THE WOBBLY DIVISORS OF THE MODULI SPACE OF RANK-2 VECTOR BUNDLES

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Abstract. Let $X$ be a smooth projective complex curve of genus $g \geq 2$ and let $M_X(2, \Lambda)$ be the moduli space of semi-stable rank-2 vector bundles over $X$ with fixed determinant $\Lambda$. We show that the wobbly locus, i.e., the locus of semi-stable vector bundles admitting a non-zero nilpotent Higgs field is a union of divisors $W_k \subset M_X(2, \Lambda)$. We show that on one wobbly divisor the set of maximal subbundles is degenerate. We also compute the class of the divisors $W_k$ in the Picard group of $M_X(2, \Lambda)$.

1. Introduction

Let $X$ be a smooth projective complex curve of genus $g \geq 2$ and let $K$ be its canonical line bundle. Fixing a line bundle $\Lambda$ we consider the coarse moduli space $M_X(2, \Lambda)$ parameterizing semi-stable rank-2 vector bundles of fixed determinant $\Lambda$ over $X$. In this note we study the locus in $M_X(2, \Lambda)$ of wobbly, or non-very stable, vector bundles over $X$. We recall that a vector bundle $E$ is called very stable if $E$ has no non-zero nilpotent Higgs field $\phi \in H^0(X, \text{End}(E) \otimes K)$. Laumon [Lau, Proposition 3.5] proved, assuming $g \geq 2$, that a very stable vector bundle is stable and that the locus of very stable bundles is a non-empty open subset of $M_X(2, \Lambda)$. Hence the locus of wobbly bundles is a closed subset $W \subset M_X(2, \Lambda)$.

It was announced in Laumon [Lau, Remarque 3.6 (ii)] that $W$ is of pure codimension 1. The term “wobbly” was introduced in the paper [DP].

Our first result proves this claim. Since the isomorphism class of the moduli space $M_X(2, \Lambda)$ depends only on the parity of the degree $\lambda = \deg \Lambda$, it will be enough to study two cases, $\lambda = 0$ and $\lambda = 1$. For $1 \leq k \leq g - \lambda$ we define $W_k$ to be the closure in $M_X(2, \Lambda)$ of the locus of all semi-stable vector bundles arising as extensions

$$0 \rightarrow L \rightarrow E \rightarrow \Lambda L^{-1} \rightarrow 0,$$

with $\deg L = 1 - k$ and $\dim H^0(X, KL^2\Lambda^{-1}) > 0$. We denote by $[x]$ the ceiling of the real number $x$. With this notation we have the following results.

Theorem 1.1. The wobbly locus $W \subset M_X(2, \Lambda)$ is of pure codimension 1 and we have the following decomposition for $\lambda = 0$ and $\lambda = 1$

$$W = W_{\frac{\lambda + 1}{2}} \cup \ldots \cup W_{g - \lambda}.$$

In particular, all loci $W_k$ appearing in the above decomposition are divisors. They are all irreducible, except $W_0$ for $\lambda = 0$, which is the union of $2^{2g}$ irreducible divisors.

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This theorem completes the results obtained in [P] showing that \( W \) is of codimension 1 for \( \lambda = 1 \). The idea of the proof is to consider the rational forgetful map (forgetting the non-zero Higgs field) from the equidimensional nilpotent cone in the moduli space of semi-stable Higgs bundles to \( \mathcal{M}_X(2, \Lambda) \). It turns out that roughly half of the irreducible components of the nilpotent cone gets contracted by the forgetful map with one-dimensional fibers to the above mentioned divisors \( W_k \), and the other half gets contracted with fibers of dimension > 1 to subvarieties of these disisors \( W_k \).

Our second result studies the relationship between very stable vector bundles \( E \) and the loci of maximal line subbundles of \( E \). We recall here the main results on maximal line subbundles of rank-2 bundles (see e.g. [O], [LN]). Under the assumption that \( g + \lambda \) is odd, i.e., \( g \) odd if \( \lambda = 0 \) and \( g \) even if \( \lambda = 1 \), the Quot-scheme

\[
M(E) := \text{Quot}^{1,1-[\frac{g-\lambda}{2}]}(E)
\]

parameterizing subsheaves of rank 1 and degree \( 1 - [\frac{g-\lambda}{2}] \) of \( E \) is a zero-dimensional, reduced scheme of length \( 2^g \) for a general bundle \( E \in \mathcal{M}_X(2, \Lambda) \). In that case \( M(E) \) parameterizes line subbundles of \( E \) of maximal degree. We say that \( M(E) \) is non-degenerate if \( \dim M(E) = 0 \) and \( M(E) \) is reduced, and degenerate if the opposite holds.

**Theorem 1.2.** Under the assumption that \( g + \lambda \) is odd, the following holds.

1. If \( E \) is very stable, then \( M(E) \) is non-degenerate.
2. The subscheme \( M(E) \) is degenerate for any \( E \in W_{\lceil \frac{g-\lambda}{2} \rceil} \).

The case \( g = 2, \lambda = 1 \) was already worked out in [P]. In Remark 3.1 we show that part (2) does not hold on other components of the wobbly locus.

Our last result computes the class \( \text{cl}(W_k) \) of the wobbly divisors \( W_k \) in the Picard group of the moduli space \( \mathcal{M}_X(2, \Lambda) \), which is isomorphic to \( \mathbb{Z} \) (see e.g. [DN]).

**Theorem 1.3.** We have the following equality for \( \lambda = 0 \) and \( \lambda = 1 \)

\[
\text{cl}(W_k) = 2^{2k} \left( \frac{g}{2g-2k-\lambda} \right) \text{ for } \left[ \frac{g-\lambda}{2} \right] \leq k \leq g - \lambda.
\]

In the case \( \lambda = 0 \) the computations of the class \( \text{cl}(W_k) \) were already carried out in [F2], but due to some typos the final result in loc.cit. is not correct. For the convenience of the reader we include a detailed presentation of the computations in the case \( \lambda = 1 \).

In the last section we give a description of these divisors for low genus.

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2. Proof of Theorem 1.1

Let \( \text{Higgs}_X(2, \Lambda) \) be the moduli space of semi-stable Higgs bundles of rank 2 with fixed determinant \( \Lambda \). The Hitchin map defined by mapping a Higgs field \( (E, \phi) \) to its determinant \( \text{det}(\phi) \) is a proper surjective map

\[
h : \text{Higgs}_X(2, \Lambda) \to H^0(X, K^2).
\]

It is easy to see that the nilpotent cone decomposes as

\[
h^{-1}(0) = \mathcal{M}_X(2, \Lambda) \cup \mathcal{E},
\]
where $M_X(2, \Lambda)$ denotes here pairs $(E, 0)$ with zero Higgs field and $\tilde{\mathcal{E}}$ consists of semi-stable pairs $(E, \phi)$ with non-zero nilpotent Higgs field — note that the underlying bundle $E$ is not necessarily semi-stable. In other words, the image of $\mathcal{E}$ under the rational forgetful map $h^{-1}(0) \to M_X(2, \Lambda)$ is the locus of wobbly bundles. In the case $\lambda = 1$ the nilpotent cone was already described in [2]. For the convenience of the reader we recall now the description.

By [2] Lemma 3.1 a vector bundle $E$ admits a non-zero nilpotent Higgs field if and only if it contains a line subbundle $L$ with $H^0(X, KL^2 \Lambda^{-1}) \neq 0$. Thus any such bundle can be written as an extension

$$0 \to L \to E \to L^{-1} \Lambda \to 0,$$

where $\Lambda$ is a line bundle of degree $\lambda \in \{0, 1\}$ and the nilpotent Higgs field is given as the composition $\phi = u \circ \pi$ with a non-zero $u \in H^0(X, KL^2 \Lambda^{-1}) = \text{Hom}(L^{-1} \Lambda, LR)$. Then we have the following inequalities:

- Since $\phi(L) = 0$, $L$ is invariant under $\phi$ and by semi-stability of the pair $(E, \phi)$, we have $\text{deg}(L) = d \leq \frac{\lambda}{2}$.
- We also have $u \neq 0$. This implies that $\text{deg}(KL^2 \Lambda^{-1}) \geq 0 \Leftrightarrow d \geq \frac{\lambda}{2} + 1 - g$.

Hence we obtain the inequalities

$$\frac{\lambda}{2} + 1 - g \leq d \leq \frac{\lambda}{2}.$$

We set $k = 1 - d$ and we distinguish two cases:

- $\lambda = 0 : 1 - g \leq d \leq 0 \Leftrightarrow 1 \leq k \leq g$.
- $\lambda = 1 : \frac{1}{2} - g \leq d \leq \frac{1}{2} \Leftrightarrow 1 \leq k \leq g - 1$.

We introduce the subloci

$$W_k^0 := \{ E \in W : E \text{ contains a line subbundle } L \text{ of degree } 1-k \text{ with } H^0(X, KL^2 \Lambda^{-1}) \neq 0\}$$

and denote by $W_k$ the Zariski closure of $W_k^0$ in $M_X(2, \Lambda)$. We therefore deduce from the above considerations the following decompositions $W = \bigcup_{k=1}^{g} W_k$ for $\lambda = 0$ and $W = \bigcup_{k=1}^{g-1} W_k$ for $\lambda = 1$.

**Remark 2.1.** We observe that for $\lambda = 0$ the locus $W_1$ coincides with the semi-stable boundary of $M_X(2, \Lambda)$, which equals the Kummer variety of $X$.

Now we decompose $\tilde{\mathcal{E}}$ as $\bigcup_{k=1}^{g} \tilde{\mathcal{E}}_k$ for $\lambda = 0$ and $\bigcup_{k=1}^{g-1} \tilde{\mathcal{E}}_k$ for $\lambda = 1$ such that the image of $\tilde{\mathcal{E}}_k$ under the forgetful map is $W_k$. The construction goes as follows (we omit the construction of $\tilde{\mathcal{E}}_1$ for $\lambda = 0$ — see Remark 2.1):

We introduce the subvarieties $Z_k \subset \text{Pic}^{1-k}(X)$ for $1 \leq k \leq g - \lambda$ defined by

$$Z_k := \{ L \in \text{Pic}^{1-k}(X) \text{ such that } h^0(X, KL^2 \Lambda^{-1}) \neq 0 \}.$$

Then one can construct $Z_k$ as the pre-image of the Brill-Noether locus $W_{2g - 2k - \lambda}(X)$ under the map

$$\mu_k : \text{Pic}^{1-k}(X) \to \text{Pic}^{2g - 2k - \lambda}(X)$$

taking $L$ to $KL^2 \Lambda^{-1}$. Then

$$\dim Z_k = 2g - 2k - \lambda \text{ if } 2g - 2k - \lambda \leq g$$

$$= g \text{ if } 2g - 2k - \lambda \geq g.$$
Note that in the latter case $Z_k = \text{Pic}^{-k}(X)$. Consider the fiber product $\tilde{Z}_k = Z_k \times_{W_{2g-2k-\lambda}} S^{2g-2k-\lambda}(X)$

\begin{equation}
\begin{array}{c}
\tilde{Z}_k \\
\downarrow q \\
Z_k
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\mu_k \\
\downarrow \\
W_{2g-2k-\lambda}
\end{array} \quad (2.2)
\end{equation}

where the right vertical map is the natural map from the symmetric product of the curve to its Picard variety. Then the projection map $p : \tilde{Z}_k \to S^{2g-2k-\lambda}(X)$ is a $2^g$-fold étale covering of $S^{2g-2k-\lambda}(X)$ and $\tilde{Z}_k$ parameterizes line bundles $L$ and effective divisors in $|\mathcal{K}L^2\Lambda^{-1}|$. There exists a unique line bundle $\mathcal{L}$ over $S^{2g-2k-\lambda}(X)$ whose fiber at a divisor $D$ is canonically isomorphic to the space of sections of the line bundle determined by $D$ which vanish precisely on $D$. Furthermore, it can be shown that this line bundle is trivial.

Excluding the case $\lambda = 0$ and $k = 1$ (see Remark 2.1), we observe that the dimension of $\text{Ext}^1(\mathcal{L}L^{-1}, \mathcal{L}) = H^1(\Lambda^{-1}L^2)$ depends only on the degree of the line bundle $L$. Therefore there exists a vector bundle $\mathcal{V}_k$ over $Z_k$ whose fiber at a point $L \in Z_k$ is canonically isomorphic to $\text{Ext}^1(\mathcal{L}L^{-1}, \mathcal{L})$. The rank of the vector bundle $\mathcal{V}_k$ is $g + 2k + \lambda - 3$ and a general extension class $v$ in the fiber $(\mathcal{V}_k)_L$ defines a stable rank-2 vector bundle $E_v$.

**Proposition 2.2.** We have the following:

1. The total space of the vector bundle $q^*\mathcal{V}_k \oplus p^*\mathcal{L}$ over $\tilde{Z}_k$ parameterizes triples $(L, v, u)$, where $L$ is a line bundle in $\tilde{Z}_k$, $v$ is an extension class in the fiber $(\mathcal{V}_k)_L$ and $u$ is a global section of $\mathcal{K}L^2\Lambda^{-1}$.

2. There exists a rational map $\phi_k$ from the projectivized bundle $\mathbb{P}(q^*\mathcal{V}_k \oplus p^*\mathcal{L})$ to $\text{Higgs}_X(2, \Lambda)$ defined by sending $(L, v, u)$ to the Higgs bundle $(E_v, i \circ u \circ \pi)$ as defined by the exact sequence (2.2).

3. We have a commutative diagram

\begin{equation}
\begin{array}{c}
\mathbb{P}(q^*\mathcal{V}_k \oplus p^*\mathcal{L}) \\
\downarrow \phi_k \\
\text{Higgs}_X(2, \Lambda)
\end{array} \quad \begin{array}{c}
\mathbb{P}(\mathcal{V}_k) \\
\downarrow \psi_k \\
\text{M}_X(2, \Lambda),
\end{array}
\end{equation}

where all arrows are rational maps. The vertical maps are forgetful maps of the global section of $\mathcal{K}L^2\Lambda^{-1}$ and of the Higgs field respectively.

4. The restriction of $\phi_k$ to the vector bundle $q^*\mathcal{V}_k \subset \mathbb{P}(q^*\mathcal{V}_k \oplus p^*\mathcal{L})$ is an injective morphism.

**Proof.** Part (1) follows immediately from the previous description of $\mathcal{V}_k$ and $\mathcal{L}$. As for part (2) it will be enough to show that the Higgs bundle associated to the triple $(L, \lambda v, \lambda u)$ does not depend on the scalar $\lambda \in \mathbb{C}^*$. But this follows from the observation that the extension class of the exact sequence obtained from (2.2) by replacing either $i$ by $\frac{1}{\lambda}i$ or $\pi$ by $\frac{1}{\lambda^2}\pi$ equals $\lambda v \in \text{Ext}^1(\mathcal{L}L^{-1}, \mathcal{L})$ for any $\lambda \in \mathbb{C}^*$. Part (3) and part (4) are straightforward.

Now we define $\mathcal{E}_k^0$ to be the image of the rational map $\phi_k$ and $\mathcal{E}_k$ its Zariski closure. Then clearly $\mathcal{E}_k \subset \mathcal{E}$ and $\mathcal{W}_k^0$ is the image of $\mathcal{E}_k^0$ under the forgetful map. Clearly $\mathcal{E}_k$ is irreducible, except if $\dim Z_k = 0$ which is equivalent to $\lambda = 0$ and
\( k = g \), and since \( \phi_k \) is generically injective

\[
\dim \mathcal{E}_k = \dim \mathcal{Z}_k + \text{rk} \mathcal{V}_k = (2g - 2k - \lambda) + (g + 2k + \lambda - 3) = 3g - 3.
\]

We note that the fiber over a general element \( E \in \mathcal{W}_k^{0} \) of the forgetful map \( \mathcal{E}_k^{0} \to \mathcal{W}_k^{0} \) is \( H^0(X, KL^2 \Lambda^{-1}) \), where \( L \) is a line bundle of degree \( -k + 1 \) contained in \( E \). If \( 2g - 2k - \lambda \leq g \) and \( L \) is a general line bundle of degree \( -k + 1 \) with \( H^0(X, KL^2 \Lambda^{-1}) \neq 0 \), then \( h^0(X, KL^2 \Lambda^{-1}) = 1 \). Therefore \( \dim \mathcal{W}_k^{0} = 3g - 3 - 1 = 3g - 4 \) for \( 2g - 2k - \lambda \leq g \). Thus \( W_k \) is an irreducible divisor in \( M_X(2, \Lambda) \) for \( k \geq \frac{3g-4}{2} \) and \( k \neq g \).

**Proposition 2.3.** We have the inclusions

\[
\left\{ \frac{2g-2k-\lambda}{2} \right\}_{k=1}^{\dim \mathcal{E}_k} W_k \subset \mathcal{W}_{\left\{ \frac{2g-2k-\lambda}{2} \right\}}.
\]

**Proof.** We put \( k_0 = \left\{ \frac{2g-2k-\lambda}{2} \right\} \) and consider the rational map

\[ \psi_{k_0} : \mathbb{P}(\mathcal{V}_{k_0}) \to M_X(2, \Lambda) \]

introduced in Proposition 2.2 (3). Let \( L \in \mathcal{Z}_{k_0} \). The restriction of \( \psi_{k_0} \) on the fiber of \( \mathbb{P}(\mathcal{V}_{k_0}) \) over \( L \) is not defined at the points where the associated bundles are not semi-stable. Let

\[ \psi_L : \mathbb{P}_L := \mathbb{P}(H^1(X, L^2 \Lambda^{-1})) \to M_X(2, \Lambda) \]

be the restriction of \( \psi_{k_0} \) at the fiber over \( L \). Then by [3] Theorem 1 there is a natural sequence \( \sigma \) of blow-ups along smooth centers resolving \( \psi_L \) into a morphism \( \tilde{\psi}_L : \tilde{\mathbb{P}}_L \to M_X(2, \Lambda) \). The image of \( \tilde{\psi}_L \) is contained in closure of the image of the rational map \( \tilde{\psi}_L \). Here \( \sigma \) is the blow-up morphism \( \sigma : \tilde{\mathbb{P}}_L \to \mathbb{P}_L \).

Now \( X \) is embedded in \( \mathbb{P}_L \) via the natural map. Let \( x \in X \) and \( E \) be the bundle associated to \( x \). Then \( E \) fits into the exact sequence (2.1) and by [3] Observation (2) page 451 the bundle \( E \) is not semi-stable. Furthermore by [3] Theorem 1 (2) there is a natural isomorphism \( \sigma^{-1}(x) \cong \tilde{\mathbb{P}}_{L(x)} \) and, when restricted to \( \sigma^{-1}(x) \), the morphism \( \tilde{\psi}_L \) coincides with the morphism

\[ \tilde{\psi}_{L(x)} : \tilde{\mathbb{P}}_{L(x)} \to M_X(2, \Lambda). \]

Now \( \tilde{\mathbb{P}}_{L(x)} \) is the blow-up of \( \mathbb{P}_L(x) \) and the bundles corresponding to extension classes in \( \mathbb{P}_{L(x)} \) fit in the exact sequence of the form

\[ 0 \to L(x) \to V \to L^{-1}(-x) \Lambda \to 0. \]

Hence we deduce that the image of \( \tilde{\psi}_{L(x)} \) is contained in \( \mathcal{W}_{k_0} \). Next we observe that if \( L \in \mathcal{Z}_{k_0} \), then \( L(x) \in \mathcal{Z}_{k_0-1} \) for any \( x \in X \). Hence we obtain a morphism

\[ \mu : \mathcal{Z}_{k_0} \times X \to \mathcal{Z}_{k_0-1}, \quad \mu(L, x) = L(x). \]

It will be enough to show that \( \mu \) is surjective to conclude that \( \mathcal{W}_{k_0-1}^{0} \subset \mathcal{W}_{k_0} \). Hence \( \mathcal{W}_{k_0-1} \subset \mathcal{W}_{k_0} \) since \( \mathcal{W}_{k_0} \) is closed.

If \( g - \lambda \) is even, then \( Z_{k_0-1} = \text{Pic}^{2-k_0}(X) \) et \( Z_{k_0} = \text{Pic}^{1-k_0}(X) \) and \( \mu \) is obviously surjective. If \( g - \lambda \) is odd, then \( Z_{k_0-1} = \text{Pic}^{2-k_0}(X) \) and \( Z_{k_0} \) is an irreducible divisor in \( \text{Pic}^{1-k_0}(X) \). If on the contrary \( \mu \) is not surjective, then \( Z_{k_0} \) would be invariant by a translation by an line bundle of the form \( \mathcal{O}_X(x - y) \) for \( x, y \in X \), hence by any translation. This is a contradiction.

More generally let \( D \) be a general effective divisor of degree \( 1 \leq d \leq k_0 - 1 \). Then again by [3] Observation 2 any point \( x \in \overline{D} \) corresponds to a non-semi-stable
bundle, except if $\lambda = 0$ and $d = k_0 - 1$ (see below for a discussion of this exceptional case). Furthermore if $x$ is general in $D$ then by [2] Theorem 1 (2) there is a natural
isomorphism $\sigma^{-1}(x) \cong \mathbb{P}_L(D)$ and the restriction of $\psi_L$ to $\sigma^{-1}(x)$ coincides with the map
$$\tilde{\psi}_{L(D)} : \mathbb{P}_L(D) \to M_X(2, \Lambda).$$
As before, the natural multiplication morphism
$$\mu : Z_{k_0} \times S^d(X) \to Z_{k_0 - d}$$
is easily seen to be surjective, which implies that $\mathcal{W}_{k_0 - d}^{0} \subset \mathcal{W}_{k_0}$, hence $\mathcal{W}_{k_0 - d} \subset \mathcal{W}_{k_0}$ since $\mathcal{W}_{k_0}$ is closed.
Finally, the case $\lambda = 0$ and $d = k_0 - 1$ corresponds to $\mathcal{W}_1$, which is the image of the $(k_0 - 1)$-th secant variety to $X \subset \mathbb{P}_L$ under the rational map $\psi_L$, when $L$ varies in $Z_{k_0}$.

We will need the following lemma in section 4.

Lemma 2.4. With the notation of Proposition 2.2 and for $\left\lceil \frac{2 - \lambda}{2} \right\rceil \leq k \leq g - \lambda$, a general line bundle $L \in Z_k$ and a general extension class $v \in (\mathcal{V}_k)_L = \text{Ext}^1(\Lambda L^{-1}, L)$ defining a bundle $E_v$ as in (2.1), we have
$$\dim \text{Hom}(E_v, L^{-1}\Lambda) = 1.$$

Proof. Since $L$ is general in $Z_k$ we have $h^0(KL^{-1}\Lambda) = 1$, or equivalently by Riemann-Roch and Serre duality $h^1(\Lambda L^{-1} = 1) = h^0(L^{-2}\Lambda) = \lambda + 2k - g$. Applying the functor $\text{Hom}(\cdot, L^{-1}\Lambda)$ to the short exact sequence (2.1) we see that $\dim \text{Hom}(E_v, L^{-1}\Lambda) = 1$ if and only if the coboundary map given by the cup product $\cup v$ with the extension class $v$
$$\cup v : H^0(L^{-2}\Lambda) = \text{Hom}(L, L^{-1}\Lambda) \longrightarrow H^1(\mathcal{O}_X) = \text{Ext}^1(L^{-1}\Lambda, L^{-1}\Lambda)$$
is injective. Given a non-zero section $s \in H^0(L^2\Lambda)$ it is well-known that $s \cup v = 0$ if and only if the extension class $v \in \text{Ext}^1(\Lambda L^{-1}, L) = \text{Ext}^1(L^2\Lambda^{-1}) = H^0(KL^{-2}\Lambda)^*$ lies in the linear span $(D) \subset |KL^{-2}\Lambda|^*$, where $D$ is the zero divisor of $s$. But $\dim(D) = 2k - 4 + \lambda$, so
$$\dim \bigcup_{D \in |L^{-2}\Lambda|} (D) \leq (\lambda + 2k - g - 1) + (2k - 4 + \lambda) = 4k - g + 2\lambda - 5,$$
which is $< \dim \mathbb{P}(\mathcal{V}_k)_L = g + 2k + \lambda - 4$. So for a general extension class $v$ we see that $s \cup v \neq 0$ for any non-zero $s \in H^0(L^{-2}\Lambda)$, which is equivalent to
$$\dim \text{Hom}(E_v, L^{-1}\Lambda) = 1.$$ 

3. Proof of Theorem 1.2

We only consider the case when $\lambda = 1$ and $g$ is even, i.e., $g = 2a$ for some integer $a$. The proof in the other case can be carried out similarly. If $g = 2a$, then
$$\left\lceil \frac{2 - \lambda}{2} \right\rceil = a.$$ In this situation, a general rank-2 vector bundle of degree 1 has a line subbundle of maximal degree $1 - a$ (see e.g. [O], [LN]).

Proof of (1): Let $E$ be a very stable rank-2 vector bundle of degree 1. Suppose on the contrary that $\dim M(E) > 0$ or $M(E)$ is non-reduced. Let $L_0 \in M(E)$. If $L_0 \to E$ is not saturated, then we have a sequence of maps
$$L_0 \to L \to E \to L^{-1}\Lambda \to L_0^{-1}\Lambda,$$
where $\deg L \geq \deg L_0 + 1 = 2 - a$. Then $\chi(L^{-2}\Lambda) \leq -2$. Therefore $h^1(X, L^{-2}\Lambda) = h^0(X, KL^{-2}\Lambda^{-1}) > 0$, which implies that $E$ contains a line subbundle $L$ with
$h^0(X, KL^2 \Lambda^{-1}) \neq 0$. Then by [L] Lemma 3.1 the bundle $E$ is not very stable, a contradiction.

On the other hand, if $L_0 \rightarrow E$ is saturated, then $E$ fits in the exact sequence

$$0 \rightarrow L_0 \rightarrow E \rightarrow L_0^{-1} \Lambda \rightarrow 0.$$ 

Note that the Zariski tangent space at $L_0$ is given by $\text{Hom}(L_0, L_0^{-1} \Lambda)$. Therefore if $\text{dim } M(E) \geq 1$ or if $L_0$ is a non-reduced point in $M(E)$, then $\text{dim } \text{Hom}(L_0, L_0^{-1} \Lambda) > 0$. But $\chi(L_0^{-1} \Lambda) = 0$. Thus if $\text{dim } \text{Hom}(L_0, L_0^{-1} \Lambda) > 0$, then $h^1(X, L_0^{-1} \Lambda) = h^0(X, KL^2 \Lambda) > 0$. Therefore $E$ is not very stable, a contradiction.

**Proof of (2):** Let $E \in \mathcal{W}_0$. Then $E$ contains a line subbundle $L$ of degree $1-a$ such that $h^0(X, KL^2 \Lambda^{-1}) \neq 0$ and we have the exact sequence (2.1). Since $\chi(L^{-2} \Lambda) = 0$, we obtain that $h^0(X, L^{-2} \Lambda) = h^0(X, KL^2 \Lambda^{-1}) > 0$. Therefore the dimension of the tangent space at $L$ to the Quot-scheme $M(E)$ is $h^0(X, L^{-2} \Lambda) > 0$. Therefore $M(E)$ is degenerate at $L$. Finally, since being degenerate is a closed condition, $M(E)$ is degenerate for any $E \in \mathcal{W}_0$.

**Remark 3.1.** The statement in Theorem 1.2 (2) is not valid for the points in other components $\mathcal{W}_k$ for $k > \left[ \frac{2a}{3} \right]$.

**Example:** Take $\lambda = 0$ and $g = 2a + 1 = 3$, i.e., $a = 1$. For simplicity we assume that the curve $X$ is non-hyperelliptic. Then the wobbly locus has two components, $\mathcal{W}_2$ and $\mathcal{W}_3$, where $\mathcal{W}_2$ is an irreducible divisor, but $\mathcal{W}_3$ is a union of $64$ hyperplane sections. The $64$ hyperplane sections are indexed by the $64$ theta-characteristics $\theta$, i.e. line bundles satisfying $\theta^2 = K$. We claim that a general extension class $\xi \in \mathbb{P}(H^1(X, \theta^{-2})) \rightarrow \mathcal{W}_2 \subset M_X(2, \mathcal{O})$ is such that $M(E_\xi)$ is non-degenerate. Note that $\mathbb{P}(H^1(X, \theta^{-2})) = \mathbb{P}^5 = \mathbb{P}(H^0(X, K^2)^*) = |K^2|^*$ contains the following secant varieties to the curve $X \hookrightarrow |K^2|^*$

$$X = \text{Sec}^1(X) \hookrightarrow |K^2|^*, \quad \text{Sec}^2(X) \hookrightarrow |K^2|^*,$$

which are of dimension $1$ and $3$ respectively, and we have an equality (see e.g. [Lan])

$$\text{Sec}^3(X) = |K^2|^*.$$ 

(3.1)

It is well-known, see e.g. [Lan] Proposition 1.1, that an extension class $\xi \in |K^2|^*$ lies on $\text{Sec}^i(X)$ for $1 \leq i \leq 3$ if and only if the associated vector bundle $E_\xi$ contains a line subbundle of degree $2-i$. Let $\xi \in \text{Sec}^3(X) \setminus \text{Sec}^2(X)$. Then the maximal degree of line subbundles of $E_\xi$ is $-1$. Let $D = x_1 + x_2 + x_3$ be a general effective divisor on $X$ of degree $3$ such that $\xi \in (D)$, the linear span of three points $x_1, x_2, x_3$ of $D$ in $|K^2|^*$. Then $E_\xi$ contains the line subbundle $\theta(-D)$. Note that $E_\xi$ is an extension

$$0 \rightarrow \theta^{-1} \rightarrow E_\xi \rightarrow \theta \rightarrow 0.$$ 

We also note that $L = \theta(-L)$ is a reduced point of the Quot-scheme $M(E_\xi)$ if and only if $\text{Hom}(L, E_\xi/L) = \text{Hom}(L, L^{-1}) = H^0(X, L^{-2}) = \{0\}$.

Now consider the map

$$\Phi : S^3(X) \rightarrow \text{Pic}^2(X)$$

which takes a point $(x_1, x_2, x_3)$ to $\mathcal{O}(2x_1 + 2x_2 + 2x_3) \otimes \theta^{-2}$. Clearly the map $\Phi$ is surjective. Let $Z$ denote the divisor $\Phi^{-1}(\Theta)$, where $\Theta$ denotes the theta divisor in $\text{Pic}^2(X)$. Then $D \in Z$ if and only if $h^0(X, \mathcal{O}(2D) \otimes \theta^{-2}) \neq 0$. We denote by $\tilde{Z}$ the span of all projective planes $(D) \subset |K^2|^*$ when $D$ varies in $Z$. Then $\tilde{Z}$ is a divisor in $|K^2|^*$ and we have $\xi \in \tilde{Z}$ if and only if $E_\xi$ contains a line subbundle $L$ such that $H^0(X, L^{-2}) \neq \{0\}$. So for general $\xi \notin \tilde{Z}$ the set $M(E_\xi)$ is reduced and consists of $8$ line subbundles.
A similar computation can be done for λ = 1 and g = 4 to show that the statement in Theorem 1.2 (2) is not valid for the points in the component \( W_3 \).

4. PROOF OF THEOREM 1.3

In this section we compute the class \( cl(W_k) \) of the wobly divisor \( W_k \) in the case \( \lambda = 1 \) for \( \left\lfloor \frac{g}{2} \right\rfloor \leq k \leq g - 1 \) following closely the method used in [Fa] Section 5 Example 1. Note that in [Fa] the case \( \lambda = 0 \) is worked out.

Let \( S \) be a smooth connected variety and let \( E \) be a rank-2 vector bundle over \( S \times X \) such that \( \det E = \pi_X^* (\Lambda) \), where \( \pi_S \) and \( \pi_X \) denote the projections onto \( S \) and \( X \) respectively, and such that \( E_s := E_{(s) \times X} \) is stable for any \( s \in S \). Then the family \( E \) determines a classifying map

\[ f : S \to M_X(2, \Lambda). \]

Our first task is to compute the first Chern class of the pull-back under \( f \) of the ample generator \( D \) of the Picard group of \( M_X(2, \Lambda) \), i.e.,

\[ \Theta_S := c_1(f^* D) \in H^2(S), \]

in terms of Chern classes of \( E \). We recall [DN] Théorème B that the line bundle \( f^* D \) is defined as the inverse of the determinant line bundle

\[ \det R\pi_{S*} (E \otimes \pi_X^* H), \]

where \( H \) is a rank-2 vector bundle of degree \( 2g - 3 \). Note that the condition on the degree is equivalent to \( \chi(E_s \otimes H) = 0 \). Then the Grothendieck-Riemann-Roch theorem gives the equalities

\[
\begin{align*}
\Theta_S &= -\frac{1}{2} \pi_{S*} \left[ c_1(E \otimes \pi_X^* H) \right] - 2c_2(E \otimes \pi_X^* H) - c_1(E \otimes \pi_X^* H) \cdot \pi_X^*(c_1(K)) \\
&= 2\pi_{S*} c_2(E \otimes \pi_X^* H) \\
&= 2\pi_{S*} c_2(E) \in H^2(S).
\end{align*}
\]

Since we need to compute the class in \( H^2(S) \) of the \( k \)-th wobly divisor \( f^{-1}(W_k) \subset S \) in terms of \( \Theta_S \), it will be enough to do the computations modulo classes in \( H^i(S) \) for \( i \geq 3 \). Hence the above relation allows to write

\[
c_2(E) = \frac{1}{2} \Theta_S \otimes \eta \in H^4(S \times X),
\]

where \( \eta \in H^2(X) \) denotes the class of a point in \( X \) — note that we omit classes in \( H^4(S) \otimes H^0(X) \) and \( H^3(S) \otimes H^1(X) \). Since \( c_1(E) = 1 \otimes \eta \in H^0(S) \otimes H^2(X) \subset H^2(S \times X) \) we get the following expression for the Chern character of \( E \)

\[
ch(E) = 2 + 1 \otimes \eta - \frac{1}{2} \Theta_S \otimes \eta + \text{h.o.t.},
\]

where all h.o.t. are contained in \( \oplus_{i \geq 3} H^i(S) \otimes H^*(X) \).

We also need to recall some standard facts on the first Chern class of a Poincaré bundle \( L \) over \( P \times X \), with \( P := \text{Pic}^{1-k}(X) \) (see e.g. [ACGH] page 335). We have

\[
c_1(L) = (1-k)1 \otimes \eta + \gamma \in H^2(P \times X),
\]

where \( \gamma \) denotes a class in \( H^1(P) \otimes H^1(X) \) — for a more precise description of \( \gamma \) see [ACGH] — with the property

\[
\gamma^2 = -2\Theta_P \otimes \eta.
\]

Here \( \Theta_P \in H^2(P) \) denotes the class of a theta divisor in \( P \). The rest of the computations goes exactly as in the case \( \lambda = 0 \). For the convenience of the reader we include the details.
The main idea is to realize the $k$-th wobbly divisor
\[ f^{-1}(W_k) = \pi_S(\Delta_k \cap (S \times Z_k)) \subset S \]
as the projection onto $S$ of the intersection of $S \times Z_k$ with the determinantal subvariety $\Delta_k \subset S \times P$ defined by
\[ \Delta_k = \{(s, L) \in S \times P \mid \text{Hom}(E_s, L^{-1}A) \neq 0\} , \]
and which is constructed by the standard technique as follows. We fix a reduced divisor $D_0$ of degree $d_0$ sufficiently large such that $h^1(X, \text{Hom}(E_s, L^{-1}A(D_0))) = 0$ for all $s \in S$ and $L \in P$. We consider the exact sequence over $S \times P \times X$
\[ 0 \rightarrow \text{Hom}(E, L^{-1}A) \rightarrow \text{Hom}(E, L^{-1}A(D_0)) \rightarrow \text{Hom}(E, L^{-1}A(D_0))|_{D_0} \rightarrow 0 . \]
We introduce the following two vector bundles over $S \times P$
\[ F := (\pi_{S \times P})_* \left( \text{Hom}(E, L^{-1}A(D_0)) \right) \quad \text{and} \quad A := \bigoplus_{x \in D_0} \text{Hom}(E, L^{-1})|_{S \times P \times \{x\}} \]
of ranks $2(d_0 + k - g) + 1$ and $2d_0$ respectively. Taking the direct image of the above exact sequence under the projection $\pi_{S \times P}$ onto $S \times P$ we obtain a map $\phi : F \rightarrow A$ over $S \times P$. Let us denote by $q : \mathbb{P}(F) \rightarrow S \times P$ the projection from the projectivized bundle $\mathbb{P}(F)$ onto the base variety $S \times P$. Then the composition of the tautological section over $\mathbb{P}(F)$ with $q^*\phi$
\[ O(-1) \rightarrow q^*F \rightarrow q^*A \]
defines a global section $s \in H^0(\mathbb{P}(F), q^*A \otimes O(1))$ whose zero set equals
\[ \Delta_k = \{(s, L, \varphi) \mid (s, L) \in \Delta_k \text{ and } \varphi \in \mathbb{P}(\text{Hom}(E_s, L^{-1}A))\} . \]
By Lemma [3], the map $\Delta_k \cap q^{-1}(S \times Z_k) \rightarrow \Delta_k \cap (S \times Z_k)$ induced by the projection $q$ is birational, which implies that
\[ \dim \Delta_k \cap q^{-1}(S \times Z_k) = \dim \Delta_k \cap (S \times Z_k) = \dim S - 1 . \]
Hence
\[ \dim \Delta_k \leq \dim S - 1 + \text{codim } Z_k = \dim \mathbb{P}(F) - 2d_0 , \]
which shows that codim $\Delta_k = 2d_0$. Hence we can conclude that its fundamental class is given by the top Chern class
\[ [\Delta_k] = c_2d_0(q^*A \otimes O(1)) . \]
Moreover, by the projection formula we have
\[ [\Delta_k] = q_*[\Delta_k] = \sum_{i=0}^{2d_0} q_*(c_i(q^*A)c_1(O(1))^{2d_0-i}) = \sum_{i=0}^{2d_0} c_i(A)q_*(c_1(O(1))^{2d_0-i}) = q_*(c_1(O(1))^{2d_0}) \mod H^i(S) \ for \ i \geq 3 . \]
The last equality follows from the facts that
\[ c_1(\text{Hom}(E, L^{-1})|_{S \times P \times \{x\}}) = 0 \quad \text{and} \quad c_2(\text{Hom}(E, L^{-1})|_{S \times P \times \{x\}}) \in H^4(S) \otimes H^0(P) , \]
which imply that
\[ c_1(A) = 0 \quad \text{and} \quad c_k(A) \in H^{2k}(S) \otimes H^0(P) \ for \ k \geq 2 . \]
In order to compute the class \( q_*(c_1(O(1))^{2d_0}) \), we compute by the Grothendieck-Riemann-Roch theorem the first terms of the Chern character of \( F \).

\[
ch(F) = \frac{(\pi_S \times P)_*(ch(E^0) \cdot ch(L^{-1} \Lambda(D_0)) \cdot Td(X))}{(\pi_S \times P)_*(\frac{2}{k} \Theta_S \otimes \eta + h.o.t) \cdot (1 + (d_0 + k) \otimes \eta + \gamma \Theta_P \otimes \eta) + h.o.t.}
\]

where we put \( e = 2d_0 - r = 2g - 2k - 1 = \text{dim } Z_k \). We only need to compute the component in \( H^2(S) \otimes H^{2e}(P) \) of this class, which equals

\[
\frac{(e + 1)\frac{1}{2} \Theta_S \otimes 2^e \Theta_P^g}{(e + 1)!} = \Theta_S \otimes \frac{2^{e-1}}{e!} \Theta_P^g.
\]

In order to conclude we will need the following fact.

**Lemma 4.1.** The fundamental class of \( Z_k \) in \( P \) equals

\[
[Z_k] = \frac{2^{2(g - e)}}{(g - e)!} \Theta_P^{g - e}.
\]

**Proof.** We recall that the duplication map of an abelian variety \( A \) acts as multiplication by \( 2^n \) on the cohomology \( H^n(A, \mathbb{C}) \). We apply this fact to the map \( \mu_k \) defining \( Z_k \) and we obtain

\[
[Z_k] = \mu_k^*[W_c(X)] = \frac{2^{2(g - e)}}{(g - e)!} \Theta_P^{g - e},
\]

where \( W_c(X) \subset \text{Pic}^c(X) \) denotes the Brill-Noether locus of line bundles \( L \) with \( h^0(L) > 0 \), whose fundamental class equals \( \Theta_P^{g - c} \) by Poincaré’s formula. \( \Box \)

We now combine the previous results and we obtain

\[
[\Delta_k][Z_k] = \Theta_S \otimes \frac{2^{g-1}2^{2(g - e)}}{e!(g - e)!} \Theta_P^g = \Theta_S 2^{2g - e - 1} \left( \frac{g}{e} \right) = \Theta_S 2^{2k} \left( \frac{g}{2} - 2k - 1 \right),
\]

which gives the class \( c_2(W_k) \) stated in Theorem 1.3.
Remark 4.2. We observe that in [Fa] page 350 the factor $(\tilde{e})$ is missing in the formula giving “the integral of $\Theta_j^\ast$ over the preimage of $C^\infty$”.

5. Examples

5.1. Genus 2.

5.1.1. $\lambda = 0$. It is known that $M_X(2, \mathcal{O}_X)$ is isomorphic to $\mathbb{P}^3$. By Theorem 1.1 the wobbly locus has two components

$$W = W_1 \cup W_2,$$

where $W_k$ is the closure of the locus $W_k^0$ for $k = 1, 2$.

Let $k = 1$. Note that for any line bundle $L$ of degree zero $h^0(X, KL^2) \neq 0$ and any bundle which contains a line subbundle of degree zero is semi-stable. Therefore, $W_1$ is precisely the locus of semi-stable bundles which are not stable. It is known that the strictly semi-stable locus is a quartic hypersurface (known as Kummer surface) in $\mathbb{P}^3$. Thus the class $\text{cl}(W_1)$ of the wobbly divisor $W_1$ in the Picard group of $M_X(2, \mathcal{O}_X)$ is $4\Theta$, where $\Theta$ is the ample generator of the Picard group of $M_X(2, \mathcal{O}_X)$.

Let $k = 2$. Then for a line bundle $L$, $h^0(X, KL^2) \neq 0$ if and only if $L$ is the inverse of a theta characteristic. There are precisely 16 such line bundles. If $L$ is such a line bundle, then any nontrivial extension of $L$ by $L^{-1}$ is stable and for each such line bundle $L$ the space of extensions gives a hyperplane in $\mathbb{P}^3$. Therefore $W_2$ is the union of 16 hyperplanes in $\mathbb{P}^3$ and the class $\text{cl}(W_2)$ of the wobbly divisor $W_2$ in the Picard group of $M_X(2, \mathcal{O}_X)$ is $16\Theta$.

5.1.2. $\lambda = 1$. Let $\Lambda$ be a line bundle of degree 1. It is known that $M_X(2, \Lambda)$ is isomorphic to a smooth intersection $Y$ of two quadrics in $\mathbb{P}^5$. By Theorem 1.1 the wobbly locus is irreducible. If a stable vector bundle $E$ in the wobbly locus, then under the identification of $M_X(2, \Lambda)$ with $Y$, it corresponds to a point $P \in Y$ such that the intersection of $Y$ with the projectivized embedded tangent space of $Y$ at $P$ contains fewer than 4 lines. Classically, it is known that the locus of such points $P \in Y$ is isomorphic to a surface in $\mathbb{P}^5$ of degree 32. In other words, the irreducible wobbly divisor is isomorphic to a surface in $\mathbb{P}^5$ of degree 32. Thus the class $\text{cl}(W_1)$ of the wobbly divisor $W_1$ in the Picard group of $M_X(2, \mathcal{O}_X)$ is $8\Theta$, where $\Theta$ is a hyperplane section of degree 4 of $M_X(2, \Lambda)$.

5.2. Genus 3, $\lambda = 0, k = 2$. It is known that $M_X(2, \mathcal{O}_X)$ is isomorphic to Coble’s quartic hypersurface in $\mathbb{P}^7$ [NR]. On the other hand, by Theorem 1.3 we have that the class $\text{cl}(W_2)$ of the wobbly divisor $W_2$ in the Picard group of $M_X(2, \mathcal{O}_X)$ is $48\Theta$, where $\Theta$ is a hyperplane section of degree 4. Therefore, we can describe $W_2$ as the cut out of Coble’s quartic by a hypersurface of degree 48.

5.3. Arbitrary genus $g$, $\lambda = 0$, $k = g$. We recall that $W_g^0 = \{ E : E$ contains a line subbundle $L$ of degree $1 - g$ with $h^0(KL^2) \neq 0 \}$. For a line bundle $L$ of degree $1 - g$, $h^0(KL^2) \neq 0$ if and only if $L$ is the inverse of a theta characteristic. For each such line bundle $L$ the space of non-trivial extension classes of $L$ by $L^{-1}$ gives a divisor of $M_X(2, \mathcal{O}_X)$, whose class is the ample generator $\Theta$ of the Picard group of $M_X(2, \mathcal{O}_X)$. Therefore the $2^{2g}$ irreducible divisors of $W_g$ correspond to the $2^{2g}$ theta characteristics of $X$. Thus the class $\text{cl}(W_g)$ of the wobbly divisor $W_g$ in the Picard group of $M_X(2, \mathcal{O}_X)$ is $2^{2g}\Theta$. 


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