A Moment-Based Safety Analysis for Stochastic Dynamical Systems

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Abstract—Given a stochastic dynamical system modelled via stochastic differential equations (SDEs), we evaluate the safety of the system through characterisations of its exit time moments. We lift the (possibly nonlinear) dynamics into the space of the occupation and exit measures to obtain a set of linear evolution equations which depend on the infinitesimal generator of the SDE. Coupled with appropriate semidefinite positive matrix constraints, this yields a moment-based approach for the computation of exit time moments of SDEs with polynomial drift and diffusion dynamics. To extend the capability of the moment approach, we propose a state augmentation method which allows us to generate the evolution equations for a broader class of nonlinear stochastic systems and apply the moment method to previously unsupported dynamics. In particular, we show a general augmentation strategy for sinusoidal dynamics which can be found in most physical systems. We employ the methodology on an Ornstein-Uhlenbeck process and stochastic spring-mass-damper model to characterise their safety via their expected exit times and show the additional exit distribution insights that are afforded through higher order moments.

I. INTRODUCTION

Safety verification is an important step in ensuring dynamical systems perform in ways that designers intend while also mitigating the risks associated with unplanned behaviour. In the deterministic scenario, reachability can be applied to the safety verification problem and produce a boolean result that predicts whether a system enters a set of unsafe configurations at some point in the future.

In the stochastic setting, we seek an analog to this boolean safety statement through the characterisation of the distribution of exit times for a system governed by stochastic dynamics. We consider a state space partitioned into two sets $S$ (safe), and $S^c$ (unsafe), and denote the exit time $\tau$ as the first time that the system, starting from $x_0 \in S$, reaches a state $x_{\tau} \in S^c$.

Our work aims to apply a moment-based approach to obtain exit time moments for controlled stochastic dynamical systems modelled via stochastic differential equations (SDEs). In particular, we base our approach on the method in [1] which formulates the computation of the exit time as an infinite-dimensional convex optimization over the space of measures through the use of a linear evolution equation on the moments of the measures. This measure based method has previously seen successful application in financial instrument pricing [2] and optimal control [3]. In this paper, we seek to make use of the method's capabilities in order to study the safety properties of dynamical systems through the distributions of their exit times from a safe region.

In the context of probabilistic systems, it is often intuitive to characterise the safety of a system through some probability $p$ of violating some set of constraints. This can further be combined with an appropriate cost function to generate a risk metric which can aid in the determination of a suitable $p$. Prior works such as [4], [5], and [6] have looked at obtaining a set of initial states that satisfy a safety requirement based on $p$. In our work, we consider the problem from an alternative perspective in which we characterise the safety of a given set of initial states through their exit time distribution.

We apply the methods of Lasserre [1] and consider a semidefinite programming (SDP) based approach for the numerical computation of exit time moments for Markov processes. More precisely, we focus our attention to dynamical systems modelled by stochastic differential equations. Applied to SDEs, the method in [1] is largely restricted to systems with polynomial drift and diffusion dynamics as it relies on considering processes whose infinitesimal generator maps the monomials into polynomials. In particular, the inability to support sinusoidal dynamics prevents application to a large number of physical systems. Sinusoidal terms can be found in the dynamics of virtually all robotic systems operating in multidimensional space where the forces on the system are applied at varying angles with respect to a chosen coordinate frame. Examples include robotic arms, tracked robots, wheeled robots, and quadcopters.

To analyse the safety of more complex dynamics, we propose a redundant state augmentation approach to extend the moment method to systems with non-polynomial drift/diffusion which violate the above property of the infinitesimal generator. As a result, we are able to handle a wider class of real world dynamics which include polynomial, natural exponential, and trigonometric functions. In contrast with safety notions defined through the violation probability $p$, we show cases where the exit time analysis provides additional information to distinguish between the behaviour of differing states that have the same "safety" level based off their probability of entering an unsafe set.

This paper is organised as follows: Section II reviews related works and techniques in stochastic system safety verification and the exit time problem. Section III gives a brief summary of the notation used. Sections IV and V present the SDE system model and the method for exit time moment computation, respectively. In Section VI, we present numerical experiments and compare with other characterisa-
tions of safety. Lastly, we conclude in Section VII.

II. RELATED WORK

Fisac et al. [7] proposes a general safety framework for controlled dynamical systems subject to deterministic, state-dependent disturbances $\tilde{d}(x)$. Hamilton-Jacobi reachability methods are combined with Bayesian inference to generate a safe control policy that ensures the system remains within a predefined safe set. The state-dependent disturbances are assumed to be drawn from a Gaussian process and new observations are incorporated as the system samples additional disturbances. Equipped with a model of $\tilde{d}(x)$, the authors construct a probabilistic bound over the space of disturbances and incorporate this into the computation of a non-static safe set and optimal safety controller.

In the stochastic dynamics setting, several studies have been proposed in which an initial set satisfying certain (static) safety conditions is obtained. Reachability of stochastic systems has been studied through stochastic viability and target problems [8], [9]. A connection between stochastic optimal control (the exit-time problem) and the reach-avoid problem for controlled diffusion processes is presented by Esfahani et al. [4]. Here, the work proposes a method for computing the set of initial states where there exists an admissible control scheme such that the system hits a desired set prior to entering an avoid set. Notably, the set of initial states are characterized by the super level sets of the viscosity solution of a suitable Hamilton-Jacobi-Bellman equation.

Occupation measure approaches for computing the region of attraction (ROA) in deterministic systems have been studied in [10] and [11]. Korda et al. [10] analyses the ROA for deterministic nonlinear (polynomial) dynamical systems and show a linear programming approach for approximating the region. The authors propose an optimisation over occupation measures and describe the nonlinear system dynamics through an equivalent linear evolution equation over measures. Using a similar analysis over measures, the notion of $p$-safety for stochastic systems is developed by Wisniewski et al. [5], [6] and is most closely related to our methodology. Under the notion of $p$-safety, one analyses the set of initial conditions for which the system is safe with probability at least $p$. A starting state $s_0$ is said to be $p$-safe if the probability of trajectories initiating from $s_0$ reaching an unsafe set is less than $1 - p$. The evolution equation of the occupation measure is applied to stochastic polynomial dynamics. Similar to the approach we use, $p$-safety employs the linear evolution to link the initial, final, and occupation measures of the stochastic system and formulates an infinite-dimensional optimization that is solved by the generalized moment method (returning the largest $p$-safe set) [6].

Numerical approaches for the exit time problem applied to Markov processes are given in [1] and [12]. At their core, the methods aim to characterise the exit time problem via an infinite-dimensional convex program subject to constraints derived from the martingale characterisation of Markov processes [13]. Relating to the characterisation found in [10], the evolution of functionals over these processes is formulated using the moments of the occupation and exit location measures (described through the basic adjoint equation), reducing the analysis to the space of moment sequences of these measures. Application of the occupation measure approach have included options pricing prediction [2] and optimal control [3], [14], [15]. In contrast with the linear programming approach found in [12], Lasserre et al. [1] replaced the LP Hausdorff moment constraints with SDP constraints and showed increased computational performance and accuracy. In this paper, we consider the SDP constraint based moment method and extend it to a broader class of stochastic dynamical systems through appropriate state augmentation.

III. NOTATION

For two values $a, b \in \mathbb{R}$, we define $a \land b := \min\{a, b\}$. Given a set $A$, we denote its complement by $A^c$ and its boundary by $\partial A$. The Borel $\sigma$-algebra on a topological space $A$ is denoted by $\mathcal{B}(A)$ and for $B \in \mathcal{B}(A)$, the indicator function is denoted by $1_B$ and defined as $1_B(x) = 1$ if $x \in B$ and 0 otherwise. The support of a measure $\mu$ on a measurable space $(A, \mathcal{B}(A))$ is denoted by $\text{supp}(\mu)$. For a process $X = (X_t)_{t \geq 0}$ described via stochastic differential equation in $\mathbb{R}^n$, we denote $X$ as the state augmented version of $X$ given by an SDE in $\mathbb{R}^{n+s}, s > 0$. We represent the $n$-dimensional multi-index $\alpha$ as a tuple such that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$. The set of such $n$-dimensional multi-indexes is denoted by $\mathbb{N}^n$. Lastly, the monomial with degree corresponding to the multi-index $\alpha$ is given by $(x_1, x_2, ..., x_n)^{\alpha}$ such that $(x_1, x_2, ..., x_n)^{\alpha} = x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}$.

IV. SYSTEM MODEL

We consider the $\mathbb{R}^n$ valued stochastic differential equation:

$$dX_t = h(X_t, t)dt + \sigma(X_t, t)dB_t \quad (1)$$

where $X_t \in E \subseteq \mathbb{R}^n$, $X_0 = x_0$ is known, $0 \leq t \leq T$, and $T > 0$. Let $B_t$ be a standard $d$-dimensional Brownian motion and $dB_t$ represent its differential form. Let the functions $h : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times [0, T] \to \mathbb{R}^{n \times d}$ represent the drift and diffusion terms of the SDE respectively. Furthermore, let the functions be measurable and satisfy the space variable growth condition:

$$|h(x, t)| + |\sigma(x, t)| \leq C(1 + |x|) \quad x \in \mathbb{R}^n, t \in [0, T]$$

for some constant $C$, as well as the space variable Lipschitz condition:

$$|h(x, t) - h(y, t)| + |\sigma(x, t) - \sigma(y, t)| \leq D|x - y|$$

$$x, y \in \mathbb{R}^n, t \in [0, T]$$

Under these circumstances, the stochastic differential equation (1) has a unique time continuous solution starting at time $t$ and state $x_0$ [16, Theorem 5.2.1]. In addition, the stochastic process $X = (X_t)_{t \geq 0}$, given by the SDE (1), with
initial condition $X_0 = x_0$ with probability one, is a Markov process with continuous sample paths [16, Theorem 7.1.2].

We consider a state space $E \subseteq \mathbb{R}^n$ that is partitioned into two sets: $S$ and $S^c$. Here, $S$ is an open and bounded safe set and $S^c = E - S$ is its complement (unsafe set). In this paper, $\tau$ is a stopping time defined with respect to $S^c$ and is the minimum of the first time that the process $X$ reaches the unsafe set:

$$\tau = \inf\{t \mid X_t \in S^c\}$$

(2)

Throughout the rest of this paper, we will be concerned with the finite exit time $\tau \wedge T$. Intuitively speaking, if $\tau \wedge T = t$ then the system has become unsafe within the time horizon we are concerned with. While if $\tau \wedge T = T$, then the system has stayed safe for the entire finite duration we care about.

V. Computation of Exit Time Moments

In this section, we present the methods in [1] and [12] for exit moment computation through an infinite-dimensional optimisation program and build on the approaches using a state augmentation method to analyse the exit times of non-polynomial nonlinear dynamics.

A. Linear Evolution Equation

We start with the time-homogeneous Itô diffusion and later extend to the time-dependent case. Let $(X_t)_{t\geq0}$ be a time-homogeneous diffusion in $\mathbb{R}^n$ such that its dynamics are given by the following SDE:

$$dX_t = h(X_t)dt + \sigma(X_t)dB_t$$

(3)

We use $P_x$ to denote the probability laws of $(X_t)_{t\geq0}$ such that $P_x$ gives the distribution of $(X_t)_{t\geq0}$ when $X_0 = x$. Furthermore, let $E^x$ denote the expectation w.r.t the probability law $P_x$. The infinitesimal generator $A$ of $X_t$ is defined as [16, Definition 7.3.1]:

$$Af(x) \equiv \lim_{t \to 0} \frac{E^x[f(X_t)] - f(x)}{t}$$

(4)

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the above limit exists for all $x \in \mathbb{R}^n$ is named as the domain $D(A)$.

The generator of a time-homogeneous Itô diffusion in $\mathbb{R}^n$ for twice differentiable continuous $f$ is [16, Theorem 7.3.3]:

$$Af(x) = \sum_i h_i(x) \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j} (\sigma\sigma^T)_{i,j}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}$$

(5)

The dynamics of (3) are now lifted to the space of measures to define a set of linear evolution equations. The process $(X_t)$ given by the SDE (3) satisfies the martingale problem where:

$$f(X_t) - f(X_0) - \int_0^t Af(X_s)ds$$

is a martingale for all test functions $f \in D(A)$. The first moment of the exit time $\tau \wedge T$ remains finite. Combined with the martingale property of (5), we have:

$$\mathbb{E}[f(X_{\tau \wedge T})] - \mathbb{E}[f(X_0)] - \mathbb{E} \left[ \int_0^{\tau \wedge T} Af(X_s)ds \right] = 0$$

(6)

Let $\mu_0$ be the expected occupation measure up to exit time $\tau \wedge T$ of the process $(X_t)$, and $\mu_1$ be its exit location distribution:

$$\mu_0(B) = \mathbb{E} \int_0^{\tau \wedge T} 1_B(X_t)dt$$

$$\mu_1(B) = \mathbb{P}(X_{\tau \wedge T} \in B)$$

The measures $\mu_0$ and $\mu_1$ are supported on the safe set $S$ and safe set boundary $\partial S$, respectively ($\text{supp}(\mu_0) = S$, $\text{supp}(\mu_1) = \partial S$). Equation (6) is now rewritten as:

$$\int_{\partial S} f(x)\mu_1(dx) - f(x_0) - \int_S Af(x)\mu_0(dx) = 0$$

(7)

for every test function $f \in D(A)$ and $X_0 = x_0 \in S$. Equation (7) represents a linear evolution equation linking the occupation and exit measures of the process $(X_t)$, also referred to as the basic adjoint equation [12].

The moments of the measures $\mu_0$ and $\mu_1$ are given by:

$$m_i = \int_E x^i \mu_0(dx) \quad \text{and} \quad b_i = \int_E x^i \mu_1(dx)$$

where each $i \in \mathbb{N}^n$ is an $n$-dimensional multi-index and $x^i = x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n}$. Notice the first moment of the exit time is $m_0$. Under processes where monomial test functions $f$ produce a polynomial infinitesimal generator $A_f$, the conditions imposed by the basic adjoint equation are further relaxed in [1], [12] from the space of all functions $f \in D(A)$ to all monomials $f$, and expressed through the sequence of moments of $\mu_0$ and $\mu_1$: $[m_i]_{i \in \mathbb{N}^n}$ and $[b_i]_{i \in \mathbb{N}^n}$. The relaxed condition:

$$\sum_{j} c_j(i) \cdot m_j + y_0^i - b_i = 0$$

(8)

is imposed for every $i \in \mathbb{N}^n$ and monomial $f(x) = x^i$. The condition (8) gives a set of linear constraints involving the moments of the exit time and exit distribution.

B. Time-Dependent Itô Diffusion

We now consider the time-dependent Itô diffusion by modifying the system shown in (3) and including the time dimension within the state. The new state $\hat{X}_t \in \mathbb{R}^{n+1}$ is given as $\hat{X}_t = [X_t, t]^T$, with dynamics:

$$d\hat{X}_t = [h(X_t), 1]^T dt + [\sigma(X_t), 0]^T dB_t$$

$$\hat{h}(\hat{X}_t)dt + \hat{\sigma}(\hat{X}_t)dB_t$$

(9)
Thus we see that \( \hat{X} = (\hat{X}_t)_{t \geq 0} \) is an Itô diffusion in \( \mathbb{R}^{n+1} \) with initial condition \( \hat{x}_0 = (x_0, 0) \). Recall that Eq. \((\ref{eq:generator})\) requires consideration of a finite exit time. In order to address system dynamics which may stay within the safe set for all time \( t \in [0, \infty) \), we make use of a finite time horizon \( T \). The safe set of the SDE \((\ref{eq:SDE})\) is then \( \hat{S} = S \times [0, T] \), which guarantees a finite exit time. Applying the evolution equation \((\ref{eq:adjoint})\) to \( \hat{X} \) yields the basic adjoint equation:

\[
\int_{\partial \hat{S}} f(x, s)\mu_1(\, \! dx \times ds) - f(x_0, 0) - \int_\hat{S} Af(x, s)\mu_0(\, \! dx \times ds) = 0
\]

The boundary of the safe set can be further divided into two parts: \( F_1 = \partial \hat{S} \times [0, T] \) and \( F_2 = \hat{S} \times \{T\} \) with their respective measures \( \mu_1^{(1)} \) and \( \mu_1^{(2)} \), and moments \( \hat{b}_i^{(1)} \) and \( \hat{b}_i^{(2)} \). In particular, \( \hat{b}_0^{(2)} \) corresponds with the mass associated with exit at the time horizon \( T \). As a result, we can use \( \hat{b}_0^{(2)} \) to inform our choice of time horizon. We wish to choose \( T \) large enough such that the majority of the exit time distribution’s mass is captured. An example where \( T \) is too small is shown in Fig. \( \ref{fig:exit_time} \). Here, the choice of \( T \) results in significant mass begin concentrated at the finite time horizon, thus skewing the exit time distribution and moments. In order to pick a suitable \( T \) such that the distribution of the original dynamics \((\ref{eq:SDE})\) are not artificially skewed, we seek \( T \) such that \( \hat{b}_0^{(2)} \) is kept small.

Fig. 1: Exit time distribution where time horizon \( T \) is chosen too small causing exit measure to concentrate on \( S \times \{T\} \).

We are now ready to generate the set of linear moment constraints \((\ref{eq:moment_constraints})\) for the time dependent Itô diffusion \((\ref{eq:SDE})\). In the prior works of \cite{1} and \cite{12}, an assumption is made on the drift and diffusion terms \( \hat{h} \) and \( \hat{\sigma} \) such that the generator \( Af(x, s) \) is closed with respect to the space of polynomials:

\[
Af(x, s) \mapsto \sum_{i \in \mathbb{N}^{n+1}} c_i(k) \cdot (x, s)^i \quad (\text{10})
\]

where \( f = (x, s)^k \) and \( k \in \mathbb{N}^{n+1} \). In other words, the process \( \hat{X} \) is such that the infinitesimal generator maps monomial functions \( f \) to polynomials.

In view of the SDE \((\ref{eq:SDE})\) and generator \((\ref{eq:generator})\) associated with the Itô diffusion, we observe that the operator \( A \) is composed of differential and summation operations (with respect to the state variables). Thus stochastic dynamics with both polynomial drift \( \hat{h}(\cdot) \) and diffusion \( \hat{\sigma}(\cdot) \) satisfy the above assumption. As in Section \( \text{V-A} \) the basic adjoint equation conditions are relaxed from all \( f \) \( \in \mathcal{D}(A) \) to all monomials \( f \) to obtain the following martingale constraints in terms of the moment sequences \([m_i]_{i \in \mathbb{N}^{n+1}} \) and \([b_i]_{i \in \mathbb{N}^{n+1}} \):

\[
\sum_{i \in \mathbb{N}^{n+1}} [c_i(k) \cdot (x, s)^i] + \hat{x}_0^i - b_k = 0 \quad (\text{11})
\]

for every monomial \( f(x, s) \in \mathcal{D}(A) \), such that \( f(x, s) = (x, s)^k, k \in \mathbb{N}^{n+1} \).

C. State Space Augmentation

The restriction to polynomial drift and diffusion dynamics in \cite{1} and \cite{12} exclude a large class of real world systems which exhibit other nonlinear behaviour in their dynamics model. In particular, sinusoidal dynamics are prevalent in physical systems operating in multidimensional space as they are used to compute the components of forces acting on the system with respect to a particular coordinate frame. Sinusoids are also often found in the rotation matrices used to transform physical agents into a global frame of reference. For example, the dynamics of the Dubins car depend on the sine/cosine of the heading angle of the car, while those of a quadcopter depend on the sine/cosine of it’s roll, pitch, and yaw. In these broader classes of dynamics for \( \hat{X} \) (including sinusoidal and natural exponential functions), the assumption of the infinitesimal generator mapping monomial test functions \( f \) to polynomials is broken. As a result, we are unable to relax the basic adjoint equation and generate a series of constraints based on the moment sequences of \( \mu_0 \) and \( \mu_1 \). In this section, we provide a state augmentation technique to restore this desired property of the generator.

Definition V.1. Given a stochastic process \( X \in \mathbb{R}^n \), we say that \( X \) is closed under infinitesimal generation if \( Af(x) \) is a polynomial with respect to the state variables for all monomial functions \( f(x) = x^i, i \in \mathbb{N}^n \).

Proposition V.1. The time dependent Itô diffusion \( \hat{X} \) in \( \mathbb{R}^{n+1} \) described via the SDE \((\ref{eq:SDE})\) is closed under infinitesimal generation if for each drift term \( \hat{h}_i(\hat{x}) \), \( 0 \leq i < n + 1 \), and diffusion term \( (\delta \hat{\sigma}^T)_{j,k}(\hat{x}) \), \( 0 \leq j, k < n + 1 \), there exists \( n + 1 \) dimensional multi-index sets \( P, Q \), such that \( \hat{h}_i(\hat{x}) = \sum_{p \in P} c_p \hat{x}^p \) and \( (\delta \hat{\sigma}^T)_{j,k}(\hat{x}) = \sum_{q \in Q} c_q \hat{x}^q \), where \( c_p, c_q \in \mathbb{R} \).

In the cases where the dynamics of the SDE violate the requirements in Proposition \( \text{V.1} \) we propose an augmentation technique where the state space is extended with redundant variables. The augmentation is chosen such that the expanded state space now includes the non-polynomial (w.r.t the state) terms of the drift and diffusion components, as well as possibly their derivatives.

Let \( \{c_j(\hat{X}_t)\}_{j=0,J-1} \) be the set of coefficients (along with possibly their derivatives) of the generator \( Af(x, s) \) that violate the polynomial assumption \((\ref{eq:moment_constraints})\). Note that the \( c_j \)’s are partially determined by the coefficients of the partial derivatives found in the infinitesimal generator and
correspond to the drift and diffusion terms of the SDE. We consider the augmented state space \( \tilde{X}_t \in \mathbb{R}^{J+n+1} \):

\[
\tilde{X}_t = [X_t, t, c_0(X_t), \cdots, c_J(X_t)]^\top
\]

with corresponding dynamics:

\[
d\tilde{X}_t = \tilde{h}(\tilde{X}_t)dt + \tilde{\sigma}(\tilde{X}_t)dB_t
\]

where

\[
\tilde{h}(X_t) = \begin{bmatrix} h_1(X_t) \\ h_2(X_t) \\ \vdots \\ h_n(X_t) \end{bmatrix}, \quad
\tilde{\sigma}(X_t) = \begin{bmatrix} \sigma_{1,1}(X_t) & \cdots & \sigma_{1,d}(X_t) \\ \sigma_{2,1}(X_t) & \cdots & \sigma_{2,d}(X_t) \\ \vdots \\ \sigma_{n,1}(X_t) & \cdots & \sigma_{n,d}(X_t) \end{bmatrix}
\]

\[
h_i(X_t) = \sum_{p} \alpha_p(\sin(\phi_px^\gamma), \cos(\psi_px^\gamma), x^\gamma)^{\beta_p}
\]

\[
\sigma_{ij}(X_t) = \sum_{q} \alpha_q(\sin(\phi_qx^\gamma), \cos(\psi_qx^\gamma), x^\gamma)^{\beta_q}
\]

where each \( \alpha(\cdot), \phi(\cdot), \psi(\cdot) \in \mathbb{R} \) is a scalar coefficient, \( \beta(\cdot) \in \mathbb{N}^n \) is a multi-index, and \( x^\gamma \) is a monomial with respect to the state \( x \) given by a multi-index \( \gamma(\cdot) \in \mathbb{N}^n \). Let \( \hat{x} \) denote the sinusoidal augmented state space such that:

\[
\hat{x} = [x, \sin(\phi_1x^\gamma_1), \sin(\phi_2x^\gamma_2), \cdots, \sin(\phi_mx^\gamma_m),
\sin(\psi_1x^\gamma_1), \sin(\psi_2x^\gamma_2), \cdots, \sin(\psi_mx^\gamma_m),
\cos(\phi_1x^\gamma_1), \cos(\phi_2x^\gamma_2), \cdots, \cos(\phi_mx^\gamma_m)
\cos(\psi_1x^\gamma_1), \cos(\psi_2x^\gamma_2), \cdots, \cos(\psi_mx^\gamma_m)]
\]

where the order of \( \gamma_m \) is greater than the order of the highest monomial \( x^\gamma \) found in the dynamics of \( X \). Then, we have that the augmented system \( \hat{X} \) with state \( \hat{x} \) satisfies Definition \( \text{VI} \).

**Proof.** Here, we show the proof for the scalar SDE case and later extend to the general \( n \)-dimensional case. The proof for \( n \)-dimensional systems follows the same logic, however, extra care regarding notation is taken in order to properly track all variables (see Appendix). Consider a scalar instance of (12) with state \( x \in \mathbb{R} \):

\[
dx_t = h(x_t)dt + \sigma(x_t)dB_t
\]

where

\[
h(x_t) = \sum_{p} \alpha_p(\sin(\phi_p x^\gamma), \cos(\psi_p x^\gamma), x)^{\beta_p}
\]

\[
\sigma(x_t) = \sum_{q} \alpha_q(\sin(\phi_q x^\gamma), \cos(\psi_q x^\gamma), x)^{\beta_q}
\]

Let the augmented states be denoted by \( f(\xi_z x^\gamma) \) such that \( 0 \leq z < m \). For an arbitrary \( z \), the dynamics of the state \( f(\xi_z x^\gamma) \) is given by:

\[
df(\xi_z x^\gamma) = f'(\xi_z x^\gamma) \cdot \xi_z \cdot dx^\gamma
\]

where

\[
dx^\gamma = \gamma_z x^\gamma - 1 dx
\]

By construction, the drift and diffusion of \( dx \) are polynomials with respect to both \( x \) and the sinusoidal terms \( \sin(\phi(\cdot)x^\gamma), \cos(\psi(\cdot)x^\gamma) \) up to \( x \) monomials of degree \( \gamma_m \) such that \( \gamma_m \) is greater in order than the highest monomial \( x^\gamma \) found in the original drift and diffusion terms. Therefore, we observe that \( dx^\gamma \) is composed of addition and multiplication operations on polynomials w.r.t. the augmented state space \( \hat{x} \). Applying the closure properties of polynomials, \( dx^\gamma \) is a polynomial w.r.t. \( \hat{x} \). The dynamics of the augmented system is:

\[
d\hat{X}_t = \begin{bmatrix} h \\ h \sin(\phi_1 x^\gamma_1) \\ \vdots \\ h \sin(\phi_m x^\gamma_m) \\ h \cos(\phi_1 x^\gamma_1) \\ \vdots \\ h \cos(\phi_m x^\gamma_m) \end{bmatrix} dt + \begin{bmatrix} \sigma \\ \sigma \sin(\phi_1 x^\gamma_1) \\ \vdots \\ \sigma \sin(\phi_m x^\gamma_m) \\ \sigma \cos(\phi_1 x^\gamma_1) \\ \vdots \\ \sigma \cos(\phi_m x^\gamma_m) \end{bmatrix} dB_t
\]
where

\[
    h_{\sin}(\xi)(x) = \cos(\xi(x)) : \xi(x) \cdot \gamma(x)^{-1} \cdot \sigma
\]

\[
    \sigma_{\sin}(\xi)(x) = \cos(\xi(x)) : \xi(x) \cdot \gamma(x)^{-1} \cdot \sigma
\]

\[
    h_{\cos}(\xi)(x) = -\sin(\xi(x)) : \xi(x) \cdot \gamma(x)^{-1} \cdot \sigma
\]

\[
    \sigma_{\cos}(\xi)(x) = -\sin(\xi(x)) : \xi(x) \cdot \gamma(x)^{-1} \cdot \sigma
\]

We observe that all terms in Eqs. (13)-(16) are polynomials or monomials w.r.t. the augmented state space \( \hat{x} \). Following the closure properties of polynomials, the resulting drift and diffusion of the augmented system are thus polynomial w.r.t. \( \hat{x} \). The monomials of \( \hat{x} \) are closed under differentiation w.r.t. \( x \) for all \( x, x_j \in \hat{x} \) and test functions \( f = (\hat{x})^\beta, \beta \in \mathbb{N}^{\hat{x}} \). Thus, we see that Eq. (4) applied to the augmented system \( \hat{X} \) for monomial test functions yields a generator comprised of the sum of products between polynomials and monomials \( (h(x), \sigma(x))^2 : \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2} \) w.r.t. \( \hat{x} \). Once again, following the closure properties of polynomials, the generator is polynomial w.r.t. the augmented state space.

**Example V.1.** We obtain the martingale constraints through state space augmentation for the time dependent SDE:

\[
    dX_t = \left[ \frac{dx}{dt} = \left[ \sin(x) \right] dt + \left[ \cos(x) \right] dB_t \right]
\]

Following (4), the generator of the system \( Af \) is given by:

\[
    Af = \sin(x) \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2} \cos^2(x) \frac{\partial^2 f}{\partial x^2}
\]

We see that due to the coefficients \( \sin(x) \) and \( \cos^2(x) \), we are unable to express the generator \( Af \) as a polynomial with respect to the state space \( X_t = [x, t]^T \). Thus we add redundant states to augment the original state space. The augmented state space \( \hat{X}_t \) is given by:

\[
    \hat{X}_t = [x, t, \sin(x), \cos(x)]^T
\]

The dynamics of the augmented SDE are:

\[
    d\hat{X}_t = \hat{h}(\hat{X}_t)dt + \hat{\sigma}(\hat{X}_t)dB_t
\]

\[
    = \left[ \begin{array}{c}
        \sin(x) \\
        \sin(x) \cos(x) \\
        -\sin^2(x)
    \end{array} \right] dt + \left[ \begin{array}{c}
        \cos(x) \\
        0 \\
        -\sin(x) \cos(x)
    \end{array} \right] dB_t
\]

The SDE now satisfies the conditions in Proposition V.1. The new generator \( \hat{A}f \) can then be obtained through (4). For monomial test functions \( f \), the generator is a polynomial with respect to the augmented state \( \hat{X}_t \). We apply (11) to derive the corresponding martingale constraints.

With the appropriate state augmentation, we may now obtain the martingale constraints (11) in terms of the moment sequences for previously unsupported nonlinear dynamics. In performing the augmentation, we have to consider the moments of these redundant states and thus trade off in additional computational/memory requirements in order to analyse a broader class of systems.

**D. SDP Moment Constraints**

The martingale constraints (11) alone are not able to guarantee the sequences \( [m_i] \) and \( [b_i] \) are moment sequences with respect to some occupation and exit measures. In order to enforce that the sequences are moment sequences, we impose additional conditions. Helmes et al. [12] considers linear moment constraints while Lasserre et al. [1] derives SDP conditions. We choose to use the SDP conditions as they have been shown to provide greater precision and reduced computational requirements.

Let \( [m_\alpha] \) be a sequence where \( \alpha \in \mathbb{N}^d \) is a multi-index. The sequence is sorted according to the graded lexicographic order where \( \alpha \) represents a monomial \( x^\alpha \).

**Definition V.2** (Graded Lexicographic Order). Let \( \alpha = [\alpha_1, \alpha_2, ..., \alpha_n], \beta = [\beta_1, \beta_2, ..., \beta_n] \in \mathbb{N}^n \) be multi-indexes. The degree of \( \alpha, \beta \) is given by:

\[
    \text{Deg}(\alpha) = \sum_{i=1}^{n} \alpha_i \quad \text{and} \quad \text{Deg}(\beta) = \sum_{i=1}^{n} \beta_i
\]

The multi-index \( \alpha \) precedes \( \beta \) in graded lexicographic order if \( \text{Deg}(\alpha) < \text{Deg}(\beta) \). If \( \text{Deg}(\alpha) = \text{Deg}(\beta) \), \( \alpha \) precedes \( \beta \) if the leftmost non-zero entry of the element wise vector difference \( \alpha - \beta \) is positive.

**Example V.2.** The 3-dimensional indexed moment sequence \( [m_\alpha], \alpha \in \mathbb{N}^3 \) is given by:

\[
    [m_\alpha] = [m_{000}, m_{100}, m_{010}, m_{001}, m_{200}, m_{110}, m_{101}, m_{020}, m_{011}, m_{002}, ...]
\]

Given a moment sequence, the moment matrix \( M_k(m) \) is defined as follows:

\[
    M_k(m)(i, j) = m_{\alpha + \beta}
\]

where

\[
    M_k(m)(1, j) = m_\alpha \\
    M_k(m)(i, 1) = m_\beta
\]

In other words, the top most row and left most column of \( M_k(m) \) (i.e. \( M_k(m)(0, \cdot) \) and \( M_k(m)(\cdot, 0) \)) consist of the elements of \( [m_\alpha] \) up to degree \( k \).
Example V.3. Let $x \in \mathbb{R}^2$. The second degree moment matrix $M_2(m)$ is given by:

$$M_2(m) = \begin{bmatrix} m_{00} & m_{10} & m_{01} & m_{20} & m_{11} & m_{02} \\ m_{10} & m_{20} & m_{11} & m_{30} & m_{21} & m_{12} \\ m_{01} & m_{11} & m_{02} & m_{21} & m_{12} & m_{03} \\ m_{20} & m_{30} & m_{21} & m_{40} & m_{31} & m_{22} \\ m_{11} & m_{21} & m_{12} & m_{31} & m_{22} & m_{13} \\ m_{02} & m_{12} & m_{03} & m_{22} & m_{03} \\ m_{02} & m_{12} & m_{03} & m_{22} & m_{03} & m_{04} \end{bmatrix}$$

Next the localizing matrix $M_k(qm)$ is defined with respect to a polynomial $q$. Let $\beta(i,j)$ be the multi-index of the $i,j$ entry of the moment matrix $M_k(m)$ and let $[q^\alpha]$ be the vector of coefficients of the polynomial $q$ in graded lexicographic order. The entries of the localizing matrix is then given by:

$$M_k(qm)(i,j) = \sum_{\alpha} q^\alpha \cdot m^\beta(i,j) + \alpha$$

Example V.4. Let $x \in \mathbb{R}$ and $q(x) := 1 + x^2 + x^4$. The first degree localizing matrix $M_1(qm)$ is given by:

$$M_1(qm) = \begin{bmatrix} m_0 + m_2 + m_4 & m_1 + m_3 + m_5 \\ m_1 + m_3 + m_5 & m_2 + m_4 + m_6 \end{bmatrix}$$

E. Optimization Program

Applying the optimisation in [1], the upper and lower bounds on the exit time moment $\mathbb{E}[\tau^n]$ of the system (1) can be computed through the following semidefinite program:

Maximize (resp. Minimize): $n \cdot m_{0,n-1}$

Subject to:

$$\dot{x}_0 + \sum_{i \in \mathbb{N}} c_i(k) \cdot \dot{x}^i - b_k = 0$$

$$M_k(m) \succ 0$$

$$M_k(b) \succeq 0$$

$$M_k(q_0m) \succ 0$$

$$M_k(q_1b) \succeq 0$$

$$\forall k \leq K$$

where $M_k(m)$ and $M_k(q_0m)$ are the moment and localising matrices corresponding to the moment sequence of $\mu_0$, and $M_k(b)$ and $M_k(q_1b)$ are the moment and localising matrices corresponding to the moment sequence of $\mu_1$. The polynomials $q_0, q_1$ with which the localising matrices are defined with respect to are derived from the semi-algebraic sets $E_1 := \{x \in \mathbb{R}^d \mid q_0(x) \geq 0\}$, $E_2 := \{x \in \mathbb{R}^d \mid q_1(x) \geq 0\}$, such that the measures $\mu_0, \mu_1$ are supported on $E_1, E_2$, respectively.

In order to make the program numerical tractable, we restrict the optimisation to a finite number of moments $K$ and consider both the martingale constraints (11) and positive semidefinite matrix constraints on all moment/localising matrices for the sequences $[m_i]$ and $[b_i]$.

With this, we may now obtain numerical approximations for the expected exit time and higher order moments of the SDE (1) subject to a broad class of real world nonlinear dynamics. The first moment of the exit time provides a safety assessment under an “averaged” case analysis while the usage of the second moment to compute variance provides increased insight regarding the spread of the system’s safety behaviour. Additional higher order moments further allow us to characterise the exit time through distribution bounding on both the upper and lower ends.

VI. Experiments

In this section, we apply the moment-based exit time method to analyse the safety properties of two examples of stochastic dynamics modelled via SDE.

A. Ornstein-Uhlenbeck (OU) Process

We consider an example of the Ornstein-Uhlenbeck process demonstrating comparable $p$-safety but distinct exit time distributions. The exit or first hitting times of OU-processes have received a significant amount of research attention as characterisations and approximations of their exit time distributions have been studied from various perspectives [17]–[19].

The OU process $X = (X_t)_{t \geq 0}$ is defined as follows:

$$dX_t = -\alpha X_t dt + \sigma dB_t \quad X_0 = x_0$$

We consider three sets of parameters for the process as shown in Table 1 and evaluate their corresponding $p$-safety and exit time metrics. Let $S = [1, 10]$ be a set of allowable states and $S^c = (-\infty, 1) \cup (10, \infty)$ be a set of forbidden states. We denote $\tau_U$ as the first hitting time of $S^c$ and $T_p$ as a safety horizon representing the duration of time where safety is critical. A point $x \in S$ is $p$-safe if

$$\mathbb{P}^x[\tau_U < T_p] \leq p$$

We modify this definition slightly to introduce the notion of a $p$-cost such that the $p$-cost of a point $x \in S$ is given by $\mathbb{P}^x[\tau_U < T_p]$. The $p$-cost corresponding to the three parameter sets and starting points are approximated through Monte Carlo simulations ($N_{\text{trials}} = 50000$) and shown in Table 1. We observe that all points are $p$-safe (for $p = 0.1$) and furthermore have similar $p$-costs ($\approx 0.1$). Intuitively, the three points can be said to lie near the boundary of the $p$-safe set parameterized by $p = 0.1$. In this scenario, a

| Parameter Set | $\alpha$ | $\sigma$ | $x_0$ |
|---------------|----------|----------|-------|
| 1             | 0.525    | 0.688    | 9.600 |
| 2             | 0.0105   | 0.011    | 1.060 |
| 3             | 0.235    | 1.400    | 8.500 |

1We exclude the condition $\tau_U < \zeta_S$ found in [6] (where $\zeta_S$ is the first time the process leaves $S$) as we consider allowable states encompassing the full state space.
safety analysis following the notion of \( p \)-safety yields three similar configurations where the system is safe with > 90% probability for a safety horizon of \( T_p = 3s \).

**TABLE II: \( p \)-costs and Moments of OU Process**

| Parameter Set | \( p \)-cost (\( T_p = 3s \)) | First Moment | Second Moment |
|---------------|------------------------------|--------------|--------------|
| 1             | 0.0992                       | 3.90         | 16.93        |
| 2             | 0.0983                       | 5.39         | 33.07        |
| 3             | 0.0992                       | 6.00         | 42.08        |

Next, we approximate (lower bound) the first and second moments of the exit time using the SDP formulation. The time augmented state space is \( \hat{X}_t = [X_t, t]^\top \) with dynamics:

\[
d\hat{X}_t = [-\alpha X_t, 1]^\top dt + [\sigma, 0]dB_t \quad \hat{X}_0 = [x_0, 0]^\top
\]

We define the safe set of the system as \( S \), the moments of the exit time using the SDP formulation. The exit boundary is made up of three components: \( \{ 1 \} \times [0, 30], \{ 10 \} \times [0, 30], \{ 10 \} \times [0, 30] \). Thus, we can represent the exit measure \( \mu_1 \) as the sum of three 1-dimensional measures \( \mu_1^{(1)}, \mu_1^{(2)}, \mu_1^{(3)} \); each corresponding to the boundary listed above. The \( i \)-th moment of each is denoted by \( b_i^{(1)}, b_i^{(2)}, b_i^{(3)} \) respectively. The martingale moment constraints for monomial test functions \( f(x, t) = x^j t^k \) are given by:

\[
m_{j,k-1} - \alpha jm_{j,k} + \frac{j(j-1)\sigma^2}{2} m_{j-2,k} + x_j^2 - b_k^{(1)} - b_k^{(2)} - b_k^{(3)} = 0 \tag{17}
\]

Table II shows the first and second moments of the exit time obtained through the semidefinite program with moment constraints (17). Fig. 2 overlays the first moment and variance given by the SDP with an exit time distribution obtained through Monte Carlo simulation. In contrast with their \( p \)-safety characterisations, the three parameter sets show distinct safety differences through their exit time moments.

**B. Spring Mass Damper with Variable Damping Rate**

1) **System Model:** We consider a spring mass damper system where the mass sits vertically above the spring and damper with the following parameters:

- Spring constant \( k_s = 5.0 \)
- Object mass \( m_s = 1.0 \)
- Static damper constant \( k_c = 1.0 \)

To demonstrate the redundant state augmentation technique, we consider a variable damper force subject to noise and proportional to both the static damper constant and a sinusoidal term with respect to the position of the mass. A diagram of the setup is shown in Fig. 3. The state space \( X_t \) is defined as follows:

\[
X_t = [x, v, t]^\top
\]

Here, \( x \) is the vertical position of the mass, \( v \) is its velocity, and \( t \) is the time. The system dynamics are given by:

\[
dX_t = \left[ \begin{array}{c} \frac{v}{m_s} - g + \frac{k_s}{m_s} \sin(x) \\ 1 \\ v \cos(x) \\ v \sin(x) \end{array} \right] dt + \left[ \begin{array}{c} 0 \\ \frac{k_c}{m_s} \\ 0 \\ 0 \end{array} \right] dB_t
\]

In order to produce a generator that maps monomial test functions \( f \) to polynomials with respect to the state variables, we consider the following state augmentation:

\[
\hat{X}_t = [x, v, t, \sin(x), \cos(x)]^\top
\]

\[
d\hat{X}_t = \left[ \begin{array}{c} \frac{v}{m_s} - g + \frac{k_s}{m_s} \sin(x) \\ 1 \\ v \cos(x) \\ v \sin(x) \end{array} \right] dt + \left[ \begin{array}{c} 0 \\ \frac{k_c}{m_s} \\ 0 \\ 0 \end{array} \right] dB_t
\]

We define the safe set as all states where \( x \in [-2.3, -1.0] \) and approximate the moments of the exit time of the augmented SDE through the SDP formulation. The first, second,
and third moments are given in Table III while Fig. 4 overlays the first moment and variance obtained through the moment method with the simulated exit time distribution.

Fig. 3: Spring mass damper system with variable damping rate. Damper force $F_d$ is proportional to a sinusoidal function of mass position.

Fig. 4: Normalised histogram and first moment/variance of exit time for variable spring mass damper system. Histogram obtained through Monte Carlo simulation ($N_{\text{trials}} = 50000$).

2) Distribution Bounding: To make use of higher moments, we apply the numerical method presented in [20] to generate a distribution bound based on a finite number of moments. We consider the upper and lower bounds of the distribution at an arbitrary point $C$ and use a proper transformation such that $C$ is shifted to 0:

$$\bar{\mu}_i = \sum_{k=0}^{i} \binom{i}{k} (-C)^{i-k} \mu_k$$

The transformed moments of Table III with respect to $C$ is given by:

$$\bar{\mu}_0 = \mu_0$$
$$\bar{\mu}_1 = -C\mu_0 + \mu_1$$
$$\bar{\mu}_2 = C^2\mu_0 - 2C\mu_1 + \mu_2$$

Fig. 5 shows the upper distribution bound. With three moments $\mu_0, \mu_1, \mu_2 > 0$ and $\mu_0\mu_2 - \mu_1^2 > 0$, the lower bound $L$ at each point $C$ is given by $L = 0$, while the upper bound $U$ is given by:

$$U = \frac{\mu_0\mu_2 - \mu_0\mu_1^2}{\mu_0^2\mu_2 - \mu_0\mu_2^2 + \mu_1^2}$$

Fig. 5: Upper bound of distribution with first three moments from Table III

VII. CONCLUSION

In this paper we considered a safety analysis of stochastic systems through a moment based exit time method. Our formulation first considers the martingale problem and uses it to define a linear evolution equation linking the occupation and exit measures of the stochastic process under study. When considering processes with appropriate generators, this evolution can be relaxed to a series of conditions involving the moments of the occupation and exit measures for all monomial test functions. Together with appropriate SDP moment conditions, we form a convex optimization problem to approximate the exit time moments of the process. We extend the stochastic processes considered from SDEs with purely polynomial drift and diffusion terms, to include a broader class of nonlinear dynamics through appropriate state space augmentation. Lastly, we applied our methods to two systems with both polynomial and non-polynomial dynamics, and showed scenarios where a consideration of exit time moments grants useful insight into the safety of the system.

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**APPENDIX**

We present here the proof of Theorem [1] for the general $n$-dimensional SDE.

**Proof.** We denote each additional state in the augmented state space as $f(\xi_h x^{\gamma_h})$. For an arbitrary $h, 0 \leq h < m$, the dynamics of the state $f(\xi_h x^{\gamma_h})$ is given by:

$$df(\xi_h x^{\gamma_h}) = f'(\xi_h x^{\gamma_h}) \cdot \xi_h \cdot dx^{\gamma_h}$$

where

$$dx^{\gamma_h} = \sum_{i=1}^{n} \left( \mathbb{1}_{x_i \in x^{\gamma_h}} \cdot \prod_{j \neq i} x_j^{\gamma_j[i]} \right) \cdot \sum_{i=1}^{n} \left( \mathbb{1}_{x_i \in x^{\gamma_h}} \cdot \prod_{j \neq i} x_j^{\gamma_j[i]} \cdot \gamma_i x_i^{\gamma_i[i]-1} \cdot dx_i \right)$$

We note that by construction, $dx_i$ has drift ($h_i$) and diffusion ($\sigma_i(x)$) that is polynomial w.r.t. the sinusoidal terms $\sin(\phi_1 X^{(1)})$, $\cos(\phi_1 X^{(1)})$, $\sin(\phi_2 X^{(1)})$, $\cos(\phi_2 X^{(1)})$, and state $x$. Furthermore, the augmented state space $\tilde{x}$ includes all sinusoidal terms $\sin(\phi_1 X^{(1)})$, $\cos(\phi_1 X^{(1)})$, $\sin(\phi_2 X^{(1)})$, $\cos(\phi_2 X^{(1)})$, up to monomials of degree $\gamma_m$ where $\gamma_m$ is greater than the order of the highest monomial $X^{(1)}$ in the original dynamics of $X$. Therefore $dx_i$ is polynomial w.r.t. the augmented state space $\tilde{x}$. We denote where the drift and diffusion of $dx_i$ as $p_{i}^{(1)}$ and $p_{i}^{(2)}$, respectively:

$$dx^{\gamma_h} = \sum_{i=1}^{n} \left( \mathbb{1}_{x_i \in x^{\gamma_h}} \cdot \gamma_i x_i^{\gamma_i[i]-1} \cdot \prod_{j \neq i} x_j^{\gamma_j[i]} \cdot \right) \cdot \sum_{i=1}^{n} \left( \mathbb{1}_{x_i \in x^{\gamma_h}} \cdot \gamma_i x_i^{\gamma_i[i]-1} \cdot \prod_{j \neq i} x_j^{\gamma_j[i]} \cdot \right) \cdot \left( p_{i}^{(1)} dt + (p_{i}^{(2)}, dB_i) \right)$$

$$= \sum_{i=1}^{n} \left( \mathbb{1}_{x_i \in x^{\gamma_h}} \cdot \gamma_i x_i^{\gamma_i[i]-1} \cdot \prod_{j \neq i} x_j^{\gamma_j[i]} \cdot \right) \cdot \sum_{i=1}^{n} \left( \mathbb{1}_{x_i \in x^{\gamma_h}} \cdot \gamma_i x_i^{\gamma_i[i]-1} \cdot \prod_{j \neq i} x_j^{\gamma_j[i]} \cdot \right) \cdot \left( p_{i}^{(1)} dt + (p_{i}^{(2)}, dB_i) \right)$$

The dynamics of the augmented SDE is now given by:

$$d\tilde{X}_i = \begin{bmatrix} h_1 \\ \vdots \\ h_n \\ h_{\sin(\phi_1 X^{(1)})} \\ \vdots \\ h_{\sin(\phi_2 X^{(1)})} \\ h_{\cos(\phi_1 X^{(1)})} \\ \vdots \\ h_{\cos(\phi_2 X^{(1)})} \\ \vdots \\ h_{\cos(\phi_1 X^{(1)})} \\ \vdots \\ h_{\cos(\phi_2 X^{(1)})} \end{bmatrix} dt + \begin{bmatrix} \sigma_{1,1} & \cdots & \sigma_{1,d} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,d} \\ \sigma_{1,1} & \cdots & \sigma_{1,d} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,d} \\ \sigma_{1,1} & \cdots & \sigma_{1,d} \\ \vdots & \ddots & \vdots \\ \sigma_{n,1} & \cdots & \sigma_{n,d} \end{bmatrix} dB_t$$
\[ h_{\sin(\xi \cdot \gamma)} = \cos(\xi \cdot \gamma) \xi \]

\[
\sum_{i=1}^{n} \left( \mathbb{1}_{x_i \in \gamma} \cdot \gamma[i][x_i \gamma[i][1] \cdot \prod_{j \neq i} \gamma[j][x_j \gamma[j][1] \cdot h_i \right]
\]

\[
\sigma_{\sin(\xi \cdot \gamma)} = \cos(\xi \cdot \gamma) \xi
\]

\[
\sum_{i=1}^{n} \left( \mathbb{1}_{x_i \in \gamma} \cdot \gamma[i][x_i \gamma[i][1] \cdot \prod_{j \neq i} \sigma[j][x_j \gamma[j][1] \cdot \sigma[i,k] \right]
\]

\[
\sigma_{\cos(\xi \cdot \gamma)} = -\sin(\xi \cdot \gamma) \xi
\]

\[
\sum_{i=1}^{n} \left( \mathbb{1}_{x_i \in \gamma} \cdot \gamma[i][x_i \gamma[i][1] \cdot \prod_{j \neq i} \sigma[j][x_j \gamma[j][1] \cdot \sigma[i,k] \right]
\]

We note that all terms in Eqs. (18)-(21) are polynomial w.r.t. the augmented state space \( \hat{x} \), in addition, as polynomials are closed under addition and multiplication, the resulting drift and diffusion terms corresponding to the augmented state is also polynomial w.r.t. \( \hat{x} \). The remaining dynamics \( h_1, ..., h_n \) and \( \sigma_{1,1}, ..., \sigma_{n,n} \) have already been shown to be polynomial w.r.t. \( \hat{x} \). We now apply Eq. (4) to obtain the generator \( A\hat{f}(\hat{x}) \) of the augmented system. The monomials of \( \hat{x} \) are closed under differentiation w.r.t. \( x \in \hat{x} \) which gives us that \( \frac{\partial f}{\partial x_i} \) and \( \frac{\partial^2 f}{\partial x_i \partial x_j} \) are monomials of \( \hat{x} \) for all \( x_i, x_j \in \hat{x} \), and test functions \( f = (\hat{x})^\beta, \beta \in \mathbb{N}^{\hat{x}} \). As a result, we see that Eq. (4) applied to the augmented system \( \hat{X} \) for monomial test functions yields a generator consisting of the sum of products between polynomials and monomials \( (h_\xi, \sigma_\xi \sigma^T \xi), \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j} \) w.r.t. the augmented state space. Again, following the closure properties of polynomials, the resulting generator is polynomial w.r.t. the augmented state space. \[\square\]