Global Convergence of the ODE Limit for Online Actor-Critic Algorithms in Reinforcement Learning

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Abstract

Actor-critic algorithms are widely used in reinforcement learning, but are challenging to mathematically analyse due to the online arrival of non-i.i.d. data samples. The distribution of the data samples dynamically changes as the model is updated, introducing a complex feedback loop between the data distribution and the reinforcement learning algorithm. We prove that, under a time rescaling, the online actor-critic algorithm with tabular parametrization converges to an ordinary differential equation (ODE) as the number of updates becomes large. The proof first establishes the geometric ergodicity of the data samples under a fixed actor policy. Then, using a Poisson equation, we prove that the fluctuations of the data samples around a dynamic probability measure, which is a function of the evolving actor model, vanish as the number of updates become large. Once the ODE limit has been derived, we study its convergence properties using a two time-scale analysis which asymptotically de-couples the critic ODE from the actor ODE. The convergence of the critic to the solution of the Bellman equation and the actor to the optimal policy are proven. In addition, a convergence rate to this global minimum is also established. Our convergence analysis holds under specific choices for the learning rates and exploration rates in the actor-critic algorithm, which could provide guidance for the implementation of actor-critic algorithms in practice.

1 Introduction

Actor-critic (AC) algorithms [16, 18] have become some of the most successful and widely-used methods in reinforcement learning (RL) [31]. AC algorithms are typically implemented in two ways: either as a batch or online algorithm. In the batch setting, there is a “double for loop” where one update of the actor in the outer for loop is followed by a large number of critic updates in the inner for loop to obtain a good approximation of the value function for the current policy. The convergence of batch AC has recently been studied in [19, 35, 39]. An online, two time-scale AC algorithm was first proposed in [16], where the actor and critic are updated simultaneously with i.i.d. data samples. In this paper, we study a class of online actor-critic [13, 37, 38] algorithms where the data samples arrive from a Markov chain [15] (instead of i.i.d. data samples) and prove the actor/critic converge to the solution of an ODE as the number learning steps becomes large. It is then proven that the solution of the ODE converges to the optimal policy.

We consider an actor-critic algorithm where the actor and critic are updated simultaneously at each new time step by using the data samples from simultaneous simulations of two different Markov decision processes (MDPs). Specifically, the data samples used to update the critic are from the original MDP while the samples for the actor are from an artificial MDP with a slightly different transition probability (which will be clearly defined in Section 2) such that the update direction of the actor asymptotically converges to the unbiased policy gradient direction (see the algorithm in [39] for details). The data samples from the MDPs are non-i.i.d. and the transition probability function depends upon the action selected at each time step. Actions are selected using the actor’s current policy. Therefore, the stationary distributions of the MDPs change as the actor evolves during learning. In order for the critic converge to the value function, an exploration component is included in the selection of the actions, where the exploration decays to zero as the

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number of learning steps becomes large. We find that carefully choosing the decay rate for the exploration as well as the learning rate is crucial for proving global convergence of the limit ODEs to the optimal policy and the learning rates we use can be easily implemented in practice.

1.1 Related literature

Policy gradient The policy gradient (PG) method [32] is one of the most important concepts in RL and has achieved great empirical success [28, 29]. However, PG algorithms involve non-convex optimization problems for tabular policy parameters [1] and are thus difficult to analyze mathematically. Recently, [1, 3, 14, 22, 23] have established the convergence and convergence rate to the global optimum for the standard PG method by assuming the value function is known. [3] proved that projected PG on the simplex does not suffer from spurious local optima. [1] proves that all of the stationary points of PG for a softmax tabular policy are actually the global optimum and natural PG converges at rate $O \left( \frac{1}{t} \right)$. [23] proves the convergence rate $O \left( \frac{1}{t} \right)$ for the PG method with a softmax tabular policy.

Actor-critic The AC algorithm was first developed in [32] and then extended to the Natural Actor-Critic (AC) algorithm in [27]. Batch AC algorithms [19, 35, 39, 40] involve a “double for loop” where the outer iteration updates the actor and, for each update of the actor, there is a large sub-iteration to solve the critic. [39] studied the global convergence of AC algorithms under the Linear Quadratic Regulator. [40] analyzed the finite-sample performance of the batch AC algorithm. [19] considered the sample complexity for the “decoupled” AC methods with i.i.d. data samples. [35], under the over-parametrized two-layer neural-network proved that the neural AC algorithm converges to a global optimum at a sub-linear rate. In online AC [13, 16, 37, 38], the actor and critic models update simultaneously but with two time-scales. The actor updates at a slower rate while critic updates more quickly to provide the actor an accurate policy gradient. [16] studies an online AC algorithm with Markovian data samples without using the ODE method and prove convergence to a stationary point. [37] proves that two time-scale algorithms with non-i.i.d. data samples and linear function approximation finds an $\epsilon$-stationary point with $O(\epsilon^{-2})$ samples, where $\epsilon$ measures the squared norm of the policy gradient. [38] under the compatibility condition [11, 32] between actor and critic, shows that two time-scale AC requires sample complexity at order $O(\epsilon^{-2.5} \log^5(\epsilon^{-1}))$ to converge to an $\epsilon$-stationary point. By carefully decreasing the exploration rate, [13] shows that the two time-scale natural AC algorithm has sample complexity of $O \left( \frac{1}{\delta^6} \right)$ for convergence to the global optimum. [12] proposes an off-policy variant of the natural AC algorithm based on Importance Sampling, where they use the Q-trace algorithm for the critic and provide a sample complexity of $O \left( \epsilon^{-3} \log \left( \frac{1}{\delta} \right) \right)$.

Stochastic approximation in RL Stochastic approximation [4, 5, 7] can be seen as a general framework to analyze RL algorithms. Two time-scale stochastic approximations [8, 9] are one of the most popular methods for AC [6, 13, 18, 37, 38] algorithms. [4, 5, 7] establish the classical ODE method and use it for the stability and convergence analysis of the (two time-scale) stochastic approximation where the stochastic error is a martingale difference sequence. [8, 9] proved convergence rate and finite time analysis for the two time-scale linear stochastic approximation in RL under an i.i.d. assumption. [6, 18] use the ODE method for two time-scale stochastic approximation in AC algorithms where the actor is updated by a policy iteration algorithm.

Our paper studies a different class of algorithms than this previous literature. We consider the global convergence of the ODE limit for the online tabular AC algorithm. First, we use a time re-scaling [30] of the algorithm (2) to map it into a time interval $[0, T]$, and the mathematical analysis required to prove convergence to the ODE limit is different from the classical ODE method in stochastic approximation theory [4, 5, 7, 17]. Second, unlike the batch AC with a nested loop structure [19, 35], our online algorithm updates the actor and critic simultaneously with dynamic Markovian sampling. Third, [16, 37, 38] also studies an online AC algorithm with non-i.i.d. data samples. However, they only prove convergence to a stationary point while we prove global convergence for the tabular AC algorithm by analyzing the limit ODE. The algorithm in [17] uses data samples which arrive from a time non-homogeneous Markov chain (non-i.i.d). However, the value function in [17] is the averaged reward while ours is the discounted sum of the rewards and [17] uses policy iteration to update the actor while we use the policy gradient theorem.
In our paper, we include exploration in the policy so that the Markov chain visits all states and actions. The exploration decays to zero at a certain rate as the number of learning steps become large. A careful choice of the exploration rate and the learning rate is necessary in order to prove global convergence. In particular, the exploration rate does not satisfy the standard conditions (sum of the squares is finite) in stochastic approximation theory in [4, 5, 6, 7, 18]. However, by using the time-rescaling limit, we are still able to establish an ODE limit for a class of actor-critic algorithms.

2 Actor-Critic Algorithms

Let \( \mathcal{M} = (\mathcal{X}, \mathcal{A}, p, \mu, r, \gamma) \) be an MDP, where \( \mathcal{X} \) is a finite discrete state space, \( \mathcal{A} \) is a finite discrete action space, \( p(x'|x,a) \) is the transition probability function, \( \mu \) is the initial probability distribution of the Markov chain, \( r(x,a) \) is a bounded reward function, and the discount factor is \( \gamma \in (0,1) \). Let the policy \( f(x,a) \) be the probability of selecting action \( a \) in state \( x \). The state and action-value functions \( V^f(x) : \mathcal{X} \to \mathbb{R} \) and \( Q^f(x,a) : \mathcal{X} \times \mathcal{A} \to \mathbb{R} \) are defined as the expected discounted sum of future rewards when actions are selected from the policy \( f \):

\[
V^f(x) = \mathbb{E} \left[ \sum_{k=0}^{\infty} \gamma^k \cdot r(x_k,a_k) \mid x_0 = x \right], \quad V^f(x,a) = \mathbb{E} \left[ \sum_{k=0}^{\infty} \gamma^k \cdot r(x_k,a_k) \mid x_0 = x, a_0 = a \right], \tag{2.1}
\]

where \( x_k \sim f(x_k,\cdot), x_{k+1} \sim p(\cdot \mid x_k,a_k) \) for all \( k \in \mathbb{Z}^+ \). Note that the transition kernel \( p \) and policy \( f \) induce a Markov chain on the state-action space \( \mathcal{X} \times \mathcal{A} \). Then for any \((x,a) \in \mathcal{X} \times \mathcal{A}\), define the state and state-action visiting measures respectively as \( \nu^f_{\mu} \) and \( \sigma^f_\mu \), where

\[
\nu^f_{\mu}(x) = \sum_{k=0}^{\infty} \gamma^k \cdot P(x_k = x), \quad \sigma^f_{\mu}(x,a) = \sum_{k=0}^{\infty} \gamma^k \cdot P(x_k = x, a_k = a) \tag{2.2}
\]

and \( x_0 \sim \mu(\cdot), a_k \sim f(x_k,\cdot), x_{k+1} \sim p(\cdot \mid x_k,a_k) \) for all \( k \geq 0 \). The goal of reinforcement learning is to learn the optimal policy \( f^* \) which maximizes the expected discounted sum of the future rewards:

\[
\max_f J(f),
\]

where the objective function \( J(f) \) is defined as

\[
J(f) = \mathbb{E} \left[ \sum_{k=0}^{\infty} \gamma^k \cdot r(x_k,a_k) \right] = \sum_{x \in \mathcal{X}} \mu(x) V^f(x) = \sum_{(x,a) \in \mathcal{X} \times \mathcal{A}} \sigma^f_\mu(x,a) r(x,a). \tag{2.3}
\]

Policy-based reinforcement learning methods optimize the objective function over a class of policies \( \{f_\theta \mid \theta \in \mathcal{B}\} \) using the policy gradient theorem \cite{71}. In practice, the value function in the policy gradient theorem is unknown and must therefore also be estimated by a statistical learning algorithm. Online actor-critic algorithms simultaneously estimate the value function using a critic model and the optimal policy using an actor model. In this paper, we specifically study a class of online actor-critic algorithms where the “actor” is a tabular softmax policy

\[
f_\theta(x,a) = \frac{e^{\theta(x,a)}}{\sum_{a' \in \mathcal{A}}} \tag{2.4}
\]

with parameters \( \theta = (\theta(x,a))_{(x,a) \in \mathcal{X} \times \mathcal{A}} \). The “critic” \( Q = (Q(x,a))_{(x,a) \in \mathcal{X} \times \mathcal{A}} \) is also tabular with a separate parameter for each state-action pair. The policy \( f_\theta(x) = (f_\theta(x,a))_{a \in \mathcal{A}} \) is a probability distribution on the set of actions \( \mathcal{A} \).

Define a new MDP \( \tilde{\mathcal{M}} = (\mathcal{X}, \mathcal{A}, \tilde{p}, \mu, r, \gamma) \) with the transition probability function

\[
\tilde{p}(x' \mid x,a) = \gamma \cdot p(x' \mid x,a) + (1 - \gamma) \cdot \mu(x'), \tag{2.5}
\]

\(^1\)Note that the series in equation (2.1) converge since \( \gamma \in (0,1) \) and \( r(x,a) \) is bounded.
Note that (2.5) is similar to the transition probability of MDP $\mathcal{M}$ except that with probability $1-\gamma$ the state will be randomly re-initialized with distribution $\mu$ \cite{23, 27, 28}. \cite{23} proved that the stationary distribution of $\mathcal{M}$ under policy $f$ is the $\frac{1}{1-\gamma} \sigma_\mu$ in (2.2). At the learning step $k$, we use $\theta_k$ to denote the estimate for the policy parameters while $Q_k$ is the estimate for the value function under the policy $f_{\theta_k}$. At step $k$, the sample $(x_k, a_k)$ used to update the actor parameters $\theta_k$ is generated from MDP $\mathcal{M}$ by policy $f_{\theta_k}$. Then we use the policy gradient theorem \cite{35} to update the actor and get new policy $f_{\theta_{k+1}}$. The sample $(x_k, a_k)$ is sampled from MDP $\mathcal{M}$ by the exploration policy $g_k$ (see equation (2.10)). We then update the critic by temporal difference learning \cite{36} to obtain the new critic approximation $Q_{k+1}$. An exploration policy is used to guarantee that the policy will have a positive probability to visit all states and actions. For notational convenience, we will sometimes use $f_k$ and $g_k$ to denote $f_{\theta_k}$ and $g_{\theta_k}$.

In summary, the samples $\{x_k, a_k\}_{k \geq 1}$ used to train the critic model are sampled from $\mathcal{M}$ under the exploration policy $g_k$:

$$
\begin{align*}
  x_0, a_0 & \xrightarrow{p(\cdot|x_0, a_0)} x_1 \xrightarrow{g_0(\cdot|x_0, \cdot)} x_2 \xrightarrow{p(\cdot|x_1, a_1)} x_2 \xrightarrow{g_1(\cdot|x_1, \cdot)} x_3 \cdots 
\end{align*}
$$

(2.6)

The samples $\{\tilde{x}_k, \tilde{a}_k\}_{k \geq 1}$ for the actor model are sampled from $\tilde{\mathcal{M}}$ under the policy $f_k$:

$$
\begin{align*}
  \tilde{x}_0, \tilde{a}_0 & \xrightarrow{\tilde{p}(\cdot|\tilde{x}_0, \tilde{a}_0)} \tilde{x}_1 \xrightarrow{f_0(\tilde{x}_0, \cdot)} \tilde{x}_2 \xrightarrow{\tilde{p}(\cdot|\tilde{x}_1, \tilde{a}_1)} \tilde{x}_2 \xrightarrow{f_1(\tilde{x}_1, \cdot)} \tilde{x}_3 \cdots 
\end{align*}
$$

(2.7)

and $\theta_k, Q_k$ are updated according to the actor-critic algorithm:

$$
\begin{align*}
  Q_{k+1}(x, a) &= Q_k(x, a) + \frac{\alpha}{N} \left( r(x_k, a_k) + \gamma \sum_{a''} Q_k(x_{k+1}, a'') g_k(x_{k+1}, a'') - Q_k(x_k, a_k) \right) \partial_{x,a} Q_k(x_k, a_k) \\
  \theta_{k+1}(x, a) &= \theta_k(x, a) + \frac{c_k}{N} Q_k(\tilde{x}_k, \tilde{a}_k) \partial_{x,a} \log f_k(\tilde{x}_k, \tilde{a}_k),
\end{align*}
$$

(2.8)

for $k = 0, 1, \ldots, TN$ and the notation $\partial_{x,a}$ is defined as the derivative with respect to the location $(x, a)$ in the tabular variable, that is:

$$
\begin{align*}
  \partial_{x,a} Q_k(x_k, a_k) := \partial_{Q(x,a)} Q_k(x_k, a_k) &= 1_{x_k = x, a_k = a}, \\
  \partial_{x,a} \log f_k(x_k, a_k) := \partial_{Q(x,a)} \log f_k(x_k, a_k) &= 1_{x_k = x} \left[ 1_{x_k = a} - f_k(x, a) \right]
\end{align*}
$$

(2.9)

The actions $a_k$ in (2.8) are selected from the distribution

$$
\begin{align*}
  g_k(x, a) &= \frac{\eta_k^N}{d_A} + (1 - \eta_k^N) \cdot f_k(x, a), \quad \forall (x, a) \in \mathcal{X} \times \mathcal{A}.
\end{align*}
$$

(2.10)

where $0 \leq \eta_k^N \leq 1$ and $d_A = |\mathcal{A}|$. That is, with probability $\eta_k^N$, we select an action uniformly at random and, with probability $1 - \eta_k^N$, we select an action from the current estimate for the optimal policy. We let the exploration rate decay during training, i.e., $\eta_k^N \rightarrow 0$ as $k \rightarrow \infty$. Note that the step-size for the online actor-critic algorithm (2.8) is $\frac{1}{N}$ and the number of learning steps is $TN$. We will later show that as $N \rightarrow \infty$ the critic and actor models converge to the solution of an ODE on the time interval $[0, T]$. Here we highlight that in order for the Q-learning algorithm converge, the policy needs to have positive probability to choose every action (see \cite{24, 33} for details) and this is why we add exploration in the policy used to generate data samples.

Finally, we remark that there are two limitations of our algorithm. First, in order to calculate an unbiased policy gradient to update the actor in the algorithm (2.8), we actually sample from two MDPs, which will be computationally expensive in practice (although this is standard in the literature \cite{23, 27, 28}). Second, the exploration combined with the two time scales of the online learning algorithm (2.8) will lead to a slightly slower convergence rate for our algorithm.

**Challenges for mathematical analysis:** Convergence analysis of the actor-critic algorithm (2.8) must address several technical challenges. The data samples are non-i.i.d. and their distribution depends upon the actor model, which changes as the parameters are updated. Actions are selected using the actor model,
which influences the states visited in the Markov chain and affects the actor model’s evolution in the learning algorithm. Thus, actor-critic algorithms introduce a complex feedback loop between the distribution of the data samples and the model updates. Another challenge is that the learning algorithm is not guaranteed to update the model in a descent direction for the objective function, which is an obstacle for proving global convergence to the optimal policy. Finally, due to the softmax policy, the objective function is non-convex.

**Overview of the proof:** In our mathematical approach, we prove that the actor-critic algorithm (2.8) converges to an ODE under an appropriate time re-scaling. We address the challenge of non-i.i.d. data depending upon the actor model in two steps. The proof first establishes the geometric ergodicity of the data samples to a stationary distribution \( \pi_{f^\theta} \) under a fixed actor policy \( f^\theta \). Then, using a Poisson equation, we prove that the fluctuations of the data samples around a dynamic probability measure \( \pi_{\theta^k} \), which is a function of the evolving actor model, vanish as the number of updates become large.

Once the ODE limit has been derived, we study its convergence properties using a two time-scale analysis which asymptotically de-couples the critic ODE from the actor ODE. The convergence of the critic to the solution of the Bellman equation and the actor to the optimal policy are proven. In addition, a convergence rate to this global minimum is also established. In order to prove the global convergence, the learning rate and exploration rate for the actor-critic algorithm must be carefully chosen.

### 3 Main Result

We prove that the actor and critic models converge to the solution of a nonlinear ODE system as the learning steps become large. Our results are proven under the following assumptions.

**Assumption 3.1.** The reward function \( r \) is bounded in \([0, 1]\). \( \mathcal{X} \) and \( \mathcal{A} \) are finite, discrete spaces.

In addition, an assumption regarding the ergodicity of the Markov chains (2.6) and (2.7) is required.

**Assumption 3.2.** For any finite \( \theta \), the Markov chain \((X, A)\) for the MDP \( M \) under exploration policy \( g^\theta \) and the Markov chain \((\tilde{X}, \tilde{A})\) for the MDP \( \tilde{M} \) under policy \( f^\theta \) are irreducible and non-periodic. Their stationary distributions \( \pi^f, \sigma^f \) (which exist and are unique by Section 1.3.3 of [20]) are globally Lipschitz in policy \( f \).

The global convergence proof also requires a careful choice for the learning rate and exploration rate.

**Assumption 3.3.** The learning rate and exploration rate are:

\[
\zeta^N_k = \frac{1}{1 + \frac{k}{N}}, \quad \eta^N_k = \frac{1}{1 + \log^2(\frac{k}{N}) + 1},
\]

thus \( \zeta^N_{\lfloor Nt \rfloor} \rightarrow \zeta_t = \frac{1}{1 + t}, \quad \eta^N_{\lfloor Nt \rfloor} \rightarrow \eta_t = \frac{1}{1 + \log^2(t + 1)} \).

**Remark 3.4.** The learning rate and exploration rate in (3.1) satisfy the following properties for any integer \( n \in \mathbb{N} \):

\[
\int_0^\infty \zeta_s ds = \infty, \quad \int_0^\infty \zeta_s^2 dt < \infty, \quad \int_0^\infty \zeta_s \eta_s ds < \infty, \quad \lim_{t \to \infty} \frac{\zeta_t}{\eta_t} = 0.
\]

These properties are verified in the Appendix A.

The main results of this paper are the following theorems.

**Theorem 3.5 (Limit Equations).** For any \( T > 0 \),

\[
\lim_{N \to \infty} \mathbb{E} \sup_{t \in [0, T]} \left[ \left\| \theta_{\lfloor Nt \rfloor} - \tilde{\theta}_t \right\| + \left\| Q_{\lfloor Nt \rfloor} - \tilde{Q}_t \right\| \right] = 0,
\]

(3.3)
where $\bar{Q}_t$ and $\bar{\theta}_t$ satisfy the nonlinear system of ODEs:

$$\frac{d\bar{Q}_t}{dt}(x,a) = \alpha \pi g_{\bar{\theta}_t}(x,a) \left( r(x,a) + \gamma \sum_{z,a''} \bar{Q}_t(z,a'')g_{\bar{\theta}_t}(z,a'')p(z|x,a) - \bar{Q}_t(x,a) \right),$$

$$\frac{d\bar{\theta}_t}{dt}(x,a) = \xi \sigma \mu^f_{\bar{\theta}_t}(x,a) \left[ \bar{Q}_t(x,a) - \sum_{a'} \bar{Q}_t(x,a')f_{\bar{\theta}_t}(x,a') \right].$$

(3.4)

with initial condition $(\bar{Q}_0, \bar{\theta}_0) = (Q_0, \theta_0)$.

Thus, the critic converges to the limit variable $\bar{Q}_t$ while the actor converges to the limit variable $\bar{\theta}_t$, where $\bar{Q}_t$ and $\bar{\theta}_t$ are solutions to a nonlinear system of ODEs. We then prove the convergence of the limit ODEs (3.4) to the value function and optimal policy. The convergence analysis also allows us to obtain convergence rates.

**Theorem 3.6** (Global Convergence). The limit critic model converges to the value function:

$$\|\bar{Q}_t - V^{f_{\theta_t}}\| = O\left(\frac{1}{\log^2 t}\right).$$

(3.5)

For an initial distribution $\mu(x) > 0, \forall x \in X$, the limit actor model converges to the optimal policy:

$$J(f^*) - J(f_{\theta_t}) = O\left(\frac{1}{\log t}\right).$$

(3.6)

where $f^*$ is any optimal policy.

**Remark 3.7.** We highlight several points regarding the convergence results in Theorem 3.5 and 3.6:

1. The limit ODEs in Theorem 3.5 have some similarities to the literature of stochastic approximation and ODE method by Borkar [4, 5]. In these previous articles, the long-time behaviour of the discrete-time stochastic algorithm will closely follow deterministic ODEs. The derivation of the ODEs and the convergence analysis in our paper are different than in [4, 5]. In our algorithm the learning rate for both actor and critic have the re-scaling $1/T$ and we study the algorithm under a time re-scaling. Thus when $N \to \infty$, for any time interval $[t, t + \Delta t]$, the number of parameters updates $\to \infty$. Therefore, the random fluctuations in the algorithm vanish and the algorithm will converge to the limit ODE in Theorem 3.5.

2. The polylog convergence rate in Theorem 3.6 is a consequence of the specific choice of the exploration rate $\eta_t = \frac{1}{T + \log^2(1 + t)}$. The effect of the exploration rate $\eta_t$ is similar to the effect of a learning rate. The specific function $\eta_t$ necessary to guarantee convergence is a consequence of our mathematical analysis.

3. Theorem 3.5 and 3.6 imply a convergence result for the discrete actor-critic algorithm: for any $\epsilon > 0$, there exists a $T$ and $N$ such that $\mathbb{E}[\|f_{\theta(t_N)} - f^*\|] < \frac{\epsilon}{T}$ for all $n \geq N$. The proof follows directly from Theorems 3.5 and 3.6. Specifically, for any $\epsilon > 0$, by (3.6) we can select a $T$ large enough such that $f_{\theta_T}$ is within $\frac{\epsilon}{T}$ of the global optimal policy. Then, we can apply (3.3) to select an $N$ large enough to ensure that the discrete algorithm $f_{\theta(nT)}$ is within $\frac{\epsilon}{T}$ of the ODE limit $f_{\theta_T}$.

4. The global convergence to the optimal policy strongly relies on the sufficient exploration initial distribution $\mu(x) > 0$ for any state [1]. However, as in [1] when the numbers of states becomes large, even with the uniformly random action selection mixed in, the visiting measure of the policy could place exponentially small probability (in the size of the state space) on particular states, which will significantly decrease the convergence rate in (3.6).
4 Derivation of the limit ODEs

We use the following steps to prove convergence to the limit ODEs:

- Prove a priori bounds for the actor and critic models.
- Derive random ODEs for the evolution of the actor and critic models. The ODEs will contain stochastic remainder terms from the non-i.i.d. data samples.
- Use a Poisson equation to estimate the fluctuations of the remainder terms around zero.
- Use Gronwall’s inequality to obtain the convergence to the limit ODEs.

4.1 A Priori Bounds

In order to prove convergence to the limit equation, we first establish a priori bounds for the parameters. In our proof, we will use $C, C_0$ and $C_T$ to denote generic constants. For notational convenience, we will sometimes use $\xi, \xi'$ and $\xi_k, \xi_k'$ to denote the elements $(x, a), (x', a')$ and data samples $(x_k, a_k), (\tilde{x}_k, \tilde{a}_k)$, respectively.

First, we establish a priori estimates for the actor and critic models.

**Lemma 4.1.** For any fixed $T > 0, N \in \mathbb{N}$, there exists a constant $C_T$ which only depends on $T$ such that

\[
\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |Q_k(x,a)| \leq C_T < \infty, \quad \forall k \leq NT
\]

and

\[
\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |\theta_k(x,a)| \leq C_T < \infty, \quad \forall k \leq NT.
\]

**Proof.** For the update algorithm in (2.8)

\[
Q_{k+1}(\xi) = Q_k(\xi) + \frac{\alpha}{N} \left( r(\xi_k) + \gamma \sum_{a''} Q_k(x_{k+1}, a'')g_k(x_{k+1}, a'') - Q_k(\xi_k) \right) \partial \xi Q_k(\xi_k),
\]

and we have the bound

\[
\sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_{k+1}(\xi)| \leq \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_k(\xi)| + \frac{C}{N} \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_k(\xi)| + \frac{C}{N}.
\]

Then, using a telescoping series, we have

\[
\sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_k(\xi)| = \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_0(\xi)| + \sum_{j=1}^{k} \left( \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_j(\xi)| - \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_{j-1}(\xi)| \right)
\]

\[
\leq \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_0(\xi)| + \sum_{j=1}^{k} \left( \frac{C}{N} \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_{j-1}(\xi)| + \frac{C}{N} \right)
\]

\[
\leq \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_0(\xi)| + \frac{C}{N} \sum_{j=1}^{k} \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_{j-1}(\xi)| + C
\]

\[
\leq C + \frac{C}{N} \sum_{j=1}^{k} \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |Q_{j-1}(\xi)|,
\]

where the last inequality follows from the fact that $Q_0$ is a fixed finite vector. Then, by the discrete Gronwall lemma and using $\frac{k}{N} \leq T$, we have

\[
\sup_{(x,a) \in \mathcal{X} \times \mathcal{A}} |Q_k(x,a)| \leq C \exp \left( \frac{Ck}{N} \right) \leq C \exp(CT) = C_T, \quad \forall k \leq NT.
\]
Recall that the update for the actor model is

$$
\theta_{k+1}(\xi) = \theta_k(\xi) + \frac{C_T}{N} Q_k(\xi_k) \partial_\xi \log f_k(\xi_k)
$$

$$
= \theta_k(\xi) + \frac{C_T}{N} Q_k(\xi_k) \mathbb{1}_{\{\tilde{a}_k = a\}} \left[ \mathbb{1}_{\{\tilde{a}_k = a\}} - f_k(\tilde{x}_k, a) \right],
$$

(4.6)

which together with the bound for the critic in (4.5) leads to

$$
\sup_{\xi \in \mathcal{X} \times \mathcal{A}} |\theta_{k+1}(\xi)| \leq \sup_{\xi \in \mathcal{X} \times \mathcal{A}} |\theta_k(\xi)| + \frac{C_T}{N}.
$$

(4.7)

Then, using a telescoping series, we immediately obtain the bound in the statement of the lemma. \(\square\)

## 4.2 Evolution of the Pre-limit Process

From their definitions in (2.1), \(V^f(x)\) and \(V^f(x, a)\) are related via the formula

$$
V^f(x) = \sum_a V^f(x, a) f(x, a).
$$

(4.8)

Define the state and state-action visiting measures, respectively, as \(\nu^f_\mu\) and \(\sigma^f_\mu\), where

$$
\nu^f_\mu(x) = \sum_{k=0}^\infty \gamma^k \cdot P(x_k = x), \quad \sigma^f_\mu(x, a) = \sum_{k=0}^\infty \gamma^k \cdot P(x_k = x, a_k = a),
$$

(4.9)

where \(x_0 \sim \mu(\cdot), a_k \sim f(x_k, \cdot)\) and \(x_{k+1} \sim p(\cdot | x_k, a_k)\) for all \(k \geq 0\). By definition, we have \(\sigma^f_\mu(x, a) = f(x, a) \cdot \nu^f_\mu(x)\) and, by [15], the stationary distribution of \(\mathcal{M}\) is the corresponding visitation measure of \(\mathcal{M}\).

**Notation** We first clarify some of the notation that will be used in the analysis.

(a) For any \(k \geq 0\), let \(P_{\theta_k}\) denote the transition probability for the Markov chain \((X, A)\) induced by \(\mathcal{M}\) under softmax policy \(f_k\) and let \(\Pi_{\theta_k}\) denote the transition probability for the Markov chain \((\tilde{X}, \tilde{A})\) induced by \(\mathcal{M}\) under exploration policy \(g_k\). That is,

$$
P_{\theta_k}(x, a; x', a') = p(x' | x, a) g_k(x', a'),
$$

$$
\Pi_{\theta_k}(x, a; x', a') = \overline{p}(x' | x, a) f_k(x', a').
$$

(4.10)

(b) Let \(\sigma^{f_k}\) and \(\pi^{g_k}\) denote the stationary distributions (whose existence and uniqueness are given by Assumption 3.2) for the transition probability \(\Pi_{\theta_k}\) and \(P_{\theta_k}\), respectively.

(c) Define the \(\sigma\)-field of events generated by the samples \(\xi_1, \cdots, \xi_n, \tilde{\xi}_1, \ldots, \tilde{\xi}_n\) in (2.6) and (2.7) to be \(\mathcal{F}_n\). Then, for any Borel function \(h(\theta, \xi)\),

$$
E \left[ h \left( \theta_n, \tilde{\xi}_{n+1} \right) \mid \mathcal{F}_n \right] = \sum_{y \in \mathcal{X} \times \mathcal{A}} h(\theta_n, y) \Pi_{\theta_n}(\xi_n; y).
$$

(4.11)

For any function \(h(\theta, \xi)\), we shall denote the partial mapping \(\xi \to h(\theta, \xi)\) by \(h_\theta\) and define the function

$$
\Pi_\theta h_\theta(\xi) := \sum_{y \in \mathcal{X} \times \mathcal{A}} h(\theta, y) \Pi_\theta(\xi; y).
$$

Using the visiting measures in (4.9), the policy gradient can be evaluated using the following formula.

**Theorem 4.2** (Policy Gradient Theorem [32]). For the MDP starting from \(\mu\), the policy gradient for \(f_\theta\) is

$$
\nabla_\theta J(f_\theta) = \sum_{x, a} \sigma^f_\mu(x, a) V^f_\mu(x, a) \nabla_\theta \log f_\theta(x, a),
$$

(4.12)
Let the advantage function of policy \(f\) denoted by
\[
A^f(x, a) = V^f(x, a) - V^f(x), \quad \forall (x, a) \in \mathcal{X} \times \mathcal{A},
\]  
(4.13)
and the gradient \(\nabla_{\theta} J(f_\theta)\) can be evaluated using the following formula when \(f_\theta\) satisfies the softmax policy (2.4).

**Lemma 4.3.** Define \(\partial_{x,a} J(f_\theta) := \frac{\partial J(f_\theta)}{\partial f_\theta(x,a)}\) and then for the tabular policy (2.4), by policy gradient theorem (4.12), we have
\[
\partial_{x,a} J(f_\theta) = \sigma^{f_\theta}_\mu(x, a) A^{f_\theta}(x, a).
\]  
(4.14)

**Proof.** By the policy gradient theorem, we have
\[
\partial_{x,a} J(f_\theta) = \sum_{x', a'} \nu^{f_\theta}_\mu(x') f_\theta(x', a') \mathbbm{1}_{\{a' = a\}} \left[ f^f_\theta(x', a') - f_\theta(x', a) \right] V^{f_\theta}(x', a')
\]  
\[
= \sum_{a'} \nu^{f_\theta}_\mu(x) f_\theta(x, a') \left[ f^f_\theta(x, a') - f_\theta(x, a) \right] V^{f_\theta}(x, a')
\]  
\[
= \nu^{f_\theta}_\mu(x) f_\theta(x, a) V^{f_\theta}(x, a) - \nu^{f_\theta}_\mu(x) f_\theta(x, a') \left[ \sum_{a'} f_\theta(x, a') V^{f_\theta}(x, a') \right]
\]  
\[
= \nu^{f_\theta}_\mu(x) f_\theta(x, a) A^{f_\theta}(x, a)
\]  
\[
= \sigma^{f_\theta}_\mu(x, a) A^{f_\theta}(x, a).
\]

\( \square \)

Using a telescoping series and the update equation for the actor (2.8),
\[
\theta_{\{Nt\}}(x, a) = \theta_0(x, a) + \frac{1}{N} \sum_{k=0}^{[Nt]-1} \zeta_k^N Q_k(\bar{x}_k, \bar{a}_k) \partial_{x,a} \log f_k(\bar{x}_k, \bar{a}_k).
\]  
(4.16)

Note that \(\xi = (x, a), \bar{\xi}_k = (\bar{x}_k, \bar{a}_k)\) and define
\[
M^N_t(\xi) = \frac{1}{N} \sum_{k=0}^{[Nt]-1} \zeta_k^N Q_k(\bar{x}_k, \bar{a}_k) \partial_{x,a} \log f_k(\bar{x}_k) - \frac{1}{N} \sum_{k=0}^{[Nt]-1} \sum_{\xi' \in \mathcal{X} \times \mathcal{A}} \zeta_k^N Q_k(\xi') \partial_{x,a} \log f_k(\xi') \sigma^{f_k}_\mu(\xi'),
\]  
(4.17)

where \(\sigma^{f_k}_\mu\) is the visiting measure for \(\mathcal{M}\) under policy \(f_k\). Combining (4.16) and (4.17), we obtain the following pre-limit equation for the actor parameters:
\[
\begin{align*}
\theta_{\{Nt\}}(x, a) - \theta_0(x, a) \\
\quad = \frac{1}{N} \sum_{k=0}^{[Nt]-1} \sum_{\xi' \in \mathcal{X} \times \mathcal{A}} \zeta_k^N Q_k(\xi') \partial_{x,a} \log f_k(\xi') \sigma^{f_k}_\mu(\xi') + M^N_t(x, a)
\end{align*}
\]  
\[
= \frac{1}{N} \sum_{k=0}^{[Nt]-1} \sum_{a'} \nu^{f_k}_\mu(x) f_k(x, a') \left[ \mathbbm{1}_{\{a' = a\}} - f_k(x, a) \right] Q_k(x, a') + M^N_t(x, a)
\]  
\[
= \frac{1}{N} \sum_{k=0}^{[Nt]-1} \zeta_k^N \sigma^{f_k}_\mu(x, a) \left[ Q_k(x, a) - \sum_{a'} Q_k(x, a') f_k(x, a') \right] + M^N_t(x, a)
\]  
\[
= \left. \frac{\partial}{\partial t} \right|_{t=0} \zeta_{\{Ns\}}^N \sigma^{f_{\{Ns\}}}_\mu(x, a) \left[ Q_{\{Ns\}}(x, a) - \sum_{a'} Q_{\{Ns\}}(x, a') f_{\{Ns\}}(x, a') \right] ds + M^N_t(x, a) + O(N^{-1}),
\]  
(4.18)

where step (a) uses the a priori bound for the critic \(Q_k\) in Lemma 4.1.
Using a similar method, we can also prove the convergence of the Markov chains in our paper under the time-evolving actor policy updated using the actor-critic algorithm \( (\tilde{N}, T) \in \mathcal{X} \times \mathcal{A} \).

Define

\[
M^{1,N}_t(x, a, k) = \frac{1}{N} \sum_{k=0}^{N-1} Q_k(x_k, a_k) \frac{d}{d\tau} Q_k(x_k, a_k).
\]

\[
M^{2,N}_t(x, a, k) = \frac{1}{N} \sum_{k=0}^{N-1} \gamma Q_k(x_k, a_k) \frac{d}{d\tau} Q_k(x_k, a_k).
\]

\[
M^{3,N}_t(x, a, k) = \frac{1}{N} \sum_{k=0}^{N-1} \frac{d}{d\tau} Q_k(x_k, a_k).
\]

Let \( \Pi_{\theta}^{n_0} \) denote the n-step transition matrix under the policy \( f_{\theta} \). Then, for any fixed \( T > 0 \), there exists an integer \( n_0 \) such that the following uniform estimates hold for all \( \{\theta_k\}_{0 \leq k \leq NT} \) and \( N \in \mathbb{N} \) for the algorithm (2.8).

**Lemma 4.4.** Let \( \Pi_{\theta}^{n_0} \) denote the n-step transition matrix under the policy \( f_{\theta} \). Then, for any fixed \( T > 0 \), there exists an integer \( n_0 \) such that the following uniform estimates hold for all \( \{\theta_k\}_{0 \leq k \leq NT} \) and \( N \in \mathbb{N} \) for the algorithm (2.8).

- **Lower bound for the stationary distribution:**
  \[
  \inf_{k \leq NT} \sigma_{\theta_k}^F(x, a) \geq C \epsilon_T^{n_0}, \quad \forall (x, a) \in \mathcal{X} \times \mathcal{A}.
  \]
  \[ (4.22) \]

where \( C, \epsilon_T > 0 \) are positive constants.

- **Uniform geometric ergodicity:**
  \[
  \sup_{k \leq NT} \|\Pi_{\theta_k}^n (\xi) - \sigma_{\theta_k}^F (\cdot)\| \leq (1 - \beta_T)^{\frac{n_0}{\epsilon_T}}, \quad \forall \xi \in \mathcal{X} \times \mathcal{A}.
  \]
  \[ (4.23) \]

where \( \beta_T \in (0, 1) \) is a positive constant.
Proof. By Assumption 3.2 and Lemma 1.8.2 of [26], for any fixed $\tilde{\theta} \in \mathbb{R}^d$, there exists an $n_0 = n_0(\tilde{\theta}) \in \mathbb{N}$ such that

\[
\Pi_{\tilde{\theta}}^{n_0}(\xi; \xi') > 0 \quad \forall \xi, \xi'.
\] (4.24)

For any $(\xi, \xi')$,

\[
\Pi_{\tilde{\theta}}^{n_0}(\xi; \xi') = \sum_{\xi_1, \cdots, \xi_{n_0-1}} \Pi_{\tilde{\theta}}(\xi; \xi_1) \cdots \Pi_{\tilde{\theta}}(\xi_{n_0-1}; \xi')
\]
\[
= \sum_{\xi_1, \cdots, \xi_{n_0-1}} \bar{p}(x_1 | x, a) f_{\tilde{\theta}}(x_1, a_1) \cdots \bar{p}(x_{n_0-1} | x_{n_0-1}, a_{n_0-1}) f_{\tilde{\theta}}(x_{n_0-1}, a'),
\] (4.25)

where the constant $C$ is defined as

\[
C = C(n_0) := \inf_{x, a, x'} \sum_{\xi_1, \cdots, \xi_{n_0-1}} \bar{p}(x_1 | x, a) \cdots \bar{p}(x_{n_0-1} | x_{n_0-1}, a_{n_0-1}) > 0,
\] (4.26)

where $C > 0$ is because (4.24).

Due to $f_{\tilde{\theta}}$ being a softmax policy and the bound from Lemma 4.1, there exists a constant $\epsilon_T > 0$ such that

\[
\inf_{k \leq \epsilon_T} f_k(x, a) > \epsilon_T, \quad \forall (x, a) \in \mathcal{X} \times \mathcal{A}.
\] (4.27)

Then, using similar analysis as in (4.25) with constant $n_0 = n_0(\tilde{\theta})$ and $C = C(n_0)$, we have for all $k \leq \epsilon_T$

\[
\Pi'^{n_0}_{\tilde{\theta}}(\xi; \xi') \geq C e_{\epsilon_T}^{n_0}, \quad \forall \xi, \xi'.
\] (4.28)

Thus, we can derive a lower bound for the stationary distribution

\[
\inf_{k \leq \epsilon_T} \sigma^f_{\mu}(x', a') = \inf_{k \leq \epsilon_T} \sum_{x, a} \sigma^f_{\mu}(x, a) \Pi'^{n_0}_{\tilde{\theta}}(x, a; x', a')
\]
\[
\geq \inf_{k \leq \epsilon_T} \sum_{x, a} \sigma^f_{\mu}(x, a) C e_{\epsilon_T}^{n_0}
\]
\[
\geq (a) C e_{\epsilon_T}^{n_0}
\]
\[
> 0,
\] (4.29)

where step (a) is because $\sigma^f_{\mu}$ is a probability and thus the summation equals to 1. We can now establish the uniform geometric ergodicity of the Markov chain. Let us choose $\beta_T = \inf_{k \leq \epsilon_T} \min_{\xi, \xi'} \Pi'^{n_0}_{\tilde{\theta}}(\xi, \xi') > 0$ in (4.23), where $\beta_T > 0$ is by (4.28). Thus, for $\forall k \leq \epsilon_T$, the Markov chain with transition probability $\Pi_{\theta_k}$ satisfies Doeblin’s condition. In particular, we can show that

\[
\Pi'^{n_0}_{\theta_k}(\xi, \xi') \geq \beta_T > 0, \quad \forall \xi, \xi'.
\] (4.30)

Since $n_0$ and $\beta_T$ are independent of $\theta_k$, we can apply Theorem 16.2.4 of [25] to prove that for all $k \leq \epsilon_T$

\[
||\Pi'^{n_0}_{\theta_k}(\xi; \cdot) - \sigma^f_{\mu} (\cdot)|| \leq (1 - \beta_T)^{|\mathcal{K}|} \quad \forall \xi \in \mathcal{X} \times \mathcal{A},
\] (4.31)

which proves the uniform geometric ergodicity (4.23). \qed

Then, using the same method as in Lemma 4.4, we can prove a similar result for the MDP $\mathcal{M}$ with exploration policy $g_k$.

**Corollary 4.5.** Let $P^\theta_n$ denote the $n$-step transition matrix under policy $g_k$. Then, for any fixed $T < \infty$, there exists an integer $n_0$ and a constant

\[
C = C(n_0) := \inf_{x, a, x'} \sum_{\xi_1, \cdots, \xi_{n_0-1}} p(x_1 | x, a) \cdots p(x_{n_0-1} | x_{n_0-1}, a_{n_0-1}) > 0,
\] (4.32)

such that the following uniform estimate holds for all $\{\theta_k\}_{0 \leq k \leq \epsilon_T}$ and $N \in \mathbb{N}$ for the update algorithm (2.8):
• **Lower bound for the stationary distribution:**

\[
\inf_{k \leq NT} \pi^{\theta_k}(x, a) \geq C \left( \eta^{N}_{[NT]} \right)^{n_0}, \quad \forall (x, a) \in X \times A.
\]  

(4.33)

• **Uniform geometric ergodicity:**

\[
\sup_{k \leq NT} \| \Pi_{\theta_k}^{n}(\xi; \cdot) - \pi^{\theta_k}(\cdot) \| \leq (1 - \beta_T)^{\frac{n_0}{\eta}} \quad \forall \xi \in X \times A,
\]  

(4.34)

where \( \beta_T = C \left( \eta^{N}_{[NT]} \right)^{n_0} \in (0, 1) \) is a positive constant.

**Remark 4.6.** Without loss of generality, we suppose the integer \( n_0 \) in Lemma 4.4 and Corollary 4.5 are the same. The proof of Corollary 4.5 is the same as the proof of Lemma 4.4 and the detailed proof can be found in Appendix B.

In order to prove the stochastic fluctuation term vanishes as \( N \to \infty \), we first introduce a Poisson equation with a uniformly bounded solution.

**Lemma 4.7.** For any \( N \in \mathbb{N} \), state-action pair \( \xi = (x, a), T > 0 \) and \( k \leq NT \), the Poisson equation

\[
\nu_{\theta_k}(\xi') - \Pi_{\theta_k} \nu_{\theta_k}(\xi') = \mathbb{I}_{\{\xi' = \xi\}} - \sigma^{f_k}(\xi), \quad \xi' \in X \times A
\]  

(4.35)

has a solution\(^2\)

\[
\nu_{\theta_k}(\xi') := \sum_{n \geq 0} \left[ \Pi_{\theta_k}^{n}(\xi'; \xi) - \sigma^{f_k}(\xi) \right],
\]  

(4.36)

and there exists a constant \( C_T \) (which only depends on \( T \)) such that

\[
\sup_{k \leq NT} |\nu_{\theta_k}(\xi')| \leq C_T, \quad \forall \xi' \in X \times A.
\]  

(4.37)

**Proof.** Due to the uniform geometric convergence rate (4.23) for all \( k \leq NT \) in Lemma 4.4, there exists a \( \beta_T > 0 \) (independent with \( k \)) such that for any \( \xi' \in X \times A \)

\[
|\Pi_{\theta_k}^{n}(\xi'; \xi) - \sigma^{f_k}(\xi)| \leq (1 - \beta_T)^{\frac{n_0}{\eta}}, \quad \forall k \leq NT
\]  

(4.38)

which can be used to show the convergence of the series in (4.36). Consequently, \( \nu_{\theta_k} \) is well-defined. The uniform bound (4.37) follows from

\[
|\nu_{\theta_k}(\xi')| \leq \sum_{n \geq 0} |\Pi_{\theta_k}^{n}(\xi'; \xi) - \sigma^{f_k}(\xi)| \leq \sum_{n \geq 0} (1 - \beta_T)^{\frac{n_0}{\eta}} \leq C_T.
\]  

(4.39)

Finally, we can verify that \( \nu_{\theta_k} \) is a solution to the Poisson equation by observing that

\[
\Pi_{\theta_k} \nu_{\theta_k}(\xi') = \sum_{y} \nu_{\theta_k}(y) \Pi_{\theta_k}(\xi'; y)
\]

\[
= \sum_{y} \left( \sum_{n \geq 0} \left[ \Pi_{\theta_k}^{n}(y; \xi) - \sigma^{f_k}(\xi) \right] \right) \Pi_{\theta_k}(\xi'; y)
\]

\[
= \left( \sum_{n \geq 0} \left[ \Pi_{\theta_k}^{n}(y; \xi) - \sigma^{f_k}(\xi) \right] \right) \Pi_{\theta_k}(\xi'; y)
\]

\[
= \sum_{n \geq 1} \left[ \Pi_{\theta_k}^{n}(\xi'; \xi) - \sigma^{f_k}(\xi) \right]
\]

\[
= \nu_{\theta_k}(\xi') - (\mathbb{I}_{\{\xi' = \xi\}} - \sigma^{f_k}(\xi)),
\]

where the step (a) uses (4.38) and the Dominated Convergence Theorem.

\[^{2}\text{We do not prove uniqueness of the solution to the Poisson equation (4.35). For the purposes of our later analysis, it is only necessary to find a uniformly bounded solution } \nu_{\theta_k} \text{ which satisfies (4.36).}\]
Using the Poisson equation (4.7), we can prove that the fluctuations of the data samples around a dynamic visiting measure $\sigma^f_\mu$ decay when the iteration steps becomes large.

**Lemma 4.8.** For any fixed state action pair $\xi = (x, a)$ and $T > 0$,

$$
\lim_{N \to \infty} E \left\{ \frac{1}{N} \sum_{k=0}^{[NT]-1} \zeta_k \left[ \mathbb{1}_{(\xi_k = \xi)} - \sigma^f_\mu (\xi) \right] \right\} = 0. \tag{4.41}
$$

**Proof.** We define the error $\epsilon_k$ to be

$$
\epsilon_k := \zeta_k \left[ \mathbb{1}_{(\xi_{k+1} = \xi)} - \sigma^f_\mu (\xi) \right] = \zeta_k \left[ \nu_{\theta_k} (\xi_{k+1}) - \Pi_{\theta_k} \nu_{\theta_k} (\bar{\xi}_{k+1}) \right] + \zeta_k \left[ \Pi_{\theta_k} \nu_{\theta_k} (\xi_k) - \Pi_{\theta_k} \nu_{\theta_k} (\bar{\xi}_{k+1}) \right], \tag{4.42}
$$

where we have used the definition of the Poisson equation (4.35). Let

$$
\psi_\theta(y) = \Pi_{\theta} \nu_{\theta}(y). \tag{4.43}
$$

Then, we have that

$$
\sum_{k=0}^{[NT]-1} \epsilon_k = \sum_{k=0}^{[NT]-1} \zeta_k \left[ \nu_{\theta_k} (\xi_{k+1}) - \Pi_{\theta_k} \nu_{\theta_k} (\bar{\xi}_{k+1}) \right] + \sum_{k=0}^{[NT]-1} \zeta_k \left[ \psi_{\theta_k} (\xi_k) - \psi_{\theta_k} (\bar{\xi}_{k+1}) \right]
$$

$$
= \sum_{k=0}^{[NT]-1} \zeta_k \left[ \nu_{\theta_k} (\xi_{k+1}) - \Pi_{\theta_k} \nu_{\theta_k} (\bar{\xi}_{k+1}) \right] + \zeta_k \psi_{\theta_0} (\bar{\xi}_0) + \sum_{k=1}^{[NT]-1} \zeta_k \left[ \psi_{\theta_k} (\xi_k) - \psi_{\theta_{k-1}} (\xi_k) \right]
$$

$$
+ \sum_{k=1}^{[NT]-1} \left( \zeta_k - \zeta_{k-1} \right) \psi_{\theta_{k-1}} (\xi_k) - \zeta_{[NT]-1} \psi_{\theta_{[NT]-1}} (\bar{\xi}_{[NT]}) \tag{4.44}
$$

Define the error term

$$
\sum_{k=0}^{[NT]-1} \epsilon_k = \sum_{k=0}^{[NT]-1} \epsilon_k^{(1)} + \sum_{k=1}^{[NT]-1} \epsilon_k^{(2)} + \sum_{k=1}^{[NT]-1} \epsilon_k^{(3)} + \rho_{[NT]:0}, \tag{4.45}
$$

where

$$
\epsilon_k^{(1)} = \zeta_k \left[ \nu_{\theta_k} (\bar{\xi}_{k+1}) - \Pi_{\theta_k} \nu_{\theta_k} (\bar{\xi}_{k}) \right],
$$

$$
\epsilon_k^{(2)} = \zeta_k \left[ \psi_{\theta_k} (\bar{\xi}_{k}) - \psi_{\theta_{k-1}} (\bar{\xi}_{k}) \right],
$$

$$
\epsilon_k^{(3)} = \left( \zeta_k - \zeta_{k-1} \right) \psi_{\theta_{k-1}} (\bar{\xi}_{k}),
$$

$$
\rho_{[NT]:0} = \zeta_{0} \psi_{\theta_0} (\bar{\xi}_0) - \zeta_{[NT]-1} \psi_{\theta_{[NT]-1}} (\bar{\xi}_{[NT]}). \tag{4.46}
$$

To prove the convergence (4.41), it suffices to appropriately bound the fluctuation term $\sum_{k=0}^{[NT]-1} \epsilon_k$. Actually, the first term can be bound due to the martingale property while the second term can be bounded using the uniform geometric ergodicity and Lipschitz continuity. The third and fourth terms are uniformly bounded by (4.37).

For the first term in (4.45), note that

$$
E \left\{ \nu_{\theta_k} (\bar{\xi}_{k+1}) \mid \mathcal{F}_k \right\} = \Pi_{\theta_k} \nu_{\theta_k} (\bar{\xi}_k), \tag{4.47}
$$
Thus
\[
\left\{ Z_n = \sum_{k=0}^{n-1} \gamma_k(1), \ F_n \right\}_{n \geq 0}
\]
is a martingale and since the conditional expectation is a contraction in $L^2$, we have
\[
E \left[ \Pi \nu_{\theta_k} \left( \overline{\xi} \right) \right]^2 \leq E \left[ \nu_{\theta_k} \left( \overline{\xi}_{k+1} \right) \right]^2.
\]
(4.48)

Then,
\[
E \left[ \frac{1}{N} \sum_{k=0}^{\lfloor NT \rfloor - 1} \epsilon_k(1)^2 \right] = \frac{1}{N^2} \sum_{k=0}^{\lfloor NT \rfloor - 1} \left( \frac{N}{NT} \right)^2 E \left[ \Pi \nu_{\theta_k} \left( \overline{\xi}_k \right) - \nu_{\theta_k} \left( \overline{\xi}_{k+1} \right) \right]^2
\]
\[
\leq \frac{4}{N^2} \sum_{k=0}^{\lfloor NT \rfloor - 1} \left( \frac{N}{NT} \right)^2 E \left[ \nu_{\theta_k} \left( \overline{\xi}_{k+1} \right) \right]^2
\]
(4.49)
\[
\leq \frac{4C_T N^2}{N^2} \sum_{k=0}^{\lfloor NT \rfloor - 1} \left( \frac{N}{NT} \right)^2,
\]
where the step (a) is by the uniform boundedness (4.37). Thus, for any $T > 0$,
\[
\lim_{N \to \infty} E \left[ \frac{1}{N} \sum_{k=0}^{\lfloor NT \rfloor - 1} \epsilon_k(1)^2 \right] = 0.
\]
(4.50)

For the second term of (4.45), by the uniform geometric ergodicity (4.23), for any fixed $\gamma_0 > 0$ we can choose $N_0$ large enough such that
\[
\sup_{k \leq NT} \sum_{n=\lfloor N_0 T \rfloor}^{\infty} \left| \Pi_{\theta_k}^n (y, \xi) - \sigma_{\mu}(\xi) \right| < \gamma_0, \ \forall y \in X \times A
\]
(4.51)

\[
\left| \frac{1}{N} \sum_{k=1}^{\lfloor NT \rfloor - 1} \epsilon_k(2) \right| = \left| \frac{1}{N} \sum_{k=1}^{\lfloor NT \rfloor - 1} \zeta_k \left[ \psi_{\theta_k} \left( \overline{\xi}_k \right) - \psi_{\theta_{k-1}} \left( \overline{\xi}_k \right) \right] \right|
\]
\[
\leq \left| \frac{1}{N} \sum_{k=1}^{\lfloor NT \rfloor - 1} \zeta_k \left[ \sum_{n=1}^{\lfloor N_0 T \rfloor - 1} \left( \Pi_{\theta_k}^n \left( \overline{\xi}_k, \xi \right) - \sigma_{\mu}(\xi) \right) - \sum_{n=1}^{\lfloor N_0 T \rfloor - 1} \left( \Pi_{\theta_{k-1}}^n \left( \overline{\xi}_{k-1}, \xi \right) - \sigma_{\mu}(\xi) \right) \right] \right| + 2C_T \gamma_0
\]
\[
\leq \left| \frac{1}{N} \sum_{k=1}^{\lfloor NT \rfloor - 1} \zeta_k \left[ \sum_{n=1}^{\lfloor N_0 T \rfloor - 1} \left( \Pi_{\theta_k}^n \left( \overline{\xi}_k, \xi \right) - \Pi_{\theta_{k-1}}^n \left( \overline{\xi}_{k-1}, \xi \right) \right) \right] \right| + \frac{\lfloor N_0 T \rfloor - 1}{N} \left| \sum_{k=1}^{\lfloor NT \rfloor - 1} \zeta_k^N \left[ \sigma_{\mu}(\xi) - \sigma_{\mu}(\xi) \right] \right| + 2C_T \gamma_0
\]
\[
:= I_1^N + I_2^N + 2C_T \gamma_0.
\]
(4.52)

By Lemma 4.1, for any $k \leq NT$ we have
\[
\| \theta_k - \theta_{k-1} \| \leq \sum_{x, a \in X \times A} \left| \theta_k(x, a) - \theta_{k-1}(x, a) \right| \leq \frac{C_T}{N}
\]
For any finite \( n \), \( \Pi_n^\theta \) is Lipschitz continuous in \( \theta \). Consequently,

\[
I_1^N \leq \frac{|N_0T|}{N} \sum_{k=1}^{[NT]-1} \zeta_k^N C \|\theta_k - \theta_{k-1}\| \leq \frac{C_T}{N},
\]

\[
I_2^N \leq \frac{|N_0T|}{N} \sum_{k=1}^{[NT]-1} \zeta_k^N C \|\theta_k - \theta_{k-1}\| \leq \frac{C_T}{N},
\]

where the constant \( C_T \) only depends on the fixed \( N_0, T \). Thus, when \( N \) is large enough,

\[
1 \leq \sum_{k=1}^{[NT]-1} \zeta_k^N C \|\theta_k - \theta_{k-1}\| \leq 4\gamma_0 (4.54)
\]

where step (a) is by the uniform bound (4.37). Therefore,

\[
\lim_{N \to \infty} \mathbb{E} \left| \frac{1}{N} \sum_{k=1}^{[NT]-1} \epsilon_k^{(2)} \right| = 0
\]

(4.55)

For the third term of (4.45),

\[
\mathbb{E} \left| \frac{1}{N} \sum_{k=1}^{[NT]-1} \epsilon_k^{(3)} \right| = \frac{1}{N} \mathbb{E} \left| \sum_{k=1}^{[NT]-1} (\zeta_k^N - \zeta_{k-1}^N) \psi_{\theta_k-1} (\xi_k) \right| \leq \frac{C_T}{N} \sum_{k=1}^{[NT]-1} (\zeta_k^N - \zeta_{k-1}^N) \leq \frac{C_T}{N}
\]

(4.56)

which together with (4.50), (4.55) and (4.57) derive the convergence of \( \frac{1}{N} \sum_{k=0}^{[NT]-1} \epsilon_k \) and therefore proving (4.41).

\[\Box\]

4.4 Identification of the Limit ODEs

We next prove the convergence of \( M_t^N \), which will allow us to prove the convergence to the limit ODEs (3.4).

**Lemma 4.9.** For any \( \xi = (x, a) \) and the stochastic error \( M_t^N \) defined in (4.17), we have

\[
\lim_{N \to \infty} \sup_{t \in (0, T]} \mathbb{E} |M_t^N(\xi)| = 0.
\]

(4.58)
Proof. For any $K \in \mathbb{N}$ and $\Delta = \frac{1}{K}$, we have
\[
M^N_t(\xi) = \sum_{j=0}^{K-1} \frac{1}{[\Delta N]} \sum_{k=0}^{(j+1)[\Delta N] - 1} \zeta^N_k \left( Q_k(\tilde{\xi}_k) \partial_k \log f_k(\tilde{\xi}_k) - \sum_{\xi' \in \mathcal{X} \times \mathcal{A}} Q_k(\xi') \partial_k \log f_k(\xi') \sigma^f_k(\xi') \right) + o(1)
\]
\[
= \sum_{j=0}^{K-1} \frac{1}{[\Delta N]} \sum_{k=0}^{(j+1)[\Delta N] - 1} \zeta^N_k \left( Q_{j \Delta N}(\tilde{\xi}_k) \partial_k \log f_{j \Delta N}(\tilde{\xi}_k) - \sum_{\xi' \in \mathcal{X} \times \mathcal{A}} Q_{j \Delta N}(\xi') \partial_k \log f_{j \Delta N}(\xi') \sigma^f_k(\xi') \right) + o(1)
\]
\[
+ \sum_{j=0}^{K-1} \frac{1}{[\Delta N]} \sum_{k=0}^{(j+1)[\Delta N] - 1} \zeta^N_k \left[ \left( Q_{j \Delta N}(\tilde{\xi}_k) \partial_k \log f_{j \Delta N}(\tilde{\xi}_k) - \sum_{\xi' \in \mathcal{X} \times \mathcal{A}} Q_{j \Delta N}(\xi') \partial_k \log f_{j \Delta N}(\xi') \sigma^f_k(\xi') \right) - \left( Q_{j \Delta N}(\tilde{\xi}_k) \partial_k \log f_{j \Delta N}(\tilde{\xi}_k) - \sum_{\xi' \in \mathcal{X} \times \mathcal{A}} Q_{j \Delta N}(\xi') \partial_k \log f_{j \Delta N}(\xi') \sigma^f_k(\xi') \right) \right] + o(1)
\]
\[
:= \sum_{j=0}^{K-1} \Delta I^N_{1,j} + \sum_{j=0}^{K-1} \Delta I^N_{2,j} + o(1). \tag{4.59}
\]
where the term $o(1)$ goes to zero, at least, in $L^1$ as $N \to \infty$.

To prove the convergence of the first term, note that
\[
Q_{j \Delta N}(\tilde{\xi}_k) \partial_k \log f_{j \Delta N}(\tilde{\xi}_k) - \sum_{\xi'} Q_{j \Delta N}(\xi') \partial_k \log f_{j \Delta N}(\xi') \sigma^f_k(\xi')
\]
\[
= \sum_{\xi'} Q_{j \Delta N}(\xi') \partial_k \log f_{j \Delta N}(\xi') \mathbb{1}_{\{\tilde{\xi}_k = \xi'\}} - \sum_{\xi'} Q_{j \Delta N}(\xi') \partial_k \log f_{j \Delta N}(\xi') \sigma^f_k(\xi')
\]
\[
= \sum_{\xi'} Q_{j \Delta N}(\xi') \partial_k \log f_{j \Delta N}(\xi') \left[ \mathbb{1}_{\{\tilde{\xi}_k = \xi'\}} - \sigma^f_k(\xi') \right]. \tag{4.60}
\]
Thus, for any $j \in \{0, 1, \ldots, K\}$,
\[
|I^N_{1,j}| = \frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N] - 1} \zeta^N_k \left| \sum_{\xi'} Q_{j \Delta N}(\xi') \partial_k \log f_{j \Delta N}(\xi') \left[ \mathbb{1}_{\{\tilde{\xi}_k = \xi'\}} - \sigma^f_k(\xi') \right] \right|
\]
\[
= \sum_{\xi'} Q_{j \Delta N}(\xi') \partial_k \log f_{j \Delta N}(\xi') \left[ \mathbb{1}_{\{\tilde{\xi}_k = \xi'\}} - \sigma^f_k(\xi') \right]
\]
\[
\leq C_T \sum_{\xi'} \left| \frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N] - 1} \zeta^N_k \left[ \mathbb{1}_{\{\tilde{\xi}_k = \xi'\}} - \sigma^f_k(\xi') \right] \right|
\]
\[
\leq C_T \sum_{\xi'} \left| \frac{1}{[\Delta N]} \sum_{k=j[\Delta N]}^{(j+1)[\Delta N] - 1} \zeta^N_k \left[ \mathbb{1}_{\{\tilde{\xi}_k = \xi'\}} - \sigma^f_k(\xi') \right] \right|, \tag{4.61}
\]
which together with Lemma 4.8 proves
\[
\lim_{N \to \infty} \mathbb{E} |I^N_{1,j}| = 0. \tag{4.62}
\]
Thus,
\[
\sum_{j=0}^{K-1} \Delta I^N_{1,j} = \Delta \sum_{j=0}^{K-1} O_P(1) = t \sum_{j=0}^{K-1} O_P(1) = K,
\]
which derives the convergence of the first term.

For the second term, using the bound in Lemma 4.1, we have for any $k \leq TN$
\[
\sup_{\xi' \in \mathcal{X} \times \mathcal{A}} |Q_k(\xi')| \leq C_T,
\]
\[
\sup_{\xi'} |Q_k(\xi') - Q_{k-1}(\xi')| \leq \frac{C_T}{N}. \tag{4.64}
\]
Noting that
\[ \partial_{\xi} \log f_k(\xi') = \mathbb{1}_{\{x'=x\}} \left[ \mathbb{1}_{\{a'=a\}} - f_k(x',a) \right], \]
then by the Lipschitz continuity of the softmax transformation and (4.64) we have
\[ |Q_k(\xi')\partial_{\xi} \log f_k(\xi') - Q_{k-1}(\xi')\partial_{\xi} \log f_{k-1}(\xi')| \]
\[ = |Q_k(\xi') - Q_{k-1}(\xi')||\partial_{\xi} \log f_k(\xi') - \partial_{\xi} \log f_{k-1}(\xi')| \]
\[ \leq C \frac{T}{N} + C_T |\partial_{\xi} f_k(\xi') - \partial_{\xi} f_{k-1}(\xi')| \]
\[ \leq C \frac{T}{N} + C_T |\theta_k - \theta_{k-1}| \leq \frac{C_T}{N}. \] (4.65)

Then, for any \( j \in 0,1,\ldots,K - 1 \) and any \( k \in [j|\Delta N|,(j+1)|\Delta N| - 1] \),
\[ |Q_k(\xi')\partial_{\xi} \log f_k(\xi') - Q_{j|\Delta N|}(\xi')\partial_{\xi} \log f_{j|\Delta N|}(\xi')| \leq \frac{C(k-j|\Delta N|)}{N}. \] (4.66)

Thus,
\[ \sum_{j=0}^{K-1} \Delta I_{2,j}^N \leq C \sum_{j=0}^{K-1} \Delta \left[ \frac{1}{|\Delta N|} \sum_{k=j|\Delta N|}^{(j+1)|\Delta N| - 1} \right] \zeta_k^N k - j|\Delta N| \]
\[ = C \sum_{j=0}^{K-1} \Delta \left[ \frac{1}{|\Delta N|} \sum_{k=0}^{(j+1)|\Delta N| - 1} k \right] \]
\[ \leq C \sum_{j=0}^{K-1} \Delta \left[ \frac{1}{|\Delta N|} \frac{|\Delta N| - 1}{2} \right] \]
\[ \leq C \sum_{j=0}^{K-1} \Delta \left[ \frac{|\Delta N|}{2} \right] \]
\[ \leq C \sum_{j=0}^{K-1} \Delta^2 \]
\[ \leq C \Delta. \] (4.67)

Collecting our results, we have shown that
\[ \lim_{N \to \infty} \sup_{t \in (0,T]} E|M^N_t| \leq C \frac{T}{K} \] (4.68)

Note that \( K \) was arbitrary. Consequently, we obtain
\[ \lim_{N \to \infty} \sup_{t \in (0,T]} E|M^N_t| = 0, \] (4.69)
concluding the proof of the lemma.

Following the same method, we can finish proving the convergence of the stochastic fluctuation terms and the detailed proof can be found in Appendix C.

**Lemma 4.10.** For \( t \in [0,T] \), \( M^1,N_t, M^2,N_t, M^3,N_t \) \( L^1 \) as \( N \to \infty \).

Using Lemma 4.9 and 4.10, we can now finish the derivation of the limit ODEs.

**Proof of Theorem 3.5:** Due to Assumption 3.2 and the Lipschitz continuity of softmax transformation, we know \( \sigma^N, \pi^N \) is Lipschitz continuous in \( \theta_t \). By Theorem 2.2 and Theorem 2.17 of [34], for any initial value, there exists a unique solution on \((0,\infty)\) for the ODE system (3.4). Let \( Q_t(x,a), \theta_t(x,a) \) be the solution of
and the Lipschitz continuity from Assumption 3.2, we have for \( t \in [0, T] \)
\[
|\sigma^{f^{\nu_0}}_{\theta}(x, a)Q^{\nu_0}(x, a) - \sigma^{f_{\theta_0}}_{\theta}(x, a)Q_0(x, a)|
\leq C_T \left[ \|\theta^{\nu_0}_t - \tilde{\theta}_t\| + \|Q^{\nu_0}_t - \tilde{Q}_t\| \right],
\]
and we can also show for the exploration policy from (2.10) that
\[
|g^{\nu_0}_t(x, a) - g_0(x, a)|
\leq \frac{|\eta^{\nu_0}_t - \eta_0|}{\alpha} + \left| \left(1 - \eta^{\nu_0}_t\right) \cdot f_{\theta^{\nu_0}_t}(x, a) - \left(1 - \eta_0\right) \cdot f_{\theta_0}(x, a) \right| + |\eta^{\nu_0}_t f_{\theta^{\nu_0}_t}(x, a) - \eta f_{\theta_0}(x, a)|
\leq C \frac{|\eta^{\nu_0}_t - \eta_0|}{\alpha} + C \|\theta^{\nu_0}_t - \tilde{\theta}_t\|.
\]
Combining (4.18), (4.20), and (3.4) and using the same decomposition method as in (4.70), we have for \( t \in [0, T] \)
\[
\|\theta^{\nu_0}_t - \tilde{\theta}_t\| + \|Q^{\nu_0}_t - \tilde{Q}_t\|
\leq \sum_{(x, a) \in X \times A} \left[ \|\theta^{\nu_0}_t(x, a) - \tilde{\theta}_t(x, a)\| + \|Q^{\nu_0}_t(x, a) - \tilde{Q}_t(x, a)\| \right]
\leq C_T \int_0^t \left[ \|\theta^{\nu_0}_{t_s} - \tilde{\theta}_t\| + \|Q^{\nu_0}_{t_s} - \tilde{Q}_t\| \right] ds + |M^{\nu_0}_t| + \sum_{i=1}^3 |M^{i, \nu_0}_t| + O(N^{-1})
+ C_T \int_0^t \left[ |\zeta^{\nu_0}_{t_s} - \zeta_s| + |\eta^{\nu_0}_{t_s} - \eta_s| \right] ds.
\]
Define
\[
\varphi^{\nu_0}_t := \|\theta^{\nu_0}_t - \tilde{\theta}_t\| + \|Q^{\nu_0}_t - \tilde{Q}_t\|
\]
\[
B^{\nu_0}_t := |M^{\nu_0}_t| + \sum_{i=1}^3 |M^{i, \nu_0}_t| + O(N^{-1}) + C_T \int_0^t \left[ |\zeta^{\nu_0}_{t_s} - \zeta_s| + |\eta^{\nu_0}_{t_s} - \eta_s| \right] ds.
\]
Due to Lemma 4.9 and 4.10,
\[
\lim_{N \to \infty} \mathbb{E} \sup_{t \in [0, T]} B^{\nu_0}_t = 0.
\]
Taking the supremum and expectation of (4.72),
\[
\mathbb{E} \sup_{s \in [0, t]} \varphi^{\nu_0}_s \leq C_T \int_0^t \mathbb{E} \sup_{r \in [0, s]} \varphi^{\nu_0}_r ds + \mathbb{E} \sup_{s \in [0, t]} B^{\nu_0}_s, \quad \forall t \in [0, T]
\]
By Gronwall’s lemma, we have
\[
\mathbb{E} \sup_{t \in [0, T]} \varphi^{\nu_0}_t \leq \mathbb{E} \sup_{t \in [0, T]} B^{\nu_0}_t + C_T \int_0^T \mathbb{E} \sup_{s \in [0, t]} B^{\nu_0}_s dt \leq C_T \mathbb{E} \sup_{t \in [0, T]} B^{\nu_0}_t,
\]
which together with (4.74) proves the convergence (3.3).

5 Convergence of Limit ODEs
We now study the convergence of the limit actor-critic algorithm, which satisfies the ODE system (3.4).
5.1 Critic convergence

Now we prove convergence of the critic (3.5), which states that the critic model will converge to the state-action value function during training. We first derive an ODE for the difference between the critic and the value function. Then, we use a comparison lemma, a two time-scale analysis, and the properties of the learning and exploration rates (3.2) to prove the convergence of the critic to the value function.

Recall that the value function $V^g_t$ satisfies the Bellman equation

$$r(x, a) + \gamma \sum_{z, a''} V^{g_t}(z, a'') g_t(z, a'') p(z|x, a) - V^{g_t}(x, a) = 0. \quad (5.1)$$

Define the difference

$$\phi_t = \bar{Q}_t - V^{\bar{g}_t}, \quad (5.2)$$

As a first step, we prove an a priori uniform bound for the critic in the update (3.4). Without loss of generality, we initialize the ODE as $\bar{Q}_0 = 0$ (we can always define $\bar{Q}_t = \bar{Q}_t - \bar{Q}_0$ and prove the uniform bound for $Q'_t$).

**Lemma 5.1.** For any state $x$ and action $a$, we have

$$\max_{x, a} |\bar{Q}_t(x, a)| \leq \frac{2}{1 - \gamma}, \quad t \geq 0. \quad (5.3)$$

**Proof.** We first prove $\max_{x, a} \bar{Q}_t(x, a)$ cannot become larger than $\frac{2}{1 - \gamma}$. Actually, if $\max_{x, a} \bar{Q}_t(x, a)$ ever attains $\frac{2}{1 - \gamma}$, that is for some $t_0 \geq 0$

$$\max_{x, a} \bar{Q}_{t_0}(x, a) = \frac{2}{1 - \gamma}, \quad (5.4)$$

then for any state-action pair $(x_0, a_0)$ such that $Q_{t_0}(x_0, a_0) = \frac{2}{1 - \gamma}$ we have

$$\left. \frac{d\bar{Q}_t}{dt} (x_0, a_0) \right|_{t = t_0} \leq \alpha \pi^{g_{t_0}}(x_0, a_0) \left[ \frac{1 + 2 \gamma}{1 - \gamma} - \frac{2}{1 - \gamma} \right] = -\alpha \pi^{g_{t_0}}(x_0, a_0) \leq 0, \quad (5.5)$$

and therefore $\max_{x, a} \bar{Q}_t(x, a)$ can never exceed $\frac{2}{1 - \gamma}$. Similarly, we can prove

$$\min_{x, a} \bar{Q}_t(x, a) \geq -\frac{2}{1 - \gamma}, \quad t \geq 0, \quad (5.6)$$

which concludes the proof of the lemma. $\square$

We now develop an ODE comparison principle which will help us to prove the convergence (3.5).

**Lemma 5.2.** Suppose a non-negative function $Y_t$ satisfies

$$\frac{dY_t}{dt} \leq -\frac{C}{\log^{2n_0} t} Y_t + \frac{1}{t}, \quad t \geq t_0, \quad (5.7)$$

where $C, n_0$ are constant and $t_0 \geq 0$. Then,

$$Y_t = O\left( \frac{1}{\log^2 t} \right). \quad (5.8)$$

**Proof.** First, we establish a comparison principle with the following ODE:

$$\frac{dZ_t}{dt} = -\frac{C}{\log^{2n_0} t} Z_t + \frac{1}{t}, \quad t \geq t_0, \quad (5.9)$$

Define

$$V_t = Y_t - Z_t.$$
Then, we have $V_{t_0} = 0$ and for any $t \geq t_0$

\[
\frac{dV_t}{dt} = \frac{dY_t}{dt} \frac{dV_t}{dt} \leq -\frac{C}{\log^{2n_0} t} Y_t + \frac{1}{t} - \left( -\frac{C}{\log^{2n_0} t} Z_t + \frac{1}{t} \right) = -\frac{C}{\log^{2n_0} t} (Y_t - Z_t)
\]

\[
= -\frac{C}{\log^{2n_0} t} V_t. \tag{5.10}
\]

Then, using an integrating factor,

\[
\frac{d}{dt} \left[ \exp \left\{ \int_{t_0}^{t} \frac{C}{\log^{2n_0} \tau} d\tau \right\} V_t \right] = \exp \left\{ \int_{t_0}^{t} \frac{C}{\log^{2n_0} \tau} d\tau \right\} \left[ \frac{dV_t}{dt} + \frac{C}{\log^{2n_0} t} V_t \right] \leq 0. \tag{5.11}
\]

Thus we have $V_t \leq \exp \left\{ -\int_{t_0}^{t} \frac{C}{\log^{2n_0} \tau} d\tau \right\} V_{t_0} = 0, \quad t \geq t_0$. Therefore,

\[
Y_t \leq Z_t, \quad t \geq t_0. \tag{5.12}
\]

Then, if we can establish a convergence rate for $Z_t$, we have a convergence rate for $Y_t$.

To solve the ODE (5.9), note that

\[
\frac{d}{dt} \left[ \exp \left\{ \int_{t_0}^{t} \frac{C}{\log^{2n_0} \tau} d\tau \right\} Z_t \right] = \exp \left\{ \int_{t_0}^{t} \frac{C}{\log^{2n_0} \tau} d\tau \right\} \left[ \frac{dZ_t}{dt} + \frac{C}{\log^{2n_0} t} Z_t \right] = \frac{1}{t} \exp \left\{ \int_{t_0}^{t} \frac{C}{\log^{2n_0} \tau} d\tau \right\}. \tag{5.13}
\]

Then,

\[
Z_t = \frac{Z_{t_0}}{\exp \left\{ \int_{t_0}^{t} \frac{C}{\log^{2n_0} \tau} d\tau \right\} + \frac{1}{t} \exp \left\{ \int_{t_0}^{t} \frac{C}{\log^{2n_0} \tau} d\tau \right\} \int_{t_0}^{t} \frac{1}{s} \exp \left\{ \int_{t_0}^{s} \frac{C}{\log^{2n_0} \tau} d\tau \right\} ds} \exp \left\{ \int_{t_0}^{t} \frac{C}{\log^{2n_0} \tau} d\tau \right\} \right. + I_3^t + I_4^t. \tag{5.14}
\]

Note that for any integer $n$ and constant $\gamma > 0$,

\[
\lim_{t \to \infty} \frac{\log^n t}{t^\gamma} = 0. \tag{5.15}
\]

Thus, without loss of generality, we can suppose $t_0$ is large enough such that

\[
\log^{2n_0} t \leq t, \quad t \geq t_0. \tag{5.16}
\]

Then, we can show that

\[
I_3^t \leq \frac{Z_{t_0}}{\exp \left\{ \int_{t_0}^{t} \frac{C}{\log^{2n_0} \tau} d\tau \right\}} = \frac{Z_{t_0} t^C}{t^C}. \tag{5.17}
\]
By L'Hospital's Rule, we have
\[
\lim_{t \to \infty} \log^4 t \cdot I_t^4 = \lim_{t \to \infty} \frac{\int_{t_0}^t \frac{1}{\log^4 t} \exp \left\{ \int_{t_0}^s \frac{C}{\log^4 \tau} d\tau \right\} ds}{Ct} + \lim_{t \to \infty} \frac{\log^{2n_0 + 4} t}{Ct}
\]
\[
\overset{(a)}{=} \lim_{t \to \infty} \frac{\int_{t_0}^t \frac{1}{\log^4 t} \exp \left\{ \int_{t_0}^s \frac{C}{\log^4 \tau} d\tau \right\} ds}{Ct}
\]
\[
= \lim_{t \to \infty} \frac{\frac{1}{t} \exp \left\{ \int_{t_0}^t \frac{C}{\log^{2n_0} \tau} d\tau \right\} \log^{2n_0} t}{Ct}
\]
\[
= \lim_{t \to \infty} \frac{\log^{2n_0} t}{Ct}
\]
\[
= 0,
\]
where step (a) is by (5.15). Therefore, we can let \( t_0 \) be large enough such that
\[
I_t^4 \leq \frac{1}{\log^4 t}, \quad \forall t \geq t_0.
\]

Combining our results, we have
\[
Y_t \leq Z_t \leq Y_t \frac{\log^4 t}{t} + \frac{1}{\log^4 t}, \quad t \geq t_0,
\]
which together with (5.15) proves (5.8).

Using Lemma 5.2, now we prove the critic convergence (3.5).

**Proof of (3.5):** Combining (3.4) and (5.1),
\[
\frac{d\phi_t}{dt} (x, a) = -\alpha \pi^{g_{\theta_t}} (x, a) \phi_t (x, a) + \alpha \gamma \pi^{g_{\theta_t}} (x, a) \sum_{z, a''} \phi_t (z, a'') g_{\theta_t} (z, a'') p(z|x, a) + \frac{dV^{g_{\theta_t}}}{dt} (x, a).
\]
Let \( \odot \) denote element-wise multiplication. Then,
\[
\frac{d\phi_t}{dt} = -\alpha \pi^{g_{\theta_t}} \odot \phi_t + \alpha \gamma \pi^{g_{\theta_t}} \odot \phi_t + \frac{dV^{g_{\theta_t}}}{dt},
\]
where \( \Gamma_t(x', a') = \sum_{z, a''} \phi_t (z, a'') g_{\theta_t} (z, a'') p(z|x', a'). \)

Define the process
\[
Y_t = \frac{1}{2} \phi_t^T \phi_t.
\]
Differentiating yields
\[
\frac{dY_t}{dt} = \phi_t^T \frac{d\phi_t}{dt} = -\alpha \phi_t^T \pi^{g_{\theta_t}} \odot \phi_t + \alpha \gamma \phi_t^T \pi^{g_{\theta_t}} \odot \Gamma_t + \phi_t^T \frac{dV^{g_{\theta_t}}}{dt}.
\]
The second term on the last line of (5.24) becomes:

\[
\left| \phi_t^\top \pi \gamma_{\theta_{\mu}} \otimes \Gamma_t \right| = \left| \sum_{x',a'} \phi_t(x',a') \pi \gamma_{\theta_{\mu}}(x',a') \sum_{x,a''} \phi_t(z,a'') g_{\theta_{\mu}}(z,a'') p(z|x',a') \right| = \left| \sum_{x',a',z,a''} \phi_t(z,a'') \phi_t(x',a') g_{\theta_{\mu}}(z,a'') p(z|x',a') \pi \gamma_{\theta_{\mu}}(x',a') \right| \leq \frac{1}{2} \sum_{x',a',z,a''} \left( \phi_t(z,a'')^2 + \phi_t(x',a')^2 \right) g_{\theta_{\mu}}(z,a'') p(z|x',a') \pi \gamma_{\theta_{\mu}}(x',a') = \frac{1}{2} \sum_{x',a',z,a''} \phi_t(z,a'')^2 \sum_{x',a'} g_{\theta_{\mu}}(z,a'') p(z|x',a') \pi \gamma_{\theta_{\mu}}(x',a') + \frac{1}{2} \sum_{x',a'} \phi_t(x',a')^2 \pi \gamma_{\theta_{\mu}}(x',a') \sum_{z,a''} g_{\theta_{\mu}}(z,a'') p(z|x',a') = \frac{1}{2} \sum_{x',a',z,a''} \phi_t(z,a'')^2 \pi \gamma_{\theta_{\mu}}(x',a') \]

where we have used Young's inequality, the fact that \( \sum_{x,a'} g_{\theta_{\mu}}(z,a'') p(z|x',a') = 1 \) for each \((x',a')\), and

\[
\sum_{x',a'} g_{\theta_{\mu}}(z,a'') p(z|x',a') \pi \gamma_{\theta_{\mu}}(x',a') = \pi \gamma_{\theta_{\mu}}(z,a'').
\]

Therefore,

\[
\frac{dY_t}{dt} \leq -\alpha(1 - \gamma) \pi \gamma_{\theta_{\mu}} \cdot \phi_t^\top + \phi_t^\top \frac{dV}{dt},
\]  

(5.25)

where \( \phi_t^2 \) is an element-wise square.

By the limit ODEs in (3.4) and the uniform boundedness in Lemma 5.1, we have for any \((x,a)\)

\[
\left| \frac{d\theta_t}{dt}(x,a) \right| = \left| \zeta_t \sigma^f_{\mu}(x,a) \left[ Q_t(x,a) - \sum_{a'} Q_t(x,a') f_{\theta_{\mu}}(x,a') \right] \right| \leq C\zeta_t
\]  

(5.26)

For any state \(x_0\), define \( \partial_{x,a} V^{f_{\theta_{\mu}}}(x_0) = \frac{\partial V^{f_{\theta_{\mu}}}(x_0)}{\partial(x,a)} \). Then, for the exploration policy (2.10), by the policy gradient theorem (4.12) we have

\[
|\partial_{x,a} V^{\gamma_{\theta_{\mu}}}(x_0)| = \left| \sum_{x',a'} \sigma_{\theta_{\mu}}(x',a') V^{\gamma_{\theta_{\mu}}}(x',a') \partial_{x,a} \log g_{\theta_{\mu}}(x',a') \right| \leq C \sum_{x',a'} \left| \partial_{x,a} \log g_{\theta_{\mu}}(x',a') \right| \leq C(1 - \eta_t) \sum_{x',a'} \frac{f_{\theta_{\mu}}(x',a')}{g_{\theta_{\mu}}(x',a')} \left| \partial_{x,a} \log f_{\theta_{\mu}}(x',a') \right| \leq C,
\]  

(5.27)

where step (a) is by

\[
\frac{f_{\theta_{\mu}}(x',a')}{g_{\theta_{\mu}}(x',a')} = \frac{f_{\theta_{\mu}}(x',a')}{\pi_{x,a}} + (1 - \eta_t) \cdot \frac{f_{\theta_{\mu}}(x',a')}{g_{\theta_{\mu}}(x',a')} \leq C
\]  

(5.28)

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and
\[ |\partial_{x,a} \log f_{\theta_i}(x', a')| = |\mathbb{I}_{x'=x} [\mathbb{I}_{a'=a} - f_{\theta_i}(x', a)]| \leq 2. \tag{5.29} \]

The relationship between the value functions
\[ V^{f_{\theta_i}}(x_0, a_0) = r(x_0, a_0) + \gamma \sum_{x'} V^{f_{\theta_i}}(x') p(x'|x_0, a_0), \quad \forall (x_0, a_0), \tag{5.30} \]
can be combined with (5.27) to derive
\[ \|\nabla V^{g_{\phi_i}}(x, a)\| \leq C, \quad \forall (x, a). \tag{5.31} \]

Combining (5.26) and (5.31),
\[ \left| \frac{dV^{g_{\phi_i}}}{dt}(x, a) \right| = \left| \nabla V^{g_{\phi_i}}(x, a) \cdot \frac{d\theta_i}{dt} \right| \leq \|\nabla V^{g_{\phi_i}}(x, a)\| \cdot \left| \frac{d\theta_i}{dt} \right| \leq C \zeta_t, \tag{5.32} \]
where \( C > 0 \) is a constant independent with \( T \).

Combining (5.25), (5.32) and (4.33), we have
\[
\frac{dY_t}{dt} \leq -\alpha(1 - \gamma) \min_{x, a} \{\pi^{\phi_i}(x, a)\} Y_t + C \phi_t^T \zeta_t
\leq -\alpha C \eta_t^{n_0} (1 - \gamma) Y_t + C \phi_t^T \zeta_t
\leq -C \eta_t^{n_0} Y_t + \eta_t^{n_0} \|\phi_t\| C \zeta_t
\leq -C \eta_t^{n_0} Y_t + \frac{\|\phi_t\|^2 \eta_t^{n_0}}{\eta_t^{2n_0}} + \frac{C \zeta_t^2}{\eta_t^{2n_0}}
= -\eta_t^{n_0} (C - 2 \eta_t^{n_0}) Y_t + \frac{C \zeta_t}{\eta_t^{2n_0}}. \tag{5.33} \]

Since \( \frac{\zeta}{\eta_t^{n_0}} \to 0 \) as \( t \to \infty \), there exists \( t_0 \geq 2 \) such that \( \forall t \geq t_0 \)
\[ \frac{dY_t}{dt} \leq -C \eta_t^{n_0} Y_t + \zeta_t \leq -\frac{C}{\log^{2n_0} t} Y_t + \frac{1}{t}, \tag{5.34} \]
where the \( C \) is a constant independent with \( t \). Then, by Lemma 5.2, there exists \( t_1 \geq t_0 \) such that
\[ Y_t = O \left( \frac{1}{\log^2 t} \right) = O \left( \eta_t^2 \right). \tag{5.35} \]

By the policy gradient theorem (4.12), we have
\[ \frac{\partial V^f}{\partial f(x, a)}(x_0) = V^f(x, a) \sigma^f_{x_0}(x). \tag{5.36} \]

Thus, by the relationship (5.30),
\[ \frac{\partial V^{f_{\theta_i}}(x_0, a_0)}{\partial f(x, a)} = \gamma \sum_{x'} V^{f_{\theta_i}}(x, a) \sigma^f_{x_0}(x)p(x'|x_0, a_0) \leq C. \tag{5.37} \]

Then, for any \((x, a) \in \mathcal{X} \times \mathcal{A}\), there exists \( \bar{t} \in [0, 1] \) such that
\[ |V^{g_{\phi_i}}(x, a) - V^{f_{\theta_i}}(x, a)| = \left| \nabla_f \tilde{V}^{f_{\theta_i}}(1 - \bar{t}) g_{\phi_i}(x, a) \cdot [g_{\phi_i} - f_{\theta_i}] \right| \leq C \eta_t, \tag{5.38} \]

Finally, combining (5.35) and (5.38), we obtain (3.5).
5.2 Actor convergence

5.2.1 Convergence to stationary point

In order to prove global convergence, we first show that the actor converges to a stationary point. We introduce the following notation:

\[
\hat{\nabla}_\theta J(f_{\theta_t}) := \sum_{x,a} \sigma^f_{\theta_t} (x,a) \dot{Q}_t(x,a) \nabla_{\theta} \log f_{\theta_t}(x,a),
\]

\[
\tilde{\nabla}_{x,a} J(f_{\theta_t}) := \sum_{x,a} \sigma^f_{\theta_t} (x,a) \dot{Q}_t(x,a) \partial_{x,a} \log f_{\theta_t}(x,a).
\] 

(5.39)

Then, the limit ode for \( \theta \) in (3.4) can be written as

\[
\frac{d\theta_t}{dt} = \zeta_t \hat{\nabla}_\theta J(f_{\theta_t}).
\] 

(5.40)

By direct calculations,

\[
\nabla_{\theta} \log f_{\theta}(x,a) = \nabla_{\theta} \left[ \theta(x,a) - \log \sum_{a'} e^{\theta(x,a')} \right] \\
= \nabla_{\theta} \theta(x,a) - \sum_{a'} e^{\theta(x,a')} \nabla_{\theta} \theta(x,a') \\
= \nabla_{\theta} \theta(x,a) - \sum_{a'} f_{\theta}(x,a') \nabla_{\theta} \theta(x,a') \\
= \nabla_{\theta} \theta(x,a) - E_{a' \sim f_{\theta}(x,a)}[\nabla_{\theta} \theta(x,a')] \\
= e_{x,a} - \sum_{a'} e_{x,a',\theta} f_{\theta}(x,a'),
\]

where \( e_{x,a} \) is the unit vector where only the \( x, a \) element is 1 and all other elements are 0. Then, the difference is

\[
\nabla_{\theta} J(f_{\theta_t}) - \hat{\nabla}_{\theta} J(f_{\theta_t}) = \sum_{x,a} \sigma^f_{\theta_t} (x,a) \left( \dot{Q}_t(x,a) - V_{\theta} (x,a) \right) \nabla_{\theta} \log f_{\theta_t}(x,a),
\]

\[
= \sum_{x,a} \sigma^f_{\theta_t} (x,a) \left( \dot{Q}_t(x,a) - V_{\theta} (x,a) \right) \left( e_{x,a} - \sum_{a'} e_{x,a',\theta} f_{\theta}(x,a') \right),
\] 

(5.42)

which together with (3.5) derives

\[
\| \nabla_{\theta} J(\theta_t) - \hat{\nabla}_{\theta} J(\theta_t) \|_2 \leq C \| \dot{Q}_t - V_{\theta_t} \|_2 \leq C \eta_t.
\] 

(5.43)

Thus we re-write the gradient flow (5.40) as

\[
\frac{d\theta_t}{dt} = \zeta_t \hat{\nabla}_{\theta} J(f_{\theta_t}) + \zeta_t \sum_{x,a} \sigma^f_{\theta_t} (x,a) \left[ (Q_t(x,a) - V_{\theta_t} (x,a)) \right] \nabla_{\theta} \log f_{\theta_t}(x,a).
\] 

(5.44)

Now we can adapt the proof in [2] to show the gradient flow converges to a stationary point. We first provide a useful lemma.

Lemma 5.3. Let \( Y_t, W_t \) and \( Z_t \) be three functions such that \( W_t \) is nonnegative. Assume that

\[
\frac{dY_t}{dt} \geq W_t + Z_t, \quad t \geq 0
\] 

(5.45)

and that \( \int_0^\infty Z_t dt \) converges. Then, either \( Y_t \rightarrow \infty \) or else \( Y_t \) converges to a finite value and \( \int_0^\infty W_t dt < \infty \).
Proof. For any \( t > 0 \). By integrating the relationship \( \frac{dY_t}{dt} \geq Z_t \) from \( t \) to \( t \geq \bar{t} \) and taking the limit inferior as \( t \to \infty \), we obtain

\[
\liminf_{t \to \infty} Y_t \geq Y_{\bar{t}} + \int_{\bar{t}}^{\infty} Z_t dt > -\infty. \tag{5.46}
\]

By taking the limit superior of the right-hand side as \( t \to \infty \) and using the fact \( \lim_{t \to \infty} \int_{\bar{t}}^{\infty} Z_t dt = 0 \), we obtain

\[
\liminf_{t \to \infty} Y_t \geq \limsup_{t \to \infty} Y_t > -\infty. \tag{5.47}
\]

This proves that either \( Y_t \to \infty \) or \( Y_t \) converges to a finite value. If \( Y_t \) converges to a finite value, we can integrate the relationship (5.45) to show that

\[
\int_{0}^{t} W_s ds \leq Y_t - Y_0 - \int_{0}^{t} Z_s ds,
\]

which implies that \( \int_{0}^{\infty} W_s ds \leq \lim_{t \to \infty} Y_t - Y_0 - \int_{0}^{\infty} Z_s ds < \infty \).

Next we can prove convergence to the stationary point under the learning rate (3.1).

**Theorem 5.4.** Suppose the learning rate \( \zeta_t \) satisfies (3.1). Then, for the gradient flow (5.40), we have that \( J(\theta_t) \) converges to a finite value and

\[
\lim_{t \to +\infty} \nabla_{\theta} J(f_{\theta_t}) = 0. \tag{5.49}
\]

**Proof:** First we note that by the proof of Lemma 7 in [23], we know that the eigenvalues of the Hessian matrix of \( J(f_\theta) \) are smaller than \( L := \frac{8}{(1-\gamma)^2} \) and thus \( \nabla_{\theta} J(f_\theta) \) is \( L \)-Lipschitz continuous with respect to \( \theta \).

Then, by the gradient flow (5.40), (5.43), and chain rule, we can show that

\[
\begin{align*}
\frac{dJ(f_{\theta_t})}{dt} &= \zeta_t \nabla_{\theta} J(f_{\theta_t}) \nabla_{\theta} J(f_{\theta_t}) \\
&= \zeta_t \| \nabla_{\theta} J(f_{\theta_t}) \|^2 + \zeta_t \nabla_{\theta} J(f_{\theta_t}) \left( \nabla_{\theta} J(f_{\theta_t}) - \nabla_{\theta} J(f_{\theta_t}) \right) \\
&\geq \zeta_t \| \nabla_{\theta} J(f_{\theta_t}) \|^2 - C_\zeta \| \nabla_{\theta} J(f_{\theta_t}) \| \cdot \| \nabla_{\theta} J(f_{\theta_t}) - V f_{\theta_t} \|_2 \\
&\geq (\zeta_t - C_\zeta \eta_t) \| \nabla_{\theta} J(f_{\theta_t}) \|^2 - C_\zeta \eta_t \\
&\geq C_\zeta \zeta_t \| \nabla_{\theta} J(f_{\theta_t}) \|^2 - C_\zeta \eta_t.
\end{align*}
\tag{5.50}
\]

where the step (a) follows (5.43). Step (b) is by using the relationship \( \| \nabla_{\theta} J(f_{\theta_t}) \| \leq 1 + \| \nabla_{\theta} J(f_{\theta_t}) \|^2 \) and step (c) is because \( \eta_t \to 0 \) and \( C_1, C_2 \) are some sufficiently large enough constants. Then, by Lemma 5.3 and the assumption in (3.1), we can show that either \( J(f_{\theta_t}) \to \infty \) or \( J(f_{\theta_t}) \) converges to a finite value and

\[
\int_{0}^{+\infty} \zeta_t \| \nabla_{\theta} J(f_{\theta_t}) \|^2 dt < \infty. \tag{5.51}
\]

Note that \( J(f_{\theta}) = E_{f_{\theta}} \left[ \sum_{k=0}^{+\infty} \gamma^k r(x_k, a_k) \right] \). Therefore, the objective function \( J \) is bounded by Assumption 3.1 and thus we know \( J(\theta_t) \) converges to a finite value and (5.51) is valid.

If there existed an \( \epsilon_0 > 0 \) and \( \bar{t} > 0 \) such that \( \| \nabla_{\theta} J(f_{\theta_t}) \| \geq \epsilon_0 \) for all \( t \geq \bar{t} \), we would have

\[
\int_{\bar{t}}^{+\infty} \zeta_t \| \nabla_{\theta} J(f_{\theta_t}) \|^2 dt \geq \epsilon_0^2 \int_{\bar{t}}^{+\infty} \zeta_t dt = \infty,
\tag{5.52}
\]

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which contradicts (5.51). Therefore, \( \liminf \| \nabla_\theta J(f_{\tilde{\theta}_i}) \| = 0 \). To show that \( \lim \| \nabla_\theta J(f_{\tilde{\theta}_i}) \| = 0 \), assume the contrary; that is \( \limsup \| \nabla_\theta J(f_{\tilde{\theta}_i}) \| > 0 \). Then we can find a constant \( \epsilon_1 > 0 \) and two increasing sequences \( \{a_n\}_{n \geq 1}, \{b_n\}_{n \geq 1} \) such that

\[
a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < \ldots, \\
\| \nabla_\theta J(f_{\tilde{\theta}_{a_n}}) \| < \frac{\epsilon_1}{2}, \quad \| \nabla_\theta J(f_{\tilde{\theta}_{b_n}}) \| > \epsilon_1.
\]

(5.53)

Define the following cycle of stopping times:

\[
t_n := \sup \{ s | s \in (a_n, b_n), \| \nabla_\theta J(f_{\tilde{\theta}_s}) \| < \frac{\epsilon_1}{2} \}, \\
i(t_n) := \inf \{ s | s \in (t_n, b_n), \| \nabla_\theta J(f_{\tilde{\theta}_s}) \| > \epsilon_1 \}.
\]

(5.54)

Note that \( \| \nabla_\theta J(f_{\tilde{\theta}_s}) \| \) is continuous against \( t \), thus we have

\[
a_n \leq t_n < i(t_n) \leq b_n \\
\| \nabla_\theta J(f_{\tilde{\theta}_{a_n}}) \| = \frac{\epsilon_1}{2}, \quad \| \nabla_\theta J(f_{\tilde{\theta}_{i(t_n)}}) \| = \epsilon_1 \\
\frac{\epsilon_1}{2} \leq \| \nabla_\theta J(f_{\tilde{\theta}_s}) \| \leq \epsilon_1, \quad s \in (t_n, i(t_n)).
\]

(5.55)

Then, by the \( L \)-Lipschitz property of the gradient, we have for any \( t_n \)

\[
\frac{\epsilon_1}{2} = \| \nabla_\theta J(f_{\tilde{\theta}_{i(t_n)}}) \| - \| \nabla_\theta J(f_{\tilde{\theta}_{a_n}}) \| \\
\leq \| \nabla_\theta J(f_{\tilde{\theta}_{i(t_n)}}) - \nabla_\theta J(f_{\tilde{\theta}_{a_n}}) \| \\
\leq L \| \tilde{\theta}_{i(t_n)} - \tilde{\theta}_{a_n} \| \\
\leq L \int_{t_n}^{i(t_n)} \zeta_s \| \nabla_\theta J(f_{\tilde{\theta}_s}) \| ds + L \int_{i(t_n)}^{a_n} \zeta_s \| \nabla_\theta J(f_{\tilde{\theta}_s}) \| ds \\
\leq L \epsilon_1 \int_{t_n}^{i(t_n)} \zeta_s ds + C L \int_{i(t_n)}^{a_n} \zeta_s \eta_s ds.
\]

(5.56)

From this and by (3.2) it follows that

\[
\frac{1}{2L} \leq \liminf_{n \to \infty} \int_{t_n}^{i(t_n)} \zeta_s ds. \quad \text{(5.57)}
\]

Using (5.50) and (5.55), we see that

\[
J(f_{\tilde{\theta}_{i(t_n)}}) - J(f_{\tilde{\theta}_{a_n}}) \geq C_1 (\frac{\epsilon_1}{2})^2 \int_{t_n}^{i(t_n)} \zeta_s ds - C_2 \int_{t_n}^{i(t_n)} \zeta_s \eta_s ds.
\]

(5.58)

Due to the convergence of \( J(f_{\tilde{\theta}_{a_n}}) \) and the assumption of the learning rate, this implies that

\[
\lim_{n \to \infty} \int_{t_n}^{i(t_n)} \zeta_s ds = 0, \quad \text{(5.59)}
\]

which contradicts (5.57) and thus the convergence to the stationary point is proven.

\[ \square \]

5.2.2 Global convergence

We now prove the global convergence rate (3.6) for the actor dynamic using the following steps:

- Derive non-uniform Łojasiewicz inequalities.
- Adapt the method in [1] to obtain the global convergence.
Set up the uniform Lojasiewicz inequalities and the ODE for actor convergence.

Analyse the ODE by a comparison lemma to get the convergence rate.

Since the objective function $J(f_\theta)$ is non-concave, the convergence to a stationary point in Theorem 5.4 does not guarantee global convergence to the optimal policy. As a first step, we establish the following non-uniform Lojasiewicz inequalities that show that the gradient of the objective function for any parameter value dominates the sub-optimality of the parameter. Actually, (5.60) is used for the case that the best action at any state $x$ is unique, while (5.63) is for the non-unique optimal action case.

**Lemma 5.5** (Non-uniform Lojasiewicz Bound). Choose any deterministic optimal policy $f^*$.

- Suppose for any state $\forall x \in \mathcal{X}$, there exists unique optimal action, then we have

$$
\| \nabla J(f_\theta) \| \geq \frac{1}{\sqrt{|\mathcal{X}|}} \cdot \left( \frac{\nu_\mu^f}{\nu_\mu^f} \right)_\infty^{-1} \cdot \min_x f_\theta(x, a^*(x)) \cdot [J(f^*) - J(f_\theta)]
$$

where $a^*(x) = \arg \max_a Vf^*(x, a), \forall x \in \mathcal{X}$.

- When under some state $x \in \mathcal{X}$, there is an “optimal action set”:

$$
\mathcal{A}^*(x) := \left\{ a^*(x) \in \mathcal{A} : Vf^*(x, a^*(x)) = \max_a Vf^*(x, a) \right\},
$$

i.e. all actions $a^*(x) \in \mathcal{A}^*(x)$ are the greedy action w.r.t. the optimal state-action value function $Vf^*$.

Given any policy $f_\theta$, construct the following optimal policy

$$
f^*_\theta(x, a) = \begin{cases} 
\frac{f_\theta(x, a)}{\sum_{a' \in \mathcal{A}^*(x)} f_\theta(x, a')}, & \text{if } a \in \mathcal{A}^*(x), \\
0, & \text{otherwise}
\end{cases}
$$

It is obvious that $f^*_\theta$ is an optimal policy, since for all $x \in \mathcal{X}$,

$$
\sum_{a \in \mathcal{A}^*(x)} f^*_\theta(x, a) = \frac{\sum_{a \in \mathcal{A}^*(x)} f_\theta(x, a)}{\sum_{a' \in \mathcal{A}^*(x)} f_\theta(x, a')} = 1.
$$

Now we have

$$
\| \nabla J(f_\theta) \| \geq \frac{1}{\sqrt{|\mathcal{X}|}} \cdot \left( \frac{\nu_\mu^f}{\nu_\mu^f} \right)_\infty^{-1} \cdot \left[ \min_x \sum_{a^*(x) \in \mathcal{A}^*(x)} f_\theta(x, a^*(x)) \right] \cdot [J(f^*) - J(f_\theta)].
$$

**Remark 5.6.** As the proof of Lemma 5.5 is similar as in [23], we move the detailed proof into Appendix D.

Lemma 5.5 is not sufficient to prove a global convergence rate (or even global convergence). For example, the term $\min_{x \in \mathcal{X}} f^*_{\tilde{\theta}_t}(x, a^*(x))$ in (5.60) could converge to zero as $t \to \infty$. Thus to obtain (3.6), we follow the steps.

(i) Prove the global convergence

$$
J(f^*) - J(f^*_{\tilde{\theta}_t}) \to 0, \quad t \to \infty,
$$

This global convergence can be proven by adapting the method in [1] to the setting in our paper.

(ii) Due to the convergence (5.64), if for each state $x$ the best action $a^*(x)$ is unique, we will have

$$
\lim_{t \to \infty} f^*_{\tilde{\theta}_t}(x, a^*(x)) = 1, \quad \forall x \in \mathcal{X}
$$

(5.65)
and thus
\[ \inf_{x \in \mathcal{X}, t \geq 0} f_{\hat{\theta}_t}(x, a^*(x)) > 0. \] (5.66)

If for some state \( x \), the best action is not unique, then the convergence (5.64) implies that
\[ \lim_{t \to \infty} \sum_{a^*(x) \in \mathcal{A}^*(x)} f_{\hat{\theta}_t}(x, a^*(x)) = 1, \quad \forall x \in \mathcal{X} \] (5.67)

and thus
\[ \inf_{x \in \mathcal{X}, t \geq 0} \sum_{a^*(x) \in \mathcal{A}^*(x)} f_{\hat{\theta}_t}(x, a^*(x)) > 0. \] (5.68)

(iii) The lower bound for \( \min_x f_{\hat{\theta}_t}(x, a^*(x)) \), \( \min_x \sum_{a^*(x) \in \mathcal{A}^*(x)} f_{\hat{\theta}_t}(x, a^*(x)) \) and (5.60), (5.63) can be used to derive the uniform Lojasiewicz inequality for MDP with unique or non-unique optimal action. By analysing the gradient flow, we can prove the convergence rate (3.6).

Now we adapt the method in [1] to obtain the global convergence (5.64). For the gradient flow
\[ \frac{d \hat{\theta}_t}{dt} = \zeta_t \nabla_{\hat{\theta}} J(f_{\hat{\theta}_t}), \] (5.69)

where \( \nabla_{\hat{\theta}} J(f_{\hat{\theta}_t}) := \sum_{x,a} \sigma^{f_{\hat{\theta}_t}}_{\mu}(x,a) Q_t(x,a) \nabla_{\hat{\theta}} \log f_{\hat{\theta}_t}(x,a) \), with the similar calculations in (4.15), it can be shown that
\[ \frac{d}{dt} \hat{\theta}_t(x,a) = \zeta_t \nabla_{\hat{\theta}} J(f_{\hat{\theta}_t}) \]
\[ = \zeta_t \sum_{x',a'} \nu_{\mu}(x') f_{\hat{\theta}_t}(x',a') 1_{x'=x} [1_{a'=a} - f_{\hat{\theta}_t}(x',a)] \bar{Q}_t(x',a') \]
\[ = \zeta_t \sum_{a'} \nu_{\mu}(x) f_{\hat{\theta}_t}(x,a') [1_{a'=a} - f_{\hat{\theta}_t}(x,a)] \bar{Q}_t(x,a') \]
\[ = \zeta_t \nu_{\mu}(x) f_{\hat{\theta}_t}(x,a) \bar{Q}_t(x,a) - \zeta_t \nu_{\mu}(x) f_{\hat{\theta}_t}(x,a) \sum_{a'} f_{\hat{\theta}_t}(x,a') \bar{Q}_t(x,a') \]
\[ = \zeta_t \sigma^{f_{\hat{\theta}_t}}_{\mu}(x,a) \bar{Q}_t(x,a) - \sum_{a'} \bar{Q}_t(x,a') f_{\hat{\theta}_t}(x,a') \] (5.70)

The following lemma is important in our proof.

**Lemma 5.7** (The performance difference lemma ([10])). For all policies \( f, f' \) and state \( x_0 \),
\[ V^f(x_0) - V^{f'}(x_0) = \sum_{x,a} \sigma^f_{x_0}(x,a) A^{f'}(x,a), \] (5.71)

where \( \sigma^f_{x_0} \) is the visiting measure for the MDP \( \mathcal{M} \) with initial distribution \( \delta_{x_0} \) and policy \( f \).

We first prove the following convergence lemma for value functions \( V^{f_{\hat{\theta}_t}}(x) \) and \( V^{f_{\hat{\theta}_t}}(x,a) \).

**Lemma 5.8.** There exists value \( V^{\infty}(x) \) and \( V^{\infty}(x,a) \) for every state \( x \) and action \( a \) such that
\[ \lim_{t \to \infty} V^{f_{\hat{\theta}_t}}(x) = V^{\infty}(x), \quad \lim_{t \to \infty} V^{f_{\hat{\theta}_t}}(x,a) = V^{\infty}(x,a). \]

Then, by the critic convergence (3.5), we immediately have when \( t \to \infty \)
\[ \bar{Q}_t(x,a) \to V^{\infty}(x,a) \]
\[ \bar{Q}_t(x) := \sum_a \bar{Q}_t(x,a) f_{\hat{\theta}_t}(x,a) \to V^{\infty}(x). \] (5.72)
Define
\[
\Delta = \min_{x,a: A^\infty(x,a) \neq 0} |A^\infty(x,a)|,
\]
where \(A^\infty(x,a) = V^\infty(x,a) - V^\infty(x)\). Then there exists a \(T_0\) such that \(\forall t > T_0, (x,a) \in \mathcal{X} \times \mathcal{A}\), we have
\[
V^\infty(x,a) - \frac{\Delta}{4} \leq Q_t(x,a) \leq V^\infty(x,a) + \frac{\Delta}{4}
\]
(5.74)

Remark 5.9. Here we can suppose that \(\Delta > 0\) because if \(\Delta = 0\), then we have for any states and actions \(A^\infty(x,a) = 0\). By Lemma 5.7,
\[
\lim_{t \to \infty} [J(f^*) - J(f_{\theta_t})] = \lim_{t \to \infty} \sum_{x_0} \mu(x_0) \left[ V^f(x_0) - V^{f_{\theta_t}}(x_0) \right]
\]
\[
= \lim_{t \to \infty} \sum_{x_0} \mu(x_0) \left[ \sum_{x,a} \sigma^f_{x_0}(x,a) \left[ V^{f_{\theta_t}}(x,a) - V^{f_{\theta_t}}(x) \right] \right]
\]
(5.75)
\[
= \lim_{t \to \infty} \sum_{x,a} \sigma^f_{\mu}(x,a) A^{f_{\theta_t}}(x,a)
\]
\[
= 0,
\]
which immediately concludes the global convergence.

Proof. For any fixed state \(x_0\), treat the state value \(V^{f_{\theta_t}}(x_0)\) as the objective function for an MDP whose initial distribution is \(\delta_{x_0}\), and, by the policy gradient theorem (4.12), we have
\[
\nabla_\theta V^{f_{\theta_t}}(x_0) = \sum_{x,a} \sigma^f_{x_0}(x,a) V^{f_{\theta_t}}(x,a) \nabla_\theta \log f_{\theta_t}(x,a),
\]
(5.76)
where \(\sigma^f_{x_0}(x,a)\) denotes the visiting measure of the MDP starting from \(x_0\) under the policy \(f_{\theta_t}\). Thus, using the same calculations as in (4.14), we have
\[
\frac{\partial}{\partial \theta(x,a)} V^{f_{\theta_t}}(x_0) = \sigma^f_{x_0}(x,a) A^{f_{\theta_t}}(x,a)
\]
(5.77)

Let \(\beta_t(x,a) = \bar{Q}_t(x,a) - V^{f_{\theta_t}}(x,a)\) denote the critic error. Due to (3.5), we know that for any state-action pair \((x,a)\), \(|\beta_t(x,a)| \leq C_\eta_t\). Combining (5.70) with (5.77) and using the chain rule, we have
\[
\frac{d}{dt} V^{f_{\theta_t}}(x_0) = \nabla_\theta V^{f_{\theta_t}}(x_0) \cdot \frac{d}{dt} \bar{\theta}_t
\]
\[
= \sum_{x,a} \frac{\partial}{\partial \theta(x,a)} V^{f_{\theta_t}}(x_0) \frac{d}{dt} \bar{\theta}_t(x,a)
\]
\[
= \zeta_t \sum_{x,a} \sigma^{f_{\theta_t}}(x,a) A^{f_{\theta_t}}(x,a) \sigma^{f_{\theta_t}}(x,a) \left[ \bar{Q}_t(x,a) - \sum_{a'} \bar{Q}_t(x,a') f_{\theta_t}(x,a') \right]
\]
(5.78)
\[
= \zeta_t \sum_{x,a} \sigma^{f_{\theta_t}}(x,a) A^{f_{\theta_t}}(x,a) \sigma^{f_{\theta_t}}(x,a) \left[ \beta_t(x,a) - \sum_{a'} \beta_t(x,a') f_{\theta_t}(x,a') + A^{f_{\theta_t}}(x,a) \right]
\]
\[
\geq \zeta_t \sum_{x,a} \sigma^{f_{\theta_t}}(x,a) A^{f_{\theta_t}}(x,a) (A^{f_{\theta_t}}(x,a))^2 - C_\eta_t \zeta_t \eta_t,
\]
where the last inequality follows from (3.5). Thus, by Lemma 5.3 and the boundedness of the value functions, we obtain the convergence for the state value function. Then, due to
\[
V^{f_{\theta_t}}(x,a) = r(x,a) + \gamma \sum_{x'} V^{f_{\theta_t}}(x') p(x'|x,a),
\]
(5.79)
the convergence for the state action value function is concluded. The convergence for $Q_t$ is immediately follows from the critic convergence (5.33). Combining the convergence for value functions, $\Delta > 0$, and the finiteness of the action space, we obtain (5.74).

Next, partition the action space $A$ into three sets according to the value $V^\infty(x)$ and $V^\infty(x,a)$,

$$
I^x_0 := \{a|V^\infty(x,a) = V^\infty(x)\}
$$

$$
I^x_+ := \{a|V^\infty(x,a) > V^\infty(x)\}
$$

$$
I^x_- := \{a|V^\infty(x,a) < V^\infty(x)\}.
$$

(5.80)

The following steps can be used to prove the global convergence (5.64).

- Show that the probabilities
  $$
  \lim_{t \to \infty} f_\bar{\theta}_t(x,a) = 0, \quad \forall a \in I^x_+ \cup I^x_-
  $$

- Show that for actions $a \in I^x_-$, $\lim_{t \to \infty} \bar{\theta}_t(x,a) = -\infty$ and, for all actions $a \in I^x_+$, $\bar{\theta}_t(x,a)$ is bounded below as $t \to \infty$.

- Prove that the set $I^x_+$ is empty by contradiction for all states $x$ and conclude the global convergence (5.64).

**Lemma 5.10.** Define the advantage function for the critic as

$$
A_t(x,a) := \bar{Q}_t(x,a) - \bar{Q}_t(x).
$$

(5.81)

Then, there exists a $T_1$ such that $\forall t \geq T_1, x \in X$, we have

$$
A_t(x,a) < -\frac{\Delta}{4} \quad \forall a \in I^x_+; \quad A_t(x,a) > \frac{\Delta}{4} \quad \forall a \in I^x_-.
$$

(5.82)

**Proof.** Since $\bar{Q}_t(x) \to V^\infty(x)$, we have that there exists $T_1 > T_0$ such that for all $t \geq T_1$,

$$
V^\infty(x) - \frac{\Delta}{4} < Q_t(x) < V^\infty(x) + \frac{\Delta}{4}.
$$

(5.83)

Then, for any actions $a \in I^x_-$, we have for any $t \geq T_1 > T_0$

$$
A_t(x,a) = \bar{Q}_t(x,a) - \bar{Q}_t(x)
$$

$$
\leq V^\infty(x,a) + \frac{\Delta}{4} - \bar{Q}_t(x) \quad \text{(a)}
$$

$$
\leq V^\infty(x,a) + \frac{\Delta}{4} - V^\infty(x) + \frac{\Delta}{4} \quad \text{(b)}
$$

$$
\leq -\Delta + \frac{\Delta}{2} \quad \text{(c)}
$$

$$
< -\frac{\Delta}{4},
$$

(5.84)

where step (a) is by (5.74), step (b) is by (5.83) and step (c) is by the definition of $I^x_-$ in (5.80) and $\Delta$ in (5.73). Similarly, for $a \in I^x_+$,

$$
A_t(x,a) = \bar{Q}_t(x,a) - \bar{Q}_t(x)
$$

$$
\geq V^\infty(x,a) - \frac{\Delta}{4} - \bar{Q}_t(x)
$$

$$
\geq V^\infty(x,a) - \frac{\Delta}{4} - V^\infty(x) - \frac{\Delta}{4}
$$

$$
\geq \Delta - \frac{\Delta}{2}
$$

$$
> \frac{\Delta}{4}.
$$

(5.85)
Lemma 5.11. For any state action pair \((x, a) \in \mathcal{X} \times \mathcal{A}\), we have \(\lim_{t \to \infty} \hat{\theta}_{x,a} J(f_{\theta_t}) = 0\). This implies that
\[
\lim_{t \to \infty} f_{\theta_t}(x, a) = 0, \quad \forall a \in I^+ x I^-,
\]
and thus
\[
\lim_{t \to \infty} \sum_{a \in I^+_0} f_{\theta_t}(x, a) = 1.
\] (5.86)

Lemma 5.12 (Monotonicity in \(\hat{\theta}_t(x, a)\)). For all \(a \in I^+_x\), \(\hat{\theta}_t(x, a)\) is strictly increasing for \(t \geq T_1\). For all \(a \in I^- x \), \(\hat{\theta}_t(x, a)\) is strictly decreasing for \(t \geq T_1\).

Lemma 5.13. For any state \(x\) with the set \(I^+_x \neq \emptyset\), we have:
\[
\max_{a \in I^+_0} \hat{\theta}_t(x, a) \to \infty, \quad \min_{a \in \mathcal{A}} \hat{\theta}_t(x, a) \to -\infty.
\] (5.87)

The proofs of Lemmas 5.11, 5.12, and 5.13 are the same as in [1] and therefore are omitted.

Lemma 5.14. For all states \(x\) with the set \(I^+_x \neq \emptyset\), choose any \(a_+ \in I^+_x\). Then, for any \(a \in I^+_0\), if there exists \(t \geq T_0\) such that \(f_{\hat{\theta}_t}(x, a) \leq f_{\hat{\theta}_t}(x, a_+)\), we have
\[
f_{\hat{\theta}_t}(x, a) \leq f_{\hat{\theta}_t}(x, a_+), \quad \forall \tau \geq t.
\] (5.88)

Proof. If \(f_{\hat{\theta}_t}(x, a) \leq f_{\hat{\theta}_t}(x, a_+)\), we know \(\hat{\theta}_t(x, a) \leq \hat{\theta}_t(x, a_+)\) and there exists a small \(\epsilon_0 > 0\) such that \(f_{\hat{\theta}_t}(x, a_+) \geq \epsilon_0\). Therefore,
\[
\hat{\theta}_t(x, a) = \nu_{\mu} f_{\hat{\theta}_t}(x) f_{\hat{\theta}_t}(x, a) \left[\hat{Q}_t(x, a) - \hat{Q}_t(x)\right]
\leq \nu_{\mu} f_{\hat{\theta}_t}(x, a) \left[\hat{Q}_t(x, a) - \hat{Q}_t(x) - \frac{\Delta}{4}\right]
\leq \nu_{\mu} f_{\hat{\theta}_t}(x, a_+) \left[\hat{Q}_t(x, a_+) - \hat{Q}_t(x)\right] - \epsilon_0 \nu_{\mu} f_{\hat{\theta}_t}(x) \left[\hat{Q}_t(x, a) - \hat{Q}_t(x)\right]
\leq \hat{\theta}_t(x, a) - \nu_{\mu} f_{\hat{\theta}_t}(x) \frac{\Delta \epsilon_0}{4},
\] (5.89)

where the step (a) follows from \(t > T_0\), \(a \in I^+_0\) and \(a_+ \in I^+_x\),
\[
\hat{Q}_t(x, a_+) \geq V^\infty(x, a_+) - \frac{\Delta}{4} \geq V^\infty(x) + \Delta - \frac{\Delta}{4} = V^\infty(x, a) + \frac{3}{4} \Delta > \hat{Q}_t(x, a) + \frac{\Delta}{4},
\] (5.90)

and the fact that \(\beta_t(x, a)\) decay exponentially. Let \(C = \nu_{\mu} f_{\hat{\theta}_t}(x) \frac{\Delta \epsilon_0}{4}\) and note that
\[
\hat{\theta}_t(x, a) - C \geq 0.
\] (5.91)

Then, we have
\[
\hat{\theta}_t(x, a) \leq \theta_t(x, a_+) - C \zeta_t.
\] (5.92)

By the gradient flow (5.69), Theorem 5.4, and (5.43), we have for any action \(a\)
\[
\frac{\partial}{\partial t}\hat{\theta}_t(x, a) = \hat{\theta}_t(x, a) J(f_{\theta_t}) \to 0, \quad t \to \infty.
\] (5.93)

Thus, without lose of generality, we can suppose that constant \(T_0\) is large enough such that for any \(t \geq T_0\) and any action \(a \in \mathcal{A}\),
\[
-\frac{C}{3} \zeta_t \leq \hat{\theta}_t(x, a) \leq \frac{C}{3} \zeta_t.
\] (5.94)

Thus, for any \(s > t > T_0\),
\[
\hat{\theta}_s(x, a) = \hat{\theta}_s(x, a) - \hat{\theta}_t(x, a) + \hat{\theta}_t(x, a)
\leq \frac{C}{3} \zeta_t + \frac{C}{3} \zeta_t + \hat{\theta}_t(x, a_+) - C \zeta_t
\leq \hat{\theta}_s(x, a) - \frac{C}{3} \zeta_t,
\] (5.95)
where step (a) use $\zeta_t$ is decreasing. Finally, we have for any $T_0 < t \leq \tau$,
\[
\bar{\theta}_t(x,a) = \bar{\theta}_t(x,a) + \int_t^\tau \bar{\theta}_s(x,a)ds \\
\leq \bar{\theta}_t(x,a) + \int_t^\tau \bar{\theta}_s(x,a)ds \\
= \bar{\theta}_\tau(x,a_+).
\]
and therefore (5.88) is true. \[
\]
For any $a_+ \in I_+^\tau$, we divide the set $I_0^\tau$ into two sets $B_0^\tau(a_+)$ and $\bar{B}_0^\tau(a_+)$ as follows: $B_0^\tau(a_+)$ is the set of all $a \in I_0^\tau$ such that for all $t \geq T_0$, $f_{\bar{\theta}_t}(x,a_+) < f_{\bar{\theta}_t}(x,a)$ and $\bar{B}_0^\tau(a_+)$ contains the remainder of the actions from $I_0^\tau$. By the definition of $B_0^\tau(a_+)$, we immediately have two Lemmas.

**Lemma 5.15.** Suppose for a state $x \in X$, $I_+^\tau \neq \emptyset$. Then, $\forall a_+ \in I_+^\tau$ we have that $B_0^\tau(a_+) \neq \emptyset$ and that
\[
\lim_{t \to \infty} \sum_{a \in B_0^\tau(a_+)} f_{\bar{\theta}_t}(x,a) = 1,
\]
which also derives
\[
\max_{a \in B_0^\tau(a_+)} \bar{\theta}_t(x,a) \to \infty.
\]

**Lemma 5.16.** Consider any $x$ with $I_+^\tau \neq \emptyset$. Then, for any $a_+ \in I_+^\tau$, there exists an $T_{a_+}$ such that for all $a \in B_0^\tau(a_+) \nexists \bar{\theta}_t(x,a) \geq f_{\bar{\theta}_t}(x,a), \forall t > T_{a_+}$. The proofs of Lemmas 5.15 and 5.16 are the same as in [1] and therefore are omitted.

**Lemma 5.17.** For all actions $a \in I_+^\tau$, we have that $\bar{\theta}_t(x,a)$ is bounded from below as $t \to \infty$. For all actions $a \in I_0^\tau$, we have that $\bar{\theta}_t(x,a) \to -\infty$ as $t \to \infty$.

**Proof.** From Lemma 5.12, we know that when $t \geq T_1$ and for any $a \in I_+^\tau$, $\bar{\theta}_t(x,a)$ is strictly increasing. Thus $\bar{\theta}_t(x,a)$ is bounded from below for any $a \in I_0^\tau$. For the second claim, from Lemma 5.12 we know that when $t \geq T_1$, $\bar{\theta}_t(x,a)$ is strictly decreasing for $a \in I_0^\tau$. Therefore, by monotone convergence theorem, $\lim_{t \to \infty} \bar{\theta}_t(x,a)$ exists and is either $-\infty$ or some constant $\epsilon_0$. Next, we prove the convergence to $-\infty$ by contradiction. Suppose for some $a \in I_0^\tau$ that there exists a $\epsilon_0$ such that $\bar{\theta}_t(x,a) > \epsilon_0, \forall t > T_1$. By Lemma 5.13, we know that there exists an action $a' \in A$ such that
\[
\liminf_{t \to \infty} \bar{\theta}_t(x,a') = -\infty.
\]
Choose a constant $\delta > 0$ such that $\bar{\theta}_{T_1}(x,a') \geq \epsilon_0 - \delta$. Then, we can find an increasing sequence $\{t_n\}_{n \geq 0}$ larger than $T_1$ and converging to $\infty$ such that
\[
\theta_{t_n}(x,a') < \epsilon_0 - \delta, \lim_{n \to \infty} \bar{\theta}_{t_n}(x,a') = -\infty.
\]
Define $\tau_n$ as
\[
\tau_n := \sup \{s | s \in [T_1,t_n], \bar{\theta}_x(s,a') \geq \epsilon_0 - \delta\}
\]
where
\[
\mathcal{T}^{(n)} := \{s | s \in [\tau_n,t_n], \bar{\theta}_x(s,a') \geq \epsilon_0 - \delta\}
\]
By the continuity of $\nabla_y J(f_\bar{\theta})$, we know $\mathcal{T}^{(n)}$ is a Lebesgue measurable set. Note that the Lebesgue measure of $\mathcal{T}^{(n)}$ should be positive for all $n$. Suppose there is a constant $n$ such that $\mathcal{L}(\mathcal{T}^{(n)}) = 0$, then by $\bar{\theta}_{t_n}(x,a') \geq \epsilon_0 - \delta$, we will have
\[
\bar{\theta}_{t_n}(x,a') = \bar{\theta}_{t_n}(x,a') + \int_{t_n}^{\tau_n} \zeta_s \bar{\theta}_x(s,a') J(f_{\bar{\theta}_s})ds \\
\geq \bar{\theta}_{t_n}(x,a') + \int_{[\tau_n,t_n] \setminus \mathcal{T}^{(n)}} \zeta_s \bar{\theta}_x(s,a') J(f_{\bar{\theta}_s})ds \\
\geq \epsilon_0 - \delta,
\]
where
which contradicts (5.100).

Define the sequence \( \{Z_n\}_{n \geq 0} \) as

\[
Z_n := \int_{T(n)} \zeta_s \partial_{x,a} J(f_{\theta_s}) ds.
\]

Then,

\[
Z_n \leq \int_{T(n)} \zeta_s \partial_{x,a} J(f_{\theta_s}) ds \leq \tilde{\theta}_t(x, a') - (\epsilon_0 - \delta).
\]

By (5.100), this implies that

\[
\lim_{n \to \infty} Z_n = -\infty
\]

(5.105)

For the positive measure set \( T^{(n)} \), we have for any \( t' \in T^{(n)} \),

\[
\left| \frac{\partial_{x,a} J(f_{\theta_s})}{\partial_{x,a'} J(f_{\theta_{t'}})} \right| = \left| \frac{f_{\theta_s'}(x, a) A_{\epsilon'}(x, a)}{f_{\theta_{t'}}(x, a') A_{\epsilon'}(x, a')} \right| \geq \exp \left( \epsilon_0 - \bar{\theta}_{\epsilon'}(x, a') \right) \frac{(1 - \gamma)\Delta}{2} \geq \exp(\delta) \frac{(1 - \gamma)\Delta}{2}
\]

(5.106)

where we have used that \( |A^{f_{\theta_s'}}(x, a')| \leq \frac{1}{\epsilon_0} \), \( |A^{f_{\theta_{t'}}}(x, a')| \rightarrow 0 \) and \( |A^{f_{\theta_s'}}(x, a)| \geq \frac{\Delta}{1} \) for all \( t' > T_1 \) (from Lemma 5.10). Note that since \( \partial_{x,a} J(f_{\theta_s}) < 0 \) and \( \partial_{x,a'} J(f_{\theta_{t'}}) < 0 \) for all \( t' \in T^{(n)} \), we have

\[
\partial_{x,a} J(f_{\theta_s}) \leq \exp(\delta) \frac{(1 - \gamma)\Delta}{2} \partial_{x,a'} J(f_{\theta_{t'}}).
\]

(5.107)

Thus

\[
\tilde{\theta}_t(x, a) = \tilde{\theta}_{T_1}(x, a) + \int_{T_1}^{t_n} \zeta_s \partial_{x,a} J(f_{\theta_s}) ds \leq \tilde{\theta}_{T_1}(x, a) + \int_{T(n)} \zeta_s \partial_{x,a} J(f_{\theta_s}) ds \leq \tilde{\theta}_{T_1}(x, a) + \exp(\delta) \frac{(1 - \gamma)\Delta}{2} \int_{T(n)} \zeta_s \partial_{x,a'} J(f_{\theta_s}) ds = \tilde{\theta}_{T_1}(x, a) + \exp(\delta) \frac{(1 - \gamma)\Delta}{2} Z_n.
\]

(5.108)

where the step (a) follows from \( \partial_{x,a} J(f_{\theta_s}) < 0 \) for any \( s \geq T_1 \) (Lemma 5.12) and step (b) is from (5.107). Since (5.105) and (5.108) contradict that \( \tilde{\theta}_t(x, a) \) is bounded from below, the proof is completed.

\[ \square \]

Lemma 5.18. Consider any state \( x \) with \( I^+_x \neq 0 \). We have for any \( a_+ \in I^+_x \),

\[
\lim_{t \to \infty} \sum_{a \in B^0_{\tilde{\theta}_t}(a_+)} \tilde{\theta}_t(x, a) = \infty
\]

(5.109)

Proof. By definition of \( B^0_{\tilde{\theta}_t}(a_+) \), we know when \( t \geq T_0 \),

\[
f_{\tilde{\theta}_t}(x, a_+) < f_{\tilde{\theta}_t}(x, a), \quad \forall a \in B^0_{\tilde{\theta}_t}(a_+),
\]

which implies \( \tilde{\theta}_t(x, a_+) < \tilde{\theta}_t(x, a) \). By Lemma 5.17, we know \( \tilde{\theta}_t(x, a_+) \) is lower bounded as \( t \to \infty \), and thus for all \( a \in B^0_{\tilde{\theta}_t}(a_+) \), \( \tilde{\theta}_t(x, a) \) is lower bounded as \( t \to \infty \), which together with \( \max_{a \in B^0_{\tilde{\theta}_t}(a_+)} \tilde{\theta}_t(x, a) \to \infty \) in Lemma 5.15 derive (5.109).

\[ \square \]

We are now ready to prove the global convergence of tabular actor-critic algorithm by following the same method in [1].

Lemma 5.19 (Global convergence). For any optimal policy \( f^* \),

\[
J(f^*) - J(f_{\tilde{\theta}_t}) \to 0, \quad t \to \infty.
\]

(5.110)
Proof. We only need to prove $I^x_+$ is empty for any $x$. If so, by (5.75)

$$0 \leq \lim_{t \to \infty} [J(f^\star) - J(f_{\bar{\theta}})] = \lim_{t \to \infty} \sum_{x,a} \sigma^f_{\mu}(x,a) A f_{\bar{\theta}}(x,a) = \sum_{x,a} \sigma^f_{\mu}(x,a) [V^\infty(x,a) - V^\infty(x)] \leq 0,$$

(5.111)

which implies the global convergence (5.110).

Now we prove $I^x_+ = \emptyset, \forall x \in \mathcal{X}$ by contradiction. Suppose $I^x_+$ is non-empty for some state $x \in \mathcal{X}$ and let $a_+ \in I^x_+$. Then, from Lemma 5.18, we must have

$$\sum_{a \in \bar{B}_{\bar{\theta}}(a_+)} \bar{\theta}_t(x,a) \to \infty.$$  

(5.112)

By Lemma 5.17, we know for any $a \in I^x_-$, $\bar{\theta}_t(x,a) \to -\infty$ and $\bar{\theta}_t(x,a_+)$ is bounded from below. Thus we have

$$\frac{f_{\bar{\theta}}(x,a)}{f_{\bar{\theta}}(x,a_+)} = \exp\{\bar{\theta}_t(x,a) - \bar{\theta}_t(x,a_+)\} \to 0,$$

and there exists $T_2 > T_0$ such that $\forall t \geq T_2$

$$\frac{f_{\bar{\theta}}(x,a)}{f_{\bar{\theta}}(x,a_+)} < \frac{(1 - \gamma)\Delta}{16 |A|},$$

(5.113)

or equivalently

$$-\sum_{a \in I^x_-} \frac{f_{\bar{\theta}}(x,a)}{1 - \gamma} > -f_{\bar{\theta}}(x,a_+) \frac{\Delta}{16}.$$  

(5.115)

Noting that $\bar{B}_{\bar{\theta}} \subset I_{\bar{\theta}}$, we have

$$\lim_{t \to \infty} A_t(x,a) = 0, \quad \forall a \in \bar{B}_{\bar{\theta}}(a_+).$$  

(5.116)

By Lemma 5.16,

$$\frac{f_{\bar{\theta}}(x,a_+)}{f_{\bar{\theta}}(x,a)} \geq 1, \quad \forall t > T_{a_+},$$

which together (5.116) derives that there exists $T_3 > T_2, T_{a_+}$ such that

$$|A_t(x,a)| < \frac{f_{\bar{\theta}}(x,a_+)}{f_{\bar{\theta}}(x,a)} \frac{\Delta}{16 |A|} \quad \forall t \geq T_3.$$  

(5.117)

Thus we have

$$\sum_{a \in \bar{B}_{\bar{\theta}}(a_+)} f_{\bar{\theta}}(x,a) |A_t(x,a)| < f_{\bar{\theta}}(x,a_+) \frac{\Delta}{16},$$

(5.118)

or equivalently

$$-f_{\bar{\theta}}(x,a_+) \frac{\Delta}{16} < \sum_{a \in \bar{B}_{\bar{\theta}}(a_+)} f_{\bar{\theta}}(x,a) A_t(x,a) < f_{\bar{\theta}}(x,a_+) \frac{\Delta}{16}.$$  

(5.119)
Then, we have for $t > T_3$,
\begin{align}
0 \leq & \sum_{a \in I^*_+} f_{\tilde{\theta}_t}(x, a)A_t(x, a) + \sum_{a \in I^*_L} f_{\tilde{\theta}_t}(x, a)A_t(x, a) + \sum_{a \in I^*_F} f_{\tilde{\theta}_t}(x, a)A_t(x, a) \\
\leq & \sum_{a \in B^*_0(a_{a+})} f_{\tilde{\theta}_t}(x, a)A_t(x, a) + \sum_{a \in B^*_0(a_{a+})} f_{\bar{\theta}_t}(x, a)A_t(x, a) + f_{\tilde{\theta}_t}(x, a_{a+})A_t(x, a_{a+}) + \sum_{a \in I^*_F} f_{\tilde{\theta}_t}(x, a)A_t(x, a) \\
\leq & \sum_{a \in B^*_0(a_{a+})} f_{\tilde{\theta}_t}(x, a)A_t(x, a) + \sum_{a \in B^*_0(a_{a+})} f_{\bar{\theta}_t}(x, a)A_t(x, a) + f_{\tilde{\theta}_t}(x, a_{a+})A_t(x, a_{a+}) + \sum_{a \in I^*_F} f_{\tilde{\theta}_t}(x, a)A_t(x, a) \\
> & \sum_{a \in B^*_0(a_{a+})} f_{\tilde{\theta}_t}(x, a)A_t(x, a),
\end{align}

(5.120)

where step (a) is from (5.70) and in the step (b) we used $A_t(x, a) > 0$ for all actions $a \in I^*_+$. From Lemma 5.10, the fact $A^{f_t}(x, a) \geq -\frac{1}{\gamma}t$ and the critic convergence $|A^{f_t}(x, a) - A_t(x, a)| \to 0$, while step (d) is by (5.115) and the left inequality in (5.119). This implies that for all $t > T_3$
\[ \sum_{a \in B^*_0(a_{a+})} \tilde{\theta}_{t,a}J(f_{\tilde{\theta}_t}) < 0. \]

Then,
\[ \lim_{t \to \infty} \sum_{a \in B^*_0(a_{a+})} (\bar{\theta}_t(x, a) - \tilde{\theta}_{t,a}(x, a)) \leq \int_{T_3}^{\infty} \zeta_t \sum_{a \in B^*_0(a_{a+})} \tilde{\theta}_{x,a}J(f_{\tilde{\theta}_t}) dt < \infty, \]

(5.121)

which contradicts (5.112). Therefore, the set $I^*_+$ must be empty for all $x \in X$ and then the proof is completed.

The global convergence in Lemma 5.19 can also allow one to prove the global convergence of the policy.

**Lemma 5.20.** For any deterministic optimal policy $f^*$, let $a^*(x) = \arg \max_a f^*(x, a)$, $\forall x \in X$. Recall that the optimal actions set
\[ \mathcal{A}^*(x) := \left\{ a^*(x) \in \mathcal{A} : V^{f^*}(x, a^*(x)) = \max_a V^{f^*}(x, a) \right\}, \quad \forall x \in X. \]

Then, by the convergence (5.110), if for each state $x$ the best action $a^*(x)$ is unique, we will have
\[ \lim_{t \to \infty} f_{\tilde{\theta}_t}(x, a^*(x)) = 1, \quad \forall x \in X \]

(5.122)

and thus
\[ \inf_{x \in X, t \geq 0} f_{\tilde{\theta}_t}(x, a^*(x)) > 0. \]

(5.123)

If for some state $x$, the best action is not unique, then the convergence (5.110) will imply
\[ \lim_{t \to \infty} \sum_{a^*(x) \in \mathcal{A}^*(x)} f_{\tilde{\theta}_t}(x, a^*(x)) = 1, \quad \forall x \in X \]

(5.124)

and thus
\[ \inf_{x \in X, t \geq 0} \sum_{a^*(x) \in \mathcal{A}^*(x)} f_{\tilde{\theta}_t}(x, a^*(x)) > 0. \]

(5.125)
Proof. As in (5.75), we have
\[ J(f^*) - J(f_{\bar{\theta}_t}) = \sum_x \nu^*_\mu(x) \sum_a f^*(x, a) A^{f_{\bar{\theta}_t}}(x, a) \]
\[ = \sum_x \nu^*_\mu(x) A^{f_{\bar{\theta}_t}}(x, a^*(x)) \]
\[ = \sum_x \nu^*_\mu(x) \left[ V^{f_{\bar{\theta}_t}}(x, a^*(x)) - \sum_{a'} V^{f_{\bar{\theta}_t}}(x, a') f_{\bar{\theta}_t}(x, a') \right] \]
\[ (5.126) \]

By (5.110), we have the convergence
\[ 0 = \lim_{t \to \infty} [J(f^*) - J(f_{\bar{\theta}_t})] = \sum_x \mu(x) \left[ V^{f^*}(x) - V^{f_{\bar{\theta}_t}}(x) \right] , \]
\[ (5.127) \]
which together with \( \mu(x) > 0 \), \( V^{f^*}(x) - V^{f_{\bar{\theta}_t}}(x) \geq 0, \forall x \in \mathcal{X} \) and the relationship (5.79) leads to
\[ \lim_{t \to \infty} V^{f^*}(x) - V^{f_{\bar{\theta}_t}}(x) = 0, \forall x \in \mathcal{X} \]
\[ \lim_{t \to \infty} V^{f^*}(x, a) - V^{f_{\bar{\theta}_t}}(x, a) = 0, \forall (x, a) \in \mathcal{X} \times \mathcal{A}. \]
\[ (5.128) \]

Combining (5.126) and (5.128), we have
\[ 0 = \lim_{t \to \infty} [J(f^*) - J(f_{\bar{\theta}_t})] \]
\[ = \lim_{t \to \infty} \sum_x \nu^*_\mu(x) \left[ V^{f_{\bar{\theta}_t}}(x, a^*(x)) - \sum_{a'} V^{f_{\bar{\theta}_t}}(x, a') f_{\bar{\theta}_t}(x, a') \right] \]
\[ = \lim_{t \to \infty} \sum_x \nu^*_\mu(x) \left[ \max_a V^{f^*}(x, a) - \sum_{a'} V^{f^*}(x, a') f_{\bar{\theta}_t}(x, a') \right] \]
\[ \geq (a) \lim_{t \to \infty} \sum_x \mu(x) \left[ \max_a V^{f^*}(x, a) - \sum_{a'} V^{f^*}(x, a') f_{\bar{\theta}_t}(x, a') \right] \]
\[ (5.129) \]

where step (a) is due to
\[ \max_a V^{f^*}(x, a) - \sum_{a'} V^{f^*}(x, a') f_{\bar{\theta}_t}(x, a') \geq 0, \forall x \in \mathcal{X}. \]

Then we have
\[ \lim_{t \to \infty} \left[ V^{f^*}(x, a^*(x)) - \sum_{a'} V^{f^*}(x, a') f_{\bar{\theta}_t}(x, a') \right] = 0, \forall x \in \mathcal{X}. \]
\[ (5.130) \]

Thus if the best action \( a^*(x) \) for any state \( x \in \mathcal{X} \) is unique, (5.130) derives
\[ \lim_{t \to \infty} f_{\bar{\theta}_t}(x, a^*(x)) = 1, \forall x \in \mathcal{X}. \]

When there exist multiple optimal actions in \( \mathcal{A}^*(x) \), (5.130) derives
\[ \lim_{t \to \infty} \sum_{a^*(x) \in \mathcal{A}^*(x)} f_{\bar{\theta}_t}(x, a^*(x)) = 1, \forall x \in \mathcal{X}. \]

Finally, noting that \( f_\theta \) being a softmax policy and the bound in Lemma 4.1, for any finite \( t > 0 \), the policy is positive. Thus (5.123) and (5.125) are direct corollary of (5.122) and (5.124). \( \square \)

Finally, combining Lemma 5.5 and Lemma 5.20, we can obtain the uniform Lojasiewicz inequality, which will prove the convergence rate (3.6).
Proof of (3.6): Define the actor error 
\[ Y_t := J(f^*) - J(f_{\theta_t}) \] 

Then, by chain rule, 
\[
\frac{dY_t}{dt} = -\zeta_t \nabla_\theta J(f_{\theta_t}) \hat{\nabla}_\theta J(f_{\theta_t}) \\
\leq -\zeta_t \| \nabla_\theta J(f_{\theta_t}) \|^2 + C \zeta_t \| \nabla_\theta J(f_{\theta_t}) \| \cdot \| \hat{\nabla}_\theta J - V f_{\theta_t} \|_2 \\
\leq -\zeta_t \| \nabla_\theta J(f_{\theta_t}) \|^2 + C \zeta_t \eta_t \| \nabla_\theta J(f_{\theta_t}) \|
\]

By Lemma 5.5 and Lemma 5.20, there exists a constant \( C > 0 \) such that 
\[
\| \nabla_\theta J(f_{\theta_t}) \| \geq C [J(f^*) - J(f_{\theta_t})] = CY_t,
\]

which together with (5.131) derives 
\[
\frac{dY_t}{dt} \leq -C\zeta_t Y_t^2 + C\zeta_t \eta_t \leq -\frac{C}{t} Y_t^2 + \frac{C}{t \log^2 t}
\]

Consider the comparison ODE: 
\[
\frac{dZ_t}{dt} = -\frac{C}{t} Z_t^2 + \frac{C}{t \log^2 t}, \quad t \geq 2. \\
Z_2 > Y_2
\]

By the Basic Comparison Theorem in [21], we have 
\[
0 \leq Y_t < Z_t \quad t \geq 2.
\]

Then, if we can establish a convergence rate for \( Z_t \), we will have a convergence rate for \( Y_t \).

Without loss of generality, we suppose the constant \( C = 1 \) and define function 
\[
0 \leq X_t = Z_t \log t, \quad t \geq 2.
\]

Thus, 
\[
\frac{dX_t}{dt} = \frac{1}{t} Z_t \log t + \frac{1}{t} Z_t^2 + \frac{1}{t \log^2 t} \\
= \frac{1}{t \log t} (Z_t \log t - Z_t^2 \log^2 t + 1) \\
= \frac{1}{t \log t} (X_t - X_t^2 + 1), \quad t \geq 2.
\]

Noting that \( \frac{1}{2 - \sqrt{2}} \) and \( \frac{1}{2 + \sqrt{2}} \) are two stationary solution of (5.136), the solution \( X_t \) will decrease if it is larger than \( \frac{1}{2 + \sqrt{2}} \) and it will increase for \( X_t \in [0, \frac{1}{2 + \sqrt{2}}] \). Thus, for a solution \( X_t \) starting from \( X_2 \geq 0 \), there are two cases:

1. If the starting point \( X_2 \geq \frac{1}{2 + \sqrt{2}} \), the solution \( X_t \) will decrease and always be larger than \( \frac{1}{2 + \sqrt{2}} \) by the uniqueness theorem for ODEs (Theorem 2.2 of [34]).

2. If the starting point \( X_2 \in [0, \frac{1}{2 + \sqrt{2}}] \), the solution \( X_t \) will increase and always be smaller than \( \frac{1}{2 + \sqrt{2}} \) by the uniqueness theorem for ODEs (Theorem 2.2 of [34]).

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Thus, no matter where $X_t$ starts from, we always have

$$0 \leq X_t \leq \max\{X_2, \frac{1 + \sqrt{5}}{2}\}, \quad t \geq 2,$$

which shows that

$$0 \leq Y_t < Z_t \leq \frac{C}{\log t}, \quad t \geq 2,$$

and therefore the convergence rate (3.6) is proven.

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**Appendix**

**A Verification of (3.2)**

$$\int_0^\infty \zeta_s \eta_s ds = \int_0^2 \zeta_s \eta_s ds + \int_2^\infty \zeta_s \eta_s ds$$

$$\leq C + \int_2^\infty \frac{1}{t \log^2 t} dt$$

$$= C - \frac{1}{\log t} \Big|_2^\infty < \infty,$$

$$\lim_{t \to \infty} \frac{\zeta_t}{\eta_t^n} = \lim_{t \to \infty} \frac{\log^{2n} t}{t} (a) = 0$$

where step (a) is by L’Hospital’s Rule.

**B Proof of Corollary 4.5**

Proof. Recall the exploration policy in (2.10) with the decreasing exploration rate $\eta_k^N$. Then, we have for $\forall k \leq NT$,

$$g_k(x, a) \geq \frac{\eta_k^N}{d_A}, \quad \forall x, a \in \mathcal{X} \times \mathcal{A}.$$  

Then, for any $\xi, \xi'$ and $k \leq NT$, with the constant $C$ from (4.32),

$$P_{\theta_k}^n(\xi; \xi') = \sum_{\xi_1, \cdots, \xi_{n_0}} P_{\theta_k}(\xi; \xi_1) \cdots P_{\theta_k}(\xi_{n_0-1}; \xi')$$

$$= \sum_{\xi_1, \cdots, \xi_{n_0-1}} p(x_1|x, a) g_k(x_1, a_1) \cdots p(x'|x_{n_0-1}, a_{n_0-1}) g_k(x', a')$$

$$\geq C \left(\eta_{NT}^N\right)^{n_0}.$$
Thus, we can derive a lower bound for the stationary distribution
\[ \inf_{k \leq NT} \pi^{g_k}(x', a') = \inf_{k \leq NT} \sum_{x,a} \pi^{g_k}(x, a) P_{\theta_k}^n(x, a; x', a') \]
\[ \geq \inf_{k \leq NT} \sum_{x,a} \pi^{g_k}(x, a) C \left( \frac{\eta_N}{|\eta_N|} \right)^n \]
\[ \overset{(a)}{=} C \left( \frac{\eta_N}{|\eta_N|} \right)^n > 0, \]
where the step (a) is because \( \pi^{g_k} \) is a probability and thus the summation equals to 1. For the uniform geometric ergodicity, we can choose \( \beta_T = \inf_{k \leq NT} \min P_{\theta_k}(\xi, \xi') > 0 \) in (4.34), where \( \beta_T > 0 \) is by (B.2). Thus for \( \forall k \leq NT \), the Markov chain with transition probability \( P_{\theta_k} \) satisfies the Doeblin’s condition, then by Theorem 16.2.4 of [25], we can derive the uniform geometric ergodicity (4.34).

C Proof of Lemma 4.10

Proof. As in the proof for the decay of \( M_t^N \), we use two steps to prove the result.

(i) Prove that the fluctuations of the data samples around a dynamic stationary distribution \( \pi^{g_k} \) decay when the number of iteration steps becomes large.

(ii) Use the same method as in Lemma 4.9 to prove the stochastic fluctuation terms vanish as \( N \to \infty \).

(i) To prove that for any fixed state action pair \( \xi = (x, a), \forall T > 0 \)

\[ \lim_{N \to 0} \mathbb{E} \left[ \frac{1}{N} \sum_{k=0}^{NT-1} \left[ 1_{\{\xi_k = \xi\}} - \pi^{g_k}(\xi) \right] \right] = 0, \]  

(C.1)

we first introduce a similar Poisson equation for any fixed state-action pair \( \xi = (x, a), N \in \mathbb{N}, T < \infty \) and \( k \leq NT \),

\[ \tilde{\nu}_{\theta_k}(\xi') - P_{\theta_k} \tilde{\nu}_{\theta_k}(\xi') = 1_{\{\xi' = \xi\}} - \pi^{g_k}(\xi), \quad \xi' \in \mathcal{X} \times \mathcal{A}. \]  

(C.2)

A solution of (C.2) can be expressed as

\[ \tilde{\nu}_{\theta_k}(\xi') = \sum_{n \geq 0} \left[ P_{\theta_k}^n(\xi'; \xi) - \pi^{g_k}(\xi) \right]. \]  

(C.3)

By Corollary 4.5, there exists a constant \( C_T \) (which only depends on \( T \)) such that

\[ \sup_{k \leq NT} |\tilde{\nu}_{\theta_k}(\xi')| \leq C_T, \quad \forall \xi' \in \mathcal{X} \times \mathcal{A}. \]  

(C.4)

Then, as in the proof of Lemma 4.8, we define the error \( \tilde{\epsilon}_k \)

\[ \tilde{\epsilon}_k := 1_{\{\xi_k+1 = \xi\}} - \pi^{g_k}(\xi) \]

\[ = \tilde{\nu}_{\theta_k}(\xi_k+1) - P_{\theta_k} \tilde{\nu}_{\theta_k}(\xi_k+1) \]

\[ = [\tilde{\nu}_{\theta_k}(\xi_k+1) - P_{\theta_k} \tilde{\nu}_{\theta_k}(\xi_k)] + [P_{\theta_k} \tilde{\nu}_{\theta_k}(\xi_k) - P_{\theta_k} \tilde{\nu}_{\theta_k}(\xi_k+1)]. \]  

(C.5)

Let

\[ \tilde{\psi}_{\theta}(y) = P_{\theta} \tilde{\nu}_{\theta}(y). \]  

(C.6)

Then, we have

\[ \sum_{k=0}^{NT-1} \tilde{\epsilon}_k = \sum_{k=0}^{NT-1} [\tilde{\nu}_{\theta_k}(\xi_k+1) - P_{\theta_k} \tilde{\nu}_{\theta_k}(\xi_k)] + \sum_{k=0}^{NT-1} [\tilde{\psi}_{\theta_k}(\xi_k) - \tilde{\psi}_{\theta_k}(\xi_k+1)] \]

\[ = \sum_{k=0}^{NT-1} [\tilde{\nu}_{\theta_k}(\xi_k+1) - P_{\theta_k} \tilde{\nu}_{\theta_k}(\xi_k)] + \sum_{k=1}^{NT-1} [\tilde{\psi}_{\theta_k}(\xi_k) - \tilde{\psi}_{\theta_k-1}(\xi_k)] \]  

(C.7)
Define the error term as

\[
\sum_{k=0}^{\lfloor NT \rfloor - 1} \bar{\epsilon}_k = \sum_{k=0}^{\lfloor NT \rfloor - 1} \bar{\epsilon}_k^{(1)} + \sum_{k=1}^{\lfloor NT \rfloor - 1} \bar{\epsilon}_k^{(2)} + \bar{\rho}_{\lfloor NT \rfloor;0} \tag{C.8}
\]

where

\[
\bar{\epsilon}_k^{(1)} = \bar{\nu}_{\theta_k}(\xi_{k+1}) - \bar{\nu}_{\theta_k}(\xi_k)
\]

\[
\bar{\epsilon}_k^{(2)} = \bar{\psi}_{\theta_k}(\xi_k) - \bar{\psi}_{\theta_{k-1}}(\xi_k)
\]

\[
\bar{\rho}_{\lfloor NT \rfloor;0} = \bar{\psi}_{\theta_0}(\xi_0) - \bar{\psi}_{\theta_{\lfloor NT \rfloor-1}}(\xi_{\lfloor NT \rfloor}).
\]

To prove the convergence (C.1), it suffices to appropriately bound the fluctuation term \(\sum_{k=0}^{\lfloor NT \rfloor - 1} \bar{\epsilon}_k\). The first term can be bounded using the martingale property while the second term can be bounded using the uniform geometric ergodicity and Lipschitz continuity. The third term is bounded using (C.4).

For the first term in (C.8), note that

\[
E \{ \bar{\nu}_{\theta_k}(\xi_{k+1}) \mid \mathcal{F}_k \} = \bar{\nu}_{\theta_k}(\xi_k).
\]

Therefore,

\[
\left\{ \bar{Z}_n = \sum_{k=0}^{n-1} \bar{\epsilon}_k^{(1)} , \mathcal{F}_n \right\}_{n \geq 0}
\]

is a martingale and since the conditional expectation is a contraction in \(L^2\), we have

\[
E |P_{\theta_k} \bar{\nu}_{\theta_k}(\xi_k)|^2 \leq E |\bar{\nu}_{\theta_k}(\xi_{k+1})|^2.
\]

Then,

\[
E \left| \frac{1}{N} \sum_{k=0}^{\lfloor NT \rfloor - 1} \bar{\epsilon}_k^{(1)} \right|^2 = \frac{1}{N^2} \sum_{k=0}^{\lfloor NT \rfloor - 1} E |\bar{\nu}_{\theta_k}(\xi_{k+1}) - P_{\theta_k} \bar{\nu}_{\theta_k}(\xi_k)|^2
\]

\[
\leq \frac{4}{N^2} \sum_{k=0}^{\lfloor NT \rfloor - 1} E |\bar{\nu}_{\theta_k}(\xi_{k+1})|^2
\]

\[
\leq (a) \frac{4C_T}{N},
\]

where the step (a) is by the uniform boundedness (C.4). Thus we have for any \(T > 0\)

\[
\lim_{N \to \infty} E \left| \frac{1}{N} \sum_{k=0}^{\lfloor NT \rfloor - 1} \bar{\epsilon}_k^{(1)} \right| = 0. \tag{C.13}
\]

For the second term of (C.8), by the uniform geometric ergodicity (4.34), for any fixed \(\gamma_0 > 0\) we can choose \(N_0\) large enough such that

\[
\sup_{k \leq NT} \sum_{n=[N_0 T]}^{\infty} \left| P_{\theta_k}^n (y, \xi) - \pi^g(\xi) \right| < \gamma_0, \quad \forall y \in \mathcal{X} \times \mathcal{A} \tag{C.14}
\]
Since \( \gamma \), Thus, when \( N \) is large enough, which together with (C.13) and (C.20) derive the convergence of \( \frac{1}{N} \sum_{k=0}^{NT-1} \tilde{e}_k \) and (C.1).
(ii) Following the same method in Lemma 4.9, we can prove the convergence of the stochastic error $M_t^{1,N}$ for \( i = 1, 2, 3 \).

For any \( K \in \mathbb{N} \) and \( \Delta = \frac{t}{K} \), we have

\[
-M_t^{1,N}(\xi) = \sum_{j=0}^{K-1} \Delta \left[ \sum_{k=j+1 \Delta N}^{(j+1) \Delta N - 1} \left[ \sum_{\xi' \in X \times A} (Q_k(\xi_k) \partial \xi_k Q_k(\xi_k) - \sum_{\xi' \in X \times A} Q_k(\xi') \partial \xi_k Q_k(\xi') \pi^{g_k}(\xi') \right) \right] + o(1)
\]

\[
= \sum_{j=0}^{K-1} \Delta \left[ \sum_{k=j+1 \Delta N}^{(j+1) \Delta N - 1} \left[ \sum_{\xi' \in X \times A} (Q_j(\Delta N)(\xi_k) \partial \xi_k Q_j(\Delta N)(\xi_k) - \sum_{\xi' \in X \times A} Q_j(\Delta N)(\xi') \partial \xi_k Q_j(\Delta N)(\xi') \pi^{g_k}(\xi') \right) \right] + o(1)
\]

\[
= \sum_{j=0}^{K-1} \Delta \left[ \sum_{k=j+1 \Delta N}^{(j+1) \Delta N - 1} \left[ \left( Q_j(\Delta N)(\xi_k) \partial \xi_k Q_j(\Delta N)(\xi_k) - \sum_{\xi' \in X \times A} Q_j(\Delta N)(\xi') \partial \xi_k Q_j(\Delta N)(\xi') \pi^{g_k}(\xi') \right) \right] + o(1)
\]

\[
:= \sum_{j=0}^{K-1} \Delta I_{5,j}^{N} + \sum_{j=0}^{K-1} \Delta I_{6,j}^{N} + o(1),
\]

where the term \( o(1) \) goes to zero, at least, in \( L^1 \) as \( N \to \infty \).

To prove the convergence of the first term, note that

\[
Q_j(\Delta N)(\xi_k) \partial \xi_k Q_j(\Delta N)(\xi_k) - \sum_{\xi'} Q_j(\Delta N)(\xi') \partial \xi_k Q_j(\Delta N)(\xi') \pi^{g_k}(\xi')
\]

\[
= \sum_{\xi'} Q_j(\Delta N)(\xi') \partial \xi_k Q_j(\Delta N)(\xi') [I_{\{\xi_k = \xi'\}} - \pi^{g_k}(\xi')]
\]

\[
= \sum_{\xi'} Q_j(\Delta N)(\xi') \partial \xi_k Q_j(\Delta N)(\xi') \left[ I_{\{\xi_k = \xi'\}} - \pi^{g_k}(\xi') \right].
\]

Thus, for any \( j \in 0, 1, \ldots, K \),

\[
|I_{5,j}^{N}| = \left| \sum_{k=j+1 \Delta N}^{(j+1) \Delta N - 1} \sum_{\xi'} Q_j(\Delta N)(\xi') \partial \xi_k Q_j(\Delta N)(\xi') \left[ I_{\{\xi_k = \xi'\}} - \pi^{g_k}(\xi') \right] \right|
\]

\[
= \sum_{\xi'} Q_j(\Delta N)(\xi') \partial \xi_k Q_j(\Delta N)(\xi') \left[ I_{\{\xi_k = \xi'\}} - \pi^{g_k}(\xi') \right]
\]

\[
\leq C \sum_{\xi'} \left[ \sum_{k=j+1 \Delta N}^{(j+1) \Delta N - 1} \left[ I_{\{\xi_k = \xi'\}} - \pi^{g_k}(\xi') \right] \right]
\]

which together with Lemma 4.8 proves

\[
\lim_{N \to \infty} \mathbb{E} \left| I_{5,j}^{N} \right| = 0.
\]

Thus,

\[
\sum_{j=0}^{K-1} \Delta I_{5,j}^{N} = \Delta \sum_{j=0}^{K-1} O(1) = t \sum_{j=0}^{K-1} O(1) \frac{1}{K},
\]

which proves the convergence of the first term.
For the second term, by the bound in Lemma 4.1, for any \( k \leq TN \) we have
\[
\sup_{\xi' \in X \times A} |Q_k(\xi')| \leq C,
\]
\[
\sup_{\xi' \in X \times A} |Q_k(\xi') - Q_{k-1}(\xi')| \leq \frac{C}{N}.
\] (C.26)

Note that
\[
\partial \xi Q_k(\xi') = 1_{\{\xi' = \xi\}}.
\]
Then, by the Lipschitz continuity of the softmax transformation and the bound in Lemma 4.1,
\[
|Q_k(\xi')\partial \xi Q_k(\xi') - Q_{k-1}(\xi')\partial \xi Q_{k-1}(\xi')| = 1_{\{\xi_k = \xi\}} |Q_k(\xi') - Q_{k-1}(\xi')| \leq \frac{C}{N}.
\] (C.27)

Then, for any \( j \in 0, 1, \ldots, K - 1 \) and any \( k \in [j|\Delta N], (j + 1)|\Delta N] - 1 \),
\[
|Q_k(\xi')\partial \xi Q_k(\xi') - Q_{j|\Delta N]}(\xi')\partial \xi Q_{j|\Delta N}](\xi')| \leq \frac{C(k - j|\Delta N}){N}.
\] (C.28)

Therefore,
\[
\sum_{j=0}^{K-1} \Delta I_{6,j}^N \leq C \sum_{j=0}^{K-1} \Delta \left( \frac{1}{|\Delta N]} \sum_{k=j|\Delta N]}^{(j+1)|\Delta N]} \frac{k - j|\Delta N]}{N} \right)
\leq C \sum_{j=0}^{K-1} \Delta \frac{1}{|\Delta N]} \sum_{k=0}^{(j+1)|\Delta N]} - 1 \frac{k}{N}
\leq C \sum_{j=0}^{K-1} \Delta \frac{|\Delta N]}{N^2}
\leq C \sum_{j=0}^{K-1} \Delta \frac{|\Delta N]}{N}
\leq C \sum_{j=0}^{K-1} \Delta^2
\leq C \Delta.
\] (C.29)

Collecting our results, we have shown that
\[
\lim_{N \to \infty} \sup_{t \in [0, T]} \mathbb{E} \left| M_{t,0,N}^1 \right| \leq C \frac{T}{K}.
\] (C.30)

Note that \( K \) was arbitrary. Consequently, we obtain
\[
\lim_{N \to \infty} \sup_{t \in [0, T]} \mathbb{E} \left| M_{t,1,N}^1 \right| = 0,
\] (C.31)

Using the same approach, one can prove the claim for \( M_{t,2,N}^2 \) and \( M_{t,3,N}^3 \). The details of the proof are omitted due to the similarity of the argument.
D Proof of Lemma 5.5

Proof. To prove (5.60), note that

\[
\| \nabla_{\theta} J(f_{\theta}) \| = \left[ \sum_{x,a} (\partial_{x,a} J(f_{\theta}))^2 \right]^{1/2}
\]

\[
\geq \left[ \sum_{x} \left( \frac{\partial J(f_{\theta})}{\partial \theta(x,a^*(x))} \right)^2 \right]^{1/2}
\]

\[
\overset{(a)}{\geq} \frac{1}{\sqrt{|A|}} \sum_{x} \left| \frac{\partial J(f_{\theta})}{\partial \theta(x,a^*(x))} \right|
\]

\[
\overset{(b)}{=} \frac{1}{\sqrt{|A|}} \sum_{x} \left| \nu_{\mu_{\theta}}^{f_{\theta}}(x) \cdot f_{\theta}(x,a^*(x)) \cdot A^{f_{\theta}}(x,a^*(x)) \right|
\]

where step (a) is by Cauchy-Schwarz inequality and step (b) is by Lemma 4.3.

Define the coefficient as

\[
\frac{\| \nu_{\mu_{\theta}}^{f_{\star}} \|}{\| \nu_{\mu_{\theta}}^{f_{\star}} \|_{\infty}} = \max_{x} \frac{\nu_{\mu_{\theta}}^{f_{\star}}(x)}{\nu_{\mu_{\theta}}^{f_{\star}}(x)}.
\]

We then have the inequality:

\[
\| \nabla_{\theta} J(f_{\theta}) \| \geq \frac{1}{\sqrt{|A|}} \sum_{x} \nu_{\mu_{\theta}}^{f_{\theta}}(x) \cdot \nu_{\mu_{\theta}}^{f_{\star}}(x) \cdot f_{\theta}(x,a^*(x)) \cdot |A^{f_{\theta}}(x,a^*(x))|
\]

\[
\overset{(a)}{\geq} \frac{1}{\sqrt{|A|}} \left\| \frac{\nu_{\mu_{\theta}}^{f_{\star}}}{\nu_{\mu_{\theta}}^{f_{\star}}} \right\|^{-1} \cdot \min_{x} f_{\theta}(x,a^*(x)) \cdot \sum_{x} \nu_{\mu_{\theta}}^{f_{\star}}(x) \cdot |A^{f_{\theta}}(x,a^*(x))|
\]

\[
\overset{(b)}{=} \frac{1}{\sqrt{|A|}} \left\| \frac{\nu_{\mu_{\theta}}^{f_{\star}}}{\nu_{\mu_{\theta}}^{f_{\star}}} \right\|^{-1} \cdot \min_{x} f_{\theta}(x,a^*(x)) \cdot \sum_{x} \nu_{\mu_{\theta}}^{f_{\star}}(x) \cdot A^{f_{\theta}}(x,a^*(x))
\]

\[
\overset{(a)}{=} \frac{1}{\sqrt{|A|}} \left\| \frac{\nu_{\mu_{\theta}}^{f_{\star}}}{\nu_{\mu_{\theta}}^{f_{\star}}} \right\|^{-1} \cdot \min_{x} f_{\theta}(x,a^*(x)) \cdot \sum_{x,a} f^*(x,a) \cdot A^{f_{\theta}}(x,a)
\]

\[
= \frac{1}{\sqrt{|A|}} \left\| \frac{\nu_{\mu_{\theta}}^{f_{\star}}}{\nu_{\mu_{\theta}}^{f_{\star}}} \right\|^{-1} \cdot \min_{x} f_{\theta}(x,a^*(x)) \cdot [J(f^*) - J(f_{\theta})]
\]

where step (a) uses the fact that \( f^* \) is deterministic and in state \( x \) selects \( a^*(x) \) with probability one. The last equality uses Lemma 5.7.

To prove the second claim, given a policy \( f \), recall the greedy action set for each state \( x \):

\[
\mathcal{A}^*(x) = \left\{ a^*(x) \in \mathcal{A} : V^{f^*}(x,a^*(x)) = \max_{a} V^{f^*}(x,a) \right\},
\]
By similar arguments as before, we can show that
\[
\|\nabla_{\theta} J(f_{\theta})\| \geq \frac{1}{\sqrt{\mathcal{X}|\mathcal{A}|}} \sum_{x,a} \left| \frac{\partial J(f_{\theta})}{\partial \theta(x,a)} \right|
\]
\[
= \frac{1}{\sqrt{\mathcal{X}|\mathcal{A}|}} \sum_{x} \nu_{\theta} f_{\theta}(x) \sum_{a} f_{\theta}(x,a) \cdot |A_{\theta} f_{\theta}(x,a)| \quad \text{(by Lemma 4.3)}
\]
\[
\geq \frac{1}{\sqrt{\mathcal{X}|\mathcal{A}|}} \sum_{x} \nu_{\theta} f_{\theta}(x) \sum_{a}(x) \sum_{a'} f_{\theta}(x,a') \cdot |A_{\theta} f_{\theta}(x,a')| \cdot \left[ \sum_{a' \in \mathcal{A}^*(x)} f_{\theta}(x,a') \right] \cdot |A_{\theta} f_{\theta}(x,a^*(x))|
\]
\[
= \frac{1}{\sqrt{\mathcal{X}|\mathcal{A}|}} \sum_{x} \nu_{\theta} f_{\theta}(x) \sum_{a}(x) \sum_{a'} f_{\theta}(x,a') \cdot \left[ \sum_{a' \in \mathcal{A}^*(x)} f_{\theta}(x,a') \right] \cdot |A_{\theta} f_{\theta}(x,a^*(x))|
\]
\[
\geq \frac{1}{\sqrt{\mathcal{X}|\mathcal{A}|}} \cdot \left\| \frac{\nu_{\theta} f_{\theta}}{\nu_{\theta} f_{\theta}} \right\|_{\infty}^{-1} \left[ \min_{a' \in \mathcal{A}^*(x)} \sum_{a} f_{\theta}(x,a') \right] \cdot \left[ \sum_{a} \nu_{\theta} f_{\theta}(x,a) \sum_{a} f_{\theta}(x,a) A_{\theta} f_{\theta}(x,a) \right]
\]
\[
\geq \frac{1}{\sqrt{\mathcal{X}|\mathcal{A}|}} \cdot \left\| \frac{\nu_{\theta} f_{\theta}}{\nu_{\theta} f_{\theta}} \right\|_{\infty}^{-1} \left[ \min_{a' \in \mathcal{A}^*(x)} \sum_{a} f_{\theta}(x,a') \right] \cdot \left[ J(f_{\theta}) - J(f_{\theta}) \right]
\]
\[
\geq \frac{1}{\sqrt{\mathcal{X}|\mathcal{A}|}} \cdot \left\| \frac{\nu_{\theta} f_{\theta}}{\nu_{\theta} f_{\theta}} \right\|_{\infty}^{-1} \left[ \min_{a' \in \mathcal{A}^*(x)} \sum_{a} f_{\theta}(x,a') \right] \cdot \left[ J(f^*) - J(f_{\theta}) \right],
\]
\]
(D.3)

where step (a) and (b) are by the definition of the optimal policy (5.62), step (c) by due to difference lemma 5.7 and step d is because of \( f_{\theta}^* \) is optimal policy and thus \( J(f_{\theta}^*) = J(f^*) \).

\[\square\]

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