Asymptotic symmetry and conservation laws in 2d Poincaré gauge theory of gravity

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Abstract. The structure of the asymptotic symmetry in the Poincaré gauge theory of gravity in 2d is clarified by using the Hamiltonian formalism. The improved form of the generator of the asymptotic symmetry is found for very general asymptotic behaviour of phase space variables, and the related conserved quantities are explicitly constructed.
1. Introduction

General relativity is a successful theory of macroscopic gravitational phenomena, but all attempts to quantize the theory encounter serious difficulties [1]. It seems natural to try to understand the structure of gravity on the basis of the concept of gauge symmetry, which has been very successful in describing other fundamental interactions in nature. Poincaré gauge approach to gravity leads to a linear connection with torsion and the tetrad field as independent variables [2]. The investigation of the general action of this type, which is at most quadratic in the curvature and the torsion, shows that the theory does not contain ghosts or tachions, but the power–counting renormalizability is unfortunately lost [3].

One expects that investigations of two–dimensional theories of gravity may provide a better understanding of quantum properties of higher dimensional gravity, as well as a deeper understanding of string theory. The presence of torsion in string theory considerably modifies the theory and changes its dynamical content and quantum properties [4]. The investigation of classical solutions of the two–dimensional gravity with dynamical torsion in the conformal gauge shows that the system is completely integrable [4]. In the light–cone gauge, which is more appropriate for the separation of true dynamical degrees of freedom, general analytic solutions of the classical equations of motion are found [5]. The origin of the classical integrability is traced back to a specific local symmetry of the theory found in the Hamiltonian formalism [6]. It is interesting to note that the first order formulation of the theory coincides with the gauge theory based on the nonlinear extension of the Poincéré algebra [7]. Both classical structure of the theory and its quantum properties show very interesting features [8,9], which might be helpful in our attempts to understand more realistic four–dimensional theory.

The Hamiltonian approach was very useful in clarifying symmetry properties of the four–dimensional Poincéré gauge theory [10–12]. In the present paper we shall use this method to analyse the local symmetries of the Poincéré gauge theory in 2d, and study the structure of the symmetry in the asymptotic region. The correct definition of the asymptotic generators is important for several reasons: it can be used to study the related conservation laws, to find an explicit form of the conserved quantities, and to study the stability problem.

We begin our consideration in section 2 by developing the basic Hamiltonian formalism, and constructing the generator of the local symmetries. Then we clarify the important notion of the asymptotic symmetry, which is determined by the behaviour of phase space variables in the asymptotic region, and discuss the question of surface terms and the conservation laws. In sections 3 and 4 we use this formalism to study the asymptotic structure of the theory when the solutions in the asymptotic region
define the space–time of constant curvature and vanishing torsion, and the Minkowski space–time, respectively. These specific examples motivate the central consideration in sections 5 and 6, where we derive general rules for constructing the asymptotic generators and the conserved quantities under rather general asymptotic conditions. We find that the conserved quantities are expressed in terms of the constant of motion $Q$, known from earlier investigations [6]. Section 7 is devoted to concluding remarks. Some technical details are displayed in Appendices.

2. Hamiltonian structure of Poincaré gauge theory in 2d

**Classical action.** The basic dynamical variables of this theory are the diad $b^a_\mu$ and the connection $A^{ab}_\mu$, associated with the translation and Lorentz subgroup of the Poincaré group, respectively. Here, $a, b, \ldots = 0, 1$ are the local Lorentz indices, while $\mu, \nu, \ldots = 0, 1$ are the coordinate indices. The structure of the Poincaré group is reflected in the existence of two kinds of gauge field strengths: the torsion $T^a_{\mu \nu}$, and the curvature $R^{ab}_{\mu \nu}$. The geometrical structure of the theory corresponds to the Riemann–Cartan geometry $U_2$. The most general action of the Poincaré gauge theory in 2d, which is at most quadratic in gauge field strengths, has the form [4]:

$$I = \int d^2 x b \left( \frac{1}{16 \alpha} R^{ab}_{\mu \nu} R^{\mu \nu}_{ab} - \frac{1}{8 \beta} T^a_{\mu \nu} T^a_{\mu \nu} - \gamma \right),$$  \hspace{1cm} (2.1)

where $b = \text{det}(b^a_\mu)$, and $\alpha, \beta, \gamma$ are constants. The term linear in curvature is dynamically trivial, as it represents a topological invariant. In 2d the Lorentz connection can be parametrized as $A^{ab}_\mu = \varepsilon^{ab} A_\mu$, so that

$$R^{ab}_{\mu \nu} = \varepsilon^{ab} F_{\mu \nu}, \quad F_{\mu \nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu,$$

$$T^a_{\mu \nu} = \nabla_\mu b^a_\nu - \nabla_\nu b^a_\mu,$$

and $\nabla_\mu b^a_\nu = \partial_\mu b^a_\nu + \varepsilon^a_c A_\mu b^c_\nu$ is the covariant derivative of the diad field. The action (2.1) is invariant under the local Poincaré transformations with parameters $\omega^{ab} = \varepsilon^{ab} \omega$ and $\xi^\lambda$:

$$\delta_0 b^a_\mu = \omega \varepsilon^a_c b^c_\mu - \xi^\lambda \mu b^a_\lambda - \xi^\lambda \partial_\lambda b^a_\mu,$$

$$\delta_0 A_\mu = -\partial_\mu \omega - \xi^\lambda \cdot \mu A_\lambda - \xi^\lambda \partial_\lambda A_\mu.$$  \hspace{1cm} (2.2)

The action (2.1) leads to the following equations of motion [4]:

$$\frac{1}{2 \alpha} D_\mu F^{\mu \nu} + \frac{1}{2 \beta} \varepsilon^{ac} T^c_{\mu \nu} = 0,$$

$$\frac{1}{2 \alpha} F_{ac} F^{\mu \nu} + \frac{1}{2 \beta} (T_{ca \nu} T^{\mu \nu} + D_\nu T^a_{\mu \nu}) - h^a_\mu \left( \frac{1}{8 \alpha} F_{bc} F^{b \nu} + \frac{1}{8 \beta} T_{bce} T^{bce} + \gamma \right) = 0.$$

Here, $D_\mu$ denotes the covariant derivative acting on both local Lorentz and coordinate indices.
It is interesting to note that these equations have a solution describing the space of constant curvature and zero torsion, $F = \text{const.}, T_{abc} = 0$. Indeed, in this case the first equation is automatically satisfied, while the second one yields $(1/4\alpha)F_{bc}F^{bc} - 2\gamma = 0$. In particular, $F_{bc} = 0$ for $\gamma = 0$.

**Constraints and the Hamiltonian.** The Hamiltonian analysis of the theory will help us to understand its symmetry properties and clarify the meaning of the conservation laws [13].

Let us denote the momenta conjugate to the basic Lagrangian variables by $\pi^a_\mu$, $\pi^\mu$. Since the curvature and the torsion do not depend on the velocities $\dot{b}_a^0, \dot{A}_0$, one easily finds the primary constraints,

$$
\pi_a^0 \approx 0, \quad \pi^0 \approx 0. \quad (2.3)
$$

The canonical Hamiltonian density has the form

$$
\mathcal{H}_c = b^a_0 G_a - A_0 G, \quad (2.4a)
$$

where

$$
G_a = -\varepsilon_{ab} b_1^b E + (\pi_a^1)' - \varepsilon_{ab} A_1 \pi^{b1},
$$

$$
G = \varepsilon^{ab} b_1^b \pi_a^1 + (\pi^1)', \quad E \equiv \beta \pi^{e1} \pi_e^1 - \alpha \pi^{1} \pi^1 + \gamma, \quad (2.4b)
$$

and prime means the space derivative. The general Hamiltonian dynamics is described by the total Hamiltonian:

$$
H_T = \int dx^1 (\mathcal{H}_c + u^a \pi_a^0 + u^0 \pi^0). \quad (2.5)
$$

We shall now demand that all constraints be conserved during the time evolution of the system governed by the total Hamiltonian. The consistency of the primary constraints yields the relations

$$
\{\pi_a^0, H_T\} = -G_a, \quad \{\pi^0, H_T\} = G,
$$

and we conclude that $G$ and $G_a$ are secondary constraints. Their algebra is of the form

$$
\{G_a, G_b\} = -2\varepsilon_{ab} (\beta \pi^{e1} G_c + \alpha \pi^1 G) \delta, \quad \{G_a, G\} = \varepsilon_a^b G_b \delta, \quad \{G, G\} = 0, \quad (2.6)
$$

so that further consistency conditions are automatically satisfied.

All the constraints of the system are of the first class. Observe the parallel between the algebra (2.6) and the related structure in $d = 4$ [12].

**The gauge generators.** The Hamiltonian theory described above possesses, by construction, the local Poincaré symmetry. The general method for constructing
the generators of local symmetries has been developed by Castellani [11]. The gauge generators of the local Poinceré symmetry have the form

$$\Gamma = \int dx^1 [\dot{\varepsilon}(t)\Gamma^{(1)} + \varepsilon(t)\Gamma^{(0)}],$$

where $\varepsilon(t)$ are arbitrary parameters, phase space functions $\Gamma^{(0)}, \Gamma^{(1)}$ satisfy the conditions

$$\Gamma^{(1)} = C_{PFC},$$
$$\Gamma^{(0)} + \{\Gamma^{(1)}, H_T\} = C_{PFC},$$
$$\{\Gamma^{(0)}, H_T\} = C_{PFC},$$

and $C_{PFC}$ means primary first class (PFC) constraint [11]. Here, the equality sign denotes an equality up to constants and squares (or higher powers) of constraints.

It is clear that the construction of gauge generators is based on the Poisson bracket algebra of constraints. Starting with a suitable combination of primary first class constraints, one obtains the following form of the gauge generator:

$$\Gamma = \int dx^1 [\dot{\xi}^\mu\Gamma^{(1)}_{\mu} + \xi^\mu P_\mu + \dot{\omega}\Gamma^{(1)} + \omega S], \quad (2.7a)$$

where

$$\Gamma^{(1)}_{\mu} = b^a_{\mu}\pi^0_a + A_\mu\pi^0, \quad \Gamma^{(0)} = \pi^0,$$
$$P_0 = H_T, \quad P_1 = b^a_{1}G_a - A_1G + (b^a_0)\pi^0_a + A_0\pi^0,$$
$$S = -G + \varepsilon^{ab}b_{a0}\pi^0_b. \quad (2.7b)$$

It is easy to check that the action of the generator (2.7a) on the basic dynamical variables gives a good description of the local Poinceré symmetry. This form of the generator will be the starting point for our study of the asymptotic structure of the theory and the related conservation laws.

In order to simplify the notation we shall use $(x^0, x^1) = (\tau, \sigma)$.

**Asymptotic symmetry.** The physical content of the notion of symmetry is determined not only by the symmetry of the action, but also by the symmetry of the boundary conditions.

To clarify this statement let us consider a set of solutions of the field equations which have the following behaviour at large distances:

$$b^a_{\mu} = \tilde{b}^a_{\mu} + O(\tilde{b}^a_{\mu}), \quad A_\mu = \tilde{A}_\mu + O(\tilde{A}_\mu). \quad (2.8)$$

This notation means that $O(\tilde{b}^a_{\mu})/\tilde{b}^a_{\mu}$ tends to zero as $\sigma \to \infty$, and similarly for $A_\mu$. The quantities $\tilde{b}^a_{\mu}$ and $\tilde{A}_\mu$ are asymptotic values of $b^a_{\mu}$ and $A_\mu$ which define an Einstein–Cartan space $\tilde{U}$. Let us, further, assume that the space $\tilde{U}$ has some isometries...
described by the Killing vectors $K_i^\mu \ (i \leq 3 \ in \ d = 2)$. The form–invariance of the metric under the local translation with $\tilde{\xi}^\mu = c^i K_i^\mu \ (c^i = \text{const.})$ does not imply the form–invariance of the diad field $\tilde{b}^a_\mu$; instead, the diad may undergo an additional Lorentz rotation defined by a parameter $-\tilde{\omega}$, which is linear in $c^i$. Thus, there exists a combination of the isometry transformation and a Lorentz rotation by $\tilde{\omega}$, such that $\tilde{b}^a_\mu$ is form–invariant,

$$\tilde{\delta}_0 \tilde{b}^a_\mu = 0, \quad \tilde{\delta}_0 \equiv \delta_0(\tilde{\omega}, \tilde{\xi}^\mu).$$

If the same transformation leaves $\tilde{A}_\mu$ invariant, it represents the symmetry of the space $\tilde{U}$, or the asymptotic symmetry of a given set of solutions.

The choice of the asymptotic behaviour for dynamical variables defines the asymptotic structure of spacetime $\tilde{U}$, which usually represents the ground state or the vacuum of the theory. The symmetry of the action spontaneously breaks down to the symmetry of $\tilde{U}$, which is responsible for the existence of conservation laws.

The existence of isometries of the space $\tilde{U}$ does not guarantee the existence of asymptotic symmetries. If these symmetries exist, they can be obtained from the corresponding local Poincaré transformations by the replacement

$$\xi^\mu(x) \rightarrow \tilde{\xi}^\mu = c^i K_i^\mu, \quad \omega(x) \rightarrow \tilde{\omega}.$$  \hspace{1cm} (2.9a)

The generator of these transformations can be obtained from the gauge generator (2.7) in the same manner. It has the general form

$$\Gamma_{as} = c^i T_i,$$ \hspace{1cm} (2.9b)

where the explicit form of $T_i$ depends on the structure of $\tilde{U}$.

The symmetry transformation of a dynamical variable in the asymptotic region is defined by its Poisson bracket with the generator $\Gamma_{as}$. Therefore, in order to give a precise meaning to the notion of asymptotic symmetry it is necessary to examine the existence of functional derivatives of $\Gamma_{as}$. A functional $G[\varphi, \pi] = \int d\sigma g(\varphi, \partial_1 \varphi, \pi, \partial_1 \pi)$ is said to have well defined functional derivatives if its variation can be written in the form

$$\delta G = \int d\sigma \left[ A(\sigma) \delta \varphi(\sigma) + B(\sigma) \delta \pi(\sigma) \right],$$

where $\delta \varphi,_1$ and $\delta \pi,_1$ are absent. In general this is not the case with the generator $\Gamma_{as}$, so that its form should be improved by adding a nontrivial surface term:

$$\Gamma_{as} \rightarrow \tilde{\Gamma} = \Gamma_{as} + \Omega.$$ \hspace{1cm} (2.10)

The possibility of constructing the improved generator $\tilde{\Gamma}$ depends on the structure of the phase space in which the generator $\Gamma_{as}$ acts.
We now wish to see how the asymptotic symmetry implies the existence of certain conserved quantities. Although Castelani’s method has been originally designed to study local symmetries, it can be also used to obtain useful information about global symmetries. One can easily prove that a phase space functional $G[\varphi, \pi; \tau]$ is the generator of global symmetries if and only if the following relations hold true:

$$\{G, H_T\} + \partial G/\partial \tau = C_{PFC},$$

$$\{G, \phi_s\} \approx 0,$$  \hspace{1cm} (2.11a)

where $\phi_s$ are all constraints in the theory, and both $G$ and $H_T$ are corrected by surface terms, if necessary. As we mentioned before, the equality in the first equation is an equality up to terms which act trivially as symmetry generators, such as constants, squares of constraints and surface terms, so that its left-hand side need not vanish. Note that we restrict ourselves to global symmetries obtained from the corresponding local ones by some process of parameter fixing, so that $G$ and $H_T$ are first class constraints, and their Poisson bracket vanishes weakly. Consequently, the first equation, representing the Hamiltonian form of the conservation law, implies a weak equality

$$dG/d\tau \equiv \{G, H_T\} + \partial G/\partial \tau \approx \partial G/\partial \tau \approx \partial \Omega/\partial \tau,$$  \hspace{1cm} (2.11b)

where $\Omega$ is a possible surface term, so that $G$ is conserved if the time derivative of $\Omega$ vanishes. By calculating $d\tilde{\Gamma}/d\tau$ one can check on the existence of the conserved quantities in the asymptotic region.

The conservation law for the generator $\tilde{\Gamma}$ without surface terms is rather trivial: such generator is given as an integral of constraints, so that its on–shell value always remains zero. **Nontrivial conservation law can be obtained only in the presence of non-vanishing surface terms.** In that case the on–shell value of the conserved quantity is equal to the value of the related surface term: $\tilde{\Gamma} \approx \Omega$.

### 3. Solutions with constant curvature and zero torsion at large distances

We shall first study the asymptotic symmetry when the solutions of the gravitational field equations in the asymptotic region represent the space of constant curvature and zero torsion, $\tilde{V}$.

The symmetries of the Riemann space $\tilde{V}$ can be obtained from the corresponding local Poincaré transformations by the replacement of parameters $\omega(x)$ and $\xi(x)$ as in Eq.(A.9):

$$\xi^\mu(x) \rightarrow c^i K_i^\mu, \quad \omega(x) \rightarrow -\xi^0,1,$$

where $c^i$ are constants and $K_i^\mu$ are the Killing vectors of $\tilde{V}$, defined in (A.7). The generator of these transformations can be obtained from the gauge generator (2.7) in
the same manner. An explicit calculation yields

\[ \Gamma_{as} = c^0 T_0 + c^1 T_1 + c^2 T_2, \quad (3.1) \]

where

\[ T_0 = \int d\sigma P_1, \]
\[ T_1 = \cos \tau \int d\sigma \left[ \cos \sigma \Gamma_0^{(1)} + \sin \sigma (P_1 + \Gamma^{(1)}) \right] + \sin \tau \int d\sigma \left[ \cos \sigma P_0 - \sin \sigma (\Gamma_1^{(1)} - S) \right], \]
\[ T_2 = \cos \tau \int d\sigma \left[ \sin \sigma \Gamma_0^{(1)} - \cos \sigma (P_1 + \Gamma^{(1)}) \right] + \sin \tau \int d\sigma \left[ \sin \sigma P_0 + \cos \sigma (\Gamma_1^{(1)} - S) \right]. \]

In order to examine the existence of well defined functional derivatives of the generator \( \Gamma_{as} \) one should first define the phase space in which the generator acts.

The structure of the space \( \tilde{V} \) is described in Appendix A. Starting from Eqs. (A.8), which describe the form of the diad field and the connection in a suitable coordinate system, we are led to consider solutions of the theory (2.1) with the following asymptotic behaviour:

\[ b^a_\mu = \frac{r}{\sin \tau} \delta^a_\mu + \mathcal{O}_\alpha, \quad A_\mu = \ctg \tau \delta^1_\mu + \mathcal{O}_\beta, \quad (3.2) \]

where \( \mathcal{O}_\alpha \) denotes a term that decreases like \( |\sigma|^{-\alpha} \) or faster for large \( \sigma \), \( \alpha, \beta > 0 \), and \( r = (-4\alpha \gamma)^{-1/4} \).

We shall demand that all expressions that vanish on shell have an arbitrarily fast asymptotic decrease, as no solutions of the equations of motion are thereby lost. Thus, the asymptotic behaviour of the momentum variables will be determined by requiring \( \pi - \partial \mathcal{L}/\partial \dot{\varphi} \sim \dot{\mathcal{O}} \), where \( \dot{\mathcal{O}} \) denotes a term that decreases sufficiently fast, e. g. like \( \mathcal{O}_3 \). From the definition of momenta and the accepted asymptotic behaviour (3.2) of the Lagrangian variables one finds:

\[ \pi_a^0 = \dot{\mathcal{O}}, \quad \pi_a^1 = \mathcal{O}_\alpha + \mathcal{O}_\beta, \]
\[ \pi^0 = \dot{\mathcal{O}}, \quad \pi^1 = \frac{1}{2\alpha r^2} + \mathcal{O}_\alpha + \mathcal{O}_\beta. \quad (3.3) \]

In a similar manner one can determine the asymptotic behaviour of the Hamiltonian multipliers \( u^a \) and \( u \).

After having determined the asymptotic behaviour of all phase space variables, we are now ready to check on the existence of well defined functional derivatives of the generator (3.1). Let us first check on the differentiability of \( T_0 \). By varying \( P_1 \) one finds

\[ \delta P_1 = b^a_1 \delta G_a - A_1 \delta G + \delta b^a_{0,1} \pi_a^0 + \delta A_{0,1} \pi^0 + R \]
\[ = b^a_1 \delta G_a - A_1 \delta G + (\delta b^a_0 \pi_a^0 + \delta A_0 \pi^0),_1 + R. \]
Here, terms that contain unwanted variations are explicitly displayed, while the remaining, regular terms are denoted by $R$. Taking into account the relations $\delta G_a = -\delta \pi_a^{1,1} + R$, $\delta G = \delta \pi^{1,1} + R$, and the asymptotic conditions (3.3), the above relation becomes

$$
\delta \mathcal{P}_1 = -(b^a_1 \delta \pi_a^{1} + A_1 \delta \pi^{1} + \hat{O}),_{1} + R.
$$

After that the integration over $\sigma$ leads to

$$
\delta T_0 = -(b^a_1 \delta \pi_a^{1} + A_1 \delta \pi^{1}),\bigg|_{\sigma \to \pm\infty} + R = R .
$$

(3.4a)

The surface term vanishes since the asymptotic values of all dynamical variables are fixed constants, so that their variations tend to zero. Therefore, the generator $T_0$ has well defined functional derivatives.

In a similar way we find

$$
\delta T_1 = R, \quad \delta T_2 = R .
$$

(3.4b)

Compared to $T_0$, the expressions for $T_1$ and $T_2$ contain additional $\sigma$-dependent terms, but these are of the form $\sin \sigma$ or $\cos \sigma$, i.e. they are bounded, and the structure of surface terms remains unchanged.

Vanishing of surface terms means that $T_0$, $T_1$ and $T_2$ are well defined generators in the asymptotic region. The structure of the asymptotic symmetry is described by the corresponding Poisson bracket algebra. Up to PFC terms one finds:

$$
\{T_0, T_1\} = -T_2, \quad \{T_0, T_2\} = T_1 , \quad \{T_1, T_2\} = T_0 .
$$

(3.5)

Not surprisingly we obtain the $SL(2, R)$ algebra — the symmetry algebra of constant curvature Riemann spaces.

This symmetry implies, as usual, the existence of certain conserved quantities. An explicit calculation with the help of Eq.(2.11) shows that the generators $T_0$, $T_1$ and $T_2$ are conserved quantities. These conservation laws are, however, rather trivial, as all surface terms vanish.

The above analysis of surface terms shows some features which are important for a deeper understanding of the structure of these terms. By a slight generalization of the previous discussion one can reach the following conclusion:

if $a)$ the parameters $\omega(x), \xi^\mu(x)$ are bounded as $\sigma \to \pm \infty$, and

$b)$ the variations of all dynamical variables vanish when $\sigma \to \pm \infty$,

than all surface terms vanish.

(3.6)

In the case of asymptotically flat space we shall relax the condition $a)$ by considering parameters which are linear functions of $\sigma$, and see the consequences on the structure of surface terms. After that we shall study more general case in which none of the conditions $a), b)$ is fulfilled.
4. Asymptotically flat space

We assume that the asymptotic structure of space–time is described by the global Poinceré symmetry. Global Poinceré transformations can be obtained from the corresponding expressions for local transformations by the following replacement of parameters:

\[ \omega(x) \to -\omega, \]
\[ \xi^\mu(x) \to -\omega^\mu x^\nu - \varepsilon^\nu, \]

where \( \omega^{\mu\nu} = \varepsilon^{\mu\nu}\omega \) and \( \varepsilon^\nu \) are constants. After that, the generators of global transformations take the form

\[ \Gamma_{\alpha\beta} = \omega T - \varepsilon^\nu T_\nu, \] (4.2a)

where

\[ T_\mu = \int d\sigma \mathcal{P}_\mu, \]
\[ T = \int d\sigma (\tau \mathcal{P}_1 + \sigma \mathcal{P}_0 - S + b^a_1 \pi^0_a + A_1 \pi^0). \] (4.2b)

We assume that the asymptotic structure of space–time is determined by the following behaviour of diads and connection:

\[ b^a_\mu = \delta^a_\mu + \mathcal{O}_1, \quad A_\mu = \mathcal{O}_1. \] (4.3a)

The first condition describes the rule by which the metric approaches the Minkowskian form, and the second one ensures the absolute paralelism in the asymptotic region.

From the definition of momenta and the relations (4.3a) we find

\[ \pi^0_a = \hat{\mathcal{O}}, \quad \pi^1_a = \mathcal{O}_1, \]
\[ \pi^0 = \hat{\mathcal{O}}, \quad \pi^1 = \mathcal{O}_1. \] (4.3b)

Since the vacuum values of all dynamical variables are constant and the parameters \( \varepsilon^\mu \) multiplying \( T_\mu \) are also constants, it is clear from (3.6) that

\[ \delta T_\mu = R. \] (4.4)

An explicit calculation shows the parallel with Eq.(3.4a).

In the variation of \( T \) the only nontrivial contribution comes from \( \sigma \mathcal{P}_0 \). By observing the relation \( \delta \mathcal{P}_0 = -(b^a_0 \delta \pi^1_a + A_0 \delta \pi^1)_1 + R \), and taking into account the asymptotic behaviour displayed in (4.3) one finds

\[ \delta T = \int d\sigma [\sigma \delta \mathcal{P}_0] + R = -\delta \Omega + R, \]
\[ \Omega \equiv (\sigma \pi^1_0)|_{\sigma \to +\infty}. \] (4.5a)
This result indicates that there might be asymptotically flat solutions for which the surface term $\Omega$ is nonvanishing. However, an inspection of the constraint $G \approx 0$ easily shows that $\pi_0^1$ must decrease as $O_2$, so that

$$\Omega = 0.$$ (4.5b)

The asymptotic conditions (4.3) are not the most general conditions corresponding to the flat space at large distances. It is, therefore, natural to try to find out whether one can change these conditions in a way consistent with the Minkowskian structure in the asymptotic region, and obtain nonvanishing surface terms. The general discussion in the next section will show that this is not possible.

5. General asymptotic structure

We have seen that the conditions $a)$ and $b)$ in (3.6) are of special importance in considerations of the structure of surface terms, as they imply vanishing of these terms. The investigation of the asymptotically flat solutions, which are characterized by linearly rising parameters when $\sigma \to \pm \infty$ and, therefore, violate the condition $a)$, also leads to vanishing surface terms. This situation provides a rationale for trying to understand what happens if the variations of dynamical variables in the asymptotic region do not vanish, i.e. when the condition $b)$ is also violated.

5.1 The light–cone gauge

The generators of the local symmetry (2.7) are constructed so as to act on both physical and unphysical dynamical variables. In order to simplify further discussion we shall fix the gauge and eliminate unphysical variables from the theory. It is clear that this procedure does not change physical properties of the theory.

It is convenient to choose the local Lorentz frame in the form of the light–cone basis:

$$a = (+, -), \quad \eta_{+-} = \eta_{-+} = 1, \quad \epsilon^{+-} = -1, \quad u \cdot v = u_+ v_- + u_- v_+ .$$

The existence of the first–class constraints (2.3) enables us to fix the values of $(b^a_0, A_0)$ by imposing the light–cone gauge conditions:

$$b^+_0 = 1, \quad b^-_0 = 0, \quad A_0 = 0 .$$ (5.1)

The consistency of the gauge conditions implies

$$u^a = 0, \quad u = 0 .$$
After introducing the preliminary Dirac brackets corresponding to the gauge conditions (5.1) and the constraints (2.3), we can easily see that the variables \((b_0^a, A_0, \pi_0^a, \pi_0^0)\) become ignorable: they can be replaced by their values given in Eqs. (2.3) and (5.1), while the Dirac brackets for the remaining variables coincide with the Poisson brackets.

In the light–cone gauge the constraints and the Hamiltonian take the simpler form:

\[
\begin{align*}
G_+ &= -b^{-1}_1 E - \pi_+^1 A_1 - \pi_+^{-1,1}, \\
G_- &= b^+_1 E + \pi_-^1 A_1 - \pi_-^{-1,1}, \\
G &= b^{-1}_1 \pi_-^{-1} - b^+_1 \pi_+^1 + \pi_1^1, \\
H_T &= \int d\sigma G_+, \quad E \equiv 2\beta \pi_+^1 \pi_-^{-1} - \alpha \pi_1^1 + \gamma, 
\end{align*}
\] (5.2)

Observe also that now \(b = -\varepsilon_{ab} b_0^a b_1^b = -b^{-1}_1\), so that the condition of the nondegeneracy of the metric becomes \(b^{-1}_1 \neq 0\).

The equations of motion for the remaining set of variables take the form:

\[
\begin{align*}
\dot{\pi}_+^1 &= 0, \\
\dot{\pi}_1^1 &= \pi_+^1, \\
\dot{b}^{-1}_1 &= -2\beta b^{-1}_1 \pi_+^1, \\
\dot{b}^+_1 &= -2\beta b^{-1}_1 \pi_1^1 - A_1, \\
\dot{A}^+_1 &= 2\alpha b^{-1}_1 \pi_1^1, \\
\dot{\pi}_-^1 &= E.
\end{align*}
\]

The first three equations are easily solved:

\[
\begin{align*}
\pi_+^1 &= A(\sigma), \\
\pi_1^1 &= A(\sigma)\tau + B(\sigma), \\
b^{-1}_1 &= C(\sigma)e^{-2\beta A(\sigma)\tau},
\end{align*}
\]

where \(A(\sigma), B(\sigma)\) and \(C(\sigma)\) are three arbitrary functions. The remaining variables \(\pi_1^1, b^+_1\) and \(A_1\) can be determined from the constraints.

From the form of the local Poincaré symmetry one concludes that the light–cone gauge is preserved if the parameters \(\xi^1, \dot{\omega}\) and \(\xi_0^0 + \omega\) vanish. After that, there remains the residual symmetry defined by parameters

\[
\xi^1 = \varepsilon^1(\sigma), \quad \omega = \varepsilon(\sigma), \quad \xi_0^0 = -\varepsilon(\sigma)\tau + \varepsilon^0(\sigma). \quad (5.3)
\]

The corresponding generator is obtained by replacing these parameters into the general expression (2.7) for \(\Gamma\):

\[
\Gamma = \int d\sigma \left[ \varepsilon^0 \mathcal{P}_0 + \varepsilon^1 \mathcal{P}_1 - \varepsilon \tilde{S} \right], \quad (5.4a)
\]

where

\[
\begin{align*}
\mathcal{P}_0 &= G_+, \\
\mathcal{P}_1 &= b^+_1 G_+ + b^{-1}_1 G_- - A_1 G = -(b^+_1 \pi_+^1,1 + b^{-1}_1 \pi_-^1,1 + A_1 \pi_1^1,1), \\
\tilde{S} &= \tau \mathcal{P}_0 + G.
\end{align*}
\]
To examine the differentiability of $\Gamma$ we calculate $\delta \Gamma$:

$$
\delta \Gamma = \int d\sigma \left[ -\epsilon^0 \delta \pi^+ \right]_1 - \epsilon^1 \left( b^+ \delta \pi^+ + b^- \delta \pi^- + A_1 \delta \pi^1 \right)_1 \\
-\epsilon (\delta \pi^1 - \tau \delta \pi^+)_1 + R = L|_{\pm \infty} + R, \tag{5.5a}
$$

where the surface term $L$ is given by the expression

$$
L \equiv -\epsilon^0 \delta \pi^+ - \epsilon^1 \left( b^+ \delta \pi^+ + b^- \delta \pi^- + A_1 \delta \pi^1 \right) - \epsilon (\delta \pi^1 - \tau \delta \pi^+). \tag{5.5b}
$$

Explicit form of $L$ depends on asymptotic values of both dynamical variables and multipliers. If $L$ can be brought into the form $L = -\delta \Omega$, the improved generator in the asymptotic region will have the form (2.10).

5.2 Asymptotic symmetry

General relation between boundary conditions and the structure of the asymptotic symmetry will be given in two steps.

A. Let us first consider the asymptotic conditions

$$
b^- = a^- + O(a^-), \quad \pi^- = c^- + O(c^-),
$$

$$
b^+ = a^+ + O(a^+), \quad \pi^+ = c^+ + O(c^+),
$$

$$
A_1 = a + O(a), \quad \pi^1 = c + O(c), \tag{5.6}
$$

where $\lambda_\alpha = (a^\mp, a, c^\pm, c)$ are real parameters independent of $\sigma$ and $\tau$ which may take different values as $\sigma \to \pm \infty$, and $O(\lambda)$ are terms that vanish when $\sigma \to \pm \infty$. We assume that the phase space contains a collection of solutions of the type (5.6), where the real parameters $\lambda_\alpha$ are not fixed constants, but belong to certain subsets of real line. This, in particular, means that these parameters can be varied: *the variations of dynamical variables in the asymptotic region do not vanish.*

The restriction to fixed parameters $\lambda_\alpha$ would convert the phase space into a configuration which is usually called vacuum. Here we are working with a phase space consisting of a collection of different vacua. This situation is possible as the notion of gauge symmetry enables us to consider a symmetry transformation that maps a solution of the type (5.6) into another solution of the same type, but with different $\lambda_\alpha$’s.

It is natural to assume that the asymptotic behaviour is restricted by the demand that all the constraints decrease sufficiently fast:

$$
G_+, G_-, G = \hat{O}. \tag{5.7a}
$$

In particular, the values of the parameters $\lambda_\alpha$ are restricted by demanding the consistency with the constraints. Thus, one can take the values of $a^-$, $c_+$ and $c$ arbitrarily,
while $a^+, a$ and $c_-$ are restricted to satisfy the constraints $G, G_+$ and $G_-:
\begin{align}
a^+ c_+ &= a^- c_- , \\
as_+ &= a^- (2 \beta c_+ c_- - a c^2 + \gamma) ,
\end{align}
(5.7b)

Let us note that the condition (5.7) also restricts the form of different factors $O$ appearing in (5.6).

We will restrict our discussion to the case
\begin{equation}
b^{-1} \neq 0 ,
\end{equation}
so as to ensure the nondegeneracy of the metric in the whole space–time.

By a simple inspection of the expression (5.5) for the surface term one concludes that the Hamiltonian and the generators $\int d\sigma \varepsilon^0 p_0 , \int d\sigma S$ can easily be made well defined (finite and differentiable) by adding a suitable surface term, if $\varepsilon^0$ and $\varepsilon$ satisfy the conditions
\begin{equation}
\varepsilon^0 , \varepsilon = \text{const.} + O .
\end{equation}

In order to investigate the generator $\int d\sigma \varepsilon^1 \bar{p}_1$, we note that
\begin{equation}
L = -\delta \left[ \varepsilon^0 \pi^1 + \varepsilon (\pi^1 - \tau \pi^1) \right] - \varepsilon^1 \left(b^1 - b^1 - b^- A^1 \delta \pi^1 \right) .
\end{equation}
(5.10)

The first part has the form of the variation of a surface term, while the second one requires further discussion. Let us first consider the simple case when $c_+ \neq 0$. Then by using the constraints (5.7) one finds (Appendix B)
\begin{align}
b^1 + b^1 - b^- A^1 \delta \pi^1 &= -\frac{\alpha}{8 \beta^3 c^2} e^{2 \beta c} \delta \tilde{Q} , \\
\tilde{Q} &\equiv e^{-2 \beta c} \left[ 1 + (1 + 2 \beta c)^2 - \frac{4 \beta^2}{\alpha} (2 \beta c c_+ + \gamma) \right] ,
\end{align}
and the expression for $L$ becomes
\begin{equation}
L = -\delta \left[ \varepsilon^0 \pi^1 + \varepsilon (\pi^1 - \tau \pi^1) \right] + \frac{\alpha}{8 \beta^3} \varepsilon^1 \frac{a^-}{c^2} e^{2 \beta c} \delta \tilde{Q} .
\end{equation}
The consistency of the above discussion requires $\varepsilon^1 = \text{const.} + O$.

The above result for $L$ does not have the form of a total variation, so that the form of the generator $\Gamma_{as}$ cannot be improved. The problem can be solved by redefining the parameter $\varepsilon^1$ as follows:
\begin{equation}
\varepsilon^1 = \frac{\pi^1}{b^-} e^{-2 \beta \pi^1} \eta , \quad \eta = \text{const.} + O .
\end{equation}
(5.11)
Then, without using the assumption $c_+ \neq 0$, the calculation analogous to the one given above leads to

$$L = -\delta \left[ \varepsilon^0 \pi^+_1 + \varepsilon \left( \pi^1 - \tau \pi^+_1 \right) - \frac{\alpha}{8\beta^3} \eta \tilde{Q} \right].$$

It is now easy to see from Eq.(5.5) that the variation of the generator can be written in the form

$$\delta \Gamma_{as} = -\delta \Omega + R,$$

$$\Omega \equiv \left[ \varepsilon^0 \pi^+_1 + \varepsilon \left( \pi^1 - \tau \pi^+_1 \right) - \frac{\alpha}{8\beta^3} \eta \tilde{Q} \right]^{+\infty}_{-\infty},$$

where the surface term $\Omega$ is defined on the boundary of the one-dimensional space. Now we can redefine the generator $\Gamma_{as}$ as in Eq.(2.10), $\tilde{\Gamma} = \Gamma_{as} + \Omega$, so that $\tilde{\Gamma}$ has well defined functional derivatives. The assumed asymptotic behaviour ensures finitness of $\Omega$. The generator $\tilde{\Gamma}$ does not vanish on–shell as $\Gamma_{as}$ does — it takes on the value $\Omega$. The related conservation law will be discussed in the next section.

The above result for the surface term is obtained in the phase space defined by (5.6). The residual symmetry of the theory is defined by the parameters that are bounded at large distances. It is clear from the form of the general surface term (5.5) that the asymptotic symmetry, defined by the behaviour of parameters at large distances, is closely related to the form of the asymptotic conditions. More general asymptotic symmetries can be obtained by considering more general asymptotic conditions.

**B.** The previous considerations can be easily generalized so as to include solutions with more general asymptotic behaviour.

(i) Let us first generalize the asymptotics (5.6) by assuming that for large $\sigma$ the dynamical variables behave as

$$b^{-1}_1 = M^- + O(M^-), \quad \pi^{-1}_- = N_+ + O(N_-),$$
$$b^+_1 = M^+ + O(M^+), \quad \pi^+_1 = N_+ + O(N_+),$$
$$A_1 = M + O(M), \quad \pi^1 = N + O(N),$$

where $(M^\mp, M, N^\pm, N)$ are certain functions of $\sigma$, not necessarily bounded in $\sigma$.

(ii) We shall demand that all constraints decrease sufficiently fast, as in Eq.(5.7a).

(iii) The condition of the nondegeneracy of the metric (5.8) will be also retained.

(iv) Leaving the question of the finitness of various integrals for the end, one can redefine the parameter $\varepsilon^1$ as in Eq.(5.11). Then, starting from the expression (5.10) for $L$ and making use of the constraints (following the calculation outlined in Appendix B) one obtains the result

$$\delta \Gamma_{as} = -\delta \Omega + R,$$

$$\Omega \equiv \left[ \varepsilon^0 \pi^+_1 + \varepsilon \left( \pi^1 - \tau \pi^+_1 \right) - \frac{\alpha}{8\beta^3} \eta \tilde{Q} \right]^{+\infty}_{-\infty},$$

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where \( Q \) is defined in (B.3). This leads to the correctly defined generator \( \tilde{\Gamma} = \Gamma_{as} + \Omega \) if \( \eta \sim \text{const.} \). The result (5.14) is obtained provided the following condition is satisfied:

\[
\eta e^{-2\beta \pi^1}(b_1^{-1})^{-1}(\pi^1, \delta \pi^1 + \pi^1, \delta \pi^1) = O.
\]

This condition is ensured if we demand

\[
\pi^1, \pi^1 \sim \text{const.}, \quad (5.15a)
\]

\[
b_1^{-1} \text{ does not decrease faster than } O_1. \quad (5.15b)
\]

The geometrical meaning of these conditions is discussed in Appendix C.

The last term in \( \Omega \) gives a nonzero contribution if \( \eta \neq 0 \) and the values of \( \eta \) when \( \sigma \to \pm \infty \) are different.

5.3 Asymptotically flat space — general consideration

The general structure of asymptotic symmetries obtained so far gives us a clue to understand, on more general grounds, the properties of space–time which at large distances has the Minkowskian structure:

\[
b_1^+ = 0, \quad b_1^- = 1, \quad A_\mu = 0, \\
\pi^1_+ = 0, \quad \pi^1_- = 0, \quad \pi^1 = 0. \quad (5.16)
\]

The symmetries of the Minkowski vacuum (5.16) are obtained by the following restriction on the parameters \( \varepsilon^\mu(\sigma), \varepsilon(\sigma) \):

\[
\varepsilon(\sigma) = \varepsilon = \text{const.}, \quad \varepsilon^0(\sigma) = \varepsilon^0 = \text{const.}, \\
\varepsilon^1(\sigma) = \varepsilon \sigma + \varepsilon^1, \quad \varepsilon^1 = \text{const.} \quad (5.17a)
\]

They define the global Poincaré transformations, with the generator

\[
\Gamma_{as} = \varepsilon^0 T_0 + \varepsilon^1 T_1 - \varepsilon T, \\
T_\mu = \int d\sigma P_\mu, \quad T = \int d\sigma (\sigma P_0 - \sigma P_1 + G). \quad (5.17b)
\]

We choose the following asymptotic behaviour:

\[
b_1^+ = O, \quad b_1^- = 1 + O, \quad A_1 = O, \\
\pi^1_+ = O, \quad \pi^1_- = O, \quad \pi^1 = O. \quad (5.18)
\]

where \( O \) stands for the general term of the \( O_\alpha \) type, with \( \alpha > 0 \).

The expression for the surface term \( L \) simplifies:

\[
L = -\varepsilon \sigma (b_1^+ \delta \pi^1 + b_1^- \delta \pi^1 + A_1 \delta \pi^1) + O = -\varepsilon \sigma \pi^1 \delta \ln \pi^1 + O,
\]

where we used \( \delta Q = \hat{O} \) (in vacuum \( Q = 2 \); since \( Q' \) is a constraint, one concludes that \( Q = 2 + \hat{O} \), therefore \( \delta Q = \hat{O} \)). By using the relation

\[
\sigma \pi^1 \delta \ln \pi^1 = \delta (\sigma \pi^1 \ln \pi^1) - \sigma (\delta \pi^1) \ln \pi^1,
\]

we easily see that \( L = 0 \) under the asymptotic conditions (5.18).
6. Conserved quantities — the construction

Now that we have the general expression (5.14) for the surface term we can improve the symmetry generators and make them finite and differentiable. Thus, we define

\[ \tilde{\Gamma} = c^0 \tilde{T}_0 + c^1 \tilde{T}_1 - c \tilde{T}, \]  

(6.1)

where \( c^0, c^1 \) and \( c \) are constants, and the improved generators are given by

\[ \begin{align*}
\tilde{T}_0 &= \int d\sigma (\varepsilon^0 \bar{P}_0) + \Omega_0, \quad \Omega_0 \equiv (\varepsilon^0 \pi^1)|_{-\infty}^{+\infty}, \\
\tilde{T}_1 &= \int d\sigma \left( \eta \frac{\pi^1}{b-1} e^{-2\beta \pi^1 \bar{P}_1} \right) + \Omega_1, \quad \Omega_1 \equiv -\frac{\alpha}{8\beta^3} (\eta Q)|_{-\infty}^{+\infty}, \\
\tilde{T} &= \int d\sigma (\varepsilon \bar{S}) + \Omega, \quad \Omega \equiv [\varepsilon (\pi^1 \tau - \pi^1)]|_{-\infty}^{+\infty}.
\end{align*} \]

(6.2)

Here, \( \varepsilon^0(\sigma) \), \( \varepsilon^1(\sigma) \) and \( \eta(\sigma) \) are arbitrarily given functions of \( \sigma \), the asymptotic behaviour of which is chosen to ensure

a) the finitness of the generators, and
b) the invariance of the asymptotic conditions (5.13).

For a given set of parameters \( \varepsilon^0(\sigma) \), \( \varepsilon^1(\sigma) \) and \( \eta(\sigma) \), the functionals (6.2) are the global symmetry generators and, consequently, satisfy the equations (2.11), where the Hamiltonian is also corrected by a surface term:

\[ \tilde{H}_T = \int d\sigma \bar{P}_0 + (\pi^1)|_{-\infty}^{+\infty}, \]

(6.3)

The Poisson bracket \( \{ \tilde{\Gamma}, \tilde{H}_T \} \), being basically the commutator of two first class constraints, vanishes on–shell. It follows then that \( d\tilde{\Gamma}/d\tau \approx \partial \tilde{\Gamma}/\partial \tau \), which means that only those generators whose surface terms have an explicit time dependence may not be conserved.

For the generators \( \tilde{T}_0 \) and \( \tilde{T}_1 \) one easily finds the relations

\[ \begin{align*}
\frac{d\tilde{T}_0}{d\tau} &\approx \frac{\partial \Omega_0}{\partial \tau} = 0, \\
\frac{d\tilde{T}_1}{d\tau} &\approx \frac{\partial \Omega_1}{\partial \tau} = 0,
\end{align*} \]

(6.4a)

showing that the surface terms \( \Omega_0 \) and \( \Omega_1 \) represent the values of \( \tilde{T}_0 \) and \( \tilde{T}_1 \) as the conserved charges. Any particular choice of \( \varepsilon^0 \) and \( \varepsilon^1 \) defines the related expressions for \( \Omega_0 \) and \( \Omega_1 \), respectively. In particular, if \( \varepsilon^0 = 1 \) then \( \tilde{T}_0 = \tilde{H}_T \), and \( \Omega_0 \) represents the conserved value of the Hamiltonian. Taking into account the existence of parameters \( \varepsilon^0 \) and \( \varepsilon^1 \) with various asymptotic behaviour, we find that the following quantities are also conserved:

\[ \begin{align*}
\Omega_0^\pm &\equiv \pi^1(\sigma \to \pm \infty), \\
\Omega_1^\pm &\equiv Q(\sigma \to \pm \infty) \equiv Q^\pm.
\end{align*} \]

(6.4b)
Note also that from $Q' \approx 0$ it follows that $Q(\sigma)$ is a constant of motion with the value $Q^+ = Q^-$ for all $\sigma$'s.

As concerns the generator $\tilde{T}$, we find

$$\frac{d\tilde{T}}{d\tau} \approx \frac{\partial \Omega}{\partial \tau} = (\varepsilon \pi^1_+)^{\pm \infty},$$

so that $\tilde{T} \approx \Omega$ will be conserved if $\Omega_0^{\pm} = 0$.

Close inspection of the equations of motion justifies the results of the above symmetry considerations.

7. Concluding remarks

We presented here an analysis of the relation between the asymptotic symmetries and the conservation laws in the Poincaré gauge theory in 2d. The generators of the asymptotic symmetries are constructed from the corresponding gauge generators. By demanding that the asymptotic generators have well defined functional derivatives we obtained their improved form, containing certain surface terms. These surface terms represent the values of the conserved quantities, whose physical meaning depend on the nature of the asymptotic symmetry. The analysis is carried out for very general asymptotic conditions of phase space variables.

It can be generalized to other gravitational theories in 2d, such as the dilaton gravity, and used to study the stability problem.

Appendix A: The Riemann space of constant curvature in 2d

1. The Riemann space of constant curvature is maximally symmetric space $\tilde{V}$ [14]. By using the fact that maximally symmetric spaces are essentially unique, one can find the metric of such space in 2d by considering a two–dimensional hypersphere embedded in a three–dimensional flat space:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \kappa^{-1} dz^2, \quad \kappa \eta_{\mu\nu} x^\mu x^\nu + z^2 = 1, \quad \kappa = \text{const.}$$

After eliminating $z$, the metric of $\tilde{V}$ takes the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad g_{\mu\nu} \equiv \eta_{\mu\nu} + \frac{\kappa x^\mu x^\nu}{1 - \kappa x^2},$$

where $x^2 = \eta_{\mu\nu} x^\mu x^\nu$. The isometries of $\tilde{V}$ are defined by the following Killing vectors:

$$\xi_{(a)}^\mu = \delta_a^\mu \sqrt{1 - \kappa x^2} \quad (a = 0, 1), \quad \xi_{(2)}^\mu = \varepsilon^\mu_{\nu} x^\nu.$$
2. For \( \kappa \leq 0 \) it is convenient to introduce new coordinates,

\[
x^0 = r \sinh \tau, \quad r^2 = 1/|\kappa|, \\
x^1 = r \cosh \tau \cos \sigma, \\
z = \cosh \tau \sin \sigma,
\]

where \( \tau \in (-\infty, +\infty), \sigma \in (0, 2\pi) \). The new form of the metric is given by

\[
ds^2 = r^2(d\tau^2 - \cosh^2 \tau d\sigma^2).
\]

It is interesting to note that here we can enlarge the domain for \( \sigma \) by assuming \( \sigma \in (-\infty, +\infty) \). The Killing vectors in new coordinates are given as:

\[
\xi^{(0)} = \frac{1}{r} \begin{pmatrix} \sin \sigma \\ \tgh \tau \cos \sigma \end{pmatrix}, \quad \xi^{(1)} = \frac{1}{r} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} -\cos \sigma \\ \tgh \tau \sin \sigma \end{pmatrix}.
\]

They satisfy the Killing equation in the whole region \( \tau \in (-\infty, +\infty), \sigma \in (-\infty, +\infty) \).

3. Let us now introduce another coordinate transformation,

\[
\tau = \ln \tgn (\tau'/2), \quad \sigma = \sigma', \quad \tau' \in (0, \pi), \quad \sigma' \in (-\infty, +\infty),
\]

which transforms the metric into conformally flat form:

\[
g'_{\mu\nu}(\tau', \sigma') = \frac{r^2}{\sin^2 \tau'} \eta_{\mu\nu}.
\]

The Killing vectors are given as

\[
\xi^{(0)} = \frac{1}{r} \begin{pmatrix} \sin \tau' \sin \sigma' \\ -\cos \tau' \cos \sigma' \end{pmatrix}, \quad \xi^{(1)} = \frac{1}{r} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \xi^{(2)} = \begin{pmatrix} \sin \tau' \cos \sigma' \\ \cos \tau' \sin \sigma' \end{pmatrix}.
\]

After changing the basis

\[
K_0 = -r \xi^{(1)}, \quad K_1 = -\xi^{(2)}, \quad K_2 = r \xi^{(0)},
\]

one easily finds

\[
[K_0, K_1] = -K_2, \quad [K_0, K_2] = K_1, \quad [K_1, K_2] = K_0.
\]

The symmetry of the space is characterized by the \( SL(2, R) \) algebra, as expected.

4. We now discuss the form of the diad and the connection by using the conformally flat form of the metric, and dropping primes for simplicity. The fields \( b^a_\mu \) and \( A_\mu \) will be defined by demanding \( T^a_{\mu\nu} = 0 \) and \( R^{ab}_{\mu\nu} = \text{const} \).
From $T^a_{\mu\nu} = 0$ one finds the relation $A^\mu = \varepsilon^{ab} h^a_{\nu} h^a_{\mu,\nu}$. After that one concludes from (A.5) that the diad and the connection can be chosen in the form:

$$b^a_\mu = \frac{r}{\sin\tau} \delta^a_\mu, \quad A_\mu = \cotg \tau \delta^1_\mu.$$  

(A.8)

Explicit check leads to $F = h^a_\mu h^b_{\nu} R^a_{\mu\nu} = -2/r^2$.

The isometry transformations $\xi^\mu = c^i K^\mu_i$, where $c^i$ are constants, do not change the form of the metric (A.5). What happens with the fields $b^a_\mu$ and $A_\mu$? Introducing the notation $\delta^0_0 = \delta^0_0(\omega = 0, \xi^\mu = a^i K^\mu_i)$ one finds

$$\delta^0_0 b^a_\mu = -\sin\tau (a^1 \sin\sigma - a^2 \cos\sigma) \varepsilon^a_c b^c_\mu = \xi^0_{\cdot1} \varepsilon^a_c b^c_\mu.$$  

Thus, the diad field is not invariant under the isometries. However, the change $\delta^0_0 b^a_\mu$ has the form of a local Lorentz transformation. Combining the isometry transformation with the local Lorentz transformation defined by $\omega = -\xi^0_{\cdot1}$ one obtains

$$\delta_0 b^a_\mu = 0. \quad \delta_0 A_\mu = 0.$$  

Therefore, the Poinceré transformations defined by the global parameters

$$\xi^\mu = a^i K^\mu_i, \quad \omega = -\xi^0_{\cdot1},$$  

(A.9) leave the diad and the connection (A.8) of the space $\tilde{V}$ invariant.

**Appendix B: Surface term and constant of motion**

In this Appendix we shall derive the expression (5.12) for the surface term, and show that it is given in terms of a constant of motion. We start from the relation

$$L_1 \equiv b^+ b \delta^+ - b^- b^{-1} \delta^- 1 + A_1 \delta^1 - a^+ \delta c + a^- \delta c - a^\dagger c^\dagger = \frac{a^-}{c^+} c^+ c^+ c^+ c^+ \delta c + O$$  

(B.1)

obtained on the basis of the asymptotic behaviour (5.6) and the constraints (5.7). The expression on the right hand side can be written in the form

$$\frac{a^-}{c^+} \left[ \delta(c^- c^+) - (2\beta c^+ c^- - \alpha c^2 + \gamma) \delta c \right] = -\frac{\alpha}{8\beta^2 c^+} \left[ \delta \Lambda - 2\beta \Lambda \delta c \right],$$

$$\Lambda \equiv 1 + (1 + 2\beta c)^2 - \frac{4\beta^2}{\alpha} (2\beta c^+ c^- + \gamma).$$

It is now easy to see that

$$L_1 = -\frac{\alpha}{8\beta^2 c^+} \frac{a^-}{c^+} e^{2\beta c} \delta \tilde{Q} + O, \quad \tilde{Q} \equiv e^{-2\beta c} \Lambda.$$  

(B.2)
whereafter the result (5.12) follows directly.

It is interesting to note that $\tilde{Q}$ represents the asymptotic value of

$$Q \equiv e^{-2\beta \pi^1} \left[ 1 + (1 + 2\beta \pi^1)^2 - \frac{4\beta^2}{\alpha} (\gamma + 2\beta \pi^1 \pi^{-1}) \right]. \quad (B.3)$$

The quantity $Q$ is related to an important feature of the dynamics of the theory, which consists in the existence of a general constant of motion in the phase space. To see that we observe the relation:

$$\pi^a G_a + \frac{\alpha}{8\beta^3} e^{2\beta \pi^1} Q', \quad (B.4)$$

Since $Q' \approx 0$, it follows that $Q$ is a function of time only, $Q = Q(\tau)$. It is easy to check that $\{Q, G_a\} = \{Q, G\} = 0$, therefore $\dot{Q} = \{Q, H_T\} = 0$, i. e. $Q$ is a constant of motion:

$$Q = Q_0 = \text{const.} \quad (B.5)$$

The constraints $G, G_+$ and $Q'$ are equivalent to $G, G_+$ and $G_-$. 

**Appendix C: The geometrical meaning of the conditions (5.15)**

Let us note, first, that the value of the variable $\pi^1$ coincides with the curvature of the space–time. The equation of motion $\dot{\pi}^1 = \pi_{+1}$ then tells us that the asymptotic behaviour $\pi^1, \pi_{+1} \sim \text{const.}$ is necessary if we want the space–time to have asymptotically finite curvature:

$$(5.15a) \iff \text{asymptotically finite curvature.} \quad (C.1)$$

At the same time, the condition $\pi_{+1} \sim \text{const.}$ ensures finiteness and differentiability of the Hamiltonian.

To understand the asymptotic behaviour of $b_{-1}$ we shall consider the quantity

$$D \equiv \int d\sigma \left( \frac{b_{-1}}{\pi_{+1}} \right) e^{2\beta \pi^1},$$

which is proportional to the length of the lines along which the space–time curvature is constant (see the third reference in [9]). The asymptotic condition (5.15b) then implies that $D$ is infinite:

$$(5.15b) \iff \text{infinite lines of constant curvature.} \quad (C.2)$$

The integral $D$ is a natural measure of the linear size of space–time, since it is obviously a gauge invariant quantity.
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