Interference phenomena in scalar transport induced by a noise finite correlation time

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Abstract

The role played on the scalar transport by a finite, not small, correlation time, $\tau_u$, for the noise velocity is investigated, both analytically and numerically. For small $\tau_u$'s a mechanism leading to enhancement of transport has recently been identified and shown to be dominating for any type of flow. For finite non-vanishing $\tau_u$'s we recognize the existence of a further mechanism associated with regions of anticorrelation of the Lagrangian advecting velocity. Depending on the extension of the anticorrelated regions, either an enhancement (corresponding to constructive interference) or a depletion (corresponding to destructive interference) in the turbulent transport now takes place.

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The understanding of the mechanisms leading to transport enhancement/depletion is of great interest in various domains of science [1]. In the theory of scalar transport, the main regime of interest corresponds to the long-time behavior of the concentration field (e.g. the density of passive particles injected into the flow) for time scales much longer than the characteristic time-scale of the advecting velocity field. In this case, long-wave disturbances governs the slow modes of concentration and the asymptotic dynamics is described by a diffusive equation with renormalized diffusion coefficients, $D_{\alpha\beta}^E$, namely:

$$\partial_t \langle \theta \rangle = D_{\alpha\beta}^E \frac{\partial^2}{\partial x_{\alpha} \partial x_{\beta}} \langle \theta \rangle$$  (1)
where $\langle \theta \rangle$ is the concentration field averaged locally over a volume of linear dimensions much larger than the typical length $l_0$ of the advecting (incompressible) velocity field. The latter equation can be rigorously proved by using, for instance, asymptotic methods like multiscale techniques \[2\]. From the Lagrangian viewpoint, the large-scale diffusion regime is a consequence of the central limit theorem, stating (roughly) that, for particle displacements weakly correlated, their sum tends to a Gaussian and, as an immediate consequence, particles undergo diffusion.

An exact (i.e. non-perturbative) expression of the Green-Kubo type obtained by Taylor \[3\] allows one to express the effective diffusivities as time-integrals of Lagrangian correlations. For Lagrangian chaotic flow the correlations decay rapidly, the integral converges and gives a finite effective diffusivity $D_E$.

In many geophysical domains (for instance, to study the dispersion processes either in atmospheric or oceanic flows), Lagrangian trajectories, $x(t)$, of particles advected by a prescribed velocity field, $U(x(t))$, and plunged in a noise, small-scale velocity, $u(t)$, are properly modelled by the Langevin equation:

$$\frac{dx(t)}{dt} = U(x(t)) + u(t) \tag{2}$$

where the small-scale turbulence, $u(t)$, is described by a Markovian process with finite correlation time, $\tau_u$, rather than by a white-noise process, on account of experimental observations \[4, 5\] suggesting the presence of some form of finite correlation for the smallest scales of motion. In the following, the advecting velocity field $U(x(t))$ will be assumed incompressible, the noise term $u(t)$ a Gaussian random process having zero average value and the colored-noise correlation function given by:

$$\langle u_\alpha(t) u_\beta(t') \rangle = \frac{D^e}{\tau_u} \delta_{\alpha \beta} \delta e^{-\frac{|t-t'|}{\tau_u}} , \tag{3}$$

where $D^e$ is the isotropic eddy-diffusivity arising from the smallest scales of turbulent motion (not explicitly solved) and $\tau_u$ is their correlation time.

The main question addressed in the present Letter, concerns the role played by the finite (not necessarily small) correlation time, $\tau_u$, of the noise velocity in the large-scale dynamics and thus in the behavior of the effective diffusivity, $D_E$. Specifically, we are interested in the difference

$$I(D^e, \tau_u) \equiv D_E(D^e, \tau_u) - D_E(0, 0) - D^e \tag{4}$$

for small, fixed, $D^e$ and different values of $\tau_u$. The argument of $D_E$ is meant to stress that the eddy-diffusivity depends on both $D^e$ and $\tau_u$ due to the dependence of Lagrangian
trajectories on the colored-noise, as a whole. Physically, positive (negative) values of $I$ correspond to constructive (destructive) interference induced by the dependence of Lagrangian trajectories on the colored-noise.

In Refs. [6] and [7] the interference problem was investigated focusing the attention on the variation of the sign of $I$ for small $D^E$ and $\tau_u = 0$ (i.e. for white-in-time processes). It was conjectured in Ref. [6] that the interference process should always be destructive, and actually shown in Ref. [7] that it can be either constructive or destructive, depending on the extension of the anticorrelated regions of the velocity field.

The effect of a small $\tau_u$ on the variation of $I$ was investigated in Ref. [8]. Accordingly, splitting $I$ into two distinct parts, $I = I_{D^E} + I_{\tau_u}$, the first one being the interference contribution of $D^E$ when $\tau_u = 0$, and the second the correction due to a non-vanishing $\tau_u$, it was shown perturbatively in the parameter $\tau_u/\tau_U \ll 1$, that $I_{\tau_u} > 0$ for any type of flow, $\tau_U$ being the correlation time of the advecting component $U$. Physically, the mechanism leading to positive interference, and thus to enhanced diffusion, was identified in the augmentation of coherence in the diffusion process induced by the small correlation time $\tau_u$. Indeed, the latter makes the particles of the diffusing substance to forget less rapidly their past than in the presence of white-noise turbulence. This fact increases the Lagrangian correlation time and therefore the eddy-diffusivity.

In this Letter, we want to remark that the above mechanism is certainly present for all values of $\tau_u$, but when the latter is large enough another effect can be even stronger. To show this, let us consider the case when the Lagrangian correlation function presents negative correlated regions. The contribution given by such regions in the integral along the trajectories required in order to obtain the eddy-diffusivities is negative and a reduction of the diffusion may occur. The role played by $\tau_u$ is thus twofold. On one hand, according to the results of Ref. [8], it permits the particles to go away from their initial positions more slowly. This effect increases the Lagrangian correlation time and therefore $D^E$. On the other hand, it works to maintain the particles inside the trapping zones where the velocity is anticorrelated. The slowing-down due to these anticorrelated regions is thus enhanced and diffusion is reduced. Due to the well-known sensitivity of transport to slow parts of the trajectories, the latter mechanism can become stronger than the former and a reduction in the transport may occur (i.e. $I_{\tau_u} < 0$).

In the following we show that the destructive effect associated with anticorrelated regions can be stronger than the constructive effects. To that end, in order to simplify the problem, let us consider a very simple model for the advecting velocity, $U$, whose stream function is defined as:

$$\psi(y, t) = \psi_0 \left[ \alpha(t) \cos(k_0 y) + \beta(t) \sin(k_0 y) \right].$$

For this 2-d random parallel flow, as in Ref. [7], the (random) processes $\alpha(t)$ and $\beta(t)$ are
assumed Gaussian, independent and with auto-correlation function given by
\[ \mathcal{C}(t) = e^{-|t|/\tau_U} \cos(\omega t) \tag{6} \]
where \( \tau_U \) is the correlation time of \( U \) and \( \omega \) is the parameter controlling the extension of anticorrelated regions. The form of (6) permits us to obtain a finite and non-vanishing effective diffusivity also for \( D^e = 0 \) and, moreover, represents the simplest way to introduce recirculating zones where the scalar tends to be homogenized and trapped.

The following example of cellular flow defined as
\[ U(x, y) = U_0(\cos(y), \cos(x)), \tag{7} \]
the well-known ‘BC’ flow \[9, 10, 11\] whose streamlines form a closed structure made of four cells in each periodicity box, shows indeed that auto-correlation functions like (6) are associated to recirculation. In order to show this point, we have integrated the Langevin equation by using a second-order Runge–Kutta scheme, with the advecting velocity field given by eq. (7) superimposed to a white-in-time noise. The averages necessary to evaluate the auto-correlation function \( C(t) = \langle U(x(t)) \cdot U(x(0)) \rangle / U^2 \), \( U^2 = \langle U \cdot U \rangle \) is the energy of the advecting velocity field) have been made over different realizations and performed by uniformly distributing \( 10^6 \) particles in the four cells of the periodicity box. The obtained auto-correlation function is shown in Fig. 1. The qualitative resemblance between the BC auto-correlation function and that given by the simple expression (6) makes meaningful the use of the latter to mimic the statistical effect of recirculation in the diffusion process. Notice that, although possible, the main disadvantage in studying the interference problem by using the BC flow is that only numerical analysis seems to be feasible. This is not the case for the random parallel flow \[5\] with the auto-correlation (6), where the interference mechanisms can be investigated analytically. To that end, we use asymptotic methods since we are interested in the dynamics of the field \( \theta \) on scales of the order of \( L \), larger than \( l_0 \), the typical scale of the advecting velocity field, \( U \). The ratio \( l_0/L \sim O(\epsilon) \) with \( \epsilon \ll 1 \), the parameter controlling the scale separation, naturally suggests to look for a perturbative approach. The perturbation is however singular \[12\] and multiscale techniques \[2\] are thus needed to treat the singularities. For the random parallel flow, the auxiliary equation (a partial differential equation which needs to be solved to obtain the eddy-diffusivity) can be tackled analytically and a simple expression for the component of the effective diffusivity tensor parallel to the direction of the velocity \( U \) (the normal component being equal to \( D^e \)) can be obtained \[13\] and its expression reads:
\[ D^E(D^e, \tau_U) = D^e + u_0^2 \int_0^\infty \mathcal{C}(t) \left\{ e^{-k_0^2 D^e \tau_U} \left[ e^{-\frac{t}{\tau_U}} \left( e^{-\frac{t}{\tau_U}} - 1 \right) \right] \right\} dt, \tag{8} \]
where \( u_0^2 \equiv \psi_0 k_0 \) and \( \mathcal{C}(t) \) has a typical correlation time \( \tau_U \) as in (6).
The velocity auto-correlation function $\langle U(x(t)) \cdot U(x(0)) \rangle / U^2$ as a function of the time $t$ for the ‘BC’ flow (7). The curve has been obtained by integrating, with a second-order Runge–Kutta scheme, the Langevin equation for the coordinate of scalar particles in the presence of a white-in-time noise.

The two opposite limits $\tau_u / \tau_U \ll 1$ and $\tau_u / \tau_U \gg 1$ can be easily investigated from (8). About the former, it is immediately checked that the results obtained in Ref. [8] are here reproduced by expanding at the first order in both $\tau_u$ and $D^e$ the term inside the curly brackets in the r.h.s. of (8). As a result, the interference term $I_{\tau_u}$ reads:

$$I_{\tau_u} = D^e k_0^2 u_0^2 \tau_u \int_0^\infty \mathcal{C}(t) \, dt \quad .$$

(9)

The second limit can be easily exploited by expanding the same term inside the curly brackets but now for $\tau_u / \tau_U \gg 1$. The expression for $D^E$ corrected at the order $1/\tau_u$ is thus obtained:

$$D^E = D^e + u_0^2 \int_0^\infty \mathcal{C}(t) \left[ 1 - \frac{k_0^2 D^e \tau_u}{2} \left( \frac{t}{\tau_u} \right)^2 \right] \, dt \quad ,$$

(10)

from which the interference term $I$ immediately follows:

$$I = -\frac{u_0^2 k_0^2 D^e}{2 \tau_u} \int_0^\infty t^2 \mathcal{C}(t) \, dt \quad .$$

(11)

It is easy to verify from (8) that, for both $\tau_u = 0$ and small $D^e$ (i.e. $D^e k_0^2 \tau_U \ll 1$), the interference term associated to $D^e$ is given by:

$$I_{D^e} = -D^e u_0^2 k_0^2 \int_0^\infty t \mathcal{C}(t) \, dt \quad .$$

(12)
and thus, from (11), it follows:

\[ I_{\tau_u} = -u_0^2 k_0^2 D^e \left( \frac{1}{2\tau_u} \int_0^\infty t^2 C(t) \, dt - \int_0^\infty t \, C(t) \, dt \right). \tag{13} \]

For large \( \tau_u \)'s, the character of the interference due to the effect of \( \tau_u \) on the Lagrangian trajectories is thus related to the variation of \( I_{\tau_u} \) with \( \tau_u \).

By inserting the expression (6) for \( C(t) \) into (13) we obtain, after simple integrations, the following condition for destructive interference:

\[ I_{\tau_u} < 0 \quad \text{when} \quad \omega \tau_U > 1. \tag{14} \]

According to our previous discussions, destructive interference appears when the extension of anticorrelated regions (controlled by \( \omega \)) is large enough. Such phenomenon, found in the limit of large \( \tau_u \)'s, is actually present for a wide range of values of \( \tau_u \)'s provided that \( \tau_u/\tau_U \ll 1 \).

To show this point, we have computed numerically the integral in the r.h.s. of (8) from which we have subtracted the contribution \( I_{D^e} + D^E(0,0) \) to obtain \( I_{\tau_u} \). The behavior of \( I_{\tau_u} \) as a function of \( \omega \tau_U \) for \( \tau_u/\tau_U = 0.05 \) (long-dashed line), 0.5 (dashed), 1.0 (dotted) and 5.0 (dot-dashed) is shown in Fig. 2 for \( k_0^2 D^e \tau_U = 0.05 \) and \( u_0^2 = 1 \). From this figure we observe that the smaller \( \tau_u/\tau_U \), the larger the values of \( \omega \tau_U \) at which the crossover from
constructive to destructive interference takes place. For all \( \tau_u \)'s, we can then conclude that the condition (14) should be replaced by:

\[
I_{\tau_u} < 0 \quad \text{when} \quad \omega \tau_U > K_c
\]

with \( K_c \) depending on \( \tau_u/\tau_U \), \( K_c \geq 1 \) (on account of (14)) and \( \lim_{\tau_u \to 0} K_c = \infty \) (on account of the positive character of the interference when \( \tau_u \to 0 \)). Condition (15) is corroborated by the results of Fig. 3, where the values, \( K_c \)'s, of \( \omega \tau_U \) at which the crossover takes place, have been plotted for different values of \( \tau_u/\tau_U \).

![Figure 3: Values \( K_c \) of \( \omega \tau_U \) corresponding to the crossover from constructive to destructive interference (i.e. from \( I_{\tau_u} > 0 \) to \( I_{\tau_u} < 0 \)), as a function of the ratio \( \tau_u/\tau_U \). Values of \( K_c \) reported in figure have been obtained by numerical computation of integral (8) with the same parameters of Fig. 2.](image)

In conclusion, the effect on the scalar transport due to a finite correlation time \( \tau_u \) of the noise velocity has been investigated. We have shown, both analytically and numerically, that for \( \tau_u \) large enough two mechanisms exist and act in competition to determine the interference character. The first mechanism works to enhance the transport due to an augmentation of coherence induced in the diffusion process. The second mechanism maintains particles inside the zones of trapping (where the auto-correlation of the advecting velocity field is negative) for a time longer than the time relative to the white-noise case. Depending on the extension of anti-correlated regions, constructive or destructive interference take place. For finite values of \( \tau_u \), the symmetry with the scenario presented in Ref. [7] is thus perfectly restored.
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References

[1] H.K. Moffatt, Rep. Prog. Phys., 46, 621 (1983).

[2] A. Bensoussan, J.-L. Lions and G. Papanicolaou, Asymptotic Analysis for Periodic Structures (North-Holland, Amsterdam, 1978)

[3] G.I. Taylor, Proc. Lond. Math. Soc. Ser. 2, 20, 196 (1921).

[4] R.E Davis, J. Geophys. Res., 90, 4756 (1985).

[5] P.M. Poulain and P.P. Niiler, J. Phys. Oceanogr., 19, 1588 (1989).

[6] P.G. Saffman, J. Fluid Mech., 8, 273 (1960).

[7] A. Mazzino and M. Vergassola, Europhys. Lett., 37, 535 (1997).

[8] P. Castiglione and A. Mazzino, Europhys. Lett., 43, 522 (1998).

[9] V.I. Arnold, C.R. Acad. Sci. Paris A, 261, 17 (1965).

[10] M. Hénon, C. R. Acad. Sci. Paris A, 262, 312 (1966).

[11] T. Dombre, U. Frisch, J. M. Greene, M. Hénon, A. Mehr and A.M. Soward, J. Fluid Mech., 167, 353 (1986).

[12] C.M. Bender and S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers (McGraw–Hill, 1978).

[13] P. Castiglione and A. Crisanti, “Dispersion of passive tracers for a Langevin equation with non delta-correlated noise”, submitted to Phys. Rev. E, (1998).