ON EXCESS OF THE ODIOUS PRIMES

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ABSTRACT. We give a more strong heuristic justification of our conjecture on the excess of the odious primes.

1. INTRODUCTION

This note is a continuation of the author's paper [4]. Until now the author did not know about the Moser’s digit conjecture and its solutions in [3] and [1]. In fact, we gave in [4] a new combinatorial proof of this conjecture (see Theorem 3[4]) and proved also an addition to the Moser-Newman theorem for the excess of the evil nonnegative odd integers less than $n$ and divisible by 3 (see. Theorem 4 and 5 in [4]).

The aim of the present note - to give a more strong heuristic justification ("almost strong proof") of our Conjectures 1 and 2 [4].

Recently in their excellent research [2], Mauduit and Rivat solved the Gelfond digit problem for primes. In particular, they proved that

$$\lim_{n \to \infty} \frac{\pi^0(n)}{\pi(n)} = \lim_{n \to \infty} \frac{\pi^e(n)}{\pi(n)} = \frac{1}{2}.$$  

Moreover, if $\pi_{3,i}^0(n)(\pi_{3,i}^e(n)), \ i = 1, 2,$ is the number of the odious (evil) primes $p \equiv i(mod3)$ not exceeding $n$ and

$$\pi_{3,i}(n) = \pi_{3,i}^0(n) + \pi_{3,i}^e(n), \ i = 1, 2,$$

they also proved that

$$\lim_{n \to \infty} \frac{\pi_{3,i}^0(n)}{\pi_{3,i}(n)} = \lim_{n \to \infty} \frac{\pi_{3,i}^e(n)}{\pi_{3,i}(n)} = \frac{1}{2}.$$  

These results mean that the events "$n$ is a prime" and "$n$ is an odious integer" are asymptotic independent for large $n$.

In turn, this means that the odious-evil asymptotic behavior of the primes of the form $3k + 1(3k + 2)$ is proportionally similar to the odious-evil asymptotic behavior of all odd integers of the form $3k + 1(3k + 2)$. 

1991 Mathematics Subject Classification. 11N05
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2. PROOF OF CONJECTURES

Let \( \mu^0(n)(\mu^e(n)) \) be the number of odd odious (evil) integers less than \( n \).

**Lemma 1.** \(|\mu^0(n) - \mu^e(n)| \leq 1.\)

*Proof.* The lemma follows from the identity

\[
(3) \quad \mu^0(4m + 1) = \mu^e(4m + 1), \quad m \in \mathbb{N},
\]

which is proved by induction.

Notice that (3) is valid for \( m = 1 \). Assuming that it is valid for \( 4m + 1 \) we prove (3) for \( 4(m + 1) + 1 \). Indeed, let \( m \) have \( k \) ones in the binary expansion. Then taking into account that for odd \( k \) the number \( 4m + 1 \) is evil and for even \( k \) the number \( 4m + 1 \) is odious, and using the induction conjecture we have

\[
\mu^0(4m + 3) - \mu^e(4m + 3) = (-1)^k.
\]

Furthermore, \( 4m + 3 \) is evil if \( k \) is even and is odious if \( k \) is odd. Therefore,

\[
\mu^0(4m + 5) - \mu^e(4m + 5) = (-1)^k + (-1)^{k+1} = 0. \]

Let \( \Delta_{3,i}(n)(\Delta_{3,i}^{\text{odd}}(n)) \), \( i = 0, 1, 2 \) be the excess of the (odd) odious integers \( m \in [0,n) \) such that \( m \equiv i \pmod{3} \).

In particular, according to the notations of [4]

\[
(4) \quad \Delta_{3,0}^{\text{odd}}(n) = -\Delta_3^{\text{odd}}([0,n)) < 0, \quad \Delta_{3,0}(n) = -\Delta_3([0,n)) < 0.
\]

Let, furthermore, \( \Delta_{3,i}^{\text{primes}} \), \( i = 1, 2 \) be the excess of odious odd primes \( p \in [0,n) \) such that \( p \equiv i \pmod{3} \). Then by (1),(2) taking into account the independence of the above mentioned events, in the case of \( |\Delta_{3,i}^{\text{odd}}(n)| >> \ln n \) we have

\[
(5) \quad \Delta_{3,i}^{\text{primes}} \sim \frac{3\Delta_{3,i}^{\text{odd}}(n)}{\ln n}, \quad i = 1, 2.
\]

So, for \( i = 1 \) and for even \( n \) we have

\[
(6) \quad \Delta_{3,1}^{\text{odd}}(n) = -\Delta_{3,0} \left( \frac{n}{2} \right) = \Delta_3 \left( [0, \frac{n}{2}) \right).
\]
Newman showed [3], that for all \( n \in \mathbb{N} \)

\[
(0.05 \cdot 3^\alpha)n^\alpha \leq \Delta_3([0, n)) \leq (5 \cdot 3^\alpha)n^\alpha \quad \text{with} \quad \alpha = \frac{\ln 3}{\ln 4}.
\]

Therefore, by (6)

\[
\Delta_{3,1}^{\text{odd}}(n) \geq 0.05(1.5)^\alpha n^\alpha >> \ln n.
\]

In the case of \( i = 2 \) the absolute value of the excess \( \Delta_{3,2}^{\text{odd}}(n) \) is small for some \( n \). Indeed, by Lemma 1

\[
\Delta_{3,0}^{\text{odd}} + \Delta_{3,1}^{\text{odd}} + \Delta_{3,2}^{\text{odd}} = \delta_n,
\]

where \( |\delta_n| \leq 1 \).

Thus by (5) and (6)

\[
\Delta_{3,2}^{\text{odd}} = \delta_n - \Delta_{3,0}^{\text{odd}} - \Delta_{3,1}^{\text{odd}} = \Delta_3^{\text{odd}}([0, n)) - \Delta_3\left([0, \frac{n}{2})\right) + \delta_n.
\]

With help of (10) and the exact formulas for \( \Delta_n^{\text{odd}}(n), \Delta_3(n) \) [4] we obtain in particular that

\[
\Delta_{3,2}^{\text{odd}}([0, 2^{2n-1})) = -3n^{-2}, \quad \Delta_{3,2}^{\text{odd}}([0, 2^{2n})) = 0.
\]

Nevertheless, it is sufficient for us to understand (5) for small \( |\Delta_{3,2}^{\text{odd}}(n)| \) by the following way: if

\[
|\Delta_{3,2}^{\text{odd}}(n)| \leq \sqrt{n} \quad \text{then} \quad |\Delta_{3,2}^{\text{prime}}(n)| = O \left( \frac{\sqrt{n}}{\ln n} \right).
\]

Now if \( |\Delta_{3,2}^{\text{odd}}| > \sqrt{n} \) by (5), (6) and (10) we have

\[
\pi^0(n) - \pi^e(n) = \Delta_{3,1}^{\text{prime}}(n) + \Delta_{3,2}^{\text{prime}}(n) \sim \frac{3\Delta_3^{\text{odd}}([0, n))}{\ln n}.
\]

Note that, according to [4]

\[
\lim_{n \to \infty} \frac{\ln \Delta_3^{\text{odd}}([0, n))}{\ln n} = n^\alpha.
\]
If $|\Delta_{3,2}^{\text{odd}}| \leq \sqrt{n}$ then by (3), (6) and (12) we have

$$\pi^0(n) - \pi^e(n) = \frac{3\Delta_{3,1}^{\text{odd}}(n)}{\ln n}(1 + o(1)) + O(\sqrt{n}) \sim \frac{3\Delta_{3}(\lfloor 0, \frac{n}{2} \rfloor)}{\ln n}.$$  

Now by (8), (13)-(15) we find as the final result that

$$\ln (\pi^0(n) - \pi^e(n)) = \frac{\ln 3}{\ln 4} \ln n + o(\ln n)$$

and our Conjecture 2 follows.\[\Box\]

Note that from Conjecture 2 evidently follows the statement of Conjecture 1 but only for sufficiently large $n \geq n_0$. Unfortunately, until now we are not able to estimate $n_0$.

Note that by the way we obtain the limits

$$\lim_{n \to \infty} \frac{\ln (\pi^0_{3,1}(n) - \pi^e_{3,1}(n))}{\ln n} = \frac{\ln 3}{\ln 4},$$

for $n_k = 2^{2k-1}$, $k \in \mathbb{N}$,

$$\lim_{k \to \infty} \frac{\ln (\pi^e_{3,2}(n_k) - \pi^0_{3,2}(n_k))}{\ln n_k} = \frac{\ln 3}{\ln 4}.$$

The author is grateful to D. Berend, R. K. Guy, G. Martin and T. D. Noe which show an interest in the considered conjectures.

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