Simulation of convective transport during frequency chirping of a TAE using the MEGA code

H. Hezaveh1,∗, Y. Todo2, Z.S. Qu1, B.N. Breizman3 and M.J. Hole1,4

1 Mathematical Sciences Institute, The Australian National University, Canberra ACT 2601, Australia
2 National Institute for Fusion Science, National Institutes of Natural Sciences, Toki, Gifu 502-5292, Japan
3 Institute for Fusion Studies, The University of Texas at Austin, Austin, TX 78712, United States of America
4 Australian Nuclear Science and Technology Organisation, Locked Bag 2001, Kirrawee DC, NSW, 2232, Australia

E-mail: hooman.hezaveh@anu.edu.au

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Abstract

We present a procedure to examine energetic particle phase-space during long range frequency chirping phenomena in tokamak plasmas. To apply the proposed method, we have performed self-consistent simulations using the MEGA code and analyzed the simulation data. We demonstrate a traveling wave in phase-space and that there exist specific slices of phase-space on which the resonant particles lie throughout the wave evolution. For non-linear evolution of an \( n = 6 \) toroidicity-induced Alfvén eigenmode (TAE), our results reveal the formation of coherent phase-space structures (holes/clumps) after coarse-graining of the distribution function. These structures cause a convective transport in phase-space which implies a radial drift of the resonant particles. We also demonstrate that the rate of frequency chirping increases with the TAE damping rate. Our observations of the TAE behavior and the corresponding phase-space dynamics are consistent with the Berk–Breizman theory.

Keywords: energetic particle instabilities in tokamaks, frequency chirping of Alfvén waves, convective transport in tokamaks, fast particles conservation law in toroidal systems, wave-particle interaction, self-consistent simulations using the MEGA code

(Some figures may appear in colour only in the online journal)

1. Introduction

The physics of energetic particles (EPs) plays an essential role in fusion plasmas. It has very attractive diagnostic applications but, on the other hand, it involves the possibility of unacceptably fast particle losses. A famous example is the destabilization of weakly damped plasma waves inside the gaps of the shear Alfvén continuum, which entails redistribution or ejection of EPs either through diffusive transport or a convective transport where an isolated resonance moves radially like a bucket that carries resonant particles. The latter is associated with long range frequency chirping and has been observed for a variety of modes in experiments [1–5]. References [6–11] show a correlation between wave-particle resonant interactions and fast ion loss and redistribution.

The formation of coherent structures in fast electrons phase-space was observed in non-linear simulations of a 1D electrostatic wave in reference [12]. These structures (holes and clumps) are BGK-type modes [13] with a chirping frequency. They evolve adiabatically and carry the trapped particles. Non-perturbative adiabatic models [14–18] suggest the slow evolution of a Langmuir wave as a 1D paradigm of the more general wave-particle interactions in realistic geometries. In references [19, 20], the theory has been extended to tokamak applications where the frequency chirping of Alfvénic pertur-
bations are studied. Reference [21] demonstrates the formation of holes and clumps during frequency chirping of the \( n = 0 \) EGAM modes, where the toroidal momentum \( (P_\varphi) \) of the EPs is conserved in the presence of the electrostatic perturbations. The impact of EP beta value \((\beta_{EP})\) on chirping of a toroidicity-induced Alfvén eigenmode (TAE) mode was studied in reference [22] and it has been shown that as the frequency of the wave changes, the dominant perturbation occurs at different slices of phase-space (\( P_\varphi \) vs \( E \) with \( \mu = \text{const} \)). In reference [23], the phase-space dynamics of EPs are studied during the long range frequency chirping of a TAE with a fixed eigen-function, where phase-space slices are determined using two constants of motion, namely \( \mu \) and \( C = \omega_{\text{TAE}} P_\varphi - nE \) (see references [24, 25]) with \( \mu, \omega_{\text{TAE}}, n \) and \( E \) being the magnetic moment, linear eigenfrequency, toroidal mode number and the EP energy, respectively. Still the question of how the chirping wave transports particles in phase-space deserves more detailed analysis. Technically speaking, best suited constants of motion for EPs dynamics need to be defined as the frequency evolves.

In this work, we describe an appropriate procedure to observe the EPs dynamics on sub-slices of the phase-space during frequency chirping of a TAE mode. Subsequently, we validate this method by applying the corresponding analysis to the results of EP simulations with the MEGA code [26, 27]. We also show that the rate of frequency chirping is directly related to the damping rate of the mode in the bulk plasma. We demonstrate the latter by altering the dissipation coefficients when the mode has already evolved into chirping regime. In order to increase the resolution in phase-space, we have added test particles to the code. These test particles are pushed by the fields but do not contribute to the total EP current.

The rest of the paper is structured as follows: in section 2, we introduce a set of equations implemented in the hybrid MEGA code. Section 3 describes the appropriate coordinates and constants of motion needed to analyze the guiding center dynamics of EPs in phase-space during the non-linear frequency chirping. This involves canonical action-angle variables. Subsequently, we apply our phase-space analysis to the simulation data of the MEGA code in section 4 and report on the evolution of the TAE parameters. We identify resonant particles and exhibit their convective transport in phase-space. Section 5 contains concluding remarks.

2. The simulation model in MEGA

We simulate the evolution of the energetic particle driven mode within a hybrid model implemented in the MEGA code, where the bulk plasma particles are described as a fluid by the non-linear MHD equations and the fast particles are treated in a drift-kinetic approach. The MEGA code solves the following set of equations:

The momentum balance equation given by

\[
\frac{\partial \rho \nu}{\partial t} = -\rho \nu \cdot \nabla \nu - \nabla p + \left( \frac{1}{\mu_0} \nabla \times B - j_\alpha \right)
\times B + \frac{4}{3} \nabla \left( \nu p \nabla \cdot \nu \right) - \nabla \times \left( \nu p \nabla \times \nu \right), \tag{1}
\]

where \( j_\alpha \) denotes the EP current, \( \nu \) is the viscosity coefficient and \( \rho \) and \( p \) are the density and scalar pressure of the bulk plasma, respectively. In hybrid models, the contribution of fast particles is coupled to the MHD equations either through the current term or the pressure term. Here, a current coupling approach has been implemented. It is noteworthy that the perpendicular component of the EP current arises from curvature drift, grad-B drift and magnetization current. This component of the EP current can be expressed in terms of the parallel and perpendicular EP pressure.

The continuity equation for the bulk plasma

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \nu) + \nu_\alpha \Delta (\rho - \rho_0), \tag{2}
\]

where \( \nu_\alpha \) is the mass diffusivity. The energy balance equation for the evolution of the bulk plasma pressure

\[
\frac{\partial p}{\partial t} = -\nabla \cdot (p \nu) - (\gamma - 1) p \nabla \cdot \nu + (\gamma - 1) \times \left[ \nu_\rho (\nabla \cdot \nu)^2 + \frac{4}{3} \nu \rho (\nabla \cdot \nu)^2 + v_j \cdot (j - j_0) \right] + \lambda \Delta (p - p_0), \tag{3}
\]

where \( \gamma \) is the adiabatic constant and \( \lambda \) represents the heat conductivity.

The set of Maxwell’s equations and the Ohm’s law given by

\[
\frac{\partial B}{\partial t} = -\nabla \times E, \tag{4a}
\]

\[
j = \frac{1}{\mu_0} \nabla \times B, \tag{4b}
\]

\[
E = -\nu \times B + \eta (j - j_0), \tag{4c}
\]

where \( \eta \) represents resistivity.

In the above equations, all the other quantities are conventional. The subscript 0 represents the equilibrium values of the parameters and the corresponding terms, as the source terms, have been used to enforce MHD equilibrium and compensate the diffusion and dissipation of the equilibrium fields. This set of equations is discretized using the method of finite difference and the fields are solved in an Eulerian scheme where the computational domain is gridded.

The EPs are treated kinetically in a Lagrangian picture. A particle-in-cell method is applied to project the impact of EPs (EPs charge) on the grid points and update the fields in a self-consistent manner at each time step. The perturbation of the EPs, due to the wave, is calculated using the \( \delta f \) approach as the time evolution of the weight of each particle. This gives the following expression for the EPs current

\[
j_\alpha = \sum_{i=1}^{N} e Z_i u_i \left( v_{i,\|} + v_{i,\perp} \right) S(x - x_i)
- \nabla \times \left[ b \sum_{i=1}^{N} \mu_i u_i S(x - x_i) \right], \tag{5}
\]
where the second term on the right-hand side is the magnetization current, the subscript \( i \) represents the \( i \)th EP, \( eZ_i \) is the charge of the EPs, \( v_i \) is the weight, \( \nu_{by} \) is the drift due to the gradient of the magnetic field, \( \beta \) is the shape factor and \( \mu = E_i (1 - \lambda^2)/B \) is the magnetic moment with \( E_i \), \( \lambda \) and \( B \) being the kinetic energy, pitch angle and the magnetic field at the guiding center, respectively, \( v_i^\parallel \) contains the parallel velocity \( v_i^\parallel \) to the magnetic field and magnetic curvature drift, and is given by

\[
v_i^\parallel = \frac{v_i}{B^\parallel}[B^\parallel + \rho_i B^A \nabla \times b],
\]

where \( \rho_i = \frac{m_i v_i^\parallel}{eB^\parallel} \) is the parallel gyro-radius, \( B^\parallel = B(1 + \rho_i b \cdot \nabla \times b) \) [28] and \( m_i \) is the ion mass. It is noteworthy that \( \rho_i \) does not contain \( E \times B \) drift due to quasi-neutrality [26].

The EPs current is coupled to the MHD equations through equation (1).

### 3. Phase-space study

In \((E, p_z, \mu)\) coordinate, the conservation of the conventional constant of motion \( C = \omega_T m P_z - nE \) will break down as the frequency of the wave chirps. Therefore, in order to study the phase-space during chirping, we need to identify generalized canonical momenta that remain constant not only in the perturbative linear phase of the TAE evolution but also during the long range frequency chirping. This is done in this section, where we introduce a set of coordinates using which the dynamics of EPs interacting with a chirping wave is reduced to essentially 1D. We start from the Littlejohn’s Lagrangian [28] for singly-charged ions given by

\[
L_{\text{Littlejohn}} = e(A + \rho_i B) \cdot \dot{X} + \frac{m_i}{e} \mu \dot{\Omega} - H,
\]

where \( e \) is the electron charge, \( X \) is the guiding center position, \( \Omega \) is the gyro angle, \( A \) is the total vector potential and \( B = \nabla \times A \) and \( H = \frac{1}{2} m v_i^2 + \mu B \) is the Hamiltonian. It should be noted that a gauge is considered where the perturbed electrostatic potential is zero. The expression of \( (7) \) contains both the particle/upperturbed and the interaction Lagrangian.

For common choices of magnetic field line coordinates e.g. Boozer [29], PEST [30], Hamada [31] and etc, the guiding center Lagrangian does not immediately reveal three canonical pairs of the Hamiltonian structure. This is due to the fact that the Lagrangian contains the time derivative of four variables as opposed to three. There have been several attempts to tackle this issue [29, 32, 33] but each has its own disadvantages. In reference [34], the problem is resolved by introducing canonical angles, namely \((\theta_c, \xi_c)\), which give a new type of global coordinates called canonical straight field line coordinates.

Using the new coordinates, a Legendre transformation can be implemented to find the unperturbed Hamiltonian

\[
H_0(\theta_c, \xi_c, \Omega_c, \phi) = P_{\theta_c} \dot{\theta}_c + P_{\xi_c} \dot{\xi}_c + P_{\Omega_c} \dot{\Omega} - L_{eq},
\]

where \( L_{eq} \) is the unperturbed part of the Little John’s Lagrangian. This Hamiltonian describes the unperturbed guiding center dynamics of EPs with

\[
\begin{align*}
P_{\theta_c} &= \frac{e \psi + m v_i^\parallel b_i}{e}, \\
P_{\xi_c} &= -e \chi + m v_i^\parallel b_i, \\
P_{\Omega_c} &= \frac{m}{e} \mu.
\end{align*}
\]

where \( \chi \) and \( \psi \) denote the poloidal and toroidal flux, respectively. The set \((P_{\theta_c}, P_{\xi_c}, P_{\Omega_c})\) are the canonical momenta conjugated to \((\theta_c, \xi_c, \Omega_c)\). For this completely integrable system, the \( \theta_c \)-dependence of the Hamiltonian can be eliminated by using a canonical transformation to action-angle variables. In these variables, we have

\[
H_0 = H_0(P_{\theta_c}, P_{\xi_c}, P_{\Omega_c}),
\]

where the action variables \((P_{\theta_c}, P_{\xi_c}, P_{\Omega_c})\) correspond to the angles \((\theta_c, \xi_c, \Omega_c)\) that are linear functions of time in the unperturbed motion, i.e.

\[
\begin{align*}
\dot{\theta}_c &= \frac{\partial H_0}{\partial P_{\theta_c}}, \\
\dot{\xi}_c &= \frac{\partial H_0}{\partial P_{\xi_c}}, \\
\dot{\Omega}_c &= \frac{\partial H_0}{\partial P_{\Omega_c}}.
\end{align*}
\]

To describe the perturbed motion of the particles, we write their total Hamiltonian \(H_{\text{total}}\) as a sum of the unperturbed Hamiltonian \(H_0\) and a perturbation \(U\) associated with the wave. This gives

\[
H_{\text{total}} = H_0 + U.
\]

We use the following representation for the perturbation \(U\)

\[
U = \sum_{n=1}^\infty \alpha(n) e^{i(n \theta_c + m \xi_c - \alpha(t))},
\]

where \( \alpha \) denotes the \( n \)th harmonic of the non-linear wave and \( r_c \) is the generalised radial coordinate corresponding to \( \theta_c \) and \( \xi_c \). This representation corresponds to a single chirping wave formed and evolved as a BGK-type wave through excitation of sideband/secondary oscillations of a single eigenmode in an isolated resonance. BGK modes are long-term non-linear solutions to the Vlasov–Poisson system which propagate steadily and in reference [12] chirping waves are described as BGK nonlinear waves that last much longer than the inverse linear damping rate while they are upshifting and downshifting in frequency. We rewrite \( U \) in terms of the action-angle variables of the unperturbed motion to have

\[
H_{\text{total}} = H_0 + U(P_{\theta_c}, P_{\xi_c}, P_{\Omega_c}, \theta_c) + \frac{e}{\Omega_c} \left( n \xi_c - \alpha(t) \right).
\]
sional. This 1D description of wave–particle interaction can be represented by transferring the coordinates canonically to a frame co-moving with the chirping wave. A type-2 generating function for such a transformation is

\[ G_2(\mathbf{q}, p_{\text{new}}, t) = P_1 \left[ \dot{\theta}_c + n\dot{\xi}_c - \alpha(t) \right] + P_2 \ddot{\xi}_c + P_3 \dot{\Omega}. \]  \hspace{1cm} (15)

It can be used to write the explicit expressions for the new variables and constants of motion as

\[
\begin{align*}
    P_1 &= \frac{1}{T} p_{\dot{\theta}_c} \quad Q_1 = \dot{\theta}_c = \theta_c + n\xi_c - \alpha(t) \\
    P_2 &= p_{\dot{\xi}_c} + \frac{n}{2} p_{\dot{\theta}_c} \quad Q_2 = \dot{\xi}_c \\
    P_3 &= p_{\dot{\Omega}} \quad Q_3 = \dot{\Omega},
\end{align*}
\]

after which the new Hamiltonian takes the form

\[
H_{\text{new}} = H_0(P_1, P_2, P_3) + U(\xi_c, P_1, P_2, P_3) + \frac{\partial G_2}{\partial t},
\]

where \( P_2 \) and \( P_3 \) are constants of motion and a generalized momentum (\( P_1 \)) and its corresponding coordinate (\( \xi_c \)), to which the momentum is conjugated, constitute the dynamical variables. We thereby follow the EPs dynamics in \( P_1-\xi_c \) on sub-slices of \( P_2 = \text{const} \) and \( P_3 = \text{const} \). The distinctive feature of the chosen variables is that \( P_2 \) remains conserved as the frequency chirps. In what follows, we focus on the first harmonic of the nonlinear wave \( l = 1 \) and we drop the subscript \( l = 1 \) from \( \xi \) for simplicity.

So far, we have introduced proper coordinates for our phase-space analysis, and the next step is to identify how the six-dimensional coordinate transformation of \( (P_{\theta_c}, P_{\xi_c}, P_{\Omega}, \theta_c, \xi_c, \Omega) \rightarrow (P_{\dot{\theta}_c}, P_{\dot{\xi}_c}, P_{\dot{\Omega}}, \theta_c, \xi_c, \Omega) \) is carried out. We do that by relating the EPs frequencies to \( P_{\dot{\theta}_c} \) and \( P_{\dot{\xi}_c} \) using equations (11a), (11b) and (10).

Given (10), \( H_0 \) is known as the particle energy \( E \), and \( P_{\dot{\xi}_c} \) and \( P_{\dot{\Omega}} \) are also known quantities and can be evaluated using (9b) and (9c), respectively, for \( \xi_c \) and \( \Omega \) being ignorable coordinates in \( H_0 \) of (8). Hence, equation (10) can be inverted to write

\[
P_{\dot{\theta}_c} = P_{\dot{\xi}_c} (H_0 = E, P_{\dot{\xi}_c}, P_{\dot{\Omega}}).
\]

We use the following procedure to implement this inversion. For a slice of \( \mu = \text{const} \), we write

\[
P_{\dot{\theta}_c} = G(E, P_{\dot{\xi}_c}),
\]

where \( G \) is a 2D polynomial of \( \sqrt{E} \) and \( P_{\dot{\xi}_c} \). The reason we take \( G \) as a function of \( \sqrt{E} \) is that in this work we focus on the highly passing particles (\( \mu = 0 \)), as in a neutral beam injection (NBI) scenario, for which \( \omega_{\dot{\theta}_c} = \sqrt{\frac{\mu}{\mu_0}} \propto \sqrt{E} \) and \( \omega_{\dot{\xi}_c} = \sqrt{\frac{n}{\mu_0}} \propto \sqrt{E} \). Therefore, we have \( H_0 = G(E, P_{\dot{\xi}_c}, P_{\dot{\Omega}}) \). Applying the derivative operator to both sides of (19) with respect to \( P_{\dot{\theta}_c} \) and \( P_{\dot{\xi}_c} \) gives

\[
\frac{\partial G}{\partial E} = \frac{1}{\omega_{\dot{\theta}_c}} \quad \text{and} \quad \frac{\partial G}{\partial P_{\dot{\xi}_c}} = -\frac{1}{\omega_{\dot{\xi}_c}},
\]

respectively, where equations (11a) and (11b) are used and \( \omega \) denotes the frequencies calculated using the fitting function \( G \). To fit \( G \), we use the method of least squares with the following minimization function

\[
M = \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{1}{\omega_{\dot{\theta}_c}^2} - \frac{1}{\omega_{\dot{\xi}_c}^2} \right]^2 + \sum_{i=1}^{N} \left[ \frac{1}{\omega_{\dot{\xi}_c}^2} - \frac{1}{\omega_{\dot{\xi}_c}^2} \right] \}
\]

(21)

where \( N \) is the total number of EPs on a \( \mu = \text{const} \) slice. In order to evaluate \( M \), the equilibrium frequencies \( (\omega_{\dot{\theta}_c}, \omega_{\dot{\xi}_c}) \) must be determined from simulation. These are computed by tracing particle trajectories for different \( P_{\dot{\xi}_c} \) and \( E \). Once known, the polynomial coefficients of \( G \) are varied until equation (21) is minimised. This determines \( G \).

Considering the set \((R, z, \varphi)\) as the cylindrical coordinate, we consider \( \theta_c = 0 \) and \( \xi_c = \varphi \) on the \( z = 0 \) plane with largest \( R \) where we also record the particle data. On this plane, the canonical angles \((\theta_c, \xi_c)\) equal geometrical angles (see [34] and equation (23) of reference [36]). As a convenient choice, this plane can also be used to show \( P_{\dot{\xi}_c} = P_{\varphi_c} \), where \( P_{\varphi_c} \) is the toroidal angular momenta conjugated to \( \varphi \).

The above approach gives an essentially 1D representation of the wave-particle interaction using phase-space plots in \( P_1-\xi_c \) space. A notable advantage of this method is that \( P_{\dot{\theta}_c} \) is conserved even when the frequency experiences long deviation from the initial eigenfrequency. This has important implications when resolving the question of whether the EPs trapped inside the chirping wave are carried with the wave (consistent with the adiabatic theory of frequency chirping) or different particles are perturbed by the wave as the frequency chirps. The mapping technique of transferring to the canonical coordinates, introduced above, is not restricted to TAE-type perturbations and can be implemented to study the phase dynamics of particles interacting with other types of perturbations such as tearing modes, energetic particle modes and fishbone-type oscillations, among others.

4. Analysis of the simulations

The MEGA code uses an equilibrium configuration constructed by a Grad–Shafranov solver for a given \( q \)-profile. In this work, we use a linear \( q \)-profile depicted in figure 1(a). The novel phase-space analysis tool described in section 3 is not restricted to the shape of the \( q \)-profile and the choice of the linear \( q \)-profile here is just for the purpose of illustration and simplicity. We choose the inverse aspect ratio \( \epsilon = 3.2 \). The density and pressure are uniform throughout the plasma. The corresponding shear Alfven continuum for \( n = 6 \) is plotted in figure 1(b). The accumulation points of the first gap are located at \( r/a = 0.26 \). For a TAE, the \( q \)-profile has a rational value at the cylindrical cross-over points, where \( q = (2m + 1)/2n \). As shown in figure 1(a), the first gap corresponds to \( m = 6 \) coupled to \( m = 17 \), and the second gap located at \( r/a = 0.82 \) corresponds to \( m = 7 \) that is coupled to \( m = 8 \). The equilibrium phase-space density of EPs is initialized using a slowing
Figure 1. (a) The safety factor and (b) the corresponding shear Alfvén continuum in a circular cross section configuration for $n = 6$. The red dashed line represents the linear frequency of the toroidal Alfvén eigenmode.

Figure 2. Time evolution of the TAE envelope (a) and the frequency spectrogram (b) for $n = 6, m = 6$ oscillations at $r/a = 0.27$. The $y$-axis of panel a shows the normalised radial component of the plasma velocity. The color bar of panel (b) represents an estimate of the short-term time-localized frequency content of cos component of $v_r$. The dissipative coefficients are kept the same as given in expression (23) throughout the mode evolution.

down distribution given by

$$F_{eq,\alpha} = \frac{\kappa}{E^3 + E_{\text{crit}}^3} \left[ 1 + \text{erf} \left( \frac{E_0 - E}{\Delta E} \right) \right] \exp \left( -\frac{\langle \psi \rangle}{\Delta \psi} \right),$$

where $E_{\text{crit}}$ and $E_0$ represent the critical and birth energies of the alpha particles, respectively, $\psi$ is the poloidal magnetic flux, $\langle \rangle$ denotes averaging over the particle orbit, $\Delta E$ and $\Delta \psi$ specify the characteristic width of the equilibrium phase-space density in energy and $\psi$, respectively. For the purpose of this work, the values are set as $E_0 = 1.44E_A$, $E_{\text{crit}} = 0.25E_A$, $\Delta E = 0.0144E_A$ and $\Delta \psi = 0.148\psi_{\text{max}}$, where $\psi_{\text{max}}$ is the maximum value of $\psi$ and $E_A = \frac{1}{2}mv_A^2$ with $v_A$ being the Alfvén velocity at the center of the plasma. The EPs pressure is set to give an EP beta value of $\beta_{\text{EP}} = 0.6\%$ on the magnetic axis. The damping coefficients are

$$\nu = \eta = 0.3 \times 10^{-7} v_A R_0, \quad \nu_n = \lambda = 0.$$  (23)

4.1. Evolution of the driven eigenmode

By solving the initial value problem with the MEGA code we find that the dominant perturbation is a TAE excited above the lower tip of the first gap with a linear frequency $\omega_{\text{TAE}}/\omega_A = 0.4553$, where $\omega_A$ is the Alfvén frequency on the axis. Figure 2
Figure 3. (a) A scan of the net growth rate versus EP pressure on axis from simulation data (the black circles) and a linear fit to the data (dashed line) and (b) the structure of the radial component of the bulk plasma velocity ($v_r$), normalised to the Alfvén velocity on the axis ($v_A$), versus the normalised minor radius at $t/t_A = 217.6$. Here, $t_A$ is the Alfvén time on the axis.

Figure 4. Time evolution of the TAE envelope (a) and the frequency spectrogram (b) for $n = 6$, $m = 6$ oscillations at $r/a = 0.27$. The two vertical dashed lines on panel (b) denote the times, namely $t/t_A = 1119.1$ and $1243.4$, at which the damping coefficients has been increased. The color bar represents an estimate of the short-term time-localized frequency content of cos component of $v_r$. The absolute value of the plasma radial velocity $v_r$ is depicted in figure 2(a) as a function of time. Using an exponential fit to the early/linear stage data of figure 2(a), we find that the net growth rate of the mode is $\left(\gamma_l - |\gamma_d|\right)/\omega_A = 0.0067$. Similarly, we perform a scan of the net growth rate over $\beta_{0,EP}$ to find the damping rate ($\gamma_d$) of the mode. This is depicted in figure 3(a) where a linear polynomial, fitted to the simulation data, identifies the intercept with the vertical axis. This gives a damping rate of $\gamma_d/\omega_A = 0.0053$ which corresponds to the energy being dissipated due to viscosity and resistivity in the coupled set of equations (1) to (5). It does not account for any damping mechanism related with kinetic effects by the thermal ions. Subsequently, the linear growth rate of the TAE is $\gamma_l/\omega_A = 0.012$. Hence, in this simulation we have $\gamma_d/\gamma_l = 0.44$, $\gamma_l/\omega_{TAE} = 2.64\%$.

The two dominant radial profiles of the TAE corresponding to the poloidal mode numbers $m = 6$ and 7 are shown in figure 3(b). We observe that the peak lies around the location of the first gap of the shear Alfvén continuum. Figure 2(b) shows an evolving spectrum of the cosine part of $v_r$. It reveals the primary up-ward and down-ward branches...
Figure 5. Frequency spectrum across the radial coordinate at different time slices. The color bar represents the absolute value of power(dB) per frequency. Panels (a), (b), (c) and (d) correspond to $t/t_A = 196.1$, $t/t_A = 469.7$, $t/t_A = 1713.1$ and $t/t_A = 3570.4$ respectively. The dotted line shows the shear Alfvén continuum.

We change the dissipation coefficients at $t/t_A = 1119.1$ and 1243.4, from their initial values of (23) to $\nu = \eta = 6 \times 10^{-3}v_A R_0$ and $\nu = \eta = 1.2 \times 10^{-6}v_A R_0$, respectively. We note that this change will not affect the linear evolution of the TAE. Figure 4 shows the resulting amplitude and frequency of the TAE as functions of time that we analyze subsequently. The times at which the damping rate has increased are denoted by vertical dashes in figure 4(b). A comparison of figures 2 and 4 shows that besides an expected drop in the amplitude of the signals, the rate of frequency chirping has increased in figure 4(b) after increasing the dissipation coefficients. Figure 4 confirms the essential role of dissipation in the chirping mechanism. The above technique of increasing the damping coefficients during the non-linear process of chirping provides a useful probing tool for nonlinear simulations. It can also save computational resources in large-scale simulations.
Figure 6. The resonance curve (a) and the resonance line (b) in \(E \text{ vs } P_{\varphi}\) and \(\omega^\xi \text{ vs } \omega^\theta\) plane, respectively. Each panel shows a \(\mu = 0\) slice of the phase-space for co-passing EPs. The color bar represents the particle weights (perturbed distributions). The dashed line is a fit using the resonance condition of (25).

Figure 7. The action corresponding to the poloidal angular momenta for \(\mu = 0\) in (a) \(E \text{ vs } P_{\varphi}\) plane and (b) \(\omega^\xi \text{ vs } \omega^\theta\).

Figure 5 shows the frequency content at each radial location at four different stages of the wave evolution. The linear mode structure of figure 5(a) is comparable to the one shown in figure 3(b). Figure 5(b) corresponds to the early stages of frequency chirping where the sideband/secondary waves have just formed inside the toroidicity gap. In figures 5(c) and (d), the frequencies of the chirping waves deviate further from the initial eigenfrequency toward the tips of the gap which leads to the excitation of continuum waves. Finally, the frequencies of the chirping waves enter the shear Alfvén continuum and exhibit different frequencies at different radial locations as they follow the continuum.

4.2. Resonance condition

In tokamak plasmas, the resonance condition between the particle guiding center motion and a wave with a toroidal mode number \(n\) reads [37].

\[
\omega = n\omega^\xi + p\omega^\theta.
\]  

(25)

where \(p\) is an integer. In the case of TAE, the mode has two dominant poloidal components of the field (\(m\) and \(m + 1\)). These two components have opposite phase velocities along the magnetic field. Consequently, the strongly co-passing particles resonate at \(p = -m\), whereas the strongly counter-passing particles resonate at \(p = -(m + 1)\) [26]. For our modes of interest, we expect the co-passing particle resonance to be at \(n = 6\) and \(p = -6\) in the simulations.

Figure 6 shows two images of the perturbed particle distribution function: the color-coded particle weights on the \(E - P_{\varphi}\) plane and on the \(\omega^\xi - \omega^\theta\) plane at the same time. As expected, the perturbed distribution is strongly localized around the resonance line with \(n = 6\) and \(p = -6\).
plane with $R > R_0$. The corresponding plots of $P_1 - \zeta$ are essentially Poincaré plots generated for $P_2 = \text{const}$ lines in figure 8. We focus on the particles with $P_2 = 39$. This value is chosen to present EPs with the most perturbed phase-space density (see figure 8). To improve numerical resolution, we record the particle data in the narrow interval $|P_2 - 39| < 0.2$.

The aforementioned Poincaré plots are shown in figure 9 at different stages of the TAE evolution. The colors in figures 9(a), (c), (e), (g) and (i) represent the perturbed weight/phase-space density of each particle. In the unperturbed state, each EP is assigned a color label according to its corresponding value of $P_1$ (see figure 9(b)). This label/color is kept the same throughout the simulations. Using this label, we produce a set of snapshots of the phase-space i.e. figures 9(b), (d), (f), (h) and (j) where the color bar denotes the particle label. The importance of these color labeled plots i.e. right panels of figure 9, in identifying whether the convective transport occurs can be explained as follows: in the left panels of figure 9, the color denotes the weight of the EPs related to the perturbed phase-space density. After the saturation of the wave, the plots demonstrate a group of detached perturbed particles at either side of the flattened area. One might imagine that the chirping waves cause local perturbations in phase-space and leave the particles behind and perturb another new set of particles. However, to reject this idea, we use the color labeled panels to show that the particles inside the chirping wave are the ones initially located around the linear resonance and are being carried by the BGK-type chirping waves in a moving phase-space bucket in a convective way.

Figures 9(a) and (b) correspond to the linear stages of the TAE excitation i.e. $t/t_\lambda = 205.4$. Figures 9(c) and (d) demonstrate the coarse graining of the distribution function around $P_1 \approx 19.5$ in phase-space just before the non-linear saturation of the TAE. Figures 9(e) and (f) demonstrate the phase-space dynamics during the frequency chirping of the wave at $t/t_\lambda = 2477.8$. At this point, the up-chirping and down-chirping waves have experienced a frequency sweep of 13.55% and 11.33%, respectively. We observe the holes (blue) and clumps (red) centered around $P_1 \approx 19.91$ and $P_1 \approx 18.95$, corresponding to the down-chirping and up-chirping waves, respectively. They form at either side of the flattened region and move in the phase-space of EPs as the frequencies chirp. The rest of the panels correspond to further evolution of the frequencies. It is worth mentioning that the dashed ovals in figure 9(g) mark the detachment of a second set of phase-space holes. We attribute these structures to the second branch of down-chirping waves illustrated in figure 4(b).

Since EPs remain on the same sub-layer of the phase-space, which on $P_2$ is a constant of motion, the constructed phase-space plots ascertain the mechanism under which the phase-space density is being perturbed. As figures 9(f), (h) and (j) clearly demonstrate, the phase-space islands act like buckets that carry particles in phase-space and lead to radial convection of the EPs.

Conservation of the generalised momentum $P_2$, given by (16), is the key part of this understanding. Although the constancy of $P_2$ is evident in phase-space plots of figure 9, we investigate the value of $P_2$ as a function of time for an EP which is transported by the up-chirping wave. This particle is denoted in figures 9(a), (c), (e), (g) and (i) by a purple circle. Simultaneously, we calculate the value of $C$, introduced in section 1, for the same EP. This comparison is depicted in figure 10 where the value of $C$, unlike $P_2$, changes as the mode frequency begins to chirp. It is worth noting that $P_2$ is comparable to $\omega_{\text{wave}}$ in terms of units. Hence, slices of $C = \text{const}$ do not represent the most appropriate sub-layers of the phase-space to study/observe the dynamics during the long range
Figure 9. A $\mu = 0$ and $P_2 \approx 39$ slice of the EPs phase-space as a function of EP weights (panels (a), (c) and (e)) and EPs color label (panels (b), (d) and (f)) at different stages of the wave evolution. The purple circle denotes a particle that is convected by the phase-space clump. A $\mu = 0$ and $P_2 \approx 39$ slice of the EPs phase-space as a function of EP weights (panels (g) and (i)) and EPs color label (panels (h) and (j)) at different stages of the wave evolution. The purple circle denotes a particle that is convected by the phase-space clump.
frequency chirping. Here, we explain why $C$ cannot be an appropriate constant during frequency chirping. In $(E, p_\varphi, n, t)$ coordinate, the $\varphi$ and $t$ dependency of the perturbation ($\delta H$) can be represented by $\delta H(n\varphi - \omega t)$. Then, Hamilton’s equations give

$$\dot{p}_\varphi = \frac{\partial \delta H}{\partial \varphi} = -n\delta H'$$  \hspace{1cm} (26a)

$$\dot{E} = \frac{\partial \delta H}{\partial t} = -\omega \delta H', \tag{26b}$$

A simple arrangement gives

$$\frac{1}{n} \frac{\Delta n}{\Delta t} = -\delta H' \tag{27a}$$

$$\dot{E} \frac{1}{\omega} = \delta H'. \tag{28}$$

This gives $\frac{\Delta n}{\Delta t} - \dot{E} \frac{1}{\omega} = 0$. Therefore, $\frac{\partial (n\varphi - \omega t)}{\partial t} = 0$. Hence, $C = \omega p_\varphi - nE$ is a constant of motion. Nevertheless, all the above calculations are based on a fixed frequency ($\omega$) and when $\omega(t)$, $\varphi(t)$ will not be zero anymore and hence the conventional constant of motion ($C$) for electromagnetic perturbations will not be an appropriate choice to analyze the phase-space of chirping waves.

5. Summary

We have refined the formalism for the phase-space analysis of the chirping modes driven by resonant energetic particles in a tokamak. As an application of this refinement, we analyze the results of self-consistent simulations performed with the MEGA code (an initial value problem solver in a hybrid MHD-kinetic model). The initial perturbation under study is a shear Alfvén eigenmode in the toroidicity-induced gap of the Alfvén continuum (TAE). The initial population of the energetic particles has an isotropic slowing down distribution. The EPs current provides a linear growth drive of $\gamma_\text{i}/\omega_{\text{TAE}} = 2.64\%$ to the mode in the presence of background dissipation at a rate of $\gamma_d/\gamma_\text{i} = 0.44$.

Subsequent to the non-linear saturation of the eigenmode, the sideband (secondary) oscillations appear inside the toroidicity gap. These modes evolve into chirping waves. In this case, we observe both up-ward and down-ward trends as the frequency chirps. We demonstrate that the rate of frequency sweeping increases with the damping rate of the eigenmode. As the chirping waves enter the shear Alfvén continuum, the radial structure of the perturbation experiences different frequencies at different radii. This is consistent with the theoretical model of reference [20].

A new conservation law is introduced which remains valid as the frequency of the wave chirps. This allows defining sub-layers of the EPs distribution function on which the particles are expected to remain even during the frequency chirping stage. Investigation of the energetic particle dynamics reveals that these particles lie on the same sub-layer of the phase-space throughout the simulations. Contingent on the formation and evolution of the chirping waves, phase-space islands form and evolve adiabatically. This means that the same particles are carried inside the coherent phase-space islands providing a convective or bucket transport in phase-space. Once formed in the gap, the phase-space holes and clumps survive even in the shear Alfvén continuum.

In fusion plasmas, Alfvénic chirping waves are commonly observed in the non-linear phase of wave-particle interaction. This work clearly demonstrates that apart from diffusive transport of particles due to overlap of multiple resonances, a single isolated resonance of TAEs can also lead to the transport of particles in phase-space through convection. Such convective transports demonstrated in this work will result in a change in the particles flux surface label i.e. an inward or outward drift of the particles. This can lead to the ejection of the particles from the hot core of the plasma and negatively influence the machine performance by degrading particle confinement.

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ORCID IDs

H. Hezaveh @ https://orcid.org/0000-0002-1164-4798
Y. Todo @ https://orcid.org/0000-0001-9323-8285
Z.S. Qu @ https://orcid.org/0000-0003-4628-6983
B.N. Breizman @ https://orcid.org/0000-0002-7908-6497
M.J. Hole @ https://orcid.org/0000-0002-9550-8776
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