-energy densities and the pressures of the components is assumed
in each space. When the number of the non Ricci-flat factor spaces and the number of the perfect fluid
components are both equal to 2, the Einstein equations for the model are reduced to the generalized Emden-
Fowler (second-order ordinary differential) equation, which has been recently investigated by Zaitsev and
Polyanin within discrete-group analysis. Using the integrable classes of this equation one generates the
integrable cosmological models. The corresponding metrics are presented. The method is demonstrated for
the special model with Ricci-flat spaces $M_1, M_2$ and the 2-component perfect fluid source.

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1 Introduction

Following the purpose to study the early universe we develop the multidimensional generalization
\cite{8,13,16,21,22,23,28} of the standard Friedman-Robertson-Walker world model. If the extra dimensions of
the space-time manifold really exist, the unique conceivable site, where they might become dynamically
important, seems to be possible. This is some early stage of the evolution. Usually within multidimensional
cosmology (see, for instance, \cite{1,2,5,13,15,21,28,31,36} and references therein) it is assumed the
occurrence of the topological partition for the multidimensional space-time on the external 3-dimensional
space and additional so called internal space (or spaces) due to the quantum processes at the beginning
of this stage. In correspondence with such partition the space-time acquires the topology
$M = R \times M_1 \times \ldots \times M_n$, where $R$ is the time axis, one part of the manifolds
$M_1, \ldots, M_n$ is interpreted as 3-dimensional external space and the other part stands for internal spaces. Usually the internal spaces are compact, however the models with noncompact internal spaces are also discussed \cite{14,20,29,30}. The subsequent
evolution of the multidimensional Universe is considered as classical admitting the description by means of
the multidimensional Einstein equations. Achieving the integrability of these equations is the main goal of
our investigation. As the present world seems to be 4-dimensional, there is the assumption that the internal
space(s) had contracted to extremely small sizes, which are inaccessible for experiment. This contraction
accompanied by the expansion of the external space is described by some models (the first model of such
type has been found in \cite{6}) within multidimensional cosmology and is called dynamical compactification.

We consider a mixture of several perfect fluid components as a source for the multidimensional Einstein
equations. Such multicomponent systems are usually employed in 4-dimensional cosmology and are quite
adequate type of matter for description some early epochs in the history of the universe \cite{4}.

The paper is organized as follows. In section 2 we describe the multidimensional cosmological model and
obtain the Einstein equations in the form of the Lagrange-Euler equations following from some Lagrangian.
Here we develop the $n$-dimensional vector formalism for the integrating of the equations of motion. Conclud-
ing Section 2 we present the review of the all known integrable models. In Section 3 we suggest the method
for obtaining the new class of the integrable models on the manifold $M = R \times M_1 \times M_2$. The method is based
on the reducing of the Einstein equations to the generalized Emden-Fowler (second-order ordinary differential)
equation. The method is useful for any 2-component model on the manifold $M = R \times M_1 \times M_2$ except for
the cases admitting the integration by more simple way. The total number of the model components is
equal to the sum of the number of the non Ricci-flat spaces with the number of the perfect fluid components.
The integrable classes recently derived by Zaitsev and Polyanin of the generalized Emden-Fowler equation allow to generate the new integrable cosmological models. Their metrics are presented. In Section 3 the method is applied for the models with Ricci-flat spaces $M_1, M_2$ and 2-component perfect fluid.

2 The model and the equations of motion

Within $n$-factor spaces cosmological model $D$-dimensional space-time manifold $M$ is considered as a product of the time axis $R$ and $n$ manifolds $M_1, \ldots, M_n$, i.e.

$$M = R \times M_1 \times \ldots \times M_n,$$

(2.1)

The product of one part of the manifolds gives the external 3-dimensional space and the remaining part stands for so called internal spaces. The internal spaces are supposed to be compact. Further, for sake of generality, we admit that dimensions $N_i = \dim M_i$ for $i = 1, \ldots, n$ are arbitrary.

The manifold $M$ is equipped with the metric

$$g = -e^{2\gamma(t)} dt \otimes dt + \sum_{i=1}^n \exp[2x^i(t)] g^{(i)}(t),$$

(2.2)

where $\gamma(t)$ is an arbitrary function determining the time $t$ and $g^{(i)}$ is the metric on the manifold $M_i$. We suppose that the manifolds $M_1, \ldots, M_n$ are the Einstein spaces, i.e.

$$R_{k_i l_i}[g^{(i)}] = \lambda_i g^{(i)}_{k_i l_i}, \quad k_i, l_i = 1, \ldots, N_i, \quad i = 1, \ldots, n,$$

(2.3)

where $\lambda_i$ is constant. In the special case, when $M_i$ is a space of constant Riemann curvature $K_i$ the constant $\lambda_i$ reads: $\lambda_i = K_i (N_i - 1)$ (here $N_i > 1$).

Using the assumptions (2.3) we obtain the following non-zero components of the Ricci tensor for the metric (2.2) [13]

$$R_{00}^M = e^{-2\gamma} \left( \sum_{i=1}^n N_i (x^i)^2 + \gamma_0 - \dot{\gamma} \gamma_0 \right),$$

(2.4)

$$R^{m_i}_{n_i} = \{ \lambda_i \exp[-2x^i] + [\ddot{x}^i + \dot{x}^i (\gamma_0 - \dot{\gamma})] e^{-2\gamma} \} \delta_{n_i}^{m_i},$$

(2.5)

where we denoted

$$\gamma_0 = \sum_{i=1}^n N_i x^i.$$

(2.6)

Indices $m_i$ and $n_i$ in (2.4),(2.5) for $i = 1, \ldots, n$ run over from $(D - \sum_{j=1}^n N_j)$ to $(D - \sum_{j=1}^n N_j + N_i)$ ($D = 1 + \sum_{i=1}^n N_i = \dim M$).

We consider a source of gravitational field in the form of multicomponent perfect fluid. The energy-momentum tensor of such source under the comoving observer condition reads

$$T^{M}_{N} = \sum_{\mu=1}^{\bar{m}} T^{M(\mu)}_{N},$$

(2.7)

$$\left( T^{M(\mu)}_{N} \right) = \text{diag} \left( -\rho^{(\mu)}(t), p_1^{(\mu)}(t) \delta_{l_1}^{k_1}, \ldots, p_n^{(\mu)}(t) \delta_{l_n}^{k_n} \right),$$

(2.8)

Furthermore we suppose that for any $\mu$-th component of the perfect fluid the barotropic equation of state holds

$$p_i^{(\mu)}(t) = \left( 1 - h_i^{(\mu)} \right) \rho^{(\mu)}(t), \quad \mu = 1, \ldots, \bar{m},$$

(2.9)

where $h_i^{(\mu)} = \text{const}$. It should be noted that each $\mu$-th component admits different barotropic equations of state in the different spaces $M_1, \ldots, M_n$. From the physical viewpoint this follows from the separation of the internal spaces with respect to the external one and with respect to each others.
One easily shows that the equation of motion $\nabla_M T^M_{\mu} = 0$ for the $\mu$-th component of the perfect fluid described by the tensor (2.8) reads

$$\dot{\rho}^{(\mu)} + \sum_{i=1}^{n} N_i \dot{x}^i (\rho^{(\mu)} + p^{(\mu)}_i) = 0. \quad (2.10)$$

Using the equations of state (2.9), we obtain from (2.10) the following integrals of motion

$$A^{(\mu)} = \rho^{(\mu)} \exp \left[ 2\gamma_0 - \sum_{i=1}^{n} N_i h^{(\mu)}_i x^i \right] = \text{const.} \quad (2.11)$$

The Einstein equations $R^M_N - R^M_{\mu N}/2 = \kappa^2 T^M_{\mu}$ ($\kappa^2$ is the gravitational constant), can be written as $R^M_N = \kappa^2 [T^M_N - T^M_{\mu N} / (D - 2)]$. Further we employ the equations $R^n^0 - R/2 = \kappa^2 T^n_0$ and $R^n_m = \kappa^2 [T^n_m - T^M_{\mu m} / (D - 2)]$. Using (2.4)-(2.9), we obtain for them

$$\frac{1}{2} \sum_{i,j=1}^{m} G_{ij} \dot{x}^i \dot{x}^j + V = 0, \quad (2.12)$$

$$\lambda^i e^{-2\gamma} + [\dot{x}^i + \dot{x}^j (\gamma_0 - \dot{\gamma})] e^{-2\gamma} = -\kappa^2 \sum_{\mu=1}^{m} A^{(\mu)} \left( h^{(\mu)}_i - \frac{\sum_{k=1}^{n} N_k h^{(\mu)}_i}{D - 2} \right)$$

$$\times \exp \left[ \sum_{i=1}^{n} N_i h^{(\mu)}_i x^i - 2\gamma_0 \right]. \quad (2.13)$$

Here

$$G_{ij} = N_i \delta_{ij} - N_i N_j \quad (2.14)$$

are the components of the minisuperspace metric,

$$V = e^{2\gamma} \left( \frac{1}{2} \sum_{i=1}^{n} \lambda^i N_i e^{-2\gamma} + \kappa^2 \sum_{\mu=1}^{m} A^{(\mu)} \exp \left[ \sum_{i=1}^{n} N_i h^{(\mu)}_i x^i - 2\gamma_0 \right] \right). \quad (2.15)$$

The dependence on the densities $\rho^{(\mu)}$ in (2.12),(2.13) has been canceled according to the relations (2.11).

It is not difficult to verify that after the gauge fixing $\gamma = F(x^1, \ldots, x^n)$ the equations of motion (2.13) may be considered as the Lagrange-Euler equations obtained from the Lagrangian

$$L = e^{\gamma_0} - \gamma \left( \frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j - V \right) \quad (2.16)$$

under the zero-energy constraint (2.12).

Now we introduce n-dimensional real vector space $R^n$. By $e_1, \ldots, e_n$ we denote the canonical basis in $R^n$ ($e_1 = (1, 0, \ldots, 0)$ etc.). Hereafter we use the following vectors:

the vector we need to obtain

$$x = x^1(t)e_1 + \ldots + x^n(t)e_n, \quad (2.17)$$

the vector induced by the curvature of the space $M_k$

$$v_k = -\frac{2}{N_k} e_k = \sum_{i=1}^{n} -\frac{2}{N_k} \delta_k e_i, \quad (2.18)$$

the vector induced by $\mu$-th component of the perfect fluid

$$u_\mu = \sum_{i=1}^{n} \left( h^{(\mu)}_i - \frac{\sum_{k=1}^{n} N_k h^{(\mu)}_k}{D - 2} \right) e_i. \quad (2.19)$$
Let $<.,.>$ be a symmetrical bilinear form defined on $\mathbb{R}^n$ such that

$$<e_i,e_j> = G_{ij}. \quad (2.20)$$

The form is nongenerated and the inverse matrix to $(G_{ij})$ has the components

$$G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D}. \quad (2.21)$$

The form $<.,.>$ endows the space $\mathbb{R}^n$ with the metric, which signature is $(-,+,+,\ldots,+) \ [17], \ [18]$. By the usual way we may introduce the covariant components of vectors. For the vectors $v_k$ and $u_\mu$ we have

$$v^i_{(k)} = -2\frac{\delta^i_k}{N_k}, \quad v^i_{(k)} = \sum_{i=1}^{n} G_{ij}v^j_{(k)} = 2(N_i - \delta^i_k), \quad (2.22)$$

$$u^i_{(\mu)} = h^i_{(\nu)} - \sum_{k=1}^{n} N_k h^i_{(\nu)}, \quad u^i_{(\mu)} = \sum_{i=1}^{n} G_{ij}u^j_{(\mu)} = N_i h^i_{(\mu)}. \quad (2.23)$$

The values of $<v_k,v_i>$, $<v_k,u_\mu>$ and $<u_\mu,u_\nu>$ are presented in Table 1.

| $<.,.>$ | $v_j$ | $u_\nu$ |
|--------|------|--------|
| $v_i$  | $4\left(\frac{\delta^i}{N_i} - 1\right)$ | $-2h^i_{(\nu)}$ |
| $u_\mu$ | $-2h^i_{(\nu)}$ | $\sum_{i=1}^{n} N_i h^i_{(\nu)} + \frac{1}{2-D} \left[ \sum_{i=1}^{n} h^i_{(\nu)} N_i \right]$ |

TABLE I. Values of the bilinear form $<.,.>$ for the vectors $v_i$ and $u_\mu$, induced by curvature of the space $M_i$ and $\mu$-th component of the perfect fluid correspondingly.

A vector $y \in \mathbb{R}^n$ is called time-like, space-like or isotropic, if $<y,y>$ has negative, positive or null values correspondingly. Vectors $y$ and $z$ are called orthogonal if $<y,z> = 0$. It should be noted that the curvature induced vector $v_i$ is always time-like, while the perfect fluid induced vector $u_\mu$ admits any value of $<u_\mu,u_\mu>$ (see Table 1).

Using the notation $<.,.>$ and the vectors (2.17)-(2.19), we may write the zero-energy constraint (2.12) and the Lagrangian (2.16) in the form

$$E = \frac{1}{2} <\dot{x},\dot{x}> + V = 0, \quad (2.24)$$

$$L = e^{\gamma - \gamma_0} \left( \frac{1}{2} <\dot{x},\dot{x}> - V \right), \quad (2.25)$$

where

$$V = e^{2(\gamma - \gamma_0)} \left[ -\frac{1}{2} \sum_{i=1}^{n} N_i e^{<v_i,x>} + \kappa^2 \sum_{\mu=1}^{\tilde{m}} A^{(\mu)} e^{<u_\mu,x>} \right]. \quad (2.26)$$

It is obviously from (2.26) that the term induced in the potential by the non-Ricci flat space $M_i$ is similar to the term induced by $\mu$-component of the perfect fluid. Due to this fact the non-zero curvature of the manifold $M_i$ may be also called a component and now we use the notion of the component in such new sense. Further we employ the so called harmonic time gauge, which implies

$$\gamma(t) = \gamma_0 = \sum_{i=1}^{n} N_i x^i. \quad (2.27)$$
From the mathematical viewpoint the problem consist in integrability of the system with \( n \geq 2 \) degrees of freedom, described by the Lagrangian of the form

\[
L = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle - \sum_{\mu=1}^{m} a^{(\mu)} e^{\langle b_{\mu}, x \rangle},
\]

(2.28)

where \( x, b_{\mu} \in \mathbb{R}^n \). In (2.28) \( m \) denotes the total number of the components including the curvatures and the perfect fluid components. It should be noted that the kinetic term \( \langle \dot{x}, \dot{x} \rangle \) is not positively definite bilinear form as it usually takes place in classical mechanics. Due to the pseudo-Euclidean signature \((-+,+,...,+)\) of the form \( \langle ., . \rangle \) such systems may be called pseudo-Euclidean Toda-like systems as the potential like that given in (2.28) defines well known in classical mechanics Toda lattices \([34]\). In the papers \([8],[19],[20]\) the following classes of the integrable pseudo-Euclidean Toda-like systems have been found

1. \( m = 0 \). This case corresponds to the vacuum multidimensional cosmological model on the manifold \( M = R \times M_1 \times \ldots \times M_n \) with all Ricci-flat spaces \( M_i \). The corresponding metric is a multidimensional generalization of the well-known Kasner solution \([15]\).

2. \( m = 1 \), the vector \( b_1 \) is arbitrary. The metrics for this 1-component case were obtained in \([20]\). This integrable class may be enlarged by the addition of the new components inducing the vectors collinear to the vector \( b_1 \).

3. \( m \geq 2 \), \( n = 2 \), \( b_{\mu} = b + C_{\mu} b_0 \), where \( b \) is an arbitrary vector and \( b_0 \) is an arbitrary isotropic vector, \( C_{\mu} = \text{const} \). This class was integrated in \([20]\) only under the zero energy constraint.

4. \( m \geq 2 \), the vectors \( b_1, \ldots, b_m \) are linear independent and satisfy the conditions \( \langle b_{\mu}, b_{\nu} \rangle = 0 \) for \( \mu \neq \nu \). This integrable class may be enlarged by the addition of the new components inducing the vectors collinear to one from the orthogonal set \( b_1, \ldots, b_m \). The corresponding cosmological models are studied in \([3]\).

5. \( m \geq 2 \), the vectors \( b_1, \ldots, b_m \) are space-like and may be interpreted as a set of admissible roots \([3]\) of a simple complex Lie’s algebra \( G \). In this case the pseudo-Euclidean Toda-like system is trivially reducible to the Toda lattice associated with the Lie algebra \( G \) \([34]\). For \( G = A_2 \equiv \text{sl}(3,C) \) the metric of the corresponding cosmological model was explicitly written in \([8]\).

In the present paper we consider only 2-component (\( m = 2 \)) pseudo-Euclidean Toda-like systems with 2 degrees of freedom \( (n = 2) \) under the zero energy constraint. The corresponding multidimensional cosmological models are 2-factor spaces, i.e.

\[
M = R \times M_1 \times M_2
\]

(2.29)

and admit the following combinations of the components: curvature of \( M_1 \) and curvature of \( M_2 \) (vacuum models); curvature of \( M_1 \) or \( M_2 \) and 1-component perfect fluid; 2-component perfect fluid in Ricci-flat spaces \( M_1 \) and \( M_2 \). In our recent paper \([10]\) we have integrated the vacuum model of the type (2.29) with 2 curvatures for the dimensions \( (\dim M_1, \dim M_2) = (6,3),(8,2),(5,5) \). Now we develop more general procedure useful for any combination of the 2 components.

### 3 Reducing to the generalized Emden-Fowler equation

Let us consider the equations of motion following from the Lagrangian (2.28) with \( n = m = 2 \) under the zero energy constraint. If the vectors \( b_1 \) and \( b_2 \) satisfy one of the following conditions

1. \( b_1 \) and \( b_2 \) are linearly dependent,

2. \( \langle b_1, b_2 \rangle = 0 \), i.e. \( b_1 \) and \( b_2 \) are orthogonal,

3. \( \langle b_1 - b_2, b_1 - b_2 \rangle = 0 \), i.e. vector \( b_1 - b_2 \) is isotropic,
the equations of motion are easily integrable and the corresponding exact solutions have been obtained in the papers [8, 24]. Now we aim to develop the integration procedure just for all remaining cases. Then, further we suppose that the vectors $b_1$ and $b_2$ do not satisfy any condition from 1-3.

Let us introduce in $R^2$ an orthogonal basis forming by the following two vectors

$$f_1 = (u_{22} - u_{12})b_1 + (u_{11} - u_{12})b_2, \quad f_2 = b_2 - b_1,$$

(3.1)

where we denoted

$$u_{\mu\nu} = < b_\mu, b_\nu >, \quad \mu, \nu = 1, 2.$$  
(3.2)

According to the admission accepted $f_2$ is not isotropic vector, i.e.

$$< f_2, f_2 > = u_{11} + u_{22} - 2u_{12} \neq 0.$$  
(3.3)

One may easily check that $u_{12}^2 - u_{11}u_{22} \geq 0$ for any vectors $b_1, b_2 \in R^2$ and $u_{12}^2 - u_{11}u_{22} = 0$ if and only if $b_1$ and $b_2$ are linearly dependent. Then in the case under consideration

$$< f_1, f_1 > = -(u_{12}^2 - u_{11}u_{22})(u_{11} + u_{22} - 2u_{12}) \neq 0,$$

(3.4)

$$< f_1, f_1 > / < f_2, f_2 > = -(u_{12}^2 - u_{11}u_{22}) < 0,$$

(3.5)

i.e. one from the orthogonal vectors $f_1$ and $f_2$ is space-like and the other is time-like.

The vector $x(t)$ we have to find decomposes as follows

$$x = \frac{< x, f_1 >}{< f_1, f_1 >} f_1 + \frac{< x, f_2 >}{< f_2, f_2 >} f_2.$$  
(3.6)

For the new configuration variables

$$z(t) = \frac{< x, f_2 >}{2} + \ln \sqrt{\frac{|a^{(2)}|}{|a^{(1)}|}},$$

(3.7)

$$y(t) = \frac{1}{2} \sqrt{-\frac{< f_2, f_2 >}{< f_1, f_1 >}} < x, f_1 >$$

(3.8)

the Lagrangian (2.28) and the corresponding zero-energy constraint look as follows

$$L = 2\beta (\dot{z}^2 - \dot{y}^2) - V(z, y),$$

(3.9)

$$E = 2\beta (\dot{z}^2 - \dot{y}^2) + V(z, y) = 0,$$

(3.10)

where the potential $V(z, y)$ has the form

$$V(z, y) = V_0 e^{2\alpha \beta y} \left( \text{sgn} \left[ a^{(1)} \right] e^{2\beta_1 z} + \text{sgn} \left[ a^{(2)} \right] e^{2\beta_2 z} \right).$$

(3.11)

In formulas (3.9)-(3.11) the following constants are used

$$\alpha = \sqrt{u_{12}^2 - u_{11}u_{22}}, \quad \beta = (u_{11} + u_{22} - 2u_{12})^{-1},$$

(3.12)

$$\beta_1 = -(u_{11} - u_{12})\beta, \quad \beta_2 = \beta_1 + 1 = (u_{22} - u_{12})\beta,$$

(3.13)

$$V_0 = |a^{(1)}|^{\beta_1}|a^{(2)}|^{-\beta_1}.$$  
(3.14)

It should be mentioned that using of a basis in the form (3.1) provides the factorization of the potential (3.11) with respect to the coordinates of the vector $x(t)$ (the additional linear transformation (3.7),(3.8) does not matter in this situation). Such factorization of the potential is essential under the developing of the following procedure proposed in [8]. Using the equation of motion following from the Lagrangian (3.9)

$$\ddot{z} = -\frac{1}{2\beta} V_0 e^{2\alpha \beta y} \left( \beta_1 \text{sgn} \left[ a^{(1)} \right] e^{2\beta_1 z} + \beta_2 \text{sgn} \left[ a^{(2)} \right] e^{2\beta_2 z} \right),$$

(3.15)

$$\dot{y} = \frac{\alpha}{4} V(z, y),$$

(3.16)
the zero-energy condition (3.10) written in the form
\[ \dot{z}^2 = \frac{1}{2\beta} \frac{V(z, y)}{(\dot{y}/\dot{z})^2 - 1} = \frac{1}{2\beta} \frac{V(z, y)}{(dy/dz)^2 - 1} \] (3.17)
and the relation
\[ \frac{d^2 y}{d\dot{z}^2} = \frac{\dot{y} - \dot{z} \frac{dy}{dz}}{\dot{z}^2} \] (3.18)
we obtain the following second-order ordinary differential equation
\[ \frac{d^2 y}{d\dot{z}^2} = \left[ \frac{d y}{d\dot{z}} \right]^{-2} - 1 \left\{ \frac{1}{2} \left( \beta_1 + \beta_2 + e^{2\dot{z} - \varepsilon} \right) \frac{dy}{dz} + \alpha \beta \right\}, \] (3.19)
where
\[ \varepsilon = \text{sgn} \left[ a^{(1)}a^{(2)} \right]. \] (3.20)
We notice that due to the factorization of the potential the right side of the equation (3.19) does not contain $y$, so, in fact, the equation is the first-order one with respect to $dy/dz$.

This procedure is valid for the solutions such that $\dot{z} \neq 0$. Under the zero energy constraint the solutions of (3.15),(3.16) with $\dot{z} \equiv 0$ gives the following vector $x(t)$
\[ x(t) = p \ln |t - t_0| + q, \] (3.21)
where the constant vectors $p, q \in \mathbb{R}^2$ are such that
\[ p = \frac{2}{\alpha^2} f_1, \] (3.22)
\[ e^{<q, b_1>} = \frac{\beta_2}{a^{(1)} \alpha^2 \beta} > 0, \quad e^{<q, b_2>} = -\frac{\beta_1}{a^{(2)} \alpha^2 \beta} > 0. \] (3.23)
We note that the exceptional solution (3.21) exists only if the inequalities in (3.23) are satisfied. It should be mentioned, that the set of the equations (3.10),(3.15),(3.16) does not admit static solutions $\dot{z} = y \equiv 0$ due to the condition (3.3). The solutions with $\dot{z} = \pm \dot{y}$ are also impossible, so using the relation (3.17) we do not lose any solutions of the set (3.10),(3.15),(3.16) except, possibly, the solution (3.21).

Let us suppose that one is able to obtain the general solution of the equation (3.19) in the parametrical form $z = z(\tau), y = y(\tau)$, where $\tau$ is a parameter. Then using (3.6)-(3.8) we obtain the vector $x$ as the function of the parameter $\tau$
\[ x(\tau) = \frac{2y(\tau)}{\alpha} (-\beta_2 b_1 + \beta_1 b_2) + 2\beta \left[ z(\tau) - \ln \sqrt{\frac{a^{(2)}}{a^{(1)}}} \right] (b_2 - b_1). \] (3.24)
We recall that coordinates of the vector $x(\tau)$ in the canonical basis are the logarithms of the scale factors for the spaces $M_1, M_2$. The relation between the harmonic time $t$ and the parameter $\tau$ may be always derived by integration of the zero-energy constraint written in the form of the separable equation
\[ dt^2 = 2\beta \left( \frac{dx}{d\tau} \right)^2 \frac{2}{V(z(\tau), y(\tau))} d\tau^2. \] (3.25)
Thus the problem of the integrability by quadrature of the pseudo-Euclidean Toda-like systems with 2 degrees of freedom under the zero-energy constraint is reduced to the integrability of the equation (3.19).

For $dy/dz$ the equation (3.19) represents the first-order nonlinear ordinary differential equation. Its right side is third-order polynom (with the coefficients depending on $z$) with respect to the $dy/dz$. An equation of such type is called Abel’s equation (see, for instance, [27],[57]). There are no methods to integrate arbitrary Abel’s equation, however the equation (3.19) may be integrated for some values of the parameters $\alpha \beta$ and $\beta_1 + \beta_2$. First of all let us notice that the equation (3.19) has the partial integrals $y \pm z = \text{const}$,
which make the relation (3.17) singular and as was already mentioned are not partial integrals of the set (3.10),(3.15),(3.16). Existence of this partial solution of the Abel equation (3.19) allows to find the following nontrivial transformation

\[ e^{2z} = -\frac{X}{Y} \left( \frac{dY}{dX} \right), \quad (3.26) \]

\[ y = -\delta \left[ z + \ln \left( \frac{Y}{X} \right) + \ln C \right], \quad \delta = \pm 1, \quad C > 0, \quad (3.27) \]

which reduces the Abel equation (3.19) to the following integrable Emden-Fowler equation

\[ \frac{d^2Y}{dX^2} = X^n Y^m \left( \frac{dY}{dX} \right)^l, \quad (3.28) \]

where the constant parameters \( n, m \) and \( l \) read

\[ n = \frac{1}{2} \left( \beta_1 + \beta_2 - 2\delta \alpha \beta - 3 \right) = \frac{-2u_{11} - u_{22} + 3u_{12} - \delta \sqrt{u_{12}^2 - u_{11}u_{22}}}{u_{11} + u_{22} - 2u_{12}}, \quad (3.29) \]

\[ m = -\frac{1}{2} \left( \beta_1 + \beta_2 - 2\delta \alpha \beta + 3 \right) = \frac{-u_{11} - 2u_{22} + 3u_{12} + \delta \sqrt{u_{12}^2 - u_{11}u_{22}}}{u_{11} + u_{22} - 2u_{12}}, \quad (3.30) \]

\[ l = -\frac{1}{2} \left( \beta_1 + \beta_2 + 2\delta \alpha \beta - 3 \right) = \frac{2u_{11} + u_{22} - 3u_{12} - \delta \sqrt{u_{12}^2 - u_{11}u_{22}}}{u_{11} + u_{22} - 2u_{12}}, \quad (3.31) \]

For our models the parameters in the generalized Emden-Fowler equation are not independent. It follows from (3.29),(3.30) that

\[ n + m = -3. \quad (3.32) \]

In the special case \( l = 0 \) the equation (3.28) is known as the Emden-Fowler equation.

If the parameters \( l \) and \( m \) given by (3.31),(3.30) are such that \( l = 0, \ m \neq 1 \) there exists one more transformation

\[ 1 + e^{2z} = \frac{2}{m - 1} \frac{X}{Y} \left( \frac{dY}{dX} \right), \quad (3.33) \]

\[ y = \delta \left[ z - \frac{1}{m - 1} \ln Y^2 + C \right], \quad \delta = \pm 1, \quad C \in R, \quad (3.34) \]

which reduces the Abel equation (3.19) to the following integrable Emden-Fowler equation

\[ \frac{d^2Y}{dX^2} = Y^{\frac{n+m}{m-1}}. \quad (3.35) \]

There are no methods for integrating of the generalized Emden-Fowler equation with arbitrary independent parameters \( n, m \) and \( l \). However, the discrete-group methods recently developed by Zaitsev and Polyanin [2] allows to integrate by quadrature 3 two-parametrical classes, 11 one-parametrical classes and about 90 separated points in the parametrical space \( (n, m, l) \) of the generalized Emden-Fowler equation. For instance, the two-parametrical integrable classes arise when \( m \) and \( l \) are arbitrary and \( n = 0 \) or when \( n \) and \( l \) are arbitrary and \( m = 0 \). The one-parametrical class with \( l = 0 \) and \( n + m = -3 \) is also integrable by quadrature.

Let us suppose that the two components of the 2-factor spaces cosmological model under consideration induce such vectors \( b_1 \) and \( b_2 \) that the corresponding to the model generalized Emden-Fowler equation (3.28) with the parameters defined by (3.2),(3.29)-(3.31) is integrable in the parametrical form \( X = X(\tau), \ Y = Y(\tau) \), where \( \tau \) is a parameter. Then, using the parameter \( \tau \) as the new time coordinate we obtain by the formulas (3.26),(3.27),(3.24),(3.25) the following final result for the metric (2.2)

\[ g = -f^2(\tau)[a_1(\tau)]^{2N_1}[a_2(\tau)]^{2N_2}d\tau \otimes d\tau + [a_1(\tau)]^2g^{(1)} + [a_2(\tau)]^2g^{(2)}, \quad (3.36) \]
On the other hand the most known cases, when the generalized Emden-Fowler equation (3.28) is integrable, arise for the rational parameters. By \( \gamma_i \) for \( i = 1, 2 \) we denoted the following constants

\[
\gamma^i = 2\beta \left\{ \ln \frac{C}{n + l} \left[ (n - l + 4)b^{(1)}_{(1)} - (n - l + 2)b^{(2)}_{(2)} \right] - \ln \left( \frac{a^{(2)}}{a^{(1)}} \right) \left[ b^{(2)}_{(2)} - b^{(1)}_{(1)} \right] \right\}
\]

(3.39)

We recall that \( b^{(\mu)}_i \) are coordinates of the vector \( b_\mu \) in the canonical basis. In the special case \( l = 0 \) one may also use by the similar manner the transformation (3.33),(3.34) and the result of integration of the equation (3.35) to write the metric. This transformation was used in [10] for integrating of the models with two curvatures.

Thus the method described allows to integrate the cosmological models if the corresponding generalized Emden-Fowler equation is integrable. Note that if the model with some vectors \( b_1 \) and \( b_2 \) is integrable by such manner then any model with the vectors \( \alpha b_1 \) and \( \alpha b_2 \) (\( \alpha \) is an arbitrary non-zero constant) is also integrable as the parameters \( n, m \) and \( l \) do not change under such transformation of the vectors. Taking into account the classes 1-4 (the class 5 does not arise for \( n = 2 \)) and the additional to them class, which may be integrated by the method described, we obtain the quite large variety of the integrable 2-factor spaces cosmological models with 2 components.

### 4 Examples of the integrable models

Now we apply the method proposed in Section 3 to the cosmological models on the manifold (2.29) with both Ricci-flat spaces \( M_1, M_2 \) and the 2-component perfect fluid source. Let us represent such model by Table 2

| manifold/source | external space \( M_1^{N_1} \) | internal space \( M_2^{N_2} \) |
|-----------------|-------------------------------|-------------------------------|
| 1-st component of the perfect fluid | \( h^{(1)}_1 \) | \( h^{(1)}_2 \) |
| 2-nd component of the perfect fluid | \( h^{(2)}_1 \) | \( h^{(2)}_2 \) |

**TABLE 2.** Representation of the model on the manifold \( M = R \times M_1 \times M_2 \) with Ricci-flat spaces \( M_1, M_2 \) for the 2-component perfect fluid.

We recall that \( N_i = \text{dim} M_i \) and \( h^{(\mu)}_i \) are the constant parameters in the barotropic equation of state (2.9). The model is entirely defined by these 6 parameters. One easily shows [12] that the dominant energy condition applied to the stress-energy tensor (2.8) implies \( 0 \leq h^{(\mu)}_i \leq 2 \). Usually rational values of the parameter \( h^{(\mu)}_i \) are employed in cosmology, for instance, \( h^{(\mu)}_i = (N_i - 1)/N_i \) - radiation, \( h^{(\mu)}_i = 1 \) - dust, \( h^{(\mu)}_i = 0 \) - Zeldovich (stiff) matter, \( h^{(\mu)}_i = 2 \) - false vacuum (A-term), \( h^{(\mu)}_i = (D - 1)/D \) - superradiation etc. On the other hand the most known cases, when the generalized Emden-Fowler equation (3.28) is integrable, arise for the rational parameters \( n, m \) and \( l \). So if one demands the rationality of the parameters \( n, m \) and \( l \)
in the equation (3.28) corresponding to the model under the condition of the rationality for the parameters $h^{(1)}_1$, then due to the following relation

$$\alpha^2 = u_{12}^2 - u_{11}u_{22} = \frac{N_1N_2}{N_1+N_2-1} \left( h^{(1)}_1h^{(2)}_2 - h^{(1)}_2h^{(2)}_1 \right)^2,$$

(4.1)

the dimensions $N_1, N_2$ with integer value of the expression $R \equiv \sqrt{N_1N_2(N_1+N_2-1)}$ are singled out. For instance, the expression $R$ is integer for the following dimensions: \((N_1, N_2) = (3, 6), (2, 8), (5, 5), (7, 8), (3, 25), (N_1, 1)\). From the physical viewpoint the following cases may be of interest: \((2, 1), (3, 1), (3, 6), (3, 25)\).

Let us consider the models of the type represented in Table 2 leading to the generalized Emden-Fowler equation (3.28), arising when $n = 0$ or $m = 0$, describe the same cosmological models. It follows easily from (3.29),(3.31) that if the model is such that $\delta = 1$ (or $\delta = -1$) then $n = 0$ for $\delta = 1$ (correspondingly, $\delta = 1$). It is easy to see also from (3.30),(3.31) that the condition $l = 0$ transforms to the condition $m = 0$ under the inverse numbering of the components. Thus, from 3 integrable classes, arising for $n = 0$, $m = 0$ and $l = 0$, correspondingly, of the generalized Emden-Fowler equation (3.28) for our models it is enough to study any one from them, let it be the class with $l = 0$. In this case the equation (3.28) has the form

$$\frac{d^2Y}{dX^2} = X^{-m-3}Y^m, \quad m = -\beta_1 - \beta_2.$$

(4.3)

By the following transformation

$$Y = \frac{\tau}{\xi}, \quad X = \frac{1}{\xi},$$

(4.4)

it reduces to the equation

$$\frac{d^2\tau}{d\xi^2} = \tau^m,$$

(4.5)

which is easily integrable. Then the general solution of the equation (4.3) has the form

$$Y = \pm \frac{\tau}{F(\tau)}, \quad X = \pm \frac{1}{F(\tau)},$$

(4.6)

where

$$F(\tau) = \int \left[ \frac{2}{m+1} \tau^{m+1} + C_1 \right]^{-1/2} d\tau + C_2, \quad m \neq -1,$$

(4.7)

$$= \int \left[ 2\ln|\tau| + C_1 \right]^{-1/2} d\tau + C_2, \quad m = -1.$$  

(4.8)

We suppose that the both components of the perfect fluid have the positive mass-energy densities given by (2.11). It means $a^{(\mu)} = \kappa^2A^{(\mu)} > 0$ for $\mu = 1, 2$, so $\varepsilon = \text{sgn} \left[ a^{(1)}a^{(2)} \right] = 1$. Then taking into account the formula (3.26), we must consider the general solution (4.6) on such interval of the variable $\tau$ where

$$G(\tau) = \frac{X(\tau)Y'(\tau)}{Y(\tau)X'(\tau)} = \frac{F(\tau)}{\tau F'(\tau)} - 1 > 0.$$ 

(4.9)

Finally using the results of Section 3 we obtain the following exact solution for the cosmological model represented by Table 2 in the special case $l = 0$; the metric is given by the formula (3.36), where

$$f^2(\tau) = \frac{2|\beta|C^{-\alpha_1\beta}}{[A^{(1)}]^{\beta_2} [A^{(2)}]^{-\beta_1}} \left[ \frac{F'(\tau)}{G(\tau)\tau^2} \right]^2,$$

(4.10)
the scale factors of the spaces $M_1, M_2$ have the form

$$a_i(\tau) = \left[ \frac{A^{(1)}}{A^{(2)}} \right]^{\beta_2 - u_{i(1)}} \left\{ [G(\tau)]^{u_{i(2)} - 2u_{i(1)}} [C^2\tau^2]^{\beta_1 u_{i(2)} - \beta_2 u_{i(1)}} \right\}^{\delta/\alpha}, \quad (4.11)$$

the mass-energy-densities of the components read

$$\rho^{(1)}(\tau) = \left[ \frac{A^{(1)}}{a_1(\tau)} \right]^{\beta_2} \left[ \frac{A^{(2)}}{a_2(\tau)} \right]^{-\beta_1} \left\{ [G(\tau)]^{u_{12} - 2u_{11}} [C^2\tau^2]^{\alpha^2\beta} \right\}^{\delta/\alpha}, \quad (4.12)$$

$$\rho^{(2)}(\tau) = \left[ \frac{A^{(1)}}{a_1(\tau)} \right]^{\beta_2} \left[ \frac{A^{(2)}}{a_2(\tau)} \right]^{-\beta_1} \left\{ [G(\tau)]^{u_{22} - 2u_{12}} [C^2\tau^2]^{\alpha^2\beta} \right\}^{\delta/\alpha}. \quad (4.13)$$

The functions $F(\tau)$ and $G(\tau)$ are defined in (4.7)-(4.9); components $u_{i(\mu)}^i$ of the vectors in the canonical basis are given in (2.23); the parameters $\alpha, \beta, \beta_1, \beta_2$ are defined in (3.12), (3.13); the values $u_{\mu\nu} = u_{\mu}, u_{\nu}$ for $\mu, \nu = 1, 2$ may be calculated from the parameters $N_i, h^{(i)}_{\mu}$ by the formula given in Table 1.

The following relation is valid for the densities (4.12)-(4.13)

$$\rho^{(2)}(\tau)/\rho^{(1)}(\tau) = G(\tau). \quad (4.14)$$

We recall the possible existence of the special solution (3.21).

Concluding the paper, we mention in Table 3 some interesting from our viewpoint special models with $l = 0$ for the dimensions $N_1 = 3$ and $N_2 = 6$. One easily shows that for these dimensions and given in Table 3 values of the parameters $h^{(i)}_{\mu}$ the parameter $l$ given by (3.31) is equal to zero, so the special model is described by the exact solution (4.10)-(4.13). It follows from (2.22), (2.23) that for these dimensions the 2 perfect fluid components with the parameters 1-st: $h^{(1)}_{\mu} = 4/3, h^{(2)}_{\mu} = 2$, 2-nd: $h^{(1)}_{\mu} = 2, h^{(2)}_{\mu} = 5/3$ induce the vectors, which coincide with the vectors induced by the curvatures of $M_1^3$ and $M_2^6$, correspondingly. Then the adding of such 2 components to the integrable vacuum model on $R \times M_1^3 \times M_2^6$ with 2 curvatures (see investigation of this model in [10]) provides with the integrable model for the 2 non Ricci-flat spaces and the 2-component perfect fluid.

| manifold/source | external space $M_1^3$ | internal space $M_2^6$ |
|-----------------|----------------------|----------------------|
| 1-st component of the perfect fluid | radiation | radiation |
| 2-nd component of the perfect fluid | radiation | Zeldovich matter |
| 1-st component of the perfect fluid | dust | radiation |
| 2-nd component of the perfect fluid | radiation | dust |
| 1-st component of the perfect fluid | radiation | radiation |
| 2-nd component of the perfect fluid | dust | radiation |
| 1-st component of the perfect fluid | radiation | radiation |
| 2-nd component of the perfect fluid | false vacuum | false vacuum |
| 1-st component of the perfect fluid | false vacuum | false vacuum |
| 2-nd component of the perfect fluid | Zeldovich matter | false vacuum |
TABLE 3. Examples of the integrable models for dimensions $N_1 = 3$ and $N_2 = 6$. The corresponding exact solutions are given by the formulas (4.10)-(4.13).

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References

[1] Bleyer U, Liebscher D-E and Polnarev A G 1991 *Class. Quantum Grav.* **8** 477

[2] Bleyer U, Ivashchuk V D, Melnikov V N and Zhuk A I 1994 *Nucl. Phys.* **B429** 177

[3] Bogoyavlensky O I 1976 *Comm. Math. Phys.* **51** 201

[4] Börner G 1992 *The Early Universe. Facts and Fiction* (Berlin: Springer)

[5] Forgacs P and Horvath Z 1979 *Gen. Relativ. Grav.* **11** 205

[6] Chodos A and Detweiler S 1980 *Phys. Rev.* **D21** 2167

[7] Demiansky M and Polnarev A G 1990 *Phys. Rev. D* **41** 3003

[8] Gavrilov V R, Ivashchuk V D and Melnikov V N 1995 *J. Math. Phys.* **36** 5829

[9] Gavrilov V R, Melnikov V N and Novello M 1995 *Gravitation and Cosmology* **1** 149

[10] Gavrilov V R, Ivashchuk V D and Melnikov V N, 1996 *Class. Quantum Grav.* **13** 3039

[11] Gavrilov V R, Melnikov V N and Novello M 1996 *Gravitation and Cosmology* **4(8)** 325

[12] Gavrilov V R, Melnikov V N and Triay R Exact solutions in multidimensional cosmology with bulk and shear viscosity *Preprint* CPT-96/P.3396, Marseille, France; to be appeared in *Class. Quantum Grav.*

[13] Gavrilov V R, Ivashchuk V D, Kasper U and Melnikov V N, 1997 *Gen. Relativ. Grav.* **29** 599

[14] Gibbons G W and Wiltshire D L 1987 *Nucl. Phys.* **B287** 717

[15] Gleiser M, Rajpoot S and Teylor J G 1985 *Ann. Phys.* (NY) **160** 299

[16] Ivashchuk V D and Melnikov V N 1988 *Nuovo Cimento* **B102** 131

[17] Ivashchuk V D and Melnikov V N 1989 *Phys. Lett.* **A135** 465

[18] Ivashchuk V D, Melnikov V N and Zhuk A I 1989 *Nuovo Cimento B* **104** 575

[19] Ivashchuk V D 1992 *Phys. Lett.* **A 170** 16

[20] Ivashchuk V D and Melnikov V N 1994 *Int. J. Mod. Phys.* **D3** 795

[21] Ivashchuk V D and Melnikov V N 1995 *Class. Quantum Grav.* **12** 809

[22] Koikawa T and Yoshimura M 1985 *Phys. Lett. B* **155** 137

[23] Lorenz-Petzold D 1984 *Phys. Lett. B* **149** 79

[24] Lorenz-Petzold D 1985 *Phys. Lett. B* **158** 110

[25] Melnikov V N 1994 *Multidimensional Classical and Quantum Cosmology and Gravitation: Exact Solutions and Variations of Constants* In: Cosmology and Gravitation. Ed. M.Novello. Editions Frontiers, Singapore, p. 147

[26] Melnikov V N, *Multidimensional Cosmology and Gravitation*, CBPF-MO-002/95, Rio de Janeiro, Brasil

[27] Polyanin A D and Zaitsev V F 1995 *Handbook on Exact Solutions for Ordinary Differential Equations* (Roca Raton, CRC Press)
[28] Rainer M 1996 *Gravitation and Cosmology* **1**(5) 27

[29] Rainer M and Zhuk A I 1996 *Phys. Rev. D* **54** 6186

[30] Rubakov V A, Shaposhnikov M E 1983 *Phys. Lett.B* **125**, 136

[31] Sahdev D 1984 *Phys. Rev. D* **30** 2495

[32] Szydlowski M 1988 *Gen Relativ. Grav.* **20** 221

[33] Szydlowski M and Pajdosz G 1989 *Class. Quant. Grav.* **6** 1391

[34] Toda M 1981 *Theory of Nonlinear Lattices* (Springer-Verlag, Berlin)

[35] Wesson P S and Ponce de Leon J 1994 *Gen. Rel. Gravit.* **26** 555

[36] Wiltshire D L 1987 *Phys. Rev. D* **36** 1634

[37] Zaitsev V F and Polyanin A D 1994 *Discrete-Groups Methods for Integrating Equations of Nonlinear Mechanics* (Roca Raton, CRC Press-Begel House)