A note on homological systems

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Abstract

We give an elementary short proof of the known fact that the category \( \mathfrak{F}(\Delta) \) of \( \Delta \)-filtered modules, associated to a given finite homological system \( (\Delta; \Omega, \leq) \), is closed under direct summands.

Homological systems are generalizations of the standard modules, associated to quasi-hereditary algebras, and so they have similar features (see [2] and [1]).

Homological systems in module categories over pre-ordered sets were introduced in [1]. There they show, see Corollary 3.16, that the associated category of the \( \Delta \)-filtered modules is closed under direct summands: if \( M \) is a \( \Delta \)-filtered module and there is a decomposition \( M = M_1 \oplus M_2 \), then \( M_1 \) and \( M_2 \) are \( \Delta \)-filtered modules.

In this note we use a height map associated to a pre-ordered set defined below, a function determined by the relative position of the elements of the pre-ordered set, to provide a direct proof of that property.

From now on, \( \Lambda \) is an Artin \( R \)-algebra, where \( R \) is an artinian commutative ring, \( \Lambda \)-mod is the full subcategory of the finitely generated left \( \Lambda \)-modules, and all modules and homomorphisms considered in this note belong to \( \Lambda \)-mod.

Definition 0.1. A homological system \( (\Delta; \Omega, \leq) \) consists of the following:

HS1: A finite pre-ordered set \( (\Omega, \leq) : \Omega \) is a finite not empty set and \( \leq \) is a reflexive and transitive relation on \( \Omega \).

HS2: The set \( \Delta = \{ \Delta_\omega \}_{\omega \in \Omega} \), where \( \Delta_\omega \) is indecomposable in \( \Lambda \)-mod and \( \omega \neq \omega' \) implies \( \Delta_\omega \not\cong \Delta_{\omega'} \).

HS3: If \( \text{Hom}_\Lambda (\Delta_\omega, \Delta_{\omega'}) \neq 0 \) then \( \omega \leq \omega' \).

HS4: If \( \text{Ext}_\Lambda (\Delta_\omega, \Delta_{\omega'}) \neq 0 \) then \( \omega \leq \omega' \) and it is not true that \( \omega' \leq \omega \).

Definition 0.2. Given the set \( \Delta = \{ \Delta_\omega \}_{\omega \in \Omega} \), as in HS2, denote by \( \mathfrak{F}(\Delta) \) the full subcategory of \( \Lambda \)-mod of those \( M \) having a \( \Delta \)-filtration, that is a filtration of the form

\[ \{0\} = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M \]

such that \( M_i/M_{i-1} \cong \Delta_\omega \), for some \( \omega \) that depends of \( i \), and for each \( i \in \{1, 2, \ldots, t\} \). So \( \mathfrak{F}(\Delta) \) is closed under isomorphisms, extensions and it contains zero objects. The modules in \( \mathfrak{F}(\Delta) \) are called \( \Delta \)-filtered modules.

Given a pre-ordered set \( (\Omega, \leq) \), the relation \( \omega \sim \omega' \) if and only if \( \omega \leq \omega' \) and \( \omega' \leq \omega \) is an equivalence relation, and so there is a canonical surjective order-preserving function \( \pi : \Omega \rightarrow \Omega/\sim \), where \( (\Omega/\sim, \leq) \) is a poset and \( \leq \) is the order induced by \( \leq \).

Definition 0.3. Let \( (\Omega', \leq) \) be a finite poset. The height function \( h' : \Omega' \rightarrow \mathbb{N} \) of the poset is defined recursively: the elements of height 1 are the minimal elements of the poset, and the elements of height \( n + 1 \) are the minimal elements of the induced poset on \( \Omega - \{ \omega \in \Omega \mid h'(\omega) \in \{1, 2, \ldots, n\} \} \). Given a finite pre-ordered set \( (\Omega, \leq) \), the composition \( h : \Omega \xrightarrow{\pi} \Omega'/\sim \xrightarrow{h'} \mathbb{N} \) is the height function of the pre-ordered set.

Remark 0.4. Given a finite pre-ordered set \( (\Omega, \leq) \) and its height function \( h \), if \( \omega \leq \omega' \) and \( \omega' \not\leq \omega \) then \( h(\omega) < h(\omega') \), and if \( h(\omega) < h(\omega') \) then \( \omega' \not\leq \omega \).

So we get, for \( (\Delta; \Omega, \leq) \) a homological system, that if \( h(\omega) > h(\omega') \) then \( \text{Hom}_\Lambda (\Delta_\omega, \Delta_{\omega'}) = 0 \), and if \( h(\omega) \geq h(\omega') \) then \( \text{Ext}_\Lambda (\Delta_\omega, \Delta_{\omega'}) = 0 \).

Definition 0.5. Let \( F = \{M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t\} \) be a \( \Delta \)-filtration of \( M \in \mathfrak{F}(\Delta) \). We will denote by \( \ell(F) = t \) the length of \( F \), and by \( \ell_\omega(F) \) the number of factors isomorphic to \( \Delta_\omega \), and so \( \ell(F) = \sum_{\omega \in \Omega} \ell_\omega(F) \).
Proposition 0.6. Let $\Delta; \Omega, \leq$ be a homological system, $M \in \mathfrak{F}(\Delta) - \{0\}$, and $F = \{M_0 \subseteq M_1 \subseteq \cdots \subseteq M_t\}$ be a $\Delta$–filtration. Then there exists a $\Delta$–filtration $F' = \{M'_0 \subseteq M'_1 \subseteq \cdots \subseteq M'_t\}$ of $M$ with $\ell_\omega (F) = \ell_\omega (F')$, for each $\omega \in \Omega$, and such that: $1 \leq i \leq j \leq t$, $M'_i / M'_{i-1} \cong \Delta_\omega$ and $M'_j / M'_{j-1} \cong \Delta_{\omega'}$, imply $h(\omega) \geq h(\omega')$.

**Proof:** With the notation of the statement, but starting with the filtration $F$, let us assume that for a fixed $i$ and $j = i + 1$ we have $h(\omega) < h(\omega')$. Then, from HS4, we have that the exact sequence

$$0 \longrightarrow M_i / M_{i-1} \longrightarrow M_{i+1} / M_{i-1} \longrightarrow M_{i+1} / M_i \longrightarrow 0$$

splits, so there exists $N$ submodule of $M_{i+1} / M_{i-1}$ such that $N \cong \Delta_\omega$ and $(M_{i+1} / M_{i-1}) / N \cong \Delta_\omega$.

From the third theorem of isomorphism we get for $N$, the pre-image of $N$ under the canonical epimorphism $p : M_{i+1} \rightarrow M_{i+1} / M_{i-1}$, that $F_1 = \{M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{i-1} \subseteq N \subseteq M_{i+1} \subseteq \cdots \subseteq M_t\}$ is a $\Delta$–filtration with $\ell_\omega (F) = \ell_\omega (F_1)$ for each $\omega$, satisfying the order condition in positions $i$ and $i + 1$.

We can repeat this process generating new $\Delta$–filtrations, and in a finite number of steps to obtain one as in the statement. \hfill \Box

For the proof of the following proposition, let us recall that given $L, N \in \Lambda\mathcal{M}$, the *trace* $\text{tr}_L (N)$ of $L$ on $N$ is the sum of all the images of the homomorphisms from $L$ to $N$.

**Proposition 0.7.** Let $\Delta; \Omega, \leq$ be a homological system and $M \in \mathfrak{F}(\Delta)$. Let $\{1, 2, \ldots, a\}$ be the image of the height function $h$. There exists a filtration $\{0\} = W_{a+1} \subseteq W_a \subseteq \cdots \subseteq W_1 = M$ such that

$$W_i / W_{i+1} \cong \bigoplus_{\omega \in \Omega} n_\omega \Delta_\omega$$

with $n_\omega \in \{0\} \cup \mathbb{N}$. Let us call a such filtration an $h$–filtration. Also, the numbers $n'_\omega$ of any other $h$–filtration are the same.

**Proof:** The existence of at least one $h$–filtration follows directly from the Proposition 0.6 and HS4.

Now assume that $\{0\} = W'_{a+1} \subseteq W'_a \subseteq \cdots \subseteq W'_1 = M$ is another $h$–filtration. By HS3 and the definition of trace we have

$$W_a = \text{tr}_{W_a} (W_a) = \text{tr}_{W_a} (M) = \text{tr}_{W_a} (W'_a) = W'_a$$

and so, by the Krull-Schmidt-Remak Theorem, it follows that $n_\omega = n'_\omega$ for any $\omega \in \Omega$ of height $a$.

We repeat the argument in the quotients $M / W_a$, $M / W_{a-1}$, ..., $M / W_2$ in order to verify the statement. \hfill \Box

**Corollary 0.8.** Let $\Delta; \Omega, \leq$ be a homological system.

1. Let $M \in \mathfrak{F}(\Delta)$ and $F$ and $F'$ be $\Delta$–filtrations of $M$. Then $\ell_\omega (F) = \ell_\omega (F')$ for each $\omega \in \Omega$. It follows that the $\Delta$–length and the number of $\Delta$–factors are well defined for $M$.

2. Let $L, N \in \mathfrak{F}(\Delta)$ and $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence. Then $\ell_\omega (M) = \ell_\omega (L) + \ell_\omega (N)$ for each $\omega \in \Omega$.

3. $\mathfrak{F}(\Delta)$ is closed under direct summands.

**Proof:**

1.- If $F$ is a $\Delta$–filtration of $M$, and $F'$ is an $h$–filtration obtained from $F$, as in the proofs of the propositions 0.6 and 0.7, then $\ell_\omega (F) = n_\omega$ for each $\omega$, where $n_\omega$ are the coefficients associated to $F$ as in 0.7. Then the claim follows by the uniqueness of those coefficients.

2.- It is a direct consequence of the previous item.

3.- Consider an $h$–filtration of $M$ and assume that $M = M_1 \oplus M_2$. 

\hfill \Box
Then, by additivity of the trace, we have $W_a = \text{tr}_{W_a}(M) = \text{tr}_{W_a}(M_1) \oplus \text{tr}_{W_a}(M_2)$, and so $\text{tr}_{W_a}(M_j)$ is a submodule of $M_j$ which is in $\mathcal{F}(\Delta)$, for $j \in \{1, 2\}$.

We can repeat this argument for the quotients $M/W_a \cong (M_1/W_a(M_1)) \oplus (M_2/W_a(M_2))$ in order to get bigger submodules of $M_1$ and $M_2$ that are in $\mathcal{F}(\Delta)$.

Repeating the procedure we obtain $h$–filtrations of $M_1$ and $M_2$, so they belong to $\mathcal{F}(\Delta)$. □

References

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