A Combinatorial Description of the Dormant Miura Transformation

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Abstract
The aim of the present paper is to describe Miura \(\mathfrak{sl}_2\)-opers and Miura transformations in terms of graph-theoretic objects. We construct a bijective correspondence between dormant generic Miura \(\mathfrak{sl}_2\)-opers on a totally degenerate curve in positive characteristic and certain branch numberings on a 3-regular graph. This correspondence allows us to completely identify dormant generic Miura \(\mathfrak{sl}_2\)-opers on totally degenerate curves. Also, we investigate how this result can be related to the combinatorial description of dormant \(\mathfrak{sl}_2\)-opers given by S. Mochizuki, F. Liu, and B. Osserman.

Keywords  Dormant oper · Miura oper · Miura transformation · 3-Regular graph

Mathematics Subject Classification  14H70 · 05C30

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1 Introduction

1.1 The purpose of the present paper is to give combinatorial descriptions of dormant generic Miura $\mathfrak{sl}_2$-opers. The combinatorial objects that we use are certain branch numberings of 3-regular (i.e., trivalent) graphs. Our description will be thought of as an analogue of the combinatorial description of dormant $\mathfrak{sl}_2$-opers given by S. Mochizuki, F. Liu, and B. Osserman (cf. [12, Introduction, Sect. 1.2, Theorem 1.3]; [10, Theorem 3.9]). By using this result, we explicitly construct Miura transformations of dormant generic Miura $\mathfrak{sl}_2$-opers on a totally degenerate curve.

Here, recall that the celebrated Miura transformation concerns the Korteweg-de Vries (KdV) and the modified KdV (mKdV) equations. The KdV equation was derived as an equation modeling the behavior of shallow water waves moving in one direction by Korteweg and his student de Vries. The KdV equation reads

$$v_t - 6vv_x + v_{xxx} = 0,$$

while its modified counterpart, the mKdV equation, equals

$$u_t - 6u^2u_x + u_{xxx} = 0.$$

Let us take a solution $u \in \mathbb{C}[t, x]$ to the mKdV equation. Then, the function $v$ characterized by the equality of differential operators

$$(\partial_x - u)(\partial_x + u) = \partial_x^2 - v \quad \text{(1.1)}$$

(i.e., $v = u^2 - u_x$) specifies a solution to the KdV equation. The assignment

$$u \mapsto v \left( = u^2 - u_x \right)$$

is nothing but the Miura transformation.

The differential operator $\partial_x^2 - v$ (resp., $(\partial_x - u)(\partial_x + u)$) in the right-hand (resp., the left-hand) side of (1.1) corresponds, in the usual manner, to the connection on a vector bundle, i.e., the first order matrix differential operator, of the form

$$\nabla = \partial_x - \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix} \quad \text{(resp., } \nabla_{\text{Miura}} = \partial_x - \begin{pmatrix} u & 0 \\ 1 & -u \end{pmatrix} \text{)}.$$

A connection of this form is called an $\mathfrak{sl}_2$-oper (resp., a generic Miura $\mathfrak{sl}_2$-oper). If the underlying space of the vector bundle is a Riemann surface, then such a connection may be identified with a projective connection (resp., an affine connection) on this Riemann surface (cf. [3]). The connection $\nabla_{\text{Miura}}$ becomes $\nabla$ after gauge transformation by some upper triangular matrix. This implies that the Miura transformation may be thought of as the assignment from generic Miura $\mathfrak{sl}_2$-opers $\nabla_{\text{Miura}}$ to $\mathfrak{sl}_2$-opers $\nabla$ induced by gauge transformations in this way.
1.2 In the present paper, we study the case of positive characteristic. Just as in the complex case mentioned above, $\mathfrak{sl}_2$-opers and Miura $\mathfrak{sl}_2$-opers can be defined in characteristic $p > 0$ because of the algebraic nature of their formulations. However, unlike the complex case, there exist generic Miura $\mathfrak{sl}_2$-opers on some entire (i.e., proper) smooth curve in positive characteristic of genus $> 1$ (cf. [17, Sect. 6, Remark 6.3.3, (i)]; [18, Theorem B]). One of the key ingredients in the study of them is the $p$-curvature of a connection. Recall that the $p$-curvature of a connection is an invariant that measures the obstruction to the compatibility of $p$-power structures appearing in certain associated spaces of infinitesimal symmetries. We say that an $\mathfrak{sl}_2$-oper (resp., a generic Miura $\mathfrak{sl}_2$-oper) is dormant if it has vanishing $p$-curvature. We refer to [20] (resp., [17]) for various discussions and results concerning dormant opers (resp., dormant Miura opers) on pointed stable curves. Each pointed stable curve $\mathcal{X}$ gives rise to the set

$$\text{Op}_{\mathcal{X}}^{zzz...} \quad (\text{resp., } M\text{Op}_{\mathcal{X}}^{zzz...})$$

of isomorphism classes of dormant $\mathfrak{sl}_2$-opers (resp., dormant generic Miura $\mathfrak{sl}_2$-opers) on $\mathcal{X}$. The Miura transformation described in terms of opers, i.e., $\nabla_{\text{Miura}} \mapsto \nabla$ as discussed in the previous subsection, gives a map of sets

$$\mu : M\text{Op}_{\mathcal{X}}^{zzz...} \rightarrow \text{Op}_{\mathcal{X}}^{zzz...}.$$  

We shall refer to it as the dormant Miura transformation. Our main interest of the present paper is to understand, by using graph-theoretic objects, the behavior of this map in the case where the underlying curve is totally degenerate (cf. the discussion preceding Proposition 3.3).

1.3 To this end, we first recall the combinatorial description of dormant $\mathfrak{sl}_2$-opers on such a curve, which was obtained by S. Mochizuki in the study of $p$-adic Teichmüller theory and developed by F. Liu and B. Osserman (cf. Sect. 3.2). Let $(g, r)$ be a pair of nonnegative integers with $2g - 2 + r > 0$ and $\mathcal{X}$ an $r$-pointed totally degenerate curve of genus $g$ over an algebraically closed field $k$ of characteristic $p > 2$. The curve $\mathcal{X}$ associates its dual marked semi-graph $\Gamma^+$ (cf. the discussion in Remark 2.7), which is 3-regular. A balanced $p$-edge numbering (cf. Definition 3.1, (i)) on $\Gamma^+$ is a numbering $(a_e)_{e \in E}$ on the set of edges $E$ of $\Gamma^+$ satisfying certain triangle inequalities with respect to each triple of numbers on branches incident to one vertex. Denote by

$$Ed_{\Gamma^+}$$

the set of balanced $p$-edge numberings on $\Gamma^+$. Then, Mochizuki showed that there exists a canonical bijection

$$Ed_{\Gamma^+} \sim \text{Op}_{\mathcal{X}}^{zzz...} \quad (1.2)$$
(cf. the discussion preceding Proposition 3.3 for the precise construction). The inverse of this bijection can be obtained by taking the \textit{radii} of dormant $\mathfrak{sl}_2$-opers on $\mathfrak{X}$ restricted to the various rational components of its normalization.

A balanced $p$-edge number (with $p >> 0$)

A strict $p$-branch number (with $p = 11$)

Next, we introduce the notion of a \textbf{strict $p$-branch numbering} (cf. Definition 3.4) on $\Gamma^+$, which is defined to be a certain numbering $(\varepsilon_b)_{b \in B}$ on the set of branches $B$ of $\Gamma^+$ satisfying some conditions, including the condition that the three numbers on branches incident to one vertex amount precisely to $p + 1$. Each strict $p$-branch numbering $(\varepsilon_b)_{b \in B}$ on $\Gamma^+$ induces a balanced $p$-edge numbering $(\varepsilon^\mu_e)_{e \in E}$ on $\Gamma^+$ which is well-defined in such a way that for each $e \in E$ having a branch $b \in B$, $\varepsilon^\mu_e$ equals $\frac{p - \varepsilon_e - 1}{2}$ if $\varepsilon_e$ is even and $\varepsilon^\mu_e$ equals $\frac{\varepsilon_e - 1}{2}$ if $\varepsilon_e$ is odd. We shall denote by

$$Br_{\Gamma^+}$$

the set of strict $p$-branch numberings on $\Gamma^+$. Then, the assignment $(\varepsilon_b)_{b \in B} \mapsto (\varepsilon^\mu_e)_{e \in E}$ defines a map of sets

$$\mu^\text{comb} : Br_{\Gamma^+} \rightarrow Ed_{\Gamma^+}.$$  

then the main result of the present paper is described as follows (cf. Theorem 3.5 for the full statement).

\textbf{Theorem A} \textit{Let $\mathfrak{X}$ and $\Gamma^+$ be as above. Then, there exists a canonical bijection

$$Br_{\Gamma^+} \sim \rightarrow MOp_{\mathfrak{X}}$$

making the following square diagram commute:

\begin{align*}
Br_{\Gamma^+} & \sim \rightarrow MOp_{\mathfrak{X}}^\text{zzz...} \\
\mu^\text{comb} \downarrow & \quad \downarrow \mu \\
Ed_{\Gamma^+} & \sim \rightarrow Op_{\mathfrak{X}}^\text{zzz...} \quad (1.4)
\end{align*}

1.4 By the above theorem, one may consider the morphism $\mu^\text{comb}$ as a combinatorial realization of the dormant Miura transformation. Also, note that the correspondence $\mu^\text{comb}$ helps us to study pathological phenomena that occur in algebraic geometry in positive characteristic. According to the discussion in [17, Sect. 7], we can associate
to a dormant generic Miura $\mathfrak{sl}_2$-opers an algebraic surface satisfying some completely different properties from complex algebraic varieties, e.g., violating the Kodaira vanishing theorem. However, the existence of a dormant generic Miura $\mathfrak{sl}_2$-oper implies a strong restriction to the genus $g$ of the underlying curve, i.e., $p$ must divide $2g - 2$. Hence, it will be natural to ask how many curves admitting such an object, and this question leads us to investigate the moduli space classifying pointed stable curves equipped with a dormant generic Miura $\mathfrak{sl}_2$-oper. In that respect, the result in the present paper will be useful because the geometric structure of the locus classifying singular curves gives, via degeneration and deformation techniques, a lot of information on the entire moduli space (cf. Remark 4.7). For example, Theorem A provides a complete determination of dormant generic Miura $\mathfrak{sl}_2$-opers on a totally degenerate curve, as described in the following assertion.

**Theorem B** (cf. Corollaries 4.4 and 4.6 for the full statements) Let $\mathcal{X}$ be a totally degenerate $r$-pointed stable curve over $k$ of genus $g$. Then, $\mathcal{X}$ admits a dormant generic Miura $\mathfrak{sl}_2$-oper if and only if $g \leq 1$. Moreover, if $g = 0$ (resp., $g = 1$), then the set $\text{MOp}^{-}_{\mathcal{X}}$ is finite and its cardinality is given by the formula

$$\sharp(\text{MOp}^{-}_{\mathcal{X}}) = p - 1 \quad (\text{resp., } \sharp(\text{MOp}^{-}_{\mathcal{X}}) = \binom{r - 1}{p + 1} \text{)}).$$

### 1.6 Notation and Conventions

Let us introduce some notation and conventions used in the present paper. Throughout the present paper, we fix an algebraically closed field $k$ of characteristic $p > 2$ and a pair of nonnegative integers $(g, r)$ with $2g - 2 + r > 0$.

For a log scheme indicated, say, by $Y^{\log}$, we shall write $Y$ for the underlying scheme of $Y^{\log}$. If, moreover, $Z^{\log}$ is a log scheme over $Y^{\log}$, then we shall write $\Omega_{Z^{\log}/Y^{\log}}$ for the sheaf of logarithmic 1-forms on $Z^{\log}$ over $Y^{\log}$ and write $T_{Z^{\log}/Y^{\log}} := \Omega_{Z^{\log}/Y^{\log}}^\vee$ for its dual. (Basic references for the notion of a log scheme are [7], [5], and [6].)

Given a set $S$ and a positive integer $r$, we shall denote by $S^{\times r}$ the set of $r$-tuples of elements in $S$.

## 2 Dormant Generic Miura $\mathfrak{sl}_2$-Opers

### 2.1 Semi-graphs

First, recall from [13, Appendix] (or [14, §1], [20, Sect. 7.1, Definition 7.1]) the definition of a semi-graph, as follows.

**Definition 2.1** A semi-graph is a triple

$$\Gamma = (V, E, \xi)$$

consisting of the following data:
• a set $V$, whose elements are called vertices;
• a set $E$, whose elements are called edges, consisting of sets with cardinality 2 such that $e \neq e' \in E$ implies $e \cap e' = \emptyset$;
• a map $\zeta : \bigsqcup_{e \in E} e \to V \cup \{\otimes\}$ (where “$\otimes$” denotes an abstract symbol with $\otimes \notin V$), called a coincidence map, such that $\zeta(e) \neq \{\otimes\}$ for any $e \in E$.

Each edge $e \in E$ with $\otimes \in \zeta(e)$ (resp., $\otimes \notin \zeta(e)$) is called open (resp., closed). Let $e$ be an edge of $\Gamma$, i.e., $e \in E$. Then, we shall refer to any element $b \in e$ as a branch of $e$. If $b$ is a branch of $e$, then we shall denote by

$$b^*(\in e)$$

the branch of $e$ with $\{b, b^*\} = e$. We write $B := \bigsqcup_{e \in E} e$ and, for each $v \in V \cup \{\otimes\}$, we write

$$B_v := \zeta^{-1}\{\{v\}\}$$

In particular, we have $B = \bigsqcup_{v \in V \cup \{\otimes\}} B_v$.

Hereinafter, we fix a semi-graph $\Gamma := (V, E, \zeta)$.

**Definition 2.2** (i) We shall say that $\Gamma$ is finite if both $V$ and $E$ are finite.
(ii) Let $m$ be a positive integer. We shall say that $\Gamma$ is $m$-regular if for any vertex $v \in V$, the cardinality of $B_v$ is precisely $m$.

**Definition 2.3** (i) Let $u, v$ be vertices of $\Gamma$. A path from $u$ to $v$ is an ordered collection of branches

$$(b_j)_{j=1}^l \in B^l,$$

where $l$ denotes some positive integer, such that $u = \zeta(b_1)$, $v = \zeta(b_j^*)$, and $\zeta(b_j^*) = \zeta(b_{j+1})$ for every $j \in \{1, \ldots, l-1\}$.
(ii) We shall say that $\Gamma$ is connected if for any two distinct vertices $u, v \in V$, there exists a path from $u$ to $v$.

**Definition 2.4** Suppose that $\Gamma$ is finite, connected, and 3-regular. Then, we shall say that $\Gamma$ is of type $(g, r)$ if the following equalities hold:

$$g = 1 - \#(V) + \#(E) - \#(B_{\otimes}), \quad r = \#(B_{\otimes}).$$

**Remark 2.5** We have defined the notion of a semi-graph in a purely set-theoretic formulation, but each semi-graph $\Gamma := (V, E, \zeta)$ can be regarded as a topological space in the usual way (cf. [14, Sect. 1], [20, Sect. 7.1, Remark 7.3]). Indeed, we realize each closed edge $e := \{b_1, b_2\} \in E$ as a copy of the closed unit interval $[0, 1] (\subseteq \mathbb{R})$ whose endpoints are indexed by $b_1$ and $b_2$, realize each open edge $e := \{b_1, b_2\}$ (with $\zeta(b_1) \neq \otimes$) as a copy of $[0, 1] (\subseteq \mathbb{R})$, where the point $0 \in \mathbb{R}$ is indexed by $b_1$. Then, we can construct a topological space obtained from these constituents in such a way
that, for each \( v \in V \), the points indexed by elements of \( \zeta^{-1}([v]) \) are attached to form a single point. The connectedness for semi-graphs defined in Definition 2.3, (ii), is equivalent to the usual definition of connectedness for the corresponding topological space. Also, it makes sense to speak of the Betti number \( b_1(\Gamma) := \dim_{\mathbb{Q}}(H_1(\Gamma, \mathbb{Q})) \) of the semi-graph \( \Gamma \). If \( \Gamma \) is finite, connected, and 3-regular, then the equality \( g = b_1(\Gamma) \) holds.

**Definition 2.6** (i) Suppose that \( \Gamma \) is finite. A **marking** on \( \Gamma \) is a bijection of sets

\[
\lambda : B_\oplus \rightarrow \{1, \ldots, r\},
\]

where \( r := \sharp(B_\oplus) \). We consider any semi-graph \( \Gamma := (V, E, \zeta) \) with \( \sharp(B_\oplus) = 0 \) as being equipped with a unique marking \( B_\oplus \rightarrow \emptyset \).

(ii) A **marked semi-graph** is a quadruple

\[
\Gamma^+ := (V, E, \zeta, \lambda),
\]

where \( \Gamma := (V, E, \zeta) \) is a finite semi-graph and \( \lambda \) is a marking on \( \Gamma \). We shall refer to \( \Gamma \) as the underlying semi-graph of \( \Gamma^+ \).

**Remark 2.7** Let \( \mathcal{X} := (X/k, \{\sigma_i\}_{i=1}^r) \) be an \( r \)-pointed stable curve of genus \( g \) over \( k \), which consists of a prestable curve \( X \) over \( k \) of genus \( g \) and ordered \( r \) marked points \( \sigma_i \) \((i = 1, \ldots, r)\). Then, in the usual manner, one can associate to \( \mathcal{X} \) a marked semi-graph

\[
\Gamma^+_\mathcal{X} := (V_\mathcal{X}, E_\mathcal{X}, \zeta_\mathcal{X}, \lambda_\mathcal{X})
\]

defined as follows. \( V_\mathcal{X} \) is taken as the set of irreducible components of \( X \) and \( E_\mathcal{X} \) is the disjoint union \( N_\mathcal{X} \cup \{\sigma_i\}_{i=1}^r \) of the set of nodal points \( N_\mathcal{X} \subseteq X(k) \) and the set of marked points \( \{\sigma_i\}_{i=1}^r \). Here, note that any node \( e \in X(k) \) has two distinct branches \( b_1 \) and \( b_2 \), each of which lies on some well-defined irreducible component of \( X \). (Even when the two branches \( b_1, b_2 \) lie on the same component, they are still treated differently from each other.) We identify \( e \) with the set \( \{b_1, b_2\} \), and moreover, identify each marked point \( \sigma_i \) with the set \( \{\sigma_i, i\} \).

In this way, we regard each element of \( E_\mathcal{X} \) as a set with cardinality 2. Also, we define \( \zeta_\mathcal{X} \) to be the map \( \bigsqcup_{e \in E_\mathcal{X}} e \rightarrow V_\mathcal{X} \cup \{\oplus\} \) determined as follows:

- If \( b \in e \) (for some \( e \in N_\mathcal{X} \)) or \( b = \sigma_i \) (for some \( i \in \{1, \ldots, r\} \)), then \( \zeta_\mathcal{X}(b) \) is the irreducible component (i.e., an element of \( V_\mathcal{X} \)) in which \( b \) lies;
- \( \zeta_\mathcal{X}(i) = \oplus \) for any \( i = 1, \ldots, r \).

In particular, we have \( B_\oplus = \{1, \ldots, r\} \). Finally, \( \lambda_\mathcal{X} \) is defined as the identity map of \( \{1, \ldots, r\} \). We shall refer to \( \Gamma^+_\mathcal{X} \) as the dual marked semi-graph associated with \( \mathcal{X} \).

We shall write

\[
\tilde{\mathbb{F}}_p := \{0, 1, \ldots, p - 1\} (\subseteq \mathbb{Z}),
\]
and write $\tau$ for the natural composite bijection

$$\tau : \tilde{\mathbb{F}}_p \leftrightarrow \mathbb{Z} \rightarrow \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}.$$  

Let us define an involution $(-)^{\vee}$ on $\tilde{\mathbb{F}}_p$ to be the map given as follows:

$$m^{\vee} := \begin{cases} 
  p - m & \text{if } m \in \tilde{\mathbb{F}}_p \setminus \{0\}, \\
  0 & \text{if } m = 0.
\end{cases}$$

In particular, we have $m^{\vee \vee} = m$, $\tau(m^{\vee}) = -\tau(m)$.

**Definition 2.8**

(i) A $p$-branch numbering on $\Gamma$ is a collection $\vec{m} := (m_b)_{b \in B} \in \tilde{\mathbb{F}}_p$ of elements of $\tilde{\mathbb{F}}_p$ indexed by the set $B$ such that for any edge $e := \{b, b^*\}$, the equality $m_{b^*} = m_b^{\vee}$ holds.

(ii) Assume further that $\Gamma$ is finite and $B_\oplus \neq \emptyset$. Also, assume that we are given a marking $\lambda : B_\oplus \sim \{1, \ldots, r\}$ (where $r := \sharp(B_\oplus)$) on $\Gamma$ and an element $\vec{\varepsilon} := (\varepsilon_i)_{i=1}^r$ of $\mathbb{F}_p^r$. Then, we shall say that a $p$-branch numbering $\vec{m} := (m_b)_{b \in B} \in \tilde{\mathbb{F}}_p$ is of exponent $\vec{\varepsilon}$ if $\tau(m_b) = \varepsilon_{\lambda(b)}$ for any $b \in B_\oplus$. For convenience, (regardless of whether $B_\oplus$ is empty or not) we shall refer to any $p$-branch numbering as being of exponent $\emptyset$.

2.2 Opers and Miura opers on a Pointed Stable Curve

In this subsection, we recall the notion of a dormant generic Miura oper defined on a pointed stable curve. We refer to [17] and [20] for various definitions and notation used in this section.

First, let

$$\mathcal{X} := (f : X \rightarrow \operatorname{Spec}(k), \{\sigma_i : \operatorname{Spec}(k) \rightarrow X\}_{i=1}^r)$$

be an $r$-pointed stable curve over $k$ of genus $g$, consisting of a prestable curve $X$ over $k$ of genus $g$ and ordered $r$ marked points $\sigma_i$ ($i = 1, \ldots, r$) of $X$. Note that there exists natural log structures on $X$ and $\operatorname{Spec}(k)$ (cf. [20, Sect. 1.5.5], [17, Sect. 1.1]); we denote the resulting log schemes by $\operatorname{Spec}(k)^{\log}$ and $X^{\log}$, respectively. The structure morphism $f : X \rightarrow \operatorname{Spec}(k)$ extends to a morphism $f^{\log} : X^{\log} \rightarrow \operatorname{Spec}(k)^{\log}$ of log schemes, by which $X^{\log}$ determines a log-curve over $\operatorname{Spec}(k)^{\log}$ (cf. [1, Definition 4.5] for the definition of a log-curve).

Next, let us recall briefly the definitions of an $\mathfrak{sl}_2$-opers and a Miura $\mathfrak{sl}_2$-oper. Denote by $\operatorname{PGL}_2$ the projective linear group over $\mathbb{F}_p$ of rank 2, where we occasionally consider it as an algebraic group over $k$ via base-change by the field extension $k/\mathbb{F}_p$. Also, denote by $B_2$ the Borel subgroup of $\operatorname{PGL}_2$ consisting of the images via the
quotient $GL_2 \twoheadrightarrow PGL_2$ of invertible upper triangular $2 \times 2$ matrices. By the condition $p > 2$, we can identify the Lie algebra of $PGL_2$ with $sl_2$, i.e., the Lie algebra consisting of $2 \times 2$ matrices with vanishing trace. An $sl_2$-oper on $\mathcal{X}$ is a pair

$$\mathcal{E}^\bullet := (\mathcal{E}_B, \nabla)$$

consisting of a $B_2$-torsor $\mathcal{E}_B$ over $X$ and a $k^{\log}$-connection $\nabla$ on the $PGL_2$-torsor $\mathcal{E}_{PGL_2} := \mathcal{E}_B \times^{B_2} PGL_2$ induced by $\mathcal{E}_B$ such that $\mathcal{E}_B$ is transversal to $\nabla$. (We refer to [17, Sect. 1.3] or [20, Sect. 1.3, Definition 1.17] for the definition of a connection on a torsor in the logarithmic sense, and refer to [17, Sect. 3, Definition 3.1.1 (i)], [20, Sect. 2.1, Definition 2.1], or [12, Chap. I, Sect. 2, Definition 2.2] for the precise definition of an $sl_2$-oper). Also, a Miura $sl_2$-oper on $\mathcal{X}$ is defined to be a collection of data

$$\hat{\mathcal{E}}^\bullet := (\mathcal{E}_B, \nabla, \mathcal{E}'_B, \eta),$$

where $(\mathcal{E}_B, \nabla)$ is an $sl_2$-oper on $\mathcal{X}$, $\mathcal{E}'_B$ is another $B$-torsor $\mathcal{E}'_B$ over $X$, and $\eta$ is an isomorphism $\mathcal{E}'_B \times^{B_2} PGL_2 \sim \mathcal{E}_{PGL_2}$ of $PGL_2$-torsors via which $\mathcal{E}'_B$ is preserved by $\nabla$ (cf. [17, Sect. 3, Definition 3.2.1] for the precise definition of a Miura $sl_2$-oper). We shall say that a Miura $sl_2$-oper $\hat{\mathcal{E}}^\bullet := (\mathcal{E}_B, \nabla, \mathcal{E}'_B, \eta)$ is generic (cf. [17, Definition 3.3.1]) if $\mathcal{E}_B$ and $\mathcal{E}'_B$ are in generic relative position. Moreover, we shall say that an $sl_2$-oper $\mathcal{E}^\bullet := (\mathcal{E}_B, \nabla)$ (resp., a Miura $sl_2$-oper $\hat{\mathcal{E}}^\bullet := (\mathcal{E}_B, \nabla, \mathcal{E}'_B, \eta)$) is dormant if $\nabla$ has vanishing $p$-curvature (cf. [20, Sect. 3.3, Definition 3.8] for the definition of $p$-curvature). Denote by

$$Op_\mathcal{X} \ (\text{resp., } MOp_\mathcal{X})$$

the set of isomorphism classes of $sl_2$-opers (resp., generic Miura $sl_2$-opers) on $\mathcal{X}$. Also, denote by

$$Op^{zzz...}_\mathcal{X} \ (\text{resp., } MOp^{zzz...}_\mathcal{X})$$

the subset of $Op_\mathcal{X}$ (resp., $MOp_\mathcal{X}$) consisting of dormant $sl_2$-opers (resp., dormant generic Miura $sl_2$-opers). The assignment $(\mathcal{E}_B, \nabla, \mathcal{E}'_B, \eta) \mapsto (\mathcal{E}_B, \nabla)$ determines a map of sets

$$MOp_\mathcal{X} \to Op_\mathcal{X},$$

which is nothing but the Miura transformation discussed in Introduction. This map restricts to a map

$$\mu : MOp^{zzz...}_\mathcal{X} \to Op^{zzz...}_\mathcal{X},$$

referred to as the dormant Miura transformation for $\mathcal{X}$.
2.3 Radius and Exponent

Next, let $t$ be the Lie algebra associated with the maximal torus of $\text{PGL}_2$ consisting of the images via the quotient $\text{GL}_2 \twoheadrightarrow \text{PGL}_2$ of invertible diagonal $2 \times 2$ matrices. Denote by $c$ the GIT quotient $\mathfrak{sl}_2//\text{PGL}_2$ of $\mathfrak{sl}_2$ by the adjoint action of $\text{PGL}_2$ and by $\chi : t \twoheadrightarrow c$ the composite $t \hookrightarrow \mathfrak{sl}_2 \twoheadrightarrow c$. Since both $t$ and $c$ can be defined over $\mathbb{F}_p$, it makes sense to speak of the sets $t(\mathbb{F}_p)$, $c(\mathbb{F}_p)$ of the $\mathbb{F}_p$-rational points of $t$, $c$ respectively. Let us write

$$\tilde{\rho} := \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \in t(\mathbb{F}_p).$$

Then, the assignment $\varepsilon \mapsto \varepsilon \cdot \tilde{\rho}$ determines a bijection $k \sim \rightarrow t(k)$.

According to [20, Sect. 2.8, Definition 2.32], the notion of radii of a given $\mathfrak{sl}_2$-oper is defined as an element of $c(k)^{\times r}$ (if $r > 0$). Also, the notion of exponent (cf. [17, Sect. 3, Definition 3.6.1]) of a given generic Miura $\mathfrak{sl}_2$-oper is defined as an element of $t(k)^{\times r}$ (if $r > 0$). For convenience, we shall refer to any $\mathfrak{sl}_2$-oper (resp., generic Miura $\mathfrak{sl}_2$-oper) as being of radius $\emptyset$ (resp., of exponent $\emptyset$). For each $\tilde{\rho} \in c(k)^{\times r}$ (resp., $\tilde{\varepsilon} \in t(k)^{\times r}$), where $\tilde{\rho} := \emptyset$ (resp., $\tilde{\varepsilon} := \emptyset$) if $r = 0$, we denote by

$$O_{\mathcal{X}, \tilde{\rho}}^{\text{ZZ...}} \left( \text{resp., } MO_{\mathcal{X}, \tilde{\rho}}^{\text{ZZ...}} \right)$$

the subset of $O_{\mathcal{X}, \tilde{\rho}}^{\text{ZZ...}}$ (resp., $MO_{\mathcal{X}, \tilde{\rho}}^{\text{ZZ...}}$) consisting of dormant $\mathfrak{sl}_2$-opers of radii $\tilde{\rho}$ (resp., dormant generic Miura $\mathfrak{sl}_2$-opers of exponents $\tilde{\varepsilon}$). It follows from [20, Theorem C] and [17, Sect. 3, Theorem 3.8.3] that these sets are finite. Moreover, the sets $O_{\mathcal{X}, \tilde{\rho}}^{\text{ZZ...}}$ and $MO_{\mathcal{X}, \tilde{\rho}}^{\text{ZZ...}}$ decompose into the disjoint unions

$$O_{\mathcal{X}, \tilde{\rho}}^{\text{ZZ...}} = \bigsqcup_{\tilde{\rho} \in c(\mathbb{F}_p)^{\times r}} O_{\mathfrak{sl}_2, \mathcal{X}, \tilde{\rho}}^{\text{ZZ...}}, \quad MO_{\mathcal{X}, \tilde{\rho}}^{\text{ZZ...}} = \bigsqcup_{\tilde{\varepsilon} \in t(\mathbb{F}_p)^{\times r}} MO_{\mathcal{X}, \tilde{\varepsilon}}^{\text{ZZ...}}, \quad (2.2)$$

respectively. The map $\mu$ restricts to a map of sets

$$\mu_{\tilde{\varepsilon}} : MO_{\mathcal{X}, \tilde{\varepsilon}}^{\text{ZZ...}} \rightarrow O_{\mathcal{X}, \chi(\tilde{\varepsilon})}^{\text{ZZ...}},$$

where $\chi(\tilde{\varepsilon}) := (\chi(\varepsilon_i))_{i=1}^r$ if $r > 0$ (resp., $\chi(\tilde{\varepsilon}) := \emptyset$ if $r = 0$).

2.4 Pre-Tango Structures on a Pointed Stable Curve

In this subsection, we recall the definition of a pre-Tango structure given in [17, Sect. 5, Definition 5.3.1]. Let $\mathcal{X} := (X, \{\sigma_i\}_{i=1}^r)$ be as above.

**Definition 2.9** Let $\tilde{\nabla}$ be a $k^{\text{log}}$-connection on the line bundle $\Omega_{X^{\text{log}}/k^{\text{log}}}$, i.e., a $k$-linear morphism $\tilde{\nabla} : \Omega_{X^{\text{log}}/k^{\text{log}}} \rightarrow \Omega_{X^{\text{log}}/k^{\text{log}}}^{\otimes 2}$ satisfying the Leibniz rule, meaning that $\tilde{\nabla}(a \cdot$
\( v) = da \otimes v + a \cdot \tilde{\nabla}(v) \) for any local sections \( a \in O_X, v \in \Omega^1_{X} \). Then, we shall say that \( \tilde{\nabla} \) is a \textbf{pre-Tango structure} on \( X \) if it satisfies the following two conditions:

- It has vanishing \( p \)-curvature;
- Let \( C_{X}^{\log/k_{\log}} \) denote the Cartier operator \( \Omega^1_{X} \otimes k_{\log} \to \Omega^1_{X} \) induced by \( \tilde{\nabla} \) (the inverse of) \( C^{-1} \) resulting from [7, Theorem 4.12 (1)], where \( X^{(1)} \) is the Frobenius twist of \( X \) over \( k \). Then, the inclusion relation \( \text{Ker}(\tilde{\nabla}) \subseteq \text{Ker}(C_{X}^{\log/k_{\log}}) \) holds.

Note that it makes sense to speak of the \textit{monodromy (operator)} of a pre-Tango structure on \( X \) at each marked point \( \sigma_i \) of \( X \) (cf. [20, Sect. 1.6, Definition 1.46] for the definition of monodromy). The monodromy of a pre-Tango structure lies in \( k = \mathbb{E}nd_{O_X}(\sigma_i^*(\Omega^1_{X} \otimes k_{\log})) \). For convenience, we shall refer to any pre-Tango structure as being \textit{of monodromy} \( \emptyset \).

Let \( \tilde{\varepsilon} := (\varepsilon_i)_{i=1}^r \) be an element of \( \mathbb{F}_p^r \), where \( \tilde{\varepsilon} := \emptyset \) if \( r = 0 \). Denote by

\[ Tan_{\mathcal{X}} \quad (\text{resp., } Tan_{\mathcal{X}, \tilde{\varepsilon}}) \]

the set of pre-Tango structures on \( \mathcal{X} \) (resp., pre-Tango structures on \( \mathcal{X} \) monodromies \( \tilde{\varepsilon} \)). The set \( Tan_{\mathcal{X}} \) decomposes into the disjoint union

\[ Tan_{\mathcal{X}} = \bigsqcup_{\tilde{\varepsilon} \in \mathbb{F}_p^r} Tan_{\mathcal{X}, \tilde{\varepsilon}} \quad (2.3) \]

(cf. [17, (190)]). If \( Con_{\mathcal{X}} \) denotes the set of \( k_{\log} \)-connections on \( \Omega^1_{X} \), then there exists a natural bijection of sets

\[ Con_{\mathcal{X}} \sim \to MOp_{\mathcal{X}} , \quad (2.4) \]

which induces, by restriction, a bijection

\[ Tan_{\mathcal{X}} \sim \to MOp_{\mathcal{X}}^{zzz...} \quad (2.5) \]

(cf. [17, Theorems 4.4.1 and 5.4.1]).

Next, let us write \( \tilde{\varepsilon} \cdot \tilde{\rho} := (\varepsilon_i \cdot \tilde{\rho}_i)_{i=1}^r \), where \( \tilde{\varepsilon} \cdot \tilde{\rho} := \emptyset \) if \( r = 0 \). Then, (2.5) restricts to a bijection

\[ Tan_{\mathcal{X}, \tilde{\varepsilon}} \sim \to MOp_{\mathcal{X}, \tilde{\varepsilon} \cdot \tilde{\rho}}^{zzz...} \quad (2.6) \]

\textbf{2.5 Gluing Pre-Tango Structures}

Let us discuss the procedure for gluing pre-Tango structures by means of clutching data. To begin with, we shall recall the notion of clutching data (cf. [20, Sect. 7.1, Definition 7.4]).
Definition 2.10  By clutching data of type \((g, r)\), we shall mean a collection of data:

\[
\mathcal{J} := (\Gamma^+, \{(g_v, r_v)\}_{v \in V}, \{\lambda_v\}_{v \in V}),
\]

where

- \(\Gamma^+ := (V, E, \zeta, \lambda)\) denotes a marked semi-graph with \(r = \sharp(B_\otimes)\) whose underlying semi-graph is finite and connected;
- \((g_v, r_v)\) (for each \(v \in V\)) is a pair of nonnegative integers with \(2g_v - 2 + r_v > 0\), \(r_v > 0\), and \(g = b_1(\Gamma) + \sum_{j=1}^n g_j\) (cf. Remark 2.5);
- \(\lambda_v\) (for each \(v \in V\)) denotes a bijection of sets \(\lambda_v : B_v \sim \{1, \ldots, r_v\}\).

Let \(\mathcal{J}\) be clutching data of type \((g, r)\) as in (2.7). Also, let us take a collection \(\{\mathcal{J}_v\}_{v \in V}\) of pointed stable curves over \(k\) indexed by the elements of \(V\), where \(\mathcal{J}_v := (X_v, \{\sigma_i\}_{i=1}^{r_v})\) is \(r_v\)-pointed and of genus \(g_v\), and moreover the underlying curve \(X_v\) is irreducible. Then, we can use \(\mathcal{J}\) to glue together \(\{\mathcal{J}_v\}_{v \in V}\) and obtain an \(r\)-pointed stable curve \(\mathcal{J}^- := (X, \{\sigma_i\}_{i=1}^{r})\) in such a way that

- the dual marked semi-graph associated with \(\mathcal{J}^-\) is given by \(\Gamma^+\), where each vertex \(v \in V\) corresponds to the irreducible component \(X_v\);
- if \(e = \{b_1, b_2\}\) is a closed edge of \(\Gamma\) with \(\zeta(b_1) = v_1, \zeta(b_2) = v_2\) (for some \(v_1, v_2 \in V_\Gamma\)), then \(e\) corresponds to the node of \(X\) obtained by gluing together \(X_{v_1}\) at the \(\lambda_{v_1}(b_1)\)-th marked point \(\sigma_{v_1, \lambda_{v_1}(b_1)}\) to \(X_{v_2}\) at the \(\lambda_{v_2}(b_2)\)-th marked point \(\sigma_{v_2, \lambda_{v_2}(b_2)}\);
- for each \(v \in V\), the set of open edges \(e \in E\) with \(\zeta(e) = \{v, \emptyset\}\) is in bijection with the set of marked points of \(\mathcal{J}^-\) lying on \(X_v\);
- for each \(i = 1, \ldots, r\), the \(i\)-th marked point of \(\mathcal{J}^-\) is chosen as the \(\lambda_{\zeta(\lambda^{-1}(i)^*)}(\lambda^{-1}(i)^*)\)-th marked point of \(\mathcal{J}_{\zeta(\lambda^{-1}(i)^*)}^-\).

By the construction of \(\mathcal{J}^-\), we see that the genus of \(X\) equals \(g\) and there exist natural morphisms \(\text{Clut}_v : X_v \to X (v \in V)\). Also, one may extend, in an evident way, this construction to the case where the \(X_v\)’s are (possibly reducible) pointed stable curves.

As mentioned in Sect. 2.2, the pointed stable curve \(\mathcal{J}^-\) associates a log-curve \(X^\log \to \text{Spec}(k)^\log\) extending the underlying prestable curve \(X \to \text{Spec}(k)\). Also, \(\mathcal{J}^-\) for each \(v \in V\) associates a log-curve \(X^\log_v \to \text{Spec}(k)^\log_v\) extending \(X_v \to \text{Spec}(k)\), where we use the notation “\(\text{Spec}(k)^\log\)” so as not to be confused with \(\text{Spec}(k)^\log\) defined just above (in particular, we have \(\text{Spec}(k)^\log_v = \text{Spec}(k)\)). We shall write \(X^\log|_X\) for the log scheme obtained by equipping \(X_v\) with the log structure pulled-back from the log structure of \(X^\log\) via \(\text{Clut}_v\). Then, the natural morphism \(X^\log|_X \to X_v\) extends naturally to a commutative square diagram

\[
\begin{array}{ccc}
X^\log|_X & \longrightarrow & X^\log \\
\downarrow & & \downarrow \\
\text{Spec}(k)^\log & \longrightarrow & \text{Spec}(k)^\log_v.
\end{array}
\]
This diagram induces a morphism

\[ e_v : X^\log|X \rightarrow X^\log_v \times_{\Spec(k)^\log} \Spec(k)^\log \]

over \( \Spec(k)^\log \) whose underlying morphism of \( k \)-schemes coincides with the identity morphism of \( X_v \). The differential of this morphism yields an isomorphism of \( O_{X_v} \)-modules

\[ \Omega^1_{X^\log|X} / \Omega^1_{X^\log} \cong \Omega^1_{X^\log_v} / \Omega^1_{k^\log} \cong \Clut^*_v(\Omega^1_{X^\log / k^\log}) . \] (2.8)

Now, let \( \tilde{\nabla} \) be a \( k^\log \)-connection on \( \Omega^1_{X^\log / k^\log} \), and let us fix \( v \in V \). By using the identification given by (2.8), we can pull-back \( \tilde{\nabla} \) via \( \Clut^*_v \) to obtain a \( k^\log_v \)-connection

\[ \tilde{\nabla}^v : \Omega^1_{X^\log_v / k^\log_v} \rightarrow \Omega^2_{X^\log_v / k^\log_v} \]

on \( \Omega^1_{X^\log_v / k^\log_v} \). We shall refer to \( \tilde{\nabla}^v \) as the restriction of \( \tilde{\nabla} \) to \( X_v \).

**Proposition 2.11** Let us keep the above notation.

(i) Suppose that \( \tilde{\nabla} \) specifies a pre-Tango structure on \( X \). Then, for each \( v \in V \), the restriction \( \tilde{\nabla}^v \) specifies a pre-Tango structure on \( X_v \). If, moreover, \( \tilde{\nabla}^v \) \( (v \in V) \) is of monodromies \( \vec{\epsilon}^v := (\tau \lambda - 1)^r v \), then the collection

\[ (\tau^{-1}(\xi(\lambda(b)))) b \in B \in \tilde{\Gamma}^p \] (where \( \xi^p(b) := -\xi^p(b^*) \) for any \( b \in B^p \)) forms a \( p \)-branch numbering on \( \Gamma \).

(ii) Conversely, let \( \vec{m}_G := (m_b) b \in B \in \tilde{\Gamma}^p \) be a \( p \)-branch numbering on \( \Gamma \) of exponent \( \vec{e} \in \mathbb{F}^\times_p \), where \( \vec{e} := \emptyset \) if \( r = 0 \). Suppose that on each \( X_v \), we are given a pre-Tango structure \( \tilde{\nabla} \) of monodromies \( \vec{e} \), which is uniquely determined by the condition that for any \( v \in V \) the restriction of \( \tilde{\nabla} \) to \( X_v \) coincides with \( \tilde{\nabla}^v \).

**Proof** Assertions (i) and (ii) follow immediately from [20, Sect. 7.2, Proposition 7.6], and the fact that for each \( v \in V \), the restriction of the Cartier operator \( C_{X^\log / k^\log} \) to \( X_j \) may be identified, via (2.8), with \( C^\log_{X^\log / k^\log} \).

Let \( \vec{e} \) be an element of \( \mathbb{F}^\times_p \) (where \( \vec{e} := \emptyset \) if \( r = 0 \)), \( G \) clutching data of type \( (g, r) \) with underlying marked semi-graph \( \Gamma^+ := (V, E, \xi, \lambda) \), and \( \vec{m}_G := (m_b) b \in B \in \tilde{\Gamma}^p \) a \( p \)-branch numbering on \( \Gamma \) of exponent \( \vec{e} \). For each \( v \in V \), we shall write

\[ \vec{e}^v := (\tau \lambda^{-1}(i)) r_v \]. Then, it follows from Proposition 2.11 that a collection of
pre-Tango structures \((\hat{V}_v)_{v \in V}\), where each \(\hat{V}_v\) is of monodromies \(\tilde{e}_v\), induces a pre-Tango structure \(\hat{V}\) on \(\mathcal{X}\) of monodromies \(\tilde{e}\). The assignment \((\hat{V}_v)_{v \in V} \mapsto \hat{V}\) defines a map of sets

\[
\text{Clut}_{G, \vec{m}} : \prod_{v \in V} \text{Tan}_{\mathcal{X}, \tilde{e}_v} \to \text{Tan}_{\mathcal{X}, \tilde{e}}.
\]

**Proposition 2.12** Let \(G\) and \(\tilde{e}\) be as above. Then, the following map is bijective:

\[
\bigcup \prod_{\vec{m} \in \tilde{F}_B} \prod_{v \in V} \text{Tan}_{\mathcal{X}, \tilde{e}_v} \overset{\text{Clut}_{G, \vec{m}_{\vec{m}}}}{\longrightarrow} \text{Tan}_{\mathcal{X}, \tilde{e}},
\]

where the disjoint union in the domain is taken over the set of \(p\)-branch numberings on \(\Gamma\) of exponent \(\vec{e}\).

**Proof** The assertion follows immediately from (2.3), Proposition 2.11, and the definition of \(\text{Clut}_{G, \vec{m}_{\vec{m}}}\). \(\square\)

### 3 Combinatorial Description of Dormant Miura Opers

#### 3.1 Balanced \(p\)-Edge Numberings

In this section, we shall study a combinatorial description of dormant Miura \(\mathfrak{sl}_2\)-opers (equivalently, pre-Tango structures) on a totally degenerate curve. We first recall the combinatorial description of dormant \(\mathfrak{sl}_2\)-opers on a totally degenerate curve, which was essentially given in the previous work of \(p\)-adic Teichmüller theory due to S. Mochizuki (cf. [12, Chap. V, Sect. 1, (3), p. 232] or [20, Sect. 7.8]). In the present paper, the combinatorial objects used to describe dormant \(\mathfrak{sl}_2\)-opers will be referred to as balanced \(p\)-edge numberings (cf. Definition 3.1 below).

Let us fix a marked semi-graph \(\Gamma^+ := (V, E, \xi, \lambda)\) whose underlying semi-graph is connected, 3-regular, and of type \((g, r)\).

**Definition 3.1** (i) A balanced \(p\)-edge numbering on \(\Gamma^+\) is a collection

\[
\vec{m} := (m_b)_{b \in B} \in \tilde{F}_B
\]

of elements of \(\tilde{F}_B\) indexed by \(B\) satisfying the following two conditions:

- For each edge \(e := \{b, b^*\} \in E\), the equality \(m_b = m_{b^*}\) holds.
- For each vertex \(v \in V\) (where we write \(B_v := \{b_1, b_2, b_3\}\) and \(m_l := m_{b_l}\) for each \(l = 1, 2, 3\)), the inequalities in \(\Delta_{m_1, m_2, m_3}\) displayed below are satisfied:

\[
\Delta_{m_1, m_2, m_3} : |m_2 - m_3| \leq m_1 \leq m_2 + m_3, m_1 + m_2 + m_3 \leq p - 2.
\]

(By the first condition, any balanced \(p\)-edge numbering may be thought of as a numbering written on the set of edges \(E\), as its name suggests. The first figure in Introduction illustrates an example of a balanced \(p\)-edge numbering for any sufficiently large \(p\).)
ticular, we have

\[ Ed_{\Gamma^+} \]

the set of balanced \( p \)-edge numberings on \( \Gamma^+ \). Also, for each \( \vec{e} \in \mathbb{P}^r_p \) (where \( \vec{e} := \emptyset \) if \( r = 0 \), we shall write

\[ Ed_{\Gamma^+, \vec{e}} \]

for the subset of \( Ed_{\Gamma^+} \) consisting of balanced \( p \)-edge numberings of radii \( \vec{e} \). In particular, we have

\[ Ed_{\Gamma^+} = \bigsqcup_{\vec{e} \in \mathbb{P}^r_p} Ed_{\Gamma^+, \vec{e}}. \tag{3.1} \]

3.2 Combinatorial Description of Dormant \( sl_2 \)-Opers

Next, let us construct a bijective correspondence between the set of dormant \( sl_2 \)-opers on a totally degenerate curve and the set of balanced \( p \)-branch numberings on the dual semi-graph of this curve. First, let us consider the case where \( (g, r) = (0, 3) \). Denote by \( [0], [1], \) and \( [\infty] \) the \( k \)-rational points of the projective line \( \mathbb{P}^1 \) over \( k \) determined by the values 0, 1, and \( \infty \), respectively. After ordering the points \( [0], [1], [\infty] \) suitably (say, \( \sigma_1 := [0], \sigma_2 := [1], \) and \( \sigma_3 := [\infty] \)), we obtain a unique (up to isomorphism) 3-pointed stable curve

\[ \mathcal{P} := (\mathbb{P}^1, \{\sigma_i\}_{i=1}^3) \]

of genus 0 over \( k \). The dual marked semi-graph \( \Gamma^+_\mathcal{P} := (V, E, \xi, \lambda) \) associated with \( \mathcal{P} \) is given as follows:

- \( V := \{v := [\mathbb{P}^1]\}; \)
- \( E := \{e_1, e_2, e_3\} \), where \( e_i := \{\sigma_i, i\} \) (\( i = 1, 2, 3 \));
- \( \xi : B \to V \sqcup \{\emptyset\} \) is given by \( \xi(\sigma_i) = v \) and \( \xi(i) = \emptyset \) (\( i = 1, 2, 3 \)).
- \( \lambda : B := \{1, 2, 3\} \to \{1, 2, 3\} \) is defined as the identity map.

The assignment \( (m_b)_{b \in B} \mapsto (m_{\sigma_1}, m_{\sigma_2}, m_{\sigma_3}) \) gives a bijective correspondence between the set of balanced \( p \)-edge numberings on \( \Gamma^+_\mathcal{P} \) and the set of triples of integers \( (m_1, m_2, m_3) \in \mathbb{Z}^3 \) satisfying the condition \( \Delta_{m_1, m_2, m_3} \). The inverse assignment is given by \( (m_1, m_2, m_3) \mapsto (m_b)_{b \in B} \), where \( m_{\sigma_i} := m_i \) and \( m_i := m_i \) (\( i = 1, 2, 3 \)). By using this correspondence, we shall identify each balanced \( p \)-edge numbering on \( \Gamma^+_\mathcal{P} with such a triple \( (m_1, m_2, m_3) \).

Now, denote by \( \iota \) the bijection of sets defined as
$$\iota: k \rightsquigarrow \iota(k)$$
$$\psi \quad \psi$$
$$a \mapsto \chi \left( \left( \begin{array}{c}
 a + \frac{p+1}{2} \\
 0
\end{array} \right) \left( \begin{array}{c}
 b + \frac{p+1}{2}
\end{array} \right) \right),$$

where $\iota$ denotes the image of $\iota$ via the composite $\mathbb{Z} \rightarrow \mathbb{F}_p \hookrightarrow k$. Recall from [12, Chap. I, § 4.3, p. 117, Theorem 4.4] (or [20, Theorem A]) that for each $(\rho_1, \rho_2, \rho_3) \in \mathcal{C}(k)^3$ there exists a unique $\mathfrak{sl}_2$-oper $\mathcal{E}_{\rho_1, \rho_2, \rho_3}$ on $\mathcal{P}$ of radii $(\rho_1, \rho_2, \rho_3)$. That is to say, we obtain a bijection

$$k^3 \sim \rightarrow Op_{\mathcal{P}}$$
$$\psi \quad \psi$$
$$(a_1, a_2, a_3) \mapsto \mathcal{E}_{\iota(a_1), \iota(a_2), \iota(a_3)}.$$

**Lemma 3.2** The composite $\mathbb{Z}^3 \rightarrow \mathbb{F}_p^3 \hookrightarrow k^3 \rightarrow Op_{\mathcal{P}}$ restricts to a bijection

$$Ed_{\Gamma_+} \sim \rightarrow Op_{\mathcal{P}}.$$

**Proof** The assertion follows from [12, Chap. V, Sect. 1, (3), p. 232] (cf. [20, Sect. 7.8]).$$\square$$

Next, we shall extend the above result to the case where the underlying curve is an arbitrary totally degenerate curve. To this end, let us recall the definition of a totally degenerate curve. Let $X := (X, \{\sigma_i\}_{i=1}^r)$ be an $r$-pointed stable curve over $k$ of genus $g$. Write $v: \bigsqcup_{l=1}^L X_l \rightarrow X$ for the normalization of $X$, where $L$ denotes some positive integer and each $X_l$ $(l = 1, \ldots, L)$ is a proper smooth connected curve over $k$. Then, we shall say that $X$ is totally degenerate if, for any $l = 1, \ldots, L$, the pointed stable curve $X_l := (X_l, v^{-1}(E_{\mathcal{Y}}) \cap X_l(k))$ is isomorphic to $\mathcal{P}$, where we regard each element of $E_{\mathcal{Y}} := N_{\mathcal{Y}} \sqcup \{\sigma_i\}_{i=1}^r$ (cf. (2.1)) as a $k$-rational point of $X$.

Now, let $\mathcal{Y}$ be a totally degenerate $r$-pointed stable curve over $k$ of genus $g$. Then, $\mathcal{Y}$ may be obtained by gluing together finite copies of $\mathcal{P}$ by means of some clutching data whose underlying marked semi-graph is $\Gamma_+^{\mathcal{Y}}$. For each $v \in V_\mathcal{Y}$, we shall denote by $\mathcal{P}_v$ the 3-pointed projective line corresponding to $v$ (in particular, $\mathcal{P}_v \cong \mathcal{P}$). Also, for each $v \in V_{\mathcal{Y}}$ and $b \in B_v$, we shall denote by $\sigma_b$ the marked point of $\mathcal{P}_v$ corresponding to $b$.

Let $(m_p)_{p \in B}$ be a balanced $p$-edge numbering on $\Gamma_+^{\mathcal{Y}}$. For each $v \in V_{\mathcal{Y}}$ with $B_v := \{b_1, b_2, b_3\}$, the triple $(m_{b_1}, m_{b_2}, m_{b_3})$ specifies a balanced $p$-edge numbering on $\Gamma_+^{\mathcal{Y}}$. This triple corresponds, via (3.3), to a dormant $\mathfrak{sl}_2$-oper $\mathcal{E}_v$ on $\mathcal{P}_v$. One may
assume, without loss of generality, that each $E_v$ is normal, in the sense of [20, Sect. 4.9, Definition 4.53]. If $e := \{b, b^*\}$ is a closed edge of $\Gamma^+_\mathcal{X}$, then the radius of $E_v(b)$ at $\sigma_b$ coincides with the radius of $E_v(b^*)$ at $\sigma_{b^*}$. It follows from [20, Sect. 7.3, Proposition 7.12] that $E_v$'s may be glued together to obtain a dormant (normal) $\mathfrak{sl}_2$-oper $E_v$ on $\mathcal{X}$. The bijectivity of (3.3) implies the following proposition.

**Proposition 3.3** Let $\mathcal{X}$ be as above. Then, the assignment $(m_b)_{b \in B} \mapsto E_v$ discussed above defines a bijection

$$Ed_{\Gamma^+_\mathcal{X}} \sim \to Op_{\mathcal{X}}.$$  

(3.4)

If, moreover, $r > 0$, then for each $\vec{\epsilon} := (\epsilon_i)_{i=1}^r \in \mathbb{Z}^r_p$, (3.4) restricts to a bijection

$$Ed_{\Gamma^+_\mathcal{X}},\vec{\epsilon} \sim \to Op_{\mathcal{X}},\iota(\vec{\epsilon}),$$  

(3.5)

where $\iota(\vec{\epsilon}) := (\iota(\epsilon_i))_{i=1}^r$.

### 3.3 Strict $p$-Branch Numberings

Next, in order to construct a combinatorial description of dormant generic Miura $\mathfrak{sl}_2$-opers (or equivalently, pre-Tango structures), we shall introduce the notion of a strict $p$-branch numbering. Let $\Gamma^+ := (V, E, \zeta, \lambda)$ be a marked semi-graph whose underlying semi-graph $\Gamma$ is connected, 3-regular, and of type $(g, r)$. We write $B := \bigsqcup_{e \in E} e$ and $\Gamma := (V, E, \zeta)$.

**Definition 3.4** A strict $p$-branch numbering on $\Gamma^+$ is a $p$-branch numbering $\vec{m} := (m_b)_{b \in B} \in \mathbb{Z}^r_p$ on $\Gamma$ such that, for each vertex $v \in V$, the equality $\sum_{j=1}^3 m_{b_j} = 1 + p$ holds, where $\{b_1, b_2, b_3\} := B_v$. (The second figure in Introduction illustrates an example of a strict $p$-branch numbering for $p = 11$.)

Let $\vec{\epsilon}$ be an element of $\mathbb{Z}^r_p$, where $\vec{\epsilon} := \emptyset$ if $r = 0$. Denote by

$$Br_{\Gamma^+} \quad (\text{resp., } Br_{\Gamma^+, \vec{\epsilon}})$$

the set of strict $p$-branch numberings on $\Gamma^+$ (resp., strict $p$-branch numberings on $\Gamma^+$ of exponent $\vec{\epsilon}$). The set $Br_{\Gamma^+}$ decomposes into the disjoint union

$$Br_{\Gamma^+} = \bigsqcup_{\vec{\epsilon} \in \mathbb{Z}^r_p} Br_{\Gamma^+, \vec{\epsilon}}.$$  

(3.6)

Now, let us construct an assignment from each strict $p$-branch numbering to a balanced $p$-edge numbering. Given an element $m \in \mathbb{Z}^r_p$, we shall write $m^\mu$ for the element of $\mathbb{Z}^r_p$ defined as follows:
For each strict \( p \)-branch numbering \( \vec{m} := (m_b)_{b \in B} \) on \( \Gamma^+ \), the collection \( \vec{m}^\mu := (m_b^\mu)_{b \in B} \) is verified to specify a balanced \( p \)-edge numbering on \( \Gamma^+ \). That is to say, the assignment \( \vec{m} \mapsto \vec{m}^\mu \) defines a map of sets

\[
\mu^{\text{comb}} : Br^{+} \rightarrow Ed^{+}.
\]

We shall refer to this map as the **combinatorial dormant Miura transformation** for \( \Gamma \). Moreover, given \( \vec{\epsilon} := (\epsilon_i)_{i=1}^r \in \mathbb{F}_p^\times r \) (where \( \vec{\epsilon} := \emptyset \) if \( r = 0 \)), the map \( \mu^{\text{comb}} \) restricts to a map

\[
\mu^{\text{comb}}_{\vec{\epsilon}} : Br^{+}_{\vec{\epsilon}} \rightarrow Ed^{+}_{\vec{\epsilon}^\mu},
\]

where \( \vec{\epsilon}^\mu := (\epsilon_i^\mu)_{i=1}^r \) (and \( \vec{\epsilon}^\mu := \emptyset \) if \( r = 0 \)).

### 3.4 Combinatorial Description of Dormant Generic Miura \( \mathfrak{sl}_2 \)-opers

We shall give a combinatorial description of pre-Tango structures (as well as dormant generic Miura \( \mathfrak{sl}_2 \)-opers) on a totally degenerate curve by using strict \( p \)-branch numberings. The following assertion is the main result of this subsection.

**Theorem 3.5** Let \( \mathcal{X} \) be a totally degenerate \( r \)-pointed stable curve over \( k \) of genus \( g \), and let \( \vec{\epsilon} \) be an element of \( \mathbb{F}_p^\times r \), where \( \vec{\epsilon} := \emptyset \) if \( r = 0 \). Then, there exists a canonical bijection

\[
Br^{+}_{\mathcal{X}, \vec{\epsilon}} \sim \Tan_{\mathcal{Y}, \vec{\epsilon}}
\]

which makes the following diagram commute:

\[
\begin{array}{ccc}
Br^{+}_{\mathcal{X}, \vec{\epsilon}} & \xrightarrow{(3.7)} & \Tan_{\mathcal{Y}, \vec{\epsilon}} \\
\mu^{\text{comb}}_{\vec{\epsilon}} \downarrow & & \downarrow \mu_{\vec{\epsilon}} \\
Ed^{+}_{\mathcal{X}, \vec{\epsilon}^{\mu}} & \xrightarrow{(3.5)} & Op^{zzz}_{\mathcal{Y}, \chi(\vec{\epsilon}, \vec{\rho})}
\end{array}
\]

In particular, the set \( Op^{zzz}_{\mathcal{Y}, \chi(\vec{\epsilon}, \vec{\rho})} \) is in bijection with the set \( Br^{+}_{\mathcal{X}, \vec{\epsilon}} \).

To prove the above theorem, let us first consider the case where \( \mathcal{X} = \mathcal{P} \). By the correspondence \( (m_b)_b \leftrightarrow (m_{\sigma_1}, m_{\sigma_2}, m_{\sigma_3}) \), we shall identify each \( p \)-branch numbering on \( \Gamma \mathcal{P} \) with an element of \( \mathbb{F}_p^{3} \).
Lemma 3.6  Let us consider the set
\[ Br'_{\Gamma^+} := \left\{ (m_1, m_2, m_3) \in \mathbb{Z}_{\geq 0}^3 \mid p\left( \sum_{i=1}^{3} m_i - 1 \right) \right\}. \]

Then, an element \((m_1, m_2, m_3)\) of \( Br'_{\Gamma^+} \) specifies a strict \( p \)-branch numbering on \( \Gamma^+ \)(under the identification mentioned above) if and only if the triple \((m_1^\mu, m_2^\mu, m_3^\mu)\) specifies a balanced \( p \)-edge numbering on \( \Gamma^+ \)(i.e., satisfies the inequalities in \( \Delta_{m_1^\mu, m_2^\mu, m_3^\mu} \)).

Proof  Let \((m_1, m_2, m_3)\) be an element of \( Br'_{\Gamma^+} \) such that \((m_1^\mu, m_2^\mu, m_3^\mu)\) satisfies the inequalities in \( \Delta_{m_1^\mu, m_2^\mu, m_3^\mu} \). To complete the proof, it suffices to verify that this triple specifies a strict \( p \)-branch numbering on \( \Gamma^+ \). Define \( m \) to be the integer with \( mp = \sum_{i=1}^{3} m_i - 1 \). Since \( m_i \leq p - 1 \), the inequality \( m < 3 \) holds. Consider the case where \( m = 0 \). After possibly changing the ordering, we may assume that \( m_1 = m_2 = 0 \) and \( m_3 = 1 \). Then, we obtain a contradiction since
\[ p - 2 \geq \sum_{i=1}^{3} m_i^\mu = \frac{p - 1}{2} + \frac{p - 1}{2} + 0 = p - 1, \]
where the first inequality follows from \( \Delta_{m_1^\mu, m_2^\mu, m_3^\mu} \). Next, let us assume that \( m = 2 \). As the sum \( \sum_{i=1}^{3} m_i \) is odd by this assumption, either one of the following two cases (a), (b) is satisfied:

(a) the three integers \( m_1, m_2, m_3 \) are all odd;
(b) two of the three integers \( m_1, m_2, m_3 \) are even and the remaining one is odd.

In the case (a), we obtain a contradiction since
\[ p - 2 \geq \sum_{i=1}^{3} m_i^\mu = \sum_{i=1}^{3} m_i - 1 \geq \frac{p - 1}{2} + \frac{p - 1}{2} + 0 = p - 1. \]

On the other hand, to consider the case (b), we may assume, without loss of generality, that \( m_1 \) is odd and both \( m_2 \) and \( m_3 \) are even. Then, we have
\[ 0 \leq -m_1^\mu + m_2^\mu + m_3^\mu = \frac{2p - 1 - \sum_{i=1}^{3} m_i}{2} = -1, \]
so this is a contradiction. As a consequence, the equality \( m = 1 \) (i.e., \( \sum_{i=1}^{3} m_i = p + 1 \)) must be satisfied. This completes the proof of the lemma. \( \square \)

Next, denote by \( Con_{\mathcal{P}}^{\psi=0} \) the set of (logarithmic) \( k \)-connections on \( \Omega_{\mathcal{P}^{1log/k}} \) with vanishing \( p \)-curvature. Hence, \( Tan_{\mathcal{P}} \) is a subset of \( Con_{\mathcal{P}}^{\psi=0} \). Let \((m_1, m_2, m_3)\) be a triple in \( Br_{\Gamma^+} \) and let \( m \) be the integer with \( mp = \sum_{i=1}^{3} m_i - 1 \). Denote by \( \mathcal{O}_{\mathcal{P}^{1(1)}}(\mathbb{Z}) \) Springer
a unique (up to isomorphism) line bundle of degree \(-m\) on the Frobenius twist \(\mathbb{P}^1(1)\) of \(\mathbb{P}^1\). We obtain the pull-back \(F^* (\mathcal{O}_{\mathbb{P}^1(1)}(-m))\) of \(\mathcal{O}_{\mathbb{P}^1(1)}(-m)\) via the relative Frobenius morphism \(F : \mathbb{P}^1 \rightarrow \mathbb{P}^1(1)\). As is well-known, there exists a (logarithmic) \(k\)-connection \(\nabla^\text{can}\) on \(F^* (\mathcal{O}_{\mathbb{P}^1(1)}(-m))\) with vanishing \(p\)-curvature determined uniquely by the condition that the sections of the subsheaf \(F^{-1} (\mathcal{O}_{\mathbb{P}^1(1)}(-m)) \subseteq F^* (\mathcal{O}_{\mathbb{P}^1(1)}(-m))\) are contained in \(\text{Ker}(\nabla^\text{can})\). Also, one may construct uniquely a \(k\)-connection \(\nabla^\text{can}_{m_1, m_2, m_3}\) on \(F^* (\mathcal{O}_{\mathbb{P}^1(1)}(-m))\) whose restriction to \(F^* (\mathcal{O}_{\mathbb{P}^1(1)}(-m))\) coincides with \(\nabla^\text{can}\). The monodromy of \(\nabla^\text{can}_{m_1, m_2, m_3}\) at \(\sigma_i\) \((i = 1, 2, 3)\) is \(-\tau(m_i)\). Since

\[
\deg \left( F^* (\mathcal{O}_{\mathbb{P}^1(1)}(-m)) \left( \sum_{i=1}^{3} m_i \sigma_i \right) \right) = -pm + \sum_{i=1}^{3} m_i = 1 = \deg(\mathcal{O}_{\mathbb{P}^1(1)/k}),
\]

we have an isomorphism

\[
F^* (\mathcal{O}_{\mathbb{P}^1(1)}(-m)) \left( \sum_{i=1}^{3} m_i \sigma_i \right) \sim \mathcal{O}_{\mathbb{P}^1(1)/k}. \tag{3.8}
\]

The connection \(\nabla^\text{can}_{m_1, m_2, m_3}\) corresponds, via this isomorphism, to a \(k\)-connection \(\tilde{\nabla}^\text{can}_{m_1, m_2, m_3}\) on \(\Omega_{\mathbb{P}^1(1)/k}\) of monodromies \((-\tau(m_1), -\tau(m_2), -\tau(m_3)) \in k^3\) with vanishing \(p\)-curvature. Notice that this connection does not depend on the choice of (3.8). The resulting assignment \((m_1, m_2, m_3) \mapsto \tilde{\nabla}^\text{can}_{m_1, m_2, m_3}\) determines a well-defined bijection

\[
Br_{\Gamma^+_{\mathcal{P}}} \sim \text{Con}_{\mathcal{P}}^{\psi=0}. \tag{3.9}
\]

Indeed, its inverse is given by \(\nabla \mapsto (\tau^{-1}(-\mu^1), \tau^{-1}(-\mu^2), \tau^{-1}(-\mu^3))\), where each \(\mu_i \in k\) \((i = 1, 2, 3)\) denotes the monodromy of \(\nabla\) at \(\sigma_i\). In particular, a \(k\)-connection on \(\Omega_{\mathbb{P}^1/\log/k}\) with vanishing \(p\)-curvature may be uniquely determined by its monodromies.

**Lemma 3.7** The bijection displayed in (3.9) restricts to a bijection

\[
Br_{\Gamma^+_{\mathcal{P}}} \sim \text{Tan}_{\mathcal{P}}, \tag{3.10}
\]

which makes the following diagram commute:

\[
\begin{array}{ccc}
Br_{\Gamma^+_{\mathcal{P}}} & \xrightarrow{(3.10)} & \text{Tan}_{\mathcal{P}} \\
\mu_{\text{comb}} & \sim & \phi \\
E_{\Gamma^+_{\mathcal{P}}} & \xrightarrow{(3.3)} & O_{\mathcal{P}}^{\mathbb{Z}/2}
\end{array}
\]

\[
\begin{array}{ccc}
Br_{\Gamma^+_{\mathcal{P}}} & \xrightarrow{(3.11)} & MO_{\mathcal{P}}^{\mathbb{Z}/2} \\
\mu & \sim & \phi
\end{array}
\]
In particular, the set \( MOp^\mathcal{Z}_{\mathcal{P}} \) is in bijection with the set of triples of positive integers \((m_1, m_2, m_3)\) with \(\sum_{i=1}^{3} m_i = p + 1\).

**Proof** The following diagram is verified to be commutative:

\[
\begin{array}{ccc}
Br'_{\mathcal{P}} & \xrightarrow{\mu'} & Con_{\mathcal{P}} \\
\mu' & \downarrow & \downarrow \mu_{\mathcal{P}} \\
\mathbb{F}^3_p & \xrightarrow{\sim} & Op_{\mathcal{P}},
\end{array}
\]

where

- the left-hand vertical arrow \(\mu'\) denotes the map given by \((m_i)_{i=1}^{3} \mapsto (m_{\mathcal{P}}^\mu)_i = 1\);
- the upper left-hand horizontal arrow is the composite of (3.9) and the natural injection \(Con_{\mathcal{P}} \hookrightarrow Con_{\mathcal{P}}\);
- the lower horizontal arrow denotes the composite \(\mathbb{F}^3_p \xrightarrow{\rho} \mathbb{F}^3_p \hookrightarrow k^3 \xrightarrow{(3.2)} Op_{\mathcal{P}}\).

Then, the assertion follows from the fact that the desired diagram (1.4) can be obtained from (3.12) by restricting \(\mu'\) and \(\mu_{\mathcal{P}}\) over \(Ed_{\Gamma^+_{\mathcal{P}}} \subseteq \mathbb{F}^3_p\) and \(Op_{\mathcal{P}} \subseteq Op_{\mathcal{P}}\), respectively (cf. Proposition 3.3).

**Proof of Theorem 3.5** The assertion follows immediately from Proposition 2.12 and Lemma 3.7. Indeed, the isomorphisms (3.10) applied to \(\mathcal{P}_v\) for the various \(v \in V_{\mathcal{P}}\) (with the notation following Lemma 3.2) may be glued together to obtain the desired isomorphism (3.7).

Moreover, Theorem A is a direct consequence of Theorem 3.5 together with the decompositions (2.2), (2.3), (3.1), and (3.6).

### 4 Dormant Miura \(\mathfrak{sl}_2\)-Opers on Totally Degenerate Curves

#### 4.1 A Necessary Condition for the Existence of a Dormant Generic Miura \(\mathfrak{sl}_2\)-Oper

In this final section, we consider the complete determination of strict \(p\)-branch numberings, which gives a classification of dormant generic Miura \(\mathfrak{sl}_2\)-opers on totally degenerate curves.

Let \(\Gamma^+ := (V, E, \xi, \lambda)\) be a marked semi-graph whose underlying semi-graph \(\Gamma\) is connected, 3-regular, and of type \((g, r)\). Then, we obtain the following assertion.

**Proposition 4.1** If there exists a strict \(p\)-branch numbering on \(\Gamma^+\), then the inequality \(g \leq 1\) holds.

**Proof** Let \(\vec{m} := (m_b)_{b \in B} \in \mathbb{F}^B_p\) be a strict \(p\)-branch numbering on \(\Gamma^+\). Let us assume that \(g > 0\). Since \(b_1(\Gamma) \neq 0\), one may find a vertex \(v_0\) of \(\Gamma\), a positive integer \(l\), and
a path \((b_j^l)_{j=1}^l\) from \(v_0\) to \(v_0\) itself such that \(b_j \neq b_{j-1}^*\) for any \(j \in \{1, \ldots, l\}\), where \(b_0^* := b_l^*\). (We shall refer to such a path as a **reduced loop** based at \(v_0\).) For each \(j \in \{1, \ldots, l\}\), there exists a unique branch \(b_j^c \in B\) with \(B_{\zeta(b_j)} := \{b_j, b_{j-1}^*, b_j^c\}\). The assumption that \(\vec{m}\) specifies a strict \(p\)-branch numbering implies the following equalities

\[
\begin{align*}
m_{b_1} &= p + 1 - (m_{b_0^*} + m_{b_1^l}), \\
m_{b_2} &= p + 1 - (m_{b_1^*} + m_{b_2^l}) \\
&= p + 1 - ((p - m_{b_1}) + m_{b_2^c}) \\
&= p + 2 - (m_{b_0^*} + m_{b_1^l} + m_{b_2^c}).
\end{align*}
\]

The last equality is equivalent to the equality \(l = \sum_{j=1}^l m_{b_j^c}\); this implies the equality \(m_{b_j^c} = 1\) for any \(j \in \{1, \ldots, l\}\).

Now, suppose further that \(g \geq 2\). Then, one verifies from the topological structure of \(\Gamma\) that (after replacing \(v_0\) with another vertex) there exist two reduced loops \((b_j'')_{j=1}^l\), \((b_j''')_{j=1}^l\) based at \(v_0\) such that \(\{b_1', b_l^*\} = \{b_1^*, b_1^c\}, \{b_1'', b_l^{**}\} = \{b_1^c, b_1\}\). By applying the above discussion to \((b_j'')_{j=1}^l\) and \((b_j''')_{j=1}^l\), respectively, we obtain the equalities \((m_{b_j^c} = m_{b_j^*} = 1\). Hence, \(3 (= m_{b_1} + m_{b_1^c} + m_{b_1^*}) = 1 + p\), and this contradicts the assumption \(p > 2\). Consequently, the inequality \(g \leq 1\) holds. □

By the above proposition and Theorem 3.5, we obtain the following assertion; this is reminiscent of [3, Lemma 1], which asserts the emptiness of the space of complex affine structures (i.e., generic Miura sl\(_2\)-opers) on a closed Riemann surface of genus \(> 1\).

**Corollary 4.2** Let \(\mathcal{X}\) be a totally degenerate \(r\)-pointed curve over \(k\) of genus \(g\). If there exists a dormant generic Miura sl\(_2\)-oper \(\hat{\mathcal{E}}\) on \(\mathcal{X}\), then the inequality \(g \leq 1\) holds.

**4.2 Case of \(g = 0\)**

According to Proposition 4.1 proved above, we can assume the inequality \(g \leq 1\) when looking for a strict \(p\)-branch numbering on \(\Gamma^+\). First, let us consider the case where \(g = 0\). Note that since \(\Gamma\) is 3-regular, strict \(p\)-branch numberings on it are uniquely determined by their exponents.

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Theorem 4.3 Suppose that \( g = 0 \) (and \( r \geq 3 \)). Then, for \( \bar{e} := (e_i)_{i=1}^r \in \mathbb{P}^r \), there exists a strict \( p \)-branch numbering on \( \Gamma^+ \) of exponent \( \bar{e} \) if and only if the following equality holds:

\[
\sum_{i=1}^r \tau^{-1}(e_i) = p + (r - 2)(p - 1). \tag{4.1}
\]

(This implies that the existence of a strict \( p \)-branch numbering of prescribed exponents depends neither on the topological structure of the underlying graph \( \Gamma \) nor on the marking \( \lambda \).) In particular, there always exists a strict \( p \)-branch numbering on \( \Gamma^+ \). Finally, the cardinality of the set \( Br_{\Gamma^+} \) is given by

\[
\#(Br_{\Gamma^+}) = \binom{(r - 1)p + 1}{r - 1}.
\]

**Proof** It suffices to consider the first assertion because it deduces the remaining assertions. To prove the first assertion, we use induction on \( r \). The base step, i.e., the case of \( r = 3 \) is clear from the definition of a strict \( p \)-branch numbering. Next, let us consider the induction step (under the assumption \( r \geq 4 \)). For each \( i = 1, \ldots, r \), we shall write \( b_i := \lambda^{-1}(i) \). After possibly replacing \( \lambda \) with another, we can assume, without loss of generality, that \( \zeta(b_{r-1}^*) = \zeta(b_r^*) := v \). Denote by \( e_{r-1} \) and \( e_r \) the edges of \( \Gamma \) defined as \( e_{r-1} := \{b_{r-1}, b_{r-1}^*\} \) and \( e_r := \{b_r, b_r^*\} \) respectively. Also, denote by \( b^c \) the unique branch in \( \Gamma \) with \( B_v = \{b_{r-1}^*, b_r^*, b_r^c\} \). Then, we obtain the unique marked semi-graph \( \bar{\Gamma}^+ := (\bar{V}, \bar{E}, \bar{\zeta}, \bar{\lambda}) \) determined by the following conditions:

- \( \bar{V} := V \setminus \{v\} \).
- \( \bar{E} := E \setminus \{e_{r-1}, e_r\} \).
- \( \bar{\zeta} \) is the map \( \bigcup_{e \in E} e \mapsto \bar{B} \rightarrow \bar{V} \cup \{\oplus\} \) given by \( \bar{\zeta}(b^c) = \oplus \) and \( \bar{\zeta}(b) = \zeta(b) \) for \( b \neq b^c \).
- \( \bar{\lambda} \) is the bijection \( \{b_1, \ldots, b_{r-1}, b^c\} := \bar{B}_\oplus \sim \{1, \ldots, r - 1\} \) given by \( \bar{\lambda}(b^c) = r - 1 \) and \( \bar{\lambda}(b_i) = i \) for \( i = 1, \ldots, r - 2 \).

Now, let \( \bar{m} := (m_b)_{b \in B} \) be a \( p \)-branch numbering on \( \Gamma \) of exponent \( \bar{e} \). We shall write \( m := (m_b)_{b \in \bar{B}} \) for the collection of elements of \( \mathbb{Z} \) indexed by \( \bar{B} \) defined by \( m_{b^c} = m_{b_{r-1}} + m_{b_r} + 1 - p \) and \( m_b = m_b \) for \( b \neq b^c \). It follows from the construction of \( \bar{\Gamma}^+ \) that \( \bar{m} \) forms a strict \( p \)-branch numbering on \( \Gamma^+ \) if and only if \( \bar{m} \) forms a strict \( p \)-branch numbering on \( \bar{\Gamma}^+ \). On the other hand, observe that

\[
\sum_{b \in \bar{B}_\oplus} m_b = m_{b^c} + \sum_{i=1}^{r-2} m_{b_i} = 1 - p + \sum_{i=1}^r m_{b_i} = 1 - p + \sum_{i=1}^r \tau^{-1}(e_i).
\]

Hence, by inductive hypothesis, \( \bar{m} \) (or equivalently \( \bar{m} \) by the above discussion) forms a strict \( p \)-branch numbering if and only if \( (4.1) \) holds. This completes the induction step, and consequently, completes the proof of the assertion. \( \square \)
The following assertion is a direct consequence of Theorems 3.5 and 4.3.

**Corollary 4.4**

(i) Each totally degenerate r-pointed curve $\mathcal{X}$ over k of genus 0 (with $r \geq 3$) admits a dormant generic Miura $\mathfrak{sl}_2$-oper, and the cardinality $\sharp(M\mathcal{O}_p^{\mathcal{Z}_{xz\ldots}X})$ of the set $M\mathcal{O}_p^{\mathcal{Z}_{xz\ldots}X}$ is given by the following formula:

$$\sharp(M\mathcal{O}_p^{\mathcal{Z}_{xz\ldots}X}) = \binom{(r - 1)p + 1}{r - 1}.$$  

(ii) Let $\mathcal{X}$ be as in (i). Then, for each $\vec{v} := (\epsilon_i)_{i=1}^{r} \in \mathbb{F}_p^r$, $\mathcal{X}$ admits a dormant generic Miura $\mathfrak{sl}_2$-oper of exponents $\vec{v} \cdot \hat{\rho}$ if and only if the following equality holds:

$$\sum_{i=1}^{r} r^{-1}(\epsilon_i) = p + (r - 2)(p - 1).$$

4.3 Case of $g = 1$

Next, we shall consider the case where $g = 1$.

**Theorem 4.5** Suppose that $g = 1$. Then, the inclusion

$$B_r \Gamma^+, \vec{v} \hookrightarrow B_{r} \Gamma^+$$

is bijective, where $\vec{v} := (-1, -1, \ldots, -1) \in \mathbb{F}_p^r$ if $r > 0$ (resp., $\vec{v} := \emptyset$ if $r = 0$). Moreover, the following equalities hold:

$$\sharp(B_{r} \Gamma^+, \vec{v}) = \sharp(B_{r} \Gamma^+) = p - 1.$$  

**Proof** Let $v_0$ and $(b_j)_{j=1}^{l}$ be as in the proof of Proposition 4.1 (under the assumption that $g = 1$). If we are given a strict p-branch numbering $\vec{m} := (m_b)_{b \in B}$ on $\Gamma^+$, then since $m_{b_{j}} = 1$ and $m_{b_j} + m_{b_{j}}^* + m_{b_{j-1}}^* = 1 + p$ for any $j$, there exists a unique $a \in \{1, \ldots, p - 1\}$ satisfying the condition $(\ast)_a$ described as follows: $(\ast)_a m_{b_j} = a$ and $m_{b_j}^* = p - a$ for any $j$. Conversely, for each $a \in \{1, \ldots, p - 1\}$, one may construct a unique strict p-branch numbering $\vec{m} := (m_b)_{b \in B}$ on $\Gamma^+$ satisfying the condition $(\ast)_a$ in such a way that for each $v \in B \setminus \{b_j\}_{j=1}^{l}$, the multiset $[m_{b_1}, m_{b_2}, m_{b_3}]$ (where $B_v = \{b_1, b_2, b_3\}$) coincides with $[1, 1, p - 1]$. If $r > 0$, then each such strict p-branch numbering is verified to be of exponents $\vec{v} \in \mathbb{F}_p^r$. Thus, this completes the proof of the assertion. \[\Box\]

**Corollary 4.6**

(i) Each totally degenerate r-pointed curve $\mathcal{X}$ over k of genus 0 (with $r \geq 1$) admits a dormant generic Miura $\mathfrak{sl}_2$-oper, and the cardinality $\sharp(M\mathcal{O}_p^{\mathcal{Z}_{xz\ldots}X})$ of the set $M\mathcal{O}_p^{\mathcal{Z}_{xz\ldots}X}$ is given by the following formula:

$$\sharp(M\mathcal{O}_p^{\mathcal{Z}_{xz\ldots}X}) = p - 1.$$  

(4.2)
(ii) Let \( \mathcal{X} \) be as in (i). Then, any dormant generic Miura \( \mathfrak{sl}_2 \)-oper is of exponents \( \vec{e} \cdot \vec{p} \) (cf. Theorem 4.5 for the definition of \( \vec{e} \)).

**Proof** The assertion follows from Theorems 3.5 and 4.5. \( \square \)

**Remark 4.7** In this remark, we shall consider a connection of the above result and the previous study concerning the moduli space of Miura \( \mathfrak{sl}_2 \)-opers in positive characteristic. Denote by \( \overline{\mathcal{M}}_{g,r} \) the moduli stack classifying \( r \)-pointed stable curves of genus \( g \) over \( k \). Also, for each \( \vec{e} := (e_i)_{i=1}^r \in \mathbb{P}^{r-1} \), we shall write \( \mathcal{M}O p_{g,r,\vec{e} \cdot \vec{p}}^{\mathfrak{sl}_2} \) for the moduli stack classifying such curves equipped with a dormant generic Miura \( \mathfrak{sl}_2 \)-oper of exponents \( \vec{e} \cdot \vec{p} \). Recall from [17, § 6, Theorem 6.3.2 and § 3, Theorem 3.8.3, (i)] that \( \mathcal{M}O p_{g,r,\vec{e} \cdot \vec{p}}^{\mathfrak{sl}_2} \) may be represented by a (possibly empty) smooth proper Deligne-Mumford stack over \( k \), and the natural projection \( \mathcal{M}O p_{g,r,\vec{e} \cdot \vec{p}}^{\mathfrak{sl}_2} \to \overline{\mathcal{M}}_{g,r} \) is finite; moreover, if \( N_{\vec{e}} := 2g - 2 + \frac{2g-2+2 \sum_{i=1}^{r} e_i}{p} \) is a positive integer, then any irreducible component of \( \mathcal{M}O p_{g,r,\vec{e} \cdot \vec{p}}^{\mathfrak{sl}_2} \) that contains a point classifying a smooth curve is equidimensional of dimension \( N_{\vec{e}} \).

Now, we shall consider the case where \( g = 1 \) and \( \vec{e} = \vec{e} \). Let \( \mathcal{X} := (X/k, \{\sigma_i\}_{i=1}^r) \) be an \( r \)-pointed smooth curve of genus 1 over \( k \). It follows from [19, Theorem A, (ii), and § 7, Proposition 7.5.1] that the unpointed genus 1 curve \( X \) admits a dormant generic Miura \( \mathfrak{sl}_2 \)-oper. Moreover, by [17, § 6, Proposition 6.4.1], it yields a dormant generic Miura \( \mathfrak{sl}_2 \)-oper on \( \mathcal{X} \) of exponents \( \vec{e} \cdot \vec{p} \). Hence, \( \mathcal{M}O p_{1,r,\vec{e} \cdot \vec{p}}^{\mathfrak{sl}_2} \) contains a point classifying a smooth curve, and hence, forms a nonempty smooth Deligne-Mumford stack of dimension \( N_{\vec{e}} := 2 \cdot 1 - 2 + \frac{2 \sum_{i=1}^{r} e_i}{p} - (p-1) = r \). Since dim(\( \overline{\mathcal{M}}_{1,r} \)) = \( r \), the finite morphism \( \mathcal{M}O p_{1,r,\vec{e} \cdot \vec{p}}^{\mathfrak{sl}_2} \to \overline{\mathcal{M}}_{1,r} \) turns out to be surjective. In particular, the fiber of this morphism over any point classifying a totally degenerate curve is nonempty. This means that any \( r \)-pointed totally degenerate curve of genus 1 admits a dormant generic Miura \( \mathfrak{sl}_2 \)-oper of exponents \( \vec{e} \cdot \vec{p} \). This result is nothing but a part of Corollary 4.6. Moreover, since both \( \mathcal{M}O p_{1,r,\vec{e} \cdot \vec{p}}^{\mathfrak{sl}_2} \) and \( \overline{\mathcal{M}}_{1,r} \) are smooth, the morphism \( \mathcal{M}O p_{1,r,\vec{e} \cdot \vec{p}}^{\mathfrak{sl}_2} \to \overline{\mathcal{M}}_{1,r} \) is verified to be flat (cf. [4, Chap. III, Exercise 10.9]). By comparing (4.2) and the computation of the degree deg(\( \mathcal{M}O p_{1,r,\vec{e} \cdot \vec{p}}^{\mathfrak{sl}_2} / \overline{\mathcal{M}}_{1,r} \)) obtained in [17, § 6, Corollary 6.4.2], we see that this morphism is étale over the points classifying totally degenerate curves.

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**References**

1. Abramovich, D., Chen, Q., Gillam, D., Huang, Y., Olsson, M., Satriano, M., Sun, S.: Logarithmic Geometry and Moduli. Handbook of Moduli, Vol I, Advanced Lectures in Mathematics, vol. 24, pp. 1–61. International Press, Somerville (2013)
2. Deligne, P., Mumford, D.: The irreducibility of the space of curves of given genus. Publ. Math. I.H.E.S. 36, 75–110 (1969)
3. Gunning, R.C.: Special coordinate covering of Riemann surfaces. Math. Ann. **170**, 67–86 (1967)
4. Hartshorne, R.: Algebraic Geometry. Graduate Texts in Mathematics, vol. 52. Springer, New York (1977)
5. Illusie, L.: An Overview of the Work of K. Fujiwara, K. Kato and C. Nakamura on Logarithmic Etale Cohomology. Astérisque **279**, 271–322 (2002)
6. Kato, F.: Log smooth deformation and moduli of log smooth curves. Int. J. Math. **11**, 215–232 (2000)
7. Kato, K.: Logarithmic structures of Fontaine-Illusie. Algebraic analysis, geometry, and number theory, pp. 191–224. John Hopkins University Press, Baltimore (1989)
8. Kiehl, R., Weissauer, R.: Weil conjectures, Perverse Sheaves and l’adic Fourier Transform. Ergeb. Math. Grenzgeb. (3), vol. 42. Springer, New York (2001)
9. Knudsen, F.F.: The projectivity of the moduli space of stable curves. II. The stacks $M_{g,r}$. Math. Scand. **52**, 161–199 (1983)
10. Liu, F., Osserman, B.: Mochizuki’s indigenous bundles and Ehrhart polynomials. J. Algebraic Combin. **23**, 125–136 (2006)
11. Mochizuki, S.: A theory of ordinary $p$-adic curves. Publ. RIMS **32**, 957–1151 (1996)
12. Mochizuki, S.: Foundations of $p$-adic Teichmüller Theory. American Mathematical Society, London (1999)
13. Mochizuki, S.: The Absolute Anabelian Geometry of Hyperbolic Curves. Galois Theory and Modular Forms, pp. 77–122. Kluwer Academic Publishers, New York (2003)
14. Mochizuki, S.: Semi-graphs of Anabelioids. Publ. RIMS **42**, 221–322 (2006)
15. Ngô, B.C.: Le lemme Fondamental pour les Algèbres de Lie. Publ. Math. IHES **111**, 1–271 (2010)
16. Ogus, A.: $F$-Crystals, Griffiths Transversality, and the Hodge Decomposition. Astérisque **221**, Soc. Math. de France, (1994)
17. Wakabayashi, Y.: Moduli of Tango structures and dormant Miura opers. Moscow Math. J. **20**, 575–636 (2020)
18. Wakabayashi, Y.: Gaudin Model modulo $p$, Tango Structures, and Dormant Miura opers. arXiv:1905.03364 [math.AG], (2020)
19. Wakabayashi, Y.: Frobenius Projective and Affine Geometry of Varieties in Positive Characteristic. arXiv:2011.04846 [math.AG], (2020)
20. Wakabayashi, Y.: A Theory of Dormant opers on Pointed Stable Curves. Astérisque **432**, Soc. Math. de France, (2022)

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