Surreal Birthdays and Their Arithmetic

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I used to feel guilty in Cambridge that I spent all day playing games, while I was supposed to be doing mathematics. Then, when I discovered surreal numbers, I realized that playing games IS math.

John Horton Conway

This is a paper about the structure of surreal numbers. These are not numbers as we are taught in grade school, but they have many of the same properties. The tricky thing is that they are defined recursively from the very start.

Recursion is like the joke: “an American, an Englishman and an Australian walk into a bar, and one of them says

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A recursive definition means that a surreal number is defined in terms of other surreals, and so on. The break in this circularity that makes it possible to get a foothold is that each turn of the circle progresses inexorably toward 0, which can be defined ab initio. To continue the Latin, the surreal numbers violate the dictum ex nihlo nihil fit, or “from nothing comes nothing.” From 0 springs forth all other numbers.

The definition that performs this miracle is as follows: a surreal number \( x \) consists of an ordered pair of two sets of surreal numbers (call them the left and right sets, \( X_L \) and \( X_R \), respectively) such that no member of the left set is \( \geq \) any of the members of the right set. We write \( x = \{X_L | X_R\} \) for such a surreal.

This seems a difficult definition. We have not even defined \( \geq \), and yet are using it in the definition. Everything resolves because we can always work with empty sets. The starting point—the first surreal number—is \( \{\emptyset | \emptyset\} \) (where \( \emptyset \) is the empty set). A careful reading of the definition says that no elements of one can be \( \geq \) to the other, but as there are no elements, the comparison is automatically true. We call this number \( 0 \).

Then, on the “first day” a new generation of surreals can be created, building on \( 0 \). On the second day, we create a second generation, and so on. Each has a meaning corresponding to traditional numbers in order to have a consistent interpretation with respect to standard mathematical operators such as addition. The construction is elegant and surprisingly general, and leads naturally to the idea of the “birthday” of a surreal numbers being literally the day on which it is born.

A natural question then is, when we perform arithmetic on surreal numbers, what is the birthday of the result? This paper answers that question.

The surreal numbers and their birthdays

Surreal numbers were invented by John Horton Conway [3] as a side project in the study of games. However, they have an elegant and complex structure that is worthy

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of study in its own right. Most people examining the surreals see them as numbers first, and the underlying structure as purely an issue of construction. This is interesting because the construction requires only set theory, and yet can construct the reals, rationals, hyperreals and ordinals, thus providing underpinning idea of what a number really is. In this paper we want to examine properties of the construction.

We will not describe every facet of the surreals here; there are several good books and tutorials, such as Conway [3], Grimm [7], Knuth [9], Simons [14], Stack Exchange [15], and Tondering [16]. In particular, the name for these numbers was coined by Knuth in his book *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness* [9]. However, we do need to provide some background.

For instance, the notation varies somewhat in different sources: here we denote numbers in lower case and sets in upper case, with the convention that $X_L$ and $X_R$ are the left and right sets of $x$. We write $x = \{X_L \mid X_R\}$ to denote the form of the surreal.* It is common to omit empty sets, but we prefer writing $\emptyset$ explicitly because it is a little clearer when writing complicated sequences.

The resulting objects are called numbers because we can use a consistent mapping from these objects to values that corresponds to our everyday numbers. The mapping preserves properties such as addition, so that, for instance, if $v(x)$ were the value of a surreal in conventional number terms, then for two surreals $x$ and $y$ we would have $v(x + y) = v(x) + v(y)$, and so on for all the standard arithmetic operators. The association is such that we might just write 1 to denote the surreal number whose conventional value is 1, rather than bothering to write out its complete form.

The value of a surreal number is interwoven with its form. This is best explained by an analogy to rational numbers. We can write a rational number in many ways, for example $1/2 = 2/4$. That is, we have many forms of the same number. Likewise, a surreal number can have many forms. This requires a distinction to clarify the notion of equality. Following Keddie, we call two forms identical if they are the same form (that is, they have identical left and right sets), and equal if they have the same value (that is, they denote the same number) [8]. We shall distinguish these two cases by writing equality of value as equivalence, $\equiv$, and identity by $==$. A single equal sign will be reserved for conventional numbers.

The first surreal number form to be defined is $\bar{0} \equiv \{\emptyset \mid \emptyset\}$. We call this number $\bar{0}$ because it will turn out to be the additive identity (the 0 of conventional arithmetic). All other numbers are defined from this point, following a construction to be laid out below. The line over the 0 denotes that this is a special, canonical form of zero.

The second two surreal number forms, the numbers we can define on Day 1, immediately after creating $\bar{0}$, are

$$\bar{1} \equiv \{\emptyset \mid \emptyset\} \quad \text{and} \quad \bar{-1} \equiv \{\emptyset \mid \bar{0}\}.\]

This notation, however, hides some of the structure of the surreals. To see them in all their glory, we should write

$$\bar{1} \equiv \{\{\emptyset \mid \emptyset\} \mid \emptyset\} \quad \text{and} \quad \bar{-1} \equiv \{\emptyset \mid \{\emptyset \mid \emptyset\}\}.\]

*I’d like to make a surrealist/computer-science joke here, namely ‘|’ is not a pipe, but a conjunction between Magritte and Unix might be considered too obscure even for a paper on surreal numbers.
but no doubt you can see that this will quickly result in very complicated expressions. We will resolve this by drawing pictures such as in Figure 1(a). The figure shows each surreal as a node in a graph. It is a connected, labeled, Directed Acyclic Graph (DAG) with links showing how each surreal is constructed from its parents, that is, the surreals that comprise its left and right sets. A DAG, by itself, would lose information. The graph would only specify parents, not left and right parents (sets), so each edge needs a label. We show these in the DAGs using a box for each surreal number, with the value given in the top section and the left and right sets shown in the bottom left and right sections, respectively. From each member of each set, we show a link from its parents. Viewed in color, a red link indicates a left parent, and blue right. The DAG shows the whole recursive structure of a surreal, with 0 always at the root.

Most aspects of surreals are defined recursively. For instance, \( x \geq y \) (which we need even in the definition) means that no member of \( X_L \) is greater than or equal to \( y \), and no member of \( Y_R \) is less than or equal to \( x \). It might be hard to see how to use this in the definition when it is also defined in terms of surreals (which in turn use the definition), but this is the nature of surreal operations: they are recursive, not just in terms of themselves, but also in the sense that each definition in turn uses others at lower levels. In any case, it is now relatively easy to check that \( -1 \leq 0 \leq 1 \), and we can define further comparisons, for instance, \( x \equiv y \) means \( x \geq y \) and \( y \geq x \).

Once we have defined \( \pm 1 \), we can proceed to define yet more surreals. Figure 1(b) shows another form equivalent to zero, that is, \( \{ -1 | 1 \} \equiv \{ \emptyset | \emptyset \} \equiv 0 \). The graph shows that a value can reappear at multiple places in the structure of the form: in this case, the value 0 appears both at the top and the bottom of the DAG. The information characterizing the surreal form is not its value, or even the values of its subsets, but the structure of the whole DAG that describes it. So the two “0” nodes in the graph are different (nonidentical) surreal forms that just happen to have the same value.

Each number is actually an infinite equivalence class of forms, so we need to have standard canonical forms, at least to bootstrap later work. The standard construction (called the Dali function by Tøndering [16]) maps dyadic numbers

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Note that the online version of this article has color diagrams.
The dyadic DAG, i.e., the recursive structure of the canonical form of dyadic surreal numbers, up to birthday/generation 3.

\[
\mathbb{D} := \{n/2^k \mid n, k \text{ integers}\}
\]
to (finite) surreals and is defined recursively by 
\[
d : \mathbb{D} \to S
\]
where 
\[
d(x) = \begin{cases} 
\{\emptyset | \emptyset\}, & \text{if } x = 0, \\
\{d(n - 1) | \emptyset\}, & \text{if } x = n, \text{ a positive integer}, \\
\{\emptyset | d(n + 1)\}, & \text{if } x = n, \text{ a negative integer}, \\
\{d\left(\frac{n-1}{2^k}\right) | d\left(\frac{n+1}{2^k}\right)\}, & \text{if } x = n/2^k \text{ for } k > 0 \text{ and } n \text{ odd}.
\end{cases}
\] 

(1)

For convenience, we denote canonical forms by the shorthand of placing a line above the number. All other forms are defined by their DAG, or in terms of canonnals.

This recursive construction is often illustrated as a tree showing the numbers that are created in each generation and their position on the real number line. However, that is misleading. The (non-integer) Dali surreals have two parents, and the resultant structure of dependency in the recursion is the DAG shown in Figure 2.

Knuth’s description of Conway’s construction of surreals leads to the notion of birthdays: taking 0 to be born on Day 0, ±1 to be born on Day 1, and so on, we can assign a birthday to all surreals. We prefer the term generation over birthday, if only because it links up to the notion of parents and children more cleanly. The birthday can also be seen as how deeply you must recurse through the DAG structure to get to 0. We say a surreal is older if it comes from an earlier generation, meaning it has a smaller (earlier) birthday. We formalize these notions below.

**Definition 1.** We refer to the elements of the left and right sets of a surreal form \(x\) as its parents (note there may be some number other than two parents), and \(x\) as their child. The birthday or generation of \(\{\emptyset | \emptyset\}\) is 0, and the generation of all other finite surreal forms is 1 greater than that of their youngest parent.

If we denote the parents of \(x\) by \(X_P = X_L \cup X_R\), then the generation/birthday function of a surreal number form \(x\) is given by 
\[
g(x) = \sup_{x_p \in X_P} g(x_p) + 1,
\]

(2)

where \(g(\{\emptyset | \emptyset\}) = 0\). For example, in Figure 1(b), we have 
\[
g\left(\{-1 | 1\}\right) = g(1) + 1 = 2.
\]
That brings us to the nub of our problem: can we calculate the generation/birthday of surreals, such as the sum of two surreals, purely from the generation of the inputs? Simons proves that $g(x + y) \leq g(x) + g(y)$ [14, p. 25], but we present a small set of examples in Table 1, and in all of these (and every other case tested) we find $g(x + y) = g(x) + g(y)$. The question then is, can we prove that equality always holds? The answer can be found in the following sections.

There is one side-issue we must discuss first: the above definition is not the only way to construct the surreals. There is also a "sign-sequence" definition, discussed extensively by Gonshor [6]. The construction provides an alternative means to construct and work with surreal numbers which is useful in some respects. Sign-sequence lengths are related to birthdays, but results on lengths are not exactly equivalent to those for birthdays. We will indicate some of the connections as we proceed, but will primarily concern ourselves with Conway’s original construction.

Surreal forms with equivalent values can come from different generations, so knowing the value of a surreal tells us only a lower bound on its generation (namely the generation of the canonical form of that surreal). Thus, some questions arise in regard to birthdays. For example, can we derive birthdays for standard surreal constructs? We call these problems in birthday arithmetic.

We will start by deriving the birthday of the canonical forms since this derivation provides an example of the standard inductive proof structure for many surreal arguments. Simons states the result, but only as a minor note within a larger result [14, p. 27]. Throughout, we use $g(x)$ as shorthand for $g(d(x))$.

**Lemma 1.** The generation/birthday of the canonical form of dyadic $x = n/2^k$, which is in irreducible form (or lowest terms) is

$$g(x) = \left\lceil |x| \right\rceil + k,$$

where $\left\lceil x \right\rceil$ denotes the ceiling function of $x$ (the smallest integer at least as large as $x$).

**Proof.** The statement is true for $x = 0$ because $g(0) = 0$ by definition. The negative case can be treated by considering $g(-x)$ (see p. 337), so we only consider $x > 0$ here. The integer case is trivial (see Figure 2) so we focus on the case $n$ odd and $k > 0$.

Assume for the purpose of induction that the lemma is true for all parents of $x$.

From equation (1) a non-integer dyadic has exactly two parents, and hence equation (2) reduces to

$$g \left( \frac{n}{2^k} \right) = \max \left\{ g \left( \frac{n - 1}{2^k} \right), g \left( \frac{n + 1}{2^k} \right) \right\} + 1.$$

Since we take $x = n/2^k$ to be irreducible, $n$ must be odd, and so can write it as $2a + 1$, where $a$ is an integer. Thus, we can write

$$g \left( \frac{n}{2^k} \right) = \max \left\{ g \left( \frac{a}{2^{k-1}} \right), g \left( \frac{a + 1}{2^{k-1}} \right) \right\} + 1,$$

but note that it cannot be the case that both of the parents here are in irreducible form and so we cannot apply the recursion directly. There are two possible cases.

In the first case, when $a$ is even, we have that $(a + 1)/2^{k - 1}$ is irreducible, and

- for non-integer $x$ we have $\left\lceil (a + 1)/2^{k-1} \right\rceil = \left\lceil x \right\rceil$;
- we have $\left\lfloor a/2^{k-1} \right\rfloor \leq \left\lceil x \right\rceil$; and
we can further reduce $a/2^{k-1} = b/2^{k-2}$ (at least), so that under the inductive hypothesis, we have

$$g \left( \frac{a}{2^{k-1}} \right) < g \left( \frac{a + 1}{2^{k-1}} \right) = [x] + k - 1.$$ 

In the second case, when $a$ is odd, we have that $a/2^{k-1}$ is irreducible, thus,

- $[a/2^{k-1}] = [(a + 1)/2^{k-1}] = [x]$ for non-integer $x$; and
- we can further reduce $(a + 1)/2^{k-1} = b/2^{k-2}$ (at least), so that under the inductive hypothesis, we have

$$g \left( \frac{a + 1}{2^{k-1}} \right) < g \left( \frac{a}{2^{k-1}} \right) = [x] + k - 1.$$ 

In both cases, the recursive expression for $g(n/2^k)$ reduces to

$$g \left( \frac{n}{2^k} \right) = ([x] + k - 1) + 1 = [x] + k.$$

Intrinsic to this proof (and others) is the fact that $\emptyset$ is the starting point for the construction of the surreals, and therefore is the ultimate ancestor of all surreals.

**Addition and subtraction**

In order for the surreals to fulfill their role as “numbers,” they must be able to play all the tricks of numbers. For instance, we must be able to do arithmetic. Conway defined addition and subtraction for surreal numbers and showed his operations satisfy the required conditions, but they are actually operations on the forms. Let us examine them in detail.

**Addition**

The standard definition of addition on surreal forms $[3, 16]$ is

$$x + y \overset{\text{def}}{=} \{X_L + y \cup x + Y_L \mid X_R + y \cup x + Y_R \}.$$ 

Notation is often abbreviated, and so you will sometimes set operations simplified: for example, $\{x, y\} \cup A$ is written $\{x, y, A\}$. Also, in this and other definitions, we implicitly extend the operators to sets, or to combinations of sets with surreals: for example, $x + y$ is comprised of terms like $X_L + y$. Such operations on sets are applied to each member $[16]$:

$$\{x_1, x_2, \ldots, x_n\} + y \overset{\text{def}}{=} \{x_1 + y, x_2 + y, \ldots, x_n + y\},$$

and operations on empty sets result in empty sets, that is, $x + \emptyset = \emptyset$.

Many of the texts on surreals provide proofs that addition satisfies all of the usual requirements: associativity, commutativity and so on (see Conway [3] and Tondering [16], for example). For instance, 0 is the additive identity since

$$x + 0 \equiv 0 + x \equiv x,$$
but this is a statement about values, not forms: \( x + 0 \) is not (in general) identical to \( x \), except for the canonical zero. That is, \( x + 0 == x \). For instance, consider the following addition of \( \frac{1}{2} + 0 \), noting carefully the bars indicating which terms are canonical.

\[
(0 \mid 1) + (-\frac{1}{2} \mid 1) == (0 + 0, \ \frac{1}{2} + (-\frac{1}{2}) \mid 1 + 0, \ \frac{1}{2} + 1) \\
== (0, -1/2 \mid 1, 3/2),
\]

which is not identical to \( \{0 \mid 1\} \).

Another instructive example is \( \overline{2} + \overline{2} \). In this case, \( X_R = Y_R = \emptyset \), and so the right-set of \( \overline{2} + \overline{2} \) will also be \( \emptyset \). Thus,

\[
\overline{2} + \overline{2} == (\overline{1} + \overline{2}, \overline{2} + \overline{1} \mid \emptyset) \\
== (\overline{3} \mid \emptyset) \\
== \overline{4}.
\]

The result is the canonical form of 4. Naively, we might expect that addition of canonical forms would always lead to the same. However, this is not true. We can play with such hypotheses using the \texttt{SurrealNumbers} package in the programming language Julia [13]. For instance, the above calculation can be performed using the commands

1  julia> using \texttt{SurrealNumbers} \\
2  julia> x = dali(2)   // set x to the canonical form of 2 \\
3  julia> y = x + x      // calculate \( \overline{2} + \overline{2} \)

A more complicated example with a non-canonical result is

\[
\overline{1} + \overline{\frac{1}{2}} == \{(\emptyset \mid \emptyset) \mid (\emptyset \mid \emptyset \mid \emptyset)\}, \{(\emptyset \mid \emptyset) \mid \emptyset\} \mid (\{(\emptyset \mid \emptyset) \mid \emptyset\} \mid \emptyset)\}.
\]

Figure 3 shows this, and two other forms with value \( 3/2 \). The figure shows the differences that can occur in form and in generation.

That brings us to the nub of the problem: can we calculate the generation/birthday of the sum purely from the generation of the inputs? Simons proves that \( g(x + y) \leq g(x) + g(y) \) [14, p. 25], but we present a small set of examples in Table 1, and in all of these (and every other case tested) we find \( g(x + y) = g(x) + g(y) \). The question then is, can we prove that equality is always the case? The answer follows.

| \( y \) | \( g(y) \) | \( \emptyset \) | \( \frac{1}{2} \) | \( \frac{3}{4} \) | \( 1 \) | \( 2 \) | \( x \) | \( g(x) \) |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|
| \( 0 \) | 0      | 0      | 2      | 3      | 1      | 2      | \( y \) | \( g(y) \) |
| \( \frac{1}{2} \) | 2      | 2      | 4      | 5      | 3      | 4      | \( \emptyset \) | 0      |
| \( \frac{3}{4} \) | 3      | 3      | 5      | 6      | 4      | 5      | \( \frac{1}{2} \) | 2      |
| \( 1 \) | 1      | 1      | 3      | 4      | 2      | 3      | \( \frac{3}{4} \) | 3      |
| \( \overline{2} \) | 2      | 2      | 4      | 5      | 3      | 4      | \( 1 \) | 1      |

TABLE 1: The birthday table \( g(x + y) \), that is, the birthdays of the sums of the \( x \) and \( y \) values in the columns and rows. Note that they are additive: \( g(x + y) = g(x) + g(y) \).
(a) Canonical form $3/2$. (b) The form of $1 + 1/2$. (c) The form of $3/4 + 3/4$.

Figure 3  Graphs depicting three forms for the surreal number $3/2$. It is interesting that the form in (b) appears as a subgraph of the form in (c). An open question concerns whether this is true in general: is the canonical form always a subgraph of any alternative form of the same number?

**Theorem 1 (Birthday addition theorem).** For two surreal numbers $x$ and $y$, we have

$$g(x + y) = g(x) + g(y),$$

where $g(\cdot)$ is the birthday/generation function.

**Proof.** Birthday addition is trivially true if $x$ or $y = 0$. For induction, presume that the theorem is true for all combinations of the parents of summands $x$ and $y$ and the summands themselves (excepting $x + y$). That is, the theorem is true for all terms like $x + y_p$, where $y_p \in Y_P = Y_L \cup Y_R$ and $y + x_p$, where $x_p \in X_P = X_L \cup X_R$.

Applying equation (2) to the definition of addition gives us

$$g(x + y) = \sup_{x_p \in X_P, y_p \in Y_P} \left\{ g(x + y_p) + 1, g(y + x_p) + 1 \right\}.$$  

By the inductive hypothesis, birthday addition is true for all parents and their combinations. Thus, the above reduces to

$$g(x + y) = \sup_{x_p \in X_P, y_p \in Y_P} \left\{ g(x) + g(y_p) + 1, g(y) + g(x_p) + 1 \right\}$$

$$= \max \left\{ g(x) + \sup_{y_p \in Y_P} \{ g(y_p) \} + 1, g(y) + \sup_{x_p \in X_P} \{ g(x_p) \} + 1 \right\}$$

$$= \max \left\{ g(x) + g(y), g(x) + g(y) \right\}$$

$$= g(x) + g(y).$$

Thus, the result follows by induction.  

$\blacksquare$
Negation and subtraction  Negation and subtraction are defined together as
\[ -x \overset{\text{def}}{=} \{-X_R \mid -X_L\} \quad \text{and} \quad x - y \overset{\text{def}}{=} x + (-y). \]

Subtraction looks as simple as addition, but can complicate matters more than one might think with respect to surreal forms because the number of empty sets in the results change, and we end up with more complicated expressions. For instance, \( \overline{1} - \overline{1} \) is equivalent to \( 0 \) (this is the form that is illustrated in Figure 1(b)), but is not identical to \( \overline{0} \). Thus, \( -x \) is the additive inverse of \( x \) in terms of equivalence, but not identity.

However, here we are interested in the birthday/generation of the output. Using the definition above, we can construct a very simple inductive proof that \( g(-x) = g(x) \). Incidentally, this concludes the proof of Lemma 1.

From this, the definition of subtraction, and Theorem 1, we immediately get the following corollary:

**Corollary 1 (Birthday subtraction corollary).** For two surreal numbers \( x \) and \( y \)
\[ g(x - y) = g(x) + g(y), \]
where \( g(\cdot) \) is the birthday/generation function.

**Multiplication**

If you thought addition and subtraction were complicated, then fasten your seat belts. Multiplication is defined by
\[ xy \overset{\text{def}}{=} \left\{ \begin{array}{l} \{X_L y + xY_L - X_L Y_L \} \cup \{X_R y + xY_R - X_R Y_R \} \\ \{X_L y + xY_R - X_L Y_R \} \cup \{X_R y + xY_L - X_R Y_L \} \end{array} \right\}. \tag{3} 

We need to be clear about exactly what each term in this definition means because it does not follow the typical convention for expressions of this nature. Consider the term \( \{X_L y + xY_L - X_L Y_L \} \); this means in detail:
\[ \{X_L y + xY_L - X_L Y_L \} \overset{\text{def}}{=} \{x_i y + x y_j - x_i y_j \mid \forall x_i \in X_L \text{ and } y_j \in Y_L\}, \]
where each of the products in the set above is another surreal multiplication and the additions are surreal additions.

A quick example is informative: let us work through \( \overline{2} \times \overline{2} \). Once again, \( X_R = Y_R = \emptyset \), and so only one term of the four in the multiplication is non-empty:
\[ \overline{2} \times \overline{2} == \{ \overline{1} \times \overline{2} + \overline{2} \times \overline{1} - \overline{1} \times \overline{1} \mid \emptyset \} \\
== \{ \overline{2} + \overline{2} - \overline{1} \mid \emptyset \} \\
== \{ \overline{4} - \overline{1} \mid \emptyset \} \\
== \{ 3 \mid \emptyset \}, \]
where we exploit the fact that \( \overline{1} \) is the multiplicative identity for surreal forms, and \( \overline{0} \) is the multiplicative annihilator, that is, \( \overline{0} x = \overline{0} \) for all \( x \). This can be seen by considering that
\[ \overline{0} x == \{ \emptyset y + \overline{0} Y_L - \emptyset Y_L, \emptyset y + \overline{0} Y_R - \emptyset Y_R \mid \emptyset y + \overline{0} Y_L - \emptyset Y_L, \emptyset y + \overline{0} Y_R - \emptyset Y_R \}, \]
and noting that operations with $\emptyset$ result in $\emptyset$. The proof of the identity of $\bar{1}$ is very similar, though it requires an inductive step.

We also use a minor result we have not bothered to prove: that addition of canonical forms of nonnegative integers results in canonical forms, but subtraction does not. The result of $\bar{2} \times \bar{2}$ looks like the canonical form for 4, but we are taking liberties by reducing “3” to its short hand. This is not the canonical 3, and so the shorthand is misleading. In fact, if we write this out in full we get (as in Chu-Carroll [2])

$$\bar{2} \times \bar{2} = \{\{\{\emptyset | \emptyset | \emptyset\} | \{\emptyset | \emptyset\} | \emptyset\} | \{\emptyset | \emptyset | \emptyset\} | \{\emptyset | \emptyset\} | \emptyset\} | \{\emptyset | \emptyset | \emptyset\} | \{\emptyset | \emptyset\} | \emptyset\} | \emptyset\} | \emptyset\}.$$ 

Gonshor proves a weak birthday multiplication theorem [6, Theorem 6.2], namely that

$$g(xy) \leq 3^{g(x) + g(y)}.$$ 

Simons [14] conjectures a much tighter bound that $g(xy) \leq g(x)g(y)$ and states that there are no obvious counter-examples. It was this hypothesis that largely motivated this investigation.

The problem with Simons’ statement is that there are very few examples at all. Multiplication is very complex to do in practice. As far as I am aware, only a few multiplications have been explicitly tabulated! Several places show that simple multiplicative identities hold, but few go any further. With the help of the SurrealNumbers package [13] we can calculate other products (they quickly become too complicated to do with pen and paper). The result of $\bar{3} \times \bar{2}$ is shown in DAG form in Figure 4. From this, we can calculate $g(\bar{3} \times \bar{2}) = 12$. Unfortunately, this breaks Simons’ conjecture, which suggests the bound should be 6. It also suggests that Gonshor’s bound is very weak.

Table 2 shows a table for the birthday/generation of products. The black numbers are those that we have calculated (in multiple ways) using the Julia package. Note that the only surreal forms with birthdays 0 and 1 are $\emptyset$ and $\pm \emptyset$, so the first two rows and columns of this table are trivially true.

Patterns appear in the results, for example, $g(\bar{2} \times m) = m(m + 1)$, but the general pattern is not so trivial. For instance, the product $g(\bar{3} \times 3) = 31$ is prime, and so the resulting birthdays are not even the product of a function the underlying birthdays!

The results below explain the pattern observed in the black values of the table and can be extrapolated to provide the gray values.

**Lemma 2.** If the generation of $xy$ is given by a function $f$ of the generations of $x$ and $y$, that is, $g(xy) = f(g(x), g(y))$, then the function will be symmetric ($f(n, m) = f(m, n)$), and strictly increasing.

**Proof.** The function must be symmetric by the commutativity of multiplication. Now find the “maximal” parent of $x$, meaning the parent $x_p^{(\text{max})}$ that has the maximum generation, so that

$$g(x) = g(x_p^{(\text{max})}) + 1.$$ 

If the generation of $xy$ is given by a function $f(g(x), g(y))$, then

$$g(xy) = f(g(x_p^{(\text{max})}) + 1, g(y)).$$
Figure 4  A graph depicting the form of $3 \times 2$. Inside this we frequently see subforms that might superficially appear identical, e.g., $4 = \{3 \mid 5\}$ appears three times, but these are not identical forms, which is implicit in their dependent DAG. Interestingly, the DAG also includes even some negative values, which might be unexpected in a product of positive integers.
TABLE 2: The values of $g(xy)$ in relation to $n = g(x)$ and $m = g(y)$. Roman values have been verified through multiple instances of the products of (not just canonical) forms from the same generation. Numbers in italics are extrapolated using Theorem 2. The right-hand column expresses the pattern, where known.

| $n = g(x)$ | 0  | 1  | 2  | 3  | 4  | 5  | 6  | $m = g(y)$ |
|------------|----|----|----|----|----|----|----|------------|
| 0          | 0  | 0  | 0  | 0  | 0  | 0  | 0  | = $m$      |
| 1          | 0  | 1  | 2  | 3  | 4  | 5  | 6  | = $m(m + 1)$ |
| 2          | 0  | 2  | 6  | 12 | 20 | 30 | 42 | $f(n, m)$  |
| 3          | 0  | 3  | 12 | 31 | 64 | 115| 188|            |
| 4          | 0  | 4  | 20 | 64 | 160| 340| 644|            |
| 5          | 0  | 5  | 30 | 115| 340| 841| 1826|          |
| 6          | 0  | 6  | 42 | 188| 644| 1826|4494|            |

Also, $xy$ contains parents containing products of all pairs of parents of $x$ and $y$, and so $g(xy) > g(xpy)$ for all $x_p \in X_p$. Choosing the maximal parent, we get

$$f(g(x_p^{\text{max}}) + 1, g(y)) > f(g(x_p^{\text{max}}^\cdot), g(y)),$$

and since we can find examples $x$ for any $n = g(x)$, this must be true for all values of $n$, and likewise $m$. Hence, the function is increasing. ■

Finally, we can derive the generation of forms after multiplication. The result is a kind of two-dimensional Fibonacci array of numbers.

**Theorem 2** (Birthday multiplication theorem). The generation of $xy$ is given by a function $f(\cdot, \cdot)$ of the generations of $x$ and $y$, that is, $g(xy) = f(g(x), g(y))$, where the function $f(\cdot, \cdot)$ satisfies the recurrence relation

$$f(n, m) = \begin{cases} 
0, & \text{if } m = 0, \\
0, & \text{if } n = 0, \\
f(n, m - 1) + f(n - 1, m) + f(n - 1, m - 1) + 1, & \text{otherwise,}
\end{cases}$$

for $n, m \in \mathbb{Z}^+$. 

**Proof.** The only number $x$ with $g(x) = 0$ is $\overline{0}$ which is the multiplicative annihilator, meaning that $\overline{0} \times x = \overline{0}$ for all $x$, and hence the bounding case $n = 0$ and by symmetry $m = 0$.

For $n, m > 0$ we start from the definition of multiplication, equation (3), which contains terms $x_p y + xy_p - x_p y_p$ for all pairs of parents $(x_p, y_p)$, and hence

$$g(xy) = \sup_{(x_p, y_p)} g(x_p y + xy_p - x_p y_p) + 1$$

$$= \sup_{(x_p, y_p)} \left[ g(x_p y) + g(xy_p) + g(x_p y_p) \right] + 1,$$

by the birthday addition theorem, and its subtraction corollary.

Assume (for inductive purposes) that

$$g(x_p y) = f(g(x_p), g(y)),$$

and
and similarly for the other such terms, leading to

\[ f(g(x), g(y)) = \sup_{(x_p, y_p)} \left[ f(g(x_p), g(y)) + f(g(x), g(y_p)) + f(g(x), g(y_p)) \right] + 1, \]

We can choose the respective parents \( x_p \) and \( y_p \) independently. By the previous theorem, the function \( f(n, m) \) must be increasing. Hence, the above sum will be maximized when we choose \((x_p, y_p)\) such that \( g(x_p) \) and \( g(y_p) \) are both individually maximized. In this case, note the definition of \( g(x) \) in equation (2), and hence

\[ f(g(x), g(y)) = \sup_{x_p} f(g(x_p), g(y)) + \sup_{y_p} f(g(x), g(y_p)) + 1, \]

\[ = f(g(x) - 1, g(y)) + f(g(x), g(y) - 1) + f(g(x) - 1, g(y) - 1) + 1, \]

where existence of the suprema is required by the construction of the surreals.

The two-dimensional Fibonacci-like arrays defined by the recurrence relationship of Theorem 2 have been studied by Fredman [5] and are summarized in [11, a047662]. Table 2 shows values that have been derived empirically via multiplication of surreal forms and (in grey italics) values that have been derived from the recurrence. Unfortunately, since the depth of recursion is given by the birthday, and multiplication is built from the recursive application of multiple recursive operations combined across the two surreal forms, we have not been able to pursue complicated surreal multiplications with resulting generations beyond roughly 100, though it is noteworthy that many of the empirical values in the table were first extrapolated using the recursion before being verified computationally.

We conclude by noting that Theorem 2 leads to a number of immediate corollaries regarding the asymptotic growth of surreals birthdays in particular cases.

**Corollary 2.** The birthday/generation of \( n^2 \) takes values

\[ 0, 1, 6, 31, 160, 841, 4494, \ldots, \]

which grow as

\[ g(n^2) \sim a\lambda^n / \sqrt{n}, \]

where

\[ \lambda = 3 + 2\sqrt{2} \approx 5.83 \quad \text{and} \quad a = 2^{-9/4} \sqrt{\lambda / \pi} \approx 0.29. \]

**Proof.** The values \( g(n^2) \) comprise the main diagonal of \( f(n, m) \), which are given in Fredman [5] and the OEIS [11, a047665].

**Corollary 3.** The birthday/generation of powers of \( 2 \) takes values

\[ 2, 6, 42, 1806, \ldots, \]

which grow as \( |c(2^n)| \) for

\[ c = 1.597910218031873178338070118157 \ldots. \]
Proof. The generation of $2^n$ is

$$g(2 \times 2^{n-1}) = f(2, g(2^{n-1})),$$

where $f(2, n) = n(n + 1)$ from Theorem 2, and hence

$$g(2^n) = g(2^{n-1}) \left(g(2^{n-1}) + 1\right).$$

Now, the equation

$$g_n = g_{n-1}(g_{n-1} + 1)$$

is known (see Aho [1] or the OEIS [11, a007018], and follows $g_n = \lfloor c(2^n) \rfloor$. ■

The result is derived from the techniques of Aho [1] with extended precision provided by Cloitre [11, a007018]. The value

$$c = \exp\left(\sum_{n=0}^\infty \log(1 + 1/a_n)/2^{n+1}\right),$$

where $a_n = \lfloor c(2^n) \rfloor$, is the sequence itself. The series arises as a special case of certain nonlinear recurrence problems, for instance, the number of ordered trees having nodes of out-degree 0, 1, or 2 such that all leaves are at level $n$ [11, a007018]. The result also highlights that powers of $2$ are not canonical numbers themselves: $2^n \neq 2^n$ for $n > 1$.

The outstanding feature of these corollaries is the very high rate of growth, for example, the generation of the ninth power of $2$ has 209 digits. The size and complexity of calculations involving these forms grows even faster than the generation of the output due to the complicated set of recursive operations built on top of each other, and hence the complexity of larger computations.

Conclusion

This paper derived rules for birthday arithmetic (in particular, addition, subtraction and multiplication) for surreal number forms.

The notable absentee from this list is division. Naively, division is the reciprocal of multiplication and so should be no harder, that is, $x/y = x \times (1/y)$. But division is in fact quite different. Only dyadic surreals have finite representations. Thus, we can represent $1/2$, $15/16$ and so on exactly with a finite form. However, non-dyadic numbers do not have finite forms. Thus, numbers even as simple as $1/3$ do not have a finite representation. So, to apply the multiplication theorem to division, we need a formula to calculate the birthday of a reciprocal. This remains to be found.

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Summary. This paper is about the structure underlying surreal numbers, namely, the birthdays of surreal number forms. The results are intriguing because (i) addition of surreal forms results in simple addition of their birthdays, but (ii) multiplication results in a much more complicated structure of birthdays, given by a kind of two-dimensional Fibonacci sequence. The paper also provides many examples and illustrations designed to help a student become familiar with this interesting set of numbers.

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