GENERALIZATION OF A CONJECTURE OF MUMFORD

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ABSTRACT. A conjecture of Mumford predicts a complete set of relations between the generators of the cohomology ring of the moduli space of rank 2 semi-stable sheaves with fixed odd degree determinant on a smooth, projective curve of genus at least 2. The conjecture was proven by Kirwan [24]. In this article, we generalize the conjecture to the case when the underlying curve is irreducible, nodal. In fact, we show that these relations (in the nodal curve case) arise naturally as degeneration of the Mumford relations shown by Kirwan in the smooth curve case. As a byproduct, we compute the Hodge-Poincaré polynomial of the moduli space of rank 2, semi-stable, torsion-free sheaves with fixed determinant on an irreducible, nodal curve.

1. Introduction

The underlying field will always be $\mathbb{C}$. Let $C$ be a smooth, projective curve of genus $g \geq 2$, $d$ be an odd integer and $\mathcal{L}_0$ be an invertible sheaf on $C$ of degree $d$. Denote by $M_C(2,d)$ the moduli space of stable, locally-free sheaves of rank 2 and degree $d$ over $C$ and by $M_C(2,\mathcal{L}_0)$ the sub-moduli space of $M_C(2,d)$ parametrizing locally-free sheaves with determinant $\mathcal{L}_0$. Since it was first constructed by Mumford, almost all aspects of the moduli spaces $M_C(2,d)$ and $M_C(2,\mathcal{L}_0)$ have been extensively studied. Several different methods ranging from topology, number-theory, gauge theory as well as algebraic geometry have been used to study its cohomology ring $H^*(M_C(2,\mathcal{L}_0),\mathbb{Q})$. The generators of this ring were given by Newstead in [28] and a complete set of relations between these generators was conjectured by Mumford. We briefly recall the conjecture. Choose a symplectic basis $e_1, e_2, \ldots, e_{2g}$ of $H^1(C,\mathbb{Z})$ such that $e_i \cup e_j = 0$ for $|j-i| \neq g$ and $e_i \cup e_{i+g} = -[C]^\vee$, where $[C]^\vee$ is the Poincaré dual of the fundamental class of $C$. Mumford and Newstead [26] showed that there exists an isomorphism of pure Hodge structures

$$\phi : H^3(C,\mathbb{Z}) \to H^3(M_C(2,\mathcal{L}_0),\mathbb{Z}),$$

induced by the second Chern class of the universal vector bundle $\mathcal{U}$ over $C \times M_C(2,\mathcal{L}_0)$. Denote by $\gamma_i := \phi(e_i)$ for $1 \leq i \leq 2g$ and $\gamma = \sum_{i=1}^{9} \gamma_i \gamma_{i+g}$. Newstead in [28] showed that there exists $\alpha \in H^2(M_C(2,\mathcal{L}_0),\mathbb{Z})$ and $\beta \in H^4(M_C(2,\mathcal{L}_0),\mathbb{Z})$ (again arising from Chern classes of $\mathcal{U}$) such that the cohomology ring $H^*(M_C(2,\mathcal{L}_0),\mathbb{Q})$ is generated by $\alpha, \beta$ and $\gamma_i$ for $1 \leq i \leq 2g$. Mumford conjectured that there is a decomposition

$$H^*(M_C(2,\mathcal{L}_0),\mathbb{Q}) \cong \bigoplus_{k=0}^{g} P_k \otimes \mathbb{Q}[\alpha, \beta, \gamma]/I_{g-k}$$

where $I_k$ is an ideal of relations between $\alpha, \beta$ and $\gamma$ and $P_k$ is the primitive component of $\wedge^k H^3(M_C(2,\mathcal{L}_0),\mathbb{Q})$ with respect to $\gamma$ (see §5.2 for precise definitions). The conjecture was proved by Kirwan [24]. In [40] Zagier showed that in fact the relations between the generators

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can be determined recursively. In particular, \( I_k \subset \mathbb{Q}[\alpha, \beta, \gamma] \) is generated by \( (\xi_k, \xi_{k+1}, \xi_{k+2}) \), where \( \xi_0 = 1 \) and recursively,

\[
\xi_{k+1} := \alpha \xi_k + k^2 \beta \xi_{k-1} + 2(k - 1) \gamma \xi_{k-2}.
\]

This was also proven independently by Baranovskii [3], Siebert and Tian [24], later by Herrera and Salamon [21], Earl [14] and also by King and Newstead [23]. Although the obvious generalization of Mumford’s conjecture to the cases when rank is greater than 2 is false, Earl in [14] for \( n = 3 \) and Earl, Kirwan in [17] for arbitrary \( n \), give additional relations such that together with the Mumford relations they form a complete set of relations between the generators of the cohomology ring of the moduli space of rank \( n \) semi-stable sheaves with coprime degree \( d \) over \( C \). However, none of the existing literature studies the above conjecture for a singular curve, even in the case of rank 2.

Let \( X_0 \) be an irreducible nodal curve with exactly one node. Denote by \( U_{X_0}(2, \mathcal{L}_0) \) the moduli space of rank 2 semi-stable sheaves on \( X_0 \) with determinant \( \mathcal{L}_0 \) (here \( \mathcal{L}_0 \) is also an invertible sheaf of odd degree) as defined by Sun in [36] (we use a different notation for the moduli space as the definition of determinant in this case is different from the classical definition, see also Bhosle [5]). The main difficulty in generalizing the above results to the cohomology ring of \( U_{X_0}(2, \mathcal{L}_0) \) arises from the fact that \( U_{X_0}(2, \mathcal{L}_0) \) is singular. The moduli space \( M_C(2, \mathcal{L}_0) \) was non-singular. As a result, most of the techniques in the literature fail. We remark that the moduli space considered by Martinez [19] is singular (as \( \deg(\mathcal{L}_0) \) is even in his case) but arises as a GIT quotient of a non-singular variety, which is also false in our setup.

As there is no straightforward way to generalize the techniques in the literature, we instead embed the nodal curve \( X_0 \) in a regular family \( \pi : \mathcal{X} \to \Delta \) (here \( \Delta \) denotes the unit disc), smooth over \( \Delta^* := \Delta \setminus \{0\} \) and central fiber isomorphic to \( X_0 \) (the existence of such a family follows from the completeness of the moduli space of stable curves, see [2 Theorem B.2]). Note that, the invertible sheaf \( \mathcal{L}_0 \) on \( X_0 \) lifts to a relative invertible sheaf \( \mathcal{L}_X \) over \( \mathcal{X} \). There is a well-known relative Simpson’s moduli space, denoted \( U_X(2, \mathcal{L}_X) \) of rank 2 semi-stable sheaves with determinant \( \mathcal{L}_X \) over \( \mathcal{X} \) (see [22 Theorem 4.3.7]). The (relative) moduli space \( U_X(2, \mathcal{L}_X) \) is flat over \( \Delta \) and central fiber \( U_{X_0}(2, \mathcal{L}_0) \). It is well-known that for any \( s \in \Delta^* \), the fiber \( U_X(2, \mathcal{L}_s) \) is isomorphic to \( M_{\mathcal{X}}(2, \mathcal{L}|_{\mathcal{X}_s}) \), hence non-singular. Substituting \( C \) by \( \mathcal{X}_s \) and \( \mathcal{L}_0 \) by \( \mathcal{L}^s|_{\mathcal{X}_s} \) in the above discussion, we rewrite Mumford’s conjecture for the cohomology ring \( H^*(U_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \) as follows:

**Conjecture** (Generalized Mumford conjecture). Denote by \( P^\text{mon}_k \) the subspace of \( P_k \) (as before) consisting of all elements that are monodromy invariant (under the natural monodromy action on \( H^*(M_{\mathcal{X}}(2, \mathcal{L}|_{\mathcal{X}_s}), \mathbb{Q}) \)). Then, \( P^\text{mon}_k \) is independent (upto isomorphism) of the choice of the family \( \pi \) and the cohomology ring \( H^*(U_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \) decomposes as

\[
H^*(U_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \cong \bigoplus_{k=0}^g P^\text{mon}_k \otimes \mathbb{Q}[\alpha, \beta, \gamma]/I_{g-k}.
\]

Apart from the obvious motivation of Mumford’s conjecture, one can also use the conjecture to compute the Hodge-Poincaré polynomial of the cohomology ring of \( M_{\mathcal{X}}(2, \mathcal{L}_s) \) for \( s \in \Delta^* \), where \( \mathcal{L}_s := \mathcal{L}_X|_{\mathcal{X}_s} \) (see [23]). Recall, the Hodge-Poincaré polynomial for \( M_{\mathcal{X}}(2, \mathcal{L}_s) \):

\[
\sum_{p,q} h^{p,q}(M_{\mathcal{X}}(2, \mathcal{L}_s), \mathbb{C}) u^p v^q.
\]

One of the first results in this direction was by Newstead [27], where he gives a recursive formula for Betti numbers of \( M_C(2, \mathcal{L}_0) \). This was generalized by Harder, Narasimhan [20], Desale and Ramanan [13] using number-theoretic methods, in the case of any coprime rank and degree. Similar results were discovered by Atiyah and Bott [1], using methods from gauge theory.
The formula was solved by Zagier in [39]. Later, Bifet, Ghione and Letizia [7] gave the same formula but using methods from algebraic geometry. Their methods were used by Baño [9,10] to compute the motivic Poincaré polynomial of $M_{X_1}(2, d)$ and $M_{X_1}(2, L_s)$. Using methods from gauge theory, Earl and Kirwan [16] gave the Hodge-Poincaré polynomial for $M_{X_1}(2, d)$ and $M_{X_1}(2, L_s)$. Unfortunately, an analogous Hodge-Poincaré polynomial for $U_{X_0}(2, L_0)$ is unknown. In this article we prove that:

**Theorem 1.1.** The generalized Mumford conjecture holds true. Furthermore, the cohomology ring $H^*(U_{X_0}(2, L_0), \mathbb{Q})$ is generated by $\alpha, \beta, \gamma_i$ for $1 \leq i \leq 2g - 1$ and $\gamma g\gamma_2$.

See Theorem 7.2 and Remark 7.3 for the precise statements. We also obtain the Hodge-Poincaré formula: As $U_{X_0}(2, L_0)$ is singular, the associated cohomology groups do not have a pure Hodge structure. Then, the Hodge-Poincaré formula for $U_{X_0}(2, L_0)$ is defined as

$$\sum_{p,q,i} \dim H^p,q Gr^W_i(U_{X_0}(2, L_0), \mathbb{C}) x^p y^q.$$

We prove that:

**Theorem 1.2 (see Theorem 7.2).** The Hodge-Poincaré polynomial associated to the moduli space $U_{X_0}(2, L_0)$ is

$$\frac{(1 + xy^2)^{g-1}(1 + x^2y^{g-1}(1 + xy + x^3y)) - x^g y^g(1 + x)^{g-1}(x + y)^{g-1}(2 + xy)}{(1 - xy)(1 - x^2y^2)}.$$

We now discuss the strategy of the proof. The idea is to use the theory of variation of mixed Hodge structures by Steenbrink [35] and Schmid [31] to relate the mixed Hodge structure on the central fiber of the relative moduli space to the limit mixed Hodge structure on the generic fiber. Unfortunately, the singularity of the central fiber of $U_{X}(2, \mathcal{L}_X)$ is not a normal crossings divisor, hence not a suitable candidate for using tools from [31,35]. However, there is a different construction of a relative moduli space of rank 2 semi-stable sheaves with determinant $\mathcal{L}_X$ over $X$, due to Gieseker [15]. The advantage of the latter family of moduli spaces is that, in this case the central fiber, denoted $G_{X_0}(2, L_0)$ is a simple normal crossings divisor, hence compatible with the setup in [31,35]. Moreover, the generic fiber of the two relative moduli spaces coincide. Denote by $G_{X}(2, \mathcal{L}_X)_{\infty}$ the generic fiber of the relative moduli space. By Steenbrink [35], $H^i(G_{X}(2, \mathcal{L}_X)_{\infty}, \mathbb{Q})$ is equipped with a (limit) mixed Hodge structure such that the specialization morphism

$$sp_i : H^i(G_{X_0}(2, L_0), \mathbb{Q}) \to H^i(G_{X}(2, \mathcal{L}_X)_{\infty}, \mathbb{Q})$$

is a morphism of mixed Hodge structures for all $i \geq 0$. Using the Mumford relations on $M_{X_1}(2, L_s)$ shown by Kirwan, we obtain a complete set of relations between the generators of the cohomology ring $H^*(G_{X_0}(2, L_0), \mathbb{Q})$ of the generic fiber $G_{X}(2, \mathcal{L}_X)_{\infty}$. However, not all elements in $H^i(G_{X}(2, \mathcal{L}_X)_{\infty}, \mathbb{Q})$ are monodromy invariant. As a result the specialization morphism $sp_i$ is neither injective, nor surjective (see Corollary 4.2). To compute explicitly the kernel and cokernel of $sp_i$ we study the Gysin morphism from the intersection of the two components of $G_{X_0}(2, L_0)$ to the irreducible components (see Theorem 4.2). Using this we prove:

**Theorem 1.3 (see Theorem 5.1).** We have the following isomorphism of graded rings:

$$H^*(G_{X_0}(2, L_0), \mathbb{Q}) \cong \left( \bigoplus_i P^\text{mon}_i \otimes \frac{\mathbb{Q}[\alpha, \beta, \gamma]}{I_{g-i}} \right) \oplus \left( \bigoplus_i \tilde{P}_{i-2} \otimes \frac{Q[\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, X, Y]}{(I_{g-i-3}, X^2, Y^2, X - Y)} \right),$$

where $\tilde{P},\tilde{\alpha},\tilde{\beta},\tilde{\gamma}$ and $I_{g-i}$ are objects analogous to $P,\alpha,\beta,\gamma$ and $I_{g-i}$ defined earlier after replacing $C$ by $X_0$ and $L_0$ by $\tilde{L}_0 := \pi_0^*L_0$, $\pi_0 : \tilde{X}_0 \to X_0$ is the normalization morphism.
Similarly, we obtain the Hodge-Poincaré formula for \( \mathcal{G}_{X_0}(2, \mathcal{L}_0) \) (see Theorem 6.2). Finally, there exists a proper morphism from \( \mathcal{G}_{X_0}(2, \mathcal{L}_0) \) to \( U_{X_0}(2, \mathcal{L}_0) \). Using this morphism we obtain from Theorem 1.3, the relations between the generators of the cohomology ring of \( U_{X_0}(2, \mathcal{L}_0) \) and compute the Hodge-Poincaré polynomial for \( U_{X_0}(2, \mathcal{L}_0) \).

**Outline:** In §2 we recall the preliminaries on the limit mixed Hodge structures applicable in our setup and use it to compute the limit mixed Hodge structure associated to the degenerating family \( \pi \) of curves, mentioned above. In §3 we recall the relative Mumford-Newstead isomorphism as mentioned in [4] which gives us an isomorphism (of mixed Hodge structures) between the limit mixed Hodge structure coming from \( \pi \) and that coming from the associated family of moduli space of semi-stable sheaves. In §4 we compute a Gysin morphism to relate the cohomology ring of \( \mathcal{G}_{X_0}(2, \mathcal{L}_0) \) to that of the generic fiber of the family of moduli spaces. In §5 we prove the generalized Mumford conjecture for \( \mathcal{G}_{X_0}(2, \mathcal{L}_0) \). In §6 we compute the Hodge-Poincaré polynomial for \( \mathcal{G}_{X_0}(2, \mathcal{L}_0) \). In §7 we prove the generalized Mumford conjecture for \( U_{X_0}(2, \mathcal{L}_0) \) and compute the associated Hodge-Poincaré polynomial.

**Notation:** Given any morphism \( f : \mathcal{Y} \to S \) and a point \( s \in S \), we denote by \( \mathcal{Y}_s := f^{-1}(s) \). The open unit disc is denoted by \( \Delta \) and \( \Delta^* := \Delta \setminus \{0\} \) denotes the punctured disc.

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## 2. Preliminaries: Limit mixed Hodge structure

We recall basic results on limit mixed Hodge structures relevant to our setup. See [30] for a detailed treatment of the subject.

Let \( \rho : \mathcal{Y} \to \Delta \) be a flat family of projective varieties, smooth over \( \Delta^* \) and \( \rho' : \mathcal{Y}_{\Delta^*} \to \Delta^* \) the restriction of \( \rho \) to \( \Delta^* \).

### 2.1. Hodge bundles.

Using Ehresmann’s theorem (see [38] Theorem 9.3)), we have for all \( i \geq 0 \),

\[
H^i_{\mathcal{Y}_{\Delta^*}} := H^i(\mathcal{Y}_t, \mathbb{Z})
\]

the local systems over \( \Delta^* \) with fiber \( H^i(\mathcal{Y}_t, \mathbb{Z}) \), for \( t \in \Delta^* \). One can canonically associate to these local systems, the holomorphic vector bundles \( H^i_{\mathcal{Y}_{\Delta^*}} := H^i_{\mathcal{Y}_{\Delta^*}} \otimes \mathcal{O}_{\Delta^*} \) called the *Hodge bundles*. There exist holomorphic sub-bundles \( F^p H^i_{\mathcal{Y}_{\Delta^*}} \subset H^i_{\mathcal{Y}_{\Delta^*}} \) defined by the condition: for any \( t \in \Delta^* \), the fibers

\[
(F^p H^i_{\mathcal{Y}_{\Delta^*}})_t \subset (H^i_{\mathcal{Y}_{\Delta^*}})_t
\]
can be identified respectively with \( F^p H^i(\mathcal{Y}_t, \mathbb{C}) \subset H^i(\mathcal{Y}_t, \mathbb{C}) \), where \( F^p \) denotes the Hodge filtration (see [38] §10.2.1)).

### 2.2. Canonical extension of Hodge bundles.

The Hodge bundles and their holomorphic sub-bundles defined above can be extended to the entire disc. In particular, there exists a *canonical extensions*, \( \overline{H}_Y \) of \( H^i_{\mathcal{Y}_{\Delta^*}} \) to \( \Delta \) (see [30] Definition 11.4)). Note that, \( \overline{H}_Y \) is locally-free over \( \Delta \). Denote by \( j : \Delta^* \to \Delta \) the inclusion morphism, \( F^p \overline{H}_Y := j_* \left( F^p H^i_{\mathcal{Y}_{\Delta^*}} \right) \cap \overline{H}_Y \). Note that, \( F^p \overline{H}_Y \) is the *unique largest* locally-free sub-sheaf of \( \overline{H}_Y \) which extends \( F^p H^i_{\mathcal{Y}_{\Delta^*}} \).

Consider the universal cover \( \mathfrak{h} \to \Delta^* \) of the punctured unit disc. Denote by \( e : \mathfrak{h} \to \Delta^* \) the composed morphism and \( \mathcal{Y}_\infty \) the base change of the family \( \mathcal{Y} \) over \( \Delta \) to \( \mathfrak{h} \), by the morphism
e. There is an explicit identification of the central fiber of the canonical extensions $\overline{Y}_Y$ and the cohomology group $H^i(Y_\infty, \mathbb{C})$, depending on the choice of the parameter $t$ on $\Delta$ (see [30, XI-8]):

$$g_Y^i : H^i(Y_\infty, \mathbb{C}) \to \left(\overline{Y}_Y\right)_0.$$  

This induce (Hodge) filtrations on $H^i(Y_\infty, \mathbb{C})$ as $F^p H^i(Y_\infty, \mathbb{C}) := (g_Y^i)^{-1} \left(F^p \overline{Y}_Y\right)_0$.

2.3. Monodromy transformations. For any $s \in \Delta^*$ and $i \geq 0$, denote by

$$T_{s, i} : H^i(Y_s, \mathbb{Z}) \to H^i(Y_s, \mathbb{Z})$$

and $T_{s, i}^Q : H^i(Y_s, \mathbb{Q}) \to H^i(Y_s, \mathbb{Q})$

the local monodromy transformations defined by parallel transport along a counterclockwise loop about $0 \in \Delta$ (see [30, §11.1.1]). By [12, Theorem II.1.17] (see also [25, Proposition I.7.8.1]) the automorphism extends to a $\mathbb{Q}$-automorphism

$$T_i : H^i(Y_\infty, \mathbb{Q}) \to H^i(Y_\infty, \mathbb{Q}).$$  

Denote by $T_{i, \mathbb{C}}$ the induced automorphism on $H^i(Y_\infty, \mathbb{C})$.

2.4. Schmid’s limit mixed Hodge structures. The natural specialization morphism from the cohomology on the central fiber of the family $\mathcal{Y}$ to a general fiber $Y_s$, $s \in \Delta^*$ is not in general a morphism of mixed Hodge structures, if one considers the cohomology of $Y_s$ with the natural pure Hodge structure. However, one can define a mixed Hodge structure on the cohomology of $Y_s$, such that the specialization morphism is a morphism of mixed Hodge structures. More precisely,

**Remark 2.1.** Let $N_i$ the logarithm of the monodromy operator $T_i$. By [30, Lemma-Definition 11.9], there exists an unique increasing monodromy weight filtration $W_\bullet$ on $H^i(Y_\infty, \mathbb{Q})$ such that,

1. for $j \geq 2$, $N_i(W_j H^i(Y_\infty, \mathbb{Q})) \subset W_{j-2} H^i(Y_\infty, \mathbb{Q})$ and
2. the map $N_i^l : \text{Gr}^W_{i,l} H^i(Y_\infty, \mathbb{Q}) \to \text{Gr}^W_{i-1,l} H^i(Y_\infty, \mathbb{Q})$ is an isomorphism for all $l \geq 0$.

Now, [31, Theorem 6.16] states that the induced filtrations on $H^i(Y_\infty, \mathbb{C})$ defines a mixed Hodge structure $(H^i(Y_\infty, \mathbb{Z}), W_\bullet, F^\bullet)$.

In the case, the central fiber $Y_0$ is a reduced simple normal crossings divisor of $\mathcal{Y}$, we have the following description of the specialization morphism.

**Remark 2.2.** Suppose that the central fiber $Y_0$ is a reduced simple normal crossings divisor of $\mathcal{Y}$. By the local invariant cycle theorem [30, Theorem 11.43], we have the following exact sequence of mixed Hodge structure:

$$H^i(Y_0, \mathbb{Q}) \xrightarrow{sp_i} H^i(Y_\infty, \mathbb{Q}) \xrightarrow{N_i/(2\pi \sqrt{-1})} H^i(Y_\infty, \mathbb{Q})(-1)$$

where $sp_i$ denotes the specialization morphism.

2.5. Steenbrink’s limit mixed Hodge structures. It is not always easy to compute the monodromy weight filtration defined by Schmid. As a result we will use the Steenbrink spectral sequences below. Note that, we do not give the general form of the spectral sequence, instead we restrict to the case relevant to this article.

**Proposition 2.3** ([30, Corollaries 11.23 and 11.41] and [35, Example 3.5]). Suppose $\mathcal{Y}$ is regular and $Y_0$ is a reduced simple normal crossings divisor of $\mathcal{Y}$, consisting of exactly two irreducible components, say $Y_1$ and $Y_2$. The limit weight spectral sequence $E^p,q_w \Rightarrow H^{p+q}(Y_\infty, \mathbb{Q})$ consists of the following terms:

1. if $|p| \geq 2$, then $\lim_w E^{p,q}_1 = 0$. 

The spectral sequence \( W \) is the Gysin morphism.

The limit weight spectral sequence \( \overset{\infty}{w}E_{p,q}^{1} \) degenerates at \( E_2 \) and the induced filtration on \( H^{p+q}(Y_{\infty}, \mathbb{Q}) \) coincides with the monodromy weight filtration as in Remark [23] above. Similarly, the weighted spectral sequence \( wE_{1}^{p,q} \Rightarrow H^{p+q}(Y_{0}, \mathbb{Q}) \) on \( Y_{0} \) consists of the following terms:

1. for \( p > 2 \) or \( p < 0 \), we have \( wE_{1}^{p,q} = 0 \),
2. \( wE_{1}^{1,q} = H^{q}(Y_{1} \cap Y_{2}, \mathbb{Q})(0) \) and \( wE_{0}^{0,q} = H^{q}(Y_{1}, \mathbb{Q})(0) \oplus H^{q}(Y_{2}, \mathbb{Q})(0) \),
3. the differential map \( d_{1} : wE_{1}^{0,q} \to wE_{1}^{1,q} \) is the restriction morphism and \( d_{1} : wE_{1}^{-1,q} \to wE_{1}^{0,q} \) is the Gysin morphism.

The spectral sequence \( wE_{p,q}^{1} \) degenerates at \( E_2 \) and induces a weight filtration on \( H^{p+q}(Y_{0}, \mathbb{Q}) \).

We note that the resulting weight filtrations on \( H^{p+q}(Y_{\infty}, \mathbb{Q}) \) and \( H^{p+q}(Y_{0}, \mathbb{Q}) \) are given by:

\[ wE_{2}^{p,q} = \text{Gr}_{q}^{W} H^{p+q}(Y_{\infty}, \mathbb{Q}) \quad \text{and} \quad wE_{2}^{p,q} = \text{Gr}_{q}^{W} H^{p+q}(Y_{0}, \mathbb{Q}). \]

**Corollary 2.4.** Let \( Y \) and \( Y_{0} \) be as in Proposition [23]. Then, we have the following exact sequence of mixed Hodge structures:

\[ H^{i-2}(Y_{1} \cap Y_{2}, \mathbb{Q})(-1) \xrightarrow{f_{i}} H^{i}(Y_{0}, \mathbb{Q}) \xrightarrow{\text{sp}_{p}} H^{i}(Y_{\infty}, \mathbb{Q}) \xrightarrow{\text{sp}_{i}} \text{Gr}_{i+1}^{W} H^{i}(Y_{\infty}, \mathbb{Q}) \to 0, \tag{2.4} \]

where \( f_{i} \) is the natural morphism induced by the Gysin morphism from \( H^{i-2}(Y_{1} \cap Y_{2}, \mathbb{Q})(-1) \) to \( H^{i}(Y_{1}, \mathbb{Q}) \oplus H^{i}(Y_{2}, \mathbb{Q}) \) (use the Mayer-Vietoris sequence associated to \( Y_{1} \cup Y_{2} \)), \( \text{sp}_{i} \) is the specialization morphism (see [30] Theorem 11.29) and \( g_{i} \) is the natural projection.

**Proof.** The corollary is an immediate consequence of Proposition [23]. \( \square \)

### 2.6. The curve case.

Consider the flat family \( \rho : \tilde{\mathcal{X}} \to \Delta \) of projective curves with \( \tilde{\mathcal{X}} \) regular, \( \tilde{\mathcal{X}}_{t} \) is smooth of genus \( g \) for all \( t \in \Delta^{*} \) and \( \tilde{\mathcal{X}}_{0} = Y_{1} \cup Y_{2} \) with \( Y_{1} \cong \mathbb{P}^{1}, \) \( Y_{2} \) smooth, irreducible and intersecting \( Y_{1} \) transversally at two points, say \( y_{1}, y_{2} \). We compute the limit mixed Hodge structure associated to this family of curves. This description will be used later in the article to give the generators of the weight filtration on the cohomology ring of the moduli space of semi-stable sheaves with fixed determinant over an irreducible nodal curve.

**Theorem 2.5.** Denote by \( f' \in H^{2}(Y_{2}, \mathbb{Z}) \), the Poincaré dual of the fundamental class of \( Y_{2} \) and 

\[ sp_{2} : H^{2}(Y_{1}, \mathbb{Z}) \oplus H^{2}(Y_{2}, \mathbb{Z}) \cong H^{2}(\tilde{\mathcal{X}}_{0}, \mathbb{Z}) \xrightarrow{sp_{2}} H^{2}(\tilde{\mathcal{X}}_{\infty}, \mathbb{Z}) \]

the specialization morphism as in Corollary [24] composed with the isomorphism arising from the Mayer-Vietoris sequence. Then, there exists a basis \( e_{1}, e_{2}, ..., e_{2g} \) of \( H^{1}(\tilde{\mathcal{X}}_{\infty}, \mathbb{Z}) \) such that

1. \( e_{g} \) (resp. \( e_{2g} \)) generates \( \text{Gr}_{0}^{W} H^{1}(\tilde{\mathcal{X}}_{\infty}, \mathbb{Q}) \) (resp. \( \text{Gr}_{2}^{W} H^{1}(\tilde{\mathcal{X}}_{\infty}, \mathbb{Q}) \)),
2. \( e_{1}, e_{2}, ..., e_{g-1}, e_{g+1}, e_{g+2}, ..., e_{2g-1} \) form a basis of \( \text{Gr}_{1}^{W} H^{1}(\tilde{\mathcal{X}}_{\infty}, \mathbb{Q}) \),
3. \( e_{i} \cup e_{i+g} = sp_{2}(0 \oplus -f') \) and \( e_{i} \cup e_{j} = 0 \) for \( |j - i| \neq g \).

Before proving the theorem, we note that when we say “\( e_{i_{1}}, ..., e_{i_{r}} \) generate \( \text{Gr}_{j}^{W} H^{1}(\tilde{\mathcal{X}}_{\infty}, \mathbb{Q}) \)” we always mean that the image of \( e_{i_{1}}, ..., e_{i_{r}} \) in \( \text{Gr}_{j}^{W} H^{1}(\tilde{\mathcal{X}}_{\infty}, \mathbb{Q}) \) (under the natural projection morphism) generate it.
Proof. The Mayer-Vietoris sequence associated the central fiber $\tilde{X}_0$ is

$$0 \to H^0(\tilde{X}_0, \mathbb{Z}) \to H^0(Y_1, \mathbb{Z}) \oplus H^0(Y_2, \mathbb{Z}) \to H^0(Y_1 \cap Y_2, \mathbb{Z}) \to H^1(\tilde{X}_0, \mathbb{Z}) \to H^1(Y_1, \mathbb{Z}) \oplus H^1(Y_2, \mathbb{Z}) \to 0.$$  

Since $H^1(Y_1, \mathbb{Z}) = 0$, this gives us the short exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{\rho} H^1(\tilde{X}_0, \mathbb{Z}) \xrightarrow{\rho_g} H^1(Y_2, \mathbb{Z}) \to 0, \quad (2.5)$$

inducing isomorphisms $\mathbb{Q} \cong \text{Gr}_1^W H^1(\tilde{X}_0, \mathbb{Q})$ and $\text{Gr}_1^W H^1(\tilde{X}_0, \mathbb{Q}) \cong H^1(Y_2, \mathbb{Q})$. Since $\rho$ is a flat family of projective curves with $\rho^{-1}(t)$ of genus $g$ for $g \in \Delta^*$, we have

$$g = \rho_0(Y_1 \cup Y_2) = \rho_0(Y_1) + \rho_0(Y_2) + Y_1.Y_2 - 1 = \rho_0(Y_2) + 1,$$

where $\rho_0$ denotes the arithmetic genus (use $\rho_0(Y_1) = 0$). In other words, $\rho_0(Y_2) = g - 1$. There exists a symplectic basis $e'_1, e'_2, \ldots, e'_{g-1}, e'_{g+1}, e'_{g+2}, \ldots, e'_{2g-1}$ of $H^1(Y_2, \mathbb{Z})$ such that $e'_i \cup e'_{g+i} = -f'$ and $e'_i \cup e'_j = 0$ for $|i - j| \neq g$, where $f' \in H^2(Y_2, \mathbb{Z})$ is the dual of fundamental class of $Y_2$ (see §1.2). Let $e''_i \in H^1(\tilde{X}_0, \mathbb{Z})$ such that $q(e''_i) = e'_i$. Denote by $e_i := sp_1(e''_i)$, where

$$sp_1 : H^1(\tilde{X}_0, \mathbb{Z}) \to H^1(\tilde{X}_\infty, \mathbb{Z}),$$

is the specialization morphism. Since $sp_1$ maps $\text{Gr}_1^W H^1(\tilde{X}_0, \mathbb{Z})$ isomorphically to $\text{Gr}_1^W H^1(\tilde{X}_\infty, \mathbb{Z})$ (Corollary 2.4), we conclude that $e_1, e_2, \ldots, e_{g-1}, e_{g+1}, e_{g+2}, \ldots, e_{2g-1}$ is a basis of $\text{Gr}_1^W H^1(\tilde{X}_\infty, \mathbb{Q})$.

Denote by $i_1 : Y_1 \hookrightarrow \tilde{X}_0$ and $i_2 : Y_2 \hookrightarrow \tilde{X}_0$ the natural inclusions. Since cup-product commutes with pull-back, we have $i_1^*(e''_i \cup e''_j) = 0$ for all $i, j$ (use $H^1(\mathbb{P}^1) = 0$), $i_2^*(e''_i \cup e''_{g+i}) = e'_i \cup e'_{g+i} = -f'$ and $i_2^*(e''_i \cup e''_j) = e'_i \cup e'_j = 0$ for any $|i - j| \neq g$. This implies $e_i \cup e_j = 0$ for $|i - j| \neq g$ and

$$e_i \cup e_{i+g} = sp_2(0 \oplus -f')$$

under the morphism

$$H^2(Y_1, \mathbb{Z}) \oplus H^2(Y_2, \mathbb{Z}) \overset{(i_1^*, i_2^*)}{\sim} H^2(\tilde{X}_0, \mathbb{Z}) \overset{sp_2}{\to} H^2(\tilde{X}_\infty, \mathbb{Z}), \quad (2.6)$$

where the first isomorphism follows from the Mayer-Vietoris sequence.

Denote by $e_g := sp_1 \circ p(1)$. Note that, $e_g$ generates $W_0 H^1(\tilde{X}_\infty, \mathbb{Q})$ (use Corollary 2.4). Since the cup-product morphism

$$H^1(\tilde{X}_\infty, \mathbb{Z}) \otimes H^1(\tilde{X}_\infty, \mathbb{Z}) \to H^2(\tilde{X}_\infty, \mathbb{Z})$$

is a morphism of mixed Hodge structures and $H^2(\tilde{X}_\infty, \mathbb{Q})$ is pure, we conclude that $e_g \cup e_i = 0$ for all $1 \leq i \leq 2g - 1$. Choose $e_{2g} \in H^1(\tilde{X}_\infty, \mathbb{Z})$ such that $e_1, e_2, \ldots, e_{2g}$ generates $H^1(\tilde{X}_\infty, \mathbb{Z})$. Let $e_i \cup e_{2g} = a_i sp_2(0 \oplus -f')$. As the cup-product morphism above is a perfect pairing, we have $|a_g| = 1$. Replace $e_{2g}$ by

$$\frac{1}{a_g} (e_{2g} - \sum_{i \leq g} a_i e_{i+g} + \sum_{i > g} a_i e_{i-g} - (a_{2g}/a_g)e_g).$$

Note that $e_{2g} \in H^1(\tilde{X}_\infty, \mathbb{Z})$, generates $\text{Gr}_2^W H^1(\tilde{X}_\infty, \mathbb{Q})$, $e_i \cup e_{2g} = 0$ for $i \neq g$ and $e_g \cup e_{2g} = sp_2(0 \oplus -f')$. This proves the theorem. \square

3. Relative Mumford-Newstead isomorphism

In this section we review results on relative Mumford-Newstead isomorphisms as described in §4.

**Notation 3.1.** Let $X_0$ be an irreducible nodal curve of genus $g \geq 2$, with exactly one node, say at $x_0$. Denote by $\pi_0 : \tilde{X}_0 \to X_0$ be the normalization map. Since the moduli space of stable curve is complete, there exists a regular, flat family of projective curves $\pi_1 : \mathcal{X} \to \Delta$ smooth over $\Delta^*$ and central fiber isomorphic to $X_0$ (see [2] Theorem B.2]). Fix an invertible sheaf $\mathcal{L}$ on
\( \mathcal{X} \) of relative odd degree, say \( d \). Set \( \mathcal{L}_0 := \mathcal{L}|_{X_0} \), the restriction of \( \mathcal{L} \) to the central fiber. Denote by \( \tilde{\mathcal{L}}_0 := \pi_0^*(\mathcal{L}_0) \).

3.1. Relative Gieseker moduli space. There exists a regular, flat, projective family

\[
\pi_2 : \mathcal{G}(2, \mathcal{L}) \to \Delta
\]
called the \textit{relative Gieseker moduli space of rank 2 semi-stable sheaves on} \( \mathcal{X} \) \textit{with determinant} \( \mathcal{L} \), such that for all \( s \in \Delta^s \), \( \mathcal{G}(2, \mathcal{L})_s := \pi_2^{-1}(s) = M_{\mathcal{X}_s}(2, \mathcal{L}_s) \) and the central fiber \( \pi_2^{-1}(0) \), denoted \( \mathcal{G}_{X_0}(2, \mathcal{L}_0) \), is a reduced simple normal crossings divisor of \( \mathcal{G}(2, \mathcal{L}) \) (see [4] Notation A.5, Theorem A.7 and Remark A.8]). In particular, \( \mathcal{G}(2, \mathcal{L}) \) is smooth over \( \Delta^s \) and satisfies the conditions of Proposition [2,3]

Denote by \( M_{\mathcal{X}_0}(2, \tilde{\mathcal{L}}_0) \) the fine moduli space of semi-stable sheaves of rank 2 and with determinant \( \tilde{\mathcal{L}}_0 \) over \( \tilde{\mathcal{X}}_0 \) (see [22, Theorem 4.3.7 and 4.6.6]). By [4] Theorem A.7, \( \mathcal{G}_{X_0}(2, \mathcal{L}_0) \) can be written as the union of two irreducible components, say \( \mathcal{G}_0 \) and \( \mathcal{G}_1 \), and \( \mathcal{G}_0 \) (resp. \( \mathcal{G}_0 \cap \mathcal{G}_1 \)) is isomorphic to a \( \mathbb{P}^3 \) (resp. \( \mathbb{P}^1 \times \mathbb{P}^1 \))-bundle over \( M_{\mathcal{X}_0}(2, \tilde{\mathcal{L}}_0) \).

3.2. Mumford-Newstead isomorphism in families. Let us consider the relative version of the construction in [26]. Denote by

\[
\mathcal{W} := \mathcal{X}_{\Delta^s} \times_{\Delta^s} \mathcal{G}(2, \mathcal{L})_{\Delta^s} \quad \text{and} \quad \pi_3 : \mathcal{W} \to \Delta^s
\]
the natural morphism. Recall, for all \( t \in \Delta^s \), the fiber
\[
\mathcal{W}_t := \pi_3^{-1}(t) = \mathcal{X}_t \times \mathcal{G}(2, \mathcal{L})_t \cong \mathcal{X}_t \times M_{\mathcal{X}_t}(2, \mathcal{L}_t).
\]
There exists a (relative) universal bundle \( U \) over \( \mathcal{W} \) associated to the (relative) moduli space \( \mathcal{G}(2, \mathcal{L})_{\Delta^s} \) i.e., \( U \) is a vector bundle over \( \mathcal{W} \) such that for each \( t \in \Delta^s \), \( U|_{\mathcal{W}_t} \) is the universal bundle over \( \mathcal{X}_t \times M_{\mathcal{X}_t}(2, \mathcal{L}_t) \) associated to fine moduli space \( M_{\mathcal{X}_t}(2, \mathcal{L}_t) \) (use [29, Theorem 9.1.1]). Denote by \( \mathbb{H}^i_{\mathcal{W}} := R^i\pi_3^*\mathcal{Z} \) the local system associated to \( \mathcal{W} \). Using Künneth decomposition, we have (see [23] for notations)

\[
\mathbb{H}^i_{\mathcal{W}} = \bigoplus_i \left( \mathbb{H}^i_{\mathcal{X}_{\Delta^s}} \otimes \mathbb{H}^{4-i}_{\mathcal{G}(2, \mathcal{L})_{\Delta^s}} \right).
\] (3.1)

Denote by \( c_2(U)^{1,3} \in \Gamma \left( \mathbb{H}^1_{\mathcal{X}_{\Delta^s}} \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^s}} \right) \) the image of the second Chern class \( c_2(U) \in \Gamma(\mathbb{H}^4_{\mathcal{W}}) \) under the natural projection from \( \mathbb{H}^4_{\mathcal{W}} \) to \( \mathbb{H}^1_{\mathcal{X}_{\Delta^s}} \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^s}} \). Using Poincaré duality applied to the local system \( \mathbb{H}^1_{\mathcal{X}_{\Delta^s}} \) (see [25, §I,2.6]), we have

\[
\mathbb{H}^1_{\mathcal{X}_{\Delta^s}} \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^s}} \overset{\text{PD}}{\cong} \left( \mathbb{H}^1_{\mathcal{X}_{\Delta^s}} \right)^\vee \otimes \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^s}} \cong \text{Hom} \left( \mathbb{H}^1_{\mathcal{X}_{\Delta^s}}, \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^s}} \right).
\] (3.2)

Therefore, \( c_2(U)^{1,3} \) induces a homomorphism \( \Phi_{\Delta^s} : \mathbb{H}^1_{\mathcal{X}_{\Delta^s}} \to \mathbb{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^s}} \).

By [26, Lemma 1 and Proposition 1], we conclude that the homomorphism \( \Phi_{\Delta^s} \) is an isomorphism that the induced isomorphism on the associated vector bundles:

\[
\Phi_{\Delta^s} : \mathcal{H}^1_{\mathcal{X}_{\Delta^s}} \cong \mathcal{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^s}} \quad \text{satisfies} \quad \Phi_{\Delta^s} (F^p\mathcal{H}^1_{\mathcal{X}_{\Delta^s}}) = F^{p+1}\mathcal{H}^3_{\mathcal{G}(2, \mathcal{L})_{\Delta^s}} \quad \text{for all} \quad p \geq 0.
\]

3.3. Limit Mumford-Newstead isomorphism. We now extend the isomorphism \( \Phi_{\Delta^s} \) to the entire disc \( \Delta \) and show that the induced morphism on the central fibers is an isomorphism of limit mixed Hodge structures. We do this using the monodromy operator (see [22]). In order to guarantee that the monodromy operator is unipotent, we want the central fiber of the relevant families of projective varieties to be reduced simple, normal crossings divisor. The family \( \pi_2 \) of moduli spaces already satisfy this criterion. Unfortunately, the central fiber of \( \pi_1 \) is \( X_0 \), which
is not a simple normal crossings divisor. We can easily rectify this problem by blowing up $\mathcal{X}$ at the point $x_0$. Denote by $\tilde{\mathcal{X}} := \text{Bl}_{x_0} \mathcal{X}$ and by

$$\tilde{\pi}_1 : \tilde{\mathcal{X}} \to \mathcal{X} \xrightarrow{\pi_1} \Delta.$$  \hspace{1cm} (3.3)

Note that the central fiber of $\tilde{\pi}_1$ is the union of two irreducible components, the normalization $\tilde{X}_0$ of $X_0$ and the exceptional divisor $F \cong \mathbb{P}^1_{x_0}$ intersecting $\tilde{X}_0$ at the two points over $x_0$.

Let $\mathcal{H}^1_{\tilde{\mathcal{X}}_*}$ and $\mathcal{H}^3_{G(2, \mathcal{L})_*}$ be the canonical extensions of $\mathcal{H}^1_{\mathcal{X}_*}$ and $\mathcal{H}^3_{G(2, \mathcal{L})_*}$, respectively. Then, the morphism $\Phi_{\Delta_*}$ extend to the entire disc:

$$\tilde{\Phi} : \mathcal{H}^1_{\tilde{\mathcal{X}}} \to \mathcal{H}^3_{G(2, \mathcal{L})}.$$  

Using the identification (2.1) and restricting $\tilde{\Phi}$ to the central fiber, we have an isomorphism:

$$\tilde{\Phi}_0 : H^1(\tilde{X}_\infty, \mathbb{Q}) \xrightarrow{\sim} H^3(G(2, \mathcal{L})_\infty, \mathbb{Q}).$$ \hspace{1cm} (3.4)

We now prove that $\tilde{\Phi}_0$ is an isomorphism of mixed Hodge structures in some sense (see Theorem 3.2 below). We briefly discuss the idea of proof. The first step is to check that the monodromy action commutes with $\tilde{\Phi}_0$. Then, we use Remark 2.1 and Theorem 2.5 to determine the monodromy invariant part of $H^3(G(2, \mathcal{L})_\infty, \mathbb{Q})$. Finally, using the invariant cycle theorem (Remark 2.2) we conclude:

**Theorem 3.2.** For the extended morphism $\tilde{\Phi}$, we have $\tilde{\Phi}(F^p \mathcal{H}^1_{\tilde{\mathcal{X}}}) = F^{p+1} \mathcal{H}^3_{G(2, \mathcal{L})}$ for $p = 0, 1$ and $\tilde{\Phi}(\mathcal{H}^1_{\tilde{\mathcal{X}}}) = \mathcal{H}^3_{G(2, \mathcal{L})}$. Moreover, $\tilde{\Phi}_0(W_i H^1(\tilde{X}_\infty, \mathbb{Q})) = W_{i+2} H^3(G(2, \mathcal{L})_\infty, \mathbb{Q})$ for all $i \geq 0$.

**Proof.** See [4, Proposition 4.1] for a proof of the statement. \hfill $\square$

### 4. Computing the kernel of the Gysin morphism

Notations as in Notation 3.1 and 3.1. The goal of this section is to compute the kernel of the Gysin morphism $f_!$ as in (2.4), in the case when the flat family $\rho$ is the relative Gieseker moduli space of rank 2 semi-stable sheaves with fixed determinant associated to a degenerating family of smooth curves (Theorem 4.2).

#### 4.1. Cohomology of the fibers of $P_0, \mathcal{G}_1$ and $\mathcal{G}_0 \cap \mathcal{G}_1$.

Recall, $\overline{SL}_2 \subset \mathbb{P}^4 \cong \mathbb{P}((\mathbb{C}^2) \oplus \mathbb{C})$ consists of points $[x_0 : x_1 : ... : x_4]$ such that $x_0 x_3 - x_1 x_2 = x_4^2$ (see discussion before [4, Proposition A.4]). Denote by $j_1 : \overline{SL}_2 \hookrightarrow \mathbb{P}^4$ the natural inclusion as a quadric hypersurface. Recall, $\mathcal{G}_1$ (resp. $\mathcal{G}_0 \cap \mathcal{G}_1$) is a $\mathbb{P}^3$ (resp. $\mathbb{P}^1 \times \mathbb{P}^1$)-bundle over $M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0)$. Denote by

$$\rho_1 : \mathcal{G}_0 \cap \mathcal{G}_1 \longrightarrow M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0) \quad \text{and} \quad \rho_2 : \mathcal{G}_1 \longrightarrow M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0)$$

the natural projections. Recall, by [4, Proposition A.6], there exists an $\overline{SL}_2$-bundle $P_0$ over $M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0)$ and natural inclusions

$$i_1 : \mathcal{G}_0 \cap \mathcal{G}_1 \hookrightarrow P_0, \quad i_2 : \mathcal{G}_0 \cap \mathcal{G}_1 \hookrightarrow \mathcal{G}_1 \quad \text{and} \quad i_3 : \mathcal{G}_0 \cap \mathcal{G}_1 \hookrightarrow \mathcal{G}_0$$

such that for the natural projection $\rho_1 : P_0 \longrightarrow M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0)$, we have for any $y \in M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0)$, identifying the fiber $\rho_1^{-1}(y)$ (resp. $\rho_2^{-1}(y), \rho_3^{-1}(y)$) with $\mathbb{P}^1 \times \mathbb{P}^1$ (resp. $\mathbb{P}^3, \overline{SL}_2$), the natural
inclusions $i_{1,y}$ and $i_{2,y}$ induced by $i_1$ and $i_2$ respectively, sits in the following diagram:

\[
\begin{array}{ccc}
P^1 \times P^1 & \xrightarrow{i_{1,y}} & \mathcal{S}_{2} \\
\downarrow & & \downarrow j_1 \\
P^3 & \xrightarrow{j_2} & P^4 \\
\end{array}
\]

where $j_2([x_0 : ... : x_3]) = [x_0 : ... : x_3 : 0]$ and $i_{1,y}$ (resp. $i_{2,y}$) is the Segre embedding sending $[s : t] \times [u : v] \in P^1 \times P^1$ to $[su : sv : tu : tv] \in \mathcal{S}_{2}$ (resp. $[su : sv : tu : tv] \in P^3$).

See discussion before Proposition A.4.

Let $\xi_0$ be a generator of $H^2(P^1, Q)$, pr$_i$ the natural projections from $P^1 \times P^1$ to $P^1$ and $\xi_i := pr_i^*(\xi_0)$. Using the Kunneth decomposition, we have

$$H^1(P^1 \times P^1, Q) = 0 = H^3(P^1 \times P^1, Q), H^0(P^1 \times P^1, Q) = Q, H^2(P^1 \times P^1, Q) = Q\xi_1 \oplus Q\xi_2$$

and $$H^4(P^1 \times P^1, Q) = Q\xi_1 \xi_2.\quad \text{Since } \mathcal{S}_{2} \text{ is a quadric hypersurface in } P^4, \text{ the Lefschetz hyperplane section theorem implies that } H^{2i}(\mathcal{S}_{2}, Q) \cong Q \text{ for } 0 \leq i \leq 3 \text{ and } H^1(\mathcal{S}_{2}, Q) = 0 = H^5(\mathcal{S}_{2}, Q).$$

It is also known that $H^3(\mathcal{S}_{2}, Q) = 0$. Denote by $\xi \in H^2(P^1, Z)$ a generator, $\xi' := j_1^*(\xi)$ and $\xi'' := j_2^*(\xi)$.

### 4.2. Kernel of the Gysin morphisms

We now compute the kernel of the Gysin morphisms $i_{2,*}$ and $i_{3,*}$. The first step is to determine the kernel of $i_{1,*}$ and $i_{2,*}$ (Proposition 4.1). This is done using the Leray-Hirsch theorem.

**Proposition 4.1.** We have,

$$\ker((i_{1,*}, i_{2,*}) : H^{i-2}(G_0 \cap G_1, Q)(-1) \to H^i(P_0, Q) \oplus H^i(G_1, Q)) \cong H^{i-4}(M_{X_0}(2, L_0), Q)(\xi_1 \oplus -\xi_2).$$

**Proof.** It is easy to check that,

$$\ker(((i_{1,y}), (i_{2,y})) : H^{2j}(P^1 \times P^1, Q) \to H^{2j+2}(\mathcal{S}_{2}, Q) \oplus H^{2j+2}(P^3, Q)) \cong Q \quad \text{if } j = 1 \quad (4.1)$$

$$\quad = 0 \quad \text{if } j \neq 1. \quad (4.2)$$

By the Deligne-Blanchard theorem [11] (the Leray spectral sequence degenerates at $E_2$ for smooth families), we have

$$H^i(G_0 \cap G_1, Q) \cong \oplus_j H^{i-j}(R^j\rho_1, Q), H^i(G_1, Q) \cong \oplus_j H^{i-j}(R^j\rho_2, Q) \text{ and } H^i(P_0, Q) \cong \oplus_j H^{i-j}(R^j\rho_3, Q).$$

Since $M_{X_0}(2, L_0)$ is smooth and simply connected, the local systems $R^j\rho_1, Q$, $R^j\rho_2, Q$ and $R^j\rho_3, Q$ are trivial. Therefore, for any $y \in M_{X_0}(2, L_0)$, the natural morphisms

$$H^i(G_0 \cap G_1, Q) \to H^0(R^j\rho_1, Q) \to H^i(G_0 \cap G_1, Q),$$

$$H^i(G_1, Q) \to H^0(R^j\rho_2, Q) \to H^i(G_1, Q) \text{ and } H^i(P_0, Q) \to H^0(R^j\rho_3, Q) \to H^i(P_0, Q),$$
are surjective. Then, Leray-Hirsch theorem (see [38, Theorem 7.33]), implies that for any closed point \( y \in M_{\tilde{X}_0}(2, \tilde{L}_0) \),

\[
H^i(G_0 \cap G_1, \mathbb{Q}) \cong \bigoplus_{j \geq 0} (H^j((G_0 \cap G_1)_y, \mathbb{Q}) \otimes H^{i-j}(M_{\tilde{X}_0}(2, \tilde{L}_0)))
\]

\[
\cong H^i(M_{\tilde{X}_0}(2, \tilde{L}_0)) \oplus H^{i-2}(M_{\tilde{X}_0}(2, \tilde{L}_0)) \otimes (\mathbb{Q}\xi_1 \oplus \mathbb{Q}\xi_2) \oplus H^{i-4}(M_{\tilde{X}_0}(2, \tilde{L}_0))\xi_1\xi_2,
\]

(4.3)

\[
H^i(G_1, \mathbb{Q}) \cong \bigoplus_{j \geq 0} (H^j(G_1, \mathbb{Q}) \otimes H^{i-j}(M_{\tilde{X}_0}(2, \tilde{L}_0))) \cong \bigoplus_{j \geq 0} H^{i-2j}(M_{\tilde{X}_0}(2, \tilde{L}_0)) \otimes (\xi''^2)^j
\]

(4.4)

\[
H^i(P_0, \mathbb{Q}) \cong \bigoplus_{j \geq 0} (H^j(P_0, \mathbb{Q}) \otimes H^{i-j}(M_{\tilde{X}_0}(2, \tilde{L}_0))) \cong \bigoplus_{j \geq 0} H^{i-2j}(M_{\tilde{X}_0}(2, \tilde{L}_0)) \otimes (\xi')^j
\]

(4.5)

Using the projection formula (see [30, Lemma B.26]) and the identifications described in the Leray-Hirsch theorem (identifying certain cohomology classes with its restrictions to \( y \)), we have for any \( \alpha \in H^{i-2}((G_0 \cap G_1)_y, \mathbb{Q}) \) and \( \beta \in H^{i-2}(M_{\tilde{X}_0}(2, \tilde{L}_0), \mathbb{Q}) \),

\[
i_{1,*}(\alpha \cup \rho_1^*\beta) = i_{1,*}(\alpha \cup i_1^*\rho_1^*\beta) = (i_1)_y(\alpha \cup \rho_1^*\beta) \quad \text{and} \quad i_{2,*}(\alpha \cup \rho_1^*\beta) = i_{2,*}(\alpha \cup i_2^*\rho_1^*\beta) = (i_2)_y(\alpha \cup \rho_1^*\beta).
\]

Note that for \( \beta \neq 0 \), \((i_1)_y(\alpha \cup \rho_1^*\beta) = 0\) (resp. \((i_2)_y(\alpha \cup \rho_1^*\beta) = 0\)) if and only if \((i_1)_y(\alpha) = 0\) (resp. \((i_2)_y(\alpha) = 0\)). Using the decompositions above and (4.1), (4.2), this implies \( \ker((i_{1,*}, i_{2,*})) \) is isomorphic to \( H^{i-2}(M_{\tilde{X}_0}(2, \tilde{L}_0), \mathbb{Q})(\xi_1 + -\xi_2) \). This proves the proposition. \( \square \)

**Theorem 4.2.** The kernel of the Gysin morphism

\[(i_{3,*}, i_{2,*}) : H^{i-2}(G_0 \cap G_1, \mathbb{Q})(-1) \to H^i(G_0, \mathbb{Q}) \oplus H^i(G_1, \mathbb{Q})\]

is isomorphic to \( H^{i-4}(M_{\tilde{X}_0}(2, \tilde{L}_0), \mathbb{Q})(\xi_1 + -\xi_2) \).

**Proof.** By [4, Proposition A.4], there exist closed subschemes \( Z \subset P_0 \) and \( Z' \subset G_0 \) such that \( P_0 \setminus Z \cong G_0 \setminus Z' \). Using [18] (see also [37, P. 27] or [33, Remark 6.5(c), Theorem 6.2]), one can observe that \( Z \cap \text{Im}(i_3) = \emptyset = Z' \cap \text{Im}(i_3) \) and there exists a smooth, projective variety \( W \) along with proper, birational morphisms \( \tau_1 : W \to P_0 \) and \( \tau_2 : W \to G_0 \) such that

\[
W \setminus \tau_1^{-1}(Z) \cong P_0 \setminus Z \cong G_0 \setminus Z' \cong W \setminus \tau_2^{-1}(Z').
\]

Therefore, there exists a natural closed immersion \( l : G_0 \cap G_1 \to W \) such that \( i_1 = \tau_1 \circ l \) and \( i_3 = \tau_2 \circ l \). We claim that, given any \( \xi \in H^{k-2}(G_0 \cap G_1, \mathbb{Q}) \), we have \( \tau_1^*i_{1,*}(\xi) = l_*(\xi) = \tau_2^*i_{3,*}(\xi) \). Indeed, since \( \text{Im}(i_1) \) (resp. \( \text{Im}(i_3) \)) does not intersect \( Z \) (resp. \( Z' \)), the pullback of \( l_*(\xi) \) to \( \tau_1^{-1}(Z) \) and \( \tau_2^{-1}(Z') \) vanish. Using the (relative) cohomology exact sequence (see [30, Proposition 5.54]), we conclude that there exists \( \beta_1 \in H^k(P_0) \) and \( \beta_2 \in H^k(G_0) \) such that \( \tau_1^*(\beta_1) = l_*(\xi) = \tau_2^*(\beta_2) \). Applying \( \tau_1_* \) and \( \tau_2_* \) to the two equalities respectively and using [30, Proposition B.27], we get

\[
\beta_1 = \tau_{1,*}\tau_1^*(\beta_1) = \tau_{1,*}l_*(\xi) = i_{1,*}(\xi) \quad \text{and} \quad \beta_2 = \tau_{2,*}\tau_2^*(\beta_2) = \tau_{2,*}l_*(\xi) = i_{3,*}(\xi).
\]

In other words, \( \tau_1^*i_{1,*}(\xi) = l_*(\xi) = \tau_2^*i_{3,*}(\xi) \). This proves the claim. Using [30, Theorem 5.41], we then conclude that \( i_{1,*}(\xi) = 0 \) (resp. \( i_{3,*}(\xi) = 0 \)) if and only if \( l_*(\xi) = 0 \). In other words, \( \ker(i_{1,*}) \cong \ker(i_{3,*}) \). Using Proposition 4.1, we have that \( \ker(f_1) \cong H^{i-4}(M_{\tilde{X}_0}(2, \tilde{L}_0), \mathbb{Q})(\xi_1 + -\xi_2) \). This proves the theorem. \( \square \)

5. ON A CONJECTURE OF MUMFORD

In Theorem 5.1 we give a complete set of relations between the generators of the cohomology ring of the moduli space of rank 2 semi-stable sheaves with fixed determinant over an irreducible nodal curve, analogous to a classical conjecture of Mumford as proved in [24]. We use notations as in Notation 3.1 and 3.1.
5.1. Generators of the cohomology ring \( H^*(G(2, \mathcal{L}_s), \mathbb{Q}) \). We first use the classical result of Newstead [28] to determine the generators of the cohomology ring \( H^*(G(2, \mathcal{L}_s), \mathbb{Q}) \). Let \( \psi_i := \Phi_0(e_i) \), where \( e_i \in H^1(\tilde{X}_s, \mathbb{Z}) \) as in Theorem 2.1, \( 1 \leq i \leq 2g \) and \( \Phi_0 \) as in [3.1]. Fix \( s \in \Delta^* \). Let \( \psi_{i,s} \in H^3(G(2, \mathcal{L}_s), \mathbb{Z}) \) the image of \( \psi_i^\infty \) under the natural isomorphism

\[
\eta_j : H^j(G(2, \mathcal{L}_s), \mathbb{Z}) \simto H^j(G(2, \mathcal{L}_s), \mathbb{Z}), \quad \text{for all } j \geq 0. \tag{5.1}
\]

Using [28, Theorem 1], one can observe that for any \( s \in \Delta^* \), there exist elements

\[
\alpha_s \in H^{1,1}(G(2, \mathcal{L}_s), \mathbb{Z}) \quad \text{and} \quad \beta_s \in H^{2,2}(G(2, \mathcal{L}_s), \mathbb{Z})
\]

such that the cohomology ring \( H^*(G(2, \mathcal{L}_s), \mathbb{Q}) \) is generated by \( \alpha_s, \beta_s, \psi_{1,s}, \psi_{2,s}, \ldots, \psi_{2g,s} \). Let

\[
\alpha_\infty \in H^2(G(2, \mathcal{L}_s), \mathbb{Z}) \quad \text{and} \quad \beta_\infty \in H^4(G(2, \mathcal{L}_s), \mathbb{Z})
\]

the preimage of \( \alpha_s \) and \( \beta_s \), respectively, under the natural isomorphism \([5.1]\). It is immediate that the cohomology ring \( H^*(G(2, \mathcal{L}_s), \mathbb{Q}) \) is generated by \( \alpha_\infty, \beta_\infty, \psi_1^\infty, \psi_2^\infty, \ldots, \psi_{2g}^\infty \).

5.2. Relations on the cohomology rings \( H^*(M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0), \mathbb{Q}) \) and \( H^*(G(2, \mathcal{L}_s), \mathbb{Q}) \). Denote by \( \psi_i := \Phi_1(\psi_i^\infty) \) for \( 1 \leq i \leq g - 1 \) and \( \psi_i := \Phi_1(\psi_{i+1}^\infty) \) for \( g \leq i \leq 2g - 2 \). Let \( \alpha \in H^2(M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0), \mathbb{Z}) \) (resp. \( \beta \in H^2(M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0), \mathbb{Z}) \)) such that \( \alpha \) (resp. \( \alpha_\infty \) and \( \beta \) generate \( H^2(M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0), \mathbb{Q}) \) (resp. \( H^4(M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0), \mathbb{Q}) \)). Denote by \( \psi_\infty := \sum_{i=1}^g \psi_i^\infty \psi_i^\infty \),

\[
\psi := \sum_{i=1}^{g-1} \psi_i \psi_i + g \quad \text{and} \quad P_i := \ker \left( \psi^{g-i} : \bigcup_{j=1}^{i} H^3(M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0), \mathbb{Q}) \to \bigcup_{j=1}^{2g-i} H^3(M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0), \mathbb{Q}) \right).
\]

Given an ordered set \( (i_1, \ldots, i_k) = I \) with \( 1 \leq i_1 < i_2 < \ldots < i_k \leq 2g \), denote by \( \psi_I^\infty := \psi_{i_1}^\infty \ldots \psi_{i_k}^\infty \). For \( i \leq g \), denote by \( P_i^\infty \) the \( \mathbb{Q} \)-vector space generated by elements of the form

\[
\sum_I a_I \psi_I^\infty \in \ker \left( \psi^{g-i+1} : \bigcup_{j=1}^{i} H^3(G(2, \mathcal{L}_s), \mathbb{Q}) \to \bigcup_{j=1}^{2g-i+2} H^3(G(2, \mathcal{L}_s), \mathbb{Q}) \right), \quad a_I \in \mathbb{Q},
\]

such that for all \( 2g, I \) then \( g \in I \). Define the ideals \( I_k \subset \mathbb{Q}[\alpha, \beta, \psi] \) and \( I_k^\infty \subset \mathbb{Q}[\alpha_\infty, \beta_\infty, \psi_\infty] \) generated by \( (\zeta_k, \zeta_{k+1}, \zeta_{k+2}) \) and \( (\zeta_k^\infty, \zeta_{k+1}^\infty, \zeta_{k+2}^\infty) \) respectively, where \( \zeta_i \) and \( \zeta_i^\infty \) are recursively defined by \( \zeta_0 = 1, \zeta_0^\infty = 1, \zeta_i = 0 = \zeta_i^\infty \) for \( i < 0 \),

\[
\zeta_{k+1} := \alpha \zeta_k + k^2 \beta \zeta_{k-1} + 2k(k-1) \psi \zeta_{k-2} \quad \text{and} \quad \zeta_{k+1}^\infty := \alpha_\infty \zeta_k^\infty + k^2 \beta_\infty \zeta_{k-1}^\infty + 2k(k-1) \psi_\infty \zeta_{k-2}^\infty.
\]

Using [28, Theorem 3.2], the natural morphism \( \nu_0 \) from \( \bigoplus_{k=0}^{g-1} P_k \otimes \mathbb{Q}[\alpha, \beta, \psi] \) to \( H^*(M_{\tilde{X}_0}(2, \tilde{\mathcal{L}}_0), \mathbb{Q}) \) is surjective with kernel \( \otimes \mathbb{Q}[\alpha, \beta, \psi] \). Denote by

\[
\psi_s := \sum_{i=1}^{g} \psi_i \psi_i + g \quad \text{and} \quad P_i^0 := \ker \left( \psi_s^{g-i+1} : \bigcup_{j=1}^{i} H^3(G(2, \mathcal{L}_s), \mathbb{Q}) \to \bigcup_{j=1}^{2g-i+2} H^3(G(2, \mathcal{L}_s), \mathbb{Q}) \right).
\]

Similarly as before, the obvious map

\[
\nu : \bigoplus_{k=0}^{g-1} P_k^0 \otimes \mathbb{Q}[\alpha, \beta, \psi] \to H^*(G(2, \mathcal{L}_s), \mathbb{Q})
\]

is surjective with kernel isomorphic to \( \otimes P_k^0 \otimes I_{g-k,s} \), where \( I_{g-k,s} \) is defined identically as \( I_{g-k}^\infty \) above, after replacing \( \alpha_\infty, \beta_\infty \) and \( \psi_\infty \) by \( \alpha_s, \beta_s \) and \( \psi_s \), respectively.
5.3. Comparing pure Hodge structures on $\text{Gr}_3^W \mathcal{H}^3(G(2, \mathcal{L})_\infty, \mathbb{Q})$ and $\mathcal{H}^3(M_{\check{X}_0}(2, \check{L}_0), \mathbb{Q})$.

The cohomology ring of $M_{\check{X}_0}(2, \check{L}_0)$ will also play an important role in this section and the next. Recall, [26] Proposition 1] states that there exists an isomorphism of pure Hodge structures:

$$\Phi_0 : H^1(\check{X}_0, \mathbb{Z}) \xrightarrow{\sim} H^3(M_{\check{X}_0}(2, \check{L}_0), \mathbb{Z}).$$

Using the short exact sequence (2.5) and Theorem 3.2 we have the composed morphism

$$\Phi_1 : \text{Gr}_3^W \mathcal{H}^3(G(2, \mathcal{L})_\infty, \mathbb{Q}) \rightarrow H^3(M_{\check{X}_0}(2, \check{L}_0), \mathbb{Q})$$

defined by

$$\text{Gr}_3^W \mathcal{H}^3(G(2, \mathcal{L})_\infty, \mathbb{Q}) \xrightarrow{\phi} \text{Gr}_1^W \mathcal{H}^1(\check{X}_\infty, \mathbb{Q}) \xrightarrow{\text{sp}_1} \text{Gr}_1^W \mathcal{H}^1(\check{X}_0, \mathbb{Q}) \xrightarrow{\phi_0} H^1(\check{X}_0, \mathbb{Q}) \xrightarrow{\Phi_0} H^3(M_{\check{X}_0}(2, \check{L}_0), \mathbb{Q}),$$

where the first isomorphism is given by (3.1), the second and third isomorphisms are given in the proof of Theorem 2.5. By Theorem 3.2, $\Phi_0$ is an isomorphism of pure Hodge structures. Also, note that the last three morphisms in the composed morphism $\Phi_1$ are morphisms of pure Hodge structures. Therefore, $\Phi_1$ is an isomorphism of pure Hodge structures.

5.4. Generalized Mumford’s conjecture on $H^*(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q})$. We briefly discuss the idea of the proof of Theorem 5.1 below. Combining (2.4) with Theorem 1.2 we prove that $\oplus_i \ker(\text{sp}_i)$ is generated as a polynomial ring over $H^* (M_{\check{X}_0}(2, \check{L}_0), \mathbb{Q})$ by two variables $X$ and $Y$ satisfying $X^2 - Y^2 = X - Y - 0$. Then, \[5.2\] gives us the relations between the generators of $\oplus_i \ker(\text{sp}_i)$. To obtain the relations between the generators of $\oplus_i \ker(\text{sp}_i)$, we use the isomorphism $\Phi_1$ above, along with the description of the generators of $H^*(G(2, \mathcal{L}), \mathbb{Q})$ for all $i \geq 0$, as given in [23] Remark 5.3. The theorem would then follow immediately.

**Theorem 5.1.** We have the following isomorphism of graded rings:

$$H^*(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \cong \left( \bigoplus_i P_i^\infty \otimes \frac{\mathbb{Q}[\alpha_i, \beta_i, \psi_i]}{I_{g-i}} \right) \oplus \left( \bigoplus_i P_i^{2-2} \otimes \frac{\mathbb{Q}[\alpha, \beta, \psi, X, Y]}{(I_{g-i-3}, X^2, Y^2, X - Y)} \right).$$

(Note that, the multiplicative identity of the ring $H^*(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q})$ lies in the first summand.)

**Proof.** For any $s \in \Delta^*$, consider the natural isomorphism $\phi_s : H^1(\check{X}_\infty, \mathbb{Z}) \xrightarrow{\sim} H^1(X_s, \mathbb{Z})$ induced by the closed immersion of $X_s$ as a fiber of $\check{X}_\infty$. Denote by $e_i := \phi_s(e_i)$, where $e_i$ as in Theorem 2.5. Consider the composed morphism (use (2.6)):

$$\bar{\phi}_s : H^2(\mathbb{P}^1, \mathbb{Z}) \oplus H^2(\check{X}_0, \mathbb{Z}) \xrightarrow{\text{sp}_2} H^2(\check{X}_\infty, \mathbb{Z}) \xrightarrow{\phi} H^2(X_s, \mathbb{Z}).$$

Let $f' \in H^2(\check{X}_0, \mathbb{Z})$ be the Poincaré dual of the fundamental class of $\check{X}_0$. Since $H^2(\check{X}_\infty, \mathbb{Z}) \cong \mathbb{Z}$ is pure, $\text{sp}_2$ is surjective. It follows from the definition of the Gysin morphism that

$$\text{Im}(f) : H^2(\check{X}_0, \mathbb{Z}) \rightarrow H^2(\check{X}_0, \mathbb{Z}) \oplus H^2(\mathbb{P}^1, \mathbb{Z})) = \mathbb{Z}(f' \oplus [\mathbb{P}^1]^\vee),$$

hence does not intersect $\mathbb{Z}(0 \oplus f')$ non-trivially (here $f$ as in Corollary 2.4). Using Corollary 2.4 this implies $\text{Im}(\text{sp}_2) = \text{sp}_2(\mathbb{Z}(0 \oplus f'))$. As $\phi_s$ is an isomorphism, this implies $\bar{\phi}_s(0 \oplus f')$ is the Poincaré dual of the fundamental class of $X_s$ (upto a sign), denoted $f_s$. Since pullback commutes with cup-product, Theorem 2.5 implies that $e_{1,s}, \ldots, e_{2g,s}$ is a symplectic basis of $H^1(X_s, \mathbb{Z})$ with $e_{i,s}e_{i+g,s} = -f_s$ for $1 \leq i \leq g$. Using [23] Remark 5.3, $H^1(G(2, \mathcal{L}), \mathbb{Q})$ has a $\mathbb{Q}$-basis consisting of monomials of the form $\alpha_1^{j_1} \beta_1^{j_2} \psi_{i_1} \psi_{i_2} \ldots \psi_{i_k}$ such that $j_1 + k < g$, $j_2 + k < g$, $i_1 < i_2 < \cdots < i_k$ and $2j_1 + 4j_2 + 3k = i$. By Theorems 2.5 and 3.2 $\psi_0^\infty$ (resp. $\psi_1^\infty$) generates $\text{Gr}_4^W H^3(G(2, \mathcal{L}), \mathbb{Q})$ (resp. $W_2 H^3(G(2, \mathcal{L}), \mathbb{Q})$). Since cup-product is a morphism of mixed Hodge structures, we have a basis of $W_i H^3(G(2, \mathcal{L}), \mathbb{Q})$ consisting of monomials of the form $\alpha_1^{j_1} \beta_1^{j_2} \psi_{i_1} \psi_{i_2} \ldots \psi_{i_k}$ with
$j_1 + k < g$, $j_2 + k < g$, $i_1 < i_2 < \ldots < i_k$, $2j_1 + 4j_2 + 3k = i$ satisfying: if $2g \in \{i_1, \ldots, i_k\}$ then $g \in \{i_1, \ldots, i_k\}$. Using the isomorphism $\eta_j$, we then obtain the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \bigoplus_{k=0}^{g} \mathcal{P}_k^\infty \otimes \mathcal{I}_{g-k}^\infty \\
& \downarrow \eta_j & \\
0 & \longrightarrow & \bigoplus_{k=0}^{g} \mathcal{P}_k^\infty \otimes \mathbb{Q}[\alpha_\infty, \beta_\infty, \psi_\infty] \longrightarrow H^*(G(2, \mathcal{L})_\infty, \mathbb{Q})
\end{array}
\]

where $\nu_\infty$ is the natural morphism and the two rows are exact. Using the description of $W_i H^i(G(2, \mathcal{L})_\infty, \mathbb{Q})$ above, it is easy to check that $\text{Im}(\nu_\infty) = W_i H^i(G(2, \mathcal{L})_\infty, \mathbb{Q})$. Therefore,

\[
\bigoplus_i W_i H^i(G(2, \mathcal{L})_\infty, \mathbb{Q}) \cong \bigoplus_i \mathcal{P}_i^\infty \otimes \mathbb{Q}[\alpha_\infty, \beta_\infty, \psi_\infty].
\]  

Similarly as before, under the identification \[(5.1)\], Theorem \[(4.2)\] implies that:

\[
\ker(sp_i) \cong \frac{H^{i-2}(M_{X_0}(2, \bar{L}_0), \mathbb{Q}) \oplus H^{i-4}(M_{X_0}(2, \bar{L}_0), \mathbb{Q})(\xi_1 \oplus \xi_2) \oplus H^{i-6}(M_{X_0}(2, \bar{L}_0), \mathbb{Q})\xi_1\xi_2}{H^{i-4}(M_{X_0}(2, \bar{L}_0), \mathbb{Q})(\xi_1 \oplus -\xi_2)}
\]

(see the proof of Proposition \[(4.1)\] to note that $\ker(f_i) = H^{i-4}(M_{X_0}(2, \bar{L}_0), \mathbb{Q})(\xi_1 \oplus -\xi_2)$). Hence, $\oplus_i \ker(sp_i) \cong H^*(M_{X_0}(2, \bar{L}_0), \mathbb{Q})(-2)[X,Y]/(X^2, Y^2, X - Y)$ defined by sending $\xi_1$ (resp. $\xi_2$) to $X$ (resp. $Y$), where $(X^2, Y^2, X + Y)$ is the ideal in the ring $H^*(M_{X_0}(2, \bar{L}_0), \mathbb{Q})[X,Y]$ generated $X^2, Y^2$ and $X - Y$. Using \[(23)\] and \[(3.2)\], we conclude that

\[
\bigoplus_i \ker(sp_i) \cong \bigoplus_i \mathcal{P}_{i-2} \otimes \frac{\mathbb{Q}[\alpha, \beta, \psi, X, Y]}{(I_{g-i-3}, X^2, Y^2, X - Y)}.
\]

By Corollary \[(2.4)\] we have $H^*(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \cong (\oplus_i W_i H^i(G(2, \mathcal{L})_\infty, \mathbb{Q})) \oplus (\oplus_i \ker(sp_i))$. The theorem follows immediately.

### 6. Hodge-Poincaré Formula

The Hodge-Poincaré formula for moduli spaces of semi-stable sheaves on smooth, projective curves is well-known and was classically computed by Earl and Kirwan \[(16)\]. It records the Hodge decomposition of the cohomology ring of the moduli space. In this section, we compute the Hodge-Poincaré formula for the Gieseker’s moduli space of rank 2 semi-stable sheaves with fixed determinant. One important difference with the classical case is that the cohomology of the Gieseker’s moduli space is not pure, which make computations more complicated.

We use notations as in Notation \[(3.1)\] and \[(3.1)\]. We briefly discuss the idea of the proof of Theorem \[(6.2)\]. Fix $s \in \Delta^*$. The first step is to use the isomorphism $\eta_s$ as in \[(5.1)\], to prove that there exists $\xi \in H^{2,1}(G(2, \mathcal{L})_s, \mathbb{C})$ such that

\[
H^3(G(2, \mathcal{L})_s, \mathbb{C}) \cong \eta_s(G^n W^3 H^3(G(2, \mathcal{L})_\infty, \mathbb{C})) \oplus \mathbb{C} \xi \circ \Phi^C_s \text{ and } (\xi^2, \xi^3, \xi^4) \in \eta_s(G^n W^6 H^6(G(2, \mathcal{L})_\infty, \mathbb{C})).
\]

Next, we use Newstead’s classical result \[(28)\] Theorem 1, which states that the cohomology ring of $G(2, \mathcal{L})_s$ is generated is degrees 2, 3 and 4. Since cup-product is a morphism of mixed Hodge structures, we get Proposition \[(6.1)\]. We then use the isomorphism $\Phi_1$ to prove that $G^n W H^{i+1}(G(2, \mathcal{L})_\infty, \mathbb{Q})$ can be identified with $H^{i+2}(M_{X_0}(2, \bar{L}_0), \mathbb{Q})$ as pure Hodge structures. Theorem \[(6.2)\] then follows from the exact sequence \[(2.4)\] combined with Theorem \[(4.2)\].
For any \( s \in \Delta^\ast \), we have the following equality:
\[
h^{p,q}Gr_{p+q}^W H^{p+q}(G(2, L)_s, \mathbb{C}) = h^{p,q}(G(2, L)_s) - h^{p-2,q-1}(M_{\bar{X}_0}(2, \tilde{\mathcal{L}}_0)) - h^{p-1,q-2}(M_{\bar{X}_0}(2, \tilde{\mathcal{L}}_0)).
\]

**Proof.** For any \( s \in \Delta^\ast \), we have the natural isomorphism
\[
\eta_i : H^i(G(2, L), \mathbb{Q}) \sim H^i(G(2, L)_s, \mathbb{Q})
\]
for all \( i \geq 0 \), as in (5.1). Given a subspace \( W \) of \( H^i(G(2, L)_s, \mathbb{Z}) \) (resp. \( H^i(G(2, L), \mathbb{Q}) \)), denote by \( W^\prime(G(2, L)_s) \) (resp. \( W^\prime(G(2, L)) \)) the subspace of \( W \) consisting of elements that are invariant under the action of the monodromy operator \( T_{G(2, L)_s} \) (resp. \( T_{G(2, L)} \)), where the monodromy operators are described in (2.2) and the discussion before that. We claim that there exists \( \xi \in F^2 H^3(G(2, L)_s, \mathbb{C}) \) such that \( T_{G(2, L)_s}(\xi) \neq \xi \). Indeed, \( H^3(G(2, L)_s, \mathbb{C}) = F^2 H^3(G(2, L)_s, \mathbb{C}) \oplus F^3 H^3(G(2, L)_s, \mathbb{C}) \). Since \( T_{G(2, L)_s} \) commutes with conjugation, we observe that if the entire \( F^2 H^3(G(2, L)_s, \mathbb{C}) \) is invariant under \( T_{G(2, L)_s} \), then so is \( H^3(G(2, L)_s, \mathbb{C}) \). Since \( T_{G(2, L)} \) is a canonical extension of \( T_{G(2, L)_s} \), this implies \( H^3(G(2, L), \mathbb{C}) \) is \( T_{G(2, L)} \)-invariant. But, Propositions (2.3) and (1.1) imply that \( Gr_1^W H^3(G(2, L), \mathbb{C}) \cong \mathbb{C} \), which is not \( T_{G(2, L)} \)-invariant by the invariant cycle theorem (Remark 2.2). This proves the claim.

Since \( Gr_1^W H^1(G(2, L)_s, \mathbb{C}) \) is \( T_{G(2, L)} \)-invariant, [25, p. 66, Lemma 2.14.12 and p. 69, Theorem 6.6] implies that \( \eta_i(H^{p,i}Gr_1^W H^1(G(2, L)_s, \mathbb{C})) \subset H^{p,i-p}(G(2, L)_s) \). Since
\[
\mathbb{C} \cong Gr_1^W H^3(G(2, L)_s, \mathbb{C}) \cong F^2 Gr_1^W H^3(G(2, L), \mathbb{C}) \text{ and}
\]
\[
H^{p-2, q-1}(M_{\bar{X}_0}(2, \tilde{\mathcal{L}}_0)),
\]
\[
H^{p-1, q-2}(M_{\bar{X}_0}(2, \tilde{\mathcal{L}}_0)),
\]
\[
H^{p,q}Gr_{p+q}^W H^{p+q}(G(2, L)_s, \mathbb{C}) \subset H^{p,q}(G(2, L)_s) - h^{p-2,q-1}(M_{\bar{X}_0}(2, \tilde{\mathcal{L}}_0)) - h^{p-1,q-2}(M_{\bar{X}_0}(2, \tilde{\mathcal{L}}_0)).
\]

**Theorem 6.2.** The Hodge-Poincaré formula for the cohomology ring \( H^*(G_{X_0}(2, \mathcal{L}_0), \mathbb{C}) \) is
\[
(1 + xy)^{g-1}(1 + x^2 y^2) + (1 + x^2 y^2) - x^g y^g(1 + x + y)^{g-1}(2 + (1 + xy)^2).
\]

**Proof.** Using Theorem 1.12 and the identification 1.13, the exact sequence 2.14 become the following short exact sequence:
\[
0 \to \bigoplus_{j=1}^2 H^{p,q-j}(M_{\bar{X}_0}(2, \tilde{\mathcal{L}}_0), \mathbb{C})(-j) \xrightarrow{\text{Li}} H^{p,q}Gr_{p+q}^W H^{p+q}(G_{X_0}(2, \mathcal{L}_0), \mathbb{C})
\] (6.1)
and extend linearly, where  is an isomorphism of pure Hodge structures which sends Hodge type 

\[ \Phi_{p+q+1} : Gr_{p+q}^W H^{p+q+1}(G(X_0, 2, L_0), \mathbb{C}) \to H^{p+q-2}(M_{\tilde{X}_0}(2, \tilde{L}_0), \mathbb{Q}) \]

and  is the isomorphism defined in [5] Since  is of Hodge type (1, 1) and  is an isomorphism of Hodge structures, [23] Remark 5.3 implies that  is an isomorphism of pure Hodge structures which sends Hodge type \((i, p+q-i)\) to \((i-1, p+q-1-i)\).

Note that, the \((p, q)\)-th part of the cohomology ring \(H^*(G(X_0, 2, L_0), \mathbb{C})\), denoted

\[ h^{p,q}(G(X_0, 2, L_0), \mathbb{C}) := h^{p,q}Gr_{p+q}^W (G(X_0, 2, L_0), \mathbb{C}) + h^{p,q}Gr_{p+q}^W (G(X_0, 2, L_0), \mathbb{C}), \]

Using the isomorphism  and Proposition 6.1 we have

\[ h^{p,q}(G(X_0, 2, L_0)) = \left( h^{p,q}(G(2, L)_s) - h^{p-2,q-1}(M_{\tilde{X}_0}(2, \tilde{L}_0)) - h^{p-1,q-2}(M_{\tilde{X}_0}(2, \tilde{L}_0)) + \right. \]

\[ \left. + \sum_{j=1}^{3} h^{p-j,q-j}(M_{\tilde{X}_0}(2, \tilde{L}_0)) \right) + h^{p-1,q-1}(M_{\tilde{X}_0}(2, \tilde{L}_0)). \]

By [10] Corollary 2.9], the Hodge-Poincaré formula for  and  are given by

\[ P_g(x, y) := \frac{(1 + x^2 y)^g(1 + x y)^g - x^g y^g (1 + x)^g (1 + y)^g}{(1 - x y)(1 - x^2 y^2)} \]

and \(Q(x, y) := P_{g-1}(x, y)\), (6.2)

respectively. This implies, the Hodge-Poincaré formula for  is given by

\[ P_g(x, y) + Q(x, y)(-x^2 y - x y^2 + 2 x y + x^2 y^2 + x^3 y^3) = \]

\[ \frac{(1 + x^2 y)^g-1(1 + x^2 y)^g-1(1 + 2 x y)(1 + x y)^2 - x^g y^g (1 + x)^g (1 + y)^g-1(2 + (1 + x y)^2)}{(1 - x y)(1 - x^2 y^2)}, \]

This proves the theorem.

7. Simpson’s moduli space with fixed determinant

In this section we prove the analogue of the Mumford conjecture for the Simpson’s moduli space of rank 2 semi-stable sheaves with fixed odd degree determinant on an irreducible nodal curve and compute the associated Hodge-Poincaré formula (Theorem 7.2). We use Notation 3.1 and notations as in [4.1]
7.1. **Simpson’s moduli spaces with fixed determinant.** Let $E$ be a rank 2, torsion-free sheaf on $X_0$ of degree $d$ and $L_0$ an invertible sheaf on $X_0$ of degree $d$. We say that $E$ has determinant $L_0$ if there is a $\mathcal{O}_{X_0}$-morphism $\wedge^2(E) \to L_0$ which is an isomorphism outside the node $x_0$. Note that if $E$ is locally free then this condition implies $\wedge^2 E \cong L_0$.

Let $U_{X_0}(2, d)$ be the moduli space of stable rank 2, degree $d$ torsion free sheaves on $X_0$ (see [32]). Denote by $U_{X_0}(2, d)^0$ the sublocus parametrizing locally free sheaves. Note that, $U_{X_0}(2, d)^0$ is an open subvariety of $U_{X_0}(2, d)$. We have a well defined morphism $\det: U_{X_0}(2, d)^0 \to \text{Pic}(X_0)$ defined by $E \mapsto \wedge^2 E$. Denote by $U_{X_0}(2, L_0)^0 := \det^{-1}([L_0])$.

Denote by $U_{X_0}(2, L_0) := \{[E] \in U_{X_0}(2, d) \mid E \text{ has determinant } L_0\}$. See [35] for a modular interpretation of $U_{X_0}(2, L_0)$. By [36, Theorem 1.10], the Zariski closure $\bar{U}_{X_0}(2, L_0)^0$ of $U_{X_0}(2, L_0)^0$ in $U_{X_0}(2, d)$ is $U_{X_0}(2, L_0)$.

7.2. **Stratification on the moduli space.** Let $m_{x_0}$ denote the maximal ideal of $\mathcal{O}_{X, x_0}$. We have a stratification of $U_{X_0}(2, d)$ by locally closed subschemes $U_0 \subset U_1 \subset U_2 := U_{X_0}(2, d)$, where

$$U_i := \{[E] \in U_{X_0}(2, d) \mid E \simeq \mathcal{O}_{x_0}^{\oplus j} \oplus m_{x_0}^{\geq j} \text{ for } j \leq i\}.$$ 

This induces the stratification $U_0(L_0) \subset U_1(L_0) \subset U_{X_0}(2, L_0)$, where $U_i(L_0) := U_i \cap U_{X_0}(2, L_0)$.

Denote by $\pi : \bar{X}_0 \to X_0$ the normalization morphism. By [31, Proposition 4.9], there exists a natural isomorphism from $M_{X_0}^\sim(2, d - 2)$ to $U_0$, sending $[E]$ to $[\pi_* E]$. Denote by $\bar{L}_0 := \pi^* L_0$ and $D := \pi^{-1}(x_0)$. Then, $M_{\bar{X}_0}(2, \bar{L}_0(-D))$ maps isomorphically to $U_0(L_0)$ (see proof of [37, (6.1)])).

7.3. **Comparison between Gieseker’s and Simpson’s moduli spaces.** Recall, there exists a natural proper morphism

$$\theta : \mathcal{G}_{X_0}(2, L_0) \to U_{X_0}(2, L_0) \quad (7.1)$$

defined by pushing forward a rank 2, locally-free sheaf defined over a curve semi-stably equivalent to $X_0$, via the natural contraction map to $X_0$ (see [36, Theorem 3.7(3)]). Denote by $\mathcal{G}_{X_0}(2, L_0)^0 \subset \mathcal{G}_0$ the sub-locus of $\mathcal{G}_{X_0}(2, L_0)$ parametrizing locally-free sheaves on $X_0$. Using [33, Remark 5.2], we conclude that $\theta$ maps the irreducible component $\mathcal{G}_1$ into $U_0(L_0)$ and $\mathcal{G}_{X_0}(2, L_0)^0$ maps isomorphically to $U_{X_0}(2, L_0)^0$ (use the description of the irreducible components of $\mathcal{G}_{X_0}(2, L_0)$ as given in [37, §6] and [36, Theorem 3.7]). By the properness of the $\theta$, we note that $\theta$ maps $\mathcal{G}_1$ surjectively to $U_0(L_0)$. Moreover, since $U_0(L_0) \cong M_{\bar{X}_0}(2, \bar{L}_0(-D))$, it is non-singular (see [22, Corollary 4.5.5]).

7.4. **Generalized Mumford’s conjecture and Hodge-Poincaré formula for Simpson’s moduli space.** We briefly discuss the idea of the proof of Theorem 7.2. Using the restriction of the proper morphism $\theta$ to $\mathcal{G}_1$ and the identifications (1.3) and (1.4), we compute the kernel and the cokernel of the pull-back morphism $\theta^*$ (see Proposition 7.1). Combining (2.4) and Proposition 4.1, we obtained an explicit description of the kernel of the specialization morphism $\text{sp}_i$. We use this to show that $H^i(U_{X_0}(2, L_0), \mathbb{Q})$ can be identified with the image of $\text{sp}_i$ as mixed Hodge structures. Theorem 7.2 then follows from Theorems 5.1 and 6.2.

**Proposition 7.1.** The natural morphism $\theta^* : H^i(U_{X_0}(2, L_0), \mathbb{Q}) \to H^i(\mathcal{G}_{X_0}(2, L_0), \mathbb{Q})$ is injective with $\text{Gr}_{i-1}^W H^i(U_{X_0}(2, L_0), \mathbb{Q}) \xrightarrow{\theta^*} \text{Gr}_{i-1}^W H^i(\mathcal{G}_{X_0}(2, L_0), \mathbb{Q})$ and cokernel isomorphic to

$$\bigoplus_{j=1}^3 H^{i-2j}(M_{\bar{X}_0}(2, \bar{L}_0), \mathbb{Q})(\xi^*)^j.$$ 

such that the resulting (cokernel) morphism factors as

$$\text{Gr}_i^W H^i(\mathcal{G}_{X_0}(2, L_0), \mathbb{Q}) \xrightarrow{\theta^*} H^i(\mathcal{G}_1, \mathbb{Q}) \xrightarrow{(1.1)} \bigoplus_{j=1}^3 H^{i-2j}(M_{\bar{X}_0}(2, \bar{L}_0), \mathbb{Q})(\xi^*)^j, \quad (7.2)$$
where \( r : G_1 \to G_{X_0}(2, \mathcal{L}_0) \) is the natural inclusion.

**Proof.** Recall, \( U_{X_0}(2, \mathcal{L}_0) \setminus U_0(\mathcal{L}_0) \cong G_{X_0}(2, \mathcal{L}_0) \setminus G_1 \cong G_0 \setminus G_1 \). Since \( G_0 \) and \( G_1 \) are excessive couple (30 Example B.5(2)), we have for all \( i \geq 0 \),

\[
H^i(G_{X_0}(2, \mathcal{L}_0), G_1) \cong H^i(G_0, G_1 \cap G_0) \cong H^i(G_0 \cap G_1) \cong H^i(U_{X_0}(2, \mathcal{L}_0), U_0(\mathcal{L}_0)),
\]

where the last two isomorphisms follow from [30 Corollary B.14]. The morphism \( \theta \) induces the following commutative diagram where every morphism is a morphism of mixed Hodge structures (use [30 Proposition 5.46]):

\[
\begin{array}{cccc}
H^i(U_{X_0}(2, \mathcal{L}_0), U_0(\mathcal{L}_0)) & \to & H^i(U_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) & \to & H^i(U_0(\mathcal{L}_0), \mathbb{Q}) & \to & H^{i+1}(U_{X_0}(2, \mathcal{L}_0), U_0(\mathcal{L}_0)) \\
\cong & \circ & \theta^* & \circ & (\theta^*)^* & \circ & \cong \\
H^i(G_{X_0}(2, \mathcal{L}_0), G_1) & \to & H^i(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) & \to & H^i(G_1, \mathbb{Q}) & \to & H^{i+1}(G_{X_0}(2, \mathcal{L}_0), G_1)
\end{array}
\]

Using the identification (4.4) and \( U_0(\mathcal{L}_0) \cong M_{\tilde{X}}(2, \tilde{\mathcal{L}}_0) \), we have the following short exact sequence of pure Hodge structures:

\[
0 \to H^i(U_0(\mathcal{L}_0), \mathbb{Q}) \xrightarrow{(\theta^*)^*} H^i(G_1, \mathbb{Q}) \to \bigoplus_{j=1}^{3} H^{i-2j}(M_{\tilde{X}}(2, \tilde{\mathcal{L}}_0), \mathbb{Q})(\xi'')^j \to 0.
\]

Applying Snake lemma to the commutative diagram (7.3), we get the following short exact sequence of mixed Hodge structures:

\[
0 \to H^i(U_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \xrightarrow{\theta^*} H^i(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \to \bigoplus_{j=1}^{3} H^{i-2j}(M_{\tilde{X}}(2, \tilde{\mathcal{L}}_0), \mathbb{Q})(\xi'')^j \to 0. \tag{7.4}
\]

such that the morphism from \( H^i(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \) to \( \bigoplus_{j=1}^{3} H^{i-2j}(M_{\tilde{X}}(2, \tilde{\mathcal{L}}_0), \mathbb{Q})(\xi'')^j \) factors as in (7.2). Note that, the last term of this short exact sequence is pure of weight \( i \). The proposition then follows immediately from this short exact sequence (7.4). \( \square \)

**Theorem 7.2.** Let \( P_i^{\infty} \) and \( I_k^{\infty} \) be as in [30]. Then,

\[
H^*(U_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \cong \bigoplus_i P_i^{\infty} \otimes \frac{\mathbb{Q}[\alpha_\infty, \beta_\infty, \psi_\infty]}{I_i^{g-i}}.
\]

Moreover, the Hodge-Poincaré polynomial associated to the moduli space \( U_{X_0}(2, \mathcal{L}_0) \) is

\[
(1 + xy^2)^{g-1}(1 + x^2y^2)^{g-1}(1 + xy + x^3y^3) - x^g y^{g-1}(1 + x)^{g-1}(x + y)^{g-1}(2 + xy).
\]

\[
(1 - xy)(1 - x^2y^2)
\]

**Proof.** Notations as in Corollary 2.4 Using the identifications (4.3) and (4.4) one can easily observe that the image of the composition

\[
H^{i-2}(G_0 \cap G_1, \mathbb{Q}) \xrightarrow{\imath} G_1 \xrightarrow{\psi^*} H^i(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \xrightarrow{\rho^*} H^i(G_1, \mathbb{Q})
\]

is isomorphic to \( \bigoplus_{j=1}^{3} H^{i-2j}(M_{\tilde{X}}(2, \tilde{\mathcal{L}}_0), \mathbb{Q})(\xi'')^j \) with kernel \( H^{i-4}(M_{\tilde{X}}(2, \tilde{\mathcal{L}}_0), \mathbb{Q})(-2) \) (see proof of Theorem 1.2). Proposition 7.1 then implies that the natural morphism

\[
\text{ker}(\text{sp}_1 : H^i(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \to H^i(G(2, \mathcal{L}_\infty), \mathbb{Q})) \to \text{coker}(\theta^* : H^i(U_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \to H^i(G_{X_0}(2, \mathcal{L}_0), \mathbb{Q}))
\]

is an isomorphism of pure Hodge structures. Therefore, by Corollary 2.4 we have

\[
H^i(U_{X_0}(2, \mathcal{L}_0), \mathbb{Q}) \cong W_i H^i(G(2, \mathcal{L}_\infty), \mathbb{Q}) \text{ as mixed Hodge structures} \tag{7.5}
\]
Remark 7.3. This proves the theorem. □

Let $P_{p+q+1}(\mathcal{G}(2, \mathcal{L}), \mathbb{C}) \to H^{p+q+2}(\mathcal{M}_{X_0}(2, \mathcal{L}), \mathbb{C})$ sending a class of type $(p, q)$ to that of type $(p-1, q-1)$. Then, Proposition 6.1 along with (7.5) implies that

$$h^{p,q}(U_{X_0}(2, \mathcal{L}), \mathbb{C}) = h^{p,q}(\mathcal{G}(2, \mathcal{L})_s, \mathbb{C}) - h^{p-2,q-1}(\mathcal{M}_{X_0}(2, \mathcal{L}), \mathbb{C}) - h^{p-1,q-2}(\mathcal{M}_{X_0}(2, \mathcal{L}), \mathbb{C}) +$$

$$+ h^{p-1,q-1}(\mathcal{M}_{X_0}(2, \mathcal{L}), \mathbb{C}).$$

Let $P_0(x, y)$ and $Q(x, y)$ be the Hodge-Poincaré polynomial $\mathcal{G}(2, \mathcal{L})_s$ and $\mathcal{M}_{X_0}(2, \mathcal{L})$ respectively, defined in (6.2). Then, the Hodge-Poincaré polynomial of $U_{X_0}(2, \mathcal{L})$ is given by

$$P_0(x, y) + Q(x, y)(-x^2y - xy^2 + xy) =$$

$$= \frac{(1 + xy^2)g_1(1 + x^2y)(1 + xy + x^3y^3) - xy^2g_1(1 + x)g_1(1 + y)}{(1 - xy)(1 - x^2y^2)}.$$ 

This proves the theorem. □

**Remark 7.3.** Under the identification (7.5) above, the cohomology ring $H^*(U_{X_0}(2, \mathcal{L}), \mathbb{Q})$ is generated by $\alpha^i, \beta^i, \psi^i$ for $1 \leq i \leq 2g - 1$ and $\psi^i \psi^j$.

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