Schur Polynomials and Plücker degree of Schubert Varieties

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Abstract

The following is an informal report on the contributed talk given by the author during the INPANGA 2020[+1] meeting on Schubert Varieties.

The polynomial ring $B$ in infinitely many indeterminates $(x_1, x_2, \ldots)$, with rational coefficients, has a vector space basis of Schur polynomials, parametrized by partitions. The goal of this note is to provide an explanation of the following fact. If $\lambda$ is a partition of weight $d$, then the partial derivative of order $d$ with respect to $x_1$ of the Schur polynomial $S_\lambda(x)$ coincides with the Plücker degree of the Schubert variety of dimension $d$ associated to $\lambda$, equal to the number of standard Young tableaux of shape $\lambda$. The generating function encoding all the degree of Schubert varieties is determined and some (known) corollaries are also discussed.

1 Introduction

1.1 Let $G(r,n)$ be the complex Grassmann variety parametrizing $r$-dimensional vector subspaces of $\mathbb{C}^n$, $\mathbb{Q}_r \rightarrow G(r,n)$ be its universal quotient bundle and $c_i(Q_r)$ its Chern polynomial. Following [5, p. 271], let $F_\bullet(\Lambda) : 0 \subseteq F_1 \subset F_2 \subset \cdots \subset F_r \subset \mathbb{C}^n$ be a flag of $r \geq 1$ subspaces of $\mathbb{C}^n$ such that $\dim F_i = i + \lambda_{r-i}$. Then $\Lambda := (\lambda_1 \geq \cdots \geq \lambda_r) \in \mathcal{P}_{r,n}$, a partition whose Young diagram is contained in a $r \times (n-r)$ rectangle. Let $\Omega^\lambda$ be the class of the closed (Schubert) irreducible variety of dimension $|\lambda| := \lambda_1 + \cdots + \lambda_r$ ([5, Example 14.7.11]).

$$\Omega^\lambda(F_\bullet) := \Omega(F_1, \ldots, F_r) := \{ \Lambda \in G(r,n) \mid \dim(\Lambda \cap F_i \geq i) \},$$

Its Plücker degree $f^\lambda$ (which coincides, by [14, Theorem 2.39], with the number of standard Young tableaux of shape $\lambda$), does not depend on $n \geq r$.

1.2 Let now $B := \mathbb{Q}[x]$ be the polynomial ring in the infinitely many indeterminates $x := (x_1, x_2, \ldots)$. It possesses a basis parametrized by the set $\mathcal{P}$ of all the partitions

$$B := \bigoplus_{\lambda \in \mathcal{P}} \mathbb{Q} \cdot S_\lambda(x).$$

If each indeterminate $x_i$ is given weight $i$, then $S_\lambda(x)$ is a homogeneous polynomial of weighted degree $|\lambda|$. Consider the vector subspace $\tilde{B}_{r,n} := \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \mathbb{Q} \cdot S_\lambda(x)$ of $B$. The map

$$\begin{cases}
\pi_{r,n} : \tilde{B}_{r,n} &\rightarrow H_*(G(r,n), \mathbb{Q}) \\
S_\lambda(x) &\mapsto \Omega^\lambda
\end{cases}$$

is a vector space isomorphism for trivial reasons. The main result of this note is the following
1.3 Theorem.

\[ c_t(Q_r) \cap \Omega^\lambda = \pi_{r,n} \left( \exp \left( \sum_{i \geq 1} t^i \frac{\partial}{\partial x_i} \right) S_\lambda(x) \right) \quad (3) \]

In particular, equating the coefficients of the power of \( t \) of same degree:

\[ c_t(Q_r) \cap \Omega^\lambda = S_i(\tilde{\partial}) \cap \Omega^\lambda \quad (4) \]

where \( S_i(\tilde{\partial}) \) is an explicit polynomial expression in \( \tilde{\partial} := \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \ldots \right) \) corresponding to the coefficient of \( t^i \) in the expansion of \( \exp \left( \sum_{i \geq 1} t^i \frac{\partial}{\partial x_i} \right) \). Theorem 1.3 will be shortly proven in Section 5, basing upon the notion of Schubert derivation on an exterior algebra as in \([6, 9, 8]\) along with its extension to an infinite wedge power, as in \([10, 13]\).

1.4 Theorem 1.3 has a number of corollaries, all collected in Section 3. The most important is:

**Corollary 3.1.** For all \( \lambda \in \mathcal{P} \)

\[ f^\lambda = \frac{\partial^d S_\lambda(x)}{\partial x_1^d} \quad (5) \]

For example \( f^{(2,2)} = \frac{\partial^4 S_{(2,2)}}{\partial x_1^4} = 2 \), which is the Plücker degree of the Grassmannian \( G(1, \mathbb{P}^3) \) of lines in the three dimensional projective space. We emphasize that we have not been able to find any reference to (5), it looks new and is the main motivation of this note.

Recall that the classical way to compute \( f^\lambda \) is to rely on a formula due to Schubert, accounted for in \([5, \text{Example 14.7.11}]\). Because of its combinatorial interpretation in terms of Young tableaux, it is also computed by the celebrated hook length formula (see e.g. \([4, \text{p. 53}] \) or \([11, \text{Theorem 4.33}] \))

\[ f^\lambda := \frac{|\lambda|!}{\prod_{x \in Y(\lambda)} h(x)}, \]

proved in \([12]\), where \( Y(\lambda) \) is the Young diagram of \( \lambda \) and \( h(x) \) is the hook length of the box \( x \in Y(\lambda) \).

**Corollary 3.2.**

\[ f^\lambda := |\lambda|! \cdot \Delta_\lambda(\exp(t)) \quad (6) \]

Formula (6) has been first observed by O. Behzad during the investigations which lead to her Ph. D. Thesis \([1]\). See also the forthcoming \([2]\).

**Corollary 3.4.** Let \( \lambda \in \mathcal{P} \) and \( Y(\lambda) \) its Young diagram. Then

\[ \prod_{x \in Y(\lambda)} h(x) = \frac{1}{\Delta_\lambda(\exp(t))} \quad (7) \]

where \( h(x) \) denotes the hook length of the box \( x \) in the Young diagram of \( \lambda \).
Corollary 3.5 Let $\mathcal{P}_r$ be the set of all partitions of length at most $r$. For all $\lambda \in \mathcal{P}_r$, let $s_{\lambda}(z_r)$ denote the Schur symmetric polynomial in the $r$ indeterminates $(z_1, \ldots, z_r)$, i.e.

$$s_{\lambda}(z_r) = \frac{\det(z_j^{\lambda_j - j + i})}{\Delta_0(z_r)}$$

Then

$$\sum_{d \geq 0} \frac{t^d}{d!} \sum_{\lambda \vdash n} f_{\lambda}^d s_{\lambda}(z_r) = \exp(tp_1(z)) = \exp(t \cdot (z_1 + \cdots + z_r))$$

(8)

In particular, for all $d \geq 0$

$$(z_1 + \cdots + z_r)^d = \sum_{\lambda \vdash r - d} f_{\lambda} \cdot s_{\lambda}(z_r)$$

(9)

We additionally observe that evaluating the equality at $z_i = 1$, formula (9) turns into

$$x^d = \sum_{\lambda \vdash r - d} s_{\lambda}(1, \ldots, 1) f_{\lambda}.$$ 

(10)

Comparing (10) with [4, Formula (5), p. 52], one deduces that $s_{\lambda}(1, \ldots, 1)$ is precisely the number $d_{\lambda}(r)$ of standard Young tableaux of shape $\lambda$, whose entries are taken from the alphabet $\{1, 2, \ldots, r\}$.

1.5 Let $\sum S_j(\partial^i) t^j = \exp \left( \sum_{i \geq 1} \frac{t^i}{i} \frac{\partial}{\partial x_i} \right)$. It is not difficult to see that

$$\langle P(S_i(x)), S_{\lambda}(x) \rangle = P(S_i(\partial)) S_{\lambda}(x)$$

(11)

where by $P(S_i(\partial))$ is the evaluation of $P$ at $x_i = \frac{1}{i} \frac{\partial}{\partial x_i}$. In particular

$$x_1^n = \sum_{\lambda \vdash n} < x_1^n, S_{\lambda}(x) > S_{\lambda}(x) = \sum_{\lambda \vdash n} \frac{\partial^n S_{\lambda}(x)}{\partial x_1^n} \cdot S_{\lambda}(x),$$

which, due to Corollary 1.4, gives:

$$x_1^n = \sum_{\lambda \vdash n} f_{\lambda} S_{\lambda}(x)$$

from which, taking the derivative with respect to $x_1$ of order $n$, and again by Corollary ?? gives:

$$n! = \sum_{\lambda \vdash n} (f_{\lambda})^2$$

(12)

which is [4, Formula (4), p. 50].

In Section 2 we recall a few preliminaries. Then we will state and prove the main corollaries in Section 3. In Section 4 the notion of Schubert derivation (as in [6], [7], [9]) is reviewed. The short proof of Theorem 1.3 will conclude this short note.
2 Preliminaries

The content of this section is very well known and easily available in many common textbooks and its only purpose is to introduce the notation adopted in the sequel.

2.1 Partitions. Let $\mathcal{P}$ be the set of all partitions, namely the monoid of all non-increasing sequences $\lambda := (\lambda_1 \geq \lambda_2 \geq \cdots)$ of non-negative integers with finite support (all terms zero but finitely many). The non zero terms of $\lambda$ are called parts, the number $\ell(\lambda)$ of parts is called length. Let $\mathcal{P}_r := \mathcal{P} \cap \mathbb{N}^r_+$: it is the set of all partitions with at most $r$-parts and $\mathcal{P}_\infty = \mathcal{P}$. The Young diagram of a partition is the left justified array of $r$-rows such that the $i$th row has $\lambda_i$ boxes. For all $\leq r \leq n$ we denote by $\mathcal{P}_{r,n}$ the set of partitions whose Young diagram is contained in a $r \times (n-r)$ rectangle. Then $\mathcal{P}_{r,\infty} = \mathcal{P}$ and $\mathcal{P}_{\infty} = \mathcal{P}$. If $\lambda \in \mathcal{P}_{r,n}$, we denote by $\lambda^c$ the partition whose Young diagram is the complement of the Young diagram of $\lambda$ in the $r \times (n-r)$ rectangle. For example the complement of the partition $(3,3,2,1)$ in the $4 \times 3$ rectangle is $(2,1)$. Its complement in the $5 \times 4$ rectangle is $(4,3,2,1,1)$.

2.2 Schur Determinants. Let $A$ be any commutative algebra. To each pair

$$\left( f(t) = \sum_{n \geq 0} f_n t^n, \lambda \right) \in A[t^{-1}, t] \times \mathcal{P}_r$$

one attachesthe Schur determinant:

$$\Delta_\lambda(f(t)) = \det(f_{\lambda_i+1-j+i})_{1 \leq i,j \leq r} \in A$$

(13)

If $f(t) \in A[[t]]$, one think of it as a formal Laurent series with $f_j = 0$ for $j < 0$.

Putting $S_\lambda(x) := \det(S_{\lambda_i+1-j+i})$, it is well known that

$$B := \bigoplus_{\lambda \in \mathcal{P}} \mathbb{Q} \cdot S_\lambda(x)$$

2.3 For all $(i, \lambda) \in \mathbb{N} \times \mathcal{P}_r$, define

$$PF_i(\lambda) := \{ \mu \in \mathcal{P}_r \mid |\mu| = |\lambda| + i \quad \text{and} \quad \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_r \geq \lambda_r \}$$

and

$$PF_{-i}(\lambda) := \{ \mu \in \mathcal{P}_r \mid |\mu| = |\lambda| - i \quad \text{and} \quad \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_r \geq \mu_r \}$$

2.4 Proposition. Pieri’s rule for Schur $S$-function holds:

$$S_i(x) \cdot S_\lambda(x) = \sum_{\mu \in PF_i(\lambda)} S_\mu(x)$$

(14)

2.5 Proposition. The “dual” Pieri’s rule for $\Omega^\lambda$ holds:

$$c_i(Q_r) \cap \Omega^\lambda = \sum_{\mu \in PF_{-i}(\lambda)} \Omega_\mu(x)$$

(15)

Proposition 2.5 is basically another phrasing of [5, Example 14.7.1] It is very well known that $H_\pi(G(r,n), \mathbb{Q}) := \bigoplus_{\lambda \in \mathcal{P}_{r,n}} \mathbb{Q} \cdot \Omega_\lambda$. Moreover, Giambelli’s formula says that

$$\Omega^\lambda = \Delta_\lambda(c_\lambda(Q_r)) \cap [G(r,n)] := (c_{\lambda_i+1-j+i}(Q_r))_{1 \leq i,j \leq r},$$

where $\lambda^c$ denotes the complement of $\lambda$ in the $r \times (n-r)$ rectangle.
3 Corollaries to Theorem 1.3

In this section we assume Theorem 1.3, to find a few corollaries. As it is customary to do we set \( \sigma_i := c_i(Q_r) \).

3.1 Corollary Let \( \lambda \in P_{r,n} \). Then

\[
\sigma_1 \cap \Omega^\lambda = \pi_{r,n} \left( \frac{\partial S_\lambda}{\partial x_1} \right)
\]  
(16)

In particular, if \( d = |\lambda| \):

\[
f^\lambda := \sigma_1^d \cap \Omega^\lambda = \frac{\partial^d}{\partial x_1^d} S_\lambda(x)
\]  
(17)

**Proof.** Formula (16) comes from equating the coefficients of the linear terms on both sides of (3). Iterating it \( d \) times one obtains (17), where the projection \( \pi_{r,n} \) can be omitted (\( \pi_{r,n} \) is the identity on constants).

Formula (16) generalizes to

\[
\sigma_i \cap \Omega^\lambda = \pi_{r,n} \left( S_i(\overline{\partial})S_\lambda(x) \right)
\]

For example

\[
\sigma_3 \cap \Omega^\lambda = \pi_{r,n} \left[ \left( \frac{1}{6} \frac{\partial^3}{\partial x_1^3} + \frac{1}{2} \frac{\partial^2}{\partial x_1 \partial x_2} + \frac{1}{3} \frac{\partial}{\partial x_3} \right) S_\lambda(x) \right]
\]

3.2 Let \( \Delta_\lambda(\exp(t)) \) be the Schur determinant as in (13), attached to the exponential formal power series. For example

\[
\Delta_{(3,2,2)}(\exp(t)) = \begin{vmatrix}
1 & 1 & 1 \\
3! & 1! & 0! \\
1 & 1 & 1 \\
3! & 3! & 2!
\end{vmatrix} = 15
\]

3.2 Corollary.

\[
f^\lambda := |\lambda|! \cdot \Delta_\lambda(\exp(t))
\]  
(18)

**Proof.** Let \( d := |\lambda| \). Then

\[
f^\lambda = \frac{\partial^d S_\lambda(x)}{\partial x_1^d} = \begin{vmatrix}
S_{\lambda_1}(x) & S_{\lambda_2-1}(x) & \cdots & S_{\lambda_{r+1}}(x) \\
S_{\lambda_1-1}(x) & S_{\lambda_2}(x) & \cdots & S_{\lambda_{r+2}}(x) \\
\vdots & \vdots & \ddots & \vdots \\
S_{\lambda_1+r-1}(x) & S_{\lambda_2+r-2}(x) & \cdots & S_{\lambda_r}(x)
\end{vmatrix}
\]

Now \( S_i(x) = \frac{x_i^i}{i!} + g_i \), where

\[g_i := g_i(x_1, x_2, \ldots, x_i)\]
is a polynomial in which $x_1$ occurs with degree strictly smaller than $i$. Therefore the determinant occurring in (3.2) can be written as

$$
\begin{vmatrix}
\frac{x_1^{\lambda_1}}{\lambda_1!} + g_{\lambda_1} & \frac{x_1^{\lambda_2-1}}{(\lambda_2 - 1)!} + g_{\lambda_2-1} & \cdots & \frac{x_1^{\lambda_r+r-1}}{(\lambda_r + r - 1)!} + g_{\lambda_r+r-1} \\
\frac{x_2^{\lambda_1+1}}{(\lambda_1 + 1)!} + g_{\lambda_1+1} & \frac{x_2^{\lambda_2}}{\lambda_2!} + g_{\lambda_2} & \cdots & \frac{x_2^{\lambda_r+r-2}}{(\lambda_r + r - 2)!} + g_{\lambda_r+r-2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{x_r^{\lambda_1+r-1}}{(\lambda_1 + r - 1)!} + g_{\lambda_1+r-1} & \frac{x_r^{\lambda_2+r-2}}{(\lambda_2 + r - 2)!} + g_{\lambda_2+r-2} & \cdots & \frac{x_r^{\lambda_r}}{\lambda_r!} + g_{\lambda_r}
\end{vmatrix}
$$

(19)

Easy manipulations with determinants show that (19) can be written as

$$
x_1^d \Delta_\lambda(e^t) + F(x_1, x_2, \ldots, x_{\lambda_1})
$$

where $F$ is a polynomial in which $x_1$ occurs in degree smaller than $d$. Therefore

$$
f^\lambda := \frac{\partial^d S_\lambda(x)}{\partial x_1^d} = \frac{\partial^d}{\partial x_1^d} \left( x_1^d \Delta_\lambda(e^t) + F(x_1, x_2, \ldots, x_{\lambda_1}) \right) = d! \cdot \Delta_\lambda(e^t)
$$

3.3 Example. The degree of the Schubert variety $\Omega^{(3,2,1)}(F^\bullet)$ is

$$
f^{(3,2,1)} = 6! \begin{vmatrix}
\frac{1}{3!} & \frac{1}{1!} & 0 \\
\frac{1}{4!} & \frac{1}{2!} & 1 \\
\frac{1}{5!} & \frac{1}{3!} & 1
\end{vmatrix} = 16
$$

which is also the number of standard Young Tableaux of shape $(3, 2, 1)$.

3.4 Corollary. Let $Y(\lambda)$ denote the Young tableau of the partition $\lambda$. Let $h(x)$ denote the hook length of the box $x$ of $Y(\lambda)$. Then

$$
\prod_{x \in Y(\lambda)} h(x) = \frac{1}{\Delta_\lambda(e^t)}
$$

Proof. As in [14], the degree of a Schubert variety coincides with the number of standard Young tableaux of shape $\lambda$. Then one invoke the celebrated hook length formula proven by \textit{qed}

3.5 Corollary.

$$
\sum_{d \geq 0} \frac{t^d}{d!} \sum_{\lambda \in \mathcal{P}, |\lambda| = d} f^\lambda s_\lambda(z_1, \ldots, z_r) = \exp(t(z_1 + \cdots + z_r))
$$

(20)
as a consequence, for each $d \geq 0$, the Plücker coordinates of the symmetric polynomial $e_1(z_r)$ are the degree of the Schubert varieties $\Omega^\lambda(F^*)$
\[
(z_1 + \cdots + z_r)^d = \sum_{\lambda} f^\lambda s_\lambda
\]

**Proof.** Let $z_r := (z_1, \ldots, z_r)$ and consider the generating function
\[
\sum_{\lambda \in P_r} X^\lambda(z_r) = \exp \left(\sum_{i \geq 0} x_i p_i(z_r)\right)
\]
of the basis $X^\lambda$ of $B_r$. By [3], this is equal to
\[
\sigma_+(z_1, \ldots, z_r) = \exp \left(\sum_{i \geq 0} x_i p_i(z_r)\right)
\]
Then
\[
\sum_{d \geq 0} \frac{t^d}{d!} \sum_{|\lambda|=d} f^\lambda s_\lambda(z_r) = \exp \left(t \frac{\partial}{\partial x_1}\right) |_{t=0} \exp \left(\sum_{i \geq 0} x_i p_i(z_r)\right) = \exp((x_1 + t)p_1(z_r)) |_{x=0} = \exp(t p_1(z_r))
\]
which concludes the proof because $p_1(z_r) = e_1(z_r)$.

More generally, one can consider Schur polynomials in infinitely many indeterminates $z := (z_1, z_2, \ldots)$. We apply the translation operator along the $x_1$ direction to the generating function
\[
S_\lambda(x, z) = \exp(\sum_{i \geq 0} x_i p_i(z))
\]
of the basis elements of $B$, obtaining:
\[
\exp \left(\frac{t}{\partial x_1}\right) S_\lambda(x, z) = \exp \left(\exp(t + x_1)p_1(z) + \sum_{j \geq 2} x_j p_j(z)\right)
\]
Evaluating at $x = 0$ one gets
\[
\sum_{d \geq 0} \frac{t^d}{d!} \sum_{\lambda \vdash d} S_\lambda(x) = \exp(tp_1(z))
\]

**4 Review on Schubert Derivations**

Consider the following vector spaces over the rationals:
\[
V := \mathbb{Q}[X^{-1}, X], \quad V = V_\infty = \mathbb{Q}[X], \quad V_n := \frac{V}{X^n - V}
\]
along with their restricted duals

\[ \mathcal{V}^* = \bigoplus_{i \in \mathbb{Z}} \mathbb{Q} \cdot \partial^i, \quad V^* := \bigoplus_{i \geq 0} \mathbb{Q} \cdot X^i, \quad V_n^* = \bigoplus_{0 \leq i < n} \mathbb{Q} \cdot \partial^i. \]

where \( \partial^i \) stands for the unique linear form on \( \mathcal{U} \) such that \( \partial^i(X^i) = \delta_{ij} \). There is a natural chain of inclusions

\[ V_n \hookrightarrow V \hookrightarrow \mathcal{V} \]

where the first map is the natural section \( X^i + (X^n) \mapsto X^i \) associated to the canonical projection \( \mathcal{V} \hookrightarrow V_n \) and the second is by seeing a polynomial as a Laurent polynomial with no singular part.

### 4.1 Exterior Algebras

For all \( r \geq 0 \) and all \( \lambda \in \mathcal{P}_r \) let

\[ X^r(\lambda) = X^{r-1+\lambda_1} \wedge \cdots \wedge X^{\lambda_r}. \tag{22} \]

The exterior algebra of \( V_n \) (\( n \in \mathbb{N} \cup \{\infty\} \)) is:

\[ \wedge V_n = \bigoplus_{r \geq 0} r \wedge V_n \quad \text{where} \quad r \wedge V_n = \bigoplus_{\lambda \in \mathcal{P}_r,n} \mathbb{Q} \cdot X^r(\lambda) \]

For each \( m \in \mathbb{Z} \), let \( [X]^{m+(0)} := X^m \wedge X^{m-1} \wedge X^{m-2} \wedge \cdots \). The fermionic Fock space of charge 0 is

\[ F_0 := \bigoplus_{\lambda \in \mathcal{P}} \mathbb{Q} \cdot [X]^{0+\lambda} \]

where

\[ [X]^{0+\lambda} := X^{\lambda_1} \wedge X^{-1+\lambda_2} \wedge X^{-2+\lambda_2} \wedge \cdots \wedge X^{-r+1+\lambda} \wedge [X]^{r-1+(0)} \tag{23} \]

expression which does not depend on \( r \geq \ell(\lambda) \). The boson-fermion correspondence can be phrased by saying that \( F_0 \) is an invertible \( B \)-module generated by \( [X]^0 := [X]^{0+(0)} \), such that

\[ [X]^{0+\lambda} = S_\lambda(\chi)[X]^{0}. \]

Let now \( \mathcal{U} \) denote anyone of the spaces listed in (21).

### 4.2 Definition

A Hasse-Schmidt (HS) derivation on \( \bigwedge \mathcal{U} \) is a \( \mathbb{Q} \)-linear map \( \mathcal{D}(z) : \bigwedge \mathcal{U} \to \bigwedge \mathcal{U}[z] \) such that

\[ \mathcal{D}(z)(u \wedge v) = \mathcal{D}(z)u \wedge \mathcal{D}(z)v, \quad \forall u, v \in \bigwedge \mathcal{U} \tag{24} \]

Writing \( \mathcal{D}(z) = \sum_{i \geq 0} D_i z^i \in \text{End}_\mathbb{Q}(\bigwedge \mathcal{U})[z] \), equation (24) is equivalent to

\[ D_j(u \wedge v) = \sum_{i=0}^j D_i u \wedge D_{j-i} v. \tag{25} \]

If \( A \in \text{End}(\mathcal{U}) \), denote by \( \delta(A) \in \text{End}_\mathbb{Q}(\bigwedge \mathcal{U}) \) the unique derivation of \( \bigwedge \mathcal{U} \) such that

\[ \delta(A)u = A \cdot u, \quad \forall u \in \mathcal{U} = \bigwedge \mathcal{U}. \tag{26} \]
4.3 Proposition. The plethystic exponential

\[ D^A(z) = \text{Exp}(\delta(A^i)z) = \exp \left( \sum_{i \geq 1} \frac{1}{i} \delta(A)z^i \right) : \bigwedge \mathcal{U} \rightarrow \bigwedge \mathcal{U}[z] \]  

is the unique Hasse-Schmidt (HS) derivation on \( \bigwedge \mathcal{U} \), such that \( D^A(z)u = \sum_{i \geq 0} (A^i)u z^i \).

Proof. Based on the general fact that the exponential of a derivation of an algebra is an algebra homomorphism.

Abusing notation \( X \) will also stand for the endomorphism of \( \mathcal{U} \) given by \( u \mapsto Xu \), which is nilpotent if \( \mathcal{U} = V_n \) and \( n < \infty \).

4.4 Definition. The Schubert derivation \( \sigma_+(z) : \bigwedge V_n \rightarrow \bigwedge V_n[[z]] \) is

\[ \sigma_+(z) := \sum_{i \geq 0} \sigma_i z^i = \text{Exp}(\delta(X)z). \]  

Its inverse is

\[ \sigma_+(z) := \sum_{i \geq 0} (-1)^i \sigma_i z^i = \text{Exp}(-\delta(X)z). \]

They are clearly the unique HS derivation such that \( \sigma_+(z)u = \sum_{i \geq 0} X^i u \cdot z^i \) and \( \sigma_+(z)u = u - Xu \), for all \( u \in V_n \).

4.5 If \( u = \sum_{\lambda \in P_{r,n}} a_{\lambda} \cdot X^r(\lambda) \in \bigwedge^r V_n \), we denote by \( u^c \) the sum \( \sum_{\lambda \in P_{r,n}} a_{\lambda} X^r(\lambda^c) \).

4.6 Proposition. Let \( \sigma_-(z) := \sigma_+(z)^* \) be the \( \langle \cdot, \cdot \rangle \)-adjoint of the Schubert derivation \( \sigma_+(z) \), i.e.

\[ \langle \sigma_-(z)u, v \rangle = \langle u, \sigma_+(z)v \rangle. \]  

Then \( \sigma_-(z) = \text{Exp}(\delta(X^{-1})z) \), where \( X^{-1} \) is the unique endomorphism of \( V_n \) mapping \( X^j \) to \( X^{j-1} \) if \( j \geq 1 \) and to 0 otherwise.

Proof. To show that \( \sigma_-(z) \) is a HS-derivation one first identifies \( V^*_n \) with \( V_n \) through the isomorphism \( u \mapsto \langle u, \cdot \rangle \) and then argues as in [8, p.]. Then one observes that

\[ \langle \sigma_-(z)X^i, X^k \rangle = \langle X^i, \sigma_-(z)X^k \rangle = \langle X^i, X^{j+k} \rangle = \delta^{i,j+k} = \delta^{i-j,k} = \langle X^{i-j}, X^k \rangle \]

which proves that \( \sigma_-(z)X^i = X^{i-j} \). Thus \( \sigma_-(z) = \text{Exp}(\delta(X^{-1})z) \), because both sides restrict to the same endomorphism of \( V_n \).

Recall the notation 2.3. The \( \langle \cdot, \cdot \rangle \)-adjoint \( \sigma_-(z) \) of \( \sigma_+(z) \) will be called Schubert derivation as well. The reason is due to:

4.7 Theorem. Schubert derivations \( \sigma_{\pm}(z) \) satisfy Pieri’s rule, i.e.

\[ \sigma_+ X^r(\lambda) = \sum_{\mu \in PF_r(\lambda)} X^r(\mu) \]  

and

\[ \sigma_- X^r(\lambda) = \sum_{\mu \in PF_-(\lambda)} X^r(\mu) \]  

9
Proof. Formula 31 is, up to the notation, \[6\], Theorem 2.4. To prove (32), due to the fact that \((X^r(\lambda))_{\lambda \in \mathcal{P}_{r,n}}\) is an orthonormal basis of \(\bigwedge^r V_n\), one has
\[
\sigma_{-i}X^r(\lambda) = \sum_{\mu \in \mathcal{P}_{r,n}} \langle \sigma_{-i}X(\lambda), X^r(\mu) \rangle X^r(\mu)
\]
\[
= \sum_{\mu \in \mathcal{P}_{r,n}} \langle X^r(\lambda), \sigma_iX^r(\mu) \rangle X^r(\mu)
\]

Using Pieri’s formula (31) one has
\[
\langle X^r(\lambda), \sigma_iX^r(\mu) \rangle = \langle X^r(\lambda), \sum_{\nu \in PF_{i}(\mu)} X^r(\nu) \rangle
\]
and \(\nu \in PF_{i}(\mu)\) if and only if \(|\nu| = |\mu| + i\) and \(\nu_1 \geq \mu_1 \geq \cdots \geq \nu_r \geq \mu_r\). Thus
\[
\langle X^r(\lambda), \sigma_iX^r(\mu) \rangle = \delta_{\lambda,\nu}
\]
i.e. the only non zero coefficients are those for which \(\nu_1 = \lambda_1, \ldots, \nu_r = \lambda_r\), which are then the summands of \(\sigma_{-i}X^r(\lambda)\).

5 Proof of Theorem 1 and one generalization

By Proposition 2.4, the product \(S_i(x)S_\lambda(x)\) obeys Pieri’s formula. With respect to the inner product \((,\) for which \((S_\lambda(x))_{\lambda \in \mathcal{P}}\) is an orthonormal basis of \(B\), one has
\[
\left\langle \exp\left(\sum_{i \geq 0} x_i t^i\right) S_\lambda(x), S_\mu(x) \right\rangle = \left\langle S_\lambda(x), \exp\left(\sum_{i \geq 1} \frac{t^i}{i} \frac{\partial}{\partial x^i}\right) \right\rangle
\]
(33)

Because of (33), it follows that the coefficients \(S_i(\vec{\partial})\) of \(\exp\left(\sum_{i \geq 1} \frac{t^i}{i} \frac{\partial}{\partial x^i}\right)\) satisfy the dual Pieri formula as in (32). Therefore
\[
\pi_{r,n}(S_i(\vec{\partial})S_\lambda(x)) = \pi_{r,n}\left(\sum_{\mu \in PF_{-i}(\lambda)} S_\lambda(x)\right) = \sum_{\mu \in PF_{-i}(\lambda)} \pi_{r,n}(S_\lambda(x))
\]
\[
= \sum_{\mu \in PF_{-i}(\lambda)} \Omega^\mu = \sigma_i \cap \Omega^\lambda.
\]

Let
\[
F(u, t) := \exp\left(\sum_{i \geq 1} t_i \frac{\partial}{\partial x_i}\right)_{x_i = 0} \exp\left(\sum_{i \geq 1} u_i S_i(x)\right) \quad \text{.} \quad \text{(34)}
\]
5.1 Proposition.

\[ F(u, t) = \exp \left( \sum_{i \geq 1} u_i S_i(t) \right), \]  

where \( S_i(t) \) is the Schur polynomial in the variable \( t \).

**Proof.** It amounts to straightforward manipulation with the Taylor formula. \( \blacksquare \)

5.2 Remark. Formula (35) is the generating function of the “integrals” of product of special Schubert cycles. Putting \( h_i := S_i(\lambda(x)) \), it is the generating functions of

\[ \left( \frac{\partial}{\partial x_1} \right)^{|\mu|} h_\mu \]

where if \( \mu = (\mu_1, \ldots, \mu_r) \in \mathcal{P}_r \), one sets \( h_\mu := h_{\mu_1} \cdots h_{\mu_r} \).

5.3 A remarkable special case is obtained by setting \( t_1 = t \), and \( t_j = 0 \) for all \( j \geq 2 \):

\[ F(u, t) = \exp \left( \sum_{i \geq 1} u_i \frac{t^i}{i!} \right). \]  

(36)

If \( \mu = (1^{m_1} \cdots r^{m_r}) \) it is easy to see that

\[ \left( \frac{\partial}{\partial x_1} \right)^{|\mu|} h_\mu = \frac{(m_1 + 2m_2 + \cdots + rm_r)!}{1!(2!)^{m_2} \cdots (r!)^{m_r}} \]

because it is merely the coefficient of \( \frac{t^{m_1+\cdots+rm_r}}{(m_1 + \cdots + rm_r)!} \) in the expansion of \( F(u, t) \). In particular

\[ h_\mu = \sum_{\lambda^r = \mu, S_\lambda(x) > S_\lambda(x)} < h_\mu, S_\lambda(x) > S_\lambda(x) \]  

(37)

from which, iterating \( (m_1 + 2m_2 + \cdots + rm_r) \) times the derivative with respect to \( x_1 \) of (37):

\[ \frac{(m_1 + 2m_2 + \cdots + rm_r)!}{1!(2!)^{m_2} \cdots (r!)^{m_r}} = \sum_{\lambda^r = \mu, S_\lambda(x) > S_\lambda(x)} S_\mu(\partial) S_\lambda(x) \cdot f^\lambda \]  

(38)

which generalizes (12). Special cases of (38) are studied explicitly in [2].

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