Scaling properties of one-dimensional Anderson models in an electric field: Exponential vs. factorial localization

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We investigate the scaling properties of eigenstates of a one-dimensional (1D) Anderson model in the presence of a constant electric field. The states show a transition from exponential to factorial localization. For infinite systems this transition can be described by a simple scaling law based on a single parameter $\lambda_{\infty} = l_{\infty}/l_4$, the ratio between the Anderson localization length $l_{\infty}$ and the Stark localization length $l_4$. For finite samples, however, the system size $N$ enters the problem as a third parameter. In that case the global structure of eigenstates is uniquely determined by two scaling parameters $\lambda_N = l_{\infty}/N$ and $\lambda_{\infty} = l_{\infty}/l_4$.

I. INTRODUCTION

In recent years several studies have investigated the influence of constant uniform electric fields on the localization of electrons in one-dimensional (1D) systems with on-site randomness. In the absence of dc-fields it is by now well known that even small amounts of disorder lead to an exponential localization of all eigenstates. On the other hand, in the case of a single-orbital tight-binding model of an electron in a periodic potential, application of a static electric field is known to generate a discrete, uniformly spaced eigenvalue spectrum, known as Stark localization and has been observed experimentally in superlattices, which are commonly used for such measurements.

In infinite samples the localization of eigenstates can be characterized in terms of the localization length; the latter is commonly defined from the amplitude decay of eigenstates in the limit $|n| \rightarrow \infty$, where $n$ labels the site in the tight-binding picture. The most powerful and informative method available for such studies is the transfer matrix method. In the presence of a strong electric field, however, it appears to be less efficient due to the factorial nature of the Stark localization. Moreover, for finite systems the structure of eigenvectors cannot be characterized in the same way. One then needs to use other quantities (such as the inverse participation ratio), that are valid both for finite and infinite samples. Through the use of scaling conjectures, one can link then the properties of eigenstates in infinite samples to those of finite samples. Since the scaling approach proved to be extremely useful in describing conductance and its fluctuations (see, e.g., [13]) in the theory of disordered solids, it seems natural to use this approach also in order to describe localization properties of eigenfunctions of 1D disordered systems in the presence of constant electric field.

In this paper we study the 1D Anderson model in the presence of a constant electric field in view of scaling properties of its eigenstates. The main question that we want to answer is whether the up to now known equivalence between quasi-1D and 1D disordered models can be extended in order to include also systems with constant electric field. For this purpose we analyze the scaling properties of information lengths for infinite and finite samples, which were used successfully in the studies of one and quasi-one dimensional systems. Contrary to the standard Anderson case, where the ratio of the Anderson localization length $l_{\infty}$ and the sample size $N$, i.e. $\lambda_N = l_{\infty}/N$, is the only relevant scaling parameter, we find in the present case an additional scaling parameter $\lambda_{\infty} = l_{\infty}/l_4$. Here $l_4$ is the Stark localization length, which arises from the applied constant electric field. Hence, the structure of eigenvectors in our model is characterized by two scaling parameters $\lambda_N$, $\lambda_{\infty}$.

The structure of the paper is as follows. In Section 2 we describe the mathematical model and briefly summarize the known facts about the two limiting cases that appear for our model. In Section 3 we discuss different definitions of localization length, which are used in our numerical simulations. In Section 4 we present numerical data on scaling of localization lengths of eigenstates in infinite and finite systems. Finally, in Section 5 we study the scaling of the whole distribution of eigenvectors. Our conclusions are summarized in Section 6.

II. THE MODEL

Our starting point is the 1D Schrödinger equation in the tight-binding approximation
\[
\frac{d\psi_n(t)}{dt} = (\epsilon_n + neF)\psi_n(t) + V\psi_{n+1}(t) + V\psi_{n-1}(t),
\]  

(1)

where \(\psi_n(t)\) denotes the probability amplitude for an electron to be at site \(n\). Moreover, \(\epsilon_n\) is the local site energy, \(V\) is the hopping element, \(e\) is the electron charge and \(F\) the strength of the applied dc field. By applying the transformation \(\psi_n(t) = \exp(-iEt)\varphi_n\) one obtains the stationary equation

\[
E\varphi_n = V\varphi_{n+1} + (\epsilon_n + neF)\varphi_n + V\varphi_{n-1},
\]  

(2)

for the eigenvalues \(E\) and the corresponding eigenstates \(\varphi_n(E)\). We can distinguish two limiting cases which are relevant for our study: (a) perfect system (i.e. \(\epsilon_n = \epsilon\)) with non-zero electric field \(F \neq 0\), and (b) zero field (i.e. \(F = 0\)) with random on-site potential.

In the case of a perfect system with \(F \neq 0\), one can prove that the corresponding eigenstates \(\varphi(E)\), known as Wannier-Stark states, show a generic factorial decay i.e.\[ \varphi_n(E) = J_{m-n}(2/eF) \to (1/eF)^{|n|}/(|n|!); n \to \pm \infty \]  

(3)

where \(J_n\) is the Bessel function of order \(n\). Wannier-Stark states constitute a complete set of energy eigenstates. Their eigenvalues \(E_m = meF \) form the so-called Wannier-Stark ladder. A particular Wannier-Stark state \(\varphi_n\) is factorially localized around the \(n\)-th site, with a localization length of the order of \(l_{\lambda} \approx 1/eF\), i.e. the electric field appears in the denominator of the localization length in Eq. (3). This underscores the fact that \(F\) cannot be treated as a small perturbation to the field-free Hamiltonian. An example of such a state is presented in Fig. 1a.

The other limit of interest corresponds to zero electric field with \(\epsilon_n\) random and \(\delta\)-correlated, chosen from a distribution \(P_\delta\) with mean zero and variance \(\sigma^2\). Below, in our numerical investigation we will always assume that \(P_\delta\) is a uniform distribution in \([-W/2, W/2]\). Such a model is known in the literature as the Anderson model and has been studied in great detail in the context of disordered materials. It was shown with mathematical rigor that in the limit of infinite samples this model displays exponentially localized eigenfunctions, no matter how small the disorder is (see Fig. 1b). The rate of decay is measured by the Lyapunov exponent \(\gamma\) which may be evaluated by the transfer matrix method. To this end, one has to study the asymptotic behavior of the random matrix product \(\prod M_n\), where \(M_n\) is defined through the relation

\[
\xi_{n+1} = M_n\xi_n; \quad M_n = \left(\begin{array}{cc} v_n & -1 \\ 1 & 0 \end{array} \right); \quad v_n = \frac{E - \epsilon_n}{V} \]  

(4)

for the vector \(\xi_n = (x_n, x_{n-1})\) with the matrix \(M_n\) known as the transfer matrix. The localization length \(l_\infty\) is hence the inverse Lyapunov exponent \(\gamma\); the latter is evaluated as the exponential rate of increase of an initial vector \(\xi_1\),

\[
l_\infty^{-1} = \gamma = \lim_{N \to \infty} \frac{1}{N} \ln \left( \prod_{n=1}^{N} |M_n\xi_n| \right) / |\xi_1|. \]  

(5)

Although the Lyapunov exponent \(\gamma\) for finite \(N\) depends on the particular realization of disorder, for \(N \to \infty\) it converges to its mean value. For the calculations below we have used samples of length \(5 \times 10^5\) for relatively large values of \(W\) and up to \(4 \times 10^6\) for small values of \(W\).

### III. SCALING APPROACH FOR THE EIGENSTATES

Our interest is dedicated to the structure of eigenstates for infinite as well as finite samples, as we tune the disorder and the electric field strength. Unlike the simpler case of infinite samples, however, the meaning of a localization length for finite samples is not clear. Below we follow the approach developed in the theory of quasi-1D disordered solids which is based on the evaluation of multifractal localization lengths (see, e.g.,\[ \text{[1]} \]). The great advantage of this approach is the applicability in both finite and infinite samples.

One of the commonly used quantities in this approach is the so-called entropic localization length, defined through the information entropy \(\mathcal{H}_N\) of eigenstates,

\[
\mathcal{H}_N = - \sum_{n=1}^{N} w_n \ln w_n; \quad w_n = |\varphi_n|^2 \]  

(6)

where \(\varphi_n\) is the \(n\)-th component of an eigenstate in a given basis. For eigenstates normalized as \(\sum_n w_n = 1\),
the simplest case of \( \varphi_n = N^{-1/2} \) results in an entropy equal to the maximum value: \( H_N = \ln(N) \). Therefore, we define the localization length \( L \) as the number of basis states occupied by the eigenstate \( \varphi_n \); the latter is equal to \( \exp(H_N) \). We note that in general, the amplitudes \( \varphi_n \) fluctuate strongly with \( n \) and thus the coefficient of proportionality between \( L \) and \( l_\infty \) depends on the type of fluctuations.

In order to study the properties of eigenstates in quasi-1D solids, localized on some scale in the finite basis, it was suggested in [4] to normalize the localization length \( L \) in such a way that in the extreme case of fully extended states the quantity \( L \) is equal to the size of the basis \( N \). In such an approach, the entropic localization length \( L_1 \) is defined as

\[
L_1 = N \exp(<H_N> - H_{\text{ref}}) \tag{7}
\]

In Eq. (7) the ensemble average \(< ... >\) is performed over the number of eigenstates with the same structure and over realizations of the disorder potential. The normalization factor \( H_{\text{ref}} \) has the meaning of an average entropy of the completely extended random eigenstates in the finite basis. For the quasi-1D case the distribution of amplitudes \( \varphi_n \) is assumed to correspond to the Gaussian Orthogonal Ensemble (GOE) [5].

Analogously, a whole set of localization lengths \( L_q \) can be defined in the following way [6]:

\[
L_q = N \left( \frac{\langle P_q \rangle}{P_{q(q)}^{\text{ref}}} \right)^{1/q} ; P_q = \sum_{n=1}^{N} (w_n)^q ; q \geq 2 \tag{8}
\]

where \( P_{q(q)}^{\text{ref}} \) is the average value of \( P_q \) for the reference ensemble of completely extended states. One should note that for the particular case \( q = 2 \) the quantity \( P_q \) is known as the participation ratio.

In the context of quasi-1D disordered models in the presence of constant electric field [6], it was shown that all global properties of eigenfunctions are described by the following localization parameters

\[
\beta^\infty_q = \frac{L_\infty}{l_{\text{el}}} ; \beta^N_q = \frac{L_q}{N} \tag{9}
\]

where the superscript \( \infty \) (\( N \)) denotes infinite (finite) samples, respectively. Moreover it was found that \( \beta^{\infty,N}_q \) obey some scaling law, i.e. they depend only on the ratio of the characteristic lengths of the system. In the case of infinite samples only two length scales, i.e. \( l_\infty \) and \( l_{\text{el}} \), are relevant. For finite samples, however, a third length \( N \), which is the actual size of the sample, comes into play and has to be taken into account in the scaling theory. According to the scaling conjecture in the modern theory of disordered solids, it was found that for quasi-1D systems in the presence of electric field \( F \), the \( \beta^{\infty,N}_q \) follow the scaling laws:

\[
\beta^\infty_q = \beta^\infty_q (\lambda_\infty) ; \beta^N_q = \beta^N_q (\lambda_\infty, \lambda_N),
\]

where \( \lambda_\infty = \frac{l_\infty}{l_{\text{el}}} ; \lambda_N = \frac{l_\infty}{N} \tag{10} \)

Our main question is whether relations of the type of Eq. (10) are also applicable for our 1D Anderson tight-binding model with electric field. In Ref. [4] it was shown analytically that the eigenstates in 1D and quasi-1D disordered systems \textit{without} electric field, possess the same gross structure (envelope) on scales comparable with the localization length, while their statistical properties are quite different on a much finer scale of the order of the lattice spacing. That is the reason why many scaling laws, which are dominated by the fluctuations of the envelope, hold both for 1D and quasi-1D. The validity of such a statement is however questionable in the presence of electric fields. We will show that such a similarity between quasi-1D and 1D disordered systems persist also in this case.

The first nontrivial question in this context arises about the reference ensemble for the computation of the average entropy \( H_{\text{ref}} \). Indeed, in application to 1D Anderson type models [4], the reference ensemble cannot be chosen as an ensemble of full random matrices, like the GOE. This point is related to the fact that in the 1D tight-binding case fully extended states are not Gaussian random functions but just plane waves which arise for zero disorder. In the presence of electric field, the situation is even more complicated due to strong dependence of the eigenstates on the electric field. However, and this is our expectation, in spite of the above mentioned differences, scaling properties of the eigenstates of the 1D model [4] are of the generic type discovered for quasi-1D systems.

For this reason and in the spirit of Refs. [4,12], we define the normalization factors \( H_{\text{ref}} \) and \( P_{q(q)}^{\text{ref}} \) from the solution of Eq. (2) for zero disorder and electric field, i.e. \( \epsilon_n = 0 \) and \( F = 0 \)

\[
E^k = 2V \cos \left( \frac{k\pi}{N+1} \right) ; \varphi^k_n = \sqrt{\left( \frac{2}{N+1} \right)} \sin \frac{n k \pi}{N+1} , \tag{11}
\]

with \( k, n = 1, \ldots, N \). The entropy \( H_{\text{ref}} \) and the participation ration \( P_{2}^{\text{ref}} \) of the above eigenstates in the large \( N \) limit has the same value for every eigenvalue \( E^k \), i.e.

\[
H_{\text{ref}} = \ln(2N) - 1 ; P_{2}^{\text{ref}} = \frac{3}{2N} . \tag{12}
\]

IV. SCALING PROPERTIES OF LOCALIZATION LENGTHS

A. Infinite samples

In this Section we analyze the scaling properties of eigenstates of infinite systems. In numerical studies the matrices are obviously of finite size \( N \). However in our
analysis below we will always consider the case that 
\( N \gg l_{\infty}, l_{\text{el}} \), and thus the finite (but large) size of the 
matrix becomes irrelevant. We therefore have used these 
data to investigate our scaling assumption for \( \beta_q^\infty \) (see Eq. (8)).

As was mentioned in Section II the introduction of a 
non-zero electric field \( F \neq 0 \), results in an additional 
length scale \( l_{\text{el}} \). This length arises when we consider a 
cross section of the energy band locally tilted by the 
electric field: \(-V/2 + F n \leq E \leq V/2 + F n\) for an energy level 
\( E \). Therefore the scaling parameter \( \lambda_{\infty} = l_{\infty}/l_{\text{el}} \) enters 
the problem. Furthermore, if we consider the localization 
lengths \( L(q) \) of Eqs. (7), (9) as the typical length, which 
contains most of an eigenvector normalization, we expect 
that

\[
L_q \simeq \frac{l_{\infty}}{l_{\text{el}}} \lambda_{\infty} \ll 1 \quad \text{if} \quad \lambda_{\infty} \ll 1,
\]

i.e. the exponentially localized states progressively be-
comes localized factorially as the field increases. This 
is due to the fact that, for weak electric field we have 
\( \lambda_{\infty} \ll 1 \), and thus the dominant localization mechanism, 
i.e. the one that produces the shortest localization length 
scale, is the one related to the randomness. From now 
onwards we will refer to this as the “Anderson regime”. In 
the opposite limit \( \lambda_{\infty} \gg 1 \), the dominant localization 
mechanism is due to the electric field. We will refer to 
this regime as the “\( \lambda_{\infty} \gg 1 \)” regime. Based on the previous 
considerations we expect that the \( L_q \)'s have the following 
scaling form, (see also Ref.4 for an equivalent reasoning 
for quasi-1D systems)

\[
L_q = l_{\infty} f(\lambda_{\infty}) \quad \text{with} \quad f(\lambda_{\infty}) \simeq \frac{1}{\lambda_{\infty}} \lambda_{\infty} \ll 1 \quad (14)
\]

where \( f(\lambda_{\infty}) \) is related to the scaling function \( \beta_q^\infty \) as 
\( \beta_q^\infty = \lambda_{\infty} f(\lambda_{\infty}) \).

Our aim in this paragraph is to support the above men-
tioned scaling law based on numerical data and to extend 
our knowledge on the structure of the eigenstates in the 
intermediate regime between the two discussed limits. In 
order to study scaling properties of the localized eigen-
states we have used the transfer matrix method for the 
calculation of \( l_{\infty} \) as well as the direct diagonalization of the 
Hamiltonians that are associated with Eq. (5), for 
finite but long chains of size \( N = 10^4 \). In all numerical 
calculations below we used \( V = 1 \). We then varied the 
disorder strength \( W \) as well as the dc field strength in a 
regime, where always \( l_{\infty}, l_{\text{el}} \ll N \). One should stress here 
that both localization lengths \( l_{\infty} \) and \( L_q \) are functions of 
the energy \( E \). For this reason, in our numerical experi-
ments we consider ensembles of states specified by the 
values of the energy \( E \) in a small window \( E \in [0.95, 1.05] \). The size of the energy window was chosen in such a 
way that the localization length \( l_{\text{el}} \) is approximately constant 
inside this window (in all the cases \( \lambda_{\infty} l_{\text{el}} \) is \( < 0.06 \)). The 
values of \( \beta_1^\infty \) and \( \beta_2^\infty \) are then obtained by averaging over 
all eigenstates which are found inside the energy window 
for a set of different realizations of disorder. As a result, 
the total number of eigenstates used for the calculation 
of \( \beta_q^\infty \) were more than 1500.

A detailed analysis of the numerical data gives evi-
dence for a scaling behaviour of the form of Eqs. (10), (14) 
with the scaling function

\[
\beta_q^\infty \approx a_q^0 (1 - \exp(-a_q^1 \lambda_{\infty})) \quad (15)
\]

where the parameters \( a_q^0, a_q^1 \) are determined from a least 
squares fit. We have found that \( a_q^0 = 4.45 \) (4.43) and 
\( a_q^1 = 0.55 \) (0.37) for \( q = 1 \) (2). We notice here, that a 
similar scaling function was found in the framework of 
quasi-1D systems for \( q = 1 \). Our data together with 
a fit according to Eq. (15) are presented in Fig. 2. We 
observe a nice agreement with the scaling assumption of 
Eqs. (10), (14).

**B. Finite samples**

In realistic situations one always deals with finite sam-
ple. In such cases the understanding of the statistical 
properties of conductance are of major importance. Since 
these properties are directly related to the structure of 
eigenstates, it is important to investigate the statistical 
properties of eigenstates for finite systems. This is 
the goal of the present subsection.

For finite \( N \) and zero electric field, it was shown in 
Ref. 4 that the statistical properties of 1D Anderson models are characterized by a single scaling parameter 
\( \lambda_N = l_{\infty}/N \). Moreover, the scaling relation for the 
eigenvectors was found to be very simple.
\[
\beta_q^N = \beta_q^N(\lambda_N) = \frac{c_q \lambda_N}{1 + c_q \lambda_N}
\]

where the constants \(c_q\) were found to be \(c_1 \approx 2.6\) and \(c_2 \approx 1.5\). In fact, this scaling relation is exact only for \(q = 2\). For other cases of small values \(q\), including \(q = 1\), however, it is very close to the correct one (see details in [3]).

![Figure 3](image-url)

**FIG. 3.** Finite sample scaling of \(\beta_q^N\) as a function of \(\lambda_N\) with \(\lambda_N = 1\) and \(\lambda_N = 10\). Different symbols correspond to various sample sizes \(N \in \{200, 1000\}\). These were chosen appropriately \((\epsilon F \in [5 \cdot 10^{-4}, 5 \cdot 10^{-1}]\)). Dashed lines correspond to Eq. (15), where the values \(c_q\) were taken from a least squares fit of Fig. 4a,b (see below). Full lines represent the scaling law derived in Eq. (19), where the values \(c_q\) were taken from Fig. 2. (a) Scaling of \(\beta_1^N\) \((a_1^0 = 4.45, c_1 = 2.59\). (b) Scaling of \(\beta_2^N\) \((a_2^0 = 4.43, c_2 = 1.45\).

Once the electric field is turned on, however, a new length scale \(l_{sd}\) (with respect to the standard Anderson models) appears. This can be seen already from the previous paragraph, where on the basis of numerical results we were able to conclude that for infinite 1D Anderson models in the presence of an electric field the scaling properties of the eigenvectors are characterized by the parameter \(\lambda_{\infty}\). Since the sample size now enters as a third length scale, the second scaling parameter \(\lambda_N\) is likely to show up. Thus we expect that the statistical properties of the eigenstates, and accordingly the \(\beta_q^N\)'s, are going to be determined by the two parameters \(\lambda_{\infty}\) and \(\lambda_N\) which arise due to the competition between the characteristic lengths \(l_{\infty}\) and \(l_{sd}\) of the corresponding infinite sample and the actual size \(N\) of the sample. In the rest of the Section we are going to present numerical evidence that for finite 1D Anderson models in the presence of an electric field, the statistical properties of the eigenstates are characterized by the two scaling parameters \(\lambda_{\infty}\) and \(\lambda_N\). To this end, we will concentrate on the localization measures \(\beta_q^N\) (see Eq. (1)), which are the finite size counterparts of \(\beta_q^\infty\).

To find the localization lengths \(L_q\) for finite samples of size \(N\), we have used the same approach as in the previous subsection. The average values of \(L_q\) were calculated by choosing only the eigenstates which had eigenvalues within a small energy window \(E \in [0.95, 1.05]\). Additionally, we performed an ensemble averaging over at least 100 realizations of the disorder potential. For each \(N\) the total averaging thus involved several hundreds up to several thousands of eigenstates. In all these calculations the sample size was varied from \(N = 200\) up to 1000.

To test the scaling assumption (10) for finite systems we first analyze the behaviour of the localization measures \(\beta_{1,2}^N\) once \(\lambda_N\) is fixed. This is the finite sample counterpart of the scaling analysis presented in the previous subsection. In Fig. 3 we report our numerical results. Different symbols correspond to various sample sizes \(N\) and disorder strengths \(W\) such that always \(\lambda_N = 1\). The good overlap confirms the scaling dependence \(\beta_q^N = \beta_q^N(\lambda_{\infty})\) conjectured in Eq. (10).

Let us now try to gain some insight in the asymptotic form of the scaling law of \(\beta_q^N(\lambda_{\infty}, \lambda_N = \text{const.})\). For \(\lambda_{\infty} \ll 1\), the system is in the Anderson regime, where \(\beta_q^N\) is given by Eq. (10). The latter expression does not depend on \(\lambda_{\infty}\), and thus we can conclude that \(\beta_q^N\) has to saturate to a constant which is given by

\[
\beta_q^N(\lambda_{\infty} \ll 1, \lambda_N) \approx \frac{c_q \lambda_N}{1 + c_q \lambda_N}
\]

where \(\lambda_N = \text{const.}\). The agreement with the numerical data is very good.

In the opposite limit \(\lambda_{\infty} \gg 1\), Stark localization sets in. Since we can always choose the strength of the electric field \(F\) such that \(N, l_{\infty} \gg l_{sd}\), and assuming continuity in the form of \(\beta_q^N\), we can approximate the latter with the help of Eq. (13). For \(\lambda_{\infty} \gg 1\) this formula yields \(\beta_q^N \approx a_q^0\). Next, by changing variables and going from \(\beta_q^\infty\) to \(\beta_q^N\) we get

\[
\beta_q^N = \frac{l_{sd}^N}{N} = \frac{\lambda_N a_q^0}{\lambda_{\infty}}.
\]

Displaying \(\beta_q^N\) versus \(\lambda_{\infty}\) in a double logarithmic fashion, this yields

\[
\ln(\beta_q^N) \approx \ln(\lambda_N a_q^0) - \ln(\lambda_{\infty})\,.
\]

This linear behaviour (13) is shown by solid lines in Fig. 3, it describes approximately the numerical data. Deviations are due to the fact that the approximation via the scaling law of Eq. (13) actually requires not only \(l_{sd} \ll N\) but also \(l_{\infty} \ll N\), which is not fulfilled in our case. Nevertheless it gives a reasonable estimate.

We now turn to the case, where \(\lambda_{\infty}\) is fixed and \(\lambda_N\) varies. Our numerical results, for \(\lambda_{\infty} = 0.01, 1, 20\), corresponding to the Anderson, intermediate and Stark regime, respectively, are shown in Fig. 4 where now we refer to the new variable

\[
Y_q = \frac{\beta_q^N}{1 - \beta_q^N}.
\]
The points corresponding to the same $\lambda_\infty$ (but different $l_\infty$ and $l_\alpha$) fall onto the same smooth curve with a good accuracy, confirming the scaling hypothesis (21). From Fig. 4 one can see that the behaviour of $Y_q$ is different in the two regimes $\lambda_\infty \gg 1$ ($\lambda_\infty \ll 1$) where localization is due to the Stark (Anderson) mechanism. As a matter of fact, as we are increasing $\lambda_\infty$ two asymptotic regimes start to build up which have the same slope and differ only by a constant shift.

To understand, the behaviour of $Y_q(\lambda_N)$ as a function of $\lambda_\infty$, we first try to illuminate the limiting cases. Let us start with the limit $\lambda_\infty \ll 1$. This condition, defines the Anderson regime, where Eq. (16) holds and thus a behaviour

$$\ln(Y_q) = \ln(\lambda_N) + \ln c_q$$

(21)
in terms of the new variable $Y_q$ is expected over the whole range of $\lambda_N$. This expectation is shown in Fig. 4 by solid lines. Since $\lambda_\infty$ is small but still finite we can estimate the $c_q$ also from Eq. (15). The limit $\lambda_N \ll \lambda_\infty \ll 1$ corresponds to the Anderson regime of a nearly infinite sample. Expanding Eq. (15) to first order yields $\beta_q^\infty \approx a_q^0 a_q^1 \lambda_\infty$. Substituting this expression into Eq. (18) we end up with the following term for $\beta_q^N$

$$\beta_q^N \approx a_q^0 a_q^1 \lambda_N .$$

(22)

Inserting (23) into the definition of $Y_q$ and assuming $\lambda_N \ll 1$ we get

$$\ln(Y_q) \approx \ln(\lambda_N) + \ln(a_q^0 a_q^1) ,$$

(23)

which implies $c_q \approx a_q^0 a_q^1$. A comparison of $c_q = 2.59 (1.55)$ and $a_q^0 a_q^1 = 2.45 (1.64)$ shows a very good agreement. Thus, we conclude that our scaling function (23) is consistent with Eq. (21) as it should be in the above limit.

In the opposite limit of $\lambda_\infty \gg 1$, we have to distinguish between the following two cases. When $\lambda_N \gg \lambda_\infty$, the sample size $N$ sets the smallest length scale. In this limit the eigenstates extend over the whole sample. Then by continuity we expect that the behaviour of $Y_q(\lambda_N)$ for $\lambda_N \gg 1$ will be given by Eq. (21). Our numerical data (see Figs. 4e,f) support this hypothesis. The second case, in which $\lambda_N \ll \lambda_\infty$ holds, can be viewed as the extreme Stark regime of an infinite sample. In that limit one obtains Eq. (19) again, which yields

$$\ln(Y_q) \approx \ln(\lambda_N) + \ln \left( \frac{a_q^0}{\lambda_\infty} \right) .$$

(24)
The asymptotic behaviour (24) is reported in Figs. 4e,f with dashed lines and agrees quite well with our numerical data.

From the above analysis we conclude that at
\[ \lambda_\infty \simeq \lambda_N \]  

(25)

two asymptotic regimes are created due to the interplay between Anderson and Stark localization mechanisms. Although these estimations are made only on a very rough level, they describe our numerical findings rather well.

V. SCALING OF THE DISTRIBUTION OF EIGENVECTORS

As a complement to the above analysis, we show in this section, that the distributions of squared components of eigenvectors are parametrized in the same fashion by \( \lambda_N \) and \( \lambda_\infty \). Again we restrict ourselves to a definite energy window \( E \in [0.95, 1.05] \), where all eigenvectors corresponding to eigenvalues in that window were computed for several realizations of disorder. The total number of eigenstates were in all cases more than 1000.

Before examining the scaling properties of the distribution let us distinguish between the various cases that appear due to the competition between the three characteristic length scales \( N, l_{el}, l_\infty \) (see previous section). For simplicity we define these regimes only by their limiting cases, which read as follows:

1. \( l_\infty < l_{el} < N \)  &  \( l_{el} < l_\infty < N \)  (infinite sample)
2. \( l_\infty < N < l_{el} \)  &  \( N < l_\infty < l_{el} \)  (Anderson regime)
3. \( N < l_{el} < l_\infty \)  &  \( l_{el} < N < l_\infty \)  (Stark regime)

The first pair in this list corresponds to scaling behaviour of the infinite sample, since the sample size is always larger than the other two competing lengths. For this case and \( q = 1, 2 \) we have shown already in Section IV.A that \( \beta_{\infty} \) follows a single parameter scaling with respect to \( \lambda_\infty \). We will show here that also the whole distribution of eigenvector components is parametrized according to the same scaling parameter. The first question which arises for the infinite sample, is the proper normalization of the squared entries of the eigenstates \( w_n = |\varphi_n|^2 \). A normalization with respect to the number of sites \( N \) does not seem appropriate since we are interested in the limit \( N \to \infty \). Since the \( w_n \) have to scale with some length, however, following our previous strategy (see Eq. (23)), we introduce the variable

\[ r = \ln(w_n l_{el}) \]  

(26)

and investigate the distribution \( p(r) \). For our calculations we consider matrices of size \( N = 10^4 \), while we \( l_{el}, l_\infty \ll N \) and varied \( \lambda_\infty \). For each \( \lambda_\infty \) we considered two different disorder strengths \( (l_\infty \approx 6, 10) \) and adjusted the dc field strength appropriately. The results for \( \lambda_\infty = 10, 1, 0.1 \) are shown in Fig. 5a-c in a semilogarithmic plot. The assumed scaling of \( p(r) \) with \( \lambda_\infty \) is clearly visible.

![FIG. 5. Scaling of the entire distribution of squared eigenvector components \( p(r) \) with \( \lambda_\infty \) in the case of nearly infinite samples \( (N = 10^4) \). Two different pairs of \( l_\infty, l_{el} \ll N \) are denoted by full lines \( (W = 3.49, eF = 1.666, 0.166, 0.016) \) and symbols \( (W = 2.62, eF = 1, 0.1, 0.01) \), while keeping \( \lambda_\infty \) fixed. (a) \( \lambda_\infty = 10 \) (b) \( \lambda_\infty = 1 \) (c) \( \lambda_\infty = 0.1 \).

The second and third pair of conditions always involve the sample size \( N \). Therefore scaling according to \( \lambda_\infty \) and \( \lambda_N \) has to be taken into account. For these cases renormalization with respect to the sample size is meaningful; hence we define the rescaled squared entries of eigenstates as

\[ r = \ln(w_n N) \]  

(27)

Before turning to the analysis of our numerical data, let us first qualitatively analyze the form of \( p(r) \). In the limit \( N \ll l_\infty, l_{el} \) the system does not feel any localization due to disorder or electric field. Therefore all eigenstates are given approximately by Eq. (11). The distribution \( p(r) \) is then given by

\[ p(r) = \frac{e^r}{\pi \sqrt{e^r(2 - e^r)}} \]  

(28)

Plotting \( \ln(p(r)) \) versus \( r \) for \( r \ll 0 \) yields a curve with slope 1/2, as can be verified by expanding Eq. (28). Moreover Eq. (28) shows a sharp peak around \( r = 0 \).

In the case where \( l_\infty \ll N, l_{el} \) the disorder sets the relevant length scale and the system resembles an infinite Anderson model with exponentially localized eigenstates \( w_n = \exp(-|n - m|/l_\infty) \). For small \( r \) this particular form of eigenstates leads to

\[ p(r) = l_\infty / N \]  

(29)

In a semilogarithmic representation this results in a nearly horizontal curve of height \( \ln(l_\infty / N) \) for \( r \ll 0 \), which drops rapidly for some \( r > 0 \).
FIG. 6. Scaling of the entire distribution of squared eigenvector components \( p(r) \) in the Stark regime \((\lambda_\infty = 20)\) in the case of finite samples \( (N \in [230, 1200], W \in [0.01, 2], eF \in [10^{-5}, 1])\). Full lines and symbols denote different sample sizes and thus different strengths of disorder, while \( \lambda_N \) is kept fixed to a chosen value. (a) \( \lambda_N = 100 \) (the dashed line has slope 1/2) (b) \( \lambda_N = 2 \) (c) \( \lambda_N = 0.05 \).

For the second pair of conditions, i.e. the Anderson regime, the scaling properties of the distribution were already analyzed in [1]. A good agreement with the limiting Eqs. (28), (29) was found.

The new and more interesting case is the pair of conditions with label 3, where the competition between \( N \) and \( l_{el} \) is dominant. In this case \( \lambda_\infty \gg 1 \) and thus the localization mechanism is due to the Stark effect. The resulting distribution for some representative values of \( \lambda_\infty \) and \( \lambda_N \) are shown in Fig. 6. One can clearly see that distributions corresponding to different sample sizes \( N \) and disorder strengths \( W \), but having the same \( \lambda_N \) and \( \lambda_\infty \) coincide. Moreover, as we move from higher to smaller values of \( \lambda_N \) the shape of the distributions changes drastically. In the case \( \lambda_N \gg 1 \) (corresponding to \( N \ll l_{el} \)) the eigenstates can be considered as extended with respect to the sample size, and thus we obtain again Eq. (28) (see Fig. 6a). The peak of the distribution broadens and moves to the right upon an increase of \( \lambda_N \) as can be seen from Fig. 6b. At the same time, for negative values of \( r \) the distribution possesses long tails. This becomes more and more apparent as we are moving towards the Stark regime (Fig. 6c). In the strong field limit the eigenstates are essentially localized at one site i.e. \( w_n \simeq \delta_{nm} \). In this case, one has \( p(r \simeq ln(N)) \sim 1/N \), while long tails appear due to factorial localization.

We conclude this Section by noticing that the scaling of the distributions of squared eigenvector components with \( \lambda_\infty \) and \( \lambda_N \) implies scaling of the localization parameters (11) for arbitrary \( q \).

VI. SUMMARY

We have studied a 1D tight-binding model in the presence of a constant electric field. For such a model we can distinguish between two regimes where localization is due to totally different mechanisms. The first regime is the Anderson regime, which is defined through the condition \( \lambda_\infty \ll 1 \). Here the localization due to disorder is the dominant mechanism that controls the statistical properties of the eigenstates. In the opposite limit, \( \lambda_\infty \gg 1 \), the localization is due to the presence of the electric field. This is the Stark regime. Our numerical study deals with the scaling properties of the eigenstates both for infinite and finite samples. This study was motivated by the remarkable scaling law that has been found for quasi-1D models with electric field [4,12]. Our results indicate that in both infinite and finite samples with disorder and electric field the eigenstates have generic properties, regardless of the dimensionality of the system, provided that an appropriate renormalization (with respect to the corresponding "extended" states) is done. Thus we show that the similarity between 1D and quasi-1D eigenstates should persist also for systems with electric field.

We found that for infinite systems the statistical properties of the eigenstates are characterized by the single parameter \( \lambda_\infty \). This conclusion was based on an extensive numerical analysis of both the localization measures \( \delta(q) \) and of the whole distribution of squared eigenvector components. Moreover for \( \beta_{q=1,2}^{\infty} \) we have found a simple scaling law (15) that describes quite nicely our numerical data. This expression can be used in order to find the strength of the applied dc electric field once \( l_\infty \) is known for the field free model.

Moreover, we have performed a finite length scaling analysis. Our numerical analysis showed that for finite systems the statistical properties of the eigenstates are characterized by two parameters, namely \( \lambda_\infty \), as in the case of infinite systems, and \( \lambda_N \). The latter parameter involves the actual size of the sample which enters in the scaling analysis as a third length scale. We found that the localization measures \( \beta_{q=1,2}^{\infty} \) show a totally different asymptotic behaviour in \( \lambda_N \rightarrow 0, \infty \) as we are increasing the parameter \( \lambda_\infty \). Based on some analytical arguments we estimated that this occurs approximately at \( \lambda_N \simeq \lambda_\infty \). This can be used as a criterion to identify which localization mechanism (Anderson or Stark localization) is responsible for the structure of eigenstates. It will be interesting to investigate if the same asymptotic behaviour also holds for higher moments \( q \). Finally, we studied the whole distribution of squared eigenvector components and showed that it also follows the same scaling behaviour with respect to the two scaling parameters \( \lambda_\infty \) and \( \lambda_N \).

The main result of our work is the fact that scaling properties of eigenstates of infinite systems are described by one parameter scaling \( \lambda_\infty \), whereas for finite systems an additional parameter \( \lambda_N \) is also needed. In partic-
ular, both localization lengths, the entropy localization length as well as the one defined via the inverse participation ratio, follow the universal scaling law of Eq. (10) after appropriate normalization. This is in contrast to the standard Anderson models without electric field, where only one parameter ($\lambda N$) is needed to describe the scaling properties of eigenstates.

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