Remarks on the $L^p$ convergence of Bessel–Fourier series on the disc

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Abstract

The $L^p$ convergence of eigenfunction expansions for the Laplacian on planar domains is largely unknown for $p \neq 2$. After discussing the classical Fourier series on the 2-torus, we move onto the disc, whose eigenfunctions are explicitly computable as products of trigonometric and Bessel functions. We summarise a result of Balodis and Córdoba (1999) regarding the $L^p$ convergence of the Bessel–Fourier series in the mixed norm space $L^p_{rad}(L^2_{ang})$ on the disk for the range $\frac{4}{3} < p < 4$. We then describe how to modify their result to obtain $L^p(D, r dr dt)$ norm convergence in the subspace $L^p_{rad}(L^q_{ang}) (\frac{1}{p} + \frac{1}{q} = 1)$ for the restricted range $2 \leq p < 4$.

1 Introduction

For a function $f \in L^2(\mathbb{T}^n)$, we can truncate its Fourier series by “spherical modes”

$$S_N f := \sum_{|k| \leq N} \hat{f}(k) e^{2\pi i k \cdot x}, \quad (1)$$

or by “cubic modes”

$$S_{[N]} f := \sum_{|k| \leq N} \hat{f}(k) e^{2\pi i k \cdot x}, \quad (2)$$

where

$$k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \quad \text{and} \quad |k|^2 = \sum_{j=1}^n |k_j|^2.$$ 

It is well known that $S_N f$ from Eq. (1) fails in general to converge to $f$ when $p \neq 2$. This follows, by standard transference arguments (see Grafakos,

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2014a), from Fefferman’s (1971) result that the indicator function of the ball is not an $L^p$-bounded Fourier multiplier for any $p \neq 2$. (See Grafakos, 2014b, for a detailed discussion and related results.) On the other hand, the square truncations from Eq. (2) are perfectly well-behaved for all $1 < p < \infty$ (see again Grafakos, 2014a).

Recently, Fefferman et al. (2021) have asked whether, given a differential operator with an orthonormal family $w_k$ of eigenfunctions, there is a choice \{\(w \in E_N : N \in \mathbb{N}\)\} of eigenfunctions such that the “truncations”

\[
S_{E_N} f = \sum_{w \in E_N} \langle f, w \rangle w
\]

are “well-behaved” in $L^p$ for all $1 < p < \infty$.

That this is not possible in general can be shown by considering the disc $\mathbb{D} \subset \mathbb{R}^2$. The eigenfunctions for the Laplacian on $\mathbb{D}$ are of the form

\[
e^{2\pi i \theta m} J_m(j_m^r \theta) \quad \text{for} \quad (r, \theta) \in [0, 1]^2, (m, n) \in \mathbb{Z} \times \mathbb{N}
\]

corresponding to the respective eigenvalues $4\pi^2 m^2 + (j_m^r)^2$. Here $J_m := J_{|m|}$ denotes a Bessel function of the first kind and $j_m^r := j_{|m|}$ its non-negative zeros (see Watson, 1995).

Consider the function $f(r) = r^{-3/2}$, which lies in the space $L^p([0, 1], r \, dr)$ for $1 \leq p < 4/3$. Wing (1950) proved that, for any choice of $J_m$, the 1-dimensional Bessel series of $f$ fails to converge in $L^p([0, 1], r \, dr)$. By letting $g(r, t) := f(r)$ in $L^p(\mathbb{D})$, it follows that the 2-dimensional Bessel–Fourier series of $g$ is

\[
\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{N}} a_m n J_m(j_n^m r)e^{2\pi i m t} = \sum_{n \in \mathbb{N}} a_n J_m(j_n^m r)
\]

and so does not converge to $g$ for any $m \geq 0$, regardless of the truncation method.

Thus, restrictions on the range of $p$ are to be expected. A natural range is $4/3 < p < 4$, since this is precisely the range that works for the 1-dimensional Bessel series (Wing, 1950). (It is instructive to compare this to the ranges of $L^p$ convergence for the Bochner-Riesz means on $\mathbb{R}^2$; see Grafakos, 2014a, for details. We will return to this question later.)

The best result known at this time is due to Balodis and Córdoba (1999), who reduced the problem of convergence on the disc to extant results on the convergence of Fourier and 1-dimensional Bessel series, albeit with a modified norm. We will exploit their argument to obtain $L^p$ convergence in a certain subspace of $L^p$. 

2
2 Mixed-norm convergence

Define the space $L^p_{\text{rad}}(L^2_{\text{ang}})$ by the inequality
\[
\|f\|_{p,2} := \left[ \int_0^1 \left( \sum_m |f_m(r)|^2 \right)^{p/2} r \, dr \right]^{1/p} = \|\| (f_m(r)) \|_{L^p(r \, dr)} < \infty, \tag{3}
\]
where $f_m(r)$ are the Fourier coefficients of the angular function $t \mapsto f(r,t)$:
\[
f_m(r) := \int_0^1 f(r, \theta) e^{-2\pi i m \theta} \, d\theta.
\]

Denote by $S_{N,M}f$ the partial sums of the Bessel–Fourier series of $f : \mathbb{D} \to \mathbb{C}$:
\[
S_{N,M}f(r,t) := M \sum_{m=-M}^M \sum_{n=1}^N a_{m,n} J_m(j^m_n r)e^{2\pi i mt}.
\]
(We drop the superscript “(d)” present in Balodis and Córdoba, 1999, since $d = 2$ will remain fixed in our discussion.)

**Theorem 1** (Balodis and Córdoba, 1999). *The operators $S_{N,M}$ are uniformly bounded on $L^p_{\text{rad}}(L^2_{\text{ang}})$ if, and only if, $\frac{4}{3} < p < 4$ when $N \geq AM + 1$ for an absolute constant $A > 0$.*

The norm convergence of the series to $f$ follows by the usual uniform boundedness argument. (See Balodis and Córdoba, 1999; cf. the analogous Fourier series argument in Grafakos, 2014a.)

To attack the proof of Theorem 1, they exploited the presence of the Fourier coefficients in the norm (3). Indeed, $S_{N,M}f$ is a trigonometric polynomial whose $m^{th}$ Fourier mode $(|m| \leq M)$ is
\[
S_{N,m}f_m(r) \equiv \sum_{n=1}^N a_{m,n} J_m(j^m_n r), \tag{5}
\]
which is precisely the 1-dimensional Bessel series summation operator for the radial function $r \mapsto f_m(r)$ in terms of the $m^{th}$ order Bessel function $J_m$. Thus,
\[
\|S_{N,M}f\|_{p,2} = \left[ \int_0^1 \left( \sum_{m=-M}^M |S_{N,m}f_m(r)|^2 \right)^{p/2} r \, dr \right]^{1/p} = \|\| (S_{N,m}f_m)_m \|_{L^p(r \, dr; \ell^2)}.
\]
so the boundedness of $S_{N,M}$ in $L^p_{\text{rad}}(L^2_{\text{ang}})$ is reduced to a uniform bound for vector-valued inequalities. Such bounds must be independent of the length, $2M + 1$, of the vector

$$(S_{N,m} f_{-m}, \ldots, S_{N,m} f_m).$$

Note that $S_{N,-m} = S_{N,m}$ by our convention that $J_m = J_{|m|}$ for $m \in \mathbb{Z}$. The functions $f_m$ and $f_{-m}$, however, are distinct in general, as they correspond to distinct Fourier coefficients.

Let us now turn to $L^p$ convergence on the disc, where the relevant norm is

$$\|f\|_{L^p(D)} = \left\| \|f(r,t)\|_{L^p(dt)} \right\|_{L^p(dr)}.$$

For $p \neq 2$ we cannot replace the inner “angular” $L^p$ norm by a sum of Fourier coefficients (see Chap. IV of Katznelson, 2004). However, for $p \geq 2$ we have the Reverse Hausdorff-Young Inequality:

$$\|g\|_{L^p(T)} \leq \|\hat{g}(k)\|_{C(\mathbb{Z})} \quad \text{whenever } p \geq 2 \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

We therefore have

$$\|f\|_{L^p(D)} = \left[ \int_0^1 \|f(r,t)\|_{L^p(T,dt)}^p r \, dr \right]^{1/p} \leq \left[ \int_0^1 \left( \sum_k |f_k(r)|^q \right)^{p/q} r \, dr \right]^{1/p} =: \|f\|_{p,q}.$$

Using this norm, we define the space

$$L^p_{\text{rad}}(\ell^q_{\text{ang}}) := \left\{ f \in L^p(D) : \|f\|_{p,q} < \infty \right\}.$$

Careful inspection of the proofs in Balodis and Córdoba (1999) shows that the space $\ell^2$ can be replaced by $\ell^q$ throughout when

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad 2 \leq p < 4.$$

The kernel of the 1-dimensional summation operators $S_{N,m}$ in Eq. (5) are controlled by weighted, vector-valued norms on the operators

$$\int_0^1 \frac{f(t)}{2 - x - t} \, dt \quad \text{and} \quad \int_0^1 \frac{f(t)}{x + t} \, dt,$$
the Hilbert Transform and the Hardy-Littlewood Maximal Functional. The weight \( r^{1-p/2} \) satisfies the 1-dimensional Muckenhoupt \( A_p \) condition if, and only if,

\[-1 < 1 - \frac{p}{2} < p - 1 \quad \text{that is} \quad \frac{4}{3} < p < 4\]

(see Grafakos, 2014a, Example 7.1.7) and, when this is the case, we have the inequalities

\[
\| (\mathcal{M} f_k)_k \|_{L^p(r^{1-p/2};\ell^q)} + \| (H f_k)_k \|_{L^p(r^{1-p/2};\ell^q)} \lesssim_{p,q} \| (f_k)_k \|_{L^p(r^{1-p/2};\ell^q)}.
\]

(See Córdoba, 1989, (B) on p. 25.) Furthermore, the kernels of the operators in Eq. (6) are nice enough that both are bounded on the space \( L^p([0,1], r^{1-p/2} \, dr) \) and, since they are positive, they admit vector-valued extensions to \( \ell^q \) too (Grafakos, 2014a, Theorem 5.5.10).

There is one more operator to consider on p. 280 of Balodis and Córdoba (1999):

\[
T_{N,m} f(x) := \sqrt{x} f_{\nu}(A_N^\nu x) \int_0^1 \sqrt{t} f_{\nu}(A_N^\nu t) f(t) \, dt
\]

with \( f_{\nu} \) as in Lemma 1 of Balodis and Córdoba (1999). For the stated range of \( p, p/q \geq 1 \), so we can apply Jensen’s Inequality as on p.280 of Balodis and Córdoba (1999), and we obtain the \( L^p_{\text{rad}}(\ell^q_\text{ang}) \)-boundeness of \( T_{N,\nu} \) too.

**Theorem 2.** The operators \( S_{N,M} \) are uniformly bounded on \( L^p_{\text{rad}}(\ell^q_\text{ang}) \) if \( 2 \leq p < 4 \), when \( N \geq AM + 1 \) for an absolute constant \( A > 0 \).

**Corollary 3.** Let \( 2 \leq p < 4 \). Then, if \( N_k, M_k \) are sequences of natural numbers such that \( N_k \geq AM_k + 1 \) and \( M_k \to \infty \), we have

\[
\lim_{k \to \infty} \| S_{N_k,M_k} f - f \|_{p,q} = 0
\]

for all \( f \in L^p_{\text{rad}}(\ell^q_\text{ang}) \). In particular,

\[
\lim_{k \to \infty} \| S_{N_k,M_k} f - f \|_{L^p(\mathbb{D})} = 0
\]

for all \( f \in L^p_{\text{rad}}(\ell^q_\text{ang}) \).

Note, however, that this method does not allow us to conclude \( L^p \) norm convergence for all \( f \in L^p \), but only for the smaller space \( L^p_{\text{rad}}(\ell^q_\text{ang}) \subset L^p(\mathbb{D}) \). The following function lies in \( L^p \) (since it is continuous, see Zygmund and Fefferman, 2003) but its Fourier coefficients are not \( \ell^q \) summable for any \( q > 2 \):

\[
g(t) := \sum_{k=2}^{\infty} \frac{e^{ik \log k}}{\sqrt{k(\log k)^2}^2} e^{2\pi i t}.
\]
In other words, $g \in L^p(\mathbb{D}) \setminus L^p_{\text{rad}}(L^q_{\text{ang}})$.

The above example, which is a counterexample to Plancherel’s theorem in $L^p$, hints at the underlying issue: we did not obtain $L^p$ bounds for the partial sum operators, so we cannot apply the proof of Corollary 3 to $L^p(\mathbb{D})$ directly.

\section{Concluding remarks}

To summarise, we have the following pieces of the convergence puzzle:

| $p$ range          | $[1, 4/3]$ | $[4/3, 2)$ | $2$   | $(2, 4)$ | $4$   | $(4, \infty)$ |
|--------------------|------------|------------|-------|---------|-------|--------------|
| $\|\cdot\|_{L^p(\mathbb{D})}$-convergence | No         | ?          | Yes   | $f \in L^p_{\text{rad}}(L^q_{\text{ang}})$ | ?     | No           |

Table 1: $L^p$ convergence of Bessel–Fourier series for various ranges of $p$.

In light of the existing results (Balodis and Córdoba, 1999; Wing, 1950; Benedek and Panzone, 1972/73), we might offer the following conjecture.

**Conjecture 1.** For all $4/3 < p < 4$ and $f \in L^p(\mathbb{D})$,

$$\lim_{k \to \infty} \|S_{N_k, M_k} f - f\|_{L^p(\mathbb{D})} = 0,$$

for some appropriate choice of $N_k, M_k \in \mathbb{N}$, $k \geq 1$. For $p$ outside of this range, convergence fails in general.

However, it is not completely clear that we can expect this. Córdoba (1989) proved that the ball multiplier is bounded on the mixed norm space $L^p_{\text{rad}}(L^2_{\text{ang}})$ in the range $2n/(n+1) < p < 2n/(n-1)$ that was originally conjectured for $L^p(\mathbb{R}^n)$, but disproved in Fefferman (1971). As we saw above, the $L^p_{\text{rad}}(L^2_{\text{ang}})$ argument works by essentially “eliminating” the angular eigenfunctions and reducing the problem to bounds on the one-dimensional Bessel–Fourier series. This “trick” is not available in $L^p(\mathbb{D})$, owing to the failure of Plancherel’s Theorem for $p \neq 2$, so obtaining uniform bounds on the operators $S_{N,M}$ is considerably more difficult.

**Acknowledgments**

I owe a special thanks to my doctoral advisor, Prof James C. Robinson, for many insightful discussions and his helpful criticism of early drafts. This work was supported by the EPSRC/2443915 studentship and the Warwick Mathematics Institute.
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