Laplacian on fuzzy de Sitter space

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Received 19 November 2021
Accepted for publication 25 March 2022
Published 18 May 2022

Abstract
We study details of geometry of noncommutative de Sitter space: we determine the Riemann and Ricci curvature tensors, the energy and the Laplacian. We find, in particular, that fuzzy de Sitter space is an Einstein space, $R_{ab} = -3\zeta\delta_{ab}$. The Laplacian, defined in the noncommutative frame formalism, is not Hermitian and gives nonunitary evolution. When symmetrically ordered, it has the usual quadratic form $\Delta = \Pi_a \Pi^a$ (when acting on functions in representation space, $\Psi \in \mathcal{H}$): we find its eigenstates and discuss its spectrum. This result is a first step in a study of the scalar field Laplacian, $\Delta = [\Pi_a, [\Pi^a, ]$, and its propagator.

Keywords: noncommutative geometry, models of quantum gravity, de Sitter cosmology

1. Introduction

The relevance of noncommutative or fuzzy spaces is, both in mathematics and theoretical physics, many-fold. Mathematically, the concept is interesting as it provides an extension of geometry to algebraic structures like $C^*$ or matrix algebras: in particular, it is important to generalize the notion of smoothness to discrete structures such as algebras of matrices. Physically, discreteness of observables is usually related to their quantization: it is commonly expected that spacetime is quantized at the Planck scale, i.e. that coordinates are represented by operators. If we describe fields as functions of spacetime variables (that is, as elements of the algebra of coordinates, $A$), then in order to write their equations of motion we need to define Laplace and Dirac operators. Further, if additional differential-geometric quantities like connection and curvature are defined, we have achieved a description of gravity at the Planck scale, or more precisely, at the noncommutativity scale.

An alternative way to obtain quantum spacetime is to express coordinates, metric etc through a set of fields which are assumed to be elementary, for example strings. Then, if the fundamental structure is a linear (quantum field) theory, the effective geometric structure will likely be linear too, presumably described by operators. It is perhaps in such case less natural to expect that
quantum spacetime has a differential-geometric structure. However, if the classical limit exists, effective quantities corresponding to metric, connection and curvature exist as well, with their usual geometric properties.

Fuzzy de Sitter space is a noncommutative space defined in the framework of the noncommutative frame formalism [1], using the Lie-algebra structure of the de Sitter group $SO(1,4)$. The formalism gives a general definition of noncommutative differential geometry adjusted to gravity, and thus has a potential to describe gravity (or effective gravity) as geometry, in the quantum regime. However, it had been mostly applied to lower-dimensional models, while the real advance would be to find noncommutative versions of physical solutions to Einstein equations like the Schwarzschild black hole or the FLRW cosmologies (with unbroken spherical symmetry and beyond the tensor-product constructions).

One of distinctive properties of the noncommutative frame formalism is that it singles out a set of derivations $e_a$ and the corresponding dual one-forms $\theta^a$ that define the free-falling frame. Metric $g$ is a linear function with constant values in the frame basis,

$$g^{ab} = g(\theta^a \otimes \theta^b) = \eta^{ab}. \quad (1)$$

A sufficient condition that the metric components be constant is

$$[f, \theta^a] = 0, \quad (2)$$

where $f$ is an arbitrary function of noncommutative coordinates, $f \in \mathcal{A}$; this condition is a part of definition of the moving frame. Frame derivations $e_a$ are generated by momenta $\Pi_a$,

$$e_a f = [\Pi_a, f], \quad (3)$$

and $\Pi_a$ are usually taken to be antihermitian elements of $\mathcal{A}$. From these initial assumptions one can build a differential geometry in close analogy to the commutative one.

In this paper we investigate geometric properties of fuzzy de Sitter space additional to those already discussed in [2, 3]; immediate consequences of fuzzy de Sitter geometry to cosmology were previously analyzed in [3, 4]. After reviewing the representations of the $SO(1,4)$ which we use in section 2, in section 3 we introduce the operator of energy of fuzzy de Sitter space and determine its spectrum. In section 4 we calculate the Riemann and Ricci curvature tensors and the Laplacian. In section 5 we find the spectrum and the eigenfunctions of the quantum-mechanical Laplacian in $(\rho, s = 0, \frac{1}{2})$ unitary irreducible representations (UIR) of the principal continuous series. A discussion of the presented results, of properties and limitations that they give and of possible further research directions is given in the concluding section.

2. Review of the representation

Fuzzy de Sitter space is defined using the Lie algebra of the de Sitter group $SO(1,4)$. Its generators $M_{\alpha \beta}$ satisfy

$$[M_{\alpha \beta}, M_{\gamma \delta}] = -i(\eta_{\alpha \gamma} M_{\beta \delta} - \eta_{\alpha \delta} M_{\beta \gamma} - \eta_{\beta \gamma} M_{\alpha \delta} + \eta_{\beta \delta} M_{\alpha \gamma}), \quad (4)$$

$\alpha, \beta = 0, 1, 2, 3, 4$ and the signature is $\eta_{\alpha \beta} = \text{diag}(1, -1, -1, -1, -1)$. In unitary representations $M_{\alpha \beta}$ are Hermitian. Vector $\mathcal{V}^\rho$, a generalization of the Pauli–Lubanski vector, is defined by

\[1\] We replace here the usual notation for the operators of momenta $p_a$ by $\Pi_a$ to avoid possible confusion, as in the representations we are using wave functions $\Psi$ are functions of the momentum variables, $\Psi(p_0, \vec{p})$. 

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The $SO(1, 4)$ has a quadratic and a quartic Casimir operator,

$$C_2 = Q = -\frac{1}{2} M_{\alpha \beta} M^{\alpha \beta}, \quad C_4 = W = -\eta_{\alpha \beta} W^\alpha W^\beta.$$  \hspace{1cm} (6)

In UIR of the $SO(1, 4)$, labelled by quantum numbers $(\rho, s)$, values of the Casimir operators are

$$Q = -s(s+1) + \frac{9}{4} + \rho^2, \quad W = s(s+1) \left( \frac{1}{4} + \rho^2 \right).$$  \hspace{1cm} (7)

There are three series of UIR: the principal continuous series with $\rho \in \mathbb{R}$, $\rho \geq 0$, $s = 0, \frac{1}{2}, 1, \ldots$, the complementary continuous series, $i \rho = \nu \in \mathbb{R}_0$, $|\nu| < \frac{1}{2}$, $s = 0, 1, 2, \ldots$, and two discrete series, $\frac{1}{2} + i \rho = q$, $q = s, s-1, \ldots$ or $\frac{1}{2}, s = \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

In analogy with the fuzzy sphere [5], fuzzy de Sitter space can be defined as a ‘noncommutative embedding’ or hypersurface in five-dimensional noncommutative space. The latter is generated by coordinates

$$x^\alpha = \ell W^\alpha,$$  \hspace{1cm} (8)

while the embedding is realized through the second Casimir relation,

$$\eta_{\alpha \beta} x^\alpha x^\beta = -\ell^2 W = -\frac{3}{\Lambda},$$  \hspace{1cm} (9)

where $\Lambda$ is the cosmological constant. As in the case of fuzzy anti-de Sitter space [6], not every UIR gives a physically meaningful realization of de Sitter space (assuming the identification (8)). For example, (7) implies that $\Lambda$ is positive just in: all representations of the principal continuous series, representations of complementary series with $|\nu| < \frac{1}{2}$, and representations of discrete series with $q = s = 0, \frac{1}{2}$. Therefore, discreteness of the cosmological constant (which appears naturally in discrete series) is in fact not generic. We will, in the following, discuss properties of fuzzy de Sitter space defined as principal continuous series representations $(\rho, s)$: concrete calculations are done for values $s = 0$ and $s = \frac{1}{2}$.

The linear space of vector fields $\mathcal{X}(\mathcal{A})$ on a noncommutative algebra $\mathcal{A}$ is infinite-dimensional. In order to restrict dimension of the tangent space $T(\mathcal{A})$ associated to $\mathcal{A}$, it is usual to define it as a subspace $T(\mathcal{A}) \subset \mathcal{X}(\mathcal{A})$ by giving it a finite basis, for example $\{e_a\}$ defined by (3). There are two sets of momenta $\Pi_a$ that give, in application of the frame formalism to fuzzy de Sitter space, the de Sitter metric. One is

$$i\Pi_a = \sqrt{\zeta \Lambda} M_{\alpha \beta},$$  \hspace{1cm} (10)

where $\zeta$ is a numerical factor fixed by the value of the curvature scalar and index $a$ denotes antisymmetric pairs $\{\alpha \beta\}$. In this case we obtain the metric with coordinate components equal to

$$g^{\alpha \beta} = e_a^\alpha e_b^\beta \eta^{ab} = 3\eta^{\alpha \beta} - \Lambda x^\beta x^\alpha.$$  \hspace{1cm} (11)

In the commutative limit $g^{\alpha \beta}$ projects to the de Sitter hypersurface.
The tangent space is ten-dimensional. Second, ‘compact’ choice which gives a four-dimensional tangent space is to define momenta as operators proportional to dilation and translations, \( [2] \)

\[
i\Pi_0 = \sqrt{\zeta}M_{0\delta}, \quad i\Pi_i = \sqrt{\zeta}(M_{i4} + M_{i0}), \quad i = 1, 2, 3. \tag{12}
\]

It implies the line element

\[
dx^2 = d\tau^2 - e^{2\tau}(dx')^2, \tag{13}
\]

with cosmic time \( \tau \) and conformal time \( \eta \) given by

\[
\tau = \ell \log(\mathcal{V}_0 - \mathcal{V}_1), \quad \eta = -\ell e^{-\tau/\ell} = \mathcal{V}_0 - \mathcal{V}_1. \tag{14}
\]

Apparently, the most viable Hilbert space representation of the principal continuous series and the one that we used in calculations was given by Moylan \([8]\). It is defined on the space of UIR’s of the Poincaré group of mass \( m \) and spin \( s \) \([9]\): the corresponding Hilbert space is a direct sum of two Hilbert spaces of states with positive and negative energies, \( \mathcal{H}(m, s, +) \oplus \mathcal{H}(m, s, -) \).

The group generators, denoted by \( \mathcal{M}_{\alpha\beta} \), are

\[
\mathcal{M}_{\alpha\beta} = \begin{pmatrix} M_{\alpha\beta} & 0 \\ 0 & M_{\beta\alpha} \end{pmatrix}, \tag{15}
\]

\[
M_{ij} = i\left( p^i \frac{\partial}{\partial p^j} - p^j \frac{\partial}{\partial p^i} \right) + S_{ij}, \quad M_{i0} = -ip_0 \frac{\partial}{\partial p^i} + S_{i0},
\]

\[
M_{4j} = -\frac{p_j}{m} - \frac{1}{2m} \left( p^0, M_{ij} \right) - \frac{1}{2m} \left( p^i, M_{j0} \right), \quad M_{40} = -\frac{p_0}{m} p_0 + \frac{1}{2m} \left( p^0, M_{0i} \right). \tag{16}
\]

To simplify, we will in the following rescale \( p_\mu \) to be dimensionless, \( p_\mu/m \rightarrow p_\mu \). The \( S_{\mu\nu} \) in (16), \( \mu, \nu = 0, 1, 2, 3 \), are the usual spin generators: for the scalar UIR \( (\rho, s = 0) \), \( S_{\mu\nu} = 0 \); for the spinor representation \( (\rho, s = \frac{1}{2}) \), \( S_{\mu\nu} = \frac{i}{2} \{ \gamma_\mu, \gamma_\nu \} \).

Though the Moylan Hilbert-space representation is, at the level of the algebra, block-diagonal, it is not reducible: this is a very fine point which shows up (as noted and commented in \([8]\)) in the non-self-adjointness of operators \( M_{\alpha\beta} \) that appear diagonally in (15).\(^4\) However, the \( \mathcal{M}_{\alpha\beta} \) are Hermitian in \( \mathcal{H}(m, s, +) \oplus \mathcal{H}(m, s, -) \). This can be best seen by mapping the given representation to an equivalent one, defined in the direct sum of two positive-energy Hilbert spaces,

\[
\theta : \mathcal{H}(m, s, +) \oplus \mathcal{H}(m, s, -) \rightarrow \mathcal{H}(m, s, +) \oplus \mathcal{H}(m, s, +) \equiv \mathcal{H}_1 \oplus \mathcal{H}_1. \tag{17}
\]

We will use the latter in the calculations; thus we denote

\[
\theta \tilde{\Psi} \equiv \Psi, \quad \theta \mathcal{M}_{\mu\nu} \theta^{-1} \equiv \mathcal{M}_{\mu\nu}, \tag{18}
\]

for \( \tilde{\Psi} \in \mathcal{H}(m, s, +) \oplus \mathcal{H}(m, s, -) \) and \( \Psi \in \mathcal{H}(m, s, +) \oplus \mathcal{H}(m, s, +) \). We shall however not discuss in details the difference in hermiticity properties of \( M_{\alpha\beta} \) and \( \mathcal{M}_{\alpha\beta} \); we only comment the case of spherically symmetric operators, where the transition from \( \mathcal{H}(m, s, +) \) to

\[^3\]The given sign of \( \eta \) corresponds to the upper half of the de Sitter manifold, \([7]\).

\[^4\]We thank Ilija Buric for discussions of the Gårding domain which brought this fact into our focus: in the previous paper \([4]\) we erroneously assumed that the only consequence of the direct sum is doubling of states.

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\(H(m,s,+) \oplus H(m,s,+\) reduces to the change of interval of radial variable from \(z \in (0, 1)\) to \(z \in (0, \infty)\). This change affects hermiticity of the corresponding operators in a clear way, through the boundary conditions imposed on functions generating the Hilbert space.

The action of \(\theta\) is defined by, [8]

\[
\theta \begin{pmatrix} |\bar{\psi}_3+\rangle \\ |\bar{\psi}_3-\rangle \end{pmatrix} = \begin{pmatrix} |\bar{\psi}_3+\rangle \\ |\bar{\psi}_3+\rangle \end{pmatrix}, \quad \theta \begin{pmatrix} p_\mu \\ 0 \end{pmatrix} \theta^{-1} = \begin{pmatrix} p_\mu \\ 0 \end{pmatrix},
\]

and implies

\[
\mathcal{M}_{\mu\nu} = \begin{pmatrix} M_{\mu\nu} \\ 0 \\ M_{\mu\nu} \end{pmatrix}, \quad \mathcal{M}_{\mu4} = \begin{pmatrix} M_{\mu4} \\ 0 \\ -M_{\mu4} \end{pmatrix}.
\]

The scalar product is invariant to \(\theta\) and positive-definite. However, due to specific form of (19), it acquires a somewhat counter-intuitive form after the \(\theta\)-mapping. Introducing

\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]

the scalar product can be expressed as

\[
(\Psi, \Psi^\prime) = (\psi_1, \psi_1^\prime) + (-1)^s(\psi_2, \psi_2^\prime),
\]

where \((\psi, \psi^\prime)\) in the last formula depends on the spin: for cases we discuss we have [8, 9],

\[
(\psi, \psi^\prime) = \int \frac{d^3p}{2|p_0|} \bar{\psi}\gamma^0 \psi^\prime, \quad s = 0,
\]

\[
(\psi, \psi^\prime) = \int \frac{d^3p}{2|p_0|} \bar{\psi} \gamma^0 \psi^\prime, \quad s = \frac{1}{2}.
\]

It is perhaps at this point appropriate to note that the Moylan representation, though with a number of advantages, is in several aspects inconvenient to use: apart from the problem with hermiticity (appearing because of the splitting of the Hilbert space into a direct sum), the scalar product is different for each value of spin. More practical seems to be the Hilbert space representation of the principal continuous series that is commonly used in conformal field theory,[10].

3. Energy

Let us discuss the spectrum of energy of fuzzy de Sitter space. In conformal field theory energy \(E\) is often identified with the dilation generator \(\mathcal{M}_{04}\); here we have

\[
[i\mathcal{M}_{04}, \mathcal{W}_0 - \mathcal{W}_4] = \mathcal{W}_0 - \mathcal{W}_4,
\]

that is, \(\mathcal{M}_{04}\) is canonically conjugate to the cosmic time \(\tau\). Therefore we define energy as

\[
\mathcal{E} = \frac{\hbar}{\ell} \mathcal{M}_{04} = \frac{i\hbar}{\sqrt{\zeta}} \Pi_0.
\]
We solve the energy eigenvalue equation in representations \((\rho, s = 0)\). In subspace \(\mathcal{H}_\uparrow\), \(M_{04}\) reduces to
\[ M_{04\uparrow} = M_{04} = p_0 \left( \rho - \frac{3i}{2} - ip\frac{\partial}{\partial p} \right). \] (27)
Here \(p\) is the radial component of \(\vec{p}\), \(p^2 = \vec{p}^2 = -p_0 p'\), and \(p_0\) denotes the positive square root, \(p_0 = \sqrt{p^2 + 1}\). The \(M_{04}\) commutes with the angular momentum \(M_{jl}\), thus the angular variables can be separated in equation
\[ M_{04\uparrow}\psi = \lambda\psi. \] (28)
Using the ansatz
\[ \psi_{\lambda\ell\ell'}(\vec{p}) = \psi_{\lambda\ell}(p)Y_{\ell\ell'}(\theta, \varphi) = \frac{f_{\lambda\ell}(p)}{p} Y_{\ell\ell'}(\theta, \varphi), \] (29)
we obtain the radial equation
\[ i\left(p_0^2 - 1\right) \frac{d\psi_{\lambda\ell}}{dp_0} + \left(\frac{3i}{2} - \rho\right) p_0\psi_{\lambda\ell} = -\lambda\psi_{\lambda\ell}. \] (30)
It has a solution
\[ \psi_{\lambda\ell} = c\lambda p^{\frac{1}{2} - ip} \left(\frac{p_0 - 1}{p_0 + 1}\right)^\lambda. \] (31)
or, written in variable \(z\) defined in (61),
\[ f_{\lambda\ell} = C_\lambda (1 - z^2)^{\frac{1}{2} + ip} z^{-\frac{1}{2} - ip + i\lambda}. \] (32)
The radial equation is the same in subspace \(\mathcal{H}_\downarrow\), with \(p_0\) replaced by \(-p_0\) or \(\lambda\) by \(-\lambda\); therefore
\[ f_{-\lambda\ell}(p) = f_{-\lambda\ell}(-p). \] (33)

The radial solutions behave, upon integration, as plane waves in \(\log z\). The eigenvalue \(\lambda\) is not restricted: \(\lambda \in \mathbb{R}\), and \(\mathcal{E}\) has continuous spectrum. It is perhaps instructive to check explicitly the norm of the eigenfunctions. From (23), for solutions (31) we obtain [11],
\[ (\psi_{\lambda\ell}, \psi_{\lambda'\ell'}) = \delta_{\ell\ell'} \delta_{\lambda\lambda'} C_\lambda C_{\lambda'} \int_0^1 \frac{dz}{z} e^{i(\lambda - \lambda')}, \]
\[ = \delta_{\ell\ell'} \delta_{\lambda\lambda'} C_\lambda C_{\lambda'} \begin{cases} \pi\delta(\lambda' - \lambda), & \lambda' = \lambda \\ \frac{1}{i(\lambda' - \lambda)}, & \lambda' \neq \lambda. \end{cases} \] (34)
In fact, the eigenfunctions are properly normalized only in the full Hilbert space \(\mathcal{H}_\uparrow \oplus \mathcal{H}_\downarrow\):

\[ (\psi_{\lambda\ell}, \psi_{\lambda'\ell'}) = (\psi_{\lambda\ell}, \psi_{\chi'\ell'}) + (\psi_{-\lambda\ell}, \psi_{\chi'\ell'}) \]

\[ = (\psi_{\lambda\ell}, \psi_{\chi'\ell'}) + (\psi_{-\lambda\ell}, \psi_{-\chi'\ell'}) = \delta_{\ell\ell'} \delta_{\lambda\lambda'} \delta(\lambda' - \lambda) \]
for \(C_\lambda = \sqrt{1/2\pi}\).

In the \((\rho, s = 1/2)\) UIR’s the radial eigenfunctions have a slightly different form because the measure in the scalar product differs; the energy spectrum and normalization are the same.
4. Laplacian, general aspects

In order to find expression for the Laplacian on fuzzy de Sitter space we give a few definitions; details can be found in reference [1] which is dedicated to the noncommutative frame differential calculus.

Differential geometry of fuzzy de Sitter space was partly discussed in [2]. It is, in the frame formalism, completely defined by the operators of momenta $\Pi_a$ and their algebra. Intuitively speaking, momenta generate infinitesimal translations. They need not belong to spacetime algebra $A$ (indeed in particular case of commutative spaces they do not); $\Pi_a$ are usually taken to be antihermitian. This assumption is in [1] called the ‘reality condition’: it means that frame derivations map real functions (i.e. Hermitian operators) to real functions.

Differential of a function $f \in A$ is defined as $df = (e_a f) \theta^a$. Assuming that $d^2 = 0$ and that condition (2) is consistent with the action of $d$, we lastly that $\theta^a$ form a base to the cotangent space, one finds that momenta satisfy relations of the form

$$2P_{ab}^{cd} \Pi_a \Pi_b + C^a_{cd} \Pi_a + K_{cd} = 0,$$

(35)

where all coefficients are constant: $\Pi_a$’s cannot be chosen arbitrarily. The action of $d$ can be extended to one-forms, the exterior multiplication can be introduced. Clearly, one-forms in general do not anti-commute, not even the $\theta^a$’s. However, the structure of the exterior algebra $\Omega^*(A)$ is closely related to the structure of momentum algebra (35). As shown in [1], a consistent way to introduce the exterior product is to define

$$\theta^a \wedge \theta^b \equiv \theta^a \theta^b = P_{ab}^{cd} \theta^c \theta^d,$$  

$$d \theta^a = -\frac{1}{2} C^a_{bc} \theta^b \theta^c.$$ (36)

The momentum algebra in the fuzzy de Sitter case is a Lie subalgebra of the $SO(1,4)$,

$$[\Pi_0, \Pi_i] = \sqrt{\zeta} \Lambda \Pi_i, \quad [\Pi_i, \Pi_j] = 0,$$

(37)

so the nonvanishing structure constants are

$$C^i_{0j} = -C^i_{ji} = \sqrt{\zeta} \Lambda \delta^i_j, \quad P_{ab}^{cd} = \frac{1}{2} \left( \delta_a^c \delta_b^d - \delta_a^d \delta_b^c \right).$$ (38)

The Lie-algebra structure of (37) implies that the frame one-forms anticommute, $\theta^a \theta^b = -\theta^b \theta^a$. Furthermore, the connection one-form $\omega^a_{\ b} = \omega^a_{\ cb} \theta^c$ can be taken to be

$$\omega_{abc} = \frac{1}{2} (C_{abc} + C_{cab} - C_{ba})$$ (39)

as in the commutative case, so we have

$$\omega^0_0 = 0, \quad \omega^0_j = \sqrt{\zeta} \Lambda \eta_j \theta^i, \quad \omega^i_0 = -\sqrt{\zeta} \Lambda \theta^i, \quad \omega^i_j = 0.$$ (40)

This connection is metric-compatible and torsionless. For the Riemann curvature, $\Omega^a_{\ b} = \frac{1}{4} R^a_{\ cde} \theta^c \theta^d$, we find

$$\Omega^0_0 = 0, \quad \Omega^0_j = -\zeta \Lambda \eta_j \theta^0 \theta^i, \quad \Omega^i_0 = \zeta \Lambda \theta^0 \theta^i, \quad \Omega^j_0 = -\zeta \Lambda \eta_j \theta^0 \theta^k.$$ (41)
\[ R_{00} = -3\zeta \Lambda \eta_{00}, \quad R_{ij} = -3\zeta \Lambda \eta_{ij}, \quad R = 6\zeta \Lambda, \]  
\hspace{1cm} (42)

where \( R_{0d} = R_{d0} \) and \( R = R^a_{\ a} \). The Ricci tensor satisfies relation \( R_{ab} = -3\zeta \Lambda \eta_{ab} \), that is, fuzzy de Sitter space is a (noncommutative) Einstein manifold.

In order to define the Laplacian \( \triangle \) we need the Hodge-dual * and the codifferential \( \delta \). Noncommutative \( \delta \) is defined as the corresponding commutative operator: its action on \( p \)-forms is 
\[ \delta = (-1)^{p+n+1} d^* , \]
where \( n \) is dimension of the cotangent space. Noncommutative Hodge-dual differs from the standard one only when the frame one-forms do not anticommute, like e.g. on the truncated Heisenberg algebra \cite{12}; here it is straightforward. The Laplacian is then given by
\[ \triangle = d\delta + \delta d. \]  
\hspace{1cm} (43)

Applying (43) to 0-forms we find the action of the Laplacian on scalar fields, i.e. scalar functions of noncommutative coordinates:
\[ \triangle f = \left[ \Pi_0, \left[ \Pi_0, f \right] \right] + \left[ \Pi_i, \left[ \Pi^i, f \right] \right] - 3\sqrt{\zeta \Lambda} [\Pi_0, f]. \]  
\hspace{1cm} (44)

Analogously, its action on wave functions, elements of the representation space, is given by
\[ \triangle \Psi = \left( \Pi_0 \Pi^0 + \Pi_i \Pi^i - 3\sqrt{\zeta \Lambda} \Pi_0 \right) \Psi. \]  
\hspace{1cm} (45)

Neither of the two actions is Hermitian: this property is unusual and probably unwanted. In the commutative case, hermiticity of the Laplacian is guaranteed by uniqueness of the de Rham calculus and by invariance properties of the integral measure. In the noncommutative case differential calculus is not unique, and the integral (trace) is defined only in concrete representation. A nonhermitian Laplacian similar to (44) was obtained for the \( \hbar \)-deformed Lobachevsky plane, \cite{13, 14}. The problem of hermiticity was there solved by changing the operator ordering in \cite{13}, and by changing the definition of the adjoint in \cite{14}. Here we act similarly: we define the Laplacian by
\[ \Delta \Psi = \frac{1}{2} (\triangle + \triangle^\dagger) \Psi = (\Pi_0 \Pi^0 + \Pi_i \Pi^i) \Psi. \]  
\hspace{1cm} (46)

It is perhaps worth mentioning that, for the first set of momentum operators (10) (that we do not consider), the Laplacian reduces to quadratic Casimir operator \( C_2 \). This means that, acting on the wave functions \( \Psi \), Laplacian is a constant. The spectrum of scalar field \( \Delta \) is then given by the branching rules for the tensor product of the utilized representations.

5. Laplacian, representation

In commutative curved spacetime the classical equation of motion for the scalar field \( f \) is
\[ (\Delta + \mu^2 + \xi R) f = 0, \]  
\hspace{1cm} (47)

where \( \mu \) is mass of the field and \( \xi \) is the coupling to curvature. Since in de Sitter space the scalar curvature \( R \) is constant, (47) has the form of the eigenvalue equation for the Laplacian,
\[ (\Delta + M^2) f = 0, \quad M^2 = \mu^2 + \xi R. \]  
\hspace{1cm} (48)
Particular solutions to (47) constitute a basis to the Hilbert space of solutions $\mathcal{H}$ that is adapted for quantization: positive-energy solutions define the one-particle Hilbert space of states $\mathcal{H}_+$ that gives quantum-mechanical description of the scalar particles.

It is important to emphasize that in the noncommutative framework the classical equation of motion for the scalar field $f \in \mathcal{A}$,

$$[\Pi_0, [\Pi^0, f]] + [\Pi_, [\Pi^-, f]] + M^2 f = 0,$$

differs from the quantum-mechanical equation for the scalar particle described by the wave function $\Psi \in \mathcal{H}$,

$$(\Pi_0 \Pi^0 + \Pi_- \Pi^-) \Psi + M^2 \Psi = 0.$$  

In simple cases solutions to these two equations are related. Another important observation is that, in fuzzy de Sitter space, eigenstates of the Laplacian (46) do not have definite values of energy because the two observables are not compatible, $[\mathcal{E}, \Delta] \neq 0$. This obstructs direct interpretation of the positive-energy subspace $\mathcal{H}_+$ as space of one-particle excitations of the quantum field.

Fuzzy de Sitter Laplacian, mapped to $\mathcal{H}_+ \oplus \mathcal{H}_-$, is block-diagonal

$$\Delta = \begin{pmatrix} \Delta_+ & 0 \\ 0 & \Delta_- \end{pmatrix}.$$  

To simplify already cumbersome notation, we rescale in equations that follow $\Delta \to \zeta \Delta$ and $\zeta M^2 \to M^2$. We solve (47) in $(\rho, s = \frac{1}{2})$ representations. The wave functions are of the form (21), where $\psi^\pm, (\vec{p})$ are bispinors, solutions to the momentum-space Dirac equation. They are given, in accordance with [4], (19), by

$$\psi^\pm (\vec{p}) = \begin{pmatrix} \varphi^\pm (\vec{p}) \\ \frac{\rho_0 \sigma^k}{1 + \rho_0} \varphi^\pm (\vec{p}) \end{pmatrix}, \quad \psi^\pm (\vec{p}) = \begin{pmatrix} \varphi^\pm (-\vec{p}) \\ \frac{\rho_0 \sigma^k}{1 - \rho_0} \varphi^\pm (-\vec{p}) \end{pmatrix}.$$  

The $\varphi^\pm, (\vec{p})$ are spinors, and the scalar product is given by (24),

$$\langle \Psi, \Psi' \rangle = \langle \psi^\pm, \psi'^\pm \rangle = \int \frac{d^3 p}{2 \rho_0 \rho_0} \frac{1}{1 + \rho_0^2} \varphi^\dagger_+ \varphi_+ + \int \frac{d^3 p}{2 \rho_0 \rho_0} \frac{1}{1 - \rho_0^2} \varphi^\dagger_- \varphi_-.$$  

The momentum operators are

$$i\Pi_0 = \begin{pmatrix} i\Pi_{0,\dagger} \\ 0 \end{pmatrix} = \sqrt{\zeta \Lambda} \begin{pmatrix} M_{04} & 0 \\ 0 & -M_{04} \end{pmatrix},$$  

$$i\Pi_\pm = \begin{pmatrix} i\Pi_{\pm,\dagger} \\ 0 \end{pmatrix} = \sqrt{\zeta \Lambda} \begin{pmatrix} M_{00} + M_{44} & 0 \\ 0 & M_{00} - M_{44} \end{pmatrix},$$  

with

$$M_{04} = \begin{pmatrix} \rho - \frac{3i}{2} \\ \frac{i}{2} \sigma_i p^j \end{pmatrix} \rho_0 - i \rho_0 p_0 \frac{\partial}{\partial p_i} + \frac{i}{2} \sigma_i p^j \begin{pmatrix} \rho - \frac{3i}{2} \\ \frac{i}{2} \sigma_i p^j \end{pmatrix} \rho_0 - i \rho_0 p_0 \frac{\partial}{\partial p_i}.$$  

Division to positive- and negative-energy states can be done in static spacetimes. However, partitioning $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (which is not unique and depends on the choice of coordinates) can be done generally, [15].
$M_a + M_d = $

$$= \left( (\rho - \frac{3i}{2}) \rho + i(p_0 + 1) \frac{\partial}{\partial p} - ip_0 \frac{\partial}{\partial p_\sigma} + \frac{1}{2} \epsilon_{ijk} p^j \sigma^k \right) - \frac{i}{2} (p_0 + 1) \chi_i \left( (\rho - \frac{3i}{2}) \rho + i(p_0 + 1) \frac{\partial}{\partial p} - ip_0 \frac{\partial}{\partial p_\sigma} + \frac{1}{2} \epsilon_{ijk} p^j \sigma^k \right).$$

We first solve equation (48) in the upper subspace, $\mathcal{H}_u$. Since we are dealing with spinors, it takes more steps to come to the nontrivial part, to radial equation: some details of the derivation are given in appendix A. The Laplacian has the form

$$\Delta \equiv \left( \begin{array}{cc} A_1 & B_1 \\ B_1 & A_1 \end{array} \right), \quad \text{with}$$

$$A_1 = \frac{15}{4} + \rho^2 + 3i(\rho - 2i)p_0 - i(p_0 + 1)\epsilon_{ijk} p^j \frac{\partial}{\partial p_k} \sigma^k + \frac{p_0 + 1}{p_0 - 1} L_1 L' + \frac{2 p_0^2}{p_0 - 1} \left( \frac{p_0}{\partial p_1} \right)^2 + \left( 2 i p_0 p + 2 \frac{3 p_0^2}{p_0 - 1} p_0 - 1 \right) \frac{p_0}{\partial p_1},$$

$$B_1 = -i(\rho - 2i)\sigma_i p^i + (p_0 + 1)^2 \sigma_i \frac{\partial}{\partial p_i} - p_0 \sigma_i p_0 \frac{\partial}{\partial p_i}. \quad (55)$$

$L_i$ is the orbital part of the angular momentum, $L_3 = i\epsilon_{ijk} p^j \partial^k, L^2 = -L_i L_i$. Introducing (52) to eigenvalue equation, after some transformations we obtain

$$\Delta_{\text{eff}} \chi_1 = -M^2 \chi_1, \quad (57)$$

where the effective operator $\Delta_{\text{eff}}$ is given by

$$\Delta_{\text{eff}} \equiv \frac{1 + p_0}{2} \left( A_1 + \frac{p_0}{1 + p_0} A_1 + \left[ B_1 + \frac{p_0}{1 + p_0} \right] \right)$$

$$= \left( \rho + \frac{i}{2} \right)^2 + 2i(p_0 + 1) \frac{\partial}{\partial p} p + 2 \frac{\partial}{\partial p} p + \frac{1 + p_0}{1 - p_0} L^2 - \frac{2 p_0^2}{1 - p_0} \frac{\partial}{\partial p} p \frac{\partial}{\partial p}. \quad (58)$$

Analogously, in the lower subspace $\mathcal{H}_l$, we find

$$\Delta_{\text{eff}} \equiv \frac{1 - p_0}{2} \left( A_1 - \frac{p_0}{1 - p_0} A_1 + \left[ B_1 - \frac{p_0}{1 - p_0} \right] \right)$$

$$= \left( \rho + \frac{i}{2} \right)^2 - 2i(p_0 + 1) \frac{\partial}{\partial p} p + 2 \frac{\partial}{\partial p} p + \frac{1 - p_0}{1 + p_0} L^2 - \frac{2 p_0^2}{1 + p_0} \frac{\partial}{\partial p} p \frac{\partial}{\partial p}. \quad (59)$$

The two effective operators differ by the change of sign, $p_0 \rightarrow -p_0$, that is $\Delta_{\text{eff}}(p_0) = -\Delta_{\text{eff}}(-p_0)$.

The $\Delta_{\text{eff}}$ is spherically symmetric and has no terms that include spin: it commutes with $\vec{J}^2, J_3$ and $\vec{L}^2$, so we can label its solutions by the corresponding quantum numbers $j, m$ and $l$. Linearly independent solutions to (57), denoted by $\varphi_1$ and $\chi_1$, can be written as
\[ \varphi_j(p) = \frac{f_j(p)}{p} \varphi_{jm}(\theta, \varphi), \quad j = l + \frac{1}{2}, \quad \chi_j(p) = \frac{b_j(p)}{p} \chi_{jm}(\theta, \varphi), \quad j = l - \frac{1}{2}. \tag{60} \]

where \( \varphi_{jm} \) and \( \chi_{jm} \) are the spin spherical harmonics, [16]. Introducing the variable \( z \),

\[ z = \sqrt{\frac{p_0 - 1}{p_0 + 1}} \in (0, 1), \tag{61} \]

from (57) we find the radial equation for \( f_j \),

\[ (1 - z^2) \frac{d^2 f_j}{dz^2} + (2i\rho - 1)z \frac{df_j}{dz} + \left( \left( \rho + \frac{i}{2} \right)^2 - M^2 - \frac{l(l + 1)}{z^2} \right) f_j = 0. \tag{62} \]

The equation for \( h_j \) is identical (with different \( l \), (60)), so in the following we will discuss only \( f_j \). Solutions to (62) are

\[ f_{j,1} = z^{-l} F \left( \frac{1}{4} - \frac{l}{2} - \frac{i\rho}{2} - \frac{iM}{2} \frac{l}{2} - \frac{i\rho}{2} + \frac{iM}{2} \frac{l}{2} - l; z^2 \right), \tag{63} \]

\[ f_{j,2} = z^{l+1} F \left( \frac{3}{4} + \frac{l}{2} - \frac{i\rho}{2} + \frac{iM}{2} \frac{3}{2} + \frac{l}{2} - \frac{i\rho}{2} + \frac{iM}{2} \frac{3}{2} + l; z^2 \right), \tag{64} \]

where \( F(a, b; c, z) \equiv \, _2F_1(a, b; c, z) \) is the hypergeometric function. As at \( z = 0 \) first solution is divergent and second finite, \( f_{j,2} \) is the physical one.

Equation for \( \Delta_j \) in \( \mathcal{H}_j \) is completely analogous. For the radial part we obtain

\[ (1 - w^2) \frac{d^2 f_j}{dw^2} + (2i\rho - 1)w \frac{df_j}{dw} + \left( \left( \rho + \frac{i}{2} \right)^2 - M^2 - \frac{l(l + 1)}{w^2} \right) f_j = 0, \tag{65} \]

that is, an equation identical to (62) only written in variable \( w \),

\[ w = \sqrt{\frac{p_0 + 1}{p_0 - 1}} = \frac{1}{z} \in (1, \infty). \tag{66} \]

Therefore we have

\[ f_{j,1} = w^{-l} F \left( \frac{1}{4} - \frac{l}{2} - \frac{i\rho}{2} - \frac{iM}{2} \frac{l}{2} - \frac{i\rho}{2} + \frac{iM}{2} \frac{l}{2} - l; w^2 \right), \tag{67} \]

\[ f_{j,2} = w^{l+1} F \left( \frac{3}{4} + \frac{l}{2} - \frac{i\rho}{2} + \frac{iM}{2} \frac{3}{2} + \frac{l}{2} - \frac{i\rho}{2} + \frac{iM}{2} \frac{3}{2} + l; w^2 \right). \tag{68} \]

Analyzing the asymptotics of these functions for \( w \to \infty \), we find that they behave in the same way,

\[ f_{j,1}(w), f_{j,2}(w) \bigg|_{w \to \infty} \sim w^{i\rho - \frac{1}{2}} (C w^{iM} + C^* w^{-iM}). \tag{69} \]

For real \( M \), both solutions are asymptotically combinations of plane waves in \( \log w \) (up to a multiplicative function); for \( M^2 < 0 \), solutions diverge.
This asymptotics eventually results in the δ-function normalisation of the solutions. To show that, let us consider the scalar product in more details. Calculating the product of two functions of the form (60) we obtain

\[
(\psi_1, \psi_1') = \int \frac{d^3p}{2p_0(1 + p_0)} \varphi_1^{\dagger} \varphi_1' = \delta_{jj'} \delta_{mm'} \int_0^1 dz(f_1^{*} f_1' + h_1^{*} h_1').
\] (70)

\[
(\psi_1, \psi_1') = \int \frac{d^3p}{2p_0(1 - p_0)} \varphi_1^{\dagger} \varphi_1' = -\delta_{jj'} \delta_{mm'} \int_1^\infty dw(f_1^{*} f_1' + h_1^{*} h_1').
\] (71)

Therefore taking into account (24) we have

\[
(\Psi, \Psi') = \delta_{jj'} \delta_{mm'} \int_0^1 dz \left( f_1^{*} f_1' + h_1^{*} h_1' \right) + \delta_{jj'} \delta_{mm'} \int_1^\infty dw \left( f_1^{*} f_1' + h_1^{*} h_1' \right)
\] (72)

\[
= \delta_{jj'} \delta_{mm'} \int_0^\infty dz (f^{*} f' + h^{*} h').
\] (73)

In the last line we extended the radial functions \(f_{1,2}(z), h_{1,2}(z)\) to complete semi-axis \(z > 0\) by gluing them smoothly at \(z = 1\):

\[
f(z) = \begin{cases} 
  f_1(z), & z \in (0, 1) \\
  f_2(z), & z \in (1, \infty)
\end{cases}, \quad h(z) = \begin{cases} 
  h_1(z), & z \in (0, 1) \\
  h_2(z), & z \in (1, \infty)
\end{cases}.
\] (74)

Hence, the physical solution of (65) is the smooth continuation of (64) over \(z = 1\), that is \(f_{2,\pm}\).

In summary, the eigenfunctions of the Laplace operator \(\Delta\) are given by

\[
\varphi_{M,\mu}(\theta, \varphi) = C(1 - z^2)^{\mu/2} \times \left( \frac{3}{4} + \frac{l}{2} + \frac{i\rho}{2} - \frac{iM}{2} + \frac{3}{4} + \frac{l}{2} + \frac{i\rho}{2} + \frac{iM}{2} + l; z^2 \right) \varphi_{\mu}(\theta, \varphi)
\]

\[
= C(1 - z^2)^{\mu/2} \times \left( j + 1 - \frac{i\rho - iM}{2}, j + 1 - \frac{i\rho + iM}{2}; j + 1, z^2 \right) \varphi_{\mu}(\theta, \varphi),
\] (75)

\[
\chi_{M,\mu}(\theta, \varphi) = C'(1 - z^2)^{\mu/2} \times \left( \frac{3}{4} + \frac{l}{2} + \frac{i\rho}{2} + \frac{iM}{2} + \frac{3}{4} + \frac{l}{2} + \frac{i\rho}{2} + \frac{iM}{2} + l; z^2 \right) \chi_{\mu}(\theta, \varphi)
\]

\[
= C'(1 - z^2)^{\mu/2} \times \left( j + 2 - \frac{i\rho - iM}{2}, j + 2 - \frac{i\rho + iM}{2}; j + 2, z^2 \right) \chi_{\mu}(\theta, \varphi),
\] (76)

with \(z \in (0, \infty)\). The spectrum of the Laplacian is continuous, \(M^2 \in (0, \infty)\); the mass parameter \(M\) can be taken to be positive, \(M > 0\), as \(\Psi_{M,\mu} = \Psi_{-M,\mu}\). Each value of \(M\) is infinitely degenerate, with degeneracy \(2 \times \sum_{j} (2j + 1)\), \(j = \frac{1}{2}, \frac{3}{2}, \ldots\). Normalization of the eigenfunctions is
essentially determined by their asymptotics at $z \to \infty$, which implies

$$(\varphi_{Mjn}, \varphi_{M'jn'}) \sim \delta_{jj'}\delta_{nn'}\delta(M - M').$$

(77)

Completeness and orthogonality of the radial functions can be shown exactly, see appendix B.

The trick (that is, the method) of writing the sum of two scalar products in different subspaces as one integral provides in fact a proof of hermiticity for spherically-symmetric operators on $\mathcal{H}_s \oplus \mathcal{H}_l$. Let us outline the proof: to be concrete, we consider the Laplace operator, but the reasoning is very similar in other cases. Assume that we have two radial functions of the type (60) with

$$\varphi(p) = \frac{f(p)}{p}, \quad \varphi'(p) = \frac{g(p)}{p}. \quad (78)$$

The matrix elements of the Laplacian are defined by (70) and (72). We find that in the radial subspace

$$(\varphi, \Delta_1\varphi') - (\Delta_1\varphi, \varphi') = 2i\rho\zeta f^*g\left|_{0}^{1}\right. + (1 - z^2)\left(f\frac{d\rho}{dz} + \frac{df^*}{dz}\frac{d\rho}{dz}\right)\left|_{0}^{1}\right..$$

(79)

Evaluating this expression for the eigenfunctions (75) and (76) we can verify that the boundary contribution at $z = 1$ does not vanish. This means that $\Delta_1$ is only formally self-adjoint, that is, the difference $(\varphi, (\Delta_1 - \Delta_1)\varphi')$ is not always zero. To achieve self-adjointness, additional boundary conditions at $z = 1$ have to be imposed: they restrict the initial $\mathcal{H}_s$ in such way that (79) vanishes in all states, [17]. A more detailed analysis shows that the appropriate boundary conditions in fact imply discreteness of the spectrum of $\Delta_1$.

The matrix elements of $\Delta$ defined on $\mathcal{H}_s \oplus \mathcal{H}_l$ are given by exactly same expression (79), except that the boundary terms are evaluated at 0 and $\infty$. The reason is that $\Delta_1$ is obtained from $\Delta_1$ by replacement $p_\mu \to -p_\mu$, that is (for spherically symmetric operators) by $z \to w$; otherwise the form of the operator is exactly the same. Therefore, when evaluating (79) at $0$ there will be no special conditions at $z = 1$, or more precisely, the boundary terms for $\Delta_1$ at $z = 1 - \epsilon$ will cancel the boundary terms of $\Delta_1$ at $z = 1 + \epsilon$, when evaluated for continuous functions with continuous first derivative at $z = 1$. One can check that conditions at $z = \infty$ coming from (79) do not impose further restrictions compared to those that come from normalization.

For comparison and future reference, we add the results for $(\rho, s = 0)$. The Laplacian is

$$\Delta_1 = \rho^2 + \frac{1}{4} + \frac{1}{4} (1 + 2i\rho) p_0 + (2i\rho p_0 + 4p_0 + 2) \frac{\partial}{\partial p}p + \frac{1 + p_0}{1 - p_0} \frac{2p_0^2}{1 - p_0^2} \frac{\partial}{\partial p} \frac{\partial}{\partial p}p,$$

(80)

and the eigenfunctions are of the form

$$\psi_{\text{Min},1}(\vec{p}) = \frac{f_1(p)}{p} \varphi_1^0(\theta, \varphi).$$

(81)

We obtain the radial equation

$$(1 - z^2)\frac{d^2f_1}{dz^2} + 2i\zeta \frac{df_1}{dz} + \left(\rho - \frac{i}{2}\right)^2 - M^2 - \frac{\zeta(l + 1)}{z^2} + \frac{1 + 2i\rho}{1 - z^2} f_1 = 0,$$

(82)
with solutions that are almost identical to (63) and (64),

\[
f_{\uparrow,1} = z^{-1} (1 - z^2)^{1/2} F \left( \frac{1}{4} - \frac{l}{2} - \frac{i \rho}{2} - \frac{i M}{2}, \frac{1}{4} - \frac{l}{2} - \frac{i \rho}{2} + \frac{i M}{2} ; 1 - l^2 \right) \tag{83}
\]

\[
f_{\uparrow,2} = z^{l+1} (1 - z^2)^{1/2} F \left( \frac{3}{4} + \frac{l}{2} + \frac{i \rho}{2} + \frac{i M}{2}, \frac{3}{4} + \frac{l}{2} + \frac{i \rho}{2} + \frac{i M}{2} ; 1 + l^2 \right). \tag{84}
\]

As before, the second one is physical. Extension to \( H_1 \) and properties of \( \Delta_1 \) are analogous to \( s = \frac{1}{2} \); complete solution is obtained by continuation of \( f_{\uparrow,2} \) to the interval \( z \in (0, \infty) \).

We will in appendix B use the fact that this solution is real, so let us prove it. Using the properties of hypergeometric function [18], we find

\[
F \left( \frac{3}{4} + \frac{l}{2} + \frac{i \rho}{2} + \frac{i M}{2}, \frac{3}{4} + \frac{l}{2} + \frac{i \rho}{2} + \frac{i M}{2} ; 1 + l^2 \right) = (1 - z^2)^{i \rho} F \left( \frac{3}{4} + \frac{l}{2} + \frac{i \rho}{2} + \frac{i M}{2}, \frac{3}{4} + \frac{l}{2} + \frac{i \rho}{2} + \frac{i M}{2} ; 1 + l^2 \right),
\]

that is,

\[
f_{\downarrow,2,\uparrow} = z^{l+1} \sqrt{1 - z^2} (1 - z^2)^{i \rho} F \left( \frac{3}{4} + \frac{l}{2} + \frac{i \rho}{2} + \frac{i M}{2}, \frac{3}{4} + \frac{l}{2} + \frac{i \rho}{2} + \frac{i M}{2} ; 1 + l^2 \right) = f_{\downarrow,2,\uparrow}' \tag{85}
\]

The spectrum of \( \Delta \) for \( s = 0 \) is, up to the spin degeneracy, the same as the spectrum of \( \Delta \) for \( s = \frac{1}{2} \).

6. Discussion

To the comments and discussion given in the text already we add the following.

The main idea in this paper was to find the Laplace operator on fuzzy de Sitter space and study its properties. The immediate motivation was a possible application to cosmology and inflation, but the overall idea is more general: to understand whether the behavior of matter on noncommutative spaces can provide some explanations of semiclassical or quantum effects in gravity. In the commutative case, Laplacian is one of the main geometric characteristics of a manifold; it is in the noncommutative frame formalism defined in a systematic way as well. Concrete expression for the Laplacian depends on degree of differential form which it acts upon: here, in addition, we consider the action of the Laplacian on wave functions that define the representation space \( \mathcal{H} \) (the ‘quantum-mechanical’ Laplacian). In fact, the main results of the paper concern noncommutative quantum mechanics on fuzzy de Sitter space; properties of the scalar field will be discussed separately.

We obtained that the fuzzy de Sitter Laplacian, defined straightforwardly, is a non-Hermitian operator, both when acting on wave functions \( \Psi \), (45) and on scalar fields \( f \), (44). The meaning of this result is not quite clear. It might be an implication of the (unnecessary) generality of the frame formalism, which (among other assumptions) postulates that the frame derivations \( e_a \) obey the ‘reality condition’, that is that momenta \( \Pi_a \) are defined as antihermitian operators.
On the other hand, this assumption seems to be quite logical and works very well in a number of examples. The commutative Laplacian is always Hermitian: we therefore symmetrized (45) to obtain a Hermitian expression. We then solved the corresponding quantum-mechanical equation and found that the spectrum of the Laplacian is continuous, given by eigenvalue $M \in (0, \infty)$, and that the eigenfunctions behave as spherical waves. The eigenfunctions $\Psi_{M,jm}$ are given by (75), (76) and (84) for, respectively, $(\rho, s = 1)$ and $(\rho, s = 0)$ Moylan representations; each eigenvalue has a degeneracy in the angular momentum. In comparison with the analogous result for commutative de Sitter space we see the (expected) reduction of the number of degrees of freedom: the corresponding commutative solutions, along with $M$, $j$ and $m$, have the eigenvalue of energy $\omega$ as a quantum number, [7]. It is, however, interesting to observe that radial functions which appear in commutative and noncommutative solutions can both be expressed as Jacobi functions, with different values of parameters $\alpha$ and $\beta$, [19].

It is possible to solve the eigenvalue equation for the non-Hermitian Laplacian (46) as well. For $s = 0$, the operator $\triangle$ is in $\mathcal{H}_\uparrow$ given by

$$\triangle = \rho^2 + \frac{1}{4} - (1 + 2i\rho) p_0 + (2i\rho p_0 + p_0 + 2) \frac{\partial}{\partial p} p + \frac{1 + p_0}{p_0 - 1} L^2 - \frac{2p_0^2}{1 - p_0} \frac{\partial}{\partial p} \frac{\partial}{\partial p} p. \quad (86)$$

The radial part of the corresponding eigenvalue equation is

$$(1 - z^2) \frac{d^2 f_\uparrow}{dz^2} + (2i\rho - 3)z \frac{df_\uparrow}{dz} + \left((\rho + i)^2 + \frac{9}{4} - M^2 - \frac{l(l + 1)}{z^2} - \frac{2(1 + 2i\rho)}{1 - z^2}\right) f_\uparrow = 0 \quad (87)$$

and has solutions

$$f_{\uparrow,2} = \frac{z^{-l}}{1 - z^2} F\left(\frac{l}{2} - \frac{i\rho}{2} - \frac{i}{2} \sqrt{M^2 - \frac{9}{4} - \frac{l}{2}}; \frac{i}{2} \sqrt{M^2 - \frac{9}{4} + \frac{3}{2}} + l; z^2\right) \quad (88)$$

$$f_{\uparrow,2} = \frac{z^{l+1}}{1 - z^2} F\left(\frac{l}{2} - \frac{i\rho}{2} - \frac{i}{2} \sqrt{M^2 - \frac{9}{4} - \frac{l}{2}}; \frac{i}{2} \sqrt{M^2 - \frac{9}{4} + \frac{3}{2}} + l; z^2\right). \quad (89)$$

The solutions are of course not normalizable: they all diverge at $z = 1$.

Still, it is interesting and perhaps important that the Laplacian on the fuzzy de Sitter space $\triangle$ obtained from ‘the first principles’ that is, directly from definitions, is non-Hermitian. It gives a model of non-unitary evolution in the gravitational field (which Hawking considered as possible solution to the information paradox). Indeed, antihermitian part of the Laplacian, $V = -3\sqrt{\zeta}\Pi_0$, implies non-conservation of probability; non-Hermitian terms have been used before, in nuclear physics, to describe the $\alpha$-decay. It would be interesting to see whether the transition amplitudes $\langle \Psi_{M,jm} | V | \Psi_{M',jm} \rangle$, calculated in perturbation theory, could in any way be related to the Hawking flux.

If we stick to conservative interpretation and to Hermitian ordering, there is another interesting problem that one can pose in the given framework: to solve the quantum-mechanical equation for the scalar particle in the potential of inflatory type, and compare the results with the effects obtained in the standard de Sitter cosmology.

An important problem which needs further investigation is to solve classical equation of motion (49), and then find the propagator of the noncommutative scalar field $f$. It would seem that this problem is not easily reducible to the pure representation theory. The scalar field can
be expanded as
\[ f = \sum_{M,j,m,M',j',m'} c_{M,j,m,M',j',m'} \langle \Psi_{Mj m} | \Psi_{M'j' m'} \rangle. \]  
(90)

and (49) becomes an equation for the coefficients \( c_{M,j,m,M',j',m'} \). As
\[ [\Pi_\mu, [\Pi_\nu, f]] = \Pi_\mu \Pi_\nu f + f \Pi_\nu \Pi_\mu - 2 \Pi_\mu f \Pi_\nu, \]  
(91)

(denoting the mass of the scalar field by \( M \)) we have
\[ \sum (M^2 + M'^2 + M'^2) c_{M,j,m,M',j',m'} \langle \Psi_{Mj m} | \Psi_{M'j' m'} \rangle = 2 \sum c_{M,j,m,M',j',m'} \Pi_\mu | \Psi_{M'j' m'} \rangle \langle \Psi_{Mj m} | \Pi_\nu. \]  
(92)

Since the action of the Laplacian is off-diagonal, this equation is (in principle) complicated; however, even if not fully solved, it might provide an insight into the structure of the scalar field propagator.

**Acknowledgment**

This work was supported by the Serbian Ministry of Education, Science and Technological Development Grant 451-03-9/2021-14/200162.

**Data availability statement**

No new data were created or analysed in this study.

**Appendix A**

In this appendix we give details of how to solve (48) for \( s = \frac{1}{2} \). In the signature that we use we have
\[ \vec{p} = (p_i), \quad \vec{r} = (x^i) = i \nabla^i = i \frac{\partial}{\partial p_i}, \]
\[ \vec{L} = (L_i) = \vec{r} \times \vec{p}, \quad L_d = i \epsilon_{ijk} p^j \frac{\partial}{\partial p_k}, \quad \vec{\sigma} = (\sigma_i), \]  
(93)

\[ \sigma_i \sigma_j = -\eta_{ij} - i \epsilon_{ijk} \sigma^k, \quad \epsilon_{ijk} \epsilon^{lmn} = -(\delta^m_j \delta_n^k - \delta^m_k \delta_n^j), \]
\[ \epsilon_{ijk} \epsilon^{ijn} = -2 \delta^m_k. \]

In \( \mathcal{H}_\uparrow \) the Laplacian has the form \( \Delta_{\uparrow} = \begin{pmatrix} A_\uparrow & B_\uparrow \\ B_\uparrow & A_\uparrow \end{pmatrix} \) and the scalar product is given by (24) and (53). Applying \( \Delta_{\uparrow} \) to bispinor (52) in the eigenvalue equation, we obtain
\[ A_\uparrow \varphi_\uparrow + B_\uparrow \frac{p_i \sigma^k}{1 + p_0} \varphi_\uparrow = -M^2 \varphi_\uparrow \]  
(94)
\[ B_\uparrow \varphi_\uparrow + A_\uparrow \frac{p_i \sigma^k}{1 + p_0} A_\uparrow \varphi_\uparrow = -M^2 \frac{p_i \sigma^k}{1 + p_0} \varphi_\uparrow. \]  
(95)
After some transformations, introducing
\[ \Delta_{\text{eff}} \equiv \frac{1 + p_0}{2} \left( A_\uparrow - \frac{p_0 \sigma^k}{1 + p_0} A_\uparrow \frac{p_i \sigma^i}{1 + p_0} + \left[ B_\uparrow, \frac{p_i \sigma^i}{1 + p_0} \right] \right) \] (96)
these two equations can be reduced to one
\[ \Delta_{\text{eff}} \varphi_\uparrow = -M^2 \varphi_\uparrow. \] (97)

Using \( A_\uparrow \) and \( B_\uparrow \) (55) and (56), we arrive at
\[ \Delta_{\text{eff}} = \left( \rho + \frac{i}{2} \right)^2 + 2(i \rho + 1)p_0 \left( 1 + p_i \frac{\partial}{\partial p_i} \right) + 2 \left( 1 + p_i \frac{\partial}{\partial p_i} \right)
+ \frac{1 + p_0 - 2p_0}{1 - p_0} \frac{L^2}{1 - p_0} \frac{\partial}{\partial p_i} \left( 1 + p_i \frac{\partial}{\partial p_i} \right), \]
and by passing to the spherical coordinate system, the effective Laplacian adopts the form (58) given in the text.

**Appendix B**

In this appendix we outline the proof of orthogonality and completeness of eigenfunctions of the Laplacian: as simpler, we discuss eigenfunctions in the \((\rho, s = 0)\) representations
\[ \Psi_{Ml}(\vec{p}) = C_{Ml} f_{Ml}(p) Y^m_l(\theta, \varphi) \] (98)
\[ = \frac{C_{Ml}}{2} \sqrt{1 - z^2} \frac{1}{1 - z^2} \phi_{\lambda}(\alpha, \beta) \lambda(\rho) Y^m_l. \] (99)

The scalar product of two functions of the form (98) is given by
\[ (\Psi, \Psi') = C^* C' \delta_{p_0} \delta_{\text{radial}} \int_0^{\infty} \frac{dz}{1 - z^2} f_{Ml} f'_{M'l}. \] (100)

Orthogonality can be shown by expressing the hypergeometric function in (99) as the Jacobi function, and using the properties of the Jacobi transform and its inverse. In the following, we use the notation of [19] and the results presented there. The Jacobi functions are defined as
\[ \phi_{\lambda}(\alpha, \beta)(t) = F \left( \frac{1}{2}(\alpha + \beta + 1 - i \lambda), \frac{1}{2}(\alpha + \beta + 1 + i \lambda); \alpha + 1; -\sinh^2 t \right), \] (101)
for \( \alpha, \beta, \lambda \in \mathbb{C}, \alpha \neq -1, -2, \ldots \). Therefore radial eigenfunctions entering (98) can be rewritten as
\[ f_{Ml} = f'_{Ml} = C_{Ml}^* \sqrt{1 - z^2} \left( 1 - z^2 \right)^{l+1/2} \phi_{\lambda}(\alpha, \beta)(t), \] with
\[ \alpha = l + \frac{1}{2}, \quad \beta = i \rho, \quad \lambda = M, \quad i \sinh t = z. \] (103)
Clearly, integration in formulas for the Jacobi transform will in our case be done along the imaginary axis \((-\infty, \infty)\), using the analytic continuation of the hypergeometric function. A rigorous derivation of the necessary steps is beyond the scope of this text; we shall use the continued to complex values of parameters \(\beta, \alpha \in \mathbb{C}, \alpha \neq -1, -2, \ldots\) We start by referring to some definitions and theorems.

For \(\alpha > -1, |\beta| < \alpha + 1\), the Jacobi function is the kernel of the Jacobi transformation

\[
\hat{f}(\lambda) = \int_0^\infty dt (2 \sinh t)^{2\alpha+1} (2 \cosh t)^{2\beta+1} \phi^{(\alpha,\beta)}_\lambda(t) f(t).
\] (104)

The inverse transformation is given by

\[
f(t) = \frac{1}{2\pi} \int_0^\infty d\lambda |c_{\alpha,\beta}(\lambda)|^{-2} \phi^{(\alpha,\beta)}_{\lambda}(t) \hat{f}(\lambda),
\] (105)

where the constants \(c_{\alpha,\beta}(\lambda)\) are

\[
c_{\alpha,\beta}(\lambda) = \frac{2^{\alpha+\beta+1-1}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma\left(\frac{1}{2}(\alpha+\beta+1+i\lambda)\right)\Gamma\left(\frac{1}{2}(\alpha+\beta+1-i\lambda)\right)}. \tag{106}
\]

Equation (105) holds when \(c_{\alpha,\beta}(\lambda), c^{-1}_{\alpha,\beta}(\lambda)\) are finite i.e. have no poles, which is true in our case, (103). Applying the Jacobi transformation to Jacobi functions we obtain that they are continuously orthogonal,

\[
\int_0^\infty dt (2 \sinh t)^{2\alpha+1} (2 \cosh t)^{2\beta+1} \phi^{(\alpha,\beta)}_\lambda(t) \phi^{(\alpha,\beta)}_{\lambda'}(t) = 2\pi |c_{\alpha,\beta}(\lambda)|^2 \delta(\lambda - \lambda'). \tag{107}
\]

The last formula gives in fact orthogonality of the eigenfunctions of the Laplacian. Indeed, since the eigenfunctions are real, we have

\[
\int_0^\infty \frac{dz}{|1-z^2|} \hat{f}_M \hat{f}_{M'} = \int_0^\infty \frac{dz}{|1-z^2|} z^{2\ell+2} |1-z^2| (1-z^2)^{\nu} \phi^{(\alpha,\beta)}_M \phi^{(\alpha,\beta)}_{M'} \tag{108}
\]

\[
= \int_0^\infty dt (\sinh t)^{2\alpha+1} (\cosh t)^{2\beta+1} \phi^{(\alpha,\beta)}_M \phi^{(\alpha,\beta)}_{M'} \tag{109}
\]

for values of parameters given by (103). Therefore a possible choice of normalization constants \(C_M\),

\[
|C_M| = 2^{-l-1} \sqrt{\pi} |c_{\alpha,\beta}(M)| \tag{110}
\]

is

\[
C_M = \frac{\sqrt{2\pi} \Gamma \left(\frac{3}{2} + \frac{i\rho}{2} + iM\right)}{\Gamma \left(\frac{1}{2} + \frac{i\rho}{2} + iM\right) \Gamma \left(\frac{1}{2} \left(l + \frac{1}{2} + i\rho + iM\right)\right)} \tag{111}
\]

\(^6\) Jacobi functions are even, and the Jacobi transformation is a generalization of the Fourier cosine transformation, as \(\phi^{1/2, -1/2}_\lambda(t) = \cos \lambda t\). Therefore the integrals in (104) and (105) can be extended to interval \((-\infty, \infty)\), assuming that \(f, \hat{f}\) are even functions.
Completeness of the set of functions (102) is basically the formula for the inverse Jacobi transformation (105).

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References

[1] Madore J 2000 An Introduction to Noncommutative Differential Geometry and its Physical Applications (London Mathematical Society Lecture Note Series vol 257) (Cambridge: Cambridge University Press)
[2] Buric M and Madore J 2015 Eur. Phys. J. C 75 502
[3] Buric M, Latas D and Nenadovic L 2018 Eur. Phys. J. C 78 953
[4] Buric M and Latas D 2019 Phys. Rev. D 100 024053
[5] Madore J 1992 Class. Quantum Grav. 9 69
[6] Jurman D and Steinacker H 2014 J. High Energy Phys. JHEP01(2014)100
[7] Birrell N D and Davies P C W 1982 Quantum Fields in Curved Space (Cambridge Monographs on Mathematical Physics) (Cambridge: Cambridge University Press)
[8] Moylan P 1983 J. Math. Phys. 24 2706
[9] Bargmann V and Wigner E P 1948 Proc. Natl Acad. Sci. USA 34 211
[10] Dobrev V K, Mack G, Petkova V B, Petrova S G and Todorov I T 1977 Harmonic Analysis on the n-Dimensional Lorentz Group and its Application to Conformal Quantum Field Theory (Lecture Notes in Physics) (Berlin: Springer)
[11] Skinner D 2015 Mathematical methods http://damtp.cam.ac.uk/user/dbs26/1Bmethods.html
[12] Buric M, Grosse H and Madore J 2010 J. High Energy Phys. JHEP07(2010)010
[13] Madore J and Steinacker H 1999 J. Phys. A: Math. Gen. 33 327
[14] Cho S 1999 J. Phys. A: Math. Gen. 32 2091
[15] Wald R M 1995 Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics (Chicago Lectures in Physics) (Chicago: University of Chicago Press)
[16] Bjorken J D and Drell S D 1965 Relativistic Quantum Mechanics (New York: McGraw-Hill)
[17] Hutson V, Pym J and Cloud M J 2005 Applications of Functional Analysis and Operator Theory (Amsterdam: Elsevier)
[18] Abramowitz M and Stegun I A 1965 Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables (New York: Dover)
[19] Koornwinder T H 1984 Jacobi functions and analysis on noncompact semisimple Lie groups Special Functions: Group Theoretical Aspects and Applications (Berlin: Springer)