Abstract

In this paper we provide a procedure to obtain a non-trivial HHS structure on a hyperbolic space. In particular, we prove that given a finite collection $\mathcal{F}$ of quasi-convex subgroups of a hyperbolic group $G$, there is an HHG structure on $G$ that is compatible with $\mathcal{F}$. We will use this to provide explicit descriptions of the Gromov Boundary of hyperbolic HHS and HHG, and we recover results from Hamenstädt, Manning, Trang for the case when $G$ is hyperbolic relative to $\mathcal{F}$. Further applications in the construction of new HHG will be presented in a subsequent paper.

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1 Introduction

The study of hierarchically hyperbolic spaces and groups (respectively HHS and HHG) was introduced by Behrstock, Hagen and Sisto in [BHS17c], and led to a number of remarkable results explored, for instance, in [BHS15, BHS17a, BHS17b, DHS17, HSS16, Osi16]. Hierarchically hyperbolic spaces and groups form a very large class of spaces which contains several examples of interest including mapping class groups, right-angled Artin groups, proper cocompact CAT(0) cube complexes, most 3-manifold groups, Teichmüller space (in any of the standard metrics), etc. Important properties of HHG include that, for example, any HHG admits an acylindrical action on a hyperbolic space that depends on the HHG structure ([BHS17c]). This recovers the result that Mapping Class Groups of non-sporadic surfaces, and non-cyclic, non directly indecomposable RAAGs are acylindrically hyperbolic ([MM99, Bow12, Osi06]). Another remarkable result that was unknown before is that, under very mild conditions, every top-dimensional quasi-flat in an HHS lies within finite distance of a union of standard orthants. This proves open conjectures of Farb in the case of Mapping Class Group, and Brock in the case of Teichmüller space.

Work of Bowditch and Osin ([Bow12, Osi06]) shows that given a hyperbolic group $G$ and a collection $\mathcal{F}$ of proper subgroups, $G$ is hyperbolic relative to $\mathcal{F}$ if and only if $\mathcal{F}$ is an almost malnormal collection of quasi-convex subgroups.

The extra structure on $G$ given by the family $\mathcal{F}$ provides useful information on $G$. An example of this can be found in the Appendix of [Ago13], where Dehn fillings in hyperbolic relatively hyperbolic groups are used in a step of the proof of the virtual Haken conjecture. Moreover, independent works of Manning and Trang ([Man15, Tra13]) provide an explicit decomposition of the boundary of $G$ in terms of $\mathcal{F}$. We will elaborate on this result later.

The natural question that arises is: What happens if the malnormality condition is weakened (or removed)? The theory of hierarchically hyperbolic spaces provides some answers in these cases. We recall that an HHS structure on a quasi-geodesic space $X$ consists of a collection of hyperbolic spaces $\{CU | U \in \mathcal{S}\}$ indexed by a set $\mathcal{S}$, and projections that relate them, satisfying some axioms and relations (see Section 3 for more details). An HHG is a group that acts on an HHS in a compatible way. The key example is when a Cayley graph of a group $G$ admits an HHS structure and $G$ preserves the set $\{CU\}$.

It is showed in [BHS15] that a space $G$ which is hyperbolic relative to a (uniform) family $\mathcal{F}$ of HHS is itself an HHS. In particular, the result of Bowditch implies that given an almost malnormal family $\mathcal{F}$ of quasi-convex subgroups of $G$, the family $\mathcal{F}$ provides an HHS structure on the Cayley graph of $G$, where each element $CU$ is either a coset of an element of $\mathcal{F}$, or the cone-off of $G$ with respect to $\mathcal{F}$.

In this paper, we show that the malnormality condition can be completely dropped if we are looking for an HHS structure.

Theorem 1 ([5,24]). Let $G$ be a hyperbolic group and let $\mathcal{F} = \{F_1, \ldots, F_N\}$ be a finite family of infinite quasi-convex subgroups. Let $\sim$ be the equivalence relation between subset of $G$ given by having finite Hausdorff distance in Cay($G$). Then there exists a finite family of quasi-convex subgroups $\mathcal{F}'$ that contains $\mathcal{F}$ such that if $\mathcal{F}'_{\text{cos}}$ is the set of cosets of $\mathcal{F}'$, then $(G, \mathcal{F}'_{\text{cos}}/\sim)$ is a hierarchically hyperbolic structure on $G$, and each $CU$ is a cone-off of a coset of an element $F \in (\mathcal{F}' \cup G)$.

The family $\mathcal{F}'$ of the Theorem is obtained from the family $\mathcal{F}$ by considering an appropriate set of intersections of conjugates of the elements of $\mathcal{F}$.

The key ingredient in the above result is the concept of weak factor system. A weak factor system consists of a family $\mathcal{F}$ of uniformly quasi-convex subspaces such that there exist constants $c, q, \xi, B, D$ such that the following hold:

1. Every chain of proper coarse inclusions $H_n \preceq \cdots \preceq H_1$ of elements of $\mathcal{F}$ has length at most $c$.
2. Given $V, W \in \mathcal{F}$, then either $\text{diam}_V(p_V(W)) < \xi$, or there exists $U \in \mathcal{F}$ such that $d_{\text{Haus}}(U, p_V(W)) \leq B$.


3. For each $V \in \mathcal{F}$ and every $v \in V$ there is an arbitrarily long $q$-quasi-geodesic segment with endpoints on $V$ whose midpoint lies at distance at most $D$ from $v$.

The first two condition can be reinterpreted as: the closure process given by coarse projection terminates, and the third is always true in an infinite quasi-convex subgroup of a hyperbolic group.

The first three sections of this paper will investigate the geometry of cone-offs of hyperbolic spaces. If $\hat{\mathcal{X}}$ is the cone-off of $\mathcal{X}$ with respect to a family $\mathcal{F}$, given a path $\hat{\gamma}$ of $\hat{\mathcal{X}}$, we can obtain a path $\gamma$ of $\mathcal{X}$ substituting each connected component of $\gamma - \mathcal{X}$ with a geodesic of $\mathcal{X}$ connecting the endpoints of such connected component. Such a path $\gamma$ is called a \textit{de-electrification} of $\hat{\gamma}$.

We show (2.28) that if the family $\mathcal{F}$ satisfies some mild hypotheses, which are implied by the fact that $\mathcal{F}$ consists of uniformly-quasi-convex subsets of $\mathcal{X}$, then for every pair of points $x, y$ of $\mathcal{X}$, there is a $\tau_1$-quasi-geodesic $\hat{\gamma}$ of $\hat{\mathcal{X}}$ connecting them, such that its de-electrification is a $\tau_2$-quasi-geodesic of $\mathcal{X}$. An analogous result was previously established by Hamenstäd in the case where $\mathcal{X}$ is hyperbolic relative to $\mathcal{F}$ (Ham16).

Using this result (although we will need it in full generality only in a subsequent paper), and other considerations, we will pin down a handful of properties that the family $\mathcal{F}$ needs to satisfy in order for $(\mathcal{X}, \mathcal{F})$ to be an HHS. In this case we say that $\mathcal{F}$ is a factor system.

In Section 4, we will show that such properties can be weakened to a weak-factor system, adding the extra cost of an equivalence relation, as can be seen in Theorem 1.

As an application, in the last section we provide an explicit description of the Gromov boundary for a hyperbolic HHS. It was proved in [DHS17] that for a hyperbolic HHS $(\mathcal{X}, \mathcal{F})$ the following holds:

$$\partial \mathcal{X} = \bigcup_{U \in \mathcal{F}} \partial C U,$$

where $\partial Z$ denotes the Gromov boundary of $Z$, for a hyperbolic space $\mathcal{X}$. Combining this with our construction, we get a very explicit description of the Gromov boundary in a variety of cases. Indeed, given a (weak) factor system $\mathcal{F}$ on a hyperbolic space $\mathcal{X}$, we can decompose the boundary of $\mathcal{X}$ as the union of boundaries of the various cone-offs with respect to elements of $\mathcal{F}$.

Theorem 2 (6.13). Let $G$ be a hyperbolic group and let $\mathcal{F}$ be a finite family of quasi-convex subgroups of $G$. Then there is a family $\mathcal{F}'$ such that

$$\partial G = \bigcup_{U \in \mathcal{F}} \partial C U,$$

where each $C U$ is as in Theorem 4.

This recovers two results of Hamenstäd (Ham16), namely the case when $\mathcal{X}$ is hyperbolic relative to $\mathcal{F}$, and the case of the disk graph of a handlebody. Indeed, the proof of the latter implies that the family of electrified disk graphs of a certain class of subsurfaces of the boundary handlebody is a factor system for the disk graph of the handlebody.

In the case where $\mathcal{X}$ is hyperbolic relative to $\mathcal{F}$, we also recover a result proved independently by Manning and Tran (respectively, [Man15, Theorem 1.3], [Tra13, Section 6]). Namely, in this case the Bowditch boundary of $(\mathcal{X}, \mathcal{F})$ can be expressed as the quotient

$$\partial (\mathcal{X}, \mathcal{F}) = \partial \mathcal{X}/\sim,$$

where $x \sim y$ if there is $U \in \mathcal{F}$ such that $x, y \in \partial C U$.

In a follow-up paper, we will use Theorem 4 to prove a combination Theorem for HHG. Indeed, the flexibility coming from the mild hypotheses on the family $\mathcal{F}$ allows to turn a large class of graphs of groups into graphs of HHG. Roughly speaking, given a graph $\mathcal{G}$ of HHG, we can add a new vertex $v$ with vertex group a hyperbolic group $G$. Then, the HHG structure on $G$ will be the one induced choosing $\mathcal{F}$ to be the set of images of the edge groups adjacent to $G$. As an application we get that if $H$ is a quasi-convex subgroup of a hyperbolic
group $G$, and if, moreover, $H$ is hyperbolically embedded in the Mapping Class Group of a surface $\Sigma$, then $G \ast_H \text{MCG}(\Sigma)$ is an HHG.

Outline

Subsections 2.1, 2.2 and 2.3 contain some background on hyperbolic spaces and approximating graphs, subsection 2.4 contains the bulk of the theory on the cone-off procedure that is going to be used in Sections 3 and 4. Section 3 contains the definition of factor system (3.1) and the proof of the fact that a factor system on a hyperbolic space determines a non-trivial HHS structure (Theorem 3.14). Section 4 provides some coarser conditions (4.2) that are sufficient to obtain an HHS structure (Theorem 4.7). In Section 5, we prove that a finite family of infinite quasi-convex subgroups induces a weak factor system for a hyperbolic group, providing a variety of new HHG structures on hyperbolic groups (Corollary 5.23). In Section 6, we use the HHS structure to give an explicit description of the Gromov boundary of $X$, where $X$ is a hyperbolic space equipped with a factor system (Proposition 6.11). We use this result to give an explicit description of the Bowditch boundary of a hyperbolic relative hyperbolic space (Theorem 6.17).

Acknowledgments

The author would like to thank Alessandro Sisto for suggesting the topic and for very helpful comments and suggestions.

2 Introductory tools

2.1 Approximating metric spaces

It is a well known fact that every quasi-geodesic metric space (Definition 2.3) is quasi-isometric to a geodesic one. Geodesic spaces are significantly better behaved than quasi-geodesic ones. In this subsection we will show how to approximate a quasi-geodesic space with a geodesic one.

Notation. Let $X$ be a metric space. For a subset $Y$ of $X$, we define $p_Y : X \to 2^Y$ to be the shortest distance projection from $X$ to $Y$.

Note that for a general subspace $Y$, the projection $p_Y(x)$ of a point may be empty. In order to avoid this, we will abuse notation and denote in the same way the $\varepsilon$-projection $p^\varepsilon_Y$ which is defined as $p^\varepsilon_Y(x) = \{ y \in Y \mid d(y, x) - d(Y, x) < \varepsilon \}$ for some very small $\varepsilon$. From now on, with an abuse, we will implicitly assume that all projections are, in fact, $\varepsilon$-projections.

Definition 2.1 (Quasi-isometric embedding, quasi-isometry). Let $(X, d_X)$ and $(Y, d_Y)$ be geodesic metric spaces. A $(C, \varepsilon)$-quasi-isometric embedding is a map $f : X \to Y$ such that for each pair of points $x, y \in X$, the following holds:

$$\frac{1}{C} d_Y(f(x), f(y)) - \varepsilon \leq d_X(x, y) \leq C d_Y(f(x), f(y)) + \varepsilon.$$

A $(C, \varepsilon)$ quasi-isometric embedding $f : X \to Y$ is a $(C, \varepsilon)$-quasi-isometry if $N_\varepsilon(f(X)) = Y$.

We say that a map $f$ is a quasi-isometric embedding (respectively quasi-isometry) if there exist $(C, \varepsilon)$ such that $f$ is a $(C, \varepsilon)$-quasi-isometric embedding (respectively $(C, \varepsilon)$-quasi-isometry).

Definition 2.2 (Quasi-geodesic). A $(C, \varepsilon)$-quasi-geodesic (respectively ray, segment) of a metric space $X$ is a $(C, \varepsilon)$-quasi-isometric embedding of $\mathbb{R}$ (respectively $[0, \infty)$, $[a, b]$ for some $a, b \in \mathbb{R}$). We say that a map $f$ is a quasi-geodesic if there exist $(C, \varepsilon)$ such that $f$ is a $(C, \varepsilon)$-quasi-geodesic.
Definition 2.3 (Quasi-geodesic metric space). A metric space $X$ is $(C,\varepsilon)$-quasi-geodesic if any two points can be joined by a $(C,\varepsilon)$-quasi-geodesic. A metric space $X$ is a quasi-geodesic metric space if there exists $(C,\varepsilon)$ such that $X$ is a $(C,\varepsilon)$-quasi-geodesic metric space.

Definition 2.4 (Maximal nets and approximation graphs). Let $X$ be a metric space, and let $\zeta>0$. A $\zeta$-net for $X$ is a collection $N$ of points of $x$ such that for each pair of elements $x,y,N$, we have $d(x,y)\leq k$. A $\zeta$-net is maximal if it is not possible to add more points to it.

Given a metric space $X$, a $\zeta$-net $N$ and $\lambda>0$, the $(\zeta,\lambda)$-approximation graph on $N$ is the graph $\Omega(X)$ whose vertex set is the set $N$ with the condition that two vertices are connected if and only if their distance as points of $X$ is at most $\lambda$.

Let $\omega: X \to 2^{2^{|X|}}$ be the map that associates to each point of $X$ its closest point projection on $N$, seen as the set of vertices of $\Omega(X)$. For each subspace $Y \subseteq X$, we denote by $\Omega(Y)$ the subgraph of $\Omega(X)$ induced by $\omega(Y)$.

The approximation graph will be the geodesic space that approximates a quasi-geodesic one.

Definition 2.5. Let $X$ be a $\varrho$-quasi-geodesic metric space and $Y$ a subspace of $X$. We say that $Y$ is $K$-quasi-convex if for every pair of points $x,y \in Y$ and $\varrho$-quasi-geodesic $\gamma$ between them, we have that $\gamma$ is contained in the $K$-neighborhood of $Y$.

Lemma 2.6. Let $q = (C,\varepsilon)$ and $X$ be a $q$-quasi-geodesic space and let $F$ be a family of $K$-quasi-convex subspaces of $X$. Let $\zeta = \max\{C+\varepsilon,K\}$, $N$ be a $\zeta$-maximal net for $X$ and $\Omega(X)$ be a $(\zeta,5\lambda)$-approximation graph for $X$ obtained from $N$. Then for each $W \in F \cup \{X\}$ and $x,y \in W$, we have:

$$\frac{1}{5\lambda}d_X(x,y) - 5\lambda \leq d_{\Omega(W)}(\omega(x),\omega(y)) \leq C d_X(x,y) + \varepsilon.$$

Proof. We note that all the above is implied by the following: for each $W \in F \cup \{X\}$, $x,y \in W$ and $x' \in \omega(x)$, $y' \in \omega(y)$, there is a path $\gamma \subseteq \Omega(W)$ between $x'$ and $y'$ such that

$$d_X(x,y) - \zeta \leq L(\gamma) \leq 5d_X(x,y) + 5\zeta.$$

Let $\eta: [a,b] \to X$ be a $(C,\varepsilon)$ quasi-geodesic in $X$ between $x$ and $y$. We observe that $\eta((a,b)) \subseteq N_{2C}(p_X(W))$, where $p_X(W)$ represents the closest point projection of $W$ on the net $N$. Indeed, by quasi-convexity, each point of $\eta((a,b))$ lies at distance at most $K$ from a point of $W$, and every point of $W$ lies at distance at most $\zeta$ from $N$, since $N$ is a maximal $\zeta$-net.

Let $0 = t_0 < \cdots < t_s = b$ be a partition of $[a,b]$ such that for each $i < s$, we have $|t_i - t_{i+1}| \leq 1$. Then for each $i$ we have $d(\eta(t_i), \eta(t_{i+1})) \leq C + \varepsilon \leq \zeta$. For each $0 < i < s$, choose a representative $v_i \in \omega(\eta(t_i))$ and set $v_0 = x'$, $v_s = y'$. Then for each $i < s$, we have $d_X(v_i, v_{i+1}) \leq 5\zeta$. Thus, $v_0 \cdots v_s$ is a path in $\Omega(W)$ joining $x'$ and $y'$. This implies that there is path of length at most $s$ in $\Omega(W) \subseteq \Omega(X)$ joining $x'$ and $y'$. Since $s \geq b - a$ and $\eta: [a,b] \to X$ is a $(C,\varepsilon)$-quasi-geodesic, we get $d_{\Omega(W)}(\omega(x),\omega(y)) \leq C d_X(x,y) + \varepsilon$. On the other hand, by triangular inequality we have $d_X(x,y) \leq 5\zeta d_{\Omega(W)}(\chi(x),\chi(y)) + 5\zeta$.

Corollary 2.7. If we equip the family $F$ of Lemma 2.6 with the induced metric coming from $X$, we obtain that each element $W \in F \cup \{X\}$ is uniformly quasi-isometric to $\Omega(W)$.

In particular, since the spaces $W$ with the induced metric are isometrically embedded in $X$, we obtain that the graphs $\Omega(W)$ are uniformly quasi-isometrically embedded in $\Omega(X)$, with respect to the path metric.

We will now prove a technical fact that will be useful in future, but before we recall the, so called, Morse Lemma for hyperbolic spaces.
Lemma 2.8 (Morse Lemma). Let $X$ be a geodesic Gromov hyperbolic space. Then there exists a function $H : \mathbb{R}^2 \to \mathbb{R}$ such that for each geodesic $\alpha$ and $(C, \varepsilon)$-quasi-geodesic $\gamma$ with the same endpoints of $\alpha$, we have:

$$d_{\text{Haus}}(\gamma, \alpha) \leq H(C, \varepsilon).$$

We recall that the midpoint of a (quasi) geodesic $\gamma : [0, a] \to X$ is defined as $\gamma \left( \frac{a}{2} \right)$.

Lemma 2.9. Let $X$ be a Gromov hyperbolic $(C, \varepsilon)$-quasi-geodesic space, and let $V$ be a quasi-convex subspace of $X$ such that the following holds. There is $D \geq 0$ such that for each $v \in V$ there is an arbitrarily long $(C, \varepsilon)$-quasi-geodesic segment with endpoints on $V$ whose midpoint lies at distance at most $D$ from $v$.

Suppose that $Y$ is a geodesic metric space and $\phi : X \to Y$ is a quasi-isometry. Then there exists $D'$ such that for each $w \in \phi(V)$ there is an arbitrarily long geodesic segment with endpoints on $V$, whose midpoint lies at distance at most $D'$ from $w$.

Proof. Considering a preimage of $w$ in $X$, we can find a quasi-geodesic that satisfies the requirements, and we can map it on $Y$ using $\phi$. The Morse Lemma (Lemma 2.8) allows to uniformly bound the distance between $w$ and a geodesic with endpoints on $\phi(V)$. Using quasi-convexity, we can modify the geodesic obtained to uniformly bound the distance between $w$ and the midpoint of such a geodesic.

2.2 Basics on hyperbolic spaces

Convention. From now on, we will always assume that

- spaces are geodesic (see Subsection 2.1);
- quasi-geodesic are continuous (see [BH99, Lemma 1.11, Chapter III.H]);

In particular, this allows to define the length of quasi-geodesic regardless of the parametrization. Or equivalently, we always assume that quasi-geodesics are parametrized by arc length. Moreover, we will freely identify (quasi) geodesics with their image. Given a (quasi) geodesic $\gamma$ and points $a, b$ of $\gamma$, we denote by $\gamma_{[a,b]}$ the restriction of $\gamma$ from $a$ to $b$. Given two points $x, y$ in $X$, we denote by $[x, y]$ a geodesic segment connecting them.

Definition 2.10 (Projection geodesics). Let $X$ be a geodesic metric space and $Y \subseteq X$ be a subspace of $X$. We say that a geodesic $\gamma$ with endpoints $x, y$ is a projection geodesic for $Y$ if $y \in p_Y(x)$.

The following lemma is well-known. We recall it as it will be used extensively.

Lemma 2.11 (Quadrilateral argument). Let $X$ be a geodesic $\delta$-hyperbolic space, and let $H$ be a $K$-quasi-convex subspace. Let $\gamma, \gamma'$ be projection geodesics for $H$ and let $a, b, a', b'$ be the endpoints. Up to changing names, we may assume $b, b' \in H$. Consider geodesic quadrilateral $[b, b'] \gamma'[a', a]$, and let $I$ be the set of points in the image of $[b, b']$ consisting of points at distance at least $4\delta + K$ from $\{b \cup b'\}$. Then for each $s \in I$, we have that $d(s, [a, a']) < 2\delta$.

Proof. We may assume that $I$ is non empty, otherwise the lemma is trivially true. Let $s \in I$ and consider the diagonal $[b, a']$. By hyperbolicity, $d(s, [b, a'] \cup [a, a']) < \delta$. If $d(s, [a, a']) < \delta$, then we would get a contradiction. In fact, let $m$ be a point in the image of $[a, a']$ witnessing the distance. Then, by triangular inequality, $d(m, a') \geq d(s, a') - d(a', m) > 3\delta + K$. But since $H$ is $K$-quasi-convex, there is a point $t \in H$ at distance less than $K$ from $s$. Then $d(m, t) < d(m, a')$ contradicting $\gamma'$ being a projection geodesic. Hence there is a point $q \in [b, a']$ with $d(q, s) < \delta$. The same argument on the triangle $\gamma[b, a']$ shows that $d(q, [a, a]) < \delta$, and hence $d(s, [a', a]) < 2\delta$. 

\]
Lemma 2.12 (Closest point projections are quasi-Lipschitz). Let $X$ be a $\delta$-hyperbolic space and $H$ a $K$-quasi-convex subspace. Then there exists $\rho = \rho(K, \delta)$ such that the map $p_H$ is $(1, \rho)$-quasi-Lipschitz.

Proof. Consider $x, y$ in $X$, and their projections $p_W(x), p_W(y)$ on $W$. Pick points $p \in p_W(x), q \in p_W(y)$ realizing $d(p_W(x), p_W(y))$. Consider the sub-interval $I$ of a geodesic $[p, q]$ consisting of points that have distance at least $4\delta + K$ from both $p$ and $q$. A hyperbolic quadrilateral argument (Lemma 2.11) gives that there exist a (non-empty) sub-interval $\[s, b\]$ witnessing the distance. Since $d(p_W(x), p_W(y)) \leq d(x, y) + 12\delta + 2K$.

Corollary 2.13. Let $X$ be $\delta$-hyperbolic and $H \subseteq X$ be $K$-quasi-convex. Then for each $R$ there exists $S = S(R, K, \delta)$ such that for each quasi-convex $Y \subseteq X$, we have $p_H(N_R(Y)) \subseteq N_S(p_H(Y))$.

Proof. This is an easy consequence of the fact that the map $p_H$ is quasi-Lipschitz. In fact, let $x \in N_R(Y)$, and let $y \in Y$ such that $d(x, y) < R$. Then, by Lemma 2.12, $d(p_H(x), p_H(y)) < R + 12\delta + 2K$. Since $\text{diam}(p_H(x))$ is uniformly bounded, we get the claim.

2.3 Behrstock’s inequalities

Proposition 2.14. Let $X$ be a $\delta$-hyperbolic space and $V, W$ be $K$-quasi-convex subspaces of $X$. Then there exists $\kappa_1 = \kappa_1(K, \delta)$ such that for each $x$ in $X$

$$\min\{d(p_V(x), p_V(W)), d(p_W(x), p_W(V))\} \leq \kappa_1.$$ 

Proof. Suppose that $d(p_W(x), p_W(V)) > 8\delta + 2K$. We claim that this will imply an uniform bound on $d(p_V(x), p_V(W))$. Choose points $a \in p_V(x), b \in p_W(a) \subseteq p_W(V)$ and $c \in p_W(x)$. By assumption, $d(b, c) > 8\delta + 2K$. Consider a geodesic quadrilateral between $a, b, c, d$. Observe that $[a, b], [a, c]$ and $[a, b]$ are projection geodesics. Since $L([b, c]) > 8\delta + 2K$, a quadrilateral argument (Lemma 2.11) gives that there exists a (non-empty) sub-interval $I$ of $[b, c]$ with $d(I, [a, c]) < 2\delta$. Quasi-convexity of $W$ gives $[a, b] \cap N_{K+2\delta+1}(W) \neq \emptyset$. Since $[a, c]$ is a projection geodesic for $V$, we have that $p_V([a, c]) \cap (N_{K+2\delta+1}(W)) \neq \emptyset$. Lemma 2.13 gives an uniform $\kappa' = \kappa'(K + 2\delta + 1, K, \delta) = \kappa'(K, \delta)$ such that $p_V(N_{K+2\delta+1}(W)) \subseteq N_{\kappa'}(p_V(W))$. In particular, $p_V(x) \cap N_{\kappa'}(p_V(W)) \neq \emptyset$. Since $\kappa_1 = \max\{\kappa', 8\delta + 2K\}$ depends only on $K$ and $\delta$, we get the claim. 

Proposition 2.15. Let $X$ be a $\delta$-hyperbolic space and $V \subseteq W$ be $K$-quasi-convex subspaces of $X$. Then

$$\text{diam}(p_V(x) \cup p_V(p_W(x))) \leq 12\delta + 4K.$$ 

Proof. Let $a, b \in p_V(x) \cup p_V(p_W(x))$ be witnessing the diameter. We can assume that $a \in p_V(x)$ and $b \in p_W(p_W(x))$, otherwise we would have a uniform bound since projections are quasi-Lipschitz. We claim that $d(a, b) \leq 12\delta + 4K$. Suppose that it is not the case. Let $c \in p_W(x)$ be such that $b \in p_V(c)$. Consider the quadrilateral $a, b, c, x$. By assumption, there is a point $s \in [a, b]$ such that $d(s, a) > 4\delta + K$ and $d(s, b) > 8\delta + 3K$. Since $[a, c]$ and $[c, b]$ are projection geodesics onto $V$, we have that $d(s, [c, b]) \leq 2\delta$. Let $m$ be a point on $[x, c]$ witnessing the distance. Since $V \subseteq W$ and $c \in p_W(m)$, we have that $d(c, v) < 2\delta$. By triangular inequality, $d(c, b) < 4\delta + 2K$. Again by triangular inequality, we get that $d(b, s) < 8\delta + 3K$, obtaining a contradiction. 


2.4 Coning-off

**Definition 2.16** (Coning-off). Let $\Gamma$ be a graph, $H$ a connected subgraph of $\Gamma$. We define the cone-off of $\Gamma$ with respect to $H$, and denote it by $\tilde{\Gamma}$, as the graph obtained from $\Gamma$ adding an edge connecting each pair of vertices in $(H \times H) - \Delta_{H \times H}$, where $\Delta_{H \times H}$ denotes the diagonal. We call the edges added in such a way $H$-components. Similarly, the cone-off with respect to a family of connected subgraphs $H = \{H_i\}$ is obtained adding the $H_i$-components for each $H_i \in H$. An edge is an $H$-component if it is a $H_i$-component for some $H_i \in H$.

Given a graph $\Gamma$ and its cone-off $\tilde{\Gamma}$ with respect to some family $H$, we have that $V(\tilde{\Gamma}) = V(\Gamma)$. In particular, if we regard $\Gamma$ and $\tilde{\Gamma}$ as metric spaces, this implies that there is a bijection $i : \Gamma \to \tilde{\Gamma}$ such that for each pair of points $x, y \in \Gamma$, we have that $d_\Gamma(x, y) \geq d_{\tilde{\Gamma}}(i(x), i(y))$ and for each pair of points $x', y' \in \tilde{\Gamma}$, $d_{\tilde{\Gamma}}(x', y') \leq d_{\tilde{\Gamma}}(i^{-1}(x), i^{-1}(y))$. To simplify notation, we will identify points of $\Gamma$ and $\tilde{\Gamma}$.

**Proof.** We can assume that $\Gamma$ is quasi-convex. Indeed, this will guarantee that for each $H$-component $\gamma$, the corresponding geodesic segment $\eta$ will be coarsely contained in $H$. Without this property (or maybe some different property of the same flavor), there is almost no relation between the original path and the corresponding geodesic of $\Gamma$ between $x$ and $y$.

**Notation.** If $\tilde{\Gamma}$ is the cone-off of $\Gamma$, we use the notation $[x, y]$ for a geodesic of $\Gamma$ between $x$ and $y$.

**Definition 2.17** (Pieces). Let $\eta = \eta_1 \ast \cdots \ast \eta_n$ be a concatenation of geodesic segments. Then we call each of the non-trivial $\eta_i$ a piece of $\eta$.

Note that the subdivision is part of the data.

**Definition 2.18** (De-electrifications). Let $\tilde{\Gamma}$ be the cone-off of a graph $\Gamma$ with respect to a family of subgraphs $H$. Let $\gamma = e_1 * e_2 * \cdots * e_n * u_{n+1}$ be a path of $\tilde{\Gamma}$, where the $e_i$ are $H$-components and the $u_i$ are (possibly trivial) segments of $\Gamma$. The total de-electrification (or simply de-electrification) $\tilde{\gamma}$ of $\gamma$ is the concatenation $u_1 * \eta_1 * \cdots * \eta_n * u_{n+1}$ where each $\eta_i$ is a geodesic segment of $\Gamma$ connecting the endpoints of $e_i$. If $e_i$ was an $H$-component, we say that $\eta_i$ is an $H$-piece. A piece of $\tilde{\gamma}$ is an $H$-piece if it is an $H$-piece for some $H \in H$.

Even though the definition of de-electrification makes formal sense for any family of subgraphs $H$, in practice we will be interested in the case of the elements of $H$ being quasi-convex. Indeed, this will guarantee that for each $H$-component $\gamma$, the corresponding geodesic segment $\eta$ will be coarsely contained in $H$. Without this property (or maybe some different property of the same flavor), there is almost no relation between the original path and the de-electrification of it. However, it is possible to establish at least some mild result about the combinatorial properties of de-electrifications.

**Lemma 2.19** (Pigeonhole for cone off). Let $\Gamma$ be a graph and $\tilde{\Gamma}$ be the cone off with respect to a family of graphs $H$. Then for each $\theta > 1$ there exists a $T = T(\theta)$ such that if $d(x, y)_\Gamma \geq T$, then for each $\tilde{\Gamma}$-path $\gamma$ connecting $x$ and $y$, either $L_{\tilde{\Gamma}}(\gamma) \geq \theta$ or $\tilde{\gamma}$ has an $H$-piece that has $\Gamma$-length greater or equal $\theta$.

**Proof.** We can assume that $\theta > 1$. We claim that $T = 2\theta^2$ does the job. Consider $x, y$ with $d(x, y)_\Gamma = T_0 \geq T$, and fix a $\tilde{\Gamma}$-path $\gamma$ between them. Assume that $L_{\tilde{\Gamma}}(\gamma) = \varepsilon < \theta$, and let $P$ be the number of $H$-components of $\gamma$. Consider a de-electrification $\tilde{\gamma}$ of $\gamma$ and let $A = \sum_{\gamma} L(u_i)_\Gamma$, where the $u_i$ are the $H$-pieces of $\tilde{\gamma}$ and $B = L(\tilde{\gamma}) - A$. Then we have that $A + B = T_0$ and $B + \varepsilon = \varepsilon$. Hence $A = P + T_0 - \varepsilon$. Since $A$ is the sum of $P$ terms, we have that there is at least one that has value greater or equal $1 + \frac{1}{P}(T_0 - \varepsilon)$. Since $P \leq \varepsilon < \theta$, we get

$$1 + \frac{1}{P}(T_0 - \varepsilon) \geq 1 + \frac{1}{P}(2\theta^2 - \varepsilon) \geq 1 + \frac{2\theta^2 - \theta}{\theta} \geq \theta.$$ 

$\square$
It is easily seen that de-electrification of geodesics of $\tilde{\Gamma}$ consist of concatenations of geodesic segments in $\Gamma$. We will now record a basic fact about concatenations of geodesic segments.

**Lemma 2.20.** Let $X$ be a $\delta$-hyperbolic space, $\eta$ be a geodesic segment of $X$ and $\sigma$ the concatenation of $n$ geodesic segments. If $\sigma \cap N_{\eta \cdot 1}(\eta) = \emptyset$, then $\text{diam}(p_{\eta}(\sigma)) < 8\delta$.

**Proof.** Let $a, b \in p_{\eta}(\sigma)$ be two points realizing the diameter, and let $a', b'$ be preimages in $\sigma$. Then $\sigma_{\vert [a', a'']} = [a', b']$ is an $m$-gon, with $m \leq n + 1$. In particular, we get that $[a', b']$ is contained in the $(n - 1)\delta$-neighborhood of $\sigma$, and hence $N_{2\delta} \cap [a', b'] = \emptyset$. A quadrilateral argument gives the claim. \hfill \Box

**Proposition 2.21.** Let $\Gamma$ be a $\delta$-hyperbolic graph, $\mathcal{H}$ a family of connected subgraphs and $\hat{\Gamma}$ the cone-off of $\Gamma$ with respect to $\mathcal{H}$. Then there exist $D', p$ depending only on $\delta$ such that the following holds. For each pair of points $x, y$ of $\Gamma$, for each geodesic $\gamma$ of $\hat{\Gamma}$ connecting $x$ and $y$ and for each geodesic $[x, y]$ of $\Gamma$ connecting $x$ and $y$, every connected component of $\hat{\gamma} - N_{D'}([x, y])$ has at most $p$ pieces.

**Proof.** To simplify notation, throughout this proof we will drop the superscript $\Gamma$ and assume that $N_{D'}(Y)$ denotes the $D'$-neighborhood of $Y$ with respect to the metric of $\Gamma$. Let $\xi = 8\delta + 1$, $D' = \delta(\xi + 1)$, and suppose that $\hat{\gamma}$ leaves the $D'$-neighborhood of $[x, y]$. Let $a, b$ be the endpoints of one of the connected components of $\hat{\gamma} - N_{D'}([x, y])$. To simplify notation let $\sigma = \hat{\gamma}_{\vert [a, b]}$. Let $P$ be the number of pieces of $\sigma$, and let $q$ be the maximal integer such that $P = q\xi + r$, for some $r < \xi$. We subdivide the concatenation $\sigma$ into sub-concatenations $\sigma_i$ such that:

- each $\sigma_i$ is the concatenation of consecutive pieces of $\sigma$;
- each $\sigma_i$ contains at most $\xi$ pieces of $\sigma$;
- the subdivision is chosen in such a way that the number of $\sigma_i$ is minimal (in fact, it is at most $q + 1$).

To simplify notation, let $q \leq Q \leq q + 1$ be the number of the sub-concatenations $\sigma_i$. Our goal is to give a uniform bound on $Q$ and hence on the number of pieces. Let $a', b'$ be closest point projections in $\Gamma$ of $a$ and $b$ on $[x, y]$. We want to argue that $L([a, a'] \ast [a', b'] \ast [b', b], [x, y]) < L([a, a], [x, y])$. In particular, this will imply that $[a', b']$ is a short cut for $\gamma$ in $\hat{\Gamma}$. Note that $L([a, a'])_{\Gamma} \leq D'$ and $L([b, b'])_{\Gamma} \leq D'$. Moreover, $L([a, b])_{\Gamma} \leq \sum_{i=1}^{Q} \text{diam}(p_{[x, y]}(\sigma_i))$. Note, however, that Lemma 2.20 guarantees that for each $i \leq Q$, $\text{diam}(p_{[x, y]}(\sigma_i)) < 8\delta < \xi - 1$. Hence we have that $L([a, a'] \ast [a', b'] \ast [b', b])_{\Gamma} < 2D(\xi + 1) + Q(\xi - 1)$. On the other hand, since $\sigma$ has $P$ pieces, we have that $L(\sigma)_{\hat{\Gamma}} \geq P = Q\xi + r \geq Q\xi$. But it is clear that for a large enough $Q_0$, the following holds: $2D(\xi + 1) + Q_0(\xi - 1) < Q_0\xi$, That is, for large enough values of $Q$, we get a short-cut. Hence we get the desired bound on $Q$, and thus a bound on the maximum number of pieces that $\sigma$ can have. \hfill \Box

**Remark 2.22.** In the above proof, the quantities $D', p$ depends on $\xi$, which, ultimately, only needs to satisfy $\xi - 1 > 8\delta$. In particular, the bounds on $p$ and $Q$ of the above proof are surely not optimal, and it is possible to minimize and explicitly compute them via varying $\xi$.

The proof of Proposition 2.21 gives us two important corollaries:

**Corollary 2.23.** In the hypotheses and notations of Proposition 2.21 we have the following: there exist $D = D(\delta)$ such that for each connected component $\sigma$ of $\gamma - N_D([x, y])$, the distance in $\Gamma$ between the endpoints of $\sigma$ is uniformly bounded by $2D + 8\delta$.

**Proof.** Set $D = \max\{p + 1, D'\}$, where $p, D'$ are as in Proposition 2.21. Then we have that each connected component $\gamma - N_D([x, y])$ has at most $p$ pieces. But then Lemma 2.20 gives that the projection of $\sigma$ on $[x, y]$ has diameter at most $8\delta$. Hence, by triangular inequality, the distance between the endpoints of $\sigma$ is at most $2D + 8\delta$. \hfill \Box
Corollary 2.24. In the hypotheses and notations of the proof of Proposition 2.23, we have that there exists $D = D(\delta)$ such that $[x,y] \subseteq N_{10\delta + D}(\gamma)$.

Proof. Let $z$ be a point of $[x,y]$. It is a well known fact of $\delta$-hyperbolic spaces that, if $\eta$ is a geodesic and $\gamma$ a path with the same endpoints of $\eta$, then the projection of $\gamma$ onto $\eta$ is $2\delta$-dense. Thus, there is a point $z' \in [x,y]$, with $d(z,z') < 2\delta$ that is contained $p_{\eta}(\gamma)$. If there is a preimage of $z'$ in $N_{\delta}(x,y) \cap \gamma$, then we are done. So suppose this is not the case, and let $a$ be a preimage of $z'$ in $\gamma$. Let $\sigma$ be the connected component of $\gamma - N_{\delta}(x,y)$ that contains $a$. Let $D$ as in Corollary 2.24 that is $D = \max\{\delta(p + 1), D'\}$, where $p, D'$ are as in Proposition 2.21. Since $\sigma$ has at most $p$ components, Lemma 2.20 gives that that $\text{diam}(p_{\eta}(\sigma)) < 8\delta$. Let $\sigma^\pm$ be the endpoints of $\sigma$. Since both $p_{\eta}(\sigma^\pm)$, $z'$ are contained in the projection $p_{\eta}(\sigma)$, it follows that $d(z', p_{\eta}(\sigma^\pm)) < 8\delta$. By triangular inequality, we get that $d(z, \sigma^\pm) < 10\delta + D$, which proves the claim. \hfill \qed

Now we would like to prove the other inclusion, namely that there is a constant $D'$ such that $\gamma \subseteq N_{D'}(x,y)$. Without further assumptions on the family $H$, this is easily seen to be hopeless. This motivates to consider a family $\mathcal{H}$ with more structure, in particular, we will require the elements of $\mathcal{H}$ to satisfy a slightly more general version of uniform-quasi-convexity.

Definition 2.25 (Cone-off quasi-convexity). Let $\Gamma$ be a connected graph, $\mathcal{H}$ a family of subgraphs of $\Gamma$ and $\hat{\Gamma}$ the cone-off of $\Gamma$ with respect to $\mathcal{H}$. We say that a subset $S \subseteq \Gamma$ is $K$-cone-off quasi-convex (K-COQC) with respect to $(\Gamma, \mathcal{H})$ if the following holds. For each two points $s, t$ of $S$, each geodesic $[s, t]$ of $\Gamma$ between them and each point $z \in [s, t]$, we have:

$$d_{\hat{\Gamma}}(z, S) \leq K.$$ 

Similarly, we say that the family $\mathcal{H}$ is $K$-cone-off quasi-convex (K-COQC) if for each $H$ in $\mathcal{H}$, $H$ is K-COQC with respect to $(\Gamma, \mathcal{H})$.

We emphasize that a $K$-quasi-convex subset is also K-COQC, regardless of the coning-off family.

We will now introduce the definition of interruption. The idea is the following: consider a path $\gamma = u_1e_1u_2$ of $\Gamma$ and assume that $e$ is an $H$-component (for simplicity assume the only one) of $\gamma$, for some $K$-quasi-convex $H$. If $x, y$ are the endpoints of the edge $e$, then $\gamma = u_1[x, y]u_2$ is a de-electrification for $\gamma$, for some geodesic $[x, y]$ between $x, y$ in $\Gamma$. It may happen that we are interested in some point $z \in [x, y]$. In general, such a $z$ will be an element of $\gamma$, but not of $\gamma$. In order to solve this, we will modify $\gamma$ into a path whose length is comparable to the one of $\gamma$, but contains $z$. A pictorial illustration of this can be found in Figure 22.

Definition 2.26 (Interruption). Let $\Gamma$ be a graph, $\hat{\Gamma}$ be the cone-off of $\Gamma$ with respect to a $K$-COQC family $\mathcal{H}$ and let $\gamma$ be a path of $\Gamma$. Suppose that $\gamma = u_1e_1u_2$, where $e$ is a $H$-component for some $H \in \mathcal{H}$, and let $u_1\gamma u_2$ be a de-electrification for $u_1e_2u_2$. Let $z$ be a point in $\eta$. We define the interruption of $\gamma$ in $z$ to be the path $u_1e_1[z', z]r[z, z']r'\gamma u_2$ of $\hat{\Gamma}$, where $z'$ realizes the shortest distance projection of $z$ on $H$ and $e_1, e_2$ are defined as the edges $(e'^-, z)$, $(z, e'^+)$ respectively. If $\hat{\gamma}$ is a de-electrification of $\gamma$ and $S$ is a set of points belonging to $\mathcal{H}$-pieces of $\hat{\gamma}$, we similarly define the interruption of $\gamma$ in $S$.

Remark 2.27. Let $S$ be as in Definition 2.26. If $\sigma^\delta$ is the interruption of $\gamma$ with respect to $\gamma$, we have that $L(\sigma^\delta) \leq L(\gamma) + |S|2K + 1$. This is easily seen because, in the notation of Definition 2.26, we substitute $\mathcal{H}$ components $e$ with concatenations of the form $e_1[z', z]r[z, z']r\gamma e_2$. By definition of $K$-COQC, $L_\mathcal{H}([z, z']) \leq K$.

It is quite remarkable that the estimate of Remark 2.27 is the only step of the proof or Proposition 2.24 that uses quasi-convexity.

We recall that given a graph $\Gamma$, a point $p \in \Gamma$ and a number $R \in \mathbb{R}$, the ball of center $p$ and radius $R$, is the set $\{x \in \Gamma \mid d(x, p) < R\}$, and the sphere of center $p$ and radius $R$ is the set $\{x \in \Gamma \mid d(x, p) = |R|\}$.
Proposition 2.28. Let $\Gamma$ be a $\delta$-hyperbolic graph, $\mathcal{H}$ a family of $K$-COQC subgraphs and $\hat{\Gamma}$ the cone-off of $\Gamma$ with respect to $\mathcal{H}$. Then there exist $\tau_1 = \tau_1(\delta, K)$ and $\tau_2 = \tau_2(\delta, K)$ such that for each pair of points $x, y \in \Gamma$ there exists a $\tau_1$-quasi-geodesic $\gamma'$ of $\hat{\Gamma}$ with the property that for each de-electrification $\tilde{\gamma}'$ of $\tau_1$-quasi-geodesic, $\tilde{\gamma}'$ is a $\tau_2$-quasi-geodesic of $\Gamma$.

Proof. Consider a geodesic $\gamma$ of $\hat{\Gamma}$ between $x$ and $y$ and fix a de-electrification $\tilde{\gamma}$. We will stick to the convention that $[x, y]$ denotes a (fixed once and for all) geodesic segment of $\Gamma$ between $x$ and $y$.

Our goal is to modify $\gamma$ into a $\tau_1$-quasi-geodesic $\gamma'$ with the same endpoints of $\gamma$ such that each de-electrification of $\gamma'$ is a $\tau_2$-quasi-geodesic, where $\tau_1$ and $\tau_2$ do not depend on $x, y$. Since $\gamma$ is a geodesic of $\hat{\Gamma}$, if $\tilde{\gamma}$ was already uniformly a quasi-geodesic of $\Gamma$, then this will conclude the proof. However, it is easily seen that the de-electrification $\tilde{\gamma}$ can, in general, be arbitrarily far away in $\Gamma$ from $[x, y]$. Since $\Gamma$ is $\delta$-hyperbolic, this clearly constitutes an obstruction for $\tilde{\gamma}$ to be uniformly a quasi-geodesic. So, our first step will be to modify $\gamma$ to some quasi-geodesic $\gamma''$ such that each de-electrification of $\gamma''$ is contained in a uniform $\Gamma$-neighborhood of $[x, y]$. After this step, it is indeed possible to show that each de-electrification of $\gamma''$ is a $\tau_1$-quasi-geodesic of $\Gamma$, but $\tau_1$ will be depending on the distance between the points $x$ and $y$. This is because the de-electrifications may have backtracking, which, in general, can only be estimated in terms of $d_{\Gamma}(x, y)$. So, the second step will be to further modify $\gamma''$ into a new path $\gamma'$ such that the de-electrifications will (coarsely) not backtrack. The path $\gamma'$ will be the desired one. The last two steps will consist in showing that $\gamma'$ is still uniformly a quasi-geodesic of $\hat{\Gamma}$, and that the de-electrifications are uniformly quasi-geodesics of $\Gamma$.

**Step 1:** Obtain that $\tilde{\gamma} \subseteq N_{[D+4\delta]}([x, y])$.

If $\tilde{\gamma} \subseteq N_{D}([x, y])$, this step is trivial. So suppose $\tilde{\gamma}$ exits the $D$-neighborhood of $[x, y]$, and let $\Sigma = \{\sigma_{n}\}$ be the set of connected components of $\tilde{\gamma} - N_{D}([x, y])$ that have at least two pieces. We can restrict to such components because if a connected component $\sigma$ consists of only one piece, then it is easy to see that $\sigma \subseteq N_{D+4\delta}([x, y])$. In order to simplify notation, let $\Delta = [D+4\delta] \in \mathbb{Z}$. Suppose, then, that $\Sigma$ is non empty. Let $S$ be the set of endpoints of elements of $\Sigma$ that are not contained in $\gamma$. That is, the set of endpoints that are contained in $\mathcal{H}$-pieces of $\gamma$. We note that $|S| \leq 2P \leq 2L_{\Gamma}(\gamma)$, where $P$ is the number of $\mathcal{H}$-components of

Figure 1: The point $z$ is an element of the de-electrification, but not of the original path. To solve this, we substitute $e$ with the concatenation $e_1[z', z][z, z']e_2$ (blue in the picture). Since the space $H$ is $K$-quasi-convex, the geodesic $[z, z']$ has length smaller or equal $K$. Thus we have that the total length of $e_1[z', z][z, z']e_2$ is bounded above by $2 + 2K$. 

Figure 2: The de-electrification \( \tilde{\gamma} \) may exit the \( \Delta \)-neighborhood of \([x, y]\). In this case we will "cut-off" the parts that exits the \( \Delta \)-neighborhood with more than two pieces, and replace them with a geodesic segment \( \Gamma \). In the picture, there is only one such connected component, which is replaced by the segment \( \eta_l \).

Let \( \gamma^S \) be the interruption of \( \gamma \) with respect to \( S \). Note that \( L(\gamma^S) \geq L(\gamma) + |S|(2K+1) \).

For each \( \sigma_i \in \Sigma \), let \( s^+ \) be the endpoints of \( \sigma_i \), and let \( \eta_i \) be a geodesic in \( \Gamma \) connecting \( s^+ \) and \( s^- \). Applying Corollary 2.23 to the endpoints \( \chi \) of \( \eta_i \), we get:

\[
L(\eta_i) \leq 2\Delta - 2.
\]

Let \( \gamma'' \) be the path in \( \hat{\Gamma} \) obtained from \( \gamma^S \) substituting each \( \gamma^S_{(s^+,s^-)} \) with \( \eta_i \), and let \( \tilde{\gamma}'' \) be the path in \( \Gamma \) obtained from \( \gamma'' \) substituting all the \( \sigma_i \) with \( \eta_i \). It is clear from the construction that \( \tilde{\gamma}'' \subseteq N_\Delta([x, y]) \), and that it is a de-electrification of \( \gamma'' \).

We want now to show that \( \gamma'' \) is a quasi-geodesic of \( \hat{\Gamma} \). Consider a subpath \( \chi \) of \( \gamma'' \). We want produce a uniform estimate of the length of \( \chi \). Since \( \chi \) is arbitrary, this will guarantee that \( \gamma'' \) is a quasi-geodesic. First, suppose that the endpoints of \( \chi \) are contained in \( \gamma^S \), that is, they are not part of the geodesic segments \( \eta_i \). Note that this implies that \( s^+ \) is a point of \( \chi \) if and only if \( s^- \) is. Finally \( S_\chi = S \cap \chi \), and let \( \gamma'' \) be the restriction of \( \gamma'' \) between the endpoints of \( \chi \).

\[
L(\chi) \leq L(\gamma^S) + \sum_{i=1}^m L(\eta_i) \leq |S_\chi|(2K+1) + \sum_{i=1}^m (L(\eta_i)) \leq |S_\chi|(2\Delta - 2) = |S_\chi|(2\Delta + 2K - 1).
\]

Since \( |S_\chi| \leq 2L(\gamma^S) \), we have the desired estimate. Since, by Corollary 2.23, we can uniformly bound the length of the segments \( \eta_i \), up to increasing the additive constant, we get the estimate for general endpoints of \( \chi \).

Step 2: Removing backtracking.

The general strategy will be similar to the one of Step 1. First we remark that there is a natural order on a geodesic induced by the choice of an order on the endpoints. From now on, we will always consider the order on the geodesics induced by the choice \( x \leq y \).

We want now to further modify \( \gamma'' \) and \( \tilde{\gamma}'' \). We will do this inductively, obtaining a sequence \( \{t_i\}_{i=1}^m \) of points in \( \tilde{\gamma}'' \), a sequence \( \{\gamma'_{i}^{s}\}_{i=1}^m \) of paths in \( \Gamma \), and a sequence \( \{\tilde{\gamma}'_{i}^{s}\}_{i=1}^m \) of de-electrifications in \( \Gamma \). We start the process setting \( t_0 = x, \gamma'_{i-1} = \gamma'' \) and \( \tilde{\gamma}'_{i-1} = \tilde{\gamma}'' \).

Suppose that \( t_i, \gamma'_{i-1} \) and \( \tilde{\gamma}'_{i-1} \) are defined, and that \( t_i \in \gamma'_{i-1} \). Let \( B \) be the ball of center \( t_i \) and radius \( 3\Delta \) in \( \Gamma \). For the rest of Step 2, we will consider only connected components of \( \gamma'_{i-1} - B \) that:
Figure 3: In order to remove backtracking, we connect with a geodesic of $\Gamma$ the first and the last point that intersects the sphere of radius $3\Delta$ around $t_i$.

- have at least two pieces or contain $y$,
- come after the point $t_i$ in $\tilde{\gamma}'_{i-1}$.

Let $T_i$ be the set of points obtained intersecting each connected component as above with the sphere of same radius and center as $B$. If the set $T_i$ is empty or contains only one element, then no action is performed. Otherwise, let $a$ and $b$ be the first and the last points of $T_i$. Then we obtain $\tilde{\gamma}'_i$ from $\tilde{\gamma}'_{i-1}$ by removing $\tilde{\gamma}'_{i[a,b]}$ and adding a geodesic $[a,b]$ of $\Gamma$ connecting $a$ and $b$. We modify $\tilde{\gamma}'_{i-1}$ accordingly (namely, interrupting it on $[a,b]$ and then substituting $\tilde{\gamma}'_{i[a,b]}$ with $[a,b]$). Note that $\tilde{\gamma}'_i$ is a de-electrification of $\tilde{\gamma}'_{i-1}$, and that $L([a,b]_0) - L(\tilde{\gamma}'_{i[a,b]}_0) \leq 2\Delta - 2$. If $y \in B$, then we stop. Otherwise, we repeat the whole procedure with $t_{i+1} = b$. Note that if $y \notin B$, then $T_i$ is not empty and $\tilde{\gamma}'_{i[b,y]} \cap B = \emptyset$. This implies that $y$ and $b$ lie on the same connected component of $N_{\Delta}([x,y]) - B$. This, and the fact that the radius of $B$ is $3\Delta$, implies the following two:

$$dr(p_{[x,y]}(a), p_{[x,y]}(b)) \geq \Delta, \quad (1)$$
$$dr(p_{[x,y]}(a), y) \geq dr(p_{[x,y]}(b), y) + \Delta, \quad (2)$$

where $p_{[x,y]}(a), p_{[x,y]}(b)$ denote the shortest distance projections of $a$ and $b$ on $[x,y]$. In particular, there is $m < \infty$ such that $y$ is contained in the ball of radius $3\Delta$ around $t_m$, namely, the process stops after finitely many steps. We set $\gamma' = \gamma'_{tm}$ and $\gamma'' = \gamma''_{tm}$. We remark that the sequence $\{t_i\}_{i=1}^m$, may contain points that are not elements of $\gamma'$. In fact, in the cases where the set $T_i$ is empty, no action is performed, and in particular no interruption on $\gamma''$.

**Step 3:** $\gamma'$ is a quasi-geodesic in $\Gamma$.

We claim that the sequence $t_0, \ldots, t_m$ provides a quasi-isometric embedding of $\{0, \ldots, m\}$ into $\Gamma$. For each $i < m$, since $t_{i+1}$ is on the sphere of radius $3\Delta$ around $t_i$, clearly $dr(t_i, t_{i+1}) \leq 3\Delta$. Hence, we get the bound $dr(t_i, t_j) \leq 3\Delta|i - j|$. We want to provide a lower bound. Given $t_i, t_j$, and applying inequality (2) to the sequence $p_{[x,y]}(t_i), p_{[x,y]}(t_{i+1}), \ldots, p_{[x,y]}(t_j)$, we get that $dr(p_{[x,y]}(t_i), y) \geq \Delta$ since all the points considered are on a geodesic, we have that $dr(p_{[x,y]}(t_i), p_{[x,y]}(t_j)) \geq \Delta$. The fact that the distances $dr(t_i, p_{[x,y]}(t_j))$ are uniformly bounded, provides the desired bound. Now we claim that for each $i$, the path $\tilde{\gamma}_{[t_i, t_{i+1}]}$ is uniformly a quasi-geodesic. Note that it consists of a concatenation of pieces contained in a ball of radius $\Delta$. Hence each piece has length at most $2\Delta$. Moreover, it is clear by construction that $\tilde{\gamma}_{[t_i, t_{i+1}]} = \tilde{\gamma}_{[t_i, a]} * [a, t_{i+1}]$, where $a$ is as in Step 1. Let $q_1, q_2$ be any two points on $\tilde{\gamma}_{[t_i, a]}$. Since: $\gamma''$ is a quasi-geodesic of $\tilde{\Gamma}$, $\gamma''$ is a de-electrification for $\gamma''$, and $\tilde{\gamma}_{[q_1, q_2]}$ is a quasi-geodesic, arguing as in Step 1, we get that the number of pieces between them can be bounded by a linear function of the distance between $q_1$ and $q_2$. In particular, the bound on the length of each piece gives an upper bound on $L(\tilde{\gamma}''_{[q_1, q_2]})$. Now consider the case of general endpoints: since the length of $[a, t_{i+1}]$ is uniformly bounded by $2\Delta$, up to increasing the additive constant by $4\Delta$, we get the desired bound. Hence, we
get the claim. But a concatenation obtained joining a set of quasi-isometrically embedded points with uniform quasi-geodesic is a quasi-geodesic.

We remark that our construction of $\gamma'$ depends on the choice of de-electrification $\tilde{\gamma}$ of $\gamma$, and in order to perform the construction, we used the auxiliary sequence $\{\gamma_i\}$ of paths in $\Gamma$. This gave us a particular choice for the de-electrification of $\gamma'$. However, we remark that after $\gamma'$ is constructed, the argument of Step 3 applies to any de-electrification of $\gamma'$, and not only to the de-electrification obtained during the construction.

Step 4: $\gamma'$ is a quasi-geodesic in $\tilde{\Gamma}$.

Consider the set $X = \{t_i\}_{i=1}^n \cap \gamma'$. As remarked in Step 2, it may happen that $X \subseteq \{t_i\}$. Note that it is not possible that two elements $t_i, t_j$ of $X$ belong to the same $\mathcal{H}$-piece of $\gamma'$. Indeed, suppose that this holds for $t_i$ and $t_j$, with $j > i$. Then, since $\gamma'_{t_i-t_j}$ restricted to $t_i, t_{i+1}$ is a geodesic segment, it follows that the set $T_i$ contains only one element. Hence no action is performed. Since the point $t_{i+1}$ is on an $\mathcal{H}$-piece and no interruption is performed, it follows that $t_{i+1} \notin \gamma'$. Repeating the above reasoning for $t_{i+1}, \ldots, t_{j-1}$ gives the claim.

Since the number $P$ of $\mathcal{H}$-components of $\gamma$ is greater of equal to the number of $\mathcal{H}$-components of $\gamma'$, we have that $|X| \leq P \leq L_F(\gamma) = d_F(x, y)$. Consider a restriction $\chi$ of $\gamma'$ with endpoints $q_1, q_2$ and suppose that $q_1, q_2 \in \gamma''$. If $\chi = \gamma \cap \chi$, we similarly have that $|X| \leq d_F(q_1, q_2)$. For each element of $X$, let $a_i$ and $b_i$ be the endpoints of the corresponding $\mathcal{H}$-component.

Finally, we observe that by Step 2, $\gamma''$ is uniformly a quasi-geodesic. In particular, there exists $C$ such that $L(\gamma''_{(q_1, q_2)}) \leq C d_F(q_1, q_2) + C$. Then we have:

$$L(\chi) = L(\gamma''_{(q_1, q_2)}) + \sum_{X} \left( L([a_i, b_i])_{\tilde{\Gamma}} - L(\gamma''_{[a_i, b_i]})_{\tilde{\Gamma}} \right) \leq$$

$$\leq d_F(q_1, q_2) + C + |X| \left( 2K + 1 \right) + \sum_{X} \left( 22 \Delta + 2K + 1 \right) =$$

$$= d_F(q_1, q_2) + C + |X| \left( 12 \Delta^2 + 22 \Delta + 2K + 1 \right) \leq$$

$$\leq C d_F(q_1, q_2) + C + d_F(q_1, q_2) \left( 12 \Delta^2 + 22 \Delta + 2K + 1 \right) =$$

$$= d_F(q_1, q_2) \left( 12 \Delta^2 + 22 \Delta + 2K + 1 \right) + C.$$

Since the $\Gamma$-geodesic segments $[a_i, b_i]$ have uniformly bounded length in $\tilde{\Gamma}$, we get that, up to uniformly increase the additive constant, we can bound the length of each subsegment of $\gamma'$. Hence, $\gamma'$ is uniformly a quasi-geodesic of $\tilde{\Gamma}$.

Suppose that $\tilde{\Gamma}$ is the cone-off of $\Gamma$ with respect to a general family of uniformly quasi-isometrically embedded subgraphs $\mathcal{H}$. Sometimes it may be useful to consider a slightly different definition of de-electrification.

**Definition 2.29.** Let $\tilde{\Gamma}$ be the cone-off of a graph $\Gamma$ with respect to a family of uniformly quasi-isometrically embedded subgraphs $\mathcal{H}$. Let $\gamma = u_1 * e_1 * \cdots * e_n * u_{n+1}$ be a path of $\Gamma$, where each $e_i$ is an $H$-component for some $H_i \in \mathcal{H}$, and the $u_i$ are (possibly trivial) segments of $\Gamma$. The **embedded-de-electrification** $\tilde{\gamma}$ of $\gamma$ is the concatenation $u_1 * \eta_1 * \cdots * \eta_n * u_{n+1}$ where each $\eta_i$ is a geodesic segment of $H_i$ connecting the endpoints of $e_i$. We define $H$-pieces and $\mathcal{H}$-pieces as in the case of de-electrifications.

Note that if $\Gamma$ is $\delta$-hyperbolic and the elements of $\mathcal{H}$ are uniformly quasi-isometrically embedded, the two definitions are coarsely the same. Indeed, given a path $\gamma$ of $\tilde{\Gamma}$ it is easy to see that $\tilde{\gamma}$ is a quasi-geodesic of $\tilde{\Gamma}$ if and only if $\gamma$ is.

As a corollary, we get the following version of Proposition 2.25.

**Corollary 2.30.** Let $\Gamma$ be a $\delta$-hyperbolic graph, $\mathcal{H}$ a family of uniformly quasi-isometrically embedded subgraphs and $\Gamma$ the cone-off of $\Gamma$ with respect to $\mathcal{H}$. Then there exist $\eta_1 = \eta_1(\delta, K)$
Lemma 3.5 (Partial pigeonhole for cone-offs). Let \( \Gamma \) be a \( \delta \)-hyperbolic graph and \( \mathcal{H} \) be a factor system for \( \Gamma \). Then for each \( \theta \) there exists a \( T = T(K,c) \) and \( \tau = \tau(\delta,\mathcal{H}) \) such that for any pair of points \( x, y \) satisfying \( d(x,y)_{\Gamma} \geq T \), there is a path \( \gamma \) of \( \Gamma \), a number \( n \) and \( H \in \mathcal{H} \cup \{ \Gamma \} \) such that:

- the de-electrification \( \tilde{\gamma} \) is uniformly a quasi-geodesic of \( \Gamma \),
- the path \( \tilde{\gamma}\gamma^{(n)} \cap \tilde{H} \) is uniformly a quasi-geodesic of \( \tilde{H} \),
- \( L(\tilde{\gamma}\gamma^{(n)} \cap \tilde{H})_{\tilde{H}} \geq \theta \).

Proof. We induct on the complexity of \( \mathcal{H} \). So suppose now that the claim holds for \( c \leq m - 1 \). Let \( \gamma \) be the path provided by Proposition 2.28. Lemma 2.19 guarantees that for each \( T \), there exists \( T' = T'(T) \) such that if \( d(x,y) \geq T' \), then either there is a \( G \)-piece \( \sigma \) of \( \tilde{\gamma} \) with \( L(\sigma)_{\tilde{G}} \geq T \), or \( L_{\tilde{H}}(\gamma) \geq T \). In the second case, we are done choosing \( n = 0 \), because
\( \gamma^{(0)} = \gamma \) is uniformly a quasi-geodesic and, by the choice of \( \gamma \), every de-electrification is uniformly a quasi-geodesic. So suppose that there is \( G \in \mathcal{H} \) with \( L(\tilde{\gamma} \cap G) \geq T \). Let \( a, b \) be the endpoints \( \tilde{\gamma} \cap G \). By Remark \( \text{[33]} \), \( G \) is a hyperbolic graph with a factor system \( \mathcal{H}_G \) of complexity strictly less than the complexity of \( \mathcal{H} \). Since \( G \) is uniformly quasi-isometrically embedded in \( \Gamma \), we can control the \( G \)-distance between \( a, b \). In particular, we get a path \( \eta \) of \( \tilde{G} \) as in the statement of the Lemma. We claim that the path obtained substituting \( \eta \) to \( \gamma^{[a,b]} \) satisfies the original requirements. This is because \( \eta \) is uniformly a quasi-geodesic of \( \Gamma \), and thus there is a uniform \( \tau \) such that \( \eta \) is part of a \( \tau \)-partial de-electrification of \( \gamma \). Since the original complexity was finite, and at each step all the constants can be chosen uniformly, this proves the claim. \( \square \)

### 3.2 Hierarchical structure

In this section, we will prove that given a \( \delta \)-hyperbolic graph \( \Gamma \) and a factor system \( \mathcal{H} \), that the pair \((\Gamma, \mathcal{H})\) provides a hierarchically hyperbolic structure for \( \Gamma \). We emphasize that the non trivial part of the claim is the one concerning the indexing set. In fact, every hyperbolic space \( X \) admits \((X, \{X\})\) as a hierarchically hyperbolic structure.

First, we recall the definition of hierarchically hyperbolic space.

**Definition 3.6** (Hierarchically hyperbolic space, \([BHLS15]\)).

The \( q \)-quasigeodesic space \((X, d_X)\) is a **hierarchically hyperbolic space** if there exists \( \delta \geq 0 \), an index set \( \mathcal{S} \), and a set \( \{CW : W \in \mathcal{S}\} \) of \( \delta \)-hyperbolic spaces \((CU, d_U)\), such that the following conditions are satisfied:

1. **(Projections.)** There is a set \( \{\pi_W : X \to 2^{CW} \mid W \in \mathcal{S}\} \) of projections sending points in \( X \) to sets of diameter bounded by some \( \xi \geq 0 \) in the various \( CW \in \mathcal{S} \). Moreover, there exists \( K \) so that each \( \pi_W \) is \((K, K)\)-coarsely Lipschitz.

2. **(Nesting.)** \( \mathcal{S} \) is equipped with a partial order \( \subseteq \), and either \( \mathcal{S} = \emptyset \) or \( \mathcal{S} \) contains a unique \( \subseteq \)-maximal element; when \( V \subseteq W \), we say \( V \) is nested in \( W \). We require that \( W \subseteq W \) for all \( W \in \mathcal{S} \). For each \( W \in \mathcal{S} \), we denote by \( \mathcal{S}_W \) the set of \( V \in \mathcal{S} \) such that \( V \subseteq W \). Moreover, for all \( V,W \in \mathcal{S} \) with \( V \) properly nested in \( W \) there is a specified subset \( \rho^W_V \subset CW \) with \( \text{diam}_{CW}(\rho^W_V) \leq \xi \). There is also a projection \( \rho^W_V : CW \to 2^{CV} \). (The similarity in notation is justified by viewing \( \rho^W_V \) as a coarsely constant map \( CV \to 2^{CW} \).

3. **(Orthogonality.)** \( \mathcal{S} \) has a symmetric and anti-reflexive relation called orthogonality: we write \( V \perp W \) when \( V,W \) are orthogonal. Also, whenever \( V \subseteq W \) and \( U \subseteq W \), we require that \( V \perp U \). Finally, we require that for each \( T \in \mathcal{S} \) and each \( U \in \mathcal{S}_T \) for which \( \{V \in \mathcal{S}_T \mid V \perp U\} \neq \emptyset \), there exists \( W \in \mathcal{S}_T - \{T\} \), so that whenever \( V \perp U \) and \( V \subseteq T \), we have \( V \subseteq W \). Finally, if \( V \perp W \), then \( V,W \) are not \( \subseteq \)-comparable.

4. **(Transversality and consistency.)** If \( V,W \in \mathcal{S} \) are not orthogonal and neither is nested in the other, then we say \( V,W \) are transverse, denoted \( V \pitchfork W \). There exists \( \kappa_0 \geq 0 \) such that if \( V \pitchfork W \), then there are sets \( \rho^W_V \subseteq CW \) and \( \rho^V_W \subseteq CV \) each of diameter at most \( \xi \) and satisfying:

\[
\min \left\{ d_W(\pi_W(x), \rho^W_V), d_V(\pi_V(x), \rho^V_W) \right\} \leq \kappa_0
\]

for all \( x \in X \).

For \( V,W \in \mathcal{S} \) satisfying \( V \subseteq W \) and for all \( x \in X \), we have:

\[
\min \left\{ d_W(\pi_W(x), \rho^W_V), \text{diam}_{CV}(\pi_V(x) \cup \rho^V_W(\pi_W(x))) \right\} \leq \kappa_0.
\]

The preceding two inequalities are the consistency inequalities for points in \( X \).

Finally, if \( U \subseteq V \), then \( d_W(\rho^W_U, \rho^V_W) \leq \kappa_0 \) whenever \( W \in \mathcal{S} \) satisfies either \( V \subseteq W \) and \( V \neq W \) or \( V \pitchfork W \) and \( W \perp U \).
5. (Finite complexity.) There exists $n \geq 0$, the complexity of $\mathcal{X}$ (with respect to $\mathcal{S}$), so that any set of pairwise-\(\leq\)-comparable elements has cardinality at most $n$.

6. (Large links.) There exist $\lambda \geq 1$ and $E \geq \max\{\xi, \kappa_0\}$ such that the following holds. Let $W \in \mathcal{S}$ and let $x, x' \in \mathcal{X}$. Let $N = \lambda d_W(\pi_W(x), \pi_W(x')) + \lambda$. Then there exists $\{T_i\}_{i=1}^{\cdots} \subseteq \mathcal{S} - \{W\}$ such that for all $T \in \mathcal{S} - \{W\}$, either $T \in \mathcal{S} - \{W\}$ for some $i$, or $d_T(\pi_T(x), \pi_T(x')) < E$. Also, $d_W(\pi_W(x), \pi_W(x')) \leq N$ for each $i$.

7. (Bounded geodesic image.) For all $W \in \mathcal{S}$, all $V \in \mathcal{S} - \{W\}$, and all geodesics $\gamma$ of $CW$, either $\text{diam}_V(\rho^W_1(\gamma)) \leq E$ or $\gamma \cap N_E(\rho^W_1) \neq \emptyset$.

8. (Partial Realization.) There exists a constant $\alpha$ with the following property. Let $\{V_j\}$ be a family of pairwise orthogonal elements of $\mathcal{S}$, and let $p_j : \pi_V(\mathcal{X}) \subseteq CV_j$. Then there exists $x \in \mathcal{X}$ so that:
   - $d_V(x, p_j) \leq \alpha$ for all $j$,
   - for each $j$ and each $V \in \mathcal{S}$ with $V_j \subseteq V$, we have $d_V(x, \rho^j_1) \leq \alpha$, and
   - if $W \cap V_j$ for some $j$, then $d_W(x, \rho^j_1) \leq \alpha$.

9. (Uniqueness.) For each $\kappa \geq 0$, there exists $\theta_n = \theta_n(\kappa)$ such that if $x, y \in \mathcal{X}$ and $d(x, y) \geq \theta_n$, then there exists $V \in \mathcal{S}$ such that $d_V(x, y) \geq \kappa$.

We often refer to $\mathcal{S}$, together with the nesting and orthogonality relations, the projections, and the hierarchy paths, as a hierarchically hyperbolic structure for the space $\mathcal{X}$.

**Convention.** As before, we will assume that a hyperbolic graph $\Gamma$ and a factor system $H$ are fixed.

**Definition 3.7.** Let $\hat{\Gamma}$ be the cone-off of $\Gamma$ with respect to $H$, and let $i_\Gamma : \Gamma \to \hat{\Gamma}$ be the bijection on the vertex set. As remarked before, $i_\Gamma$ is distance-non-increasing, and $i_\Gamma^{-1}$ is distance-non-decreasing.

1. For each $W \in H$, let $\hat{W}$ be the cone-off of $W$ with respect to the family $H_W = \{H \in H \mid H \subseteq W\}$. Since $\hat{W} \subseteq \hat{\Gamma}$, the maps $i_\Gamma$ and $i_\Gamma^{-1}$ restrict to maps $i_W, i_W^{-1}$. As before, the maps $i_W$ are distance-non-increasing and $i_W^{-1}$ distance-non-decreasing. Note that $\hat{W} = i_W(W)$.

2. We denote with $\pi_W : \Gamma \to 2^{\hat{W}}$ the map $i_W \circ p_W$, where $p_W$ denotes the shortest distance projection in $\Gamma$.

3. For $V, W \in H$ such that $V \not\subseteq W$, we define $\rho^W_V$ to be $p_V(W)$. Condition 2 of the definition of factor system yields that the sets $\rho^W_V$ have uniformly bounded diameter.

4. For $V, W \in H$ such that $V \subseteq W$, we define a map $\rho^W_V : \hat{W} \to \hat{V}$ as $\rho^W_V = \pi_V \circ i_W^{-1}$.

The following result is proved in [KR14, Proposition 2.6]. It is an application of the Bowditch criterion for hyperbolicity (see [Bow10]).

**Proposition 3.8** (Kapovich-Rafi, Bowditch). Let $\Gamma$ be a connected graph with simplicial metric $d_\Gamma$ such that $(\mathcal{X}, d_\mathcal{X})$ is $\delta$-hyperbolic. Let $K > 0$ and $H$ be a family of $K$-quasiconvex subgraphs of $\Gamma$. Let $\hat{\Gamma}$ be the cone-off of $\Gamma$ with respect to the family $H$. Then $\hat{\Gamma}$ is $\delta'$-hyperbolic (with respect to the path metric) for some constant $\delta' > 0$ depending only on $K$ and $\delta$. Moreover there exists $H = H(K, \delta) > 0$ such that whenever $x, y \in V(\Gamma)$, $[x, y]_\Gamma$ is a $d_\Gamma$-geodesic from $x$ to $y$ in $\Gamma$ and $[x, y]_{\hat{\Gamma}}$ is a $d_{\hat{\Gamma}}$-geodesic from $x$ to $y$ in $\hat{\Gamma}$, then $[x, y]_{\hat{\Gamma}}$ and $[x, y]_{\hat{\Gamma}}$ are $H$-Hausdorff close in $(\hat{\Gamma}, d_{\hat{\Gamma}})$.

**Corollary 3.9.** There is a uniform $\delta'$ such that the spaces $\hat{W}$, for $W \in H \cup X$, are $\delta'$-hyperbolic.

**Proposition 3.10** (Bounded projections). There exists $\Theta$ such that for each pair $F, W \in H$, one of the following holds:

- $\text{diam}_F(\rho^F_V(\hat{W})) \leq \Theta$.

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F ⊆ W.

Proof. Let Θ = 2B + ξ + 2 and suppose that the first case does not hold. Unraveling the definitions, we get:

\[ \rho_F^W(\tilde{W}) = \pi_F \circ i_W^{-1}(i_W(W)) = \pi_F(W) = i_F(p_F(W)). \]

Since \( i_F \) is distance-non-increasing, we get that \( \text{diam}_F(p_F(W)) > \Theta > \xi \). By definition of factor system, this implies that there is \( U \subseteq \mathcal{H} \) such that \( U \subseteq F \) and \( \text{diam}_F(p_F(W), U) \leq B \). Since \( \Theta = 2B + \xi + 2 \), we have that \( \text{diam}_F(U) \geq 2 \), thus \( U \) is not coned-off in \( \tilde{F} \). Since each element of \( \mathcal{H}_F \) is distance-non-increasing, we get that \( U \notin \mathcal{H}_F \) and hence \( U = F \). Thus, \( \text{diam}_F(F, p_F(W)) \leq B \). The definition of factor system then implies that \( F \subseteq W \).

Lemma 3.11 (Bounded geodesic image). There exists \( B \) such that for all \( W \in \mathcal{H} \) and \( V \in \mathcal{H}_W \) and all geodesics \( \gamma \) of \( \tilde{W} \), either \( \text{diam}_V(\rho_W^W(\gamma)) < B \), or \( \gamma \cap N_B(\rho^W_V) \neq \emptyset \).

Proof. Suppose that \( \text{diam}(\rho^W_W(\gamma)) > 8\delta + K \), where \( K \) is the uniform quasi-convexity constant of the elements of \( \mathcal{H} \). We claim that this implies that \( \gamma \cap N_{2B + K + \rho^W_V(\gamma)} \neq \emptyset \), where \( \rho^W_V \) is the constant of Proposition 3.12. Let \( V \in \mathcal{H} \), and \( x, y \in \gamma \) be in the preimage under \( \rho^W_V \) of some \( a, b \). Note that \( d_V(a, b) \geq d_V(a, b) = 8\delta + K \). Then a quadrilateral argument gives that \( d_V([x, y]) < 2\delta + K \), where \([x, y]\) is a geodesic in \( \Gamma \). But since, by Proposition 3.12, \( \text{diam}_V(p_V(x), p_V(y)) \leq N \) for each \( V \). We will show that if \( E \) is chosen large enough, the proposition holds. We will give a more precise characterization of large during the proof. For now, just assume it is much larger than \( \delta, K \) and \( \Theta \), where \( K \) is the quasi-convexity constant of the family \( \mathcal{H} \), and \( \Theta \) is as in Proposition 3.10. Since \( W \) is uniformly quasi-isometrically embedded in \( \tilde{W} \), and since \( \mathcal{H}_W \) is uniformly a factor system, to simplify notation we can assume that \( W = \Gamma \). In fact, for each \( U \subseteq W \) and \( x \in \Gamma \), we have that \( p_V(x) \) and \( p_V(p_W(x)) \) (and hence \( \pi_V(x) \) and \( \pi_V(p_W(x)) \)) coarsely coincide (see Proposition 2.28). Thus we can substitute \( x \) and \( y \) with \( \pi_W(x) \) and \( \pi_W(y) \).

Let \( \gamma \) be a \( \gamma \)-quasi-geodesic of \( \tilde{\Gamma} \) between \( x \) and \( y \), such that for every de-electrification \( \tilde{\gamma} \), we have \( \tilde{\gamma} \subseteq N_{\Delta}(x, y) \). Proposition 2.28 guarantees that such a quasi-geodesic exists and that \( \gamma \), \( \Delta \) can be chosen uniformly. Suppose that there is \( F \) with \( d_F(\pi_F(x), \pi_F(y)) \geq E \). We claim that if \( \tilde{\gamma} \) does not have a \( F \)-piece, then there is \( U \in \mathcal{H} \) such that \( \tilde{\gamma} \) has a \( U \)-piece and \( F \subseteq U \). Note that the number of \( \mathcal{H} \)-pieces of \( \gamma \) is at most \( 2\hat{L}_F(\gamma) \leq \tau_1 d_F(x, y) + \tau_1 \).

Since \( \gamma \subseteq N_{\Delta}(x, y) \), we can find points \( a, b \) of \( \tilde{\gamma} \) such that:

\[
\begin{align*}
&d_F(a, F) < 2\delta + K \quad \text{and} \quad d_F(b, F) < 2\delta + K, \quad \text{where} \quad K \quad \text{is the quasi-convexity constant of} \quad F. \\
&d_F(p_F(a'), p_F(x)) \leq 4\delta + 2K, \quad \text{and} \quad d_F(p_F(b'), p_F(y)) \leq 4\delta + 2K.
\end{align*}
\]

Since \( \tilde{\gamma} \subseteq N_{\Delta}(x, y) \), we can find points \( a, b \) of \( \tilde{\gamma} \) such that:

\[
\begin{align*}
&d_F(a, F) \leq 2\delta + K + \Delta \quad \text{and} \quad d_F(b, F) \leq 2\delta + K + \Delta. \\
&d_F(p_F(a), p_F(x)) \leq 2(4\delta + 2K + \Delta) \quad \text{and} \quad d_F(p_F(b), p_F(y)) \leq 2(2K + 4\delta + \Delta).
\end{align*}
\]
Since \( F \) is uniformly quasi-isometrically embedded in \( \Gamma \), and since distances in \( F \) are larger than distances in \( \hat{\Gamma} \), the last set of inequalities gives that we can find a uniform \( \rho \) such that \( d_{\hat{\Gamma}}(\pi_{\Gamma}(a), \pi_{\Gamma}(x)) \leq \rho \), and similarly \( d_{\hat{\Gamma}}(\pi_{\Gamma}(b), \pi_{\Gamma}(y)) \leq \rho \). Since, by assumption, \( d_{\hat{\Gamma}}(\pi_{\Gamma}(x), \pi_{\Gamma}(y)) \geq E \), we obtain that \( d_{\hat{\Gamma}}(\pi_{\Gamma}(a), \pi_{\Gamma}(b)) \leq E - 2\rho \). We would like to assume that \( a \) and \( b \) are points of \( \gamma \). Since this is not true in general, replace \( \gamma \) with its \( a,b \)-interruption \( \tilde{\gamma} \). Note that \( \tilde{\gamma} \) is a \((1, 4K + 2\delta)\)-quasi geodesic of \( CX \). We want now to estimate \( L(\tilde{\gamma}|_{[a,b]}) \) _\( \hat{\Gamma} \_. Since \( F \) is coned-off in \( \hat{\Gamma} \), we have that

\[
d_{\hat{\Gamma}}(a,b) \leq d_{\hat{\Gamma}}(a,F) + d_{\hat{\Gamma}}(b,F) + 1 \leq 2(2\delta + K + \Delta) + 1.
\]

Thus, \( L(\tilde{\gamma}|_{[a,b]}) \) _\( \hat{\Gamma} \_ \leq 2(2\delta + 2K + \Delta) + 1 = B \). In particular, \( \tilde{\gamma}|_{[a,b]} \) has at most \( B \) pieces. Choosing \( E > B\Theta + 2\rho \), we get that there is at least one \( U \)-piece \( \eta \) of \( \tilde{\gamma}|_{[a,b]} \), for some \( U \in \hat{\mathcal{H}} \), such that \( \text{diam}_{\hat{\Gamma}}(\hat{\rho}_{\hat{\Gamma}}(\eta)) > \Theta \). But then by proposition 3.10, we have that \( F \subseteq U \), which proves the claim.

**Proposition 3.13 (Uniqueness).** For each \( \theta \) there exists \( T_0 \) such that if \( x, y \in \Gamma \) and \( d_r(x, y) \geq T_0 \), then there exists \( V \in \mathcal{H} \) such that \( d_{\hat{\Gamma}}(\pi_{\Gamma}(x), \pi_{\Gamma}(y)) \geq \theta \).

**Proof.** We stick to the convention that \( [x, y] \) indicates a geodesic in \( \Gamma \) between \( x \) and \( y \). We will proceed by induction on the complexity of \( \mathcal{H} \). If the complexity is 0 or 1, the result is respectively trivial or follows from Lemma 2.10. Suppose the result holds for complexity \( n - 1 \). Applying Lemma 3.5, we have that there exists \( T_C \) such that if \( d_r(x, y) \geq T_C \), then there is \( V \in \mathcal{H} \), a quasi-geodesic \( \tilde{\gamma} \) of \( \Gamma \) and points \( s, t \) in \( V \), such that:

- \( s, t \in \tilde{\gamma} \),
- there is a quasi-geodesic \( \sigma \) of \( \tilde{\Gamma} \) connecting \( s \) and \( t \),
- \( L_\tilde{\Gamma}(\sigma) \geq C \), in particular we can estimate \( d_{\tilde{\Gamma}}(s, t) \),
- \( \tilde{\gamma}|_{[s,t]} = \tilde{\sigma} \) is a de-electrification of \( \sigma \).

Since \( \tilde{\gamma} \) is a quasi-geodesic of \( \Gamma \), there is \( E \) such that \( \tilde{\gamma} \subseteq N_E^E([x,y]) \).

Let \( s' \in \pi_{\Gamma}(x), t' \in \pi_{\Gamma}(y) \) realizing the distance \( d_{\tilde{\Gamma}}(\pi_{\Gamma}(x), \pi_{\Gamma}(y)) \). Let \( q_s \) (resp. \( q_t \)) be witnessing the closest point projection in \( \Gamma \) of \( s \) (resp. \( t \)) on \([x, y]\). Since \( d_r(q_s, s) \leq E \) and \( d_r(q_t, t) \leq E \), triangular inequality gives that

\[
d_{\Gamma}(x, s) + d_{\Gamma}(s, t) + d_{\Gamma}(t, y) - 4E \leq d_{\Gamma}(x, y).
\]

Moreover, by the choice of \( s', t' \), we have that \( d_{\Gamma}(x, s') \leq d_{\Gamma}(x, s) \) and \( d_{\Gamma}(t', y) \leq d_{\Gamma}(t, y) \). Thus, we get that the left-hand-side of the above inequality is bounded below by \( d_{\Gamma}(x, s') + d_{\Gamma}(s, t) + d_{\Gamma}(t', y) - 4E \). Moreover, by triangular inequality, the right-hand-side is bounded above by \( d_{\Gamma}(x, s') + d_{\Gamma}(s', t') + d_{\Gamma}(t', y) \). Thus we get

\[
d_{\Gamma}(s, t) - 4E \leq d_{\Gamma}(s', t').
\]

In particular, up to increasing \( d_{\Gamma}(x, y) \), we can arbitrarily increase \( d_{\Gamma}(s', t') \). Since \( \mathcal{H}_V \) is a factor system for \( V \), of complexity \( n - 1 \), we can apply the induction hypothesis on \( s', t' \). Thus we will find a space \( U \subseteq V \) such that \( d_{\hat{\Gamma}}(\pi_{\Gamma}(s'), \pi_{\Gamma}(t')) \leq d_{\hat{\Gamma}}(\pi_{\Gamma}(x), \pi_{\Gamma}(y)) + \delta \). Since, by Proposition 2.10, the projection \( \pi_{\Gamma}(\pi_{\Gamma}(x)) \) and \( \pi_{\Gamma}(\pi_{\Gamma}(y)) \) coarsely coincide, for \( U \subseteq V \), up to further increasing \( d_{\Gamma}(x, y) \), we get the claim. We remark that, in order to get uniform bound on the constants, it is crucial for the complexity of \( \mathcal{H} \) to be finite.

**Verifying the axioms**

We will now verify that given a \( \delta \)-hyperbolic graph \( \Gamma \) and a factor system \( \mathcal{H} \) of \( \Gamma \), the set \( \mathcal{H} \cup \{\Gamma\} \) provides and hierarchically hyperbolic structure on \( \Gamma \). In particular, we have:

**Theorem 3.14.** Let \( \Gamma \) be a \( \delta \)-hyperbolic graph and \( \mathcal{H} \) a factor system for \( \Gamma \). Then there is a hierarchically hyperbolic space structure on \( \Gamma \) with indexing set \( \mathcal{H} \cup \{\Gamma\} \).
Proof. We claim that the set \( \{ \hat{W} \mid W \in H \cup \{ \Gamma \} \} \) satisfies the required conditions. Clearly the set is indexed by \( H \cup \{ \Gamma \} \). First note that the spaces \( \hat{W} \) for \( W \in H \cup \{ \Gamma \} \) are uniformly \( \delta' \)-hyperbolic metric spaces (Corollary 3.9). Then we have:

1. There is a set of projections \( \{ \pi_W : \Gamma \to \hat{W} \} \) (definition 3.7) that are uniformly coarsely Lipschitz. In fact, by definition, \( \pi_W = i_W \circ p_W \) which is the composition of a coarsely Lipschitz map (Lemma 2.14) with a distance-non-increasing map (Definition 3.7).

2. The set \( H \cup \{ \Gamma \} \) is naturally equipped with the partial order \( \subseteq \) induced by inclusion with maximal element \( \Gamma \). The sets and projections \( \rho_{WV} \) are defined in Definition 3.7. Since, for \( V \not\subseteq W \), \( \rho_{WV} \in H_V \), we get that \( \rho_{WV} \) is coned-off in \( V \), and thus \( \text{diam}_V(\rho_{WV}) = 2 \).

3. There are no orthogonality relations.

4. \( \cdot \) If \( W \not\subseteq V \not\subseteq W \), then there exists a uniform \( \kappa_0 \) such that, for each \( x \in \Gamma \),

   \[
   \min(\hat{d}(\pi_V(x), \rho_{WV}^V), \hat{d}(\pi_W(x), \rho_{WV}^W)) \leq \kappa_0.
   \]

   This is guaranteed by Lemma 2.14 and the fact that the maps \( i_W \) are distance-non-increasing.

   \( \cdot \) If \( V \subseteq W \), then for all \( x \in \Gamma \):

   \[
   \min(\hat{d}(\pi_W(x), \rho_{WV}^W), \hat{d}(\pi_V(x) \cup \rho_{WV}^V(\pi_V(x)))) \leq \kappa_0.
   \]

   This is guaranteed by Lemma 2.15 and again the distance-non-increasingness.

5. Finite complexity is clear by the definition of factor system.

6. The large link lemma is proved in Proposition 3.12.

7. The bounded geodesic image property is proved in Proposition 3.11.

8. The partial realization is trivial since there are no orthogonality relations, hence every family of pairwise orthogonal elements consists of a single element.

9. The uniqueness property is proved in Proposition 3.13.

\( \square \)

4 Obtaining a factor system

4.1 The case of a general metric space

The definition of factor system of Section 3 is not coarse and, as stated, works only for graphs. The goal of this section is to address this issue, namely, to provide weaker conditions that a metric space needs to satisfy in order to be quasi-isometric to a graph equipped with a factor system.

The first ingredient that we will need is a coarse definition of inclusion.

**Definition 4.1 (Coarse inclusion).** Let \( X \) be a metric space and \( V, W \) be two subspaces. We say that \( V \) is coarsely contained in \( W \), and denote it by \( V \preceq W \), if there exists \( R \) such that \( V \subseteq N_R(W) \). We say that \( V \) is properly coarsely contained in \( W \), and denote it by \( V \subsetneq W \), if \( V \preceq W \) and for each \( R, W \not\subseteq N_R(V) \). The relation \( \preceq \) will be called coarse inclusion.

**Definition 4.2 (Weak Factor System).** Let \( X \) be a quasi-geodesic Gromov-hyperbolic space. A weak factor system for \( X \) is a family \( F \) of \( J \)-quasi-convex subspaces such that there exist constants \( c, \xi, D', B', q \) such that the following holds:

1. Every chain of proper coarse inclusions \( H_n \preceq \cdots \preceq H_1 \) of elements of \( F \) has length at most \( c \).
2. Given \( V, W \in \mathcal{F} \), then either \( \text{diam}_V(p_V(W)) < \xi' \), or there exists \( U \in \mathcal{F} \) such that \( d_{\text{Haus}}(U, p_V(W)) \leq B' \).

3. For each \( V \in \mathcal{F} \) and every \( v \in V \) there is an arbitrarily long \( q \)-quasi-geodesic segment with endpoints on \( V \) whose midpoint lies at distance at most \( D' \) from \( v \).

If, in addition, the elements of \( \mathcal{F} \) are uniformly quasi-isometrically embedded in \( X \) and the following stronger condition is satisfied:

3'. For each \( V \in \mathcal{F} \) and every \( v \in V \) there is an arbitrarily long geodesic segment with endpoints on \( V \) whose midpoint lies at distance at most \( D' \) from \( v \),

we say that \( \mathcal{F} \) is a geodesic factor system for \( X \).

**Remark 4.3** (Up to quasi-isometry, all weak factor systems are geodesic.). Let \( X \) be a \((C, \varepsilon)\)-quasi-geodesic metric space and \( \mathcal{F} \) be a weak factor system for \( X \). Then Corollary 2.7 and Lemma 3.19 provide a uniform quasi-isometry \( \omega : X \to \Omega(X) \) to a geodesic metric space \( \Omega(X) \) such that the images \( \omega(F) \) of the elements of \( \mathcal{F} \) constitute a geodesic weak factor system for \( \Omega(X) \).

If the space \( X \) is already a geodesic space, it is easily seen that condition (3) implies condition (3'). However, the elements \( F \in \mathcal{F} \) may not be quasi-isometrically embedded. Thus in order to obtain a geodesic weak factor system, it may still be necessary to pass to an appropriate approximation graph as above.

**Remark 4.4.** The condition (3) of the definition of weak factor system is more restrictive than necessary. Indeed, Example 4.5 shows a space \( X \) and a factor system for \( X \) that does not satisfy condition (3). However, condition (3) can be easily verified for a large class of natural examples, such as infinite quasi-convex subgroups of hyperbolic groups.

**Example 4.5.** Let \( m \in \mathbb{N} \cup \{\infty\} \), and for each \( 0 \leq i \leq m \) let \( I_i \) be a copy of the ray \([0, \infty)\) indexed by \( i \). Let \( X \) be the space obtained identifying together the point 0 of each of the rays \( I_i \). It is clear that we can regard \( X \) as a graph, in particular as a tree. For each \( n > 0 \), let \( F_n \) be the union \( I_0 \cup I_n \). Then \( \{F_n : n > 0\} \cup I_0 \) is a factor system for \( X \), but the space \( I_0 \) does not satisfy condition (3) of the definition of weak factor system.

**Lemma 4.6** (Main consequence of condition (3')). Given a geodesic \( \alpha \)-hyperbolic space \( X \) and a geodesic weak factor system \( \mathcal{F} \) for \( X \), there exists \( \zeta = \zeta(\alpha, \mathcal{H}) \) such that for each \( V, W \in \mathcal{F} \), if \( V \preceq W \), then \( V \preceq N_{\zeta}(W) \).

**Proof.** Let \( \zeta = 2\alpha + D' + J \) and \( v \in V \). Then there exists \( v' \in V \) with \( d(v, v') \leq D' \) and a geodesic segment \( \gamma \) with endpoints in \( V \) such that \( \gamma \) has length \( 2R + 4\alpha + 1 \) and \( v' \) is the midpoint of \( \gamma \). Let \( a, b \) be the endpoints of \( \gamma \) and \( a', b' \) points in \( W \) satisfying \( d(a, a') \leq R \), and \( d(b, b') \leq R \). By hyperbolicity, \( v' \preceq N_{2\alpha}([a, a'] \cup [a', b'] \cup [b', b]) \). We claim that \( v' \preceq N_{2\alpha}([a', b']) \). Then, by \( J \)-quasi-convexity of \( W \), the first property follows. So suppose that the claim does not hold, and assume that \( d(v', [a, a']) < 2\alpha \). Let \( m \in [a, a'] \) be a point realizing the distance. Since \( d(v', a) = R + 2\alpha + 1 \) and \( d(m, a') \leq d(a, a') \leq R \), we get a contradiction.

We consider the following equivalence relation on \( \mathcal{F} \): we say that \( V \) is in relation with \( W \) if \( V \preceq W \preceq V \), that is, if \( V \) and \( W \) have bounded Hausdorff distance. Let \( \mathcal{H}(\mathcal{F}) \) be the set of equivalence classes of \( \mathcal{F} \). It is easily checked that if \( V \preceq W \), then for each \( V' \in [V] \) and \( W' \in [W] \), one has \( V' \preceq W' \). Thus, the relation \( \preceq \) descends to a partial order on the set \( \mathcal{H} \) that we will denote by \( \subseteq \). The goal of this section is to prove the following:

**Theorem 4.7.** Let \( X \) be a Gromov hyperbolic quasi-geodesic space space and \( \mathcal{F} \) a weak factor system for \( X \). Then \( (X, \{\mathcal{X} \cup \mathcal{H}(\mathcal{F})\}) \) is a hierarchically hyperbolic structure for \( X \).

The proof of Theorem 17 will a corollary of Proposition 113, Remark 13 and the fact that hierarchically hyperbolic structures are invariant under quasi-isometry. The rest of this section is devoted in proving Proposition 114.
Proof. Since the first condition implies that for each \( V \) the definition does not depend on representatives. For \( \text{Lemma 4.11.} \) The following are equivalent: \( \text{Corollary 4.10.} \) The spaces \( \text{Lemma 4.9.} \) provides that for each \( W \in \mathcal{F} \), the spaces \( \text{Lemma 4.8.} \) Given \( \mathcal{F} \), the following hold:

- \( V \preceq W \) if and only if \( \Omega(V) \preceq \Omega(W) \).
- If \( \Omega(V) \preceq \Omega(W) \), then \( \Omega(V) \subseteq N_1(\Omega(W)) \).

Proof. Since \( X \) and \( \Omega(X) \) are quasi-isometric, the first condition clearly holds. So suppose that \( \Omega(V) \preceq \Omega(W) \). By the first condition we get that \( V \preceq W \). In particular, by Lemma \( 4.6 \) we get that \( V \preceq N_1(W) \). Let \( x \) be a point of \( \Omega(V) \). Seeing it as a point of \( X \), since \( N \) is a maximal \( \zeta \)-net, we get that there is \( v \in V \) that has distance at most \( \zeta \) from \( x \). But since \( V \subseteq N_1(W) \), there is a point \( w \in W \) that has distance at most \( \zeta \) from \( v \). Again by the properties of \( N \), there exists \( y \in \Omega(W) \) such that \( d(y, w) \leq \zeta \). In particular, \( d(x, y) \leq 2\zeta + \zeta < 5\zeta \), thus they are connected by an edge in \( \Omega(X) \).

For a class \( [V] \in \mathcal{H} \), define the set \( \preceq V = \{ U \in \mathcal{F} \mid U \preceq V \} \). It is easily checked that the definition does not depend on representatives. For \( [V] \in \mathcal{H} \), we define:

\[
CV = \bigcup_{U \in [V]} \Omega(U),
\]

and we also define \( CX = \Omega(X) \).

As a consequence of Lemma \( 4.8 \) we have that for each \( V \in [V] \),

\[ d_{\text{Haus}}(\Omega(V), CV) \leq 1. \]

Lemma 4.9. The spaces \( CW \) equipped with the induced path metric are uniformly quasi-isometrically embedded in \( CX \).

Proof. This is because for each \( W \in [W] \), we have that \( CW \subseteq N_1(\Omega(W)) \). Since the spaces \( \Omega(W) \) are uniformly quasi-isometrically embedded in \( \Omega(W) \) (see Corollary \( 2.7 \)), the claim follows.

Corollary 4.10. The spaces \( CW \) are uniformly quasi-convex and Gromov hyperbolic.

Lemma 4.11. The following are equivalent:

- \( CV \preceq CW \),
- \( [V] \subseteq [W] \),
- \( CV \subseteq CW \).

Proof. Since \( CV \) is the union of sets that have Hausdorff distance at most 1, it is clear that the first condition implies that for each \( V' \in [V] \) and \( W' \in [W] \), we have \( V' \approx W' \), which is the definition of the second condition. The second implies the third because of the definition of the spaces \( CV \). The third trivially implies the first.
Lemma 4.12. Given $CV$ and $CW$, if $CV \lesssim p_{CV}(CW)$, then $[V] \subseteq [W]$.

Proof. We will show that $CV \lesssim CW$, which implies the claim. Let $x \in p_{CV}(CW)$ be any point. By definition of $CV$, there is $V' \subseteq [V]$ and $v \in V'$ such that $d_X(x,v) \leq \zeta$. By condition (3') of geodesic weak factor system, there is an arbitrarily long geodesic segment $\gamma$ with endpoints in $V'$ and midpoint $v'$ such that $d(v,v') \leq D'$, where $D'$ is an uniform constant. Since the map $\chi$ is a quasi-isometry, we get that there is a quasi-geodesic segment $\eta$ with endpoints in $CV$, such that $x$ is uniformly close to the midpoint of $\chi(\gamma)$.

Lemma 4.13 (Coarse commutativity of projections and quasi-isometries). Let $f : X \to Y$ be a $C$-quasi-isometry. Let $H, J$ be $K$-quasi-convex subspaces of $X$. Then there exists $M$ such that $d_{\text{Haus}}(f(p_H(J)), p_{f(H)}(f(J))) \leq M$.

Proof. Let $\delta_X$ be the hyperbolicity constant of $X$ and $\delta_Y$ the one of $Y$. We claim there exists $M_1 = M_1(\delta_X, \delta_Y, C)$ such that for any $x \in J$, $d(f(p_H(x)), p_{f(H)}(f(x))) \leq M_1$. In order to simplify notation we put $y = f(p_H(x))$ and $z = p_{f(H)}(f(x))$. Consider a geodesic triangle between $f(x), y$ and $z$. Since $[f(x), z]$ is a projection geodesic, it is easily seen that there is a point $m \in [f(x), y]$ that has distance at most $2\delta_Y$ from $f(H)$. However, since geodesics are uniformly near to quasi-geodesics, $m$ is uniformly close to $f([x, f^{-1}(y)])$. Thus, there is a point $m' \in f([x, f^{-1}(y)])$ such that it is possible to uniformly estimate the distances $d(m', z)$ and $d(m', H)$. Since $f([x, f^{-1}(y)])$ is uniformly a quasi-geodesic, this proves the claim.

Lemma 4.14. There exists $\xi, B$ such that if $\text{diam}(p_{CV}(CW)) \geq \xi$, then there is $U \in \mathcal{F}$ such that $[U] \subseteq [V]$ and $d_{\text{Haus}}(CU, p_{CV}(CW)) \leq B$.

Proof. Since projections are quasi-Lipschitz and since, for each $V' \subseteq [V]$, $d_{\text{Haus}}(\chi(V'), CV) \leq 1$, it is easily seen that is possible to uniformly bound the Hausdorff distance between $p_{CV}(CW)$ and $\bigcup_{V' \subseteq [V], W' \subseteq [W]} p_{\chi(V')}(\chi(W'))$. Thus it suffices to show that there exists $\beta$ such that for each $V \subseteq [V]$ and $W \subseteq [W]$, there exists $U \in \mathcal{F}$ with $d_{\text{Haus}}(\chi(U), p_{\chi(V')}(\chi(W))) \leq \beta$. But this is an easy consequence of Lemma 4.13.

Proposition 4.15. The family $\{CH | [H] \in \mathcal{H}\}$ is a factor system for $CX$.

Proof. We will show that all the items of Definition 5.1 are satisfied.

1. Lemma 4.9 gives that the spaces $CH$ are quasi-isometrically embedded in $CX$, with constants that can be chosen uniformly.
2. The second condition follows from Corollary 5.1 and Lemma 4.11.
3. The third follows from Lemma 4.12.
4. The fourth follows the requirement of weak factor system and Lemma 4.8.
5. The fifth follows from Corollary 5.1 and Lemma 4.6.

As a corollary, we get a version of Remark 5.5 for weak factor systems.
Corollary 4.16. Let $X$ be a Gromov hyperbolic space, $\mathcal{F}$ be a weak factor system for $X$ and let $F \in \mathcal{F}$. Let $\mathcal{H}(\mathcal{F})_F = \{[G] \in \mathcal{F} \mid [G] \subseteq [F]\}$. Then $(CF, \{CG \mid [G] \in \mathcal{H}(\mathcal{F})_F\})$ is a hierarchically hyperbolic space.

Proof. This is because, by Remark 3.3, $\{GC \mid [G] \in \mathcal{H}(\mathcal{F})_F\} - CF$ is a factor system for $CF$. 

4.2 The case of a graph

The construction described before can be simplified when $X$ is already a graph and $\mathcal{F}$ is a family of uniformly quasi-isometrically embedded subgraphs that satisfy the requirements of Definition 4.2. The hierarchy result is going to be unchanged, but we will introduce a construction that allows to choose $CX = X$ and consequently $CW \subseteq CX$. This is helpful for instance in the case when a group is acting on the space $X$, such as in the case when $X = \text{Cay}(G)$, for some group $G$. Indeed, this will produce an action on the $\subseteq$-maximal space of the hierarchy, instead of just a quasi-action. This will simplify some of the constructions in the next sections.

Convention. For the rest of this section, we will assume that an $\alpha$-hyperbolic graph $X$ and a family of connected, uniformly quasi-isometrically embedded subgraphs $\mathcal{F}$ that form a geodesic weak factor system (Definition 4.2) is fixed. As before, let $H(F)$ denote the quotient of $F$ by the equivalence relation given by finite Hausdorff distance. Recall that for a class $[V] \in H$, we have $[\preceq V] = \{U \in F \mid U \preceq V\}$. For $[V] \in H$, we define:

$$\overline{CV} = \bigcup_{U \in \preceq V} U.$$ 

The key property we gained is that for each class $[V]$, we have that $\overline{CV} \subseteq X$.

Lemma 4.17. There exists $k$ such that the spaces $\overline{CV}$ are uniformly $k$-quasi-convex.

Proof. Using Lemma 4.6, we obtain that each space $CV$ has uniformly bounded distance from a uniformly quasi-convex subgraph $V$.

However, in general the spaces $\overline{CV}$ will not be connected, which is an obstruction to quasi-isometric embeddedness. In order to solve this issue, we introduce the following definition:

Definition 4.18. Let $\Gamma$ be a graph and $Q$ be a subgraph of $\Gamma$. For $r > 0$ let $P_r(Q)$ to be the set of $\Gamma$-geodesic segments of length at most $r$ connecting two points in $Q$. Define

$$\text{Approx}_r(Q) = \bigcup_{\gamma \in P_r(Q)} \gamma.$$ 

The reader should think of $\text{Approx}_r(Q)$ as the graph-version of the approximation graph. Given $[V] \in H(\mathcal{F})$, we set

$$CW = \text{Approx}_{\zeta + k}(\overline{CW}).$$ 

Note that by quasi-convexity of the spaces $\overline{CW}$ we have that $CW \subseteq N_k(\overline{CW})$.

Lemma 4.19. The spaces $CW$ equipped with the path metric are quasi-isometrically embedded in $X$.

Proof. Let $x, y$ be two points of $CW$. Since they lie on a geodesic segment with endpoints in $\overline{CW}$, $k$-quasi-convexity of $\overline{CW}$ gives that $x, y \in N_k(\overline{CW})$. In particular, by Lemma 4.6 there is $W \in [W]$ such that $x, y \in N_{k+\zeta}(W)$. Then the same reasoning of Lemma 4.17 and the fact that the space $W$ is uniformly quasi-isometrically embedded in $X$ gives the claim. 

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It is easy to see that Lemmas 4.11, 4.12 and 4.14 still hold in this setting.
As before, we have the following proposition.

Proposition 4.20. The family \( \{CH | [H] \in H \} \) is a factor system for \( X \).

Proof. The proof follows as before by verifying the axioms. Alternatively, one can show that the spaces \( CW \) obtained with the two constructions are uniformly quasi-isometric. \( \Box \)

5 Groups with quasi-convex subgroups

The goal of this section is to prove that given a hyperbolic group \( G \) and a family of infinite quasi-convex subgroups \( F = \{F_1, \ldots, F_n\} \) of \( G \), there is an HHS structure on \( G \) that contains the elements of the family \( F \) in the indexing set. In particular, we will show that we can extend the family \( F \) to a weak factor system for \( G \). The key idea is the following: we will consider a "closure of \( F \) under projection". However, in this way we will only obtain a weak factor system, the main obstruction being the fact that there can be two distinct elements that have finite Hausdorff distance. Thus, we will proceed as in the case of a weak factor system, namely identifying such elements. Since the action of \( G \) on itself is compatible with the HHS structure, we can equip \( G \) with an HHG structure. We will start this section recalling the definition of HHG and then we will proceed in finding a weak factor system.

5.1 Hierarchically hyperbolic group structure

We recall the definition of hierarchically hyperbolic group.

Definition 5.1 (Hieromorphism, [BHS15], Definition 1.19). Let \( (\mathcal{X}, \mathcal{S}) \) and \( (\mathcal{X}', \mathcal{S}') \) be hierarchically hyperbolic structures on the spaces \( \mathcal{X}, \mathcal{X}' \) respectively. A hieromorphism, consists of a map \( f: \mathcal{X} \to \mathcal{X}' \), an injective map \( f^\circ: \mathcal{S} \to \mathcal{S}' \) preserving nesting, transversality, and orthogonality, and maps \( f^*(U): CU \to C(f^\circ(U)) \), for each \( U \in \mathcal{S} \), which are uniformly quasi-isometric embeddings. The three maps should preserve the structure of the hierarchically hyperbolic space, that is, they coarsely commute with the maps \( \pi_U \) and \( \rho^G_\mathcal{S} \), for \( U, V \) in either \( \mathcal{S} \) or \( \mathcal{S}' \), associated to the hierarchical structures.

Definition 5.2 (Automorphism, hierarchically hyperbolic group, [BHS15], Definition 1.20). An automorphism of the hierarchically hyperbolic space \( (\mathcal{X}, \mathcal{S}) \) is a hieromorphism \( f: (\mathcal{X}, \mathcal{S}) \to (\mathcal{X}, \mathcal{S}) \) such that \( f^\circ \) is bijective and each \( f^*(U) \) is an isometry.

The finitely generated group \( Q \) is hierarchically hyperbolic if there exists a hierarchically hyperbolic space \( (\mathcal{X}, \mathcal{S}) \) on which \( Q \) acts by automorphisms of hierarchically hyperbolic spaces, so that the uniform quasi-action of \( G \) on \( \mathcal{X} \) is metrically proper and cobounded and \( \mathcal{S} \) contains finitely many \( Q \)-orbits. Note that if \( Q \) is hierarchically hyperbolic by virtue of its action on the hierarchically hyperbolic space \( (\mathcal{X}, \mathcal{S}) \), then \( (Q, \mathcal{S}) \) is a hierarchically hyperbolic structure with respect to any word-metric on \( Q \); for any \( U \in \mathcal{S} \) the projection is the composition of the projection \( \mathcal{X} \to CU \) with a \( Q \)-equivariant quasi-isometry \( Q \to \mathcal{X} \).

In this case, \( (Q, \mathcal{S}) \) (with the implicit hyperbolic spaces and projections) is a hierarchically hyperbolic group structure.

In what follows, we will construct a weak factor system for the Cayley graph of a hyperbolic group \( G \). The weak factor system will be \( G \) equivariant, indeed, it will consists of cosets of subgroups of \( G \). Thus, the HHS structure on the Cayley graph of \( G \) will induce a HHG structure on \( G \).

5.2 Finding a factor system

In order to have a metric on a group \( G \), we need to fix a finite set of generators. In what follows, we always consider a fixed generating set. It is clear that all the following results are quasi-isometric invariant and hence holds for all finite sets of generators. We start by
recalling some well known properties of quasi-convex subgroups of hyperbolic groups. All the following facts and lemmas are proven in [BH99].

**Lemma 5.3.** If $G$ is a hyperbolic group and $H, J$ are $K$-quasi-convex subgroups, then $H \cap J$ is quasi-convex, with quasi-convexity constant depending on $G$ and $K$.

**Lemma 5.4.** If $G$ is a hyperbolic group and $H$ is a $K$-quasi-convex subgroup, then, for each $c \in G$, $cHc^{-1}$ is quasi-convex, with quasi-convexity constant depending on $G$, $K$ and $c$.

**Proposition 5.5.** An infinite hyperbolic group contains an infinite order element.

**Lemma 5.6.** If $G$ is a hyperbolic group and $g$ is an infinite-order element of $G$, then $(g^n)_{n \in \mathbb{Z}}$ is a quasi-geodesic.

**Remark 5.7.** As a consequence of Proposition 5.5 and Lemma 5.6 we have that given a hyperbolic group $G$, and $H_1, H_2$ $E$-quasi-isometrically embedded subgroups of $G$, there exists $R = R(\delta, E)$ such that if $d_{\text{Haus}}(H_1, H_2) < \infty$, then $d_{\text{Haus}}(H_1, H_2) \leq R$.

**Lemma 5.8.** Let $G$ be a group and $H, J$ be two subgroups. Then for any $g \in G$, we have that $H \cap gJg^{-1} \subseteq N_{2|g|}(p_H(gJ))$.

Proof. Let $ggg^{-1} \in H \cap gJg^{-1}$. Then $d(gyg^{-1}, gy) \leq |g|$. Thus, $d(gy, H) \leq |g|$. There is a point $x \in p_H(gJ)$ with $d(x, gy) \leq |g|$. By triangular inequality, $d(gyg^{-1}, x) \leq 2|g|$. \qed

We want now to establish some relations between conjugates and cosets.

**Lemma 5.9.** Let $G$ be a $\delta$-hyperbolic group with respect to a fixed generating set. Let $H, J$ be $K$-quasi-convex subgroups. Then there exists $D = D(\delta, K)$ such that the following holds. For each $a, b \in G$ for which diam$(p_{\text{H}}(bJ)) \geq 2(8\delta + 2K) + 1$, then

$$d_{\text{Haus}}(p_{\text{H}}(bJ), a(\{H \cap a^{-1}bJb^{-1}a\})) \leq D.$$

Proof. Since left multiplication by $a^{-1}$ is an isometry, the above is equivalent to showing that for each $c \in G$ such that diam$(p_H(cJ)) \geq 2(8\delta + 2K) + 1$, we have:

$$d_{\text{Haus}}(p_H(cJ), N_D(H \cap cJc^{-1})) \leq D.$$

By a quadrilateral argument, if $d(H, cJ) > 2\delta + 2K$, we get that diam$(p_H(cJ) \leq 8\delta + 2K)$. Thus $d(H, cJ) \leq 2\delta + 2K$ and hence $|c| \leq 2\delta + 2K$. By Lemma 5.3 we get that $H \cap cJc^{-1} \subseteq N_{2|c|}(p_H(cJ))$. Now we want to show that there is a uniform $D_1$ such that $p_H(cJ) \subseteq N_{2|c|}(H \cap cJc^{-1})$. This will prove the Lemma. Let $a \in p_H(cJ)$. By the assumptions on the diameter, there is $a' \in p_H(cJ)$ with $d(a, a') > 8\delta + 2K$. Let $C = 4\delta + K$. A quadrilateral argument shows that there is a point $x \in H$ with $d(a, x) \leq C$ and $d(x, cJ) \leq 2\delta + 2K$. That is:

$$p_H(cJ) \subseteq N_C(p_H(N_{2\delta + 2K}(H \cap cJ))).$$

We want to show that there is $R$ such that $N_{2\delta + 2K}(H \cap cJ) \subseteq N_R(H \cap cJc^{-1})$. Since projections are quasi-Lipschitz (see Lemma 2.43), this will imply the claim.

Let $B = B_{2\delta + 2K}(1)$ be the ball of radius $2\delta$ around the identity. Note that $N_{2\delta + 2K}(H) = \bigcup_{g \in B} Hg$. For each $g \in B$ such that $Hg \cap cJ \neq \emptyset$, choose once and for all an element $y_g \in Hg \cap cJ$. Let $R = \max \{|y_g| \mid g \in B\}$. This is well defined since $B$ is finite. Consider $x \in N_{2\delta + 2K}(H) \cap cJ$. Then there exists $g \in B$, $h \in H$ and $j \in J$ such that $x = hg = cj$. Similarly there are $h' \in H$ and $j' \in J$ such that $y_g = h'g = cj'$. Then it is easily seen that $xyg^{-1} \in H \cap cJc^{-1}$, and thus

$$x \in (H \cap cJc^{-1})y_g = N_{|y_g|}(H \cap cJc^{-1}).$$

Maximizing over $B$ yields $N_{2\delta + 2K}(H) \cap cJ \subseteq N_R(H \cap cJc^{-1})$. \qed

We recall the following Theorem.
Theorem 5.10 (Bravo). Let $G$ be a group that is $\delta$-hyperbolic with respect to a fixed finite set of generators. Then there exists $R = R(\delta)$ such that every finite subgroup can be conjugated to lie in the $R$-ball around the identity.

Corollary 5.11. Given a hyperbolic group $G$, there is $\Delta$ such that every finite subgroup of $G$ has at most $\Delta$ elements.

Given two quasi-convex subgroups $H$ and $J$, there are only finitely many cosets of $J$ such that $8\delta + 2K < \text{diam}(\rho_H(J)) < \infty$, and this extends to any finite family. In particular, the next Lemma is trivially satisfied whenever it is applied to a finite family, which will turn out to be the case in our situation. However, we record it because the result is slightly more general and the proof does not rely on the above fact.

Lemma 5.12. Let $G$ be a $\delta$-hyperbolic group, and let $H,J$ be $K$-quasi-convex subgroups of $G$. Then there exists $\xi = \xi(\delta,K,G)$ such that for each $c \in G$, if $\text{diam}(\rho_H(cJ)) \geq \xi$, then $\text{diam}(\rho_H(cJ)) = \infty$.

Proof. Let $B$ be the ball of radius $2\delta + 2K$ around the identity. Consider the elements of $cJ$ that are at distance at most $2\delta + 2K$ from $H$, that is the set $\bigcup_{c \in B} cJg \cap H$. By hyperbolicity, increasing $\xi$ we can assume that this set is arbitrarily large. Note that for every $g \in B$ and every pair of elements $x,y \in cJg \cap H$, one has that $xy^{-1} \in cJ^{-1} \cap H$. Since the ball $B$ contains only finitely many elements, by the pigeonhole principle we can assume that there is $g \in B$ such that $|cJg \cap H| > \Delta + 1$, where $\Delta$ is the constant provided by Corollary 5.11. In particular, we can find $x,y \in cJg \cap H$ such that $xy^{-1}$ has infinite order. Thus $|H \cap cJ^{-1}| = \infty$ and then $\text{diam}(\rho_H(cJ)) = \infty$. \hfill \qed

We want to establish how subgroups relates with coarse inclusion.

Lemma 5.13. Let $H,J$ be infinite quasi-convex subgroups of a $\delta$-hyperbolic group $G$ and let $a,b$ be elements of $G$. Then if $aH \not\subseteq bJ$, then $|aHa^{-1} \cap bJb^{-1}| = \infty$.

Proof. Let $h$ be an infinite order element of $H$. By Lemma 5.9, $(h^n)_n$ is a quasi-geodesic of $H$, and thus $(ah^n)_n$ is a quasi-geodesic of $aH$. By hypothesis, $(ah^n)_n$ is contained in a uniform neighborhood of $bJ$. In particular, there is $C = C(G,H)$ such that for each $n \in \Z$, there is an element $g \in B_C(1)$ such that $ah^ng \in bJ$. Since there are only finitely many such $g$, we get that there exist different numbers $n,m$ and $g_0 \in B_C(1)$ such that $ah^ng \in bJ$ and $ah^mg \in bJ$. Thus $ah^ng_0^{-1}h^{-m}a^{-1} \in aHa^{-1} \cap bJb^{-1}$ is an element of the intersection that has infinite order. \hfill \qed

Definition 5.14 (Proximal pair). Let $\mathcal{F} = \{F_1,\ldots,F_n\}$ be a finite family of subgroups of a group $G$. Let $F_1,F_2$ be elements of $\mathcal{F}$, and let $g \in G$. We say that $(F_i, gF_ig^{-1})$ form a proximal pair if $|F_i \cap gF_ig^{-1}| = \infty$. We define $\text{Prox}(\mathcal{F})$ to be the set of intersections of proximal pairs, namely:

$$\text{Prox}(\mathcal{F}) = \{F_i \cap gF_ig^{-1} \mid |F_i \cap gF_ig^{-1}| = \infty\}$$

Let $(F_i, gF_ig^{-1})$ be a proximal pair. It is clear that for each $f \in \mathcal{F} - \{F_i\}$, we have that $(F_i, fgF_i(fg)^{-1})$ is also a proximal pair, thus the set $\text{Prox}(\mathcal{F})$ is, in general, infinite. However, it contains only finitely many conjugacy classes.

Lemma 5.15. Let $G$ be a hyperbolic group and $\mathcal{F} = \{F_1,\ldots,F_n\}$ be a finite family of quasi-convex subgroups of $G$. Then $\text{Prox}(\mathcal{F})$ contains finitely many conjugacy classes.

Proof. Fix a finite set of generators on $G$, and let $\delta$ be the hyperbolicity constant.

We recall that for any two $F_i,F_j$, we have $F_i \cap gF_ig^{-1} \subseteq N_{2|g|}(p_{F_i}(gF_j))$. Then we will show that, up to left multiplication by an element of $F_i$, there are only finitely many choices for $gF_j$ such that $\text{diam}(p_{F_i}(gF_j)) = \infty$, which implies the result.

So let $gF_j \subseteq F_i \cap gF_ig^{-1}$. Then $d(gF_j,q) \leq |g|$. Thus, $d(gF_i,F_i) \leq |g|$. Thus there is a point $x \in p_{F_i}(F_j)$ with $d(x,gy) \leq |g|$. By triangular inequality, $d(ggy^{-1},x) \leq 2|g|$, 27
which proves the first claim. Let $K$ be such that both $F_i$ and $F_j$ are $K$-quasi-convex. By a quadrilateral argument, if $d(F_i, gF_j) > 2δ + 2K$, we get that $\text{diam}(p_{F_i}(gF_j)) \leq 8δ + 2K$. Thus, up to left multiplication by an element of $F_i$, there are only finitely many $gF_j$ such that $\text{diam}p_{F_i}(gF_j)$ is infinite, and thus finitely many conjugacy classes.

This motivates the following definition:

**Definition 5.16.** For each conjugacy class $[H]$ of elements of $\text{Prox}(F)$, choose once and for a representative whose quasi-convexity constant is minimal among the elements of the class. Then we define $\text{Prox}(F)$ to be the set of such representatives.

Note that by Lemma 5.13, if $F$ is a finite family, so is $\text{Prox}(F)$.

In what follows, we will describe an inductive “closure process” for the family $F$.

**Definition 5.17.** Given a finite family of infinite quasi-convex subgroups $F = \{F_1, \ldots, F_n\}$, we will inductively describe a sequence of families as follows:

b) Set $F^{(0)} = F$.

d) Given $F^{(i)}$, define $F^{(i+1)}$ as $\text{Prox}(F^{(i)})$.

**Remark 5.18.** Since $F$ consisted only of infinite elements, it is easy to see that for each $i$, $F^{(i)} \subseteq F^{(i+1)}$. Moreover, by Lemma 5.16, we have that if $F^{(i)}$ is finite, so is $F^{(i+1)}$. In particular, for each finite set of generators of $G$, there exists $K = K(i)$ such that all the elements of the family $F^{(i)}$ are $K$-quasi-convex.

The motivating property of the above definition is the following.

**Lemma 5.19.** Let $F^{(i)}$ be constructed as in 5.17. There exists $D = D(F^{(i)})$ such that the following holds. For every $a, b \in G$ and $H, J \in F^{(i)}$ satisfying $|p_a H(bJ)| = \infty$, there exists $c \in G$ and $E \in F^{(i+1)}$ such that

$$d_{\text{Haus}}(p_a H(bJ), cE) \leq D.$$ 

**Proof.** As previously remarked, there is $K$ such that all elements of $F^{(i)}$ are $K$-quasi-convex. In particular, we can apply Lemma 5.10 to get that there is a uniform $D$ such that

$$d_{\text{Haus}} (p_a H(bJ), a (H \cap a^{-1} bJb^{-1} a)) \leq D.$$ 

In particular, $|H \cap a^{-1} bJb^{-1} a| = \infty$, thus $(H, a^{-1} bJb^{-1} a)$ is a proximal pair for the family $F^{(i)}$. In particular, there exists a representative $E = H \cap gJg^{-1} \in \text{Prox}(F^{(i)}) = F^{(i+1)}$ and an element $h \in G$ such that 

$$H \cap a^{-1} bJb^{-1} a = hEh^{-1}.$$ 

Thus, we have an uniform estimate of the Hausdorff between $p_a H(bJ)$ and a conjugate of an element of $F^{(i+1)}$. Since $d_{\text{Haus}}(hE, hEh^{-1}) \leq |h|$, by Remark 5.7 we get that there exists an uniform $R$ such that $d_{\text{Haus}}(hE, hEh^{-1}) \leq R$. Setting $c = ah$, we get the claim.

**Theorem 5.20 ([GMRS98]).** Let $G$ be a hyperbolic group, and $\{F_1, \ldots, F_n\}$ be a finite family of quasi-convex subgroups. Then there exists $c$ such that for each collection $\{g_{\alpha}, F_{\alpha}, g_{\alpha}^{-1}\}_{\alpha=1}^{c}$ of $c$ distinct conjugates, the intersection

$$\bigcap_{\alpha=1}^{c} g_{\alpha}F_{\alpha}g_{\alpha}^{-1}$$

is finite.

**Corollary 5.21.** Given a hyperbolic group $G$ and a family $\{F_1, \ldots, F_n\}$ of quasi-convex subgroups, there is $M \in \mathbb{N}$ such that for each $n > M$, $F^{(n)} = F^{(M)}$. 

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Theorem 5.22. Let $G$ be a $\delta$-hyperbolic space, and let $\{F_1, \ldots, F_n\}$ be a family of quasi-convex subgroups. Let $M$ be the constant of Corollary 5.21 and let $F_{\text{cos}}$ be set of all left-cosets of the family $F^{(M)}$. Then $F_{\text{cos}}$ is a weak-factor system for $G$ with respect to each finite set of generators of $G$. Moreover, the natural action of $G$ on itself extend to an action on $F_{\text{cos}}$ with finitely many orbits.

Proof. As remarked, for each choice of generators of $G$, the elements of the family $F^{(M)}$ are uniformly quasi-convex. In particular, this is preserved by left-multiplication. Thus $F_{\text{cos}}$ is a family of uniformly quasi-convex subspaces of $G$. It is clear that the $G$ action preserves $F_{\text{cos}}$. Moreover, since $F^{(M)}$ is finite, there are only finitely many $G$-orbits.

1. By Lemma 5.13 we have that $|a_1 F_1 a_1^{-1} \cap a_2 F_2 a_2^{-1}| = \infty$. Since $a_1 F_1 a_1^{-1} \subseteq N_{|a_1|}(a_1 F_1)$, and since $\leq$ is a transitive relation, we get that $(a_1 F_1 a_1^{-1} \cap a_2 F_2 a_2^{-1}) \leq a_3 F_3$. Proceeding in this way, we get that $|a_1 F_1 a_1^{-1} \cap \cdots \cap a_n F_n a_n^{-1}| = \infty$.

Since we require the coarse inclusions to be proper, all the conjugates are distinct. Hence, by Theorem 5.20 we get that the length of every such chain is uniformly bounded.

2. Lemma 5.14 and Lemma 5.19 give that there are $\xi$ and $B$ depending on the choice of generators such that given $H, J \in F_{\text{cos}}$, either $\text{diam}(p_H(J)) < \xi$, or there exists $L \in F_{\text{cos}}$ such that $d_{\text{Haus}}(L, p_H(J)) \leq B$.

3. For each coset $aH \in F_{\text{cos}}$ and element $x \in aH$, if $h$ is a generator of $H$ that is not a torsion element, then $xa^n$ is a quasi-geodesic that contains $x$. Since the elements of $F^{(M)}$ are finite, one can uniformly estimate the distance between $x$ and an arbitrarily long geodesic segments with endpoints in $aH$.

As a corollary we get the main result of this section.

Corollary 5.23. Let $G$ be a hyperbolic group and let $F = \{F_1, \ldots, F_N\}$ be a finite family of infinite quasi-convex subgroups. Let $\sim$ be the equivalence relation between subset of $G$ given by having finite distance in $\text{Cay}(G)$ (note that does not depend on the choice of generators). Then there exists a finite family of quasi-convex subgroups $F^{(M)}$ that contains $F$ such that if $F_{\text{cos}}$ is the set of cosets of $F^{(M)}$, then $(G, F_{\text{cos}}/\sim)$ is a hierarchically hyperbolic group structure on $G$.

Proof. By Theorem 5.22 $F_{\text{cos}}$ is a weak factor system for the Cayley graph of $G$ on which $G$ acts equivariantly. Proposition 4.20 yields an HHS structure on the Cayley graph of $G$ on which $G$ acts equivariantly, with indexing set $F_{\text{cos}}/\sim$. Thus $(G, F_{\text{cos}}/\sim)$ is a hierarchically hyperbolic group.

6 Boundaries

Let $\mathcal{X}$ be a hyperbolic space and let $\mathcal{F}$ be a factor system for $\mathcal{X}$. We will exploit the HHS structure on $\mathcal{X}$ induced by $\mathcal{F}$ to get a more precise description of the boundary of $\mathcal{X}$.

This result was established by Hamenst"adt in [Ham10] for the case when the space $\mathcal{X}$ is relatively hyperbolic with respect to a set of uniformly hyperbolic spaces $\mathcal{F}$. Indeed, in this case the space $\mathcal{X}$ has to be hyperbolic (Theorem 2.4 of [Ham10]) and, due to a result of Sisto [St14], the Hausdorff distance between any two elements of $\mathcal{F}$ which have infinite diameter must be infinite. Thus, up to discarding those elements of $\mathcal{F}$ that are bounded, we have that $\mathcal{F}$ is a factor system for $\mathcal{X}$. Since the Gromov boundary of a bounded set is empty, our result recovers the previous result in this case.
Moreover, in the case when $\mathcal{X}$ is hyperbolic with respect to $\mathcal{F}$, the explicit description of the Gromov boundary allows us to obtain an explicit description of the Bowditch boundary (under the mild hypothesis of $\mathcal{X}$ being a proper metric space). In the case when $\mathcal{X}$ is a hyperbolic group and $\mathcal{F}$ a family of peripheral subgroups, this recovers a well known result, proved by Tran in [Tra13] and, independently, by Manning in [Man15]. The former points out that a proof of this result can also be obtained from [Ger12, GP13] or [MOY12].

6.1 Fixing notations: The Gromov boundary of a hyperbolic space

It is a well known fact that the Gromov boundary of a hyperbolic space has several different characterizations. We will briefly recall the definitions and conventions used. For the proofs of the statements and a more precise exposition, we refer to [BH99, Chapter III.H], or to the survey [KB02].

Firstly, however, we will recall the definition of hierarchy path for an HHS.

Definition 6.1 (Coarse map, unparametrized quasi-geodesic). Let $\mathcal{X}$ be a metric space. A coarse map $f$ from $Y$ to $\mathcal{X}$ is a map $f: [0, l] \to \mathcal{X}$ such that the image of each point has uniformly bounded diameter. A coarse map $f: [0, l] \to \mathcal{X}$ is an unparametrized $(D, D)$-quasi-geodesic if there exists an increasing function $g: [0, L] \to [0, l]$ such that $f \circ g$ is a $(D, D)$-quasi-isometric embedding, and for each $x, y \in [0, L]$ with $|x - y| \leq 1$, we have $\text{diam}(f(g(x, y))) \leq D$.

Definition 6.2 (Hierarchy path, [BHS15], Definition 4.2). Let $(\mathcal{X}, \mathcal{E})$ be a hierarchically hyperbolic space. A path $\gamma$ of $\mathcal{X}$ is a $D$-hierarchy path if

1. $\gamma$ is a $(D, D)$-quasi-geodesic of $\mathcal{X}$;
2. for each $U \in \mathcal{E}$ the projection $\pi_U(\gamma)$ is an unparametrized $(D, D)$-quasi-geodesic.

Theorem 6.3 (Existence of hierarchy paths, [BHS15], Theorem 4.4). Let $\mathcal{X}$ be a hierarchically hyperbolic space. Then there exists $D_0$ so that any $x, y \in \mathcal{X}$ are joined by a $D_0$-hierarchy path.

Let $\mathcal{X}$ be a $\delta$-hyperbolic HHS and let $\gamma_1, \gamma_2$ be two quasi-geodesic rays of $\mathcal{X}$. We say that $\gamma_1$ and $\gamma_2$ are equivalent if their Hausdorff distance is finite. Let $D$ be such that any two points of $\mathcal{X}$ can be joined by a $D$-hierarchy path. Since $\mathcal{X}$ is hyperbolic, there is a constant $H_D$ such that given two $D$-quasi-geodesic segments that share the same endpoints, their Hausdorff distance is at most $H_D$. If $\gamma, \eta$ are quasi-geodesic rays that represent the same point in the Gromov boundary, then there is $n = n(d(\gamma(0), \eta(0)))$ such that the Hausdorff distance between $\gamma$ and $\eta$ outside the ball of radius $n$ around $\gamma(0)$ is at most $H_D$. Let $k = D + H_D$, and let $H_k$ be such that if the Hausdorff distance between 2 $k$-hierarchy rays is finite, then it is at most $H_k$. We can identify the Gromov boundary with the set of equivalence classes of $k$-hierarchy rays. The reason why we choose $k$ to be larger than $D$ is that in this way we are allowed to "perturb hierarchy rays". Given a point $x \in \mathcal{X}$ and $p \in \partial_\infty \mathcal{X}$, there is always a $D$-hierarchy ray $\gamma \in p$ such that $\gamma(0) = x$. In that case we say that $\gamma$ connects $x$ and $p$.

Let $\mathcal{X}_\infty$ be the union $\partial_\infty \mathcal{X} \cup \mathcal{X}$. There is a natural topology on $\mathcal{X}_\infty$ such that $\mathcal{X}$ is embedded in $\mathcal{X}_\infty$, and the latter is compact in case $\mathcal{X}$ is proper. We will briefly recall how the topology of $\mathcal{X}_\infty$ is defined via prescribing a neighborhood base for each point. If $x \in \mathcal{X}$, then we consider the standard base obtained by the metric of $\mathcal{X}$. For a point at infinity, we recall the following lemma, that can be taken as a definition.

Lemma 6.4 (Neighborhoods at infinity [BH99], Chapter III.H: Theorem 1.7, Lemma 3.6). Let $\mathcal{X}$ be a $\delta$-hyperbolic space and $x_0$ be a point of $\mathcal{X}$. Let $r > 2H_k$. Let $\eta$ be a $k$-quasi-geodesic ray representing a point $p \in \partial_\infty \mathcal{X}$ and for each positive integer $n$ let $V_n(\eta)$ be the set of points $q \in \mathcal{X}_\infty$ such that for all $k$-quasi-geodesic rays $\gamma$ connecting $x_0$ and $q$ there is a point $x$ on $\eta$ such that $d(x, \eta(0)) \geq n$ and $\gamma \cap B_r(x) \neq \emptyset$. Then $\{V_n(\eta) \mid n \in \mathbb{N}\}$ is a fundamental system of neighborhoods for $p$ in $\partial_\infty \mathcal{X} \cup \mathcal{X}$.
6.2 The HHS-boundary of a hyperbolic space

We recall the definition of HHS-boundary of a space introduced in [DHS17] and some important results.

**Definition 6.5** (Support set, boundary point, HHS-boundary; [DHS17]). Let \((X, S)\) be a hierarchically hyperbolic space. A **support set** \(U \subseteq S\) is a set with \(U_i \perp U_j\) for all distinct \(U_i, U_j \in U\). A **boundary point** with support \(U\) is a formal sum \(p = \sum_{U \in U} a_U p_U\), where \(p_U \in \partial C U\) and \(a_U > 0\), with the requirement that \(\sum_U a_U = 1\). The **HHS-boundary** \(\partial X\) of \((X, S)\) is the set of all boundary points.

It is a well-known fact that, in a hierarchically hyperbolic space \((X, S)\), each family of pairwise orthogonal elements has uniformly bounded size. See for instance [DHS17, Lemma 1.4]. In particular, the boundary points consist of uniformly finite sums.

**Theorem 6.6** ([DHS17]). There is a topology on \(\partial X\) such that the following holds:

- For each \(U \in S\), the inclusion \(\partial C U \hookrightarrow \partial X\) is an embedding.
- The boundary \(\partial X\) is closed in \(X\).
- The space \(X = \partial X \cup X\) is
  - Hausdorff,
  - separable, in case \(X\) is separable,
  - compact, in case \(X\) is proper.

There is an explicit description for the topology of the HHS-boundary of a general hierarchically hyperbolic space, and we refer to [DHS17] for the precise definition.

In the case of hyperbolic HHS, however, the HHS-boundary and its topology admit a significantly simpler description. This is because, under the mild assumption that for all \(U \in S\), \(C U\) has infinite diameter, there are no pairwise orthogonal elements (see [DHS17, Lemma 4.1], or [BHS15, Subsection 5.1]). In this case, Definition 6.5 translates as:

\[
\partial X = \bigcup_{U \in S} \partial C U.
\]

We will describe the topology of \(\partial X\) via prescribing neighborhood basis at each point.

**Definition 6.7** (Boundary projections, [DHS17]). Let \((X, S)\) be a hyperbolic HHS and let \(U \in S\). Let \(x\) be a point of \(X\). We define the projection \(\partial \pi_U : X \to \partial C U \cup C U\) as:

1. The projection \(\pi_U\) if \(x \in X\).
2. The identity map if \(x \in \partial C U\).
3. The set \(\rho_U^V \) if \(x \in \partial C V\) and either \(V\) is properly nested in \(U\) or \(V \nsubseteq U\).
4. The coarse map \(\rho_{\partial V}^U : \partial C V \to 2^{CU}\) defined as follows in all the other cases. Consider a \(D\)-hierarchy ray \(\gamma\) connecting \(\rho_U^V\) with \(q \in \partial C V\). Let \(E\) be the constant of the bounded geodesic image property. We set \(\rho_{\partial V}^U (q) = \rho_V^U (\gamma - (N_E (\rho_U^V) ))\), which, by the bounded geodesic image property, has uniformly bounded diameter.

**Definition 6.8** (Neighborhood basis for the topology, [DHS17]). Let \(p\) be a point in \(\partial C U\), and let \(O_p\) be an open neighborhood for \(p\) in the cone-topology for \(C U \cup \partial C U\). We define \(N_{O_p}(p) \subseteq \partial X\) to be the set:

\[
N_{O_p}(p) = \{ q \in \partial C X \mid \partial \pi_U (q) \cap O_p \neq \emptyset\}.
\]

We declare the sets \(N_{O_p}(p)\) to be a neighborhood basis at \(p\), and this topology coincides (indeed, it is just a special case of the definition) with the topology of the HHS-boundary.

However, in this case much more it is true:
Theorem 6.9 (DHS17, Theorem 4.3). Let $(\mathcal{X}, \mathcal{G})$ be a hyperbolic HHS and let $\overline{\mathcal{X}}_\infty$ be the union $\partial_\infty \mathcal{X} \cup \mathcal{X}$. Then the identity map $\mathcal{X} \to \mathcal{X}$ extends uniquely to a homeomorphism $\overline{\mathcal{X}}_\infty \to \overline{\mathcal{X}}$.

6.3 Explicit description when the HHS structure comes from a factor system

If $(\mathcal{X}, \mathcal{G})$ is the hierarchically hyperbolic space structure obtained applying Theorem 6.11, we have a very explicit description of the projections $\pi_U$ and the spaces $\rho_U^\infty$. This allows us to give a more explicit description of the boundary, and thus of the Gromov boundary.

Unraveling the construction of Theorem 6.11, we observe that if $F \subset \mathcal{X}$ is a factor system for $\mathcal{X}$, then the elements of the set $\mathcal{G}$ are in bijection with $F \cup \{\mathcal{X}\}$. In particular, each index $U \in \mathcal{G}$ is naturally associated to a subspace $F_U \in F \cup \{X\}$ of $\mathcal{X}$ and the associated $\delta$-hyperbolic space $\mathcal{X}U$ is an appropriate cone-off of $F_U$. We also recall that the maps $\pi_U$ and $\rho_U^\infty$ are defined as closest point projection in $\mathcal{X}$ on the subspace $F_U$. Indeed, this defines a map with image in $2^{\mathcal{X}}$. Since $\mathcal{X}$ is the cone-off of $F_U$, the Hausdorff distance between $p_{F_U}(p)$ and $\partial p_{F_U}(p)$ is uniformly bounded, where $\partial p_{F_U}(x)$ is as in Definition 6.7.

We claim that substituting the projections $\partial p_{F_U}$ with the shortest distance projections $p_{F_U}$ in Definition 6.8 does not change the resulting topology. This will allow to give a non-trivial decomposition of the Gromov boundary as the union of the boundaries of the various $\mathcal{X}U$, with $U \in \mathcal{G}$.

For brevity, given $q \in \partial_\infty \mathcal{X}$, $U \in \mathcal{G}$ and a set $O \in \mathcal{X}$, we say that $p_{F_U}(q) \cap O \neq \emptyset$ if there exists a representative $\gamma \in q$ and $N \in \mathbb{N}$ such that for each $n > N$, the projection $p_{F_U}(\gamma([n, \infty)))$ intersects $O$.

Definition 6.10 (Alternative neighbourhood basis for the topology). Let $p$ be a point in $\partial_\infty \mathcal{X}$, and let $O_p$ be an open neighborhood for $p$ in the cone-topology for $\mathcal{X}$ and $\partial_\infty \mathcal{X}$. We define $F_{O_p}(p) \subseteq \partial \mathcal{X}$ to be the set:

$$F_{O_p}(p) = \{q \in \partial_\infty \mathcal{X} | p_{F_U}(q) \cap O_p \neq \emptyset\}.$$

The main result of this section is the following.

Proposition 6.11. Let $X$ be a hyperbolic space and let $\mathcal{F}$ be a factor system for $X$. For each element $U \in \mathcal{F} \cup \{\mathcal{X}\}$ let $\mathcal{X}U$ be the cone-off of $U$ with respect to all the elements $V \in \mathcal{F}$ that are strictly contained in $U$. Then the Gromov boundary of $X$ decomposes as

$$\partial_\infty \mathcal{X} = \bigcup_{U \in \mathcal{F} \cup \{\mathcal{X}\}} \partial_\infty \mathcal{X}_U,$$

with the topology described in Definition 6.10.

The proof of Proposition 6.11 is an easy consequence of Theorem 6.9 and the following proposition.

Proposition 6.12. The topologies for $\overline{\mathcal{X}}$ of Definition 6.8 and Definition 6.10 agree.

Before proving Proposition 6.12, we present one useful application.

Corollary 6.13. Let $G$ be a hyperbolic group and let $\mathcal{F}$ be a finite family of quasi-convex subgroups of $G$. Let $(G, \mathcal{G})$ be the HHG structure on $G$ induced by $\mathcal{F}$ (Corollary 5.23). Then

$$\partial_\infty G = \bigcup_{U \in \mathcal{G}} \partial_\infty \mathcal{X}_U.$$
Proof of Proposition 6.12. We need to show that every neighborhood of one of the two basis contains a neighborhood of the other.

For all $p$ and $O_p$, there exist $O'_p$ such that $M_{O'_p}(p) \subseteq N_{O_p}(p)$.

Set $O'_p = O_p$ and let $q \in M_{O_p}$. Suppose that $p \in \partial_{\infty}CU$. If $q \in \partial_{\infty}CU$, then the conclusion follows. So suppose that $q \in \partial_{\infty}CV$ for some $V \neq U$. If $V \cap U$, then, since $q \in M_{O_p}(p)$, we have that there is $N$ and $N \cap q$ such that for each $n \geq N$, we have $p_{F(V)}(\gamma([n, \infty))) \cap O_p \neq \emptyset$. In particular, $(p_{F(V)}(F_V) = \rho_l^V) \cap O_p \neq \emptyset$, thus $q \in N_{O_p}$.

So suppose that $U \subseteq V$, that is $F_U \subseteq F_V$. Since $q \in M_{O_p}(p)$, there is $\gamma \in q$ and $N$ such that for each $n > N$, we have $p_{F(V)}(\gamma([n, \infty))) \cap O_p \neq \emptyset$. Let $\eta \in q$ be a $D$-hierarchy ray connecting $\rho_l^V$ with $q$. There is $n$ such that outside a ball of radius $n$ around $\eta(0)$, the Hausdorff distance between $\gamma$ and $\eta$ is at most $H_D$. Thus there is a $(D + H_D)$-hierarchy ray $\eta' \in q$ such that $\eta'(0) = \eta(0)$ and $\eta'$ and $\gamma$ coincide outside a ball around $\eta'(0)$. In particular, this implies that $\partial_{\infty}F(V) \cap O_p \neq \emptyset$, and hence that $q$ is an element of $\subseteq N_{O_p}(p)$.

For all $p$ and $O_p$, there exist $O'_p$ such that $N_{O'_p}(p) \subseteq M_{O_p}(p)$.

We recall that there are constants $\xi$ and $E$ such that for each pair of transverse $U$ and $V$, one has $\operatorname{diam}(\rho_l^U) \leq \xi$, and for each pair of nested $U \subseteq V$ and $k$-hierarchy path $\gamma$ of $CV$, one has that $\gamma \cap N_{E}(\rho_l^U) = \emptyset$, then $\operatorname{diam}(\rho_l^U(\gamma) \leq E)$ (See Definition 3.6). Let $s$ be much larger than $\xi$ and $E$, for instance $s = 10(2\xi + 2E + 2\delta)$.

Note that $O_p \cap CU$ is an open set in $CU$, that we will still denote $O_p$.

Let $L$ be the $s$-neighborhood in $CU$ of the complementary of $O_p$, that is $L = \bigcup_{x \in CU \cap O_p} B_s(x)$. Note that $L$ is open in $CU$. We claim that $L = L \cup (O_p \cap \partial_{\infty}CU)$ is an open neighborhood of $p$ in $CU \cap \partial_{\infty}$. In order to show this, we will prove that $L$ contains a neighborhood of each of its points. Let $x \in L$ and suppose that $x \in CU$. Then, since $L$ is open, there is an open neighborhood of $x$ contained in $L$ and thus in $L$. So let $q$ be a point in $L \cap \partial_{\infty}CU$. Since $O_p$ is open, there is $m$ and a representative $\eta \in q$ for each $V_m(\eta) \subseteq O_p$, where $V_m(\eta)$ is defined as in [4.3].

We claim that there exists $n$ large enough such that $V_m(\eta) \subseteq L$, which implies the claim. In what follows, we will often use the following fact. Let $\alpha$ and $\beta$ be $k$-quasi-geodesic starting at a point $\beta(0)$. Suppose that there is a point $b$ of $\beta$ at distance more than $n$ from $\beta(0)$, such that $d(h, \alpha) \leq s$. Then, inside a ball of radius approximately $n - s$ around $\beta(0)$, the Hausdorff distance between $\alpha$ and $\beta$ is at most $H_k$.

Suppose that the claim does not hold, that is, suppose that there is $y \in V_m(\eta)$ that does not belong to $L$. Since $L$ and $O_p$ coincide on the boundary, we must have that $y \in CU$. This means that there is a point $z \in CU$ such that $d(y, z) < s$ and $z \notin O_p$. In particular, $z \notin V_m(\eta)$.

Let $\alpha$ be a $k$-hierarchy path joining $\eta(0)$ and $y$, and $\beta$ a $k$-hierarchy path joining $\eta(0)$ and $z$. Since $y$ and $z$ are $s$-near, $\alpha$ and $\beta$ are $H_k$ near approximately inside a ball of radius $d(\eta(0), y) - s \geq n - s$, since $d(\eta(0), y) \geq n$. But, since $y \in V_m(\eta)$, inside a ball of radius $n - r$, $\alpha$ is $H_k$ near to $\eta$. Choosing $n$ large enough such that both $n - s$ and $n - r$ are much larger than $m + r$, we get that in a ball of radius $m + r$, $\beta$ and $\eta$ are $2H_k$ near. Thus $z \in V_m(\eta)$, which is a contradiction.

Thus $L$ is an open set. We claim that $N_L(p) \subseteq M_{O_p}$. As before, let $q \in N_L$ and suppose that $q \in \partial_{\infty}CV$. If $V = U$, the conclusion follows since $L \subseteq \partial_{\infty}$. If $V \cap U$, then $\rho_l^V \cap L \neq \emptyset$. Since $L \subseteq \partial_{\infty}$ and $\operatorname{diam}(\rho_l^V) \leq s$, we have that $\rho_l^V \subseteq O_p$. In particular, $\pi_{F(V)}(q) \subseteq O_p$ and thus $q \in M_{O_p}$.

Finally, suppose that $U \subseteq V$. The fact that $q$ is an element of $N_L$ implies that there is $\gamma \in q$ with endpoint on $\rho_l^U$ such that $p_{F(V)}(\gamma([E, \infty))) \cap L \neq \emptyset$. Since $\operatorname{diam}(p_{F(V)}(\gamma([E, \infty]))) \leq E$, we get that $p_{F(V)}(\gamma([E, \infty))) \subseteq O_p$. In particular, $q \in M_{O_p}$. □

6.4 Bowditch Boundary

Definition 6.14. Let $F$ be a connected graph. The combinatorial horoball associated to $F$ is the graph $\Gamma(F)$ with vertices $V(F) \times \mathbb{N}$ and the following edges:
- For each $v \in V(F)$ and $n \in \mathbb{N}$, there is an edge between $(v, n)$ and $(v, n + 1)$.
- For each pair of vertices $v, w \in V(F)$ such that $d_F(v, w) \leq 2^n$, there is an edge between $(v, n)$ and $(w, n)$.

It is easily seen that vertical rays in $\Gamma(F)$ are infinite geodesic rays, and it is not hard to see that they are the only ones. In particular, for each $F$ the Gromov boundary of $\Gamma(F)$ consists of a single point that we denote $\ast_F$.

**Definition 6.15.** Let $X$ be a hyperbolic geodesic space, and let $\mathcal{F}$ be a family of subspaces of $X$. For each $F \in \mathcal{F}$, let $\Omega(F)$ be a connected approximation graph for $F$, with constants chosen uniformly for all elements of $\mathcal{F}$. The **Bowditch space** $\Gamma(X, \mathcal{F})$ (or simply $\Gamma(X)$) is defined as the space obtained from $X$ attaching to it the combinatorial horoballs $\Gamma(\Omega(F))$ under the identification $(x, 0) \sim x$.

**Definition 6.16.** A space $X$ is said to be **hyperbolic relative to the family** $\mathcal{F}$ if the Bowditch space $\Gamma(X, \mathcal{F})$ is Gromov hyperbolic.

**Convention.** From now, let $X$ be a proper hyperbolic space hyperbolic relative to a family $\mathcal{F}$, where all the elements of $\mathcal{F}$ have infinite diameter, and suppose that, in addition, $X$ is hyperbolic.

The request that the elements of $\mathcal{F}$ to have infinite diameter is because, for an $F$ of finite diameter, we would have $\partial F = \emptyset$, but $\partial \Gamma(F) = \{ \ast_F \}$. There are several ways to fix this, but they all boil down to considering only the elements of $\mathcal{F}$ that have infinite diameter.

It is an easy consequence of [Sis12] that the family $\mathcal{F}$ forms a system for $X$ and, for any two different $F_1, F_2 \in \mathcal{F}$, we have $F_1 \not\subseteq F_2 \not\subseteq F_1$, where $\subseteq$ denotes the coarse inclusion (Definition 11). It is not hard to see that this implies that $\Gamma(F) = \{ \Gamma(F) \mid F \in \mathcal{F} \}$ is a system for $\Gamma(X)$. Note, moreover, that the cone-off of $X$ with respect to $\mathcal{F}$ is quasi-isometric to the cone-off of $\Gamma(X)$ with respect to $\Gamma(\mathcal{F})$. Thus, we will identify the latter with $CX$.

In particular, applying Proposition 6.11 we obtain the following decomposition of the Gromov boundary of $\Gamma(X)$:

$$\partial \Gamma(X) = \partial CX \cup_{F \in \mathcal{F}} \partial \Gamma(F) = \partial CX \cup_{F \in \mathcal{F}} \ast_F,$$

where the topology is as in Definition 6.8. Let $\partial X$ be the quotient of $\partial X = \partial CX \cup_{F \in \mathcal{F}} \partial CF$ obtained collapsing each $\partial CF$ to a point, equipped with the quotient topology. We claim that

$$\partial \mathcal{F} X = \partial \Gamma(X),$$

which amounts to saying that the Bowditch boundary of $X$ can be easily described as the quotient of $\partial X$ by a suitable set of subspaces.

It is clear by the above description that there is a bijection $\phi: \partial \mathcal{F} X \to \partial \Gamma(X)$. We want now to show that $\phi$ is a homeomorphism. Since $X$ is proper, so is $\Gamma(X)$. In particular, $X \cup \partial \mathcal{F} X$ and $\Gamma(X)$ are compact, and so are $\partial \mathcal{F} X$ and $\partial \Gamma(X)$. Thus, it suffices to show that $\phi$ is continuous.

**Theorem 6.17.** Let $X$ be a proper hyperbolic space, which is hyperbolic relative to a family $\mathcal{F}$, where all the elements of $\mathcal{F}$ have infinite diameter. Let $CX$ be the cone-off of $X$ with respect to $\mathcal{F}$. Then $\partial X = \partial CX \cup_{F \in \mathcal{F}} \partial F$ and the Bowditch boundary $\partial \Gamma(X, \mathcal{F})$ is obtained from $\partial X$ collapsing each $\partial F$ to a point.

**Proof.** This is an immediate consequence of the Lemmas 6.18 and 6.19 which describe the behavior of the fundamental neighborhoods under the map $\phi$. \qed

**Notation.** We fix notations as follows: let $\psi: \partial X \to \partial \mathcal{F} X$ be the quotient map. Given a point $p \in \partial CX$ and an open set $O_p \subseteq CX$, that contains $p$, we denote by $N_{2p}^\psi$ the neighborhood of $p$ defined by $O_p$ in $\partial X$, and by $N_{O_p}^{\psi(G)}$ the neighborhood in $\partial \Gamma(X)$. 34
Lemma 6.18. Let $p \in \partial CX$. Then for each open set $O_p \subseteq CX$ that contains $p$, one has 
$$\phi \circ \psi \left( N^X_{O_p} \right) = N^T(\Gamma(X)) \quad \text{and} \quad \psi^{-1} \circ \phi^{-1} \left( N^T(\Gamma(X)) \right) = N^X_{O_p}.$$

Proof. It is clear that $\phi \circ \psi$ is a bijection on $\partial CX$. Thus it is easy to see that a point $q \in \partial CX$ belongs to $N^X_{O_p}$ if and only if $\phi(q)$ belongs to $N^T(\Gamma(X))$. So, consider the point $\ast_F \in \partial \Gamma(F)$ for some $F \in \mathcal{F}$. We have that $\ast_F$ belongs to $N^T(\Gamma(X))$ if $\rho^F$ intersects $O_p$, where $\rho^F$ is defined to be the projection of $\Gamma(F)$ on $CX$. But this projection coincides with the projection of $\Gamma$ on $CX$. Thus $\ast_F$ belongs to $N^T(\Gamma(X))$ if and only if $\partial CF = \psi^{-1} \circ \phi^{-1}(\ast_F)$ belongs to $N^X_{O_p}$. \hfill \qed 

Lemma 6.19. Let $F \in \mathcal{F}$. For each $O_F \subseteq \Gamma(F)$ there is $O'_F \subseteq \Gamma(F)$ such that $N^X_{O'_F} \subseteq \psi^{-1} \circ \phi^{-1} \left( N^T_{O_F} \right)$ and $\partial CF \subseteq N^X_{O'_F}$.

Proof. Let $O_F$ be open in $\Gamma(F)$. Then, there is a representative $\eta$ of $\ast_F$ and a number $n$ such that $V_n(\eta) \subseteq O_F$, where $V_n(\eta)$ is defined as in Lemma 6.4. Let $\eta(0)$ be the starting point of $\eta$. Without loss of generality we can assume that $\eta(0)$ is a point of $\partial CF$.

Let $V_n(\eta)^c$ be the complement of $V_n(\eta)$ in $\Gamma(F)$. It is easy to see that $V_n(\eta)^c$ is contained in a closed ball of finite radius around $\eta(0)$. Let $K$ be such a ball. Thus we have $K^c \subseteq O_F$.

The image of $K$ in $CF$ has also finite diameter. Thus we have that $K^c$ is an open of $\Gamma(F)$. Since $K$ has finite diameter, it is easy to see that $\partial CF \subseteq N^X_{K^c}$. Moreover, since by construction $K^c \subseteq O_F$, we have that for each point $q \in \partial CX$, if the projection of $q$ on $CF$ intersects $K^c$, then it intersects $O_F$, which shows $N^X_{O'_F} \subseteq \psi^{-1} \circ \phi^{-1} \left( N^T(\Gamma(X)) \right)$. \hfill \qed 

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