Trace maps under weak regularity assumptions. *

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In Memory of Erik Balslev

Abstract

We study bounded trace maps on hypersurfaces for Sobolev spaces from a point of view that is fundamentally different from the one in the classical theory. This allows us to construct bounded trace maps under weak regularity assumptions on the hypersurfaces. In the case of bounded domains in $\mathbb{R}^n$ we only require the continuity of the boundary. For hypersurfaces in the whole space $\mathbb{R}^n$ we only assume that the hypersurfaces are Lebesgue measurable. As an application of our trace maps we consider the Dirichlet problem and we prove a coarea formula where the level sets are only assumed to be Lebesgue measurable hypersurfaces.

1 Introduction

The study of trace maps on hypersurfaces, for Sobolev spaces, is a fundamental problem that has been considered extensively. It plays an essential role in many areas of analysis. For example, it is crucial in the theory of Sobolev spaces and in the formulation and the solution of boundary value problems. See for example [1], [2], [4], [12], [16], and [17]. The more general results in these references require that the hypersurface is $C^1$ or Lipschitz continuous. This is a strong restriction. Note that the derivatives of Lipschitz continuous functions are bounded and that this is a condition that often is not satisfied in the applications. Moreover, in [15] there are results on trace maps in non-Lipschitz domains bounded by Lipschitz surfaces, in domains with cusps and with peaks and in capacity criteria for trace maps. Furthermore, in [14] there are results on trace maps for function of bounded variation on domains with finite perimeter and with a normal vector at the boundary. Moreover, [14] gives capacity criteria for the existence of trace maps. Furthermore, [3] gives trace maps for functions of bounded variation, assuming that the boundary is a $(n-1)$-rectifiable set and that it has a normal vector. For further results see [9] and [18].

We consider hypersurfaces that are star shaped about the origin, but we impose weak conditions on the smoothness of the hypersurfaces. In the case when the hypersurface is the boundary of a bounded star shaped domain we only require that the function that characterizes the hypersurface is continuous. Further, when we take the trace of functions in Sobolev spaces in $\mathbb{R}^n$ in a star shaped hypersurface we only require that the function that characterizes the hypersurface is Lebesgue measurable. Star shaped are a special type of hypersurfaces, but note that this condition is often satisfied in the applications.

The paper is organized as follows. In Section 2 we construct trace maps in the boundary of bounded star shaped domains, and we apply our results to the Dirichlet problem. In Section 3 we construct trace maps in star shaped hypersurfaces, and we discuss the relation of our results with our previous results in [20], where we constructed trace maps in the boundary of bounded star shaped domains.
maps in the slowness surface of strongly propagative systems of equations. The trace maps in [20] were applied to the spectral and the scattering theory of strongly propagative systems of equations. In Section 4 we consider the relation of our trace maps of Section 3 with the coarea formula. We prove a coarea formula where the level sets are star shaped hypersurfaces that are only assumed to be Lebesgue measurable. Note that using the the usual localization arguments, with partitions of unity, we can extend our results to domains that are locally star shaped.

2 Domains with star shaped boundary

We first introduce some standard definitions and notations. For any open set \( \Omega \subset \mathbb{R}^n \) we denote by \( C^1(\Omega) \) the set of all continuously differentiable functions in \( \Omega \), and by \( C^1(\overline{\Omega}) \) the set of all functions in \( C^1(\Omega) \) that together with all its first derivatives have continuous extensions to \( \overline{\Omega} \). Furthermore, we designate by \( C^\infty(\Omega) \) the set of all infinitely differentiable functions in \( \Omega \), by \( C^\infty_0(\Omega) \) the set of all infinitely differentiable functions with compact support in \( \Omega \) and by \( C^\infty(\overline{\Omega}) \) the set of all functions in \( C^\infty(\Omega) \) that together with all its derivatives have a continuous extension to \( \overline{\Omega} \). By \( C^\infty_0(\mathbb{R}^n)|_{\Omega} \) we denote the set of all the restrictions to \( \Omega \) of functions in \( C^\infty_0(\mathbb{R}^n) \). We denote by \( \mathcal{S} \) the space of Schwartz of all infinitely differentiable function on \( \mathbb{R}^n \) that together with all its derivatives remain bounded when multiplied by any polynomial. We denote by \( C \) a generic constant that can take different values when it appears in various places. The symbol \( L^p(\Omega), 1 \leq p < \infty \) denotes the standard space of Lebesgue measurable functions in \( \Omega \) whose absolute value to the power \( p \) is integrable, with norm,

\[
\|f\|_{L^p(\Omega)} := \left( \int_{\Omega} |f(x)|^p \, dx \right)^{1/p}.
\]

\( L^2(\Omega) \) is a Hilbert space with the standard scalar product,

\[
(f,g)_{L^2(\Omega)} := \left( \int_{\Omega} f(x) \overline{g(x)} \, dx \right)^{1/2}.
\]

The Sobolev space \( W^{1,1}_p(\Omega), 1 \leq p < \infty \), is the Banach space of all functions \( f(x) \in L^p(\Omega) \) such that the derivatives in distribution sense \( \frac{\partial}{\partial x_i} f(x), 1 \leq i \leq n \), are functions in \( L^p(\Omega) \). The norm of \( W^{1,1}_p(\Omega) \) is given by,

\[
\|f\|_{W^{1,1}_p(\Omega)} := \left( \int_{\Omega} |f(x)|^p \, dx + \sum_{i=1}^n \int_{\Omega} \left| \frac{\partial}{\partial x_i} f(x) \right|^p \, dx \right)^{1/p}.
\]  

We denote by \( W^{1,1}_{p,0}(\Omega) \) the closure of \( C^\infty_0(\Omega) \) in the norm of \( W^{1,1}_p(\Omega) \).

A domain \( \Omega \) is an open set in \( \mathbb{R}^n, n \geq 2 \). Let \( \partial \Omega \) be its boundary. Let us denote by \( S_{n-1} \) the unit sphere in \( \mathbb{R}^n \). We consider domains that are star shaped with respect to the origin. They are defined as follows.

**DEFINITION 2.1.** The domain \( \Omega \) is star shaped with respect to the origin with a continuous function that characterizes the boundary, if there is a continuous function \( b(\nu) > 0 \) defined for \( \nu \in S_{n-1} \) such that,

\[
\Omega = \left\{ x \in \mathbb{R}^n : |x| < b \left( \frac{x}{|x|} \right) \right\}.
\]  

Note that

\[
\partial \Omega = \left\{ x \in \mathbb{R}^n : |x| = b \left( \frac{x}{|x|} \right) \right\}.
\]  

In this section we always assume that \( \Omega \) is star shaped with respect to the origin with a continuous function that characterizes the boundary. By Theorem 3.2 in page 67 of [16], or by Theorem 1 in page 10 of [14], if \( \Omega \) is star shaped with respect to the origin with a continuous boundary, \( C^\infty_0(\mathbb{R}^n)|_{\Omega} \) is dense in \( W^{1,1}_p(\Omega) \) and as \( C^\infty_0(\mathbb{R}^n)|_{\Omega} \subset C^\infty(\overline{\Omega}) \), also \( C^\infty(\overline{\Omega}) \) is dense in \( W^{1,1}_p(\Omega) \).
In the classical theory the trace maps are defined as bounded operators from $W_p^{(1)}(\Omega)$, $1 \leq p < \infty$, into $L^p(\partial \Omega)$, where $L^p(\partial \Omega)$ is the standard Banach space of all Lebesgue-measurable functions in $\partial \Omega$ whose absolute value to the power $p$ is Lebesgue integrable on $\partial \Omega$. This requires that $\partial \Omega$ is Lipschitz. To understand the origin of this restriction it is instructive to consider the case of domains in two dimensions. Let us use polar coordinates in $\mathbb{R}^2$, $x_1 = r \cos \theta, x_2 = r \sin \theta$, with $r := |x|, 0 \leq \theta < 2\pi$. In these coordinates the unit vector $\nu$ in $S_{n-1}$ can be written as follows

$$\nu = \nu(\theta) = \cos \theta e_{x_1} + \sin \theta e_{x_2},$$

(2.4)

with $e_{x_1}, e_{x_2}$, respectively, the unit vectors along the $x_1$ and the $x_2$ axis. Moreover, for simplicity we denote,

$$b(\theta) := b(\nu(\theta)).$$

Furthermore, by (2.3) $\partial \Omega$ is given in parametric form in the following way,

$$\partial \Omega = \{ x \in \mathbb{R}^2 : x = \varphi(\theta), 0 \leq \theta < 2\pi \},$$

where

$$\varphi(\theta) := b(\theta) \nu(\theta), \quad 0 \leq \theta < 2\pi.$$ 

Assume that $\partial \Omega$ is Lipschitz. For any set $O \subset \partial \Omega$, that is Lebesgue measurable, let us denote by $m_{L,S}(O)$ its Lebesgue measure. Then, for any function $f \in L^p(\partial \Omega)$ the integral over $\partial \Omega$ of its absolute value to the power $p$ is given by,

$$\int_{\partial \Omega} |f(\omega)|^p \, dm_{L,S}(\omega) = \int_0^{2\pi} |f(b(\theta) \nu(\theta))|^p |\varphi'(\theta)| \, d\theta.$$ 

(2.5)

Equation (2.3) shows clearly why in the classical theory of trace maps from $W_p^{(1)}(\Omega)$ into $L^p(\partial \Omega)$ it is necessary to impose the Lipschitz condition on $\partial \Omega$. Namely, if $\varphi(\theta)$ is Lipschitz, we have that $\varphi'(\theta)$ exists for almost every $\theta$ and it is bounded. In consequence, the right-hand side of (2.3) is well defined. In our case we only assume that $\varphi(\theta)$ is a continuous function, that does not have to be differentiable. Hence, in our situation the right-hand side of (2.3) makes no sense. So, in our case there no hope of constructing bounded trace maps with target space the standard $L^p(\partial \Omega)$, and we have to look for a different target space. This is certainly a radical departure from the standard point of view. There is, of course, a long tradition of taking the standard $L^p(\partial \Omega)$ as target space. However, on spite of this, there is no fundamental reason to take as target space the standard $L^p(\partial \Omega)$. What is important is to obtain trace maps that are useful tools in applications, for example, to boundary value problems, to spectral and scattering theory, or to a coarea formula. So, what are the options that we have in our case where $\varphi(\theta)$ is only assumed to be continuous. The simplest solution to our problem is just to eliminate $|\varphi'(\theta)|$ from the right-hand side of (2.3) and propose for the integral of $|f(\omega)|^p$ over $S$ the quantity

$$\int_0^{2\pi} |f(b(\theta) \nu(\theta)))|^p \, d\theta.$$ 

Going back to the $n$ dimensional case, for a function $f(\omega), \omega \in S$ we propose for the integral of $|f(\omega)|^p$ over $S$ the quantity

$$\int_{S_{n-1}} |f(\nu(\nu))|^p \, dm_{S_{n-1}}(\nu),$$

(2.6)

where for any set $A$ that is Lebesgue measurable in $S_{n-1}$, by $m_{S_{n-1}}(A)$ we denote its Lebesgue measure. It can be said that in (2.4) we take the pull back of the Lebesgue measure under the projection of $\partial \Omega$ onto $S_{n-1}$. We show below that with our proposal we obtain trace maps that have the main properties that make the standard trace maps useful, when the later exist. To accomplish this task we first have to precisely define our target spaces for the trace maps.

We proceed to define a measure in $\partial \Omega$ that is appropriate for our purposes. We define the following bijective function, $\mathcal{P}$, from $\partial \Omega$ onto $S_{n-1}$,

$$\mathcal{P}(\omega) := \nu = \frac{\omega}{|\omega|}, \text{ for } \omega \in \partial \Omega.$$ 

(2.7)

The inverse function is given by,

$$\mathcal{P}^{-1}(\nu) := \omega = b(\nu) \nu, \text{ for } \nu \in S_{n-1}.$$ 

(2.8)

Clearly $\mathcal{P}$ is onto and $\mathcal{P}$ and $\mathcal{P}^{-1}$ are continuous. As usual for $O \subset \partial \Omega$, we denote, $\mathcal{P}(O) := \{ \nu \in S_{n-1} : \nu = \mathcal{P}(\omega), \text{ for some } \omega \in O \}$. 

3
DEFINITION 2.2. We say that a set $O \subset \partial \Omega$ is $\partial \Omega$-measurable if $\mathcal{P}(O)$ is Lebesgue measurable in $S_{n-1}$. Further, for any $\partial \Omega$-measurable set $O$, we define its measure, in symbols $m_{\partial \Omega}(O)$, by

$$m_{\partial \Omega}(O) := m_{S_{n-1}}(\mathcal{P}(O)). \quad (2.9)$$

We designate by $\mathcal{M}_{\partial \Omega}$ the set of all $\partial \Omega$-measurable sets. Clearly, $\mathcal{M}_{\partial \Omega}$ is a $\sigma$-algebra and $m_{\partial \Omega}$ is a $\sigma$-additive measure on $\mathcal{M}_{\partial \Omega}$.

Let $f(\omega)$ be a function defined on $\partial \Omega$. We denote by $f_{S_{n-1}}(\nu)$ the function,

$$f_{S_{n-1}}(\nu) := f(b(\nu)\nu), \ \nu \in S_{n-1}. \quad (2.10)$$

Note that for any set $A \subset \mathbb{R}$,

$$\mathcal{P}(f^{-1}(A)) = f_{S_{n-1}}^{-1}(A).$$

Observe that $f(\omega)$ is $\partial \Omega$-measurable if and only if $f_{S_{n-1}}(\nu)$ is Lebesgue measurable. For $\partial \Omega$-integrable functions $f(\omega)$, we have that,

$$\int_{\partial \Omega} f(\omega) \, dm_{\partial \Omega}(\omega) = \int_{S_{n-1}} f_{S_{n-1}}(\nu) \, dm_{S_{n-1}}(\nu). \quad (2.11)$$

DEFINITION 2.3. We denote by $\mathcal{L}^p(\partial \Omega), 1 \leq p < \infty$, the Banach space of all complex valued, $\partial \Omega$-measurable functions, $f(\omega)$, such that $|f(\omega)|^p$ is integrable over $\partial \Omega$ with respect to the measure $m_{\partial \Omega}$. The norm in $\mathcal{L}^p(\partial \Omega)$ is given by

$$\|f\|_{\mathcal{L}^p(\partial \Omega)} := \left( \int_{\partial \Omega} |f(\omega)|^p \, dm_{\partial \Omega}(\omega) \right)^{1/p}. \quad (2.12)$$

We have that,

$$\|f\|_{\mathcal{L}^p(\partial \Omega)} = \left( \int_{S_{n-1}} |f_{S_{n-1}}(\nu)|^p \, dm_{S_{n-1}}(\nu) \right)^{1/p}. \quad (2.13)$$

We now state our trace map theorem.

THEOREM 2.4. Suppose that the domain $\Omega$ is star shaped with respect to the origin with a continuous function that characterizes the boundary. Then, there is a trace map $T_p$ that is bounded from $W^{(1)}_p(\Omega)$ into $\mathcal{L}^p(\partial \Omega), 1 \leq p < \infty$, such that

$$(T_p f)(\omega) = f(\omega), \ f(\omega) \in C^1(\overline{\Omega}). \quad (2.14)$$

Furthermore, the range of $T_p$ is dense in $\mathcal{L}^p(\partial \Omega), 1 \leq p < \infty.

$$\overline{T_p W^{(1)}_p(\Omega)} = \mathcal{L}^p(\partial \Omega), 1 \leq p < \infty. \quad (2.15)$$

Proof: Assume that $f(x) \in C^1(\overline{\Omega})$. Let $C_1 > 0$ be such that $0 < C_1 < b(\nu) < 1/C_1, \nu \in S_{n-1}$. Take $h(r) \in C_0^\infty((0, \infty))$, such that $h(r) = 0, 0 < r < C_1/4, h(r) = 1, C_1/2 < r < 2/C_1, h(r) = 0, r > 3/C_1$. Denote $g(x) = h(|x|) f(x)$. Then,

$$f(b(\nu)\nu) = g(b(\nu)\nu) = \int_0^{b(\nu)} \frac{\partial}{\partial \mu} g(\mu \nu) \, d\mu. \quad (2.16)$$

Hence, by the Hölder inequality,

$$|f(b(\nu)\nu)|^p \leq C \int_0^{b(\nu)} \left| \frac{\partial}{\partial \mu} g(\mu \nu) \right|^p \, d\mu \leq C b(\nu) \int_0^{b(\nu)} \left| \frac{\partial}{\partial \mu} g(\mu \nu) \right|^p \mu^{n-1} \, d\mu. \quad (2.17)$$
where in the last inequality we used that as \( g(x) = 0 \), for \( |x| \leq C_1/4 \), we have that \( \frac{1}{\mu} \leq C \) on the support of \( g(x) \). Moreover, by (2.11) and (2.17)

\[
\int_{\partial\Omega} |f(\omega)|^p \, d\mu_{\partial\Omega}(\omega) = \int_{S_{n-1}} |f(b(\nu))|^p \, d\mu_{S_{n-1}}(\nu) \leq \\
C \int_{S_{n-1}} d\mu_{S_{n-1}}(\nu) \int_{p_0} \left| \frac{\partial}{\partial x} g(\mu \nu) \right|^p \mu^{n-1} \, d\mu \leq \\
C \int_{S_{n-1}} d\mu_{S_{n-1}}(\nu) \int_{p_0} \left| (\nabla g)(\mu \nu) \right|^p \mu^{n-1} \, d\mu = \\
C \int_{\Omega} |\nabla g(x)|^p \, dx.
\]

But since \( g(x) = h(|x|) f(x) \), by (2.18),

\[
\|f\|_{L^p(\partial\Omega)} \leq C \|f\|_{W^{(1)}_p(\Omega)}.
\]

Finally, the existence of \( T_p \) follows since as \( C^\infty(\overline{\Omega}) \subset C^1(\overline{\Omega}) \), we have that \( C^1(\overline{\Omega}) \) is dense in \( W^{(1)}_p(\Omega) \).

Let us now prove (2.15). Suppose that \( f(\omega) \in L^p(\partial\Omega) \). Then, \( f_{S_{n-1}}(\nu) := f(b(\nu)) \in L^p(S_{n-1}) \). Since \( C^\infty(S_{n-1}) \) is dense on \( L^p(S_{n-1}) \) there is sequence \( g_m(\nu) \in C^\infty(S_{n-1}) \) such that,

\[
\lim_{m \to \infty}\| f_{S_{n-1}} - g_m \|_{L^p(S_{n-1})} = 0.
\]

Let us designate,

\[
f_m(x) := h(|x|) g_m \left( \frac{x}{|x|} \right) \in C^\infty_0(\mathbb{R}^n).
\]

Denote by \( f_m|_{\Omega}(x) \) the restriction of \( f_m(x) \) to \( \Omega \). Then, \( f_m|_{\Omega}(x) \in C^\infty(\overline{\Omega}) \). Further,

\[
\lim_{m \to \infty}\| f - T_p f_m \|_{L^p(\partial\Omega)} = \lim_{m \to \infty}\int_{S_{n-1}} |f(b(\nu)) - h(b(\nu)) g_m(\nu)|^p \, d\mu_{S_{n-1}}(\nu) = \\
\lim_{m \to \infty}\int_{S_{n-1}} |f(b(\nu)) - g_m(\nu)|^p \, d\mu_{S_{n-1}}(\nu) = \lim_{m \to \infty}\int_{S_{n-1}} |f_{S_{n-1}}(\nu) - g_m(\nu)|^p \, d\mu_{S_{n-1}}(\nu) = 0.
\]

\[\square\]

Note that the proof of Theorem 2.4 only requires that \( \frac{\partial}{\partial x} f(\mu \nu) \in L^p(\Omega \setminus B_{C/4}) \), where \( B_{C/4} \) is the ball of center zero and radius \( C/4 \). This allows for a more general formulation of Theorem 2.4.

The next theorem shows that our trace map \( T_p \) characterizes \( W^{(1)}_{p,0}(\Omega) \) as the Banach space of all functions in \( W^{(1)}_p(\Omega) \) that are zero on \( \partial\Omega \) in trace sense.

**THEOREM 2.5.** Suppose that the domain \( \Omega \) is star shaped with respect to the origin with a continuos function that characterizes the boundary. Then, for \( 1 \leq p < \infty \),

\[
W^{(1)}_{p,0}(\Omega) = \left\{ f(\omega) \in W^{(1)}_p(\Omega) : (T_p f)(\omega) = 0 \right\}.
\]

**Proof:** Clearly all the functions in \( W^{(1)}_{p,0}(\Omega) \) are zero on \( \partial\Omega \) in trace sense. Suppose that \( f(x) \in W^{(1)}_p(\Omega) \) and that \( (T_p f)(\omega) = 0 \). Then, eventually after redefining \( f(x) \) as equal to zero in a set of measure zero, we have that \( f(\omega) = 0, \omega \in \partial\Omega \). By Theorem 2.2 in page 61 and Theorem 2.3 in page 63 of [10], or Theorem 1 in page four of [14], (see also [11]), after redefining, if necessary, \( f(x) \) as equal to zero in a set of measure zero, we can assume that \( f(x) \) is absolutely continuous on all the lines parallel to the \( x_1 \) axis, and that the classical derivative, \( \frac{\partial}{\partial x_1} f(x) \), coincides with the distribution derivative, i.e., \( \frac{\partial}{\partial x_1} f(x) = \frac{\partial}{\partial x_1} f(x) \in L^p(\Omega) \). We extend \( f(x) \) by zero to a function defined in \( \mathbb{R}^n \) by setting \( f(x) = 0, x \in \mathbb{R}^n \setminus \Omega \) and we also denote by \( f(x) \) the function extended to \( \mathbb{R}^n \). We proceed to prove that \( \frac{\partial}{\partial x_1} f(x) \in L^p(\mathbb{R}^n) \). For any \( (x_2, \cdots, x_n) \in \mathbb{R}^{n-1} \) we denote by \( L(x_2, \cdots, x_n) \) the line parallel to the \( x_1 \) axis that passes through the point \( (0, x_2, \cdots, x_n) \), i.e.,

\[
L(x_2, \cdots, x_n) = \{(x_1, x_2, \cdots, x_n), x_1 \in \mathbb{R}\}.
\]
Take any $\varphi(x) \in C^0_0(\mathbb{R}^n)$. Since $f(x)$ is absolutely continuous on the line $L(x_2, \cdots, x_n)$ and $f(x) = 0$, for $x \in \partial \Omega \cap L(x_2, \cdots, x_n)$, integrating by parts, we obtain that,

$$
\int_{\mathbb{R}^n} f(x) \frac{\partial}{\partial x_1} \varphi(x) \, dx = \int_{\mathbb{R}^n-1} dx_2 \cdots dx_n \int_{L(x_2, \cdots, x_n)} dx_1 f(x_1, x_2, \cdots, x_n) \frac{\partial}{\partial x_1} \varphi(x_1, x_2, \cdots, x_n) = \int_{\mathbb{R}^n} \left[ \frac{\partial}{\partial x_1} f(x) \right] \varphi(x) \, dx.
$$

(2.24)

Then, $\frac{\partial}{\partial x_1} f(x) = \{ \frac{\partial}{\partial x_i} f(x) \} \in L^p(\mathbb{R}^n)$. We prove in the same way that $\frac{\partial}{\partial x_i} f(x) \in L^p(\mathbb{R}^n), i = 2, \cdots, n$. Hence $f(x) \in W^{(1)}_p(\mathbb{R}^n)$. For $\lambda > 1$ we define $f_\lambda(x) := f(\lambda x)$. Then, $f_\lambda(x) \in W^{(1)}_p(\mathbb{R}^n)$ and

$$
\lim_{\lambda \to 1} \| f(x) - f_\lambda(x) \|_{W^{(1)}_p(\mathbb{R}^n)} = 0.
$$

Finally, as for any fixed $\lambda > 1$, we have that $f_\lambda(x) = 0$, for $x \in \Omega \setminus \Omega_\lambda$, where $\Omega_\lambda := \{ x = r \nu, \nu \in S_{n-1}, 0 \leq r < \frac{b(r)}{\lambda^2} \}$, and $\overline{\Omega_\lambda} \subset \Omega$ it follows that $f_\lambda(x) \in W^{(1)}_{p,0}(\Omega)$.

By theorems 2.2 and 2.5 our trace map $T_p$ have the following properties,

1. For every $f \in C^1(\overline{\Omega})$, $T_p(f)(\omega) = f(\omega), \quad \omega \in S$.

2. The range of $T_p$ is dense in $L^p(\partial \Omega)$.

3. $W^{(1)}_{p,0}(\Omega) = \left\{ f(x) \in W^{(1)}_p(\Omega) : (T_p f)(\omega) = 0 \right\}$.

Properties (1), (2) and (3) are the main three properties that make useful the standard trace maps in the case of Lipschitz boundaries. This shows that our trace maps effectively replace the standard trace maps in the case where the boundaries are not Lipschitz.

We consider now a simple application of our trace theorem to the Dirichlet problem. Following Section 2 of Chapter 8 of [8] we formulate the Dirichlet problem as follows. Let $\Omega$ be any bounded domain in $\mathbb{R}^n$. Let $L$ be the formal differential operator,

$$
Lf := D_i \left( a^{ij}(x) D_j f + b^i(x)f \right) + c^i(x)D_i f + d(x)f,
$$

(2.25)

with $D_i := \frac{\partial}{\partial x_i}, i = 1, \cdots, n$, and where we use the convention of summation over repeated indices. We assume that the coefficients $a^{ij}, b^i, c^i$ and $d, i, j = 1, \cdots, n$ are Lebesgue measurable and bounded in $\Omega$, that $a^{ij}$ is strongly elliptic, i.e., that for some $\lambda > 0$,

$$
a^{ij}(x) \zeta_i \zeta_j \geq \lambda |\zeta|^2, \quad \forall x \in \Omega, \zeta \in \mathbb{R}^n.
$$

(2.26)

Furthermore, we assume the following negativity condition

$$
\int_{\Omega} (d \varphi - b^i D_i \varphi) \, dx \leq 0, \quad \forall \varphi \in C^0_0(\Omega), \varphi \geq 0.
$$

(2.27)

Moreover, let $g, h^i, i = 1, \cdots, n$, be locally integrable functions in $\Omega$. Formally the Dirichlet problem can be written as,

$$
Lf(x) = g(x) + D_i h^i(x), \quad x \in \Omega,
$$

$$
f(x) = h(x), \quad x \in \partial \Omega,
$$

(2.28)

for some function $h$ defined in $\partial \Omega$. In a precise mathematical sense the Dirichlet problem is formulated as follows. By a weak (or generalized) solution to the equation,

$$
Lf(x) = g(x) + D_i h^i(x), \quad x \in \Omega,
$$

(2.29)

we mean a function $f \in W^{(1)}_2(\Omega)$ such that,

$$
\int_{\Omega} \left[ (a^{ij} D_j f + b^i f) \right] D_i \varphi - (c^i D_i f + d f) \varphi \, dx = \int_{\Omega} \left( h^i D_i \varphi - g \varphi \right) \, dx \quad \forall \varphi \in C^0_0(\Omega).
$$

We have the following theorem
THEOREM 2.6. Suppose that the functions \(a^{ij}, b^j, c^i, 1 \leq i, j \leq n\) and \(d\) are Lebesgue measurable and bounded in \(\Omega\). Further, assume that (2.26) and (2.27) hold, and that the functions \(g, h^i \in L^2(\Omega), i = 1, \ldots, n\). Then, for every \(h \in W_2^1(\Omega)\), the equation (2.29) has a unique weak solution \(f \in W_2^1(\Omega)\) such that, \(f - h \in W_{2,0}^1(\Omega)\).

Proof: This result is Theorem 8.3 in page 181 of [6] where the proof is given.

\[ \square \]

Theorem 2.6 is a general existence and uniqueness result. However, it gives no information in how the solution \(f\) takes the values of \(h\) in \(\partial \Omega\). In fact, the situation is even worse, because as we only know that \(f, h \in W_2^1(\Omega)\), in the absence of a trace map, we can not even give a mathematical meaning to the values of \(f\) and \(h\) in \(\partial \Omega\). In the next theorem we provide a precise answer to this issue using our trace map.

THEOREM 2.7. Assume that the domain \(\Omega\) is star shaped with respect to the origin with a continuos function that characterizes the boundary. Further, suppose that (2.26) and (2.27) hold, and that the functions \(g, h^i \in L^2(\Omega), i = 1, \ldots, n\). Then, for every \(h \in W_2^1(\Omega)\), the unique weak solution \(f \in W_2^1(\Omega)\) to (2.29) such that, \(f - h \in W_{2,0}^1(\Omega)\), satisfies \(T_2f = T_2h\) in \(L^2(\partial \Omega)\). In particular, \(f(\omega) = h(\omega)\) for \(m_{\partial \Omega}\)- almost every \(\omega \in \partial \Omega\). Equivalently, \(f(b(\nu)\nu) = h(b(\nu)\nu)\) for Lebesgue almost every \(\nu \in S_{n-1}\).

Proof: The theorem follows from Theorem 2.6 and 2.8 since as \(f - h \in W_{2,0}^1(\Omega)\), we have that \(T_2f = T_2h\). \[ \square \]

Being able to give a precise mathematical meaning to the values of the data and the solution on the boundary, and proving that they coincide up to a set of measure zero, is an important piece of information, not only from the mathematical point of view, but also from the numerical perspective and furthermore, in order to use the solution to our problem in concrete applications.

3 The case of \(\mathbb{R}^n\)

We consider now traces on star-shaped surfaces of functions in \(W_{2}^{(s)}(\mathbb{R}^n), n \geq 2\). We already discussed this problem in 2.8. However, here we studied the traces within the context of the spectral and scattering theory of strongly propagative systems of equations and in the particular case of the slowness surface of these systems. Here we present our results in all its generality and in a more detailed way.

We define star shaped hypersurfaces as in the case of the boundary of a bounded domain, but we only assume that the function \(b(\nu)\) is Lebesgue measurable in \(S_{n-1}\).

DEFINITION 3.1. We say that a set \(S\) in \(\mathbb{R}^n\) is a star shaped with respect to the origin hypersurface characterized by a Lebesgue measurable function, if there is a function \(b(\nu) > 0\) defined for \(\nu \in S_{n-1}\) that is Lebesgue measurable in \(S_{n-1}\), such that,

\[ S = \left\{ x \in \mathbb{R}^n : |x| = b \left( \frac{x}{|x|} \right) \right\}. \tag{3.1} \]

Furthermore, we suppose that there is a constant \(C_1 > 0\) such that, \(0 < C_1 < b(\nu) < 1/C_1\), for \(\nu \in S_{n-1}\).

Let us define the Fourier transform on \(L^2(\mathbb{R}^n)\) as follows,

\[ (F_n f)(k) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ik \cdot x} f(x) \, dx. \]

Recall that, for \(l = 1, \ldots,\)

\[ k_j^l (F_n f)(k) = \left( F_n (-i)^l \frac{\partial^l}{\partial x^l_j} f(x) \right)(k), \quad j = 1, \ldots, n, \quad l = 1, \ldots, n, \tag{3.2} \]

and that,

\[ \frac{\partial^l}{\partial k^l_j} (F_n f)(k) = \left( F_n (-i)^l x^l_j f(x) \right)(k), \quad j = 1, \ldots, n. \tag{3.3} \]
As usual, the Sobolev space $W^{(s)}_2(\mathbb{R}^n), s > 0$, is defined as the set of all functions $f(x) \in L^2(\mathbb{R}^n)$ such that the Fourier transform $(F_n f)(k)$ satisfies, $(1 + k^2)^{s/2} (F_n f)(k) \in L^2(\mathbb{R}^n)$, with norm given by,

$$
\|f\|_{W^{(s)}_2(\mathbb{R}^n)} := \left\| (1 + k^2)^{s/2} F_n f \right\|_{L^2(\mathbb{R}^n)}.
$$

Clearly, $S$ is dense on $W^{(s)}_2(\mathbb{R}^n)$. Observe that when $s = 1$ the norm $(3.4)$ is equivalent to the norm $(2.3)$ with $\Omega = \mathbb{R}^n$ and $p = 2$. We use the norm $(3.4)$ for the simplicity of notation.

We define a measure in $S$ and the space $\mathcal{L}^2(S)$ as in the case of star shaped domains in Section 2. We first define the measure in $S$. Let $\mathcal{P}_S$ be the following function from $S$ onto $S_{n-1}$.

$$
\mathcal{P}_S(\omega) := \nu = \frac{\omega}{|\omega|}, \text{ for } \omega \in S.
$$

(3.5)

The inverse function is given by,

$$
\mathcal{P}_S^{-1}(\nu) := \omega = b(\nu) \nu, \text{ for } \nu \in S_{n-1}.
$$

(3.6)

As before $\mathcal{P}_S$ is onto and one-to-one.

**DEFINITION 3.2.** We say that a set $O \subset S$, is $S$-measurable if $\mathcal{P}_S(O)$ is Lebesgue measurable in $S_{n-1}$. Further, for any $S$-measurable set $O$, we define its measure, $m_S(O)$, by

$$
m_S(O) := m_{S_{n-1}}(\mathcal{P}_S(O)).
$$

(3.7)

By $\mathcal{M}_S$ we denote the $\sigma$-algebra of all $S$-measurable sets. Note that $m_S$ is a $\sigma$-additive measure on $\mathcal{M}_S$.

As before, for a function, $f(\omega)$, defined on $S$, we designate by $f_{S_{n-1}}(\nu)$ the function,

$$
f_{S_{n-1}}(\nu) := f(b(\nu) \nu), \nu \in S_{n-1}.
$$

(3.8)

Then, for any set $A \subset \mathbb{R}$,

$$
\mathcal{P}_S(f^{-1}(A)) = f_{S_{n-1}}^{-1}(A).
$$

We have that $f(\omega)$ is $m_S$-measurable if and only if $f_{S_{n-1}}(\nu)$ is Lebesgue measurable. For any $m_S$-integrable function $f(\omega)$, we have that,

$$
\int_S f(\omega) dm_S(\omega) = \int_{S_{n-1}} f_{S_{n-1}}(\nu) dm_{S_{n-1}}(\nu).
$$

(3.9)

**DEFINITION 3.3.** We denote by $\mathcal{L}^2(S)$ the Hilbert space of all complex valued, $m_S$-measurable functions, $f(\omega)$ such that $|f(\omega)|^2$ is integrable over $S$ with respect to the measure $m_S$. The scalar product in $\mathcal{L}^2(S)$ is given by

$$
\int_S f(\omega) \bar{g}(\omega) dm_S(\omega) = \int_{S_{n-1}} f_{S_{n-1}}(\nu) \bar{g}_{S_{n-1}}(\nu) dm_{S_{n-1}}(\nu).
$$

(3.10)

We find it convenient to define the following weighted space.

**DEFINITION 3.4.** We denote by $\mathcal{L}^2_\omega(S)$ the Hilbert space of all complex valued $m_S$-measurable functions that are square integrable on $S$ with the scalar product,

$$
(f,g)_{\mathcal{L}^2_\omega(S)} := \int_S f(\omega) \bar{g}(\omega) b^n(\omega/|\omega|) dm_S(\omega) = \int_{S_{n-1}} f_{S_{n-1}}(\nu) \bar{g}_{S_{n-1}}(\nu) b^n(\nu) dm_{S_{n-1}}(\nu).
$$

As there is a constant $C_1$ such that, $0 < C_1 < |b(\nu)| < 1/C_1, \nu \in S_{n-1}$, the norms of $\mathcal{L}^2(S)$ and of $\mathcal{L}^2_\omega(S)$ are equivalent.

The basic idea to prove our trace theorem is to parametrize $x \in \mathbb{R}^n$ as $x = \rho \omega, \rho > 0, \omega \in S$. In this parametrization taking the restriction to $S$ means to take a sharp value of $\rho$, and this should not require regularity in $\omega$. Hence, assuming that $b(\nu)$ is Lebesgue measurable should be sufficient. As we will show this is actually true.
We prepare the following results that we need for the proof of our trace theorem. Suppose that \( f(x) \in C_0^\infty(\mathbb{R}^n) \). Then, using spherical coordinates, we have that,
\[
\int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_{S_{n-1}} d\nu \int_0^\infty r^{n-1} |f(r\nu)|^2 = \int_{S_{n-1}} d\nu \, h(\nu),
\]
where
\[
h(\nu) := \int_0^\infty dr \, r^{n-1} |f(r\nu)|^2.
\]
(3.12)

For each fixed \( \nu \in S_{n-1} \), we perform the change of variable \( \rho = r/b(\nu) \) in the one dimensional integral on the right-hand side of (3.12) and we obtain that,
\[
h(\nu) = b^n(\nu) \int_0^\infty d\rho \, \rho^{n-1} |f(\rho b(\nu)\nu)|^2.
\]
(3.13)

Further, by (3.11), (3.13),
\[
\int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_{S_{n-1}} d\nu \, b^n(\nu) \int_0^\infty d\rho \, \rho^{n-1} |f(\rho b(\nu)\nu)|^2.
\]
(3.14)

We define the function,
\[
g(\rho, \nu) := \rho^{n-1} |f(\rho b(\nu)\nu)|^2.
\]
(3.15)

By (3.14) and (3.15), it follows that,
\[
\int_{\mathbb{R}^n} |f(x)|^2 \, dx = \int_{S_{n-1}} d\nu \, b^n(\nu) \int_0^\infty d\rho \, g(\rho, \nu).
\]
(3.16)

Since \( d\nu \times d\rho \) is a product measure, by Fubini’s theorem we can exchange the order of the integration of the function \( g(\rho, \nu) \) in the right-hand side of (3.16), to obtain that,
\[
\int_0^\infty d\rho \int_{S_{n-1}} d\nu \, b^n(\nu) g(\rho, \nu) = \int_{S_{n-1}} d\nu \int_0^\infty d\rho \, g(\rho, \nu).
\]
(3.17)

Let \( U \) be the following operator, first defined for \( f(x) \in C_0^\infty(\mathbb{R}^n) \),
\[
(Uf)(\rho, \omega) := \rho^{n-1} f(\rho \omega), \quad \rho > 0, \omega \in S.
\]
(3.19)

Then, by (3.15)
\[
\int_0^\infty d\rho \int_S |(Uf)(\rho, \omega)|^2 b^n(\omega/|\omega|) \, d\omega = \int_0^\infty d\rho \, \rho^{n-1} \int_{S_{n-1}} b^n(\nu) |f(\rho b(\nu)\nu)|^2 \, d\nu.
\]
(3.20)

By (3.15), (3.18) and (3.20)
\[
\int_0^\infty d\rho \int_S |(Uf)(\rho, \omega)|^2 b^n(\omega/|\omega|) \, d\omega = \int_{\mathbb{R}^n} |f(x)|^2 \, dx.
\]
(3.21)

Hence, \( U \) extends to a unitary operator from \( L^2(\mathbb{R}^n) \) onto \( L^2((0, \infty); L^2_0(S)) \). The unitarity of \( U \) plays an important role in the spectral and scattering theory of strongly propagative systems of equations in [20].

Let \( h(\lambda) \) be a function in \( C^\infty(\mathbb{R}) \) such that, \( 0 \leq h(\lambda) \leq 1, h(\lambda) = 0, \lambda < 1/4, h(\lambda) = 1, 1/2 < \lambda < \infty \). For \( \rho > 0 \) let us denote \( h_\rho(\lambda) = h(\lambda/\rho) \). Note that, \( h_\rho(\lambda) = 0, \lambda < \rho/4, h_\rho(\lambda) = 1, \rho/2 < \lambda < \infty \). For \( f(x) \in S \) we define the operator,
\[
(L_\rho(z)f)(x, \omega) = (1 + x^2)^{z/2} 1 \sqrt{2\pi} \int_{\mathbb{R}} d\lambda e^{-ix\lambda} h_\rho(\lambda) \left( F_n(1 + k^2)^{-z/2} F_n f \right)(\lambda \omega),
\]
where \( x \in \mathbb{R}, \omega \in S, z = \alpha + i\beta \in \mathbb{C}, 0 \leq \alpha \leq 4 \).

For any pair of Banach spaces, \( X, Y \), we denote by \( \mathcal{B}(X, Y) \) the Banach space of all bounded operators from \( X \) into \( Y \).
LEMMA 3.5. The operator $L_\rho(z), 0 \leq Re z \leq 4$, extends to a bounded operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}; L^2(S))$. Furthermore, for any $\rho_0 > 0$, and any $0 \leq \alpha \leq 4$, there is a constant, $C_{\rho_0}(\alpha)$, that depends only on $\rho_0$, and $\alpha$, such that,

$$
\|L_\rho(\alpha + i\beta)\|_{B(L^2(\mathbb{R}^n), L^2(\mathbb{R}; L^2(S)))} \leq C_{\rho_0}(\alpha) \frac{1}{\rho^{(n-1)/2}}, \quad \rho \geq \rho_0, 0 \leq \alpha \leq 4. \tag{3.23}
$$

Proof: Denote,

$$
\psi(\lambda, \omega) := \left(F_n^{-1}(1 + k^2)^{-i\beta/2} F_n f\right) (\lambda, \omega).
$$

Using Parseval’s identity for the one dimensional Fourier transform, taking into account that for some constant $C_1$, $0 < C_1 < b(\omega/|\omega|) < 1/C_1, \omega \in S$, as in the support of $h_\rho(\lambda), \lambda > \rho/4$, and using (3.24), we have that,

$$
\int_S dm_S(\omega) \int_\mathbb{R} dx |(L_\rho(i\beta)f)(x, \omega)|^2 = \int_S dm_S(\omega) \int_0^\infty d\lambda f_\rho(\lambda)(\lambda) \omega)^2 \leq C \frac{1}{\rho^{(n-1)/2}} \|U F_n^{-1}(1 + k^2)^{-i\beta/2} F_n f\|^2_{L^2(\mathbb{R}^n; L^2(S))} = \tag{3.24}
$$

where in the last equality we used that $U$ is unitary and Parseval’s identity for the Fourier transform on $\mathbb{R}^n$. It follows that $T_\rho(i\beta), \beta \in \mathbb{R}$, extends to a bounded operator from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}; L^2(S))$ and,

$$
\|L_\rho(i\beta)f\|_{L^2(\mathbb{R}; L^2(S))} \leq C \frac{1}{\rho^{(n-1)/2}} \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.25}
$$

Furthermore, for $f$ in Schwartz space denote,

$$
(M_\rho(z)f)(x, \omega) := (1 + x^2)^{-z/2} (L_\rho(z)f)(x, \omega). \tag{3.26}
$$

Then,

$$
\|L_\rho(4 + i\beta)f\|_{L^2(\mathbb{R}; L^2(S))} \leq C \|M_\rho(4 + i\beta)f\|_{L^2(\mathbb{R}; L^2(S))} + C \|x^4 M_\rho(4 + i\beta)f\|_{L^2(\mathbb{R}; L^2(S))}. \tag{3.27}
$$

We denote,

$$
\varphi(\lambda, \omega) := \left(F_n^{-1}(1 + k^2)^{-4+i\beta/2} F_n f\right) (\lambda, \omega). \tag{3.28}
$$

Arguing as (3.24), we prove that,

$$
\|M_\rho(4 + i\beta)f\|^2_{L^2(\mathbb{R}; L^2(S))} \leq C \frac{1}{\rho^{(n-1)/2}} \int_S dm_S(\omega) \omega^{n}(\omega/|\omega|) \int_0^\infty d\lambda f_\rho(\lambda) \omega)^2 \leq C \frac{1}{\rho^{(n-1)/2}} \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{3.29}
$$

Moreover, using (3.2) with $n = 1$, with $k$ replaced by $\lambda$ and $x$ replaced by $\lambda$, we obtain that,

$$
\|x^4 M_\rho(4 + i\beta)f\|^2_{L^2(\mathbb{R}; L^2(S))} \leq C \frac{1}{\rho^{(n-1)/2}} \int_S dm_S(\omega) \omega^{n}(\omega/|\omega|) \int_0^\infty d\lambda f_\rho(\lambda) \omega)^2 \leq C \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{3.30}
$$

Furthermore,

$$
\frac{\partial}{\partial \lambda} \varphi(\lambda, \omega) = i \sum_{l=1}^n \omega_j \left(2\pi\right)^{n/2} \int_{\mathbb{R}^n} dk e^{i\lambda \omega \cdot k} (1 + k^2)^{-4+i\beta/2} (F_n f)(k). \tag{3.31}
$$

Then, using (3.20), (3.31) as well as its derivatives with respect to $\lambda$, and Parseval’s identity on $L^2(\mathbb{R}^n)$, we obtain that,

$$
\int_S dm_S(\omega) \omega^{n}(\omega/|\omega|) \int_0^\infty d\lambda f_\rho(\lambda) \omega)^2 \leq C \|f\|^2_{L^2(\mathbb{R}^n)}. \tag{3.32}
$$

Further, by (3.27), (3.29), (3.30) and (3.32)

$$
\|L_\rho(4 + i\beta)f\|_{L^2(\mathbb{R}; L^2(S))} \leq C \frac{1}{\rho^{(n-1)/2}} \|f\|_{L^2(\mathbb{R}^n)}. \tag{3.33}
$$

Note that it follows from our estimates that the constants $C$ in (3.24) and (3.33) are uniform for $\rho \geq \rho_0$, for any $\rho_0 > 0$. By (3.25), (3.31) and Hadamard three lines theorem, [17], page 33, $L_\rho(\alpha + i\beta), 0 \leq \alpha \leq 4$, is bounded from $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}; L^2(S))$, and (3.23) holds. \qed

10
REMARK 3.6. Suppose that $s > 1/2$. Then, for any $\rho, \rho_1 > 0$ there is a constant $C_s$ that depends only on $s$ such that,

$$\int_{\mathbb{R}} dx \frac{|e^{ix\rho} - e^{ix\rho_1}|^2}{(1 + |x|)^{2s}} \leq C_s \begin{cases} 
|\rho - \rho_1|^{2s-1}, & \frac{1}{2} < s < \frac{3}{2}, \\
|\rho - \rho_1|^2 (1 + |\ln|\rho - \rho_1||), & s = \frac{3}{2}, \\
|\rho - \rho_1|^2, & s > \frac{3}{2}.
\end{cases}$$

(3.34)

Proof: We can assume that $|\rho - \rho_1| \leq 1$. Then, for any $R > 1$,

$$\int_{\mathbb{R}} dx \frac{|e^{ix\rho} - e^{ix\rho_1}|^2}{(1 + |x|)^{2s}} = \int_{|x| \geq R-1} dx \frac{|e^{ix\rho} - e^{ix\rho_1}|^2}{(1 + |x|)^{2s}} + \int_{|x| \leq R-1} dx \frac{|e^{ix\rho} - e^{ix\rho_1}|^2}{(1 + |x|)^{2s}} \leq$$

$$C_s \begin{cases} 
R^{1-2s} + R^{3-2s} |\rho - \rho_1|^2, & \frac{1}{2} < s < \frac{3}{2}, \\
R^{1-2s} + |\rho - \rho_1|^2 \ln R, & s = \frac{3}{2}, \\
R^{1-2s} + |\rho - \rho_1|^2, & s > \frac{3}{2}.
\end{cases}$$

Finally, (3.34) follows taking $R = |\rho - \rho_1|^{-1}$. ~\(\square\)

In the following theorem we state our trace map.

THEOREM 3.7. Let the set $S$ in $\mathbb{R}^n$ be a star shaped with respect to the origin hypersurface characterized by a Lebesgue measurable function. Then, for every $\rho > 0$ and every $s > 1/2$, there is a trace map $T_s(\rho)$ that is bounded from $W_2^{(s)}(\mathbb{R}^n)$ into $L^2(S)$ such that for every function $f(x)$ in the space of Schwartz,

$$(T_s(\rho)f)(\omega) = f(\rho \omega).$$

(3.35)

Moreover, for every $\rho_0 > 0$ there is constant $C_{\rho_0}$ such that,

$$\|T_s(\rho)\|_{B(W_2^{(s)}(\mathbb{R}^n), L^2(S))} \leq C_{\rho_0} \frac{1}{\rho^{(n-1)/2}}, \quad \rho \geq \rho_0,$$

(3.36)

and, furthermore,

$$\|T_s(\rho) - T_s(\rho_1)\|_{B(W_2^{(s)}(\mathbb{R}^n), L^2(S))} \leq C_{\rho_0} \frac{1}{\min[\rho^{(n-1)/2}, \rho_1^{(n-1)/2}] \times}$$

$$\begin{cases} 
|\rho - \rho_1|^{s-1/2}, & \frac{1}{2} < s < \frac{3}{2}, \\
|\rho - \rho_1|(1 + \sqrt{|\ln|\rho - \rho_1||}), & s = \frac{3}{2}, \\
|\rho - \rho_1|, & s > \frac{3}{2},
\end{cases}$$

(3.37)

for $\rho, \rho_1 \geq \rho_0$. Moreover, the range of $T(\rho)$ is dense in $L^2(S)$,

$$T(\rho)W_2^{(s)}(\mathbb{R}^n) = L^2(S).$$

(3.38)

Proof: For $f(x) \in S$ we define,

$$(T(\rho)f)(\omega) := f(\rho \omega), \quad \omega \in S.$$  \hspace{1cm} (3.39)

Let us denote,

$$g(x) := \left(F_n^{-1}(1 + k^2)^{s/2}(F_n f)(k)\right)(x).$$

(3.40)

Note that,

$$f(\rho \omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx e^{i\rho x}(1 + x^2)^{-s/2}(L_{\rho}(s)g)(x, \omega).$$

(3.41)
Then, by Schwarz inequality, for $s > 1/2$,

$$|f(\rho \omega)|^2 \leq C \int_{\mathbb{R}} dx \ |(L_\rho(s)g)(x, \omega)|^2.$$ 

Hence, for any $\rho_0 > 0$, and for any $\rho \geq \rho_0$,

$$\int_\mathcal{S} dm_\mathcal{S}(\omega) \ |f(\rho \omega)|^2 \leq C \int_\mathcal{S} dm_\mathcal{S}(\omega) \int_{\mathbb{R}} dx \ |(L_\rho(s)g)(x, \omega)|^2 =$$

$$C \|L_\rho(s)g\|_{L^2(\mathbb{R}; L^2(S))}^2 \leq C C_{\rho_0}(s) \frac{1}{\rho^{s-1}} \|g\|_{L^2(\mathbb{R}^n)}^2 = C C_{\rho_0}(s) \frac{1}{\rho^{s-1}} \|f\|_{W^{s,2}_s(\mathbb{R}^n)}^2,$$

(3.42)

where we used (3.23). It follows that $T(\rho)$ is bounded and hence, it extends uniquely to a bounded operator from $W^{s}_2(\mathbb{R}^n)$ into $L^2(S)$ and (3.39) holds. Let us now prove (3.37). Without loss of generality we can assume that $\rho_1 > \rho \geq \rho_0$. Suppose that $f(x) \in \mathcal{S}$. Since $h_\rho(\lambda) = 1, \lambda > \rho/2$, we have that $h_\rho(\rho_1) = 1$. In consequence, it follows that,

$$(T(\rho_1)f)(\omega) = f(\rho_1 \omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \ e^{ip_1 x} (1 + x^2)^{-s/2} (L_\rho(s)g)(x, \omega),$$

(3.43)

where $g(x)$ is defined in (3.40). Further, by (3.39), (3.41) and (3.43)

$$(T(\rho)f)(\omega) - (T(\rho_1)f)(\omega) = f(\rho \omega) - f(\rho_1 \omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \ (e^{ip \omega x} - e^{ip_1 \omega x}) (1 + x^2)^{-s/2} (L_\rho(s)g)(x, \omega).$$

(3.44)

Hence, by Schwarz inequality,

$$\|(T(\rho) - T(\rho_1))f\|_{L^2(S)}^2 \leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dm_\mathcal{S}(\omega) \int_{\mathbb{R}} dx \ \frac{|e^{ip \omega x} - e^{ip_1 \omega x}|^2}{(1 + |x|)^{2s}} \int_{\mathbb{R}} dx \ |(L_\rho(s)g)(x, \omega)|^2 =$$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \ \frac{|e^{ip \omega x} - e^{ip_1 \omega x}|^2}{(1 + |x|)^{2s}} L^2(S) \leq C_{\rho_0}(s) \frac{1}{\rho^{s-1}} \|f\|_{W^{s,2}_s(\mathbb{R}^n)}^2,$$

(3.45)

where we used (3.29) and (3.40). Then, (3.37) follows from (3.44) and (3.45).

We proceed now to prove (3.38) as in the case of Theorem 2.4. Take any $f(\omega) \in L^2(S)$. It follows that, $f_S(\nu) := f(b(\nu)\nu) \in L^2(S_{n-1})$. As $C^\infty(S_{n-1})$ is dense on $L^2(S_{n-1})$, there is sequence $g_m(\nu) \in C^\infty(S_{n-1})$ such that,

$$\lim_{m \to \infty} \|f_S - g_m\|_{L^2(S_{n-1})} = 0.$$ 

(3.46)

Let $C_1$ be such that, that $0 < C_1 < b(\nu) < 1/C_1, \nu \in S_{n-1}$. Take $h(r) \in C^\infty_0((0, \infty))$, such that $h(r) = 0, 0 < r < C_1/4, h(r) = 1, C_1/2 < r < 2/C_1, h(r) = 0, r > 3/C_1$, and denote, $h_\rho(r) := h(r/\rho)$. We define,

$$f_m(x) := h_\nu(|x|) g_m \left( \frac{x}{|x|} \right) \in C^\infty(\mathbb{R}^n) \subset W^{s}_2(\mathbb{R}^n).$$ 

(3.47)

Finally,

$$\lim_{m \to \infty} \|f - T(\rho)f_m\|_{L^2(S)}^2 = \lim_{m \to \infty} \int_{S_{n-1}} |f(b(\nu)\nu) - h_\rho(\rho b(\nu)) g_m(\nu)|^2 dm_{S_{n-1}}(\nu) =$$

$$\lim_{m \to \infty} \int_{S_{n-1}} |f(b(\nu)\nu) - g_m(\nu)|^2 dm_{S_{n-1}}(\nu) = \lim_{m \to \infty} \int_{S_{n-1}} |f_{S_{n-1}} - g_m(\nu)|^2 dm_{S_{n-1}}(\nu) = 0.$$

(3.48)

In Theorem A.1 of [20] we proved a result like Theorem 3.7 in the case where $S$ is the slowness surface of a strongly propagative system of equations. A slowness surface is defined in terms of a continuous function, $\lambda(k), k \in \mathbb{R}^n \setminus \{0\}$ with values in $(0, \infty)$ that is homogeneous of order one, i.e. $\lambda(\rho k) = \rho \lambda(k), \rho > 0, k \in \mathbb{R}^n \setminus \{0\}$. The slowness surface is given by $S := \{k \in \mathbb{R}^n : \lambda(k) = 1\}$. This corresponds in Definition 3.1 to $b(\nu) = 1/\lambda(\nu), \nu \in S_{n-1}$. Note that in Theorem A.1 of [20] the precise estimates (3.38), (3.39) and the density in $L^2(S)$ of the range of the trace map $T(\rho)$ are not proved. In [20] the trace map in Theorem A.1 is applied to the spectral and the scattering theory of the strongly propagative systems of equations. Note that the Hölder continuity of the trace map plays an essential role.
in the spectral and scattering theory of strongly propagative systems of equations. Furthermore, the boundedness of the operator $T_s(\rho)$ was proven in Proposition 1.11 in page 115 of [21]. In fact, estimate (1.35) in page 115 of [21], with $\kappa = 1$, is equivalent to the boundedness of the operator $T_s(\rho)$ that we prove in Theorem 3.7. Remark that in Proposition 1.11 in page 115 of [21] a notation that is different from ours is used. Moreover, in Proposition 1.11 in page 115 of [21] the Hölder continuity and the precise estimates (3.36) and (3.37) for the norm of $T_s(\rho)$ are not obtained. The proof of Proposition 1.11 in page 115 of [21] is different from ours. It is based in the Mourre method. For the use of the Mourre method in this context see [19].

4 The coarea formula

For the coarea formula in the case where the level sets are given by a Lipschitz function see [11]. For the case where the level sets are given by a function in a Sobolev space look to [13]. See also [5]. Let us consider the coarea formula in a simple case. Denote,

$$a(x) := \frac{|x|}{b(x/|x|)},$$

and suppose that $b(\nu), \nu \in S_{n-1}$ is smooth and that for some $C_1 > 0, C_1 < b(\nu) < 1/C_1, \nu \in S_{n-1}$. Denote,

$$S_\lambda = \{x \in \mathbb{R}^n : a(x) = \lambda\},$$

and

$$\Omega_\lambda := \{x \in \mathbb{R}^n : a(x) < \lambda\}.$$

Note that $S$ in (3.1) satisfies $S = S_1$. Denote by $V_\lambda$ the volume of $\Omega_\lambda$ with respect to the Lebesgue measure. Then, by the coarea formula,

$$V_\lambda = \int_0^\lambda d\rho \int_{S_\rho} \frac{1}{|\nabla a(x)|} d\beta_\rho(x), \quad (4.1)$$

where $d\beta_\rho(x)$ denotes the Lebesgue measure of $S_\rho$. Furthermore,

$$\frac{d}{d\lambda} V(\lambda) = \int_{S_\lambda} \frac{1}{|\nabla a(x)|} d\beta_\lambda(x). \quad (4.2)$$

The right hand-side of (4.2) is a trace with the standard trace map in the surface $S_\lambda$.

We now prove that with our trace map we can generalize (4.1) and (4.2) to the case where $b(\nu)$ is only assumed to be Lebesgue measurable and, of course to satisfy $0 < C < b(\nu) < 1/C, \nu \in S_{n-1}$. Using spherical coordinates, we have that,

$$V_\lambda = \int_{S_{n-1}} dm_{S_{n-1}}(\nu) \int_0^{\lambda b(\nu)} dr r^{n-1} = \frac{\Lambda^n}{n} \int_{S_{n-1}} dm_{S_{n-1}}(\nu) b^n(\nu) = \int_0^\lambda d\rho \rho^{n-1} \int_{S_{n-1}} b^n(\nu) dm_{S_{n-1}}(\nu). \quad (4.3)$$

As in the case of $S = S_1$ we define a measure on $S_\rho$ as follows. By $\mathcal{P}_{S_\rho}$ we denote the function from $S_\rho$ onto $S_{n-1}$ given by,

$$\mathcal{P}_{S_\rho}(\omega) := \nu = \frac{\omega}{|\omega|}, \text{ for } \omega \in S_\rho.$$

A set $O \subset S_\rho$, is $S_\rho$-measurable if $\mathcal{P}_{S_\rho}(O)$ is Lebesgue measurable in $S_{n-1}$, and the measure of $O$ is given by,

$$m_{S_\rho}(O) := \rho^{n-1} m_{S_{n-1}}(\mathcal{P}_{S_\rho}(O)).$$

As in Section 2, for a function, $f(\omega)$, defined on $S_\rho$, we denote,

$$f_{S_{n-1}}(\nu) := f(\rho b(\nu) \nu), \nu \in S_{n-1},$$

13
and $f(\omega)$ is $S_\nu$-measurable if and only if $f_{S_{n-1}}(\nu)$ is Lebesgue measurable. For any $S_\nu$-integrable functions $f(\omega)$, we have that,

$$\int_{S_\nu} f(\omega) \, dm_{S_\nu}(\omega) = \rho^{n-1} \int_{S_{n-1}} f_{S_{n-1}}(\nu) \, dm_{S_{n-1}}(\nu). \tag{4.4}$$

With this definition, (4.3) reads,

$$V_\lambda = \int_0^\lambda d\rho \int_{S_\nu} b^n(\omega/|\omega|) \, dm_{S_\nu}(\omega). \tag{4.5}$$

Furthermore, by (4.5)

$$\frac{d}{d\lambda} V_\lambda = \int_{S_\nu} b^n(\omega/|\omega|) \, dm_{S_\nu}(\omega). \tag{4.6}$$

Formulae (4.5) and (4.6) generalize, respectively, (4.1) and (4.2) to the case where $b(\nu)$ is only assumed to be Lebesgue measurable.

Moreover, in the case where $b(\nu)$ is smooth, for any function $f(x) \in C^\infty_0(\mathbb{R}^n)$, the coarea formula implies that,

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_0^\infty d\rho \int_{S_\nu} f(x) \frac{1}{|\nabla a(x)|} \, d\beta_\rho(x). \tag{4.7}$$

With our method, we prove as in the proof of (3.18) that,

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_0^\infty d\rho \rho^{n-1} \int_{S_{n-1}} f(\rho b(\nu) \nu) b^n(\nu) \, dm_{S_{n-1}}(\nu). \tag{4.8}$$

Then, using (4.1) and (4.8) we obtain that

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_0^\infty d\rho \int_{S_\nu} f(\omega) b^n(\omega/|\omega|) \, dm_{S_\nu}(\omega). \tag{4.9}$$

Formula (4.9) generalizes the coarea formula (4.7) to the case where $b(\nu)$ is only assumed to be Lebesgue measurable. Note that with the notation of Theorem 5.7, with $S = S_1$, equation (4.9) can be written as follows,

$$\int_{\mathbb{R}^n} f(x) \, dx = \int_0^\infty d\rho \rho^{n-1} \int_{S} (T(\rho)f)(\omega) b^n(\omega/|\omega|) \, dm_{S}(\omega). \tag{4.10}$$

Since $T(\rho)$ is bounded, (4.10) extends by continuity to functions in $L^1(\mathbb{R}^n) \cap W_2^{1,s}(\mathbb{R}^n)$, $s > 1/2$.

Actually, it can be verified directly that our formulae coincide with the ones given by the coarea formula when $b(\nu)$ is smooth. For simplicity, we do the calculations for $n = 2$, but the result is also true for $n \geq 3$ with a similar computation. As in Section 2, let us take polar coordinates $x_1 = r \cos \theta, x_2 = r \sin \theta$, with $r = |x|, 0 \leq \theta < 2\pi$, and we denote $b(\theta) := b(\nu(\theta)), e(\theta) := 1/b(\theta)$, with $\nu(\theta)$ as in (2.4). In these coordinates,

$$a(r, \theta) := a(r \nu(\theta)) = r \, e(\theta).$$

Then, on $S_\rho$,

$$|\nabla a(r, \theta)| = \sqrt{(e(\theta))^2 + (e'(\theta))^2}. \tag{4.11}$$

Let us now write $S_\rho$ in parametric form,

$$S_\rho = \{ x \in \mathbb{R}^2 : x = \varphi(\theta), 0 \leq \theta \leq 2\pi \}.$$

Recall that, $e_{x_1}, e_{x_2}$ are, respectively, the unit vectors along the $x_1$ and the $x_2$ axis. Then, in polar coordinates,

$$\varphi(\theta) = \frac{\rho}{e(\theta)} (\cos \theta e_{x_1} + \sin \theta e_{x_2}).$$

It follows that,

$$|\varphi'(\theta)| = \frac{\rho}{(e(\theta))^2} \sqrt{(e(\theta))^2 + (e'(\theta))^2}.$$
Then,
\[ d\beta_\rho(x) \equiv d\beta_\rho(\theta) = \frac{\rho}{e^2(\theta)} \sqrt{\left(e(\theta)\right)^2 + \left(e'(\theta)\right)^2} \, d\theta. \]

Finally, using (4.11) and that \( e(\theta) = 1/b(\theta) \), we obtain that,
\[ \frac{1}{|\nabla a(x)|} \, d\beta_\rho(x) = \rho b^2(\theta) \, d\theta. \quad (4.12) \]

By (4.12) the right-hand sides of (4.1) and (4.5) are the same, the right-hand sides of (4.2) and (4.6) coincide, and the right-hand sides of (4.7) and (4.9) take the same value.

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References

[1] R. A. Adams, J. J. F. Fournier, *Sobolev Spaces*, Elsevier Science, Oxford, U.K., 2003.

[2] S. Agmon, L. Hörmander, Asymptotic properties of solutions of differential equations with simple characteristics, J. Analyse Math. 30, 1-38 (1976).

[3] Yu. Burago, N. Kosovsky, Boundary trace for BV functions in regions with irregular boundary, in *Analysis, Partial Differential Equations and Applications, The Vladimir Maz’ya Anniversary Volume, Operator Theory, Advances and Applications*, Birkhäuser, Basel, 2009, 1-14.

[4] H. Federer, *Geometric Measure Theory*, Springer, Berlin 1969.

[5] W. H. Fleming, R. W. Rishel, An integral formula for total gradient variation, Arch. Math. 11, 218-222 (1960).

[6] D. Gilberg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin 2001.

[7] L. Hörmander, *The Analysis of Linear Partial Differential Operators II Diff erential Operators with Constant Coefficients*, Springer, Berlin, 2005.

[8] D. Jerison, C. E. Kenig, The inhomogeneous Dirichlet problem in Lipschitz domains, J. Funct. Anal 130, 161-219 (1995).

[9] A. Jonsson, H. Wallin, *Functions Spaces on Subsets of R^n*, Mathematical Reports 2, Harwood Academic, London, 1984.

[10] T. Kato, *Perturbation Theory of Linear Operators*, Springer, Berlin, 1995.

[11] B. Levi, Sul principio di Dirichlet, Rend. Palermo 22, 293-359 (1906).

[12] J.L. Lions, E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications Volume I*, Springer, Berlin, 1972.

[13] J. Malý, D. Swanson, W. P. Ziemer, The coarea formula for Sobolev mappings, Trans. Am. Math. Soc. 355, 477-492. (2003).

[14] V. G. Maz’ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, Springer, Berlin, 2011.

[15] V. G. Maz’ya, S. V. Podorchi, *Dif ferentiable Functions on Bad Domains*, World Scientific, Singapore, 2011.

[16] J. Néčas, *Les Méthodes Directes en Théorie des Équations Elliptiques*, Masson, Paris, Academie, Prague, 1967.

[17] M. Reed, B. Simon, *Methods of Modern Mathematical Physics II Fourier Analysis, Self-Adjointness*, Academic, San Diego, 1975.
[18] P. A. Shvartsman, Sobolev $W^1_p$–spaces on closed subsets of $R^n$, Adv. Math. 220, 1842-1922 (2009).

[19] R. Weder, Spectral analysis of strongly propagative systems, J. Reine Angew. Math. (Crelles) 354, 95-122 (1984).

[20] R. Weder, Analyticity of the scattering matrix for wave propagation in crystals, J. Math. Pures et Appl., 64, 121-148 (1985).

[21] D. R. Yafaev, *Mathematical Scattering Theory Analytic Theory*, AMS, Providence, Rhode Island, 2000.