Entanglement-annihilating and entanglement-breaking channels

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Abstract

We introduce and investigate a family of entanglement-annihilating channels. These channels are capable of destroying any quantum entanglement within the system they act on. We show that they are not necessarily entanglement breaking. In order to achieve this result we analyze the subset of locally entanglement-annihilating channels. In this case, the same local noise applied on each subsystem individually is less entanglement annihilating (with respect to multi-partite entanglement) as the number of subsystems is increasing. Therefore, the bipartite case provides restrictions on the set of local entanglement-annihilating channels for the multipartite case. The introduced concepts are illustrated on the family of single-qubit depolarizing channels.

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1. Introduction

The phenomenon of quantum entanglement [1] was recognized as an important resource in many applications of quantum information theory [2]. The parallel power of quantum computers, as well as the security of quantum cryptosystems, relies on the peculiar properties of entangled states of composite systems. For example, Shor’s algorithm [3], or quantum teleportation [4], could not be invented and successful without the puzzling properties of quantum entanglement.

By definition, entanglement is a property assigned to multipartite quantum states. Following Werner [5], we say that a state $\omega$ of some bipartite system is separable, if it can be expressed as a convex combination of factorized states, i.e. written in the form $\omega = \sum p_j \xi_j \otimes \zeta_j$. If it cannot, we say it is entangled. Entangled states exhibit their nonlocal origin in the following sense. Spatially separated experimentalists cannot create
entanglement without some exchange of quantum systems, i.e. only by local actions and classical communication.

The concepts of entanglement and separability can be directly generalized to the multipartite case. Moreover, in this case a more subtle ‘entanglement-induced’ separation of the state space is possible. For example, the so-called GHZ state $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ is an example of an entangled three-partite state, but no pair of subsystems is mutually entangled, because each pair is described by the classically correlated state $\frac{1}{\sqrt{2}}(|00\rangle\langle 00| + |11\rangle\langle 11|)$. Qualitatively different family [6] of three-partite states is represented by a state $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. In this case, each pair of subsystems is entangled. This illustrates that the entanglement theory of multipartite systems is more complex and represents an interesting field of research.

The tasks related to detection and characterization of entanglement represent prominent problems of quantum entanglement theory [7, 8]. In this paper, we will pay attention to entanglement dynamics induced by the evolution of individual quantum subsystems. Since perfectly isolated quantum systems are very difficult to achieve in real experiments, unavoidable noise affects the states and potentially causes changes in the shared entanglement. It is of importance to understand the robustness of entanglement with respect to these (local) processes.

For example, in order to perform the GHZ experiment [9] the three-partite entangled state $|\text{GHZ}\rangle$ must be distributed to the laboratories of Gina, Helen and Zoe. However, it is very likely that the transmissions over long distances are not perfect and Gina, Helen and Zoe will actually receive and work with systems described by a modified quantum state $\omega_{\text{GHZ}}$. Also, the storage of quantum systems and performing the experiments themselves represent additional sources of noise. Depending on the particular type of overall noise the modified state may or may not be still used to perform the intended experiment and observe a certain phenomenon.

The better we understand the dynamical robustness of entanglement, the better we can perform multipartite experiments. Entanglement is the quantum resource, but it seems to be very fragile, which reduces our abilities to conduct scalable (in any sense) experiments. For this purpose its distribution, storage and careful local manipulation are important. Loosely speaking, our goal is to separate the ‘bad’ noise from noise that is relatively ‘nice’. In other words, which types of environmental influences must Gina, Helen and Zoe try to avoid, and which ones are acceptable? Intriguing questions are related to experiments with an increasing number of parties. Namely, are there local channels destroying any entanglement completely? How does it depend on the number of parties? For a given local channel is there always a number of parties, for which its action on each individual subsystem completely destroys any shared entanglement? In the presence of arbitrary local noise is there any limit on the number of particles that can be entangled? Such questions are partially addressed in this paper.

The interplay between the entanglement we created and its dynamical stability under particular sources of noise is currently a vivid field of research. Zanardi et al [10, 11] asked the question: Which unitary transformations are better in creating entanglement? Linden et al have shown in [12] that capacities of unitary channels to create and destroy entanglement are not the same. In particular, there are unitary channels that can create (on average) more entanglement than they can destroy. Zyczkowski et al [13] analyzed the dynamics of entanglement for various different models of nonunitary evolutions. They showed the basic qualitative features how entanglement evolves in time. Since then the phenomenon known as entanglement sudden death attracted relatively many researchers (see [14] and references therein) who analyzed many dynamical models and made many observations concerning the entanglement dynamics. In [15–17] it was shown that local channels do not preserve
entanglement-induced ordering. After its action originally more entangled states can become less entangled than states coming from some originally less entangled states. Recently, an evolution equation (in fact inequality) for the entanglement affected by local independent noise has been formulated [18–20].

In this paper we focus on channels that completely destroy any entanglement. Clearly, unitary channels do not possess such property. For them entanglement creation goes always in hand with entanglement annihilation. However, the situation becomes more interesting when general nonunitary evolutions are considered. In section 2 we present our definitions and list the basic properties of the so-called entanglement-breaking (EB) and entanglement-annihilating (EA) channels. From the perspective of these concepts, in section 3 we investigate a family of depolarizing channels in more detail. Finally, we summarize our observations in section 4.

2. Preliminaries

A composite quantum system $Q$ consisting of $n$ quantum systems is associated with a Hilbert space $H_Q \equiv H^{(n)} = H_1 \otimes \cdots \otimes H_n$. Its states are represented by so-called density operators, i.e. positive operators with a unit trace. Let us denote by $S(H) = \{\varrho : \varrho \geq 0, \text{tr}[\varrho] = 1\}$ the set of all states of a system associated with the Hilbert space $H$. We divide the system $Q$ into two subsystems $A$ and $B$ consisting of $k$ and $n-k$ particles with Hilbert spaces $H_A = H^{(k)}$, $H_B = H^{(n-k)}$, respectively. When denoting the total Hilbert space as $H_{AB} = H_A \otimes H_B$ we mean that the whole system is understood as a bipartite system consisting of subsystems $A$ and $B$.

Any Hilbert space $H$ with a defined tensor structure can be divided into two subsets $S_{\text{ent}}(H), S_{\text{sep}}(H)$ of entangled and separable states. In particular, $S_{\text{sep}}(H_Q)$ is the set of all separable states with respect to the division of $H_Q$ into $n$ particles (n-partite separability), i.e. it consists of states of the form $\varrho = \sum_j p_j \varrho^{(j)}_1 \otimes \cdots \otimes \varrho^{(j)}_n$. Similarly, the set $S_{\text{ent}}(H_A)$ contains separable states of $k$ particles forming the subsystem $A$. However, we will use $S_{\text{sep}}(H_{AB})$ to denote the set of separable states with respect to division of $H_{AB}$ into subsystems $A$ and $B$ (bipartite separability), i.e. this set consists of states that can be expressed as $\varrho = \sum_j p_j \varrho^{(j)}_A \otimes \varrho^{(j)}_B$. In such a case the internal structure of the composite subsystems $A$ and $B$ is irrelevant and $S_{\text{sep}}(H_Q) \subsetneq S_{\text{sep}}(H_{AB})$ (meaning $S_{\text{sep}}(H_Q)$ is a strict subset of $S_{\text{sep}}(H_{AB})$). The analogous notation will be used for subsets of the entangled states $S_{\text{ent}}(H_Q), S_{\text{ent}}(H_A)$ and $S_{\text{ent}}(H_{AB})$.

The evolution of quantum systems is described by means of quantum channels, i.e. completely positive trace-preserving linear maps $\mathcal{E}$ defined on the set of all linear operators $\mathcal{L}(H)$ on the considered Hilbert space $H$. A linear mapping $\mathcal{E} : \mathcal{L}(H) \rightarrow \mathcal{L}(H)$ defines a quantum channel if $\text{tr}[\mathcal{E}[X]] = \text{tr}[X]$ for all $X \in \mathcal{L}(H)$ and $(\mathcal{E} \otimes I)[X]$ remains a positive operator for all positive operators $X \in \mathcal{L}(H \otimes \mathcal{H}_{\text{anc}})$, where $\mathcal{H}_{\text{anc}}$ is the Hilbert space associated with anancillary system of arbitrary size. We use $I$ to denote the identity (trivial) channel on the ancillary system. In what follows we will denote the ancillary system by $B$; thus, in our further consideration the subsystem $B$ can be of arbitrary size and structure.

Definition 1. We say that the channel $\mathcal{E}_A$ acting on the subsystem $H_A$ is

- entanglement annihilating (EA) if

$$\mathcal{E}_A[S(H_A)] \subset S_{\text{sep}}(H_A).$$
• entanglement breaking (EB) if
\[ \mathcal{E}_A \otimes I_B[S(\mathcal{H}_{AB})] \subset S_{\text{sep}}(\mathcal{H}_{AB}) \]
for the arbitrary ancillary system B.

Thus, the EA channels are defined as the ones that completely destroy/annihilate any entanglement within the subset A of the composite system (see figure 1). In contrast, the EB channels are those that completely disentangle the subsystem they are acting on from the rest of the system. Note that by definition the EA channels (acting on the subsystem A) do not necessarily disentangle the subsystems A and B. Similarly, the EB channels do not necessarily destroy entanglement within the subsystem A. The two concepts are thus (by definition) different and our aim is to investigate their mutual relationship.

Let us denote by \( T(\mathcal{H}) \) the set of all linear maps \( \mathcal{E} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H}) \). Let \( T_{\text{chan}}(\mathcal{H}) \) be the set of all channels on the system associated with a Hilbert space \( \mathcal{H} \), i.e. \( T_{\text{chan}}(\mathcal{H}) \) are all channels defined on the subsystem \( A \). Let \( T_{\text{EA}}(\mathcal{H}_A) \) and \( T_{\text{EB}}(\mathcal{H}_A) \) denote the subsets of the EA and EB channels, respectively.

2.1. Basic properties

In quantum theory, measurements are associated with so-called positive operators valued measures (POVMs), i.e. collections of positive operators \( F_1, \ldots, F_n \) such that \( \sum_j F_j = I \).

As was shown in [21] any EB channel can be understood as a measure and prepare procedure. That is, each EB channel can be expressed in the form
\[ E_A[\cdot] = \sum_j \text{tr}[\cdot F_j] \varrho_j, \]
for some POVM \( \{F_j\} \) and some fixed states \( \varrho_1, \ldots, \varrho_n \).

To verify whether a given channel is EB or not is equivalent to detecting whether a specific bipartite quantum state is separable, or entangled. Let us denote by \( P_\psi = |\psi\rangle\langle\psi| \) a projector onto the maximally entangled vector state \( |\psi\rangle = \frac{1}{\sqrt{d}} \sum_j |\psi_j\rangle \otimes |\psi_j\rangle \) of the Hilbert space \( \mathcal{H}_A \otimes \mathcal{H}_B \), where \( \mathcal{H}_B \equiv \mathcal{H}_A \), and the vectors \( \{|\psi_j\rangle\} \) form an orthonormal basis of the Hilbert space \( \mathcal{H}_A \) of dimension \( d \). A mapping \( J : T(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \) defined via the identity [22–24]
\[ J(E_A) = (E_A \otimes I_B)[P_\psi] \equiv \Omega e, \]
Lemma 2. If $T$ determines a unique operator $\Omega_T$ for each linear mapping $E \in \mathcal{T}(\mathcal{H}_A)$. It is known as the Choi–Jamiolkowski isomorphism. The Choi–Jamiolkowski operator $\Omega_T$ provides an alternative representation of a quantum channel (acting on the system associated with the Hilbert space $\mathcal{H}_A$) as a specific linear operator on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. The complete positivity of $E_A$ is translated to the positivity of $\Omega_T$ and the trace-preserving condition is equivalent to $\text{tr}_A \Omega_T = \frac{1}{d} I$; thus $\Omega_T$ is a valid density operator on $\mathcal{H}_A \otimes \mathcal{H}_B$. Clearly, if $E_A$ is EB, then $\Omega_T$ is a separable state on $\mathcal{H}_A \otimes \mathcal{H}_B$. Surprisingly, the inverse implication is also true [21], i.e. the separability of $\Omega_T$ is necessary and sufficient for $E_A$ being EB. This significantly simplifies the analysis of channels with respect to EB, because it is sufficient to test the action of the channel only on a single state—the maximally entangled state—and test whether $\Omega_T$ is separable, or not.

While EB channels had been already investigated, the concept of EA channels is new. An interesting question is how to test whether a given channel is EA, or not. If $E[|\psi\rangle \langle \psi|]$ with $P_{|\psi\rangle} = |\psi\rangle \langle \psi|$ is separable for all pure states $|\psi\rangle \in \mathcal{H}$, then for any state $\omega \in \mathcal{S}(\mathcal{H})$ the state $E[\omega]$ is separable. This follows from the fact that the set of separable states is convex and any state $\omega$ can be decomposed into a convex combination of pure states, i.e. $\omega = \sum p_i P_{|\psi_i\rangle}$. That is, whether or not the channel is EA, it is sufficient to test its action only on pure states. Unfortunately, this is still not an easy task. We left open whether there exists a simpler way of testing for the EA property.

Let us continue with simple observations on the elementary properties of the sets of the EA and EB channels.

**Lemma 1.** $T_{EA}(\mathcal{H}_A), T_{EB}(\mathcal{H}_A)$ are convex.

**Proof.** By definition, if $E_1, E_2 \in T_{EA}$, then $E_j[S(\mathcal{H}_A)] \subset \mathcal{S}_{sep}(\mathcal{H}_A)$ for $j = 1, 2$. Due to convexity of $E_j[S(\mathcal{H}_A)]$ and $\mathcal{S}_{sep}(\mathcal{H}_A)$ it follows that also $(\lambda E_1 + (1 - \lambda)E_2)[S(\mathcal{H}_A)] \subset \mathcal{S}_{sep}(\mathcal{H}_A)$. Similarly for the case of EB channels.

**Lemma 2.** If $E \in T_{EA}(\mathcal{H}_A)$ and $F \in T(\mathcal{H}_A)$, then $E \cdot F \in T_{EA}(\mathcal{H}_A)$.

**Proof.** Defining the property of $E$ implies that $E[F[S(\mathcal{H}_A)] \subset \mathcal{S}_{sep}(\mathcal{H}_A)$; hence $E \cdot F$ is an EA channel.

**Lemma 3.** If $F \in T_{EB}(\mathcal{H}_A)$ and $F \in T(\mathcal{H}_A)$, then $E \cdot F, F \cdot E \in T_{EB}(\mathcal{H}_A)$.

**Proof.** Since $E \in T_{EB}(\mathcal{H}_A)$ it follows that $(E \otimes I)[F[S(\mathcal{H}_{AB})]] \subset \mathcal{S}_{sep}(\mathcal{H}_{AB})$. For any $F \in T(\mathcal{H}_A)$ the channel $F \otimes I$ cannot create entanglement (out of the separable state) between the subsystems $A$ and $B$. Therefore, $(F \otimes I)[(E \otimes I)[S(\mathcal{H}_{AB})]] \subset \mathcal{S}_{sep}(\mathcal{H}_{AB})$; hence both the channels $E \cdot F, F \cdot E$ are EB providing that one of them is.

In what follows we will investigate the set relation between $T_{EA} \equiv T_{EA}(\mathcal{H}_A)$ and $T_{EB} \equiv T_{EB}(\mathcal{H}_A)$ (see figure 2), both defined on the same Hilbert space $\mathcal{H}_A$. As a consequence of the above lemmas we get that a composition $E \cdot F$ of the EB channel $F$ and of the EA channel $E$ belongs to the intersection $T_{EA} \cap T_{EB}$. That is, there are channels which are simultaneously EB and EA. On the other hand, although the channel $F \cdot E$ is necessarily EB, it does not have to be EA. For example, a single-point contraction $F$ of the whole state space into a single entangled state $\omega \in \mathcal{S}_{ent}(\mathcal{H}_A)$ is EB, because it can be expressed in the form $F[\cdot] = \sum \text{tr}[F_j |\omega\rangle\langle \omega|$. However, the channel $F \cdot E$ is not EA for arbitrary $E \in T_{EA}$, because $(F \cdot E)[S(\mathcal{H}_A)] = \omega \in \mathcal{S}_{ent}(\mathcal{H}_A)$. This means there are EB channels which are not EA, i.e. $T_{EB} \nsubseteq T_{EA}$. Later on we shall also return to the inverse question whether $T_{EA} \subset T_{EB}$, or not.
Figure 2. This figure schematically illustrates the subsets of EA and EB channels. The family of two-qubit local depolarizing channels $E_\lambda \otimes E_\lambda$ is depicted, too.

Lemma 4. Let $\mathcal{E}[-] = \sum_j \text{tr}[F_j] \varrho_j$, i.e. $\mathcal{E} \in \mathcal{T}_{\text{EB}}$. Then the following statements hold.

(i) If $\varrho_j \in S_{\text{sep}}(\mathcal{H}_A)$ for all $j$, then $\mathcal{E} \in \mathcal{T}_{\text{EA}}$.

(ii) If there exists $|\psi\rangle \in \mathcal{H}_A$ such that $F_j |\psi\rangle = |\psi\rangle$ for some $j$, then $\mathcal{E} \in \mathcal{T}_{\text{EA}}$ only if $\varrho_j$ is separable.

Proof. The first part (i) is obvious, because the convex sum of separable states is necessarily a separable state. The second part (ii) follows from the formula $\mathcal{E}[|\psi\rangle \langle \psi|] = \varrho_j$, which implies that being EA requires that $\varrho_j$ is separable. □

The second half of this lemma can be used to show that its first half cannot be if and only if statement. Consider an entangled state $\omega$. Let us define $\kappa$ as the largest value of $x \in [0, 1]$ for which the state $x \omega + (1-x) \frac{1}{d} I$ is separable. This value is strictly larger than 0. Let $F$ be a positive operator with all the eigenvalues smaller than $\kappa$. Then $\mathcal{E}[-] = \text{tr}[F] \omega + \text{tr}[(I-F)] \frac{1}{d} I$ defines an EA channel, because $\mathcal{E}[\varrho] = x \omega + (1-x) \frac{1}{d} I$ with $x = \text{tr}[\varrho F] < \kappa$. Thus, $\mathcal{E} \in \mathcal{T}_{\text{EA}} \cap \mathcal{T}_{\text{EB}}$ does not imply that all $\varrho_j$ are necessarily separable.

3. Local channels

We distinguish two basic types of channels acting on a composite system of $k$ particles: global and local. We say a channel $\mathcal{F}$ is local if it has a tensor product form $\mathcal{F} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_k$, where $\mathcal{E}_j$ are the channels acting on individual particles. If a channel does not have the factorized form, we say it is global. In what follows we will investigate entanglement dynamics under local channels. Moreover, we will assume that Hilbert spaces of all particles are isomorphic and each particle undergoes the same evolution, i.e. $\mathcal{E}_j = \mathcal{E}$ for all $j$. Although this is not the most general case, under certain circumstances it is of physical relevance.

We say a single-particle channel $\mathcal{E}$ is a $k$-locally EA channel ($k$-LEA), if $\mathcal{E} \otimes I \in \mathcal{T}_{\text{EA}}(\mathcal{H}^{\otimes k})$. Similarly, $\mathcal{E}$ is a $k$-locally EB channel ($k$-LEB) if $\mathcal{E} \otimes I \in \mathcal{T}_{\text{EB}}(\mathcal{H}^{\otimes k})$. By $\mathcal{T}_{k-\text{LEA}}, \mathcal{T}_{k-\text{LEB}}$ we shall denote the subsets of $k$-LEA and $k$-LEB channels, respectively. Since elements of these sets are uniquely associated with single-particle channels $\mathcal{E} \in \mathcal{T}_{\text{chan}}(\mathcal{H})$, we can understand these sets as subsets of $\mathcal{T}_{\text{chan}}(\mathcal{H})$, i.e. $\mathcal{T}_{k-\text{LEA}}, \mathcal{T}_{k-\text{LEB}} \subset \mathcal{T}_{\text{chan}}(\mathcal{H})$. Moreover, let us denote by $\mathcal{T}_{\text{EB}}^1 \subset \mathcal{T}_{\text{chan}}(\mathcal{H})$ the subset of EB channels acting on the single system $\mathcal{H}$, i.e. $\mathcal{E} \in \mathcal{T}_{\text{EB}}^1$ means that $(\mathcal{E} \otimes \mathcal{I}_{\text{anc}})[\omega]$ is separable for all $\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}_{\text{anc}})$. Our goal is to analyze the relation between the subsets $\mathcal{T}_{k-\text{LEA}}, \mathcal{T}_{k-\text{LEB}}$ and $\mathcal{T}_{\text{EB}}^1$. The ultimate question is which single-particle channels $\mathcal{E}$ if applied to a suitable number of particles necessarily destroy any entanglement within the $k$-partite system.
By definition single-particle EB channels disentangle each particle from the rest of the system. Therefore, they are simultaneously locally EA and locally EB channels for any value of $k$; hence

$$T^1_{EB} \subset T_{k-LEB}; \quad T^1_{EB} \subset T_{k-LEA}. \quad (3)$$

Further, consider a channel $\mathcal{E} \in T_{k-LEA}$. It means $\mathcal{E} \otimes k$ transforms any state of $k$ particles into some separable state of $k$ particles, i.e. $\mathcal{E} \otimes k[\omega] = \sum_p \xi_j^{(1)} \otimes \cdots \otimes \xi_j^{(k)}$. Setting $\omega = \rho_0 \otimes \omega'$ we get that $\mathcal{E} \otimes (k-1)[\omega']$ is separable for any state $\omega' \in S(\mathcal{H}^{\otimes (k-1)})$. That is, $\mathcal{E}$ is also a $(k-1)$-LEA channel. Consequently, we can write the relation

$$T^1_{EA} \subset T_{j-LEA} \quad \text{for} \quad k > j, \quad (4)$$

which implies

$$T^1_{EB} \subset T_{\infty-LEB} \subset \cdots \subset T_{3-LEB} \subset T_{2-LEB}. \quad (5)$$

On the other hand, for a channel $\mathcal{E} \otimes k$ the corresponding Choi operator takes the form $\Omega_{\mathcal{E} \otimes k}$, where $\Omega_{\mathcal{E} \otimes k} = (\mathcal{E} \otimes I)[P_+]$. If $\mathcal{E} \otimes k$ is EB, i.e. $\mathcal{E} \in T_{k-LEB}$, then $\Omega_{\mathcal{E} \otimes k}$ is separable with respect to a bipartite splitting into $k$ principal systems and $k$ ancillary systems. However, this implies that $\Omega_{\mathcal{E}}$ itself is separable; hence, the single-particle channel $\mathcal{E}$ is EB, i.e. $\mathcal{E} \in T^1_{EB}$. As a result we get the following set identities:

$$T^1_{EB} = T_{2-LEB} = T_{3-LEB} = \cdots = T_{\infty-LEB}. \quad (6)$$

4. A case study: depolarizing channels

In this section we will address the question whether the EB channels are not the only locally EA channels. We will give explicit example of qubit channels that are not EB, but completely destroy entanglement if applied on individual particles.

Consider a one-parametric family of depolarizing channels

$$\mathcal{E}_\lambda[X] = \lambda X + (1 - \lambda) \text{tr}[X] \frac{1}{d} I, \quad (7)$$

where $\lambda \in [0, 1]$. Applying this channel to the maximally entangled state $P_+$ we get the so-called Werner states [5]

$$\Omega_\lambda = \lambda P_+ + (1 - \lambda) \frac{1}{d} I \otimes \frac{1}{d} I. \quad (8)$$

For qubit ($d = 2$) the states $\Omega_\lambda$ are separable for $\lambda \leq 1/3$. Therefore, if $\lambda \leq 1/3$ the qubit depolarizing channel $\mathcal{E}_\lambda$ is EB.

4.1. 2-LEA channels

The local channel $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$ acts as follows:

$$\omega'_{12} = (\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda)[\omega_{12}] = \lambda^2 \omega_{12} + (1 - \lambda)^2 \frac{1}{d} I \otimes \frac{1}{d} I$$

$$+ \lambda (1 - \lambda) (\omega_1 \otimes \frac{1}{d} I + \frac{1}{d} I \otimes \omega_2), \quad (9)$$

where $\omega_1 = \text{tr}_2[\omega_{12}]$ and $\omega_2 = \text{tr}_1[\omega_{12}]$. Unlike in the analysis of EB channels we need to verify the separability for all input states $\omega_{12}$ in order to conclude that the channel $\mathcal{E}_\lambda$ is 2-LEA. Fortunately, it is sufficient to analyze the separability for the pure states $\omega_{12} = |\psi\rangle \langle \psi|$, only, because the set of separable states is convex and channels preserve the convexity. Let us note that $\mathcal{E}_\lambda = \lambda \mathcal{I} + (1 - \lambda) C_0$, where $C_0$ denotes the contraction of the whole state space into the
complete mixture state $\frac{1}{2}I$. Both the channels $\mathcal{I}, \mathcal{C}_0$ commute with unitary transformations, i.e., $\mathcal{I}[UXU^\dagger] = UXU^\dagger$ and $\mathcal{C}_0[UUXU^\dagger] = U\mathcal{C}_0[U]U^\dagger$. Consequently, $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$ commutes with all unitary channels $U \otimes V$. Using such local unitary channels any vector $|\psi\rangle$ can be written in its Schmidt form

$$|\psi\rangle = \sum_j \sqrt{q_j} |\psi_j\rangle \otimes |\psi'_j\rangle,$$

where $\{|\psi_j\rangle\}, \{|\psi'_j\rangle\}$ are the suitable orthonormal bases of the first and the second particle, respectively. Because of the unitary invariance of $\mathcal{E}_\lambda \otimes \mathcal{E}_\lambda$ it is sufficient to consider only vectors expressed in a single fixed Schmidt basis. The reduced states take the diagonal form

$$\omega_1 = \sum_j q_j |\psi_j\rangle \langle \psi_j|, \quad \omega_2 = \sum_j q'_j |\psi'_j\rangle \langle \psi'_j|.$$  

Further, let us analyze the case of qubit, i.e. $d = 2$ and $|\psi\rangle = \sqrt{q_0} |\psi_0\rangle \otimes |\psi_0\rangle + \sqrt{q_1} |\psi_1\rangle \otimes |\psi_1\rangle$; thus,

$$\omega_{12}' = \frac{1}{4} \begin{pmatrix} \lambda^2 + 4\lambda q_0 & 0 & 0 & 4\lambda^2 \sqrt{q_0q_1} \\ 0 & \lambda_+ \lambda_- & 0 & 0 \\ 0 & 0 & \lambda_+ \lambda_- & 0 \\ 4\lambda^2 \sqrt{q_0q_1} & 0 & 0 & \lambda^2 + 4\lambda q_1 \end{pmatrix},$$

where we set $\lambda_\pm = 1 \pm \lambda$. The eigenvalues of the partially transposed operator $\omega_{12}'$ read

$$\mu_1 = \frac{1}{4}(1 - \lambda^2) + \lambda q_0,$$  
$$\mu_2 = \frac{1}{4}(1 - \lambda^2) + \lambda q_1,$$  
$$\mu_\pm = \frac{1}{4}(1 - \lambda^2 \pm 4\lambda^2 \sqrt{q_0q_1}).$$

where $\lambda \in [0, 1]$, $q_0 \in [0, 1]$ and $q_1 = 1 - q_0$. According to the Peres–Horodecki criterion [25, 26] a two-qubit state $\omega$ is separable if and only if $\omega^T$ is positive. For larger systems this separability criterion is not sufficient to unambiguously distinguish between entangled and separable states. From the nonpositivity of $\omega^T$ we can conclude that the state is entangled, but the inverse implication does not hold.

All the eigenvalues of $\omega_{12}'$ except $\mu_-$ are always positive; hence it is sufficient to analyze only this one. In particular, if for a fixed value of $\lambda$ the eigenvalue $\mu_-$ is positive for all values $q_0, q_1$, then the corresponding depolarizing channel $\mathcal{E}_\lambda$ is a 2-locally EA channel. Thus, we want to minimize $\mu_-$ over the interval $q_0 \in [0, 1]$ for each depolarizing channel $\mathcal{E}_\lambda$. Fortunately, for each $\lambda$ the minimum is achieved for the same value $q_0 = q_1 = 1/2$. Consequently, the partially transposed operator $\omega_{12}'$ is positive if and only if $1 - 3\lambda^2 \geq 0$, i.e.

$$\lambda \leq \frac{1}{\sqrt{3}} \approx 0.577.$$  

In summary, the qubit depolarizing channel $\mathcal{E}_\lambda$ is 2-locally EA if and only if $\lambda \in [0, 1/\sqrt{3}]$, whereas it is EB for $\lambda \in [0, 1/3]$; hence $T_{2,\text{EA}}^T \neq T_{2,\text{EA}}$.

### 4.2. 3-LEA channels

We have already shown (see equation (5)) that $T_{1,\text{LEA}} \subset T_{2,\text{LEA}}$. In this section we will investigate the inverse relation, namely whether 2-LEA depolarizing channels $\mathcal{E}_\lambda$ are
necessarily also 3-LEA channels. Under the action of $\mathcal{E}_3 \otimes \mathcal{E}_3 \otimes \mathcal{E}_3$, a general three-partite state $\omega_{123}$ is transformed into the state

$$\omega'_{123} = \lambda^3 \omega_{123} + \frac{1}{d^3} (1 - \lambda)^3 I_1 \otimes I_2 \otimes I_3$$

$$+ \frac{1}{d^2} \lambda^2 (1 - \lambda) (\omega_{12} \otimes I_3 + \omega_{13} \otimes I_2 + \omega_{23} \otimes I_1)$$

$$+ \frac{1}{d^2} \lambda^2 (1 - \lambda)^2 (\omega_{1} \otimes I_{23} + \omega_{2} \otimes I_{13} + \omega_{3} \otimes I_{12}),$$

where $I_{jk} = I_j \otimes I_k$ and $I_j$ stands for the identity operator on the $j$th particle. Since $\mathcal{E}_3 \in T_{2\text{-LEA}}$, the reduced bipartite states $\omega'_{12}$, $\omega'_{13}$, $\omega'_{23}$ are separable. If some entanglement has left in the composite system, then it must be visible with respect to bipartite partitionings $1|23$, or $2|13$, or $3|12$.

As before let us assume the case of qubits ($d = 2$) and set $\omega_{123} = |\text{GHZ}\rangle \langle \text{GHZ}|$, i.e. $\omega_{12} = \omega_{13} = \omega_{12} = \frac{1}{2}(|00\rangle \langle 00| + |11\rangle \langle 11|) \equiv \Theta$ and $\omega_1 = \omega_2 = \omega_3 = \frac{1}{2} I$. Thus,

$$\omega'_{\text{GHZ}} = \lambda^3 |\text{GHZ}\rangle \langle \text{GHZ}| + \frac{(1 - \lambda)^2}{8} I \otimes I \otimes I$$

$$+ \frac{1}{2} \lambda^2 (1 - \lambda) (\Theta_{12} \otimes I + \Theta_{13} \otimes I + \Theta_{23} \otimes I).$$

Let us consider the splitting $1|23$ and define the basis elements of the system (23) as follows:

$$|0\rangle = |00\rangle, \quad |1\rangle = |11\rangle, \quad |2\rangle = |01\rangle, \quad |3\rangle = |10\rangle.$$  \hspace{1cm} (16)

In this basis

$$\omega'_{\text{GHZ}} = \frac{1}{8} (1 - \lambda^2) |0\rangle \langle 0| \otimes (I_{23} - |0\rangle \langle 0|) + \frac{1}{8} (1 - \lambda^2) |1\rangle \langle 1| \otimes (I_{23} - |1\rangle \langle 1|)$$

$$+ \frac{1}{8} (1 + 3 \lambda^2) (|00\rangle \langle 00| + |11\rangle \langle 11|) + \frac{1}{2} \lambda^2 (|00\rangle \langle 11| + |11\rangle \langle 00|).$$

The last term plays the crucial role from the point of partial transposition criterion applied with respect to splitting $1|23$. Let us note that due to symmetry for different splitting $2|13$ and $3|12$ we will derive qualitatively the same bipartite density matrix. That is, if the state $\omega'_{\text{GHZ}}$ is entangled with respect to the splitting $1|23$, then it is also entangled with respect to remaining bipartite splittings.

Among all the eigenvalues of $\omega'_{\text{GHZ}}$, only

$$\mu_- = \frac{1}{2} \left( \frac{3}{4} (1 - \lambda^2) - \lambda^3 \right)$$  \hspace{1cm} (17)

is negative when $\lambda > 0.5567$. Important for us is that for $\lambda = 1/\sqrt{3}$ this eigenvalue is negative; hence the state remains entangled although the depolarizing channel is 2-LEA. Therefore, we can conclude that

$$T_{3\text{-LEA}} \subseteq T_{2\text{-LEA}}.$$  \hspace{1cm} (18)

Since the partial transposition criterion is not sufficient to conclude the separability, it cannot be used to decide for which $\lambda$ the channel $\mathcal{E}_3$ is 3-LEA and for which it is not. If $\lambda > 0.5567$ we can safely say that $\mathcal{E}_3$ is not the 3-locally EA channel. However, for smaller values we cannot exclude the possibility that the channel $\mathcal{E}_3$ does not belong to $T_{3\text{-LEA}}$ unless $\lambda \leq 1/3$, when the channel is EB.

4.3. EA versus EB

In this section we shall return to the question on relation between EA and EB channels. We have already shown that there are EB channels which are not EA. In what follows we are
interested in whether the opposite case is also possible, i.e. whether there are EA channels that are EB. Mathematically, we are asking which of the relations $T_{EA} \subset T_{EB}$, $T_{EA} \not\subset T_{EB}$ hold. We can use the derived results to argue whether 2-LEA channels $E_\lambda \otimes E_\lambda$ considered as channels acting on the bipartite system are EB, or not. Certainly, they are EA because $T_{2-LEA}(H \otimes H) \subset T_{EA}(H \otimes H)$. We have seen that for the value $\lambda = \frac{1}{\sqrt{3}}$ the channel $E_\lambda \otimes E_\lambda$ is EA. More importantly, we have also shown that for the same value $E_\lambda \otimes E_\lambda \otimes E_\lambda$ is not EA. Let us formally write $E_\lambda \otimes E_\lambda \otimes E_\lambda = (I \otimes I \otimes E_\lambda)(E_\lambda \otimes E_\lambda \otimes I)$. Since channels of the form $I \otimes I \otimes E$ can only decrease the entanglement and since $E_\lambda^{(3)}[|\text{GHZ}\rangle\langle\text{GHZ}|]$ is entangled for $\lambda = \frac{1}{\sqrt{3}}$, we can conclude that also $(E_\lambda \otimes E_\lambda) \otimes I[|\text{GHZ}\rangle\langle\text{GHZ}|]$ is entangled. But this is in contradiction with the assumption that $E_\lambda \otimes E_\lambda$ is EB. Based on this example we can conclude that

$$T_{EA} \not\subset T_{EB},$$

that is, there are EA channels which are not EB.

5. Summary

In this paper we introduced the concept of entanglement-annihilating (EA) channels as the channels that completely destroy any entanglement within the systems they act on. We investigated the structural properties of the set of these channels and its relation to the set of entanglement-breaking (EB) channels $T_{EB}$, i.e. channels that completely destroy entanglement between the subsystem they act on and the rest of the composite system (see figure 1). In particular, we have shown that

$$T_{EA} \cap T_{EB} \neq \emptyset,$$

$$T_{EB} \not\subset T_{EA} \not\subset T_{EB}.$$

That is, there are channels which are simultaneously EB and EA, but also channels possessing only one of this features. The set of EA channels $T_{EA}$ is convex. Moreover, a composition of an EA channel and an arbitrary channel results in an EA channel, i.e. the property of being EA is preserved under channel composition. We were able to prove one of the above relations by analyzing the family of local depolarizing channels. We defined the so-called $k$-local channels as channels of the form $E \otimes \cdots \otimes E$. That is, the same noise $E$ is applied on each individual subsystem forming a composite $k$-partite system. We investigated when a single-particle channel $E$ constitutes a $k$-locally EA channel ($k$-LEA), or a $k$-locally EB channel ($k$-LEB). In particular, for depolarizing qubit channels we found that for $\lambda \leq \frac{1}{\sqrt{3}}$ the channel is 2-locally EA, while for $\lambda > \frac{1}{3}$ it is not locally EB for any $k$. Moreover, for $\lambda > 0.5567$ the qubit depolarizing channel is not $k$-LEA for all $k \geq 3$. We found the following set relations:

$$T_{EB}^{1} = T_{2-LEB} = T_{3-LEB} = \cdots = T_{\infty-LEB},$$

$$T_{EB}^{1} \subset T_{\infty-LEA} \subset \cdots \subset T_{3-LEA} \subset T_{2-LEA},$$

where $T_{EB}^{1}$ is the set of EB channels of a single particle.

The introduced concept of EA channels opens several interesting mathematical and physical questions related to generic properties of entanglement dynamics. For example, we left open the problem of complete characterization of EA channels. For practical purposes, it would also be of interest to find an efficient testing algorithm for EA channels.
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