Critical phenomena of social complex network

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We propose a multi-state spin model in order to describe equilibrial behavior of a society. Our physical spin system is inspired by the Axelrod model used in social network studies. In the framework of the statistical mechanics language, we analyze phase transitions of our model in which the spin interactions are interpreted as a mutual communication among individuals forming a society. The thermal fluctuations introduce a noise into the communication which suppresses long-range correlations. Below a certain phase transition point, large-scale groups of the individuals, who share a specific dominant property, are formed. The measure of the group sizes is an order parameter after the spontaneous symmetry breaking mechanism has occurred. By means of the Corner transfer matrix renormalization group algorithm, we treat our model in the thermodynamic limit and classify the phase transitions with respect to inherent degrees of freedom. Each individual is chosen to have two independent features and each feature has \(q\) traits (e.g., interests). A single first order phase transition is detected in our model if \(q > 2\), whereas two distinct continuous phase transitions are found if \(q = 2\). Evaluating the free energy, order parameters, specific heat, and the entanglement entropy, we classify the phase transitions in detail. The permanent existence of the ordered phase (the large-scale group formation) is conjectured below a non-zero transition point \(T_t \approx 0.5\) in the asymptotic regime \(q \to \infty\).

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I. INTRODUCTION

Most of the real-world systems can be represented as a complex network [1,2]. The complex network is composed of elements, where interactions lead to a collective emergent behavior of the whole system [2]. The collective behavior of a particular interest is social behavior, where the interacting elements are represented by individuals with relations of different kinds. The studied forms of social behavior include an opinion formation, cultural dynamics, language dynamics, formation of hierarchies, etc. [3,4].

In this paper we consider a social complex network as a system treated from the point of view of the statistical mechanics. We focus the attention on behavior of a large society in equilibrium. The society is represented by individuals who mutually interact via communication channels (e.g., sharing interests) with the nearest neighbors. The society is subject to special rules given by a model of the statistical mechanics we have introduced for this purpose. A noise plays an important role in this study. The noise interferes with the communication channels. With the increasing noise, the communicating individuals become weakly correlated on larger distances. Thus, the noise acts against formation of larger groups of individuals with a particular character, i.e., a set of shared features. In such a group, the individuals share a similar social background. One can quantify the size of the groups by calculating an appropriate order parameter, correlation length, etc., which are widely used in the statistical physics. If a phase transition point exists in a given model, this point frequently separates an ordered phase from a disordered one. The two phases can be determined by the order parameter being non-zero for the ordered phase and zero for the disordered, provided that the system is infinitely large, and a spontaneous symmetry-breaking mechanism has occurred below the transition point.

We, therefore, propose a multi-state spin model on a two-dimensional regular square lattice of the infinite size. Each vertex of the lattice is represented by a multi-state spin variable (being an individual with a certain cultural setting). We define special nearest-neighbor interactions among spins as to represent a conditional communication among individuals. The statistical Gibbs distribution introduces thermal fluctuations into our model acting on the given spin Hamiltonian. Here, the temperature can be identified as the noise introduced above. Imposing a magnetic field on given spin states has the same effect as, for instance, the mass media or advertisements. Having calculated effects of a magnetic field, no new (physically interesting) implications have been observed in our model. For this reason, we do not consider these effects in this paper.

The model we proposed mimics features of the well-known Axelrod model [6] which has been used in the studies of social behavior on the complex networks. Our studies go beyond the Axelrod model conjectures since we intend to study phase transitions on the complex networks, where number of the individuals required is infinite. Therefore, in such model, the spontaneous symmetry-breaking mechanism causes that a certain preferred character is selected resulting in the large group formation which is characterized by a non-zero order parameter.

This task is certainly nontrivial since our model has probably no analytical solution. Therefore, we apply the Corner Transfer Matrix Renormalization Group (CTMRG) algorithm [7] which is a powerful numerical...
tool in the statistical mechanics. The CTMRG can calculate all thermodynamic functions in high accuracy and enables to analyze the phase transitions as well as to control the spontaneous symmetry breaking. As discussed later, the phase transition point decreases with increasing number of the traits of the individuals. We intend to investigate the asymptotic case in this paper, when the number of the traits of each individual is infinite in order to locate the phase transition point, i.e., whether the ordered phase is permanently present or not.

The paper is organized as follows. In Sec. II we define the Hamiltonian of our model and introduce the thermodynamic functions, which are used in the analysis of the phase transition. The Sec. III contains numerical calculations explained in the language of statistical physics. In Sec. IV we discuss and interpret our results in terms of the communicating individuals influenced by the presence of the noise.

II. LATTICE MODEL

A. Hamiltonian and density matrix

A classical spin lattice model is considered on a regular two-dimensional square lattice, where the nearest-neighbor multi-state spins sitting on the lattice vertices can interact. Let $\sigma_{i,j} = 0, 1, \ldots, n - 1$ be a generalized spin with integer degrees of freedom $n$. The subscript indices $i$ and $j$, respectively, denote a position of each lattice vertex, where the spins are placed, in the $X$ and $Y$ coordinate system on the underlying square lattice of the infinite size, i.e., $-\infty < i, j < \infty$. We select the $n$-state clock (vector) model \[ \mathcal{H} = -J \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=0}^{1} \cos(\theta_{i,j} - \theta_{i+k,j-k+1}) \] for this purpose with the Hamiltonian

\[ \mathcal{H} = -J \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \sum_{k=0}^{1} \cos(\theta_{i,j} - \theta_{i+k,j-k+1}). \quad (1) \]

The interaction term $J$ acts between the nearest-neighbor vector spins $\theta_{i,j} = 2\pi\sigma_{i,j}/n$. The $k$ summation includes the horizontal and the vertical directions on the square lattice.

Let us generalize the spin clock model so that the interaction term $J$ contains a special attribute, i.e., extra spins are added. We, therefore, introduce additional degrees of freedom for each vertex. The Hamiltonian in Eq. (1) can be further modified into the form $\mathcal{H} = \sum_{i,j,k} J_{ijk} \cos(\theta_{i,j} - \theta_{i+k,j-k+1})$. The position-dependent term $J_{ijk}$ describes the spin interactions of the $n$-state clock model controlled by additional $q$-state Potts model $\delta$-interactions. Then, the total number of the spin states is $qn$ on each vertex $i,j$. We study the simplified case when $q = n$ starting from the case of $q = 2$ up to $q = 6$ which is computationally feasible. (For the more general case when $q \neq n$, we do not expect substantially different physical consequences as those studied in this work.)

Hence, our multi-state spin model contains two independent $q$-state spins $\sigma_{i,j}^{(1)} = 0, 1, 2, \ldots, q - 1$ and $\sigma_{i,j}^{(2)} = 0, 1, 2, \ldots, q - 1$, which are distinguished by the superscripts (1) and (2); note that the superscripts are not meant to be the power exponents. It is instructive to introduce a $q^2$-variable $\xi_{i,j} = q\sigma_{i,j}^{(1)} + \sigma_{i,j}^{(2)} = 0, 1, \ldots, q^2 - 1$. The Hamiltonian of our model has its final form

\[ \mathcal{H} = \sum_{i,j=-\infty}^{\infty} \sum_{k=0}^{1} \left\{ J_{ijk}^{(1)} \cos(\theta_{i,j}^{(1)} - \theta_{i+k,j-k+1}^{(1)}) + J_{ijk}^{(2)} \cos(\theta_{i,j}^{(2)} - \theta_{i+k,j-k+1}^{(2)}) \right\}, \]

noticing that $\theta_{i,j}^{(\alpha)} = 2\pi\sigma_{i,j}^{(\alpha)}/q$, where

\[ J_{ijk}^{(\alpha)} = -J \delta(\sigma_{i,j}^{(\alpha)}, \sigma_{i+k,j-k+1}^{(\alpha)}) \]

\[ = \begin{cases} -J, & \text{if } \sigma_{i,j}^{(\alpha)} = \sigma_{i+k,j-k+1}^{(\alpha)}, \\ 0, & \text{otherwise}. \end{cases} \quad (3) \]

The superscript $(\alpha)$ can take two values only as mentioned above. The Potts-like interaction $J_{ijk}^{(\alpha)}$ is represented by a diagonal $q \times q$ matrix with the elements $-J$ on the diagonal.

Thus defined model can also describe a conditionally communicating (interacting) individuals of a society. The society is modeled by individuals $(\xi_{i,j})$ and each individual has two distinguished features $\sigma_{i,j}^{(1)}$ and $\sigma_{i,j}^{(2)}$. Each of the features can assume $q$ different values (traits). In particular, an individual positioned at $(i,j)$ vertex of the square lattice communicates with the nearest neighbor, say at $(i+1,j)$, by comparing values of the first feature $\sigma_{i,j}^{(1)}$. This comparison is carried out by means of the $q$-state Potts interaction $\sigma_{i,j}^{(1)} + \sigma_{i+k,j-k+1}^{(1)}$. If the Potts interaction is evaluated as non-zero, i.e. $J_{ijk}^{(1)} = -J$, (they share the identical trait), then the individuals continue in the communication via the $q$-state clock interaction term of the remaining feature $\cos(\sigma_{i,j}^{(2)} - \sigma_{i+k,j-k+1}^{(2)})$. The cosine term enables a broader communication spectrum than the Potts term. Such Potts-clock conditional communication should be symmetric, therefore, we add the terms exchanging (swapping) the role of the feature (1) for (2) in our model. In particular, the Potts-like communication first compares the feature $J_{ijk}^{(2)}$ and the cosine term with the feature (1) follows. (Allowing extra interactions between the two features within each individual and/or the cross-interactions of the two adjacent individuals is to be studied elsewhere.) The total number of all individuals is considered to be infinite in order to analyze the phase transition critical phenomena with the spontaneous symmetry breaking.

In the framework of the statistical mechanics, we investigate a combined $q$-state Potts and $q$-state clock model which is abbreviated as the $q^2$-state spin model. Recall that the spin interactions are related to the communication between the individuals. Each $q^2$-state spin variable
leisure-time interests while the other feature $\sigma^{(2)}$ involves working duties. In the former feature of the leisure-time interests, one could list $q = 3$ properties, such as reading books, listening to music, and hiking, whereas the second feature, could consist of manual activities, intellectual activities, and creative activities. The thermal fluctuations, induced by thermodynamic temperature $T$ of the Gibbs distribution, are meant to describe the noise which weakens the communication between the individuals as $T$ increases. The higher the noise, the stronger suppression of the communication is resulted.

The classification of the phase transitions in our model is performed by numerical calculation of the partition function

$$Z = \sum_{\{\sigma\}} \exp \left( -\frac{\mathcal{H}}{k_B T} \right),$$

where the configuration sum has to be taken through all spin configurations $\{\sigma\}$ on the infinite lattice. Here, Boltzmann constant and temperature, respectively, are denoted by $k_B$ and $T$. The partition function is evaluated by means of the CTMRG algorithm [2] which is a classical generalization of the Density Matrix Renormalization Group [10]. In the CTMRG language, the whole square lattice is divided into four identical corners, the so-called corner transfer matrices, and the RG transformations carry out the numerical calculations with efficiency and high accuracy [2].

A typical formulation of an observable (averaged thermodynamic function) $\langle \hat{X} \rangle$ obeys the standard expression

$$\langle \hat{X} \rangle = Z^{-1} \sum_{\{\sigma\}} \hat{X} \exp \left( -\frac{\mathcal{H}\{\sigma\}}{k_B T} \right) = \text{Tr}_s \left( \hat{X} \hat{\rho}_s \right),$$

where we introduced a matrix $\hat{\rho}_s$ which is commonly called the reduced density matrix

$$\hat{\rho}_s = Z^{-1} \sum_{\{\sigma_i\}} \exp \left( -\frac{\mathcal{H}\{\sigma\}}{k_B T} \right).$$

It is a classical counterpart of the one-dimensional quantum reduced density matrix. It is defined for a subsystem $s$ in contact with an environment $e$. The configuration sum is taken over all spins of the environment $\{\sigma_e\}$ but those of the subsystem $\{\sigma_s\}$. Then, we obtain the reduced density matrix of the subsystem $\{\sigma_s\}$. The subsystem also contains spins, where the observable $\hat{X}$ acts. Notice that $\text{Tr}_s \hat{\rho}_s = 1$, and in classical physics, it has the meaning of the partition function $Z$ normalized to unity. Notice that our model can be effectively thought of as a system with two non-trivially coupled sub-lattices, where either sub-lattice is composed of the $q$-state variables distinguished by the feature $\alpha$.

### B. Thermodynamic functions

The CTMRG algorithm is used to obtain the thermodynamic functions. We select only the most relevant thermodynamic functions to specify the type of the phase transition unambiguously. An order parameter $\langle O \rangle$ in the simplest case is non-zero in an ordered spin phase while it becomes zero in the disordered phase. The phase transition occurs exactly at a point where the non-zero order parameter turns to zero. A continuous transition usually leads to the second-order phase transition, and the discontinuous behavior signals the first-order phase transition. However, a detailed analysis of the free energy and other thermodynamic functions is required to confirm the order of the transition. The sub-site order parameter for a given feature $\alpha$ is

$$\langle O_\alpha \rangle = \text{Tr}_s \left( \hat{O}_s^{(\alpha)} \hat{\rho}_s \right) = \text{Tr}_s \left[ \cos \left( \frac{2\pi q}{q^2} \right) \hat{\rho}_s \right]$$

with $\hat{\rho}_s$ being the reduced density matrix of the subsystem $s$, where the sub-site order parameter $\hat{O}_s^{(\alpha)}$ is measured. For simplicity, we exclude the subscripts $i, j$ from the order parameter notation. If we are interested in evaluating the order parameter on the same vertex for both spins at once, the complete order parameter has the definition

$$\langle O \rangle = \text{Tr}_s \left( \hat{O}_s \hat{\rho}_s \right) = \text{Tr}_s \left[ \cos \left( \frac{2\pi x_{i,j}}{q^2} \right) \hat{\rho}_s \right].$$

Analogously, we simplify the expression $\xi = q\sigma^{(1)} + \sigma^{(2)}$ and omit the position subscripts $i, j$. As shown later, both of the definitions provide a useful and different insight into our model.

Another important thermodynamic function to calculate is the entanglement von Neumann entropy $S_v$. It follows the standard quantum-mechanical definition

$$S_v = -\text{Tr}_s \left( \hat{\rho}_s \log_2 \hat{\rho}_s \right).$$

This quantity reflects the correlation effects, which are maximal at the phase transition point. We also define the Helmholtz free energy $F$ per spin

$$F = -k_B T \ln (Z).$$

The derivatives of the free energy determine other thermodynamic functions used in the classification of the phase transition, namely, the first derivative with respect to temperature $T$ results in the internal energy

$$U = -T^2 \frac{\partial (F/T)}{\partial T},$$

which is equivalent to the nearest-neighbor correlation function. Taking the consequent derivative of the internal energy with respect to $T$ yields the specific heat

$$C = \frac{\partial U}{\partial T}.$$
which has a non-analytic (divergent) behavior at a phase transition. We calculate these thermodynamic functions within a high accuracy keeping the number of the CTMRG states kept 100 ≤ \( m \) ≤ 200. Such choice leads to the truncation error as small as \( \varepsilon \lesssim 10^{-8} \) around the phase transition, otherwise the error is many orders lower.

### III. NUMERICAL RESULTS

The phase transitions in the classical spin systems are induced by the thermal fluctuations while varying the temperature \( T \) in Eq. (1). We use dimensionless units, in which \( J = k_B = 1 \). Then, \( J = 1 \) corresponds to the ferromagnetic spin ordering. We begin by considering the simplest non-trivial case of \( q = 2 \). Figure 1 shows the sub-site order parameter \( \langle O \rangle \) with respect to temperature \( T \) which is identical for both \( \alpha = 1 \) and \( \alpha = 2 \). The second order phase transition is resulted at the critical temperature \( T_c = 2.1973 \). The associated universality scaling \( \langle O \rangle \propto (T - T_c)^\beta \) results in the critical exponent \( \beta = 0.1113 \approx \frac{1}{9} \). The inset shows the linear behavior of \( \langle O \rangle^{1/\beta} \) when approaching the critical temperature \( T \) from the ferromagnetic ordered phase. The critical exponent \( \beta = \frac{1}{9} \) in our model at \( q = 2 \) belongs to the 3-state Potts model universality class [9]. This exponent differs from the well-known Ising universality, where \( \beta = \frac{1}{4} \) and does not belong to the simple 2- and/or 4-state Potts and/or clock model universality classes.

The sub-site order parameter \( \langle O_{\alpha} \rangle \) for \( q = 3, 4, \) and 5 is depicted in Fig. 2. It gradually decreases with increasing temperature, but at certain temperature it discontinuously jumps to zero. Such behavior usually suggests the first order phase transition. To confirm this statement, the free energy \( F \) per spin is plotted with respect to \( T \) for two different boundary conditions (BCs). The fixed (open) BCs are imposed at the very beginning of the iterative CTMRG scheme in order to enhance (suppress) spontaneous symmetry breaking resulting in the ordered (disordered) phase in a small vicinity of the phase transition point. In particular, if the fixed BCs are applied, the spontaneous symmetry-breaking mechanism selects one of \( q^2 \) free energy minima with the details specified by the fixed BCs. On the contrary, the open BCs prevent the spontaneous symmetry breaking from falling into a minimum and makes the system be in a metastable state below the phase transition. Since the first-order phase transition is known to exhibit the coexistence of two phases in a small temperature interval around the phase transition, such analysis with the two different BCs is inevitable to locate the phase transition accurately. The insets for the three cases, \( q = 3, 4, 5 \), show the normalized Helmholtz free energy around the transition temperature. The red and blue symbols of the free energy, respectively, correspond to the fixed and the open BCs. The temperature interval, in which both of the symbols are present, is the region, where the two phases can coexist. The true phase transition point \( T_t(q) \) is located at the free energy crossover. The correct equilibrium free energy is shown by the black-green dashed line (being always the lower one). The free energy is a non-analytical function at \( T_t(q > 2) \), as it exhibits a kink for the first-order phase transitions. If taking the derivatives of \( F \) with respect to \( T \), the discontinuity of the thermodynamic functions in Eqs. (11) and (12) has to be detected.

The phase transition temperatures for \( q > 2 \) are calculated with a high precision resulting \( T_t(3) = 1.60909, T_t(4) = 1.30175, T_t(5) = 1.12684, \) and \( T_t(6) = 1.03234 \) (not plotted). It is obvious that \( T_t(q) \) gradually decreases.
with increasing $q$. It is worth to mention that the first-order phase transition is not critical in such sense that the correlation length is always finite and does not diverge at the phase transition temperature (not shown) in contrast to the second order phase transition, when the correlation length diverges at the phase transition (after the spontaneous symmetry breaks). For this reason, we reserve the term critical temperature, $T_c$, for the second-order phase transition only which is resulted in our model only if $q = 2$. Otherwise, we have used the notation transition temperature $T_t$ above. The free energy is not sensitive to the application of the different BCs if the critical second order phase transition is present provided that the system is in the thermodynamic limit.

The entanglement von Neumann entropy $S_v$ when $q = 2$ is plotted in Fig. 3. Evidently, our calculations of $S_v$ result in two maxima, not the only one maximum as one expects for the single phase transition observed in Fig. 1. Hence, the entanglement entropy can indicate the existence of the second phase transition which cannot be detected by the sub-site order parameter $\langle O_{\alpha} \rangle$. The first phase transition at lower temperature, $T_1(q) = 2.1973$, is identical to $T_c$ plotted in Fig. 1 whereas the higher phase transition temperature appears at $T_2(q) = 2.57$. To support this result, we calculated the specific heat $C$, as shown in the inset. The two evident maxima in $C$ remain present in our model at the identical critical temperatures $T_1(q)$ and $T_2(q)$. The sub-site order parameter $\langle O_{\alpha} \rangle$ in Fig. 1 does not reflect the higher phase transition temperature at all. Thus, we have achieved a new phase transition point, which is likely pointing to a topological ordering. Having analyzed both of the phase transitions for $q = 2$, we exclude presence of the Berezinski-Kosterlitz-Thouless (BKT) phase transition in this case [11].

The entanglement entropy $S_v$ has just one maximum for any $q > 2$ as seen in Fig. 4. The discontinuity of $S_v$ at the phase transition temperature $T_1(q)$ is also characteristic for the first order phase transition. The three insets display the specific heat with the single maximum for each $q > 2$ at the transition temperature, which is in full agreement with the sub-site order parameter. Therefore, we conclude the existence of the single phase transition point of the first order if $q > 2$.

Figure 4 shows the complete order parameter for the case of $q = 2$ as defined in Eq. 8. Obviously, the non-analytic behavior of $\langle O \rangle$ points to the two critical temperatures $T_1(2)$ and $T_2(2)$, which coincide with the critical temperatures depicted in Fig. 5. Since the $q^2$-state spin $\xi$
FIG. 6: (Color online) The complete order parameter acting on the whole $3^2$-state spin $\xi$ when plotted with respect to all of the nine reference states $\phi = 0, 1, \ldots, 8$. For comparison, the insets shows all nine order parameter projections for the standard 9-state clock model (the upper panel) and for the standard 9-state Potts model (the lower panel).

has four degrees of freedom, the mechanism of the spontaneous symmetry breaking at low temperatures causes that the free energy may have up to four minima with respect to the complete order parameter. To access any of the four free energy minima can be numerically controlled by the changing the reference spin state $\phi = 0, 1, \ldots, q − 1$. Let us denote the four spin state at the vertex by the notation $[\sigma^{(1)} \sigma^{(2)}]$. There are four possible scenarios for the order parameter $\langle O \rangle$ as shown in Fig. 6. These scenarios are depicted by the black circles ($\phi = 0$), the red diamonds ($\phi = 1$), the blue squares ($\phi = 2$), and the green triangles ($\phi = 3$), and they correspond to the following vertex configurations $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow\rangle$, $|\downarrow\uparrow\rangle$, and $|\downarrow\downarrow\rangle$, respectively.

At zero temperature there are three minima of the free energy leading to the three different complete order parameters $\langle O \rangle$ being $−1, 0,$ and $+1$. There are four minima of the free energy if $0 < T < T_1(2)$ so that the order parameter has four different values $\langle O \rangle = −1 + \varepsilon, −\varepsilon, +\varepsilon,$ and $+1 − \varepsilon$ with the condition $0 < \varepsilon < \frac{1}{2}$. That means that two states share the same free energy minimum in zero order parameter at $T = 0$ when $\varepsilon = 0$. In the temperature interval $T_1(2) \leq T < T_2(2)$, there are only two free energy minima present and the pair $\phi = 0$ and $\phi = 3$ become indistinguishable as well as for the pair $\phi = 1$ and $\phi = 2$. The only single free energy minimum is resulted at $T \geq T_2(2)$ when the order parameter is zero which is typical for the disordered phase.

It is worth to mention that at the temperatures in between $T_1(2)$ and $T_2(2)$, a pair of the site configurations $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ is indistinguishable for the complete order parameter (i.e. the black and green symbols coincide), and the same topological uniformity happens for the pair of the site configurations $|\uparrow\downarrow\rangle$ and $|\down\uparrow\rangle$. In other words, the anti-parallel alignment for the first spin $\sigma^{(1)}$ and the identical anti-parallel alignment for the second spin $\sigma^{(2)}$ are resulted in between the two critical phase transition temperatures $T_1(2) \leq T < T_2(2)$. This is also equivalent to the anti-parallel coupling between the two non-interacting spins on the same vertex.

In the same analogy, we plotted the complete order parameter if $q = 3$ in Fig. 6. The free energy has five free energy minima at zero temperature or nine minima at $0 < T < T_1(3)$. Just the single free energy minimum is characteristic for the disordered phase at $T \geq T_1(3)$. In order to compare and stress differences of the order parameter in our model with respect to the standard 9-state clock model and the 9-state Potts models, we plot the respective order parameter in the insets of Fig. 6. In the former case (the clock model) there are always five free energy minima resulting in the five order parameters out of the nine below the higher phase transition temperature. We remark that the BKT phase transitions of the infinite order present in the $q \geq 6$-state clock models. In the latter case (the Potts model), there are only two free energy minima below the first order phase transition.

FIG. 7: (Color online) The there variants of the extrapolated transition temperature $T_1(q \to \infty)$ by the power-law fitting (the green long-dashed line), the exponential fitting (the blue full line), and the inverse proportionality (the red short-dashed line).

If the transition temperature $T_1(q)$ is extrapolated toward the asymptotic limit $q \to \infty$, a non-zero phase transition temperature $T_1(\infty) \approx 0.5$ is resulted. We carried out the three independent extrapolations as shown in Fig. 7 by means of the least square fitting. In particular, the power-law $T_1(q) = T_1(\infty) + a_0 q^{-a_1}$, the exponential $T_1(q) = T_1(\infty) + a_0 (1 - e^{-a_1 q})$, and the inverse proportional $T_1(q) = T_1(\infty) + a_0 q^{-1}$ fitting functions are used to find $T_1(\infty)$, $a_0$, and $a_1$ parameters. All of them give the non-zero transition temperature $T_1(\infty) \approx 0.5$. Hence, the existence of the ordered phase is conjectured.
IV. DISCUSSION AND CONCLUSION

Having been motivated by the Axelrod model, we studied a multi-state spin model we have proposed for this purpose. Our model is defined on the two-dimensional infinite square lattice. The spin model is analyzed by the numerical tools of the statistical physics in order to deduce equilibrial properties of the complex networks modeling behavior of a society. We focused on analyzing phases and the phase transitions. Our spin model can be mapped onto mutually communicating individuals subject to a noise which prevents them to communicate. The gradual increase of the noise disables the formation of larger groups of the individuals who share specific cultural features, e.g., interests (the group size is quantified by the order parameter). The raising noise breaks correlations at longer distances as it has the same character as the thermal fluctuations. Each individual is characterized by two independent features (1) and (2) and each feature assumes \( q \) different traits (e.g. interests) resulting in \( q^2 \) cultural settings of each individual.

We have found out that such complex social network exhibits the two phase transitions when \( q = 2 \). Using the above-mentioned examples, let the first two-state leisure-time feature represent the ‘reading of books’ \( \sigma^{(1)} = \uparrow \) and the ‘listening to music’ \( \sigma^{(1)} = \downarrow \), whereas the second two-state feature involves the ‘manual activity’ \( \sigma^{(2)} = \uparrow \) and the ‘intellectual activity’ \( \sigma^{(2)} = \downarrow \). Both of the phase transitions are continuous separating the three phases, which are equivalent to the (i) low-noise regime, (ii) the medium-noise regime, and (iii) the high-noise regime.

(i) In the low-noise regime, the individuals tend to form one dominant group, where the complete order parameter can have four values, see Fig. 5. The statistical probability of forming the dominant groups is proportional to the complete order parameter. (ii) In the medium-noise regime, an interesting topological regime reveals two equally likely traits of the individuals. In the language of the social network, the pairing of the cultural settings coincides either with (1) the equal mixture of those individuals who ‘read books’ and ‘do manual activity’ \( (\uparrow \downarrow) \) and the individuals who ‘listen to music’ and ‘do intellectual activity’ \( (\downarrow \uparrow) \) or (2) the equal mixture of those who ‘listen to music’ and ‘do manual activity’ \( (\downarrow \uparrow) \) and those who ‘read books’ and ‘do intellectual activity’ \( (\uparrow \downarrow) \). (iii) In the high-noise regime, the groups are not significant (small correlation length), and the individuals behave in a completely uncorrelated way.

The discontinuous phase transition of the first order is present when the number of the traits \( q > 2 \). In the low-noise regime, larger groups of individuals with a given cultural setting (any out of \( q^2 \) ) are formed. The selected cultural setting of the dominant group sizes is proportional to the order parameter \( \langle O \rangle \). This is equivalent to the ordered multi-state spin phase below the phase transition noise \( T_1(q) \). The regime of the uncorrelated individuals (disordered phase) appears above the phase transition noise. The low-noise regime is separated from the high-noise regime by a discontinuous jump of the group size (order parameter).

If the phase transition noise is extrapolated to the asymptotic number of the traits (cultural settings) \( q \to \infty \), we conjectured that the phase transition noise \( T_1(\infty) \) remains finite (being approximately 0.5). We interpret this result as the permanent existence of the correlated groups, in which the individuals share interests.

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