Observation of SLE($\kappa, \rho$) on the critical statistical models

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Abstract
Schramm–Loewner evolution (SLE) is a stochastic process that helps classify critical statistical models using one real parameter $\kappa$. Numerical study of SLE often involves curves that start and end on the real axis. To reduce numerical errors in studying the critical curves which start from the real axis and end on it, we have used hydrodynamically normalized SLE($\kappa, \rho$) which is a stochastic differential equation governing such curves. In this paper, we directly verify this hypothesis and numerically apply this formalism to the domain wall curves of the Abelian sandpile model ($\kappa = 2$) and critical percolation ($\kappa = 6$). We observe that this method is more reliable than previously used methods in the literature for analyzing interface loops.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Classification of probability measures on random curves in simply connected domains of the complex plane was first proposed by Schramm [1]. According to this idea one can describe the interfaces of two-dimensional statistical models via growth processes named as Schramm–Loewner evolution (SLE). This technique has become a powerful tool to study the macroscopic interfaces of two-dimensional systems. Another powerful tool for classification of 2d critical phenomena is conformal field theory (CFT). Hence a connection between the two methods is natural. The relation between CFT and SLE was found in the pioneering works of Bauer and Bernard [2]. Parallel to the analytical achievements, a large amount of numerical work has been performed using SLE to study the statistics of the critical interfaces of various statistical models. These numerical studies are often based on curves such as iso-height lines and thus we are faced with ensembles of loops, whereas for study of SLE we need paths starting at the real axis and extending to infinity. To resolve the problem one divides the loops into two halves. Taking one of the two halves, one sends one of the end point of the cut curves to
the origin and the other end point to infinity (infinity mapping). This method is widely used, e.g. in turbulence [3], abelian sandpile model (ASM) avalanche frontier [4] iso-height lines of KPZ [5] and WO3 [6], Ising model [7], etc. As this algorithm was first introduced in [3], we name it BBCF algorithm. But there is an important source of numerical error in a BBCF algorithm due to the infinity map which sends the end point to the infinity. This map enlarges the lattice constant in the regions close to the end point of the curve and this causes numerical large errors. To avoid these errors, we propose a method that is free of infinity mapping.

Our proposed method is based on a variant of SLE_κ known as SLE(κ, ρ), a generalization of SLE_κ which describes the curves that are self-similar but are not conformally invariant, i.e. there are some preferred points on the domain of growing SLE curves. It is proved that the hydrodynamically normalized stochastic differential equation, governing the mappings of the critical curves that start from the real axis and also end on it, is the SLE(κ, ρ) where ρ = κ − 6 [8]. Therefore, we not only introduce a new more reliable numerical method to analyze SLE curves, but also we have a numerical check for the application of SLE(κ, ρ) formalism to the curves going from the real axis to the real axis as stated above (at least for the case ρ = κ − 6).

We show that this formalism is properly applicable and obtains more precise results for the ASM and percolation model. To this end, we use the slit uniformizing map to obtain the driving function and use maximum likelihood estimation (MLE) to fit the resulting process to Brownian motion, estimating parameters ρ and κ and their stochastic errors. In section 2, we briefly introduce chordal SLE. Section 3 is a brief introduction to the critical percolation and ASM respectively and SLE(κ, ρ). In section 4, we present our numerical results.

2. SLE

Critical behavior of the two-dimensional statistical models can be described by their geometrical features. In fact instead of studying the local observables, we can focus on the interfaces of two-dimensional models. These domain walls are some non-intersecting curves which directly reflect the status of the system in question and are supposed to have two properties: conformal invariance and the domain Markov property. SLE is the candidate to analyze these random curves by classifying them as the one-parameter classes SLE_κ. We offer a very brief introduction below; for good introductory reviews see references [11, 12].

2.1. Chordal SLE

Chordal SLE_κ is a growth process defined via conformal maps which are solutions of Loewner’s equation:

\[ \frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \xi_t}, \]

where the initial condition is \( g_t(z) = z \) and \( \xi_t = \sqrt{\kappa} B_t \) is a real-valued function and \( B_t \) is a one-dimensional standard Brownian motion. Let us denote the upper half-plane by \( H \) and \( \gamma \) as the SLE trace. For fixed \( z \), \( g_t(z) \) is well defined up to time \( \tau_z \) for which \( g_t(z) = \xi_t \). Then, the hull is defined as \( K_t = \{ z \in H : \tau_z \leq t \} \) and the complement \( H_t := H \setminus K_t \) is simply connected. The map \( g_t(z) \) is the unique conformal mapping \( H_t \to H \) with \( g_t(z) = z + \frac{z}{t} + O\left(\frac{1}{t}\right) \) as \( z \to \infty \) known as hydrodynamical normalization. One can retrieve the SLE trace by \( \gamma_t = \lim_{\epsilon \to 0} g_t^{-1}(\xi_t + i\epsilon) \). There are phases for these curves, for \( 2 \leq \kappa \leq 4 \) the trace is non-self-intersecting and it does not hit the real axis; in this case the hull and the trace are identical: \( K_t = \gamma_t \). This is called ‘dilute phase’. For \( 4 < \kappa \leq 8 \), the trace touches itself and the real axis so that a typical point is surely swallowed as \( t \to \infty \) and \( K_t \neq \gamma_t \). This phase is
called the ‘dense phase’. There is a connection between the two phases: for $4 \leq \kappa \leq 8$ the frontier of $K_t$, i.e. the boundary of $H_t$ minus any portions of the real axis is a simple curve which is locally a SLE$_{\kappa}$ curve with $\kappa = \frac{16}{3}$, i.e. it is in the dilute phase [9].

2.2. SLE($\kappa, \rho$)

We define SLE($\kappa, \rho$) in the upper half-plane. The parameter $\kappa$ identifies the local properties of the model in hand and corresponds directly to the central charge of the corresponding CFT, the parameter $\rho$ has to do with the boundary conditions (BCs) imposed. The origin and the other point $x_\infty$ on the real axis mark the locations where BCs change; these two points are the starting and end points of the curve we are interested in. Of course, generically the trace of an SLE($\kappa, \rho$) process terminates on the interval ($x_\infty$, $+\infty$). The actual behavior depends on the concrete values of $\kappa$ and $\rho$, e.g. $\rho = (\kappa - 6)/2$ yields dipolar SLE($\kappa$) [10]. The stochastic equation governing such curves is the same as formula (1) but the driving function has a different form

$$d\xi_t = \sqrt{\kappa} \, dB_t + \frac{\rho}{\xi_t - g_t(x_\infty)} \, dt. \tag{2}$$

Now, consider a curve that starts from origin and ends on a point on real axis ($x_\infty$). Then using the map $\phi = x_\infty z/(x_\infty - z)$, one can send the end point of the curve to the infinity. In this respect, the function $h_t = \phi \circ g_t \circ \phi^{-1}$ describes chordal SLE. It is easy to show that the equation governing $h_t$ is $\partial_t g_t = 2/\{\phi'(g_t) (\phi(g_t) - \xi_t)\}$. But it is explicit that this function is not hydrodynamically normalized. It has been shown that if one uses another mapping $\tilde{g}_t = v_t \circ h_t \circ u^{-1}$, where $u = \phi^{-1}$ and $v_t$ is a linear fractional transformation that makes the corresponding map hydrodynamically normalized, then the stochastic equation of $\tilde{g}_t$ is the same as equation (1) [13]. In fact, this procedure leaves equation (1) unchanged but leads the driving function to have a drift term [13]

$$d\xi_t = \sqrt{\kappa} \, dB_t + \frac{\kappa - 6}{\xi_t - g_t(x_\infty)} \, dt. \tag{3}$$

In other words, this stochastic function is the driving function of the SLE($\kappa, \rho$) with $\rho = \kappa - 6$. Thus for the critical curves from boundary to boundary, the corresponding driving function acquires a drift term. This generalization of SLE can be generalized further to have multiple preferred real axis points. For a review, refer to [11, 12].

3. Statistical models

In this section, we briefly introduce some statistical models to be simulated in the following section. We have chosen two models from two regimes of $\kappa$: ASM with $\kappa = 2$ (dilute phase) and percolation with $\kappa = 6$ (dense phase). In the case of percolation, there is an analytical proof that the interface curves are SLE$_{\kappa=6}$ [14]. Also, percolation has the property of locality and a fictitious line can be taken as a boundary [15]. Therefore our numerical simulation is only a check. However in the case of ASM wave frontiers, there is no rigorous proof that their statistics is the same as SLE$_2$. It has only numerically been checked that the curves are SLE$_{\kappa=2}$ [4]. Also it does not have the locality property and introducing a fictitious line to the system as a boundary does not seem to be reasonable. This means that we are not sure if the curves in the case of ASM have the statistics implied from SLE($\kappa, \rho$). One might think that the curves do not have the statistics of SLE, but if we note that many properties of the SLE curves are local, it may be understandable that introducing such curve may have little effect on the basic properties. In the case of SLE($\kappa, \rho$) the story is more complicated as the effects
of BCs and some specific points are more explicitly seen in the evolution. This may mean that we do not see the SLE($\kappa, \rho$) statistics with the new fictitious BC we imposed. However, we will see that these curves show correct statistics, showing that SLE properties are more local than predicted. Let us first give a short review of the two models and then bring our numerical results.

3.1. Critical percolation

Let $(H, a, b)$ be a hexagonal lattice on the upper half-plane domain with the BC changes on two points on the real axis, namely $a$ and $b$, i.e. color the hexagons on the boundary in black and white so that the resulting boundaries have two BC changes on points $a$ and $b$ as indicated in figure 1. A configuration is a choice of color for inner hexagons which becomes black or white with the probability $p$ and $1 - p$, respectively. One can easily identify unique domain walls separating white and black sites from each other. Each configuration defines an interface, i.e. the unique path from $a$ to $b$ in $H$ such that the hexagon on the left (right) of any of its edges is black (white); see figure 1. Hence, the probability distribution on configurations induces a probability distribution on paths from $a$ to $b$ in $H$. There is a critical probability $p = p_c = 0.5$ in which the size of the resulting clusters is of the order of the size of lattice and the clusters become self-similar in the scaling limit. We consider here the critical interfaces of percolation which are known as SLE($\kappa = 6$). Percolation also enjoys an important property namely locality, that is, the evolution of the SLE curve is insensitive to the BCs and events that take place in the $H \setminus K_t$ up to time $t$ [15]. Due to the locality, the interfaces of the percolation can be generated by a simple growth process: in the $n$th step, when the tip of the curve reaches an inner hexagon, using a fair coin, color this site in black or white. The upshot is that the $(n+1)$th step of the resulting domain wall will be the edge of the hexagon whose adjacent faces have different colors. By this simple rule, we can generate domain wall samples that start from the origin and end on an arbitrary point on the lattice boundary. Locality implies that this growth process is insensitive to the fact that ‘where the curve will end’. It is guaranteed in equation (3) which implies $\rho = \kappa - 6 = 0$, saying that the drift term in this equation vanishes. So, these stochastic curves evolve like the chordal one and are insensitive to the BCs as expected.

![Figure 1. Typical configuration of the percolation starting and ending on the real axis.](image)
3.2. Abelian sandpile model

Sandpile models have been introduced by Bak et al [16] as an example for a class of models that show self-organized criticality. These models show critical behaviors, without tuning external parameters such as temperature. The Abelian structure of this model was first discovered by Dhar and named as the ASM [17]. Despite its simplicity, the ASM has various interesting features and many different analytical and numerical works have been performed on this model, for example, different height and cluster probabilities [19], its connection with spanning trees [18], ghost models [20], q-state Potts model [22], etc. For a good review, refer to [21].

Consider the ASM on a two-dimensional square lattice \( L \times L \). To each site \( i \), a height variable \( h_i \) is assigned taking its values from the set \( 1, 2, 3 \), and \( 4 \), the number of sand grains on this site. The dynamics of this model is as follows; in each step, a grain is added to a random site \( i \), i.e. \( h_i \rightarrow h_i + 1 \); if the resulting height becomes more than 4, the site topples and loses 4 sand grains, each of which is transferred to one of the four neighbors of the original site. As a result, the neighboring sites may become unstable and topple and a chain of topplings may occur in the system. At the boundary sites, the toppling causes one or two sand grains leave the system. This process continues until the system reaches a stable configuration. Now another random site is selected and the sand is released on this site and the process continues. After a finite number of steps, the ASM reaches a well-defined distribution of states. Some configurations will not occur (transient configurations) and other configurations occur with equal probability (recurrent configurations); for details, refer to [21]. This model is related to Potts model with \( q \rightarrow 0 \) and CFT with central charge \( c = -2 \). For a lattice with \( d \) neighboring sites, the toppling occurs when \( h_i > d \), then the original site will lose \( d \) grains and the height of each of its neighbors will increase by 1. It is obvious that for a triangular lattice \( d = 6 \).

4. Numerical methods and results

To use equation (1) for the statistical models on the lattice, one has to discretize this equation. For this purpose, we assume a special function form for the driving function in some discrete intervals and find the corresponding uniformizing map in each interval. We approximate the driving function in each interval \([t_{n-1} , t_n] \) by the constant function \( \xi_n = \xi_n \). Using the composite map \( G_n = G_{\delta t_n, \xi_n} \cdots G_{\delta t_0, \xi_0} \) (\( \delta t_n = t_n - t_{n-1} \)), one can then send every point on the \( \gamma \) to the real axis step-by-step. In this map \( G_{\delta t_n, \xi_n}(z) \) is the solution of the Loewner equation (1) for the constant driving function \( \xi_n \):

\[
G_{\delta t_n, \xi_n}(z) = \xi_n + \sqrt{(z - \xi_n)^2 + 4\delta t_n}.
\]

Note that \( G_{\delta t_n, \xi_n}(z) \approx z + \frac{2\delta t_n}{4} + O(\frac{1}{4}) \) as \( z \rightarrow \infty \), so the resulting composite map is also hydrodynamically normalized. The method that is widely used to apply this formalism to the interface loops for the statistical models on lattice is as follows [3–5]: one cuts the loops horizontally from its middle to generate curves starting from the origin and ending at some point on the real axis. Then by mapping the end point of the curve to infinity and applying the chordal SLE, i.e. equation (1), one can obtain the underlying driving function. For the simulation, we consider the triangular lattice and its dual lattice (honeycomb lattice) and also consider two sublattices A and B as indicated in figure 2. Thus, the position of each point can be coded in \((n, m, A/B)\) where \(n\) and \(m\) represents the position of the lattice point and A or B represent its basis. Consider that we have ensembles of interface loops of some model, then by the process described above we have ensembles of the curves that start and end on the real axis. Thus, when we have the set of discrete points \( \{z_1, z_2, z_3, \ldots, z_n\} \), the first uniformizing map is of the form of
equation (4) with the parameters $\delta t_1 = \frac{1}{2}(\text{Im}[z_1])^2$ and $\xi_1 = \text{Re}[z_1]$ and after applying this mapping we continue this process for $z_2$. The process continues until every point of the curve is sent to the real axis. To analyze the resulting driving function and use equation (4), we discretize this equation.

$$\delta \xi_n = \sqrt{\kappa} \delta B_n + \frac{\rho}{\xi_n - G_t(x_\infty)} \delta t_n. \quad (5)$$

To having a pure Brownian motion, we should write equation (6) in the following form:

$$\xi_n = \sum_{i=1}^{n} \left[ \frac{\rho \delta t_i}{\xi_n - G_t(x_\infty)} \right] = B_n, \quad (6)$$

where $\rho = \kappa - 6$. After doing this process, we can analyze the resulting quantity on the right-hand side of equation (6) to see if this stochastic process is a Brownian motion. The best fit to the Brownian motion gives the value of $\kappa$. For this, we used the MLE method [23]. Given a model, MLE would find the best 'fit' by treating the mean and variance as parameters and finding particular values (here $\kappa$ and $\rho_0 \equiv -\rho$) that make the observed results the most probable. In this method, the best values for parameters are those which minimize the function and the precisions of the parameters can be obtained from the distribution function of the $\chi^2$ function by calculating the ratio of the area under the resulting diagram to the whole area under it.

4.1. Critical percolation

Consider the interfaces of critical percolation. A typical critical percolation curve is shown in figure 3 that starts from origin and ends at the point 5200 on the real axis. Due to its locality property, percolation is very hard to control. We have to wait a long time before a certain point is hit, in this figure it reaches the real axis after 850 000 steps. We simulated over 10 000 curves with typical length of 50 000. After simulating such curves we obtained $\xi_t$ as described above. Figure 4 shows the behavior and the minimum of the $\chi^2$ function versus $\kappa$ which is obtained by fitting $\langle B_t^2 \rangle$ versus $t$. The global minimum of this curve is at the point $\kappa = 6.08$.
Figure 3. Sample graph of the percolation going from real axis to itself.

Figure 4. $\chi^2$ graph versus $\kappa$, which has a minimum at $\kappa = 6.08$ and $\rho_0 = -0.08$. Using the distribution of $\chi^2$, we obtain the precision of the resulting parameters: with the probability 60% $\kappa = 6.08 \pm 0.07$ and $\rho_0 = -0.08 \pm 0.07$. The plot of $\langle B_t^2 \rangle$ versus $t$ for the obtained $\kappa$ and $\rho_0$ is shown in figure 5. We see that this graph is linear with the slope $0.98 \pm 0.02$. The inner graph shows the distribution of $B_t$ for $t = 0.625$ that is properly fitted to the Gaussian function with $\sigma = 0.625$, this ensures that this process is also a Brownian motion.
4.2. Abelian sandpile model

Consider avalanche frontiers in the ASM. By adding a grain to the saturated sandpile and making it unstable, avalanches occur and their frontiers form loops with discrete points. Figure 6 shows a typical avalanche frontier in the ASM which starts from origin and ends at the real axis. From the theory one expects that $\rho_0 = 6 - \kappa = 4$. To test this, we let $\rho_0$ be a free parameter in this case. The resulting values for the ASM on the $2048 \times 2048$ triangular lattice over 10 000 interface curves are $\kappa = 1.95 \pm 0.05$ and $\rho_0 = 3.5 \pm 0.5$ with a 60% probability. Figure 7 indicates the contour plot of $\chi^2$ in terms of $\kappa$ and $\rho_0$ in which the probability of
Figure 7. Contour graph for $\chi^2$ (for fitting $\langle B_t^2 \rangle$ with $t$) as a function of $\kappa$ and $\rho_0$.

Figure 8. $\langle B_t^2 \rangle$ versus $t$ graph that is linear with a good slope $1.00 \pm 0.05$ for ASM. The inner graph is the distribution of $B_t$ for $t = 0.125$. The dots correspond to the numerical result and the solid line is the Gaussian distribution $\exp\left[ -\frac{B_t^2}{2\sigma} \right]$ with $\sigma = 0.125$.

Finding parameter values in various regions are indicated. In figure 8 we have shown the graph $\langle B_t^2 \rangle$ versus $t$ for the parameters considered above ($\kappa = 1.95$ and $\rho_0 = 3.80$). It is seen that the graph is properly linear with the slope $0.98 \pm 0.05$ ($\langle B_t^2 \rangle \simeq t$) a well-known property of the Brownian motion. In the inner graph of figure 8, the distribution of $B_t$ at time $t = 0.125$ and also the Gaussian distribution $\exp\left[ -\frac{B_t^2}{2\sigma} \right]$ with $\sigma = 0.125$ are indicated together.
Figure 9. Lattice deformations due to applying the previous uniformizing maps.

Table 1. Comparison between the amount of $\kappa$ for the previous ($\kappa_1$) and the present ($\kappa_2$) numerical approaches.

|        | $\kappa_1$ (BBCF algorithm) | $\kappa_2$ (new algorithm) | Theoretical value |
|--------|-----------------------------|------------------------------|-------------------|
| Perc.  | 6.0 ± 0.2                   | 6.08 ± 0.08                  | 6                 |
| ASM    | 2.1 ± 0.2                   | 1.98 ± 0.07                  | 2                 |

As seen in the graph, they fit each other showing that the distribution of $B_t$ are Gaussian. Based on this and the fact that $\langle B_t \rangle \simeq 0$ we declare that this process is Brownian and the driving function obtained from hydrodynamically normalized uniformizing map has the form of equation (3).

We can now compare our result with the BBCF algorithm. Table 1 shows numerical values of the obtained diffusivity parameter $\kappa$ by simulation, for the BBCF algorithm ($\kappa_1$) and for the new algorithm ($\kappa_2$). It is notable that the conditions under which the numerical calculations were performed are the same for the two columns. For percolation the typical curve lengths are $\simeq 6000$. The calculations show that the standard deviation of $\kappa_1$ becomes as large as 0.2 when the curve lengths increase to 30000, whereas the standard deviation of $\kappa_2$ does not change significantly. The same phenomenon is observed for the ASM. So we have reported the precision of $\kappa_1$ as 0.2.
As mentioned before, the large lattice deformations existing in the BBCF algorithm may lead to a lack of accuracy in the evaluation of the diffusivity parameter $\kappa$. We have presented an example of this deformation in figure 9, which shows the deformations of an ASM curve with length $\simeq 2000$ on the triangular lattice (the number $n$ under each curve shows the number of points on the curve which has been uniformized).

As is seen, for some $n$, the distance between successive points on the curves becomes extremely large which leads to large errors. This is explicit in the figure for $n = 503$ after...
which $\xi_t$ crosses the point $x_\infty$. This effect becomes more significant for smaller curves and is more pronounced in the simulations with cubic lattice. The effect of this error can be seen in the following simple test: let us call the length of the curve going from the real axis to itself $L$. If we apply the uniformizing map only to the points on the curve up to a maximum length $\alpha L$ with $\alpha \in (0, 1]$; that is, we take into account only a fraction of the whole curve, we expect that the resulting value for $\kappa$ does not depend strongly on $\alpha$, at least for the values of $\alpha$ not close to zero. Figure 10 shows the results for $\alpha = 1$ and $\alpha = 0.8$. It is remarkable that this dependence is more considerable for small curves. This is due to stronger dependence of small curves to the lattice deformations. We have performed the same test with our new method; the result is shown in figure 11. Hence, we believe that the error in the BBCF algorithm is more than what has been previously reported.

5. Conclusion

In this paper, we analyzed the statistics of the curves that start and end on the real axis, using the formalism of hydrodynamically normalized SLE($\kappa, \rho$). We numerically examined the validity of the formalism and proposed a new numerical method to study conformally invariant curves which reveals more reliable and more exact results. To this end we considered avalanche frontiers of critical ASM and critical percolation. Then using the maximum likelihood estimation, we estimated the best values of parameters $\kappa$ and $\rho_0 = -\rho$ and their errors. Summing up, we have a method with much smaller fitting SLE curves.

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