The superspace geometry of gravitational Chern-Simons forms and their couplings to linear multiplets: a review

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Abstract

The superspace geometry of Chern-Simons forms is shown to be closely related to that of the 3-form multiplet. This observation allows to simplify considerably the geometric structure of supersymmetric Chern-Simons forms and their coupling to linear multiplets. The analysis is carried through in $U_K(1)$ superspace, relevant at the same time for supergravity-matter couplings and for chirally extended supergravity.

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1 Introduction

1.1 Context and possible motivations

The purpose of this article is to provide a concise introductory review of the superspace geometry relevant for $N = 1$ supersymmetric theories in four dimensions which go beyond the by now commonly used standard formulation of the general supergravity-matter system [1], [2], [3]. Beyond is meant here in the sense that, in addition to the general couplings of chiral matter multiplets and of Yang-Mills multiplets to supergravity, couplings of linear supermultiplets in the presence of Chern-Simons forms will be included. Relying on the mechanisms used successfully for the implementation of Chern-Simons forms of the Yang-Mills type in supersymmetric theories [4], [5], an attempt is made to clarify, as concisely as possible, the geometrical structures and the special features occurring in the corresponding descriptions of gravitational Chern-Simons forms.

As the most popular motivation for this work we recall that couplings of antisymmetric
tensor gauge fields to supersymmetric theories, and the appearance of Chern-Simons forms, are important ingredients in the construction of low energy effective approximations of some underlying fundamental superstring theory.

As usual in this kind of approach it is prohibitively complicated, if not impossible, to explicitly derive the effective from the exact theory. There are, however, criteria which allow nevertheless to obtain nontrivial information on the form of the low energy theory, one of them being the requirement of absence of anomalies in the fundamental theory.

In the case of superstring theory, this kind of reasoning leads, among other things, to the coupling of the antisymmetric tensor gauge field together with Yang-Mills and gravitational Chern-Simons forms via the so-called Green-Schwarz mechanism in the ten-dimensional effective theory. At this point, however, supersymmetry of the mechanism is far from evident, not only due to technical complications but also at a more fundamental level due to the lack of complete understanding of higher-dimensional and/or extended supersymmetries.

In attempts of relating such kinds of theories to effective four-dimensional ones with \( N = 1 \) supersymmetry the remnants of anomaly cancellation mechanisms should show up in one way or another. As, in particular, one requires these couplings to appear in a supersymmetric way, a more profound understanding of the general structure of supersymmetric theories in themselves is important, irrespective of the motivation put forward in relation with superstring theory. Stated differently, one might turn the argument around and study the general form of \( N = 1 \) four-dimensional supersymmetric theories as a framework into which any of the candidates of such low energy approximations should fit. It is actually this point of view which will be adopted in our investigations.

Hence turning to \( N = 1 \) supersymmetry in four dimensions, we recall that an antisymmetric tensor appears in the so-called linear multiplet, together with a real scalar and a Majorana spinor. The lesson to be learnt is then to couple this multiplet to the general supergravity-matter system, which, together with its intrinsic Kähler invariance is by now rather well understood in geometric terms.

The structure of chiral Kähler transformations, inherent in any coupling of supergravity with matter has led to a unified geometric description of such theories: chiral Kähler transformations appear together with Lorentz transformations in the structure group of superspace.

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It is well-known from the standard superspace formulations, that the spin connection in supergravity theories is expressed as a function of the vierbein field, its derivatives and of quadratic Rarita-Schwinger field terms. Likewise, in the new approach, the Kähler connection (not to be confused with the Christoffel connection on the Kähler manifold itself) is given in terms of the bosonic matter component fields, their space-time derivatives and their fermionic supersymmetric partners (for this reason it is sometimes referred to as composite connection).
Both these geometric objects appear naturally in the framework of so-called Kähler superspace geometry. Moreover, this geometric formulation gives rise to a unified dynamical description: the supersymmetric action of the kinetic terms of the complete supergravity-matter system (with canonically normalized Einstein term) is given by one single term in superspace, namely the superdeterminant of the frame of Kähler superspace.

Given this powerful and elegant formulation it is natural to search for a generalization which allows to accommodate the couplings of linear multiplets as well. In addition, such a construction should be able to embody the additional structures arising from the supersymmetric inclusion of Chern-Simons terms in the field strength of the antisymmetric tensor.

A promising way to implement more general couplings of any number of linear multiplets to the supergravity-matter system consists in generalizing the notion of Kähler superfield potential: in addition of being exclusively a function of the chiral and antichiral matter superfields it is allowed to depend on the linear superfields as well. This approach has given already a number of interesting results in particular cases:

- The coupling of a single linear multiplet to supergravity (without matter) came under the disguise of the so-called 16 − 16 supergravity \[13\], \[14\], \[15\], \[16\].
- This coupling was subsequently amended to include Chern-Simons forms of supersymmetric Yang-Mills theory \[4\].
- Based on the geometric description in superspace, a particularly interesting special coupling \[17\] of one linear multiplet with Chern-Simons form of Yang-Mills type to the complete supergravity-matter system has been worked out in full detail \[3\]. Moreover the extension of this construction to the case of general couplings of an arbitrary number of linear multiplets has been indicated.
- The superspace geometry relevant for gravitational Chern-Simons forms has been developed and a superspace action has been proposed \[18\], \[19\].

In a slightly different language, making use of the duality between linear and chiral multiplet formulations, component field expressions arising from couplings of gravitational Chern-Simons forms in ”higher derivative supergravity theories” have been reported in a series of publications of S. Ferrara et al. \[20\], \[21\], \[22\], \[23\], \[24\], clearly demonstrating the complexity of the subject and pointing out the appearance of new unorthodox structures (related to higher derivative couplings of matter fields and the rôle of previously auxiliary fields which may become dynamical in some sense), questions which clearly deserve further study.

Recall also that the issue of linear multiplet couplings and supersymmetric Chern-Simons forms is relevant in the context of string loop corrections to effective gauge coupling functions
and its relations to Kähler sigma model anomaly cancellation mechanisms \[28, 29, 30, 31\].

From the point of view of general properties of supersymmetric theories, as alluded to above, the situation may be interpreted as follows. In distinction to the traditional approaches, \textit{i.e.} without linear multiplets and Chern-Simons forms, where the effective gauge coupling functions are restricted to be the sum of a holomorphic and an anti-holomorphic function of the scalar matter superfields, the explicit calculations of string loop corrections yielded non-holomorphic gauge coupling functions. This apparent contradiction can be explained in the framework of theories which include linear superfield and Chern-Simons couplings \[31, 32, 33\], already contained in the general formulation proposed in \[3\].

Again, in this kind of investigations, the Yang-Mills case is comparatively well understood, whereas in the corresponding mechanisms involving supersymmetric gravitational Chern-Simons forms many questions are still awaiting a satisfactory answer.

The present paper is intended to contribute to a clarification and a better understanding of the structure of supersymmetric theories describing couplings of linear multiplets and Chern-Simons forms, in particular gravitational ones.

The basic idea of the approach presented here is the use of methods of superspace geometry, that is to proceed as far as possible in terms of superfields in order to encode compactly the embarrassing complications of explicit component field expressions. Done in an appropriate way this allows to analyse concisely the principal features of a supersymmetric theory, in particular when it comes to the formulation of the invariant action used to describe the supersymmetric dynamics. Only after having completely set the stage in geometrical terms, the transition to the description in terms of component fields (\textit{viz.} invariant action, supersymmetry transformation laws, etc.) is performed, using standard textbook methods and without any further ambiguities.

The basic strategy pursued in the present approach will be to generalize the description which works very well in the Yang-Mills case, to the gravitational problems. Without pretending that this approach is the only possible one, we feel nevertheless that for the time being it is the only realistic viable one.

One of the purposes of this paper is to show that the gravitational case differs from the Yang-Mills case not only in being technically more involved, but also in certain conceptual respects. Loosely speaking this may be assigned to the fact that contrary to Yang-Mills given as superspace geometry in the supergravity-matter "background", the gravitational Chern-Simons forms are to be included concisely in the supergravity geometry itself.
1.2 Linear multiplet without supergravity

The linear supermultiplet is the supersymmetric extension of the antisymmetric tensor gauge potential

\[ b_{mn} = -b_{nm}. \]  

(1.1)

Historically the antisymmetric tensor was studied already some time ago by V. I. Ogievetsky and I. V. Polubarinov [34], later on it appeared in the context of string theory in the work of Kalb and Ramond [35]. The linear multiplet [36], [37] is the prototype supermultiplet which contains an antisymmetric tensor gauge field, but there are other ones, in \( N = 1 \) four dimensional supergravity, the new-minimal multiplet, as well as in extended and higher dimensional supersymmetry. It was its ten-dimensional incarnation which was used by Green and Schwarz in their anomaly-cancellation mechanism [3]. In four dimensional effective theories this mechanism is expected to result, among other things, in couplings of a modified linear multiplet to the supergravity-matter system. Modified means here that the effects of Chern-Simons forms of Yang-Mills and gravitational types should be taken into account.

To begin with, we present in this section the basic features of a linear multiplet with Yang-Mills Chern-Simons forms in global supersymmetry. This discussion is intended to provide a first impression of the geometric methods used in the more general and complicated case when the traditional supergravity-matter couplings and gravitational Chern-Simons forms will be taken into account.

Consider first the non-supersymmetric case, \( i.e. \) the simple case of the antisymmetric tensor gauge potential \( b_{mn} \) in four dimensions with gauge transformations generated by a four vector \( \rho_m \) such that

\[ b_{mn} \rightarrow b_{mn} + \partial_m \rho_n - \partial_n \rho_m, \]  

(1.2)

and with invariant field strength given as

\[ h_{0l mn} = \partial_l b_{mn} + \partial_m b_{nl} + \partial_n b_{lm}. \]  

(1.3)

The subscript 0 denotes here the absence of Chern-Simons forms. As a consequence of its definition the field strength satisfies the Bianchi identity

\[ \varepsilon^{klmn} \partial_k h_{0l mn} = 0. \]  

(1.4)

The invariant kinetic action is given as

\[ \mathcal{L} = \frac{1}{2} \tilde{h}_0^m \tilde{h}_0^m, \]  

(1.5)

with \( \tilde{h}_0^k = \frac{1}{3!} \varepsilon^{klmn} h_{0l mn} \) denoting the dual of the field strength tensor.
Consider next the case where a Chern-Simons term for a Yang-Mills potential $a_m$ is added such that

$$h_{lmn} = h_{0	ext{lmn}} + k Q_{lmn}. \tag{1.6}$$

Here $k$ is a constant which helps keeping track of the terms induced by the inclusion of the Chern-Simons combination

$$Q_{lmn} = - \text{tr} \left( a_l \partial_m a_n - \frac{2i}{3} a_l [a_m a_n] \right), \tag{1.7}$$

with $[lmn] = lmn + mnl + nlm - lnm - nm$. The gauge transformations of the Chern-Simons term are compensated by assigning suitably adjusted Yang-Mills gauge transformations to the antisymmetric tensor, thus rendering the modified field strength invariant. The presence of the Chern-Simons term modifies the Bianchi identity as well, it reads now

$$\varepsilon^{klmn} \partial_k h_{lmn} = - \frac{3}{2} k \varepsilon^{klmn} \text{tr}(f_{kl} f_{mn}). \tag{1.8}$$

A dynamical theory may then be obtained from the invariant action

$$\mathcal{L} = \frac{1}{2} \tilde{h}^m \tilde{h}_m - \frac{1}{4} \text{tr}(f_{mn} f_{mn}), \tag{1.9}$$

with $\tilde{h}^k = \frac{1}{3!} \varepsilon^{klmn} h_{lmn}$, and Yang-Mills field strength

$$f_{mn} = \partial_m a_n - \partial_n a_m - i [a_m, a_n]. \tag{1.10}$$

This action describes the dynamics of Yang-Mills potentials $a_m(x)$ and an antisymmetric tensor gauge potential $b_{mn}$ with effective $k$-dependent couplings induced through the Chern-Simons form.

This theory is dual to another one where the antisymmetric tensor is replaced by a real scalar $a(x)$ in the following sense: one starts from a first order action describing a vector $X^m(x)$, a scalar $a(x)$ and the Yang-Mills gauge potential $a_m(x)$,

$$\mathcal{L} = (X^m - k \tilde{Q}^m) \partial_m a + \frac{1}{2} X^m X_m - \frac{1}{4} \text{tr}(f_{mn} f_{mn}), \tag{1.11}$$

where the gauge Chern-Simons form is included as

$$\tilde{Q}^k = \frac{1}{3!} \varepsilon^{klmn} Q_{lmn} = - \varepsilon^{klmn} \text{tr} \left( a_l \partial_m a_n - \frac{2i}{3} a_l [a_m a_n] \right). \tag{1.12}$$

Variation of the first order action with respect to the scalar field $a$ gives rise to the equation of motion

$$\partial_m (X^m - k \tilde{Q}^m) = 0, \tag{1.13}$$
which is solved in terms of an antisymmetric tensor such that

\[ X^k - k \tilde{Q}^k = \frac{1}{2} \varepsilon^{klmn} \partial_k b_{mn}. \]  

(1.14)

Substituting back shows that the first term in (1.11) becomes a total derivative and one ends up with the previous action (1.9) with \( \tilde{h}^m = X^m \), describing an antisymmetric tensor gauge field coupled to a gauge Chern-Simons form.

On the other hand, varying the first order action with respect to \( X^m \) yields

\[ X^m = - \partial_m a. \]  

(1.15)

In this case, substitution of the equation of motion, together with the divergence equation for the Chern-Simons form, i.e.

\[ \partial_k \tilde{Q}^k = - \frac{1}{4} \varepsilon^{klmn} \text{tr} (f_{kl} f_{mn}). \]  

(1.16)

gives rise to a theory describing a real scalar field with an axion coupling term:

\[ L = - \frac{1}{2} \partial^m a(x) \partial_m a(x) - \frac{1}{4} \text{tr}(f_{mn} f_{mn}) - \frac{1}{4} a(x) \varepsilon^{klmn} \text{tr}(f_{kl} f_{mn}). \]  

(1.17)

It is in this sense the two actions (1.9) and (1.17) derived here from the first order one (1.11) are dual to each other. They describe the dynamics of an antisymmetric tensor gauge field and of a real scalar, respectively, with special types of Yang-Mills couplings. Observe that the kinetic term of the Yang-Mills sector is not modified in this procedure.

We come now to the discussion of the globally supersymmetric case. The linear supermultiplet consists of an antisymmetric tensor, a real scalar and a Majorana spinor. In superfield language it is described by a superfield \( L_0 \), subject to the constraints\[^1\]

\[ D^2 L_0 = 0, \quad \bar{D}^2 L_0 = 0. \]  

(1.18)

Again, the subscript 0 means that we do not include, for the moment, Chern-Simons forms. The linear superfield \( L_0 \) contains the antisymmetric tensor only through its field strength \( h_{0lmn} \).

Indeed, the superfield \( L_0 \) is the supersymmetric analogue of \( h_{0lmn} \) (it describes the multiplet of field strengths) and the constraints (1.18) are the supersymmetric version of the Bianchi identities. The particular form of these constraints implies that terms quadratic in \( \theta \) resp. in \( \bar{\theta} \) are irrelevant (they are not independent component fields), it is for this reason that \( L_0 \) has been called a linear superfield\[^2\].

Instead of writing down explicitly the power series expansion in \( \theta, \bar{\theta} \) of the superfields and to identify the component fields as the respective coefficient functions (keeping in mind the

\[^1\] with the usual notations \( D^2 = D^\alpha D_\alpha \) and \( \bar{D}^2 = D_\alpha D^\alpha \)

\[^2\]
constraint equations!), we shall use here suitable projections to lowest superfield components for the identification of component fields. This is reminiscent of the geometric superspace description and convenient for keeping track of constraints and deriving supersymmetry transformations (in particular later on in the case of local supersymmetry i.e. coupling to supergravity).

To begin with we identify the real scalar \( L(x) \) of the linear multiplet as the lowest component

\[ L_0|_{\theta=\bar{\theta}=0} = L_0(x). \quad (1.19) \]

The spinor derivatives of superfields are again superfields and we define the Weyl components \((\Lambda_\alpha(x), \bar{\Lambda}^{\dot{\alpha}}(x))\) of the Majorana spinor of the linear multiplet as

\[ D_\alpha L_0|_{\theta=\bar{\theta}=0} = \Lambda_\alpha(x), \quad D^{\dot{\alpha}} L_0|_{\theta=\bar{\theta}=0} = \bar{\Lambda}^{\dot{\alpha}}(x). \quad (1.20) \]

The antisymmetric tensor appears in \( L_0 \) via its field strength identified as

\[ [D_\alpha, D^{\dot{\alpha}}] L_0|_{\theta=\bar{\theta}=0} = -\frac{1}{3} \sigma_{k\alpha\dot{\alpha}} \varepsilon^{klmn} h_{0lmn}, \quad (1.21) \]

thus completing the identification of the independent component fields contained in \( L_0 \). The canonical supersymmetric kinetic action for the linear multiplet is then given by the square of the linear superfield integrated over superspace. In more explicit terms and in the language of projections to lowest superfield components it is obtained from

\[ \mathcal{L} = -\frac{1}{32} \left( D^2 \bar{D}^2 + \bar{D}^2 D^2 \right) (L_0)^2|_{\theta=\bar{\theta}=0}. \quad (1.22) \]

Evaluated in terms of component fields, it reads simply

\[ \mathcal{L} = \frac{1}{2} \bar{h}_0^m \bar{h}_0^m - \frac{1}{2} \partial^m L_0 \partial_m L_0 - \frac{i}{2} \sigma^m_{\alpha\dot{\alpha}} (\Lambda^\alpha \partial_m \bar{\Lambda}^{\dot{\alpha}} + \bar{\Lambda}^{\dot{\alpha}} \partial_m \Lambda^\alpha), \quad (1.23) \]

generalizing the purely bosonic action \((1.5)\) given above and showing that there is no auxiliary field in the linear multiplet.

We proceed now to introduce Chern-Simons forms in the supersymmetric case, in other words to construct the supersymmetric version of \((1.9)\). As a prerequisite we recall first some basic properties of the Yang-Mills gauge multiplet. It consists of the gauge potentials \( a_m(x) \), the gauginos \( \lambda(x), \bar{\lambda}(x) \), which are Majorana spinors and the auxiliary scalars \( D(x) \). All of these component fields are Lie-algebra valued. They are identified in the gaugino superfields \( W^\alpha, W^{\dot{\alpha}} \), which are Lie-algebra valued as well, subject to the chirality conditions

\[ D_\alpha W^{\dot{\alpha}} = 0, \quad D^{\dot{\alpha}} W_\alpha = 0, \quad (1.24) \]

and to the additional constraints

\[ D^\alpha W_\alpha = D_\alpha W^{\dot{\alpha}}. \quad (1.25) \]
The spinor derivatives occurring here are defined to be covariant with respect to Yang-Mills transformations. Again, these constraint equations have a geometric interpretation as Bianchi identities in superspace.

To be more precise, the gaugino component fields are defined as the lowest components of the gaugino superfields themselves,

\[ W_\alpha |_{\theta = \bar{\theta} = 0} = -i \lambda_\alpha, \quad W_\dot{\alpha} |_{\theta = \bar{\theta} = 0} = i \bar{\lambda}_{\dot{\alpha}}, \]  

(1.26)

whereas the usual Yang-Mills field strengths \( f_{mn} \) and the auxiliary fields \( D(x) \) occur at the linear level in the superfield expansion, :

\[
\begin{align*}
D_\beta W_\alpha |_{\theta = \bar{\theta} = 0} &= -i (\sigma^{mn} \epsilon)_{\beta\alpha} f_{mn} - \epsilon_{\beta\alpha} D(x), \\
D_\dot{\beta} W_\dot{\alpha} |_{\theta = \bar{\theta} = 0} &= -i (\dot{\epsilon}^{mn})_{\dot{\beta}\dot{\alpha}} f_{mn} + \dot{\epsilon}_{\dot{\beta}\dot{\alpha}} D(x).
\end{align*}
\]  

(1.27)

We come now to the supersymmetric description of the corresponding Chern-Simons forms, that is the supersymmetric extension of (1.7). As discussed in detail in appendix A, it is described in terms of the Chern-Simons superfield \( \Omega \), which has the properties

\[
\begin{align*}
\text{tr}(W^\alpha W_\alpha) &= \frac{1}{2} \bar{D}^2 \Omega, \\
\text{tr}(W_\dot{\alpha} W^{\dot{\alpha}}) &= \frac{1}{2} D^2 \Omega,
\end{align*}
\]  

(1.28)

(1.29)

in accordance with the constraint equations (1.24) and (1.25): the appearance of the differential operators \( D^2 \) and \( \bar{D}^2 \) is due to the chirality constraint whereas the additional constraint (1.25) is responsible for the fact that one and the same superfield \( \Omega \) appears in both equations. The component field Chern-Simons form (1.7) is then identified in the lowest superfield component

\[
[D_\alpha, D_\dot{\alpha}] \Omega |_{\theta = \bar{\theta} = 0} = -\frac{1}{3} \sigma_{k\dot{\alpha}\dot{\alpha}} \epsilon^{klmn} Q_{lmn} - 4 \text{tr}(\lambda_\alpha \bar{\lambda}_{\dot{\alpha}}),
\]  

(1.30)

with \( Q_{lmn} \) given in eq. (1.7).

Since the terms on the left-hand sides in (1.28) and (1.29) are gauge invariant, it is clear that a gauge transformation adds a linear superfield to \( \Omega \) (the explicit construction is given in appendix A). As a consequence, and in analogy with the non-supersymmetric case discussed before, the linear superfield \( L_0 \) can be assigned Yang-Mills transformations such that the combination

\[ L = L_0 + k \Omega, \]  

(1.31)

is gauge invariant. However, this superfield \( L \) satisfies now the modified linearity conditions

\[
\begin{align*}
\bar{D}^2 L &= 2k \text{tr}(W^\alpha W_\alpha), \\
D^2 L &= 2k \text{tr}(W_\dot{\alpha} W^{\dot{\alpha}}),
\end{align*}
\]  

(1.32)

(1.33)
Again, these equations together with
\[ [D_\alpha, D_{\dot{\alpha}}]L = \frac{1}{3} \sigma_{d\epsilon a} \epsilon^{d\epsilon b c} H_{c\alpha} - 4k \text{tr}(W_\alpha W_{\dot{\alpha}}), \]  
(1.34)

have an interpretation as Bianchi identities in superspace geometry. The last one shows how the usual field strength of the antisymmetric tensor together with the component field Chern-Simons form appears in the superfield expansion of \( L \):
\[ [D_\alpha, D_{\dot{\alpha}}]L|_{\theta = \overline{\theta} = 0} = \sigma_{k\alpha \dot{a}} \epsilon^{klmn} \left( \partial_n b_{ml} + \frac{k}{3} Q_{nml} \right) - 4k \text{tr}(W_\alpha W_{\dot{\alpha}}). \]  
(1.35)

The invariant action for this supersymmetric system is given as the lowest component of the superfield
\[ \mathcal{L} = -\frac{1}{32} \left( D^2 \bar{D}^2 + \bar{D}^2 D^2 \right) L^2 - \frac{1}{16} D^2 \text{tr}(W^2) - \frac{1}{16} \bar{D}^2 \text{tr}(\bar{W}^2). \]  
(1.36)

This action describes the supersymmetric version of the purely bosonic action (1.9). Its explicit component field gestalt will be displayed and commented on in a short while.

As is well known [38], the notion of duality as described above in the non-supersymmetric case, can be extended to supersymmetric theories as well. This is most conveniently done in the language of superfields. The supersymmetric version of the first order action (1.11) is given as
\[ \mathcal{L} = -\frac{1}{32} \left( D^2 \bar{D}^2 + \bar{D}^2 D^2 \right) \left( X^2 + \sqrt{2}(X - k\Omega)(S + \bar{S}) \right) - \frac{1}{16} D^2 \text{tr}(W^2) - \frac{1}{16} \bar{D}^2 \text{tr}(\bar{W}^2). \]  
(1.37)

Here, \( X \) is a real but otherwise unconstrained superfield, whereas \( S \) and \( \bar{S} \) are chiral,
\[ D_\alpha \bar{S} = 0, \quad \bar{D}^\dot{\alpha} S = 0. \]  
(1.38)

Of course, the chiral multiplets are going to play the part of the scalar field \( a(x) \) in the previous non-supersymmetric discussion.

Varying the first order action with respect to the superfield \( S \), or, more correctly with respect to its unconstrained prepotential \( \Sigma \), defined as \( S = D^2 \Sigma \), the solution of the chirality constraint, shows immediately (upon integration by parts using spinor derivatives) that the superfield \( X \) must satisfy the modified linearity condition. It is therefore identified with \( L \) and we recover the action (1.36) above.

On the other hand, varying the first order action (1.37) with respect to \( X \) yields the superfield equation of motion
\[ X = -\frac{1}{\sqrt{2}} (S + \bar{S}). \]  
(1.39)

Substituting for \( X \) in (1.37) and neglecting terms \( S^2 \) and \( \bar{S}^2 \) which are trivial upon superspace integration, we arrive at
\[ \mathcal{L} = \frac{1}{32} \left( D^2 \bar{D}^2 + \bar{D}^2 D^2 \right) \left( \bar{S}S + k\sqrt{2} \Omega (S + \bar{S}) \right) - \frac{1}{16} D^2 \text{tr}(W^2) - \frac{1}{16} \bar{D}^2 \text{tr}(\bar{W}^2). \]  
(1.40)
It is already obvious to recognize the usual superfield kinetic term for the chiral multiplet and the Yang-Mills kinetic terms, it remains to have a closer look at the terms containing the Chern-Simons superfield. Taking into account the chirality properties for $S$ and $\bar{S}$ and the derivative relations (1.28) and (1.29) for the Chern-Simons superfields we obtain

$$L = \frac{1}{32} \left( D^2 \bar{D}^2 + \bar{D}^2 D^2 \right) \bar{S} S - \frac{1}{16} D^2 \text{tr}(W^2) - \frac{1}{16} \bar{D}^2 \text{tr}(\bar{W}^2) + \frac{k\sqrt{2}}{8} D^2 \left( S \text{tr}(W^2) \right) + \frac{k\sqrt{2}}{8} \bar{D}^2 \left( \bar{S} \text{tr}(\bar{W}^2) \right).$$

(1.41)

This action is now the supersymmetric version of the action (1.17).

We display now the component field expressions for the two dual versions (1.36) and (1.41) of the supersymmetric construction. In the antisymmetric tensor version, the complete invariant component field action deriving from (1.36) is given as

$$L = \frac{1}{2} \hat{h}^{m} \hat{h}_{m} - \frac{k}{2} \sigma^{m}_{\alpha\dot{\alpha}} \left( \Lambda^{\alpha} \partial_{m} \bar{\Lambda}^{\dot{\alpha}} + \bar{\Lambda}^{\dot{\alpha}} \partial_{m} \Lambda^{\alpha} \right) + (1 + 2kL) \text{tr} \left[ -\frac{1}{4} f^{mn} f_{mn} - \frac{i}{2} \sigma^{m}_{\alpha\dot{\alpha}} \left( \lambda^{\alpha} \partial_{m} \bar{\lambda}^{\dot{\alpha}} + \bar{\lambda}^{\dot{\alpha}} \partial_{m} \lambda^{\alpha} \right) + \frac{1}{2} \hat{D} \hat{D} \right] - k \hat{h}^{m} \text{tr}(\lambda \sigma_{m} \bar{\lambda}) - k \Lambda \sigma^{mn} \text{tr}(\lambda f_{mn}) - k \bar{\Lambda} \bar{\sigma}^{mn} \text{tr}(\bar{\lambda} f_{mn}) - \frac{k^2}{4} (1 + 2kL)^{-1} \left( \Lambda^2 \text{tr}(\lambda^2) + \bar{\Lambda}^2 \text{tr}(\bar{\lambda}^2) - 2 \Lambda \sigma^{m} \bar{\lambda} \text{tr}(\lambda \sigma_{m} \bar{\lambda}) \right) - \frac{k^2}{2} \left( \text{tr}(\lambda^2) \text{tr}(\lambda^2) - \text{tr}(\lambda \sigma_{m} \bar{\lambda}) \text{tr}(\lambda \sigma_{m} \bar{\lambda}) \right).$$

(1.42)

This is the supersymmetric version of (1.3). The redefined auxiliary field

$$\hat{D} = D + \frac{ik}{1 + 2kL} (\Lambda \lambda - \bar{\Lambda} \bar{\lambda}),$$

(1.43)

has trivial equation of motion.

On the other hand, in order to display the component field Lagrangian in the chiral superfield version, we recall the definition of the component field content of the chiral superfields

$$S|_{\theta = \bar{\theta} = 0} = S(x), \quad D_{\alpha} S|_{\theta = \bar{\theta} = 0} = \sqrt{2} \chi_{\alpha}(x), \quad \bar{D}^{\dot{\alpha}} S|_{\theta = \bar{\theta} = 0} = -4F(x),$$

(1.44)

and

$$\bar{S}|_{\theta = \bar{\theta} = 0} = \bar{S}(x), \quad \bar{D}^{\dot{\alpha}} \bar{S}|_{\theta = \bar{\theta} = 0} = \sqrt{2} \bar{\chi}^{\dot{\alpha}}(x), \quad \bar{D}^{\dot{\alpha}} \bar{S}|_{\theta = \bar{\theta} = 0} = -4\bar{F}(x).$$

(1.45)

The component field action in the dual formulation, derived from the superfield action (1.41) takes then the form

$$L = -\partial^{m} \bar{S} \partial_{m} S - \frac{i}{2} \sigma^{m}_{\alpha\dot{\alpha}} \left( \chi^{\alpha} \partial_{m} \bar{\chi}^{\dot{\alpha}} + \bar{\chi}^{\dot{\alpha}} \partial_{m} \chi^{\alpha} \right) + \hat{F} \hat{F}.$$


\[ + \left(1 - k\sqrt{2} (S + \bar{S})\right) \text{tr} \left[ -\frac{1}{4} f^{mn} f_{mn} - \frac{i}{2} \sigma^{m}_{\alpha\dot{\alpha}} \left( \lambda^{\alpha} D_m \bar{\lambda}^{\dot{\alpha}} + \bar{\lambda}^{\dot{\alpha}} D_m \lambda^{\alpha} \right) + \frac{1}{2} \hat{D} \hat{D} \right] \]

\[-\frac{k}{4\sqrt{2}} (S - \bar{S}) \left[ \varepsilon^{klmn} \text{tr}(f_{kl} f_{mn}) + 4 \partial_n \text{tr}(\lambda \sigma^m \bar{\lambda}) \right] \]

\[+ k \chi \sigma^{mn} \text{tr}(\lambda f_{mn}) + k \bar{\chi} \sigma^{mn} \text{tr}(\bar{\lambda} f_{mn}) - \frac{k^2}{4} \text{tr} \lambda^2 \text{tr} \bar{\lambda}^2 \]

\[-\frac{k^2}{8} \left(1 - k\sqrt{2} (S + \bar{S})\right)^{-1} \left( \chi^2 \text{tr} \lambda^2 + \bar{\chi}^2 \text{tr} \bar{\lambda}^2 - 2(\chi \sigma^m \bar{\chi}) \text{tr}(\lambda \sigma^m \bar{\lambda}) \right). \quad (1.46) \]

This is the supersymmetric version of (1.17). Again, we have introduced the diagonalized combinations for the auxiliary fields

\[ \hat{F} = F + \frac{k\sqrt{2}}{4} \text{tr} \lambda^2, \quad \hat{\bar{F}} = \bar{F} + \frac{k\sqrt{2}}{4} \text{tr} \bar{\lambda}^2, \quad (1.47) \]

and

\[ \hat{D} = D - \frac{ik}{1- k\sqrt{2} (S + \bar{S})} (\chi \lambda - \bar{\chi} \bar{\lambda}). \quad (1.48) \]

The two supersymmetric actions (1.42) and (1.46) are dual to each other, in the precise sense of the construction performed above. In both cases the presence of the Chern-Simons form induces \( k \)-dependent effective couplings, in particular quadrilinear spinor couplings. Also, one recognizes easily the axion term in the second version already encountered in the purely bosonic case discussed before.

A striking difference to the non-supersymmetric case, however, is the appearance of a \( k \)-dependent gauge coupling function, multiplying the Yang-Mills kinetic terms. This shows that supersymmetrization of (1.9) and (1.17) results not only in supplementary fermionic terms, but induces also genuinely new purely bosonic terms.

### 1.3 3-form multiplet

Before turning to supergravity and to our main subject, gravitational Chern-Simons forms, let us close this introduction with some remarks on the 3-form gauge supermultiplet. This is, besides the chiral and linear multiplet, yet another supermultiplet describing helicity \((0,1/2)\). It consists of a three-index antisymmetric gauge potential \( C_{lmn}(x) \), a complex scalar \( T(x) \), a Majorana spinor with Weyl components \( \eta_\alpha(x) \), \( \eta^{\dot{\alpha}}(x) \) and a real scalar auxiliary field \( H(x) \). In superfield language [39], [40] it is described by a chiral superfield

\[ \bar{D}^\alpha T = 0, \quad D_\alpha \bar{T} = 0, \quad (1.49) \]

which is subject to the additional constraint

\[ D^2 T - \bar{D}^2 \bar{T} = \frac{\hat{g}}{3} \varepsilon^{klmn} \Sigma_{klmn}, \quad (1.50) \]
with
\[ \Sigma_{klmn} = \partial_k C_{lmn} - \partial_l C_{kmn} + \partial_m C_{nkl} - \partial_n C_{klm}, \]  
the field strength tensor of the three-index gauge potential superfield. It is invariant under the transformation
\[ C_{lmn} \rightarrow C_{lmn} + \partial_l \xi_{mn} + \partial_m \xi_{nl} + \partial_n \xi_{lm}, \]
where the gauge parameter \( \xi_{mn} = -\xi_{nm} \) is a 2-form.

An explicit realization of this multiplet structure is provided by the composite superfield \( \text{tr}(W^2) \) and its complex conjugate \( \text{tr}(\bar{W}^2) \). As the gaugino superfield appearing here is chiral (1.24), these composites are chiral, resp. antichiral as well,
\[ \bar{D}^\alpha \text{tr}(W^2) = 0, \quad D_\alpha \text{tr}(\bar{W}^2) = 0. \]  
On the other hand the gaugino superfields are subject to an additional constraint (1.25), which translates into an additional equation for the composites as well, namely
\[ D^2 \text{tr}(W^2) - \bar{D}^2 \text{tr}(\bar{W}^2) = i\epsilon_{klmn} \text{tr}(f_{kl} f_{mn}), \]
where the topological density
\[ \epsilon_{klmn} \text{tr}(f_{kl} f_{mn}) = -\frac{2}{3} \epsilon_{klmn} \partial_k Q_{lmn}, \]
plays now the rôle of the field-strength and the Chern-Simons form (which, under Yang-Mills transformations changes indeed by the derivative of a 2-form) the rôle of the 3-form gauge potential. In other words, supersymmetric Chern-Simons forms fit perfectly in the framework of the 3-form multiplet. It is this analogy which will be exploited in this paper for the description of supersymmetric Chern-Simons forms, in particular in the gravitational case.

2 The basic superspace structures

2.1 Outline

The purpose of this paper is to discuss the properties of locally supersymmetric theories which contain gravitational Chern-Simons forms coupled via the fieldstrengths of antisymmetric tensor gauge fields to the standard supergravity-matter system.

So far, in spite of a number of efforts, no satisfactory answer has been given to this problem. This is, in part, due to the formidable technical complexity of such a theory. Not only does one have to understand the structure of the multiplets involved, but one should also be able to
identify the component fields and to derive the complete structure of their supersymmetry transformation laws, not to forget the construction of invariant actions describing supersymmetric dynamics.

In order to cope with the technical complexities, we propose to employ methods of superspace geometry. One of the advantages in using this approach is that a great deal of the investigations can be carried out at a purely geometrical level. This is in particular true for the structure of the supersymmetry transformations, but also for issues like Kähler transformations which arise as a consequence of supersymmetry in supergravity-matter coupling.

Moreover, when it comes to supersymmetric dynamics, superspace provides methods to determine invariant actions and to discuss their properties in a concise way. Finally, component field results, in particular complete supersymmetric component field actions with all their embarrassing wealth of couplings can be derived.

We begin, in subsection 2.2, with the description of the superspace structure relevant for the supergravity-matter system, namely the general coupling of chiral superfields to supergravity and supersymmetric Yang-Mills theory. In this formulation the Kähler structure is properly taken into account ab initio: Kähler transformations appear in the structure group of superspace as field dependent chiral $U_K(1)$ transformations. This kind of superspace geometry is called $U_K(1)$ superspace.

Moreover, the kinetic terms for the supersymmetric sigma-model appear through a $D$-term construction for a superfield Kähler potential, with chiral superfields taking the rôle of the complex coordinates.

In subsection 2.3 we present a short reminder of the generic method for constructing actions, invariant at the same time with respect to supersymmetry and $U_K(1)$ transformations.

This then provides the geometrical background for the description of the linear multiplet and of the various types of Chern-Simons forms. The bare linear multiplet (i.e. in the absence of Chern-Simons forms) arises from the superspace geometry of the 2-form gauge potential. In subsection 2.4 this superspace geometry will be presented in $U_K(1)$ superspace, featuring the corresponding linearity conditions.

In subsection 2.5 we first display the salient properties of supersymmetric Yang-Mills theory in $U_K(1)$ superspace, in particular the corresponding Chern-Simons forms. In combination with the linear superfield geometry this gives then rise to the coupling of Chern-Simons form and antisymmetric tensor gauge field, summarized, at the superfield level, in terms of the so-called modified linearity conditions. In the course of this construction we also point out the close relation between Chern-Simons forms and the superspace geometry of a 3-form gauge potential.
2.2 $U_K(1)$ superspace

Supergravity is a generalization of general relativity. Since supersymmetry, by definition, brings in fermionic degrees of freedom, the relevant formulation of Einstein gravity is in terms of vierbein field (local Lorentz frames) and spin connection. As is well known, the supersymmetric extension consists then in adding the Rarita-Schwinger field as the supersymmetry partner of the vierbein together with certain auxiliary (usually non-propagating) fields which serve to establish an off-shell realization of the local supersymmetry algebra.

Supergravity may be viewed as the gauged theory of supersymmetry: the anticommuting parameters of supersymmetry transformations become space-time dependent and it is the Rarita-Schwinger field, which, under supersymmetry transformations, acquires an inhomogeneous term proportional to the (covariant) space-time derivative of the local supersymmetry parameter.

In superspace one generalizes the notions of local frame and spin connection in extending them to the anticommuting directions of superspace equipped with a full-fledged graded differential geometry.

In some more detail, the usual frame, viewed as a differential form over space-time,
\[
e^a = dx^m e^a_m(x),
\]
is extended to a differential form over superspace,
\[
E^A = dz^M E_M^A(z),
\]
where transition from lower case to upper case indices signifies the passage from ordinary to superspace geometry, based on coordinates $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$. Accordingly,
\[
E^A = (E^a, E^\alpha, E^{\dot{\alpha}}),
\]
has vectorial and spinor indices, the latter in Weyl-spinor notation. In this general set-up the usual vierbein and Rarita-Schwinger fields are identified as lowest superfield components, i.e.
\[
e_m^a(x) = E_m^a(x, 0, 0) = E_m^a|,
\]
and
\[
\frac{1}{2}\psi_m^\alpha(x) = E_m^\alpha|, \quad \frac{1}{2}\bar{\psi}_{m\dot{\alpha}}(x) = E_{m\dot{\alpha}}|.
\]

The symmetries in this superspace description are general supercoordinate transformations, unifying the usual general coordinate transformations and the local supersymmetry transformations in their vector and spinor parts, respectively, and local Lorentz transformations, in turn acting through vector and spinor representations on $E^a$ and $E^\alpha, E^{\dot{\alpha}}$.  

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As to the first, covariance is achieved through the use of differential form and inverse frame language in superspace, whereas covariance with respect to local Lorentz transformations is ensured by covariant derivatives using the spin connection.

These are the symmetries of pure supergravity. If one wants to include supersymmetric matter, described in terms of chiral superfields and their complex conjugates, a new additional geometric structure shows up: the chiral matter superfields in the general supergravity-matter system are coordinates of a superfield Kähler manifold. This property arises from the requirement of supersymmetry of the dynamical theory, as pointed out by Zumino already for the supersymmetric sigma model without supergravity. When coupled to supergravity the sigma model action was recognized to be Kähler invariant provided the spinor fields (including those of supersymmetric Yang-Mills theory) transform under well-prescribed chiral phase transformations whose parameters are given as the imaginary part of the field dependent Kähler transformations.

It is precisely the Kähler superspace formulation which clarifies this situation. In particular it puts the structure of the chiral Kähler transformations on a sound geometrical basis, without reference to the dynamical construction. The key mechanism of this formulation is to include the chiral Kähler phase transformations into the structure group of superspace, on the same footing as the local Lorentz transformations.

The Kähler potential appears then quite naturally as a prepotential for these chiral transformations. Moreover, its $D$-term provides an action invariant at the same time (and for the same reasons) under supersymmetry and under (superfield) Kähler transformations.

This formulation of the supergravity/matter/Yang-Mills system is presented in full detail in [8]. At present, in this section we will consider a generic chiral $U_K(1)$ and identify its prepotential $K$ with the Kähler potential only afterwards. More generally, as explained in [3], [32], [33], it may be allowed to depend on linear superfields as well, in which case we will refer to it as kinetic prepotential.

Coming back to the basic object of the superspace formulation, namely the frame $E^A$ in superspace, this means that in addition to the aforementioned general supercoordinate and Lorentz transformations we assign chiral transformations (in terms of a chiral superfield $F$ and its complex conjugate) such that

$$E^A \mapsto E^A \exp \left[ -\frac{i}{2} w(E^A) \Im F \right], \quad (2.6)$$

with chiral weights $w(E^A)$ defined as

$$w(E^a) = 0, \quad w(E^\alpha) = 1, \quad w(E_{\dot{\alpha}}) = -1. \quad (2.7)$$

\footnote{Given the explicit form of these chiral transformations, the issue of gauged $R$-transformation comes immediately to ones mind.}
In view of all this, superspace torsion is defined as
\[ T^A = dE^A + E^B \phi_B^A + w(E^A)E^A A, \] (2.8)
that is, just the covariant exterior derivative of the frame in superspace. The first two terms on the right are standard, in particular, the spin connection is a one form in superspace,
\[ \phi_B^A = dz^M \phi_{MB}^A(z), \] (2.9)
taking values in the Lie-algebra of the Lorentz group such that its spinor components are given in terms of the vector ones as
\[ \phi_{\beta}^{\alpha} = -\frac{1}{2}(\sigma^{ba})^{\beta}_{\alpha} \phi_{ba}, \quad \phi_{\dot{\beta}}^{\dot{\alpha}} = -\frac{1}{2}(\bar{\sigma}^{ba})^{\dot{\beta}}_{\dot{\alpha}} \phi_{ba}. \] (2.10)
The abelian gauge potential
\[ A = dz^M A_M(z), \] (2.11)
is new: it serves to covariantize the chiral \( U_K(1) \) transformations,
\[ A \mapsto A + \frac{1}{2} d \text{Im} F. \] (2.12)

We do not intend here to give a complete and detailed review of this geometrical structure. For this we refer to our earlier work. Here we try to concentrate on the crucial points which will be of relevance later on in the discussion of the structure of linear superfield geometry and of Chern-Simons forms in superspace. Recall nevertheless the definitions of the fieldstrengths
\[ R_B^A = d\phi_B^A + \phi_B^C \phi_C^A, \] (2.13)
\[ F = dA. \] (2.14)

Torsion, curvature and \( U_K(1) \) fieldstrength are 2-forms in superspace, their expansion in the covariant frame basis being defined as
\[ T^A = \frac{1}{2} E^B E^C T_{CB}^A, \] (2.15)
\[ R_B^A = \frac{1}{2} E^C E^D R_{DC}^A, \] (2.16)
\[ F = \frac{1}{2} E^C E^D F_{DC}. \] (2.17)

Recall that superspace torsion is subject to covariant constraints which imply that all the coefficients of torsion, curvature and \( U_K(1) \) fieldstrength are given in terms of the few covariant supergravity superfields
\[ R, \quad R^\dagger, \quad G_a, \quad W_{\gamma \beta \alpha}, \quad W_{\dot{\gamma} \dot{\beta} \dot{\alpha}}, \] (2.18)
and their covariant derivatives. As all these basic superfields are identified in the torsion coefficients, with chiral weights

\[ w(T^A_{CB}) = w(E^A) - w(E^B) - w(E^C), \]

their chiral weights are fixed to be

\[ w(R) = 2, \quad w(R^\dagger) = -2, \quad w(G_\alpha) = 0, \]
\[ w(W_{\gamma\beta\alpha}) = 1, \quad w(W_{\gamma_{\dot{\beta}\dot{\alpha}}}) = -1. \]

To be more explicit, the nonvanishing components of superspace torsion are

\[ T^\gamma_{\dot{\beta}\alpha} = -2i(\sigma^a e)_{\gamma_{\dot{\beta}}}^\alpha, \] (2.21)
\[ T^{\gamma}_{b\alpha} = -i\sigma_{b\gamma} R^\dagger, \quad T^{\dot{\gamma}}_{b\alpha} = -i\sigma_{b}^\gamma R, \] (2.22)
\[ T^{\gamma}_{b\alpha} = \frac{i}{2}(\sigma_c \bar{\sigma}_b)_{\gamma}^\alpha G^c, \quad T^{\dot{\gamma}}_{b\alpha} = -\frac{i}{2}(\bar{\sigma}_c \sigma_b)_{\dot{\gamma}}^\alpha G^c, \] (2.23)
and \( T^a_{cb} \) and \( T^{a}_{cb\dot{\alpha}} \), the covariant Rarita-Schwinger fieldstrength superfields. The superfields \( W_{\gamma\beta\alpha} \) and \( W_{\gamma_{\dot{\gamma}_{\dot{\beta}\dot{\alpha}}}^\dagger} \) are called Weyl spinor superfields, because they occur in the decomposition of these Rarita-Schwinger superfields in very much the same way as the usual Weyl tensor occurs in the decomposition of the covariant curvature tensor. For a detailed account of basic superspace geometry see [11], [8].

Consistency of the superspace Bianchi identities with the special form of the torsion components displayed so far implies the chirality conditions:

\[ D_\alpha R^\dagger = 0, \quad D_{\dot{\alpha}} R = 0, \]
\[ D_\alpha W_{\gamma\beta\alpha} = 0, \quad D_{\dot{\alpha}} W_{\gamma_{\dot{\beta}\dot{\alpha}}} = 0. \]

(2.24) \hspace{1cm} (2.25)

Leaving aside, for the moment, the description of the explicit form of the curvatures we turn to the fieldstrengths of the chiral \( U_K(1) \) sector, which are described in terms of the superfields \( X_\alpha \) and \( \bar{X}^{\dot{\alpha}} \), defined as spinor derivatives of the basic superfields \( R, R^\dagger \) and \( G_\alpha \) as follows:

\[ X_\alpha = D_\alpha R - D^{\dot{\alpha}} G_{\alpha\dot{\alpha}}, \]
\[ \bar{X}^{\dot{\alpha}} = D^{\dot{\alpha}} R^\dagger + D_\alpha G^{\alpha\dot{\alpha}}. \]

(2.26) \hspace{1cm} (2.27)

Again, consistency with the Bianchi identities implies chirality, i.e.

\[ D_\alpha \bar{X}^{\dot{\alpha}} = 0, \quad D^{\dot{\alpha}} X_\alpha = 0, \]
\[ D^{\alpha} X_\alpha - D_{\dot{\alpha}} \bar{X}^{\dot{\alpha}} = 0. \]

(2.28) \hspace{1cm} (2.29)
As mentioned earlier, the coefficients of the $U_K(1)$ gauge potential $A$,

$$A = E^\alpha A_\alpha = E_a A_a + E^\alpha A_\alpha + E_{\dot{\alpha}} A_{\dot{\alpha}},$$

are given in terms of the $U_K(1)$ prepotential superfield $K$ as

$$A_\alpha = +\frac{1}{4} E_\alpha{}^M \partial_M K,$$

$$A_{\dot{\alpha}} = -\frac{1}{4} E^{\dot{\alpha}}{}^M \partial_M K,$$

$$A_{\alpha\dot{\alpha}} - \frac{3}{2} G_{\alpha\dot{\alpha}} = \frac{i}{2}(\mathcal{D}_\alpha A_{\dot{\alpha}} + \mathcal{D}_{\dot{\alpha}} A_\alpha).$$

In the last equation the $G_{\alpha\dot{\alpha}}$ term appears due to the special choice $F_{\alpha\dot{\alpha}} = -3G_{\alpha\dot{\alpha}}$ of conventional constraint for the $U_K(1)$ field strength. Using this explicit form of $A$ yields then

$$X_\alpha = -\frac{1}{8} (\mathcal{D}^2 - 8R) \mathcal{D}_\alpha K,$$

$$\tilde{X}_{\dot{\alpha}} = -\frac{1}{8} (\mathcal{D}^2 - 8R^\dagger) \mathcal{D}_{\dot{\alpha}} K.$$

As a consequence one has then

$$\mathcal{D}^\alpha X_\alpha = \mathcal{D}_{\dot{\alpha}} \tilde{X}_{\dot{\alpha}},$$

for the $D$-term pertaining to the $U_K(1)$ factor. As long as $K$ is an independent superfield, the lowest component of the superfield $\mathcal{D}^\alpha X_\alpha$ is an independent component, as usually in supersymmetric gauge theory. On the other hand if one allows $K$ to be a function of chiral and linear superfields, this $D$-term yields the corresponding kinetic terms after successive applications of the spinorial covariant derivatives to $K$ and due to the chirality and linearity conditions.

The superfield $\mathcal{D}^\alpha X_\alpha$ is related to the basic supergravity superfields such that

$$\mathcal{D}^2 R + \mathcal{D}^2 R^\dagger = -\frac{2}{3} \mathcal{R} - \frac{2}{3} \mathcal{D}^\alpha X_\alpha + 4 G^a G_a + 32 R^\dagger R,$$

where $\mathcal{R}$ is the curvature scalar (see appendix B for notational details). This relation is at the heart of the construction of the supersymmetric component field action, as will become clear in the following subsection. On the other hand, the orthogonal combination

$$\mathcal{D}^2 R + \mathcal{D}^2 R^\dagger = 4i \mathcal{D}_a G^a,$$

has an intriguing resemblance to the 3-form superspace, as reviewed in appendix A.

2.3 Construction of generic invariant actions

An important topic which can already be addressed here is the question of constructing actions invariant under supersymmetry transformations. As in the geometrical formulation
outlined so far, supersymmetry transformations occur in the general supercoordinate transformations, the method of constructing invariant actions proceeds along the same lines as in usual general relativity, namely with the help of invariant densities. In general relativity the basic object is the determinant $e(x)$ of the vierbein field $e^m_a(x)$. It was pointed out by Wess and Zumino that in supergravity this should be generalized to the superdeterminant $E(x, \theta, \bar{\theta})$ of $E^A_M(x, \theta, \bar{\theta})$, the frame in superspace, as introduced above, together with the the usual concept of superspace integration.

It is not hard to see that the construction of Wess and Zumino, originally performed in traditional superspace, holds in $U_K(1)$ superspace as well. This is due to the fact that the geometric framework concerning general superspace coordinate transformations remains the same in $U_K(1)$ superspace, what changes is the structure group - but the determinant of the frame is invariant under the new chiral phase transformations.

Already in the original formulation it is quite intriguing that the complete action for supergravity is given by the superspace volume element

$$\int d^4x\,d^4\theta\,E.$$  

Even more amazing, when matter fields are included through $U_K(1)$ superspace, this action describes the kinetic terms for the complete supergravity-matter system, the differences to the previous case arising of course from the different geometric structures to be taken into account in evaluating the corresponding component field expressions. In this sense one might speak of a unified description of gravity and matter fields, the complete action arising from one single distinguished geometrical object - the $U_K(1)$ superspace volume element.

In both cases the component field Lagrangians deduced from these superspace expressions contain the Einstein curvature scalar term

$$-\frac{1}{2}\,e\,\mathcal{R}$$  

with the usual canonical normalization. The use of $U_K(1)$ superspace avoids the cumbersome component field rescalings of the original constructions of Cremmer et. al..

In the construction of the supersymmetric superpotential term or the kinetic action for supersymmetric Yang-Mills theory one employs the so-called chiral volume elements of superspace,

$$\int d^4\theta\,\frac{E}{R}\,r \sim \int d^2\theta\,\mathcal{E}r, \quad \text{and} \quad \int d^4\theta\,\frac{E}{R^\dagger}\,\bar{r} \sim \int d^2\theta\,\bar{\mathcal{E}}\bar{r},$$

where $R$ and $R^\dagger$ are the chiral supergravity superfields appearing in the torsion as explained above and $r$ and $\bar{r}$ are generic chiral superfields of weights $w(r) = +2$ and $w(\bar{r}) = -2$, respectively, which should be specified according to the kind of invariant action one intends to construct. Observe that in particular the choice $r = R$ gives back the kinetic supergravity actions discussed above.
In other words, this chiral density construction is the generalization to local supersymmetry of what is called the $F$-term construction in rigid supersymmetry, applied to the generic chiral superfield $r$. The explicit algorithm consists then simply in writing out the supersymmetric completion of the component field expression of the $F$-term $D^2 r$ according to

$$L_{\text{gen}} = -\frac{1}{4} e \left( D^2 - 24 R^\dagger \right) r| + \frac{i}{2} e (\bar{\psi}_m \bar{\sigma}^m)^\alpha D_\alpha r| - e (\bar{\psi}_m \bar{\sigma}^m \bar{\psi}_n) r| + \text{h.c.}.$$  \hfill (2.39)

Here, the lowest components of the supergravity superfields $R$ and $R^\dagger$ are defined as

$$R| = -\frac{1}{6} M(x), \quad R^\dagger| = -\frac{1}{6} \bar{M}(x).$$  \hfill (2.40)

Hence, in this prescription for the construction of a supersymmetric component field action one has to choose appropriately some chiral superfield $r$ with $w(r) = 2$, work out the projections to component fields of $r$, $D_\alpha r$ and $D^2 r$ and substitute in the equation for $L_{\text{gen}}$. The supergravity-matter action (with properly normalized curvature scalar term) is then obtained in taking the superfields $R$ and $R^\dagger$, thus justifying the remark at the end of the previous subsection, whereas the Yang-Mills action is obtained from the superfield $W^\alpha W_\alpha$ and its complex conjugate.

### 2.4 Linear multiplet geometry and supergravity

The linear multiplet has a geometrical interpretation as a 2-form gauge potential in superspace geometry. Since we wish to construct theories where the linear multiplet is coupled to the supergravity-matter system, we will formulate this 2-form geometry in the background of $U_K(1)$ superspace. The basic object is the 2-form gauge potential defined as

$$B = \frac{1}{2} dz^M dz^N B_{NM}.$$

It is subject to gauge transformations of parameters $\xi = dz^M \xi_M$ which are themselves one forms in superspace:

$$B \mapsto B + d\xi,$$

or, in more detail,

$$B_{NM} \mapsto B_{NM} + \partial_N \xi_M - (-)^{d(N)d(M)} \partial_M \xi_N.$$

The invariant fieldstrength is a 3-form, defined as

$$H = dB.$$

As a 3-form in superspace, $H$ is given as

$$H = \frac{1}{3!} E^A E^B E^C H_{CBA},$$
with $E^A$ the frame of $U_K(1)$ superspace. As a consequence of $dd = 0$ one obtains the Bianchi identities

$$dH = 0.$$  

Fully developed this reads

$$\frac{1}{3!} E^A E^B E^C E^D \big( 4\mathcal{D}_DH_{CBA} + 6T_{DC}^F H_{FBA} \big) = 0. \tag{2.41}$$

The linear superfield is recovered from this general structure in imposing covariant constraints on the fieldstrength coefficients $H_{CBA}$, a rather common procedure in the superspace formulation of supersymmetric theories. The constraints to be chosen here are

$$H_{\bar{\gamma}\beta\alpha} = 0, \quad H_{\gamma\beta\alpha} = 0, \quad H_{\dot{\gamma}\bar{\beta}\alpha} = 0, \tag{2.42}$$

where as usual underlined indices serve to denote both dotted and undotted ones, $\underline{a} = (\alpha, \dot{\alpha})$. The consequences of these constraints on the other coefficients are obtained either by explicitly solving the constraints in terms of (unconstrained) prepotentials or else by working through the covariant Bianchi identities. As a result one finds that all the fieldstrength components of the 2-form are expressed in terms of one superfield $L$ which is identified in

$$H_{\gamma\alpha}^\beta = -2i (\sigma_\alpha^\beta)_{\gamma} L. \tag{2.43}$$

Furthermore one obtains

$$H_{\beta a}^\gamma = 2(\sigma_{ba})^\gamma_\phi \mathcal{D}_\phi L, \tag{2.44}$$

$$H_{\dot{\gamma} \bar{a}}^\beta = 2(\tilde{\sigma}_{ba})^{\dot{\gamma}}_{\dot{\phi}} \mathcal{D}_{\dot{\phi}} L. \tag{2.45}$$

Compatibility of the constraints imposed above with the structure of the Bianchi identities then implies the linearity conditions

$$\left(D^2 - 8R\right) L = 0, \tag{2.46}$$

$$\left(D^2 - 8R^\dagger\right) L = 0, \tag{2.47}$$

for a linear superfield in interaction with the supergravity-matter system. Finally, the vector component $H_{cba}$ appears at the level

$$\left([\mathcal{D}_\alpha, \mathcal{D}_\dot{\alpha}] - 4\sigma_{\alpha\dot{\alpha}G_a}\right) L = -\frac{1}{3} \sigma_{d\alpha\dot{\alpha}} \varepsilon^{dcb\alpha} H_{cba}. \tag{2.48}$$

In terms of component fields this means that $H_{cba}$ is identified in the $\theta\bar{\theta}$ component (in the language where superfield expansion is defined through successive application of covariant spinor derivatives).
Let us close this subsection with a few remarks concerning the definition of the component fields of the linear multiplet as obtained from this superspace formulation, in particular the identification of the antisymmetric tensor gauge potential:

\[ B|| = b = \frac{1}{2} dx^m dx^n b_{nm}(x). \]  

(2.49)

The so-called double bar construction \[41\], used here, projects at the same time superspace differentials on their purely vector parts, \( dz^M \mapsto dx^m \), and the corresponding superfield coefficients on their lowest superfield components at \( \theta = \dot{\theta} = 0 \). For the covariant fieldstrength, this gives:

\[ H|| = h = db = \frac{1}{3!} dx^l dx^m dx^n h_{nml}, \]

(2.50)

or

\[ h_{nml} = \partial_n b_{ml} + \partial_m b_{ln} + \partial_l b_{nm}. \]

(2.51)

We also shall frequently make use of the dual, defined as

\[ \tilde{h}^k = \frac{1}{3!} \varepsilon^{klmn} h_{lmn} = \frac{1}{2} \varepsilon^{klmn} \partial_l b_{mn}. \]

(2.52)

On the other hand, if the double bar projection is applied to the expansion of \( H \) in terms of the covariant frame, as given above, we have to use the projection

\[ E^a|| = e^a(x) = dx^m e_m^a(x), \]

(2.53)

for the vector part and

\[ E_\dot{a}|| = e_\dot{a}(x) = \frac{1}{2} dx^m \psi_m^\dot{a}(x), \]

(2.54)

for the spinor ones. Using then the decomposition

\[ H|| = \frac{1}{3!} e^a e^b e^c H_{cba} + \frac{1}{2} e^a e^b e^\gamma H_{\gamma ba} + \frac{1}{2} e^a e^b e_\dot{\gamma} H^{\dot{\gamma}} ba + e^a e^\beta e_\dot{\gamma} H^{\dot{\gamma}} \beta a, \]

(2.55)

one derives in a straightforward way the expression

\[ -\frac{1}{3} \sigma_{\dot{d}a} \varepsilon^{\dot{d}eba} H_{cba} = \sigma_{kaa} \varepsilon^{klmn} \partial_h b_{ml} \]

\[ + i L \sigma_{\dot{k}a\dot{a}} \varepsilon^{klmn} (\psi_n \sigma_m \dot{\psi}_{\dot{t}}) + 2 i L \sigma_{k\dot{a}\dot{a}} \left( \psi_m \sigma^m \Lambda - \psi_m \sigma^m \bar{\Lambda} \right) \],

(2.56)

where we have used the definitions

\[ L = L(x), \]

(2.57)

and

\[ D\alpha L = \Lambda_\alpha(x), \quad D\dot{\alpha} L = \bar{\Lambda}^{\dot{\alpha}}(x). \]

(2.58)

This short excursion was made to show how the superspace construction provides in a rather straightforward and compact way the basic building blocks which will be used later on in the evaluation of supersymmetry invariant actions. In the example worked out here the supercovariant component fieldstrength \( H_{cba} \) exhibits terms linear and quadratic in the Rarita-Schwinger field when coupled to supergravity.
2.5 Super Chern-Simons forms: the Yang-Mills case

We define the Yang-Mills gauge potential as a Lie algebra valued one-form in the background of $U_K(1)$ superspace, i.e.

$$\mathcal{A} = E^A A_A^{(r)} T_r = A^{(r)} T_r. \quad (2.59)$$

Latin indices in parentheses are used here to denote the basis of the Lie algebra, the commutation relations of the generators $T_r$ being defined as

$$[T_r, T_s] = ic_{(r)(s)}(t) T_t, \quad (2.60)$$

Gauge transformations are parametrized by group elements $g$ in the usual way except that now the parameters of the gauge transformations are promoted from real functions to real superfields.

$$\mathcal{A} \rightarrow g \mathcal{A} = g^{-1} A g - g^{-1} d g. \quad (2.61)$$

The covariant fieldstrength

$$\mathcal{F} = d \mathcal{A} + \mathcal{A} \mathcal{A}, \quad (2.62)$$

is a 2-form in superspace defined as

$$\mathcal{F} = \frac{1}{2} E^A E^B \mathcal{F}_{BA}, \quad (2.63)$$

with coefficients

$$\mathcal{F}_{BA} = D_B A_A - (-)^{ab} D_A A_B - (A_B, A_A) + T_{BA} C A_C. \quad (2.64)$$

Note the appearance of the supergravity torsion terms. The derivatives occurring here covariantize Lorentz and $U_K(1)$ transformations, following the usual prescriptions, with chiral weights $w(A_A) = -w(E^A)$. We use the notation

$$(A_B, A_A) = A_B A_A - (-)^{ba} A_A A_B, \quad (2.65)$$

for the graded commutation relations. Of course, the fieldstrength is Lie algebra valued as well,

$$\mathcal{F} = \mathcal{F}^{(r)} T_r, \quad (2.66)$$

and it is sometimes useful to display it in the form

$$\mathcal{F}^{(r)} = d A^{(r)} + \frac{i}{2} A^{(p)} A^{(q)} c_{(p)(q)}^{(r)}. \quad (2.67)$$

Based on these definitions one can then go ahead with the construction of the superspace analogue of Chern-Simons forms \[18\]. In the present context we restrict ourselves to the case of the Chern-Simons 3-form. Following the notation of \[12\] we define

$$Q^{(y,m)} = \text{tr} \left( A d A + \frac{2}{3} A A A \right), \quad (2.68)$$

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which, as a 3-form in superspace, has the decomposition
\[ Q^{(YM)} = \frac{1}{4!} dz^K d\bar{z}^L dz^M Q_{MLK}^{(YM)} = \frac{1}{4!} F^A E^B E^C Q_{CBA}^{(YM)}. \]  

(2.69)

Clearly, the exterior derivative of this superspace Chern-Simons form yields the fieldstrength squared term
\[ dQ^{(YM)} = \text{tr} (\mathcal{F}\mathcal{F}). \]  

(2.70)

The coupling to the antisymmetric tensor multiplet is then obtained by incorporating this Chern-Simons forms in the fieldstrength of 2-form gauge potential as follows:
\[ H^{(YM)} = dB + k Q^{(YM)}. \]  

(2.71)

The superscript \(^{(YM)}\) indicates the presence of the Chern-Simons form in the definition of the fieldstrength. Since the Chern-Simons 3-form \( Q^{(YM)} \) changes under gauge transformations of the Yang-Mills connection \( A \) with the exterior derivative of a 2-form,
\[ Q^{(YM)} \mapsto gQ^{(YM)} = Q^{(YM)} + d\Delta(g), \]  

(2.72)

covariance of \( H^{(YM)} \) can be achieved in assigning an inhomogeneous compensating gauge transformation
\[ B \mapsto gB = B - \Delta(g), \]  

(2.73)

to the 2-form gauge potential. Finally, the addition of the Chern-Simons forms gives rise to the modified Bianchi identities
\[ dH^{(YM)} = k \text{tr} (\mathcal{F}\mathcal{F}). \]  

(2.74)

We discuss now the restrictions on the covariant fieldstrengths \( H^{(YM)} \) and \( \mathcal{F} \). As we have pointed out in the preceding subsection, the linear multiplet corresponds to a 2-form geometry with constraints on the fieldstrength. On the other hand it is well known that in supersymmetric Yang-Mills theory the fieldstrength \( \mathcal{F} \) is constrained as well. As a consequence, a question of compatibility arises when these two superspace structures are combined in the way we propose here.

The answer to this question is that the coupling of Chern-Simons forms to the antisymmetric tensor multiplet is indeed consistent. The most immediate way to see this is to investigate explicitly the structure of the modified Bianchi identities in the presence of the constraints.

To this end let us first recall that supersymmetric Yang-Mills theory is defined by the covariant constraints
\[ \mathcal{F}^{\dot{a}\dot{b}} = 0, \quad \mathcal{F}_{\dot{b}a} = 0, \quad \mathcal{F}_{\dot{b}a} = 0, \]  

(2.75)

which can be understood as compatibility conditions for the covariant chirality constraints on the matter superfields and, the third one, as a covariant redefinition of the vector component \( A_a \) of the superspace Yang-Mills connection.
These constraints severely restrict the form of the remaining components of the Yang-Mills fieldstrength, as can be seen from their explicit solution or by a simple analysis of the Bianchi identities. In any case one finds

\[ F^{\beta a} = +i(\sigma_a \epsilon)_{\beta} \tilde{W}_{\beta} , \]
\[ F^{\beta} = -i(\bar{\sigma}_{\epsilon})_{\beta} W_{\beta} , \]
\[ F_{ba} = \frac{1}{2}(\epsilon \sigma_{ba})^{\beta\bar{\alpha}} D_{\alpha} W_{\beta} + \frac{1}{2}(\bar{\sigma}_{ba} \epsilon)^{\bar{\beta}\bar{\alpha}} \bar{D}_{\bar{\alpha}} \bar{W}_{\bar{\beta}} . \]  

(2.76) \hspace{1cm} (2.77) \hspace{1cm} (2.78)

All the superspace fieldstrength components are given in terms of the covariant Yang-Mills superfields

\[ \bar{\Phi}^{\dot{\alpha}} = \tilde{\Phi}^{(r)\dot{\alpha}} T_{(r)} , \quad W_{\alpha} = \Phi^{(r)\alpha} T_{(r)} , \]

which, with respect to \( U_K(1) \) superspace, have chiral weights

\[ w(\bar{\Phi}^{\dot{\alpha}}) = -1, \quad w(W_{\alpha}) = +1. \]  

(2.79)

Moreover, as a consequence of the constraints the Bianchi identities boil down to the equations

\[ D_{\alpha} \bar{\Phi}^{\dot{\alpha}} = 0, \quad D^{\dot{\alpha}} W_{\alpha} = 0, \]
\[ D^{\bar{\alpha}} W_{\bar{\alpha}} = D_{\tilde{\alpha}} \bar{W}^{\dot{\alpha}} . \]  

(2.81) \hspace{1cm} (2.82)

We also define the D-term superfield \( D^{(r)} \) as

\[ D^{(r)} = -\frac{1}{2} D^{\bar{\alpha}} W^{(r)\bar{\alpha}} , \]

which, by construction, has vanishing chiral weight,

\[ w(D^{(r)}) = 0. \]  

(2.83) \hspace{1cm} (2.84)

Observe that in solving the Yang-Mills Bianchi identities the complete structure of \( U_K(1) \) superspace as presented earlier has been taken into account. Derivatives are covariant with respect to Lorentz, chiral \( U_K(1) \) and Yang-Mills gauge transformations.

We now turn back to the modified Bianchi identities for the fieldstrength of the 2-form gauge potential in the presence of Yang-Mills Chern-Simons forms. Assuming for \( H^{(\gamma,M)} \) the same constraints as in the preceding subsection for \( H \) on the one hand and taking into account the special properties arising from the Yang-Mills constraints in the fieldstrength squared terms on the other hand one arrives, after some algebra, at the result that the general modified Bianchi identities are simply replaced by the modified linearity conditions

\[ \left( D^2 - 8 R^I \right) L^{(\gamma,M)} = 2k \text{ tr} \left( \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right) , \]
\[ \left( \bar{D}^2 - 8 R \right) L^{(\gamma,M)} = 2k \text{ tr} \left( W^{\alpha} W_{\alpha} \right) . \]  

(2.85) \hspace{1cm} (2.86)

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written in $U_K(1)$ superspace, together with the relation

$$
([\mathcal{D}_\alpha, \mathcal{D}_\bar{\alpha}] - 4\sigma^{\alpha}_{a\dot{d}} G_a) L^{(YM)} = -\frac{1}{32}\delta_{d\alpha\dot{d}} \varepsilon^{dcb\dot{a}} H^{(YM)}_{cb\dot{a}} - 4k \text{tr} (W_\alpha \tilde{W}_\alpha),
$$

(2.87)

which identifies the fieldstrength tensor $H^{(YM)}_{cb\dot{a}}$ in the superfield expansion of

$$
L^{(YM)} = L + k \Omega^{(YM)},
$$

(2.88)

The Chern-Simons superfield $\Omega^{(YM)}$ will be discussed in detail shortly. The compatibility of the two superspace structures involved in this construction has an explanation in the language of the superspace geometry of the so-called 3-form gauge potential. To see this in some more detail, we denote

$$
\Sigma^{(YM)} = \text{tr} (\mathcal{F}\mathcal{F}),
$$

(2.89)

the fieldstrength squared term. From the explicit decomposition

$$
\Sigma^{(YM)} = \frac{1}{12} E^A E^B E^C E^D \Sigma^{(YM)}_{DCBA} = \frac{1}{12} E^A E^B E^C E^D 6 F_{DC} \mathcal{F}_{BA},
$$

(2.90)

and from the constraints on $\mathcal{F}$ it is immediate to deduce that

$$
\Sigma^{(YM)} \delta_{\gamma} A^A = 0.
$$

(2.91)

These are just the constraints which characterize the fieldstrength of the 3-form gauge potential.

Let us therefore open here a parenthesis and digress shortly on the features of the corresponding superspace formulation. In the generic case we have a 3-form gauge potential $B^3$ with covariant fieldstrength $\Sigma = dB^3$ subject to precisely this set of constraints. In appendix A we point out in some detail how the explicit solution of these constraints can be described in terms of one single real scalar superfield $\Omega$. This means that, up to pure gauge contributions, all the coefficients $B^3_{CBA}$ of the 3-form gauge potential $B^3$ are expressible in terms of the prepotential $\Omega$.

On the other hand, at the level of the covariant fieldstrengths, this implies restrictions on the other coefficients of $\Sigma$. As usual in constrained superspace geometry the explicit structure of the fieldstrength components may be obtained from the Bianchi identities, in this case $d\Sigma = 0$. It turns out that they are completely determined by superfields $S$ and $T$ subject to chirality conditions

$$
\mathcal{D}_\alpha S = 0, \quad \mathcal{D}^\dot{\alpha} T = 0,
$$

(2.92)

and appearing as follows in the coefficients of the 4-form fieldstrength:

$$
\Sigma_{\dot{\gamma} \gamma} ba = \frac{1}{2} (\sigma_{ba} \varepsilon)_{\dot{\gamma} \gamma} S,
$$

(2.93)

$$
\Sigma^{\dot{\gamma} \dot{\gamma}} ba = \frac{1}{2} (\bar{\sigma}_{ba} \varepsilon)^{\dot{\gamma} \dot{\gamma}} T.
$$

(2.94)

3. In the generic case the superscript $(YM)$ is omitted.
As another consequence of the constraints one finds that

\[ \Sigma_{\delta}^{\dot{\gamma}c} \Sigma_{cba} = T_{\delta}^{\dot{\gamma}c} \Sigma_{cba}, \quad (2.95) \]

where \( \Sigma_{cba} \) is totally antisymmetric in its three indices. But this means that it can be absorbed in a redefinition of the coefficient \( B_{cba}^3 \) of 3-form gauge potential \( B^3 \). This is easily deduced from the explicit expression

\[ \frac{1}{4!} E^A E^B E^C E^D \Sigma_{DCBA} = \frac{1}{4!} E^A E^B E^C E^D \left( 4 D_D B_{CBA}^3 + 6 T_{DCF}^F B_{FBA}^3 \right), \quad (2.96) \]

where for the fieldstrength coefficient we are interested in one has

\[ \Sigma_{\delta}^{\dot{\gamma}c} B_{cba}^3 + \text{derivative and other torsion terms} \quad (2.97) \]

This shows that the modified 3-form gauge potential

\[ B_{cba}^3 = B_{cba}^3 - \Sigma_{cba} \quad (2.98) \]

corresponds to the modified fieldstrength coefficient

\[ \Sigma_{\delta}^{\dot{\gamma}c} = 0 \quad (2.99) \]

Since this equation is obtained from a covariant and linear redefinition of the gauge potential, it is sometimes referred to as conventional constraint.

Taking, from now on, into account this modification, the remaining coefficients, at canonical dimensions 3/2 and 2, i.e. \( \Sigma_{\delta}^{\dot{\gamma}c} \) and \( \Sigma_{dcba} \), respectively, are given in terms of spinor derivatives of the basic superfields \( S \) and \( T \). To be more precise, at dimension 3/2 one obtains

\[ \Sigma_{\delta}^{\dot{\gamma}c} = -\frac{1}{16} \sigma_{\delta}^{d} \varepsilon_{dcba} D_{\delta}^d S, \quad (2.100) \]
\[ \Sigma_{\delta}^{\dot{\gamma}c} = +\frac{1}{16} \sigma_{\delta}^{d} \varepsilon_{dcba} D_{\delta}^d T, \quad (2.101) \]

and the Bianchi identity at dimension two takes the simple form

\[ \left( D^2 - 24 R^I \right) T - \left( \bar{D}^2 - 24 R \right) S = \frac{8i}{3} \varepsilon_{dcba} \sum_{dcba}. \quad (2.102) \]

This equation should be understood as another condition which serves to further restrict the chiral superfields \( S \) and \( T \), thus describing the supermultiplet of 3-form gauge potential in \( U_K(1) \) superspace.

Correspondingly, from the explicit solution of the constraints one finds that \( S \) and \( T \) are given as the chiral projections of \( U_K(1) \) superspace geometry acting on one and the same prepotential \( \Omega \):

\[ S = -4 \left( D^2 - 8 R^I \right) \Omega, \quad (2.103) \]
\[ T = -4 \left( \bar{D}^2 - 8 R \right) \Omega. \quad (2.104) \]
To discuss the relevance of this geometric structure for the discussion of supersymmetric Chern-Simons forms, we come back to the explicit expression of the superspace 4-form as field-strength squared term and interprete it as the fieldstrength of the 3-form gauge potential identified in turn with the Chern-Simons 3-form,

\[ \Sigma^{(YM)} = \text{tr}(\mathcal{F}\mathcal{F}) = dQ^{(YM)}. \] (2.105)

First of all, it is straightforward to convince oneself that in this case the previously generic superfields \( S \) and \( T \) are given as

\[ S^{(YM)} = -8 \text{tr}(\tilde{W}_\dot{\alpha}\tilde{W}^{\dot{\alpha}}), \quad T^{(YM)} = -8 \text{tr}(W^\alpha W_\alpha). \] (2.106)

On the other hand, the explicit solution of the constraints shows the existence of a so-called Chern-Simons superfield \( \Omega \) such that

\[ \text{tr}(W_\dot{\alpha}W^{\dot{\alpha}}) = \frac{1}{2}(\mathcal{D}^2 - 8R^i)\Omega^{(YM)}, \] (2.107)

\[ \text{tr}(W^\alpha W_\alpha) = \frac{1}{2}(\mathcal{\bar{D}}^2 - 8\bar{R})\Omega^{(YM)}. \] (2.108)

By definition, the Chern-Simons superfield is given in terms of the (pre)potentials which define supersymmetric Yang-Mills theory. In order to present an explicit expression for \( \Omega^{(YM)} \) one has to take into account the solution of the constraints in terms of prepotentials.

We have tried to make clear in this section that the superspace geometry of the 3-form gauge potential provides a generic framework for the discussion of Chern-Simons forms in superspace. Established in full detail for the Yang-Mills case, this property will be advantageously exploited in the more complicated gravitational case.

3 Gravitational Chern-Simons forms in superspace

3.1 Some general considerations

We come now to the description of gravitational Chern-Simons forms. The discussion will proceed in two steps: in this present section we display the geometric structure in the framework of superspace geometry, whereas in the next section we will discuss a relatively simple example including gravitational Chern-Simons forms via the antisymmetric tensor coupling à la Green and Schwarz.

In the geometrical description we will exploit what we have learned in the case of supersymmetric Yang-Mills theory. There, the geometric structure of Chern-Simons forms in superspace
is quite well understood and invariant actions for the antisymmetric tensor with Chern-Simons form in its fieldstrength, coupled to the general supergravity-matter system, can be obtained by means of the standard chiral density construction.

It is natural to ask whether the techniques which work quite well in the Yang-Mills case can be generalized to include gravitational Chern-Simons forms. We will see in the following that this is indeed true to a large extent, although substantially more involved technically. But we shall also see that novel features on the conceptual level appear, in particular when it comes to the construction of supersymmetric dynamics.

In order to cope with this new situation we shall investigate the structure of Chern-Simons forms in superspace in a more systematic way, based on the observation that the structure of Chern-Simons forms fits remarkably well into the superspace geometry of the 3-form gauge potential.

As a starting point we take a number of 2-form gauge potentials \( B^I \) numbered by \( I = 1, ..., n \) and certain types of Chern-Simons forms \( Q^\Delta \) with constant couplings \( k^I_\Delta \). The corresponding fieldstrengths are then defined as

\[
H^I = dB^I + k^I_\Delta Q^\Delta.
\]  

In practice we will include here Yang-Mills, gravitational (two chiralities) and \( U_K(1) \) Chern-Simons forms with \( \Delta \) taking values

\[
\Delta \in \{ (+), (-), (1), (YM) \}.
\]

We denote the derivative of the Chern-Simons form

\[
dQ^\Delta = \Psi^\Delta.
\]

In more explicit terms

\[
\begin{align*}
\Psi^{(YM)} &= \text{tr}(\mathcal{F}\mathcal{F}), & \Psi^{(1)} &= FF, \\
\Psi^{(+)} &= R_\beta^\alpha R_\alpha^\beta, & \Psi^{(-)} &= R_\beta^\dot{\alpha} R_\dot{\alpha}^\beta.
\end{align*}
\]

The Bianchi identity for the 2-form gauge field is then given as

\[
dH^I = k^I_\Delta \Psi^\Delta.
\]

This fixes our notations. The first, and crucial, nontrivial point in this approach is the observation that \( \Psi^\Delta \) allows for the decomposition

\[
\Psi^\Delta = \Sigma^\Delta + dM^\Delta,
\]

such that

\footnote{One should distinguish carefully indices \( I \) counting linear superfields from indices \( A = a, \alpha, \dot{\alpha} \) denoting superspace.}
• the coefficients of the 3-form $M^\Delta$, as well as those of the 4-form $\Sigma^\Delta$, are covariant expressions in terms of the corresponding fieldstrength, torsion and curvature superfields.

• the tensorial structure of the coefficients of the 4-form $\Sigma^\Delta$ corresponds exactly to that of the constraints in the 3-form geometry.

Of course, this decomposition must be explicitly established in every particular case. Before doing so we recall however a number of generic features valid in all cases.

First of all, upon substitution into the Bianchi identity for $H^I$, one arrives at

$$dH^I = k^I_\Delta \Sigma^\Delta,$$  \hspace{1cm} (3.8)

where we use the definition

$$H^I = H^I - k^I_\Delta M^\Delta = dB^I + k^I_\Delta (Q^\Delta - M^\Delta).$$  \hspace{1cm} (3.9)

It is in this form that the analogy with the 3-form gauge potential shows up. Defined as a differential 4-form in superspace,

$$\Sigma^\Delta = \frac{1}{4!} E^A E^B E^C E^D \Sigma^\Delta_{DCBA},$$  \hspace{1cm} (3.10)

is subject to Bianchi identities

$$d\Sigma^\Delta = 0.$$  \hspace{1cm} (3.11)

We shall show below, that the coefficients of $\Sigma^\Delta$ can be determined such that

$$\Sigma^\Delta_{\delta\gamma \beta \alpha} = 0,$$  \hspace{1cm} (3.12)

(where $\underline{\alpha} = \alpha, \dot{\alpha}$ and $A = a, \alpha, \dot{\alpha}$). This is the tensorial structure of the constraints of the 3-form geometry. We can then exploit our knowledge of the supersymmetric 3-form gauge potential to gain more insight into the structure of curvature-squared terms without needing to know all the details of the explicit form of the decomposition (which may be rather complicated and which are given below).

The Bianchi identities, given the property (3.12), determine the tensorial structure of the remaining fieldstrength components and give rise to certain relations involving covariant spinor derivatives. These general features of the 4-form $\Sigma^\Delta$ do not depend on the particular properties of the type of Chern-Simons forms under consideration.

Let us briefly recall the outcome of the analysis of the Bianchi identities. The components of $\Sigma^\Delta$ are completely described in terms of two superfields $S^\Delta$ and $T^\Delta$, appearing as follows in the tensor decomposition:

$$\Sigma^\Delta_{\delta\gamma ba} = \frac{1}{2} (\sigma^\Delta_{ba} \epsilon)_{\delta\gamma} S^\Delta,$$  \hspace{1cm} (3.13)

$$\Sigma^\Delta_{\dot{\delta}\dot{\gamma} ba} = \frac{1}{2} (\tilde{\sigma}^\Delta_{ba} \epsilon)_{\dot{\delta}\dot{\gamma}} T^\Delta.$$  \hspace{1cm} (3.14)
By a special choice of conventional constraints, it is possible to impose
\[ \Sigma^\Delta \delta \gamma_{ba} = 0. \] (3.15)

As to the superfields \( S^\Delta \) and \( T^\Delta \) the Bianchi identities reduce to the chirality conditions
\[ D_\alpha S^\Delta = 0, \quad D^\dot{\alpha} T^\Delta = 0. \] (3.16)

The remaining components at higher (canonical) dimensions are then
\[ \Sigma^\Delta \delta cba = -\frac{1}{16} \sigma^d_{\delta\delta} \varepsilon_{dcba} D^\delta S^\Delta, \] (3.17)
\[ \Sigma^\Delta \delta cba = +\frac{1}{16} \bar{\sigma}^d_{\delta\delta} \varepsilon_{dcba} D^\delta T^\Delta, \] (3.18)
and, finally,
\[ \Sigma^\Delta_{dcba} = \varepsilon_{dcba} \Sigma^\Delta. \] (3.19)

The boldscript scalar superfields \( \Sigma^\Delta \) appearing at this level are given as the second order spinor derivatives of the basic superfields:
\[ 2i \Sigma^\Delta = -\frac{1}{32} \left( D^2 - 8 R^I \right) T^\Delta + \frac{1}{32} \left( \bar{D}^2 - 8 \bar{R} \right) S^\Delta. \] (3.20)

In conclusion, we have seen that all the coefficients of the superspace 4-form \( \Sigma^\Delta \), subject to the constraints (3.12), are given in terms of the superfields \( S^\Delta \) and \( T^\Delta \) and their spinor derivatives. As to the analysis of the curvature-squared terms, this shows that it is sufficient to identify the superfields \( S^\Delta \) and \( T^\Delta \) in terms of the underlying geometry (Yang-Mills, supergravity or \( U_K(1) \)) for full knowledge of the corresponding superspace 4-form \( \Sigma^\Delta \).

This decomposition is particularly useful in the determination of the modified linearity conditions. To this end we observe first of all that the redefined fieldstrengths \( H^I \) are subject to the same constraints as before, i.e. without Chern-Simons forms:
\[ H^I_{\gamma\beta a} = 0, \quad H^I_{\gamma\beta a} = 0, \quad H^I_{\dot{\gamma}\dot{\beta} a} = 0. \] (3.21)

In other words, whereas the redefined quantities \( H^I \) have a very simple form, the original \( H^I \) can be quite complicated [18]. The linear superfield is then identified in
\[ H^I_{\gamma\dot{\beta} a} = -2 i (\sigma_a \epsilon)_{\gamma\dot{\beta}} L^I. \] (3.22)

Furthermore, one still has
\[ H^I_{\gamma ba} = 2 (\sigma_{ba})_{\gamma} \varphi D_\varphi L^I, \] (3.23)
\[ H^I_{\dot{\gamma} \dot{b} a} = 2 (\bar{\sigma}_{ba})_{\dot{\gamma}} \bar{\varphi} D\bar{\varphi} L^I. \] (3.24)
The Bianchi identities boil then down to the modified linearity conditions

\[
\begin{align*}
(D^2 - 8R)\mathcal{L}^I &= -\frac{1}{4}S^I, & S^I &= k^I_\Delta S^\Delta, \quad (3.25) \\
(D^2 - 8R)\mathcal{T}^I &= -\frac{1}{4}\mathcal{T}^I, & \mathcal{T}^I &= k^I_\Delta T^\Delta. \quad (3.26)
\end{align*}
\]

Note that we allow in general \(S^I\) and \(\mathcal{T}^I\) to be linear combinations of terms pertaining to Yang-Mills, gravitational or \(U_K(1)\) Chern-Simons forms.

Finally, the vector component \(\mathcal{H}^I_{cba}\) appears in the same way as before in the \(\theta\bar{\theta}\)-component of \(\mathcal{L}^I\):

\[
([D_\alpha, D_\alpha] - 4\sigma^a_{\dot{a}a}G_a) \mathcal{L}^I = -\frac{1}{3}\sigma^a_{\dot{a}a}\varepsilon^{cba}\mathcal{H}^I_{cba}. \quad (3.27)
\]

The difference is of course, that now, as a consequence of the decomposition, \(\mathcal{H}^I_{cba}\) contains additional terms,

\[
\mathcal{H}^I_{cba} = H^I_{cba} - k^I_\Delta M^\Delta_{cba}. \quad (3.28)
\]

So far the discussion was quite general, it applied for any particular case subsumed in the index \(\Delta\). In the following we will discuss the various different cases separately.

The Yang-Mills case, as already discussed in section 2 is reproduced in the general formulation presented here with the identifications

\[
\begin{align*}
S^{(\text{YM})} &= -8 \text{tr} (\bar{W}_a W^a), \quad & T^{(\text{YM})} &= -8 \text{tr} (W^a W_a). \quad (3.29) \\
M^{(\text{YM})}_{cba} &= \text{tr} (W^a dW_{\dot{a}}) \varepsilon^{dcba}. \quad (3.30)
\end{align*}
\]

Before proceeding to the explicit and detailed presentation of the covariant decomposition for the gravitational curvature-squared terms \(\Psi^{(\pm)}\) introduced above, we would like to draw attention to another feature of the formulation presented here which we found quite useful in the analysis of the gravitational curvature-squared terms.

It corresponds to a certain freedom in the identification of \(\Sigma^\Delta\) and \(M^\Delta\) without changing \(\Psi^\Delta\). In other words, the replacements \(\Sigma^\Delta \mapsto \Sigma^\Delta + \sigma^\Delta\) and \(M^\Delta \mapsto M^\Delta + m^\Delta\) do not affect \(\Psi^\Delta\) as long as they satisfy the superspace equation

\[
\sigma^\Delta + dm^\Delta = 0. \quad (3.31)
\]

A particularly useful solution is given in terms of an arbitrary unconstrained superfield \(\mu^\Delta\) such that the nonvanishing components of \(m^\Delta\) are

\[
m^\Delta_{\gamma}{}^\dot{\alpha}_a = T_{\gamma}{}^\dot{\alpha}_a \mu^\Delta, \quad (3.32)
\]
and
\[
m^\Delta_{\gamma ba} = 2(\sigma_{ba})_\gamma^\phi \mathcal{D}_\phi \mu^\Delta,
\]
(3.33)
\[
m^\Delta_{\hat{\gamma} ba} = 2(\bar{\sigma}_{ba})_{\hat{\gamma}}^\phi \mathcal{D}_\phi \mu^\Delta,
\]
(3.34)
as well as
\[
\varepsilon^{dcba} m^\Delta_{cba} = \frac{3}{2} \bar{\sigma}^d \alpha \beta \left( [\mathcal{D}_\alpha, \mathcal{D}_\beta] - 4G_{\alpha\beta} \right) \mu^\Delta.
\]
(3.35)
In \(\sigma^\Delta\), on the other hand, the unconstrained superfields \(\mu^\Delta\) appear (in obvious notations) as
\[
s^\Delta = (\mathcal{D}^2 - 8R^a) \mu^\Delta,
\]
(3.36)
\[
t^\Delta = (\mathcal{D}^2 - 8\mathcal{R}) \mu^\Delta.
\]
(3.37)
Of course, requiring \(\sigma^\Delta = 0\) would impose the linearity constraints of curved superspace on \(\mu^\Delta\), in accordance with our discussion of the linear superfields in previous sections. As already mentioned the freedom in assigning particular values to the arbitrary superfield \(\mu^\Delta\) may turn out to be useful in the gravitational case.

### 3.2 Covariant decomposition of curvature-squared terms

We come now back to the covariant decomposition of curvature-squared terms mentioned above. As we have already stressed, it is established by an explicit calculation. As a consequence, the present subsection will be rather technical. In order to give an impression of the actual procedure employed we present explicitly the \((+)-\) sector. The method consists in successive rearrangements of terms appearing in the curvature-squared terms
\[
\Psi^{(+)} = R_\beta^\alpha R_\alpha^\beta,
\]
(3.38)
to arrive at a decomposition
\[
\Psi^{(+)} = \Sigma^{(+)} + dM^{(+)},
\]
(3.39)
such that the coefficients of the differential forms \(\Sigma^{(+)}\) and \(M^{(+)}\),
\[
M^{(+)} = \frac{1}{\pi} E^A E^B E^C M^{(+)}_{CBA},
\]
(3.40)
\[
\Sigma^{(+)} = \frac{1}{\pi} E^A E^B E^C E^D \Sigma^{(+)}_{DCBA},
\]
(3.41)
can be completely expressed in terms of the covariant supergravity superfields and their covariant derivatives and that \(\Sigma^{(+)}\) can be chosen such that \(\Sigma^{(+)}_{\beta \gamma \alpha} = 0\), \(\text{i.e. the 3-form constraints}\)
\[
\Sigma^{(+)}_{\beta \gamma \alpha} = 0.
\]
(3.42)
\(^5\text{Note however that } M^{(+)} \text{ is only determined up to the exterior derivative of a 2-form, this means that our decomposition allows for the replacements } M^{(+)} \rightarrow M^{(+)} + dm^{(+)} \text{ as explained above.}\)
To begin with we should have a closer look to the coefficients of the 4-form
\[ \Psi^{(+)} = \frac{1}{4!} E^A E^B E^C E^D \Psi^{(+) DCBA}, \] (3.43)
which are expressed in terms of the (constrained) superspace curvatures,
\[ \Psi^{(+) DCBA} = 2 \oint_{DCB} R_{DC} \varphi^\varepsilon R_{BA} \varepsilon^\varphi. \] (3.44)

From the explicit form of the supergravity curvatures one finds immediately
\[ \Psi^{(+)} \delta \gamma \beta A = 0, \quad \Psi^{(+)} \delta \gamma \beta a = 0, \] (3.45)
in accordance with the corresponding coefficients in (3.42). However, a non-trivial contribution arises in
\[ \Psi^{(+)} \delta \gamma \beta a = 2 \oint \delta \gamma \beta R_{\delta \gamma} \varphi^\varepsilon R_{\delta \beta a} \varepsilon^\varphi = -16 R^\dagger \oint \delta \gamma \beta R_{\delta \gamma a \gamma \beta}. \] (3.46)
Replacing \( a \) by \( a\dot{a} \) and using supergravity information leads then to
\[ \Psi^{(+)} \delta \gamma \beta a\dot{a} = -8i \oint \nabla \delta \left( \epsilon_{\gamma \alpha} R^\dagger G_{\beta \dot{a}} + \epsilon_{\beta \alpha} R^\dagger G_{\gamma \dot{a}} \right). \] (3.47)
Inspection shows then that the desired decomposition for the coefficients considered so far can be established by the choice
\[ M^{(+)} \gamma \beta \dot{a} = 0, \quad M^{(+)} \gamma \beta \alpha = 0, \] (3.48)
and
\[ M^{(+)} \gamma \beta a \dot{a} = -8i R^\dagger (\epsilon_{\gamma \alpha} G_{\beta \dot{a}} + \epsilon_{\beta \alpha} G_{\gamma \dot{a}}), \] (3.49)
with spinor notation,
\[ M^{(+)} \gamma \beta a \dot{a} = \sigma^a_{\dot{a} a} M^{(+)} \gamma \beta a, \] (3.50)
understood. This means that so far we have established
\[ \Sigma^{(+)} \delta \gamma \beta A = 0, \quad \Sigma^{(+)} \delta \gamma \beta a = 0. \] (3.51)

In the next step we consider
\[ \Psi^{(+)} \delta \gamma \beta \dot{a} = 2 \sum_{\delta \gamma} R_{\delta \beta} \dot{\varphi}^\varepsilon R_{\gamma \dot{a}} \dot{\varphi}^\varepsilon = 4 \sum_{\delta \gamma} G_{\delta \beta} G_{\gamma \dot{a}}, \] (3.52)
which serves to identify \( M^{(+)} \gamma \beta \alpha \) as
\[ M^{(+)} \gamma \beta a \dot{a} = -i \epsilon_{\gamma \alpha} \epsilon_{\beta \dot{a}} \mu^{(+)} - \frac{i}{2} \left( G_{\gamma \beta} G_{a \dot{a}} + G_{a \beta} G_{\gamma \dot{a}} \right). \] (3.53)
Observe that here we have, in view of the discussion following eq.(3.31), allowed for the appearance of the arbitrary superfield \( \mu^{(+)} \).

36
We continue with
\[ \Psi^{(+)}_{\delta \gamma} \beta_a = -16 R^\dagger R a \delta_\gamma + 4 \left( R_{\gamma a} \delta^\phi + R_{\delta a} \gamma^\phi \right) G^\beta, \]  
(3.54)
and
\[ \Psi^{(+)}_{\delta \gamma} \beta a = 4 R^\gamma a \beta^\phi G^\delta + 4 R^\gamma \beta a \phi^\gamma. \]  
(3.55)
At this level the coefficients \( M^{(+)}_{\gamma b a} \) come in. Using spinor notation
\[ M^{(+)}_{\gamma \beta \gamma b a} = \sigma^b_{\beta \gamma} c^a_{\gamma b} M^{(+)}_{\gamma b a}, \]  
(3.56)
with standard tensor decomposition
\[ M^{(+)}_{\gamma \beta \alpha} = 2 \epsilon_{\beta \alpha} M^{(+)}_{\gamma \gamma \beta \alpha} - 2 \epsilon_{\beta \alpha} M^{(+)}_{\gamma \beta \gamma \alpha}, \]  
(3.57)
and
\[ M^{(+)}_{\gamma \beta \gamma b a} = \epsilon_{\beta \gamma} M^{(+)}_{\gamma \gamma \beta b a} + \epsilon_{\gamma \beta} M^{(+)}_{\gamma \gamma \beta \gamma a}, \]  
(3.58)
the different irreducible tensors defined here are then expressed in terms of the basic supergravity fields as follows:
\[ M^{(+)}_{\gamma \beta \alpha} = 4 \sum_{\beta \alpha} \left( 16 R^\dagger D \gamma G_{\gamma \beta} + 4 G_{\gamma \beta} D \gamma R^\dagger - G^\phi \beta D \gamma G_{\phi \alpha} - 4 G_{\beta \gamma} D \phi G_{\gamma \alpha} \right), \]  
(3.60)
\[ M^{(+)}_{\gamma \beta \alpha} = -8 R^\dagger W_{\gamma \beta \alpha} + \frac{1}{24} \int_{\gamma \beta \alpha} G^\phi \left( D \beta G_{\phi \alpha} + D \alpha G_{\phi \beta} \right), \]  
(3.61)
\[ 12 M^{(+)}_{\alpha} = -3 D_\alpha \mu^{(+)} - 16 D_\alpha \left( R R^\dagger \right) - 8 R^\dagger D \phi G_{\alpha \phi} - 18 G_{\alpha \phi} D \phi R^\dagger + 2 G^\phi \phi D_\alpha G_{\phi \phi} + 5 G^{\phi \phi} D \phi G_{\alpha \phi}, \]  
(3.62)
and
\[ M^{(+)}_{\gamma \beta \alpha} = -4 G^\phi \beta W_{\gamma \beta \alpha} + \frac{1}{8} \sum_{\beta \alpha} \left( G_{\beta \gamma} D \gamma G_{\alpha \phi} + \frac{3}{4} G_{\beta \gamma} \left( D_\alpha R - D \phi G_{\alpha \phi} \right) \right), \]  
(3.63)
as well as
\[ M^{(+)}_{\gamma \beta \alpha} = \frac{1}{8} \int_{\gamma \beta \alpha} G^\phi \beta \left( D \gamma G_{\phi \alpha} + D \alpha G_{\phi \beta} \right), \]  
(3.64)
\[ 4 M^{(+)}_{\alpha} = D_\alpha \mu^{(+)} + 2 G^\phi \beta D_\phi R + G^{\phi \phi} D_\phi G_{\phi \alpha}. \]  
(3.65)
These identifications establish
\[ \Sigma^{(+)}_{\delta \gamma} \beta a = 0, \quad \Sigma^{(+)}_{\delta \gamma} \beta a = 0, \]  
(3.66)
which completes the derivation of the 3-form constraint structure, eq. (3.42), for \( \Sigma^{(+)} \). Recall that this was the crucial goal we wanted to achieve in this section: the tensorial structures of the remaining coefficients of \( \Sigma^{(+)}_{DCBA} \) are now determined from the 3-form geometry, as for instance (see (3.13), (3.14))

\[
\begin{align*}
\Sigma^{(+)}_{\delta \gamma \, ba} &= \frac{1}{2}(\sigma_{ba} \epsilon)_{\delta \gamma} \, S^{(+)}, \\
\Sigma^{(+)}_{\hat{\delta} \hat{\gamma} \, ba} &= \frac{1}{2}(\sigma_{ba} \epsilon)_{\hat{\delta} \hat{\gamma}} \, T^{(+)},
\end{align*}
\]

(3.67)

(3.68)

where the chiral superfields \( S^{(+)} \), \( T^{(+)} \) are now identified in terms of the supergravity superfields as follows:

\[
\begin{align*}
S^{(+)} &= \left( \mathcal{D}^2 - 8R \right) \left( \mu^{(+)} + 16R \bar{R} - \frac{12}{4} G^{\varphi \bar{\varphi}} G_{\varphi \bar{\varphi}} \right) - 4\bar{X} \cdot \bar{X}, \\
T^{(+)} &= \left( \mathcal{D}^2 - 8R \right) \left( \mu^{(+)} + \frac{2}{4} G^{\varphi \bar{\varphi}} G_{\varphi \bar{\varphi}} \right) + 32W \gamma_{\bar{\gamma} \beta} \Sigma^{(+)}_{\alpha \beta} + \frac{4}{4} X^2 X_{\varphi}.
\end{align*}
\]

(3.69)

(3.70)

These chiral superfields will be among the basic ingredients in the construction of the supersymmetric extension of the various kinds of gravitational curvature-squared terms.

In the \( \Psi^{(+)}_{\hat{\delta} \hat{\beta}} \) \( ba \) - sector, one finds that the coefficient

\[
\Sigma^{(+)}_{\gamma \hat{\gamma} \beta \bar{\beta} \, \alpha \bar{\alpha}} = 4 \epsilon_{\gamma \beta} \epsilon_{\bar{\gamma} \bar{\beta}} \Sigma^{(+)}_{\alpha \beta} - 4 \epsilon_{\gamma \alpha} \epsilon_{\bar{\gamma} \bar{\beta}} \Sigma^{(+)}_{\beta \bar{\alpha}}.
\]

(3.71)

(3.72)

is expressed in terms of just one vector such that

\[
\Sigma^{(+)}_{\gamma \hat{\gamma} \beta \bar{\beta} \, \alpha \bar{\alpha}} = 4 \epsilon_{\gamma \beta} \epsilon_{\gamma \alpha} \Sigma^{(+)}_{\alpha \beta} - 4 \epsilon_{\gamma \alpha} \epsilon_{\gamma \beta} \Sigma^{(+)}_{\beta \alpha}.
\]

In the same equation, the component \( M^{(+)}_{eba} \), totally antisymmetric in its indices appears. Written in spinor notation

\[
M^{(+)}_{\gamma \hat{\gamma} \beta \bar{\beta} \, \alpha \bar{\alpha}} = \sigma^{e}_{\gamma \hat{\gamma}} \sigma^{b}_{\beta \bar{\beta}} \sigma^{a}_{\alpha \bar{\alpha}} M^{(+)}_{eba},
\]

(3.73)

it has the decomposition

\[
M^{(+)}_{\gamma \hat{\gamma} \beta \bar{\beta} \, \alpha \bar{\alpha}} = 2i \epsilon_{\gamma \bar{\beta}} \epsilon_{\gamma \alpha} M^{(+)}_{\beta \bar{\alpha}} - 2i \epsilon_{\hat{\gamma} \bar{\beta}} \epsilon_{\gamma \bar{\alpha}} M^{(+)}_{\alpha \beta},
\]

(3.74)

reflecting the antisymmetry property in terms of spinor indices. Following ref. [43], the combination \( M^{(+)}_{eab} + \Sigma^{(+)}_{eab} \) is then expressed in terms of the supergravity superfields as follows:

\[
M^{(+)}_{eab} + \Sigma^{(+)}_{eab} + \frac{1}{8} \left( [\mathcal{D}_a, \mathcal{D}_b] - 4G_{eab} \right) \mu^{(+)} = \\
+ \frac{1}{16} G_{\varphi \bar{\varphi}} \left( 4 [\mathcal{D}_\varphi, \mathcal{D}_{\bar{\varphi}}] G_{a\bar{a}} + [\mathcal{D}_a, \mathcal{D}_{\bar{a}}] G_{\varphi \bar{\varphi}} \right) - 8R \bar{R} G_{a\bar{a}} - \frac{15}{24} G_{eab} G_{\varphi \bar{\varphi}} G_{\varphi \bar{\varphi}} \\
- \mathcal{D}_a R \mathcal{D}_b R^\dagger - \frac{3}{8} D_{\bar{\varphi}} G_{a\bar{a}} D^\varphi G_{\varphi \bar{\varphi}} + 2 T_{\varphi \bar{\varphi}} T_{\gamma \alpha} \varphi + 8 T_{\varphi \bar{\varphi}} W_{\gamma \alpha} \\
+ T_{\varphi \bar{\varphi}} \left( \frac{3}{4} D^\varphi R + \frac{3}{8} D_{\bar{\varphi}} G_{\varphi \bar{\varphi}} \right) - T_{\varphi \bar{\varphi}} \left( 4 D^\varphi R + \frac{3}{8} D_{\bar{\varphi}} G_{\varphi \bar{\varphi}} \right) \\
+i G_{\varphi \bar{\varphi}} \left( \mathcal{D}_\varphi G_{a\bar{a}} - \frac{1}{2} \mathcal{D}_{a\bar{a}} G_{\varphi \bar{\varphi}} - \frac{3}{4} \mathcal{D}_{a\bar{a}} G_{\varphi \bar{\varphi}} - \frac{1}{4} \mathcal{D}_{a\bar{a}} G_{\varphi \bar{\varphi}} \right) - 4i R^\dagger \mathcal{D}_{a\bar{a}} R,
\]

(3.75)
Observe that $\Sigma^{(+)}_{a\dot{a}} = 0$ corresponds to the conventional constraint (3.15). The remaining coefficients of the 4-form $\Sigma^{(+)}$, i.e.

$$\Sigma^{(+)}_{\delta cba}, \Sigma^{(+)}_{dcba},$$

(3.76)

are obtained as spinor derivatives of the superfields $S^{(+)}$ and $T^{(+)}$ as explained in the discussion of the generic properties of the 3-form geometry in the previous subsection (eqs. (3.17), (3.18) and (3.20)). This completes our discussion of the $(+)$ - sector. The corresponding decomposition in the $(-)$ - sector are listed in appendix D.

Admittedly, the presentation in this subsection was notationally quite heavy, the coefficients of the 3-form $M^{(+)}$ are rather complicated expressions in terms of the basic supergravity superfields and their covariant derivatives. But once established, we then use these coefficients from now on precisely as a shorthand notation for otherwise complicated expressions, which may be expanded when necessary.

### 3.3 3-form geometry and modified linearity conditions

We have seen that supersymmetric Chern-Simons forms can be described in the framework of the superspace geometry of a 3-form gauge potential: on the one hand we have explicitly established the covariant decomposition

$$\Psi^\Delta = \Sigma^\Delta + dM^\Delta,$$

(3.77)

as anticipated in (3.7). On the other hand, from our starting point (3.3) we know

$$\Psi^\Delta = dQ^\Delta,$$

(3.78)

and therefore

$$\Sigma^\Delta = d\left(Q^\Delta - M^\Delta\right).$$

(3.79)

Here, the combination $Q^\Delta - M^\Delta$ corresponds to the 3-form gauge potential, which, under a gauge transformation changes with the exterior derivative of a 2-form. $\Sigma^\Delta$ is subject to the 4-form constraints, which ensures the existence of the Chern-Simons superfield $\Omega^\Delta$, corresponding to the prepotential of the 3-form, such that the combinations $S^\Delta$ and $T^\Delta$ as identified in (3.13) and (3.14), are given as

$$S^\Delta = -4\left(\bar{\nabla}^2 - 8\bar{R}\right)\Omega^\Delta,$$

(3.80)

$$T^\Delta = -4\left(D^2 - 8R\right)\Omega^\Delta.$$

(3.81)
Under gauge-, Lorentz-, Kähler transformations the corresponding Chern-Simons superfields change as

$$\Omega^\Delta \mapsto \Omega^\Delta + \lambda^\Delta,$$

(3.82)

where $$\lambda^\Delta$$ are linear superfields, subject to the conditions

$$\left(D^2 - 8 R^\dagger\right) \lambda^\Delta = 0, \quad \left(\bar{D}^2 - 8 R\right) \lambda^\Delta = 0.$$  

(3.83)

Of course, this reflects the above-mentioned fact that the full Chern-Simons form changes with the exterior derivative of a 2-form.

This structure is now coupled to the 2-form gauge potential in the way defined above in the first section of this chapter, the relevant definitions being (3.1), (3.3), (3.6), (3.8) and (3.9). As we have pointed out there, the presence of Chern-Simons forms gives rise to modified linearity equations. As these conditions intervene crucially in the construction of invariant actions, it will be convenient for our subsequent investigations to summarize them here.

The basic objects of interest are the superfields $$L^I$$, identified in (3.22) and subject to the modified linearity conditions

$$\left(D^2 - 8 R^\dagger\right) L^I = -\frac{i}{4} S^I,$$

(3.84)

$$\left(\bar{D}^2 - 8 R\right) L^I = -\frac{i}{4} T^I.$$  

(3.85)

Recall that we have allowed $$S^I$$ and $$T^I$$ to be linear combinations of terms pertaining to Yang-Mills, gravitational or $$U_K(1)$$ Chern-Simons forms as described in the preceding chapters:

$$S^I = k^I_\Delta S^\Delta, \quad T^I = k^I_\Delta T^\Delta,$$  

(3.86)

with $$k^I_\Delta$$ constants, indices $$I$$ referring to the 2-form multiplets under consideration and $$\Delta$$ to the different types of Chern-Simons forms,

$$\Delta \in \{ (YM), (+), (-), U_K(1) \}.$$  

(3.87)

In the Yang-Mills case we had

$$S^{(YM)} = -8 \text{tr}(\bar{W}_\alpha \bar{W}^\alpha), \quad T^{(YM)} = -8 \text{tr}(W^\alpha W_\alpha).$$

(3.88)

In the case of gravity, things were slightly more complicated. We found

$$S^{(+)} = \left(D^2 - 8 R^\dagger\right) \left(\mu^{(+) + 16 RR^\dagger} - \frac{12}{4} G^{\alpha\dot{\alpha}} G_{\alpha\dot{\alpha}}\right) - 4 \bar{X}_{\alpha} \bar{X}^{\alpha},$$

(3.89)

$$T^{(+)} = \left(\bar{D}^2 - 8 R\right) \left(\mu^{(+) + 32 W^{\gamma\beta\alpha} W_{\gamma\beta\alpha} + \frac{3}{2} X^\alpha X_\alpha. \right.$$  

(3.90)
in one chiral sector and
\[
S^{(-)} = \left( D^2 - 8 R^\dagger \right) \left( \mu^{(-)} + \frac{3}{4} G^{\alpha \dot{\alpha}} G_{\alpha \dot{\alpha}} \right) + 32 W_{\gamma \dot{\alpha}} W^{\gamma \dot{\alpha}} + \frac{4}{3} \tilde{X}_\alpha \tilde{X}^\alpha,
\]
\[
T^{(-)} = \left( \bar{D}^2 - 8 R \right) \left( \mu^{(-)} + 16 R R^\dagger - \frac{13}{4} G^{\alpha \dot{\alpha}} G_{\alpha \dot{\alpha}} \right) - 4 X^\alpha X_\alpha,
\]
in the opposite one. Finally, in the chiral $U_K(1)$ sector we identified
\[
S^{(1)} = -2 \tilde{X}_\alpha \tilde{X}^\alpha, \quad T^{(1)} = -2 X^\alpha X_\alpha.
\]

Clearly, taking into account the relations (3.80) and (3.81), one may define the truly linear superfields
\[
\mathcal{L}^I - k^I_\Delta \Omega^\Delta,
\]
which are, however, not gauge invariant, in view of (3.82). The gauge invariant superfields $\mathcal{L}^I$, subject to the modified linearity conditions, will be relevant for the description of component fields. In particular, the field strengths of the antisymmetric tensors are identified in the covariant superfield expansion of $\mathcal{L}^I$ as
\[
([D_\alpha, D_\dot{\alpha}] - 4 \sigma_{\alpha \dot{\alpha}} G_a) \mathcal{L}^I = -\frac{1}{2} \sigma_{\alpha \dot{\alpha} \epsilon}^{dca} \mathcal{H}_{cba}^I,
\]
with
\[
\mathcal{H}_{cba}^I = H_{cba}^I - k^I_\Delta M_{cba}^\Delta.
\]

4 Dynamics : Green-Schwarz for Gauss-Bonnet

The geometric formulation presented in this paper is quite general and suitable to be employed in the construction of quite a variety of dynamical theories involving any kind of gravitational Chern-Simons forms in the presence of arbitrary matter and linear multiplet couplings. We shall leave the discussion of such general constructions to a separate publication and concentrate here on the description of a simple, illustrative example.

This will be obtained from a number of simplifying assumptions. First of all we shall restrict ourselves to one single antisymmetric tensor gauge field which is coupled to the Chern-Simons form relevant for the Gauss-Bonnet combination of curvature-squared terms.

Moreover, and in order to exhibit as clearly as possible the various contributions which arise in the linear superfield formulation of this Green-Schwarz coupling we shall neglect here completely the matter and Yang-Mills sector. In technical terms this means that we may discard the $U(1)$-sector, i.e. work in the framework of traditional superspace geometry.

The salient features of this dynamical theory will first be presented in the linear superfield formulation. A supersymmetric duality transformation, taking into account the gravitational
Chern-Simons superfield, will then be employed to establish the relation with the dual theory where the antisymmetric tensor multiplet is replaced by a chiral multiplet.

4.1 From $U(1)$ to traditional superspace

The traditional superspace geometry is recovered from the $U(1)$ superspace, as presented in section 2, by simply taking the kinetic prepotential superfield to vanish. i.e. putting

$$K = 0.$$ (4.1)

As a consequence (see eqs. (2.31), (2.32) and (2.33)) one obtains

$$A_\underline{a} = 0, \quad A_a = \frac{3i}{2} G_a.$$ (4.2)

The equation for the vectorial component of the gauge potential is a particular artefact of the choice of conventional constraint for $F_{\dot{\alpha}\dot{\alpha}}$. On the level of the covariant supergravity superfields this choice implies

$$X_\alpha = 0, \quad \bar{X}_{\dot{\alpha}} = 0,$$ (4.3)

which in turn is tantamount to

$$\mathcal{D}_a R = \mathcal{D}_{\dot{\alpha}} G_{a\dot{\alpha}}, \quad \mathcal{D}^{\dot{\alpha}} R^{\dot{\alpha}} = \mathcal{D}_a G^{a\dot{\alpha}}.$$ (4.4)

Moreover, for convenience, we give the additional terms containing $G_{a\dot{\alpha}}$ in the basic torsion components,

$$\hat{T}^\alpha_{\gamma \beta} = \tilde{T}^\alpha_{\gamma \beta} + \delta^\alpha_{\gamma} A_{\beta} = + \frac{3i}{2} \delta^\alpha_{\gamma} G_{\beta} + \frac{i}{2} G^c (\sigma_c \bar{\sigma}_b) \gamma_{\alpha},$$

$$\hat{T}_{\dot{\gamma} \dot{\beta} \dot{\alpha}} = \tilde{T}_{\dot{\gamma} \dot{\beta} \dot{\alpha}} - \delta^A_{\dot{\alpha}} A_{\beta} = - \frac{3i}{2} \delta^A_{\dot{\alpha}} G_{\beta} - \frac{i}{2} G^c (\sigma_c \bar{\sigma}_b) \gamma_{\dot{\alpha}},$$

where now the hatted quantities refer to the traditional superspace geometry of the so-called old-minimal supergravity multiplet. Likewise, for the vectorial covariant derivative of a generic superfield of $U(1)$-weight $w(X)$ we employ the notation

$$\mathcal{D}_a X = \tilde{D}_a X + \frac{3i}{2} w(X) G_a X.$$ (4.7)

This last formula will be useful in identifying the additional $G_a$ contributions arising from the covariant vectorial derivatives in our equations derived in $U(1)$ superspace.
4.2 Identification of Gauss-Bonnet

Applying the analysis of chapter 3 to the case of one single antisymmetric tensor, and using notations of appendix C, the Gauss-Bonnet combination is identified in eq.(3.6), i.e.

$$dH = k_\Delta \Psi^\Delta,$$

(4.8)
in taking $k_{(+)} = -k_{(-)} = k$. Recall that after taking into account the covariant decomposition of the curvature-squared terms this becomes

$$dH = k \left( \Sigma^{(+)} \right. - \left. \Sigma^{(-)} \right) = k \Sigma^{(GB)}.$$  

(4.9)

In what follows the superscript $^{(GB)}$ will be used systematically to denote quantities referring to the Gauss-Bonnet combination. In order to write down the modified linearity conditions for the single superfield $L$, identified in eq.(3.22),

$$H_{\gamma\dot{\beta}a} = -2i (\sigma_a^\epsilon)_{\gamma\dot{\beta}L},$$

(4.10)

we define (taking $\mu^{(+)} = \mu^{(-)}$)

$$S^{(GB)} = S^{(+)} - S^{(-)} = +8 \left( D^2 - 8 R^1 \right) \left( 2 R^1 R + G^a G_a \right) - 32 W_{\dot{\gamma}\dot{\beta}\dot{\alpha}} W_{\dot{\gamma}\dot{\beta}\dot{\alpha}},$$

(4.11)

$$T^{(GB)} = T^{(+)} - T^{(-)} = -8 \left( \bar{D}^2 - 8 \bar{R} \right) \left( 2 \bar{R}^1 R + G^a G_a \right) + 32 W_{\gamma\beta\alpha} W_{\gamma\beta\alpha}.$$  

(4.12)

As a consequence, the modified linearity conditions, eqs.(3.84) to (3.92), reduce to

$$\left( D^2 - 8 R^1 \right) L = -2k \left( D^2 - 8 R^1 \right) \left( 2 R^1 R + G^a G_a \right) + 8k W_{\dot{\gamma}\dot{\beta}\dot{\alpha}} W_{\dot{\gamma}\dot{\beta}\dot{\alpha}},$$

(4.13)

$$\left( \bar{D}^2 - 8 \bar{R} \right) L = +2k \left( \bar{D}^2 - 8 \bar{R} \right) \left( 2 \bar{R}^1 R + G^a G_a \right) - 8k W_{\gamma\beta\alpha} W_{\gamma\beta\alpha}.$$  

(4.14)

4.3 Superfield towards component field action

Having specified the underlying geometric framework, the action which describes the coupling of the linear multiplet to supergravity and gravitational Chern-Simons forms of the Gauss-Bonnet type will be of the generic form

$$\int E F(L).$$

(4.15)

As matter fields (i.e. chiral multiplets) are absent we do not have to care, for the moment, about Kähler transformations. Note, however, that this action will exhibit a field dependent

\(^{6}\)beware, however, of eventual duality transformation from linear superfield formalism to chiral superfield formalism
normalization function of the Einstein term - we will come back to this issue later on. For the time being we are interested in evaluating explicitly the component field version and in determining the curvature-squared contributions.

The starting point for the construction of the component field action will be the expression for the chiral density as defined in eq.(2.39) with the generic superfield \( r \) and its complex conjugate \( \bar{r} \) identified as

\[
\begin{align*}
\bar{r} &= (\bar{D}^2 - 8 \bar{R}) F^L, \\
\bar{r} &= (\bar{D}^2 - 8 \bar{R}) F^L.
\end{align*}
\] (4.16)

Inspection of (2.39) shows then that the bosonic contributions to the action will be contained in the projection to lowest components of the superfield expression

\[
\Box^+ F^L \equiv \left[ (D^2 - 24R^l) \left( \bar{D}^2 - 8 \bar{R} \right) + \left( \bar{D}^2 - 24 \bar{R}^l \right) \left( D^2 - 8 R^l \right) \right] F^L. 
\] (4.17)

In the following we shall restrict ourselves to the discussion of the purely bosonic terms in the action. To do this appropriately at the notational level we introduce the symbol \( \text{bos} = \) which means that only bosonic terms should be retained in the explicit evaluation (and projection to lowest superfield components is understood). Applying successively the spinorial covariant derivatives using explicitly the modified linearity conditions

\[
\begin{align*}
(D^2 - 8R^l) \mathcal{L} &= -\frac{k}{4} S^{(GB)}, \\
(\bar{D}^2 - 8 \bar{R}) \mathcal{L} &= -\frac{k}{4} T^{(GB)},
\end{align*}
\] (4.18)

relevant for the Gauss-Bonnet combination gives rise to

\[
\Box^+ F^L \overset{\text{bos}}{=} -4 F'' D_{\alpha \dot{\alpha}} \mathcal{L} D^{\alpha \dot{\alpha}} \mathcal{L} + \mathcal{F}'' [D_\alpha, D_{\dot{\alpha}}] \mathcal{L} [D^\alpha, D^{\dot{\alpha}}] \mathcal{L}
- 8 (\mathcal{F} - \mathcal{L} F') \left( D^2 R + \bar{D}^2 \bar{R}^l \right) + 16 \left( 24 (\mathcal{F} - \mathcal{L} F') + 8 \mathcal{L}^2 F'' \right) R^l \bar{R}
+ \mathcal{F}' \Box^+ \mathcal{L} + \frac{k^2}{8} F'' S^{(GB)} T^{(GB)} - 4k R \mathcal{L} F'' S^{(GB)} - 4k R^l \mathcal{L} F'' T^{(GB)},
\] (4.19)

where primes denote derivatives of the function \( F \) with respect to \( \mathcal{L} \). Note that this equation contains, in a compact form, the totality of the bosonic terms in the supersymmetric action. It remains to work out, in some more detail, the various contributions of the individual terms.

4.4 The basic building blocks and the complete bosonic action

We shall now discuss one by one the individual building blocks for the bosonic part of the action, as they arise in eq.(4.19), with particular emphasis on the contributions linear and quadratic in the Gauss-Bonnet coupling \( k \). The first term,

\[
-4 F'' D_{\alpha \dot{\alpha}} \mathcal{L} D^{\alpha \dot{\alpha}} \mathcal{L},
\]
describes simply the kinetic action for the scalar field which is identified as the lowest component of the superfield $L$. The second term,

$$\mathcal{F}'' [\mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}}] \mathcal{L} [\mathcal{D}^\alpha, \mathcal{D}^{\dot{\alpha}}] \mathcal{L},$$

is slightly more complicated: among other things it contains the kinetic action for the antisymmetric tensor gauge field via its covariant field strength $\mathcal{H}_{cba}$, which appears in

$$[\mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}}] \mathcal{L} = \sigma_{\alpha \dot{\alpha}} \left( \tilde{\mathcal{H}}_a + 4 \mathcal{L} G_a \right), \quad (4.20)$$

through its dual defined as

$$\tilde{\mathcal{H}}^d = \frac{1}{3!} \varepsilon^{d c b a} \mathcal{H}_{cba}. \quad (4.21)$$

Recall that this field strength is defined as the purely vectorial component of the 3-form

$$\mathcal{H} = d\mathcal{B} + k \left( Q^{(GB)} - M^{(GB)} \right), \quad (4.22)$$

with the corresponding component of $M^{(GB)}$ given as

$$M_a^{(GB)} \overset{bos}{=} - 4i \mathcal{D}_a (R^1 R) - 2i \mathcal{D}^b(G_b G_a), \quad (4.23)$$

using obvious notations concerning the Gauss-Bonnet combinations. The third term,

$$-8 \left( \mathcal{F} - \mathcal{L} \mathcal{F}' \right) \left( \mathcal{D}^2 R + \bar{\mathcal{D}}^2 \bar{R}^1 \right),$$

brings in the curvature scalar with field dependent normalization function $\mathcal{F} - \mathcal{L} \mathcal{F}'$, due to the supergravity identity

$$\mathcal{D}^2 R + \bar{\mathcal{D}}^2 \bar{R}^1 = - \frac{2}{3} \mathcal{R} + 32 R^1 R + 4 G^a G_a, \quad (4.24)$$

whilst the fourth term does not need any comment. The four remaining terms are either linear or quadratic in the Gauss-Bonnet coupling $k$. Taking into account the relation

$$\mathcal{D}^2 R - \bar{\mathcal{D}}^2 \bar{R}^1 = 4i \mathcal{D}_a G^a, \quad (4.25)$$

one finds immediately that the bosonic contribution from the last two terms is simply given as

$$- 4k \mathcal{L} \mathcal{F}'' S^{(GB)} - 4k R^1 \mathcal{L} \mathcal{F}'' T^{(GB)} \overset{bos}{=} - 256i k \mathcal{L} \mathcal{F}'' \mathcal{D}^a \left( R^1 R G_a \right). \quad (4.26)$$

In the next step we consider the explicit form of the bosonic contributions to $S^{(GB)}$ and $T^{(GB)}$,

$$S^{(GB)} \overset{bos}{=} - 16 R^1 \left( \frac{1}{3} \mathcal{R} - 8 R^1 R + 14 G^a G_a \right) + 32i \left( 2 G^a \bar{\mathcal{D}}_a R + R \bar{\mathcal{D}}_a G^a \right), \quad (4.27)$$

$$T^{(GB)} \overset{bos}{=} + 16 R \left( \frac{1}{3} \mathcal{R} - 8 R^1 R + 14 G^a G_a \right) + 32i \left( 2 G^a \mathcal{D}_a R + R \mathcal{D}_a G^a \right), \quad (4.28)$$
which will be used to establish the explicit expression for the term
\[
\frac{k^2}{8} F'' S^{(GB)} T^{(GB)},
\]
at order \(k^2\). Finally we have to substitute for \(\Box^{\perp} \mathcal{L}\) in the last remaining term. Using the explicit definitions and relations from appendix \(\text{C}\) we find
\[
\Box^{\perp} \mathcal{L} = -\frac{k}{4} \left( D^2 - 24 R^\dagger \right) T^{(GB)} - \frac{k}{4} \left( \bar{D}^2 - 24 R \right) S^{(GB)},
\]
with bosonic part
\[
\Box^{\perp} \mathcal{L}^{\text{bos}} = +2i k \varepsilon_{fedc} W^{fe, ba} W^{dc, ba} + 48i k \varepsilon^{dcba}(\hat{D}_d G_c)(\hat{D}_b G_a) + 2k \Box^{-} \left( 2R^\dagger R + G^a G_a \right).
\]
The last term in this equation is basically a space-time divergence whose explicit form is irrelevant for the present discussion. If desired, it can be evaluated using the definition of \(\Box^{-}\) in appendix \(\text{C}\).

Putting all the information concerning the individual terms in \(\mathcal{L}^{(LR)}\) together one obtains
\[
\Box^{\perp} \mathcal{F}(\mathcal{L}) \equiv \frac{16}{3} \left( F - \mathcal{L} F' \right) \mathcal{R} + 16 \left( F - \mathcal{L} F' + \mathcal{L}^2 F'' \right) (8R^\dagger R - 2G^a G_a)
\]
\[
-2F'' \bar{H}^a \hat{H}_a + 8F'' D^a \mathcal{D}_a \mathcal{L} - 16\mathcal{L} F'' G_a \hat{H}^a
\]
\[
+2i k F' \varepsilon_{fenced} W^{fe, ba} W^{dc, ba} + 16i k F' \varepsilon^{dcba}(\hat{D}_d G_c)(\hat{D}_b G_a)
\]
\[
+16i k F' D^a \left[ 4i(R^\dagger \hat{D}_a R - \hat{R} \hat{D}_a R^\dagger) + 2G^b \bar{R}^\dagger ba + \frac{1}{6} G_a \mathcal{R} - 32G_a R^\dagger R - 2G_a G^b G_b \right]
\]
\[
-\frac{64}{3} k^2 F'' \mathcal{R} \left[ 2iG^a(R^\dagger \hat{D}_a R - \hat{R} \hat{D}_a R^\dagger) + (14G^a G_a - 8R^\dagger R) R^\dagger R + \frac{1}{6} R R^\dagger R \right]
\]
\[
-32k^2 F'' \left[ R^\dagger R(14G^a G_a - 8R^\dagger R)^2 + 4i(14G^b G_b - 8R^\dagger R) G^a(R^\dagger \hat{D}_a R - \hat{R} \hat{D}_a R^\dagger)
\]
\[
+16G^b G^a(\hat{D}_b R^\dagger)(\hat{D}_a R) + 8G^a(\hat{D}_b G^b) \hat{D}_a (R^\dagger R) + 4R^\dagger R(\hat{D}_b G^b)(\hat{D}_a G^a) \right].
\]

Taken as the lowest superfield component this expression describes, up to a factor \(e = \det e^m_a\), the bosonic part of the lagrangian density for the coupling of the antisymmetric tensor multiplet to supergravity in the presence of Gauss-Bonnet Chern-Simons forms. Observe that in the absence of Chern-Simons forms, \(i.e.\) for \(k = 0\), we are just left with the first two lines.

On the other hand, coupling of Chern-Simons form does not just mean modify the field-strength of the antisymmetric tensor: supersymmetry enforces quite a number of additional couplings, linear and quadratic in the parameter \(k\). This illustrates once more the striking fact that supersymmetrization of the Green-Schwarz mechanism does not only introduce new fermionic terms but necessitates genuine new bosonic contributions, as for instance the square of the Weyl-tensor in the example discussed here.
The Gauss-Bonnet combination itself is not explicitly present in this action, it appears only after discussion of the equation of motion for the antisymmetric tensor gauge field. Instead of discussing the explicit form of the component field equations of motion, which are quite complex, we shall turn immediately to the dual theory with the antisymmetric tensor replaced by a scalar field.

4.5 Duality transformation and the dual theory

Taking $X$ to be some real, covariant and unconstrained superfield we consider

$$\int E \left[ \mathcal{F}(X) + b (X - k \Omega^{(GB)}) (\phi + \bar{\phi}) \right].$$

(4.32)

Variation with respect to $\phi$ and $\bar{\phi}$, or rather with respect to their unconstrained prepotentials, entail linearity conditions for $X - k \Omega^{(GB)}$ and thus allow to identify $X$ with $\mathcal{L}$, subject to modified linearity conditions (4.13), (4.14), thus getting back the theory already discussed.

On the other hand, variation with respect to $X$ results in

$$\mathcal{F}' + b (\phi + \bar{\phi}) = 0,$$

(4.33)

giving rise to a theory expressed in terms of the chiral resp. antichiral superfields $\phi$ and $\bar{\phi}$. The theory obtained in this way is said to be dual to the one described previously.

After taking into account the duality relation (4.33) i.e. the (algebraic) equation of motion of the first order action, one obtains the dual theory described in terms of one single chiral superfield $\phi$ and its conjugate $\bar{\phi}$ as

$$\Lambda = \int E \left[ \mathcal{F}(X[\phi + \bar{\phi}]) + b (\phi + \bar{\phi}) X[\phi + \bar{\phi}] - bk \Omega^{(GB)} (\phi + \bar{\phi}) \right].$$

(4.34)

This action obviously consists of two parts: the usual action for the kinetic terms of the chiral multiplet and a new one, containing the Chern-Simons superfield, which will, at the component field level, give rise to the curvature-squared terms.

In order to match the traditional notations we parametrize

$$-3e^{-\frac{1}{3}K(\phi,\bar{\phi})} = \mathcal{F}(X[\phi + \bar{\phi}]) + b (\phi + \bar{\phi}) X[\phi + \bar{\phi}],$$

(4.35)

to establish the first part of the dual theory in the usual notation of standard supergravity matter coupling with Kähler potential $K(\phi, \bar{\phi})$ i.e.

$$\Lambda_0 = -3 \int E e^{-\frac{1}{3}K(\phi,\bar{\phi})}.$$

(4.36)
The explicit supersymmetric component field action may then be obtained from

\[ r_0 = \frac{3}{8} \left( \mathcal{D}^2 - 8R \right) e^{-\frac{4}{3}K(\phi, \bar{\phi})}, \quad \bar{r}_0 = \frac{3}{8} \left( \mathcal{D}^2 - 8R^\dagger \right) e^{-\frac{4}{3}K(\phi, \bar{\phi})}, \]  

(4.37)
in terms of the chiral density construction (2.39). It leads to the usual supergravity-matter action with field dependent normalization of the curvature scalar term, which we refrain however to reproduce here in detail.

Instead we concentrate on the second part of the action,

\[ \Lambda_1 = -bk \int E \Omega^{(GB)} (\phi + \bar{\phi}), \]  

(4.38)
which reflects the modifications of the standard theory due to the presence of the Green-Schwarz mechanism for the Gauss-Bonnet combination. In the following we will discuss the component field evaluation of this additional part of the action.

In a first step, still completely in terms of superfields, we make use of integration by parts in superspace, to write

\[ \int E \Omega^{(GB)} \phi = -\frac{1}{8} \int \frac{E}{R} \phi \left( \mathcal{D}^2 - 8R \right) \Omega^{(GB)} = \frac{1}{32} \int \frac{E}{R} \phi T^{(GB)}, \]  

(4.39)
where we used (3.81) and \( T^{(GB)} \) is given in eq.(4.12). Reasoning in the same manner for the complex conjugate term in \( \Lambda_1 \) we then arrive at

\[ \Lambda_1 = -\frac{bk}{32} \int E \frac{R}{R^\dagger} \phi T^{(GB)} - \frac{bk}{32} \int E \frac{R^\dagger}{R^\dagger} \bar{\phi} S^{(GB)}. \]  

(4.40)

Employing the language of the chiral density construction this means that the component field expression is derived from the chiral, resp. antichiral superfields

\[ r_1 = -\frac{bk}{16} \phi T^{(GB)}, \quad \bar{r}_1 = -\frac{bk}{16} \bar{\phi} S^{(GB)}, \]  

(4.41)
following the usual procedure.

\[ \Lambda_1 \overset{\text{bos}}{=} -i bk \left( \phi + \bar{\phi} \right) \left( \frac{1}{4} \epsilon^{dc'd'e'} R_{dc,b} R_{d'e',a} b + 4 \epsilon^{dcb'a} D_d G_c D_b G_a \right) \]

\[ -\frac{bk}{8} \left( \phi + \bar{\phi} \right) \epsilon^{dc'd'e'} \epsilon_{b'a'b'd'} R_{dc,b} R_{d'e',b'd'} \]

\[ + 8 bk \left( \phi - \bar{\phi} \right) D^a D_a \left( R^\dagger R \right) + 4 bk \left( \phi + \bar{\phi} \right) D^a \left( R^\dagger \tilde{D}a R - R \tilde{D}a R^\dagger \right) \]

\[ -\frac{bk}{16} \left( \phi + \bar{\phi} \right) D^a \left\{ G^b \left( R_{ab} - \frac{1}{6} \eta_{ab} R \right) - G_a \left( 2 R^\dagger R + G^b G_b \right) \right\} \]

\[ + 4 bk \left( \phi - \bar{\phi} \right) D^a D^b (G_a G_b) - bk T^{(GB)} D^2 \phi - bk S^{(GB)} D^2 \bar{\phi}. \]

Observe the analogy of this expression with the Yang-Mills case (1.46) discussed in the introductory section: the curvature-squared term which corresponds to the initial Chern-Simons
combination, in our case Gauss-Bonnet, appears with a factor $\phi - \bar{\phi}$, whilst the orthogonal one (in the sense of Hodge duality) acquires a factor $\phi + \bar{\phi}$ (in the Yang-Mills case this was just the kinetic term with field dependent gauge coupling function).

As this example is merely intended as an illustration of the methods of superspace geometry at work, we will not pursue here this discussion any further. A more detailed study of this and similar, but more general theories, is left to forthcoming research, see also the remarks in the concluding section.

5 Conclusion and outlook

The main emphasis of this review was on a concise and complete presentation of the superspace geometric description of gravitational Chern-Simons forms. Making use of the structural analogy of Chern-Simons forms in supersymmetric theories with the geometry of the 3-form multiplet (coupled to supergravity) makes it possible to cope with otherwise highly complex technicalities.

The second important point we wanted to exhibit is the relevance of this geometrical description for the construction of supersymmetric dynamical theories with gravitational Chern-Simons forms.

As an illustration of this point we described in section 4 the basic features of a particular and relatively simple example, dealing with Chern-Simons forms of the Gauss-Bonnet type. Starting from the linear superfield mechanism and performing then the duality transformation to the chiral superfield mechanism we displayed the bosonic parts of the component field action in both versions.

This example was particularly simple in two respects. First of all it was formulated in the traditional superspace geometry approach (i.e. no explicit $U_K(1)$ factor present in the structure group). Secondly only one linear, resp. chiral, superfield was taken into account, without any additional matter or Yang-Mills sectors.

As this specific example was chosen to illustrate the interplay of geometry and dynamics, we did not analyse the sector of previously auxiliary fields, which now appear with terms containing space-time derivatives - a subject which should more conveniently studied in more general situations.

Likewise, the question of the normalization of the curvature scalar term, the interpretation of its field dependent normalization function as well as questions of field redefinitions which have the form of Weyl rescalings were not further pursued. These issues are more conveniently
adressed in the full-fledged $U_K(1)$ superspace geometry framework.

We wish to emphasize, however, that the geometric superspace formulation of gravitational Chern-Simons forms, which was the main purpose of this review, is very well-suited for the discussion of the questions alluded to and which will be the subject of future investigations.

Appendices

A 3-form gauge potential and Chern-Simons forms

A.1 3-form gauge potential in $U_K(1)$ superspace

We consider

$$B^3 = \frac{1}{3!} E^A E^B E^C B^3_{CBA},$$

(A.1)

a 3-form gauge potential subject to gauge transformations

$$B^3 \mapsto \Lambda B^3 = B^3 + d\Lambda,$$

(A.2)
described in terms of a 2-form in superspace,

$$\Lambda = \frac{1}{2} E^A E^B \Lambda_{BA}.$$  

(A.3)

In some more detail

$$\frac{1}{3!} E^A E^B E^C \Lambda B^3_{CBA} = \frac{1}{3!} E^A E^B E^C \left( B^3_{CBA} + 3 \mathcal{D}_C \Lambda_{BA} + 3 T_{CB} F^{FA} \Lambda_{FA} \right),$$

(A.4)

or

$$\Lambda B^3_{CBA} = B^3_{CBA} + \oint_{CBA} \left( \mathcal{D}_C \Lambda_{BA} + T_{CB} F^{FA} \Lambda_{FA} \right),$$

(A.5)

where the graded cyclic sum is defined as

$$\oint_{CBA} = CBA + (-)^{c(b+a)} BAC + (-)^{a(b+c)} ACB,$$

(A.6)

with $a = 0$ for vectorial and $a = 1$ for spinorial values of the superspace indices.

The covariant fieldstrength

$$\Sigma = dB^3,$$

(A.7)

is a 4-form in superspace,

$$\Sigma = \frac{1}{4!} E^A E^B E^C E^D \Sigma_{DCBA},$$

(A.8)

with coefficients

$$\frac{1}{4!} E^A E^B E^C E^D \Sigma_{DCBA} = \frac{1}{4!} E^A E^B E^C E^D \left( 4 \mathcal{D}_D B^3_{CBA} + 6 T_{DC} F^F B^3_{FBA} \right).$$

(A.9)
A.2 Explicit solution of the constraints

We recall the superspace constraints for the 3-form gauge potential:

\[ \Sigma_{\delta \gamma \beta A} = 0. \]  

(A.10)

In a first step we solve

\[ \Sigma_{\delta \gamma \beta A} = 0, \]  

(A.11)

by

\[ B^{3 \gamma \beta A} = D_A U_{\gamma \beta} + \oint_{\gamma \beta} \left( D_A U_{\beta A} + T_{A \gamma} F_{F \beta} \right), \]  

(A.12)

and the complex conjugate

\[ \Sigma^{\hat{\delta} \hat{\gamma} \hat{\beta} A} = 0, \]  

(A.13)

by

\[ B^{3 \hat{\gamma} \hat{\beta} A} = D_A V^{\hat{\gamma} \hat{\beta}} + \oint^{\hat{\gamma} \hat{\beta}} \left( D_A V^{\hat{\beta} A} + T_{A \hat{\gamma}} F_{F \hat{\beta}} \right). \]  

(A.14)

Since the prepotentials \( U_{\beta A} \) and \( V^{\hat{\beta} A} \) should reproduce the gauge transformations of the gauge potentials \( B^{3 \gamma \beta A} \) and \( B^{3 \hat{\gamma} \hat{\beta} A} \) we assign

\[ U_{\beta A} \mapsto U_{\beta A} + \Lambda_{\beta A}, \]  

(A.15)

and

\[ V^{\hat{\beta} A} \mapsto V^{\hat{\beta} A} + \Lambda^{\hat{\beta} A}, \]  

(A.16)

as gauge transformation laws for the prepotentials. On the other hand, the so-called pregauge transformations are defined as the zero-modes of the gauge potentials themselves, that is transformations which leave \( B^{3 \gamma \beta A} \) and \( B^{3 \hat{\gamma} \hat{\beta} A} \) invariant. They are given as

\[ U_{\beta A} \mapsto U_{\beta A} + D_{\beta \chi A} - (-)^a D_{A \chi \beta} + T_{\beta A} F_{\chi F}, \]  

(A.17)

and

\[ V^{\hat{\beta} A} \mapsto V^{\hat{\beta} A} + D^{\hat{\beta} \psi A} - (-)^a D_{A \psi \hat{\beta}} + T^{\hat{\beta} A} F_{\psi F}. \]  

(A.18)

We parametrize the prepotentials now as follows:

\[ U_{\beta \hat{\alpha}} = W_{\beta \hat{\alpha}} + T_{\beta \hat{\alpha} f} K_f, \]  

(A.19)

\[ V_{\alpha \hat{\beta}} = W_{\alpha \hat{\beta}} - T_{\alpha \hat{\beta} f} K_f, \]  

(A.20)

and

\[ U_{\beta a} = W_{\beta a} - D_{\beta K_a}, \]  

(A.21)

\[ V_{\hat{\beta} a} = W_{\hat{\beta} a} + D^{\hat{\beta} K_a}. \]  

(A.22)
Explicit substitution shows that the $K_a$ terms drop out in $B^3_{\gamma \beta A}$ and $B^{3\dot{\gamma}\dot{\beta}}_A$. Denoting furthermore

$$U_{\beta\alpha} = W_{\beta\alpha}, \quad \text{and} \quad V^{\dot{\beta}\dot{\alpha}} = W^{\dot{\beta}\dot{\alpha}}, \quad (A.23)$$

we arrive at

$$B^3_{\gamma \beta A} = \mathcal{D}_A W_{\gamma \beta} + \oint_{\gamma \beta} \left( \mathcal{D}_\gamma W_{\beta A} + T_A F_{\gamma \beta} \right), \quad (A.24)$$
$$B^{3\dot{\gamma}\dot{\beta}}_A = \mathcal{D}_A W^{\dot{\gamma}\dot{\beta}} + \oint_{\dot{\gamma}\dot{\beta}} \left( \mathcal{D}_{\dot{\gamma}} W^{\dot{\beta}A} + T_A F_{\dot{\gamma} \dot{\beta}} \right), \quad (A.25)$$

i.e. a pure gauge form for the coefficients $B^3_{\gamma \beta A}$ and $B^{3\dot{\gamma}\dot{\beta}}_A$ with the 2-form gauge parameter $\Lambda$ replaced by the prepotential 2-form

$$W = \frac{1}{2} E^A E^B W_{BA}, \quad \text{with} \quad W_{ba} = 0. \quad (A.26)$$

We take advantage of this fact to perform a redefinition of the 3-form gauge potentials, which has the form of a gauge transformation, in the following way:

$$\hat{B}^3 := - W B^3 = B^3 - dW. \quad (A.27)$$

This leaves the fieldstrength invariant and leads in particular to

$$\hat{B}^3_{\gamma \beta A} = 0, \quad \text{and} \quad \hat{B}^{3\dot{\gamma}\dot{\beta}}_A = 0, \quad (A.28)$$

whereas the coefficient $B^3_{\gamma \dot{\beta} a}$ is replaced by

$$\hat{B}^3_{\gamma \dot{\beta} a} = B^3_{\gamma \dot{\beta} a} - \mathcal{D}_\gamma W^{\dot{\beta} a} - \mathcal{D}^{\dot{\beta}} W_{\gamma a} - \mathcal{D}_a W_{\gamma \dot{\beta}}. \quad (A.29)$$

We define the tensor decomposition

$$\hat{B}^3_{\gamma \dot{\beta} a} = T_{\gamma \dot{\beta} f} \left( \eta_{fa} \Omega + \hat{W}_{[fa]} + \hat{\Omega}_{(fa)} \right), \quad (A.30)$$

where $\hat{W}_{[fa]}$ is antisymmetric and $\hat{\Omega}_{(fa)}$ symmetric and traceless, and perform another redefinition which has again the form of a gauge transformation, this time of parameter

$$\hat{W} = \frac{1}{2} E^a E^b \hat{W}_{[ba]}, \quad (A.31)$$

such that

$$\Omega := - W \hat{B}^3 = \hat{B}^3 - d\hat{W}. \quad (A.32)$$

Note that this reparametrisation leaves $\hat{B}^3_{\gamma \beta A}$ and $\hat{B}^{3\dot{\gamma}\dot{\beta}}_A$ untouched, they remain zero.

Let us summarize the preceding discussion: we started out with the 3-form gauge potential $B^3$. The constraints on its fieldstrength led us to introduce prepotentials. By means of prepotential dependent redefinitions of $B^3$, which have the form of gauge transformations (and which,
therefore, leave the fieldstrength invariant), we arrived at the representation of the 3-form gauge potential in terms of \( \Omega \), with the particularly nice properties

\[
\Omega_{\gamma\beta A} = 0, \quad \Omega^{\dot{\gamma}\dot{\beta} A} = 0,
\]

and

\[
\Omega^{\dot{\gamma} A}_{\gamma \beta} = T^{\dot{\gamma} \beta f}_\gamma \left( \eta_{fa} \Omega + \tilde{\Omega}(fa) \right),
\]

Clearly, in this representation, calculations simplify considerably. We shall, therefore, from now on pursue the solution of the constraints in terms of \( \Omega \) and turn to the equation

\[
\Sigma_{\delta \gamma} \delta_{\dot{\gamma} \dot{\beta} A} = 0 = \oint_{\delta \gamma} \delta_{\dot{\gamma} \dot{\beta} A} T^{\dot{\gamma} \beta f}_\delta \Omega_{\dot{\gamma} \beta A},
\]

which tells us simply that \( \tilde{\Omega}(ba) \) is zero. Hence,

\[
\Omega^{\dot{\gamma} A}_{\gamma \beta} = T^{\dot{\gamma} \beta A}_\gamma \Omega.
\]

We turn next to the constraints

\[
\Sigma_{\delta} \delta_{\dot{\gamma} \dot{\beta} A} = 0 = \oint_{\delta \gamma} \delta_{\dot{\gamma} \dot{\beta} A} \left( \mathcal{D}_{\delta} \Omega_{\dot{\gamma} \dot{\beta} A} + T^{\dot{\gamma} \beta f}_{\delta} \Omega_{\dot{\gamma} \beta A} \right),
\]

and

\[
\Sigma^{\dot{\gamma} \dot{\beta} a}_{\gamma \beta} = 0 = \oint_{\gamma \beta} \delta_{\dot{\gamma} \dot{\beta} a} \left( \mathcal{D}_{\gamma} \Omega^{\dot{\gamma} \dot{\beta} a} + T^{\dot{\gamma} \beta f}_{\gamma} \Omega^{\dot{\gamma} \beta a} \right),
\]

which, after some straightforward spinorial index gymnastics give rise to

\[
\Omega_{\gamma ba} = 2(\sigma_{ba})\Omega_{\gamma \varphi} \mathcal{D}_{\varphi} \Omega,
\]

\[
\Omega^{\dot{\gamma} b a} = 2(\bar{\sigma}_{ba})\Omega_{\dot{\gamma} \varphi} \mathcal{D}_{\varphi} ^{\dot{\gamma}} \Omega.
\]

This completes the discussion of the solution of the constraints, we discuss next the consequences of this solution for the remaining components in \( \Sigma \) i.e. \( \Sigma_{\delta_{\gamma} ba}, \Sigma_{\delta_{cb} a} \) and \( \Sigma_{dcba} \). As a first step we consider

\[
\Sigma_{\delta_{\gamma} ba} = \oint_{\delta \gamma} \left( \mathcal{D}_{\delta} \Omega_{\gamma ba} - T^{\dot{\gamma} \beta f}_{\delta} \Omega_{\dot{\gamma} \beta a} + T^{\dot{\gamma} \beta f}_{\gamma} \Omega_{\dot{\gamma} \beta a} \right),
\]

and

\[
\Sigma_{\delta_{\gamma} ba} = \oint_{\delta \gamma} \left( \mathcal{D}_{\delta} \Omega^{\dot{\gamma} b a} - T^{\dot{\gamma} \beta f}_{\delta} \Omega^{\dot{\gamma} \beta a} + T^{\dot{\gamma} \beta f}_{\gamma} \Omega^{\dot{\gamma} \beta a} \right).
\]

Substituting for the 3-form gauge potentials as determined so far, and making appropriate use of the supergravity Bianchi identities yields

\[
\Sigma_{\delta_{\gamma} ba} = -2(\sigma_{ba} \epsilon)\delta_{\gamma} \left( D^2 - 8 R^{\dot{\gamma}} \right) \Omega,
\]
and
\[ \Sigma^{\dot{\gamma}}_{\dot{\alpha} \dot{\beta} a} = -2(\bar{\sigma}_{ba} \epsilon)^{\dot{\gamma}} \left( \bar{D}^2 - 8 \bar{R} \right) \Omega, \] (A.44)

The appearance of the chiral projection operators suggests to define
\[ S = -4 \left( D^2 - 8 R \right) \Omega, \] (A.45)
\[ T = -4 \left( \bar{D}^2 - 8 \bar{R} \right) \Omega. \] (A.46)

The gauge invariant superfields \( S \) and \( T \) have chirality properties
\[ D_{\alpha} S = 0, \quad D^{\dot{\alpha}} T = 0, \] (A.47)
and we obtain
\[ \Sigma^{\dot{\gamma}}_{\dot{\alpha} \dot{\beta} a} = \frac{1}{2} (\sigma_{ba} \epsilon)_{\dot{\alpha} \dot{\beta}} S, \] (A.48)
\[ \Sigma^{\dot{\gamma}}_{\dot{\alpha} \dot{\beta} a} = \frac{1}{2} (\bar{\sigma}_{ba} \epsilon)^{\dot{\alpha} \dot{\beta}} T. \] (A.49)

In the next step we observe that, due to the information extracted so far from the solution of the constraints, the fieldstrength
\[ \Sigma^{\dot{\gamma}}_{\dot{\alpha} \dot{\beta} a} = \Sigma_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}, \] (A.50)
is determined such that \( \Sigma_{c_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}} \) is totally antisymmetric in its three vectorial indices. As, in its explicit definition a linear term appears (due to the constant torsion term), \( i.e. \)
\[ \Sigma^{\dot{\gamma}}_{\dot{\alpha} \dot{\beta} a} = T^{\dot{\gamma} \dot{\alpha} \dot{\beta} c} \Omega_{c_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}} + \text{derivative and other torsion terms}, \] (A.51)
we can absorb \( \Sigma_{c_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}} \) in a modified 3-form gauge potential
\[ \Omega_{c_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}} = \Omega_{c_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}} - \Sigma_{c_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}}, \] (A.52)
such that the corresponding modified fieldstrength vanishes:
\[ \Sigma^{\dot{\gamma}}_{\dot{\alpha} \dot{\beta} a} = 0. \] (A.53)

The outcome of this equation is then the relation
\[ (\lbrack D_{\alpha}, D^{\dot{\alpha}}] - 4 G^{\alpha_{\dot{\dot{a}}} \dot{\dot{b}}} \rbrack \Omega = \frac{1}{3} \sigma_{\alpha_{\dot{\dot{a}}} \dot{\dot{b}}} \epsilon_{\dot{\dot{c}}_{\dot{\dot{d}} \dot{\dot{e}}}} \Omega_{c_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}}, \] (A.54)
which identifies \( \Omega_{c_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}} \) in the superfield expansion of the unconstrained prepotential \( \Omega \).

Working, from now on, in terms of the modified quantities, the remaining coefficients, at canonical dimensions 3/2 and 2, \( i.e. \) \( \Sigma_{c_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}} \) and \( \Sigma_{d_{\dot{\gamma} \dot{\alpha} \dot{\beta} a}} \), respectively, are quite straightforwardly
obtained in terms of spinorial derivatives of the basic gauge invariant superfields $S$ and $T$. To be more precise, at dimension $3/2$ one obtains
\begin{align}
\Sigma_{\delta cba} &= -\frac{1}{16} \sigma^d_{\delta \delta} \epsilon_{dcba} D^\delta S, \\
\Sigma^\delta_{cba} &= +\frac{1}{16} \bar{\sigma}^{d \delta} \epsilon_{dcba} D_\delta T,
\end{align}
and the Bianchi identity at dimension 2 takes the simple form
\begin{equation}
\left(D^2 - 24 R^\dagger\right) T - \left(D^2 - 24 R\right) S = \frac{8}{3} \epsilon_{dcba} \Sigma_{dcba}. \tag{A.57}
\end{equation}

As to the gauge structure of the 3-form gauge potential we note that in the transition from $B^3$ to $\Omega$, the original 2-form gauge transformations have disappeared, $\Omega$ is invariant under those. In exchange, however, as already mentioned earlier, $\Omega$ transforms under so-called pregauge transformations (which, in turn, leave $B^3$ unchanged. As a result, the residual pregauge transformations of the unconstrained prepotential superfield,
\begin{equation}
\Omega \mapsto \Omega' = \Omega + \lambda, \tag{A.58}
\end{equation}
are parametrized in terms of a linear superfield $\lambda$ which satisfies
\begin{align}
\left(D^2 - 8 R^\dagger\right) \lambda &= 0, \\
\left(D^2 - 8 R\right) \lambda &= 0. \tag{A.59}
\end{align}
In turn, $\lambda$ can be expressed in terms of an unconstrained superfield, as we know from the explicit solution of the superspace constraints of the 2-form gauge potential, actually defining the linear superfield geometrically. In other words, the pregauge transformations should respect the particular form of the coefficients of the 3-form $\Omega$.

### A.3 Yang-Mills in $U_K(1)$ superspace

In section 2.5 we have presented supersymmetric Yang-Mills theory in terms of a Lie-algebra valued gauge potential
\begin{equation}
A = E^A A^{(r)}_A T^{(r)}_r = \mathcal{A}^{(r)} T^{(r)}, \tag{A.60}
\end{equation}
whose fieldstrength
\begin{equation}
\mathcal{F} = dA + AA, \tag{A.61}
\end{equation}
is a 2-form in superspace defined as
\begin{equation}
\mathcal{F} = \frac{1}{2} E^A E^B \mathcal{F}_{BA}, \tag{A.62}
\end{equation}
with coefficients
\begin{equation}
\mathcal{F}_{BA} = D_B A_A - (-)^a b D_A A_B - (A_B, A_A) + T_{BA}^C A_C. \tag{A.63}
\end{equation}
The field strength is covariant with respect to superfield gauge transformations

\[ \mathcal{A} \rightarrow g \mathcal{A} = g^{-1} \mathcal{A} g - g^{-1} d g. \quad (A.64) \]

In the present subsection we point out in some detail the solution of the constraints in terms of prepotentials. In a first step we observe that the constraints

\[ F_{\bar{\beta} \alpha} = 0, \quad F_{\dot{\beta} \dot{\alpha}} = 0, \quad (A.65) \]

have the solution

\[ \mathcal{A}_{\alpha} = -T^{-1} D_{\alpha} T, \quad \mathcal{A}^{\dot{\alpha}} = -\mathcal{U}^{-1} D_{\dot{\alpha}} \mathcal{U}. \quad (A.66) \]

We use here \( D_{\underline{\alpha}} = E_{\underline{\alpha}} M \partial_{M} \). The unconstrained prepotential superfields \( \mathcal{T}, \mathcal{U} \) are related through

\[ \mathcal{T}^{-1} = \mathcal{U}^{\dagger}. \quad (A.67) \]

The gauge transformations of the prepotentials should be defined such that they reproduce those of the gauge potentials as given above. On the other hand, there are additional non-trivial gauge transformations which act on the prepotentials but which do not show up in those of the potentials. These transformations are called pregauge transformations. Altogether, gauge and pregauge transformations of the prepotentials are defined as follows:

\[ \mathcal{T} \rightarrow \bar{\Lambda}^{-1} \mathcal{T} g, \quad \mathcal{U} \rightarrow \Lambda^{-1} \mathcal{U} g, \quad (A.68) \]

the parameters of the pregauge transformations being chiral resp. antichiral superfields, \( i.e. \)

\[ D_{\dot{\alpha}} \Lambda = 0, \quad D_{\underline{\alpha}} \bar{\Lambda} = 0. \quad (A.69) \]

We emphasize that the combination

\[ \Upsilon = \mathcal{T} \mathcal{U}^{-1}, \quad (A.70) \]

is inert under the \( g \)-transformations and changes under pregauge transformations as

\[ \Upsilon \rightarrow \bar{\Lambda}^{-1} \Upsilon \Lambda. \quad (A.71) \]

The constraint

\[ F_{\bar{\beta} \dot{\alpha}} = 0, \quad (A.72) \]

is a so-called \textit{conventional} one, it expresses the vector component of the superspace gauge potential in terms of the spinorial ones:

\[ \mathcal{A}_{\alpha \dot{\alpha}} = \frac{1}{2} (D_{\alpha} \mathcal{A}_{\dot{\alpha}} + D_{\dot{\alpha}} \mathcal{A}_{\alpha} - \{ \mathcal{A}_{\alpha}, \mathcal{A}_{\dot{\alpha}} \}). \quad (A.73) \]

As usual in the explicit definition of the field strength, the derivatives are covariant only with respect to local Lorentz und \( U_{K}(1) \) transformations.

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The component fields of supersymmetric Yang-Mills theory are the usual gauge potentials as identified in the real representation,

\[ A|| = i dx^m a_m(x), \]  

(A.74)

the covariant ”gaugino” fields

\[ \mathcal{W}_\beta| = i \bar{\lambda}_\beta(x), \quad \mathcal{W}_\dot{\beta}| = -i \lambda_{\dot{\beta}}(x), \]  

(A.75)

and the auxiliary bosonic field

\[ \mathcal{D}^a W_\alpha| = -2 \mathcal{D}(x), \]  

(A.76)

all of them defined in the real representation as well. As to the last one, by a slight abuse of notation, we use the same symbol for the superfield itself and its lowest component.

The appearance of the prepotential superfields and the particular form of their gauge transformations allows to dispose completely of the original \( g \) gauge transformations. This is achieved by prepotential dependent redefinitions of the gauge potential \( A \) which have the form of a gauge transformation. In more technical terms, we define

\[ \varphi = UAU^{-1} - UdU^{-1} = U^{-1} A, \]  

(A.77)

\[ \bar{\varphi} = TAT^{-1} -TdT^{-1} = T^{-1} A. \]  

(A.78)

The new gauge potentials are inert under \( g \) gauge transformations but change under pregauge transformations, as induced from the redefinitions, \( i.e. \)

\[ \varphi \mapsto \Lambda \varphi = \Lambda^{-1} \varphi \Lambda - \Lambda^{-1} d \Lambda, \]  

(A.79)

\[ \bar{\varphi} \mapsto \bar{\Lambda} \bar{\varphi} = \bar{\Lambda}^{-1} \bar{\varphi} \bar{\Lambda} - \bar{\Lambda}^{-1} d \bar{\Lambda}, \]  

(A.80)

We call \( \bar{\varphi} \) the chiral and \( \varphi \) the antichiral representation because the corresponding gauge transformations are parametrized in terms of chiral resp. antichiral superfields. Also, \( A \) is called the real representation. The gauge potentials in the chiral and antichiral representations are related by

\[ \varphi = \Upsilon^{-1} \bar{\varphi} \Upsilon - \Upsilon^{-1} d \Upsilon = \Upsilon \bar{\varphi}. \]  

(A.81)

The connections \( \varphi = E^A \varphi_A \) and \( \bar{\varphi} = E^A \bar{\varphi}_A \) depend in a very simple way on \( \Upsilon \):

\[ \varphi_\alpha = -\Upsilon^{-1} D_\alpha \Upsilon, \quad \varphi^\dot{\alpha} = 0, \quad \varphi_{\alpha\dot{\alpha}} = \frac{i}{2} D_{\alpha\dot{\alpha}} \varphi, \]  

(A.82)

\[ \bar{\varphi}_\alpha = 0, \quad \bar{\varphi}^{\dot{\alpha}} = -\Upsilon D^{\dot{\alpha}} \Upsilon^{-1}, \quad \bar{\varphi}_{\alpha\dot{\alpha}} = \frac{i}{2} D_{\alpha\dot{\alpha}} \bar{\varphi}. \]  

(A.83)

Likewise, for the gaugino superfields one finds immediately

\[ \mathcal{W}_\alpha(\varphi) = -\frac{1}{8} (\mathcal{D}^2 - 8 R) \varphi_\alpha, \]  

(A.84)

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\[
\dot{W}^{\dot{a}}(\tilde{\varphi}) = \frac{1}{2} \left( \mathcal{D}^2 - 8\mathcal{R}^I \right) \tilde{\varphi}^{\dot{a}}.
\] (A.85)

Clearly, gauge invariant expressions are independent of the representation chosen to describe the gauge potentials.

### A.4 Chern-Simons forms in superspace

In this paper we deal with Chern-Simons forms of the Yang-Mills and gravitational types. Under gauge transformations these Chern-Simons 3-forms change by the exterior derivative of a 2-form, which depends on the gauge parameter and the gauge potential.

Due to this property one may view the Chern-Simons form as a special case of a generic 3-form gauge potential, as discussed in the first two subsections of this appendix. This point of view is particularly useful for the supersymmetric case.

To make this point as clear as possible we first recall, in this subsection, some general properties of Chern-Simons forms in superspace.

To begin with we consider two gauge potentials \( A_0 \) and \( A_1 \) in superspace. Their field-strength-squared invariants are related through

\[
\text{tr} (F_0 F_0) - \text{tr} (F_1 F_1) = dQ(A_0, A_1).
\] (A.86)

This is the superspace version of the Chern-Simons formula, where

\[
F_0 = dA_0 + A_0 A_0, \quad F_1 = dA_1 + A_1 A_1.
\] (A.87)

On the right appears the superspace Chern-Simons form,

\[
Q(A_0, A_1) = 2 \int_0^1 dt \text{ tr } \{(A_0 - A_1) F_t\},
\] (A.88)

where

\[
F_t = dA_t + A_t A_t,
\] (A.89)

is the fieldstrength for the interpolating gauge potential

\[
A_t = (1 - t)A_0 + tA_1.
\] (A.90)

The Chern-Simons form is antisymmetric in its arguments, \( i.e. \)

\[
Q(A_0, A_1) = -Q(A_1, A_0).
\] (A.91)

In the particular case \( A_0 = A, A_1 = 0 \), one obtains

\[
Q(A) := Q(A, 0) = \text{ tr } \left( AF - \frac{1}{3} AAA \right),
\] (A.92)
We shall also make use of the identity
\[ Q(A_0, A_1) + Q(A_1, A_2) + Q(A_2, A_0) = d\chi(A_0, A_1, A_2), \] (A.93)
with
\[ \chi(A_0, A_1, A_2) = \text{tr}(A_0 A_1 + A_1 A_2 + A_2 A_0). \] (A.94)

This last relation (the so-called triangular equation) is particularly useful for the determination of the gauge transformation of the Chern-Simons form. The argument goes as follows: first of all, using the definition given above, one observes that
\[ Q(gA, 0) = Q(A, dg^{-1}g). \] (A.95)

Combining this with the triangular equation for the special choices
\[ A_0 = 0, \quad A_1 = A, \quad A_2 = dg^{-1}, \] (A.96)
one obtains
\[ Q(0, A) + Q(gA, 0) + Q(dg^{-1}, 0) = d\text{tr}(A dg^{-1}), \] (A.97)
or, using the antisymmetry property
\[ Q(gA) - Q(A) = d\text{tr}(A dg^{-1}) - Q(dg^{-1}). \] (A.98)

The last term in this equation is an exact differential form in superspace as well, it can be written as
\[ Q(dg^{-1}) = d\sigma, \] (A.99)
where the 2-form \( \sigma \) is defined as
\[ \sigma = \int_0^1 dt \text{tr}(\partial_t g_t g_t^{-1} g_t g_t^{-1} g_t g_t^{-1}), \] (A.100)
with the interpolating group element \( g_t \) parametrized such that for \( t \in [0, 1] \)
\[ g_0 = 1, \quad g_1 = g. \] (A.101)

This shows that the gauge transformation of the Chern-Simons form, which is a 3-form in superspace, is given as the exterior derivative of a 2-form,
\[ Q(gA) - Q(A) = d\Delta(g, A), \] (A.102)
with \( \Delta = \chi - \sigma. \)

The discussion so far was quite general and valid for some generic gauge potential. It does not only apply to the Yang-Mills case but to gravitational Chern-Simons forms as well.
A.5 The Chern-Simons superfield

We specialize here to the Yang-Mills case, i.e. we shall now take into account the covariant constraints on the fieldstrength, which define supersymmetric Yang-Mills theory.

It is the purpose of the present subsection to elucidate the relation between the unconstrained prepotential, which arises in the constrained 3-form geometry, and the Chern-Simons superfield. Moreover, based on this observation and on the preceding subsections we present a geometric construction of the explicit form of the Yang-Mills Chern-Simons superfield in terms of the unconstrained prepotential of supersymmetric Yang-Mills theory.

In this construction of the Chern-Simons superfield we will combine the knowledge acquired in the discussion of the 3-form gauge potential with the special features of Yang-Mills theory in superspace.

Recall that the Chern-Simons superfield \( \Omega^{(YM)} \) is identified in the relations

\[
\begin{align*}
\text{tr} \left( W_\alpha W^\alpha \right) &= \frac{1}{2} \left( D^2 - 8 R^\dagger \right) \Omega^{(YM)}, \\
\text{tr} \left( W^\alpha W_\alpha \right) &= \frac{1}{2} \left( \bar{D}^2 - 8 R \right) \Omega^{(YM)}.
\end{align*}
\]

The appearance of one and the same superfield under the projectors reflects the fact that the gaugino superfields \( W_\alpha \) are not only subject to the chirality constraints (2.81) but satisfy the additional condition (2.82). It is for this reason that the Chern-Simons form can be so neatly embedded in the geometry of the 3-form.

As explained in section 3 the terms on the left hand side are located in the superspace 4-form

\[
\Sigma^{(YM)} = \text{tr}(FF).
\]

Of course, the constraints on the Yang-Mills fieldstrength induces special properties on the 4-form coefficients, in particular

\[
\Sigma_{\gamma \beta A}^{(YM)} = 0,
\]

which is just the same tensorial structure as the constraints on the fieldstrength of the 3-form gauge potential. As a consequence the Chern-Simons geometry can be regarded as a special case of that of the 3-form gauge potential. Keeping in mind this fact we obtain

\[
\begin{align*}
\Sigma_{\delta \gamma b a}^{(YM)} &= \frac{1}{2} (\sigma_{ba} \epsilon)_{\delta \gamma} S^{(YM)}, \\
\Sigma_{\delta \gamma b a}^{(YM)} &= \frac{1}{2} (\bar{\sigma}_{ba} \epsilon)_{\delta \gamma} T^{(YM)},
\end{align*}
\]

with

\[
\begin{align*}
S^{(YM)} &= -8 \text{tr}(W_\alpha W^\alpha), \\
T^{(YM)} &= -8 \text{tr}(W^\alpha W_\alpha).
\end{align*}
\]
These facts imply the existence and provide a method for the explicit construction of the Chern-Simons superfield: comparison of these equations with those obtained earlier in the 3-form geometry clearly suggests that the Chern-Simons superfield $\Omega^{(YM)}$ will be the analogue of the unconstrained prepotential superfield $\Omega$ of the 3-form. In order to establish this correspondence in full detail we translate the procedure developed in the case of the 3-form geometry to the Chern-Simons form which, as a 3-form in superspace, has the decomposition

$$Q = \frac{1}{3!}dz^Kdz^Ldz^M Q_{MLK} = \frac{1}{3!}E^A E^B E^C Q_{CBA}, \quad (A.111)$$

with

$$\frac{1}{3!}E^A E^B E^C Q_{CBA}(A) = \frac{1}{3!}E^A E^B E^C \text{tr} (3A_C \mathcal{F}_{BA} - 2A_C A_B A_A). \quad (A.112)$$

In order to extract the explicit form of the Chern-Simons superfield we shall now exploit the equation

$$\text{tr}(\mathcal{F}\mathcal{F}) = dQ(A). \quad (A.113)$$

In the 3-form geometry we know unambiguously the exact location of the prepotential in superspace geometry. Since we have identified Chern-Simons as a special case of the 3-form, it is now rather straightforward to identify the Chern-Simons superfield following the same strategy.

To this end we recall that the prepotential was identified after certain field dependent redefinitions which had the form of a gauge transformation, simplifying considerably the form of the potentials. For instance, the new potentials had the property

$$\Omega_{\gamma\beta A} = 0, \quad \Omega_{\dot{\gamma}\dot{\beta} A} = 0, \quad (A.114)$$

Note, en passant, that these redefinitions are not compulsory for the identification of the unconstrained prepotential. They make, however, the derivation a good deal more transparent.

Can these features be reproduced in the Chern-Simons framework? To answer this question we bring in the particularity of Yang-Mills in superspace, namely the existence of different types of gauge potentials corresponding to the different possible types of gauge resp. pre gauge transformations as described in the previous subsection. These gauge potentials are superspace one-forms denoted by $A$, $\varphi$ and $\bar{\varphi}$ with gauge transformations parametrized in terms of real, chiral and antichiral superfields, respectively. Moreover, the chiral and antichiral sectors are related by a redefinition which has the form of a gauge transformation involving the prepotential superfield $\Upsilon$:

$$\varphi = \Upsilon^{-1} \bar{\varphi} \Upsilon - \Upsilon^{-1}d\Upsilon = \Upsilon \bar{\varphi}. \quad (A.115)$$

Writing the superspace Chern-Simons form in terms of $\varphi$ shows immediately that

$$Q_{\dot{\gamma}\dot{\beta} A}(\varphi) = 0, \quad (A.116)$$
due to $\varphi^\dot{a} = 0$, but
\[ Q_{\gamma\beta A}(\varphi) \neq 0. \] (A.117)

Of course, in the antichiral basis, things are just the other way round, there we have
\[ Q_{\gamma\beta A}(\bar{\varphi}) = 0. \] (A.118)

On the other hand, due to the relation between $\varphi$ and $\bar{\varphi}$ and the transformation law of the Chern-Simons form given above we have
\[ Q(\varphi) - Q(\bar{\varphi}) = d\Delta(\Upsilon, \bar{\varphi}), \] (A.119)

where now the group element is replaced by the prepotential superfield $\Upsilon$. In some more detail, in $\Delta = \chi - \sigma$, we have
\[ \chi = \chi(0, \bar{\varphi}, Y) = \text{tr}(\bar{\varphi} Y), \] (A.120)

where
\[ Y = d\Upsilon \Upsilon^{-1} = E^A Y_A, \] (A.121)

has zero fieldstrength
\[ dY + YY = 0. \] (A.122)

As a 2-form in superspace its coefficients, identified in
\[ \chi = \frac{1}{2} E^A E^B \chi_{BA}, \] (A.123)

are given as
\[ \chi_{BA} = \text{tr} \left( Y_B \bar{\varphi}_A - (-)^{ab} Y_B \bar{\varphi}_A \right). \] (A.124)

For the interpolating prepotential $\Upsilon_t$ we define
\[ Y_t = d\Upsilon_t \Upsilon_t^{-1}, \] (A.125)

to write accordingly
\[ \sigma = \frac{1}{2} E^A E^B \sigma_{BA}, \] (A.126)

with
\[ \sigma_{BA} = \int_0^1 dt \text{tr} \left( \partial_t \Upsilon_t \Upsilon_t^{-1}(Y_t B, Y_t A) \right). \] (A.127)

Consider now
\[ Q_{\gamma\beta A}(\varphi) = \mathcal{D}_A \Delta_{\gamma\beta} + \oint_{\gamma\beta} \left( \mathcal{D}_\gamma \Delta_{\beta A} - (-)^a T_{\gamma A} F_{\Delta F\beta} \right). \] (A.128)

We perform next a redefinition
\[ \hat{Q} := Q(\varphi) - d\Lambda, \] (A.129)

where we determine the 2-form $\Lambda$ in terms of the coefficients of the 2-form $\Delta$ such that
\[ \hat{Q}^\dot{\gamma} \dot{\beta}_A = 0, \] (A.130)
and maintain, at the same time, 
\[ \hat{Q}_{\gamma \beta A} = 0. \]  
(A.131)

This is achieved with the following identification:
\[ \Lambda_{\beta A} = \Delta_{\beta A}, \quad \Lambda_{\beta a} = -\frac{i}{2} D^\beta \Delta_a, \quad \Lambda_{\beta \dot{a}} = 0, \]  
(A.132)

and, for later convenience, we put also
\[ \Lambda_{ba} = \frac{i}{2} (D_b \Delta_a - D_a \Delta_b). \]  
(A.133)

Here \( \Delta_a \) is identified using spinorial notation such that
\[ \Delta_{\gamma \dot{\beta}} = -\frac{i}{2} T_{\gamma \dot{\beta}} \Delta_a. \]  
(A.134)

We have, of course, to perform this redefinition on all the other coefficients, in particular we obtain,
\[ \hat{Q}_{\gamma \dot{\beta} a} = Q_{\gamma \dot{\beta} a} (\varphi) - D^\beta \Xi_{\gamma a}. \]  
(A.135)

In the derivation of this equation one uses the anticommutation relation of spinorial derivatives and suitable supergravity Bianchi identities together with the definition
\[ \Xi_{\gamma a} = \Delta_{\gamma a} + \frac{i}{2} D_{\gamma} \Delta_a \]  
(A.136)

We parametrize
\[ \hat{Q}_{\gamma \dot{\beta} a} = T_{\gamma \dot{\beta} a} \Omega^{(\mathcal{V}, \mathcal{M})} + T_{\gamma \dot{\beta} b} \hat{Q}_{[ba]}^{(\mathcal{V}, \mathcal{M})}, \]  
(A.137)

where we can now identify the explicit form of the Chern-Simons superfield
\[ \Omega^{(\mathcal{V}, \mathcal{M})} = Q(\varphi) - \frac{i}{16} D^a \Xi_{\gamma a}. \]  
(A.138)

The first term is obtained from the spinorial contraction of
\[ Q_{\gamma \dot{\beta} a} (\varphi) = \text{tr} \left( \varphi_\gamma F^{\dot{\beta} a} (\varphi) \right) = -i (\bar{\sigma}_a \epsilon)^{\dot{\beta} \beta} \text{tr} \left( \varphi_\gamma W^\beta (\varphi) \right). \]  
(A.139)

\[ i.e. \]
\[ Q(\varphi) = \frac{i}{16} Q^{\alpha \dot{\alpha}} (\varphi) = -\frac{1}{4} \text{tr} (\varphi^\alpha W_\alpha (\varphi)). \]  
(A.140)

It remains to read off the explicit form of the second term from the definitions above.

In closing we note that a more symmetrical form of the Chern-Simons superfield may be obtained in exploiting the relation
\[ Q_{\gamma \dot{\beta} a} (\varphi) - Q_{\gamma \dot{\beta} a} (\bar{\varphi}) = D_\gamma \Xi^{\dot{\beta} a} + D^\beta \Xi_{\gamma a} + T_{\gamma \dot{\beta} b} \left( \Delta_{ba} + \frac{i}{2} (D_b \Delta_a - D_a \Delta_b) \right), \]  
(A.141)

with
\[ \Xi^{\dot{\beta} a} = \Delta^{\dot{\beta} a} + \frac{i}{2} D^{\dot{\beta}} \Delta_a. \]  
(A.142)

Observe that different appearances of the Chern-Simons superfields should be equivalent modulo linear superfields. To establish the explicit relation of the Chern-Simons superfield presented here and that given by Ferrara et. al. in [23] is left as an exercise.
B Riemann tensor and its squared - notations

This appendix contains some notational information concerning the Riemann tensor, its tensor decomposition, spinor notation and curvature squared combinations.

B.1 Vector notation

The Riemann tensor

\[ R_{dc, ba} \]  \hspace{1cm} (B.1)

is separately antisymmetric in the indices \( d, c \), due to the fact that it is a differential 2-form, and in the indices \( b, a \), because it takes its values in the Lie algebra of the Lorentz group. As a consequence of these symmetry properties there are 36 independent components. But the Riemann tensor is also subject to the Bianchi identities

\[ R_{dc, ba} + R_{cb, da} + R_{bd, ca} = 0, \]  \hspace{1cm} (B.2)

which constitute 16 independent equations, thus reducing to 20 the number of components. An equivalent way to write the Bianchi identities is

\[ R_{dc, ba} = R_{ba, dc}, \quad \varepsilon^{dcba} R_{dc, ba} = 0. \]  \hspace{1cm} (B.3)

The irreducible tensors contained in the Riemann tensor are the curvature scalar, Ricci tensor and the Weyl tensor. We define the contractions

\[ R_{ca} = R_{dc, da}, \quad \mathcal{R} = R_{dc, dc}. \]  \hspace{1cm} (B.4)

The once contracted Riemann tensor gives rise to a symmetric tensor of 10 components,

\[ \mathcal{R}_{ba} = \mathcal{R}_{ab}, \]  \hspace{1cm} (B.5)

whose traceless part

\[ \tilde{\mathcal{R}}_{ba} = \mathcal{R}_{ba} - \frac{1}{6} \eta_{ba} \mathcal{R}, \]  \hspace{1cm} (B.6)

is called called the Ricci tensor, while its trace, \( \mathcal{R} \), is called the curvature scalar. The remaining 10 components are arranged in the Weyl tensor

\[ W_{dc, ba} = R_{dc, ba} - \frac{1}{2} (\eta_{db} \mathcal{R}_{ca} - \eta_{da} \mathcal{R}_{cb} - \eta_{cb} \mathcal{R}_{da} + \eta_{ca} \mathcal{R}_{db}) + \frac{1}{6} (\eta_{db} \eta_{ca} - \eta_{da} \eta_{cb}) \mathcal{R}, \]  \hspace{1cm} (B.7)

which is completely traceless. Altogether we have defined the decomposition of the Riemann tensor

\[ R_{dc, ba} = W_{dc, ba} + \frac{1}{2} \left( \eta_{db} \tilde{\mathcal{R}}_{ca} - \eta_{da} \tilde{\mathcal{R}}_{cb} - \eta_{cb} \tilde{\mathcal{R}}_{da} + \eta_{ca} \tilde{\mathcal{R}}_{db} \right) + \frac{1}{12} (\eta_{db} \eta_{ca} - \eta_{da} \eta_{cb}) \mathcal{R}, \]  \hspace{1cm} (B.8)
in terms of the Weyl tensor, the Ricci tensor and the curvature scalar. The Weyl tensor can be further decomposed into self-dual and anti-self-dual parts,

\[ W_{dc, ba}^{\oplus} = \frac{1}{2} \left( W_{dc, ba} + \frac{i}{2} \varepsilon_{def} W_{fe, ba} \right), \quad (B.9) \]
\[ W_{dc, ba}^{\ominus} = \frac{1}{2} \left( W_{dc, ba} - \frac{i}{2} \varepsilon_{def} W_{fe, ba} \right), \quad (B.10) \]
each consisting of 5 components.

B.2 Spinor notation

The basic definitions for the Riemann tensor are

\[ R_{dc, \beta \dot{\beta} \alpha \dot{\alpha}} = \sigma^b_{\beta \dot{\beta}} \sigma^a_{\alpha \dot{\alpha}} R_{dc, ba}, \quad (B.11) \]

with

\[ R_{dc, \beta \dot{\beta} \alpha \dot{\alpha}} = 2 \epsilon_{\beta \dot{\beta}} R_{dc \beta \alpha} - 2 \epsilon_{\beta \alpha} R_{dc \dot{\beta} \dot{\alpha}}. \quad (B.12) \]

For the 2-form indices \( d, c \) an analogous decomposition holds and one defines altogether

\[ R_{\delta \dot{\delta} \gamma \dot{\gamma}, \beta \dot{\beta} \alpha \dot{\alpha}} = \sigma^d_{\delta \dot{\delta}} \sigma^c_{\gamma \dot{\gamma}} \sigma^b_{\beta \dot{\beta}} \sigma^a_{\alpha \dot{\alpha}} R_{dc, ba}, \quad (B.13) \]

with

\[ R_{\delta \dot{\delta} \gamma \dot{\gamma}, \beta \dot{\beta} \alpha \dot{\alpha}} = 4 \epsilon_{\delta \dot{\delta}} \epsilon_{\gamma \dot{\gamma}} \chi_{\beta \dot{\beta} \alpha \dot{\alpha}} - 4 \epsilon_{\delta \gamma} \epsilon_{\dot{\delta} \dot{\gamma}} \psi_{\beta \dot{\beta} \alpha \dot{\alpha}} - 4 \epsilon_{\delta \gamma} \epsilon_{\dot{\beta} \dot{\gamma}} \psi_{\beta \dot{\beta} \alpha \dot{\alpha}} + 4 \epsilon_{\delta \gamma} \epsilon_{\beta \dot{\alpha}} \chi_{\dot{\delta} \dot{\beta} \alpha \dot{\alpha}} + 4 \epsilon_{\delta \gamma} \epsilon_{\dot{\beta} \dot{\alpha}} \chi_{\delta \dot{\beta} \alpha \dot{\alpha}}, \quad (B.14) \]
and

\[ \chi_{\delta \gamma \beta \dot{\alpha}} = \chi_{\delta \gamma \beta \dot{\alpha}} + (\epsilon_{\delta \beta} \epsilon_{\gamma \alpha} + \epsilon_{\delta \alpha} \epsilon_{\gamma \beta}) \chi, \quad (B.15) \]
\[ \tilde{\chi}_{\delta \gamma \beta \dot{\alpha}} = \tilde{\chi}_{\delta \gamma \beta \dot{\alpha}} + (\epsilon_{\dot{\delta} \beta} \epsilon_{\gamma \dot{\alpha}} + \epsilon_{\dot{\delta} \dot{\alpha}} \epsilon_{\gamma \beta}) \chi. \quad (B.16) \]

The relation of these spinor coefficients with curvature scalar, Ricci and Weyl tensor is easily established. For the curvature scalar one has

\[ \chi = \frac{1}{24} \mathcal{R}. \quad (B.17) \]
For the Ricci tensor we define

\[ \mathcal{R}_{\beta \dot{\beta} \alpha \dot{\alpha}} = \sigma^b_{\beta \dot{\beta}} \sigma^a_{\alpha \dot{\alpha}} \mathcal{R}_{ba}, \quad (B.18) \]
and the like for \( \tilde{\mathcal{R}}_{ba} \). The identification is then

\[ \mathcal{R}_{\beta \dot{\beta} \alpha \dot{\alpha}} = -4 \psi_{\beta \dot{\alpha} \dot{\beta} \alpha} - \frac{1}{2} \epsilon_{\beta \alpha} \epsilon_{\beta \dot{\alpha}} \mathcal{R}, \quad (B.19) \]
\[ \tilde{\mathcal{R}}_{\beta \dot{\beta} \alpha \dot{\alpha}} = -4 \psi_{\beta \dot{\alpha} \dot{\beta} \alpha}. \quad (B.20) \]
For the Weyl tensor we define
\[
W_{\delta \dot{\gamma}, \beta \dot{\alpha} a \dot{\alpha}} = \sigma^d_{\dot{\delta}} \sigma^c_{\dot{\gamma}} \sigma^b_{\beta} \sigma^a_{\alpha} W_{dc, ba},
\] (B.21)
and the same for the self-dual and anti self-dual parts. For the Weyl tensor itself on has then
\[
W_{\delta \dot{\gamma}, \beta \dot{\alpha} a \dot{\alpha}} = 4 \epsilon_{\delta \dot{\gamma}} \epsilon_{\beta \dot{\alpha}} \chi_{\delta \gamma \beta \alpha} + 4 \epsilon_{\delta \dot{\gamma}} \epsilon_{\beta \dot{\alpha}} \bar{\chi}_{\delta \gamma \beta \alpha},
\] (B.22)
whereas for the self-dual and anti self-dual components the corresponding relations are
\[
W_{\delta \dot{\gamma}, \beta \dot{\alpha} a \dot{\alpha}} = 4 \epsilon_{\delta \dot{\gamma}} \epsilon_{\beta \dot{\alpha}} \chi_{\delta \gamma \beta \alpha},
\] (B.23)
\[
W_{\delta \dot{\gamma}, \beta \dot{\alpha} a \dot{\alpha}} = 4 \epsilon_{\delta \dot{\gamma}} \epsilon_{\beta \dot{\alpha}} \bar{\chi}_{\delta \gamma \beta \alpha},
\] (B.24)

### B.3 Curvature-squared combinations

Frequently occurring curvature-squared combinations are
\[
\varepsilon^{dc} R_{dc, f} R_{ba, e} = 2i W^{\oplus dc, ba} W^{\oplus dc, ba} - 2i W^{\ominus dc, ba} W^{\ominus dc, ba},
\] (B.25)
\[
\varepsilon^{dc} \varepsilon^{h g} R_{h g, dc} R_{f e, ba} = -4 W^{dc, ba} W_{dc, ba} + 8 \mathcal{R}^{ba} \mathcal{R}_{ba} - \frac{2}{3} \mathcal{R} \mathcal{R},
\] (B.26)
and
\[
R_{dc, ba} R_{dc, ba} = W^{dc, ba} W_{dc, ba} + 2 \mathcal{R}^{ba} \mathcal{R}_{ba} + \frac{1}{6} \mathcal{R} \mathcal{R}.
\] (B.27)

The first two correspond to 4-form coefficients, while the last one appears in conformal gravity theories. These expressions can be rewritten in terms of the spinor decomposition using the relations
\[
\psi_{\beta \alpha \dot{\beta} \dot{\alpha}} = \frac{1}{4} \mathcal{R}^{ba} \mathcal{R}_{ba},
\] (B.28)
\[
\chi_{\dot{\gamma} \beta \alpha} \chi_{\dot{\gamma} \beta \alpha} = \frac{1}{4} W^{\oplus dc, ba} W^{\oplus dc, ba},
\] (B.29)
\[
\bar{\chi}_{\dot{\gamma} \beta \alpha} \bar{\chi}_{\dot{\gamma} \beta \alpha} = \frac{1}{4} W^{\ominus dc, ba} W^{\ominus dc, ba}.
\] (B.30)

In the language of differential forms the Riemann tensor appears as coefficient of the curvature 2-form
\[
R_{\delta}^{\alpha} = \frac{1}{2} e^{e} e^{d} R_{dc, b}^{\alpha},
\] (B.31)
which takes its values in the Lie algebra of the Lorentz group and has the standard decomposition with respect to \( SL(2, C) \),
\[
R^{\beta} \alpha = \frac{1}{2} e^{e} e^{d} R_{dc}^{\beta} \alpha, \quad R^{\dot{\beta}} \dot{\alpha} = \frac{1}{2} e^{e} e^{d} R_{dc}^{\dot{\beta}} \dot{\alpha}.
\] (B.32)
Correspondingly, for the two curvature-squared combinations one obtains
\[ \frac{1}{2} R_b^{\alpha} R_a^{\beta} = R_\beta^{\alpha} R_\alpha^{\beta} + \tilde{R}_\beta^{\alpha} R_\alpha^{\beta}, \]  
(\text{B.33})
\[ -\frac{i}{4} \varepsilon^{dcba} R_{dc} R_{ba} = R_\beta^{\alpha} R_\alpha^{\beta} - \tilde{R}_\beta^{\alpha} R_\alpha^{\beta}, \]  
(\text{B.34})

We use the notations
\[ \Psi^{(+)} = R_\beta^{\alpha} R_\alpha^{\beta}, \quad \Psi^{(-)} = \tilde{R}_\beta^{\alpha} R_\alpha^{\beta}. \]  
(\text{B.35})

for these 4-forms, their coefficients are defined as
\[ \Psi^{(\pm)} = \frac{1}{4} e^{a} e^{b} e^{c} e^{d} \varepsilon^{dcba} = -\frac{1}{4} v \varepsilon^{dcba} \Psi^{(\pm)} \]  
(\text{B.36})

with \(v\) the fundamental 4-form
\[ v = \frac{1}{4!} e^{a} e^{b} e^{c} e^{d} \varepsilon^{dcba} = e^{0} e^{1} e^{2} e^{3}. \]  
(\text{B.37})

More explicitly, using
\[ \Psi^{(+)} = -\frac{1}{4} v \varepsilon^{dcba} R_{dc} \varepsilon \varphi R_{ba} \varphi, \]  
(\text{B.38})
\[ \Psi^{(-)} = -\frac{1}{4} v \varepsilon^{dcba} R_{dc} \dot{\varphi} R_{ba} \dot{\varphi}, \]  
(\text{B.39})

and taking into account the spinor decomposition gives rise to
\[ \frac{1}{4} \varepsilon^{dcba} R_{dc} \varepsilon \varphi R_{ba} \varphi = +i \chi^{\gamma \beta \alpha} \chi^{\gamma \beta \alpha} - i \psi^{\beta \alpha} \tilde{\psi}^{\beta \alpha} + 12 i \chi^2, \]  
(\text{B.40})
\[ \frac{1}{4} \varepsilon^{dcba} R_{dc} \dot{\varphi} R_{ba} \dot{\varphi} = -i \chi^{\gamma \beta \alpha} \chi^{\gamma \beta \alpha} + i \psi^{\beta \alpha} \tilde{\psi}^{\beta \alpha} - 12 i \chi^2. \]  
(\text{B.41})

Another useful relation is (Gauss-Bonnet)
\[ \varepsilon^{dcba} \left( R_{dc} \varepsilon \varphi R_{ba} \varphi - R_{dc} \dot{\varphi} R_{ba} \dot{\varphi} \right) = \frac{1}{16} \varepsilon^{dcba} \left( R_{dc} \varepsilon^{d'}, \varphi R_{ba} \varphi^{d'} - R_{dc} \dot{\varphi} R_{ba} \dot{\varphi}^{d'} \right) \]  
(\text{B.42})

C Supersymmetry and curvature-squared terms

Supersymmetric curvature-squared terms arise naturally in conformal supergravity \[44, 45\] and in the study of supersymmetric extensions of anomalies and topological invariants \[46, 47\], as well as in attempts at new mechanism of supersymmetry breaking \[48, 49, 50\]. So far these structures have been investigated mostly \[51, 52\] in the absence of supersymmetric matter and
gauge couplings. As is well known, the general supergravity-matter system reveals interesting relations between Kähler phase transformations, super-Weyl rescalings and the normalization of the usual Einstein curvature scalar action, which have a concise geometrical interpretation in the framework of \(U_K(1)\) superspace (which is also suitable for the description of variant supergravity theories \[42\]).

It should therefore be useful to discuss the supersymmetric extensions of curvature-squared terms in a geometrical framework which allows to take care of the matter sector as well. This will be achieved in this appendix in working directly in generic \(U_K(1)\) superspace, which is slightly more general: the supergravity-matter system is obtained from it if one replaces the \(U_K(1)\)-prepotential with the Kähler potential superfield.

But the generic \(U_K(1)\) superspace, giving rise to so-called chirally extended supergravity \[53\], is also interesting in its own case - curvature-squared terms in this context have been investigated in \[54\].

As is well known, curvature-squared component field expressions are identified in the highest components of the products of the basic supergravity superfields \(R^\dagger R\) and \(G^aG_a\) as well as the squared of the Weyl superfield, \(W^\gamma\bar{\alpha}W_\gamma\beta\alpha\), and its complex conjugate. In the presence of the \(U_K(1)\) factor in the structure group, additional terms must be considered, arising from the square of the superfield \(X_\alpha\) and its complex conjugate. Different curvature-squared combinations are then obtained from appropriately chosen linear combinations of these basic superfield products. In principal, the highest components of these superfield products can be obtained through a explicit, though somewhat painful, calculation.

In this appendix we will take advantage of the geometric description in superspace, in particular the covariant decomposition in terms of the 3-form geometry, to present a more systematic construction of supersymmetric completions of curvature-squared terms.

We shall start from the Gauss-Bonnet combination of curvature-squared terms. In our notations it appears in the purely vectorial coefficient of the superspace 4-form (see (3.5))

\[
\Psi^+ - \Psi^- ,
\]

which, in some more detail, is given as (see also appendix B) for the relation between vector and spinor notation of the curvature-squared terms

\[
i_{24} \varepsilon^{dcba} \Psi^+_{dcba} - i_{24} \varepsilon^{dcba} \Psi^-_{dcba} = -\chi_{\delta\gamma\beta\alpha}^{\delta\gamma\beta\alpha} - \chi_{\delta\gamma\beta\alpha}^{\delta\gamma\beta\alpha} + 2 \psi_{\beta\alpha}^{\beta\alpha} - 24 \chi^2 .
\]

On the other hand, from the covariant decomposition established in the main text we have

\[
\varepsilon^{dcba} \Sigma^{\pm}_{dcba} = \varepsilon^{dcba} \left( \Psi^{\pm}_{dcba} - 4 D_d M^{\pm}_{eba} - 6 T_{de} \mathcal{M}^{\pm}_{eba} \right) ,
\]
where the expression on the left is related to the 3-form structure by (see [3.21])

\[
\frac{8i}{3} \varepsilon^{\alpha \beta \alpha} \Sigma^{(\pm)} \,_{\alpha \beta \alpha} = \left( D^2 - 24 R^\dagger \right) T^{\pm} - \left( \bar{D}^2 - 24 R \right) S^{(\pm)}. \tag{C.4}
\]

This shows how the supersymmetric completion of the Gauss-Bonnet combination is identified in the leading term of a supersymmetric chiral density construction. Using the explicit form of the superfields \( T^{(\pm)} \) and \( S^{(\pm)} \) as defined in eqs. (3.89) - (3.92) one obtains

\[
8 \chi \delta_{\beta \alpha} \bar{\chi} \delta_{\beta \alpha} + 8 \bar{\chi} 
\]

\[
- 16 \psi \bar{\psi} + 192 \chi^2 + \text{Div}^{(+)} - \text{Div}^{(-)} =
\]

\[
= -4 \left( D^2 - 24 R^\dagger \right) \left( W\gamma_{\beta \alpha} W_{\gamma \beta \alpha} + \frac{i}{6} X^\alpha X_\alpha \right) - 4 \left( \bar{D}^2 - 24 R \right) \left( W_{\gamma \beta \alpha} W_{\gamma \beta \alpha} + \frac{i}{6} \bar{X}_\alpha \bar{X}^\alpha \right) - \Box^\pm \left( G^a G_a + 2 R^\dagger R \right). \tag{C.5}
\]

Here we subsumed a number of supercovariant derivative and nonlinear terms under the symbols

\[
\text{Div}^{(\pm)} = + \frac{1}{8} \Box^- \left( \mu^{(\pm)} + 8 R^\dagger R + \frac{5}{2} G^a G_a \right) + \frac{i}{3} \varepsilon^{\alpha \beta \alpha} \left( 4 \mathcal{D}_d \bar{M}^{(\pm)} \,_{\alpha \beta \alpha} + 6 T_{dc} \bar{\mathcal{M}}^{(\pm)} \,_{\alpha \beta \alpha} \right), \tag{C.6}
\]

and used the notations

\[
\Box^\pm = \left( D^2 - 24 R^\dagger \right) \left( \bar{D}^2 - 8 R \right) \pm \left( \bar{D}^2 - 24 R \right) \left( D^2 - 8 R^\dagger \right), \tag{C.7}
\]

for the generalized fourth order covariant derivative operators, understood to act on superfields of vanishing \( U_K(1) \) weight. In particular, the combination \( \Box^- \) amounts to a generalized covariant divergence term. In some more detail, its action on a generic superfield \( X \) of vanishing \( U_K(1) \) weight may be written as

\[
\Box^- X = -4 i \mathcal{D}^{\alpha \dot{\alpha}} \left( [\mathcal{D}_\alpha, \mathcal{D}_\dot{\alpha}] - 4 G_{\alpha \dot{\alpha}} \right) X + 32 S^\alpha \mathcal{D}_\alpha X + 32 S_{\dot{\alpha}} \mathcal{D}^{\dot{\alpha}} X. \tag{C.8}
\]

Using this equation together with the explicit form of the coefficients of \( M^{(\pm)} \) in (C.5) one verifies directly that \( \mu^{(\pm)} \) drop out in the expression for \( \text{Div}^{(\pm)} \). This property allows to cast \( \text{Div}^{(\pm)} \) in the form

\[
\text{Div}^{(\pm)} = \frac{i}{3} \varepsilon^{\alpha \beta \alpha} \left( 4 \mathcal{D}_d \bar{M}^{(\pm)} \,_{\alpha \beta \alpha} + 6 T_{dc} \bar{\mathcal{M}}^{(\pm)} \,_{\alpha \beta \alpha} \right), \tag{C.9}
\]

with \( \bar{M}^{(\pm)} \) defined as \( M^{(\pm)} \) evaluated at the special values \( \mu^{(\pm)} = -8 R^\dagger R - \frac{5}{2} G^a G_a \), i.e.

\[
\bar{M}^{(\pm)} = M^{(\pm)} \left( \mu^{(\pm)} = -8 R^\dagger R - \frac{5}{2} G^a G_a \right). \tag{C.10}
\]

In practical calculations it is sometimes more convenient to use this same expression in spinor notation :

\[
\text{Div}^{(\pm)} = -4 i \mathcal{D}^{\alpha \dot{\alpha}} \bar{M}^{(\pm)} \,_{\alpha \dot{\alpha}} + 8 T^{\beta \dot{\alpha}} \bar{\mathcal{M}}^{(\pm)} \,_{\beta \dot{\alpha}} - 8 T^{\dot{\alpha} \dot{\beta}} \bar{\mathcal{M}}^{(\pm)} \,_{\dot{\alpha} \dot{\beta}}. \tag{C.11}
\]
Turning back to eq. (C.5), we observe that it identifies the Gauss-Bonnet combination in the expansion of the basic superfields of $U_K(1)$ superspace\footnote{On the other hand, the description of the Gauss-Bonnet combination in the traditional superspace is obtained from this expression by simply turning off the $U_K(1)$ sector, \textit{i.e.} taking $X_\alpha$ to be zero.} that is a particular combination of the $D$-term of the superfield $G^a G_a + 2 R^i R$ and the $F$-terms of the chiral superfields $W^2$ and $X^2$ and their conjugates. The term $\text{Div}^{(+)} - \text{Div}^{(-)}$, which arises naturally from the geometric construction, is necessary for the supersymmetric completion once the projection to component fields of (C.5) is employed in the chiral density construction as explained in section 2.3 of the main text.

Note also that the particular combination of $W^2$ and $X^2$ occurs in (C.5) in order to ensure the absence of the square of the $U_K(1)$ field strength.

Instead of reading eq. (C.5) as an expression for the leading term of the supersymmetric Gauss-Bonnet combination we shall now turn the argument the other way round and use the same equation to determine the highest superfield component of $G^a G_a + 2 R^i R$. To do so, we make use of the explicit expressions for the $F$-terms of the chiral superfields $W^2$ and $X^2$ and their conjugates.

First of all, the square of the Weyl \textit{spinors} provide the leading terms in the supersymmetric completion of the square of the Weyl \textit{tensor}, as can be seen explicitly from the equations

$$
\left(D^2 - 24 R^i\right)W^{\gamma^a} W_{\gamma^a} = -2 \chi^{\delta^b \gamma^a} \chi_{\delta^b \gamma^a} - \frac{8}{3} f^{\beta_\alpha} f_{\beta_\alpha} + 16 R^i \chi^{\gamma^a} \chi_{\gamma^a} \\
+ 8i \chi^{\gamma^a} \left(\partial_\alpha - i G_\alpha \right) T_{\gamma^a},
$$

(C.12)

and

$$
\left(\bar{D}^2 - 24 R\right) W_{\tilde{\gamma}^a} W_{\tilde{\gamma}^a} = -2 \chi^{\tilde{\beta}^a \tilde{\gamma}^a} \chi_{\tilde{\beta}^a \tilde{\gamma}^a} - \frac{8}{3} f_{\tilde{\beta}^a} f_{\tilde{\beta}^a} + 16 R \chi_{\tilde{\gamma}^a} W_{\tilde{\gamma}^a} \\
+ 8i \chi_{\tilde{\gamma}^a} \left(\partial^a - i G^a \right) T_{\tilde{\gamma}^a}.
$$

(C.13)

Note the presence of the squares of the $U_K(1)$ field strength, \textit{i.e.} the terms $f^{\beta_\alpha} f_{\beta_\alpha}^{\prime}$ and $f_{\tilde{\beta}^a} f_{\tilde{\beta}^a}^{\prime}$ in these equations.

In turn, the $F$-term of $X^2$ and its conjugate are given as

$$
\left(D^2 - 24 R^i\right) X^\alpha X_\alpha = -32 \chi^{\tilde{\beta}^a} \tilde{f}_{\tilde{\beta}^a} + 8i \chi_{\alpha} \partial_{\alpha} \tilde{X}^{\tilde{\alpha}} - \left(\partial^\alpha X_\alpha\right)^2, \tag{C.14}
$$

$$
\left(\bar{D}^2 - 24 R\right) \tilde{X}_{\tilde{\alpha}} \tilde{X}^{\tilde{\alpha}} = -32 \tilde{f}_{\tilde{\beta}^a} \tilde{f}^{\tilde{\beta}^a} + 8i \tilde{X}^{\tilde{\alpha}} \partial_{\alpha} X^\alpha - \left(\partial_{\tilde{\alpha}} X^{\tilde{\alpha}}\right)^2. \tag{C.15}
$$

Notations which intervene here are

$$
\tilde{f}_{\tilde{\beta}^a} = f_{\tilde{\beta}^a} - \frac{3i}{2} g_{\tilde{\beta}^a}, \quad \tilde{f}^{\tilde{\beta}^a} = f^{\tilde{\beta}^a} - \frac{3i}{2} g^{\tilde{\beta}^a}, \tag{C.16}
$$
\[ g_{\beta\alpha} = \frac{1}{4} \left( D_\beta \hat{\phi} G_{\alpha \phi} + D_\alpha \hat{\phi} G_{\beta \phi} \right), \quad g_{\beta\hat{\alpha}} = -\frac{1}{4} \left( D^\phi \hat{\phi} G_{\gamma \hat{\alpha}} + D^\phi \hat{\phi} G_{\delta \phi} \right), \] (C.17)

where \( G_{ba} = D_b G_a - D_a G_b \) has the same standard spinorial decomposition as \( F_{ba} \), that is

\[ G_{\beta\beta \alpha \hat{\alpha}} = 2 \epsilon_{\beta\hat{\alpha}} g_{\beta\alpha} - 2 \epsilon_{\beta\alpha} g_{\beta\hat{\alpha}}, \] (C.18)

and \( \hat{F}_{ba} = F_{ba} - \frac{4}{3} G_{ba} \). Putting all this information together, one finally obtains

\[
\Box^+ \left( G^a G_a + 2 R^i R \right) = \\
-16 \psi_{\beta\alpha \beta\hat{\alpha}} - \psi_{\beta\alpha \beta\hat{\alpha}} + 192 \chi^2 + \left( \text{Div}^+ - \text{Div}^- \right) \\
+32i W^{\gamma \beta \alpha} \left( D_\beta \hat{\alpha} - i G_{\alpha \hat{\alpha}} \right) T_{\gamma \beta \hat{\alpha}} + 64 R^i W^{\gamma \beta \alpha} W_{\gamma \beta \alpha} \\
+32i W^{\gamma \beta \alpha} \left( D_\alpha \hat{\alpha} + i G^\alpha \hat{\alpha} \right) T_{i \beta \alpha} + 64 RW^{\gamma \beta \alpha} W_{\gamma \beta \alpha} \\
-\frac{32}{3} \left( f_{\beta \alpha} f_{\beta \hat{\alpha}} + 2 f_{\beta \alpha} f_{\beta \hat{\alpha}} \right) - \frac{32}{3} \left( f_{\beta \hat{\alpha}} f_{\beta \hat{\alpha}} + 2 f_{\beta \hat{\alpha}} f_{\beta \hat{\alpha}} \right) \\
+16i \frac{X^\alpha D_{\alpha \alpha} X^\alpha + 16i}{3} X^\alpha D_{\alpha \hat{\alpha}} X^\alpha - \frac{4}{3} \left( D^\alpha X^\alpha \right)^2 \] (C.19)

One sees that the squares of the Weyl tensor drop out and we are left with the combination

\[ 4 \psi^{\beta\alpha \beta\hat{\alpha}} - \psi_{\beta\alpha \beta\hat{\alpha}} - 48 \chi^2 = \hat{R}_{ba} \hat{R}_{ba} - \frac{1}{12} R^2 = \hat{R}_{ba} \hat{R}_{ba} - \frac{1}{3} R^2. \] (C.20)

Recall from the previous discussion, that \( \text{Div}^+ - \text{Div}^- \) hides a number of derivative and nonlinear terms which are not very illuminating for the present discussion. Their explicit form may be inferred from the results presented in appendix D, if desired. Finally, we display the contribution arising from \( R^i R \) alone:

\[
\Box^+ R^i R = +\frac{2}{3} \left( \mathcal{R} + D^\alpha X^\alpha \right)^2 - \frac{8}{3} \mathcal{R} \left( G^a G_a + 2 R^i R \right) \\
+16 R^i D^\alpha D_a R + 16 R D^a D_a R^i - 64 i G^\alpha \left( R^i D_a R - R D_a R^i \right) + 8 \left( D^a G_a \right)^2 \\
-8i \sigma_{\alpha \alpha} \left( D^a R D_a D^\alpha R^i + D^\alpha R^i D_a D^\alpha R + 4i D^a R G_a D^\alpha R \right) \\
-\frac{2}{3} \left( G^a G_a + 8 R^i R \right) D^\alpha X^\alpha + 8 \left( G^a G_a - 4 R^i R \right)^2 \\
-8 R D_a R^i D^\alpha R^i - 8 R D^a R D_a R - 24 R X^\alpha D^\alpha R^i - 24 \left( X^\alpha D_a R \right) \] (C.21)

In conclusion, the formulas derived in this appendix provide the starting point for a constructive procedure to describe the supersymmetric completion of any combination of curvature-squared terms by means of the generic chiral density construction.
The covariant decompositions $\Psi^\Delta = \Sigma^\Delta + dM^\Delta$

An important point in our investigation of gravitational Chern-Simons forms was the covariant decomposition $\Psi^\Delta = \Sigma^\Delta + dM^\Delta$, relating the curvature (resp. fieldstrength)-squared 4-form $\Psi^\Delta$ to the geometrical structure of the 3-form multiplet of supersymmetry in the four cases $\Delta \in \{ (+), (-), (1), (YM) \}$. As explained in section 3.1, the components of the 4-forms $\Sigma^\Delta$ reflect the constraint structure of the geometry of the 3-form multiplet in their tensor structure and the nonvanishing components are expressed in terms of the basic covariant superfields and their (covariant) derivatives. Moreover, the difference between the original complete curvature-squared 4-form $\Psi^\Delta$ and the constrained 4-form $\Sigma^\Delta$ can be cast in the form of an exterior superspace derivative of the 3-form $M^\Delta$, which is expressed in terms of the basic covariant superfields and their covariant derivatives as well. The resulting expressions, which are obtained by an explicit calculation in each individual case, are rather involved in particular in the gravitational $(+)$- and $(-)$- sectors. It seems therefore preferable to give a compendium of the corresponding formulae in the four subsections of this appendix.

Although the covariant decomposition have been established by an explicit calculation in each sector separately, there is a number of features they have in common.

First of all, resuming the discussion after eq.(3.7), the components of $\Sigma^\Delta$ reflect the 3-form constraints, i.e.

$$\Sigma^\Delta_{\delta\gamma\beta A} = 0.$$ (D.1)

Furthermore, given these restrictions, the Bianchi identities $d\Sigma^\Delta = 0$ imply a number of consequences for the remaining components. Most importantly it turns out that all the components of $\Sigma^\Delta$ are completely described in terms of two superfields $S^\Delta$ and $T^\Delta$, appearing in

$$\Sigma^\Delta_{\delta\gamma} ba = \frac{1}{2}(\sigma_{ba}\epsilon)_{\delta\gamma} S^\Delta, \quad \Sigma^\Delta_{\delta\gamma} ba = \frac{1}{2}(\bar{\sigma}_{ba}\epsilon)_{\delta\gamma} T^\Delta,$$ (D.2)

which are subject to the chirality conditions

$$D^a S^\Delta = 0, \quad D^\dagger T^\Delta = 0.$$ (D.3)

Furthermore one finds (cf. eqs.(3.17), (3.18) and (3.20)).

$$\Sigma^\Delta_{\delta} cba = -\frac{1}{16} \sigma^d_{\delta\bar{d}} \varepsilon_{deba} D^d S^\Delta, \quad \Sigma^\Delta_{\delta} cba = +\frac{1}{16} \bar{\sigma}^d_{\delta\bar{d}} \varepsilon_{deba} D^\dagger T^\Delta,$$ (D.4)

and

$$2i\Sigma^\Delta = -\frac{1}{32} (D^2 - 8R^l) T^\Delta + \frac{1}{32} (\bar{D}^2 - 8R) S^\Delta,$$ (D.5)

where the boldscript scalar superfields $\Sigma^\Delta$ is defined as

$$\Sigma^\Delta_{deba} = \varepsilon_{deba} \Sigma^\Delta.$$ (D.6)
As already explained above the superfields $S^\Delta$ and $T^\Delta$ have a different form in each sector, but once they are known (and they will be given explicitly below), the 4-form $\Sigma^\Delta$ is completely determined.

Recall also that the component $\Sigma^\Delta_{\hat{\gamma}^c b a}$ is found to have the tensor structure

$$\Sigma^\Delta_{\hat{\gamma}^c b a} = \sigma^b_c \sigma^a_{\hat{\gamma}^c} M^\Delta_{\hat{\gamma}^c b a} = 4 \epsilon_{\hat{\gamma}^c \hat{\gamma}^c} \Sigma^\Delta_{\hat{\gamma}^c b a} - 4 \epsilon_{\gamma^\alpha} \epsilon_{\hat{\gamma}^c} \Sigma^\Delta_{\hat{\gamma}^c b a}. \quad (D.7)$$

In other words this means that $\Sigma^\Delta_{\hat{\gamma}^c b a}$ is completely antisymmetric in the three vector indices $c$, $b$, and $a$. A closer look at the decomposition $\Psi^\Delta = \Sigma^\Delta + dM^\Delta$ shows then that this component appears always in a particular linear combination with $M^\Delta_{cba}$. More precisely, using the decomposition

$$M^\Delta_{\gamma^\beta \hat{\gamma}^c} = \sigma^a_{\hat{\gamma}^c} \sigma^b_{\gamma^\beta} M^\Delta_{cba} = 2 i \epsilon_{\gamma^\beta} \epsilon_{\hat{\gamma}^c} M^\Delta_{\hat{\gamma}^c \beta a} - 2 i \epsilon_{\gamma^\alpha} \epsilon_{\gamma^\beta} M^\Delta_{\gamma^\alpha \beta a}, \quad (D.8)$$

this combination is $M^\Delta_{\hat{\gamma}^c b a} + \Sigma^\Delta_{\hat{\gamma}^c b a}$. As a consequence, one has the freedom to redefine individually $M^\Delta_{\hat{\gamma}^c b a}$ and $\Sigma^\Delta_{\hat{\gamma}^c b a}$, provided their sum remains unchanged. Such special assignments are called *conventional constraints* and one might, for instance absorb $\Sigma^\Delta_{\hat{\gamma}^c b a}$ completely in a redefinition of $M^\Delta_{\hat{\gamma}^c b a}$, such establishing the conventional constraint mentioned in eq. (3.15).

In the remaining part of this preamble we give the definitions of the tensor decompositions of the components of the 3-form $M^\Delta$ which are the same in the four cases. One has

$$M^\Delta_{\gamma^\beta \gamma^\alpha} a = \sigma^a_{\hat{\gamma}^c} M^\Delta_{\gamma^\beta \gamma^\alpha} a, \quad (D.9)$$

and

$$M^\Delta_{\gamma^\alpha \beta \gamma^\alpha} a = 2 \epsilon_{\gamma^\alpha} M^\Delta_{\gamma^\alpha \beta \gamma^\alpha} a - 2 \epsilon_{\gamma^\alpha} M^\Delta_{\gamma^\alpha \beta \gamma^\alpha} a, \quad (D.10)$$

with

$$M^\Delta_{\gamma^\alpha \beta \alpha} = M^\Delta_{\gamma^\alpha \beta \alpha} + \epsilon_{\gamma^\beta} M^\Delta_{\gamma^\alpha \alpha} + \epsilon_{\gamma^\alpha} M^\Delta_{\gamma^\beta \alpha}, \quad (D.11)$$

$$M^\Delta_{\gamma^\alpha \beta \hat{\gamma}^c} = M^\Delta_{\gamma^\alpha \beta \hat{\gamma}^c} + \epsilon_{\gamma^\beta} M^\Delta_{\beta \alpha} + \epsilon_{\gamma^\alpha} M^\Delta_{\beta \hat{\gamma}^c}. \quad (D.12)$$

In the following the explicit expressions for these superfields in the different sectors will be given.

**D.1 \ \Psi^{(+)} = \Sigma^{(+)} + dM^{(+)}**

Although part of these results have already been exposed in section 3, the complete set of expressions is displayed here. For the components of the 3-form $M^{(+)}$ one obtains

$$M^{(+)}_{\gamma^\beta \gamma^\alpha} a = 0, \quad M^{(+)}_{\gamma^\alpha \hat{\gamma}^c} a = 0, \quad (D.13)$$
\[ M^{(+)}_{\gamma \beta \alpha \dot{\alpha}} = -8 R^1 (\epsilon_{\gamma \alpha} G_{\beta \dot{\alpha}} + \epsilon_{\beta \dot{\alpha}} G_{\gamma \alpha}), \]  
\[ M^{(+)}_{\gamma \beta \alpha \dot{\alpha}} = -i \epsilon_{\gamma \alpha} \epsilon_{\beta \dot{\alpha}} \mu^{(+)} - \frac{i}{2} \left( G_{\gamma \beta} G_{\alpha \dot{\alpha}} + G_{\alpha \beta} G_{\gamma \dot{\alpha}} \right), \]  
\text{at dimension } 3/2 \text{ and } 2, \text{ whereas at dimension } 5/2 \text{ the various irreducible components are given as}
\[ M^{(+)}_{\gamma \beta \alpha \dot{\alpha}} = -8 R^1 W_{\gamma \beta \alpha} + \frac{i}{2 R} \int_{\gamma \beta \alpha} G_{\gamma \beta} (D_{\beta} G_{\alpha \dot{\alpha}} + D_{\alpha} G_{\beta \dot{\alpha}}), \]  
\[ 12 M^{(+)}_{\alpha} = -3 D_\alpha \mu^{(+)} - 16 D_\alpha (RR^\dagger) \]  
\[ -8 R^1 D^\phi G_{\alpha \dot{\phi}} - 18 G_{\alpha \dot{\phi}} D^\phi R^\dagger + 2 G_{\alpha \dot{\phi}} D_\alpha G_{\dot{\phi}} + 5 G_{\alpha \dot{\phi}} D_\phi G_{\dot{\phi}}, \]
and
\[ M^{(+)}_{\gamma \beta \alpha \dot{\alpha}} = -4 G_{\gamma \beta \alpha \dot{\alpha}} \]  
\[ M^{(+)}_{\gamma \beta \alpha \dot{\alpha}} = \frac{1}{8} \int_{\gamma \beta \alpha} G_{\gamma \beta} (D_{\beta} G_{\alpha \dot{\alpha}} + D_{\alpha} G_{\beta \dot{\alpha}}), \]  
\[ 4 M^{(+)}_{\dot{\alpha}} = D_{\dot{\alpha}} \mu^{(+)} + 2 G_{\alpha \dot{\phi}} D_\phi R + G_{\alpha \dot{\phi}} D_\phi G_{\dot{\phi}}. \]

The superfields \( S^{(+)} \) and \( T^{(+)} \) are defined as
\[ S^{(+)} = \left( D^2 - 8 R^1 \right) \left( \mu^{(+)} + 16 R^1 R - \frac{13}{4} G_{\alpha \dot{\phi}} G_{\dot{\phi}} \right) - 4 \bar{X} \dot{\phi} \bar{X} \dot{\phi}, \]  
\[ T^{(+)} = \left( D^2 - 8 R \right) \left( \mu^{(+)} + \frac{3}{4} G_{\alpha \dot{\phi}} G_{\dot{\phi}} \right) + 32 W_{\gamma \beta \alpha \dot{\alpha}} + \frac{4}{3} \bar{X} \dot{\phi} \bar{X} \dot{\phi}. \]

For the remaining components at dimension 3 one finds
\[ M^{(+)}_{\alpha \dot{\alpha}} + \Sigma^{(+)}_{\alpha \dot{\alpha}} + \frac{i}{8} \left( [D_{\alpha}, D_{\dot{\alpha}}] - 4 G_{\alpha \dot{\alpha}} \right) \mu^{(+)} = \]  
\[ + \frac{1}{16} G_{\alpha \dot{\phi}} \left( 4 [D_{\dot{\phi}}, D_{\phi}] G_{\alpha \dot{\alpha}} + [D_{\alpha}, D_{\dot{\alpha}}] G_{\dot{\phi}} \right) - 8 R^1 R G_{\alpha \dot{\alpha}} - \frac{15}{24} G_{\alpha \dot{\alpha}} R G_{\alpha \dot{\phi}} \]  
\[ - D_{\alpha} R D_{\dot{\alpha}} R^1 - \frac{3}{32} D^2 G_{\alpha \dot{\phi}} G_{\dot{\phi}} D_{\phi} G_{\dot{\phi}} + \frac{1}{2} T_{\phi \alpha \dot{\phi}} T_{\dot{\phi} \alpha \phi} + 8 T_{\dot{\beta} \alpha} W_{\gamma \beta \alpha \dot{\alpha}} \]  
\[ + T_{\phi \alpha \dot{\alpha}} \left( \frac{3}{32} D^2 G_{\alpha \dot{\phi}} + \frac{2}{3} D_{\phi} G_{\alpha \dot{\phi}} \right) - T_{\dot{\phi} \alpha \phi} \left( 4 D^2 R^1 + \frac{3}{8} D_{\phi} G_{\alpha \dot{\phi}} \right) \]  
\[ + i G_{\alpha \dot{\phi}} \left( D_{\phi \alpha \dot{\phi}} G_{\alpha \dot{\alpha}} + \frac{1}{2} D_{\alpha} G_{\phi \dot{\phi}} - \frac{1}{2} D_{\alpha} G_{\phi \dot{\phi}} - \frac{1}{4} D_{\alpha \dot{\phi}} G_{\phi \dot{\phi}} \right) - 4 i R^1 D_{\alpha \dot{\alpha}} R. \]

The remaining coefficients of the 4-form \( \Sigma^{(+)} \), \textit{i.e.,}
\[ \Sigma^{(+)}_{\alpha \beta \gamma \delta}, \quad \Sigma^{(+)}_{\alpha \beta \gamma \delta \epsilon}, \]  
are obtained as spinor derivatives of the superfields \( S^{(+)} \) and \( T^{(+)} \) as explained in the preamble to this appendix.

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D.2 $\Psi^{(-)} = \Sigma^{(-)} + dM^{(-)}$

Although the basic structure in this sector follows the same pattern as in the previous subsection (it is basically the complex conjugate), we will be slightly more explicit and try to give a flavour of the sequence of arguments used to establish the 3-form constraints

$$\Sigma^{(-)} \delta_{\gamma \beta} A = 0,$$

starting from the curvature-squared 4-form

$$\Psi^{(-)} = R^\beta_a R_\beta^\alpha.$$

In more detail, and in analogy to eqs. (3.45) and (3.46), one has

$$\Psi^{(-)} \delta_{\gamma \beta} A = 0, \quad \Psi^{(-)} \delta_{\dot{\gamma} \dot{\beta}} = 0,$$

$$\Psi^{(-)} \delta_{\dot{\gamma} \dot{\beta}} a \dot{\alpha} = -8i \oint_{\delta_{\dot{\gamma} \dot{\beta}}} D_\delta \left( \epsilon_{\dot{\gamma} \dot{\alpha}} R G_{\dot{\alpha} \dot{\beta}} + \epsilon_{\dot{\beta} \dot{\alpha}} R G_{\dot{\alpha} \dot{\gamma}} \right),$$

due to the constraints on the curvatures themselves. Identifying (modulo the discussion at the end of section 3.1, eqs. (3.31) - (3.37))

$$M^{(-)} \gamma \beta \alpha = 0, \quad M^{(-)} \gamma \beta a = 0,$$

and

$$M^{(-)} \gamma \dot{\beta} a \dot{\alpha} = -8iR \left( \epsilon_{\gamma \dot{\alpha}} G_{\dot{\alpha} \dot{\beta}} + \epsilon_{\dot{\beta} \dot{\alpha}} G_{\dot{\alpha} \gamma} \right),$$

estd establishing

$$\Sigma^{(-)} \delta_{\gamma \beta} A = 0, \quad \Sigma^{(-)} \delta_{\dot{\gamma} \dot{\beta}} = 0.$$

In the next step, taking into account

$$\Psi^{(-)} \delta_{\gamma \dot{\beta}} \dot{\alpha} = 2 \sum_{\delta} R_{\delta} \beta_{\dot{\varphi}} R_{\gamma} \dot{\alpha}_{\dot{\varphi}} = 4 \sum_{\delta \gamma} G_{\delta} \beta G_{\gamma} \dot{\alpha},$$

one is lead to parametrize

$$M^{(-)} \gamma \dot{\beta} a \dot{\alpha} = -i \epsilon_{\gamma \alpha} \epsilon_{\dot{\beta} \dot{\alpha}} \mu^{(-)} - \frac{i}{2} \left( G_{\gamma \dot{\beta}} G_{a \dot{\alpha}} + G_{a \dot{\beta}} G_{\gamma \dot{\alpha}} \right),$$

where we note the appearance of the arbitrary superfield $\mu^{(-)}$.

At dimension 5/2 the components of the curvature-squared 4-form are given as

$$\Psi^{(-)} \dot{\gamma} \dot{\beta} a = -16R R_{\delta a} \dot{\gamma} \dot{\beta} - 4 \left( R_{\delta a} \dot{\gamma} \dot{\beta} + R_{\delta a} \dot{\gamma} \dot{\beta} \right) G_{\delta} \dot{\varphi},$$

and

$$\Psi^{(-)} \dot{\gamma} \dot{\beta} a = -4R_{\delta a} \dot{\gamma} \dot{\beta} G_{\dot{\varphi}} - 4R_{\gamma a} \dot{\beta} \dot{\varphi} G_{\delta} \dot{\varphi}.$$

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This in turn determines the components $M^{(-)}_{\gamma\beta\alpha}$. Employing the decompositions defined in the preamble to this appendix, the different irreducible tensors appearing here are given as

\begin{align}
M^{(-)}_{\dot{\gamma}\dot{\beta}\dot{\alpha}} &= -\frac{1}{8} \sum_{\beta\alpha} \left( 16 R \mathcal{D}_\beta G_{\alpha\dot{\gamma}} + 4 G_{\beta\alpha} \mathcal{D}_\alpha R - G_{\beta\alpha} \dot{\mathcal{D}}_\gamma G_{\alpha\dot{\phi}} - 4 G_{\beta\dot{\phi}} \mathcal{D}_\phi G_{\alpha\dot{\gamma}} \right), \quad (D.37)
\end{align}

\begin{align}
M^{(-)}_{\dot{\gamma}\dot{\beta}\dot{\alpha}} &= -8 R W_{\dot{\gamma}\dot{\beta}\dot{\alpha}} - \frac{1}{16} \oint_{\gamma\beta\alpha} G^{\gamma\beta\alpha} \left( \mathcal{D}_\gamma G_{\phi\alpha} + \mathcal{D}_\alpha G_{\phi\beta} \right), \quad (D.38)
\end{align}

\begin{align}
12 M^{(-)}_{\dot{\alpha}} &= 3 \mathcal{D}_\alpha \mu^{(-)} + 16 \mathcal{D}_\alpha (RR^\dagger) \\
&+ 8 R \mathcal{D}^\phi G_{\phi\dot{\alpha}} + 18 G_{\phi\dot{\alpha}} \mathcal{D}^\phi R - 2 G_{\phi\dot{\alpha}} \mathcal{D}_\alpha G_{\phi\dot{\phi}} - 5 G_{\phi\dot{\phi}} \mathcal{D}_\phi G_{\phi\dot{\alpha}}, \quad (D.39)
\end{align}

\begin{align}
M^{(-)}_{\gamma\beta\alpha} &= -4 G_{\gamma\beta} W_{\gamma\beta\alpha} - \frac{1}{8} \sum_{\beta\alpha} \left( G^{\gamma\beta} \mathcal{D}_\gamma G_{\phi\alpha} + \frac{4}{3} G_{\gamma\beta} \left( \mathcal{D}_\alpha R - \mathcal{D}^\phi G_{\phi\alpha} \right) \right), \quad (D.40)
\end{align}

\begin{align}
M^{(-)}_{\gamma\beta\alpha} &= -\frac{1}{8} \oint_{\gamma\beta\alpha} G_{\gamma\beta} \left( \mathcal{D}_\beta G_{\alpha\phi} + \mathcal{D}_\alpha G_{\beta\phi} \right), \quad (D.41)
\end{align}

\begin{align}
4 M^{(-)}_{\alpha} &= -\mathcal{D}_\alpha \mu^{(-)} - 2 G_{\alpha\phi} \mathcal{D}_\phi R^\dagger - G_{\phi\dot{\phi}} \mathcal{D}_\phi G_{\alpha\dot{\phi}}. \quad (D.42)
\end{align}

In this way one ensures that

\begin{align}
\Sigma^{(-)}_{\delta\gamma\dot{\beta}\alpha} &= 0, \quad \Sigma^{(-)}_{\delta\gamma\dot{\beta}\alpha} = 0. \quad (D.43)
\end{align}

The two chiral superfields $S^{(-)}$ and $T^{(-)}$ are then determined to be

\begin{align}
S^{(-)} &= \left( \mathcal{D}^2 - 8 R^\dagger \right) \left( \mu^{(-)} + \frac{3}{4} G^{\alpha\dot{\alpha}} G_{\alpha\dot{\alpha}} \right) + 32 W_{\gamma\beta\alpha} W_{\gamma\dot{\alpha}} + \frac{4}{3} X_{\alpha} X_{\dot{\alpha}}, \quad (D.44)
\end{align}

\begin{align}
T^{(-)} &= \left( \mathcal{D}^2 - 8 R \right) \left( \mu^{(-)} + 16 RR^\dagger - \frac{13}{4} G^{\alpha\dot{\alpha}} G_{\alpha\dot{\alpha}} \right) - 4 X_{\alpha} X_{\dot{\alpha}}. \quad (D.45)
\end{align}

The analysis of the $\Psi^{(-)}_{\beta\alpha}$ sector shows that the component $\Sigma^{(-)}_{\delta\gamma\beta\alpha}$, is expressed in terms of one single vector $\Sigma^{(-)}_{\gamma\alpha\dot{\alpha}}$ (see preamble to this appendix again) which in turn combines with the purely vectorial component of $M^{(-)}_{\alpha\dot{\alpha}}$ such that

\begin{align}
M^{(-)}_{\alpha\dot{\alpha}} + \Sigma^{(-)}_{\alpha\dot{\alpha}} + \frac{1}{8} \left( [\mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}}] - 4 G_{\alpha\dot{\alpha}} \right) \mu^{(-)} &= \\
&+ \frac{1}{16} G_{\phi\dot{\phi}} \left( 4 [\mathcal{D}_\phi, \mathcal{D}_\phi] G_{\alpha\dot{\alpha}} + [\mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}}] G_{\phi\dot{\phi}} \right) - 8 R^\dagger R G_{\alpha\dot{\alpha}} - \frac{15}{24} G_{\alpha\dot{\alpha}} G_{\phi\dot{\phi}} G_{\phi\dot{\phi}} \mathcal{D}_\phi G_{\alpha\dot{\phi}}, \\
&- \mathcal{D}_\alpha R \mathcal{D}_\alpha R^\dagger - \frac{3}{2} \mathcal{D}^\phi G_{\alpha\phi} \mathcal{D}^\phi G_{\alpha\dot{\phi}} + 3 T_{\phi\alpha} \mathcal{T}_{\phi\alpha} + 3 T_{\phi\alpha} \mathcal{T}_{\phi\alpha} \mathcal{T}_{\phi\alpha} - 8 T_{\phi\alpha} W_{\gamma\beta\alpha}, \\
&- T_{\phi\alpha} \left( 4 \mathcal{D}^\phi R + \frac{3}{8} \mathcal{D}^\phi G_{\phi\dot{\phi}} \right) + T_{\phi\alpha} \alpha \left( \frac{3}{4} \mathcal{D}^\phi R^\dagger + \frac{23}{24} \mathcal{D}^\phi G_{\phi\dot{\phi}} \right), \\
&- i G_{\phi\dot{\phi}} \left( \mathcal{D}_{\phi\phi} G_{\alpha\dot{\alpha}} + \frac{1}{2} \mathcal{D}_{\alpha\alpha} G_{\phi\dot{\phi}} - \frac{1}{4} \mathcal{D}_{\alpha\dot{\alpha}} G_{\phi\phi} - \frac{3}{4} \mathcal{D}_{\alpha\phi} G_{\phi\dot{\phi}} \right) + 4i R \mathcal{D}_\alpha R^\dagger. \quad (D.46)
\end{align}

Again, the remaining coefficients of the 4-form $\Sigma^{(-)}$, i.e. $\Sigma^{(-)}_{\delta\epsilon\beta\alpha}$ and $\Sigma^{(-)}_{\epsilon\delta\alpha\beta}$, are obtained as spinor derivatives of the superfields $S^{(-)}$ and $T^{(-)}$ as explained in the preamble.
D.3 $\Psi^{(1)} = \Sigma^{(1)} + d M^{(1)}$

The $U_K(1)$ - sector with $\Psi^{(1)} = FF$ is slightly less involved than the preceding gravitational sectors. The components of the 3-form $M^{(1)}$ at dimension 3/2 and 2 are given as

$$M^{(1)}_{\gamma \beta \alpha} = 0, \quad M^{(1)}_{\gamma \beta a} = 0, \quad M^{(1)}_{\gamma \beta \ a} = 0,$$

and

$$M^{(1)}_{\gamma \beta \ a a} = -i \epsilon_{\gamma a} \epsilon_{\beta a} \mu^{(1)} - \frac{g}{8} (G_{\gamma \beta} G_{a a} + G_{\alpha \beta} G_{\gamma a}).$$

At dimension 5/2 the components $M^{(1)}_{\gamma \beta \ a} ba$ have irreducible components given as

$$M^{(1)}_{\gamma \beta \alpha} = \frac{-3}{8} \sum_{\beta \alpha} G_{\beta \alpha} (3 T_{\beta \alpha} + \epsilon \phi (X_{\alpha} - S_{\alpha})),$$

$$M^{(1)}_{\gamma \beta \alpha} = \frac{3}{4} \sum_{\beta \alpha} G_{\gamma \beta} T_{\beta \alpha} + \frac{3}{2} G_{\phi \gamma} (3 T_{\phi \gamma} + \epsilon \phi (X_{\gamma} - S_{\gamma})),$$

$$M^{(1)}_{\gamma \beta \gamma \alpha} = \frac{3}{4} \sum_{\gamma \alpha} G_{\gamma \beta} T_{\beta \gamma \alpha},$$

$$M^{(1)}_{\gamma \beta \gamma \alpha} = \frac{3}{4} \sum_{\beta \alpha} G_{\gamma \beta} T_{\beta \gamma \alpha} + \frac{3}{2} G_{\phi \gamma} (3 T_{\phi \gamma} + \epsilon \phi (X_{\gamma} + S_{\gamma})).$$

The chiral superfields $S^{(1)}$ and $T^{(1)}$ are given as

$$S^{(1)} = (D^2 - 8R) (\mu^{(1)} + \frac{g}{4} G_{b} G_{b}) - 2 X_{\hat{a}} X_{\hat{a}},$$

$$T^{(1)} = (D^2 - 8R) (\mu^{(1)} + \frac{g}{4} G_{b} G_{b}) - 2 X_{a} X_{a}.$$}

Finally, one obtains

$$M^{(1)}_{\alpha \alpha} + \Sigma^{(1)}_{\alpha \alpha} + \frac{1}{8} ([D_{\alpha}, D_{\alpha}] - 4G_{\alpha \alpha}) (\mu^{(1)} + \frac{g}{4} G_{b} G_{b}) =$$

$$-\frac{1}{2} X_{\alpha} X_{\hat{a}} - 3 (G_{\alpha \phi} f_{\phi \hat{a}} + G_{\phi \alpha} f_{\phi \hat{a}}) + \frac{g_{\phi}}{8} G_{\phi \phi} (D_{\phi \alpha} G_{\alpha \phi} - D_{\alpha \phi} G_{\phi \alpha}).$$

D.4 $\Psi^{(YM)} = \Sigma^{(YM)} + d M^{(YM)}$

In the Yang-Mills sector the components of $M^{(YM)}$ and $\Sigma^{(YM)}$ are given as

$$M^{(YM)}_{\gamma \beta \alpha} = 0, \quad M^{(YM)}_{\gamma \beta a} = 0, \quad M^{(YM)}_{\gamma \beta \ a} = 0,$$

$$M^{(YM)}_{\gamma \beta \ a a} = 0, \quad M^{(YM)}_{\gamma \beta \ a} = 0, \quad M^{(YM)}_{\gamma \beta \ a} = 0,$$

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\[ M^{(YM)}_{\gamma \dot{\beta} \alpha \dot{\alpha}} = -i \epsilon_{\gamma \alpha} \epsilon_{\dot{\beta} \dot{\alpha}} \mu^{(YM)}, \quad (D.59) \]

\[ M^{(YM)}_{\gamma \beta \dot{\beta} \dot{\alpha}} = -\frac{1}{2} \epsilon_{\beta \dot{\alpha}} \left( \epsilon_{\gamma \beta} D_{\alpha} \mu^{(YM)} + \epsilon_{\gamma \alpha} D_{\beta} \mu^{(YM)} \right), \quad (D.60) \]

\[ M^{(YM)}_{\gamma \dot{\beta} \dot{\beta} \dot{\alpha}} = -\frac{1}{2} \epsilon_{\beta \dot{\alpha}} \left( \epsilon_{\gamma \dot{\beta}} D_{\dot{\alpha}} \mu^{(YM)} + \epsilon_{\gamma \dot{\alpha}} D_{\dot{\beta}} \mu^{(YM)} \right), \quad (D.61) \]

as well as

\[ S^{(YM)} = \left( D^2 - 8R^\hat{y} \right) \mu^{(YM)} - 8\bar{W}_\dot{\alpha} \dot{W}_{\dot{\alpha}}, \quad (D.62) \]

\[ T^{(YM)} = \left( \bar{D}^2 - 8R \right) \mu^{(YM)} - 8W^\alpha \dot{W}_\alpha, \quad (D.63) \]

where \( S^{(YM)} \) and \( T^{(YM)} \) are as usual related to the components \( \Sigma^{(YM)}_{\delta \gamma \beta a} \) and \( \Sigma^{(YM)}_{\delta \dot{\gamma} \dot{\beta} \dot{a}} \), and

\[ M^{(YM)}_{\alpha \dot{\alpha}} + \Sigma^{(YM)}_{\alpha \dot{\alpha}} + \frac{1}{8} \left( [D_\alpha, D_\dot{\alpha}] - 4G_{\alpha \dot{\alpha}} \right) \mu^{(YM)} = -2W_\alpha \dot{W}_\dot{\alpha}. \quad (D.64) \]

It is clear, that in this case the decomposition is trivial in the sense that one can take \( \mu^{(YM)} = 0 \) and \( M^{(YM)} = 0 \) as a superspace 3-form.

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