Truncated Random Measures

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with: T. Campbell, J. How, T. Broderick
What leads to a statistical method being used for science?
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1. Conceptually clear
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1. **Conceptually clear**
   - Bayesian methods are *conceptually clear*…
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2. Easy to use
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   • …but often not easy to use…
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3. Reliable
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   - ...which makes them *less reliable*
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     - v2.0: Stan (HMC or variational inference or MAP estimation)
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     - v2.0: Stan (HMC or variational inference or MAP estimation)
   - Goal: integrate BNP priors into PPLs like Stan
BNP: awesome, but challenging to use
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Need models that can extract new, useful information from infinite streams of data
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**e.g.** keep learning new topics from a stream of documents
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movie  text  medicine  robotics  genetics

finance  astronomy  traffic  agriculture  pathology

[Gopalan 2014] [Teh 2006] [Huang 2014] [Michini 2015] [Lennox 2010] [Prunster 2014] [Yang 2015] [Yu 2012] [Ozaki 2008] [Kottas 2008]
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automate inference with probabilistic programming

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Inference in BNP models
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• Option #1: Integrate out the parameter (CRP, IBP, etc.)
  **issues:** care about the parameters, using approximations (HMC/VB),
  distributed computation
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[Blei 06; Neal 10]
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Wide variety of priors in
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All BNP priors

Previously studied priors
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**Contributions:**

All BNP priors

Previously studied priors
with finite approx (past work)
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**Contributions:**
• 2 representation forms (7 reps total) that allow finite approximation
  of *(normalized) completely random measures ( (N)CRMs )*
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• Approximation error analysis
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**Contributions:**
• 2 representation forms (7 reps total) that allow finite approximation of *(normalized)* completely random measures *(N)*CRMs
• Approximation error analysis
• Computational complexity analysis *(not in this talk)*
Past work: finite approximations to BNP priors

|       | Finite Approximation | Approximation Error Bounds | Computational Complexity |
|-------|----------------------|----------------------------|--------------------------|
| DP    | ✓                    |                            | ✓                        |
| BP    | ✓                    |                            | ✓                        |
| BPP   |                      |                            |                          |
| ΓP    | ✓                    | ✓                          |                          |
| (N)CRM|                      |                            |                          |
### Past work: finite approximations to BNP priors

|       | Finite Approximation | Approximation Error Bounds | Computational Complexity |
|-------|----------------------|----------------------------|--------------------------|
| **DP** | ✓ [Sethuraman 94]   | [Roychowdhury 15]         | ✓ [Ishwaran 01]          |
| **BP** | ✓ [Teh 07] [Paisley 12] [Thibaux 07] | ✓ [Doshi-Velez 09] [Paisley 12] | ✓ |
| **BPP** | ✓ [Broderick 14] | | |
| **(N)CRM** | ✓ [Bondesson 82] [Roychowdhury 15] | ✓ [Roychowdhury 15] | ✓ |
| **(N)CRM** | ✓ [Broderick 14] | | |
Past work: finite approximations to BNP priors

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| DP  | ✓                    |                           |                          |
|     | [Sethuraman 94]      |                           | [Ishwaran 01]            |
|     | [Roychowdhury 15]    |                           |                          |
| BP  | ✓                    |                           | ✓                        |
|     | [Teh 07]             |                           | [Doshi-Velez 09]         |
|     | [Paisley 12]         |                           | [Roychowdhury 15]        |
| BPP | ✓                    |                           | ✓                        |
|     | [Broderick 14]       |                           |                          |
| GP  | ✓                    | ✓                         |                          |
|     | [Bondesson 82]       |                           |                          |
|     | [Roychowdhury 15]    |                           |                          |
| (N)CRM | ✓                  |                           |                          |
|     | [Broderick 14]       |                           |                          |

Sparse results for a few priors in BNP
Past work: finite approximations to BNP priors

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|       | [Bondesson 82]       | [Roychowdhury 15]         |                          |
| (N)CRM| ✓                    |                            |                          |
|       | [Broderick 14]       | [Roychowdhury 15]         |                          |

Sparse results for a few priors in BNP

No general theory
Truncation Roadmap
Truncation Roadmap

Tractable models
in BNP
Truncation Roadmap

Tractable models in BNP

two forms for sequential representations

\[ \sum_{k=1}^{\infty} \theta_k \delta \psi_k \]
Truncation Roadmap

Tractable models in BNP

two forms for sequential representations

$$\sum_{k=1}^{\infty} \theta_k \delta \psi_k$$

Truncation and error analysis

$$\sum_{k=1}^{K} \theta_k \delta \psi_k$$
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Tractable models in BNP

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The Standard Model in BNP (By Example)
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| Topic  | Frequency | Doc 1 (532 words) | Doc 2 (210 words) | Doc 3 (854 words) | Doc 4 (926 words) |
|--------|-----------|-------------------|-------------------|-------------------|-------------------|
| sports | 343       | 210               | 854               | 342               |                   |
| politics | 189      |                   |                   |                   |                   |
| food   |           |                   |                   |                   |                   |

...
The Standard Model in BNP (By Example)

|        | sports | politics | food | ... |
|--------|--------|----------|------|-----|
| Doc 1  | 343    | 189      |      |     |
| Doc 2  |        | 210      |      |     |
| Doc 3  | 854    |          |      |     |
| Doc 4  | 342    | 584      |      |     |

0.7 0.5 0.2
The Standard Model in BNP (By Example)

- **Topic Space**
  - Sports
  - Politics
  - Food
  -...

- **Frequency Space**

- **Docs**
  - Doc 1 (532 words)
    - 343
    - 189
    - 0.7
  - Doc 2 (210 words)
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  -...

- **Dimensions**
  - Topic Space
  - Frequency Space
The Standard Model in BNP (By Example)

Doc 1 (532 words)
- Sports: 343
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Topic space

Frequency space

Sports

0.7
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| Doc   | Frequency | Sports | Politics | Food | ... |
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frequency space

0.7

sports
topic space
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Frequency space

Sports

Topic space
The Standard Model in BNP (By Example)

θ is a random discrete measure on the topics.

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| Topic  | sports | politics | food | ... |
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$\Theta$ is a random discrete measure on the topics.
The Standard Model in BNP (By Example)

θ is a random discrete measure on the topics

"traits"

"rates"
The Standard Model in BNP (By Example)

| Obs 1 | Obs 2 | Obs 3 | Obs 4 |
|-------|-------|-------|-------|
| \( \psi_1 \) | 343   | 210   | 854   | 342   | 584   |
| \( \psi_2 \) | 189   |       |       |       |       |
| \( \psi_3 \) |       |       |       |       |       |
| ...   |       |       |       |       |       |

\( \Theta \) is a random discrete measure on the topics traits

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Poisson processes and (N)CRMs

How do we generate infinitely many trait/rate points \((\psi, \theta)\)?
Poisson processes and (N)CRMs

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**Poisson point process** with measure \(\nu(d\theta \times d\psi)\):

![Diagram showing points in rate space and trait space]
Poisson processes and (N)CRMs

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\[
\Theta
\]

**completely random measure** (CRM) (e.g. BP, \(\Gamma P\))

[Kingman 93]
Poisson processes and (N)CRMs

How do we generate infinitely many trait/rate points \((\psi, \theta)\)?

**Poisson point process** with measure \(\nu(\mathrm{d}\theta \times \mathrm{d}\psi)\):

- Completely random measure (CRM) (e.g. BP, GP)
- Normalize rates: **normalized CRM** (NCRM) (e.g. DP)

[Kingman 93]
Poisson processes and (N)CRMs

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Captures a large class of useful priors in BNP

[Kingman 93]
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How do we generate infinitely many trait/rate points \((\psi, \theta)\)?

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\[ \Theta \]

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Captures a large class of useful priors in BNP

How do we pick a finite subset of the points? [Kingman 93]
Truncation Roadmap

Tractable models in BNP

two forms for sequential representations

\[
\sum_{k=1}^{\infty} \theta_k \delta \psi_k
\]

Truncation and error analysis

\[
\sum_{k=1}^{K} \theta_k \delta \psi_k
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Truncation Roadmap

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\[ \sum_{k=1}^{K} \theta_k \delta \psi_k \]

Truncation and error analysis
Sequential representation & truncation

We pick a finite subset of atoms \((\psi, \theta)\) by:
Sequential representation & truncation

We pick a finite subset of atoms \((\psi, \theta)\) by:

1) ordering the atoms **(sequential representation)**
Sequential representation & truncation

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Sequential representation & truncation

We pick a finite subset of atoms $(\psi, \theta)$ by:

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We pick a finite subset of atoms \((\psi, \theta)\) by:

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We pick a finite subset of atoms \((\psi, \theta)\) by:

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\[
\Theta = \sum_{k=1}^{\infty} \theta_k \delta_{\psi_k}
\]
Sequential representation & truncation

We pick a finite subset of atoms \((\psi, \theta)\) by:
1) ordering the atoms (sequential representation)
2) removing any atoms beyond the K-th (truncation)

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1) ordering the atoms \textbf{(sequential representation)}

2) removing any atoms beyond the K-th \textbf{(truncation)}

\[
\Theta = \sum_{k=1}^{K} \theta_k \delta \psi_k
\]
Ordering of (N)CRM atoms

We describe 2 forms for sequential representations
Ordering of (N)CRM atoms

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Series representation
function of a homogenous
Poisson point process

(4 versions)
Ordering of (N)CRM atoms

We describe 2 forms for sequential representations

**Series representation**
function of a homogenous Poisson point process
*(4 versions)*

**Superposition representation**
infinite sum of homogenous CRMs, each with finite # of atoms
*(3 versions)*
Ordering of (N)CRM atoms

We describe 2 forms for sequential representations

**Series representation**
function of a homogenous Poisson point process
(4 versions)

**Superposition representation**
infinite sum of homogenous CRMs, each with finite # of atoms
(3 versions)

Theorem (H., Campbell, How, Broderick).
Can generate (N)CRMs using all 7 sequential representations
Sequential representation comparison

Why so many representations?
Sequential representation comparison

Why so many representations?

They’re all useful in different circumstances
Sequential representation comparison

Why so many representations?

They’re all useful in different circumstances

| Error Bound Decay | Series Reps | Superposition Reps |
|-------------------|-------------|---------------------|
|                   | B-Rep | IL-Rep | R-Rep | T-Rep | DB-Rep | PL-Rep | SB-Rep |
| ✓ (exp)           | ✓     | ✓     | ✓/x   | x     | ✓      | ✓      | x      |

| Ease of Analysis | Series Reps | Superposition Reps |
|------------------|-------------|---------------------|
| x                | x          | ✓                   |
| xx               | xx         | ✓                   |
| x                | x          | ✓                   |
| x                | x          | ✓                   |

| Generality       | Series Reps | Superposition Reps |
|------------------|-------------|---------------------|
| ✓                | ✓           | ✓                   |
| ✓                | ✓           | ✓                   |
| ✓                | ✓           | ✓                   |
| ✓                | ✓           | ✓                   |

| Known # Atoms    | Series Reps | Superposition Reps |
|------------------|-------------|---------------------|
| ✓                | ✓           | x                   |
| ✓                | ✓           | x                   |
| x                | x           | x                   |
| x                | x           | x                   |
| x                | x           | x                   |
Sequential representation example

Given Gamma process: \( \nu(\theta) = \gamma \lambda \theta^{-1} e^{-\lambda \theta} \)
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Step 1: compute \( c := \lim_{\theta \to 0} \theta \nu(\theta) \)
Sequential representation example

**Given** Gamma process: \( \nu(\theta) = \gamma \lambda \theta^{-1} e^{-\lambda \theta} \)

**Step 1:** compute \( c := \lim_{\theta \to 0} \theta \nu(\theta) = \gamma \lambda \)
Sequential representation example

**Given** Gamma process: \( \nu(\theta) = \gamma \lambda \theta^{-1} e^{-\lambda \theta} \)

**Step 1:** compute \( c := \lim_{\theta \to 0} \theta \nu(\theta) = \gamma \lambda \)

**Step 2:** compute \( f(\theta) := -c^{-1} \frac{d}{d\theta} [\theta \nu(\theta)] \)
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Exponential(\(\lambda\)) density!
Sequential representation example

**Given** Gamma process: \( \nu(\theta) = \gamma \lambda \theta^{-1} e^{-\lambda \theta} \)

**Step 1:** compute \( c := \lim_{\theta \to 0} \theta \nu(\theta) = \gamma \lambda \)

**Step 2:** compute \( f(\theta) := -c^{-1} \frac{d}{d\theta} [\theta \nu(\theta)] = \lambda e^{-\lambda \theta} \)

**Step 3:** plug in!

\[ \Theta = \sum_{k=1}^{\infty} V_k e^{-\Gamma_k} \delta_{\psi_k}, \quad V_k \overset{iid}{\sim} f, \quad \Gamma \sim \text{Poisson} P(c) \]
Truncation Roadmap

Tractable models in BNP

two forms for sequential representations

\[ \sum_{k=1}^{\infty} \theta_k \delta \psi_k \]

Truncation and error analysis

\[ \sum_{k=1}^{K} \theta_k \delta \psi_k \]
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Truncation and error analysis
Choosing between the seven representations

How close is our finite approximation?
Choosing between the seven representations

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**Truncation error:** \[ \| p_{N,\infty} - p_{N,K} \|_1 = \frac{1}{2} \int | p_{N,\infty}(y) - p_{N,K}(y) | \, dy \]
Choosing between the seven representations

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full infinite \( \Theta \)  

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\[
\begin{align*}
\text{full infinite} & \quad \Theta \\
\downarrow & \\
generated\ data
\end{align*}
\quad
\begin{align*}
\text{truncated} & \quad \Theta_K \\
\downarrow & \\
generated\ data
\end{align*}
\]
Choosing between the seven representations

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Compare the distribution of the data under full vs. truncated
Choosing between the seven representations

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**Truncation error:** \[ \|p_{N,\infty} - p_{N,K}\|_1 = \frac{1}{2} \int |p_{N,\infty}(y) - p_{N,K}(y)| \, dy \]

Depends on **number of observations** \(N\) and **truncation level** \(K\).
Choosing between the seven representations

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As \( N \) gets larger, error increases
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Depends on **number of observations** \( N \) and **truncation level** \( K \)

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As \( K \) gets larger, error decreases

**Cannot evaluate exactly**, so we develop **new upper bounds**
Lemma (H., Campbell, How, Broderick).

$$\|p_{n,\infty} - p_{n,K}\|_1 \leq P(\text{any datum selects a removed trait})$$

The truncation error

i.e. P( whoops! )
Protobound

Leads to all the other truncation error bounds in this work

**Lemma (H., Campbell, How, Broderick).**

\[ \|p_{N,\infty} - p_{N,K}\|_1 \leq P \text{ (any datum selects a removed trait)} \]

The truncation error

**Theorem (HCHB).** The series rep error is bounded by

\[
\|p_{N,\infty} - p_{N,K}\|_1 \leq 1 - e^{-\int_0^\infty \mathbb{E}[\pi(V,u+G_K)^N] du}
\]

i.e. P( whoops! )
Lemma (H., Campbell, How, Broderick).
\[ \| p_{N,\infty} - p_{N,K} \|_1 \leq \mathbb{P} \text{ (any datum selects a removed trait)} \]

The truncation error

Theorem (HCHB). The series rep error is bounded by
\[
\| p_{N,\infty} - p_{N,K} \|_1 \leq 1 - e^{-\int_0^\infty \mathbb{E}[\bar{\pi}(\tau(V,u+G_K))^N] du}
\]

Theorem (HCHB). The superposition rep error is bounded by
\[
\| p_{N,\infty} - p_{N,K} \|_1 \leq 1 - e^{-\int_0^\infty \bar{\pi}(\theta)^N v_K^+(d\theta)}
\]
Error bound example

**Given** Gamma-Poisson process: \( \nu(\theta) = \gamma \lambda \theta^{-1} e^{-\lambda \theta} \) \( \pi(\theta) = e^{-\theta} \)
Error bound example

**Given** Gamma-Poisson process: $\nu(\theta) = \gamma \lambda \theta^{-1} e^{-\lambda \theta}$  
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**Step 1:** bound the integral, where $G_K \sim \text{Gamma}(K, c)$:

$$\int_0^{\infty} (1 - \mathbb{E} [\pi(\theta e^{-G_K})]) \nu(d\theta)$$
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\int_{0}^{\infty} (1 - \mathbb{E}[\pi(\theta e^{-G_K})]) \nu(d\theta) = \gamma \lambda \mathbb{E}[\log(1 + e^{-G_K}/\lambda)]
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Integration by parts
Error bound example

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\int_0^\infty (1 - \mathbb{E}[\pi(\theta e^{-G_K})]) \nu(d\theta) = \gamma \lambda \mathbb{E}[\log(1 + e^{-G_K} / \lambda)] \quad \text{Integration by parts}
\]

\[
\leq \gamma \mathbb{E}[e^{-G_K}] \quad \text{log}(1 + x) \leq x
\]
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&\leq \gamma \mathbb{E} [e^{-G_K}] \\
&= \gamma \left( \frac{\gamma \lambda}{1 + \gamma \lambda} \right)^K
\end{align*}
\]

Integration by parts
log(1 + x) \leq x
Gamma expectation
**Error bound example**

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\[
\int_0^\infty \left( 1 - \mathbb{E} \left[ \pi(\theta e^{-G_K}) \right] \right) \nu(d\theta) = \gamma \lambda \mathbb{E} \left[ \log(1 + e^{-G_K} / \lambda) \right] 
\leq \gamma \mathbb{E} \left[ e^{-G_K} \right] 
= \gamma \left( \frac{\gamma \lambda}{1 + \gamma \lambda} \right)^K
\]

- Integration by parts
- \( \log(1 + x) \leq x \)
- Gamma expectation

**Step 2:** plug in!

\[
\frac{1}{2} \| p_{N,\infty} - p_{N,K} \|_1 \leq 1 - \exp \left\{ -N \gamma \left( \frac{\gamma \lambda}{1 + \gamma \lambda} \right)^K \right\}
\]
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Truncation and error analysis
## Previous Work

|                | Finite Approximation | Approximation Error Bounds | Computational Complexity |
|----------------|-----------------------|---------------------------|--------------------------|
| **DP**         | ✓                     | ✓                         | ✓                        |
| **BP**         | ✓                     | ✓                         | ✓                        |
| **BPP**        | ✓                     |                           |                          |
| **GP**         | ✓                     | ✓                         | ✓                        |
| **(N)CRM**     | ✓                     |                           |                          |
## Our Work

|       | Finite Approximation | Approximation Error Bounds | Computational Complexity |
|-------|----------------------|---------------------------|-------------------------|
| DP    | ✓                    |                           | ✓                       |
| BP    | ✓                    | ✓                         | ✓                       |
| BPP   | ✓                    |                           |                         |
| ΓP    | ✓                    | ✓                         | ✓                       |
| (N)CRM| ✓                    | ✓                         | ✓                       |
|            | Finite Approximation | Approximation Error Bounds | Computational Complexity |
|------------|----------------------|----------------------------|--------------------------|
| DP         | ✓                    | ✓                          | ✓                        |
| BP         | ✓                    | ✓                          | ✓                        |
| BPP        | ✓                    | ✓                          | ✓                        |
| ΓP         | ✓                    | ✓                          | ✓                        |
| (N)CRM     | ✓                    | ✓                          | ✓                        |
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• Trade off computational efficiency and statistical accuracy of truncated model
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J. Huggins*, T. Campbell*, J. How, T. Broderick.  
**Truncated Random Measures.** Submitted, 2016.  
Available online: [https://arxiv.org/abs/1603.00861](https://arxiv.org/abs/1603.00861)