A dichotomy for $D$-rank 1 types in simple theories

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Abstract

We prove a dichotomy for $D$-rank 1 types in simple theories that generalizes Buechler’s dichotomy for $D$-rank 1 minimal types: every such type is either 1-based or its algebraic closure, by a single formula, almost contains a non-algebraic formula that belongs to a non-forking extension of the type. In addition we prove that a densely-1 based type of $D$-rank 1 is 1-based. We also observe that for a hypersimple unidimensional theory the existence of a non-algebraic stable type implies stability (and thus superstability).

1 Introduction

In 1985 Buechler proved [B] a remarkable dichotomy between model theoretic simplicity and geometric simplicity; it says that any minimal $D$-rank 1 type in a stable theory is either 1-based or has Morley rank 1. In this paper we give a generalization of this result for any $D$-rank 1 type of arbitrary simple theory. As a special case we get Buechler’s dichotomy for any $D$-rank 1 minimal type in exactly the form mentioned above. The proof applies certain properties of the forking topology, a topology that introduced in [S1] and is a variant of the topologies introduced in [H0,P]. In these papers, this topology (and generalizations of it in [S1]) has been used to obtain certain approximations of definable sets of finite rank for proving supersimplicity of countable hypersimple/hypersimple low/stable unidimensional theories. In addition we show that the notion of an essentially 1-based type, that introduced in [S1], coincide with the notion of a 1-based type in the $D$-rank.
1 case. A posteriori, this shows that the case handled in [S1] in which the theory is essentially-1 based is in fact just the case in which it is 1-based (for which the proof is much easier) but, of course, that does mean we can simplify the proof (we don’t know, of course, a $D$-rank 1 type exists).

The notations are standard, and throughout the paper we work in a highly saturated, highly strongly-homogeneous model $C$ of a complete first-order theory $T$ in a language $L$ with no finite models. We will often work in $C^eq$.

2 Preliminaries

We assume basic knowledge of simple theories as in [K],[KP],[HKP] as well as some knowledge on hyperimaginaries, internality and analyzability in simple theories as in [W]. In this section, we recall some basic facts related to the forking topology and to pairs of models in a simple theory that are relevant for this paper. $T$ will denote a simple theory.

2.1 The forking topology

Definition 2.1 Let $A \subseteq C$ and let $x$ be a finite tuple of variables. An invariant set $U$ over $A$ is said to be a basic $\tau^f$-open set over $A$ if there is a $\phi(x,y) \in L(A)$ such that
\[ U = \{ a | \phi(a,y) \text{ forks over } A \}. \]

Note that the family of basic $\tau^f$-open sets over $A$ is closed under finite intersections, thus form a basis for a unique topology on $S_x(A)$ which we call the the $\tau^f$-topology or the forking-topology.

Definition 2.2 We say that the $\tau^f$-topologies over $A$ are closed under projections ($T$ is PCFT over $A$) if for every $\tau^f$-open set $U(x,y)$ over $A$ the set $\exists y U(x,y)$ is a $\tau^f$-open set over $A$. We say that the $\tau^f$-topologies are closed under projections ($T$ is PCFT) if they are over every set $A$.

Fact 2.3 [S0] Let $U$ be a $\tau^f$-open set over a set $A$ and let $B \supseteq A$ be any set. Then $U$ is $\tau^f$-open over $B$.

We say that an $A$-invariant set $U$ has $SU$-rank $\alpha$ and write $SU(U) = \alpha$ if $\text{Max}\{SU(p) | p \in S(A), p^C \subseteq U\} = \alpha$. 

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Fact 2.4 [S0] Let \( \mathcal{U} \) be an unbounded \( \tau^f \)-open set over some set \( A \). Assume \( \mathcal{U} \) has bounded finite \( SU \)-rank. Then there exists a set \( B \supseteq A \) and \( \theta(x) \in L(B) \) of \( SU \)-rank 1 such that \( \theta^c \subseteq \mathcal{U} \cup acl(B) \). In case \( SU(\mathcal{U}) = 1 \), the set \( acl_s(A) \cup \mathcal{U} \) is Stone-open, where \( s \) is the sort of (elements of) \( \mathcal{U} \).

2.2 The extension property being first-order and PCFT

We recall some natural extensions of notions from [BPV]. By a pair \((M, P_M)\) of \( T \) we mean an \( L_P = L \cup \{P\} \)-structure, where \( M \) is a model of \( T \) and \( P \) is a new predicate symbol whose interpretation is an elementary submodel of \( M \). For the rest of this subsection, by a \( |T| \)-small type we mean a complete hyperimaginary type in \( \leq |T| \) variables over a hyperimaginary of length \( \leq |T| \).

Definition 2.5 Let \( \mathcal{P}_0, \mathcal{P}_1 \) be \( \emptyset \)-invariant families of \( |T| \)-small types.
1) We say that a pair \((M, P^M)\) satisfies the extension property for \( \mathcal{P}_0 \) if for every \( L \)-type \( p \in S(A), \ A \in dcl(M) \) with \( p \in \mathcal{P}_0 \) there is \( a \in P^M \) such that \( a \downarrow P^M \).
2) Let
\[
T_{E_{\mathcal{P}_0}} = \bigcap \{Th_{L_P}(M, P^M)| \text{the pair } (M, P^M) \text{ satisfies the extension property w.r.t. } \mathcal{P}_0 \}.
\]
3) We say that \( \mathcal{P}_0 \) dominates \( \mathcal{P}_1 \) w.r.t. the extension property if \((M, P^M)\) satisfies the extension property for \( \mathcal{P}_1 \) for every \(|T|^+\)-saturated pair \((M, P^M) \models T_{E_{\mathcal{P}_0}}\). In this case we write \( \mathcal{P}_0 \geq_{E_{\mathcal{P}_0}} \mathcal{P}_1 \).
4) We say that the extension property is first-order for \( \mathcal{P}_0 \) if \( \mathcal{P}_0 \geq_{E_{\mathcal{P}_0}} \mathcal{P}_0 \). We say that the extension property is first-order if the extension property is first-order for the family of all \(|T|\)-small types (equivalently, for the family of all real types over sets of size \( \leq |T| \)).

Fact 2.6 [S1] Let \( \mathcal{P}_0 \) be an \( \emptyset \)-invariant family of \(|T|\)-small types. Assume \( \mathcal{P}_0 \) is extension-closed and that the extension property is first-order for \( \mathcal{P}_0 \). Let \( \mathcal{P}^* \) be the maximal class of \(|T|\)-small types such that \( \mathcal{P}_0 \geq_{E_{\mathcal{P}_0}} \mathcal{P}^* \). Then \( \mathcal{P}^* \supseteq An(\mathcal{P}_0) \), where \( An(\mathcal{P}_0) \) denotes the class of all \(|T|\)-small types analyzable in \( \mathcal{P}_0 \) by hyperimaginaries.

Fact 2.7 [S1] Suppose the extension property is first-order in \( T \). Then \( T \) is PCFT.
3 Dichotomies for rank 1 types

We first prove a dichotomy between essential 1-basedness and strong-minimality for any minimal type with possibly no ordinal \( D \)-rank; this is in fact a special case, in general \( p \) may be any \( SU \)-rank 1 type but then we don't get a strongly-minimal set. Then we prove a strong version of this for \( D \)-rank 1 types: any such type is either 1-based or its algebraic closure (by a single formula) almost contains a non-algebraic formula; in the special case when \( p \) is in addition minimal we conclude that if \( p \) is not 1-based then it has Morley rank 1. In this section \( T \) is assumed to be a simple theory and we work in \( C^{eq} \).

The following definability result [S1, Proposition 4.4] will be useful.

Fact 3.1 Let \( q(x, y) \in S(\emptyset) \) and let \( \chi(x, y, z) \in L \) be such that \( \models \forall y \forall z \exists x \chi(x, y, z) \). Then the set

\[
U = \{ (e, c, b, a) \mid e \in acl(Cb(a)) \}
\]

is relatively Stone-open inside the type-definable set

\[
F = \{ (e, c, b, a) \mid b \downarrow a, \models \chi(c, b, a), tp(cb) = q \}.
\]

(where \( e \) is taken from a fixed sort too).

First, we prove certain version of the dichotomy theorem from [S1] (and generalizations of it in [S2]) that is closely related to Buechler's dichotomy. Here we assume that \( p \) itself is not essentially 1-based (rather than some type that is internal in \( p \)) and find a non-algebraic definable set contained in the algebraic closure of \( p \) (rather than almost-internal in \( p \)).

Proposition 3.2 Let \( T \) be a countable hypersimple theory and assume \( T^{eq} \) has PCFT. Let \( p \in S(\emptyset) \) be a type of \( SU \)-rank 1 that is not essentially 1-based by means of the forking-topology. Then \( acl(p^c) \) contains a weakly-minimal definable set defined over \( acl(p^c) \). If, in addition, \( p \) is minimal then \( acl(p^c) \) contains a strongly-minimal definable set.

Proof: By the assumption, there exists a type-definable forking-open set \( U \) over \( \bar{c} \) and \( \bar{c} \)-invariant Stone-dense subset \( D \subseteq U \) such that \( Cb(\bar{c}/a) \notin bdd(\bar{c}) \) for all \( a \in D \). By Baire category theorem for the Stone topology of \( U \) there are disjoint tuples \( \bar{c}_0, \bar{c}_1 \) such that \( \bar{c} = \bar{c}_0 \cup \bar{c}_1 \), and \( \chi(\bar{x}, \bar{x}_0, y) \in L \) with \( \forall \bar{x}_0 \exists \chi^{<\infty} x_1 \chi(\bar{x}_1, \bar{x}_0, y) \) such that
contains a non-empty invariant set over \( \bar{c} \) that is relatively Stone-open in \( \mathcal{U} \). So, let \( \mathcal{U}_0 \subseteq \mathcal{U}' \) be a non-empty type-definable forking-open set over \( \bar{c} \). Let \( a^* \in D \cap \mathcal{U}_0 \). Then \( e = Cb(\bar{c}/a^*) \notin \text{bdd}(\bar{c}) \). Let \( e^* \in \text{acl}^a(e) \setminus \text{bdd}(\bar{c}) \). Let \( s^* \) be the sort of \( e^* \). Now, by Fact 2.3 (we may assume that \( e \in \text{acl}(Cb(\bar{c}/a^*))) \) is relatively Stone-open over \( \bar{c} \) inside \( C^{*} \times \mathcal{U}_0 \). Since \( T^{eq} \) has PCFT,

\[
E = \{ e \in C^{*} | \exists a \in \mathcal{U}_0[ e \in \text{acl}(Cb(\bar{c}/a))] \}
\]

is an unbounded forking-open set over \( \bar{c} \) and moreover there exists an unbounded forking-open type-definable set \( E' \subseteq E \) over \( \bar{c} \). Since every element of \( E' \) is in the algebraic closure of some finite tuple of realizations of \( p \), we may assume, by Baire category theorem for the Stone topology of \( E' \), that \( SU(E') = n^* \) for some \( 0 < n^* < \omega \). Let \( e' \in E' \) be such that \( SU(e'/\bar{c}) = n^* \) and let \( A \supseteq \bar{c} \) be such that \( SU(e'/A) = n^*-1 \). By passing to the canonical base of \( \text{Ltp}(e'/A) \) we may assume that \( A \subseteq \text{acl}^a(p^c) \) (we may assume \( A \) is finite). Let \( E'_1 = \{ e' \in E' | \phi(e',a) \} \), where \( \phi(x,a) \in L(A) \) is a formula in \( tp(e'/A) \) that forks over \( \bar{c} \). By Fact 2.3, \( E'_1 \) is a forking-open (type-definable) set over \( A \), and \( SU(E'_1) = n^* - 1 \). Repeating this we get a forking-open (type-definable) set \( E^* \subseteq acl(p^c) \) over a finite set \( A^* \subseteq acl(p^c) \) of \( SU \)-rank 1. By Fact 2.4, there exists a non-algebraic formula \( \theta(x) \in L(A^*) \) such that \( \theta(C) \subseteq E^* \cup acl(A^*) \subseteq acl(p^c) \). Clearly, \( \theta(x) \) is weakly-minimal. If \( p \) is minimal then \( \theta(x) \) has ordinal Morley rank (since the language is countable, every realization of \( \theta(x) \) is in the algebraic closure of some tuple of realizations of \( p \) and thus by minimality of \( p \), for every countable set \( A \), the number of types of realizations of \( \theta \) over \( A \) is countable). Thus there exists a strongly-minimal \( \theta^*(x) \models \theta(x) \).

**Theorem 3.3** Let \( p \in S(\emptyset) \) be a type of \( D \)-rank 1. Then either \( p \) is 1-based or there exists \( \chi(x,\bar{z}) \in L \) with \( \forall \bar{z} \exists x < \infty \chi(x,\bar{z}) \) and an \( \emptyset \)-independent tuple \( \bar{c} \) of realizations of \( p \), and a (non-algebraic) formula \( \theta(x) \in L(\bar{c}) \) in some non-forking extension \( \bar{p} \) of \( p \) such that for any non-algebraic realization \( a \) of \( \theta(x) \) there is an \( \emptyset \)-independent tuple \( \bar{c}' \) of realizations of \( p \) such that \( \chi(a,\bar{c}') \).

Before presenting the proof we recall some standard terminology.

**Definition 3.4** A type \( p \in S(\emptyset) \) is 1-based if for every set \( C \) and tuple \( \bar{a} \subseteq p^c \) we have \( Cb(\bar{a}/C) \in \text{bdd}(\bar{a}) \).
Definition 3.5 An SU-rank 1 type $p \in S(\emptyset)$ is called linear if for every set $C$ and all $a, b \in p^C$ with $SU(ab/C) = 1$ we have $SU(Cb(ab/C)) \leq 1$.

Fact 3.6 [V,DK] Assume $p \in S(\emptyset)$ is a type of SU-rank 1. Then $p$ is 1-based iff $p$ is linear.

Lemma 3.7 Let $p \in S(\emptyset)$ be a type of SU-rank 1. Then $p$ is 1-based iff for every $a, b \in p^C$ and finite tuple $\bar{c}$ of realizations of $p$ we have $ab \downarrow bdd(ab) \cap bdd(\bar{c})$.

Proof: First note the following general observation (an easy SU-rank calculation).

Claim 3.8 Assume $SU(a) = 2$ and $SU(a/C) = 1$. Then $a \downarrow bdd(a) \cap bdd(C)$ iff $SU(Cb(a/C)) = 1$.

Now, clearly we only need to prove right to left. Assume the right hand side holds. By Claim 3.8 and Fact 3.6, it will be sufficient to show that for every $a, b \in p^C$ and set $C$ such that $SU(ab/C) = 1$ we have $ab \downarrow bdd(ab) \cap bdd(C)$. Indeed, otherwise $e \equiv Cb(ab/C) \notin bdd(ab)$. Let $(a_i b_i | i < \omega)$ be a sequence such that $(a_i b_i | i < \omega)^{ab}$ is a Morley sequence of $Lstp(ab/C)$. Then $e \in dcl(a_i b_i | i < \omega)$, and therefore $ab \downarrow bdd(ab) \cap bdd(a_i b_i | i < \omega)$.

Thus for some $i^* < \omega$, $ab \downarrow bdd(ab) \cap bdd(a_i b_i | i < i^*)$, a contradiction to our assumption.

Remark 3.9 Let $T$ be a 1-sorted theory of SU-rank 1. Then the extension property is first order in $T$ and in $T^{eq}$; thus $T$ and $T^{eq}$ have PCFT.

Proof: By [H1], any 1-sorted theory of SU-rank 1 eliminates the $\exists^\infty$ quantifier. Thus the extension property is first-order for 1-types (see [V, Proposition 2.15]). Since every non-algebraic type is non-orthogonal to a 1-type, the extension property is first-order in $T$ and in $T^{eq}$ by Fact 2.6. By Fact 2.7, $T$ and $T^{eq}$ have PCFT.
Notation 3.10 Let $D$ be a definable set over $\emptyset$. Let $D_*$ be the induced structure on $D$, that is, the universe of $D_*$ is $D$ and it is equipped with all $\emptyset$-definable subsets of $C$ that are subsets of $D^n$ for some $n < \omega$. $D_*$ is saturated. Note that $D_*^{eq}$ can be interpreted as the induced structure of $C^{eq}$ on $D$ and appropriate disjoint $\emptyset$-definable subsets of $C^{eq}$.

Claim 3.11 Let $T$ be any simple theory. Let $D$ be a non-algebraic $\emptyset$-definable set. Then for every tuples $\bar{a}, \bar{b}$ from $D$, we have $C \models \bar{a} \downarrow \bar{b}$ iff $D_* \models \bar{a} \downarrow \bar{b}$.

Proof: Let $e_C = Cb^C(\bar{a}/\bar{b})$ be the canonical base of $Lstp(\bar{a}/\bar{b})$ in the sense of $C$, and let $e_{D_*} = Cb^{D_*}(\bar{a}/\bar{b})$ be the canonical base of $Lstp(\bar{a}/\bar{b})$ in the sense of $D_*$. We need to show that $C \models e_C \in bdd(\bar{a})$ iff $D_* \models e_{D_*} \in bdd(\bar{a})$. Now, note that for every partial type $p(x, c)$ of $D_*$ and small set $A$ of $D_*$, $p(x, c)$ doesn’t fork over $A$ in the sense of $D_*$ iff $p(x, c)$ doesn’t fork over $A$ in the sense of $C$. Thus $e_C = e_{D_*}$, and the claim follows.

Proof of Theorem 3.3 Assume $p$ is not 1-based. Let $D \models p$ be of $D$-rank 1. By Lemma 3.7, there exists $a, b \in p^2$ and a finite tuple $\bar{c} = c_0c_1...c_n$ of realizations of $p$ such that $ab \downarrow bdd(ab) \sqcap bdd(\bar{c})$. By Claim 3.11, $D_* \models ab \downarrow bdd(ab) \sqcap bdd(\bar{c})$. From now on we work in $D_*$. As $SU(p) = 1$, we may clearly assume $\bar{c}$ is an $\emptyset$-independent sequence of realizations of $p$. Moreover, we may assume $a \downarrow \bar{c}$, and $\chi_0(b, a, \bar{c})$ for some $\chi_0(x, y, \bar{z}) \in L$ such that $\forall y \forall \bar{z} \exists z^< x \chi_0(x, y, \bar{z})$, and $\chi_1(z_n, z_{n-1}, ..., c_0, a, b)$ for some $\chi_1 \in L$ such that $\forall z_0 z_1 ... z_{n-1} \forall x \forall y \exists z^< z_n \chi_1(z_n, z_{n-1}, ..., z_0, x, y)$. Therefore $\{c_0, c_1, ..., c_{n-1}, a, b\}$ is $\emptyset$-independent (as the dimension of $\{c_0, c_1, c_2, ..., c_n, a, b\}$ in the pregeometry $(p^2, acl)$ is $n + 2$.) Let $\tilde{D} = \{(a', b') \in D^2| a' \notin acl(\bar{c}), \chi_0(b', a', \bar{c}), \chi_1(c_n, c_{n-1}, ..., c_0, a', b')\}$. Clearly, $\tilde{D}$ is a forking-open set over $\bar{c}$, and $(a, b) \in \tilde{D}$.

Subclaim 3.12 Let $\alpha$ be any ordinal. There is a partial type $\Lambda(xy, \langle \bar{z}_i| i < \alpha\rangle)$ over $\bar{c}$ such that for every $(a', b') \in \tilde{D}$, for every sequence $\langle \bar{c}_i| i < \alpha\rangle$, we
have \( \Lambda(a'b', \langle \bar{c}_i | i < \alpha \rangle) \) iff \( \langle \bar{c}_i | i < \alpha \rangle \) is a Morley sequence of \( tp(\bar{c}/a'b') \) that starts at \( \bar{c} \).

**Proof:** By the definition of \( \tilde{D} \), for every \( (a', b') \in \tilde{D} \), a sequence \( \langle \bar{c}_i | i < \alpha \rangle \) that starts at \( \bar{c} \) is a Morley sequence of \( tp(\bar{c}/a'b') \) iff it is indiscernible over \( \bar{c} \) and \( \{a'b'\} \cup \{\bar{c}_i^{< n} | i < \alpha \} \) is independent over \( \emptyset \). Since the type of \( \bar{c}_i^{< n} \) in such a sequence is fixed (equal to \( tp(\bar{c}^{< n}) \)) the required condition is a type-definable condition over \( \bar{c} \) on \( (a', b') \in \tilde{D} \) and sequence \( \langle \bar{c}_i | i < \alpha \rangle \).

**Subclaim 3.13** There exists a \( \chi^*(xy, \bar{z}) \in L \) and \( m^* < \omega \) such that \( \forall \bar{z} \exists x^< \infty \chi^*(xy, \bar{z}) \) and such that for every Morley sequence \( \langle \bar{c}_i | i \leq m^* \rangle \) of \( tp(\bar{c}/ab) \) that starts at \( \bar{c} \) we have

\[
\bigvee_{0 \leq i < j \leq m^*} \chi^*(ab, \bar{c}_0^{< n}, \bar{c}_1^{< n}, ..., \bar{c}_m^{< n}, \bar{c}_i, \bar{c}_j).
\]

**Proof:** We first show that there exists a \( \chi'(xy, \bar{z}) \in L \) and \( m^* < \omega \) such that \( \forall \bar{z} \exists x^< \infty \chi'(xy, \bar{z}) \) and such that for every Morley sequence \( \langle \bar{c}_i | i \leq m^* \rangle \) of \( tp(\bar{c}/ab) \) that starts at \( \bar{c} \) we have \( \chi'(ab, \bar{c}_0, ..., \bar{c}_m^*) \). Indeed, otherwise by Subclaim 3.12 and compactness there is a Morley sequence \( \langle \bar{c}_i^* | i < \omega \rangle \) of \( tp(\bar{c}/ab) \) that starts at \( \bar{c} \) such that \( ab \not\in acl(\bar{c}_i | i < \omega) \). But note that if \( e = Cb(\bar{c}/ab) \) then by our assumption \( e \) is interbounded with \( ab \) which is a contradiction to the fact that \( e \in acl(\bar{c}_i^* | i < \omega) \). The Subclaim follows now by compactness and dimension considerations.

Let \( S = \{(a', b') \in \tilde{D} \} \) for every Morley sequence \( \langle \bar{c}_i | i \leq m^* \rangle \) of \( tp(\bar{c}/a'b') \) that starts at \( \bar{c} \)

\[
\bigvee_{0 \leq i < j \leq m^*} \chi^*(a'b', \bar{c}_0^{< n}, \bar{c}_1^{< n}, ..., \bar{c}_m^{< n}, \bar{c}_i, \bar{c}_j).
\]

**Subclaim 3.14** Let \( S_1 \) be the projection of \( S \) on the first coordinate. Then \( S_1 \cup acl(\bar{c}) \) is an unbounded Stone-open set over \( \bar{c} \) and for every \( a' \in S_1 \) there exists an independent tuple \( \bar{c}' \) of realizations of \( p \) such that \( \chi_1^*(a', \bar{c}') \), where \( \chi_1^*(x, \bar{z}) \equiv \exists y \chi^*(xy, \bar{z}) \).

**Proof:** By Subclaim 3.13 \( (a, b) \in S \), and clearly \( a \not\in acl(\bar{c}) \). We conclude that \( S_1 \) is unbounded. Now, the set \( S \) is a forking-open set over \( \bar{c} \) by Subclaim 3.12 (and the fact that \( \{a \in C^s | a \not\in acl(\bar{c}) \} \) is a forking-open set over \( \bar{c} \) for any given sort \( s \)). Now, by Remark 3.9 \( D_* \) has PCFT and hence \( S_1 \) is
a forking-open set \( \bar{c} \). Now, as \( SU(S_1) = 1 \), Fact \[2.4\] implies that \( S_1 \cup acl(\bar{c}) \) is a Stone-open set over \( \bar{c} \). Assume now that \( a' \in S_1 \). Then for some \( b', (a',b') \in S \). So, \( \chi^*(a'b',\bar{c}_0^{<n},\bar{c}_1^{<n},...,\bar{c}_m^{<n}) \) for some (all) Morley sequence \( (\bar{c}_k|k \leq m^*) \) of \( tp(\bar{c}/a'b') \) that starts at \( \bar{c} \) and some \( 0 \leq i < j \leq m^* \). As \( \{a'b',\bar{c}_0^{<n},\bar{c}_1^{<n},...,\bar{c}_m^{<n}\} \) is \( \emptyset \)-independent we conclude, by counting dimensions in the pregeometry \( (D, acl) \), that \( \{\bar{c}_0^{<n},\bar{c}_1^{<n},...,\bar{c}_m^{<n}\} \) is \( \emptyset \)-independent and clearly \( \chi^*(a',\bar{c}_0^{<n},\bar{c}_1^{<n},...,\bar{c}_m^{<n}) \).

Now, let \( \theta(x) \in L(\bar{c}) \) be any formula such that \( a \models \theta(x) \) and \( \theta^c \subseteq S_1 \cup acl(\bar{c}) \). Then \( \theta(x) \) is the required formula.

As a special case we get Buechler’s dichotomy for minimal \( D \)-rank 1 types in any simple theory.

**Corollary 3.15** Let \( p \in S(\emptyset) \) be a minimal type with \( D(p) = 1 \). Then either \( p \) is 1-based or there exists \( \check{\chi}(x,z) \in L \) with \( \forall \bar{z} \exists z \in x \check{\chi}(x,z) \) and \( \theta^*(x) \in p(x) \) such that for any non-algebraic realization \( a \) of \( \theta^*(x) \) there is an \( \emptyset \)-independent tuple \( \bar{c} \) of realizations of \( p \) such that \( \check{\chi}(a,\bar{c}) \). In particular, \( RM(\theta^*(x)) = 1 \).

**Proof:** Assume \( p \) is not 1-based. Let \( \theta(x) \in L(\bar{c}), \check{\chi}(x,z) \in L \) be the formulas given by Theorem 3.3, so \( \theta(x) \in \bar{p} \), where \( \bar{p} \in S(\bar{c}) \) is the unique non-algebraic complete extension of \( p \) over \( \bar{c} \). Now, \( p^c \subseteq acl(\bar{c}) \cup \theta^c \). By compactness, there exists \( \theta^*(x) \in p \) with \( \theta^*(x)^c \subseteq acl(\bar{c}) \cup \theta^c \). It follows that for every non-algebraic realization \( a \) of \( \theta^*(x) \) there is an \( \emptyset \)-independent tuple \( \bar{c}' \) of realizations of \( p \) such that \( \check{\chi}(a,\bar{c}') \). To see that the latest implies \( RM(\theta^*) = 1 \), note that for any set \( A \), the formula \( \theta^*(x) \) has finitely many complete non-algebraic extensions over \( A \) (as \( tp(a_0,...a_n) \) has a unique non-algebraic complete extension over \( A \) for every \( \emptyset \)-independent realizations \( a_0,...,a_n \) of \( p \)).

**Lemma 3.16** Let \( D \) be a weakly-minimal definable set over \( \emptyset \). Let \( \bar{c} \subseteq D \) be any tuple. Then the set \( D^2_{\check{\chi}(\bar{c})} = \{(a,b) \in D^2| \bigwedge_{y \in bdd(ab) \cap bdd(\bar{c})} \check{\chi} \} \) is a forking-open set over \( \bar{c} \).

**Proof:** Assume \( (a,b) \in D^2_{\check{\chi}(\bar{c})} \). It will be sufficient to show that there exists a forking-open set \( U \) over \( \bar{c} \) such that \( (a,b) \in U \subseteq D^2_{\check{\chi}(\bar{c})} \). By Claim 3.11 and the fact that \( D_* \) is supersimple (and thus eliminates hyperimaginaries),
we may work in $D^e_\textsf{eq}$ and replace $bdd$ by $acl=acl^{eq}$ in the definition of $D^2_{\text{NO}}(\bar{c})$ (note that every forking-open set over some set $A$ in $D^e_\textsf{eq}$ is a forking open set over $A$ in $C^{eq}$.) So, from now on we work in $D^e_\textsf{eq}$. Clearly we may assume that $\bar{c}=c_0c_1...c_n$ is an $0$-independent. As $(D,acl)$ is a pregeometry, we may assume, as in the proof of Theorem 3.3 that there are $\chi_0(x,y,z)\in L$, $\chi_1(z_n,z_{n-1},...,z_0,x,y)\in L$ such that $a\perp\bar{c}$, $\chi_0(b,a,c)$, $\forall y\forall z\exists x<\infty \chi_0(x,y,z)$ and $\chi_1(c_n,c_{n-1},...,c_0,a,b)$, and

$$\forall z_0z_1...z_{n-1}\forall xy\exists x<\infty z_n\chi_1(z_n,z_{n-1},...,z_0,x,y).$$

Let

$$\tilde{D} = \{(a',b') \in D^2 | a' \not\in acl(\bar{c}), \chi_0(b',a',\bar{c}), \chi_1(c_n,c_{n-1},...,c_0,a',b')\}.$$  

Clearly, $\tilde{D}$ is a forking-open set over $\bar{c}$, and $(a,b) \in \tilde{D}$. Now, since $ab \subseteq acl(ab) \cap acl(\bar{c})$, there exists $e \in dcl(Cb(\bar{c}/ab)) \cap acl(\bar{c})$. Let $s$ be the sort of $e$. Let

$$U = \{(a',b') \in \tilde{D} | \exists e' \in (D^{eq}_s)[e' \not\in acl(\bar{c}) \land e' \in acl(Cb(\bar{c}/a'b'))]\}.$$  

To finish the proof it remains to show the following.

**Subclaim 3.17** $(a,b) \in U$ and $U \subseteq D^2_{\text{NO}}(\bar{c})$, and $U$ is a forking-open set over $\bar{c}$.

**Proof**: By the definition of $e$ and $U$, $(a,b) \in U$. By the definition $D^2_{\text{NO}}(\bar{c})$ and $U, U \subseteq D^2_{\text{NO}}(\bar{c})$. To prove that $U$ is a forking open set first note that for every $(a',b') \in \tilde{D}$, the dimension of $\{c_n,...c_1,c_0,a',b'\}$ in the pregeometry $(D,acl)$ is $n+2$, and therefore $\{c_{n-1},...c_1,c_0,a',b'\}$ is $0$-independent. In particular, $a'b'$ is independent from $\{c_{n-1},...c_1,c_0\}$ for all $(a',b') \in \tilde{D}$. By Fact 3.1 we conclude that the set $\{(e',a'b')| e' \in acl(Cb(\bar{c}/a'b'))\}$ is relatively Stone-open over $\bar{c}$ in $F = \{(e',a'b')| e' \in (D^{eq}_s)(a',b') \in \tilde{D}\}$. By Remark 3.9 $D^{eq}_s$ has PCFT, thus $U$ is a forking-open set over $\bar{c}$.

**Definition 3.18** A type $p \in S(\emptyset)$ is said to be densely $1$-based if for every finite tuple $\bar{c}$ of realizations of $p$ and every forking-open set $U$ over $\bar{c}$ with $(p^n)^c \cap U \neq \emptyset$ for some $n < \omega$ there exist $a \in U$ such that $a^{\downarrow \text{bdd}(\bar{c}) \cap \text{bdd}(a)}$.

**Remark 3.19** Clearly, if $p \in S(\emptyset)$ is $s$-essentially $1$-based then $p$ is densely $1$-based.

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Theorem 3.20 Let $p \in S(\emptyset)$ be a type of $D$-rank 1. If $p$ is densely 1-based then $p$ is 1-based. In particular, if $p \in S(\emptyset)$ is an s-essentially 1-based type of $D$-rank 1 then $p$ is 1-based.

Proof: Assume $p \in S(\emptyset)$ is a type of $D$-rank 1 and $p$ is densely 1-based. If $p$ is not 1-based then by Lemma 3.7, there are $a, b \in p^c$ and a finite tuple $\bar{c}$ of realizations of $p$ such that $\overset{\not\sim}{ab} \text{bdd}(ab) \cap \text{bdd}(\bar{c})$. Let $D \in p$ be of $D$-rank 1. By Lemma 3.16, we conclude that $D^{2\mathcal{NO}}(\bar{c})$ is a non-empty forking-open set over $\bar{c}$. Contradiction to the assumption that $p$ is densely 1-based.

Corollary 3.21 A supersimple unidimensional theory that is s-essentially 1-based is 1-based. In particular, any countable hypersimple unidimensional theory that is s-essentially 1-based is 1-based.

Proof: First, recall the following result [W1].

Fact 3.22 Let $T$ be any simple theory and work with hyperimaginaries. Assume $p \in S(A)$ is analyzable in an $A$-invariant family of 1-based types. Then $p$ is 1-based.

Let $T$ be a unidimensional supersimple theory that is s-essentially 1-based. Let $D$ be a weakly-minimal set (a non-algebraic definable set of minimal $D$-rank). Then any non-algebraic completion of $D$ is a type of $D$-rank 1 and in particular of $SU$-rank 1. By Theorem 3.20 $p$ is 1-based. By Fact 3.22 $T$ is 1-based. The last statement follows by supersimplicity of countable hypersimple unidimensional theories [S1].

4 Stable types in hypersimple unidimensional theories

In this section we observe that the existence of a non-algebraic stable partial type in a hypersimple unidimensional theory implies superstability. $T$ will denote an arbitrary complete theory.

The following definition is standard.
Definition 4.1 Let \( p(x) \) be a partial type over \( C \).
1) \( p(x) \) is called stable for \( \phi(x,y) \) if there does not exist a sequence \( (a_i,b_i|i < \omega) \) such that \( a_i \models p \) and such that \( \phi(a_i,b_j) \) iff \( i < j \).
2) \( p(x) \) is said to be stable if \( p(x) \) is stable for any formula \( \phi(x,y) \in L \).

In this section we show the following.

Proposition 4.2 Let \( T \) be a hypersimple unidimensional theory. Assume there exists a non-algebraic stable partial type. Then \( T \) is superstable.

The following claim follows easily from well known results.

Claim 4.3 Let \( p(x) \) be a partial type. Then the following are equivalent.
1) \( p = p(x) \) is stable.
2) For every infinite cardinal \( \lambda \) such that \( \lambda^{\lvert T \rvert} = \lambda \), for every set \( A \supseteq \text{dom}(p) \) with \( \lvert A \rvert = \lambda \), we have \( \lvert \{ q \in S(A) \mid p \subseteq q \} \rvert \leq \lambda \).
3) For some infinite cardinal \( \lambda \geq \lvert \text{dom}(p) \rvert \), for every set \( A \supseteq \text{dom}(p) \) with \( \lvert A \rvert = \lambda \), \( \lvert \{ q \in S(A) \mid p \subseteq q \} \rvert \leq \lambda \).
4) For every model \( M \), every type \( q \in S(M) \) that extends \( p \) is definable.
5) Every \( q \in S(B) \) that extends \( p \) is definable.
6) For every formula \( \phi = \phi(x,y) \), \( R(p,\phi,2) < \omega \).

Proof: 1) \( \Rightarrow \) (4 follows by the usual proof of definability of \( \phi \)-types over a model, using the fact that if \( p(x) \) is stable for \( \phi(x,y) \) then there is a \( \psi(x) \in p(x) \) such that \( \psi(x) \) is stable for \( \phi(x,y) \) (compactness). 4) \( \Rightarrow \) (2 is clear.
2) \( \Rightarrow \) (3 is trivial. 3) \( \Rightarrow \) (6: otherwise let \( \lambda \) be as given in 3) and let \( \mu \) be the minimal cardinal such that \( 2^\mu > \lambda \) and let \( \{a_\eta \mid \eta \in 2^{<\mu} \} \) be such that for every \( \bar{\eta} \in 2^\mu \), \( p(x) \wedge \bigwedge_{i<\mu} \phi(x,a_\eta[i])^{\bar{\eta}(i)} \) is consistent, contradicting the assumption in 3). 6) \( \Rightarrow \) (5: given \( q(x) \in S(B) \) extending \( p(x) \) and any \( \phi = \phi(x,y) \in L \) let \( r = R(q(x),\phi,2) \) and let \( \psi(x) \in q(x) \) such that \( r = R(\psi(x),\phi,2) \). Then, for any \( b \in B \) we have \( \phi(x,b) \in q \) iff \( R(\psi(x) \wedge \phi(x,b),\phi,2) = r \). This is a definable condition on \( b \) over \( B \). 5) \( \Rightarrow \) (2 is clear. 6) \( \Rightarrow \) (1 Otherwise there exists a sequence \( (a_i,b_i|i \in \mathbb{Q} \} \) such that \( a_i \models p \) and such that \( \phi(a_i,b_j) \) iff \( i < j \). The required consistency is now obvious.

Remark 4.4 1) Assume \( p_i(x_i) \) for \( i < n \ (n < \omega) \) are stable partial types over \( C \). Then so is \( \bigwedge_{i<n} p_i(x_i) \).
2) Assume \( p(x),q(y) \) are partial types over \( C \) such that for some small set \( B \) we have \( q^C \subseteq \text{acl}(p^C \cup B) \). Then, if \( p(x) \) is stable so is \( q(y) \).
3) Assume \( \Gamma(x) \equiv \bigvee_i p_i(x) \), where each of \( \Gamma(x) \) is a partial types over \( C \), and assume \( p_i(x) \) are stable. Then \( \Gamma(x) \) is stable.
**Proof:** 1) We may clearly assume \( n = 2 \). In this case let \( B \) be a set containing the domains of both \( p_0(x_0) \) and \( p_1(x_1) \), so the type of \((x_0, x_1)\) over \( B \) is determined by the type of \( x_0 \) over \( B \) and of \( x_1 \) over \( Bx_0 \). By stability of \( p_0(x_0) \) and \( p_1(x_1) \) we are done.

2) Let \( A \) be a sufficiently large superset of \( B \) and of the domains of \( p, q \) with \( |A|^{|T|} = |A| \). Then for every \( c \in q^c \) there is an algebraic formula \( \chi(y, x_0, \ldots, x_n, b) \in L(B) \) in \( y \), and realizations \( a_0, \ldots, a_n \) of \( p \) such that \( \models \chi(c, a_0, \ldots, a_n, b) \). By 1) and Claim 4.3 the number of possible types of the tuple \((a_0, \ldots a_n)\) over \( A \) is \( \leq |A| \) and thus so is the number of possible types of \( c \) over \( A \).

3) is immediate.

**Claim 4.5** Let \( T \) be simple. Assume \( p \in S(A) \) and almost \( q \)-internal, where \( q \) is a stable type over \( A \). Then \( p \) is stable.

**Proof:** There exists a small set \( B \) such that \( p^c \subseteq acl(q^c \cup B) \). By Remark 4.4(2), we are done.

**Lemma 4.6** Let \( T \) be simple. Assume \( q = q(x, b) \) is a stable partial type. Let \( A \supseteq b \) be a small infinite set such that \( |A|^{|T|+|b|} = |A| \). Then

\[
|\{ p \in S(A) \mid p \text{ is almost } q \text{-internal} \}| \leq |A|.
\]

**Proof:** Assume \( p \in S(A) \) is almost \( q \)-internal. Let \( A_0 \subseteq A \) be such that \( p \) doesn’t fork over \( A_0 \) and with \( b \subseteq A_0, |A_0| \leq |T| + |b| \). Now, \( p_0 = p|A_0 \) is almost \( q \)-internal and thus by Claim 4.5, \( p_0 \) is stable. In particular, every \( p \in S(A) \) that is almost \( q \)-internal extends a stable type over a subset of \( A \) of size \( \leq |T| + |b| \). By Claim 4.3(2) and the fact that the number of types over subsets of \( A \) of size \( \leq |T| + |b| \) is \( \leq |A| \), we are done.

**Corollary 4.7** Let \( T \) be hypersimple. Assume \( q = q(x, b) \) is a stable partial type and let \( A \supseteq b \) be a small set.

1) If \( A \) is infinite such that \( |A|^{|T|+|b|} = |A| \), then

\[
|\{ p \in S(A) \mid p \text{ is } q \text{-analyzable} \}| \leq |A|.
\]

2) Assume \( p \in S(A) \) is analyzable in \( q \) (by an imaginary sequence). Then \( p \) is stable.
Proof: To prove 1), it will be sufficient to note that for every \( \alpha < |T|^+ \)

\[ |\{tp((a_i|i \leq \alpha)/A) | (a_i|i \leq \alpha) \text{ is an analysis in } q \text{ over } A\}| \leq |A| \]

(we say that \((a_i|i \leq \alpha)\) is an analysis in \(q\) over \(A\) if \(tp(a_i/(a_i|j < i) \cup A)\)

is \(q\)-internal for all \(i \leq \alpha\)). Indeed, this follows by repeated applications of

Lemma 4.6 and the assumption that \(A\) is infinite and \(|A||T|^+|b| = |A|\) (which
implies \(|A|^{|T|} = |A|\)). Now, as in particular \(|A| \geq |T|^+\), we get the required
statement.

To prove 2), assume \(p \in S(A)\) is analyzable in \(q\) and let \(B \supseteq A\) be any
infinite small set such that \(|B||T|^+|b| = |B|\). It will be sufficient to show that
the number of complete extensions of \(p\) over \(B\) is \(\leq |B|\). Indeed, let \(a\) be any
realization of \(p\). Then there is an analysis \((a_i|i \leq \alpha)\) in \(q\) over \(A\) for some
\(\alpha < |T|^+\) such that \(a_\alpha = a\). Therefore \((a_i|i \leq \alpha)\) is an analysis in \(q\) over \(B\).
Thus, \(tp(a/B)\) is analyzable in \(q\) over \(B\). By part 1) we conclude that the
number of extensions of \(p\) over \(B\) is \(\leq |B|\), thus \(p\) is stable.

Proof of Proposition 4.2: Let \(q(x,b)\) be a stable non-algebraic partial type. By the assumption that \(T\) is a hypersimple unidimensional theory, every complete type over a superset of \(b\) is analyzable in \(q\). By Corollary 4.7(2), every complete type over a superset of \(b\) is stable. Thus \(T\) is stable over \(b\) and so \(T\) is stable. By \([H]\), \(T\) is superstable.

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