ADDITIVE MAPS PRESERVING IDEMPOTENCY OF PRODUCTS OR JORDAN PRODUCTS OF OPERATORS

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Abstract. Let $H$ and $K$ be infinite dimensional Hilbert spaces, while $B(H)$ and $B(K)$ denote the algebras of all linear bounded operators on $H$ and $K$, respectively. We characterize the forms of additive mappings from $B(H)$ into $B(K)$ that preserve the nonzero idempotency of either Jordan products of operators or usual products of operators in both directions.

1. Introduction And Statement of the Results

The study of maps on operator algebras preserving certain properties or subsets is a topic which attracts much attention of many authors. Some of the problems are concerned with preserving a certain property of usual product or other products of operators. For example see [3, 5–9, 11, 13, 14].

Let $R$ and $R'$ be two rings and $\phi : R \to R'$ be a map. Denote by $P_R$ and $P_{R'}$ the set of all idempotent elements of $R$ and $R'$, respectively. We say that $\phi$ preserves the idempotency of product of two elements, the idempotency of triple Jordan product of two elements and the idempotency of Jordan product of two elements, whenever we have

\[ AB \in P_R \Rightarrow \phi(A)\phi(B) \in P_{R'}, \]

\[ ABA \in P_R \Rightarrow \phi(A)\phi(B)\phi(A) \in P_{R'}, \]

and

\[ \frac{1}{2}(AB + BA) \in P_R \Rightarrow \frac{1}{2}(\phi(A)\phi(B) + \phi(B)\phi(A)) \in P_{R'}, \]

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respectively. The triple Jordan product and the Jordan product of two elements $A$ and $B$ are defined as $ABA$ and $\frac{1}{2}(AB + BA)$, respectively. Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional Hilbert spaces, while $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ denote the algebras of all linear bounded operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. In [7], authors characterized the forms of unital surjective maps on $\mathcal{B}(X)$ preserving the nonzero idempotency of product of operators in both directions. Also in [13], authors characterized the forms of linear surjective maps on $\mathcal{B}(X)$ preserving the nonzero idempotency of either products of operators or triple Jordan products of operators.

In this paper, we determine the form of additive mapping $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ such that the range of $\phi$ contains all minimal idempotents and $I$ and also $\phi$ preserves the nonzero idempotency of Jordan products of operators in both directions. Moreover, we determine the form of surjective additive mapping $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ that preserves the nonzero idempotency of usual products of operators in both directions. Our main result are as follows.

**Theorem 1.1.** Let $\mathcal{H}$ and $\mathcal{K}$ be two infinite dimensional real or complex Hilbert spaces and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be an additive map such that the range of $\phi$ contains all minimal idempotents and $I$. If $\phi$ preserves the nonzero idempotency of Jordan products of operators in both directions, then $\phi$ either annihilates minimal idempotents or there exists a bounded linear or conjugate linear bijection $A : \mathcal{H} \to \mathcal{K}$ such that $\phi(T) = \xi AT A^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$ or $\phi(T) = \xi A^t T^t A^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$, where $\xi = \pm 1$ (in the case that $\mathcal{H}$ and $\mathcal{K}$ are real, $A$ is linear).

**Theorem 1.2.** Let $\mathcal{H}$ and $\mathcal{K}$ be two infinite dimensional complex Hilbert spaces and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a surjective additive map. If $\phi$ preserves the nonzero idempotency of products of operators in both directions, then there exists a bounded linear or conjugate linear bijection $A : \mathcal{H} \to \mathcal{K}$ such that $\phi(T) = \xi AT A^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$ or $\phi(T) = \xi A^t T^t A^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$, where $\xi = \pm 1$. 
2. Proofs

In this section we prove our results. First we recall some notations. Let $X$ and $Y$ be Banach spaces. Recall that a standard operator algebra on $X$ is a norm closed subalgebra of $B(X)$ which contains the identity and all finite rank operators. Denote the set of all idempotent operators of $B(H)$ by $I(H)$ and the Jordan product of $A, B$ by $A \circ B = \frac{1}{2}(AB + BA)$. Also denote the dual space $X$ by $X^*$. For every nonzero $x \in X$ and $f \in X^*$, the symbol $x \otimes f$ stands for the rank one linear operator on $X$ defined by

$$(x \otimes f)y = f(y)x \quad (y \in X)$$

If $x, y \in H$, then $x \otimes y$ stands for the rank one linear operator on $H$ defined by

$$(x \otimes y)z = < z, y > x \quad (z \in H)$$

where $< z, y >$ denotes the inner product of $z$ and $y$. We need some lemmas to prove our main result. Let $A \subseteq B(X)$ and $B \subseteq B(Y)$ be standard operator algebras.

The proof of the following lemma is similar to that of Lemma 2.2 in [13].

**Lemma 2.1.** [13] Let $\phi : A \rightarrow B$ be an additive map such that preserves the nonzero idempotency of Jordan products of operators. If $N \in A$ is a finite rank operator such that $N^2 = 0$, then $\phi(N)^4 = 0$.

**Lemma 2.2.** Let $\phi : A \rightarrow B$ be an additive map. Then the following statements are hold.

(i) If $\phi$ preserves the nonzero idempotency of Jordan products of operators, then $\phi$ is injective.

(ii) If $I \in \text{rng} \phi$ and $\phi$ preserves the nonzero idempotency of Jordan products of operators in both directions, then $\phi(I) = I$ or $\phi(I) = -I$.

**Proof.** (i) Assume $\phi(A) = 0$. We assert that $A$ satisfies a quadratic polynomial equation. Otherwise, by the discussion in [10], there exists an $x \in X$ such that $x, Ax$ and $A^2x$ are linear independent. Then there is a linear functional $f$ such that $f(x) = f(A^2x) = 0$ and $f(Ax) = 2$,
because $\dim X \geq 3$. Setting $B = x \otimes f$, we have $A \circ B \in \mathcal{P}_A \setminus \{0\}$, implying that 
\[ \phi(A) \circ \phi(B) \in \mathcal{P}_B \setminus \{0\}. \]
This is a contradiction, because $\phi(A) \circ \phi(B) = 0$. So by the discussion in [10], $A$ satisfies a quadratic polynomial equation.

Assume on the contrary that $A$ is a nonzero operator. For any $B \in \mathcal{A}$ we have 
\[ \phi(A) \circ \phi(B) = 0. \]
However, there exists $B = x \otimes f$ such that $A \circ B = Ax \otimes f + x \otimes fA$ is a nonzero idempotent, a contradiction. We construct such $B$.

By the proved assertion, $A$ satisfies a quadratic polynomial equation. The spectrum of such $A$ consists only of eigenvalues. If $A^2 \neq 0$, then $A$ has a nonzero eigenvalue $\lambda$, because in this case there exist $r, s \in \mathbb{C}$ such that $rs \neq 0$ and $\lambda$ satisfies a quadratic polynomial equation $\alpha^2 = r\alpha + s$. Since $rs \neq 0$, $\alpha^2 = r\alpha + s$ has a nonzero root. Let $x$ be its eigenvector. Choose a bounded functional $f$ with $f(x) = \frac{1}{2\lambda}$ to form $B = x \otimes f$ with the desired properties.

The remaining case is $A^2 = 0$. Since $A$ is nonzero, we can find a vector $x$ so that $Ax \neq 0$ and a functional $f$ with $f(x) = 0$ and $f(Ax) = 1$ to form $B = x \otimes f$ with the desired properties. The proof is complete.

(ii) Since $I \in \mathrm{rng}\phi$, there exists a nonzero operator $U \in \mathcal{A}$ such that $\phi(U) = I$. We show that $U = I$ or $-I$. We have $\phi(U) \circ \phi(U) \in \mathcal{P}_B \setminus \{0\}$. Hence we obtain

\[ U^2 = U \circ U \in \mathcal{P}_A \setminus \{0\}. \]

This implies that for any $x \in X$, $x, Ux$ and $U^2x$ are linear dependent. Thus $U$ satisfies a quadratic polynomial equation, by the discussion in [10]. This together with (1) yields that there exist $a, b \in \mathbb{C}$ such that we have

\[ U^2 = aU + bI. \]

From (1) and (2), we obtain the answers $(0, 1)$, $(1, 0)$ and $(-1, 0)$ for $(a, b)$ which imply that $U^2 = I$, $U^2 = U$ and $U^2 = -U$. 
Let $U^2 = U$. We assert that $U = I$. Assume on the contrary that $U \neq I$. From $U \circ I \in \mathcal{P}_A \setminus \{0\}$, we obtain

\[
(3) \quad \phi(I) = I \circ \phi(I) \in \mathcal{P}_B \setminus \{0\}.
\]

On the other hand, there exists an idempotent operator $T$ such that $U - T$ is not idempotent. In fact, $U + S - I$ isn’t idempotent, where $S = I - T$. Thus

\[(U + S - I) \circ I = U + S - I \not\in \mathcal{P}_A \setminus \{0\}\]

which implies that

\[\phi(U + S - I) \circ \phi(I) \not\in \mathcal{P}_B \setminus \{0\}\]

This together with (3) yields that

\[\phi(I) \circ \phi(S) \not\in \mathcal{P}_B \setminus \{0\}\]

which implies that

\[S = I \circ S \not\in \mathcal{P}_A \setminus \{0\}\]

This is a contradiction, because $S$ is idempotent. So the proof of assertion is completed.

With a similar proof, the assumption $U^2 = -U$ yields that $U = -I$.

Now let $U^2 = I$. We assert that $U$ is a multiple of $I$. Assume on the contrary that $U$ is a non-scalar operator. Since $I$ and $U$ are linear independent, there is a nonzero vector $x \in X$ such that $x$ and $Ux$ are linear independent. Hence there exists $f \in X^*$ such that $f(x) = 0$ and $f(Ux) = 1$. Setting $B = x \otimes f$, we obtain

\[U \circ B \in \mathcal{P}_A \setminus \{0\}\]

which implies that

\[\phi(B) = \phi(U) \circ \phi(B) \in \mathcal{P}_B \setminus \{0\}\]

This is a contradiction, because $B$ is a nilpotent such that $B^2 = 0$ and so by Lemma 2.1, $\phi(B)$ is a nilpotent operator. So the proof of assertion is completed. By the proved assertion, there exists a nonzero complex number $\lambda$ such that $U = \lambda I$. Since $U^2 = I$, we obtain $\lambda^2 = 1$ and this completes the proof. $\square$
Theorem 2.3. [4] Let $\mathcal{H}$ and $\mathcal{K}$ be two infinite dimensional real or complex Hilbert spaces and $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be an additive map preserving idempotents. Suppose that the range of $\phi$ contains all minimal idempotents. Then $\phi$ either annihilates minimal idempotents or there exists a bounded linear or conjugate linear bijection $A : \mathcal{H} \to \mathcal{K}$ such that $\phi(T) = AT A^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$ or $\phi(T) = AT^\dagger A^{-1}$ for every $T \in \mathcal{B}(\mathcal{H})$ (in the case that $\mathcal{H}$ and $\mathcal{K}$ are real, $A$ is linear).

Proof of Theorem 1.1. Since by Lemma 2.2, $\phi(I) = I$ or $\phi(I) = -I$, from $P = I \circ P \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ we obtain that $\phi(P)$ or $-\phi(P)$ belongs to $\mathcal{I}(\mathcal{H}) \setminus \{0\}$. This together with $\phi(0) = 0$ implies that $\phi$ or $-\phi$ preserves the idempotent operators in both directions. Hence the forms of $\phi$ follows from Theorem 2.3.

Proposition 2.4. Let $\dim \mathcal{H} \geq 3$. Let $A$ be an arbitrary operator of $\mathcal{B}(\mathcal{H})$ and $P$ be a rank one idempotent operator. Then $A \in C^*P$ if and only if for every $T \in \mathcal{B}(\mathcal{H})$ such that $PT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ we have $AT \not\in \mathcal{I}(\mathcal{H}) \setminus \{0\}$, where $C^* = C \setminus \{0, 1\}$.

Proof. If $A \in C^*P$, then there exists a $\lambda \in C^*$ such that $A = \lambda P$. hence it is trivial that for every $T$ such that $PT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ then $\lambda PT \not\in \mathcal{I}(\mathcal{H}) \setminus \{0\}$.

Conversely, Let $A \not\in C^*P$. Since $P$ is rank one, by [2], there exists either an $x \in \mathcal{H}$ such that $Px$ and $Ax$ are linear independent or an $x \in \mathcal{H}$ and linear independent vectors $z_1, z_2 \in \mathcal{H}$ such that $P = x \otimes z_1$ and $A = x \otimes z_2$.

If $Px$ and $Ax$ are linear independent, then there exists $y \in \mathcal{H}$ such that $\langle Px, y \rangle = \langle Ax, y \rangle = 1$, because $\dim \mathcal{H} \geq 3$. Setting $T = x \otimes y$ follows that $PT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ and also $AT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$. This is a contradiction.

If $P = x \otimes z_1$ and $A = x \otimes z_2$, then there exist $y, z_3 \in \mathcal{H}$ such that $\langle x, z_3 \rangle = \langle y, z_1 \rangle = \langle y, z_2 \rangle = 1$. Setting $T = y \otimes z_3$ follows that $PT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$ and also $AT \in \mathcal{I}(\mathcal{H}) \setminus \{0\}$. This is a contradiction.

These contradictions yields that $A \in C^*P$ and this completes the proof.
Lemma 2.5. Let \( \phi : A \rightarrow B \) be a surjective additive map such that
\[
AB \in \mathcal{P}_A \setminus \{0\} \iff \phi(A)\phi(B) \in \mathcal{P}_B \setminus \{0\}
\]
for every \( A \in A \) and \( B \in B \). Then the following statements are hold.

(i) \( \phi(I) = I \) or \( \phi(I) = -I \).

(ii) If \( A = B(\mathcal{H}) \) with \( \dim \mathcal{H} \geq 3 \) and \( B = B(\mathcal{K}) \), then \( \phi(\mathbb{C}P) \subseteq \mathbb{C}\phi(P) \), for every rank one idempotent \( P \).

Proof. (i) It is proved by using Lemma 2.1 and similar to the proof of Lemma 2.2 in [13].

(ii) By (i), \( \phi(I) = I \) or \( \phi(I) = -I \). Since \( \phi(0) = 0 \), we can conclude from (i) that \( \phi \) or \( -\phi \) preserves the idempotent operators in both directions. By Lemma 2.6 in [12], \( \phi \) or \( -\phi \) preserves the rank one idempotent operators in both directions. If \( A \in \mathbb{C}^*P \), then by Proposition 2.4, for every \( T \in B(\mathcal{H}) \) such that \( PT \in \mathcal{I}(\mathcal{H}) \setminus \{0\} \) we have \( AT \notin \mathcal{I}(\mathcal{H}) \setminus \{0\} \) which by surjectivity of \( \phi \) imply that for every \( T' \in B(\mathcal{H}) \) such that \( \phi(P)T' \in \mathcal{I}(\mathcal{H}) \setminus \{0\} \) we have \( \phi(A)T' \notin \mathcal{I}(\mathcal{H}) \setminus \{0\} \). Since \( \phi(P) \) is a rank one idempotent, by Proposition 2.4 we can conclude that \( \phi(A) \in \mathbb{C}^*\phi(P) \). This together with (i) and \( \phi(0) = 0 \) follows that \( \phi(\mathbb{C}P) \subseteq \mathbb{C}\phi(P) \). This completes the proof. \qed

Proposition 2.6. Let \( \mathcal{H} \) and \( \mathcal{K} \) be two infinite dimensional real or complex Hilbert spaces and \( \phi : B(\mathcal{H}) \rightarrow B(\mathcal{K}) \) be an additive map. If \( \phi \) preserves the idempotent operators, then \( \phi \) preserves the square zero operators.

Proof. Let \( N \in B(\mathcal{H}) \) be a square zero operator. Then we have \( \mathcal{H} = \ker N \oplus M \) for some closed subspace \( M \) of \( \mathcal{H} \). Thus by this decomposition \( N \) has the following operator matrix
\[
N = \begin{pmatrix} 0 & N_1 \\ 0 & 0 \end{pmatrix}.
\]
If
\[
A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}
\]
then \( A + nN \in \mathcal{I}(\mathcal{H}) \) for every natural number \( n \). It implies that 
\[
\phi(A) + n\phi(N) \in \mathcal{I}(\mathcal{K}) \quad \text{for every natural number} \quad n.
\]
That is,
\[
\phi(A) + n\phi(N) = \phi(A)^2 + n(\phi(A)\phi(N) + \phi(N)\phi(A)) + n^2\phi(N)^2
\]
for all \( n \). Setting \( n = 1 \) and \( n = 2 \) yield
\[
\phi(N) = \phi(A)\phi(N) + \phi(N)\phi(A) + \phi(N)^2,
\]
\[
2\phi(N) = 2(\phi(A)\phi(N) + \phi(N)\phi(A)) + 4\phi(N)^2
\]
which imply that \( \phi(N)^2 = 0 \) and this completes the proof. \( \square \)

**Theorem 2.7.** [1] Let \( \mathcal{H} \) and \( \mathcal{K} \) be two infinite dimensional complex Hilbert spaces and \( \phi : B(\mathcal{H}) \to B(\mathcal{K}) \) be a surjective additive map such that \( \phi(\mathbb{C}P) \subseteq \mathbb{C}\phi(P) \) holds for every rank one operator \( P \). Then \( \phi \) preserves square zero in both directions if and only if there exists a nonzero scalar \( c \) and a bounded linear or conjugate linear bijection \( A : \mathcal{H} \to \mathcal{K} \) such that \( \phi(T) = cATA^{-1} \) for every \( T \in B(\mathcal{H}) \) or \( \phi(T) = cAT^tA^{-1} \) for every \( T \in B(\mathcal{H}) \).

**Proof of Theorem 1.2.** By a similar proof to that of Lemma 2.3 in [13], we obtain that \( \phi \) is injective. Since \( \phi(0) = 0 \), we can conclude from part (i) of Lemma 2.5 that \( \phi \) or \( -\phi \) preserves the idempotent operators in both directions. This together with Proposition 2.6 and the injectivity of \( \phi \) implies that \( \phi \) preserves the square zero operators in both directions. Moreover, by part (i) of Lemma 2.5, \( \phi(\mathbb{C}P) \subseteq \mathbb{C}\phi(P) \), for every rank one idempotent \( P \). Therefore the forms of \( \phi \) follow from Theorem 2.7. The scalar \( c \) in Theorem 2.7 is the scalar that \( \phi(I) = cI \) (by the proof of this theorem in [1]). This together with the part (i) of Lemma 2.5 completes the proof.

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