AN OPEN SINGULARITY-FREE COSMOLOGICAL MODEL WITH INFLATION

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In the light of recent observations which point to an open universe \( \Omega_0 < 1 \), we construct an open singularity-free cosmological model by reconsidering a model originally constructed for a closed universe. Our model starts from a nonsingular state called prematter, governed by an inflationary equation of state \( P = (\gamma_p - 1)\rho \) where \( \gamma_p \approx 10^{-3} \) is a small positive parameter representing the initial vacuum dominance of the universe. Unlike the closed models universe cannot be initially static hence, starts with an initial expansion rate represented by the initial value of the Hubble constant \( H(0) \). Therefore, our model is a two-parameter universe model \((\gamma_p, H(0))\). Comparing the predictions of this model for the present properties of the universe with the recent observational results, we argue that the model constructed in this work could be used as a realistic universe model.

We started the evolution of a flat universe from a nonsingular state called prematter which is governed by an inflationary equation of state \( P = (\gamma_p - 1)\rho \) where \( \gamma_p \) represents the initial vacuum dominance of the universe. The evolution of the universe except in the prematter era is affected neither by the initial vacuum dominance nor the initial expansion rate of the universe. On the other hand, present properties of the universe such as Hubble constant, age and density are sensitive to the temperature at the decoupling.

Over a range between value between 50 and 80 \( Km \cdot sec^{-1} \cdot Mpc^{-1} \) for the present value of the Hubble constant \((H_0)\). Assuming that the thermal history of the universe is independent from its geometry, the above range could be considered as transition range for the decoupling temperature.

Keywords: inflation; flatness; singularity-free.

1. Introduction

Although the standard cosmological model has been successful in explaining the homogenous expansion of the universe and the 2.7 K cosmic microwave background
radiation, it has some shortcomings like the initial singularity, horizon (or causality),
flatness, homogeneity and isotropy problems. Among these problems the initial
singularity problem may be the one which weakens the model much more than the
others, in the sense that it causes infinities in the physical quantities such as the
density, pressure and temperature. Since these infinities cannot be accepted by any
model which claims to be physical, during the past two decades, several authors
have considered the possibility of describing the universe with a singularity-free
cosmological model.

Recent observations point out that the universe has an $\Omega_0(\equiv \rho/\rho_c , \rho_c \equiv
3H^2/(8\pi))$ value lower than unity which means that the universe is spatially open.
In the light of these, we propose an open singularity-free cosmological
model with the same motivations as in Ref. 5. However, initially static condition
($\dot{a}(0) = 0$) used by the closed universe models can no longer be used in the open
universe case since it leads to negative energy densities. For this reason in our model
universe starts expanding with an initial velocity. Hence, we have a two-parameter
universe model in which one of the parameters characterizes the equation of state
used to describe the prematter era, and the other one corresponds to the initial
expansion rate of the universe.

2. Description of the Model

2.1. Field Equations

Our model describes an open, spatially homogenous and isotropic universe with a
space-time geometry given by the Robertson-Walker (RW) line element:

$$ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 + r^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right], \quad (1)$$

where $(t, r, \theta, \phi)$ are the comoving coordinates, $a(t)$ is the scale factor which repres-
ents the size of the universe.

For the RW metric, Einstein field equations lead us to the following differential
equations:

$$\left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{a^2} = \frac{8\pi}{3} \rho, \quad (2)$$

$$2\frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 - \frac{1}{a^2} = -8\pi P, \quad (3)$$

where $\rho$ and $P$ are the energy density and pressure in the universe respectively,
where we have used perfect fluid energy momentum tensor in comoving coordinates.
Here a dot denotes differentiation with respect to the cosmic time $t$ and we use the
units so that $M_{pl} = G^{-1/2} = 1$, where $M_{pl}$ is the Planck mass. Combining Eqs. (2)
and (3) and using an equation of state given as

$$P = (\gamma - 1) \rho, \quad (4)$$
one obtains the following relation:

$$\frac{\ddot{a}}{a} + 4\pi \left( \gamma - \frac{2}{3} \right) \rho = 0. \quad (5)$$

Where, $\gamma$ is assumed to be a constant parameter during each era in the history of the universe.

We can eliminate $\rho$ between Eqs. (2) and (5) and obtain an equation involving only the scale factor $a(t)$:

$$\left( \frac{\ddot{a}}{a} \right) + \left( \frac{3}{2} \gamma - 1 \right) \left( \frac{\dot{a}^2 - 1}{a^2} \right) = 0. \quad (6)$$

To solve this equation for any $\gamma$, it is advantageous to work in conformal time $\eta$ which is defined as

$$dt = a(\eta) d\eta. \quad (7)$$

Using this transformation, Eq. (6) is solved to give two linearly independent solutions given as

$$a(\eta) = a_0 \left[ \sinh (\eta c + \delta) \right]^{1/c}, \quad (8)$$

$$a(\eta) = a_0 \left[ \cosh (\eta c + \delta) \right]^{1/c}, \quad (9)$$

where

$$c = \frac{3}{2} \gamma - 1. \quad (10)$$

and $a_0$ and $\delta$ are the integration constants.

These solutions correspond to different signs of the energy density $\rho$. Assuming that the universe is filled with positive energy, we eliminate Eq. (9) and consider only Eq. (8) as the solution for the scale factor of the universe. In this work, we model the universe as a series of perfect fluid eras connected by first order phase transitions. According to the considerations of this scenario, the universe starts expanding with a period of rapid expansion called inflationary era. This period is characterized by an equation of state in the form $P \simeq -\rho$ and inflation arises due to this vacuum-like characteristic of the equation of state. Since matter would be under extreme conditions during this period and behaves very differently from the ordinary matter, it is called “prematter”. During this period due to the unusual characteristic of the equation of state, temperature increases although the universe expands enormously. We assume that inflation continues until the temperature reaches the maximum allowed temperature i.e. the Planck temperature $T_{pl} = 1.4169 \cdot 10^{32} \text{ K}$. This behavior, which does not necessitate a “re-heating mechanism” as in the other models of the universe, follows from the vacuum like characteristic of the equation of state used to describe the universe in this era.
2.2. Boundary Conditions and the Solutions For the Scale Factor

Initially, we assume that the universe is in a vacuum-like state and has a limiting density called the Planck density ($\rho_{pl}$), which is first formulated by Markov as a universal law of nature. Since our model describes a universe starting from a finite size and density, the initial expansion rate must be taken as positive. Hence we take the initial expansion rate as

$$a'(0) = v,$$  \hspace{1cm} (11)

where $v$ is some positive constant.

Solutions for the scale factor in different eras are:

$$a(\eta) = \begin{cases} a_0^{(p)}[\sinh(c_p\eta + \delta_p)]^{\frac{1}{1/c_p}} & 0 \leq \eta \leq \eta_r, \\ a_0^{(r)}[\sinh(\eta + \delta_r)] & \eta_r \leq \eta \leq \eta_m, \\ a_0^{(m)}[\sinh(\eta/2 + \delta_m)]^{2} & \eta_m \leq \eta. \end{cases}$$  \hspace{1cm} (12)

where $p$, $r$, and $m$ denote the prematter, radiation and matter eras respectively. We next impose the boundary condition that the scale factor and its derivative are continuous at points $\eta_r$ and $\eta_m$, where the phase transitions take place. This determines the integration constants as,

$$a_0^{(p)} = \left[ \sqrt{v^2 - a_0^2(0)} \right]^{\frac{1}{2}/c_p} d^{-1/2},$$  \hspace{1cm} (13)

$$\delta_p = \ln \left( \frac{v + a_0(0)}{v - a_0(0)} \right),$$  \hspace{1cm} (14)

$$a_0^{(r)} = a_0^{(p)}[\sinh(c_p\eta_r + \delta_p)]^{\frac{1}{1/c_p}} - 1,$$  \hspace{1cm} (15)

$$\delta_r = (c_p - 1) \eta_r + \delta_p,$$  \hspace{1cm} (16)

$$a_0^{(m)} = \frac{a_0^{(r)}}{\sinh(\eta_m + \delta_r)},$$  \hspace{1cm} (17)

$$\delta_m = \eta_m/2 + \delta_r.$$  \hspace{1cm} (18)

As seen from the solutions for the scale factor, the model that we propose is a two-parameter model. The parameters are the $c_p$ value and the initial value of the Hubble constant ($H(0)$). The former determines the amount of inflation that the universe has experienced during the prematter era and the latter is related to the initial expansion rate of the universe.

We can find the comoving times corresponding to the conformal times $\eta_r$, $\eta_m$, and $\eta_{now}$ by using the definition given in Eq. (7). Assuming that $t = 0$ at $\eta = 0$ we get from Eq. (8)

$$t(\eta) = a_0 \int_0^\eta [\sinh(\eta'c + \delta)]^{1/c} d\eta'.$$  \hspace{1cm} (19)
This integral depends on the values of $c$ and has to be computed numerically in the prematter era. Whereas, for the radiation ($c_r = 1$) and matter ($c_m = 1/2$) eras, Eq. (18) could be integrated to yield analytical expressions. The expressions for the comoving times corresponding to $\eta_r$, $\eta_m$ and $\eta_{now}$ are

$$t_r = a_0^{(p)} \int_0^{\eta_r} [\sinh (c_p \eta + \delta_p)]^{1/c_p} d\eta,$$

$$t_m = t_r + a_0^{(r)} [\cosh (\eta_m + \delta_r) - \cosh (\eta_r + \delta_r)],$$

$$t_{now} = t_m + \frac{a_0^{(m)}}{2} [\sinh (\eta_{now} + 2\delta_m) - \sinh (\eta_m + 2\delta_m) + (\eta_m - \eta_{now})].$$

Hubble constants at $\eta_r$, $\eta_m$ and $\eta_{now}$ could now be obtained by using

$$H(\eta) = \frac{a'(\eta)}{a^2(\eta)},$$

as

$$H(\eta_r) = \frac{a'(\eta_r)}{a^2(\eta_r)} = \frac{(9.2503 \cdot 10^{29}) \cosh (c_p \eta_r + \delta_p)}{a_0^{(p)} [\sinh (c_p \eta_r + \delta_p)]^{2 + 1}} \text{Km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1},$$

$$H(\eta_m) = \frac{a'(\eta_m)}{a^2(\eta_m)} = \frac{(9.2503 \cdot 10^{29}) \cosh (\eta_m + \delta_r)}{a_0^{(r)} \sinh^2 (\eta_m + \delta_r)} \text{Km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1},$$

$$H(\eta_{now}) = \frac{a'(\eta_{now})}{a^2(\eta_{now})} = \frac{(9.2503 \cdot 10^{29}) \cosh (\eta_{now}/2 + \delta_m)}{a_0^{(m)} \sinh^3 (\eta_{now}/2 + \delta_m)} \text{Km} \cdot \text{s}^{-1} \cdot \text{Mpc}^{-1}.$$ (26)

The evolution of the energy density is described by

$$\rho' + 3\gamma \frac{a'}{a} \rho = 0.$$ (27)

Eq. (26) could then be solved to yield

$$\frac{\rho(\eta_f)}{\rho(\eta_i)} = \left( \frac{a(\eta_f)}{a(\eta_i)} \right)^{3\gamma},$$ (28)

where $\eta_i$ and $\eta_f$ are the initial and final instants of any conformal time interval in a given era. Choosing $\eta_i, \eta_f = (0, \eta_r), (\eta_r, \eta_m), (\eta_m, \eta_{now})$ we obtain respectively

$$\rho(\eta_r) = \left( \frac{a(0)}{a(\eta_r)} \right)^{3\gamma_p} \rho_{pl},$$ (29)

$$\rho(\eta_m) = \left( \frac{a(\eta_r)}{a(\eta_m)} \right)^4 \rho(\eta_r),$$ (30)

$$\rho(\eta_{now}) = \left( \frac{a(\eta_m)}{a(\eta_{now})} \right)^3 \rho(\eta_m).$$ (31)
3. Numerical Results

In order to see this dependence and the development of the universe in this model, we changed one of the parameters while fixing the other and provided some numerical results. This allowed us to identify the character of the dependency of the model to each parameter. First, we fix \( H(0) \) to \( 2.3427 \cdot 10^6 \) \( Km \cdot s^{-1} \cdot Mpc^{-1} \), and changed \( \gamma_p \) in such a way that it took values between \( 2.0000 \cdot 10^{-3} \) and \( 2.0500 \cdot 10^{-3} \). It is to be noted that in this broad range, the value of Hubble constant has a wide spectrum between \( 49.2304 \) \( Km \cdot s^{-1} \cdot Mpc^{-1} \) and \( 96.5196 \) \( Km \cdot s^{-1} \cdot Mpc^{-1} \). Next, we set \( \gamma_p \) to \( 2.0350 \cdot 10^{-3} \) and varied \( H(0) \) in the range between \( 2.6941 \cdot 10^6 \) \( Km \cdot s^{-1} \cdot Mpc^{-1} \) and \( 2.1084 \cdot 10^6 \) \( Km \cdot s^{-1} \cdot Mpc^{-1} \). The corresponding values for the Hubble constant vary between \( 50.1328 \) \( Km \cdot s^{-1} \cdot Mpc^{-1} \) and \( 91.6791 \) \( Km \cdot s^{-1} \cdot Mpc^{-1} \).

4. Conclusion

We construct an open singularity-free cosmological model with the assumptions that the universe is initially in a vacuum like state and the physical quantities are limited by their Planck values. Evolution of the temperature of the universe is governed by its expansion. During the prematter era, temperature increases due to inflation and the universe is characterized by a vacuum like equation of state in the form \( P = (\gamma_p - 1)\rho \) where \( \gamma_p \sim 10^{-3} \) is a parameter which determines the vacuum dominance of the early universe. Hence, the model we construct is a two-parameter universe model, the parameters being \( H(0) \) and \( \gamma_p \). This era ends when the temperature reaches the maximum allowed temperature \( T_{pl} \). Then the universe enters the radiation era and starts cooling down. This cooling continues during the matter era described by the standard model.

The so-called flatness problem is not present in this model since the initial value of \( \Omega \) is not set to unity. In the standard model, such a precise initial condition has to be assumed without explanation to produce a universe resembling the actual one. In this model, the universe starts its journey with an \( \Omega \) value no matter how close to unity and during the prematter era \( \Omega \) is driven toward unity. At the end of this era, the universe is nearly flat since its \( \Omega \) value is very close to one. This fact could be attributed to the inflation mechanism which causes the universe to expand enormously in a very small time interval (Planck time) and thus become flatter than at the beginning. During the radiation and matter eras, \( \Omega \) is driven away unity which apparently displays the open character of the space-time geometry of the model.

In this work, we have not been interested in the microphysics of the inflationary era. This includes quantum mechanical investigation of the material content of the early universe and the form of the scalar field potential responsible for the inflation. However, with this simple form, the scenario proposed in this work might provide a guidance for the future more complete versions of the open inflation.
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