ON THE SOLVABILITY OF FREDHOLM BOUNDARY-VALUE PROBLEMS IN FRACTIONAL SOBOLEV SPACES

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We study systems of linear ordinary differential equations with the most general inhomogeneous boundary conditions in fractional Sobolev spaces on a finite interval. The Fredholm property of these problems in the corresponding pairs of Banach spaces is proved. Their indices and dimensions of the kernels and cokernels are determined. We also present examples showing the constructive character of the obtained results.

1. Introduction and Statement of the Problems

Investigation of solutions of systems of ordinary differential equations is an important part of numerous problems of contemporary analysis and its applications (see, e.g., the monograph [1] and the references therein). Unlike the Cauchy problems, the solutions of these problems may be absent or not unique.

For inhomogeneous boundary-value problems on a finite interval of the form

\[ Ly := y'(t) + A(t)y(t) = f(t), \quad t \in (a, b), \]

\[ By = c, \]

where the matrix function \( A(\cdot) \) and vector function \( f(\cdot) \) are summable on \([a, b]\) and

\[ B : C([a, b]; \mathbb{R}^m) \to \mathbb{R}^m, \]

is a linear continuous operator, the problem of correct solvability and continuous dependence of the solutions on a parameter in the space \( C([a, b]; \mathbb{R}^m) \) was studied by Kiguradze [2, 3] and his followers [4–6]. These problems cover all classical types of boundary conditions (two-point, multipoint, integral, and mixed). However, they do not cover the problems with derivatives of an unknown function of integer or fractional order in the boundary conditions. These boundary conditions are connected with function spaces in which the analyzed problem is investigated. Their analysis requires new approaches and methods of investigation. In the case of Sobolev spaces of integer order, they were analyzed in [7–10]. At the same time, in the case of Hölder spaces, they were studied in [11]. Note that these investigations were essentially based on the analytic description of linear operators that continuously act from a Sobolev space or \( C^{(n)} \) into the space \( C^m \).

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In the present paper, we study the case of fractional Sobolev spaces. Note that the description of linear continuous operators acting from these spaces into $\mathbb{C}^m$ is absent, which significantly complicates the investigation of boundary-value problems.

We now introduce the notation required to state the analyzed problems. Consider a given finite interval $(a, b) \subset \mathbb{R}$ and numerical parameters
\[ \{m, n, r\} \subset \mathbb{N}, \quad s \in (1, \infty) \setminus \mathbb{N}, \quad 1 \leq p < \infty. \]

By $W^m_p := W^m_p([a, b]; \mathbb{C})$ we denote a complex Sobolev space and set $W^0_p := L^p$. In a similar way, we denote the Sobolev spaces of vector functions $(W^m_p)^m := W^m_p([a, b]; \mathbb{C}^m)$ and matrix functions $(W^m_p)^{m \times m} := W^m_p([a, b]; \mathbb{C}^{m \times m})$ whose elements belong to the function space $W^m_p$. The norms in these spaces are denoted by $\| \cdot \|_{n,p}$. These are sums of the corresponding norms in $W^m_p$ of all elements of the vector- or matrix-valued functions. It is always clear from the context what kind of norm is considered in each case (in the spaces of scalar, vector, or matrix-valued functions). For $m = 1$, all these spaces coincide. It is known that the spaces $W^m_p$ are Banach and separable for $p < \infty$.

By $W^s_p := W^s_p([a, b]; \mathbb{C})$, where $1 \leq p < \infty$ and $s > 1$ is a noninteger number, we denote the Sobolev–Slobodettsky space of all complex-valued functions that belong to the Sobolev space $W^s_{[s]}$ and satisfy the condition
\[
\| f \|_{s,p} := \| f \|_{[s],p} + \left( \int_a^b \int_a^b \frac{|f^{[s]}(x) - f^{[s]}(y)|^p}{|x - y|^{1 + \{s\} p}} dxdy \right)^{1/p} < +\infty,
\]
where $[s]$ and $\{s\}$ are, respectively, the integer and fractional parts of the number $s$. Recall that $\| \cdot \|_{[s],p}$ is the norm in the Sobolev space $W^s_{[s]}$. This equality gives the norm $\| f \|_{s,p}$ in the space $W^s_p$.

On a finite interval $(a, b)$, we consider the following linear boundary-value problem for a system of $m$ differential equations of the first order:
\[
(Ly)(t) := y'(t) + A(t)y(t) = f(t), \quad t \in (a, b), \quad By = c,
\]
where
- $A(\cdot)$ is a matrix function that belongs the space $(W^{s-1}_p)^{m \times m}$,
- $f(\cdot)$ is a vector function that belongs to the space $(W^{s-1}_p)^m$,
- $c$ is a vector from the space $\mathbb{C}^r$,
and
- $B$ is a linear continuous operator $B : (W^s_p)^m \to \mathbb{C}^r$. 
The boundary condition (2) gives \( r \) scalar boundary conditions for a system of \( m \) differential equations of the first order. The vectors and vector functions are written in the form of columns. Note that the boundary-value problem (1), (2) is \textit{overdetermined} for \( r > m \) and \textit{underdetermined} for \( r < m \). A solution of the boundary-value problem (1), (2) is understood as a vector function \( y(\cdot) \in (W^s_p)^m \) satisfying both Eq. (1) (everywhere for \( s > 1 + 1/p \) and almost everywhere for \( s \leq 1 + 1/p \) on \((a, b)\)) and equality (2) specifying \( r \) scalar boundary conditions.

The solution of Eq. (1) fills the space \((W^s_p)^m\) if its right-hand side \( f(\cdot) \) runs through the space \((W^{s-1}_p)^m\). Therefore, the boundary condition (2) is the most general condition for this equation. It includes all known types of classical boundary conditions (namely, the Cauchy problem, two-point, multipoint, integral, and mixed problems) and numerous nonclassical problems. The last class of problems may contain derivatives of the required vector function of integer or fractional orders \( \beta \), where \( 0 \leq \beta < s - \frac{1}{p} \).

The aim of the present paper is to establish the Fredholm property for the boundary-value problem (1), (2) and to find its index and dimensions of the kernel and cokernel of the operator of inhomogeneous boundary-value problem via the same characteristics of a special rectangular numerical matrix. For Sobolev spaces of integer order, similar results were obtained in [12].

2. Main Results

We rewrite the inhomogeneous boundary-value problem (1), (2) in the form of a linear operator equation

\[
(L, B)y = (f, c),
\]

where \((L, B)\) is the linear operator in a pair of Banach spaces

\[
(L, B) : (W^s_p)^m \to (W^{s-1}_p)^m \times \mathbb{C}^r. \tag{3}
\]

Recall that a linear continuous operator \( T : X \to Y \), where \( X \) and \( Y \) are Banach spaces, is called Fredholm if its kernel \( \ker T \) and cokernel \( Y/T(X) \) are finite-dimensional. If this operator is Fredholm, then its range \( T(X) \) is closed in \( Y \) and the index

\[
\text{ind } T := \dim \ker T - \dim(Y/T(X)) \in \mathbb{Z}
\]

is finite (see, e.g., [13], Lemma 19.1.1).

\textbf{Theorem 1.} The linear operator (3) is a bounded Fredholm operator with index \( m - r \).

By \( Y(\cdot) \in (W^s_p)^m \) we denote the unique solution of the linear homogeneous matrix equation with an initial Cauchy condition

\[
Y'(t) + A(t)Y(t) = O_m, \quad t \in (a, b), \quad Y(a) = I_m. \tag{4}
\]

Here, \( O_m \) is the \((m \times m)\) null matrix and \( I_m \) is the \((m \times m)\) identity matrix. The unique solution of the Cauchy problem (4) belongs to the space \((W^s_p)^m\).

\textbf{Definition 1.} A rectangular numerical matrix

\[
M(L, B) \in \mathbb{C}^{m \times r} \tag{5}
\]
is characteristic for the boundary-value problem (1), (2) if its jth column is obtained as a result of action of the operator B upon the jth column of the matrix function $Y(\cdot)$. Here, $m$ is the number of scalar differential equations in system (1) and $r$ is the number of scalar boundary conditions.

**Theorem 2.** The dimensions of the kernel and cokernel of operator (3) are equal to the dimensions of the kernel and cokernel of the characteristic matrix, respectively:

$$\dim \ker (L, B) = \dim \ker (M(L, B)), \quad (6)$$

$$\dim \text{coker} (L, B) = \dim \text{coker} (M(L, B)). \quad (7)$$

Theorem 2 yields a criterion for invertibility of the operator $(L, B)$, i.e., the condition under which problem (1), (2) has a unique solution that continuously depends on the right-hand sides of the differential equation and the boundary condition.

**Theorem 3.** The operator $(L, B)$ is invertible if and only if $r = m$ and the square matrix $M(L, B)$ is nondegenerate.

3. Examples

**Example 1.** Consider a linear one-point boundary-value problem for the differential equation

$$Ly(t) := y'(t) + Ay(t) = f(t), \quad t \in (a, b), \quad (8)$$

$$By = \sum_{k=0}^{n-1} \alpha_k y^{(k)}(a) = c, \quad (9)$$

where

- $A$ is an $(m \times m)$ constant matrix,
- $f(\cdot)$ is a vector function from the space $(W^{s-1}_p)^m$,
- the matrices $\alpha_k \in \mathbb{C}^{r \times m}$, the vector $c \in \mathbb{C}^r$,
- $B : (W^s_p)^m \to \mathbb{C}^r$ and $(L, B) : (W^s_p)^m \to (W^{s-1}_p)^m \times \mathbb{C}^r$ are continuous linear operators,
- the vector function $y(\cdot) \in (W^s_p)^m$, and $s > n + \frac{1}{p} - 1$.

By $Y(\cdot) \in (W^s_p)^{m \times m}$ we denote the unique solution of the matrix Cauchy problem

$$Y'(t) + AY(t) = O_m, \quad t \in (a, b), \quad Y(a) = I_m.$$  

Then the matrix function $Y(\cdot)$ and its $k$th derivative take the form

$$Y(t) = \exp (-A(t - a)), \quad Y(a) = I_m, \quad Y^{(k)}(t) = (-A)^k \exp (-A(t - a)), \quad Y^{(k)}(a) = (-A)^k, \quad k \in \mathbb{N}.$$
Substituting these expressions in equality (9), we get

\[ M(L, B) = \sum_{k=0}^{n-1} \alpha_k (-A)^k. \]

It follows from Theorem 1 that

\[ \text{ind} (L, B) = \text{ind} (M(L, B)) = m - r. \]

Hence, by virtue of Theorem 2, we obtain

\[ \dim \ker (L, B) = \dim \ker \left( \sum_{k=0}^{n-1} \alpha_k (-A)^k \right) = m - \text{rank} \left( \sum_{k=0}^{n-1} \alpha_k (-A)^k \right), \]

\[ \dim \text{coker} (L, B) = -m + r + \dim \ker \left( \sum_{k=0}^{n-1} \alpha_k (-A)^k \right) = r - \text{rank} \left( \sum_{k=0}^{n-1} \alpha_k (-A)^k \right). \]

This implies that the Fredholm numbers of the problem are independent of the choice of the right end of the interval \( b > a \).

**Example 2.** Consider a two-point boundary-value problem for the system of differential equations (8) with a coefficient \( A(t) \equiv O_m \) and boundary conditions specified at points \( \{t_0, t_1\} \subset [a, b] \) containing the derivatives of integer and/or fractional orders (in the Caputo sense; see, e.g., [14]). They are given by the equality

\[ B y = \sum_{j} \alpha_{0j} y^{(\beta_{0j})}(t_0) + \sum_{j} \alpha_{1j} y^{(\beta_{1j})}(t_1), \]

where both sums are finite, \( \alpha_{kj} \in \mathbb{C}^{r \times m} \) are numerical matrices, and \( \beta_{kj} \) are nonnegative numbers such that, for all \( k \in \{1, 2\} \),

\[ \beta_{k,0} = 0, \quad \beta_{kj} < s - \frac{1}{p}. \]

Then, we easily show that the linear operator

\[ B : (W^s_p)^m \rightarrow \mathbb{C}^r \]

is continuous.
By Theorem 1, the index of the operator \((L, B)\) is equal to \(m - r\). We now find its Fredholm numbers. In this case, the matrix function \(Y(\cdot) = I_m\) and the characteristic matrix has the form

\[
M(L, B) = \sum_j \alpha_{0j} I^{(\beta_{0j})}_m + \sum_j \alpha_{1j} I^{(\beta_{1j})}_m = \alpha_{0,0} + \alpha_{1,1}
\]

because the derivatives \(I^{(\beta_{kj})}_m = 0\) for \(\beta_{kj} > 0\) [14]. Hence, by Theorem 2,

\[
\dim \ker(L, B) = \dim \ker(\alpha_{0,0} + \alpha_{1,1}) = m - \text{rank} (\alpha_{0,0} + \alpha_{1,1}),
\]

\[
\dim \coker(L, B) = -m + r + \dim \ker(\alpha_{0,0} + \alpha_{1,1}) = r - \text{rank} (\alpha_{0,0} + \alpha_{1,1}).
\]

These formulas imply that the Fredholm numbers of the problem are independent of the choice of an interval \((a, b)\), points \(\{t_0, t_1\} \subset [a, b]\), and matrices \(\alpha_{0,j}\) and \(\alpha_{1,j}\) with \(j \geq 1\).

4. Previous Results

To prove Theorems 1–3, we need two auxiliary statements.

We introduce a metric space of matrix functions

\[
\mathcal{Y}_s := \{ Y(\cdot) \in (W^s_p)^{m \times m} : Y(a) = I_m, \; \det Y(t) \neq 0 \}
\]

with the following metric:

\[
d^s_p(Y, Z) := ||Y(\cdot) - Z(\cdot)||_{s,p}.
\]

**Theorem 4.** The nonlinear mapping \(\gamma : A \mapsto Y\), where \(A(\cdot) \in (W^{s-1}_p)^{m \times m}\) and \(Y(\cdot) \in (AC[a, b])^{m \times m}\) is a solution of the Cauchy problem (4), is a homeomorphism of the Banach space \((W^{s-1}_p)^{m \times m}\) onto the metric space \(\mathcal{Y}_s\).

The proof of Theorem 4 can be found in [15]. We set

\[
[BY] := \begin{pmatrix}
B(\begin{pmatrix}
y_{1,1}(\cdot)
\vdots
y_{m,1}(\cdot)
\end{pmatrix}) \ldots B(\begin{pmatrix}
y_{1,m}(\cdot)
\vdots
y_{m,m}(\cdot)
\end{pmatrix})
\end{pmatrix} = M(L, B).
\]

(10)

**Lemma 1.** For any matrix function \(Y(\cdot) \in (W^s_p)^{m \times m}\), a vector \(q \in \mathbb{C}^m\), and a linear continuous operator \(B : (W^s_p)^{m \times m} \to \mathbb{C}^r\), the following equality is true:

\[
B(Y(\cdot)q) = [BY] q,
\]

where \([BY]\) is a matrix given by equality (10).

**Proof.** Let \(i \in \{1, 2, \ldots, m\}, \; k \in \{1, 2, \ldots, m\}, \; j \in \{1, 2, \ldots, r\}\). Consider a matrix function \(Y(\cdot) = (y_{ik}(\cdot))\) and a column vector \(q = (q_i)\). Denote

\[
(\alpha_j) = [BY] q \quad \text{and} \quad (\beta_j) = B(Y(\cdot)q).
\]
Further, let

\[ B(y_k(\cdot)) =: (c_j). \]

As a result of action of the operator \( B \) upon the matrix function \( Y(\cdot) \), we obtain a matrix

\[ [BY] = (c_{ji}). \]

This yields

\[ (\alpha_j) = (c_{ji}) \cdot (q_j) = \left( \sum_i c_{ji} q_j \right). \]

Thus, any element \( \alpha_i \) has the form

\[ \alpha_j = \sum_j c_{ji} q_j. \]

However, the following equalities hold:

\[ (\beta_j) = B((y_{ik}(\cdot)) \cdot (q_k)) = B \left( \sum_k y_{ik}(\cdot) q_k \right) = \sum_k (By_{ik}(\cdot)) q_k = \sum_k (c_{jk}) q_k = \left( \sum_j c_{jk} q_j \right). \]

This implies that \( \alpha_j = \beta_j \).

Lemma 1 is proved.

5. Proofs of the Theorems

**Proof of Theorem 1.** We first show that the operator \((L, B)\) is continuous. Since, by condition, the operator \( B \) is linear and continuous, it suffices to prove the continuity of the operator \( L \), which is equivalent to its boundedness. The boundedness of the linear operator \( L \):

\[ (W^s_p)^m \rightarrow (W^{s-1}_p)^m \]

follows from the definition of norms in the Sobolev spaces \( W^{s-1}_p \) and the well-known fact that each of these spaces forms a Banach algebra.

We now prove the Fredholm property of the operator \((L, B)\) and determine its index. We choose a fixed linear bounded operator

\[ C_{r,m} : (W^s_p)^m \rightarrow \mathbb{C}^r. \]

The operator \((L, B)\) admits the representation

\[ (L, B) = (L, C_{r,m}) + (0, B - C_{r,m}), \]
where the operator

\[(L, C_r, m) : (W^s_p)^m \to (W^{s-1}_p)^m \times \mathbb{C}^r,\]

and the second term is a finite-dimensional operator. By the second theorem on stability (see, e.g., [16], Sec. 3.1), the operator \((L, B)\) is Fredholm if the operator \((L, C_r, m)\) is Fredholm and, in addition,

\[\text{ind}(L, B) = \text{ind}(L, C_r, m).\]

Thus, it suffices to prove the Fredholm property of the operator \((L, C_r, m)\) and determine its index by choosing the proper operator \(C_r, m\). To this end, consider the following three cases:

1. Let \(r = m\). We set

\[C_{m, m} y := (y_1(a), \ldots, y_m(a)).\]

We determine the null space and range of this operator. Let \(y(\cdot)\) belong to \(\ker(L, C_r, m)\). Then \(Ly = 0\) and

\[C_{m, m} y = (y_1(a), \ldots, y_m(a)) = 0.\]

By the theorem on unique solvability of the Cauchy problem, we get \(y(\cdot) = 0\). Hence,

\[\ker(L, C_m, m) = \{0\}.\]

Further, let \(h \in (W^{s-1}_p)^m \times \mathbb{C}^m\) and \(c \in \mathbb{C}^m\) be arbitrarily chosen. It follows from Theorem 4 that there exists a vector function \(y(\cdot) \in (W^s_p)^m\) such that

\[Ly = h, \quad (y_1(a), \ldots, y_m(a)) = c.\]

Therefore,

\[\text{ran}(L, C_r, m) = (W^{s-1}_p)^m \times \mathbb{C}^m.\]

2. Let \(r > m\). We set

\[C_{r, m} y := (y_1(a), \ldots, y_m(a), 0, \ldots, 0) \in \mathbb{C}^r.\]

We determine the null space of the operator \((L, C_r, m)\). Let \(y(\cdot)\) belong to \(\ker(L, C_r, m)\). Then \(Ly = 0\) and \((y_1(a), \ldots, y_m(a)) = 0\). By the theorem on uniqueness of solution of the Cauchy problem, we get \(y(\cdot) = 0\). We represent the set of values of the operator \((L, C_r, m)\) in the form of direct sum of two subspaces as follows:

\[\text{ran}(L, C_r, m) = \text{ran}(L, C_m, m) \oplus (0, \ldots, 0) \in \mathbb{C}^r.\]

However, as shown above, \(\text{ran}(L, C_m, m) = (W^{s-1}_p)^m \times \mathbb{C}^m\). Thus, \(\text{def ran}(L, C_r, m) = r - m\).
3. Let $r < m$. We set

$$C_{r,m}(y) := (y_1(a), \ldots, y_r(a)) \in \mathbb{C}^r.$$ 

It is necessary to prove that

$$\dim \ker(L, C_{r,m}) = m - r,$$

$$\def \ran \text{ran}(L, C_{r,m}) = 0.$$ 

Let $y(\cdot)$ belong to $\ker(L, C_{r,m})$. Then

$$L y = 0 \quad \text{and} \quad (y_1(a), \ldots, y_r(a)) = 0.$$ 

We consider $m - r$ Cauchy problems

$$L y_k = 0, \quad C_{m,m} y_k = e_k,$$

where $k \in \{r + 1, r + 2, \ldots, m\}$, $e_k := (0, \ldots, 0, \frac{1}{k}, 0, \ldots, 0) \in \mathbb{C}^m$. 

It follows from Theorem 4 that the solutions of these problems are linearly independent and form a basis in the subspace $\ker(L, C_{r,m})$.

The surjectivity of the operator $(L, C_{r,m})$ follows from the already established surjectivity of the operator $(L, C_{m,m})$.

Thus, in each of the analyzed three cases, $(L, B)$ is a Fredholm operator with index $m - r$.

Theorem 1 is proved.

**Proof of Theorem 2.** We prove equality (6). To do this, we introduce the notation

$$\dim \ker(L, B) = n',$$

$$\dim \ker(M(L, B)) = n'',$$

and establish the validity of the equality

$$n' = n''.$$ \hfill (11)

Let $\dim \ker(L, B) = n'$. Then there exist $n'$ linearly independent solutions of the homogeneous equation $(L, B)y = (0, 0)$ such that, by Lemma 1,

$$y_k(\cdot) \in \ker(L, B) \iff (\exists q_k \in \mathbb{C}^m : y_k(t) = Y(t)q_k, [BY] q_k = 0)$$

where the vectors $q_k \neq 0$ and $k \in \{1, \ldots, n'\}$. This means that $r - n'$ columns of matrix (5) are linearly dependent and $n' \leq n''$. 

Conversely, let \( \dim \ker (M(L, B)) = n'' \). Then \( r - n'' \) columns of the matrix are linearly independent. This means that, for some vectors \( q_k \neq 0, \ k \in \{1, \ldots, n'\} \),

\[
[BY] q_k = 0.
\]

We set

\[
y_k(\cdot) := Y(\cdot)q_k.
\]

Then \( y_k(\cdot) \neq 0, \ Ly_k(\cdot) = 0, \) and

\[
By_k(\cdot) = B(Y(\cdot)q_k) = [BY] q_k = 0
\]

by Lemma 1. Hence, \( y_k(\cdot) \in \ker(L, B) \) and, therefore, \( n' \geq n'' \). Thus, equality (6) is true.

According to the definition, the characteristic matrix \( M(L, B) \) belongs to the space \( \mathbb{C}^{m \times r} \). It is known that the dimension of the matrix kernel is the difference between the number of its rows and the rank, while the dimension of cokernel is the difference between the number of its columns and the rank. Thus, for the matrix \( M(L, B) \), we get the following difference:

\[
\dim \coker (M(L, B)) = r - m + \dim \ker (M(L, B)). \tag{12}
\]

By using the formula for the index of the operator \( (L, B) \)

\[
\text{ind} (L, B) := \dim \ker (L, B) - \dim \coker (L, B),
\]

we obtain

\[
\dim \coker (L, B) = r - m + \dim \ker (L, B). \tag{13}
\]

By using equalities (11)–(13), we arrive at equality (7).

Theorem 2 is proved.

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