New fermionic soft theorems

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Soft limits of massless S-matrix are known to reflect symmetries of the theory. In particular for theories with Goldstone bosons, the double-soft limit of scalars reveals the coset structure of the vacuum manifold. In this letter, we propose that such universal double-soft behavior is not only true for scalars, but also for spin-1/2 particles in four dimensions and fermions in three dimensions. We first consider Akulov-Volkov theory, and demonstrate the double-soft-limit of goldstinos yields the supersymmetry algebra. More surprisingly we also find amplitudes in 4 ≤ N ≤ 8 supergravity theories in four dimensions as well as N = 16 supergravity in three dimensions behave universally in the double-soft-fermion limit, analogue to the scalar ones. The validity of the new soft theorems at loop level is also studied. The results for supergravity are beyond what is implied by SUSY Ward identities, and may impose non-trivial constraints on the possible counter terms for supergravity theories.

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The connection between soft behavior of S-matrix and symmetries of the theory has been exemplified in various examples. The most famous being Weinberg’s soft graviton theorem [1], which states that the leading divergence of gravity S-matrix is constrained by Ward identities and is in fact universal. Similar results, based on other symmetries, were shown to be applicable for the subleading divergences in gauge and gravity theories as well [2-6]. Another famous example, which is more relevant to this letter, is in the context of Goldstone bosons of a spontaneous broken symmetry. In particular, taking the momenta of the Goldstone boson to near zero, which correspond to a constant scalar field, the S-matrix should vanish due to the scalars being derivative coupled. This is the well-known Adler’s zero [7,8].

As discussed in [9], Adler’s zero can also be understood from the structure of the vacuum. Perturbative scattering amplitudes should be identical when computed at any point in the vacuum moduli. On the other hand, one can in principle use the operator $e^{i\theta \cdot T}$ to relate one vacuum to another $|\theta\rangle = e^{i\theta \cdot T}|0\rangle$, where $T^a$ represents the broken generators, and $\theta$ is a constant that is the vacuum expectation value (vev) of the soft scalar. The amplitude evaluated in the $|\theta\rangle$ vacuum can be written as a perturbative series by expanding out the exponential,

$$|\theta\rangle = e^{i\theta \cdot T}|0\rangle = |0\rangle + \theta_i |\pi^i\rangle + \frac{1}{2} \theta_i \theta_j |\pi^i \pi^j\rangle + \cdots \quad (1)$$

where $|\pi^i\rangle, |\pi^i \pi^j\rangle$ etc represents vacuums with one and two additional soft scalars respectively. Since the amplitudes are the same in either $|\theta\rangle$ or $|0\rangle$ vacuum, this implies that the amplitude with one or more soft scalars must vanish. The vanishing of the single soft scalar is precisely Adler’s zero. For two soft scalars, it turns out that the amplitude is non-zero due to the non-commutativity of the broken generators. It was shown in [9] that taking two Goldstone bosons to have soft momenta, the amplitude behaves as:

$$M_n [\phi^i(p_1^2), \phi^i(p_2^2), \cdots | t \to 0 \sum_{a=3}^n \mathcal{B}_a f^{ij} K H_{aK} M_{n-2} \cdots ]$$

where $\mathcal{B}_a \equiv \frac{p_a \cdot (p_1 - p_2)}{2 p_a \cdot (p_1 + p_2)}$ and $H_{aK}$ is the generator of the invariant subgroup in coset $G/H$, while $f^{ij} K$ is the structure constant for $[T^i, T^j] = f^{ij} K H_K$.

In extended supergravity theories, the scalars are elements of coset. Thus it is expected that the double-soft-scalar limit of the S-matrix must behave as eq. (2). The fact that the symmetry has such non-trivial imprint on the S-matrix is extremely useful in the discussion of ultra-violet behavior of supergravity theories. In particular, modulo quantum anomalies, any possible counter terms for the theory must respect this non-linearly realized symmetry. This implies that the S-matrix elements generated by such counter terms must behave in the double-soft scalar limit according to eq. (2), and thus provide a direct check of the compatibility of counter terms with the symmetry of the theory. Indeed such analysis was carried out for maximal supergravity in four-dimensions [10,11].

In this paper, we demonstrate that remarkably, the same single and double-soft behavior also applies to fermions in Akulov-Volkov theory as well as supergravity theories both in three and four dimensions. For Akulov-Volkov theory, we show that the amplitudes vanish in the single-soft-fermion limit, while in the double-soft-limit they exhibit a similar form as that of the scalars in eq. (2) with $H_{aK}$ replaced by $[2] p_a |1\rangle$, which is precisely the
anti-commutator of super charge, \( \{Q, Q\} = p \), with wave functions of the soft fermions.

For the supergravity theories, we show that the amplitudes vanish in the single-soft limit of spin-1/2 particles in four-dimensions as well as all fermions in three-dimensions. In the double-soft limit, amplitudes in four-dimensional supergravity theories again behave in a analogous way as that of the scalars in eq.\((2)\), now with the replacement: \( p_{\alpha} \cdot (p_1 - p_2) \to 2|p_\alpha|\), curiously enough the same factor \(2|p_\alpha|\) appears. The invariant subgroup is now SU(\(N\)) instead of U(\(N\)) for \(N < 8\) supergravity. In three dimensions, we show that for \(N = 16\) supergravity \([17]\), where the 128 scalars parametrize an \(E_{8(8)}/SO(16)\) coset, the double-soft limit of any pair of the 128 fermions exhibits the same behavior as the spin-\(\frac{1}{2}\) fermions in four dimensions. This result is not implied by supersymmetric Ward identities, and thus possibly hint at the existence of a new symmetry.

DOUBLE SOFT LIMITS AND SPONTANEOUS (SUPER)SYMMETRY BREAKING

Scattering amplitudes involving Goldstones bosons have interesting soft-behaviour that reveals the algebra its corresponding coset space. In \([9]\) it was explicitly shown that for \(N = 8\) supergravity, where the 70 scalars parameterize \(E_{7(7)}/SU(8)\), the double-soft scalar limit is given precisely by:

\[
\left. M_n \left[ \phi^I_1 t_2^I_1 t_3^I_3 (t^2 p_1), \phi^J_4 t_5^J_4 t_6^J_6 (t^2 p_2) \cdots \right] \right|_{t \to 0} = 4 \sum_{a=3}^{n} B_a c_{t_1^I_1 t_2^I_2 t_3^I_3 t_4^I_4 t_5^I_5 t_6^I_6} (R_a)_{t_7}^J \left. M_{a-2} \right|
\]

where square bracket \([\ ]\) means antisymmetrization. And \(I_i = 1, \cdots, 8\) are the fundamental indices of SU(8) symmetry, while \((R_a)^J_{\ i}\) is the single site SU(8) generator: \((R_a)^J_{\ i}\ \ = n_a^I \frac{\partial}{\partial \eta^I}\), with \(n_a^I\) the Grassmann variables that parametrize the on-shell degrees of freedom in the supermultiplet. To realize the soft limit \(p_{1,2} \to t^2 p_{1,2}\), we rescale\(^1\)

\[
\lambda_{1,2} \to t\lambda_{1,2}, \quad \tilde{\lambda}_{1,2} \to t\tilde{\lambda}_{1,2}.
\]

This analysis was later extended to \(D = 4, 4 \leq N < 8\) and \(D = 3\ N = 16\) supergravity in \([12]\). One new subtlety is the presence of U(1) factors in the isotropy group \(H_i\) which produces soft-graviton singularities in the double soft-limit. To extract the finite piece one instead considers the anti-symmetrized double-soft limit \(M_n^{[ij]}\), defined as:

\[
M_n^{[ij]} = M_n (\phi^i (t^2 p_1), \phi^j (t^2 p_2), \cdots, n) - (1 \leftrightarrow 2) \bigg|_{t \to 0}.
\]

Indeed for all \(4 \leq N < 8\) supergravity in four dimensions and \(N = 16\) supergravity in three dimensions, the anti-symmetrized double-soft-limit is given by:

\[
M_n^{[ij]} = \sum_{a=3}^{n} p_a \cdot (p_1 + p_2) f^{ijk} (H_a) K M_{a-2}
\]

where the \((H_a)K\) are single site U(\(N\)) generators in \(D = 4\) and SO(\(N\)) generators in \(D = 3\).

As another example, consider the Akulov-Volkov action \([13]\) which is the low energy effective theory of fermions associated to spontaneous symmetry breaking of supersymmetry. The action for Weyl fermions \(\lambda^a\) in four-dimensions is

\[
S_N = -\frac{1}{2g^2} \int d^4x \det(1 + ig^2 \bar{\sigma}\partial^\mu \bar{\psi}^\mu),
\]

where \(\sigma^\mu = (1, \bar{\sigma})\). One can expand the determinant and read out the Feynman rules to compute the six-point amplitude. The relevant vertices are

\[
\lambda^a_{\ i} \rightarrow \lambda^b_{\ j} \quad g^2[V_{jk} - V_{kj}]_{\ ab\bar{a}} - ([\iota_a, k_b] \leftrightarrow \iota_{a\bar{b}}) + ([\iota_b, \ell_k] \leftrightarrow [\iota_{a\bar{o}}])
\]

\[
= \sum_{\sigma \in \text{perm.}} (-\sigma)^\sigma V_6 (\sigma (\iota_a, \iota_{a\bar{o}}, k_b, \ell_k, m_c, n_{\bar{c}}))
\]

\[
= \sum_{\lambda^a_{\ i}} \lambda^b_{\ j} \sum_{\sigma \in \text{perm.}} (-\sigma)^\sigma V_6 (\sigma (\iota_a, \iota_{a\bar{o}}, k_b, \ell_k, m_c, n_{\bar{c}}))
\]

where

\[
V_6 (\iota_a, \iota_{a\bar{o}}, k_b, \ell_k, m_c, n_{\bar{c}}) \equiv ig^2[V_{\ell m} - V_{\ell m} - \frac{1}{2}V_{\ell m} + V_{\ell m} + V_{\ell m} + V_{\ell m} - \frac{1}{2}V_{\ell m} - \frac{1}{2}V_{\ell m} - \frac{1}{2}V_{\ell m} - \frac{1}{2}V_{\ell m}]_{\ ab\bar{a}c\bar{b}},
\]

with \(V_{ijk} \equiv \partial_{\iota_a} \bar{\psi}^\iota_{a\bar{b}} (\iota_{a\bar{o}}) \psi_{j\bar{b}} (\iota_{j\bar{o}}) \psi_{k\bar{c}} (\iota_{k\bar{o}}) \psi_{n\bar{c}} (\iota_{n\bar{o}})\). Straightforward computation shows that the relative coefficient between the quartic and sextic interactions are precisely that needed for the leading term in the single-soft limit to cancel, such that the six-point amplitude is of order \(t^2\). In the double soft limit one finds:

\[
A_6 (\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3, \bar{\psi}_4, \bar{\psi}_5, \bar{\psi}_6) \lambda^a_{\ i} \lambda^{\bar{b}}_{\ j} \rightarrow \lambda^{a\bar{b}}_{\ i\ j}
\]

\[
= t^2 g^2 \sum_{a=3}^{6} B_a (1|p_a|2) A_4 (\bar{\psi}_3, \bar{\psi}_4, \bar{\psi}_5, \bar{\psi}_6) + O(t^4),
\]

where \(A_4 (\bar{\psi}_3, \bar{\psi}_4, \bar{\psi}_5, \bar{\psi}_6) = 2g^2 s_{46} \langle 35 \rangle [46]\). The above results are consistent with the interpretation that the

\(^1\) We use standard spinor-helicity formalism: \(p_{\alpha\bar{b}} = \lambda^a_{\ i} \lambda^{\bar{b}}_{\ j} \gamma^{\alpha\bar{b}}\), and scalar products \(\lambda^a_{\ i} \lambda^{\bar{b}}_{\ j} \gamma^{\alpha\bar{b}} = (ij), \lambda^a_{\ i} \lambda^{\bar{b}}_{\ j} \epsilon^{\alpha\bar{b}} = [ij], s_{ij} = (ij)[ij].\)
fermions are goldstinos. Following the single- and double-soft behavior of Goldstone bosons, one would expect that the single soft limit of a goldstino should vanish as $O(t)$, due to the corresponding broken generators are the SUSY charge $Q$, with $(Q, Q) \sim P$, and the double-soft-limit should be finite and proportional to $B_{a\hat{p}a}M_{n-2}$. The extra factor of $t$ for the single-soft, and $t^2\lambda_1\lambda_2$ for the double-soft, is simply due to the presence of external line-factors for fermions.

**NEW DOUBLE-SOFT THEOREMS IN FOUR-DIMENSIONS**

We now consider soft fermions in four-dimensions. As discussed in ref. [12], due to the fact that the single-soft-scalars vanish as $O(t^2)$, SUSY Ward identities [14] requires that the single-soft fermions vanish as $O(t)$. The same result can alternatively be deduced from BCFW recursion [15]. However, for the double-soft limit, Ward identities are no longer sufficient since it will be mixed with the double-soft-scalar and soft-scalar-fermion limits. Instead we will proceed using recursion.

To treat all the hard particles democratically, we follow [9] and add an auxiliary negative-helicity graviton, which at the end is taken away by sending its momentum be soft$^2$. Now, we choose the shifted legs in the recursive formula as one of the soft legs and the added graviton. The remaining soft leg will be in one of the factorized amplitudes which generally vanishes by the above analysis. Thus all diagrams vanish except for the following two special cases:

![Diagram](image)

We will consider the case where the two fermions do not form a singlet, for which the second diagram does not contribute. The contribution for diagram (I) is given as [12] (with $\mathcal{N} = 8$):

$$M(\hat{1}, a, \hat{P}) \frac{\langle \hat{1} P \rangle^{8} \delta^{8} (\eta_{1} + \frac{\langle P \rangle}{\langle P \rangle} \eta_{2} + t \frac{\langle P \rangle}{\langle P \rangle} \eta_{n})}{2p_{a} \cdot (p_{1} + p_{2}) t^{2}} \times \exp \left[-\frac{t}{\langle P \rangle} \eta_{2} \frac{\partial}{\partial \eta_{2}} \exp \left[-t^{2} z p_{1} \frac{\partial}{\partial \lambda_{p}} \right] M_{n-1}(g) \right].$$

(11)

Here $z_{P}$ is the solution to $(p_{1} + p_{2} + p_{a})^{2} = 0$ with BCFW shift $|\hat{1}| = |\hat{1}| + z|n|$, and $M_{n-1}(g)$ is an unshifted $(n-1)$-point amplitude with the additional soft graviton. Note that we have explicitly scaled out the leading $t$ dependence. For soft fermions, we choose the component in eq. (11) that is degree 3 in $\eta_{i}$ and degree 5 in $\eta_{j}$. For the fermion pair $(\psi_{1}^{1}, \psi_{2}^{I_{1} I_{2} I_{3} I_{4} I_{5}}, \tilde{\psi}_{i}^{I_{1} I_{2} I_{3} I_{4} I_{5}})$, diagram (I) then yields:

$$\frac{5(2|p_{a}|)}{2p_{a} \cdot (p_{1} + p_{2}) t^{2} \sqrt{t_{1} l_{2} l_{3} I_{5} I_{6} I_{7} J}} (R_{a}) I_{5} J M_{n-1}(g).$$

Thus after remove the auxiliary graviton using the universal behavior of its soft limit, and sum over all relevant BCFW channels, we obtain:

$$M_{n} \left[ \langle \psi^{I_{1} I_{2} I_{3} (t^{2} p_{1})}, \tilde{\psi}_{i}^{I_{1} I_{2} I_{3} I_{4} I_{5}} (t^{2} p_{2}) \ldots \rangle \right]_{t \to 0} = \sum_{a=3}^{n} 5F_{a} t^{I_{1} I_{2} I_{3} I_{5} I_{6} I_{7} J} (R_{a}) I_{5} J M_{n-2}$$

(12)

where $F_{a} = \frac{2|p_{a}|}{2p_{a} \cdot (p_{1} - p_{2})}$. Note the remarkable similarity to the double-soft limit of scalars. One simply replaces the factor $p_{a} \cdot (p_{1} - p_{2})$ in the numerator of $B_{a}$ with $\langle 2|p_{a}|1 \rangle$. Again we stress that this result is not implied by SUSY Ward identities.

It is straightforward to reduce the above result to 4 $\mathcal{N} < 8$, by applying supersymmetry truncation discussed in [16]. Note that this only produces the SU($\mathcal{N}$) part of the invariant subgroup. The U(1) is inaccessible due to the soft-graviton divergence, for which we can no-longer use (anti)-symmetric extraction scheme to project out.

**NEW DOUBLE-SOFT THEOREMS IN THREE-DIMENSIONS**

Next we consider double-soft limits of fermions for amplitudes in 3D $\mathcal{N} = 16$ supergravity [17]. The on-shell degrees of freedom, 128 bosons and 128 fermions, can be packaged into one superfield

$$\Phi = \xi + \sum_{i=1}^{8} \xi_{I_{i} J} \eta^{|I_{i}|} \eta^{J} \ldots \eta^{|I_{i}|}, \quad I_{i} = 1, \ldots, 8.$$  

(13)

The grassmann variables $\eta^{|I_{i}|}$ transforms as fundamentals of SU(8)$\subset$SO(16), thus only part of the SO(16) are linearly realized in these variables. The remaining generators, SO(16)/U(8), are realized non-linearly. The 128 bosons parametrize the coset space E_8/SO(16), and the commutation relations among the broken generators are also reflected in the double-soft-scalars limit [12].

As discussed in ref. [12] due to the presence of U(1) in U(8), the double-soft limit is polluted by soft-graviton divergences whenever the two soft-particles form a U(1) singlet. Similar results apply to double-soft fermions as

\[\text{NEW DOUBLE-SOFT THEOREMS IN THREE-DIMENSIONS}\]
well. Thus inspired by [12], we consider the symmetrized double-soft-limit:

\[ \mathcal{M}^{(i,j)}_n \equiv M_n \left( \psi^i(t^2 p_1), \psi^j(t^2 p_2), \ldots, n \right) + (1 \leftrightarrow 2) \bigg|_{t \to 0} . \]

Using the BCFW representation of three-dimensional supergravity amplitudes, we find that remarkably the symmetrized double-soft-limit of fermions also behaves universally and is given by:

\[ \mathcal{M}^{(i,j)}_n = - \sum_{a=3}^n \langle S_a \rangle_{i,j} M_{n-2} + O(t), \]

where \( \langle S_a \rangle_{i,j} \) are the corresponding soft factor acting on the \((n-2)\)-point amplitude, and are given by:

\[ (S_a)_{i_1 \ldots i_n, j_1 \ldots j_n} = -3 \mathcal{F}_a \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n} (R_a)_{j_1 \ldots j_n}, \]

\[ (S_a)_{i_1 \ldots i_n, j_1 \ldots j_n} = 10 \mathcal{F}_a \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n} (R_a)_{j_1 \ldots j_n}, \]

\[ (S_a)_{i_1 \ldots i_n, j_1 \ldots j_n} = \mathcal{F}_a (2 \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n} (R_a)_{j_1 \ldots j_n} - \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n} (R_a)_{j_1 \ldots j_n}), \]

\[ (S_a)_{i_1 \ldots i_n, j_1 \ldots j_n} = -\mathcal{F}_a (6 \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n} (R_a)_{j_1 \ldots j_n} - \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n} (R_a)_{j_1 \ldots j_n}), \]

\[ (S_a)_{i_1 \ldots i_n, j_1 \ldots j_n} = -\mathcal{F}_a (4 \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n} (R_a)_{j_1 \ldots j_n} - \epsilon_{i_1 \ldots i_n} \epsilon_{j_1 \ldots j_n} (R_a)_{j_1 \ldots j_n}). \]

Here \( (R_a)_{i,j} \equiv \eta^I_{i,j} \partial_{\eta^I_{i,j}} - 4, (R_a)_{I,J} \equiv \partial_{\eta^I_{i,j}} \partial_{\eta^J_{i,j}}, (R_a)_{I,J} \equiv \eta^I_{i,j} \eta^J_{i,j} \) and \( \mathcal{F}_a \equiv \frac{2 \log(p_1 p_2)}{2 p_1 p_2 (p_1 + p_2)} \). Note that while the double-soft fermion limit behaves in a similar fashion as that of the bosons, their detailed algebra is different. This is reflected in the fact that they form distinct representations under the on-shell SU(8) symmetry.

**NEW SOFT THEOREMS AT LOOP LEVEL**

It is interesting to see if the new soft-theorems in supergravity theories are subject to loop corrections. We begin with \( N = 16 \) supergravity in three dimensions, whose one-loop amplitudes can be expressed in terms of scalar triangle integrals with coefficients determined by generalized unitarity cuts in 3D. We then can apply tree-level soft theorems since it is the tree-level amplitude that enters the cuts. Following the same proof of their scalar partner [12], it is straightforward to see that single- and double-soft-fermion theorems do not receive any one-loop corrections. For the theories in 4D, one can in principle also expressed the amplitudes in terms of one-loop integral basis. However, unlike the 3D case where only amplitudes with even-number external legs exist, new complication arises due to discontinuity of these integral functions [18]. Here we provide some evidence that the new soft theorems are not corrected by loops by considering the leading IR-divergent part of a L-loop amplitude, which is given by:

\[ M^{L-\text{loop}}_{n} \big|_{\text{lead. IR}} = \frac{1}{L!} \left( \sum_{i,j} s_{ij} \log(s_{ij}) \right)^L M^{\text{tree}}_n. \]

Thanks to the kinematics factor \( s_{ij} \), applying the tree-level soft theorems \( M^{\text{tree}}_n \) we find, at least, the leading IR-divergent part of loop amplitudes satisfy the same soft theorems as tree-level amplitudes.

**CONCLUSIONS**

In this letter, we propose new soft theorems by studying soft-fermion limits for the amplitudes in a wide range of theories, include Akulov-Volkov theory and supergravity theories in 4D and 3D. We find that they all vanish in the single-soft limit, and behave universally in the double-soft limit, as with the case of soft scalars in theories with spontaneously broken symmetry. The results for Akulov-Volkov theory precisely reflects that this is the effective theory for the spontaneously supersymmetry breaking. To our surprises, remarkably the amplitudes in supergravity theories also behave universally in the double-soft-fermion limit, in a very similar form of that of scalars in the theories. We also provide evidence that the soft theorems do not receive loop corrections. Finally we like to emphasize that the results are not a simple consequence of supersymmetric Ward identities, which could only provide relations between amplitudes with double-soft fermions and those with double-soft scalars through amplitudes with one soft scalar and one soft fermion. Indeed, a straightforward calculation shows that amplitudes in supergravity also behave universally in the limit with one soft scalar and one soft fermion (spin-1/2 for four-dimensional theories), and the results of the limit are again in the form of a simple soft factor acting a lower-point amplitude. It would be of great interest to clarify the implications of all those new soft theorems, in particular whether there are new hidden symmetries behind them, and their possible application for constraining potential UV counter terms.

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