Right-Handed Neutrino Production at Finite Temperature:
Radiative Corrections, Soft and Collinear Divergences

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Abstract

The production and decay rate of massive sterile neutrinos at finite temperature receives next-to-leading order corrections from the gauge interactions of lepton and Higgs doublets. Using the Closed-Time-Path approach, we demonstrate that the perturbatively obtained inclusive rate is finite. For this purpose, we show that soft, collinear and Bose divergences cancel when adding the tree-level rates from $1 \leftrightarrow 3$ and $2 \leftrightarrow 2$ processes to vertex and wave-function corrections to $1 \leftrightarrow 2$ processes. These results hold for a general momentum of the sterile neutrino with respect to the plasma frame. Moreover, they do not rely on non-relativistic approximations, such that the full quantum-statistical effects are accounted for to the given order in perturbation theory. While the neutrino production rate is of relevance for Leptogenesis, the proposed methods may as well be suitable for application to a more general class of relativistic transport phenomena.
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1 Introduction

Scattering processes between particles at finite temperature play an important role in Early Universe Cosmology in view of applications such as Baryogenesis and the production of Dark Matter. Besides, certain properties of hot and dense strongly interacting matter, such as the photon production rate and transport coefficients rely on the knowledge of these scattering rates.

It turns out useful to distinguish between the scatterings of four massless particles, which are at leading order restricted to $2 \leftrightarrow 2$ processes and the case where one of these particles is massive, which opens up the phase space for $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$ processes as well. In the case of the scattering of massless particles only, an important complication can arise from the $t$-channel exchange of a massless particle, leading to a Coulomb divergence at tree level. This divergence is mitigated by the screening in the plasma, and technically, it can be removed by a resummation of thermal self-energies within the propagator of the mediating particle $[1, 2]$. When one of the scattering partners is massive, this divergence no longer occurs, but instead, there are soft and collinear divergences.

The soft and collinear divergences are of the same type as $e.g.$ those familiar from QCD corrections to the quark pair production from an off-shell photon. While the production rate for quark pairs and a soft or collinear gluon becomes non-perturbative below a certain transversal momentum scale between the gluon and the emitting quark, the inclusive cross section for pair production with and without gluons is perturbatively well defined, due to the cancellation of soft and collinear divergences between tree-level and loop diagrams. In the following, we refer to the soft and collinear divergences collectively as infrared (IR) divergences.

It is natural to consider the same problem at finite temperature. Recently, this matter has received attention in the context of Leptogenesis $[3, 4]$. The quantity of interest is the relaxation rate of sterile right-handed neutrinos $N$ toward equilibrium from decays and inverse decays of Higgs bosons and leptons. When the mass $M$ of $N$ is small compared to the temperature $T$, radiative corrections are of leading importance, because the $1 \leftrightarrow 2$ processes are kinematically suppressed. In the opposite case, $M \gg T$, radiative corrections are subdominant to the leading $1 \leftrightarrow 2$ rates, but yet, it may be interesting to accurately know the size of the next-to-leading order (NLO) corrections. Furthermore, the situation $M \gg T$ corresponds to strong washout, which is in type-I see-saw scenarios perhaps favoured in the light of the observed mass-scale of the active neutrinos. Whether the $1 \leftrightarrow 2$ processes are kinematically suppressed due to $M \ll T$ or not, we refer to the radiative corrections from Standard Model gauge interactions as NLO in this paper. The problem of the NLO corrections to $N$ production for $M \gg T$ has recently been resolved $[3, 4]$, making use of the fact that Maxwell statistics is a good approximation in the non-relativistic regime and that one may assume that the scale of the three momentum of $N$ is much below the temperature $T$. Related to this matter are also works on the Drell-Yan production of dileptons from a strongly interacting plasma $[7, 10]$, and some useful calculational techniques are developed in Ref. $[11, 12]$. 

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where the spectral function of hot Yang-Mills theory is studied.

However, it still remains interesting to demonstrate the cancellation of IR divergences between tree-level and loop corrections at NLO, keeping full account of relativistic effects and of quantum statistics. This is the main result of this work. The proof of the cancellation of the IR divergences that is presented here relies partly on analytically isolating them from the integrals (for the wave-function corrections) and partly on a rearrangement of the integrand such that only integrable singularities remain (for the vertex corrections). Therefore, this proof is already suggestive of a method of numerically calculating the production rate, a task that we will pursue in a separate work.

In order to compute the relaxation rate for $N$ at finite temperature, we choose to use the Closed-Time-Path (CTP) method \cite{13, 14}. In this formulation, Schwinger-Dyson equations can be derived \cite{13, 17} (a particular subset of which are known as the Kadanoff-Baym equations) that describe the time evolution of the Quantum-Field-Theoretical non-equilibrium system. For this reason, the CTP approach has recently been applied to Leptogenesis in different parametric regimes, and various new effects and corrections have been derived \cite{18–34}. In the CTP formulation, the relaxation rate is directly related to the spectral (or anti-hermitian) self energy. As we assume that the fields that participate in gauge interactions are in thermal equilibrium (which applies to all fields within the loop diagrams that contribute to the self energy of $N$ in the present approximations), we notice that the relaxation rate can be obtained from equilibrium field theory as well. Here, we choose the CTP approach because of its connection to non-equilibrium field theory formulations for Leptogenesis. Moreover, the relation of the spectral self energy to the relaxation rate can be easily inferred from the kinetic equations that are derived from the Schwinger-Dyson equations, (cf. Ref. \cite{24}), and finally, the CTP approach provides very efficient Feynman rules that are perhaps easier to apply than certain finite-temperature cutting rules used in Ref. \cite{5}.

The methods that we employ in order to demonstrate the cancellation of IR divergences are very explicit, which has the disadvantage that they rely on a number of technical details on the evaluation of phase-space and loop integrals. On the other hand, the explicit demonstration of the cancellation readily suggests a method of practical evaluation of the neutrino relaxation rate to be pursued in a future work. In order to communicate the main points of this work to a reader who is not interested in the technical details and to give a first overview to a reader who is at least partly interested in these details and their reproduction, we provide in Section 2 a summary of our method and of the cancellation of the IR divergences, that gives reference to the main results that are worked out in the technical Sections. We begin Section 2 with a part containing general prerequisites and remarks about the CTP approach and the neutrino relaxation rate (Section 2.1). As suggested by the different diagrammatic contributions to the relaxation rate, which also lead to different calculational methods, we further present separate parts about wave-function type (Section 2.2) and vertex-type (Section 2.3) corrections. In Section 3, we present the main results for the IR- and ultraviolet-finite neutrino production rate, and the the full calculational details are then presented in Section 4 for the wave-function corrections and in Section 5 for the vertex corrections. We conclude with
2 Summary of the Calculation and Cancellation of Collinear, Soft and Bose Divergences

2.1 Kinetic Equations, Spectral Self Energy and the Relaxation rate for Right-Handed Neutrinos

We are interested in the production rate of right-handed neutrinos $N$, or more precisely, in their relaxation rate toward thermal equilibrium. The self energy for $N$ is given by $\Sigma_N$. At one-loop order, it originates from Yukawa interactions $Y$ (that we choose here to be real) with lepton doublets $\ell$ and Higgs-boson doublets $\phi$. Besides, $\Sigma_N$ receives higher order corrections due to the SU(2) and U(1) gauge interactions in the symmetric Electroweak phase at high temperatures. In particular, the model we consider is given by the Lagrangian

$$L = L_{\text{SM}} + \frac{1}{2} \bar{\psi}_N (i\partial - M) \psi_N - Y \bar{\psi}_\ell \phi^\dagger P_R \psi_N - Y \bar{\psi}_N \phi P_L \psi_\ell,$$  

where $L_{\text{SM}}$ is the Lagrangian of the Standard Model, $\psi_{N,\ell}$ are the spinors associated with $N$ and $\ell$ and $P_{LR} = (1 \mp \gamma^5)/2$. (Group indices of SU(2) and antisymmetric tensors that may be necessary to form invariant products are suppressed throughout this paper.) On the spinor associated with $N$, we impose the Majorana condition $\psi^c_N = \bar{\psi}_N = \psi_N$, where $C$ is the charge-conjugation matrix. The gauge couplings are given by $g_2$ for SU(2) and $g_1$ for U(1), and moreover, we denote by $g_w = 2$ the dimension of the fundamental doublet representation of SU(2). As the Higgs doublet and the lepton doublet have the same Electroweak charges, it is useful to define the recurring factor

$$G = \frac{3}{4} g_2^2 + \frac{1}{4} g_1^2.$$  

Besides, there are sizeable corrections from top-quark loops and self-interactions of the Higgs boson. While we do not include the latter explicitly, their treatment should be straightforward, given the discussion of the wave-function corrections in Sections 2.2 and 4.

Since we assume the leptons, Higgs bosons and gauge bosons to be in thermal equilibrium, the relaxation term in the kinetic equation for the right handed neutrinos is proportional to their deviation from equilibrium [24]:

$$\frac{d}{dt} f_N(k) = \frac{1}{4} \int \frac{d k_0}{2\pi} \text{sign}(k_0) \text{tr} \left[ i \Sigma_N^\gtrless(k) i S_N^\leq(k) - i \Sigma_N^\leq(k) i S_N^\gtrless(k) \right]$$  

$$= -\frac{1}{2} \int \frac{d k_0}{2\pi} \text{sign}(k_0) \text{tr} \left[ k \Sigma_N^A(k) \right] 2\pi \delta(k^2 - m^2_N) \delta f_N(k)$$  

$$= -\frac{1}{2k^3} \text{tr} \left[ k \Sigma_N^A(k) \right] \delta f_N(k),$$  

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Figure 1: The self energy $[\Sigma_N]^{\text{LO}}$.

where $k^0 = \pm \sqrt{k^2 + M^2}$ in the last term and where $\delta f_N(k)$ is the deviation of the distribution function $f_N(k)$ of right-handed neutrinos from the Fermi-Dirac distribution. We assume spatially homogeneous conditions and work in Wigner space, where the two-point functions are Fourier transforms with respect to the relative coordinate. Therefore, there remains a time dependence, but we however suppress the explicit time coordinate in the arguments of the Wigner functions and distribution functions. As we calculate the self energies in this paper assuming thermal equilibrium for the internal propagators, we make the remark that the approach we use here is effectively identical to the real-time formulation of equilibrium field-theory. In Eq. (3), we have made use of the definition of the spectral self-energy $A_{1d}$ and of the KMS relation for equilibrium two-point functions $A_{4}$. Notice that when $\Sigma_N^A(k)$ is known in equilibrium, the KMS relations immediately yield $\Sigma_N^A(k)$ and therefore the relaxation rate.

In order to calculate $\Sigma_N^A(k)$ when the mass $M$ of $N$ is non-zero, we perform a loop expansion. For this expansion to converge, it is crucial that we calculate an inclusive production rate with or without the production or absorption of a real gauge boson, such that IR-divergent contributions from real emissions (2 ↔ 2 scatterings and 1 ↔ 3 decays and inverse decays) and loop corrections (to the 1 ↔ 2 decays and inverse decays) cancel. Note however that the perturbation expansion breaks down in the limit $M \to 0$, because of the $t$-channel exchange of massless fermions [1–3]. We will deal with this matter in a separate publication [35].

The Yukawa coupling between the right-handed neutrino $N$, the lepton doublet $\ell$ and the Higgs doublet $\phi$ gives rise to the leading-order contribution to the self energy of the right-handed neutrino, that is represented diagrammatically in Figure 1,

$$
\left[ i\Sigma_{ab}^N(p) \right]^{\text{LO}} = g_w Y^2 \int \frac{d^4 p}{(2\pi)^4} \left( iS_{ab}(p + k)i\Delta_{ab}(-k) + C[iS_{ba}(-p - k)]t C\dagger i\Delta_{ba}(k) \right),
$$

where $a, b = \pm$ are CTP indices (we follow the conventions used in Ref. [16]) and $t$ stands for a transposition of Dirac matrices. (We indicate the various contributions to $\Sigma_N$ by superscripts on square brackets, for the sake of readability. Notice that we deviate from the conventions of Ref. [16] in the detail that we define the fermionic self-energies as $\Sigma$, while in Ref. [16] the same quantities are defined without a slash.)

\footnote{Here it is understood that $\Sigma$ may involve chiral projection matrices $P_{L,R}$, for instance for left-chiral}
propagators for the lepton and the Higgs fields are given by Eqs. (A5) and (A6). Explicit expressions for $\left[\Sigma^{A}_N(p)\right]^{LO}$ can be found in Ref. [24].

The main purpose of this work is to provide a method for calculating the NLO contributions. These may be categorised in two-particle-reducible wave-function type corrections (superscript WV, Figure 3) and two-particle-irreducible vertex-type corrections (superscript VERT, Figure 4). Due to the $t$-channel divergences, it turns out that the two-particle-reducible contributions diverge for $M \to 0$, and instead, a two-particle-irreducible one-loop diagram with resummed propagators should be calculated [35].

In all our calculations, we denote by

$$p^\mu = (M, 0, 0, 0)$$

(5)

the momentum of the right-handed neutrino in the Centre of Mass System (CMS). The relative motion with respect to the plasma is accounted for by a plasma vector

$$u^\mu = \frac{1}{M}(p^0, \tilde{p})$$,

(6)

where $p^0$ and $\tilde{p}$ are energy and three-momentum of the right-handed neutrino in the plasma frame (where the plasma is at rest). Moreover we note that

$$\text{tr}[p\Sigma^{ab}_N(p)] = \text{tr}[p\Sigma^{ba}_N(-p)]$$,

(7)

which follows from the relation (A8), such that our choice for $p$ with $p^0 > 0$ does not imply a restriction of the results.

### 2.2 Summary of Cancellation of IR Divergences in the Wave-Function Diagrams

The wave-function type contributions can be written as

$$\left[\Sigma^{A}_N(p)\right]^{\text{WV}} = g_w Y^2 \int \frac{d^4k}{(2\pi)^4} \left[ iS^{(1)}_\ell(p+k)i\Delta^>_\phi(-k) + iS^>_\ell(p+k)i\Delta^{(1)>}_\phi(-k) \right] - <\leftrightarrow>, $$

(8)

where $iS^{(1)}_\ell$ and $i\Delta^{(1)}_\phi$ are the one-loop corrections to the propagators, i.e. the self-energies with external legs (cf. Figure 2). The particular diagrams contributing to $[\Sigma_N]^{\text{WV}}$ are therefore two-particle reducible, as apparent from the diagrammatic expression for $[\Sigma_N]^{\text{WV}}$ in Figure 3. It would be a simple matter to use fully resummed propagators rather than the single-loop insertions, however it is interesting to verify that perturbation theory is well defined without resorting to a resummation.

fermions $\Sigma = P_R\gamma^\mu\Sigma_\mu$. 

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The amputated diagrams in this Figure represent the gauge contributions to $\Sigma_\ell$ (first diagram) and $\Pi_\phi$ (the sum of the second and third diagram). When these diagrams are not amputated, they correspond to $iS^{(1)}_\ell$ (first) and $i\Delta^{(1)}_\phi$ (sum of second and third). We refer to the first and second diagram as sunset diagrams, to the third as seagull diagram. Due to the CTP Feynman-rules, the seagull diagram contributes to $\Pi^H_\phi$, but not to $\Pi^A_\phi$.

The rate can be decomposed into a contribution $F$ from gauge-boson radiation from the lepton and a contribution $B$ from gauge-boson radiation from the Higgs boson, such that

$$\text{tr} \left[ \hat{p} \Sigma^A_N(p) \right]_{\text{WV}} = F + B,$$

$$F = g_w Y^2 \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \hat{p} \left( iS^{(1)}_\ell(p + k) i\Delta^{(1)}_\phi(k) - iS^{(1)}_\ell(p + k) i\Delta^{(1)}_\phi(-k) \right) \right],$$

$$B = g_w Y^2 \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \hat{p} \left( iS^{(1)}_\ell(p + k) i\Delta^{(1)}_\phi(k) - iS^{(1)}_\ell(p + k) i\Delta^{(1)}_\phi(-k) \right) \right].$$

The diagrams that represent $[\Sigma_N]_{\text{WV}}$ and its decomposition in $F$ and $B$ are shown in Figure 3.

In order to explain how the cancellation of IR divergences works, we focus on emissions of gauge radiation from the Higgs boson, that is captured by $B$. Radiation from the lepton can be treated along the same lines, but is technically slightly more involved due to the spinor structure. In Section 5 we present all the necessary details on gauge radiation from the lepton. The one-loop correction to the scalar propagator can be written as

$$i\Delta^{(1)}_{\phi}^{<<}(k) = 2\Delta^A_{\phi}(k) f^{<<}_{\phi}(k \cdot u),$$

$$f^{<}_{\phi}(k \cdot u) = f_{\phi}(k \cdot u), \quad f^{>}_{\phi}(k \cdot u) = 1 + f_{\phi}(k \cdot u),$$

where $\Delta^A_{\phi}(k)$ is the one-loop correction to the spectral function, cf. Figure 11 and Eqs. (A1). As we assume that thermal equilibrium is forced by the gauge interactions, throughout this paper, we take for the distributions of Higgs bosons and gauge bosons $f_{\phi}$ and $f_A$ the Bose-Einstein distribution and for the distribution of leptons $f_\ell$ the Fermi-Dirac distribution. Nonetheless, we keep the subscripts $A$, $\phi$ and $\ell$ in order to indicate the origin of the individual distributions.
Figure 3: The self-energy term $[\Sigma_N]^{WV}$. The first and second terms constitute the contribution $\mathcal{F}$, the third to sixth the contribution $\mathcal{B}$ to $\text{tr}[P^{\mathcal{A}}N]^{WV}$.

Now in thermal equilibrium, the one-loop correction to the spectral function can be expressed as \cite{36,37}

$$\Delta^{A(1)}_\phi(k) = \frac{1}{2} \left( ([\Delta^R_\phi(k)]^2 [i\Pi^H_\phi + \Pi^A_\phi] - [\Delta^A_\phi(k)]^2 [i\Pi^H_\phi - \Pi^A_\phi]) \right)$$

(11)

$$= \frac{1}{2} \left( ([\Delta^R_\phi(k)]^2 - [\Delta^A_\phi(k)]^2) i\Pi^H_\phi + \frac{1}{2} ([\Delta^R_\phi(k)]^2 + [\Delta^A_\phi(k)]^2) \Pi^A_\phi \right),$$

where $\Pi^{H,A}_\phi$ are the hermitian and spectral self-energies of the Higgs boson with a gauge boson in the loop. \cite{Cf. Eqs. (A1) for the definition of spectral and hermitian two-point functions.} In principle, it is important to assume here thermal equilibrium, because otherwise, there would also occur the product of a retarded and an advanced propagator, \textit{i.e.}
a pinch singularity, which leads to an ill-defined integral because of a double pole above and below the real axis. However, it is demonstrated in Ref. [36] that a resummation of all loop insertions (in contrast to the single loop insertion performed here) removes the pinch singularities and leaves behind only the equilibrium contributions to the propagator. In Ref. [37], it is shown that when performing the gradient expansion, that occurs in the Wigner space formulation, to all orders, the non-equilibrium part of the propagator takes the intuitively expected resummed form as well. Recently, in Ref. [38], it has been proposed instead to deal with this issue by using a two-momentum representation of the two-point function in contrast to the single-momentum representation in Wigner space.

The retarded and advanced propagators that appear in Eq. (11) are given by

\[ i\Delta_{R,A}^{\phi} = \frac{i}{k^2 \pm i \text{sign} k^0 \varepsilon}, \]

such that in the distributional sense,

\[ [\Delta_{R}^{\phi}(k)]^2 - [\Delta_{A}^{\phi}(k)]^2 = -2\pi i \sum_{\pm} \delta(k^0 \pm |k|) \left( \frac{1}{4k^0^2} \frac{d}{dk^0} - \frac{1}{4k^0^3} \right), \]

\[ [\Delta_{R}^{\phi}(k)]^2 + [\Delta_{A}^{\phi}(k)]^2 = \frac{2}{k^4}. \]

Using above identities together with the expressions for the tree-level propagators (A6,A7), the one-loop corrections due to real and virtual gauge bosons attaching to the Higgs boson are

\[ \mathcal{B} = g_{\omega} Y^2 \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ piS^{\phi}_{\ell}(p + k)i\Delta^{(1)>}_{\phi}(-k) - piS^{\phi}_{\ell}(p + k)i\Delta^{(1)<}_{\phi}(-k) \right] \]

\[ = g_{\omega} Y^2 \int \frac{d^3k}{(2\pi)^3} \sum_{k^0 = -p^0 \pm |k|} p^0 2\Delta^{(1)A}_{\phi}(k) \left[ f_{\ell}((p + k) \cdot u) + f_{\phi}(k \cdot u) \right] \]

\[ = g_{\omega} Y^2 \int \frac{d^3k}{(2\pi)^3} \sum_{k^0 = -p^0 \pm |k|} p^0 \left[ 2\pi \sum_{\pm'} \delta(k^0 \pm' |k|) \left( \frac{1}{4k^0^2} \frac{d}{dk^0} - \frac{1}{4k^0^3} \right) \right] \Pi^H_{\phi}(k) \]

\[ + \frac{2}{k^4} \Pi^A_{\phi}(k) \times \left[ f_{\ell}((p + k) \cdot u) + f_{\phi}(k \cdot u) \right]. \]

In order to identify the IR-divergent contributions, it is useful to split the self-energy into parts that are present for \( T = 0 \) with a superscript vac and parts that vanish for \( T = 0 \) with a superscript \( T \neq 0 \):

\[ \Pi^H_{\phi} = \Pi^{H,A,vac}_{\phi} + \Pi^{H,A,T \neq 0}_{\phi}, \]

The self-energies \( \Pi^H_{\phi} \) and \( \Pi^A_{\phi} \) give rise to two different contributions to \( \mathcal{B} \). To those from \( \Pi^H_{\phi} \), we refer to as wave-function contributions, and to those from \( \Pi^A_{\phi} \) as scattering
contributions. Notice that the scattering contributions include besides the tree-level $2 \leftrightarrow 2$ scattering rates the $1 \leftrightarrow 3$ decays and inverse decays as well.

We now briefly discuss the various contributions to $\mathcal{B}$ resulting from this splitting and explain how the IR divergences present in the particular contributions can be seen to cancel eventually:

- **Finite temperature contributions:** The contributions from $\Pi_{\phi}^{H,T\neq0}$ to Eq. (14) lead to IR-finite integrals. It is well known that the hermitian part of the thermal self-energy $\Pi_{\phi}^{H,T\neq0}$ is finite, cf. the explicit expression (27) below. Therefore, it gives rise to a finite contribution to the integral (14). For the contribution from $\Pi_{\phi}^{A,T\neq0}$, we show that we can express $\Pi_{\phi}^{A,T\neq0}(k) = k^2 h(k)$, where $h(k)$ is a continuous function, cf. Eqs. (36) below. Therefore, the first order singularity at $|k| = k^0$ (or, equivalently, $|k| = p^0/2$) is integrable in the principal value sense.

- **Vacuum wave-function contributions:** It is well-known that the vacuum self-energy $\Pi_{\phi,vac}^{H}$ for a scalar field with a gauge boson and a scalar boson in the loop is IR-divergent. This divergence can be regulated by a fictitious gauge-boson mass $\lambda$. More precisely, the $\lambda$ dependence is $\propto \log \lambda$ and given by Eqs. (24b,25) below.

- **Real gauge boson emission (scatterings) in the vacuum:** On the other hand, $\Pi_{\phi,vac}^{A}(k) \propto k^2$, such that the resulting singularity at $|k| = k^0$ is again first order. Now however, this gives an IR-divergent contribution to the integral, because $\Pi_{\phi,vac}^{A}(k) / k^2$ is not continuous for $k^2 = 0$. Therefore, we introduce for this term the gauge boson mass $\lambda$ as well. The contribution from $\Pi_{\phi,vac}^{A}$ to $\mathcal{B}$ adds to the rate of $1 \rightarrow 3$ particle decays of $N$. Because $\Pi_{\phi,vac}^{A}(k) / k^0$ itself corresponds to a decay rate of a scalar of mass-square $k^2$ into a massless scalar and a gauge boson of mass $\lambda$, there is a kinematic threshold that can be expressed in terms of a Heaviside $\vartheta$-function, cf. Eq. (28) below. As a result, this contribution to the integral has an IR divergence $\propto \log \lambda$ as well. When we perform an integration by parts, we can isolate the IR divergence and compare it with the one from wave-function corrections.

- **Cancellation of IR divergences from wave-function and scattering corrections:** Once the IR divergences are isolated in terms $\propto \log \lambda$ by performing the integral over the $\delta$ function in Eq. (14) (for the virtual contributions from $\Pi_{\phi,vac}^{H}$) or through integration by parts (for the real contributions from $\Pi_{\phi,vac}^{A}$), one can see that the dependences on $\log \lambda$ cancel in the total result. For this purpose, compare Eqs. (33) and (38). This can be viewed as a consequence of the fact that

$$\vartheta(k^2 - \lambda^2) \pi \frac{d}{d \log \lambda^2} \Pi_{\phi,vac}^{H}(k) = \text{sign}(k^0) \Pi_{\phi,vac}^{A}(k) ,$$

which is no accident, because $\Pi_{\phi,vac}^{H}$ and $\Pi_{\phi,vac}^{A}$ are real and imaginary part of the same analytic self-energy.
The contributions from gauge-boson radiation from the lepton are discussed in detail in Section 4.2. For the vacuum contributions to the spectral and hermitian self-energies, the discussion goes along the same lines as for the radiation from the Higgs boson. In addition, there are IR divergences when the thermal part of the hermitian self-energy is evaluated on shell, which are matched by IR-divergent contributions from the thermal part of the spectral self-energy. We note here that these thermal contributions which lead to IR divergences can be identified with the self energies that one obtains in the hard thermal loop (HTL) approximation. Otherwise, the method of demonstrating the cancellation of these divergences is very similar to the one applied to the cancellation of the IR-divergent vacuum-contributions.

2.3 Summary of Cancellation of IR Divergences in the Vertex Diagram

The method relying on the relation (16) between the spectral and hermitian self-energies that we employ for the two-particle-reducible wave-function contributions obviously cannot be applied to the present case of the vertex diagram, Figure 4. One may again use a gauge-boson mass in order to regulate the soft and collinear divergences. Due to the multiple integration boundaries that appear for virtual corrections and real emissions, one yet faces the problem of matching the particular contributions that lead to a cancellation of the IR divergences, i.e., of the dependence on the fictitious gauge-boson mass. This task can be facilitated by transforming the integrals and by arranging and adding the particular integrands in such a manner that there remains a single integrand that manifestly evaluates to a finite answer. This is the method pursued in the present work. One could then either go back to evaluate the particular divergent contributions with a regulating gauge-boson mass, since it is then clear that the dependence on the regulator eventually cancels from the finite result. Alternatively, one may directly integrate the total integrand without introducing an IR regulator, what again leads to a finite result.

In order to accomplish the task of arranging the particular contributions to the vertex-type self-energy into a manifestly IR-finite integral, we focus on the reduced Wightman self-energy $\text{tr}[\pi \Sigma_N^N(\vec{p})]$. This quantity is proportional to the production rate of right-handed neutrinos $\text{tr}[\pi \Sigma_N^<(<\vec{p})]/(4\tilde{p}^0)$ (for $\tilde{p}^0 = \sqrt{\vec{p}^2 + M^2}$) [cf. Eq (3) and the definitions (A1)]. The thermal decay rate $\text{tr}[\pi \Sigma_N^>(<\vec{p})]/(4\tilde{p}^0)$ may be directly inferred from the KMS relation (A4), $i\Sigma_N^>(p) = -e^{\beta p \cdot u} i\Sigma_N^<(<p)$, and the relaxation rate $\text{tr}[\pi \Sigma_N^A(\vec{p})]/(2\tilde{p}^0)$ can then be found using the definitions (A1). Besides, we assume first that $\tilde{p}^0 = M > 0$, while the solution in the case $\tilde{p}^0 < 0$ can be obtained from relation (7).

The CTP expression for the 2-particle-irreducible vertex-type correction to the right-handed neutrino self-energy, represented graphically in Figure 4, is

$$\left[ i\Sigma_N^{ab} \right]_{\text{VERT}}(p) = 2g_w G Y^2 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \sum_{c,d} cd i S^{ac}_{\ell}(k+q) \gamma^\mu i \Delta^{cd}_{\mu
u}(q)$$

$$\times [2(k-p) + q]^\nu i S^{ab}_{\ell}(k)i \Delta^{dh}_{\phi}(-k+p)i \Delta_{\phi}(-k+p-q).$$

12
The self energy term $[\Sigma_N]^{\text{VERT}}$.

Figure 5: Vertex ($[\Sigma_N]^{\text{vert}}$) and scattering ($[\Sigma_N]^{\text{sca}}$) contributions to $[\Sigma_N]^{\text{VERT}}$.

where the tree-level propagators are given by Eqs. (A5, A6, A7).

The reduced correction to the Wightman self-energy $\text{tr} [\not{p} \Sigma_N^<(p)]^{\text{VERT}}$ follows from the $a, b = +, -$ contribution. Furthermore it is useful to shift in the two terms where $c = -$ the momenta as $k \rightarrow k + q$ and $q \rightarrow -q$, to make use of the cyclicity of the trace and of the definitions (A1d, A1e), such that one obtains

$$\text{tr} [\not{p} \Sigma_N^<(p)]^{\text{VERT}} = 4g_w G y^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} [2(k - p) + q] \text{tr} \left[ \not{p} \right]$$

$$\times \left\{ i S^H_\ell (k) \gamma^\mu i S^<_\ell (k + q) i \Delta^<_\phi (-k + p) i \Delta^H_{\mu\nu} (q) i \Delta^H_{\phi} (-k + p - q) \right. $$

$$+ i S^H_\ell (k) \gamma^\mu i S^<_\ell (k + q) i \Delta^<_\phi (-k + p - q) i \Delta^H_{\mu\nu} (-k + p) i \Delta^H_{\phi} (q) $$

$$+ i S^H_\ell (k) \gamma^\mu i S^<_\ell (k + q) i \Delta^<_\phi (-k + p - q) i \Delta^F_{\mu\nu} (-k + p) i \Delta^H_{\phi} (q) $$

$$+ i S^F_\ell (k) \gamma^\mu i S^<_\ell (k + q) i \Delta^<_\phi (-k + p - q) i \Delta^H_{\mu\nu} (-k + p) i \Delta^H_{\phi} (q) \right\} .$$

Notice that the Wightman and statistic propagators (superscripts $<, >, F$) are purely on shell, as they are proportional to an on-shell $\delta$ function, whereas the hermitian propagators (superscript $H$) correspond to principal values. It is therefore easy to see that in this decomposition, the four terms in the integrand correspond to the following rates:

- The first term describes the interference of two tree-level $2 \leftrightarrow 2$ scattering or $1 \leftrightarrow 3$ decay diagrams, where in one of these, the gauge boson radiates from the
lepton, in the other one from the Higgs boson. We denote these contributions collectively as scatterings (even though for $1 \leftrightarrow 3$ processes this term does not strictly apply) with the superscript $\text{sca}$, $\text{tr}[\hat{\Sigma}_N^< (p)]^{\text{sca}}$. Diagrammatically, this contribution corresponds to a cut through the Wightman propagators [those with superscripts $\langle, \rangle$, cf. Eq. (A2)], which are purely on shell, cf. Figure 5.

- The second term yields a correction to the $1 \leftrightarrow 2$ process. It is the interference of the part of the vertex correction where the gauge boson is on shell with the tree-level amplitude. We denote this contribution by $\text{tr}[\hat{\Sigma}_N^< (p)]^{\text{vert}}_1$.

- The third term yields a correction to the $1 \leftrightarrow 2$ process as well. It is the interference of the part of the vertex correction where the Higgs boson is on shell with the tree-level amplitude. We denote this contribution by $\text{tr}[\hat{\Sigma}_N^< (p)]^{\text{vert}}_2$.

- The fourth term yields a correction to the $1 \leftrightarrow 2$ process as well. It is the interference of the part of the vertex correction where the lepton is on shell with the tree-level amplitude. We denote this contribution by $\text{tr}[\hat{\Sigma}_N^< (p)]^{\text{vert}}_3$.

The vertex corrections can be represented by cuts through the Wightman propagators as well, cf. Figure 5. We furthermore emphasise that all of these individual contributions contain the full quantum statistical factors for the on-shell particles.

For all of these four individual contributions, IR divergences can occur whenever the hermitian propagators go on shell. In order to identify the location of the IR-divergent contributions to $\text{tr}[\hat{\Sigma}_N^< (p)]^{\text{sca}}$, it is therefore useful to parametrise the integrals by variables that are proportional to the invariant momentum squares of these propagators. Here, we choose $x = (-k + p - q)^2 / M$ and $y = k^2 / M$ with the momenta as parametrised in Eq. (18). [Cf. Eqs. (77) for the parametrisation in terms of the momenta chosen in Figure 6] In addition, as we are working in a homogeneous finite-temperature background, there eventually remain two non-trivial angular integrations. The kinematic constraints that are forced within the CTP approach by the on-shell delta functions within the Wightman and statistic propagators then imply that the integrand for $\text{tr}[\hat{\Sigma}_N^< (p)]^{\text{sca}}$ has a limited support, which is illustrated in Figure 6. Notice that the resulting sub-domains have simple kinematic interpretations that are given in the Figure caption. The lines where $x = 0$ or $y = 0$ correspond to the location of collinear divergences, while the point where $x = y = 0$ to the soft divergence, where the emission of a zero-momentum gauge boson leads to a Sudakov logarithm square. Because the collinear fringes where $x = 0$ and $y = 0$ are also the boundaries of the integration domain for the real emissions, the singularities from the hermitian propagators cannot be integrated in the principal value sense but lead to logarithmic IR divergences. In addition, there are Bose singularities from the gauge boson distribution at $x = y = 0$ and from the Higgs boson distribution at $(x, y) = (0, M/2)$.

These divergences cancel with IR divergences from the corrections $\text{tr}[\hat{\Sigma}_N^< (p)]^{\text{vert}}_{1,2,3}$. What we demonstrate in this work is that the integrals defining $\text{tr}[\hat{\Sigma}_N^< (p)]^{\text{vert}}_{1,2,3}$ can be
Figure 6: Regions of integration for \(\text{tr}[\psi \Sigma_N^<(p)]^{\text{sca}}\), corresponding to tree-level 1 \(\leftrightarrow\) 3 and 2 \(\leftrightarrow\) 2 processes. The collinear singularities coincide with the solid blue boundaries, \(x = 0\) or \(y = 0\). Region I corresponds to 1 \(\leftrightarrow\) 3 processes, Region II to the interference of two \(t\) channel amplitudes, the lower branch of Region III to the interference of one amplitude with a lepton in the \(s\) channel with one with a Higgs boson in the \(t\) channel and the upper branch of Region III to the interference of one amplitude with a lepton in the \(t\) channel and one with a Higgs boson in the \(s\) channel.

transformed and added to \(\text{tr}[\psi \Sigma_N^<(p)]^{\text{sca}}\) in such a manner, that the resulting total integrand in \(\text{tr}[\psi \Sigma_N^<(p)]^{\text{VERT}}\) has only integrable singularities after all. This procedure is also suggestive of a practical method for numerical integrations or analytical approximations, that can be described as follows:
• Parametrise the integral $\text{tr}[\pi \Sigma_N^{<}(p)]^{\text{sca}}$ in terms of $x, y$ as described above and as illustrated in Figure 6 (Section 5.1).

• Parametrise the integrals $\text{tr}[\pi \Sigma_N^{<}(p)]^{\text{vert}}_{1,2,3}$ by $x$ and $y$ as well, such that these correspond to $1/M$ times the momentum squares of the Higgs and the lepton propagators in the sub-diagrams that correspond to virtual vertex corrections. Add various contributions such that the integration area for the virtual corrections covers the quadrant $x > 0, y > 0$ (Sections 5.2, 5.3 and 5.4).

• For the real corrections, fold Regions II and III (either or both $x, y < 0$) to the quadrant where $x, y > 0$.

• Now add all contributions. The resulting integrand $\mathcal{J}^{\text{total}}(x, y)$, Eq. (116b), only contains integrable singularities (Section 5.5).

• The integral is not yet convergent for large values of $x, y$, what corresponds to an ultraviolet (UV) divergence. We obtain a convergent integral by subtracting a term that corresponds to the vertex correction to the vacuum decays $N \rightarrow \ell \phi$ weighted by the thermal distributions for $\ell$ and $\phi$, $\mathcal{J}^{\text{vert, vac}}$, Eq. (130a). As this contribution is IR divergent, we cancel it by adding a correspondingly weighted rate for $1 \rightarrow 3$ decays, $\mathcal{J}^{\text{sca, vac}}$, Eq. (132). The subtracted contributions must be added again to the final results, but for these particular terms, we can perform the integration over $dx dy$ analytically and thus isolate and renormalise the UV divergence (Section 5.6).

The point stating the presence of integrable singularities only requires a proof, that we present in Section 5 and that at least in parts elucidates how the cancellation of IR divergences works in the finite-temperature background. The various contributions to the integrand obviously factorise into kinematic and quantum statistical terms. We perform an expansion of the kinematic factors and the arguments of the statistical functions that applies to the collinear fringes, where either $|x| \ll M$ or $|y| \ll M$ as well as close to the point of the soft divergence, where both $|x|, |y| \ll M$. Using these expansions, in Section 5.5, we demonstrate the following points:

• On the collinear fringes, where $x = 0$ or $y = 0$, the total integrand takes finite values and is hence integrable, as it is expressed by Eqs. (118) and (119).

• The point $x = y = 0$ requires special care, since the soft gauge-boson singularity coincides with the Bose divergence of the gauge-boson distribution function. We show that the integrand behaves $1/\sqrt{x^2 + y^2}$ for $(x, y) \rightarrow 0$, such that this isolated singularity is integrable as well. A technical point is here that contributions from different regions of the angular integrations must be averaged.

• The divergence from the Higgs boson distribution functions is integrable in the principal value sense.
In order to show the cancellation of the singularities on the collinear fringes, we need to assume that Higgs bosons, leptons and gauge bosons are in thermal equilibrium. As a consequence, the collinear splitting processes are in equilibrium as well, what leads to the detailed balance relations (117), that are essential in order to show that the integrand is finite on the fringes.

3 NLO Neutrino Production Rate

In this Section, we present the final results for the neutrino relaxation rate in terms of manifestly IR- and UV-finite integrals. The explicit expressions for the various terms are given in the subsequent Sections. The total relaxation rate in the plasma frame is [cf. Eq. (3)]

\[
\frac{1}{2\tilde{p}^0} \text{tr} \left[ \hat{\Sigma}^A_N(\tilde{p}) \right] = \frac{1}{2\tilde{p}^0} \left( \text{tr} \left[ \hat{\Sigma}^A_N(\tilde{p}) \right]^{\text{LO}} + \text{tr} \left[ \hat{\Sigma}^A_N(\tilde{p}) \right]^{\text{WV}} + \text{tr} \left[ \hat{\Sigma}^A_N(\tilde{p}) \right]^{\text{VERT}} \right).
\]

(19)

Expressions for the leading order term \left[ \Sigma^A_N(\tilde{p}) \right]^{\text{LO}} can be found in Ref. [24]. The wave-function-type contributions are given by \[2\]

\[
\text{tr} \left[ \hat{\Sigma}^A_N(p) \right]^{\text{WV}} = (B^{\text{wv,vac}} + B^{\text{sca,vac}}) + B^{\text{wv,T} \neq 0} + B^{\text{sca,T} \neq 0}
\]

\[
+ (F^{\text{vac,col,wv}} + F^{\text{vac,col,sca}}) + (F^{\text{HTL,col,wv}} + F^{\text{HTL,col,sca}})
\]

\[
+ F^{\text{vac,fin}} + F^{\text{HTL,fin}} + F^{T \neq 0},
\]

(20)

and where the \( B \) terms correspond to radiation from the Higgs boson and \( F \) to radiation from the fermion. The various wave function, scattering, vacuum, collinear and HTL (sub)contributions are isolated and collected in a way that the IR divergences cancel within each of the parentheses. The UV divergences cancel among the vacuum parts against the counter terms. The expressions for the various \( B \) and \( F \) contributions are given in Section 4.

\[2\]Due to Lorentz covariance: \text{tr} \left[ \hat{\Sigma}^A_N(\tilde{p}) \right] = \text{tr} \left[ \hat{\Sigma}^A_N(p) \right].
The vertex-type contribution to the relaxation rate is given by

\[
\begin{align*}
    f_N^{\text{eq}}(p \cdot u) \text{tr} \left( \mathbb{P}^{\text{A}}_{p_N}(p) \right)^{\text{VERT}} &= -\frac{2g_w G Y^2}{32(2\pi)^4} \int_0^{2\pi} d(\varphi - \psi) \int_{-1}^1 d\cos \vartheta \int_0^\infty dx \int_0^\infty dy J_{\text{UV}}^{\text{total}}(x, y) \\
    &- 2g_w G Y^2 \frac{1}{32(2\pi)^3} \int_{-1}^1 d\cos \vartheta \left( J^{\text{vert, vac}} + J^{\text{sca, vac}} \right) \\
    &- \frac{M^2}{8\pi} Y \delta Y \int_{-1}^1 d\cos \vartheta f_\ell(E_+) f_\phi(E_-),
\end{align*}
\]

with

\[
J_{\text{UV}}^{\text{total}}(x, y) = \frac{1}{4} \left( J^{\text{sca}}(x, y) + J^{\text{sca}}(-x, -y) + J^{\text{sca}}(-x, y) + J^{\text{sca}}(x, -y) \right) + J^{\text{vert}}(x, y) + J^{\text{vert}}_1(x, y) + J^{\text{vert}}_2(x, y) + J^{\text{vert}}_3(x, y)
\]

where the \( J^{\text{sca}}(x, y) \) terms correspond to 2 \( \leftrightarrow \) 2 scatterings and 1 \( \leftrightarrow \) 3 decay and inverse decay processes with the kinematic regions\(^3\) indicated in Figure 6 and \( J^{\text{vert}}(x, y) \), \( J^{\text{vert}}_1(x, y) \) and \( J^{\text{vert}}_2(x, y) \) correspond to vertex corrections with an on-shell gauge boson, Higgs boson and lepton, respectively. The vacuum-type contributions \( J^{\text{sca, vac}}(x, y) \) and \( J^{\text{vert, vac}}(x, y) \) are subtracted within \( J_{\text{UV}}^{\text{total}}(x, y) \) to make it UV finite (in addition to being manifestly IR finite), and they are cancelled by the vertex counter-term \( \delta Y \). The Fermi-Dirac distribution \( f_N^{\text{eq}} \) in the front of Eq. (21) results from using the KMS relation (A4). The explicit expressions for the integrands are presented in Section 5. In addition to the kinematic \( x, y \)-variables, the integrands depend non-trivially also on the angles \( \varphi - \psi \) and \( \rho \) depicting the orientation between loop momenta and the plasma vector \( u \), and \( E_\pm \) are given by Eq. (82).

4 Wave-Function Contributions

4.1 Radiation from the Higgs Boson

We now present the technical details that are omitted in Section 2.2, where a more qualitative overview of the present approach to the regulation and cancellation of the IR divergences is provided.

\(^3\)We have recast the integration area to positive \( x, y > 0 \), hence the explicit minus signs in the arguments of contributions corresponding to regions II and III.
For a massless scalar radiating a gauge boson with the IR-regulating mass \( \lambda \), the vacuum and finite-temperature contributions to the hermitian self-energy are

\[
\Pi_{\phi, \text{vac}}^H(p) = -\frac{G}{2} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} (2\pi)^d \delta^4(p - k - q)(p + k)^2
\]

\[
\times \left[ \text{PV} \frac{1}{k^2} 2\pi \delta(q^2 - \lambda^2) + \text{PV} \frac{1}{q^2 - \lambda^2} 2\pi \delta(k^2) \right]
\]

\[
+ 4G \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - \lambda^2 + i\varepsilon},
\]

\[
\Pi_{\phi, \text{T}\neq 0}^H(p) = -G \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} (2\pi)^d \delta^4(p - k - q)(p + k)^2
\]

\[
\times \left[ \text{PV} \frac{1}{k^2} 2\pi \delta(q^2 - \lambda^2) f_A(|q \cdot u|) + \text{PV} \frac{1}{q^2 - \lambda^2} 2\pi \delta(k^2) f_\phi(|k \cdot u|) \right]
\]

\[
+ 4G \int \frac{d^4k}{(2\pi)^4} 2\pi \delta(k^2 - \lambda^2) f_A(k^0),
\]

where \( G \) as defined by Eq. (2) encompasses the gauge coupling constants and \( u \) is the plasma vector defined in Eq. (6). The first of the integrals for \( \Pi_{\phi, \text{vac}}^H \) and \( \Pi_{\phi, \text{T}\neq 0}^H \) correspond to the sunset diagrams, the second of the integrals to the seagull diagrams (cf. Figure 2).

A possible way of evaluating \( \Pi_{\phi, \text{vac}}^H \) is to go to the frame where \( p = 0 \) and to introduce a momentum cutoff \( \Lambda \). Then,

\[
\frac{\partial \Pi_{\phi, \text{vac}}^H(p^0, 0)}{\partial (p^0)^2} = \frac{G}{4\pi^2} \left( \log \frac{\Lambda}{\lambda} + \log 2 + \frac{5}{8} \right),
\]

\[
\Rightarrow \Pi_{\phi, \text{vac}}^H(p) = \frac{G}{4\pi^2} p^2 \left( \log \frac{\Lambda}{\lambda} + C_1 \right) + C_2 + p^2 \delta Z_\phi + \delta m^2_\phi,
\]

where \( \delta Z_\phi \) and \( \delta m^2_\phi \) are field-strength and mass counterterms. Alternatively, of course, an effectively equivalent result may be obtained using other regularisation procedures, e.g. dimensional regularisation in \( 4 - \epsilon \) dimensions, such that the leading terms in \( p^2 \) are

\[
\Pi_{\phi, \text{vac}}^H(p) = \frac{G}{4\pi^2} p^2 \left( \log \frac{\mu}{\lambda} + \frac{1}{2} \Delta_\epsilon + \frac{1}{2} \right) + p^2 \delta Z_\phi,
\]

where

\[
\Delta_\epsilon = \frac{2}{\epsilon} - \gamma_E + \log(4\pi).
\]

Notice that, as it is well known, the seagull graph vanishes when evaluated using dimensional regularisation. For both regularisation procedures, the UV divergences should
be cancelled by the field-strength renormalisation \( \delta Z \). In the following, we effectively account for this by replacing \( \Lambda \to \bar{\Lambda} \), which takes a finite, renormalisation-scheme dependent value.

For the part that vanishes as we take \( T \to 0 \), we set \( \lambda = 0 \) and obtain

\[
\Pi_{\phi}^{H,T \neq 0}(p) = G \frac{T^2 - p^2}{8} - G \frac{p^2}{4\pi^2 |p|} \times \int d|k| \left[ \log \left| \frac{p^2 - 2|k|^2 + 2|k| |p|}{p^2 - 2|k|^2 - 2|k| |p|} \right| + \log \left| \frac{p^2 + 2|k|^2 + 2|k| |p|}{p^2 + 2|k|^2 - 2|k| |p|} \right| \right] f_B(|k|).
\]

This is IR finite, such that it is indeed justified to take \( \lambda = 0 \) for this contribution.

For the spectral self-energy, we perform the split into vacuum and finite-temperature parts accordingly. Notice that due to the CTP Feynman-rules, the spectral self-energy only receives sunset and no seagull contributions. The zero-temperature part is

\[
\Pi_{\phi}^{A,\text{vac}}(k) = -G \frac{2k^2 - \lambda^2 k^2 - \lambda^2}{16\pi} \vartheta(k^2 - \lambda^2) \text{sign} k^0 \approx -G \frac{1}{8\pi} k^2 \vartheta(k^2 - \lambda^2) \text{sign} k^0,
\]

where for the approximation, we neglect numerator terms of order \( \lambda^2 \). The Heaviside \( \vartheta \)-function occurs due to the mass threshold of the would-be process of a scalar boson of mass square \( k^2 \) decaying into a massless scalar boson and a gauge boson of mass \( \lambda \). (The minus sign in front of the rate is not problematic, because an on-shell scalar particle cannot actually change its mass by radiating a gauge boson when the gauge symmetry is unbroken.) For the finite temperature contribution, we obtain

\[
\Pi_{\phi}^{A,T \neq 0}(k) = \frac{G}{8\pi} \frac{k^2}{|k|} \left( 2|k| - \frac{2}{\beta} \log \frac{1 - e^{\beta k^2/2}}{1 - e^{\beta |k|^2/2}} \right) \quad \text{for } k^2 \geq 0, \quad (29a)
\]

\[
\Pi_{\phi}^{A,T \neq 0}(k) = \frac{G}{8\pi} \frac{k^2}{|k|} \left( 2k^0 - \frac{2}{\beta} \log \frac{1 - e^{\beta k^0/2}}{1 - e^{\beta |k|^2/2}} \right) \quad \text{for } k^2 < 0. \quad (29b)
\]

We aim to calculate the correction \( B \) [defined in Eq. (14)] by gauge-boson emission from the Higgs boson to the relaxation rate of right-handed neutrinos. The term that depends on \( \Pi_{\phi}^{H} \) can be identified as the wave-function correction (superscript \text{wv}), the term depending on \( \Pi_{\phi}^{A} \) as the correction from \( 2 \leftrightarrow 2 \) scatterings and \( 1 \leftrightarrow 3 \) decays and inverse decays, that we collectively denote as scatterings (superscript \text{sca}), such that we may write

\[
B = B^{\text{wv}} + B^{\text{sca}}.
\]

For the following calculations, we choose \( M = p^0 > 0 \), for definiteness, and note that \( B \)
should be odd in \( p^0 \). The wave-function correction is given by

\[
B^{\text{wv}} = g_w Y^2 \int \frac{d^3 k}{(2\pi)^3} \sum_{k^0 = -p^0 \pm |k|} p^0 2\pi \sum_{\pm'} \delta(-p^0 \pm |k| \pm' |k|) \frac{1}{4k^0^2} \tag{31}
\]

\[
\times \left( \frac{\partial}{\partial k^0} - \frac{1}{k^0} \right) \Pi_H^H(k) [f_\ell((p + k) \cdot u) + f_\phi(k \cdot u)] ,
\]

where we have made use of Eq. (13a). Clearly, we obtain only contributions for \( \pm = \pm' = + \) when \( p^0 > 0 \). According to our decomposition above, this can be written as a sum of a term depending on \( \Pi_{H,\text{vac}}^H \) and \( \Pi_{H,T}^H \neq 0 \).

\[
B^{\text{wv}} = B^{\text{wv,vac}} + B^{\text{wv,T} \neq 0} . \tag{32}
\]

The contribution \( B^{\text{wv,T} \neq 0} \) can be evaluated numerically, while \( B^{\text{wv,vac}} \) inherits the IR divergence from \( \Pi_{H,\text{vac}}^H \), Eq. (24b). We can cast it to the form \(|p| = 0\)

\[
B^{\text{wv,vac}} = -g_w Y^2 GM^2 \int d^3 k \left( f_\ell((p + k) \cdot u) + f_\phi(k \cdot u) \right) . \tag{33}
\]

The scattering correction is

\[
B^{\text{sca}} = g_w Y^2 \int \frac{d^3 k}{(2\pi)^3} \sum_{k^0 = -p^0 \pm |k|} 2p^0 (k^2)^2 \Pi_A^A(k) [f_\ell((p + k) \cdot u) + f_\phi(k \cdot u)] . \tag{34}
\]

Following the splitting of \( \Pi_A^A \) into vacuum and \( T \neq 0 \) contributions, we decompose as well

\[
B^{\text{sca}} = B^{\text{sca,vac}} + B^{\text{sca,T} \neq 0} . \tag{35}
\]

First, we show that \( B^{\text{sca,T} \neq 0} \) yields a finite result when \( \lambda \to 0 \). For this purpose, note that for small \(|k^0 - |k||\), one may expand for \( k^2 > 0 \)

\[
\Pi_{A,T}^{A,T \neq 0} (k) = \frac{G}{8\pi} \frac{k^2}{|k|} \left[ 2|k| - \frac{2}{\beta} \log \left| 1 - e^{\beta k^0 + |k|} \right| + \frac{2}{\beta} \log \left( \frac{\beta (k^0 - |k|)}{2} \right) + \frac{k^0 - |k|}{2} + \ldots \right] \tag{36a}
\]

and for \( k^2 < 0 \)

\[
\Pi_{A,T}^{A,T \neq 0} (k) = \frac{G}{8\pi} \frac{k^2}{|k|} \left[ 2k^0 - \frac{2}{\beta} \log \left| 1 - e^{\beta k^0 + |k|} \right| + \frac{2}{\beta} \log \left( \frac{\beta (|k| - k^0)}{2} \right) + \frac{|k| - k^0}{2} + \ldots \right] . \tag{36b}
\]

Therefore, within \( B^{\text{sca,T} \neq 0} \), we can integrate over the singularity at \( k^2 = 0 \) in the principal value sense.
Next, we demonstrate that $B_{\text{sca,vac}}^\text{+}$ depends on $\lambda$ in a way that cancels the $\lambda$ dependence within $B_{\text{wv,vac}}^\text{+}$. For this purpose, we evaluate ($p=0$)

$$B_{\text{sca,vac}}^\text{+} = -g_w Y^2 \int \frac{d^3 k}{(2\pi)^3} \sum_{k^0 = -M^0} \frac{\lambda}{\sqrt{\lambda^2}} \frac{G}{8\pi} \theta(p^0 + 2|k| - \lambda^2) \text{sign} k^0$$

\[ \times [f_t((p + k) \cdot u) + f_\phi(k \cdot u)] =: B_{\text{sca,vac}}^\text{+} + B_{\text{sca,vac}}^-.
\]

While for $B_{\text{sca,vac}}^\text{+}$, we can set $\lambda = 0$ and perform the integral numerically, we proceed with $B_{\text{sca,vac}}^-$. We emphasise that this cancellation works out, because $\Pi_{\text{wv}}^\text{+}$ and $\Pi_{\text{wv}}^\text{+}$ are imaginary and real part of the same analytic self-energy, which implies relation (16).

While these expressions are already defined within the text above, we finally collect the explicit expressions for the various IR-finite contributions:

$$B_{\text{sca,vac}}^- = g_w Y^2 \int \frac{d^3 k}{(2\pi)^3} \sum_{k^0 = -M^0} \frac{2M}{k^0 - M^0} [f_t((p + k) \cdot u) + f_\phi(k \cdot u)]$$

where $k^0 = -M - |k|$, 

$$B_{\text{sca,T} \neq 0} = g_w Y^2 \int \frac{d^3 k}{(2\pi)^3} \sum_{k^0 = -M^0} \frac{2M}{k^0} \Pi_{\text{sca,T} \neq 0}(k) [f_t((p + k) \cdot u) + f_\phi(k \cdot u)]$$

with $\Pi_{\text{sca,T} \neq 0}$ given by Eqs. (29) and

$$B_{\text{wv,T} \neq 0} = g_w Y^2 \int \frac{d^3 k}{(2\pi)^3} 2\pi \delta(M - 2|k|) \frac{G T^2}{4 M^2} \left(1 + \frac{M}{2} \frac{d}{dk^0}\right) [f_t((p + k) \cdot u) + f_\phi(k \cdot u)]$$

where $k^0 = -M + |k| = -M/2$. In the last expression, we have substituted $\Pi_{\text{wv,T} \neq 0}(k)$, Eq. (27) and have made use of the fact that the non-HTL part of $\Pi_{\text{wv,T} \neq 0}(k)$ [i.e. the
integral term in Eq. (27) and its derivative with respect to $k^0$ vanish for $k^2 = 0$. We also note that $B^{wv,T\neq 0}$ corresponds to the phase-space suppression due to the asymptotic thermal Higgs boson mass (note the minus sign from the statistical factors in the square brackets, as $k^0 < 0$). Together with the IR-divergent pieces, these expressions above can be used in order to calculate the the NLO wave-function corrections to the right-handed neutrino production rate summarised in Eq. (20).

4.2 Radiation from the Fermion

We now adapt the same strategy as for radiation from the scalar propagator to radiation from the fermionic one. As a complication, besides the cancellation between scattering and wave-function corrections from vacuum loops, there is also a cancellation between scattering and wave-function hard thermal loop (HTL) contributions.

We express the leptonic self-energies as \( \Sigma = P_R \gamma^\mu \Sigma_{\mu} \). In the approximation of massless particles in the loop, the spectral self-energy is given by

\[
\Sigma^{A0}_\ell(k) = \frac{GT^2}{8\pi|k|} I_1 \left( \frac{k^0}{T}, \frac{|k|}{T} \right),
\]

\[
\Sigma^{Ai}_\ell(k) = \frac{GT^2}{8\pi|k|} \left[ \frac{k^0}{|k|} I_1 \left( \frac{k^0}{T}, \frac{|k|}{T} \right) - \frac{(k^0)^2 - k^2}{2|k|T} I_0 \left( \frac{k^0}{T}, \frac{|k|}{T} \right) \right] \frac{k^i}{|k|},
\]

where

\[
I_0(y^0, y) = I^{vac}_0(y^0, y) + I^{T\neq 0}_0(y^0, y),
\]

\[
I^{T\neq 0}_0(y^0, y) = -\vartheta((y^0)^2 - y^2) y - \vartheta((y^0)^2 - y^2) y \text{sign}(y^0)
\]

\[
+ \log \left| \frac{1 + e^{\frac{1}{2}((y^0)^2 - y^2)}}{1 - e^{\frac{1}{2}((y^0)^2 - y^2)}} \right| + \log \left| \frac{1 - e^{\frac{1}{2}((y^0)^2 - y^2)}}{1 + e^{\frac{1}{2}((y^0)^2 - y^2)}} \right|
\]

\[
I^{vac}_0(y^0, y) = \vartheta((y^0)^2 - y^2) y \text{sign}(y^0),
\]

\[
I_1(y^0, y) = I^{vac}_1(y^0, y) + I^{T\neq 0}_1(y^0, y) + I^{HTL}_1(y^0, y),
\]

\[
I^{T\neq 0}_1(y^0, y) = -\vartheta((y^0)^2 - y^2) \frac{((y^0)^2 - y^2)^2}{2} - \frac{1}{2} \vartheta((y^0)^2 - y^2) |y^0|^2 y
\]

\[
+ \text{Re} \left[ x(\log(1 + e^x) - \log(1 - e^{-x}y^0)) + \text{Li}_2(-e^x) - \text{Li}_2(e^{-x}y^0) \right]_{x=\frac{1}{2}((y^0)^2 - y^2)},
\]

\[
I^{HTL}_1(y^0, y) = \vartheta(y^2 - (y^0)^2) \frac{\pi^2}{2},
\]

\[
I^{vac}_1(y^0, y) = \frac{1}{2} \vartheta((y^0)^2 - y^2) |y^0|^2 y.
\]

The bar on $I^{T\neq 0}_1(y^0, y)$ indicates that from this finite temperature term, the HTL contribution, which is purely thermal as well, is subtracted.
To this end, we also need the HTL-type contribution to the hermitian self-energy

$$\Sigma^{H,HTL_0}_\ell = \frac{G T^2}{16|k|} \log \left| \frac{k^0 + |k|}{k^0 - |k|} \right|, \quad (44a)$$

$$\Sigma^{H,HTL_i}_\ell = \frac{G T^2 k^0 k^i}{16|k|^3} \log \left| \frac{k^0 + |k|}{k^0 - |k|} \right| - \frac{G T^2 k^i}{8|k|^2}, \quad (44b)$$

and the vacuum contribution

$$\Sigma^{H,\text{vac}}_\ell (k) = \frac{G}{8\pi^2} P_R k \log \frac{\lambda}{\Lambda} + k \delta Z_\ell, \quad (45)$$

or, in dimensional regularisation,

$$\Sigma^{H,\text{vac}}_\ell (k) = \frac{G}{8\pi^2} P_R k \left( \log \frac{\lambda}{\mu} - \frac{1}{2} \Delta_\epsilon \right) + k \delta Z_\ell. \quad (46)$$

The term $\delta Z_\ell$ is a field-strength renormalisation that cancels the divergences as $\Lambda \to \infty$ or $\epsilon \to 0$, but is scheme dependent otherwise. In the following, we again implement its effect by replacing $\Lambda \to \bar{\Lambda}$, where $\bar{\Lambda}$ is finite. The non-HTL contribution to the finite-temperature hermitian self-energy $\Sigma^{H,T\neq 0}_\ell$ is IR finite and can be expressed in terms of a one-dimensional integral, a classic result that can be found in Ref. [39].

The leading correction to the fermionic spectral function is

$$S^{(1),A}_\ell = -\frac{1}{2} \left( i S^{(0)R}_\ell i S^{(0)A}_\ell - i S^{(0)A}_\ell i S^{(0)A}_\ell \right) \quad (47)$$

$$= \frac{1}{2i} \left( i S^{(0)R}_\ell \Sigma^{H}_\ell i S^{(0)R}_\ell - i S^{(0)A}_\ell \Sigma^{H}_\ell i S^{(0)A}_\ell \right)$$

$$- \frac{1}{2} \left( i S^{(0)R}_\ell \Sigma^{H}_\ell i S^{(0)R}_\ell + i S^{(0)A}_\ell \Sigma^{H}_\ell i S^{(0)A}_\ell \right),$$

and the corrections to the Wightman functions are

$$i S^{(1)<}_\ell = 2 S^{(1),A}_\ell (-f_\ell), \quad (48a)$$

$$i S^{(1)>}_\ell = 2 S^{(1),A}_\ell (1 - f_\ell). \quad (48b)$$

In the following, we attach the same superscripts as for the functions $I_{1,2}$ to various quantities, in order to indicate which loop terms they originate from.
The correction to the neutrino relaxation rate follows from Eq. (9b)

\[ F = \text{tr} \left[ \hat{p} \Sigma^{A}_\ell(p) \right] = g_A Y^2 \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \hat{p}(p) \Sigma^{(1)}_{\ell}(k) i \Delta_\phi^>(p-k) - \hat{p}(p) \Sigma^{(1)}_{\ell}(k) i \Delta_\phi^<(p-k) \right] \]

\[ = g_A Y^2 \int \frac{d^3k}{(2\pi)^3} \sum_{k_0 = p^0 \pm |k_0|} \frac{\mp 1}{2|p^0 - k_0|} \text{tr} \left[ \hat{p} S^{(1),A}_\ell(k) \right] \]

\[ \times \left[ 1 - f_\ell(k \cdot u) + f_\phi((p-k) \cdot u) \right] \]

\[ =: F^{\text{vac,fin}} + F^{\text{vac,col}} + F^{\text{HTL,fin}} + F^{\text{HTL,col}} + F^{T \neq 0}, \]

where the particular contributions to \( F \) are defined below.

The Dirac trace is evaluated as

\[ \text{tr} \left[ \hat{p} S^{(1),A}_\ell(k) \right] = \left[ \left( \frac{i}{k^2 + i \text{sign} k_0 \epsilon} \right)^2 - \left( \frac{i}{k^2 - i \text{sign} k_0 \epsilon} \right)^2 \right] \]

\[ \times \left[ - (p^2 + k^2) k \cdot i \Sigma^H_\ell(k) + k^2 p \cdot i \Sigma^H_\ell(k) \right] \]

\[ + \left[ \left( \frac{i}{k^2 + i \text{sign} k_0 \epsilon} \right)^2 + \left( \frac{i}{k^2 - i \text{sign} k_0 \epsilon} \right)^2 \right] \]

\[ \times \left[ - (p^2 + k^2) k \cdot \Sigma^A_\ell(k) + k^2 p \cdot \Sigma^A_\ell(k) \right] \].

As a consequence of Lorentz invariance, \( \Sigma^{A,H,\text{vac}}(k) \propto k^\mu \), which may be verified explicitly by inspection of Eqs. (42,43,45). Therefore, we define

\[ \Sigma^{A,H,\text{vac}}_\ell(k) = P_R \hat{p} \Sigma^{A,H,\text{vac}}_\ell(k), \]

such that

\[ \hat{\Sigma}^{A,\text{vac}}_\ell(k) = \frac{G}{16\pi} \theta(k^2 - \lambda^2) \text{sign}(k_0), \]

\[ \hat{\Sigma}^{H,\text{vac}}_\ell(k) = \frac{G}{8\pi^2} \log \frac{\lambda}{\Lambda}. \]

We have included here the threshold condition form the gauge-boson mass \( \lambda \), that is needed for the infrared regularisation. We obtain

\[ \left[ - 4(p^2 + k^2) k \cdot \Sigma^{A,H,\text{vac}}_\ell(k) + 4k^2 p \cdot \Sigma^{A,H,\text{vac}}_\ell(k) \right] = -2(k^2 p^2 + (k^2)^2) \hat{\Sigma}^{A,H,\text{vac}}_\ell(k). \]
from $\Sigma_{\ell}^{A,\text{vac}}$. These terms individually contain collinear divergences that are regulated by the gauge-boson mass $\lambda$ and that cancel when added to $F^{\text{vac, col}}$, as we demonstrate now. We find

$$F^{\text{vac, col, wv}} = g_w Y^2 \int\frac{d^3 k}{(2\pi)^3} \frac{1}{2|m|} \frac{1}{2\pi} \delta(p^0 - 2|m|) \frac{1}{4k^0} \frac{\partial}{\partial k^0} \frac{G}{8\pi^2} p^0 k^2 \log \frac{\lambda}{k}$$

where $k^0 = p^0 - |m|$. The scattering corrections yield

$$F^{\text{vac, col, sca}} = g_w Y^2 \int\frac{d^3 k}{(2\pi)^3} \frac{1}{2|m|} \sum_{k^0 = p^0 \pm |m|} \frac{2p^2}{16\pi} \lambda^2 \delta(k^0 - \lambda^2) \text{sign}k^0$$

Again, we analytically isolate the logarithmic dependence on $\lambda$, which is for the $-$ contribution

$$F^{\text{vac, col, sca, -}} = g_w Y^2 G \int \frac{d^3 k}{128\pi^4} \int \frac{p^0}{p^0 - 2|m|} [1 - f_{\ell}(k \cdot u) + f_{\phi}((p - k) \cdot u)]$$

The logarithmic dependence on $\lambda$ is therefore cancelled cancelled with $F^{\text{vac, col, wv}}$, Eq. (54). This is again a consequence of the fact that $\Sigma_{\ell}^{A,\text{H, vac}}$ are the anti hermitian and hermitian parts of the same analytic self energy, evaluated at the two-particle branch cut.

The HTL-type contributions are decomposed into terms originating from $\Sigma_{\ell}^{H}$ and $\Sigma_{\ell}^{A}$ as well, $F_{\text{HTL}} = F_{\text{HTL, wv}} + F_{\text{HTL, sca}}$. We observe that $k \cdot \Sigma_{\ell}^{A,\text{HTL}}(k) = 0$ and $k \cdot \Sigma_{\ell}^{H,\text{HTL}}(k)$ is finite for $k^2 = 0$, whereas $p \cdot \Sigma_{\ell}^{H,\text{HTL}}(k)$ has a logarithmic divergence for $k^2 \to 0$. The terms that are $\propto p \cdot \Sigma_{\ell}^{H,\text{HTL}}(k)$ and $\propto p \cdot \Sigma_{\ell}^{A,\text{HTL}}(k)$ in Eq. (50) therefore lead to collinearly divergent contributions to $F_{\text{HTL}}$, and we denote these by $F_{\text{HTL, col, wv}}$ and
\( \mathcal{F}_{\text{HTL,\text{col,sca}}} \) whereas the terms \( \propto k \cdot \Sigma_{H,\text{HTL}}^H(k) \) and \( \propto k \cdot \Sigma_{A,\text{HTL}}(k) \) are finite and referred to as \( \mathcal{F}_{\text{HTL,\text{fin,wv}}} \) and \( \mathcal{F}_{\text{HTL,\text{fin,sca}}} \), such that

\[
\mathcal{F}_{\text{HTL,\text{col}}} = \mathcal{F}_{\text{HTL,\text{col,wv}}} + \mathcal{F}_{\text{HTL,\text{col,sca}}},
\]

\[
\mathcal{F}_{\text{HTL,\text{fin}}} = \mathcal{F}_{\text{HTL,\text{fin,wv}}} + \mathcal{F}_{\text{HTL,\text{fin,sca}}}. \tag{57b}
\]

The finite terms can be integrated in a straightforward manner, while the collinearly divergent contributions must be regularised, with a cancellation of the dependence on the regulator in the sum of wave-function and scattering contributions. In order to regulate these terms, we set the gauge-boson mass \( \lambda = 0 \), but we introduce a lepton mass \( m_\ell \). Then, we find that

\[
\mathcal{F}_{\text{HTL,\text{col,wv}}} = -g_w Y^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \frac{1}{2\pi} \delta(p^0 - |k| - \sqrt{k^2 + m_\ell^2}) \frac{1}{4k^0} \frac{\partial}{\partial k^0} k^2 p \cdot \Sigma_{\ell}^{H,\text{HTL}}(k) \left[ 1 - f_\ell(k \cdot u) + f_\phi((p - k) \cdot u) \right]
\]

\[
= \frac{g_w Y^2 G T^2}{27 \pi^2} \log \frac{m_\ell^2}{M^2} \int d\Omega \left[ 1 - f_\ell(k \cdot u) + f_\phi((p - k) \cdot u) \right] \tag{58}
\]

and

\[
\mathcal{F}_{\text{HTL,\text{col,sca}}} = -g_w Y^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{|k|} \frac{1}{2\pi} \delta(p^0 - |k| - \sqrt{k^2 + m_\ell^2}) k^2 p \cdot \Sigma_{\ell}^{A,\text{HTL}}(k) \left[ 1 - f_\ell(k \cdot u) + f_\phi((p - k) \cdot u) \right]
\]

\[
= \frac{g_w Y^2 G T^2}{27 \pi^2} \int d\Omega \left\{ \log \left( m_\ell^2 + 2|k| - M \right) M \right\} \left[ 1 - f_\ell(k \cdot u) + f_\phi((p - k) \cdot u) \right] \bigg|_{|k| = \frac{M}{2}}^{\infty}
\]

\[
- \int \frac{d|k| \log \left( m_\ell^2 + 2|k| - M \right) M \frac{\partial}{\partial |k|} \left[ 1 - f_\ell(k \cdot u) + f_\phi((p - k) \cdot u) \right]}{\frac{M}{2}} \bigg|_{|k| = \frac{M}{2}}^{\infty}. \tag{59}
\]

Notice that there is no + contribution from the sum over \( \pm \) in Eq. (49), because \( \Sigma_{\text{HTL}}^A(k) \) vanishes for positive \( k^0 \). The logarithmic dependence on \( m_\ell^2 \) therefore cancels when adding scattering and wave-function corrections, Eqs. (59) and (58). Once more, the prefactors for the cancellation match because \( \Sigma_{\text{HTL}}^A(k) \) are anti hermitian and hermitian part of an analytic self-energy evaluated at the branch cut. Notice, that the HTL-type contributions diverge logarithmically when \( p^2 \to 0 \). In that situation, one should use the resummed lepton propagator \([1, 3, 35]\).

The result for \( \mathcal{F}^{T \neq 0} \) that is defined as the contribution to \( \mathcal{F} \) from \( \Sigma_{H,T \neq 0}^H \) and from \( \Sigma_{A,T \neq 0}^A \), which in turn results from replacing \( I_0 \to I_0^{T \neq 0} \) and \( I_1 \to I_1^{T \neq 0} \) in Eq. (42) can
be obtained by integrating over the pole at \( k^2 = 0 \) in the principal value sense, what can be verified by checking the limiting behaviour of \( I_0^{T \neq 0}(y^0,y) \) and \( I_1^{T \neq 0}(y^0,y) \) for \( y^0 \rightarrow y \), i.e. that these functions are continuous at that point.

For completeness, we again list the explicit expressions for the various IR-finite terms:

\[
\mathcal{F}^{\text{vac,fin},wv} = 0, \tag{60}
\]

\[
\mathcal{F}^{\text{vac,fin},sca} = g_w Y^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2|k|} \sum_{k^0 = M \pm |k|} \frac{G}{8\pi} \vartheta(k^2) \text{sign}(k^0)
\]

\[
\times [1 - f_\ell(k \cdot u) + f_\phi((p - k) \cdot u)], \tag{61}
\]

\[
\mathcal{F}^{\text{vac,col},sca+} = -g_w Y^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2|k|} \frac{M}{M + 2|k|} \frac{G}{8\pi} [1 - f_\ell(k \cdot u) + f_\phi((p - k) \cdot u)]_{k^0 = M + |k|}, \tag{62}
\]

\[
\mathcal{F}^{\text{HTL,fin},wv} = -g_w Y^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2|k|} 2\pi \delta(M - 2|k|) \left( -\frac{1}{4k^0} \frac{d}{dk^0} + \frac{1}{4k^0} \right)
\]

\[
\times \left( M^2 + k^0 - k^2 \right) \frac{GT^2}{8} [1 - f_\ell(k \cdot u) + f_\phi((p - k) \cdot u)], \tag{63}
\]

where \( k^0 = \frac{M}{2} \),

\[
\mathcal{F}^{\text{HTL,fin},sca} = 0, \tag{64}
\]

\[
\mathcal{F}^{T \neq 0} = g_w Y^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2|k|} \left\{ \sum_{k^0 = M \pm |k|} \frac{1}{(k^2)^2} \right[ 2(p^2 + k^2)k \cdot \Sigma_\ell^{A,T \neq 0}(k) - 2k^2 p \cdot \Sigma_\ell^{A,T \neq 0}(k) \\
\times [1 - f_\ell(k \cdot u) + f_\phi((p - k) \cdot u)]
\right. \]

\[
+ 2\pi \delta(M - 2|k|) \left[ \left( \frac{1}{2k^0} \frac{d}{dk^0} - \frac{1}{2k^0} \right) \left( p^2 + k^2 \right) k \cdot \Sigma_\ell^{H,T \neq 0}(k) - k^2 p \cdot \Sigma_\ell^{H,T \neq 0}(k) \right)
\]

\[
\times [1 - f_\ell(k \cdot u) + f_\phi((p - k) \cdot u)] \right\} |k^0 = \frac{M}{2}|, \tag{65}
\]

where \( \mathcal{F}^{\text{vac,fin}} = \mathcal{F}^{\text{vac,fin},wv} + \mathcal{F}^{\text{vac,fin},sca} \). These expressions can be substituted into Eq. (20), in order to obtain the production rate for right-handed neutrinos. Notice also that the sign of \( \mathcal{F}^{\text{HTL,fin},wv} \) is consistent with the phase-space suppression due to the asymptotic thermal lepton mass.
5 Vertex Contribution

5.1 Scatterings or Real Emissions

The $2 \leftrightarrow 2$ and $1 \leftrightarrow 3$ contributions (involving the real emission of gauge bosons, here collectively referred to as scatterings) to the thermal production rate of $N$ follow from the first term of Eq. (18),

$$\text{tr}[\hat{p}i\Sigma_N^{\leq}(p)]^{\text{sca}} = 4g_wGY^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} [2(k - p) + q]^{\nu} \text{tr}\left[\hat{p}\right.$$ (66)

$$\times iS^H_\ell(k)\gamma^\mu iS^\leq_\ell(k + q)i\Delta^\leq_\phi(-k + p)i\Delta^H_\mu\nu(q)i\Delta^H_\phi(-k + p - q).$$

In order to bring the integrand to a symmetric form, we replace the momenta

$$q \rightarrow -r, \quad k \rightarrow \frac{p}{2} - q + \frac{r}{2}, \quad (67)$$

which takes the effect

$$-k + p \rightarrow \frac{p}{2} + q - \frac{r}{2}, \quad k + q \rightarrow \frac{p}{2} - q - \frac{r}{2}, \quad -k + p - q \rightarrow \frac{p}{2} + q + \frac{r}{2}. \quad (68)$$

Furthermore, the on-shell delta-functions of the Wightman propagators force

$$\left(\frac{p}{2} + q - \frac{r}{2}\right)^2 = 0, \quad (69a)$$

$$\left(\frac{p}{2} - q - \frac{r}{2}\right)^2 = 0, \quad (69b)$$

$$r^2 = \lambda^2, \quad (69c)$$

where $\lambda$ again is a fictitious gauge-boson mass, such that this contribution reads

$$\text{tr}[\hat{p}i\Sigma_N^{\leq}(p)]^{\text{sca}} = 4g_wGY^2 \int \frac{d^4q}{(2\pi)^4} \frac{d^4r}{(2\pi)^4}$$

$$\times \left(\frac{p}{2}^2 - 2p^2 - 2\lambda^2 + 2\left(\frac{p}{2} + q + \frac{r}{2}\right)^2 + \left(\frac{p}{2} - q - \frac{r}{2}\right)^2\right)$$

$$+ \left[\left(\frac{p}{2} + q + \frac{r}{2}\right)^2 - \left(\frac{p}{2} - q + \frac{r}{2}\right)^2\right] \text{PV} \left[\frac{1}{\left(\frac{p}{2} - q + \frac{r}{2}\right)^2}\right] \times 2\pi\delta \left(\left(\frac{p}{2} - q - \frac{r}{2}\right)^2\right) \times 2\pi\delta \left(\left(\frac{p}{2} + q - \frac{r}{2}\right)^2\right)$$

$$\times 2\pi\delta \left(\left(\frac{p}{2} - q - \frac{r}{2}\right)^2\right) \times 2\pi\delta \left(\left(\frac{p}{2} + q - \frac{r}{2}\right)^2\right) \times 2\pi\delta \left(\left(\frac{r}{2} - \lambda^2\right)\right)$$

$$\times \text{sign} \left(\frac{M}{2} - q^0 - \frac{r^0}{2}\right) \text{sign} \left(\frac{M}{2} + q^0 - \frac{r^0}{2}\right) \text{sign} \left(r^0\right)$$

$$\times f_\ell \left(\left(\frac{p}{2} - q - \frac{r}{2}\right) \cdot u\right) f_\phi \left(\left(\frac{p}{2} + q - \frac{r}{2}\right) \cdot u\right) \left[1 + f_A(-r \cdot u)\right].$$
We integrate first over $dr^0, dq^0$ and $d\cos \vartheta$ using the three on-shell delta-functions and then change the integration variables $|r|$ and $|q|$ back to $r^0$ and $q^0$, i.e. the integration measure changes as

$$\int \frac{d^4q}{(2\pi)^4} \frac{d^4r}{(2\pi)^4} 2\pi \delta \left( \left( \frac{p}{2} - q - r \right)^2 \right) 2\pi \delta \left( \left( \frac{p}{2} + q - r \right)^2 \right) 2\pi \delta \left( r^2 - \lambda^2 \right) \quad (71)$$

$$\rightarrow \sum_{r^0 = \pm |r^0|, \varphi = \pm |q^0|} \frac{1}{(2\pi)^5} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \int_{-1}^1 d\cos \varrho \int_{I'_r} q^2 d|q| \int_{I'_q} r^2 d|r| \frac{1}{2\sqrt{r^2 + \lambda^2}} \frac{1}{4|q^0| |q||r|} \quad (72)$$

$$\rightarrow \frac{1}{8} \frac{1}{(2\pi)^5} \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \int_{-1}^1 d\cos \varrho \int_{I'_r} dr^0 \int_{I'_q} dq^0,$$

where $\varrho$ denotes the angle between $q$ and $u$. More specifically, we choose to parametrise the angular dependences as

$$r = |r| \begin{pmatrix} \sin \vartheta \sin \varphi \\ \sin \vartheta \cos \varphi \\ \cos \vartheta \end{pmatrix}, \quad q = |q| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad u = \frac{|p|}{M} \begin{pmatrix} \sin \varrho \sin \varphi \\ \sin \varrho \cos \varphi \\ \cos \varrho \end{pmatrix}. \quad (72)$$

The integration domains $I$ are best determined from distinguishing between the various kinematic situations, which are sketched in Figure 6. Besides, we take within these regions a positive integration measure, such that we do not obtain explicit minus signs from the Jacobians. The regions follow from the three on-shell constraints. Notice that these conditions may be combined to obtain

$$|q| = \frac{1}{2} \sqrt{M^2 + 4q^0_2 - 2Mr^0 + \lambda^2}, \quad (73)$$

and when using this as a relation for $\cos \vartheta$,

$$\left( \frac{p}{2} + q - \frac{r}{2} \right)^2 = \left( \frac{M}{2} + q^0 \right)^2 - \left( \frac{M}{2} + q^0 \right) r^0 + |q||r| \cos \vartheta + \frac{\lambda^2}{4} - q^2 = 0. \quad (74)$$

The relation $-1 \leq \cos \vartheta \leq 1$ is then one of the constraints that must be imposed in order to determine the regions of integration in Figure 6. Indeed, for $\lambda = 0$ this inequality reduces to:

$$\left( r^0 - \frac{M}{2} \right) \left( q^0_2 - \frac{1}{4} r^0_2 \right) \geq 0. \quad (75)$$

Besides, we note that the on-shell constraints imply that the invariant momentum squares of the off-shell propagators are

$$\left( \frac{p}{2} + q + \frac{r}{2} \right)^2 = M \left( 2q^0 + r^0 \right), \quad (76a)$$

$$\left( \frac{p}{2} - q + \frac{r}{2} \right)^2 = M \left( -2q^0 + r^0 \right). \quad (76b)$$
This suggests a change of variables to

\[ x = q^0 + \frac{1}{2} r^0 \tag{77a} \]
\[ y = -q^0 + \frac{1}{2} r^0 \tag{77b} \]

which has the Jacobian \(-1\). The integration regions in terms of these variables are indicated in Figure 6 as well. The collinear divergences are located on the fringes of the integration region where either \(x\) or \(y\) vanish, soft divergences are at the coincident point \(x = y = 0\).

We can therefore recast the scattering contributions to the neutrino production as

\[
\text{tr}[\hat{\rho} \Sigma_N^{<} (p)] = 4 g_w G Y^2 \frac{1}{32} \frac{1}{(2\pi)^4} \int_0^{2\pi} d(\varphi - \psi) \int d \cos \varrho \int dx \int dy \int \left[ 1 + f_A(-r \cdot u) \right] \times f_\ell ((p/2 - q - r/2) \cdot u) f_\varphi ((p/2 + q - r/2) \cdot u) \left[ 1 + f_A(-r \cdot u) \right].
\]

The statistical functions can be evaluated using above relations. In particular, the scalar products that appear in the arguments are given by

\[
p \cdot u = p^0 \tag{79a},
\]
\[
q \cdot u = \frac{1}{2} (x - y) \frac{p^0}{M} - |q| \frac{p}{M} \cos \varrho \tag{79b},
\]
\[
r \cdot u = (x + y) \frac{p^0}{M} - (x + y) \frac{p}{M} \left[ \cos \varrho \cos \vartheta + \sin \varrho \sin \vartheta \cos(\varphi - \psi) \right] \tag{79c}.
\]

with \(|q|\) as in Eq. (73) and \(\cos \vartheta\) given by Eq. (74). Close to the collinear edges, where
\( x = 0 \lor y = 0 \), we can evaluate
\[
\left( \frac{p}{2} - q - \frac{r}{2} \right) \cdot u \approx \left( \frac{1}{2} - \frac{x}{M} \right) \left( \vec{p}^0 + |\vec{p}| \cos \varrho \ \text{sign}(M - x)\text{sign}(M - y) \right) + \sqrt{xy} M |\vec{p}| \sin \varrho \cos(\varphi - \psi),
\]
\[
\left( \frac{p}{2} + q - \frac{r}{2} \right) \cdot u \approx \left( \frac{1}{2} - \frac{y}{M} \right) \left( \vec{p}^0 - |\vec{p}| \cos \varrho \ \text{sign}(M - x)\text{sign}(M - y) \right) + \sqrt{xy} M |\vec{p}| \sin \varrho \cos(\varphi - \psi),
\]
\[
r \cdot u \approx \frac{x + y}{M} \vec{p}^0 + \frac{x - y}{M} |\vec{p}| \cos \varrho \ \text{sign}(M - x)\text{sign}(M - y) - 2 \sqrt{xy} M |\vec{p}| \sin \varrho \cos(\varphi - \psi).
\]

Note that Eqs. (80) are understood as leading order expansions and hence the \( \sim \sqrt{xy} \) corrections are applicable only when both \( x \sim y \approx 0 \). As we aim for rearranging the various contributions to the vertex-type correction into a manifestly finite integral, we have dropped here the dependence on the regulating gauge-boson mass \( \lambda \). We have also let \( \cos \varrho \rightarrow \text{sign}(M - x)\text{sign}(M - y) \cos \varrho \) for later convenience.

In order to arrange for an IR-finite integral, where the collinear divergences on the fringes \( x = 0 \) and \( y = 0 \) cancel, it is useful to define the integrand
\[
\mathcal{J}_{\text{sca}}(x, y) = K_{\text{sca}}(x, y) f_\ell ((p/2 - q - r/2) \cdot u) f_\phi ((p/2 + q - r/2) \cdot u) \times [1 + f_A (-r \cdot u)] S(x, y)
\]
\[
\approx \left|_{|x|, |y| \ll M} \right. - K_{\text{sca}}(x, y) f_\ell \left( \left( 1 - \frac{2x}{M} \right) E_s \right) f_\phi \left( \left( 1 - \frac{2y}{M} \right) E_{-s} \right) \times f_A \left( \frac{2x}{M} E_s + \frac{2y}{M} E_{-s} \right) S(x, y),
\]
where \( s \equiv \text{sign}(M - x)\text{sign}(M - y), \)
\[
E_\pm \equiv \frac{1}{2} (\vec{p}_0 \pm |\vec{p}| \cos \varrho),
\]
and \( S(x, y) \) is a step function defining the support of the integral in accordance with Figure [6]
\[
S(x, y) = \vartheta(x) \vartheta(y) \vartheta(M/2 - x - y) + \vartheta(-x) \vartheta(-y) + \vartheta(x) \vartheta(-y) \vartheta(x + y - M/2).
\]
For later convenience, we have separated the kinematic part in the factor
\[ K^\text{sca}(x, y) = \text{sign}(x + y) \text{sign} \left( \frac{M}{2} - x \right) \text{sign} \left( \frac{M}{2} - y \right) \text{PV} \frac{-2M^2 + 4Mx + 2My + 4xy}{xy}. \] (84)

In the approximate form for \( J^\text{sca} \), we have made the restriction to \( |x| \ll M \dot{\vee} |y| \ll M \), where \( \dot{\vee} \) denotes the exclusive or, which justifies dropping the \( \sim \sqrt{xy} \) corrections. We are going to use this expression in order to show the cancellation of collinear divergences. For the soft divergence at \( x = y = 0 \), there occurs a complication due to the Bose divergence of the gauge boson distribution, and we will provide a separate discussion.

### 5.2 Vertex Correction with on-Shell Gauge Boson

We now bring the vertex contribution with an on-shell gauge boson to a form that can be matched with the contributions from scatterings. It is useful to reparametrise in Eq. (18)
\[ k \rightarrow p - q, \quad q \rightarrow q - k, \] (85)
such that the second term in Eq. (18) turns into
\[
\text{tr}[\mathcal{P} \Sigma^<(p)]^\text{vert}_1 = (3g_2^2 + g_1^2)Y^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} [-k - q]^\nu \text{tr} [\not{p} \not{q}]
\]
\[
\times iS^H(p - q) \gamma^\mu iS^<_\nu(p - q - k) i\Delta^<_\phi(k) i\Delta^H_\phi(q) i\Delta^F_\mu(q - k). \]

We perform the integrations over \( dk^0 \) and \( dq^0 \) making use of the on-shell delta-functions. When we define
\[ k_\pm = (\pm |k|, k), \quad q_{1\pm} = (k^0 \pm \omega, q), \quad \omega = \sqrt{(q - k)^2 + \lambda^2}, \] (87)
we obtain
\[
\text{tr}[\mathcal{P} \Sigma^<(p)]^\text{vert}_1 = 4g_wGY^2 \frac{1}{16(2\pi)^4} \sum_{\pm} \int_0^{2\pi} d(\varphi - \psi) \int_1^\infty \frac{d\cos \theta}{\lambda} \int_0^{q_{\text{max}}} \frac{d|q|M}{\lambda} \int |q| \] (88)
\[
\times \text{PV} \left[ \left( \frac{M}{2} \mp \omega \right)^2 - q^2 \right] \left( \frac{M}{2} \mp \omega \right)^2 - q^2 \right] \left( \frac{M}{2} \mp \omega \right)^2 - q^2 \right]
\]
\[
\times \left[ 1 + 2f_A(\{q_{1\pm} - k_\pm \cdot u\}) f_\ell((p - k_\pm) \cdot u) f_\phi(k_\pm \cdot u) \right],
\]
where
\[ q_{\text{max, min}} = \frac{M}{2} \pm |\sqrt{\omega^2 - \lambda^2}|. \] (89)
Here, we parametrise the angular dependences as
\[
\begin{align*}
\mathbf{q} &= |\mathbf{q}| \begin{pmatrix} \sin \vartheta \sin \varphi \\ \sin \vartheta \cos \varphi \\ \cos \vartheta \end{pmatrix}, \quad \mathbf{k} &= |\mathbf{k}| \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{u} = \frac{|\mathbf{p}|}{M} \begin{pmatrix} \sin \vartheta \sin \psi \\ \sin \vartheta \cos \psi \\ \cos \vartheta \end{pmatrix}.
\end{align*}
\]
(90)

In order to evaluate the distribution functions, we use
\[
\begin{align*}
\mathbf{k}_+ \cdot \mathbf{u} &= \frac{1}{2}(\bar{p}_0 - |\mathbf{p}| \cos \vartheta) = E_-, \\
(\mathbf{p} - \mathbf{k}_+) \cdot \mathbf{u} &= \frac{1}{2}(\bar{p}_0 + |\mathbf{p}| \cos \vartheta) = E_+, \\
q_{1\pm} \cdot \mathbf{u} &= \bar{p}_0 \left( \frac{1}{2} \pm \frac{\omega}{M} \right) - |\mathbf{p}| \frac{|\mathbf{q}|}{M} \left( \cos \vartheta \cos \varphi + \sin \vartheta \sin \varphi \cos(\varphi - \psi) \right), \\
\cos \vartheta &= \frac{q_2^2 + M^2 - \omega^2 + \lambda^2}{M|\mathbf{q}|}.
\end{align*}
\]
(91a, 91b, 91c, 91d)

Next, we change variables to
\[
\begin{align*}
x &= -\frac{1}{2M} \left[ \left( \frac{M}{2} - \omega \right)^2 - q^2 \right], \quad y = \frac{1}{2M} \left[ \left( \frac{M}{2} + \omega \right)^2 - q^2 \right].
\end{align*}
\]
(92)

Notice that \( x = 0 \) corresponds to the pole of the lepton (Higgs) propagator for \( \pm \to + \) \( (\pm \to -) \) and that vice versa, \( y = 0 \) corresponds to the pole of the lepton (Higgs) propagator for \( \pm \to - \) \( (\pm \to +) \). The inverse transformation gives
\[
\begin{align*}
|\mathbf{q}| &= \frac{1}{2} \sqrt{M^2 + 4M(x - y) + 4(x + y)^2}, \\
\omega &= x + y,
\end{align*}
\]
(93a, 93b)

and the Jacobian is \(-M/|\mathbf{q}|\). We express the vertex contributions as (we take \( \lambda \to 0 \) at this point):
\[
\begin{align*}
\text{tr}[\bar{p}_i \Sigma^\pm(p)_i]^{\text{vert}} = 4g_\omega G Y^2 \frac{1}{32(2\pi)^4} \int_0^{2\pi} d(\varphi - \psi) \int_{-1}^1 d \cos \varphi \int_0^\infty dy \int_0^\infty dx \\
\times (J^{\text{vert}}_{i+}(x, y) + J^{\text{vert}}_{i-}(x, y)),
\end{align*}
\]
(94)

where for the present terms
\[
J^{\text{vert}}_{1\pm}(x, y) = K^{\text{vert}}_{1\pm}(x, y) [1 + 2f_A(|(q_{1\pm} - k_+) \cdot \mathbf{u}|)] f_\ell(E_+) f_\phi(E_-),
\]
(95)
\[ K_{\pm}^{\text{vert}}(x, y) = - \text{PV} \left[ \vartheta(\mp) \frac{M^2 + 2Mx - My}{xy} + \vartheta(\pm) \frac{M^2 - 2My + Mx}{xy} \right]. \] (96)

In this form, we can directly compare with the terms from real emissions. When either or both, \( x, y \) are close to zero, we can approximate
\[ |q| \approx \frac{1}{2} |M + 2x - 2y|, \] (97)
\[ (q_{1+} - k_+) \cdot u \approx \pm \frac{1}{M} |\mathbf{p}| \cos \frac{x - y}{M} - 2 \frac{\sqrt{xy}}{M} |\mathbf{p}| \sin \varphi \cos(\varphi - \psi). \] (98)

Expanding the arguments of the distribution functions close to the surfaces \( x = 0 \) \( \check{\varpi} y = 0 \), we obtain
\[ J_{\pm}^{\text{vert}}(x, y) \approx K_{\pm}^{\text{vert}}(x, y) \left[ 1 + 2f_A \left( \frac{2x}{M} E_+ + \frac{2y}{M} E_- \right) \right] f_\ell(E_+)f_\phi(E_-). \] (99)

### 5.3 Vertex Correction with on-Shell Higgs Boson

We now consider the third term of Eq. (18), which using the reparametrisation \( (85) \) becomes
\[
\text{tr}[\hat{p} \Sigma_N^{\leq}(p)]_{12}^{\text{vert}} = 4g_w GY^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} [-k - q]^\nu \text{tr}[\hat{p}]
\times iS^\nu(p - q)\gamma^\mu iS^\leq(p - k)i\Delta^\leq(\phi)(k)i\Delta^F(q)i\Delta^H_{\mu\nu}(q - k)].
\] (100)

Making use of the on-shell delta-functions, we find
\[
\text{tr}[\hat{p} \Sigma_N^{\leq}(p)]_{12}^{\text{vert}} = 4g_w GY^2 \frac{1}{32(2\pi)^4} \sum_{\pm} \int_0^{2\pi} d(\varphi - \psi) \int_{-1}^1 d\cos \varphi \int_0^\infty dqM \int \frac{\omega}{q_{\min}} dq_{\max}^{\omega} \frac{M^3}{(M^2 - \sigma^2 q)(M/2 \pm \sigma q + \omega)(M/2 \pm \sigma q - \omega)}
\times \text{PV}\left[ \frac{4M^3}{M^2 - \sigma^2 q + \omega}(M/2 \pm \sigma q - \omega) \right]
\times [1 + 2f_\phi(\{q_{2+\sigma} - k_+\} \cdot u)] f_\ell((p - k_+) \cdot u)f_\phi(k_+ \cdot u),
\] (101)

where we define
\[
\sigma = \text{sign} \left( \frac{M}{2} - y \right). \] (102)
Note that in the expression (101), \( q \) is a real number, not a four-vector. In terms of the original integration variables, the new variables are given here by

\[
\omega = \sqrt{q^2 + \lambda^2}, \quad q = \sqrt{(k - q)^2}.
\] (103)

We again change variables according to Eqs. (92) and with \( |q| \rightarrow q \) and decompose the present contribution as in Eqs. (94). Notice that the IR divergences only occur for the contributions where \( \pm \rightarrow - \), such that apart from the Bose divergence in the Higgs boson distribution function \( \text{tr}[\mathcal{P}(p)_N]^{\text{vert}} \) is finite. The integrand for Eq. (94) with \( i = 2 \) then is

\[
J^{\text{vert}}_{2 \pm}(x, y) = K^{\text{vert}}_{2 \pm}(x, y) \left[ 1 + 2 f_\phi(|(q_{2 \pm \sigma} - k_+) \cdot u|) \right] f_\phi(E_+) f_\phi(E_-),
\] (104)

where \( k_+ \cdot u \) and \( (p - k_+) \cdot u \) are again given by Eq. (91) and

\[
K^{\text{vert}}_{2 \pm}(x, y) = \frac{\omega}{q} \text{PV} \left[ \frac{4M^3}{(M + \sigma q + \omega)(M + \sigma q - \omega)} + \frac{4M}{M + \sigma q} \right] - \frac{2M^2}{(M + \sigma q + \omega)(M + \sigma q - \omega)}. \] (105)

We do not explicitly substitute \( x \) and \( y \) here for \( q \) and \( \omega \) as this would give rise to a lengthy expression in this case.

In order to compare with the other contributions, we notice that when either or both, \( x \) and \( y \) are small, we may approximate

\[
\left( \frac{M}{2} - q + \omega \right) \left( \frac{M}{2} - q - \omega \right) \approx -4xy,
\] (106a)

\[
M - 2q \approx 2(y - x).
\] (106b)

The relations for constructing the arguments of the distribution functions are

\[
q_{2 \pm} \cdot u = \tilde{p}_0 \left( \frac{1}{2} \pm \frac{q}{M} \right) - |\tilde{p}| \sqrt{\omega^2 - \lambda^2} M (\cos \vartheta \cos \varphi + \sin \vartheta \sin \varphi \cos(\varphi - \psi)),
\] (107a)

\[
\cos \vartheta = \frac{\omega^2 + M^2 - q^2 - \lambda^2}{M \sqrt{\omega^2 - \lambda^2}}.
\] (107b)

When either \( x, y \) or both are close to zero, we may approximate (again, we take \( \lambda \rightarrow 0 \) here)

\[
(q_{2 \pm} - k_+) \cdot u \approx \left( \frac{1}{2} + \frac{x - y}{M} \right) (\pm \tilde{p}_0 + |\tilde{p}| \cos \vartheta) - 2\frac{\sqrt{xy}}{M} |\tilde{p}| \sin \vartheta \cos(\varphi - \psi).
\] (108)
Using these approximations, we find for \(x \ll M \vee y \ll M\)

\[
K_{2+}^{\text{vert, appr}}(x, y) = \text{PV} \frac{4M(x + y)}{(M + 2x - 2y)^2}, \tag{109a}
\]

\[
K_{2-}^{\text{vert, appr}}(x, y) = \text{PV} \left[ \frac{M(M + x - y)}{xy} - \frac{4M}{M + 2x - 2y} \right] \text{sign}(x - y), \tag{109b}
\]

\[
J_{2\pm}^{\text{vert}}(x, y) \approx K_{2\pm}^{\text{vert, appr}}(x, y) \text{sign}(M - 2y)
\]

\[
 \times \left[ 1 + 2 f_\phi \left( \left| \frac{2x - 2y}{M} \right| E_\pm \right) \right] f_\ell(E_+) f_\phi(E_-). \tag{109c}
\]

\[5.4\] \textbf{Vertex Correction with on-Shell Lepton}

The third term of Eq. (18) in the reparametrisation (85) is

\[
\text{tr}[\mathbf{p}^i \Sigma_N^3(p)]^{\text{vert}} = 4 g_w G Y^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} [-k - q]^\nu \text{tr}[\mathbf{p}^i S_\ell^F(p - q) \gamma^\mu S_\ell^< (p - k)i\Delta^\leph(k) i\Delta^H_\phi(q) i\Delta^H_\mu\nu(q - k)]. \tag{110}
\]

The kinematic situation is essentially the same as for the on-shell Higgs boson, which is why the relations (106, 107, 108) apply to the present contribution as well. The integrand can be expressed as

\[
J_{3\pm}^{\text{vert}}(x, y) = \text{PV} \left[ 1 - 2 f_\ell (\left| (q_{2\pm} - k_+) \cdot u \right|) \right] f_\ell(E_+) f_\phi(E_-), \tag{111}
\]

where

\[
K_{3\pm}^{\text{vert}}(x, y) = \frac{\omega}{q} \text{PV} \left[ \frac{4M^3}{(M \pm \sigma 2q)(M/2 \pm \sigma q + \omega)(M/2 \pm \sigma q - \omega)} + \frac{4M}{M \pm \sigma 2q} \right. \tag{112}
\]

\[
- \left. \frac{4M^2}{(M/2 \pm \sigma q + \omega)(M/2 \pm \sigma q - \omega)} \right].
\]

The arguments for the distribution functions can be constructed using Eqs. (107), with the important exception that we replace \(\cos \theta \rightarrow - \cos \theta\) in order to interchange \(E_+ \leftrightarrow E_-\). This change of variable leaves the value of the integral unaffected, but it is helpful in order to to combine the integrands to a manifestly IR-convergent expression for the right-handed neutrino production rate.

The portion of the integrand that leads to IR divergences is approximated for \(x \ll \)
\[ M \vee y \ll M \]

as
\[ \dot{K}_{3+}^{\text{vert.appr}}(x, y) = 0, \]  
\[ K_{3-}^{\text{vert.appr}}(x, y) = \text{PV} \left[ \frac{M(M + 2x - 2y)}{xy} - \frac{4M}{M + 2x - 2y} \right] \text{sign}(x - y), \]  
\[ \mathcal{J}_{3\pm}^{\text{vert}}(x, y) \approx K_{3\pm}^{\text{vert.appr}}(x, y) \text{sign}(M - 2y) \times \left[ 1 - 2f_\ell \left( \frac{2x - 2y}{M} \right) \right] f_\ell(E_+)f_\phi(E_-). \]  

\section{5.5 Infrared Finite Integral}

The remaining task is to combine the various pieces to an integral, that can be evaluated with a finite result (in the principal value sense) for each subset of the complete integration region. This is achieved by the combination
\[ \text{tr}[\phi \Sigma_N^<(p)]^{\text{VERT}} = 4g_\omega G Y^2 \int_0^{2\pi} d(\varphi - \psi) \int_{-1}^1 d\cos \varphi \int_0^\infty dx \int_0^\infty dy \mathcal{J}^{\text{total}}(x, y), \]  
\[ \mathcal{J}^{\text{total}}(x, y) = \frac{1}{8} \left( \mathcal{J}^{\text{sca}}(x, y) + \mathcal{J}^{\text{sca}}(-x, y) + \mathcal{J}^{\text{sca}}(x, -y) + \mathcal{J}_{1-}^{\text{vert}}(x, y) + \mathcal{J}_{2-}^{\text{vert}}(x, y) ight. \]  
\[ + \mathcal{J}^{\text{sca}}(-x, -y) + \mathcal{J}^{\text{sca}}(-x, y) + \mathcal{J}_{1+}^{\text{vert}}(y, x) + \mathcal{J}_{3-}^{\text{vert}}(y, x) \]  
\[ + \mathcal{J}^{\text{sca}}(-y, -x) + \mathcal{J}_{1-}^{\text{vert}}(y, x) + \mathcal{J}_{3-}^{\text{vert}}(y, x) \]  
\[ + 2\mathcal{J}_{2+}^{\text{vert}}(x, y) + 2\mathcal{J}_{3+}^{\text{vert}}(y, x) + x \leftrightarrow y \left. \right) + \varphi \rightarrow \varphi + \pi. \]  

(Recall that the support of \( \mathcal{J}^{\text{sca}} \) is given by Eq. (113) and shown in Figure 6.) This expression may be integrated without introducing an infrared regulator.

In order to show that this is integrable on the fringes where \( x \to 0 \vee y \to 0 \), we make use of the detailed balance relations applicable to collinear splittings
\[ f_\phi(E) + f_\phi(E)f_\phi(\alpha E) + f_\phi(E)f_A((1 - \alpha)E) = f_\phi(\alpha E)f_A((1 - \alpha)E), \]  
\[ f_\ell(E) - f_\ell(E)f_\ell(\alpha E) + f_\ell(E)f_A((1 - \alpha)E) = f_\ell(\alpha E)f_A((1 - \alpha)E). \]  

It then follows that for \( x \to 0 \) (divergences in the Higgs boson [lepton] propagator for Eq. (118a) [Eq. (118b)])
\[ xy \left( \mathcal{J}^{\text{sca}}(x, y) + \mathcal{J}^{\text{sca}}(-x, y) + \mathcal{J}_{1-}^{\text{vert}}(x, y) + \mathcal{J}_{2-}^{\text{vert}}(x, y) \right) \to 0, \]  
\[ xy \left( \mathcal{J}^{\text{sca}}(-y, -x) + \mathcal{J}_{1-}^{\text{vert}}(y, x) + \mathcal{J}_{3-}^{\text{vert}}(y, x) \right) \to 0 \]
and that for \( y \to 0 \) (divergences in the lepton [Higgs boson] propagator for Eq. (119a) [Eq. (119b)]

\[
xy(J^{\text{sca}}(x,y) + J^{\text{sca}}(x,-y) + J^{\text{vert}}(y,x) + J^{\text{vert}}_3(y,x)) \to 0, \tag{119a}
\]

\[
xy(J^{\text{sca}}(-y,-x) + J^{\text{vert}}_1(y,x) + J^{\text{vert}}_2(y,x)) \to 0. \tag{119b}
\]

The sum \( J^{\text{sca}}(x,y) + J^{\text{sca}}(-x,y) \) in Eq. (118a) makes it sure that the cancellation with vertex contributions takes place on the whole collinear edge \( x = 0 \) (for both \( y < M/2 \) and \( y \geq M/2 \)), and similarly in Eq. (119a) for \( y = 0 \) in accordance with the kinematic regions, cf. Figure 6. In order to obtain the manifestly IR-finite result for the vertex-type part of the neutrino production rate (116a), we need to add these contributions to those terms, that are integrable from the outset and finally symmetrise \( x \leftrightarrow y \), in order to make sure that the sum of all divergent terms cancels on both edges \((x = 0 \text{ and } y = 0)\). Furthermore, we have symmetrised over the polar angles.

The criteria (118,119) are in general not sufficient in order to show the integrability at the point \( x = y = 0 \), because the detailed balance relations only hold on the collinear fringes where \( x = 0 \lor y = 0 \). We must therefore show in addition that \( xyJ^{\text{total}}(x,y) \) vanishes on each trajectory for \( (x, y) \to 0 \). The problematic terms are the Bose divergences in \( J^{\text{sca}} \) and \( J^{\text{vert}}_\pm \). When using Eqs. (81,95) together with the approximations for the arguments of the distribution functions (80,98), we notice the cancellation of the divergences in \( xy(J^{\text{sca}}(x,y) + J^{\text{vert}}_\pm(x,y)) \). There is however a finite remainder that moreover depends on the direction of approach of \( (x, y) \) toward \((0,0)\). This finite contribution follows from the terms that are linear in \( x, y \) and \( \sqrt{xy} \) in the arguments of the distribution functions (80,98). For this finite term to cancel as well, we must add \( J^{\text{sca}}(-x,-y) \), symmetrise over \( x \) and \( y \) (as it is already necessary for the cancellation of the collinear divergences) and moreover, add the contributions from positive and negative values of \( \cos(\varphi - \psi) \). In summary, it follows that

\[
\lim_{x,y \to 0} xyJ^{\text{total}}(x,y) = 0, \tag{120}
\]

such that \( J^{\text{total}}(x,y) \) grows more slowly than \( 1/(x^2 + y^2) \), and therefore it is integrable at \( x = y = 0 \). In fact, we have verified by numerical evaluation that \( J^{\text{total}}(x,y) \sim 1/\sqrt{x^2 + y^2} \) for \((x, y) \to 0\).

Similarly, care must be taken about the points \((x,y) = (0,M/2)\) and \((x,y) = (M/2,0)\), where the Bose divergence of the Higgs scalar coincides with the collinear fringes. Suppose we are close to the fringe where \( x \to 0 \). Inspecting the approximation (109) for \( J^{\text{vert}}_\pm \), we notice that the \( y \) integration can be performed at \( y = M/2 \) in the principal value sense. The same is true for the collinearly divergent terms \( 1/x \) in \( J^{\text{sca}}(\text{sign}(M/2 - y)x,y) \), Eq. (81). Therefore, we can perform the \( d\Delta y \) integral over \( J^{\text{total}}(x,M/2 - \Delta y) + J^{\text{total}}(x,M/2 + \Delta y) \) as well \((\Delta y > 0)\). Due to the detailed balance relations (118,119), \( x \) times this integral vanishes for \( x \to 0 \), which is why the remaining \( x \) integration yields a finite result as well. We have also checked numerically that for \((x,y) \to (0,M/2)\)

\[
x(y - M/2)J^{\text{total}}(x,y) \to 0, \tag{121}
\]
and moreover that \( J_{\text{total}}(x, y) \sim 1/\sqrt{x^2 + (y - M/2)^2} \) irrespective of the limiting direction, which further confirms the integrability at \((0, M/2)\) and \((M/2, 0)\) by \(x \leftrightarrow y\) symmetry. Furthermore, we have verified that the Bose divergences in \( J_{\text{vert}}^2 \) on the line \( y = M/2 + x \) away from fringe \( x = 0 \) are actually cancelled in that \((y - M/2 - x)J_{\text{total}}(x, y) \rightarrow 0 \) and \( J_{\text{total}}(x, y) \rightarrow \text{finite} \) for \( y \rightarrow M/2 + x \), and similarly for \( x \leftrightarrow y \).

### 5.6 Ultraviolet Regularisation

The vertex-type correction (116a) to the neutrino production contains UV divergences that need to be isolated and subtracted in order to obtain a remainder that is suitable for numerical integration. The UV divergences originate from the vertex contributions. In particular, the integrands (95, 104, 111) are not exponentially suppressed for large values of \( x \) or \( y \) because of the constant term in the square brackets. The contributions arising from this constant term are easily interpreted as the vacuum vertex-corrections to the \(2 \rightarrow 1\) processes \( \ell, \phi \rightarrow N \) or \( \bar{\ell}\phi^\dagger \rightarrow N \), weighted by the statistical distributions for \( \ell \) and \( \phi \). These UV-divergent contributions are analytically calculable, and are given by the following sum of terms:

\[
J_{\text{vert, vac}}^\text{vert}(x, y) = \sum_{\pm} (K_{1\pm}^\text{vert}(x, y) + K_{2\pm}^\text{vert}(x, y) + K_{3\pm}^\text{vert}(x, y)) f_\ell(E_+) f_\phi(E_-) .
\]

(122a)

Now, we need to subtract this expression from the integrand that we aim to evaluate numerically, while adding terms that balance the collinear divergences that this subtraction may induce. The vacuum-vertex corrections should only have IR divergences that cancel with those for real emissions in \(1 \rightarrow 3\) processes. (Scatterings of the \(2 \leftrightarrow 2\) type do not occur in the vacuum, in absence of incoming gauge bosons, leptons and Higgs bosons.) This implies that outside of Region I, \( J_{\text{vert}}^\text{vert}(x, y) \) should not yield collinearly divergent contributions, which can be verified as

\[
xy(J_{\text{vert, vac}}^\text{vert}(x, y) + J_{\text{vert, vac}}^\text{vert}(y, x)) = 0 \quad \text{for} \quad (x = 0 \wedge y > M/2) \vee (y = 0 \wedge x > M/2) .
\]

(123)

This is no longer true for Region I. In order to cancel collinear divergences there, we define

\[
J_{\text{sca, vac}}^\text{sca}(x, y) = K_{\text{sca}}^\text{sca}(x, y) \delta(x + y - M/2) f_\ell(E_+) f_\phi(E_-) ,
\]

(124)

which we subtract in addition. This is just \(1 \rightarrow 3\) rate weighted with the same statistical factors for leptons and Higgs bosons as \( J_{\text{vert, vac}}^\text{vert}(x, y) \), and it has the desired property that

\[
xy(J_{\text{sca, vac}}^\text{sca}(x, y) + J_{\text{vert, vac}}^\text{vert}(x, y) + x \leftrightarrow y) = 0 \quad \text{for} \quad x = 0 \vee y = 0 .
\]

(125)

Now the integrand

\[
J_{\text{UV}}^\text{total}(x, y) = J_{\text{total}}(x, y) - \frac{1}{2} (J_{\text{sca, vac}}^\text{sca}(x, y) + J_{\text{vert, vac}}^\text{vert}(x, y) + x \leftrightarrow y)
\]

(126)
is IR and UV finite and suitable for numerical evaluation.

Finally, the subtracted terms must be calculated analytically and then added again. For this purpose, we note that the virtual $T = 0$ corrections to the decay rate, evaluated with a cutoff $\omega < \Lambda$ [from the parametrisations in Eqs. (88,101)] are

$$\text{tr}[\not p \Sigma^A_N(p)]_{T=0}^{\text{vert}} = \frac{4g_w G Y^2}{16(2\pi)^3} M^2 \left[ -2 \log^2 \frac{\lambda}{M} + \frac{\pi^2}{6} - 3 \log \frac{\Lambda}{M} - \frac{3}{2} + \log 2 \right]$$

$$+ Y \delta Y \frac{M^2}{4\pi}.$$  \hspace{1cm} (127)

Here, $\delta Y$ is a counter term that must remove the logarithmic divergence in $\Lambda$ and the further details of which are determined by a particular renormalisation scheme.

Alternatively, we can use dimensional regularisation in order to calculate the reduced vertex function

$$V(p) = 2g_w G Y^2 \mu' \int \frac{d^{4-\epsilon} q}{(2\pi)^4} \frac{1}{(p-q)^2 + i\epsilon} \frac{1}{[(q-k)^2 + i\epsilon][q^2 + i\epsilon]} + Y \delta Y 2M^2$$

$$= 2g_w G Y^2 \frac{M^2}{8\pi^2} \left[ -2 \log^2 \frac{\lambda}{M} + \frac{\pi^2}{6} - 3 \log \frac{\lambda}{\mu} + 2 \log \frac{M}{\mu} - \Delta_\epsilon - \frac{1}{2} \right] + Y \delta Y 2M^2. \hspace{1cm} (128)$$

In this form, $\delta Y$ must cancel the $1/\epsilon$ divergence and may otherwise again be renormalisation-scheme dependent. Integrating over the $T = 0$ phase space, we obtain the zero-temperature spectral-function for the right-handed neutrino

$$\text{tr}[\not p \Sigma^A_N(p)]_{T=0}^{\text{vert}} = \int \frac{d^{4} k}{(2\pi)^4} \frac{d^{4} q}{(2\pi)^4} 4\pi \delta^4(p - k - q) 2\pi \delta(k^2) 2\pi \delta(q^2) V(p) = \frac{V(p)}{8\pi}$$

$$= 4g_w G Y^2 \frac{M^2}{2^7\pi^3} \left[ -2 \log^2 \frac{\lambda}{M} + \frac{\pi^2}{6} - 3 \log \frac{\lambda}{\mu} + 2 \log \frac{M}{\mu} - \Delta_\epsilon - \frac{1}{2} \right]$$

$$+ Y \delta Y \frac{M^2}{4\pi}.$$ \hspace{1cm} (129)

Note the factors from the two different diagrammatic contributions in the CTP approach (cut on the left or on the right). Comparing with the prefactors in Eq. (94) and using the definitions (A1), we can conclude that

$$\bar{J}^{\text{vert, vac}} = \int_0^\infty \int_0^\infty \bar{J}^{\text{vert, vac}}(x, y), \hspace{1cm} (130a)$$

$$\bar{J}^{\text{vert, vac}}(x, y) = - 2 M^2 \left[ -2 \log^2 \frac{\lambda}{M} + \frac{\pi^2}{6} - 3 \log \frac{\lambda}{\mu} + 2 \log \frac{M}{\mu} - \Delta_\epsilon - \frac{1}{2} \right]$$

$$\times f_\epsilon(E_+) f_\phi(E_-). \hspace{1cm} (130b)$$

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The IR divergences are again isolated in terms involving log $\lambda$, which cancel with corresponding terms in the vacuum vertex contribution (130a).

Next, we need to obtain a matching expression for $J_{\text{sca,vac}}(x,y)$, which we have introduced in order to cancel the IR-divergent $\lambda$-dependence. For this purpose, we evaluate the integral

$$
\int_{x_0}^{x} dx \int_{y_0}^{y} dy \left[ -2 \frac{M^2}{xy} + 4 \frac{M}{y} + 2 \frac{M}{x} + 4 \right] = 2M^2 \left[ -2 \log \frac{\lambda}{M} + \frac{\pi^2}{6} + 3 \log \frac{M}{\lambda} - \frac{11}{2} M^2 \right],
$$

where the boundaries of integration derive from the condition $-1 \leq \cos \vartheta \leq 1$ with Eq. (74). Hence,

$$
\tilde{J}_{\text{sca,vac}} = \int_{0}^{\infty} dx \int_{0}^{\infty} dy J_{\text{sca,vac}}(x,y)
$$

$$
= 2M^2 \left[ -2 \log \frac{\lambda}{M} + \frac{\pi^2}{6} + 3 \log \frac{M}{\lambda} - \frac{11}{2} M^2 \right] f_{\ell}(E_+) f_{\phi}(E_-).
$$

The final expression for the reduced self-energy is

$$
\text{tr} \left[ \hat{\rho} \Sigma_{N}(p) \right]_{\text{VERT}} = 4g_w GY^2 \frac{1}{32(2\pi)^4} \int_{0}^{2\pi} d(\varphi - \psi) \int_{-1}^{1} d \cos \vartheta \int_{0}^{\infty} dx \int_{0}^{\infty} dy J_{\text{total}}(x,y)
$$

$$
+ 4g_w GY^2 \frac{1}{32(2\pi)^3} \int_{-1}^{1} d \cos \vartheta \left( \tilde{J}_{\text{vert,vac}} + \tilde{J}_{\text{sca,vac}} \right)
$$

$$
+ \frac{M^2}{4\pi} Y \delta Y \int_{-1}^{1} d \cos \vartheta f_{\ell}(E_+) f_{\phi}(E_-).
$$

Each of the explicit integrals in this expression is convergent, while the remaining UV and IR divergences are isolated within simple factors that cancel in the sum, as discussed above in this present Section.

6 Discussion and Conclusions

The results of this work show that the relaxation rate of right-handed neutrinos $N$ toward thermal equilibrium is finite to NLO in perturbation theory, i.e. when including the leading corrections from Standard Model gauge radiation. As we perform a fully
relativistic calculation, we generalise earlier results that are based on non-relativistic approximations \[5, 6\] that are valid when \(M \gg T\). In particular, the relative motion of the neutrino \(N\) with respect to the plasma is kept general here and the full quantum statistical Bose-Einstein and Fermi-Dirac effects are accounted for.

We have developed two somewhat different methods in order to handle the wave-function- and the vertex-type corrections. The treatment of the wave-function corrections is greatly facilitated, because well known and relatively simple analytical expressions for the thermal self-energies are available. We can isolate the IR divergences in terms of logarithmic dependences on the regulating gauge-boson mass \(\lambda\) for both, the scattering contributions that rely on the spectral self-energy and the wave-function contributions originating from the hermitian self-energy. Eventually, we find that the IR divergences cancel [Eq. (33) with Eq. (38), Eq. (54) with Eq. (56) and Eq. (58) with Eq. (59)] because the hermitian and the spectral self-energies are the real and imaginary parts of the same analytic function evaluated at the two-particle branch cut.

For the two-particle irreducible vertex-type corrections, the situation appears more complicated. Rather than isolating the IR divergences in analytic expressions for various contributions, we therefore manipulate these in such a manner that we obtain the integrand (116b) that is manifestly free from non-integrable IR and Bose divergences.

Since the methods that we use in order to demonstrate the cancellation are very explicit and as we also discuss the subtraction of UV divergences, they readily suggest a method for analytic or numerical evaluation of the relaxation rates of right-handed neutrinos \(N\). In a future work, it would be of interest to verify the results of Refs. \[5, 6\] by taking the non-relativistic limit of the present results and moreover, to perform a numerical evaluation of the production rate of \(N\) without non-relativistic approximations.

While the present work provides a method for evaluating the relaxation rate of right-handed neutrinos \(N\), that derives from \(\text{tr}[\Sigma^A_{LM}]\), cf. Eq. (3), the rates of non-equilibrium \(CP\)-violation from decays and inverse decays of \(N\) typically rely on different components of the self energy \[24, 29, 31–33\]. Another future task would therefore be the evaluation of \(CP\)-violating rates, in generalisation of the methods presented here.

While we have focused in the present work on the production of right-handed neutrinos, the basic topology of the self-energy diagrams that describe the decay and scattering rates and their NLO corrections is obviously shared with diagrams for other high-energy reactions in finite-temperature backgrounds. For that reason, it may be useful to formulate our method in terms of master integrals in such a way that the results could be more directly applied to other situations with different kinds of particles in the loops (different Lorentz tensor structures). One may investigate for example the processes that underlay the transport coefficients used in calculations for Electroweak Baryogenesis or corrections to the production rate of Dark Matter particles. It will therefore be interesting to explore the possibilities of applying the present methods to additional topics in Early Universe Cosmology.
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Appendix

We follow the conventions of Ref. [16]. The relations between the CTP two-point functions, \((G\) stands for a propagator \(\Delta\) or \(S\) or for a self-energy \(\Pi\) or \(\Sigma\)) are given by:

\[
G^A = G^T - G^> = G^<- G^T \quad \text{(advanced)},
\]
\[
G^R = G^T - G^< = G^> - G^T \quad \text{(retarded)},
\]
\[
G^H = \frac{1}{2}(G^R + G^A) \quad \text{(Hermitian)},
\]
\[
G^A = \frac{1}{2i}(G^A - G^R) = \frac{i}{2}(G^> - G^<) \quad \text{(anti-Hermitian, spectral)},
\]
\[
G^F = \frac{1}{2}(G^> + G^<) = \frac{1}{2}(G^T + G^\bar{T}) \quad \text{(statistic)}.
\]

In terms of the functions bearing CTP indices, these can be expressed as

\[
G^T = G^{++}, \quad G^\bar{T} = G^{--} \quad \text{(time ordered, anti time-ordered)},
\]
\[
G^< = G^{+-}, \quad G^> = G^{-+} \quad \text{(Wightman)}.
\]

The Wigner transform is defined by

\[
G(p, x) = \int d^4re^{ipr}G(x + r/2, x - r/2).
\]

Under spatially homogeneous conditions, there is no dependence on the average spatial coordinate \(x\) and furthermore, we suppress the explicit average time coordinate \(t = x^0\).

Equilibrium Green functions or self energies observe the Kubo-Martín-Schwinger (KMS) relation

\[
G^>(p) = \pm e^{\beta p \cdot u}G^<(p),
\]

where the plus sign applies to bosonic, the minus sign to fermionic two-point functions, and \(u\) is the plasma four-velocity.


The tree-level equilibrium propagators for the lepton doublet that we use here are

\[ iS^\prec_\ell(p) = -2\pi\delta(p^2)\text{sign}(p^0)P_L\hat{p}_\ell(p \cdot u), \quad (A5a) \]
\[ iS^\succ_\ell(p) = 2\pi\delta(p^2)\text{sign}(p^0)P_L\hat{p}_\ell(1 - f_\ell(p \cdot u)), \quad (A5b) \]
\[ iS^T_\ell(p) = P_L\frac{i\hat{p}}{p^2 + i\varepsilon} - 2\pi\delta(p^2)P_L\hat{p}_\ell(|p \cdot u|), \quad (A5c) \]
\[ iS^{\tilde{T}}_\ell(p) = -P_L\frac{i\hat{p}}{p^2 - i\varepsilon} - 2\pi\delta(p^2)P_L\hat{p}_\ell(|p^0|), \quad (A5d) \]

where \( f_\ell \) is the Fermi-Dirac distribution. The generalisation to non-equilibrium distributions can be found in Ref. [24]. We suppress explicit SU(2) indices, which are contracted with that of the Higgs doublet \( \phi \) in the usual manner, using the two-dimensional anti-symmetric tensor.

Similarly, we use the scalar equilibrium propagators

\[ i\Delta^\prec_\phi(p) = 2\pi\delta(p^2)\text{sign}(p^0)f_\phi(p \cdot u), \quad (A6a) \]
\[ i\Delta^\succ_\phi(p) = 2\pi\delta(p^2)\text{sign}(p^0)(1 + f_\phi(p \cdot u)), \quad (A6b) \]
\[ i\Delta^T_\phi(p) = \frac{i}{p^2 + i\varepsilon} + 2\pi\delta(p^2)f_\phi(|p \cdot u|), \quad (A6c) \]
\[ i\Delta^{\tilde{T}}_\phi(p) = -\frac{i}{p^2 - i\varepsilon} + 2\pi\delta(p^2)f_\phi(|p \cdot u|), \quad (A6d) \]

where \( f_\phi \) is a Bose-Einstein distribution. The gauge field propagators are

\[ i\Delta^\prec_{\mu\nu}(p) = 2\pi\delta(p^2)\text{sign}(p^0)(-g_{\mu\nu})f_A(p \cdot u), \quad (A7a) \]
\[ i\Delta^\succ_{\mu\nu}(p) = 2\pi\delta(p^2)\text{sign}(p^0)(-g_{\mu\nu})(1 + f_A(p \cdot u)), \quad (A7b) \]
\[ i\Delta^T_{\mu\nu}(p) = \frac{-ig_{\mu\nu}}{p^2 + i\varepsilon} + 2\pi\delta(p^2)(-g_{\mu\nu})f_A(|p \cdot u|), \quad (A7c) \]
\[ i\Delta^{\tilde{T}}_{\mu\nu}(p) = -\frac{-ig_{\mu\nu}}{p^2 - i\varepsilon} + 2\pi\delta(p^2)(-g_{\mu\nu})f_A(|p \cdot u|), \quad (A7d) \]

and \( f_A \) is a Bose-Einstein distribution.

As a consequence of the Majorana condition, the self energy of the right-handed neutrinos inherits the property

\[ \Sigma_N(x, y) = C\Sigma^t_N(y, x)C^†, \quad (A8) \]

where the transposition \( t \) acts here on both the CTP and the Dirac indices, cf. Ref. [29] for a more detailed discussion that also includes the mixing of several right-handed neutrinos and the effect of non-zero chemical potentials for the leptons \( \ell \).
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