Research Article

Global Existence of Solutions for the Viscoelastic Kirchhoff Equation with Logarithmic Source Terms

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Received 24 January 2020; Revised 16 February 2020; Accepted 27 February 2020; Published 4 April 2020

Guest Editor: Karthikeyan Rajagopal

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In this paper, a nonlinear viscoelastic Kirchhoff equation in a bounded domain with a time-varying delay term and logarithmic nonlinearity in the weakly nonlinear internal feedback is considered, where the global and local existence of solutions in suitable Sobolev spaces by means of the energy method combined with Faedo-Galerkin procedure is proved with respect to the condition of the weight of the delay term in the feedback and the weight of the term without delay and the speed of delay. Furthermore, a general stability estimate using some properties of convex functions is given. These results extend and improve many results in the literature.

1. Introduction

1.1. Model. In this paper, we consider the global existence and decay properties of solutions for the initial boundary value problem of the following viscoelastic nondegenerate Kirchhoff equation of the form:

\[
\begin{align*}
|u_t|^\nu u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + & \int_0^t h(t-s)\Delta u(s)ds \\
+ & \mu_1 g_1(u_t(x,t)) + \mu_2 g_2(u_t(x,t-\tau(t))) = \nu\ln|u|, \quad \text{in } \Omega \times [0, +\infty[,

u(x, t) = 0, \quad \text{on } \partial\Omega \times [0, +\infty[,

u(x, 0) = u_0(x), u_t(x,0) = u_1(x), \quad \text{in } \Omega,

u_t(x, t-\tau(0)) = f_0(x, t-\tau(0)), \quad \text{in } \Omega \times [0, \tau(0[),
\end{align*}
\]

(1)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\), \(n \in \mathbb{N}^*\), with a smooth boundary \(\partial\Omega\), \(l > 0\), \(v, \mu_1\), and \(\mu_2\) are positive real numbers, \(h\) is a positive function which decays exponentially, \(\tau(t) > 0\) is a time-varying delay, \(g_1\) and \(g_2\) are two functions, and the initial data \((u_0, u_1, f_0)\) are in a suitable function space. \(M(r) = a + br^\gamma\) is a \(C^1\)-function for \(r \geq 0\), with \(a, b > 0\) and \(\gamma \geq 1\).
In the absence of delay term (i.e., \( \mu = 2 \)), Han and Wang in [1] considered the following nonlinear viscoelastic equation with damping:

\[
\begin{align*}
|u_i|u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t h(t-s)\Delta u(s)\,ds \\
+ u_x(x,t) = 0, \text { in } \Omega \times ]0, +\infty[.
\end{align*}
\]  

Time delay is often present in applications and practical problems. In recent years, the control of PDEs with time delay effects has become an active area of research (see, for example, [2–4]). For example, in [5], it has been proven that a small delay in a boundary control could turn a well-behaved hyperbolic system into a wild one, thus showing that delay can be a source of instability.

Wu [6] treated problem (1) for a constant time delay \( \tau \) and \( g_1(x) = g_2(x) = x \). He proved the local existence result using the Faedo-Galerkin method and established the decay result employing suitable Lyapunov functionals under appropriate conditions on \( \mu_1 \) and \( \mu_2 \), and on the kernel \( h \).

Benaissa et al. [7] considered the case of constant time delay \( \tau \), with \( l = 0 \) and \( M(r) = 1 \). They proved the global existence and uniform decay for the following problem:

\[
\begin{align*}
u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)\,ds + \mu_1 g_1(u_t(x,t)) \\
+ \mu_2 g_2(u_x(x,t-\tau)) = 0, \text { in } \Omega \times ]0, +\infty[.
\end{align*}
\]  

The same problem (3) was also treated by Kirane and Said-Houari [8] for \( g_1(x) = g_2(x) = x \) and a homogeneous right hand side with \( \tau \), a constant time delay. Daewook [9] considered a viscoelastic Kirchhoff equation, with a time-varying delay and a nonlinear source term, given as

\[
\begin{align*}
u_{tt} - M(x,t,\|u\|^2)\Delta u + \int_0^t h(t-s)\text{div}(a(x)u(x,s))ds + |u|^mu \\
+ \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)) = 0, \text { in } \Omega \times ]0, +\infty[,
\end{align*}
\]  

This equation describes axially moving viscoelastic materials. Using the smallness condition with respect to Kirchhoff coefficient and the relaxation function and by assuming \( 0 \leq m \leq (2/(n-2)) \) if \( n > 2 \) or \( 0 \leq m \leq 2 \) if \( n \leq 2 \), he obtained the uniform decay rate of the Kirchhoff-type energy.

In [10], the authors studied homogeneous problem (1) without the viscoelastic term, with \( l = 0 \) and \( M(r) = 1 \). In addition, \( \mu_1 g_1 \) and \( \mu_2 g_2 \) are multiplied by a positive non-increasing function \( \sigma \) of \( C^1(R_+) \) satisfying \( \int_0^\infty \sigma(s)\,ds = +\infty \) and \( |\sigma'(t)| \leq c \sigma(t) \). They proved the global existence, and using a multiplier method with some properties of convex functions to get decay rate of the energy (when \( t \) goes to infinity) depends on the function \( \sigma \) and on the function \( H \) which represents the growth at the origin of \( g_1 \).

Apart from the aforesaid attention given to polynomial nonlinear terms, logarithmic nonlinearity has also received a great deal of interest from both physicists and mathematicians. This type of nonlinearity was introduced in the nonrelativistic wave equations describing spinning particles moving in an external electromagnetic field and also in the relativistic wave equation for spinless particles [11]. Moreover, the logarithmic nonlinearity appears in several branches of physics such as inflationary cosmology [12], nuclear physics [13], optics [14], and geophysics [15]. With all this specific underlying meaning in physics, the global-time well-posedness of solution to the problem of evolution equation with such logarithmic-type nonlinearity captures lots of attention. Birula and Mycielski [16, 17] studied the following problem:

\[
\begin{align*}
u_{tt} - u_{xx} + u - \varepsilon \ln|u|^2 = 0, \quad &\text { in } [a,b] \times (0, T), \\
u(a,t) = u(b,t) = 0, \quad &\text { in } (0, T), \\
u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \quad &\text { in } [a,b],
\end{align*}
\]  

which is a relativistic version of logarithmic quantum mechanics and can also be obtained by taking the limit \( p \) goes to 1 for the \( p \)-adic string equation [18, 19]. In [20], Cazenave and Haraux considered

\[
u_{tt} - \Delta u = u \ln|u|^k, \quad \text { in } \mathbb{R}^3,
\]  

and they established the existence and uniqueness of the solution for the Cauchy problem. Gorka [21] used some compactness arguments and obtained the global existence of weak solutions, for all

\[
(u_0, u_1) \in H^1_0(\Omega) \times L^2([a, b]),
\]  

to initial boundary value problem (5) in the one-dimensional case. Bartkowski and Gorka [22] proved the existence of classical solutions and investigated the weak solutions for the corresponding one-dimensional Cauchy problem for equation (6). Hiramatsu et al. [23] introduced the following equation:

\[
u_{tt} - \Delta u + u + u_t + |u|^mu = \varepsilon \ln|u|,
\]  

to study the dynamics of Q-ball in theoretical physics and presented a numerical study. However, there was no theoretical analysis for the problem. In [24], Han proved the global existence of weak solutions, for all

\[
(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega),
\]  

to initial boundary value problem (8) in \( \mathbb{R}^3 \).

In the present paper, we investigate the stabilization of a dynamic model describing a string with a rigid surface and an interior somehow permissive to slight deformations. This leads to a varying material density \( |u_t|^2 \) and a Kirchhoff term \( M(|\nabla u|^2) \) that depends on \( |\nabla u|^2 \). We prove the existence of global solutions in suitable Sobolev spaces by combining the energy method with the Faedo-Galerkin procedure. We also establish an explicit and general decay result using a perturbed energy method with some techniques due to Mustafa and Messaoudi [25], as well as some properties of convex functions. These convexity arguments were introduced and developed by Lasiecka et al. [26–28] and used, with appropriate modifications, by Liu and Zuazua [29], Alabau-Boussouira [30], and others.

The paper is organized as follows: In Section 2, we give some hypotheses and state our main result. Then, in Section 3,
we prove the global existence of weak solutions. Furthermore, in Section 4, the uniform decay of the energy is derived.

1.2. Formulation of the Results. We denote by $\langle \cdot , \cdot \rangle$ the inner product in $L^2(\Omega)$ and the corresponding norm by $\| \cdot \|^2$. Now, we introduce, as in [31], the new variable:

$$
\begin{align*}
z(x, \rho, t) &= u_t(x, t - \rho \tau(t)), \quad x \in \Omega, \rho \in (0, 1), t > 0.
\end{align*}
$$

Then, we have

$$
\tau(t)z_t(x, \rho, t) + (1 - \rho \tau'(t))z_p(x, \rho, t) = 0,
$$

in $\Omega \times (0, 1) \times (0, +\infty)$. Therefore, problem (1) is equivalent to

$$
\begin{align*}
|u_t|^2u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_{tt} + \int_0^\tau h(t-s)\Delta u(s)ds
+ \mu_1g_1(u_t(x, t)) + \mu_2g_2(z(x, 1, t)) &= \nu u_n|u|
\end{align*}

in $\Omega \times (0, +\infty)$,

$$
\begin{align*}
\tau(t)z_t(x, \rho, t) + (1 - \rho \tau'(t))z_p(x, \rho, t) = 0,
\end{align*}

in $\Omega \times [0, 1] \times [0, +\infty]$,

$$
\begin{align*}
u(x, t) = 0,
\end{align*}

on $\partial \Omega \times (0, +\infty)$,

$$
\begin{align*}
z(x, 0, t) &= u_t(x, t),
\end{align*}

on $\Omega \times (0, +\infty)$,

$$
\begin{align*}
u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x),
\end{align*}

in $\Omega$,

$$
\begin{align*}
z(x, \rho, 0) &= f_0(x, -\rho \tau(0)),
\end{align*}

in $\Omega \times [0, 1]$.

To state and prove our result, we need some assumptions.

(A1) Assume that $l$ satisfies

$$
0 < l \leq \frac{2}{n - 2} \quad \text{if} \ n > 2,
$$

$$
0 < l < \infty, \quad \text{if} \ n \leq 2.
$$

(A2) The relaxation function $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded $C^1$ function, such that

$$
a - \int_0^\infty h(s)ds = k > 0, \quad h(0) > 0,
$$

and suppose that there exists a positive constant $\zeta$ satisfying

$$
h'(t) \leq -\zeta h(t).
$$

(A3) $g_1: \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing function of class $C^1$ and $H: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is convex, increasing and of class $C^1(\mathbb{R}_+) \cap C^2([0, +\infty))$, satisfying

$$
\begin{align*}
H(0) &= 0 \quad \text{and} \ H \text{ is linear, \quad on} \ [0, e] \ \text{or}
\end{align*}

$$
\begin{align*}
H'(0) &= 0 \quad \text{and} \ H'' > 0, \quad \text{on} \ [0, e]\ \text{such that}
\end{align*}

$$
\begin{align*}
c_1|s| \leq |g_1(s)| \leq c_2|s|,
\end{align*}

\begin{align*}
s^2 + g_1(s) \leq H^{-1}(sg_1(s), s), \quad \text{if} \ |s| \geq \varepsilon
\end{align*}

\begin{align*}
s^2 + g_1(s) \leq H^{-1}(sg_1(s), s), \quad \text{if} \ |s| \leq \varepsilon,
\end{align*}

where $e, c_1$, and $c_2$ are positive constants. $g_2: \mathbb{R} \rightarrow \mathbb{R}$ is an odd nondecreasing function of class $C^1(\mathbb{R})$ such that there exist $c_3, a_1$, and $a_2 > 0$.

We define the energy associated to the solution of system (12) by

$$
\begin{align*}
E(t) &= \frac{1}{2} \|u_t\|_{H^2}^2 + \frac{b}{2(y + 1)}\|\nabla u\|^{2(y + 1)}
\end{align*}

+ \frac{1}{2} \left( a - \int_0^\tau h(s)ds \right)\|\nabla u\|^2 + \frac{1}{2}\|\nabla u_t\|^2

- \frac{y}{2} \int_\Omega u^2\ln|u|dx + \frac{y}{4}\|u\|^2 + \frac{1}{2}(ho\nabla u)(t)

+ \xi\tau(t) \int_\Omega \int_0^1 G(z(x, \rho, t))d\rho dx,
\end{align*}

(22)
where $\xi$ is a positive constant such that
\[ \frac{\mu_2(1 - \alpha_4)}{\alpha_1(1 - d)} < \xi < \frac{\mu_1 - \alpha_5 \mu_2}{\alpha_2}. \]  
(23)

\[(h^{ov})(t) = \int_{0}^{t} h(t - s)[(v, t) - v(s, s)]^2 ds. \]  
(24)

**Theorem 1** (Global Existence). Let $(u_0, u_1, f_0) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega) \times H^1_0((\Omega), H^1(0, 1))$ satisfy the compatibility condition:
\[ f_0(., 0) = u_1. \]  
(25)

Assume that (A1)–(A6) hold under smallness condition on the initial data $(u_0, u_1)$. Then, problem (1) admits a weak solution:
\[ u \in L^{\infty}((0, \infty); H^2(\Omega) \cap H^1_0(\Omega)), \]
\[ u_t \in L^{\infty}((0, \infty); H^1_0(\Omega)), \]
\[ u_{tt} \in L^{2}((0, \infty); L^2(\Omega)) \]  
(26)

**Theorem 2** (Uniform Decay Rates of Energy). Assume that (A1)–(A6) hold and if $E(0)$ is positive and bounded, then for every $t_0 > 0$, there exist positive constants $\omega_1, \omega_2, \omega_3$, and $\epsilon_0$ such that the solution energy of (1) satisfies
\[ E(t) \leq \omega_1 H_t^{-1}(\omega_1 t + \omega_2), \quad \forall t \geq t_0, \]  
(27)

where
\[ H_1(t) = \int_{1}^{t} \frac{1}{H_2(s)} ds, \]  
(28)

\[ H_2(t) = t H'(\epsilon_0 t). \]

Here, $H_1$ is strictly decreasing and convex on $(0, 1]$ with
\[ \lim_{t \to 0^+} H_1(t) = +\infty. \]

**2. Preliminaries**

**Lemma 1** (Sobolev–Poincaré's Inequality). Let $q$ be a number with
\[ 2 \leq q \leq +\infty (n = 1, 2) \]
\[ or \quad \frac{2 \leq q \leq 2n}{(n - 2)(n - 3)} \]  
(29)

Then, there exists a constant $C_q = C_q(\Omega, q)$ such that
\[ \|u\|_q \leq C_q \|\nabla u\| \quad for \; u \in H^1_0(\Omega). \]  
(30)

**Lemma 2** (see [32, 33]) (Logarithmic Sobolev Inequality). Let $u$ be any function in $H^1_0(\Omega)$ and $\sigma > 0$ be any number. Then,
\[ \int_{\Omega} u^2 \ln|u| dx \leq \frac{1}{2} \|u\|^2 \ln|u|^2 + \frac{\sigma^2}{2n} \|\nabla u\|^2 - (1 + \ln\sigma) \|u\|^2. \]  
(31)

**Lemma 3** (see [20]) (Logarithmic Gronwall Inequality). Let $C > 0$ and $\varphi \in L^{1}(0, T; \mathbb{R}^n)$ and assume that the function
\[ w : [0, T] \to [1, \infty) \]  
(32)

Then,
\[ w(t) \leq C \exp \left( C \int_{0}^{t} \varphi(s) ds \right), \quad \forall t \in [0, T]. \]  
(33)

**Lemma 4.** Let $\epsilon_0 \in (0, 1)$. Then, there exists $d_\epsilon > 0$ such that
\[ s \ln s \leq s^3 + d_\epsilon s^{3-\epsilon}, \quad \forall s > 0. \]  
(34)

**Proof.** Let $r(s) = s^\alpha (\ln s - s)$. Notice that $r$ is continuous on $(0, \infty)$, and its limit at 0 is 0 and its limit at $\infty$ is $-\infty$. Then, $r$ has a maximum at $d_\epsilon$ on $(0, \infty)$, so the proof is complete.

The following lemma states an important property of the convolution operator.

**Remark 1.** Let us denote by $\Phi^*$ the conjugate function of the differentiable convex function $\Phi$, i.e.,
\[ \Phi^*(s) = \sup_{r \in \mathbb{R}} (st - \Phi(t)). \]  
(36)

Then, $\Phi^*$ is the Legendre transform of $\Phi$, which is given by (see Arnold [35], p. 61-62)
\[ \Phi^*(s) = s \left( \Phi^{-1} - 1 \right)(s) - \Phi \left( \Phi^{-1}(s) \right), \quad if \; s \in (0, \Phi'(r)], \]  
(37)

and $\Phi^*$ satisfies the generalized Young inequality:
\[ AB \leq \Phi^*(A) + \Phi(B), \quad if \; A \in (0, \Phi'(r)] \land B \in (0, r]. \]  
(38)

**Lemma 6.** Let $(u, z)$ be a solution of problem (12). Then, the energy functional defined by (22) satisfies
\[ E'(t) \leq -\lambda \int_{\Omega} u_t g_1(u_t) dx - \beta \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) dx \]
\[ -\frac{1}{2} h(t) \|\nabla u(t)\|^2 + \frac{1}{2} (h'(t) v(t)) \leq 0, \]  
(39)

where $\lambda = \mu_1 - \xi \alpha_2 - \mu_2 \alpha_3$ and $\beta = \xi (1 - d) \alpha_1 - \mu_2 (1 - \alpha_1)$. 

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**Proof.** Multiplying the first equation in (12) by $u_t$, integrating over $\Omega$, and using integration by parts, we get

\[
\frac{d}{dt} \left[ \frac{1}{2} \|u\|_{\Omega, 2}^2 + \frac{b}{2(\gamma + 1)} \|\nabla u\|_{\Omega, 2}^{2(\gamma + 1)} + \frac{1}{2} d \|\nabla u_t\|^2 \right] + \frac{\gamma}{2} \int_{\Omega} |u| |\ln |u| | dx + \frac{b}{4} \|u_t\|^2 + \frac{1}{2} \int_{\Omega} h(t - s) \nabla u(s) \nabla u_t(t) dx
+ \mu_1 \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) dx + \mu_2 \int_{\Omega} u_t(x, t) g_2(z(x, 1, t)) dx = 0.
\]

(40)

Consequently, by applying Lemma 5, equation (40) becomes

\[
\frac{d}{dt} \left[ \frac{1}{2} \|u\|_{\Omega, 2}^2 + \frac{b}{2(\gamma + 1)} \|\nabla u\|_{\Omega, 2}^{2(\gamma + 1)} + \frac{\gamma}{4} \|u_t\|^2 + \frac{1}{2} (h' \nabla u)(t) \right] + \frac{1}{2} \int_{\Omega} h(t) \nabla u(t) \nabla u_t(t) dx
+ \mu_1 \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) dx + \mu_2 \int_{\Omega} u_t(x, t) g_2(z(x, 1, t)) dx = 0.
\]

(41)

Multiplying the second equation in (12) by $\xi g_2(z)$ and integrating the result over $\Omega \times (0, 1)$, we obtain

\[
\xi \tau(t) \int_{\Omega} \int_{0}^{1} z_t(x, \rho, t) g_2(z(x, \rho, t)) d\rho \, dx \leq \xi \int_{\Omega} \int_{0}^{1} \left(1 - \rho \tau'(t) \right) \frac{\partial}{\partial \rho} G(z(x, \rho, t)) d\rho \, dx.
\]

(42)

Consequently,

\[
\frac{d}{dt} \left( \xi \tau(t) \int_{\Omega} \int_{0}^{1} G(z(x, \rho, t)) d\rho \, dx \right)
= \xi \tau'(t) \int_{\Omega} \int_{0}^{1} G(z(x, \rho, t)) d\rho \, dx
- \xi \int_{\Omega} \int_{0}^{1} \left(1 - \rho \tau'(t) \right) \frac{\partial}{\partial \rho} G(z(x, \rho, t)) d\rho \, dx
= -\xi \int_{\Omega} \int_{0}^{1} \left(1 - \rho \tau'(t) \right) \frac{\partial}{\partial \rho} G(z(x, \rho, t)) d\rho \, dx
= -\xi (1 - \tau'(t)) \int_{\Omega} G(z(x, 1, t)) dx + \xi \int_{\Omega} G(u_t(x, t)) dx.
\]

(43)

Combining (41) and (43), we obtain

\[
E'(t) = -\xi (1 - \tau'(t)) \int_{\Omega} G(z(x, 1, t)) dx
+ \frac{1}{2} \int_{\Omega} G(u_t(x, t)) dx - \frac{1}{2} h(t) \nabla u(t) \nabla u_t(t) dx
+ \frac{1}{2} (h' \nabla u_u)(t) - \mu_1 \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) dx
- \mu_2 \int_{\Omega} u_t(x, t) g_2(z(x, 1, t)) dx.
\]

(44)

From (18) and (44), we get

\[
E'(t) \leq -\left(\mu_1 - \xi \alpha_2 \right) \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) dx
- \xi (1 - d) \alpha_1 \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) dx
- \mu_2 \int_{\Omega} u_t(x, t) g_2(z(x, 1, t)) dx - \frac{1}{2} h(t) \nabla u(t) \nabla u_t(t) dx
+ \frac{1}{2} (h' \nabla u_u)(t).
\]

(45)

Using (18) and Remark 1, we obtain

\[
G^*(s) = s g_2^{-1}(s) - G\left(g_2^{-1}(s)\right), \quad \forall s \geq 0.
\]

(46)

Hence,

\[
G'(g_2(z(x, 1, t))) = z(x, 1, t) g_2((x, 1, t)) - G(z(x, 1, t)) \leq (1 - \alpha_1) z(x, 1, t) g_2(z(x, 1, t)).
\]

(47)

Using (18) and (38) with $A = g_2(z(x, 1, t))$ and $B = u_t(x, t)$, we have from (45) that

\[
E'(t) \leq -\left(\mu_1 - \xi \alpha_2 - \mu_2 \alpha_1 \right) \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) dx
- \xi (1 - d) \alpha_1 \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) dx
+ \mu_2 \int_{\Omega} (G(u_t(x, t)) + G'(g_2(z(x, 1, t)))) dx
- \frac{1}{2} h(t) \nabla u(t) \nabla u_t(t) dx
\leq -\left(\mu_1 - \xi \alpha_2 - \mu_2 \alpha_1 \right) \int_{\Omega} u_t(x, t) g_1(u_t(x, t)) dx
- \xi (1 - d) \alpha_1 - \mu_2 (1 - \alpha_1) \int_{\Omega} z(x, 1, t) g_2(z(x, 1, t)) dx
- \frac{1}{2} h(t) \nabla u(t) \nabla u_t(t) dx
+ \frac{1}{2} (h' \nabla u_u)(t).
\]

(48)

This completes the proof.

\[\square\]

3. **Proof of Theorem 1**

3.1. **Local Existence.** Throughout this section, we assume $u_0 \in H^2(\Omega) \cap H^1_0(\Omega), u_1 \in H^1(\Omega)$, and $f_0 \in H^1(\Omega, H^1(0, 1))$. We will use the Faedo-Galerkin method to prove the existence of a solution to problem (1). Let $T > 0$ be fixed and let $\omega^k, k \in \mathbb{N}$, be a basis of $H^2(\Omega) \cap H^1_0(\Omega)$, and let $V_k$ be the space generated by $\{\omega^j\}$. Now, we define, for $1 \leq j \leq k$, the sequence $\phi^j(x, \rho)$ as follows:

\[
\phi^j(x, 0) = \omega^j.
\]

(49)
Then, we may extend $\phi^i(x, 0)$ by $\phi^i(x, \rho)$ over $L^2(\Omega \times (0, 1))$ such that $(\phi^i)$, forms a basis of $L^2(\Omega, H^1(0, 1))$ and denote $Z_k$ as the space generated by $(\phi^i)$. We construct approximate solutions $(u^k, z^k)$, $k = 1, 2, 3, \ldots$, in the form
\begin{align}
  u^k(t) &= \sum_{j=1}^{k} c^{jk}(t) u^j, \\
  z^k(t) &= \sum_{j=1}^{k} d^{jk}(t) \phi^i,
\end{align}
(50)
where $c^{jk}$ and $d^{jk}$ ($j = 1, 2, \ldots, k$) are determined by the following ordinary differential equations:
\begin{align}
  \left( \left| u^k\right|^2 u^l, w^j \right) + M \left( \| \nabla u^k(t) \|^2 \right) (\nabla u^k, \nabla w^j) + (\nabla u^k, \nabla w^j) \\
  - \int_0^t h(t - s) \left( \nabla u^k(s), \nabla w^j \right) ds + \mu_1 \left( g_1(u^k), w^j \right) \\
  + \mu_2 \left( g_2(z^k(\cdot, 1)), w^j \right) = \nu \int_\Omega w^j \ln |u^l| dx,
\end{align}
(51)
\begin{align}
  u^k(0) &= u^k_0 = \sum_{j=1}^{k} (u_0, w^j) \rightarrow u_0, \\
  \text{in } H^2(\Omega) \cap \text{H}^1_0(\Omega) \text{as } k \rightarrow + \infty,
\end{align}
(52)
\begin{align}
  u^k_0 &= u^k_0 = \sum_{j=1}^{k} (u_1, w^j) \rightarrow u_1, \\
  \text{in } H^1_0(\Omega) \text{as } k \rightarrow + \infty,
\end{align}
(53)
\begin{align}
  \left\{ \begin{array}{l}
  (\tau(t)x^i_j) + (1 - \rho \tau^i_j, \phi^i) = 0, \\
  1 \leq j \leq k,
  \end{array} \right.
\end{align}
(54)
\begin{align}
  z^k(\rho, 0) &= z^k_0 = \sum_{j=1}^{k} (f_0, \phi^i) \rightarrow f_0, \\
  \text{in } H^1_0(\Omega, H^1(0, 1)) \text{as } k \rightarrow + \infty.
\end{align}
(55)
Noting that $(L/(2(1 + 1))) + (1/(2(1 + 1))) + (1/2) = 1$, from the generalized Hölder inequality, we obtain
\begin{align}
  \left( |u^k_1|^2 |u^l_1, w_j| \right) = \int_{\Omega} |u^k_1|^2 u^l_1 w_j dx \\
  \leq \left( \left\| u^k_1 \right\|_{L^2(\Omega)} \right)^2 \left\| u^l_1 \right\|_{L^2(\Omega)} \left\| w_j \right\|_{L^2(\Omega)}.
\end{align}
(56)
Since (A1) holds, according to Sobolev, embedding the nonlinear term $(|u^k|^2 |u^l, w_j|)$ in (51) makes sense.

The standard theory of ODE guarantees that systems (51)–(55) have a unique solution in $[0, t_k]$, with $0 < t_k < T$, by Zorn lemma since the nonlinear terms in (51) are locally Lipschitz continuous. Note that $u^k(t)$ is of class $C^2$. In the next step, we obtain a priori estimate for the solution of systems (51)–(55), so that it can be extended to $[0, T)$ and that the local solution is uniformly bounded independently of $k$ and $t$.

### 3.1.1. The First Estimate

Since the sequences $u^k_0, u^k_1$, and $z^k_0$ converge and from Lemma 6, we can find a positive constant $C_1$ independent of $k$ such that
\begin{align}
  E^k(t) - E^k(0) &\leq -\lambda \int_\Omega u^k_1 g_1(u^k) dx ds \\
  - \beta \int_\Omega \int_0^t \int_\Omega z^k(x, s) g_2(z^k(x, s)) dx ds ds \\
  - \frac{1}{2} \int_0^t h(s) \| \nabla u^k(s) \|^2 ds ds + \frac{1}{2} \int_0^t (h' \nabla u^k)(s) ds ds \\
  \leq -\lambda \int_\Omega u^k_1 g_1(u^k) dx ds \\
  - \beta \int_\Omega \int_0^t \int_\Omega z^k(x, s) g_2(z^k(x, s)) dx ds ds.
\end{align}
(57)
As $h$ is a positive nonincreasing function, we get
\begin{align}
  E^k(t) + \lambda \int_\Omega u^k_1 g_1(u^k) dx ds \\
  + \beta \int_\Omega \int_0^t \int_\Omega z^k(x, s) g_2(z^k(x, s)) dx ds ds \leq E^k(0) \leq C_1,
\end{align}
(58)
where
\begin{align}
  E^k(t) &= \frac{1}{L + 2} ||u^k||^2_{H^1} + \frac{b}{2(y + 1)} \| \nabla u^k \|^2 \| \nabla u^k \|^2 + \frac{1}{2} \left( h_{\nu} \nabla u^k(t) \right) \\
  + \frac{1}{2} \left( a - \int_0^t h(s) ds \right) \| \nabla u^k \|^2 + \frac{1}{2} \| \nabla u^k \|^2 + 2 \left( h_{\nu} \nabla u^k \right) \| \nabla u^k \|^2 \\
  - \frac{\nu}{2} \int_\Omega |u^k|^2 \ln |u^k| dx + \frac{\nu}{4} |u^k|^2 \\
  + \xi \tau \int_{\Omega} \int_0^1 G(z^k(x, \rho, t)) d\rho dx.
\end{align}
(59)
By applying the Logarithmic Sobolev inequality, (58) yields
\begin{align}
  ||u^k||^2_{L^2} + \left( k \frac{\nu^2}{2\pi} \| \nabla u^k \|^2 + \| \nabla u^k \|^2 + \left[ \frac{\nu}{2} + \nu(1 + \ln \sigma) \right] \| u^k \|^2 \\
  + \left( h_{\nu} \nabla u^k \right) + \int_\Omega \int_0^t G(z^k(x, \rho, t)) d\rho dx \\
  + \int_0^t \int_\Omega u^k_1 g_1(u^k) dx ds + \int_0^t \int_\Omega z^k(x, s) g_2(z^k(x, s)) dx ds ds \\
  \leq C_2 + \| u^k \|^2 \ln |u^k|^2,
\end{align}
(60)
where $C_2$ is a positive constant depending only on $\|u_0\|_{H^1}, \|u_1\|_{H^1}, I, l, \xi, \tau_1, \lambda,$ and $\beta$.

By choosing
\[
eg 2\pi k \sqrt{\frac{2c\eta}{l}} < \sigma < \sqrt{\frac{2\pi k}{\nu}},
\] (61)
we obtain $k - (\nu a^2/2\pi) > 0$ and $(\nu/2) + (\nu (1 + \ln \sigma)) > 0$.

This selection is possible thanks to (A6). So we get
\[
\begin{aligned}
\|u^k\|_{l^2} &+ \|\nabla u^k\|^2 + \|\nabla u^k\|^2 + (\nu \nabla \nabla u^k)(t) \\
&\quad + \int_0^t \int_0^1 G(z^k (x, \rho, t)) \rho \, dx \, ds \\
&\quad + \int_0^t \int_0^1 z^k(x, 1, s) g_2(z^k(x, 1, s)) \, dx \, ds \\
\leq c \left( 1 + \|u^k\|^2 \ln \|u^k\|^2 \right).
\end{aligned}
\] (62)

Let us note that
\[
\begin{aligned}
u^k(t) &= \nu^k(0) + \int_0^t \nu^k(s) \, ds.
\end{aligned}
\] (63)

Then, by using Cauchy Schwarz’s inequality, we get
\[
\begin{aligned}
\|u^k\|^2 &\leq \|u^k(0)\|^2 + 2 \int_0^t \|u^k(s)\| \, ds \\
&\leq \|u^k(0)\|^2 + 2 T \int_0^t \|u^k(s)\| \, ds.
\end{aligned}
\] (64)

Hence, (62) gives
\[
\|u^k\|^2 \leq C \left( 1 + \int_0^t \|u^k\|^2 \ln \|u^k\|^2 \, ds \right).
\] (65)

where $C = \max \{2T, 2\|u^k(0)\|^2\}$. Applying the Logarithmic Gronwall inequality to (65), we obtain
\[
\|u^k\|^2 \leq C e^{CT}.
\] (66)

Hence, from (58), we obtain the first estimate:
\[
\begin{aligned}
\|u^k\|_{l^2} &+ \|\nabla u^k\|^2 + \|\nabla u^k\|^2 + (\nu \nabla \nabla u^k)(t) \\
&\quad + \int_0^t \int_0^1 G(z^k (x, \rho, t)) \rho \, dx \, ds \\
&\quad + \int_0^t \int_0^1 z^k(x, 1, s) g_2(z^k(x, 1, s)) \, dx \, ds \\
\leq c \left( 1 + C e^{CT} \ln (C e^{CT}) \right) = A_1,
\end{aligned}
\] (67)

The estimate implies that the solution $(u^k, z^k)$ exists in $[0, T]$ and it yields
\[
\begin{aligned}
u^k &\text{ is bounded in } L^\infty_{loc}(0, \infty, H^1(\Omega)), \\

u^k &\text{ is bounded in } L^\infty_{loc}(0, \infty, H^1(\Omega)), \\
G(z^k (x, \rho, t)) &\text{ is bounded in } L^\infty_{loc}(0, \infty, L^1(\Omega \times (0, 1))).
\end{aligned}
\] (72)
To estimate the term on the right-hand side of (76), we apply Lemma 4 with $\epsilon_0 = (1/2)$ and use repeatedly Young's, Cauchy-Schwartz's, and the embedding inequalities as follows:

\[
\left| \int_{\Omega} \Delta \mathbf{u}^k \cdot \mathbf{u}^k \ln | \mathbf{u}^k | \, dx \right| \leq \eta \int_{\Omega} \left| \nabla \mathbf{u}^k \right| \left( \left| \mathbf{u}^k \right|^2 + d_{\mathbf{u}} \sqrt{| \mathbf{u}^k |} \right) \, dx \\
\leq \eta \int_{\Omega} \left| \nabla \mathbf{u}^k \right|^2 \, dx + \frac{\eta}{4} \int_{\Omega} \left( \left| \mathbf{u}^k \right|^2 + d_{\mathbf{u}} \sqrt{| \mathbf{u}^k |} \right)^2 \, dx \\
\leq \eta \| \Delta \mathbf{u}^k \|^2 + \frac{\epsilon}{4\eta} \left( \| \nabla \mathbf{u}^k \|^4 + \| \mathbf{u}^k \|^4 \right), \quad \eta > 0.
\]  
(77)

Combining (76) and (77) to have

\[
\frac{1}{2} \frac{d}{dt} \left[ \int_{\Omega} | \nabla \mathbf{u}^k |^2 \, dx + \left( a - \int_{t_0}^t h(\mathbf{s}) \, d\mathbf{s} \right) \| \Delta \mathbf{u}^k \|^2 + \| \Delta \mathbf{u}^k \|^2 + (\mathbf{h} \Delta \mathbf{u}^k) \right] \\
- (l + 1) \int_{\Omega} | \nabla \mathbf{u}^k | | \nabla \mathbf{u}^k \| \, dx + \frac{1}{2} h(t) \| \Delta \mathbf{u}^k \|^2 + \frac{1}{2} \left( \mathbf{h} \Delta \mathbf{u}^k \right) + \mu_1 \int_{\Omega} | \nabla \mathbf{u}^k |^2 \, dx + \frac{\epsilon}{4\eta} \left( \| \nabla \mathbf{u}^k \|^4 + \| \mathbf{u}^k \|^4 \right) \\
+ \mu_2 \int_{\Omega} \nabla \mathbf{u}^k \cdot \nabla \mathbf{z}^k_{(x, t)} \, dx \leq \eta \| \Delta \mathbf{u}^k \|^2 + \frac{\epsilon}{4\eta} \left( \| \nabla \mathbf{u}^k \|^4 + \| \mathbf{u}^k \|^4 \right) \\
\]  
(78)

Replacing $\phi^j$ by $-\Delta \phi^j$ in (54), multiplying by $d\mathbf{k}$, and summing over $j$ from 1 to $k$, it follows that

\[
\frac{\tau(t)}{1 - r' (t) \rho} \int_{\Omega} \nabla \mathbf{z}^k \cdot \nabla \mathbf{z}^k \, dx + \int_{\Omega} \nabla \mathbf{z}^k \cdot \nabla \mathbf{z}^k \, dx = 0. 
\]  
(79)

Then, we get

\[
\frac{1}{2} \left[ \frac{d}{dt} \left( \frac{\tau(t)}{1 - r'(t) \rho} \| \nabla \mathbf{z}^k \|^2 \right) - \left( \frac{\tau(t)}{1 - r'(t) \rho} \right) \| \nabla \mathbf{z}^k \|^2 \right] \\
+ \frac{1}{2} \frac{d}{d\rho} \| \nabla \mathbf{z}^k \|^2 = 0. 
\]  
(80)

We integrate over $(0, 1)$, and we find

\[
\frac{1}{2} \frac{d}{dt} \int_{0}^{1} \frac{\tau(t)}{1 - r'(t) \rho} \| \nabla \mathbf{z}^k (x, \rho, t) \|^2 \, d\rho + \frac{1}{2} \| \nabla \mathbf{z}^k (x, 1, t) \|^2 = 0. 
\]  
(81)

Combining (78) and (81) and using (A2), we get
From the first estimate (67) and Young’s inequality, we get
\[
\int_\Omega |u^k_1|^2 \|\nabla u^k_1(t)\|_2 dx + \int_0^t h(s) ds \leq A_1(1^{(l+2)/2})^{+\|\nabla u^k_1\|_2}.
\] (83)

Using (17) and Cauchy-Schwarz’s inequality, we obtain
\[
\int_\Omega \nabla u^k_1 \nabla z^k(x, 1, t) dx \leq c\varepsilon \int_\Omega |\nabla u^k_1|^2 dx + \int_\Omega |\nabla z^k(x, 1, t)|^2 dx.
\] (84)

Taking into account (83) and (84) into (82) yields
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |u^k_t(t)|^2 \|\nabla u^k_t\|_2 + (a - \int_0^t h(s) ds) \|\nabla u^k_t\|_2 + \|\Delta u^k_t\|_2 + (\sigma \Delta u^k_t) + \int_\Omega |\nabla z^k(x, 1, t)|^2 dx.
\] (85)

Multiplying (51) by $c_{i_1}^k$ and summing over $j$ from 1 to $k$, it follows that
\[
\int_\Omega |u^k_1|^2 \|\nabla u^k_1\|^2 dx + \int_\Omega M \|\nabla u^k_1\|^2 \nabla u^k_1 \nabla u^k dx + \int_\Omega |\nabla u^k_1|^2 dx - \int_0^t h(t - s) \int_\Omega \nabla u^k_1(s) \nabla u^k_1(t) dx ds
\] (86)

Then,
\[
\int_\Omega |u^k_1|^2 \|\nabla u^k_1\|^2 dx + \int_\Omega \nabla u^k_1 \nabla u^k dx + \mu_1 \int_\Omega u^k g_1(u^k) dx + \mu_2 \int_\Omega u^k g_2(z^k(x, 1, t)) dx = v \int_\Omega u^k \ln|u^k| dx.
\] (87)

Differentiating (54) with respect to $t$, we get
\[
\left( \frac{\tau(t)}{1 - \rho_2(t)} \right) z^k + \frac{\tau(t)}{1 - \rho_2(t)} v_2 + \frac{\tau(t)}{1 - \rho_2(t)} \phi = 0.
\] (88)

Multiplying by $a_{i_1}^k$ and summing over $j$ from 1 to $k$, it follows that
\[
\int_\Omega \frac{\tau(t)}{1 - \rho_2(t)} \|z^k\|^2 + \frac{\tau(t)}{2} \frac{d}{dt} \|z^k\|^2 + \frac{1}{2} \frac{d}{dt} \|v_2\|^2 = 0.
\] (89)

Then, we have
\[
\int_\Omega \frac{\tau(t)}{1 - \rho_2(t)} \|v_2\|^2 + \frac{1}{2} \frac{d}{dt} \|z^k\|^2 + \frac{1}{2} \frac{d}{dt} \|v_2\|^2 = 0.
\] (90)

Integrating over $(0, 1)$ with respect to $\rho$, we obtain
\[
\int_0^1 \frac{\tau(t)}{1 - \rho_2(t)} \|v_2\|^2 d\rho + \frac{1}{2} \frac{d}{dt} \int_0^1 \frac{\tau(t)}{1 - \rho_2(t)} \|z^k\|^2 d\rho
\] (91)
Summing (87) and (91), we get

\[
\int_{\Omega} \left| u_{i}^{k} \right|^{2} dx + \left\| \nabla u_{i}^{k} \right\|^{2} + \frac{1}{2} \int_{0}^{1} \left( \frac{\tau(t)}{1 - \rho_{tt}(t)} \right) \| x_{\eta}^{2} \| d\rho + \frac{1}{2} \| x_{\eta}^{2} \| (1, t)^{2} \\
= \frac{1}{2} \| u_{i}^{k}(x, t) \|^{2} + \frac{1}{2} \int_{0}^{1} \left( \frac{\tau(t)}{1 - \rho_{tt}(t)} \right) \| x_{\eta}^{2} \| d\rho - \int_{\Omega} M \left( \| u_{i}^{k} \|^{2} \right) \nabla u_{i}^{k} \nabla u_{i}^{k} dx \\
+ \int_{0}^{t} h(t-s) \int_{\Omega} \nabla u_{i}^{k}(s) \nabla u_{i}^{k}(t) dx ds + \mu \int_{\Omega} u_{i}^{k} \ln | u_{i}^{k} | dx - \mu \int_{\Omega} u_{i}^{k} g_{i}(u_{i}^{k}) dx \\
- \mu \int_{\Omega} u_{i}^{k} g_{2}(x, 1, t) dx.
\] (92)

By Cauchy-Schwarz's, Sobolev's, and Young's inequalities, the right hand side of (92) can be estimated as follows:

\[
\left| \int_{\Omega} M \left( \| u_{i}^{k} \|^{2} \right) \nabla u_{i}^{k} \nabla u_{i}^{k} dx \right| \leq \left( a + b \| u_{i}^{k} \|^{2} \right) \int_{\Omega} \nabla u_{i}^{k} \nabla u_{i}^{k} dx \\
\leq \left( a + bE(0)^{(2y)/(y+1)} \right) \int_{\Omega} \nabla u_{i}^{k} \nabla u_{i}^{k} dx \leq \eta \| u_{i}^{k} \|^{2} + \frac{m_{0}^{2}}{4\eta} \| u_{i}^{k} \|^{2},
\] (93)

\[
\left| \int_{0}^{t} h(t-s) \int_{\Omega} \nabla u_{i}^{k}(s) \nabla u_{i}^{k}(t) dx ds \right| \leq \eta \| u_{i}^{k} \|^{2} + \frac{1}{4\eta} \left| \int_{0}^{t} h(t-s) \nabla u_{i}^{k}(s) ds \right|^{2} \\
\leq \eta \| u_{i}^{k} \|^{2} + \frac{1}{4\eta} \left| \int_{0}^{t} h(s) ds \right| \left| \int_{0}^{t} h(t-s) \| u_{i}^{k}(s) \|^{2} ds \right| dx \\
\leq \eta \| u_{i}^{k} \|^{2} + \frac{1}{4\eta} (a-k) \left| \int_{0}^{t} h(t-s) \| u_{i}^{k}(s) \|^{2} ds \right| dx \\
\leq \eta \| u_{i}^{k} \|^{2} + \frac{1}{4\eta} (a-k) \left| \int_{0}^{t} h(t-s) \| u_{i}^{k}(s) \|^{2} ds \right| \\
\leq \eta \| u_{i}^{k} \|^{2} + \frac{1}{4\eta} (a-k) (t) \left| \| u_{i}^{k} \|^{2} dx, \right.
\] (94)

and from (16),

\[
\left| \int_{\Omega} u_{i}^{k} g_{i}(u_{i}^{k}) dx \right| \leq \frac{1}{2} \int_{\Omega} \left| u_{i}^{k} \right|^{2} dx + \frac{1}{2} \int_{\Omega} \left| g_{i}(u_{i}^{k}) \right|^{2} dx \\
\leq \frac{1}{2} \int_{\Omega} \left| u_{i}^{k} \right|^{2} dx + \frac{1}{2} \int_{|u_{i}^{k}| \geq \epsilon} \left| g_{i}(u_{i}^{k}) \right|^{2} dx + \frac{1}{2} \int_{|u_{i}^{k}| \leq \epsilon} \left| g_{i}(u_{i}^{k}) \right|^{2} dx \\
\leq \frac{1}{2} \int_{\Omega} \left| u_{i}^{k} \right|^{2} dx + \frac{1}{2} \int_{|u_{i}^{k}| \geq \epsilon} \left| u_{i}^{k} g_{i}(u_{i}^{k}) \right| dx + \frac{1}{2} \int_{\Omega} H^{-1}(u_{i}^{k} g_{i}(u_{i}^{k})) dx.
\] (95)
Complexity

Using Lemma 6, Jensen’s inequality, and the concavity of $H^{-1}$, we obtain

\begin{equation}
\left| \int_{\Omega} u_{t}^{k} g_{1}(u_{t}^{k}) \, dx \right| \leq \frac{1}{2} \int_{\Omega} |u_{t}^{k}|^{2} \, dx + \frac{1}{2} \int_{|u_{t}^{k}| \geq c} u_{t}^{k} g_{1}(u_{t}^{k}) \, dx + c H^{-1} \left( \int_{\Omega} u_{t}^{k} g_{1}(u_{t}^{k}) \, dx \right) + k \int_{\Omega} u_{t}^{k} u_{t}^{k} \ln |u_{t}^{k}| \, dx \\
\leq \frac{1}{2} \int_{\Omega} |u_{t}^{k}|^{2} \, dx + \frac{1}{2} \int_{|u_{t}^{k}| \geq c} u_{t}^{k} g_{1}(u_{t}^{k}) \, dx + c^{'} H^{*}(1) + c^{''} \int_{\Omega} u_{t}^{k} g_{1}(u_{t}^{k}) \, dx \\
\leq \frac{1}{2} \|u_{t}^{k}\|_{2}^{2} + c H^{*}(1) + c^{'} (\text{-}E'). \tag{96}
\end{equation}

From (17) (that is, $|g_{2}(s)| \leq c|s|, \forall s \in \mathbb{R}$), we get

\begin{equation}
\left| \int_{\Omega} u_{t}^{k} g_{2}(\varepsilon^{k}(x, 1, t)) \, dx \right| \\
\leq \frac{1}{2} \int_{\Omega} |u_{t}^{k}|^{2} \, dx + \frac{1}{2} \int_{\Omega} g_{2}(\varepsilon^{k}(x, 1, t))^{2} \, dx \\
\leq \frac{1}{2} \|u_{t}^{k}\|_{2}^{2} + c_{3} \int_{\Omega} \varepsilon^{k}(x, 1, t) g_{2}(\varepsilon^{k}(x, 1, t)) \, dx \\
\leq C_{j} \|\nabla u_{t}^{k}\|_{2}^{2} + c^{'} (\text{-}E'). \tag{97}
\end{equation}

Similar to (77), we get

\begin{equation}
\gamma \int_{\Omega} u_{t}^{k} u_{t}^{k} \ln |u_{t}^{k}| \, dx \leq c_{\eta} \|\nabla u_{t}^{k}\|_{2}^{2} + C_{\eta} \left( \|\nabla u_{t}^{k}\|_{1}^{4} + \|u_{t}^{k}\| \right). \tag{98}
\end{equation}

Substituting (93)–(98) into (92) yields

\begin{equation}
\frac{1}{2} \frac{d}{d\tau} \left[ \int_{\Omega} |u_{t}^{k}(\tau)|^{2} \|\nabla u_{t}^{k}\|_{2}^{2} \, dx + \left( a - \int_{0}^{\tau} h(s) \, ds \right) \|\Delta u_{t}^{k}\|_{2}^{2} + \|\Delta u_{t}^{k}\|_{2}^{2} + \left( h\Delta u_{t}^{k} \right) \right] \\
+ \int_{0}^{\tau} \frac{\tau(t)}{1 - \tau'(t)} \|\nabla z^{k}(x, \rho, t)\|_{2}^{2} \, d\rho + \int_{0}^{\tau} \frac{\tau(t)}{1 - \rho(t)} \|z^{k}(\tau')\|_{2}^{2} \, d\rho + c E(t) \right] \\
+ \frac{1}{2} \|z^{k}(x, 1, t)\|_{2}^{2} + \left( 1 - (l + 3)\eta - 3C_{\eta} \right) \|\nabla u_{t}^{k}\|_{2}^{2} + C_{\nu} \int_{\Omega} \nabla u_{t}^{k} \cdot g_{1}(u_{t}^{k}) \, dx + c \|\nabla z^{k}(x, 1, t)\|_{2}^{2} \\
\leq C_{\nu} A_{1} + \frac{1}{2} \int_{0}^{\tau} \left( \frac{\tau(t)}{1 - \rho(t)} \right) \|\nabla z^{k}(x, \rho, t)\|_{2}^{2} \, d\rho + c' \|\nabla u_{t}^{k}\|_{2}^{2} + \frac{m_{2}^{2}}{4\eta} \|\nabla u_{t}^{k}\|_{2}^{2} \tag{100}
\end{equation}

\begin{equation}
+ c_{\eta} \|\nabla u_{t}^{k}\|_{2}^{2} + C_{\eta} \left( \|\nabla u_{t}^{k}\|_{1}^{4} + \|u_{t}^{k}\| \right) + C_{\nu} \|\Delta u_{t}^{k}\|_{2}^{2}.
\end{equation}

Combining (85) and (99), we get
Then, from (67) and by integration over \((0, t)\), (100) yields

\[
\left( a - \int_0^t h(s) \, ds \right) \| \Delta u_k^t \|^2 + \| \Delta u_k^t \|^2 + (ho \Delta u_k^t) + \int_0^t \frac{\tau(t)}{1 - \tau(t)} \| \nabla z^k(x, \rho, t) \|^2 \, d\rho + \int_0^1 \frac{\tau(t)}{1 - \rho \tau(t)} \| z_k^t \|^2 \, d\rho \\
+ cE(t) + 2 \int_\Omega |u_k^t| |\nabla u_k^t| \, dx + 2 \int_0^t \int_\Omega |u_k^t| |u_k^t| \, dx \, ds + c \int_0^1 \| z_k^t(x, 1, t) \|^2 \, ds
\]

\[
+ 2(1 - (l + 3 - c) \eta - 3C_\eta) \int_0^t \| \nabla u_k^t \|^2 \, ds + c_* \int_0^t \| \nabla z^k(x, 1, t) \|^2 \, ds
\]

\[
\leq \left( C_\eta A_1 + c + \frac{m_\eta^2}{2} + \frac{1}{2\eta} (a - k) h(0) T \right) \tilde{T} E(0) + A_2 + \eta \int_0^t \| \Delta u_k^t \|^2 \, ds
\]

\[
+ c_* \int_0^t \int_0^1 \left( \frac{\tau(t)}{1 - \tau(t)} \right) \| \nabla z^k(x, \rho, t) \|^2 \, d\rho \, ds + c_* \int_0^t \| \nabla z_k^t(x, 1, t) \|^2 \, ds.
\]

(101)

For a suitable \( \eta \), we get

\[
\| \Delta u_k^t \|^2 + \| \Delta u_k^t \|^2 + (ho \Delta u_k^t) + \int_0^t \frac{\tau(t)}{1 - \tau(t)} \| \nabla z^k(x, \rho, t) \|^2 \, d\rho + \int_0^1 \frac{\tau(t)}{1 - \rho \tau(t)} \| z_k^t \|^2 \, d\rho + \int_0^1 \| \nabla u_k^t \|^2 \, ds
\]

\[
\leq \left( C_\eta A_1 + A_2 + c_* \int_0^t \int_0^1 \left( \frac{\tau(t)}{1 - \rho \tau(t)} \right) \| \nabla z^k(x, \rho, t) \|^2 \, d\rho \, ds \right)
\]

\[
+ c_* \int_0^t \left( \frac{\tau(t)}{1 - \rho \tau(t)} \right) \| z_k^t \|^2 \, d\rho + \int_0^t \| \Delta u_k^t \|^2 \, ds.
\]

(102)

Using Gronwall lemma, we obtain

\[
\| \Delta u_k^t \|^2 + \| \Delta u_k^t \|^2 + (ho \Delta u_k^t) + \int_0^t \frac{\tau(t)}{1 - \tau(t)} \| \nabla z^k(x, \rho, t) \|^2 \, d\rho + \int_0^1 \frac{\tau(t)}{1 - \rho \tau(t)} \| z_k^t \|^2 \, d\rho + \int_0^t \| \nabla u_k^t \|^2 \, ds \leq C_3.
\]

(103)

We observe from the estimates (67) and (103) that there exists a subsequence \( \{ u_m^m \} \) of \( \{ u_k^k \} \) and functions \( u, z, \chi, \) and \( \psi \) such that

\[
u_m \rightharpoonup u \text{ weakly in } L^\infty(0, T, H^1(\Omega)) \cap H^1(\Omega),
\]

(104)

\[
u_i^m \rightharpoonup u_i \text{ weakly in } L^\infty(0, T, H^2(\Omega)),
\]

(105)

\[
\chi_i(u_m^m) \rightharpoonup \chi \text{ weakly in } L^2(\Omega \times (0, T)),
\]

(106)

\[
u_i^m \rightharpoonup u_i \text{ weakly in } L^2(0, T, H^2(\Omega)),
\]

(107)

\[
z_m \rightharpoonup z \text{ weakly in } L^\infty(0, T, H^1(\Omega, L^2(0, 1))),
\]

(108)

\[
z_i^m \rightharpoonup z_i \text{ weakly in } L^\infty(0, T, L^2(\Omega \times (0, 1))),
\]

(109)

\[
\psi_i(z^m(x, 1, t)) \rightharpoonup \psi \text{ weakly in } L^2(\Omega \times (0, T)),
\]

(110)

Now, we will prove that \( u \) is the solution of (1). First, we will treat the nonlinear terms.

(1) Term \( |u_i^m|^t u_i^m \): from the first estimate (67) and Lemma 1, we deduce

\[
\| u_i^m \|^2 \leq \int_0^1 \| u_i^m \|^2 \, dt \leq \tilde{C}_i \| u_i^m \|^2 \|
\]

\[
\leq \tilde{C}_i \| u_i^m \|^2 \leq \tilde{C}_i \| u_i^m \|^2 \|
\]

(111)

On the other hand, from Aubin–Lions theorem (see Lions [36]), we deduce that there exists a subsequence of \( \{ u_m^m \} \), still denoted by \( \{ u_m^m \} \) such that

\[
u_i^m \rightharpoonup u_i \text{ strongly in } L^2(0, T, L^2(\Omega)),
\]

(112)

which implies that

\[
u_i^m \rightharpoonup u_i \text{ almost everywhere in } \mathcal{A}.
\]

Hence,

\[
|u_i^m| \leq |u_i|^t u_i \text{ almost everywhere in } \mathcal{A},
\]

(113)

where \( \mathcal{A} = \Omega \times (0, T) \). Thus, using (117), (114), and Lions Lemma, we derive
which implies $z^m \to z$ almost everywhere in $\mathcal{A}$.

(2) Term $\int \! u^k \ln |u^k|$: using (103), we have $|u^k|$ being bounded in $L^\infty(0, T; H^2_0(\Omega))$ which implies the boundedness of $|u^k|$ in $L^2(\mathcal{A})$. Similarly, $\{u_1^k\}$ is bounded in $L^2(\mathcal{A})$. Then, from Aubin-Lions theorem, we find a subsequence such that

$$u^m \to u \text{ strongly in } L^2(\mathcal{A}).$$

which implies

$$u^m \to u \text{ almost everywhere in } \mathcal{A}. \quad (118)$$

Since the map $s \to \nu \ln |s|$ is continuous, we have the following convergence:

$$\nu u^m \ln |u^m| \to \nu u \ln |u| \text{ almost everywhere in } \mathcal{A}. \quad (119)$$

Using the embedding of $H^1_0(\Omega)$ in $L^\infty(\Omega)$, it is clear that $\nu |u^m \ln |u^m||$ is bounded in $L^\infty(\Omega \times (0, T))$. Next, taking into account the Lebesgue bounded convergence theorem ($\nu$ is bounded), we get

$$\nu u^m \ln |u^m| \to \nu u \ln |u| \text{ strongly in } L^2(0, T; L^2(\Omega)). \quad (120)$$

**Lemma 7.** For each $T > 0$, $g_1(u_1), g_2(z(x, 1, t)) \in L^1(\mathcal{A})$, and $\|g_1(u_1)\|_{L^1(\mathcal{A})} \|g_2(z(x, 1, t))\|_{L^1(\mathcal{A})} \leq K$, where $K$ is a constant independent of $t$.

**Proof.** By (A2) and (118), we have

$$g_1(u_1^m(x, t)) \to g_1(u_1(x, t)) \text{ almost everywhere in } \mathcal{A}, \quad 0 \leq u_1^k(x, t)g_1(u_1^m(x, t)) \to u_1(x, t)g_1(u_1(x, t))$$

almost everywhere in $\mathcal{A}$. \quad (121)

Hence, by (71) and Fatou’s Lemma, we have

$$\int_0^T \int_\Omega u_1(x, t)g_1(u_1(x, t))\,dx\,dt \leq K_1, \quad \text{for } T > 0. \quad (122)$$

By using Cauchy-Schwarz’s inequality, (96), and (122), we have

$$\int_0^T \int_\Omega |g_1(u_1(x, t))|\,dx\,dt \leq c|\mathcal{A}|^{(1/2)} \left(\int_0^T \int_\Omega u_1(x, t)g_1(u_1(x, t))\,dx\,dt\right)^{(1/2)} \leq c|\mathcal{A}|^{(1/2)} K_1^{(1/2)} = K. \quad (123)$$

**Lemma 8.** We have $g_1(u_1^k) \to g_1(u_1)$ weak in $L^2(\Omega \times (0, T))$ and $g_2(z^k) \to g_2(z)$ weak in $L^2(\Omega \times (0, T))$.

**Proof.** Let $E \subset \Omega \times [0, T]$ and set

$$E_1 = \{(x, t) \in E, |g_1(u_1^k(x, t))| \leq \frac{1}{|E|}\}, E_2 = \frac{E}{E_1}, \quad (124)$$

where $|E|$ is the measure of $E$. If $M(r) = \inf \{|s|, \ s \in \mathbb{R} \text{ and } |g(s)| \geq r\}$,

$$\int_E |g_1(u_1^k)|\,dx\,dt \leq c\sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}}\right)\right)^{-1} \int_{E_2} |u_1^k(g_1(u_1^k)|\,dx\,dt. \quad (125)$$

By applying (71), we deduce that $\sup_k \int_E |g_1(u_1^k)|\,dx\,dt \to 0$ as $|E| \to 0$. From Vitali’s convergence theorem, we deduce that

$$g_1(u_1^k) \to g_1(u_1) \text{ in } L^1(\Omega \times (0, T)). \quad (126)$$

Hence,

$$g_1(u_1^k) \to g_1(u_1) \text{ weak in } L^2(\Omega \times (0, T)). \quad (127)$$

Similarly, we have

$$g_2(z^k) \to g_2(z) \text{ weak in } L^2(\Omega \times (0, T)). \quad (128)$$

**Remark 2.** By using (103) and from (104) and (105) combined with the Aubin-Lions compactness lemma, we deduce

$$\int_0^T \left(M\left(\|\nabla u^k(t)\|\right)\right)^2 \Delta u^k(t), w\theta(t)\,dt$$

$$\to \int_0^T \left(M\|\nabla u(t)\|^2 \Delta u(t), w\theta(t)\right)\,dt, \quad \text{as } k \to \infty. \quad (129)$$

By multiplying (51) and (54) by $\theta(t) \in \mathcal{D}(0, T)$ and by integrating over $(0, T)$, it follows that
3.2. Global Existence. To state and prove our global existence, we introduce the following functionals:

\[ I(t) = \left( a - \int_0^t h(s) ds \right) \| \nabla u \|^2 + \| \nabla u_t \|^2 + (h \circ \nabla u)(t) \]
\[ - 3 \nu \int_\Omega u^2 \ln |u| dx, \]
\[ J(t) = \left( a - \int_0^t h(s) ds \right) \| \nabla u \|^2 + \frac{1}{2} \| \nabla u_t \|^2 + \frac{1}{2} (h \circ \nabla u)(t) \]
\[ - \frac{\nu}{2} \int_\Omega u^2 \ln |u| dx + \frac{\nu}{4} \| u \|^2 \]
\[ = \int_\Omega \left( a - \int_0^t h(s) ds \right) \| \nabla u \|^2 + \| \nabla u_t \|^2 \]
\[ + (h \circ \nabla u)(t) \] 
\[ + \frac{\nu}{4} \| u \|^2 + \frac{1}{6} I(t). \]

We note that

\[ E(t) = \frac{1}{l + 2b} \| u \|_{H^2}^2 + \frac{b}{2(y + 1)} \| \nabla u \|^{2(y + 1)} + J(t) \]
\[ + \xi \tau(t) \int_0^1 G(z(x, \rho, t)) d\rho dx. \]

for all \( j = 1, \ldots, k. \)

The convergence of (104)–(110), (115), and (124)–(129) is sufficient to pass to the limit in (130) in order to obtain

\[ \int_0^T \int_\Omega \left( |u_{tt}^j| u_{tt} - M(\| \nabla u \|^2) \Delta u - \Delta u_{tt} + \int_0^t h(t-s) \Delta u(s) ds + \mu_1 g_1(u_t) \right. \]
\[ + \mu_2 g_2(z(\cdot), 1)) u_t \theta(t) dr = \nu \int_0^T (u(s) \ln |u(s)|, u) \theta(t) dr, \]
\[ \int_0^T \int_\Omega (\tau(t) z_{1}^{j} + (1 - \rho \tau(t)) z_{2}^{j}) \phi(t) dx d\rho dt = 0. \]

Lemma 9. The following inequalities hold:

\[ -d_0 \sqrt{\| \Omega \| c_1^3 \| \nabla u \|^3} \leq \int_\Omega u^2 \ln |u| dx \leq c_3^3 \| \nabla u \|^3, \]
\[ \forall u \in H^1_0(\Omega), \]

where \( d_0 = \sup_{s \in \Omega} \sqrt{s} \ln s = (2/e), |\Omega| \) is the Lebesgue measure of \( \Omega \), and \( c_3 \) is the smallest embedding constant:

\[ \| u \|_3 \leq c_3 \| \nabla u \|, \quad \forall u \in H^1_0(\Omega). \]

Proof. Let \( \Omega_1 = \{ x \in \Omega : |u| \leq 1 \} \) and \( \Omega_2 = \{ x \in \Omega : |u| > 1 \}. \)

By using (136), we have

\[ \int_\Omega u^2 \ln |u| dx = \int_{\Omega_1} u^2 \ln |u| dx + \int_{\Omega_2} u^2 \ln |u| dx \]
\[ \leq \int_{\Omega_1} u^2 \ln |u| dx \leq \int_{\Omega_1} |u|^3 dx \]
\[ \leq \int_{\Omega_2} |u|^3 dx \leq c_3^3 \| \nabla u \|^3. \]

On the other hand, using Hölder’s inequality and (136), we get
\[-\int_\Omega u^2 \ln |u| \, dx = -\int_{\Omega_1} u^2 \ln |u| \, dx - \int_{\Omega_2} u^2 \ln |u| \, dx \leq -\int_{\Omega_1} u^2 \ln |u| \, dx = \int_{\Omega_1} u^2 \ln |u| \, dx \leq d_0 \int_{\Omega_1} |u|^{3(1/2)} \leq d_0 \sqrt{|\Omega|} \left( \int_{\Omega_1} |u|^3 \, dx \right)^{1/2} \leq d_0 \sqrt{|\Omega|} c_1 \|u\|^{3/2}. \tag{138}\]

Hence, (135) is obtained.

Lemma 10. Assume that (A1)-(A6) hold. Let $(u_0, u_1) \in H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ such that $I(0) > 0, \sqrt{27n_0^3} \left( \frac{E(0)}{k} \right)^{(1/2)} \leq k. \tag{139}\]

Then, $I(t) > 0, \forall t \in [0, T). \tag{140} \]

Proof. Since $I(0) > 0$ and $I$ is continuous on $[0, T]$, there exists $t_* \in (0, T]$ such that $I(t) > 0$, for all $t \in [0, t_*].$ Let us denote by $t_0$ the largest real number in $(0, T]$ such that $I > 0$ on $[0, t_0).$ We assume by contradiction that $t_0 \neq T,$ so we have $I(t_0) = 0$ and from (133), we have

$$\|\nabla u(t_0)\|^2 \leq \frac{3}{K} I(t) \leq \frac{3}{K} E(t) \leq \frac{3}{K} E(0), \quad \forall t \in [0, t_0). \tag{141}\]

The last inequality is obtained from Lemma 6. If $\|\nabla u(t_0)\|^2 = 0$, then (132) and (135) give

$0 = I(t_0) = \left( a - \int_0^{t_0} h(s) \, ds \right) \|\nabla u(t_0)\|^2 + \|\nabla u(t_0)\|^2 + (hoV u)(t_0) - 3\| \Omega \| \|u(t_0)\| \|u(t_0)\|^2 \, ds.

Consequently, if $h > 0$ on $[0, t_0)$, we get $\|\nabla u(s)\|^2 = 0, \forall s \in [0, t_0). \tag{143}\]

Then, $I(t) = 0, \forall s \in [0, t_0), \tag{144}\]

which is not true since $I > 0$ on $[0, t_0).$ If there exists $t^* \in [0, t_0)$ such that $h(t^*) = 0$, then let $t_1 \in [0, t_0)$ be the smallest real number such that $h(t_1) = 0.$ Because $h(0) > 0$ and $h$ is positive, nonincreasing, and continuous on $\mathbb{R}_+$, then $t_1 > 0$ and $h = 0$ on $[t_1, +\infty).$ Therefore, from (142), we deduce that

$$0 = \int_0^{t_1} h(s) \|\nabla u(s)\|^2 \, ds = \int_0^{t_1} h(s) \|\nabla u(s)\|^2 \, ds. \tag{145}\]

Then, $\|\nabla u(s)\|^2 = 0, \forall s \in [0, t_1). \tag{146}\]

As given above, we get a contradiction with the fact that $I > 0$ on $[0, t_0).$ Then, we conclude that $\|\nabla u(t_0)\|^2 > 0.$ From (132), we have

$$I(t_0) \geq k \|\nabla u(t_0)\|^2 - 3\| \Omega \| \|u(t_0)\|^2 \tag{147}\]

By using (135) and (141), we have

$$I(t_0) \geq \left( k - 3\| \Omega \| \left( \frac{3}{K} E(0) \right)^{(1/2)} \right) \|\nabla u(t_0)\|^2. \tag{148}\]

By recalling (139), we arrive at $I(t_0) > 0$, which contradicts the assumption that $I(t_0) = 0.$ Hence, $t_0 = T$ and then $I > 0$ on $[0, T).$

This completes the proof of Theorem 1.

\[\square\]

4. Uniform Decay of the Energy Proof of Theorem 2

In this section, we study the solution’s asymptotic behavior of system (1).

To prove our main result, we construct a Lyapunov functional $F$ equivalent to $E$. For this, we define some functionals which allow us to obtain the desired estimate.

Lemma 11. Let $(u, z)$ be a solution of problem (12). Then, the functional

$$\chi(t) = \tau(t) \int_0^1 \int_\Omega e^{-2\tau(t)} G(z(x, \rho, t)) \, d\rho \, dx \tag{149}\]

satisfies the estimates

\[(i) |\chi(t)| \leq \frac{1}{\xi} E(t), \]

\[(ii) \chi'(t) \leq -2\tau(t) e^{-2\tau(t)} \int_0^1 \int_\Omega G(z(x, \rho, t)) \, d\rho \, dx \tag{150}\]

\[-\alpha_1 (1 - d) e^{-2\tau(t)} \int_\Omega z(x, 1, t) g_2(z(x, 1, t)) \, dx + \alpha_2 \int_\Omega u_t(x, t) g_1(u_t(x, t)) \, dx. \]

Proof.

\[\square\]

(ii) Differentiating (149) with respect to $t$ and using (16), (11), and (A4), we get

\[\square\]
\[
\frac{d}{dt} \chi(t) = \tau'(t) \int_0^1 e^{-2t(1+\rho)} (Gz(x, \rho, t)) d\rho \, dx \\
+ \tau(t) \int_0^1 \int_0^1 e^{-2t(1+\rho)} \frac{\partial G(z(x, \rho, t))}{\partial t} - 2\tau'(t) \rho e^{-2t(1+\rho)} G(z(x, \rho, t)) \, d\rho \, dx \\
= \int_0^1 \int_0^1 e^{-2t(1+\rho)} \left[ \tau'(t) G(z(x, \rho, t)) + \tau(t) \frac{\partial G(z(x, \rho, t))}{\partial t} \right] d\rho \, dx \\
- 2 \int_0^1 \int_0^1 \tau(t) \tau'(t) \rho e^{-2t(1+\rho)} G(z(x, \rho, t)) d\rho \, dx \\
= -\int_0^1 \int_0^1 e^{-2t(1+\rho)} \frac{\partial}{\partial \rho} \left( (1 - \rho \tau'(t)) G(z(x, \rho, t)) \right) d\rho \, dx \\
- 2 \int_0^1 \int_0^1 \tau(t) \tau'(t) \rho e^{-2t(1+\rho)} G(z(x, \rho, t)) d\rho \, dx \\
= -\int_0^1 \int_0^1 \left( \frac{\partial}{\partial \rho} (e^{-2t(1+\rho)} (1 - \tau'(t)\rho)) G(z(x, \rho, t)) \right) d\rho \, dx \\
+ 2\tau(t) e^{-2t(1+\rho)} (1 - \tau'(t)\rho) G(z(x, \rho, t)) d\rho \, dx \\
- 2\tau(t) \tau'(t) \int_0^1 \int_0^1 \rho e^{-2t(1+\rho)} G(z(x, \rho, t)) d\rho \, dx \\
= \int_0^1 G(u(t, x), t) dx - e^{-2t(1+\rho)} (1 - \tau'(t)) \int_0^1 G(z(x, 1, t)) dx \\
- 2\tau(t) \left[ \int_0^1 (1 - \tau'(t)\rho + \tau'(t) \rho) e^{-2t(1+\rho)} G(z(x, \rho, t)) d\rho \right] dx \\
\leq -2\chi(t) + \alpha_2 \int_0^1 u_i(x, t) g_1(u_i(x, t)) dx \\
- e^{-2t(1+\rho)} (1 - d) u_1 \int_0^1 z(x, 1, t) g_2(z(x, 1, t)) dx. 
\]

Since \(e^{-2t(1+\rho)}\) is a decreasing function for \(\rho \in [0, 1]\) and \(\tau(t) \in [\tau_0, \tau_1]\), we deduce

\[
\chi(t) \geq \tau(t) \int_0^1 e^{-2t(1+\rho)} G(z(x, \rho, t)) d\rho \, dx. 
\]

Thus, our proof is completed.

**Lemma 12.** Let \((u, z)\) be a solution of problem (12). Then, the functional

\[
\phi(t) = \frac{1}{I + 1} \int_\Omega |u_i|^I u_i \, dx + \int_\Omega \nabla u_i \nabla u \, dx
\]

satisfies the estimates

\[
(\text{I}) \, |\phi(t)| \leq \frac{1}{I + 2} \|u_i\|_{L^2}^{I + 2} + \left( \frac{I + 1}{I + 2} \right)^{I + 2} \left( \frac{2 E(0)}{a} \right)^{(I + 2)} \|\nabla u\|^2 + \frac{1}{2} \|\nabla u_i\|^2, \\
(\text{II}) \, \phi'(t) \leq \frac{1}{I + 1} \|u_i\|_{L^2}^{I + 2} - Ma(\|\nabla u\|^2 + (1 + \eta)(a - k)\|\nabla u\|^2) \\
+ \frac{1}{4\eta} (ho\nabla u)(t) + \|\nabla u_i\|^2 - \mu_1 \int_\Omega u(x, t) g_1(u_i(x, t)) \, dx \\
- \mu_2 \int_\Omega u(x, t) g_2(z(x, 1, t)) \, dx + \gamma \int_\Omega u^2 |u| \, dx.
\]
where $\eta > 0$ and $c_s$ is the Sobolev embedding constant.

Proof.

(i) From Young’s inequality, Sobolev embedding, and Lemma 6, we deduce

\[
|\phi(t)| \leq \frac{1}{l+1} \left\| u_l \right\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2} \left\| u_l \right\|_{l+2}^{l+2} + \frac{1}{2} \left\| \nabla u_l \right\|^2 + \frac{1}{2} \left\| \nabla u \right\|^2
\]

\[
\leq \frac{1}{l+1} \left\| u_l \right\|_{l+2}^{l+2} + \frac{(l+1)^{-1}}{l+2} c_s \left\| u_l \right\|_{l+2}^{l+2} + \frac{1}{2} \left\| \nabla u_l \right\|^2 + \frac{1}{2} \left\| \nabla u \right\|^2
\]

\[
\leq \frac{1}{l+1} \left\| u_l \right\|_{l+2}^{l+2} + \left( \frac{(l+1)^{-1}}{l+2} c_s \left( \frac{2E(0)}{a} \right)^{(l/2)} + \frac{1}{2} \right) \left\| \nabla u \right\|^2 + \frac{1}{2} \left\| \nabla u_l \right\|^2. \tag{155}
\]

(ii) Differentiating $\phi(t)$ with respect to $t$ and using the first equation of (12), we get

\[
\phi'(t) = \frac{1}{l+1} \int_\Omega \left( \left[ u_l \right]_t \right)_t^t u dx + \frac{1}{l+1} \int_\Omega \left[ u_l \right]_{t+2}^t dx + \int_\Omega \nabla u_0 \nabla u dx + \int_\Omega \nabla u_0 \nabla u_t dx
\]

\[
= \int_\Omega \left[ u_l \right]_t \nabla u dx + \frac{1}{l+1} \left\| u_l \right\|_{l+2}^{l+2} - \int_\Omega \Delta u_0 \nabla u dx + \left\| \nabla u_0 \right\|^2
\]

\[
= \frac{1}{l+1} \left\| u_l \right\|_{l+2}^{l+2} + \int_\Omega \left[ u_l \right]_t \nabla u dx + \left\| \nabla u_0 \right\|^2
\]

\[
= \frac{1}{l+1} \left\| u_l \right\|_{l+2}^{l+2} - \left( \left\| \nabla u_0 \right\|^2 \right) + \int_\Omega \nabla u_0 \sqrt{h(t-s)\nabla u(s)} ds
\]

\[
- \mu_1 \int_\Omega u_0 \sqrt{h(t-s)\nabla u(s)} ds dx + \nabla u_0 \nabla u_t dx + \frac{1}{2} \int_\Omega u^2 \ln u dx. \tag{156}
\]

By using Young’s inequality and Sobolev embedding, we can estimate the third term in the right side as follows:

\[
\int_\Omega \nabla u_0 \sqrt{h(t-s)\nabla u(s)} ds dx \leq \int_0^t h(t-s) \int_\Omega \left| \nabla u_0 \nabla u(s) - \nabla u(t) \right|^2 ds + \left\| \nabla u(t) \right\|^2 \int_0^t h(t-s) ds
\]

\[
\leq \eta \left\| \nabla u(t) \right\|^2 \int_0^t h(s) ds + \frac{1}{4\eta} \int_0^t h(t-s) \left\| \nabla u(s) - \nabla u(t) \right\|^2 ds
\]

\[
+ \left\| \nabla u(t) \right\|^2 \int_0^t h(s) ds
\]

\[
\leq \left( 1 + \eta \right) \left( a - k \right) \left\| \nabla u(t) \right\|^2 + \frac{1}{4\eta} \left( h \nabla u \right)(t). \tag{157}
\]

Thus, our proof is completed. □
Lemma 13. Let \((u, z)\) be a solution of problem (12). Then, the functional

\[
\psi (t) = \int_{\Omega} \left( \Delta u_t - \frac{1}{t + 1} u_t \right) \int_0^t h(t-s) (u(t)-u(s)) ds \, dx
\]

satisfies the estimates

\[
(158)
\]

(i) \(|\psi (t)| \leq \frac{1}{2} \|\nabla u\|_2^2 + \left( \frac{1}{2} (a-k) + \frac{(l+1)^{-1}}{l+2} (a-k)^{l+2} c_s^2 \right) \left( \frac{4E(0)}{a} \right)^{(l+2)/2} \|\nabla \nu u\|_2^2 (\nu \nu u)(t) + \frac{1}{l+2} \|u\|_2^{l+2} \]

(ii) \(\psi' (t) \leq \delta ((a-k) + b_0) M(\|\nabla u\|_2^2) \|\nabla u\|_2^2 + 2\delta (a-k) \|\nabla u\|_2^2 \]

\[
+ \left( \frac{M_0}{4\delta} + \left( \frac{2\delta}{1 \delta} + (\mu_1 + \mu_2 + \nu) \frac{c_s^2}{4\delta} \right) (a-k) \right) (\nu \nu u)(t) - \frac{h(0)}{4\delta} \left( 1 + \frac{c_s^2}{l+1} \right) (\nu \nu u)(t) - \int_0^t h(s) ds \|u\|_2^{l+2}, \]

where

\[
M_0 = a + b \left( \frac{2E(0)}{a} \right) \gamma, \quad \alpha_0 = c_s^2 (l+1) \left( \frac{2E(0)}{a} \right) \gamma, \quad \eta > 0, \quad b_0 = \frac{\gamma}{2\gamma} \max \left\{ \frac{\sqrt{l+1}}{a}, \frac{c_s^2}{b} \right\}, \]

and \(c_s\) is the Sobolev embedding constant.

Proof.

(1) We have

\[
\left| -\int_{\Omega} \frac{1}{l+1} u_t \int_0^t h(t-s) (u(t)-u(s)) ds \, dx \right|
\\leq \frac{1}{l+2} \|u\|_2^{l+2} + \frac{(l+1)^{-1}}{l+2} \int_{\Omega} \left( \int_0^t (h(t-s))^{(l+1)/(l+2)} (h(t-s))^{(l+1)/(l+2)} |u(t)-u(s)| ds \right)^{l+2} dx
\\leq \frac{1}{l+2} \|u\|_2^{l+2} + \frac{(l+1)^{-1}}{l+2} \int_{\Omega} \left( \int_0^t h(t-s) ds \right)^{l+1} \int_0^t h(t-s) |u(t)-u(s)|^{l+2} ds dx
\\leq \frac{1}{l+2} \|u\|_2^{l+2} + \frac{(l+1)^{-1}}{l+2} (a-k)^{l+1} c_s^2 \left( \frac{4E(0)}{a} \right)^{(l+2)/2} (\nu \nu u)(t).
\]

We get the last inequality from (22) and Lemma 6. Similarly, we use Young’s and Hölder’s inequalities with \(p = q = 2\) to get

\[
\left| -\int_{\Omega} \nabla u_t \int_0^t h(t-s) (\nabla u(t)-\nabla u(s)) ds \, dx \right| \leq \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} \int_{\Omega} \left( \int_0^t h(t-s) \|\nabla u(t)-\nabla u(s)\|^2 ds \right)^2 dx
\\leq \frac{1}{2} \|\nabla u_t\|_2^2 + \frac{1}{2} (a-k) (\nu \nu u)(t).
\]

(162)

We use Young’s and Hölder’s inequalities with the conjugate exponents \(p = ((l+2)/(l+1))\) and \(q = l+2\); the second term in the right hand side can be estimated as
Combining (162) and (163), we deduce

\[
|\psi(t)| \leq \frac{1}{2} \|\nabla u\|^2 + \left( \frac{1}{2} (a-k) + \frac{(l+1)^{-1}}{(l+2)} (a-k)^{1/2} \right)^{(l/2)} \left( \frac{4E(0)}{a} \right)^{(l/2)} (h \nabla u)(t) + \frac{1}{l+2} \| u \|_{l+2} |
\]

(ii) We use the Leibniz formula and the first equation of (12), and we have

\[
\psi'(t) = \int_\Omega (\Delta u_t - |u_t|^l u_t) \int_0^t h(t-s) (u(t) - u(s))ds \, dx \\
+ \int_\Omega (\Delta u_t - \frac{1}{l+1} |u_t|^l u_t) \left( \int_0^t (h^l(t-s)(u(t) - u(s)) + h(t-s)u_t(s))ds \right) \, dx \\
= \int_\Omega M(\|u\|)^2 \nabla u(t) \int_0^t h(t-s) (\nabla u(t) - \nabla u(s))ds \, dx \\
- \int_\Omega \int_0^t h(t-s) \nabla u(s)ds \int_0^t h(t-s) (\nabla u(t) - \nabla u(s))ds \, dx \\
+ \mu_1 \int_\Omega g_1(u_t(x,t)) \int_0^t h(t-s) (u(t) - u(s))ds \, dx \\
+ \mu_2 \int_\Omega g_2(z(x,1,t)) \int_0^t h(t-s) (u(t) - u(s))ds \, dx \\
- \int_\Omega \mu \ln|u| \int_0^t h(t-s) (u(t) - u(s))ds \, dx \\
- \int_\Omega \nabla u_t \int_0^t h^l(t-s)(u(t) - u(s))ds \, dx \\
- \|\nabla u_t\|^2 \int_0^t h(s)ds \ - \frac{1}{l+1} \| u_t \|_{l+2} \int_0^t h(s)ds \\
= I_1 + I_2 + \mu_1 I_3 + \mu_2 I_4 + I_5 + I_6 + I_7 - \|\nabla u_t\|^2 \int_0^t h(s)ds - \frac{1}{l+1} \| u_t \|_{l+2} \int_0^t h(s)ds.
\]

In what follows, we will estimate \(I_1, \ldots, I_7\). For \(I_1\), we use Hölder’s and Young’s inequalities with \(p = q = 2\), and we get

\[
|I_1| \leq M(\|u\|^2) \int_\Omega |\nabla u(t)| \left( \int_0^t h(s)ds \right)^{(l/2)} \left( \int_0^t h(t-s) |\nabla u(t) - \nabla u(s)|^2 ds \right)^{(l/2)} \, dx \\
\leq M(\|u\|^2) \left[ \delta \int_\Omega |\nabla u(t)|^2 \int_0^t h(s)ds \, dx + \frac{1}{4\delta} \int_\Omega \int_0^t h(t-s)|\nabla u(t) - \nabla u(s)|^2 ds \, dx \right] \\
\leq M(\|u\|^2) \left( \delta \|\nabla u(t)\|^2 \int_0^t h(s)ds + \frac{1}{4\delta} (h \nabla u)(t) \right) \\
\leq \delta M(\|u\|^2) \|\nabla u(t)\|^2 (a-k) + \frac{M_q}{4\delta} (h \nabla u)(t),
\]

where \(M_q = (a + b)(2E(0)/a)^l\) obtained by recalling (22) and Lemma 6. Similarly,
\[ |I_2| \leq \delta \int_\Omega \left( \int_0^t h(t-s)|\nabla u(s)| \, ds \right)^2 \, dx + \frac{1}{4\delta} \int_\Omega \left( \int_0^t h(t-s)|\nabla u(t) - \nabla u(s)| \, ds \right)^2 \, dx \]

\[ \leq \delta \int_\Omega \left( \int_0^t h(t-s)(|\nabla u(s) - \nabla u(t)| + |\nabla u(t)|) \, ds \right)^2 \, dx + \frac{1}{4\delta} \int_0^t h(s) \, ds \, (ho\nabla u)(t) \]

\[ \leq 2\delta \|
abla u(t)\|^2 \left( \int_0^t h(s) \, ds \right)^2 \, dx + \left( 2\delta + \frac{1}{4\delta} \right) \left( \int_0^t h(s) \, ds \right) (ho\nabla u)(t) \]

\[ \leq 2\delta \|
abla u(t)\|^2 (a-k)^2 + \left( 2\delta + \frac{1}{4\delta} \right) (a-k) (ho\nabla u)(t), \]

\[ |I_3| \leq \delta \|g_1(u(x,t))\|^2 + \frac{c_2^2}{4\delta} (a-k) (ho\nabla u)(t), \]

\[ |I_4| \leq \delta \|g_2(z(x,1,t))\|^2 + \frac{c_2^2}{4\delta} (a-k) (ho\nabla u)(t). \]

To estimate \( I_5 \), we apply Lemma 4 with \( \epsilon = 0 \) and use repeatedly Young's, Cauchy-Schwartz's, and the embedding inequalities, as follows:

\[ |I_5| \leq \nu \int_\Omega \left( u^2 + d_n \sqrt{|d|} \right) \left( \int_0^t h(t-s)(u(t) - u(s)) \, ds \right) \, dx \]

\[ \leq \nu \left( \frac{1}{4\delta} \int_\Omega \left( \int_0^t h(t-s)(u(t) - u(s)) \, ds \right)^2 \, dx + \delta \int_\Omega \left( u^2 + d_n \sqrt{|d|} \right)^2 \, dx \right) \]

\[ \leq \frac{c_2^2 (a-k)\nu}{4\delta} (ho\nabla u)(t) + \frac{\nu}{2} \delta \left( \int_\Omega |u|^4 \, dx + \int_\Omega |u| \, dx \right) \]

\[ \leq \frac{c_2^2 (a-k)\nu}{4\delta} (ho\nabla u)(t) + \frac{\nu}{2} \delta \left( \|
abla u\|^4 + \sqrt{|d|} \|u\| \right) \]

\[ \leq \frac{c_2^2 (a-k)\nu}{4\delta} (ho\nabla u)(t) + b_0 \delta M(\|
abla u\|^2) \|u\|^2, \]

where \( b_0 = (\nu/2)c_2^2 \max\{ \sqrt{|d|}/a \}, (c_2^2/b) \}. \) Also,

\[ |I_6| \leq \delta \int_\Omega \|
abla u_t\|^2 \, dx + \frac{1}{4\delta} \int_\Omega \left( \int_0^t |h'(t-s)|\|
abla u(t) - \nabla u(s)\| \, ds \right)^2 \, dx \]

\[ \leq \delta \|
abla u_t\|^2 + \frac{1}{4\delta} \int_0^t h'(t-s) \, ds \int_\Omega \int_0^t |h'(t-s)|\|
abla u(t) - \nabla u(s)\|^2 \, ds \, dx. \]

As \( h \) is a positive decreasing function, \( |h'(t-s)| = -h'(t-s) \) and then,
where \( a_0 = c_1^{2(l+1)} (2E(0)/\alpha)^l \) obtained by recalling (22) and Lemma 6. Combining (165) and (166)—(173), we finish the proof. Now, for \( M, \varepsilon_1 > 0 \), we introduce the following functional:

\[
F(t) = ME(t) + \varepsilon_1 \phi(t) + \psi(t) + \chi(t). \tag{174}
\]

Lemma 14. Let \((u, z)\) be a solution of problem (12). Assume that (A1)–(A6) hold and

\[
0 < E(0) < \min \left\{ \frac{ek\pi}{4r^2}, \frac{k^3}{27r^2e^6} \right\}. \tag{175}
\]

Then, \( F(t) \) satisfies, along the solution and for some positive constants \( m, c > 0 \), the following estimate:

\[
F'(t) \leq -mE(t) + c \left[ \|g_1(u_t(x, t))\|^2 + \|g_2(z(x, 1, t))\|^2 \right] + \int_{\Omega} |u(x, t)g_1(u_t(x, t))| dx + \int_{\Omega} |u(x, t)g_2(z(x, 1, t))| dx \tag{176}
\]

and \( F(t) \sim E(t) \).

Proof. By (ii) of Lemmas 11–13 and Lemma 6 and by (A2), we deduce that for \( t \geq t_0 > 0 \):

\[
F'(t) = ME'(t) + \varepsilon_1 \phi'(t) + \psi'(t) + \chi'(t)
\]

\[
\leq -(M\lambda - a_2) \int_{\Omega} u_t(x, t)g_1(u_t(x, t)) dx - (M\beta + a_1(1 - d) e^{-2\tau_1}) \int_{\Omega} z(x, 1, t)g_2(z(x, 1, t)) dx
\]

\[
- 2\tau(1-2\tau_1) \int_{\Omega} 1 G(z(x, p, t)) dp dx - (\varepsilon_1 - \delta(a - k + h_0)) M \| \nabla u \|^2 \| \nabla u \|^2 - \frac{1}{l+1} (h_0 - \varepsilon_1) \| u_t \|^2 + 
\]

\[
\left[ \frac{M \lambda}{2} - \frac{h(0)}{4\delta} \left( \frac{1}{1 + \frac{c_1^2}{l + 1}} - \frac{\varepsilon_1}{4\eta} + \frac{M_0}{4\delta} + \left( \frac{\delta}{4\delta} + \frac{\delta_1}{4\delta} \right) \right) \right] \left( ho\nabla u \right) (t)
\]

\[
+ \mu_1 \| g_1(u_t(x, t)) \|^2 + \mu_2 \delta \| g_2(z(x, 1, t)) \|^2 - \varepsilon_1 \mu_1 \int_{\Omega} u(x, t)g_1(u_t(x, t)) dx - \varepsilon_1 \mu_2 \int_{\Omega} u(x, t)g_2(z(x, 1, t)) dx
\]

\[
+ \varepsilon_1 \varepsilon_1 \int_{\Omega} u_t^2 dx,
\]

(177)
where

\[
\begin{align*}
    h_0 &= \int_0^{t_0} h(s)ds, \\
    h_1 &= \min\{h(t), t \geq t_0 > 0\}.
\end{align*}
\]  

(178)

We take \( h_0 > \varepsilon_1 \) and \( \delta > 0 \) sufficiently small such that

\[
    a_1 = M\lambda - a_2 > 0,
\]

\[
    a_5 = \frac{Mh_1}{2} - \varepsilon_1(1 + \eta)(a - k) - 2\delta(a - k)^2 > 0,
\]

\[
    a_6 = \xi \left(\frac{M}{2} - \frac{h(0)}{4\delta} \left(1 + \frac{c^2}{l + 1}\right)\right) - \left(\frac{\varepsilon_1}{4\eta} + \frac{M_0}{4\delta} + (2\delta + \frac{1}{4\delta} + (\mu_1 + \mu_2 + \gamma)\frac{c^2}{4\delta})(a - k)\right) > 0.
\]

(180)

Thus,

\[
F'(t) \leq -a_1 \frac{1}{t+1} \|u_t\|_{L^2}^2 - a_2 M \left(\|\nabla u\|^2\right) \|\nabla u\|^2 - a_4 \|\nabla u\|^2
\]

\[
-a_6 (h(u)\nabla u)(t) - 2\tau(t)e^{-2t} \int_0^1 \int_\Omega G(\rho, \rho, t) d\rho dx
\]

\[
+ c \left[\|g_1(u_1(t))\|^2 + \|g_2(z, 1, t)\|^2\right] + \int_\Omega |u(t)g_1(u_1(t))| dx
\]

\[
+ \int_\Omega |u(t)g_2(z, 1, t)| dx + \varepsilon_1 \int_\Omega u^2 \ln|u| dx
\]

\[
\leq - m_1 E(t) + \left(\frac{\varepsilon_1}{2} - m_1\right) \int_\Omega u^2 \ln|u| dx + m_1 \frac{\nu}{4} \|u\|^2
\]

\[
+ c \left[\|g_1(u_1(t))\|^2 + \|g_2(z, 1, t)\|^2\right] + \int_\Omega |u(t)g_1(u_1(t))| dx + \int_\Omega |u(t)g_2(z, 1, t)| dx,
\]

(181)

where

\[
m_1 = \min\{2a_2, 2e^{-2t_1} \xi, 2a_4, a_3\}.
\]

(182)

Using the Logarithmic Sobolev inequality, we get

\[
F'(t) \leq - m_1 E(t) - \left(\frac{\varepsilon_1}{2} - m_1\right) \left(2(1 + \ln\sigma) - \ln\|u\|^2\right) \|u\|^2 + \left(\frac{\varepsilon_1}{2} - m_1\right) \frac{\sigma^2}{2\pi} \|\nabla u\|^2 + m_1 \frac{\nu}{4} \|u\|^2
\]

\[
+ c \left[\|g_1(u_1(t))\|^2 + \|g_2(z, 1, t)\|^2\right] + \left(\frac{\varepsilon_1}{2} - m_1\right) \int_\Omega u^2 \ln|u| dx + \int_\Omega |u(t)g_2(z, 1, t)| dx,
\]

(183)
From (61) and for
\[ m_1 \leq \varepsilon_1 \leq \frac{m_1}{2} \,(v+1), \tag{184} \]
we have
\[ m_2 = \frac{m_1}{2} \left( a - \int_0^t h(s) \, ds \right) - \left( \varepsilon_1 - \frac{m_1}{2} \right) \frac{\sigma^2}{2\pi} \]
\[ \geq \frac{m_1}{2} k - \left( \varepsilon_1 - \frac{m_1}{2} \right) \frac{k}{\gamma} > 0 \tag{185} \]
\[ \frac{m_1 \gamma}{4} \leq \left( \varepsilon_1 - \frac{m_1}{2} \right) \frac{\gamma}{2}. \]

This selection is possible thanks to (A6). So we get
\[ F'(t) \leq -mE(t) - \left( \varepsilon_1 - \frac{m_1}{2} \right)^2 \left( 1 + 2\ln \sigma - \ln \|u\|^2 \right) \|u\|^2 \]
\[ F'(t) \leq -mE(t) - \left( \varepsilon_1 - \frac{m_1}{2} \right)^2 \left( 1 + 2\ln \sigma - \ln \|u\|^2 \right) \|u\|^2 \]
\[ + \int_\Omega |u(x,t)g_1(u_t(x,t))| \, dx \leq \int_\Omega |u(x,t)g_2(z(x,1,t))| \, dx \tag{186} \]
where \( m = \min\{m_1, m_2\} \). By recalling that \( E' \leq 0 \) and \( I(t) > 0 \) and using (133), (134), and (175), we obtain
\[ \ln \|u\|^2 \leq \ln \left( \frac{4}{\gamma} E(t) \right) \leq \ln \left( \frac{4}{\gamma} E(0) \right) \leq \ln \left( \frac{e\gamma \pi}{\gamma} \right). \tag{187} \]

Taking \( \sigma \) satisfies
\[ \max \left\{ e^{-\gamma/2}, \sqrt{\frac{\kappa\pi}{\gamma}} \right\} \leq \sigma \leq \frac{2\kappa\pi}{\gamma}. \tag{188} \]

(So (61) is satisfied), and we guarantee
\[ 1 + 2 \ln \sigma - \ln \|u\|^2 \geq 0, \tag{189} \]
which completes the proof of (176). To prove \( F(t) \sim E(t) \), we show that there exist two positive constants \( \kappa_1 \) and \( \kappa_2 \) such that
\[ \kappa_1 E(t) \leq F(t) \leq \kappa_2 E(t). \tag{190} \]

From (i) of Lemmas 11–13, (140), (133), and (134), we get \( \kappa > 0 \) depending on \( \varepsilon_1, a, l, c_s, E(0), k, \) and \( \xi \) such that
\[ |\varepsilon_1 \phi(t) + \psi(t) + \chi(t)| \leq \kappa E(t). \tag{191} \]

For a choice of \( M \) large enough such that \( \kappa_1 = M - \kappa > 0 \) and \( \kappa_2 = M + \kappa > 0 \), we get our result. By the proof of Theorem 2 as given by Komornik [37], we consider the following partition of \( \Omega \):
\[ \Omega_1 = \{ x \in \Omega : |u_t| < \varepsilon \}, \]
\[ \Omega_2 = \{ x \in \Omega : |u_t| > \varepsilon \}. \tag{192} \]

We use Young’s inequality (with \( p = q = 2 \), (22), and Lemma 6, and we have
\[ \int_\Omega |u g_1(u_t)| \, dx + \|g_1(u_t)\|^2 \leq \delta \|u\|^2 + \left( \frac{1}{4\delta} + 1 \right) \|g_1(u_t)\|^2 \]
\[ \leq \delta C^2 \|\nabla u\|^2 + \left( \frac{1}{4\delta} + 1 \right) \left( \int_\Omega H^{-1}(u_t g_1(u_t)) \, dx + c_1 \int_{\Omega_1} u_t g_1(u_t) \, dx \right) \tag{193} \]
\[ \leq \frac{2\delta C^2}{a} E(t) + c_1 \int_{\Omega_1} H^{-1}(u_t g_1(u_t)) \, dx - C\delta E'(t). \]

Similarly and by application of (17), we obtain
\[ \int_\Omega |u g_2(z(x,1,t))| \, dx + \|g_2(z(x,1,t))\|^2 \leq \delta C^2 \|\nabla u\|^2 \]
\[ + \left( \frac{1}{4\delta} + 1 \right) c_1 \int_\Omega z(x,1,t) g_2(z(x,1,t)) \, dx \]
\[ \leq \frac{2\delta C^2}{a} E(t) - C\delta E'(t). \tag{194} \]

Combining (193) and (194), (176) becomes
\[ F'(t) \leq - \left( m - \frac{4\delta C^2}{a} \right) E(t) - C\delta E'(t) + c_3 \int_{\Omega_1} H^{-1}(u_t g_1(u_t)) \, dx, \tag{195} \]
where \( C\delta = C^2 + C_b \). Now, for \( \delta \) small enough such that \( d = m - (4\delta C_a/a) > 0 \), the function \( L(t) = F(t) + C\delta E(t) \) satisfies
\[ L'(t) \leq -d E(t) + c_3 \int_{\Omega_1} H^{-1}(u_t g_1(u_t)) \, dx, \tag{196} \]
\[ L(t) \sim E(t). \tag{197} \]
Case 1. $H$ is linear on $[0, \varepsilon]$; using (16) and Lemma 6, we deduce that

$$L'(t) \leq -dE(t) - cE'(t).$$  \hspace{1cm} (198)

Thus, $R = L + cE \sim E$ satisfies

$$R(t) \leq R(0)e^{-c't}.$$  \hspace{1cm} (199)

Hence,

$$E(t) \leq C(E(0))e^{-c't}.$$  \hspace{1cm} (200)

Case 2. $H$ is nonlinear on $[0, \varepsilon]$; so we exploit Jensen’s inequality (see [2]) and the concavity of $H^{-1}$ to obtain

$$H^{-1}\left(\frac{1}{|\Omega|} \int_{\Omega} u_i g_i(u_i)dx\right) \geq c \int_{\Omega} H^{-1}(u_i g_i(u_i))dx.$$  \hspace{1cm} (201)

Using Remark 1 with $H^*$, the convex conjugate of $H$ in the sense of Young, we obtain

$$L'_0(t) = \varepsilon_0 E'(t) E''(t) H'(E(t)/E(0)) L(t) + H'(E(t)/E(0)) L'(t) + w_0 E'(t)$$

$$\leq -dE(t) H'(E(t)/E(0)) + c H'(E(t)/E(0)) H^{-1}\left(\frac{1}{|\Omega|} \int_{\Omega} u_i g_i(u_i)dx\right) + w_0 E'(t).$$  \hspace{1cm} (205)

By recalling (28), we deduce

$$L'_1(t) \leq w_1 H'_1(L_1(t)),$$  \hspace{1cm} (210)

which gives

$$[H'_1(L_1(t))]' \leq w_1.$$  \hspace{1cm} (211)

A simple integration leads to

$$H_1(L_1(t)) \leq w_1 t + H_1(0).$$  \hspace{1cm} (212)

Consequently,

$$L_1(t) \leq H^{-1}_1(w_1 t + w_2).$$  \hspace{1cm} (213)

Using (208) and (213), we obtain (27). The proof is completed.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

**Authors’ Contributions**

All authors contributed equally to this article. They have all read and approved the final manuscript.
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