Superintegrable quantum mechanical systems with position dependent masses invariant with respect to three parametric Lie groups

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Abstract

Quantum mechanical systems with position dependent masses (PDM) admitting four and more dimensional symmetry algebras are classified. Namely, all PDM systems are specified which, in addition to their invariance w.r.t. a three parametric Lie group, admit at least one second order integral of motion. The presented classification is partially extended to the more generic systems which admit one or two parametric Lie groups.

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1 Introduction

The title of the present paper is a bit conventional. The results presented there are more generic than it declares. In addition to the PDM systems admitting three parametric Lie groups and second order integrals of motion we give the classification of the systems invariant with respect to selected two- and one parametric groups.

Let us start with short historical comments related to symmetries of quantum mechanical systems with constant masses.

Symmetry is one of the most fundamental concept of theoretical and mathematical physics, especially of quantum mechanics. The fundamentals of the science of continuous symmetries were created long time ago by the great Norwegian mathematician Sophus Lie. In particular, he discovered all such symmetries admitted by the fundamental equation of quantum mechanics. More exactly, Lie found the maximal continuous invariance group of the heat equation, which in the main coincides with the symmetry group of the free Schrödinger equation.

A systematic search for Lie symmetries of Schrödinger equation started in papers [1, 2, 3] and [4] where the maximal invariance groups of this equation with arbitrary scalar potential were presented. For symmetries of this equation with scalar and vector potentials and corrected results of classical paper [4] see papers [5, 6]. Lie symmetries of Schrödinger equation with matrix potentials are classified in [7, 8].

The more general symmetries, namely, the second order symmetry operators for 2d and 3d Schrödinger equation have been classified in [9, 10] and [11, 12] correspondingly. The extended (in particular, second order) symmetries are requested for description of systems admitting solutions in separated variables [13]. Just such symmetries characterize integrable and superintegrable systems [14]. Let us mention also the nice conjecture of Ian Marquette and Pavel Winternitz [15] which can give a surprising connection of higher order superintegrability in the quantum case with soliton theory of infinite-dimensional integrable nonlinear systems.

An important research field is formed by superintegrable systems with spin whose systematic investigation was started with paper [16, 17, 18] where the systems with spin-orbit interaction were classified. Superintegrable systems with Pauli type interactions were studied in [19, 20] and [21].

Let us note that the first example of a superintegrable system with spin 1/2 was presented earlier in paper [22]. Superintegrable systems with arbitrary spin were discussed in [23], [20], [24] and [19], the relativistic systems were elaborated in [21] and [25].

The modern trend is to study the superintegrable systems admitting integrals of motion of the third and even arbitrary orders [13, 26], see also [27] where the determining equations for such symmetries were deduced, and [28] where symmetry operators of arbitrary order for the free Schrödinger equation had been enumerated.

Thus the amazing world of symmetries of Schrödinger equation is an important and interesting research field which attracts the attention of numerous investigators. The same is true for the Schrödinger equation with position dependent mass whose symmetries are studied much less. The latter equation is requested in many branches of modern theoretical physics, whose list can be found, e.g., in [29, 30].

Symmetries of various PDM Schrödinger equations with respect to the continuous groups have been classified in papers [30] - [32]. More exactly, the symmetries of the stationary equation are presented in [30] while the time dependent equations with two and three spatial variables are studied in [31] and [32] correspondingly.
The situation with the higher symmetries of the PDM quantum mechanical systems is much more complicated. There is a lot of paper devoted to symmetries of particular equations or of the restricted classes of such equations, see, e.g., [33, 34, 35, 36, 37, 38, 39]. However, the completed classification of 3d superintegrable systems with PDM is still missing.

On the other hand the 2d classical systems with position dependent mass which admit second order integrals of motion are known and well studied [40, 41, 42, 44, 45] and there is a correspondence between classical and quantum superintegrable systems [43].

The main stream in studying of superintegrable systems with PDM is the investigation of classical Hamiltonian systems. And there are effective tools for such business created in classical works of Bernard, Stäckel, Koenigs and Perlick.

Surely there exist the analogous quantum mechanical systems which in principle can be obtained starting with the classical ones and applying the second quantization procedure. However, the mentioned procedure is not unique, and in general it is possible to generate few inequivalent quantum systems which have the same classical limit. In addition, a part of symmetries and integrals of motion of quantum mechanical systems can disappear in the classical limit $h \to 0$ [46].

Thus it is desirable to classify superintegrable quantum systems directly. However, to obtain the completed classification of such systems is very and very difficult, and it is reasonable to solve this problem step by step, restricting ourselves to some well defined subclasses of such equations. And this is just the strategy which we will follow.

In the present paper the complete classification of a special class of superintegrable PDM Schrödinger equations is presented. This class includes equations which admit three parametric symmetry groups. In addition, we will specify a certain subclass of such equations which admit the symmetry groups including two parameters.

## 2 PDM Schrödinger equations

We will search for superintegrable stationary Schrödinger equations with position dependent mass of the following generic form:

$$H \psi = E \psi, \quad (1)$$

where

$$H = p_a f(x) p_a + V(x). \quad (2)$$

Here $x = (x^1, x^2, x^3)$, $p_a = -i \partial_a$, $V(x)$ and $f(x) = \frac{1}{2m(x)}$ are functions associated with the effective potential and inverse PDM, and summation from 1 to 3 is imposed over the repeating index $a$.

A more general form of the PDM Hamiltonian is [47]

$$H = \frac{1}{4} \left( m^\alpha p_a m^\beta p_a m^\gamma + m^\gamma p_a m^\beta p_a m^\alpha \right) + \hat{V} \quad (3)$$

where $\alpha, \beta$ and $\gamma$ are the so called ambiguity parameters satisfying the condition $\alpha + \beta + \gamma = -1$. Physically, representation (3) is more consistent but mathematically it is completely equivalent to (2) [32].
In paper [30] all equations (1) admitting at least one first order integral of motion has been classified. Such integrals of motion are nothing but generators of Lie groups which leave the related equations invariant. The list of such equations includes three representatives which accept three parametrical invariance groups. The corresponding inverse masses $f$ and potentials $V$ are presented in the following formulae:

$$f = F(r), \quad V = V(r),$$

$$f = F(x_3), \quad V = V(x_3),$$

$$f = \tilde{r}^2 F(\varphi), \quad V = V(\varphi)$$

where $F(\cdot)$ and $V(\cdot)$ are arbitrary functions whose arguments are fixed in the brackets, $r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$, $\tilde{r} = (x_1^2 + x_2^2)^{\frac{1}{2}}$, $\varphi = \text{arctan} \left( \frac{x_2}{x_1} \right)$.

Equations (1), (2) whose arbitrary parameters are fixed by formulae (4), (5) and (6) admit the following integrals of motion

$$L_1 = x_2 p_3 - x_3 p_2, \quad L_2 = x_3 p_1 - x_1 p_3, \quad L_2 = x_3 p_1 - x_1 p_3,$$

$$P_1 = p_1, \quad P_2 = p_2, \quad L_3$$

and

$$P_3 = p_3, \quad D = x_a p_a - \frac{3i}{2}, \quad K_3 = x^2 p_3 - 2x_3 D$$

Correspondingly, which form bases of Lie algebras so(3), e(2) and so(1,2) respectively. In other words, equations (1), (2), (4) and (1), (2), (5) are invariant w.r.t. the rotation group SO(3) and Euclid group E(2) correspondingly while equations (1), (2), (4) are invariant w.r.t. the three parametrical Lie group isomorphic to Lorentz group SO(1,2) in (1+2) - dimensional space.

3 Determining equations

Let us search for second order integrals of motion for equation (1), i.e., for second order differential operators commuting with $H$. We will represent these integrals of motion in the following form:

$$Q = \partial_a \mu^{ab} \partial_b + \eta$$

where $\mu^{ab} = \mu^{ba}$ and $\eta$ are unknown functions of $x$ and summation from 1 to 3 is imposed over all repeating indices.

By definition, operators $Q$ should commute with $H$:

$$[H, Q] \equiv HQ - QH = 0.$$  

Evaluating the commutator and equating to zero the coefficients for the linearly independent differential operators $\partial_a \partial_b \partial_c$ and $\partial_a$ we come to the following determining equations:

$$(f \mu_c^{ab} - \mu^{ac} f_a \delta^{bc}) + \text{cycle}(a, b, c) = 0,$$
\[ \mu^{ab} V_b - f \eta_a = 0 \]  

(13)

where \( \delta^{bc} \) is the Kronecker delta, \( f_n = \frac{\partial f}{\partial x_n} \), \( \mu^{an}_n = \frac{\partial \mu^{an}}{\partial x_n} \), etc., and summation is imposed over the repeating indices \( n, n = 1, 2, 3 \).

Equations (12) and (13) present the necessary and sufficient conditions for commutativity of operators \( H \) and \( Q \).

It is an element of common knowledge that a commutator of two second order differential operators is a linear combination of the third, second, first and an zero order operators. The beauty of the representations (2) for \( H \) and (10) for \( Q \) is that we are not supposed to collect and nullify the coefficients for second and zero order differentials which do not appear in the commutator (11).

Let us present also the traceless part and the trace of the tensorial equations (12):

\[
5 \left( \mu^c_{ab} + \mu^{ac}_b + \mu^{bc}_a \right) = \delta^{ab} \left( \mu^m_{cn} + 2 \mu^{cn}_n \right) + \delta^{bc} \left( \mu^m_{an} + 2 \mu^{an}_n \right) + \delta^{ac} \left( \mu^m_{bn} + 2 \mu^{bn}_n \right),
\]

(14)

\[
(\mu^m_{an} + 2 \mu^{na}_n) f - 5 \mu^{an} f_n = 0.
\]

(15)

Thus to classify Hamiltonians (2) admitting second order integrals of motion (10) we are supposed to find inequivalent solutions of rather complicated system (14–13).

The autonomous subsystem (14) defines the conformal Killing tensor. Its general solution is a linear combination of the following tensors (see, e.g., [48])

\[
\begin{align*}
\mu^0_{ab} &= \delta^{ab} g_0(x), \\
\mu^1_{ab} &= \lambda^1_{ab} + \delta^{ab} g_1(x), \\
\mu^2_{ab} &= \lambda^2_{ab} x^b + \tilde{\lambda}^b_{ab} x^a - \delta^{ab} (2 \lambda^c_{a} x^c - g_2(x)), \\
\mu^3_{ab} &= (\epsilon^{acd} \lambda^c_{ab} + \epsilon^{bcd} \lambda^c_{ba}) x^d + \delta^{ab} g_3(x), \\
\mu^4_{ab} &= (x^a \epsilon_{bcd} + x^b \epsilon_{acd}) x^c \lambda^d_{ab} + \delta^{ab} g_4(x), \\
\mu^5_{ab} &= \delta^{ab} (r^2 + g_5(x)) + k(x^a x^b - \delta^{ab} r^2), \\
\mu^6_{ab} &= \lambda^6_{ab} r^2 - (x^a \lambda^b_{ac} + x^b \lambda^a_{ac}) x^c - \delta^{ab} (\tilde{\lambda}^c_{ad} x^c x^d - g_6(x)), \\
\mu^7_{ab} &= (x^a \lambda^b_{a} + x^b \lambda^a_{a}) r^2 - 4x^a x^b \lambda^c_{ac} x^c + \delta^{ab} (\tilde{\lambda}^c_{ad} x^c r^2 + g_7(x)), \\
\mu^8_{ab} &= 2(x^a \epsilon_{bcd} + x^b \epsilon_{acd}) \lambda^d_{ab} x^d x^n - (\epsilon^{aok} \lambda^k_{bk} + \epsilon^{lok} \lambda^k_{bk}) x^c x^2 + \delta^{ab} g_8(x), \\
\mu^9_{ab} &= \lambda^9_{ab} r^4 - 2(x^a \lambda^b_{ac} + x^b \lambda^a_{ac}) x^c r^2 + (4x^a x^b + \delta^{ab} r^2) \lambda^d_{ab} x^c x^d \\
&\quad + \delta^{ab} (\tilde{\lambda}^c_{ad} x^c x^d r^2 + g_9(x)).
\end{align*}
\]

(16)

(17)

where \( r = \sqrt{x^1_x^2 + x^2^2 + x^3^2} \), \( \lambda^{ab} = \lambda^{ba}_n \), \( \tilde{\lambda}^{ab} = \tilde{\lambda}^{ba}_n \) and \( \lambda^n_a \) are arbitrary parameters, and \( g_1, \ldots, g_9 \) are arbitrary functions of \( x \).

Thus our classification problem is reduced to finding inequivalent solutions of equations (13) and (14) where \( \mu^{ab} \) are linear combinations of tensors (16), and generic form of functions \( f \) and \( V \) is specified in (14–9).

The mentioned linear combinations are the fourth order polynomials in \( x_n \) and include nine arbitrary functions and as many as 50 arbitrary parameters, and so in this stage the classification problems looks huge indeed. Fortunately, for the systems whose inverse masses are specified in (7–9) this problem can be reduced to the series of relatively simple subproblems corresponding to particular linear combinations of these tensors.
4 Scale invariant PDM systems

Let us start with the systems admitting three dimensional symmetry algebra isomorphic to so(1,2). The corresponding Hamiltonians are specified by equations (1), (2) and (6) while the related symmetries are given in (9). The mentioned systems admit second order symmetry operators (10) provided equations (15) and (13) are satisfied. In particular these systems by definition should be invariant w.r.t. the dilatation transformations whose generator $D$ is present in the list (9).

We will consider even a more generic problem. Namely, let us temporarily forget about symmetries generated by the shift generator $P_a$ and generator $K_a$ of the conformal transformations, and solve the determining equations for the masses and potentials admitting only the dilatation symmetry. Such problem has its own value and is an important subproblem of classification of PDM systems with Lie symmetry groups including the dilatation as a subgroup.

In this case we have a bit more general forms of $f$ and $V$ than ones fixed in (18), namely

$$f = r^2 F(\varphi, \theta), \quad V = V(\varphi, \theta)$$

(18)

where $F(.)$ and $V(.)$ are arbitrary functions, $\varphi$ and $\theta$ are the Euler angles. After finding all inequivalent symmetries for systems with the inverse masses and potentials specified in (18) we will impose the additional conditions $\frac{\partial f}{\partial \theta} = 0$ and $\frac{\partial V}{\partial \theta} = 0$ and obtain the systems with $SO(1,2)$ symmetry. In addition, asking for the solutions of the determining equations satisfying $\frac{\partial f}{\partial \varphi} = 0$ and $\frac{\partial V}{\partial \varphi} = 0$ we come to the systems, admitting the two parametric Lie group including dilatations and rotations around the third coordinate axis, etc.

4.1 Equivalence relations and reduction of the determining equations

Changes of dependent and independent variables are called the equivalence transformations provided they keep the generic form of the differential equation (in our case of equation (1)) up to the changes of the explicit form of arbitrary elements (in our case functions $f$ and $V$). The set of the equivalence transformations includes equivalence groups extended by some discrete elements.

In accordance with the results presented in [30], the maximal continuous equivalence group of equation (1) is C(3), i.e., the group of conformal transformations of the 3d Euclidean space. The basis elements of the corresponding Lie algebra can be chosen in the following form:

$$P_a = p^a = -i \frac{\partial}{\partial x^a}, \quad L_a = \varepsilon^{abc} x^b p^c,$$

$$D = x_n p^n - \frac{3i}{2}, \quad K_a = r^2 p^a - 2x^a D,$$

(19)

where $r^2 = x_1^2 + x_2^2 + x_3^2$ and $p_a = -i \frac{\partial}{\partial x^a}$. Operators $P^a$, $L^a$, $D$ and $K^a$ generate shifts, rotations, dilatations and pure conformal transformations respectively. The corresponding group transformations (whose explicit form can be found, e.g., in [30]) keep the generic form of equations (1), (2) but can change the explicit form of $f$ and $V$.

In addition to the invariance with respect to dilatation transformations the considered equations admit the discrete inverse transformation:

$$x_a \rightarrow \tilde{x}_a = \frac{x_a}{r^2}, \quad \psi(x) \rightarrow \tilde{\psi}(\tilde{x}), \quad \tilde{x} = \sqrt{\tilde{x}_1^2 + \tilde{x}_2^2 + \tilde{x}_3^2}$$

(20)
which acts on operators \( \{19\} \) in the following manner:

\[
P_a \rightarrow K_a, \quad K_a \rightarrow P_a, \quad L_a \rightarrow L_a, \quad D \rightarrow D
\]  

(21)

For the class of equations considered in the present section the equivalence group is reduced to the direct product of the rotations group and dilatation transformations since \( L_a \) and \( D \) commute with \( D \) while the remaining operators \( \{19\} \) do not have this property.

In the following we will use the rotations and the inverse transformation \( \{20\} \) for optimisation of calculation.

Since the considered systems by definition should be invariant w.r.t. the scaling transformations (whose generator \( D \) is present in the list \( \{9\} \)), the related Killing tensors cannot include linear combinations of all polynomials listed in \( \{16\} \) but are reduced to homogeneous polynomials with \( n = 0, 1, 2, 3, 4 \), and arbitrary functions \( g_1, g_2, ..., g_9 \) should satisfy the following equations:

\[
x_a g(x) = n g(x).
\]  

(22)

Moreover, since Hamiltonians \( \{2\} \) with arbitrary elements \( \{6\} \) are invariant with respect to the inverse transformation \( \{20\} \), we can restrict ourselves to the polynomials of order \( n < 3 \), since symmetries with \( n = 3 \) and \( n = 4 \) appears to be equivalent to ones with \( n = 1 \) and \( n = 0 \) correspondingly.

Thus it is sufficient to solve determining equations \( \{15\} \) and \( \{13\} \) with the following versions of functions \( \mu^{ab} \):

\[
\mu^{ab} = \tilde{\mu}^{ab} + \delta^{ab} g
\]  

(23)

where \( g = g(x) \) and

\[
\tilde{\mu}^{ab} = \lambda^{ab},
\]

(24)

\[
\tilde{\mu}^{ab} = \lambda^a x^b + \lambda^b x^a - 2 \delta^{ab} x^c + \mu^a x^b + \mu^b x^a + (\varepsilon^{acd} \lambda^b + \varepsilon^{bcd} \lambda^a) x^d,
\]

(25)

\[
\tilde{\mu}^{ab} = \kappa x^a x^b + (x^a \varepsilon^{bcd} + x^b \varepsilon^{acd}) \lambda^c x^d + \delta^{ab} \tilde{\lambda} x^c x^d + \lambda^{ab} \nu^2 - (x^a \lambda^b + x^b \lambda^a) x^c.
\]

(26)

Formula \( \{25\} \) represents tensor \( \mu_2^{ab} \) from \( \{16\} \) with slightly modernized notations \( \lambda^2 = \lambda^a \) and \( \tilde{\lambda}^2 = \mu^a + \lambda^a \). We also omit the sub indices for \( \lambda^a \) and \( \lambda^{ab} \).

Let us note note that the integrals of motion \( \{10\} \) corresponding to the Killing tensors \( \{23\} \) can be expressed via generators \( \{19\} \):

\[
Q = \lambda P_a P_b + \eta + P_a g P_a,
\]

(27)

\[
Q = \lambda \{ P_a, D \} + \lambda^{ab} \{ P_a, L_b \} + \eta + P_a g P_a
\]

(28)

and

\[
Q = \nu^{ab}(\{ K_a, P_b \} + \{ P_b, K_a \}) + \tilde{\lambda}^{ab} Q^{ab},
\]

(29)

where \( \nu^{ab} = \lambda^{ab} + \varepsilon_{abc} \lambda^c \), \( \tilde{\lambda}^{ab} \) are arbitrary koefficients, \( Q^{ab} = P_a x_a x_b P_c \), and the symbols \( \{.,.\} \) denote anticommutators. Representations \( \{27\}, \{28\} \) and \( \{29\} \) correspond to tensors \( \{24\}, \{25\} \).
and (26) respectively. These representations are not unique since we can indicate the following identities:

\[ \{P_a, D\} + \varepsilon_{abc}\{P_b, L_c\} = 2P_c x_a P_c, \]
\[ \{L_a, L_b\} + \{P_a, K_b\} = 2Q^{ab}, \quad a \neq b, \]
\[ \{P_1, K_1\} + \{P_2, K_2\} + L_3^2 = 2Q^{33}. \]  

Notice also that any second order symmetry corresponding to \( n = 0 \) and \( n = 1 \) is accompanied by the addition symmetry generated by the changes of variables (20) and transformations (21).

### 4.2 Evolution of the determining equations

The first step of our analysis is to evolve determining equations (15) for the inverse mass functions and functions \( g \) included to Killing tensors.

For the systems invariant w.r.t. the dilatation transformations function \( f \) satisfies one more condition

\[ x_a f_a = 2f \]  

which is obviously correct in view of (18). However, this condition enables to reduce (13) to the following homogeneous system of linear algebraic equations for derivatives \( f_a \):

\[ M^{ab} f_b = 0 \]  

where

\[ M^{ab} = \tilde{\mu}^{ab} - g_a x_b, \]
\[ M^{ab} = \tilde{\mu}^{ab} - \lambda^a x^b - \mu^a x_b - g_a x_b \]

and

\[ M^{ab} = \tilde{\mu}^{ab} - \lambda^c x_c x_b - g_a x_b \]  

for Killing vectors (24), (25) and (26) correspondingly.

Notice that for the Killing vectors (24) and (25) functions \( g(x) \) can be expressed via \( f \) by the following equations:

\[ g(x) = \frac{1}{2f} - x_a M^{ab} f_b \]

and

\[ g(x) = \frac{1}{f} - x_a M^{ab} f_b f \]

correspondingly, while for Killing vectors (26) we have:

\[ g(x) = fG(g, \theta) \]
where $G(\varphi, \theta)$ is yet unknown function of Euler angles, satisfying the equation

$$G(\varphi, \theta) = \frac{1}{f^2}(x_aM^{bc}f_c - x_bM^{ac}f_c)$$  \hspace{1cm} (37)

 Equations (34) - (37) are algebraic consequences of (22), (31) and (32), obtained by multiplication on $x_a$ and summing up with respect to the repeating index $a$.

 Equation (32) admits nontrivial solution iff the determinant of the matrix whose entries are $M^{ab}$ is equal to zero. Thus we have to specify the admissible combinations of arbitrary constants nullifying this determinant and then find solutions of the corresponding equations (32) and (13). For the Killing tensors presented in (24)-(26) the latter equation is simplified to the following form

$$f\eta^a - \mu^{ab}V_b = 0.$$  \hspace{1cm} (38)

 Thus our classification problem is reduced to solving the system of equations (32) and (38) for unknowns $f, g$ and $V$.

 We will not present all the related routine calculations whose details can be found in [49], but restrict ourselves to one special case which is missing there.

### 4.3 Polynomial potentials

Consider the most complicated case when the symmetry operator is the reduced to the following bilinear form of generators of group C(3):

$$Q = \mu\{K_1, P_1\} + \kappa\{K_2, P_2\} + 2\eta$$  \hspace{1cm} (39)

 where $\mu$ and $\kappa$ are arbitrary coefficients, which, up to normalization, are supposed to satisfy the condition $\mu^2 + \kappa^2 = 1$.

 The corresponding matrix $M$ is degenerated, and its nonzero entries take the following form:

$$M^{11} = \mu(x_3^2 + x_2^2), \quad M^{12} = -\kappa x_1 x_2, \quad M^{21} = -x_1 x_2,$$

$$M^{22} = \kappa(x_1^2 + x_3^2), \quad M^{31} = -\mu x_1 x_3, \quad M^{32} = -\kappa x_2 x_3.$$

 The related equations (32) are solved by $f = x_3^2$, and the corresponding equations (13) take the following form:

$$\mu(x_2^2 + x_3^2)V_1 - \kappa x_1 x_2 V_2 = x_3^2\eta_1,$$

$$\kappa(x_1^2 + x_3^2)V_2 - \mu x_1 x_2 V_1 = x_3^2\eta_2.$$  \hspace{1cm} (40)

 Notice that the third component of equations (40) in our case is a consequence of the system (40) since $\eta$ should satisfy the condition $x_a\eta_a = 0$.

 By definition potential $V$ should be scale invariant and so can be treated as a function of two scale invariant variables

$$y_1 = \frac{x_1}{x_3}, \quad \text{and} \quad y_2 = \frac{x_2}{x_3}.$$  \hspace{1cm} (41)

 The system (40) is compatible provided the following second order equation for $V$ is satisfied:

$$(\mu y_2^2 - \kappa y_1^2 + \mu - \kappa)V_{y_1y_2} + y_1 y_2(\mu V_{y_1y_1} - \kappa V_{y_2y_2}) + 3(\mu y_2 V_{y_1} - \kappa y_1 V_{y_2}) = 0.$$  \hspace{1cm} (42)
The system (40) can be easily solved for special combinations of parameters $a$ and $b$, namely, $\mu = \kappa$ and $\mu = 0$ (or $\kappa = 0$ which is the same up to rotation). However, to solve this system for $\mu \kappa (\mu - \kappa) \neq 0$ a rather spectacular approach is requested.

In the latter case we can restrict ourselves to the parameters values satisfying the following conditions:

$$\mu \kappa (\mu - \kappa) > 0, \quad \mu^2 + \kappa^2 = 1. \quad (43)$$

It can be done without lost of generality up to normalization of the symmetry operator (39) and the equivalence transformations which are reduced to the rotations with respect to the third coordinate axis.

To simplify the related equation (42) it is convenient to use the following variables:

$$x = \mu y_1^2 - \kappa y_2^2 + \mu \kappa (\mu - \kappa), \quad y = \frac{\kappa y_1^2 + \mu y_2^2 + (\mu - \kappa)(\mu^2 - \kappa^2)}{\sqrt{\mu \kappa (\mu - \kappa)}} \quad (44)$$

which reduce it to the following form:

$$V_{yy} = y V_{xy} + x V_{xx} + 2V_x = 0. \quad (45)$$

By construction the latter equation has to have polynomial solutions which we find in the following form:

$$V^{(s)} = y^s + (s - 1)xy^{s-2} + \frac{(s - 2)(s - 3)}{2} x^2 y^{s-4} + ...$$
$$+ \frac{(s - k)(s - k + 1)(s - k + 2)...(s - 2k + 1)}{k!} x^k y^{s-2k} + ...$$

$$+ \left( \delta \left( \frac{s - 1}{2} \right) + 1 \right) x^{\frac{s+1}{2}} y^\delta$$

where $\delta = 1$ for even $s$ and $\delta = 0$ for $s$ odd. In particular,

$$V^{(1)} = y,$$
$$V^{(2)} = y^2 + x,$$
$$V^{(3)} = y^3 + 2xy,$$
$$V^{(4)} = y^4 + 3xy^2 + x^2,$$
$$V^{(5)} = y^5 + 4xy^3 + 3x^2y,$$
$$V^{(6)} = y^6 + 5y^4x + 6y^2x^2 + x^3 \quad (47)$$

where $x$ and $y$ have to be expressed via the initial variables $x_1, x_2$ and $x_3$ by formulae (44) and (41).

Of course, a linear combination of generic polynomials (46) and their particular cases presented in (47) also solves equation (45). In addition, we can fix a multi parametric parametric solution which cannot be expressed via linear combinations of polynomials (16):

$$V = \frac{\alpha x_2^2}{\kappa^2 x_1^2 + \mu^2 x_2^2 - \kappa \mu x_3^2}. \quad (48)$$
Thus we find a countable set of integrable PDM systems, admitting second order integrals of motion. The next step is to find the corresponding functions $\eta$. For any fixed potential $V$ enumerated in \cite{17} and \cite{18} it can be easily done solving equations \cite{10}. In particular, for potentials \cite{18} we obtain

$$\eta = \frac{\kappa x_1^2 + \mu x_2^2}{V}.$$

Notice that for some particular values of arbitrary parameters the PDM systems with potentials \cite{18} have more extended symmetries which are indicated in Items 3 and 4 of Table 2.

In analogous way we can solve the remaining inequivalent compatible systems (32) and (13) which in fact are more easy to handle. To save a room we will not present the calculation details here since they can be found in \cite{49}.

Thus we have classified the PDM systems which are scale invariant and admit second order integrals of motion. These integrals belong to one out of two subclasses. The first of them includes integrals which belong to the enveloping algebra of the conformal algebra $c(3)$ up to constant terms including functions of $x$. The other subclass includes integrals of motions which do not belong to this enveloping algebra. We have found all of them, but in the following Tables 1 and 2 just the systems belonging to the first subclass are presented. In contrary, Table 3 collects the systems which belong to the second class and are invariant with respect to the algebra so(1,2). The presented list of PDM systems admitting second order integrals of motion is complete up to rotation transformations.

In the tables $F(\cdot), G(\cdot)$ and $R(\cdot)$ are arbitrary functions of the arguments specified in brackets, $V^{(s)}$ are polynomials \cite{15}, $c, c_1, c_2, \mu$ and $\nu$ are arbitrary real parameters $\varphi$ and $\theta$ are Euler angles, $\bar{r}^2 = x_1^2 + x_2^2 + x_3^2$, $\hat{r}^2 = x_1^2 + x_2^2$, $P_a, K_a, D$ and $L_3$ are operators defined in \cite{19}, and the summation is imposed over the repeating indices $a$ by values 1, 2 and 3. The symbol $\{A, B\}$ denotes the anticommutator of operators $A$ and $B$, i.e., $\{A, B\} = AB + BA$.

Table 1. Inverse masses, potentials and the related integrals of motion defined up to arbitrary functions.

| No | $f$         | $V$                              | Integrals of motion                  |
|----|------------|----------------------------------|--------------------------------------|
| 1  | $\hat{r}^2 F(\theta)$ | $G(\varphi)F(\theta) + R(\theta)$ | $L_3^2 + G(\varphi)$ |
| 2  | $r^2 F(\theta)$      | $cF(\theta)\varphi + G(\theta)$ | $\{L_3, D\} + 2c\ln(r)$            |
| 3  | $\tilde{r}^2 F(\varphi)$ | $F(\varphi)G(\theta) + R(\varphi)$ | $\{P_3, K_3\} + 2G(\theta)$ |
| 4  | $\tilde{r}^2 F(\varphi)$ | $cF(\varphi)x_3^2 r + G(\varphi)$ | $\{P_3, D\} - \frac{2c}{r}$, $\{P_3, K_3\} + \frac{2ex_3}{r}$, $\{K_3, D\} - 2cr - 12x_3$ |
| 5  | $\tilde{r}^2 F(\varphi)$ | $cF(\varphi)x_3^2 r F(\varphi) + R(\varphi)$ | $P_3^2 + \frac{\bar{r}^2}{x_3}$, $K_3^2 + 3(4x_3^2 - \hat{r}^2) + \frac{\bar{r}^4}{x_3}$, $\{P_3, K_3\} + \frac{2ex_3^2}{x_3}$ |
| 6  | $\tilde{r}^2$          | $G(\theta)$                       | $\{P_3, K_3\} + 2G(\theta)$, $L_3$ |
| 7  | $x_3^2$             | $x_3^2 \frac{F(\varphi)}{r^2}$       | $P_1^2 + P_2^2 + \frac{F(\varphi)}{r^4}$, $K_1^2 + K_2^2 + 3(3\bar{r}^2 - 2x_3^2) + \frac{F(\varphi)x_3^4}{r^2}$, $L_3^2 + F(\varphi)$ |
| 8  | $x_3^2$             | $V^{(s)}(\varphi, \theta)$       | $b\{P_1, K_1\} + a\{P_2, K_2\} + 2\tilde{\eta}$ |
4.4 Algebraic structure of integrals of motion

It is an element of common knowledge that the commutator of integrals of motion is the integral of motion too. In other words integrals of motion form a Lie algebra which, however, can be infinite dimensional. Indeed, a commutator of $n$ order (in our case second order) differential operators is the operator whose order is generally speaking $2n - 1$. The next commutator will have the order $3n - 2$, etc., and the discussed algebra can include infinite number of integrals of motion of arbitrary order.

However, for some special symmetries the algebra of integrals of motion appears to be finite dimensional. First, these integrals can simple commute. Secondly, the well known example is the Laplace-Runge-Lenz vector which form the algebra so(4) provided the representation space of this algebra is the set of solutions of Schrödinger equation for the Hydrogen atom and some more general quantum mechanical systems [21].

For the systems considered in the above the algebras of integrals of motion are infinite dimensional, but their structure is rather transparent. Namely, let $Q_1, Q_2, ..., Q_n$ are second order integrals of motion for one of the system. Then they satisfy the following generic commutation relations:

$$[Q_a, Q_b] = c_{ab} Q_k Q^{(1)}$$

where $C_{ab}$ are structure constants and $Q^{(1)}$ is the dilatation operator specified in (51) or some other first order integral of motion. However, this rule has one exception.

By definition integrals of motion commute with the Hamiltonian $H$. Thus $H$ and $Q^{(1)}$ have the same set of eigenfunctions, and relations (19) specify the Lie algebra whose representation space is an eigenvector of $Q^{(1)}$.

The structure constants $c_{ab}$ can be easily found by the direct calculation. We will not do this routine job it for all sets of integrals of motion presented in the tables, but restrict ourselves to the systems presented in Table 1.

Let us denote the integrals of motion presented in Table 1 as

$$Q_1 = \{P_3, K_3\} - 2D^2 + \frac{2cx_3}{r}, \quad Q_2 = \{P_3, D\} - \frac{2c}{r}, \quad Q_3 = \{K_3, D\} - 2cr,$$

$$Q_4 = P_3^2 + \frac{c}{x_3}, \quad Q_5 = K_3^2 + \frac{c\varphi^4}{x_3^2}, \quad Q_6 = \{P_3, K_3\} + \frac{2c\tilde{r}^2}{x_3},$$

$$Q_7 = P_1^2 + P_2^2 + \frac{F(\varphi)}{r^2}, \quad Q_8 = K_1^2 + K_2^2 + \frac{F(\varphi)\tilde{r}^4}{r^2},$$

$$Q_9 = L_3^2 + H + F(\varphi)$$

where $H$ is Hamiltonian and $D^2$ is the squared dilatation. We add the latter obvious symmetries into (50) to simplify the coupling constants. Calculating their commutators we specify the following algebraic structures:

$$[Q_7, Q_8] = 2iQ_9 D, \quad [Q_7, Q_9] = [Q_8, Q_9] = 0,$$

$$[Q_1, Q_2] = -2iQ_2 D, \quad [Q_1, Q_3] = 2iQ_3 D, \quad [Q_3, Q_2] = 2iQ_1 D,$$

$$[Q_4, Q_5] = 4iQ_6 D, \quad [Q_4, Q_6] = 2iQ_4 D, \quad [Q_5, Q_6] = -2iQ_5 D.$$

The presented commutators are proportional to the dilatation generator. Acting by the operators in the l.h.s and r.h.s. on the eigenvectors of this generator we recognize that relations
specify the Heisenberg algebra while relations (55) specify the Lie algebras isomorphic to so(1,2).

An example of the situation when the commutator of the second order integrals of motion is not proportional to $D$ but to another first order symmetry (namely, $L_3$) can be found in Item 1 of Table 2. The only case when such commutators are not proportional to the first order symmetry is present in Item 8 of the same table.

Table 2. Inverse masses, potentials and integrals of motion defined up to arbitrary coefficients.

| No | $f$   | $V$                        | Integrals of motion                           |
|----|-------|---------------------------|-----------------------------------------------|
| 1  | $r^2$ | $c_{x^1}^2 x_3^2$         | $L_2^2 - L_1^2 + c_f x_3^2 - x_1^2$,          |
|    |       |                            | $\{L_1, L_2\} + 2c_{x^1}^2 x_3^2$, $L_3$    |
| 2  | $x_3^2$ | $\frac{\alpha x_1}{\kappa^2 x_1^2 + \mu^2 x_2^2 - \kappa \mu x_3^2}$ | $\mu \{P_1, K_1\} + \kappa \{P_2, K_2\} + \frac{\mu^2 x_1^2 + \kappa \mu x_3^2}{x_1^2} V$ |
|    |       |                            | $\{P_1, D\} - \{P_3, L_2\} + c_1 x_1^2 - 2c_1 x_3^2$, $L_3$, $P_1$ | |
| 3  | $x_3^2$ | $c_1 x_2 x_3 + c_2 x_3^3$ | $K_1^2 + K_2^2 + \frac{\alpha x_1}{r} V + 3(3\mu^2 - 2x_3^2)$, | |
|    |       |                            | $L_3^2 + C_1 x_2 x_3 + C_2 x_3^3$, $P_1^2 + P_2^2 + \frac{c_2}{x_3}$ | |
| 4  | $x_3^2$ | $c_{x_1}^2 x_3^2 + c_2 x_3^2 x_3^3$ | $\frac{\alpha x_1}{x_1^2}$ | |
| 5  | $x_3^2$ | $c_{x_1}^2 x_3^2$         | $\frac{\alpha x_1}{x_1^2}$ | |
| 6  | $x_3^2$ | $c_1 x_2 x_3 + c_2 x_3^3$ | $K_2^2 + 3(5x_2^2 - r^2) + \frac{c_{x_1}^4}{x_3^2}$, $\{K_1, P_1\} + \frac{2c_{x_1}^2}{x_3^2}$, $\{P_2, K_2\} + \frac{2c_{x_1}^2}{x_3^2}$ | |
| 7  | $x_3^2$ | $c_{x_1}^2 x_3^2$         | $\frac{\alpha x_1}{x_1^2}$ | |
| 8  | $x_3^2$ | $c_{x_1}^2 x_3^2$         | $\frac{\alpha x_1}{x_1^2}$ | |
| 9  | $r^2$  | $c_1 e^{-2\mu x_3 + x_3^2}$ | $\{P_3, (L_3 + D)\} + c_1 e^{-2\mu x_3 + x_3^2}$ + $c_2 e^{-\phi} x_3^3$, $\{K_3, (L_3 + D)\} - 12x_3 + c_1 e^{-2\mu x_3 + x_3^2}$ + $c_2 e^{-\phi} x_3^3$ | |
| 10 | $r^2$  | $c_1 e^{2\mu x_3 + x_3^2}$ | $\{P_3, (L_3 - D)\} - c_1 e^{2\mu x_3 + x_3^2}$ - $c_2 e^{-\phi} x_3^3$, $\{K_3, (L_3 - D)\} + 12 x_3 - c_1 e^{2\mu x_3 + x_3^2}$ - $c_2 e^{-\phi} x_3^3$ | |
| 11 | $r^2$  | $c_{x_1}$                   | $\frac{\alpha x_1}{x_1^2}$ | |
| 12 | $r^2$  | $c_{x_1}^2 x_3^2$         | $\frac{\alpha x_1}{x_1^2}$ | |

12
5 PDM systems invariant with respect to algebra so(1, 2)

The results obtained in Section 4 can be generalized to the cases of PDM systems invariant w.r.t. multi parametric symmetry groups including the subgroup of dilatations. In this section we specify the systems which are invariant with respect to the Lie group isomorphic to SO(1,2) and admit second order integrals of motion. The generic form of the corresponding Hamiltonians and the related first order symmetries are fixed in (6) and (9).

The considered systems are invariant with respect to dilatations, and so the related inverse masses and potentials should be present in Tables 1 and 2. Thus our task is to select such systems specified in these tables with at least for some particular arbitrary functions or parameters are invariant with respect to shifts and dilatations generated by operators $P_3$ and $K_3$ presented in (9). In fact it is sufficient to ask for the symmetries with respect to the shifts since symmetry with respect to the conformal transformations is a consequence of the symmetry with respect to shifts and dilatations, see equations (20) and (21).

In other words everything we need is to select such potentials and inverse masses presented in the Tables 1 and 2 which satisfy the additional conditions

$$\frac{\partial f}{\partial x_a} = 0, \quad \frac{\partial V}{\partial x_a} = 0$$

for some fixed $a$ not necessary equal to $3$. Whenever conditions (56) are satisfied, we can transform it to the case $a = 3$ by a rotation transformation.

In addition, it is necessary to consider second order symmetries which do not belong to the enveloping algebra of c(3). The related Killing tensors initially include arbitrary functions $g(x)$ which are connected with the inverse mass functions $f$ via relations (34)-(37).

Let us consider the most important symmetry which is possessed by all analysed systems. It has the following generic form:

$$Q = L_3^2 + P_a\phi P_a + \eta$$

where $L_3 = -i(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1})$, $P_a = -i \frac{\partial}{\partial x_a}$, $\phi$ and $\eta$ are unknown functions of $x$.

The corresponding matrix $\tilde{\mu}^{ab}$ is degenerated, and its nonzero entries are:

$$\tilde{\mu}^{11} = x_2^2, \quad \tilde{\mu}^{22} = x_1^2, \quad \tilde{\mu}^{12} = \tilde{\mu}^{21} = -x_1 x_2.$$  

To obtain these entries from formula (26) we have to set $\lambda^{11} = \tilde{\lambda}^{11} = \tilde{\lambda}^{22} = \lambda^{33} = 1$ while the remaining entries $\lambda^{ab}$ and $\tilde{\lambda}^{ab}$ are equal to zero.

In accordance with (58), the related equations (32) are reduced to the following system:

$$x_2 f_\varphi - g f_1 + f g_1 = 0, \quad x_1 f_\varphi + g f_2 - f g_2 = 0, \quad g f_3 - f g_3 = 0$$

where $f_\varphi = \frac{\partial f}{\partial \varphi}$ and $f_a = \frac{\partial f}{\partial x_a}$.

Since we are dealing with the Killing tensor quadratic in $x_a$ functions $f$ and $\varphi$ are connected by relation (37) which reduce equations (59) to the following relations:

$$f_\varphi = f^2 g_\varphi, \quad g_\theta = 0$$
whose generic (up to constant multiplier) solution is

\[ f = \frac{\tilde{r}^2}{F(\varphi) + G(\theta)}. \]  

(61)

Substituting (37) and (61) into equation (38) and integrating it we obtain

\[ V = \frac{M(\theta) + N(\varphi)}{F(\varphi) + G(\theta)}, \eta = -F(\varphi)V + G(\theta). \]  

(62)

The corresponding Hamiltonian commutes with \( P_3 \) provided functions \( G(\theta) \) and \( M(\theta) \) are constants, and we come to the results connected in Item 1 of Table 3, where we use the notations

\[ (F(x) \cdot H) = p_a F(x) f p_a + F(x)V. \]  

(63)

For some particular functions \( F(\varphi) \) and \( N(\varphi) \) the related Hamiltonians admit additional symmetries presented in Items 2-7 of the mentioned table. They can be calculated in analogy with the above.

Table 3. Inverse masses and potentials for equations admitting symmetry algebra so(1, 2)

| No | \( f \) | \( V \) | Integrals of motion |
|----|--------|-------|--------------------|
| 1  | \( \tilde{r}^2 F(\varphi) \) | \( F(\varphi)G(\varphi) \) | \( L_3^2 - \left( \frac{1}{r(\varphi)} \cdot H \right) + G(\varphi) \) |
| 2  | \( x_2^2 \) | \( F(\varphi) \) | \( \{K_1, P_1\} - 2L_2^2 - 2F(\varphi), \) \( L_3^2 - \left( \frac{r}{x_2^2} \cdot H \right) + \frac{x_2^2}{r} F(\varphi) \) |
| 3  | \( \frac{x_2^2 r}{c_1 r + c_2 x_2} \) | \( \frac{c_1 r + c_2 x_2}{c_1 r + c_2 x_2} \) | \( \{K_2, D\} + \{K_3, L_1\} + 2 \left( \frac{c_1 r + c_2 x_2}{x_1^2} \cdot H \right) - 2\frac{c_1^2 r^2}{x_1^2} - 15x_2, \) \( L_3^2 - \left( \frac{r}{x_2^2} \cdot H \right) + \frac{r}{x_1^2} F(\varphi) \) |
| 4  | \( \tilde{r}^2 + \varepsilon x_1 \tilde{r} \) | \( \frac{c_1 x_1 + c_2 x_2}{c_1 x_1 + c_2 x_2} \) | \( \{K_2, D\} + \{K_3, L_1\} + 2 \left( \frac{r^2}{H} - 2\frac{c_1^2 r^2}{x_1^2} - 15x_2, \right) \) \( L_3^2 - \left( \frac{r}{x_2} \cdot H \right) + \frac{r}{x_1^2} F(\varphi) \) |
| 5  | \( \frac{x_1^2 x_2^2}{c_1 x_2^2} \) | \( \frac{c_1 x_1^2 + c_2 x_2^2}{c_1 x_2^2} \) | \( K_1^2 - \left( \frac{c_1 r}{x_1^2} \cdot H \right) + \frac{c_1 r}{x_1^2} + 3(5x_1^2 - r^2), \) \( K_2^2 - \left( \frac{c_1 r}{x_1^2} \cdot H \right) + \frac{c_1 r}{x_1^2} + 3(5x_1^2 - r^2), \) \( L_3^2 - \left( \frac{r^2 (c_1 x_1^2 + c_2 x_2^2)}{x_1^2 x_2^2} \cdot H \right) + \frac{r^2 (c_1 x_1^2 + c_2 x_2^2)}{x_1^2 x_2^2} \) |
| 6  | \( x_2^2 \) | \( \frac{x_2^2}{x_1^2} \) | \( \{L_2, P_1\} + 2c_1 \frac{x_2^3}{x_1^2}, \{L_2, K_1\} + 2c_2 \frac{x_2^3}{x_1^2} - 3x_3, \) \( \{P_1, K_1\} + 2c_2 \frac{x_2^3}{x_1^2}, \)  \( P_2 + \frac{x_2^3}{x_1^2}, K_2^2 + \frac{x_2^3}{x_1^2} + 3(5x_1^2 - r^2) \) |
| 7  | \( \tilde{r}^2 \) | \( \frac{x_2^2}{x_1^2} \) | \( P_3, L_3, D, K_3 \) |
| 8  | \( x_3^2 \) | \( \frac{x_3^2}{x_1^2} \) | \( P_3, P_2, K_1, K_2, D, L_3 \) |
The last two items of Table 3 include systems which admit only the first order integrals of motion and their bilinear combinations.

6 PDM systems invariant w.r.t. the rotation group

Let us discuss the rotationally invariant systems which are specified in equations (1), (2) and (4), and present such of them which admit second order integrals of motion. In contrast with the scale invariant systems classified in Section 4 in this case we cannot decouple the determining equations with respect to the order in $x^i$ of the related Killing tensors. However, these equation are reduced to systems of ordinary differential equations with well defined tensorial properties. And it is possible (and necessary) to make the another type of decoupling corresponding to the scalar, vector, and tensor integrals of motion. The related Killing tensors looks as follows:

$$\mu_{ab} = \mu_{5}^{ab},$$
$$\mu_{ab} = \nu \mu_{3}^{ab} + \lambda \mu_{4}^{ab},$$
$$\mu_{ab} = \mu_{8}^{ab},$$
$$\mu_{ab} = \nu \mu_{2}^{ab} + \mu \mu_{4}^{ab} = \nu(\lambda x^b + \lambda x^a - 2\delta^{ab}\lambda^c x^c) + \mu(x^a\lambda^b + x^b\lambda^a)x^2 - 4x^a x^b \lambda^c x^c + 2\delta^{ab}x^2 \lambda^c x^c) + \delta^{ab}\lambda x^c g(r),$$
$$\mu_{ab} = \nu \lambda^{ab} + \omega (\lambda x^a x^b - (x^2 \lambda_{bc} + x^b \lambda_{ac})x^c - 2\delta^{ab}\lambda^{cd} x^c x^d + \delta^{ab}\lambda^{cd} x^c x^d g(r).$$

where $\mu_{n}^{ab}$ are tensors represented in (17).

Tensors (64), (65) and (66) correspond to the scalar, pseudovector and pseudotensor integrals of motion respectively. Notice that to keep the correct transformation properties of the integrals of motion with respect to rotations arbitrary functions $g_n$ present in (17) should be reduced to zero for the case of pseudo vectors (tensors) and to functions of $r$ for the true ones.

The scalar integral is the squared orbital momentum and is admitted by any of the considered systems. The pseudovector and pseudotensor integrals are not admitted by any system. The technical reason of the latter situation is just the triviality of the arbitrary functions $g_n$.

For rotationally invariant inverse masses and potentials $\Omega$ the determining equations (15) and (13) are reduced to the following forms:

$$(\mu_{a}^{aa} + 2\mu_{n}^{na})f = 10\mu_{a}^{an} x_n f'$$
and

$$fy^a - 2\mu_{ab} x_b V' = 0.$$ 

where $f' = \frac{df}{dr}$ and $V' = \frac{dV}{dr}$.

Let us consider the tensor integrals of motion generated by the Killing tensors (68). Substituting (68) into (69) we come to the following system:

$$(g + \mu r^2)f' - (2\mu + g)f = 0,$$
$$(\mu r^4 - \nu)g' + 2(g - \mu r^2)f = 0$$
whose generic solutions are

\[ f = c_1 r^4, \quad g = 0 \]  

and

\[ f = \frac{(r^4 - \nu)^2}{c_1(\mu r^4 + \nu) + c_2 r^2}, \quad g = -\frac{c_2(\mu r^4 + \nu) + 4\mu \nu c_1 r^2}{c_1(\mu r^4 + \nu) + c_2 r^2} \]  

where \( c_1 \) and \( c_2 \) are integration constants. Substituting \( (72) \) and \( (73) \) into \( (70) \) we easily find the corresponding potentials \( V \) and functions \( \eta \) which are presented in Items 4-6 of Table 4. Notice that the zero value of arbitrary constant \( c_1 \) correspond to the special solution for \( V \).

In analogous way we find the solutions of the determining equations \( (69) \) and \( (70) \) for the integrals of motion and their bilinear combinations.

### Table 4. Inverse masses and potentials for equations admitting symmetry algebra \( \text{so}(3) \)

| No | \( f \) | \( V \) | Integrals of motion |
|----|--------|-------|--------------------|
| 1. | \((\mu r^2 - \nu)^2\) | \(\frac{\alpha(\mu r^2 + \nu)}{r}\) | \(Q_a = \varepsilon_{abc}\{(\nu P_b + \mu K_b), L_c\} + \frac{x_a(\alpha - 6r)}{r}\) |
| 2. | \(\frac{r(\mu r^2 - \nu)^2}{\kappa r - 2(\mu r^2 + \nu)}\) | \(\frac{\alpha r}{\kappa r - 2(\mu r^2 + \nu)}\) | \(Q_a = \varepsilon_{abc}\{(\nu P_b + \mu K_b), L_c\} - \frac{\alpha x_a}{2(\mu r^2 + \nu) - \kappa r} - 6x_a\) |
| 3. | \((\mu \tilde{\mu} x^4 - \nu \tilde{\nu})^2\) | \(\frac{\alpha^2}{(\mu r^2 + \nu)^2}\) | \(Q_{ab} = \{\mu K_a + \nu P_a, \mu K_b + \nu P_b\} + 6\mu \tilde{\mu}(5x_a x_b - \delta^{ab}r^2) + \frac{2\alpha^2 x_a x_b}{(\mu r^2 + \nu)^2}\) |
| 4. | \((\mu r^4 - \nu)^2\) | \(\frac{\alpha(\mu r^4 + \nu)}{r^2}\) | \(Q_{ab} = \frac{\alpha}{2}\{K_a, K_b\} + \nu P_a P_b + \frac{x_a x_b (\mu r^4 + \nu)}{(\mu r^4 - \nu)^2} \cdot \frac{\alpha}{\mu r^4 + \nu} + \frac{\alpha}{\mu r^2 + \nu}\) |
| 5. | \(\frac{(\mu r^4 - \nu)^2}{\mu r^4 + \nu}\) | \(\frac{\alpha x}{\mu r^4 + \nu}\) | \(Q_{ab} = \frac{\alpha}{2}\{K_a, K_b\} + \nu P_a P_b + \frac{x_a x_b (\mu r^4 + \nu)}{(\mu r^4 - \nu)^2} \cdot \frac{\alpha}{\mu r^4 + \nu} + \frac{\alpha}{\mu r^2 + \nu}\) |
| 6. | \((\mu r^4 - \nu)^2\) | \(\alpha\) | \(\mu K_a + \nu P_a\) |

In Table 4 the Greek letters denote arbitrary real coefficients. Any of them (except \( \alpha \)) either can be normalized to \( \pm 1 \) or be equal to zero. The last of Table 4 includes systems which admit only the first order integrals of motion and their bilinear combinations.

Notice that the systems presented in Items 1 and 2 of Table 4 are Stäckel equivalent between themselves. The same is true for the systems fixed in Items 3, 4 and 5. We remind that the Stäckel transform consists in the multiplication of the Hamiltonian by the inverse potential (i.e., \( H - E \rightarrow (V \cdot H) - \tilde{V} \)) where the operation \((V \cdot H)\) should be treated in the sense defined by equation \((63)\) combined with the conformal transformations and changing the roles played by coupling constants and Hamiltonian eigenvalues. Moreover, at any step the potentials can be added by a constant.
7 PDM systems invariant w.r.t. the 2d Euclid group E(2)

Thus we have specified inequivalent superintegrable systems invariant with respect groups $SO(1, 2)$ and $SO(3)$. The last task is to describe the systems invariant with respect to the Euclid group $E(2)$, whose inverse masses and potentials have the form (5).

Since the arbitrary elements do not depend on $x_1$ and $x_2$ we can make rather restrictive a priori predictions about the possible second order integrals of motion.

Indeed, let $Q$ be an integral of motion. By definition $P_a$ with $a = 1, 2$ are integrals of motion too, the same is true for the commutators $[P_a, Q]$, $[P_a, [P_b, Q]]$ and $[P_a, [P_b, [P_c, Q]]]$. Since the Killing tensors are fourth order polynomials in $x$, and the derivation of the Killing tensor w.r.t. $x_a$ is again the Killing tensor, we conclude that any second order symmetry induces the symmetry generated by $\mu_{0}^{ab}$ (refer to (16), i.e.,

$$Q = P_a (\lambda_{ab} + \delta_{ab}G(x)) P_b + \eta.$$

We can specify the following qualitatively different versions of coefficients $\lambda_{ab}$ in formula (74):

$$\lambda^{3\mu} \neq 0, \quad \lambda^{\mu\nu} = \lambda^{33} = 0, \quad \mu, \nu = 1, 2,$$

$$\lambda^{3\mu} = 0, \quad \text{some of coefficients } \lambda^{\mu\nu} \text{ or } \lambda^{33} \text{ are nontrivial}$$

In the case (75) the systems admitting second order constants of motion necessary admit the following constants

$$Q^{31} = P_3 P_1 + \eta^{31}, \quad Q^{32} = P_3 P_2 + \eta^{32},$$

moreover, both of them, in view of the symmetry with respect to rotations around the third coordinate axis.

In the case (76) the related integrals of motion are trivial since the Hamiltonians considered in this section commute with them by definition. It is evident for $Q^{\mu\nu} = P_{\mu} P_{\nu} + \eta^{\mu\nu}$ with $\mu, \nu < 3$. On the other hand, $Q^{33} = P_3^2 + \eta^{33}$ is equal to $1 - Q^{11} - Q^{22}$.

In the case (76) we have to consider integrals of motion whose commutators with $P_1$ and $P_2$ are reduced to $Q^{\mu\nu}$ with $\mu, \nu < 3$ and $Q^{33}$. They are listed in the following formula:

$$Q^{a} = \{P_a, D\} + \eta^{a}, \quad \tilde{Q}^{a} = \{P_3, L_a\} + \tilde{\eta}^{a}.$$

Thus to find the PDM systems which are invariant with respect to algebra $e(2)$ and admit second order integrals of motion it is sufficient to fix such of them which admit the integrals (77) and (78). Such systems can admit some additional symmetries whose calculation for known system is a rather simple problem.

We realize the presented algorithm and find three systems with rather extended sets of integrals of motion which are presented in Table 5. The calculations requested of solving the determining equations corresponding to (77) and (78) are rather straightforward. In particular, for the case of operator $Q^{31}$ (77) equations (15) and (35) are reduced to the following systems:

$$f_3 = 2f g_1, \quad fg_2 = 0, \quad g f_3 = fg_3$$
and

$$V_3 = f \eta_3^{21}, \quad f \eta_2^{31} = 0, \quad g f_3 = f \eta_3^{31}$$
correspondingly. These systems are solved by the following functions:

$$f = \frac{c_1}{x_3 + c_2}, \quad g = \frac{x_1 + c_3}{x_3 + c_2}, \quad V = \frac{c_4}{x_3 + c_2} + c_5, \quad \eta = 2(2 + c_6)V + c_6$$  \hspace{1cm} (79)$$

where \(c_1, c_2, \ldots, c_6\) are integration constants. Scaling and shifting the spatial variables we can reduce \(c_1\) to the unity and \(c_2, c_3\) to zero. Constants \(c_5\) and \(c_6\) also can be nullified since they are added to the functions which are defined up to constant shifts. As a result we come to the system presented in Item 1 of Table 5. However, it is necessary to search for the additional integrals of motion admitted by this system, i.e., to solve the determining equations including the generic Killing tensor and functions \(f\) and \(V\) fixed in (79).

**Table 5.** Inverse masses and potentials for equations admitting symmetry algebra \(e(2)\)

| No | \(f\) | \(V\) | Integrals of motion |
|----|------|------|-------------------|
| 1  | \(\frac{1}{x_3}\) | \(\frac{c}{x_3}\) | \(2P_3P_2 - (x_2 \cdot H) - c \frac{x_2}{x_3}, \quad 2P_3P_1 - (x_1 \cdot H) - c \frac{x_1}{x_3}, \quad \{P_3, D\} = \frac{1}{2}((\dot{r}^2 + 4r^2) \cdot H) - \frac{c(\dot{r}^2 - 4x_3^2)}{2x_3}, \quad \{P_3, L_2\} + \{P_2, L_1\} + \frac{1}{2}((x_1^2 - x_2^2) \cdot H) + \frac{c(x_1^2 - x_2^2)}{2x_3}, \quad \{P_1, L_1\} - \{P_2, L_2\} + (x_1x_2 \cdot H) + \frac{c_1x_1x_3}{x_3}\) |
| 2  | \(\frac{x_3^2}{x_3^2 - 1}\) | \(\frac{cx_3^2}{x_3^2 - 1}\) | \(\{P_1, D\} - (x_1 \cdot H) + \frac{c_1(x_3^2 - 2)}{x_3^2 - 1}, \quad \{P_2, D\} - (x_2 \cdot H) + \frac{c_2(x_3^2 - 2)}{x_3^2 - 1}, \quad \{P_3, K_3\} + (\frac{x_3^2 + 2x_3^2}{x_3^2 - 1}) \cdot H + \frac{c_3x_3^2}{x_3^2 - 1}, \quad \{P_1, K_1\} + (x_1^2 \cdot H) - \frac{c_3x_3^2}{x_3^2 - 1}, \quad \{P_2, K_2\} + (x_2^2 \cdot H) - \frac{c_3x_3^2}{x_3^2 - 1}, \quad \{P_1, K_2\} + \{P_2, K_1\} + 2(x_1x_2 \cdot H) - \frac{2c_1x_1x_2(x_3^2 - 2)}{x_3^2 - 1}\) |
| 3  | \(\frac{x_3^2}{x_3^2 + 1}\) | \(\frac{cx_3^2}{x_3^2 + 1}\) | \(\{P_3, L_1\} - (\frac{x_3^2}{x_3^2 + 1}) \cdot H - \frac{c_1x_3^2}{x_3^2 + 1}, \quad \{P_3, L_2\} + (\frac{x_3^2}{x_3^2 + 1}) \cdot H + \frac{c_1x_3^2}{x_3^2 + 1}, \quad \{P_1, D\} - (x_1 \cdot H) + \frac{c_1(x_3^2 + 2)}{x_3^2 + 1}, \quad \{P_2, D\} - (x_2 \cdot H) + \frac{c_2(x_3^2 + 2)}{x_3^2 + 1}, \quad \{P_3, K_3\} + (\frac{x_3^2 + 2x_3^2}{x_3^2 + 1}) \cdot H + \frac{c_3x_3^2}{x_3^2 + 1}, \quad \{P_2, K_2\} + (x_2^2 \cdot H) - \frac{c_3x_3^2}{x_3^2 + 1}, \quad \{P_1, K_1\} + (x_1^2 \cdot H) - \frac{c_3x_3^2}{x_3^2 + 1}, \quad \{P_1, K_2\} + \{P_2, K_1\} + 2(x_1x_2 \cdot H) - \frac{2c_1x_1x_2(x_3^2 + 2)}{2(x_3^2 + 1)}\) |
| 4  | \(x_3^2\) | \(c\) | \(P_1, P_2, K_1, K_2, D, L_3\) |
8 Discussion

We classify inequivalent quantum mechanical systems with position dependent masses which admit second order integrals of motion and three parametric symmetry groups. The classification results are summarized in Tables 3, 4 and 5. In addition, we present the superintegrable systems which are supposed to admit at least one Lie symmetry, namely, the symmetry with respect to scaling of the dependent and independent variables, see Tables 1 and 2.

As it was indicated in [30] there are three inequivalent three parametric Lie groups which can be admitted by the PDM Schrödinger equation, namely, the rotation group SO(3), the Lorentz group in (1+2)-dimensional space SO(1,2) and the Euclid group in 2d space E(2).

We believe that the PDM systems invariant with respect to groups SO(1,2) and E(2) are classified in the present paper for the first time.

Superintegrable PDM systems with the rotational symmetries have been discussed in numerous papers, see [33], [34], [35], [36] and references cited therein. A formal complete classification of such quantum mechanical systems admitting second order integrals of motion was presented in [38]. In the present paper we revise the results of this classification and present its results in a compact form and in the only table, namely, Table 4 whenever in [38] you can find two rather extended tables which, however, include a lot of useful information concerning the supersymmetry and integrability of the discussed systems.

Notice that the systems presented in the same item of Table 4 and differ only by the value of arbitrary parameters in fact are essentially different. In particular they can possess different supersymmetry [38].

To solve the classification problems we use a specific representation of the Hamiltonians and integrals of motion fixed in equations (2) and (10). Being mathematically equivalent to other representations with another orders of differentials and functions (compare (3) and (2)) they led to maximally compact and simple systems of the determining equations for the arbitrary elements $V$ and $f$.

The next natural steps are to classify superintegrable systems admitting two-parametric symmetry groups and at least a one one-parametric symmetry group. Just such systems but in two dimensions are used and studied in numerous papers, see, e.g., [44, 45].

Notice that the present paper includes some important elements of such generalized analysis. Indeed, in Tables 1 and 2 the results of the classification of superintegrable systems invariant with respect to the one parametric group of dilatation transformations is presented. Among them are rather exotic systems whose potentials are arbitrary order polynomials in $x_1^2$ and $x_2^2$ presented in Item 8 of Table 1. However, this classification is restricted to the integrals of motion which, up to scalar terms belong to the enveloping algebra of algebra $c(3)$.

The total number of the inequivalent one- and two-parametric Lie groups which can be admitted by quantum mechanical PDM systems is not too large. In accordance with the results of paper [30] there exist five two parametric and five one parametric groups which can be accepted by the 3d quantum mechanical systems with PDM. Among them there are the groups generated by the following pairs of infinitesimal operators belonging to the list presented in [49]:

$$\langle D, P_3 \rangle, \quad \langle D, L_3 \rangle, \quad \langle P_1, P_2 \rangle.$$ (80)

The superintegrable systems invariant with respect to these groups are partially classified in the present paper. Indeed, the systems admitting the algebras spanned on $\langle D, P_3 \rangle$ and
are presented in Tables 3 and 5. Moreover Tables 1 and 2 include the systems admitting the algebra $<D, L_3>$, see Item 6 of Table 1 and Items 1, 7, 8, 11, 12 of Table 2. In other words, we present an essential part of superintegrable systems admitting two parametric symmetry groups and the systems admitting one out of five possible one parametric groups. We plan to complete this classification in the following paper.

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