Vector and Axial-vector form factors in radiative kaon decay and flavor SU(3) symmetry breaking

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We study the vector and axial-vector form factors of radiative kaon decay within the framework of the gauged nonlocal effective chiral action from the instanton vacuum, focusing on the effects of flavor SU(3) symmetry breaking. The general tendency of the results are rather similar to those of radiative pion decays: The nonlocal contributions make the results of the vector form factor increased by about 20%, whereas they reduce those of the axial-vector form factor by almost 30%. suppressing the prefactors consisting of the kaon mass and the pion decay constant, we scrutinize how the kaon form factors undergo changes as the mass of the strange current quark is varied. Those related to the vector and second axial-vector form factors tend to decrease monotonically as the strange quark mass increases, whereas that for the axial-vector form factor decreases. When $K \to e\nu\gamma$ decay is considered, both the results of the vector and axial-vector form factors at the zero momentum transfer are in good agreement with the experimental data. The results are also compared with those from chiral perturbation theory to $p^6$ order.

Keywords: Radiative kaon decay, vector and axial-vector form factors of the kaon, instanton vacuum

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I. INTRODUCTION

Radiative kaon decay ($K_{2\gamma}$) provides essential information on the structure of the kaon. Though the structure of the radiative kaon form factors is very similar to that of the pion decay, the effects of flavor SU(3) symmetry breaking, which arises from the current mass of the strange quark, makes the kaon distinguished from the pion. As in the case of the pion, the radiative decay amplitude for the $K^+ \rightarrow l^+ + \nu_l + \gamma$ can be decomposed into two parts, i.e., the structure-dependent (SD) part, and the inner Bremsstrahlung (IB) one or the QED corrections \cite{1, 2}. The decay rate is given by the squared modulus of the amplitude, so that it consists of three different terms, that is, the IB one, the mixed one, and the pure SD one. The IB and mixed terms are proportional to the squared ratio of the lepton and kaon masses, i.e., $(m_l/m_K)^2$, which is called the helicity suppression factor, the radiative kaon decay to the electron ($K_{e2\gamma}$) is governed by the SD term. On the other hand, $K_{\mu2\gamma}$ is sensitive to the IB and mixed terms. Neville \cite{3} proposed that by choosing the angle between the neutrino and the photon with the helicities of both the lepton and neutrino fixed one could measure the vector and axial-vector form factors of $K_{\mu2\gamma}$.

The suggestion of Ref. \cite{4} being considered, the radiative kaon decay $K_{e2\gamma}$ was measured several decades ago \cite{8, 9, 10}. Heintze et al. \cite{10} extracted the following results: $|F_V + F_A| = 0.148 \pm 0.010$ and $|F_V - F_A| < 0.49$ in the standard notation \cite{7}, where $F_V$ and $F_A$ denote the vector and axial-vector form factors of the radiative kaon decay. In Refs. \cite{11, 12, 13}, the $K_{\mu2\gamma}$ decay was experimentally studied but the form factors were not extracted, since they investigated mainly the IB-dominant region. The E787 Collaboration \cite{14} performed the first measurement of an SD component in the radiative kaon decay $K_{\mu2}$, and extracted $|F_V + F_A| = 0.165 \pm 0.007$ (stat) $\pm 0.011$ (syst) and the limit $-0.04 < F_V - F_A < 0.24$ at 90% confidence level. The E865 experiment at the Brookhaven National Laboratory (BNL) \cite{15} studied experimentally the kaon radiative decays $K^+ \rightarrow \mu^+\nu\gamma$ and $K^+ \rightarrow e^+\nu\gamma$, where the photon is in a virtual state. When $\gamma$ is virtual, yet an additional form factor $R$ is involved. Though the analysis of Ref. \cite{15} inevitably contains the model dependence, the results were obtained to be $F_V = 0.112 \pm 0.015 \pm 0.010 \pm 0.003$ (model), $F_A = 0.035 \pm 0.014 \pm 0.013 \pm 0.003$, and $R = 0.227 \pm 0.013 \pm 0.010 \pm 0.009$, when both the data of radiative decays $K^+ \rightarrow \mu^+\nu\gamma$ and $K^+ \rightarrow e^+\nu\gamma$ were combined. The KLOE Collaboration \cite{16} measured the ratio $\Gamma(K_{e2}(\gamma)/\Gamma(K_{\mu2}(\gamma))$ and obtained also the sum of the vector and axial-vector form factors as $F_V + F_A = 0.125 \pm 0.007 \pm 0.001$. Some years ago, ISTRA+ Collaboration \cite{17} reported the extraction of the form factors from the $K^- \rightarrow \mu^-\nu\gamma$ decay: $F_V - F_A = 0.21 \pm 0.04 \pm 0.04$ with the sign also determined. We want to mention that the value of $F_V - F_A$ extracted from the E865 experiment is in disagreement with that from the ISTRA+ Collaboration.

The vector and axial-vector form factors for the radiative kaon decay were studied in chiral perturbation theory (\chiPT) \cite{19, 20, 21, 22, 23}, since the experimental data on the axial-vector form factors can be used to determine a part of the low-energy constants (LECs) that reflect certain features of nonperturbative quark-gluon dynamics. Geng et al. examined the form factors for the radiative kaon decay to order $O(\rho^0)$. The analysis of $K_{e2\gamma}$ was also carried out within the light-front quark model \cite{24}, in which the dependence of $F_V$ and $F_A$ on the momentum transfer squared was presented. Duk et al. \cite{17} showed explicitly that the results of $O(\rho^0)$ \chiPT are found to be slightly away from the 3 $\sigma$ ellipse. It is also interesting to see that the radiative kaon decay could be used for understanding a possible new physics such as the massless dark photon \cite{25, 26}. The radiative kaon decay provides yet another theoretically important aspect on the effects of flavor SU(3) symmetry breaking. The structure of the radiative transition amplitude of the kaon is the same as that of the pion except for the difference of their masses. It indicates that by suppressing the kinematical factors in the expressions of the form factors one can scrutinize how the current quark mass comes into play in describing the radiative kaon and pion decays.

In this work, we investigate the vector and axial-vector form factors for the radiative kaon decay within the framework of the gauged effective chiral action (E\chiA) from the instanton vacuum \cite{27, 33}, emphasizing the effects of the flavor SU(3) symmetry breaking. The instanton vacuum offers a natural realization of the spontaneous breakdown of chiral symmetry (SB\chiS). Consequently, the kaon appears as a pseudo-Goldstone boson like the pion and the mass of the quark is dynamically generated, which depends on the quark virtuality by the quark zero modes in the instanton background. Thus the dynamical quark mass becomes momentum-dependent. There are two parameters characterizing the diluteness of the instanton liquid, namely, the average instanton size $\bar{\rho}$ $\approx 1/3$ fm and average interinstanton distance $\bar{R} \approx 1$ fm. In particular, the average size of instantons is considered as a normalization point equal to $\bar{\rho}^{-1} \approx 0.6$ GeV. It implies that the results of the scale-dependent quantities from the model can be easily compared with those from other theoretical framework such as \chiPT and lattice QCD. These values of the $\bar{\rho}$ and $\bar{R}$ were determined many years ago by a variational method \cite{27, 28} as well as phenomenologically \cite{34, 35}. Note that these values were also confirmed within lattice QCD \cite{36, 37}. Reference \cite{38} simulated the QCD vacuum in the interacting instanton liquid model and derived $\bar{\rho} \approx 0.32$ fm and $\bar{R} \approx 0.76$ fm with the finite current quark mass taken into account.

When the finite current quark mass is considered, one needs to modify the E\chiA, which was performed by Musakhanov \cite{40, 41}. In particular, it is essential to include the current quark mass in the E\chiA, when one deals with properties of the kaon. It was shown in Ref. \cite{42} that the improved E\chiA explained very well the dependence of the
quark and gluon condensates on the current quark mass. Furthermore, the momentum-dependent dynamical quark mass brings about the breakdown of the Ward-Takahashi (WT) identities, that is, the current nonconservation\textsuperscript{44, 47}. Musakhanov and Kim\textsuperscript{29, 30} showed how the gauge-invariant $E\chi A$ can be constructed, which satisfies the WT identities. We will use this gauged $E\chi A$ in the present work to investigate the vector and axial-vector form factors for the radiative kaon decay.

The present work is outlined as follows: In Section II, we define one vector and two axial-vector form factors for the radiative kaon decay. In Section III, we explain the gauged $E\chi A$, focusing on the radiative kaon decay. In Section IV, we compute the vector and axial-vector form factors. In Section V, we first present the numerical results of the two dynamical quantities as functions of the mass of the strange current quark and discuss the effects of the flavor SU(3) symmetry breaking. Then we show the main results of the three form factors of the kaon. We also compare the numerical results with the various experimental data and those from $\chi$PT. In the final Section we summarize the results and draw conclusions.

**II. VECTOR AND AXIAL-VECTOR FORM FACTORS OF THE $K^+ \rightarrow l^+ \nu \gamma$ DECAY**

The SD part of the kaon radiative decay amplitude is expressed in terms of the vector and axial-vector form factors of the kaon, i.e., $F_V(q^2)$ and $F_A(q^2)$, and the second axial-vector form factor $R_A(q^2)$. The transition matrix elements of the vector and axial-vector currents are parametrized by these three form factors

\begin{equation}
\langle \gamma(k) | V_\mu^{45}(0) | K^+(p) \rangle = \frac{e}{m_K} \epsilon^{*\alpha} F_V(q^2) \epsilon_{\mu\rho\sigma} p^\rho k^\sigma, \quad (1)
\end{equation}

\begin{equation}
\langle \gamma(k) | A_\mu^{45}(0) | K^+(p) \rangle = i e \epsilon^{*\alpha} \sqrt{2} f_K \left[ -g_{\mu\alpha} + q_\mu q_\alpha + p_\alpha \right] \left( F_K(k^2) \frac{q^2 - m_K^2}{q^2 - m_K^2} \right) + i \epsilon^{*\alpha} \frac{e}{m_K} \left[ F_A(q^2) (k_\mu q_\alpha - g_{\mu\alpha} q \cdot k) + R_A(q^2) (k_\mu k_\alpha - g_{\mu\alpha} k^2) \right], \quad (2)
\end{equation}

where $|K^+(p)\rangle$ and $|\gamma(k)\rangle$ denote the initial kaon and the final photon states. The $\Delta S = 1$ transition vector and axial-vector currents are defined respectively as

\begin{equation}
V_\mu^{45} = \bar{\psi} \gamma_\mu \lambda^4 - i \lambda^5 \psi, \quad A_\mu^{45} = \bar{\psi} \gamma_\mu \gamma_5 \frac{\lambda^4 - i \lambda^5}{2} \psi, \quad (3)
\end{equation}

where $\psi = (u, d, s)$ represents the quark field, $\gamma_\mu$ and $\gamma_5$ the Dirac matrices, and $\lambda_i$ the flavor SU(3) Gell-Mann matrices. $p$ and $k$ stand for the momenta of the kaon and the photon respectively, whereas $q$ is the momentum of the lepton pair. The value of the kaon mass is taken from the experimental data $m_K = 493.68$ MeV. The second axial-vector form factor, $R_A(q^2)$ is considered only when the outgoing photon is virtual ($k^2 \neq 0$). $F_K(k^2)$ designates the electromagnetic (EM) form factor which becomes unity at $k^2 = 0$, i.e., $F_K(0) = 1$. Note that the EM charge radius $\langle r_s^2 \rangle$ and $F_K(k^2)$, were already computed in this model\textsuperscript{48}. In the present work, we concentrate on the derivation of the three form factors $F_V(q^2)$, $F_A(q^2)$, and $R_A(q^2)$ within the gauged $E\chi A$.

**III. GAUGED EFFECTIVE CHIRAL ACTION IN THE PRESENCE OF EXTERNAL FIELDS**

Since the EM, vector and axial-vector currents are involved in computing the form factors of kaon radiative decay, it is essential to preserve the relevant gauge invariance. So, we introduce the corresponding external fields into the $E\chi A$ as follows

\begin{equation}
S_{\text{eff}}[v^\text{em}, v, a, M] = -\text{Sp} \ln \left[ i \mathcal{D} + i \tilde{m} + i \sqrt{M f(i \mathcal{D}^L) U^{\gamma_5} \sqrt{M f(i \mathcal{D}^R)}} \right], \quad (4)
\end{equation}

where the functional trace $\text{Sp}$ runs over the space-time, color, flavor, and spin spaces. The current quark mass $\tilde{m}$ is written as $\text{diag}(m_u, m_d, m_s) = \tilde{m} 1 + m_3 \lambda_3 + m_8 \lambda_8$ with $\tilde{m} = (m_u + m_d + m_s)/3$ and $m_3 = (m_u - m_d)/2$ and $m_8 = (m_u + m_d - 2m_s)/2\sqrt{3}$. $\lambda_i$ denotes the $i$-th Gell-Mann matrix. Since isospin symmetry is assumed, $m_3 = 0$. The covariant derivative $\mathcal{D}_\mu$ is defined as

\begin{equation}
i \mathcal{D}_\mu = i \partial_\mu + e \tilde{Q} v^\text{em}_\mu + \frac{\lambda^a}{2} v^a_\mu + \gamma_5 \frac{\lambda^a}{2} a^a_\mu, \quad (5)
\end{equation}
where $v_{\mu}^e$, $v_{\mu}^\alpha$, and $a_{\mu}^\alpha$ denote the external EM field, vector field and axial-vector field, respectively. The charge operator $\hat{Q}$ for the quark fields is defined by

$$\hat{Q} = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} = \frac{1}{6} \lambda^3 - \frac{1}{6\sqrt{3}} \lambda^8.$$  

(6)

The left-handed and right-handed covariant derivatives in the momentum-dependent dynamical quark mass $M(i\mathcal{D}_{L,R})$ are defined respectively as

$$i\mathcal{D}_{\mu}^L = i\partial_{\mu} + e\hat{Q}v_{\mu}^e + \frac{\lambda^a}{2}v_{\mu}^\alpha - \gamma_5 \frac{\lambda^a}{2}a_{\mu}^\alpha; \quad i\mathcal{D}_{\mu}^R = i\partial_{\mu} + e\hat{Q}v_{\mu}^e + \frac{\lambda^a}{2}v_{\mu}^\alpha + \gamma_5 \frac{\lambda^a}{2}a_{\mu}^\alpha.$$  

(7)

The covariant derivatives ensure the gauge invariance of Eq. (4) in the presence of the external fields [29, 30]. The nonlinear pseudo-Nambu-Goldstone boson field is expressed as

$$U^{\gamma_5} = U(x)\frac{1 + \gamma_5}{2} + U^\dagger(x)\frac{1 - \gamma_5}{2} = \exp\left(\frac{\gamma_5}{f_{\pi}}\lambda^a\mathcal{M}^a\right),$$  

(8)

where $f_{\pi}$ denotes the pion decay constant and $\mathcal{M}^a$ represent the pseudoscalar meson fields which can be written as

$$\lambda^a\mathcal{M}^a = \sqrt{2}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta \\ \pi^- \\ K^- \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta \\ -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta \\ K^0 \end{pmatrix}.$$  

(9)

The momentum-dependent dynamical quark mass, which arises from the quark zero-modes of the Dirac equation in the instanton background fields, is expressed as

$$M^f(k) = M_0 F^2(k)f(m_f),$$  

(10)

where $M_0$ is the constituent quark mass at zero quark virtuality, and is determined to be around 350 MeV by the saddle-point equation [27, 28]. The form factor $F(k)$ arises from the Fourier transform of the quark zero-mode solution for the Dirac equation

$$F(k) = 2\tau \left[I_0(\tau)K_1(\tau) - I_1(\tau)K_0(\tau) - \frac{1}{\tau}I_1(\tau)K_1(\tau)\right],$$  

(11)

where $\tau \equiv |k|^2$. $I_i$ and $K_i$ designate the modified Bessel functions. For simplicity, we employ the dipole-type parametrization of $F(k)$ defined by

$$F(k) = \frac{2\Lambda^2}{2\Lambda^2 + k^2}.$$  

(12)

with $\Lambda = 1/\bar{\rho}$ instead of the modified Bessel functions. We already have shown in Ref. [51] that the results of the form factors for the radiative pion decay are not much changed by the dipole-type parametrization in place of the original form. Since $\rho = 0.33$ fm in the large $N_c$ limit, we take $\Lambda = 600$ MeV.

The presence of the current quark mass also affects the dynamical one, which was studied in Refs. [40, 42] in detail. The additional factor $f(m_f)$ describes the $m_f$ dependence of the dynamical quark mass, which is defined as [41, 43]

$$f(m_f) = \sqrt{1 + \frac{m_f^2}{d^2} - \frac{m_f}{d}}.$$  

(13)

This $m_f$-dependent dynamical quark mass produces the gluon condensate that does not depend on $m_f$ [43]. Pobylitsa considered the sum of all planar diagrams, expanding the quark propagator in the instanton background in the large $N_c$ limit [45]. Taking the limit of $N/(VN_c) \to 0$ leads to $f(m_f)$. The parameter $d$ is given by 198 MeV. The $m_f$-dependent dynamical quark mass also explains a correct hierarchy of the chiral quark condensates: $\langle \bar{u}u \rangle \approx \langle \bar{d}d \rangle > \langle \bar{s}s \rangle$ [43]. The explicit value of the strange current quark mass is actually not a free parameter. It can be determined by computing the kaonic two-point correlation function of the axial-vector currents and fixing the pole mass by the experimental...
data. However, the expression for the kaon mass will be rather complicated, in particular, when the quark mass is momentum-dependent. Instead, one can also use the relation \[19\]

\[
\frac{m_K^2}{m_n^2} = \frac{m_s + \bar{m}}{2\bar{m}},
\]

(14)

where \(\bar{m}\) is the average of the up and down current quark masses whose values are chosen as \(m_u = m_d = 5\text{ MeV}\). Though Eq. (14) is valid only at the leading order in the large \(N_c\) expansion and \(m_s\) expansion, we will adopt it to scrutinize the dependence of the form factors on the current quark mass for simplicity. The value of the strange current quark mass is set to be \(m_s = 120\text{ MeV}\), which satisfies Eq. (14) very well.

**IV. DERIVATION OF THE KAON FORM FACTORS**

The matrix elements of the vector and axial vector currents given in Eq.(2) are derived by computing the following three-point correlation function

\[
\langle \gamma(k)|W^a_\mu(0)|M^b(p)\rangle = \epsilon_\alpha^* \int d^4x e^{-ik\cdot x} \int d^4ye^{ip\cdot y} G^{-1}_\alpha(k) G^{-1}_M(p)(0)\{V_{\rho}(x)W^a_\mu(0)P^b(y)\}|0\rangle,
\]

(15)

where \(W^a_\mu\) denotes generically either the vector current or the axial-vector current defined in Eq.(3). The operators \(V_{\rho}\) and \(P^b\) in the correlation function correspond to the EM current, and the field operator for the kaon, respectively. \(G^{-1}_\alpha(k), G^{-1}_M(p)\) stand for the propagators of the photon and the meson, respectively. The matrix element (15) can be obtained by differentiating the gauged \(\chi A\) in Eq.(4) with respect to the external EM field, the vector (axial-vector) field, and the kaon field

\[
\langle \gamma(k)|W^a_\mu(0)|M^b(p)\rangle = \epsilon_\alpha^* \int d^4x e^{-ik\cdot x} \int d^4ye^{ip\cdot y} \frac{\delta^3 S_{\text{eff}}[v_{\rho\alpha}, w, M]}{\delta v_{\rho\alpha}(x)\delta w^a_\mu(0)\delta M^b(y)}\bigg|_{v_{\rho\alpha}, w, M=0},
\]

(16)

which results in five Feynman diagrams drawn in Fig. 1. The situation is very similar to the case of the pion. Only

\[\text{FIG. 1. The Feynman diagrams for the vector and axial-vector form factors for the radiative kaon decay. The dashed line depicts the kaon, the dashed double line and the wavy line describe the vector (axial-vector) field and the photon field, respectively. Diagram (a) contains both the local and nonlocal contributions, whereas diagrams (b)-(e) arise solely from the nonlocal interaction due to the momentum-dependent dynamical quark mass.}\]

\[\text{\footnotesize \textsuperscript{1} In the previous works \[48, 49\] within the same framework, the mass of the strange current quark is taken to be } m_s = 150\text{ MeV as a free parameter.}\]
diagram (a) contributes to the vector form factor, while all other diagrams vanish because of the trace over spin space. On the other hand, all the diagrams come into play when it comes to the axial-vector form factors. Diagram (a) includes both the local and nonlocal terms, while all other diagrams come only from the nonlocal terms with the derivatives of the momentum-dependent quark mass.

A. Vector form factor

We first derive the vector form factor of the kaon. Having computed Eq. (16) explicitly, we obtain the matrix element of the vector current \((W = V)\)

\[
\langle \gamma(k)|V_\mu^{45}|K^+(p)\rangle = i \frac{4\sqrt{2}eN_c}{3f_\pi} \epsilon^*_\alpha \int \frac{d^4l}{(2\pi)^4} \sqrt{M^u(k_a)|M^s(k_b)} \left[ \varepsilon_{\mu\rho\sigma} \left( M^u_{a}k_{b\rho}k_{c\sigma} + M^s_{c}k_{b\rho}k_{a\sigma} + M^s_{c}k_{a\rho}k_{b\sigma} \right) \\
+ \varepsilon_{\alpha\beta\rho\sigma}k_{a\beta}k_{b\rho}k_{c\sigma} \left( \sqrt{M^u(k_a)}\sqrt{M^u(k_b)} + \sqrt{M^s(k_c)}\sqrt{M^s(k_c)} \right) \\
- \varepsilon_{\mu\beta\rho\sigma}k_{a\beta}k_{b\rho}k_{c\sigma} \left( \sqrt{M^s(k_b)}\sqrt{M^s(k_a)} + \sqrt{M^s(k_c)}\sqrt{M^s(k_c)} \right) \right] - 2(u \leftrightarrow s), \tag{17}
\]

where \(N_c\) is the number of colors and \(M^d_f\) is defined by the sum of the dynamical and current quark masses \(M_f^d = m_f + \Delta F(k_i)\). The momenta \(k_i\) are expressed as \(k_a = l + \frac{q}{2} + \frac{k}{2}, \ k_b = l - \frac{q}{2} - \frac{k}{2},\) \(k_c = l - \frac{q}{2} + \frac{k}{2}\), and \(q = p - k\). \(\Delta F_f^d\) are given as \(\Delta F_f^d = (k_i^2 + \Delta F^d_f)\). \(\sqrt{M_f^d(k_i)}\) represents \(\sqrt{M_f^d(k_i)} = \partial \sqrt{M_f^d(k_i)} / \partial k_i\). Equation (17) corresponds to diagram (a) in Fig. 1, and diagrams (b)-(e) do not contribute at all to the vector form factor. Using the transverse condition \(\epsilon^*_\alpha \cdot p = \epsilon \cdot p = 0\), we are able to extract the vector form factor, comparing Eq. (1) with Eq. (17). The final expression of the vector form factor is written as

\[
F_V(Q^2) = F^\text{local}_V(Q^2) + F^\text{NL}_V(Q^2), \tag{18}
\]

where \(F^\text{local}_V(Q^2)\) and \(F^\text{NL}_V(Q^2)\) stand for the local and nonlocal contributions respectively

\[
F^\text{local}_V(Q^2) = \frac{4\sqrt{2}M_K}{f_\pi} G^\text{local}_V(Q^2), \quad F^\text{NL}_V(Q^2) = \frac{4\sqrt{2}M_K}{f_\pi} G^\text{NL}_V(Q^2). \tag{19}
\]

Here, \(G^\text{local}_V(Q^2)\) and \(G^\text{NL}_V(Q^2)\) are defined respectively by

\[
G^\text{local}_V(Q^2) = -\frac{N_c}{3(p \cdot k)^2} \int \frac{d^4l}{(2\pi)^4} \sqrt{M^u(k_a)|M^s(k_b)} \left[ M^u_{a}(k_{b\rho}k_{c\sigma} - k_{c\rho}k_{b\sigma}) \\
+ \overline{M}_b(k_{c\rho}k_{a\sigma} - k_{a\rho}k_{c\sigma}) + \overline{M}_s^{-}(k_{b\rho}k_{c\sigma} - k_{b\rho}k_{c\sigma}) \right] - 2(u \leftrightarrow s), \tag{20}
\]

\[
G^\text{NL}_V(Q^2) = -\frac{4\sqrt{2}N_cM_K}{3f_\pi(p \cdot k)^2} \int \frac{d^4l}{(2\pi)^4} \sqrt{M^u(k_a)|M^s(k_b)} \left[ - (M_{a}^{u} + M_{a}^{s}) (p \cdot k)^2 (\epsilon^*_\alpha \cdot l)(\epsilon \cdot l) \\
+ (M_{a}^{u} + M_{a}^{s}) (\varepsilon_{\mu\nu\rho\sigma}(l\cdot p)k_{\mu\nu\rho\sigma}) \right] - 2(u \leftrightarrow s), \tag{21}
\]

where \(M_{a}^{f}\) denotes the derivative of the dynamical quark mass with respect to the squared momentum, \(M_{a}^{f} = \partial M_f^d(k_i) / \partial k_i^2\). We use the positive definite \(Q^2\) defined as \(Q^2 = -q^2\). In fact, \(G^\text{local}_V(Q^2)\) and \(G^\text{NL}_V(Q^2)\) are exactly same as those for the pion form factors given explicitly in Ref. [33] except for the current mass of the strange quark. Thus, the results of \(G^\text{local}_V(Q^2)\) and \(G^\text{NL}_V(Q^2)\) will exhibit how the vector form factor gets changed, if one considers the mass of the strange current quark as a parameter.

B. Axial-vector form factors

Similarly, the transition matrix element of the axial-vector current \((W = A)\) in Eq. (2) can be obtained as

\[
\langle \gamma(k)|A_\mu^{45}|K^+(p)\rangle = -\frac{4\sqrt{2}eN_c}{3f_\pi} \epsilon^*_\alpha \int \frac{d^4l}{(2\pi)^4} \sum_{i \in \alpha} \left( \left( f_{\mu\alpha}^{(i)} + 2(u \leftrightarrow s) \right) + \left( \tilde{f}_{\mu\alpha}^{(i)} - (u \leftrightarrow s) \right) \right), \tag{22}
\]
where $F^a_{\mu\alpha}$ and $\tilde{F}^a_{\mu\alpha}$ correspond to diagram (i) in which $i = a, b, c, d, e$. Explicitly, they can be written as

$$
F^a_{\mu\alpha} = \frac{\sqrt{\bar{M}^u(k_a)M^s(k_b)}}{D^u_a D^s_b} \left[ \delta_{\mu\alpha} \left\{ \bar{M}^u_{\mu} k_b \cdot k_c - \bar{M}^u_{\mu} k_c \cdot k_a + \bar{M}^s_{\mu} k_a \cdot k_b + \bar{M}^u_{\mu} \right\} ight. \\
+ \left\{ -\bar{M}^u_{\mu} (k_a k_c + k_a k_b) + \bar{M}^s_{\mu} (k_a k_c + k_b k_a) + \bar{M}^s_{\mu} (k_a k_b - k_b k_a) \right\} \\
+ \left( M^u_{\mu} k_{\alpha} + M^s_{\mu} k_{\alpha} \right) \left\{ -(k_b \cdot k_c - \bar{M}^u_{\mu} k_b + \bar{M}^u_{\mu} k_c) \right\} k_{\mu} \\
- \left( M^u_{\mu} k_{\alpha} - M^s_{\mu} k_{\alpha} \right) \left\{ (k_b \cdot k_c + \bar{M}^u_{\mu} k_b - \bar{M}^u_{\mu} k_c) \right\} k_{\mu} \\
+ \left( M^s_{\mu} k_{\alpha} - M^s_{\mu} k_{\alpha} \right) \left\{ (M^u_{\mu} k_{\alpha} - M^u_{\mu} k_{\alpha}) \right\} k_{\mu} \\
\left. \right\} \left( \bar{M}^u_{\mu} k_b \cdot k_c - \bar{M}^u_{\mu} k_c \cdot k_a + \bar{M}^u_{\mu} k_a \cdot k_b - \bar{M}^u_{\mu} \right),
$$

and

$$
\tilde{F}^a_{\mu\alpha} = 0,
$$

$$
F^a_{\mu\alpha} = -\frac{M^u(k_b)}{2D^u_a D^s_b} \sqrt{\bar{M}^u(k_a)M^s(k_b)} (k_a \cdot k_b + \bar{M}^u),
$$

$$
\tilde{F}^a_{\mu\alpha} = -\frac{M^u(k_b)}{2D^u_a D^s_b} \sqrt{\bar{M}^u(k_a)M^s(k_b)}. \tag{24}
$$

Here, we have introduced some short-handed notations $\bar{M}^{f_1 f_2} = \bar{M}^{f_1} M^{f_2}$, $\bar{M}^{f_1 f_2 f_3} = \bar{M}^{f_1} M^{f_2} M^{f_3}$, and $\sqrt{M^{f_\mu}(k_i)} = \partial^2 \sqrt{M(k_i)} / \partial k_{\mu} \partial k_{i\alpha}$.

As done in the case of the pion axial-vector form factors, the kaon axial-vector form factors can be easily obtained by introducing an arbitrary vector $\xi^\perp$ that satisfies the following properties: $\xi^\perp \cdot \xi^\perp = 0$, $\xi^\perp \cdot q = 0$, and $\xi^\perp \cdot k \neq 0$. Hence, the axial-vector form factors $F_A(Q^2)$ and $R_A(Q^2)$ are obtained as

$$
F^A_K(Q^2) = \frac{4\sqrt{2}m_K}{f_\pi} G_A(Q^2), \quad R^A_K(Q^2) = \frac{4\sqrt{2}m_K}{f_\pi} H_A(Q^2) \tag{25}
$$

where

$$
G_A(Q^2) = \frac{N_c}{3(q \cdot k)^2} \int \frac{d^4 l}{(2\pi)^4} \sum_{\alpha = 0}^{\infty} \left[ \left( F^{(i)}_{\mu\alpha} + 2u \leftrightarrow s \right) \right] \frac{\xi^\perp_{\mu} k_{\alpha}}{\xi^\perp \cdot k - \epsilon_{\mu} \epsilon_{\alpha}}, \tag{26}
$$

$$
H_A(Q^2) = \frac{N_c}{3(\xi^\perp \cdot k)^2} \int \frac{d^4 l}{(2\pi)^4} \sum_{\alpha = 0}^{\infty} \left[ \left( F^{(i)}_{\mu\alpha} + 2u \leftrightarrow s \right) \right] \frac{\xi^\perp_{\mu} \xi^\perp_{\alpha}}{\xi^\perp \cdot k}. \tag{27}
$$

The local contribution to the axial-vector form factors comes from the first and second terms of $F^{(a)}_{\mu\alpha}$ in Eq. (23). In the flavor SU(3) symmetric case, that is, $m_u = m_d = m_s$, $\tilde{F}^{(d,e)}_{\mu\alpha}$ would have been exactly canceled by the exchange terms expressed as $(u \leftrightarrow s)$. This will lead to the same expressions of the form factors for the radiative pion decay [51]. Thus, $\tilde{F}^{(d,e)}_{\mu\alpha}$ become finite only in the case of radiative kaon decays.\(^2\)

\(^2\) When isospin symmetry breaking is taken into account, $\tilde{F}^{(d,e)}_{\mu\alpha}$ contribute also to the pion axial-vector form factors, though their effects are tiny.
V. RESULTS AND DISCUSSION

In this Section, we begin by examining the effects of the flavor SU(3) symmetry breaking. Before we discuss the results, we want to emphasize that the nonlocal $E_A$ does not contain any additional parameters to fit. The average size of the instanton $\rho = 4^{-1}$ and the interdistance between instantons $R$ were taken from the original work [27]. The dynamical quark mass at the zero quark quark virtuality was determined by the saddle-point equation [27]. The strange current quark mass was also fixed by the leading-order mass relation in the large $N_c$ limit and in the current quark mass expansion. Thus, there is no fitting procedure in the present work.

The expressions of both the pion and kaon form factors contain the prefactors $m_\pi/f_\pi \approx 1.5$ and $m_K/f_\pi \approx 5.3$ respectively, which makes a large difference in the magnitudes of the pion and kaon form factors. If we factor out these kinematical factors and release the value of the strange current quark mass from these kinematical factors and release the value of the strange current quark mass, respectively, which makes a large difference in the magnitudes of the pion and kaon form factors. If we factor out these kinematical factors and release the value of the strange current quark mass from $m_s = 120$ MeV, then we can more closely explore the effects of the flavor SU(3) symmetry breaking. Thus, we first compute $G_V$, $G_A$, and $H_A$ defined in Eqs. (20), (21), (26), and (27) respectively as functions of the variable strange current quark mass which is denoted by $m$ between the nominal values of up and strange current quark masses. Note that in doing so, we fix the up or down current quark masses fixed at 5 MeV.

![Numerical results of $G_V(0)$, $G_A(0)$, and $H_A(0)$ multiplied by $10^3$ as functions of the variable current quark mass $m$.](image)

FIG. 2. Numerical results of $G_V(0)$, $G_A(0)$, and $H_A(0)$ multiplied by $10^3$ as functions of the variable current quark mass $m$ drawn in the left, middle, and right panels, respectively. The dashed and dot-dashed lines depict the local and nonlocal contributions, respectively. The solid line represents the total contribution.

Figure 2 draws the numerical results of $G_V(0)$, $G_A(0)$, and $H_A(0)$ as functions of the variable current quark mass $m$. In the case of $G_V$, the local contribution increases monotonically as $m$ increases. The magnitude of the nonlocal part gets only slightly larger when $m$ increases. Hence, the $m$ dependence is governed by the local terms. $H_A$ depicted in the right panel of Fig. 2 shows a similar tendency to $G_V$. On the other hand, $G_A$ decreases as $m$ increases. As displayed in Fig. 2, the effects of the SU(3) symmetry breaking increase the vector form factor $F_V(0)$ and the second axial-vector form factor $R_A(0)$ for the radiative kaon decay by about 16% and 19%, compared to the corresponding pion form factors. On the other hand, they lessen the axial-vector form factor $F_A(0)$ for the radiative kaon decay in comparison with the corresponding pion axial-vector form factor. These kinds of tendencies of $G_V(0)$, $G_A(0)$, and $H_A(0)$ determine the pion and kaon weak form factors at $Q^2 = 0$ with corresponding current quark masses, $m = 5$ MeV and $m = 120$ MeV.

In Table I we list the results of the form factors at $Q^2 = 0$, various combinations of them, and slope parameters $a_V$ and $a_A$, $a_A$ in (26), and (27) respectively as functions of the variable strange current quark mass $m$. The slope parameters are defined from the following parametrizations of the vector and axial-vector form factors for the radiative kaon decay

$$F_V(Q^2) = \frac{F_V(0)}{1 + a_V Q^2 m_K^2}, \quad F_A(Q^2) = \frac{F_A(0)}{1 + a_A Q^2 m_K^2} \quad m = 120 \text{ MeV}$$

where $a_V$ and $a_A$ denote the slope parameters for the vector and axial-vector form factors, respectively.

Generally, the present numerical results are in very good agreement with the experimental data taken from Ref. [15], where kaon radiative decays $K^+ \rightarrow \mu^+\nu e^-\bar{e}^-$ and $K^+ \rightarrow e^+\nu e^-\bar{e}^-$ were experimentally studied. The experimental data presented in the third column of Table I are those from the combined fit including both radiative decays

\[ 15 \to 17 \to 21 \]
TABLE I. Results of the form factors at $Q^2 = 0$ and slope parameters in comparison with those from CHPT to order $O(p^0)$ and the experimental data.

| CHPT $p^0$ | $K \to e(\mu)\nu e^{-}$ | Experimental data | Present results |
|------------|-------------------------|------------------|----------------|
| $F_V(0)$  | 0.078(5) 0.112(28) 15  | $K \to e\nu\gamma$ | 0.114 |
| $F_A(0)$  | 0.034 0.035(30)         | $K \to \mu\nu\gamma$ | 0.033 |
| $R_A(0)$  | 0.227(32)           |                  | 0.20 |
| $F_V(0) + F_A(0)$ | 0.112(5) 0.147(40) | The experimental data indicate consistently that the decay $K \to \mu\nu\gamma$ yields the larger values of $F_V + F_A$ than the experimental channel $K \to e\nu\gamma$. For example, Ref. 15 reported the mean value of $F_V + F_A = 0.155$ from the $K^+ \to \mu^+\nu e^-e^-$ data whereas $F_V + F_A = 0.125$ from the $K^+ \to e^+\nu e^-e^-$ data and the weighted average value is in TABLE I. The results of $F_V - F_A$ show similar tendencies. The comparison of the KLOE data 14 with those of the E787 Experiment 13 leads to the same conclusion. The data on $K^- \to \mu^-\nu\gamma$ from the ISTRA+ Collaboration 17 gives a rather large value of $F_V - F_A = 0.21$ that is almost three times larger than that from Ref. 14. Another analysis from the ISTRA+ Collaboration yields $F_V - F_A = 0.126$ with the exotic tensor interaction excluded 18. The present results lie between those from the $\mu$ and electron channels. In general, the results from $O(p^0)$ CHPT are underestimated, compared with the present ones except for $F_A$ of which the value is almost the same as our result. |
| $F_V(0) - F_A(0)$  | 0.044(5) 0.077(45)     |                  | 0.147 |
| $R_A(0) + F_V(0)$  | 0.338(45) |                  | 0.081 |
| $R_A(0) - F_V(0)$  | 0.114(42) |                  | 0.314 |
| $R_A(0) + F_A(0)$  | 0.262(21) |                  | 0.086 |
| $R_A(0) - F_A(0)$  | 0.191(61) |                  | 0.233 |
| $a_V$ | 0.3(1) | $K \to e\nu\gamma$ | 0.167 |
| $a_A$ | 0.38(4) 16 | $K \to \mu\nu\gamma$ | 0.383 |
|           |           |                  | 0.194 |

$K^+ \to \mu^+\nu e^-e^-$ and $K^+ \to e^+\nu e^-e^-$ 15. Experimentally, more plausible quantities are $F_V(0) + F_A(0)$ and $F_V(0) - F_A(0)$. The experimental data indicate consistently that the decay $K \to \mu\nu\gamma$ yields the larger values of $F_V + F_A$ than the electron channel $K \to e\nu\gamma$. For example, Ref. 15 reported the mean value of $F_V + F_A = 0.155$ from the $K^+ \to \mu^+\nu e^-e^-$ data whereas $F_V + F_A = 0.125$ from the $K^+ \to e^+\nu e^-e^-$ data and the weighted average value is in TABLE I. The results of $F_V - F_A$ show similar tendencies. The comparison of the KLOE data 14 with those of the E787 Experiment 13 leads to the same conclusion. The data on $K^- \to \mu^-\nu\gamma$ from the ISTRA+ Collaboration 17 gives a rather large value of $F_V - F_A = 0.21$ that is almost three times larger than that from Ref. 14. Another analysis from the ISTRA+ Collaboration yields $F_V - F_A = 0.126$ with the exotic tensor interaction excluded 18. The present results lie between those from the $\mu$ and electron channels. In general, the results from $O(p^0)$ CHPT are underestimated, compared with the present ones except for $F_A$ of which the value is almost the same as our result.

The vector slope parameter $a_V$ is experimentally known to be $a_V = 0.38(4)$ from Ref. 16 where as $a_A$ is still at large experimentally. The present result of $a_V$ is 0.383 that is in very good agreement with the KLOE data. We predict $a_A = 0.194$.

In Fig. 3, we show the numerical results of the vector and axial-vector form factors of the kaon radiative decays. All these three form factors fall off monotonically as $Q^2$ increases. In fact, the results of the form factors for the radiative kaon decays show the same tendency as those for the radiative pion decays 51, since the expressions of the form factors are the same as those for the pion decays except for the strange current quark mass, as already discussed in Fig. 2. Nevertheless let us recapitulate briefly what we have found. The nonlocal terms appear from the gauged

\[ F_{V}(0) = 0.078(5) \quad \text{and} \quad F_{A}(0) = 0.034. \]**
E\(\chi\)A that was constructed in such a way that the relevant gauge invariance is preserved. In the left panel of Fig. 3, we find that the nonlocal contribution enhances the vector form factor by almost about 20%. On the other hand, the nonlocal terms reduce the axial-vector form factors almost by 50%. It implies that it is essential to preserve the gauge invariance not only theoretically but also quantitatively. The nonlocal contribution turns out to be marginal to the second axial-vector form factor.

Figure 4 illustrates the 3\(\sigma\)-ellipse taken from Fig. 22 of Ref. [17], where \(F_V(0)\) is fixed from \(\mathcal{O}(p^6)\) \(\chi\)PT and \(a_V\) and \(F_A\) are regarded as fitting parameters. We can put the present results \(a_V = 0.383, F_A = 0.033\) as a red blob in Fig. 4. Interestingly, the red blob turns out to be slightly higher than that of \(\mathcal{O}(p^6)\) \(\chi\)PT, which indicates that the present ones are closer to the 3\(\sigma\) boundary. However, note that the value of \(F_V\) obtained in the present work is \(F_V = 0.114\) as shown in Table II which is almost the same as the experimental data given in Ref. [15].

Finally, we extract the parameters for the \(p\)-pole parametrizations of the vector and axial-vector form factors for the kaon radiative decays. In lattice QCD, the \(p\)-pole parametrization for a form factor is often introduced to fit various lattice data [52, 53]. Although there is no the results from lattice QCD yet, it will be useful to provide the parameters here so that one can easily compare the present results with those of lattice QCD in near future. The \(p\)-pole parametrizations of the vector and axial-vector form factors are expressed as

\[
F_V(Q^2) = \frac{F_V(0)}{\left(1 + \frac{Q^2}{M_{pV}^2}\right)^{p_V}}, \quad F_A(Q^2) = \frac{F_A(0)}{\left(1 + \frac{Q^2}{M_{pA}^2}\right)^{p_A}}, \quad R_A(Q^2) = \frac{R_A(0)}{\left(1 + \frac{Q^2}{M_{pR}^2}\right)^{p_R}},
\]

where the results of the parameters \(p_V, p_A, p_R, M_{pA}, M_{pA}, M_{pR}\) are listed in Table II.

### Table II. The results of the \(p\)-pole parameters.

| \(p_V\)  | \(M_{pV}\)  | \(p_A\)  | \(M_{pA}\)  | \(p_R\)  | \(M_{pR}\)  |
|----------|-------------|----------|-------------|----------|-------------|
| Present work | 1.26  | 0.832 GeV | 1.20  | 1.15 GeV | 0.779  | 1.07 GeV   |
VI. SUMMARY AND CONCLUSION

In the present work, we investigated the vector and axial-vector form factors for kaon radiative decays within the framework of the gauged nonlocal effective chiral action, which constitute the essential part of the structure-dependent decay amplitude. We scrutinized the effects of the flavor SU(3) symmetry breaking, releasing the strange current quark mass from its fixed value $m_s = 120$ MeV. The results showed how the vector and axial-vector form factors undergo changes when the current quark mass $m$ is varied. We found that the vector form factor and the second axial-vector form factor increase monotonically as $m$ increases. On the other hand, the axial-vector form factor lessens as $m$ increases.

The numerical results of the form factors are in good agreement with the experimental data. In general, the experimental data extracted from the radiative decay of the kaon to the electron are smaller than those from the radiative pion decay to the muon. The present results are found to lie between the data taken from the electron and muon channels. The slope parameter for the axial-vector form factor was predicted. The $Q^2$ dependences of all the three form factors were presented and the general tendency is almost the same as in the case of pion radiative decay. The nonlocal contributions enhance the vector form factor while they reduce the axial-vector one. However, their effects are marginal on the second axial-vector form factors. We compared the present results of the vector slope parameter and the axial-vector form factor with the $3\sigma$-ellipse taken from the ISTRA+ Collaboration. Finally, we provided the parameters for the $p$-pole parametrization of the vector and axial-vector form factors.

In the present work, we concentrated only on the vector and axial-vector form factors for kaon radiative decays. However, it is of great importance to consider the tensor form factors, though they must be small experimentally. Since the nonlocal chiral quark model from the instanton vacuum is a well-defined theoretical framework and furthermore it does not have any additional free parameter to handle, it is very interesting to consider the tensor form factors for the kaon radiative decay. There are at least two important physical implications on them. Firstly, it offers a possible new physics beyond the standard model, in particular, related to dark photons. Secondly, the transition tensor form factors allow one to examine the spin structure of the kaon in the course of its radiative decay. The relevant works are under way.

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