Localization properties of random-mass Dirac fermions from real-space renormalization group

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Localization properties of random-mass Dirac fermions for a realization of mass disorder, commonly referred to as Cho-Fisher model, is studied on the D-class chiral network. We show that a simple RG description captures accurately three phases: thermal metal and two insulators with quantized Hall conductances, as well as transitions between them (including critical exponents).

We find that, with no randomness in phases on the links, transmission via the RG block exhibits a sizable portion of perfect resonances. Delocalization occurs by proliferation of these resonances to larger scales. Evolution of the thermal conductance distribution towards metallic fixed point is synchronized with evolution of signs of transmission coefficients, so that delocalization is accompanied with sign percolation.

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**Introduction.** As it was pointed out in Ref. [1], there exists a profound connection between 2D electron motion in random potential, \(V(x, y)\), in a quantizing magnetic field and the motion of 2D Dirac fermions with random mass, \(M(x, y)\), in a zero field. This connection can be traced to the analogy of the drift of electron along closed equipotentials, \(V(x, y) = E\), and chiral motion of Dirac fermion along contours, \(M(x, y) = 0\). Due to this similarity, conventional quantum Hall transition, which takes place as equipotentials merge, has its counterpart for Dirac fermions. However, critical behaviors for these two transitions are different due to the different underlying symmetries. In simple terms, the difference comes from the fact that electron drifting along equipotential acquires a random phase, while Dirac fermion moving along a line \(M = 0\) does not. According to general classification [2], delocalization transition of Dirac fermions with mass randomness belongs to the D-class [3-17], time-reversal and spin-rotational symmetries are broken. It was realized that delocalization transition in the D-class is related to criticality in the random-bond Ising model [5, 7], thermal quantum Hall effect in superconductors [8, 9, 14-17] with either spin-orbit scattering or with special type of pairing, and lately, to in-gap transport in graphene with broken sublattice symmetry [10-18].

Not only the critical exponent of localization radius for the D-class is different from the quantum Hall critical exponent, but the very picture of delocalization in the D-class is much less intuitive than in the quantum Hall transition. This is because transformation of the wave function corresponding to the drift along \(V(x, y) = E\) upon merging of two equipotentials is different from transformation of chiral Dirac wave functions upon merging of two contours \(M(x, y) = 0\). In both cases, the reflection and transmission coefficients, \(r\) and \(t\), at the point of merging can be chosen real with \(r^2 + t^2 = 1\), ensuring the current conservation. In the quantum Hall case, the signs of \(r\) and \(t\) can be chosen arbitrary, since change of the sign can be absorbed into the random drift-phase.

By contrast, there is no such freedom in the D-class. Indeed, the phase accumulated by Dirac edge state along a contour \(M(x, y) = 0\) is always equal to \(\pi\), which is the consequence of the pseudospin structure of the eigenfunction [1]. If scattering matrices

\[
S_i = \left( \begin{array}{cc} t_i & r_i \\ r_i & -t_i \end{array} \right)
\]  

are the same for all points of contact \(i\) (without any randomness in signs), it is obvious that delocalization will take place at \(r = t = 1/\sqrt{2}\), which corresponds to \(\langle M(x, y) \rangle = 0\). Then the critical exponent is \(\nu = 1\), which is a consequence of in-gap tunneling of the Dirac fermion.

An amazing property of the D-class is that disorder in signs of \(r_i\) and \(t_i\) has a drastic effect on delocalization...
FIG. 2: (Color online) Left: Chiral network for the CF model. 
Sign arrangement of $t$ and $r$ at the nodes ensures $\pi$ flux per 
plaquette [14]. Red lines single out a group of five nodes
constituting an RG block. Right: Illustration of the RG transformation Eqs. [4, 9] for elements of renormalized node
scattering matrix on the red sublattice. Truncation procedure (dashed lines on the original lattice) enforces, via sign 
structure in the nodes, the factors 1 and $-1$ in the corners. These factors are taken into account in Eqs. [4, 9].

transformation [14]. Moreover, for a given degree of the net
sign disorder, a particular way in which it is introduced
[3, 6, 22, 23] can turn the system either into metal (M)
or into insulator (I). In particular, if disorder is intro-
duced by randomly changing signs of columns in Eq. [11]
with probability, $W$, then arbitrarily small $W$ leads to de-
localization at any average mass, $\langle M(x,y) \rangle \propto (t^2) - 1/2$. 
This fact was established numerically in Ref. [23], where 
the corresponding randomness was dubbed $O(1)$. For 
two other types of sign disorder: random-bond Ising
model [3, 7] and Cho-Fisher (CF) model [22], the metallic
phase is either absent or emerges when $W$ exceeds 
certain $W_c$, correspondingly. The above two facts were 
also established numerically by studying transmission of
finite-width (up to $M = 256$) stripes [12, 23, 25].

In conventional quantum Hall effect, emergence of an 
isolated delocalized state at $\langle t^2 \rangle = 1/2$ is qualitatively
transparent, since the Chalker-Coddington model [20] is
a quantum version of a classical percolation, which pos-
sesses self-duality. By contrast, there is no classical
version of the D-class. Thus there is no qualitative expla-
nation how a particular local sign disorder in $t_i$ and $r_i$
results either in metal or in insulator at large distances.
Moreover, the fact that critical disorder in CF model is
weakly sensitive to $\langle M \rangle$ [23, 24] seems counterintuitive.
Indeed, it implies that delocalization takes place no mat-
ner how weakly the $M = 0$ contours are coupled to each
other.

In this paper we demonstrate that above puzzles of the
D-class delocalization find a natural explanation within
a simple real-space renormalization group (RG) descrip-
tion. This description is developed below by adjusting
to the D-class the approach of Ref. [27] developed for
conventional quantum Hall effect. We focus on the CF
model most general in the sense that its phase diagram
in the $(\langle t^2 \rangle, W)$-plane contains all three (metal and two quantum Hall insulator) phases.

The RG procedure [27] prescribes how the magnitude
and sign of effective transmission coefficient, $\hat{t}$, evolve
with the system size, $L$. Within RG we demonstrate that
I-M transition (finite $\hat{t}$ at large $L$) occurs at the same $W_c$
where the distribution of the amplitude, $\hat{t}$, becomes sym-
metric with respect to $\hat{t} = 0$. For smaller $W$ (insulating
phase), the initial signs of $t_i$ are "forgotten" with increas-
ing $L$. In this sense, the I-M transition can be viewed as
sign percolation. Remarkably, RG description appears to
be very accurate on the quantitative level. In fact, it re-
produces the entire phase diagram found in [23, 25]. In
addition, we find the following universal distribution of
the conductance, $G = \hat{P}$, of the thermal metal

$$P(G) = 0.414[G(1-G)]^{-0.6}.$$  (2)

Within RG, insensitivity to $\langle M \rangle$ emerges as a result of
high (resonant) transmission via certain local disorder
configurations; importantly, these resonances proliferate
to large distances.

**RG transformation.** In the "clean" case, the network
model of D-class is shown in Fig. [2]. The signs of trans-
mission and reflection at each node ensure the $\pi$-flux per
plaquette. With respect to signs, scattering matrices of
the nodes are different for two sublattices of the square
lattice, so that $S$-matrix has the form Eq. [11] for red
nodes and

$$S' = \begin{pmatrix} t_i & -r_i \\ r_i & t_i \end{pmatrix}$$  (3)

for green nodes. The RG superblock shown in Fig. [2]
consists of four red and one green nodes. With signs in the
$S$-matrices chosen accordance with Fig. [2] scattering
matrix of this superblock reproduces the form Eq. [11]
with effective transmission coefficient given by

$$\hat{t} = \frac{t_1t_5(r_2r_3r_4 + 1) + t_2t_4(r_1r_3r_5 + 1) + t_3(t_1t_4 + t_2t_5)}{(r_3 + r_1r_5)(r_3 + r_2r_4) + (t_3 + t_1t_2)(t_3 + t_4t_5)}.$$  (4)

At the next RG step, values of $t_i$ generated by Eq. [4],
including the signs, should be placed into red nodes of
renormalized lattice, which has a doubled lattice constant.

The limit $W = 0$. As a first test of applicability of the
RG description we apply Eq. [4] to an ordered system.
Upon setting all $t_i$ equal to $t$ and all $r_i$ equal to $\sqrt{1-t^2}$,
we see that $t = t = 1/\sqrt{2} \equiv 1$ is indeed a fixed point, cor-
responding to zero mass. Finite $|t^2 - 1/2|$ plays the role of
a finite mass, $M$, in a clean system. From Dirac equation
it follows that transmission disappears at lengths exceed-
ing tunneling length $1/M$. This means that the $W = 0$

critical exponent is $\nu = 1$. On the other hand, Eq. [4]
transforms finite $|t^2 - 1/2|$ into $|t^2 - 1/2| = \tau|t^2 - 1/2|$,
where

$$\tau = \sum_{i=1}^{5} c_i = \sum_{i=1}^{5} \frac{\partial^2}{\partial t^2} \bigg|_{t^2=1/2}. \quad (5)$$

Elementary calculation yields \(c_1 = c_2 = c_4 = c_5 = \sqrt{3} - 1\), \(c_3 = 3 - 2\sqrt{2}\), so that \(\tau = 2\sqrt{2} - 1\). Since \(\tau > 1\), transmission disappears upon \(n\) subsequent steps, where \(n\) is determined by the condition, \(|t^2 - 1/2|^{\nu n} \sim 1\). For such \(n\) the unit cell is \(\xi = 2^n\) and should be identified with the in-gap tunneling length. From the above two conditions we find that \(\xi \sim |t^2 - 1/2|^{-\nu} = |M|^{-\nu}\), where \(\nu = \ln 2 / \ln 2 \approx 1.15\), i.e., the RG value of the exponent is only 15% different from the exact value.

Before introducing disorder in signs of \(t_i\) and \(r_i\), we elucidate another property of \(t^2 = 1/2\) fixed point: initial symmetric distribution of \(t^2\) around \(1/2\) not only remains symmetric but narrows upon renormalization. Indeed, the width of distribution after one RG step evolves as \(\langle \delta t^2 \rangle = \sum_{i=1}^{5} c_i^2 (\delta t^2) \approx 0.7 (\delta t^2)\). This means that the critical states at the above fixed point do not exhibit mesoscopic fluctuations, in contrast to conventional quantum Hall critical state. We will see below that thermal metal does exhibit strong mesoscopic fluctuations.

\textbf{Resonances.} Suppose that in Eq. (4) all \(t_i\) are small. Corresponding \(r_i\) are close to 1. It might seem that \(t\) is even smaller, \(\propto t^2\). At this point we divulg the following remarkable property of the transformation Eq. (4). Let us choose the following sign combination:

$$r_1 = r_2 = r_3 = r_4 = r, \quad r_5 = -r. \quad (6)$$

Substituting the above values into Eq. (4), we find that \(t\) is identically equal to \(\pm 1\), i.e., resonant tunneling takes place. It might also seem that such resonant configurations are irrelevant due to vanishing statistical weight. On the contrary, we will see that these resonances play a key role in delocalization, since they persist even when \(r_i\) are not equal. Evidence to this fact can be found in Fig. 3b. Bare values of \(t_i^2\) are homogeneously distributed in the interval, \(0.1 < t_i^2 < 0.3\). For a modest randomness in signs, Eq. (4) transforms the box-like distribution into a three-peak distribution: the left peak corresponds to resonant reflection, the middle peak corresponds to expected overall reduction of transmission, while the right peak comes from resonant transmission. We see that, with spread in the bare \(t_i^2\) values, the portion of resonant transmission is considerable, about 8%. Figs. 3b, 3d, and 3h illustrate that the competition between the resonant transmission peak and low-\(t_i^2\) peaks decides whether the system evolves into a metal or an insulator. In the metallic phase, Figs. 3a, 3e, and 3g, the resonant transmission peak continuously broadens and grows until the distribution becomes symmetric around \(t^2 = 1/2\). In the insulating phase, Figs. 3c, 3f, and 3h, the resonant tunneling peak gets gradually suppressed with the system size, and large-scale distribution flows to \(t^2 \to 0\). Details of determination of the I-M boundary are described below.

\textbf{CF sign disorder.} According to prescription of Ref. [14] for the isotropic version of the CF model, for a given positive \(t\) the values of \(t_i\) are chosen to be

$$t_i = \begin{cases} 
 t, & \text{with probability } \left(1 - \frac{W}{2}\right), \\
 -t, & \text{with probability } \frac{W}{2}.
\end{cases} \quad (7)$$

Correspondingly,

$$r_i = \begin{cases} 
 \sqrt{1-t^2}, & \text{with probability } \left(1 - \frac{W}{2}\right), \\
 -\sqrt{1-t^2}, & \text{with probability } \frac{W}{2}.
\end{cases} \quad (8)$$

The RG transformation for \(r_i\) has a form

$$\tilde{r} = \frac{r_1 r_2 (t_3 t_4 t_5 + 1) + r_4 r_5 (t_1 t_2 t_3 + 1) + r_3 (r_1 r_4 + r_2 r_5)}{(r_3 + r_1 r_5)(r_3 + r_2 r_4) + (t_3 + t_1 t_2)(t_3 + t_4 t_5)}, \quad (9)$$

which is in accord with Eq. (4). With prescription Eqs. (7), (8), and Eqs. (3), (4), Mathematica performs each subsequent RG step almost instantaneously even for a...
dense sample of $10^6 t_i$ and $r_i$. We started with checking that for initial box-like distribution of $t_i^2$ in the interval, $t^2_{\text{min}} < t^2 < t^2_{\text{max}}$, the RG evolution depends only on the average, $\langle t^2 \rangle = (t^2_{\text{min}} + t^2_{\text{max}})/2$, but not on the width, $(t^2_{\text{max}} - t^2_{\text{min}})$. For particular box, $t^2_{\text{min}} = 0.1, t^2_{\text{max}} = 0.3$, results shown in Fig. 3 illustrate that distinction between two disorder values, $W = 0.15$ and $W = 0.2$, emerges at system size, $L = 2^5$. At the same time, the difference in signs distribution of $r_i$ develops already at the second step. This is illustrated in Figs. 3c-d insets. This behavior is generic: memory about initial sign disorder in $r_i$ is quickly forgotten in the insulating phase (all $r_i$ have are positive after 3-4 steps), whereas in the metallic phase both signs of $r_i$ are completely equilibrated after 3-4 steps. Nevertheless, complete convergence to the symmetric fixed point distribution $P(t^2)$ given by Eq. (2) takes place at very large size, $L = 2^{25}$.

RG evolutions depicted in Fig. 3 suggest that a M-I transition point lies within $0.15 < W_c < 0.2$. By gradually shrinking this interval, one can find $W_c$ with high accuracy. The results for different $\langle t^2 \rangle$, which are in general agreement with numerics of Refs. 23-25, are shown in Fig. 1. We see that the I-M boundary is approximately horizontal, except for the interval, $0.4 < \langle t^2 \rangle < 0.6$. Within this interval the boundary rapidly drops to the value, $W_{tr} = 0.06$, which we identify with tricritical point. Insets in Fig. 1 illustrate our statement concerning sign percolation. They show how the portion, $\kappa$, of negative $r_i$-values Eq. (3) evolves with the sample size. Equilibration of signs in metal, $W > W_c$, implies that $\kappa$ grows towards $\kappa = 1/2$. We see that the bigger is $(W - W_c)$ the faster is the growth. By contrast, a decrease of $\kappa$ with $L$ in insulator implies that all $r_i$ become positive after several steps. It is seen that erasing of signs is more efficient at smaller $W$. As signs are erased, the magnitude, $\langle t^2 \rangle$ approaches 1. We used the rate of this approach to estimate the critical exponent of the I-M transition. Assuming that $(1 - \langle t^2 \rangle)$ is a function of a single parameter, $(W_c - W)^\nu L$, we found $\nu \approx 1.2$ for initial distribution with $\langle t^2 \rangle = 0.2$ (red triangle mark in Fig. 1).

**Tricritical point.** In the RG language, tricritical point on the vertical axis, $t^2 = 1/2$, suggests that a symmetric initial distribution of $t^2_i$ evolves to $P(t^2) = \delta(t^2 - 1/2)$ for $W < W_{tr}$, and to metallic fixed point, Eq. (2) for $W > W_{tr}$. Fig. 1 shows the evolution of symmetric box-like distribution $0.25 < t^2 < 0.75$ for two $W$-values. We see that for $W = 0.05$ and $W = 0.07$ these evolutions are different. For $W = 0.07$ the distribution at $L = 2^5$ is flat. After a few steps the histogram bends down at the center and distribution flows towards metallic fixed point Eq. (2). For $W = 0.05$, the box-like initial distribution develops a maximum at $L = 2^5$, Fig. 4. This narrowing suggests a flow towards the ordered fixed point, $t = r = 1/\sqrt{2}$. By gradually shrinking the $W$-interval we locate the value $W_{tr} \approx 0.06$, which separates the behavior Fig. 3 (maximum at the center) and Fig. 4 (minimum at the center). We identify $W_{tr}$ with tricritical point 13, 14, 23, 25. The smaller is $W$ the more pronounced is the shrinking of initial distribution with $L$. Note, however, that this shrinking eventually stops: at large $L$ maximum at the center is accompanied by satellite peaks at $t^2 = 0$ and $t^2 = 1$, which gradually take over and drive the system to metal. Similar complications were pointed out in Ref. 14. The behavior of signs of $t_i$ ($r_i$) is synchronized with distribution of $t^2_i$ ($r^2_i$). Namely, as the distribution shrinks, the portion, $\kappa$, changes from $W$ down monotonically. As the distribution turns back to metal, $\kappa$ starts to grow towards $\kappa = 1/2$.

**Discussion** The value, $W_c \approx 0.2$, and the fact that it depends weakly on $t^2_i$, can be inferred from the calculation of likelihood, $P_W$, of resonant configurations of the type Eq. (4). These configurations occur when denominator in Eq. (1) is small. When all $t_i$ are small and $|r_i| < 3W$, this condition requires that the product, $(r_3 + r_1r_5)(r_3 + r_2r_4)$, is small. On the other hand, if both brackets are small, we will have $t \sim t_i$, i.e., the resonance is absent (due to suppression of the numerator). The probability that only one of the brackets is small is given by

$$P_W = 4W(1 - W)^3 + 4W^3(1 - W) = \frac{1 - (1 - 2W)^4}{2}.$$  

(10)

It turns out that $P_W$ is almost flat and close to $1/2$ in the wide interval, $0.2 < W < 0.8$. This is the consequence of the fact that first three derivatives of $P_W$ are all zero at $W = 1/2$. Therefore if the resonances do not proliferate at $W \approx 0.2$, they will not proliferate upon further increase of $W$. This explains why, at small $t_i$, the critical

**FIG. 4:** (Color online) RG evolution with sample size, $L$, of the symmetric distribution of $0.25 < t^2 < 0.75$ (purple box) is shown for two magnitudes of disorder, $W = 0.05$ (left) and $W = 0.07$ (right).
Another feature of the phase diagram Fig. 1 is weak dependence of \( W_c \) on \( \langle t^2 \rangle \), can be traced to the fact that for resonant configuration Eq. (6) we have \( \hat{t}^2 = 1 \) regardless of the value of \( r \).

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[1] A. W. W. Ludwig, M. P. A. Fisher, R. Shankar, and G. Grinstein, Phys. Rev. B 50, 7526 (1994).
[2] A. Altland and M. R. Zirnbauer, Phys. Rev. B 55, 1142 (1997).
[3] M. Bocquet, D. Serban, and M. R. Zirnbauer, Nucl. Phys. B 578, 628 (2000).
[4] C. -M. Ho and J. T. Chalker, Phys. Rev. B 54, 8708 (1996).
[5] N. Read and A. W. W. Ludwig, Phys. Rev. B 63, 024404 (2000).
[6] I. A. Gruzberg, N. Read, and A. W. W. Ludwig, Phys. Rev. B 63, 104422 (2001).
[7] F. Merz and J. T. Chalker, Phys. Rev. B 65, 054425 (2002).
[8] T. Senthil and M. P. A. Fisher, Phys. Rev. B 61, 9690 (2000).
[9] N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
[10] J. Snyman, J. Tworzydlo, and C. W. J. Beenakker, Phys. Rev. B 78, 045118 (2008).
[11] J. Tworzydlo, C. W. Groth, and C. W. J. Beenakker, Phys. Rev. B 78, 235438 (2008).
[12] J. H. Bardarson, M. V. Medvedyeva, J. Tworzydlo, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. B 81, 121414 (2010).
[13] M. V. Medvedyeva, J. Tworzydlo, and C. W. J. Beenakker, Phys. Rev. B 81, 214203 (2010).
[14] A. Mildenerger, F. Evers, A. D. Mirlin, and J. T. Chalker, Phys. Rev. B 75, 245321 (2007).
[15] F. Evers and A. D. Mirlin, Rev. Mod. Phys. 80, 1355 (2008).
[16] P. W. Brouwer, A. Furusaki, I. A. Gurzberg, and C. Mudry, Phys. Rev. Lett. 85, 1064 (2000).
[17] I. A. Gruzberg, N. Read, and S. Vishveshwara, Phys. Rev. B 71, 245124 (2005).
[18] K. Ziegler, Phys. Rev. Lett. 102, 126802 (2009); Phys. Rev. B 79, 195424 (2009).
[19] In the language of Ref. [15] (see also Refs. [20, 21]) the randomness in signs is related to the small-scale behavior (vorticity) near the saddle point in \( M(x, y) \).
[20] V. Gurarie and L. Radzihovsky, Phys. Rev. B 75, 212509 (2007).
[21] M. Wimmer, A. R. Akhmerov, M. V. Medvedyeva, J. Tworzydlo, and C. W. J. Beenakker, Phys. Rev. Lett. 105, 046803 (2010).
[22] S. Cho and M. P. A. Fisher, Phys. Rev. B 55, 1025 (1997).
[23] J. T. Chalker, N. Read, V. Kagalovsky, B. Horovitz, Y. Avishai, and A. W. W. Ludwig, Phys. Rev. B 65, 012506 (2001).
[24] V. Kagalovsky and D. Nemirovsky, Phys. Rev. Lett. 101, 127001 (2008).
[25] V. Kagalovsky and D. Nemirovsky, Phys. Rev. B 81, 033406 (2010).
[26] J. T. Chalker and P. D. Coddington, J. Phys. C 21, 2665 (1988).
[27] A. G. Galshtyan and M. E. Raikh, Phys. Rev. B 56, 1422 (1997); P. Cain, R. A. Römer, M. Schreiber, and M. E. Raikh, Phys. Rev. B 64, 235326 (2001); P. Cain, R. A. Römer, and M. E. Raikh, Phys. Rev. B 67, 075307 (2003).
[28] Scattering matrix of superblock consisting of four green and one red nodes reproduces the form Eq. (3).
[29] P. J. Reynolds, W. Klein, and H. E. Stanley, J. Phys. C 10, L167 (1977).