Dynamical Stability of Cantilevered Pipe Conveying Fluid with Inerter-Based Dynamic Vibration Absorber

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Abstract: Cantilevered pipe conveying fluid may become unstable and flutter instability would occur when the velocity of the fluid flow in the pipe exceeds a critical value. In the present study, the theoretical model of a cantilevered fluid-conveying pipe attached by an inerter-based dynamic vibration absorber (IDVA) is proposed and the stability of this dynamical system is explored. Based on linear governing equations of the pipe and the IDVA, the effects of damping coefficient, weight, inerter, location and spring stiffness of the IDVA on the critical flow velocities of the pipe system is examined. It is shown that the stability of the pipe may be significantly affected by the IDVA. In many cases, the stability of the cantilevered pipe can be enhanced by designing the parameter values of the IDVA. By solving nonlinear governing equations of the dynamical system, the nonlinear oscillations of the pipe with IDVA for sufficiently high flow velocity beyond the critical value are determined, showing that the oscillation amplitudes of the pipe can also be suppressed to some extent with a suitable design of the IDVA.

Keywords: Cantilevered pipe conveying fluid; inerter-based dynamic vibration absorber; dynamic vibration absorber; critical flow velocity; nonlinear oscillation

1 Introduction

The dynamical stability of cantilevered pipes conveying fluid has been widely studied by many researchers [1–6] in the past decades. The main reason why so many people are devoting on this problem is because the dynamical system of pipes conveying fluid can be found in many engineering fields, such as, heat exchanger tubes, concrete pump truck, locomotive braking system, oil pipeline, aerial refueling, ocean risers, and etc.

It is generally known that a cantilevered pipe may become unstable and flutter instability would occur when the velocity of the fluid flow in the pipe exceeds a critical value. Indeed, a lot of researchers have endeavored to enhance the stability of cantilevered pipes conveying...
fluid [7–17], by using either passive or active methods. For instance, Tani et al. [12,13] applied a torsional moment to a certain position of the cantilevered fluid-conveying pipe, and then experimentally and numerically studied the suppression effect of the torque on the oscillation responses of the pipe. Yau et al. [2] added a piezoelectric layer to a certain position of the cantilevered pipe, and explored the effect of the mounting position and the length of the piezoelectric layer on the vibration responses of the pipe. Lin et al. [9,14,15] further demonstrated that the cantilevered pipe could obtain remarkable suppression effect when the voltage imposed on the piezoelectric layer was within a suitable range. Khajehpour et al. [16] utilized piezoelectric layers to control the vibrations of a rotating cantilever conveying fluid. Hussein et al. [17] analyzed the effect of hydraulic damper position, base width of hydraulic damper, damping and flow pressure on the dynamic responses of slender pipes by using state space technique.

Since active control methods for cantilevered pipes conveying fluid are always restricted by the reliability and service life of the controller, passive control methods [18–22] have attracted more attentions. For instance, Wang et al. [23] derived the governing equations of a pipe conveying fluid on elastic foundation and showed that an elastic foundation can increase the critical flow velocity for static and dynamical instabilities of the pipe. DoarÊ et al. [24] compared the stability characteristics of finite- and infinite-length pipes conveying fluid on elastic foundations. Hiramoto et al. [20] improved the stability of cantilevered pipes conveying fluid by optimizing the outer diameter distribution of the pipe with a closed-loop device. Pisarski et al. [25] applied electromagnetic devices of a motional type to cantilevered pipes conveying fluid to improve the dynamical stability of the pipe system. It was demonstrated that the electromagnetic devices of the motional type can remarkably increase the critical flow velocity by fifty percent comparing to the same pipe but without the electromagnetic actuator.

At the year of 2013, Yang et al. [26] initiated to numerically investigate the nonlinear responses of simply supported pipes conveying fluid with an attached nonlinear energy sink (NES). A cubic spring linked with a mass was used to model the effect of NES on the pipe system. It was indicated that the vibrational energy of the simply supported pipe conveying fluid could be robustly absorbed by the NES. Based on the work of Yang et al. [26], Mamaghani et al. [27] used an attached NES to suppress the oscillation responses of clamped–clamped pipe conveying fluid subjected to an external harmonic force. It was demonstrated that the pipe could achieve excellent suppression effect by attaching the NES at the middle point of the pipe. Song et al. [28] explored the vibration control performance of a Pounding Tuned Mass Damper (PTMD) for pipe structures by installing a PTMD on an M-shaped pipeline, using both experimental and numerical methods. Rechenberger et al. [29] used Microsoft Excel spreadsheet calculations to establish a mathematical model of Tuned Mass Damper (TMD). Their studies were of a practical guidance on the design of TMD for suppressing the oscillations of pipeline structures. Zhou et al. [30] installed an NES attachment somewhere along the length of a cantilevered pipe conveying fluid to enhance the stability of the pipe. The effects of mass ratio, spring stiffness, damping and location of the NES on the stability and nonlinear responses of the pipe were explored. Very recently, Liu et al. [31] analyzed the dynamical stability of a cantilevered pipe with an additional linear dynamic vibration absorber (DVA) attachment. It was shown that the damping coefficient, spring stiffness, location and weight of the DVA can remarkably affect the dynamical behaviors of the pipe.

In the current work, we introduce an inerter-based dynamic vibration absorber (IDVA) [32–35] to adjust the dynamical stability of cantilevered pipes conveying fluid. Based on the proposed governing equations, the effect of several key parameters of the additional IDVA
on the dynamics of the pipe are investigated. It will be shown that the IDVA can enhance the stability and suppress the oscillations of the pipe in many cases.

2 Governing Equations

Fig. 1 shows the schematic diagram of a cantilevered pipe conveying fluid with an additional IDVA. The spring-mass attachment is attached at \( x = x_b \leq L \), where \( L \) is the pipe length. The pipe is horizontal and its motion is limited in a horizontal plane by embedded a steel strip in the pipe.

In Fig. 1, the lateral displacement of the pipe along the \( y \) axis is denoted by \( W(s,t) \), with \( s \) being the curvilinear coordinate along the pipe length and \( t \) being the time. Before giving the nonlinear governing equations of the pipe system, several basic assumptions were made [36,37]: (1) The internal axial fluid is incompressible; (2) The centreline of the pipe is inextensible; (3) The Euler–Bernoulli beam theory is acceptable for the pipe; (4) The pipe’s axial strain is sufficiently small, although its lateral deflection may be relatively large. Following the derivation of Semler [38] and Liu et al. [31] and considering the effect of IDVA, the equation of motion of the pipe may be written as

\[
(m + M) \dddot{W} + 2MU \dddot{W} + MU^2 (1 + W')W'' + \psi W''' = 0
\]

in which the overdots and primes denote the derivative with respect to \( t \) and \( s \), respectively; \( \psi \) is the Kelvin–Voigt damping coefficient of the pipe, \( EI \) is the flexural rigidity of the pipe; \( m \) is...
the mass of the empty pipe per unit length, \( U \) is the steady flow velocity, \( M \) is the mass of the internal fluid per unit length; \( W_b \) is the lateral deflection of the pipe at the location of the IDVA attachment; \( C \) is the damping coefficient of the damper, \( K \) is the stiffness of the spring, \( I_0 \) denotes the inerter of the IDVA, \( V \) is the displacement of the additional mass; \( \delta (s - s_b) \) is the Dirac delta function with \( s_b \) being the location of the IDVA.

The governing equation of the IDVA takes the form

\[
m_1 \ddot{V} + K (V - W_b) + C (\dot{V} - \dot{W}_b) + I_0 (\ddot{V} - \ddot{W}_b) = 0
\]

where \( m_1 \) is the mass of the attached rigid body.

Introducing the following dimensionless quantities

\[
\xi = s / L, \quad w = W / L, \quad v = V / L, \quad \tau = \left( EI / (m + M) \right)^{1/2} t / L^3, \quad u = \left( M / EI \right)^{1/2} UL, \quad \beta = M / (M + m),
\]

\[
\phi = \left( EI / (m + M) \right)^{1/2} \psi / L^2, \quad \alpha = m_1 / (M + m) L, \quad k = KL^3 / EI, \quad \gamma = (m + M) EI / L^3 g, \quad c = CL / [(m + M) EI]^{1/2},
\]

\[
\theta = I_0 / (M + m) L
\]

Eqs. (1) and (2) may be written in dimensionless forms as

\[
w'''' + \phi w''' + \dot{w} + 2u \sqrt{\beta} \dot{w}' + u^2 w'' + N (w) - [k (v - w_b) + c (\dot{v} - \dot{w}_b) + \theta (\ddot{v} - \ddot{w}_b)] \delta (\xi - \xi_b) = 0
\]

and

\[
\alpha \ddot{v} + k (v - w_b) + c (\dot{v} - \dot{w}_b) + \theta (\ddot{v} - \ddot{w}_b) = 0
\]

where the prime and overdot on each variable now denote the derivative with respect to \( \xi \) and \( \tau \), respectively. The nonlinear term \( N(w) \) in Eq. (3) is given by

\[
N (w) = 2u \sqrt{\beta} \dot{w}' w'^2 + w'' w'^2 + 3w' w'' w' + w'' + w' \int_0^\xi \left[ \dot{w}'^2 - 2u \sqrt{\beta} \dot{w}' - u^2 w' w'' + w'' w''' \right] d\xi
\]

\[
- w'' \int_0^\xi \left[ \dot{w}'^2 - 2u \sqrt{\beta} \dot{w}' - u^2 w' w'' + w'' w''' \right] d\xi d\xi
\]

\[
- w'' \int_\xi^1 \left( 2u \sqrt{\beta} \dot{w}' w' + u^2 w' w'' + w''' \right) d\xi
\]

3 Galerkin Discretization

The governing equations for the pipe and IDVA are in partial differential forms, which can be discretized by using several effective methods including Galerkin approach [39–41] and differential quadrature method [42–44]. In the following calculations, the Galerkin approach is used to discretize the partial differential equations. Based on this method, the displacements of the pipe can be given by

\[
w (\xi, \tau) = \sum_{r=1}^{N} \varphi_r (\xi) q_r (\tau)
\]
where $\varphi_r(\xi)$ is the base eigenfunctions of a plain cantilevered beam, and $q_r(\tau)$ is the corresponding generalized coordinates; $N$ is the number of base functions used in the discretization. Substituting expression (6) into Eqs. (3) and (4), multiplying by $\varphi_i(\xi)$ and integrating from 0 to 1, the following ordinary differential equations can be obtained

$$\begin{bmatrix} M \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \ddot{v} \end{bmatrix} + \begin{bmatrix} C \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{v} \end{bmatrix} + \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} q \\ v \end{bmatrix} + \{f_{nonl}\} = \{0\}$$

where the overdots now denote the total derivative with respect to dimensionless time $\tau$. In Eq. (7), $[M]$, $[C]$ and $[K]$ represent the mass, damping and stiffness matrices for the linear parts and $f_{nonl}$ denotes the nonlinear term associated with various nonlinearities of the pipe system. In the following calculations, a four-mode Galerkin approximation will be utilized ($N = 4$) because the instability of the pipe system is usually associated with the lowest several modes.

By neglecting the nonlinear terms in Eq. (7), the eigenvalues of the pipe with IDVA can be obtained. According to the obtained eigenvalues for each mode, the stability of the pipe with the IDVA can be evaluated. When the dimensionless inerter in Eq. (7) is set as $\theta = 0$ and the nonlinear term $f_{nonl}$ is absent, the eigenvalues of the pipe with DVA may be obtained, referring to [31]. The nonlinear oscillations of the pipe with IDVA can be predicted by numerically solving the nonlinear governing equations via a fourth-order Runge–Kutta iteration algorithm.

4 Results

In this section, the effect of IDVA on the dynamical stability and nonlinear responses of the pipe system is explored. For that purpose, the evolution of eigenvalues for the pipe and the attached mass as a function of the flow velocity will be shown first. Based on the stability analysis of the linear system, the nonlinear oscillations of the pipe and the attached mass will be further studied. Results for the dynamical behaviors of the cantilevered pipe and the attached IDVA for various system parameters will be presented mainly in the form of Argand diagrams, bifurcation diagrams and displacement-time diagrams.

4.1 Model Validation and Simple Comparison

The Argand diagram of a cantilevered fluid-conveying pipe without IDVA for $\phi = 0.001$ and $\beta = 0.213$ is reproduced first to check the correctness of our approximately analytical solutions. The evolution of lowest four eigenvalues of the pipe without IDVA with increasing dimensionless flow velocity, $u$, is illustrated in Fig. 2. In this figure, it should be noted that Re($\omega$) is the dimensionless oscillation frequency, while Im($\omega$) is related to the dimensionless damping of the whole system. It is obvious that the flutter instability of the pipe occurs in the second mode, at $u_{cr} \approx 5.8$. It is also seen that the results plotted in Fig. 2 agree well with those obtained by Gregory et al. [45] and Paidoussis et al. [46], demonstrating that the approximately analytical solutions obtained in this work are reliable.

In order to explore the effect of IDVA on the basic dynamics of the fluid-conveying cantilever, typical results of Argand diagrams for a cantilevered pipe conveying fluid with DVA and IDVA are shown in Figs. 3 and 4. It is seen from Fig. 3 that flutter instability of the pipe with DVA occurs at $u_{cr} \approx 6.4$. This dimensionless critical flow velocity is much larger than that shown in Fig. 2, indicating that the DVA can improve the stability of the pipe system. It is also noted from Fig. 3 that the present result for the pipe with DVA agrees well with that reported in [31]. Furthermore, as shown in Fig. 4, once the inerter is added to the spring-mass attachment, the dimensionless critical flow velocity of the pipe system would increase further to $u_{cr} \approx 6.9$. 
In order to make the calculation results easier to understand, the dimensional values of critical flow velocity for the pipe with IDVA or without IDVA are briefly discussed. Taking a silicone tube [47] as an example, several key physical and geometrical parameters are chosen as: the flexural rigidity of the pipe $EI = 0.0217$ Nm², the mass of the empty pipe per unit length $m = 0.19$ kg/m, the density of the internal fluid per unit length $\rho = 1000$ kg/m³, the pipe length $L = 0.35$ m, the outer diameter $D = 0.0155$ m, and the inner diameter of the pipe $d = 0.00794$ m. According to the definition of the dimensionless flow velocity $u = \left[M/(EI)\right]^{1/2}UL$, the dimensional critical flow velocity of the pipe with IDVA is found to be about $U_{cr} = 7.78$ m/s.
Figure 4: Argand diagram for a cantilevered pipe conveying fluid with IDVA for $k = 20$, $c = 0.5$, $\phi = 0.001$, $\alpha = 0.1$, $\theta = 0.02$, $\beta = 0.213$, $\xi_b = 0.5$. It is seen that the critical flow velocity for the whole system is about $u_{cr} = 6.9$

4.2 Effect of IDVA on the Critical Flow Velocity

In this subsection, the stability of the cantilevered fluid-conveying pipe for various system parameters of the IDVA will be investigated. Figs. 5–12 show the critical flow velocities of the pipe system for various physical and geometrical parameters of the IDVA. In these figures, unless otherwise stated, the system parameters were chosen as: $\phi = 0.001$, $\alpha = 0.1$, $\beta = 0.213$, $\theta = 0.02$, $k = 20$, $c = 0.5$ and $\xi_b = 0.5$.

Figure 5: Dimensionless critical flow velocities $u_{cr}$ of the system as a function of dimensionless inerter and mass ratio of the IDVA for $\phi = 0.001$, $k = 28$, $\beta = 0.213$, $\xi_b = 0.5$: (a) $c = 0.1$, (b) $c = 0.3$ and (c) $c = 0.5$

The dimensionless critical flow velocities $u_{cr}$ of the pipe system as a function of dimensionless inerter and mass ratio of the IDVA for $\phi = 0.001$, $k = 28$, $\beta = 0.213$, $\xi_b = 0.5$ are indicated in Fig. 5. The results shown in this figure demonstrate that the critical flow velocities of the
dynamical system increase with the increase of the damping coefficient. As shown in Figs. 5a–5c, with the increase of dimensionless mass ratio, a smaller value of dimensionless inerter of the IDVA is required to achieve higher critical flow velocity.

Fig. 6 shows that the critical flow velocities of the pipe system as a function of dimensionless inerter and mass ratio for three different values of IDVA location ($\xi_b$). It is found that the critical flow velocities of the system in the case of $\xi_b = 0.75$ change slightly only. The peak value of the critical flow velocity of the dynamical system occurs at $\xi_b \approx 0.5$, as can be seen in Figs. 6a–6c.

![Figure 6](image)

**Figure 6:** Dimensionless critical flow velocities $u_{cr}$ of the system as a function of dimensionless inerter and mass ratio of the IDVA for $\phi = 0.001$, $k = 28$, $c = 0.5$, $\beta = 0.213$: (a) $\xi_b = 0.25$, (b) $\xi_b = 0.5$ and (c) $\xi_b = 0.75$

![Figure 7](image)

**Figure 7:** Dimensionless critical flow velocities $u_{cr}$ of the system as a function of dimensionless stiffness and mass ratio of the IDVA for $\phi = 0.001$, $\theta = 0.02$, $\beta = 0.213$, $\xi_b = 0.5$: (a) $c = 0.1$, (b) $c = 0.3$ and (c) $c = 0.5$

Fig. 7 shows that the critical flow velocities of the pipe system as a function of dimensionless stiffness and mass ratio of the IDVA, for three different values of damping coefficient ($c$). It is obvious that the peak value of the critical flow velocity increases with the increase of damping
coefficient. As shown in Figs. 7a–7c, with the increase of dimensionless damping coefficient of the IDVA, the cantilivered pipe conveying fluid with IDVA can achieve higher critical flow velocity.

The critical flow velocities of the pipe system as a function of dimensionless stiffness and mass ratio of the IDVA are shown in Fig. 8, for three given values of inerter ($\theta$). By inspecting Figs. 8a–8c, a remarkable feature can be found: with the increase of the inerter of IDVA, the critical flow velocities of the pipe with IDVA would reduce. That is to say, a larger value of inerter is detrimental to the system’s stability.

**Figure 8:** Dimensionless critical flow velocities $u_{cr}$ of the system as a function of dimensionless stiffness and mass ratio of the IDVA for $\phi = 0.001$, $c = 0.5$, $\beta = 0.213$, $\xi_b = 0.5$: (a) $\theta = 0.01$, (b) $\theta = 0.03$ and (c) $\theta = 0.05$

**Figure 9:** Dimensionless critical flow velocities $u_{cr}$ of the system as a function of dimensionless stiffness and inerter of the IDVA for $\phi = 0.001$, $c = 0.5$, $\alpha = 0.16$, $\beta = 0.213$: (a) $\xi_b = 0.25$, (b) $\xi_b = 0.5$ and (c) $\xi_b = 0.75$

Figs. 9a–9c plot the results of critical flow velocities for two independent parameters, stiffness and inerter, with three different values of IDVA location ($\xi_b$). It is found that when the IDVA location is closer to the free end of the pipe, the stiffness of the IDVA needs to be decreased to
obtain higher critical flow velocity. Among the three cases shown in Figs. 9a–9c, it is noted that the maximum critical flow velocity appears at $\xi_b = 0.5$.

Figs. 10a–10c show the critical flow velocities of the pipe with the IDVA being attached at $\xi_b = 0.5$, for three different values of mass ratio ($\alpha$). It is observed that with the increase of mass ratio, the stiffness of the IDVA needs to be increased to achieve higher critical flow velocity. Upon comparing the three diagrams of Fig. 10, again, it is found that higher critical flow velocity can be realized in the case of $\alpha = 0.1$.

![Figure 10](image1.png)

**Figure 10:** Dimensionless critical flow velocities $u_{cr}$ of the system as a function of dimensionless stiffness and inerter of the IDVA for $\phi = 0.001$, $c = 0.5$, $\beta = 0.213$, $\xi_b = 0.5$: (a) $\alpha = 0.05$, (b) $\alpha = 0.1$ and (c) $\alpha = 0.15$

![Figure 11](image2.png)

**Figure 11:** Dimensionless critical flow velocities $u_{cr}$ of the system as a function of dimensionless stiffness and damping of the IDVA for $\phi = 0.001$, $\alpha = 0.16$, $\beta = 0.213$, $\xi_b = 0.5$: (a) $\theta = 0.01$, (b) $\theta = 0.03$ and (c) $\theta = 0.05$

In Fig. 11, the critical flow velocities as a function of dimensionless stiffness and damping coefficient of the IDVA for three different values of inerter are shown. Once again, it is found
that the stability of the pipe can be better enhanced by using the IDVA with smaller values of inerter.

The critical flow velocities of the pipe system as a function of dimensionless stiffness and damping of the IDVA for $\phi = 0.001$, $\theta = 0.02$, $\beta = 0.213$, $\xi_b = 0.5$ and three different values of mass ratio of the IDVA are plotted in Fig. 12. Among the three cases shown in Fig. 12, it is noted that with the increase of mass ratio, the spring stiffness of the IDVA needs to be increased to achieve higher critical flow velocity, and the peak value of the critical flow velocity of the system appears at $\alpha = 0.1$.

![Figure 12: Dimensionless critical flow velocities $u_{cr}$ of the system as a function of dimensionless stiffness and damping of the IDVA for $\phi = 0.001$, $\theta = 0.02$, $\beta = 0.213$, $\xi_b = 0.5$: (a) $\alpha = 0.05$, (b) $\alpha = 0.1$ and (c) $\alpha = 0.15$](image)

**4.3 Effect of IDVA on Nonlinear Oscillations of the Pipe**

In this subsection, the nonlinear oscillations of the cantilevered pipe conveying fluid with IDVA will be studied when the flow velocity is successively increased. Some fascinating dynamical behaviors will be shown by analyzing this modified system. In order to illustrate the effect of IDVA on the pipe, nonlinear responses of the cantilevered pipe conveying fluid with and without IDVA are examined.

Before embarking some numerical calculations, it is recalled that the argand diagrams for the cantilevered pipe conveying fluid attached with DVA and IDVA show some difference (see Figs. 3 and 4). Therefore, it is expected that the nonlinear responses of the pipe with DVA and with IDVA are also different. To illustrate this, two bifurcation diagrams for the pipe with DVA and with IDVA are plotted in Fig. 13. It is immediately seen that the flutter instability of the pipe with DVA occurs at a higher flow velocity if compared with that of the pipe without IDVA. Furthermore, the pipe with IDVA shows a much higher critical flow velocity. These critical flow velocities for flutter instability based on nonlinear theory are consistent with the linear results shown in Figs. 3 and 4. It is noted that the oscillation amplitudes of the pipe with DVA or with IDVA are generally smaller than that of the plain pipe without any attachments, for a wide range of flow velocity. Interestingly, the pipe with a DVA loses instability at about $u_{cr} = 6.4$, then regains stability at about $u = 7.2$, and finally becomes unstable with further increasing flow velocity. When the flow velocities are higher than $u = 10$, the oscillation amplitudes of the pipe with and without...
mass attachment has no obvious difference. The dynamic responses of the DVA and IDVA show similar behaviors as the pipe, which can be observed in Fig. 13b.

![Figure 13](image1.png)

**Figure 13:** Bifurcation diagrams of the oscillation amplitudes with internal flow velocity $u$ being the variable parameter for: $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 20$, $\alpha = 0.1$, $\theta = 0.02$, $\beta = 0.213$. (a) Oscillation amplitudes of the pipe at $\xi = 0.5$ and (b) oscillation amplitudes of the IDVA and DVA

When the value of the mass ratio of the IDVA is increased to $\alpha = 0.15$, it is indicated that the dynamical behaviors of the pipe shown in Fig. 14 exhibit some difference from those given in Fig. 13 for $\alpha = 0.1$. It is noted that the pipe with either DVA or IDVA becomes unstable at a critical flow velocity higher than the flow velocity for flutter instability of the pipe without IDVA. One can see that the pipe with DVA for $\alpha = 0.15$ no longer switches between stable and unstable states when the flow velocity is successively increased. Similar phenomenon can be observed in Fig. 14b for the dynamic responses of the IDVA.

![Figure 14](image2.png)

**Figure 14:** Bifurcation diagrams of the oscillation amplitudes with internal flow velocity $u$ being the variable parameter for: $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 20$, $\alpha = 0.15$, $\theta = 0.02$, $\beta = 0.213$. (a) Oscillation amplitudes of the pipe at $\xi = 0.5$ and (b) oscillation amplitudes of the IDVA and DVA
In the case of $\alpha = 0.1$ and $\theta = 0.04$, the bifurcation diagrams for the pipe system with flow velocity as the variable parameter are plotted in Fig. 15. It is seen from this figure that the oscillation amplitudes of the pipe with IDVA at the location of the attachment are larger than the counterpart of the pipe with DVA for most flow velocities, while the oscillation amplitudes of the IDVA are generally smaller than that of the DVA. For both cases, the attachment mass can absorb some energy of the whole system, resulting in the decrease of the oscillation amplitudes of the cantilevered pipe.

![Bifurcation diagrams of the oscillation amplitudes with internal flow velocity $u$ being the variable parameter for: $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 20$, $\alpha = 0.1$, $\theta = 0.04$, $\beta = 0.213$: (a) displacements of the pipe at $\xi = 0.5$ and (b) displacements of the IDVA and DVA.](image1)

**Figure 15:** Bifurcation diagrams of the oscillation amplitudes with internal flow velocity $u$ being the variable parameter for: $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 20$, $\alpha = 0.1$, $\theta = 0.04$, $\beta = 0.213$.

Bifurcation diagrams of the oscillation amplitudes as a function of the flow velocity for $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 20$, $\alpha = 0.15$, $\theta = 0.04$, $\beta = 0.213$ are plotted in Fig. 16. In this case, the pipe with DVA has smaller oscillation amplitudes than that of the pipe with IDVA. This implies that the inerter has a negative effect on the vibration suppression of the pipe in such a case.

![Bifurcation diagrams of the oscillation amplitudes with internal flow velocity $u$ being the variable parameter for: $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 20$, $\alpha = 0.15$, $\theta = 0.04$, $\beta = 0.213$: (a) Oscillation amplitudes of the pipe at $\xi = 0.5$ and (b) oscillation amplitudes of the IDVA and DVA.](image2)

**Figure 16:** Bifurcation diagrams of the oscillation amplitudes with internal flow velocity $u$ being the variable parameter for: $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 20$, $\alpha = 0.15$, $\theta = 0.04$, $\beta = 0.213$. (a) Oscillation amplitudes of the pipe at $\xi = 0.5$ and (b) oscillation amplitudes of the IDVA and DVA.
The bifurcation diagrams for the pipe system with flow velocity as the variable parameter for $k = 28$, $\alpha = 0.1$ and two different values of inerter are shown in Figs. 17 and 18. It is seen that the critical flow velocity of the pipe with IDVA for $\theta = 0.04$ is slightly larger than that of the pipe with IDVA for $\theta = 0.02$, indicating that the inerter has a positive effect on the stability of the system in this case. It is also observed that, with increasing flow velocity, the oscillation amplitudes of the pipe with IDVA (or DVA) at $\xi = 0.5$ would increase gradually to relatively large values and thereafter decrease to relatively small values. When the flow velocity is sufficiently high (e.g., $u = 10.5$), the oscillation amplitude of the pipe at $\xi = 0.5$ would tend to a constant value.

### Figure 17: Bifurcation diagrams of the oscillation amplitudes with internal flow velocity $u$ being the variable parameter for: $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 28$, $\alpha = 0.1$, $\theta = 0.02$, $\beta = 0.213$. (a) Oscillation amplitudes of the pipe at $\xi = 0.5$ and (b) oscillation amplitudes of the IDVA and DVA

### Figure 18: Bifurcation diagrams of the oscillation amplitudes with internal flow velocity $u$ being the variable parameter for: $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 28$, $\alpha = 0.1$, $\theta = 0.04$, $\beta = 0.213$. (a) Oscillation amplitudes of the pipe at $\xi = 0.5$ and (b) oscillation amplitudes of the IDVA and DVA
In order to further compare the dynamic responses of the cantilevered pipe conveying fluid with IDVA and without IDVA, some displacement-time curves (time traces) for several typical flow velocities are demonstrated in Figs. 19–22. The displacement-time curves shown in Fig. 19 are for $u = 6.2$. It is obvious that the pipe without IDVA undergoes a periodic oscillation while the displacements of the pipe with DVA or IDVA is towards to zero. In this case, therefore, the pipe with IDVA or DVA becomes more stable than the pipe without IDVA. It is clearly seen from Fig. 20 that the pipe undergoes a periodic motion, either with DVA or without IDVA, while the pipe with IDVA keeps still for $u = 6.8$. Moreover, the oscillation amplitudes of the pipe without IDVA are much larger than the counterpart of the same pipe with DVA. In the case of $u = 7.3$, the displacement-time curves are shown in Fig. 21. It is fascinating that the pipe without and with IDVA occurs flutter instability while the pipe with DVA keeps stable. That is to say, the
pipe with DVA shows a better stability performance than the same pipe but with IDVA in case of $u = 7.3$. When the flow velocity is up to $u = 8$, the result shown in Fig. 22 indicates that the pipe undergoes a periodic motion, even if the pipe is installed with IDVA or DVA. It is also noted that the oscillation amplitude of the pipe without IDVA is larger than that of the pipe with IDVA. Unfortunately, the oscillation amplitude of the pipe with IDVA is larger than that of the pipe with DVA.

**Figure 21:** Tip-end displacements of the cantilevered pipe with IDVA, with DVA and without IDVA, for $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 20$, $\alpha = 0.1$, $\theta = 0.02$, $\beta = 0.213$ and $u = 7.3$

**Figure 22:** Tip-end displacements of the cantilevered pipe with IDVA, with DVA and without IDVA, for $\phi = 0.001$, $\xi_b = 0.5$, $c = 0.5$, $k = 20$, $\alpha = 0.1$, $\theta = 0.02$, $\beta = 0.213$ and $u = 8$
5 Conclusions

The dynamical stability and nonlinear responses of a cantilevered pipe conveying fluid with an IDVA added somewhere along the pipe length are explored in the present study. For a plain pipe without IDVA, the dynamical system loses stability via flutter when the flow velocity exceeds a certain critical value. For the same pipe but with an IDVA, the flutter instability would occur at a higher critical flow velocity. By constructing Argand diagrams for eigenvalues of the dynamical system, it is found that the damping coefficient, stiffness, location, weight and inerter of the additional IDVA do have effect on the stability of the pipe. Under certain conditions, the critical flow velocity of the pipe can be remarkably increased by adding the IDVA, and hence the stability of the pipe can be enhanced. The underlying reason of the enhanced stability of the fluid-conveying pipe is associated with the transfer of energy from the pipe to the IDVA. To evaluate the effect of IDVA on the nonlinear behaviors of the pipe, the oscillations of the pipe and the IDVA are also calculated based on nonlinear theories. It is shown that the oscillation amplitudes of the pipe with IDVA are sometimes smaller than that of the pipe with DVA, and the oscillation amplitudes of the IDVA are always larger than that of the DVA. Therefore, the results obtained in this paper may be expected to be useful for the design of energy absorbers (or energy transfer devices) of fluid-conveying pipes by adding IDVAs somewhere along the pipe length. However, this does not mean that the IDVA is always better than a DVA for enhancing the stability and suppressing the oscillations of cantilevered pipes conveying fluid. Within some ranges of flow velocity, indeed, the DVA has a better performance, as shown in Figs. 13–18.

Furthermore, for a pipe with both ends positively supported, buckling is the preferred form of instability since it is a conservative system in the absence of dissipation. When an inerter-based DVA is added to the supported pipe, the linear spring of the IDVA could change the effective bending stiffness of the whole pipe system. For this reason, it can be foreseen that the IDVA can affect the stability of pipes with both ends supported.

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