Can one hear the density of a drum? Weyl’s law for inhomogeneous media

P. Amore

Facultad de Ciencias, CUICBAS, Universidad de Colima - Bernal Díaz del Castillo 340, Colima, México

received 13 July 2010; accepted in final form 24 September 2010
published online 1 November 2010

PACS 02.30.Mv – Approximations and expansions
PACS 11.15.Bt – General properties of perturbation theory
PACS 11.15.Tk – Other nonperturbative techniques

Abstract – We generalize Weyl’s law to inhomogeneous bodies in \( d \) dimensions. Using a perturbation scheme recently obtained by us (Amore P., J. Math. Phys., 51 (2010) 052105), we have derived an explicit formula, which describes the asymptotic behavior of the eigenvalues of the negative Laplacian on a closed \( d \)-dimensional cubic domain, either with Dirichlet or Neumann boundary conditions. For homogeneous bodies, the leading term in our formula reduces to the standard expression for Weyl’s law. We have also used Weyl’s conjecture to obtain a non-perturbative extension of our formula and we have compared our analytical results with the precise numerical results obtained using the Conformal Collocation Method (CCM) (see Amore P., J. Math. Phys., 51 (2010) 052105; J. Phys. A, 41 (2008) 265206).

In this paper we revisit the question posed by Weyl with the help of a density perturbation approach developed in a previous paper, ref. [6], obtaining a formula which holds also for systems of variable density, with Dirichlet or Neumann boundary conditions in two dimensions, for the cases where the density can be obtained from a conformal transformation, our result reproduces the well known dependence of the energies on the reciprocal of the area of the drum.

According to Weyl’s law the largest frequencies of the sound of a uniform drum (membrane) are primarily determined by the area of the drum and not by its shape. The same result also applies to the electromagnetic field inside a waveguide, to the eigenmodes of a particle in a quantum billiard, or in general to the normal modes of the Laplacian operator in a closed domain in \( d \) dimensions, with Dirichlet or Neumann boundary conditions (bc).

Weyl conjectured that the number of modes in a two-dimensional drum of area \( A \) and perimeter \( L \), of energy lower than a given energy \( E \), is

\[ N(E) = \frac{A}{4\pi} E \pm \frac{L}{4\pi} \sqrt{E} + \cdots, \]

where the – and + signs hold for Dirichlet and Neumann bc, respectively. This formula is known as the Weyl conjecture and, if true, it implies

\[ E = \frac{4\pi N}{A} \pm \frac{L}{A\sqrt{4\pi N/A}} + \cdots, \tag{1} \]

which describes quite well the spectrum of a homogeneous drum. This conjecture has also been extended to \( d \) dimensions. Reference [1], which contains a detailed account of Weyl’s law, provides a historical perspective. Additional references on this problem are [2–4]. We also mention ref. [5], where a different attempt of generalization of Weyl’s result to arbitrary dimensions is discussed.

In ref. [7] the inhomogeneous media problem is briefly mentioned in the section concerning “open problems”. Incidentally, the answer to this question is negative as for Kac question, since the counterexamples already known in one case may also be seen as inhomogeneous drums via a conformal transformation.

1E-mail: paolo@ucol.mx

Copyright c⃝ EPLA, 2010
Let us describe our approach. We focus for the moment on two dimensional homogeneous membranes of arbitrary shape. In a recent paper, ref. [6], we have devised a perturbative approach to the calculation of the normal modes of these membranes (or quantum billiards), that involves a conformal map which sends the original shape into a reference shape.

To be more explicit, \( w = f(z) \) is a conformal transformation that maps a region \( \Omega \), which for example could be chosen to be a circle or a rectangle, into the region \( \mathcal{D} \), representing the shape of the drum. Under such transformation the homogeneous Helmholtz equation transforms to an inhomogeneous Helmholtz equation:

\[
\Delta \psi(x, y) + E \Sigma(x, y) \psi(x, y) = 0, \tag{2}
\]

where

\[
\Sigma \equiv \left| \frac{df}{dz} \right|^2. \tag{3}
\]

We will refer to \( \Sigma \) as the conformal density. Notice that \( \Sigma \) could also be interpreted from the start as a physical density, although the reverse is not necessarily true, as we shall soon see. Notice that the operator corresponding to eq. (2) fits in the class of problems which can be treated with the heat-kernel expansion (see eq. (2.1) of ref. [4] with suitable identifications). The procedure that we will describe here, however, is not equivalent to a heat-kernel expansion on the inhomogeneous operator, as we shall soon see.

If we have in mind shapes which are obtained by small deformations of the reference shape \( \Omega \), we may express \( \Sigma = 1 + \sigma \), where \( \sigma \) is the perturbation density generated by the mapping. In ref. [6] we have obtained an explicit form for the perturbative corrections to the energy up to third order in \( \sigma \):

\[
E_n^{(0)} = \epsilon_n, \tag{4}
\]

\[
E_n^{(1)} = -\epsilon_n \langle n|\sigma|n \rangle, \tag{5}
\]

\[
E_n^{(2)} = \epsilon_n \langle n|\sigma|n \rangle^2 + \epsilon_n^2 \sum_{k \neq n} \frac{\langle n|\sigma|k \rangle^2}{\omega_{nk}}, \tag{6}
\]

\[
E_n^{(3)} = -\epsilon_n \langle n|\sigma|n \rangle^3 + \epsilon_n^3 \sum_{k \neq n} \sum_{m \neq n} \frac{\langle n|\sigma|k \rangle \langle k|\sigma|m \rangle \langle m|\sigma|n \rangle}{\omega_{nk} \omega_{nm}}. \tag{7}
\]

Clearly, \( \epsilon_n \) and \( \langle n \rangle \) are the exact eigenvalues and eigenstates on \( \Omega \) (\( n \) is the set of quantum numbers which define the state). We have defined \( \omega_{nk} \equiv \epsilon_n - \epsilon_k \).

As we have observed in ref. [6] the first terms appearing in each of the equations above correspond to the terms of a geometric series:

\[
\epsilon_n \left( 1 - \langle n|\sigma|n \rangle + \langle n|\sigma|n \rangle^2 + \langle n|\sigma|n \rangle^3 + \cdots \right),
\]

which can be resummed as

\[
E_n \approx \frac{\epsilon_n}{1 + \langle n|\sigma|n \rangle} = \frac{\epsilon_n}{\langle n|\Sigma|n \rangle}. \tag{8}
\]

It is important to realize that, although the interpretation of \( \Sigma \) as a conformal density is limited to two-dimensional problems, the perturbative scheme of ref. [6] is general and it can be applied to a problem in \( d \) dimensions, where \( \Sigma \) is a physical density. In what follows we will therefore work in \( d \) dimensions, assuming \( \Omega_d \) to be a \( d \)-dimensional cube of side \( 2L \) centered in the origin and \( \Sigma \) a physical density inside \( \Omega_d \).

The eigenfunctions of \( \Omega_d \) corresponding to Dirichlet and Neumann boundary conditions are obtained with the direct product of the functions on each orthogonal direction. We have

\[
\Psi_{n_x}^{(D)}(x) = \frac{1}{\sqrt{L}} \sin \left( \frac{n_x \pi x}{2L} \right) \tag{9}
\]

for the Dirichlet modes \( n_x = 1, 2, \ldots \), and

\[
\Psi_{n_x}^{(N)}(x) = \frac{1}{\sqrt{L}} \cos \left( \frac{n_x \pi x}{2L} \right) \tag{10}
\]

for the Neumann modes (this expression holds for \( n_x = 1, 2, \ldots \); for \( n_x = 0 \), \( \Psi_0^{(N)}(x) = 1/\sqrt{2L} \)).

Let us now go back to eq. (8); we first concentrate on the matrix elements \( \langle n|\Sigma|n \rangle \), and look for a suitable approximation for \( n \to \infty \). As we have already mentioned here \( n \) stands for the full set of quantum numbers specifying the states, i.e. \( n = (n_{x_1}, \ldots, n_{x_d}) \).

Therefore

\[
\langle n|\Sigma|n \rangle = \int_{\Omega_d} \Psi_{n_{x_1}}^{(D,N)}(x_1) \cdots \Psi_{n_{x_d}}^{(D,N)}(x_d) \Sigma(x_1, \ldots, x_d) \, d\Omega_d,
\]

where \( d\Omega_d \equiv dx_1 \cdots dx_d \). For \( n_{x_i} \to \infty \) one may approximate the highly oscillatory functions with their average value, \( \Psi_{n_{x_i}}^{(D,N)}(x_i) \approx 1/2L \), and therefore write

\[
\langle n|\Sigma^{(D,N)}|n \rangle \approx \frac{1}{(2L)^d} \int_{\Omega_d} \Sigma(x_1, \ldots, x_d) \, d\Omega_d + \delta \Sigma_n. \tag{11}
\]

We have estimated that the contribution of \( \delta \Sigma_n \) can be neglected for the present calculation. To prove this statement let us focus on the matrix element for Dirichlet boundary conditions (the Neumann case can be treated similarly):

\[
\langle n|\Sigma^{(D)}|n \rangle = \frac{1}{(2L)^d} \int_{\Omega_d} \left( 1 - \sum_{i=1}^{d} \cos \left( \frac{n_i \pi(x_i + L)}{L} \right) \right) - \sum_{i \neq j=1}^{d} \cos \left( \frac{n_i \pi(x_i + L)}{L} \right) \cos \left( \frac{n_j \pi(x_j + L)}{L} \right) + \cdots \times \Sigma(x_1, \ldots, x_d). \tag{12}
\]
Clearly, the first term in the parenthesis generates the leading contribution discussed above; the second term, on the other hand, can be manipulated with repeated integrations by parts leading to surface terms involving derivatives of \( \Sigma \) evaluated on all the sides of the hypercube:

\[
-\frac{1}{(2L)^d} \int d\Omega \sum_{i=1}^d \cos \left( \frac{n_i \pi (x_i + L)}{L} \right) \Sigma(x_1, \ldots, x_d) = \]

\[
\frac{L^2}{\pi^2(2L)^d} \sum_{i=1}^d \frac{1}{n_i^2} \int d\Omega^{(i)}_d \left[ \frac{\partial \Sigma(x_1, \ldots, +L, \ldots, x_d)}{\partial x_i} \right] + \ldots,
\]

where \( d\Omega^{(i)}_d \equiv dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_d \). Each time that an integration by part is performed a factor \( 1/n_i \) appears, as well as derivatives of the density along the \( i \)-th direction. This holds for all the terms appearing in eq. (12), and therefore we may conclude that, for \( n \to \infty \), the leading behaviour is the one shown in eq. (11). We now would like to relate the energy of a state in \( \Omega_d \), with quantum numbers \( n_1, \ldots, n_d \), to the number of states with equal or lower energy \( N(\epsilon) \). As shown in ref. [9] the number of states with Dirichlet bc of energy less than \( \epsilon_{n_1}, \ldots, n_d \) may be written as

\[
N^{(D)}(\epsilon) = \frac{1}{(4\pi)^{d/2}} \frac{(2L)^d}{\Gamma(d/2 + 1)} \epsilon^{d/2}
- \frac{(2L)^{d-1} d \sqrt{\pi}}{\Gamma(d/2 + 1/2)} \epsilon^{d/2 - 1/2} + \ldots.
\]

For states obeying Neumann bc we have

\[
N^{(N)}(\epsilon) \approx N^{(D)}(\epsilon) + \frac{2}{(4\pi)^{d/2}} \frac{(2L)^{d-1} d \sqrt{\pi}}{\Gamma(d/2 + 1/2)} \epsilon^{d/2 - 1/2},
\]

where the second term counts the states with one of the quantum numbers vanishing. Substituting these results in eq. (8) we obtain a relation for the energy as a function of \( N \):

\[
E_N^{(D,N)} \approx \frac{\pi}{L^2} \frac{(2L)^d}{\Gamma(d/2 + 1)} \left( \frac{N}{V_d} \right)^{2/d}
\times \left[ 1 \pm \frac{\Gamma\left(\frac{d}{2} + 1\right)}{\Gamma\left(\frac{d}{2} + 1\right)} N^{-1/d} \right] + \ldots,
\]

which is a generalization of Weyl’s law for a \( d \)-cube filled with density \( \Sigma(x_1, \ldots, x_d) \). The + and − signs hold for Dirichlet and Neumann bc, respectively.

We would like to point out that the procedure that we have described is not equivalent to a heat-kernel expansion carried out on the inhomogeneous equation: as a matter of fact the counting function used in our calculation is the one of an homogeneous hypercube, given in ref. [9]. It is eq. (8) which allows to express \( N \) directly in terms of the energies of the inhomogeneous domain.

It is interesting to check some particular limits of this expression. For example, for \( \Sigma = 1 \), one should recover the standard form of Weyl’s law for an homogeneous \( d \)-cube: in this case the first term of eq. (15) reduces to the correct expression

\[
E_N^{(D,N)} \approx 4\pi \frac{\Gamma(d/2 + 1) N^{2/d}}{V_d},
\]

where \( V_d = (2L)^d \) is the volume of the \( d \)-cube.

Let us now focus on the case \( d = 2 \). In this case we have that

\[
E_N^{(D,N)} \approx \frac{4\pi N}{\int_\Omega \Sigma(x, y) \, dx \, dy} \pm \frac{8\sqrt{\pi} N}{\int_\Omega \Sigma(x, y) \, dx \, dy} + \ldots
\]

where \( \int_\Omega \Sigma(x, y) \, dx \, dy = V \) and \( \int_\Omega \Sigma(x, y) \, dx \, dy \approx 4 \) (the ratio of the perimeter to square root of the area for a square), from which eq. (17) follows.

We will now use both eqs. (1) and (17) to study two-dimensional drums of arbitrary shape and density. The transformed Helmholtz equation now reads

\[
\Delta \psi(x, y) + E \Sigma(x, y) \rho(u, v) \psi(x, y) = 0,
\]

where \( \rho(u, v) \) is the physical density of the drum and \( \psi(x, y) = (Re f(z), Im f(z)) \). Therefore eqs. (1) and (17) must now be used substituting \( \Sigma(x, y) \) with \( \Sigma(x, y) \equiv \Sigma(x, y) \rho(u, v) \).

We have tested eqs. (17) and (18) on a cardioid drum with density \( \rho(u, v) = 1/(1 + 4(u^2 + v^2)) \). In this case we have that

\[
L \equiv \int_{\partial \Omega} \Sigma^{1/2}(x, y) \, ds \approx 3.00112,
\]

\[
A \equiv \int_{\Omega} \Sigma(x, y) \, dx \, dy \approx 1.21205.
\]
Notice that \( L \) and \( A \) do not have a geometric interpretation of perimeter and area of a drum since the ratio \( L/A = 2.48 \) is smaller than the corresponding ratio between circumference and area of a circle with area \( \tilde{A} \), \( 2/\sqrt{\pi \tilde{A}} = 3.22 \). Therefore \( \tilde{\Sigma}(x, y) \) cannot be obtained from a conformal transformation.

An independent confirmation of this observation comes from the Payne-Polya-Weinberger conjecture [11,12] (later proved by Ashbaugh and Benguria in ref. [13]), according to which the ratio between the first two eigenvalues of the Dirichlet Laplacian is maximal for the circle, i.e.

\[
\frac{E_2^{(D)}}{E_1^{(D)}} \leq \left. \frac{E_2}{E_1} \right|_{disk} = \left( \frac{j_{1,1}}{j_{0,1}} \right)^2 \approx 2.539, 
\]

where \( j_{0,1} \) and \( j_{1,1} \) are the first positive zeros of the Bessel functions \( J_0(x) \) and \( J_1(x) \).

Using the Conformal Collocation Method (CCM) with a grid of 99 points in each direction we have found for the inhomogeneous Helmholtz equation with density \( \rho(u, v) = 1/(1 + 4(u^2 + v^2)) \) the values

\[
E_1^{(D)} = 10.6769, \quad E_2^{(D)} = 29.7008, 
\]

corresponding to a ratio \( r = E_2^{(D)}/E_1^{(D)} \approx 2.78 \). Since this ratio violates the theorem proved by Ashbaugh and Benguria, \( \tilde{\Sigma} \) cannot be interpreted as a conformal density. In other words, it is not possible to build an homogeneous drum of appropriate shape so that it sounds precisely as this inhomogeneous drum.

As we see in fig. 1, eq. (18) describes quite precisely the behavior of the energies of this inhomogeneous drum. In particular from the left plot we learn that \((E_N^{(D)} + E_N^{(N)})/2\) essentially behaves as the first term in eq. (18), i.e. Weyl’s law for inhomogeneous drums.

For bodies of dimension \( d \geq 3 \), our formulas only apply at present to cubical shapes of arbitrary density, since it is not clear to the author if a generalization of a conformal transformation to higher dimensions exists. When there is an angle-preserving map which relates an arbitrary \( d \)-dimensional region to a \( d \)-cube, the formula obtained in the present paper applies.

The results obtained here also allow to give a sense to the question contained in the title: after the discovery made by Gordon, Webb and Wolpert in 1992, ref. [14], of two different homogeneous drums which are isospectral, it is now known that the answer to Kac’s question is negative. In the present case, it makes sense to wonder if there are genuinely inhomogeneous drums (which cannot be reduced to homogeneous drums one by a conformal transformation), which are isospectral although corresponding to different (inequivalent) domains and/or densities.

We conclude by observing that an improvement of the analytical formulas contained in this paper necessarily involves taking into account the terms in the perturbation expansion which have been neglected here and keeping the subleading terms in the expansion of the matrix elements of \( \Sigma \). As these terms involve sums over internal states, we expect them to be more difficult to evaluate; moreover, in this case the possible degeneration of the levels must also be taken into account, as discussed in ref. [6]. In a recent paper, ref. [15], we have shown that these contributions cannot be neglected in one dimension for a string of variable density and that the asymptotic law obtained in this case is different from the Weyl’s law in higher dimensions and coincides with the leading WKB result. The study of these terms in higher dimensions may therefore be an interesting topic for future work.
**

It is a pleasure to thank Prof. A. Aranda and Prof. G. Strang for reading the manuscript. I acknowledge support of Conacyt through the SNI fellowship.

REFERENCES

[1] ARENDT W., NITTKA R., PETER W. and STEINER F., Mathematical Analysis of Evolution, Information, and Complexity, edited by ARENDT W. and SCHLEICH W. P. (Wiley) 2009.
[2] BORDA M. et al., Advances in the Casimir Effect (Oxford University Press) 2009.
[3] KIRSTEN K., Spectral Functions in Mathematics and Physics (Chapman and Hall) 2001.
[4] VASSILEVICH D. V., Phys. Rep., 388 (2003) 279.
[5] DAI W. S. and XIE M., J. Math. Phys., 48 (2007) 123302.
[6] AMORE P., J. Math. Phys., 51 (2010) 052105.
[7] BALTES H. P. and HILF E. R., Spectra of Finite Systems: A Review of Weyl’s Problem: the Eigenvalues Distribution of the Wave Equation for Finite Domains and its Application to Small Systems (Bibliographisches Institut Wissenschaftsverlag, Mannheim) 1976.
[8] KAC M., Am. Math. Mon., 73 (1966) 1.
[9] DAI W. S. and XIE M., JHEP, 02 (2009) 033.
[10] AMORE P., J. Phys. A, 41 (1991) 265206.
[11] PAYNE L. E., POLYA G. and WEINBERGER H. F., C.R. Acad. Sci. Paris, 241 (1955) 917.
[12] PAYNE L. E., POLYA G. and WEINBERGER H. F., J. Math. Phys., 35 (1956) 289.
[13] ASHBAUGH M. S. and BENGURIA R. D., Bull. Am. Math. Soc., 25 (1991) 19.
[14] GORDON C., WEBB D. and WOLPERT S., Invent. Math., 110 (1992) 1.
[15] AMORE P., Ann. Phys. (N.Y.), 325 (2010) 2679.