Interaction of a brane with a moving bulk black hole

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Abstract

We study the interaction of an $n$-dimensional topological defect ($n$-brane) described by the Nambu-Goto action with a higher-dimensional Schwarzschild black hole moving in the bulk spacetime. We derive the general form of the perturbation equations for an $n$-brane in the weak field approximation and solve them analytically in the most interesting cases. We specially analyze applications to brane world models. We calculate the induced geometry on the brane generated by a moving black hole. From the point of view of a brane observer, this geometry can be obtained by solving $(n+1)$-dimensional Einstein’s equations with a non-vanishing right hand side. We calculate the effective stress-energy tensor corresponding to this ‘shadow-matter’. We explicitly show that there exist regions on the brane where a brane observer sees an apparent violation of energy conditions. We also study the deflection of light propagating in the region of influence of this ‘shadow matter’.

1 Introduction

Topological defects play an important role in modern physics. Monopoles, strings, domain walls etc. are known as classical solutions of field theory models which admit non-trivial topology. They can arise in a wide class of phenomenological models including GUT theories [1] and some extensions of the electroweak standard model [2]. The interaction of
various topological defects with black holes has been studied before. In \([3]\) and \([4]\) interaction of black holes with cosmic strings and domain walls in \((3 + 1)\)-dimensional universe was studied. Here we generalize this consideration to a case where both a brane and a bulk space in which the brane is moving may have an arbitrary number of dimensions. We assume that the higher-dimensional brane can be described by the Nambu-Goto action. We will study motion of such \((n + 1)\)-dimensional object in a background of \((n + k + 1)\)-dimensional Schwarzschild black hole. We consider situations where the black hole is far from the brane so that one can use a weak field approximation.

We will keep our analysis as general as possible, however we will devote special attention to the so-called brane world model in which the brane has three spatial dimensions while the number of bulk dimensions is arbitrary. It was proposed recently that the whole universe could just be a three dimensional domain wall (a brane) embedded in a higher dimensional space. In this model, all the standard model particles are localized on the brane while gravity can propagate everywhere. In particular, black holes being gravitational solitons can propagate in a higher dimensional bulk space. To make the consideration more concrete we use the so called ADD model \([5]\) in which the gravitational field of the brane is neglected and extra dimensions are flat. An important generic feature of this model is that the fundamental quantum gravity mass scale \(M_*\) may be very low (of order TeV) and the size of the extra spatial dimensions may be much larger than the Planck length (\(\sim 10^{-33}\) cm). In the most interesting version of the ADD model there exist two extra spatial dimensions with the size \(L\) of the order of 0.1 mm, which is still allowed by the experiments testing the Newton law at small distances. The gravitational radius \(R_0\) of a black hole of mass \(M\) in the spacetime with \(k\) extra dimensions is defined by the relation \(G^{(4+k)} M \sim R_0^{k+1}\), where \(G^{(4+k)} = 1/M_\star^{(k+2)}\) is the \((4 + k)\)-dimensional Newton coupling constant. The minimal mass of the black hole is determined by the condition that its gravitational radius coincides with its Compton length \(\sim 1/M\). The mass of such an elementary black hole is \(M_*\). For \(M_* \sim TeV\) one has \(R_* \sim 10^{-17}\) cm. When \(M \gg M_*\), the higher dimensional mini black holes can be described by the classical solutions of vacuum Einstein’s equations. We assume that the size of a black hole \(R_0\) is much smaller than the characteristic size of extra dimensions, \(L\), and neglect the effects of the black hole deformation connected with this size.

Black holes in the bulk space may exist as a result the early stage of the evolution of the universe, for example as a result of the recoil during the quantum evaporation of primordial black holes created on the brane \([6]\). The final state of such evaporating mini black holes is not known. Similarly to the early studied case of evaporation of the mini black holes in the 4-dimensional theory with the Planck scale quantum gravity, two different final states are possible. Either a stable remnant of mass of the order of \(M_*\) is formed or the evaporation is complete. We assume that bulk black holes are either those stable remnants or black hole with mass \(M \gg M_*\), so that they are relatively long living.

We start with the general solution of an unperturbed flat n-brane in \((n + k + 1)\)-dimensional flat spacetime. We then introduce a moving bulk black hole as a source of perturbation. Using coordinate transformations we work in the reference frame in which the n-brane is moving in the background spacetime of a static black hole. We then derive the equations of motion of a perturbed brane in the weak field approximation. Using the method of Green’s functions, we give the form of the general solution for an n-brane in
arbitrary number of spacetime dimensions. We specially analyze the most interesting case of a 3-brane in 4 + 1 and 5 + 1 dimensional spacetime where we give the explicit solutions and discuss their properties. The solutions for higher number of extra dimensions can easily be generated from these solutions. These solutions allow us to calculate the energy transferred from the black hole to brane excitations in this process of interaction.

We also calculate the induced geometry on the brane generated by a moving black hole. As considered by a brane observer this geometry can be obtained by solving the \((n + 1)\)-dimensional Einstein’s equations with a non-vanishing right hand side. We calculate the effective stress-energy tensor corresponding to this ‘shadow-matter’. We show that there exist regions where a brane observer sees an apparent violation of energy conditions, namely, there exist regions with negative energy density and negative pressure. We also derive formulas for deflection of light propagating in the induced spacetime metric on the brane.

\section{Black hole metric}

The metric of a static multi-dimensional black hole in \((K + 1)\)-dimensional spacetime is

\[
dS^2 = -FdT^2 + \frac{dR^2}{F} + R^2 d\Omega^2_{K-1}, \quad F = 1 - \left(\frac{R_0}{R}\right)^{K-2},
\]

where \(K\) is the number of spatial dimensions, \(R_0\) is the gravitational radius and \(d\Omega^2_{K-1}\) is the line element on a unit \((K - 1)\)-dimensional sphere. For \(K = 3\) this is a usual 4-dimensional Schwarzschild metric. We shall use the metric in the isotropic coordinates

\[
dS^2 = -FdT^2 + A^2 \left[ d\sigma^2 + \sigma^2 d\Omega^2_{K-1} \right],
\]

where

\[
\ln \sigma = \int \frac{dR}{R\sqrt{F(R)}}, \quad A = \frac{R}{\sigma}.
\]

In the weak field approximation when \(R_0/R \ll 1\) one gets

\[
R \sim \sigma + \frac{R_0^{K-2}}{2(K-2)\sigma^{K-3}}, \quad A \sim 1 + \frac{R_0^{K-2}}{2(K-2)\sigma^{K-2}},
\]

and the metric takes the following asymptotic form

\[
dS^2 = g_{AB} dX^A dX^B,
\]

\[
g_{AB} = (1 + \Psi)\eta_{AB} + (K - 1)\Psi \delta^0_0 \delta^0_B = \eta_{AB} + \Psi h_{AB},
\]

where

\[
h_{AB} = \eta_{AB} + (K - 1) \delta^0_0 \delta^0_B, \quad \Psi = \frac{R_0^{K-2}}{(K-2)\sigma^{K-2}}.
\]
It should be emphasized that the metric (2.5)-(2.6) describes gravitational field of any compact static distribution of matter \(^1\) since we are considering only the leading terms. The metric (2.6) is a perturbation over the background flat metric
\[
dS_0^2 = -dT^2 + dL_K^2, \tag{2.8}
\]
where
\[
dL_K^2 = (dX^i)^2 + (dY^m)^2 = d\sigma^2 + \sigma^2 d\Omega_{K-1}^2, \tag{2.9}
\]
\[
\sigma^2 = (X^i)^2 + (Y^m)^2. \tag{2.10}
\]
We denote by \(X^i, i = 1, \ldots, n\) the ‘standard’ \(n\) Cartesian coordinates, and by \(Y^m, m = n+1, \ldots, n+k = K\) the Cartesian coordinates in ‘extra-dimensions’. This splitting will be convenient later when we consider an \(n\)-dimensional brane in the spacetime of the \((K+1)\)-dimensional black hole.

### 3 A flat brane in a spacetime with a fixed point

Our goal is to study the equations of motion of a brane in the gravitational field of a black hole. We keep the number of brane spatial dimensions, \(n\), and the number of extra dimensions, \(k\), arbitrary.

In the lowest order, when the gravitational field of the black hole is neglected, \(g_{AB} = \eta_{AB}\). \(\eta_{AB}\) is the \((n+k+1)\)-dimensional Minkowski metric, and the brane is flat. It is convenient to introduce an orthonormal ennumple, \(N_B^A\):
\[
g_{AB} N_C^A N_D^B = \eta_{\hat{C}\hat{D}} \tag{3.1}
\]
so that its first \(n+1\) vectors, \(N_{\hat{\mu}}^A (\mu = 0, \ldots, n)\), are tangent to the brane, while the other \(k\) vectors, \(N_{\hat{m}}^A (m = n+1, \ldots, n+k)\), are orthogonal to the brane. A world-line \(\Gamma\) representing a position of uniformly moving black hole is described by the equation
\[
X^A_T = (x_0^B + U^B T) N_B^A. \tag{3.2}
\]
Here \(T\) is the proper time parameter along the world line, \(U^A\) is the \((n+k+1)\)-velocity, \(\eta_{\hat{A}\hat{B}} U^\hat{A} U^\hat{B} = -1\), and \(X^A_0 \equiv x_0^B N_B^A\) are the coordinates of the black hole position at \(T = 0\).

Consider first the case of one extra dimension. There are two possibilities.

1) The black hole crosses the brane. In this case we use the ambiguity \(T \rightarrow T + \text{const}\) to put \(T = 0\) at the moment when the black hole meets the brane. Using the Poincare group \(P(n+1)\) of coordinate transformations preserving the position of the brane in the bulk space one can always put
\[
x_0^B = 0, \quad U^B = \cosh \beta \delta_0^B + \sinh \beta \delta_{n+1}^B. \tag{3.3}
\]
That is a projection of the point representing the black hole onto the brane surface is located at the origin of the brane coordinates, at \(x^1 = \ldots = x^n = 0\). The black hole

\(^1\)See for example non-topological solitons in brane world models in [7].
crosses the brane at the moment \( x^0 = 0 \) of the brane time, and is moving orthogonally to the brane surface. \( \beta \) is a rapidity parameter related to the velocity \( v \) as \( v = \tanh \beta \).

(2) The black hole never crosses the brane. This is a special case when the velocity of the black hole relative to the brane vanishes. In this case by using transformations from \( P(n+1) \) one can put

\[
x^0 = 0, \quad x^{n+1} = b, \quad U^B = \delta_0^B,
\]

where \( b \) is the distance between the black hole and the brane.

Similarly, in the case with two or more extra dimensions two cases are possible.

(1) The black hole crosses the brane. We choose \( T \) so this happens at the moment \( T = 0 \). It means that \( x^0 = 0 \). Using the Poincare group \( P(n+1) \) of coordinate transformations preserving the position of the brane in the bulk space we put \( x^0 = 0 \) and \( U^\mu = \cosh \beta \delta_0^B \).

We use the group of rotations \( O(k) \) which preserve the position of the brane to put \( U^B = \sinh \beta \delta_{n+1}^B \). Thus we have

\[
x^0 = 0, \quad U^B = \cosh \beta \delta_0^B + \sinh \beta \delta_{n+1}^B.
\]

(2) The black hole never crosses the brane. There exists a minimal distance, \( b \), between the black hole and the brane, which we call the impact parameter. As earlier we can put \( x^0 = 0 \). We also can choose the \( N^A_{n+2} \) to be directed from the brane to the position of the black hole when it is at the minimal distance from the brane. There still exists a group \( O(k-1) \) of rotations which preserves the position of the brane and the direction of \( N^A_{n+2} \). We use this freedom to choose the vector \( N^A_{n+1} \) to coincide with the direction of the black hole velocity. For this choice we have

\[
x^0 = b \delta_{n+2}^B, \quad U^B = \cosh \beta \delta_0^B + \sinh \beta \delta_{n+1}^B.
\]

It is easy to see that the expression (3.6) is in fact the most general one. The relations (3.3–3.5) for the other cases can be obtained from (3.6) by either taking the limit \( \beta = 0 \) or putting \( b = 0 \).

The gravitational potential \( \Psi \) entering the expression (2.6) for the gravitational field of the black hole depends on the interval \( \sigma \) between the position of the black hole and a point in a spacetime, calculated along \( T=\text{const} \) surface. Let us calculate \( \sigma \) for a point on the brane. Denote by \( V^B \) a vector

\[
V^B = \sinh \beta \delta_0^B + \cosh \beta \delta_{n+1}^B.
\]

This vector is orthogonal to \( U^B \) and hence it is tangent to \( T=\text{const} \) plane. The brane time \( x^0 \) corresponding to a given \( T \) can be found from the equations

\[
TU^B + \lambda V^B = x^0 \delta_0^B.
\]

This equation for \( \hat{B} = n + 1 \) gives

\[
\lambda = -\tanh \beta T,
\]

while for \( \hat{B} = 0 \) it gives

\[
T = \cosh \beta x^0, \quad \lambda = \sinh \beta x^0.
\]
Using this results we easily find that
\[
\sigma^2 = \rho^2 + b^2 + \sinh^2 \beta (x^0)^2 ,
\] (3.11)
where \( \rho^2 = x^i x_i \). From this expression for \( \sigma \) it follows that the induced metric on the brane will be ‘spherically symmetric’, that is, it possesses the group \( O(n) \) of symmetry.

Let \( X^A \) be Cartesian coordinates in the reference frame where the black hole is at rest. Then the components of the ensemble \( N^A_B \) in this frame are
\[
N^A_0 = \cosh \beta \delta^A_0 + \sinh \beta \delta^A_{n+1} , \quad N^A_i = \delta^A_i , \quad N^A_{n+1} = \sinh \beta \delta^A_0 + \cosh \beta \delta^A_{n+1} , \quad N^A_m = \delta^A_m , \quad m > n + 1 .
\] (3.12)
(3.13)
In the reference frame of the black hole the brane equation is
\[
X^A_0 = x^\bar{\mu} N^A_{\bar{\mu}} + b N^A_{n+2} .
\] (3.14)

4 Brane perturbation equation of motion

Consider a brane \( \mathcal{X}^A(x^{\bar{\mu}}) \) moving in the spacetime with a metric \( g_{\bar{A}\bar{B}}(X^C) \). The induced metric on the brane (which we assume can be described by the Nambu-Goto action) is
\[
\gamma_{\bar{\mu}\bar{\nu}} = g_{\bar{A}\bar{B}}(X^C) \mathcal{X}^A_{\bar{\mu}} \mathcal{X}^B_{\bar{\nu}} .
\] (4.1)
The brane equation of motion is
\[
\left( \sqrt{-\gamma} \gamma^{\bar{\mu}\bar{\nu}} \mathcal{X}^A_{\bar{\mu}} \right)_{\bar{\nu}} + \sqrt{-\gamma} \gamma^{\bar{\mu}\bar{\nu}} \Gamma^A_{\bar{\mu}\bar{\nu}} \mathcal{X}^B_{\bar{\rho}} \mathcal{X}^C_{\bar{\rho}} = 0 .
\] (4.2)
When the brane is far from the black hole \( g_{\bar{A}\bar{B}} \) has the form (2.5).

In the absence of gravity the unperturbed brane is described by the equation (3.14). We write the solution for a perturbed brane in the form
\[
\mathcal{X}^A = \mathcal{X}^A_0 + \chi^\hat{m}(x) N^A_{\hat{m}} + \zeta^{\hat{\mu}}(x) N^A_{\hat{\mu}} .
\] (4.3)
Let us show first that by changing the coordinates on the brane one can put \( \zeta^{\hat{\mu}} = 0 \). Indeed, a change of Cartesian coordinates on the brane \( x^{\bar{\mu}} \rightarrow x^{\bar{\mu}} + \xi^{\hat{\mu}}(x) \) generates in (4.3) an extra term \( \xi^{\hat{\mu}}(x) N^A_{\hat{\mu}} \). That is why, by using the diffeo-invariance of the brane equations and choosing \( \xi^{\hat{\mu}} = -\zeta^{\hat{\mu}} \) one can always take
\[
\mathcal{X}^A = \mathcal{X}^A_0 + \chi^\hat{m}(x) N^A_{\hat{m}} .
\] (4.4)
In this gauge \( \chi^\hat{m}(x) \) are the ‘physical’ degrees of freedom of the brane. In a general case they describe both types of the brane perturbations, free waves and brane deformations induced by an external force. We focus our attention on the perturbations induced by the brane motion in the weak gravitational field. We restrict ourselves by considering only first order effects.

Using the relation
\[
\mathcal{X}^A_{\hat{\mu}} = N^A_{\hat{\mu}} + \chi_{\hat{m}}^\hat{m} N^A_{\hat{m}} ,
\] (4.5)
and keeping only the first order terms we obtain
\[ \gamma_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} + \Psi h_{\hat{\mu}\hat{\nu}} , \quad h_{\hat{\mu}\hat{\nu}} = h_{AB} N_{\hat{\mu}}^A N_{\hat{\nu}}^B = \eta_{\hat{\mu}\hat{\nu}} + (K - 1) \cosh^2 \beta \delta_{\hat{\mu}\hat{\nu}}. \] (4.6)

We also have
\[ \sqrt{-\gamma} = 1 + \frac{1}{2} \left[ n + 1 - (K - 1) \cosh^2 \beta \right] \Psi , \] (4.7)
\[ \hat{\gamma}_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} - \Psi h_{\hat{\mu}\hat{\nu}} , \quad h_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\alpha}} \eta_{\hat{\nu}\beta} h_{\hat{\alpha}\hat{\beta}}. \] (4.8)

In the chosen coordinate system, the Christoffel symbols \( \Gamma^A_{BC} \) are first order quantities
\[ \Gamma^A_{ABC} = \eta^{AD} \Gamma^D_{DBC} , \quad \Gamma^A_{DBC} = \frac{1}{2} (\Psi^0_{,B} h^0_{CD} + \Psi^0_{,C} h^0_{BD} - \Psi^0_{,D} h^0_{BC}) . \] (4.9)

By multiplying \( (4.2) \) by \( N^0_{\hat{m}A} \) we obtain the following equations
\[ \Box^{(n+1)} \chi_{\hat{m}} = f_{\hat{m}} , \] (4.10)
\[ f_{\hat{m}} = \frac{1}{2} \Psi_{,\hat{m}} h + \frac{1}{2} (K - 1) \sinh(2\beta) \Psi_{,\hat{\alpha}} \delta^0_{\hat{m}} + \eta_{\hat{\mu}\hat{\nu}} h_{\hat{\mu}\hat{\nu}} = n + 1 - (K - 1) \cosh^2 \beta . \] (4.11)

Here \( \Psi_{,\hat{\mu}} = N^A_{\hat{\mu}} \Psi_{,A} , \quad \Psi_{,\hat{m}} = N^A_{\hat{m}} \Psi_{,A} , \) and the \( (n + 1) \)-dimensional flat ‘box’-operator is defined as
\[ \Box^{(n+1)} = \eta_{\hat{\mu}\hat{\nu}} \partial_{\hat{\mu}} \partial_{\hat{\nu}} . \] (4.12)

It is easy to check that the other equations obtained by multiplication of \( (4.2) \) by \( N^0_{\hat{m}A} \) are trivially satisfied.

To calculate \( f_{\hat{m}} \) we note that the function \( \Psi \) which enters this expression depends on \( \sigma^2 \), i.e. \( \Psi(\sigma^2) \), so that
\[ \Psi_{,\hat{\alpha}} = \Psi' \sigma^2_{,\hat{\alpha}} , \quad \Psi_{,\hat{m}} = \Psi' \sigma^2_{,\hat{m}} , \] (4.13)
where
\[ \Psi' = \frac{\partial \Psi}{\partial \sigma^2} = -\frac{\sigma_{,\hat{\alpha}K} - 2}{2\sigma^K} , \] (4.14)
\[ \sigma^2 = \rho^2 + (\sinh^2 \beta)t^2 + b^2 . \] (4.15)

We denoted \( t = x^0 , \) \( \rho^2 = x^i x^i \). We also have
\[ \sigma^2_{,\hat{\alpha}} = 2t \sinh^2 \beta , \quad \sigma^2_{,\hat{m}+1} = 2t \sinh \beta \cosh \beta , \quad \sigma^2_{,\hat{m}+2} = 2b . \] (4.16)

The other terms \( \sigma^2_{,\hat{m}+p} \) with \( p > 2 \) vanish.

The induced metric on the unperturbed brane in the spherical coordinates is
\[ ds_0^2 = \eta_{\hat{\mu}\hat{\nu}} dx^\hat{\mu} dx^\hat{\nu} = -dt^2 + d\rho^2 + \rho^2 d\Omega_{n-2}^2 . \] (4.17)

Because of the symmetry of the problem, the ‘force’ terms \( f_{\hat{m}} \) on the right hand side of \( (4.10) \) are functions of \( t \) and \( \rho \). Thus the induced perturbations of the brane are ‘spherically symmetric’ and can be written in the form
\[ \chi_{\hat{m}} = \frac{P_{\hat{m}}(t, \rho)}{\rho^{(n-1)/2}} . \] (4.18)
By substituting this expression into (4.10) one obtains

\[
-\partial_t^2 + \partial_\rho^2 - \frac{(n-1)(n-3)}{4\rho^2} \right] P_m(t, \rho) = F_m(t, \rho), \tag{4.19}
\]

where

\[
f_m = \frac{F_m(t, \rho)}{\rho^{(n-1)/2}}. \tag{4.20}
\]

From this equation we see that the cases of a string \((n = 1)\) and a three-brane \((n = 3)\) are particularly easy to study. Solutions for the general case are discussed in the appendix A.

5 Solutions of the brane perturbation equations

5.1 Generators of solutions

In the chosen coordinate system and imposed gauge, the ‘force’ \(f_m\) which enters the right hand side of (4.11) has only two non-vanishing components, \(f_- = f_{n+1}\) and \(f_+ = f_{n+2}\). Since we consider only solutions which are induced by the ‘force’ acting on the brane, we shall also have only two non-trivial functions, \(\chi_- = \chi_{n+1}\) and \(\chi_+ = \chi_{n+2}\) which describe the brane excitations. We write the equation (4.11) as

\[
\Box^{(n+1)} \chi^{(n,k)} = f^{(n,k)}_. \tag{5.1}
\]

We included an upper index \((n, k)\) to make clear the dependence of the functions on the number of spatial dimensions, \(n\), of the brane, and on the number of extra dimensions, \(k\). Simple calculations give

\[
f^{(n,k)}_\pm = A^{(n,k)}_\pm \tilde{f}^{(n,k)}_\pm, \tag{5.2}
\]

where

\[
\tilde{f}^{(n,k)}_- = \frac{\sigma}{\sigma_{n+k}}, \quad \tilde{f}^{(n,k)}_+ = \frac{1}{\sigma_{n+k}}, \tag{5.3}
\]

\[
A^{(n,k)}_- = -\frac{1}{4} R_0^{m+k-2} \left[ 2 - k + (n + k - 1) \sinh^2 \beta \right] \sinh(2\beta), \tag{5.4}
\]

\[
A^{(n,k)}_+ = -\frac{1}{2} R_0^{m+k-2} \left[ n + 1 - (n + k - 1) \cosh^2 \beta \right] b. \tag{5.5}
\]

If we write

\[
\chi^{(n,k)}_\pm = A^{(n,k)}_\pm \tilde{\chi}^{(n,k)}_\pm, \tag{5.6}
\]

then

\[
\Box^{(n+1)} \tilde{\chi}^{(n,k)}_\pm = \tilde{f}^{(n,k)}_\pm. \tag{5.7}
\]

Let us note now that

\[
\tilde{f}^{(n,k+2)}_\pm = -\frac{2}{n+k} \frac{\partial}{\partial(b^2)} \tilde{f}^{(n,k)}_\pm, \tag{5.8}
\]

and therefore

\[
\tilde{\chi}^{(n,k+2)}_\pm = -\frac{2}{n+k} \frac{\partial}{\partial(b^2)} \tilde{\chi}^{(n,k)}_\pm. \tag{5.9}
\]
This relation shows that one can generate solutions for an arbitrary \( k > 2 \) if the solutions for \( k = 1, 2 \) are known. It should be noted that for \( k = 1 \) there is only one transverse excitation of the brane, so that a solution \( \chi_{+}^{(n,1)} \) does not have a direct physical meaning. Only its derivatives corresponding to higher values of \( k \) are physical. We call the functions \( \chi_{+}^{(n,1)} \) and \( \chi_{+}^{(n,2)} \) generating solutions.

### 5.2 Generating solutions for \( n = 3 \) brane

As an important example we consider now a special case when the brane has three spatial dimensions. This case is interesting for brane world models. The generating solutions for this case can be written as follows (see appendix A)

\[
\tilde{\chi}_{\pm}^{(k)} = -\frac{1}{2\rho} \int_{-\infty}^{t} dt' \int_{0}^{\infty} d\rho' \rho' \tilde{f}_{\pm}^{(k)} (\vartheta(\lambda_{+}) - \vartheta(\lambda_{-})) ,
\]

(5.10)

where

\[
\lambda_{\pm} = (t - t')^2 - (\rho \mp \rho')^2 .
\]

(5.11)

In order to make notations more compact we omitted index \( n = 3 \) in the superscript.

Since the \( \tilde{f}_{\pm}^{(k)} \) are even functions of \( \rho \) the integral over the two ‘mirror’ regions can be rewritten as an integral over one region characterized only by the \( \vartheta(\lambda_{+}) \) without the restriction of \( \rho \) being greater than zero (see figure 1).

After calculating the integrals we obtain

\[
\tilde{\chi}_{-}^{(1)} = \frac{1}{\cosh^2 \beta} \left[ (t + \rho) S_{+}^{(1)} - (t - \rho) S_{-}^{(1)} \right] ,
\]

(5.12)

\[
\tilde{\chi}_{+}^{(1)} = S_{+}^{(1)} - S_{-}^{(1)} ,
\]

(5.13)

where

\[
S_{\pm}^{(1)} = \frac{1}{4\rho R_{\pm}} \left[ \arctan \left( \frac{t \sinh \beta \mp \rho}{R_{\pm}} \right) + \frac{\pi}{2} \right] ,
\]

(5.14)

and

\[
R_{\pm}^2 = (t \pm \rho)^2 \sinh^2 \beta + b^2 \cosh^2 \beta .
\]

(5.15)
We also have
\[
\tilde{\chi}^{(2)}_\pm = \frac{1}{\cosh^2 \beta} \left[ (t + \rho) S^{(2)}_\pm - (t - \rho) S^{(2)}_\mp \right],
\]
(5.16)
\[
\tilde{\chi}^{(2)}_+ = S^{(2)}_+ - S^{(2)}_-,
\]
(5.17)
where
\[
S^{(2)}_\pm = \frac{1}{6 \rho R^2_\pm} \left( \frac{t \sin^2 \beta \mp \rho}{\sqrt{\rho^2 + t^2 \sin^2 \beta + b^2}} + \cosh \beta \right).
\]
(5.18)

Let us illustrate the motion of the brane in the case \( n = 3, \) and \( k = 2. \) The parametric equations for the brane in isotropic coordinates are
\[
X^0 = t \cosh \beta - 2 R^3_0 \sinh^4 \beta \cosh \beta \tilde{\chi}^{(2)}_-, \\
X^1 = x^1, \quad X^2 = x^2, \quad X^3 = x^3, \\
X^4 = t \sinh \beta - 2 R^3_0 \sinh^2 \beta \cosh^2 \beta \tilde{\chi}^{(2)}_-, \\
X^5 = b + 2b R^3_0 \sinh^2 \beta \tilde{\chi}^{(2)}_+.
\]
(5.19)
(5.20)
(5.21)
(5.22)

Figure 2 shows plots of \( \chi_\pm \) for a particular choice of parameters. Both plots depict a disturbance of the brane developed around "time" \( t = 0 \) which travels at a speed of light outward from the point of the brane closest to the black hole.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Plots of \( \chi_- \) and \( \chi_+ \) for \( \beta = 1, \) \( R_0 = 1, \) \( b = 10, \) \( k = 2. \)}
\end{figure}

### 5.3 Energy loss

As a result of the black hole action, the brane transforms from its initial state without excitations to a new excited state. The energy gained by the brane in this process is equal to the loss of the kinetic energy of the black hole. We calculate now this energy loss.
In the limit $t \to \infty$, $u = t - \rho = \text{const}$, the brane excitation amplitudes $\chi_{\pm}$ take the following form

$$\chi_{\pm} = \frac{\Phi_{\pm}(u)}{\rho},$$

(5.23)

$$\Phi_- = \frac{2}{3} R_0^3 \sinh^3 \beta \frac{u}{u^2 \sinh^2 \beta + b^2 \cosh^2 \beta},$$

(5.24)

$$\Phi_+ = -\frac{2}{3} R_0^3 b \cosh \beta \sinh^2 \beta \frac{1}{u^2 \sinh^2 \beta + b^2 \cosh^2 \beta}.$$  

(5.25)

Figure 3 shows the typical shape of the functions $\Phi_{\pm}$ around $u = 0$.

\[\text{Figure 3: Asymptotic shape of the fields } \Phi_- \text{ and } \Phi_+.\]

In the asymptotic regime, when the gravitational field of the black hole can be neglected, the induced metric (4.1) is

$$\gamma_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} + \chi_{\hat{m}} \chi_{\hat{n}} \gamma_{\hat{m}\hat{n}}.$$  

(5.26)

The Nambu-Goto action is

$$I = \mu \int \sqrt{-\gamma} d^n x = I_0 + I_2,$$  

(5.27)

where $\mu$ is the tension of the brane, $I_0$ is constant and

$$I_2 = -\frac{1}{2} \mu \sum_{\hat{m} = n+1}^{n+k} \int (\nabla \chi_{\hat{m}})^2.$$  

(5.28)

In our case the asymptotic excitations are described by two massless scalar fields with the Lagrangian density

$$\mathcal{L}_{\pm} = -\frac{1}{2} \mu \partial_{\mu} \chi_{\pm} \partial^{\mu} \chi_{\pm}.$$  

(5.29)

The energy flux $\mathcal{E}_{\pm}$ calculated at the future null infinity is

$$\mathcal{E}_{\pm} = 4\pi \mu \int_{-\infty}^{\infty} du \left( \Phi_{\pm}(u) \right)^2.$$  

(5.30)
Simple calculations give
\[ E_- = E_+ = \frac{4 \pi^2 \mu \sinh^5 \beta}{9 \cosh^3 \beta} \frac{R_0^6}{b^3}. \]

Thus the total energy lost by the black hole and gained by the brane is
\[ \Delta E = E_- + E_+ = \frac{8 \pi^2 \mu \sinh^5 \beta}{9 \cosh^3 \beta} \frac{R_0^6}{b^3}. \]

Since extra dimensions are compactified, the black hole will be passing near the brane again and again. Because of the friction force connected with the energy loss, the black hole will be slowing down until finally stops with respect to the brane.

6 ‘Shadow matter’ effect

The metric induced on the brane by a black hole moving in the bulk space is
\[ ds^2 = -\left[ 1 + [1 - (k + 2) \cosh^2 \beta] \Psi \right] dt^2 + (1 + \Psi) \left[ d\rho^2 + \rho^2 d\Omega_2^2 \right], \]
with
\[ \Psi = \frac{R_0^{k+1}}{(k + 1) \sigma^{k+1}} \quad \text{and} \quad \sigma^2 = \rho^2 + (\sinh^2 \beta) t^2 + b^2. \]

Particles and light on the brane are moving along geodesics in this metric. If an observer on the brane does not know about the existence of extra dimension and uses the standard 4-dimensional Einstein’s equations he would arrive to the conclusion that there exists some distribution of matter on the brane responsible for this gravitational field. Since this matter is not connected with any usual 4-dimensional physical fields and particles we call it a ‘shadow matter’. We discuss now the properties of this matter.

The Einstein tensor \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} g_{\mu\nu} \) for the metric (6.1) takes the form (in spherical coordinates \((t, \rho, \theta, \phi)\))
\[ G_{00} = \frac{R_0^{k+1}}{\sigma^{k+5}} \left[ 3t^2 \sinh^2 \beta - \rho^2 k + 3b^2 \right], \]
\[ G_{11} = \frac{R_0^{k+1}}{\sigma^{k+5}} \left[ kt^2 \sinh^2 \beta + (\rho^2 + b^2)((3 + k) \cosh^2 \beta - 3) \right], \]
\[ G_{01} = -\frac{t \rho R_0^{k+1}(3 + k) \sinh^2 \beta}{\sigma^{k+5}}, \]
\[ G_{22} = \frac{G_{33}}{\sin^2 \theta} = \frac{R_0^{k+1} \rho^2}{2\sigma^{k+5}} \left[ k[2t^2 \sinh^2 \beta + \rho^2(2 - (3 + k) \cosh^2 \beta)] + 2\rho^2((3 + k) \cosh^2 \beta - 3) \right]. \]

Figure 4 shows plots of the various non-zero components of the Einstein tensor \( G_{\mu\nu} \) which is proportional to the stress-energy tensor \( T_{\mu\nu} \) measured by observers living on the brane. The thick line on figure 4 marks the border between the positive and negative energy density.
In the simplest case when the bulk black hole is not moving, $\beta = 0$, these expressions simplify and non-vanishing components of the Einstein tensor are (at $t = 0$)

$$G_{00} = \frac{R_0^{k+1}}{\sigma^{k+5}} \left[ -\rho^2 k + 3b^2 \right], \quad (6.7)$$

$$G_{11} = \frac{R_0^{k+1}}{\sigma^{k+5}} k(\rho^2 + b^2), \quad (6.8)$$

$$G_{22} = \frac{G_{33}}{\sin^2 \theta} = \frac{R_0^{k+1} \rho^2 k}{2\sigma^{3+k}} \left[ -\rho^2 (k+1) + 2b^2 \right]. \quad (6.9)$$

Suppose a brane observer uses the Einstein’s equations to describe the gravitational field on the brane. In this case he would come to a conclusion that there exists some form of matter for which

$$T_{\hat{\mu}\hat{\nu}} = \frac{1}{8\pi G(4)} G_{\hat{\mu}\hat{\nu}}, \quad (6.10)$$

where $G(4)$ is a 4-dimensional Newtonian coupling constant. For a static black hole this spherically symmetric distribution of matter is of the form

$$T^{\nu,\mu} = \text{diag}(-\varepsilon, p_\rho, p_\perp, p_\perp). \quad (6.11)$$

Since the total number of spatial dimensions is greater than three, i.e. $k > 0$, the energy density $\varepsilon$ is positive at the center and changes its sign and becomes negative at $\rho > \rho_b = b\sqrt{3/k}$. The radial pressure $p_\rho$ is always positive, while the transverse pressures $p_\perp$ are positive at the center and become negative at $\rho > b\sqrt{2/(k+1)}$.

If there is a distribution of physical matter on the brane, it will give an additional contribution to the metric (6.1). In the weak field approximation this contribution will be additive.

Let us estimate the total positive mass $m_b$ of the ‘shadow matter’ inside the sphere of the radius $\rho_b$. To define this mass one can use the relation

$$m(r) = 4\pi \int_0^r dr r^2 T_{00}. \quad (6.12)$$

The radius coordinate $r$ is related to the isotropic coordinate as

$$(1 + \Psi) \rho^2 = r^2. \quad (6.13)$$

Since we are considering only first order terms we can write

$$m(\rho) = \frac{1}{2G(4)} \int_0^\rho d\rho \rho^2 G_{00} = \frac{1}{2G(4)} \frac{R_0 \rho^3}{(\rho^2 + b^2)^{3/2}}. \quad (6.14)$$

In particular, we have

$$m_b = m(\rho_b) = \alpha(k) \frac{R_0}{G(4)} \left( \frac{R_0}{b} \right)^k, \quad (6.15)$$

where

$$\alpha(k) = \frac{3\sqrt{3}}{2} \frac{k^{k/2}}{(k + 3)^{(k+3)/2}}. \quad (6.16)$$
It is convenient to rewrite (6.15) as

\[ m_b = \alpha(k) m_* \left( \frac{R_0}{R_*} \right)^{k+1} \left( \frac{R_*}{b} \right)^k. \]  

(6.17)

Here \( R_* = 1/M_* \) is the gravitational radius of the fundamental mini black hole, and \( m_* = R_*/G^{(4)} \). For \( M_* \sim \text{TeV} \) one has \( m_* \sim 10^{11} \text{g} \). Thus, for \( b \sim \text{TeV} \), we have 100000 tons of “shadow matter” concentrated in the region of the size \( \text{TeV}^{-1} \). However, this feature is visible only for test particles whose wavelength is of order \( \text{TeV}^{-1} \).

The mass \( m_b \) is surrounded by the negative mass distribution \( \varepsilon \). For infinite size of extra dimensions, at far distances it exactly cancels the mass \( m_b \), so that the total mass as measured at infinity vanishes. It happens because the induced gravitational field potential falls down at infinity as it is required by \((3 + k)\)-dimensional Newton’s law (i.e. \( \sim G^{(4+k)} M/r^{1+k} \)), while the standard 3-dimensional Newtonian gravitational potential of mass \( M \) at far distance is \( G^{(4)} M/r \). For a finite size \( L \) of extra dimensions this cancellation is not complete. The gravitational field of the bulk mass \( M \) as measured by the brane observer at \( r \gg L \) is \( \sim G^{(4+k)} M/(L^k r) = G^{(4)} M/r \). In other words, the bulk masses at \( r \gg L \) contribute to the gravitational field on the brane similarly to the matter on the brane.

One can easily check that for \( \rho > \rho_b \) the ‘shadow matter’ distribution violates the weak energy condition \(^2\): \( \varepsilon \geq 0 \) and \( \varepsilon + p_i \geq 0 \), \((i = 1, 2, 3)\). This violation is apparent only for an observer located on the brane. The complete system (brane + bulk) does not violate any of the energy conditions. Similar effects can be present in many different frameworks (see for example [8]). Our situation is distinguishable by the fact that the source of apparent violation of energy conditions is a simple distribution of bulk matter.

The violation of the weak energy condition in particular implies that an \( n \)-dimensional pencil of initially parallel null rays propagating on the brane and passing through the region with \( \rho > \rho_b \) will be defocused, that is the \( n \)-dimensional area of its cross-section will increase. If on the other hand one consider the \((n + k)\)-dimensional pencil of initially parallel null rays propagating in the bulk, the area of its \((n + k)\)-dimensional cross-section will decrease. There is no contradiction between these two results, since the Weyl tensor of the bulk gravitational field does not vanish. As a result a shear is generated, and the expansion of the beam in the direction of the brane is compensated by the contraction of the beam in the bulk dimensions.

7 Deflection of light

The ‘shadow-matter’ can affect the propagation of test particles and light on the brane. We consider now the deflection of a light ray passing in the region of influence of the ‘shadow-matter’. For simplicity we assume that the bulk black hole velocity vanishes.

Since the propagation of light is invariant under conformal transformation of the metric, we divide the metric \((6.1)\) by \( 1 + \Psi \) and keep the leading order terms

\[ ds^2 = -(1 + A\Psi) dt^2 + d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2, \]

\( (7.1) \)

\(^2\)In fact the dominant and strong energy conditions are also violated.
Figure 4: Plots of the components of the Einstein tensor of the induced geometry on the brane. The parameters are $R_0 = 1, \beta = 1, b = 10, k = 2$

where $A = -(k + 2)$. Since the metric is spherically symmetric the light trajectory lies in a plane and we can set $\theta = \pi/2$. The eiconal equation which describes the motion of light in some background metric is

$$
g^{\mu \nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = -(1 - A \psi) \left( \frac{\partial S}{\partial t} \right)^2 + \left( \frac{\partial S}{\partial \rho} \right)^2 + \frac{1}{\rho^2} \left( \frac{\partial S}{\partial \phi} \right)^2 = 0. \tag{7.2}$$

Here $S$ is the Hamilton-Jacobi action for massless particle (often called eiconal).

The action $S$ can be written as

$$
S = -\omega t + m\phi + S_\rho(\rho), \tag{7.3}
$$
where ω is the energy and m is the angular momentum. Substituting (7.3) into (7.2) we get

\[ S_\rho = \int \sqrt{(1 - A \Psi) \omega^2 - \frac{m^2}{\rho^2}} \, d\rho . \]  
(7.4)

When the spacetime is flat the light ray is a straight line. The corresponding \( S_\rho^{(0)} \) is given by the integral (7.4) with \( A = 0 \). Keeping only the first order terms we can write

\[ S_\rho = S_\rho^{(0)} + \Delta S_\rho , \]  
(7.5)

where

\[ \Delta S_\rho = \int \frac{A \omega^2 \Psi}{2 \sqrt{\omega^2 - \frac{m^2}{\rho^2}}} \, d\rho . \]  
(7.6)

After substituting the value for \( A \) and

\[ \Psi = \frac{R_0^{k+1}}{(k + 1)(\rho^2 + b^2)^{\frac{k+1}{2}}} \]  
(7.7)

we obtain

\[ \Delta S_\rho = - \int_R^{b_0} \frac{(k + 2) \omega R_0^{k+1} \rho d\rho}{(k + 1)^2 \sqrt{\rho^2 + b^2}^{k+1}(\rho^2 - \frac{m^2}{\omega^2})} . \]  
(7.8)

Note that the impact parameter of the light ray in terms of integrals of motion is just \( b_0 = \frac{m}{\omega} \). Light propagates from some large distance \( \rho = R \) to the point \( \rho = b_0 \) nearest to the center of the projected ‘shadow matter’ and then back to the distance \( R \). Therefore the integration goes from \( R \) to \( b_0 \) and the result should be multiplied by 2. For a ray coming from infinity \( R \to \infty \). After integration we get

\[ \Delta S_\rho = \frac{(k + 2)\sqrt{\pi} \omega \Gamma(\frac{k}{2})}{2(k + 1)\Gamma(\frac{k+1}{2}) \left( b_0^2 + b^2 \right)^{\frac{k}{2}}} R_0^{k+1} . \]  
(7.9)

From eq. (7.3) we obtain the total deflection angle

\[ \Delta \phi = - \frac{\partial}{\partial m}(\Delta S_\rho) . \]  
(7.10)

The calculation gives

\[ \Delta \phi^{(k)} = \frac{(k + 2) \sqrt{\pi} k \Gamma(k/2)}{2(k + 1) \Gamma((k + 1)/2)} \frac{R_0^{k+1} b_0}{(b_0^2 + b^2)^{\frac{k+1}{2}}} . \]  
(7.11)

In particular, for the case of one and two extra dimensions one has

\[ \Delta \phi^{(1)} = \frac{3}{4} \pi R_0^2 \frac{b_0}{(b^2 + b_0^2)^{3/2}} , \quad \Delta \phi^{(2)} = \frac{8}{3} R_0^3 \frac{b_0}{(b^2 + b_0^2)^{3/2}} . \]  
(7.12)

For comparison, the standard \((3 + 1)\)-dimensional case \((k = 0, b = 0)\) is

\[ \Delta \phi = 2 \frac{R_0}{b_0} . \]  
(7.13)

The different functional dependence of the deflection angle on the impact parameter of light \( b_0 \) gives us opportunity to distinguish between the real matter on the brane and the bulk ‘shadow matter’ as a cause of deflection.
8 Conclusions

We studied the interaction of an \( n \)-dimensional brane described by the Nambu-Goto action with a higher-dimensional Schwarzschild black hole moving in the \((n + k + 1)\)-dimensional spacetime. The \( n = 1 \) case corresponds to a cosmic string, \( n = 2 \) to a domain wall, while \( n = 3 \) can be interpreted as the observable universe in the context of brane world models. We derived the general form of the perturbation equations for an \( n \)-brane in the background of a \((n + k)\)-dimensional black hole in the weak field approximation.

For odd number of spatial brane dimensions by using convenient mathematical transformations we derived the closed form solution of the D’Alambert Equation with a spherically symmetric source (see appendix A). We applied this result to the most interesting case of a 3-brane in a spacetime with extra dimensions where we obtained a general solution.

We calculated the induced geometry on the brane generated by a moving black hole. As considered by a brane observer this geometry can be obtained by solving \((n + 1)\)-dimensional Einstein’s equations with a non-vanishing right hand side. We calculated the effective stress-energy tensor corresponding to this ‘shadow-matter’. We showed that there exist regions where a brane observer sees an apparent violation of energy conditions. The ‘shadow-matter’ also affects the propagation of test particles and light on the brane. We demonstrated this by deriving results for deflection of light propagating in the induced spacetime metric on the brane. As expected, results are quite different from the \((3 + 1)\)-dimensional results. It would be interesting to study eventual observational tests which would indicate that the ‘shadow-matter’ and thus the extra dimensions influence the physics of our \((3 + 1)\)-dimensional world.

One of the possible interesting application of the ‘shadow matter’ effect might be the following. If there exists a diluted non-relativistic gas of stable elementary mini black holes in the extra dimensions, its gravitational action on the brane would be similar to the “observable” dark matter, provided the \((3 + k)\)-dimensional density of this gas is \( n_k \) where

\[
\epsilon_{DM} \sim M_s n_k L_k. \tag{8.14}
\]

Here \( \epsilon_{DM} \) is the mass density of the dark matter. For \( \epsilon_{DM} \approx 10^{-29} \text{g/cm}^3 \) one has

\[
n_k \sim 10^{-6+2k} 1/\text{cm}^{3+k}. \tag{8.15}
\]

Since the average distance between the bulk black holes, \( n_k^{-1/(3+k)} \), is much larger than the black hole radius \( R_s \), with a very high accuracy one can consider such gas to be very diluted and neglect the gravitational interaction between the black holes. A distinguishing property of this model is that there are no physical particles on the brane responsible for the dark matter.

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A Spherically-Symmetric Solutions of D’Alambert Equation in Higher Dimensional Spacetimes

A solution of the \((n+1)\)-dimensional D’Alambert equation

\[ \Box^{(n+1)} p = f \] (A.1)

can be written by using a retarded Green’s function

\[ \Box^{(n+1)} G_{n+1}^{ret}(x, x') = -\delta^{(n+1)}(x - x') . \] (A.2)

We assume that in the infinite past there were no incoming waves. The solution \(p(x)\) is then completely generated by the external ‘force’ \(f(x)\),

\[ p(x) = -\int G_{n+1}^{ret}(x, x') f(x') dx' . \] (A.3)

The Green’s function for odd \(n = 2\nu + 1, \ \nu \geq 1\), can be written as [9]

\[ G_{n+1}^{ret}(x, x') = \left(-1\right)^{\nu} \frac{\partial(t - t')}{(2\pi)^{\nu}} \left[ \frac{d^{\nu-1}}{(RdR)^{\nu-1}} \delta((t - t')^2 - R^2) \right] , \] (A.4)

where \(R = |x - x'|\). For our convenience, we rewrite it in a slightly different form

\[ G_{n+1}^{ret}(x, x') = \frac{\partial(t - t')}{(2\pi)^{\nu}} \left[ \frac{d^{\nu-1}}{d\alpha^{\nu-1}} \delta(\lambda + \alpha) \right]_{\alpha=0} . \] (A.5)

Here

\[ \lambda = (t - t')^2 - |x - x'|^2 , \] (A.6)

\[ |x - x'|^2 = \rho^2 + \rho'^2 - 2\rho\rho' z , \quad z = \cos \theta , \] (A.7)

and \(\theta\) is the angle between \(n\)-dimensional vectors \(x\) and \(x'\).

We are interested in spherically symmetric solutions of D’Alambert equation. Integrating over the angular variables we get

\[ \hat{G}(t, \rho; t', \rho') = \Omega_{n-2} \int_{-1}^{1} dz \left(1 - z^2\right)^{(n-3)/2} G_{n+1}^{ret}(t, \rho; t', \rho' ; z) , \] (A.8)

where \(\Omega_k = 2\pi^{\frac{k+1}{2}}/\Gamma\left(\frac{k+1}{2}\right)\) is a volume of a \(k\)-dimensional unit sphere \(S^k\). Here we made explicit that \(G_{n+1}^{ret}\) depends on the angle variables only through the parameter \(z\).

Let us denote

\[ F = \rho^{\nu} f , \quad P = \rho^{\nu} p . \] (A.9)

Then in the absence of incoming waves the spherically symmetric solution of the equation [A.1] is

\[ P(t, \rho) = -\int dt' \ d\rho' G(t, \rho; t', \rho') F(t', \rho') , \] (A.10)

where

\[ G(t, \rho; t', \rho') = (\rho' \rho)^{\nu} \hat{G}(t, \rho; t', \rho') . \] (A.11)
Let us show that for odd $n$ the representation (A.4) allows one to obtain the Green’s function for this reduced equation. The reduced Green’s function $G$ can be written as

$$G(t, \rho; t', \rho') = \vartheta(t - t') \frac{(\rho')^{\nu}}{\Gamma(\nu)} B,$$

where

$$B = \left[ \frac{d^{\nu-1}}{d\alpha^{\nu-1}} \int_{-1}^{1} dz \left(1 - z^2\right)^{\nu-1} \delta(b(z + z_*)) \right]_{\alpha=0}, \tag{A.13}$$

$$\lambda_0 = (t - t')^2 - \rho^2 - \rho'^2, \quad b = 2\rho\rho', \quad z_* = \frac{\lambda_0 + \alpha}{b}. \tag{A.14}$$

By calculating the integral in (A.13) one gets

$$B = B_+ - B_-, \tag{A.15}$$

$$B_{\pm} = \frac{1}{b} \left[ \frac{d^{\nu-1}}{d\alpha^{\nu-1}} \left\{(1 - z_2^2)^{\nu-1}\vartheta(z_{*\pm} \pm 1)\right\} \right]_{\alpha=0}. \tag{A.16}$$

Now let us note that if any of the derivative over $\alpha$ is acting on the $\vartheta$-function, the result vanishes because of the remaining factor $1 - z_*^2$. Using Rodrigues’ formula for Legendre polynomials ([10], relation 22.11.5)

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left[(1 - x^2)^n\right], \tag{A.17}$$

one obtains

$$B_{\pm} = \vartheta(\lambda_{\pm}) \left(\frac{-2^{\nu-1}}{b^{\nu}}(\nu - 1)! P_{\nu-1}(\lambda_0/b)\right), \tag{A.18}$$

where

$$\lambda_{\pm} = (t - t')^2 - (\rho \mp \rho')^2. \tag{A.19}$$

Combining the above expressions we obtain the following representation for the reduced Green’s function

$$G(t, \rho; t', \rho') = C_\nu P_{\nu-1} \left(\frac{\lambda_0}{b}\right) \left[\vartheta(\lambda_+) - \vartheta(\lambda_-)\right], \tag{A.20}$$

where

$$C_\nu = \frac{(-1)^{\nu-1}(\nu - 1)!}{2\Gamma(\nu)}. \tag{A.21}$$

For $\nu = 1$, which corresponds to a $n = 3$ brane we have

$$p(t, \rho) = -\frac{1}{2} \int dt' d\rho' \frac{\rho'}{\rho} f(t', \rho') \left[\vartheta(\lambda_+) - \vartheta(\lambda_-)\right]. \tag{A.22}$$

A similar procedure can be applied for the case of even $n$, i.e., odd number of spacetime dimensions. The expressions are more complicated and will not be discussed here.
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