ON SUBCONTINUA AND CONTINUOUS IMAGES OF $\beta\mathbb{R}\setminus\mathbb{R}$

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Dedicated with great fondness to the memory of Mary Ellen Rudin

Abstract. We prove that the Čech-Stone remainder of the real line has a family of $2^\mathfrak{c}$ mutually non-homeomorphic subcontinua.
We also exhibit a consistent example of a first-countable continuum that is not a continuous image of $H^*$.

Introduction

This paper contains two disparate results on $H^*$, the Čech-Stone remainder of the half line $H = [0, \infty)$.

We prove that $H^*$ has a family of $2^\mathfrak{c}$ many mutually non-homeomorphic subcontinua. This completes the proof of this fact begun in [4]: in that paper the first-named author showed that that $\neg\text{CH}$, the negation of the Continuum Hypothesis, implies that such a family exists, consisting of decomposable continua.

We prove that $\text{CH}$ also implies the existence of a family of $2^\mathfrak{c}$ many mutually nonhomeomorphic subcontinua as well: in fact, we construct, in one fell swoop, two families: one consisting of indecomposable, the other of decomposable continua.

This suggests the obvious question whether one construct from ZFC, or even $\text{ZFC} + \neg\text{CH}$, a family of $2^\mathfrak{c}$ many mutually non-homeomorphic indecomposable subcontinua of $H^*$.

Our second result concerns continuous images of $H^*$. There are various parallels between $H^*$ and $\omega^*$ as regards their continuous images. Some of these can be found in [7]: every continuum of weight $\aleph_1$ or less is a continuous image of $H^*$ and the Continuum Hypothesis implies that the continuous images of $H^*$ are exactly the continua of weight $\mathfrak{c}$ or less (parallel to Parovićenko’s results from [12] on continuous images of $\omega^*$). That not all results carry over was shown in [8]: there is a continuum that is a continuous image of $\omega^*$ (it is even separable) that is consistently not a continuous image of $H^*$. Also, the Open Colouring Axiom implies that $H^*$ itself is not a continuous image of $\omega^*$, see [6].

We present another parallel, this one of Bell’s result from [3] that, consistently, not every first-countable compact space is a continuous image of $\omega^*$. We give a consistent example of a first-countable continuum that is neither a continuous image of $\omega^*$ nor one of $H^*$. The interest in such examples stems from Arhangel’skii’s theorem in [1] that compact first-countable spaces have cardinality and hence weight at most $\mathfrak{c}$ and thus are continuous images of $\omega^*$ if one assumes $\text{CH}$.

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In this section we collect the necessary results on the subcontinua of $H^*$ that we shall need. We refer to [10] for the necessary proofs and further information.

1. Preliminaries

1.1. An auxiliary space. A useful space to have is the product $\omega \times I$, which we denote by $M$. Its Čech-Stone compactification, $\beta M$, and its remainder, $M^*$, are very useful in the study of $\beta H$ and $H^*$ because there are many continuous maps from both onto their respective counterparts.

The natural projection $\pi : M \rightarrow \omega$ extends to a surjection $\beta \pi : \beta M \rightarrow \beta \omega$; because $\pi$ is monotone the extension $\beta \pi$ is monotone as well. For $u \in \beta \omega$ we denote the preimage $\beta \pi^{-1}(u)$ by $I_u$. For $n \in \omega$ we simply have $I_n = \{n\} \times I$ but if $u \in \omega^*$ then $I_u$ is a continuum that has a few properties that make it resemble $I$ somewhat.

It has two end points, $0_u$ and $1_u$; these are obtained by intersecting $I_u$ with the closures of $\omega \times \{0\}$ and $\omega \times \{1\}$ respectively. The continuum $I_u$ is irreducible between these end points and thus it is divided into layers by the following quasi-order: $x \lessdot y$ if every subcontinuum of $I_u$ that contains $0_u$ and $y$ also contains $x$. These layers are the equivalence classes under the equivalence relation `$x \equiv y$ and $y \lessdot x$' and they form an upper semicontinuous decomposition of $I_u$ with an ordered continuum as its decomposition space.

Many of these layers are one-point sets, for instance: every sequence $\langle x_n : n \in \omega \rangle$ in $I$ determines a point $x_u$; the unique point of $I_u$ that is in the closure of the set $\{\langle n, x_n \rangle : n \in \omega \}$. Each such point is a cut point and the set of these is dense in $I_u$, and linearly ordered by $\lessdot$. If $\langle x_n : n \in \omega \rangle$ is an increasing sequence in $I_u$ then its `supremum' is a single layer that is non-trivial since it contains the accumulation points of $\langle x_n : n \in \omega \rangle$ and these form a set that is homeomorphic to $\omega^*$, because $H^*$ is an $F$-space. Also, every layer is an indecomposable continuum; this fact will make some verifications in our construction relatively painless.

1.2. Subcontinua of $H^*$. We now describe a general construction of subcontinua of $H^*$. To this end let $\langle [a_n, b_n] : n \in \omega \rangle$ be a sequence of closed intervals in $H$ such that $b_n = a_{n+1}$ for all $n$ and $\lim_{n \to \infty} a_n = \infty$. Take the map $q : M \rightarrow H$ defined by $q(n,t) = a_n + t(b_n - a_n)$ for all $n$ and $t$. This map is almost everywhere one-to-one; the exceptions are at the end points: we always have $q(n,1) = q(n+1,0)$. This behaviour persists when we take $\beta q$; this map is also almost injective, the exceptions are those of $\beta \langle 0_u \rangle = \beta q(1_u)$ for all $u$, where $u+1$ is the image of $u$ under the extension of the shift map $n \mapsto n + 1$.

For every $u \in \omega^*$ the restriction of $\beta q$ to $I_u$ is injective and hence an embedding. We shall denote the image by $[a_u, b_u]$ and refer to such a continuum as a standard subcontinuum of $H^*$.

These continua determine the structure of the other continua completely: every subcontinuum of $H^*$ is both the intersection and the union of families of standard subcontinua.

Some work is needed to establish the following fundamental facts:

**Lemma 1.1** ([10, Theorem 5.8]). Every decomposable subcontinuum of $H^*$ is a non-trivial interval in some standard subcontinuum. □

**Lemma 1.2** ([10, Theorem 5.9]). If $K$ and $L$ are subcontinua of $H^*$ that intersect and if one of these is indecomposable then $K \subseteq L$ or $L \subseteq K$. □

In particular: if a standard subcontinuum $K$ intersects an indecomposable subcontinuum $L$ then either $K \subseteq L$ and $K$ is nowhere dense in $L$, or $L$ is contained in a layer of $K$ and hence nowhere dense in $K$. 

Lemma 1.3 ([10, Theorem 5.10]). If $K$ and $L$ are subcontinua of $\mathbb{H}^*$ such that $K$ is a proper subset of $L$ and $L$ is indecomposable then there is a standard subcontinuum $M$ such that $K \subseteq M \subseteq L$. \hfill $\Box$

2. GETTING THE CONTINUA

In this section we describe a general construction of indecomposable continua in $\mathbb{H}^*$; in the next section we show that we can actually find $2^\omega$ many such continua.

We let $\Gamma$ denote the collection of all sequences $\langle [a_n, b_n] : n \in \omega \rangle$ of closed intervals in $\mathbb{H}$ with integer end points and such that $b_n = a_{n+1}$ for all $n$.

As we have seen above, if $A = \langle [a_n, b_n] : n \in \omega \rangle$ is such a sequence then for every free ultrafilter, $u$, on $\omega$ we obtain the standard subcontinuum $[a_n, b_n]$.

We can also associate another subcontinuum to $A$ and an ultrafilter $u$, as follows. If $q$ is the map from $\mathcal{M}$ to $\mathbb{H}$ associated to $A$ as above then the restriction $\beta q$ of $\mathcal{M}^*$ maps $\mathcal{M}^*$ onto $\mathbb{H}^*$. Therefore there is an ultrafilter $v$ on $\omega$ such that $u \in [a_v, b_v]$; this continuum we shall denote by $A_u$.

Thus each ultrafilter $u$ determines a whole family of continua in $\mathbb{H}^*$, to wit $\mathcal{S}_u = \{ A_u : A \in \Gamma \}$.

We shall find $2^\omega$ many ultrafilters on $\omega$ and for each such ultrafilter $u$ a chain $\mathcal{C}_u$ in $\mathcal{S}_u$. Each chain $\mathcal{C}_u$ gives us an indecomposable continuum, $K_u = \text{cl} \bigcup \mathcal{C}_u$, and our ulterior motive is to have all $K_u$ be mutually non-homeomorphic.

To this end we shall find for each linear order $(T, \prec)$ of cardinality $\aleph_1$ an ultrafilter $u_T$, in fact a $P$-point, such that $T$ embeds in $\mathcal{S}_{u_T}$ in a special way: there will be a family $\{ A_t : t \in T \}$ in $\Gamma$ such that

(1) $t \prec s$ if $A_{u_T}^s$ is contained in a layer of $A_{u_T}^t$.

(2) every $A \in \Gamma$ is equivalent to some $A_t$, in a manner to be specified presently.

These two conditions will ensure that a homeomorphism between $K_{u_T}$ and $K_{u_S}$ will give rise to an isomorphism between final segments of $T$ and $S$. Thus the proof will be finished once we exhibit $2^\varnothing$ many linearly ordered sets without isomorphic final segments.

As mentioned before, the construction proceeds under the assumption of the Continuum Hypothesis.

2.1. Bad triples. The central notion will be that of a bad triple.$^1$

A bad triple has three coordinates:

- a free filter base $\mathcal{F}$ on $\omega$,
- a linear order $(T, \prec)$, and
- a subset $A_T = \{ A^t : t \in T \}$ of $\Gamma$.

These should satisfy the following properties, where, in the interest of readability we write $A(t, n)$ for $[a_n^t, b_n^t]$.

(1) if $s \prec t$ in $T$ then there is $F \in \mathcal{F}$ such that for every $k$ there is an $l$ for which $A(s, k) \cap F \subseteq A(t, l)$

(2) for every decreasing sequence $\{ t_i : i < \ell \}$ in $T$, for every $m \in \omega$ and every $F \in \mathcal{F}$ there is a function $\varphi : \leq^i m \to \omega$ such that

(a) if $\rho \in \leq i m$ then $\varphi(\rho) \in F$;

(b) if $\rho \in \leq^i m$ then $i \mapsto \varphi(\rho \upharpoonright i)$ is increasing

(c) if $k < \ell$ and $\rho \in \leq^i m$ then $A(k+1, \varphi(\rho \upharpoonright i)) \subseteq A(t_k, \varphi(\rho))$ for all $i < m$.

If $\mathcal{F}$ is an ultrafilter then property (1) translates into $A_{\mathcal{F}}^s \subseteq A_{\mathcal{F}}^t$ and property (2) implies that the inclusion is as described above: the (possibly partial) function $\psi$ that satisfies $\psi(k) = l$ iff $A(s, k) \subseteq A(t, l)$ is finite to one, but its fibers have

$^1$The word ‘good’ seems overused and, especially in the vernacular, ‘bad’ may carry a positive connotation.
unbounded cardinality, even when restricted to an arbitrary element of \( F \) and this implies that \( A_\omega \) is a subset of a layer of \( A_\omega \).

Condition (2) will also be seen to keep our recursive constructions alive. To be able to keep our formulations readable we shall say that the function \( \varphi \) in this condition is \( m \)-dense for \( F \) and \( \langle t_i : i < l \rangle \), or for \( F \) and \( \{ t_i : i < l \} \) (set rather than sequence). We will abbreviate \( \{ \varphi(p) : p \in {}^i m \} \) as \( \text{Im } \varphi \) and refer to it as the image of \( \varphi \).

The following is a sketch of the construction. Let \( \langle T, \prec \rangle \) be a linear order of cardinality \( \aleph_1 \) and let \( \langle t_\alpha : \alpha \in \omega_1 \rangle \) be an enumeration of \( T \). By transfinite recursion we construct a sequence \( \langle F_\alpha : \alpha \in \omega_1 \rangle \) of infinite subsets of \( \omega \) and a map \( t \mapsto A^t \) from \( T \) to \( \Gamma \) such that

1. \( F_\beta \subseteq^* F_\alpha \) whenever \( \alpha < \beta \)
2. \( \langle F_\alpha, T_\alpha, A_\alpha \rangle \) is a bad triple, where \( F_\alpha = \{ F_\beta : \beta < \alpha \} \), \( T_\alpha = \{ t_\beta : \beta < \alpha \} \), and \( A_\alpha = \{ A^t : \beta < \alpha \} \)
3. \( \{ F_\alpha : \alpha \in \omega_1 \} \) generates an ultrafilter on \( \omega \) that by (1) will be a \( P \)-point.

For technical reasons we add a minimum and a maximum to \( T \), if not already present.

We will formulate and prove a series of lemmas about bad triples that will facilitate such a construction; the standing assumptions in the lemmas will be

1. \( F \) and \( T \) are countable, and \( F \) extends the cofinite filter,
2. \( T \) has a minimum and a maximum, denoted 0 and 1 respectively, and
3. \( \langle [a_n, b_n] : n \in \omega \rangle = \langle [n, n+1] : n \in \omega \rangle \).

To begin we show that at any time during our construction we can assume that \( F \) is a principal filter, or rather, the restriction of the cofinite filter to a single set; in the next lemma \( \text{cof}_G \) denotes the filter \( \{ F : G \subseteq^* F \} \).

**Lemma 2.1.** If \( \langle F, T, A_T \rangle \) is a bad triple then there is a single infinite \( G \) such that \( G \subseteq^* F \) for all \( F \in F \) and such that \( \langle \text{cof}_G, T, A_T \rangle \) is a bad triple.

**Proof.** Let \( \langle T_n : n \in \omega \rangle \) be an increasing sequence of finite sets whose union is \( T \) and let \( \langle F_n : n \in \omega \rangle \) be a decreasing sequence in \( F \) such that for every \( F \in F \) there is an \( n \) such that \( F_n \subseteq F \). Recursively let \( \varphi_m \) be \( m \)-dense for \( F_m \) and \( T_m \) and such that \( \text{Im } \varphi_m \) is disjoint from \( \text{Im } \varphi_i \) for \( i < m \). Then \( G = \bigcup_{m \in \omega} \text{Im } \varphi_m \) is as required.

This lemma is used at limit steps of our construction, basically to make them look like successor steps. For the remainder of the paper we shall write \( \langle G, T, A_T \rangle \) for \( \langle \text{cof}_G, T, A_T \rangle \).

At some steps in the construction the following technical fact will be useful.

**Lemma 2.2.** A triple \( \langle F, T, A_T \rangle \) satisfies (2) in the definition of a bad triple if and only if for every (some) increasing sequence \( \langle m_n : n \in \omega \rangle \) in \( \omega \) and every (some) increasing sequence \( \langle T_n : n \in \omega \rangle \) finite subsets of \( T \) such that \( T = \bigcup_{n \in \omega} T_n \) there is a sequence \( \langle \varphi_n : n \in \omega \rangle \) of functions such that \( \varphi_n \) is \( m_n \)-dense for \( F \) and \( T_n \), and max \( \text{Im } \varphi_n \) < min \( \text{Im } \varphi_{n+1} \) for all \( n \).

**Proof.** For the non-trivial implication we find the functions \( \varphi_n \) by recursion: \( \varphi_0 \) exists by assumption and if \( \varphi_n \) is found then we let \( M = \max \text{Im } \varphi_n \) and we choose a function \( \varphi \) that is \( M + m_{n+1} + 1 \)-dense for \( F \) and \( T_{n+1} \). By condition (2b) in the definition of a bad triple we have \( \varphi(M + 1 + \rho) > M \) whenever \( \rho \in {}^i m_{n+1} \) for some \( i \leq \lceil T_{n+1} \rceil \) (here \( M + 1 + \rho \) denotes the sequence obtained by adding \( M + 1 \) to all values of \( \rho \)). Thus defining \( \varphi_{n+1}(\rho) = \varphi(M + 1 + \rho) \) gives us our next function.

The next lemma ensures that we can make our final filter an ultrafilter.
Lemma 2.3. Let \( \langle F, T, A_T \rangle \) be a bad triple and assume \( F = F_0 \cup F_1 \); then at least one of \( \langle F_0, T, A_T \rangle \) and \( \langle F_1, T, A_T \rangle \) is a bad triple.

Proof. We show by induction on \( l \): if \( (t_i : i < l) \) is decreasing and \( \varphi \) is \( 2m \)-dense for \( F \) and \( (t_i : i < l) \) then \( \varphi \) induces an \( m \)-dense function for \( F_0 \) or \( F_1 \) and \( (t_i : i < l) \).

If \( l = 1 \) then \( \text{Im } \varphi \) is just a \( 2m \)-element subset of \( F \) and its intersection with one of \( F_0 \) and \( F_1 \) has at least \( m \) elements; the increasing enumeration of that intersection is \( m \)-dense.

In the step from \( l \) to \( l + 1 \) we let \( (t_i : i < l) \) and a \( 2m \)-dense \( \varphi \) be given. For each \( j < 2m \) the function \( \varphi_j : \leq 2m \rightarrow \omega \), defined by \( \varphi_j(\rho) = \varphi(j - \rho) \), is \( 2m \)-dense for \( F \) and \( (t_i : 1 \leq i \leq l) \) and so induces an \( m \)-dense function \( \varphi'_j \) for \( F_{t_j} \) and \( (t_i : 1 \leq i \leq l) \), where \( j \in \{0, 1\} \). Take \( \epsilon \) such that \( A = \{ j : \epsilon_j = \epsilon \} \) has size at least \( m \) and define \( \varphi : \leq 2m + 1 \rightarrow \omega \) by \( \varphi((j)) \) is the \( j \)-th element of \( \text{dom } A' \) and \( \varphi(j - \rho) = \varphi'_j((j))((\rho)) \) for \( \rho \in 2m \).

Now enumerate \( T \) as \( (t_n : n \in \omega) \) and apply the above for each \( m \) to the pair \( (t_i : i < m) \) and \( m \). Whichever of \( F_0 \) and \( F_1 \) appears infinitely often in the conclusion is the set that we seek.

Now we show how to extend the ordered set \( T \) by one element.

Lemma 2.4. Let \( \langle F, T, A_T \rangle \) be a bad triple and let \( t^* \) be a point not in \( T \). Assume \( T \cup \{ t^* \} \) is ordered so that \( T \) retains its original order and \( 0 < t^* < 1 \). Then there are \( G \subseteq F \) and \( A_{t^*} \subseteq \Gamma \) such that \( \langle G, A_T \cup \{ A_{t^*} \}, T \cup \{ t^* \} \rangle \) is a bad triple.

Proof. We write \( T \) as an increasing union of finite sets \( T_m \), with \( 0, 1 \in T_0 \) and we construct \( G \) and \( A_{t^*} \) as follows. We apply Lemma 2.2 to find a sequence \( \langle \varphi_m : m \in \omega \rangle \) (such that \( \varphi_m \) is \( m \)-dense for \( F \) and \( T_m \), and max \( \text{Im } \varphi_m \leq \min \text{Im } \varphi_{m+1} \) for all \( m \).

We fix \( m \) for the moment and let \( (t_i : i < l) \) enumerate \( T_m \) in decreasing order and let \( i \) be such that \( t_{i+1} \prec t^* \prec t_i \). Our task is to convert \( \varphi_m \) into an \( m \)-dense function, \( \psi_m \), for our future \( G \) and \( T_m \cup \{ t^* \} \). The idea is simple — we use level \( i+1 \) in dom \( \varphi_m \) to create two levels in dom \( \psi_m \) — but the notation is a bit messy: we take the following subset of the domain of \( \varphi_m \):

\[
D = \{ \rho \in \text{dom } \varphi_m : (\forall j \in \text{dom } \rho)(j \neq i \Rightarrow \rho(j) < m) \}
\]

Using the \( m \)-values for all \( \rho \) we transform \( D \) into the tree \( \leq i+1 \):

- if \( \text{dom } \rho \leq i \) then \( \rho \) does not change;
- if \( \text{dom } \rho = i+1 \) then \( \rho = \rho^*\setminus (km+j) \) for some \( \rho^* \in i^m \) and \( k, j < m \); in this case \( \rho \) determines two nodes: \( \rho^+ = \rho^*\setminus k \) and \( \rho^{++} = \rho^*\setminus k\setminus j \);
- if \( i+1 \leq \text{dom } \rho \) then \( \rho = \rho^*\setminus (km+j)^* \) for some \( \rho^* \in i^m \), some \( k, j < m \) and some sequence \( \sigma \); then \( \rho \) determines \( \rho^+y = \rho^*\setminus k \setminus j \setminus \sigma \).

We define \( \psi_m : \leq i+1 \rightarrow \omega \) by

\[
\psi_m(\rho) = \begin{cases} 
\varphi_m(\rho) & \text{if } \text{dom } \rho \leq i \\
\varphi_m(\rho) & \text{if } \rho = \rho^{++} \text{ for some } \omega \in i+1 \\
\varphi_m(\rho) & \text{if } \rho = \rho^+ \text{ for some } \rho \text{ with } \text{dom } \rho > i+1
\end{cases}
\]

This leaves \( \psi_m(\rho) \) undefined in case \( \text{dom } \rho = i+1 \), that is, if \( \rho = \rho^*\setminus k \) for some \( \rho^* \in i^m \) and \( k < m \), and it is here that we build and insert part of \( A_{t^*} \).

In words: for each \( \rho \in i^m \) we bundle the \( m \)-values \( \{ \varphi_{i+1}^{t\setminus(km+j)} \, \rho \in i^m \} \) into groups of \( m \) consecutive ones and among each group we take the smallest interval that surrounds its members.

In symbols: for each \( k < m \) the interval \( [\varphi_{i+1}^{t\setminus(km+j)}] A_{t^*}^{\varphi_{i+1}^{t\setminus((k+1)m-1)}} \) will be a term of \( A_{t^*} \) and its index will be the value of \( \psi_m \) at \( \rho^*\setminus k \).
In the end we set \( G = \bigcup_m \psi_m \) and thus ensure that for every \( m \) the function \( \psi_m \) will be \( m \)-dense for \( G \) and \( T_m \cup \{ t^* \} \).

We now turn to the task of avoiding having to add points to our linear order when we do not want to, that is, we want ensure that we can achieve property (2) (on page 3) of the embedding. It is here that we define the notion of equivalence, promised in that property.

We introduce some notation: let \( F \subseteq \omega \) and let \( A, B \in \Gamma \).

We say that \( A \) refines \( B \) modulo \( F \), and we write \( A \preceq_F B \), if for every term, \([a, b], \) of \( A \) with \([a, b] \cap F \neq \emptyset \) there is a term \([c, d], \) of \( B \) such that \([a, b] \cap F \subseteq [c, d] \).

We say that \( A \) and \( B \) are equivalent modulo \( F \), written \( A \equiv_F B \), if for every \( n \in F \) there are terms \([a, b], \) of \( A \) and \([c, d], \) of \( B \) such that \( n \in [a, b] \cap F \) and \([a, b] \cap F = [c, d] \cap F \).

**Lemma 2.5.** Let \((F, T, A_T)\) be a bad triple, let \( t \in T \) and \( A \in \Gamma \). Then there is \( F_1 \subseteq F \) such that \((F_1, T, A_{T_1})\) is a bad triple and \( A \preceq_{F_1} A' \) or \( A' \preceq_{F_1} A \); in addition if \( t \) has a direct \( \prec\)-predecessor \( s \) then we can even achieve “\( A \preceq_{F_1} A^* \) or \( A^* \preceq_{F_1} A \)”.

**Proof.**
Write \( T \) as the union of an increasing sequence \((T_m : m \in \omega)\) of finite sets such that 0, 1, \( t \in T_0 \) (and also \( s \) if present). Upon applying Lemmas 2.3 and 2.2 we may assume that \( F \) does not meet consecutive intervals of \( A' \), and that we have a sequence \((\varphi_m : m \in \omega)\) of functions such that \( \varphi_m \) is \((m + 1)(m + 2)\)-dense for \( F \) and \( T_m \), and max \( \text{Im} \varphi_m \leq \min \text{Im} \varphi_{m+1} \) for all \( m \). Also assume \( F = \bigcup_{m \in \omega} \text{Im} \varphi_m \).

Enumerate \( T_m \) in decreasing order as \( \{ t^m_n < t^m_m \} \), and for every \( m \) let \( i_m \) be the index of \( t \). Abbreviate \( t^m_{i_m} \) as \( t_m \) and \( t^m_{i_m+1} \) as \( s_m \) (so \( s_m = s \) for all \( m \) if \( s \) is present).

We fix \( m \) for a moment and for every \( \rho \in \iota^m((m + 1)(m + 2)) \) we take a term \([a_{\rho, m}, b_{\rho, m}] \) of \( A \) such that
\[
J^m_{\rho} = \{ j < (m + 1)(m + 2) : A(s_m, \varphi_m(\rho - j)) \subseteq [a_{\rho, m}, b_{\rho, m}] \}
\]
has maximum cardinality. Divide \( \iota^m((m + 1)(m + 2)) \) into two parts: \( R_m = \{ \rho : |J^m_{\rho}| \geq m \} \) and its complement \( S_m \).

The proof of Lemma 2.3 gives us a subfunction \( \phi_m \) of \( \varphi_m \) whose domain is \((m + 1)(m + 2)/2\)-branching and such that \( X_m = \text{dom} \phi_m \cap \iota^m((m + 1)(m + 2)) \) is a subset of \( R_m \) or of \( S_m \).

In case \( X_m \subseteq R_m \) we define a set \( F^m_t \) as follows:
\[
F^m_t = F \cap \bigcup \{ A(s_m, \varphi(\rho - j)) : \rho \in X_m \text{ and } j \in J^m_{\rho} \}
\]
We extend \( \phi_m \) to a subfunction \( \psi_m \) of \( \varphi_m \) by adding
\[
\{ \rho \in \text{dom} \varphi_m : (\exists \sigma \in X_m)(\exists j \in J^m_{\rho})(\sigma - j \subseteq \rho) \}
\]
to its domain and using the values of \( \varphi_m \) at those points. The resulting function is (more than) \( m \)-dense for \( F^m_t \) and \( T_m \). Also, if \( n \in F^m_t \) then there are \( \rho \in X_m \) and \( j \in J^m_{\rho} \) such that \( n \in A(s_m, \varphi(\rho - j)) \subseteq A(t, \varphi(\rho \in t)) \) and, by definition, \( F^m_t \cap A(t, \varphi(\rho)) \subseteq [a_{\rho, m}, b_{\rho, m}] \). This shows that if \( F^m_t \) were to contribute to \( F_t \) it would also witness \( A' \preceq_{F_1} A \).

Thus, if the situation \( X_m \subseteq R_m \) occurs infinitely often then we can build an \( F_t \) such that \( A' \preceq_{F_1} A \).

In the other case we get \( X_m \subseteq S_m \) infinitely (even cofinitely) often. We shall build an \( F_t \) that will satisfy \( A \preceq_{F_1} A' \) and even \( A \preceq_{F_1} A^* \) if \( s \) is present.

Consider an \( m \) such that \( X_m \subseteq S_m \) and fix \( \rho \in X_m \). For each term \([a, b] \) of \( A \) the set \( \{ (s_m, \varphi(\rho - j)) : j \} \subseteq [a, b] \) has at most \( m - 1 \) elements; as \([a, b] \) is an interval these are consecutive elements. This means that \([a, b] \) can intersect
at most \( m + 1 \) of these intervals: at most \( m - 1 \) in the interior and possibly two more that merely overlap at the ends. We use the intervals indexed by \( X_m \) and \( I = \{(m + 2)(j + 1) : j < m\} \) to define \( F^m \):

\[
F^m_i = F \cap \bigcup \{A(s_m, \varphi(\rho - i)) : \rho \in X_m \text{ and } i \in I\}
\]

the same formula as in the case \( X_m \subseteq R_m \) with \( J^m_\rho \) replaced by \( I \). Now if \([a, b]\) is a term of \( A \) and \( n \in F^m_i \cap [a, b] \) then there are one \( \rho \in X_m \) and one \( i \in I \) such that \( n \in A(s_m, \varphi(\rho - i)) \) and the latter is also the only interval of that form that \([a, b]\) intersects. It follows automatically that

\[
F^m_i \cap [a, b] \subseteq A(s_m, \varphi(\rho - i)) \subseteq A(t, \varphi_m(\rho)).
\]

Thus, if we let \( F_t \) be the union of these \( F^m_i \) then we achieve \( A \preceq F_t \) and even \( A \preceq F_t \) if \( s \) is present.

**Lemma 2.6.** Let \( \langle F, T, A_T \rangle \) be a bad triple and \( A \in \Gamma \). Then there are \( G \subseteq F \) and an extension \( T^* \) of \( T \) by at most one point \( t^* \) such that \( \langle G, T^*, A_T \rangle \) is a bad triple and \( A \equiv A^t \) for some \( t \in T^* \).

**Proof.** We apply Lemma 2.5 countably many times and Lemma 2.1 once so that we can assume that for every \( t \in T \) there is a cofinite subset \( F_t \) of \( F \) such that \( A \not\preceq F_t \), \( A^t \) or \( A^t \not\preceq F_t \) and \( A \) is \( \preceq F_t \) (\( \preceq F_t \) and \( A \) are identical, say \( \omega \), \( S \)).

Case 2: \( \psi \) verifies that \( A \preceq A^t \) and even \( A \preceq A^t \) if \( s \) is present.

We need to consider several cases.

Case 1: \( S_0 \) has a maximum and \( S_1 \) has a minimum. Note that by the condition on direct predecessors these must be identical, say \( t = \max(S_0) = \min(S_1) \). Then one verifies that \( A \equiv F_t \), \( A^t \).

Case 2: \( S_1 \) is empty. In this case we have \( A^1 \preceq F_t \) and we can thin out \( F_t \) to a set \( G \) such that \( A^1 \equiv G \); then \( \langle G, T, A_T \rangle \) is a bad triple.

For the other cases we write \( T \) as the union of an increasing sequence \( \langle T_m : m \in \omega \rangle \) of finite sets such that \( 0, 1 \in T_0 \); as before we take the decreasing enumeration \( \langle t_m^1 : 1 < t_m^1 \rangle \) of \( T_m \). For each \( m \) we let \( i_m \) be such that \( t_m^1 = s_{i_m} \) and \( t_m^1 + 1 \in S_0 \); we denote these two points by \( t_m^1 \) and \( s_{i_m} \) respectively.

Furthermore we choose \( \langle \varphi_m : m \in \omega \rangle \) as in Lemma 2.2 so that \( \varphi_m \) is \( m \)-dense for \( F \) and \( T_m \) and such that \( \Im \varphi_m \subseteq F_{s_m} \cap F_{t_m} \).

Fix \( m \) for a moment. We know that \( A^{s_m} \preceq F_{s_m} \preceq F_{t_m} \preceq A^{t_m} ; \) this implies that for every \( \rho \in t^m \) and every \( j < m \) there is a term \( [a, b] \) of \( A \) such that

\[
A(s_m, \varphi(\rho - j)) \cap F \subseteq [a, b] \cap F \subseteq A(t_m, \varphi(\rho)) \cap F
\]

indeed, \([a, b]\) is found by an application of \( A^{s_m} \preceq F_{s_m} \) and \( A(t_m, \varphi(\rho)) \) is the only possible term of \( A^{t_m} \) that can help witness \( A \preceq F_{s_m} A^{t_m} \).

We put \( G_m = F \cap \bigcup \rho A(s_m, \varphi(\rho - 0)) \), where \( \rho \) runs through \( t^m \). We can define two functions \( \phi_m \) and \( \psi_m \) on \( s_{t^m} \), as follows.

1. If \( |\rho| < i_m \) then \( \phi_m(\rho) = \psi_m(\rho) = \varphi_m(\rho) \).
2. If \( |\rho| = i_m \) then \( \phi_m(\rho) = \varphi_m(\rho) \) and \( \psi_m(\rho) = \psi_m(\rho) = \varphi_m(\rho - 0) \).
3. If \( |\rho| > i_m \), say \( \rho = \rho - \sigma \), with \( |\sigma| = i_m \), then \( \phi_m(\rho) = \psi_m(\rho) = \varphi_m(\rho - 0 - \sigma) \).

So, in \( \phi_m \) we skip level \( i_m + 1 \) of the domain of \( \varphi_m \) and in \( \psi_m \) we skip level \( i_m \).

The effect is that \( \phi_m \) is \( m \)-dense for \( T_m \setminus \{s_m\} \) and \( G_m \), whereas \( \psi_m \) is \( m \)-dense for \( T_m \setminus \{t_m\} \) and \( G_m \).

In addition we have made sure that \( A^{s_m} \equiv G_m \preceq G_m A^{t_m} \).

We let \( G = \bigcup_m G_m \) and consider the remaining cases in turn.
Case 3: $S_0$ has no maximum and $S_1$ has a minimum, say $s = \min S_1$. In this case we know that $s_m = s$ cofinitely often. If we drop the finitely many $G_m$ for which $s_m \neq s$ then we achieve $A \equiv_G A'$. Moreover $(G,T,A_T)$ is a bad triple, as witnessed by the functions $\phi_m$.

Case 4: $S_0$ has a maximum and $S_1$ has no minimum, say $t = \max S_0$. In this case we know that $t_m = s$ cofinitely often. If we drop the finitely many $G_m$ for which $s \neq t_m$ then we achieve $A^* \equiv_G A$. Moreover $(G,T,A_T)$ is a bad triple, as witnessed by the functions $\psi_m$.

Case 5: $S_0$ has no maximum and $S_1$ has no minimum. This case necessitates adding a new point, $t^*$, to $T$ and inserting it into the gap formed by $S_0$ and $S_1$ to form $T^*$.

We then redefine $\phi_m$ on level $i_m$ so that its value at $\rho$ becomes the index of the term of $A$ that was chosen to satisfy inclusions ($\ast$). The new $\phi_m$ is $m$-dense for $\{t^*\} \cup T_m \setminus \{s_m,t_m\}$ and $G_m$; this establishes that $\langle G,T^*,A_{T^*} \rangle$ is a bad triple. \hfill \Box

Repeated application of these lemmas will prove the following theorem, where we extend the notion of equivalence to (ultra)filters: if $p$ is an (ultra)filter on $\omega$ then $A \equiv_p B$ means that $A \equiv_F B$ for some $F \in p$.

**Theorem 2.7 (CH).** Let $T$ be a linear order of cardinality at most $\aleph_1$ that has a maximum and no $(\omega,\omega)$-gaps. Then one can find a subcollection $A_T = \{A_t^j : t \in T\}$ of $\Gamma$ and a $P$-point ultrafilter $p$ on $\omega$ such that

1. $\langle p,T,A_T \rangle$ is a bad triple
2. for all $A \in \Gamma$, there is a $t \in T$ such that $A \equiv_p A^t$.

\hfill \Box

### 3. Finding many different continua

In this section we shall use Theorem 2.7 (and hence the Continuum Hypothesis) to find $2^\omega$ many different subcontinua of $\mathbb{H}^*$.

We shall apply the theorem to the following type of linearly ordered sets

1. cardinality at most $\aleph_1$
2. no $(\omega,\omega)$-gaps
3. cofinality $\aleph_0$ (in particular: no maximum)

In keeping with our use of the vernacular we shall call this a mean linear order.

#### 3.1. One continuum.

Let $T$ be a mean linear order. We order $T^+ = T \cup \{T\}$ ordered by stipulating that $t \prec T$ for all $t \in T$. We apply Theorem 2.7 to $T^+$ to obtain a family $A_T = \{A_p^t : t \in T^+\}$ and a $P$-point $p$ satisfying the conditions of that theorem. We define

$$K_T = \text{cl} \bigcup_{t \in T} A_p^t,$$

as announced in the beginning of Section 2.

We list some properties of $K_T$ and the individual continua $A_p^t$.

**Lemma 3.1.** For every $t \neq \min T$ there is a layer $L_p^t$ of $A_p^t$ such that $\bigcup_{s < t} A_p^s \subseteq L_p^t$.

**Proof.** Lemma 6.2 of [10] establishes that $A_p^s$ is contained in a layer of $A_p^t$ whenever $s < t$; because $A_T$ is a chain this layer is independent of $s$. We need the assumption $t \neq \min T$ to ensure that we actually have points below $t$. \hfill \Box

**Lemma 3.2.** Every $A_p^t$ is nowhere dense in $K_T$ and $\bigcup_{t \in T} L_p^t = \bigcup_{t \in T} A_p^t$.

**Proof.** Given $t \in T$ there is $s \in T$ such that $t < s$. Then $A_p^t \subseteq L_s$, which establishes the equality of the two unions.

Because $L_s$ is nowhere dense in $A_p^t$ this also implies that $A_p^t$ is nowhere dense in $K_T$. \hfill \Box
Lemma 3.3. \( K_T \) is indecomposable.

Proof. The proof is implicit in [14] and [10] as part of a construction of an indecomposable subcontinuum of \( \mathbb{H}^* \) called \( K_9 \) in the latter paper.

Let \( L \) be a proper subcontinuum of \( K_T \). Note that because each \( L^*_p \) is indecomposable we know that \( L^*_p \subset L \) or \( L \subset L^*_p \) for all \( t \) such that \( L \) and \( L^*_p \) is nonempty. Since it is impossible that \( L^*_p \subset L \) for all \( t \) (otherwise \( L = K_T \)) it follows that \( L \cap \bigcup_{t \in T} L^*_p = \emptyset \) or \( L \subset L^*_p \) for some \( t \). In either case \( L \) is nowhere dense in \( K_T \). \( \square \)

Lemma 3.4. Every \( A^*_p \) is a P-set in \( \mathbb{H}^* \) as is every \( L^*_p \), for \( t \neq min T \).

Proof. The preimage of \( A^*_p \) under the parametrizing map \( q : M^* \to \mathbb{H}^* \) consists of \( \mathbb{I}_v \), the point \( 1 \in \mathbb{I}_u \) and the point \( 0_{u+1} \), where \( v \) is such that \( A^*_p = [a^*_v, b^*_v] \). This makes the preimage a P-set, as \( \pi \) is closed this implies that \( A^*_p \) is a P-set as well.

It suffices to show that \( L^*_p \) is not a countable cofinality layer in \( A^*_p \) if \( t \neq min T \). If \( L^*_p \) were such a layer then one of the open intervals with \( L^*_p \) as its end layer, call it \( I \), would be a \( F_\sigma \) set such that \( I \cap L^*_p = \emptyset \) and \( L^*_p \subset cl I \). Now let \( s < t \); then \( A^*_p \) is a P-set and \( A^*_p \cap I = \emptyset \). It follows that \( A^*_p \cap cl I = \emptyset \) as well, which contradicts \( L^*_p \subset cl I \). \( \square \)

3.2. Consequences of homeomorphism. Let \( T \) and \( S \) be two mean linear orders.

We assume we have families \( A_T \) and \( A_S \) and P-points \( p \) and \( q \) respectively as in Theorem 2.7. We write \( F_T = \bigcup_{t \in T} A^*_p \) and \( F_S = \bigcup_{s \in S} A^*_p \) and let \( K_T = cl F_T \) and \( K_S = cl F_S \). We retain the notations \( L^*_p \) and \( L^*_q \) respectively for the layers from Lemma 3.1. We assume that \( K_T \) and \( K_S \) are homeomorphic and let \( f : K_T \to K_S \) be a homeomorphism.

Lemma 3.5. \( f[F_T] = F_S \).

Proof. Let \( t \in T \). Because the P-set \( f[A^*_p] \) is in the closure of the \( F_\sigma \)-set \( F_S \) it must actually intersect that set. Thus there is an \( s \in S \) such that \( f[A^*_p] \cap A^*_p \neq \emptyset \) and hence \( f[A^*_p] \cap L^*_q \neq \emptyset \) whenever \( s \prec r \) in \( S \). It follows that \( f[A^*_p] \subset L^*_q \) or \( L^*_q \subset f[A^*_p] \) for all \( r \succ s \) and because \( f[A^*_p] \) is nowhere dense in \( K_S \) we must have \( f[A^*_p] \subset L^*_q \) for a final segment of \( r \) in \( S \).

This shows that \( f[F_T] \subset F_S \) and, using \( f^{-1} \) instead of \( f \), we can also deduce that \( F_S \subset f[F_T] \). Thus we find that \( F_T \) is mapped onto \( F_S \) by \( f \). \( \square \)

Our aim is now to show that \( T \) and \( S \) have isomorphic final segments.

Let \( T' = \{ t \in T : (\exists s \in S)(A^*_p \subset f[L^*_q]) \} \) and, symmetrically, let \( S' = \{ s \in S : (\exists t \in T)(f[A^*_p] \subset L^*_q) \} \). We shall show that \( T' \) and \( S' \) are isomorphic by showing that \( f \) induces an isomorphism between the families \( \{ L^*_q : t \in T' \} \) and \( \{ L^*_q : s \in S' \} \) (ordered by inclusion).

Let \( t \in T' \) and consider \( f[A^*_p] \); this is a decomposable continuum and hence it is an interval of some standard subcontinuum. We shall find \( A \in \Gamma \) such that \( f[A^*_p] \) is in fact an interval of \( A_q \). To this end let \( \langle [c_n, d_n] : n \in \omega \rangle \) be a sequence of closed intervals with \( d_n = c_{n+1} \) for all \( n \) and let \( r \in \omega^* \) be such that \( f[A^*_p] \) is an interval of \( [c_r, d_r] \). For every \( n \) let \( i_n = [c_n] \) and \( j_n = [d_n] \).

There is a member \( R \) of \( r \) such that if \( n < m \) in \( R \) then \( j_n < i_m \) and in this case we can assume that \( \langle [i_n, j_n] : n \in R \rangle \) is a subsequence of some \( A \in \Gamma \). It is clear that \( [c_r, d_r] \subset [i_r, j_r] \) and it is also true that \( q \in f[A^*_p] \subset [c_r, d_r] \); together these statements imply that \( A_q = [i_r, j_r] \), so that \( f[A^*_p] \) is indeed an interval of \( A_q \).

Now let \( s_1 \in S \) be such that \( A \equiv q \) and fix some \( s \subset S \) such that \( A^*_q \subset f[L^*_q] \). We claim that \( s \prec s_1 \). Indeed, if \( s_1 \not\prec s \) then we find that \( A^*_q \subset f[L^*_q] \) and hence that \( A^*_q \) is nowhere dense in \( f[A^*_p] \) and hence in \( A_q \), which contradicts
formally we let $A^s = A^q$. Thus we find that $A^s_q \subseteq L^s_q$ and hence that $f[L^s_p] \cap L^s_q \neq \emptyset$. But $f[L^s_p]$ is a layer of $f[A^s_q]$ and hence of $A^s_q \cup A^s_0$, as is $L^s_q$ of course. But then we must have $f[L^s_p] = L^s_q$.

Since $L^s_p$ is nowhere dense in $L^s_q$, whenever $t_1 < t_2$ in $T$, the map $t \mapsto s_t$ from $T'$ to $S'$ is strictly increasing; that it is surjective follows by interchanging $S'$ and $T'$ and considering $f^{-1}$.

This shows that $T'$ and $S'$ are isomorphic.

### 3.3. Many ordered sets

We define a family of $2^{\aleph_1}$ many linear orders of countable cofinality and without isomorphic final segments.

For a set $X$ of countable limit ordinals we define a linear order $L_X$ by inserting upside-down copies of $\omega$ into $\omega_1$, one between $\alpha$ and $\alpha + 1$ for every $\alpha \in X$. More formally we let

$$L_X = \{ (\alpha, m) \in \omega_1 \times \omega : \alpha \notin X \rightarrow m = 0 \}$$

ordered by $(\alpha, m) < (\beta, n)$ if 1) $\alpha \in \beta$, or 2) $\alpha = \beta$ and $m = 0 < n$, or 3) $\alpha = \beta$ and $m > n > 0$.

**Proposition 3.6.** $L_X$ and $L_Y$ are isomorphic iff $X = Y$.

**Proof.** Let $f : L_X \rightarrow L_Y$ be an isomorphism. We show by induction that $f((\alpha, 0)) = (\alpha, 0)$ for every limit ordinal $\alpha$ as well as $\alpha \in X$ iff $\alpha \in Y$.

In both $L_X$ and $L_Y$ the point $(\omega, 0)$ has $\omega \times \{0\}$ as its set of predecessors and so $f((\omega, 0)) = (\omega, 0)$. Assume $\alpha$ is a limit and that $f((\beta, 0)) = (\beta, 0)$ for all limits below $\alpha$. If $\alpha$ is a limit of limits then in both ordered sets we have $(\alpha, 0) = \sup\{(\beta, 0) : \beta \in \alpha \text{ and } \beta \text{ is a limit}\}$ and hence $f((\alpha, 0)) = (\alpha, 0)$.

Next assume $\alpha = \beta + \omega$ for a limit $\beta$. If $\beta \notin X$ then $(\beta + 1, 0)$ is the direct successor in $L_X$ of $(\beta, 0)$, hence $(\beta, 0)$ must have a direct successor in $L_Y$ as well. From this it follows that $\beta \notin Y$ and $f((\beta + n, 0)) = (\beta + n, 0)$ for all $n \in \omega$ and hence also $f((\alpha, 0)) = (\alpha, 0)$.

If $\beta \in X$ then the interval $(\langle \beta, 0 \rangle, \langle \alpha, 0 \rangle)$ has the same order type as $\mathbb{Z}$, the set of integers. Now the interval $(\langle \beta, 0 \rangle, \langle \beta, 1 \rangle)$ is infinite and every point in it has a direct predecessor. This means that $f((\langle \beta, 1 \rangle)) < (\alpha, 0)$ and hence that $(\beta, 0)$ does not have a direct successor in $L_Y$ and hence that $\beta \in Y$. It follows that $f$ maps the interval $(\langle \beta, 0 \rangle, \langle \alpha, 0 \rangle)$ isomorphically onto the corresponding interval of $L_Y$ and that $f((\alpha, 0)) = (\alpha, 0)$. 

From $L_X$ we define $T_X$ to be the ordered sum of $\omega$ copies of $L_X$:

$$T_X = \omega \times L_X$$

ordered lexicographically. Now note that the points $\langle n, (0, 0) \rangle$ are the only ones in $T_X$ whose sets of predecessors have cofinality $\aleph_1$.

Thus, if $f$ is an isomorphism between final segments of some $T_X$ and $T_Y$ then there is an isomorphism $g$ between final segments of $\omega$ such that $f((\langle n, (0, 0) \rangle)) = (\langle g(n), (0, 0) \rangle)$ for all $n$ in the final segment on the $T_X$-side. For each such $n$ the map $f$ then maps $\{n\} \times L_X$ isomorphically onto $\{g(n)\} \times L_Y$. It follows that $X = Y$.

This then provides us with our family of $2^{\aleph_1}$ many linear orders, indexed by the family of sets of countable limit ordinals.

This proves the following theorem and with it the existence of a family of $2^\omega$ many mutually nonhomeomorphic subcontinua of $\mathbb{R}$.

**Theorem 3.7 (CH).** There is a family of $2^\omega$ mean linear orders such that no two members have isomorphic final segments.
3.4. **Summary: two families of continua.** The combination of subsection 3.2 and Theorem 3.7 tells us that \{KTX : X a set of countable limit ordinals\} is a family of 2\(^c\) many indecomposable subcontinua of \(H^+\) that are mutually non-homeomorphic.

To get a family of 2\(^c\) many decomposable continua use Lemma 1.3 to deduce that in our construction the continuum \(KT\) is actually a layer of the ‘top continuum’ \(AT^+\). Indeed, \(KT\) is a subset of some layer \(L\) of \(AT^+\); if it were a proper subset then there would be a standard subcontinuum \(M\) with \(KT \subseteq M \subseteq L\). As in subsection 3.2 we could then find \(A \in \Gamma\) such that \(M\) is an interval of \(A\); yet there would be no \(t \in T^+\) such that \(A \equiv_p A'\).

Our second family is now obtained by taking for every set \(X\) of countable limit ordinals the interval \([a_X, KT_X]\) of the standard subcontinuum \(AT^+_X\), where \(a_X\) is the initial point of \(A_{T^+_X}\) as described in subsection 1.2. These decomposable continua are mutually non-homeomorphic because a homeomorphism between \([a_X, KT_X]\) and \([a_Y, KT_Y]\) will have to map \(a_X\) to \(a_Y\) (as these are the unique end points) and \(KT_X\) onto \(KT_Y\); the latter is not possible if \(X \neq Y\).

**Remark 3.1.** The family in [4] consists of standard subcontinua. By one of the results in [5] CH implies that all standard subcontinua are homeomorphic. Thus there is a striking difference between the effects of CH and ¬CH on the structure of family of standard subcontinua.

Our result shows that under CH each standard subcontinuum has a rich variety of layers and intervals. We leave as an open question how rich this variety is in ZFC alone.

### 4. A FIRST-COUNTABLE CONTINUUM

As mentioned in the introduction CH implies that the continuous images of \(H^+\) are precisely the continua of weight \(\omega\) or less — in particular every first-countable continuum is such an image. In this section we show that in the absence of CH there may be a first-countable continuum that is neither an image of \(\omega^*\) nor of \(H^+\).

#### 4.1. Bell’s graph. A major ingredient in our construction is Bell’s graph, constructed in [2]. It is a graph on the ordinal \(\omega_2\), represented by a symmetric subset \(E\) of \((\omega_2)^2\). The crucial property of this graph is that there is no map \(\varphi: \omega_2 \to \mathcal{P}(\omega)\) that represents this graph, where \(\varphi\) represents \(E\) if \(\langle \alpha, \beta \rangle \in E\) if and only if \(\varphi(\alpha) \cap \varphi(\beta)\) is infinite.

Bell’s graph exists in any forcing extension in which \(\aleph_2\) Cohen reals are added; for the reader’s convenience we shall, in subsection 4.5 below, describe the construction of \(E\) and adapt Bell’s proof so that it applies to continuous maps defined on \(H^+\). The proof shows that a similar graph also exists in the extension by \(\aleph_2\) random reals.

#### 4.2. Building \(C_E\). Our starting point is a connected version of the Alexandroff double of the unit interval, devised by Saalfrank [13]. We topologize the unit square as follows.

1. a local base at points of the form \((x, 0)\) consists of the sets
   
   \[ U(x, 0, n) = (x - 2^{-n}, x + 2^{-n}) \times [0, 1] \setminus \{x\} \times [2^{-n}, 1] \]

2. a local base at points of the form \((x, y)\), with \(y > 0\) consists of the sets
   
   \[ U(x, y, n) = \{x\} \times (y - 2^{-n}, y + 2^{-n}) \]

We call the resulting space the **connected comb** and denote it by \(C\). It is straightforward to verify that \(C\) is compact, Hausdorff and connected; it is first-countable by definition.
For each $x \in [0,1]$ and positive $a$ we define the following cross-shaped closed subset of $C^2$:

$$D_{x,a} = \{(x) \times [a,1] \times C \} \cup \{ C \times \{ x \} \times [a,1]\}$$

We note the following two properties of the sets $D_{x,a}$

1. if $a < b$ then $D_{x,b}$ is in the interior of $D_{x,a}$, and
2. if $x \neq y$ then $D_{x,a} \cap D_{y,a}$ is the union of two squares: $\{ x \} \times [a,1] \times \{ y \} \times [a,1]$ and $\{ y \} \times [a,1] \times \{ x \} \times [a,1]$.

Next take any $\aleph_2$-sized subset of $[0,1]$ and index it (faithfully) as $\{ x_\alpha : \alpha < \omega_2 \}$. We use this indexing to identify $E$ with the subset $\{ \langle x_\alpha, x_\beta \rangle : \langle \alpha, \beta \rangle \in E \}$ of the unit square. We remove from $C^2$ the following open set:

$$\bigcup_{(x,y) \notin E} \left( \{(x) \times (0,1] \times \{ y \} \times [0,1]\} \cup \{ \{ y \} \times (0,1] \times \{ x \} \times (0,1)\}\right)$$

The resulting compact space we denote by $C_E$. Observe that the intersections $D_{x,a} \cap C_E$ represent $E$ in the sense that $D_{x,a} \cap D_{x,a} \cap C_E$ is nonempty if and only if $\langle \alpha, \beta \rangle \in E$. We write $D_{x,a}^E = D_{x,a} \cap C_E$.

4.3. $C_E$ is (arcwise) connected. To begin: the square $S$ of the base line of $C$ is a subset of $C_E$ and homeomorphic to the unit square so that it is (arcwise) connected.

Let $(x, a, y, b)$ be a point of $C_E$ not in $S$. If, say, $a = 0$ then $\{ (x,0) \} \times \{ \{ y \} \times [0,b]\}$ is an arc in $C_E$ that connects $\langle x, 0, y, b \rangle$ to the point $(x,0,y,0)$ in $S$. If $a, b > 0$ then $\langle x, y \rangle \in E$, so the whole square $\{ x \} \times [0,1] \times \{ y \} \times [0,1]$ is in $C_E$ and it provides us with an arc in $C_E$ from $\langle x, a, y, b \rangle$ to $\langle x, 0, y, 0 \rangle$.

We find that $C_E$ is a first-countable continuum.

4.4. $C_E$ is not an $\mathbb{H}^+\text{-image}$. Assume $h : \mathbb{H}^+ \to C_E$ is a continuous surjection and consider, for each $\alpha$, the sets $D_{x,a}^E_{\alpha} \cap \mathbb{H}^+$.

Using standard properties of $\mathbb{H}^+$, see [10, Proposition 3.2], we find for each $\alpha$ a sequence $\langle (a_{\alpha,n}, b_{\alpha,n}) : n \in \omega \rangle$ of open intervals with rational endpoints, and with $b_{\alpha,n} < a_{\alpha,n}+1$ for all $n$, such that $h^E(D_{x,a}^E \cap \mathbb{H}^+ \subseteq \text{Ext} O_{\alpha} \cap \mathbb{H}^+ \subseteq h^E(D_{x,a}^E \cap \mathbb{H}^+$, where

$$O_{\alpha} = \bigcup_{n \in \omega} (a_{\alpha,n}, b_{\alpha,n})$$

and Ext $O_{\alpha} = \beta \mathbb{H} \setminus \text{Cl}(\beta \mathbb{H} \setminus O_{\alpha})$.

Because the intersections of the sets $D_{x,a}^E$ represent $E$ the intersections of the $O_{\alpha}$ will do this as well: the conditions ‘$O_{\alpha} \cap O_{\beta}$ is unbounded’ and ‘$\langle \alpha, \beta \rangle \in E$’ are equivalent.

In the next subsection we show that for (many) $\langle \alpha, \beta \rangle$ this equivalence does not hold and that therefore $C_E$ is not a continuous image of $\mathbb{H}^+$.

Note also that our continuum is not an $\omega^*\text{-image}$ either: if $g : \omega^* \to C_E$ were continuous and onto we could use clopen subsets of $\omega^*$ and their representing infinite subsets of $\omega$ to contradict the unrepresentability property of $E$.

4.5. Building the graph. We follow the argument from [2] and we rely on Kunen’s book [11, Chapter VII] for basic facts on forcing. We let $L = \{ \langle \alpha, \beta \rangle \in (\omega_2)_2 : \alpha \leq \beta \}$ and we force with the partial order $\text{Fn}(L,2)$ of finite partial functions with domain in $L$ and range in $\{0,1\}$. If $G$ is a generic filter on $\text{Fn}(L,2)$ then we let $E = \{ \langle \alpha, \beta \rangle : \bigcup G(\alpha, \beta) = 1 \text{ or } \bigcup G(\beta, \alpha) = 1 \}$.

To show that $E$ is as required we take a nice name $\dot{F}$ for a function, $F$, from $\omega_2$ to $(Q^2)^{\omega}$ that represents a choice of open sets $\alpha \mapsto O_{\alpha}$ as above in that $F(\alpha) = \langle (a_{\alpha,n}, b_{\alpha,n}) : n \in \omega \rangle$ for all $\alpha$. As a nice name $\dot{F}$ is a subset of $\omega_2 \times \omega \times Q^2 \times \text{Fn}(L,2)$, where for each point $\langle \alpha, n, a, b, p \rangle$ the set $\{ p : \langle \alpha, n, a, b, p \rangle \in \dot{F} \}$ is a maximal antichain in the set of conditions that forces the $n$th term of $\dot{F}(\alpha)$ to be $\langle a, b \rangle$. 
For each \( \alpha \) we let \( I_\alpha \) be the set of ordinals that occur in the domains of the conditions that appear as a fifth coordinate in the elements of \( \mathcal{F} \) with first coordinate \( \alpha \). The sets \( I_\alpha \) are countable, by the ccc of \( \text{Fn}(L,2) \). We may therefore apply the Free-Set Lemma, see [9, Corollary 44.2], and find a subset \( A \) of \( \omega_2 \) of cardinality \( \aleph_2 \) such that \( \alpha \notin I_\beta \) and \( \beta \notin I_\alpha \) whenever \( \alpha, \beta \in A \) and \( \alpha \neq \beta \).

Let \( p \in \text{Fn}(L,2) \) be arbitrary and take \( \alpha \) and \( \beta \) in \( A \) with \( \alpha < \beta \) and such that \( \alpha > \eta \) whenever \( \eta \) occurs in \( p \). Consider the condition \( q = p \cup \{ (\alpha, \beta, 1) \} \). If \( q \) forces \( O_\alpha \cap O_\beta \) to be bounded in \([0, \infty)\) then we are done: \( q \) forces that the equivalence fails at \( \langle \alpha, \beta \rangle \).

If \( q \) does not force the intersection to be bounded we can extend \( q \) to a condition \( r \) that forces \( O_\alpha \cap O_\beta \) to be unbounded. We define an automorphism \( h \) of \( \text{Fn}(L,2) \) by changing the value of the conditions only at \( \langle \alpha, \beta \rangle \): from 0 to 1 and vice versa. The condition \( p \) as well as the values \( \mathcal{F}(\alpha) \) and \( \mathcal{F}(\beta) \) are invariant under \( h \). It follows that \( h(r) \) extends \( p \) and

\[
h(r) \models \bigcup \mathcal{G}(\alpha, \beta) = 0 \quad \text{and} \quad O_\alpha \cap O_\beta \text{ is unbounded}
\]

so again the equivalence is forced to fail at \( \langle \alpha, \beta \rangle \).

Remark 4.1. The argument above goes through almost verbatim to show that Bell’s graph can also be obtained adding \( \aleph_2 \) random reals. When forcing with the random real algebra one needs only consider conditions that belong to the \( \sigma \)-algebra generated by the clopen sets of the product \( \{ 0, 1 \}^L \); these all have countable supports so that, again by the ccc, one can define the sets \( I_\alpha \) as before. The rest of the argument remains virtually unchanged.

Remark 4.2. Bell’s original example from [2] was not easily made connected. One obtains an essentially equivalent example by taking the square of the Alexandroff double of the unit interval (the subspace \( \{ (x, i) : x \in [0, 1], i \in \{ 0, 1 \} \} \) of \( C \)) and removing the points \( \langle (x, 1), (y, 1) \rangle \) with \( \langle x, y \rangle \notin E \).

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