Stabilization of frictional sliding by normal load modulation:

A bifurcation analysis

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Abstract

This paper presents the stability analysis of a system sliding at low velocities (< 100\mu m.s\(^{-1}\)) under a periodically modulated normal load, preserving interfacial contact. Experiments clearly evidence that normal vibrations generally stabilize the system against stick-slip oscillations, at least for a modulation frequency much larger than the stick-slip one. The mechanical model of Bureau et al. (2000), validated on the steady-state response of the system, is used to map its stability diagram. The model takes explicitly into account the finite shear stiffness of the load-bearing asperities, in addition to a classical state- and rate-dependent friction force. The numerical results are in excellent quantitative agreement with the experimental data obtained from a multicontact frictional system between glassy polymer materials. Simulations at larger amplitude of modulation (typically 20% of the mean normal load) suggest that the non-linear coupling between normal and sliding motion could have a destabilizing effect in restricted regions of the parameter space.

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I. INTRODUCTION

It is well known that non-linearities in the constitutive laws of dry friction may lead to the instability of steady frictional sliding against stick-slip oscillations, even for a single degree-of-freedom system driven at nominal constant velocity through a compliant stage. Sliding instability is an important issue in mechanical engineering since it is an ultimate limitation to the positioning accuracy for precision structures and machine tools. Moreover, stick-slip oscillations may be strongly non-linear and make servo-control extremely difficult. When designing a sliding mechanism, it is therefore of primary importance to characterize accurately the variations of the friction coefficient with, e.g., sliding velocity, keeping in mind that even slight variations may have a destabilizing effect. This requires to go beyond Amontons-Coulomb’s law which assumes a constant friction coefficient. It might be legitimately feared that a more detailed constitutive law would have a restricted scope, e.g., in terms of materials and range of sliding velocities. It is thus remarkable that in the limit of low velocities (typically lower that 100\(\mu\text{m.s}^{-1}\)), and light enough loads so that the interface is made of a sparse set of micro-contacts between load-bearing asperities, a relatively material-independent frictional behaviour is found which can be accounted for by a simple set of non-linear constitutive equations. Such studies have been initiated in the field of rock mechanics by Dieterich [1] and Rice and Ruina [2], motivated by the need for low velocity friction models to investigate fault dynamics and earthquake nucleation. They have put on a firm phenomenological basis the idea, already suggested by the work of Rabinowicz [3], that friction does not depend only on the instantaneous sliding velocity \(v\) but also on the whole sliding history. An experimental signature is the hysteretic frictional response of the interface when the slider is driven at a non-steady rate. Rice and Ruina [2] proposed a family of dynamical equations coupling the sliding velocity to a set of state variables. Subsequent experimental investigations have shown that a single state variable \(\phi\) is sufficient for most purposes. These experimental studies were performed on a wide range of materials, such as granite [1], paper [4], polymer glasses [5] and elastomers [6]. The friction force in this model is \(F = W \mu(v, \phi)\) with \(W\) the normal load and \(\mu\) the friction coefficient. Moreover, it has been possible [4, 8] to give a physical interpretation of \(\phi\) as the average “age” of the microcontacts which grow while the material creeps under normal load, until sliding interrupts the process by renewing the load-bearing contact population. The dynamical model is
closed by specifying a differential equation coupling $v$ to $\phi$ so as to account for the renewal of the microcontact population after a slip length $D_0$ of micrometric order. This length is of order the mean radius of the microcontacts between surfaces of micrometric roughness\[^9\]. The resulting state- and rate-dependent friction laws will be hereafter referred to as SRF. Among several SRF expressions proposed originally, the one that we use in this paper are:

$$\mu(v, \phi) = \mu_0 + A \ln \left( \frac{v}{V_0} \right) + B \ln \left( 1 + \frac{V_{sat} \phi}{D_0} \right)$$

for the friction coefficient and

$$\frac{d\phi}{dt} = 1 - \frac{v \phi}{D_0}$$

for the evolution of the state variable, where $\mu_0$, $A$, $B$, $V_{sat}$ and $V_0$ are constants.

This SFR model has been extensively validated by testing against numerous experimental situations involving transient dynamical responses of the system. The most stringent test relies upon the non-linear characteristics of the bifurcation from steady-sliding to stick-slip oscillations\[^10\]. The model can be understood as resulting from two distinct physical mechanisms, the effect of which can be summarized in the following decomposition of the friction force, proposed by Bowden and Tabor\[^11\], in terms of the real area of contact $\Sigma_r$, and an interfacial shear strength $\sigma_s$:

$$F(v, \phi) = \sigma_s(v)\Sigma_r(\phi) .$$

Here, the real area of contact depends on the interfacial age because it grows due to the creep of the load-bearing asperities\[^7\]. The velocity dependent interfacial shear strength has been ascribed to the adhesive, nanometer thick junctions between microasperities. A simple microscopic model has been proposed for the elasto-visco-plastic rheology of the junctions, compatible with the existence of a finite friction threshold\[^5\].

Recently, attention has been paid to the effect of a time-dependent normal load on the response of a single degree-of-freedom sliding system. This situation is of practical interest when the mechanical design allows cross-talking between the normal (loading) and the tangential (driving) forces\[^12\], or when external vibrations contribute to the loading of the interface, as it may be the case for seismic faults\[^13\, 14\, 15\]. The response of the system
is not intuitive. First, since the friction force is directly proportional to the normal load, the sliding velocity is dynamically coupled to the normal load modulation, hence feeds back the friction force. Moreover, it has been shown that more subtle interplays must be taken into account. Linker and Dieterich \[13\] have interpreted the transient response to a step in normal load by coupling directly the time-variations of $\phi$ and $W$, thus adding a term $-\text{const } \phi \, d \ln(W)/dt$ in Eq. (2). The physical motivation for this extension of the SRF ageing equation is the fact that, according to \[9\], a change in normal force creates fresh load-bearing contact area. This certainly influences directly the age $\phi$, though probably in a weaker measure than proposed by Linker and Dieterich \[13\], as briefly discussed in \[16\]. More recently, Bureau et al. \[16\] have studied the response of a sliding system to a periodically modulated normal load $W(t) = W_0[1 + \epsilon \cos(\omega t)]$ with $\epsilon < 1$. They found that the friction force, averaged on a modulation period, is significantly lowered with respect to the situation under constant load $W_0$. The oscillating part of the force, primarily reduced to its spectral component at $\omega$ in the limit of vanishing $\epsilon$, becomes quickly anharmonic as $\epsilon$ is increased while still remaining much smaller than 1. They have shown that the SRF equations can fit accurately all their results provided that the model is modified to account for the finite interfacial shear stiffness $\kappa$ resulting from the elastic deformation of the load-bearing asperities. This means that the sliding velocity differs from the velocity of the center of mass of the slider, a statement which is clearly illustrated in the static state, i.e., for tangential forces well below the static threshold, where the interface responds elastically without sliding \[17\], \[18\]. Under constant normal load and constant driving velocity $V$, this “hidden” interfacial degree of freedom manifests itself only for non-steady motion, and plays no significant role at the circular frequency $\Omega_c \sim V/D_0$ of the oscillations at the onset of the stick-slip instability.

However, under a modulated load at $\omega \gg \Omega_c$, one must take the finite interfacial compliance $\kappa$ into account, all the more so since the latter is known to be itself proportional to $W$ \[18\]. This results in a non-trivial and efficient coupling between the normal load and the sliding velocity.

Of particular interest is the effect of load modulation on the sliding stability of the system. Dupont and Bapna \[12\] have computed the critical stiffness of the drive below which stick-slip occurs for a slider-spring loaded at a constant angle with respect to the sliding plane. This configuration would provide a direct test for the coupling between $\phi$ and $W$ proposed
by Linker and Dieterich [13], but the experimental study has not been performed so far.

The present paper addresses the problem of the stability of a slider-spring system under an externally and harmonically modulated normal load. The experimental arrangement is described in section 2 and it is shown that for a circular frequency $\omega \gg \Omega_c$ the modulation generally stabilizes the system against stick-slip. This spectacular effect is accounted for by the SRF model with modulated interfacial stiffness, as shown by the numerical study of the bifurcation which is detailed in section 3.

II. EXPERIMENTS

A. Apparatus

The apparatus (Fig. 1) consists of a slider of mass $M$ driven along a track through a loading spring of stiffness $K = 0.21 \text{ N.}\mu\text{m}^{-1}$. The loading end of the spring is moved at constant velocity $V$ in the range 1–100 $\mu\text{m.s}^{-1}$ by means of a translation stage driven by a stepping motor. The spring elongation is measured by an inductive probe (Electro, sensor 4937, module PBA200), with a 0.1 $\mu\text{m}$ resolution over the 10 kHz bandwidth. The average normal load $W_0$ can be set in the range 3–23 N with the help of a vertical spring attached to a remote point itself translated horizontally at the pulling velocity $V$ through a second translation stage, in order to prevent any tangential coupling. The normal load modulation is achieved by means of a vibration exciter (LDS, model V100) rigidly attached to the slider: a harmonic voltage input of given amplitude and frequency $f$ results in a harmonic vertical motion of the moving element of the exciter on which an accelerometer (Brüel & Kjær, type 4371 V) is fixed. An acceleration of amplitude $\gamma$ of this moving element of mass $m$ induces a normal load modulation on the slider of amplitude $m\gamma$ at frequency $f$. We thus obtain a normal load $W(t) = W_0[1 + \epsilon \cos(\omega t)]$ with $\omega = 2\pi f$ and $\epsilon = m\gamma/W_0$ in the range 0.01–0.5. A fixed frequency $f = 120$ Hz has been used for the whole study. Two poly(methylmethacrylate) (PMMA) samples are glued, respectively, on the slider and the track. They have nominally flat surfaces which have been lapped together with 400-grit SiC powder and water to obtain a rms roughness $R_q = 1.3 \mu\text{m}$ [13]. The interface between the two blocks is made of a sparse set of load-bearing microcontacts [13]. An air layer of micrometric thickness is therefore trapped between the surfaces and acts as a viscoelastic
element, in parallel with the microcontacts, which partially bears the normal load. This effect has been studied in details in Bureau et al. [10] who concluded that the remaining effect of the load modulation on the asperities can be described by an effective amplitude \( \varepsilon_{\text{eff}} = \rho \varepsilon \), with \( \rho \) a constant close to 0.5, taken in the following as \( \rho = 0.43 \), a value which will be justified in section III C.

B. Localization of the stick-slip bifurcation, effect of the modulation

The bifurcation between steady sliding and stick-slip oscillations under constant load (\( \varepsilon = 0 \)) has been extensively described (see e.g. [4]). When \( K \) and \( V \) are kept constant, steady sliding occurs for values of the remaining control parameter \( W_0 < W_0^c(\varepsilon) \) where the \( K \)-dependency has been omitted here since the value of \( K \) is fixed in this study. The bifurcation is of the direct Hopf kind, which means that the amplitude of oscillation of the slider velocity is vanishing \(^1\) when approaching \( W_0^c \) from below (i.e. increasing \( W_0 \) up to \( W_0^c \)), while the circular frequency tends to a finite value \( \Omega_\varepsilon \). In addition, the characteristic time of the oscillating transients diverges when approaching \( W_0^c \) from above (i.e. decreasing \( W_0 \) towards \( W_0^c \)). A practical consequence is that as the bifurcation is tracked down, it becomes increasingly difficult to distinguish between steady stick-slip oscillations and transient relaxation towards steady sliding, resulting from the perturbating effect of friction force fluctuations along the track.

For \( \varepsilon = 0 \), the ratio \( K/W_0 \) is the relevant control parameter, at least in the low velocity region where inertia of the slider oscillating at the circular frequency \( \Omega_\varepsilon \) can be neglected [1]. Henceforth, although the external stiffness \( K \) is kept constant for the whole set of experiments reported in this paper, we will keep on representing the stability domain of the system in the parameter plane \((K/W_0, V)\) where it is bounded by the experimental bifurcation curve (Fig. 2). The experimental uncertainty on the critical value \( K/W_0^c \), determined from the standard deviation over at least 10 measurements at a given velocity \( V \), is indicated by the error bars in Fig. 2. It is typically \( \pm 3\% \), except at the larger velocity, since the results are more sensitive to long wavelength irregularities along the track for large sliding distances.

When a harmonic modulation at \( \omega \) is superimposed to a value of \( W_0 \) corresponding to

\(^1\) Note that the term “slick-slip” is therefore a misnomer since the sliding velocity does not reach zero, i.e., the slider does not “stick” during an oscillation period.
steady sliding at $V$, the velocity of the slider oscillates about $V$, possibly in a anharmonic way, with a fundamental component at $\omega$. The motion of the slider, when averaged over a period $2\pi/\omega$ is therefore steady. For given $V$ and $K$, one has now to consider two control parameters, namely $W_0$ and $\epsilon_{\text{eff}}$.

When the d.c. load $W_0$ is increased while keeping constant $\epsilon_{\text{eff}}$, the average slider motion is found to become of the stick-slip kind above a critical value, $W_0^c(V, \epsilon_{\text{eff}}) > W_0^c(V)$ (Fig. 2). A normal load modulation of even very small effective amplitude may therefore stabilize the system against stick-slip as illustrated directly in Fig. 3 where a modulation with $\epsilon_{\text{eff}} = 4.5 \times 10^{-2}$ is enough to suppress well developed, strongly anharmonic, large amplitude and low frequency stick-slip oscillations (the force signal then only shows the the remaining small amplitude modulation at the forcing higher frequency).

The effect of $\epsilon_{\text{eff}}$ on the critical value of $K/W_0(V)$ is characterized in Fig. 4. The higher the velocity, the stronger the stabilizing effect of the normal load modulation. The effect is spectacular when described in terms of the velocity domain corresponding to steady sliding at constant $K/W_0$. For instance, the critical velocity at $K/W_0 = 0.026 \mu m^{-1}$ is decreased by more than a factor of ten by applying a modulation with $\epsilon_{\text{eff}} = 0.09$.

The empirical study indicates that, as a rule of thumb, steady-sliding is promoted by high velocity, high amplitude of normal load modulation, low average normal load and large stiffness. This is tested in the following against a numerical study of the SRF model including normal load modulation.

### III. NUMERICAL STUDY

#### A. The state- and rate-dependent fiction (SRF) model equations

The SRF laws (Eqs. [1], [2]) are incorporated into the equation of motion of the slider, according to the simple model sketched in Fig. 5. The proportionality constant between the normal load $W$ and the interfacial stiffness $\kappa$ is a length $\lambda$ of micrometric order: $\kappa = W/\lambda$ [18]. The equation of motion of the slider thus reads:

$$Mx'' = K(Vt - x) - \kappa(x - x_{\text{pl}})$$  \hspace{1cm} (4)
where here and henceforth the prime denotes time derivative. Taking a massless interfacial zone (a reasonable assumption, see appendix), we also have, according to Eq. (1):

\[
\frac{W}{\lambda}(x - x_{pl}) = W\left[\mu_0 + A\ln\left(\frac{v}{V_0}\right) + B\ln\left(1 + \frac{V_{sat}\phi}{D_0}\right)\right],
\]

where \( v = x'_{pl} \) is the relative sliding velocity at the interface and \( \phi \) follows the evolution law rewritten here for the sake of clarity:

\[
\dot{\phi}(t) = 1 - \frac{v\phi(t)}{D_0}.
\]

For numerical purposes, we wish to recast those equations in the form of a system of first order ordinary differential equations (ODEs). Noting \( z = x' \), \( u = Vt - x \), further differentiating \( x - x_{pl} \) with respect to time in (5) using the explicit expression for \( W(t) = W_0[1 + \epsilon_{eff}\cos(\omega t)] \) and solving for \( v' \), we get the following ODE system, which we will use for the numerical bifurcation analysis:

\[
u' = V - z
\]
\[
z' = \frac{K}{M}u - \frac{W_0(1 + \epsilon_{eff}\cos(\omega t))}{M}\left[\mu_0 + A\ln\left(\frac{v}{V_0}\right) + B\ln\left(1 + \frac{V_{sat}\phi}{D_0}\right)\right]
\]
\[
v' = \frac{v}{\lambda A}\left[z - v - \frac{\lambda B V_{sat}(1 - v\phi/D_0)}{D_0(1 + V_{sat}\phi/D_0)}\right]
\]
\[
\phi' = 1 - \frac{v\phi}{D_0}.
\]

B. Determination of the SRF parameters

In order to analyse the data within the SRF framework, we need to determine a set of values for the relevant parameters of the model. This is performed under constant normal load, according to a well established procedure. The values, which will be used in the numerical analysis, some of them as trial ones, are gathered in Table I. The useful formulae are established in the appendix.

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2 This equation, including the actual sliding velocity \( v \) in place of \( x' \) is consistent with our physical understanding of the age \( \phi \). It has been checked that mistaking \( x' \) for \( v \), as in [16], has no significant effect on their results, at least for the modulation frequencies much larger than \( V/D_0 \) used in their study.
(i) First, the steady sliding friction coefficient $\mu_d(V)$ is measured to be velocity-weakening with an almost constant logarithmic slope over the $1 – 100 \ \mu m.s^{-1}$ range. This indicates that $V_{\text{sat}}$, above which $\mu_d(V)$ increases with increasing $V$ according to Eqs. (1, 2), is certainly larger than $100 \ \mu m.s^{-1}$ and allows to extract a value for $B - A$ and $\mu_0$ at $V_0 = 1 \ \mu m.s^{-1}$, i.e. far below the saturation of the ageing term.

(ii) Next, the critical value of $K/W_0$, here taken at midrange ($10 \ \mu m.s^{-1}$) where inertial terms can be neglected in Eq. (21), yields a determination of the memory length $D_0$.

(iii) From the value of the critical stick-slip period at $10 \ \mu m.s^{-1}$ (Eq. (22)), we obtain a value for $A$.

(iv) A determination of $V_{\text{sat}}$ is finally obtained by a best fit of the whole bifurcation curve in the plane $(K/W_0, V)$ for $V$ in the $1 – 50 \ \mu m.s^{-1}$ range, treating the inertial term in Eq. (21) as a perturbative one. Since the value of $V_{\text{sat}}$ is out of the experimental velocity window, this determination is not very accurate. Treating several data set corresponding to different runs yields an uncertainty as large as $\pm 25\%$ on the value of $V_{\text{sat}}$.

(iv) The value of the length $\lambda$, defined by the ratio of the load $W_0$ and the interfacial shear stiffness $\kappa$, has been obtained in [16] from a best fit of the a.c. response of the slider position to the normal load modulation.

It is clear that this procedure, though systematic, generates cumulative errors which are difficult to evaluate (the uncertainties on $A$, $B$ and $D_0$ given in Table (I) are conservative values). In view of the high sensitivity of the bifurcation to small variations of the parameters, we have chosen in the following numerical analysis to use the set determined above as a trial one. Namely, the parameters which are left free are $A$, $B$, $D_0$, $\lambda$, $V_{\text{sat}}$, and the ratio $\rho = \epsilon_{\text{eff}}/\epsilon$.

C. Bifurcation analysis

Technically, the transition from steady sliding to stick-slip, both states being modulated by the forcing (when $\epsilon \neq 0$), is a Neimark-Saker bifurcation (also called secondary Hopf), which corresponds to two complex conjugate values of the fundamental matrix of the ODE
system crossing the unit circle. The fundamental matrix $H$ is defined as $dH/dt = J(x(t))H$ with $H(x, 0) = I$, with $J$ the Jacobian matrix of the ODE system and $I$ the identity. The numerical software CANDYS/QA \cite{19} has been used to track this bifurcation. For a given parameter set a bifurcation curve like in Fig. 2 can be obtained as follows: for a given driving velocity, one starts from a low enough normal load $W_0$ in order to be in the steady sliding regime. Once such a “first point” is indeed found by the software, one varies $W_0$ only (1-parameter continuation) until a bifurcation is detected ($W_0 = W^c_0$); one then follows this bifurcation curve by further varying the driving velocity too (2-parameter continuation).

The procedure to detect the bifurcation has been automated. Starting from the parameter values determined in \textsection{III.B}, the critical values $W^c_0$ are determined and compared to the experimental ones $W_{0 \exp}^c$. A systematic procedure (Powell’s method, as described in \textsection{20}) is then used that attempts to minimize $\sum_{\nu, \epsilon} [W^c_0(\rho \epsilon) - W^c_{0 \exp}]^2$, with $\rho = \epsilon_{\text{eff}} / \epsilon$, for a representative set of experimental data. The set of parameters hence determined, which corresponds to a local minimum of the cost function, reducing it by a factor of fifteen \footnote{When several critical values for $W_0$ can be detected, the one retained in the evaluation of the cost function is the closest chosen among the odd ones (first, third, etc.), corresponding to a transition from steady sliding to stick-slip when increasing $W_0$, while even ones correspond to a transition from steady sliding to stick-slip.}, is given in Table I.

The full bifurcation curves are then determined as described above, by a 2-parameter continuation. The results are shown in figures 2 and 4 together with experimental data covering a range of $\epsilon_{\text{eff}}$ values wider than the one involved in the adjustment procedure.

\section{IV. DISCUSSION}

The curves displayed in Figs. 2 and 4 have been calculated with the optimized set of parameters. However, it is worth noting that the trial set yields numerical results in good qualitative agreement with the experimental data as well. Namely, the main effect of normal load modulation, at least for moderate values of $\epsilon_{\text{eff}}$, which is to stabilize sliding against stick-slip oscillations, is well reproduced by the SRF model. Moreover, the enhanced efficiency of the modulation on increasing the sliding velocity is correctly accounted for.

The set of optimized data differs from the trial set essentially for three parameters, namely...
the elastic length $\lambda$, the saturation velocity $V_{\text{sat}}$, and the ratio $\rho = \epsilon_{\text{eff}}/\epsilon$ accounting for the air-cushion effect.

The final value for $\lambda$ lies within the error bars estimated in [16].

As already mentioned in Sec. III B, the large variation of $V_{\text{sat}}$ during the optimization procedure is attributable to the fact that the crossover from a velocity-weakening regime to a strengthening one for steady sliding friction lies well above the upper experimental velocity, whence the goodness of the fit is only weakly sensitive to $V_{\text{sat}}$.

The ratio $\rho = \epsilon_{\text{eff}}/\epsilon$ was determined in [16] by comparison of the experimental shift of the steady friction level $\Delta \mu_0$ at 120 Hz and the value predicted by the SRF model. The value taken in this reference was $\rho = 0.48$. Taking into account the error bars on $\Delta \mu_0$ one finds that the relative uncertainty on $\rho$ is about $\pm 20\%$. The optimized value for this parameter lies therefore within this range.

Thus, the SRF model with its set of parameters as determined from the dynamical study of the system under constant normal load is fully predictive as regards the sliding stability of the system under modulated load, at least for the values of $\epsilon_{\text{eff}}$ probed by the data of Fig. 2. In turn, the sensitivity of this experiment enabled us to refine the determination of the parameters.

The quantitative overall agreement between the experimental data and the numerical curves in Fig. 4 is excellent for, say, $\epsilon_{\text{eff}} < 0.15$. Above this value the calculated curves tend to fold and correspond to a re-entrant stability diagram; namely, for given $\epsilon_{\text{eff}}$ and $V$, increasing $K/W_0$ yields successive bifurcations from stick-slip to steady sliding then back to steady-sliding, etc. No experimental evidence of such an unexpected behaviour has been encountered so far. It is clearly the result of the non-linear coupling between the normal load modulation and the stick-slip oscillations. As such, it is expected to depend drastically on the details of the SRF laws. The importance of the terms which ultimately cut-off the logarithmic variations in the SRF laws has been stressed in several studies [10, 21]. The existence of $V_{\text{sat}}$, which accounts for a short time cut-off in the creep deformation of the load-bearing asperities [7], yields one of these terms. It should be kept in mind that the SRF constitutive law [1] retains only the leading terms in the expansion of the friction force in powers of $\ln(v)$. For instance, Eq. (3) with physically sounded expressions for $\sigma_s(v)$ and $\Sigma_r(\phi)$ would lead to terms of order $AB \ln(v) \ln(\phi)$ which, though negligible for most purposes, would probably affect the critical behaviour of the system under a strongly
modulated load. For these reasons, we think that a full quantitative agreement between the experimental bifurcation at large $\epsilon_{\text{eff}}$ and the SRF model predictions would be illusive.

In addition, we have investigated numerically the effect of the extra term in Eq. (2) proposed by Linker and Dieterich [13] and already discussed in [16]. No significant effect has been found at 120 Hz, where we conclude that this term, if it exists as such, is not relevant to the present relatively high frequency study.

V. CONCLUSION

The stability of a sliding system with a few degrees of freedom, submitted to a periodically modulated normal load, has been studied experimentally. The study clearly evidences the role of load modulation, even at moderate amplitude, as a stabilizer against stick-slip oscillations. Though this effect of vibrations is seemingly part of the empirical culture in mechanical engineering, it is the first time, as far as we know, that it is investigated experimentally. This effect should be of great interest in fault mechanics as well as in the control of precision structures. The results have been compared to the numerical predictions of a model of the SRF type, relevant to multicontact friction at low velocities and low loads, including finite interfacial shear stiffness as a key parameter. Excellent quantitative agreement has been found as long as the amplitude of load modulation is restricted to about 10% of the dead load.

Although, as discussed above, the main effect of the normal load modulation is stabilization, the numerical study strongly suggests that destabilization may also occur, due to the highly non-linear features of the model which also gives rise to rentrent stability diagram in Fig. 4. More precisely, it can be seen in this figure that the $\epsilon_{\text{eff}} = 0.18$ curve crosses the $\epsilon_{\text{eff}} = 0.13$ one around $V = 7 \mu \text{m.s}^{-1}$. For a $(V, K/W_0)$ point slightly on the right of this crossing, in between the two curves, increasing $\epsilon_{\text{eff}}$ would result in a bifurcation from stable sliding to stick-slip. This effect has not been observed directly so far, probably because it corresponds to small regions of the parameter space, strongly dependent on the value of the parameters. Clearly, this point would deserve further experimental study.

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4 It has been shown in [16] that the relevant perturbation parameter is actually $\mu_0 \epsilon_{\text{eff}} / A \gg \epsilon_{\text{eff}}$ which is already larger than 1 for $\epsilon_{\text{eff}} = 0.1$. 

12
LINEAR STABILITY ANALYSIS FOR $\epsilon = 0$

The linear stability analysis of the SRF equations has been performed previously [2, 4]. However, difference between sliding velocity and the velocity of the center of mass of the slider was disregarded in these works. Since the interfacial stiffness is of paramount importance when the normal load is modulated at relatively high frequency, it is necessary to evaluate its role on the location of the bifurcation under constant load. Moreover, we will derive in this appendix expressions for the critical stiffness and the critical pulsation that hold for any state- and rate-dependent friction force.

Let us consider a general expression:

$$F = W_0\mu(v, \phi) \ .$$

The time-evolution of the age variable $\phi$ is ruled by:

$$\dot{\phi} = 1 - \frac{v\phi}{D_0} \ .$$

When the slider is driven at constant velocity $V$, the steady sliding values of the dynamical variables are $v = V$ and $\phi = D_0/V$. We define:

$$
\begin{cases}
\mu_v = \frac{\partial \mu}{\partial \ln v}(V, D_0/V) > 0 \\
\mu_\phi = \frac{\partial \mu}{\partial \ln \phi}(V, D_0/V) > 0
\end{cases}
$$

The position of the center of mass of the slider is $x(t)$ so that the elongation of the loading spring is $x - Vt$. At frequencies of interest the interfacial zone can be assumed massless and essentially elastic with a frequency independent, real stiffness $\kappa$. The following relation thus holds:

$$v = x' + \frac{d}{dt} \left( \frac{Mx'' - K(Vt - x)}{\kappa} \right) \ .$$

We will make use in the following of the ratio $\eta$ of the loading spring stiffness $K$ to the equivalent stiffness of the loading spring in parallel with the interface $K \parallel \kappa$:

$$\eta = \frac{K}{K \parallel \kappa} = \frac{K + \kappa}{\kappa} \ .$$

Finally, the dynamical equation for the motion of the slider reads:

$$Mx'' = K(Vt - x) - W_0\mu(v, \phi) \ .$$
The set of dynamical equations (12,14,16) is closed and can be linearized about the steady sliding state, setting:

\[
\begin{align*}
    x &= V t - F(V, D_0/V)/K + \delta x, \quad |\delta x| \ll F(V, D_0/V)/K \\
    \phi &= D_0/V + \delta \phi, \quad |\delta \phi| \ll D_0/V .
\end{align*}
\]

The linearized system becomes:

\[
\begin{align*}
    M \delta x'' &= -K \delta x - W_0 \left[ (\mu_v/V) \left( \eta \delta x' + \frac{M}{\kappa} \delta x''' \right) + (\mu_\phi V/D_0) \delta \phi \right] \\
    \delta \phi' &= -\eta \delta x'/V - \frac{M}{\kappa} \delta x'''/V - \delta \phi V/D_0
\end{align*}
\]

The solutions are the real parts of the complex \( \tilde{\delta x} = \tilde{\delta x}_0 \exp(i \Omega t) \) and \( \tilde{\delta \phi} = \tilde{\delta \phi}_0 \exp(i \Omega t) \) with \( \Omega \) a complex number. Replacing into Eq. (18) and writing the condition for non-trivial amplitudes \( \tilde{\delta x}_0 \) and \( \tilde{\delta \phi}_0 \), one finds the dispersion relationship

\[
C_4 \Omega^4 + C_3 \Omega^3 + C_2 \Omega^2 + C_1 \Omega + C_0 = 0
\]

with:

\[
\begin{align*}
    C_0 &= \frac{KV}{D_0} \\
    C_1 &= i \frac{W_0}{D_0} \left[ \frac{K D_0}{W_0} - \eta (\mu_\phi - \mu_v) \right] \\
    C_2 &= - \left( \frac{M V}{D_0} + \eta \mu_v W_0/V \right) \\
    C_3 &= -i M \frac{W_0}{\kappa D_0} \left[ \kappa D_0 \frac{W_0}{W_0} + (\mu_v - \mu_\phi) \right] \\
    C_4 &= M \mu_v \frac{W_0}{\kappa V}.
\end{align*}
\]

The critical value of the control parameters and the critical pulsation \( \Omega_c \) are obtained by expressing that at the Hopf bifurcation (at least) one root of Eq. (19) crosses the imaginary axis. Setting that \( \Omega \) is purely imaginary and extracting the real and imaginary components from Eq. (19) yield the requested values.

Let us first solve for an infinitely stiff interface, i.e. for \( \eta = 1 \) and \( \kappa \to \infty \). It is then straightforward to find:

\[
\frac{K D_0}{W_0^c} = (\mu_\phi - \mu_v) \left( 1 + \frac{M V^2}{W_0^c D_0 \mu_v} \right)
\]

and

\[
\frac{D_0 \Omega_c}{V} = \sqrt{\frac{\mu_\phi - \mu_v}{\mu_v}}.
\]
These relations make sense only for \( \mu_\phi - \mu_v > 0 \), i.e. when the steady sliding friction coefficient \( \mu^{ss}(V) \) is velocity weakening: \( \partial \mu^{ss}/\partial \ln V < 0 \). For the particular expression of \( \mu(v, \phi) \) used in the numerical analysis, this reads:

\[
-\frac{\partial \mu^{ss}}{\partial \ln V} = \mu_\phi - \mu_v = \frac{B}{1 + V/V_{sat}} - A > 0 .
\]  

(23)

Now, let us evaluate the contribution of the finite interfacial stiffness \( \kappa \) to Eq. (19) by estimating the order of magnitude of \( c = |C_4 \Omega_c^4/C_0| \) with \( \Omega_c \) given by Eq. (22), i.e. \( \Omega_c \simeq V/D_0 \). This reads:

\[
c \simeq \mu_v M V^2 / K D_0^2
\]

(24)

where we have expressed that \( W_0/(\kappa D_0) = \lambda/D_0 \simeq 1 \). One can estimate \( c < 10^{-3} \) within the experimental velocity range, hence the fourth order term in Eq. (19) can be safely discarded.

Next, a finite \( \kappa \) introduces perturbative terms in \( C_3 \) which are of order \( \mu_v, \mu_\phi \simeq 10^{-2} \), still well below the relative uncertainty on the experimental determination of the critical parameters. Since the correction to the drive stiffness due to the interfacial elastic element is of order \( \eta - 1 = K/\kappa \simeq 10^{-2} \), it can be concluded that for the purpose of calculating the values of the critical parameters, the finite interfacial stiffness has no practical effect, and one can make use of Eqs. (21) and (22).

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FIG. 1: Main elements of the experimental setup: Translation stage (Drv); Loading leaf spring (Lsp); Displacement gauge (Gg); Vibration exciter (Vb); Weighting spring (Spr); Accelerometer (Acc).

FIG. 2: Stability diagram for different values of the modulation amplitude. For given $V$ and $\epsilon_{\text{eff}}$, bifurcation from stick-slip to stable sliding occurs when the control parameter $K/W_0$ overcomes the plotted critical value: $\epsilon_{\text{eff}} = 0$ (\(\bullet\)); 0.045 (\(\triangle\)); 0.09 (\(\blacksquare\)); 0.13 (\(\square\)); 0.18 (\(\blacktriangle\)). For the sake of clarity, typical standard deviations are plotted as error bars only for $\epsilon_{\text{eff}} = 0$. The solid line curves are the output of the numerical study (see section III.C). The larger $\epsilon_{\text{eff}}$ the lower the curve at $V = 1\mu\text{m.s}^{-1}$. 

18
FIG. 3: Time evolution of the loading spring elongation for $V = 8 \mu m.s^{-1}$ and different modulation amplitudes $\epsilon_{\text{eff}}$ indicated at the right end of each trace. A vertical offset has been added to each trace in order to display clearly the bifurcation sequence from stick-slip to stable sliding. The inset is a blow-up of the stable sliding trace showing the remaining oscillating response at the frequency of the load modulation ($f = 120$ Hz, much higher than the stick-slip frequency).

FIG. 4: Reduced critical load vs. $\epsilon_{\text{eff}}$ for different driving velocities: $V(\mu m.s^{-1}) = 1 (\triangle); 5 (\bullet); 10 (\square); 30 (\blacksquare); 50 (\circ)$. The curves are the output of the numerical study (see section III C), labeled with the corresponding velocities in $\mu m.s^{-1}$.
FIG. 5: Equivalent mechanical circuit of the slider/track system. $K$ is the stiffness of the loading spring, $\kappa$ is the one of the interface.

TABLES

| parameter | trial value | optimized value |
|-----------|-------------|-----------------|
| $K$ (N.µm$^{-1}$) | 0.21         |                 |
| $M$ (kg)   | 2.37         |                 |
| $\mu_0(V_0=1\mu m.s^{-1})$ | 0.33 |                 |
| $A$        | 0.012 ± 0.002 | 0.0126          |
| $B$        | 0.023 ± 0.002 | 0.0241          |
| $D_0$ (µm) | 0.40 ± 0.04  | 0.402           |
| $V_{sat}$ (µm.s$^{-1}$) | 280 ± 70 | 256             |
| $\lambda$ (µm) | 0.62 ± 0.15 | 0.56            |
| $\rho = \epsilon_{eff}/\epsilon$ | 0.48 ± 0.10 | 0.43            |