HARVEY LAWSON MANIFOLDS AND DUALITIES

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Abstract. The purpose of this paper is to introduce Harvey-Lawson manifolds and review the construction of certain “mirror dual” Calabi-Yau submanifolds inside a $G_2$ manifold. More specifically, given a Harvey-Lawson manifold $HL$, we explain how to assign a pair of tangent bundle valued 2 and 3-forms to a $G_2$ manifold $(M, HL, \varphi, \Lambda)$, with the calibration 3-form $\varphi$ and an oriented 2-plane field $\Lambda$. As in [AS2] these forms can then be used to define different complex and symplectic structures on certain 6-dimensional subbundles of $T(M)$. When these bundles are integrated they give mirror CY manifolds (related thru HL manifolds).

1. Introduction

Let $(M^7, \varphi)$ be a $G_2$ manifold with the calibration 3-form $\varphi$. If $\varphi$ restricts to be the volume form of an oriented 3-dimensional submanifold $Y^3$, then $Y$ is called an associative submanifold of $M$. In [AS2] the authors introduced a notion of mirror duality in any $G_2$ manifold $(M^7, \varphi)$ based on the associative/coassociative splitting of its tangent bundle $TM = E \oplus V$ by the non-vanishing 2-plane fields provided by $[T]$. This duality initially depends on the choice of two non-vanishing vector fields, one in $E$ and the other in $V$. In this article we give a natural form of this duality where the choice of these vector fields are made more canonical, in the expense of possibly localizing this process to the tubular neighborhood of the 3-skeleton of $(M, \varphi)$.

2. Basic Definitions

Let us recall some basic facts about $G_2$ manifolds (e.g. [BT], [HL], [AS1]). Octonions give an 8 dimensional division algebra $\mathbb{O} = \mathbb{H} \oplus i\mathbb{H} = \mathbb{R}^8$ generated by $\langle 1, i, j, k, li, lj, lk \rangle$. The imaginary octonions $im\mathbb{O} = \mathbb{R}^7$ is equipped with the cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ defined by $u \times v = im(\bar{v}u)$. The exceptional Lie group $G_2$ is the linear automorphisms of $im\mathbb{O}$ preserving this cross product. Alternatively:

\[(1) \quad G_2 = \{(u_1, u_2, u_3) \in (\mathbb{R}^7)^3 \mid \langle u_i, u_j \rangle = \delta_{ij}, \langle u_1 \times u_2, u_3 \rangle = 0 \}.
\]

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where $\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$ with $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$. We say a 7-manifold $M^7$ has a $G_2$ structure if there is a 3-form $\varphi \in \Omega^3(M)$ such that at each $p \in M$ the pair $(T_p(M), \varphi(p))$ is (pointwise) isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$. This condition is equivalent to reducing the tangent frame bundle of $M$ from $GL(7, \mathbb{R})$ to $G_2$. A manifold with $G_2$ structure $(M, \varphi)$ is called a $G_2$ manifold (integrable $G_2$ structure) if at each point $p \in M$ there is a chart $(U, p) \rightarrow (\mathbb{R}^7, 0)$ on which $\varphi$ equals to $\varphi_0$ up to second order term, i.e. on the image of the open set $U$ we can write $\varphi(x) = \varphi_0 + O(|x|^2)$.

One important class of $G_2$ manifolds are the ones obtained from Calabi-Yau manifolds. Let $(X, \omega, \Omega)$ be a complex 3-dimensional Calabi-Yau manifold with Kähler form $\omega$ and a nowhere vanishing holomorphic 3-form $\Omega$, then $X^6 \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a $G_2$ manifold. In this case $\varphi = \text{Re} \Omega + \omega \wedge dt$. Similarly, $X^6 \times \mathbb{R}$ gives a noncompact $G_2$ manifold.

**Definition 1.** Let $(M, \varphi)$ be a $G_2$ manifold. A 4-dimensional submanifold $X \subset M$ is called coassociative if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M$ is called associative if $\varphi|_Y \equiv \text{vol}(Y)$; this condition is equivalent to the condition $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form defined by the identity:

$$
\langle \chi(u,v,w), z \rangle = *\varphi(u,v,w, z)
$$

The equivalence of these conditions follows from the ‘associator equality’ of [HL]

$$
\varphi(u,v,w)^2 + |\chi(u,v,w)|^2/4 = |u \wedge v \wedge w|^2
$$

Similar to the definition of $\chi$ one can define a tangent bundle 2-form, which is just the cross product of $M$ (nevertheless viewing it as a 2-form has its advantages).

**Definition 2.** Let $(M, \varphi)$ be a $G_2$ manifold. Then $\psi \in \Omega^2(M, TM)$ is the tangent bundle valued 2-form defined by the identity:

$$
\langle \psi(u,v), w \rangle = \varphi(u,v,w) = \langle u \times v, w \rangle
$$

On a local chart of a $G_2$ manifold $(M, \varphi)$, the form $\varphi$ coincides with the form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$ up to quadratic terms, we can express the tangent valued forms $\chi$ and $\psi$ in terms of $\varphi_0$ in local coordinates. More generally, if $e_1, ..., e_7$ is any local orthonormal frame and $e^1, ..., e^7$ is the dual frame, from definitions we get:
\[ \chi = (e^{256} + e^{247} + e^{346} - e^{357})e_1 \\
+ (-e^{156} - e^{147} - e^{345} - e^{367})e_2 \\
+ (e^{157} - e^{146} + e^{245} + e^{267})e_3 \\
+ (e^{127} + e^{136} - e^{235} + e^{657})e_4 \\
+ (e^{126} - e^{137} + e^{234} + e^{467})e_5 \\
+ (-e^{125} - e^{134} - e^{237} - e^{457})e_6 \\
+ (-e^{124} + e^{135} + e^{236} + e^{456})e_7. \]

\[ \psi = (e^{23} + e^{45} + e^{67})e_1 \\
+ (e^{46} - e^{13} - e^{57})e_2 \\
+ (e^{12} - e^{47} - e^{56})e_3 \\
+ (e^{37} - e^{15} - e^{26})e_4 \\
+ (e^{14} + e^{27} + e^{36})e_5 \\
+ (e^{24} - e^{17} - e^{35})e_6 \\
+ (e^{16} - e^{25} - e^{34})e_7. \]

Here are some useful facts:

**Lemma 1.** ([AS1]) To any 3-dimensional submanifold \( Y^3 \subset (M, \varphi) \), \( \chi \) assigns a normal vector field, which vanishes when \( Y \) is associative.

**Lemma 2.** ([AS1]) To any associative manifold \( Y^3 \subset (M, \varphi) \) with a non-vanishing oriented 2-plane field, \( \chi \) defines a complex structure on its normal bundle (notice in particular that any coassociative submanifold \( X \subset M \) has an almost complex structure if its normal bundle has a non-vanishing section).

**Proof.** Let \( L \subset \mathbb{R}^7 \) be an associative 3-plane, that is \( \varphi_0|_L = vol(L) \). Then for every pair of orthonormal vectors \( \{u, v\} \subset L \), the form \( \chi \) defines a complex structure on the orthogonal 4-plane \( L^\perp \), as follows: Define \( j : L^\perp \rightarrow L^\perp \) by

\[ j(X) = \chi(u, v, X) \]

This is well defined i.e. \( j(X) \in L^\perp \), because when \( w \in L \) we have:

\[ \langle \chi(u, v, X), w \rangle = *\varphi_0(u, v, X, w) = -*\varphi_0(u, v, w, X) = \langle \chi(u, v, w), X \rangle = 0 \]

Also \( j^2(X) = j(\chi(u, v, X)) = \chi(u, v, \chi(u, v, X)) = -X \). We can check the last equality by taking an orthonormal basis \( \{X_j\} \subset L^\perp \) and calculating

\[ \langle \chi(u, v, \chi(u, v, X_i)), X_j \rangle = *\varphi_0(u, v, \chi(u, v, X_i), X_j) = -*\varphi_0(u, v, X_j, \chi(u, v, X_i)) \]
\[ = -\langle \chi(u, v, X_j), \chi(u, v, X_i) \rangle = -\delta_{ij} \]
The last equality holds since the map \( j \) is orthogonal, and the orthogonality can be seen by polarizing the associator equality, and by noticing \( \varphi_0(u, v, X_i) = 0 \). Observe that the map \( j \) only depends on the oriented 2-plane \( \Lambda = \langle u, v \rangle \) generated by \( \{u, v\} \) (i.e. it only depends on the complex structure on \( \Lambda \)).

### 3. Calabi-Yau’s hypersurfaces in \( G_2 \) manifolds

In [AS2] authors proposed a notion of mirror duality for Calabi-Yau submanifold pairs lying inside of a \( G_2 \) manifold \((M, \varphi)\). This is done first by assigning almost Calabi-Yau structures to hypersurface induced by hyperplane distributions. The construction goes as follows. Suppose \( \xi \) be a nonvanishing vector field \( \xi \in \Omega^0(M, TM) \), which gives a codimension one integrable distribution \( V_\xi := \xi^\perp \) on \( M \). If \( X_\xi \) is a leaf of this distribution, then the forms \( \chi \) and \( \psi \) induce a non-degenerate 2-form \( \omega_\xi \) and an almost complex structure \( J_\xi \) on \( X_\xi \) as follows:

\[
\omega_\xi = \langle \psi, \xi \rangle \quad \text{and} \quad J_\xi(u) = u \times \xi.
\]

\[
\text{Re } \Omega_\xi = \varphi|_{V_\xi} \quad \text{and} \quad \text{Im } \Omega_\xi = \langle \chi, \xi \rangle.
\]

where the inner products, of the vector valued differential forms \( \psi \) and \( \chi \) with vector field \( \xi \), are performed by using their vector part. So \( \omega_\xi = \xi \wedge \varphi \), and \( \text{Im } \Omega_\xi = \xi \wedge \ast \varphi \). Call \( \Omega_\xi = \text{Re } \Omega_\xi + i \text{ Im } \Omega_\xi \). These induce almost Calabi-Yau structure on \( X_\xi \), analogous to Example 1.

**Theorem 3.** ([AS2]) Let \((M, \varphi)\) be a \( G_2 \) manifold, and \( \xi \) be a unit vector field such that \( \xi^\perp \) comes from a codimension one foliation on \( M \), then \((X_\xi, \omega_\xi, \Omega_\xi, J_\xi)\) is an almost Calabi-Yau manifold such that \( \varphi|_{X_\xi} = \text{Re } \Omega_\xi \) and \( \ast \varphi|_{X_\xi} = \ast_3 \omega_\xi \). Furthermore, if \( \mathcal{L}_\xi(\varphi)|_{X_\xi} = 0 \) then \( d\omega_\xi = 0 \), and if \( \mathcal{L}_\xi(\ast \varphi)|_{X_\xi} = 0 \) then \( J_\xi \) is integrable; when both conditions are satisfied \((X_\xi, \omega_\xi, \Omega_\xi, J_\xi)\) is a Calabi-Yau manifold.

Here is a brief discussion of [AS2] with explanation of its terms: Let \( \xi^\# \) be the dual 1-form of \( \xi \), and \( e_{\xi^\#} \) and \( i_\xi = \xi \lrcorner \) denote the exterior and interior product operations on differential forms, then

\[
\varphi = e_{\xi^\#} \circ i_\xi(\varphi) + i_\xi \circ e_{\xi^\#}(\varphi) = \omega_\xi \wedge \xi^\# + \text{Re } \Omega_\xi.
\]

This is the decomposition of the form \( \varphi \) with respect to \( \xi \oplus \xi^\perp \). The condition that the distribution \( V_\xi \) to be integrable is \( d\xi^\# \wedge \xi^\# = 0 \). Also it is clear from definitions that \( J_\xi \) is an almost complex structure on \( X_\xi \), and the 2-form \( \omega_\xi \) is non-degenerate on \( X_\xi \), because

\[
\omega_\xi^2 = (\xi \lrcorner \varphi)^3 = \xi \lrcorner [ (\xi \lrcorner \varphi) \wedge (\xi \lrcorner \varphi) \wedge \varphi ] = \xi \lrcorner (6|\xi|^2 \mu) = 6\mu_\xi
\]
where $\mu_\xi = \mu|_{V_\xi}$ is the induced orientation form on $V_\xi$. For $u, v \in V_\xi$,
\[
\omega_\xi(J_\xi(u), v) = \omega_\xi(u \times \xi, v) = \langle \psi(u \times \xi, v), \xi \rangle = \varphi(u \times \xi, v, \xi)
\]
\[
= -\varphi(\xi, \xi \times u, v) = -\langle \xi \times (\xi \times u), v \rangle
\]
\[
= -\langle -|\xi|^2 u + \langle \xi, u \rangle \xi, v \rangle = |\xi|^2(u, v) - \langle \xi, u \rangle \langle \xi, v \rangle
\]
\[
= \langle u, v \rangle.
\]
implies $\langle J_\xi(u), J_\xi(v) \rangle = -\omega_\xi(u, J_\xi(v)) = \langle u, v \rangle$. By a calculation of $J_\xi$, one checks that the 3-form $\Omega_\xi$ is a $(3,0)$ form, furthermore it is non-vanishing because
\[
\frac{1}{2i} \Omega_\xi \wedge \overline{\Omega_\xi} = Im \Omega_\xi \wedge Re \Omega_\xi = (\xi \wedge \varphi) \wedge [\xi \wedge (\xi^\# \wedge \varphi)]
\]
\[
= -\xi \wedge [(\xi \wedge \varphi) \wedge (\xi^\# \wedge \varphi)]
\]
\[
= \xi \wedge [*(\xi^\# \wedge \varphi) \wedge (\xi^\# \wedge \varphi)]
\]
\[
= |\xi^\# \wedge \varphi|^2 \xi \wedge vol(M)
\]
\[
= 4|\xi^\#|^2 (*\xi^\#) = 4 vol(X_\xi).
\]

It is easy to see $*Re \Omega_\xi = -Im \Omega_\xi \wedge \xi^\#$ and $*Im \Omega_\xi = Re \Omega_\xi \wedge \xi^\#$. 

$*_3 Re \Omega_\xi = Im \Omega_\xi$.

Notice that $\omega_\xi$ is a symplectic structure on $X_\xi$ when $d\varphi = 0$ and $L_\xi(\varphi)|_{V_\xi} = 0$, 
($L_\xi$ is the Lie derivative along $\xi$), since $\omega_\xi = \xi \wedge \varphi$ and:
\[
d\omega_\xi = L_\xi(\varphi) - \xi \wedge d\varphi = L_\xi(\varphi)
\]

$J_\xi$ is integrable complex structure if $d^*\varphi = 0$ and $L_\xi(*\varphi)|_{V_\xi} = 0$ since
\[
d(Im\Omega_\xi) = d(\xi \wedge *\varphi) = L_\xi(*\varphi) - \xi \wedge d(*\varphi) = 0
\]

Also notice that $d\varphi = 0 \implies d(Re \Omega_\xi) = d(\varphi|_{X_\xi}) = 0$.

4. HL manifolds and Mirror duality in $G_2$ manifolds

By [T] any 7-dimensional Riemanninan manifold admits a non-vanishing orthonormal 2-frame field $\Lambda = \langle u, v \rangle$, in particular $(M, \varphi)$ admits such a field. $\Lambda$ gives a section of the bundle of oriented 2-frames $V_2(M) \to M$, and hence gives an associative/coassociative splitting of the tangent bundle $TM = E \oplus V$, where $E = E_\Lambda = \langle u, v, u \times v \rangle$ and $V = V_\Lambda = E^\perp$. When there is no danger of confusion we will denote the 2-frame fields and the 2-planes fields which they induce by the same symbol $\Lambda$. Also, any unit section $\xi$ of $E \to M$ induces a complex structure $J_\xi$ on the bundle $V \to M$ by the cross product $J_\xi(u) = u \times \xi$.

In [AS2] any two almost Calabi-Yau’s $X_\xi$ and $X_{\xi'}$ inside $(M, \varphi)$ were called dual if the defining vector fields $\xi$ and $\xi'$ are chosen from $V$ and $E$, respectively. Here we
Lemma 4. By definitions, the following hold.

Theorem 5. For \((a, b, c) \in \mathbb{R}^3\) with \(a + b + c = 0\), then

(a) \(TX_R = [au + bv + cw, R', R', a(v \times w) + b(w \times u) + c(u \times v)]\)
\(J_R(au + bv + cw) = -a(v \times w) - b(w \times u) - c(u \times v)\)
\(J_R(R') = -R'\)

(b) \(TX_{R'} = [au + bv + cw, R'', R, a(v \times w) + b(w \times u) + c(u \times v)]\)
\(J_{R'}(au + bv + cw) = -((b - c)u + (c - a)v + (a - b)w)/\sqrt{3}\)
\(J_{R'}(a(v \times w) + b(w \times u) + c(u \times v)) = (b - c)(v \times w) + (c - a)(w \times u) + (a - b)(u \times v))/\sqrt{3}\)
\(J_{R'}(R'') = R\)

(c) \(TX_{R''} = [au + bv + cw, R, R', a(v \times w) + b(w \times u) + c(u \times v)]\)
\[ J_{R''}(au + bv + cw) = \]
\[ ((b - a)(u \times v) + (c - b)(v \times w) + (a - c)(w \times u))/\sqrt{3} \]
\[ J_{R''}(R) = R' \]

(d) \( \{u, v, w, R, u \times v, v \times w, w \times u\} \) is an orthonormal frame field.

Proof. To show (a) by using (4) we calculate:
\[ R \times u = \chi(u, v, w) \times u = - [u \times (v \times w)] \times u = u \times [u \times (v \times w)] \]
\[ = -\chi(u, u, v \times w) - <u, u > (v \times w) + <u, v \times w > u \]
\[ = -(v \times w) + \varphi(u, v, w)u \]
(8)
Therefore \( R \times u = -(v \times w) \)

Similarly, \( R \times v = -(w \times u) \) and \( R \times w = -(u \times v) \). Therefore we have \( J_R(au + bv + cw) = -a(v \times w) - b(w \times u) - c(u \times v) \), and \( J_R(R'') = -R' \).

\[ \sqrt{3} R' \times u = (u \times v + v \times w + w \times u) \times u \]
\[ = -u \times (u \times v) - u \times (v \times w) - u \times (w \times u) \]
\[ = <u, u > v - <u, v > u \]
\[ + \chi(u, v, w) + <u, v > w - <u, w > v \]
\[ + <u, w > u - <u, u > w \]
(9)
Therefore \( \sqrt{3} R' \times u = R + (v - w) \)

Similarly \( \sqrt{3} R' \times v = R + (w - u) \), and \( \sqrt{3} R' \times w = R + (u - v) \), which implies the first part of (b), and \( J_{R'}(R''') = R \).

For the second part of (b) we need the compute the following:
(10)
\[ \sqrt{3} R' \times [a(v \times w) + b(w \times u) + c(u \times v)] = \]
\[ (u \times v + v \times w + w \times u) \times [a(v \times w) + b(w \times u) + c(u \times v)] \]

For this first by repeatedly using (4) and \( \varphi(u, v, w) = 0 \) we calculate:
\[ (v \times u) \times (w \times v) = -\chi(v, u, w, v) - <v \times u, w > v + <v \times u, v > w \]
\[ = -\chi(v \times u, w, v) = -\chi(w, v, v \times u) \]
\[ = w \times (v \times (v \times u)) + <w, v > (v \times u) - <w, v \times u > v \]
\[ = w \times (v \times (v \times u)) \]
\[ = w \times (\chi(v, v, u) - <v, v > u + <v, u > v) = -(w \times u) \]

Then by plugging in (9) gives (b). Proof of (c) is similar to (a)

In particular from the above calculations we get can express \( \varphi \) as:
Corollary 6.
\[ \varphi = u^\# \wedge v^\# \wedge (u^\# \times v^\#) + v^\# \wedge w^\# \wedge (v^\# \times w^\#) + w^\# \wedge u^\# \wedge (w^\# \times u^\#) + u^\# \wedge R^\# \wedge (v^\# \times w^\#) + (v^\# \wedge R^\#) \wedge (w^\# \times u^\#) + w^\# \wedge R^\# \wedge (u^\# \times v^\#) - (u^\# \times v^\#) \wedge (v^\# \times w^\#) \wedge (w^\# \times u^\#) \]

Recall that in an earlier paper we proved the following facts:

**Proposition 7.** [AS2] Let \( \{\alpha, \beta\} \) be orthonormal vector fields on \((M, \varphi)\). Then on \(X_\alpha\) the following hold

(i) \( \text{Re} \ \Omega_\alpha = \omega_\beta \wedge (\beta^\# \wedge \varphi) + \text{Re} \ \Omega_\beta \)

(ii) \( \text{Im} \ \Omega_\alpha = \alpha \lrcorner \ (\ast \omega_\beta) = (\alpha \lrcorner \text{Im} \ \Omega_\beta) \wedge \beta^\# \)

(iii) \( \omega_\alpha = \alpha \lrcorner \text{Re} \ \Omega_\beta + (\alpha \lrcorner \omega_\beta) \wedge \beta^\# \)

**Proof.** Since \( \text{Re} \ \Omega_\alpha = \varphi|_{X_\alpha} \) (i) follows. Since \( \text{Im} \ \Omega_\alpha = \alpha \lrcorner \varphi \), following gives (ii)

\[
\alpha \lrcorner (\ast \omega_\beta) = \alpha \lrcorner \left[ \beta \lrcorner (\beta \varphi) \right] = \alpha \lrcorner \beta \lrcorner (\beta^\# \wedge \ast \varphi) = \alpha \lrcorner \varphi + \beta^\# \wedge (\alpha \lrcorner \beta \varphi) = \alpha \lrcorner \varphi - (\alpha \lrcorner \omega_\beta) \wedge \beta^\#
\]

(iii) follows from the following computation

\[
\alpha \lrcorner \text{Re} \ \Omega_\beta = \alpha \lrcorner \beta \lrcorner (\beta^\# \wedge \varphi) = \alpha \lrcorner \varphi + \beta^\# \wedge (\alpha \lrcorner \beta \varphi) = \alpha \lrcorner \varphi - (\alpha \lrcorner \omega_\beta) \wedge \beta^#
\]

\( \Box \)

Note that even though the identities of this proposition hold only after restricting the right hand side to \(X_\alpha\), all the individual terms are defined everywhere on \((M, \varphi)\). Also, from the construction, \(X_\alpha\) and \(X_\beta\) inherit vector fields \(\beta\) and \(\alpha\), respectively.

**Corollary 8.** [AS2] Let \( \{\alpha, \beta\} \) be orthonormal vector fields on \((M, \varphi)\). Then there are \(A_{\alpha\beta} \in \Omega^3(M)\), and \(W_{\alpha\beta} \in \Omega^2(M)\) satisfying

(a) \( \varphi|_{X_\alpha} = \text{Re} \ \Omega_\alpha \) and \( \varphi|_{X_\beta} = \text{Re} \ \Omega_\beta \)

(b) \( A_{\alpha\beta}|_{X_\alpha} = \text{Im} \ \Omega_\alpha \) and \( A_{\alpha\beta}|_{X_\beta} = \alpha \lrcorner (\ast \omega_\beta) \)

(c) \( W_{\alpha\beta}|_{X_\alpha} = \omega_\alpha \) and \( W_{\alpha\beta}|_{X_\beta} = \alpha \lrcorner \text{Re} \ \Omega_\beta \)

Now we can choose \( \alpha \) as \( R \) and \( \beta \) as \( R' \) of the given HL manifold. That concludes that given a HL submanifold of a \(G_2\) manifold, it will determine a “canonical” mirror pair of Calabi-Yau manifolds (related thru the HL manifold) with the complex and symplectic structures given above.
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