Continuity of some non-local functionals with respect to a convergence of the underlying measures

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Abstract
We study some non-local functionals on the Sobolev space $W^{1,p}_0(\Omega)$ involving a double integral on $\Omega \times \Omega$ with respect to a measure $\mu$. We introduce a suitable notion of convergence of measures on product spaces which implies a stability property in the sense of $\Gamma$-convergence of the corresponding functionals.

Keywords: non-local functionals, $\Gamma$-convergence, Mosco convergence, graphons, cut norm
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1 Introduction
In this paper we give a contribution to the study of $\Gamma$-limits of non-local integral functional, for which only few results are available in the literature (see for instance [13]). We consider sequences of integrals of the type
\[
\int_{\Omega \times \Omega} f(u(x), u(y)) d\mu_k(x,y) + \int_{\Omega} g_k(x, \nabla u(x)) dx,
\]
where $\Omega$ is a bounded open subset of $\mathbb{R}^d$, with $d \geq 1$. These functionals have a non-local term
\[
F_k(u) := \int_{\Omega \times \Omega} f(u(x), u(y)) d\mu_k(x,y)
\]
depending on a fixed function $f: \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ and varying positive bounded measures $\mu_k$ on $\Omega \times \Omega$, while the local term
\[
G_k(u) := \int_{\Omega} g_k(x, \nabla u(x)) dx
\]
depends on a function $g_k : \Omega \times \mathbb{R}^d \to [0, +\infty)$. These functionals are defined for $u$ in the Sobolev space $W^{1,p}_0(\Omega)$.

We assume that the functions $g_k$ satisfy usual growth conditions and that the integral functionals $G_k$ Γ-converge in the weak topology in $W^{1,p}_0(\Omega)$ to a functional $G$ of the same form, with integrand $g$.

We address the question of the stability for functionals in (1.1); more precisely, we focus on a notion of convergence on $\mu_k$ such that the functionals $F_k + G_k$ Γ-converge with respect to the weak topology in $W^{1,p}_0(\Omega)$ to a functional of the form

$$\int_{\Omega \times \Omega} f(u(x), u(y)) d\mu(x, y) + \int_{\Omega} g(x, \nabla u(x)) dx$$

for a limit measure $\mu$. Under some additional assumptions, we also obtain the convergence of $F_k + G_k$ in the sense of Mosco convergence in $W^{1,p}_0(\Omega)$.

If $p = 2$, $g_k(x, \cdot)$ are quadratic forms, and $f(s, t) = |t - s|^2$, then the study of such functionals can be framed within the theory of Dirichlet Forms [8], where the Beurling-Deny formula ensures, under suitable assumptions, that the Γ-limit of $F_k + G_k$ can be represented analogously (see [13]). The extension of that theory is not immediate in a non-quadratic setting or when $f$ is an arbitrary continuous function.

Note that, under suitable growth conditions on $f$, stability is easily proved under the strong assumption of convergence of $\mu_k$ to $\mu$ in the space $W^{-1,q}(\Omega \times \Omega)$ dual to $W^{1,p}_0(\Omega \times \Omega)$. However, this result is not satisfactory, since such a space may fail to contain relevant measures $\mu$, depending on the value of $p$, such as Dirac deltas if $p < 2d$.

We introduce a wider space of measures on $\Omega \times \Omega$, together with a new notion of norm, inspired by a convergence that is used in the theory of graphons [5, 11]. The latter can be be seen as limits of Dirichlet forms on dense graphs (for an interpretation in terms of Γ-convergence we refer to [2]).

We prove that, if $\mu_k$ are non-negative measures that converge to $\mu$ with respect to that ‘graphon’ norm and $\mu_k(\Omega \times \Omega) \to \mu(\Omega \times \Omega)$, then the functionals defined by (1.1) Γ-converge under the only assumption that $f$ be continuous and a very mild technical assumption (see (5.2)).

We now describe more in detail the content of the paper. In Section 2 we recall some preliminaries on Sobolev functions and introduce the quasicontinuous representative $\tilde{u}$ of a function $u \in W^{1,p}_0(\Omega)$, which is needed in the precise definition of the functionals (1.1) and (1.2), when $\mu_k$ or $\mu$ are not absolutely continuous with respect to the Lebesgue measure.

In Section 3 we introduce the space $\mathcal{M}^{1,p}(\Omega \times \Omega)$ of Radon measures on $\Omega \times \Omega$ with finite ‘Sobolev cut norm’, defined as

$$\| \mu \| := \sup \left\{ \int_{\Omega \times \Omega} (\varphi(x)\psi(y) d\mu(x, y)) : \varphi, \psi \in C^\infty_c(\Omega), \| \varphi \|_{1,p}, \| \psi \|_{1,p} \leq 1 \right\},$$

where $\| \cdot \|_{1,p}$ is the norm in $W^{1,p}_0(\Omega)$. We prove that, if $\mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$ and $u, v \in
$W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with compact support in $\Omega$, we have

$$\left| \int_{\Omega \times \Omega} \tilde{u}(x)\tilde{v}(y) d\mu(x,y) \right| \leq \|\mu\|_\square \|u\|_{1,p} \|v\|_{1,p},$$

and the integral does not depend on the choice of the quasicontinuous representatives (see Theorem 3.4).

In Section 4 we prove some continuity results for double integrals. We consider $\mu_k, \mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$ with $\mu_k(\Omega \times \Omega) < +\infty$, $\mu(\Omega \times \Omega) < +\infty$, and $\|\mu_k - \mu\|_\square \to 0$ and $\mu_k(\Omega \times \Omega) \to \mu(\Omega \times \Omega)$. If $u_k, v_k$ are sequences in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ converging to $u, v$ weakly in $W_0^{1,p}(\Omega)$ and with $\sup_k (\|u_k\|_\infty + \|v_k\|_\infty) < +\infty$, then we have

$$\lim_{k \to +\infty} \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x,y) = \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y)) d\mu(x,y)$$

for all continuous functions $f: \mathbb{R}^2 \to \mathbb{R}$ (see Corollary 4.5). This result is obtained by considering first the special case $f(s,t) = st$ (see Corollary 4.2), from which the result can be obtained for a general polynomial (see Proposition 4.4), and eventually for an arbitrary continuous function by approximation.

Finally, in Section 5 we consider non-negative continuous functions $f$, and prove the above-mentioned $\Gamma$-convergence results (see Theorem 5.1). Note that no convexity assumption on $f$ is needed.

## 2 Preliminaries on fine properties of Sobolev functions

Throughout the paper we fix $1 < p < +\infty$, $d \geq 1$, and a bounded open subset $\Omega$ of $\mathbb{R}^d$. We consider the Sobolev space $W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\|_{1,p} = \left( \int_\Omega |\nabla u|^p dx \right)^{1/p}.$$

Its dual is denoted by $W^{-1,q}(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, and this norm is denoted by $\|\cdot\|_{-1,q}$.

For all subset $A \subset \Omega$ the capacity of in $\Omega$ is defined as

$$C_{1,p}(A) := \inf \left\{ \int_\Omega |\nabla u|^p dx : u \in W_0^{1,p}(\Omega), \ u \geq 1 \ \text{a.e. in a neighbourhood of} \ A \right\}.$$ 

A function $f: \Omega \to \mathbb{R}$ is quasicontinuous if for every $\varepsilon > 0$ there exists an open set $U \subset \Omega$ with $C_{1,p}(U) < \varepsilon$ such that the restriction of to $\Omega \setminus U$ is continuous. It is known (see e.g. [10] [7]) that each function $u \in W_0^{1,p}(\Omega)$ has a Borel quasicontinuous representative.
which we denote by \( \tilde{u} \), in the sense that \( \tilde{u} \) is Borel measurable and quasicontinuous and \( \tilde{u} = u \) almost everywhere in \( \Omega \). Such a representative is unique up to sets of zero capacity.

Let \( \mathcal{M}(\Omega) \) denote the space of signed Radon measures on \( \Omega \), which can be identified with the dual of \( C_c^0(\Omega) \). We say that \( \mu \in \mathcal{M}(\Omega) \) belongs to \( W^{-1,q}(\Omega) \) if there exists \( C \geq 0 \) such that
\[
\left| \int_\Omega \varphi \, d\mu \right| \leq C \| \varphi \|_{1,p} \quad \text{for all } \varphi \in C_c^\infty(\Omega).
\]
In this case there exists a unique \( T_\mu \in W^{-1,q}(\Omega) \) such that
\[
\langle T_\mu, \varphi \rangle = \int_\Omega \varphi \, d\mu \quad \text{for all } \varphi \in C_c^\infty(\Omega),
\]
where \( \langle \cdot, \cdot \rangle \) denotes the duality product between \( W^{-1,q}(\Omega) \) and \( W_{1,p}^0(\Omega) \). In other words \( \mu \) and \( T_\mu \) coincide as distributions on \( \Omega \). For notational convenience, we shall sometimes directly write \( \mu \) in the place of \( T_\mu \) when the distinction between the two is not relevant.

In the following theorem we recall a property of sets of zero \( C_{1,p} \)-capacity (see [9]) and an integral representation of \( T_\mu \) that can be deduced from a result of Brezis and Browder [3].

In order to simplify the notation, we introduce the space
\[
W_{1,p}^c(\Omega) := \{ u \in W_{1,p}^0(\Omega) : u \text{ has compact support in } \Omega \}.
\]

**Theorem 2.1.** Let \( \mu \in \mathcal{M}(\Omega) \cap W^{-1,q}(\Omega) \). Then for every \( A \subset \Omega \) with \( C_{1,p}(A) = 0 \)-capacity \( A \) is \( |\mu| \)-measurable and
\[
|\mu|(A) = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega).
\]

If \( u \in W_{1,p}^c(\Omega) \cap L^\infty(\Omega) \), then
\[
\langle T_\mu, u \rangle = \int_\Omega u \, d\mu.
\]

If, in addition, \( \mu \geq 0 \), then, for all \( u \in W_{1,p}^0(\Omega) \), we have \( \tilde{u} \in L^1(\Omega; \mu) \) and (2.3) holds.

## 3 Sobolev graphons

**Graphons** are functions \( \rho \) defined on \((0, 1) \times (0, 1)\) introduced to study functionals of the form
\[
\int_{(0,1) \times (0,1)} f(u(x), u(y)) \rho(x, y) \, dx \, dy,
\]
defined for \( u \in L^\infty(0, 1) \), especially when \( f(u, v) = |u - v|^2 \). The functionals above are introduced as a generalization, and in some sense a limit, of energies on dense graphs [5, 11]. To that end, the space of graphons is equipped with the so-called cut norm
\[
\| \rho \|_{\square} := \sup_{\varphi, \psi : (0,1) \to [0,1]} \left| \int_{(0,1) \times (0,1)} \varphi(x) \psi(y) \rho(x, y) \, dx \, dy \right|.
\]

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We extend the definition of cut norm to arbitrary measures defined on $\Omega \times \Omega$, with $\Omega$ in the place of $(0,1)$. For our purpose it is convenient to take test functions in the Sobolev space $W_0^{1,p}(\Omega)$.

**Definition 3.1.** The space $M^{1,p}(\Omega \times \Omega)$ is defined as

$$M^{1,p}(\Omega \times \Omega) := \{ \mu \in M(\Omega \times \Omega) : \| \mu \|_\square < +\infty \},$$

where the Sobolev cut norm of $\mu$ is defined as

$$\| \mu \|_\square := \sup \left\{ \left| \int_{\Omega \times \Omega} \varphi(x)\psi(y) d\mu(x,y) \right| : \varphi, \psi \in C_c^\infty(\Omega), \| \varphi \|_{1,p}, \| \psi \|_{1,p} \leq 1 \right\}. \quad (3.3)$$

We let $M_+^{1,p}(\Omega \times \Omega)$ denote the cone of the positive measures in $M^{1,p}(\Omega \times \Omega)$.

**Remark 3.2.** We make some observations on the convergence in the space $M^{1,p}(\Omega \times \Omega)$, and in particular we compare it with weak* convergence in $M(\Omega \times \Omega)$ and with the convergence $W^{-1,q}(\Omega \times \Omega)$.

(i) $\| \cdot \|_\square$ defines a norm on $M^{1,p}(\Omega \times \Omega)$.

(ii) if $\mu$ belongs to $W^{-1,q}(\Omega \times \Omega)$, then $\| \mu \|_\square \leq \| \mu \|_{-1,q}$. Indeed in such a case the function $\Phi(x,y) = \varphi(x)\psi(y)$ belongs to $W_0^{1,p}(\Omega \times \Omega)$ and

$$\int_{\Omega \times \Omega} |\nabla \Phi|^p dx \, dy = \int_{\Omega \times \Omega} |\psi(y)\nabla \varphi(x) + \varphi(x)\nabla \psi(y)|^p dx \, dy$$

$$\leq C \int_{\Omega \times \Omega} |\psi(y)|^p |\nabla \varphi(x)|^p + |\varphi(x)|^p |\nabla \psi(y)|^p dx \, dy \leq C$$

so that

$$\left| \int_{\Omega \times \Omega} \varphi(x)\psi(y) d\mu(x,y) \right| \leq C \| \mu \|_{-1,q};$$

(iii) if $\mu_k, \mu \in M^{1,p}(\Omega \times \Omega)$ and $\| \mu_k - \mu \|_\square \to 0$, then

$$\int_{\Omega \times \Omega} \varphi(x)\psi(y) d\mu_k(x,y) \to \int_{\Omega \times \Omega} \varphi(x)\psi(y) d\mu(x,y) \quad (3.4)$$

for all $\varphi, \psi \in C_c^\infty(\Omega)$. If in addition $\sup_k |\mu_k|_{\Omega \times \Omega} < +\infty$, then $\sup_k |\mu_k|_{\Omega \times \Omega} < +\infty$ and a density argument imply that $\mu_k$ converges to $\mu$ weakly* in $M(\Omega \times \Omega)$.

(iv) if $\mu_k^1$ and $\mu_k^2$ are such that $\mu_k^j$ converge to $\mu^j$ in $W^{-1,q}(\Omega)$, then the measures $\mu_k = \mu_k^1 \otimes \mu_k^2$ converge to $\mu = \mu^1 \otimes \mu^2$;

(v) if $\sup_k |\mu_k|_{\square} < +\infty$ and $\mu_k$ converges to some $\mu$ weakly* in $M(\Omega \times \Omega)$, then $\mu \in M^{1,p}(\Omega \times \Omega)$ and $\| \mu \|_\square \leq \lim \inf_k \| \mu_k \|_\square$.

Note that $M^{1,p}(\Omega \times \Omega)$ is strictly larger than $M(\Omega \times \Omega) \cap W^{-1,q}(\Omega \times \Omega)$, and in particular its convergence is weaker than convergence in $W^{-1,q}(\Omega \times \Omega)$. Some examples of this inclusion are given below.
Example 3.3. (i) The first example is simply a Dirac delta $\mu = \delta(x_0, y_0)$ for $x_0, y_0 \in \Omega$. Indeed if $d < p < 2d$, then $\mu$ does not belong to $W^{-1,q}(\Omega \times \Omega)$, while

$$\left| \int_{\Omega \times \Omega} \varphi(x)\psi(y) d\mu(x,y) \right| = |\varphi(x_0)\psi(y_0)| \leq C\|\varphi\|_{1,p}\|\psi\|_{1,p}$$

where the inequality follows from the embedding of $W_{0}^{1,p}(\Omega)$ into $L^\infty(\Omega)$.

(ii) We can also exhibit an example where $\mu$ is absolutely continuous with respect to $L^d$ with density of the form $m(x)m(y)$. We choose $d = 1, p = 2, \Omega = (-1,1)$, and

$$m(x) = \frac{1}{|x| \log |x| \log |\log x| \log |\log x|}.$$ 

Note that $m \in L^1(0,1)$ so that $\mu \in M^{1,2}(\Omega \times \Omega)$. If we take

$$w(x,y) = \begin{cases} \log |\log \sqrt{x^2 + y^2}| & \text{if } \sqrt{x^2 + y^2} < 1/e, \\ 0 & \text{otherwise}, \end{cases}$$

then $w \in W^{1,2}_0((-1,1)^2)$ but $\int_{(-1,1)^2} w d\mu = +\infty$. By Theorem 2.1 this shows that $\mu \notin W^{-1,2}((-1,1)^2)$.

Theorem 3.4. Let $\mu \in M^{1,p}(\Omega \times \Omega)$. Then for every $A \subset \subset \Omega$ with $C_{1,p}(A) = 0$ the sets $A \times \Omega$ and $\Omega \times A$ are $|\mu|$-measurable and

$$|\mu|(A \times \Omega) = |\mu|(\Omega \times A) = 0.$$ 

Moreover, for all $u, v \in W_c^{1,p}(\Omega) \cap L^\infty(\Omega)$ we have

$$\left| \int_{\Omega \times \Omega} \tilde{u}(x)\tilde{v}(y) d\mu(x,y) \right| \leq \|\mu\|_\square\|u\|_{1,p}\|v\|_{1,p}.$$ 

Proof. Fix $\psi \in C_c^\infty(\Omega)$, and let $\mu_\psi \in M(\Omega)$ be defined by

$$\mu_\psi(B) = \int_{B \times \Omega} \psi(y) d\mu(x,y) \text{ for all Borel sets } B \subset \subset \Omega.$$ 

Since

$$\int_{\Omega} \varphi(x) d\mu_\psi(x) = \int_{\Omega \times \Omega} \varphi(x)\psi(y) d\mu(x,y) \text{ for all } \varphi \in C_c^\infty(\Omega),$$

by the definition of $\|\mu\|_\square$ we have

$$\left| \int_{\Omega} \varphi(x) d\mu_\psi(x) \right| \leq \|\mu\|_\square\|\psi\|_{1,p}\|\varphi\|_{1,p} \text{ for all } \varphi \in C_c^\infty(\Omega).$$
Hence, $\mu_\psi \in \mathcal{M}(\Omega) \cap W^{-1,q}(\Omega)$.  

Let $B \subset \Omega$ be a Borel set with $C_{1,p}(B) = 0$. Let $\Omega'$ be an open set with $\Omega' \subset \subset \Omega$, and let $\psi_k$ be a non-decreasing sequence of non-negative functions in $C_c^\infty(\Omega')$ converging to the constant 1 in $\Omega$. By the definition of $\mu_\psi$ and Theorem 2.1, we have
\[
\int_{B \times \Omega} \psi_k(y) d\mu(x,y) = \mu_\psi(B) = 0.
\]

Letting $k \to +\infty$ we deduce $\mu(B \times \Omega') = 0$. Since this holds for all Borel subsets of $B$ we deduce that $|\mu|(B \times \Omega') = 0$. By the arbitrariness of $\Omega' \subset \subset \Omega$ we finally obtain $|\mu|(B \times \Omega) = 0$. For a general $A \subset \Omega$ with $C_{1,p}(A) = 0$ it is sufficient to observe that there exists a Borel set $B$ such that $A \subset B \subset \Omega$ and $C_{1,p}(B) = 0$. This implies that $A \times \Omega$ and is $|\mu|$-measurable and $|\mu|(A \times \Omega) = 0$. A similar argument proves the same result for $\Omega \times A$.

Let $T_{\mu_\psi} \in W^{-1,q}(\Omega)$ be defined as in [2.1] with $\mu$ replaced by $\mu_\psi$. Fix $u \in W^{1,p}_c(\Omega) \cap L^\infty(\Omega)$. Then, by Theorem 2.1 and the definition of $\mu_\psi$, we have
\[
\langle T_{\mu_\psi}, u \rangle = \int_{\Omega} \bar{u}(x) d\mu_\psi(x) = \int_{\Omega \times \Omega} \bar{u}(x) \psi(y) d\mu(x,y).
\]

By (3.8) we have $\|T_{\mu_\psi}\|_{1,q} \leq \|\bar{\psi}\|_{1,p}$, so that the previous equality gives
\[
\left| \int_{\Omega \times \Omega} \bar{u}(x) \psi(y) d\mu(x,y) \right| \leq \|\bar{\psi}\|_{1,p} \|u\|_{1,p}
\]
for all $u \in W^{1,p}_c(\Omega) \cap L^\infty(\Omega)$ and every $\psi \in C_c^\infty(\Omega)$.

Fix $u \in W^{1,p}_c(\Omega) \cap L^\infty(\Omega)$ and let $\mu^u \in \mathcal{M}(\Omega)$ be defined by
\[
\mu^u(B) := \int_{\Omega \times B} \bar{u}(x) d\mu(x,y) \text{ for all Borel sets } B \subset \subset \Omega. \tag{3.10}
\]

Since
\[
\int_{\Omega} \psi(y) d\mu^u(y) = \int_{\Omega \times \Omega} \bar{u}(x) \psi(y) d\mu(x,y) \text{ for all } \psi \in C_c^\infty(\Omega),
\]
thanks to (3.9) we then have
\[
\left| \int_{\Omega} \psi(y) d\mu^u(y) \right| \leq \|\bar{\psi}\|_{1,p} \|u\|_{1,p} \text{ for all } \psi \in C_c^\infty(\Omega).
\]

Hence, $\mu^u \in \mathcal{M}(\Omega) \cap W^{-1,q}(\Omega)$, and, using the notation above, we obtain
\[
\|T_{\mu^u}\|_{1,q} \leq \|\bar{\psi}\|_{1,p} \|u\|_{1,p}.
\]

By Theorem 2.1 and the definition of $\mu^u$, we then have
\[
\langle T_{\mu^u}, v \rangle = \int_{\Omega} \bar{v}(y) d\mu^u(y) = \int_{\Omega \times \Omega} \bar{u}(x) \bar{v}(y) d\mu(x,y).
\]
Together with the previous inequality this gives
\[
\left| \int_{\Omega \times \Omega} \tilde{u}(x)\tilde{v}(y)d\mu(x,y) \right| \leq \|\mu\|_1 \|u\|_{1,p} \|v\|_{1,p},
\]
which concludes the proof of (3.6).

\[\square\]

4 Continuity properties of some double integrals

In this section we find conditions on \(f, u_k, v_k, u, v, \mu_k\), and \(\mu\) which imply the convergence
\[
\lim_{k \to +\infty} \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y))d\mu_k(x,y) = \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y))d\mu(x,y).
\]
We begin with the case \(f(s,t) = st\). Subsequently, we consider the case when \(f\) is a polynomial, and finally an arbitrary continuous function by approximation.

**Lemma 4.1.** Let \(\mu \in \mathcal{M}^{1,p}_{\ast}(\Omega \times \Omega)\) and let \(u_k, v_k\) be sequences in \(W^{1,p}_{\ast}(\Omega) \cap L^\infty(\Omega)\) converging to \(u, v\) weakly in \(W^{1,p}(\Omega)\). Assume that there exist a compact set \(K \subset \Omega\) and a constant \(M\) such that
\[
\text{supp}(u_k) \cup \text{supp}(v_k) \subset K \quad \text{and} \quad \|u_k\|_\infty + \|v_k\|_\infty \leq M, \tag{4.1}
\]
where \(\|\cdot\|_\infty\) denotes the norm in \(L^\infty(\Omega)\). Then
\[
\lim_{k \to +\infty} \int_{\Omega \times \Omega} \tilde{u}_k(x)\tilde{v}_k(y)d\mu_k(x,y) = \int_{\Omega \times \Omega} \tilde{u}(x)\tilde{v}(y)d\mu(x,y). \tag{4.2}
\]

**Proof.** We write
\[
\int_{\Omega \times \Omega} \tilde{u}_k(x)\tilde{v}_k(y)d\mu_k(x,y) - \int_{\Omega \times \Omega} \tilde{u}(x)\tilde{v}(y)d\mu(x,y)
= \int_{\Omega \times \Omega} (\tilde{u}_k(x) - \tilde{u}(x))\tilde{v}_k(y)d\mu_k(x,y) + \int_{\Omega \times \Omega} \tilde{u}(x)(\tilde{v}_k(y) - \tilde{v}(y))d\mu(x,y). \tag{4.3}
\]
The first integral in (4.3) is estimated by
\[
\left| \int_{\Omega \times \Omega} (\tilde{u}_k(x) - \tilde{u}(x))\tilde{v}_k(y)d\mu_k(x,y) \right| \leq M \int_{\Omega \times \Omega} |\tilde{u}_k(x) - \tilde{u}(x)|\psi(y)d\mu(x,y)
= M \langle T_{\mu_\psi}, |u_k - u| \rangle = o(1) \tag{4.4}
\]
as \(k \to +\infty\), where \(\psi\) is any function in \(C^\infty_c(\Omega)\) with \(0 \leq \psi \leq 1\) in \(\Omega\) and \(\psi = 1\) on \(K\), and \(\mu_\psi\) is defined in (3.7), and the convergence to 0 follows from the fact that \(|u_k - u| \rightharpoonup 0 \) weakly in \(W^{1,p}_0(\Omega)\). Moreover, if \(\mu^u\) is defined in (3.10), we have
\[
\int_{\Omega \times \Omega} \tilde{u}(x)(\tilde{v}_k(y) - \tilde{v}(y))d\mu(x,y) = \langle T_{\mu^u}, v_k - v \rangle = o(1) \tag{4.5}
\]
as \(k \to +\infty\). The convergence to 0 follows from the fact that \(v_k - v \rightharpoonup 0 \) weakly in \(W^{1,p}_0(\Omega)\). \[\square\]
Corollary 4.2. Under the same hypotheses of Lemma 4.1 let \( \mu_k \in \mathcal{M}^{1,p}(\Omega \times \Omega) \) with \( \| \mu_k - \mu \|_\square \to 0 \). Then
\[
\lim_{k \to +\infty} \int_{\Omega \times \Omega} \tilde{u}_k(x)\tilde{v}_k(y)d\mu_k(x,y) = \int_{\Omega \times \Omega} \tilde{u}(x)\tilde{v}(y)d\mu(x,y). \tag{4.6}
\]

Proof. It suffices to remark that
\[
\left| \int_{\Omega \times \Omega} \tilde{u}_k(x)\tilde{v}_k(y)d\mu_k(x,y) - \int_{\Omega \times \Omega} \tilde{u}_k(x)\tilde{v}_k(y)d\mu(x,y) \right| \leq \| \mu_k - \mu \|_\square \| u_k \|_{1,p} \| v_k \|_{1,p}
\]
and that the right-hand side tends to 0. The conclusion then follows from Lemma 4.1. \( \square \)

Proposition 4.3. Let \( P: \mathbb{R}^2 \to \mathbb{R} \) be a polynomial function and let \( \varphi, \psi \in C_\infty(\Omega) \). Let \( \mu_k, \mu \in \mathcal{M}^{1,p}_+(\Omega \times \Omega) \) with \( \| \mu_k - \mu \|_\square \to 0 \), let \( u_k, v_k \) be sequences in \( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) converging to \( u, v \) weakly in \( W^{1,p}_0(\Omega) \) and equibounded in \( L^\infty(\Omega) \). Then
\[
\lim_{k \to +\infty} \int_{\Omega \times \Omega} \varphi(x)\psi(y)P(\tilde{u}_k(x),\tilde{v}_k(y))d\mu_k(x,y) = \int_{\Omega \times \Omega} \varphi(x)\psi(y)P(\tilde{u}(x),\tilde{v}(y))d\mu(x,y). \tag{4.7}
\]

Proof. It is sufficient to consider \( P(s,t) = s^m t^n \) for some non-negative integers \( m, n \). We define \( w_k(x) = \varphi(x)u_k(x)^m \) and \( z_k(y) = \psi(y)v_k(y)^n \). Observe that \( w_k, z_k \) satisfy the hypotheses of Lemma 4.1 weakly converging in \( W^{1,p}_0(\Omega) \) to \( w, z \) given by \( w(x) = \varphi(x)u(x)^m \) and \( z(y) = \psi(y)v(y)^n \). Since
\[
\varphi(x)\psi(y)P(\tilde{u}_k(x),\tilde{v}_k(y)) = \tilde{w}_k(x)\tilde{z}_k(y), \quad \varphi(x)\psi(y)P(\tilde{u}(x),\tilde{v}(y)) = \tilde{w}(x)\tilde{z}(y),
\]
the claim then follows by Corollary 4.2. \( \square \)

In the next proposition we consider a stronger condition on the convergence of \( \mu_k \) to \( \mu \) which allows to avoid the multiplication by the cut-off functions in the previous proposition. Note that the second condition in (4.8) is necessary for the validity of (4.9) when \( P \) is a constant.

Proposition 4.4. Let \( P: \mathbb{R}^2 \to \mathbb{R} \) be a polynomial function. Let \( \mu_k, \mu \in \mathcal{M}^{1,p}_+(\Omega \times \Omega) \) with \( \mu_k(\Omega \times \Omega) < +\infty \) and \( \mu(\Omega \times \Omega) < +\infty \). Suppose that
\[
\| \mu_k - \mu \|_\square \to 0 \quad \text{and} \quad \mu_k(\Omega \times \Omega) \to \mu(\Omega \times \Omega). \tag{4.8}
\]
Let \( u_k, v_k \) be sequences in \( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \) converging to \( u, v \) weakly in \( W^{1,p}_0(\Omega) \) and with \( \sup_k(\| u_k \|_\infty + \| v_k \|_\infty) = M < +\infty \). Then
\[
\lim_{k \to +\infty} \int_{\Omega \times \Omega} P(\tilde{u}_k(x),\tilde{v}_k(y))d\mu_k(x,y) = \int_{\Omega \times \Omega} P(\tilde{u}(x),\tilde{v}(y))d\mu(x,y). \tag{4.9}
\]
Proof. By Remark 3.2(iii) the first condition in (4.8) implies that \( \mu_k \rightarrow \mu \) weakly* in \( \mathcal{M}(\Omega \times \Omega) \). This, together with the second condition in (4.8), gives \( \mu_k(B) \rightarrow \mu(B) \) for all Borel sets in \( \Omega \times \Omega \) such that \( \mu(\partial B \cap (\Omega \times \Omega)) = 0 \). Then, for every \( \varepsilon > 0 \) there exists a compact set \( K_\varepsilon \) of \( \Omega \) such that

\[
\mu((\Omega \times \Omega) \setminus (K_\varepsilon \times K_\varepsilon)) < \varepsilon \quad \text{and} \quad \mu_k((\Omega \times \Omega) \setminus (K_\varepsilon \times K_\varepsilon)) < \varepsilon \quad \text{for every } k. \tag{4.10}
\]

Let \( \varphi_\varepsilon \in C_c^\infty(\Omega) \) with \( 0 \leq \varphi_\varepsilon \leq 1 \) in \( \Omega \) and \( \varphi_\varepsilon = 1 \) on \( K_\varepsilon \), and let

\[
C_M = \max\{P(s,t) : s,t \in [-M,M]\}.
\]

With this choice

\[
\left| \int_{\Omega \times \Omega} P(\tilde{u}_k(x), \tilde{v}_k(y))d\mu_k(x,y) - \int_{\Omega \times \Omega} \varphi_\varepsilon(x)\varphi_\varepsilon(y)P(\tilde{u}_k(x), \tilde{v}_k(y))d\mu_k(x,y) \right| \leq C_M \varepsilon
\]

and

\[
\left| \int_{\Omega \times \Omega} P(\tilde{u}(x), \tilde{v}(y))d\mu(x,y) - \int_{\Omega \times \Omega} \varphi_\varepsilon(x)\varphi_\varepsilon(y)P(\tilde{u}(x), \tilde{v}(y))d\mu(x,y) \right| \leq C_M \varepsilon.
\]

By (4.7) with \( \varphi = \psi = \varphi_\varepsilon \) we then deduce

\[
\limsup_{k \to +\infty} \left| \int_{\Omega \times \Omega} P(\tilde{u}_k(x), \tilde{v}_k(y))d\mu_k(x,y) - \int_{\Omega \times \Omega} P(\tilde{u}(x), \tilde{v}(y))d\mu(x,y) \right| \leq 2C_M \varepsilon,
\]

from which we obtain (4.9) by the arbitrariness of \( \varepsilon \).

We are now ready to prove the result for an arbitrary continuous function \( f \).

**Corollary 4.5.** Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a continuous function. Under the assumptions of Proposition 4.4 we have

\[
\lim_{k \to +\infty} \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y))d\mu_k(x,y) = \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y))d\mu(x,y). \tag{4.11}
\]

**Proof.** It suffices to approximate uniformly \( f \) by polynomials on \([-M,M]^2\) and apply the previous proposition. \( \square \)

Finally, if \( f \) is bounded we can remove the hypothesis that \( u_k \) and \( v_k \) are bounded in \( L^\infty \).

**Theorem 4.6.** Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a bounded continuous function. Let \( \mu_k, \mu \in \mathcal{M}^1_{+p}(\Omega \times \Omega) \) with \( \mu_k(\Omega \times \Omega) < +\infty \) and \( \mu(\Omega \times \Omega) < +\infty \). Suppose that

\[
||\mu_k - \mu||_\square \rightarrow 0 \quad \text{and} \quad \mu_k(\Omega \times \Omega) \rightarrow \mu(\Omega \times \Omega). \tag{4.12}
\]

Let \( u_k, v_k \) be sequences in \( W^{1,p}_0(\Omega) \) converging to \( u, v \) weakly in \( W^{1,p}_0(\Omega) \), then

\[
\lim_{k \to +\infty} \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y))d\mu_k(x,y) = \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y))d\mu(x,y). \tag{4.13}
\]

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\textbf{Proof.} For all $\lambda > 0$ we define the truncation operator
\begin{equation}
\tau^\lambda(s) := (s \vee (-\lambda)) \wedge \lambda.
\end{equation}
With fixed $\lambda > 1$ we set $u_k^\lambda(x) := \tau^\lambda(u_k(x))$ and $v_k^\lambda(y) := \tau^\lambda(v_k(y))$, and correspondingly, $u^\lambda(x) := \tau^\lambda(u(x))$ and $v^\lambda(y) := \tau^\lambda(v(y))$. By the uniqueness of the quasicontinuous representatives, we deduce from (3.5) that $\tilde{u}_k^\lambda(x) = \tau^\lambda(\tilde{u}_k(x))$ and $\tilde{v}_k^\lambda(x) = \tau^\lambda(\tilde{v}_k(x))$ for $\mu_k$-almost every $(x, y) \in \Omega \times \Omega$. Similarly, we have $\tilde{u}^\lambda(x) = \tau^\lambda(\tilde{u}(x))$ and $\tilde{v}^\lambda(x) = \tau^\lambda(\tilde{v}(x))$ for $\mu$-almost every $(x, y) \in \Omega \times \Omega$.

Since $u_k^\lambda \rightharpoonup u^\lambda$ and $v_k^\lambda \rightharpoonup v^\lambda$ weakly in $W_0^{1,p}(\Omega)$, we have
\begin{equation}
\lim_{k \to +\infty} \int_{\Omega \times \Omega} f(\tau^\lambda(\tilde{u}_k(x)), \tau^\lambda(\tilde{v}_k(y))) d\mu_k(x, y) = \int_{\Omega \times \Omega} f(\tau^\lambda(\tilde{u}(x)), \tau^\lambda(\tilde{v}(y))) d\mu(x, y)
\end{equation}
by Corollary 4.5. To conclude the proof of the result is suffices to estimate
\begin{align}
A_k^\lambda := & \left| \int_{\Omega \times \Omega} f(\tau^\lambda(\tilde{u}_k(x)), \tau^\lambda(\tilde{v}_k(y))) d\mu_k(x, y) - \int_{\Omega \times \Omega} f(\tilde{u}_k(x), \tilde{v}_k(y)) d\mu_k(x, y) \right|,
A^\lambda := & \left| \int_{\Omega \times \Omega} f(\tau^\lambda(\tilde{u}(x)), \tau^\lambda(\tilde{v}(y))) d\mu(x, y) - \int_{\Omega \times \Omega} f(\tilde{u}(x), \tilde{v}(y)) d\mu(x, y) \right|.
\end{align}
Let $M_0 = \sup |f|$. Since
\begin{equation}
A_k^\lambda \leq 2M_0 \mu_k(\{(x, y) \in \Omega \times \Omega : (\tilde{u}_k(x), \tilde{v}_k(y)) \notin [-\lambda, \lambda]^2\}),
\end{equation}
it is enough to separately estimate
\begin{align}
\mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{u}_k(x) > \lambda\}), & \quad \mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{u}_k(x) < -\lambda\}), \quad \mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{v}_k(y) > \lambda\}), \quad \mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{v}_k(y) < -\lambda\}).
\end{align}
For fixed $\varepsilon > 0$ let $K_\varepsilon$ be the compact sets introduced at the beginning of the proof of Proposition 4.4. By (4.10) we have
\begin{equation}
\mu_k(\{(x, y) \in \Omega \times \Omega : \tilde{u}_k(x) > \lambda\}) \leq \mu_k(\{(x, y) \in K_\varepsilon \times K_\varepsilon : \tilde{u}_k(x) > \lambda\}) + \varepsilon.
\end{equation}
Let $\psi \in C_c^\infty(\Omega)$ with $0 \leq \psi \leq 1$ and $\psi = 1$ on $K_\varepsilon$. Let $T_\varepsilon$ be the element in $W^{-1,q}(\Omega)$ defined by
\begin{equation}
\langle T_\varepsilon, v \rangle := \int_{\Omega \times \Omega} \tilde{v}(x)\psi(x) d\mu(x, y) \text{ for every } v \in W_0^{1,p}(\Omega).
\end{equation}
Since $u_k$ are equibounded in $W_0^{1,p}(\Omega)$, by (3.6) there exists $C_\varepsilon > 0$ and $k_\varepsilon \in \mathbb{N}$ such that
\begin{align}
\mu_k(\{(x, y) \in K_\varepsilon \times K_\varepsilon : \tilde{u}_k(x) > \lambda\}) \leq & \int_{\Omega \times \Omega} [\tilde{u}_k(x) - \lambda + 1]^+ \psi(x)\psi(y) d\mu_k(x, y),
\leq & \int_{\Omega \times \Omega} [\tilde{u}_k(x) - \lambda + 1]^+ \psi(x)\psi(y) d\mu_k(x, y) + \int_{\Omega \times \Omega} [\tilde{u}_k(x) - \lambda + 1]^+ \psi(x)\psi(y) d\mu_k(x, y),
\leq & C_\varepsilon \|\mu_k - \mu\|_\square + \langle T_\psi, [u_k - \lambda + 1]^+ \rangle = \varepsilon + \langle T_\psi, [u - \lambda + 1]^+ \rangle.
\end{align}
for all $k \geq k_\varepsilon$. Now, since $\lim_{\lambda \to +\infty} \langle T_{\psi_k}, [u - \lambda + 1]^+ \rangle = 0$, we can choose $\lambda_\varepsilon > 0$ such that

$$\mu_k(\{(x, y) \in K_\varepsilon \times K_\varepsilon : \bar{u}_k(x) > \lambda\}) \leq 2\varepsilon$$

for all $k \geq k_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$. By (4.20) this in turn gives

$$\mu_k(\{(x, y) \in \Omega \times \Omega : \bar{u}_k(x) > \lambda\}) \leq 3\varepsilon$$

for all $k \geq k_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$.

In the same way, we can prove analogue estimates for the other measures in (4.18) and (4.19), which we may assume to hold for the same $K_\varepsilon$ and $\lambda_\varepsilon$, and conclude that $A^\lambda_k \leq 24M_0\varepsilon$ for all $k \geq k_\varepsilon$ and $\lambda \geq \lambda_\varepsilon$. Similarly we can prove that $A^\lambda \leq 24M_0\varepsilon$ for $\lambda \geq \lambda_\varepsilon$. From these estimates, by (4.15)–(4.17) the claim follows by the arbitrariness of $\varepsilon$.

We finally prove a lower bound for limits of double integrals.

**Corollary 4.7.** Let $f : \mathbb{R}^2 \to [0, +\infty)$ be a continuous function. Let $\mu_k, \mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$ with $\mu_k(\Omega \times \Omega) < +\infty$, $\mu(\Omega \times \Omega) < +\infty$, satisfying (4.12). Let $u_k, v_k$ be sequences in $W^{1,p}_0(\Omega)$ converging to $u, v$ weakly in $W^{1,p}_0(\Omega)$. Then

$$\liminf_{k \to +\infty} \int_{\Omega \times \Omega} f(\bar{u}_k(x), \bar{v}_k(y)) d\mu_k(x, y) \geq \int_{\Omega \times \Omega} f(u(x), v(y)) d\mu(x, y).$$

(4.21)

**Proof.** For every $\lambda > 0$ let $f^\lambda := \tau^\lambda(f)$, where $\tau^\lambda$ is the truncation operator as in (4.14). By Theorem 4.6 we then have

$$\liminf_{k \to +\infty} \int_{\Omega \times \Omega} f(\bar{u}_k(x), \bar{v}_k(y)) d\mu_k(x, y) \geq \lim_{k \to +\infty} \int_{\Omega \times \Omega} f^\lambda(\bar{u}_k(x), \bar{v}_k(y)) d\mu_k(x, y) = \int_{\Omega \times \Omega} f^\lambda(\bar{u}(x), \bar{v}(y)) d\mu(x, y).$$

We then conclude by letting $\lambda \to +\infty$.

\[\square\]

\section{\(\Gamma\)-convergence}

In this final section we shall prove the $\Gamma$-convergence and the Mosco convergence of sequences of functionals as in (1.1).

Let $f : \mathbb{R}^2 \to [0, +\infty)$ be a continuous function, and let $\mu_k, \mu \in \mathcal{M}^{1,p}(\Omega \times \Omega)$. We define $F_k, F : W^{1,p}_0(\Omega) \to [0, +\infty]$ by

$$F_k(u) := \int_{\Omega \times \Omega} f(\bar{u}(x), \bar{u}(y)) d\mu_k(x, y) \quad \text{and} \quad F(u) := \int_{\Omega \times \Omega} f(\bar{u}(x), \bar{u}(y)) d\mu(x, y).$$

(5.1)
We assume that $f$ satisfies the following condition: there exists an unbounded set $\Lambda \subset [0, +\infty)$ and two constants $a, b \geq 0$ such that

$$f(\tau^\lambda(s), \tau^\lambda(t)) \leq a f(s, t) + b \quad \text{for all } s, t \in \mathbb{R} \text{ and for all } \lambda \in \Lambda,$$

(5.2)

where $\tau^\lambda$ is the truncation operator defined in (4.14). Note that this condition is valid if $f$ is bounded (with $a = 0$) or when $f$ is decreasing by truncations (with $a = 1$ and $b = 0$).

Let $g_k, g : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be Carathéodory functions satisfying the growth conditions

$$c_0|\xi|^p \leq g_k(x, \xi) \leq c_1|\xi|^p + a(x), \quad c_0|\xi|^p \leq g(x, \xi) \leq c_1|\xi|^p + a(x) \quad \text{for all } (x, \xi) \in \Omega \times \mathbb{R}^d,$$

(5.3)

for some constants $c_0, c_1 > 0$ and some function $a \in L^1(\Omega)$. Let $G_k, G : W^{1,p}_0(\Omega) \to [0, +\infty)$ be defined by

$$G_k(u) := \int_{\Omega} g_k(x, \nabla u)dx \quad \text{and} \quad G(u) := \int_{\Omega} g(x, \nabla u)dx.$$

(5.4)

**Theorem 5.1.** Let $\mu_k, \mu \in M^+_1(\Omega \times \mathbb{R})$ with $\mu_k(\Omega \times \mathbb{R}) < +\infty$ and $\mu(\Omega \times \mathbb{R}) < +\infty$. Let $F_k, F$ be defined as in (5.1) with $f : \mathbb{R}^2 \to [0, +\infty)$ a continuous function satisfying (5.2). Let $G_k, G$ be defined as in (5.4) with $g_k, g : \Omega \times \mathbb{R}^d \to \mathbb{R}$ Carathéodory functions satisfying (5.3). Suppose that

$$\|\mu_k - \mu\|_{\infty} \to 0 \quad \text{and} \quad \mu_k(\Omega \times \mathbb{R}) \to \mu(\Omega \times \mathbb{R}),$$

(5.5)

Then $F + G = \Gamma- \lim_{k \to +\infty} (F_k + G_k)$ with respect to the weak convergence in $W^{1,p}_0(\Omega)$.

**Proof.** By [6, Proposition 8.10] we have to prove that for all $u \in W^{1,p}_0(\Omega)$ the following properties hold:

(i) for all $u_k$ converging to $u$ weakly in $W^{1,p}_0(\Omega)$ we have

$$F(u) + G(u) \leq \liminf_{k \to +\infty} (F_k(u_k) + G_k(u_k));$$

(ii) there exist $u_k$ converging to $u$ weakly in $W^{1,p}_0(\Omega)$ such that

$$F(u) + G(u) = \lim_{k \to +\infty} (F_k(u_k) + G_k(u_k)).$$

Claim (i) follows from Corollary 4.7 with $v_k = u_k$ and from the liminf inequality for $G_k$, which follows from (5.6).

If $u \in W^{1,p}_0(\Omega) \cap L^{\infty}(\Omega)$, we deduce from (5.6) that there exists a sequence $u_k$ converging to $u$ weakly in $W^{1,p}_0(\Omega)$ such that $G(u_k) \to G(u)$ and $\|u_k\|_{\infty} \leq M$ for some constant $M$ and
for all $k$ (see for instance [4, Proposition 2.5]). Setting $\lambda = \max\{f(s, t) : |s| \leq M, |t| \leq M\}$, we may apply Theorem 4.6 with $v_k = u_k$ and $\tau^\lambda(f)$ in the place of $f$, obtaining (ii).

Let now $u \in W_0^{1,p}(\Omega)$. If $F(u) = +\infty$, then claim (ii) follows from claim (i). Suppose then that $F(u) < +\infty$. By the validity of claim (ii) for $\tau^\lambda(u)$ we obtain

$$F(\tau^\lambda(u)) + G(\tau^\lambda(u)) = \left(\Gamma-\limsup_{k \to +\infty} (F_k + G_k)\right)(\tau^\lambda(u)).$$

(5.7)

By (5.3) we have $G(\tau^\lambda(u)) \to G(u)$ as $\lambda \to +\infty$. Since $F(u) < +\infty$, by (5.2) and the Dominated Convergence Theorem we have $F(\tau^\lambda(u)) \to F(u)$ as $\lambda \to +\infty$ with $\lambda \in \Lambda$. By (5.7), using the lower semicontinuity of the $\Gamma$-limsup we get

$$F(u) + G(u) \geq \left(\Gamma-\limsup_{k \to +\infty} (F_k + G_k)\right)(u).$$

By [6, Proposition 8.10] we then obtain an inequality in claim (ii). The proof is completed by using claim (i). \qed

**Remark 5.2** (Mosco convergence). If the functionals $G_k$ converge to $G$ in the sense of the Mosco convergence in $W_0^{1,p}(\Omega)$; that is, for all $u \in W_0^{1,p}(\Omega)$ we have

(i) for all $u_k$ converging to $u$ weakly in $W_0^{1,p}(\Omega)$ we have

$$G(u) \leq \liminf_{k \to +\infty} G_k(u_k);$$

(ii) there exist $u_k$ converging to $u$ strongly in $W_0^{1,p}(\Omega)$ such that

$$G(u) = \lim_{k \to +\infty} G_k(u_k),$$

then also $F_k + G_k$ converges in the sense of the Mosco convergence in $W_0^{1,p}(\Omega)$.

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