NONDEGENERACY OF THE ENTIRE SOLUTION FOR THE N-LAPLACE LIOUVILLE EQUATION

FUTOSHI TAKAHASHI

Abstract. In this note, we prove the nondegeneracy of the explicit finite-mass solution to the N-Laplace Liouville equation on the whole space, which is recently shown to be unique up to scaling and translation.

1. Introduction

Let $N \geq 2$ be an integer. In this note, we concern the following quasi-linear Liouville equation:

$$
\begin{cases}
-\Delta_N U = e^U & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} e^U\,dx < \infty,
\end{cases}
$$

(1.1)

here $\Delta_N U = \text{div}(\nabla |\nabla U|^{N-2} \nabla U)$ denotes the N-Laplacian of a function $U$. Problem (1.1) has the explicit solution (Liouville bubble)

$$
U(x) = \log \frac{C_N}{\left(1 + |x|^{N-1}\right)^{N}}, \quad x \in \mathbb{R}^N,
$$

(1.2)

where $C_N = N \left(\frac{N^2}{N-1}\right)^{N-1}$. Thanks to the scaling and translation invariance of the problem, the functions

$$
U_{\lambda,a}(x) = U(\lambda(x - a)) + N \log \lambda, \quad \lambda > 0, \, a \in \mathbb{R}^N
$$

(1.3)

constitute a $(N + 1)$-dimensional family of solutions to (1.1) with

$$
\int_{\mathbb{R}^N} e^{U_{\lambda,a}}\,dx = \left(\frac{\omega_{N-1}}{N}\right)C_N,
$$

where $\omega_{N-1}$ denotes the area of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$. Indeed, all the solutions of (1.1) are of the form (1.3). This fact is first proven by Chen and Li [2] when $N = 2$ by the method of moving planes. Recently, P. Esposito [4] proves the same classification result for (1.1) when $N \geq 3$. His method exploits a weighted Sobolev estimates at infinity for any solution to (1.1),

Date: November 4, 2022.

2010 Mathematics Subject Classification. Primary 35J60; Secondary 35J20, 35B08.

Key words and phrases. Liouville equation, nondegeneracy, N-Laplacian.
an isoperimetric argument, and the Pohozaev identity, and does not use the moving plane arguments.

We are interested in the linear nondegeneracy of the explicit solution $U$ in (1.2). Thus we consider the linearized operator around $U$:

$$L(\phi) = \frac{d}{dt} \bigg|_{t=0} N(U + t\phi)$$

where $N(U + t\phi) = \Delta_N(U + t\phi) + e^{U+t\phi}$ for $t \in \mathbb{R}$ and a function $\phi$. Then we compute directly that

$$L(\phi) = \text{div}(|\nabla U|^N - 2 \nabla U \cdot \nabla \phi) + e^{U+t\phi}$$

(1.4)

here and henceforth, “·” denotes the standard inner product in $\mathbb{R}^N$. Since $U_{\lambda,a}$ in (1.3) solves the equation

$$\Delta_N U_{\lambda,a} + e^{U_{\lambda,a}} = 0 \quad \text{in} \quad \mathbb{R}^N,$$

by differentiating the above equation with respect to the parameters $\lambda$ and $a_1, \ldots, a_N$ at $\lambda = 1$ and $a = 0$, we obtain the bounded solutions

$$Z_0(x) = \frac{d}{d\lambda} \bigg|_{\lambda=1, a=0} U_{\lambda,a} = x \cdot \nabla U + N,$$

(1.5)

$$Z_i(x) = \frac{d}{da_i} \bigg|_{\lambda=1, a=0} U_{\lambda,a} = \frac{\partial U}{\partial x_i}, \quad (i = 1, \ldots, N)$$

(1.6)

to the linearized equation $L(\phi) = 0$.

The aim of this note is to prove the following nondegeneracy of $U$:

**Theorem 1.** Let $U$ be as in (1.2) and let $\phi$ be a solution in $L^\infty \cap C^2(\mathbb{R}^N)$ to the linearized equation $L(\phi) = 0$, here $L$ is as in (1.4). Then $\phi$ can be written as a linear combination of $Z_0, Z_1, \ldots, Z_N$ defined by (1.5), (1.6).

The above theorem was known already when $N = 2$, see [1], [3]. In this note, we extend the result to $N \geq 3$.

Our proof is similar to that of [6], in which the authors study the critical $p$-Laplace equation

$$-\Delta_p U = U^{p^*-1} \quad \text{in} \quad \mathbb{R}^N, \quad U > 0,$$

where $p^* = \frac{Np}{N-p}$, $1 < p < N$, is the critical Sobolev exponent. They prove the linear nondegeneracy of the explicit entire solution (Aubin-Talenti bubble)

$$U(x) = \left(\frac{\alpha_{N,p}}{1 + |x|^{N-p}}\right)^{\frac{N-p}{p-1}}$$

where $\alpha_{p,N} = N^\frac{1}{p} \left(\frac{N-p}{p-1}\right)^{\frac{p-1}{p}}$, extending the former result by Rey [5] for $p = 2$. 
2. Proof of Theorem 1

In this section, we prove Theorem 1. We follow the method by [6]. See also [1], [3].

First, we prove the next proposition:

**Proposition 1.** Let $L$ be as in (1.4). Then $\phi \in C^2(\mathbb{R}^N)$ solves $L(\phi) = 0$ if and only if $\phi$ is a solution to

$$
|x|^2 \Delta \phi + N(N - 2) \left( \frac{x \cdot \nabla \phi}{1 + |x|^\frac{N-2}{N}} + (N - 2) \sum_{i,j=1}^{N} \frac{\partial^2 \phi}{\partial x_i \partial x_j} x_i x_j \right) \tag{2.1}
$$

$$
+ \left( \frac{N^3}{N - 1} \right) \left( \frac{|x|^\frac{N}{N-1}}{1 + |x|^\frac{N}{N-1}} \right)^2 = 0.
$$

**Proof.** We rewrite the equation $L(\phi) = 0$ as

$$
L(\phi) = \text{div}(\nabla U^{N-2} \nabla \phi) + (N - 2) \text{div}(\nabla U^{N-4} (\nabla U \cdot \nabla \phi) \nabla U) + e^U \phi
$$

$$
= |\nabla U|^{N-2} \Delta \phi
$$

$$
+ \nabla (|\nabla U|^{N-2}) \cdot \nabla \phi
$$

$$
+ (N - 2) |\nabla U|^{N-4} (\nabla U \cdot \nabla \phi) \Delta U
$$

$$
+ (N - 2) (\nabla U \cdot \nabla \phi) \nabla (|\nabla U|^{N-4}) \cdot \nabla U
$$

$$
+ (N - 2) |\nabla U|^{N-4} (\nabla (\frac{1}{2} |\nabla U|^2) \cdot \nabla \phi
$$

$$
+ (N - 2) |\nabla U|^{N-4} (D^2 \phi)(\nabla U, \nabla U)
$$

$$
+ e^U \phi
$$

$$
= A + B + C + D + E + F + G = 0,
$$

where we have used

$$
\nabla (\nabla U \cdot \nabla \phi) \cdot \nabla \phi = \nabla (\frac{1}{2} |\nabla U|^2) \cdot \nabla \phi + D^2 \phi(\nabla U, \nabla U)
$$

with the notation that $(D^2 \phi)(\nabla U, \nabla U) = \sum_{i,j=1}^{N} \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j}$.

Now, we compute that

$$
\nabla U = - \left( \frac{N^2}{N - 1} \right) \frac{|x|^\frac{N-2}{N}}{1 + |x|^\frac{N-2}{N}} \frac{x}{|x|},
$$

$$
|\nabla U|^k = \left( \frac{N^2}{N - 1} \right)^k \frac{|x|^\frac{N-2}{N}}{(1 + |x|^\frac{N}{N-1})^k}, \quad (k \in \mathbb{Z})
$$

$$
\nabla (|\nabla U|^k) = \left( \frac{N^2}{N - 1} \right)^k \left( \frac{k}{N - 1} \right) \frac{|x|^\frac{N-2}{N} - 1}{(1 + |x|^\frac{N}{N-1})^{k+1}} \left( 1 + (1 - N)|x|^\frac{N}{N-1} \right)^{-1} \frac{x}{|x|}, \quad (k \in \mathbb{Z}).
$$
Thus we have
\[ \nabla U \cdot \nabla \phi = -\left( \frac{N^2}{N - 1} \right) \frac{|x|^\frac{N-4}{N-2}}{1 + |x|^\frac{N}{N-2}} (x \cdot \nabla \phi), \]
\[ \nabla \left( |\nabla U|^N \right) \cdot \nabla U = -\left( \frac{N^2}{N - 1} \right)^{N-3} \left( \frac{N - 4}{N - 1} \right) \frac{|x|^\frac{N^2}{N-2}}{(1 + |x|^\frac{N}{N-2})^{N-2}} \left\{ 1 + (1 - N)|x|^\frac{N}{N-2} \right\}, \]
\[ (D^2 \phi)(\nabla U, \nabla U) = \left( \frac{N^2}{N - 1} \right)^2 \frac{|x|^\frac{N}{N-2}}{(1 + |x|^\frac{N}{N-2})^2} \sum_{i,j=1}^N \frac{\partial^2 \phi}{\partial x_i \partial x_j} x_i x_j. \]

Also we see
\[ \Delta U = -\left( \frac{N^2}{N - 1} \right) \frac{|x|^\frac{N-4}{N-2}}{(1 + |x|^\frac{N}{N-2})^2} \left\{ \left( N - 1 + \frac{1}{N - 1} \right) + (N - 2)|x|^\frac{N}{N-2} \right\}. \]

From these, we obtain
where
\[ A = |\nabla U|^N \Delta \phi = \left( \frac{N^2}{N - 1} \right)^N \frac{|x|^\frac{N}{N-2}}{(1 + |x|^\frac{N}{N-2})^N} \Delta \phi, \]
\[ B = \nabla (|\nabla U|^N) \cdot \nabla \phi \]
\[ = \left( \frac{N^2}{N - 1} \right)^{N-2} \left( \frac{N - 2}{N - 1} \right) \frac{|x|^\frac{N}{N-2}}{(1 + |x|^\frac{N}{N-2})^{N-2}} \left\{ 1 + (1 - N)|x|^\frac{N}{N-2} \right\} (x \cdot \nabla \phi), \]
\[ C = (N - 2)|\nabla U|^N (\nabla U \cdot \nabla \phi) \Delta U \]
\[ = (N - 2) \left( \frac{N^2}{N - 1} \right)^{N-2} \frac{|x|^\frac{N}{N-2}}{(1 + |x|^\frac{N}{N-2})^{N-1}} \left\{ \left( N - 1 + \frac{1}{N - 1} \right) + (N - 2)|x|^\frac{N}{N-2} \right\} (x \cdot \nabla \phi), \]
\[ D = (N - 2)(\nabla U \cdot \nabla \phi) \nabla (|\nabla U|^N) \cdot \nabla U \]
\[ = (N - 2) \left( \frac{N^2}{N - 1} \right)^{N-2} \left( \frac{N - 4}{N - 1} \right) \frac{|x|^\frac{N}{N-2}}{(1 + |x|^\frac{N}{N-2})^{N-1}} \left\{ 1 + (1 - N)|x|^\frac{N}{N-2} \right\} (x \cdot \nabla \phi), \]
\[ E = (N - 2)|\nabla U|^N \nabla \left( \frac{1}{2} |\nabla U|^2 \right) \cdot \nabla \phi \]
\[ = \left( \frac{N^2}{N - 1} \right)^{N-2} \left( \frac{N - 2}{N - 1} \right) \frac{|x|^\frac{N}{N-2}}{(1 + |x|^\frac{N}{N-2})^{N-1}} \left\{ 1 + (1 - N)|x|^\frac{N}{N-2} \right\} (x \cdot \nabla \phi), \]
\[ F = (N - 2)|\nabla U|^N (D^2 \phi)(\nabla U, \nabla U) \]
\[ = (N - 2) \left( \frac{N^2}{N - 1} \right)^{N-2} \frac{|x|^\frac{N}{N-2}}{(1 + |x|^\frac{N}{N-2})^{N-2}} \sum_{i,j=1}^N \frac{\partial^2 \phi}{\partial x_i \partial x_j} x_i x_j, \]
\[ G = e^U \phi = \frac{C_N}{(1 + |x|^\frac{N}{N-2})^N} \phi. \]
Returning to the equation $L(\phi) = 0$ with these expressions and after some manipulations, we obtain that $L(\phi) = 0$ is equivalent to that $\phi$ satisfies (2.1).

□

Proof of Theorem 1.

As in [1], [3], and [6], we decompose a solution $\phi$ to (2.1) by using spherical harmonics. Let us denote $x = r\omega$, $r = |x|$, $\omega = \frac{x}{|x|} \in S^{N-1}$ for a point $x \in \mathbb{R}^N$. We write

$$
\phi(x) = \phi(r, \omega) = \sum_{k=0}^{\infty} \psi_k(r) Y_k(\omega), \quad \psi_k(r) = \int_{S^{N-1}} \phi(r, \omega) Y_k(\omega) dS_{\omega}, \quad (2.2)
$$

where $Y_k(\omega)$ denote the $k$-th spherical harmonics, that is, the $k$-th eigenfunctions for the Laplace-Beltrami operator $\Delta_{S^{N-1}}$ on $S^{N-1}$ associated with the $k$-th eigenvalue $\lambda_k$:

$$-\Delta_{S^{N-1}} Y_k = \lambda_k Y_k \quad \text{on } S^{N-1}
$$

where

$$\lambda_k = k(k + N - 2), \quad k = 0, 1, 2, \ldots,
$$

denotes the $k$-th eigenvalue. It is known that the multiplicity of $\lambda_k$ is $\frac{(2k+N-2)(N+k-3)!}{k!(N-2)!}$, especially, $\lambda_0 = 0$ has the multiplicity 1 and $\lambda_1 = N - 1$ has the multiplicity $N$.

We derive the equation satisfied by $\psi_k$ for $k = 0, 1, 2, \ldots$. Let $\nabla_\omega$ denote the spherical gradient operator on $S^{N-1}$. Since the decomposition of the gradient operator

$$\nabla = \omega \frac{\partial}{\partial r} + \frac{1}{r} \nabla_\omega, \quad \omega \cdot \nabla_\omega \equiv 0
$$

holds, for a function $\phi$ of the form $\phi(x) = \psi(r) Y(\omega)$, we have

$$x \cdot \nabla \phi = x \cdot \nabla (\psi(r) Y(\omega)) = r \psi'(r) Y(\omega),
$$

$$\sum_{i,j=1}^{N} \frac{\partial^2 \phi}{\partial x_i \partial x_j} x_i x_j = \sum_{i,j=1}^{N} \frac{\partial^2 (\psi(r) Y(\omega))}{\partial x_i \partial x_j} x_i x_j = r^2 \psi''(r) Y(\omega).
$$

Also recall the formula

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N - 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^{N-1}}.$$

Thus we have, for $\phi$ of the form $\phi(x) = \psi(r)Y(\omega)$, the equation \((2.1)\) becomes
\[
\begin{aligned}
&\quad \quad \quad r^2 \left( \psi''(r) + \frac{N - 1}{r} \psi'(r) \right) Y(\omega) + \psi(r) \Delta S_{N - 1} Y(\omega) \\
&\quad + N(N - 2) \frac{r \psi'(r) Y(\omega)}{1 + r^{\frac{N}{N - 1}}} + (N - 2) r^2 \psi''(r) Y(\omega) + \left( \frac{N^3}{N - 1} \right) \frac{r^{\frac{N}{N - 1}}}{\left( 1 + r^{\frac{N}{N - 1}} \right)^2} \psi(r) Y(\omega) = 0.
\end{aligned}
\]
Thus inserting \((2.2)\) into \((2.1)\), we see that each $\psi_k$ must be a solution to
\[
L_k(\psi) := \psi''(r) + \left( 1 + \frac{N(N - 2)}{N - 1} \frac{1}{1 + r^{\frac{N}{N - 1}}} \right) \frac{\psi'(r)}{r} - \frac{\lambda_k}{N - 1} r^{\frac{N}{N - 1}} \psi(r) + \frac{N^3}{(N - 1)^2} \frac{r^{\frac{N}{N - 1}}}{\left( 1 + r^{\frac{N}{N - 1}} \right)^2} \psi(r) = 0.
\]
Also note that, by using the expression $U(r) = \log \frac{C_{\phi}}{x^{\frac{N}{N - 1}}}$, we see that the equation
\[
L(\phi) = \text{div}(|\nabla U|^{N - 2} \nabla \phi) + (N - 2) \text{div}(|\nabla U|^{N - 4} (\nabla U \cdot \nabla \phi) \nabla U) + e^U \phi = 0
\]
for $\phi(x) = \psi(r)Y(\omega)$ is equivalent to that $\psi$ satisfies
\[
\left\{ r^{N - 1} \psi'(r)|U'(r)|^{N - 2} \right\} - \lambda_k r^{N - 3} \left( \frac{1}{N - 1} \right) |U'(r)|^{N - 2} \psi(r) + \frac{e^{U(r)}}{N - 1} r^{N - 1} \psi(r) = 0.
\]
In the following, we treat the equation $L_k(\psi) = 0$ in \((2.3)\) for $k = 0, k = 1$, and $k \geq 2$ separately.

**The case $k = 0$.**

By the invariance under the scaling, we know that $Z_0(x)$ defined in \((1.5)\) satisfies \((2.1)\). Since
\[
Z_0(x) = x \cdot \nabla U(x) + N = \left( \frac{N}{N - 1} \right) \frac{(N - 1) - |x|^{\frac{N}{N - 1}}}{1 + |x|^{\frac{N}{N - 1}}},
\]
we see that
\[
\psi_0(r) = \frac{(N - 1) - r^{\frac{N}{N - 1}}}{1 + r^{\frac{N}{N - 1}}}
\]
is a solution of $L_0(\psi) = 0$, which is bounded on $[0, +\infty)$. We claim that any other bounded solution of $L_0(\psi) = 0$ must be a constant multiple of $\psi_0$. Indeed, assume the contrary that there existed the second
linearly independent, bounded solution \( \psi \) satisfying \( L_0(\psi) = 0 \). We may always assume that \( \psi \) is of the form

\[
\psi(r) = c(r)\psi_0(r)
\]

for some \( c = c(r) \). Inserting this into (2.3) and noting \( \lambda_0 = 0 \), we obtain

\[
c''(r)\psi_0(r) + c'(r)\left[2\psi_0'(r) + \frac{\psi_0(r)}{r}\left(1 + \frac{N(N-2)}{N-1} \frac{1}{1 + \frac{N}{N-1} r}\right)\right]
+ c \left[\psi''_0(r) + \left(1 + \frac{N(N-2)}{N-1} \frac{1}{1 + \frac{N}{N-1} r}\right) \frac{\psi'_0(r)}{r} + \frac{N^3}{(N-1)^2} \frac{r^{\frac{N}{N-1}}}{1 + \frac{N}{N-1} r^2} \psi_0(r)\right] = 0,
\]

which leads to

\[
\frac{c''(r)}{c'(r)} = -2\frac{\psi_0'(r)}{\psi_0(r)} - \frac{1}{r}\left(1 + \frac{N(N-2)}{N-1} \frac{1}{1 + \frac{N}{N-1} r}\right).
\]

This can be written as

\[
(\log |c'(r)|)' = -2(\log |\psi_0(r)|)' - \left(1 + \frac{N(N-2)}{N-1}\right)(\log r)' + (N-2)\left[\log \left(1 + \frac{N}{N-1} r\right)\right] ',
\]

so we have that

\[
c'(r) = A\frac{\psi_0^2(r)r^{1+\frac{2(N-2)}{N-1}}}{\psi_0^2(r)}^N N^{-2}
\]

for some \( A \neq 0 \). Since \( \psi_0(r) \sim -1 \) near \( r = \infty \), we have

\[
c'(r) \sim A\frac{r^{\frac{N}{N-1}}}{r^{1+\frac{2(N-2)}{N-1}}} = A \frac{r}{r} \text{ as } r \to \infty
\]

which implies \( c(r) \sim A \log r + B \) as \( r \to \infty \) for some \( A \neq 0 \) and \( B \in \mathbb{R} \). However, in this case, \( |\psi(r)| \sim |(A \log r + B)\psi_0(r)| \to +\infty \) as \( r \to +\infty \), which contradicts to the assumption that \( \psi \) is bounded. Therefore, we obtain the claim.

The case \( k = 1 \).

By the invariance under the translation, we know that \( Z_i(x) \) \((i = 1, \ldots, N)\) defined in (1.6) satisfies (2.1). Since

\[
Z_i(x) = \frac{\partial U}{\partial x_i} = -\left(\frac{N^2}{N-1}\right) \frac{r^{\frac{1}{N-1}}}{1 + \frac{N}{N-1} r^\frac{N}{N-1}} x_i \Bigg|_{x = \psi_1(r)}, \quad (i = 1, \ldots, N),
\]

we see that

\[
\psi_1(r) = \frac{r^{\frac{1}{N-1}}}{1 + \frac{N}{N-1} r^\frac{N}{N-1}}
\]
is a solution of $L_1(\psi) = 0$, which is bounded (decaying) on $[0, +\infty)$. \(\psi_1(r) \sim \frac{1}{r}\) as \(r \to +\infty\).

As before, we claim that any other bounded solution of \(L_1(\psi) = 0\) must be a constant multiple of \(\psi_1\). Indeed, assume the contrary that there existed the second linearly independent, bounded solution \(\psi\) satisfying \(L_1(\psi) = 0\). We may always assume that \(\psi\) is of the form

\[\psi(r) = c(r)\psi_1(r)\]

for some \(c = c(r)\). Inserting this into (2.3) and noting \(\lambda_1 = N - 1\), we obtain

\[
c''(r)\psi_1(r) + c'(r) \left[2\psi_1'(r) + \frac{\psi_1(r)}{r} \left(1 + \frac{N(N-2)}{N-1} \frac{1}{1 + \frac{N}{N-1}}\right)\right] + c \left[\frac{\psi_1''(r)}{r} + \left(1 + \frac{N(N-2)}{N-1} \frac{1}{1 + \frac{N}{N-1}}\right) \frac{\psi_1'(r)}{r} - \frac{\psi_1(r)}{r^2} + \frac{N^3}{(N-1)^2} \frac{r^{\frac{N}{N-1}}}{1 + \frac{N}{N-1}} \frac{\psi_1(r)}{r^2}\right] = 0,
\]

which leads to

\[
\frac{c''(r)}{c'(r)} = \frac{-2\psi_1'(r)}{\psi_1(r)} - \frac{1}{r} \left(1 + \frac{N(N-2)}{N-1} \frac{1}{1 + \frac{N}{N-1}}\right).
\]

Again we have that

\[
c'(r) = A \left(1 + \frac{N}{N-1}\right)^{\frac{N-2}{N-1}} \psi_1'(r) r^{\frac{N(N-2)}{N-1} - \frac{2}{N-1}}\]

for some \(A \neq 0\). Since \(\psi_1(r) \sim \frac{1}{r}\) as \(r \to +\infty\), we obtain

\[
c'(r) \sim A \frac{N}{r^{1 + \frac{N(N-2)}{N-1}}} = Ar \quad \text{as } r \to +\infty
\]

which implies \(c(r) \sim \frac{Ar}{r^{2 + \frac{2}{N-1}}} + B\) as \(r \to +\infty\) for some \(A \neq 0\) and \(B \in \mathbb{R}\). However, in this case, \(\psi(r) \sim (\frac{N}{2} r^2 + B)\psi_1(r) \sim \frac{Ar}{r^{\frac{3}{N-1}}} \psi_1(r)\) as \(r \to +\infty\), which contradicts to the assumption that \(\psi\) is bounded. Therefore, we obtain the claim.

**The case \(k \geq 2\).**

In this case, we claim that all the bounded solutions of \(L_k(\psi) = 0\) are identically zero. Assume the contrary that there existed \(\psi \neq 0\) satisfying \(L_k(\psi) = 0\). We may assume that there exists \(R_k > 0\) such that \(\psi(r) > 0\) on \((0, R_k)\) and \(\psi'(R_k) \leq 0\). Now, \(\psi\) satisfies (2.4):

\[
\left[r^{N-1}\psi'(r)|U'(r)|^{N-2}\right] - \lambda_k r^{N-3} \left(\frac{1}{N-1}\right)|U'(r)|^{N-2}\psi(r) + \frac{e^{U(r)}}{N-1} r^{N-1}\psi(r) = 0.
\]

(2.5)
Also $\psi_1$ is a solution of (2.4) for $k = 1$:
\[
\left\{ r^{N-1}\psi_1'(r)|U'(r)|^{N-2}\right\}' - \lambda_1 r^{N-3} \left( \frac{1}{N-1} \right) |U'(r)|^{N-2} \psi_1(r) + \frac{e^{U(r)}}{N-1} r^{N-1} \psi_1(r) = 0.
\]

(2.6)

Multiply (2.5) by $\psi_1$ and multiply (2.6) by $\psi_k$ and subtracting, we have
\[
\left\{ r^{N-1}\psi_k'(r)|U'(r)|^{N-2}\right\}' \psi_1 - \left\{ r^{N-1}\psi_1'(r)|U'(r)|^{N-2}\right\}' \psi_k = \frac{\lambda_k - \lambda_1}{N-1} r^{N-3} |U'(r)|^{N-2} \psi_k \psi_1.
\]

Integrating both sides of the above from $r = 0$ to $r = R_k$ and using $\psi_k(R_k) = 0$, we obtain
\[
R_k^{N-1} |U'(r)|^{N-2} \psi_k'(R_k) \psi_1(R_k) = \frac{\lambda_k - \lambda_1}{N-1} \int_0^{R_k} r^{N-3} |U'(r)|^{N-2} \psi_k(r) \psi_1(r) dr.
\]

(2.7)

Since $\lambda_k > \lambda_1$ for $k \geq 2$, $\psi_k(r) > 0$ on $(0, R_k)$, and $\psi_1(r) > 0$, the right-hand side of (2.7) is positive. On the other hand, the left-hand side of (2.7) is non positive since $\psi_k'(R_k) \leq 0$. This contradiction implies the claim.

Combining all these facts, we have finished the proof of Theorem 1. □

Acknowledgments.

Part of this work was supported by JSPS Grant-in-Aid for Scientific Research (B), No.19H01800. This work was partly supported by Osaka Central Advanced Mathematical Institute; MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849.

References

[1] S. Baraket, and F. Pacard: Construction of singular limits for a semilinear elliptic equation in dimension 2, Calc. Var. Partial Differential Equations 6 (1998), no. 1, 1–38

[2] W. X. Chen, and C. Li: Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (1991), no. 3, 615–622.

[3] K. El Mehdi, and M. Grossi: Asymptotic estimates and qualitative properties of an elliptic problem in dimension two, Adv. Nonlinear Stud. 4 (2004), no. 1, 15–36.

[4] P. Esposito: A classification result for the quasi-linear Liouville equation, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 35 (2018), no. 3, 781–801.

[5] O. Rey: The role of the Green’s function in a nonlinear elliptic equation involving the critical Sobolev exponent, J. Funct. Anal. 89 (1990), no. 1, 1–52.

[6] A. Pistoia, and G. Vaira: Nondegeneracy of the bubble for the critical p-Laplace equation, Proc. Royal Soc. Edinburgh 151 (2021), 151–168.

Department of Mathematics, Osaka Metropolitan University & OCAMI, Sumiyoshi-ku, Osaka, 558-8585, Japan

Email address: futoshi@omu.ac.jp