A model of the collapse and evaporation of charged black holes

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Abstract
In this paper, a natural generalization of KMY model is proposed for the evaporation of charged black holes. Within the proposed model, the back reaction of Hawking radiation is considered. More specifically, we consider the equation $G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle$, in which the matter content $\langle T_{\mu\nu} \rangle$ is assumed spherically symmetric. We also assume the radiation to be massless and chargeless. With this equation of motion, the asymptotic behavior of the model is analyzed.

Two kinds of matter contents are taken into consideration in this paper. In the first case (the thin-shell model), the infalling matter is simulated by a null-like charged sphere collapsing into its center. In the second case, we consider a continuous distribution of spherical symmetric infalling null-like charged matter. It is simulated by taking the continuous limit of many co-centric spheres collapsing into the center.

We find that in the thin-shell case, an event horizon forms and the shell passes through the horizon before becoming extremal, provided that it is not initially near-extremal. In the case with continuous matter distribution, we consider explicitly an extremal center covered by neutral infalling matter, and find that the event horizon also forms. The black hole itself will become near-extremal eventually, leaving possibly a non-electromagnetic energy residue less than the order of $e^{-3\alpha}$.

The details of the behavior of these models are explicitly worked out in this paper.
Keywords: information paradox, semi-classical Einstein equation, Reissner–Nordström–Vaidya metric, Hawking radiation, KMY model, charged black holes

(Some figures may appear in colour only in the online journal)

1. Introduction

As to the study of Hawking radiation, it is often assumed that the background geometry is that of the classical black holes, such as Schwarzschild metric for neutral non-rotating black holes, Reissner–Nordström metric for charged non-rotating black holes, etc. The back reaction of Hawking radiation is usually neglected in these cases, assuming that it has little impact on the relevant physical features of the systems [1–4].

In the paper of Kawai et al a new consistent model of black hole evaporation was proposed [5–7]. This model (KMY model) is a consistent solution to the semi-classical Einstein equation:

\[ G_{\mu\nu} = 8\pi \langle T_{\mu\nu} \rangle \]  

where \( \langle T_{\mu\nu} \rangle \) consists of Hawking radiation and the infalling matter. As a special case of KMY model, one single null-like shell of infalling matter is considered in Schwarzschild background [5]. It was found out that, in this case, the black hole would never evaporate completely, and event horizon and singularity are formed. Similar results are obtained by other authors with different models [16–20]. In the paper, a more general case with a continuum of infalling null shells was also taken into consideration [5]. In this general case, the black hole evaporates completely, and each layer of collapsing matter never touches its corresponding Schwarzschild radius.

This model gives us many useful insights to the problem of information loss paradox [8, 9], since in KMY model, either a permanent event horizon forms, or the infalling matter never reaches event horizon. It is unnecessary to introduce any non-local interaction to maintain the unitarity of the initial state of infalling matter and the final state of Hawking radiation.

It is therefore natural to ask whether this model can be generalized to more complex cases, such as charged or rotating black holes. We also hope that the beautiful properties of KMY model, such as that the infalling matter will never pass through event horizon, or the formation of a permanent event horizon, will persist after this generalization. These properties imply that the information will not be lost through event horizon, or will be locked in a permanent horizon.

In this paper, we generalized KMY model to describe the evaporation of charged black holes, considering the thin-shell model (a charged null-like collapsing sphere) and the a more general model of continuously collapsing matter (the continuous limit of many co-centered charged null-like shells). A detailed analysis showed that the information of infalling matter will not be lost in either case. During the analysis, we had assumed that Hawking radiation is both chargeless and massless. The motivation of applying such assumptions goes as the following: if the Hawking temperature is low, no matter what the field content is, the Hawking radiation should be dominated by massless radiation. Namely, for the approximation of massless radiation to hold, the Hawking radiation temperature \( T_H \) needs to be sufficiently small, such that \( kT_H \ll m_0c^2 \), in which \( m_0 \) is the smallest mass of any other massive field in the model. In the real world, the only known massless long range field is photons, so it is expected that the Hawking radiation is dominated by chargeless photons. Therefore, the chargeless and massless field approximation holds as least at the part of evaporation process in which Hawking temperature is low. Actually, we will show that, for this model, the Hawking
temperature is indeed very low at any time during the evaporation, assuming that it is initially very small. Therefore, this approximation would hold during any stage of evaporation\(^1\).

We also showed in this paper that, an event horizon will always exist when a black hole is charged, whether in the thin-shell case or in the case of continuously collapsing matter. The difference is that, in the thin-shell model, the black hole only emits certain amount of radiation before the thin shell enters the horizon, and the black hole remains non-extremal. On the other hand, in a model of continuously collapsing matter, there could be two cases depending on the initial condition. In the first case, all layers of shells become extremal as they pass the horizon. In the second case, there exists a critical layer. A shell above the critical layer will become extremal as it enters the horizon, while a shell below the critical layer will not become extremal as it enters the horizon\(^2\). One can see this better if we present the black hole in the Penrose diagram (see figures 1 and 2).

This paper is presented in the following way. In sections 1.1 and 1.2, we introduce some background related to KMY model and Reissner–Nordström–Vaidya metric (RNV metric). RNV metric describes a radiating charged black hole, and is crucial in the formulation of our model.

In section 2, we give out the formulation of the thin-shell model, and work out the set of equations of motions of the sphere ((22) and (26)). We analyze this set of equations in section 2.1, and find out that under suitable conditions, the sphere will only move for a finite small distance in an infinite amount of time for a distant observer, implying that the black hole will not become extremal as the sphere crosses the event horizon.

Then, we turn to the case of continuously collapsing matter in section 3. In section 3.1, we give out the basic formulation and write down the metric of the such a system (52), and in section 3.2, we work out the flux formula of the black hole ((60) and (61)). In section 4, we use the equations we derive from the previous section to gain some insight of the asymptotic behavior of the black hole.

In section 4.1, we consider the simpler case in which the black hole is not near-extremal and have an extremal center followed by the collapsing of a neutral outer layer. Based on a few assumptions (most importantly, the quasi-static approximation), the evaporation equation (74) can be deduced. It is further analyzed and found that this equation is only valid under the condition \( x \gg e_0^{3/2} \), where \( x = a_0 - e_0 \). (For the definition of \( a_0 \) and \( e_0 \), please refer to section 4.1, and without other specification, all of the quantities are written in plank unit.) Therefore, the approximation can be applied to almost all ranges of \( a_0 \). In section 4.2, we turn to focus on the near-extremal case, when the criteria of the previous section does not hold, but every shell is near its extremal limit. Therefore, we can apply perturbation to the model. We show that we can deduce some properties of the asymptotic behavior by only considering the null condition \( \dot{r}_\alpha = -\frac{1}{\sqrt{2}}(u_\alpha, r_\alpha) \). Also, by assuming the weak energy condition and some suitable initial conditions, we can get an estimation of the redshift factor \( \tilde{\psi}_\alpha := \log(dU/du_\alpha) \)

\(^1\) Some readers may ask: if we assume the radiating field is both chargeless and massless, why is it reasonable to assume that the infalling matter is charged but massless? After all, there is no known (classical) matter that is both null and containing charge. The reason for this approximation to be valid is that the Hawking radiation is determined by the kinematics rather than the content of infalling matter. Note that in a thin-shell model, the relation of retarded time inside the shell (\( U \)) and outside the shell (\( u \)) is determined by the trajectory of the shell, and therefore, the trajectory determines the energy flux (see equation (6)). If the shell is very close to the horizon (this is the part where we are interested in), any free falling trajectory will be very close to the null trajectory. Thus, any free falling trajectory can be viewed as a perturbation of the null trajectory.

\(^2\) Under this case, there is still a possibility that the critical layer is the outermost layer, and the black hole does not become totally extremal. We will show in this paper that even if this is the case, the residue of the non-electromagnetic energy is at the order of \( e_0^{3/2} \).
and we find that it will approach to $-\infty$ as $u_\alpha$ approaches to infinity. Also, the general Penrose diagram is given in figures 1 and 2.

In section 4.3, we calculate the flux and temperature of Hawking radiation of the black hole based on the result we derived previously, and find that the Hawking temperature is always small, which justifies our assumption of chargeless field. In section 4.4, we give out the energy-momentum tensor of the model of the black hole. We observe that it can be decomposed into an outgoing radiation part, an infalling matter part, an electromagnetic part and a tangential part.

1.1. KMY model

In the thin-shell KMY model, we consider a sphere collapsing from the past null-like infinity at the speed of light. Suppose that at any moment, the geometry outside the sphere of radius $R(u)$ can be described as the outgoing Vaidya metric: [5, 10]

$$ds^2 = -\left(1 - \frac{2a(u)}{r}\right)du^2 - 2dudr + r^2d\Omega^2, \quad \text{where } r \gg R(u) \quad (2)$$

(92), and we find that it will approach to $-\infty$ as $u_\alpha$ approaches to infinity. Also, the general Penrose diagram is given in figures 1 and 2.

In section 4.3, we calculate the flux and temperature of Hawking radiation of the black hole based on the result we derived previously, and find that the Hawking temperature is always small, which justifies our assumption of chargeless field. In section 4.4, we give out the energy-momentum tensor of the model of the black hole. We observe that it can be decomposed into an outgoing radiation part, an infalling matter part, an electromagnetic part and a tangential part.

**Figure 1.** The Penrose diagram that exhibits the critical layer $\alpha^*$ is depicted as the following. The crucial difference of this figure with figure 2 is the existence of $\alpha^*$, such that any layer $\alpha_2$ which is below $\alpha^*$ will only become extremal inside the horizon $H$ (at $P_2$). Therefore, it will still emit some radiation into an asymptotic region other than the region of the observer. The solid curves are constant $r$ curves and the dashed curves represent constant $a$ curves. More precisely, $S_1, S_2$ and $S_3$ respectively represent $r = e^{\alpha_1}$, $r = e^{\alpha_2} = e^0$ and $r = \text{const.} > e^0$. On the other hand, $C_1$ and $C_2$ respectively represent $a = e^{\alpha_1}$ and $a = e^{\alpha_2} = e^0$. 

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which corresponds to a black hole emitting massless radiation. The energy-momentum tensor is

$$T_{uu} = \frac{\dot{a}(u)}{4\pi Gr^2}$$

with all the other components being equal to zero. On the other hand, the geometry is flat inside the sphere:

$$ds^2 = -dU^2 - 2dUdr + r^2d\Omega^2, \quad \text{where } r < R(u).$$

Imposing the continuous relation between the coordinate between the inside and the outside of the sphere and the condition of the null-matter, one can obtain:

$$dU = -2dR = \left(1 - \frac{2a(u)}{R(u)}\right)du.$$
Thus, the relationship between \( u \) and \( U \) and the trajectory of \( R(u) \) can be found.

In order to find out the effect of Hawking radiation on the metric, one has to find the radiation flux \( J \), or more precisely, the component \( \langle T_{uu} \rangle \). In [5], they calculated \( J \) via Eikonal approximation

\[
J(u) = 4\pi r^2 \langle T_{uu} \rangle = \frac{\hbar}{8\pi} \left( \frac{\ddot{U}(u)^2}{U(u)^2} - \frac{2\dddot{U}(u)}{3U(u)} \right) \equiv \frac{\hbar}{8\pi} \{u, U\}. \tag{6}
\]

Thus,

\[
\frac{da}{du} = -GJ = -\frac{1}{8\pi} \left( \frac{\ddot{R}(u)^2}{R(u)^2} - \frac{2\dddot{R}(u)}{3R(u)} \right). \tag{7}
\]

The first equality in (5) was used to replace \( U(u) \) with \( R(u) \) in the second equality in (7).

Combining the second equality of (5) and (7), we can solve \( R(u), a(u) \) in a self-consistent way, and hence the geometry and the dynamics of the shell are found.

In [5], they found out that \( a(u) \) approaches an asymptotic value as \( u \) approaches to infinity, forming an event horizon [5, 8].

In the generic spherical symmetric case, the geometry is considered to be outgoing Vaidya between layers with their own coordinates \( u_n \), requiring every coordinate of the layers to satisfy continuity at their own junctions. One can obtain the transformation law between the coordinate of different layers. More precisely,

\[
- \left(1 - \frac{2a_n(u_n)}{R_n}\right) du_n^2 - 2du_n dR_n = - \left(1 - \frac{2a_{n+1}(u_{n+1})}{R_{n+1}}\right) du_{n+1}^2 - 2du_{n+1} dR_{n+1} \tag{8}
\]

where \( R_n \) (or \( R_{n+1} \)) is the radius of \( n \)th (or \( n + 1 \)th) sphere and \( a_n \) (or \( a_{n+1} \)) is the Schwarzschild radius of the corresponding sphere. In the continuous limit, the index \( n \) becomes a continuous parameter. In [5], the relation between \( u_{\alpha_1} \) and \( u_{\alpha_2} \) is calculated:

\[
\frac{du_{\alpha_1}}{du_{\alpha_2}} = \exp \left(2 \int_{\alpha_2}^{\alpha_1} \frac{da_\alpha}{R_\alpha(a_\alpha) - 2a_\alpha(a_\alpha)} \right). \tag{9}
\]

The Hawking radiation at each sphere also governs the evolution of the sphere:

\[
\frac{da_\alpha}{du_\alpha} = -\frac{1}{8\pi} \{u_\alpha, U\}. \tag{10}
\]

\( U \) is the flat coordinate inside the innermost sphere. Also, by inserting (9) into (10), and using the null condition on each shell, one can obtain the evolution of the Schwarzschild radius of each shell [5, 8]:

\[
\frac{da_\alpha}{du_\alpha} = -\frac{1}{384\pi a_\alpha^4} + O(a_\alpha^{-4}). \tag{11}
\]

Therefore, the black hole will evaporate at about the time order of \( a^3 \), where \( a \) is the initial Schwarzschild radius of the black hole.

1.2. Reissner–Nordström–Vaidya black hole

In order to generalize KMY model to describe charged black holes, one needs to find an appropriate metrics to describe a charged black hole simultaneously emitting radiation. In the papers of Ibohal and Wang [11–14], a series of metrics are proposed by letting the mass or charge parameter of Reissner–Nordström metric or Kerr metric to have a dependence on \( u \) and
The paper suggested that the dependence of those parameters contributes to the charge and mass loss of the black hole by Hawking radiation.

In particular, we will focus on one specific type of metric called Reissner–Nordström–Vaidya metric (RNV metric), which can be considered as a charged black hole emitting massless and chargeless radiation: [12, 13]

\[
ds^2 = - \left( 1 - \frac{2a(u)}{r} + \frac{e^2}{r^2} \right) du^2 - 2du dr + r^2 d\Omega^2
\]

(12)

where \(a(u)\) and \(e\) are parameters representing its mass and charge respectively. Including all the dimensionful constants, they are

\[
a(u) = \frac{2GM(u)}{c^2}, \quad e^2 = \frac{GQ}{4\pi\epsilon_0 c^4}.
\]

(13)

\(M\) and \(Q\) could be seen as the mass and the charge of the black hole. Both \(a\) and \(e\) have the dimension of length.

The energy-momentum tensor of RNV black holes is:

\[
T_{ab} = \mu n_a n_b + 2\rho (n_a n_b) + 2p (\bar{m}_a \bar{m}_b).
\]

(14)

The energy-momentum tensor can be decomposed into the radiation part and the electromagnetic field part:

\[
T_{ab}^{(n)} = \mu n_a n_b
\]

(15)

\[
T_{ab}^{(E)} = 2\rho (n_a n_b) + 2p (\bar{m}_a \bar{m}_b)
\]

(16)

where \(\mu\) can be considered as the null radiation density, and \(\rho\) and \(p\) are the energy density and pressure of the electromagnetic field respectively

\[
\mu = -\frac{\dot{a}(u)}{4\pi Gr^2}, \quad \text{and} \quad p = \rho = \frac{e^2}{8\pi Gr^2}.
\]

(17)

\(l_a\) and \(n_a\) are real null vectors; \(m_a\) are complex null vectors, which satisfy \(n_a^\mu l_\mu = -1 = -m_a\bar{m}_a^\mu\)

\[
l_a = \delta_a^\mu
\]

(18)

\[n_a = \frac{1}{2} \left( 1 - \frac{2a(u)}{r} + \frac{e^2}{r^2} \right) \delta_a^\mu + \delta_a^\gamma
\]

(19)

\[m_a = -\frac{r}{\sqrt{2}} (\delta_a^\gamma + i \sin \theta \delta_a^\delta)
\]

(20)

With the radiation part of the energy-momentum tensor being the same as that of Vaidya metric, we will only equate this part \((T_{\mu\nu}^{(n)})\) with the quantum field expectation value \(\langle T_{\mu\nu} \rangle\) of a massless scalar field, assuming that the electromagnetic field can be treated classically, thus it has no contribution to Hawking radiation.

2. The collapse of a thin-shell RNV black hole

We shall now consider a single infalling null-like sphere which is charged and collapses from the past null-like infinity. The geometry can be described as RNV metric outside the sphere and flat Minkowski inside the sphere.
\[ ds^2 = \begin{cases} -\frac{4U}{r^2} - 2\frac{dU}{dr} + r^2 d\Omega^2 & \text{when } r < r_s(u) \\ -F(u, r)dr^2 - 2dudr + r^2 d\Omega^2 & \text{when } r \geq r_s(u) \end{cases} \]

\( r_s(u) \) is the radius of the sphere, and \( F(u, r) = 1 - 2a(u)/r + e^2/r^2 \). Then, imposing the continuity condition between the inside and the outside of the sphere and the light-like condition of the sphere, we get

\[
\frac{dr_s}{du} = -\frac{1}{2} \frac{dU}{du} - \frac{1}{2} F(u, r_s).
\] (22)

By directly solving (22) we can get:

\[
(1 + 2r_s)^2 - 2ar_s + e^2 = 0, \Rightarrow r_s = \frac{a + \sqrt{a^2 - (1 + 2r_s)e^2}}{1 + 2r_s}. \] (23)

Note that only the positive solution is considered because the radius of the shell is larger than the outer horizon of the black hole. Namely, \( r_s > r_h \).

We are mainly interested in the case when the sphere is very close to its apparent horizon, namely \( |\dot{r}_s| \ll 1 \). We can expand (23) up to first order of \( \dot{r} \)

\[
r_s = r_h - 2r_h\dot{r}_s - \frac{e^2\dot{r}_s}{\sqrt{a^2 - e^2}} + O(\dot{r}_s^2). \] (24)

We can see that \( r_s > r_h \) if \( \dot{r}_s < 0 \). In the limit of chargeless black hole \( (e \ll a) \), the third term of the right hand side of (24) can be neglected. Otherwise, we need to include both the second term and the third term of (24). Also, it can be observed that the first order approximation only holds if \( \frac{\dot{a}^2 - e^2}{a^2} \gg |\dot{r}_s| \).

Now we consider the flux formula for Hawking radiation. Since we are considering a spherical symmetric metric, the Eikonal approximation discussed in appendix A of [5] still applies

\[
J(u) = \frac{\hbar}{8\pi} \left( \frac{\dot{U}(u)^2}{U(u)^2} - \frac{2\ddot{U}(u)}{3U(u)} \right) \] (25)

but the relation of \( U \) and \( u \) is determined by (22). With (3) and the first equality of (22), we can deduce an equation that is similar to (7):

\[
\dot{a}(u) = -\frac{1}{8\pi} \left( \frac{\dot{r}_s(u)^2}{\dot{r}_s(u)^2} - \frac{2\ddot{r}_s(u)}{3\dot{r}_s(u)} \right). \] (26)

With (22) and (26) together, \( a(u) \) and \( r_s(u) \) can be solved in a consistent way.

2.1. The asymptotic behavior of thin-shell RNV black holes

In this section, we shall consider the asymptotic behavior of non extremal infalling null shell in its late time limit.

The process of the collapse of shell is divided into two stages. In the first stage, the radius of the shell \( r_s \) is not near its apparent horizon \( r_h = a + \sqrt{a^2 - e^2} \). Form (22) we know that \( |\dot{r}_s| = O(1) \). Therefore, \( \dot{a} \) is negligible compared to \( \dot{r}_s \), and \( a(u) \) can be regarded as constant at this stage. This is true until \( r_s \) is close enough to \( r_h \) such that \( \dot{a} \) is comparable to \( \dot{r}_s \). In the second stage, both \( \dot{a} \) and \( \dot{r}_s \) is small but comparable (i.e. \( O(\dot{a}) = O(\dot{r}_s) \)), and \( r_s \) is close to the apparent horizon. This is the stage we are interested at, and the rest of the section will focus on analyzing the behavior at this stage.
Assume that the black hole is not near extremal, so (24) can apply. If \( a(u), r_s(u) \) changes rather smoothly, from (24) one can show that

\[
r_s = r_h, \dot{r}_s = \ddot{r}_h, \dddot{r}_s = \dddot{r}_h, \cdots 
\]

When we say that \( a(u), r_s(u) \) change smoothly, we mean that \( a(u), r_s(u) \) satisfy the condition

\[
r_s \gg |\dot{r}_s| \gg |\ddot{r}_s| \cdots \tag{27}
\]

at late time limit. (It is the same for \( a \)). This condition can be accomplished, for example, if \( r_s \) is dominated by some negative power of \( u \). Equation (27) and be shown by applying this condition to (24).

Therefore, using the relation

\[
a = e^2 + \frac{r_h^2}{2r_h} \Rightarrow \dot{a} = \frac{\dot{r}_h}{2} - \frac{e^2}{2r_h^2} \tag{29}
\]

and inserting it into (26), we get

\[
\ddot{r}_h = \frac{3}{2} \frac{\dot{r}_h^2}{r_h} + 6\pi \left( 1 - \frac{e^2}{r_h^2} \right) \dot{r}_h^2. \tag{30}
\]

Consider that after some late time \( u^* \), the equation (30) holds, and then we shall assume that the last term in the RHS of (30) could be neglected. Solving \( \ddot{r}_h = 3\frac{\dot{r}_h^2}{2r_h} \), one can find,

\[
\dot{r}_h = -\frac{C_2}{(u-u_0)^2}, \text{ for } u > u^* \tag{31}
\]

where \( u_0(< u^*) \) and \( C_2 \) are integration constants. Therefore, the total distance that \( r_h \) changes over an infinite time is finite:

\[
r_h(\pm \infty) = r_h(u^*) + \frac{C_2}{u^* - u_0}. \tag{32}
\]

The total distance \( r_s \) travelled after \( u > u^* \) is \( C_2 / (u^* - u_0) \), while

\[
\frac{2\ddot{r}_h(u^*)}{r_h(u^*)} = \frac{C_2}{u^* - u_0}. \tag{33}
\]

This value is very small. Using the null equation (22), we find

\[
\frac{2\ddot{r}_h(u^*)}{r_h(u^*)} = O(\ddot{r}_s(u^*)). \tag{34}
\]

We have use the assumption \( \dot{a} = O(\dot{r}_s) \). Thus, if \( r_h \) is not very close to \( a \) (not near extremal), \( r_h \) will approach a certain value at late time limit. It will not become extremal before the sphere falls into the horizon.

Finally, we shall verify that the last term in the RHS of (30) can indeed be neglected. By inserting the solution \( \dot{r}_h = C_2(u - u_0)^{-2} \) into the first term and the second term of the equation, we find that

\[
\frac{3}{2} \frac{\dot{r}_h^2}{r_h} \sim C_2 (u - u_0)^{-4}, \text{ and } 6\pi \left( 1 - \frac{e^2}{r_h^2} \right) \dot{r}_h^2 \sim C_2^2 (u - u_0)^{-4}. \tag{35}
\]

It is observed that \( C_2 = \frac{4\ddot{r}_s(u^*)}{r_h(u^*)} \) will also be a small value since again form the null condition (22) and \( \dot{a} = O(\dot{r}_s) \).
\[
\frac{4r_s^3(u^*)}{r_n(u^*)} = O(\dot{r}_n(u^*)).
\] (36)

Therefore, the second term which is proportional to \(C_2^2\) is generally much smaller than the first term (which is proportional to \(C_2\)). Thus, the second term can be neglected.

3. The continuously collapsing RNV black hole

3.1. The continuous limit

We now turn to the case of continuously infalling matter. We consider many concentric infalling null shells, each shell to be labelled by an index \(n\), with radius \(R_n\). By denoting \(R_0 = r_s\) to be the radius of the outermost shell, the geometry between shell \(n\) and shell \(n-1\) can be described by Reissner–Nordström–Vaidya metric:

\[
d s^2 = -F_n du_n^2 - 2du_n dr + r^2 d\Omega^2
\] (37)

\[
= -\left(1 - \frac{2a_n(u_n)}{r} + \frac{\epsilon_n^2}{r^2}\right) du_n^2 - 2du_n dr + r^2 d\Omega^2.
\] (38)

\(a_n(u_n)\) generally describes the Bondi mass that is inside the \(n\)th shell, and \(\epsilon_n\) is the charge inside the shell which is constant if we assume that the radiation is chargeless, and the charge is fixed on each shell. The continuity condition between shells implies

\[-F_n(R_n, u_n) du_n^2 - 2du_n dR_n = -F_{n+1}(R_n, u_{n+1}) du_{n+1}^2 - 2du_{n+1} dR_n\] (39)

or,

\[
\frac{du_{n+1}}{du_n} = \frac{1}{F_{n+1}} \left[ \sqrt{\left(\frac{dR_n}{du_n}\right)^2 + F_{n+1} \left(F_n + 2\frac{dR_n}{du_n}\right) - \frac{dR_n}{du_n}} \right]
\] (40)

\[
= 1 - \frac{dF_n}{2(R_n + F_n)} + O(dF_n^2)
\] (41)

where \(\frac{dR_n}{du_n} = \dot{R}_n\), and

\[dF_n = F_{n+1}(u_{n+1}, R_n) - F_n(u_n, R_n).\] (42)

Note that we only consider the positive solution in (40) since we need the retarded time between the \(n\)th shells (\(u_n\) and \(u_{n+1}\)) to be flowing in the same direction. Otherwise, the continuous limit of infinite shells would not make sense.

For \(m > n\)

\[
\frac{du_m}{du_n} = \prod_{k=n}^{m-1} \left(1 - \frac{dF_k}{2(R_k + F_k)}\right).\] (43)

In the continuous limit, we change the parameter from \(n\) to \(\alpha\)

\[
\frac{du_{\alpha_2}}{du_{\alpha_1}} = \exp\left(-\frac{1}{2} \int_{\alpha_2}^{\alpha_1} dF_{\alpha} \frac{d\alpha}{F_{\alpha} + R_{\alpha}}\right).\] (44)
We adapt the convention that $u_0 = u$ is the retarded time of the outermost layer, and $u_{\alpha_{\text{max}}} = U$ is the retarded time of the innermost layer, which can also be considered as the retarded time from the past null infinity [15]. For later convenience, we rewrite the index $\alpha$ in terms of $r$. That is:

$$R_\alpha(u_\alpha(u)) = r.$$  \hfill (45)

Using (45), we can redefine $\alpha$ in terms of $u$ and $r$. Thus,

$$F_\alpha(u_\alpha, r) \equiv F(u, r) = 1 - \frac{2a(u, r)}{r} + \frac{e^2(u, r)}{r^2}$$  \hfill (46)

$$\dot{R}_\alpha \equiv V(u, r).$$  \hfill (47)

The redshift factor from $r$ to $r_s$ is

$$\frac{du_\alpha}{du} \equiv \exp \psi(u, r)$$  \hfill (48)

$$\psi(u, r) = \frac{1}{2} \int_{r}^{r_s(u)} \frac{dF}{F(u, r') + V(u, r')}$$  \hfill (49)

$$= \int_{r}^{r_s(u)} \frac{ee' - a'd'r'}{(1 + V)r'^2 - 2ar' + e^2} \, dr'.$$  \hfill (50)

where $dF = - \frac{2dr}{r} + \frac{2de}{r}$ and $a' = \frac{\partial a}{\partial r'}$, $e' = \frac{\partial e}{\partial r'}$. If the infalling matter is null, $V(u, r) = - \frac{1}{2} F(u, r)$. Thus,

$$\psi(u, r) = 2 \int_{r}^{r_s(u)} \frac{ee' - a'd'r'}{r'^2 - 2ar' + e^2} \, dr'.$$  \hfill (51)

Then, the full metric of the system is obtained:

$$ds^2 = \begin{cases} -F(u, r)e^{2\psi} \, du^2 - 2e^\psi \, du \, dr + r^2 \, d\Omega^2 & \text{when } r < r_s(u) \\ -F_0(u, r) \, du^2 - 2du \, dr + r^2 \, d\Omega^2 & \text{when } r \geq r_s(u) \end{cases}$$  \hfill (52)

where

$$F_0(u, r) = 1 - \frac{2a_0(u)}{r} + \frac{e_0^2}{r^2}$$  \hfill (53)

$a_0$ and $e_0$ are respectively the total Bondi mass and the total electrical charge.

3.2. Hawking radiation

Let $f_\alpha(u_\alpha)$ be any function of $u_\alpha$, we can construct the 'covariant derivative' of $f_\alpha$:

$$\frac{df_\alpha}{du_\alpha} = \frac{du}{du_\alpha} \frac{\partial f(u, r)}{\partial u} + \frac{dr}{du_\alpha} \frac{\partial f(u, r)}{\partial r}$$  \hfill (54)

$$= e^{-\psi(u, r)} \frac{\partial f(u, r)}{\partial u} + V(u, r) \frac{\partial f(u, r)}{\partial r}.$$  \hfill (55)
The operation
\[
\frac{d}{du_\alpha} \rightarrow D \equiv e^{-\psi} \frac{\partial}{\partial u} + V \frac{\partial}{\partial r}
\]  \hspace{1cm} (56)

is called the covariant derivative with respect to the moving shells.

In order to find the evolution of each shell, note that (26) can be modified to be
\[
\frac{da_\alpha}{du_\alpha} = -\frac{1}{8\pi} \{u_\alpha, U\}. \hspace{1cm} (57)
\]

\(U\) is the light cone parameter from the past infinity [15], which can be identified as the retarded time of the innermost shell. By (44), the relation between \(u_\alpha\) and \(U\) is
\[
\frac{dU}{du_\alpha} = e^{\tilde{\psi}(u,r)} \hspace{1cm} (58)
\]

where
\[
\tilde{\psi}(u,r) = \psi(u,0) - \psi(u,r) = \int_0^r \frac{ee' - a'r'}{(1 + V)r'^2 - ar' + e^2} dr'. \hspace{1cm} (59)
\]

By (57) and (58), the equation of the dynamics of \(a(u,r)\) is
\[
Da(u,r) = -\frac{1}{24\pi} \left[(D\tilde{\psi}(u,r))^2 - 2D^2\tilde{\psi}(u,r)\right], r < r_s(u). \hspace{1cm} (60)
\]

In particular, if we consider the evolution of the outermost shell
\[
\dot{a}_0(u) = -\frac{1}{24\pi} \left[\tilde{\psi}_0(u,r)^2 - 2\tilde{\psi}_0(u,r)\right], r \geq r_s(u) \hspace{1cm} (61)
\]

in which \(a_0\) is the Bondi mass of the whole black hole, and
\[
\tilde{\psi}_0(u) = \int_{r_s(u)}^r \frac{ee' - a'r'}{(1 + V)r'^2 - ar' + e^2} dr'. \hspace{1cm} (62)
\]

Due to the conservation of charges on each shell
\[
\frac{de_\alpha}{du_\alpha} = 0 \Rightarrow De(u,r) = 0. \hspace{1cm} (63)
\]

At this point, if one imposes the null matter condition of \(V(u,r) = -\frac{1}{2}F(u,r)\), the geometry and the dynamics will be completely determined in a consistent way by (50), (52), (59), (60) and (63).

4. The asymptotic analysis

In this section, we analyze the behavior of continuously collapsing RNV black hole under various circumstances. As an assumption, every shell obeys the null condition \(\dot{r}_\alpha = -\frac{1}{2}F(r_\alpha, r_\alpha)\). We will consider the case in which every collapsing shell is near its apparent horizon. The shell being near its apparent horizon implies that \(|\dot{r}_\alpha| \ll 1\) and the dependence of time in \(a_\alpha(u_\alpha)\) cannot be ignored.

The discussion will be divided into two parts. The first one is the non-extremal case in which \(a_0\) is not too close to \(e_0\), in which (24) can hold. We also consider a region of near extremal inner layers covered by a region of collapsing neutral outer layers. Under this situation, quasi-static approximation can be applied.
In the second part, we consider the near-extremal case, in which every shell is near its extremal limit in the initial condition. In this case, (24) does not hold. A more sophisticated treatment is needed under this situation.

4.1. The non-extremal behavior

Being parallel to the notation used in [5], we define

$$\xi(u) := \frac{d}{du} \log(dU/du) = \tilde{\psi}_0.$$  \hspace{1cm} (64)

By inserting (62), the definition of $\tilde{\psi}_0$, into (64) with $V = -\frac{1}{2}F$, we get

$$\xi = \frac{\tilde{r}_h}{\tilde{F}} \left( -\frac{2a'(u,r_s)}{r_s} + \frac{2e'(u,r_s)q_0}{r_s^2} \right) + 2 \int_0^{r_s(u)} \frac{d}{du} \left( \frac{ee' - d'r}{r^2 - 2ar + e^2} \right) dr.$$  \hspace{1cm} (65)

$a'$ denotes $\frac{\partial a}{\partial r}$, and $e'$ is $\frac{\partial e}{\partial r}$. The right hand side of (65) consists of two terms. The first term of (65) can be regarded as the ‘surface term’ which contributes to Hawking radiation emitted due to the curvature on the surface of the collapsing star, and the second term (65) can be considered as the ‘internal term’, which contributes to Hawking radiation emitted due to the geometry inside the collapsing star.

It is observed that the redshift from the internal of the collapsing star to the external observer is very large, since each layer is close to its apparent horizon. If some Hawking radiation is emitted from an internal shell, it is expected to experience huge redshift. This means that the energy flux from the internal shells is small compared to the outermost shell, especially when we set the initial condition of inner shells to be already near extremal. Therefore, the internal term in (65) can be neglected, by assuming the surface term to be much larger.

To be more precise, we adapt the quasi-static approximation. The quasi-static approximation means that, due to the huge redshift, the inner shells ‘freeze’ when compared to the outermost shell. $a(u,r)$ can be seen as solely dependent on $r$ for $r < r_s$. Therefore,

$$a(u,r) = \begin{cases} a(r) & \text{when } r < r_s(u) \\ a_0(u) & \text{when } r \geq r_s(u). \end{cases}$$ \hspace{1cm} (66)

In order to calculate $\xi$, we have to impose some boundary condition for $a(u,r)$ and $e(u,r)$ on the junction $r = r_s(u)$. Namely, we have to find $a'(u,r_s)$ and $e'(u,r_s)$.

To find $a'(u,r_s)$, the equation of Hawking radiation (60) and (61) should be consistent. Therefore, the continuity condition of $Da(u,r)$ on $r = r_s$ can be imposed, which can be used to find $a'(u,r_s)$

$$\dot{a}_0(u) = \lim_{r \rightarrow r_s} Da(u,r) = r_s \frac{\partial a}{\partial r} \bigg|_{r = r_s}.$$ \hspace{1cm} (67)

To find $e'(u,r_s)$, we shall assume that $e'(u,r_s) = 0$. Since the charge is fixed on each shell, this can be achieved by manipulating the initial distribution of charge on each shell. For example, it can be done by letting a region of the outermost shells contain no charge.

Equation (67) can give the form of $a(r)$ in the region of $r$ where the quasi-static approximation can apply. More precisely, the form of $a(r)$ can be solved by the equation

$$a'(r_s(u)) = \frac{\dot{a}_0}{r_s} = \frac{\sqrt{a_0^2 - e_0^2}}{r_h}.$$ \hspace{1cm} (68)
We have adopted the assumption that the shells are near their apparent horizon in the second equality of (68). Therefore, for the quasi-static approximation to hold, a specific initial condition of \( a(u_0, r_0) \), which is the solution of (68), have to be set. \( u_0 \) is some initial time such that each shell is sufficiently close to its apparent horizon for \( u \geq u_0 \).

Under this circumstance, the configuration of \( a(u, r) \) and \( e(u, r) \) corresponds to neutral outer layer in the region of \( r_0 \leq r \leq r_s(u) \). \( r_0 \) is the minimum of \( r_s(u) \) such that the quasi-static approximation holds. Since each shell is close to its horizon, \( e(u, r) = \sqrt{2a(u, r)r - r^2} \). It implies that \( e(u, r) \) is also independent of \( u \) for \( r < r_0 \). We have set the boundary condition to \( e'(u, r_s) = 0 \). Therefore, we conclude that \( e(u, r) = e_0 \) for \( r \geq r_0 \). Since we are assuming that the collapsing star has a near extremal inner shells, we have \( r_0 \approx e_0 \).

We precede to evaluate \( \xi(u, r) \). Since the second term of (65) and \( e'(u, r_s) \) vanish under our assumptions

\[
\xi(u, r) = -\frac{1}{F(u, r_s)} \frac{2d_0}{r_s} . \tag{69}
\]

Since the non-extremal condition (24) holds, we can use the assumptions \( r_s = r_h \) and \( \dot{r}_h = r_h \). The latter assumption holds if \( r_s \) changes rather smoothly as discussed in section 2.1. With the null condition \( F(u, r_s) = -2r_h \), \( F(u, r_s) \) can be rewritten as

\[
F(u, r_s) = -\frac{2\pi\dot{d}_0}{\sqrt{a_0^2 - e_0^2}} . \tag{70}
\]

By inserting (70) into \( \xi \), we get

\[
\xi = -\frac{1}{F(u, r_s)} \frac{2d_0}{r_s} = \frac{\sqrt{a_0^2 - e_0^2}}{r_h} . \tag{71}
\]

In order to find the evolution of \( d_0 \), we need to recall the previous result (61)

\[
\dot{d}_0 = \frac{1}{24\pi}(2\dot{\xi} - \xi^2) . \tag{72}
\]

Inserting the relation of \( \xi \) (71) into the above, the evaporation equation of \( d_0 \) can be derived:

\[
\dot{d}_0 = -\frac{1}{24\pi} \left( 1 + \frac{1}{12\pi r_h^2} \left( 2 - \frac{a_0}{\sqrt{a_0^2 - e_0^2}} \right) \right)^{-1} \frac{a_0^2 - e_0^2}{r_h^3} . \tag{73}
\]

For \( a_0 \) which is not too close to \( e_0 \), (73) can be simplified to

\[
\dot{d}_0 = -\frac{1}{24\pi} \frac{a_0^2 - e_0^2}{r_h^3} . \tag{74}
\]

It can be observed that, in the limit \( a_0 \gg e_0 \), the equation becomes \( \dot{a}_0 = -1/(24\pi a_0^2) \), which is consistent with the result derived by Kawai et al [5].

On the other hand, if we let \( a_0 \rightarrow e_0 \) in (73), \( \dot{a}_0 \) becomes positive. It implies an unnatural negative energy flux, but it should be noted that we have made the non-extremal assumption which may not hold in the late time limit when deriving the result. Therefore, if we prove that \( \dot{a}_0 \) is always positive under the range of \( a_0 \) where (24) holds, the problem can be resolved.

Actually, we can show that the evaporation equation (74) and the non-extremal condition (24) are consistent under the same condition \( \frac{1}{e_0} \approx x \) where \( a_0 = e_0 + x \).
To see this, we begin from the fact that (24) holds if and only if $\frac{a_0^2 - e_0^2}{e_0^2} \gg |\dot{r}|$. Under the condition in which (74) holds, $\dot{r}$ can be rewritten as

$$\dot{r} \approx \dot{r}_h = \frac{r_0 a_0}{\sqrt{a_0^2 - e_0^2}} = -\frac{1}{24\pi} \frac{\sqrt{a_0^2 - e_0^2}}{r_h^3}. \quad (75)$$

Letting $a_0 = e_0 + x$ and $x \ll e_0$, the condition turns out to be

$$|\dot{r}| \ll \frac{a_0^2 - e_0^2}{e_0^2} \Rightarrow \sqrt{\frac{x}{e_0^2}} \ll \frac{x}{e_0}, \text{ or } e_0^{-3} \ll x. \quad (76)$$

We should now check the consistency with (74)

$$\frac{a_0}{\sqrt{a_0^2 - e_0^2 r_{0 \text{hd}}}} \approx \sqrt{\frac{1}{e_0^4 x}} \ll 1. \quad (77)$$

Indeed, the approximation from (73) to (74) is consistent with the non-extremal assumption. We see that the valid range for (74) is $\frac{1}{e_0^3} \ll x$, which is sub-Planck scale for $e_0 > 1$.

4.2. Comments on the near-extremal behavior

In the previous section, we demonstrated that, for non-extremal RNV black holes, the evaporation process is governed by (74) under suitable assumptions: the non-extremal conditions (24) and the quasi-static approximation. We checked that (74) is valid when $1/e_0^3 \ll x$, where $a_0 = e_0 + x$. In the late time limit, both of the conditions may not hold. Also, we found that the quasi-static assumption leads to a particular configuration of $(u, r)$.

We consider a more general situation in this section. Both the quasi-static and (24) are not assumed. To avoid complexity in later calculations, we will adopt a general parameterization $\alpha$ on different layers. $u_0 = u$ is the retarded time for the outermost layer, and $a_{\alpha_{\text{max}}} = U$ is the retarded time for the innermost layer. Also, we will consider the near-extremal limit, in which each shell is very close to its extremal limit. Therefore, $r_\alpha \approx e_\alpha$ and $a_\alpha \approx e_\alpha$, and perturbation on $a_\alpha$ and $e_\alpha$ can be evoked.

In order to analyze the asymptotic behavior of the near-extremal black hole, we need to replace (24) with (23), and the second term of (65) cannot be neglected. Generally speaking, the equation is highly non-linear. This section provides an analysis of the behavior of $a_\alpha$, $r_\alpha$, or the redshift factor $\tilde{\psi}_\alpha$ under different assumptions, and we give an illustration of how the general Penrose diagram of the RNV black hole looks like.

Before invoking the equation for Hawking radiation (60), there are many things we can say about by solely considering the null condition. The null condition for an arbitrary layer $\alpha$ is

$$\dot{r}_\alpha = -\frac{1}{2} F_\alpha (u_\alpha, r_\alpha). \quad (78)$$

Since $r_\alpha \approx e_\alpha$ and $a_\alpha \approx e_\alpha$, we can make perturbation on $a_\alpha = e_\alpha + x_\alpha$ and $r_\alpha = e_\alpha + y_\alpha$. To obtain consistency, the perturbation of (78) needs to be taken to the second order, which is

$$y_\alpha = \left(\frac{x_\alpha}{e_\alpha}\right) - \frac{1}{2} \left(\frac{y_\alpha}{e_\alpha}\right)^2 = \left(\frac{x_\alpha}{e_\alpha}\right) \left(\frac{y_\alpha}{e_\alpha}\right) + O(x_\alpha y_\alpha^2). \quad (79)$$
Note that we will assume there exists some late time $u_\alpha^*$ such that the perturbation (79) will hold for all $u_\alpha > u_\alpha^*$.

Rearranging (79), we get

$$\frac{x_\alpha}{e_\alpha} = \frac{\gamma_\alpha + \frac{1}{2} \left( \frac{\gamma_\alpha}{e_\alpha} \right)^2}{1 - \gamma_\alpha/e_\alpha} \approx \gamma_\alpha + \frac{1}{2} \left( \frac{\gamma_\alpha}{e_\alpha} \right)^2.$$  \hspace{1cm} (80)

One important constraint is that we need to have $x_\alpha > 0$, or in other words, $\gamma_\alpha + \frac{1}{2} \left( \frac{\gamma_\alpha}{e_\alpha} \right)^2 > 0$. To further analyze this, we can rewrite this inequality to be

$$\frac{d}{du_\alpha} \left( -\frac{2e_\alpha^2}{u_\alpha} + u_\alpha \right) > 0 \Rightarrow -\frac{2e_\alpha^2}{u_\alpha} + u_\alpha = f(u_\alpha).$$  \hspace{1cm} (81)

$f(u_\alpha)$ is a monotonic increasing function, which is $f(u_\alpha) = \int_0^{u_\alpha} \frac{2e_\alpha^2}{y_\alpha} dy_\alpha$. Therefore, we find that

$$y_\alpha = \frac{2e_\alpha^2}{u_\alpha - f(u_\alpha)} > \frac{2e_\alpha^2}{u_\alpha - f(u_\alpha^*)}.$$  \hspace{1cm} (82)

If we match up the initial condition, we will get $f(u_\alpha^*) = -u_\alpha^* + 2e_\alpha^2/y_\alpha(u_\alpha^*)$. Another constraint is $\gamma_\alpha < 0$. Since the shell is outside the apparent horizon $r_{\alpha_0} := a_\alpha + \sqrt{a_\alpha^2 - e_\alpha^2}$, from the null condition, we can see that $\gamma_\alpha$ will always remain negative ($F_\alpha > 0$). From (82), this implies

$$\frac{2e_\alpha x_\alpha}{y_\alpha^2} = f(u_\alpha) < 1.$$  \hspace{1cm} (83)

Now, we shall consider the asymptotic behavior $x_\alpha/y_\alpha^2$ as $u_\alpha \to \infty$. Suppose that $x_\alpha/y_\alpha^2$ approaches some limit value denoted as $\eta_\alpha$, and we know from (83) that $\eta_\alpha \leq 1/2e_\alpha$.

Then, we consider three different cases of the value of $\eta_\alpha$:

1. $\eta_\alpha = 0$
2. $0 < \eta_\alpha < 1/2e_\alpha$
3. $\eta_\alpha = 1/2e_\alpha$

**4.2.1. Case (1).** In case (1), it can be observed that $\eta_\alpha = 0$ implies $\dot{f}(u_\alpha) \to 0$ as $u_\alpha \to \infty$. Therefore, for any $\epsilon > 0$ and large enough $u_\alpha$, the inequality $f(u_\alpha) < \epsilon u_\alpha$ can be achieved. Thus, we can expand (82) with respect to $f(u_\alpha)/u_\alpha$ and find that

$$y_\alpha = \frac{2e_\alpha^2}{u_\alpha} + o(u_\alpha^{-1}).$$  \hspace{1cm} (84)

Also, since $x_\alpha/y_\alpha^2 \to 0$, we can conclude that $x_\alpha = o(u_\alpha^{-2})$.

**4.2.2. Case (2).** In case (2), similarly, we observe that $\dot{f}(u_\alpha) \to 2\eta_\alpha e_\alpha < 1$ as $u_\alpha \to \infty$. Therefore, $f(u_\alpha) = 2\eta_\alpha e_\alpha u_\alpha + o(u_\alpha)$, and this implies

$$y_\alpha = \frac{A_\alpha}{u_\alpha} + o(u_\alpha^{-1}),$$  \hspace{1cm} (85)

where $A_\alpha$ is some constant which can be written explicitly as $A_\alpha = \frac{2e_\alpha^2}{1 - 2\eta_\alpha e_\alpha}$. On the other hand, since $x_\alpha = \eta_\alpha y_\alpha^2$ for $u_\alpha \to \infty$
\[
x_\alpha = \frac{B_\alpha}{u_\alpha^3} + o(u_\alpha^{-2}),
\]
where \( B_\alpha = \frac{4n_\alpha e_\alpha^3}{(1-2\rho_\alpha e_\alpha)^{3/2}} \).

Also, we can observe that \( A_\alpha \) and \( B_\alpha \) satisfy
\[
B_\alpha = \frac{A_\alpha^2}{2\epsilon_\alpha} - A_\alpha e_\alpha,
\]
(87)

4.2.3. Case (3). In case (3), we can only conclude that \( y_\alpha = \sqrt{2\epsilon_\alpha x_\alpha} \) at the late time limit.

By observing that the apparent horizon is just \( r_{\alpha} = a_\alpha + \sqrt{u_\alpha^2 - e_\alpha^2} \approx e_\alpha + \sqrt{2\epsilon_\alpha x_\alpha} \), this is the case in which the distance between each shell and its horizon is much smaller than the distance of each shell and its extremal limit \( e_\alpha \). Therefore, \( r_\alpha = n_h \) still holds under this situation. In this case, the evolution of \( y_\alpha \) is fully determined by the evolution of \( x_\alpha \), which itself is determined by the evaporation process.

It should be noted that, in case (1) and case (2), \( y_\alpha \) and \( x_\alpha \) will approach to zero as \( u_\alpha \to \infty \), but in case (3) it is not obvious whether \( x_\alpha \) will approach to zero as \( u_\alpha \to \infty \).

In order to obtain more information, we will focus on the equation of evaporation (60).

Written in the general parameterization \( \alpha \), it is
\[
\dot{\alpha} = \frac{1}{24\pi}(2\xi_\alpha - \xi_\alpha^2),
\]
(88)

and \( \xi_\alpha \) is defined as
\[
\xi_\alpha = \frac{d}{dt_\alpha} \log \left( \frac{dU}{dt_\alpha} \right).
\]
(89)

We shall assume that the energy momentum tensor holds under the weak energy condition. This demands the total energy flux of the black hole to be positive, which corresponds to \( \dot{\alpha} < 0 \). From the same technique used in deriving (82), this criteria can be reformulate as
\[
2\dot{\xi}_\alpha - \xi_\alpha^2 < 0 \Rightarrow \frac{d}{dt_\alpha} \left( \frac{2}{\xi_\alpha} + u_\alpha \right) > 0.
\]
(90)

Therefore, if we define \( g(u_\alpha) = -\int \frac{24\pi x_\alpha}{\xi_\alpha} du_\alpha \), which is monotonic increasing, we get
\[
\frac{2}{\xi_\alpha} + u_\alpha = g(u_\alpha), \Rightarrow \xi_\alpha = \frac{2}{u_\alpha - g(u_\alpha)} < \frac{2}{u_\alpha - g(u_\alpha)}.
\]
(91)

Matching up the initial condition, we get \( g(u_\alpha^*) = u_\alpha^* + \frac{2}{\xi_\alpha^*} \).

In general, we can assume that \( \xi_\alpha^*(u_\alpha^*) < 0 \), and there is no singularity in \( \xi_\alpha \). Therefore, an estimation of the redshift factor \( \psi_\alpha := \frac{du}{dt_\alpha} \) can be obtained:
\[
\tilde{\psi}_\alpha(u_\alpha) - \tilde{\psi}_\alpha(u_\alpha^*) < -2 \log \left( \frac{u_\alpha - g(u_\alpha)}{u_\alpha^* - g(u_\alpha^*)} \right).
\]
(92)

3 In general, it is natural to assume that \( \xi_\alpha < 0 \) in the late time limit. This can be seen as the following: from the definition, \( \xi_\alpha = \frac{\psi_\alpha}{\dot{\alpha}^*} \), \( \xi_\alpha < 0 \) implies \( \frac{\psi_\alpha}{\dot{\alpha}^*} < 0 \). This is reasonable because ones would expect the redshift between the innermost layer and \( \alpha \) to increase with time due to the approaching of each layer to its apparent horizon.

Also, since there are little Hawking radiation emitted in the near-extremal limit, there is no mass loss of the black hole to reduce its redshift factor. This implies that \( \frac{\psi_\alpha}{\dot{\alpha}^*} \) should decrease with time.
Therefore, \( \bar{\psi}_\alpha \) will diverge to \(-\infty\) as \( u_\alpha \to \infty \).

In case (1) and case (2), we have \( x_\alpha = O(u_\alpha^{-2}) \). Also, since that
\[
\frac{24\pi \dot{u}_\alpha}{c_\alpha} = \frac{24\pi \dot{x}_\alpha}{c_\alpha^2} > 6\pi c_\alpha (u_\alpha - g(u_\alpha^*))^2.
\]
we can get an estimation of the growth of \( g(u_\alpha) \):
\[
g(u_\alpha) - g(u_\alpha^*) < -6\pi \int_{u_\alpha^*}^{u_\alpha} \dot{x}_\alpha (u' - g(u_\alpha^*))^2 du'
\]
\[
= -6\pi x_\alpha (u' - g(u_\alpha^*))^2 u_\alpha^* + 12\pi \int_{u_\alpha^*}^{u_\alpha} x_\alpha (u' - g(u_\alpha^*)) du'.
\]
The first term of (95) is \( O(-1) \), and the second term is \( O(\log(u_\alpha^*)) \). Therefore, under the circumstances of case (1) and case (2), \( g(u_\alpha) = O(\log(u_\alpha)) \). Thus, \( \xi_\alpha = -2/u_\alpha + O(\log(u_\alpha)/u_\alpha^*) \).

Using (89), the general relationship between the retarded time of two arbitrary layers \( u_\alpha \) and \( u_{\alpha_2} \) can be obtained:
\[
\frac{d u_{\alpha_1}}{d u_{\alpha_2}} = \frac{d U}{d u_{\alpha_2}} = e^{\int_{\xi_{\alpha_2}}^{\xi_{\alpha_1}} \xi_{\alpha_1} d u_{\alpha_1} - \int_{\xi_{\alpha_2}}^{\xi_{\alpha_1}} d u_{\alpha_1}} = \frac{c_{\alpha_1} u_{\alpha_1}^2}{c_{\alpha_2} u_{\alpha_2}^2},
\]
in which \( c_{\alpha_1} \) and \( c_{\alpha_2} \) are integral constants. Integrating this result, the relation between \( u_{\alpha_1} \) and \( u_{\alpha_2} \) is
\[
\frac{1}{c_{\alpha_1} u_{\alpha_1}} + c_{\alpha_1}' = -\frac{1}{c_{\alpha_2} u_{\alpha_2}} + c_{\alpha_2}',
\]
in which \( c_{\alpha_1}' \) and \( c_{\alpha_2}' \) are also integral constants. If \( \alpha_2 > \alpha_1 \) (\( \alpha_2 \) is below \( \alpha_1 \)), by letting \( u_{\alpha_1} \to \infty \), \( u_{\alpha_2}(u_{\alpha_1}) \) approaches to its limiting value \( u_{\alpha_2}(u_{\alpha_1} = \infty) = \frac{1}{c_{\alpha_2}(c_{\alpha_2}' - c_{\alpha_1}')}. \) For consistency, it is required that \( u_{\alpha_2}(\infty) > u_{\alpha_2}', \) and \( c_{\alpha}' \) needs to be monotonic increasing with respect to \( \alpha \) since \( u_{\alpha_2}(u_{\alpha_1} = \infty) > 0 \).

As we can see, \( c_{\alpha}' \) determine the fate of black hole form a distance observer. If \( c_{\alpha}' = c_{\alpha}' \) for all \( \alpha \), then \( u_\alpha(u) \to \infty \) as \( u \to \infty \). This correspond to the case when every shell become extremal in the same time. This correspond to the Penrose diagram in figure 2, as we will discuss later.

On the other hand, let \( \alpha^* \) be some value such that \( c_{\alpha} = c_{\alpha}^* \) for \( \alpha \leq \alpha^* \), otherwise \( c_{\alpha} > c_{\alpha}^* \). \( \alpha^* \) become a critical shell such that very layer above \( \alpha^* \) will become extremal in \( u \to \infty \) while the shells under it would still remain non extremal in the \( u \to \infty \) limit. This correspond to the Penrose diagram in figure 1.

Noted that the analysis of this section is based on the assumption that \( r_\alpha \) and \( u_\alpha \) are already near-extremal for all layers initially. It is possible that in a general collapse of a charged black hole, some or all layers will not approach their extremal limit in the late time limit. In this case, the back hole will not radiate all of the non-electromagnetic energy in the late time limit. Whether the black hole will become extremal depends on its initial energy and charge distribution. For example, in the thin shell model, the distribution of mass and charge are delta functions, and in section 2 we know that the collapsing of a single charged shell will not become extremal in the late time limit.

We can still give some treatment to this most general case. Considering the outermost layer \( \alpha = 0 \), from the previous derived result, we know that under the assumption of weak energy condition \( \psi_0 = \log(\frac{d U}{d u}) \) will approach to \(-\infty\) in the late time limit. This implies that there
exists some layer $\alpha'$ such that the redshift factor $\psi_\alpha := \log(u_{\alpha}'/du)$ approaches to negative infinity at late time limit for all $\alpha < \alpha'$, and approaches to a finite value for $\alpha > \alpha'$.

It is important to note that $\psi_\alpha \to -\infty$ does not necessarily imply $u_{\alpha} \to \text{const.} < \infty$ as $u \to \infty$. For example, if $\psi_\alpha = -\log(u_{\alpha})$, $u_{\alpha} = \sqrt{2u + C_{\alpha}}$, which approaches infinity as $u \to \infty$.

Therefore, there are two possible cases. The first case is the existence of a critical shell $\alpha^* \geq \alpha'$, such that any layer above $\alpha^*$ will become extremal as the time of the observer approaches infinity; otherwise it will not (figure 1).

For the second case, such a critical layer does not exist. This implies that an observer with a distance to the black hole will find every layer of the shell becomes extremal as the time approaches to infinity (figure 2). The case when black hole does not become near extremal correspond to the critical shell being the outermost shell.

We can see this more clearly with the help of the Penrose diagrams (figures 1 and 2). In the figures, $\alpha_1$ and $\alpha_2$ are the trajectories of two infalling null shells (thus they are also constant $e$ curves), and $S_1$, $S_2$ and $S_3$ are respectively the constant $r$ curves with $r = e_{\alpha_1}$, $r = e_{\alpha_2}$, and $r = \text{const.} > e_0$. The dash curves $C_1$ and $C_2$ are respectively the constant $a$ curves for $a = e_{\alpha_1}$ and $a = e_{\alpha_2}$. In figure 2, we can see that $S_1$, $S_2$, $C_1$ and $C_2$ cross the null shells $\alpha_1$, $\alpha_2$ as they approach the horizon $H$ (at $P_1$, $P_2$), while in figure 1, $S_2$ and $C_2$ cross the null shell inside the horizon $H$, meaning that an outside observer cannot see this event.

### 4.3. The Hawking temperature

In this section, we study the Hawking temperature of the collapsing RNV black hole, subjected to the assumptions of section 4.1 in more detail. Since the evaporation equation (74) is derived under the condition $1/e_0 \ll x$, the Hawking temperature is determined for all $a_0$ subjected under this condition.

We aim at finding the character of the Hawking temperature. We find that, being opposed to the chargeless case, the Hawking temperature will have a maximum at some point, and it will approach to 0 as $a_0 \to e_0$ and $a_0 \to \infty$.

According to section 4.1, the energy flux density is

$$J = \frac{1}{4\pi r^2} \int d\Omega r uu = -\frac{\dot{a}_0}{4\pi r^2} = \frac{1}{96\pi^2} \frac{a^2_0 - e^2_0}{r^6_H}$$

and the Hawking temperature is $T_H$

$$T_H = \left(\frac{J}{\sigma}\right)^{1/4}$$

$\sigma = \frac{\pi^2}{60}$ is the Stefan–Boltzmann constant. In the chargeless limit, $e_0 \ll a_0$, $T_H \propto 1/a_0$, which coincides with the calculation of Kawai et al [5].

On the other hand, the Hawking temperature approaches to 0 as the black hole becomes extremal; therefore, it reaches a maximum at some point of evaporation. (See figure 3).

In order to find this temperature, one has to solve

$$\partial_a J = 0.$$  \hspace{1cm} (100)

The solution is

$$a_{\text{max}} = \frac{3e_0}{2\sqrt{2}}$$  \hspace{1cm} (101)
and,

\[
T_{\text{max}} = \left( \frac{5}{2} \right)^{1/4} \frac{1}{4\pi e_0}. \tag{102}
\]

Thus, for large \( e_0 \), the Hawking temperature will remain small for all periods of evaporation under the range \( x \gg 1/e_0^3 \). In section 1.1, we have stated that the approximation for chargeless radiation is reasonable because we assume that the Hawking temperature will always remain small during the entire evaporation process. We have showed that this is true if \( x \gg 1/e_0^3 \), making this assumption consistent under this condition.

Now we turn for the near-extremal case. Since there is little non-electromagnetic energy left, we can expect \( \dot{a}_0 \), and therefore, the Hawking temperature is very small.

4.4. The energy momentum tensor

We shall calculate the energy momentum tensor \( T_{\mu\nu} \) by inserting the RNV metric (52) into Einstein tensor.

The results are:

\[
G_{uu} = \frac{e^\psi}{r} (r \partial_r F - e^\psi F (F - 1 + r \partial_r F)) \tag{103}
\]

\[
G_{ur} = \frac{-e^\psi}{r} (F - 1 + r \partial_r F) \tag{104}
\]

\[
G_{rr} = \frac{2 \partial_r \psi}{r} \tag{105}
\]

\[
G_{\theta\theta} = r (\partial_\theta F + F \partial_\psi) + r^2 \left( \frac{3}{2} \partial_\theta F \partial_\psi + F (\partial_\psi)^2 + \frac{1}{2} \partial_\theta^2 F + F \partial_\psi^2 - e^{-\psi} \partial_\theta \partial_\phi \psi \right) \tag{106}
\]

\[
G_{\phi\phi} = \sin^2 \theta G_{\theta\theta}. \tag{107}
\]
All the other components of Einstein tensor are zero.

Incorporating the condition of null infalling matter and obtaining from (51)
\[
\psi' = 2 \frac{rd' - ee'}{r^2 F},
\]
and using the fact that the charge does not radiate out (i.e. $De = 0$), we can decompose the energy momentum potential into:
\[
T_{\mu\nu} = T_{\mu\nu}^{(\text{out})} + T_{\mu\nu}^{(\text{in})} + T_{\mu\nu}^{(T)} + T_{\mu\nu}^{(EM)}.
\]

To express the energy momentum tensor, we adopt a set of null atlases being parallel to the one defined in (18)
\[
\begin{align*}
I_a &= e^\psi \delta_a^0, \\
N_a &= \frac{1}{2} Fe^\psi \delta_a^0 + \delta_a^1, \\
M_a &= -\frac{r}{\sqrt{2}} \left( \delta_a^2 + i \sin \theta \delta_a^3 \right)
\end{align*}
\]
where $T_{\mu\nu}^{(\text{out})}$ represents the massless radiation:
\[
T_{\mu\nu}^{(\text{out})} = -\frac{Da}{4\pi Gr^2} I_\mu I_\nu,
\]
and $T_{\mu\nu}^{(\text{in})}$ is the energy momentum tensor of the infalling matter:
\[
T_{\mu\nu}^{(\text{in})} = T_{\text{in}} N_\mu N_\nu,
\]
where,
\[
T_{\text{in}} = \frac{d'r - ee'}{2G\pi r(r^2 - 2ar + e^2)}.
\]

$T_{\mu\nu}^{(\text{EM})}$ is the contribution of the electromagnetic field
\[
T_{\mu\nu}^{(\text{EM})} = \rho n_{(\mu} l_{\nu)} + p m_{(\mu} m_{\nu)}
\]
where
\[
\rho = p = \frac{e^2}{4\pi Gr^2}.
\]

Finally, $T_{\mu\nu}^{(T)}$ is the tangential component induced by the collapse and the radiation
\[
T_{\mu\nu}^{(T)} = T_{\text{tan}} 2m_{(\mu} m_{\nu)} = T_{\text{tan}} r^2 d\Omega.
\]
\[
T_{\text{tan}} = \frac{1}{8\pi r^2 G} \left( e^2 \left( 2 + ee' + r\psi' + a\psi' \right) - 2e^2 \frac{\psi'}{r} + 3ra\psi' + 3ee' \psi' + r^2 F\psi'^2 + ra'' + ee'' + r^2 F\psi'' - e^{-\psi} \psi' \right).
\]
5. Conclusion

In this paper, we have proposed a model of the evaporation of charged black holes, which is an analog of KMY model proposed by Kawai et al. We consider two scenarios: the thin-shell case and the generic spherical symmetric case.

In the thin-shell model, a charged sphere collapses at the speed of light. We demonstrated that an event horizon will form, and the shell will eventually fall into this horizon. Also, the black hole will not become extremal before it falls into the event horizon, provided that \( a_0 \) is not very close to \( e_0 \). That is, it does not radiate the maximal possible amount of energy before it falls into the event horizon.

We then considered a more realistic case: a continuous flow of charged matter that collapses spherically at the speed of light, which can be simulated by the continuous limit of many co-centered shells collapsing inward. The dynamics of the system is given at (52), (60), (61) and (63).

Next we consider the evaporation process of a specific initial condition: an near extremal center covered by neutral shell on the outside. Using the quasi-static approximation, the dynamics of the system is derived in (74). The condition for this equation to hold is \( a_0 - e_0 \gg e_0^{-3} \). It is consistent with KMY model in the chargeless limit. (\( e_0 \ll a_0 \)) The Hawking temperature under this situation is depicted in figure 3. Unlike KMY model, the temperature is always very small for large \( e_0 \). More specifically, \( T_{\text{max}} \sim e_0^{-1} \). This suggested that the field being chargeless is a reasonable assumption, since all charged fields have mass.

We also give the asymptotic analysis in the case when each shells are extremal initially, in which the assumption for deriving (74) does not hold anymore. By only considering the null condition of each infalling shell, some statements about the the asymptotic behavior of \( r_\alpha \) and \( a_\alpha \) can be made. We classified different asymptotic behaviors by their limit values of \( x_\alpha^2 / y_\alpha \) (case (1), case (2) and case (3)), and analyzed them separately. We found that, for case (1) and case (2), \( y_\alpha = A_\alpha / u_\alpha + o(u_\alpha^{-1}) \), and \( x_\alpha = O(u_\alpha^{-2}) \) (84)–(86), while for case (3), there is a possibility that \( x_\alpha \) will not approach to 0 in the late time limit.

Also more generally, even if we don’t assume the shells are near extremal initially, from assuming the weak energy condition, we can get an estimation of the redshift factor \( \tilde{\psi}_\alpha \) in (92), and \( \tilde{\psi}_\alpha \to -\infty \) in the late time limit. From this we derived the possibility of a critical shell. The shells above the critical shell would become extremal in the late time limit, while the shells below the critical shell will not. The general Penrose diagrams are also proposed (figures 2 and 1). In figure 1 the critical shell exists and in figure 2 the critical shell does not.

In conclusion, since in both cases (thin-shell and continuous), a permanent horizon forms, it implies that there will be no information loss problem, since the information can be kept inside the horizon.

However, there are still some questions that can be investigated in further researches. First, our analysis depended on a very specific regularization in calculating \( \langle T_{\mu\nu} \rangle \), the point-splitting regularization, which was used to derive (6) in [5]. It is natural to ask how many of the important features (becoming extremal in late time limit, etc) will remain if the regularization is changed to a different or more general regularization method. From the paper of Ho [8], we know that the information will not be lost if \( \dot{a}(u) < 0 \). Therefore, it is also worth investigating whether there will be a negative energy flux for some regularization method if back reaction is considered (just as in KMY model).

We can also generalize some of the assumptions in our analysis, such as the chargelessness of fields or the spherical symmetry. For example, we can consider a Kerr-like metric for a rotating black hole. The problem is that in our analysis, we simulated the infalling matter as a
series of co-centered spheres, and by making Bogolubov transformation between the retarded times between connecting shells, one can get the flux equation (6). There is no simple generalization if the system is not spherical symmetric, and thus new ways of calculating the flux equation are needed.

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