Two Phase Transitions in Two-way Bootstrap Percolation

Ahad N. Zehmakan

1 ETH Zurich, Switzerland
abdolahad.noori@inf.ethz.ch

Abstract

Consider the $d$-dimensional torus $T^d_L$ and an initial random configuration, where each node is black with probability $p$ and white otherwise, independently. In discrete-time rounds, each node gets black if it has at least $r$ black neighbors and white otherwise, for some $1 \leq r \leq d$. This basic process is known as two-way $r$-bootstrap percolation. We prove the process exhibits a threshold behavior with two phase transitions. More precisely, for $p \ll p^{(1)}$ the process reaches fully white configuration, $p^{(1)} \ll p \ll p^{(2)}$ results in the stable coexistence of both colors, and $p^{(2)} \ll p$ outputs fully black configuration asymptotically almost surely.

Keywords and phrases bootstrap percolation, cellular automata, phase transition, $d$-dimensional torus, $r$-threshold model, biased majority.

Digital Object Identifier 10.4230/LIPIcs...
length of the cycle and number of rounds the process needs to reach it are respectively called the period and the consensus time of the process. In \( r\)-BP, the periodicity is always one and the consensus time is upper-bounded by \( |V| - 1 \), which is tight (for instance consider a path \( P_n \) and 1-BP, where initially all nodes are white except one of the leaves). However, the situation in two-way \( r\)-BP is a bit more complicated. Goles and Olivos \[25\] proved the periodicity of the two-way \( r\)-BP is always one or two. Regarding the consensus time, Fogelman, Goles, and Weisbuch \[19\] proved the upper-bound of \( \mathcal{O}(|E|) \), which is asymptotically tight (for instance, consider a cycle \( C_n \) which is fully white except two adjacent black nodes for \( r = 1 \)).

Another interesting question concerning the behavior of these models is: what is the minimum \( p \) for which the process gets fully black with a probability close to one? This question has been studied on different classes of graphs like hypercube \[5\], the binomial random graph \[30\] \[13\], random regular graphs \[9\] \[24\] \[30\], infinite trees \[32\], and many others. A substantial amount of attention has been devoted when the underlying graph is the \( d \)-dimensional torus, motivated from the literature of statistical physics due to the study of the behavior of certain interacting particle systems like fluid flow in rocks \[1\] and dynamics of glasses \[21\], and also motivated from the literature of cellular automata to model different biological interactions \[34\]. The \( d \)-dimensional torus \( T^d_L \) is the graph with node set \([L]^d\), where two vertices are adjacent if and only if they differ by 1 \( \mod L \) in exactly one coordinate. Notice we always assume \( d \) is a constant while let \( L \) tend to infinity.

To address the above question regarding the minimum \( p \) for black color to take over fully in \( T^d_L \) and \( r\)-BP, firstly Aizenman and Lebowitz \[2\] studied the case of \( r = 2 \) and proved actually the process exhibits a weak threshold behavior. Weak threshold means there exits a threshold value \( p^* \) such that for \( p \geq p^* \) the process gets fully black and for \( p \leq p^* \) it does not asymptotically almost surely \[7\] where we shortly write \( f \approx g \) instead of \( f = o(g) \) for two functions \( f, g \). Cerf and Cirillo \[10\] made one step further by proving the aforementioned weak threshold behavior for \( d = r = 3 \). Finally, Cerf and Manzo \[11\] extended this result to all dimensions, building on the work by Schonmann \[10\]. Later on, it was proven that the process actually exhibits a sharp threshold behavior for \( d = r = 2 \) by Holroyd \[29\], that is the process a.a.s. gets fully black for \( p \geq (1 + \epsilon)p^* \) and does not for \( p \leq (1 - \epsilon)p^* \) and any constant \( \epsilon > 0 \). This sharp threshold behavior was extended to \( d = r = 3 \) by Balogh, Bollobas, and Morris \[7\], and finally to all dimensions by Balogh, Bollobas, Duminil-Copin, and Morris \[3\]. Along the way, as an intermediate step the behavior of a similar process which assumed to be easier to handle was also studied. In modified \( r\)-bootstrap percolation on \( T^d_L \), by starting from a random initial configuration in every round each white node gets black if it has black neighbor in at least \( r \) distinct dimensions and black nodes stay unchanged. See \[23\] \[28\] by Holroyd regarding the sharp threshold behavior of modified \( r\)-BP for \( r = d \).

In contrast to \( r\)-BP, the behavior of two-way \( r\)-BP on \( T^d_L \) has remained unknown, despite several attempts \[30\] \[3\] \[14\] \[34\]. Intuitively speaking, the main difficulty regarding the analysis of two-way \( r\)-BP is that it is not monotone, unlike \( r\)-BP. The first rigorous result was provided by Schonmann \[39\], where by applying the results from \[2\] he proved if \( d = r = 2 \), for \( p \gg 1/\log L \) the process gets fully black and it does not for \( p \ll 1/\log L \) a.a.s., i.e. it exhibits a weak threshold behavior. Balister, Bollobas, Johnson, and Walters \[3\] tried to shed some light on the behavior of the process by finding lower and upper bounds on the minimum number of black nodes needed initially to make the whole torus black eventually, for \( r = d \). Coker and Gunderson \[14\] proved a phase transition behavior at \( 1/\sqrt{\log L} \) similar.

---

1 For a graph \( G = (V, E) \), we say an event happens asymptotically almost surely (a.a.s.) if its probability is at least \( 1 - o(1) \) as a function of \( |V| \).
to the one by Schonmann from above on the two-dimensional torus and a slightly different variant of two-way $r$-BP.

By providing several new techniques, using some ideas inspired from [39, 22] (where they covered the special case of $r = d = 2$), and applying some prior results regarding $r$-BP and modified $r$-BP [11, 6, 28], we extend the above threshold behavior in two-way $r$-BP to all dimensions.

One might relax the above question a bit and asks what is the minimum $p$ for which black color survives (but does not make the whole torus black necessarily). In $r$-BP and $T_d^L$ as will be discussed, it is easy to show that a.a.s. for $p \gg 1/L$ black color survives and it does not for $p \ll 1/L$. For two-way $r$-BP, the situation is more involved. Relying on some new techniques, we prove a similar threshold behavior in the two-way setting, which leads into some interesting insights regarding the behavior of the process.

All in all, we prove two-way $r$-BP on the $d$-dimensional torus $T_d^L$ exhibits two phase transitions. More precisely, asymptotically almost surely

(i) $p \ll L^{-\frac{d}{d+r-1}}$ outputs fully white configuration (phase 1)
(ii) $L^{-\frac{d}{d+r-1}} \ll p \ll (\log_{(r-1)} L)^{-\frac{d}{d+r-1}}$ outputs stable coexistence of both colors (phase 2)
(iii) $(\log_{(r-1)} L)^{-\frac{d}{d+r-1}} \ll p \ll (\log_{(r-1)} L)^{-\frac{d-r+1}{d+r-1}}$ results in fully black configuration (phase 3)

where $\log_r L := \log \log_{(r-1)} L$ for $r \geq 1$ and $\log_0 L = L$.

So far we considered the random setting, but one might approach these processes from an extremal point of view and asks for the minimum number of nodes which must be black initially to guarantee that the process gets fully black (it relates to the concept of dynamo which we introduce formally in the next section). Although this problem was studied earlier, e.g. by Schonmann [10] and Balogh and Pete [8], it was formally defined and studied in the seminal work by Kempe, Kleinberg, and Tardos [33] and independently by Peleg [37]. There is a massive body of work concerning this problem in different classes of graphs, for instance hypercube [27, 35], the binomial random graph [12, 17, 41], random regular graphs [24], power-law graphs [16], planar graphs [38], and many others. Again, a considerable amount of effort has been associated to the $d$-dimensional torus, e.g. see the results by Balogh and Pete [8], Flocchini, Lodi, Luccio, Pagli, and Santoro [18], and also [3, 35, 23, 27], and finally very recent results by Jeger and Zehmakan [31], which provide a complete picture of the minimum size of a dynamo in the $d$-dimensional torus.

The outline of the paper is as follows. After setting up some basic definitions in Section 1.1, we provide some insights and the main ideas behind our proof techniques in Section 1.2. Finally in Section 2 our main results regarding two phase transitions of two-way $r$-BP on the $d$-dimensional torus are provided.

1.1 Definitions and Preliminaries

Let for a graph $G = (V, E)$ and a node $v \in V$, the neighborhood of $v$ be $N(v) := \{u \in V : \{v, u\} \in E\}$. For a set $S \subseteq V$ we have $N_S(v) := N(v) \cap S$ and $N(S) := \bigcup_{v \in S} N(v)$. Furthermore, for two nodes $v, u \in V$ we define the distance $d(v, u)$ to be the length of the
shortest path between \( v \) and \( u \), in terms of the number of edges. Let \( G^2 \) be the second power of graph \( G = (V, E) \), where two nodes are adjacent if their distance in \( G \) is at most 2. Two nodes \( v, u \in V \) are called semi-connected if they are connected in \( G^2 \).

Formally, a configuration is a function \( C : V \to \{b, w\} \), where \( b, w \) stands for black and white. For a configuration \( C \), node \( v \in V \), and color \( c \in \{b, w\} \), we define \( N^C(v) := \{u \in N(v) : C(u) = c\} \) which is the set of neighbors of \( v \) which have color \( c \) in configuration \( C \). Finally, for a color \( c \in \{b, w\} \) and a set \( S \subseteq V \), we write \( C|_S = c \) if \( \forall v \in S, C(v) = c \).

Given a graph \( G = (V, E) \) and an initial random configuration \( C_0 \), where each node is black with probability \( p \) and white otherwise independently, in discrete-time rounds, simultaneously each node selects black color if it has at least \( r \) black neighbors and white otherwise, for some \( r > 0 \). More formally, \( C_t(v) = b \) if \( |N^C(v)| \geq r \) and \( C_t(v) = w \) otherwise for \( t \geq 1 \), where \( C_t \) is the \( t \)-th configuration. This model is called two-way \( r \)-bootstrap percolation. If in the same setting, we require that a black node stays black forever, the model is called \( r \)-bootstrap percolation.

Another similar and well-studied model in this setting is the majority model, where in each round all nodes update their color to the most frequent color in their neighborhood, and in case of a tie, a node keeps its current color. It is easy to see for \( \Delta \)-regular graphs with odd \( \Delta \), the majority model is the same as two-way \( r \)-BP for \( r = \lfloor \Delta/2 \rfloor + 1 \).

For a graph \( G \) and two configurations \( C \) and \( C' \) we write \( C \leq C' \) if all black nodes in \( C \) are also black in \( C' \). A model \( M_1 \) is stronger than model \( M_2 \) if for any graph \( G \) and any configuration \( C \), we have \( C_0 \leq C_1 \) where \( C_1 \) and \( C_2 \) are configurations obtained after one round of respectively \( M_1 \) and \( M_2 \) from \( C \). For instance, \( r \)-BP is stronger than two-way \( r \)-BP. Furthermore, \( M \) is a monotone model if for any graph \( G \) and any two configurations \( C_1 \leq C_2 \), we have \( C_1' \leq C_2' \) where \( C_1' \) and \( C_2' \) are respectively the configurations obtained from \( C_1 \) and \( C_2 \) after one round of \( M \). All models introduced in this paper are monotone.

For any process, like (two-way) \( r \)-BP or the majority model, and graph \( G = (V, E) \) a set \( S \subseteq V \) is called a \( c \)-stable set for \( c \in \{b, w\} \) whenever the following holds: if all nodes in \( S \) share color \( c \) in some configuration during the process, then they will all keep it in all upcoming configurations. Furthermore, a set \( S \subseteq V \) is \( c \)-eternal for \( c \in \{b, w\} \) means if all nodes in \( S \) have color \( c \) in some configuration, then color \( c \) survives, that is for any upcoming configuration there is a node which has color \( c \). Clearly, a \( c \)-stable set is also a \( c \)-eternal set, but not necessarily the other way around. Furthermore, a \( c \)-dynamo for \( c \in \{b, w\} \) is a subset of nodes which “take over” if they share color \( c \), meaning the whole graph will have color \( c \) after some rounds. For example in any connected graph and two-way 1-BP, any two adjacent nodes are a \( b \)-dynamo. Notice one node might not be enough, for instance in an even cycle. For a node set \( S \) which is \( c \)-stable, \( c \)-eternal, or a \( c \)-dynamo in two-way \( r \)-BP, we say \( S \) is \((r, c)\)-stable, \((r, c)\)-eternal, and \((r, c)\)-dynamo, respectively. Clearly, in an \((r, b)\)-stable set (analogously \((r, w)\)-stable set) for each node \( v \in S \), \(|N_S(v)| \geq r \) (resp. \(|N_{V \setminus S}(v)| < r \)).

The \( d \)-dimensional torus \( T^d_L \) is the graph with the node set \( V = \{(i_1, \cdots, i_d) : 1 \leq i_1, \cdots, i_d \leq L\} \) and the edge set \( E = \{(i, i') : |i_j - i'_j| \text{ mod } L = 1 \text{ for some } j \text{ and } i_k = i'_k \forall k \neq j\} \), where mod \( L \) means that the sum is reduced modulo \( L \). To lighten the notation, we skip the modulo \( L \) operation in the rest of the paper, when it is clear from the context. Notice each node in \( T^d_L \) has \( 2d \) neighbors, two neighbors in each dimension. For a node \( v = (i_1, \cdots, i_d) \) and \( 1 \leq j \leq d \), we call \((i_1, \cdots, i_j + 1, \cdots, i_d)\) and \((i_1, \cdots, i_j - 1, \cdots, i_d)\) the neighbors of \( v \) in the \( j \)-th dimension.

A hyper-rectangle of size \( l_1 \times \cdots \times l_d \) starting from node \((i_1, \cdots, i_d)\) is the node set \( \{(i'_1, \cdots, i'_d) : i_j \leq i'_j \leq i_j + l_j \forall 1 \leq j \leq d\} \). An \( r \)-dimensional hyper-square \( HS \) starting at node \( i \) is the hyper-rectangle starting at \( i \) such that \( l_j \)'s are one in exactly \( r \) dimensions and
zero in others, where we define $J_{HS} := \{ j : l_j \neq 0 \}$. We denote the odd-part (analogously even-part) of $HS$ by $HS^{(1)}$ (resp. $HS^{(2)}$), which are the nodes that differ in odd (resp. even) number of coordinates with $i$. Now, as a warm-up let us make the following simple, however crucial, observation.

Lemma 1. For an $r$-dimensional hyper-square $HS$ in $\mathbb{T}_L^r = (V, E)$ and two-way $r$-BP, $HS^{(1)}$ and $HS^{(2)}$ are $(r, b)$-eternal sets.

Proof. It suffices to show each node in $HS^{(1)}$ has exactly $r$ neighbors in $HS^{(2)}$ and vice versa because it implies that if $C_{t}^{i}_{HS^{(1)}} = b$, we will have $C_{t+2t'+1}^{i}_{HS^{(2)}} = b$ and $C_{t+2t'}^{i}_{HS^{(1)}} = b$ for any $t' \geq 0$ (a similar argument for $C_{t}^{i}_{HS^{(2)}} = b$). Let $i'$ be a node in $HS^{(1)}$ and assume $HS$ starts at $i$. There is an odd-size subset $J \subseteq J_{HS}$ of coordinates in which $i'$ is larger than $i$ by one. Now, by decrementing any coordinate in $J$ or incrementing any coordinate in $J_{HS} \setminus J$, we reach a node in $HS^{(2)}$ which is a neighbor of $i'$. Thus, $i'$ has $r$ neighbors in $HS^{(2)}$. The proof of the other direction is analogous. See Figure 2 for an example.

![Figure 2](left) A $(2, b)$-eternal set (right) a $(3, b)$-eternal set.

Notice the even/odd-part of an $r$-dimensional hyper-square is of size $2^{r-1}$ which implies there exists an $(r, b)$-eternal set of size $2^{r-1}$. Furthermore, we always assume $2 \leq r \leq d$. The case of $d + 1 \leq r \leq 2d$ is the same as $1 \leq r \leq d$ by swapping the black and white color. The case of $r = 1$ is trivial in the sense that any two black nodes take over the whole torus.

In the $r$-dimensional torus $\mathbb{T}_L^r$, the set of $r$-dimensional hyper-squares whose starting node is in $\{(2i_1 - 1, \ldots , 2i_r - 1) : 1 \leq i_1, \ldots , i_r \leq \lfloor L/2 \rfloor \}$ divide the node set (except the nodes with value $L$ in one of their coordinates when $L$ is odd) into $\lfloor L/2 \rfloor^r$ pair-wise disjoint hyper-squares. Furthermore, if we divide the nodes in the $d$-dimensional torus $\mathbb{T}_L^d$ into $L^{d-r}$ pair-wise disjoint subsets according to their last $d - r$ coordinates, the induced subgraph by each of these subsets is an $r$-dimensional torus. Now, if we partition the node set of each of these $r$-dimensional tori into $\lfloor L/2 \rfloor^r$ hyper-squares as above, we will have $L^{d-r}[L/2]^r = \Theta(L^d)$ pair-wise disjoint $r$-dimensional hyper-squares. We call this procedure the tiling of $\mathbb{T}_L^d$ into $r$-dimensional hyper-squares.

1.2 Proof Techniques and Some Insights

Intuitively speaking, one might expect any monotone model to exhibit some sort of threshold behavior with two phase transitions. Assume the initial probability $p$ is very close to zero, then black nodes probably disappear in a few number of rounds, but if we gradually increase the initial probability, at some point it would suffice to guarantee their survival, however probably it is not high enough to let them take over the whole graph. Finally, if we keep increasing the initial probability, suddenly it should be sufficient to guarantee not only the survival of black color but also the disappearance of white color. Another way of seeing these two phase transitions is in terms of the $b$-eternal set and $b$-dynamo. One might think of the first threshold as the threshold value for having a fully black $b$-eternal set and the second one as the threshold for having a fully black $b$-dynamo since black color survives if and only if there is a black $b$-eternal set initially and it will take over if and only if there is a black $b$-dynamo in the initial configuration. Notice the first and the second threshold
values might match, which means the process goes actually through one phase transition; for instance in 1-BP the existence of a black node is the necessary condition for survival of black color and at the same time sufficient condition to take over the whole graph; i.e., any $b$-eternal set is also a $b$-dynamo. Although this threshold behavior might seem conceptually simple, determining the exact threshold values and their sharpness is typically very hard; as we discussed even for the very special case of $r$-BP on $T_2^d$ the answer was known after a large series of papers over more than three decades.

Following our intuitive argument from above, in $r$-BP and $T_2^d$ for $p \ll p_1$ the process gets fully white, $p_1 \ll p \ll p_2$ results in the stable coexistence of both colors, and $p_2 \ll p$ outputs fully black configuration a.a.s. where $p_1 := L^{-d}$ and $p_2 := (\log((r-1)L)^{-d-r+1})$. The first transition has not been considered before, but it is very easy to handle. For $p \ll p_1$, by a simple union bound the probability that there exists a black node in the initial configuration is upper-bounded by $L^d o(p_1) = o(1)$, which implies a.a.s. the initial configuration is already fully white. For $p \gg p_1$, the expected number of black nodes in the initial configuration is equal to $L^d \omega(p_1) = \omega(1)$; applying Chernoff bound [15] yields a.a.s. there exists at least a black node initially, which guarantees the survival of black color. As argued before, by prior results we know that actually the second phase transition exhibits a sharp threshold behavior, meaning if $p \geq (1 + \epsilon)\lambda p_2$ (analogously $p \leq (1 - \epsilon)\lambda p_2$) for some fixed constant $\lambda > 0$ the process gets (resp. does not get) fully black a.a.s. for any constant $\epsilon > 0$. However, the first threshold cannot be sharp because for $p = c' p_1$, black color survives for some non-zero constant probability for any constant $c' > 0$ (which implies there is no constant $c''$ such that $p \ll c'' p_1$ results in fully white configuration a.a.s.). The probability that all nodes are white initially is $(1 - p)^{L^d}$ which is smaller than $e^{-p L^d} = e^{-c'}$ for $p = c' p_1$. This probability is a constant smaller than one.

Let us get back to two-way $r$-BP and recall that in Lemma 1 we argued there is an $(r,b)$-eternal set of size $2^{r-1}$; in Lemma 2 we will show that actually there is no smaller $(r,b)$-eternal set. Therefore, by switching from $r$-BP to two-way $r$-BP, the minimum size of a $b$-eternal set increases from 1 to $2^{r-1}$. Recall we claimed the threshold values for two-way variant are $\epsilon^{-1} \sqrt{\Pi}$ and $\epsilon^{-1} \sqrt{\bar{\Pi}}$; this might now explain where is the origin of the exponent $1/2^{r-1}$. In modified $r$-BP and $T_2^d$, the first phase transition happens at $p_1$ similar to $r$-BP because a single black node suffices for the survival of black color and based on [23] the second phase transition occurs at $p_2$. Notice for $r$-BP, two-way $r$-BP, and modified $r$-BP the two transitions happen at $\sqrt{\Pi}$ and $\sqrt{\bar{\Pi}}$, where $s$ is the minimum size of a $b$-eternal set in the corresponding model. Can this statement be generalized to capture a bigger class of models? We do not know the answer, however it would be very interesting to characterize such class of models. Let us discuss an interesting example which also follows this pattern. As an intermediate step from the analysis of $r$-BP to the analysis of two-way $r$-BP, Coker and Gunderson [14] studied the following variant on $T_2^d$, which is called 2-bootstrap percolation with recovery. In this model, a white node gets black if it has at least two black neighbors, but a black node stays black, except all its four neighbors are white. Coker and Gunderson [14] proved that it exhibits two phase transitions respectively in $\sqrt{\Pi}$ and $\sqrt{\bar{\Pi}}$. Notice this is consistent with the above observation because in 2-BP with recovery the smallest $b$-eternal set is of size two. Clearly one black node disappears in one round but two adjacent black nodes survive forever.

Now, we discuss the high-level ideas of the proof techniques applied. In phase one, we want to show for $p \ll \epsilon^{-1} \sqrt{\Pi}$, black color disappears a.a.s. We exploit a technique which we call clustering, where roughly speaking we show $p$ is so small that a.a.s. one can cluster all black nodes in the initial configuration in small clusters which are far from each other.
This distance lets us treat each cluster independently since there is no interaction among them. Furthermore, the number of black nodes in each cluster is less than $2^{r-1}$, that is the minimum size of an $(r, b)$- eternal set, which then results in the disappearance of black color.

For the second phase, we must show both colors survive a.a.s. For black color, since there are $\Theta(L^d)$ pair-wise disjoint $(r, b)$- eternal sets of size $2^{r-1}$, namely the even-part of $\Theta(L^d)$ pair-wise disjoint $r$- dimensional hyper-squares, applying Chernoff bound implies there is a black $(r, b)$- eternal set initially a.a.s. This guarantees the survival of black color. We also need to show for $p \ll \sqrt{\frac{1}{2}}$, white color survives a.a.s. For that, we rely on the threshold behavior of r-BP. More precisely, applying the fact that the minimum size of an eternal set is equal to $2^{r-1}$, we show the probability that an arbitrary node is black after $t^*$ rounds, for some constant $t^*$, is $o(p_2)$. Using the fact that the stronger model of r-BP a.a.s. results in the survival of white color in this case, we discuss that it is also the case for two-way r-BP.

Regarding phase 3, we want to show for $\sqrt{\frac{r-1}{r}} \ll p$, the process a.a.s. gets fully black. We utilize a method, which we call scaling. The idea is to tile the torus $T^d_L$ into $r$- dimensional hyper-squares and treat each of the hyper-squares as a single node. Let us say two hyper-squares are neighbors if there is at least one edge between them, then each hyper-square has $2d$ neighbors, two in each dimension. Furthermore, we say a hyper-square is occupied in configuration $C_t$ for even $t$ (analogously odd $t$) if its even-part (resp. odd-part) is black. We prove if in some configuration, a hyper-square has occupied neighbors in $r$ distinct dimensions, then it gets occupied in constantly many rounds. Furthermore, each hyper-square is occupied initially with probability $p^{2^{r-1}} \gg p_2$. Therefore, we can see the process scaled to the hyper-squares is at least as strong as modified r-BP, where initially each hyper-square is occupied with probability $\omega(p_2)$. We know modified r-BP for initial probability $\omega(p_2)$ results in fully black configuration a.a.s. This implies two-way r-BP on $T^d_L$ reaches a configuration where the even-part of each of the hyper-squares is black a.a.s. We can do the same argument by switching the terms of odd and even in the definition of occupation, and then by a union bound, a.a.s. the process gets fully black. However, in this argument, in addition to several technical details, we were hiding two crucial points. Firstly, the fact that in modified r-BP the initial probability $\omega(p_2)$ results in fully black configuration a.a.s. is known only for $r = d$ [25]; however, we show the upper-bound proof by Cerf and Manzo [11] regarding r-BP can be easily adapted to prove our desired upper bound for modified r-BP. Furthermore, our definition for occupation works only for $d = r$, and for $r < d$ we need to redefine it in a slightly more complicated way, where the occupation of a hyper-square is defined based on its starting node.

Notice our results demonstrate the weak threshold behavior of two-way r-BP. For the first transition, this is the best that one can hope for because there is no sharp threshold. One can prove for any constant $c' > 0$ and $p = c' \sqrt[2]{\frac{1}{2}}$, black color has a constant non-zero probability to survive. The proof idea is very simple and similar to one from r-BP, which was argued above. There are $L^{d/c}$ pair-wise-disjoint $r$- dimensional hyper-squares in $T^d_L$ for some constant $c > 0$ and based on Lemma [1] the even-part of an $r$- dimensional hyper-square is an $(r, b)$- eternal set. The probability that the even-part of at least one of these disjoint hyper-square is black initially is lower-bounded by $1 - (1 - p^{2^{r-1}})^{L^d/c}$. By applying the estimate $1 - x \leq e^{-x}$, this probability is a constant larger than zero.

Let us finish this section, by mentioning a very interesting observation. Two-way r-BP on $T^d_L$ for $r = d$ is sometimes called the biased majority model because each node selects the most frequent color in its neighborhood and in case of a tie, it chooses black (notice each node has degree $2d$). Therefore, the majority model on $T^d_L$ is the same as the biased variant except in case of a tie, a node keeps its color. We claim in the majority model if $p \leq 1 - \delta$
Two Phase Transitions in Two-way Bootstrap Percolation

for any arbitrary constant $\delta > 0$, then black color does not take over fully a.a.s. (we provide the proof in the extended version of the paper). On the other hand as we discussed, in the biased model $p \gg \frac{\delta}{\sqrt{d}}$, consequently $p \geq \delta$ for an arbitrarily small constant $\delta > 0$, results in fully black configuration a.a.s. Putting these two results in parallel, we observe in the majority model $p$ should be very close to 1 to have a high chance of final complete occupancy by black, but by just changing the tie-breaking rule in favor of black, the process ends up in fully black configuration a.a.s. even for initial probability very close to 0. This comparison illustrates how small alternations in local behavior, like changing the tie-breaking rule, can result in considerable changes in global behavior.

2 Two Phase Transitions

2.1 Phase 1

The idea of the proof is to show $p$ is so small that a.a.s. black nodes in $C_0$ are contained in a group of hyper-rectangles which are sufficiently far from each other and each hyper-rectangle includes less than $2^{-r}$ black nodes. We argue since the hyper-rectangles are far from each other, the nodes out of the hyper-rectangles, which are all white initially, stay white forever (i.e., create a white $(r,w)$-stable set). Furthermore, the black nodes inside each hyper-rectangle die out after some rounds because they are less than the size of the smallest $(r,b)$-eternal set. We prove our claim in Theorem 3 building on Lemma 2 (whose proof will be presented in the extended version of the paper). The proof of Lemma 2 is built on two basic propositions. Firstly, we show a non-empty $(r,b)$-stable set intersects at least $2^{-r}$ pair-wise disjoint $(r,w)$-stable sets. Furthermore, based on [25] we know that two-way $r$-BP always reaches a cycle of configurations of length one or two.

Lemma 2. In $\mathbb{Z}^d_L$ and two-way $r$-BP, a configuration with less than $2^{-1}$ black nodes gets fully white in a constant number of rounds.

Theorem 3. Two-way $r$-BP with $p \ll L^{-d/2^{-r}}$ on $\mathbb{Z}^d_L$ gets fully white a.a.s.

Proof. Let the distance between two hyper-rectangles $HR_1$ and $HR_2$ be $d(HR_1, HR_2) := \min_{v \in HR_1, u \in HR_2} d(v, u)$. We show for the initial configuration a.a.s. there is a set of hyper-rectangles which are pair-wise in distance at least three from each other and any black node belongs to one of these hyper-rectangles and the number of black nodes in each hyper-rectangle is less than $2^{-r}$. Each node which is not in any of the hyper-rectangles is adjacent to at most one of them (otherwise, there are two hyper-rectangles whose distance is less than three). Thus, each of these nodes has at least $2d - 1$ white neighbors which implies they all stay white forever. On the other hand, in each of these isolated hyper-rectangles there are less than $2^{-r}$ black nodes which die out after some rounds by Lemma 2.

It remains to prove such a set of hyper-rectangles a.a.s. exist. For each black connected component in $C_0$, consider the smallest hyper-rectangle which includes all its node. Let $A_0$ be the set of these (not necessarily disjoint) hyper-rectangles. There is no black connected component of size $2^{-r}$ or larger in $C_0$ a.a.s. Let $X$ denote the number of black connected subgraphs of size $2^{-r}$ in $C_0$. The number of connected subgraphs of size $2^{-r}$ which include an arbitrary node $v$ is a constant (notice $d$, thus also $r$, is fixed), then the number of connected subgraphs of size $2^{-r}$ is of order $\Theta(L^d)$. Then, $E[X] = \Theta(L^d)p^{2^{-r}} = \Theta(L^d)\delta(L^{-d}) = o(1)$. By Markov’s inequality \[15\] a.a.s. there is no black connected subgraph of size $2^{-r}$, which implies there is no black connected component of this size or larger. Therefore, for any hyper-rectangle of size $l_1 \times \cdots \times l_d$ in $A_0$, $l_j < 2^{-r}$ for all $1 \leq j \leq d$ a.a.s.
Consider the following procedure. By starting from $A = A_0$, in each iteration if all hyper-rectangles in $A$ are pair-wise in distance at least three from each other, the procedure is over, otherwise there are two hyper-rectangles $HR_1, HR_2 \in A$ such that $d(HR_1, HR_2) \leq 2$. In this case, we set $A = A \setminus \{HR_1, HR_2\} \cup \{HR\}$, where $HR$ is the smallest hyper-rectangle which includes all black nodes in both $HR_1$ and $HR_2$. See Figure 3 (a)(b) for an example, where the boundaries of the smallest hyper-rectangles are distinguished by green. The process definitely terminates, because in each round $|A|$ decreases and when the process is over, the hyper-rectangles in $A$ satisfy our desired distance property. We still have to show each of them contains less than $2^{r-1}$ black nodes. Let us make the three following observations.

\begin{align*}
(1) & \quad \text{Let } HR \text{ of size } l_1 \times \cdots \times l_d \text{ be the smallest hyper-rectangle which contains all black nodes in both } HR_1 \text{ and } HR_2 \text{ respectively of size } l_1^{(1)} \times \cdots \times l_d^{(1)} \text{ and } l_1^{(2)} \times \cdots \times l_d^{(2)} \text{ in the above procedure, we have } l_j \leq 3 \max_{i \in \{1,2\}}^{l_j^{(i)}} \text{ for all } 1 \leq j \leq d \text{ because } d(HR_1, HR_2) \leq 2. \\
(2) & \quad \text{Assume hyper-rectangle } HR \text{ of size } l_1 \times \cdots \times l_d \text{ starting in } (i_1, \cdots, i_d) \text{ is in } A, \text{ at some iteration in the above procedure, then it contains at least } \max_{1 \leq j \leq d} l_j/2 \text{ black nodes. Intuitively speaking, this is true since initially each hyper-rectangle in } A = A_0 \text{ is the smallest rectangle which contains all nodes in a connected component and in each iteration we combine two hyper-rectangle whose distance is at most two. However, for a formal proof let us first show for any two black nodes } v, u \text{ in a hyper-rectangle } HR \text{ in } A, \text{ there is a semi-connected path between these two nodes which includes at least } l_j/2 \text{ black nodes. Consider the semi-connected path between these two nodes which includes at least } l_j/2 \text{ black nodes.}

(3) & \quad \text{W.h.p. in } C_4 \text{ there is no hyper-rectangle } HR \text{ of size } l_1 \times \cdots \times l_d \text{ in } T_d^d \text{ which includes at least } 2^{r-1} \text{ black nodes and } l_j < 6 \cdot 2^{r-1} \text{ for all } 1 \leq j \leq d. \text{ Let random variable } X \text{ denote the number of such hyper-rectangles. The number of hyper-rectangles of the aforementioned sizes starting from a fixed node } i \text{ is bounded by constant } K = (6 \cdot 2^{r-1})^d, \text{ which implies there are at most } KL^d \text{ hyper-rectangles of such sizes. Thus, } E[X] \leq KL^d(2K)^p2^{r-1} = o(1) \text{ for } p = o(L^{-\frac{d}{r^2}}), \text{ which implies } X = 0 \text{ a.a.s. by applying Markov’s inequality.}

\text{At the beginning of the proof we showed all the sides of any hyper-rectangle in } A_0 \text{ are smaller than } 2^{r-1} \text{ a.a.s. Putting this fact in parallel with (a), we conclude if the process does not terminate, the number of black nodes in each iteration is smaller than } 2^{r-1}. \text{ Hence, the number of iterations is at most } 2^{2r-2}, \text{ which is a contradiction; hence the process terminates.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure3.png}
\caption{(a) the smallest hyper-rectangles (b) after two iterations (c) a linear-time example for $r = d = 2$ (d) the inner and outer neighbors.}
\end{figure}
Two Phase Transitions in Two-way Bootstrap Percolation

not terminate while all the sides of any hyper-rectangle in \( \mathcal{A} \) are smaller than or equal to \( 2 \cdot 2^{-r-1} \), then it generates a hyper-rectangle \( HR' \) of size \( l'_1 \times \cdots \times l'_d \) such that \( \forall 1 \leq j \leq d, l'_j \leq 6 \cdot 2^{-r-1} \) and there exists \( 1 \leq j' \leq d \) such that \( 2 \cdot 2^{-r-1} < l'_{j'} \). Furthermore, based on (b) \( HR' \) must include at least \( l'_{j'}/2 \geq 2^{-r-1} \) black nodes; however, based on (c) such an \( HR' \) does not exist a.a.s. Therefore a.a.s. the process terminates while all the sides of any hyper-rectangle in \( \mathcal{A} \) are upper-bounded by \( 2 \cdot 2^{-r-1} \) a.a.s. By applying (c) another time, none of the hyper-rectangles includes \( 2^{-r-1} \) or more black nodes a.a.s. ▶

2.2 Phase 2

In Theorem 5, we prove phase 2, where we exploit some prior result regarding \( r \)-BP [11, 6, 12, 2] in one part of the proof.

**Theorem 4.** \([11, 6, 12, 2]\) In \( \mathbb{T}^d_L \) and \( r \)-BP, for \( p \ll (\log_{(r-1)} L)^{-(d-r+1)} \) white color will survive forever a.a.s.

**Theorem 5.** Two-way \( r \)-BP with \( L^{-\frac{d}{2-\alpha}} \ll p \ll (\log_{(r-1)} L)^{-(d-r+1)} \) on \( \mathbb{T}^d_L = (V, E) \) results in the stable coexistence of both colors a.a.s.

**Proof.** Let us first show that black color a.a.s. will survive for \( L^{-\frac{d}{2-\alpha}} \ll p \). As argued, in \( \mathbb{T}^d_L \) there are \( L^d/c \) pair-wise disjoint \( r \)-dimensional hyper-squares, for a constant \( c \simeq 2^r \). Consider an arbitrary labeling from 1 to \( L^d/c \) on these hyper-squares and define Bernoulli random variable \( x_k \) for \( 1 \leq k \leq L^d/c \) to be one if the even-part of \( k \)-th hyper-square is fully black in \( C_0 \) and let \( X := \sum_{k=1}^{L^d/c} x_k \). We show \( X \neq 0 \) a.a.s., which implies there is a hyper-square whose even-part is fully black initially and since the even-part of an \( r \)-dimensional hyper-square is an \((r, b)\)-eternal set (see Lemma 4), it guarantees the survival of black color. Since the even-part of an \( r \)-dimensional hyper-square is of size \( 2^{r-1} \), we have \( \mathbb{E}[X] = \frac{L^d}{c} p^{2^{r-1}} = \frac{L^d}{c} \omega(\frac{1}{L^d}) = \omega(1) \). Therefore, by using the fact that \( X \) is the sum of independent Bernoulli random variables and applying Chernoff bound, we have \( \mathbb{P}[X = 0] \leq e^{-\omega(1)} = o(1) \).

Now, we argue that for \( p \ll (\log_{(r-1)} L)^{-(d-r+1)} \) white color survives a.a.s. Based on Lemma 2, there is a constant \( t_c \) so that by starting from an initial configuration with less than \( 2^{-r-1} \) black nodes, we have no black nodes after \( t_c \) rounds. We claim this implies for an arbitrary node \( v \) to be black in round \( t_c \), it needs at least \( 2^{r-1} \) black nodes in its \( t_c \)-neighborhood (i.e. nodes in distance at most \( t_c \) from \( v \)) in the initial configuration. For the sake of contradiction, assume there is an initial configuration \( C_0 \) in which \( v \) has less than \( 2^{r-1} \) black nodes in its \( t_c \)-neighborhood and it is black in round \( t_c \). Then, we consider the initial configuration \( C'_0 \) in which all nodes in \( v \)'s \( t_c \)-neighborhood have the same color as \( C_0 \) and all others are white. \( C'_0 \) has less than \( 2^{r-1} \) black nodes. Furthermore, \( v \) must be black after \( t_c \) rounds by starting from \( C'_0 \) because the color of \( v \) in round \( t_c \) is only a function of the initial color of nodes in its \( t_c \)-neighborhood (this is easy to see; however, for a formal proof one can simply apply induction) and the color of all nodes in the \( t_c \)-neighborhood of \( v \) is the same as \( C_0 \). However, this is in contradiction with Lemma 2.

So far, we know for a node \( v \) to be black in round \( t_c \), it needs at least \( 2^{r-1} \) black nodes in its \( t_c \)-neighborhood initially. This immediately implies for an arbitrary node, the probability of being black in round \( t_c \) is upper-bounded by \( Kp^{2^{r-1}} = o(\log_{(r-1)} L)^{-(d-r+1)} \), where constant \( K \) is an upper-bound on the number of possibilities of choosing \( 2^{r-1} \) nodes in the \( t_c \)-neighborhood of an arbitrary node in \( \mathbb{T}^d_L \); notice since \( d \) and \( t_c \) both are constant, the number of nodes in \( t_c \)-neighborhood of a node is bounded by a constant. Therefore, in round \( t_c \) each node is black with probability \( o(\log_{(r-1)} L)^{-(d-r+1)} \), regardless of the color all other nodes. Based on Theorem 4, the stronger model of \( r \)-BP results in the survival of white
color from such a configuration a.a.s., so does two-way r-BP. (The reader might already have noticed that we are ignoring the dependence between the states of nearby nodes. In the extended version of the paper, we discuss that how one can get rid of it, in details.)

2.3 Phase 3

In this section, we prove in $T^d_L$ and two-way $r$-BP, $\frac{r}{\sqrt{p_2}} \ll p$ results in fully black configuration a.a.s. where $p_2 = (\log_{(r-1)} L)^{-(d-r+1)}$. For the sake of simplicity, assume $L$ is even (we discuss at the end, how our argument easily carries on the odd case). As argued in Section 1.1 tiling procedure partitions the node set of $T^d_L$ into $L^d/2^r$ pair-wise disjoint $r$-dimensional hyper-squares. We say two hyper-squares are neighbors if their distance is equal to one, meaning there is an edge between them. More precisely, the neighbors of an $r$-dimensional hyper-square $HS$ starting from $i = (i_1, \cdots, i_d)$ are divided into two groups. First, $2r$ $r$-dimensional hyper-squares whose starting nodes differ with $i$ only in one of the first $r$ coordinates and exactly by two, which are called the inner neighbors. The second group are $2(d-r)$ $r$-dimensional hyper-squares whose starting nodes differ with $i$ in only one of the last $d-r$ coordinates and exactly by one, which are called the outer neighbors. The two hyper-squares whose starting nodes differ with $i$ in the $j$-th coordinate are called the neighbors in $j$-th dimension. See Figure 3 (d) for an example of the inner (red) and outer (green) neighbors of a two-dimensional hyper-square in $T^3_L$. Furthermore, let us define the parity of $HS$ to be the parity of the sum of the last $d-r$ coordinates of $i$. Clearly, the inner neighbors have the same parity as $HS$ but the outer neighbors have different parity.

From now on, we only look at the even configurations, meaning we only consider $C_t$ for even $t$. For an $r$-dimensional hyper-square of even parity (similarly odd parity), we say it is occupied in $C_t$ if its even-part (resp. odd-part) is black. Notice based on Lemma 1 an occupied hyper-square stays occupied forever. In Lemma 6, we state that if in some configuration in two-way $r$-BP on $T^d_L$, an $r$-dimensional hyper-square has occupied neighbors in at least $r$ distinct dimensions then it gets occupied in constantly many rounds. The proof is technical and is presented in the extended version of the paper. The idea is to apply induction on $r$. See Figure 4 for two examples of how a two-dimensional hyper-square gets occupied with occupied neighbors in two distinct dimensions (regarding the selection of black nodes, recall the parity of a hyper-square is the same as its inner neighbors but different with outer ones).

![Figure 4](top) two inner occupied neighbors (bottom) one inner and one outer occupied neighbor.

Lemma 6. In two-way $r$-BP, if an $r$-dimensional hyper-square has occupied neighbor in at least $r$ distinct dimensions, it gets occupied in $t'$ rounds for some even constant $t'$.

For our proof, we also need that modified $r$-BP on $T^d_L$ with initial probability $\omega(p_2)$ results in fully black configuration a.a.s. However, this is known only for $d=r$ by Holroyd [28]. He showed the process exhibits a sharp threshold behavior at $\lambda p_1$ for some constant $\lambda > 0$. We
Two Phase Transitions in Two-way Bootstrap Percolation

require a much weaker statement, meaning the initial probability \( \omega(p_2) \) a.a.s. results in fully black configuration, but for all values of \( r \). The good news is that the upper-bound proof by Cerf and Manzo [11] regarding \( r \text{-BP} \) can be easily adapted to prove our desired upper bound for modified \( r \text{-BP} \). Actually, exactly the same proof works because everywhere they apply \( r \text{-BP} \) rule, modified \( r \text{-BP} \) suffices. However, it is interesting by its own sake to study the sharp threshold behavior of modified \( r \text{-BP} \) also for \( r \neq d \), in future work.

\[ \text{Theorem 7. (derived from [11]) In } T^d_L, \text{ modified } r \text{-BP with initial probability } \omega(p_2) \text{ results in fully black configuration a.a.s. for } p_2 = (\log(r-1)L)^{(d-r+1)}. \]

Now, it is time to put the aforementioned claims together to finish the proof. If we tile \( T^d_L \) into hyper-squares as above, in two-way \( r \text{-BP} \) with \( p = \omega(\frac{r}{\sqrt{p_2}}) \) each hyper-square is occupied initially with probability \( \omega(p_2) \). Furthermore, based on Lemma 6 if a hyper-square has occupied neighbors in at least \( r \) distinct dimensions, it gets occupied, which implies the occupation process among the hyper-squares is at least as strong as modified \( r \text{-BP} \). Based on Theorem 7, we know modified \( r \text{-BP} \) with initial probability \( \omega(p_2) \) gets fully black a.a.s. Thus, all the hyper-squares get occupied in our process a.a.s. We can do the same argument by just switching the terms of even and odd in the definition of occupation. Thus by a union bound, a.a.s. for two-way \( r \text{-BP} \) on \( T^d_L \) with \( p = \omega(\frac{r}{\sqrt{p_2}}) \) eventually both the even-part and odd-part of all the hyper-squares are black, which implies the process gets fully black.

We assumed at the beginning that \( L \) is even. Theorem 7 also works for the \( d \)-dimensional lattice \( \{1, \cdots, L\}^d \). Therefore, for odd \( L \) we can do the same argument for the lattice, attained by skipping the nodes with at least one coordinate equal to \( L \), and also the lattice, attained by skipping the nodes with at least one coordinate equal to one. Then, again a union bound finishes the proof.

Future Work

The natural next step in this line of research is to find out that whether two-way \( r \text{-BP} \) also exhibits a sharp threshold behavior at the second transition.

As we discussed, by prior results there is a linear upper bound (in terms of the number of nodes) on the consensus time of two-way \( r \text{-BP} \) in \( T^d_L \). This is asymptotically tight; see Figure 3(c) for the case of \( r = d = 2 \). This snake-shape structure can be extended to any \( r \) and \( d \) with a small trick. However, the point is since we start from an initial random configuration, such configurations are very unlikely. Thus, one might ask what is the expected consensus time, which is the expected number of rounds the process needs to reach a cycle of configurations for an initial random configuration, which is a function of \( p \). We are not aware of any result regarding the expected consensus time of two-way \( r \text{-BP} \) on \( T^d_L \), and the picture for \( r \text{-BP} \) is also very incomplete, meaning it is known only for \( d = 2 \) (by Balister, Bollobas, and Smith [2]) and some other very special cases.

Acknowledgments

The author likes to thank Raphael Cerf, Bernd Gärtner, and Roberto H. Schonmann for several stimulating discussions.

References

1. Joan Adler and Amnon Aharony. Diffusion percolation. i. infinite time limit and bootstrap percolation. *Journal of Physics A: Mathematical and General*, 21(6):1387, 1988.
2. Michael Aizenman and Joel L Lebowitz. Metastability effects in bootstrap percolation. *Journal of Physics A: Mathematical and General*, 21(19):3801, 1988.
3 Paul Balister, Béla Bollobás, J Robert Johnson, and Mark Walters. Random majority percolation. *Random Structures & Algorithms*, 36(3):315–340, 2010.
4 Paul Balister, Béla Bollobás, and Paul Smith. The time of bootstrap percolation in two dimensions. *Probability Theory and Related Fields*, 166(1-2):321–364, 2016.
5 József Balogh and Béla Bollobás. Bootstrap percolation on the hypercube. *Probability Theory and Related Fields*, 134(4):624–648, 2006.
6 József Balogh, Béla Bollobás, Hugo Duminil-Copin, and Robert Morris. The sharp threshold for bootstrap percolation in all dimensions. *Transactions of the American Mathematical Society*, 364(5):2667–2701, 2012.
7 József Balogh, Béla Bollobás, and Robert Morris. Bootstrap percolation in three dimensions. *The Annals of Probability*, pages 1329–1380, 2009.
8 József Balogh and Gábor Pete. Random disease on the square grid. *Random Structures and Algorithms*, 13(3-4):409–422, 1998.
9 József Balogh and Boris G Pittel. Bootstrap percolation on the random regular graph. *Random Structures & Algorithms*, 30(1-2):257–286, 2007.
10 Raphaël Cerf and Emilio NM Cirillo. Finite size scaling in three-dimensional bootstrap percolation. *Annals of probability*, pages 1837–1850, 1999.
11 Raphaël Cerf and Francesco Manzo. The threshold regime of finite volume bootstrap percolation. *Stochastic Processes and their Applications*, 101(1):69–82, 2002.
12 Ching-Lueh Chang and Yuh-Dauh Lyuu. Spreading of messages in random graphs. *Theory of Computing Systems*, 48(2):389–401, 2011.
13 Ching-Lueh Chang and Yuh-Dauh Lyuu. Bounding the sizes of dynamic monopolies and convergent sets for threshold-based cascades. *Theoretical Computer Science*, 468:37–49, 2013.
14 Tom Coker and Karen Gunderson. A sharp threshold for a modified bootstrap percolation with recovery. *Journal of Statistical Physics*, 157(3):531–570, 2014.
15 Devdatt P Dubhashi and Alessandro Panconesi. *Concentration of measure for the analysis of randomized algorithms*. Cambridge University Press, 2009.
16 MohammadAmin Fazli, Mohammad Ghodsi, Jafar Habibi, Pooya Jalaly, Vahab Mirrokni, and Sina Sadeghian. On non-progressive spread of influence through social networks. *Theoretical Computer Science*, 550:36–50, 2014.
17 Uriel Feige, Michael Krivelevich, Daniel Reichman, et al. Contagious sets in random graphs. *The Annals of Applied Probability*, 27(5):2675–2697, 2017.
18 Paola Flocchini, Elena Lodì, Fabrizio Luccio, Linda Pagli, and Nicola Ro. Dynamic monopolies in tori. *Discrete applied mathematics*, 137(2):197–212, 2004.
19 F Fogelman, Eric Goles, and Gérard Weisbuch. Transient length in sequential iteration of threshold functions. *Discrete Applied Mathematics*, 6(1):95–98, 1983.
20 Silvio Frischknecht, Barbara Keller, and Roger Wattenhofer. Convergence in (social) influence networks. In *International Symposium on Distributed Computing*, pages 433–446. Springer, 2013.
21 Juan P Garrahan, Peter Sollich, and Cristina Toninelli. Kinetically constrained models. *Dynamical heterogeneities in glasses, colloids, and granular media*, 150:111–137, 2011.
22 Bernd Gärtner and Ahad N Zehmakan. (biased) majority rule cellular automata. *arXiv preprint arXiv:1711.10920*, 2017.
23 Bernd Gärtner and Ahad N Zehmakan. Color war: Cellular automata with majority-rule. In *International Conference on Language and Automata Theory and Applications*, pages 393–404. Springer, 2017.
24 Bernd Gärtner and Ahad N Zehmakan. Majority model on random regular graphs. *Latin American Symposium on Theoretical Informatics*, pages 572–583, 2018.
Eric Goles and Jorge Olivos. Periodic behaviour of generalized threshold functions. *Discrete mathematics*, 30(2):187–189, 1980.

Mark Granovetter. Threshold models of collective behavior. *American journal of sociology*, 83(6):1420–1443, 1978.

Lianna Hambardzumyan, Hamed Hatami, and Yingjie Qian. Polynomial method and graph bootstrap percolation. *arXiv preprint arXiv:1708.04640*, 2017.

Alexander Holroyd et al. The metastability threshold for modified bootstrap percolation in $d$ dimensions. *Electronic Journal of Probability*, 11:418–433, 2006.

Alexander E Holroyd. Sharp metastability threshold for two-dimensional bootstrap percolation. *Probability Theory and Related Fields*, 125(2):195–224, 2003.

Svante Janson, Tomasz Łuczak, Tatyana Turova, Thomas Vallier, et al. Bootstrap percolation on the random graph $g_{n,p}$. *The Annals of Applied Probability*, 22(5):1989–2047, 2012.

Clemens Jeger and Ahad N Zehmakan. Dynamic monopolies in reversible bootstrap percolation. *arXiv preprint arXiv:1805.07392*, 2018.

Yashodhan Kanoria, Andrea Montanari, et al. Majority dynamics on trees and the dynamic cavity method. *The Annals of Applied Probability*, 21(5):1694–1748, 2011.

David Kempe, Jon Kleinberg, and Éva Tardos. Maximizing the spread of influence through a social network. In *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 137–146. ACM, 2003.

Jane Molofsky, Richard Durrett, Jonathan Dushoff, David Griffeath, and Simon Levin. Local frequency dependence and global coexistence. *Theoretical population biology*, 55(3):270–282, 1999.

Natasha Morrison and Jonathan A Noel. Extremal bounds for bootstrap percolation in the hypercube. *Journal of Combinatorial Theory, Series A*, 156:61–84, 2018.

Elchanan Mossel, Joe Neeman, and Omer Tamuz. Majority dynamics and aggregation of information in social networks. *Autonomous Agents and Multi-Agent Systems*, 28(3):408–429, 2014.

David Peleg. Local majority voting, small coalitions and controlling monopolies in graphs: A review. In *Proc. of 3rd Colloquium on Structural Information and Communication Complexity*, pages 152–169, 1997.

David Peleg. Local majorities, coalitions and monopolies in graphs: a review. *Theoretical Computer Science*, 282(2):231–257, 2002.

Roberto H Schonmann. Finite size scaling behavior of a biased majority rule cellular automaton. *Physica A: Statistical Mechanics and its Applications*, 167(3):619–627, 1990.

Roberto H Schonmann. On the behavior of some cellular automata related to bootstrap percolation. *The Annals of Probability*, pages 174–193, 1992.

Ahad N Zehmakan. Opinion forming in binomial random graph and expanders. *arXiv preprint arXiv:1805.12172*, 2018.