Gravitational collapse with non-vanishing tangential stresses: II. A laboratory for cosmic censorship experiments

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Abstract. The general exact solution describing the dynamics of anisotropic elastic spheres supported only by tangential stresses is reduced to a quadrature using Ori’s mass–area coordinates. This leads to the explicit construction of the root equation governing the nature of the central singularity. Using this equation, we formulate and motivate on physical grounds a conjecture on the nature of this singularity. The conjecture covers a large sector of the space of initial data; roughly speaking, it asserts that addition of a tangential stress cannot undress a covered dust singularity. The root equation also allows us to analyse the case of self-similar spacetimes and gain some insight into the role of stresses in deciding the nature of the singularities in this case.

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1. Introduction

The present status of research in black-hole formation and cosmic censorship is quite intriguing. Indeed, both analytical results in dust collapse (see, e.g., Jhingan and Joshi 1998 and references therein) and numerical results in scalar field collapse (see, e.g., Gundlach 1997 and references therein) indicate the existence of a critical behaviour governing the formation of black holes or naked singularities. In principle, this behaviour should be the ‘remnant’ of some hypothesis of a—still unknown—cosmic censorship theorem and, as such, should be related to the properties of the collapsing matter such as fulfillment of energy conditions and local stability. If we want to understand the physics of such phenomena in the case of ‘ordinary’ matter (i.e. not boson stars), we are enforced to approach analytically the gravitational collapse of non-pressureless matter, since the dust equation of state is ‘trivial’ from the viewpoint of matter properties: all energy conditions, causality conditions and stability conditions reduce to positivity of energy. However, this problem is extremely difficult if approached in full generality (see, e.g., Joshi 1996 and references therein).

Recently (Magli 1997, henceforth referred to as [I]), we discussed a class of solutions of the Einstein field equations describing spherically-symmetric, non-static elastic spheres supported only by tangential stresses (elastic matter is of interest in strongly collapsed situations; for instance neutron stars typically have solid regions, see e.g. Haensel (1995)). Previous investigations on systems having vanishing radial stresses trace back to Einstein

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(1939) and Florides (1974) in the static case, while non-static models have been considered by Datta (1970), Bondi (1971), and Herrera and Santos (1995) (for a general review on anisotropic systems in general relativity see Herrera and Santos (1997)).

From the viewpoint of cosmic censorship, understanding the nature of singularities for such solutions can be considered as a first step toward a full understanding of spherical gravitational collapse with general stresses. Indeed the investigation of some particular models carried out recently by Singh and Witten (1997) already shows behaviour which can be drastically different from that for dust. The purpose of this paper is to contribute to a programme which should hopefully lead to a full understanding of the final fate of gravitational collapse with tangential stresses.

This paper is organized as follows. The first problem we have to face is the fact that, due to the presence of stresses, the comoving time differs from the proper time of the shells of particles. As a consequence, the first-order ‘energy equation’, which arises from mass conservation, is coupled, for generic equations of state, to the equation giving the acceleration of the worldlines of the particles. This coupling has the effect that the equation for null geodesics (whose behaviour near the central singularity governs the nature of the collapse) cannot be written in explicit form. Here, we overcome this difficulty (section 2) using mass–area coordinates. These coordinates were originally introduced by Ori (1990) to obtain the general exact solution for charged dust. Here we obtain the general solution for gravitational collapse with non-vanishing tangential stresses in a very simple form, in which only an integral remains to be performed.

Using the line element in the new coordinates, it is possible (section 3) to write the ‘root equation’, namely the algebraic equation governing the nature of the central singularity, in explicit form. This equation depends only on the choice of the equation of state and on the initial distributions of density and velocity.

It turns out to be quite a difficult task to study this equation in full generality; it is, however, worth mentioning that a full understanding of the nature of the singularities for dust spacetimes has been achieved only very recently (Singh and Joshi 1996). To get some insight into the physical content of this equation, we carry out a detailed examination of small deviations from the dust equation of state. This leads us to formulate a conjecture on the final fate of the collapse and to give a plausibility argument supporting it (section 4). Finally, in section 5, we analyse the case of self-similar spacetimes exploring some qualitative effects of the tangential stress on the nature of the central singularity.

The paper ends with some concluding remarks in section 6.

2. Mass–area coordinates and the general solution

Recently [I] we discussed a class of solutions of the Einstein field equations describing spherically symmetric, non-static elastic spheres supported only by tangential stresses. Using comoving coordinates, the line element reads

$$ds^2 = -e^{2
u} \, dt^2 + \frac{(Y')^2 h^2}{1 + f} \, dr^2 + Y^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),$$

where $\nu$ and $Y$ are functions of $r$ and $t$ satisfying

$$\nu' = -\frac{1}{h} \frac{\partial h}{\partial Y} Y',$$

$$Y^2 e^{-2\nu} = -1 + \frac{2F}{Y} + \frac{1 + f}{h^2}.$$  (3)
In the above formulae, $F(r)$ and $f(r)$ are the ‘conserved mass’ and the ‘binding function’ familiar from the Tolman–Bondi solutions, while $h = h(r, Y)$ is the internal elastic energy per unit volume (it plays the role of the equation of state of the material). For physical reasons, $F$, $1 + f$ and $h$ must be chosen as positive functions (positivity of mass, of $g_{rr}$ and of elastic energy, respectively). Moreover, the function $h$ is severely constrained by the local stability of matter, which requires this function to have a minimum at $Y = r$ (see [I] for details). If $h$ is equal to one (more precisely, is a constant which may be rescaled to unity) there is no dependence on the strain: the material is a dust cloud and the line element reduces to the Tolman–Bondi one. This fact will be very important in what follows; we shall call the dust limit of any equation depending on the choice of $h$ the same equation written with $h = 1$.

The energy–momentum tensor of the material is diagonal in the comoving frame and has only three non-vanishing components, namely the energy density $\epsilon = -T^0_0$ and the tangential stress $\Pi = T^\theta_\theta = T^\phi_\phi$. These quantities are given by

$$\epsilon = \frac{F'}{4\pi Y^2 Y'},$$

$$\Pi = \mathcal{H}\epsilon,$$

where the ‘generalized adiabatic index’ $\mathcal{H}$ is defined as follows:

$$\mathcal{H} := -\frac{Y}{2h} \frac{\partial h}{\partial Y}.$$  

In [I] the physical properties of these solutions are thoroughly discussed. We recall here that the metric is well behaved at the centre if the equation of state satisfies the ‘minimal stability requirement’ ($h$ has a minimum at $Y = r$) and the conditions $Y(0, t) = 0$, $f(0) = h^2(0, 0) - 1$ hold. The behaviour of the energy density at $r = 0$ is the same as that familiar from the Tolman–Bondi models, namely $\epsilon$ is initially regular if $F(r)$ is of the form $r^3 \tilde{F}(r)$ with $\tilde{F}(0) < +\infty$. Matching our solutions with the Schwarzschild vacuum is possible on any chosen boundary surface $r = r_b$, provided that the value of $F$ at $r_b$ is identified with the Schwarzschild mass $M$. However, the transformation between comoving and Schwarzschild coordinates is highly non-trivial. The energy conditions lead to inequalities on the function $\mathcal{H}$, and therefore to differential inequalities on the state function $h$. In particular WEC holds if $\mathcal{H} \geq -1$. Once this is satisfied, DEC requires $\mathcal{H} \leq 1$, while SEC is satisfied if $\mathcal{H} \geq \frac{1}{2}$. Analysing the behaviour of the function appearing on the right-hand side of equation (3) for a fixed shell of particles ($r =$ constant), one can give a qualitative analysis of the possible motions. In particular, it is shown in [I] that there exist physically valid models of oscillating elastic spheres as well as of finite-bouncing spheres.

The metrics (1) are not completely explicit due to the coupling between (2) and (3); physically, this is simply a reflection of the fact that the comoving time differs from the proper time since the particles are not in geodesic motion. The two equations decouple only if $h$ depends uniquely on $Y$ ($h = w(Y)$, say). In this case we have

$$e^{2\nu} = \frac{1}{w^2},$$

$$\dot{Y}^2 = \frac{1}{w^2} \left[ \frac{1 + f}{w^2} - \left( 1 - \frac{2F}{Y} \right) \right],$$

and the function $\mathcal{H}$ defined in (6) reads

$$\mathcal{H} = -\frac{Y}{2} \frac{d}{dY} \log w(Y).$$
This particular class of solutions contains that discussed by Singh and Witten (1997), in which the tangential stress is proportional to the density. In this case the stress–strain relation has the ‘barotropic form’ \( \Pi = k \epsilon \) with constant ‘adiabatic index’ \( \mathcal{H} = k \), and the equation of state is \( w(Y) = Y^{-2k} \).

A consequence of the above-described coupling problem is that, in general, it is not possible to write explicitly the null geodesic equation in comoving coordinates. As will be recalled in the next section, this equation governs the nature (naked or black hole) of the central singularity, and it is therefore very difficult to investigate censorship using the comoving frame.

A system which is mathematically very similar to ours is charged, spherically-symmetric dust (Vickers 1973). Indeed for such a system the mass is conserved and it is possible to analyse the dynamics in comoving coordinates; however the coupling between ‘times’ does not allow explicit integration. To get rid of this problem Ori (1990) introduced a system of coordinates which removes the coupling and obtained the general exact solution for charged dust. Ori’s system is obtained by replacing the comoving time \( t \) and the radial label \( r \) with the ‘area coordinate’ \( R = Y(r, t) \) and the ‘mass coordinate’ \( m = F(r) \); the mass coordinate, being conserved, is comoving (if \( u^\mu \) denotes the velocity of matter, one has \( u^\mu = u_\delta^\mu \) where \( u := u^R \)). The line element in these coordinates has the form

\[
\text{ds}^2 = -A \, \text{d}m^2 - 2B \, \text{d}R \, \text{d}m - C \, \text{d}R^2 + R^2 (\text{d}\theta^2 + \sin^2 \theta \, \text{d}\phi^2) ,
\]

where \( A, B \) and \( C \) are functions of \( m \) and \( R \).

We are now going to show that Ori’s technique can also be applied in the case of solutions with vanishing radial stresses. This is essentially due to the fact that the mass is also conserved in this case, and therefore gives an unambiguous ‘comoving label’ for the shell of particles (for simplicity, the general solution is presented here in the non-charged case; however our results can be easily extended to the case of charged materials, as briefly reported in the appendix).

Since \( m \) is comoving, we have

\[
C = \frac{1}{u^2}.
\]

The energy density can now be written as

\[
\epsilon = \frac{h}{4\pi u R^2 E \sqrt{H}} ,
\]

where \( h \) is to be considered as a function of \( m \) and \( R \), the quantity \( H \) is defined by

\[
H := B^2 - AC ,
\]

and \( E \) is an arbitrary function of \( m \) corresponding to \( \sqrt{1 + f} \) (we have introduced this notation in order to facilitate the comparison with Ori (1990)).

There are four (compatible) Einstein equations for the three unknowns \( A, B, C \). We start by considering the equations \( G_m = 8\pi T_m^m, G_R^m = 8\pi T_m^R \) and \( G_R^m = 8\pi T_R^m \):

\[
\frac{1}{R^2} \left[ 1 - \frac{A}{H} - R \left( \frac{A}{H} \right)_R \right] = 0 ,
\]

\[
\frac{1}{R} \left( \frac{A}{H} \right)_m = 8\pi B u^2 \epsilon ,
\]

\[
B \frac{H}{H} = C_m = 0 .
\]
Equation (8) can be integrated:

\[ A = H \left(1 - \frac{2m}{R}\right). \] (11)

Using \( B^2 = H + A/u^2 \) and (7), from (9) we get

\[ u = \pm \sqrt{-1 + \frac{2m}{R} + \frac{E^2}{h^2}}, \] (12)

and the metric function \( B \) can be written as

\[ B = -\frac{E \sqrt{H}}{hu}. \]

Therefore, we have solved for \( u \) (and thus for \( C \)) in terms of the arbitrary functions and expressed \( A \) and \( B \) in terms of a single unknown \( H \). To complete the solution, we plug the above results in (10), obtaining

\[ \left(\sqrt{H}\right)_{R} = \frac{h}{E} \left(\frac{1}{u}\right)_{,m}. \]

We have, therefore, reduced the problem to the calculation of an indefinite integral:

\[ \sqrt{H(m, R)} = g(m) \pm \int G(m, R) \, dR, \quad (13) \]

where the \( \pm \) sign is the same as that of \( u \), \( g(m) \) is an arbitrary function, and

\[ G(m, R) := \frac{h}{RE} \left[ 1 + \frac{R}{2} \left(\frac{E^2}{h^2}\right)_{,m} \right] \left(-1 + \frac{2m}{R} + \frac{E^2}{h^2}\right)^{-3/2}. \] (14)

If \( h = 1 \) the above formulae give the Tolman–Bondi line element in mass–area coordinates (Ori 1990).

It is easy to check that the remaining field equation \( G_{\theta}^{\theta} = 8\pi T_{\theta}^{\theta} \) is identically satisfied once (8)–(10) are.

3. A laboratory for cosmic censorship

In this section we use the general exact solution derived in the previous section to build up a ‘laboratory’ for studying cosmic censorship. The key instrument which is needed in this laboratory is already known from the work by Dwivedi and Joshi (1994) and may be called the ‘root equation’; as recalled below, it is an algebraic equation arising from the behaviour of outgoing null geodesics near the singularity. In the present section we construct this equation explicitly for the case at hand. We shall also derive the conditions for shell-crossing singularities in terms of an integral equation.

3.1. Physical content of the arbitrary functions: initial data

In order to approach the problem of singularities, we first need to identify the physical content of the arbitrary functions, so that regular initial data can be chosen.

There are three arbitrary functions, namely the equation of state \( h = h(m, R) \) and the functions \( E(m), g(m) \). To understand the physical meaning of \( E \) and \( g \) observe that, physically, such functions must be related to the ‘initial distributions’ of density and velocity (here quotation marks are due to the fact that we shall take care of the initial data always referring to the ‘original’ (comoving) coordinates). Consider, therefore, regular initial data
at some comoving time \( t \) (\( t = 0 \), say). We use the scaling freedom in the choice of the \( r \) coordinate to identify the Lagrangian and the Eulerian label initially, so that \( Y(r, 0) = r \).

In mass–area coordinates, to the equation \( Y(r, 0) = r \) corresponds some curve \( R = R_0(m) \), where \( R_0 = F^{-1} \) and we assume the mass to be a monotonically increasing function. Introducing the initial distribution of velocity \( (V(m), \text{say}) \) from (12) follows

\[
V^2(m) = \frac{E^2(m)}{h^2(m, R_0)} - 1 + \frac{2m}{R_0}.
\]

The above formula gives the relationship between \( E \) and the initial velocity profile.

The relationship between \( g \) and the initial data is, in general, quite complicated. To obtain it, observe that the following formula may be easily proved:

\[
\frac{\partial Y}{\partial m} = \frac{E}{h} u \sqrt{H}.
\] (15)

The above equation evaluated ‘at \( t = 0 \)’ yields

\[
g(m) = \pm \left[ \frac{R_0^0(m) h(m, R_0)}{V(m) E(m)} - \int_{R_0^0}^{R(m)} G(m, R) \, dR \right],
\] (16)

(the \pm sign is the same as that of \( u \)).

### 3.2. Shell-focusing singularities

The energy density (4) becomes singular whenever \( Y(r, t) \) or \( Y'(r, t) \) vanish during the dynamics. Physically, such singularities correspond to those occurring in dust models: \( Y = 0 \) corresponds to ‘crushing to zero size’ (shell-focusing singularities) while \( Y' = 0 \) corresponds to the shell-crossing phenomenon: the worldlines of the (shells of) particles intersect each other and the ‘Lagrangian labelling’ description breaks down.

In contrast to what happens in the dust case, where shell-focusing collapse is unavoidable, within our solutions there exist globally regular models of oscillating or bouncing back materials. However, equation (3) can be used (see [I]) to show that for any physically valid choice of the equation of state it is possible to choose initial data leading to continued gravitational collapse and therefore to shell-focusing singularities (it is worth mentioning that the remark made by Singh and Witten (1997) that regularity conditions explicitly disallow the formation of a singularity at \( r = 0 \) is incorrect: these conditions imply only regularity on the initial data surface and non-preferredness of the centre).

To analyse the nature of shell-focusing singularities, we first observe that the relation \( Y(r, t) = 0 \) defines a ‘singularity curve’ \( t^*(r) \); in general, different shells become singular at different times, and it is customary to call the central singularity that occurring at \( r = 0 \).

This singularity plays a distinguished role because it is possible to show that non-central singularities are always covered. To see this, notice that the shell labelled \( r \) becomes trapped at a time \( t^*(r) \) such that \( Y(r, t^*) = 2F(r) \). For each fixed shell, consider the function \( \tilde{Y}(r) = Y(r, t) \). We have \( \tilde{Y}(t^*) = 2F(r) > 0 \), but \( d\tilde{Y}(t)/dt \) is negative in a collapsing situation, so that \( \tilde{Y}(t) \) is decreasing and we must have \( t^*(r) > t^*(r) \). It follows that the shell becomes trapped before becoming singular, so that the singularity is covered (under certain conditions, it is also possible to prove this result in the presence of non-vanishing radial stresses, see Cooperstock et al (1997) for details).

The above argument does not work for the central singularity, at which \( Y(0, t^*) = 2F(0) = 0 \). To study this singularity, we translate in the mass–area formalism and make use of the method developed by Dwivedi and Joshi (1994) and successfully applied to the dust case in a series of recent papers (see, e.g., Singh and Joshi 1996, Jhingan et al 1996).
Consider the equation for radial, outgoing null geodesics in mass–area coordinates:

\[
\frac{dR}{dm} = -\frac{B + \sqrt{B^2 - AC}}{C} = -\sqrt{Hu} \left( |u| - \frac{E}{h} \right).
\]  \hspace{1cm} (17)

This is an ordinary differential equation with a singular point at the central singularity \( R = 0, m = 0 \). This singularity is (at least locally) naked if there are geodesics starting at it with a definite value of the tangent. If no such geodesics exist, the singularity is not naked and (strong) cosmic censorship holds. To investigate the behaviour near the singular point, define

\[
x := \frac{R}{2m^\alpha},
\]

where \( \alpha > \frac{1}{3} \). If the singularity is naked, there exist some \( \alpha \) such that at least one finite positive value \( x_0 \) exists which solves the algebraic equation

\[
x_0 := \lim_{R,m \to 0} x = \lim_{R,m \to 0} \frac{R}{2m^\alpha}.
\]

Applying L'Hôpital's rule we have

\[
x_0 = \lim_{R,m \to 0} \frac{m^{1-\alpha}}{2\alpha} \frac{dR}{dm} = \lim_{R,m \to 0} -\frac{m^{1-\alpha}}{2\alpha} \sqrt{Hu} \left( |u| - \frac{E}{h} \right).
\]

Using (12) and (13), the above equation can be written in explicit form as

\[
x_0 = \frac{1}{2\alpha} \lim_{R,m \to 0} \frac{m^{3(1-\alpha)/2}}{2^\alpha} \int_{2m^\alpha x}^{2m^\alpha \alpha} G(m, R) \, dR \left( \sqrt{\left( -1 + \frac{E^2}{h^2} \right) m^{\alpha - 1} + \frac{1}{x}} \right) \times \left( \sqrt{\frac{m^{1-\alpha}}{x} - 1 + \frac{E^2}{h^2} - \frac{E}{h}} \right). \hspace{1cm} (18)
\]

This equation depends only on the initial data \( g \) and \( E \), as in the dust case, and on the choice of the material we are dealing with, i.e. the equation of state \( h \). This means that the dynamics has been completely 'gauged away'; such a simplification cannot be achieved in comoving coordinates since in such coordinates the general exact solution is not available in explicit form. In the next two sections, we shall illustrate a simple way to extract physically interesting information from this equation without solving it explicitly.

3.3. Remarks on shell-crossing singularities

As recalled above, shell-crossing singularities correspond to zeros of \( Y' \), so that in the mass–area description a shell crossing occurs when \( Y_m \) vanishes. Generally speaking, we do not expect a zero of \( Y_m \) to occur at a turning point \( (u = 0) \) so that equation (15) implies that shell-crossing singularities correspond to zeros of \( H \). Let \( R^\text{sc}(m) \) be the curve on which such singularities eventually occur. Using (13) and (16), we obtain that \( R^\text{sc}(m) \) must satisfy

\[
\frac{R^\text{sc}_m(m) h(m, R^0)}{V(m) E(m)} = -\int_{R^\text{sc}(m)}^{R^0(m)} G(m, R) \, dR.
\]  \hspace{1cm} (19)

This equation \textit{may} have physically meaningful solutions. For instance, consider the Tolman–Bondi case. The solutions of (19) are physically meaningful only if the vanishing of \( Y_m \) happens \textit{before} (in comoving time terms) the singularity at \( Y = 0 \) is reached. It is possible to characterize fully in terms of differential inequalities the set of initial data such that no shell crossings occur in physically allowed 'times' (Hellaby and Lake 1985, Newman
In contrast, Ori (1990) used the charged dust counterpart of this equation (see the appendix) to show that the characteristic ‘bounce in a new universe’ process (De La Cruz and Israel 1967), which is typical in such solutions, always occurs after a shell crossing, thereby casting serious doubts on its physical realizability (see also Ori 1991).

In the general case of non-vanishing tangential stresses, the analysis is also possible in full generality and will be presented elsewhere.

4. The nature of the central singularity: a conjecture

As we have seen, the nature of the central singularity depends on the existence of solutions of (18). A complete study of this equation requires a detailed investigation of the behaviour of the equation of state in the limit of approach to the singularity in physically valid situations, and goes far beyond the scope of the present paper. However, some insights into this problem can be obtained by a careful analysis and comparison with the (already well known) results holding for dust. For our considerations it will be sufficient to consider the case of marginally bound collapse; we are, therefore, going to give a simple derivation in mass–area coordinates of the results on the nature of the central singularity in this case (we refer the reader to the original paper by Singh and Joshi (1996) for complete details).

The marginally bound dust case corresponds to $E = h = 1$. From (16) we get

$$g(m) = \pm \sqrt{\frac{R^0}{2m}} \frac{d}{dm} (m^{-1/3} R^0). \quad (20)$$

The behaviour of this function as $m$ tends to zero can be obtained as follows. Consider regular initial data in comoving coordinates. Then the function $F$ will be of the form

$$F(r) = F_0 r^3 + F_q r^q + \cdots$$

where $q$ is the order of the first non-vanishing derivative of the initial density profile at the centre, and dots stand for higher-order terms. Therefore, we have

$$R^0(m) = F^{-1}(m) = \left( \frac{m}{F_0} \right)^{1/3} - \frac{F_q}{3 F_0} \left( \frac{m}{F_0} \right)^{(1+q)/3} + \cdots.$$

Considering now (20) and recalling that we are considering collapse (so that the negative sign must be chosen), we obtain

$$g \approx P_q m^{(q/3) - 1},$$

where

$$P_q := \frac{q F_q}{9 \sqrt{2} F_0^{(q/3) + 3/2}}.$$

Thus $g$ exhibits a ‘critical’ behaviour: it diverges (respectively, goes to a finite non-zero limit, vanishes) if $q < 3$ (respectively, $q = 3$, $q > 3$). Surprisingly enough, it is this behaviour that governs the nature of the singularity. In fact, (18) yields

$$x_0 = \frac{1}{2 \alpha} \lim_{r \to 0} \left[ P_q m^{(q/3) + (1-3\alpha)/2} - \frac{2}{3} x^{3/2} \right] \frac{1}{\sqrt{x}} \left( \frac{m^{1-\alpha/2}}{\sqrt{x}} - 1 \right).$$

The first term in square brackets goes to a finite, non-zero limit iff

$$\alpha = \frac{1}{3} \left( 1 + \frac{2}{3} q \right),$$
so we get

\[ x_0 = \frac{1}{\frac{2}{3} \left(1 + \frac{2q}{3}\right)} \left( P - \frac{2}{3} x_0^{3/2} \right) \lim_{R,m \to 0} \frac{m^{(1-q/3)/3}}{\sqrt{x}} - 1 \]  

(21)

If \( q \) is ‘super-critical’ (\( q > 3 \)) the limit diverges: there are no null geodesics escaping and therefore the singularity is not naked. If \( q \) is ‘sub-critical’ (\( q = 1, 2 \)) the limit goes to minus one and (21) gives a real positive solution for \( x_0 \) (provided that the initial density is decreasing outwards): the singularity is naked. At the critical value \( q = 3 \) (21) becomes a quartic equation. This equation has no real positive roots (and therefore the singularity is covered) if the quantity \( \zeta = F_3/(2\sqrt{2}F_0^{5/2}) \) is greater than a certain numerical value, otherwise the singularity is naked.

The qualitative features of the general (i.e. non-marginally bound) case are similar to those recalled above, namely nakedness depends on the ‘critical’ behaviour of some parameter \( \tilde{q} \) (which reduces to the parameter \( q \) in the marginally bound case); if the singularity is censored the limit diverges for any \( \tilde{q} \) greater than the critical value (Singh and Joshi 1996).

Consider now a generic solution with tangential stresses. To identify it uniquely, we need to chose the equation of state and the initial distribution of density and velocity. This means that the space of the free functions can be visualized as follows: to any fixed choice of the initial data \( g(m) \) and \( E(m) \) corresponds a family of solutions \( S_h \). Each member of this family corresponds to a different material (a different choice of \( h(m,R) \) within the physically allowed range) and each family contains one and only one Tolman–Bondi solution \( S_1 \) (‘dust limit’) corresponding to \( h = 1 \). Having chosen a family \( S_h \), we can immediately infer from the work of Singh and Joshi whether the dust limit \( S_1 \) corresponds to a naked singularity or to a black hole.

We conjecture that, if the central singularity of \( S_1 \) is not naked and it is not critical (i.e. the limit in (21) is divergent), the central singularity of \( S_h \) is also not naked for any physically valid choice of \( h \). Roughly speaking, this means that one cannot use a physically valid tangential stress to undress a covered dust singularity.

The above conjecture is based on arguments of physical plausibility as follows. Consider a small deviation from the dust equation of state. This can be represented as

\[ h = 1 + \mu(R,m) \],

(22)

where the function \( \mu \) is positive and vanishes at \( R = R^0(m) \) (a reasonable choice for \( \mu \) could be the ‘quasi-Hookean’ equation of state \( \mu = \mu_0(m)(R - R^0)^2 \) with positive \( \mu_0 \)). Then each term in (18) can be expanded to first order in \( \mu \); if this function is physically valid (i.e. is chosen as described above) every such term will be quadratic in \( R - R^0 \).

In particular, the last factor in parentheses will have this behaviour. Now, it is easy to check that quadratic terms in \( R - R^0 \) (or higher-order terms) cannot regularize a diverging behaviour of the zero-order term in this factor. Since divergence occurs in the non-critical covered case, the conjecture is proved at least for small deviations from the dust equation of state.

5. Self-similar spacetimes

The above-described proposal on the nature of the central singularity, although covering a large sector of the space of initial data, leaves completely open the problem of the interpretation of critical behaviour. For instance: what happens to marginally bound solutions having a naked dust limit with \( q = 3 \)? Do such solutions remain naked with
the addition of any (physically valid) tangential stress? Do they always become covered? It seems likely that none of the above would hold, but rather that the threshold of black-hole formation for fixed initial data should depend on the equation of state (i.e. on the choice of the function \( h \)), hopefully in a physically reasonable and understandable way. We do not have the answer to this question yet. However, we are now going to present some (again, qualitative) evidence that behaviour like this really should occur.

Difficulties in studying critical cases arise because we must investigate finite values of the limit (18) and, therefore, existence of positive solutions of the root equation. To get some insight into this we consider, among the solutions presented above, a particularly simple case which, however, is not devoid of physics, namely the case of self-similar spacetimes (see, e.g., Carr 1997).

In mass–area coordinates we can use as the self-similar variable the quantity \( x = R/2m \). It is easy to check that the spacetimes are self-similar if the following conditions hold (see Magli (1993) for a discussion of self-similarity in the case of anisotropic matter):

(i) \( h \) is a function of \( x \) only;
(ii) \( g(m) \) and \( E(m) \) are constant.

In what follows, it will again be sufficient to consider the marginally bound case \( E = 1 \).

The above conditions imply that \( R^0(m) \) is a linear function, and indeed it is well known that for self-similar spacetimes the mass function (the inverse of \( R^0 \)) is linear. We therefore set \( F(r) = \lambda r/2 (\lambda = \text{constant}) \) so that

\[
R^0(m) = \frac{2m}{\lambda},
\]

this implies that the initial value \( \bar{x} \) of \( x \) is \( 1/\lambda \). Using equation (16) with \( h(\bar{x}) = 1 \) the root equation (18) can be written as

\[
x = -\frac{1}{2} \left[ \frac{1}{6y} + \int_T^x G(z) \, dz \right] \sqrt{-1 + \frac{1}{h^2} + \frac{1}{x} \left( \sqrt{-1 + \frac{1}{h^2} + \frac{1}{x} - \frac{1}{h}} \right)}, \tag{23}
\]

where \( \gamma := \lambda^{3/2}/12 \) and

\[
G(x) := \frac{h}{x} \left( 1 + 2\frac{x^2}{h^3} \frac{dh}{dx} \right) \left( -1 + \frac{1}{x} + \frac{1}{h^2} \right)^{-3/2} . \tag{24}
\]

To extract from (23) some qualitative information, we again start from the dust limit. What happens in this limit is already well known (Joshi and Singh 1995), and we refer the reader to this paper for details.

Setting \( h = 1 \), equation (23) simplifies to

\[
x = -\frac{1}{2} \left( \frac{1}{6y} + \frac{2}{3} x^{3/2} \right) \sqrt{\frac{T}{x} \left( \sqrt{\frac{T}{x}} - 1 \right)} . \tag{25}
\]

To facilitate the comparison with the paper by Joshi and Singh, we change variables to

\[
y := \frac{1}{3} \left( 1 - 12\gamma x^{3/2} \right)
\]

then (25) becomes a quartic equation in the variable \( y \):

\[
y^3 \left( \frac{4}{3} - y \right) = \gamma(2 - y)^3 , \tag{26}
\]

where \( 0 < y < \frac{2}{3} \). Rewriting this equation in canonical form as \( ay^4 + 4by^3 + 6cy^2 + 4dy + e = 0 \), it can be shown that real, positive solutions exist only if the quantity

\[
\Delta := (ae - 4bd + 3c^2)^3 - 27(ae + 2bcd - ad^2 - eb^2 - c^3)^2 ,
\]
Gravitational collapse with non-vanishing tangential stresses: II

is negative (in this case there are two such solutions). This happens if $\gamma < \gamma_1 \simeq 6.41 \times 10^{-3}$ or $\gamma > \gamma_2 \simeq 17.32$. Thus the collapse leads to black-hole formation if $\gamma_1 < \gamma < \gamma_2$, and to naked singularities otherwise. The range $\gamma > \gamma_2$ is, however, unphysical since it would correspond to imaginary values of $x$. Therefore self-similar dust spacetimes exhibit a ‘phase transition’ between naked singularities and black holes. The transition depends on the value of $\gamma$ which is the remaining free parameter. This quantity is related to the central density of the material; using a fiducial model, Joshi and Singh have shown that $\gamma$ typically belongs to the range of black-hole formation for near-nuclear densities.

We now want to investigate, at least qualitatively, the changes introduced in the above picture by the presence of tangential stresses. Consider, once again, a small deviation from the dust equation of state of the form (22) (obviously, for self-similar spacetimes $\mu$ has to be considered as a function of $x$ only). In what follows, we shall systematically discard terms of order higher than one in $\mu$. Expanding the root equation (23), we obtain

$$y^3 \left(y - \frac{2}{3}\right) = \gamma(y - 2 + \tilde{K})^3.$$  

(27)

In the above formula, $\tilde{K}$ is the value at order one of the following function

$$K(y) := \frac{9}{2y} \left(\frac{2}{3} - y\right) \left[\frac{(2 - y)(\frac{2}{3} - y)}{4y^2}\mu + \gamma \int_{(1-3y/2)/\delta}^{1/\delta} \sqrt{z} \left(1 + 3z\mu + 2z^2 \frac{d\mu}{dz}\right) dz\right].$$  

(28)

Notice that this function is strictly positive. Linearity in $\mu$ also implies $\tilde{K} = K(y_0)$ where $y_0$ is the solution of the dust quartic (26) in the neighbourhood of which we want to study the deviation from the dust case.

Expanding also equation (27) to first order, we finally obtain a quartic with ‘displaced’ parameters $a, b, c, d, e$. A quite long, but straightforward, calculation gives that the range of black-hole formation $\gamma_1 < \gamma (\gamma < \gamma_2)$ is altered by the perturbation to $\tilde{\gamma}_1 < \gamma (\tilde{\gamma}_2)$, where

$$\tilde{\gamma}_1 = \gamma_1 - \tilde{K}\delta_1,$$

$$\tilde{\gamma}_2 = \gamma_2 + \tilde{K}\delta_2,$$

and $\delta_1 \simeq 17 \times 10^{-3}$, $\delta_2 \simeq 12.8$ (as usual in any perturbative approach, the above results also give the condition of applicability of the approximation: $\tilde{K}$ must not be greater than about $\frac{1}{3}$).

Formulae (29) show, at least at a qualitative level, that black-hole formation is facilitated by the presence of tangential stresses. Indeed, since $\tilde{K}$ is positive, the upper bound $\gamma_2$ becomes higher and certainly remains unphysical, while the lower bound $\gamma_1$ tends to decrease. It might happen that the addition of tangential stress dresses the singularity, which would be equivalent to $\tilde{\gamma}_1 < 0$, but of course to draw a conclusion of this kind it will be necessary to investigate the root equation without approximations.

6. Concluding remarks

The results of the present paper can be summarized as follows. First of all, we have shown that Ori’s mass–area formalism can be used to bring the general spherically symmetric solution of the Einstein field equations with non-vanishing tangential stresses into a very explicit form, in which only the calculation of an integral remains to be performed (this can be done independently whether electromagnetic coupling is present or not; the generalization to charged materials is indeed straightforward and is briefly reported in the appendix).
The introduction of mass–area coordinates proves to be a very powerful tool as far as
the analysis of the existence and nature of singularities in such solutions is concerned. In
fact it allows one to obtain the root equation governing the nature of the central singularity
in explicit form.

We presented first results coming from the investigation of the dust limit of this equation.
Such results give some insight into the nature of the final fate of collapse with tangential
stresses. In particular, we proposed a conjecture which, roughly speaking, asserts that
‘tangential stress cannot undress covered dust’. In the last section, we discussed self-
similar spacetimes and showed, at least qualitatively, that the effect of the stress can be an
enlargement of the black-hole initial data space.

Both the above results depend on the structure of the state equation, and therefore
show once again and in a clear way that a connection should exist between a (still lacking)
mathematically rigorous formulation of cosmic censorship and the conditions of physical
acceptability of the equations of state. Such conditions obviously include the energy
conditions but also the existence of an absolute minimum of the internal energy, which
is intimately related to stability issues; a relevant improvement in our understanding of this
topic could come from the knowledge of the explicit structure of the black-hole threshold
in terms of the derivatives of the state equation evaluated near the singularity. Work in this
direction is now in progress.

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Appendix. The charged case

Consider a material carrying a non-vanishing charge density \( \sigma \). We keep the description
of the mechanical and gravitational degrees of freedom as in the body of the paper, and simply
introduce the Maxwell tensor which, due to spherical symmetry, has only one independent
component \( m^R := \mathcal{E}(m, R) \). Maxwell’s equations yield

\[
\mathcal{E} = \frac{Q}{R^2 \sqrt{H}}.
\]

\[
\sigma = \frac{Q_m}{4 \pi R^2 \sqrt{H} u}.
\]

where \( Q = Q(m) \) is arbitrary. The field equation (8) now has a source term and reads

\[
\frac{1}{R^2} \left[ 1 - \frac{A}{H} - R \left( \frac{A}{H} \right)_R \right] = \frac{Q^2}{R^4},
\]

while (9) and (10) remain unchanged. Since \( Q = Q(m) \), the equation above can be
integrated at once and gives

\[
A = H \left( 1 - \frac{2m}{R} + \frac{Q^2}{R^2} \right).
\]
Using $B^2 = H + A/u^2$ and equation (7), from (9) we get

$$u = \pm \sqrt{-1 + \frac{2m}{R} - \frac{Q^2}{R^2} + \frac{1}{h^2} \left( E - \frac{Q}{R} \right)^2},$$

where the ‘specific charge’ $\xi := EQ_m$. The metric function $B$ can be written as

$$B = -\frac{1}{hu} \left( E - \frac{\xi}{R} \right) \sqrt{H},$$

so that (10) gives

$$\left( \sqrt{H} \right)_R = \frac{h}{E - \frac{\xi}{R}} \left( \frac{1}{u} \right)_m.$$

It follows that

$$\sqrt{H(m, R)} = g(m) \pm \int G(m, R) dR,$$

where $g(m)$ is arbitrary and

$$G := \frac{h}{R \left( E - \frac{\xi}{R} \right)} \left\{ 1 - \frac{Q Q_m}{R} + \frac{R}{2} \left[ \frac{1}{h^2} \left( E - \frac{\xi}{R} \right)^2 \right]_m \right\}
\times \left[ -1 + \frac{2m}{R} - \frac{Q^2}{R^2} + \frac{1}{h^2} \left( E - \frac{\xi}{R} \right)^2 \right]^{-3/2}.$$

For $h = 1$ (dust case) the function $G$ is a rational fraction and its integral may be carried out explicitly (Ori 1990), while for $Q = 0$ (non-charged case) the above function coincides with that defined in (14).

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