Phase bistability and phase bistable patterns in self-oscillatory systems under a resonant periodic forcing with spatially modulated amplitude

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Abstract

I consider the problem of self-oscillatory systems undergoing a homogeneous Hopf bifurcation when they are submitted to an external forcing that is periodic in time, at a frequency close to the system’s natural frequency (1:1 resonance), and whose amplitude is slowly modulated in space. Starting from a general, unspecified model and making use of standard multiple scales analysis, I show that the close-to-threshold dynamics of such systems is universally governed by a generalized, complex Ginzburg-Landau (CGL) equation. The nature of the generalization depends on the strength and of other features of forcing: (i) For generic, sufficiently weak forcings the CGL equation contains an extra, inhomogeneous term proportional to the complex amplitude of forcing, as in the usual 1:1 resonance with spatially uniform forcing; (ii) For stronger perturbations, whose amplitude sign alternates across the system, the CGL equation contains a term proportional to the complex conjugate of the oscillations envelope, like in the classical 2:1 resonance, responsible for the emergence of phase bistability and of phase bistable patterns in the system. Finally I show that case (ii) is retrieved from case (i) in the appropriate limit so that the latter can be regarded as the generic model for the close-to-threshold dynamics of the type of systems considered here. The kind of forcing studied in this work thus represents an alternative to the classical parametric forcing at twice the natural frequency of oscillations and opens the way to new forms of pattern formation control in self-oscillatory systems, what is especially relevant in the case of systems that are quite insensitive to parametric forcing, such as lasers and other nonlinear optical cavities.
I. INTRODUCTION

The temporal periodic forcing of spatially extended, self-oscillatory systems is a classical method to control and excite the formation of spatial patterns in such systems. This kind of forcing admits a universal description when the system is operated near the oscillation threshold and forcing acts on an \( n : m \) resonance, defined by the relation

\[ \omega_f = \left( \frac{n}{m} \right) (\omega_0 + \delta \omega) \]

between the external forcing frequency \( \omega_f \) and the natural frequency of oscillations \( \omega_0 \), where \( n/m \) is an irreducible integer fraction and \( \delta \omega \) is a small mistuning. In such case oscillations settle down in the system so that any of its variables takes the form

\[ \text{Re} \left[ \kappa u(x, t) e^{i(\omega_0 + \delta \omega)t} \right] \]

to the leading order, where \( \kappa \) is a complex constant and the slowly varying complex amplitude of the oscillations \( u \) verifies the following generalized complex Ginzburg-Landau equation (CGLE) \(^1, ^2\)

\[
\partial_t u = a_1 u + a_2 \nabla^2 u + a_3 |u|^2 u + a_4 \overline{u}^{n-1},
\]

where \( \overline{u} \) stands for the complex conjugate of \( u \), \( a_{i=1,2,3} \) are complex coefficients and \( a_4 \) is proportional to the \( m \)-th power of the forcing amplitude. Equation \(^1\) is valid in principle for perfectly periodic forcings and has been introduced making use of elegant symmetry arguments \(^1\) as well as has been derived in specific contexts, such as in chemistry \(^3\), making use of standard multiple scales analysis \(^4\).

As commented the validity of \(^1\) requires that the system is close to the Hopf bifurcation and, additionally that the amplitude of the external forcing be either constant or slowly varying in time and/or space, see \(^5\) for the case \( n = 1 \). Here I show that Eq. \(^1\) with \( n = 2 \) applies as well to systems forced at the 1:1 resonance (hence at \( n = 1 \)) when the forcing amplitude varies on space on a long (but not too long) scale, to be defined formally below, and the spatial variation involves a sign alternation of the amplitude. This implies that the system becomes phase bistable, which is one of the salient features of Eq. \(^1\) with \( n = 2 \), in contrast to the phase locking (to just one phase value) predicted in the case \( n = 1 \). The intuitive picture is that, as the system is resonantly forced at a 1:1 resonance the phase of the oscillations tend to lock to a single value that depends in particular on the phase of the forcing. In our case however forcing displays two opposite phases (two signs) that are alternating across the system. The system’s oscillations have then two reference phases and can lock to two dynamically equivalent values, leading to phase bistability. Clearly this must require that the typical spatial scale over which forcing varies is short as compared
with the typical spatial scale of the unforced system as otherwise the system could lock locally to the forcing phase at that region. The kind of forcing considered in this work gets inspiration from another 1:1 resonant forcing, called rocking [6], in which the amplitude of forcing is uniform in space but its sign alternates along time. Rocking has been considered both theoretically and experimentally in several systems [7–10].

The analysis presented here is based on the technique of multiple scales and generalizes the concept of resonant forcing as it considers periodic forcings in time that are nonuniform across the system on an apparently nonresonant scale. The derivation allows a rigorous simplified study of both spatially periodic forcings as well as spatially noisy forcings within a 1:1 resonance, whenever they verify the conditions imposed below. Generalizations to other resonances \((n \neq 1 \text{ or } m \neq 1)\), although cumbersome, can be made straightforwardly following the lines of the present derivation.

The kind of forcing studied here thus represents an alternative to the classical parametric (or 2:1 resonant) forcing, what is especially relevant in systems that are insensitive to the latter, like most nonlinear optical systems. In particular this allows the emergence of phase bistability in the system (as the term \(a_4\pi\) breaks the usual continuous phase symmetry of the classical CGLE down to the discrete one \(u \rightarrow -u\)) and of phase bistable patterns, like phase domains and phase domain walls, rolls, hexagons and localized structures.

II. MODEL AND NOTATION

We consider a generic one-dimensional system described by \(N\) real dynamical variables \(\{U_i(x,t)\}_{i=1}^N\) whose time evolution is governed by the following set of real equations written in vector form,

\[
\partial_t U(x,t) = f(\mu, \alpha; U) + \mathcal{D}(\mu, \alpha) \cdot \nabla^2 U, \tag{2}
\]

where \(f\) is a sufficiently differentiable function of its arguments, \(\nabla^2 = \partial_x^2\) is the one-dimensional Laplacian and \(\mathcal{D}(\mu, \alpha)\) is a diffusion matrix. This is the simplest dependence on derivatives in space-translation invariant systems at the time it corresponds to actual systems of most relevance like, \(e.g.,\) reaction-diffusion and nonlinear optical systems. \(\mu\) is the bifurcation parameter and \(\alpha(x,t)\) is the forcing parameter, which is allowed to vary on time and space. Physically \(\alpha\) may represent either an independent parameter, or the modulated part of any other parameter.
We assume that in the absence of forcing \((\alpha = 0)\) Eq. (2) supports a steady, spatially homogeneous state \(U = U_s(\mu)\) \((\partial_t U_s = \partial_x U_s = 0)\), which loses stability at \(\mu = \mu_0\) giving rise to a self-oscillatory, spatially homogeneous state. In other words, we assume that the reference state \(U_s\) suffers a homogeneous Hopf bifurcation at \(\mu = \mu_0\). We wish to study the small amplitude oscillations that form in the system close to the bifurcation when the arbitrary parameter \(\alpha\) is periodically modulated in time with a frequency close to that of the free oscillations and is modulated in space on a long spatial scale to be defined below.

For the sake of convenience we introduce a new vector

\[
u (r, t) = U(x, t) - U_s,
\]

which measures the deviation of the system from the reference state, in terms of which we rewrite Eq. (2) as a Taylor series,

\[
\partial_t u = F(\mu, \alpha) + J(\mu, \alpha) \cdot u + D(\mu, \alpha) \cdot \nabla^2 u + K(\mu, \alpha; u, u) + L(\mu, \alpha; u, u, u) + h.o.t.,
\]

where \(h.o.t.\) denotes terms of higher order than 3 in \(u\). As usual, and as we show below, these \(h.o.t.\) have no influence near the bifurcation whenever it is supercritical, what we assume.

The different elements of the expansion (4) are defined as

\[
F(\mu, \alpha) = f_s,
\]

\[
J_{ij}(\mu, \alpha) = \left[ \frac{\partial f_i}{\partial U_j} \right]_s,
\]

\[
K(\mu, \alpha; a, b) = \frac{1}{2} \sum_{i,j=1}^N \left[ \frac{\partial^2 f}{\partial U_i \partial U_j} \right]_s a_i b_j,
\]

\[
L(\mu, \alpha; a, b, c) = \frac{1}{6} \sum_{i,j,k=1}^N \left[ \frac{\partial^3 f}{\partial U_i \partial U_j \partial U_k} \right]_s a_i b_j c_k,
\]

where \(a, b, c\) are arbitrary vectors and the subscript ”s” denotes \(U = U_s(\mu)\). Vector \(F(\mu, \alpha)\) is subjected to the condition

\[
F(\mu, 0) = 0,
\]

since in the absence of forcing \((\alpha = 0)\) the reference state \(u = 0\) is a steady state of Eq. (1) by hypothesis. \(J\) is a matrix and vector \(K\) \((L)\) is a symmetric and bilinear \((\text{symmetric and trilinear})\) form of its two \((\text{three})\) last arguments.
III. THE HOPF BIFURCATION OF THE UNFORCED SYSTEM

In the absence of forcing \((\alpha = 0)\) the stability of the reference state against small perturbations \(\delta u\) is governed by the following equation

\[
\partial_t \delta u = J(\mu, 0) \cdot \delta u + D(\mu, 0) \cdot \nabla^2 \delta u,
\]

obtained upon linearizing Eq. (4) for \(\alpha = 0\) with respect to \(\delta u\). The general solution to Eq. (6) is a superposition of plane waves of the form

\[
\delta u(r,t) = \sum_j w_j \exp(\Lambda_j t) \exp(ik_j x),
\]

with

\[
\Lambda_j w_j = M(\mu, k^2_j) \cdot w_j,
\]

\[
M(\mu, k^2) = J(\mu, 0) - k^2 D(\mu, 0),
\]

hence eigenvalues and eigenvectors of matrix \(M\) depend on \(k\) only through \(k^2\) as \(M\) does (this is a consequence of the assumed spatial-translation invariance of the unforced system).

As we are assuming that the reference state looses stability at \(\mu = \mu_0\) via a homogeneous Hopf bifurcation in the absence of forcing, matrix \(M(\mu, k^2)\) must have a pair of complex-conjugate eigenvalues \(\{\Lambda_1, \Lambda_2\} = \{\lambda(\mu, k^2), \overline{\lambda(\mu, k^2)}\}\) (the overbar denotes complex conjugation) governing the instability, \(i.e.\):

(i) Close to the bifurcation \(\text{Re } \Lambda_i \geq 3 < 0\), while \(\text{Re } \lambda\) can become positive for some \(k\)'s,

(ii) At the bifurcation \(\text{Re } \lambda\) is maximum and null at \(k = 0\) (the perturbation with largest growth rate is spatially homogeneous):

\[
\text{Re } \lambda_0 = 0, (\partial_k \text{Re } \lambda)_0 = 0, (\partial^2_k \text{Re } \lambda)_0 < 0.
\]

where, here and in the following,

subscript 0 affecting functions denotes \(\{\mu = \mu_0, \alpha = 0, k = 0\}\).

(iii) The instability is oscillatory, \(i.e.\),

\[
\text{Im } \lambda_0 \equiv \omega_0 \neq 0.
\]
Finally, the preceding properties imply that:

(iv) All eigenvalues of $\mathcal{M}_0 = \mathcal{J}_0$, see Eq. (9), have negative real part but $\{\lambda_0, \overline{\lambda}_0\} = \{i\omega_0, -i\omega_0\}$.

For the sake of later use we introduce the right and left eigenvectors of $\mathcal{J}_0$ associated with eigenvalues $\{i\omega_0, -i\omega_0\}$,

$$
\mathcal{J}_0 \cdot \mathbf{h} = i\omega_0 \mathbf{h}, \quad \mathcal{J}_0 \cdot \overline{\mathbf{h}} = -i\omega_0 \overline{\mathbf{h}},
$$

$$
\mathbf{h} \dagger \cdot \mathcal{J}_0 = i\omega_0 \mathbf{h} \dagger, \quad \overline{\mathbf{h}} \dagger \cdot \mathcal{J}_0 = -i\omega_0 \overline{\mathbf{h}} \dagger,
$$

where the short-hand notation $\mathbf{h} = w_1 (\mu = \mu_0, k^2 = 0), \overline{\mathbf{h}} = w_2 (\mu = \mu_0, k^2 = 0)$ has been introduced. These vectors verify the following orthonormality relations:

$$
\mathbf{h} \dagger \cdot \overline{\mathbf{h}} = 0, \mathbf{h} \dagger \cdot \mathbf{h} = 1,
$$

as is trivially to be checked.

IV. SCALES

We are interested in determining the small amplitude solutions that emerge in the system close to the Hopf bifurcation, a parametric region that we define by

$$
\mu = \mu_0 + \varepsilon^2 \mu_2,
$$

where $\varepsilon$ is a smallness parameter ($0 < \varepsilon \ll 1$). The study is based on the widely used technique of multiple scales [4]. These spatial and temporal scales appear naturally close to the bifurcation and are those on which the asymptotic dynamics of the unforced system naturally evolves. As is well known, in a homogeneous Hopf bifurcation these slow scales are given by

$$
T = \varepsilon^2 t, X = \varepsilon x,
$$

which follow from the behaviour of $\lambda$ close to the bifurcation, Eq. (14), for values of $k$ close to the most unstable mode $k = 0$:

$$
\lambda \left(\mu_0 + \varepsilon^2 \mu_2, k^2\right) = \lambda_0 + (\partial_\mu \lambda)_0 \varepsilon^2 \mu_2 + \frac{1}{2} (\partial_k^2 \lambda)_0 k^2 + \max \left\{O(\varepsilon^4), O(k^2)\right\},
$$
where a term \((\partial_k \lambda)_0 k\) has not been included since \((\partial_k \lambda)_0 = 0\) as \(\lambda\) is an even function of \(k\). From this equation we obtain, making use of Eq. (10),

\[
\text{Re} \lambda \left( \mu_0 + \varepsilon^2 \mu_2, k^2 \right) = \left( \partial_\mu \text{Re} \lambda \right)_0 \mu_2 - \frac{1}{2} \left| \partial^2_k \text{Re} \lambda \right|_0 k^2 + \max \{ O (\varepsilon^4) ; O (k^2) \},
\]

what indicates that the only modes which can experience linear growth verify \(k = O (\varepsilon)\), hence the asymptotic dynamics of the system exhibits spatial variations on a scale \(x \sim k^{-1} \sim \varepsilon^{-1}\) and the slow spatial scale \(X = \varepsilon x\) follows. Thus, setting \(k = \varepsilon k_1\),

\[
\text{Re} \lambda \left( \mu_0 + \varepsilon^2 \mu_2, \varepsilon^2 k_1^2 \right) = \varepsilon^2 \left[ \left( \partial_\mu \text{Re} \lambda \right)_0 \mu_2 - \frac{1}{2} \left| \partial^2_k \text{Re} \lambda \right|_0 k_1^2 \right] + O (\varepsilon^4),
\]

which shows that the growth (or decay) of the perturbations occurs on a scale \(t \sim (\text{Re} \lambda)^{-1} \sim \varepsilon^{-2}\) and the slow timescale \(T = \varepsilon^2 t\) follows. On the other hand, making use of Eq. (11) and setting \(k = \varepsilon k_1\) again, Eq. (16) yields

\[
\text{Im} \lambda \left( \mu_0 + \varepsilon^2 \mu_2, \varepsilon^2 k_1^2 \right) = \omega_0 + \varepsilon^2 \left[ \left( \partial_\mu \text{Im} \lambda \right)_0 \mu_2 + \frac{1}{2} \left( \partial^2_k \text{Im} \lambda \right)_0 k_1^2 \right] + O (\varepsilon^4),
\]

whose first term, \(\omega_0 = O (\varepsilon^0)\), indicates that the original timescale \(t\) must be retained, while the rest of terms do not introduce other relevant timescales.

### A. Scales for the forcing

As for the external forcing \(\alpha (x, t)\) we assume that it is weak, of order \(\varepsilon^2\), periodic in time at a frequency \(\omega_t\) close to the Hopf frequency \(\omega_0\), and depends on space on a scale of order \(\varepsilon^{-1/2}\), which is shorter that the typical spatial scale \(X = \varepsilon x\) of the system. We note that this choice introduces another spatial scale

\[
\xi = \varepsilon^{1/2} x. \quad (17)
\]

For the sake of definiteness we assume that \(\omega_t = \omega_0 + \varepsilon^2 \omega_2\) and adopt the following expression for \(\alpha\):

\[
\alpha (x, t) = \varepsilon^2 \alpha_2 (x, t) = \varepsilon^2 A (\xi) \exp (i \omega_2 T) \exp (i \omega_0 t) + c.c. \quad (18)
\]

We note that the final result of the derivation does not depend on the harmonic character of the forcing: Any periodic (in fact almost periodic) \(\alpha\) at frequency \(\omega_t\) close to \(\omega_0\) can be expressed as a Fourier series (with slowly varying coefficients) with fundamental frequency
ω₀, of which (18) is its first term, and the final result depends only on it \[5\]. We note for later use that

\[ F(μ, α) = F(μ₀ + ε²μ₂, ε²α₂) = ε²α₂(∂₄F)₀ + O(ε⁴), \] (19)
as follows from \[5\].

**B. Scales for the system’s oscillations**

Under all the previous conditions a multiple scale analysis is possible \[4\] and we look for asymptotic solutions to Eq. (4) in the form

\[ u(x, t) = \sum_{m=2}^{∞} ε^{m/2} u_{m/2}(X, T, ξ, t), \] (20)
so that the expansion starts at order ε, as usual.

We finally introduce Eqs. (14) and (18–20) into Eq. (4) making use of the following chain rules for differentiation

\[ \partial_t u = \sum_{m=2}^{∞} ε^{m/2} (\partial_t + ε²∂_T) u_{m/2}, \] (21)
\[ \nabla^2 u = \sum_{m=2}^{∞} ε^{m/2} \left( ε∂_ξ² + 2ε³/²∂_ξ∂_X + ε²∂_X² \right) u_{m/2}, \] (22)
and solve at increasing orders in ε. Note that the occurrence of a term \(2ε³/²∂_ξ∂_X\) in the Laplacian operator suggests performing the expansion in terms of \(ε^{1/2}\), and not of ε, see (20).

**V. THE ASYMPTOTIC ANALYSIS**

The general form of Eq. (4) at any order \(ε^m\) is found to be

\[ ￦(u_m) = ㏒_m(X, T, ξ, t), \] (23)
where

\[ ￦(u_m) ≡ ∂_t u_m - ￦₀ · u_m, \] (24)
and \(㏒_m\) does not depend on \(u_m\) but on \(u_{m<m}\). Clearly, as \(￦(\exp(iω₀t) h) = ￦(\exp(-iω₀t) h) = 0\), see Eq. (12), the solvability of Eq. (23) requires

\[ \omega₀ \int_{t}^{t+2π/ω₀} dt'[hᵀ · ㏒_m(X, T, ξ, t') \exp(-iω₀t')] = 0, \] (25)
(or its equivalent complex-conjugate) which ensures that $g_m$ does not contain secular terms (proportional to $\exp (i\omega_0 t) \ h$ or to $\exp (-i\omega_0 t) \ \overline{h}$). Once condition (25) is verified, the asymptotic solution to Eq. (23) reads

$$u_m (X,T,\xi,t) = u_m (X,T,\xi) \exp (i\omega_0 t) \ h + \overline{u}_m (X,T,\xi) \exp (-i\omega_0 t) \ \overline{h} + u_m^\perp (X,T,\xi,t),$$

where $u_m (X,T,\xi)$ is not fixed at this order and the last term is the particular solution. Note that the solution (26) should involve, in principle, terms proportional to all the eigenvectors of $J (\cdot)$ [which are those of $J_0$, see Eq. (8)]. However all of them (but the first two) are damped according to $\exp [-|\text{Re} \Lambda (\mu_0,0)| \ t]$, since $\text{Re} \Lambda_{i\geq 3} (\mu = \mu_0, k = 0) < 0$ (and of order $\varepsilon^0$ by hypothesis), except those associated with $(h,\overline{h})$.

A. Order $\varepsilon$

This is the lowest nontrivial order and

$$g_1 = 0,$$

so the solvability condition (25) at this order is automatically fulfilled. Then, according to Eq. (26),

$$u_1 = u_1 (X,T,\xi) \exp (i\omega_0 t) \ h + \overline{u}_1 (X,T,\xi) \exp (-i\omega_0 t) \ \overline{h},$$

where the scaled, slowly varying complex amplitude of oscillations $u_1 (X,T,\xi)$ is undetermined at this stage. We need to continue the analysis in order to meet solvability conditions that fix an equation for $u_1$, which is our goal.

B. Order $\varepsilon^{3/2}$

At this order we get

$$g_{3/2} = 0,$$

hence the solvability condition is automatically fulfilled again and $u_{3/2}$ reads, according to (26),

$$u_{3/2} = u_{3/2} (X,T,\xi) \exp (i\omega_0 t) \ h + \overline{u}_{3/2} (X,T,\xi) \exp (-i\omega_0 t) \ \overline{h}.$$
C. Order $\varepsilon^2$

Trivially one has

$$g_2 = \alpha_2 (\partial_{\alpha} F)_0 + D_0 \cdot \partial_\xi^2 u_1 + K (\mu_0; u_1, u_1). \tag{31}$$

Making use of Eq. (28) and taking into account that $K$ is symmetric and bilinear in its two last arguments, Eq. (31) can be written as

$$g_2 = \alpha_2 (\partial_{\alpha} F)_0 + (D_0 \cdot h) \partial_\xi^2 u_1 (X, T, \xi) \exp (i\omega_0 t) + (D_0 \cdot \overline{h}) \partial_\xi^2 \overline{u}_1 (X, T, \xi) \exp (-i\omega_0 t)$$

$$+ 2K (\mu_0; h, \overline{h}) |u_1|^2 + K (\mu_0; h, h) u_1^2 \exp (i\omega_0 t) + K (\mu_0; \overline{h}, h) \overline{u}_1^2 \exp (-i\omega_0 t). \tag{32}$$

The solvability condition (25) reads now

$$\left( h^\dagger \cdot D_0 \cdot h \right) \partial_\xi^2 u_1 (X, T, \xi) + \left[ h^\dagger \cdot (\partial_{\alpha} F)_0 \right] \frac{\omega_0}{2\pi} \int_{t}^{t+2\pi/\omega_0} dt' \alpha_2 \exp (-i\omega_0 t') = 0, \tag{33}$$

which making use of (18) becomes

$$c_2 \partial_\xi^2 u_1 (X, T, \xi) + c_4 A (\xi) e^{i\omega_2 T} = 0, \tag{34}$$

where we defined

$$c_2 = h^\dagger \cdot D_0 \cdot h, \tag{35}$$

$$c_4 = h^\dagger \cdot (\partial_{\alpha} F)_0. \tag{36}$$

Condition (34) implies that $u_1$ can be expressed as

$$u_1 (X, T, \xi) = \frac{c_4}{c_2} u_A (\xi) e^{i\omega_2 T} + U_1 (X, T), \tag{37}$$

where $u_A (\xi)$ is the particular solution to

$$\frac{d^2 u_A (\xi)}{d\xi^2} = -A (\xi), \tag{38}$$

and $U_1 (X, T)$ is a yet undetermined function of the slow scales $X$ and $T$. Note that the solvability of Eq. (38) requires that $A$ does not contain a constant term. Hence we assume in the following that

$$\langle A (\xi) \rangle = 0, \tag{39}$$
where the angular brackets denote averaging over the spatial scale $\xi$.

Finally, according to Eq. (26),

$$
\mathbf{u}_2 = u_2(X,T,\xi) \exp (i\omega_0 t) \mathbf{h} + \overline{w}_2(X,T,\xi) \exp (-i\omega_0 t) \overline{\mathbf{h}} \\
+ v_0 |u_1|^2 + v_2 u_1^2 \exp (2i\omega_0 t) + \nabla_2 \overline{u}_1^2 \exp (-2i\omega_0 t) \\
+ [v_A A(\xi) \exp (i\omega_2 T) + (D_0 \cdot \mathbf{h}) \partial_\xi^2 u_1(X,T,\xi)] \exp (i\omega_0 t) \\
+ [\nabla_A \overline{A}(\xi) \exp (-i\omega_2 T) + (D_0 \cdot \overline{\mathbf{h}}) \partial_\xi^2 \overline{u}_1(X,T,\xi)] \exp (-i\omega_0 t) 
$$

(40)

where,

$$
v_0 = -2 J_0^{-1} \cdot \mathbf{K} (\mu_0,0; \mathbf{h},\overline{\mathbf{h}}),
$$

(41)

$$
v_2 = -(J_0 - i2\omega_0 I)^{-1} \cdot \mathbf{K} (\mu_0,0; \mathbf{h},\mathbf{h}),
$$

(42)

$$
v_A = -(J_0 - i\omega_0 I - i\omega_0 \mathbf{h} \otimes \mathbf{h}^\dagger + i\omega_0 \overline{\mathbf{h}} \otimes \overline{\mathbf{h}}^\dagger)^{-1} \cdot (\partial_\alpha \mathbf{F})_0
$$

(43)

are constant vectors, and $I$ is the $N \times N$ identity matrix. Note that both $J_0$ and $(J_0 - i2\omega_0 I)$ are invertible since neither 0 nor $2i\omega_0$ are eigenvalues of $J_0$ by hypothesis: otherwise other eigenvalues different from $\{i\omega_0,-i\omega_0\}$ would have null real part at the bifurcation. For the same reason $(J_0 - i\omega_0 I - i\omega_0 \mathbf{h} \otimes \mathbf{h}^\dagger + i\omega_0 \overline{\mathbf{h}} \otimes \overline{\mathbf{h}}^\dagger)$ is invertible too since $i\omega_0$ is not an eigenvalue of $(J_0 - i\omega_0 \mathbf{h} \otimes \mathbf{h}^\dagger + i\omega_0 \overline{\mathbf{h}} \otimes \overline{\mathbf{h}}^\dagger)$ because we removed the subspaces spanned by $\{\mathbf{h},\overline{\mathbf{h}}\}$.

D. Order $\varepsilon^{5/2}$

At this order we get

$$
\mathbf{g}_{5/2} = 2\partial_\xi \partial_X \mathbf{u}_1 + \partial_\xi^2 \mathbf{u}_{3/2} + \mathbf{K} (\mu_0,0; \mathbf{u}_1,\mathbf{u}_{3/2}) = \partial_\xi^2 \mathbf{u}_{3/2} + \mathbf{K} (\mu_0,0; \mathbf{u}_1,\mathbf{u}_{3/2}),
$$

(44)

where the last equality comes from (28). The solvability condition (25) reduces in this case to

$$
\int_t^{t+2\pi/\omega_0} dt' \mathbf{h}^\dagger \cdot \partial_\xi^2 \mathbf{u}_{3/2} \exp (-i\omega_0 t') = 0,
$$

as $\mathbf{K} (\mu_0,0; \mathbf{u}_1,\mathbf{u}_{3/2})$ does not contain terms oscillating as $\exp (\pm i\omega_0 t)$. Making use of (30) we get

$$
\partial_\xi^2 \mathbf{u}_{3/2} (X,T,\xi) = 0.
$$

(45)
Then, either \( u_{3/2} = 0 \) or \( u_{3/2} \) does not depend on \( \xi \). In the following we will consider the more general case,\[ u_{3/2} = u_{3/2} (X, T) \exp (i\omega_0 t) \mathbf{h} + \overline{u}_{3/2} (X, T) \exp (-i\omega_0 t) \overline{\mathbf{h}}. \] (46)

Once the solvability condition has been verified, \( u_{5/2} \) can be written, according to (26), as
\[ u_{5/2} = u_{5/2} (X, T, \xi) \exp (i\omega_0 t) \mathbf{h} + \overline{u}_{5/2} (X, T, \xi) \exp (-i\omega_0 t) \overline{\mathbf{h}} + u_{5/2} (X, T, \xi, t), \] (47)
where the last term can be computed easily but we will not do as we need not knowing its expression.

E. Order \( \varepsilon^3 \)

This is the final order to be considered as it provides the sought equation for the small amplitude of oscillations. At this order we find\[ g_3 = -\partial_T u_1 + \mu_2 (\partial_n \mathcal{J})_0 \cdot u_1 + \mathbf{D}_0 \cdot \partial_X^2 u_1 + 2\mathbf{D}_0 \cdot \partial_X \partial_\xi u_{3/2} + \mathbf{D}_0 \cdot \partial_\xi^2 u_2 + 2\mathbf{K} (\mu_0, 0; u_1, u_2) + \mathbf{K} (\mu_0, 0; u_{3/2}, u_{3/2}) + \mathbf{L} (\mu_0, 0; u_1, u_1, u_1). \] (48)

Note in the previous expression that \( \partial_X \partial_\xi u_{3/2} = 0 \) according to (46). Application of the solvability condition (25) yields, after substituting Eqs. (28), (46) and (40) into Eq. (48), making use of the symmetry and linearity properties of vectors \( \mathbf{K} \) and \( \mathbf{L} \), and after simple but tedious algebra,
\[ \partial_T u_1 = c_1 \mu_2 u_1 + c_2 \partial_X^2 u_1 + c_3 |u_1|^2 u_1 + c_4 \partial_\xi^2 u_2 + c_4 \partial_\xi^2 A (\xi) e^{i\omega_2 T} + c_5 \partial_\xi^4 u_1, \] (49)
where

\[ c_1 = h^\dagger \cdot (\partial_\mu J)_0 \cdot h, \tag{50a} \]
\[ c_2 = h^\dagger \cdot D_0 \cdot h, \tag{50b} \]
\[ c_3 = 2h^\dagger \cdot K (\mu_0, 0; h, v_0) + 2h^\dagger \cdot K (\mu_0, 0; \overline{h}, v_2) + 3h^\dagger \cdot L (\mu_0, 0; h, h, \overline{h}), \tag{50c} \]
\[ c_4 = h^\dagger \cdot (\partial_\alpha F)_0, \tag{50d} \]
\[ c_A = h^\dagger \cdot D_0 \cdot v_A, \tag{50e} \]
\[ c_5 = h^\dagger \cdot D_0 \cdot D_0 \cdot h, \tag{50f} \]

are constant coefficients.

VI. THE COMPLEX GINZBURG-LANDAU EQUATION

There remains just substituting (37) into (49). Once this is done, one can see that in Eq. (49) there are terms depending on the slow spatial scale \( X \) as well as terms depending on the "fast" spatial scale \( \xi \) so that Eq. (49) can be split into two equations: one containing just functions of \( X \) alone, and one containing functions of \( \xi \) (and possibly \( X \)). The terms depending on \( \xi \) determine partially the value of \( u_2 \). The terms depending only on \( X \) determine the equation of motion for \( U_1 \), see Eq. (37), which is the leading order amplitude of oscillations.

The result is

\[ \partial_T U_1 = c_1 \mu_2 U_1 + c_2 \partial_X^2 U_1 + c_3 |U_1|^2 U_1 + c_3 (c_4/c_2)^2 \gamma e^{2i\omega_2 T} \overline{U_1} + 2c_3 |c_4/c_2|^2 \gamma' U_1, \tag{51} \]
\[ \gamma = \langle |u_A(\xi)|^2 \rangle, \quad \gamma' = \langle |u_A(\xi)|^2 \rangle, \tag{52} \]

where \( \langle \rangle \) denotes a spatial average (over the scale \( \xi \)) as already introduced. In order to arrive to this equation we assumed that \( \langle |u_A(\xi)|^2 u_A(\xi) \rangle = 0 \).

It is convenient to remove the explicit time dependence in (51) by performing the following change,

\[ U = U_1 e^{-i\omega_2 T}, \tag{53} \]

and the equation becomes

\[ \partial_T U = (c_1 \mu_2 + i\omega_2) U + c_2 \partial_X^2 U + c_3 |U|^2 U + c_3 (c_4/c_2)^2 \gamma U + 2c_3 |c_4/c_2|^2 \gamma' U, \tag{54} \]
which is of the type (1) with \( n = 2 \) as anticipated.

Summarizing, for a system like (2) close to a homogeneous Hopf bifurcation, defined as

\[
\mu = \mu_0 + \varepsilon^2 \mu_2, \tag{55}
\]

and under the type of forcing analyzed along this paper, namely

\[
\alpha (x, t) = \varepsilon^2 A (\xi) \exp (i \omega_2 T) \exp (i \omega_0 t) + c.c., \tag{56}
\]

small oscillations emerge in the form

\[
\mathbf{u}_1 = \varepsilon u_1 (X, T, \xi) \exp (i \omega_0 t) \mathbf{h} + \varepsilon \overline{u}_1 (X, T, \xi) \exp (-i \omega_0 t) \overline{\mathbf{h}} + \mathcal{O} (\varepsilon^{3/2}), \tag{57}
\]

where

\[
u_1 = [(c_4/c_2) u_A (\xi) + U (X, T)] e^{i \omega_2 T}, \tag{58}
\]

\( u_A (\xi) \) just follows passively the forcing via

\[
\frac{d^2 u_A (\xi)}{d \xi^2} = -A (\xi), \tag{59}
\]

where the forcing amplitude must verify \( \langle A (\xi) \rangle = 0 \), and \( U \) is governed by (54), in which the different coefficients are defined in (50) and in (52). Finally the validity of (54) requires that \( \langle |u_A (\xi)|^2 u_A (\xi) \rangle = 0 \). Note that all the conditions imposed on \( A \) (or on \( u_A \)) imply that the sign of \( A \) should alternate in space. As an example, all previous conditions on the forcing are met for the simple case \( A (\xi) = A_0 \cos (q \xi) \), in which case \( u_A (\xi) = (A_0/q^2) \cos (q \xi) \), and \( \gamma = \gamma' = \frac{1}{2} (A_0/q^2)^2 \).

Finally, Eq. (54) is valid whenever \( \text{Re} c_3 \leq 0 \) (supercritical bifurcation) since otherwise it would lead to unbounded solutions. If \( \text{Re} c_3 > 0 \) the bifurcation is subcritical and the analysis must incorporate higher orders in the \( \varepsilon \)-expansion.

**VII. A REMARK ON THE VALIDITY OF THE CLASSICAL COMPLEX GINZBURG-LANDAU EQUATION WITH RESONANT FORCING TO THE PRESENT CASE**

Equation (54) is a CGLE with broken phase invariance, because of the occurrence of the term proportional to \( \overline{U} \). This term is typical of self-oscillatory systems forced at twice the natural frequency (2:1 resonance), as stated in the Introduction, but here it has been
obtained for a forcing resonant (1:1 resonant in fact) with the oscillations, whose amplitude is spatially modulated at a “short” spatial scale (shorter than the typical one in the unforced case). Why has this happened?

Coming back to the 1:1 resonance its universal description is given by the CGLE (1) with \( n = 1 \), i.e.

\[
\partial_t u = a_1 u + a_2 \nabla^2 u + a_3 |u|^2 u + a_4.
\]  

(60)

The derivation of this equation requires formally that

\[
\alpha = \varepsilon a_3 = \varepsilon A_3 (X, T) e^{i\omega t} + c.c.
\]  

(61)

of order \( \varepsilon^3 \), not \( \varepsilon^2 \) as we assumed up to here, and then \( a_4 \) is proportional to \( A_3 (X, T) \), see (5). The question is then: Is Eq. (60) valid even when forcing is a bit stronger (of order \( \varepsilon^2 \)) and acts on shorter spatial scales, as we are considering in this paper? In order to make a closer analysis we consider the version of Eq. (60) adapted to our notation, as derived in [5],

\[
\partial_T u_1 = c_1 \mu_2 u_1 + c_2 \partial_X^2 u_1 + c_3 |u_1|^2 u_1 + c_4 A_3 (X, T),
\]  

(62)

where all coefficients have the same meaning as in Eq. (54).

Assume now that \( A_3 \) is ”large” (this does not mean that the actual forcing \( \alpha \) is) and that it varies on a ”short” spatial scale \( \xi = \varepsilon^{-1/2} X \), see (17) and (15). In particular we consider formally that

\[
A_3 = \varepsilon^{-1} A (\varepsilon^{-1/2} X) e^{i\omega_2 T},
\]  

(63)

where the exponential \( e^{i\omega_2 T} \) has been included in order to consider the same case we have been dealing with, namely that the frequency of forcing is \( \omega_f = \omega_0 + \varepsilon^2 \omega_2 \), see (61) and (18). Note that with scaling (63) we are formally in the same situation as in (18). What we will show next is that, if in Eq. (62) we consider formally the scaling (63) for the forcing, even if, apparently, this scaling is at odds with the validity conditions applicable to that equation, one obtains the very same Eq. (54) we have obtained before. This means that Eq. (62) can be regarded as valid even when forcing is ”strong” and varies on a ”short” spatial scale.

The derivation is reasonably simple: As a new spatial scale has been introduced we assume that \( u_1 \) in (62) can be written as

\[
u_1 (X, T) = u_1^{(0)} (X, T, \xi) + \varepsilon^{1/2} u_1^{(1/2)} (X, T, \xi) + \varepsilon^1 u_1^{(1)} (X, T, \xi) + O (\varepsilon^{3/2}), \]  

(64)
what implies that the Laplacian will act as $\partial^2_X \rightarrow \partial^2_X + 2\varepsilon^{-1/2}\partial_X \partial_\xi + \varepsilon^{-1}\partial^2_\xi$. Then the asymptotic analysis starts. At the leading, $\varepsilon^{-1}$ order we obtain $\partial^2_\xi u_1^{(0)} = -(c_4/c_2) A(\xi) e^{i\omega_2 T}$, what implies that

$$u_1^{(0)}(X, T, \xi) = (c_4/c_2) u_A(\xi) e^{i\omega_2 T} + U_1(X, T), \quad (65)$$

where

$$\frac{d^2 u_A(\xi)}{d\xi^2} = -A(\xi), \quad (66)$$

which are nothing but Eqs. (37) and (38), respectively.

The next, $\varepsilon^{-1/2}$, order reads

$$0 = 2\partial_X \partial_\xi u_1^{(0)}, \quad (67)$$

which is identically fulfilled because of (65). Then, at order $\varepsilon^0$ we get

$$\partial_T u_1^{(0)} = c_1\mu_2 u_1^{(0)} + c_2 \left( \partial^2_X u_1^{(0)} + 2\partial_X \partial_\xi u_1^{(1/2)} + \partial^2_\xi u_1^{(1)} \right) + c_3 \left| u_1^{(0)} \right|^2 u_1^{(0)}. \quad (68)$$

Substitution of (65) leads to

$$\partial_T U_1 - c_1\mu_2 U_1 - c_2 \partial^2_X U_1 - c_3 \left| U_1 \right|^2 U_1 - 2c_3 \left| c_4/c_2 \right|^2 \gamma' U_1 - c_3 \left( c_4/c_2 \right)^2 \gamma e^{2i\omega_2 T} U_1$$

$$= c_3 \left[ 2 \left| c_4/c_2 \right|^2 \left( \left| u_A \right|^2 - \gamma' \right) U_1 + \left( c_4/c_2 \right)^2 \left( u_A^2 - \gamma \right) e^{2i\omega_2 T} U_1 \right]$$

$$+ (c_1\mu_2 - i\omega_2) \left( c_4/c_2 \right) u_A e^{i\omega_2 T} + c_2 \left( 2\partial_X \partial_\xi u_1^{(1/2)} + \partial^2_\xi u_1^{(1)} \right)$$

$$+ c_3 \left[ c_4/c_2 \right]^2 \left( c_4/c_2 \right) \left| u_A \right|^2 U_A e^{i\omega_2 T} + (\overline{c_4/c_2}) U_A e^{-i\omega_2 T} U_1^2 + 2 \left( c_4/c_2 \right) u_A e^{i\omega_2 T} \left| U_1 \right|^2, \quad (69)$$

where $\gamma$ and $\gamma'$ have been defined as in (52). We note that in this equation the left hand side is independent of the short spatial scale $\xi$. On the other hand no term on the right hand side depends solely on $X$ (we assume that $\langle \left| u_A \right|^2 u_A \rangle = 0$ as we have done in the rest of this paper). Then the solution to (69) is simple: equate to zero both sides. Doing it with the right hand side gives a condition on $2\partial_X \partial_\xi u_1^{(1/2)} + \partial^2_\xi u_1^{(1)}$, which we are not interested in. Equating to zero the left hand side yields the sought equation,

$$\partial_T U_1 = c_1\mu_2 U_1 + c_2 \partial^2_X U_1 + c_3 \left| U_1 \right|^2 U_1 + c_3 \left( c_4/c_2 \right)^2 \gamma e^{2i\omega_2 T} U_1 + 2c_3 \left| c_4/c_2 \right|^2 \gamma' U_1, \quad (70)$$

which coincides exactly with (51).

This demonstrates that the usual CGLE describing the 1:1 resonant forcing of self-oscillatory systems, Eq. (62), is valid even when the forcing term is "large" and varies on a short spatial scale. The reason for this is quite easy to understand: Equation (62) is
valid for positive $\mu_2$ (above the bifurcation), for negative $\mu_2$ (below the bifurcation), but even for $\mu_2 = 0$, as is trivial to be checked. Then one should consider that Eq. (62) is valid whichever the value of $\mu_2$ be (it must be however, at most, of order $\varepsilon^0$). Then a trivial rescaling of (62) with $A_3 (X, T) = A(X) e^{i\omega_2 T}$,

$$\tau = \eta T, \quad \xi = \eta^{1/2} X, \quad \psi (\xi, \tau) = \eta^{-1/2} u_1 (X, T), \quad \eta = \mu_2 \text{Re } c_1,$$

leads to

$$\partial_\tau \psi = (1 + i\theta) \psi + c_2 \partial_\xi^2 \psi + c_3 |\psi|^2 \psi + c_4 \eta^{-3/2} \eta^{-1/2} A_3 (\eta^{-1/2} \xi) e^{i\eta^{-1} \omega_2 \tau}, \quad (72)$$

where $\theta = \frac{\text{Im } c_1}{\text{Re } c_1}$. Then, we see that there exists a normalization telling us that, if $\mu_2$ is small (then $\eta$ is too) the forcing term in (62) can be effectively large and vary on a short spatial scale. In this case the detuning $\omega_2$ must be accordingly small, of order $\eta$, so that $\eta^{-1} \omega_2 = O (\eta^0)$, as otherwise the inhomogeneous term is highly nonresonant and produces no effect.

VIII. CONCLUSIONS

Starting from a general, unspecified model for a spatially extended system affected by a homogeneous Hopf bifurcation and forced externally by a perturbation that is resonant in time with the system’s oscillations (1:1 resonance) and is spatially modulated (involving sign alternations) at a scale shorter that the typical one of the unforced system, I have shown that the close to threshold dynamics of the system is governed by a complex Ginzburg-Landau equation with a phase symmetry breaking term, proportional to the complex conjugate of the oscillations amplitude. This term appears in the universal description of the 2:1 resonance of self-oscillatory systems (forcing at twice the system’s natural frequency) and is responsible for the emergence of phase bistability and phase bistable patterns, including phase domain walls, bright solitonic structures and periodic patterns. Thus the kind of forcing put forward in this paper represents an alternative to the classical 2:1 periodic forcing.

Finally I have shown that even the classical complex Ginzburg-Landau equation valid in a 1:1 resonance, which contains an inhomogeneous term (that can vary, in principle, only on long spatial and time scales), is valid to describe the phenomenon analyzed here, i.e., in that equation one can consider that the forcing is "large" and that varies on a "short" spatial scale. Even if not shown here, the same applies if the forcing term varies on a "short"
temporal scale, a case that has been analyzed in the context of ”rocking” [6], a 1:1 resonant forcing technique in which the amplitude of forcing is spatially uniform but varies in time, which has been studied theoretically and experimentally in different contexts [7–10].

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