AN EXTENSION OF
UNIQUENESS THEOREMS
FOR MEROMORPHIC MAPPINGS

Gerd Dethloff and Tran Van Tan

Abstract
In this paper, we give some results on the number of meromorphic
mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ under a condition on the inverse images
of hyperplanes in $\mathbb{C}P^n$. At the same time, we give an answer for an
open question posed by H. Fujimoto in 1998.

1 Introduction

In 1926, R. Nevanlinna showed that for two nonconstant meromorphic func-
tions $f$ and $g$ on the complex plane $\mathbb{C}$, if they have the same inverse images
for five distinct values, then $f = g$, and that $g$ is a special type of a linear
fractional transformation of $f$ if they have the same inverse images, counted
with multiplicities, for four distinct values.

In 1975, H. Fujimoto [2] generalized Nevanlinna’s result to the case of
meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$. This problem continued to be stud-
ied by L. Smiley [9], S.Ji [5] and others.

Let $f$ be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $H$ be a hyperplane
in $\mathbb{C}P^n$ such that $\text{im} f \not\in H$. Denote by $v_{(f,H)}$ the map of $\mathbb{C}^m$ into $\mathbb{N}_0$ such
that $v_{(f,H)}(a)$ ($a \in \mathbb{C}^m$) is the intersection multiplicity of the image of $f$ and
$H$ at $f(a)$. Let $k$ be a positive integer or $+\infty$. We set

2000 Mathematics Subject Classification: 32H 30.
Key words and phrases: uniqueness theorem, meromorphic mapping, linearly degenerate.
Let $f$ be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and \{ $H_j$ \}$_{j=1}^q$ be $q$ hyperplanes in general position with

(a) \[ \dim \left\{ z : v^k_{(f,H_i)}(z) > 0 \text{ and } v^k_{(f,H_j)}(z) > 0 \right\} \leq m - 2 \text{ for all } 1 \leq i < j \leq q. \]

For each positive integer $p$, denote by $F_{k}^{q}($ \{ $H_j$ \}$_{j=1}^q$, $f$, $p$ $)$ the set of all linearly nondegenerate meromorphic mappings $g$ of $\mathbb{C}^m$ into $\mathbb{C}P^n$ such that:

(b) \[ \min\left\{ v^k_{(g,H_j)}, p \right\} = \min\left\{ v^k_{(f,H_j)}, p \right\}, \]

(c) \[ g = f \text{ on } \bigcup_{j=1}^q \left\{ z : v^k_{(f,H_j)}(z) > 0 \right\}. \]

In [5], S.Ji showed the following

**Theorem J.** ([5]) If $q = 3n + 1$ and $k = +\infty$, then for three mappings $f_1, f_2, f_3 \in F_{k}^{q}($ \{ $H_j$ \}$_{j=1}^q$, $f$ $)$, the mapping $f_1 \times f_2 \times f_3 : \mathbb{C}^m \rightarrow \mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n$ is algebraically degenerate, namely, \{ $(f_1(z), f_2(z), f_3(z))$, $z \in \mathbb{C}^m$ \} is contained in a proper algebraic subset of $\mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n$.

In 1929, H. Cartan declared that there are at most two meromorphic functions on $\mathbb{C}$ which have the same inverse images (ignoring multiplicities) for four distinct values. However in 1988, N. Steinmetz ([10]) gave examples which showed that H. Cartan’s declaration is false. On the other hand, in 1998, Fujimoto ([4]) showed that H. Cartan’s declaration is true if we assume that meromorphic functions on $\mathbb{C}$ share four distinct values counted with multiplicities truncated by 2. He gave the following theorem

**Theorem F.** ([4]) If $q = 3n + 1$ and $k = +\infty$ then $F_{k}^{q}($ \{ $H_j$ \}$_{j=1}^q$, $f$, $2$ $)$ contains at most two mappings.

He also proposed an open problem asking if the number $q = 3n + 1$ in Theorem F can be replaced by a smaller one. Inspired by this question, in this paper we will generalize the above results to the case where the number $q = 3n + 1$ is in fact replaced by a smaller one. We also obtain an improvement concerning truncating multiplicities.

Denote by $\Psi$ the Segre embedding of $\mathbb{C}P^n \times \mathbb{C}P^n$ into $\mathbb{C}P^{n^2 + 2n}$ which is
defined by sending the ordered pair \(((w_0, ..., w_n), (v_0, ..., v_n))\) to \((..., w_i v_j, ...)\) (in lexicographic order).

Let \( h : \mathbb{C}^m \rightarrow \mathbb{C}P^n \times \mathbb{C}P^n \) be a meromorphic mapping. Let \( (h_0 : ... : h_{n^2+2n}) \) be a representation of \( \Psi \circ h \). We say that \( h \) is linearly degenerate (with the algebraic structure in \( \mathbb{C}P^n \times \mathbb{C}P^n \) given by the Segre embedding) if \( h_0, ..., h_{n^2+2n} \) are linearly dependent over \( \mathbb{C} \).

Our main results are stated as follows:

**Theorem 1.** There are at most two distinct mappings in \( F_k(\{H_j\}_{j=1}^q, f, p) \) in each of the following cases:

i) \( 1 \leq n \leq 3, q = 3n + 1, p = 2 \) and \( 23n \leq k \leq +\infty \)

ii) \( 4 \leq n \leq 6, q = 3n, p = 2 \) and \( \frac{(6n-1)n}{n-3} \leq k \leq +\infty \)

iii) \( n \geq 7, q = 3n - 1, p = 1 \) and \( \frac{(6n-4)n}{n-6} \leq k \leq +\infty \).

**Theorem 2.** Assume that \( q = \frac{5(n+1)}{2}, (65n + 171)n \leq k \leq +\infty \), where \([x] := \max\{d \in \mathbb{N} : d \leq x\}\) for a positive constant \( x \). Then one of the following assertions holds:

i) \( \#F_k(\{H_j\}_{j=1}^q, f, 1) \leq 2 \).

ii) For any \( f_1, f_2 \in F_k(\{H_j\}_{j=1}^q, f, 1) \), the mapping \( f_1 \times f_2 : \mathbb{C}^m \rightarrow \mathbb{C}P^n \times \mathbb{C}P^n \) is linearly degenerate (with the algebraic structure in \( \mathbb{C}P^n \times \mathbb{C}P^n \) given by the Segre embedding).

We finally remark that we obtained similar uniqueness theorems with moving targets in [11], but only with a bigger number of targets and with much bigger truncations.

**Acknowledgements:** The second author would like to thank Professor Do Duc Thai for valuable discussions, the Université de Bretagne Occidentale for its hospitality and support, and the PICS-CNRS ForMathVietnam for its support.

## 2 Preliminaries

We set \( \|z\| := (|z_1|^2 + \cdots + |z_m|^2)^{1/2} \) for \( z = (z_1, \ldots, z_m) \in \mathbb{C}^m \), \( B(r) := \{z : \|z\| < r\} \), \( S(r) := \{z : \|z\| = r\} \), \( d^c := \frac{\sqrt{-1}}{4\pi} (\overline{\partial} - \partial) \), \( v := (dd^c \|z\|^2)^{m-1} \) and \( \sigma := d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{m-1} \).
Let $F$ be a nonzero holomorphic function on $\mathbb{C}^m$. For an $m$-tuple $\alpha := (\alpha_1, \ldots, \alpha_m)$ of nonnegative integers, set $|\alpha| := \alpha_1 + \cdots + \alpha_m$ and $D^\alpha F := \frac{\partial^{\alpha_1} \cdots \partial^{\alpha_m} F}{\partial z_1^{\alpha_1} \cdots \partial z_m^{\alpha_m}}$. We define the map $v_F : \mathbb{C}^m \to \mathbb{N}_0$ by $v_F(z) := \max\{p : D^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < p\}$. Let $k$ be a positive integer or $+\infty$. Define the map $v^{(k)}_F$ of $\mathbb{C}^m$ into $\mathbb{N}_0$ by
\[
v^{(k)}_F(z) := \begin{cases} 0 & \text{if } v_F(z) > k, \\ v_F(z) & \text{if } v_F(z) \leq k. \end{cases}
\]
Let $\varphi$ be a nonzero meromorphic function on $\mathbb{C}^m$. We define the map $v^{(k)}_\varphi$ as follows: For each $z \in \mathbb{C}^m$, choose nonzero holomorphic functions $F$ and $G$ on a neighbourhood $U$ of $z$ such that $\varphi = \frac{F}{G}$ on $U$ and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m - 2$. Then put $v^{(k)}_\varphi(z) := v^{(k)}_F(z)$. Set
\[
|v^{(k)}_\varphi| := \{z : v^{(k)}_\varphi(z) = 0\}.
\]
Define
\[
N^{(k)}(r, v_\varphi) := \int_1^r \frac{n^{(k)}(t)}{t^{2m-1}} dt, \quad (1 < r < +\infty)
\]
where
\[
n^{(k)}(t) := \int_{|v^{(k)}_\varphi| \cap B(t)} v^{(k)}_\varphi u \quad \text{for } m \geq 2
\]
and
\[
n^{(k)}(t) := \sum_{|z| \leq t} v^{(k)}_\varphi(z) \quad \text{for } m = 1.
\]
Set $N(r, v_\varphi) := N^{+\infty}(r, v_\varphi)$. For $l$ a positive integer or $+\infty$, set
\[
N^{(k)}_l(r, v_\varphi) := \int_1^r \frac{n^{(k)}_l(t)}{t^{2m-1}} dt, \quad (1 < r < +\infty)
\]
where $n^{(k)}_l(t) := \int_{|v^{(k)}_\varphi| \cap B(t)} \min\{v^{(k)}_\varphi, l\} u \quad \text{for } m \geq 2$ and $n^{(k)}_l(t) := \sum_{|z| \leq t} \min\{v^{(k)}_\varphi(z), l\} \quad \text{for } m = 1$. Set $N(r, v_\varphi) := N^{+\infty}_1(r, v_\varphi)$ and
\( \overline{\mathcal{N}}^k(r, v_\varphi) := N^k_1(r, v_\varphi) \). For a closed subset \( A \) of a purely \((m-1)\)-dimensional analytic subset of \( \mathbb{C}^m \), we define

\[
\overline{\mathcal{N}}(r, A) := \int_1^r \frac{\tilde{n}(t)}{t^{2m-1}} \, dt, \quad (1 < r < +\infty)
\]

where

\[
\tilde{n}(t) := \begin{cases} 
\int_{A \cap B(t)} v & \text{for } m \geq 2 \\
\#(A \cap B(t)) & \text{for } m = 1.
\end{cases}
\]

Let \( f : \mathbb{C}^m \to \mathbb{C}P^n \) be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates \((w_0 : \cdots : w_n)\) on \( \mathbb{C}P^n \), we take a reduced representation \( f = (f_0 : \cdots : f_n) \) which means that each \( f_i \) is a holomorphic function on \( \mathbb{C}^m \) and \( f(z) = (f_0(z) : \cdots : f_n(z)) \) outside the analytic set \( \{ f_0 = \cdots = f_n = 0 \} \) of codimension \( \geq 2 \).

Set \( \|f\| := (|f_0|^2 + \cdots + |f_n|^2)^{1/2} \). The characteristic function of \( f \) is defined by

\[
T_f(r) := \int_{S(r)} \log \|f\| \sigma - \int_{S(1)} \log \|f\| \sigma, \quad r > 1.
\]

For a nonzero meromorphic function \( \varphi \) on \( \mathbb{C}^m \), the characteristic function \( T_\varphi(r) \) of \( \varphi \) is defined by considering \( \varphi \) as a meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{C}P^1 \).

Let \( H = \{a_0w_0 + \cdots + a_nw_n = 0\} \) be a hyperplane in \( \mathbb{C}P^n \) such that \( \text{im} f \nsubseteq H \). Set \( (f, H) := a_0f_0 + \cdots + a_nf_n \). We define

\[
N^k_f(r, H) := N^k(r, v_{(f,H)}) \quad \text{and} \quad N^k_{l,f}(r, H) := N^k_l(r, v_{(f,H)}).
\]

Sometimes we write \( \overline{N}^k_f(r, H) \) for \( N^k_{1,f}(r, H) \), \( N_{l,f}(r, H) \) for \( N^k_{l,f}(r, H) \), and \( N_f(r, H) \) for \( N^k_{+\infty,f}(r, H) \).

Set \( \psi_f(H) := \frac{\|f\|(|a_0|^2 + \cdots + |a_n|^2)^{1/2}}{(f, H)} \). We define the proximity function by

\[
m_f(r, H) := \int_{S(r)} \log |\psi_f(H)| \sigma - \int_{S(1)} \log |\psi_f(H)| \sigma.
\]
For a nonzero meromorphic function \( \varphi \), the proximity function is defined by
\[
m(r, \varphi) := \int_{S(r)} \log^+ | \varphi | \sigma.
\]
We note that \( m(r, \varphi) = m_\varphi(r, +\infty) + O(1) \) ([4], p.135).

We state the First and the Second Main Theorem of Value Distribution Theory.

**First Main Theorem.** Let \( f : \mathbb{C}^m \to \mathbb{C}P^n \) be a meromorphic mapping and \( H \) a hyperplane in \( \mathbb{C}P^n \) such that \( \text{im } f \not\subseteq H \). Then:
\[
N_f(r, H) + m_f(r, H) = T_f(r).
\]
For a nonzero meromorphic function \( \varphi \) we have :
\[
N(r, v_\varphi) + m(r, \varphi) = T_\varphi(r) + O(1).
\]

**Second Main Theorem.** Let \( f : \mathbb{C}^m \to \mathbb{C}P^n \) be a linearly nondegenerate meromorphic mapping and \( H_1, \ldots, H_q \) be hyperplanes in general position in \( \mathbb{C}P^n \). Then:
\[
(q - n - 1)T_f(r) \leq \sum_{j=1}^q N_{n,f}(r, H_j) + o(T_f(r))
\]
except for a set \( E \subset (1, +\infty) \) of finite Lebesgue measure.

The following so-called logarithmic derivative lemma plays an essential role in Nevanlinna theory.

**Theorem 2.1.** ([5], Lemma 3.1) Let \( \varphi \) be a non-constant meromorphic function on \( \mathbb{C}^m \). Then for any \( i, \ 1 \leq i \leq m \), we have
\[
m\left(r, \frac{\partial}{\partial z_i} \varphi \right) = o(T_\varphi(r)) \quad \text{as } r \to \infty, \ r \notin E,
\]
where \( E \subset (1, +\infty) \) of finite Lebesgue measure.
Let $F, G$ and $H$ be nonzero meromorphic functions on $\mathbb{C}^m$. For each $l$, $1 \leq l \leq m$, we define the Cartan auxiliary function by

\[
\Phi^l(F, G, H) := F \cdot G \cdot H \cdot \left| \begin{array}{ccc}
\frac{1}{F} & \frac{1}{G} & \frac{1}{H} \\
\frac{\partial}{\partial z_l}(F) & \frac{\partial}{\partial z_l}(G) & \frac{\partial}{\partial z_l}(H)
\end{array} \right|.
\]

By [4] (Proposition 3.4) we have the following

**Theorem 2.2.** Let $F, G, H$ be nonzero meromorphic functions on $\mathbb{C}^m$. Assume that $\Phi^l(F, G, H) \equiv 0$ and $\Phi^l\left(\frac{1}{F}, \frac{1}{G}, \frac{1}{H}\right) \equiv 0$ for all $l$, $1 \leq l \leq m$. Then one of the following assertions holds

i) $F = G$ or $G = H$ or $H = F$.

ii) $\frac{F}{G}, \frac{G}{H}, \frac{H}{F}$ are all constant.

### 3 Proof of the Theorems

First of all, we need the following lemmas:

**Lemma 1.** Let $f_1, ..., f_d$ be linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{H_j\}_{j=1}^q$ be hyperplanes in $\mathbb{C}P^n$. Then there exists a dense subset $C \subset \mathbb{C}^{n+1}\setminus\{0\}$ such that for any $c = (c_0, ..., c_n) \in C$, the hyperplane $H_c$ defined by $c_0\omega_0 + ... + c_n\omega_n = 0$ satisfies

\[
\dim(f_i^{-1}(H_j) \cap f_i^{-1}(H_c)) \leq m - 2 \quad \text{for all } i \in \{1, ..., d\} \text{ and } j \in \{1, ..., q\}.
\]

**Proof.** We refer to [5], Lemma 5.1. \(\square\)

Let $f_1, f_2, f_3 \in F_k(\{H_j\}_{j=1}^q, f, 1)$, for $q \geq n + 1$. Set

\[
T(r) := T_{f_1}(r) + T_{f_2}(r) + T_{f_3}(r).
\]

For each $c \in C$, set $F_{ic}^j := \frac{(f_i, H_j)}{(f_i, H_c)}$ for $i \in \{1, 2, 3\}$ and $j \in \{1, ..., q\}$.

**Lemma 2.** Assume that there exist $j_0 \in \{1, ..., q\}, c \in C, l \in \{1, ..., m\}$ and a closed subset $A$ of a purely $(m - 1)$-dimensional analytic subset of $\mathbb{C}^m$
satisfying
1) \( \Phi^1_c := \Phi^l(F^{j_0}_{1c}, F^{j_0}_{2c}, F^{j_0}_{3c}) \neq 0 \), and
2) \( \min \{v^{(k)}_{(f_1, H_{j_0})}, p\} = \min \{v^{(k)}_{(f_2, H_{j_0})}, p\} = \min \{v^{(k)}_{(f_3, H_{j_0})}, p\} \) on \( \mathbb{C}^m \setminus A \),
where \( p \) is a positive integer. Then

\[
2 \sum_{j=1, j \neq j_0}^q N_{j, f_i}(r, H_j) + N_{p-1, f_i}(r, H_{j_0}) \leq N(r, v^{(k)}_{q \ell}) + (p - 1)N(r, A)
\]

\[
\leq \frac{k + 2}{k + 1} T(r) + (p + 2)N(r, A) + o(T(r))
\]

for all \( i \in \{1, 2, 3\} \).

**Proof.** Without loss of generality, we may assume that \( l = 1 \). For an arbitrary point \( a \in \mathbb{C}^m \setminus A \) satisfying \( v^{(k)}_{(f_1, H_{j_0})}(a) > 0 \), we have \( v^{(k)}_{(f_i, H_{j_0})}(a) > 0 \) for all \( i \in \{1, 2, 3\} \). We choose \( a \) such that \( a \notin \bigcup_{i=1}^3 f_i^{-1}(H_c) \). We distinguish between two cases, leading to equations (1) and (2).

**Case 1.** If \( v_{(f_1, H_{j_0})}(a) \geq p \), then \( v_{(f_i, H_{j_0})}(a) \geq p, i \in \{1, 2, 3\} \). This means that \( a \) is a zero point of \( F^{j_0}_{i_c} \) with multiplicity \( \geq p \) for \( i \in \{1, 2, 3\} \). We have

\[
\Phi^1_c = F^{j_0}_{1c} F^{j_0}_{3c} \frac{\partial}{\partial z_1} \left( \frac{1}{F^{j_0}_{1c}} \right) - F^{j_0}_{1c} F^{j_0}_{3c} \frac{\partial}{\partial z_1} \left( \frac{1}{F^{j_0}_{2c}} \right)
\]

\[
+ F^{j_0}_{2c} F^{j_0}_{1c} \frac{\partial}{\partial z_1} \left( \frac{1}{F^{j_0}_{2c}} \right) - F^{j_0}_{2c} F^{j_0}_{3c} \frac{\partial}{\partial z_1} \left( \frac{1}{F^{j_0}_{3c}} \right)
\]

\[
+ F^{j_0}_{3c} F^{j_0}_{2c} \frac{\partial}{\partial z_1} \left( \frac{1}{F^{j_0}_{3c}} \right) - F^{j_0}_{3c} F^{j_0}_{1c} \frac{\partial}{\partial z_1} \left( \frac{1}{F^{j_0}_{1c}} \right)
\]

On the other hand

\[
F^{j_0}_{1c} F^{j_0}_{3c} \frac{\partial}{\partial z_1} \left( \frac{1}{F^{j_0}_{3c}} \right) = \frac{-F^{j_0}_{1c} \frac{\partial}{\partial z_1} F^{j_0}_{3c}}{F^{j_0}_{3c}}
\]

so \( a \) is a zero point of \( F^{j_0}_{1c} F^{j_0}_{3c} \frac{\partial}{\partial z_1} \left( \frac{1}{F^{j_0}_{3c}} \right) \) with multiplicity \( \geq p - 1 \). By applying the same argument also to all other combinations of indices, we see that \( a \) is a zero point of \( \Phi^1_c \) with multiplicity \( \geq p - 1 \).

**Case 2.** If \( v_{(f_1, H_{j_0})}(a) \leq p \), then \( p_0 := v_{(f_1, H_{j_0})}(a) = v_{(f_2, H_{j_0})}(a) = v_{(f_3, H_{j_0})}(a) \leq p \). There exists a neighborhood \( U \) of \( a \) such that \( v_{(f_1, H_{j_0})} \leq p \) on \( U \). Indeed, otherwise there exist a sequence \( \{a_s\}_{s=1}^\infty \subset \mathbb{C}^m, \lim_{s \to \infty} a_s = a \) and
For each $j \geq f$ for all $a \in U$ no zero point on $v$. We can choose $U$ such that $U \cap \{a \in U \}$ exists a holomorphic function $h$ on $U$ such that $dh$ has no zero point and $F_{3c} = u_i$ on $U$, where $u_i(i = 1, 2, 3)$ are nowhere vanishing holomorphic functions on $U$ (note that $v_{F_{3c}}(a) = v_{F_{3c}}(a) = p_0$). We have

$$
\Phi_1 = e^{u_1 \frac{\partial}{\partial z_1} u_2 - u_2 \frac{\partial}{\partial z_1} u_3} + u_2 \left( u_1 \frac{\partial}{\partial z_1} u_3 - u_3 \frac{\partial}{\partial z_1} u_1 \right) u_3 u_1 + u_3 \left( u_2 \frac{\partial}{\partial z_1} u_1 - u_1 \frac{\partial}{\partial z_1} u_2 \right) u_1 u_2.
$$

So, we have $a$ is a zero point of $\Phi_1$ with multiplicity $\geq p_0$ \hspace{1cm} (2)

By (1), (2) and our choice of $a$, there exists an analytic set $M \subset \mathbb{C}^m$ with codimension $\geq 2$ such that $v_{\Phi_1} \geq \min\{v_{(f_{i,H_{j}}), p-1}\}$ on

$$
\{ z : v_{(f_{i,H_{j}})}(z) > 0 \} \setminus (M \cup A). \hspace{1cm} (3)
$$

For each $j \in \{1, \ldots, q\} \setminus \{j_0\}$, let $a$ (depending on $j$) be an arbitrary point in $\mathbb{C}^m$ such that $v_{(f_{i,H_{j}})}(a) > 0$ (if there exist any). Then $v_{(f_{i,H_{j}})}(a) > 0$ for all $i \in \{1, 2, 3\}$, since $f_1, f_2, f_3 \in \mathcal{F}_k(\{H_j\}_{j=1}^q, f, 1)$. We can choose $a \notin f_i^{-1}(H_c) \cup f_i^{-1}(H_{j_0}), i = 1, 2, 3$. Then there exists a neighborhood $U$ of $a$ such that $v_{(f_{i,H_{j}})}(a) \leq k$ on $U$ and $(f_i, H_{j_0}), (f_i, H_c)$ ( $i = 1, 2, 3$ ) have no zero point on $U$. We have $B := f_1^{-1}(H_j) \cap U = f_2^{-1}(H_j) \cap U = f_3^{-1}(H_j) \cap U$ and

$$
\frac{1}{F_{j_0}^{1c}} = \frac{1}{F_{j_0}^{2c}} = \frac{1}{F_{j_0}^{3c}} = \frac{1}{B} \quad \text{on } B. \quad \text{Choose } a \text{ such that } a \text{ is a regular point of } B. \quad \text{By shrinking } U, \text{ we may assume that there exists a holomorphic function } h \text{ on } U \text{ such that } dh \text{ has no zero point and } U \cap \{h = 0\} = B. \text{ Then } \frac{1}{F_{j_0}^{1c}} - \frac{1}{F_{j_0}^{2c}} = h \varphi_2
$$

and

$$
\frac{1}{F_{j_0}^{3c}} - \frac{1}{F_{j_0}^{1c}} = h \varphi_3 \quad \text{on } U \text{ where } \varphi_2, \varphi_3 \text{ are holomorphic functions on } U.
Hence, we get
\[
\Phi^1_c = F_{1c}^{-j_0} F_{2c}^{j_0} F_{3c}^{j_0} \begin{vmatrix}
1 & 0 & 0 \\
\frac{\partial}{\partial z_1} \left( \frac{1}{F_{1c}^{j_0}} \right) & h \varphi_2 & h \varphi_3 \\
\frac{\partial}{\partial z_2} \varphi_2 & \varphi_3 & h \varphi_3 \\
\frac{\partial}{\partial z_3} \varphi_2 & \varphi_3 & h \varphi_3 \\
\end{vmatrix}.
\]
Hence, \(a\) is a zero point of \(\Phi^1_c\) with multiplicity \(\geq 2\). Thus, for each \(j \in \{1, \ldots, q\} \setminus \{j_0\}\), there exists an analytic set \(N \subset \mathbb{C}^m\) with codimension \(\geq 2\) such that \(v_{\Phi^1_c} \geq 2\) on
\[
\{ z : v_{(f_1, H_j)}(z) > 0 \} \setminus N. \tag{4}
\]
By (3) and (4), we have
\[
2 \sum_{j=1, j \neq j_0}^{q} N_{f_1}(r, H_j) + N_{p-1, f_1}(r, H_{j_0}) \leq N(r, v_{\Phi^1_c}) + (p-1)N(r, A). \tag{5}
\]
Similarly, we have
\[
2 \sum_{j=1, j \neq j_0}^{q} N_{f_i}(r, H_j) + N_{p-1, f_i}(r, H_{j_0}) \leq N(r, v_{\Phi^1_c}) + (p-1)N(r, A), \quad i = 1, 2, 3. \tag{5}
\]
Let \(a\) be an arbitrary zero point of some \(F_{1c}^{-j_0}\), \(a \notin f_i^{-1}(H_c)\), say \(i = 1\). We have
\[
\Phi^1_c = (F_{2c}^{j_0} - F_{3c}^{j_0}) F_{1c}^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{F_{1c}^{j_0}} \right) + (F_{2c}^{j_0} - F_{3c}^{j_0}) F_{2c}^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{F_{2c}^{j_0}} \right) \\
+ (F_{1c}^{j_0} - F_{2c}^{j_0}) F_{3c}^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{F_{3c}^{j_0}} \right). \tag{6}
\]
So we have
\[
v_{\frac{1}{\Phi^1_c}}(a) \leq 1 + \max \{ v_{\frac{1}{F_{1c}^{j_0}}} (a), i = 2, 3 \} \leq 1 + v_{\frac{1}{F_{2c}^{j_0}}} (a) + v_{\frac{1}{F_{3c}^{j_0}}} (a). \]
Furthermore, if $0 < v_{F_{i \in c}^{j \in 0}}(a) \leq k$ (and, hence, $v_{(f, H_{i \in c})}^k(a) > 0$) and $a \notin \mathcal{A}$, then by (3) we may assume that $v_{\frac{1}{\Phi_c}}(a) = 0$ (outside an analytic set of codimension $\geq 2$).

Let $a$ be an arbitrary pole of all $F_{i \in c}^{j \in 0}$, $i = 1, 2, 3$. By (6) we have

$$v_{\frac{1}{\Phi_c}}(a) \leq \max \{v_{\frac{1}{F_{i \in c}^{j \in 0}}}(a), i = 1, 2, 3\} + 1 < \sum_{i=1}^{3} v_{\frac{1}{F_{i \in c}^{j \in 0}}}(a) \leq k$$

It follows from (6) that a pole of $\Phi_c^1$ is a zero or a pole of some $F_{i \in c}^{j \in 0}$. Thus, by (6), (7) and (8), we have

$$N(r, v_{\frac{1}{\Phi_c}}) \leq \sum_{i=1}^{3} N(r, v_{\frac{1}{F_{i \in c}^{j \in 0}}}) + \sum_{i=1}^{3} \left( N(r, v_{F_{i \in c}^{j \in 0}}) - N(r, v_{F_{i \in c}^{j \in 0}}) \right) + 3N(r, A)$$

$$\leq \sum_{i=1}^{3} N(r, v_{\frac{1}{F_{i \in c}^{j \in 0}}}) + \frac{1}{k + 1} \sum_{i=1}^{3} N(r, v_{F_{i \in c}^{j \in 0}}) + 3N(r, A)$$

$$\leq \sum_{i=1}^{3} N(r, v_{\frac{1}{F_{i \in c}^{j \in 0}}}) + \frac{1}{k + 1} \sum_{i=1}^{3} T_{F_{i \in c}^{j \in 0}}(r) + 3N(r, A)$$

$$\leq \sum_{i=1}^{3} N(r, v_{\frac{1}{F_{i \in c}^{j \in 0}}}) + \frac{1}{k + 1} T(r) + 3N(r, A) + O(1).$$

We have

$$\Phi_c^1 = F_{i \in c}^{j \in 0} \left[ F_{j \in c}^{j \in 0} \frac{\partial}{\partial z_1} \left( \frac{1}{F_{j \in c}^{j \in 0}} \right) - F_{j \in c}^{j \in 0} \frac{\partial}{\partial z_1} \left( \frac{1}{F_{j \in c}^{j \in 0}} \right) \right]$$

$$+ F_{j \in 2}^{j \in 0} \left[ F_{j \in c}^{j \in 0} \frac{\partial}{\partial z_1} \left( \frac{1}{F_{j \in c}^{j \in 0}} \right) - F_{j \in c}^{j \in 0} \frac{\partial}{\partial z_1} \left( \frac{1}{F_{j \in c}^{j \in 0}} \right) \right]$$

so $m(r, \Phi_c^1) \leq \sum_{i=1}^{3} m(r, F_{i \in c}^{j \in 0}) + 2 \sum_{i=1}^{3} m \left( r, F_{i \in c}^{j \in 0} \frac{\partial}{\partial z_1} \left( \frac{1}{F_{i \in c}^{j \in 0}} \right) \right) + 0(1)$. By Theorem 2.1, we have

$$m \left( r, F_{i \in c}^{j \in 0} \frac{\partial}{\partial z_1} \left( \frac{1}{F_{j \in c}^{j \in 0}} \right) \right) = o(T_{F_{i \in c}^{j \in 0}}(r))$$

Thus, we get
\[ m(r, \Phi_c^1) \leq \sum_{i=1}^{3} m(r, F^o_{ic}) + o(T(r)), \quad (10) \]

(note that \( T_{F^o_{ic}}(r) \leq T_{f}(r) + O(1) \)).

By (9), (10) and by the First Main Theorem, we have

\[ N(r, v_{\Phi_c^1}) \leq T_{\Phi_c^1}(r) + O(1) = N(r, v_{\Phi_c^1}) + m(r, \Phi_c^1) + O(1) \]

\[ \leq \sum_{i=1}^{3} \left( N(r, v_{F^o_{ic}}) + m(r, F^o_{ic}) \right) + \frac{1}{k+1} T(r) + 3N(r, A) + o(T(r)) \]

\[ \leq \sum_{i=1}^{3} T_{F^o_{ic}}(r) + \frac{1}{k+1} T(r) + 3N(r, A) + o(T(r)) \]

\[ \leq \sum_{i=1}^{3} T_{f}(r) + \frac{1}{k+1} T(r) + 3N(r, A) + o(T(r)) \]

\[ = \frac{k+2}{k+1} T(r) + 3N(r, A) + o(T(r)). \quad (11) \]

By (5) and (11) we get Lemma 2. \( \square \)

The following lemma is a variant of the Second Main Theorem without taking account of multiplicities of order \( > k \) in the counting functions.

**Lemma 3.** Let \( f \) be a linearly nondegenerate meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{C}P^n \) and \( \{H_j\}_{j=1}^{q}(q \geq n+2) \) be hyperplanes in \( \mathbb{C}P^n \) in general position. Take a positive integer \( k \) with \( \frac{qn}{q-n-1} \leq k \leq +\infty \). Then

\[ T_f(r) \leq \frac{k}{(q-n-1)(k+1) - qn} \sum_{j=1}^{q} N_{n,f}(r, H_j) + o(T_f(r)) \]

\[ \leq \frac{nk}{(q-n-1)(k+1) - qn} \sum_{j=1}^{q} N_f(r, H_j) + o(T_f(r)) \]

for all \( r > 1 \) except a set \( E \) of finite Lebesgue measure.

**Proof.** By the First and the Second Main Theorems, we have

\[ (q-n-1)T_f(r) \leq \sum_{j=1}^{q} N_{n,f}(r, H_j) + o(T_f(r)) \]

\[ 12 \]
\[
\leq \frac{k}{k+1} \sum_{j=1}^{q} N_{n,f}^{k}(r, H_j) + \frac{n}{k+1} \sum_{j=1}^{q} N_{f}(r, H_j) + o(T_f(r))
\]
\[
\leq \frac{k}{k+1} \sum_{j=1}^{q} N_{n,f}^{k}(r, H_j) + \frac{qn}{k+1} T_f(r) + o(T_f(r)), \ r \notin E,
\]
which implies that
\[
\left( q - n - 1 - \frac{qn}{k+1} \right) T_f(r) \leq \frac{k}{k+1} \sum_{j=1}^{q} N_{n,f}^{k}(r, H_j) + o(T_f(r)).
\]

Thus, we have
\[
T_f(r) \leq \frac{k}{(q - n - 1)(k + 1) - qn} \sum_{j=1}^{q} N_{n,f}^{k}(r, H_j) + o(T_f(r))
\]
\[
\leq \frac{nk}{(q - n - 1)(k + 1) - qn} \sum_{j=1}^{q} N_{f}^{k}(r, H_j) + o(T_f(r)) \quad \square
\]

**Proof of Theorem 1.** Assume that there exist three distinct mappings \( f_1, f_2, f_3 \in F_k(\{H_j\}_{j=1}^{q}, f, p) \). Denote by \( Q \) the set which contains all indices \( j \in \{1, ..., q\} \) satisfying \( \Phi^l(F_{1c}^j, F_{2c}^j, F_{3c}^j) \neq 0 \) for some \( c \in C \) and some \( l \in \{1, ..., m\} \). We now prove that
\[
\#(\{1, ..., q\} \setminus Q) \geq 3n - 1. \tag{12}
\]
For the proof of (12) we distinguish three cases:

**Case 1.** \( 1 \leq n \leq 3, q = 3n + 1, p = 2, k \geq 23n \).

Suppose that (12) does not hold, then \( \#Q \geq 3 \). For each \( j_0 \in Q \), by Lemma 2 (with \( A = \emptyset, p = 2 \)) we have
\[
2 \sum_{j=1,j \neq i_0}^{q} N_{f_i}^{k}(r, H_j) + N_{f_i}^{k}(r, H_{j_0}) \leq \frac{k + 2}{k + 1} T(r) + o(T(r)), \ i = 1, 2, 3. \tag{13}
\]
By (13) and Lemma 3 we have
\[
\left( q - n - 1 - \frac{qn}{k+1} \right) T_{f_1}(r) \leq \frac{nk}{k+1} \sum_{j=1}^{q} N_{f_1}^k(r, H_j) + o(T_{f_1}(r))
\]
\[
\leq \frac{nk(k+2)}{2(k+1)^2} T(r) + \frac{nk}{2(k+1)} \sum_{j=1}^{n} N_{f_1}^k(r, H_{j_0}) + o(T(r)), \quad i = 1, 2, 3.
\]
Thus, we obtain
\[
\left( q - n - 1 - \frac{qn}{k+1} \right) T(r) \leq \frac{3nk(k+2)}{2(k+1)^2} T(r) + \frac{nk}{2(k+1)} \sum_{i=1}^{3} N_{f_i}^k(r, H_{j_0}) + o(T(r)),
\]
which implies
\[
\left[ 2(q - n - 1)(k+1)^2 - 2qn(k+1) - 3nk(k+2) \right] T(r)
\]
\[
\leq nk(k+1) \sum_{i=1}^{3} N_{f_i}^k(r, H_{j_0}) + o(T(r)) = 3nk(k+1)N_{f_i}^k(r, H_{j_0}) + o(T(r)) .
\]
Hence, we have
\[
\liminf_{r \to \infty \atop r \notin E} \frac{N_{f_i}^k(r, H_{j_0})}{T(r)} \geq \frac{2(q - n - 1)(k+1)^2 - 2qn(k+1) - 3nk(k+2)}{3nk(k+1)}
\]
\[
= \frac{k^2 - 6nk - 6n + 2}{3k(k+1)}, \quad i \in \{1, 2, 3\}.
\]

Set
\[
A_i := \{ r > 1 : T_{f_i}(r) = \min \{ T_{f_1}(r), T_{f_2}(r), T_{f_3}(r) \} \}, \quad i \in \{1, 2, 3\}.
\]
Then \( A_1 \cup A_2 \cup A_3 = (1, +\infty) \). Without loss of generality, we may assume that the Lebesgue measure of \( A_1 \) is infinite. By (14) we have
\[
\liminf_{r \to \infty \atop r \in A_1 \setminus E} \frac{N_{f_i}^k(r, H_{j_0})}{T_{f_i}(r)} \geq \frac{k^2 - 6nk - 6n + 2}{k(k+1)}, \quad j_0 \in Q.
\]
Take three distinct indices \( j_1, j_2, j_3 \in Q \) (note that \( \#Q \geq 3 \)). Then we have
\[
\liminf_{r \to \infty \atop r \in A_1 \setminus E} \frac{N_{f_1}^k(r, H_{j_1}) + N_{f_2}^k(r, H_{j_2}) + N_{f_3}^k(r, H_{j_3})}{T_{f_1}(r)} \geq \frac{3(k^2 - 6nk - 6n + 2)}{k(k+1)},
\]
which implies that
\[
\liminf_{r \to \infty} \frac{\sum_{j=1}^{q} N_{f_1}(r, H_j)}{T_{f_1}(r)} \geq \frac{3(k^2 - 6nk - 6n + 2)}{k(k + 1)}.
\]  

(15)

Since \( f_1 \neq f_2 \) there exists \( c \in C \) such that \( \frac{f_1}{f_2} \in \mathcal{C} \). Indeed, otherwise by Lemma 1 we have that \( \frac{f_1}{f_2} \) for all hyperplanes \( H \) in \( \mathbb{C}P^n \). In particular \( \frac{f_1}{f_2} \) for all \( j = 2, \ldots, n + 1 \). We choose homogeneous coordinates \( (\omega_0 : \cdots : \omega_n) \) on \( \mathbb{C}P^n \) with \( H_j = \{ \omega_j = 0 \} \) \( 1 \leq j \leq n + 1 \) and take reduced representations: \( f_1 = (f_{11} : \cdots : f_{n+1}), f_2 = (f_{21} : \cdots : f_{n+1}) \). Then

\[
\begin{cases}
\frac{f_{1j}}{f_{2j}} = \frac{f_{1j}}{f_{2j}} \\
\text{(j = 2, \ldots, n + 1)}
\end{cases}
\]

\( \Rightarrow \frac{f_1}{f_2} = \cdots = \frac{f_{n+1}}{f_{n+1}} \Rightarrow f_1 \equiv f_2. \)

This is a contradiction.

Since \( \dim (f_1^{-1}(H_1) \cap f_2^{-1}(H_c)) \leq m - 2 \) we have
\[
T_{(f_1,H_1)}(r) = \int_{S(r)} \log \left( |(f_1, H_1)|^2 + |(f_1, H_c)|^2 \right)^{\frac{1}{2}} \sigma + O(1)
\]

\[
\leq \int_{S(r)} \log \|f_i\| \sigma + O(1) = T_{f_i}(r) + O(1), \quad i = 1, 2, 3.
\]

Since \( f_1 = f_2 \) on \( \bigcup_{j=1}^{q} \{ z : v_{(f_1,H_j)}^{(k)}(z) > 0 \} \) and
\[
\dim \left\{ z : v_{(f_1,H_i)}^{(k)}(z) > 0 \text{ and } v_{(f_1,H_j)}^{(k)}(z) > 0 \right\} \leq m - 2 \text{ for all } i \neq j, \text{? we have}
\]
\[
\sum_{j=1}^{q} N_{f_1}(r, H_j) \leq N(r, v_{(f_1,H_1)}^{(k)}, v_{(f_1,H_c)}^{(k)}) \leq T_{(f_1,H_1)}(r) + 0(1)
\]

\[
\leq T_{(f_1,H_1)}(r) + T_{(f_2,H_1)}(r) + 0(1) \leq T_{f_1}(r) + T_{f_2}(r) + 0(1),
\]

which implies
\[
\liminf_{r \to \infty} \frac{T_{f_1}(r) + T_{f_2}(r)}{\sum_{j=1}^{q} N_{f_1}(r, H_j)} \geq 1.
\]

15
On the other hand, by Lemma 3, we have

\[
(q - n - 1 - \frac{qn}{k+1}) T_{f_i}(r) \leq \frac{nk}{k+1} \sum_{j=1}^{q} \overline{N}_{f_i}(r, H_j) + o(T_{f_i}(r))
\]

\[
= \frac{nk}{k+1} \sum_{j=1}^{q} \overline{N}_{f_i}(r, H_j) + o(T_{f_i}(r))
\]

which implies

\[
\limsup_{r \to \infty, r \notin E} \frac{T_{f_i}(r)}{\sum_{j=1}^{q} \overline{N}_{f_i}(r, H_j)} \leq \frac{nk}{(q - n - 1)(k+1) - qn}, \quad i = 1, 2, 3.
\]

Hence, we obtain

\[
\limsup_{r \to \infty, r \in A_1 \setminus E} \frac{T_{f_i}(r)}{\sum_{j=1}^{q} \overline{N}_{f_i}(r, H_j)} = \limsup_{r \to \infty, r \in A_1 \setminus E} \left( \frac{T_{f_1}(r) + T_{f_2}(r)}{\sum_{j=1}^{q} \overline{N}_{f_1}(r, H_j)} - \frac{T_{f_2}(r)}{\sum_{j=1}^{q} \overline{N}_{f_1}(r, H_j)} \right)
\]

\[
\geq \liminf_{r \to \infty, r \in A_1 \setminus E} \frac{T_{f_1}(r) + T_{f_2}(r)}{\sum_{j=1}^{q} \overline{N}_{f_1}(r, H_j)} - \limsup_{r \to \infty, r \in A_1 \setminus E} \frac{T_{f_2}(r)}{\sum_{j=1}^{q} \overline{N}_{f_1}(r, H_j)} \geq 1 - \frac{nk}{(q - n - 1)(k+1) - qn}
\]

Consequently, we get

\[
\liminf_{r \to \infty, r \in A_1 \setminus E} \frac{\sum_{j=1}^{q} \overline{N}_{f_1}(r, H_j)}{T_{f_1}(r)} \leq \frac{(q - n - 1)(k+1) - qn}{(q - n - 1)(k+1) - qn - nk}
\]

\[
= \frac{2k + 1 - 3n}{k + 1 - 3n}
\]

By (15) and (16) we have

\[
\frac{3(k^2 - 6nk - 6n + 2)}{k(k+1)} \leq \frac{2k + 1 - 3n}{k + 1 - 3n}
\]

This contradicts \( k \geq 23n \). Thus, we get (12) in this case.
Case 2. \(4 \leq n \leq 6, q = 3n, p = 2, k \geq \frac{(6n-1)n}{n-3}\).

Suppose that (12) does not hold, then there exists \(j_0 \in Q\). By Lemma 2 (with \(A = \emptyset, p = 2\)) we have

\[
2 \sum_{j=1, j \neq j_0}^{3n} N_{f_i}(r, H_j) + N_{f_i}(r, H_{j_0}) \leq \frac{k + 2}{k + 1}T(r) + o(T(r)), \quad i = 1, 2, 3.
\]

On the other hand, by Lemma 3 we have

\[
\sum_{j=1}^{3n} N_{f_i}(r, H_j) + o(T_f(r)) \geq \frac{2(n-2)(k+1) - (3n-1)n}{nk}T_f(r), \quad \text{and}
\]

\[
\sum_{j=1}^{3n} N_{f_i}(r, H_j) + o(T_f(r)) \geq \frac{(2n-1)(k+1) - 3n^2}{nk}T_f(r),
\]

which implies that

\[
2 \sum_{j=1, j \neq j_0}^{3n} N_{f_i}(r, H_j) + N_{f_i}(r, H_{j_0}) + o(T_f(r)) \geq \frac{(4n-3)(k+1) - (6n-1)n}{nk}T_f(r)
\]

Hence, we have

\[
\frac{(4n-3)(k+1) - (6n-1)n}{nk}T_f(r) \leq \frac{k + 2}{k + 1}T(r) + o(T(r)) \quad , \quad i = 1, 2, 3.
\]

Consequently, we get

\[
\frac{(4n-3)(k+1) - (6n-1)n}{nk}T(r) \leq \frac{3(k + 2)}{k + 1}T(r) + o(T(r)),
\]

which implies that

\[
((4n-3)(k+1) - (6n-1)n)T(r) \leq \frac{3nk(k + 2)}{k + 1}T(r) + o(T(r)) \leq 3n(k + 1)T(r) + o(T(r)).
\]

Hence, we obtain \(k + 1 \leq \frac{(6n-1)n}{n-3}\). This is a contradiction. Thus, we get (12) in this case.
Case 3. $n \geq 7, q = 3n - 1, p = 1, k \geq \frac{(6n-4) n}{n-6}$.
Suppose that (12) does not hold, then there exists $j_0 \in Q$. By Lemma 2 (with $A = \emptyset, p = 1$) we have

$$2 \sum_{j=1, j \neq j_0}^{3n-1} N_{f_i}^k(r, H_j) \leq \frac{k + 2}{k + 1} T(r) + o(T(r)), \ i = 1, 2, 3$$

(note that $N_{f_i}^k(r, H_{j_0}) = 0$). On the other hand, by Lemma 3, we have

$$2 \sum_{j=1, j \neq j_0}^{3n-1} N_{f_i}^k(r, H_j) + o(T_f(r)) \geq 2(2n - 3)(k + 1) - (3n - 2)n n k T_f(r)$$

Hence, we get

$$\frac{2[(2n - 3)(k + 1) - (3n - 2)n] n k}{n k} T_f(r) \leq \frac{k + 2}{k + 1} T(r) + o(T(r)),$$

which implies

$$(4n - 6)(k + 1) - (6n - 4)n T_f(r) \leq \frac{n k (k + 2)}{k + 1} T(r) + o(T(r)), \ i = 1, 2, 3.$$

Hence, we have

$$((4n - 6)(k + 1) - (6n - 4)n) T(r) \leq \frac{3nk(k + 2)}{k + 1} T(r) + o(T(r))$$

$$\leq 3n(k + 1) T(r) + o(T(r)).$$

Thus, we obtain

$$((4n - 6)(k + 1) - (6n - 4)n) \leq 3n(k + 1)$$

implying

$$k + 1 \leq \frac{(6n - 4)n}{n - 6},$$

which is a contradiction. Thus, we get (12) in this case.

So, for any case we have $ #(\{1, \ldots, q\} \setminus Q) \geq 3n - 1$. Without loss of generality, we may assume that $1, \ldots, 3n - 1 \notin Q$. We have

$$\Phi^l(F_{1c}^l, F_{2c}^l, F_{3c}^l) \equiv 0 \text{ for all } c \in \mathcal{C}, \ l \in \{1, \ldots, m\}, \ j \in \{1, \ldots, 3n - 1\}.$$

18
On the other hand, $\mathcal{C}$ is dense in $\mathbb{C}^{n+1}$. Hence, $\Phi'(F_{1c}^{j}, F_{2c}^{j}, F_{3c}^{j}) \equiv 0$ for all $c \in \mathbb{C}^{n+1} \setminus \{0\}$, $l \in \{1, \ldots, m\}$, $j \in \{1, \ldots, 3n-1\}$. In particular (for $H_c = H_l$) we have

$$\Phi' \left( \frac{(f_1, H_j)}{(f_1, H_i)}, \frac{(f_2, H_j)}{(f_2, H_i)}, \frac{(f_3, H_j)}{(f_3, H_i)} \right) \equiv 0$$

for all $1 \leq i \neq j \leq 3n-1$, $l \in \{1, \ldots, m\}$.

(17)

In the following we distinguish between the cases $n = 1$ and $n \geq 2$.

**Case 1.** If $n = 1$, then $a_j := H_j(j = 1, 2, 3, 4)$ are distinct points in $\mathbb{C}P^1$. We have that

$$g_1 := \frac{(f_1, a_1)}{(f_1, a_2)}, \quad g_2 := \frac{(f_2, a_1)}{(f_2, a_2)}, \quad g_3 := \frac{(f_3, a_1)}{(f_3, a_2)}$$

are distinct nonconstant meromorphic functions. By (17) and by Theorem 2.2, there exist constants $\alpha, \beta$ such that

$$g_2 = \alpha g_1, \quad g_3 = \beta g_1, \quad (\alpha, \beta \notin \{1, \infty, 0\}, \alpha \neq \beta) \quad (18)$$

We have $v_{(f_1, a_3)} \geq k+1$ on $\{z : (f_1, a_3)(z) = 0\}$: Indeed, otherwise there exists $z_0$ such that $0 < v_{(f_1, a_3)}(z_0) \leq k$. Then $v_{(f_1, a_3)}^{(k)}(z_0) > 0$, for all $i \in \{1, 2, 3\}$. We have $(f_1, a_3)(z_0) = (f_2, a_3)(z_0) = 0 \Rightarrow f_1(z_0) = f_2(z_0) = a_3^*$, where we denote $a_j^* := (a_{j_1} : -a_{j_0})$ for every point $a_j = (a_{j_0} : a_{j_1}) \in \mathbb{C}P^1$. So $g_1(z_0) = g_2(z_0) = \left(\frac{a_3^*, a_1}{a_3^*, a_2}\right) \neq 0$, $\infty$ (note that $a_3 \neq a_1, a_3 \neq a_2$). So, by (18) we have $\alpha = 1$. This is a contradiction. Thus $v_{(f_1, a_3)} \geq k + 1$ on $\{z : (f_1, a_3)(z) = 0\}$. Similarly, $v_{(f_i, a_j)} \geq k + 1$ on $\{z : (f_i, a_j)(z) = 0\}$ for $i \in \{1, 2, 3\}, j \in \{3, 4\}$.

Set $b_1 = \alpha \left(\frac{a_3^*, a_1}{a_3^*, a_2}\right)$, $b_2 = \alpha \left(\frac{a_3^*, a_1}{a_3^*, a_2}\right)$, $b_3 = \left(\frac{a_3^*, a_1}{a_3^*, a_2}\right)$. Then we have

$$v_{g_2-b_3} = v_{(f_2, a_3)}(a_3^*, a_2) \geq k + 1 \text{ on } \{z : (g_2 - b_3)(z) = 0\},$$

$$v_{g_2-b_1} = v_{g_2-\frac{1}{\alpha} b_1} = v_{(f_2, a_3)}(a_3^*, a_2) \geq k + 1 \text{ on } \{z : (g_2 - b_1)(z) = 0\},$$

$$v_{g_2-b_2} = v_{g_3-\frac{\beta}{\alpha} b_2} = v_{(f_3, a_3)}(a_3^*, a_2) \geq k + 1 \text{ on } \{z : (g_2 - b_2)(z) = 0\}.$$
Since the points $b_1, b_2, b_3$ are distinct, by the First and the Second Main Theorem, we have

$$T_{g_2}(r) \leq \sum_{j=1}^{3} N(r, v_{g_2-b_j}) + o(T_{g_2}(r))$$

$$\leq \frac{1}{k+1} \sum_{j=1}^{3} N(r, v_{g_2-b_j}) + o(T_{g_2}(r))$$

$$\leq \frac{3}{k+1} T_{g_2}(r) + o(T_{g_2}(r)).$$

This contradicts $k \geq 23$.

**Case 2.** If $n \geq 2$, for each $1 \leq i \neq j \leq 3n - 1$, by (17) and Theorem 2.2., there exists a constant $\alpha_{ij}$ such that

$$\frac{(f_2, H_j)}{(f_2, H_i)} = \alpha_{ij} \frac{(f_1, H_j)}{(f_1, H_i)} \quad \text{or} \quad \frac{(f_3, H_j)}{(f_3, H_i)} = \alpha_{ij} \frac{(f_1, H_j)}{(f_1, H_i)} \quad \text{or} \quad \frac{(f_3, H_j)}{(f_3, H_i)} = \alpha_{ij} \frac{(f_2, H_j)}{(f_2, H_i)} \quad (19)$$

We now prove that $\alpha_{ij} = 1$ for all $1 \leq i \neq j \leq 3n - 1$. Indeed, if there exists $\alpha_{i_0 j_0} \neq 1$, without loss of generality, we may assume that $\frac{(f_2, H_{j_0})}{(f_2, H_{i_0})} = \alpha_{i_0 j_0} \frac{(f_1, H_{j_0})}{(f_1, H_{i_0})}$. On the other hand $f_1 = f_2$ on $\Omega := \bigcup_{j=1}^{q} \{z : v_{(f_1, H_j)}(z) > 0\}$. Hence, $(f_1, H_{j_0}) = (f_2, H_{j_0}) = 0$ on $\Omega \setminus f_1^{-1}(H_{i_0})$. So we have

$$\sum_{j=1,j \neq i_0}^{q} N_{f_1}^{k}(r, H_j) \leq N(r, v_{(f_1, H_{i_0})}) + \left( N(r, v_{(f_1, H_{i_0})}) - N_{f_1}^{k}(r, v_{(f_1, H_{i_0})}) \right).$$

Thus, by the First and the Second Main Theorem, we have

$$(q - n - 2)T_{f_1}(r) \leq \sum_{j=1,j \neq i_0}^{q} N_{n,f_1}(r, H_j) + o(T_{f_1}(r))$$

$$\leq n \sum_{j=1,j \neq i_0}^{q} N_{1,f_1}(r, H_j) + o(T_{f_1}(r))$$

20
\[
\leq \frac{n k}{k + 1} \sum_{j=1,j \neq i_0}^q N_{f_1}^k(r, H_j) + \frac{n}{k + 1} \sum_{j=1,j \neq i_0}^q N_{f_1}(r, H_j) + o(T_{f_1}(r))
\]

\[
\leq \frac{n k}{k + 1} N(r, v_{(f_1, u_{i_0})}) + \frac{n k}{k + 1} \left( N(r, v_{(f_1, H_{i_0})}) - N^k(r, v_{(f_1, H_{i_0})}) \right) + \frac{(q - 1)n}{k + 1} T_{f_1}(r) + o(T_{f_1}(r))
\]

\[
\leq \frac{n k}{k + 1} T_{(f_1, u_{i_0})}(r) + \frac{n k}{(k + 1)^2} N_{f_1}(r, H_{i_0}) + \frac{(q - 1)n}{k + 1} T_{f_1}(r) + o(T_{f_1}(r))
\]

\[
\leq \left( \frac{n k}{k + 1} + \frac{n k}{(k + 1)^2} + \frac{(q - 1)n}{k + 1} \right) T_{f_1}(r) + o(T_{f_1}(r))
\]

Thus, we get \((q - n - 2) \leq \frac{n k}{k + 1} + \frac{n k}{(k + 1)^2} + \frac{(q - 1)n}{k + 1} \leq n + \frac{q n}{k}\). This contradicts any of the following cases:

i) \(2 \leq n \leq 3, q = 3n + 1\) and \(k \geq 23n\),

ii) \(4 \leq n \leq 6, q = 3n\) and \(k \geq \frac{(6n - 1)n}{n - 3}\),

iii) \(n \geq 7, q = 3n - 1\) and \(k \geq \frac{(6n - 4)n}{n - 6}\).

Thus \(\alpha_{ij} = 1\) for all \(1 \leq i \neq j \leq 3n - 1\).

By (19), for \(i = 3n - 1, j \in \{1, \ldots, 3n - 2\}\), without loss of generality, we may assume that

\[
\frac{(f_1, H_j)}{(f_1, H_{3n-1})} = \frac{(f_2, H_j)}{(f_2, H_{3n-1})}, \quad j = 1, \ldots, n:
\]

(20)

For \(1 \leq s < v \leq 3\), denote by \(L_{sv}\) the set of all \(j \in \{1, \ldots, 3n - 2\}\) such that \(\frac{(f_s, H_j)}{(f_s, H_{3n-1})} = \frac{(f_v, H_j)}{(f_v, H_{3n-1})}\). By (19) we have \(L_{12} \cup L_{23} \cup L_{13} = \{1, \ldots, 3n - 2\}\). So by Dirichlet we have that one of the three sets contains at least \(n\) different indices, which are, without loss of generality, \(j = 1, \ldots, n\), which proves (20).

We choose homogeneous coordinates \((\omega_0 : \cdots : \omega_n)\) on \(\mathbb{C}P^n\) with \(H_j = \{\omega_j = 0\}\) \((1 \leq j \leq n)\), \(H_{3n-1} = \{\omega_0 = 0\}\) and take reduced representations: \(f_1 = (f_{i_0} : \cdots : f_{i_n})\), \(f_2 = (f_{j_0} : \cdots : f_{j_n})\). Then by (20) we have

\[
\begin{align*}
&\begin{cases} 
      f_{i_j} = f_{j_j} \\
      f_{i_{i_0}} = f_{j_{j_0}} \\
      f_{i_{j_0}} = f_{j_{j_0}} \\
    \end{cases} \\
&\implies \frac{f_{i_0}}{f_{j_0}} = \cdots = \frac{f_{i_n}}{f_{j_n}} \implies f_1 \equiv f_2.
\end{align*}
\]
This is a contradiction. Thus, for any case we have that \( f_1, f_2, f_3 \) can not be distinct. Hence, the Proof of Theorem 1 is complete. \( \square \)

**Proof of Theorem 2.** Assume that \#\( F_k(\{H_j\}_{j=1}^q, f, 1) \geq 3 \). Take arbitrarily three distinct mappings \( f_1, f_2, f_3 \in F_k(\{H_j\}_{j=1}^q, f, 1) \). We have to prove that \( f_s \times f_v : C^m \longrightarrow C^{P^n} \times C^{P^n} \) is linearly degenerate for all \( 1 \leq s < v \leq 3 \).

Denote by \( Q \) the set which contains all indices \( j \in \{1, \ldots, q\} \) satisfying \( \Phi^j(F_{1c}, F_{2c}, F_{3c}) \neq 0 \) for some \( c \in C \). We distinguish between the two cases \( n \) odd and \( n \) even:

**Case 1.** If \( n \) is odd, then \( q = \frac{5(n+1)}{2} \).

We now prove that: \( Q = \emptyset \). \( (21) \)

Indeed, otherwise there exist \( j_0 \in Q \). Then by Lemma 2 (with \( A = \emptyset, p = 1 \)) we have

\[
2 \sum_{j=1, j \neq j_0}^q N^k_{f_i}(r, H_j) \leq \frac{k + 2}{k + 1} T(r) + o(T(r)), \ i = 1, 2, 3.
\]

(note that \( N^k_{f_i}(r, H_{j_0}) = 0 \)). On the other hand, by Lemma 3 we have

\[
2 \sum_{j=1, j \neq j_0}^q N^k_{f_i}(r, H_j) + o(T_{f_i}(r)) \geq 2\frac{(q - n - 2)(k + 1) - (q - 1)n}{nk} T_{f_i}(r), \ i = 1, 2, 3.
\]

Hence, we have

\[
((2q - 2n - 4)(k + 1) - 2(q - 1)n) T_{f_i}(r) \leq \frac{(k + 2)nk}{k + 1} T(r) + o(T(r)), \ i = 1, 2, 3,
\]

which implies

\[
\left( (2q - 2n - 4)(k + 1) - 2(q - 1)n \right) T(r) \leq \frac{3(k + 2)nk}{k + 1} T(r) + o(T(r)) \leq 3n(k + 1) T(r) + o(T(r)).
\]

Hence, we obtain

\[(2q - 2n - 4)(k + 1) - 2(q - 1)n \leq 3n(k + 1)\]

implying

\[k + 1 \leq (5n + 3)n.\]

22
This is a contradiction. Thus, we get (21).

**Case 2.** If \( n \) is even, then \( q = \frac{5n+4}{2} \).

We now prove that \( \#Q \leq 1 \).  \hspace{1cm} (22)

Indeed, suppose that this assertion does not hold, then there exist two distinct indices \( j_0, j_1 \in Q \). By Lemma 2 (with \( A = \emptyset, p = 1 \)) we have

\[
2 \sum_{j=1,j \neq j_0}^{q} \overline{N}_{f_i}^k(r, H_j) \leq \frac{k + 2}{k + 1} T(r) + o(T(r)) \hspace{1cm}, \hspace{0.5cm} i = 1, 2, 3,
\]

which implies that, for \( i=1,2,3 \)

\[
2 \sum_{j=1,j \neq j_0}^{q} \left( \overline{N}_{f_i}^k(r, H_j) - \frac{1}{n} N_{n,f_i}^k(r, H_j) \right) \leq \frac{k + 2}{k + 1} T(r) + o(T(r))
\]

\[
- \frac{2}{n} \sum_{j=1,j \neq j_0}^{q} N_{n,f_i}^k(r, H_j), \hspace{0.5cm} i = 1, 2, 3.
\]

Hence, we get

\[
2 \sum_{j=1,j \neq j_0}^{q} \sum_{i=1}^{3} \left( \overline{N}_{f_i}^k(r, H_j) - \frac{1}{n} N_{n,f_i}^k(r, H_j) \right) \leq \frac{3(k + 2)}{k + 1} T(r) + o(T(r))
\]

\[
- \frac{2}{n} \sum_{j=1,j \neq j_0}^{q} \sum_{i=1}^{3} N_{n,f_i}^k(r, H_j), \hspace{0.5cm} (23)
\]

By Lemma 3 (with \( q = \frac{5n+4}{2} \)), we have

\[
2 \sum_{j=1,j \neq j_0}^{q} N_{n,f_i}^k(r, H_j) + o(T_{f_i}(r)) \geq \frac{3n(k + 1) - (5n + 2)n}{k} T_{f_i}(r), \hspace{0.5cm} i = 1, 2, 3.
\]

Hence, we have

\[
\frac{2}{n} \sum_{j=1,j \neq j_0}^{q} \sum_{i=1}^{3} N_{n,f_i}^k(r, H_j) + o(T(r)) \geq \frac{3n(k + 1) - (5n + 2)n}{nk} T(r) \hspace{0.5cm} (24)
\]

By (23) and (24) we have

\[
2 \sum_{j=1,j \neq j_0}^{q} \sum_{i=1}^{3} \left( \overline{N}_{f_i}^k(r, H_j) - \frac{1}{n} N_{n,f_i}^k(r, H_j) \right) \leq \frac{(5n + 2)n(k + 1) - 3n}{nk(k + 1)} T(r) + o(T(r))
\]

\[
\leq \frac{5n + 2}{k} T(r) + o(T(r)).
\]
On the other hand, we obtain
\[
N_{f_i}^k(r, H_j) - \frac{1}{n} N_{n, f_i}^k(r, H_j) \geq 0 \text{ for all } i \in \{1, 2, 3\}, j \in \{1, ..., q\}.
\]
Hence, we get
\[
\sum_{i=1}^{3} \left( N_{f_i}^k(r, H_j) - \frac{1}{n} N_{n, f_i}^k(r, H_j) \right) \leq \frac{5n + 2}{k} T(r) + o(T(r)), \quad j \in \{1, ..., q\} \setminus \{j_0\}.
\]
In particular, we get
\[
\sum_{i=1}^{3} \left( N_{f_i}^k(r, H_{j_1}) - \frac{1}{n} N_{n, f_i}^k(r, H_{j_1}) \right) \leq \frac{5n + 2}{k} T(r) + o(T(r)) \quad (25)
\]
Set \(A_i := \{z \in \mathbb{C}^m : v_{(f_i, H_{j_1})}(z) = 1\} \) for \(i = 1, 2, 3\). For each \(i \in \{1, 2, 3\}\), we have \(\overline{A_i} \setminus A_i \subseteq \text{sing} f_i^{-1}(H_{j_1})\). Indeed, otherwise there existed \(a \in (\overline{A_i} \setminus A_i) \cap \text{reg} f_i^{-1}(H_{j_1})\). Then \(p_0 := v_{(f_i, H_{j_1})}(a) \geq 2\). Since \(a\) is a regular point of \(f_i^{-1}(H_{j_1})\) we can choose nonzero holomorphic functions \(h\) and \(u\) on a neighborhood \(U\) of \(a\) such that \(dh\) and \(u\) have no zeroes and \((f_i, H_{j_1}) \equiv h p_0 u\) on \(U\). Since \(a \in \overline{A_i}\) there exists \(b \in A_i \cap U\). Then, we get \(1 = v_{(f_i, H_{j_1})}(b) = v_{h p_0 u}(b) = p_0 \geq 2\). This is a contradiction. Thus, we get that \(\overline{A_i} \setminus A_i \subseteq \text{sing} f_i^{-1}(H_{j_1})\).

Set \(B := A_1 \cup A_2 \cup A_3\). Then \(\overline{B} \setminus B \subseteq \bigcup_{i=1}^{3} \text{sing} f_i^{-1}(H_{j_1})\). This means that \(\overline{B} \setminus B\) is included in an analytic set of codimension \(\geq 2\). So we have
\[
(n - 1) \overline{N}(r, \overline{B}) \leq \sum_{i=1}^{3} \left( n \overline{N}_{f_i}^k(r, H_{j_1}) - \overline{N}_{n, f_i}^k(r, H_{j_1}) \right).
\]
By (25) we have
\[
\overline{N}(r, \overline{B}) \leq \frac{(5n + 2)n}{(n - 1)k} T(r) + o(T(r)),
\]
where we note that \(n \geq 2\), since \(n\) is even. It is clear that \(\min\{v_{(f_1, H_{j_1})}, 2\} = \min\{v_{(f_2, H_{j_1})}, 2\} = \min\{v_{(f_3, H_{j_1})}, 2\}\) on \(\mathbb{C}^m \setminus \overline{B} \subseteq \mathbb{C}^m \setminus B\).
By Lemma 2 (with $\overline{A} = \overline{B}$, $p = 2$) we have

$$2 \sum_{j=1, j \neq j_1}^{q} \overline{N}^k_{f_i}(r, H_j) + \overline{N}^k_{f_i}(r, H_{j_1}) \leq \frac{k + 2}{k + 1} T(r) + 4\overline{N}(r, \overline{B}) + o(T(r))$$

$$\leq \left( \frac{k + 2}{k + 1} + \frac{4(5n + 2)n}{(n - 1)k} \right) T(r) + o(T(r)) , \ (26)$$

(note that $j_1 \in Q$). By Lemma 3 we have

$$\sum_{j=1, j \neq j_1}^{q} \overline{N}^k_{f_i}(r, H_j) + o(T_{f_i}(r)) \geq \frac{(q - n - 2)(k + 1) - (q - 1)n}{nk} T_{f_i}(r) , \text{ and}$$

$$\sum_{j=1}^{q} \overline{N}^k_{f_i}(r, H_j) + o(T_{f_i}(r)) \geq \frac{(q - n - 1)(k + 1) - qn}{nk} T_{f_i}(r) .$$

Consequently, we obtain

$$2 \sum_{j=1, j \neq j_1}^{q} \overline{N}^k_{f_i}(r, H_j) + \overline{N}^k_{f_i}(r, H_{j_1}) + o(T_{f_i}(r)) \geq \frac{(2q - 2n - 3)(k + 1) - (2q - 1)n}{nk} T_{f_i}(r)$$

$$\geq (2q - 2n - 3)(k + 1) - (2q - 1)n T_{f_i}(r), \ (27)$$

By (26) and (27) we have

$$\frac{(2q - 2n - 3)(k + 1) - (2q - 1)n}{nk} T_{f_i}(r) \leq \left( \frac{k + 2}{k + 1} + \frac{4(5n + 2)n}{(n - 1)k} \right) T(r) + o(T(r)) ,$$

which implies

$$((3n + 1)(k + 1) - (5n + 3)n) T(r) \leq \left( \frac{3nk(k + 2)}{k + 1} + \frac{12(5n + 2)n^2}{(n - 1)} \right) T(r) + o(T(r))$$

$$\leq (3n(k + 1) + \frac{12(5n + 2)n^2}{(n - 1)}) T(r) + o(T(r)) ,$$

and, hence,

$$k + 1 \leq (5n + 3)n + \frac{12(5n + 2)n^2}{(n - 1)} .$$

This contradicts $k \geq (65n + 171)n , \ n \geq 2$. Hence, we have $\#Q \leq 1$. So we get (22).
By (21) and (22) we have \( \#(\{1, \ldots, q\} \setminus Q) \geq q - 1 \). Without loss of generality we may assume that \( 1, \ldots, q-1 \notin Q \). For any \( j \in \{1, \ldots, q-1\} \) we have

\[
\Phi^l(F_{1c}^j, F_{2c}^j, F_{3c}^j) \equiv 0 \quad \text{for all } c \in C, \ l \in \{1, \ldots, m\}.
\]

On the other hand, \( C \) is dense in \( \mathbb{C}^{n+1} \). Hence, we get that \( \Phi^l(F_{1c}^j, F_{2c}^j, F_{3c}^j) \equiv 0 \) for all \( c \in \mathbb{C}^{n+1} \setminus \{0\}, \ l \in \{1, \ldots, m\}, \ j \in \{1, \ldots, q-1\} \). In particular (for \( H_c = H_i \)), we get

\[
\Phi^l\left(\frac{(f_1, H_j)}{(f_1, H_i)}, \frac{(f_2, H_j)}{(f_2, H_i)}, \frac{(f_3, H_j)}{(f_3, H_i)}\right) \equiv 0
\]

for all \( 1 \leq i \neq j \leq q-1, \ l \in \{1, \ldots, m\} \).

For each \( 1 \leq i \neq j \leq q - 1 \), by Theorem 2.2, there exists a constant \( \alpha_{ij} \) such that

\[
\frac{(f_2, H_j)}{(f_2, H_i)} = \alpha_{ij} \frac{(f_1, H_j)}{(f_1, H_i)} \quad \text{or} \quad \frac{(f_3, H_j)}{(f_3, H_i)} = \alpha_{ij} \frac{(f_1, H_j)}{(f_1, H_i)} \quad \text{or} \quad \frac{(f_3, H_j)}{(f_3, H_i)} = \alpha_{ij} \frac{(f_2, H_j)}{(f_2, H_i)}.
\]

We now prove that

\[
\alpha_{ij} = 1 \quad \text{for all } 1 \leq i \neq j \leq q - 1.
\]

Indeed, if there exists \( \alpha_{i_0j_0} \neq 1 \), without loss of generality, we may assume that \( \frac{(f_2, H_{j_0})}{(f_2, H_{i_0})} = \alpha_{i_0j_0} \frac{(f_1, H_{j_0})}{(f_1, H_{i_0})} \). On the other hand, we have \( f_1 = f_2 \) on \( D := \bigcup_{j=1}^q \{z : v^k_{(f_1, H_j)} > 0\} \). Hence, we get \( (f_1, H_{j_0}) = (f_2, H_{j_0}) = 0 \) on \( D \setminus f_1^{-1}(H_{i_0}) \). So we have

\[
\sum_{j=1, j \neq i_0}^q \mathbb{N}_{f_1}^{(k)}(r, H_j) \leq N\left(r, v_{(f_1, H_{j_0})}\right) + \left(\mathbb{N}(r, v_{(f_1, H_{i_0})}) - \mathbb{N}^{(k)}(r, v_{(f_1, H_{i_0})})\right).
\]

Thus, by the First and the Second Main Theorem, we have

\[
(q - n - 2)T_{f_1}(r) \leq \sum_{j=1, j \neq i_0}^q N_{n, f_1}(r, H_j) + o(T_{f_1}(r))
\]

\[
\leq n \sum_{j=1, j \neq i_0}^q N_{1, f_1}(r, H_j) + o(T_{f_1}(r))
\]
\[
\frac{nk}{k+1} + \sum_{j=1, j \neq i_0}^{q} \mathcal{N}^{(k)}_{f_1}(r, H_j) + \frac{n}{k+1} \sum_{j=1, j \neq i_0}^{q} N_{f_1}(r, H_j) + o(T_{f_1}(r))
\]

\[
\leq \frac{nk}{k+1} + \frac{nk}{(k+1)^2} N_{f_1}(r, H_{i_0}) + \frac{(q-1)n}{k+1} T_{f_1}(r) + o(T_{f_1}(r))
\]

Thus, we have \((q - n - 2) \leq \frac{nk}{k+1} + \frac{nk}{(k+1)^2} + \frac{(q-1)n}{k+1} \leq n + \frac{nq}{k} \).

This contradicts \(q = \left\lfloor \frac{5(n+1)}{2} \right\rfloor, k \geq (65n + 171)n \). Thus, we get that \(\alpha_{ij} = 1 \)
for all \(1 \leq i \neq j \leq q - 1 \).

For \(1 \leq s < v \leq 3 \), denote by \(L_{sv} \) the set of all \( j \in \{1, ..., q-2 \} \) such that
\[
\frac{(f_s, H_j)}{(f_v, H_{q-1})} = \frac{(f_s, H_j)}{(f_v, H_{q-1})} .
\]

By (28), we have that \(L_{12} \cup L_{23} \cup L_{13} = \{1, ..., q-2 \} \).

If there exists some \(L_{sv} = \emptyset \), without loss of generality, we may assume
that \(L_{13} = \emptyset \). Then \(L_{12} \cup L_{23} = \{1, ..., q-2 \} \). Since \(q = \left\lfloor \frac{5(n+1)}{2} \right\rfloor \) we have that
\(\#L_{12} \geq n \) or \(\#L_{23} \geq n \). We may assume that \(\#L_{12} \geq n \), and furthermore
\(1, ..., n \in L_{12} \). Then \(\frac{(f_{s}, H_{j})}{(f_{v}, H_{q-1})} = \frac{(f_{s}, H_{j})}{(f_{v}, H_{q-1})} \) for all \( j \in \{1, ..., n \} \), so \(f_1 \equiv f_2 \) (as in the proof of Theorem 1). This is a contradiction.

Thus, we have \(L_{sv} \neq \emptyset \) for all \(1 \leq s < v \leq 3 \). Then for any \(1 \leq s < v \leq 3 \),
there exists \( j \in \{1, ..., q-2 \} \) such that \(\frac{(f_s, H_i)}{(f_v, H_{q-1})} = \frac{(f_s, H_i)}{(f_v, H_{q-1})} \). Hence, we finally get that \(f_s \times f_v : \mathbb{C}^n \to \mathbb{C}P^n \times \mathbb{C}P^n \) is linearly degenerate. We thus have completed the proof of Theorem 2. \(\square\)

References

[1] H. Cartan, *Un nouveau théorème d’unicité relatif aux fonctions méromorphes*, C. R. Acad. Sci. Paris **188** (1929), 301-330.
[2] H. Fujimoto, *The uniqueness problem of meromorphic maps into the complex projective space*, Nagoya Math. J. **58** (1975), 1-23.

[3] H. Fujimoto, *Nonintegrated defect relation for meromorphic maps of complete Kähler manifolds into $P^{N_1}(\mathbb{C}) \times \ldots \times P^{N_k}(\mathbb{C})$*, Japan. J. Math. **11** (1985), 233-264.

[4] H. Fujimoto, *Uniqueness problem with truncated multiplicities in value distribution theory*, Nagoya Math. J. **152** (1998), 131-152.

[5] S. Ji, *Uniqueness problem without multiplicities in value distribution theory*, Pacific J. Math. **135** (1988), 323-348.

[6] R. Nevanlinna, *Einige Eideutigkeitssätze in der Theorie der meromorphen Funktionen*, Acta. Math. **48** (1926), 367-391.

[7] J. Noguchi, T. Ochiai, *Geometric Function Theory in Several Complex Variables*, Translations of Mathematical Monographs **80** American Mathematical Society, Providence, RI 1990.

[8] B. Shiffman, *Introduction to the Carlson-Griffiths equidistribution theory*, Lecture Note in Math. **981**, Springer–Verlag, 1983.

[9] L. Smiley, *Geometric conditions for unicity of holomorphic curves*, Contemp. Math. **25** (1983), pp. 149-154.

[10] N. Steinmetz, *A uniqueness theorem for three meromorphic functions*, Ann. Acad. Sci. Fenn. **13** (1988), 93-110.

[11] G. Dethloff, Tran Van Tan, *Uniqueness problem for meromorphic mappings with truncated multiplicities and moving targets*, to appear in Nagoya Math. J.

Gerd Dethloff
Université de Bretagne Occidentale
UFR Sciences et Techniques
Département de Mathématiques
6, avenue Le Gorgeu, BP 452
29275 Brest Cedex, France
e-mail: gerd.dethloff@univ-brest.fr
Tran Van Tan
Department of Mathematics
Hanoi University of Education
Cau Giay, Hanoi, Vietnam
e-mail: tranvantanhn@yahoo.com