NON-ORIENTABLE FUNDAMENTAL SURFACES IN LENS SPACES

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Abstract. We give a concrete example of an infinite sequence of \((p_n, q_n)\)-lens spaces \(L(p_n, q_n)\) with natural triangulations \(T(p_n, q_n)\) with \(p_n\) tetrahedra such that \(L(p_n, q_n)\) contains a certain non-orientable closed surface which is fundamental with respect to \(T(p_n, q_n)\) and of minimal crosscap number among all closed non-orientable surfaces in \(L(p_n, q_n)\) and has \(n - 2\) parallel sheets of normal disks of a quadrilateral type disjoint from the pair of core circles of \(L(p_n, q_n)\). Actually, we can set \(p_0 = 0, q_0 = 1, p_{k+1} = 3p_k + 2q_k\) and \(q_{k+1} = p_k + q_k\).

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1. Introduction

The theory of normal surface was introduced by H. Kneser ([10]) and W. Haken ([3]), and have been playing an important role in study of topology of 3-manifolds. Almost all sorts of important surfaces, such as essential spheres, essential tori, knot spanning surfaces with maximal Euler characteristics and so on, can be deformed to normal surfaces and to fundamental surfaces. See, for example, [8], [9], [13], [12], [4] and [11].

We briefly recall the definitions of normal surfaces and fundamental surfaces. In [10], K. Kneser introduced normal surfaces. Let \(M\) be a closed 3-manifold and \(T\) a triangulation of \(M\), that is, a decomposition of \(M\) into finitely many tetrahedra, where all the faces of the tetrahedra of \(T\) are separated into pairs, and each pair of faces are identified in \(M\). Let \(F\) be a closed surface embedded in \(M\). \(F\) is called a normal surface with respect to the triangulation \(T\) if \(F\) intersects each tetrahedron in disjoint union of normal disks as below, or in the empty set. There are two kinds of normal disks, trigons called T-disks and quadrilaterals called Q-disks. Each tetrahedron contains 4 types of T-disks and 3 types of Q-disks as in illustrated in Figure 1. Two normal disks are of the same type if they have vertices in the same edges of \(T\) as in Figure 2.

In [3], W. Haken found that normal surfaces correspond to non-negative integral solutions of a certain system of simultaneous linear equations with integer coefficients, called the matching equations. First we number all the types of normal disks in the tetrahedra. Then
we let a vector \( v_F \) represent a normal surface \( F \) such that the \( i \)-th element \( x_i \) of \( v_F \) is the number of the normal disks of the \( i \)-th type \( X_i \) contained in \( F \). In each 2-simplex \( \Delta \) of the triangulation \( T \), a properly embedded arc \( \alpha \) is called a normal arc if its two endpoints are in the interior of distinct two edges of \( \Delta \). Two normal arcs are of the same type if they have their endpoints in the same pair of edges of \( \Delta \). One matching equation arises for each normal arc. For a type \( \alpha \) of a normal arc in a 2-simplex \( \Delta \), there are two tetrahedra \( \tau_1, \tau_2 \) which contain \( \Delta \), and each tetrahedron \( \tau_i \) contains a trigonal type \( X_{ij} \) of a normal disk and a quadrilateral type \( X_{ik} \) of a normal disk which have an edge of type \( \alpha \). The matching equation for \( \alpha \) is \( x_{1j} + x_{1k} = x_{2j} + x_{2k} \) where \( x_{st} \) denotes a variable corresponding the number of normal disks of type \( X_{st} \). The system of matching equations for all the types of normal arcs is simply called the matching equations.

We need some terminologies on algebra. Let \( v = t(v_1, \cdots, w_n) \), \( w = t(w_1, \cdots, w_n) \) be vectors in \( \mathbb{R}^n \), where \( t \) denotes the transposition of \( x \). In this paper, we write \( v \leq w \) if \( v_i \leq w_i \) for all \( i \in \{1, \cdots, n\} \). \( v < w \) means that both \( v \leq w \) and \( v \neq w \) hold. A vector \( u \in \mathbb{R}^n \) is non-negative if \( 0 \leq u \), and integral if all its elements are in \( \mathbb{Z} \).

The set of all the (possibly disconnected) normal surfaces is in one-to-one correspondence with the set of all the non-negative integral solutions of the matching equations satisfying the square condition as below. If two or three types of Q-disks exist in a single tetrahedron, then they intersect each other. See Figure 2 (1). Hence any normal surface intersects each tetrahedron in Q-disks of the same type and T-disks (Figure 2 (2)). This is called the square condition.

A normal surface is called a fundamental surface if it corresponds to a fundamental solution of the system of the matching equations which is defined as below. Let \( Ax = 0 \).
be a linear system of equations, where $A$ is a matrix with all the elements in $\mathbb{Z}$, and $x$ is a vector of variables. $V_A$ denotes the solution space of the linear system considered in $\mathbb{R}^n$. A non-zero non-negative integral solution $v$ is called a fundamental solution, if there is no integral solution $v' \in V_A$ with $0 < v' < v$. There are only finite number of fundamental solutions for each system. Moreover, an upper bound for elements of fundamental solutions is known ([5]). Hence there is an algorithm which determines all the fundamental solutions, though it is not practical.

Let $p$ and $q$ be positive integers such that $p$ and $q$ are coprime. We can obtain a $(p, q)$-lens space $L(p, q)$ from a suspension of a $p$-gon by gluing each trigonal face in the upper hemisphere with that in the lower hemisphere, performing $(2\pi q/p)$-rotation and taking a mirror image about the equator. See Figure 3. Precisely, the trigon $v_x v_i v_{i+1}$ is glued to
where indices are considered modulo $p$. The edge $e_i$ connects $v_+$ and $v_i$, and also $v_-$ and $v_{i+q}$. The horizontal edges connecting $v_i$ and $v_{i+1}$ for $1 \leq i \leq p$ are all glued up together into an edge $E_h$. Taking an axis $E_v$ connecting the vertices $v_+$ and $v_-$ in the suspension, we can decompose it into $p$ tetrahedra. This gives a natural triangulation $T(p, q)$ of a $(p, q)$-lens space. The $i$-th tetrahedron $\tau_i$ has vertices $v_+, v_-, v_i$ and $v_{i+1}$.

An embedded circle $C$ in a lens space $M$ is called a core circle if the exterior $M \setminus \text{int } N(C)$ is homeomorphic to the solid torus. Note that each of $E_h$ and $E_v$ forms a core circle of $L(p, q)$.

In lens spaces, non-orientable closed surfaces with maximum Euler characteristics are interesting. A formula for calculating the maximum Euler characteristic is given by G. E. Bredon and J. W. Wood in [1].

**Theorem 1.1.** Let $\{p_n\}$, $\{q_n\}$ be infinite sequences of integers such that $p_0 = 0, q_0 = 1, p_{k+1} = 3p_k + 2q_k$ and $q_{k+1} = p_k + q_k$ for any non-negative integer $k$. For $n \geq 2$, the $(p_n, q_n)$-lens space contains a non-orientable closed surface $h_{n-1}$ with maximal Euler characteristic which is fundamental with respect to Haken’s matching equations on the triangulation $T(p_n, q_n)$ and homeomorphic to the connected sum of $n$ projective planes and has $n-2$ sheets of quadrilateral normal disks of type $X_{m1}$ shown in Figure 4 for some $m$. The construction of the fundamental surface $h_{n-1}$ is described in section 3.

In [6], Hass, Snoeyink and Thurston gave an example of infinite sequence of polygonal knots $K_n$ in $\mathbb{R}^3$ such that $K_n$ has fewer than $10n + 9$ edges and its piecewise linear triangulated disk spanning $K_n$ contains at least $2^{n-1}$ flat triangles. This implies existence of fundamental surfaces with huge number of sheets of some type of a normal disk. However, no concrete example of such surface was given.
Fominykh gave a complete description of fundamental surfaces with respect to certain good handle decompositions for three manifolds including lens spaces in [2]. However, no fundamental surface there has a 2-handle with weight more than two.

**Conjecture 1.2.** The surface \( h_{n-1} \), which will be constructed in section 3 and is fundamental with respect to the Haken’s matching equations, is fundamental also with respect to the Q-matching equations on the triangulation \( T(p_n, q_n) \).

For the definition of Q-matching equation, see [14].

2. **Preliminaries**

In this section, we introduce two basic lemmas on closed non-orientable surfaces in lens spaces.

The next lemma is well-known. See ll.24-28 in p.97 in [1].

**Lemma 2.1.** Let \( M \) be a lens space, and \( C \) a core of it. Let \( F \) be a closed surface \( F \) embedded in \( M \) intersecting \( C \) transversely. Then \( F \) is non-orientable if and only if it intersects \( C \) in odd number of points.

**Lemma 2.2.** Let \( F \) be a normal surface in the \((p, q)\)-lens space with the triangulation \( T(p, q) \). If \( F \) intersects each of \( E_v \) and \( E_h \) in a single point and contains a normal disk of type \( X_{k2} \) or \( X_{k3} \), then \( F \) is fundamental with respect to Haken’s matching equations. In particular, \( F \) is connected.

**Proof.** Suppose, for a contradiction, that \( F \) is decomposed as \( F = F_1 + F_2 \). Since \( F \) intersects \( E_v \) and \( E_h \) in a single point, one of \( F_1 \) or \( F_2 \), say \( F_1 \) intersects \( E_v \) and \( E_h \) in a single point, and \( F_2 \) is disjoint from \( E_v \cup E_h \). Since \( X_{i1} \) is the only type of normal disk in \( \tau_i \) which is disjoint from \( E_v \cup E_h \) for all \( i \in \{1, 2, \ldots, p\} \), the surface \( F_2 \) intersects each tetrahedron \( \tau_i \) in copies of the quadrilateral \( X_{i1} \). Hence \( F_2 \) is a union of \( n \) parallel copies of the Heegaard splitting torus surrounding \( E_v \) and \( E_h \) for some positive integer \( n \). \( F_2 \) intersects each \( \tau_i \) in \( n \) sheets of normal disks of type \( X_{i1} \). This implies that \( F \cap \tau_i \) contains a normal disk of type \( X_{i1} \) for each \( i \in \{1, 2, \ldots, p\} \). By assumption, for some \( k \in \{1, 2, \ldots, p\} \) \( F \) has a normal disk of type \( X_{k2} \) or \( X_{k3} \) which cannot exist together with a normal disk of \( X_{k1} \) by the square condition. This is a contradiction. \( \square \)

3. **Construction of surfaces**

In this section, we construct the fundamental surface \( h_{n-1} \) in Theorem 1.1.

Tollefson introduced Q-coordinates representing normal surfaces in [14]. We first number types of Q-disks. In our case, we number the 3-types of Q-disks \( i1, i2 \) and \( i3 \) in \( \tau_i \) as in Figure 4 where the axis \( E_v \) is in front of the tetrahedron. \( X_{i1} \) separates the edges \( E_h \) and \( E_v \), \( X_{i2} \) separates \( e_{i+1} \) and \( e_{i-q} \) and \( X_{i3} \) does \( e_i \) and \( e_{i-(q-1)} \).
Let $x_{ij}$ be the number of Q-disks of type $X_{ij}$ contained in a normal surface $F$. Then we place them in a vertical line in the order of indices lexicographically, to obtain a \(Q\)-coordinate \(v_{Q}(F) = t(x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, \ldots, x_{11}, x_{12}, x_{13})\) of $F$, where $t$ is the number of tetrahedra of the triangulation. A normal surface with the \(Q\)-coordinate $0$ is called \textit{trivial}. It is composed of trigonal normal disks and has no quadrilateral normal disks, and hence is a disjoint union of 2-spheres each of which surrounds a vertex of the triangulation. Any non-zero solution of the system of \(Q\)-matching equations with non-negative integer elements determines a normal surface with no trivial component uniquely.

In addition, for any normal surface $F$ with no trivial component, and for any normal surface $F'$ with \(v_{Q}(F') = v_{Q}(F)\), there is a trivial normal surface $\Sigma$ such that $\Sigma$ is disjoint from $F$ and $F' = F + \Sigma$ (Theorem 1 in \cite{14}). He introduced there the \(Q\)-matching equations on numbers of sheets of normal disks of \(Q\)-disk types such that the set of non-trivial non-negative integral solutions satisfying the square condition is in one to one correspondence with the set of normal surfaces with no trivial component in the 3-manifold.

In our situation of \((p, q)\)-lens space, the \((3i - 2)\)-nd, \((3i - 1)\)-st and \((3i)\)-th elements $x_{i1}, x_{i2}, x_{i3}$ of a \(Q\)-coordinate together form the $i$-th block for $1 \leq i \leq p$. We often put a line "\mid" instead of a comma between every adjacent pair of blocks in such a manner as $t(x_{11}, x_{12}, x_{13} \mid x_{21}, x_{22}, x_{23} \mid \cdots \mid x_{p1}, x_{p2}, x_{p3})$.

In the previous paper, we have shown the next lemma.

\textbf{Lemma 3.1.} (Lemma 1.3 in \cite{7}) For the triangulation $T(p, q)$ of the \((p, q)\)-lens space with $p \geq 5$, $2 \leq q < p/2$ and $GCM(p, q) = 1$, the vectors $s_1, s_2, \ldots, s_p, t_1, t_2, \ldots, t_p$ as below form a basis of the solution space in $\mathbb{R}^{3p}$ of the system of \(Q\)-matching equations.

\begin{align*}
\text{(The } j\text{-th block of } s_i) & = \begin{cases} 
 t(1, 1, 1) & \text{if } j = i \\
 t(0, 0, 0) & \text{otherwise}
\end{cases} \\
\text{(The } j\text{-th block of } t_i) & = \begin{cases} 
 t(0, 1, 0) & \text{if } j = i \text{ or } i + q + 1 \\
 t(0, 0, 1) & \text{if } j = i + 1 \text{ or } i + q \\
 t(0, 0, 0) & \text{otherwise}
\end{cases}
\end{align*}

Hence a general solution $v$ in $\mathbb{R}^{3p}$ is presented as below.

\[v = a_1 s_1 + \cdots + a_p s_p + b_1 t_1 + \cdots + b_p t_p\]

\[= t(a_1, a_1 + b_1 + b_{p-q}, a_1 + b_p + b_{p-q+1} \mid \cdots \mid a_i, a_i + b_i + b_{p-q+i-1}, a_i + b_{i-1} + b_{p-q+i} \mid \cdots \mid a_p, a_p + b_p + b_{p-q-1}, a_p + b_{p-1} + b_{p-q}),\]

with $a_1, \ldots, a_p, b_1, \ldots, b_p \in \mathbb{R}$, where $t(a_i, a_i + b_i + b_{p-q+i-1}, a_i + b_{i-1} + b_{p-q+i})$ is the $i$-th block.

When $p$ is even and $q \geq 3$, in the \((p, q)\)-lens space the normal surface represented by

\[h_0 = (\sum_{m=1}^{p/2} t_{2m-1})/2 = t(0 \ 1 \ 0 \mid 0 \ 0 \ 1 \mid 0 \ 1 \ 0 \mid 0 \ 0 \ 1 \mid \cdots \mid 0 \ 1 \ 0 \mid 0 \ 0 \ 1)\]
is non-orientable by Lemma 2.1 and fundamental by Lemma 2.2. In fact, this surface consists of only Q-disks of types $X_k$ and $X_4$ for any odd integer $k$ and any even integer $\ell$, and intersects each of $E_h$ and $E_v$ in a single point. However, $h_0$ is not Q-fundamental because it is larger than $t_1$. In fact, it is of Euler characteristic $2 - p/2$, which is not maximal among all closed non-orientable surfaces in a $(p, q)$-lens space with $p$ even and $q \geq 3$ by [1]. We will perform compressing operations on the surface represented by $h_0$ to obtain one of maximal Euler characteristic.

In what follows, we consider the infinite sequence of $(p_n, q_n)$-lens spaces, where $p_0 = 0$, $q_0 = 1$, $p_n = 3p_{n-1} + 2q_{n-1}$ and $q_n = p_{n-1} + q_{n-1}$, which are derived from $\frac{p_n}{q_n} = 2 + \frac{1}{1 + \frac{p_{n-1}}{q_{n-1}}}$ and $\frac{p_1}{q_1} = \frac{2}{1}$. Thus $(p_1, q_1) = (2, 1)$, $(p_2, q_2) = (8, 3)$, $(p_3, q_3) = (30, 11)$, $(p_4, q_4) = (112, 41)$, $(p_5, q_5) = (418, 153)$, $(p_6, q_6) = (1560, 571)$, $\ldots$. Note that $p_n$ is even for any natural number $n$. We will obtain some formulae on this infinite sequences $\{p_n\}$, $\{q_n\}$ in Lemma 4.1 in section 4.

In the case of $n \geq 2$, we perform $n - 1$ steps of compressing operations on $h_0$ to obtain a desired closed non-orientable surface $h_{n-1}$ of maximal Euler characteristic. The $k$-th step is composed of $(q_{(n-(k-1))} - 1)/2$ compressing operations.

For $1 \leq i \leq (q_n - 1)/2$, the $i$-th compressing disk of the first step is placed in the union of four tetrahedra $\tau_{2i-1}, \tau_{2i}, \tau_{q_n+2i-1}$ and $\tau_{q_n+2i}$. It intersects each tetrahedron in a single trigonal disk patch. We call the pair of patches in $\tau_{2i-1} \cup \tau_{2i}$ the first pair (see Figure 5(a)), and that in $\tau_{q_n+2i-1} \cup \tau_{q_n+2i}$ the second (see Figure 5(b)). Each pair is composed of the leading trigonal disk patch in the tetrahedron assigned the smaller number $(\tau_{2i-1}, \tau_{q_n+2i-1})$ and the following one in that assigned the larger number $(\tau_{2i}, \tau_{q_n+2i})$.

The compressions of the first step deform the Q-coordinate $h_0$ into

$$h_1 = h_0 - \sum_{i=1}^{(q_n-1)/2} t_{2i-1}.$$ 

Subtracting $t_{2i-1}$ corresponds to the surgery along the $i$-th compressing disk. In fact, the four Q-disks of $t_{2i-1}$ are deformed to T-disks by the $i$-th surgery as shown in Figure 5. A single compression increases the Euler characteristic of the surface by two. Hence we obtain a closed surface of Euler characteristic $2 - p_n/2 + q_n - 1$. The resulting surface is non-orientable and fundamental with respect to Haken’s matching equations by Lemmas 2.1 and 2.2.

See Figure 6 where described the compressing disks for the case of $(p_2, q_2) = (8, 3)$. We obtain a closed non-orientable surface as in Figure 7 after the compression in the first
step. This is a Klein bottle, which is of maximal Euler characteristic among all closed non-orientable surfaces in the lens space $L(8,3)$.

We call the union of the first $q_n$ tetrahedra $\tau_1, \tau_2, \cdots, \tau_{q_n}$ and the union of corresponding $q_n$ blocks of the Q-coordinate the first region, the union of the second $q_n$ tetrahedra $\tau_{q_n+1}, \tau_{q_n+2}, \cdots, \tau_{2q_n}$ and the union of corresponding $q_n$ blocks the second region, the union of remainder $p_{n-1}$ tetrahedra $\tau_{2q_n+1}, \tau_{2q_n+2}, \cdots, \tau_{p_n}$ and the union of corresponding $p_{n-1}$ blocks the last region. In fact, the last region is composed of $p_n - 2q_n = (3p_{n-1} + 2q_{n-1}) - (p_{n-1} + q_{n-1}) = p_{n-1}$ tetrahedra or blocks.

Compressing operations of the first step have occurred in all the tetrahedra corresponding to the first and the second regions but in the ending tetrahedra $\tau_{q_n}$ and $\tau_{2q_n}$.

When $n \geq 3$, we perform the second step of compressions. This operation is performed along $(q_{n-1} - 1)/2$ compressing disks. The $i$-th compressing disk is composed of four pairs of disk patches. The first pair consists of two trigonal disk patches in $\tau_{2q_n+2i-1} \cup \tau_{2q_n+2i}$ similar as those described in Figure 5(a), the second pair consists of quadrilateral disk patches in $\tau_{3q_n+2i-1} \cup \tau_{3q_n+2i}$ as described in Figure 8(a), the third pair quadrilateral disk patches in $\tau_{4q_n+2i-1} \cup \tau_{4q_n+2i}$ as described in Figure 8(b), and the last pair trigonal disk patches in $\tau_{5q_n+2i-1} \cup \tau_{5q_n+2i}$ similar as those described in Figure 5(b). Because $2q_n + 2i - 1 > 2q_n$ and $2q_n + 2i \leq 2q_n + 2 \cdot (q_{n-1} - 1)/2 = 2q_n + q_{n-1} < 2q_n + p_{n-1} = p_n$, the first pair of two trigonal disk patches are in the last region, where compressions of the first step have not been performed. Since

![Figure 5](image-url)
the second pair of quadrilateral disk patches are contained in \( \tau(q_{n-1}+2i-1) \cup \tau(q_{n-1}+2i) \). In these tetrahedra, compressions of the first step have already occurred. In fact, \( q_{n-1}+2i-1 \geq 1 \) and \( q_{n-1}+2i \leq q_{n-1}+2\left\lfloor (q_{n-1}-1)/2 \right\rfloor = 2q_{n-1}-1 < p_{n-1}+q_{n-1} = q_n \), and hence the tetrahedra are in the first region, and are not the ending ones. The third pair of quadrilateral disk patches are contained in the \( (q_{n-1}+2i-1) \)-st and the \( (q_{n-1}+2i) \)-th tetrahedra in the second region, where compressions of the first step have been performed. The last pair of trigonal patches are in the \( (q_{n-1}+2i-1) \)-st and the \( (q_{n-1}+2i) \)-th tetrahedra in the last region, i.e., in \( \tau_{2q_n+q_{n-1}+2i-1} \cup \tau_{2q_n+q_{n-1}+2i} \), where compressions of the first step have not occurred. Moreover, the first pairs of disk patches and the last ones do not appear in the same tetrahedron because \( 2q_n+q_{n-1}+2i'-1 \geq 2q_n+q_{n-1}+1 > 2q_n+q_{n-1}-1 \geq 2q_n+2i \) for the leading one of the last pair of disk patches of the \( i' \)-th compressing disk and the following one of the 1st pair of disk patches of the \( i \)-th compressing disk.

See Figure 9 where described the compressing disks of the 2nd step for the case of \( (p_3,q_3) = (30,11) \).
Figure 7.

Figure 8.
After the second step Q-coordinate will be
\[
h_2 = h_1 - \sum_{i=1}^{(q_n-1)/2} \left( t_{2q_n+2i-1} + t_{3q_n+2i-1} + t_{4q_n+2i-1} - s_{3q_n+2i-1} - s_{4q_n+2i-1} - s_{4q_n+2i} \right).
\]

Subtracting \( t_{2q_n+2i-1} + t_{3q_n+2i-1} + t_{4q_n+2i-1} - s_{3q_n+2i-1} - s_{4q_n+2i-1} - s_{4q_n+2i} \) corresponds to the surgery along the \( i \)-th compressing disk. In fact, after surgery on the \( i \)-th compressing disk, four Q-disks in \( \tau_{2q_n+2i-1} \cup \tau_{2q_n+2i} \cup \tau_{5q_n+2i-1} \cup \tau_{5q_n+2i} \) are changed to T-disks by deformations along the first and the last pairs of trigonal disk patches. A Q-disk disjoint from \( E_h \cup E_v \) is added in \( \tau_{3q_n+2i-1} \cup \tau_{3q_n+2i} \cup \tau_{4q_n+2i-1} \cup \tau_{4q_n+2i} \) by deformations along the second and the third pairs of quadrilateral disk patches. More precisely, trigonal normal disks on the right hand sides of tetrahedra are replaced by ones on the left hand sides, and vice versa. See Figure 8.

The resulting surface is non-orientable and fundamental with respect to Haken’s matching equations by Lemmas 2.1 and 2.2. A single compression increases the Euler characteristic of the surface by two. Hence we obtain a closed surface of Euler characteristic \( 2 - p_n/2 + q_n - 1 + q_{n-1} - 1 \).

Let \( k \) be an integer equal to or larger than 3. When \( n \geq k + 1 \), we perform the \( k \)-th step of compressions after the \( (k - 1) \)-st step ones. This operation is performed along \( (q_n-(k-1)) \) compressing disks. The \( i \)-th compressing disk is composed of \( q_k + 1 \) pairs of disk patches. The first pair is composed of two trigonal disk patches which are in \( \tau_{2q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i-1} \) and \( \tau_{2q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i} \) similar as those in Figure 5 (a). For \( j \in \{2, 3, \cdots, q_k\} \), the \( j \)-th pair consists of quadrilateral disk patches in
\[
\tau_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i-1} \cup \tau_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i} \quad \text{as described in Figure 5 (b)}.
\]

We will prove that we can place disk patches of compression disks as above in section 5.

We should show that the first and the last pairs are in tetrahedra where the compressing operations of the preceding steps have not occurred, and the second through the \( q_k \)-th pairs are in tetrahedra where such compressing operations have occurred. Moreover, for each integer \( k \) with \( 1 \leq k \leq n - 1 \), every tetrahedron must contain at most one disk patch of compressing disks of the \( k \)-th step.

After the \( k \)-th step, Q-coordinate will be
\[
h_k = h_{k-1} - \sum_{i=1}^{(q_n-(k-1))/2} \left( \sum_{j=1}^{q_k} t_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i-1} \right.
\]
\[
\left. - \sum_{j=2}^{q_k} \left( s_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i} + s_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i} \right) \right)
\]
Figure 9.
Subtracting $\sum_{j=1}^{q_k} t_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i-1} - \sum_{j=2}^{q_k} (s_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i-1} + s_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i})$ corresponds to the surgery along the $i$-th compressing disk.

A single compression increases the Euler characteristic of the surface by two. Hence we obtain a closed surface $h_k$ of Euler characteristic $2 - p_n/2 + \sum_{r=1}^{k} (q_n-(r-1)-1)$. The resulting surface is non-orientable and fundamental with respect to Haken’s matching equations by Lemmas 2.1 and 2.2.

After the compressing operations of the $(n-1)$-st step, we obtain a fundamental closed non-orientable surface

$$h_{n-1} = \left( \sum_{m=1}^{n} t_{2m-1} \right)/2$$

$$- \sum_{k=1}^{n-1} (q_{n-(k-1)}-1)/2 \cdot q_k$$

$$- \sum_{i=1}^{k-1} \left( \sum_{j=1}^{q_k} t_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i-1} - \sum_{j=2}^{q_k} (s_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i-1} + s_{(j+1)q_n+2q_{n-1}+\cdots+2q_{n-(k-2)}+2i}) \right)$$
Lemma 4.1. Let \( p \) and \( q \) be finite sequences of integers defined by \( p_0 = 0, q_0 = 1 \), \( p_n = (\kappa + 1)p_{n-1} + \kappa q_{n-1} \) and \( q_n = p_{n-1} + q_{n-1} \), where \( \kappa \) is a natural number. Then the formulae below hold.

1. \( p_n = \kappa q_n + p_{n-1} \)
2. \( \kappa(q_1 + q_2 + \cdots + q_{\ell}) = p_\ell \) for \( \ell \geq 1 \).
3. \(- (q_1 + q_2 + \cdots + q_{\ell-1})p_n + q_{\ell}q_n = -(q_1 + q_2 + \cdots + q_{\ell-2})p_{n-1} + q_{\ell-1}q_{n-1} \), where \( \ell \geq 2 \) and \( n \geq 1 \). When \( \ell = 2 \), we define the sum \( q_1 + q_2 + \cdots + q_{\ell-2} \) to be equal to 0.
4. \(- (q_1 + q_2 + \cdots + q_{m-1})p_n + q_{m}q_n = q_{n-(m-1)} \), and hence \( q_{m}q_n \equiv q_{n-(m-1)} \) (mod. \( p_n \)) for \( n \geq m - 1 \).
5. \( (\kappa + 1)q_n - p_n = q_{n-1} \) for \( n \geq 1 \).
6. \( (2\kappa + 1)q_n - 2p_n = -p_{n-1} + q_{n-1} \) for \( n \geq 1 \).

Proof. 1. \( p_n = (\kappa + 1)p_{n-1} + \kappa q_{n-1} = \kappa(p_{n-1} + q_{n-1}) + p_{n-1} = \kappa q_n + p_{n-1} \).

2. We prove this formula by induction on \( \ell \).

\( \kappa(q_1 + q_2 + \cdots + q_\ell) = \kappa(q_1 + q_2 + \cdots + q_{\ell-1}) + \kappa q_{\ell} = p_{\ell-1} + \kappa q_{\ell} = p_{\ell} \)

The second equality holds by the assumption of induction. The last equality is the formula (1).

3. \(- (q_1 + q_2 + \cdots + q_{\ell-1})p_n + q_{\ell}q_n = -(q_1 + q_2 + \cdots + q_{\ell-1})(\kappa q_n + p_{n-1}) + q_{\ell}q_n \)
\(- \kappa(q_1 + q_2 + \cdots + q_{\ell-1})q_n - (q_1 + q_2 + \cdots + q_{\ell-1})p_{n-1} + q_{\ell}q_n \)
\(- p_{\ell-1}q_n - (q_1 + q_2 + \cdots + q_{\ell-1})p_{n-1} + q_{\ell}q_n \)
\(- q_{\ell-1}p_n - (q_1 + q_2 + \cdots + q_{\ell-1})p_{n-1} + q_{\ell-1}q_n \)
\(- (q_1 + q_2 + \cdots + q_{\ell-1})p_{n-1} + q_{\ell-1}q_n \)
\(- (q_1 + q_2 + \cdots + q_{\ell-1})p_{n-1} - q_{n-1}p_{n-1} + q_{\ell}p_{n-1} + q_{\ell}q_{n-1} \)
\(- (q_1 + q_2 + \cdots + q_{\ell-2})p_{n-1} + q_{\ell-1}q_{n-1} \)

The first equality holds by (1). The third equality follows from (2).

4. Applying (3) \( m - 1 \) times we have:
\( -(q_1 + q_2 + \cdots + q_{m-1})p_n + q_{m}q_n = -(q_1 + q_2 + \cdots + q_{m-2})p_{n-1} + q_{m-1}q_{n-1} \)
\[=- (q_1 + q_2 + \cdots + q_{m-3})p_{n-2} + q_{m-2}q_{n-2} = \cdots\]
\[=- (q_1 + q_2 + \cdots + q_{m-n})p_{n-(m-1)} + q_{m-(m-1)}q_{n-(m-1)} = 0 + q_1q_{n-(m-1)} = q_{n-(m-1)}.
\]
\[(5) \quad (k+1)q_n - p_n = (k+1)(p_{n-1} + q_{n-1}) - ((k+1)p_{n-1} + \kappa q_{n-1}) = q_{n-1}.
\]
\[(6) \quad (2k+1)q_n - 2p_n = (2k+1)(p_{n-1} + q_{n-1}) - 2((k+1)p_{n-1} + \kappa q_{n-1}) = -p_{n-1} + q_{n-1}. \square\]

5. Placement of Disk Patches

In this section, we show that we can place the disk patches of the compression disks as in section \[3\].

**Lemma 5.1.** Let \( n \), \( k \) and \( r \) be integers with \( n \geq 3 \), \( 2 \leq k \leq n - 1 \) and \( 1 \leq r \leq p_{n-1} \).

1. The first pair of disk patches and the last pair of disk patches of the compression disks in section \[3\] are contained in tetrahedra in the last region.

2. The tetrahedron \( \tau_{2q_{n+r}} \) in the last region of the \( (p_n, q_n) \)-lens space contains the leading (resp. following) disk patch of the \( j \)-th pair of the \( i \)-th compressing disk of the \( k \)-th step if and only if \( \tau_r \) of the \( (p_{n-1}, q_{n-1}) \)-lens space contains the leading (resp. following) disk patch of the \( j \)-th pair of the \( i \)-th compressing disk of the \( (k - 1) \)-st step.

The number of compressing disks of the \( k \)-th step for \( (p_n, q_n) \)-lens space is \( (q_n-(k-1)-1)/2 \), and that of the \( (k - 1) \)-st step for \( (p_{n-1}, q_{n-1}) \)-lens space is \( (q_{(n-1)-((k-1)-1)}-1)/2 \). Note that they are equal.

**Proof.** The leading disk patch of the 1st pair of the \( i \)-th compressing disk of the \( k \)-th step for the \( (p_n, q_n) \)-lens space is contained in \( \tau_{2(q_n+q_{n-1}+\cdots+q_{n-(k-2)})+2i-1} \). This tetrahedron is contained in the last region, which follows from Lemma \[4.1\] (2) and the fact that the number of compression disks of the \( k \)-th step is \( (q_{n-(k-1)}-1)/2 \). On the other hand, \( \tau_{2(q_{n-1}+q_{n-2}+\cdots+q_{(n-1)-((k-1)-2)})+2i-1} \) of the \( (p_{n-1}, q_{n-1}) \)-lens space contains the leading disk patch of the 1st pair of the \( i \)-th compressing disk of the \( (k - 1) \)-st step. The numbers assigned to these two tetrahedra differ by \( 2q_n \).

The leading disk patch of the last pair of the \( i \)-th compressing disk of the \( k \)-th step for the \( (p_n, q_n) \)-lens space is contained in the tetrahedron numbered
\[2(q_n + q_{n-1} + \cdots + q_{n-(k-2)}) + 2i - 1 + q_nq_k \equiv 2(q_n + q_{n-1} + \cdots + q_{n-(k-2)}) + 2i - 1 + q_{n-(k-1)} \text{ (mod. } p_n) \] since the last pair is the \( (q_k + 1) \)-st one. The congruence \( \equiv \) follows by Lemma \[4.1\] (4). This tetrahedron is in the last region by a similar argument as above. On the other hand, the leading disk patch of the last pair of the \( i \)-th compressing disk of the \( (k - 1) \)-st step for the \( (p_{n-1}, q_{n-1}) \)-lens space is contained in the tetrahedron numbered
\[2(q_{n-1} + q_{n-2} + \cdots + q_{(n-1)-((k-1)-2)}) + 2i - 1 + q_{n-1}q_{k-1} \]
\[\equiv 2(q_{n-1} + q_{n-2} + \cdots + q_{(n-1)-((k-1)-2)}) + 2i - 1 + q_{(n-1)-((k-1)-1)}. \text{ (mod. } p_{n-1})\] The numbers assigned to these two tetrahedra differ by \( 2q_n \).
Similar things hold for the following disk patches of the first and the last pairs since they are in tetrahedra next to those contain the leading disk patches.

Suppose that $\tau_{2q_n+r}$ in the last region contains the leading (resp. following) disk patch of the $j$-th pair of the $i$-th compressing disk of the $k$-th step for the $(p_n, q_n)$-lens space. Let $j'$ be the number assigned to the next pair whose leading (resp. following) disk patch is contained in a tetrahedron $\tau_{2q_n+r'}$ in the last region. Then we will show that either $r' - r = q_{n-1}$ (when $1 \leq r \leq p_{n-1} - q_{n-1}$) or $r' = q_{n-1} - p_{n-1}$ (when $p_{n-1} - q_{n-1} < r \leq p_{n-1}$) holds as if these two leading (resp. following) disk patches were of adjacent pairs of a compressing disk for $(p_{n-1}, q_{n-1})$-lens space.

If $1 \leq r \leq p_{n-1} - q_{n-1}$, then the leading (resp. following) disk patches of the $(j+1)$-st pair, the $(j+2)$-nd pair and the $(j+3)$-rd pair are contained in a tetrahedron in the first region, the second region and the last region respectively. In fact, the leading (resp. following) disk patch of the $(j+\ell)$-th pair is contained in $\tau_{(\ell+2)q_n+r} = \tau_{(\ell+2)q_n+r-p_n}$, and $(1 <)q_{n-1} + 1 \leq 3q_n + r - p_n \leq q_n - 1 (< q_n)$, $q_n + q_n - 1 \leq 4q_n + r - p_n \leq q_n + p_n - 1$ and $2q_n + q_n - 1 \leq 5q_n + r - p_n \leq 2q_n + p_n - 1$ hold by Lemma 4.1 (5) and $q \leq r \leq p_{n-1} - q_{n-1}$.

Hence $j' = j + 3$, $r' = 3q_n + r - p_n$, and $r' - r = 3q_n - p_n = q_{n-1}$ as expected.

If $p_{n-1} - q_{n-1} < r \leq p_{n-1}$, then the leading (resp. following) disk patch of the $(j+1)$-st pair, the $(j+2)$-nd pair, the $(j+3)$-rd pair the $(j+4)$-th pair, and the $(j+5)$-th pair are contained in a tetrahedron in the first region, the second region, the first region, the second region and the last region respectively. In fact, the leading (resp. following) disk patch of the $(j+\ell)$-th pair is contained in $\tau_{(\ell+2)q_n+r} = \tau_{(\ell+2)q_n+r-p_n} = \tau_{(\ell+2)q_n+r-2p_n}$, and $(1 <)p_{n-1} < 3q_n + r - p_{n-1} \leq q_n$, $q_n + p_{n-1} < 4q_n + r - p_n \leq 2q_n$, $0 < 5q_n + r - 2p_n \leq q_{n-1} (< q_n)$, $q_n < 6q_n + r - 2p_n \leq q_n + q_{n-1}$, $2q_n < 7q_n + r - 2p_n \leq 2q_n + q_{n-1}$ hold by Lemma 4.1 (5), (6) and $p_{n-1} - q_{n-1} < r \leq p_{n-1}$. Hence $j' = j + 5$, $r' = 5q_n + r - 2p_n$, and $r' - r = 5q_n - 2p_n = -p_{n-1} + q_{n-1}$ as expected.

The above arguments show the lemma since $p_{n-1}$ and $q_{n-1}$ are coprime, since the number of pairs of a compressing disk of the $k$-th step is $q_k + 1$ and since this number is smaller than $p_{n-1}$ because of the conditions $k \leq n - 1$ and $n \geq 3$.

\[\Box\]

**Lemma 5.2.**

1. No disk patch is placed in $\tau_{q_n}$, $\tau_{2q_n}$ and $\tau_{p_n}$, the last tetrahedra of the first, the second and the last regions.

2. For any integer $j$ with $1 \leq j \leq p_k + 1$ and any integer $k$ with $1 \leq k \leq n - 1$, the $j$-th pairs of disk patches of $(q_n - (k-1) - 1)/2$ compressing disks of the $k$-th step is in the same region.

**Proof.** As shown in Figure 6, the last tetrahedron $\tau_{p_n}$ does not contain a disk patch for $n = 2$. Then Lemma 5.1 (2) and an inductive argument guarantee that the last tetrahedron does not contain a disk patch. Since every compressing disk has the first pair of disk
patches and the last pair of disk patches in the last region as shown in Lemma 5.1 (1), no disk patches are in $\tau_{q_n} \cup \tau_{2q_n}$. Thus (1) holds.

Then (2) follows because the $j$-th pairs of disk patches of \((q_n-(k-1) - 1)/2\) compressing disks of the $k$-th step are contained in $q_n-(k-1) - 1$ consecutive tetrahedra.

\[ \square \]

**Lemma 5.3.**

(1) The first and the last pairs of disk patches of a compressing disk in section 3 are in tetrahedra where the compressing operations of the preceding steps have not occurred. The second through the $q_k$-th pairs of disk patches of a compressing disk in section 3 are in tetrahedra where such compressing operations have already occurred.

(2) It does not occur that a disk patch of the $j_1$-th pair of the $i_1$-th compressing disk of the $k$-th step and that of the $j_2$-th pair of the $i_2$-th compressing disk of the $k$-th step with $j_1 \neq j_2$ or $i_1 \neq i_2$ are in the same tetrahedron.

**Proof.** We show this lemma by an inductive argument on $n$. For $n = 2$, this lemma holds as shown in Figure 6. We assume it holds for the $(p_{n-1}, q_{n-1})$-lens space.

For $(p_n, q_n)$-lens space, this lemma holds in the last region by Lemma 5.1 and an inductive argument.

In the first and the second regions, disk patches are placed in the tetrahedra where compressing operations of the first step have already occurred by Lemma 5.2 (1). Moreover, Lemma 5.2 (2) implies that (2) of this lemma holds for disk patches in the first and the second regions because (2) of this lemma holds for disk patches in the last region where the first and the last patches are.

\[ \square \]

### 6. Maximal of Euler characteristic

In this section, we show that the surface $h_{n-1}$ constructed in section 3 is of maximal Euler characteristic among all closed non-orientable surfaces in the $(p_n, q_n)$-lens space.

A formula of the minimal crosscap number among those of all such surfaces is given by Bredon and Wood in [1]. We briefly recall it. We consider a $(p, q)$-lens space with $p$ even. $p/q$ can be presented as a continued fraction:

$$p/q = [a_0, a_1, \cdots, a_m] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots + \frac{1}{a_m}}}}$$

where $a_i$ are integers, $a_0 \geq 0$, $a_i > 0$ for $1 \leq i \leq m$, and $a_m > 1$. We define $b_0, b_1, \cdots, b_m$, inductively,

$$b_0 = a_0,$$
In fact, using the formula of Lemma 4.1 (2), we have
\( b_i = \begin{cases} 
  a_i & \text{if } b_{i-1} \neq a_{i-1} \text{ or if } \sum_{j=0}^{i-1} b_j \text{ is odd} \\
  0 & \text{if } b_{i-1} = a_{i-1} \text{ and } \sum_{j=0}^{i-1} b_j \text{ is even.}
\end{cases} \)

Then, the minimal crosscap number is \( x_{\max}(p, q) = \frac{1}{2} \sum_{i=0}^{m} b_i. \)

For our case of \((p_n, q_n)\)-lens space,
\[
\frac{p_1}{q_1} = 2, \quad \frac{p_2}{q_2} = 2 + \frac{1}{1 + \frac{1}{2}}, \quad \cdots, \quad \frac{p_n}{q_n} = 2 + \frac{1}{1 + \frac{1}{2} + \cdots + \frac{1}{2}}
\]
and hence, for \(0 \leq i \leq 2(n-1)\), \(a_i = \begin{cases} 
  2 \text{ (when } i \text{ is even)} \\
  1 \text{ (when } i \text{ is odd)}
\end{cases} \) and \(b_i = \begin{cases} 
  2 \text{ (when } i \text{ is even)} \\
  0 \text{ (when } i \text{ is odd)}
\end{cases} \).
Thus \(x_{\min}(p_n, q_n) = n\), and the maximal Euler characteristic is \(2 - n\).

7. \(n - 2\) PARALLEL SHEETS OF NORMAL DISKS

In this section, we will show the next lemma, which implies that the compressing operations in section 3 occur \(n - 1\) times in \(\tau_m\) with \(m = \sum_{u=1}^{n-1} q_u\), and hence \(\tau_m\) contains \(n - 2\) sheets of normal disk of type \(X_{m1}\).

**Lemma 7.1.** For an odd integer \(k\) with \(1 \leq k \leq n - 1\), the leading disk patch of the \((q_{n-k} - \sum_{j=1}^{n-(k+1)} q_j)\)-th pair of the \(((\sum_{r=1}^{k} q_r) + 1)/2\)-th compression disk of the \((n-k)\)-th step is in \(\tau_m\) with \(m = \sum_{u=1}^{n-1} q_u\). For an even integer \(k\) with \(1 \leq k \leq n - 1\), the following disk patch of the \((q_{n-k} - \sum_{j=1}^{n-(k+1)} q_j)\)-th pair of the \(((\sum_{r=1}^{k} q_r)/2\)-th compression disk of the \((n-k)\)-th step is in \(\tau_m\) with \(m = \sum_{u=1}^{n-1} q_u\).

Disk patches as in the above lemma actually appear in the construction in section 3. In fact, using the formula of Lemma 4.1 (2), we have \(q_1 + q_2 + \cdots + q_\ell + 1 = \frac{p_\ell}{2} + 1 \leq p_\ell < p_\ell + q_\ell = q_{\ell+1}\) for \(\ell \geq 1\). This implies \(1 \leq q_{n-k} - \sum_{j=1}^{n-(k+1)} q_j \leq q_{n-k} + 1\) and \(1 \leq ((\sum_{r=1}^{k} q_r) + 1)/2 \leq (q_{n-(n-k)-1}) - 1)/2\).

**Proof.** For an odd integer \(k\), the leading disk patch is contained in the tetrahedron numbered
\[
2(q_n + q_{n-1} + \cdots + q_{n-(n-k)-2}) + 2((\sum_{r=1}^{k} q_r) + 1)/2 - 1) + q_n(q_{n-k} - (\sum_{j=1}^{n-(k+1)} q_j) - 1) + 1
= 2(q_n + q_{n-1} + \cdots + q_{k+2}) + (\sum_{r=1}^{k} q_r) + 1 - 2 + q_nq_{n-k} - (\sum_{j=1}^{n-(k+1)} q_nq_j) - q_n + 1
\equiv 2(q_n + q_{n-1} + \cdots + q_{k+2}) + (\sum_{r=1}^{k} q_r) + 1 - 2 + q_{k+1} - (\sum_{j=1}^{n-(k+1)} q_{n-j+1}) - q_n + 1
= q_1 + q_2 + \cdots + q_{n-1}.
\]
where the symbol \( \equiv \) denotes congruence modulo \( p_n \), and it holds by the formula of Lemma 4.1 (4).

The proof is similar for an even integer \( k \), and we omit it.

\[ \square \]

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