STABILITY OF SPACELIKE HYPERSURFACES IN FOLIATED SPACETIMES

A. BARROS, A. BRASIL AND A. CAMINHA

Abstract. Given a generalized $\mathbb{M}^{n+1} = I \times \phi F^n$ Robertson-Walker spacetime we will classify strongly stable spacelike hypersurfaces with constant mean curvature whose warping function verifies a certain convexity condition. More precisely, we will show that given $x : M^n \to \mathbb{M}^{n+1}$ a closed spacelike hypersurfaces of $\mathbb{M}^{n+1}$ with constant mean curvature $H$ and the warping function $\phi$ satisfying $\phi'' \geq \max \{H\phi', 0\}$, then $M^n$ is either minimal or a spacelike slice $M_{t_0} = \{t_0\} \times F$, for some $t_0 \in I$.

1. Introduction

Spacelike hypersurfaces with constant mean curvature in Lorentz manifolds have been object of great interest in recent years, both from physical and mathematical points of view. In [1], the authors studied the uniqueness of spacelike hypersurfaces with CMC in generalized Robertson-Walker (GRW) spacetimes, namely, Lorentz warped products with 1-dimensional negative definite base and Riemannian fiber. They proved that in a GRW spacetime obeying the timelike convergence condition (i.e., the Ricci curvature is non-negative on timelike directions), every compact spacelike hypersurface with CMC must be umbilical. Recently, Alías and Montiel obtained, in [2], a more general condition on the warping function $f$ that is sufficient in order to guarantee uniqueness. More precisely, they proved the following

**Theorem 1.1.** Let $f : I \to \mathbb{R}$ be a positive smooth function defined on an open interval, such that $ff'' - (f')^2 \leq 0$, that is, such that $-\log f$ is convex. Then, the only compact spacelike hypersurfaces immersed into a generalized Robertson-Walker spacetime $I \times_f F^n$ and having constant mean curvature are the slices $\{t\} \times F$, for a (necessarily compact) Riemannian manifold $F$.

Stability questions concerning CMC, compact hypersurfaces in Riemannian space forms began with Barbosa and do Carmo in [4], and Barbosa, Do Carmo and Eschenburg in [5]. In the former paper, they introduced the notion of stability and proved that spheres are the only stable critical points for the area functional, for volume-preserving variations. In the setting of spacelike hypersurfaces in Lorentz manifolds, Barbosa and Oliker proved in [6] that CMC spacelike hypersurfaces are critical points of volume-preserving variations. Moreover, by computing the second variation formula they showed that CMC embedded spheres in the de Sitter space $S^{n+1}_1$ maximize the area functional for such variations. In this paper, we give a characterization of strongly stable, CMC spacelike hypersurfaces in GRW spacetimes, the essential tool for the proof being a formula for the Laplacian of a new support function. More precisely, it is our purpose to show the following
Theorem 1.2. Let $\overline{M}^{n+1} = I \times_\phi F^n$ be a generalized Robertson-Walker spacetime, and $x : M^n \to \overline{M}^{n+1}$ be a closed spacelike hypersurface of $\overline{M}^{n+1}$, having constant mean curvature $H$. If the warping function $\phi$ satisfies $\phi'' \geq \max\{H\phi', 0\}$ and $M^n$ is strongly stable, then $M^n$ is either minimal or a spacelike slice $M_{t_0} = \{t_0\} \times F$, for some $t_0 \in I$.

2. Stable spacelike hypersurfaces

In what follows, $\overline{M}^{n+1}$ denotes an orientable, time-oriented Lorentz manifold with Lorentz metric $\overline{g} = \langle \ , \ \rangle$ and semi-Riemannian connection $\overline{\nabla}$. If $x : M^n \to \overline{M}^{n+1}$ is a spacelike hypersurface of $\overline{M}^{n+1}$, then $M^n$ is automatically orientable ([8], p. 189), and one can choose a globally defined unit normal vector field $N$ on $M^n$ having the same time-orientation of $V$, that is, such that $\langle V, N \rangle < 0$ on $M$. One says that such an $N$ points to the future.

A variation of $x$ is a smooth map $X : M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}$ satisfying the following conditions:

1. For $t \in (-\epsilon, \epsilon)$, the map $X_t : M^n \to \overline{M}^{n+1}$ given by $X_t(p) = X(t, p)$ is a spacelike immersion such that $X_0 = x$.

2. $X_t|_{\partial M} = x|_{\partial M}$, for all $t \in (-\epsilon, \epsilon)$.

The variational field associated to the variation $X$ is the vector field $\frac{\partial X}{\partial t}$. Letting $f = -\langle \frac{\partial X}{\partial t}, N \rangle$, we get

$$\frac{\partial X}{\partial t}|_M = fN + \left(\frac{\partial X}{\partial t}\right)^T,$$

where $T$ stands for tangential components. The balance of volume of the variation $X$ is the function $\mathcal{V} : (-\epsilon, \epsilon) \to \mathbb{R}$ given by

$$\mathcal{V}(t) = \int_{M \times [0, t]} X^*(d\overline{M}),$$

where $d\overline{M}$ denotes the volume element of $\overline{M}$.

The area functional $\mathcal{A} : (-\epsilon, \epsilon) \to \mathbb{R}$ associated to the variation $X$ is given by

$$\mathcal{A}(t) = \int_M dM_t,$$

where $dM_t$ denotes the volume element of the metric induced in $M$ by $X_t$. Note that $dM_0 = dM$ and $\mathcal{A}(0) = \mathcal{A}$, the volume of $M$. The following lemma is classical:

Lemma 2.1. Let $\overline{M}^{n+1}$ be a time-oriented Lorentz manifold and $x : M^n \to \overline{M}^{n+1}$ a spacelike closed hypersurface having mean curvature $H$. If $X : M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}$ is a variation of $x$, then

$$\frac{d\mathcal{V}}{dt}|_{t=0} = \int_M f dM,$$

$$\frac{d\mathcal{A}}{dt}|_{t=0} = \int_M nH f dM.$$
Set $H_0 = \frac{1}{A} \int_M dM$ and $J : (-\epsilon, \epsilon) \to \mathbb{R}$ given by
\[
J(t) = A(t) - nH_0 \mathcal{V}(t).
\]
$J$ is called the \textit{Jacobi functional} associated to the variation, and it is a well known result \cite{3} that $x$ has constant mean curvature $H_0$ if and only if $J'(0) = 0$ for all variations $X$ of $x$.

We wish to study here immersions $x : M^n \to \overline{M}^{n+1}$ that maximize $J$ for all variations $X$. Since $x$ must be a critical point of $J$, it thus follows from the above discussion that $x$ must have constant mean curvature. Therefore, in order to examine whether or not some critical immersion $x$ is actually a maximum for $J$, one certainly needs to study the second variation $J''(0)$. We start with the following

\textbf{Proposition 2.2.} Let $x : M^n \to \overline{M}^{n+1}$ be a closed spacelike hypersurface of the time-oriented Lorentz manifold $\overline{M}^{n+1}$, and $X : M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1}$ be a variation of $x$. Then,
\[
(2.1) \quad n \frac{\partial H}{\partial t} = \Delta f - \{\text{Ric}(N, N) + |A|^2\} f - n\langle \left( \frac{\partial X}{\partial t} \right)^T, \nabla H \rangle.
\]

Although the above proposition is known to be true, we believe there is a lack, in the literature, of a clear proof of it in this degree of generality, so we present a simple proof here.

\textbf{Proof.} Let $p \in M$ and $\{e_k\}$ be a moving frame on a neighborhood $U \subset M$ of $p$, geodesic at $p$ and diagonalizing $A$ at $p$, with $Ae_k = \lambda_k e_k$ for $1 \leq k \leq n$. Extend $N$ and the $e_k's$ to a neighborhood of $p$ in $\overline{M}$, so that $\langle N, e_k \rangle = 0$ and $\langle \nabla_{N} e_k \rangle(p) = 0$. Then
\[
n \frac{\partial H}{\partial t} = -\text{tr} \left( \frac{\partial A}{\partial t} \right) = -\sum_k \langle \frac{\partial A}{\partial t} e_k, e_k \rangle = -\sum_k \langle \nabla_{\frac{\partial A}{\partial t}} e_k, e_k \rangle - \sum_k \langle Ae_k, e_k \rangle
\]
\[
= -\sum_k \left\{ \langle \nabla_{\frac{\partial A}{\partial t}} e_k, e_k \rangle \right\} - \sum_k \langle Ae_k, e_k \rangle
\]
\[
= \sum_k \langle \nabla_{\frac{\partial A}{\partial t}} e_k, e_k \rangle + \sum_k \langle Ae_k, e_k \rangle - \sum_k \langle \nabla_{\frac{\partial X}{\partial t}} e_k, e_k \rangle,
\]
where in the last equality we used the fact that $[\frac{\partial X}{\partial t}, e_k] = 0$. Letting
\[
I = \sum_k \langle \nabla_{\frac{\partial X}{\partial t}} e_k, e_k \rangle \quad \text{and} \quad II = \sum_k \langle \nabla_{e_k} \frac{\partial X}{\partial t}, e_k \rangle,
\]
we have
\[
I = \sum_k \left\{ \langle \nabla_{\frac{\partial A}{\partial t}} e_k, N - \nabla_{e_k} \frac{\partial A}{\partial t} N + \nabla\left[ e_k, \frac{\partial A}{\partial t} \right] N, e_k \rangle \right\} + \sum_k \langle \nabla_{e_k} \frac{\partial A}{\partial t} N, e_k \rangle
\]
\[
= \sum_k \left\{ \langle \text{Ric} \left( e_k, \frac{\partial X}{\partial t} \right) N, e_k \rangle + \langle \nabla_{e_k} \nabla_{\frac{\partial A}{\partial t}} e_k, e_k \rangle \right\}
\]
\[
= -\text{Ric} \left( \frac{\partial X}{\partial t}, N \right) + \sum_k \langle \nabla_{e_k} \nabla_{\frac{\partial A}{\partial t}} N, e_k \rangle.
\]
Since the frame $\{e_k\}$ is geodesic at $p$, it follows that
\[
\langle \nabla_{\frac{\partial A}{\partial t}} N, \nabla_{e_k} e_k \rangle = \langle \nabla_{\frac{\partial A}{\partial t}} N, N \rangle \langle \nabla_{e_k} e_k, N \rangle = 0
\]
at $p$, and hence
\[
\langle \nabla_{e_k} \nabla_{\partial_X} N, e_k \rangle = e_k \langle \nabla_{\partial_X} N, e_k \rangle = -e_k \langle N, \nabla_{e_k} \partial_X \rangle = -e_k \langle N, \nabla_{e_k} \partial_X \rangle \]
\[
= -e_k e_k \langle N, \partial_X \rangle + e_k \langle \nabla_{e_k} N, \partial_X \rangle^T \]
\[
= e_k e_k (f) + e_k \langle \nabla_{e_k} \partial_X \rangle^T \]
\[
= e_k e_k (f) + \langle \nabla_{e_k} \partial_X \rangle^T \}
\]

For $II$, we have
\[
II = \sum_k \langle Ae_k, \nabla_{e_k} \partial_X \rangle = \sum_k \langle Ae_k, \nabla_{e_k} (fN + \left( \frac{\partial X}{\partial t} \right)^T) \rangle \]
\[
= \sum_k \langle Ae_k, f \nabla_{e_k} N \rangle + \sum_k \langle Ae_k, \nabla_{e_k} \left( \frac{\partial X}{\partial t} \right)^T \rangle \]
\[
= -f|A|^2 + \sum_k \langle Ae_k, \nabla_{e_k} \left( \frac{\partial X}{\partial t} \right)^T \rangle \]

Therefore,
\[
(2.2) \quad \frac{\partial H}{\partial t} = -\overline{Ric} \left( \frac{\partial X}{\partial t}, N \right) + \Delta f - f|A|^2 + \sum_k \langle \nabla_{e_k} \nabla_{e_k} N, \left( \frac{\partial X}{\partial t} \right)^T \rangle. \]

Now, letting
\[
\frac{\partial X}{\partial t} = \sum_{l} \alpha_l e_l + fN
\]
and $Ae_k = \sum_j h_{jk} e_j$, one successively gets
\[
\overline{Ric} \left( \frac{\partial X}{\partial t}, N \right) = \sum_{l} \alpha_l \overline{Ric}(N, e_l) + f \overline{Ric}(N, N) \]
\[
= \sum_{k,l} \alpha_l (R(e_k, e_l) e_k, N) + f \overline{Ric}(N, N) \]
and, since $(\nabla_N e_k)(p) = 0$,
\[
\langle R(e_k, e_l) e_k, N \rangle = \langle \nabla_{e_k} \nabla_{e_k} e_k - \nabla_{e_k} \nabla_{e_l} e_k, N \rangle_p = e_l \langle \nabla_{e_k} e_k, N \rangle_p - e_k \langle \nabla_{e_k} e_k, N \rangle_p = e_l \langle e_k, \nabla_{e_l} N \rangle_p + e_k \langle e_k, \nabla_{e_l} N \rangle_p = e_l (h_{kk}) - e_k (h_{kl}),
\]
so that
\[
(2.3) \quad \overline{Ric} \left( \frac{\partial X}{\partial t}, N \right)_p = \sum_{k,l} \alpha_l e_l (h_{kk}) - \sum_{k,l} \alpha_l e_k (h_{kl}) + f \overline{Ric}(N, N)_p. \]
Also,
\[ \alpha_t(\nabla_{e_k} \nabla_{e_k} N, e_l) = -\alpha_t \sum_j \langle \nabla_{e_k} h_{kj} e_j, e_l \rangle \]
\[ = -\alpha_t \sum_j \{ e_k(h_{kj}) \delta_{lj} + h_{kj} \langle \nabla_{e_k} e_j, e_l \rangle \} \]
\[ = -\alpha_t e_k(h_{kl}), \]
and hence
\[ \sum_k \langle \nabla_{e_k} \nabla_{e_k} N, \left( \frac{\partial X}{\partial t} \right)^T \rangle = - \sum_{k,l} \alpha_t e_k(h_{kl}). \tag{2.4} \]

Substituting (2.3) and (2.4) into (2.2), we finally arrive at
\[ n \frac{\partial H}{\partial t} = - \sum_{k,l} \alpha_t e_l(h_{kk}) - f \text{Ric}(N, N)_p + \Delta f - f |A|^2 \]
\[ = - \left( \frac{\partial X}{\partial t} \right)^T (nH) - f \text{Ric}(N, N)_p + \Delta f - f |A|^2. \]

**Proposition 2.3.** Let \( \overline{M}^{n+1} \) be a Lorentz manifold and \( x : M^n \to \overline{M}^{n+1} \) be a closed spacelike hypersurface having constant mean curvature \( H \). If \( X : M^n \times (-\epsilon, \epsilon) \to \overline{M}^{n+1} \) is a variation of \( x \), then
\[ J''(0)(f) = \int_M f \{ \Delta f - (\text{Ric}(N, N) + |A|^2) f \} dM. \tag{2.5} \]

**Proof.** In the notations of the above discussion, set \( f = f(0) \) and note that \( H(0) = H \). It follows from lemma 2.1 that
\[ J'(t) = \int_M n \{ H(t) - H \} f(t) dM. \]
Therefore, differentiating with respect to \( t \) once more
\[ J''(0) = \int_M n H'(0) f(0) dM_0 + \int_M n \{ H(0) - H \} \frac{d}{dt} f(t) dM_\parallel \bigg|_{t=0} \]
\[ = \int_M n H'(0) f dM. \]

Taking into account that \( H \) is constant, relation 2.1 finally gives formula 2.5. \( \square \)

It follows from the previous result that \( J''(0) = J''(0)(f) \) depends only on \( f \in C^\infty(M) \), for which there exists a variation \( X \) of \( M^n \) such that \( \left( \frac{\partial X}{\partial t} \right)^\perp = fN \). Therefore, the following definition makes sense:

**Definition 2.4.** Let \( \overline{M}^{n+1} \) be a Lorentz manifold and \( x : M^n \to \overline{M}^{n+1} \) be a closed spacelike hypersurface having constant mean curvature \( H \). We say that \( x \) is strongly stable if, for every function \( f \in C^\infty(M) \) for which there exists a variation \( X \) of \( M^n \) such that \( \left( \frac{\partial X}{\partial t} \right)^\perp = fN \), one has \( J''(0)(f) \leq 0 \).
3. Conformal vector fields

As in the previous section, let $\mathcal{M}^{n+1}$ be a Lorentz manifold. A vector field $V$ on $\mathcal{M}^{n+1}$ is said to be conformal if

$$\mathcal{L}_V \langle , \rangle = 2\psi \langle , \rangle$$  \hspace{1cm} (3.1)

for some function $\psi \in C^\infty(\mathcal{M})$, where $\mathcal{L}$ stands for the Lie derivative of the Lorentz metric of $\mathcal{M}$. The function $\psi$ is called the conformal factor of $V$.

Since $\mathcal{L}_V(X) = [V, X]$ for all $X \in \mathcal{X}(\mathcal{M})$, it follows from the tensorial character of $\mathcal{L}_V$ that $V \in \mathcal{X}(\mathcal{M})$ is conformal if and only if

$$\langle \nabla_X V, Y \rangle + \langle X, \nabla_Y V \rangle = 2\psi \langle X, Y \rangle,$$  \hspace{1cm} (3.2)

for all $X, Y \in \mathcal{X}(\mathcal{M})$. In particular, $V$ is a Killing vector field relatively to $\overline{g}$ if and only if $\psi \equiv 0$.

Any Lorentz manifold $\mathcal{M}^{n+1}$, possessing a globally defined, timelike conformal vector field is said to be a conformally stationary spacetime.

**Proposition 3.1.** Let $\mathcal{M}^{n+1}$ be a conformally stationary Lorentz manifold, with conformal vector field $V$ having conformal factor $\psi : \mathcal{M}^{n+1} \to \mathbb{R}$. Let also $x : M^n \to \mathcal{M}^{n+1}$ be a spacelike hypersurface of $\mathcal{M}^{n+1}$, and $N$ a future-pointing, unit normal vector field globally defined on $M^n$. If $f = \langle V, N \rangle$, then

$$\Delta f = n \langle V, \nabla H \rangle + f \{ \overline{\text{Ric}}(N, N) + |A|^2 \} + n \{ H \psi - N(\psi) \},$$  \hspace{1cm} (3.3)

where $\overline{\text{Ric}}$ denotes the Ricci tensor of $\mathcal{M}$, $A$ is the second fundamental form of $x$ with respect to $N$, $H = -\frac{1}{n} \text{tr}(A)$ is the mean curvature of $x$ and $\nabla H$ denotes the gradient of $H$ in the metric of $M$.

**Proof.** Fix $p \in M$ and let $\{e_k\}$ be an orthonormal moving frame on $M$, geodesic at $p$. Extend the $e_k$ to a neighborhood of $p$ in $\mathcal{M}$, so that $(\nabla_N e_k)(p) = 0$, and let

$$V = \sum_{l} \alpha_l e_l - fN.$$  \hspace{1cm} (3.4)

Then

$$f = \langle N, V \rangle \Rightarrow e_k(f) = \langle \nabla_{e_k} N, V \rangle + \langle N, \nabla_{e_k} V \rangle$$

$$= -\langle A e_k, V \rangle + \langle N, \nabla_{e_k} V \rangle,$$

so that

$$\Delta f = \sum_{k} e_k(e_k(f)) = -\sum_{k} e_k\langle A e_k, V \rangle + \sum_{k} e_k\langle N, \nabla_{e_k} V \rangle$$

$$= -\sum_{k} \langle \nabla_{e_k} A e_k, V \rangle - 2 \sum_{k} \langle A e_k, \nabla_{e_k} V \rangle + \sum_{k} \langle N, \nabla_{e_k} \nabla_{e_k} V \rangle.$$
Now, differentiating $Ae_k = \sum_i h_{ki} e_l$ with respect to $e_k$, one gets at $p$

$$\sum_k \langle \nabla_{e_k} Ae_k, V \rangle = \sum_k e_k(h_{kl})\langle e_l, V \rangle + \sum_k h_{kl}\langle \nabla_{e_k} e_l, V \rangle$$

$$= \sum_k a_l e_k(h_{kl}) - \sum_k h_{kl}\langle \nabla_{e_k} e_l, N \rangle\langle V, N \rangle$$

$$= \sum_k a_l e_k(h_{kl}) - \sum_k h_{kl}^2 f$$

(3.5)

$$= \sum_k a_l e_k(h_{kl}) - f |A|^2.$$

Asking further that $Ae_k = \lambda_k e_k$ at $p$ (which is always possible), we have at $p$

$$\sum_k \langle Ae_k, \nabla_{e_k} V \rangle = \sum_k \lambda_k \langle e_k, \nabla_{e_k} V \rangle = \sum_k \lambda_k \psi = -nH \psi.$$ (3.6)

In order to compute the last summand of (3.5), note that the conformality of $V$ gives

$$\langle \nabla_N V, e_k \rangle + \langle N, \nabla_{e_k} V \rangle = 0$$

for all $k$. Hence, differentiating the above relation in the direction of $e_k$, we get

$$\langle \nabla_{e_k} \nabla_N V, e_k \rangle + \langle \nabla_N V, \nabla_{e_k} e_k \rangle + \langle \nabla_{e_k} N, \nabla_{e_k} V \rangle + \langle N, \nabla_{e_k} \nabla_{e_k} V \rangle = 0.$$ However, at $p$ one has

$$\langle \nabla_N V, \nabla_{e_k} e_k \rangle = -\langle \nabla_N V, (\nabla_{e_k} e_k, N) \rangle N = -\langle \nabla_N V, \lambda_k N \rangle$$

$$= -\lambda_k \psi \langle N, N \rangle = \lambda_k \psi$$

and

$$\langle \nabla_{e_k} N, \nabla_{e_k} V \rangle = -\lambda_k \langle e_k, \nabla_{e_k} V \rangle = -\lambda_k \psi,$$

so that

(3.7) $$\langle \nabla_{e_k} \nabla_N V, e_k \rangle + \langle N, \nabla_{e_k} \nabla_{e_k} V \rangle = 0$$

at $p$. On the other hand, since

$$[N, e_k](p) = (\nabla_{Ne_k})(p) - (\nabla_{e_k} N)(p) = \lambda_k e_k(p),$$

it follows from (3.7) that

$$\langle R(N, e_k) V, e_k \rangle_p = \langle \nabla_{e_k} \nabla_N V - \nabla_N \nabla_{e_k} V + \nabla_{[N, e_k]} V, e_k \rangle_p$$

$$= -\langle N, \nabla_{e_k} \nabla_{e_k} V \rangle_p - N \langle \nabla_{e_k} V, e_k \rangle_p + \langle \nabla_{\lambda_k e_k} V, e_k \rangle_p$$

$$= -\langle N, \nabla_{e_k} \nabla_{e_k} V \rangle_p - N \langle \psi \rangle_p + \lambda_k \psi,$$

and hence

(3.8) $$\sum_k \langle N, \nabla_{e_k} \nabla_{e_k} V \rangle_p = -nN(\psi) - nH \psi - \overline{Ric}(N, V)_p$$

Finally,

$$\overline{Ric}(N, V) = \sum_l \alpha_l \overline{Ric}(N, e_l) - f \overline{Ric}(N, N)$$

$$= \sum_{k,l} \alpha_l (R(e_k, e_l) e_k, N) - f \overline{Ric}(N, N),$$
and
\[
\langle R(e_k, e_l)e_k, N \rangle_p = \langle \nabla_{e_l} \nabla_{e_k} e_k - \nabla_{e_k} \nabla_{e_l} e_k, N \rangle_p
\]
\[= e_l(\nabla_{e_k} e_k, N)_p - (\nabla_{e_k} e_k, \nabla_{e_l} N)_p - e_k(\nabla_{e_l} e_k, N)_p
\]
\[+ (\nabla_{e_l} e_k, \nabla_{e_k} N)_p
\]
\[= -e_l(e_k, \nabla_{e_k} N)_p + e_k(e_k, \nabla_{e_l} N)_p
\]
\[= e_l(h_{kk}) - e_k(h_{kl}),
\]
so that
\[
\overline{\text{Ric}}(N, V)_p = \sum_{k,l} \alpha_l e_l(h_{kk}) - \sum_{k,l} \alpha_k e_k(h_{kl}) - f\overline{\text{Ric}}(N, N)_p,
\]
and it follows from (3.8) that
\[
\sum_{k} \langle N, \nabla_{e_k} \nabla_{e_k} V \rangle_p = -nN(\psi) - nH\psi + V^T(nH)
\]
\[+ \sum_{k,l} \alpha_l e_k(h_{kl}) + f\overline{\text{Ric}}(N, N).
\]
(3.9)

Substituting (3.4), (3.5) and (3.6) into (3.3), one gets the desired formula (3.3). □

4. Applications

A particular class of conformally stationary spacetimes is that of generalized Robertson-Walker spacetimes \(\mathbb{II}\), namely, warped products \(M^{n+1} = I \times_\phi F^n\), where \(I \subseteq \mathbb{R}\) is an interval with the metric \(-dt^2\), \(F^n\) is an \(n\)-dimensional Riemannian manifold and \(\phi : I \to \mathbb{R}\) is positive and smooth. For such a space, let \(\pi_I : M^{n+1} \to I\) denote the canonical projection onto the \(I\)-factor. Then the vector field
\[
V = (\phi \circ \pi_I) \frac{\partial}{\partial t}
\]
is conformal, timelike and closed (in the sense that its dual 1–form is closed), with conformal factor \(\psi = \phi'\), where the prime denotes differentiation with respect to \(t\). Moreover, according to [7], for \(t_0 \in I\), orienting the (spacelike) leaf \(M^n = \{t_0\} \times F^n\) by using the future-pointing unit normal vector field \(N\), it follows that \(M^n\) has constant mean curvature
\[
H = \frac{\phi'(t_0)}{\phi(t_0)}.
\]

If \(M^{n+1} = I \times_\phi F^n\) is a generalized Robertson-Walker spacetime and \(x : M^n \to M^{n+1}\) is a complete spacelike hypersurface of \(M^{n+1}\), such that \(\phi \circ \pi_I\) is limited on \(M\), then \(\pi_F|_M : M^n \to F^n\) is necessarily a covering map (\(\mathbb{II}\)). In particular, if \(M^n\) is closed, then \(F^n\) is automatically closed.

One has the following corollary of proposition 3.1

**Corollary 4.1.** Let \(M^{n+1} = I \times_\phi F^n\) be a generalized Robertson-Walker spacetime, and \(x : M^n \to M^{n+1}\) a spacelike hypersurface of \(M^{n+1}\), having constant mean curvature \(H\). Let also \(N\) be a future-pointing unit normal vector field globally defined on \(M^n\). If \(V = (\phi \circ \pi_I) \frac{\partial}{\partial t}\) and \(f = \langle V, N \rangle\), then
\[
\Delta f = \left\{ \overline{\text{Ric}}(N, N) + |A|^2 \right\} f + n \left\{ H\phi' + \phi'' \langle N, \frac{\partial}{\partial t} \rangle \right\}.
\]
(4.1)
where $\text{Ric}$ denotes the Ricci tensor of $\overline{M}$, $A$ is the second fundamental form of $x$ with respect to $N$, and $H = -\frac{1}{n}\text{tr}(A)$ is the mean curvature of $x$.

Proof. First of all, $f = \langle V, N \rangle = \phi \langle N, \frac{\partial}{\partial t} \rangle$, and it thus follows from (3.3) that
\[
\Delta f = \left\{ \text{Ric}(N, N) + |A|^2 \right\} f + n \left\{ H\phi' - N(\phi') \right\}.
\]
However,
\[
\nabla \phi' = -\langle \nabla \phi', \frac{\partial}{\partial t} \rangle \frac{\partial}{\partial t} = -\phi'' \frac{\partial}{\partial t},
\]
so that
\[
N(\phi') = \langle N, \nabla \phi' \rangle = -\phi'' \langle N, \frac{\partial}{\partial t} \rangle.
\]
\[\square\]

We can now state and prove our main result:

**Theorem 4.2.** Let $M^{n+1} = I \times_F F^n$ be a generalized Robertson-Walker spacetime, and $x : M^n \to M^{n+1}$ be a closed spacelike hypersurface of $\overline{M}^{n+1}$, having constant mean curvature $H$. If the warping function $\phi$ satisfies $\phi'' \geq \max\{H\phi', 0\}$ and $M^n$ is strongly stable, then $M^n$ is either minimal or a spacelike slice $M_{t_0} = \{t_0\} \times F$, for some $t_0 \in I$.

Proof. Since $M^n$ is strongly stable, we have
\[
0 \geq J''(0)(g) = \int_M g \left\{ \Delta g - (\text{Ric}(N, N) + |A|^2) g \right\} dM
\]
for all $g \in C^\infty(M)$ for which $gN$ is the normal component of the variational field of some variation of $M^n$. In particular, if $f = \langle V, N \rangle = \phi \langle N, \frac{\partial}{\partial t} \rangle$, where $V = (\phi \circ \pi_I) \frac{\partial}{\partial t}$, and $g = -f = -\langle V, N \rangle$, then
\[
\Delta g = \left\{ \text{Ric}(N, N) + |A|^2 \right\} g - n \left\{ H\phi' + \phi'' \langle N, \frac{\partial}{\partial t} \rangle \right\}.
\]
Therefore, $M^n$ stable implies
\[
0 \geq \int_M \phi \langle N, \frac{\partial}{\partial t} \rangle \left\{ H\phi' + \phi'' \langle N, \frac{\partial}{\partial t} \rangle \right\} dM
\]
Letting $\theta$ be the hyperbolic angle between $N$ and $\frac{\partial}{\partial t}$, it follows from the reversed Cauchy-Schwarz inequality that $\cosh \theta = -\langle N, \frac{\partial}{\partial t} \rangle$, with $\cosh \theta \equiv 1$ if and only if $N$ and $\frac{\partial}{\partial t}$ are collinear at every point, that is, if and only if $M^n$ is a spacelike leaf $M_{t_0}$ for some $t_0 \in I$. Hence,
\[
0 \geq \int_M \phi \cosh \theta \left\{ -H\phi' + \phi'' \cosh \theta \right\} dM.
\]
Now, notice that $-H\phi' + \phi'' \cosh \theta \geq -\phi'' + \phi'' \cosh \theta$, which gives
\[
\phi \cosh \theta(-H\phi' + \phi'' \cosh \theta) \geq \phi\phi'' \cosh \theta(\cosh \theta - 1).
\]
Therefore,
\[
0 \geq \int_M \phi \cosh \theta(-H\phi' + \phi'' \cosh \theta)dM \geq \int_M \phi\phi'' \cosh \theta(\cosh \theta - 1) \geq 0,
\]
and hence
\[
\phi''(\cosh \theta - 1) = 0 \quad \text{and} \quad \phi'' = H\phi'.
\]
on $M$. If, for some $p \in M$, one has $\phi''(p) = 0$, then $\phi'H = 0$ at $p$. If $H \neq 0$, then $\phi'(p) = 0$. But if this is the case, then proposition 7.35 of [8] gives that
\[
\nabla_V \frac{\partial}{\partial t} = \frac{\phi'}{\phi} V = 0
\]
at $p$ for any $V$, and $M$ is totally geodesic at $p$. In particular, $H = 0$, a contradiction. Therefore, either $\phi''(p) = 0$ for some $p \in M$, and $M$ is minimal, or $\phi'' \neq 0$ on all of $M$, whence $\cosh \theta = 1$ always, and $M$ is an umbilical leaf such that $\phi'' = H \phi'$. □

Remark 4.3. Note that $\frac{\phi''}{\phi'} = H = \frac{\phi'}{\phi}$, i.e., $\phi'' \phi - (\phi')^2 = 0$, which is a limit case of Alías and Montiel’s timelike convergent condition.

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