Guarantees for Spectral Clustering with Fairness Constraints

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Abstract

Given the widespread popularity of spectral clustering (SC) for partitioning graph data, we study a version of constrained SC in which we try to incorporate the fairness notion proposed by Chierichetti et al. (2017). According to this notion, a clustering is fair if every demographic group is approximately proportionally represented in each cluster. To this end, we develop variants of both normalized and unnormalized constrained SC and show that they help find fairer clusterings on both synthetic and real data. We also provide a rigorous theoretical analysis of our algorithms. While there have been efforts to incorporate various constraints into the SC framework, theoretically analyzing them is a challenging problem. We overcome this by proposing a natural variant of the stochastic block model where $h$ groups have strong inter-group connectivity, but also exhibit a “natural” clustering structure which is fair. We prove that our algorithms can recover this fair clustering with high probability.

1. Introduction

Machine learning (ML) has recently seen an explosion of applications in settings to guide or make choices directly affecting people. Examples include applications in lending, marketing, education, and many more. Close on the heels of the adoption of ML methods in these everyday domains have been any number of examples of ML methods displaying unsavory behavior towards certain demographic groups. These have spurred the study of fairness of machine learning algorithms. Numerous mathematical formulations of fairness have been proposed for supervised learning settings, each with their strengths and shortcomings in terms of what they disallow and how difficult they may be to satisfy (e.g., Dwork et al., 2012; Hardt et al., 2016; Chouldechova, 2017; Kleinberg et al., 2017). Somewhat more recently, the community has begun to study appropriate notions of fairness for unsupervised learning settings (e.g., Chierichetti et al., 2017; Celis et al., 2018b; Samadi et al., 2018; see Section 5).

In particular, the recent work of Chierichetti et al. (2017) proposes a notion of fairness for clustering: namely, that each cluster has proportional representation from different demographic groups. Their paper provides approximation algorithms for $k$-center and $k$-median clustering that incorporate this fairness notion. The follow-up work of Schmidt et al. (2018) extends this to $k$-means clustering. These papers open up an important line of work that aims at studying the following questions for clustering: a) How to incorporate fairness constraints into popular clustering objectives and algorithms? and b) What is the price of fairness? For example, the results of Chierichetti et al. indicate that achieving fair clusterings comes at a significant increase in the $k$-center/$k$-median objective value. While the above works focus on clustering data sets in Euclidean/metric spaces, a large body of clustering problems involve graph data. On such data, spectral clustering (SC; von Luxburg, 2007) is the method of choice in practice. In this paper, we extend the above line of work by studying the implications of incorporating the fairness notion of Chierichetti et al. into SC.

The contributions of this paper are as follows:

- We show how to incorporate the constraints that in each cluster, every group should be represented with the same proportion as in the original data set into the SC framework. For continuity with prior work (as discussed above; also see Section 5), we refer to these constraints as fairness constraints and speak of fair clusterings. However, the terms proportionality and proportional would be a more formal description of our goal. Our approach to incorporate the fairness constraints is analogous to existing versions of constrained SC that try to incorporate must-link constraints (see Section 5). In contrast to the work of Chierichetti et al. (2017), which always yields a fair clustering no matter how much the objective value increases compared to an unfair clustering, our approach does not guarantee that we end up with a fair clustering. Rather, our approach guides SC to find a good and fair clustering if such a one exists.
- Indeed, we prove that our algorithms find a good and fair clustering in a natural variant of the famous stochastic

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block model that we propose. In our variant, $h$ demographic groups have strong inter-group connectivity, but also exhibit a “natural” clustering structure that is fair. We provide a rigorous analysis of our algorithms showing that they can recover this fair clustering with high probability. To the best of our knowledge, such an analysis has not been done before for constrained versions of SC.

- We conclude by giving experimental results on real-world data sets where proportional clustering can be a desirable goal, comparing the proportionality and objective value of standard SC to our methods. Our experiments confirm that our algorithms tend to find fairer clusterings compared to standard SC. A surprising finding is that in many real data sets achieving higher proportionality often comes at minimal cost, namely, that our methods produce clusterings that are fairer, but have objective values very close to those of clusterings produced by standard SC. This complements the results of Chierichetti et al. (2017), where achieving fairness constraints exactly comes at a significant cost in the objective value, and indicates that in some scenarios fairness and objective value need not be at odds with one other.

**Notation** For $n \in \mathbb{N}$, we use $[n] = \{1, \ldots, n\}$. $I_n$ denotes the $n \times n$-identity matrix and $0_{n \times m}$ is the $n \times m$-zero matrix. $I_n$ denotes a vector of length $n$ with all entries equaling 1.

For a matrix $A \in \mathbb{R}^{n \times m}$, we denote the transpose of $A$ by $A^T \in \mathbb{R}^{m \times n}$. For $A \in \mathbb{R}^{n \times n}$, $\text{Tr}(A)$ denotes the trace of $A$, that is $\text{Tr}(A) = \sum_{i=1}^{n} A_{ii}$. If we say that a matrix is positive (semi-)definite, this implies the matrix is symmetric.

2. **Spectral Clustering**

To set the ground and introduce terminology, we review Spectral Clustering (SC). There are several versions of SC (von Luxburg, 2007). For ease of presentation, here we focus on unnormalized SC (Hagen & Kahng, 1992). In Appendix A, we adapt all findings of this section and the following Section 3 to normalized SC (Shi & Malik, 2000).

Let $G = (V, E)$ be an undirected graph on $V = [n]$. We assume that each edge between two vertices $i$ and $j$ carries a positive weight $W_{ij} > 0$ encoding the strength of similarity between the vertices. If there is no edge between $i$ and $j$, we set $W_{ij} = 0$. We assume that $W_{ii} = 0$, for all $i \in [n]$. Given $k \in \mathbb{N}$, unnormalized SC aims to partition $V$ into $k$ clusters with minimum value of the RatioCut objective function as follows (see von Luxburg, 2007, for details): for a clustering $V = C_1 \cup \ldots \cup C_k$, we have

$$\text{RatioCut}(C_1, \ldots, C_k) = \frac{\sum_{l=1}^{k} \text{Cut}(C_l, V \setminus C_l)}{|C_l|},$$  \hspace{1cm} (1)

where $\text{Cut}(C_l, V \setminus C_l)$ denotes the weight of the cut $(C_l, V \setminus C_l)$, that is

$$\text{Cut}(C_l, V \setminus C_l) = \sum_{i \in C_l, j \in V \setminus C_l} W_{ij}.$$  \hspace{1cm} (2)

Let $W = (W_{ij})_{i,j \in [n]}$ be the weighted adjacency matrix of $G$ and $D$ be the degree matrix, that is a diagonal matrix with the vertex degrees $d_i = \sum_{j \in [n]} W_{ij}$, $i \in [n]$, on the diagonal. Let $L = D - W$ denote the unnormalized graph Laplacian matrix. Note that $L$ is positive semi-definite. A key insight is that if we encode a clustering $V = C_1 \cup \ldots \cup C_k$ by a matrix $H \in \mathbb{R}^{n \times k}$ with

$$H_{il} = \begin{cases} 1/\sqrt{|C_l|}, & i \in C_l, \\ 0, & i \not\in C_l, \end{cases}$$  \hspace{1cm} (3)

then $\text{RatioCut}(C_1, \ldots, C_k) = \text{Tr}(H^T LH)$. Hence, in order to minimize the RatioCut function over all possible clusterings, we could instead solve

$$\min_{H \in \mathbb{R}^{n \times k}} \text{Tr}(H^T LH) \text{ subject to } H \geq 0.$$  \hspace{1cm} (4)

Spectral clustering relaxes this minimization problem by replacing the requirement that $H$ has to be of form (2) with the weaker requirement that $H^T H = I_k$, that is it solves

$$\min_{H \in \mathbb{R}^{n \times k}} \text{Tr}(H^T LH) \text{ subject to } H^T H = I_k.$$  \hspace{1cm} (5)

Since $L$ is symmetric, it is well known that a solution to (4) is given by the matrix $H$ that contains some orthonormal eigenvectors corresponding to the $k$ smallest eigenvalues (respecting multiplicities) of $L$ as columns (Lükepohl, 1996, Section 5.2.2). Consequently, the first step of SC is to compute such an optimal $H$ by computing the $k$ smallest eigenvalues and corresponding eigenvectors. The second step is to infer a clustering from $H$. While there is a one-to-one correspondence between a clustering and a matrix of the form (2), this is not the case for a solution $H$ to the relaxed problem (4). Usually, a clustering of $V$ is inferred from $H$ by applying $k$-means clustering to the rows of $H$.

We summarize unnormalized SC as Algorithm 1. Note that, in general, there is no guarantee on how close the RatioCut value of the clustering obtained by Algorithm 1 to the RatioCut value of an optimal clustering (solving (3)) is.

3. **Adding Fairness Constraints**

We now extend the above setting to incorporate fairness constraints. Suppose that the data set $V$ contains $h$ groups $V_s$ such that $V = \bigcup_{s \in [h]} V_s$. Chierichetti et al. (2017) proposed a notion of fairness for clustering asking that every cluster contains approximately the same number of elements from each group $V_s$. For a clustering $V = C_1 \cup \ldots \cup C_k$, define the balance of cluster $C_l$ as

$$\text{balance}(C_l) = \min_{s \neq s' \in [h]} \frac{|V_s \cap C_l|}{|V_{s'} \cap C_l|} \in [0, 1].$$  \hspace{1cm} (6)
For the higher the balance of each cluster, the fairer is the clustering according to the notion of Chierichetti et al. (2017). For any clustering, we have \( \min_{i \in [k]} \text{balance}(C_i) \leq \min_{i \neq i' \in [n]} \frac{|V_i|}{|V_i'|} \), so that this fairness notion is actually asking for a clustering in which in every cluster, each group is (approximately) represented with the same fraction as in the whole data set \( V \). The following lemma shows how to incorporate this goal into the RatioCut minimization problem (3) using a linear constraint on \( H \).

**Lemma 1** (Fairness constraints as linear constraint on \( H \)). For \( s \in [h] \), let \( f(s) \in \{0, 1\}^n \) be the group-membership vector of \( V_s \), that is \( f(s)_i = 1 \) if \( i \in V_s \) and \( f(s)_i = 0 \) otherwise. Let \( V = C_1 \cup \ldots \cup C_k \) be a clustering that is encoded as in (2). We have, for every \( l \in [k] \),
\[
\forall s \in [h-1]: \sum_{i=1}^n \left( f(s)_i - \frac{|V_s|}{n} \right) H_{il} = 0 \iff \forall s \in [h]: \frac{|V_s \cap C_l|}{|C_l|} = \frac{|V_s|}{n}.
\]

**Proof.** This simply follows from
\[
\sum_{i=1}^n \left( f(s)_i - \frac{|V_s|}{n} \right) H_{il} = \frac{|V_s \cap C_l|}{\sqrt{|C_l|}} - \frac{|V_s| \cdot |C_l|}{n \sqrt{|C_l|}},
\]
and \( |C_l| = \sum_{s=1}^h |V_s \cap C_l| \).

Consequently, if we want to find a clustering that minimizes the RatioCut objective function and is as fair as possible, we have to solve
\[
\min_{H \in \mathbb{R}^{n \times k}} \text{Tr}(H^T LH) \quad \text{subject to } H \text{ is of form (2)}
\]
and \( F^T H = 0_{(h-1) \times k} \),
\[
\min_{H \in \mathbb{R}^{n \times k}} \text{Tr}(H^T LH) \quad \text{subject to } H^T H = I_k
\]
and \( F^T H = 0_{(h-1) \times k} \).

Our proposed approach to incorporate the fairness notion by Chierichetti et al. (2017) into the SC framework consists of solving (7) instead of (4) (and, as before, applying \( k \)-means clustering to the rows of an optimal \( H \) in order to infer a clustering). Our approach is analogous to the numerous versions of constrained SC that try to incorporate must-link constraints (“vertices A and B should end up in the same cluster”) by putting a linear constraint on \( H \) (e.g., Yu & Shi, 2004; Kawale & Boley, 2013; see Section 5). Next, we describe a straightforward way to solve (7), which is also discussed by Yu & Shi (2004). It is easy to see that \( \text{rank}(F) = \text{rank}(F^T) = h-1 \). We need to assume that \( k \leq n - h + 1 \) since otherwise (7) does not have any solution. Let \( Z \in \mathbb{R}^{n \times (n-h+1)} \) be a matrix whose columns form an orthonormal basis of the nullspace of \( F^T \). We can substitute \( H = ZY \) for \( Y \in \mathbb{R}^{(n-h+1) \times k} \) and then, using that \( Z^T Z = I_{(n-h+1)} \), Problem (7) becomes
\[
\min_{Y \in \mathbb{R}^{(n-h+1) \times k}} \text{Tr}(Y^T Z^T LZY) \quad \text{subj. to } Y^TY = I_k.
\]

Similarly to problem (4), a solution to (8) is given by the matrix \( Y \) that contains some orthonormal eigenvectors corresponding to the \( k \) smallest eigenvalues (respecting multiplicities) of \( Z^T LZ \) as columns. We then set \( H = ZY \).

This way of solving (7) gives rise to our “fair” version of unnormalized SC as stated in Algorithm 2. Note that just as there is no guarantee on the RatioCut value of the output of Algorithm 1 or Algorithm 2 compared to an optimal clustering, in general, there is also no guarantee on how fair the output of Algorithm 2 is. We will still refer to Algorithm 2 as our fair version of SC. Similarly to how we proceeded here, in Appendix A, we incorporate the fairness constraints into normalized SC and state our fair version of normalized SC as Algorithm 3.

One might wonder why we do not simply run standard SC on each group separately in order to derive a fair version. In
Appendix D we show why such an idea does not work.

**Computational complexity** We provide a complete discussion in Appendix B. With the implementation as stated, the complexity of both Algorithm 2 and Algorithm 3 is \( O(n^3) \) regarding time and \( O(n^2) \) regarding space, which is the same as the worst-case complexity of standard SC when the number of clusters can be arbitrary. One could apply one of the techniques suggested in the existing literature on constrained spectral clustering to speed up computation (e.g., Yu & Shi, 2004, or Xu et al., 2009; see Section 5), but most of these techniques only work for \( k = 2 \) clusters.

## 4. Variant of the Stochastic Block Model

In this section, our goal is to model data sets that have two or more meaningful ground-truth clusterings, of which only one is fair, and show that our algorithms recover the fair ground-truth clustering. If there was only one meaningful ground-truth clustering and this clustering was fair, then any clustering algorithm that is able to recover the ground-truth clustering (e.g., standard SC) would be a fair algorithm. To this end, we define a variant of the famous stochastic block model (SBM; Holland et al., 1983). The SBM is a random graph model that has been widely used to study the performance of clustering algorithms, including standard SC (see Section 5 for related work). In the traditional SBM there is a ground-truth clustering of the vertex set \( V = [n] \) into \( k \) clusters, and in a random graph generated from the model, two vertices \( i \) and \( j \) are connected with a probability that only depends on which clusters \( i \) and \( j \) belong to.

In our variant of the SBM we assume that \( V = [n] \) comprises \( h \) groups \( V = V_1 \cup \ldots \cup V_h \) and is partitioned into \( k \) ground-truth clusters \( V = C_1 \cup \ldots \cup C_k \) such that \( |V_s \cap C_l|/|C_l| = \eta_l, s \in [h], l \in [k] \), for some \( \eta_1, \ldots, \eta_h \in (0, 1) \) with \( \sum_{s=1}^h \eta_s = 1 \). Hence, in every cluster each group is represented with the same fraction as in the whole data set \( V \) and this ground-truth clustering is fair. Now we define a random graph on \( V \) by connecting two vertices \( i \) and \( j \) with a certain probability \( \Pr(i, j) \) that only depends on whether \( i \) and \( j \) are in the same group (or not) and on whether \( i \) and \( j \) are in the same group (or not). More specifically, we have

\[
\begin{align*}
\Pr(i, j) = \\
\begin{cases} 
  a, & i \text{ and } j \text{ in same cluster and in same group,} \\
  b, & i \text{ and } j \text{ not in same cluster, but in same group,} \\
  c, & i \text{ and } j \text{ in same cluster, but not in same group,} \\
  d, & i \text{ and } j \text{ not in same cluster and in same group,} 
\end{cases}
\end{align*}
\]

and assume that \( a > b > c > d \). As in the ordinary SBM, connecting \( i \) and \( j \) is independent of connecting \( i' \) and \( j' \) for \( \{i, j\} \neq \{i', j'\} \). Every edge is assigned a weight of +1, that is no two connected vertices are considered more similar to each other than any two other connected vertices.

An example of a graph generated from our model (with \( h = k = 2 \) and \( \eta_1 = \eta_2 = 1/2 \)) can be seen in Figure 1. We can see that there are two meaningful ground-truth clusterings into two clusters: \( V = C_1 \cup C_2 \) and \( V = V_1 \cup V_2 \). Among these two clusterings, only \( V = C_1 \cup C_2 \) is fair since balance\((C_i) = 1 \) while balance\((V_i) = 0 \) for \( i \in \{1, 2\} \). Note that the clustering \( V = V_1 \cup V_2 \) has a smaller RatioCut value than \( V = C_1 \cup C_2 \) because there are more edges between \( V_1 \cap C_1 \) and \( V_2 \cap C_2 \) (\( s = 1 \) or \( s = 2 \)) than between \( V_1 \cap C_1 \) and \( V_2 \cap C_1 \) (\( l = 1 \) or \( l = 2 \)). As we will see in the experiments in Section 6 (and can also be seen from the proof of the following Theorem 1), for such a graph, standard SC is very likely to return the unfair clustering \( V = V_1 \cup V_2 \) as output. In contrast, our fair versions of SC return the fair clustering \( V = C_1 \cup C_2 \) with high probability:

**Theorem 1** (SC with fairness constraints succeeds on variant of stochastic block model). Let \( V = [n] \) comprise \( h = h(n) \) groups \( V = V_1 \cup \ldots \cup V_h \) and be partitioned into \( k = k(n) \) ground-truth clusters \( V = C_1 \cup \ldots \cup C_k \) such that for all \( s \in [h] \) and \( l \in [k] \)

\[
|V_s| = \frac{n}{h}, \quad |C_l| = \frac{n}{k}, \quad \frac{|V_s \cap C_l|}{|C_l|} = \frac{1}{h}.
\]

Let \( G \) be a random graph constructed according to our variant of the stochastic block model \((9)\) with probabilities

\[
a = a(n), \quad b = b(n), \quad c = c(n), \quad d = d(n) \quad \text{satisfying } a > b > c > d \quad \text{and } a \geq C \ln n/n \quad \text{for some } C > 0.
\]

Assume that we run Algorithm 2 or Algorithm 3 (stated in Appendix A) on \( G \), where we apply a \((1+M)\)-approximation algorithm to the \( k \)-means problem encountered in the last step of Algorithm 2 or Algorithm 3, for some \( M > 0 \). Then, for every \( r > 0 \), there exist constants \( \td{C}_1 = \td{C}_1(C, r) \) and \( \td{C}_i = \td{C}_i(C, r), i \in \{1, 2\} \), such that the following is true:
We make several remarks on Theorem 1:

1. By “misclassifies at most $x$ many vertices” we mean that, considering the index $l$ of the cluster $C_l$ that a vertex belongs to as the vertex’s class label, there exists a permutation of cluster indices $1, \ldots, k$ such that up to this permutation the clustering returned by our algorithm predicts the correct class label for all but $x$ many vertices.

2. The condition (11) is satisfied, for $n$ sufficiently large and assuming that $M \in O(\ln k)$ (see the next remark), in various regimes: assuming that $k \in O(n^s)$ for some $s \in [0, 1/3)$, it is satisfied in the dense regime $a, b, c, d \sim \text{const}$, but also in the sparse regime $a, b, c, d \sim \text{const} \cdot (\ln n/n)^q$ for some $q \in [0, 1 - 3s)$.

The same is true for condition (13), but here we require $s \in [0, 1/4)$ and $q \in [0, 1 - 4s)$. We suspect that condition (13), with respect to $k$, is stronger than necessary. We also suspect that the error bound in (14) is not tight with respect to $k$. Note that in (14), both in the dense and in the sparse regime, the term $a \cdot k^3 \cdot \ln n/(c - d)^2$ is dominating over the term $a \cdot n^2 \cdot \ln n/(\lambda_1 - a)^2$ by the factor $k^3$.

3. There are efficient approximation algorithms for the $k$-means problem in $\mathbb{R}^d$. An algorithm by Ahmadian et al. (2017) achieves a constant approximation factor and has running time polynomial in $n, k$ and $l$, where $n$ is the number of data points. There is also the famous $(1 + \epsilon)$-approximation algorithm by Kumar et al. (2004) with running time linear in $n$ and $l$, but exponential in $k$ and $1/\epsilon$. The algorithm most widely used in practice (e.g., as default method in MATLAB) is $k$-means++, which is a randomized $O(\ln k)$-approximation algorithm (Arthur & Vassilvitskii, 2007).

4. We show empirically in Section 6 that our algorithms are also able to find the fair ground-truth clustering in a graph constructed according to our variant of the SBM when (10) is not satisfied, that is when the clusters are of different size or the balance of the fair ground-truth clustering is smaller than 1 (i.e. $\eta_s \neq 1/h$ for some $s \in [h]$). For Algorithm 3 the violation of (10) can be more severe than for Algorithm 2. In general, we observe Algorithm 3 to outperform Algorithm 2. This is in accordance with standard SC, for which normalized SC has been observed to outperform unnormalized SC (von Luxburg, 2007; Sarkar & Bickel, 2015).

The proof of Theorem 1 can be found in Appendix C. It consists of two technical challenges (described here only for the unnormalized case). The first one is to compute the eigenvalues and eigenvectors of the matrix $Z^T \mathcal{L} Z$, where $\mathcal{L}$ is the expected Laplacian matrix of the random graph $G$ and $Z$ is the matrix computed in Algorithm 2. Let $Y$ be a matrix containing some orthonormal eigenvectors corresponding to the $k$ smallest eigenvalues of $Z^T \mathcal{L} Z$ as columns and $Y$ be a matrix containing orthonormal eigenvectors corresponding to the $k$ smallest eigenvalues of $Z^T \mathcal{L} Z$, where $L$ is the observed Laplacian matrix of $G$. The second challenge is to prove that with high probability, $Z Y$ is close to $Z Y$. For doing so we make use of the famous Davis-Kahan sin$\theta$ Theorem (Davis & Kahan, 1970). After that, we can use existing results about $k$-means clustering of perturbed eigenvectors (Lei & Rinaldo, 2015) to derive the theorem.

5. Related Work

Spectral clustering and stochastic block model SC is one of the most prominent clustering techniques, with a long history and an abundance of related papers. See von Luxburg (2007) or Nascimento & de Carvalho (2011) for general introductions and an overview of the literature. There are numerous papers on constrained SC, where the goal is to incorporate prior knowledge about the target clustering (usually in the form of must-link and/or cannot-link constraints) into the SC framework (e.g., Yu & Shi, 2001; 2004; Joachims, 2003; Lu & Carreira-Perpinan, 2008; Xu et al., 2009; Wang & Davidson, 2010; Eriksson et al., 2011; Maji et al., 2011; Kawale & Boley, 2013; Khoreva et al., 2014; Wang et al., 2014; Cucuringu et al., 2016). Most of
these papers are motivated by the use of SC in image or video segmentation. Closely related to our work are the papers by Yu & Shi (2004); Xu et al. (2009); Eriksson et al. (2011); Kawale & Boley (2013), which incorporate the prior knowledge by imposing a linear constraint in the RatioCut or NCut optimization problem analogous to how we derived our fair versions of SC. These papers provide efficient algorithms to solve the resulting optimization problems. However, the iterative algorithms by Xu et al. (2009); Eriksson et al. (2011); Kawale & Boley (2013) only work for \( k = 2 \) clusters. The method by Yu & Shi (2004) works for arbitrary \( k \) and could be used to speed up the computation of a solution of (7) or (18) compared to our straightforward way as implemented by Algorithm 2 and Algorithm 3, respectively, but requires to modify the eigensolver in use.

The stochastic block model (SBM; Holland et al., 1983) is the canonical model to study the performance of clustering algorithms. There exist several variants of the original model such as the degree-corrected SBM or the labeled SBM. For a recent survey see Abbe (2018). In the labeled SBM, vertices can carry a label that is correlated with the ground-truth clustering. This is quite the opposite of our model, in which the group-membership information is “orthogonal” to the ground-truth clustering. Several papers show the consistency (i.e., the capability to recover the ground-truth clustering) of different versions of SC on the SBM or the degree-corrected SBM under different assumptions (Rohe et al., 2011; Fishkind et al., 2013; Qin & Rohe, 2013; Lei & Rinaldo, 2015; Joseph & Yu, 2016; Su et al., 2017). For example, Rohe et al. (2011) show consistency of normalized SC assuming that the minimum expected vertex degree is in \( \Omega(n/\sqrt{\log n}) \), while Lei & Rinaldo (2015) show that SC based on the adjacency matrix is consistent requiring only that the maximum expected degree is in \( \Omega(\sqrt{\log n}) \). Note that these papers also make assumptions on the eigenvalues of the expected Laplacian or adjacency matrix while all assumptions and guarantees stated in our Theorem 1 directly depend on the connection probabilities \( a, b, c, d \) of our model. We are not aware of any work providing consistency results for constrained SC methods as we do in this paper.

**Fairness** By now, there is a huge body of work on fairness in machine learning. For a recent paper providing an overview of the literature on fair classification see Donini et al. (2018). Our paper adds to the literature on fair methods for unsupervised learning tasks (Chierichetti et al., 2017; Celis et al., 2018a;b;c; Samadi et al., 2018; Schmidt et al., 2018). Note that all these papers assume to know which demographic group a data point belongs to just as we do. We discuss the pieces of work most closely related to our paper.

Chierichetti et al. (2017) proposed the notion of fairness for clustering underlying our paper. It is based on the fairness notion of disparate impact (Feldman et al., 2015) and the \( p\%\)-rule (Zafar et al., 2017), respectively, which essentially say that the output of a machine learning algorithm should be independent of a sensitive attribute. In their paper, Chierichetti et al. focus on \( k \)-median and \( k \)-center clustering. For the case of a binary sensitive attribute, that is there are only two demographic groups, they provide approximation algorithms for the problems of finding a clustering with minimum \( k \)-median / \( k \)-center cost under the constraint that all clusters have some prespecified level of balance. Subsequently, Rösner & Schmidt (2018) provide an approximation algorithm for such a fair \( k \)-center problem with multiple groups. Schmidt et al. (2018) build upon the fairness notion and techniques of Chierichetti et al. and devise an approximation algorithm for the fair \( k \)-means problem, assuming that there are only two groups of the same size.

### 6. Experiments

In this section, we present a number of experiments. We first study our fair versions of spectral clustering, Algorithm 2 and Algorithm 3, on synthetic data generated according to our variant of the SBM and compare our algorithms to standard SC. We also study how robust our algorithms are with respect to a certain perturbation of our model. We then compare our algorithms to standard SC on real network data. We implemented all algorithms in MATLAB. We used the built-in function for \( k \)-means clustering with all parameters set to their default values except for the number of replicates, which we set to 10. In the following, all plots show average results obtained from running an experiment for 100 times.

#### 6.1. Synthetic Data

We run experiments on our variant of the SBM introduced in Section 4. To assess the quality of a clustering we measure the fraction of misclassified vertices w.r.t. the fair ground-truth clustering (cf. Section 4), which we refer to as error.

In the experiments of Figure 2, we study the performance of standard unnormalized and normalized SC and of our fair versions, Algorithm 2 and Algorithm 3, as a function of \( n \). Due to the high running time of Algorithm 3 (cf. Section 3), we only run it up to \( n = 4000 \). All plots show the error of the methods, except for the fourth plot in the first row, which shows their run-time. We study several parameter settings. For the plots in the first row, Assumption (10) in Theorem 1 is satisfied, that is \( |V_s \cap C_l| = \frac{n_s}{nk} \) for all \( s \in [h] \) and \( l \in [k] \). In this case, in accordance with Theorem 1, both Algorithm 2 and Algorithm 3 are able to recover the fair ground-truth clustering if \( n \) is just large enough while standard SC always fails to do so. Algorithm 3 yields significantly better results than Algorithm 2 and requires much smaller values of \( n \) for achieving zero error. This comes at the cost of a higher running time of Algorithm 3 (still it is in \( O(n^3) \) as claimed in Section 3). The run-time of Algo-
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Figure 2. Performance of standard spectral clustering and our fair versions on our variant of the stochastic block model as a function of \( n \) for various parameter settings. Error is the fraction of misclassified vertices w.r.t. the fair ground-truth clustering (cf. Section 4). **First row:** Assumption (10) in Theorem 1 is satisfied, that is \( |V_s \cap C_l| \geq 4n^{-5}\sqrt{k} / k^7 \), \( s \in [h] \), \( l \in [k] \). **Second row:** Assumption (10) is not satisfied.

2nd row, 1st plot: \( |C_l| = 7n / 10 \), \( |C_l| = 15n / 100 \) and \( |V_s \cap C_l| = 10^{-6} / n \), \( l \in \{2\} \); 3rd plot: \( |C_l| = 4n / 10 \), \( |C_l| = 3n / 10 \) and \( |V_s \cap C_l| = 10^{-5} / n \), \( l \in \{5\} \); 4th plot: \( |C_l| = 1n / 10 \), \( |C_l| = 1n / 10 \) and \( |V_s \cap C_l| = 10^{-3} / n \), \( l \in \{5\} \).

Figure 3. Error of our algorithms as a function of \( k \). We consider \( a = F \cdot \frac{25}{100} \), \( b = F \cdot \frac{25}{100} \), \( c = F \cdot \frac{15}{100} \), \( d = F \cdot \frac{4}{10} \) for various values of \( F \). **Left:** Alg. 2, \( n \approx 5000 \). **Right:** Alg. 3, \( n \approx 2000 \).

Algorithm 2 is the same as the run-time of standard normalized SC. For the plots in the second row, Assumption (10) in Theorem 1 is not satisfied. We consider various scenarios of cluster sizes \( |C_l| \) and group sizes \( |V_s| \) (however, we always have \( |V_s \cap C_l| / |C_l| = |V_s| / n \), \( s \in [h] \), \( l \in [k] \), so that \( C_1 \cup \ldots \cup C_k \) is as fair as possible). When the cluster sizes are different, but the group sizes are all equal to each other (1st plot in the 2nd row) or Assumption (10) is only slightly violated (2nd plot), both Algorithm 2 and Algorithm 3 are still able to recover the fair ground-truth clustering. Compared to the plots in the first row, Algorithm 2 requires a larger value of \( n \) than usual, even though \( k \) is smaller. Algorithm 3 achieves (almost) zero error already for \( n = 1000 \) in these scenarios. When Assumption (10) is strongly violated (3rd and 4th plot), Algorithm 2 fails to recover the fair ground-truth clustering, but Algorithm 3 still succeeds.

In the experiments shown in Figure 3, we study the error of Algorithm 2 (left plot) and Algorithm 3 (right plot) as a function of \( k \) when \( n \) is roughly fixed. More precisely, for \( k \in \{2, \ldots, 8\} \) and \( h = 4 \), we have \( n = kh[1000 / kh] \) (Alg. 2; left) or \( n = kh[2000 / kh] \) (Alg. 3; right), which allows for fair ground-truth clusterings satisfying (10). We consider connection probabilities \( a = F \cdot \frac{25}{100} \), \( b = F \cdot \frac{25}{100} \), \( c = F \cdot \frac{15}{100} \), \( d = F \cdot \frac{4}{10} \) for \( F \in \{1, 1.5, 2, 2.5, 3\} \). Unsurprisingly, for both Algorithm 2 and Algorithm 3 the error is monotonically increasing with \( k \). The rate of increase critically depends on \( F \) (or the probabilities \( a, b, c, d \)). For Algorithm 2, this is even more severe. There is only a small range in which the various curves exhibit polynomial growth, which makes it impossible to empirically evaluate whether our error guarantees (12) and (14) are tight with respect to \( k \).

In the experiments of Figure 4, we consider a perturbation of our model as follows: first, for \( n = 4000 \) (left plot) or \( n = 2000 \) (right plot), \( k = 4 \) and \( h = 2 \) we generate a graph from our model just as before (Assumption (10) is satisfied; in particular, the two groups have the same size), but then we assign some of the vertices in the first group to the other
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6.2. Real Data

In the experiments of Figure 5, we evaluate the performance of standard unnormalized and normalized SC versus our fair versions on real network data. The quality of a clustering is measured through its “Balance” (defined as the average of the balance (5) over all clusters; shown on left axis of the plots) and its RatioCut (1) or NCut (15) value (right axis). All networks that we are working with are the largest connected component of an originally unconnected network.

The first row of Figure 5 shows the results as a function of the number of clusters $k$ for two high school friendship networks (Mastrandrea et al., 2015). Vertices correspond to students and are split into two groups of males and females. FRIENDSHIPNET has 127 vertices and an edge between two students indicates that one of them reported friendship with the other one. FACEBOOKNET consists of 155 vertices and an edge between two students indicates friendship on Facebook. As we can see from the plots, compared to standard SC, our fair versions improve the output clustering’s balance (by 10% / 15% / 34% / 10% on average over $k$) while almost not changing its RatioCut or NCut value.

The second row shows the results for DRUGNET, a network encoding acquaintanceship between drug users in Hartford, CT (Weeks et al., 2002). In the left two plots, the network consists of 185 vertices split into two groups of males and females (we had to remove some vertices for which the gender was not known). In the right two plots, the network has 193 vertices split into three ethnic groups of African Americans, Latinos and others. Again, our fair versions of SC quite significantly improve the balance of the output clustering over standard SC (by 5% / 18% / 86% / 167% on average over $k$). However, in the right two plots we also observe a moderate increase of the RatioCut or NCut value.

7. Discussion

In this work, we presented an algorithmic approach towards incorporating fairness constraints into the SC framework. We provided a rigorous analysis of our algorithms and proved that they can recover fair ground-truth clusterings in a natural variant of the stochastic block model. Furthermore, we provided strong empirical evidence that often in real data sets, it is possible to achieve higher demographic proportionality at minimal additional cost in the clustering objective.

An important direction for future work is to understand the price of fairness in the SC framework if one needs to satisfy the fairness constraints exactly. One way to achieve this would be to run the fair $k$-means algorithm of Schmidt et al. (2018) in the last step of our Algorithms 2 or 3. We want to point out that the algorithm of Schmidt et al. currently does not extend beyond two groups of the same size. Second, our experimental results on the stochastic block model provide evidence that our algorithms are robust to moderate levels of perturbations in the group assignments. Characterizing this robustness rigorously is an intriguing open problem.
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Appendix

A. Adding Fairness Constraints to Normalized Spectral Clustering

In this section we derive a fair version of normalized spectral clustering (similarly to how we proceeded for unnormalized spectral clustering in Sections 2 and 3 of the main paper).

Normalized spectral clustering aims at partitioning \( V \) into \( k \) clusters with minimum value of the NCut objective function as follows (see von Luxburg, 2007, for details): for a clustering \( V = C_1 \cup \ldots \cup C_k \) we have

\[
\text{NCut}(C_1, \ldots, C_k) = \sum_{l=1}^{k} \frac{\text{Cut}(C_l, V \setminus C_l)}{\text{vol}(C_l)},
\]

where \( \text{vol}(C_l) = \sum_{i \in C_l} d_i = \sum_{i \in C_l, j \in [n]} W_{ij} \). Encoding a clustering \( V = C_1 \cup \ldots \cup C_k \) by a matrix \( H \in \mathbb{R}^{n \times k} \) with

\[
H_{il} = \begin{cases} 
1/\sqrt{\text{vol}(C_l)}, & i \in C_l, \\
0, & i \notin C_l,
\end{cases}
\]

we have \( \text{NCut}(C_1, \ldots, C_k) = \text{Tr}(H^T L H) \). Note that any \( H \) of the form (16) satisfies \( H^T D H = I_k \). Normalized spectral clustering relaxes the problem of minimizing \( \text{Tr}(H^T L H) \) over all \( H \) of the form (16) to

\[
\min_{H \in \mathbb{R}^{n \times k}} \text{Tr}(H^T L H) \text{ subject to } H^T D H = I_k.
\]

Substituting \( H = D^{-1/2} T \) for \( T \in \mathbb{R}^{n \times k} \) (we need to assume that \( G \) does not contain any isolated vertices since otherwise \( D^{-1/2} \) does not exist), problem (17) becomes

\[
\min_{T \in \mathbb{R}^{n \times k}} \text{Tr}(T^T D^{-1/2} L D^{-1/2} T) \text{ subject to } T^T T = I_k.
\]

Similarly to unnormalized spectral clustering, normalized spectral clustering computes an optimal \( T \) by computing the \( k \) smallest eigenvalues and some corresponding eigenvectors of \( D^{-1/2} L D^{-1/2} \) and applies \( k \)-means clustering to the rows of \( H = D^{-1/2} T \) (in practice, \( H \) can be computed directly by solving the generalized eigenproblem \( L x = \lambda D x, x \in \mathbb{R}^n, \lambda \in \mathbb{R} \); see von Luxburg, 2007).

Now we want to derive our fair version of normalized spectral clustering. The first step is to show that Lemma 1 holds true if we encode a clustering as in (16):

**Lemma 2** (Fairness constraint as linear constraint on \( H \) for normalized spectral clustering). For \( s \in [h], \) let \( f^{(s)} \in \{0, 1\}^n \) be the group-membership vector of \( V_s \), that is \( f^{(s)}_i = 1 \) if \( i \in V_s \) and \( f^{(s)}_i = 0 \) otherwise. Let \( V = C_1 \cup \ldots \cup C_k \) be a clustering that is encoded as in (16). We have, for every \( l \in [k],
\)

\[
\forall s \in [h - 1]: \sum_{i=1}^n \left( f^{(s)}_i - \frac{|V_s|}{n} \right) H_{il} = 0 \iff \forall s \in [h]: \frac{|V_s \cap C_l|}{|C_l|} = \frac{|V_s|}{n}.
\]

**Proof.** This simply follows from

\[
\sum_{i=1}^n \left( f^{(s)}_i - \frac{|V_s|}{n} \right) H_{il} = \frac{|V_s \cap C_l|}{\sqrt{\text{vol}(C_l)}} \frac{|V_s|}{n\sqrt{\text{vol}(C_l)}}
\]

and \( |C_l| = \sum_{s=1}^h |V_s \cap C_l| \). \hfill \Box

Lemma 2 suggests that in a fair version of normalized spectral clustering, rather than solving (17), we should solve

\[
\min_{H \in \mathbb{R}^{n \times k}} \text{Tr}(H^T L H) \text{ subject to } H^T D H = I_k \text{ and } F^T H = 0_{(h-1) \times k},
\]

where \( F \) is a matrix with all columns equal to \( f^{(s)} \) for \( s \in [h - 1] \).
Algorithm 3 Normalized SC with fairness constraints

**Input:** weighted adjacency matrix $W \in \mathbb{R}^{n \times n}$ (the underlying graph must not contain any isolated vertices); $k \in \mathbb{N}$; group-membership vectors $f^{(s)} \in \{0, 1\}^n$, $s \in [h]$.

**Output:** a clustering of $[n]$ into $k$ clusters

- compute the Laplacian matrix $L = D - W$ with the degree matrix $D$
- build the matrix $F$ that has the vectors $f^{(s)} - \frac{|V_s|}{n} \cdot 1_n$, $s \in [h-1]$, as columns
- compute a matrix $Z$ whose columns form an orthonormal basis of the nullspace of $F^T$
- compute the square root $Q$ of $Z^T D Z$
- compute some orthonormal eigenvectors corresponding to the $k$ smallest eigenvalues (respecting multiplicities) of $Q^{-1} Z^T L Z Q^{-1}$
- let $X$ be a matrix containing these eigenvectors as columns
- apply $k$-means clustering to the rows of $H = Z Q^{-1} X \in \mathbb{R}^{n \times k}$, which yields a clustering of $[n]$ into $k$ clusters.

where $F \in \mathbb{R}^{n \times (h-1)}$ is the matrix that has the vectors $f^{(s)} - (|V_s|/n) \cdot 1_n$, $s \in [h-1]$, as columns (just as in Section 3). It is $\text{rank}(F) = \text{rank}(F^T) = h - 1$ and we need to assume that $k \leq n - h + 1$ since otherwise (18) does not have any solution. Let $Z \in \mathbb{R}^{n \times (n-h+1)}$ be a matrix whose columns form an orthonormal basis of the nullspace of $F^T$. We substitute $H = Z Y$ for $Y \in \mathbb{R}^{(n-h+1) \times k}$ and then problem (18) becomes

$$\min_{Y \in \mathbb{R}^{(n-h+1) \times k}} \text{Tr}(Y^T Z^T L Z Y) \quad \text{subject to} \quad Y^T Z^T D Z Y = I_k.$$  \tag{19}$$

Assuming that $G$ does not contain any isolated vertices, $Z^T D Z$ is positive definite and hence has a positive definite square root, that is there exists a positive definite $Q \in \mathbb{R}^{(n-h+1) \times (n-h+1)}$ with $Z^T D Z = Q^2$. We can substitute $Y = Q^{-1} X$ for $X \in \mathbb{R}^{(n-h+1) \times k}$ and then problem (19) becomes

$$\min_{X \in \mathbb{R}^{(n-h+1) \times k}} \text{Tr}(X^T Q^{-1} Z^T L Z Q^{-1} X) \quad \text{subject to} \quad X^T X = I_k.$$  \tag{20}$$

A solution to (20) is given by the matrix $X$ that contains some orthonormal eigenvectors corresponding to the $k$ smallest eigenvalues (respecting multiplicities) of $Q^{-1} Z^T L Z Q^{-1}$ as columns. This gives rise to our fair version of normalized spectral clustering as stated in Algorithm 3.

**B. Computational Complexity of our Algorithms**

The costs of standard spectral clustering (e.g., Algorithm 1) are dominated by the complexity of the eigenvector computations and are commonly stated to be, in general, in $\mathcal{O}(n^3)$ regarding time and $\mathcal{O}(n^2)$ regarding space for an arbitrary number of clusters $k$, unless approximations are applied (Yan et al., 2009; Li et al., 2011). In addition to the computations performed in Algorithm 1, in Algorithm 2 and Algorithm 3 we have to compute an orthonormal basis of the nullspace of $F^T$, perform some matrix multiplications, and (only for Algorithm 3) compute the square root of an $(n-h+1) \times (n-h+1)$-matrix and the inverse of this square root. All these computations can be done in $\mathcal{O}(n^3)$ regarding time and $\mathcal{O}(n^2)$ regarding space (an orthonormal basis of the nullspace of $F^T$ can be computed by means of an SVD; see, e.g., Golub & Van Loan, 2013), and hence our algorithms have the same worst-case complexity as standard spectral clustering. On the other hand, if the graph $G$, and thus the Laplacian matrix $L$, is sparse or $k$ is small, then the eigenvector computations in Algorithm 1 can be done more efficiently than with cubic running time (Bai et al., 2000). This is not the case for our algorithms as stated. However, one could apply one of the techniques suggested in the existing literature on constrained spectral clustering to speed up computation (e.g., Yu & Shi, 2004, or Xu et al., 2009; see Section 5 of the main paper). With the implementation as stated, in our experiments in Section 6 of the main paper we observe that Algorithm 2 has a similar running time as standard normalized spectral clustering while the running time of Algorithm 3 is significantly higher.
C. Proof of Theorem 1

We split the proof of Theorem 1 into four parts. In the first part, we analyze the eigenvalues and eigenvectors of the expected adjacency matrix \( \mathcal{W} \) and of the matrix \( Z^T L Z \), where \( L \) is the expected Laplacian matrix and \( Z \) is the matrix computed in the execution of Algorithm 2 or Algorithm 3. In the second part, we study the deviation of the observed matrix \( Z^T L Z \) from the expected matrix \( Z^T L Z \). In the third part, we use the results from the first and the second part to prove Theorem 1 for Algorithm 2 (unnormalized SC with fairness constraints). In the fourth part, we prove Theorem 1 for Algorithm 3 (normalized SC with fairness constraints).

Notation. For \( x \in \mathbb{R}^n \), by \( \|x\| \) we denote the Euclidean norm of \( x \), that is \( \|x\| = \sqrt{x_1^2 + \ldots + x_n^2} \). For \( A \in \mathbb{R}^{n \times m} \), by \( \|A\| \) we denote the operator norm (also known as spectral norm) and by \( \|A\|_F \) the Frobenius norm of \( A \). It is

\[
\|A\| = \max_{x \in \mathbb{R}^n : \|x\| = 1} \|Ax\| = \lambda_{\max}(A^T A),
\]

where \( \lambda_{\max}(A^T A) \) is the largest eigenvalue of \( A^T A \), and

\[
\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} A_{ij}^2} = \sqrt{\text{Tr}(A^T A)}.
\]

Note that for a symmetric matrix \( A \in \mathbb{R}^{n \times n} \) with \( A = A^T \) we have \( \|A\| = \max\{\lambda_i : \lambda_i \text{ is an eigenvalue of } A\} \). It follows from (21) and (22) that for any \( A \in \mathbb{R}^{n \times m} \) with rank at most \( r \) we have

\[
\|A\| \leq \|A\|_F \leq \sqrt{r} \|A\|.
\]

We use \( \text{const}(X) \) to denote a universal constant that only depends on \( X \) and that may change from line to line.

Part 1: Eigenvalues and eigenvectors of \( \mathcal{W} \) and of \( Z^T L Z \)

Assuming the \( n \) vertices \( 1, \ldots, n \) are sorted in a way such that

\[
1, \ldots, \frac{n}{kh} \in C_1 \cap V_1, \quad \frac{n}{kh} + 1, \ldots, \frac{2n}{kh} \in C_1 \cap V_2, \quad \ldots \ldots, \quad \frac{(h-1)n}{kh} + 1, \ldots, \frac{n}{k} \in C_1 \cap V_h,
\]

\[
\frac{n}{kh} + 1, \ldots, \frac{n}{k} \in C_2 \cap V_1, \quad \ldots \ldots, \quad \frac{n}{kh} + 1, \ldots, \frac{2n}{kh} \in C_2 \cap V_h,
\]

\[
\vdots
\]

\[
\frac{(k-1)n}{k} + 1, \ldots, \frac{(k-1)n}{k} \in C_k \cap V_1, \quad \ldots \ldots, \quad \frac{(k-1)n}{k} + 1, \ldots, \frac{n}{k} \in C_k \cap V_h,
\]

the expected adjacency matrix \( \mathcal{W} \in \mathbb{R}^{n \times n} \) is given by the block matrix

\[
\mathcal{W} = \begin{pmatrix}
[R] & [S] & [S] & [S] & \cdots & [S] & [S] \\
[S] & [R] & [S] & [S] & \cdots & [S] & [S] \\
[S] & [S] & [R] & [S] & \cdots & [S] & [S] \\
\vdots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots \\
[S] & [S] & [S] & [S] & \cdots & [S] & [R]
\end{pmatrix} - a I_n,
\]

where \( a = \text{const}(X) \) and \( I_n \) is the \( n \times n \) identity matrix.
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where \([R] \in \{a, c\}^{\frac{n}{kh}} \times \{b, d\}^{\frac{n}{kh}}\) and \([S] \in \{b, d\}^{\frac{n}{kh}} \times \{d\}^{\frac{n}{kh}}\) are themselves block matrices

\[
[R] = \begin{pmatrix}
[a] & [c] & [c] & [c] & \cdots & [c] & [c] \\
[c] & [a] & [c] & [c] & \cdots & [c] & [c] \\
[c] & [c] & [a] & [c] & \cdots & [c] & [c] \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
[c] & [c] & [c] & [c] & \cdots & [c] & [a]
\end{pmatrix}
\quad \quad \quad
[S] = \begin{pmatrix}
[b] & [d] & [d] & [d] & \cdots & [d] & [d] \\
[d] & [b] & [d] & [d] & \cdots & [d] & [d] \\
[d] & [d] & [b] & [d] & \cdots & [d] & [d] \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
[d] & [d] & [d] & [d] & \cdots & [d] & [b]
\end{pmatrix}
\]

with \([a], [b], [c]\) and \([d]\) being matrices of size \(\frac{n}{kh} \times \frac{n}{kh}\) with all entries equaling \(a, b, c\) and \(d\), respectively. We denote the matrix \(\mathcal{W} + aI_n\) with \(\mathcal{W}\) as in (25) by \(\tilde{\mathcal{W}}\). If the vertices are not ordered as in (24), the expected adjacency matrix \(\mathcal{W}\) is rather given by \(\mathcal{W} = P^T \hat{\mathcal{W}} P - aI_n\) for some permutation matrix \(P \in \{0, 1\}^{n \times n}\) with \(PP^T = P^TP = I_n\). Note that \(v \in \mathbb{R}^n\) is an eigenvector of \(\tilde{\mathcal{W}}\) with eigenvalue \(\lambda\) if and only if \(P^TV\) is an eigenvector of \(P^T\hat{\mathcal{W}} P\) with eigenvalue \(\lambda\). Keeping track of the permutation matrices \(P\) and \(P^T\) throughout the proof of Theorem 1 does not impose any technical challenges, but makes the writing more complicated. Hence, for simplicity and without loss of generality, we assume in the following that the vertices are ordered as in (24).

The following lemma characterizes the eigenvalues and eigenvectors of \(\tilde{\mathcal{W}}\). Clearly, this also characterizes the eigenvalues and eigenvectors of \(\mathcal{W}\): \(v \in \mathbb{R}^n\) is an eigenvector of \(\tilde{\mathcal{W}}\) with eigenvalue \(\lambda\) if and only if \(v\) is an eigenvector of \(\mathcal{W}\) with eigenvalue \(\lambda - a\).

**Lemma 3.** Assuming that \(a > b > c > d \geq 0\), the matrix \(\tilde{\mathcal{W}}\) has rank \(kh\) or rank \(k + h - 1\) (the latter is true if and only if \(a - c = b - d\)). It has the following eigenvalues \(\lambda_1, \ldots, \lambda_n\):

\[
\lambda_1 = \frac{n}{kh} [(a + (h - 1)c) + (k - 1)(b + (h - 1)d)],
\]

\[
\lambda_2 = \lambda_3 = \ldots = \lambda_h = \frac{n}{kh} [(a - c) + (k - 1)(b - d)],
\]

\[
\lambda_{h+1} = \lambda_{h+2} = \ldots = \lambda_{h+k-1} = \frac{n}{kh} [(a + (h - 1)c) - (b + (h - 1)d)],
\]

\[
\lambda_{h+k}, \lambda_{h+k+1} = \ldots = \lambda_{hhk} = \frac{n}{kh} [(a - c) - (b - d)],
\]

\[
\lambda_{hhk+1} = \lambda_{hhk+2} = \ldots = \lambda_n = 0.
\]

(26)

It is \(\lambda_1 > \lambda_2 = \ldots = \lambda_h > 0\) as well as \(\lambda_1 > \lambda_{h+1} = \ldots = \lambda_{h+k-1} > 0\) and \(\lambda_2 = \ldots = \lambda_h > |\lambda_{h+k}| = \ldots = |\lambda_{hhk}|\) as well as \(\lambda_{h+1} = \ldots = \lambda_{h+k-1} > |\lambda_{hhk}| = \ldots = |\lambda_{hhk}|\).

An eigenvector corresponding to \(\lambda_1\) is given by \(v_1 = 1_n\). The eigenspace corresponding to the eigenvalue \(\lambda_2 = \ldots = \lambda_h\) is
spanned by the vectors $v_2, \ldots, v_h$ with

$$
\begin{bmatrix}
1 \\
-\frac{1}{h-1} \\
-\frac{1}{h-1} \\
\vdots \\
-\frac{1}{h-1}
\end{bmatrix},
\begin{bmatrix}
-\frac{1}{h-1} \\
1 \\
-\frac{1}{h-1} \\
\vdots \\
-\frac{1}{h-1}
\end{bmatrix},
\begin{bmatrix}
-\frac{1}{h-1} \\
-\frac{1}{h-1} \\
1 \\
\vdots \\
-\frac{1}{h-1}
\end{bmatrix},
\begin{bmatrix}
-\frac{1}{h-1} \\
-\frac{1}{h-1} \\
-\frac{1}{h-1} \\
\vdots \\
1
\end{bmatrix}
$$

where for $z \in \mathbb{R}$, by $[z]$ we denote a block of size $\frac{n}{kh}$ with all entries equaling $z$. The eigenspace corresponding to the eigenvalue $\lambda_{h+1} = \ldots = \lambda_{h+k-1}$ is spanned by the vectors $v_{h+1}, \ldots, v_{h+k-1}$ with

$$
\begin{bmatrix}
1 \\
-\frac{1}{k-1} \\
-\frac{1}{k-1} \\
\vdots \\
-\frac{1}{k-1}
\end{bmatrix},
\begin{bmatrix}
-\frac{1}{k-1} \\
1 \\
-\frac{1}{k-1} \\
\vdots \\
-\frac{1}{k-1}
\end{bmatrix},
\begin{bmatrix}
-\frac{1}{k-1} \\
-\frac{1}{k-1} \\
1 \\
\vdots \\
-\frac{1}{k-1}
\end{bmatrix},
\begin{bmatrix}
-\frac{1}{k-1} \\
-\frac{1}{k-1} \\
-\frac{1}{k-1} \\
\vdots \\
1
\end{bmatrix}
$$

where for $z \in \mathbb{R}$, by $[z]$ we denote a block of size $\frac{n}{k}$ with all entries equaling $z$. The eigenspace corresponding to the
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eigenvalue $\lambda_{h+k} = \ldots = \lambda_{h+k}$ is spanned by the vectors $v_{h+k}, \ldots, v_{h+k}$ with

\[ \begin{bmatrix} 1 \\ -\frac{1}{h-1} \\ -\frac{1}{h-1} \\ \vdots \\ -\frac{1}{h-1} \\ -\frac{1}{h-1} \end{bmatrix}, \begin{bmatrix} -\frac{1}{h-1} \\ 1 \\ -\frac{1}{h-1} \\ \vdots \\ -\frac{1}{h-1} \\ -\frac{1}{h-1} \end{bmatrix}, \ldots, \begin{bmatrix} -\frac{1}{h-1} \\ 1 \\ -\frac{1}{h-1} \\ \vdots \\ -\frac{1}{h-1} \\ -\frac{1}{h-1} \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ -\frac{1}{h-1} \\ -\frac{1}{h-1} \\ \vdots \\ -\frac{1}{h-1} \\ -\frac{1}{h-1} \end{bmatrix}, \begin{bmatrix} -\frac{1}{h-1} \\ 1 \\ -\frac{1}{h-1} \\ \vdots \\ -\frac{1}{h-1} \\ -\frac{1}{h-1} \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{1}{h-1} \\ -\frac{1}{h-1} \\ \vdots \\ -\frac{1}{h-1} \\ -\frac{1}{h-1} \end{bmatrix} \]

where for $z \in \mathbb{R}$, by $[z]$ we denote a block of size $\frac{h}{kh}$ with all entries equaling $z$.

Proof. The matrix $\tilde{W}$ has only $kh$ different columns and hence rank $\tilde{A} \leq kh$ and there are at most $kh$ many non-zero eigenvalues. The statement about the rank of $\tilde{W}$ follows from the statement about the eigenvalues of $\tilde{W}$.

It is easy to verify that any of the vectors $v_i$ is actually an eigenvector of $\tilde{W}$ corresponding to eigenvalue $\lambda_i$, $i \in [hk]$. We need to show that the vectors $v_2, \ldots, v_h$, the vectors $v_{h+1}, \ldots, v_{h+k-1}$, as well as the vectors $v_{h+k}, \ldots, v_{h+k}$ are linearly independent. For example, let us show that $v_2, \ldots, v_h$ are linearly independent: assume that $\sum_{j \in \{2, \ldots, h\}} \alpha_j v_j = 0$ for some $\alpha_j \in \mathbb{R}$. We need to show that $\alpha_j = 0$, $j \in \{2, \ldots, h\}$. Looking at the $n$-th coordinate of $\sum_{j \in \{2, \ldots, h\}} \alpha_j v_j$, we infer that
which shows that $f$ which shows $\lambda$

The next lemma provides an orthonormal basis of the eigenspace associated with eigenvalue $\lambda$

and hence $\alpha_i = 0$, $i \in \{2, \ldots, h\}$. Similarly, we can show that the vectors $v_{h+1}, \ldots, v_{h+k-1}$ as well as the vectors $v_{h+k}, \ldots, v_h$ are linearly independent.

Since we assume $a > b > c > d \geq 0$, we clearly have

$$(a + (h - 1)c + (k - 1)(b + (h - 1)d) > (a - c) + (k - 1)(b - d) > 0$$

and

$$(a + (h - 1)c + (k - 1)(b + (h - 1)d) > (a + (h - 1)c) - (b + (h - 1)d) = (a - b) + (h - 1)(c - d) > 0,$$

which shows that $\lambda_1 > \lambda_2 = \ldots = \lambda_h > 0$ and $\lambda_1 > \lambda_{h+1} = \ldots = \lambda_{h+k-1} > 0$. It is

$$(a - c) + (k - 1)(b - d) > (a - c) - (b - d) \quad \text{and} \quad (a - c) + (k - 1)(b - d) > -(a - c) + (b - d),$$

and also

$$(a + (h - 1)c) - (b + (h - 1)d) = (a - b) + (h - 1)(c - d) \not\geq (a - b) + (c - d) > (a - b) + (d - c) = (a - c) - (b - d)$$

and

$$(a + (h - 1)c) - (b + (h - 1)d) = (a - b) + (h - 1)(c - d) \not\geq (a - b) + (c - d) > (b - a) + (c - d) = -(a - c) + (b - d),$$

which shows $\lambda_2 = \ldots = \lambda_h > |\lambda_{h+k}| = \ldots = |\lambda_{hk}|$ and $\lambda_{h+1} = \ldots = \lambda_{h+k-1} > |\lambda_{h+k}| = \ldots = |\lambda_{hk}|$.

Note that we have

$$f(s) - \frac{|V_s|}{n} \cdot 1_n = f(s) - \frac{1}{h} \cdot 1_n = \frac{h - 1}{h} v_{1+s}, \quad s \in [h - 1],$$

(27)

where $f(s)$ is the group-membership vector of $V_s$ and $f(s) - \frac{|V_s|}{n}$ is the vector encountered in the second step of Algorithm 2 or Algorithm 3.

The next lemma provides an orthonormal basis of the eigenspace associated with eigenvalue $\lambda_{h+1} = \ldots = \lambda_{h+k-1}$.\n
**Lemma 4.** An orthonormal basis of the eigenspace of $\mathcal{W}$ corresponding to the eigenvalue $\lambda_{h+1} = \ldots = \lambda_{h+k-1}$ is given by $\{n_1, \ldots, n_{k-1}\}$ with

\[
n_1 = \begin{pmatrix} (k-1)q_1 \\ -q_1 \\ -q_1 \\ \vdots \\ -q_1 \\ -q_1 \end{pmatrix}, \quad n_2 = \begin{pmatrix} (k-2)q_2 \\ -q_2 \\ -q_2 \\ \vdots \\ -q_2 \\ -q_2 \end{pmatrix}, \quad \ldots, \quad n_i = \begin{pmatrix} (k-i)q_i \\ -q_i \\ -q_i \\ \vdots \\ -q_i \\ -q_i \end{pmatrix}, \quad \ldots, \quad n_{k-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ q_{k-1} \end{pmatrix}, \quad (28)
\]

where for $z \in \mathbb{R}$, by $[z]$ we denote a block of size $\frac{n}{k}$ with all entries equaling $z$ and

\[q_i = \frac{1}{\sqrt{\left(\frac{n}{k}(k-i)^2 + \frac{n}{k}(k-i)\right)}}, \quad i \in [k-1].\]
Furthermore, there exists an orthonormal basis \( \{u_i\} \) of \( \mathbb{W} \) with eigenvalue \( \lambda_{i+1} = \ldots = \lambda_{i+k-1} \), \( i \in [k-1] \). Furthermore, we have
\[
\|n_i\|^2 = q_i^2 \left( \frac{n}{k} (k-i)^2 + \frac{n}{k} (k-i) \right) = 1, \quad i \in [k-1],
\]
and
\[
\langle n_i, n_j \rangle = \frac{n}{k} (-q_i \cdot (k-j) q_j) + \frac{n}{k} (k-j)(q_i \cdot q_j) = 0, \quad 1 \leq i < j \leq n.
\]

\[\square\]

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**Proof.** It is easy to verify that any \( n_i \) is indeed an eigenvector of \( \mathbb{W} \) with eigenvalue \( \lambda_{i+1} = \ldots = \lambda_{i+k-1} \), \( i \in [k-1] \). Furthermore, we have
\[
\|n_i\|^2 = q_i^2 \left( \frac{n}{k} (k-i)^2 + \frac{n}{k} (k-i) \right) = 1, \quad i \in [k-1],
\]
and
\[
\langle n_i, n_j \rangle = \frac{n}{k} (-q_i \cdot (k-j) q_j) + \frac{n}{k} (k-j)(q_i \cdot q_j) = 0, \quad 1 \leq i < j \leq n.
\]

Let \( L \) be the expected Laplacian matrix. We have \( L = D - W \), where \( D \) is the expected degree matrix. The expected degree of vertex \( i \) in a random graph constructed according to our variant of the stochastic block model equals \( \sum_{j \in [n] \setminus \{i\}} W_{ij} = \lambda_{1-a} \) (with \( \lambda_1 \) defined in (26)) and hence \( D = (\lambda_1 - a) I_n \).

The following lemma characterizes the eigenvalues and eigenvectors of \( Z^T L Z \), where \( Z \in \mathbb{R}^{n \times (n-h+1)} \) is the matrix computed in the execution of Algorithm 2 or Algorithm 3.

**Lemma 5.** Let \( Z \in \mathbb{R}^{n \times (n-h+1)} \) be any matrix whose columns form an orthonormal basis of the nullspace of \( F^T \), where \( F \) is the matrix that has the vectors \( f^{(s)} - \frac{|V_s|}{n} \cdot 1_n, s \in [h-1] \), as columns. Then the eigenvalues of \( Z^T L Z \) are
\[
\lambda_1 - \lambda_1, \lambda_1 - \lambda_{h+1}, \lambda_1 - \lambda_{h+2}, \ldots, \lambda_1 - \lambda_n
\]
with \( \lambda_i \) defined in (26). It is
\[
\begin{align*}
\lambda_1 - \lambda_1 &= 0, \\
\lambda_1 - \lambda_{h+1} &= \lambda_1 - \lambda_{h+2} = \ldots = \lambda_1 - \lambda_{h+k-1}, \\
\lambda_1 - \lambda_{k+h} &= \lambda_1 - \lambda_{h+k+1} = \ldots = \lambda_1 - \lambda_{h+k}, \\
\lambda_1 - \lambda_{k+1} &= \lambda_1 - \lambda_{h+2} = \ldots = \lambda_1 - \lambda_n = \lambda_1
\end{align*}
\]
with
\[
\lambda_1 - \lambda_1 < \lambda_1 - \lambda_{h+1} < \min\{\lambda_1 - \lambda_{h+k}, \lambda_1 - \lambda_{h+k+1}\},
\]
so that the \( k \) smallest eigenvalues of \( Z^T L Z \) are
\[
\lambda_1 - \lambda_1, \lambda_1 - \lambda_{h+1}, \lambda_1 - \lambda_{h+2}, \ldots, \lambda_1 - \lambda_{h+k-1}.
\]
Furthermore, there exists an orthonormal basis \( \{r_1, r_{h+1}, r_{h+2}, \ldots, r_n\} \) of eigenvectors of \( Z^T L Z \) with \( r_i \) corresponding to eigenvalue \( \lambda_1 - \lambda_i \) such that
\[
Z r_1 = 1_n / \sqrt{n} \quad \text{and} \quad Z r_{h+i} = n_i, \quad i \in [k-1],
\]
with \( n_i \) defined in (28).

**Proof.** Because of \( Z^T Z = I_{(n-h+1)} \) we have
\[
Z^T L Z = Z^T (D - W) Z = Z^T D Z - Z^T (\mathbb{W} - a I_n) Z = (\lambda_1 - a) I_n - Z^T \mathbb{W} Z + a I_n = \lambda_1 I_n - Z^T \mathbb{W} Z.
\]

Let \( \{u_1, \ldots, u_n\} \) be an orthonormal basis of eigenvectors of \( \mathbb{W} \) with \( u_i \) corresponding to eigenvalue \( \lambda_i \). According to Lemma 3 and Lemma 4 we can choose \( u_1 = 1_n / \sqrt{n} \) and \( u_{h+i} = n_i \) for \( i \in [k-1] \). We can write \( \mathbb{W} \) as \( \mathbb{W} = \sum_{i=1}^n \lambda_i u_i u_i^T \).

The nullspace of \( F^T \), where \( F \) is the matrix that has the vectors \( f^{(s)} - \frac{|V_s|}{n} \cdot 1_n, s \in [h-1] \), as columns, equals the orthogonal complement of \( \{f^{(s)} - \frac{|V_s|}{n} \cdot 1_n, s \in [h-1]\} \). According to (27), the orthogonal complement of \( \{f^{(s)} - \frac{|V_s|}{n} \cdot 1_n, s \in [h-1]\} \) equals the orthogonal complement of \( \{v_{1+s}, s \in [h-1]\} \), with \( v_i \) defined in Lemma 3 and being an eigenvalue of \( \mathbb{W} \) with eigenvalue \( \lambda_i \). According to Lemma 3, \( \{v_{1+s}, s \in [h-1]\} \) is a basis of the eigenspace of \( \mathbb{W} \) corresponding to eigenvalue \( \lambda_2 = \lambda_3 = \ldots = \lambda_h \), and hence the orthogonal complement of \( \{v_{1+s}, s \in [h-1]\} \)
We want to obtain an upper bound on $\|\|Z^{T}LZ\|\|$. Because of $Z^{T}Z = I_{(n-h+1)}$, it is $\|Z\| = \|Z^{T}\| = 1$ and hence
\[
\|Z^{T}LZ - Z^{T}\hat{L}Z\| \leq \|Z^{T}\| \cdot \|L - \hat{L}\| \cdot \|Z\| \leq \|L - \hat{L}\|.
\]  
(32)

We have
\[
\|L - \hat{L}\| = \|((D - W) - (\hat{D} - \hat{W}))\| \leq \|D - \hat{D}\| + \|W - \hat{W}\|
\],
with $\hat{D} = (\lambda_1 - a)I_n$ as we have seen in Part 1. We bound both terms separately.

**Lemma 6.** Let $T \in \mathbb{R}^{n \times k}$ be a matrix that contains the vectors $1_n/\sqrt{n}, n_1, n_2, \ldots, n_{k-1}$, with $n_i$ defined in (28), as columns. For $i \in [n]$, let $t_i$ denote the $i$-th row of $T$. For all $i, j \in [n]$, we have $t_i = t_j$ if and only if the vertices $i$ and $j$ are in the same cluster $C_l$. If the vertices $i$ and $j$ are not in the same cluster, then $\|t_i - t_j\| = \sqrt{2k/n}$.

**Proof.** This simply follows from the structure of the vectors $n_i$. It is, up to a permutation of the entries,
\[
t_i = \left(\frac{1}{\sqrt{n}}, -q_1, -q_2, \ldots, -q_{l-1}, (k - l)q_l, 0, 0, \ldots, 0\right),
\]
with $q_l$ defined in (29), for all $i \in [n]$ such that vertex $i$ is in cluster $C_l$, $l \in [k - 1]$, and
\[
t_i = \left(\frac{1}{\sqrt{n}}, -q_1, -q_2, \ldots, -q_{k-1}\right)
\]
for all $i \in [n]$ such that vertex $i$ is in cluster $C_k$. It is easy to verify that $\|t_i - t_j\|^2 = 2k/n$ for all $i, j \in [n]$ such that the vertices $i$ and $j$ are not in the same cluster.

**Part 2: Deviation of $Z^{T}LZ$ from $Z^{T}\hat{L}Z$**

We want to obtain an upper bound on $\|Z^{T}LZ - Z^{T}\hat{L}Z\|$. Because of $Z^{T}Z = I_{(n-h+1)}$, it is $\|Z\| = \|Z^{T}\| = 1$ and hence
\[
\|Z^{T}LZ - Z^{T}\hat{L}Z\| \leq \|Z^{T}\| \cdot \|L - \hat{L}\| \cdot \|Z\| \leq \|L - \hat{L}\|.
\]  
(32)

We have
\[
\|L - \hat{L}\| = \|((D - W) - (\hat{D} - \hat{W}))\| \leq \|D - \hat{D}\| + \|W - \hat{W}\|
\],
with $\hat{D} = (\lambda_1 - a)I_n$ as we have seen in Part 1. We bound both terms separately.
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- Upper bound on $\|W - W\|$:

  Theorem 5.2 of Lei & Rinaldo (2015) provides a bound on $\|W - W\|$; assuming that $a \geq C \ln n/n$ for some $C > 0$, for every $r > 0$ there exists a constant $\text{const}(C, r)$ such that

  $$\|W - W\| \leq \text{const}(C, r) \sqrt{a \cdot n}$$

  with probability at least $1 - n^{-r}$.

- Upper bound on $\|D - D\|$:

  The matrix $D - D$ is a diagonal matrix and hence $\|D - D\| = \max_{i \in [n]} |D_{ii} - D_{ii}| = \max_{i \in [n]} |D_{ii} - (\lambda_1 - a)|$. The random variable $D_{ii} = \sum_{j \in [n] \setminus \{i\}} 1_{[i \sim j]}$, where $1_{[i \sim j]}$ denotes the indicator function of the event that there is an edge between vertices $i$ and $j$, is a sum of independent Bernoulli random variables. It is $\mathbb{E}[D_{ii}] = \lambda_1 - a$. For a fixed $i \in [n]$, we want to obtain an upper bound on $|D_{ii} - (\lambda_1 - a)| = |D_{ii} - \mathbb{E}[D_{ii}]|$ and distinguish two cases:

  1. $a > \frac{1}{2}$:

     Hoeffding’s inequality (e.g., Boucheron et al., 2004, Theorem 1) yields

     $$\Pr[|D_{ii} - (\lambda_1 - a)| \geq t] \leq 2 \exp\left( -\frac{2t^2}{n} \right)$$

     for any $t > 0$. Choosing $t = \sqrt{2(r + 1)\sqrt{a \cdot n \ln n}}$ for $r > 0$, we have with $\text{const}(r) = \sqrt{2(r + 1)}$ that

     $$\Pr\left[|D_{ii} - (\lambda_1 - a)| \geq \text{const}(r) \cdot \sqrt{a \cdot n \ln n} \right] \leq 2 \exp\left(-4(r + 1)a \ln n \right) \leq n^{-(r + 1)}.$$  

  2. $a \leq \frac{1}{2}$:

     Bernstein’s inequality (e.g., Boucheron et al., 2004, Theorem 3) yields

     $$\Pr[|D_{ii} - (\lambda_1 - a)| > tn] \leq 2 \exp\left( -\frac{n t^2}{2 \left( \frac{\text{Var}[D_{ii}]}{n} + \frac{t}{3} \right)} \right)$$

     for any $t > 0$. It is

     $$\text{Var}[D_{ii}] = \sum_{j \in [n] \setminus \{i\}} \text{Var}[1_{[i \sim j]}] = \sum_{j \in [n] \setminus \{i\}} \Pr[1_{[i \sim j]}](1 - \Pr[1_{[i \sim j]}]) \leq na(1 - a) \leq na$$

     since the function $x \mapsto x(1 - x)$ is monotonically increasing on $[0, 1/2]$. If we choose $t = \text{const} \cdot \sqrt{a \cdot n \ln n}$ for some $\text{const} > 0$, assuming that $a \geq C \ln n/n$ for some $C > 0$, we have

     $$\frac{\text{Var}[D_{ii}]}{n} + \frac{t}{3} \leq a \left(1 + \frac{\text{const}}{3\sqrt{C}}\right)$$

     and hence

     $$\Pr\left[|D_{ii} - (\lambda_1 - a)| > \text{const} \cdot \sqrt{a \cdot n \ln n} \right] \leq 2 \exp\left( -\frac{\text{const}^2 \cdot \ln n}{2 + \frac{2\text{const}^2}{3\sqrt{C}}} \right).$$

     Because of

     $$\frac{\text{const}^2}{2 + \frac{2\text{const}^2}{3\sqrt{C}}} \to \infty \text{ as } \text{const} \to \infty,$$

     for every $r > 0$ we can choose $\text{const} = \text{const}(C, r)$ large enough such that $\text{const}^2 / \left(2 + \frac{2\text{const}^2}{3\sqrt{C}}\right) \geq 2(r + 1)$ and

     $$\Pr\left[|D_{ii} - (\lambda_1 - a)| > \text{const}(C, r) \cdot \sqrt{a \cdot n \ln n} \right] \leq n^{-(r + 1)}.$$  

(35)
Figure 6. Average deviations $\|W - W\|$, $\|D - D\|$ and $\|L - L\|$ as a function of $n$ when $a = 0.6$, $b = 0.5$, $c = 0.4$, $d = 0.3$ are constant, $k = 5$ and $h = 2$. The average is computed over sampling the graph for 100 times.

Choosing $\text{const}(C, r)$ as the maximum of $\text{const}(r)$ encountered in (34) and $\text{const}(C, r)$ encountered in (35), we see that there exists $\text{const}(C, r)$ such that

$$
\Pr \left[ |D_{ii} - (\lambda_1 - a)| > \text{const}(C, r) \cdot \sqrt{a \cdot n \ln n} \right] \leq n^{-(r+1)},
$$

no matter whether $a > 1/2$ or $1/2 \geq a \geq C \ln n/n$. Applying a union bound we obtain

$$
\Pr \left[ \max_{i \in [n]} |D_{ii} - (\lambda_1 - a)| > \text{const}(C, r) \cdot \sqrt{a \cdot n \ln n} \right] \leq n \cdot n^{-(r+1)} = n^{-r},
$$

and hence with probability at least $1 - n^{-r}$ we have

$$
\|D - D\| \leq \text{const}(C, r) \sqrt{a \cdot n \ln n}. \quad (36)
$$

From (33) and (36) we see that for every $r > 0$ there exists $\text{const}(C, r)$ such that with probability at least $1 - n^{-r}$ we have

$$
\|W - W\| \leq \text{const}(C, r) \sqrt{a \cdot n} \quad \text{and} \quad \|D - D\| \leq \text{const}(C, r) \sqrt{a \cdot n \ln n} \quad (37)
$$

and hence

$$
\|Z^T LZ - Z^T L^2\| \leq \|L - L\| \leq \|D - D\| + \|W - W\| \leq \text{const}(C, r) \sqrt{a \cdot n \ln n}. \quad (38)
$$

For illustrative purposes, we show empirically that, in general, our upper bounds on $\|W - W\|$, $\|D - D\|$ and $\|L - L\|$ in (37) and (38), respectively, are tight, up to a factor of at most $\sqrt{\ln n}$ in case of $\|D - D\|$ and $\|L - L\|$. The plot in Figure 6 shows the observed deviations $\|W - W\|$, $\|D - D\|$ and $\|L - L\|$ as a function of $n$ when $a = 0.6$, $b = 0.5$, $c = 0.4$, $d = 0.3$ are constant, $k = 5$ and $h = 2$. The shown curves are average results, obtained from sampling the graph for 100 times.

Part 3: Proving Theorem 1 for Algorithm 2 (unnormalized SC with fairness constraints)

In the last step of Algorithm 2 we apply $k$-means clustering to the rows of the matrix $ZY$, where $Y \in \mathbb{R}^{(n-h+1) \times k}$ contains some orthonormal eigenvectors corresponding to the $k$ smallest eigenvalues of $Z^T LZ$ as columns. We want to show that up to some orthogonal transformation, the rows of $ZY$ are close to the rows of $ZY$, where $Y \in \mathbb{R}^{(n-h+1) \times k}$ contains some orthonormal eigenvectors corresponding to the $k$ smallest eigenvalues of $Z^T LZ$ as columns. According to Lemma 5, we
We want to show that we want to obtain an upper bound on \( \|Z \mathcal{Y} - Z Y U \|_F \). For any \( U \in \mathbb{R}^{k \times k} \) with \( U^T U = UU^T = I_k \), because of \( Z^T Z = I_{(n-h+1)} \) we have
\[
\|Z \mathcal{Y} - Z Y U\|_F^2 = \|Z(Y - Y U)\|_F^2 = \text{Tr}((Y - Y U)^T Z^T Z(Y - Y U)) = \|Y - Y U\|_F^2
\]
and hence
\[
\min_{U \in \mathbb{R}^{k \times k} : U^T U = UU^T = I_k} \|Z \mathcal{Y} - Z Y U\|_F = \min_{U \in \mathbb{R}^{k \times k} : U^T U = UU^T = I_k} \|Y - Y U\|_F. \tag{39}
\]
We proceed similarly to Lei & Rinaldo (2015). According to Proposition 2.2 and Equation (2.6) in Vu & Lei (2013) we have (note that the set of all orthogonal matrices \( U \in \mathbb{R}^{k \times k} \) is a compact subset of \( \mathbb{R}^{k \times k} \) and hence the infimum is indeed a minimum)
\[
\min_{U \in \mathbb{R}^{k \times k} : U^T U = UU^T = I_k} \|Y - Y U\|_F \leq \sqrt{2} \|Y^T (I_{(n-h+1)} - Y Y^T)\|_F \leq \sqrt{2} \sqrt{k} \|Y^T (I_{(n-h+1)} - Y Y^T)\|. \tag{40}
\]
According to Lemma 5 the eigenvalues of \( Z^T L Z \) are \( \lambda_1 - \lambda_1, \lambda_1 - \lambda_{h+1}, \lambda_1 - \lambda_{h+2}, \ldots, \lambda_1 - \lambda_n \). The \( k \) smallest eigenvalues are \( \lambda_1 - \lambda_1, \lambda_1 - \lambda_{h+1}, \lambda_1 - \lambda_{h+2}, \ldots, \lambda_1 - \lambda_{h+k-1} \) and the \( (k+1) \)-th smallest eigenvalue is either \( \lambda_1 - \lambda_{h+k} \) or \( \lambda_1 \). Hence, for the eigengap \( \gamma \) between the \( k \)-th and the \( (k+1) \)-th smallest eigenvalue we have
\[
\gamma = \min \left\{ (\lambda_1 - \lambda_{h+k}) - (\lambda_1 - \lambda_{h+k-1}), (\lambda_1 - \lambda_{h+k-1}) \right\} = \min \left\{ \frac{n}{k} (c - d), \frac{n}{kh} ((a - b) + (h - 1)(c - d)) \right\},
\]
It is
\[
\frac{n}{2k} (c - d) \leq \frac{n(h - 1)}{hk} (c - d) \leq \gamma \leq \frac{n}{k} (c - d). \tag{41}
\]
We want to show that
\[
\|Y^T (I_{(n-h+1)} - Y Y^T)\|_F \leq \frac{4}{\gamma} \|Z^T L Z - Z^T L Z\|. \tag{42}
\]
If \( \|Z^T L Z - Z^T L Z\| > \frac{\gamma}{4} \), then (42) holds trivially because of
\[
\|Y^T (I_{(n-h+1)} - Y Y^T)\|_F \leq \|Y^T\|_F \cdot \|I_{(n-h+1)} - Y Y^T\|_F = 1 \cdot 1 = 1.
\]
Assume that \( \|Z^T L Z - Z^T L Z\| \leq \frac{\gamma}{4} \) and let \( \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{n-h+1} \) be the eigenvalues of \( Z^T L Z \). Since \( L \) is positive semi-definite, so is \( Z^T L Z \), and hence \( \mu_1 \geq 0 \). Let \( \lambda_1', \lambda_2', \ldots, \lambda_{n-h+1}' \) be the eigenvalues \( \lambda_1 - \lambda_1, \lambda_1 - \lambda_{h+1}, \lambda_1 - \lambda_{h+2}, \ldots, \lambda_1 - \lambda_n \) of \( Z^T L Z \) in ascending order. According to Weyl’s Perturbation Theorem (e.g., Bhatia, 1997, Corollary III.2.6) it is
\[
|\mu_i - \lambda'_i| \leq \|Z^T L Z - Z^T L Z\| \leq \frac{\gamma}{4}, \quad i \in [n-h+1].
\]
In particular, we have
\[
\mu_1, \ldots, \mu_k \in \left[0, \lambda'_k + \frac{\gamma}{4}\right], \quad \mu_{k+1}, \ldots, \mu_n \in \left[\lambda'_{k+1} - \frac{\gamma}{4}, \infty\right)
\]
with \( (\lambda'_{k+1} - \frac{\gamma}{4}) - (\lambda'_k + \frac{\gamma}{4}) = \frac{\gamma}{2} \). The Davis-Kahan sin\( \Theta \) Theorem (e.g., Bhatia, 1997, Theorem VII.3.1) yields that
\[
\|Y^T (I_{(n-h+1)} - Y Y^T)\|_F \leq \frac{2}{\gamma} \|Z^T L Z - Z^T L Z\|
\]
and hence (42). Combining (39) to (42), we end up with
\[
\min_{U \in \mathbb{R}^{k \times k} : U^T U = UU^T = I_k} \|Z \mathcal{Y} - Z Y U\|_F \leq \frac{16 \sqrt{k}}{n(c - d)} \|Z^T L Z - Z^T L Z\|. \tag{43}
\]
Using (38) from Part 2, we see that with probability at least $1 - n^{-r}$

$$
\min_{U \in \mathbb{R}^{k \times k}: U^TU = I_k} \|Z\mathcal{Y} - ZYU\|_F \leq \text{const}(C, r) \cdot \frac{\sqrt{k^3}}{c - d} \cdot \sqrt{\frac{a \cdot \ln n}{n}}.
$$

We use Lemma 5.3 in Lei & Rinaldo (2015) to complete the proof of Theorem 1 for Algorithm 2. Assume that (44) holds and let $U \in \mathbb{R}^{k \times k}$ be an orthogonal matrix attaining the minimum, that is we have

$$
\|Z\mathcal{Y}U^T - ZY\|_F = \|Z\mathcal{Y} - ZYU\|_F \leq \text{const}(C, r) \cdot \frac{\sqrt{k^3}}{c - d} \cdot \sqrt{\frac{a \cdot \ln n}{n}}.
$$

As we have noted above, we can choose $\mathcal{Y}$ in such a way that $Z\mathcal{Y} = T$ with $T$ as in Lemma 6. According to Lemma 6, if we denote the $i$-th row of $T$ by $t_i$, then $t_i = t_j$ if the vertices $i$ and $j$ are in the same cluster and $\|t_i - t_j\| = \sqrt{2k/n}$ if the vertices $i$ and $j$ are not in the same cluster. Since multiplying $T$ by $U^T$ from the right side has the effect of applying an orthogonal transformation to the rows of $T$, the same properties are true for the matrix $TU^T$. Lemma 5.3 in Lei & Rinaldo (2015) guarantees that for any $\delta \leq \sqrt{2k/n}$, if

$$
\frac{16 + 8M}{\delta^2} \|TU^T - ZY\|_F^2 < |C_i|, \quad i \in [k],
$$

with $|C_i|$ being the size of cluster $C_i$, then a $(1 + M)$-approximation algorithm for $k$-means clustering applied to the rows of the matrix $ZY$ returns a clustering that misclassifies at most

$$
\frac{4(4 + 2M)}{\delta^2} \|TU^T - ZY\|_F^2
$$

many vertices. If we choose $\delta = \sqrt{2k/n}$, then for a small enough $\tilde{C}_1 = \tilde{C}_1(C, r)$, because of (45), the condition (11) implies (46). Also, for a large enough $C_1 = \tilde{C}_1(C, r)$, the expression (47) is upper bounded by the expression (12).

**Part 4: Proving Theorem 1 for Algorithm 3 (normalized SC with fairness constraints)**

According to Part 2, for every $r > 0$ there exists $\text{const}(C, r)$ such that with probability at least $1 - n^{-r}$ we have

$$
\|D - \mathcal{D}\| \leq \text{const}(C, r) \sqrt{a \cdot n \ln n}.
$$

Condition (13), with a suitable $\tilde{C}_2 = \tilde{C}_2(C, r)$, implies that in this case we also have

$$
\|D - \mathcal{D}\| \leq \frac{\lambda_1 - a}{2}.
$$

Let $\mu'_1, \ldots, \mu'_{n-h+1}$ denote the eigenvalues of $Z^TDZ$. It is $\mathcal{D} = (\lambda_1 - a)I_n$ (see Part 1) and because of $Z^TZ = I_{(n-h+1)}$, we have $Z^TDZ = (\lambda_1 - a)I_{(n-h+1)}$. According to Weyl’s Perturbation Theorem (e.g., Bhatia, 1997, Corollary III.2.6) it is

$$
|\mu'_i - (\lambda_1 - a)| \leq \|Z^TDZ - Z^TD\| \leq \|D - \mathcal{D}\|, \quad i \in [n - h + 1],
$$

where the second inequality follows analogously to (32). It follows from (49) that

$$
\mu'_i \geq \frac{\lambda_1 - a}{2} > 0, \quad i \in [n - h + 1],
$$

In particular, this shows that $Z^TDZ$ is positive definite and hence Algorithm 3 is well-defined.

Now we proceed similarly to Part 3. In the last step of Algorithm 3 we apply $k$-means clustering to the rows of the matrix $Q^{-1}X$, where $Q \in \mathbb{R}^{(n-h+1) \times (n-h+1)}$ is the positive definite square root of $Z^TDZ$ and $X \in \mathbb{R}^{(n-h+1) \times k}$ contains some orthonormal eigenvectors corresponding to the $k$ smallest eigenvalues of $Q^{-1}Z^TLZQ^{-1}$ as columns. We want to show that up to some orthogonal transformation, the rows of $ZQ^{-1}X$ are close to the rows of $ZQ^{-1}X'$, where $Q \in \mathbb{R}^{(n-h+1) \times (n-h+1)}$ is the positive definite square root of $Z^TDZ$ and $X' \in \mathbb{R}^{(n-h+1) \times k}$ contains some orthonormal
We want to obtain an upper bound on $\|Q^{-1}X' - Q^{-1}XU\|_F$. Analogously to (39) we obtain
\[
\min_{U \in \mathbb{R}^{k \times k}; U^T U = UU^T = I_k} \|Q^{-1}X' - Q^{-1}XU\|_F = \min_{U \in \mathbb{R}^{k \times k}; U^T U = UU^T = I_k} \|Q^{-1}X' - Q^{-1}XU\|_F.
\]

The rank of both $Q^{-1}X'$ and $Q^{-1}XU$ equals $k$ and hence the rank of $Q^{-1}X' - Q^{-1}XU$ is not greater than $2k$. We have
\[
\|Q^{-1}X' - Q^{-1}XU\|_F \leq \sqrt{2k} \cdot \|Q^{-1}X' - Q^{-1}XU\| \leq \sqrt{2k} \cdot \|Q^{-1}\| \cdot \|X' - XU\| + \sqrt{2k} \cdot \|Q^{-1} - Q^{-1}\| \cdot \|XU\|
\]
with $\|Q^{-1}\| = \frac{1}{\sqrt{\lambda_1 - a}}$ and $\|XU\| = 1$ because of $X^T X = I_k$ and $UU^T = I_k$. Hence
\[
\min_{U: U^T U = UU^T = I_k} \|Q^{-1}X' - Q^{-1}XU\|_F \leq \frac{\sqrt{2k}}{\sqrt{\lambda_1 - a}} \cdot \min_{U: U^T U = UU^T = I_k} \|X' - XU\| + \sqrt{2k} \cdot \|Q^{-1} - Q^{-1}\|. \tag{52}
\]

Because of (23) we have
\[
\frac{\min_{U: U^T U = UU^T = I_k} \|X' - XU\|_F}{\min_{U: U^T U = UU^T = I_k} \|X' - XU\|} \leq \frac{16\sqrt{k^3}(\lambda_1 - a)}{n(c - d)} \|Q^{-1}Z^T L ZQ^{-1} - Q^{-1}Z^T L ZQ^{-1}\|. \tag{54}
\]

Before looking at $\|Q^{-1}Z^T L ZQ^{-1} - Q^{-1}Z^T L ZQ^{-1}\|$ let us first look at the second term in (52). Because $Q^{-1}$ is symmetric and $Q^{-1} = \frac{1}{\sqrt{\lambda_1 - a}} \cdot I_{(n-h+1)}$ we have
\[
\|Q^{-1} - Q^{-1}\| = \max \left\{ \left| \nu_i - \frac{1}{\sqrt{\lambda_1 - a}} \right| : \nu_i \text{ is an eigenvalue of } Q^{-1} \right\}.
\]

It is $Q^2 = Z^T D Z$. Denoting the eigenvalues of $Z^T D Z$ by $\mu_1', \ldots, \mu_{n-h+1}'$ (note that all of them are greater than zero according to (51)), the eigenvalues of $Q^{-1}$ are $1/\sqrt{\mu_1'}, \ldots, 1/\sqrt{\mu_{n-h+1}'}$. For any $z_1, z_2 > 0$ we have
\[
\left| \sqrt{z_1} - \sqrt{z_2} \right| = \left| \frac{(\sqrt{z_1} - \sqrt{z_2})(\sqrt{z_1} + \sqrt{z_2})}{\sqrt{z_1} + \sqrt{z_2}} \right| = \left| \frac{z_1 - z_2}{\sqrt{z_1} + \sqrt{z_2}} \right| \leq \frac{|z_1 - z_2|}{\sqrt{z_2}} \tag{55}
\]
and
\[
\left| \frac{1}{\sqrt{z_1}} - \frac{1}{\sqrt{z_2}} \right| = \left| \frac{\sqrt{z_1} - \sqrt{z_2}}{\sqrt{z_1} \sqrt{z_2}} \right| \leq \frac{\sqrt{2} \cdot |\sqrt{z_1} - \sqrt{z_2}|}{z_2} \leq \frac{\sqrt{2} \cdot |z_1 - z_2|}{\sqrt{z_2}^2} \tag{56}
\]

According to (51) we have $\mu_i' \geq \frac{\lambda_1 - a}{2} > 0, i \in [n - h + 1]$, and hence
\[
\left| \frac{1}{\sqrt{\mu_i'}} - \frac{1}{\sqrt{\lambda_1 - a}} \right| \leq \frac{\sqrt{2} \cdot |\mu_i' - (\lambda_1 - a)|}{\sqrt{(\lambda_1 - a)^3}} \leq \frac{\sqrt{2} \cdot \|D - D\|}{\sqrt{(\lambda_1 - a)^3}}, \quad i \in [n - h + 1],
\]
and
\[
\|Q^{-1} - Q^{-1}\| \leq \frac{\sqrt{2} \cdot \|D - D\|}{\sqrt{(\lambda_1 - a)^3}}. \tag{57}
\]
Let us now look at \( \| Q^{-1}Z^T \mathcal{L}Q^{-1} - Q^{-1}Z^T \mathcal{L}Q^{-1} \| \). It is
\[
\| Q^{-1}Z^T \mathcal{L}Q^{-1} - Q^{-1}Z^T \mathcal{L}Q^{-1} \| \leq \| Q^{-1} \| \cdot \| Z^T \| \cdot \| Z \| \cdot \| Q^{-1} \| + \\
\| Q^{-1} \| \cdot \| Z^T \mathcal{L} - Z^T \mathcal{L}Q^{-1} \| \cdot \| Q^{-1} \| + \| Q^{-1} \| \cdot \| Z^T \mathcal{L}Q^{-1} \| \cdot \| Q^{-1} - Q^{-1} \|. 
\] (58)

It is \( \| Q^{-1} \| = \frac{1}{\sqrt{\lambda_1 - a}} \). According to Lemma 5, the largest eigenvalue of \( Z^T \mathcal{L} \) is \( \lambda_1 \) or \( \lambda_1 - \lambda_{hk} \), where \( \lambda_1 - \lambda_{hk} \leq 2\lambda_1 \) according to Lemma 3. Consequently, \( \| Z^T \mathcal{L} \| \leq 2\lambda_1 \). It is
\[
\| Q^{-1} \| \leq \| Q^{-1} - Q^{-1} \| + \| Q^{-1} \| \leq \frac{\sqrt{2} \cdot \| D - \mathcal{D} \|}{\sqrt{\lambda_1 - a}} + \frac{1}{\sqrt{\lambda_1 - a}} 
\] (57)
and
\[
\| Z^T \mathcal{L} \| \leq \| Z^T \mathcal{L} - Z^T \mathcal{L} \| + \| Z^T \mathcal{L} \| \leq \| \mathcal{L} - \mathcal{L} \| + 2\lambda_1. 
\] (32)

It follows that
\[
\| Q^{-1}Z^T \mathcal{L}Q^{-1} - Q^{-1}Z^T \mathcal{L}Q^{-1} \| \leq \frac{4\lambda_1 \cdot \| D - \mathcal{D} \|}{(\lambda_1 - a)^2} + \left( \frac{\sqrt{2} \cdot \| D - \mathcal{D} \|}{(\lambda_1 - a)} + \frac{1}{\lambda_1 - a} \right) \cdot \| \mathcal{L} - \mathcal{L} \| + \\
\left( \frac{2 \cdot \| D - \mathcal{D} \|^2}{(\lambda_1 - a)^3} + \frac{\sqrt{2} \cdot \| D - \mathcal{D} \|}{(\lambda_1 - a)^2} \right) \cdot \| \mathcal{L} - \mathcal{L} \| + 2\lambda_1 
\] (59)

If (37) and (38) hold, then, after combining (52), (53), (54), (57), (59) and using that \( \lambda_1 - a > \lambda_1/2 \), which follows from (13), we end up with
\[
\min_{U \in \mathbb{R}^{k \times k}; U^T U = U U^T = I_k} \| ZQ^{-1}X - ZQ^{-1}XU \|_F \leq \text{const}(C, r) \cdot \frac{k^2}{n(c - d)\sqrt{\lambda_1 - a}} \left( \sqrt{a \cdot n \ln n} + \frac{a \cdot n \ln n}{\lambda_1 - a} + \frac{(a \cdot n \ln n)^{3/2}}{(\lambda_1 - a)^2} \right) + \\
\text{const}(C, r) \cdot \frac{\sqrt{c} \cdot \sqrt{a \cdot n \ln n}}{\sqrt{\lambda_1 - a}} \cdot \frac{\sqrt{c} \cdot \sqrt{a \cdot n \ln n}}{\lambda_1 - a} \cdot \frac{\sqrt{\lambda_1 - a}}{\lambda_1 - a}
\]

for some \( \text{const}(C, r) \). Using that \( \sqrt{c} \cdot \sqrt{a \cdot n \ln n} \leq \sqrt{k} \cdot a \cdot n \ln n \leq \hat{C}_2(\lambda_1 - a) \leq \hat{C}_2(C, r) \) due to (13), for some \( \hat{C}_2 = \hat{C}_2(C, r) \) that we will specify shortly (we will choose it smaller than 1), we can simplify this bound such that
\[
\min_{U \in \mathbb{R}^{k \times k}; U^T U = U U^T = I_k} \| ZQ^{-1}X - ZQ^{-1}XU \|_F \leq \text{const}(C, r) \cdot \frac{k^2}{n(c - d)\sqrt{\lambda_1 - a}} \cdot \sqrt{a \cdot n \ln n} + \frac{\text{const}(C, r) \cdot \sqrt{k} \cdot \sqrt{a \cdot n \ln n}}{\sqrt{\lambda_1 - a}} \cdot \frac{\sqrt{\lambda_1 - a}}{\lambda_1 - a} \cdot \frac{\sqrt{\lambda_1 - a}}{\lambda_1 - a}
\] (60)

Similarly to Part 3, we use Lemma 5.3 in Lei & Rinaldo (2015) to complete the proof of Theorem 1 for Algorithm 3.

Assume that (60) holds and let \( U \in \mathbb{R}^{k \times k} \) be an orthogonal matrix attaining the minimum, that is we have
\[
\| ZQ^{-1}XU - ZQ^{-1}X \|_F \leq \text{const}(C, r) \cdot \frac{k^2}{n(c - d)\sqrt{\lambda_1 - a}} \cdot \sqrt{a \cdot n \ln n} + \frac{\text{const}(C, r) \cdot \sqrt{k} \cdot \sqrt{a \cdot n \ln n}}{\sqrt{\lambda_1 - a}} \cdot \frac{\sqrt{k} \cdot \sqrt{a \cdot n \ln n}}{\lambda_1 - a}.
\] (61)

As we have noted above, we can choose \( X \) in such a way that \( ZQ^{-1}X = \frac{1}{\sqrt{\lambda_1 - a}} \cdot T \) with \( T \) as in Lemma 6. According to Lemma 6, if we denote the i-th row of \( \frac{1}{\sqrt{\lambda_1 - a}} \cdot T \) by \( \hat{t}_i \), then \( \hat{t}_i = \hat{t}_j \) if the vertices \( i \) and \( j \) are in the same cluster and
\[
\| \hat{t}_i - \hat{t}_j \| = \frac{2k}{n(\lambda_1 - a)} \] if the vertices \( i \) and \( j \) are not in the same cluster. Since multiplying \( \frac{1}{\sqrt{\lambda_1 - a}} \cdot T \) by \( U^T \) from the right
Figure 7. Example of a graph for which both standard spectral clustering and our fair version are able to recover the fair meaningful ground-truth clustering while a naive approach that runs standard spectral clustering on each group separately fails to do so. It is $V = [12]$, $V_1 = \{1, 2, 3, 7, 8, 9\}$, $V_2 = \{4, 5, 6, 10, 11, 12\}$ and the fair ground-truth clustering is $V = \{1, 2, 3, 4, 5, 6\} \cup \{7, 8, 9, 10, 11, 12\}$.

side has the effect of applying an orthogonal transformation to the rows of $\frac{1}{\sqrt{\lambda_1 - a}} \cdot T$, the same properties are true for the matrix $\frac{1}{\sqrt{\lambda_1 - a}} \cdot T U^T$. Lemma 5.3 in Lei & Rinaldo (2015) guarantees that for any $\delta \leq \sqrt{\frac{2k}{n(\lambda_1 - a)}}$, if

$$\frac{16 + 8M}{\delta^2} \left\| \frac{1}{\sqrt{\lambda_1 - a}} \cdot T U^T - Z Q^{-1} X \right\|_F^2 < \frac{|C_l|}{\delta^2}, \quad l \in [k],$$

with $|C_l|$ being the size of cluster $C_l$, then a $(1 + M)$-approximation algorithm for $k$-means clustering applied to the rows of the matrix $Z Q^{-1} X$ returns a clustering that misclassifies at most

$$\frac{4(4 + 2M)}{\delta^2} \left\| \frac{1}{\sqrt{\lambda_1 - a}} \cdot T U^T - Z Q^{-1} X \right\|_F^2$$

many vertices. If we choose $\delta = \sqrt{\frac{2k}{n(\lambda_1 - a)}}$, then for a small enough $\hat{C}_2 = \hat{C}_2(C, r)$ (chosen smaller than 1 and also so small that (48) implies (49)), because of (61), the condition (13) implies (62). Also, for a large enough $\hat{C}_2 = \hat{C}_2(C, r)$ the expression (63) is upper bounded by the expression (14).

D. Why Running Standard Spectral Clustering on Each Group $V_s$ Separately is not a Good Idea

One might think that the following was a good idea for partitioning $V = V_1 \cup \ldots \cup V_h$ into $k$ clusters such that every cluster has a high balance value: we could try to run standard spectral clustering with $k$ clusters on each of the groups $V_s$, $s \in [h]$, separately and then to merge the $k \cdot h$ many clusters to end up with $k$ clusters.

The graph shown in Figure 7 illustrates that such an approach, in general, fails to recover an underlying fair ground-truth clustering, even when standard spectral clustering succeeds. We have $V = [12]$ and two groups $V_1 = \{1, 2, 3, 7, 8, 9\}$ (shown in red) and $V_2 = \{4, 5, 6, 10, 11, 12\}$ (shown in blue). We want to partition $V$ into two clusters. It can be verified that a clustering with minimum RatioCut value is given by $V = \{1, 2, 3, 4, 5, 6\} \cup \{7, 8, 9, 10, 11, 12\}$ and that this clustering is found by running standard spectral clustering. This clustering is perfectly fair with $\text{balance}([1, 2, 3, 4, 5, 6]) = \text{balance}([7, 8, 9, 10, 11, 12]) = 1$ and is also returned by our fair versions of spectral clustering. Let us now look at the idea of running standard spectral clustering on $V_1$ and $V_2$ separately: when running spectral clustering on the subgraph induced by $V_1$, we obtain the clustering $V_1 = \{1, 2, 3\} \cup \{7, 8, 9\}$ as we would hope for. However, in the subgraph induced by $V_2$ the clustering $V_2 = \{4, 5, 6\} \cup \{10, 11, 12\}$ does not have minimum RatioCut value and is not returned by spectral clustering. Consequently, no matter how we merge the two clusters for $V_1$ and the two clusters for $V_2$, we do not end up with the clustering $V = \{1, 2, 3, 4, 5, 6\} \cup \{7, 8, 9, 10, 11, 12\}$.
Note that for these findings to hold we do not require the specific graph shown in Figure 7. The key is its structure: let $V_1 = \{1, 2, 3, 7, 8, 9\}$, $V_2 = \{4, 5, 6, 10, 11, 12\}$, $C_1 = \{1, 2, 3, 4, 5, 6\}$ and $C_2 = \{7, 8, 9, 10, 11, 12\}$. Then the graph looks like a realization of the following random graph model: as in our variant of the stochastic block model introduced in Section 4, two vertices $i$ and $j$ are connected with an edge with a certain probability $\Pr(i,j)$, which is now given by

$$\Pr(i,j) = \begin{cases} a, & i,j \in C_1 \lor i,j \in C_2 \lor i,j \in V_2, \\ b, & \text{else}, \end{cases}$$

with $a$ large and $b$ small.