PROPERTIES OF DIFFERENTIAL OPERATORS WITH VANISHING COEFFICIENTS

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Abstract. In this paper, we investigate the properties of linear operators defined on $L^p(\Omega)$ that are the composition of differential operators with functions that vanish on the boundary $\partial \Omega$. We focus on bounded domains $\Omega \subset \mathbb{R}^d$ with Lipschitz continuous boundary. In this setting we are able to characterize the spectral and Fredholm properties of a large class of such operators. This includes operators of the form $Lu = \text{div}(\Phi \nabla u)$ where $\Phi$ is a matrix valued function that vanishes on the boundary, as well as operators of the form $Lu = D^\alpha(\varphi u)$ or $L = \varphi D^\alpha u$ for some function $\varphi \in C^1(\bar{\Omega})$ that vanishes on $\partial \Omega$.

1. Introduction

In this article we study the properties of linear operators when we allow the leading coefficient functions to vanish on the boundary of the domain. For example, the differential equation:

$$Lu = -\text{div}(\Phi \nabla u) = f,$$

where $f \in L^p(\Omega)$, and $\Phi(x) \in \mathbb{C}^{d \times d}$ has been extensively studied when $\Phi$ is uniformly positive definite on $\Omega$. The operator $L$ is called uniformly elliptic. For more on such operators, see [3, 6, 8, 12, 14] and the references therein. Less is known when the uniform positivity assumption on $\Phi$ is relaxed. In [6, §6.6], Trudinger and Gilberg partially relax the condition. In particular, they assume $\Phi \in C^{0,\gamma}(\bar{\Omega})$ for some $\gamma \in (0,1)$, and if $x_0 \in \partial \Omega$ then there exists a suitably chosen $y \in \mathbb{R}^d$ such that $\Phi(x_0) \cdot (x_0 - y) \neq 0$. With this restriction, they establish existence and uniqueness of solutions to (1). In [15], Murthy and Stampacchia studied the properties of weak solutions to (1) in the case where there exists a positive function $m$ with $m^{-1} \notin L^p(\Omega)$ such that

$$v \cdot \Phi(x)v \geq m(x)|v|^2, \quad \text{for a.e. } x \in \Omega, \text{ and all } v \in \mathbb{R}^d.$$

Studies on the properties of solutions to $Lu = f$, when $L$ is non-uniformly elliptic, can be found in [18, 20], as well as [2], and [5], where the authors assume restrictions on $\Phi$ that are similar to (2). The results presented here address the Fredholm properties of $L$ in the case when $\Phi = 0$ on $\partial \Omega$, and/or when $v \cdot \Phi(x)v \geq m(x)|v|$ for some positive function $m \in C^1(\bar{\Omega})$ with $m^{-1} \notin L^p(\Omega)$.

Other examples of a differential equation with vanishing coefficients arise when studying linear stability of solutions to non-linear PDE. The operator

$$Lu = (\varphi u)_{xx} + (\varphi u)_x + bu_x,$$
defined on $L^p(-1,1)$ where $\varphi(x) = a\cos^2(\frac{\pi}{2}x)$ and $a, b \in \mathbb{R}$, arose when studying compactly supported solutions to

$$u_t = (u^2)_{xxx} + (u^2)_x.$$ 

Our results can be used to accomplish two goals. The first is assessing the solvability of the boundary value problem $Lu = f$ where $u = g$ on the boundary. To that end, we analyze the Fredholm properties of $L$. Our second goal is establishing well-posedness (or ill-posedness) of the Cauchy problem $u_t = Lu$, where $u = g$ when $t = 0$. For this goal, we present results on the spectrum of $L$.

The operators studied here are linear differential operators on $L^p(\Omega)$, elliptic or otherwise, that have coefficient functions on the leading order derivative term that vanish on $\partial \Omega$. For the matrix valued function $\Phi : \bar{\Omega} \rightarrow \mathbb{C}^{d \times d}$, we only require that at least one eigenvalue vanishes on $\partial \Omega$. The operators shown in (1) and (3) are examples of operators that can be analyzed using the results presented here.

2. The main results

2.1. Preliminaries. Throughout this article we make the following assumptions:

- The domain $\Omega \subset \mathbb{R}^d$ is open and bounded and $\partial \Omega$ is Lipschitz continuous.
- The function $\varphi : \bar{\Omega} \rightarrow \mathbb{R}$ is such that $\varphi \in C^k(\bar{\Omega})$ for some positive integer $k$, $\varphi > 0$ on $\Omega \subset \mathbb{R}^d$, and $\ker \varphi = \partial \Omega$.
- The ambient function space for the differential operator $L$ is $L^p(\Omega)$ where $1 < p < \infty$.

We say the scalar valued function $\varphi$ is simply vanishing on $\partial \Omega$ if for each $y \in \partial \Omega$ there exists an $a \neq 0$ such that

$$\lim_{x \to y} \frac{\varphi(x)}{\text{dist}(x, \partial \Omega)} = a,$$

where the limit is taken in $\Omega$. For the matrix valued function $\Phi : \bar{\Omega} \rightarrow \mathbb{C}^{d \times d}$ we make restrictions on its eigenfunctions, defined as the functions $\varphi_i$ such that $\Phi(x)v = \varphi_i(x)v$ for some $v \in \mathbb{C}^d$. We say the matrix valued function $\Phi : \bar{\Omega} \rightarrow \mathbb{C}^{d \times d}$ is simply vanishing on $\partial \Omega$ if $\Phi(x)$ is positive semi-definite in $\Omega$ and for each fixed $i$, the eigenfunction $\varphi_i$ is either strictly positive on $\Omega$ or simply vanishing on $\partial \Omega$, with at least one $i$ such that $\varphi_i$ is simply vanishing on $\partial \Omega$.

2.2. Results. In this section, we summarize the results proved in this article. In the following theorem, $\lfloor a \rfloor$ denotes the integer part of $a$.

**Theorem 2.1.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $\varphi \in C^3(\bar{\Omega})$ be simply vanishing on $\partial \Omega$. Fix $m \in \mathbb{N}$ and $1 < p < \infty$. Assume $k \in \mathbb{N}$ is such that

$$k > \frac{d}{p} + (m - 1)\lfloor \frac{d}{p} \rfloor,$$

and that the boundary $\partial \Omega$ is $C^k$. If $u \in L^p(\Omega)$ and $\varphi^m u \in W^{k+m,p}(\Omega)$ then

$$u \in W^{\kappa,p}(\Omega), \quad \kappa := k - m\lfloor \frac{d}{p} \rfloor,$$

and there exists a $c > 0$, independent of $u$, such that

$$\|u\|_{W^{\kappa,p}(\Omega)} \leq c\|\varphi^m u\|_{W^{k+m,p}(\Omega)}.$$

The above result is proven in section 5.1 as Theorem 5.7, with the estimate proven in Remark 5.8. The following theorem is proven in section 6.1 as Theorem 6.6.
Theorem 2.2. Let \( \Omega \subset \mathbb{R}^d \) be an open and bounded set with \( \mathcal{C}^{0,1} \) boundary. Let \( \varphi \in \mathcal{C}^1(\Omega) \) be simply vanishing on \( \partial \Omega \). If \( A \) is Fredholm on \( L^p(\Omega) \) with domain \( W^{k,p}(\Omega) \) for some \( k > 0 \), then \( \varphi^m A, m \geq 1 \), is not closed on its natural domain,

\[
D(\varphi^m A) = \{ u \in L^p : u \in D(A), \varphi^m Au \in L^p(\Omega) \} \equiv D(A),
\]

but \( \varphi^m A \) is closable.

The above theorem tells us that even simple operators do not have the desirable property of being closed on their ‘natural domain’. For example, the operator \( Lu = \sin(\pi x)u_{xx} \) is not closed on \( W^{2,p}(0,1) \) for any \( p \in (1, \infty) \) by Theorem 2.2. We can use Theorem 2.2 to get an estimate on the properties of the domain, as demonstrated in the following example.

Example 2.3. Let \( \Omega \equiv (0,1) \) and consider the operator \( L \) acting on \( L^p(\Omega) \) defined by \( Lu = \varphi u_{xxx} \) where \( \varphi \) is simply vanishing. The operator \( L \) is of the form \( \varphi A \) where \( A \) is Fredholm. By Theorem 2.2, \( L \) is not closed on its natural domain, \( W^{3,p}(\Omega) \), but is closable. Let \( \tilde{L} \) denote the closure of \( L \), and let \( \varphi^{(k)} : L^p(\Omega) \to W^{3-k,p}(\Omega) \) denote the multiplication operator \( u \mapsto \varphi^{(k)}u \) where the function \( \varphi^{(k)} \) denotes the \( k \)-th derivative of the function \( \varphi \), and as an abuse of notation we set \( \varphi = \varphi^{(0)} \).

After rewriting \( L \) as

\[
Lu = (\varphi u)_{xxx} - 3(\varphi^{(1)} u)_{xx} + 3(\varphi^{(2)} u)_x - \varphi^{(3)} u,
\]

we see that

\[
D(\tilde{L}) = D(A^3 \varphi) \cap D(A^2 \varphi^{(1)}) \cap D(A \varphi^{(2)}) \cap D(\varphi^{(3)}),
\]

where \( A \) is the derivative operator on \( L^p(\Omega) \). Specifically, if \( \varphi \in \mathcal{C}^3(\Omega) \) is simply vanishing, then the fact that \( D(A^3) = W^{3,p}(\Omega) \) implies \( D(A^3 \varphi) \subset W^{2,p}(\Omega) \) by Theorem 2.2. By the same theorem, we have \( W^{2,p}(\Omega) \subset D(A^k \varphi^{(3-k)}) \) for \( k = 0, 1, 2 \) so the best we can do is \( D(\tilde{L}) \subset W^{2,p}(\Omega) \). More concretely, if we set \( \varphi(x) = \sin(\pi x) \) and fix \( p = 2 \), then one can construct functions \( u \in D(\tilde{L}) \) such that \( u \notin W^{3,2}(\Omega) \) but \( u \in W^{2,2}(\Omega) \). See [8] [3.1].

The following theorem speaks about the range of the multiplication operator. It is proved in section 5.2 as Theorem 5.9.

**Theorem 2.4.** Let \( \Omega \subset \mathbb{R}^d \) be open and bounded with \( \mathcal{C}^{0,1} \) boundary and assume the function \( \varphi \in \mathcal{C}^k(\Omega) \) is simply vanishing on \( \partial \Omega \). Then the range of the operator \( u \mapsto \varphi^m u \) is closed in \( W^{k,p}(\Omega) \) whenever \( k \geq m \) and is not closed when \( k < m \).

If we know the range of the multiplication operator \( u \mapsto \varphi^m u \) is closed in \( W^{k,p}(\Omega) \) then we necessarily have

\[
\|u\|_{L^p(\Omega)} \leq c\|\varphi^m u\|_{W^{k,p}(\Omega)}
\]

for some constant \( c > 0 \).

The matrix valued analogs of Theorems 2.4 and 2.4 are proved in section 5.3. One implication is illustrated in the following example.

**Example 2.5.** Let \( \Omega \subset \mathbb{R}^d \) be open and bounded with \( \mathcal{C}^{0,1} \) boundary. Let \( \Phi \in \mathcal{C}^1(\Omega; \mathbb{R}^{d \times d}) \) be simply vanishing on \( \partial \Omega \). Then by the matrix analog of Theorem 2.4 (which is Theorem 5.11) we know that the range of \( \Phi^m, m \in \mathbb{N} \), is closed in \( W^{1,2}(\Omega^d) := W^{1,2}(\Omega) \times \cdots \times W^{1,2}(\Omega), d \) copies.
if and only if $m = 1$. This implies $\Phi^m : L^2(\Omega^d) \to W^{1,2}(\Omega^d)$ is semi-Fredholm if and only if $m = 1$. Now, it is well known that the weak gradient $\nabla : W^{1,2}(\Omega) \subset L^2(\Omega) \to L^2(\Omega^d)$ and its adjoint, $\text{div}(\cdot) : L^2(\Omega^d) \to L^2(\Omega)$, are semi-Fredholm. Thus, the non-uniformly elliptic operator,

\begin{equation}
L u = \text{div}(\Phi^m \nabla u),
\end{equation}

is semi-Fredholm on $L^2(\Omega)$ if and only if $m = 1$.

**Theorem 2.6.** Let $\Omega \subset \mathbb{R}^d$ be open and bounded with $C^{0,1}$ boundary, $m, k \in \mathbb{N}$ with $0 < m < k$, and let $A$ be densely defined on $L^p(\Omega)$. Assume the following:

- The operator $A$ is closed on $L^p(\Omega)$ and $D(A) \subset W^{k,p}(\Omega)$.
- There exists a $u \in D(A)$ such that $u \notin W^{m,p}_0(\Omega) \cap W^{k,p}(\Omega)$.
- The resolvent set $\rho(A)$ is non-empty.

If $\varphi \in \mathcal{C}^k(\Omega)$ is simply vanishing then

$$
\sigma_{\text{ess}}(A \varphi^m) = \sigma_{\text{ess}}(\overline{\varphi^m A}) = \mathbb{C}.
$$

Moreover, if either $A \varphi^m$ or $\overline{\varphi^m A}$ is Fredholm then

$$
\sigma_p(\varphi^m A) = \mathbb{C}.
$$

In the above theorem, $\sigma_p(L)$ and $\sigma_{\text{ess}}(L)$ denote the point spectrum and essential spectrum of $L$ respectively. The definition of the essential spectrum is given in section 6.2 and the result is proven as Theorem 6.14. Its import is demonstrated in the following example.

**Example 2.7.** This example continues from Example 2.3 where $\Omega = (0,1)$, and $L = \varphi u_{xxx}$. For any $u \in W^{3,p}(\Omega)$, there exists $a, b \in \mathbb{R}$ such that $u + ax + b \in W^{1,p}_0(\Omega)$, which implies

$$
W^{3,p}(\Omega) = W^{3,p}(\Omega) \cap W^{1,p}_0(\Omega) \oplus \text{span}\{1, x\}.
$$

Now consider the multiplication operator $\varphi : L^p(\Omega) \to W^{3,p}(\Omega)$ given by $u \mapsto \varphi u$ where $\varphi$ is simply vanishing on $\partial \Omega$. Then we know that the range of $\varphi$ is $W^{3,p}(\Omega) \cap W^{1,p}_0(\Omega)$, which has co-dimension 2 in $W^{3,p}(\Omega)$. Thus, if we let $A$ denote three applications of the weak derivative operator on $L^p(\Omega)$, then we know that $\rho(A)$ is nonempty, and $D(A) = W^{3,p}(\Omega) \subset L^p(\Omega)$ by the Rellich-Kondrachov Theorem. Then we see that $A \varphi$ has finite dimensional nullspace and the range has finite co-dimension. This shows $A \varphi$ is Fredholm. Applying Theorem 2.6 yields $\sigma_p(L) = \sigma_p(\varphi A) = \mathbb{C}$.

More concretely, we have $\sigma_p(\overline{\sin(\pi x) u_{xxx}}) = \mathbb{C}$. The same holds if we set $\varphi(x) = \sin^2(\pi x)$, but not necessarily when we set $\varphi(x) = \sin^m(\pi x)$ where $m \in \mathbb{N}$ and $m \geq 3$.

2.3. **Outline of the article.** The article is structured as follows:

- Section 3 introduces the notation and basic definitions that are used in this article.
- Section 4 goes over some basic properties of closed and Fredholm operators.
- Section 5 covers the properties of the operators $u \mapsto \varphi u$ and $u \mapsto \Phi u$. The domain and range of the operator $u \mapsto \varphi u$ is covered in sections 5.1 and 5.2 respectively. Matrix valued operators are handled in section 5.3.
We study various properties of differential operators composed with vanishing operators in section 6. In particular, we focus on the Fredholm properties and spectra of the operators $A\varphi$ and $\varphi A$ where $A$ is a Fredholm differential operator.

3. Notation and definitions

We will review some of the basic definitions and introduce the notation used in this article. We will use capital letters, such as $W$, $X$, $Y$, or $Z$, to denote a Banach space. We use $\mathcal{B}(X, Y)$ to denote the set of bounded linear operators from $X$ to $Y$, and $\mathcal{L}(X, Y)$ to denote the set of closed and densely defined linear operators from $X$ to $Y$. The sets $\mathcal{B}(X)$ and $\mathcal{L}(X)$ denote the sets $\mathcal{B}(X, X)$ and $\mathcal{L}(X, X)$ respectively.

The domain and range of a linear operator $A$ will be denoted by $\text{D}(A)$ and $\text{R}(A)$ respectively, and we use $\text{N}(A)$ to denote the nullspace of $A$. If $A \in \mathcal{L}(X, Y)$, then $\text{D}(A)$ equipped with the graph norm,

$$\|x\|_{\text{D}(A)} := \|x\|_X + \|Ax\|_Y, \quad x \in \text{D}(A),$$

is a Banach space, and we call $\| \cdot \|_{\text{D}(A)}$ the $A$-norm. When referring to the composition of two linear operators $A$ and $B$, the subspace

$$\text{D}(AB) = \{ x \in \text{D}(B) : Bx \in \text{D}(A) \},$$

is called the natural domain of $AB$.

If $A$ is a linear operator from $X$ to $Y$, any closed operator $A_1$ where $\text{D}(A) \subset \text{D}(A_1)$ and $A = A_1$ on $\text{D}(A)$ is called a closed extension of $A$. We call $A$ closable if there exists a closed extension of $A$. We denote $\tilde{A}$ as the closure of $A$, and it is the ‘smallest’ closed extension, in the sense that $\text{D}(\tilde{A}) \subset \text{D}(A_1)$ for any operator $A_1$ that is a closed extension of $A$. An operator is closable if every sequence $\{x_n\} \subset \text{D}(A)$ where $x_n \to 0$ in $X$ and $Ax_n \to y$ in $Y$ implies $y = 0$.

A Banach space $Y$ is said to be continuously embedded in another Banach space $X$ if there exists an operator $P \in \mathcal{B}(X, Y)$ that is one-to-one. The space $Y$ is said to be compactly embedded in $X$ if $P$ is also compact and we write $Y \subset \subset X$ whenever $Y$ is compactly embedded in $X$. For Sobolev spaces, we take $P$ to be the inclusion operator, which we denote as $i$.

Most of the analysis takes place in $L^p(\Omega)$ and the Sobolev spaces $W^{k, p}(\Omega)$, where $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ is an open and bounded set, and, unless stated otherwise, $1 < p < \infty$. The closure of a set $\Omega \subset \mathbb{R}^d$ will be denoted by $\tilde{\Omega}$ and the boundary of $\Omega$ will be denoted by $\partial \Omega$. The space $\mathcal{C}^k(\Omega)$ denotes the space of all functions from $\Omega$ to $\mathbb{R}$ that are $k$-times continuous differentiable everywhere in $\Omega$, and $\mathcal{C}_0^k(\Omega) \subset \mathcal{C}^k(\Omega)$ denotes the subspace of those functions with compact support in $\Omega$. The space $W_0^{k, p}(\Omega)$ denotes the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{k, p}(\Omega)$. If $u$ is weakly differentiable, we let $D^\alpha u$ denote the $\alpha$-th weak derivative of $u$, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}_+^d$ is a multi-index, and we let $|\alpha| = \alpha_1 + \cdots + \alpha_d$ denote the order of $\alpha$. We use $\nabla^{(k)} u$ to denote the vector of all weak derivatives of $u$ with order $k$, and set $\nabla u = \nabla^{(1)} u$ to be the gradient of $u$.

We say $\partial \Omega$ is $\mathcal{C}^{k, \gamma}$ if for each point $y \in \partial \Omega$, there exists an $r > 0$ and a $\mathcal{C}^{k, \gamma}$ function $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that

$$\Omega \cap B(y, r) = \{ x \in B(y, r) : x_d > \gamma(x_1, \ldots, x_{d-1}) \},$$

where $B(x, r) := \{ y \in \mathbb{R}^d : |x - y| < r \}$, and $\mathcal{C}^{k, \gamma}$ is a Hölder space.
Definition 3.1. Let \( \Omega \subset \mathbb{R}^d \) be a bounded and open set. For any \( \varphi \in C^1(\Omega) \), we will call the multiplication operator \( u \mapsto \varphi u \) vanishing if \( \ker \varphi = \partial \Omega \). As an abuse of notation, we will use \( \varphi \) to refer to the multiplication operator \( u \mapsto \varphi u \).

Definition 3.2. Let \( \Omega \subset \mathbb{R}^d \) be a bounded and open set. Take \( \text{dist}(x, \partial \Omega) := \inf_{y \in \partial \Omega} |x - y| \) to be the distance from \( x \) to the boundary of \( \Omega \). Let \( \varphi \in C^1(\Omega) \). We say \( \varphi \) is simply vanishing on \( \partial \Omega \) if \( \varphi = \partial \Omega \), and for each \( y \in \partial \Omega \) there exists an \( a \neq 0 \) such that

\[
\lim_{x \to y} \frac{\varphi(x)}{\text{dist}(x, \partial \Omega)} = a,
\]

where the limit is taken within \( \Omega \). The multiplication operator \( u \mapsto \varphi u \) is called simply vanishing on \( \partial \Omega \) if the function \( \varphi \) is simply vanishing on \( \partial \Omega \).

Definition 3.3. Let \( \Omega \subset \mathbb{R}^d \) be a bounded and open set. We say the function \( \varphi \in C^m(\Omega) \) is vanishing of order \( m \) on \( \partial \Omega \) if \( \ker \varphi = \partial \Omega \), \( D^m \varphi = 0 \) on \( \partial \Omega \), and for each \( y \in \partial \Omega \) there exists a point \( x_n \in \Omega \) such that \( |x_n - y| = \text{dist}(x_n, \partial \Omega) \). Given \( \varphi \) is simply vanishing, there exists an \( a \neq 0 \) such that

\[
a = \lim_{n \to \infty} \frac{\varphi(x_n)}{\text{dist}(x_n, \partial \Omega)} = \lim_{n \to \infty} \frac{|\varphi(x_n) - \varphi(y)|}{|x_n - y|} = |\nabla \varphi(y)|,
\]

which shows \( \nabla \varphi(y) \neq 0 \). Since \( y \in \partial \Omega \) was arbitrary, we see that \( \nabla \varphi \neq 0 \) on \( \partial \Omega \) whenever \( \partial \Omega \) is \( C^1 \).

We have a similar definition for matrix-valued functions. Let \( \Phi : \tilde{\Omega} \to \mathbb{C}^{d \times d} \) be Hermitian for each \( x \in \tilde{\Omega} \). Then there exists a unitary matrix \( U(x) \) and a real diagonal matrix \( D(x) \) such that

\[
\Phi = UDU^*,
\]

by Schur’s decomposition theorem. If \( \Phi \in C^1(\Omega; \mathbb{C}^{d \times d}) \), we can choose \( U \) and \( D \) in \( C^1(\Omega; \mathbb{C}^{d \times d}) \) so that (7) holds. Thus, we lose no generality by assuming the operator \( D \) has the form \( D = \text{diag}(\varphi_1, \ldots, \varphi_d) \) for some functions \( \varphi_i \in C^1(\tilde{\Omega}) \) where \( i = 1, \ldots, d \).

Definition 3.4. Let \( \Omega \subset \mathbb{R}^d \) be open and bounded, and let \( \Phi \in C^m(\tilde{\Omega}; \mathbb{C}^{d \times d}) \). We say the function \( \Phi \) is vanishing of order \( m \) if \( \Phi \) is positive semi-definite on \( \tilde{\Omega} \), and the matrix \( D = \text{diag}(\varphi_1, \ldots, \varphi_d) \) in its Schur decomposition has the following property: for each \( i = 1, \ldots, d \), either \( \varphi_i > 0 \) on \( \tilde{\Omega} \) or \( \varphi_i \) is vanishing of order \( m_i \) on \( \partial \Omega \) with \( m_i \leq m \), with at least one function \( \varphi_i \) that is vanishing of order \( m \) on \( \partial \Omega \). The multiplication operator \( u \mapsto \Phi u \) is called vanishing of order \( m \) if the function \( \Phi \) is vanishing of order \( m \).

3.1. Fredholm and semi-Fredholm operators. We will utilize Fredholm operator theory when describing the properties of the multiplication operators \( u \mapsto \varphi u \) and \( u \mapsto \Phi u \). In this section, we briefly review the theory of Fredholm operators.

Let \( X \) and \( Y \) be Banach spaces. An operator \( A : X \to Y \) is called Fredholm if

(a) The domain of \( A \) is dense in \( X \),
Remark [17, p. 177].

Moreover, there exists operators \( A \) is finite dimensional, the range of \( A \) is closed in \( Y \), and the co-dimension of the range of \( A \) is finite dimensional, where the co-dimension of a closed subspace \( M \subset Y \), denoted \( \text{co-dim } M \), is the dimension of the quotient space \( Y/M \). We use \( \mathcal{F}(X, Y) \) to denote the set of Fredholm operators from \( X \) to \( Y \) and write \( \mathcal{F}(X) \) in place of \( \mathcal{F}(X, X) \). Note that property (e) is equivalent to requiring that the nullspace of the adjoint operator \( A^* \) be finite dimensional. The \textit{index} of a Fredholm operator \( A \) is defined as

\[
\text{ind}(A) := \dim N(A) - \text{co-dim R}(A).
\]

The set of \textit{semi-Fredholm} operators from \( X \) to \( Y \), denoted \( \mathcal{F}_+(X, Y) \), is the set of operators that satisfy all the properties of Fredholm operators except possibly property (e). The set of semi-Fredholm operators from \( X \) to \( X \) will be denoted by \( \mathcal{F}_+(X) \). We note that our definition for \( \mathcal{F}_+(X, Y) \) is sometimes referred to as the set of \textit{upper semi-Fredholm} operators.

The following characterization of Fredholm operators is useful.

**Theorem 3.5** ([17], Theorem 7.1). Let \( X \) and \( Y \) be Banach spaces. Then \( A \in \mathcal{F}(X, Y) \) if and only if there exists closed subspaces \( X_0 \subset X \) and \( Y_0 \subset Y \) where \( Y_0 \) is finite dimensional and \( X_0 \) has finite co-dimension such that

\[
X = X_0 \oplus N(A), \quad Y = R(A) \oplus Y_0.
\]

Moreover, there exists operators \( A_0 \in \mathcal{B}(Y, X) \), \( K_1 \in \mathcal{B}(X) \), and \( K_2 \in \mathcal{B}(Y) \) where

- The \( N(A_0) = Y_0 \),
- The \( R(A_0) = X_0 \cap D(A) \),
- \( A_0A = I - K_1 \) on \( D(A) \),
- \( AA_0 = I - K_2 \) on \( Y \),
- The \( N(K_1) = X_0 \), \( K_1 = I \) on \( N(A) \),
- The \( N(K_2) = R(A) \), \( K_2 = I \) on \( Y_0 \).

Note that \( K_1 \) and \( K_2 \) are projection operators and their ranges are finite dimensional. The operator \( A_0 \) from Theorem 3.5 will be referred to as the \textit{pseudo-inverse} of \( A \), since \( AA_0A = A \) and \( AA_0A_0 = A_0 \).

An equivalent characterization of semi-Fredholm operators is as follows: if \( X \) and \( Y \) are Banach spaces and \( A \in \mathcal{C}(X, Y) \), then \( A \) is not semi-Fredholm if and only if there exists a bounded sequence \( \{x_k\} \subset D(A) \) having no convergent subsequence such that \( \{Ax_k\} \) converges. A proof of this equivalence can be found in [16] or [17] p. 177.

**Remark 3.6.** Suppose \( \Omega \subset \mathbb{R}^d \) is open and bounded, \( \varphi \in \mathcal{C}^m(\overline{\Omega}) \) is simply vanishing on \( \partial \Omega \) and \( \zeta \in \mathcal{C}^m(\overline{\Omega}) \) is vanishing of order \( m \) on \( \partial \Omega \). Then the multiplication operator \( u \mapsto \varphi^m u \) is semi-Fredholm from \( L^p(\Omega) \) to \( W^{k,p}(\Omega) \) if and only if the mapping \( u \mapsto \zeta u \) is semi-Fredholm. Moreover, \( D(\varphi^m) = D(\zeta) \). To see why, use the fact that the multiplication operators \( u \mapsto \zeta \varphi^{-m} u \) and \( u \mapsto \varphi^m \zeta^{-1} u \) are one-to-one and onto \( L^p(\Omega) \) and that the composition of a semi-Fredholm operator with a Fredholm operator is semi-Fredholm.
4. Basic properties of closed operators

Fredholm operators are closed under composition. That is, if \( X, Y, \) and \( Z \) are Banach spaces, then \( A \in \mathcal{F}(X, Y) \) and \( B \in \mathcal{F}(Y, Z) \) implies \( BA \in \mathcal{F}(X, Z) \) with \( \text{ind}(BA) = \text{ind}(B) + \text{ind}(A) \). Moreover, if \( B \in \mathcal{C}(Y, Z) \) and \( BA \in \mathcal{F}(X, Z) \) then we necessarily have \( B \in \mathcal{F}(Y, Z) \). These claims are proved, respectively, in [17] p. 157 as Theorem 7.3, and [17] p. 162 as Theorem 7.12. As for semi-Fredholm operators, we have the following.

**Lemma 4.1** ([17], Lemma 4). Let \( X, Y, \) and \( Z \) be Banach spaces and let \( A \in \mathcal{F}(X, Y) \), and \( B \in \mathcal{C}(Y, Z) \). If \( BA \in \mathcal{F}(X, Z) \), then \( B \in \mathcal{F}(Y, Z) \).

One of the theorems that we use throughout this article is the following consequence of the Closed Graph Theorem. A proof of the Closed Graph Theorem can be found in many functional analysis textbooks, such as [17, p. 62] or [11, p. 166].

**Lemma 4.2.** Let \( X \) be a Banach space and \( Y \subset X \). If there exists a norm that converts \( Y \) into a Banach space then there exists a \( c > 0 \) such that \( \|y\|_X \leq c\|y\|_Y \) for all \( y \in Y \). If \( Y = X \) then their norms are equivalent.

**Proof.** Let \( \iota \) denote the inclusion map from \( Y \) to \( X \). It is a closed operator with domain equal to \( Y \), so by the Closed Graph Theorem it is bounded. If \( Y = X \) then apply the above argument to the inclusion map from \( X \) to \( Y \).

The above lemma is useful for showing that special subsets of \( L^p(\Omega) \) have certain properties — such as compactness — since they can inherit such properties from other Sobolev spaces. One of the most important consequences of compactness is the following theorem.

**Theorem 4.3.** Let \( X \) and \( Y \) be Banach spaces. If \( A \in \mathcal{C}(X, Y) \) with \( D(A) \subset \subset X \), then \( A \in \mathcal{F}_+(X, Y) \).

**Proof.** We prove this by contradiction. Suppose \( A \) is not semi-Fredholm. Since \( A \in \mathcal{C}(X, Y) \), this implies there exists a bounded sequence \( \{x_n\} \subset D(A) \) having no convergent subsequence, such that \( \{Ax_n\} \) convergent in \( Y \). But if \( \{x_n\} \) is bounded in \( X \) and \( \{Ax_n\} \) convergent in \( Y \), then \( \{x_n\} \) is a bounded sequence in the \( A \)-norm. Since \( D(A) \subset \subset X \) there exists a subsequence of \( \{x_n\} \) that is convergent in \( X \), the desired contradiction.

**Remark 4.4.** We know an operator \( A \in \mathcal{C}(X, Y) \) has closed range if and only if there exists a \( c > 0 \) such that

\[
\inf_{z \in N(A)} \|x - z\|_X \leq c\|Ax\|_Y, \quad \text{for all } x \in D(A).
\]

Theorem 4.3 can be used to quickly establish estimates involving differential operators. We can, for example, establish Poincaré inequalities.

It is also well known that \( W_0^{1,p}(\Omega) \subset \subset L^p(\Omega) \) for any bounded and open set \( \Omega \subset \mathbb{R}^d \). Since the weak gradient operator \( \nabla \) is closed on \( W_0^{1,p}(\Omega) \), we get \( R(\nabla) \) is closed by Theorem 4.3. This implies the existence of a \( c > 0 \) such that

\[
\|u\|_{L^p(\Omega)} = \inf_{a \in N(\nabla)} \|u - a\|_{L^p(\Omega)} \leq c\|\nabla u\|_{L^p(\Omega)},
\]

for any \( u \in W_0^{1,p}(\Omega) \).
5.1. The domain of a vanishing operator. In this section we establish properties of the domain of the vanishing operator \( \varphi : L^p(\Omega) \to W^{k,p}(\Omega) \). In particular, we will establish the embedding of the domain of \( \varphi \) in various Sobolev spaces.

If \( \varphi \in \mathcal{C}^1(\overline{\Omega}) \), then the mapping \( u \mapsto \varphi u \) is bounded from \( L^p(\Omega) \) to \( L^p(\Omega) \). This implies that a natural choice for its domain is \( L^p(\Omega) \). Whenever a vanishing operator \( \varphi \) is composed with a differential operator \( A \) to form \( A \varphi - A \) being an operator that is closed on \( W^{k,p}(\Omega) \) — it makes sense to think of \( \varphi \) as a densely defined operator that maps some subset of \( L^p(\Omega) \) to the space \( W^{k,p}(\Omega) \).

This and subsequent sections rely heavily on Hardy’s inequality, so we include the statement for the reader’s convenience.

**Theorem 5.1 (22). Hardy’s Inequality.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set with \( \mathcal{C}^{0,1} \) boundary, and let \( \delta(x) = \inf_{y \in \partial \Omega} |x - y| \). Then, for all \( u \in \mathcal{C}_0^\infty(\Omega) \),

\[
\|\delta^{-m}u\|_{L^p(\Omega)} \leq c\|\nabla^m u\|_{L^p(\Omega)},
\]

where \( c > 0 \) depends on \( \Omega, p, d, \) and \( m \).

See \([7,13]\) for recent developments on the assumptions necessary for Hardy’s inequality. The interested reader should consult \([14, \S 2.7]\) for a treatment of optimal constants for Hardy’s inequality.

We begin with basic properties of the domain and range of the multiplication operator \( u \mapsto \varphi^m u \).

**Lemma 5.2.** Let \( \Omega \subset \mathbb{R}^d \) be open and bounded and \( \varphi \in \mathcal{C}^1(\overline{\Omega}) \) be simply vanishing on \( \partial \Omega \). For each \( k, m \in \mathbb{Z}_+ \), the multiplication operator \( \varphi^m : L^p(\Omega) \to W^{k,p}(\Omega) \) defined as \( u \mapsto \varphi^m u \) is closed on \( D(\varphi^m) = \{ u \in L^p(\Omega) : \varphi^m u = W^{k,p}(\Omega) \} \).

Moreover, if \( \varphi \in \mathcal{C}^m(\overline{\Omega}) \) and \( m \leq k \) then \( R(\varphi^m) = W^{k,p}(\Omega) \cap W_0^{m,p}(\Omega) \).

**Proof.** The proof is broken into two claims.

**Claim 1:** The multiplication operator \( u \mapsto \varphi^m u \) is closed on \( D(\varphi^m) \).

We first show it is closable. Suppose \( u_n \to 0 \) in \( L^p(\Omega) \) and \( \varphi^m u_n \to y \) in \( W^{k,p}(\Omega) \). Then we know that \( \varphi^m u_n \to y \) in \( L^p(\Omega) \). But since \( \varphi \) is bounded and \( u_n \to 0 \) in \( L^p(\Omega) \) we can conclude \( \varphi^m u_n \to 0 = y \) in \( L^p(\Omega) \). This shows the multiplication operator \( \varphi^m \) is closable on its domain. But any closed extension cannot be defined on a set larger than \( D(\varphi^m) \), implying the domain of any closed extension must be \( D(\varphi^m) \). This completes the proof of the claim.

**Claim 2:** If \( \varphi \in \mathcal{C}^m(\overline{\Omega}) \) and \( m \leq k \) then \( R(\varphi^m) = W^{k,p}(\Omega) \cap W_0^{m,p}(\Omega) \).

Take \( v \in W^{k,p}(\Omega) \cap W_0^{m,p}(\Omega) \). Since \( v \in W_0^{m,p}(\Omega) \) we can apply Hardy’s inequality (Theorem 5.1) to show that \( \varphi^m v \in L^p(\Omega) \). Since this implies \( \varphi^m v \in D(\varphi^m) \) we have \( W^{k,p}(\Omega) \cap W_0^{m,p}(\Omega) \subset R(\varphi^m) \).

For the other direction, first note that since \( \varphi^m \in \mathcal{C}^m(\overline{\Omega}) \cap W_0^{m,p}(\Omega) \) there exists a sequence \( \{ \phi_n \} \subset \mathcal{C}_0^\infty(\Omega) \) such that \( \phi_n \to \varphi^m \) in \( W_0^{m,p}(\Omega) \) and \( \{ D^\alpha \phi_n \} \) is uniformly bounded when \( |\alpha| \leq m \). Let \( u \in D(\varphi^m) \) be arbitrary, and set \( v_n = \phi_n -
$\varphi^m$. Since $D^\alpha(\phi_n u)$ is bounded by $cD^\alpha(\varphi u)$ for some constant $c$ and $|D^\alpha(v_n u)|^p \to 0$ almost everywhere when $|\alpha| \leq m$ we have

$$
\lim_{n \to \infty} \|D^\alpha(\phi_n u) - D^\alpha(\varphi^m u)\|_{L^p(\Omega)} = \lim_{n \to \infty} \|D^\alpha(v_n u)\|_{L^p(\Omega)} = 0,
$$

when $|\alpha| \leq m$ by dominated convergence. Noting that $\phi_n u \in W_0^{m,p}(\Omega)$ shows $\varphi^m u \in W_0^{m,p}(\Omega)$ and completes the proof.

The following lemma establishes the relative compactness of $D(\varphi^m)$ in $L^p(\Omega)$ when $\varphi^m$ maps to $W^{k,p}(\Omega)$ for $k > m$. It uses the relative compactness of $W^{m+1,p}(\Omega)$ in $W^{m,p}(\Omega)$. This is implied by the Rellich-Kondrachov Theorem, which establishes that for $1 \leq p < \infty$, $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$ whenever $\Omega$ is a bounded domain with Lipschitz continuous boundary. See [1, p. 168] Theorem 6.3 for the full statement and proof of the Rellich-Kondrachov Theorem.

**Lemma 5.3.** Let $\Omega \subset \mathbb{R}^d$ be open and bounded with $C^{0,1}$ boundary and let $\varphi \in C^{1}(\Omega)$ be simply vanishing on $\partial \Omega$. If

$$
D(\varphi^m) = \{ u \in L^p(\Omega) : \varphi^m u \in W^{m+1,p}(\Omega) \}
$$

then $D(\varphi^m) \subset \subset L^p(\Omega)$.

**Proof.** Since $\Omega$ is bounded and $\partial \Omega$ is $C^{0,1}$, we can use the Rellich-Kondrachov Theorem to establish $W^{m+1,p}(\Omega) \subset \subset W^{m,p}(\Omega)$. Suppose $\{u_n\} \subset D(\varphi^m)$ is such that $\|u_n\|_{D(\varphi^m)} \leq 1$ for each $n$. Then $\|\varphi^m u_n\|_{W^{m+1,p}(\Omega)} \leq 1$ for each $n$, so there exists a subsequence that is convergent in $W^{m,p}(\Omega)$. After relabeling the convergent subsequence, we take this to be the entire sequence. Applying Hardy’s inequality (Theorem 5.1) yields

$$
\lim_{n,k \to \infty} \|u_n - u_k\|_{L^p(\Omega)} \leq \lim_{n,k \to \infty} c\|\varphi^m u_n - \varphi^m u_k\|_{W^{m,p}(\Omega)} = 0,
$$

completing the proof. □

Next we establish the embedding of $D(\varphi)$ in various Sobolev spaces. To do so, we will use the fact that when $\Omega \subset \mathbb{R}^d$ is open and bounded with $C^k$ boundary, the map

$$
u \mapsto u|_{\partial \Omega}
$$

from $C^k(\bar{\Omega})$ to $C^k(\partial \Omega)$ can be extended to a continuous surjective linear map from $W^{k,p}(\Omega)$ to $W^{k-1/p,p}(\partial \Omega)$ where $1 < p < \infty$ (see [3] p. 158) Theorem 3.79).

We would like to highlight the fact that the trace map $T$ on $W^{k,p}(\Omega)$ is defined on a Banach space and has range that is onto the Banach space $W^{k-1/p,p}(\partial \Omega)$. As for the nullspace of $T$, a classical result states that when $\partial \Omega$ is $C^1$, $T u = 0$ if and only if $u \in W_0^{1,p}(\Omega)$; (see [4] p. 259) Theorem 2, or [3] p. 138 Corollary 3.46).

Let $\Omega$ be an open and bounded set with $C^k$ boundary and let $T$ denote the continuous trace operator from $W^{k,p}(\Omega)$ onto $W^{k-1/p,p}(\partial \Omega)$. We will need a trace-like operator that is one-to-one. To define this new operator, first set $W_0 := W_0^{1,p}(\Omega) \cap W^{k,p}(\Omega) = N(T)$. Clearly $W_0$ is a closed subspace of $W^{k,p}(\Omega)$. Next let $W^k$ denote the quotient space $W^{k,p}(\Omega)/W_0$ and define the operator $\hat{T}$ from $\hat{W}^k$ to $W^{k-1/p,p}(\partial \Omega)$ as

$$
\hat{T}[u] = Tu, \quad [u] \in \hat{W}^k.
$$
The operator $\hat{T}$ is well-defined, linear, and one-to-one. To see that $\hat{T}$ is closed, take $\{u_n\} \subset \hat{W}^k$ such that $[u_n] \to [u]$ and $\hat{T}[u_n] \to y$ as $n \to \infty$. Then $\|[u_n] - [u]\|_{\hat{W}^k} \to 0$ implies the existence of a sequence $\{v_n\} \subset W_0$ such that

$$u_n - v_n \to u, \quad \text{in } W^{k,p}(\Omega).$$

Since $\hat{T}[u_n] \to y$, we know that $T[u_n] = T(u_n - v_n) \to y$ as $n \to \infty$. By the boundedness of $T$ we get $Tu = y$. This implies $\hat{T}[u] = y$ and proves that $\hat{T}$ is closed. Applying the Closed Graph Theorem shows that $\hat{T}$ is bounded. Moreover, the fact that $T$ is surjective implies that $\hat{T}$ is surjective as well. This tells us that $\hat{T}^{-1}$ exists and is a bounded linear operator from $W^{k-1/p,p}(\partial\Omega)$ onto $W^k$, by the Bounded Inverse Theorem.

We also need the following general Sobolev space theorem. We use $[a]$ to denote the integer part of $a$.

**Theorem 5.4** ([1] p. 85, Sobolev Imbedding). Let $\Omega \subset \mathbb{R}^d$ be open and bounded with a $C^{0,1}$ boundary. Assume $u \in W^{k,p}(\Omega)$, and that $kp > d$. Set

$$\kappa = k - \left[\frac{d}{p}\right] - 1, \quad \gamma = \begin{cases} 1 - \left(\frac{d}{p} - \left[\frac{d}{p}\right]\right), & \text{if } \frac{d}{p} \notin \mathbb{N} \\ \text{any number in } (0,1), & \text{otherwise.} \end{cases}$$

Then there exists a function $u^*$ such that $u^* = u$ a.e. and $u^* \in C^{\kappa,\gamma}(\Omega)$.

We can now establish the following lemma.

**Lemma 5.5.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open set and let $\varphi \in C^1(\bar{\Omega})$ be simply vanishing on $\partial\Omega$. Assume $k \in \mathbb{N}$ with $kp > d$ and that the boundary $\partial\Omega$ is $C^k$.

Then for every $u \in L^p(\Omega)$ where $\varphi u \in W^{k+1,p}(\Omega)$ we have

$$\varphi^{D^\alpha}u \in W^{0,1}_0(\Omega) \cap W^{k,p}(\Omega), \quad \text{for any } \alpha \text{ with } |\alpha| = 1.$$

**Proof.** The proof is divided into two claims.

**Claim 1:** If $|\alpha| = 1$ then $D^\alpha(\varphi u) = u D^\alpha \varphi$ on $\partial\Omega$.

Since $kp > d$ and $\varphi u \in W^{k+1,p}(\Omega)$, we know there exists a $\gamma \in (0,1)$ dependent on $d$ and $p$ such that

$$\varphi u \in C^{k-\left[\frac{d}{p}\right],\gamma}(\bar{\Omega}),$$

by Theorem 5.4. This shows $\varphi u \in C^1(\bar{\Omega})$. Also, since $u = \varphi^{-1}\varphi u$ whenever $\varphi \neq 0$, the fact that $\varphi \in C^1(\bar{\Omega})$ and is nonzero in $\Omega$ implies $u \in C^1(\Omega)$.

Now, since $\varphi$ is simply vanishing on $\partial\Omega$, we know that for any $y \in \partial\Omega$,

$$\lim_{x \to y} \frac{|x - y|}{|\varphi(x)|} = \lim_{x \to y} \frac{|x - y|}{|\varphi(x) - \varphi(y)|} = |\nabla \varphi(y)|^{-1},$$

where the limit is taken in $\Omega$. Note that $\nabla \varphi \neq 0$ on $\partial\Omega$, so that this is well defined. Next, given $\varphi u \in W^{1,p}(\Omega)$ and $u \in L^p(\Omega)$, we know $\varphi u \in W^{1,p}_0(\Omega)$ by Lemma 5.4.

We already know $\varphi u$ is continuous on $\Omega$, which implies $\varphi u = 0$ on $\partial\Omega$. Thus, for any $y \in \partial\Omega$ we can take any sequence in $\Omega$ that converges to $y$ and obtain

$$\lim_{x \to y} |u(x)| = \lim_{x \to y} \frac{|\varphi(x)u(x) - \varphi(y)u(y)|}{|\varphi(x) - \varphi(y)|} = |\nabla \varphi(y)||\nabla \varphi(y)|^{-1}.$$

By Leibniz’s rule, $D^\alpha(\varphi u) = u D^\alpha \varphi + \varphi D^\alpha u$ when $|\alpha| = 1$, so the above limit implies $D^\alpha(\varphi u) = u D^\alpha \varphi$ on $\partial\Omega$. 


Claim 2: The function \( \varphi D^\alpha u \) is in \( W^{1,p}_0(\Omega) \cap W^{k,p}(\Omega) \) whenever \( |\alpha| = 1 \).

Set \( W_0 := W^{1,p}_0(\Omega) \cap W^{k,p}(\Omega) \) and \( \tilde{W}^k := W^{k,p}(\Omega) / W_0 \). By assumption, \( \varphi u \in W^{k+1,p}(\Omega) \) so \( D^\alpha(\varphi u) \in W^{k,p}(\Omega) \) whenever \( |\alpha| = 1 \). This implies the coset \( [D^\alpha(\varphi u)] \) is in \( W^k \) and that its trace \( \tilde{T}[D^\alpha(\varphi u)] \) is in \( W^{k-1,p}(\partial \Omega) \). By claim 1,

\[
u D^\alpha \varphi = D^\alpha(\varphi u) \quad \text{on } \partial \Omega,
\]

which shows that \( uD^\alpha \varphi |_{\partial \Omega} \in W^{k-1,p}(\partial \Omega) \) and that \( \tilde{T}^{-1}(uD^\alpha \varphi |_{\partial \Omega}) \in \tilde{W}^k \). The one-to-one nature of \( \tilde{T}^{-1} \) implies \([D^\alpha(\varphi u)] = [u D^\alpha \varphi] \). Since these cosets are equal, there exists a function \( v \in W_0 \) such that

\[
u D^\alpha \varphi = D^\alpha(\varphi u) - v.
\]

But again, \( D^\alpha(\varphi u) = uD^\alpha \varphi + D^\alpha u \), so it must be the case that \( v = D^\alpha u \). We then conclude that \( \nu D^\alpha u \in W_0 = W^{1,p}_0(\Omega) \cap W^{k,p}(\Omega) \), completing the proof. \( \square \)

**Theorem 5.6.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set and let \( \varphi \in \mathcal{C}^1(\bar{\Omega}) \) be simply vanishing on \( \partial \Omega \). Assume \( k \in \mathbb{N} \) with \( kp > d \) and that \( \partial \Omega \) is \( \mathcal{C}^k \). Then \( u \in L^p(\Omega) \) and \( \varphi u \in W^{k+1,p}(\Omega) \) implies \( u \in W^{\kappa,p}(\Omega) \) where \( \kappa := k - \left\lfloor \frac{d}{p} \right\rfloor \).

**Proof.** Fix the multi-index \( \alpha \) with \( |\alpha| \leq \kappa \). Choose a finite sequence of multi-indices \( \{\alpha_n\}_{n \leq |\alpha|} \) each with \( |\alpha_n| = 1 \) such that \( \sum \alpha_n = \alpha \). By assumption, \( kp > d \) so we may apply Lemma 5.5 to \( u \) to obtain

\[(9) \quad \varphi D^{\alpha_1} u \in W^{1,p}_0(\Omega) \cap W^{k+1-|\alpha_1|,p}(\Omega).
\]

Applying Hardy’s inequality to \( \varphi D^{\alpha_1} u \) yields

\[(10) \quad D^{\alpha_1} u \in L^p(\Omega).
\]

Moreover, given \( |\alpha| \leq \kappa = k - \left\lfloor \frac{d}{p} \right\rfloor \), we know that

\[(11) \quad k + 1 - |\alpha_1| \geq k + 1 - |\alpha| \geq 1 + \left\lfloor \frac{d}{p} \right\rfloor > \frac{d}{p}.
\]

Since (9), (10), and (11) all hold, we may apply Lemma 5.5 to \( D^{\alpha_1} u \) to obtain \( \varphi D^{\alpha_1+\alpha_2} u \in W^{1,p}_0(\Omega) \cap W^{k-1,p}(\Omega) \). Another application of Hardy’s inequality shows \( D^{\alpha_1+\alpha_2} u \in L^p(\Omega) \). We continue inductively applying Lemma 5.5 and Hardy’s inequality at each step to finally show that

\[\varphi D^\alpha u \in W^{1,p}_0(\Omega) \cap W^{k+1-|\alpha|,p}(\Omega), \quad \text{and} \quad D^\alpha u \in L^p(\Omega).
\]

Since this applies to any multi-index \( \alpha \) with \( |\alpha| \leq \kappa \), we see that \( u \in W^{\kappa,p}(\Omega) \), completing the proof. \( \square \)

Iteratively applying the above theorem yields the following.

**Theorem 5.7.** Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set and let \( \varphi \in \mathcal{C}^1(\bar{\Omega}) \) be simply vanishing on \( \partial \Omega \). Fix \( m \in \mathbb{N} \) and \( 1 < p < \infty \). Assume \( k \in \mathbb{N} \) is such that

\[(12) \quad k > \frac{d}{p} + (m - 1)\left\lfloor \frac{d}{p} \right\rfloor,
\]

and that \( \partial \Omega \) is \( \mathcal{C}^k \). If \( u \in L^p(\Omega) \) and \( \varphi^m u \in W^{k+m,p}(\Omega) \) then

\[(13) \quad u \in W^{\kappa,p}(\Omega), \quad \text{where } \kappa := k - m\left\lfloor \frac{d}{p} \right\rfloor.
\]
Lemma 5.8 to show $D(\varphi)$ is closed in $L^p(\Omega)$. By Theorem 5.6, this implies $\varphi^{-1}u \in W^{k-1,p}(\Omega)$. If $m > 1$, we see that $\kappa_1 - 1 > p^{-1}d$, and we have $\varphi^{-2}u \in L^p(\Omega)$ and $\varphi^{-1}u \in W^{k-1,p}(\Omega)$. Thus we get $\varphi^{-2}u \in W^{k+1,p}(\Omega)$ by the same theorem. Continuing inductively, we apply Theorem 5.6 at each step to get $\varphi^{-j}u \in W^{k-j,p}(\Omega)$ for $j > 1$. When $j = m - 1$ we have $u \in L^p(\Omega)$ and $\varphi u \in W^{k-1,p}(\Omega)$. Since

$$k_{m-1} - 1 = k - (m - 1)\frac{d}{p} > \frac{d}{p},$$

we may apply Theorem 5.6 one more time to get $u \in W^{k,m,p}(\Omega)$, as desired. \hfill \Box

Remark 5.8. There is an implicit estimate accompanying Theorem 5.7. Assume $\varphi \in \mathcal{C}^1(\Omega)$ and consider the multiplication operator $\varphi^m : L^p(\Omega) \to W^{k+m,p}(\Omega)$ where $k$ satisfies (12). Theorem 5.7 tells us that $u \in D(\varphi^m)$ implies $u \in W^{k,p}(\Omega)$ where $k$ is given by (13). Thus, $D(\varphi^m) \subset W^{k,p}(\Omega)$. By the closedness of $\varphi^m$, $D(\varphi^m)$ is a Banach space with the operator norm. We can conclude that, for some constants $c_0, c_1 > 0$ and for all $u \in D(\varphi^m)$,

$$\|u\|_{W^{k,p}(\Omega)} \leq c_0 \|u\|_{L^p(\Omega)} + c_1 \|\varphi^m u\|_{W^{k+m,p}(\Omega)}$$

where (14) follows from Lemma 4.2 applied to the Banach spaces $D(\varphi^m)$ and $W^{k,p}(\Omega)$ and (15) from Hardy’s inequality.

5.2. The range of a vanishing operator. Having a closed range is a very useful property for linear operators. As we will see in section 6.2, it is often necessary for establishing basic properties of the spectrum. Showing the multiplication operator $u \mapsto \varphi u$ has closed range requires keeping track of the multiplicity of the roots of the function $\varphi$. This is formally established in the following result.

Theorem 5.9. Let $\Omega \subset \mathbb{R}^d$ be open and bounded with $\mathcal{C}^{0,1}$ boundary and assume the function $\varphi \in \mathcal{C}^k(\Omega)$ is simply vanishing on $\partial \Omega$. Then the range of the operator $u \mapsto \varphi^m u$ is closed in $W^{k,p}(\Omega)$ whenever $k \geq m$ and is not closed when $k < m$.

Proof. As usual, we treat $\varphi$ as an operator from some dense subset of $L^p(\Omega)$ to $W^{k,p}(\Omega)$. We start with the following.

Claim 1: If $k \geq m$ then the range of $\varphi^m$ is closed in $W^{k,p}(\Omega)$.

If $k = m$ then $R(\varphi^m) = W^{0,m,p}(\Omega)$ as discussed in Lemma 5.2, which clearly establishes the closedness of $R(\varphi^m)$ in $W^{m,p}(\Omega)$. If $k > m$ then we may apply Lemma 5.3 to show $D(\varphi^m) \subset \subset L^p(\Omega)$. Invoking Theorem 1.3 proves $\varphi^m$ is semi-Fredholm, which implies $R(\varphi^m)$ is closed in $W^{k,p}(\Omega)$.

Claim 2: If $k < m$ then the range of $\varphi^m$ is not closed in $W^{k,p}(\Omega)$.

We prove this claim by contradiction. Suppose $\varphi^m$ has closed range in $W^{k,p}(\Omega)$. This implies $\varphi^m$ is semi-Fredholm from $L^p(\Omega)$ to $W^{k,p}(\Omega)$. Since $\varphi^k$ is Fredholm from $L^p(\Omega)$ to $W^{k,p}(\Omega)$, and since $\varphi^m = \varphi^{m-k} \varphi^k$ is semi-Fredholm from $L^p(\Omega)$ to $W^{k,p}(\Omega)$, Lemma 1.3 implies $\varphi^{m-k}$ is semi-Fredholm from $W^{k,p}(\Omega)$ to $W^{k,p}(\Omega)$. Now, for any function $u \in \mathcal{C}_0(\Omega)$ we know that $v = \varphi^{k-m} u \in \mathcal{C}_0^k(\Omega)$, so $\varphi^{m-k} v \in W^{k,p}(\Omega)$.
\( \mathscr{C}_0^\infty(\Omega), \) implying \( \varphi^{m-k} \) is onto the subspace \( \mathscr{C}_0^\infty(\Omega) \). This implies

\[ \mathscr{C}_0^\infty(\Omega) \subset \text{R}(\varphi^{m-k}). \]

Since \( \text{R}(\varphi^{m-k}) \) is closed, we know that \( W_0^{k,p}(\Omega) \) is a subspace of \( \text{R}(\varphi^{m-k}) \). But we also know that \( \varphi^k \in W_0^{k,p}(\Omega) \), so there exists a function \( v \in W_0^{k,p}(\Omega) \) such that \( \varphi^{m-k} v = \varphi^k \), which implies \( v = \varphi^{2k-m} \). But \( \varphi^{2k-m} \) cannot be in \( W_0^{k,p}(\Omega) \) as Hardy’s inequality would then show

\[ \| \varphi^{k-m} \|_{L^p(\Omega)} = \| \varphi^{-k} \varphi^{2k-m} \|_{L^p(\Omega)} \leq c \| \varphi^{2k-m} \|_{W_0^{k,p}(\Omega)} < \infty. \]

This is our desired contradiction. \( \square \)

**Example 5.10 (The Legendre differential equation).** Set \( \Omega = (-1,1) \). Let us analyze the operator \( L \) given by

\[ Lu(x) = \frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} u(x) \right], \]

acting on \( L^p(\Omega) \), where as usual we assume \( 1 < p < \infty \). Let \( A \) denote the derivative operator on \( L^p(\Omega) \) and \( \varphi(x) = (1-x^2) \). The domain of \( A \) is \( W^{1,p}(\Omega) \), the nullspace of \( A \) is \( \text{span}\{1\} \), and the range of \( A \) is equal to \( L^p(\Omega) \). Since \( \varphi \) is simply vanishing on \( \partial \Omega \), we know the range of the multiplication operator \( u \mapsto \varphi u \) is equal to \( W_0^{1,p}(\Omega) \) by Lemma 5.2.

If \( u \in W^{1,p}(\Omega) \), then \( u \in \mathscr{C}^{0,1/p}(\Omega) \) by Sobolev Imbedding (Theorem 5.4). Thus, we can find a unique line \( l(x) \) such that \( u + l = 0 \) on \( \partial \Omega \), implying \( u + l \in W_0^{1,p}(\Omega) \). Since \( u \in W^{1,p}(\Omega) \) was arbitrary, this implies

\[ W^{1,p}(\Omega) = W_0^{1,p}(\Omega) \oplus \text{span}\{1, x\}. \]

If we take \( \tilde{A} \) to be the restriction of the derivative operator to \( W_0^{1,p}(\Omega) \), then \( \dim N(\tilde{A}) = 0 \) and \( \text{co-dim R}(\tilde{A}) = 1 \). Since \( A \), \( \tilde{A} \), and \( \varphi : L^p(\Omega) \to W_0^{1,p}(\Omega) \) are all Fredholm we know that \( L = \tilde{A} \varphi A \) is Fredholm, with

\[ \text{ind}(L) = \text{ind}(\tilde{A} \varphi A) = \text{ind}(\tilde{A}) + \text{ind}(\varphi) + \text{ind}(A) = -1 + 0 + 1 = 0, \]

and \( \text{N}(L) = \text{span}\{1\} \).

In terms of the domain of \( L \), we automatically get \( D(L) \subset D(A) = W^{1,p}(\Omega) \). The interesting thing to note is that \( L \) cannot be semi-Fredholm if \( D(L) \subset W^{2,p}(\Omega) \). To see this, first note that \( A \) maps \( W^{2,p}(\Omega) \) onto \( W^{1,p}(\Omega) \) and that the range of \( \varphi : W^{1,p}(\Omega) \to W_0^{1,p}(\Omega) \) is not closed. Since this implies that \( \varphi A : W^{2,p}(\Omega) \to W_0^{1,p}(\Omega) \) cannot be semi-Fredholm, we know that \( L = \tilde{A} \varphi A \) cannot be semi-Fredholm.

### 5.3. Matrix-valued functions.

One of our goals is to aid in the analysis of

\[ Lu = \text{div}(\Phi \nabla u), \]

when the matrix-valued function \( \Phi : \tilde{\Omega} \to \mathbb{C}^{d \times d} \) is positive semi-definite for each \( x \in \Omega \). With that in mind, this section focuses on the multiplication operator \( u \mapsto \Phi u \) where \( \Phi \in \mathscr{C}(\tilde{\Omega}; \mathbb{C}^{d \times d}) \) and \( u(x) \in \mathbb{C}^d \) for almost every \( x \in \Omega \). As we will see shortly, the properties that were established for the multiplication operator \( u \mapsto \varphi u \) apply for the multiplication operator \( u \mapsto \Phi u \) as well.

In order for the operator \( L \) defined in (16) to be uniformly elliptic, the matrix \( \Phi : \tilde{\Omega} \to \mathbb{C}^{d \times d} \) must be uniformly positive definite. This section, like the ones before it, focus on the violation of this positivity assumption. Specifically, we assume \( \Phi \)
is vanishing of order \(m\) (recall Definition 3.4). Another way to express this is as follows: for each fixed \(V \subset \subset \Omega\) we have

\[
\inf_{x \in V} \inf_{\nu \in \mathbb{C}^{d}} \tilde{v} \cdot \Phi(x)u \geq c_{V}|v|,
\]

where \(c_{V} > 0\) is the smallest eigenvalue of \(\Phi(x)\) for \(x \in V\). Moreover, the speed at which \(c_{V}\) goes to zero is proportional to \(a^{m}\) where \(a = \inf_{y \in \partial V} \text{dist}(y, \partial \Omega)\).

We take \(L^{p}(\Omega^{d})\) to be the space of all measurable functions \(u = (u_{1}, \ldots, u_{d})\) such that \(u_{i} \in L^{p}(\Omega)\) for \(i = 1, \ldots, d\). The norm of \(L^{p}(\Omega^{d})\) is taken to be

\[
\|u\|_{L^{p}(\Omega^{d})} := \sum_{i=1}^{d} \|u_{i}\|_{L^{p}(\Omega)}.
\]

In other words,

\[
L^{p}(\Omega^{d}) = \underbrace{L^{p}(\Omega) \times \cdots \times L^{p}(\Omega)}_{d \text{ copies}}.
\]

The space \(W^{k,p}(\Omega^{d})\) is defined as the subset of \(u = (u_{1}, \ldots, u_{d}) \in L^{p}(\Omega^{d})\) where \(u_{i} \in W^{k,p}(\Omega)\) for each \(i = 1, \ldots, d\). We assume \(\Phi : L^{p}(\Omega^{d}) \to W^{k,p}(\Omega^{d})\) for some \(k \in \mathbb{Z}_{+}\).

**Theorem 5.11.** Let \(\Omega \subset \mathbb{R}^{d}\) be open and bounded with \(C^{0,1}\) boundary and assume \(\Phi \in C^{k}(\bar{\Omega}; \mathbb{C}^{d \times d})\) is vanishing of order \(m\). Then the range of the operator \(u \mapsto \Phi u\) is closed in \(W^{k,p}(\Omega^{d})\) whenever \(k \geq m\) and is not closed when \(k < m\).

**Proof.** We know there exists \(U \in C^{k}(\bar{\Omega}; \mathbb{C}^{d \times d})\) and \(D \in C^{k}(\bar{\Omega}; \mathbb{R}^{d \times d})\) such that \(\Phi = UDU^{*}\), where \(D = \text{diag}(\varphi_{1}, \ldots, \varphi_{d})\) and \(\varphi_{i} \in C^{k}(\bar{\Omega})\) for each \(i = 1, \ldots, d\). Since \(U\) is one-to-one and onto \(L^{p}(\Omega^{d})\), it suffices to prove the claim for the operator \(D\). Now, by our definition of \(W^{k,p}(\Omega^{d})\), it must be the case that \(R(D)\) is closed in \(W^{k,p}(\Omega^{d})\) if and only if the multiplication operators \(u \mapsto \varphi_{i}u\) have closed range in \(W^{k,p}(\Omega)\) for each \(i = 1, \ldots, d\). With this in mind, we simply apply Theorem 5.9 for each diagonal function \(\varphi_{i}\), yielding the desired conclusion.

**Theorem 5.12.** Let \(\Omega \subset \mathbb{R}^{d}\) be a bounded open set and let \(\Phi \in C^{k}(\bar{\Omega}; \mathbb{C}^{d \times d})\) be vanishing of order \(m\). Assume \(k \in \mathbb{N}\) is such that

\[
k > \frac{d}{p} + (m - 1) \left\lfloor \frac{d}{p} \right\rfloor,
\]

and that the boundary \(\partial \Omega\) is \(C^{k}\). If \(u \in L^{p}(\Omega^{d})\) and \(\Phi u \in W^{k+m,p}(\Omega^{d})\) then

\[
u \in W^{k,p}(\Omega^{d}), \quad \text{where } \kappa := k - m \left\lfloor \frac{d}{p} \right\rfloor,
\]

and there exists a \(c > 0\), independent of \(u\), such that

\[
\|\nu\|_{W^{\kappa,p}(\Omega^{d})} \leq c \|\Phi u\|_{W^{k+m,p}(\Omega^{d})}.
\]

**Proof.** We know there exists \(U \in C^{k}(\bar{\Omega}; \mathbb{C}^{d \times d})\) and \(D \in C^{k}(\bar{\Omega}; \mathbb{R}^{d \times d})\) such that \(\Phi = UDU^{*}\), where \(D = \text{diag}(\varphi_{1}, \ldots, \varphi_{d})\) and \(\varphi_{i} \in C^{k}(\bar{\Omega})\) for each \(i = 1, \ldots, d\). As in the above theorem, it suffices to prove the claim for the operator \(D\).

Given \(u = (u_{1}, \ldots, u_{d}) \in L^{p}(\Omega^{d})\) and \(Du \in W^{k+m,p}(\Omega^{d})\) then we have \(u_{i} \in L^{p}(\Omega)\) and \(\varphi_{i}u_{i} \in W^{k+m,p}(\Omega)\) for each \(i = 1, \ldots, d\). If \(\varphi_{i} > 0\) on \(\Omega\) then \(u_{i} \in W^{k+m,p}(\Omega)\), and if \(\varphi_{i}\) is vanishing of order \(j \leq m\) we apply Theorem 5.7 to get

\[
u_{i} \in W^{\kappa_{j},p}(\Omega), \quad \text{where } \kappa_{j} := k - j \left\lfloor \frac{d}{p} \right\rfloor.
\]

In either case, \(\nu_{i} \in W^{\kappa,p}(\Omega)\) for each \(i = 1, \ldots, d\). The proof of inequality (19) mirrors that of Remark 5.8 and is omitted.
6. Differential Operators Composed with Vanishing Operators

In this section we examine differential operators that are composed with vanishing operators. By ‘differential operator’ we mean any operator that is closed on the subspace $W^{k,p}(\Omega) \subset L^p(\Omega)$, $k \geq 1$, and maps to either $L^p(\Omega)$ or $L^p(\Omega^2)$. We pay particular attention to linear differential operators that are Fredholm or semi-Fredholm. Many of these results use compactness of nested Sobolev spaces.

6.1. Compactness. One of the salient features of the Sobolev space $W^{k,p}(\Omega)$ is its compactness relationship with the ambient space $L^p(\Omega)$. In this section, we explore the implications of compactness on the composition of differential operators with vanishing functions.

We start with the following general result for Fredholm operators.

Theorem 6.1. Let $X$ and $Y$ be Banach spaces. If $A \in \mathcal{F}(X, Y)$ then $D(A) \subset \subset X$ if and only if its pseudo-inverse is compact from $Y$ to $X$.

Proof. Since $A$ is closed we may equip $D(A)$ with the $A$-norm and convert it into a Banach space, which we call $W$.

Claim 1: If $A \in \mathcal{F}(X, Y)$ with $W \subset \subset X$ then the pseudo-inverse of $A$ is compact from $Y$ to $X$.

Given that $A$ is Fredholm from $X$ to $Y$, we know it is also Fredholm from $W$ to $Y$. Let $A_0$ denote the pseudo-inverse of $A : W \to Y$, and let $\iota : W \to X$ denote the inclusion map from $W$ to $X$. The assumption that $W \subset \subset X$ tells us that $\iota$ is compact, which implies $\iota A_0 : Y \to X$ is compact as well since $\tilde{A}_0 \in \mathcal{B}(Y, W)$. If we let $A_0$ denote the pseudo-inverse of $A : X \to Y$ then we see that $A_0 = \iota \tilde{A}_0$, so $A_0$ is compact.

Claim 2: If $A \in \mathcal{F}(X, Y)$ and the pseudo-inverse of $A$ is compact from $Y$ to $X$ then $W \subset \subset X$.

We are told $A$ is Fredholm, so we know $N(A)$ is finite dimensional and that there exists a closed subspace $X_0 \subset X$ such that $X = X_0 \oplus N(A)$. Suppose \{x_n\} $\subset D(A)$ with $\|x_n\|_{D(A)} \leq c$. Then for each $n$ we have the decomposition $x_n = a_n + b_n$ where $a_n \in X_0$ and $b_n \in N(A)$. Since $\|a_n\|_{D(A)} \leq c$ for all $n$, \{Aa_n\} is a bounded sequence in $Y$. Given that $A_0$, the pseudo-inverse of $A$, is compact from $Y$ to $X$, there exists a subsequence of \{a_n\} = \{A_0 Aa_n\} that is convergent in $X$. Also, since \{b_n\} is bounded and $N(A)$ is finite dimensional, every subsequence of \{b_n\} has a further subsequence that is convergent. Thus, we can find a subsequence of \{b_n\} along the convergent subsequence of \{a_n\} that is convergent. With this we can conclude that \{x_n\} = \{a_n + b_n\} contains a convergent subsequence in $X$. This proves the claim and completes the proof of the theorem.

As a consequence of Theorem 6.1 we have the following.

Theorem 6.2. Let $X$, $Y$, and $Z$ be Banach spaces. Suppose $A \in \mathcal{F}(X, Y)$ where $D(A) \subset \subset X$. If $B \in \mathcal{C}(Y, Z)$ but is not semi-Fredholm then $BA$ is not closed on its natural domain.

Proof. The proof is by contradiction. Assume $BA$ is closed on its natural domain, $D(BA) = \{x \in D(A) : Ax \in D(B)\}$. 

$D(BA) = \{x \in D(A) : Ax \in D(B)\}$.
Claim 1: There exists a $c > 0$ such that $\|x\|_{D(A)} \leq c \|x\|_{D(BA)}$ holds for all $x \in D(BA)$. If $BA$ was closed on $D(BA)$ then $D(BA)$ would be a Banach space with the $BA$-norm. Since $A$ is Fredholm it must be closed on its domain $D(A)$, so $D(A)$ is also a Banach space with the $A$-norm. We know that $D(BA) \subset D(A)$, so by Lemma 4.2 there exists a $c > 0$ such that $\|x\|_{D(A)} \leq c \|x\|_{D(BA)}$ whenever $x \in D(BA)$.

Claim 2: There exists a sequence that converges in $D(BA)$ but does not converge in $D(A)$.

Since $B$ is not semi-Fredholm, there exists a bounded sequence $\{x_n\} \subset D(B)$ such that $\{Bx_n\}$ converges but $\{x_n\}$ has no convergent subsequence. Given that $A$ is Fredholm we know $A$ has a pseudo-inverse, which we denote by $A_0$. We then set $y_n = A_0x_n$ and notice that

$$Ay_n = AA_0x_n = (I - K)x_n,$$

where $K$ is a projection into some finite dimensional subspace of $Y$. Since $\{x_n\}$ has no convergent subsequence and $K$ projects to a finite dimensional subspace, $\{Kx_n\}$ is eventually zero. Thus, $\{Ay_n\}$ has no convergent subsequence and $\{BAy_n\}$ converges. Since $D(A) \subset X$, we know that $A_0$ is compact by Theorem 6.1 so $\{y_n\}$ has a convergent subsequence in $X$ (which, after relabeling, we take to be the entire sequence). Using claim 1, we have

$$\|y_m - y_n\|_{D(A)} = \|y_m - y_n\|_X + \|Ay_m - Ay_n\|_Y \leq c\|y_m - y_n\|_{D(BA)} = c\|y_m - y_n\|_X + c\|BAy_m - BAy_n\|_Z.$$ 

We have established that $\{BAy_n\}$ and $\{y_n\}$ converge in $Z$ and $X$ respectively, so $\{y_n\}$ is convergent in $D(BA)$. But we know that $\{Ay_n\}$ does not converge in $Y$, hence $\{y_n\}$ cannot converge in $D(A)$. This is the desired contradiction.

If $\varphi \in C^1(\Omega)$ is simply vanishing, then by Theorem 5.9 the range of the multiplication operator $u \mapsto \varphi u$ is not closed in $L^p(\Omega)$. Thus, $\varphi$ cannot be semi-Fredholm from $L^p(\Omega)$ to $L^p(\Omega)$. The above theorem then says $\varphi^mA$ is never closed on its natural domain for any $m > 0$. However, we can partially make up for this loss by showing that $\varphi^mA$ is closable. Before we begin we will need a few more tools from classical functional analysis.

The adjoint operator of $A$, denoted $A^*$, is a map from the dual $Y^*$ to $X^*$, where

$$(20) \quad A^*y^*(x) = y^*(Ax), \quad \text{ for all } x \in D(A),$$

and some $y^* \in Y^*$. The set of appropriate $y^* \in Y^*$ for which (20) holds is $D(A^*)$.

The following two results are needed.

Lemma 6.3 (41 Lemma 131, p. 137). Let $X$ be a normed vector space. Suppose that a sequence $\{x_n\} \subset X$ is bounded, and $\lim l^*(x_n) = l^*(x)$ for each $l^* \in V$ where $V$ is a dense subset of $X^*$. Then the sequence $\{x_n\}$ converges to $x$ weakly.

Theorem 6.4 (47, Theorems 7.35 and 7.36, p. 178). Let $X$, $Y$, and $Z$ be Banach spaces, and assume that $A \in \mathcal{C}(X,Y)$ where $\text{R}(A)$ is closed in $Y$ with finite codimension. Let $B$ be a densely defined operator from $Y$ to $Z$. Then $(BA)^*$ exists, $(BA)^* = A^*B^*$, and both $(BA)^*$ and $BA$ are densely defined.

With the above lemma and theorem, we can conclude the following.
Theorem 6.5. Let $X$, $Y$, and $Z$ be Banach spaces. Suppose $A \in \mathcal{F}(X,Y)$ and $B$ is a densely defined linear operator from $Y$ to $Z$. Then $BA$ is closable.

Proof. Since $A$ is Fredholm, the domains of $BA$ and $(BA)^*$ are both dense, by Theorem 6.4. Suppose we have a sequence $\{x_n\} \subset D(BA)$ where $x_n \to 0$ and $BAx_n \to z$ as $n \to \infty$. Then for each $w^* \in D((BA)^*)$,

$$w^*(z) = \lim_{n \to \infty} w^*(BAx_n) = \lim_{n \to \infty} (BA)^*w^*(x_n) = 0.$$ 

Since $D((BA)^*)$ is dense in $X^*$ and $BAx_n$ is bounded, this implies that $BAx_n \to 0$ weakly as $n \to \infty$, by Lemma 6.3. We know that weak limits must coincide with strong limits so $z = 0$. Thus $BA$ is closable.

Theorems 6.4 and 6.3 yield the following result.

Theorem 6.6. Let $\Omega \subset \mathbb{R}^d$ be an open and bounded set with $C^0,1$ boundary. Let $\varphi \in C^1(\Omega)$ be simply vanishing on $\partial \Omega$. If $A : L^p(\Omega) \to L^p(\Omega)$ is Fredholm with $D(A) \subset W^{1,p}(\Omega)$, then $\varphi^mA$, for $m \geq 1$, is not closed on its natural domain but is closable.

Proof. Theorem 5.9 tells us that the range of the multiplication operator $u \mapsto \varphi^mu$ is not closed in $L^p(\Omega)$, so $\varphi^m$ is not semi-Fredholm from $L^p(\Omega)$ to $L^p(\Omega)$. Since $\Omega$ is bounded with $C^0,1$ boundary, this implies $D(A) \subset W^{1,p}(\Omega)$ by the Rellich-Kondrachov Theorem. Theorem 6.4 then tells us that $\varphi^mA$ is not closed on its natural domain, but by Theorem 6.4 $\varphi^mA$ is closable.

Our next result makes use of the following theorem.

Theorem 6.7 (117 Theorem 7.22, p. 170). Let $X$ and $Y$ be Banach spaces. If $A \in \mathcal{F}(X,Y)$ and $Y$ is reflexive, then $A^* \in \mathcal{F}(Y^*,X^*)$ and $\text{ind}(A^*) = -\text{ind}(A)$.

We now have the following:

Theorem 6.8. Let $X$ and $Y$ be Banach spaces where $Y$ is also reflexive. If $A \in \mathcal{F}(X,Y)$ with $D(A) \subset \subset X$ then $D(A^*) \subset \subset Y^*$.

Proof. By Theorem 6.7 we know that $A^* \in \mathcal{F}(Y^*,X^*)$. Also, if $x^* \in D(A^*)$ then

$$A^*x^*(x) = A^*x^*(A_0x) = x^*(AA_0x) = x^*(x - K_2x) = (I - K_2^*)(x^*(x)).$$

Thus, from (21) and Theorems 6.4 and 5.5 we conclude:

$$A^*A_0^* = (A_0A)^* = I - K_1^*, \quad A_0^*A^* = (AA_0)^* = I - K_2^*,$$

which implies $A_0^*$ is the pseudo-inverse of $A^*$. Since $A \in \mathcal{F}(X,Y)$, and $D(A) \subset \subset X$, we know that the pseudo-inverse $A_0$ is compact from $Y$ to $X$ by Theorem 6.1. Given $A_0$ is compact from $Y$ to $X$, we know $A_0^*$ is compact from $X^*$ to $Y^*$. If we then apply Theorem 6.4 to $A^*$ we get $D(A^*) \subset \subset Y^*$.

6.2. The Spectrum. Let $X$ be a Banach space and $A$ be a densely defined linear operator from $X$ to $X$. The resolvent set of $A$, denoted $\rho(A)$, is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda I$ has a bounded inverse. The complement of $\rho(A)$ in $\mathbb{C}$ is called the spectrum of $A$, and is denoted as $\sigma(A)$. We let $\sigma_p(A)$ denote the point spectrum of $A$:

$$\sigma_p(A) := \{\lambda \in \mathbb{C} : Ax = \lambda x \text{ for some } x \in D(A)\}.$$
We define the essential spectrum as:
\[ \sigma_{\text{ess}}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K), \]
where \( \mathcal{K}(X) \) is the set of all compact operators on \( X \). This set is sometimes referred to as Schechter’s essential spectrum. Another useful characterization of the essential spectrum is given in the theorem below.

**Theorem 6.9** ([17] Theorem 7.27, p. 172). Let \( X \) be a Banach space and assume \( A \in \mathcal{C}(X) \). Then \( \lambda \notin \sigma_{\text{ess}}(A) \) if and only if \( A - \lambda \in \mathcal{F}(X) \) and \( \text{ind}(A - \lambda) = 0 \).

We will also need the notion of relatively compact operators. If \( A \in \mathcal{C}(X,Y) \), an operator \( B : X \to Z \) is called compact relative to \( A \) if \( B \) is compact from the Banach space \( D(A) \) to \( Z \). The following theorem is a more robust version of the Fredholm Alternative since it is stated for any Fredholm operator \( A \) (not just the identity operator) and the perturbations to \( A \) can be any operator that is compact relative to \( A \). A proof of this theorem can be found in [10] p. 281] Theorem 1, or [17] p. 162] Theorem 7.10.

**Theorem 6.10** ([10] Theorem 1, p. 281). If \( A \in \mathcal{F}(X,Y) \) and \( B \) is compact relative to \( A \) then \( A + B \in \mathcal{F}(X,Y) \) and \( \text{ind}(A + B) = \text{ind}(A) \).

**Remark 6.11.** With the help of Theorems 6.9 and 6.10, we see that the essential spectrum is invariant under relatively compact perturbations. Given that the identity map on \( X \) is compact relative to \( A \in \mathcal{C}(X) \) whenever \( D(A) \subset \subset X \), we know that either \( \sigma_{\text{ess}}(A) = \emptyset \) or \( \sigma_{\text{ess}}(A) = \mathbb{C} \). This fact makes calculating the essential spectrum of differential operators relatively easy whenever we can use the Rellich-Kondrachov Theorem.

**Theorem 6.12.** Let \( X \) be a Banach space, \( A \in \mathcal{C}(X) \), and \( B \in \mathcal{B}(X) \) be a one-to-one operator where \( D(A) \not\subset R(B) \). If \( D(AB) \) is dense in \( X \) with \( D(AB) \subset \subset X \) and \( \rho(A) \) is nonempty then \( \sigma_{\text{ess}}(AB) = \mathbb{C} \).

**Proof.** The proof is by contradiction. Suppose \( \sigma_{\text{ess}}(AB) \neq \mathbb{C} \). As mentioned in Remark 6.11, \( \sigma_{\text{ess}}(AB) \neq \mathbb{C} \) implies \( \sigma_{\text{ess}}(AB) = \emptyset \) since \( D(AB) \subset \subset X \). This implies \( AB \in \mathcal{F}(X) \) and \( \text{ind}(AB) = 0 \) by Theorem 6.9.

Since \( D(AB) \subset \subset X \), any bounded operator on \( X \) is compact relative to \( AB \). In particular, \( B \) is compact relative to \( AB \). By Theorem 6.10 Fredholm operators and their indices are invariant under relatively compact perturbations. Thus, for any \( \eta \in \mathbb{C} \) we have \( AB - \eta B \in \mathcal{F}(X) \) and
\[ \text{ind}(AB) = \text{ind}(AB - \eta B) = 0. \]

Now, if \( \eta \in \rho(A) \) then \( N(A - \eta) = \{0\} \) and \( R(A - \eta) = X \), implying \( A - \eta \) maps \( D(A) \) to \( X \). But since \( B \) does not map to all of \( D(A) \) we get \( R((A - \eta)B) \neq X \). In other words
\[ \text{co}-\text{dim} R(AB - \eta B) > 0. \]

Recall that \( N(B) = \{0\} \) by assumption, and \( N(A - \eta) = \{0\} \) when \( \eta \in \rho(A) \), which gives \( N((A - \eta)B) = \{0\} \). Thus,
\[ \text{ind}(AB) = \text{ind}(AB - \eta B) = \dim N(AB - \eta B) - \text{co}-\text{dim} R(AB - \eta B) < 0, \]
which contradicts the fact that \( \text{ind}(AB) = 0 \). \( \square \)
The proof of the above theorem yields the following corollary.

**Corollary 6.13.** Let $X$ be a Banach space, $A \in \mathcal{C}(X)$, and $B \in \mathcal{B}(X)$ be a one-to-one operator where the range of $B$ is not the entirety of $D(A)$. If $AB$ is Fredholm with $D(AB) \subset \subset X$ and $\rho(A)$ is nonempty then $\text{ind}(AB) < 0$.

In some cases, we are concerned with the adjoint operator $\varphi^mA^*$ instead $A\varphi^m$. For example, one might be interested in the spectral properties of the operator $L$ given by $Lu = -(1 - x^2)u_{xx}$ on $\Omega = (-1, 1)$. This operator is the adjoint of $A\varphi u = -((1 - x^2)u)_{xx}$ where $A$ is the Laplacian on $L^2(\Omega)$ and $\varphi(x) = (1 - x^2)$. In this case, we have the following theorem.

**Theorem 6.14.** Let $\Omega \subset \mathbb{R}^d$ be open and bounded with $\mathcal{C}^{0,1}$ boundary, $m, k \in \mathbb{N}$ with $m < k$, and let $A$ be densely defined on $L^p(\Omega)$. Assume

(a) $A$ is closed on $L^p(\Omega)$ and $D(A) \subset W^{k,p}(\Omega)$,
(b) There exists a $u \in D(A)$ such that $u \notin W^{m,p}_0(\Omega) \cap W^{k,p}(\Omega)$.
(c) The resolvent set $\rho(A)$ is non-empty.

If $\varphi \in \mathcal{C}^k(\Omega)$ is simply vanishing then

$$\sigma_{\text{ess}}(A\varphi^m) = \sigma_{\text{ess}}(\varphi^mA^*) = \mathbb{C}.$$ 

Moreover, if either $A\varphi^m$ or $\varphi^mA^*$ is Fredholm then

$$\sigma_{\rho}(\varphi^mA^*) = \mathbb{C}.$$ 

**Remark 6.15.** By Theorem 6.10, $\varphi^mA^*$ is never closed on its natural domain when $A$ is Fredholm. Thus, we examine its closure since statements about the essential spectrum are uninformative for operators that are not closed.

**Proof.** As usual, let $\varphi^m$ denote the multiplication operator $u \mapsto \varphi^m u$ from $L^p(\Omega)$ to $W^{k,p}(\Omega)$. The proof is broken into 4 claims.

**Claim 1:** $A\varphi^m$ is closed on its natural domain and $D(A\varphi^m) \subset \subset L^p(\Omega)$.

Given $\partial \Omega$ is $\mathcal{C}^{0,1}$, we see that $D(A) \subset W^{k,p}(\Omega) \subset \subset L^p(\Omega)$ by the Rellich-Kondrachov Theorem. Since $A$ is closed on $D(A)$, $A$ must be semi-Fredholm on $L^p(\Omega)$ by Theorem 4.3. With the assumption that $m < k$, Lemma 5.3 implies $D(\varphi^m) \subset \subset L^p(\Omega)$, and applying Theorem 6.3 yields $\varphi^m$ is semi-Fredholm from $L^p(\Omega)$ to $W^{k,p}(\Omega)$. Since both $A$ and $\varphi^m$ are semi-Fredholm, $A\varphi^m$ is closed on its natural domain. The fact that $D(A\varphi^m) \subset \subset L^p(\Omega)$ completes the proof of the claim.

**Claim 2:** $\sigma_{\text{ess}}(A\varphi^m) = \mathbb{C}$

Applying Lemma 5.2 to $\varphi^m$ shows that $R(\varphi^m) = W^{m,p}_0(\Omega) \cap W^{k,p}(\Omega)$. This and assumption (b) implies $\varphi^m$ does not map to all of $D(A)$. Given claim 1, $\rho(A)$ is non-empty, and the multiplication operator $\varphi^m : L^p(\Omega) \rightarrow L^p(\Omega)$ is bounded, we may apply Theorem 6.12 to prove $\sigma_{\text{ess}}(A\varphi^m) = \mathbb{C}$.

**Claim 3:** If $A\varphi^m$ is not Fredholm then $\sigma_{\text{ess}}(\varphi^mA^*) = \mathbb{C}$.

Fix $\lambda \in \mathbb{C}$. Claim 1 implies the identity is compact relative to $A\varphi^m$. Since $A\varphi^m$ is not Fredholm, $A\varphi^m - \lambda$ cannot be Fredholm by Theorem 6.10. By Theorem 6.7 this implies $\varphi^mA^* - \lambda$ is not Fredholm. Applying Theorem 6.9 to $\varphi^mA^* - \lambda$ shows $\lambda \in \sigma_{\text{ess}}(\varphi^mA^*)$. Noting that $\lambda \in \mathbb{C}$ was arbitrary completes the proof of the claim.
Claim 4: If $A\varphi^n$ or $\varphi^n A^*$ is Fredholm then $\sigma_{ess}(\varphi^n A^*) = \sigma_p(\varphi^n A^*) = \mathbb{C}$.

Given $L^p(\Omega)$ is reflexive, $A\varphi^n$ is Fredholm if and only if $\varphi^n A^*$ is Fredholm by Theorem 6.7. Since $\rho(A)$ is non-empty and $D(A\varphi^n) \subset L^p(\Omega)$, Corollary 6.13 then implies $\text{ind}(A\varphi^n) < 0$. Thus,

$$\text{ind}(\varphi^n A^*) = -\text{ind}(A\varphi^n) > 0.$$  

Moreover, by Theorem 6.8 and claim 1, we have that

$$D(\varphi^n A^*) \subset L^p(\Omega),$$

so for any $\lambda \in \mathbb{C}$, we can conclude

$$\text{ind}(\varphi^n A^* - \lambda) = \text{ind}(\varphi^n A^*) > 0.$$  

From (22) we have $\dim N(\varphi^n A^* - \lambda) > 0$, which implies $\lambda \in \sigma_p(\varphi^n A^*)$. By Theorem 6.9 (22) also implies $\lambda \in \sigma_{ess}(\varphi^n A^*)$. Since $\lambda \in \mathbb{C}$ was arbitrary, we are done. 

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