On the structure of the set of bifurcation points of periodic solutions for multiparameter Hamiltonian systems

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Abstract
This paper deals with periodic solutions of the Hamilton equation \( \dot{x}(t) = J \nabla_x H(x(t), \lambda) \), where \( H \in C^2([0, 2\pi]^n \times \mathbb{R}^k, \mathbb{R}) \) and \( \lambda \in \mathbb{R}^k \) is a parameter. Theorems on global bifurcation of solutions with periods \( \frac{2\pi}{j} \), \( j \in \mathbb{N} \), from a stationary point \( (x_0, \lambda_0) \in [0, 2\pi]^n \times \mathbb{R}^k \) are proved. \( \nabla_x^2 H(x_0, \lambda_0) \) can be singular. However, it is assumed that the local topological degree of \( \nabla_x H(\cdot, \lambda_0) \) at \( x_0 \) is nonzero. For systems satisfying \( \nabla_x H(x_0, \lambda) = 0 \) for all \( \lambda \in \mathbb{R}^k \) it is shown that (global) bifurcation points of solutions with periods \( \frac{2\pi}{j} \) can be identified with zeros of appropriate continuous functions \( F_j: \mathbb{R}^k \to \mathbb{R} \). If, for all \( \lambda \in \mathbb{R}^k \), \( \nabla_x^2 H(x_0, \lambda) = \text{diag}(A(\lambda), B(\lambda)) \), where \( A(\lambda) \) and \( B(\lambda) \) are \( (n \times n) \)-matrices, then \( F_j \) can be defined by \( F_j(\lambda) = \det[A(\lambda)B(\lambda) - j^2 I] \). Symmetry breaking results concerning bifurcation of solutions with different minimal periods are obtained. A geometric description of the set of bifurcation points is given. Examples of constructive application of the theorems proved to analytical and numerical investigation and visualization of the set of all bifurcation points in given domain are provided.

This paper is based on a part of the author’s thesis (Radzki 2005 Branching points of periodic solutions of autonomous Hamiltonian systems (Polish) PhD Thesis Nicolaus Copernicus University, Faculty of Mathematics and Computer Science, Toruń).

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(Some figures in this article are in colour only in the electronic version)
1. Introduction

The aim of this paper is to describe the set of stationary bifurcation points of solutions of the Hamilton equation with the condition of $2\pi$-periodicity of solutions

\[
\dot{x}(t) = J \nabla_x H(x(t), \lambda), \\
x(0) = x(2\pi),
\]

where $H \in C^{2,0}(\mathbb{R}^n \times \mathbb{R}^k, \mathbb{R})$, $J$ is the standard symplectic matrix, and $\lambda \in \mathbb{R}^k$ is a parameter. In particular, this work is intended to investigate the subsets of the set of bifurcation points consisting of global bifurcation points of solutions with periods $\frac{2\pi}{j}$, $j \in \mathbb{N}$, with determining minimal periods of these solutions near bifurcation points, and to prove theorems concerning symmetry breaking points, defined as bifurcation points of solutions with different minimal periods. Given bifurcation point is called a global bifurcation point if there exists a connected set of nontrivial solutions bifurcating from this point and it is unbounded or meets another bifurcation point, see definition 2.3.

In the case of the systems with linear dependence on one parameter problem (1.1) can be written as

\[
\dot{x}(t) = \lambda J \nabla H(x(t)), \\
x(0) = x(2\pi),
\]

where $H \in C^2(\mathbb{R}^n, \mathbb{R})$ and $\lambda \in \mathbb{R}$. One of the reasons to investigate problem (1.2) is that every connected branch of nontrivial solutions of (1.2) bifurcating (in a suitable function space) from $(x_0, \lambda_0) \in (\nabla H)^{-1}(\{0\}) \times (0, +\infty)$ can be translated to the corresponding connected (with respect to Hausdorff metric, see [30, 33]) branch of nonstationary periodic trajectories of the equation

\[
\dot{x}(t) = J \nabla H(x(t))
\]

emanating from $x_0$ with periods tending to $2\pi \lambda_0$ at $x_0$. Especially interesting systems are those for which the Hessian matrix of $H$ at $x_0$ has the block-diagonal form $\nabla^2 H(x_0) = \text{diag}(A, B)$, where $A$ and $B$ are real symmetric $(n \times n)$-matrices. This condition is satisfied in particular in the natural case of Hamiltonian function being the sum of kinetic energy dependent on generalized momenta and potential energy dependent on generalized coordinates, for example if

\[
H(x) = H(y, z) = \frac{1}{2}(M^{-1}y, y) + V(z),
\]

where $y, z \in \mathbb{R}^n$, $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and $M$ is a nonsingular real symmetric $(n \times n)$-matrix. Equation (1.3) with $H$ given by (1.4) is equivalent to the Newton equation

\[
\ddot{z}(t) = -\nabla V(z(t)).
\]

If $x_0$ is a stationary point of (1.3), $J \nabla^2 H(x_0)$ is nonsingular, and it has nonresonant purely imaginary eigenvalues then the Lyapunov centre theorem [22] ensures the existence of a one-parameter family of nonstationary periodic solutions of (1.3) emanating from $x_0$. The Lyapunov centre theorem can be derived from the Hopf bifurcation theorem [17]. Berger [5] (see also [6, 26]), Weinstein [40], Moser [27] and Fadell and Rabinowitz [14] proved the existence of a sequence of periodic solutions of (1.3) convergent to a nondegenerate stationary point $x_0$ in the case of possibly resonant purely imaginary eigenvalues of $J \nabla^2 H(x_0)$. (The theorem of Berger concerns second order equations, including (1.5) for $M = I$.) Global bifurcation theorems in nondegenerate case have been proved by Gęba and Marzantowicz [16] by using topological degree for $SO(2)$-equivariant mappings.

Zhu [41] and Szulkin [39] used Morse theoretic methods and they proved the existence of a sequence of periodic solutions of (1.3) emanating from a stationary point which can be
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Dancer and Rybicki [9] obtained a global bifurcation theorem of Rabinowitz type (see [28]) for (1.2) in the case of possibly degenerate stationary point by using the topological degree theory for $SO(2)$-equivariant gradient maps. (Maciejewski and Rybicki used a version of the theorem from [9] to investigate some concrete global bifurcation problems in celestial mechanics, see, e.g. [25].) The results from [9] were applied by the author [30] and the author with Rybicki [33] to the description of connected branches of bifurcation of (1.2) and emanation of (1.3) in possibly degenerate case under assumptions written in terms of eigenvalues of $\nabla^2 H(x_0)$ and the local topological degree of $\nabla H$ in a neighbourhood of $x_0$. The examples of applications of the results from [30, 33] were given by Maciejewski et al [24].

The structure of the set of bifurcation points of periodic solutions of the first order ordinary differential equations with many parameters was studied by Izydorek and Rybicki [20] and Rybicki [34] (see also references therein). They applied the Krasnosiel’skii bifurcation theorem (see [21]) and the results of real algebraic geometry obtained by Szafraniec [37, 38]. However, Izydorek and Rybicki assumed that the Fr’echet derivative of the right-hand side of the equation they considered was zero. In such a case there is no bifurcation of nonstationary solutions of Hamiltonian system with fixed period (see remark 3.7).

In this paper (being a revised version of preprint [32]), which presents the results of a part of the author’s PhD thesis [31] (with corollaries 3.8, 6.11, 6.12, examples 7.5, 7.6 and figures added afterwards), the stationary point $(x_0, \lambda_0)$ can be degenerate, i.e. $\nabla^2_x H(x_0, \lambda_0)$ can be singular. However, it is assumed that the local Brouwer degree of $\nabla_x H(\cdot, \lambda_0)$ in a neighbourhood of $x_0$ is well defined and nonzero. (Although theorems without this assumption and corresponding examples are also given.) The set of bifurcation points of (1.1) is investigated in the case of many parameters.

This paper is organized as follows. Section 2 concerns notation, terminology and basic tools used in the paper. Section 3 is devoted to possible minimal periods of solutions near stationary point (theorem 3.4) and to necessary conditions for bifurcation. In section 4 a generalized version of the global bifurcation theorem of Dancer and Rybicki [9] in the case of Hamiltonian systems with one parameter is proved (theorems 4.3–4.5) and some results from [30, 33] concerning unbounded branches of periodic solutions are generalized. In section 5 the theorems for one parameter are used to obtain global bifurcation theorems for Hamiltonian systems with many parameters (theorems 5.8, 5.9 and 5.11, 5.12). They are applied, in turn, in section 6 to the description of the structure of the set of bifurcation points in the case of many parameters (theorems 6.1–6.4).

The proofs exploit the topological degree for $SO(2)$-equivariant gradient mappings (see [35]). Bifurcation points of solutions of (1.1) with period $\frac{2\pi}{j}$, $j \in \mathbb{N}$ (proved to be global bifurcation points) are identified with zeros of suitable continuous functions $F_j: \mathbb{R}^k \to \mathbb{R}$, under assumptions written in terms of those functions. In the case of systems satisfying, for all $\lambda \in \mathbb{R}^k$, the condition $\nabla^2_x H(x_0, \lambda) = \text{diag}(A(\lambda), B(\lambda))$, where $A(\lambda)$ and $B(\lambda)$ are some $(n \times n)$-matrices, the functions $F_j$ are given by $F_j(\lambda) = \det[A(\lambda)B(\lambda) - j^2 I]$.

Symmetry breaking results are obtained. A geometric description of the set of bifurcation points is given (corollaries 6.8, 6.9 and 6.11, 6.12) by using results of real algebraic geometry [37, 38].

Section 7 provides examples of application of theorems proved in this paper to analytical and numerical investigation and visualization of the set of all bifurcation points in given domain (examples 7.3–7.6). They demonstrate constructive character of the results obtained in this paper by using topological degree.

Potential real-life applications are left to future papers.
2. Preliminaries

In this section notation and terminology are set up and basic results used in this paper are summarized to make the exposition self-contained.

2.1. Algebraic notation

Let $\mathbb{M}(n, \mathbb{R})$ be the set of all real $(n \times n)$-matrices and let $\mathbb{GL}(n, \mathbb{R})$, $\mathbb{S}(n, \mathbb{R})$, $\mathbb{O}(n, \mathbb{R})$ be the subsets of $\mathbb{M}(n, \mathbb{R})$ consisting of nonsingular, symmetric and orthogonal matrices, respectively. For given $n \in \mathbb{N}$ the identity $(n \times n)$-matrix is denoted by $I \equiv I_n$, whereas $J \equiv J_n$:

$$H_0 = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

For any square matrices $A_1, \ldots, A_m$ the symbol $\text{diag}(A_1, \ldots, A_m)$ stands for the block-diagonal matrix built from $A_1, \ldots, A_m$.

If $A \in \mathbb{M}(n, \mathbb{R})$ then $\sigma(A)$ denotes the spectrum of $A$, whereas $\sigma^{+}(A)$ and $\sigma^{-}(A)$ are the sets of real positive and real negative eigenvalues of $A$, respectively. If $\alpha \in \sigma(A)$ then $\mu(\alpha) \equiv \mu_A(\alpha)$ denotes the algebraic multiplicity of $\alpha$.

The negative and the positive Morse index of $A \in \mathbb{S}(n, \mathbb{R})$ are defined as

$$m^{-}(A) := \sum_{\alpha \in \sigma^{-}(A)} \mu(\alpha), \quad m^{+}(A) := \sum_{\alpha \in \sigma^{+}(A)} \mu(\alpha),$$

respectively.

Let a representation of a group $G$ on a linear space $V$ be given. For every subgroup $H$ of $G$ and every subset $\Omega$ of $V$ it is assumed

$$\Omega^H := \{v \in \Omega | h \in H \Rightarrow hv = v\} = \{v \in \Omega | H \subset G_v\},$$

$$\Omega_H := \{v \in \Omega | H = G_v\},$$

where $G_v$ is the isotropy group of $v$. Consider another representation of $G$ on a linear space $W$ and let $f: V \rightarrow W$ be a $G$-equivariant map. As well known, $G_v \subset G_{f(v)}$ for every $v \in V$.

If $H$ is a subgroup of $G$ then $f^H$ denotes the restriction of $f$ to the pair $(V^H, W^H)$.

For $j \in \mathbb{N}$ set

$$\mathbb{R}_j = \begin{cases} \mathbb{Z}_j & \text{if } j \in \mathbb{N}, \\ \mathbb{S}(2) & \text{if } j = 0. \end{cases}$$

For every $j \in \mathbb{N}$ let $\rho_j: \mathbb{S}(2) \rightarrow \mathbb{O}(2, \mathbb{R})$ be the homomorphism defined by

$$\rho_j \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} = \begin{bmatrix} \cos j \phi & -\sin j \phi \\ \sin j \phi & \cos j \phi \end{bmatrix}, \quad 0 \leq \phi < 2\pi.$$

For $m, j \in \mathbb{N}$ the representation $\mathbb{R}[m, j]$ of $\mathbb{S}(2)$ is defined as the direct sum of $m$ copies of the representation $(\mathbb{R}^2, \rho_j)$, whereas $\mathbb{R}[m, 0]$ denotes the identity representation of $\mathbb{S}(2)$ on $\mathbb{R}^m$.

Fix $k \in \mathbb{N}$. For every $j \in \mathbb{N}$, $K \in \mathbb{S}(2n, \mathbb{R})$, and every $T: \mathbb{R}^k \rightarrow \mathbb{S}(2n, \mathbb{R})$ set

$$Q_j(K) = \frac{1}{1 + j^2} \begin{bmatrix} -K & jJ' \\ jJ & -K \end{bmatrix}, \quad Q_0(K) = -K, \quad (2.1)$$

$$\Lambda_j(T) = \{\lambda \in \mathbb{R}^k | \det Q_j(T(\lambda)) = 0\},$$

$$\Lambda_0(T) = \{\lambda \in \mathbb{R}^k | \det Q_0(T(\lambda)) = 0\}, \quad (2.2)$$

where

$$J' = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$
For \( k = 1 \) let
\[
\Lambda^+_j(T) = \Lambda_j(T) \cap (0, +\infty),
\]
\[
\Lambda^-(T) = \Lambda(T) \cap (0, +\infty).
\]

Remark 2.1.
(i) For every \( j \in \mathbb{N} \) the eigenvalues of \( Q_j(K) \) have even multiplicity. Indeed, if \((v_1, v_2) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} \setminus \{(0, 0)\} \) is an eigenvector of \( Q_j(K) \) then \((v_1 - v_2, v_1 + v_2)\) is an eigenvector corresponding to the same eigenvalue.

(ii) Let \( T: \mathbb{R}^k \to S(2n, \mathbb{R}) \) be continuous. For every \( j \in \mathbb{N} \) one has \( Q_j(T(\lambda)) = \frac{1}{j^4} T(P + \frac{1}{j} Z(\lambda)), \) where
\[
P = \begin{bmatrix}
0 & J^j \\
J & 0
\end{bmatrix}, \quad Z(\lambda) = \begin{bmatrix}
-T(\lambda) & 0 \\
0 & -T(\lambda)
\end{bmatrix}.
\]

Since \( \sigma(P) = \{-1, 1\} \), there exists \( \varepsilon > 0 \) such that for every \( L \in S(4n, \mathbb{R}) \) with the operator norm \( \|L\| < \varepsilon \) one has \( \sigma(P + L) \cap (-\frac{1}{j}, \frac{1}{j}) = \emptyset \), hence \( \det(P + L) \neq 0 \). On the other hand, since \( T \) is continuous, for any bounded open set \( U \subset \mathbb{R}^k \) the number \( \sup_{x \in U} \|Z(\lambda)\| \) is finite. Thus for fixed \( U \) there exists \( m \in \mathbb{N} \) such that for every \( \lambda \in U \) and \( j \in \mathbb{N}, j > m \), one has \( \frac{1}{j} \|Z(\lambda)\| < \varepsilon \), hence \( \det Q_j(T(\lambda)) \neq 0 \).

The matrices \( Q_j(K) \) will serve as blocks forming the Hessian operator of suitable action functional and they will be involved in definition of bifurcation index. The sets \( \Lambda_j(T), \Lambda(T) \), for appropriate \( T \), will be sets of values of the parameter \( \lambda \) suspected of bifurcation.

2.2. Degree for \( S\mathbb{O}(2) \)-equivariant gradient maps

Proofs of global bifurcation theorems in this paper exploit the topological degree for \( S\mathbb{O}(2) \)-equivariant gradient mappings, which is a special case of the degree described in [35]. For earlier results concerning equivariant degree see [8, 18, 11, 19, 15] and references therein.

Consider an orthogonal representation of the group \( S\mathbb{O}(2) \) on a real inner product space \( V \) with \( \dim V < \infty \). Let \( \Omega \) be an \( S\mathbb{O}(2) \)-invariant bounded open subset of \( V \) and let \( \nabla f : V \to V \) be a continuous \( S\mathbb{O}(2) \)-equivariant gradient mapping such that \( \nabla f(x) \neq 0 \) for every \( x \in \partial \Omega \). Then
\[
\text{DEG}(\nabla f, \Omega) = \{\text{DEG}_j(\nabla f, \Omega)\}_{j \in \mathbb{N} \cup \{0\}}
\]
denotes the \( S\mathbb{O}(2) \)-degree of \( \nabla f \) in \( \Omega \) [35]. It is an element of the Euler ring of the group \( S\mathbb{O}(2) \), i.e., the ring
\[
\text{U}(S\mathbb{O}(2)) = \bigoplus_{j \in \mathbb{N} \cup \{0\}} \mathbb{K}_j
\]
with addition + and multiplication \( \star \) defined for every \( \{a_j\}_{j=0}^\infty, \{b_j\}_{j=0}^\infty \in \text{U}(S\mathbb{O}(2)) \) by
\[
\{a_j\}_{j=0}^\infty + \{b_j\}_{j=0}^\infty = \{a_j + b_j\}_{j=0}^\infty \quad \text{and} \quad \{a_j\}_{j=0}^\infty \star \{b_j\}_{j=0}^\infty = \{c_j\}_{j=0}^\infty,
\]
where \( c_0 = a_0 b_0, c_j = a_j b_j + a_j b_0, b_j, j \in \mathbb{N} \). Note that \( \Theta = (0, 0, \ldots) \) and \( (1, 0, 0, \ldots) \) are the neutral elements of addition and multiplication in this ring, respectively. The properties of the degree \( \text{DEG} \) involved in its use in this paper are described in appendix B. They are analogous to the properties of the Brouwer degree (see [35]). However, if \( \text{DEG}_j(\nabla f, \Omega) \neq 0 \) for some \( j \in \mathbb{N} \cup \{0\} \) then \( (\nabla f)^{-1}(\{0\}) \cap \Omega^K \neq \emptyset \) (not only \( (\nabla f)^{-1}(\{0\}) \cap \Omega \neq \emptyset \)).
If \( y_0 \in \mathbb{R}^m \) is an isolated zero of a continuous mapping \( g: \mathbb{R}^m \to \mathbb{R}^m \) then the topological index \( i(g, y_0) \) of \( y_0 \) with respect to \( g \) is defined as the Brouwer degree \( \deg(g, B(y_0, r), 0) \) of \( g \) in a ball \( B(y_0, r) \subset \mathbb{R}^m \) centred at \( y_0 \) with radius \( r > 0 \) such that \( g^{-1}([0]) \cap \text{cl}(B(y_0, r)) = \{y_0\} \).

Analogously, if \( x_0 \) is an isolated element of \( (\nabla f)^{-1}([0]) \) then its index

\[
I(\nabla f, x_0) = \{i(f, \nabla f, x_0) \}_{j \in \mathbb{N} \cup \{0\}} \subset U(SO(2))
\]

with respect to \( \nabla f \) is defined by the formula

\[
I(\nabla f, x_0) = \text{DEG}(\nabla f, B(x_0, r)),
\]

where \( B(x_0, r) \subset V \) is a ball such that \( (\nabla f)^{-1}([0]) \cap \text{cl}(B(x_0, r)) = \{x_0\} \).

**Lemma 2.2 ([9]).** Let \( V = \mathbb{R}^m [m, 0] \oplus \mathbb{R}[m, j_1] \oplus \cdots \oplus \mathbb{R}[m, j_r] \), where \( m, m_1, j_i \in \mathbb{N}, 0 < j_1 < \cdots < j_r \). Assume that \( f \in C^2(V, \mathbb{R}) \) is an \( SO(2) \)-equivariant map and \( x_0 \) is an isolated element of \( (\nabla f)^{-1}([0]) \) such that \( \nabla^2 f(x_0) = \text{diag}(A_0, A_1, \ldots, A_r) \) for some matrix \( A_0 \) of dimension \( m \) and for nonsingular matrices \( A_i \) of dimensions \( 2m_i \), \( i = 1, \ldots, r \). Then

\[
I_0(\nabla f, x_0) = 1(\nabla f, x_0)
\]

and for every \( j \in \mathbb{N} \) one has

\[
I_j(\nabla f, x_0) = \begin{cases} 
(i(\nabla f, x_0), x_0) & \text{if } j = j_i \text{ for some } i \in \{1, \ldots, r\}, \\
0 & \text{otherwise.}
\end{cases}
\]

Assume that \( \nabla f: V \times \mathbb{R} \to V \) is a continuous \( SO(2) \)-equivariant gradient (with respect to \( V \)) mapping such that \( \nabla f(x_0, \lambda) = 0 \) for some fixed \( x_0 \in V \) and all \( \lambda \in \mathbb{R} \). Fix \( \lambda_0 \in \mathbb{R} \) and assume that for every \( \lambda \in \mathbb{R}, \lambda \neq \lambda_0 \), from a neighbourhood of \( \lambda_0 \) there exists a neighbourhood \( W \subset V \times \mathbb{R} \) of \( (x_0, \lambda) \) such that \( (\nabla f)^{-1}([0]) \cap W \subset \{x_0\} \times \mathbb{R} \). Then for sufficiently small \( \varepsilon > 0 \) one can define the bifurcation index

\[
\text{IND}(x_0, \lambda_0) = [\text{IND}(x_0, \lambda_0)]_{j \in \mathbb{N} \cup \{0\}} \subset U(SO(2))
\]

of \( (x_0, \lambda_0) \) with respect to \( \nabla f \) by

\[
\text{IND}(x_0, \lambda_0) = I(\nabla f, (\lambda_0 + \varepsilon), x_0) - I(\nabla f, (\lambda_0 - \varepsilon), x_0).
\]

This bifurcation index will be used in the proof of theorem 4.3.

**2.3. Functional setting**

For given Hilbert spaces \( Y, E, Z \) the symbols \( C^{1,0}(Y \times E, Z) \) and \( C^{2,0}(Y \times E, Z) \) denote the sets of continuous functions from \( Y \times E \) to \( Z \) having, respectively, first partial Fréchet derivative and two first partial Fréchet derivatives with respect to \( Y \) continuous on \( Y \times E \).

Solutions \( (x, \lambda) \) of (1.1) are regarded as elements of the space \( H_{1\pi}^2 \times \mathbb{R}^k \). (The description of the Sobolev space \( H_{1\pi}^2 \equiv W^{1,2}([0, 2\pi], \mathbb{R}^{2n}) \) can be found in [26].) The inner product in \( H_{1\pi}^2 \) is defined for every \( x, y \in H_{1\pi}^2 \) by the formula

\[
(x, y)_{H_{1\pi}^2} = (x, y)_{L_{2\pi}^2} + (\dot{x}, \dot{y})_{L_{2\pi}^2} = \int_0^{2\pi} (x(t), y(t)) dt + \int_0^{2\pi} (\dot{x}(t), \dot{y}(t)) dt,
\]

where \( \dot{x} \) stands for the weak derivative of \( x \) and \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^{2n} \).

Since every \( x \in H_{1\pi}^2 \) has a continuous representative (denoted by the same symbol) satisfying the condition \( x(0) = x(2\pi) \), it can be regarded as a continuous \( 2\pi \)-periodic function on \( \mathbb{R} \).

For fixed \( \lambda \in \mathbb{R}^k \) a function \( x \in H_{1\pi}^2 \) is called a weak solution of (1.1) if the equation

\[
\dot{x}(t) = J\nabla_x H(x(t), \lambda)
\]

(where \( \dot{x} \) denotes the weak derivative of \( x \)) is satisfied for almost all \( t \in [0, 2\pi] \). However, since it is assumed that \( H \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R}) \), every such solution is
in fact a classical solution of class $C^2$ on $[0, 2\pi]$ and it has a unique extension to the classical solution on $\mathbb{R}$, which is a $2\pi$-periodic function of class $C^2$.

Let $Y, Z$ be Hilbert spaces. Consider a map $F: Y \times \mathbb{R}^k \rightarrow Z$ and a fixed set $\Delta \subset Y$ such that $F(x, \lambda) = 0$ for all $x \in \Delta, \lambda \in \mathbb{R}^k$. The set $\Delta \times \mathbb{R}^k$ is referred to as the set of trivial solutions of the equation

$$F(x, \lambda) = 0.$$  

(2.6)

The complement of $\Delta \times \mathbb{R}^k$ in the set of all solutions of (2.6) in $Y \times \mathbb{R}^k$ is called the set of nontrivial solutions.

**Definition 2.3.** Let $X \subset Y \times \mathbb{R}^k$ be a subset of the set of nontrivial solutions of (2.6). A solution $(x_0, \lambda_0) \in \Delta \times \mathbb{R}^k$ is called a bifurcation point of solutions from $X$ if it is a cluster point of $X$. It is called a branching point of solutions from $X$ if there exists a connected set $C \subset X$ such that $(x_0, \lambda_0) \in \text{cl}(C)$ (the closure in $Y \times \mathbb{R}^k$). If the connected component of $\text{cl}(X)$ containing the bifurcation point $(x_0, \lambda_0)$ is unbounded or it contains another bifurcation point of solutions from $X$ then $(x_0, \lambda_0)$ is said to be a global bifurcation point of solutions from $X$.

In the above definition bifurcation point is called global because of the existence of a global branch of nontrivial solutions bifurcating from it, being a connected set which is unbounded or meets another bifurcation point, whereas in the case of usual bifurcation point it suffices that there exists a sequence of nontrivial solutions convergent to this point, which is a local condition. Such terminology is used in the context of global bifurcation theorems of Rabinowitz type [28] (see [10, 16]).

Assuming $Y = H^1_{2\pi}, Z = L^2_{2\pi}$, and defining $F: H^1_{2\pi} \times \mathbb{R}^k \rightarrow L^2_{2\pi}$ by

$$F(x, \lambda)(t) = \dot{x}(t) - J\nabla_x H(x(t), \lambda),$$

one can write (1.1) in form (2.6), therefore definition 2.3 can be applied. If $(x, \lambda)$ is a stationary solution of (1.1), i.e. $x$ is constant, then $x$ is regarded as an element of $\mathbb{R}^{2n}$. For fixed $x_0 \in \mathbb{R}^{2n}$ such that

$$\nabla_x H(x_0, \lambda) = 0 \quad \text{for all } \lambda \in \mathbb{R}^k$$

(2.7)

one can assume $\Delta = \{x_0\}$. In such a case the set $\{x_0\} \times \mathbb{R}^k$ of trivial solutions of (1.1) is denoted by $T(x_0)$ and the symbol $NT(x_0)$ stands for the set of nontrivial solutions of (1.1). Note that $NT(x_0)$ can contain stationary solutions. The case of $\Delta$ containing more stationary points is also considered in this paper (see theorems 4.3 and 4.6).

Define the action of $SO(2)$ on $H^1_{2\pi}$ as follows. For every $x \in H^1_{2\pi}$ and

$$g = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \in SO(2), \quad 0 \leq \phi < 2\pi,$$

(2.8)

set

$$(g \chi)(t) = x(t + \phi).$$

The space $H^1_{2\pi} \times \mathbb{R}^k$ is regarded as the direct sum of the orthogonal representation of $SO(2)$ on $H^1_{2\pi}$ defined above and the identity representation of $SO(2)$ on $\mathbb{R}^k$. One has $SO(2)x = SO(2)x$ for every $(x, \lambda) \in H^1_{2\pi} \times \mathbb{R}^k$.

The subspaces of $H^1_{2\pi}$ defined as

$$E_0 := \{x \in H^1_{2\pi} \mid x(t) \equiv a, a \in \mathbb{R}^{2n}\},$$

$$E_j := \{x \in H^1_{2\pi} \mid x(t) \equiv a \cos jt + b \sin jt, \ a, b \in \mathbb{R}^{2n}\}, \quad j \in \mathbb{N},$$
are $\mathbb{SO}(2)$-equivariant. One has $E_0 \approx \mathbb{R}[2n, 0]$ and $E_j \approx \mathbb{R}[2n, j]$ for $j \in \mathbb{N}$. Obviously, $(H_{2\pi})^{\mathbb{SO}(2)} = (H_{2\pi})_{\mathbb{SO}(2)} = E_0$ and if $j \in \mathbb{N}$, $v \in E_j \setminus \{0\}$, then $\mathbb{SO}(2)_v = Z_j$. Furthermore,

$$
(H_{2\pi})^2 = \bigoplus_{i \in \mathbb{N}\cup\{0\}} E_{ij} \approx \bigoplus_{i \in \mathbb{N}\cup\{0\}} \mathbb{R}[2n, ij].
$$

Let $(e_1, \ldots, e_{2n})$ be the standard basis in $\mathbb{R}^{2n}$. For fixed $j \in \mathbb{N}$ set $\varphi_0(t) \equiv 1$, $\varphi_j(t) \equiv \cos jt$, $\psi_j(t) \equiv \sin jt$, and

$$
\hat{e}_i = \begin{cases}
  e_i \varphi_j; & 1 \leq i \leq 2n, \\
  e_{i-2n} \psi_j; & 2n + 1 \leq i \leq 4n.
\end{cases}
$$

Then $(e_1\varphi_0, \ldots, e_{2n}\varphi_0)$ and $(\hat{e}_1, \ldots, \hat{e}_{4n})$ are called the standard bases in $E_0$ and $E_j$, respectively. The standard basis in $E_{j_1} \oplus \cdots \oplus E_{j_s}$, where $j_1, \ldots, j_s \in \mathbb{N} \cup \{0\}$, $s \in \mathbb{N}$, is built from the standard bases in $E_{j_1}, \ldots, E_{j_s}$.

**Remark 2.4.** There exists $c > 0$ such that for every $x \in H_{2\pi}^1$ (identified with its continuous representative) one has $\|x\| \leq c\|x\|_H^1$, where $\|x\|_H^1 = \sup_{t \in [0, 2\pi]} |x(t)|$ (see [26]). As it was observed in [9], for given $H \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R})$ and bounded $U \subset H_{2\pi}^1 \times \mathbb{R}^k$ one can find $\xi > 0$ and $H_1 \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R})$ such that

1. for every $(x, \lambda) \in \text{cl}(U)$, $i \in \{0, 2\pi\}$ one has $(x(t), \lambda) \in B(0, \xi) \subset \mathbb{R}^{2n} \times \mathbb{R}^k$,
2. $H_{1|B(0, \xi)} = H_{1|B(0, \xi)}$, $H_{1|\mathbb{R}^{2n} \times \mathbb{R}^k \setminus B(0, 2\pi)} = 0$,
3. $(x, \lambda) \in \text{cl}(U)$ is a solution of (1.1) iff it is a solution of the problem

$$
\begin{cases}
  \dot{x}(t) = J\nabla_x H_1(x(t), \lambda), \\
  x(0) = x(2\pi).
\end{cases}
$$

Consequently, investigating bounded (in $H_{2\pi}^1 \times \mathbb{R}^k$) subsets of solutions of (1.1) one can replace $H$ by a modified Hamiltonian $H_1$ having compact support, therefore no growth conditions are needed.

Theorem 2.5 has been extracted from the proof of theorem 3.3 in [9]. It is a version of the Amann–Zehnder global reduction [3, 4]. Every point $x_0 \in \mathbb{R}^{2n}$ is identified with the constant function from $E_0 \subset H_{2\pi}^1$. The gradients $\nabla_x a(\cdot, \lambda)$, $\nabla_x H_1(\cdot, \lambda)$ and the Hessian matrices $\nabla^2_x a(x_0, \lambda)$, $\nabla^2_x H_1(x_0, \lambda)$ are computed with respect to the inner product $(\cdot, \cdot)_H^{1\pi}$ and the standard inner product in $\mathbb{R}^{2n}$, respectively. Use is made of the standard bases in $E_j$ and $\mathbb{R}^{2n}$. The matrices $Q_j(\cdot)$ are given by (2.1).

**Theorem 2.5.** If $H_1 \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R})$ has compact support then there exist $r_0 \in \mathbb{N}$ and an $\mathbb{SO}(2)$-equivariant mapping $a \in C^{2,0}(E_j \times \mathbb{R}^k, \mathbb{R})$, where

$$
E_j := \bigoplus_{j=0}^{r_0} E_j \approx \bigoplus_{j=0}^{r_0} \mathbb{R}[2n, j].
$$

such that for every $x_0 \in \mathbb{R}^{2n}$, $\lambda \in \mathbb{R}^k$ the following conditions are satisfied.

1. $a(x_0, \lambda) = -2\pi H_1(x_0, \lambda)$.
2. $\nabla_x a(x_0, \lambda) = 0$ if $\nabla_x H_1(x_0, \lambda) = 0$. Moreover, $\nabla_x a^{\mathbb{SO}(2)} = \nabla_x a|_{E_j \times \mathbb{R}^k, E_0} = -\nabla_x H_1$.
3. If $\nabla_x a(x_0, \lambda) = 0$ then

$$
\nabla^2_x a(x_0, \lambda) = \text{diag}(Q_0(\nabla^2_x H_1(x_0, \lambda)), \ldots, Q_m(\nabla^2_x H_1(x_0, \lambda))).
$$

4. For every $j > r_0$ one has $m^{-1}(Q_j(\nabla^2_x H_1(x_0, \lambda))) = m^*(Q_j(\nabla^2_x H_1(x_0, \lambda))) = 2n$ (in particular, $\det Q_j(\nabla^2_x H_1(x_0, \lambda)) \neq 0$).
Furthermore, there exists an $SO(2)$-equivariant homeomorphism $h: (\nabla, a)^{-1}(\{0\}) \to \mathcal{R}(H_1)$, where $\mathcal{R}(H_1) \subset H^1_{2\pi} \times \mathbb{R}^k$ is the set of solutions of (2.9).

Conclusion (4) in the above theorem holds true for every $x_0 \in \mathbb{R}^{2n}$ and $\lambda \in \mathbb{R}^k$, since it is assumed that $H_1$ has compact support. The fact that $h$ is a homeomorphism follows from its construction. Note that the authors of [3, 4] regard the space $H^1_{2\pi}$ as a subspace of $L^2_{2\pi} \equiv L^2([0, 2\pi], \mathbb{R}^{2n})$ and they use the inner product $\langle \cdot, \cdot \rangle_{L^2_{2\pi}}$ which generates weaker topology in $H^1_{2\pi}$ than the inner product $\langle \cdot, \cdot \rangle_{H^1_{2\pi}}$. It affects also the form of matrices $Q_j$ and changes their eigenvalues used in the reduction. The matrices used in [9] are in fact those from [3, 4] (without the factor $\frac{1}{17\pi}$). However, the change of the inner product is possible in view of the following lemma.

Lemma 2.6. Assume that $H_1 \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R})$ has compact support, $\mathcal{R}(H_1)$ is the set of solutions of (2.9), and $d_{L^2_{2\pi}}, d_{H^1_{2\pi}}$ are the metrics in $\mathcal{R}(H_1)$ induced by the product norms from $L^2_{2\pi} \times \mathbb{R}^k$ and $H^1_{2\pi} \times \mathbb{R}^k$, respectively. Then the identity mapping from $(\mathcal{R}(H_1), d_{L^2_{2\pi}})$ to $(\mathcal{R}(H_1), d_{H^1_{2\pi}})$ is a homeomorphism.

Proof. It suffices to prove that the identity mapping from the space $(\mathcal{R}(H_1), d_{L^2_{2\pi}})$ to the space $(\mathcal{R}(H_1), d_{H^1_{2\pi}})$ is continuous. Suppose that a sequence $\{(x_m, \lambda_m)\}_{m \in \mathbb{N}} \subset \mathcal{R}(H_1)$ is convergent to some $(x, \lambda) \in \mathcal{R}(H_1)$ with respect to the metric $d_{L^2_{2\pi}}$. It will be shown to be also convergent with respect to the metric $d_{H^1_{2\pi}}$. Since

$$\|x_m - x\|_{L^2_{2\pi}}^2 = \|x_m - x\|_{L^2_{2\pi}}^2 + \|\dot{x}_m - \dot{x}\|_{L^2_{2\pi}},$$

it remains to prove that $\|\dot{x}_m - \dot{x}\|_{L^2_{2\pi}} \to 0$ as $m \to \infty$. The mapping $J\nabla_x H_1$ is continuous and has compact support, hence there exist $a, b > 0$ such that for all $(y, \alpha) \in \mathbb{R}^{2n} \times \mathbb{R}^k$ the growth condition

$$|J\nabla_x H_1(y, \alpha)| \leq a + b|y| \equiv a + b|y|^\frac{1}{2}$$

is satisfied. Consequently, by a Krasnosiel’skii theorem, the mapping

$$L^2_{2\pi} \times \mathbb{R}^k \ni (z, \alpha) \mapsto J\nabla_x H_1(z(\cdot), \alpha) \in L^2_{2\pi}$$

is continuous, hence

$$\|\dot{x}_m - \dot{x}\|_{L^2_{2\pi}} = \|J\nabla_x H_1(x_m(\cdot), \lambda_m) - J\nabla_x H_1(x(\cdot), \lambda)\|_{L^2_{2\pi}} \to 0$$

as $m \to \infty$. \hfill \Box

Lemma 2.7. If $H_1 \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R})$ has compact support then the set $\mathcal{R}(H_1)$ of solutions of (2.9) is closed in $H^1_{2\pi} \times \mathbb{R}^k$ and every bounded subset of $\mathcal{R}(H_1)$ is relatively compact.

Proof. The set $\mathcal{R}(H_1)$ is closed in $H^1_{2\pi} \times \mathbb{R}^k$ as the set of critical points of the action functional of class $C^{2,0}$ defined on $H^1_{2\pi} \times \mathbb{R}^k$ (see [9]). If the action functional is defined on $H^1_{2\pi} \times \mathbb{R}^k$ (see [1, 29]) then it is still of class $C^{2,0}$, the set of its critical points is still equal to $\mathcal{R}(H_1)$, and its gradient is a compact perturbation of a selfadjoint Fredholm operator. (In the case of the space $H^1_{2\pi} \times \mathbb{R}^k$ the Hessian operator of the action functional is compact not Fredholm.) Thus $\mathcal{R}(H_1)$ is closed in $H^1_{2\pi} \times \mathbb{R}^k$ and bounded subsets of $\mathcal{R}(H_1)$ are relatively compact in $H^1_{2\pi} \times \mathbb{R}^k$. However, those subsets of $\mathcal{R}(H_1)$ that are bounded in $H^1_{2\pi} \times \mathbb{R}^k$ are also bounded in $H^\frac{1}{2}_{2\pi} \times \mathbb{R}^k$ and both topologies restricted to $\mathcal{R}(H_1)$ coincide, in view of lemma 2.6. \hfill \Box
3. Necessary conditions for bifurcation and symmetry breaking

Remark 3.1. Let \( x \in H_{1_\omega}^1 \) and \( j \in \mathbb{N} \). Then

1. \( \mathbb{SO}(2)_k = \mathbb{SO}(2) \) if \( x \) is a constant function,
2. \( \mathbb{SO}(2)_k \supset \mathbb{Z}_j \) if \( \tfrac{x}{T} \) is a period (not necessarily minimal) of \( x \),
3. \( \mathbb{SO}(2)_k = \mathbb{Z}_j \) if \( \tfrac{x}{T} \) is the minimal period of \( x \).

Definition 3.2. Let \( j \in \mathbb{N} \cup \{0\} \). A solution \( (x, \lambda) \) of (1.1) is called a \( j \)-solution if \( \mathbb{SO}(2)_{(x, \lambda)} = \mathbb{SO}(2)_{(x, \lambda)} \supset \mathbb{Z}_j \).

If \( j \in \mathbb{N} \) then \( (x, \lambda) \) is a \( j \)-solution of (1.1) iff \( \tfrac{x}{T} \) is a period (not necessarily minimal) of \( x \), whereas \( 0 \)-solutions are the stationary ones.

In the rest of this section \( x_0 \) satisfying (2.7) is fixed and \( T(x_0) = \{x_0\} \times \mathbb{R}^k \) is regarded as the set of trivial solutions of (1.1).

Definition 3.3. A point \((x_0, \lambda_0) \in T(x_0)\) is called a symmetry breaking point for (1.1) if every neighbourhood of \((x_0, \lambda_0) \in H_{1_\omega}^1 \times \mathbb{R}^k \) contains at least two nontrivial solutions of (1.1) with different isotropy groups (or, equivalently, different minimal periods—see remark 3.1).

Proofs of theorems on symmetry breaking in this paper exploit the following theorem describing possible isotropy groups of solutions near given stationary point, based on a remark from [7]. (The matrices \( Q_j(\cdot) \) are given by (2.1).)

Theorem 3.4. If \( H \in C^{2,0}(\mathbb{R}^2 \times \mathbb{R}^k, \mathbb{R}) \) then for every \( \lambda_0 \in \mathbb{R}^k \) there exists a neighbourhood \( U \subset H_{1_\omega}^1 \times \mathbb{R}^k \) of \((x_0, \lambda_0) \in T(x_0)\) such that the isotropy group \( \mathbb{SO}(2)_{(x, \lambda)} = \mathbb{SO}(2)_x \) of every nontrivial solution \((x, \lambda) \in U \cap N T(x_0)\) of (1.1) belongs to the set \( G(\lambda_0) \) of isotropy groups of nonzero elements of the finite-dimensional space \( E(\lambda_0) = \bigoplus_{j \in X(\lambda_0)} E_j \), where

\[
X(\lambda_0) = \{j \in \mathbb{N} \cup \{0\} | \det Q_j(\nabla^2 H(x_0, \lambda_0)) = 0\}.
\]

Proof. By remark 2.4 and theorem 2.5 it suffices to consider isotropy groups of solutions \((z, \lambda)\) of the equation

\[
\nabla_z a(z, \lambda) = 0,
\]

(3.1)

such that \( z \in E_\lambda \setminus \{x_0\} \) (such solutions are regarded as nontrivial solutions of (3.1)). Since \( \nabla_z^2 a(x_0, \lambda_0) \) is symmetric, one can use the decomposition

\[
E_j = (\ker \nabla_z^2 a(x_0, \lambda_0)) \oplus (\ker \nabla_z^2 a(x_0, \lambda_0))
\]

and write (3.1) as the system of equations

\[
\Pi \nabla_z a(u, (v, \lambda)) = 0,
\]

(3.2)

\[
(Id - \Pi) \nabla_z a(u, (v, \lambda)) = 0,
\]

where \( \Pi \) is a projection of \( E_j \) onto \( \text{im} \nabla_z^2 a(x_0, \lambda_0) \), \( u = \Pi(z), v = (Id - \Pi)(z) \). Write also \( x_0 = (u_0, v_0) \), where \( u_0 = \Pi(x_0), v_0 = (Id - \Pi)(x_0) \). Applying the \( \mathbb{SO}(2) \)-equivariant version of the implicit function theorem to (3.2) one obtains the existence of an open neighbourhood \( W \) of \((u_0, v_0) \) in \( \ker \nabla_z^2 a(x_0, \lambda_0) \times \mathbb{R}^k \), and an \( \mathbb{SO}(2) \)-equivariant mapping \( \gamma: V \rightarrow W \) of class \( C^{1,0} \) such that if \((u, (v, \lambda)) \in W \times V \) is a solution of (3.1) then \( u = \gamma(v, \lambda) \). (In particular, \( \gamma(v_0, \lambda) = u_0 \) for every \( \lambda \in \mathbb{R}^k \) such that \((v_0, \lambda) \in \gamma \), since \( \nabla_z a(u_0, (v_0, \lambda)) = 0 \). Thus all nontrivial solutions of (3.1) in \( U := W \times V \) are of the form \((\gamma(v, \lambda), (v, \lambda))\), where \( v \in \ker \nabla_z^2 a(x_0, \lambda_0), v \neq v_0 \). Their isotropy groups are equal to

\[
\mathbb{SO}(2)_{(\gamma(v, \lambda), (v, \lambda))} = \mathbb{SO}(2)_{(\gamma(v, \lambda))} \cap \mathbb{SO}(2)_{(v, \lambda)}.
\]
Furthermore, $SO(2)_{y(x, \lambda)} \subset SO(2)_{y(x, \lambda)}$, since the mapping $\gamma$ is $SO(2)$-equivariant, hence
\[SO(2)_{y(x, \lambda)} \subset SO(2)_{y(x, \lambda)} = SO(2)_h,\]
where $v \in \ker V^2_2 u(x_0, \lambda_0)$, $v \neq v_0$. Thus if $v_0 = 0$ then the isotropy groups of nontrivial solutions of (3.1) in a neighbourhood of $(x_0, \lambda_0)$ belong to the set of isotropy groups of nonzero elements of $\ker V^2_2 u(x_0, \lambda_0)$. The same condition is obtained for $v_0 \neq 0$ by choosing the set $V$ in such a way that $(0, \lambda) \notin V$ for all $\lambda \in \mathbb{R}^k$. Finally, observe that, by theorem 2.5,\[\ker V^2_2 u(x_0, \lambda_0) \subset E(\lambda_0),\]and that $\dim E(\lambda_0) < 4n_0 + 2n < \infty$, since $E(\lambda_0) \subset E_f$.

The set $G(\lambda_0)$ from theorem 3.4 consists of groups $\mathbb{K}_j$, $j \in X(\lambda_0)$, and all their intersections. Thus
\[G(\lambda_0) \subset \{\mathbb{K}_j, j \in \{0, \ldots, \max(X(\lambda_0))\}\}.\]
In particular, the set $G(\lambda_0)$ is finite.

As a consequence of theorem 3.4 and the definition of $j$-solution one obtains (see appendix C) the following version of necessary conditions for bifurcation formulated in [9]. (The sets $\Lambda_j(\cdot)$, $\Lambda(\cdot)$ are defined by (2.2) and (2.3).)

**Corollary 3.5.** Let $H \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R})$. If $(x_0, \lambda_0) \in T(x_0)$ is a bifurcation point of nontrivial solutions of (1.1) then $\lambda_0 \in \Lambda_0(\nabla^2_2 H(x_0, \cdot)) \cup \Lambda(\nabla^2_2 H(x_0, \cdot))$. Namely,
(1) if $(x_0, \lambda_0)$ is a bifurcation point of nontrivial stationary solutions then
\[\lambda_0 \in \Lambda_0(\nabla^2_2 H(x_0, \cdot));\]
(2) if $(x_0, \lambda_0)$ is a bifurcation point of nonstationary $j$-solutions for some $j \in \mathbb{N}$ then
\[\lambda_0 \in \bigcup_{l \in \mathbb{N}} \Lambda_{lj}(\nabla^2_2 H(x_0, \cdot)).\]

**Corollary 3.5.** derived here from stronger theorem 3.4, has been substantiated in [9] (in terms of problem (1.2)) by using the fact that if $(x_0, \lambda_0)$ is a bifurcation point of nonstationary $j$-solutions of (1.1) for some $j \in \mathbb{N}$ then the linearization $x(t) = J \nabla^2_2 H(x_0, \lambda_0) x(t)$ of (1.1) has a nonstationary $j$-solution, hence $\pm ilj \in \sigma(J \nabla^2_2 H(x_0, \lambda_0))$ for some $l \in \mathbb{N}$. (Note that
\[\pm ilj \in \sigma(J \nabla^2_2 H(x_0, \lambda_0)) \Rightarrow \det Q_{il}(\nabla^2_2 H(x_0, \lambda_0)) = 0,\]
see appendix C.)

On the other hand, as it was shown in [36, p 369] and [26, example 9.2], in nondegenerate case, a Hamiltonian system can have no nonstationary periodic solutions even if all solutions of its linearization are $2\pi$-periodic. Furthermore, an example of Dancer described in [39, p 228] proves that in degenerate case no assumptions on the Hessian matrix of Hamiltonian can ensure bifurcation in (1.2), since for every singular real symmetric $(2n \times 2n)$-matrix $A$ one can construct a Hamiltonian $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ such that $\nabla^2 H(0) = A$, $(\nabla H)^{-1}([0]) = [0]$, and $(0, \lambda_0)$ is not a bifurcation point of (1.2) for any $\lambda_0 \in \mathbb{R}$.

**Remark 3.6.** Theorem 3.4 is stronger than corollary 3.5, since it can exclude symmetry breaking in the situation when corollary 3.5 does not exclude it. For example, assume that $\det Q_{il}(\nabla^2_2 H(x_0, \lambda_0)) = 0$ and $\det Q_{jl}(\nabla^2_2 H(x_0, \lambda_0)) \neq 0$ for $j \in \mathbb{N}$, $j \neq 6$, which holds for diagonal matrix $\nabla^2_2 H(x_0, \lambda_0)$ with the $r$th and the $2n$th element of the diagonal equal to 6 and the rest of elements equal to 0 (see remark 5.7 and condition (5.3)). Then $E(\lambda_0) = E_0 \cup E_6$ and the only possible isotropy group of nonstationary solutions of (1.1) in a neighbourhood of $(x_0, \lambda_0)$ is $Z_6$, which corresponds to the minimal period $\frac{2\pi}{6}$. This excludes symmetry breaking if $(x_0, \lambda_0)$ is not a bifurcation point of nontrivial stationary solutions. However, corollary 3.5 does not exclude bifurcation of solutions of (1.1) with the minimal period $\frac{2\pi}{6}$ from $(x_0, \lambda_0)$, since $\lambda_0 \in \Lambda_6 = \Lambda_{2,3} \subset \bigcup_{l \in \mathbb{N}} \Lambda_{lj}.\]
Remark 3.7. If \((x_0, \lambda_0)\) is completely degenerate, i.e. \(\nabla^2 H(x_0, \lambda_0) = 0\), then for every \(j \in \mathbb{N}\) one has \(\det Q_j(\nabla^2 H(x_0, \lambda_0)) \neq 0\). In such a case \((x_0, \lambda_0)\) is not a bifurcation point of nonstationary solutions of (1.1), in view of corollary 3.5.

As a consequence one obtains the following corollary (for the formal proof see preprint [32]) asserting that if a completely degenerate stationary point of a Hamiltonian system without parameter is an emanation point of periodic orbits then minimal periods of these orbits tend to infinity as the orbits converge to the stationary point.

Corollary 3.8. Let \(H \in C^2(\mathbb{R}^{2n}, \mathbb{R})\). If \(x_0 \in (\nabla H)^{-1}(\{0\})\) is completely degenerate, i.e. \(\nabla^2 H(x_0) = 0\), then for every \(C > 0\) there exists \(\delta > 0\) such that every nonstationary periodic orbit of (1.3) contained in the ball in \(\mathbb{R}^{2n}\) centred at \(x_0\) with radius \(\delta\) has the minimal period greater than \(C\).

4. Dancer–Rybicki bifurcation theorem for \(j\)-solutions

In this section global bifurcation theorems for \(j\)-solutions of (1.1) are proved in the case of systems with one parameter \((k = 1)\), i.e. it is assumed that \(H \in C^2(\mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})\).

Let \(x_0 \in \mathbb{R}^{2n}\) satisfy (2.7) for \(k = 1\) and fix \(\lambda_0 \in \mathbb{R}\). Assume that for sufficiently small \(\varepsilon > 0\) and every \(\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]\) \((\lambda_0)\) there exists a neighbourhood \(W \subset \mathbb{R}^{2n} \times \mathbb{R}\) of \((x_0, \lambda)\) such that \((\nabla_x H)^{-1}(\{0\}) \cap W \subset [x_0] \times \mathbb{R}\). Set

\[\eta_0 (x_0, \lambda_0) = i(\nabla_x H(\cdot, \lambda_0 + \varepsilon), x_0) - i(\nabla_x H(\cdot, \lambda_0 - \varepsilon), x_0)\].

If \(\lambda_0 \in \mathbb{R}\) is not a cluster point of the set \(\Lambda_j(\nabla_x^2 H(x_0, \cdot))\) (see (2.2)) for some \(j \in \mathbb{N}\) then one can choose \(\varepsilon > 0\) such that \(\Lambda_j(\nabla_x^2 H(x_0, \cdot)) \cap [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] = \{\lambda_0\}\) and set

\[\eta_j (x_0, \lambda_0) = i(\nabla_x H(\cdot, \lambda_0 + \varepsilon), x_0) \cdot \frac{m^-(Q_j(\nabla_x^2 H(x_0, \lambda_0 + \varepsilon)))}{2} - i(\nabla_x H(\cdot, \lambda_0 - \varepsilon), x_0) \cdot \frac{m^-(Q_j(\nabla_x^2 H(x_0, \lambda_0 - \varepsilon)))}{2}\].

\((Q_j(\cdot))\) is given by (2.1.) The sequence

\[\eta(x_0, \lambda_0) = \{\eta_j (x_0, \lambda_0)\}_{j=0}^\infty\]

is called a bifurcation index of \((x_0, \lambda_0)\). Usually only selected coordinates of this index are needed. Note that infinitely many of them may be nonzero. However, according to theorem 2.5 and remark 2.4 there exists \(r_0 \in \mathbb{N}\) such that \(\eta_j(x_0, \lambda_0) = \eta_0(x_0, \lambda_0) \cdot n\) for \(j > r_0\). In the proof of theorem 4.3 some coordinates of \(\eta(x_0, \lambda_0)\) will be identified with coordinates of the index \(\text{IND}(x_0, \lambda_0)\) defined by (2.5) for an appropriate mapping \(\nabla_x f\).

Consider first the case of system (1.2) with linear dependence on parameter, which can be written in form (1.1) for \(H\) replaced by \(\hat{H} \in C^2(\mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})\) defined by

\[\hat{H}(x, \lambda) = \lambda H(x)\].

To define \(\eta\) in this case it suffices to assume that \(x_0\) is an isolated element of \((\nabla H)^{-1}(\{0\})\).

(For every \(a, b \in \mathbb{R}, a < b\), the set \(\Lambda(\nabla_x^2 \hat{H}(x_0, \cdot)) \cap [a, b]\) is finite.) Then

\[\eta_0 (x_0, \lambda_0) = 0\],

\[\eta_j (x_0, \lambda_0) = i(\nabla_x H(x_0) \cdot \frac{m^-(Q_j((\lambda_0 + \varepsilon)\nabla_x^2 H(x_0)))}{2} - i(\nabla_x H(\cdot, \lambda_0 - \varepsilon)\nabla_x^2 H(x_0)))\]

for every \(j \in \mathbb{N}\), as in [9]. Note that \(\eta_j(x_0, \lambda_0) = 0\) for every \(j > r_0\), hence \(\eta(x_0, \lambda_0) \in U(\mathbb{S} \ominus (2)).\)
As it was proved in [9], for every \( K > 0 \) there exists \( \delta > 0 \) such that every solution \((x, \lambda) \in H^{1}_{2\pi} \times \mathbb{R} \) of (1.2) satisfying the conditions \(|\lambda| \leq \delta \) and \(|x| \leq K\) is stationary. (In particular, \((x_0, 0)\) is not a bifurcation point of nonstationary solutions of (1.2).) Thus it suffices to consider the solutions of (1.2) for \( \lambda > 0 \).

The set \( T = (\nabla H)^{-1}(\{0\}) \times (0, +\infty) \) is regarded as the set of trivial solutions of (1.2) and nontrivial solutions are the nonstationary ones. If \((x_0, \lambda_0)\) is a global bifurcation point of nonstationary solutions of (1.2) then \( C(x_0, \lambda_0) \) denotes the connected component of the closure of the set of nonstationary solutions of (1.2) containing \((x_0, \lambda_0)\).

The following Rabinowitz-type global bifurcation theorem for Hamiltonian systems has been proved by Dancer and Rybicki [9]. \((\Lambda^+ (\cdot)\) is defined by (2.4).)

**Theorem 4.1.** Let \( H \in C^{2}(\mathbb{R}^n, \mathbb{R}) \) and let \((\nabla H)^{-1}(\{0\})\) be finite. Fix \( x_0 \in (\nabla H)^{-1}(\{0\}) \) and \( \lambda_0 \in \Lambda^+(\nabla^2 H(x_0, \cdot)) \). If \( \eta(x_0, \lambda_0) \neq \emptyset \) then \((x_0, \lambda_0)\) is a global bifurcation point of nonstationary solutions of (1.2). Moreover, if the set \( C(x_0, \lambda_0) \) is bounded in \( H^{1}_{2\pi} \times (0, +\infty) \) then \( C(x_0, \lambda_0) \cap T = \{y_1, \ldots, y_m\} \) for some \( m \in \mathbb{N}, y_1, \ldots, y_m \in T \) such that

\[
\sum_{i=1}^{m} \eta(y_i) = \emptyset.
\]

In this section generalized versions of theorem 4.1 concerning \( j \)-solutions (for systems with nonlinear dependence on parameter) are proved. To this aim, the method presented in [9] is applied to the restriction of the mapping \( \nabla_x a \) from theorem 2.5 to the subspace of fixed points of the action of the group \( \mathbb{R}_j \) for given \( j \in \mathbb{N} \cup \{0\} \).

Consider an orthogonal representation of the group \( \mathbb{SO}(2) \) on a real inner product space \( V \) with \( \dim V < \infty \) and let \( \nabla_x f : V \times \mathbb{R} \to V \) be a continuous \( \mathbb{SO}(2) \)-equivariant gradient mapping. Let \( \Delta \times \mathbb{R} \subset (\nabla_x f)^{-1}(\{0\}) \) be the set of trivial solutions of the equation

\[
\nabla_x f(x, \lambda) = 0 \quad (4.1)
\]

for some finite set \( \Delta \subset V \). If \((x_0, \lambda_0) \in \Delta \times \mathbb{R} \) is a branching point of nontrivial solutions of (4.1) then \( \Sigma(x_0, \lambda_0) \) denotes the connected component of the closure of the set of nontrivial solutions of (4.1) containing \((x_0, \lambda_0)\).

The following theorem is a slightly modified version of theorem 2.2 formulated in [9]. It will be used in the proof of theorem 4.3.

**Theorem 4.2.** Let \( \Omega \) be a bounded open subset of \( V \times \mathbb{R} \). Assume that \((\Delta \times \mathbb{R}) \cap \Omega \) contains at most finite number of bifurcation points of nontrivial solutions of (4.1) and \( \partial \Omega \) contains no bifurcation points. If \( \text{IND}(x_0, \lambda_0) \neq \emptyset \) for some \((x_0, \lambda_0) \in (\Delta \times \mathbb{R}) \cap \Omega \) then \((x_0, \lambda_0)\) is a branching point of nontrivial solutions of (4.1). Moreover, if \( \Sigma(x_0, \lambda_0) \cap \partial \Omega = \emptyset \) then \( \Sigma(x_0, \lambda_0) \cap (\Delta \times \mathbb{R}) \cap \Omega = \{z_1, \ldots, z_m\} \) for some \( m \in \mathbb{N}, z_1, \ldots, z_m \in \Delta \times \mathbb{R} \) such that

\[
\sum_{i=1}^{m} \text{IND}(z_i) = \emptyset.
\]

The proof of the above theorem proceeds analogously to that of theorem 29.1 in [10]. (It is based on Whyburn lemma and standard properties of topological degree.) The Brouwer degree \((\dim V < \infty)\) is replaced by the degree \( \text{DEG} \) in this case. To guarantee that sets over which the degree \( \text{DEG} \) is computed are \( \mathbb{SO}(2) \)-equivariant it suffices to observe that if \( D \subset V \times \mathbb{R} \) then the set \( \mathbb{SO}(2)D := \{gv \mid g \in \mathbb{SO}(2), v \in D\} \) is \( \mathbb{SO}(2) \)-equivariant and \((\nabla_x f)^{-1}(\{0\}) \cap D = (\nabla_x f)^{-1}(\{0\}) \cap \mathbb{SO}(2)D \), since the mapping \( \nabla_x f \) is \( \mathbb{SO}(2) \)-equivariant.
Assume that $H \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})$ and let $\Delta \times \mathbb{R} \subset (\nabla, H)^{-1}(0)$ be the set of trivial solutions of (1.1) for some finite set $\Delta \subset \mathbb{R}^{2n}$. (Note that some nontrivial solutions can be stationary.) Set

$$P_j(\Delta) = \bigcup_{x_0 \in \Delta} \left( [x_0] \times \bigcup_{i \in \mathbb{N}} \Lambda_{ij}(\nabla_j^2 H(x_0, \cdot)) \right)$$

for $j \in \mathbb{N} \cup \{0\}$. For a fixed bounded open set $U \subset H^1_{2n} \times \mathbb{R}$ use will be made of the following condition.

$$(N)$$ For every $(x, \lambda) \in (\Delta \times \mathbb{R}) \cap \text{cl}(U) \setminus P_j(\Delta)$ there exists its neighbourhood $W \subset \mathbb{R}^{2n} \times \mathbb{R}$ such that $(\nabla, H)^{-1}(\{0\}) \cap W \subset \Delta \times \mathbb{R}$.

If $(x_0, \lambda_0) \in P_j(\Delta)$ is a branching point of nontrivial $j$-solutions of (1.1) for some $j \in \mathbb{N} \cup \{0\}$, then $K_j(x_0, \lambda_0)$ denotes the connected component of the closure of the set of nontrivial (possibly stationary) $j$-solutions containing $(x_0, \lambda_0)$.

**Theorem 4.3.** Let $H \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})$. Fix $j \in \mathbb{N} \cup \{0\}$ and let $U$ be a bounded open subset of $H^1_{2n} \times \mathbb{R}$. Assume that the set $P_j(\Delta) \cap U$ is finite, $P_j(\Delta) \cap \partial U = \emptyset$, and condition $(N)$ is satisfied. If $\eta_j(x_0, \lambda_0) \neq 0$ for some $(x_0, \lambda_0) \in P_j(\Delta) \cap U$ then $(x_0, \lambda_0)$ is a branching point of nontrivial (possibly stationary) $j$-solutions of (1.1). Moreover, if $K_j(x_0, \lambda_0) \cap \partial U = \emptyset$ then $K_j(x_0, \lambda_0) \cap (\Delta \times \mathbb{R}) \cap U = \{z_1, \dotsc, z_m\}$ for some $m \in \mathbb{N}$, $z_1, \dotsc, z_m \in \Delta \times \mathbb{R}$ such that

$$\sum_{i=1}^m \eta_{ij}(z_i) = 0 \quad \text{for every } l \in \mathbb{N} \cup \{0\}.$$

**Proof.** $H$ can be replaced by $H_1$ from remark 2.4. One has $(\Delta \times \mathbb{R}) \cap \text{cl}(U) \subset B(0, \xi)$ and the functions $H$ and $H_1$ are equal on $B(0, \xi)$. The solutions of (1.1) in cl$(U)$ are those of (2.9). In view of theorem 2.5, $(x_0, \lambda_0)$ is a branching point of nontrivial $j$-solutions of (2.9) if and only if it is a branching point of nontrivial $j$-solutions of the equation $\nabla_x a(x, \lambda) = 0$ in the space $E_f \times \mathbb{R}$, which means that $(x_0, \lambda_0)$ is a branching point of nontrivial solutions of (4.1) in the space $V \times \mathbb{R}$, where $V = (E_f)^{\mathbb{N}} = E_f \cap \bigoplus_{i=0}^{\infty} E_{ij}$ and $\nabla_x f = (\nabla_x a)'(V \times \mathbb{R}, V)$. $(E_f$ can be regarded as a subspace of $H^1_{2n}$.) Note that the only solutions of (4.1) in $V \times \mathbb{R}$ are then $j$-solutions. The set of trivial solutions and bifurcation points of $j$-solutions remain the same as in the case of (2.9). Use will be made of theorem 4.2. According to lemma 2.2, for every $k \in \mathbb{N} \cup \{0\}$ one has

$$\text{IND}_k(x_0, \lambda_0) = \begin{cases} \eta_k(x_0, \lambda_0) & \text{if } k = lj \leq r_0 \text{ for some } l \in \mathbb{N} \cup \{0\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $\text{IND}(x_0, \lambda_0)$ is the bifurcation index (2.5). Note also that if $j > r_0$ then

$$\eta_j(x_0, \lambda_0) = \eta_0(x_0, \lambda_0) \cdot n = \text{IND}_0(x_0, \lambda_0) \cdot n.$$

(In this case, in view of theorem 4.2, $(x_0, \lambda_0)$ is a branching point of nontrivial stationary solutions, which are $j$-solutions for every $j \in \mathbb{N} \cup \{0\}$). Furthermore, in view of the assumptions and corollary 3.5, $U$ contains at most finite number of bifurcation points of nontrivial $j$-solutions of (2.9) and if $(x_0, \lambda_0) \in U$ is a branching point of nontrivial $j$-solutions of (2.9) such that $K_j(x_0, \lambda_0) \cap \partial U = \emptyset$ then $K_j(x_0, \lambda_0)$ is compact (see lemma 2.7), so also is $\Sigma(x_0, \lambda_0) := h^{-1}(K_j(x_0, \lambda_0))$, where $h$ is the homeomorphism from theorem 2.5. Thus one can find $\Omega \subset V \times \mathbb{R}$ satisfying the assumptions of theorem 4.2 and the condition $\Sigma(x_0, \lambda_0) \subset \Omega$. \qed
As a consequence of theorem 4.3 one obtains the next two theorems that will be used in subsequent sections. In the first one it is assumed that \((x_0, \lambda_0)\) is not a bifurcation point of nontrivial stationary solutions, whereas in the second one such bifurcation is allowed but it is assumed that \(\lambda_0\) is not a cluster point of \(\Lambda_0(\nabla^2 H(x_0, \cdot))\), which means that all the points from \(T(x_0) \setminus \{(x_0, \lambda_0)\}\) in a neighbourhood of \((x_0, \lambda_0)\) are nondegenerate (although \((x_0, \lambda_0)\) can be degenerate). In both cases \(T(x_0) = \{x_0\} \times \mathbb{R}\) is regarded as the set of trivial solutions, i.e. \(\Delta = \{x_0\}\).

**Theorem 4.4.** Let \(H \in C^2(\mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})\) and \(\nabla_i H(x_0, \lambda) = 0\) for some \(x_0 \in \mathbb{R}^{2n}\) and all \(\lambda \in \mathbb{R}\). Assume that \(\lambda_0\) is an isolated element of the set \(\bigcup_{j \in \mathbb{N}} \Lambda_j(\nabla^2_i H(x_0, \cdot))\) for some \(j \in \mathbb{N}\) and \((x_0, \lambda_0)\) is not a bifurcation point of nontrivial stationary solutions of (1.1). If \(\eta_j(x_0, \lambda_0) \neq 0\) then \((x_0, \lambda_0)\) is a branching point of nonstationary \(j\)-solutions of (1.1) and a global bifurcation point of nontrivial \(j\)-solutions.

**Theorem 4.5.** Let \(H \in C^2(\mathbb{R}^{2n} \times \mathbb{R}, \mathbb{R})\) and \(\nabla_i H(x_0, \lambda) = 0\) for some \(x_0 \in \mathbb{R}^{2n}\) and all \(\lambda \in \mathbb{R}\). Assume that \(\lambda_0\) is an isolated element of \(\Lambda_0(\nabla^2_i H(x_0, \cdot))\) \(\bigcup_{j \in \mathbb{N}} \Lambda_j(\nabla^2_i H(x_0, \cdot))\) for some \(j \in \mathbb{N} \cup \{0\}\). If \(\eta_j(x_0, \lambda_0) \neq 0\) then \((x_0, \lambda_0)\) is a global bifurcation point of nontrivial (possibly stationary) \(j\)-solutions of (1.1).

Recall that in the case of system (1.2) with linear dependence on parameter the set \(T = (\nabla H)^{-1}(\{0\}) \times (0, +\infty)\) is regarded as the set of trivial solutions. If \((x_0, \lambda_0)\) is a global bifurcation point of nonstationary \(j\)-solutions of (1.2) then \(C_j(x_0, \lambda_0)\) denotes the connected component of the closure of the set of nonstationary \(j\)-solutions of (1.2) containing \((x_0, \lambda_0)\). Theorem 4.3 implies the following generalized version of theorem 4.1.

**Theorem 4.6.** Let \(H \in C^2(\mathbb{R}^{2n}, \mathbb{R})\) and let \((\nabla H)^{-1}(\{0\})\) be finite. Fix \(x_0 \in (\nabla H)^{-1}(\{0\})\) and \(\lambda_0 \in (\nabla^2 H(x_0, \cdot))\). If \(\eta_j(x_0, \lambda_0) \neq 0\) for some \(j \in \mathbb{N}\) then \((x_0, \lambda_0)\) is a global bifurcation point of nonstationary \(j\)-solutions of (1.2). Moreover, if the set \(C_j(x_0, \lambda_0)\) is bounded in \(H^1_{2\pi} \times (0, +\infty)\) then \(C_j(x_0, \lambda_0) \cap T = \{z_1, \ldots, z_m\}\) for some \(m \in \mathbb{N}\), \(z_1, \ldots, z_m \in T\) such that
\[
\sum_{i=1}^m \eta_j(z_i) = 0 \quad \text{for every } l \in \mathbb{N}.
\]

The conclusion of theorem 4.6 does not seem to follow from theorem 4.1, since the formula for the sum of bifurcation indices over the branch \(C_j(x_0, \lambda_0)\) does not imply the formula for the sum of indices over the branch \(C_j(x_0, \lambda_0) \subset C(x_0, \lambda_0)\).

The results from [30, 33] provide sufficient conditions for global bifurcation of (2\(\pi\)-periodic) solutions of (1.2) and describe unbounded branches of solutions bifurcating from given points. Theorem 4.6 allows us to replace that branches by appropriate branches of \(j\)-solutions. For example, taking into account corollary 3.5 and theorem 4.6 one can generalize lemma 3.3, theorem 4.6 and corollary 5.3 from [33] as follows.

**Corollary 4.7.** Assume that \(H \in C^2(\mathbb{R}^{2n}, \mathbb{R})\) and that \((\nabla H)^{-1}(\{0\})\) is finite. Let \(x_0 \in (\nabla H)^{-1}(\{0\})\) be such that \(i(\nabla H, x_0) \neq 0\) and \(\nabla^2 H(x_0) = \text{diag}(A, B)\), \(A, B \in \mathbb{S}(n, \mathbb{R})\), where \(A\) or \(B\) is strictly positive or strictly negative definite. Then for every \(j \in \mathbb{N}\) the set of bifurcation points \((x_0, \lambda) \in \{x_0\} \times (0, +\infty)\) of nonstationary \(j\)-solutions of (1.2) is equal to the set of global bifurcation points of nonstationary \(j\)-solutions and equal to
\[
\left\{ \left( x_0, \frac{l}{\sqrt{n}} \right) \middle| v \in \sigma_+(AB), l \in \mathbb{N} \right\}.
\]
Furthermore, if \((\nabla H)^{-1}(\{0\}) = \{x_0\}, \lambda_0 = \frac{\bar{j}_0}{\sqrt{n}}, j_0 \in \mathbb{N}, v_0 \in \sigma_+(AB)\) then the set \(C_{j_0}(x_0, \lambda_0)\) is unbounded in \(H^1_{2\pi} \times (0, +\infty)\).
Remark 5.2. Fix $C(ξ, j\text{unbounded branches each eigenvalue }ν(λ))$ of $2π$-periodic solutions in corollaries 5.5–5.7 from [33] can be replaced by the unbounded branches $C_j(ξ, j)$ of $j$-solutions.

5. Global bifurcation points in multiparameter systems

In this section global bifurcation and symmetry breaking theorems for system (1.1) are proved in the case of arbitrary number $k$ of parameters. To this aim use is made of the bifurcation theorems for the system with one parameter obtained in the previous section.

Definition 5.1. Let $H \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R})$, $j \in \mathbb{N}$ and fix $x_0 \in \mathbb{R}^{2n}$. A function $F_j \in C(\mathbb{R}^k, \mathbb{R})$ is called a $j$th detecting function for system (1.1) provided that the following conditions are satisfied.

1. For every $λ \in \mathbb{R}^k$, $F_j(λ) = 0$ iff $det Q_j(∇^2 H(x_0, λ)) = 0$.
2. For every straight line $L \subset \mathbb{R}^k$ and every $λ_0 \in L$ being an isolated zero of $F_j$ on $L$, if $F_j$ changes its sign on $L$ at $λ_0$ then the function $\mathbb{R}^k \ni λ \mapsto m^-(Q_j(∇^2 H(x_0, λ)))$ changes its value on $L$ at $λ_0$.

$F_0 := det Q_0(∇^2 H(x_0, ⋅))$ is called the $0$th detecting function for (1.1). \{$(F_j)_{j∈\mathbb{N}∪\{0\}}$\} is said to be a detecting sequence for (1.1) if $F_j$ is a $j$th detecting function for (1.1) for every $j \in \mathbb{N}∪\{0\}$.

Remark 5.2. Fix $x_0 \in \mathbb{R}^{2n}$ and $j \in \mathbb{N}$. Since $H$ is of class $C^{2,0}$ and for every $λ \in \mathbb{R}^k$ each eigenvalue $ν(λ)$ of the symmetric matrix $Q_j(∇^2 H(x_0, λ))$ has even multiplicity $µ(ν(λ))$ (see remark 2.1(i)), there exist $ν_1, \ldots, ν_{2n} \in C(\mathbb{R}^k, \mathbb{R})$ such that $σ(Q_j(∇^2 H(x_0, λ))) = \{ν_1(λ), \ldots, ν_{2n}(λ)\}$ and for every $i \in \{1, \ldots, 2n\}$ the eigenvalue $ν_i(λ)$ occurs $μ(ν_i(λ))$ times in the $2n$-tuple $(ν_1(λ), \ldots, ν_{2n}(λ))$. Then the function $F_j$ defined by $F_j(λ) = ν_1(λ) \cdots ν_{2n}(λ)$ is a $j$th detecting function for (1.1). Note that the mapping $\mathbb{R}^k \ni λ \mapsto det Q_j(∇^2 H(x_0, λ))$ is nonnegative for every $j \in \mathbb{N}$, therefore it cannot be used to detect the change in the Morse index of $Q_j(∇^2 H(x_0, λ))$.

Now, explicit formulae for detecting functions (exploited in examples in section 7) will be given in the case when

$$v_{λ∈\mathbb{R}} : ∇^2 H(x_0, λ) = \begin{bmatrix} A(λ) & 0 \\ 0 & B(λ) \end{bmatrix}, \quad A(λ), B(λ) ∈ S(n, \mathbb{R}). \quad (5.1)$$

If $C, D ∈ S(n, \mathbb{R})$ and $K ∈ S(2n, \mathbb{R})$ is of the form

$$K = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}$$

then for every $j \in \mathbb{N}∪\{0\}$ define $G_j(K) ∈ S(2n, \mathbb{R}) and X ∈ O(4n, \mathbb{R})$ as follows:

$$G_j(K) = \begin{bmatrix} -C & jI \\ jI & -D \end{bmatrix} = -K + j \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \end{bmatrix},$$

where $I = I_n$. 

In the above corollary the set $C(x_0, λ_0)$ from corollary 5.3 in [33] has been replaced by $C_0(x_0, λ_0)$. Similarly, in the case when $(∇H)^{-1}(0)$ is not a singleton but it is finite, the unbounded branches $C(ξ, j)$ of $2π$-periodic solutions in corollaries 5.5–5.7 from [33] can be replaced by the unbounded branches $C_j(ξ, j)$ of $j$-solutions.
Lemma 5.3 ([33]). For every \( j \in \mathbb{N} \) one has

\[
(1) \quad X'Q_j(K)X = \frac{1}{j^2} \begin{bmatrix} G_j(K) & 0 \\ 0 & G_j(K) \end{bmatrix},
\]

\[
(2) \quad \det G_j(K) = \det [CD - j^2I].
\]

Using lemma 5.3 one obtains the following.

Lemma 5.4. Let \( H \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R}) \) and fix \( x_0 \in \mathbb{R}^{2n} \). Assume that condition (5.1) is satisfied. Define the functions \( F_j: \mathbb{R}^k \to \mathbb{R}, j \in \mathbb{N} \cup \{0\} \), by the formula

\[
F_j(\lambda) = \det G_j(\nabla^2_x H(x_0, \lambda)) = \det [A(\lambda)B(\lambda) - j^2I].
\]

(5.2)

Then \( \{F_j\}_{j \in \mathbb{N} \cup \{0\}} \) is a detecting sequence for (1.1).

Note that the functions \( F_j \) given by (5.2) multiplied by \( \frac{1}{j^2} \) are equal to the functions \( F_j \) from remark 5.2.

Clearly, for every \( j \in \mathbb{N}, \lambda \in \mathbb{R}^k \) the function \( F_j \) given by (5.2) satisfies the condition

\[
F_j(\lambda) = 0 \iff j^2 \in \sigma_+(A(\lambda)B(\lambda)) \iff 1 \in \sigma_\ast \left( \frac{1}{j^2} A(\lambda)B(\lambda) \right).
\]

(5.3)

By remark 2.1(ii) with \( T(\lambda) = \nabla^2_x H(x_0, \lambda) \), one obtains the following lemma.

Lemma 5.5. Let \( H \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R}) \), fix \( x_0 \in \mathbb{R}^{2n} \), and let \( \{F_j\}_{j \in \mathbb{N} \cup \{0\}} \subset C(\mathbb{R}^k, \mathbb{R}) \) be a detecting sequence for (1.1). Then for every bounded open set \( U \subset \mathbb{R}^k \) the set

\[
\{j \in \mathbb{N} \cup \{0\} | \exists \lambda \in U : F_j(\lambda) = 0\}
\]

is finite. Moreover, every \( \lambda_0 \in \mathbb{R}^k \) has an open neighbourhood \( U \subset \mathbb{R}^k \) such that \( F_j(\lambda) \neq 0 \) for every \( \lambda \in U \) and every \( j \in \mathbb{N} \cup \{0\} \) such that \( F_j(\lambda_0) \neq 0 \).

Let \( \lfloor a \rfloor \) denote the integer part of \( a \in \mathbb{R} \). One can use the following lemma, following from lemma 5.5, to find all the functions \( F_j \) vanishing in a neighbourhood of given \( \lambda_0 \in \mathbb{R}^k \) in the case of systems satisfying condition (5.1).

Lemma 5.6. Let the assumptions of lemma 5.4 be satisfied. Fix \( \lambda_0 \in \mathbb{R}^k \) and set

\[
N(\lambda_0) = \lfloor \sqrt{\nu} | \nu \in \sigma_+(A(\lambda_0)B(\lambda_0)) \rfloor.
\]

Then there exists an open neighbourhood \( U \subset \mathbb{R}^k \) of \( \lambda_0 \) such that \( F_j(\lambda) \neq 0 \) for every \( j > N(\lambda_0), \lambda \in U \).
The following assumptions are used in the rest of this paper.

(H1) $H \in C^{2,0}(\mathbb{R}^{2n} \times \mathbb{R}^k, \mathbb{R})$,
(H2) $x_0 \in \mathbb{R}^{2n}$ and $\nabla_x H(x_0, \lambda) = 0$ for all $\lambda \in \mathbb{R}^k$,
(H3) $\{F_j\}_{j \in \mathbb{N} \cup \{0\}} \subset C(\mathbb{R}^k, \mathbb{R})$ is a detecting sequence for (1.1).

The set $T(x_0) = \{x_0\} \times \mathbb{R}^k$ is regarded as the set of trivial solutions of (1.1), i.e. $\Delta = \{x_0\}$. In some theorems it is assumed additionally that for given $F_j$, formulated in terms of the functions $\nabla_x H(x_0, \lambda), x_0$ and for the system restricted to any straight line in the space of parameters the sum of bifurcation indices from theorem 4.3 is dropped, as in theorems 4.4 and 4.5, the subsequent bifurcation theorems can be formulated for $\lambda \in \mathbb{R}^k$ the following conditions are satisfied.

(E1($x_0, \lambda$)) There exists a neighbourhood $W \subset \mathbb{R}^{2n} \times \mathbb{R}^k$ of $(x_0, \lambda)$ such that

$$(\nabla_x H)^{-1}(\{0\}) \cap W \subset \{x_0\} \times \mathbb{R}^k$$

(i.e. $(x_0, \lambda)$ is not a bifurcation point of nontrivial stationary solutions of (1.1)),

(E2($x_0, \lambda$)) $i(\nabla_x H(\cdot, \lambda), x_0) \neq 0$.

Although in the rest of this paper, for simplicity, it is assumed that $\Delta = \{x_0\}$ and the formula for the sum of bifurcation indices from theorem 4.3 is dropped, as in theorems 4.4 and 4.5, the subsequent bifurcation theorems can be formulated for $\lambda$ containing more stationary points, and for the system restricted to any straight line in the space of parameters the sum of bifurcation indices over every bounded connected component of the closure of the set of nontrivial solutions containing finite number of bifurcation points vanishes.

Remark 5.7. If conditions (H1)–(H3) are satisfied then theorem 3.4 and corollary 3.5 can be formulated in terms of the functions $F_j$, since for every $j \in \mathbb{N} \cup \{0\}$, $\lambda_0 \in \mathbb{R}^k$ one has

$$\Lambda_j(\nabla_x^2 H(x_0, \cdot)) = F_j^{-1}(\{0\}),$$

$$X(\lambda_0) = \{j \in \mathbb{N} \cup \{0\}| F_j(\lambda_0) = 0\}.$$

In what follows $\lambda_0 \in \mathbb{R}^k$ is fixed.

Continuous curve in $\mathbb{R}^k$ means any subset of $\mathbb{R}^k$ homeomorphic to $\mathbb{R}$. A submanifold of $\mathbb{R}^k$ is called a manifold and the tangent space to such a manifold is regarded as a linear subspace of $\mathbb{R}^k$.

Theorem 5.8. Let conditions (H1)–(H3), (E1($x_0, \lambda_0$)) and (E2($x_0, \lambda_0$)) be satisfied. Assume that $M \subset \mathbb{R}^k$ is a continuous curve and $\lambda_0 \in M$ is an isolated element of the set

$$\bigcup_{i \in \mathbb{N}} F_i^{-1}(\{0\}) \cap M$$

for some $j \in \mathbb{N}$. If the restriction of $F_j$ to $M$ changes its sign at $\lambda_0$ then $(x_0, \lambda_0)$ is a branching point of nonstationary $j$-solutions of (1.1) and a global bifurcation point of nontrivial $j$-solutions.

Proof. Let $\psi: \mathbb{R} \to M$ be a homeomorphism such that $\psi(0) = \lambda_0$ and let $H_1: \mathbb{R}^{2n} \times \mathbb{R} \to \mathbb{R}$ be the Hamiltonian defined by $H_1(x, s) = H(x, \psi(s))$. It suffices to prove the conclusion for $H$ and $(x_0, 0) \in H^{2n} \times \mathbb{R}$ instead of $H$ and $(x_0, \lambda_0) \in H^{2n} \times \mathbb{R}^k$. By assumptions (E1($x_0, \lambda_0$)), (E2($x_0, \lambda_0$)) one has $i(\nabla_x H_1(\cdot, \epsilon), x_0) = i(\nabla_x H_1(\cdot, -\epsilon), x_0) = i(\nabla_x H_1(\cdot, 0), x_0) = i(\nabla_x H_1(\cdot, \lambda_0), x_0) \neq 0$, therefore

$$\eta_j(x_0, 0) = i(\nabla_x H(\cdot, \lambda_0), x_0) \cdot \left(\frac{m^+(Q_j(\nabla_x^2 H_1(x_0, \epsilon)))}{2} - \frac{m^-(Q_j(\nabla_x^2 H_1(x_0, -\epsilon)))}{2}\right).$$
Theorem 5.9. Let conditions (H1)–(H3) be satisfied. Assume that \( F_j \) changes its sign at \( \lambda_0 \), one has
\[
m^-(Q_j(\nabla^2 H(x_0, \varepsilon))) = m^-(Q_j(\nabla^2 H(x_0, \psi(\varepsilon)))) \\
\neq m^-(Q_j(\nabla^2 H(x_0, \psi(-\varepsilon)))) \\
= m^-(Q_j(\nabla^2 H(x_0, -\varepsilon))).
\]
Thus \( \eta_j(x_0, 0) \neq 0 \), which implies that \((x_0, \lambda_0)\) is a branching point of nonstationary \( j \)-solutions and a global bifurcation point of nontrivial \( j \)-solutions, according to theorem 4.4. \( \square \)

Theorem 5.9. Let conditions (H1)–(H3) be satisfied. Assume that \( M \subset \mathbb{R}^k \) is a continuous curve and \( \lambda_0 \in M \) is an isolated element of the set
\[
\left( F_0^{-1}(\{0\}) \cup \bigcup_{l \in \mathbb{N}} F_l^{-1}(\{0\}) \right) \cap M
\]
for some \( j \in \mathbb{N} \cup \{0\} \). If the restriction of \( F_j \) to \( M \) changes its sign at \( \lambda_0 \) then \((x_0, \lambda_0)\) is a global bifurcation point of nontrivial (possibly stationary) \( j \)-solutions of (1.1).

Proof. Choose the parametrization \( \varphi \) and the modified Hamiltonian \( H_1 \) as in the proof of theorem 5.8. By the assumption there exists \( \varepsilon > 0 \) such that \( F_0(\varphi(s)) \neq 0 \) for \( s \in [-\varepsilon, \varepsilon] \setminus \{0\} \).

If the restriction of \( F_0 \) to \( M \) changes its sign at \( \lambda_0 \) then
\[
\eta_0(x_0, 0) = \text{sgn det } \nabla^2 H_1(x_0, \varepsilon) - \text{sgn det } \nabla^2 H_1(x_0, -\varepsilon) \\
= \text{sgn } F_0(\varphi(\varepsilon)) - \text{sgn } F_0(\varphi(-\varepsilon)) \neq 0,
\]
hence \((x_0, 0)\) is a global bifurcation point of nontrivial stationary solutions, which are \( j \)-solutions for every \( j \in \mathbb{N} \). Thus one can assume that the restriction of \( F_0 \) to \( M \) does not change its sign at \( \lambda_0 \) (in particular, \( j \neq 0 \)). Then one has
\[
\text{sgn det } \nabla^2 H_1(x_0, \varepsilon) = \text{sgn det } \nabla^2 H_1(x_0, -\varepsilon) = \text{sgn } F(\varphi(\varepsilon)) \neq 0
\]
and
\[
\eta_j(x_0, 0) = \text{sgn } F(\varphi(\varepsilon)) \cdot \frac{m^-(Q_j(\nabla^2 H_1(x_0, \varepsilon))) - m^-(Q_j(\nabla^2 H_1(x_0, -\varepsilon)))}{2}.
\]
Thus \( \eta_j(x_0, 0) \neq 0 \), similarly as in the proof of theorem 5.8. \( \square \)

Remark 5.10. Let \( k \geq 2 \), \( \lambda_0 \in \mathbb{R}^k \), \( r \in \mathbb{N} \), \( F \in C^r(\mathbb{R}^k, \mathbb{R}) \), \( F(\lambda_0) = 0 \), \( \nabla F(\lambda_0) \neq 0 \). Then there exists a neighbourhood \( U \subset \mathbb{R}^k \) of \( \lambda_0 \), such that \( \Gamma = F^{-1}(\{0\}) \cap U \) is a \((k-1)\)-dimensional manifold of class \( C^r \). Note that if \( L \) is a one-dimensional linear subspace of \( \mathbb{R}^k \) such that \( L \not\subset T_{\lambda_0} \Gamma \) then the restriction of \( F \) to the straight line \( L_{\lambda_0} = \lambda_0 + L \) has an isolated zero at \( \lambda_0 \) and changes its sign at \( \lambda_0 \).

Set
\[
X_j(\lambda_0) := \{ l \in \mathbb{N} \cup \{0\} | F_l(\lambda_0) = 0 \}, \quad j \in \mathbb{N} \cup \{0\},
\]
\[
X^*_j(\lambda_0) := \{ l \in \mathbb{N} | F_l(\lambda_0) = 0 \}, \quad j \in \mathbb{N}.
\]

In view of remark 5.10, the next two theorems follow from theorems 5.8 and 5.9 for \( M = L_{\lambda_0} = \lambda_0 + L \), where \( L \not\subset T_{\lambda_0} F_l^{-1}(\{0\}) \cap U \) for all \( l \in X_j(\lambda_0) \) and \( l \in X^*_j(\lambda_0) \), respectively.
Theorem 5.11. Let conditions (H1)–(H3), (E1(x₀, λ₀)) and (E2(x₀, λ₀)) be satisfied. Assume that $F_j(λ₀) = 0$ for some $j \in \mathbb{N}$. If for all $l \in X_j(λ₀)$ the functions $F_l$ are of class $C^1$ in a neighbourhood of $λ₀$ and $\nabla F_j(λ₀) \neq 0$ then $(x₀, λ₀)$ is a branching point of nonstationary j-solutions of (1.1) and a global bifurcation point of nontrivial j-solutions.

Theorem 5.12. Let conditions (H1)–(H3) be satisfied. Assume that $F_j(λ₀) = 0$ for some $j \in \mathbb{N} \cup \{0\}$. If for all $l \in X_l(λ₀)$ the functions $F_l$ are of class $C^1$ in a neighbourhood of $λ₀$ and $\nabla F_j(λ₀) \neq 0$ then $(x₀, λ₀)$ is a global bifurcation point of nontrivial (possibly stationary) j-solutions of (1.1).

In view of theorem 3.4 and remark 5.7 one obtains the following two pairs of corollaries to theorems 5.11 and 5.12, concerning symmetry breaking. First consider the case of only one type of solutions in a neighbourhood of $(x₀, λ₀)$.

Corollary 5.13. Let conditions (H1)–(H3), (E1(x₀, λ₀)) and (E2(x₀, λ₀)) be satisfied. Fix $j \in \mathbb{N}$. If $F_j$ is of class $C^1$ in a neighbourhood of $λ₀$, $F_j(λ₀) = 0$, $\nabla F_j(λ₀) \neq 0$ and $F_l(λ₀) \neq 0$ for all $l \in \mathbb{N}$, $l \neq j$, then $(x₀, λ₀)$ is a global bifurcation point of nontrivial j-solutions of (1.1). Moreover, it is a branching point of nonstationary solutions with the minimal period $\frac{2π}{j}$, but it is not a symmetry breaking point.

Corollary 5.14. Let conditions (H1)–(H3) be satisfied. Fix $j \in \mathbb{N} \cup \{0\}$. If $F_j$ is of class $C^1$ in a neighbourhood of $λ₀$, $F_j(λ₀) = 0$, $\nabla F_j(λ₀) \neq 0$ and $F_l(λ₀) \neq 0$ for all $l \in \mathbb{N} \cup \{0\}$, $l \neq j$, then $(x₀, λ₀)$ is a global bifurcation point of nontrivial j-solutions of (1.1). Moreover, it is a branching point of nonstationary solutions with the minimal period $\frac{2π}{j}$ if $j \in \mathbb{N}$, and nontrivial stationary solutions if $j = 0$, but it is not a symmetry breaking point.

In the next two corollaries symmetry breaking occurs.

Corollary 5.15. Let conditions (H1)–(H3), (E1(x₀, λ₀)) and (E2(x₀, λ₀)) be satisfied. Fix $j_1, j_2 \in \mathbb{N}$ and assume that for $i = 1, 2$ the functions $F_i$ are of class $C^1$ in a neighbourhood of $λ₀$, $F_i(λ₀) = 0$, $\nabla F_i(λ₀) \neq 0$ and $F_l(λ₀) \neq 0$ for all $l \in \mathbb{N}$, $l \geq 2$. Then $(x₀, λ₀)$ is a symmetry breaking point. Namely, it is a branching point of nonstationary solutions of (1.1) with the minimal period $\frac{2π}{j_i}$ and solutions with the minimal period $\frac{2π}{j_i}$. Moreover, it is a global bifurcation point of nontrivial $j_1$-solutions and $j_2$-solutions.

Corollary 5.16. Let conditions (H1)–(H3) be satisfied. Fix $j_1, j_2 \in \mathbb{N}$ and assume that for $i = 1, 2$ the functions $F_i$ are of class $C^1$ in a neighbourhood of $λ₀$, $F_i(λ₀) = 0$, $\nabla F_i(λ₀) \neq 0$ and $F_l(λ₀) \neq 0$ for all $l \in \mathbb{N} \cup \{0\}$, $l \neq 1$. Then $(x₀, λ₀)$ is a symmetry breaking point. Namely, it is a branching point of nonstationary solutions of (1.1) with the minimal period $\frac{2π}{j_i}$ and solutions with the minimal period $\frac{2π}{j_i}$. Moreover, it is a global bifurcation point of nontrivial $j_1$-solutions and $j_2$-solutions.

6. The structure of the set of bifurcation points

In this section the results from section 5 and [37, 38] are applied to the description of the structure of the set of bifurcation points of solutions of (1.1).

Let $\text{Bif}(x₀)$ and $\text{Glbif}(x₀)$ be the sets of those $λ \in \mathbb{R}^+$ for which $(x₀, λ)$ is, respectively, a bifurcation point and a global bifurcation point of nontrivial solutions of (1.1). Similarly, for every $j \in \mathbb{N} \cup \{0\}$ let $\text{Bif}_j(x₀)$ and $\text{Glbif}_j(x₀)$ denote the sets of those $λ \in \mathbb{R}^+$ for which $(x₀, λ)$ is, respectively, a bifurcation point and a global bifurcation point of nontrivial j-solutions of (1.1). Finally, for every $j \in \mathbb{N} \cup \{0\}$ let the subsets $\text{Bif}_j^\text{min}(x₀) \subset \text{Bif}_j(x₀)$, $\text{Glbif}_j^\text{min}(x₀) \subset \text{Glbif}_j(x₀)$ consist of those $λ$ for which $(x₀, λ)$ is, respectively, a bifurcation
point and a branching point of nonstationary solutions of (1.1) with the minimal period \( \frac{2\pi}{j} \) if \( j \in \mathbb{N} \), and nontrivial stationary solutions if \( j = 0 \).

Let

\[
X(\lambda) := \{ j \in \mathbb{N} \cup \{0\} | F_j(\lambda) = 0 \},
\]

\[
X^*(\lambda) := \{ j \in \mathbb{N} | F_j(\lambda) = 0 \} = X(\lambda) \setminus \{0\},
\]

\[
X_j(\lambda) := \{ i \in \mathbb{N} \cup \{0\} | F_j(\lambda) = 0 \}, \quad j \in \mathbb{N} \cup \{0\},
\]

\[
X_j^*(\lambda) := \{ i \in \mathbb{N} | F_j(\lambda) = 0 \} = X_j(\lambda) \setminus \{0\}, \quad j \in \mathbb{N}.
\]

Set also \( P_{\text{sing}}(F) := F^{-1}(\{0\}) \cap \nabla F^{-1}(\{0\}) \).

As it is shown in the subsequent part of this paper, theorems 6.1–6.4 provide a constructive description of the set of bifurcation points of solutions of (1.1) which can be used both to obtain qualitative results by applying theorems of real algebraic geometry as well as in numerical computations for finding all bifurcation points in given domain. Note that the existence of the neighbourhood \( U \) of \( \lambda_0 \) is ensured by lemma 5.5.

**Theorem 6.1.** Let assumptions (H1)–(H3) be fulfilled and let \( U \subset \mathbb{R}^k \) be an open neighbourhood of \( \lambda_0 \in \mathbb{R}^k \) such that the conditions (E1(\( x_0, \lambda_0 \)), (E2(\( x_0, \lambda_0 \)) and \( F_m(\lambda) \neq 0 \) are satisfied for every \( \lambda \in U \) and \( m \in \mathbb{N} \). If \( X^*(\lambda_0) \neq \emptyset \) then \( \text{Bif}(x_0) \cap U = \emptyset \). If \( X^*(\lambda_0) \neq \emptyset \) and \( F_j, \ j \in X^*(\lambda_0) \), are of class \( C^1 \) in \( U \) then the following conclusions hold for \( F = \prod_{j \in X(\lambda_0)} F_j \).

1. \( \text{Bif}(x_0) \cap U \setminus P_{\text{sing}}(F) = \text{GlBif}^{\text{min}}(x_0) \cap U \setminus P_{\text{sing}}(F) \)

\[
= F^{-1}(\{0\}) \cap U \setminus P_{\text{sing}}(F) = \bigcup_{j \in X^*(\lambda_0)} F_j^{-1}(\{0\}) \cap U \setminus P_{\text{sing}}(F).
\]

2. For every \( j \in X^*(\lambda_0) \) one has

\[
\text{Bif}^{\text{min}}(x_0) \cap U \setminus P_{\text{sing}}(F) = \text{GlBif}^{\text{min}}(x_0) \cap U \setminus P_{\text{sing}}(F)
\]

\[
= F_j^{-1}(\{0\}) \cap U \setminus P_{\text{sing}}(F).
\]

The sets \( \text{GlBif}^{\text{min}}(x_0) \cap U \setminus P_{\text{sing}}(F), \ j \in X^*(\lambda_0), \) are pairwise disjoint.

3. If \( F_j, \ j \in X^*(\lambda_0), \) are analytic in \( U \) and \( \lambda \) is an isolated element of \( P_{\text{sing}}(F) \) such that \( \lambda \in \text{cl}(F_j^{-1}(\{0\}) \setminus P_{\text{sing}}(F)) \) for some \( j_0 \in X^*(\lambda_0) \) then \( \lambda \in \text{Bif}^{\text{min}} \cap \text{GlBif}_{j_0}(x_0) \).

**Proof.** If \( X^*(\lambda_0) = \emptyset \) then \( \text{Bif}(x_0) \cap U = \emptyset \), in view of corollary 3.5 and remark 5.7. Assume that \( X^*(\lambda_0) \neq \emptyset \). Conclusion (1) follows from assertion (2), corollary 3.5 and remark 5.7. To prove assertion (2) observe that

\[
\nabla F(\lambda) = \sum_{j \in X^*(\lambda_0)} \nabla F_j(\lambda) \prod_{i \in X^*(\lambda_0) \setminus \{j\}} F_i(\lambda).
\]

Fix \( j \in X^*(\lambda_0) \) and \( \lambda \in U \setminus P_{\text{sing}}(F) \) such that \( F_j(\lambda) = 0 \). Then \( \nabla F_j(\lambda) \neq 0 \) and \( F_i(\lambda) \neq 0 \) for all \( i \in X^*(\lambda_0) \setminus \{j\} \). (In particular, the sets \( F_j^{-1}(\{0\}) \cap U \setminus P_{\text{sing}}(F), \ j \in X^*(\lambda_0), \) are pairwise disjoint.) Thus corollary 5.13 with \( \lambda_0 \) replaced by \( \lambda \) implies that \( (x_0, \lambda) \in \text{GlBif}^{\text{min}}(x_0) \) and \( (x_0, \lambda) \) is not a symmetry breaking point.

Now turn to assertion (3). Note that conclusion (2) implies that \( (x_0, \lambda) \) is a bifurcation point of solutions with the minimal period \( \frac{2\pi}{j} \) as a cluster point of such bifurcation points. It remains to show that \( \lambda \in \text{GlBif}_{j_0}(x_0) \). (One cannot use corollary 5.13, since \( \nabla F(\lambda) = 0 \). In view of the curve selection lemma for semianalytic sets there exists a continuous curve \( M \) such that \( \lambda \) is an isolated element of \( F^{-1}(\{0\}) \cap M \) and the restriction of \( F_{j_0} \) to \( M \) changes
its sign at \( \bar{\lambda} \). Consequently, according to theorem 5.8, \((x_0, \bar{\lambda})\) is a global bifurcation point of \( j_{0} \)-solutions.

Applying corollary 5.14 and theorem 5.9 instead of corollary 5.13 and theorem 5.8 one obtains the following theorem in which bifurcation of nontrivial stationary solutions is allowed.

**Theorem 6.2.** Let assumptions (H1)–(H3) be fulfilled and let \( U \subset \mathbb{R}^{k} \) be an open neighbourhood of \( \lambda_{0} \in \mathbb{R}^{k} \) such that \( F_{m}(\lambda) \neq 0 \) for every \( \lambda \in U \) and \( m \in \mathbb{N} \cup \{0\} \setminus X(\lambda_{0}) \). If \( X(\lambda_{0}) = \emptyset \) then \( \text{Bif}(x_{0}) \cap U = \emptyset \). If \( X(\lambda_{0}) \neq \emptyset \) and \( F_{j} \), \( j \in X(\lambda_{0}) \), are of class \( C^{1} \) in \( U \) then conclusions (1)–(3) of theorem 6.1 hold true for \( F = \prod_{j \in X(\lambda_{0})} F_{j} \) and \( X^{*}(\lambda_{0}) \) replaced by \( X(\lambda_{0}) \).

If the functions \( F \) in theorems 6.1 and 6.2 do not satisfy the assumptions of that theorems, one can restrict the discussion to the set of bifurcation points of \( j \)-solutions for some fixed \( j \), which leads to the following two theorems.

**Theorem 6.3.** Let assumptions (H1)–(H3) be fulfilled and let \( U \subset \mathbb{R}^{k} \) be an open neighbourhood of \( \lambda_{0} \in \mathbb{R}^{k} \) such that the conditions (E1(\( x_{0}, \lambda) \)), (E2(\( x_{0}, \lambda) \)) and \( F_{mj}(\lambda) \neq 0 \) are satisfied for some fixed \( j \in \mathbb{N} \) and all \( \lambda \in U \), \( m \in \mathbb{N} \setminus X^{*}_{j}(\lambda_{0}) \). If \( X^{*}_{j}(\lambda_{0}) = \emptyset \) then \( \text{Bif}_{j}(x_{0}) \cap U = \emptyset \). If \( X^{*}_{j}(\lambda_{0}) \neq \emptyset \) and \( F_{j} \), \( l \in X^{*}_{j}(\lambda_{0}) \), are of class \( C^{1} \) in \( U \) then the following conclusions hold for \( F = \prod_{l \in X^{*}_{j}(\lambda_{0})} F_{lj} \).

1. \( \text{Bif}_{lj}(x_{0}) \cap U \setminus P_{\text{sing}}(F) = \text{GlBif}_{lj}^{\text{min}}(x_{0}) \cap U \setminus P_{\text{sing}}(F) \)
   \[ = F^{-1}_{lj}(\{0\}) \cap U \setminus P_{\text{sing}}(F) \]

The sets \( \text{GlBif}_{lj}^{\text{min}}(x_{0}) \cap U \setminus P_{\text{sing}}(F) \), \( l \in X^{*}_{j}(\lambda_{0}) \), are pairwise disjoint.

2. (3) For every \( l \in X^{*}_{j}(\lambda_{0}) \), are analytic in \( U \) and \( \bar{\lambda} \) is an isolated element of \( P_{\text{sing}}(F) \) such that \( \bar{\lambda} \in \text{cl}(F_{lj}^{-1}(\{0\}) \setminus P_{\text{sing}}(F)) \) for some fixed \( l_{0} \in X^{*}_{j}(\lambda_{0}) \) then \( \bar{\lambda} \in \text{Bif}_{lj}^{\text{min}}(x_{0}) \cap \text{GlBif}_{lj}^{\text{min}}(x_{0}) \).

**Theorem 6.4.** Let assumptions (H1)–(H3) be fulfilled and let \( U \subset \mathbb{R}^{k} \) be an open neighbourhood of \( \lambda_{0} \in \mathbb{R}^{k} \) such that \( F_{mj}(\lambda) \neq 0 \) for some fixed \( j \in \mathbb{N} \cup \{0\} \) and all \( \lambda \in U \), \( m \in \mathbb{N} \cup \{0\} \setminus X_{j}(\lambda_{0}) \). If \( X_{j}(\lambda_{0}) = \emptyset \) then \( \text{Bif}_{j}(x_{0}) \cap U = \emptyset \). If \( X_{j}(\lambda_{0}) \neq \emptyset \) and \( F_{lj} \), \( l \in X_{j}(\lambda_{0}) \), are of class \( C^{1} \) in \( U \) then conclusions (1)–(3) of theorem 6.3 hold true for \( F = \prod_{l \in X_{j}(\lambda_{0})} F_{lj} \) and \( X^{*}_{j}(\lambda_{0}) \) replaced by \( X_{j}(\lambda_{0}) \).

**Remark 6.5.** In view of lemma 5.5, theorems 6.1–6.4 remain valid for a bounded open set \( U \subset \mathbb{R}^{k} \) and the sets \( X^{*}(\lambda_{0}) \), \( X_{j}(\lambda_{0}) \), \( X^{*}_{j}(\lambda_{0}) \), \( X_{j}(\lambda_{0}) \) replaced by the sets

\[
\begin{align*}
X^{*}(U) &:= \{ j \in \mathbb{N} \cup \{0\} \mid \exists \lambda \in U : F_{j}(\lambda) = 0 \}, \\
X(U) &:= \{ j \in \mathbb{N} \cup \{0\} \mid \exists \lambda \in U : F_{j}(\lambda) = 0 \}, \\
X_{j}^{*}(U) &:= \{ l \in \mathbb{N} \cup \{0\} \mid \exists \lambda \in U : F_{j}(\lambda) = 0 \}, \\
X_{j}(U) &:= \{ l \in \mathbb{N} \cup \{0\} \mid \exists \lambda \in U : F_{j}(\lambda) = 0 \},
\end{align*}
\]

respectively, which makes that theorems independent of \( \lambda_{0} \).

Now the results from [37, 38] can be applied. In what follows the symbols \( D^{k}_{r} \) and \( S^{k-1}_{r} \) denote, respectively, the closed disc and the sphere in \( \mathbb{R}^{k} \) centred at the origin with radius \( r > 0 \).
Definition 6.6. A mapping $F: \mathbb{R}^k \to \mathbb{R}$ is called admissible if it is analytic and $0 \in \mathbb{R}^k$ is an isolated singular point of $F^{-1}(\{0\})$, i.e. it is an isolated element of the set $F^{-1}(\{0\}) \cap (\nabla F)^{-1}(\{0\})$.

Consider first the case of two parameters ($k = 2$).

Definition 6.7. Let $F: \mathbb{R}^2 \to \mathbb{R}$ be admissible. An analytic mapping $g: \mathbb{R}^2 \to \mathbb{R}$ is called a test function for $F$ if $0 \in \mathbb{R}^2$ is an isolated element of the set $g^{-1}(\{0\}) \cap F^{-1}(\{0\})$.

Set $h(g, F) := (\text{Jac}(g, F), F): \mathbb{R}^2 \to \mathbb{R}^2$ (see [37]), where $\text{Jac}(g, F): \mathbb{R}^2 \to \mathbb{R}$ is the Jacobian of the mapping $(g, F): \mathbb{R}^2 \to \mathbb{R}^2$.

Applying theorem A.1, corollary A.2 (see appendix A) and lemma 5.5 one obtains the following two corollaries to theorems 6.1 and 6.2. Determining the numbers $b_+(g, F) \mathrm{i} b_-(g, F)$ in these corollaries allows us to localize the curves forming the set of bifurcation points (see for example corollary A.3). Note that assumptions $(E1(x_0, 0))$ and $(E2(x_0, 0))$ in corollary 6.8 (and in corollary 6.11) imply that conditions $(E1(x_0, \lambda))$ and $(E2(x_0, \lambda))$ are satisfied for every $\lambda$ from a neighbourhood of the origin.

Corollary 6.8. Let conditions (H1)–(H3), $(E1(x_0, 0)$ and $(E2(x_0, 0)$ be satisfied for $k = 2$, and let $X^+(0) \neq \emptyset$. Set $F = \prod_{j \in X^+(0)} F_j$ and assume that $F, F_j$, $j \in X^+(0)$, are admissible and $g_j$ is a nonnegative test function for $F$. Then for every sufficiently small $r > 0$ the following conclusions hold.

1. Each of the sets GIbif$(x_0) \cap D^2_0(\{0\}, GIbif^{\min}(x_0) \cap D^2_0(\{0\}, j \in X^+(0)$, is a union of even (possibly zero) number of disjoint analytic curves, each of which meets the origin and crosses $S^1$ transversally in one point. The number of those curves, equal to $b(F)$ and $b(F_j)$, respectively, is determined by formula (A.2) in corollary A.2. If the number of the curves is nonzero then $0 \in GIbif(x_0)$.

2. If $g$ is an arbitrary test function for $F$ then the number of those curves forming GIbif$(x_0) \cap D^2_0(\{0\)$ and GIbif$^{\min}(x_0) \cap D^2_0(\{0\), $j \in X^+(0)$, on which $g$ is positive (negative), equal to $b_+(g, F)$ ($b_-(g, F)$) and $b_+(g, F_j)$ ($b_-(g, F_j)$), respectively, is determined by formula (A.1) in theorem A.1.

3. If $b(F) \neq b(F_j) \neq 0$ for some $j \in X^+(0)$, then $(x_0, 0)$ is a symmetry breaking point.

Corollary 6.9. Let conditions (H1)–(H3) be satisfied for $k = 2$ and let $X(0) \neq \emptyset$. Set $F = \prod_{j \in X(0)} F_j$ and assume that $F, F_j$, $j \in X(0)$, are admissible and $g_j$ is a nonnegative test function for $F$. Then for every sufficiently small $r > 0$ conclusions (1)–(3) of corollary 6.8 hold true for $X^+(0)$ replaced by $X(0)$.

Remark 6.10. One obtains two analogous corollaries to theorems 6.3 and 6.4 in the case of $k = 2$.

Now consider the case of arbitrary number $k$ of parameters.

Assume that an admissible function $F: \mathbb{R}^k \to \mathbb{R}$ is a Morse function on small spheres, i.e. there exists $r > 0$ such that $F|_{S^{k-s}}$ is a Morse function for every $0 < s \leq r$. Let $\Sigma$ be the set of critical points of $F|_{S^{k-s}}$. For $\lambda \in \Sigma$, denote by ind$(F, \lambda)$ the Morse index of $F|_{S^{k-s}}$ at $\lambda$. Set

$n_+(F) := \# \{\lambda \in \Sigma | F(\lambda) < 0 \wedge \text{ind}(F, \lambda) \text{ is even}\},$

$n_-(F) := \# \{\lambda \in \Sigma | F(\lambda) < 0 \wedge \text{ind}(F, \lambda) \text{ is odd}\},$

$p_+(F) := \# \{\lambda \in \Sigma | F(\lambda) > 0 \wedge \text{ind}(F, \lambda) \text{ is even}\},$

$p_-(F) := \# \{\lambda \in \Sigma | F(\lambda) > 0 \wedge \text{ind}(F, \lambda) \text{ is odd}\}.$
Szafraniec [38] proved theorems which can be used to verify whether \( F \) is Morse on small spheres and gave formulae for \( n_{\pm}(F), p_{\pm}(F) \) written in terms of local topological degree of mappings defined explicitly by using \( F \).

Note that if \( n_{\mu}(F) \cdot p_{\nu}(F) \neq 0 \) for some \( \mu, \nu \in \{+, -\} \) then \( F \) has zeros on \( D^k_r \) for every sufficiently small \( r > 0 \). In particular, \( F^{-1}(\{0\}) \cap D^k_r \neq \emptyset \). Thus one obtains the following two corollaries to theorems 6.1 and 6.2 (see also lemma 5.5).

**Corollary 6.11.** Let conditions (H1)–(H3), \((E1(x_0, 0))\) and \((E2(x_0, 0))\) be satisfied, and let \( X^*(0) \neq \emptyset \). Set \( F = \bigcap_{j \in X^*(0)} F_j \) and assume that \( F, F_j, j \in X^*(0), \) are admissible and Morse on small spheres. Then for every sufficiently small \( r > 0 \) the following conclusions hold.

1. If \( n_{\mu}(F) \cdot p_{\nu}(F) \neq 0 \) for some \( \mu, \nu \in \{+, -\} \) then the set \( \text{GibBif}(x_0) \cap D^k_r \) is a topological cone with vertex at the origin and base \( F^{-1}(\{0\}) \cap S^k_r \). Moreover, \( \text{GibBif}(x_0) \cap D^k_r \setminus \{0\} \) is a \((k-1)\)-dimensional manifold with boundary \( F^{-1}((0)) \cap S^k_r \).

2. Similarly, if \( n_{\mu}(F_j) \cdot p_{\nu}(F_j) \neq 0 \) for some \( j \in X^*(0) \) and \( \mu, \nu \in \{+, -\} \) then the set \( \text{GibBif}^{\text{min}}(x_0) \cap D^k_r \cup \{0\} \) is a topological cone with vertex at the origin and base \( F_j^{-1}(\{0\}) \cap S^k_r \). Moreover, \( \text{GibBif}^{\text{min}}(x_0) \cap D^k_r \setminus \{0\} \) is a \((k-1)\)-dimensional manifold with boundary \( F_j^{-1}(\{0\}) \cap S^k_r \).

3. If \( n_{\mu_1}(F_j) \cdot p_{\nu_1}(F_j) \cdot n_{\mu_2}(F_j) \cdot p_{\nu_2}(F_j) \neq 0 \) for some \( j, 1, j_2 \in X^*(0) \) and some \( \mu_1, \nu_1, \mu_2, \nu_2 \in \{+, -\} \) then \((x_0, 0)\) is a symmetry breaking point.

**Corollary 6.12.** Let conditions (H1)–(H3) be satisfied and let \( X(0) \neq \emptyset \). Set \( F = \bigcap_{j \in X(0)} F_j \) and assume that \( F, F_j, j \in X(0), \) are admissible and Morse on small spheres. Then for every sufficiently small \( r > 0 \) conclusions (1)–(3) of corollary 6.11 hold true for \( X^*(0) \) replaced by \( X(0) \).

One obtains two analogous corollaries to theorems 6.3 and 6.4.

The above corollaries allow us to use the formulae for \( n_{\pm}(F), p_{\pm}(F) \) given in [38] to detect symmetry breaking points. The results from [38] can be also used to investigate the number of the cones from the above corollaries.

7. Examples

In this section the results from section 6 are applied to examples of system (1.1) with two and three parameters. Symbolic computations of topological indices have been performed by using Łęcki’s program based on an algorithm described in [12, 23]. Other symbolic computations (solving polynomial equations, estimates, etc) have been carried out by using Maple. The graphs of curves and surfaces forming the sets of zeros of detecting functions (which are proved to be bifurcation points of given systems) in prescribed area have been obtained by using Endrass’ program **surf** [13].

Recall that \( D^k_r \) denotes the closed disc in \( \mathbb{R}^k \) centred at the origin with radius \( r > 0 \).

**Remark 7.1.** Let \( F: \mathbb{R}^k \to \mathbb{R} \) be an analytic function for some \( k \in \mathbb{N} \). Fix \( r > 0 \) and let \( 0 \in \mathbb{R}^k \) be the unique singular point of \( F \) in \( D^k_r \). Assume that for every \( \lambda \in F^{-1}(\{0\}) \cap D^k_r \setminus \{0\} \) the tangent space to \( F^{-1}(\{0\}) \) at \( \lambda \) is not equal to the tangent space at \( \lambda \) to the sphere centred at the origin. Then every connected component of \( F^{-1}(\{0\}) \cap D^k_r \) contains the origin.

**Remark 7.2.** In the next two examples, the functions \( g_i: \mathbb{R}^2 \to \mathbb{R}, i = 1, \ldots, 4, \) are defined by

\[
g_1(\lambda_1, \lambda_2) = \lambda_1^2 + \lambda_2^2, \quad g_2(\lambda_1, \lambda_2) = \lambda_1, \quad g_3(\lambda_1, \lambda_2) = \lambda_2, \quad g_4(\lambda_1, \lambda_2) = \lambda_1 \cdot \lambda_2
\]


(see [20]). If $F: \mathbb{R}^2 \to \mathbb{R}$ is an admissible mapping which has no zeros on the coordinate axes in a neighbourhood of the origin (e.g. if $F(.,0)$ and $F(0,.)$ are polynomials of nonzero degree) then $g_i, i = 1, \ldots, 4$, are test functions for $F$. For such an $F$ the symbol $b_i(F)$ denotes the number of components of $F^{-1}(\{0\}) \cap D_i^2 \setminus \{0\}$ (for sufficiently small $r > 0$) contained in the $i$th quarter of the plane $\mathbb{R}^2$ for $i = 1, \ldots, 4$.

In all examples use is made of the functions $F_j$ defined by (5.2).

**Example 7.3.** Let $H : \mathbb{R}^6 \times \mathbb{R}^2 \to \mathbb{R}$ be the Hamilton function given by

$$H(x, \lambda) \equiv H(x_1, \ldots, x_6, \lambda_1, \lambda_2) = P(x_1, \ldots, x_6, \lambda_1, \lambda_2) + Q(x_1, \ldots, x_6),$$

where

$$P(x_1, \ldots, x_6, \lambda_1, \lambda_2) = \frac{1}{2}(9 + \frac{1}{10}\lambda_1^6)x_1^2 + \frac{5}{2}x_2^2 + \frac{1}{2}x_3^2 + \frac{1}{2}(5 - \frac{5}{9}\lambda_1^6)x_6^2 + 2\lambda_2^6x_1x_3 + 8\lambda_2^6x_4x_6 + x_4^2 + x_5^2,$$

$$Q \in C^2(\mathbb{R}^6, \mathbb{R})$$

has a local minimum at the origin, and $\nabla^2 Q(0) = 0$, for example

$$Q(x_1, \ldots, x_6) = x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 + x_6^6 + (x_1^3 + x_2^3)x_2 + x_1x_2^2 + x_1^2x_3 + x_2^2x_4 + x_3^2x_5 + x_4^2x_6.$$

(7.1)

$H$ satisfies conditions (H1)–(H3) for $k = 2$ and $x_0 = 0 \in \mathbb{R}^6$.

First the set of bifurcation points in $[0] \times D^2$ will be described for sufficiently small $r > 0$ and then it will be shown that the conclusions hold for every $r \leq 0.3$.

One has

$$A(\lambda_1, \lambda_2) = \begin{bmatrix} 9 + \frac{1}{10}\lambda_1^6 & 0 & 2\lambda_2^6 \\ 0 & 0 & 0 \\ 2\lambda_2^6 & 0 & 5 \end{bmatrix},$$

$$B(\lambda_1, \lambda_2) = \begin{bmatrix} 1 & 0 & 8\lambda_2^6 \\ 0 & 0 & 0 \\ 8\lambda_2^6 & 0 & 5 - \frac{1}{9}\lambda_1^5 \end{bmatrix}.$$

Those of the functions $F_j, j \in \mathbb{N} \cup \{0\}$, defined by (5.2), which vanish at $(0,0) \in \mathbb{R}^2$ are $F_0$, $F_2$ and $F_3$, hence $X^+(0) = \{3,5\}$ (see also lemmas 5.5 and 5.6). One has

$$F_0(\lambda_1, \lambda_2) \equiv 0,$$

$$F_2(\lambda_1, \lambda_2) = 288\lambda_1^6\lambda_2^{12} - \frac{22}{3}\lambda_1^6 + \frac{9}{20}\lambda_1^{14} - 2304\lambda_2^{24} + 28692\lambda_2^{12} - \frac{9}{7}\lambda_1^5\lambda_2^{12},$$

$$F_3(\lambda_1, \lambda_2) = 800\lambda_1^6\lambda_2^{12} + \frac{5}{8}\lambda_1^{14} + 92500\lambda_2^{12} - 100\lambda_1^5 - 64000\lambda_2^{24} - 5\lambda_1^5\lambda_2^{12}.$$

Use will be made of theorems 6.1, 6.3 and corollary 6.8 (see also remark 6.10). Theorems 6.2, 6.4 and corollary 6.9 cannot be applied, since $F_0 = 0$, which means that all the points $(x_0, \lambda), \lambda \in \mathbb{R}^2$, are degenerate.

Observe that conditions (E1(0, $\lambda$)) and (E2(0, $\lambda$)) are satisfied for every $\lambda \in U := (-0.31, 0.31)^2$. (At the moment only assumptions (E1(0, 0)) and (E2(0, 0)) are needed, as in corollary 6.8.) Indeed. An appropriate estimate for the function $P$ shows that for every $\lambda \in (-0.31, 0.31)^2$ and every $v \in \mathbb{R}^6 \setminus \{0\}$ the function $[0, +\infty) \ni \epsilon \mapsto P(cv, \lambda)$ is strictly increasing (in particular, $P(\cdot, \lambda)$ has a strict local minimum at $0 \in \mathbb{R}^6$). On the other hand, $Q$ has a minimum at $0 \in \mathbb{R}^6$ and it does not depend on $\lambda$. Thus there exists $\epsilon > 0$ such that $\nabla_x H(x, \lambda) \neq 0$ for every $0 < |x| < \epsilon, \lambda \in (-0.31, 0.31)^2$. Consequently, for every
\[ \lambda \in (-0.31, 0.31)^2 \] condition \((E_1(0, \lambda))\) is fulfilled and the function \(H(\cdot, \lambda)\) has a strict local minimum at \(0 \in \mathbb{R}^6\), hence \(i(\nabla, H(\cdot, \lambda), 0) = 1 \neq 0\) (see \([2]\)).

Symbolic computations show that \((0, 0) \in \mathbb{R}^2\) is an isolated singular point of the functions \(F_3\) and \(F = F_3 \cdot F_3\). Thus they are admissible and, according to remark 7.2, \(g_i\), \(i = 1, \ldots, 4\), are test functions for them. Furthermore,

\[ i(h(g_1, F_3), 0) = 2, \quad i(h(g_1, F), 0) = 1, \quad i(h(g_1, F), 0) = 3, \quad (7.2) \]

which has been checked by using Łecki’s program. It follows from theorems 6.1, 6.3 and corollary 6.8 (for \(g_4 = g_3\)) that for every sufficiently small \(r > 0\) the following equalities hold.

\[
\begin{align*}
\text{Bif}(0) \cap D_2^2 = \text{GlBif}(0) \cap D_2^2 & = F^{-1}(\{0\}) \cap D_2^2, \\
\text{GlBif}_{\min}^3 \cap D_2^2 & = F_3^{-1}(\{0\}) \cap D_2^2, \\
\text{GlBif}_{\min}^5 \cap D_2^2 & = F_5^{-1}(\{0\}) \cap D_2^2.
\end{align*}
\]

(7.3)

The fact that \(0 \in \text{GlBif}_{\min}^3 \cap D_2^2\) and \(0 \in \text{GlBif}_{\min}^5 \cap D_2^2\) follows from theorem 3.4 and remark 5.7. (The only minimal periods of nontrivial solutions in a neighbourhood of the origin are \(\frac{2\pi}{4}\) and \(\frac{2\pi}{r}\).)

In view of corollary 6.8, the set \(\text{GlBif}(0) \cap D_2^2 \setminus \{0\}\) consists of \(b(F) = 6\) curves. The numbers of the curves forming the sets \(\text{GlBif}_{\min}^3(0) \cap D_2^2 \setminus \{0\}\) and \(\text{GlBif}_{\min}^5(0) \cap D_2^2 \setminus \{0\}\) are equal to \(b(F_3) = 4\) and \(b(F_5) = 2\), respectively.

To localize the curves in the quarters of the plane corollary A.3 will be used. Application of Łecki’s program yields

\[
\begin{align*}
i(h(g_2, F_3), 0) & = 0, \quad i(h(g_2, F), 0) = 1, \\
i(h(g_3, F_3), 0) & = 0, \quad i(h(g_3, F), 0) = 0, \\
i(h(g_4, F_3), 0) & = 0, \quad i(h(g_4, F), 0) = 0.
\end{align*}
\]

(7.4)

Taking into account (7.2), (7.4) and corollary A.3 one obtains

\[
\begin{align*}
b_1(F_3) & = 1, \quad b_2(F_3) = 1, \quad b_3(F_3) = 1, \quad b_4(F_3) = 1, \\
b_1(F_5) & = 1, \quad b_2(F_5) = 0, \quad b_3(F_5) = 0, \quad b_4(F_5) = 1.
\end{align*}
\]

The following results of additional symbolic computations and estimates ensure that the above conclusions concerning bifurcation points in \(\{0\} \times D_2^2\) hold for every \(r \leq 0.3\). One has \(F_j(\lambda) \neq 0\) for every \(j \in \mathbb{N} \setminus \{3, 5\}\), \(\lambda \in U := (-0.31, 0.31)^2\). The origin is the only singular point of \(F_3\), \(F_5\) and \(F = F_3 \cdot F_5\) in \(U\). The sets of zeros of \(F_3\), \(F_5\) restricted to \(D_2^2 \setminus \{0\}\), \(r = 0.3\), are disjoint and they have no common points with the coordinate axes. Furthermore, the functions \(F_3\), \(F_5\) and \(F\) satisfy the assumptions of remark 7.1 for \(k = 2\) and \(r = 0.3\). Thus for \(r = 0.3\) every connected component of \(F_3^{-1}(\{0\}) \cap D_2^2\), \(F_5^{-1}(\{0\}) \cap D_2^2\) and \(F^{-1}(\{0\}) \cap D_2^2\) contains the origin.

Theorem 6.1 and 6.3 have also been applied to find bifurcation points in \(\{0\} \times D_2^2\), \(r = 0.3\), numerically as zeros of the functions \(F_j\), according to formulae (7.3), which has been performed by using the program surf and presented in figure 1. The earlier conclusions ensure that the number of curves in figure 1, their localization and their relative position do not change when passing to a smaller scale.

One can summarize the above results as follows. The set of bifurcation points in \(\{0\} \times D_2^2\), \(r = 0.3\), is equal to the set of global bifurcation points in this domain and consists of six curves, for which the origin is the only common point. Apart from the origin four curves (one curve in each quarter) consist of branching points of solutions with the minimal period \(\frac{2\pi}{3}\) (and
only such solutions), whereas two curves (one curve in the first quarter and one curve in the fourth quarter) consist of branching points of solutions with the minimal period $2\pi/5$ (and only such solutions). The origin is a branching point of solutions with the minimal period $2\pi/3$ and solutions with the minimal period $2\pi/5$ (and only such solutions). In particular, the origin is a symmetry breaking point.

**Example 7.4.** Consider the Hamiltonian $H: \mathbb{R}^6 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by the formula

$$H(x, \lambda) \equiv H(x_1, \ldots, x_6, \lambda_1, \lambda_2) = P(x_1, \ldots, x_6, \lambda_1, \lambda_2) + Q(x_1, \ldots, x_6),$$

where

$$P(x_1, \ldots, x_6, \lambda_1, \lambda_2) = \frac{1}{2}(4 + 3\lambda_1^{10} + \lambda_1^{7}\lambda_2 - \lambda_1^{5}\lambda_2^3 + \lambda_1^{3}\lambda_2^5 - \lambda_1^{5}\lambda_2^3 - \lambda_1^{3}\lambda_2^5 - \lambda_1^{4}\lambda_2^4 - \lambda_2^9) + \frac{3}{2}x_1^2 + x_2^2 + \frac{3}{2}(3 + 3\lambda_1^7)x_3^2 + \frac{1}{2}\lambda_1^2\lambda_6^2 + (3\lambda_1^3 + 22\lambda_2^4)x_5x_6,$$

$$Q(x_1, \ldots, x_6) = x_1^2x_2 + (x_1 + x_4)x_3^2 + x_2^2x_3 + (x_6 - x_3)^3. \quad (7.5)$$

$H$ satisfies conditions (H1)–(H3) for $k = 2$ and $x_0 = 0 \in \mathbb{R}^6$.

Note that in this case $x_0 = 0 \in \mathbb{R}^6$ is an isolated critical point of $H(\cdot, 0)$, it is degenerate, and $i(\nabla_x H(\cdot, 0), 0) = 0$.

First the set of bifurcation points in $[0] \times D^2_0$ will be described for sufficiently small $r > 0$ and then it will be shown that the conclusions hold for every $r \leq 0.3$.

One has

$$A(\lambda_1, \lambda_2) = \begin{bmatrix} 4 + 3\lambda_1^{10} + \lambda_1^{7}\lambda_2 - \lambda_1^{5}\lambda_2^3 + \lambda_1^{3}\lambda_2^5 - \lambda_1^{5}\lambda_2^3 - \lambda_1^{3}\lambda_2^5 - \lambda_1^{4}\lambda_2^4 - \lambda_2^9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

$$B(\lambda_1, \lambda_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 + 3\lambda_1^7 & 3\lambda_1^3 + 22\lambda_2^4 \\ 0 & 3\lambda_1^3 + 22\lambda_2^4 & \lambda_1^3 \end{bmatrix}.$$

Use will be made of theorems 6.2, 6.4 and corollary 6.9 (see also remark 6.10). Theorems 6.1, 6.3 and corollary 6.8 are not suitable in this case. (It will be shown that the origin is a bifurcation point of nontrivial stationary solutions.)
Those of the functions $F_j$, $j \in \mathbb{N} \cup \{0\}$, defined by (5.2), which vanish at $(0,0)\in \mathbb{R}^2$ are $F_0$, $F_2$ and $F_3$, hence $X(0) = \{0, 2, 3\}$ (see also lemmas 5.5 and 5.6). One has

\[ F_0 = f_0 \cdot a_0, \quad F_2 = f_2 \cdot a_2, \quad F_3 = f_3 \cdot a_3, \]

where

\[
\begin{align*}
    f_0(\lambda_1, \lambda_2) &= 18\lambda_1^3 + 18\lambda_1^2 - 54\lambda_1 + 792\lambda_1^3\lambda_2^2 - 2904\lambda_2^5, \\
    f_2(\lambda_1, \lambda_2) &= 3\lambda_1^{10} + \lambda_1^7\lambda_2 - \lambda_1^7\lambda_2 - \lambda_1\lambda_2 - \lambda_1^4\lambda_2 - \lambda_2^5, \\
    f_3(\lambda_1, \lambda_2) &= 18\lambda_1^8 - 81\lambda_1^7 - 54\lambda_6 - 792\lambda_1^3\lambda_2 - 2904\lambda_2^5, \\
    a_0(\lambda_1, \lambda_2) &= f_2(\lambda_1, \lambda_2) + 4, \\
    a_2(\lambda_1, \lambda_2) &= f_0(\lambda_1, \lambda_2) - 8\lambda_1^2 - 36\lambda_1^2 - 20, \\
    a_3(\lambda_1, \lambda_2) &= a_0(\lambda_1, \lambda_2) - 9.
\end{align*}
\]

The functions $a_0$, $a_2$, $a_3$ have no zeros in $U := (-0.31, 0.31)^2$. Thus $F_0$, $F_2$, $F_3$ can be replaced by $f_0$, $f_2$, $f_3$ in computations.

It has been checked by symbolic computations that $(0,0)$ is an isolated singular point of the functions $F_0$, $F_2$, $F_3$ and $F = F_0 \cdot F_2 \cdot F_3$. Thus they are admissible and, according to remark 7.2, $g_i$, $i = 1, \ldots, 4$, are test functions for them. Application of Lęcki’s program gives

\[
\begin{align*}
    i(h(g_1, F_0), 0) &= 2, & i(h(g_1, F_2), 0) &= 3, & i(h(g_1, F), 0) &= 7. \tag{7.6}
\end{align*}
\]

In view of theorems 6.2, 6.4 and corollary 6.9 (for $g_+ = g_1$), for every sufficiently small $r > 0$ one has

\[
\begin{align*}
    \text{Bif}(0) \cap D^2_r &= \text{GlBif}(0) \cap D^2_r = F^{-1}(\{0\}) \cap D^2_r \tag{7.7} \\
    &= (\text{GlBif}_0(0) \cup \text{GlBif}_2(0) \cup \text{GlBif}_3(0)) \cap D^2_r, \\
    \text{GlBif}^\text{min}_0(0) \cap D^2_r &= \text{GlBif}_0(0) \cap D^2_r = F_0^{-1}(\{0\}) \cap D^2_r, \\
    \text{GlBif}_2(0) \cap D^2_r &= (F_0^{-1}(\{0\}) \cup F_2^{-1}(\{0\})) \cap D^2_r, \\
    \text{GlBif}_3(0) \cap D^2_r &= (F_0^{-1}(\{0\}) \cup F_3^{-1}(\{0\})) \cap D^2_r, \tag{7.8} \\
    \text{GlBif}^\text{min}_2(0) \cap D^2_r \{0\} &= F_2^{-1}(\{0\}) \cap D^2_r \{0\}, \\
    \text{GlBif}^\text{min}_3(0) \cap D^2_r \{0\} &= F_3^{-1}(\{0\}) \cap D^2_r \{0\}. \tag{7.9}
\end{align*}
\]

According to corollary 6.9, the set $\text{GlBif}(0) \cap D^2_r \{0\}$ consists of $b(F) = 14$ curves. The numbers of curves forming the sets $\text{GlBif}^\text{min}_0(0) \cap D^2_r \{0\}$, $\text{GlBif}^\text{min}_2(0) \cap D^2_r \{0\}$, $\text{GlBif}^\text{min}_3(0) \cap D^2_r \{0\}$ are equal to $b(F_0) = 4$, $b(F_2) = 6$, $b(F_3) = 4$, respectively.

With the aim of applying corollary A.3 to localize the curves in the quarters of the plane it has been checked by using Lęcki’s program that

\[
\begin{align*}
    i(h(g_2, F_0), 0) &= 0, & i(h(g_2, F_2), 0) &= 0, & i(h(g_2, F_3), 0) &= -2, \\
    i(h(g_3, F_0), 0) &= 0, & i(h(g_3, F_2), 0) &= 1, & i(h(g_3, F_3), 0) &= 0, \\
    i(h(g_4, F_0), 0) &= 0, & i(h(g_4, F_2), 0) &= -2, & i(h(g_4, F_3), 0) &= 0. \tag{7.10}
\end{align*}
\]

Taking into account (7.6), (7.10) and corollary A.3 one obtains

\[
\begin{align*}
    b_1(F_0) &= 1, & b_2(F_0) &= 1, & b_3(F_0) &= 1, & b_4(F_0) &= 1, \\
    b_1(F_2) &= 1, & b_2(F_2) &= 3, & b_3(F_2) &= 0, & b_4(F_2) &= 2, \\
    b_1(F_3) &= 0, & b_2(F_3) &= 2, & b_3(F_3) &= 2, & b_4(F_3) &= 0.
\end{align*}
\]
has not been proved that it is a branching point of solutions with the minimal periods \( a \) global bifurcation point of stationary solutions, 2-solutions and 3-solutions. However, it has minimal period 2 point of stationary solutions, solutions with the minimal periods \( \pi \)

The following results of additional symbolic computations and estimates ensure that the above conclusions concerning bifurcation points in \( [0] \times D_r^2 \) hold for every \( r \leq 0.3 \). One has \( F_j(\lambda) \neq 0 \) for every \( j \in \mathbb{N}\setminus\{0, 2, 3\}, \lambda \in U := (-0.31, 0.31)^2 \). The origin is the only singular point of \( F_0, F_2, F_3 \) and \( F = F_0 \cdot F_2 \cdot F_3 \) in \( U \). The sets of zeros of \( F_0, F_2, F_3 \) restricted to \( D_r^2 \setminus [0], r = 0.3 \), are pairwise disjoint and they have no common points with the coordinate axes. Moreover, the functions \( F_0, F_2, F_3 \) and \( F \) satisfy the assumptions of remark 7.1 for \( k = 2 \) and \( r = 0.3 \). Thus for \( r = 0.3 \) every connected component of \( F_0^{-1}([0]) \cap D_r^2, F_2^{-1}([0]) \cap D_r^2, F_3^{-1}([0]) \cap D_r^2 \) and \( F^{-1}([0]) \cap D_r^2 \) contains the origin.

Theorems 6.2 and 6.4 have also been applied to find bifurcation points in \( [0] \times D_r^2, r = 0.3 \), numerically as zeros of the functions \( F_j \), according to formulae (7.7)–(7.9), which has been performed by using the program surf and presented in figure 2. The earlier conclusions ensure that the number of curves in figure 2, their localization and their relative position do not change when passing to a smaller scale.

The above results can be summarized as follows. The set of bifurcation points in \( [0] \times D_r^2, r = 0.3 \), is equal to the set of global bifurcation points in this domain and consists of fourteen curves, for which the origin is the only common point. Apart from the origin four curves (one curve in each quarter) consist of branching points of nontrivial stationary solutions (and only such solutions), six curves (one curve in the first quarter, three curves in the second quarter and two curves in the fourth quarter) consist of branching points of solutions with the minimal period \( \pi \) (and only such solutions) and four curves (two curves in the second quarter and two curves in the third quarter) consist of branching points of solutions with the minimal period \( \frac{2}{7} \pi \) (and only such solutions). The origin is a symmetry breaking point, since it is a bifurcation point of stationary solutions, solutions with the minimal periods \( \pi \) and solutions with the minimal period \( \frac{2}{7} \pi \) (as a cluster point of branching points of such solutions). The origin is also a global bifurcation point of stationary solutions, 2-solutions and 3-solutions. However, it has not been proved that it is a branching point of solutions with the minimal periods \( \pi \) and \( \frac{2}{7} \pi \).

**Example 7.5.** Let the Hamiltonian \( H: \mathbb{R}^6 \times \mathbb{R} \to \mathbb{R} \) be of the form

\[
H(x, \lambda) \equiv H(x_1, \ldots, x_6, \lambda_1, \lambda_2, \lambda_3) = P(x_1, \ldots, x_6, \lambda_1, \lambda_2, \lambda_3) + Q(x_1, \ldots, x_6),
\]

where

\[
P(x_1, \ldots, x_6, \lambda_1, \lambda_2, \lambda_3) = \frac{1}{2}(7 - \lambda_1^4)x_1^2 + \frac{1}{2}(1 - \lambda_1^{13})x_2^2 + \frac{7}{2}x_4^2 + 8x_6^2 - \lambda_1^3x_1x_3 + \lambda_2^{13}x_4x_6 + x_2^4 + x_4^4
\]

\[
\text{Figure 2. The set of those } (\lambda_1, \lambda_2) \in D_r^2, r = 0.3, \text{ for which } (0, (\lambda_1, \lambda_2)) \in \mathbb{R}^6 \times \mathbb{R}^2 \text{ is a global bifurcation point of the system from example 7.4. The legend on the right describes the minimal periods of solutions bifurcating from the points of given curve.}
and \( Q \in C^2(\mathbb{R}^6, \mathbb{R}) \) is the same as in example 7.3, i.e. it has a local minimum at the origin and \( \nabla^2 Q(0) = 0 \), see (7.1) for instance.

\( H \) satisfies conditions (H1)–(H3) for \( k = 3 \) and \( x_0 = 0 \in \mathbb{R}^6 \).

The set of bifurcation points in \([0] \times D_3^3\) will be investigated for \( r = 0.3 \).

One has
\[
A(\lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix}
7 - \lambda_1^4 & 0 & -\lambda_3^3 \\
0 & 0 & 0 \\
-\lambda_3^2 & 0 & 1 - \lambda_1^3 \\
\end{bmatrix},
\]
\[
B(\lambda_1, \lambda_2, \lambda_3) = \begin{bmatrix}
7 & 0 & \lambda_2^3 \\
0 & 0 & 16 \\
\lambda_2^2 & 0 & 16 \\
\end{bmatrix}.
\]

Those of the functions \( F_j \), \( j \in \mathbb{N} \cup \{0\} \), defined by (5.2), which vanish at \((0, 0, 0) \in \mathbb{R}^3\) are \( F_0, F_4 \) and \( F_7 \), hence \( X^+(0) = \{4, 7\} \). One has
\[
F_0(\lambda_1, \lambda_2, \lambda_3) = 0,
\]
\[
F_3(\lambda_1, \lambda_2, \lambda_3) = -1792\lambda_1^{17} - 512\lambda_2^{17} + 8448\lambda_3^{13} - 16\lambda_3^{13} + 1792\lambda_3^8,
\]
\[
+16\lambda_2^{13} - 112\lambda_2^{13} - 16\lambda_2^4 + 112\lambda_2^6,
\]
\[
F_7(\lambda_1, \lambda_2, \lambda_3) = -11319\lambda_1^4 - 5488\lambda_1^{17} - 493\lambda_2^{13} - 4802\lambda_3^{13} + 5488\lambda_3^8,
\]
\[
+49\lambda_3^{13} - 343\lambda_3^{13} - 49\lambda_3^4 + 343\lambda_2^6.
\]

Use will be made of theorems 6.1, 6.3 (see also remark 7.7). Theorems 6.2 and 6.4 cannot be applied, since \( F_0 = 0 \), which means that all the points \((x_0, \lambda) \in \mathbb{R}^3\), are degenerate.

Analogously as in example 7.3 it has been checked that conditions (E1(0, \( \lambda \))), (E2(0, \( \lambda \))) are satisfied for every \( \lambda \in U := (-0.31, 0.31)^3 \). Other symbolic computations and estimates show what follows. One has \( F_j(\lambda) \neq 0 \) for every \( j \in \mathbb{N} \setminus \{4, 7\}, \lambda \in U \). The origin is the only singular point of \( F_4, F_7 \) and \( F = F_4 \cdot F_7 \) in \( U \). In particular, the sets of zeros of \( F_4, F_7 \) restricted to \( D_3^3 \setminus \{0\} \), \( r = 0.3 \), are disjoint. Furthermore, the functions \( F_4, F_7 \) satisfy the assumptions of remark 7.1 for \( k = 3 \) and \( r = 0.3 \). Thus for \( r = 0.3 \) every connected component of \( F_4^{-1}(\{0\}) \cap D_3^3 \) and \( F_7^{-1}(\{0\}) \cap D_3^3 \) contains the origin. It has also been checked that \( F_4 \) and \( F_7 \) do have zeros in \( D_3^3 \setminus \{0\}, r = 0.3 \).

By theorems 6.1 and 6.3 the following equalities hold for every \( r \leq 0.3 \):
\[
\text{Bif}(0) \cap D_r^3 = \text{GiBif}(0) \cap D_r^3 = F^{-1}(\{0\}) \cap D_r^3
\]
\[
= (\text{GiBif}^\text{min}(0) \cap D_r^3) \cap D_r^3,
\]
\[
\text{GiBif}^\text{min} \cap D_r^3 = F^{-1}(\{0\}) \cap D_r^3,
\]
\[
\text{GiBif}^\text{min} \cap D_r^3 = F^{-1}(\{0\}) \cap D_r^3.
\]

(7.11)

The fact that \( 0 \in \text{GiBif}^\text{min} \cap D_r^3 \) and \( 0 \in \text{GiBif}^\text{min} \cap D_r^3 \) follows from theorem 3.4 and remark 5.7. (The only minimal periods of nontrival solutions in a neighbourhood of the origin are \( 2\pi \) and \( 2\pi \).

The results of numerical application of theorems 6.1 and 6.3, consisting in finding global bifurcation points in \([0] \times D_r^3, r = 0.3 \), as zeros of the functions \( F_j \), according to formulae (7.11), have been obtained by using the program \textit{surf} and presented in figure 3. The earlier conclusions ensure that the number of the cones in figure 3 does not change when passing to a smaller scale.
Example 7.6. Let \( H: \mathbb{R}^6 \times \mathbb{R}^3 \to \mathbb{R} \) be the Hamiltonian defined by

\[
H(x, \lambda) \equiv H(x_1, \ldots, x_6, \lambda_1, \lambda_2, \lambda_3) = P(x_1, \ldots, x_6, \lambda_1, \lambda_2, \lambda_3) + Q(x_1, \ldots, x_6),
\]

where

\[
P(x_1, \ldots, x_6, \lambda_1, \lambda_2, \lambda_3) = \frac{1}{2}(16 - 85\lambda_1^0 + 11\lambda_1^2 \lambda_2^2 - 6\lambda_1^3 \lambda_3^2 - \lambda_2^4 \lambda_3^2 + 6\lambda_1^5 \lambda_3^4 + 17\lambda_2^6 + \lambda_3^6) x_1^2
\]

\[
+ \frac{5}{2} x_2^2 + x_3^2 + \frac{1}{2} x_4^2 + \frac{5}{2} (5 + \lambda_1^1 + 8\lambda_2^5) x_5^2
\]

\[
+ \frac{1}{5} (2\lambda_1^2 + 4\lambda_2^3 \lambda_3^2 + \lambda_2^4) x_6^2 + (\lambda_2^3 - \lambda_3^2) x_5 x_6,
\]

and \( Q \) is defined by formula (7.5) from example 7.4.

H satisfies conditions (H1)–(H3) for \( k = 3 \) and \( x_0 = 0 \in \mathbb{R}^6 \).

Note that in this case \( x_0 = 0 \in \mathbb{R}^6 \) is an isolated critical point of \( H(\cdot, 0) \), it is degenerate, and \( i(\nabla, H(\cdot), 0) = 0 \).

The set of bifurcation points in \([0] \times D^3\) will be investigated for \( r = 0.3 \).

Setting

\[
h(\lambda_1, \lambda_2, \lambda_3) := 16 - 85\lambda_1^0 + 11\lambda_1^2 \lambda_3^2 - 6\lambda_1^3 \lambda_2^2 - \lambda_2^4 \lambda_3^2 + 6\lambda_1^5 \lambda_3^4 + 17\lambda_2^6 + \lambda_3^6
\]

one has

\[
A(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix}
    h(\lambda_1, \lambda_2, \lambda_3) & 0 & 0 \\
    0 & 5 & 0 \\
    0 & 0 & 2
\end{pmatrix},
\]

\[
B(\lambda_1, \lambda_2, \lambda_3) = \begin{pmatrix}
    1 & 0 & 0 \\
    0 & 5 + \lambda_1^1 + 8\lambda_2^5 & \lambda_2^3 - \lambda_3^2 \\
    0 & \lambda_2^3 - \lambda_3^2 & 2\lambda_1^2 + 4\lambda_2^3 \lambda_3^2 + \lambda_3^4
\end{pmatrix}.
\]

Use will be made of theorems 6.2 and 6.4 (see also remark 7.7). Theorems 6.1 and 6.3 are not suitable in this case. (The origin is a bifurcation point of nontrivial stationary solutions.)

Those of the functions \( F_j, j \in \mathbb{N} \cup \{0\} \), defined by (5.2), which vanish at \( (0, 0, 0) \in \mathbb{R}^3 \) are \( F_0, F_4 \) and \( F_5 \), hence \( X(0) = \{0, 4, 5\} \). One has

\[
F_0 = f_0 \cdot a_0, \quad F_4 = f_4 \cdot a_4, \quad F_5 = f_5 \cdot a_5.
\]
where
\[ f_0(\lambda_1, \lambda_2, \lambda_3) = 20\lambda_1^{15} + 40\lambda_1^{11}\lambda_2^2 + 10\lambda_1^{13}\lambda_2^2 + 100\lambda_1^2 + 200\lambda_1^4\lambda_2^2 + 50\lambda_2^2 \]
\[ + 160\lambda_2^2\lambda_1^2 + 320\lambda_2^{10}\lambda_3^4 + 80\lambda_2^{12} - 10\lambda_2^2 + 20\lambda_2^3\lambda_3^2 - 10\lambda_3^4, \]
\[ f_4(\lambda_1, \lambda_2, \lambda_3) = -85\lambda_1^5 + 11\lambda_1^7\lambda_2^2 - 6\lambda_1^3\lambda_2^2 - \lambda_2^2\lambda_3^2 + 6\lambda_2^6\lambda_4^2 + 17\lambda_2^8 + \lambda_5^8, \]
\[ f_5(\lambda_1, \lambda_2, \lambda_3) = 20\lambda_1^{15} + 40\lambda_1^{11}\lambda_2^2 + 10\lambda_1^{13}\lambda_2^2 - 125\lambda_1^2 + 160\lambda_1^4\lambda_2^2 \]
\[ + 320\lambda_2^{10}\lambda_3^4 + 80\lambda_2^{12} - 10\lambda_2^2 + 20\lambda_2^3\lambda_3^2 - 10\lambda_3^4. \]
\[ a_0(\lambda_1, \lambda_2, \lambda_3) = f_4(\lambda_1, \lambda_2, \lambda_3) + 16, \]
\[ a_4(\lambda_1, \lambda_2, \lambda_3) = f_0(\lambda_1, \lambda_2, \lambda_3) - 64\lambda_1^3 + 80\lambda_1^3 - 128\lambda_1^3\lambda_2^2 \]
\[ - 32\lambda_2^2 - 640\lambda_3^3 - 144, \]
\[ a_5(\lambda_1, \lambda_2, \lambda_3) = f_4(\lambda_1, \lambda_2, \lambda_3) - 9. \]

The functions \( a_0, a_4, a_5 \) have no zeros in \( U := (-0.31, 0.31)^3 \). Thus \( F_0, F_4, F_5 \) can be replaced by \( f_0, f_4, f_5 \) in computations.

Symbolic computations and estimates show what follows. One has \( F_j(\lambda) \neq 0 \) for every \( j \in \mathbb{N}\setminus\{0, 4, 5\}, \lambda \in U := (-0.31, 0.31)^3 \). The origin is the only singular point of \( F_0, F_4, F_5 \) and \( F = F_0 \cdot F_4 \cdot F_5 \) in \( U \). In particular, the sets of zeros of \( F_0, F_4, F_5 \) restricted to \( D^3_0 \setminus \{0\}, r = 0.3 \), are pairwise disjoint. Moreover, the functions \( F_0, F_4, F_5 \) and \( F \) satisfy the assumptions of remark 7.1 for \( k = 3 \) and \( r = 0.3 \). Thus for \( r = 0.3 \) every connected component of \( F_0^{-1}(\{0\}) \cap D^3_0, F_4^{-1}(\{0\}) \cap D^3_4, F_5^{-1}(\{0\}) \cap D^3_5 \) and \( F^{-1}(\{0\}) \cap D^3_f \) contains the origin. It has also been checked that \( F_0, F_4, F_5 \) do have zeros in \( D^3_f \setminus \{0\}, r = 0.3 \).

By theorems 6.2 and 6.4 the following equalities hold for every \( r \leq 0.3 \):
\[
\text{Bif}(0) \cap D^3_f = \text{GIBif}(0) \cap D^3_f = F^{-1}(\{0\}) \cap D^3_f = (\text{GIBif}_0(0) \cup \text{GIBif}_4(0) \cup \text{GIBif}_5(0)) \cap D^3_f;
\]
\[
\text{GIBif}^0(0) \cap D^3_f = (\text{GIBif}_0(0) \cup \text{GIBif}_4(0) \cup \text{GIBif}_5(0)) \cap D^3_f;
\]
\[
\text{GIBif}_0(0) \cap D^3_f = (F_0^{-1}(\{0\}) \cup F_4^{-1}(\{0\})) \cap D^3_f;
\]
\[
\text{GIBif}_4(0) \cap D^3_f = (F_0^{-1}(\{0\}) \cup F_4^{-1}(\{0\})) \cap D^3_f;
\]
\[
\text{GIBif}_5(0) \cap D^3_f = (F_0^{-1}(\{0\}) \cup F_5^{-1}(\{0\})) \cap D^3_f;
\]
\[
\text{GIBif}_4^0(0) \cap D^3_f \setminus \{0\} = F_4^{-1}(\{0\}) \cap D^3_f \setminus \{0\},
\]
\[
\text{GIBif}_5^0(0) \cap D^3_f \setminus \{0\} = F_5^{-1}(\{0\}) \cap D^3_f \setminus \{0\}. \]

The results of numerical application of theorems 6.2 and 6.4, consisting in finding global bifurcation points in \( [0] \times D^3_f, r = 0.3 \), as zeros of the functions \( F_j \), according to formulae (7.12)–(7.14), have been obtained by using the program surf and presented in figure 4. The earlier conclusions ensure that the number of the cones in figure 4 does not change when passing to a smaller scale.

**Remark 7.7.** Corollaries 6.11, 6.12, and results from [23, 38] can be used in examples 7.5 and 7.6 to verify the number of the cones forming the set of bifurcation points and to confirm that the origin is a symmetry breaking point.

Examples analogous to examples 7.3 and 7.4 can be constructed for any number of degrees of freedom, whereas examples similar to examples 7.5, 7.6 can be given for any number of parameters and any number of degrees of freedom.
Figure 4. The set of those $\lambda_1, \lambda_2, \lambda_3 \in D^1_3$, $r = 0.3$, for which $(0, (\lambda_1, \lambda_2, \lambda_3)) \in \mathbb{R}^6 \times \mathbb{R}^1$ is a global bifurcation point of the system from example 7.6. The legend on the right describes the minimal periods of solutions bifurcating from the points of given surface.

Appendix A. Description of semianalytic sets

In this appendix the relevant results from [37], in the case they have been used in sections 6 and 7, are summarized for the convenience of the reader.

In what follows use is made of definition 6.6 of admissible function and definition 6.7 of test function.

As well known, if $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ is an admissible mapping then for sufficiently small $r > 0$ the set $F^{-1}(\{0\}) \cap D^2_3 \{0\}$ is either empty or it is a union of finitely many disjoint analytic curves, each of which meets the origin and crosses $S^1_r$ transversally in one point.

If $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a test function for an admissible mapping $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ then for sufficiently small $r > 0$ the function $g$ has a constant sign on every connected component of the set $F^{-1}(\{0\}) \cap D^2_3 \{0\}$ (i.e. on each of the analytic curves forming this set).

The following notation is used:

$$b(F) = \text{the number of components of the set } F^{-1}(\{0\}) \cap D^2_3 \{0\},$$

$$b_+(g, F) = \text{the number of components of } F^{-1}(\{0\}) \cap D^2_3 \{0\} \text{ on which } g \text{ is positive},$$

$$b_-(g, F) = \text{the number of components of } F^{-1}(\{0\}) \cap D^2_3 \{0\} \text{ on which } g \text{ is negative}.$$

Clearly, $b_+(g, F) + b_-(g, F) = b(F)$.

Let $\text{Jac}(g, F): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the Jacobian of the mapping $(g, F): \mathbb{R}^2 \rightarrow \mathbb{R}^2$, and let the mapping $h(g, F): \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by

$$h(g, F) = (\text{Jac}(g, F), F).$$

In the following theorem $i(h(g, F), 0)$ denotes the topological index of $0 \in \mathbb{R}^2$ with respect to $h(g, F)$ (see section 2.2).

**Theorem A.1 ([37]).** If $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a test function for an admissible mapping $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ then $0 \in \mathbb{R}^2$ is isolated in $h(g, F)^{-1}(\{0\})$ and

$$b_+(g, F) - b_-(g, F) = 2 \cdot i(h(g, F), 0). \quad (A.1)$$

**Corollary A.2 ([37]).** If $g+: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a nonnegative test function for an admissible mapping $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ then $0 \in \mathbb{R}^2$ is isolated in $h(g+, F)^{-1}(\{0\})$ and

$$b(F) = b_+(g+, F) = 2 \cdot i(h(g+, F), 0). \quad (A.2)$$

Let $b_i(F)$, $g_i$, $i = 1, \ldots, 4$, be such as in remark 7.2.
Corollary A.3 ([20]). If an admissible mapping $F: \mathbb{R}^2 \to \mathbb{R}$ has no zeros on the coordinate axes in a neighbourhood of the origin then

$$b_1(F) + b_2(F) + b_3(F) + b_4(F) = 2 \cdot i(h(g_1, F), 0),$$
$$b_1(F) - b_2(F) - b_3(F) + b_4(F) = 2 \cdot i(h(g_2, F), 0),$$
$$b_1(F) + b_2(F) - b_3(F) - b_4(F) = 2 \cdot i(h(g_3, F), 0),$$
$$b_1(F) - b_2(F) + b_3(F) - b_4(F) = 2 \cdot i(h(g_4, F), 0).$$

B. Properties of the degree DEG

To use the degree DEG from section 2.2 one can leave aside its construction and consider only its properties described below.

Let the space $V$, the set $\Omega \subset V$, and the mapping $\nabla f: V \to V$ be such as in section 2.2. It is assumed that $\text{DEG}(\nabla f, \emptyset) = \emptyset$. The following conditions are satisfied.

1. (Existence of solutions) If $\text{DEG}_j(\nabla f, \Omega) \neq 0$ for some $j \in \mathbb{N} \cup \{0\}$ then there exists $x \in \Omega^{\mathbb{K}}$ such that $\nabla f(x) = 0$.

2. (Additivity) If $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$ then

$$\text{DEG}(\nabla f, \Omega) = \text{DEG}(\nabla f, \Omega_1) + \text{DEG}(\nabla f, \Omega_2).$$

3. (Excision) If $\Omega_0$ is an open $SO(2)$-invariant subset of $\Omega$ and $\nabla f$ has no zeros on $\text{cl}(\Omega) \setminus \Omega_0$ then $\text{DEG}(\nabla f, \Omega) = \text{DEG}(\nabla f, \Omega_0)$.

4. (Homotopy invariance) If $\nabla_x F: V \times [0, 1] \to V$ is a continuous $SO(2)$-equivariant gradient (with respect to $V$) mapping such that $\nabla_x F(x, s) \neq 0$ for every $x \in \partial \Omega$, $s \in [0, 1]$ then $\text{DEG}(\nabla_x F(\cdot, 0), \Omega) = \text{DEG}(\nabla_x F(\cdot, 1), \Omega)$.

5. (Product formula) Consider another orthogonal representation of the group $SO(2)$ on a real inner product space $W$ with $\dim W < \infty$. Assume that $\Sigma$ is an $SO(2)$-invariant bounded open subset of $W$ and let $\nabla g: W \to W$ be a continuous $SO(2)$-equivariant gradient mapping such that $\nabla g(x) \neq 0$ for every $x \in \partial \Sigma$. Then

$$\text{DEG}(\nabla f, \nabla g), \Omega \times \Sigma) = \text{DEG}(\nabla f, \Omega) \ast \text{DEG}(\nabla g, \Sigma).$$

6. (Degree of identity) One has $\text{DEG}(Id_V, \Omega) = (1, 0, 0, \ldots)$

7. (Degree of isomorphism) Let $V = \mathbb{R}[m, 0] \oplus \mathbb{R}[m_1, j_1] \oplus \cdots \oplus \mathbb{R}[m_r, j_r]$, where $m, m_i, j_i \in \mathbb{N}$, $0 < j_1 < \cdots < j_r$. Let $L = \text{diag}(L_0, L_1, \ldots, L_r): V \to V$ be a self-adjoint $SO(2)$-equivariant linear isomorphism. Then

$$\text{DEG}_j(L, \Omega) = \begin{cases} \text{sgn}(\text{det} L_0) & \text{if } j = 0, \\ \text{sgn}(\text{det} L_0) \cdot \frac{m_i(L_i)}{2} & \text{if } j = j_i \text{ for some } i \in \{1, \ldots, r\} \\ 0 & \text{otherwise.} \end{cases}$$

8. (Index of nondegenerate point) If $f$ is of class $C^2$, $\nabla f(0) = 0$, and $\nabla^2 f(0)$ is an isomorphism then $I(\nabla f, 0) = I(\nabla^2 f(0), 0)$. (I is the index defined in section 2.2.)

The formula for the index $I$ of possibly degenerate point is given in lemma 2.2.
C. On necessary conditions for bifurcation

This appendix contains the proof of corollary 3.5 exploiting theorem 3.4 and the proof of implication (3.3).

**Proof of corollary 3.5.** If \( \lambda_0 \not\in \Lambda_0(\nabla^2 H(x_0, \cdot)) \) then there are no nontrivial stationary solutions in a neighbourhood of \((x_0, \lambda_0)\), according to theorem 3.4. To prove (2) assume that \( U, G(\lambda_0), \) and \( X(\lambda_0) \) are such as in theorem 3.4. The isotropy group of every nonstationary \( j \)-solution from the set \( U \) contains \( Z_j \), therefore it is equal to \( Z_{sj} \) for some \( s \in \mathbb{N} \), which depends on the solution. The group \( Z_{sj} \) belongs to \( G(\lambda_0) \setminus \{ SO(2) \} \), according to theorem 3.4. Thus \( Z_{sj} \subset Z_r \) for some \( r \in X(\lambda_0) \setminus \{ 0 \} \), which implies that \( r = msj \) for some \( m \in \mathbb{N} \). Setting \( l = ms \) one has \( \det Q_{lj}(\nabla^2 H(x_0, \lambda_0)) = \det Q_r(\nabla^2 H(x_0, \lambda_0)) = 0 \), hence \( \lambda_0 \in \Lambda_1(\nabla^2 H(x_0, \cdot)) \).

**Proof of implication (3.3).** Let \( K \in \mathbb{S}(2n, \mathbb{R}) \) (e.g. \( K = \nabla^2 H(x_0, \lambda_0) \)) and let

\[
Y = \begin{bmatrix} 0 & -r^{-1}I_{2n} \\ r J_r & -K J_n \end{bmatrix}.
\]

Since \( \det Y = 1 \), one has

\[
\det Q_r(K) = \det Y Q_r(K) = \frac{1}{(1 + r^2)^{3n}} \det \begin{bmatrix} -J & r^{-1}K \\ 0 & 0 & J J + K K \end{bmatrix} = \frac{1}{(1 + r^2)^{3n}} \det ([J K]^2 + r^2 I).
\]

The implication \( \pm ir \in \sigma(J K) \Rightarrow -r^2 \in \sigma((J K)^2) \) completes the proof.

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