Emergence and Stability of Vortex Clusters in Bose-Einstein Condensates: a Bifurcation Approach near the Linear Limit

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Abstract

We study the existence and stability properties of clusters of alternating charge vortices in Bose-Einstein condensates. It is illustrated that such states emerge from cascades of symmetry-breaking bifurcations that can be analytically tracked near the linear limit of the system via weakly nonlinear few-mode expansions. We present the resulting states that emerge near the first few eigenvalues of the linear limit, and illustrate how the nature of the bifurcations can be used to understand their stability. Rectilinear, polygonal and diagonal vortex clusters are only some of the obtained states while mixed states, consisting of dark solitons and vortex clusters, are identified as well.

Key words: Bose-Einstein condensates, Vortices, Dark solitons, Bifurcations

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1. Introduction

Vortices are among the most striking characteristics of nonlinear field theories in higher-dimensional settings [1]. They constitute one of the remarkable features of superfluids, while playing also a key role in critical current densities and resistances of type-II superconductors through their transport properties, and are associated with quantum turbulence in superfluid helium [2]. Vortices appear also in a wide variety of fields, ranging from fluid dynamics [3] to atomic physics [4] and optical physics [5].

A pristine setting for the study of vortices at the mesoscale has emerged after the realization of atomic Bose-Einstein condensates (BECs). In this context, so-called matter-wave vortices were experimentally observed therein [6], by using a phase-imprinting method between two hyperfine spin states of a $^{87}$Rb BEC [7]. This achievement subsequently triggered extensive studies concerning vortex formation, dynamics and interactions. For instance, stirring the BECs [8] above a certain critical angular speed [9–12] led to the production of few vortices [12], and even of robust vortex lattices [13]. Vortices can also be formed in experiments by means of other techniques, such as by dragging obstacles through the BEC [14] or by the nonlinear interference of different condensate fragments [15]. Not only unit-charged, but also higher-charged vortices were observed [16] and their dynamical instabilities have been analyzed.

While a considerable volume of work has been dedicated to individual vortices and to vortex lattices, arguably, vortex clusters consisting of only a few vortices have attracted less interest. The latter theme has become a focal point recently, through the experiments involving two-vortex states (alias vortex dipoles) [17,18], as well as three-vortex states [19]. In Ref. [17], vortex dipoles were produced by dragging a localized light beam with appropriate speed through the BEC, while in Ref. [18] they were distilled through the Kibble-Zurek mechanism [20], previously proposed and realized for vortices in Ref. [21]. For the nonlinear dynamics of the vortices in the dipoles of Ref. [18], see also the very recent analysis of Ref. [22]. In Ref. [19], different types of three-vortex configurations were produced by applying an external quadrupolar magnetic field on the BEC. The prin-
imensional (3D) interaction strength, the and Equation (1) can be expressed in the following dimension-

giving the charge vortices), by corroborating theoretical investi-

vortex states have been considered also in a number of

citations, where we will illustrate how they emerge (bifur-

Fig. 1, but we will not be concerned with it further herein,

of the nonlinear modes of Eq. (2). Notice that numerically
the relevant nonlinear states will be identified as a function
of the chemical potential $\mu$ by means of a fixed point (New-

ton iteration) scheme over a rectangular two-dimensional grid with suitably small spacing. We will also explore the linear (spectral) stability of the obtained states by means of the Bogoliubov-de Gennes (BdG) analysis. The latter involves the derivation of the BdG equations, which stem from a linearization of the GPE (2) around the stationary solution $u_0(x, y)$ via the ansatz

$$u = u_0(x, y) + [a(x, y)e^{i\omega t} + b^*(x, y)e^{-i\omega^* t}],$$

where * denotes complex conjugate. The solution of the ensuing BdG eigenvalue problem yields the eigenfunctions \{a(x, y), b(x, y)\} and eigenfrequencies $\omega$. Due to the Hamiltonian nature of the system, if $\omega$ is an eigenfrequency of the Bogoliubov spectrum, so are $-\omega$, $\omega^*$ and $-\omega^*$. Notice that a linearly stable configuration is tantamount to $\text{Im}(\omega) = 0$, i.e., all eigenfrequencies being real. It is important to mention that in what follows we only resolve the relevant eigenvalues up to $10^{-2}$ due to computational domain constraints and therefore all eigenvalues smaller than $10^{-2}$ will be omitted in the figures.

Within the BdG analysis, a relevant quantity to consider is the norm $\times$ energy product of a normal mode with eigenfrequency $\omega$, namely,

$$E = \int dx dy (|u|^2 - |b|^2) \omega.$$  

(4)

The sign of this quantity, known as Krein sign [29], is a topologi-

$\omega$ to mean a Krein sign then, typically, complex fre-

quences appear in the excitation spectrum, i.e., a dynamical instability arises [29]. We refer to this as an oscillatory instability. Furthermore, dynamical instabilities may arise due to a real mode eigenfrequency becoming imaginary. This typically coincides with a bifurcation of a new state.

In the context of Eq. (2), it is useful to consider the low-

density (linear) limit. There, eigenstates of the 2D quantum harmonic oscillator arise in the form $u_{nm}(x, y, t) = \exp(-i\mu t)H_n(x)H_m(y)$ (as well as linear combinations thereof), where $\mu = \Omega(n + m + 1)$ and $n, m$ quantify the order (and number of nodal lines) in each direction. This produces a linear limit whose first excited state has $\mu = 2\Omega$ and linear eigenstates $u_{10}$ and $u_{01}$. Notice that one of their interesting linear combinations is $u_{10} + iu_{01}$, which creates the single-charged vortex even at this linear limit; this state exists for all higher values of $\mu$ and is shown in Fig. 1, but we will not be concerned with it further herein, as our focus will be on clusters of vortices.

2. Model And Theoretical Analysis

We consider a quasi-2D (alias “disk-shaped”) condensa-
te confined in a highly anisotropic trap with frequencies $\omega_z$ and $\omega_\perp$ along the transverse and in-plane directions, respectively. In the case $\omega_\perp \ll \omega_z$ and $\mu \ll \hbar \omega_z$ (where $\mu$ is the chemical potential), and for sufficiently low temperatures, the in-plane part $u(x, y, t)$ of the macroscopic BEC wave function obeys the following (2+1)-dimensional Gross-Pitaevskii equation (GPE) (see, e.g., Ref. [4]):

$$i\hbar \partial_t u = \left[ \frac{\hbar^2}{2m} \nabla_\perp^2 + V(r) + g_{2D}|u|^2 - \mu \right] u, \tag{1}$$

where $\nabla_\perp^2$ is the in-plane Laplacian, while the potential is given by $V(r) = (1/2)m\omega_\perp^2 r^2$ (where $m$ is the atomic mass). The effective 2D nonlinearity strength is given by $g_{2D} = g_{3D}/\sqrt{2\pi a_z} = 2\sqrt{\frac{\pi}{m\omega_z}}a_z\hbar \omega_z$, with $g_{3D} = 4\pi\hbar^2 a/m$, $a$ and $a_z = \sqrt{\hbar/m\omega_z}$ denoting, respectively, the three-
dimensional (3D) interaction strength, the $s$-wave scattering length, and the transverse harmonic oscillator length. Equation (1) can be expressed in the following dimension-

less form,

$$i\hbar \partial_t u = \left[ \frac{1}{2} \nabla^2 + V(r) + |u|^2 - \mu \right] u, \tag{2}$$

where the density $|u|^2$, length, time and energy are respectively measured in units of $(2\sqrt{\pi \hbar}a_z)^{-1}, a_z, \omega_z^{-1}$ and $\hbar \omega_z$. Finally, the harmonic potential is now given by $V(r) = (1/2)\Omega^2 r^2$, with $\Omega = \omega_\perp/\omega_z$. From here on, all equations will be presented in dimensionless units for simplicity.

Below, we will analyze the existence and linear stability of the nonlinear modes of Eq. (2). Notice that numerically
Each of the above mentioned linear eigenstates, \( u_{10} \) and \( u_{01} \), represents a “stripe” i.e., a state with a nodal line. These can be continued for higher chemical potentials \( \mu \). As \( \mu \) increases, this state develops into a one-dimensional (1D) dark soliton stripe, which is an exact analytical solution of Eq. (2) in the absence of the trap [30]. However, it is well-known that such a state is dynamically unstable towards decay into vortex structures [31–33]. This decay can be understood from a symmetry-breaking bifurcation point of view [27,34]. In particular, it is possible to consider a two-mode (Galerkin-type) expansion, similar to the one used in the literature of double-well potentials (see e.g. Ref. [35]) in the form:

\[
 u(x, y, t) = c_0(t)\phi_0(x, y) + c_1(t)\phi_1(x, y),
\]

where \( c_0(t), c_1(t) \) are complex time-dependent prefactors, while \( \phi_0 = u_{10}(x, y), \phi_1(x, y) = u_{0m} \) and \( m > 1 \). The resulting equations and analysis are formally equivalent to the ones derived in Ref. [35] (see Eqs. (4)-(5) therein), with appropriate modifications of the inner products, but also with a fundamental difference. In the 1D double-well setting, only symmetry-breaking bifurcations of asymmetric real solutions are predicted (the so-called \( \pi \)-states that have recently been experimentally observed in Ref. [36]). The richer 2D case enables bifurcations even when the relative phase \( \Delta \phi \) between the complex order parameters \( c_0 \) and \( c_1 \) is \( \pi/2 \). In particular, such bifurcations are generically predicted at an atom number:

\[
 N_{cr} = \frac{\omega_0 - \omega_1}{I_0 - I_1},
\]

where \( \omega_0, \omega_1 \) are the linear state eigenvalues corresponding to \( \phi_0 \) and \( \phi_1 \), while \( I_0 = \int \phi_0^*\phi_0^*dxdy \) and \( I_1 = \int \phi_1^*\phi_1^*dxdy \); the critical chemical potential is given by \( \mu_{cr} = \omega_0 + I_1N_{cr} \).

3. Numerical Results and Comparison with Theory

The above mentioned two-mode theory provides explicit predictions for the bifurcation not only of the vortex dipole (vd) state when \( m = 2 \), but also for an aligned vortex tripole (3v) state (cf. the experimental observations of Ref. [19]) for \( m = 3 \), for an aligned vortex quadrupole (4v) state with \( m = 4 \), etc. In fact, there is an entire cascade of such bifurcations, as \( \mu \) increases, which occur progressively at \( \mu_{cr}^{vd} = 10\Omega/3, \mu_{cr}^{3v} = 86\Omega/19, \mu_{cr}^{4v} = 890\Omega/157, \) and \( \mu_{cr}^{5v} = 726\Omega/107 \) for \( m = 2, 3, 4, 5 \) etc., respectively. Notice that the number of vortices of the resulting cluster is evident by the number of intersections of the single nodal line of \( u_{10} \) with the \( m \) perpendicular nodal lines of \( u_{0m} \), as well as the \( \pi/2 \) relative phase of their complex prefactors at these \( m \) intersections. Also, it is evident that that the sign changing of \( u_{0m} \) at these intersections leads to an alternation of the ensuing vortex charges. Importantly, general bifurcation theory can be used to identify the stability characteristics of the resulting states. In particular, since the stripe is dynamically stable as it emerges from the linear limit, the vortex dipole state that arises from it upon the first symmetry-breaking “event” (\( m = 2 \)) should inherit this stability. However, now, once the stripe has become unstable, all higher bifurcations with \( m \geq 3 \) will necessarily result into dynamically unstable states.

Numerical results on the symmetry-breaking bifurcations resulting in the emergence of vortex cluster states from the first excited state (single dark soliton stripe) are summarized in Fig. 1 for \( \Omega = 0.2 \). The emergence of \( 1 \times m \) (\( m = 2, 3, 4, 5, \ldots \)) states can be observed to occur respectively at \((0.68, 0.98, 1.26, 1.54)\) while the corresponding theoretical predictions are \((\mu_{cr}^{vd},\mu_{cr}^{3v},\mu_{cr}^{4v},\mu_{cr}^{5v}) = (2/3, 0.91, 1.13, 1.36)\). Clearly, the two-mode approach cap-
Fig. 2. (Color online) Top left panel (a): Number of atoms as a function of the chemical potential for the different states bifurcating from the two dark soliton stripes (black lines) and from the dark soliton cross (orange [gray in printed version] lines) for $\Omega = 0.2$. Bottom left panel (b): Corresponding atom number difference with respect to the two dark soliton stripes branch. The occurrence of bifurcations at zero crossings of $\Delta N$ and the corresponding theoretical vertical line predictions thereof are similar to the previous (and following) figures. Middle column of panels (from top to bottom): Density and phase profiles (left and right subpanels respectively) corresponding to bifurcating states from the two dark soliton stripes (top): the diagonal six-vortex state ($6v2$), the four and doubly-charged vortex state ($5x$), and the diagonal eight-vortex state ($8x$). Density and phase profiles (left and right subpanels respectively) corresponding to bifurcating states from the X-shaped dark soliton cross (top): the diagonal six-vortex state ($6x$), the four and doubly-charged vortex state ($5x$), and the diagonal eight-vortex state ($8x$).

Importantly, this approach is not restricted to the first excited state. The advantage of the bifurcation method and of the wealth of states that can be derived from it is unveiled, e.g., when considering the next set of excited states, namely the combinations with $n + m = 2$, i.e., the degenerate states $u_{20}$, $u_{02}$ and $u_{11}$. In this setting, already a vortex quadrupole can be formed at the linear limit as $u_{20} + iu_{02}$ [37]. Interestingly, so can a doubly-charged vortex through $u_{20} - u_{02} + 2iu_{11}$. However, these states do not present symmetry breaking bifurcations and, therefore, are not considered further in what follows. Focusing on the states that do, some prototypical examples are (i) the solitonic state consisting of two stripes (i.e., a two-dark-soliton state), i.e., the $u_{20}$ state, (ii) an X-shaped dark soliton cross emerging from $u_{20} - u_{02}$, as well as (iii) a ring dark soliton state [38] (see also Refs. [39,40]) arising from $u_{20} + u_{02}$. The $u_{20}$ state is one for which the generalization of the phenomenology of Fig. 1 is most straightforward as with $\phi_0 = u_{20}$ and $\phi_1 = u_{0m}$, states with $2 \times m$ (2 lines of $m$ vortices each) are formed with $m > 2$. For example, for $\phi_1 = u_{03}$ and $\phi_1 = u_{04}$, such bifurcations are predicted at $\mu_{cr}^{6v2} = 283\Omega/67$, and $\mu_{cr}^{6v2} = 2965\Omega/551$ (see thick black vertical dashed lines in Fig. 2), respectively, leading to $6 = 2 \times 3$ and $8 = 2 \times 4$ two-line vortex clusters. On the other hand, the X-shaped dark soliton pair is subject to similar symmetry-breaking bifurcations/destabilizations due to $\phi_1 = u_{03}$, $u_{13}$, $u_{04}$ etc. These bifurcations, occurring in turn at $\mu_{cr}^{6v2} = 333\Omega/7$, $\mu_{cr}^{6v2} = 99\Omega/17$, and $\mu_{cr}^{8v2} = 369\Omega/59$ (see thin orange [gray in printed version] vertical dashed lines in Fig. 2), lead to a diagonal state with 6 vortices, a doubly-charged vortex in the center together with a four-vortex quadrupole around it, and an 8-vortex cluster of near-diagonal vortices. Finally, the ring dark soliton gets mixed with states of the form $\phi_1 = g(r)\sin(\theta)$, where $g(r) = \sqrt{\frac{\pi}{2}}(r^2)\exp(-\Phi r^2/2)$, again for $l > 3$. Remarkably, these symmetry-breaking events, which can be predicted to occur, e.g., at $\mu_{cr}^{6v2} = 5\Omega$ and $\mu_{cr}^{8v2} = 59\Omega/9$ (see, respectively, orange [gray in printed version] and blue [dark in printed version] vertical dashed lines in Fig. 3), for $l = 3$ and $4$, give rise to polygonal vortex configurations with the vortices now placed on the periphery of the circle. This way, vortex hexagons, octagons, decagons, etc. can be systematically constructed at will.
From the X-shaped dark soliton cross bifurcates a diagonal six- (6x) and eight-vortex (8x) configurations, as well as a state with a vortex quadrupole surrounding a central doubly-charged vortex (5x). We also mention in passing that interesting additional bifurcation events (collisions and disappearances into “blue sky” bifurcations) arise, e.g., between the six-vortex cluster deriving from the X-shaped dark soliton cross and that deriving from the two-stripe soliton (see blue circle denoted by $\mathbf{A}$ in Fig. 2). On the other hand, in Fig. 3 we depict the states that bifurcate from the vortex ring: vortex hexagons (6r) and octagons (8r). In this figure we also depict the vortex quadrupole state (bottom right panel) which does not bifurcate from the soliton ring. In all the cases, good quantitative agreement is found on the critical points predicted by the theory and those observed numerically.

Figures 4 and 5 complement the above existence picture with a systematic linear stability analysis of each of the corresponding states for Figs. 2 and 3 respectively. The two-stripe soliton is dynamically stable near the linear limit, a stability inherited by its first symmetry-breaking offspring, the six-vortex ($2 \times 3$) state; while the aligned eight-vortex case ($2 \times 4$) is unstable from its existence onset (see left column of panels in Fig. 4). Similarly, the X-shaped dark soliton cross state also bears imaginary eigenfrequencies for all values of $\mu$ and hence its derivative states inherit its dynamical instability (see right column of panels in Fig. 4). On the
Fig. 4. (Color online) BdG spectra for the two-dark-soliton stripe and its bifurcating states (left column) and for the dark soliton cross and its bifurcating states (right column) corresponding to the states described in Fig. 2. The inset in each of the spectra depicts a typical configuration at the chemical potential indicated. In all BdG spectra we depict the real (top subpanel) and imaginary (bottom subpanel) parts of the eigenfrequencies as a function of the chemical potential $\mu$ using the following color coding for the online (respectively, in print) version: blue (black)=positive energy (Krein sign) modes, orange (dark gray)=negative energy modes, green (gray)=oscillatory unstable modes, and pink (light gray)=non-oscillatory unstable modes. The presence of non-vanishing imaginary parts is an indication of instability. Note that the only stable solutions are the two-dark-soliton stripe (top left panel) and its first symmetry breaking offspring, the six-vortex state (left panel of second row), which are stable only for low enough chemical potentials.

On the other hand, the ring dark soliton is, as was also found earlier [37–40], always unstable, hence the polygonal vortices that derive from it inherit this instability (see Fig. 5). However, it should be noted that the instability of such states weakens as the chemical potential $\mu$ increases. Note that, the vortex quadrupole is stable in the linear limit. Since no state bifurcates from the vortex quadrupole this stability is generally preserved (apart from a small window of oscillatory instabilities).

Fig. 5. (Color online) BdG spectra for the states bifurcating from the vortex ring and for the vortex quadrupole (bottom right) that does not bifurcate from the vortex ring corresponding to the bifurcations depicted in Fig. 3. For an explanation of the color codes see Fig. 4. From the above it is clear that the most robust state is the vortex quadrupole (which is dynamically stable for small $\mu$ except for a narrow interval of oscillatory instabilities).

Fig. 7. (Color online) BdG spectra for the bifurcating states depicted in Fig. 6. From left to right and top to bottom: the nine-vortex state, the ring dark soliton plus vortex, the three-dark-soliton stripe (which is stable for low values of the chemical potential), and the mixed soliton-vortex state. For an explanation of the color codes see Fig. 4.
here. We can see that the spectra bear a large number of positive Krein sign modes (see blue (black) points is all BdG spectra) which appear to be asymptotically to appropriate corresponding values in the large chemical potential limit. In addition, there is a number of negative Krein sign eigenfrequencies (see orange (dark gray) points is all BdG spectra) which may lead to oscillatory (upon collision with positive sign ones) or exponential instabilities. Generally, we can comment that the positive Krein sign eigenmodes correspond to the ground state “background” on which a particular solitonic or vortex (or mixed) state exists. Furthermore, these eigenfrequencies have a well-defined limit when \( \mu \) is large as discussed e.g. in Ref. [41] (see also references therein). On the other hand, the negative Krein sign modes reflect the excited state nature of the considered soliton or vortex (or mixed) states and are the ones which bear the potential for dynamical instabilities. While for solitonic states, we do not have a precise count on the number of the latter eigenmodes, in the case of multi-vortex clusters consisting of individual vortices an upper bound on the maximal order of such (negative Krein) eigenstates can be given by the number of vortices in the configuration. This, in turn, also gives an upper bound on the number of potentially observable unstable eigenmodes.

Importantly, the present approach can be generalized to higher excited states. As merely a small sample of further exotic configurations that can emerge (some of which can even be structurally robust), we mention a 9-vortex cluster (a square grid of \( 3 \times 3 \) vortex “particles”), which emerges from the linear limit as \( u_{30} + i u_{03} \) and can, therefore, be regarded as a higher excited analog to the single vortex and the vortex quadrupole states and is expected to be stable in the vicinity of that limit (at least with respect to purely imaginary [i.e., non-oscillatory] instabilities, see below). On the other hand, there are bifurcations from that state including the bifurcation of a ring with a vortex in its center (which was again discussed e.g., in Ref. [40]). Another state that exists and should be robust near the linear limit is a three-soliton-stripe. In fact, from such a state bifurcations again emerge due to the mixture with states such as \( \phi_1 = u_{0m} \) with \( m \geq 4 \), but also with \( \phi_1 = u_{1m} \) with \( m \geq 3 \). We focus briefly on the latter, which is theoretically predicted to occur at \( \mu_{c3}^{\text{exp}} = 1250/239 \) (see vertical dashed line in Fig. 6), because it gives rise to yet another novel type of state, namely a mixed state between vortices and a dark soliton: this soliton-vortex mixed state has a dark soliton stripe nodal line plus two additional lines over each of which three vortices reside. Similar states with 8-, 10- etc. vortices beside the soliton also exist, arising through subsequent bifurcations.

To corroborate the above theoretical predictions we show in Figs. 6 and 7 (again for \( \Omega = 0.2 \)) some prototypical examples of states that emerge from the third excited linear branches and their corresponding BdG stability spectra. The 9-vortex “crystal” is stable close to the linear limit in the sense that its BdG spectrum possesses no imaginary mode. However there is a complex mode inducing an oscillatory instability. At \( \mu \approx 0.95 \) the vortex plus ring soliton state bifurcates from the 9-vortex “crystal” inducing a complex mode. However, the latter is small and vanishes again at \( \mu \approx 1.6 \). The BdG spectrum of the vortex plus ring soliton state contains no purely imaginary mode but many complex modes. Note that the eigenvalue spectrum looks fairly similar to the ring dark soliton case, but the imaginary part of the modes is not due to imaginary modes which are created by bifurcations. In this case, modes with positive energy cross zero and thus obtain a negative energy. These negative energy modes then collide with modes with positive energy and create the complex modes. Moreover, we show the three-soliton-stripe, as well as the mixed soliton-six-vortex cluster state emerging from its first bifurcation event in Fig. 6. The three-soliton state is stable near the linear limit (of \( \mu = 0.8 \)) but becomes destabilized at \( \mu = 1.05 \) (and then further so at 1.06, 1.1 and 1.3).

The first bifurcation gives rise to the very weakly unstable soliton-vortex state predicted above; the critical point for this bifurcation is found to be at \( \mu = 1.034 \), once again in very good agreement with the full numerical result. The present approach can naturally be extended to a multitude of additional states which, however, are expected to be dynamically unstable; therefore, having presented the most fundamental ones, we will not proceed further with such considerations.

4. Conclusions and Future Directions

In conclusion, in the present work we have shown that a detailed understanding of emergent vortex cluster states is possible through a near-linear-limit approach. This involves identifying the possible linear states and tracking systematically the symmetry-breaking bifurcations that can arise from them. This allowed us not only to discuss a cascade of bifurcations from the first excited state (in the form of aligned vortex clusters), but to also reveal a broad class of states emerging from the second- and third-excited states. These were not only rectilinear states (with “soliton-type” stripes), but also soliton rings and rings with vortices, vortex quadrupoles or nine-vortex-crystals, as well as various clusters of vortices that derive from some of these states, including vortex hexagons, octagons, decagons, \( n \times m \) states (of \( m \) vortices sitting along \( n \) stripes), soliton-vortex states, and so on.

We were also able, based on the general bifurcation structure of the problem, to reveal which ones among these states are expected to be most robust, such as the vortex dipole or quadrupole, and some of the emerging vortex clusters, such as the \( 2 \times 3 \) or the soliton-vortex state.

It would be especially interesting to extend this picture to different dimensions. On the one hand, one can consider effects of anisotropy (by changing the strengths of the two different trapping directions). This should enable a “dimension-transcending” picture, as extreme anisotropies should allow to observe how the system transitions between
quasi-one- and genuinely two-dimensional dynamics. On the other hand, it would be particularly relevant and interesting to attempt to extend such considerations into three-dimensional settings, and understand how relevant ideas generalize and potentially give rise to structures such as vortex rings that have been observed in pertinent experiments [33,42]. Such investigations are currently in progress.

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References

[1] L. M. Pismen, Vortices in Nonlinear Fields (Oxford Science Publications, Oxford, 1999).
[2] R. J. Donnelly, Quantized Vortices in Helium II (Cambridge University Press, New York, 1991); D. R. Tilley and J. Tilley, Superfluidity and Superconductivity (IOP Publishing, Philadelphia, 1990).
[3] A. J. Chorin and J. E. Marsden, A Mathematical Introduction to Fluid Mechanics (Springer-Verlag, New York, 1993).
[4] P. G. Kevrekidis, D. J. Frantzeskakis, and R. Carretero-González, Emergent Nonlinear Phenomena in Bose-Einstein Condensates (Springer-Verlag, Berlin, 2008).
[5] A. S. Desyatnikov, Yu. S. Kivshar, and L. Torner, Progress in Optics 47, 291 (2005).
[6] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. 83, 2498 (1999).
[7] J. E. Williams and M. J. Holland, Nature 401, 568 (1999).
[8] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. 84, 806 (2000).
[9] A. Recati, F. Zambelli, and S. Stringari, Phys. Rev. Lett. 86, 377 (2001).
[10] S. Sinha and Y. Castin, Phys. Rev. Lett. 87, 190402 (2001).
[11] I. Corro, R. G. Scott, and A. M. Martin, Phys. Rev. A 80, 033609 (2009).
[12] K. W. Madison, F. Chevy, V. Bretin, and J. Dalibard, Phys. Rev. Lett. 86, 4443 (2001).
[13] C. Raman, J. R. Abo-Shaeer, J. M. Vogels, K. Xu, and W. Ketterle, Phys. Rev. Lett. 87, 210402 (2001).
[14] R. Onofrio, C. Raman, J. M. Vogels, J. R. Abo-Shaeer, A. P. Chikkatur, and W. Ketterle, Phys. Rev. Lett. 85, 2228 (2000).
[15] D. R. Scherer, C. N. Weiler, T. W. Neely, and B. P. Anderson, Phys. Rev. Lett. 98, 110402 (2007).
[16] A.E. Leanhardt, A. Gorriz, A. P. Chikkatur, D. Kielpinski, Y. Shin, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. 89, 190403 (2002); Y. Shin, M. Saba, M. Vengalattore, T. A. Pasquin, C. Sanner, A. E. Leanhardt, M. Prentiss, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. 93, 160406 (2004).
[17] T. W. Neely, E. C. Samson, A. S. Bradley, M. J. Davis, and B. P. Anderson, Phys. Rev. Lett. 104, 160401 (2010).
[18] D. V. Freilich, D. M. Bianchi, A. M. Kaufman, T. K. Langin, and D. S. Hall, Science 329, 1182 (2010).
[19] J. A. Seman, E. A. L. Henn, M. Haque, R. F. Shiozaki, E. R. F. Ramos, M. Caracanhas, P. Castilho, C. Castelo Branco, K. M. F. Magalhães, and V. S. Bagnato, Phys. Rev. A 82, 033616 (2010).
[20] T. W. B. Kibble, J. Phys. A 9, 1387 (1976); W. H. Zurek, Nature 317, 505 (1985); W.H. Zurek, Phys. Rep. 276, 177 (1996).
[21] C. N. Weiler, T. W. Needy, D. R. Scherer, A. S. Bradley, M. J. Davis, B. P. Anderson, Nature 455, 948 (2008).
[22] P. Kuopanportti, J.A.M. Huhtamäki and M. Möttönen, arXiv:1011.1661.
[23] L.-C. Crasovan, V. Vekslerchik, V. M. Pérez-García, J. P. Torres, D. Mihalache, and L. Torner, Phys. Rev. A 68, 063609 (2003).
[24] M. Möttönen, S. M. M. Virtanen, T. Isoshima, and M. M. Salomaa, Phys. Rev. A 71, 033626 (2005).
[25] V. Pietilä, M. Möttönen, T. Isoshima, J. A. M. Huhtamäki and S. M. M. Virtanen, Phys. Rev. A 74, 023603 (2006).
[26] A. Klein, D. Jaksch, Y. Zhang, and W. Bao, Phys. Rev. A 76, 043602 (2007).
[27] W. Li, M. Haque and S. Komineas, Phys. Rev. A 77, 053610 (2008).
[28] J. Brand and W. P. Reinhardt, Phys. Rev. A 65, 043612 (2002).
[29] R. S. MacKay, in Hamiltonian Dynamical Systems, edited by R. S. MacKay and J. Meiss (Hilger, Bristol, 1987), p.137.
[30] D. J. Frantzeskakis, J. Phys. A: Math. Theor. 43, 213001 (2010).
[31] E. A. Kuznetsov and S. K. Turitsyn, Zh. Eksp. Teor. Fiz. 94, 119 (1988) [Sov. Phys. JEPT 67, 1583 (1988)].
[32] D. L. Feder, M. S. Pindzola, L. A. Collins, B. I. Schneider, and C. W. Clark, Phys. Rev. A 62, 053606 (2000).
[33] B. P. Anderson, P. C. Haljan, C. A. Regal, D. L. Feder, L. A. Collins, C. W. Clark, and E. A. Cornell, Phys. Rev. Lett. 86, 2926 (2001).
[34] S. Middelkamp, P. G. Kevrekidis, D. J. Frantzeskakis, R. Carretero-González, and P. Schmelcher, Phys. Rev. A 82, 013646 (2010).
[35] G. Theocharis, P. G. Kevrekidis, D. J. Frantzeskakis, and P. Schmelcher, Phys. Rev. E 74, 056608 (2006).
[36] T. Zibold, E. Nicklas, C. Gross and M.K. Oberthaler, Phys. Rev. Lett. 105, 204101 (2010).
[37] T. Kapitula, P. G. Kevrekidis and R. Carretero-González, Physica D 233, 112 (2007).
[38] G. Theocharis, D. J. Frantzeskakis, P. G. Kevrekidis, B. A. Malomed, and Yu. S. Kivshar Phys. Rev. Lett. 90, 120403 (2003).
[39] G. Theocharis, P. Schmelcher, M. K. Oberthaler, P. G. Kevrekidis, and D. J. Frantzeskakis, Phys. Rev. A 72, 023609 (2005); L. D. Carr and C. W. Clark, Phys. Rev. A 74, 043613 (2006).
[40] G. Herring, L. D. Carr, R. Carretero-González, P. G. Kevrekidis, and D.J. Frantzeskakis, Phys. Rev. A 77, 023625 (2008).
[41] P.G. Kevrekidis and D.E. Pelinovsky, Phys. Rev. A 81, 023627 (2010).
[42] N. S. Ginsberg, J. Brand and L. V. Hau, Phys. Rev. Lett. 94, 040403 (2005).