On the Well-Posedness of Two Driven-Damped Gross-Pitaevskii-Type Models for Exciton-Polariton Condensates

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Abstract

We study the well-posedness of two systems modeling the non-equilibrium dynamics of pumped decaying Bose-Einstein condensates. In particular, we present the local theory for rough initial data using the Fourier restricted norm method introduced by Bourgain. We extend the result globally for initial data in $L^2$.

Keywords: Dispersive PDE, Dissipative, Well-posedness, Restricted Norm Method, BEC

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1. Introduction

In this paper, we study the (local) well-posedness theory of two closely related models describing the (non-equilibrium) dynamics of pumped decaying condensates, e.g., the Bose-Einstein condensation of exciton-polaritons. The first model is the following driven-damped nonlinear Schrödinger equation \cite{10}:

$$i\partial_t u = -\partial_x^2 u + |u|^2 u + i \left( \xi - \sigma |u|^2 \right) u,$$

where $u = u(x,t)$, $x \in \mathbb{T}$, $\xi, \sigma$ are positive constants, and $u_0 = u(x,0) \in H^s(\mathbb{T})$, $s \geq 0$.

The second model consists of a generalized open-dissipative Gross-Pitaevskii equation for the macroscopic wave-function of the polaritons, $u = u(x,t)$, coupled to a simple rate equation for the exciton reservoir density, $n = n(x,t)$ \cite{17,18}:

$$i\partial_t u = -\partial_x^2 u + g |u|^2 u + \lambda n u + i \left( Rn - \alpha \right) u,$$

$$\partial_t n = P - \left( R |u|^2 + \beta \right) n,$$

subject to the initial data $u|_{t=0} = u_0(x), n|_{t=0} = n_0(x), x \in \mathbb{R}$. Above, $\alpha, \beta, \lambda, g, R$ are positive constants and $P = P(x) \geq 0$ (compactly supported, bounded).

In our analysis, we shall consider (1) on the one-dimensional torus. This choice is physically motivated by the fact that a stable condensate can only form in a spatially confined system. Such confinement gives rise to some technical challenges due to the loss of dispersion. Our approach is base on the Fourier restricted norm method introduced by Bourgain in \cite{2,3}. In the case of the system (2) the confinement is given by $P$. Our study of (2) requires some refinements of Bourgain’s method, in particular, the ones introduced by Kenig-Ponce-Vega in \cite{11,12} and later used by Ginibre et. al. in \cite{8} to study the well-posedness theory of the Zakharov system. On the other hand, it is important to notice that (2) does not have derivatives in the nonlinearities.

We shall refer to (1) as the complex Gross-Pitaevskii equation and to (2) as the exciton-polariton system.

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2. Well-posedness of the complex Gross-Pitaevskii equation

Using Duhamel’s principle, we consider the following integral equation associated with (1):

\[ u(t) = S(t)u_0 + \int_0^t S(t - \tau) \left( \xi u - (\sigma + i)|u|^2u \right) d\tau, \]  

where \( S(t) = e^{it\partial_x^2} \). We introduce now the basic notation and ideas related to the restricted norm method; see, e.g., [5, 7, 15] for a detailed review of this topic.

Denote by \( l^p_k L^p_t \) the Banach space \( l^p_k (\mathbb{Z} : L^p_t (\mathbb{R})) \). Let \( \hat{\cdot} \) stand for the Fourier transform with respect to space-time, i.e.,

\[ \hat{g}(k, \tau) = \int_{-\infty}^{\infty} \int_T \exp(-ikx - i\tau t) g(x, t) \, dx \, dt. \]

We denote by \( \mathcal{F}_x \) the Fourier transform with respect to the space variable

\[ \mathcal{F}_x g(k) = \int_T \exp(-ikx) g(x) \, dx. \]

**Definition 1.** Let \( \mathcal{V} \) be the space of functions \( u: \mathbb{T} \times \mathbb{R} \to \mathbb{C} \), such that \( u(x, \cdot) \in \mathcal{S} (\mathbb{R}) \) for each \( x \in \mathbb{T} \) and \( u(\cdot, t) \in \mathcal{C}^\infty (\mathbb{T}) \) for each \( t \in \mathbb{R} \). We define the space \( \mathcal{X}^{s,b} \) as the completion of \( \mathcal{V} \) with respect to the norm

\[ \|u\|_{\mathcal{X}^{s,b}} = \left\| \langle k \rangle^s (\langle \tau + k^2 \rangle^b \hat{u}(k, \tau) \right\|_{l^p_k \mathbb{L}^p_t} = \left\| e^{-it\partial_x^2} u \right\|_{H^s_x L^b_t}, \]

where \( \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2} \) (Japanese bracket).

One can verify that the dual space of \( \mathcal{X}^{s,b} \) is \( \mathcal{X}^{-s,-b} \). Moreover, \( \mathcal{X}^{s',b'} \subset \mathcal{X}^{s,b} \) for \( s' \geq s, b' \geq b \). Since we shall study the local theory using a contraction argument in a time interval \([-\delta, \delta]\) with \( \delta \leq 1 \), we define the (restricted) space \( \mathcal{X}^{s,b}_\delta \) to be the equivalent classes of functions that agree on \([-\delta, \delta]\), with the norm

\[ \|u\|_{\mathcal{X}^{s,b}_\delta} = \inf_{\tilde{u}=u, \tilde{u}\in[-\delta,\delta]} \|\tilde{u}\|_{\mathcal{X}^{s,b}}. \]

Let \( \eta \in C^\infty_0 (\mathbb{R}) \) such that \( \eta(t) = 1 \) for \( t \in [-1, 1] \). Define the operator

\[ \Gamma_{u_0}(u) = \eta(t) S(t) u_0 + \eta(t) \int_0^t S(t - \tau) \left( \xi u - (\sigma + i)|u|^2u \right) (\tau) \, d\tau, \]  

on the ball

\[ B_R = \left\{ u \in \mathcal{X}^{s,b}_\delta : \|u\|_{\mathcal{X}^{s,b}_\delta} \leq R \right\}, \]  

where \( R = C \|u_0\|_{H^s} \), \( s \geq 0 \). Note that, since \( \delta \leq 1 \), a fixed point of (4) gives a solution of the complex Gross-Pitaevskii equation on \([-\delta, \delta]\). On the other hand, we have a constraint on the value of \( b \) to ensure continuity (in time) of these solutions, as the following lemma shows (see [5, Lemma 3.9]):

**Lemma 2.** For any \( b > \frac{1}{2} \), \( \mathcal{X}^{s,b}_\delta \subset C^1_t H^s_x ([\delta, \delta] \times \mathbb{T}) \).

To handle our contraction argument, we shall use the following (see [5, Section 3.5.1]):

**Lemma 3.** Let \( 0 < \delta \leq 1, s, b \in \mathbb{R} \). Then

\[ \left\| \eta(t) e^{it\partial_x^2} u_0 \right\|_{\mathcal{X}^{s,b}_\delta} \leq C \|u_0\|_{H^s}. \]  

For any \(-\frac{1}{2} < b' < b < \frac{1}{2}\) and \( s \in \mathbb{R} \), we have

\[ \|u\|_{\mathcal{X}^{s,b}_\delta} \leq C \delta^{b-b'} \|u\|_{\mathcal{X}^{s,b}_\delta}. \]
Let \(-\frac{1}{2} < b' \leq 0\) and \(b = b' + 1\). Then
\[
\left\| \eta(t) \int_0^t e^{i(t-s)s} \partial_x^2 F(s) \, ds \right\|_{X^{s,b}_x} \leq C \left\| F \right\|_{X^{s,b'}_x}.
\] (8)

The following result by Bourgain is essential for our analysis.

Lemma 4. Let \(u\) be a smooth space-time function. Then
\[
\left\| u \right\|_{L^1_t L^s_x} \leq C \left\| u \right\|_{X^{0,3/2}}.
\]

Using the previous lemma, one can show the following (see Proposition 3.26).

Lemma 5. Let \(s \geq 0\). Then
\[
\left\| \left| \nabla \right|^2 u \right\|_{X^{s,b}_x} \leq C \left\| u \right\|_{X^{s,b}_x} \left\| u \right\|_{X^{s,b}_x}.
\]

Now we can present the main result of this section.

Proposition 6. The complex Gross-Pitaevskii equation \((1)\) is locally well-posed in \(H^s_x(\mathbb{T})\), \(s \geq 0\), i.e., for any \(u_0 \in H^s_x(\mathbb{T})\) there is a unique solution \(u \in C^0_t H^s_x([-\delta, \delta] \times \mathbb{T}) \cap X^{s,b}_x\), with \(\frac{1}{2} < b < \frac{3}{2}\). Moreover, the solution depends continuously on the data.

Proof. We run the contraction argument in \(B_R \subset X^{s,b}_x\) (with \(\frac{1}{2} < b < \frac{3}{2}\) and \(\delta\) small enough) for the operator \(\Gamma_{u_0}(u)\) defined in \((4) - (5)\). Using \((6), (7), (8)\), and the embedding \(X^{s',b'} \subset X^{s,b}\) for \(s' \geq s, b' \geq b\), we obtain
\[
\left\| \Gamma_{u_0} u \right\|_{X^{s,b}_x} \leq C \left\| u_0 \right\|_{H^s_x(\mathbb{T})} + C \delta^{1-b-\frac{2}{3}} \left( \left\| u - (\sigma + i) |u|^2 u \right\|_{X^{s,b}_x} \right) \\
\leq C \left\| u_0 \right\|_{H^s_x(\mathbb{T})} + C \delta^{1-b-\frac{2}{3}} \left( \left\| u \right\|_{X^{s,b}_x} + \left\| u \right\|_{X^{s,b}_x} \left\| u \right\|_{X^{s,b}_x} \right) \\
\leq C \left\| u_0 \right\|_{H^s_x(\mathbb{T})} + C \delta^{1-b-\frac{2}{3}} \left( \left\| u \right\|_{X^{s,b}_x} + \left\| u \right\|_{X^{s,b}_x} \right).
\]

Similar estimates hold for the difference. We omit the standard details. Note that, since \(b > \frac{1}{2}\), by Lemma 2 the solution is continuous in time with values in \(H^s_x(\mathbb{T})\), \(s \geq 0\).

□

Corollary 7. The complex Gross-Pitaevskii equation \((1)\) is globally well-posed in \(L^2_x(\mathbb{T})\).

Proof. Multiply \((1)\) by \(\bar{u}\), take the imaginary part, and use integration by parts to obtain
\[
\frac{d}{dt} \int_\mathbb{T} |u|^2 \, dx - 2\xi \int_\mathbb{T} |u|^2 \, dx + 2\sigma \int_\mathbb{T} |u|^4 \, dx = 0.
\] (9)

Since \(\left( \sqrt{\sigma s^2 - \frac{2\xi}{\sqrt{\sigma}}} \right)^2 \geq 0\), we have \(\sigma s^4 - 4\xi s^2 + \frac{4\xi^2}{\sigma} \geq 0\), for \(s \in \mathbb{R}\). Setting \(s = |u|\) and integrating over \(\mathbb{T}\) yield
\[
-\sigma \int_\mathbb{T} |u|^4 \, dx + 4\xi \int_\mathbb{T} |u|^2 \, dx \leq \frac{4\xi^2}{\sigma} |\mathbb{T}|,
\] (10)

where \(|\mathbb{T}|\) is the measure of \(\mathbb{T}\). Combining \((10)\) and \((11)\) gives
\[
\frac{d}{dt} \int_\mathbb{T} |u|^2 \, dx + 2\xi \int_\mathbb{T} |u|^2 \, dx + \sigma \int_\mathbb{T} |u|^4 \, dx \leq \frac{4\xi^2}{\sigma} |\mathbb{T}|.
\]
Using the last expression along with Gronwall’s lemma, we obtain
\[
\|u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 e^{-2\xi T} + \frac{2\xi}{\sigma} |T| \left(1 - e^{-2\xi T}\right), \quad t \geq 0.
\] (11)

From (11) we conclude that the local solution in \(L^2(T)\) can be extended globally. Note that to justify the calculations above we need to use continuous dependence on the data, approximate \(u_0\) by a sequence of smooth functions, and take the limit. See, e.g., [4, 13] for a detailed description of this procedure.

Remark 8. Letting \(t \to \infty\) in (11) gives
\[
\limsup_{t \to \infty} \|u\|_{L^2}^2 \leq \frac{2\xi}{\sigma} |T|,
\]
which guarantees the existence of an absorbing set for the complex Gross-Pitaevskii equation in \(L^2(T)\). See, e.g., [16].

The \(H^1\) theory and stationary solutions of the complex Gross-Pitaevskii equation (in the full domain with a harmonic trapping potential and \(\xi\) space dependent with compact support) have been studied in [4, 8, 13].

3. Well-posedness of the exciton-polariton system

Using Duhamel’s principle, we consider the following integral equations associated with (2):
\[
u(t) = S(t) u_0 + \int_0^t S(t-s) \left(-ig|u|^2 u + (R - i\lambda) nu - \alpha u\right) (s) \, ds,
\] (12)
\[n(t) = n_0 + \int_0^t \left(P - R |u|^2 n - \beta n\right) (s) \, ds,
\] (13)
where \(S(t) = e^{it\hat{a}^2}\). Our choice for the second expression is because the corresponding equation in (2) is an ODE in \(n\); hence, it does not have an appropriate dispersion relation for the subsequent analysis.

Definition 9. Let \(X^{s,b}\) be the Banach space of functions on \(\mathbb{R} \times \mathbb{R}\) defined by the norm
\[
\|u\|_{X^{s,b}(\mathbb{R})} = \left\|\langle \xi \rangle^s \langle \tau + \phi(\xi) \rangle^b \hat{u}(\xi,\tau)\right\|_{L_{\xi,\tau}^2},
\]
where \(\phi\) corresponds to the dispersion relation of the equation under consideration. We usually write \(\|\cdot\|_2 = \|\cdot\|_{L_{\xi,\tau}^2}\).

Similarly, we define the auxiliary spaces \(Y^{s}\) by the norm
\[
\|u\|_{Y^{s}(\mathbb{R})} = \left\|\langle \xi \rangle^s \langle \tau + \phi(\xi) \rangle^{-1} \hat{u}(\xi,\tau)\right\|_{L_{\xi,\tau}^2L^1_{\tau}}.
\]

We want to solve the Cauchy problem corresponding to (2) in the context of the previous spaces and in some time interval \([-T,T]\). To reach this goal, it is convenient to introduce a cutoff in (12)- (13). Let \(\psi \in C_0^\infty(\mathbb{R})\) be even, \(0 \leq \psi \leq 1\), such that \(\psi = 1\) on \([-1,1]\) and \(\text{supp} \, \psi \subset (-2,2)\). Furthermore, let \(\psi_T(t) = \psi(t/T), 0 < T \leq 1\). The cutoff version of (12)-(13) is given by
\[
u(t) = \psi(t) S(t) u_0 + \psi_T(t) \int_0^t S(t-s) \left(-ig|u|^2 u + (R - i\lambda) nu - \alpha u\right) (s) \, ds,
\] (14)
\[n(t) = \psi(t) n_0 + \psi_T(t) \int_0^t \left(P - R |u|^2 n - \beta n\right) (s) \, ds.
\] (15)
Like in the previous section, we define the restricted space $\|u\|_{X_{\phi(\xi)}^{s,b}}$ as the equivalent classes of functions that agree on $t \in [-T,T]$, with the norm

$$\|u\|_{X_{\phi(\xi)}^{s,b}} = \inf_{\tilde{u} = u, t \in [-T,T]} \|\tilde{u}\|_{X_{\phi(\xi)}^{s,b}}.$$ 

Similarly, we define the space $\|u\|_{Y_{\phi(\xi)}^{s,b}}$.

The following lemma will be the starting point for our contraction argument (see [8, Lemma 2.1]).

**Lemma 10.** Let $s \in \mathbb{R}$, $b' \leq 0 \leq b \leq b' + 1$, and $T \leq 1$. Then

$$\left\| \psi_T (t) \int_0^t S (t - \tau) F (\tau) d\tau \right\|_{X_{\phi(\xi)}^{s,b}} \leq C \left( T^{1-b+b'} \|F\|_{X_{\phi(\xi)}^{s,b}} + T^{1/2-b} \|F\|_{Y_{\phi(\xi)}^{s,b}} \right),$$

$$\left\| \psi_T (t) \int_0^t F (\tau) d\tau \right\|_{X_{\phi(\xi)}^{s,b}} \leq C \left( T^{1-b+b'} \|F\|_{X_{\phi(\xi)}^{s,b'}} + T^{1/2-b} \|F\|_{Y_{\phi(\xi)}^{s,b'}} \right).$$

Furthermore, if $b' > -1/2$,

$$\left\| \psi_T (t) \int_0^t S (t - \tau) F (\tau) d\tau \right\|_{X_{\phi(\xi)}^{s,b}} \leq C T^{1-b+b'} \|F\|_{X_{\phi(\xi)}^{s,b}},$$

$$\left\| \psi_T (t) \int_0^t F (\tau) d\tau \right\|_{X_{\phi(\xi)}^{s,b}} \leq C T^{1-b+b'} \|F\|_{X_{\phi(\xi)}^{s,b'}}.$$

As mentioned before, $X_{\phi(\xi)}^{s,b} \subset C (\mathbb{R}, H^n)$, $b > 1/2$. This is no longer valid if $b \leq 1/2$, and this is why we need to consider the spaces $Y_{\phi(\xi)}^{s,b}$ (see [8, Lemma 2.2]).

We now follow closely the ideas presented in [8]. As mentioned before, we want to solve the cutoff integral version of the exciton-polariton system (14)-(15) by a contraction method with $u \in X_{\phi(\xi)}^{k,a_2}$ and $n \in X_{\phi(\xi)\equiv 0}^{l,a}$ for suitable $a_1, a_2$, and $k, l$. We start by estimating the nonlinearity

$$f_1 = nu$$

in $X_{\phi(\xi)\equiv 0}^{k,-a_1}$ for suitable $a_1$.

We estimate $f_1 (\xi_1, \tau_1)$ in terms of $\hat{n} (\xi, \tau)$ and $\hat{u} (\xi_2, \tau_2)$. We have the following relations due to the convolution structure

$$\xi = \xi_1 - \xi_2,$$

$$\tau = \tau_1 - \tau_2.$$ 

We also introduce the variables

$$\sigma_1 = \tau_1 + \xi_1^2,$$

$$\sigma_2 = \tau_2 + \xi_2^2,$$

$$\sigma = \tau.$$ 

In terms of these variables, we have

$$z \equiv \xi_1^2 - \xi_2^2 = \sigma_1 - \sigma_2 - \sigma.$$ 

We use this expression to obtain estimates of $\xi_1^2$ (resp. $\xi_2^2$) in terms of $\xi_2^2$ (resp. $\xi_1^2$) and of the $\sigma$'s.

To estimate $f_1$, we define $\tilde{v}_2 = (\xi_2)^k (\sigma_2)^{a_2} \hat{u}$ and $\tilde{v} = (\xi_1)^l (\sigma)^a \hat{n}$ so that

$$\|u\|_{X_{\phi(\xi)\equiv 0}^{k,a_2}} = \|v_2\|_2,$$

$$\tilde{v}_2 \equiv \hat{v}_2 \equiv (\xi_2)^k (\sigma_2)^{a_2} \hat{u}.$$
and \[ \|v\|_{X_{\phi(\zeta)}^k} = \|v\|_2. \]

To estimate \( f_1 \) in \( X_{\phi(\zeta)=\xi^2}^{k,-a_1} \), we take its scalar product with a generic function in \( X_{\phi(\zeta)=\xi^2}^{k,-a_1} \) with Fourier transform \( \langle \xi_1 \rangle^k \langle \sigma_1 \rangle^{-a_1} \hat{v}_1 \) and \( v_1 \in L^2 \). Then the required estimate in \( X_{\phi(\zeta)=\xi^2}^{k,-a_1} \) takes the form

\[
|S| \leq C \|v\|_2 \|v_1\|_2 \|v_2\|_2, \tag{17}
\]

where

\[
S = \int \frac{\hat{v}_1 \hat{v}_2 \langle \xi_1 \rangle^k}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^{a_1} \langle \sigma_2 \rangle^{a_2} \langle \xi \rangle^l}, \tag{18}
\]

and

\[
\hat{v} = \hat{v}(\xi, \tau), \quad \hat{v}_1 = \hat{v}(\xi_1, \tau_1), \quad \hat{v}_2 = \hat{v}(\xi_2, \tau_2),
\]

constrained by

\[
\xi = \xi_1 - \xi_2, \quad \tau = \tau_1 - \tau_2,
\]

and the integral is over \( d\xi_1, d\xi_2, d\tau_1, d\tau_2 \).

We often use the following two elementary facts in our analysis.

**Lemma 11.** Let \( f \in L^q(\mathbb{R}) \), \( g \in L^{q'}(\mathbb{R}) \), \( 1 \leq q, q' \leq \infty \), \( 1/q + 1/q' = 1 \). Assume that \( f \) and \( g \) are nonnegative, even, and non-increasing for positive argument. Then, \( f * g \) has the same properties.

One can use Lemma 11 to show that \( f * g \) takes its maximum at zero. Using this fact, we can show the following

**Lemma 12.** Let \( 0 \leq a_- \leq a_+ \) and \( a_+ + a_- > 1/2 \), then the following estimate holds for all \( s \in \mathbb{R} \)

\[
J(s) = \int (y-s)^{-2a_+} (y+s)^{-2a_-} dy \leq C \langle s \rangle^{-\alpha},
\]

where \( \alpha = 2a_- - [1 - 2a_+]_+ \).

See [8] for a proof of the previous lemmata.

**Lemma 13.** Let \( k, l, a, a_1, a_2 \) satisfy

\[
l \geq -1/2, \quad k \geq 0, \quad k - l \leq 1, \tag{19}
\]

\[
a, a_1, a_2 > 1/4, \quad a + a_1 > 3/4, \quad a + a_2 > 3/4, \tag{20}
\]

\[
k - l \leq 2a_1, \tag{21}
\]

then the estimate (17) holds.

**Proof.** The principle of the proof is the following application of the Schwarz inequality. Let \( \zeta = (\xi, \tau) \), \( \zeta_i = (\xi_i, \tau_i) \), \( i = 1, 2 \) so that \( \zeta = \zeta_1 - \zeta_2 \). We want to estimate an integral of the form

\[
J = \int \hat{v}(\zeta) \hat{v}_1(\zeta_1) \hat{v}_2(\zeta_2) K(\zeta_1, \zeta_2) d\zeta_1 d\zeta_2.
\]
Note that $\zeta_1 = \zeta + \zeta_2$. Then, considering the Schwarz inequality with respect to $\zeta$ we obtain

$$\left| J \right|^2 \leq \| v \|_2^2 \int d\zeta \left| \int \hat{v}_1 (\zeta + \zeta_2) \hat{v}_2 (\zeta_2) K (\zeta + \zeta_2, \zeta_2) d\zeta_2 \right|^2$$

(Schwarz w.r.t. $\zeta_2$ and extract sup)

$$\leq \| v \|_2^2 \left\{ \sup_{\zeta} \left| K (\zeta + \zeta_2, \zeta_2) \right|^2 d\zeta_2 \right\} \int |\hat{v}_1 (\zeta + \zeta_2) \hat{v}_2 (\zeta_2)|^2 d\zeta d\zeta_2$$

(16) gives the bounds for $A$ and the last integral runs over $\zeta_2$ (or $\zeta_1$) for fixed $\zeta$. One obtains two similar estimates by circularly permuting the variables and functions 1, 2, and 1-2 (the ones with no subindex).

Moreover, we define

$$\alpha = 2 \min (a_1, a_2) - |1 - 2 \max (a_1, a_2)|_+,$$

$$\alpha_1 = 2 \min (a, a_2) - |1 - 2 \max (a, a_2)|_+,$$

$$\alpha_2 = 2 \min (a, a_1) - |1 - 2 \max (a, a_1)|_+.$$

We start by considering a particular case for $k$ and $l$.

**Case** $k = 0, l = -1/2$.

In this case, the factors containing the $\xi$’s reduce to $\langle \xi \rangle^{1/2}$. Note that

$$\langle \xi \rangle \leq 1 + |\xi|,$$

then

$$\langle \xi \rangle^{1/2} \leq (1 + |\xi|)^{1/2} \leq 1 + |\xi|^{1/2}.$$

Therefore, for this case

$$S \leq \int \frac{|\hat{\psi}_1 \hat{\psi}_2|}{\langle \sigma \rangle^{\alpha_1} \langle \sigma_1 \rangle^{a_1} \langle \sigma_2 \rangle^{a_2}} + \int \frac{|\hat{\psi}_1 \hat{\psi}_2| |\xi|^{1/2}}{\langle \sigma \rangle^{\alpha_1} \langle \sigma_1 \rangle^{a_1} \langle \sigma_2 \rangle^{a_2}} =: A + Z_0.$$  (23)

Lemma 16 gives the bounds for $A$. For $Z_0$ we consider the following subregions.

*Region $\sigma$ dominant, i.e., $|\sigma| \geq \max (|\sigma_1|, |\sigma_2|)$. We use directly (22) and obtain

$$C_2^\sigma = \sup_{\xi, \sigma} \langle \sigma \rangle^{-2a} \int |\xi|^{-2a_1} \langle \sigma_1 \rangle^{-2a_1} d\xi_2 d\sigma_2,$$

where the integral is taken at fixed $\xi, \sigma$. Now for fixed $\xi, \sigma$, and $\sigma_2$, it follows from Lemma 16 that

$$2 |\xi| d\xi_2 = dz = d\sigma_1,$$

since $z = \xi_1^2 - \xi_2^2$ and $\xi = \xi_1 - \xi_2 \Rightarrow \xi_1 = \xi + \xi_2$, which gives $dz = 2d\xi_2$. Therefore,

$$C_2^\sigma \leq C \sup_{\sigma} \langle \sigma \rangle^{-2a} \int_0^{|\sigma|} \langle \sigma_1 \rangle^{-2a_1} d\sigma_1 \int_0^{|\sigma|} \langle \sigma_2 \rangle^{-2a_2} d\sigma_2.$$
Note that $\langle u \rangle \geq |u|$, hence $\langle u \rangle^{-2a_1} \leq |u|^{-2a_1}$. Then,
\[
\int_{0}^{[\sigma]} \langle \sigma_1 \rangle^{-2a_1} d\sigma_1 \leq \int_{0}^{[\sigma]} \sigma_1^{-2a_1} d\sigma_1 \\
\leq C |\sigma|^{1-2a_1}_+. \\
\leq C \langle \sigma \rangle^{1-2a_1}_+. 
\]
Hence
\[
C_2^0 \leq C \sup_{\sigma} \langle \sigma \rangle^{-2a+|1-2a_1|_+ +|1-2a_2|_+}.
\]
The last quantity is finite provided
\[
2a - [1 - 2a_1]_+ - [1 - 2a_2]_+ \geq 0,
\]
which holds under the conditions
\[
a > 0, \ a_1 + a > 1/2, \ a_2 + a > 1/2, \ a + a_1 + a_2 > 1.
\]

Region $\sigma_1$ dominant, i.e., $|\sigma_1| \geq \max(|\sigma_1|, |\sigma_2|)$. We now use the analog of (22) with fixed $\zeta_1$ and obtain
\[
C_1^2 = \sup_{\xi_1, \sigma_1} \langle \sigma_1 \rangle^{-2a_1} \int_{0}^{[\sigma_1]} |\xi| \langle \sigma \rangle^{-2a_2} \langle \sigma_2 \rangle^{-2a_2} d\xi_2 d\sigma_2, \tag{24}
\]
where the integral is taken at fixed $\xi_1, \sigma_1$. To continue the estimate, we split the $\sigma_1$ dominant region into two subregions.

Subregion $|\xi_1| \leq 2 |\xi_2|$. Recall that $\xi = \xi_1 - \xi_2$, hence, $|\xi| = |\xi_1 - \xi_2| \leq |\xi_1| + |\xi_2| \leq 3 |\xi_2|$, the last inequality due to the subregion. Furthermore, for fixed $\xi_1, \sigma_1$, and $\sigma_2$, it follows from (16) that $2 |\xi_2| d\xi_2 = dz = d\sigma$. Therefore,
\[
C_1^2 \leq C \sup_{\sigma_1} \langle \sigma_1 \rangle^{-2a_1} \int_{0}^{[\sigma_1]} \langle \sigma \rangle^{-2a_2} d\sigma \int_{0}^{[\sigma_1]} \langle \sigma_2 \rangle^{-2a_2} d\sigma_2 \\
\leq C \sup_{\sigma_1} \langle \sigma_1 \rangle^{-2a_1+|1-2a|_+ +|1-2a_2|_+}.
\]
The last inequality is finite provided
\[
2a_1 - [1 - 2a]_+ - [1 - 2a_2]_+ \geq 0,
\]
which holds when
\[
a_1 > 0, \ a + a_1 > 1/2, \ a_1 + a_2 > 1/2, \ a + a_1 + a_2 > 1.
\]

Subregion $|\xi_1| \geq 2 |\xi_2|$. In this region, note that
\[
|\xi_1| \geq 2 |\xi_2| = 2 |\xi_1 - \xi| \geq 2 |\xi_1| - |\xi| \\
= 2 |\xi| - |\xi_1| \geq 2 |\xi_1 - \xi|_1.
\]
Then
\[
3 |\xi_1| \geq 2 |\xi| \Rightarrow |\xi| \leq \frac{3}{2} |\xi_1|.
\]
Moreover
\[
|\xi_1| \geq 2 |\xi_2| \Rightarrow \xi_1^2 \geq 4 \xi_2^2 \Rightarrow -\xi_1^2 \leq -4 \xi_2^2 \Rightarrow -\frac{1}{4} \xi_1^2 \leq -\xi_2^2.
\]
Combining the last expression with (16) and the fact that we are in the region $\sigma_1$ dominant, we obtain
\[
\frac{3}{4} \xi_1^2 = \xi_1^2 - \frac{1}{4} \xi_1^2 \leq \xi_1^2 - \xi_2^2 = \sigma_1 - \sigma_2 - \sigma \leq 3 |\sigma_1|,
\]
and therefore
\[ \xi_1^2 \leq 4|\sigma_1|. \]  \hspace{1cm} (26)

By (25), \(|\xi| \leq C\langle \xi_1 \rangle\), and by (20), \(\langle \sigma_1 \rangle^{-2\alpha_1} \leq C\langle \xi_1 \rangle^{-4\alpha_1}\). Using these facts and taking \(y = \xi_2^2\) as integration variable instead of \(\xi_2\), we obtain

\[ C_1^2 \leq \sup_{\xi, \sigma} \langle \xi_1 \rangle^{1-4\alpha_1} \int_0^{\xi_1^2/4} y^{-1/2} dy \int \langle \sigma \rangle^{-2\alpha_2} \langle \sigma_2 \rangle^{-2\alpha_2} d\sigma_2, \]  \hspace{1cm} (27)

where the boundary of the first integral is due to the subregion that we are considering: \(|\xi_1| \geq 2|\xi_2| \Rightarrow \xi_2^2 \leq \frac{1}{4}\xi_1^2\). Note that, since

\[ \xi_1^2 - \xi_2^2 = \sigma_1 - \sigma_2 - \sigma (\xi_2^2 = y), \]

then

\[ \xi_1^2 - y - \sigma = -\sigma_2 - \sigma \Rightarrow \langle \sigma_2 + (\xi_1^2 - y - \sigma_1) \rangle = (-\sigma) = \langle \sigma \rangle. \]

Hence

\[ C_1^2 \leq C \sup_{\xi, \sigma} \langle \xi_1 \rangle^{1-4\alpha_1} \int_0^{\xi_1^2/4} y^{-1/2} dy \int \langle \sigma_2 + (\xi_1^2 - y - \sigma_1) \rangle^{-2\alpha_2} \langle \sigma_2 \rangle^{-2\alpha_2} d\sigma_2. \]

We estimate the last integral for fixed \(\xi_1, \sigma, \xi_2\), by Lemma [12]. Then

\[ C_1^2 \leq C \sup_{\xi, \sigma} \langle \xi_1 \rangle^{1-4\alpha_1} \int_0^{\xi_1^2/4} \langle \xi_1^2 - y - \sigma_1 \rangle^{-\alpha_1} y^{-1/2} dy. \]

We extend the range of integration of \(y\) symmetrically to \([-\xi_1^2/4, \xi_1^2/4]\) and apply Lemma [11] with \(f(y) = |y|^{-1/2} \chi(|y| \leq \xi_1^2/4)\) and \(g(y) = \langle \xi_1^2 - y - \sigma_1 \rangle^{-\alpha_1}\) to conclude that the supremum over \(\sigma_1\) is attained for \(\sigma_1 = \xi_1^2\). Hence,

\[ C_1^2 \leq C \sup_{\xi} \langle \xi \rangle^{1-4\alpha_1} \int_0^{\xi_1^2} \langle y \rangle^{-\alpha_1} y^{-1/2} dy. \]

The last quantity is finite, provided \(a_1 \geq 1/4\) and \(a_1 > 1/2\). The latter is equivalent to

\[ a > 1/4, \quad a_2 > 1/4, \quad a + a_2 > 3/4. \]

**Region \(\sigma_2\) dominant.** This region is obtained from the previous one by exchanging 1 and 2. This has the effect of exchanging \(\alpha_2\) and \(a_1\), so that the same proof applies since the only assumption used so far, namely (20), is symmetric in \(a_2\) and \(a_1\).

**General \(k\) and \(l\), \(k \geq 0\)**

We consider separately the regions \(|\xi_1| \leq 2|\xi_2|\) and \(|\xi_1| \geq 2|\xi_2|\).

**Region \(|\xi_1| \leq 2|\xi_2|\)**

In this region

\[ \langle \xi_1 \rangle^k \langle \xi_2 \rangle^{-k} \langle \xi \rangle^{-l} \leq C \langle \xi \rangle^{-l}, \]

so that the factors with \(k\)'s disappear and the resulting expression is decreasing in \(l\). It is therefore sufficient to derive estimate (17) in the case \(l = -1/2\), which is the special case considered previously.

**Region \(|\xi_1| \geq 2|\xi_2|\)**

In this region, we have

\[ |\xi_1| \geq 2|\xi_2| \Rightarrow |\xi_2| \geq -\frac{1}{2} |\xi_1|, \]

We continue with the analysis...
and hence
\[ |\xi| = |\xi_1 - \xi_2| \geq ||\xi_1| - |\xi_2|| \geq \frac{1}{2} |\xi_1| \Rightarrow |\xi_1| \leq 2 |\xi|. \]
Moreover, from (28) we have \(3 |\xi_1| \geq 2 |\xi| \). Therefore,
\[ |\xi_1| \leq 2 |\xi| \leq 3 |\xi_1|. \]
We deduce
\[ \langle \xi_1 \rangle \leq C_1 \langle \xi \rangle \leq C_2 \langle \xi_1 \rangle. \tag{28} \]
Now, using (28), we get
\[ \int \frac{\bar{v} \bar{v}_2 (\xi_1)^k}{(\sigma)^{\alpha_1} (\sigma_2)^{\alpha_2} (\xi_2)^{k}} \leq C \int \frac{\bar{v} \bar{v}_2 (\xi_1)^{k-l}}{(\sigma)^{\alpha_1} (\sigma_2)^{\alpha_2} (\xi_2)^{k}} =: Z. \]
Note that, in this region
\[ |\xi_1| \geq 2 |\xi_2| \Rightarrow |\xi_1 - \xi_2| \geq ||\xi_1| - |\xi_2|| \geq 2 |\xi_2|. \]
Moreover, since \(|\xi_1| \geq 2 |\xi| \Rightarrow -\frac{3}{4} \xi_1 \leq -\xi_2^2\), we have
\[ \frac{3}{4} \xi_1^2 = \xi_1^2 - \frac{1}{4} \xi_1^2 \leq \xi_1 - \xi_2 = z \leq \xi_2^2. \]
On the other hand, since \(|\xi_1| \geq 2 |\xi_2| \Rightarrow \xi_1^2 \geq 4 \xi_2^2\), we have
\[ z = \xi_1^2 - \xi_2^2 \geq 3 \xi_2^2. \]
Summarizing
\[ |\xi_1| \geq 2 |\xi_2|, \ |\xi| \geq |\xi_2|, \ |\xi_1| \leq 2 |\xi| \leq 3 |\xi_1|, \tag{29} \]
\[ \frac{3}{4} \xi_1^2 \leq z \leq \xi_1^2, \ z \geq 3 \xi_2^2. \tag{30} \]
Furthermore, it follows from (10) and from \(\xi = \xi_1 - \xi_2\) that
\[ z + \xi^2 = \xi_1^2 - \xi_2^2 + \xi^2 = \xi_1^2 - \xi_2^2 + \xi_1^2 - 2 \xi_1 \xi_2 = 2 \xi_1 \xi_2 - 2 \xi_1 \xi_2 = 2 \xi_1 \xi_2. \tag{31} \]
\[ = 2 \xi_1 \xi_2 - 2 \xi_1 \xi_2 = 2 \xi_1 (\xi_1 - \xi_2) = 2 \xi_1 \xi. \]
and
\[ z - \xi^2 = \xi_1^2 - \xi_2^2 - \xi^2 = \xi_1^2 - \xi_2^2 - \xi_1^2 + 2 \xi_1 \xi_2 - \xi_2^2 = 2 \xi_1 \xi_2 - 2 \xi_2 (\xi_1 - \xi_2) = 2 \xi_2 \xi. \tag{32} \]
And therefore, by (28)
\[ z + \xi^2 = 2 \xi_1 \xi \leq 2 |\xi| |\xi_1| \leq 4 \xi^2 \Rightarrow z \leq 3 \xi^2. \]
Moreover, using (29) and (30), we obtain
\[ z + \xi^2 = 2 \xi_1 \xi \leq 2 |\xi| |\xi_1| \leq 2 \left( \frac{3}{2} |\xi_1| \right) |\xi_1| = 3 \xi_1^2 \leq 4z \Rightarrow \frac{1}{3} \xi^2 \leq z. \]
Hence,
\[ \frac{1}{3} \xi^2 \leq z \leq 3 \xi^2. \tag{33} \]
We now estimate \(Z\) by the Schwarz method.

*Estimates for \(Z\)*

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Region $\sigma_1$ dominant. By exactly the same computation as in the special case, we obtain in the same way as in (24) and (24)

$$C^2_1 \leq C \sup_{\xi_1, \sigma_1} \langle \xi_1 \rangle^{2k-2l-4\alpha_1} \int_0^{\xi_1^2/4} y^{-1/2} (y)^{-k} \, dy \int \langle \sigma \rangle^{-2a} \langle \sigma_2 \rangle^{-2a_2} \, d\sigma_2 \leq C \sup_{\xi_1} \langle \xi_1 \rangle^{2k-2l-4\alpha_1} \int_0^{\xi_1^2/4} y^{-1/2} (y)^{-k} (y)^{-\alpha_1} \, dy < \infty,$$

provided $k - l \leq 2\alpha_1$ and $\alpha_1 > 1/2$. The additional factor $(y)^{-k}$ in the integral does not provide any improvement since we need already $\alpha_1 > 1/2$ in the special case. The last integral again converges at infinity for all $k \geq 0$ but does not yield any decay in $\xi_1$. The condition $k - l \leq 2\alpha_1$ corresponds to (21).

Region $\sigma_2$ dominant. We use the analog of (22) with fixed $\xi_2$ and obtain

$$C^2_2 = \sup_{\xi_2, \sigma_2} \langle \sigma_2 \rangle^{-2a_2} \langle \xi_2 \rangle^{-2k} \int \langle \xi_2 \rangle^{2k-2l} \langle \sigma \rangle^{-2a_1} \langle \sigma_1 \rangle^{-2a_1} \, d\xi_1 d\sigma_1.$$ 

For fixed $\xi_2$, it follows from (13) that $dz = 2|\xi_1| \, d\xi_1$. Using (30) and the fact that $|z| \leq 3|\sigma_2|$ for dominant $\sigma_2$ and integrating over $\sigma_1$ by the use of Lemma (12) we get

$$C^2_2 \leq C \sup_{\xi_2, \sigma_2} \langle \sigma_2 \rangle^{-2a_2} \langle \xi_2 \rangle^{-2k} \int_0^{3|\xi_2|} \int_{3\xi_2^2} |z|^{-1/2} (z)^{k-l} (z + \sigma_2)^{-\alpha_2} \, dz.$$ 

We assume without loss of generality that $k \geq l$. We estimate the last integral by separating the region $0 \leq z \leq |\sigma_2|/2$ and $|\sigma_2|/2 \leq z \leq 3|\sigma_2|$, which in the worst case $\sigma_2 < 0$ contribute respectively

$$\langle \sigma_2 \rangle^{1/2+k-l-\alpha_2},$$

$$\langle \sigma_2 \rangle^{-1/2+k-l+[1-\alpha_2]}.$$ 

Keeping the largest contribution, namely the second one, we obtain

$$C^2_2 \leq C \sup_{\sigma_2} \langle \sigma_2 \rangle^{-2a_2-1/2+k-l+[1-\alpha_2]} + \langle \sigma_2 \rangle^{-2a_2},$$

and the last quantity is finite provided

$$k - l \leq 2\alpha_2 + 1/2 - [1 - \alpha_2]. \quad (34)$$

We shall analyze that condition below together with a similar condition coming from the region $\sigma$ dominant.

Region $\sigma$ dominant. We use (22) to get

$$C^2_2 = \sup_{\xi, \sigma} \langle \sigma \rangle^{-2a} \langle \xi \rangle^{2k-2l} \int \langle \xi \rangle^{-2k} \langle \sigma_1 \rangle^{-2a_1} \langle \sigma_2 \rangle^{-2a_2} \, d\xi d\sigma_2.$$ 

Now $\sigma$ dominant implies $|z| \leq 3|\sigma|$ and therefore $\xi^2 \leq 9|\sigma|$ by (33). We use this fact to estimate the first factor $\langle \sigma \rangle^{-2a}$ in (35). It follows again from (13) that $dz = 2|\xi| \, d\xi$ for fixed $\xi$. We furthermore express $\xi_2$ in terms of $z$ and $\xi$ by (52), and we integrate over $\sigma_2$ for fixed $z$ using Lemma (12) We obtain

$$C^2_2 = C \sup_{\xi, \sigma} \langle \xi \rangle^{2k-2l-4a} |\xi|^{-1} \int_{\xi^2/3}^{3\xi^2} \langle (z - \xi^2)/2|\xi| \rangle^{-2k} (z + \sigma)^{-\alpha} \, dz.$$ 

We next extend the range of integration of $z$ symmetrically to $-2\xi^2 \leq z - \xi^2 = y \leq 2\xi^2$ and apply Lemma (11) with $f(y) = (y/2|\xi|)^{-2k} \chi(|y| \leq 2\xi^2)$, $g(y) = (y)^{-\alpha}$ to conclude that the supremum over $\sigma$ occurs for $\sigma = -\xi^2$, so that

$$C^2_2 = C \sup_{\xi} \langle \xi \rangle^{2k-2l-4a} |\xi|^{-1} \int_0^{2\xi^2} \langle y/2|\xi| \rangle^{-2k} (y)^{-\alpha} \, dy.$$ 

(36)
The right-hand side of the last expression is bounded for $|\xi| \leq 1$, i.e., we do not need the restriction $|\xi| \geq 1$. For $|\xi| \geq 1$ we consider separately the two integration subregions $0 \leq y \leq |\xi|$ and $|\xi| \leq y \leq 2\xi^2$. The contributions of those regions are estimated respectively by

\[
\int_0^{|\xi|} \cdots \, dy \leq \int_0^{|\xi|} (y)^{-\alpha} \leq C |\xi|^{1-\alpha} + dy, \tag{37}
\]

\[
\int_{|\xi|}^{2\xi^2} \cdots \, dy \leq C |\xi|^{2k} \int_{|\xi|}^{2\xi^2} y^{-\alpha-2k} \leq C |\xi|^{1-\alpha + [1-\alpha-2k]+} dy. \tag{38}
\]

Comparing (36), (37), and (38), we see that $C_\ast$ is finite provided

\[\begin{align*}
k - l &\leq 2a + 1/2 - (1/2) [1 - \alpha]_+ , \\
l &> -(2a + \alpha) + 1/2. \tag{39}
\end{align*}\]

The last condition holds for any $l \geq -1/2$ provided $2a + \alpha > 1$, which is implied by

\[\begin{align*}
a + a_1 > 1/2 & a + a_2 > 1/2 , \quad a + a_1 + a_2 > 1. \tag{40}
\end{align*}\]

Note that the latter set of conditions has already been enforced. It only remains to ensure (34) and (39). Now we have already imposed the conditions $k - l \leq 2a_1$ and $a_1 > 1/2, a_2 > 1/2$ or equivalently

\[\begin{align*}
a, a_1, a_2 > 1/4 & a + a_1 > 3/4 , \quad a + a_2 > 3/4. \tag{41}
\end{align*}\]

The conditions (34) and (39) are implied respectively by

\[\begin{align*}
k - l &< 2a_2 + 1/2, \\
k - l &< 2a_2 + 2a - 1/2, \\
k - l &< 2a_2 + 2a_1 - 1/2, \\
k - l &< 2a_2 + 2a + 2a_1 - 3/2, \tag{42}
\end{align*}\]

and

\[\begin{align*}
k - l &< 2a + 1/2, \\
k - l &< 2a + a_1, \\
k - l &< 2a + a_2, \\
k - l &< 2a + a_1 + a_2 - 1/2. \tag{43}
\end{align*}\]

Now $k - l \leq 2a_1$ and (41) imply (42). Next, $2a + a_1 > a + 1/4 + a_1 = (1/2) (2a + 1/2 + 2a_1)$ so that $k - l \leq 2a_1$ and (46) imply (47). Furthermore, $2a + a_2 = (1/2) (2a_2 + 2a - 1/2 + 2a + 1/2)$ so that (43) and (47) imply (48). Finally, $2a + a_1 + a_2 - 1/2 > a + a_1 + a_2 - 1/2 = (1/2) (2a_1 + 2a_2 + 2a - 1/2)$ so that $k - l \leq 2a_1$ and (48) imply (49). It is therefore sufficient to ensure (42), (48), and (49). By (41), the right-hand side of those three inequalities are all $> 1$; they are implied by $k - l \leq 1$, contained in (40).

Now we have to verify the bounds for $A$ in (23), that is

\[
\int \frac{|\hat{\nu}_1 \hat{\nu}_2|}{(\sigma)^9 (\sigma_1)^{\alpha} (\sigma_2)^{\alpha}} \leq C \|v\|_2 \|v_1\|_2 \|v_2\|_2, \tag{50}
\]

provided $a, a_1, a_2 > 1/4$ (see the first condition in (20)). We use the following result for the Schrödinger equation (see [3, Lemma 2.4])

\[\begin{align*}

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\end{align*}\]
Lemma 14. Let \( \phi(\xi) = \xi^2 \) (Schrödinger equation). Assume \( b_0 > 1/2 \), \( 0 \leq b \leq b_0 \), and \( 0 < \eta \leq 1 \) \((\eta \geq 1/2 \) if \( n = 1 \), i.e., \( 1D \)). Then
\[
\|f\|_{L^2_x(L^2_t)} \leq C \|f\|_{\mathcal{C}^{0,\eta}_{\phi(\xi) = \xi^2}} ,
\]
where \( 2/q = 1 - \eta b/b_0 \), \( \delta(r) \equiv n/2 - n/r = (1 - \eta) b/b_0 \).

Using the previous lemma, we show the following

Lemma 15. Let \( a > 1/4 \), \( b_0 = 2a > 1/2 \). Consider \( n = 1 \). Let \( v \in L^2 \) and \( \alpha = \tau + \xi^2 \) (Schrödinger). Then
\[
\| F^{-1} (\langle \alpha \rangle^{-a} |\hat{v}|) \|_{L^q_t(L^r_x)} \leq C \|v\|_2 ,
\]
Proof. By Lemma 14 with \( b = a \leq b_0 = 2a \), \( f = \langle \alpha \rangle^{-a} |\hat{v}| \), and \( \eta = 1/2 \), we have
\[
\| F^{-1} (\langle \alpha \rangle^{-a} |\hat{v}|) \|_{L^q_t(L^r_x)} \leq C \|\hat{v}\|_2 ,
\]
where \( q = 8/3 \) and \( r = 4 \).

Lemma 16. Let \( a, a_1, a_2 > 1/4 \) and \( v, v_1, v_2 \in L^2 \). Then (50) holds.
Proof. Since (50) is decreasing in \( a, a_1, a_2 \), it is sufficient to consider \( a = a_1 = a_2 > 1/4 \). We apply Hölder’s inequality in space and time to obtain
\[
\int \frac{|\hat{v}\hat{v}_1\hat{v}_2|}{\langle \sigma \rangle^q \langle \sigma_1 \rangle^q_1 \langle \sigma_2 \rangle^q_2} \leq \left\| F^{-1} (\langle \sigma \rangle^{-a} |\hat{v}|) \right\|_{L^q_t(L^r_x)} \times \prod_{i=1,2} \left\| F^{-1} (\langle \sigma_i \rangle^{-a_i} |\hat{v}_i|) \right\|_{L^q_t(L^r_x)},
\]
with
\[
\frac{1}{q} + \frac{1}{q_1} + \frac{1}{q_2} = 1, \quad \frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} = 1.
\]
Let
\[
(q, r) = (4, 2), \quad (q_i, r_i) = \left(\frac{8}{3}, 4\right), \quad i = 1, 2.
\]
Then, the last two norms in (51) are estimated by Lemma 15. Now recalling the definition of \( \sigma \), we use the Hardy-Littlewood-Sobolev inequality in time to get
\[
\left\| F^{-1} (\langle \sigma \rangle^{-a} |\hat{v}|) \right\|_{L^q_t(L^r_x)} \leq C \|v\|_2 ,
\]
since \( r = 2 \) and
\[
\frac{1}{2} - \frac{1}{q} = a ,
\]
with \( q = 4, a = 1/4 \).

Our next step is to estimate the nonlinearity \( f_1 = nu \) in \( Y^k_{\phi(\xi) = \xi^2} \). For this, we divide \( \hat{f}_1 \) by \( \langle \sigma_1 \rangle \), integrate over \( \tau_1 \) (or \( \sigma_1 \)) for fixed \( \xi_1 \) and then take the scalar product with a generic function in \( H_+^k \) with Fourier transform \( \langle \xi_1 \rangle^k \hat{w}_1, w_1 \in L^2_x \). The estimate of \( f_1 \) in \( Y^k_{\phi(\xi) = \xi^2} \) becomes
\[
\hat{S} \leq C \|v\|_2 \|w_1\|_2 \|v_2\|_2 ,
\]
where
\[
\hat{S} = \int \frac{|\hat{v}\hat{v}_1\hat{v}_2| \langle \xi_1 \rangle^k}{\langle \sigma \rangle^q \langle \sigma_1 \rangle^q_1 \langle \sigma_2 \rangle^q_2 \langle \xi_1 \rangle^k \langle \xi_2 \rangle^k} ,
\]
with the same notation as in (13). Note that \( w_1 \) is a function of space only, whereas the \( v \)'s are functions of space and time.

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Lemma 17. Let $a, a_2, k,$ and $l$ satisfy (10) and 
\[ a, a_2 > 1/4, \ a + a_2 > 3/4. \]
Then (53) holds.

Proof. The proof is similar to the one of Lemma 13. However, we have to handle $w_1$ appropriately. For this, let $a_1$ satisfy 
\[ 0 < 1/2 - a_1 < \min (1/4, a - 1/4, a + a_2 - 3/4), \]
so that
\[ a_1 > 1/4, \ a + a_2 > 3/4, \ a + a_1 + a_2 > 5/4. \]
Define \( \hat{v}_1 = \langle \sigma_1 \rangle^{a_1-1} \hat{w}_1 \). It follows that \( \|v_1\|_2 \leq C (1 - 2a_1)^{-1/2} \|w_1\|_2 \). Under these conditions, one can follow the proof of Lemma 13 with just minor modifications (cf. [8, Lemma 4.5]).

Now we consider the nonlinearity \( f_2 = |u|^2 u \), which has been extensively studied in the context of the NLS equation. We want to estimate \( f_2 \) in \( X^k_{b(\xi) = \xi^2} \). Hence, we have to verify the expression
\[ |S_0| \leq C \prod_{i=1}^4 \|v_i\|_2, \] (53)
with
\[ S_0 = \int \frac{\hat{v}_1 \hat{w}_2 \hat{w}_3 \hat{w}_4 \langle \xi_1 \rangle^k}{\langle \sigma_1 \rangle^{a_1} \langle \sigma_2 \rangle^{a_2} \langle \sigma_3 \rangle^{a_3} \langle \sigma_4 \rangle^{a_4} \langle \xi_2 \rangle^k \langle \xi_3 \rangle^k \langle \xi_4 \rangle^k}, \] (54)
where \( \hat{v}_1 = \hat{v}_i (\xi_i, \tau_i) \), \( \sigma_i = \tau_i + \xi_i^2 \), \( 1 \leq i \leq 4 \). The integral is over \((\xi_1, \tau_1)\), constrained by \( \xi_1 + \xi_2 = \xi_3 + \xi_4 \) and \( \tau_1 + \tau_2 = \tau_3 + \tau_4 \). Furthermore, if either \( a_1 = 1/2 \) or \( a_2 \leq 1/2 \), we need to estimate \( f_2 \) in \( Y^k_{b(\xi) = \xi^2} \), hence, we have to verify
\[ |\hat{S}_0| \leq C \|w_1\|_2 \prod_{i=2}^4 \|v_i\|_2, \] (55)
with
\[ \hat{S}_0 = \int \frac{\hat{v}_1 \hat{v}_2 \hat{v}_3 \hat{v}_4 \langle \xi_1 \rangle^k}{\langle \sigma_1 \rangle \langle \sigma_2 \rangle^{a_2} \langle \sigma_3 \rangle^{a_2} \langle \sigma_4 \rangle^{a_2} \langle \xi_2 \rangle^k \langle \xi_3 \rangle^k \langle \xi_4 \rangle^k}, \] (56)
where we are using the same notation as before.

Lemma 18. Let \( k \geq 0 \) and
\[ \max (1/6, (1-k)/3) < a_2 < 1. \]
Then (53) and (55) hold.

See [8, Lemma 4.7] for a sketch of the proof. See, e.g., [2, 10] for additional details.

Next we consider the nonlinearity \( f_3 = |u|^3 u \). We want to estimate \( f_3 \) in \( X^k_{b(\xi) = \xi^3} \) for suitable \( a_0 \). Hence, we have to verify the expression
\[ |S_1| \leq C \prod_{i=1}^4 \|v_i\|_2, \] (57)
with
\[ S_1 = \int \frac{\hat{v}_1 \hat{v}_2 \hat{v}_3 \hat{v}_4 \langle \xi_1 \rangle^l}{\langle \sigma_1 \rangle^{a_0} \langle \sigma_2 \rangle^{a_2} \langle \sigma_3 \rangle^{a_3} \langle \sigma_4 \rangle^{a_4} \langle \xi_2 \rangle^k \langle \xi_3 \rangle^k \langle \xi_4 \rangle^k}, \] (58)
where \( \hat{v}_1 = \hat{v}_i (\xi_i, \tau_i) \) \( 1 \leq i \leq 4, \sigma_1 = \tau_1, \sigma_2 = \tau_2 + \xi_2^2, \sigma_3 = \tau_3 + \xi_3^2, \sigma_4 = \tau_4 \). The integral is over \((\xi_1, \tau_1)\), constrained by \( \xi_1 + \xi_2 = \xi_3 + \xi_4 \) and \( \tau_1 + \tau_2 = \tau_3 + \tau_4 \). Note that \( \hat{v}_2 (\xi_2, \tau_2) \) implies that \( \hat{v}_2 (-\xi_2, -\tau_2) \), the
with change of sign due to complex conjugation. Furthermore, if either \( a_0 = 1/2 \) or \( a \leq 1/2 \), we need to estimate \( f_3 \) in \( Y^k_{\phi(\xi)=0} \). Hence, we have to verify

\[
\left| \tilde{S}_1 \right| \leq C \left\| w_1 \right\|_2 \prod_{i=2}^{4} \left\| v_i \right\|_2 ,
\]

with

\[
\tilde{S}_1 = \int \frac{\hat{w}_1 \hat{v}_2 \hat{v}_3 \hat{v}_4 (\xi_1)^l}{(\sigma_1)^{a_0} (\sigma_2)^{a_2} (\sigma_3)^{a_2} (\sigma_4)^{a} (\xi_2)^k (\xi_3)^k (\xi_4)^k} ,
\]

where we are using the same notation as before. We need the following intermediate result.

Lemma 19. Let

\[
W := \int \frac{\hat{\upsilon}_1 \hat{\upsilon}_2 \hat{\upsilon}_3 \hat{\upsilon}_4}{(\sigma_1)^{a_0} (\sigma_2)^{a_2} (\sigma_3)^{a_2} (\sigma_4)^{a} (\xi_2)^k (\xi_3)^k (\xi_4)^k} .
\]

Then

\[
|W| \leq C \prod_{i=1}^{4} \left\| v_i \right\|_2 ,
\]

provided

\[
l \geq \delta_4 = 1 - 2 \left( (1 - \eta) \frac{a_2}{b_0} + k \right) , \quad l > \frac{1}{2} \text{ if } \delta_4 = \frac{1}{2} ,
\]

\[
a_0 + a + \frac{a_2}{b_0} = 1 ,
\]

with \( 1/2 \leq \eta \leq 1 , b_0 \geq a_2 , b_0 \geq 1/2 \). In particular, for \( \eta = 1/2 \) and \( b_0 = a_2 > 1/2 \), we require

\[
l \geq -2k , \quad a_0 + a = 1/2 , \quad \text{and } a_2 > 1/2 .
\]

Proof. Using Hölder’s inequality in space and time, we have

\[
|W| \leq \left\| \mathcal{F}^{-1} \left( (\sigma_1)^{-a_0} \hat{\upsilon}_1 \right) \right\|_{L^q_i(L^p_t)} \times \prod_{i=2,3} \left\| \mathcal{F}^{-1} \left( (\xi_i)^{-k} (\sigma_i)^{-a_2} \hat{\upsilon}_i \right) \right\|_{L^q_i(L^p_t)} \times \left\| \mathcal{F}^{-1} \left( (\xi_4)^{-l} (\sigma_4)^{-a} \hat{\upsilon}_4 \right) \right\|_{L^q_i(L^p_t)} ,
\]

with

\[
\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \frac{1}{q_4} = 1 ,
\]

\[
\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = 1 , \quad \text{or } \delta_1 + \delta_2 + \delta_3 + \delta_4 = 1 ,
\]

and \( r_1 = 2 \). Using an argument similar to Lemma 15 and the Hardy-Littlewood-Sobolev inequality, we obtain

\[
|W| \leq C \prod_{i=1}^{4} \left\| v_i \right\|_2 ,
\]

provided

\[
\frac{2}{q_1} = 1 - 2a_0 , \quad r_1 = 2 \Rightarrow \delta_1 = 0 ,
\]

\[
\frac{2}{q_2} = 1 - \eta \frac{a_2}{b_0} , \quad H^{k,\tilde{r}_2} \subset L^{r_2} , \quad \delta_2 = (1 - \eta) \frac{a_2}{b_0} .
\]
\[
\begin{align*}
\frac{2}{q_3} &= 1 - \eta \frac{a_2}{b_0}, \quad H^{k, r_3} \subset L^{r_3}, \quad \tilde{\delta}_3 = (1 - \eta) \frac{a_2}{b_0}, \\
\frac{2}{q_4} &= 1 - 2a, \quad H^l \subset L^r, \quad l \geq \delta_4 = \frac{1}{2} - \frac{1}{r_4} \geq 0.
\end{align*}
\] (68) (69)

and \( \frac{1}{2} \leq \eta \leq 1. \)

Considering (64) along with the \( q \)'s in (66)-(69), we get (62). From (66), we have
\[
\frac{1}{r_2} = \frac{1}{r_2} - k, \quad k < \frac{1}{r_2} \Rightarrow \delta_2 = \frac{1}{2} - \frac{1}{r_2} - k = \delta_2 - k.
\]

Hence,
\[
\delta_2 = (1 - \eta) \frac{a_2}{b_0} + k.
\]

Similarly,
\[
\delta_3 = (1 - \eta) \frac{a_2}{b_0} + k.
\]

Combining the expressions for the \( \delta \)'s with the RHS of (65), we obtain (61). (63) follows from the previous results.

**Lemma 20.** Let
\[
k \geq 0, \quad l \leq k, \quad a_0 + a = 1/2, \quad a_2 > 1/2.
\]

Then (62) and (64) hold.

**Proof.** Note that, due to the constraint \( \xi_1 + \xi_2 = \xi_3 + \xi_4 \), we get \( (\xi_1)^{\dagger} \leq C \left( (\xi_2)^{\dagger} + (\xi_3)^{\dagger} + (\xi_4)^{\dagger} \right) \). Then, considering the symmetry in the variables 2 and 3, we obtain
\[
|S_1| \leq C \int \frac{|\hat{v}_1 \hat{v}_2 \hat{v}_3 \hat{v}_4|^2 (\xi_2)^{l-k}}{(\sigma_1)^{a_0} (\sigma_2)^{a_2} (\sigma_3)^{a_2} (\sigma_4)^{a_2} (\xi_3)^{l-k} (\xi_4)^{l-k}} + C \int \frac{|\hat{v}_1 \hat{v}_2 \hat{v}_3 \hat{v}_4|^2}{(\sigma_1)^{a_0} (\sigma_2)^{a_2} (\sigma_3)^{a_2} (\sigma_4)^{a_2} (\xi_2)^{l-k} (\xi_3)^{l-k}} := A + B.
\]

To bound \( B \), we use Lemma 19 with \( l = 0 \). Hence, we require
\[
0 \geq -2k, \quad a_0 + a = 1/2, \quad a_2 > 1/2,
\]
which implies \( k \geq 0 \). For \( A \), we use the condition \( l-k \leq 0 \). Then, we consider Lemma 19 replacing \( k \mapsto k-l \), since \( A \) is increasing in \( l \) and decreasing in \( k \) (recall \( k \geq 0, \quad k \geq l \)). Then, we need
\[
-l \geq -2k, \quad a_0 + a = 1/2, \quad a_2 > 1/2.
\]

Hence, combining all the previous conditions we obtain the result for (67).

To estimate \( \tilde{S}_1 \), take \( \hat{v}_1 = (\sigma_1)^{a_0-1} \hat{w}_1 \) in (65). Since for \( a_0 < 1/2 \), we have \( \|\hat{v}_1\|_2 \leq C (a_0) \|\hat{w}_1\|_2 \) holds.

We shall use the following simple observation.

**Lemma 21.** For any \( \varepsilon > 0 \)
\[
\|u\|_{\mathcal{C}^{s}} \leq C \|u\|_{\mathcal{X}^{s-1/2+\varepsilon}_{\alpha(\xi)}}.
\] (70)
Proof. Applying the Cauchy-Schwarz inequality in \( \tau \), we get
\[
\| u \|^2_{Y_{\psi}(\tau)} = \int \left( \langle \xi \rangle^2 \int \langle \tau + \phi(\xi) \rangle^{-1} |\hat{u}(\tau, \xi)| d\tau \right)^2 d\xi \\
\leq \int \langle \xi \rangle^{2a} \left( \int \langle \tau + \phi(\xi) \rangle^{2(-\frac{1}{2} - \varepsilon)} d\tau \right) \left( \int \langle \tau + \phi(\xi) \rangle^{2(-\frac{1}{2} + \varepsilon)} |\hat{u}(\tau, \xi)|^2 d\tau \right) d\xi,
\]
for any \( \varepsilon > 0 \). Since \( -(1 + 2\varepsilon) < -1 \), \((10)\) follows.

Now we present the main result of this section

**Proposition 22.** The exciton-polariton system \((2)\) with initial data \((u_0, n_0) \in H^k \oplus H^l\) is locally well-posed in \(X_{\psi(\xi)=\xi}^{k,a_2}(\tau) \oplus X_{\psi(\xi)=0}^{l,a}(\tau)\) provided
\[
k \geq 0, \ l \leq k, \ k - l \leq 2a_1, \quad (71)\\
a = 1/4 + 3\varepsilon, \quad (72)\\na_1 = 1/2 - 2\varepsilon, \quad (73)\\na_2 = 1/2 + \varepsilon, \quad (74)
\]
with \( \varepsilon > 0 \) small enough \((\varepsilon < 1/12)\). Moreover,
\[
(u, n) \in C ([0, T]; H^k \oplus H^l),
\]
with \( T = T (\| u_0 \|_{H^k}, \| n_0 \|_{H^l}) > 0 \).

Proof. Set
\[
a_0 = 1/4 - 3\varepsilon. \quad (75)
\]
Then, one can verify that under conditions \((71)-(75)\) all the assumptions of Lemmas \([13, 17, 18, 20, 21]\) are satisfied. Moreover, we have \(a + a_0 < 1\) and \(a_2 + a_1 < 1\), hence, we can apply Lemma \([10]\) and obtain ther efrom a strictly positive power of \(T\). Then, we get the result considering the cutoff system \((13)-(15)\) and using a standard fixed point argument. Notice that we use the spaces restricted in time to deal with the term \(P = P(x)\), not to get a positive power of \(T\), which we get from Lemma \([10]\) for \(a \leq 1/2\) we have to take into account \([8, Lemma 2.2]\) to conclude continuity in time of the solution.

**Corollary 23.** Let \((u_0, n_0) \in L^2 \oplus L^2\) with \(n_0(x) \geq 0\). Then, there exists a global in time solution \((u, n) \in C ([0, \infty), L^2 \oplus L^2)\) of the exciton-polariton system \((3)\). Furthermore, the system has an absorbing set in \(L^2 \oplus L^2\).

Proof. Consider a smooth solution of \((2)\), then argue by density. Using the usual variation of constants formula in the second equation of \((2)\), we have
\[
n(t, x) = n_0(x) e^{-\int_0^t \Gamma(\tau, x) d\tau} + P \int_0^t e^{-\int_s^t \Gamma(\tau, x) d\tau} ds,
\]
where \(\Gamma(t, x) = R|u(t, x)|^2 + \beta\). Hence, if \(n_0(x) \geq 0\), then \(n(t, x) \geq 0\) for all \(t \in [0, T]\) since \(P = P(x) \geq 0\).

Now multiply the first equation in \((2)\) by \(\bar{u}\), integrate over \(\mathbb{R}\), and take the imaginary part to get
\[
\frac{d}{dt} \int_\mathbb{R} |u|^2 \, dx = \int_\mathbb{R} (Rn - \alpha) |u|^2 \, dx.
\]
Furthermore, integrate the second equation in \((2)\) over \(\mathbb{R}\) to obtain
\[
\frac{d}{dt} \int_\mathbb{R} n \, dx = \int_\mathbb{R} \left[ P - \left( R|u|^2 + \beta \right) n \right] \, dx.
\]
Combining the last two expressions gives

\[
\frac{d}{dt} \left( \frac{1}{2} \int_{\mathbb{R}} |u|^2 \, dx + \int_{\mathbb{R}} ndx \right) = \int_{\mathbb{R}} [(Rn - \alpha) |u|^2 + P - (R|u|^2 + \beta) u] \, dx \\
= \int_{\mathbb{R}} P \, dx - \alpha \int_{\mathbb{R}} |u|^2 \, dx - \beta \int_{\mathbb{R}} ndx \\
\leq \int_{\mathbb{R}} P \, dx - \gamma \left( \frac{1}{2} \int_{\mathbb{R}} |u|^2 \, dx + \int_{\mathbb{R}} ndx \right),
\]

where \( \gamma = \min (2\alpha, \beta) \). Integrating in time, we get

\[
\frac{1}{2} \int_{\mathbb{R}} |u|^2 \, dx + \int_{\mathbb{R}} ndx \leq e^{-\gamma t} \left( \frac{1}{2} \int_{\mathbb{R}} |u_0|^2 \, dx + \int_{\mathbb{R}} n_0 \, dx - \frac{1}{\gamma} \int_{\mathbb{R}} P \, dx \right) + \frac{1}{\gamma} \int_{\mathbb{R}} P \, dx.
\] (76)

Now multiply the second equation in (2) by \( 2n \) to obtain

\[
\partial_t n^2 = 2Pn - 2 (R|u|^2 + \beta) n^2 \leq \frac{P^2}{\beta} - \beta n^2,
\]

where the last inequality follows from \( \left( \frac{P}{\sqrt{\beta}} - \sqrt{\beta} n \right)^2 \geq 0 \). This implies that

\[
\partial_t (e^{t\beta} n^2) \leq e^{t\beta} \frac{P^2}{\beta}.
\]

Integrating the last expression in time, we get

\[
n^2 (\cdot, t) \leq e^{-t\beta} \left( n_0^2 - \frac{P^2}{\beta^2} \right) + \frac{P^2}{\beta^2}, \forall 0 \leq t \leq T.
\]

Hence,

\[
\int_{\mathbb{R}} n^2 \, dx \leq e^{-t\beta} \left( \int_{\mathbb{R}} n_0 \, dx - \frac{1}{\beta^2} \int_{\mathbb{R}} P^2 \, dx \right) + \frac{1}{\beta} \int_{\mathbb{R}} P^2 \, dx.
\] (77)

The result follows by combining (76), (77), the fact that \( n_0 (x) \geq 0 \), and a density argument. \( \square \)

The global existence theory of (2) in \( H^1 (\mathbb{T}) \oplus H^1 (\mathbb{T}) \) was established in [1].

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