ON CYCLICALLY SYMMETRICAL SPACETIMES

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In a recent paper Carot et al. considered the definition of cylindrical symmetry as a specialisation of the case of axial symmetry. One of their propositions states that if there is a second Killing vector, which together with the one generating the axial symmetry, forms the basis of a two-dimensional Lie algebra, then the two Killing vectors must commute, thus generating an Abelian group. In this paper a similar result, valid under considerably weaker assumptions, is derived: any two-dimensional Lie transformation group which contains a one-dimensional subgroup whose orbits are circles, must be Abelian. The method used to prove this result is extended to apply to three-dimensional Lie transformation groups. It is shown that the existence of a one-dimensional subgroup with closed orbits restricts the Bianchi type of the associated Lie algebra to be I, II, III, VII_{q=0}, VIII or IX. Some results on n-dimensional Lie groups are also derived and applied to show there are severe restrictions on the structure of the allowed four-dimensional Lie transformation groups compatible with cyclic symmetry.

1 Introduction

Following Carter\cite{Carter}, a spacetime $\mathcal{M}$ is said to have cyclical symmetry if and only if the metric is invariant under the effective smooth action $SO(2) \times \mathcal{M} \rightarrow \mathcal{M}$ of the one-parameter cyclic group $SO(2)$. A cyclically symmetric spacetime in which the set of fixed points of this isometry is not empty is said to be axially symmetric and the set of fixed points itself is referred to as the axis (of symmetry). Mars and Senovilla\cite{MarsSenovilla} proved a number of useful results on the structure of the axis. Carot, Senovilla and Vera\cite{CarotSenovilla} considered a definition of cylindrical symmetry based on the following proposition: if in an axial symmetric spacetime there is a second Killing vector which, with the Killing vector generating the axial symmetry, generates a two-dimensional isometry group then the two Killing vectors commute and the isometry group is Abelian. A similar result for stationary axisymmetric spacetimes was proved by Carter\cite{Carter}.

The proofs of all the above mentioned results rely heavily on the existence of an axis and although the assumption of the existence of an axis is reasonable in many circumstances, there are numerous situations where an axis in a cyclically symmetric manifold may not exist. The ‘axis’ may be singular due to line sources and so not part of the manifold proper or the topology of the manifold may be such that no axis exists as is the case for the standard two-dimensional torus embedded in three-dimensional Euclidean space. In the next section the
condition for the existence of an axis will be discarded and the following result will be proved: any two-dimensional Lie transformation group which acts on an \( n \)-dimensional manifold \( \mathcal{M} \) and which contains a one-dimensional subgroup acting cyclically on \( \mathcal{M} \) must be Abelian. In subsequent sections three and higher dimensional Lie transformation groups will be considered and their structure shown to be severely restricted by the existence of a one-dimensional subgroup with circular orbits.

2 Cyclically Symmetric Manifolds Admitting a \( G_2 \)

Suppose \( X_0 \) is the Killing vector field associated with the cyclic isometry acting on \( \mathcal{M} \) and let \( \mathcal{N} \) be the open submanifold of \( \mathcal{M} \) on which \( X_0 \neq 0 \). The orbit of each point of \( \mathcal{N} \) under the cyclic symmetry is a circle. Let \( \phi \) be a circular coordinate running from 0 to 2\( \pi \) which parameterises elements of \( SO(2) \) in the normal way. Then we can introduce a coordinate system \( x^i \) with \( i = 1 \ldots n \) and \( x^1 = \phi \) adapted to \( X_0 \) such that \( X_0 = \partial_\phi \).

Suppose that the isometry group of \( \mathcal{M} \) admits a two-dimensional subgroup \( G_2 \) containing the cyclic symmetry and let \( X_0 \) and \( X_1 \) be a basis of the Lie algebra of \( G_2 \). In an adapted coordinate system the commutator relation of the Lie algebra of \( G_2 \),

\[
[X_0, X_1] = aX_0 + bX_2
\]

where \( a \) and \( b \) are constants, reduces to

\[
\frac{\partial X^\mu_1}{\partial \phi} = bX^\mu_1 \quad \frac{\partial X^1_1}{\partial \phi} = a + bX^1_1
\]

where Greek indices range over the values 2 \ldots n. An elementary integration gives

\[
X^\mu_1 = B^\mu(x^\nu)e^{b\phi} \quad X^1_1 = A(x^\nu)e^{b\phi} + a/b \quad \text{for } b \neq 0
\]

\[
X^\mu_1 = B^\mu(x^\nu) \quad X^1_1 = A(x^\nu) + a\phi \quad \text{for } b = 0
\]

where \( A \) and \( B^\mu \) are arbitrary functions of integration. If \( X_1 \) is to be single-valued on \( \mathcal{N} \), then these solutions must be periodic in \( \phi \) with period 2\( \pi \). This can only occur if \( a = b = 0 \) and so from Eq. (1) \( G_2 \) must be Abelian.

The dimensionality of the manifold, the existence of a metric and the fact that the transformation group is an isometry group are not used in the proof. Hence we have shown that any two-dimensional Lie transformation group which acts on an \( n \)-dimensional manifold \( \mathcal{M} \) and which contains a one-dimensional subgroup with circular orbits, must be Abelian. Thus \textit{a fortiori} the result holds when the \( G_2 \) is group of motions, conformal motions, affine collineations or projective collineations. This remarkably simple and general
result has appeared in the literature previously (for example Bičák & Schmidt),
but is perhaps not widely known.

3 Cyclically Symmetric Manifolds Admitting a $G_3$

Suppose that $\mathcal{M}$ admits a three-dimensional Lie transformation group $G_3$ containing
a one-dimensional subgroup acting cyclically on $\mathcal{M}$ generated by the vector field $X_0$. Let $X_1$ and $X_2$ be vector fields on $\mathcal{M}$ which, with $X_0$, form
a basis of the Lie algebra of $G_3$. Now either this Lie algebra admits a two-dimensional subalgebra containing $X_0$ or there is no such subalgebra.

In the former case this subalgebra is Abelian by the result proved in the
previous section. We may assume, without loss of generality, that $X_0$ and $X_1$
form a basis of this subalgebra and consequently the commutation relations
involving $X_0$ can be written in the form

$$[X_0, X_1] = 0 \quad [X_0, X_2] = aX_0 + bX_1 + cX_2$$

where $a, b$ and $c$ are constants. In a coordinate system adapted to $X_0$ in which
$X_0 = \partial_\phi$, these equations become

$$\frac{\partial X_1}{\partial \phi} = 0 \quad \frac{\partial X_2}{\partial \phi} = a\delta^i_1 + bX_1^i + cX_2^i$$

On integrating these equations and using the fact that $X_1$ and $X_2$ must be
periodic in $\phi$ with period $2\pi$, we may deduce that $a = b = c = 0$. Thus $X_0$
commutes with both $X_1$ and $X_2$. The remaining basis freedom preserving $X_0$
is

$$\tilde{X}_1 = \alpha X_1 + \beta X_2 + \lambda X_0 \quad \tilde{X}_2 = \gamma X_1 + \delta X_2 + \mu X_0$$

subject to the condition $\alpha \delta - \beta \gamma \neq 0$. Using this basis freedom we may reduce
the commutators of the Lie algebra of $G_3$ to one of the following forms

$$[X_0, X_1] = 0 \quad [X_0, X_2] = 0 \quad [X_1, X_2] = 0 \quad \text{Bianchi type I}$$
$$[X_0, X_1] = 0 \quad [X_0, X_2] = 0 \quad [X_1, X_2] = X_0 \quad \text{Bianchi type II}$$
$$[X_0, X_1] = 0 \quad [X_0, X_2] = 0 \quad [X_1, X_2] = X_2 \quad \text{Bianchi type III}$$

These are the canonical forms for the commutators of Bianchi types I, II and
III algebras given by Petrov (apart from renumbering of the basis vectors for
type III).

If the Lie algebra of $G_3$ has no two-dimensional subalgebra containing $X_0$,
we may always choose basis vectors $X_1$ and $X_2$ such that the commutation
relations involving $X_0$ become

$$[X_0, X_1] = X_2 \quad [X_0, X_2] = aX_0 + bX_1 + cX_2$$
where \(a, b\) and \(c\) are constants. In terms of a coordinate system adapted to \(X_0 = \partial_\phi\) these become

\[
\frac{\partial X_1^i}{\partial \phi} = X_2^i \quad \frac{\partial X_2^i}{\partial \phi} = a \delta_1^i + b X_1^i + c X_2^i
\]

A straightforward integration of these equations reveals that, for solutions periodic in \(\phi\) with period \(2\pi\), we must have \(c = 0\) and \(b = -n^2\) for some positive integer \(n\). Then by a redefinition of the basis vector \(\tilde{X}_1 = n X_1 - a/n X_0\), we can set \(a = 0\). Hence the commutation relations become

\[
\left[ X_0, X_1 \right] = n X_2 \quad \left[ X_0, X_2 \right] = -n X_1 \quad \left[ X_1, X_2 \right] = d X_0 + e X_1 + f X_2
\]

where \(d, e\) and \(f\) are constants. The Jacobi identity

\[
\left[ X_0, \left[ X_1, X_2 \right] \right] + \left[ X_1, \left[ X_2, X_0 \right] \right] + \left[ X_2, \left[ X_0, X_1 \right] \right] = 0
\]

implies that

\[
\left[ X_0, \left( d X_0 + e X_1 + f X_2 \right) \right] = n \left( e X_2 - f X_1 \right) = 0
\]

Thus \(e = f = 0\). Three algebraic distinct types arise: namely Bianchi types VII\(_q=0\), VIII or IX depending on whether \(d = 0, < 0, > 0\) respectively. The commutation relations may be written in one of the following forms

- \(d = 0\)
  - \(\left[ X_0, X_1 \right] = n X_2 \quad \left[ X_0, X_2 \right] = -n X_1 \quad \left[ X_1, X_2 \right] = 0 \quad \text{Bianchi type VII}_{q=0}\)
- \(d < 0\)
  - \(\left[ X_0, X_1 \right] = n X_2 \quad \left[ X_0, X_2 \right] = -n X_1 \quad \left[ X_1, X_2 \right] = -X_0 \quad \text{Bianchi type VIII}\)
- \(d > 0\)
  - \(\left[ X_0, X_1 \right] = n X_2 \quad \left[ X_0, X_2 \right] = -n X_1 \quad \left[ X_1, X_2 \right] = X_0 \quad \text{Bianchi type IX}\)

where, in the last two types we have set \(d = \mp 1\) by the rescaling

\[
\tilde{X}_1 = 1/\sqrt{|d|} X_1 \quad \tilde{X}_2 = 1/\sqrt{|d|} X_2
\]

These commutators are closely related to the canonical forms of Bianchi types VII\(_q=0\), VIII and IX given by Petrov. To get the canonical forms we would need to scale \(X_0\) to set \(n = 1\). However this cannot be done whilst preserving both the equation \(X_0 = \partial_\phi\) and the \(2\pi\) periodicity of the coordinate \(\phi\).

Only six of the nine Bianchi types can occur; Bianchi types IV, V and VI are excluded. Note also that canonical forms of the algebras of Bianchi types VI and VII depend on an arbitrary real parameter \(q\) and so each contain an infinite number of algebraically distinct cases; those in type VI are excluded completely and of those in type VII only a single case, \(q = 0\), can occur. Moreover in all the Bianchi types that are permitted, the cyclic vector \(X_0\) is aligned with a vector of a basis in which the commutation relations take their canonical form.
4 Cyclically symmetric manifolds admitting a $G_{m+1}$

Suppose now that $\mathcal{M}$ admits an $(m+1)$-dimensional Lie transformation group $G_{m+1}$ containing a one-dimensional subgroup acting cyclically on $\mathcal{M}$ generated by the vector field $X_0$. Let $X_a$ be vector fields on $\mathcal{M}$ which, with $X_0$, form a basis of the Lie algebra of $G_{m+1}$. Here and below indices $a$, $b$ and $c$ take values in the range $1 \ldots m$. The commutators involving $X_0$ may be written in the form

$$[X_0, X_a] = C^b_a X_b + D_a X_0$$  \hspace{1cm} (2)

where $C^b_a$ and $D_a$ are constants.

If we introduce new basis vectors $\tilde{X}_a$ given by $\tilde{X}_a = P^b_a X_b$, the structure constants transform as follows

$$\tilde{C} = PCP^{-1} \hspace{1cm} \tilde{D} = PD$$

where for simplicity we have used standard matrix notation. Using these transformations we can reduce $C$ to Jordan normal form. In what follows we will work in a basis in which the structure constants $C^b_a$ are in Jordan normal form but, for typographic simplicity tildes will be omitted.

In terms of a coordinate system adapted to $X_0$ in which $X_0 = \partial_\phi$ the commutation relations in Eq. (2) become

$$\frac{\partial X^\mu_a}{\partial \phi} = C^b_a X^\mu_b \hspace{1cm} \frac{\partial X^1_a}{\partial \phi} = C^b_a X^1_b + D_a$$ \hspace{1cm} (3)

where Greek indices range over $2 \ldots n$. If the solutions of these equations are to be periodic in $\phi$ with period $2\pi$, then the eigenvalues $\lambda$ of $C$ must either be zero or of the form $\lambda = \pm in$ where $n$ is a positive integer. Moreover all of the Jordan blocks must be simple or equivalently the minimal polynomial of $C$ must have no repeated factors.

Suppose without loss of generality that $C$ has $p$ ($0 \leq 2p \leq m$) eigenvalues of the form $in_j$ ($n_j$ positive integers and $1 \leq j \leq p$) with corresponding complex eigenvectors $Z_j = X_{2j} + iX_{2j-1}$ plus $m - 2p$ zero eigenvalues with corresponding real eigenvectors $X_k$ ($2p + 1 \leq k \leq m$). Choosing these $m$ (real) vectors $X_a$ as the basis vectors, the commutators become

$$[X_0, X_{2j-1}] = n_j X_{2j} \hspace{1cm} \text{for } 1 \leq j \leq p$$
$$[X_0, X_{2j}] = -n_j X_{2j-1} \hspace{1cm} \text{and } 0 \leq 2p \leq m$$
$$[X_0, X_k] = 0 \hspace{1cm} \text{for } 2p + 1 \leq k \leq m$$ \hspace{1cm} (4)

In the above commutators the structure constants $D_a$ that appeared in Eq. (3) have been set to zero. This is valid since, for $2p + 1 \leq k \leq m$, the vanishing of
$D_k$ is a consequence of the periodicity of the solution of the second of Eqs. (1) which reduces to
\[ \frac{\partial X_k^1}{\partial \phi} = D_k \]
For $1 \leq j \leq p$, $D_{2j-1}$ and $D_{2j}$ can be set to zero by a transformation of the basis vectors of the form
\[ \tilde{X}_{2j-1} = X_{2j-1} - D_{2j}/n_j X_0 \quad \tilde{X}_{2j} = X_{2j} + D_{2j-1}/n_j X_0 \]
Thus even in the general case the existence of a one-dimensional group acting cyclically on the $M$ imposes quite strong restrictions on the allowed form of the commutation relations involving $X_0$. Also the results of sections 2 and 3 can be seen to be special cases of the general result just proved.

For the $G_4$ case ($m = 3$) two classes of algebra arise with commutators
\[
\begin{align*}
[X_0, X_1] &= 0 \quad [X_0, X_2] = 0 \quad [X_0, X_3] = 0 \\
[X_0, X_1] &= n X_2 \quad [X_0, X_2] = -n X_1 \quad [X_0, X_3] = 0
\end{align*}
\]
corresponding to the cases $p = 0$ and $p = 1$ in Eq. (4) respectively.

The Jacobi identities further restrict the structure constants appearing in the remaining commutators. In fact it is possible to enumerate completely the four-dimensional algebras of $G_4$ groups compatible with cyclic symmetry and to relate them to the eight types listed by Kruchkovchich and Petrov in their complete classification of all four-dimensional Lie algebras. Examples of all eight of the types can occur, but in many cases only a zero-parameter or one-parameter subset of a two-parameter Kruchkovchich-Petrov type is permitted and some subclasses of the Kruchkovchich-Petrov types are excluded completely. Furthermore the vector $X_0$ generating the cyclic subgroup is always nicely aligned with the canonical bases used by Kruchkovchich and Petrov. A complete account of the results for $G_4$ will appear elsewhere.

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