UNIFIED VIEW OF MULTIMODE ALGEBRAS WITH FOCK-LIKE REPRESENTATIONS

Stjepan Meljanac and Marijan Mileković†

November 13, 2018

Rudjer Bošković Institute
Bijenička c.54, 41001 Zagreb, Croatia

† Prirodoslovno-Matematički Fakultet, Zavod za teorijsku fiziku,
Bijenička c.32, 41000 Zagreb, Croatia

Short title: Unified view of multimode...
(to be published in Int.J.Mod.Phys. A)
Abstract

A unified view of general multimode oscillator algebras with Fock-like representations is presented. It extends a previous analysis of the single-mode oscillator algebras. The expansion of the $a_i a_j^\dagger$ operators is extended to include all normally ordered terms in creation and annihilation operators and we analyze their action on Fock-like states. We restrict ourselves to the algebras compatible with number operators. The connection between these algebras and generalized statistics is analyzed. We demonstrate our approach by considering the algebras obtainable from the generalized Jordan-Wigner transformation, the para-Bose and para-Fermi algebras, the Govorkov "paraquantization" algebra and generalized quon algebra.
1. Introduction

Recently, much attention has been devoted to the study of quantum groups and algebras\textsuperscript{1}, noncommutative spaces and geometries\textsuperscript{2}, generalized notions of symmetries as well as to their diverse applications in physics\textsuperscript{3}. These approaches are in close relationship with study of deformed oscillators, algebras and their Fock representations. Single-mode oscillator algebras were studied by a number of authors\textsuperscript{4}. A unified view of single-mode oscillator algebras was proposed by Bonatsos and Daskaloyannis\textsuperscript{5} and Meljanac et al.\textsuperscript{6}. Multimode oscillator algebras are much more complicated and only partial results exist in the literature\textsuperscript{7}. Particularly, the R-matrix approach to multimode algebras was followed in Refs.(8). Some of multimode algebras, but not all, can be obtained from an ordinary Bose algebra with equal number of oscillators\textsuperscript{9}.

On the other hand, there is an old additional physical motivation to study multimode oscillator algebras, connected with the problem of generalized statistics, different from Bose and Fermi statistics\textsuperscript{10}. The first consistent example of it was para-Bose and para-Fermi statistics\textsuperscript{11,12} and, recently, a new paraquantization\textsuperscript{13} satisfying trilinear commutation relations between annihilation and creation operators. These types of statistics are characterized by a discrete parameter called the order of parastatistics $p \in \mathbb{N}$, interpolating between Bose and Fermi statistics.

Recently, a new interpolation, namely infinite quon statistics characterized by a continuous parameter, has been proposed and analyzed\textsuperscript{14,15}. Its multiparameter extension was studied in Refs.(16). Parastatistics can be applied in spaces with an arbitrary number of dimensions and those statistics with a continuous parameter could be relevant to lower dimensions. Specially, in the $(2 + 1)$ dimensional space, another
type is also possible, the so-called anyonic (fractional) statistics. It is characterized by the continuous statistical parameter $\lambda$. A different type of the fractional statistics, generalizing the Pauli exclusion principle, is also proposed.

Our motivation of the present work is twofold:

(i) to try to extend the quantization defined by $a_i a_j^\dagger - R_{ij,kl} a_k^\dagger a_l = \delta_{ij}, i, j \in I$ to include all normally ordered terms in creation and annihilation operators;

(ii) to connect these algebras with the notion of generalized statistics.

In the physical world particles decay to other particles, interact and transform themselves, and, as a consequence, the number operators in these processes are not conserved. They change respecting some selection rules, characteristic of the type of interaction. However, in order to measure different sorts of particles, we always assume that any sort of physical particles is countable. Thus, the existence of the number operator is of utmost importance. Our aim is to define and describe the most general multimode oscillator algebras possessing Fock-like representations and compatible number operators.

Generally, one considers a given algebra and examines all its representations. Here we are going backwards, i.e., we start with the most general Fock-like space and look for the relations between annihilation and creation operators $\{a_i, a_i^\dagger | i \in I\}$, leading to the positive definite scalar product and to the non-negative norms of all Fock-like states.

The plan of the paper is as follows. In Sec. 2 we briefly review the main aspect of a single-mode oscillator algebra. We characterize the algebra by a set of quantities $\{\varphi(n)\}, \{c_n\}$ and $\{d_n\}$ which are generalized to a set $\{\Phi\}, \{C\}$ and $\{D\}$ for multimode oscillator algebra with Fock-like representations in Sec. 3. We also discuss the connection with generalized statistics. In Sec. 4-7 we give some examples of our approach to multimode algebras, i.e., we discuss algebras obtainable from generalized
Jordan-Wigner transformations (Sec.4), para-algebras (Sec.5), new paraquantization of Govorkov (Sec.6) and generalized quon algebra (Sec.7).

2. Single-Mode Oscillator Algebra with Fock-like Representations

In this section we briefly review single-mode oscillator algebras. For more details we refer to Ref. (6).

Let us consider a pair of operators $\bar{a}, a$ (not necessarily Hermitian conjugate to each other) with the number operator $N$. The most general commutation relation linear in the $\bar{a}a$ and $a\bar{a}$ operators is

$$a\bar{a} - F(N)\bar{a}a = G(N),$$

where $F(N)$ and $G(N)$ are arbitrary complex functions. The number operator satisfies

$$[N, a] = -a,$$
$$[N, \bar{a}] = \bar{a},$$
$$[N, \bar{a}a] = [N, a\bar{a}] = 0.$$

Hence, we can write

$$\bar{a}a = \varphi(N),$$
$$a\bar{a} = \varphi(N+1),$$

where $\varphi(N)$ is, in general, a complex function satisfying the recurrence relation

$$\varphi(N+1) - F(N)\varphi(N) = G(N).$$

If $\varphi(N)$ is the bijective mapping, then

$$N = \varphi^{-1}(\bar{a}a) = \varphi^{-1}(a\bar{a}) - 1.$$
Let us denote the Hermitian conjugate of the operator $a$ by $a^\dagger$. Then it follows that
\[
[N, a^\dagger] = a^\dagger,
\]
where $c(N)$ is a complex function of $N$. It is convenient to choose $c(N)$ to be a "phase" operator, $|c(N)| = 1$. Then we have
\[
a^\dagger a = |\varphi(N)|,
\]
\[
aa^\dagger = |\varphi(N + 1)|,
\]
\[
aa^\dagger - a^\dagger a = |\varphi(N + 1)| - |\varphi(N)| = G_1(N),
\]
\[
c(N) = e^{i \arg \varphi(N)} = \frac{\varphi(N)}{|\varphi(N)|}.
\]
If $\varphi(N) > 0$, then $\arg \varphi(N) = 0$ and $c(N) = 1$.

Let us further assume that $|0> = a |0> = 0$, $N |0> = 0$, $\varphi(0) = 0$.

One can always normalize the operators $a$ and $\bar{a}$ such that $|\varphi(1)| = 1$; then $|G(0)| = 1$. The function $\varphi(N)$ is determined by the recurrence relation (2.4) and is given by
\[
\varphi(n) = [F(n - 1)]! \sum_{j=0}^{n-1} \frac{G(j)}{[F(j)]!},
\]
where
\[
[F(j)]! = F(j)F(j - 1)...F(1),
\]
\[
[F(0)]! = 1.
\]
The excited states with unit norms are
\[
|n> = \frac{(a^\dagger)^n}{\sqrt{|\varphi(n)|!}} |0> = \frac{(c^{-\frac{1}{2}}\bar{a})^n}{\sqrt{|\varphi(n)|!}} |0>.
\]
\[
< n|m > = \delta_{mn}, \quad n, m = 0, 1, 2, \ldots, \\
< n - 1|n > = < n|a^\dagger|n - 1 > = \sqrt{|\varphi(n)|}.
\] (2.11)

The function \(|\varphi(n)|\) uniquely determines the type of deformed oscillator algebra and vice versa. If \(\varphi_1 \neq \varphi_2\) but \(|\varphi_1| = |\varphi_2|\), the corresponding algebras are isomorphic. There is a family of functions \((F, G)\) leading to the same algebra, with identical functions \(\varphi(N)\). Therefore we can fix the ”gauge”, for example:

(a) \(F(N) = 1, \quad a\bar{a} - \bar{a}a = G_1(N)\),

(b) \(G(N) = 1, \quad a\bar{a} - F_1(N)\bar{a}a = 1\),

(c) \(F(N) = q, \quad a\bar{a} - q\bar{a}a = G_q(N)\).

(2.12)

Several examples are given in Ref.(6).

The unitary irreducible representations of the single-mode oscillator algebra can be classified according to the existence (or nonexistence) of a vacuum state \(|0 >\) as Fock-like, non-Fock-like and degenerate. If \(\varphi(n) \neq 0\) for \(\forall n \in \mathbb{N}\), then there is an infinite set (”tower”) of states and, owing to the well-defined mapping to the Bose algebra

\[
a = b\sqrt{\frac{|\varphi(N)|}{N}}, \quad a^\dagger = \sqrt{\frac{|\varphi(N)|}{N}}b^\dagger,
\] (2.13)

the Fock space of the deformed algebra is identical to the Fock space of the Bose oscillator. However, if \(\varphi(n_0) = 0\) for some \(n_0\), then the state \((a^\dagger)^{n_0}|0 >\) has zero norm and, consistently, we can put \(|n_0 > \equiv 0\). The corresponding representation is finite-dimensional and the representation matrices are of the \(n_0 \times n_0\) type (2.11) (degenerate representation).

Now we present another approach to the single-mode oscillator algebras which will be generalized to the multimode case in the next section.
One can write Eq.(2.3) in an alternative form

$$\varphi(N + 1) = a\bar{a} = 1 + \sum_n c_n (\bar{a})^n (a)^n$$  \hspace{1cm} (2.14)

where the coefficients $c_n$ are defined recursively by $\varphi(n)$ as

$$c_n = \frac{\varphi(n + 1) - 1 - \sum_{k=1}^{n-1} c_k \varphi(n) \cdots \varphi(n + 1 - k)}{[\varphi(n)]!}, \quad \forall\varphi(n) \neq 0. \hspace{1cm} (2.15)$$

Knowing $c_k$ we can recursively obtain $\varphi(n)$

$$\varphi(n + 1) = \sum_{k=0}^{n} c_k \frac{[\varphi(n)]!}{[\varphi(n-k)]!}, \quad c_0 = 1. \hspace{1cm} (2.16)$$

Equivalently, one can write the expression for the number operator $N$, Eq.(2.5), as

$$N = \bar{a}a + \sum_{n=2}^{\infty} d_n (\bar{a})^n (a)^n, \hspace{1cm} (2.17)$$

where

$$d_n = \frac{n - \sum_{k=1}^{n-1} d_k \varphi(n) \cdots \varphi(n + 1 - k)}{[\varphi(n)]!}. \hspace{1cm} (2.18)$$

Note that $d_n = 0$ for $\varphi(n) = 0, \varphi(n - 1) \neq 0$. Knowing $d_n$, we can obtain $\varphi(n)$ from the same recurrent relation (2.18).

In a physical application it is important to know the vacuum matrix elements

$$A_{m,n} = \langle 0 | (a)^n (\bar{a})^m | 0 \rangle = [\varphi(n)]! \delta_{mn}. \hspace{1cm} (2.19)$$

Hence, the sets of quantities $\{\varphi(n)\}, \{c_n\}$ and $\{d_n\}$ represent a unique description of a given (deformed) single-mode oscillator algebra. Together with the generalization of the matrix $A$, Eq.(2.19), they will be particularly useful for the analysis of multimode (deformed) oscillator algebras.
3. Multimode Oscillator Algebra with Fock-like Representations

In this section we define and describe the most general multi-mode oscillator algebras possessing Fock-like representations. We start with the most general Fock-like space and look for the relations between annihilation and creation operators \( \{a_i, \bar{a}_i | i \in I \} \), leading to the positive definite scalar product and to the non-negative norms of all Fock-like states (vectors). Furthermore, we restrict ourselves to the algebras possessing number operators for all \( \forall i \in I \).

Let us first construct the most general Fock-like space. Let there be a vacuum state \( |0\rangle \) and let us assume that there exist independent annihilation and creation operators \( \{a_i, \bar{a}_i | i \in I \} \), satisfying the vacuum condition

\[
a_i |0\rangle = 0, \quad \forall i \in I.
\]

The set \( \{I\} \) can be finite or infinite, discrete or continuous and with some (partial) ordering. It may have some additional mathematical and physical structure.

Now, we build a Fock-like space starting from the vacuum state \( |0\rangle \). The space \( \mathcal{H}_0 = \{ \lambda |0\rangle | \lambda \in \mathbb{C} \} \) is one-dimensional. We assume that the states \( |0\rangle \) and \( \bar{a}_i |0\rangle \) are linearly independent states and they span the linear space \( \mathcal{H}_1 = \{ \sum_i \lambda_i \bar{a}_i |0\rangle | i \in I \} \) and \( \text{dim } \mathcal{H}_1 = |I| \). Similarly, the states \( \bar{a}_{i_1} \bar{a}_{i_2} |0\rangle \) span the linear space

\[
\mathcal{H}_2 = \{ \sum_{i_1, i_2} \lambda_{i_1 i_2} \bar{a}_{i_1} \bar{a}_{i_2} |0\rangle | i_1, i_2 \in I \}
\]
with the dimension $dim \mathcal{H}_2 \leq |I|^2$. We assume that any state belonging to $\mathcal{H}_2$ is linearly independent of the states in $\mathcal{H}_0 \oplus \mathcal{H}_1$. Generally,

$$\mathcal{H}_n = \{\sum_{i_1 \ldots i_n} \lambda_{i_1 \ldots i_n} \bar{a}_{i_1} \cdots \bar{a}_{i_n}|0\} \mid \lambda_{i_1 \ldots i_n} \in \mathbb{C}\},$$

$$\mathcal{H}_n \cap \mathcal{H}_m = \{0\}, \quad n \neq m$$

$$dim \mathcal{H}_n \leq |I|^n. \quad (3.2)$$

Hence, the full Fock-like space $\mathcal{H}$ is

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots = \bigoplus_n \mathcal{H}_n, \quad n \in \mathbb{N}_0. \quad (3.3)$$

Furthermore, we assume that the annihilation operators $a_i, i \in I$ act on the space $\mathcal{H}_n$ in such a way that

$$a_{i_1} \cdots a_{i_k} \bar{a}_{j_1} \cdots \bar{a}_{j_n}|0\rangle \in \mathcal{H}_{n-k}, \quad n \geq k,$$  

and this state is equal to zero if $n < k$. Then, the total number operator $N$ can be defined as

$$N v_n = n v_n, \quad v_n \in \mathcal{H}_n, \quad n \in \mathbb{N}_0. \quad (3.5)$$

$\mathcal{H}_n$ is an invariant subspace with respect to $N$. We call the states belonging to $\mathcal{H}_n$ $n$–particle states. The number of independent $n$-particle states is equal to $dim \mathcal{H}_n$ and is connected with the problem of generalized statistics. We discuss this point at the end of this section.

We also assume that there exist the number operators $N_i, i \in I$, for every species $i$. Namely,

$$N_i(\bar{a}_{i_1} \cdots \bar{a}_{i_n})|0\rangle = \sum_{k=1}^n \delta_{i_1 i_k}(\bar{a}_{i_1} \cdots \bar{a}_{i_k})|0\rangle; \quad i_1 \cdots i_n \in I, \quad n \in \mathbb{N}$$

$$N_i|0\rangle = 0, \quad \forall i \in I,$$

$$[N_i, \bar{a}_j] = \delta_{ij} \bar{a}_i, \quad [N_i, a_j] = -\delta_{ij} a_i,$$

$$N = \sum_{i \in I} N_i, \quad [N_i, N_j] = 0, \quad \forall i, j \in I. \quad (3.6)$$
Hence the subspace

\[ \mathcal{H}_{i_1 \cdots i_n} = \{ \sum_{\pi \in S_n} \lambda_{\pi,(i_1 \cdots i_n)} \prod (\bar{a}_{i_1} \cdots \bar{a}_{i_n}) |0\rangle | i_1 \leq i_2 \leq \cdots \leq i_n \in I \} \subset \mathcal{H} \]

is invariant with respect to \( \mathcal{N} \) and \( \mathcal{H}_{i_1 \cdots i_n} \cap \mathcal{H}_{j_1 \cdots j_n} = \{0\} \) if \( i_k \neq j_k \).

Let us also remark that if we perform a mapping from the operators \( \{ \bar{a}_i, a_i | i \in I \} \) to operators \( \{ \bar{a}'_j, a'_j | j \in I' \} \)

\[ a'_j = f_j(a_i), \quad \bar{a}'_j = \bar{f}_j(\bar{a}_i) \quad (3.7) \]

and repeat the same construction of the Fock-like space \( \mathcal{H}' \) under the vacuum state \( |0\rangle' \equiv |0\rangle \), then, generally, the number operators \( N'_j \) will not exist.

Now, let us define the dual Fock-like space \( \mathcal{H}^{(d)} \), based on the dual vacuum-state vector \( \langle 0| \), with the condition \( \langle 0|0\rangle = 1 \). The construction of \( \mathcal{H}^{(d)} \) is the same as that for \( \mathcal{H} \). Note that \( \langle 0|\bar{a} = 0, \forall i \in I \). The dual \( n - \text{particle} \) states are

\[ \langle 0| (a_{i_1} \cdots a_{i_n}) \in \mathcal{H}^{(d)}_{i_1 \cdots i_n} \quad (3.8) \]

We also have, as in Eq. (3.4),

\[ \langle 0| (a_{j_1} \cdots a_{j_n} \bar{a}_{i_1} \cdots \bar{a}_{i_k}) \in \mathcal{H}^{(d)}_{n-k}, \quad n \geq k \]

\[ \langle 0| (a_{j_1} \cdots a_{j_n} \bar{a}_{i_1} \cdots \bar{a}_{i_k}) = 0, \quad n < k. \quad (3.9) \]

From the existence of the number operators \( N_i, \text{Eq.}(3.6) \), it follows that

\[ \mathcal{H}^{(d)}_{i_1 \cdots i_n} \cap \mathcal{H}^{(d)}_{j_1 \cdots j_n} = \{0\}, \quad i_k \neq j_k. \quad (3.10) \]

Note that, so far, we have assumed no precise relation between the \( a_i \) and \( \bar{a}_i \) operators.

Our next step is to construct the scalar product and define the norm of any vector in the Fock-like space. Therefore, we construct a sesquilinear form \( \langle *, * \rangle : \)
Namely, we need to calculate the matrix $A$ with the matrix elements

$$A_{i_1 \cdots i_n;j_1 \cdots j_m} = \langle 0 | a_{i_1} \cdots a_{i_n} \bar{a}_{j_1} \cdots \bar{a}_{j_m} | 0 \rangle.$$  

(3.11)

From Eq. (3.6), i.e. the existence of the number operators $N_i$, it follows that

$$A_{i_1 \cdots i_n;j_1 \cdots j_m} = 0 \quad (i) \neq (j),$$  

(3.12)

i.e. it vanishes unless $n = m$ and the indices $(i_1 \cdots i_n)$ and $(j_1 \cdots j_m)$ are equal up to permutation.

We can always choose

$$\langle 0 | 0 \rangle = 1, \quad \langle 0 | a_i \bar{a}_j | 0 \rangle = \delta_{ij}$$  

(3.13)

From the physical point of view, we need to know how to calculate any monomial $P(a_i, \bar{a}_j)$ in $a_i, \bar{a}_j$ between vacuum states. Hence, in order to calculate all such matrix elements, it is sufficient to know the action of all annihilation operators $a_i$ on all Fock-like states

$$a_i \bar{a}_{i_1} \cdots \bar{a}_{i_n} | 0 \rangle \in \mathcal{H}_{i_1 \cdots i_n} \subset \mathcal{H}_{n-1},$$  

(3.14)

where the slash denotes the omission of the corresponding index (state) from the set $i_1 \cdots i_n$. Hence, these relations are compatible with the number operators $N_i$, Eq. (3.6), if and only if

$$a_i \bar{a}_j | 0 \rangle = \delta_{ij}$$

$$a_i \bar{a}_{i_1} \cdots \bar{a}_{i_n} | 0 \rangle = \sum_{k=1}^{n} \delta_{ii_k} \sum_{\pi \in S_{n,k}} \Phi^i_{i_1 \cdots i_n; \pi(i_1 \cdots i_n)} \pi \cdot (\bar{a}_{i_1} \cdots \bar{a}_{i_k} \cdots \bar{a}_{i_n}) | 0 \rangle \equiv$$

$$\equiv \sum_{\pi \in S_{i_1 \cdots i_n; (n-1)}} \Phi^i_{i_1 \cdots i_n; (n-1)}$$  

(3.15)

where $\{\Phi\}$ are complex coefficients, $S_{n,k}$ denotes the group of permutations acting on $(i_1 \cdots i_n)$ and the slash denotes the omission of the corresponding operator.
The state in Eq. (3.15) is zero if the index $i$ is different from any of the indices $(i_1 \cdots i_n)$. The matrix element of any monomial in $a_i \bar{a}_i$ is equal to zero if the indices in the monomial do not appear in pairs. For example,

$$A_{i_1 \cdots i_n;j_1 \cdots j_n} = \sum_{(n-1)} \sum_{(n-2)} \cdots \sum_1 \tilde{\Phi}^{i_1 \cdots i_n}_{(n-1)} \tilde{\Phi}^{j_1 \cdots j_n}_{(n-2)} \cdots \tilde{\Phi}^{j_n}_{(1),0} \equiv (\tilde{\Phi}^{(n)})!$$

is zero if $(j_1 \cdots j_n)$ is not a permutation of $(i_1 \cdots i_n)$. This is a necessary and sufficient condition for the existence of the number operators $N_i, \forall i \in I$, in a given Fock-like representation.

From Eq. (3.15), it is easy to see the action of any monomial $P(a_i, \bar{a}_j)$ on any state in a Fock-like space, particularly the action of the operators $\Gamma_{ij} = a_i \bar{a}_j$. Hence, using Eq. (3.15) we find the relation expressing $a_i \bar{a}_j$ in terms of normal ordering of operators $a$ and $\bar{a}$:

$$\Gamma_{ij} \equiv a_i \bar{a}_j = \delta_{ij} + C_{ij} a_j a_i + C_{ik} a_j a_k a_i + C_{jk} a_j a_k a_i + \cdots$$

$$\equiv \delta_{ij} + \sum_{k=1}^{\infty} \sum_{I \in 1} C_{(i\cdots j\cdots)(a \cdots a)}^i a_{(i\cdots a)} a_k (a \cdots a_i \cdots a)$$

where any allowed monomial appearing on the RHS has the $\bar{a}_j a_i$ structure while the rest of operators appear in pairs of the same indices. Equations (3.15) and Eq. (3.17) are completely equivalent. They represent the most general relations between the operators $a_i$, $\bar{a}_j$ compatible with the number operators $N_i, \forall i \in I$. Let us remark that using Eqs. (3.6) and (3.15) we can express the number operators as

$$N_i = \bar{a}_i a_i + \sum_{k=2}^{\infty} \sum_{I \in 1} D_{(i \cdots a)}^{i} \bar{a}_{i \cdots a} a_k (a \cdots a_i \cdots a)$$

(3.18)
\[
N_{ij}, a_k] = \delta_{jk}a_i, \quad [\bar{N}_{ij}, a_k] = -\delta_{ik}a_j, \quad \bar{N}_{ij} = N^T_{ji},
\]

\[
N_{ij} = \bar{a}_ia_j + \sum_{k=2}^{\infty} \sum_{I=1}^{n-1} D^{ij}_{(\cdots i\cdots)}(\bar{a} \cdots \bar{a} \bar{a} \cdots \bar{a})_k (a \cdots a_j \cdots a)_k, \quad (3.19)
\]

\[
\bar{N}_{ij} = \bar{a}_ia_j + \sum_{k=2}^{\infty} \sum_{I=1}^{n-1} \bar{D}^{ij}_{(\cdots i\cdots)}(\bar{a} \cdots \bar{a} \bar{a} \cdots \bar{a})_k (a \cdots a_j \cdots a)_k.
\]

From the above equation for \( N_{ij} \) we can obtain Eqs.(3.15) and (3.17). Hence, the sets of quantities \( \{\Phi\}, \{C\} \) and \( \{D\} \) in Eqs.(3.15),(3.17),(3.18) are completely equivalent, as the set of quantities \( \{\varphi_i\}, \{c_i\} \) and \( \{d_i\} \) are equivalent for the single-mode oscillator. Note that the relation \( \bar{a}a = \varphi(N) \) has no simple generalization for the multimode case.

In this paper we restrict ourselves to those relations between the operators \( a_i, \bar{a}_i \) compatible with the number operators \( N_i \), Eqs.(3.6). However there remains an interesting question, namely under which conditions the set of operators \( a_i, \bar{a}_i \) with the most general contractions, Eq.(3.15), without number operators, can be obtained from the operators \( c_i, \bar{c}_i \) with number operators (e.g. by mapping).

In order that the constructed Fock-like space should become physically acceptable, we have to define the norm of vectors and demand that any linear combination of vectors should have non-negative norms. Therefore, we demand a well-defined scalar product on Fock-like space. This will generally lead to additional restrictions on the coefficients \( \{\Phi\}, \{C\} \) and \( \{D\} \). In principle, we have two possibilities.

We can demand that \( \bar{a}_i \) should become Hermitian conjugate of the \( a_i \) operator, with respect to the scalar product \( <*, *> \), i.e. \( \bar{a}_i = a_i^\dagger \). In this case the block matrix \( A \) becomes Hermitian, with non-negative diagonal matrix elements, i.e.

\[
A_{(i),(j)} = A^*_{(j),(i)} \quad A_{(i),(i)} \geq 0. \quad (3.20)
\]
We define the norm $||v||$ of any vector $|v>$ as

$$||v||^2 = <v|v| > > 0 \quad |v> \neq 0.$$  \hfill (3.21)

Hence, the non-negative norms of all vectors leads us to the semi-positivity of the matrix $A$ in Eq.(3.20). This means that the matrix $A$ has no negative eigenvalues. Particularly, any zero eigenvalue implies a null-vector (the state of zero norm) which implies a relation between monomials in the operators $\bar{a}_i$.

A general null-vector $E$ is of the form

$$\sum_{\sigma \in S_n/S_t} A_{i_1 \cdots i_n, \sigma(i_1 \cdots i_n)} E_{i_1 \cdots i_n, \sigma(i_1 \cdots i_n)} = 0.$$ \hfill (3.22)

Then, the consistency conditions for null-vectors are

$$\sum_{\sigma \in S_n/S_t} E_{i_1 \cdots i_n, \sigma(i_1 \cdots i_n)} \tilde{\Phi}_{\sigma(i_1 \cdots i_n), j_1 \cdots j_{n-1}} = 0, \quad \forall j \in I.$$ \hfill (3.23)

This means that any contraction of the operator $a_j$ with a null-vector is a null-vector. Hence, starting with general contractions, Eq.(3.15), with $\bar{a}_i = a_i^\dagger$ leading to the Hermitian matrix $A$ and positive diagonal elements (Eq.(3.20)), the positivity condition and consistency of null-vectors imply a strong restriction on the possible coefficients $\{\Phi\}$ (or $\{C\}$ and $\{D\}$). Eq.(3.15) (or (3.17) or (3.18)) uniquely determines the matrix $A$, Eq.(3.20). However, the reverse is not generally true, namely, even if the matrix $A$ is Hermitian and positive definite this does not mean that the corresponding contractions (3.15) exist. An interesting question is whether the above conditions for the matrix $A$, Eqs.(3.20) and (3.23), are also sufficient for the existence of relations between $a_i$ and $a_i^\dagger$ of the type given in Eq.(3.15) or Eq.(3.17).

The second interesting possibility is that $\bar{a}_i \neq a_i^\dagger, i \in I$. In this case there is no condition on the matrix $A$ to be Hermitian and positive definite. Moreover, the matrix $A$
can have complex eigenvalues. Then, we perform a polar decomposition of the matrix $A$:

$$A = U \cdot H$$

where

$$U \cdot U^\dagger = U^\dagger \cdot U = 1, \quad A \cdot A^\dagger = H^2 \geq 0.$$  

$U$ is a unitary matrix and $H$ is a unique Hermitian, positive definite (non-negative) matrix. Now the problem is to find out in which cases the matrix $A$ can lead to a relation between $a_i$ and $a_i^\dagger$ of the type given by Eq.(3.15) or Eq.(3.17). In the following we restrict ourselves to the case $\bar{a}_i = a_i^\dagger$.

Finally, let us briefly describe the connection of the spectrum of the matrix $A$ (Eq.(3.11)) with generalized statistics, i.e. with the counting of allowed multiparticle states. As we have already stated, the appearance of null-vectors implies corresponding relations between monomials in $a_i^\dagger$. If we fix the indices $i_1 \leq i_2 \cdots \leq i_n \in I$ with multiplicities $m_1, m_2 \cdots m_s$ such that $\sum_{k=1}^{s} m_k = n$, then the number of linear independent states $\pi \cdot (a_{i_1}^\dagger \cdots a_{i_n}^\dagger) |0\rangle$, $\pi \in S_n / S_t$, is equal to

$$\text{rank}[A_{i_1 \cdots i_n}]_{\text{fixed}} \leq \frac{n!}{m_1! m_2! \cdots m_s!}. \quad (3.24)$$

If we restrict the indices $i_1, i_2 \cdots i_n \in I_d \subset I$, where $d$ denotes the number of single-mode oscillators, then the total number of $n$-particle excited states of $d$-oscillators is given by

$$W(n, d, A) = \text{rank}[A_{(n,d)}] \leq d^n. \quad (3.25)$$

For example, for $d$-Bose oscillators

$$\text{rank}[A_{i_1 \cdots i_n}]_{\text{fixed}} = 1, \quad W_B(n, d) = \frac{(d + n - 1)!}{n!(d - 1)!}. \quad (3.26)$$
For d-Fermi oscillators

\[ \text{rank}[A_{(i_1, \ldots, i_n)}_{\text{fixed}}] = \theta(2 - m_1) \cdots \theta(2 - m_s), \quad W_F(n, d) = \frac{d!}{n!(d-n)!} \] (3.27)

where \( \theta(x) \) is the step-function, i.e. \( \theta(x) = 1, x > 0 \) and \( \theta(x) = 0, x \leq 0 \).

For \( n \) quonic oscillators\(^{13,14} \) which satisfy the relation \( a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij}, |q_{ij}| < 1, q_{ij} \in \mathbb{C} \)

\[ \text{rank}[A_{(i_1, \ldots, i_n)}_{\text{fixed}}] = \frac{n!}{m_1!m_2! \cdots m_s!}, \quad W_Q(n, d) = d^n. \] (3.28)

4. Algebras from Generalized Jordan-Wigner Transformations

We assume that the number operators \( N_i, \text{Eq.}(3.6), \) exist and that

\[ a_i a_j^\dagger = q_{ij} a_j^\dagger a_i, \quad i \neq j \quad q_{ij}^* = q_{ji} \] (4.1)

Furthermore, we assume that the algebra (4.1) can be obtained by mapping from the Bose algebra

\[ [b_i, b_j^\dagger] = \delta_{ij}, \quad \forall i, j \in I \]

\[ [b_i, b_j] = 0. \] (4.2)

Then, it follows that

\[ a_i = b_i \epsilon \sum_j c_{ij} N_j \frac{\varphi_i(N_i)}{N_i}, \] (4.3)

where \( c_{ij} \) are complex numbers and \( \varphi_i(N_i) \) are arbitrary (complex) functions with the properties \( \varphi_i(0) = 0, \lim_{\epsilon \to 0} \frac{\varphi_i(\epsilon)}{\epsilon} = 1, |\varphi_i(1)| = 1, \forall i \in I \). It is important to note that the number operators are preserved, i.e.

\[ N_i^{(a)} = N_i^{(b)} \equiv N_i, \quad \forall i \in I. \] (4.4)
Then it is easy to find the corresponding deformed-oscillator algebra:

\[ a_i a_j = e^{c_{ji} - c_{ij}} a_j a_i \quad i \neq j \]
\[ a_i a_j^+ = e^{c_{ij} + c_{ji}^*} a_j^+ a_i \quad i \neq j \]
\[ a_i a_i^+ = |\varphi_i(N_i + 1)| e^{\sum_j (c_{ij} + c_{ij}^*) N_j} e^{(c_{ii} + c_{ii}^*)} \]
\[ a_i^+ a_i = |\varphi_i(N_i)| e^{\sum_j (c_{ij} + c_{ij}^*) N_j}. \]

(4.5)

Generally, there are other mappings of the Bose algebra (Eqs. (4.2)), but, in general, they do not have the number operators \( N_i^{(a)} \), and Eq. (4.4) does not hold for mappings other than those in Eq. (4.3).

We point out that the complete deformed-oscillator algebra is associative owing to the mapping of the Bose algebra. The Fock space for the deformed-oscillator algebra is spanned by powers of the creation operators \( a_i^+ \), \( i \in I \), acting on the vacuum \(|0\rangle = |0 \rangle \equiv |0 \rangle \). The states in the Fock space are specified by the eigenvalues of the number operators \( N_i \), namely \(|n_1, n_2, ... n_i, ... \rangle = |n_1, n_2, ... n_i, ... \rangle \). (If there exists a number \( n_i^{(0)} \in \mathbb{N} \), such that \( \varphi_i(n_i^{(0)}) = 0 \), then \( N_i = 0, 1, ... (n_i^{(0)} - 1) \).)

The states with unit norm are

\[ |n_1, n_2, ... n_i \rangle = (a_1^+)^{n_1} ... (a_n^+)^{n_n} \frac{1}{\sqrt{[\varphi_1(n_1)]! ... [\varphi_n(n_n)]!}} e^{-\frac{1}{2} \sum_j \theta_{ji} (c_{ij} + c_{ij}^*) n_j n_i} |0, 0, ..., 0 \rangle \]

\[ |\varphi(n)\rangle = \varphi(n) \varphi(n-1) ... \varphi(1) \]

\[ \varphi_i(n_i) = |\varphi_i(n_i)| e^{(c_{ii} + c_{ii}^*) n_i}, \]

(4.6)

where \( \theta_{ij} \) is the step function. (For anyons in \((2 + 1)\)-dimensional space, \( \theta \) is the angle function.)

Furthermore, the matrix elements of the operators \( a_i, a_i^+ \), \( i \in I \), are

\[ \langle ... (n_i - 1) ... | a_i | ... n_i ... \rangle = \langle ... n_i ... | a_i^+ | ... (n_i - 1) ... \rangle \]
\[ = \sqrt{\varphi_i(n_i)} e^{\frac{1}{2} \sum_j (c_{ij} + c_{ij}^*) n_j} \prod_{j \neq i} \delta_{n_j, n_j'}. \]

(4.7)
For any $k = 0, 1, 2, \ldots$, we also find that

\[
(a_j^+)^k (a_j)^k = \frac{[\tilde{\varphi}_j(N_j)]!}{[\tilde{\varphi}_j(N_j - k)]!} e^{k \sum_{i \neq j} (c_{ij} + c_{ji}^*) N_i} \]

\[
(a_j)^k (a_j^+)^k = \frac{[\tilde{\varphi}_j(N_j + k)]!}{[\tilde{\varphi}_j(N_j)]!} e^{k \sum_{i \neq j} (c_{ij} + c_{ji}^*) N_i}. \quad (4.8)
\]

The norms of arbitrary linear combinations of the states in Eq. (4.6) in the Fock space, corresponding to the deformed-oscillator algebra, are positive by definition owing to the mapping of the Bose algebra (Eqs. (4.2) and (4.3)). Namely, $|n_1, n_2, \ldots n_i, \ldots>^{(a)} = |n_1, n_2, \ldots n_i, \ldots>^{(b)} \equiv |n_1, n_2, \ldots n_i, \ldots>$.

This class of deformed multimode oscillator algebras comprises multimode Biedenharn-Macfarlaine, Aric-Coon, two-$(p, q)$ parameter, Fermi,genons, generalized Green's, as well as anyonic and Pusz-Woronowicz oscillators covariant under the $SU_q(n)$ $(SU_q(n|m))$ algebra (superalgebra)\textsuperscript{4,7,8,9}. Particularly, the operator algebra for the Haldane exclusion statistics\textsuperscript{18} is a special case of our mapping, Eqs. (4.3) and (4.5), with the substitutions

\[
c_{ij} = (-)c_{ji} = (-)\frac{i\pi}{m + 1} \quad i < j, m \in \mathbb{N};
\]

\[
K(N_i, g) = \frac{\varphi(N_i + 1, g)}{\varphi(N_i, g)} \quad g = \frac{1}{m}. \quad (4.9)
\]

Nonisomorphic (nonequivalent) algebras are classified by different matrix elements given by (4.7), i.e. with the functions $g_i(n_1, n_2, \ldots n_n) = |\varphi_i(n_i)| e^{\sum_j (c_{ij} + c_{ji}^*) n_j}$. It is important to mention that there are mappings of the Bose algebra which do not preserve the relation $N_i^{(a)} = N_i^{(b)}$, given by (4). Moreover, there are mappings
of Bose algebra for which the number operators $N_i^{(a)}$ do not exist. Such an example is the exchange algebra presented in Ref. (19)

5. Parabosons and parafermions

The para-algebra is defined by the trilinear commutation relation

$$[a_i a_j^\dagger + q a_j^\dagger a_i, a_k] = -\frac{2}{p} q \delta_{jk} a_i$$

$$a_i a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle = \sum_{k=1}^{n} \delta_{i_k} (-q)^{k-1} a_{i_1}^\dagger \cdots a_{i_k}^\dagger \cdots a_{i_n}^\dagger |0\rangle - \frac{2}{p} \sum_{k=2}^{n} \delta_{i_k} \sum_{l=1}^{k-1} (-q)^{l} a_{i_1}^\dagger \cdots a_{i_{k-1}}^\dagger a_{i_k}^\dagger a_{i_{k+1}}^\dagger \cdots a_{i_n}^\dagger |0\rangle.$$  \(5.1\)

Here, $p > 0$ is the order of parastatistics, $q = -1$ for the para-Fermi algebra and $q = +1$ for the para-Bose algebra.

Comparing with Eq.(3.15), one identifies the set $\{\Phi\}$ as

$$\Phi^i_{i_1 \cdots i_n, \text{id}(i_1 \cdots i_n)} = (-q)^{k-1} 1 \leq k \leq n,$$

$$\Phi^i_{i_1 \cdots i_n, \pi_{l,k-1}(i_1 \cdots i_n)} = -(\frac{2}{p}) (-q)^{l} 1 \leq l \leq k - 1,$$

where

$$\pi_{l,k-1} = \begin{pmatrix} l & k - 2 & k - 1 \\ l + 1 & k - 1 & l \end{pmatrix}$$

denotes the cyclic permutation and $\text{id}$ denotes the identity permutation.

The matrix $A$, Eq.(3.11), for the three different para-oscillators is

$$A = \begin{pmatrix} 1 & x & x & x^2 & x^2 & z \\ x & 1 & x^2 & x^2 & z & x^2 \\ x & x^2 & 1 & z & x & x^2 \\ x^2 & x & z & 1 & x^2 & x \\ x^2 & z & x & x^2 & 1 & x \\ z & x^2 & x^2 & x & x & 1 \end{pmatrix}$$

$$x = q \left(\frac{2}{p} - 1\right) \equiv qy,$$

$$z = q \left(\frac{2}{p} - q^3 \right)^2.$$ \(5.3\)
The matrix is written in the basis $a_1^\dagger a_2^\dagger a_3^\dagger|0\rangle$, $a_2^\dagger a_1^\dagger a_3^\dagger|0\rangle$, $a_1^\dagger a_3^\dagger a_2^\dagger|0\rangle$, $a_2^\dagger a_3^\dagger a_1^\dagger|0\rangle$, $a_3^\dagger a_1^\dagger a_2^\dagger|0\rangle$, and $a_3^\dagger a_2^\dagger a_1^\dagger|0\rangle$.

By inspection of the eigensystem of the matrix $A$, Eq.(5.3), one finds that the rank $A = 4$ for $q = \pm 1$ and $\forall p \in \mathbb{N}$, which means that only four of the states are linearly independent. This is in accordance with the trilinear relations which hold for parastatistics $^{11}$, namely $[a_k, [a_m, a_n]_{\pm}] = 0$. We choose the set $a_1^\dagger a_2^\dagger a_3^\dagger|0\rangle$, $a_2^\dagger a_1^\dagger a_3^\dagger|0\rangle$, $a_1^\dagger a_3^\dagger a_2^\dagger|0\rangle$ and $a_3^\dagger a_2^\dagger a_1^\dagger|0\rangle$ as linearly independent vectors. Instead of the $6 \times 6$ matrix $A$, in the following we use the $4 \times 4$ matrix, corresponding to this set of vectors. In the limit $p \to \infty$, the rank of $A$, Eq.(5.3), reduces to 1 for $q = \pm 1$, i.e. the matrix $A$ reduces to Fermi (Bose) matrix for $q = +1$ ($q = -1$). Particularly, for the two-level system $a_i^\dagger, i = 1, 2$, Ref.(12), the matrix $A$, corresponding to the $a_1^\dagger a_2^\dagger|0\rangle$ and $a_2^\dagger a_1^\dagger|0\rangle$ states is

$$A = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$$

For $q = \pm 1$, rank $A = 2, \forall p > 1$, and again, in the limit $p \to \infty$ the matrix $A$ is of the Fermi (Bose) type for $q = +1$ ($q = -1$).

For the states $(a_1^\dagger)^2 a_2^\dagger|0\rangle$ and $a_1^\dagger a_2^\dagger a_1^\dagger|0\rangle$, the matrix $A$ is

$$A = \begin{pmatrix} 1 + x & x + x^2 \\ x + x^2 & 1 + z \end{pmatrix}$$

For $q = +1, \text{rank } A = 2, \forall p > 1$. For $q = -1, \text{rank } A = 2, p \geq 3$ and rank $A = 1$ if $p = 1, 2$. For para-Fermi case, when $p = 2$ there exist additional relation between operators, what was overlooked in the paper Ref.(12). Similar statements hold for the matrix $A$ corresponding to the states $(a_1^\dagger)^r (a_2^\dagger)^s|0\rangle, r + s = n$. As a consequence, the
para-Fermi (para-Bose) system behaves as Bose (Fermi) system when $p \to \infty$.

Next, we discuss the number operators for para-Bose and para-Fermi oscillators.

Number operators $N_i, i \in I$, can be written as in Eq.(3.18). We have, up to the second order in the creation and annihilation operators ($k = 2$ in Eq.(3.18)),

$$N_i = a_i^\dagger a_i + \frac{p^2}{4(p-1)} \sum_l [Y_{il}]^\dagger [Y_{il}], \quad (5.4)$$

where $Y_{il} = a_i a_l - x a_l a_i$.

In the second order, we identify the set $\{D\}$ as

$$D_{ii,ii}^i = \frac{p^2}{4(p-1)}; \quad D_{il,il}^i = D_{i,l,li}^i = q\frac{p(p-2)}{4(p-1)}; \quad D_{i,li}^i = \frac{(p-2)^2}{4(p-1)}.$$

The computation of the terms of the higher order ($k \geq 3$) becomes more involved. For example, the generic form of the third order term is

$$\sum_j \sum_l \pi X_{\pi,ijkl} \pi \cdot (a_j a_j a_i).$$

For the parasystem, the summation is taken over those set of indices which are linearly independent, owing to the constraint implied by trilinear relation

$$[a_k, [a_m, a_n]]_\pm = 0.$$

We illustrate the procedure by calculating the third order term for the $N_1$, i.e. for $\{i, j, l\} = \{1, 2, 3\}$. The basic set of the X-operators is $\hat{X} \equiv \{X_{123}, X_{213}, X_{132}, X_{321}\}$, which is solutions of the equation

$$A \cdot \hat{X} = V \quad (5.5)$$

where

$$V = \begin{pmatrix}
0 \\
-q(\frac{2}{p})v_1 - (\frac{2}{p})v_2 + (\frac{2}{p})(\frac{2}{p}-1)v_3 + (\frac{2}{p})v_6 \\
-q(\frac{2}{p})^2v_1 + (\frac{2}{p})(\frac{2}{p}-1)v_3 + (\frac{2}{p})v_6 \\
0
\end{pmatrix}.$$
\[ q = \begin{cases} 
+1 & \text{para-Bose} \\
-1 & \text{para-Fermi} 
\end{cases} \]

\[ v_1 = a_1^\dagger a_2^\dagger a_3^\dagger \quad v_2 = a_2^\dagger a_1^\dagger a_3^\dagger \quad v_3 = a_1^\dagger a_3^\dagger a_2^\dagger \quad v_6 = a_3^\dagger a_2^\dagger a_1^\dagger \quad (5.6) \]

For the parabosons (q=+1; \( y = \left( 2 - \frac{1}{p} \right) \)) one finds

\[ X_{123} = \frac{1}{2y - y^2 - y^3} \left[ -(1 + 3y + y^2 + y^3) v_1 + (y^2 - y) v_2 + (y + 2y^2) v_3 + (1 + 2y) v_6 \right] \]

\[ X_{213} = \frac{1}{-2 + y + y^2} \left[ (1 - y) v_1 + 2v_2 - yv_3 - v_6 \right] \]

\[ X_{132} = \frac{1}{-2 + y + y^2} \left[ -(1 + 2y) v_1 - yv_2 + (y + y^2) v_3 + (1 + y) v_6 \right] \]

\[ X_{321} = \frac{1}{y} X_{132} \quad (5.7) \]

The corresponding operators \( \hat{X} \) for the parafermions are obtained from the above equations by substitutions \( X_{123} \Rightarrow (-)X_{123}(-y, -v_1), X_{213} \Rightarrow X_{213}(-y, -v_1), X_{132} \Rightarrow X_{132}(-y, -v_1), X_{321} \Rightarrow X_{321}(-y, -v_1) \).

The operators \( \hat{X} \) with some of the indices equal (e.g. \( X_{122}, X_{212}, X_{111} \) etc.) follow from the Eq.(5.7) after the suitable identification of the indices.

The transition number operators \( N_{ij}, i, j \in I \), are given, up to the second order, by

\[ N_{ij} = a_i^\dagger a_j + \frac{p^2}{4(p-1)} \sum_l [Y_{il}]^\dagger [Y_{jl}] \quad (5.8) \]

To this order, we identify the set \( \{ D \} \) as

\[ D_{li,jl}^{ij} = \frac{p^2}{4(p-1)} \quad D_{il,jl}^{ij} = D_{li,ij}^i = q \frac{p(p-2)}{4(p-1)} \quad D_{li,ij}^i = (p - 2)^2 \frac{p}{4(p-1)} \]

22
Similarly as in the case of the number operator $N_1$, we compute the basic set of the third order operators $\tilde{X} = \{\tilde{X}_{123}, \tilde{X}_{213}, \tilde{X}_{132}, \tilde{X}_{321}\}$ for the transition number operator $N_{12}$. These operators are the solutions of the equation

$$A \cdot \tilde{X} = \tilde{V}$$

where

$$\tilde{V} = \begin{pmatrix}
-(q + x)w_1 + x(q + x)w_2 \\
x(q + x - 1)w_1 + (1 - q - x - qx^2)w_2 + qxw_3 \\
(x - 1)(q + x)w_1
\end{pmatrix}$$

$$w_1 = a_1^+a_2^+a_3^+$$
$$w_2 = a_1^+a_3^+a_1^+$$
$$w_3 = a_3^+a_1^+a_1^+$$

(5.9)

For the parabosons ($x = qy = (\frac{2}{p} - 1)$) the result is:

$$\tilde{X}_{123} = \frac{1}{x^2 + x - 2}[-3xw_1 + (1 + x + x^2)w_2]$$
$$\tilde{X}_{213} = \frac{1}{x^2 + x - 2}[x^2w_1 - xw_2]$$
$$\tilde{X}_{132} = \frac{1}{x^2 + x - 2}[-2xw_1 + (x + x^2)w_2]$$
$$\tilde{X}_{321} = (-)\frac{1 + x}{x}\tilde{X}_{132}.$$  

(5.10)

For the parafermions ($x = qy = -(\frac{2}{p} - 1)$) the result is:

$$\tilde{X}_{123} = \frac{1}{x^3 - x^2 - 2x}[-5x^2 - 2x - 2w_1 + (x^3 + x^2 + 3x)w_2]$$

23
\[ \tilde{X}_{213} = \frac{1}{x^2 - x - 2}[(x^2 + 2)w_1 - 3xw_2] \]
\[ \tilde{X}_{132} = \frac{1}{x^2 - x - 2}[(2x - 2)w_1 + (-x^2 + x - 4)w_2] \]
\[ \tilde{X}_{321} = \frac{1}{x^3 - x^2 - 2x}[(2x^2 + 2x - 2)w_1 + (x^3 - 2x^2 + 3x)w_2] \quad (5.11) \]

Again, the operators \( \tilde{X} \) with some of the indices equal (e.g. \( \tilde{X}_{122}, \tilde{X}_{212}, \tilde{X}_{111} \) etc.) follow from the above result after the suitable identification of the indices.

6. **Paraquantization of Govorkov**

Govorkov\(^{13}\) has defined a new para-algebra obeying

\[ [a_i, a_j^\dagger, a_k] = \left( \frac{\lambda}{p} \right) \delta_{jk} a_i \quad (6.1) \]

\[ a_i a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_n}^\dagger |0\rangle = \delta_{i_1 i_2} a_{i_3}^\dagger \cdots a_{i_n}^\dagger |0\rangle = -\left( \frac{\lambda}{p} \right) \sum_{k=2}^{n} \delta_{i_k i_{k-1}} a_{i_{k-1}}^\dagger a_{i_{k-1}}^\dagger \cdots a_{i_k}^\dagger |0\rangle. \]

Comparing with Eq. (3.15), one identifies the set \( \Phi \) as

\[ \Phi^i_{i_1 \ldots i_n, \text{id}(i_1 \ldots i_k \ldots i_n)} = \delta_{k1} \]
\[ \Phi^i_{i_1 \ldots i_n, \text{sign}(i_1 \ldots i_k \ldots i_n)} = -\left( \frac{\lambda}{p} \right) \delta_{k1} \quad 2 \leq k \leq n. \quad (6.2) \]

With restriction to the three oscillators \( (i_k = 1, 2, 3) \), one immediately obtains
the matrix $A$ as

$$
A = \begin{pmatrix}
1 & -y & -y & y^2 & y^2 & -y \\
-y & 1 & y^2 & -y & -y & y^2 \\
-y & y^2 & 1 & 1 & y & y^2 \\
y^2 & -y & -y & 1 & y^2 & -y \\
y^2 & -y & -y & y^2 & 1 & -y \\
-y & y^2 & y^2 & -y & -y & 1
\end{pmatrix}
$$

where $y = \frac{\lambda}{p}$. The matrix $A$ is written in the same basis as the matrix $A$ for the para-Bose (para-Fermi) oscillators in Eq. (5.3).

The number operators $N_i$ can be written, up to the second order ($k = 2$ in Eq. (3.18)) as

$$
N_i = a_i^\dagger a_i + \frac{1}{(1 + \frac{\lambda}{p})(1 - \frac{\lambda}{p})} \sum_l [Y_{il}]^\dagger [Y_{il}] (6.3)
$$

where $Y_{ik} = a_i a_k + (\frac{\lambda}{p}) a_k a_i$.

To this order, we identify the set $\{D\}$ as

$$
D_{l,i,l}^i = \frac{1}{(1 + \frac{\lambda}{p})(1 - \frac{\lambda}{p})};
D_{l,i,l}^i = D_{l,i,l}^i = \frac{\lambda}{p} \frac{1}{(1 + \frac{\lambda}{p})(1 - \frac{\lambda}{p})};
D_{l,i,l}^i = (\frac{\lambda}{p})^2 \frac{1}{(1 + \frac{\lambda}{p})(1 - \frac{\lambda}{p})}.
$$

For the number operator $N_1 (\{i, j, l\} = \{1, 2, 3\})$, the basic set of the third-order operators $X_{\pi, (ijl)}$, $\{ijl\} = \{1, 2, 3\}$, is $\hat{X} = \{X_{123}, X_{213}, X_{132}, X_{231}, X_{312}, X_{321}\}$, which are the solutions of the equation

$$
A \cdot \hat{X} = V
$$

$$
V = \begin{pmatrix}
0 \\
(\frac{\lambda}{p})^2 v_3 + (\frac{\lambda}{p}) v_5 \\
0 \\
(\frac{\lambda}{p})^2 v_1 + (\frac{\lambda}{p}) v_2 + (\frac{\lambda}{p}) v_3 + v_4 \\
(\frac{\lambda}{p})^2 v_1 + (\frac{\lambda}{p}) v_2 \\
(\frac{\lambda}{p})^2 v_3 + (\frac{\lambda}{p}) v_1 + (\frac{\lambda}{p}) v_5 + v_6
\end{pmatrix}
$$
and with $v_i$'s defined as $v_1 = a_1^\dagger a_2 a_3^\dagger$, $v_2 = a_2^\dagger a_1 a_3^\dagger$, $v_3 = a_1^\dagger a_3 a_2^\dagger$, $v_4 = a_2^\dagger a_3 a_1^\dagger$, $v_5 = a_3^\dagger a_1 a_2^\dagger$, and $v_6 = a_3^\dagger a_2 a_1^\dagger$.

Explicitly,

$$X_{123} = \frac{y[(y + 4y^3)v_1 + 4y^2v_2 + 4y^2v_3 + 2yv_4 + 2yv_5 + v_6]}{1 - 5y^2 + 4y^4}$$

$$X_{213} = \frac{y[4y^2v_1 + 2yv_2 + 2yv_3 + v - 4 + v - 5 + 2yv_6]}{1 - 5y^2 + 4y^4}$$

$$X_{231} = \frac{2y^2v_1 + yv_2 + yv_3 + (1 - 2y^2)v_4 + 2y^2v_5 + yv_6}{1 - 5y^2 + 4y^4} \quad (6.5)$$

The remaining three X's are obtained from the above equation by the substitutions $v_2 \leftrightarrow v_5, v_1 \leftrightarrow v_3$ and $v_4 \leftrightarrow v_6$, i.e. $X_{123} \rightarrow X_{132}$, $X_{213} \rightarrow X_{312}$ and $X_{231} \rightarrow X_{321}$. Notice that, for $p > 2$, there are six linearly independent X's since there are no trilinear constraint as in the case of the parastatistics.

Transition number operators $N_{ij}$ are, up to the second order,

$$N_{ij} = a_i^\dagger a_j + \frac{1}{(1 + \frac{\lambda}{p})(1 - \frac{\lambda}{p})} \sum_l [Y_{il}]^\dagger [Y_{jl}], \quad (6.6)$$

and the coefficients $\{D\}$ are

$$D_{li, lj}^{ij} = \frac{1}{(1 + \frac{\lambda}{p})(1 - \frac{\lambda}{p})}; \quad D_{li, lj}^{ij} = \frac{\lambda}{p} \frac{1}{(1 + \frac{\lambda}{p})(1 - \frac{\lambda}{p})}; \quad D_{li, lj}^{ij} = (\frac{\lambda}{p})^2 \frac{1}{(1 + \frac{\lambda}{p})(1 - \frac{\lambda}{p})}$$

The basic set of the third-order operators $\tilde{X}$ for the transition number operator $N_{12}$ are

$$\tilde{X}_{123} = X_{123}(1 \equiv 2) \quad \tilde{X}_{213} = X_{213}(1 \equiv 2) \quad \tilde{X}_{132} = X_{132}(1 \equiv 2) \quad \tilde{X}_{231} = X_{231}(1 \equiv 2) \quad \tilde{X}_{312} = X_{312}(1 \equiv 2) \quad \tilde{X}_{321} = X_{321}(1 \equiv 2)$$

26
Here, the abbreviations, e.g. $X_{213}(1 \equiv 2)$, mean that one has to identify indices 1 and 2 in $v_i$'s such that $v_1 = v_2 = a_1^\dagger a_3^\dagger a_2^\dagger, v_3 = v_4 = a_1^\dagger a_3^\dagger a_1^\dagger$ and $v_5 = v_6 = a_2^\dagger a_1^\dagger a_1^\dagger$ and than read off $X$'s from Eq.(6.5).

7. Generalized Quon Algebra

The general (associative) quon algebra Ref.(16) is defined by:

\[ a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij}, \quad \forall i, j \in I, \]

\[ q_{ij} = q_{ji}, \]

\[ a_i (a_1^\dagger \cdots a_n^\dagger) |0> = \sum_{\alpha=1}^n q_{i_\alpha i_1} \cdots q_{i_{\alpha-1} i_{\alpha}} a_1^\dagger \cdots a_{\alpha}^\dagger \cdots a_n^\dagger \delta_{i_\alpha}, |0>. \]

No commutation relation between $a_i$ and $a_j$ exists if $|q_{ij}| \neq 1$ for $\forall i, j \in I$.

One identifies the set $\Phi$ as

\[ \Phi_{i_1 \cdots i_n, \text{id}(i_1 \cdots i_n)} = \prod_{k=1}^{n-1} q_{k}, \quad 1 \leq \alpha \leq n. \] (7.2)

The matrix $A$ is hermitian and block-diagonal. A generic block $A^{(i_1 \cdots i_n)}$ is characterized by mutually different ordered indices $i_1, \ldots, i_n \in I$ ($i_1 < i_2 < \cdots < i_n$) from which all other blocks in the $n$-particle sector can be obtained using a suitable specification. The $A^{(i_1 \cdots i_n)}$ matrix is an $n! \cdot n!$ matrix, whose diagonal matrix elements are equal to 1. The arbitrary matrix element $(\pi, \sigma)$, i.e., $i_{\pi(1)} \cdots i_{\pi(n)}; i_{\sigma(1)} \cdots i_{\sigma(n)}$, where $\pi$ and $\sigma$ are permutations acting on positions 1, 2, $\ldots$, $n$ ($\pi$ denotes the row and $\sigma$ the column of the matrix $A^{(i_1 \cdots i_n)}$) is given by

\[ A^{(i_1 \cdots i_n)}_{\pi, \sigma} = \prod_{\alpha, \beta} q_{i_{\alpha} i_{\beta}}. \] (7.3)
Here the product is over all pairs $\alpha, \beta = 1, \ldots, n$ satisfying $\pi^{-1}(\alpha) < \pi^{-1}(\beta)$ and $\sigma^{-1}(\alpha) > \sigma^{-1}(\beta)$.

The general simple structure of the number operator $N_k \ (|q_{ij}| < 1, \forall i, j \in S)$ is

$$N_k = a_k^\dagger a_k + \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n} \sum_{\pi \in S_n} [Y_{k,\pi(i_1, \ldots, i_n)]}^\dagger Y_{k,i_1, \ldots, i_n} (A^{(k,i_1, \ldots, i_n)})^{-1}_{k,i_1, \ldots, i_n; k, \pi(i_1, \ldots, i_n)}, \quad (7.4)$$

where

$$Y_{ki_1} = a_k a_{i_1} - q_{i_1 k} a_{i_1} a_k$$

$$Y_{k,i_1 \ldots i_n} = Y_{k,i_1 \ldots i_n-1} a_{i_n} - q_{i_n k} q_{i_n i_1} \cdots q_{i_n i_{n-1}} a_{i_n} Y_{k,i_1 \ldots i_{n-1}}$$

and $A$ is matrix given in Eq.(7.3).

Always when the parameters $q_{ij}$ tend to 1, i.e., $q_{ij} \to 1$, for $\forall i, j$, quons tend to a particular anyonic-type statistics, the matrices $(A^{(i_1 \ldots i_n)})^{-1}$ become singular and the coefficients in the number operator, Eq.(7.4), diverge. Nevertheless, the number operator $N_k$, when acting on states, is well defined. Moreover, in the exact limit it reduces to $N_k = a_k^\dagger a_k$, and additional relations between the annihilation (creation) operators $a_i, a_j (a_i^\dagger, a_j^\dagger)$ emerge. In this case, the corresponding particles are not distinguishable, i.e., they are identical in the quantum-mechanical sense. Interchanging them, we generally obtain a unit phase factor $e^{i\alpha}$ (typical of anyons).

The transition number operator $N_{ij}$ has also simple structure:

$$N_{ij} = a_i^\dagger a_j + \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n} \sum_{\pi \in S_n} [Z_{ji_1, \ldots, i_n}^i] Y_{j,\pi(i_1, \ldots, i_n)} (A^{(k,i_1, \ldots, i_n)})^{-1}_{j,\pi(i_1, \ldots, i_n); j i_1, \ldots, i_n}, \quad (7.5)$$

where

$$Z_{ki_1}^k = a_k a_i - q_{ik} a_i a_k \equiv (Y_{ki})_{a_k \to a_i}$$

$$Z_{k i_1, \ldots, i_n}^i \equiv (Y_{k i_1, \ldots, i_n})_{a_k \to a_i}$$
8. Conclusion

In this paper we have generalized our previous analysis of single-mode oscillators to multimode oscillator algebras with Fock-like representations. We have extended the quantization defined by $a_i a_j^\dagger - R_{i,j,kl} a_k^\dagger a_l = \delta_{ij}, i, j \in I$ to include all normally ordered terms in creation and annihilation operators. We have restricted ourselves to the algebras with well-defined number operators and transition number operators. In this way we have unified all approaches to multimode oscillator algebras. The connection between these algebras and generalized statistics has been pointed out.
Aknowledgment

We thank D.Svrtan for helpful discussions. This work was supported by the Scientific Fund of the Republic of Croatia.
References

[1] M.Jimbo, Lett. Math. Phys. 10, 63 (1985); V.G.Drinfeld, Quantum Groups (Proc. Int. Congr. of Math., Berkeley, CA, 1986).

[2] S.L.Woronowicz, Comm. Math. Phys. 111, 613 (1987); Yu.I.Manin, Quantum Groups and Non-Commutative Geometry (Center de Recherches, University of Montreal, 1988); J.Wess and B.Zumino, Nucl. Phys. Suppl. 318, 302 (1990); A.Connes, Noncommutative Differential Geometry (Cambridge University Press, Cambridge, 1994).

[3] For a recent review, see: C.Gomez, M.Ruiz-Altaba and G.Sierra, Quantum Groups in Two-Dimensional Physics (Cambridge University Press, Cambridge, 1993); Z.Chang, Quantum Groups and Quantum Symmetry, preprint IC/94/89 (unpublished).

[4] M.Arik and D.D.Coon, J. Math. Phys. 17, 524 (1976); L.C.Biedenharn, J. Phys. A : Math. Gen. 22, L873 (1989); A.J.Macfarlane, J. Phys. A : Math. Gen. 22, L983 (1989); P.P.Kulish and E.V.Damaskinsky, J. Phys. A : Math. Gen. 23, 415 (1990); L.de Falco et.al., Mod. Phys. Lett. A9, 3331 (1994); V.I.Man’ko et.al., Int. J. Mod. Phys. A8, 3577 (1993); T. Brzezinski, I.L.Egusquiza and A.J. Macfarlane, Phys. Lett. 311B, 202 (1993); K.Odaka, T.Kishi and S.Kamefuchi, J. Phys. A : Math. Gen. 24, L591 (1991); C.Chou, Mod. Phys. Lett. A7, 2685 (1992); N. Debergh, Mod. Phys. Lett. A8, 765 (1993); E.Celeghini, T.D.Palev and M.Tarlini, Mod. Phys. Lett. B5, 187 (1991); R. Floreanini and L.Vinet, J. Phys. A : Math. Gen. 23, L1019 (1990); S.Chaturvedi, V.Srinivasan and R.Jagannathan, Mod. Phys. Lett. A8, 3727 (1993).
[5] D.Bonatsos and C.Daskaloyannis, *J. Phys.A :Math.Gen.* **26**, 1589 (1993); *Phys.Lett.* **307B**, 100 (1993).

[6] S.Meljanac, M.Mileković and S.Pallua, *Phys.Lett.* **328B**, 55 (1994).

[7] W.Pusz and S.L.Woronowicz, *Rep.Math.Phys.* **27**, 231 (1989); D.Fairlie and C.Zachos, *Phys.Lett.* **256B**, 43 (1991); D.Fairlie and J.Nuyts, *Z.Phys.C* **56**, 237 (1992); R. Chakrabarti and R. Jagannathan, *J. Phys.A :Math.Gen.* **24**, L711 (1991); R.Jagannathan et. al., *J. Phys.A :Math.Gen.* **25**, 6429 (1992) and references therein.

[8] P.P.Kulish, *Phys.Lett.* **161A**, 50 (1990); J.Van der Jeugt, *J. Phys.A :Math.Gen.* **26**, 2405 (1993); A.Kempf, *J.Math. Phys.* **34**, 969 (1993).

[9] S.Meljanac, M.Mileković and A.Perica, **28**, 79 (1994); M.Dorešić, S.Meljanac and M.Mileković, *Fizika* **3**, 57 (1994).

[10] H.S.Green, *Phys.Rev.* **90**, 170 (1953); O.W.Greenberg and A.M.L.Messiah, *Phys.Rev.* **138B**, 1155 (1965); *J.Math.Phys.* **6**, 500 (1965).

[11] For a review, see: Y.Ohnuki and S.Kamefuchi, *Quantum Field Theory and Parastatistics* ( University of Tokio Press, Tokio, Springer, Berlin, 1982).

[12] A.Bhattacharyya et. a l., *Phys.Lett.* **224B**, 384 (1989).

[13] A.B.Govorkov, *Nucl.Phys.* **365B**, 381 (1991); *Phys.Elem.Part.Atomic Nucl.* **4**, 1341 (1993) (in russian).

[14] O.W.Greenberg, *Phys. Rev. Lett.* **64**, 705 (1990); *Phys. Rev. D* **43**, 4111 (1991); G.S.Agrawal and S.Chaturvedi, *Mod.Phys.Lett.* **A7**, 2407 (1992); R.Speicher,
Lett.Math.Phys.\textbf{27}, 97 (1993); M.Bozejko and R.Speicher, Math.Ann.\textbf{300}, 97 (1994); Commun. Math. Phys. \textbf{137}, 519 ; P.E.T.Jorgensen and R.F.Werner, Commun. Math. Phys. \textbf{164}, 455 (1994).

[15] P.E.T.Jorgensen,L.M.Schmitt and R.F.Werner, Pacific J.Math. \textbf{165},131 (1994).

[16] S.Meljanac and A.Perica, J. Phys.A :Math.Gen. \textbf{27}, 4737 (1994);
\textit{Mod.Phys.LettersA}\textbf{9}, 3293 (1994); V.Bardek,S.Meljanac and A.Perica, \textit{Phys.Lett.}\textbf{338B}, 20 (1994).

[17] J.M.Leinaas and J.Myrheim,\textit{Nuovo Cim.}\textbf{37}, 1 (1977); F.Wilczek, Phys. Rev. Lett. \textbf{48}, 1144 (1982); J.Myrheim,\textit{Anyons} preprint University of Trondheim 1993 (unpublished); V.Bardek,M.Dorešić and S.Meljanac, Phys. Rev.\textbf{D49}, 3059 (1994); V.Bardek,M.Dorešić and S.Meljanac, Int. J.Mod.Phys. \textbf{A9}, 4185 (1994); A.K.Mishra and G.Rajasekaran, Mod.Phys.Lett.\textbf{A9}, 419 (1994).

[18] F.D.M.Haldane,Phys. Rev. Lett. \textbf{67}, 937 (1991); D.Karabali and V.P.Nair, \textit{Many-Body States and Operator Algebra for Exclusion Statistics} preprint IASSNS-HEP-94/88 (to appear in Nucl.Phys.B) and references therein.

[19] A.P.Polychronakos, Phys. Rev. Lett.\textbf{69}, 703 (1991); L.Brink et.al., Nucl.Phys.\textbf{401B}, 591 (1993).