Almost Hermitian structures on tangent bundles

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Abstract

In this article, we consider the almost Hermitian structure on $TM$ induced by a pair of a metric and an affine connection on $M$. We find the conditions under which $TM$ admits almost Kähler structures, Kähler structures and Einstein metrics, respectively. Moreover, we give two examples of Kähler-Einstein structures on $TM$.

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1 Introduction

Let $D$ be an affine connection on a Riemannian manifold $(M^n, g)$ and $\pi : TM \to M$ the tangent bundle over $M$. Then the connection $D$ induces the direct decomposition

$$T_\xi(TM) = H_\xi(TM) \oplus V_\xi(TM)$$

of the tangent space at $\xi \in TM$ where $H_\xi(TM)$ is called the horizontal subspace and $V_\xi(TM)$ the vertical subspace of $T_\xi(TM)$. These subspaces are isomorphic to the tangent space $T_{\pi(\xi)}M$. Under the decomposition (1.1) and identifications $H_\xi(TM), V_\xi(TM) \cong T_{\pi(\xi)}M$, we can define an almost complex structure $J^D$ and a $J^D$-invariant metric $\tilde{g}^D$ which is known as the Sasaki metric, roughly as follows;

$$J^D = \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}, \quad \tilde{g}^D = g \oplus g.$$

We mention the detailed definition of $J^D$ and $\tilde{g}^D$ later. We call the almost Hermitian structure $(J^D, \tilde{g}^D)$ the natural almost Hermitian structure on $TM$ induced by $(g, D)$.

In this article, we find the conditions under which $TM$ admits almost Kähler structures, Kähler structures and Einstein metrics, respectively. Moreover we give some examples of Kähler-Einstein structures on $TM$. A reason why we investigate the Einstein condition is principally that the Goldberg conjecture [4], which states that a compact almost Kähler Einstein manifold is Kähler, is still unsolved completely. Now we recall definitions of almost Kähler and Kähler structures. Let $(J, g)$ be an almost Hermitian structure. If the Kähler form $\Omega = g(J\cdot, \cdot)$ is closed, we say that $(J, g)$ is almost Kähler. If $d\Omega = 0$ and $J$ is integrable, we say that $(J, g)$ is Kähler. Sekigawa [16] proved that the Goldberg conjecture is true when the scalar curvature is non-negative. In the case that the scalar curvature is negative, some partial solutions are obtained.
It is known that the assumption about the compactness is essential for the conjecture. Nurowski and Przanowski [9] gave a counter-example to non-compact version of the Goldberg conjecture by showing that $\mathbb{R}^4$ admits an almost Kähler structure which is non-Kähler and Ricci flat. The motivation of our research is to construct examples of (non-compact) non-Kähler, almost Kähler Einstein manifolds with nonzero scalar curvature, that is still under investigation.

Our main theorem is the following:

**Theorem 1.1.** Let $(g, D)$ be a pair of a metric and an affine connection on $M$ and $(J^D, \tilde{g}^D)$ be the natural almost Hermitian structure induced by $(g, D)$. Then

(i) $(J^D, \tilde{g}^D)$ is almost Kähler if and only if the dual connection $D^*$ of $D$ with respect to $g$ is torsion-free. Here the dual connection $D^*$ is the affine connection defined by the condition

$$Z(g(X, Y)) = g(D^*_Z X, Y) + g(X, D^*_Z Y)$$

for $X, Y, Z \in X(M)$ where $X(M)$ is the set of all smooth vector fields on $M$.

(ii) $(J^D, \tilde{g}^D)$ is Kähler if and only if $(M, g, D)$ is a Hessian manifold, i.e. the connection $D$ and its dual $D^*$ with respect to $g$ are both flat. Here “flat” means that its torsion and curvature both vanish.

(iii) If the Sasaki metric $\tilde{g}^D$ on $TM$ is Einstein, then the curvature tensor of $D$ vanishes.

**Remark 1.2.** (i) The cotangent bundle $T^*M$ carries a canonical symplectic form $\Omega^*$. The condition that $D^*$ is torsion-free is equivalent to the condition that the Kähler form of $(J^D, \tilde{g}^D)$ coincides with the pull-back of $\Omega^*$ by the natural isomorphism induced by $g$ (Theorem 3.1).

(ii) Dombrowski [8] shows that the almost complex structure $J^D$ on $TM$ is integrable if and only if the connection $D$ is flat. From Dombrowski’s theorem and Theorem 1.1 (i), we get immediately Theorem 1.1 (ii).

Under the assumption that $D$ is flat, it is known that $(M, g, D)$ is a Hessian manifold if and only if $(TM, J^D, \tilde{g}^D)$ is Kähler. Theorem 1.1 (ii) asserts that $(J^D, \tilde{g}^D)$ is Kähler under the assumption weaker than in [14] Proposition 2.2.4.

(iii) Theorem 1.1 (iii) is a direct consequence of Theorem 3.2. In case that $\nabla$ is the Levi-Civita connection of $g$, it is known that if $(TM, \tilde{g}^\nabla)$ is never locally symmetric unless $(M, g)$ is locally Euclidean ( [7] Theorem 2]).

From Theorem 1.1 the natural almost Hermitian structure $(J^D, \tilde{g}^D)$ on $TM$ induced by $(g, D)$ which is non-Kähler, but almost Kähler and Einstein exists only on a manifold $(M, g, D)$ which satisfies that

\begin{equation}
\begin{aligned}
\text{• the torsion of } D \text{ never vanishes, and} \\
\text{• the dual connection } D^* \text{ of } D \text{ with respect to } g \text{ is flat.}
\end{aligned}
\end{equation}

In section 4.1, we construct a family of metrics and connections which satisfy above two conditions. Moreover we show that this family includes an almost
Kähler Einstein structure. However this structure is not what we are looking for because the metric is pseudo-Riemannian and the almost complex structure is integrable.

In section 4.2, we investigate under which conditions the tangent bundle $TM$ admits a Kähler-Einstein structures. As Theorem 1.1 (ii) asserts, the natural almost Hermitian structure which is Kähler can be constructed only on Hessian manifolds. As an example of Hessian manifold, we consider the statistical model $P_{\rho}$ which is a set of probability distributions on $\mathbb{R}^n$ induced by a matrix-valued linear map $\rho$ on an open subset $U \subset \mathbb{R}^m$. For example, the tangent bundle over the manifold $P_{\rho}$ which is induced by $\rho(t) = tI_n$ for $t \in \mathbb{R}_+$ is shown to have constant holomorphic sectional curvature and consequently this is Kähler-Einstein. Here $I_n$ is the unit $n \times n$-matrix. Thus there exists many Hessian manifolds induced by matrix-valued linear maps. Do there exist other matrix-valued linear maps $\rho$ such that the tangent bundle of $P_{\rho}$ is Kähler-Einstein?

We give new examples of Hessian manifolds whose tangent bundle admits a Kähler-Einstein structure (Theorem 4.8).

2 Preliminaries

2.1 Definitions

Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold with an affine connection $D$. We denote the coefficients of the connection $D$ with respect to a local coordinate system $(U; x^1, \ldots, x^n)$ by $\{\Gamma^k_{ij}\}$:

$$D_a \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$  

Let $\pi : TM \to M$ be the tangent bundle over a manifold $M$. We define smooth functions $y^1, \ldots, y^n$ on $TM$ by $y^i(\xi) = \xi^i$ for $\xi = \sum_i \xi^i \frac{\partial}{\partial x^i}$. Then, $(\pi^{-1}(U); x^1, \ldots, x^n, y^1, \ldots, y^n)$ is a local coordinate system of $TM$.

For $X = \sum X^i \frac{\partial}{\partial x^i}$, $\xi \in T_x M$, we define the horizontal lift $X^H_\xi$ and the vertical lift $X^V_\xi$ of $X$ at $\xi$ by

$$X^H_\xi = \sum_i X^i \frac{\partial}{\partial x^i} - \sum_{i,j,k} \Gamma^k_{ij} X^i y^j(\xi) \frac{\partial}{\partial y^k},$$

$$X^V_\xi = \sum_i X^i \frac{\partial}{\partial y^i},$$

respectively. $X^H_\xi, X^V_\xi$ are tangent vectors at $\xi \in TM$. We set

$$H_\xi(TM) := \{X^H_\xi ; X \in T_x M\},$$

$$V_\xi(TM) := \{X^V_\xi ; X \in T_x M\},$$

and

$$H(TM) := \bigcup_{\xi \in TM} H_\xi(TM), \quad V(TM) := \bigcup_{\xi \in TM} V_\xi(TM).$$
We call $H(TM)$ and $V(TM)$ the horizontal and the vertical subbundles, respectively. Then we obtain the direct decomposition of the tangent bundle over $TM$;

$$T(TM) = H(TM) \oplus V(TM).$$

**Definition 2.1.** Let $(M, g)$ be a Riemannian manifold with an affine connection $D$. Then we define an almost complex structure $J^D$ by

$$J^D X^H_{\xi} = X^V_{\xi}, \quad J^D X^V_{\xi} = -X^H_{\xi},$$

and a Riemannian metric $\tilde{g}^D$ on $TM$, which is called the Sasaki metric, by

$$\tilde{g}^D(X^H_{\xi}, Y^H_{\xi}) = \tilde{g}^D(X^V_{\xi}, Y^V_{\xi}) = g(X, Y), \quad \tilde{g}^D(X^H_{\xi}, Y^V_{\xi}) = 0$$

for $X, Y, \xi \in T_xM$. We call $(J^D, \tilde{g}^D)$ the natural almost Hermitian structure on $TM$ induced by $(g, D)$.

Now we mention the dual connection. We recall that the dual connection $D^*$ of a connection $D$ with respect to a metric $g$ is defined by

$$Z(g(X, Y)) = g(D^*_Z X, Y) + g(X, D^*_Z Y). \quad (2.1)$$

The torsion tensor $T^D$ and the curvature tensor $R^D$ of a connection $D$ are $(1, 2)$- and $(1, 3)$-tensor fields respectively, defined by

$$T^D(X, Y) = D_X Y - D_Y X - [X, Y],$$

$$R^D(X, Y)Z = D_X(D_Y Z) - D_Y(D_X Z) - D_{[X,Y]}Z, \quad (2.2)$$

where $X, Y, Z \in \mathfrak{X}(M)$. The curvature and torsion tensors of a connection $D$ and its dual $D^*$ satisfy the following relation;

$$g(T^{D^*}(X, Y), Z) = (D_X g)(Y, Z) - (D_Y g)(X, Z) + g(T^D(X, Y), Z), \quad (2.3)$$

$$R^D(Z, W, X, Y) = -R^D_Y(W, Z, X, Y) \quad (2.4)$$

for $X, Y, Z, W \in T_xM$. Here $R^D$ is the $(0, 4)$-tensor field defined by

$$R^D(g)(W, Z, X, Y) = g(R^D(X, Y)Z, W).$$

### 2.2 The Levi-Civita connection and the curvature of the Sasaki metric

The bracket product of horizontal and vertical vectors are determined by the following formulae (3 Lemma 2);

$$[X^H, Y^H]_{(\xi)} = ([X, Y]_{(x)})^H_{\xi} - (R^D(X_{(x)}, Y_{(x)})\xi)^V_{\xi},$$

$$[X^H, Y^V]_{(\xi)} = (D_X Y)^V_{(\xi)},$$

$$[X^V, Y^V]_{(\xi)} = 0 \quad (2.5)$$

for $X, Y \in \mathfrak{X}(M)$ and $\xi \in T_xM$. Here $X^H$ and $X^V$ are vector fields on $TM$ defined by

$$X^H_{(\xi)} = (X_{(x)})^H_{\xi}, \quad X^V_{(\xi)} = (X_{(x)})^V_{\xi}.\quad (4)$$
Let $\tilde{\nabla}$ be the Levi-Civita connection of the Sasaki metric $\tilde{g}^D$ induced by $(g, D)$. Using (2.5) and the explicit formula

\[ 2\tilde{g}^D(\tilde{\nabla}_X Y, Z) = X(\tilde{g}^D(Y, Z)) + Y(\tilde{g}^D(X, Z)) - Z(\tilde{g}^D(X, Y)) + \tilde{g}^D([X, Y], Z) + \tilde{g}^D([Z, X], Y) + \tilde{g}^D([Z, Y], X) \]

for $X, Y, Z \in \mathfrak{x}(TM)$, we obtain the following.

**Lemma 2.2.** Let $\tilde{\nabla}$ be the Levi-Civita connection of the Sasaki metric $\tilde{g}^D$ induced by $(g, D)$. If $X, Y, Z \in \mathfrak{x}(M)$ and $\xi \in T_xM$, then

\[ (\tilde{\nabla}_X uY^H)_{(\xi)} = (\nabla_X Y)^H_{(\xi)} - \frac{1}{2} (R^D(X_\xi), Y_\xi)_{(\xi)}, \]

\[ \tilde{g}^D(\tilde{\nabla}_X vY^H, Z^H)_{(\xi)} = \tilde{g}^D(\tilde{\nabla}_Y uX^V, Z^H)_{(\xi)} = \frac{1}{2} R^g(D_\xi X_\xi, Y_\xi, Z_\xi) , \]

\[ \tilde{g}^D(\tilde{\nabla}_X uY^V, Z^V)_{(\xi)} = g(D_X Y, Z)_{(\xi)} + \frac{1}{2} (D_X g)(Y, Z)_{(\xi)}, \]

\[ \tilde{g}^D(\tilde{\nabla}_X vY^V, Z^V)_{(\xi)} = -\frac{1}{2} (D_Z g)(X, Y)_{(\xi)}, \]

\[ \tilde{g}^D(\tilde{\nabla}_X uY^V, Z^V)_{(\xi)} = 0. \]

We shall give the formulae of the curvature tensor of the Sasaki metric $\tilde{g}^D$.

**Proposition 2.3.** Let $\tilde{R}$ be the Riemannian curvature tensor of $\tilde{g}^D$. If $X, Y, Z, W, \xi \in T_xM$, then

\[ \tilde{\tilde{R}}_{\tilde{g}^D}(Z^H_\xi, W^H_\xi, X^H_\xi, Y^H_\xi) = R^g(Z, W, X, Y) - \frac{1}{2} R^g(R^D(Z, W)\xi, \xi, X, Y) \]

\[ - \frac{1}{4} \{ R^g(R^D(X, Z)\xi, \xi, Y, W) - R^g(R^D(Y, Z)\xi, \xi, X, W) \}. \]

\[ \tilde{\tilde{R}}_{\tilde{g}^D}(Z^H_\xi, W^V_\xi, X^V_\xi, Y^V_\xi) = \frac{1}{4} \sum \{ (D_{c_i}g)(Y, W) R^D(X, \xi, Z, c_i) \]

\[ - (D_{c_i}g)(X, W) R^D(Y, \xi, Z, c_i) \}, \]

\[ \tilde{\tilde{R}}_{\tilde{g}^D}(Z^V_\xi, W^V_\xi, X^V_\xi, Y^V_\xi) = -\frac{1}{4} \sum \{ (D_{c_i}g)(X, Z) (D_{c_i}g)(Y, W) \]

\[ - (D_{c_i}g)(Y, Z) (D_{c_i}g)(X, W) \}, \]

\[ \tilde{\tilde{R}}_{\tilde{g}^D}(Z^H_\xi, W^V_\xi, X^H_\xi, Y^H_\xi) = \frac{1}{2} R^g(Z, \xi, W, T^D(X, Y)) + \frac{1}{2} (D_W g)(Z, R^D(X, Y)\xi) \]

\[ + \frac{1}{2} \{ R^g(Z, \xi, \gamma(X, W), Y) - R^g(Z, \xi, \gamma(Y, W), X) \}

\[ - \frac{1}{4} \{ (D_Y g)(Z, R^D(W, X)\xi) - (D_X g)(Z, R^D(W, Y)\xi) \} \]

\[ + \frac{1}{2} g(\{D_X R^D\}(Y, W)\xi - (D_Y R^D)(X, W)\xi), Z) , \]
Here \( \{e_i\} \) is an orthonormal basis of \( T_xM \) and \( \gamma^D \) is the difference between the connection \( D \) and \( \nabla \):

\[
\gamma^D(X,Y) = D_XY - \nabla_XY.
\]

A proof of above theorem is given by direct calculations using Lemma 2.2.

**Proposition 2.4.** The Ricci tensor \( \tilde{\text{Ric}} \) of \( \tilde{g}^D \), defined by

\[
\tilde{\text{Ric}}(X,Y) = \text{Tr} \left\{ \mathcal{Z} \mapsto \tilde{R}(\mathcal{Z},Y)X \right\},
\]

is given by the following formulae:

\[
\tilde{\text{Ric}}(X^H, Y^H) = \text{Ric}^V(X,Y) - \frac{1}{4} \sum_i \text{R}^D(Y, \xi, e_i) \text{R}^D(X, \xi, Y, e_i)
\]

\[
- \frac{1}{4} \sum_i \left( (\text{D}^2_{XY})^2 + (\text{D}^2_{YX})^2 + 2 \gamma(X,Y) \gamma(Y,X) \right)(e_i, e_i) \quad (2.12)
\]

\[
\tilde{\text{Ric}}(X^V, Y^V) = \frac{1}{4} \sum_{i,j} \text{R}^D(X, \xi, e_i, e_j) \text{R}^D(Y, \xi, e_i, e_j)
\]

\[
- \frac{1}{2} \sum_i \left( (\text{D}_{e_i}^2 g)(X,Y) - (\text{D}_{e_i} g)(X,Y) \text{Tr}(T^D)(e_i) \right) \quad (2.13)
\]

where

\[
\text{R}^D(Y, \xi, e_i) \text{R}^D(X, \xi, Y, e_i) - \frac{1}{4} \sum_i \left( (\text{D}_{e_i}^2 g)(X,Y) - (\text{D}_{e_i} g)(X,Y) \text{Tr}(T^D)(e_i) \right) \quad (2.13)
\]

\[
+ \frac{1}{2} \sum_{i,j} (\text{D}_{e_i} g)(X, e_j) (\text{D}_{e_j} g)(Y, e_i).
\]
\( \widetilde{\text{Ric}}(X^H, Y^V) \)
\[= \frac{1}{2} \sum_i \{ R^D_g(Y, \xi, X, e_i) \text{Tr}(\gamma^D)(e_i) - R^D_g(Y, \xi, e_i, \gamma^D(e_i, X)) \} - \frac{1}{2} \sum_i \{ g((D_{e_i} R^D(e_i, X)\xi, Y) + (D_{e_i} g)(R^D(e_i, X)\xi, Y)) \} + \frac{1}{4} \sum_{i,j} R^D_g(Y, \xi, X, e_i) (D_{e_i} g)(e_j, e_j). \]  

(2.14)

Here \( \text{Ric}^\nabla \) is the Ricci tensor of the Levi-Civita connection \( \nabla \) of \( g \) and \( \text{Tr}(\gamma^D) \) is a 1-form defined by

\[ \text{Tr}(\gamma^D)(X) = \text{Tr}(Z \mapsto \gamma^D(Z, X)). \]

Proof. Using an orthonormal frame \( \{ E_i \} \) on \( TM \), we can express \( \widetilde{\text{Ric}} \) by

\[ \widetilde{\text{Ric}}(X, Y) = \sum_{i=1}^{2n} \widetilde{R}_{g^D}(E_i, X, E_i, Y). \]

If \( \{ e_i \} \) is an orthonormal basis of \( T_x M \) with respect to \( g \), then for \( \xi \in T_x M \) \( \{ e_iH^\xi, e_iV^\xi \} \) is an orthonormal basis of \( T_{\xi}(TM) \) with respect to \( g^D \). Hence we have

\[ \widetilde{\text{Ric}}(X^H, Y^H) = \sum_{i=1}^{n} \left\{ \widetilde{R}_{g^D}(e_iH^\xi, X^H, e_iH^\xi, Y^H) \right\}. \]  

(2.15)

Substituting (2.6) and (2.11) into (2.15), we obtain (2.12). Similarly, by simple calculations we obtain (2.13) and (2.14).

3 Proof of Theorem 1.1

(i) Using (2.3), (2.4), (2.5) and the formula

\[ d \Omega(X, Y, Z) = S_{X, Y, Z} \{ X(\Omega(Y, Z)) - \Omega([X, Y], Z) \}, \]

we have

\[ d \Omega(X^H, Y^H, Z^H) = S_{X, Y, Z} R^D_\ast(\xi, X, Y, Z), \]

(3.1)

\[ d \Omega(X^H, Y^H, Z^V) = g(T^D_\ast(X, Y), Z), \]

(3.2)

\[ d \Omega(X^H, Y^V, Z^V) = d \Omega(X^V, Y^V, Z^V) = 0. \]

Here \( S_{X, Y, Z} \) denotes the cyclic sum with respect to \( X, Y, Z \).

If we assume \( d \Omega = 0 \), from (3.2) we obtain that \( T^D_\ast = 0 \). Conversely, if \( D^\ast \) is torsion-free, then from the first Bianchi identity we find that the right hand side of (3.1) vanishes. This completes the proof of Theorem 1.1 (i).

Now we remark about the symplectic structure on \( T^\ast M \). Let \( \pi^\ast : T^\ast M \rightarrow M \) be the cotangent bundle on \( M \). We define smooth functions \( z_1, \ldots, z_n \) by
\[ z_i(\psi) = \psi_i \text{ on } T^*M \text{ for } \psi = \sum \psi_i dx^i \in T^*_x M. \] Then, \( \{x^1, \ldots, x^n, z_1, \ldots, z_n\} \) is a local coordinate system of \( T^*M \). \( T^*M \) carries a canonical symplectic structure \( \Omega^* \) locally expressed by \( \Omega^* = \sum dx^i \wedge dz_i \) (See [2]).

Then, we obtain the following result.

**Theorem 3.1.** The natural almost Hermitian structure \( (J^D, \tilde{g}^D) \) on \( TM \) induced by \( (g, D) \) is almost Kähler if and only if the Kähler form of \( (J^D, \tilde{g}^D) \) coincides with the pull-back of the symplectic form \( \Omega^* \) on \( T^*M \) by \( \varphi_g \). Here \( \varphi_g : TM \to T^*M \) is the natural isomorphism defined by \( \varphi_g(X) = g(X, \cdot) \) for \( X \in TM \).

**Proof.** Let \( \varphi_g^*(\Omega^*) \) be the pull-back of the symplectic form \( \Omega^* \) by \( \varphi_g \). Easy computations show that

\[
\varphi_g^*(\Omega^*) \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = y^k, \\
\varphi_g^*(\Omega^*) \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right) = g_{ij}, \\
\varphi_g^*(\Omega^*) \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 0,
\]

and

\[
\Omega \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \sum_{k,l} (\Gamma^l_{jk} g_{li} - \Gamma^l_{ik} g_{lj}) y^k, \\
\Omega \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right) = g_{ij}, \\
\Omega \left( \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right) = 0.
\]

Hence, from (3.3) and (3.4), \( \varphi_g^*(\Omega^*) = \Omega \) if and only if

\[
\frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} = \sum_l (\Gamma^l_{jk} g_{li} - \Gamma^l_{ik} g_{lj}).
\]

From (2.3), we find that (3.5) implies \( T^* = 0 \).

(ii) The integrability condition of the almost complex structure is equivalent to the condition that the Nijenhuis tensor \( N \) vanishes (See [6, Chapter IX]). Here \( N \) is a \((1,2)\)-tensor field defined by

\[
N(X, Y) = [J^DX, J^DY] - J^D[J^DX, Y] - J^D[X, J^DY] - [X, Y]
\]

for \( X, Y \in \mathfrak{X}(TM) \). From the definition, \( N \) satisfies

\[
N(Y, X) = -N(X, Y), \quad N(J^DX, Y) = -J^D N(X, Y).
\]

Hence, in our situation, in order to find the condition that \( N = 0 \) it is enough to compute \( N(X^H, Y^H) \). From easy computations, we have

\[
N(X^H, Y^H) = (T(X, Y))_\xi^H + (R(X, Y))_\xi^V.
\]
Hence, $N = 0$ if and only if $R^D = 0$ and $T^D = 0$, i.e. $D$ is a flat connection \[^{18}\]. From above argument and Theorem 1.1 (i), we obtain Theorem 1.1 (ii).

(iii) We show the following fact;

**Theorem 3.2.** *If the Ricci tensor $\widetilde{\text{Ric}}$ of $\tilde{g}^D$ is $J^D$-invariant or $J^D$-anti-invariant, then $R^D = 0$.***

**Proof.** The assumption implies that $\widetilde{\text{Ric}}$ satisfies

$$\text{Ric}(X^H, Y^H) = \pm \text{Ric}(X^V, Y^V) \quad (3.6)$$

for any $X, Y, \xi \in TM$, and in particular

$$-\frac{1}{2} \sum_i R_i^D(R^D(X, e_i)\xi, Y, e_i) = \pm \frac{1}{4} \sum_{i,j} R_i^D(X, \xi, e_i, e_j) R_j^D(Y, \xi, e_i, e_j) \quad (3.7)$$

which is the $\xi$-dependent part of (3.3). Substituting $X = Y = e_i$ and $\xi = e_j$ into (3.7), and summing on $i$ and $j$ we have

$$-\frac{1}{2} |R^D|_g^2 = \pm \frac{1}{4} |R^D|_g^2$$

from which we obtain that $|R^D|_g^2 = 0$, i.e. $R^D = 0$.

If $(TM, J^D, \tilde{g}^D)$ is Einstein, the Ricci tensor is $J^D$-invariant. Hence, Theorem 1.1 (iii) is obtained as the corollary of Theorem 3.2.

## 4 Examples

### 4.1 A 1-parameter family of almost Kähler structures on the tangent bundle

Let $(M, g)$ be the Riemannian product of the unit circle $(S^1, g_0)$ with the angular coordinate $\theta$ and a space of positive constant curvature $(N^{n-1}, g_N)$ and let $\omega = kd\theta$ be a 1-form on $M$ $(k \in \mathbb{R})$. Also let $D$ be a torsion-free connection $D$ on $M$ satisfying $Dg = \omega \otimes g$. Such a connection is uniquely determined for given $g$ and $\omega$.

When $k = \pm \frac{2s_N}{\sqrt{(n-1)(n-2)}}$, the curvature of $D$ vanishes, i.e. $D$ is a flat connection. Here $s_N$ is the scalar curvature of $g_N$.

Fix a constant $k$ such that $R^D = 0$. For $\lambda \in \mathbb{R}$ we define the metric $g_\lambda$ on $M$ by

$$g_\lambda := g + \frac{\lambda}{|\omega|^2} \omega \otimes \omega.$$

Let $D^\lambda_*$ be the dual connection of $D$ with respect to $g_\lambda$. Then $(g_\lambda, D^\lambda_*)$ induces a 1-parameter family of almost Kähler structures $(J_\lambda, \tilde{g}_\lambda)$ on $TM$ parametrized by $\lambda$. Moreover, $(g_\lambda, D^\lambda_*)$ satisfies the condition (1.2).
Now we find the condition that $\tilde{g}^D$ is Einstein. In our situation, from Proposition 2.4 we can express $\tilde{\text{Ric}}$ by

$$\tilde{\text{Ric}}(X^H, Y^H) = \frac{|\omega|^2 g}{8} 
\left( 2(n - 2) - \frac{\lambda^2}{\lambda + 1} \right) g_\lambda(X, Y) 
+ \frac{1}{8} (\lambda + 2) \{ \lambda - 2(n - 1) \} \omega(X) \omega(Y),$$

(4.1)

$$\tilde{\text{Ric}}(X^V, Y^V) = \frac{|\omega|^2 g}{8} \frac{\lambda^2 + 2\lambda - 2n}{\lambda + 1} g_\lambda(X, Y) - \frac{n\lambda(\lambda + 2)}{8} \omega(X) \omega(Y),$$

(4.2)

From (4.1) and (4.2), $\tilde{\text{Ric}} = k \tilde{g}_\lambda$ if and only if $\lambda = -2$. Then, $D^*_{(-2)}$ is torsion free and $g_{(-2)}$ is a pseudo-Riemannian metric. Hence $(\mathcal{T}M, J_{(-2)}, \tilde{g}_{(-2)})$ is a pseudo-Kähler Einstein manifold.

Remark 4.1. We can apply similar argument to compact flat Weyl manifolds (See section 3 in [15] for details).

At the end of this subsection we pose the following problem;

Problem 4.2. Do there exist pairs $(g, D)$ where $g$ is a positive definite metric and $D$ is an affine connection such that the natural almost Hermitian structure $(J^D, \tilde{g}^D)$ is strictly almost Kähler Einstein?

4.2 On the manifold of multivariate normal distributions on $\mathbb{R}^2$

In this subsection, we consider Kähler structures on $\mathcal{T}M$.

From Theorem 1.1, we can construct Kähler structures on tangent bundles by using Hessian structures. We recall that when an affine connection $D$ and its dual connection $D^*$ with respect to $g$ are both flat, we call $(M, g, D)$ a Hessian manifold, and in particular $g$ a Hessian metric on $(M, D)$. This condition is equivalent to the following condition: $D$ is a flat connection on $M$ and there exists a function $\varphi$ on $M$ such that $g = Dd\varphi$. We call the function $\varphi$ the potential of the Hessian metric $g$ with respect to $D$.

We define a 1-form $\alpha$ and a symmetric $(0, 2)$-tensor $\beta$, which are called the first Koszul form and the second Koszul form respectively, by

$$\alpha(X) = g(D_X dv_g, dv_g) \quad (X \in \mathcal{T}M), \quad \beta = D\alpha.$$

Here $dv_g$ is the volume form of $g$. Then we can consider the notion of “Hesse-Einstein”. If the 2nd Koszul form is proportional to the Hessian metric $g$, then we say that a Hessian manifold is Hesse-Einstein.

Remark 4.3. If $\dim M \geq 2$ and $\beta = cg$, then $c$ must be a constant.

Hesse-Einstein manifolds are characterized as follows:

Theorem 4.4 ([17] Thm. 3.1.6). Let $(M, g, D)$ be a Hessian manifold. Then the Ricci tensor $\text{Ric}$ of $\tilde{g}^D$ satisfies

$$\tilde{\text{Ric}}(X^H, Y^H) = \tilde{\text{Ric}}(X^V, Y^V) = -\beta(X, Y), \quad \tilde{\text{Ric}}(X^H, Y^V) = 0.$$

In particular, the Kähler structure on $\mathcal{T}M$ induced by $(g, D)$ is Einstein if and only if $(M, g, D)$ is Hesse-Einstein.
Using a matrix-valued linear map we can construct a Hessian structure as follows: Let $S_n$ be the set of all symmetric $n \times n$-matrices and $S_n^+$ the subset of all positive definite symmetric matrices in $S_n$. Let $\rho$ be a linear injection from a domain $U \subset \mathbb{R}^n$ to $S_n$ which satisfies that $\rho(U) \subset S_n^+$. For $(\mu, \xi) \in \mathbb{R}^n \times U$, we set a function

$$p(x; \mu, \xi) := \sqrt{\frac{\det \rho(\xi)}{(2\pi)^n}} \exp \left\{ -\frac{1}{2}(x - \mu)^T \rho(\xi)(x - \mu) \right\} \quad (x \in \mathbb{R}^n). \quad (4.3)$$

Then the set $\mathcal{P}_n^\rho = \{ p(x; \mu, \xi) \mid (\mu, \xi) \in \mathbb{R}^n \times U \}$ is a family of probability distributions on $\mathbb{R}^n$ parametrized by $(\mu, \xi)$. We call $\mathcal{P}_n^\rho$ the statistical model induced by $\rho$. $\mathcal{P}_n^\rho$ is a smooth manifold of dimension $(n + m)$ and has the Riemannian metric $g_\rho$ which is called the Fisher metric (See [I]). Moreover $\mathcal{P}_n^\rho$ admits a flat connection $D$ such that $(g_\rho, D)$ is a Hessian structure on $\mathcal{P}_n^\rho$.

**Proposition 4.5 (17 Prop. 6.2.1).** Let $\mathcal{P}_n^\rho = \{ p(x; \mu, \xi) \mid (\mu, \xi) \in \mathbb{R}^n \times U \}$ be the statistical model induced by $\rho$. We set $\theta = \rho(\xi)\mu$. Let $D$ is the standard flat connection on $\{ (\theta, \xi) \in \mathbb{R}^n \times U \}$. Then the Fisher metric $g_\rho$ on $\mathcal{P}_n^\rho$ is the Hessian metric on $(\mathcal{P}_n^\rho, D)$ whose potential is given by

$$\varphi(\theta, \xi) = \frac{1}{2} \{ \theta^T \rho(\xi)^{-1} \theta - \log \det \rho(\xi) \}.$$  

**Example 4.6.** For the linear injection $\rho : \mathbb{R}^+ \to S_n$ defined by

$$\rho(t) = tI_n \quad (I_n \text{ is the unit matrix}),$$

$(T \mathcal{P}_n^\rho, J^{D^*}, \tilde{g}_\rho^{D^*})$ has constant holomorphic sectional curvature and consequently this is Kähler-Einstein (17 Problem 6.2.1). Here $D^*$ is the dual connection of $D$ with respect to the Fisher metric $g_\rho$. In the case that $n = 1$, Sato [I3] explicitly compute the Ricci tensor of $(T \mathcal{P}_1^\rho, J^{D^*}, \tilde{g}_\rho^{D^*})$.

Then, the following problem is naturally arisen;

**Problem 4.7.** How many Hessian manifolds which are Hesse-Einstein do there exist in the class of Hessian manifolds induced by matrix-valued linear injections mentioned above?

We give a partial solution for this problem as follows.

**Theorem 4.8.** For any linear injection $\rho$ from a domain $U \subset \mathbb{R}^2$ into $S_2$ such that $\rho(U) \subset S_2^+$, $(\mathcal{P}_2^\rho, g_\rho, D^*)$ is always Hesse-Einstein. Hence, $(T \mathcal{P}_2^\rho, J^{D^*}, \tilde{g}_\rho^{D^*})$ is a Kähler Einstein manifold. Here $D^*$ is the dual connection of $D$ with respect to $g_\rho$.

**Outline of proof.** In our situation, applying certain coordinate (affine) transformations, we can reduce the linear injection $\rho$ into the form

$$\rho(\xi) = \rho(\xi_1, \xi_2) = \begin{pmatrix} \xi_1^1 & a\xi_1^1 + b\xi_2^2 \\ a\xi_1^1 + b\xi_2^2 & \xi_2^2 \end{pmatrix} \quad (a, b \in \mathbb{R}) \quad (4.4)$$

as follows:
In general, a linear map $\rho : U (\subset \mathbb{R}^2) \to \mathcal{S}_2$ is written by

$$
\rho(\xi_1, \xi_2) = \begin{pmatrix}
\rho_{11}(\xi_1, \xi_2) & \rho_{12}(\xi_1, \xi_2) \\
\rho_{21}(\xi_1, \xi_2) & \rho_{22}(\xi_1, \xi_2)
\end{pmatrix}
$$

where $\rho_{ij}(\xi_1, \xi_2)$ is a polynomial of degree $1$ ($1 \leq i, j \leq 2$).

Case 1: If $\rho_{11} \neq c\rho_{22}$ for any constant $c$, change the coordinate $(\xi_1, \xi_2; \theta) \mapsto (\rho_{11}(\xi_1, \xi_2), \rho_{22}(\xi_1, \xi_2); \theta)$.

Case 2: If $\rho_{11} = c\rho_{22}$ for a constant $c$, change the coordinate $(\xi; \theta) \mapsto (\xi; A^{-1}\theta)$ for $A \in GL(2, \mathbb{R})$ such that $\rho'(\xi) := A\rho(\xi)^t A$ satisfies the condition of the case 2 and we consider the statistical model induced by $\rho'$. Thus, we can reduce this case into the case 1.

Computing the 2nd Koszul form $\beta^*$ of the Hessian structure $(D^*, g_\rho)$ where $\rho$ is given in the form (4.4), we get

$$
\beta^* = 3g_\rho
$$

from which we obtain Theorem 4.8.

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