Construction of optimal teleportation witness operators from entanglement witnesses

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Teleportation witnesses are hermitian operators which can identify useful entanglement for quantum teleportation. Here we provide a systematic method to construct teleportation witnesses from entanglement witnesses corresponding to general qudit systems. The witnesses so constructed are shown to be optimal for qubit and qutrit systems, and therefore detect the largest set of states useful for teleportation within a given class. We demonstrate the action of the witness pertaining to different classes of states in qubits and qutrits. Decomposition of the witness in terms of spin operators facilitates experimental identification of useful resources for teleportation.

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I. INTRODUCTION

Entanglement has long been identified as a unique feature of quantum states due to the seminal work by Schrödinger [1] and EPR [2]. The notion of quantum entanglement has been extensively studied [3] and has paved the way for modern quantum information science [4] enabling tasks such as teleportation, superdense coding and cryptography [5–7], which are beyond the reach of classical physics. Since entanglement is the essential ingredient for several quantum information processing tasks, its detection is important. Experimental detection of entanglement is made possible by entanglement witnesses [8, 9]. Based upon the Hahn–Banach separation axiom from functional analysis [10], entanglement witnesses serve to demarcate entangled states from the ones which are separable. Entanglement witnesses [11–13] provide a necessary and sufficient entanglement criterion in terms of directly measurable observables [13–17] facilitating experimental detection of entanglement.

Entanglement witnesses are not universal, and hence the question as to how to maximally detect entangled states, i.e., increase the number of states detected by the witness, is of significance. The possibility of optimization of entanglement witnesses [18] has lead to the construction of optimal witnesses [19, 20]. The study of entanglement witnesses has proceeded also in the direction of Schmidt number witnesses [21–23] and common witnesses [24, 25]. Entanglement witnesses enable experimentally viable procedures to detect the presence of entanglement, a notion that has been carried forward to identify manifestations of various properties of quantum states, such as macroscopic entanglement through thermodynamical witnesses [26], as well as witnesses for quantum correlations [27], teleportation [28, 29], cryptography [30] and mixedness [31].

Although entanglement is a key ingredient for teleportation, yet not all entangled states are useful for the purpose of teleportation. The problem gets accentuated in higher dimensions where bound entangled states [32] are also present. The ability of an entangled state to perform teleportation is linked to a threshold value of the fully entangled fraction [33] which is difficult to estimate except for some known states [34]. Based upon the linkage of the threshold value of the fully entangled fraction with teleportation fidelity, and utilising again the separation axioms, the existence of hermitian operators acting as teleportation witness was demonstrated recently [28]. A teleportation witness $W_T$ is a hermitian operator with at least one negative eigenvalue and (i) $Tr(W_T \varpi) \geq 0$, for all states $\varpi$ not useful for teleportation and (ii) $Tr(W_T \vartheta) < 0$ for atleast, one entangled state $\vartheta$ which is useful for teleportation. In a following work [29] a teleportation witness with interesting universal properties was proposed, which though depends upon the choice of a unitary operator that may be difficult to find in practice, especially in higher dimensions.

The difficulty in identifying useful resources for teleportation necessitates the construction of suitable teleportation witnesses that would be possible to implement experimentally in order to ascertain whether a given unknown state would be useful as a teleportation channel. Moreover, analogous to the theory of entanglement witnesses, maximal detection of states capable for teleportation is a question of significance. The motivation of this work is to address both the above issues. In the present paper we propose an efficient method to construct teleportation witnesses for general qudit systems starting from entanglement witnesses. We next demonstrate the optimality of such a witness for the case of qubit and qutrit systems, exemplifying its action for different classes of states. We further decompose the witness in terms of spin operators, thereby taking a step towards the viability for its experimental realization.
II. OPTIMAL TELEPORTATION WITNESS

Amongst two witnesses $W_1$ and $W_2$, $W_1$ is said to be finer than $W_2$, if $DW_2 \subseteq DW_1$, where $DW_i = \{ \chi : Tr(W_i\chi) < 0 \}$, $i = 1,2$, i.e., the set of entangled states detected by $W_i$. A witness is said to be optimal if there exists no other witness finer than it [18]. Further, if the set of product vectors $|e, f\rangle$, $P_W = \{ |e, f\rangle : Tr(W|e, f\rangle\langle e, f|) = 0 \}$, spans the relevant product Hilbert space, then the witness $W$ is optimal [18]. Recently, it was shown [33] that if a witness operating on $H_m \otimes H_m$ can be expressed in the form $W = Q^T$, where $Q$ is the projector on a pure entangled state, then the witness $W$ is optimal.

On the other hand, as stated earlier, the ability of a quantum state in performing teleportation is determined by a threshold value of the fully entangled fraction, given by $F(\rho) = \max_{\lambda} Tr((U^\dagger \otimes I)\rho(U \otimes I)|\Phi\rangle\langle \Phi|)$, where $|\Phi\rangle = \frac{1}{\sqrt{2^d}} \sum_{k=1}^{2^d} |kk\rangle$ and $U$ is an unitary operator. Precisely, in $d \otimes d$ systems if $F(\rho)$ exceeds $\frac{1}{2}$, then the state is considered useful for the protocol [33].

A. Optimal teleportation witness for qubits

Consider the entanglement witness, $W^2 = \rho^{T_A}_{\phi^+}$, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, acting on two qubit systems. Since, $\rho_{\phi^+} = \frac{1}{4} (I \otimes I + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)$, one thus obtains, $W^2 = \frac{1}{4} (I \otimes I + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)$, which implies,

$$Tr((W^2 - \frac{1}{4} \sigma_y \otimes \sigma_y)\rho) = \frac{1}{4} Tr(I \otimes I + \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z)\rho)$$

for any arbitrary density matrix $\rho$. Hence,

$$F(\rho) \geq Tr(\rho|\phi^+\rangle\langle \phi^+|)$$

(2)

The r.h.s. of the above equation is given by $\frac{1}{2} Tr((I \otimes I + \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z)|\phi^+\rangle\langle \phi^+|)$, which using Eq. [1], becomes $Tr((W^2 - \frac{1}{2} \sigma_y \otimes \sigma_y)\rho)$. This in turn implies using Eq. [2] that

$$Tr((\frac{1}{2} \sigma_y \otimes \sigma_y + \frac{1}{2} I - W^2)\rho) \geq \frac{1}{2} - F(\rho)$$

(3)

If $\rho$ is not useful for teleportation, i.e., $F(\rho) \leq \frac{1}{2}$, then $Tr((\frac{1}{2} \sigma_y \otimes \sigma_y + \frac{1}{2} I - W^2)\rho) \geq 0$, implying that

$$W_{2@2} = \frac{1}{2} \sigma_y \otimes \sigma_y + \frac{1}{2} I - W^2$$

(4)

is a teleportation witness acting on two qubits.

Next, with some straightforward algebraic manipulation it is observed that the witness can be expressed as

$$W_{2@2} = (|\psi^\perp\rangle\langle \psi^\perp|)^T_A$$

(5)

where, $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Further the product vectors $|0 \rangle + i|1\rangle \otimes (|0 \rangle - i|1\rangle), (|0 \rangle + |1\rangle)\otimes (|0 \rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle$, span $C^2 \otimes C^2$ and belong to $P_{W_{2@2}}$. This establishes the optimality of the teleportation witness [18, 33].

B. Optimal teleportation witness for qutrits

The generalized Gell-Mann matrices are higher dimensional extensions of the Pauli matrices (for qubits) and are hermitian and traceless. They form an orthogonal set and basis. In particular, they can be categorized for qutrits as the following types of traceless matrices [17],

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^9 = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & -2/\sqrt{3} \end{pmatrix}$$

Now, consider the following entanglement witness in qutrits,

$$W^3 = (|\delta\rangle\langle \delta|)^T_A$$

(6)

where, $\delta = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle)$, yielding,

$$W^3 = \frac{1}{9} (I \otimes I + \frac{3}{2} \Delta)$$

(7)

with $\Delta = \sum_{i=1}^{3^2} \lambda^i \otimes \lambda^i$. Therefore, for any arbitrary density matrix $\sigma \in B(H_3 \otimes H_3)$, taking $\Delta_1 = \lambda^2 \otimes \lambda^2 + \lambda^5 \otimes \lambda^5 + \lambda^7 \otimes \lambda^7$ and $\Delta_2 = \lambda^1 \otimes \lambda^1 + \lambda^2 \otimes \lambda^2 + \lambda^3 \otimes \lambda^3 + \lambda^4 \otimes \lambda^4 + \lambda^5 \otimes \lambda^5 + \lambda^6 \otimes \lambda^6$, one gets

$$Tr(W^3 - \frac{1}{6} \Delta_1)\sigma = \frac{1}{9} Tr((I \otimes I + \frac{3}{2} \Delta_2)\sigma)$$

(8)

Hence,

$$F(\sigma) \geq Tr(\sigma|\delta\rangle\langle \delta|)$$

(9)

The r.h.s. may be expressed as $\frac{1}{9} Tr((I \otimes I + \frac{3}{2} (\Delta_2 - \Delta_1))\sigma)$ which using Eq. [6] becomes $Tr((W^3 - \frac{1}{3} \Delta_1)\sigma)$. It follows from Eq. [9] that

$$Tr((\frac{1}{3} \Delta_1 + \frac{1}{3} I - W^3)\sigma) \geq \frac{1}{3} - F(\sigma)$$

(10)

Hence, if $\sigma$ is not useful for teleportation , i.e., $F(\sigma) \leq \frac{1}{3}$, then $Tr((\frac{1}{3} \Delta_1 + \frac{1}{3} I - W^3)\sigma) \geq 0$. Thus,

$$W_{3@3} = \frac{1}{3} \Delta_1 + \frac{1}{3} I - W^3$$

(11)
is indeed a teleportation witness for qudits.

Now, let us denote by $P_{W_{3\otimes 3}}$, the set of all product vectors on which the expectation value of the witness $W_{3\otimes 3}$ vanishes, i.e., $P_{W_{3\otimes 3}} = \{ e, f \} : (e, f)|W_{3\otimes 3}\rangle = 0$. If we consider the product vectors $K_1 = |00\rangle$, $K_2 = |11\rangle$, $K_3 = |22\rangle$, $K_4 = (|0\rangle + |1\rangle + |2\rangle)/\sqrt{3}$, $K_5 = (|0\rangle + i|1\rangle) \otimes (|0\rangle - i|1\rangle)$, $K_6 = (|0\rangle + i|2\rangle) \otimes (|0\rangle - i|2\rangle)$, $K_7 = (|1\rangle + i|2\rangle) \otimes (|1\rangle - i|2\rangle)$, $K_8 = (|0\rangle - |1\rangle - |2\rangle)/\sqrt{2}$, $K_9 = (|0\rangle + |1\rangle - |2\rangle)/\sqrt{2}$, it is noticed that (i) $\langle K_i | W_{3\otimes 3} | K_i \rangle = 0$, and (ii) $K_i$’s are linearly independent, $\forall i \in \{ 1, 2, \ldots, 9 \}$. Thus it follows that $P_{W_{3\otimes 3}}$ spans $C^3 \otimes C^3$. This ascertains the optimality of the witness $W_{3\otimes 3}$.

C. Teleportation witness for qudits

For general qudit systems the construction of teleportation witnesses from entanglement witnesses may be undertaken in a manner similar to that shown above for qubits or qudits. Utilising the generalized Gell-Mann matrices for $d \otimes d$ systems, and retracing the steps of an argument similar to that used for qubits and qudits, one can obtain a teleportation witness for qudits as

$$W_{d\otimes d} = \frac{1}{d} \sum_{j=0}^{d-1} \sum_{k=0}^{d-1} (\Lambda^j_k \otimes \Lambda^k_j) + \frac{1}{d} I - \langle \Phi | \Phi \rangle T_{\alpha}$$

where, $\Lambda^j_k = -i|j\rangle \langle k| + i|k\rangle \langle j|$, $0 \leq j < k \leq d - 1$ and $\langle \Phi | = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} |l\rangle$. Here it may be remarked that there is general proof of optimality for teleportation witness for qudits, but optimality for a given dimension needs to be checked in the manner above by considering the set of all product vectors on which the expectation value of the witness vanishes.

III. ILLUSTRATIONS AND DECOMPOSITION

We now consider certain classes of states pertaining to qubits and qudits, which exemplify the action of our constructed witness. Let us first take the class of two qubit states with maximally mixed marginals, given by

$$\eta_{mix} = \frac{1}{4} (I \otimes I + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i)$$

The expectation value of the witness given by Eq. 4 on the above state gives

$$Tr(W_{2\otimes 2} \eta_{mix}) = \frac{1}{4} (1 + c_2 - c_1 - c_3)$$

implying that for $1 + c_2 - c_1 - c_3 < 0$, the witness $W_{2\otimes 2}$ detects the states as useful for teleportation. Since $W_{2\otimes 2}$ is optimal, this is the largest set of states useful for teleportation in the given class that can be detected by any witness. Next, we consider the isotropic state in qudits, given by

$$\eta_{iso} = (1/9) |\varphi^+_{12}\rangle \langle \varphi^+_{12}| + (1/9) I$$

where, $|\varphi^+_{12}\rangle = (1/\sqrt{3})(|00\rangle + |11\rangle + |22\rangle)$ and $-\frac{1}{3} \leq \alpha \leq 1$. Now applying the witness given by Eq. (11), it is observed that

$$Tr(W_{3\otimes 3} \eta_{iso}) = \frac{2 - 8\alpha}{9}$$

implying that for $\alpha > \frac{1}{4}$, the states are useful for teleportation.

Thus, the witness $W_{3\otimes 3}$ detects all entangled isotropic states as useful for teleportation, in conformity with a result already known in the literature [34]. This is a reaffirmation of the optimality of the witness $W_{3\otimes 3}$, as it detects the maximal class of isotropic states as useful for teleportation.

The practical use for teleportation witnesses is that they are experimentally realizable on account of being hermitian. For qubit systems, the decomposition of a proposed teleportation witness in terms of Pauli spin operators has been shown earlier [28]. The teleportation witness constructed here is expressed in terms of generalized Gell-Mann matrices which are hermitian. However, for $d = 3$, i.e., qudit systems the teleportation witness can also be expressed in terms of spin-1 operators [17] which are the observables $S_x, S_y, S_z, S_x^2, S_y^2, S_z^2, \{S_x, S_y\}, \{S_y, S_z\}, \{S_z, S_x\}$ of a spin-1 system, where $S^2 = \{S_x, S_y, S_z\}$ is the spin operator and $\{S_i, S_j\} = S_i S_j + S_j S_i$ (with $i, j = x, y, z$) denotes the corresponding anticommutator. They are given by

$$S_x = \hbar \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{array} \right), S_y = \hbar \frac{1}{\sqrt{2}} \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{array} \right), S_z = \hbar \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

Expressing the witness given by Eq. (11) in terms of spin-1 operators, yields

$$W_{3\otimes 3} = \frac{2}{9} (I \otimes I) + \Pi$$

where

$$\Pi = \frac{1}{6\hbar^2} (S_y \otimes S_y - S_z \otimes S_z - S_x \otimes S_x)$$

$$+ \frac{1}{6\hbar^2} (S_z \otimes S_z - \{S_z, S_z\} + S_x \otimes S_x + \{S_x, S_x\})$$

$$+ \frac{1}{3\hbar^2} (S_y \otimes S_y - \{S_y, S_y\} + \frac{2}{3\hbar^2} (I \otimes S_x^2 + I \otimes S_y^2 + S_x^2 \otimes I + S_y^2 \otimes I - \frac{2}{3\hbar^2} (S_x^2 \otimes S_y^2 + S_y^2 \otimes S_x^2)$$

$$- \frac{1}{3\hbar^2} (S_x^2 \otimes S_x^2 + S_y^2 \otimes S_y^2)$$

Thus, for an experimental outcome,

$$\langle W_{3\otimes 3} \rangle = \frac{2}{9} (I \otimes I) + \langle \Pi \rangle < 0$$
one can detect the given unknown state as useful for teleportation.

IV. SUMMARY

We have presented here a method to construct teleportation witnesses from entanglement witnesses for general qudit systems. Optimality of the witnesses that we have constructed for qubit and qutrit states ensures a broader perspective in the sense that a maximal class of entangled states can now be recognized to be useful for teleportation. Decomposition of the proposed witness in terms of spin operators authenticates its feasibility in experimental detection of entanglement. The present analysis may be extended in a few directions. One may seek to test the optimality of the witness for two-qudits of any given dimension $d > 3$. Finally, the choice of the entanglement witnesses are not limited to the ones we have taken up here, and other entanglement witnesses may be considered and checked for their viability in the construction of teleportation witnesses using similar methods.

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