QCD sum rule analysis of the field strength correlator

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Abstract

The gauge invariant two-point correlator for the gluon field strength tensor is analysed by means of the QCD sum rule method. To this end, we make use of a relation of this correlator to a two-point function for a quark-gluon hybrid in the limit of the quark mass going to infinity. From the sum rules a relation between the gluon correlation length and the gluon condensate is obtained. We briefly compare our results to recent determinations of the field strength correlator on the lattice.

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1 Introduction

The gauge invariant non-local gluon field strength correlator plays an important rôle in non-perturbative approaches to QCD \[1–5\]. It is the basic ingredient in the model of the stochastic vacuum (MSV) \[6, 7\] and in the description of high energy hadron-hadron scattering \[8–11\]. In the spectrum of heavy quark bound states it governs the effect of the gluon condensate on the level splittings \[12–15\] and it is useful for the determination of the spin dependent parts in the heavy quark potential \[16, 17\].

Its next-to-leading order correction in perturbative QCD has been calculated recently by two of the authors \[18\]. The correlator has also been measured on the lattice in pure gauge theory and full QCD using the cooling method \[19, 20\] and by making the assumptions of the MSV from lattice calculations of the heavy quark potentials \[21\]. The lattice analyses found that for distances \(z\) of the gluon field strength larger than roughly 0.4 fm an exponential decaying term dominates yielding a correlation length of approximately 0.2 fm. On the other hand the short distances are dominated by the perturbative \(1/z^4\) behaviour. Recently, the field strength correlator has also been calculated in the framework of exact renormalisation group equations \[22\].

The gauge invariant gluon field strength correlator can be related to a correlator of a colour singlet current composed of a (fictitious) infinitely heavy octet quark and the gluon field strength tensor. This fact has already been employed in ref. \[18\] in order to apply the machinery developed in the Heavy Quark Effective Theory (HQET) \[23\] for calculating the perturbative corrections. In this paper we again use this relation in order to apply QCD sum rule techniques \[1\] to the correlator in question. The sum rule analysis can be used to estimate the correlation length of the field strength correlator using as ingredients the value of the gluon condensate and the results for the perturbative calculation.

Our paper is organised as follows. In the next section we discuss again the relation of the field strength correlator and the corresponding heavy quark current correlator. In section 3 we set up the different contributions needed for the sum rule analysis and in section 4 we present our results together with a comparison with recent lattice determinations of the field strength correlator. Finally, in section 5, we end with some conclusions and an outlook.

\(^{1}\)For a review on HQET as well as original references the reader is referred to \[23\].
The field strength correlator

The gauge invariant two-point correlation function of the QCD field strength tensor $F_{\mu\nu}^a(x)$ in the adjoint representation can be defined as

$$D_{\mu\nu\rho\sigma}(z) \equiv \langle 0| T\{g_s^2 F_{\mu\nu}^a(y) P e^{g f^{abc} z^\tau \int_0^1 dt A^c_{\tau}(x + tz) F_{\rho\sigma}^b(x)}\}|0\rangle, \quad (2.1)$$

where the field strength $F_{\mu\nu}^a = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + g f^{abc} A_{\mu}^b A_{\nu}^c$, $z = y - x$ and $P$ denotes path ordering of the exponential. In general, the gauge invariant field strength correlator could be defined with an arbitrary gauge string connecting the end points $x$ and $y$, but in this work we shall restrict ourselves to a straight line. Only for that case the relation to HQET is possible. From the Lorentz structure of the field strength correlator it follows that the correlator can be parametrised in terms of two scalar functions $D(z^2)$ and $D_1(z^2)$ [7]:

$$D_{\mu\nu\rho\sigma}(z) = \left[ g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right] \left( D(z^2) + D_1(z^2) \right) + \left[ g_{\mu\rho} z_{\nu} z_\sigma - g_{\mu\sigma} z_{\nu} z_\rho - g_{\nu\rho} z_{\mu} z_\sigma + g_{\nu\sigma} z_{\mu} z_\rho \right] \frac{\partial D_1(z^2)}{\partial z^2}. \quad (2.2)$$

The invariant function $D(z^2)$ can only occur in a non-abelian gauge theory or an abelian one with monopoles. In the MSV it is responsible for confinement and the formation of a string.

The correlator $D_{\mu\nu\rho\sigma}(z)$ can be related to the correlator of a local, gauge invariant current composed of an infinitely heavy quark field in the octet representation, $h^a(x)$, and the gluon field strength tensor $F_{\mu\nu}^a$. The current in question takes the form $(g_s h^a F_{\mu\nu}^a)(x)$. Analogously to HQET the heavy octet-quark field is constructed from the field $Q^a$ with a finite mass $m_Q$ in the limit

$$h^a(x) = \lim_{m_Q \to \infty} \frac{1}{2} (1 + \nu) e^{im_Q v x} Q^a(x), \quad (2.3)$$

with $\nu$ being the four-velocity of the heavy quark. The propagator of the free heavy quark field $h_0^a(x)$ in coordinate space is given by

$$S(z) = \langle 0| T\{h_0^a(y) h_0^b(x)\}|0\rangle = \delta^{ab} \frac{1}{e^{v_0 z^0}} \theta(z^0) \delta(z - \frac{z_0}{e^0 v}), \quad (2.4)$$

where $z_0^0$ is the zero-component of the velocity. The correlator of the full field can be obtained by integrating out only the heavy quark and leaving the expectation value with respect to the gauge field:

$$\langle 0| T\{h^a(y) h^b(x)\}|0\rangle = S(z) \langle 0| P e^{g f^{abc} z^\tau \int_0^1 dt A^c_{\tau}(x + tz)}|0\rangle. \quad (2.5)$$
The gauge string is left after the elimination of the heavy quarks from the interaction term of adjoint quarks with the colour potential

\[ \mathcal{L}_{\text{int}} = -i g_s f_{abc} v^\mu \bar{h}^a(x) A^c_\mu(x) h^b(x). \]  

(2.6)

The physical picture of this result is a heavy quark moving from point \( x \) to \( y \) with a four-velocity \( v \), acquiring a phase proportional to the path-ordered exponential. The limit of \( m_Q \rightarrow \infty \) is necessary in order to constrain the heavy quark on a straight line and in order to decouple the spin interactions. The same relation also holds for quarks in the fundamental representation with the appropriate replacements in the exponential.

The equation (2.5) allows to establish a relation between the field strength correlator (2.1) and the correlator for the colourless heavy quark current. By integrating out the heavy degrees of freedom and using (2.5) we arrive at

\[ \tilde{D}_{\mu\nu\rho\sigma}(z) \equiv \langle 0 | T \{ g_s^2 F^a_{\mu\nu}(y) h^a(y) F^b_{\rho\sigma}(x) \bar{h}^b(x) \} | 0 \rangle = S(z) D_{\mu\nu\rho\sigma}(z). \]  

(2.7)

We may view the composite operator \(( g_s h^a F^a_{\mu\nu} )(x)\) as an interpolating field of colourless quark gluon hybrids and evaluate \( \tilde{D}_{\mu\nu\rho\sigma}(z) \) by introducing these as intermediate states in the absorption part of \( \tilde{D}_{\mu\nu\rho\sigma}(z) \). The lowest lying state will govern the long-range behaviour and hence the inverse of its energy is the correlation length.

Our next aim is to evaluate this correlator in the framework of QCD sum rules [1] and in that way obtain information on the correlation length of the gluon field strength correlator. For the sum rule analysis it is preferable to work with the correlator in momentum space. Thus we define

\[ \tilde{D}_{\mu\nu\rho\sigma}(w) = i \int dz e^{i q z} \langle 0 | T \{ g_s^2 F^a_{\mu\nu}(y) h^a(y) F^b_{\rho\sigma}(x) \bar{h}^b(x) \} | 0 \rangle, \]  

(2.8)

where the residual heavy quark momentum is \( w = v q \). Similar to the Lorentz decomposition of the coordinate space correlator \( D_{\mu\nu\rho\sigma}(z) \) into scalar functions \( D(z^2) \) and \( D_1(z^2) \), eq. (2.2), we can write the momentum space correlator as follows:

\[ \tilde{D}_{\mu\nu\rho\sigma}(w) = \left[ g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right] \left( \tilde{D}(w) + \tilde{D}_1(w) \right) \]

\[ + \left[ g_{\mu\rho} v_\nu v_\sigma - g_{\mu\sigma} v_\nu v_\rho - g_{\nu\rho} v_\mu v_\sigma + g_{\nu\sigma} v_\mu v_\rho \right] \tilde{D}_s(w). \]  

(2.9)
The functions $\tilde{D}(w)$ and $\tilde{D}_1(w)$ are the Fourier transforms of $S(z)D(z^2)$ and $S(z)D_1(z^2)$ respectively, the function $\tilde{D}_*(w)$ is the Fourier transform of $S(z)z^2\partial D_1(z^2)/\partial z^2$.

For our purpose of isolating intermediate states of the correlator (2.8) a decomposition according to an $O_3$ classification is more appropriate than the decomposition of eq. (2.9). Since the spin of the heavy quark decouples, we only have to consider the gluon spin. The six-component field can be decomposed into tensor structures depending on the only two external vectors in the game; the four-velocity $v_\mu$ and the polarisation vector of the gluon $e_\mu$. This leads to the two Lorentz structures for the hadronic matrix elements

\[
\langle 0 | (g_s F^a_{\mu\nu} h^a) | H^- \rangle = f^- (v_\mu e_\nu - v_\nu e_\mu),
\]

(2.10)

\[
\langle 0 | (g_s F^a_{\mu\nu} h^a) | H^+ \rangle = f^+ \varepsilon_{\mu\nu\lambda\kappa} v^\lambda e^\kappa,
\]

(2.11)

where $H^\pm$ are hadronic states with the same quantum numbers as the composite current. In the rest frame $v = 0$, the first structure transforms as a 3-vector and thus $H^-$ corresponds to a $1^-$ state whereas the second structure transforms as an axialvector and $H^+$ corresponds to a $1^+$ state.

Through appropriate projections the two quantum numbers can be singled out from the correlator $\tilde{D}_{\mu\nu\rho\sigma}(w)$. Hence, we define

\[
\tilde{D}^-(w) \equiv g^{\mu\rho} v^\nu v^\sigma \tilde{D}_{\mu\nu\rho\sigma}(w) = 3 \left( \tilde{D}(w) + \tilde{D}_1(w) + \tilde{D}_*(w) \right),
\]

(2.12)

\[
\tilde{D}^+(w) \equiv (g^{\mu\rho} g^{\sigma\tau} - 2 g^{\mu\rho} v^\nu v^\sigma) \tilde{D}_{\mu\nu\rho\sigma}(w) = 6 \left( \tilde{D}(w) + \tilde{D}_1(w) \right).
\]

(2.13)

The Fourier transforms of the functions $\tilde{D}^-(w)$ and $\tilde{D}^+(w)$ are up to the factor $S(z)$ the invariant functions $D_{\parallel}(z^2)$ and $D_{\perp}(z^2)$ respectively which have been used in the lattice calculations of refs. [19, 20].

Under the assumption of quark-hadron duality which is usually made for sum rule analyses [1], we model the correlators by a contribution from the lowest lying resonances plus the perturbative continuum above a threshold $s_0$. Inserting the matrix elements and performing the heavy quark phase space integrals one obtains

\[
\tilde{D}^\mp(w) = \frac{\kappa^\mp |f^\mp|^2}{w - E^\mp + i\epsilon} + \int_{s_0^\mp}^{\infty} d\lambda \frac{\rho^\mp(\lambda)}{\lambda - w - i\epsilon},
\]

(2.14)

where $\kappa^- = 1, \kappa^+ = -2$ and $E$ represents the energy of the glue around the heavy quark. The spectral densities are defined by $\rho^\mp(\lambda) \equiv 1/\pi \text{Im} \tilde{D}^\mp(\lambda + i\epsilon)$ and are
known at the next-to-leading order \cite{18}. Explicit expressions will be given in the next section.

After Fourier transformation to coordinate space the above representation reads:

\[
\tilde{D}(z) = -i \int \frac{d^4q}{(2\pi)^4} e^{-iqz} \tilde{D}(w) = \left\{-\kappa |f|^2 e^{-iE|z|} + \int_{s_0}^{\infty} d\lambda \rho(\lambda) e^{-i\lambda|z|}\right\} S(z) , \tag{2.15}
\]

where the factorisation of the heavy quark propagator can be seen explicitly. The inverse correlation length is found to be given by \(E\).

### 3 The sum rules

The \textit{pheno\-menological side} of the sum rules has already been given by eq. \eqref{2.14}. In this section, we shall present the \textit{theoretical side} of the sum rules which arises from calculating the correlator of eq. \eqref{2.8} in the framework of the operator product expansion \cite{1,25}.

In coordinate space the purely perturbative contribution up to the next-to-leading order in the strong coupling constant has been calculated in ref. \cite{18}. Here we give the corresponding results in momentum space for \(\tilde{D}_\pm(w)\):

\[
\tilde{D}_\pm^{PT}(w) = (-w)^3 a \left[ p_{11}^\pm + p_{11}^\pm L + a \left( p_{20}^\pm + p_{21}^\pm L + p_{22}^\pm L^2 \right) \right] , \tag{3.1}
\]

where \(a \equiv \alpha_s/\pi\), \(L = \ln(-2w/\mu)\) and the coefficients \(p_{ij}^\pm\) are given explicitly in the appendix. From this result one can immediately calculate the corresponding spectral functions:

\[
\rho^\pm(\lambda) = \lambda^3 a \left[ p_{11}^\pm + a \left( p_{21}^\pm + 2 p_{22}^\pm \ln \frac{2\lambda}{\mu} \right) \right] , \tag{3.2}
\]

where \(\lambda\) has to be greater zero. Essential for the sum rule analysis are the contributions coming from the condensates. The correlation function is expanded in powers of \(1/w\) corresponding to higher and higher dimensional condensates. In our case the dimension three condensate \(\langle \bar{h}h \rangle\) vanishes since the quark mass is infinite. The lowest nonvanishing term is the gluon condensate of dimension four:

\[
\tilde{D}_{FF}(w) = -\frac{\pi^2}{w} \langle aFF \rangle , \quad \tilde{D}_F^\pm(w) = -\frac{2\pi^2}{w} \langle aFF \rangle . \tag{3.3}
\]
The next condensate contribution would be of dimension six, but we shall neglect all higher condensate contributions in this work and restrict ourselves to the gluon condensate.

In order to suppress contributions in the dispersion integral coming from higher excited states and from higher dimensional condensates, it is convenient to apply a Borel transformation \( \tilde{B}_T \) with \( T \) being the Borel variable \([1]\). Some useful formulae for the Borel transformation are also collected in the appendix. For the phenomenological side of the sum rules, eq. (2.14), we then find

\[
\hat{D}^\mp(T) = -\kappa^\mp |f^\mp|^2 e^{-E^\mp/T} + \int_\infty^{s_0} d\lambda \rho^\mp(\lambda) e^{-\lambda/T}.
\]

(3.4)

For the perturbative contribution it is convenient to apply the following identity:

\[
\hat{B}_T \hat{D}(w) = T^4 \hat{B}_T \left( \frac{d}{dw} \right)^4 \hat{D}(w),
\]

(3.5)

from which we obtain

\[
\hat{D}_{FT}^\mp(T) = 6 T^4 a \left[ p_{11}^\mp + a \left( p_{21}^\mp + \frac{1}{3} \Gamma'(4) p_{22}^\mp + 2 p_{22}^\mp \ln \frac{2T}{\mu} \right) \right],
\]

(3.6)

where \( \gamma_E \) is Eulers constant and \( \Gamma'(4) = 11 - 6\gamma_E \). The Borel transformed expression for the gluon condensate contribution is found to be:

\[
\hat{D}_{FF} = \pi^2 \langle aF F \rangle, \quad \hat{D}_{PF} = 2\pi^2 \langle aF F \rangle.
\]

(3.7)

After Borel transformation, the correlators satisfy homogeneous renormalisation group equations. Thus we can improve the perturbative expressions by resumming the logarithmic contributions. The perturbative contribution is then expressed in terms of the running coupling \( a(2T) \):

\[
\hat{D}_{PT}^\mp(T) = 6 T^4 \left( \frac{a(2T)}{a(\mu)} \right)^{-\gamma_{1T}/\beta_1} a(2T) \left[ p_{11}^\mp + a \left( p_{21}^\mp + \frac{1}{3} \Gamma'(4) p_{22}^\mp \right) \right],
\]

(3.8)

where \( \beta_1 = 11/2 - n_f/3 \) is the first coefficient of the QCD \( \beta \)-function. Reexpanding and comparing with eq. (1.6), the anomalous dimensions \( \gamma_{1T} \) are found to be \( \gamma_{1T}^\mp = 2 p_{22}^\mp/p_{11}^\mp + \beta_1 \), or explicitly

\[
\gamma^- = 0, \quad \gamma^+ = 3.
\]

(3.9)
Let us note that the correlator \( \hat{D}^-(T) \) which corresponds to the vector intermediate state does not depend on the renormalisation scale \( \mu \) at this order.

For the continuum contribution we first evaluate the integral with the general formula [26] which makes the numerical analysis easier:

\[
\int_{s_0}^{\infty} d\lambda \lambda^{\alpha-1} \ln^{n-1} \frac{2\lambda}{\mu} e^{-\lambda/T} = T^\alpha \sum_{k=0}^{n} \binom{n}{k} \ln^k \frac{2T}{\mu} \left[ \frac{\partial^{n-k}}{\partial \alpha^{n-k}} \Gamma \left( \alpha, \frac{s_0}{T} \right) \right],
\]

some formulae for the incomplete \( \Gamma \)-function \( \Gamma(\alpha, x) \) are given in the appendix. We then obtain

\[
\chi^\mp(T, s_0) = \int_{s_0}^{\infty} d\lambda \rho^\mp(\lambda) e^{-\lambda/T} = T^4 a \left\{ p_{11}^\mp \Gamma \left( 4, \frac{s_0}{T} \right) \right. \\
+ a \left[ \left( p_{21}^\mp + 2 p_{22}^\mp \ln \frac{2T}{\mu} \right) \Gamma \left( 4, \frac{s_0}{T} \right) + 2 p_{22}^\mp \Gamma' \left( 4, \frac{s_0}{T} \right) \right] \right\},
\]

and after renormalisation group improvement

\[
\chi^\mp(T, s_0) = T^4 \left( \frac{a(2T)}{a(\mu)} \right)^{-\gamma_{\mp}/\beta_1} a(2T) \left\{ p_{11}^\mp \Gamma \left( 4, \frac{s_0}{T} \right) \right. \\
+ a \left[ p_{21}^\mp \Gamma \left( 4, \frac{s_0}{T} \right) + 2 p_{22}^\mp \Gamma' \left( 4, \frac{s_0}{T} \right) \right] \right\}. \tag{3.12}
\]

In the limit \( s_0 \to 0 \), eq. (3.12) agrees with eq. (3.8) as it should.

## 4 Numerical analysis

After equating the phenomenological and the theoretical part we end up with the sum rule

\[
K^\mp(T) \equiv -\kappa^\mp |f^{\mp}|^2 e^{-E^\mp/T} = \hat{D}^\mp_{FF} + \hat{D}^\mp_{PT}(T) - \chi^\mp(T, s_0), \tag{4.1}
\]

where \( \kappa^- = 1 \) and \( \kappa^+ = -2 \). In order to estimate the binding energy we derive as an immediate consequence of (4.1):

\[
E^\mp = -\frac{\partial}{\partial (1/T)} \ln K^\mp = -\frac{\partial}{\partial (1/T)} \left( \hat{D}^\mp_{PT}(T) - \chi^\mp(T, s_0) \right) \left( \hat{D}^\mp_{FF} + \hat{D}^\mp_{PT}(T) - \chi^\mp(T, s_0) \right). \tag{4.2}
\]
The derivative can also be given analytically if we first derive with respect to \( T \) and then perform the resummation of the logarithms. We thus find

\[
\frac{\partial}{\partial (1/T)} \left( \hat{D}_{FT}^\pm (T) - \chi^\pm (T, s_0) \right) =
\]

\[
- T^5 \left( \frac{a(2T)}{a(\mu)} \right)^{-\gamma_i^\pm /\beta_i} a(2T) \left\{ p_{11}^\pm \left( \Gamma(5) - \Gamma \left( 5, \frac{s_0}{T} \right) \right) \right.
\]

\[
+ a \left[ p_{21}^\pm \left( \Gamma(5) - \Gamma \left( 5, \frac{s_0}{T} \right) \right) + 2 p_{22}^\pm \left( \Gamma'(5) - \Gamma' \left( 5, \frac{s_0}{T} \right) \right) \right] \}. \quad (4.3)
\]

We note that the different signs of the perturbative and non-perturbative terms in the \( 1^- \) state lead to a stabilisation for the energy sum rule, whereas the equal sign in the \( 1^+ \) state destabilises the sum rule.

Let us begin our numerical analysis with the case for three light quark flavours. As our input parameters we use \( \langle a_{FF} \rangle = 0.024 \pm 0.012 \text{ GeV}^4 \) and \( \Lambda_{3fL} = 325 \text{ MeV} \). In principle, the coupling constant at next-to-leading order could be evaluated at any scale \( \mu \). As our central value in the numerical analysis we have chosen \( \mu = 2 \text{ GeV} \). For the energy \( E^- \) of the \( 1^- \) state we obtain the best stability for a continuum threshold \( s_0 = 1.7 \text{ GeV} \) in the range \( T \geq 0.7 \text{ GeV} \) with an energy \( E^- \approx 1.4 \text{ GeV} \). To estimate the errors we have varied the scale \( \mu \) as well as the continuum threshold \( s_0 \). In figure 1 we have displayed the energy \( E^- \) as a function of the Borel parameter \( T \) for \( \mu = 1 \text{ GeV} \) (dashed lines), \( 2 \text{ GeV} \) (solid lines) and \( 4 \text{ GeV} \) (dotted lines). The corresponding values of the continuum threshold are \( s_0 = 1.5 \pm 0.2 \text{ GeV} \), \( 1.7 \pm 0.2 \text{ GeV} \) and \( 1.9 \pm 0.2 \text{ GeV} \) respectively. The central values have been chosen in order to obtain maximal stability for the sum rule.

Larger values of \( s_0 \) always increase \( E \) but at the same time the stability region shrinks and goes to smaller values of \( T \). However, even at \( T = 0.7 \text{ GeV} \) the influence of the higher resonances expressed through the continuum model \( \chi^-(T) \) is very large: \( \chi^-(0.7)/D_{FT}(0.7) \approx 0.75 \). For \( s_0 = 2.1 \text{ GeV} \), we have a small stability region around \( T = 0.65 \text{ GeV} \) yielding \( E^- = 1.6 \text{ GeV} \). Here the influence of the continuum model is around tolerable 50\%, the perturbative corrections and the choice of the renormalisation scale become however more important there.

Another source of uncertainty is the value of the gluon condensate. For the value \( \langle a_{FF} \rangle = 0.012 \text{ GeV}^4 \), originally obtained by [1], we find \( E^- = 1.2 \text{ GeV} \) at \( s_0 = 1.5 \text{ GeV} \), whereas for \( \langle a_{FF} \rangle = 0.036 \text{ GeV}^4 \) we obtain \( E^- = 1.8 \text{ GeV} \) at \( s_0 = 2.4 \text{ GeV} \). We therefore conclude from the sum rules for the above mentioned
Figure 1: The energy $E^−$ as a function of the Borel-parameter $T$ for three different renormalisation scales $\mu$ and continuum thresholds $s_0$. Dashed curves $\mu = 1\, \text{GeV}$: lowest $s_0 = 1.3\, \text{GeV}$, middle $s_0 = 1.5\, \text{GeV}$, upper $s_0 = 1.7\, \text{GeV}$. Solid curves $\mu = 2\, \text{GeV}$: lowest $s_0 = 1.5\, \text{GeV}$, middle $s_0 = 1.7\, \text{GeV}$, upper $s_0 = 1.9\, \text{GeV}$. Dotted curves $\mu = 4\, \text{GeV}$: lowest $s_0 = 1.7\, \text{GeV}$, middle $s_0 = 1.9\, \text{GeV}$, upper $s_0 = 2.1\, \text{GeV}$.

parameters an energy $E^−$ and a correlation length $a^−$ of:

$$E^−_{3fl} = 1.5 \pm 0.4\, \text{GeV} \quad \text{and} \quad a^−_{3fl} = 0.13_{-0.02}^{+0.05}\, \text{fm}.$$  \hspace{1cm} (4.4)

The main sources of uncertainty are the value of the gluon condensate and the continuum contribution. Though the perturbative two-loop contributions to the sum rule are very large, their influence on the value of $E^−$ is not so dramatic. The corrections tend to cancel in the ratio of eq. (1.2). If one determines the energy from the sum rule just containing the lowest order perturbation theory and chooses as the scale for $\alpha_s$ the approximate value of the energy one finds for $\langle aFF \rangle = 0.024\, \text{GeV}^4$ the value $E^− = 1.9\, \text{GeV}$.

In a world without light quarks, i.e. $n_f = 0$, the main influence on the sum rule is the expected change of the gluon condensate which might increase by a factor two to three [27]. If we perform an analysis as above, we get for $\Lambda_{0fl} = 250\, \text{MeV}$ [28], $\langle aFF \rangle = 0.048 \pm 0.024\, \text{GeV}^4$ and $s_0 = 2.3\, \text{GeV}$ an energy
and correlation length of

\[ E_{0\text{fl}} = 1.9 \pm 0.5 \text{ GeV} \quad \text{and} \quad a_{0\text{fl}} = 0.11^{+0.04}_{-0.02} \text{ fm}. \] (4.5)

For \( E^+ \), the energy of the axial vector state, we obtain no stable sum rule. Although the expressions for \( E^- \) and \( E^+ \) are equal in lowest order perturbation theory, higher order perturbative contributions and the gluon condensate lead to a splitting in such a way that for the same values of \( s_0 \) and \( T \) the resulting value for \( E^- \) is higher than that for \( E^+ \).

## 5 Summary and conclusions

The analysis of the gauge invariant gluon field strength correlator by QCD sum rule methods allows to establish a relation between the gluon condensate and the correlation length. In order to apply the sum rule technique which consists in the comparison of a phenomenological Ansatz with a theoretical expression obtained from the operator product expansion we interpret the gluon correlator as the correlator of two colour neutral hybrid states composed of a (fictitious) heavy quark transforming under the adjoint representation and the gluon field. The former serves as the source for the gauge string in the correlator.

In this approach the decomposition in two invariant functions \( D^+ \) and \( D^- \) is more appropriate than the decomposition of eq. (2.2), since \( D^- \) receives only contributions from \( 1^- \) and \( D^+ \) from \( 1^+ \) intermediate states (ignoring the decoupled spin of the heavy octet quark). Therefore these functions show simple exponential behaviour at large distances and not \( D \) and \( D_1 \). The perturbative expressions for \( D^+ \) and \( D^- \) are nearly degenerate, but the gluon condensate contributes with different sign. It stabilises the sum rule for \( D^- \) and destabilises for \( D^+ \).

The value of the binding energy for the lowest intermediate \( 1^- \) state (the inverse correlation length of the correlator) with three flavours is determined to be \( E_{3\text{fl}} = 1/a_{3\text{fl}} \approx 1.5 \pm 0.4 \text{ GeV} \) and with zero flavours to be \( E_{0\text{fl}} = 1/a_{0\text{fl}} \approx 1.9 \pm 0.5 \text{ GeV} \). The main sources of uncertainty are the choice of the continuum threshold \( s_0 \) and the value of the gluon condensate.

Though we find no stable sum rule for the axial vector state we have from the difference of the expressions for the \( 1^- \) and \( 1^+ \) state strong evidence for the counterintuitive result that the \( 1^+ \) state is lighter than the vector state.

The gauge invariant gluon correlator has been calculated on the lattice using the cooling technique \([19, 20]\). There, the analysis has been made by assuming
at large distances an exponential behaviour for the invariant functions $D$ and $D_1$, which in light of the present investigation seems less justified than the same Ansatz for the functions $D^+$ and $D^-$. The results of the lattice calculation are in qualitative, but not quantitative agreement with the sum rule results. The lattice researchers find correlation lengths for $D$ and $D_1$, $a$ and $a_1$, which are degenerate within the errors. The computations have been done in quenched QCD and with four dynamic flavours of staggered fermions at a bare quark mass of $d \cdot m_q = 0.01$ where $d$ denotes the lattice spacing. They found \[20\]:

$$E^- = E^+ = \frac{1}{a} = 0.90 \pm 0.14 \text{ GeV} \quad \text{for 0 flavours and}$$

$$E^- = E^+ = \frac{1}{a} = 0.58 \pm 0.10 \text{ GeV} \quad \text{for 4 flavours.} \quad (5.1)$$

A preliminary analysis of the lattice data based on an exponential behaviour for $D^+$ and $D^-$ \[29\] leaves the values essentially unchanged but indicates a splitting of $E^+$ and $E^-$ in the same direction as proposed by the sum rules! The reader should also note the increase of the correlation length from zero to four flavours which is predicted by the sum rules as well where it is mainly due to the decrease of the gluon condensate.

In another approach \[21\] the exponential behaviour of the functions $D^+$ and $D^* = z^2 \partial/\partial z^2 D_1$ for quenched QCD could be extracted by analysing field insertions into a Wilson loop and assuming factorisation as in the model of the stochastic vacuum \[6, 7\]. The resulting values for the correlation lengths are smaller than those of the direct lattice calculations \[21\] and thus compare more favourably with our results:

$$E^+ = \frac{1}{a^+} = 1.64 \text{ GeV} \quad \text{and} \quad E^* = \frac{1}{a^*} = 1.04 \text{ GeV} \quad \text{for 0 flavours.} \quad (5.2)$$

The sum rule analysis shows that the state investigated here namely a gluon confined by an octet source has a much higher energy than the corresponding state in HQET. A similar analysis of a light quark bound by a source in the fundamental representation \[30\] yielded an energy which is by a factor 2 to 4 smaller. This is to be expected on general grounds \[27\] since the case treated here is nearer to a glueball than to a heavy meson.
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Appendix

The theoretical expression for the perturbative correlator up to next-to-leading order is given by:

\[
\bar{D}_{PT}(w) = (-w)^3 \left[ a(p_{10}^± + p_{11}^± L) + a^2 (p_{20}^± + p_{21}^± L + p_{22}^± L^2) \right],
\]

where \( L = \ln(-2w/\mu) \) and the \( p_{ij}^± \) have the values

\[
\begin{align*}
p_{10}^- &= \frac{40}{3} \\
p_{11}^- &= -16 \\
p_{20}^- &= \frac{2839}{9} + 18\pi^2 - 96\zeta(3) + \left( -\frac{364}{27} - \frac{4\pi^2}{9} \right) n_f \\
p_{21}^- &= -\frac{692}{3} - 16\pi^2 + \frac{104}{9} n_f \\
p_{22}^- &= 44 - \frac{8}{3} n_f \\
p_{10}^+ &= -\frac{128}{3} \\
p_{11}^+ &= 32 \\
p_{20}^+ &= -\frac{5684}{9} + 44\pi^2 + 192\zeta(3) + \left( \frac{848}{27} + \frac{8\pi^2}{9} \right) n_f \\
p_{21}^+ &= \frac{1072}{3} + 32\pi^2 - \frac{208}{9} n_f \\
p_{22}^+ &= -40 + \frac{16}{3} n_f.
\end{align*}
\]

For the convenience of the reader we also give the definition of the Borel transformation and some useful formulæ:

\[
\bar{B}_T = \lim_{-w,n \to \infty} \left( \frac{-w}{\Gamma(n+1)} \left( \frac{d}{dw} \right)^n T \right), \quad T = \frac{-w}{n} > 0 \text{ fixed}
\]

\[
\int \frac{1}{(E - w - i\epsilon)^\alpha} = \frac{1}{\Gamma(\alpha) T^{\alpha-1}} e^{-E/T}.
\]
Below, we have collected some formulae for the incomplete Gamma function which are helpful for the numerical analysis of the sum rules:

\[ \Gamma(\alpha, x) = \int_x^\infty e^{-t}t^{\alpha-1}dt \]

\[ \Gamma(n, x) = \Gamma(n) e^{-x} \sum_{k=0}^{n-1} \frac{x^k}{k!}, \quad n = 1, 2, ... \]

\[ \Gamma'(\alpha, x) \equiv \frac{\partial}{\partial \alpha} \Gamma(\alpha, x) \]

\[ \Gamma'(\alpha) - \Gamma'(\alpha, x) = \int_0^x e^{-t}t^{\alpha-1} \ln t \, dt \]

\[ \Gamma'(4) - \Gamma'(4, x) = 11 - 6\gamma_E - 6\Gamma(0, x) - e^{-x} \left(11 + 5x + x^2 + \left(6 + 6x + 3x^2 + x^3\right) \ln x\right) \]

\[ \Gamma'(5) - \Gamma'(5, x) = 50 - 24\gamma_E - 24\Gamma(0, x) - e^{-x} \left(50 + 26x + 7x^2 + x^3 + \left(24 + 24x + 12x^2 + 4x^3 + x^4\right) \ln x\right). \]
References

[1] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B147 (1979) 385, 448.

[2] M. B. Voloshin, Nucl. Phys. B 154 (1979) 365.

[3] H. Leutwyler, Phys. Lett. B 98 (1981) 447.

[4] Yu. A. Simonov, Sov. J. Nucl. Phys. 50 (1988) 878.

[5] H. G. Dosch, Prog. Part. Nucl. Phys. 33 (1994) 121.

[6] H. G. Dosch, Phys. Lett. 190 B (1987) 177.

[7] H. G. Dosch and Yu. A. Simonov, Phys. Lett. 205 B (1988) 339.

[8] O. Nachtmann and A. Reiter, Z. Phys. C 24 (1984) 283.

[9] P. V. Landshoff and O. Nachtmann, Z. Phys. C 35 (1987) 405.

[10] A. Krämer and H. G. Dosch, Phys. Lett. B 252 (1990) 669.

[11] H. G. Dosch, E. Ferreira, and A. Krämer, Phys. Rev. D 50 (1994) 1992.

[12] D. Gromes, Phys. Lett. 115 B (1982) 482.

[13] M. Campostrini, A. Di Giacomo, and S. Olejnik, Z. Phys. C 31 (1986) 577.

[14] A. Krämer, H. G. Dosch, and R. A. Bertlmann, Fortsch. Phys. 40 (1992) 93.

[15] Yu. A. Simonov, S. Titard and F. J. Yndurain, Phys. Lett. B 354 (1995) 435.

[16] M. Schiestl and H. G. Dosch, Phys. Lett. B209 (1988) 85.

[17] Yu. A. Simonov, Nucl. Phys. B 324 (1989) 67.

[18] M. Eidemüller and M. Jamin, Phys. Lett. B416 (1998) 415.
[19] A. Di Giacomo, E. Meggiolaro, and H. Panagopoulos, *Nucl. Phys. B* **483** (1997) 371.

[20] M. D’Elia, A. Di Giacomo, and E. Meggiolaro, *Phys. Lett. B* **408** (1997) 315.

[21] G. S. Bali, N. Brambilla, and A. Vairo, *Phys. Lett. B* **421** (1998) 265.

[22] U. Ellwanger, *Field strength correlator and infrared fixed point of the Wilsonian exact renormalisation group equations*, LPTHE-ORSAY **98-48**, [hep-ph/9807380](http://arxiv.org/abs/hep-ph/9807380).

[23] M. Neubert, *Phys. Rep.* **245** (1994) 259.

[24] M. Eidemüller, *Diploma thesis, Heidelberg University* (1997).

[25] K. Wilson, *Phys. Rev.* **179** (1969) 1499.

[26] M. Jamin and M. Münz, *Z. Phys. C* **66** (1995) 633.

[27] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, *Nucl. Phys. B* **191** (1981) 301.

[28] M. Lüscher, *Lectures given at the Les Houches Summer School 1997* DESY 98-017, [hep-lat/9802029](http://arxiv.org/abs/hep-lat/9802029).

[29] E. Meggiolaro, *private communications* .

[30] E. Bagan, P. Ball, V. M. Braun, and H. G. Dosch, *Phys. Lett. B* **278** (1992) 457.