Dense set of large periodic points

S.Kanmani
Material Science Division, IGCAR,
Kalpakkam, India.

V.Kannan
Department of Mathematics and Statistics,
University of Hyderabad, Hyderabad.
India.

Abstract- We say that a dynamical system \((X, f)\) has a dense set of large periodic points (abbreviated as \(DLP\)) if for every positive integer \(n\), the set of all periodic points of period \(\geq n\), is dense in \(X\). Here we provide two different proofs of the fact that every system chaotic in the sense of Devaney, has \(DLP\).

Keywords - Devaney’s chaos, periodic orbits, Topological transitivity

0.1 Introduction

A dynamical system is a pair \((X, f)\), where \(X\) is a Hausdorff space and \(f\) is a continuous self-map of \(X\). The set of all positive integers is denoted by \(N\). An element \(x \in X\) is said to be periodic if \(f^n(x) = x\) for some \(n \in N\); the smallest such \(n\) is called the period of \(x\). Here \(f^n\) denotes the \(n\)-fold composition of \(f\). The set of all periodic points is denoted \(P\). For each \(n \in N\), \(P_n\) denotes the set of all periodic points of period \(\leq n\).

Definition We say that the system has \(DLP\) if \(P \setminus P_n\) is dense in \(X\) for every \(n\) in \(N\). According to Devaney [1] \((X, f)\) is said to be chaotic if it has all the three properties below.

1. The set of all periodic points \((P)\) is dense in \(X\).
2. \(f\) is topologically transitive and
3. \(f\) has sensitive dependence on initial conditions.

But it is known [2] that 3 follows from 1 and 2. We say that \(f\) is topologically transitive if for any two non-empty open sets \(V\) and \(W\) in \(X\), there exist \(n\) in \(N\) such that \(f^n(V) \cap W\) is not empty. Roughly speaking, the requirement is that from any region \(V\) to any region \(W\), some point will go at some time.
The main result in this note is that every Devaney chaotic system possesses DLP. In the field of study like Monte carlo methods one uses a discrete dynamical system as a pseudo-random number generator. There one of the desirable features of the dynamical system is to have orbits whose period is as high as possible. Our result gives a criteria which ensures the abundance of large periodic points.

It is known [3, 4] that any continuous function on an interval of the real line, if topologically transitive, has infinitely many periodic orbits, and that the periods of these orbits forms an unbounded subset of the positive integers. Our main result allows to draw a stronger conclusion from the same hypotheses. This is possible because it is known on intervals topological transitivity implies that the set of all periodic points is dense [5].

Our arguments prove that the property of dense set of periodic points and dense set of large periodic points are equivalent among the following two classes of dynamical systems:

1. Topologically transitive dynamical systems that are infinite.
2. Dynamical systems in which the set of all periodic points has an empty interior.

However in general, dense $P$ and $DPL$ are not equivalent. Because, a dense $P$ does not necessarily imply $DLP$. Some simple examples are:

1. Cyclic permutations on finite sets. Here $P = \text{Whole set } X$ and $P \setminus P_n$ is empty if $n$ exceeds the cardinality of $X$.
2. Identity map on any $X$. Here $P = X$ and $P \setminus P_n$ is empty for $n \geq 2$
3. Reflection maps on the real line $R$. Here $P = R$ and $P \setminus P_n$ is empty if $n \geq 3$. (e.g. $f(x) = 1 - x$)

We shall later note that every continuous map on $R$ that has dense set of periodic points but without dense large periodic points, must have a subsystem that is topologically conjugate to a reflection or identity.
The main theorem is:

**Theorem-1** Every transitive map with dense set of periodic points has $DLP$, if the underlying space is infinite.

**Proof:**

Our first proof draws this as an easy corollary of the main result of Touhey [6]. We say that a collection $V_1, V_2, \ldots, V_n$ of nonempty open sets, shares a periodic orbit if there is a periodic point $x$ whose orbit meets each $V_j, 1 \leq j \leq n$. Here, the orbit of $x$ is the set of all elements of the form $f^n(x)$ where $n$ is in $N$. It follows from [6] that every transitive map with dense set of periodic points has the property that every finite collection of nonempty open sets shares a periodic orbit.

Since $X$ is infinite and $f$ is transitive, it is easily seen that $X$ has no isolated points. Next, if $V$ is any open set in $X$, then $V$ is infinite (as there are no isolated points). Let $n \in N$. Since $X$ is a Hausdorff space $n+1$ points in $V$ can be mutually separated by disjoint open sets. That is, we can find a family $V_1, V_2, \ldots, V_{n+1}$ of pairwise disjoint non-empty open sets inside $V$.

By applying the main result of [6], we get a periodic point $x$ whose orbit meets each $V_i$. Because these are pairwise disjoint, it is immediate that this orbit has at least $n + 1$ elements. If $y$ is an element in this orbit that is also in $V_1$, then $y$ is a periodic point in $V$ whose period is strictly greater than $n$. This proves that $P \setminus P_n$ is dense in $X$.

The second proof uses the following facts:

1. Each $P_n$ is closed.
2. Each $P_n$ is invariant.
3. In a topologically transitive system, every invariant set is either dense or nowhere dense.
4. When a nowhere dense set is removed from a dense set, what remains is again a dense set.

We consider two cases:

Case-1: $P_n$ is not equal to the whole space $X$ for any $n$. Then by (1) and (3) each $P_n$ is nowhere dense. By (4) $P \setminus P_n$ is dense. Thus the system has the $DPL$ property.
Case-2: $P_n = X$ for some $n$. Since the dynamical system is transitive, first we observe that if $V$ and $W$ are two nonempty open sets, then $f^m(V) \cap W$ is non-empty for some $m < n$. It follows from the proposition 1 below that $X$ has at most $n$ elements. Since $X$ is finite, transitivity demands that the dynamical system is a cyclic permutation i.e. $f^n$ is the identity map.

0.3 Speciality of $n$-cycles:

Topological transitivity requires that from every nonempty open set $V$ it is possible to reach every nonempty open set $W$ at some time or other. Can we put an upper bound to the time of reaching? That is, can we demand that there is a positive integer $k$ such that $f^n(V) \cap W$ is nonempty for some $n < k$, whatever $V$ and $W$ be? That this is not possible, except in the trivial cases, is the contention of the next proposition.

Proposition-1 Let $(X, f)$ be a dynamical system and let $k$ be a positive integer. Suppose that whenever $V$ and $W$ are non-empty open sets in $X$, there is a positive integer $n < k$ such that $f^n(V) \cap W$ is non-empty. Then $X$ has at most $k$ elements.

Proof: Let if possible, $x_0, x_1, \ldots x_k$ be $k+1$ distinct elements of $X$. Choose pairwise disjoint neighbourhoods $V_0, V_1, \ldots V_k$ of these points. This is possible if $X$ is a metric space, or more generally $X$ is a Hausdorff topological space. Then choose a neighbourhood $W_0$ of $x_0$ and a positive integer $n_0 < k$ such that $W_0 \subset V_0$ and $f^{n_0}(W_0) \subset V_1$. This is possible as $f^{n_0}$ is a continuous function. Next choose a neighbourhood $W_1$ of $x_0$ and a positive integer $n_1 < k$ such that $W_1 \subset W_0$ and $f^{n_1}(W_1) \subset V_2$. Continuing in this way we have $k$ such neighbourhoods of $x_0$, namely $W_0 \supset W_1 \supset W_2 \supset W_3 \ldots \supset W_{k-1}$. Let $W$ be their intersection, i.e., the smallest of them. Then $W$ is a neighbourhood of $x_0$ such that $f^{n_i}(W) \subset V_{i+1}$ for all $i = 0, 1, 2, \ldots, k$. But the $V_i$’s are disjoint. Therefore $n_i$’s have to be distinct. A set of $k$ positive integers each of which is strictly less than $k$. This is a contradiction.

Remark: Under the assumption of the above proposition, $f$ has to be a cyclic permutation on the finite set. We can restate the above proposition as follows: A dynamical system $(X, f)$ is an $n$-cycle if and only if

$$\sup_{V,W} \inf_{n} \{ n \in N : f^n(V) \cap W \text{ is nonempty} \} < \infty$$

where the supremum is taken over all pairs $V, W$ nonempty open sets in $X$. Here the convention is that the infimum of the empty subset of $N$ is $\infty$. 

4
Corollary: The finite cycles are the only transitive maps satisfying dense periodic points but not dense large periodic points.

0.4 Additional remarks

The second proof has a distinct advantage over the first. Its method proves some other related results stated below:

Result-1: Let \((X, f)\) be an infinite topologically transitive dynamical system. Let \(\{F_n\}\) be a sequence of closed invariant sets such that 
\[ \bigcup_{n=k}^{\infty} F_n \] is also dense, for every positive integer \(K\).

Theorem-1 becomes a particular case of Result-1.

Result-2: Let \((X, f)\) be an infinite topologically transitive dynamical system such that the set of eventually periodic points is dense. Then for every positive integer \(n\), the set of eventually periodic points having an orbit of atleast \(n\) elements, is also dense.

Result-2 is a particular case of result-1

Result-2': Let \((X, f)\) be an infinite topologically transitive dynamical system. Let \(n\) be a positive integer. If each nonempty open set contains an element with finite orbit, then each nonempty open set contains an element with finite orbit of size \(\geq n\)

Result-3: Let \((X, f)\) be topologically transitive and let \(P\) be dense. Then either \(f^n\) equals identity for some \(n\), or \(P \setminus P_n\) is dense for all \(n \in N\)

Result-4: Let \((X, f)\) be any dynamical system. Let the set \(P\) of periodic points be a dense set with empty interior. Then \(P \setminus P_n\) is also dense for every \(n\) in \(N\). Note that we are not assuming \(F\) to be topologically transitive.

When \(X = R\) or an interval in \(R\), we can say more:

Result-5: For every transitive map on \(R\) or an interval in \(R\), the set \(P\) of periodic points is a dense set with empty interior. Therefore, by Result-4, \(P \setminus P_n\) is dense for every \(n\).

Result-6: For continuous \(f : I \to I\) where \(P\) is dense, the only two possibilities are

1. \(f\) is identity or topologically conjugate to a reflection map on some nontrivial subinterval \(J \subset I\)
   or

2. \(P \setminus P_n\) is dense for every \(n\) in \(N\).

This can be proved by applying a theorem of [7] that states that
$f$ is built from transitive maps on subinterval. Note that when $f$ is transitive (1) cannot hold. Thus, for all transitive maps on intervals, (2) should hold; of course, this can be proved from result-5 also.

References

1. Devaney, R.L. An introduction to chaotic dynamical systems, Addison Wesley, (1989)
2. Banks, J. Brooks, J. Cairns, G. Davis, G. and Stacey, P. On Devaney’s definition of chaos, Amer. Math. Monthly, 99, 332-334 (1992).
3. Block, L.S. and Coppel, W.A. Dynamics in one dimension, Lecture Notes in Mathematics, 1513, Springer, (1992)
4. A. Crannell and M.Martelli, Periodic orbits from Nonperiodic orbits on an interval, Appl. Math. Lett. Vol.10, No. 6, 45-47 (1997)
5. Silverman, S. On maps with dense orbits and the definition of chaos, Rocky Mountain Jour. Math. 22,353-375 (1992)
6. Touhey, P. Yet another definition of chaos, Amer. Math. Monthly, 104, 411-414 (1997)
7. Barge, M and Martin, J. Dense periodicity on the interval, Proc. Amer. math. Soc., 94, 731-735 (1985)