We describe the structure of the extended Clifford Group (defined to be the group consisting of all operators, unitary and anti-unitary, which normalize the generalized Pauli group (or Weyl-Heisenberg group as it is often called)). We also obtain a number of results concerning the structure of the Clifford Group proper (i.e., the group consisting just of the unitary operators which normalize the generalized Pauli group). We then investigate the action of the extended Clifford group operators on symmetric informationally complete POVMs (or SIC-POVMs) covariant relative to the action of the generalized Pauli group. We show that each of the fiducial vectors which has been constructed so far (including all the vectors constructed numerically by Renes et al) is an eigenvector of one of a special class of order 3 Clifford unitaries. This suggests a strengthening of a conjecture of Zauner’s. We give a complete characterization of the orbits and stability groups in dimensions 2–7. Finally, we show that the problem of constructing fiducial vectors may be expected to simplify in the infinite sequence of dimensions $7, 13, 19, 21, 31, \ldots$. We illustrate this point by constructing exact expressions for fiducial vectors in dimensions 7 and 19.
1. Introduction

The statistics of an arbitrary quantum measurement are described by a positive operator valued measure, or POVM (Davies [1], Busch et al [2], Peres [3], Nielsen and Chuang [4] and references cited therein). Suppose the measurement has only a finite number of distinct outcomes. Then the corresponding POVM assigns to each outcome $i$ the positive operator $\hat{E}_i$ with the property that $\text{Tr}(\hat{E}_i\hat{\rho})$ is the probability of obtaining outcome $i$ (where $\hat{\rho}$ is the density operator). Since $\sum_i \text{Tr}(\hat{E}_i\hat{\rho}) = 1$ for all $\hat{\rho}$ we must have $\sum_i \hat{E}_i = 1$.

A POVM is said to be informationally complete if the probabilities $\text{Tr}(\hat{E}_i\hat{\rho})$ uniquely determine the density operator $\hat{\rho}$. The concept of informational completeness is originally due to Prugovecki [5] (also see Busch [6], Busch et al [2], d’Ariano et al [7], Flamini et al [8], Finkelstein [9], and references cited therein). It has an obvious relevance to the problem of quantum state determination. It also plays an important role in Caves et al’s [10, 11, 12, 13] Bayesian approach to the interpretation of quantum mechanics, and in Hardy’s [14, 15] proposed axiomatization.

Suppose the Hilbert space has finite dimension $d$. Then it is easily seen that an informationally complete POVM must contain at least $d^2$ distinct operators $\hat{E}_i$. An informationally complete POVM is said to be symmetric informationally complete (or SIC) if it contains exactly this minimal number of distinct operators and if, in addition,

1. $\lambda\hat{E}_i$ is a one dimensional projector for all $i$ and some fixed constant $\lambda$.
2. The overlap $\text{Tr}(\hat{E}_i\hat{E}_j)$ is the same for every pair of distinct labels $i, j$.

It is straightforward to show that this is equivalent to the requirement that, for each $i$,

$$\hat{E}_i = \frac{1}{d} |\psi_i\rangle \langle \psi_i|$$

where the $d^2$ vectors $|\psi_i\rangle$ satisfy

$$|\langle \psi_i| \psi_j \rangle| = \begin{cases} 1 & i = j \\ \frac{1}{\sqrt{d+1}} & i \neq j \end{cases}$$

SIC-POVMs were introduced in a dissertation by Zauner [16], and in Renes et al [17]. Wootters [18], Bengtsson and Ericsson [19, 20] and Grassl [21] have made further contributions. There appear to be some intimate connections with the theory of mutually unbiased bases [18, 22, 23], finite affine planes [18, 19, 20], and polytopes [19, 20].

If SIC-POVMs existed in every finite dimension (or, failing that, in a sufficiently large set of finite dimensions) they would constitute a naturally distinguished class of POVMs which might be expected to have many interesting applications to quantum tomography, cryptography and information theory generally. They would also be obvious candidates for the “fiducial” or “standard” POVMs featuring in the work of Fuchs [13] and Hardy [14, 15].

The question consequently arises: is it in fact true that SIC-POVMs exist in every finite dimension? The answer to this question is still unknown. Analytic solutions to Eqs. [2] have been constructed in dimensions 2, 3, 4, 5, 6 and 8. Moreover Renes et al [17] have constructed numerical solutions in dimensions 5 to 45 (the actual vectors can be downloaded from their website [24]). So one may plausibly speculate that SIC-POVMs exist in every finite dimension. But it has not been proved.

The SIC-POVMs which have so far been explicitly described in the literature are all covariant under the action of the generalized Pauli group (or Weyl-Heisenberg

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1Renes et al [17] mention that they have constructed numerical solutions which are covariant under the action of other groups, but they do not give any details.
group, as it is often called). It is therefore natural to investigate their behaviour under the action of the extended Clifford group. The Clifford group proper is defined to be the normalizer of the generalized Pauli group, considered as a subgroup of U(d) (the group consisting of all unitary operators in dimension d). It is relevant to a number of areas of quantum information theory, and it has been extensively discussed in the literature \cite{25, 26, 27, 28, 29, 30, 31}. Its relevance to the SIC-POVM problem has been stressed by Grassl \cite{21}. As Grassl notes, it is related to the Jacobi group \cite{32}, which has attracted some notice in the pure mathematical literature. We define the extended Clifford group to be the group which results when the Clifford group is enlarged, so as to include all anti-unitary operators which normalize the generalized Pauli group. As we will see, this enlargement is essential if one wants to achieve a full understanding of the SIC-POVM problem.

In Sections 2–4 we give a self-contained account of the structure of the extended Clifford group. In the course of this discussion we obtain a number of results concerning the structure of the Clifford group proper which, to the best of our knowledge, have not previously appeared in the literature and which may be of some independent interest.

In Section 5 we define and establish some of the properties of a function we call the Clifford trace. We also identify a distinguished class of order 3 Clifford unitaries for which the Clifford trace $= -1$. We refer to these as canonical order 3 unitaries.

In Section 6 we analyze the vectors constructed numerically by Renes et al \cite{17} (RBSC in the sequel) in dimension 5–45. We show that each of them is an eigenvector of a canonical order 3 Clifford unitary. This suggests the conjecture, that every GP fiducial vector is an eigenvector of a canonical order 3 unitary. We also show that, with one exception, the stability group of each RBSC vector is order 3 (the exception being dimension 7, where the stability group is order 6).

In Section 7 we show that RBSC’s results also support a strengthened version of a conjecture of Zauner’s \cite{16} (also see Grassl \cite{21}).

In Section 8 we use RBSC’s numerical data, regarding the total number of fiducial vectors in dimensions 2–7, to give a complete characterization of the orbits and stability groups in dimensions 2–7. Our results show that in each of these dimensions every fiducial vector covariant under the action of the generalized Pauli group is an eigenvector of a canonical order 3 Clifford unitary. We also identify the total number of distinct orbits. It was already known \cite{17, 21} that there are infinitely many orbits in dimension 3, and one orbit in dimensions 2 and 6. We show that there is, likewise, only one orbit in dimensions 4 and 5, but two distinct orbits in dimension 7. We also construct exact expressions for two fiducial vectors in dimension 7 (one on each of the two distinct orbits).

RBSC’s numerical data may suggest that, after dimension 7, the stability group of every fiducial vector has order 3. In Section 9 we show that there is at least one exception to that putative rule by constructing an exact expression for a fiducial vector in dimension 19 for which the stability group has order $\geq 18$.

Our construction of exact solutions in dimensions 7 and 19 was facilitated by the fact that in these dimensions there exist canonical order 3 unitaries having a particularly simple form. In Section 10 we show that a similar simplification occurs in every dimension $d$ for which (a) $d$ has at least one prime factor $= 1$ (mod 3), (b) $d$ has no prime factors $= 2$ (mod 3) and (c) $d$ is not divisible by 9. In other words, it happens when $d = 7, 13, 19, 21, 31, \ldots$.

2. **Fiducial Vectors for the Generalized Pauli Group**

The SIC-POVMs which have been constructed to date all have a certain group covariance property. Let $G$ be a finite group having $d^2$ elements, and suppose we
have an injective map $g \to \hat{U}_g$ which associates to each $g \in G$ a unitary operator $\hat{U}_g$ acting on $d$-dimensional Hilbert space. Suppose that for all $g, g'$

$$\hat{U}_g \hat{U}_{g'} = e^{i\xi_{g'g}} \hat{U}_{gg'}$$

(3)

where $e^{i\xi_{g'g}}$ is a phase (so the map defines a group homomorphism of $G$ into the quotient group $U(d)/U_c(d)$, where $U_c(d)$ is the centre of $U(d)$). Finally (and this, of course, is the difficult part) suppose we can find a vector $|\psi\rangle \in \mathbb{C}^d$ such that

$$\langle \psi | \psi \rangle = 1$$

and

$$|\langle \psi | U_g | \psi \rangle| = \frac{1}{\sqrt{d+1}}$$

(4)

for all $g \neq e$ ($e$ being the identity of $G$). Then the assignment

$$\hat{E}_g = \frac{1}{d} \hat{U}_g |\psi\rangle \langle \psi | \hat{U}_g^\dagger$$

(5)

defines a SIC-POVM on $\mathbb{C}^d$. The vector $|\psi\rangle$ is said to be a fiducial vector.

To date attention has been largely focussed on the case $G = (\mathbb{Z}_d)^2$, where $\mathbb{Z}_d$ is the set of integers $0, 1, \ldots, d-1$ under addition modulo $d$ (although there is numerical evidence that fiducial vectors exist for other choices of group [17]). That is also the case on which we will focus here.

To construct a suitable map $(\mathbb{Z}_d)^2 \to U(d)$, let $|e_0\rangle, |e_1\rangle, \ldots, |e_{d-1}\rangle$ be an orthonormal basis for $\mathbb{C}^d$, and let $\hat{T}$ be the operator defined by

$$\hat{T}|e_r\rangle = \omega^r |e_r\rangle$$

(6)

where $\omega = e^{2\pi i/d}$. Let $\hat{S}$ be the shift operator

$$\hat{S}|e_r\rangle = \begin{cases} |e_{r+1}\rangle & r = 0, 1, \ldots, d-2 \\ |e_0\rangle & r = d-1 \end{cases}$$

(7)

Then define, for each pair of integers $p = (p_1, p_2) \in \mathbb{Z}^2$,

$$\hat{D}_p = \tau^{p_1p_2} \hat{S}^{p_1} \hat{T}^{p_2}$$

(8)

where $\tau = e^{-i\pi/d}$ (the minus sign means that $\tau^2 = 1$ for all $d$, thereby simplifying some of the formulae needed in the sequel). We have, for all $p, q \in \mathbb{Z}^2$,

$$\hat{D}_p^\dagger = \hat{D}_{-p}$$

(9)

$$\hat{D}_p \hat{D}_q = \tau^{\langle p, q \rangle} \hat{D}_{p+q}$$

(10)

and

$$\hat{D}_{p+aq} = \begin{cases} \hat{D}_p & \text{if } d \text{ is odd} \\ (-1)^{\langle p, q \rangle} \hat{D}_p & \text{if } d \text{ is even} \end{cases}$$

(11)

where $\langle p, q \rangle$ is the symplectic form

$$\langle p, q \rangle = p_2 q_1 - p_1 q_2$$

(12)

Consequently the map $p \in (\mathbb{Z}_d)^2 \to \hat{D}_p \in U(d)$ has all the required properties. The operators $\hat{D}_p$ are sometimes called generalized Pauli matrices. So we will say that a vector $|\psi\rangle \in \mathbb{C}^d$ is a generalized Pauli fiducial vector, or GP fiducial vector for short, if it is a fiducial vector relative to the action of these operators: i.e. if

$$\langle \psi | \psi \rangle = 1$$

and

$$|\langle \psi | \hat{D}_p | \psi \rangle| = \frac{1}{\sqrt{d+1}}$$

(13)

for every $p \in \mathbb{Z}^2 \neq 0 \pmod{d}$. 


The set of operators \( \hat{D}_p \) is not a group. However, it becomes a group if we allow each \( \hat{D}_p \) to be multiplied by an arbitrary phase. We will refer to the group \( \mathrm{GP}(d) = \{ e^{i\xi} \hat{D}_p : \xi \in \mathbb{R}, p \in \mathbb{Z}^2 \} \) so obtained as the generalized Pauli group\(^2\).

We now want to investigate the normalizer of \( \mathrm{GP}(d) \): i.e. the group \( \mathrm{C}(d) \) consisting of all unitary operators \( \hat{U} \in \mathrm{U}(d) \) with the property

\[
\hat{U} \mathrm{GP}(d) \hat{U}^\dagger = \mathrm{GP}(d) \tag{14}
\]

The significance of this group for us is that it generates automorphisms of \( \mathrm{GP}(d) \) according to the prescription

\[
\hat{P} \rightarrow \hat{U} \hat{P} \hat{U}^\dagger \tag{15}
\]

Consequently, if \( |\psi\rangle \) is a \( \mathrm{GP} \) fiducial vector, then so is \( \hat{U} |\psi\rangle \) for every \( \hat{U} \in \mathrm{C}(d) \).

The group \( \mathrm{C}(d) \) is known as the Clifford group, and has been extensively discussed in the literature \[25, 26, 27, 28, 29, 30, 31\]. Its relevance to the SIC-POVM problem has been stressed by Grassl \[21\]. However, none of these accounts derive all the results needed for our analysis of the RBSC vectors. In the interests of readability we give a unified treatment in the next section.

3. THE CLIFFORD GROUP: STRUCTURE, AND CALCULATION OF THE UNITARIES

We begin with some definitions. Let

\[
d = \begin{cases} 
d & \text{if } d \text{ is odd} \
2d & \text{if } d \text{ is even} \end{cases} \tag{16}
\]

Let \( \mathrm{SL}(2, \mathbb{Z}_d) \) be the group consisting of all \( 2 \times 2 \) matrices

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

such that \( \alpha, \beta, \gamma, \delta \in \mathbb{Z}_d \) and \( \alpha\delta - \beta\gamma = 1 \pmod{d} \). Note that inverses exist in this group because the condition \( \alpha\delta - \beta\gamma = 1 \pmod{d} \) implies

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{18}
\]

in arithmetic modulo \( d \).

We then have

**Lemma 1.** For each unitary operator \( \hat{U} \in \mathrm{C}(d) \) there exists a matrix \( F \in \mathrm{SL}(2, \mathbb{Z}_d) \) and a vector \( \chi \in (\mathbb{Z}_d)^2 \) such that

\[
\hat{U} \hat{D}_p \hat{U}^\dagger = \omega^{\langle \chi, Fp \rangle} \hat{D}_{fp} \tag{19}
\]

for all \( p \in \mathbb{Z}^2 \) (where \( \omega = \tau^2 = e^{2\pi i/d} \), as before).

**Proof.** If \( \hat{U} \in \mathrm{C}(d) \) it is immediate that there exist functions \( f \) and \( g \) such that

\[
\hat{U} \hat{D}_p \hat{U}^\dagger = e^{ig(p)} \hat{D}_{f(p)} \tag{20}
\]

for all \( p \in \mathbb{Z}^2 \). It follows from Eq. \[10\] that

\[
\left( e^{ig(p)} \hat{D}_{f(p)} \right) \left( e^{ig(q)} \hat{D}_{f(q)} \right) = \tau^{\langle p, q \rangle} \left( e^{ig(p+q)} \hat{D}_{f(p+q)} \right) \tag{21}
\]

for all \( p, q \in \mathbb{Z}^2 \). Consequently

\[
e^{i(g(p+q))\tau^{\langle f(p+q), f(q) \rangle}} \hat{D}_{f(p+q)} = e^{ig(p+q)\tau^{\langle p, q \rangle}} \hat{D}_{f(p+q)} \tag{22}
\]

\(^2\)Also known as the Weyl-Heisenberg group. Our definition is, perhaps, slightly unconventional. It would be more usual to define \( \mathrm{GP}(d) = \{ \tau^n \hat{D}_p : n \in \mathbb{Z}, p \in \mathbb{Z}^2 \} \) i.e. the subgroup generated by the operators \( \hat{D}_p \).
which implies \( f(p + q) = f(p) + f(q) \) (mod \( d \)). We may therefore write

\[
  f(p) = F'p + dh(p)
\]

for some matrix \( F' \) and function \( h \). Inserting this expression in Eq. (20) gives, in view of Eq. (11),

\[
  \hat{U} \hat{D}_p \hat{U}^\dagger = e^{ig(p)} \hat{D}_{F'p + dh(p)} = \begin{cases} 
  e^{ig(p)} \hat{D}_{F'p} & d \text{ odd} \\
  e^{ig(p)}(-1)^{(p,h(p))} \hat{D}_{F'p} & d \text{ even}
\end{cases}
\]

(24)

With the appropriate definition of \( g' \) this means

\[
  \hat{U} \hat{D}_p \hat{U}^\dagger = e^{ig'(p)} \hat{D}_{F'p}
\]

(25)

for all \( p \). Repeating the argument which led to Eq. (22) we find

\[
  e^{ig'((p+q) - g'(p) - g'(q))_{\tau}(p,q) - (F'p,F'q)} = 1
\]

(26)

Interchanging \( p \) and \( q \) gives

\[
  e^{ig'(p+q) - g'(q) - g'(q)_{\tau}(p,q) + (F'p,F'q)} = 1
\]

(27)

We consequently require

\[
  \omega((p,q) - (F'p,F'q)) = \tau_{2((p,q) - (F'p,F'q))} = 1
\]

(28)

for all \( p, q \). It is readily verified that \( \langle F'p, F'q \rangle = (\text{Det} F')\langle p,q \rangle \). We must therefore have

\[
  \text{Det } F' = 1 \text{ (mod } d) \]

(29)

If \( d \) is odd, or if \( d \) is even and \( \text{Det } F' = 1 \) (mod \( d \)), we can find a matrix \( F \in \text{SL}(2,\mathbb{Z}) \) such that \( F = F' \) (mod \( d \)). It then follows from Eq. (11) that \( \hat{D}_{Fp} = \hat{D}_{F'p} \) for all \( p \).

Suppose, on the other hand, \( d \) is even and \( \text{Det } F' \neq 1 \) (mod \( d \)). Then \( \text{Det } F' = d + 1 \) (mod \( d \)). Write

\[
  F' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

(30)

We know \( \alpha \delta - \beta \gamma = \text{Det } F' \) is odd. So either \( \alpha, \delta \) are both odd, or else \( \beta, \gamma \) are both odd. If \( \alpha, \delta \) are both odd let

\[
  \Delta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

(31)

while if \( \beta, \gamma \) are both odd let

\[
  \Delta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

(32)

Then \( \text{Det}(F' + d\Delta) = 1 \) (mod \( d \)). We can therefore choose a matrix \( F \in \text{SL}(2,\mathbb{Z}) \) such that \( F = F' + d\Delta \) (mod \( d \)). Inserting this expression in Eq. (25) we have, in view of Eq. (11),

\[
  \hat{U} \hat{D}_p \hat{U}^\dagger = e^{ig'(p)} \hat{D}_{(F + d\Delta)p} = e^{ig'(p)}(-1)^{(Fp,\Delta p)} \hat{D}_{Fp}
\]

(33)

We conclude that there is, in every case, a function \( g'' \) and a matrix \( F \in \text{SL}(2,\mathbb{Z}) \) such that

\[
  \hat{U} \hat{D}_p \hat{U}^\dagger = e^{ig''(p)} \hat{D}_{Fp}
\]

(34)

for all \( p \).

It remains to establish the form of the function \( g'' \). We note, first of all, that it follows from Eqs. (8) and (10) that

\[
  (\hat{D}_p)^d = \hat{D}_{dp} = \tau d^p \phi(p) \hat{D}_{dp} \hat{D}_{dp} = 1
\]

(35)
for all \( p \) (because \( \hat{S}^d = \hat{T}^d = \tau^d = 1 \)). Consequently

\[
1 = \hat{U} (\hat{D}_p)^d \hat{U}^\dagger = (\hat{U} \hat{D}_p \hat{U}^\dagger)^d = e^{idg(p)} (\hat{D}_p)^d = e^{idg''(p)}
\]  

(36)

for all \( p \). We must therefore have \( e^{idg''(p)} = \omega^{d(p)} \) for some function \( \tilde{g} \) taking values in \( \mathbb{Z}_d \). Repeating the argument which led to Eq. (26) we find

\[
\omega^{\tilde{g}(p+q)} - \tilde{g}(p) - \tilde{g}(q) \equiv -(F^pF^q) \equiv 1 \mod d
\]

(37)

We have \( \langle p, q \rangle - \langle F^p, F^q \rangle = (1 - \text{Det } F) \langle p, q \rangle \equiv 0 \mod d \). Consequently

\[
\tau(p, q) - (F^pF^q) = 1 \quad \text{(because } \tau^d = 1 \text{ and so)}
\]

\[
\tilde{g}(p+q) = \tilde{g}(p) + \tilde{g}(q) \mod d
\]

(38)

for all \( p, q \). This implies \( \tilde{g}(p) = \langle \chi', p \rangle \mod d \) for all \( p \) some fixed \( \chi' \in (\mathbb{Z}_d)^2 \).

Setting \( \chi = F\chi' \), and using the fact that \( \langle F^{-1} \chi, p \rangle = \langle \chi, Fp \rangle \mod d \) we conclude

\[
\hat{U} \hat{D}_p \hat{U}^\dagger = \omega^{\langle \chi, Fp \rangle} \hat{D}_p
\]

(39)

for all \( p \).

We now want to prove the converse of Lemma 2. That is, we want to prove that, for each pair \( F \in SL(2, \mathbb{Z}_d) \) and \( \chi \in (\mathbb{Z}_d)^2 \) there is a corresponding operator \( \hat{U} \) in \( C(d) \). We also want to derive an explicit expression for the operator \( \hat{U} \) (this has, in effect, already been done by Hostens et al. 29; however, the formulae we derive are different, and better adapted to the questions addressed in this paper).

We begin by focussing on a special class of matrices \( F \). Let \([n_1, n_2, \ldots, n_r] \) denote the GCD (greatest common divisor) of the integers \( n_1, n_2, \ldots, n_r \). We define the class of \emph{prime matrices} to be the set of all matrices

\[
F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

(40)

\( \in SL(2, \mathbb{Z}_d) \) such that \( \beta \) is non-zero and \([\beta, \delta] = 1 \) (so that \( \beta \) has a multiplicative inverse in \( \mathbb{Z}_d \)). We then have

\textbf{Lemma 2.} Let

\[
F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

(41)

be a prime matrix \( \in SL(2, \mathbb{Z}_d) \). Let

\[
\hat{V}_F = \frac{1}{\sqrt{d}} \sum_{r,s=0}^{d-1} \tau^{\beta^{-1} (\alpha s^2 - 2rs + \delta s^2)} |e_r \rangle \langle e_s|
\]

(42)

(where \( \beta^{-1} \in \mathbb{Z}_d \) is such that \( \beta^{-1} \beta = 1 \mod d \)). Then \( \hat{V}_F \) is a unitary operator \in C(d) such that

\[
\hat{V}_F \hat{D}_p \hat{V}_F^\dagger = \hat{D}_F p
\]

(43)

for all \( p \).

\textbf{Proof.} Let

\[
\hat{S}' = \hat{D}_{(\alpha, \gamma)} \quad \text{and} \quad \hat{T}' = \hat{D}_{(\beta, \delta)}
\]

(44)

and define

\[
|f_0\rangle = \frac{1}{\sqrt{d}} \sum_{r=0}^{d-1} \langle \hat{T}' |e_0\rangle
\]

(45)

It follows from Eq. (35) that \( \langle \hat{T}' |e_0\rangle = 1 \). Consequently

\[
\hat{T}' |f_0\rangle = |f_0\rangle
\]

(46)
It follows from Eq. (10) that $\hat{T}' \hat{S}' = \omega \hat{S}' \hat{T}'$. So we can obtain a complete set of eigenvectors by laddering. Specifically, let
\[ |f_r\rangle = (\hat{S}')^r |f_0\rangle \] (47)
for $r = 1, \ldots, d - 1$. Then
\[ \hat{T}' |f_r\rangle = \omega^r |f_r\rangle \] (48)
for all $r$. Since $(\hat{S}')^d = 1$ (as follows from Eq. (35)) we also have
\[ \hat{S}' |f_r\rangle = |f_{r \oplus_d 1}\rangle \] (49)
for all $r$ (where $\oplus_d$ signifies addition modulo $d$).

We next show that the vectors $|f_r\rangle$ are orthonormal. It follows from Eqs. (6), (4), (8) and (10) that
\[ (\hat{T}')^r |e_0\rangle = \hat{D}_{(r, \beta, r\delta)} |e_0\rangle = \tau^{\beta \delta r^2} \hat{S}^{\beta r} |e_0\rangle \] (50)
and consequently
\[ |f_0\rangle = \left( \frac{1}{\sqrt{d}} \sum_{r=0}^{d-1} \tau^{\beta \delta r^2} \hat{S}^{\beta r} \right) |e_0\rangle = \left( \frac{1}{\sqrt{d}} \sum_{r=0}^{d-1} \tau^{\beta \delta r^2} \hat{S}^{\beta r} \right) |e_0\rangle \] (51)
(where we have used the fact that $\tau^d = 1$). We need to be careful at this point, due to the fact that congruence modulo $d$ need not imply congruence modulo $\overline{d}$. Let $q_r$ be the quotient of $\beta r$ on division by $d$, and let $t_r$ be the remainder. So $\beta r = q_r d + t_r$ and
\[ |f_0\rangle = \left( \frac{1}{\sqrt{d}} \sum_{r=0}^{d-1} \tau^{\beta \delta (q_r d + t_r)^2} \hat{S}^{q_r d + t_r} \right) |e_0\rangle \] (52)
We have
\[ \hat{S}^{q_r d + t_r} = \hat{S}^{t_r} \] (53)
and
\[ \tau^{\beta \delta (q_r d + t_r)^2} = \tau^{\beta \delta (t_r^2 + 2q_r d t_r + d^2 t_r)} = \tau^{\beta \delta t_r^2} \] (54)
(because $\tau^{2d} = \tau^{d^2} = 1$). Consequently
\[ |f_0\rangle = \left( \frac{1}{\sqrt{d}} \sum_{r=0}^{d-1} \tau^{\beta \delta t_r^2} \hat{S}^{t_r} \right) |e_0\rangle = \left( \frac{1}{\sqrt{d}} \sum_{r=0}^{d-1} \tau^{\beta \delta t_r^2} \right) |e_{t_r}\rangle \] (55)
The fact that $[\beta, \overline{d}] = 1$ implies that $[\beta, d] = 1$. It follows that, as $r$ runs over the integers $0, 1, \ldots, d - 1$, so does $t_r$ (though not necessarily in the same order). Consequently
\[ |f_0\rangle = \frac{1}{\sqrt{d}} \sum_{t=0}^{d-1} \tau^{\beta \delta t_r^2} |e_t\rangle \] (56)
It follows that
\[ \langle f_r | f_s \rangle = \langle f_0 | (\hat{S}')^{-r} (\hat{S}')^{-s} | f_0 \rangle = \langle f_0 | f_0 \rangle = 1 \] (57)
The fact that $\langle f_r | f_s \rangle = 0$ when $r \neq s$ is an immediate consequence of the fact that $|f_r\rangle$, $|f_s\rangle$ are eigenvectors of $\hat{T}'$ corresponding to different eigenvalues. We conclude that
\[ \langle f_r | f_s \rangle = \delta_{rs} \] (58)
as claimed.
We now want to calculate an explicit formula for $|f_r\rangle$ when $r > 0$. It follows from previous results that

$$|f_r\rangle = \hat{D}_{\alpha,\gamma} f_0 = \frac{1}{\sqrt{d}} \sum_{t=0}^{d-1} \tau^{\beta \gamma t + \alpha r t + 2 \gamma r t} (\mathcal{S})^{\gamma \alpha} |e_t\rangle$$

(59)

By an argument similar to the one leading to Eq. (56) we deduce

$$|f_r\rangle = \frac{1}{\sqrt{d}} \sum_{t=0}^{d-1} \tau^{\beta \gamma t - 2 rt + \alpha r t} |e_t\rangle$$

(60)

\[ \text{(since } \alpha \delta - \beta \gamma = 1 \text{ (mod } d) \text{).} \]

Comparing with Eq. (42) we see that $\hat{V}_F \approx \hat{D}_F$ which shows that $\hat{V}_F$ is unitary. Moreover,

$$\hat{V}_F \hat{T} \hat{V}_F^\dagger |f_r\rangle = \hat{V}_F |e_r\rangle = \omega^r |f_r\rangle$$

(63)

for all $r$. Comparing with Eq. (48) we deduce $\hat{V}_F \hat{T} \hat{V}_F^\dagger = \hat{T}'$. Similarly $\hat{V}_F \hat{S} \hat{V}_F^\dagger = \hat{S}'$. Hence

$$\hat{V}_F \hat{D}_p \hat{V}_F^\dagger = \tau^{p_1 p_2} \hat{D}_p \hat{S}^{p_1} \hat{T}^{p_2} \hat{V}_F^\dagger$$

(64)

$$= \tau^{p_1 p_2} \hat{D}_{\alpha \gamma} \gamma p_1 \hat{D}_{\beta \gamma} \delta p_2$$

(65)

$$= \tau (1 - \beta \gamma + \alpha \delta) p_1 p_2 \hat{D}_F$$

(66)

$$= \hat{D}_F$$

(67)

for all $p$. \[ \square \]

To extend this result to the case of an arbitrary matrix $\in \text{SL}(2, \mathbb{Z}_d)$ we need the following decomposition lemma, which states that every non-prime matrix can be written as the product of two prime matrices:

**Lemma 3.** Let

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

be a non-prime matrix $\in \text{SL}(2, \mathbb{Z}_d)$. Then there exists an integer $x$ such that $\delta + x \beta$ is non-zero and $[\delta + x \beta, d] = 1$. Let $x$ be any integer having that property, and let

$$F_1 = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix}$$

(69)

$$F_2 = \begin{pmatrix} \gamma + x \alpha & \delta + x \beta \\ -\alpha & -\beta \end{pmatrix}$$

(70)

Then $F_1, F_2$ are prime matrices $\in \text{SL}(2, \mathbb{Z}_d)$ such that

$$F = F_1 F_2$$

(71)

**Proof.** Suppose, to begin with, that $\beta, \delta$ are both non-zero. Let $k = [\beta, \delta]$. We then have

$$\beta = k \beta_0$$

(72)

$$\delta = k \delta_0$$

(73)
contains infinitely many primes. Consequently, there exists an integer 
δ such that 
δ + xβ_0 ≠ 0 and [δ + xβ, δ] = 1. The fact that k ≠ 0 and [k, δ] = 1 then implies that 
δ + xβ ≠ 0 and [δ + xβ, δ] = 1. The claim is now immediate.

It remains to consider the case when β, δ are not both non-zero. If δ = 0 the fact that det 
F = 1 (mod 7) would imply that β ≠ 0 and [β, δ] = 1—contrary to the assumption that the matrix F is non-prime. Suppose, on the other hand, that 
β = 0. Then the fact that det F = 1 (mod 7) implies that δ ≠ 0 and [δ, δ] = 1. So the claim is true for every choice of x. □

We can now deduce the following converse of Lemma 1

**Lemma 4.** Let (F, χ) be any pair ∈ SL(2, Z_7) \times (Z_7)^2. If F is a prime matrix define

\[ \hat{U} = \hat{D}_F \hat{V}_F \]

(where \( \hat{V}_F \) is the operator defined by Eq. (43)). If F is non-prime choose two prime matrices \( F_1, F_2 \) such that \( F = F_1 F_2 \) (the existence of such matrices being guaranteed by Lemma 3), and define

\[ \hat{U} = \hat{D}_F \hat{V}_{F_1} \hat{V}_{F_2} \]

(where \( \hat{V}_{F_1}, \hat{V}_{F_2} \) are the operators defined by Eq. (42)). Then

\[ \hat{U} \hat{D}_p \hat{U}^{†} = \omega^{(x F p)} \hat{D}_F \hat{V}_p \]

for all \( p \in Z^2 \)

**Proof.** The claim is an immediate consequence of Eqs. (68), (69) and Lemma 2 □

If \( \hat{U}, \hat{U}' \) differ by a phase, so that \( \hat{U}' = e^{iθ} \hat{U} \), they have the same action on the generalized Pauli group:

\[ \hat{U} \hat{D}_p \hat{U}^{†} = \hat{U}' \hat{D}_p \hat{U}'^{†} \]

for all \( p \). So the object of real interest is not the Clifford group itself, but the group C(d)/I(d) which results when the phases are factored out. Here I(d) is the subgroup consisting of all operators of the form \( e^{iθ} I \), where \( I \) is the identity operator and \( θ ∈ R \). The elements of C(d)/I(d) are often called Clifford operations.

Let SL(2, Z_7) \times (Z_7)^2 be the semi-direct product of SL(2, Z_7) and (Z_7)^2: i.e. the group which results when the set SL(2, Z_7) \times (Z_7)^2 is equipped with the composition rule

\[ (F_1, \chi_1) \circ (F_2, \chi_2) = (F_1 F_2, \chi_1 + F_1 \chi_2) \]

Then we have the following structure theorem, which states that C(d)/I(d) is naturally isomorphic to SL(2, Z_7) \times (Z_7)^2 when d is odd, and naturally isomorphic to a quotient group of SL(2, Z_7) \times (Z_7)^2 when d is even:

**Theorem 1.** There exists a unique surjective homomorphism

\[ f : SL(2, Z_7) \times (Z_7)^2 → C(d)/I(d) \]

with the property \( \hat{U} \hat{D}_p \hat{U}^{†} = \omega^{(x F p)} \hat{D}_F \hat{V}_p \) for each \( \hat{U} ∈ f(F, χ) \) and all \( p \in Z^2 \).

If d is odd f is an isomorphism. If d is even the kernel of f is the subgroup \( K_f ⊆ SL(2, Z_7) \times (Z_7)^2 \) consisting of the 8 elements of the form

\[ \left( \begin{array}{cc} 1 + rd & sd \\ td & 1 + rd \end{array} \right) \]

where \( r, s, t = 0 \) or 1.
Proof. An operator \( \hat{U} \in C(d) \) has the property
\[
\hat{U} \hat{D}_p \hat{U}^\dagger = \hat{D}_p
\]  
for all \( p \) if and only if it is a multiple of the identity. So it follows from results already proved that there is exactly one surjective map
\[
f : \text{SL}(2, \mathbb{Z}_d) \times (\mathbb{Z}_d)^2 \to C(d)/I(d)
\]  
such that \( \hat{U} \hat{D}_p \hat{U}^\dagger = \omega^{<\chi,p>} \hat{D}_F p \) for each \( \hat{U} \in f(F, \chi) \) and all \( p \in \mathbb{Z}^2 \). The fact that \( f \) is actually a homomorphism is then an immediate consequence of the definitions.

Let \( K_f \) be the kernel of \( f \). Then \( (F, \chi) \in K_f \) if and only if
\[
\omega^{<\chi,F,p>} \hat{D}_F p = \hat{D}_p
\]  
for all \( p \). For that to be true we must have \( F = 1 \pmod{d} \). If \( d \) is odd this implies \( \hat{D}_F p = \hat{D}_p \) for all \( p \). Eq. (84) then becomes \( \omega^{<\chi,p>} = 1 \) for all \( p \), implying \( \chi = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \). So the kernel is trivial, and \( f \) is an isomorphism as claimed.

Suppose, on the other hand, that \( d \) is even. The condition \( F = 1 \pmod{d} \) then implies that \( F = 1 + d\Delta \), where \( \Delta \) is a matrix of the form
\[
\Delta = \begin{pmatrix} r_1 & s \\ t & r_2 \end{pmatrix}
\]  
with \( r_1, r_2, s, t = 0 \) or \( 1 \). Inserting this expression in Eq. (84) we find, in view of Eqs. (91), that \( (F, \chi) \in K_f \) if and only if
\[
1 = \omega^{<\chi,F,p>} \hat{D}_F p \hat{D}_{-p} = \omega^{<\chi,p>\cdot d <p,\Delta p>}
\]  
for all \( p \). After re-arranging the condition becomes
\[
\omega^{\chi_2 p_1 - \chi_1 p_2} = (-1)^{(r_1-r_2)p_1f_1 - tp_1^2 + sp_2^2} = (-1)^{(r_1-r_2)p_1f_2 + tp_1 - sp_2}
\]  
for all \( p \). This is true if and only if \( r_1 = r_2 \), \( \chi_1 = sd/2 \) and \( \chi_2 = td/2 \).

We conclude with a result concerning the order of the group \( C(d)/I(d) \) which will be needed later on. Let \( \nu(n,d) \) be the number of distinct ordered pairs \( (x,y) \in (\mathbb{Z}_d)^2 \) such that \( xy = n \pmod{d} \). We then have

Lemma 5. The order of the group \( C(d)/I(d) \) is
\[
|C(d)/I(d)| = d^2 \left( \sum_{n=0}^{d-1} \nu(n,d) \nu(n+1,d) \right)
\]  
If \( d \) is a prime number this reduces to
\[
|C(d)/I(d)| = d^3(d^2 - 1)
\]  
Proof. We begin by showing that \( C(d)/I(d) \) and \( \text{SL}(2, \mathbb{Z}_d) \times (\mathbb{Z}_d)^2 \) have the same cardinality when considered as sets. This is true for all \( d \), notwithstanding the fact that when \( d \) is even \( C(d)/I(d) \) and \( \text{SL}(2, \mathbb{Z}_d) \times (\mathbb{Z}_d)^2 \) are not naturally isomorphic as groups.

The statement is immediate when \( d \) is odd. Suppose, on the other hand, that \( d \) is even. Let \( g : \text{SL}(2, \mathbb{Z}_{2d}) \to \text{SL}(2, \mathbb{Z}_d) \) be the natural homomorphism defined by
\[
g : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} [\alpha]_d \\ [\beta]_d \end{pmatrix}
\]  
where \( [x]_d \) denotes the residue class of \( x \pmod{d} \). It is easily seen that \( g \) is surjective. In fact, consider arbitrary
\[
F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2, \mathbb{Z}_d)
\]  

Then \(\alpha \delta - \beta \gamma = 1 + nd\) for some integer \(n\). If \(n\) is even then \(F \in \text{SL}(2, \mathbb{Z}_d)\) and \(F = g(F)\). Suppose, on the other hand, that \(n\) is odd. Then either \(\alpha\) or \(\beta\) is odd. If \(\alpha\) is odd \(F = g(F')\) where

\[
F' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta + d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}_d)
\]

(92) while if \(\beta\) is odd \(F = g(F'')\) where

\[
F'' = \begin{pmatrix} \alpha & \beta \\ \gamma + d & \delta \end{pmatrix} \in \text{SL}(2, \mathbb{Z}_d)
\]

(93)

Now let \(K_g\) be the kernel of \(g\). A matrix \(F \in K_g\) if and only if

\[
F = \begin{pmatrix} 1 + r_1d & sd \\ td & 1 + r_2d \end{pmatrix}
\]

(94) where \(r_1, r_2, s, t = 0\) or \(1\) and \((1 + r_1d)(1 + r_2d) - std^2 = 1\) (mod \(2d\)). We have

\[
(1 + r_1d)(1 + r_2d) - std^2 = 1 + (r_1 + r_2)d \pmod{2d}
\]

(bearing in mind that \(d\) is even, so \(d^2 = 0\) (mod \(2d\))). We therefore require \(r_1 = r_2\).

It follows that \(K_g\) consists of the 8 matrices of the form

\[
\begin{pmatrix} 1 + rd & sd \\ td & 1 + rd \end{pmatrix}
\]

(96) where \(r, s, t = 0\) or \(1\). The fact that \(g\) is surjective and \(|K_g| = 8\) implies \(|\text{SL}(2, \mathbb{Z}_d)| = 8|\text{SL}(2, \mathbb{Z}_d)|\). In view of Theorem 1 this means

\[
|C(d)/I(d)| = \frac{1}{8}|\text{SL}(2, \mathbb{Z}_d)| = |\text{SL}(2, \mathbb{Z}_d)\times (\mathbb{Z}_d)^2|
\]

(97) as claimed.

We have shown that \(|C(d)/I(d)| = |\text{SL}(2, \mathbb{Z}_d)\times (\mathbb{Z}_d)^2| = d^2|\text{SL}(2, \mathbb{Z}_d)|\) for all \(d\), odd or even. It remains to calculate \(|\text{SL}(2, \mathbb{Z}_d)|\). For each \(n \in \mathbb{Z}_d\) let \(M_n \subseteq \text{SL}(2, \mathbb{Z}_d)\) be the set of matrices

\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

(98) for which \(\alpha \delta = n + 1\) (mod \(d\)) and \(\beta \gamma = n\) (mod \(d\)). Clearly \(\text{SL}(2, \mathbb{Z}_d) = \bigcup_{n=0}^{d-1} M_n\) and \(|M_n| = \nu(n, d)\nu(n + 1, d)|. It follows that

\[
|\text{SL}(2, \mathbb{Z}_d)| = \sum_{n=0}^{d-1} \nu(n, d)\nu(n + 1, d)
\]

(99) Eq. (88) is now immediate.

If \(d\) is a prime number

\[
\nu(n, d) = \begin{cases} 2d - 1 & \text{if } n = 0 \pmod{d} \\ d - 1 & \text{otherwise} \end{cases}
\]

(100) implying

\[
\sum_{n=0}^{d-1} \nu(n, d)\nu(n + 1, d) = d(d^2 - 1)
\]

(101) Eq. (89) is now immediate.
4. The Extended Clifford Group

It can be seen from Eqs. (6) and (13) that, if \( |\psi\rangle = \sum_{r=0}^{d-1} \psi_r |e_r\rangle \) is a GP fiducial vector, then so is the vector \( |\psi^*\rangle = \sum_{r=0}^{d-1} \psi_r^* |e_r\rangle \) obtained by complex conjugation. So to make the analysis complete we need to consider automorphisms of GP\( (d) \) which are generated by anti-unitary operators.

An anti-linear operator is a map \( \hat{L} : \mathbb{C}^d \to \mathbb{C}^d \) with the property

\[
\hat{L} (\alpha |\phi\rangle + \beta |\psi\rangle) = \alpha^* \hat{L} |\phi\rangle + \beta^* \hat{L} |\psi\rangle
\]

for all \( |\phi\rangle, |\psi\rangle \in \mathbb{C}^d \) and all \( \alpha, \beta \in \mathbb{C} \). The adjoint \( \hat{L}^\dagger \) is defined to be the unique anti-linear operator with the property

\[
\langle \phi | \hat{L}^\dagger |\psi\rangle = \langle \psi | \hat{L} |\phi\rangle
\]

for all \( |\phi\rangle, |\psi\rangle \in \mathbb{C}^d \). An operator \( \hat{U} \) is said to be anti-unitary if it is anti-linear and \( \hat{U}^\dagger \hat{U} = 1 \) (or, equivalently, \( \hat{U} \hat{U}^\dagger = 1 \)).

We now define the extended Clifford Group to be the group \( EC(d) \) consisting of all unitary or anti-unitary operators \( \hat{U} \) having the property

\[
\hat{U} \text{ GP}(d) \hat{U}^\dagger = \text{GP}(d)
\]

Let us also define ESL\( (2, \mathbb{Z}_2) \) to be the group consisting of all \( 2 \times 2 \) matrices

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]

such that \( \alpha, \beta, \gamma, \delta \in \mathbb{Z}_2 \) and \( \alpha \delta - \beta \gamma = \pm 1 \) (mod \( d \)). In the last section we showed that there is a natural homomorphism \( f : \text{SL}(2, \mathbb{Z}_2) \times (\mathbb{Z}_2)^2 \to \text{C}(d)/\text{I}(d) \). We are going to show that this extends to a natural homomorphism \( f_E : \text{ESL}(2, \mathbb{Z}_2) \times (\mathbb{Z}_2)^2 \to \text{EC}(d)/\text{I}(d) \).

Let \( \hat{J} \) be the anti-linear operator which replaces components in the standard basis with their complex conjugates:

\[
\hat{J} : \sum_{r=0}^{d-1} \psi_r |e_r\rangle \mapsto \sum_{r=0}^{d-1} \psi_r^* |e_r\rangle
\]

Clearly \( \hat{J}^\dagger = \hat{J} \) and \( \hat{J} \hat{J}^\dagger = 1 \). So \( \hat{J} \) is an anti-unitary operator. Furthermore, it follows from Eqs. (6) and (13) that

\[
\hat{J} \hat{D}_p \hat{J}^\dagger = \hat{D}_p
\]

for all \( p \), where

\[
\hat{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

So \( \hat{J} \in EC(d) \). Note that \( \text{Det} \hat{J} = -1 \) (mod \( d \)), so \( \hat{J} \in \text{ESL}(2, \mathbb{Z}_2) \).

Now let \( C^*(d) \) be the set of anti-unitary operators \( \in EC(d) \) (so \( EC(d) \) is the disjoint union \( EC(d) = C(d) \cup C^*(d) \)). The mapping \( \hat{U} \mapsto \hat{J} \hat{U} \) defines a bijective correspondence between \( C^*(d) \) and \( C(d) \). We can use this to prove the following extension of Theorem 1.

**Theorem 2.** There is a unique surjective homomorphism

\[
f_E : \text{ESL}(2, \mathbb{Z}_2) \times (\mathbb{Z}_2)^2 \to \text{EC}(d)/\text{I}(d)
\]

such that, for each \( (F, \chi) \in \text{ESL}(2, \mathbb{Z}_2) \times (\mathbb{Z}_2)^2 \) and \( \hat{U} \in f_E(F, \chi) \),

\[
\hat{U} \hat{D}_p \hat{U}^\dagger = \omega^{<F, p>} \hat{D}_F \hat{D}_p
\]

for all \( p \). \( \hat{U} \) is unitary if \( \text{Det} F = 1 \) (mod \( 7 \)) and anti-unitary if \( \text{Det} F = -1 \) (mod \( 7 \)).
$f_E$ extends the homomorphism $f$ defined in Theorem 4, and has the same kernel. So $f_E$ is an isomorphism if $d$ is odd, while if $d$ is even its kernel is the subgroup $K_f$ defined in Theorem 4.

**Proof.** Let $\hat{U}$ be an arbitrary anti-unitary operator $\in C^*(d)$. The fact that $\hat{J}, \hat{U}$ are both anti-unitary means that $\hat{J}\hat{U}$ is unitary. So $\hat{J}\hat{U} \in C(d)$. It then follows from Theorem 1 that there exists $(F', \chi') \in \text{SL}(2, \mathbb{Z}_d) \times (\mathbb{Z}_d)^2$ such that

$$(\hat{J}\hat{U}) \hat{D}_p(\hat{J}\hat{U})^\dagger = \omega^{<\chi',F'>} \hat{D}_{F'}p$$

for all $p$. Define $F = \hat{J}F'$ and $\chi = \hat{J}\chi'$. In view of Eq. (107), and the fact that $\hat{J}^2 = 1$, we deduce

$$\hat{U} \hat{D}_p \hat{U}^\dagger = \hat{J}(\hat{J}\hat{U}) \hat{D}_p (\hat{J}\hat{U})^\dagger \hat{J}^\dagger = \omega^{<\chi',F'>} \hat{D}_{F'}p = \omega^{<\chi,F'>} \hat{D}_{F'}p$$

for all $p$ (where we have used the fact that $\langle \xi, \eta \rangle = -\langle \hat{J}\xi, \hat{J}\eta \rangle$ for all $\xi, \eta$). We have $\text{Det}(F) = (\text{Det} \hat{J})(\text{Det} F') = -1$, so $(F, \chi) \in \text{ESL}(2, \mathbb{Z}_d) \times (\mathbb{Z}_d)^2$.

Reversing the argument we deduce the converse proposition: for each $(F, \chi) \in \text{ESL}(2, \mathbb{Z}_d) \times (\mathbb{Z}_d)^2$, there exists $\hat{U} \in \text{EC}(d)$ such that $\hat{U} \hat{D}_p \hat{U}^\dagger = \omega^{<\chi,F'>} \hat{D}_{F'}p$ for all $p$. The fact that an operator commutes with $\hat{D}_p$ for all $p$ if and only if it is a multiple of the identity means that $\hat{U}$ is unique up to a phase.

This establishes the existence and uniqueness of the homomorphism $f_E$. The proof of the remaining statements is straightforward, and is left to the reader.

Finally, we have the following result which, together with Lemma 5, enables us to calculate the order of $\text{EC}(d)/\text{I}(d)$:

**Lemma 6.**

$$|\text{EC}(d)/\text{I}(d)| = 2|\text{C}(d)/\text{I}(d)|$$

for all $d$.

**Proof.** The map

$$\hat{U} \text{I}(d) \mapsto \hat{J}\hat{U} \text{I}(d)$$

defines a bijective correspondence between $C^*(d)/\text{I}(d)$ and $\text{C}(d)/\text{I}(d)$. So the set $C^*(d)/\text{I}(d)$ contains the same number of elements as $\text{C}(d)/\text{I}(d)$. The statement is now immediate.

5. The Clifford Trace

We now define the Clifford trace. The significance of this function for us is that every GP fiducial vector which has been constructed to date is an eigenvector of a Clifford unitary having Clifford trace $= -1$.

Let $[F, \chi] \in \text{EC}(d)/\text{I}(d)$ be the image of $(F, \chi)$ under the homomorphism $f_E$ defined in Theorem 4. We refer to $[F, \chi]$ as an extended Clifford operation (or Clifford operation if it $\in \text{C}(d)/\text{I}(d)$). The operators $\in [F, \chi]$ only differ by a phase. It is therefore convenient to adopt a terminology which blurs the distinction between the operation $[F, \chi]$ and the operators $\hat{U} \in [F, \chi]$. In particular, we will adopt the convention that properties which hold for each $\hat{U} \in [F, \chi]$ may also be attributed to $[F, \chi]$. Thus, we will say that $[F, \chi]$ is unitary (respectively anti-unitary) if the operators $\hat{U} \in [F, \chi]$ are unitary (respectively anti-unitary). Similarly, we will say that $|\psi\rangle \in \mathbb{C}^d$ is an eigenvector of $[F, \chi]$ if it is an eigenvector of the operators $\hat{U} \in [F, \chi]$.

It is easily verified that $\text{Tr}(F_1) = \text{Tr}(F_2)$ (mod $d$) whenever $[F_1, \chi_1] = [F_2, \chi_2]$ (note that it is not necessarily true that $\text{Tr}(F_1) = \text{Tr}(F_2)$ (mod $d$) if $d$ is even).

We therefore obtain a well-defined function $\text{EC}(d)/\text{I}(d) \rightarrow \mathbb{Z}_d$ if we assign to each operation $[F, \chi]$ the value $\text{Tr}(F)$ (mod $d$). We obtain a function $\text{EC}(d) \rightarrow \mathbb{Z}_d$ by
assigning to each \( \hat{U} \in [F, \chi] \) the value \( \text{Tr}(F) \pmod{d} \). We use the term "Clifford trace" to refer to either of these functions.

We now prove the main result of this section, which states that there is a connection between the order of a Clifford operation and its Clifford trace.

**Lemma 7.** Let \( [F, \chi] \in C(d)/I(d) \), where \( d \) is any dimension \( \neq 3 \). Then \( [F, \chi] \) is of order 3 if \( \text{Tr}(F) = -1 \pmod{d} \).

Let \( [F, \chi] \in C(d)/I(d) \), where \( d \) is any prime dimension \( \neq 3 \). Then the stronger statement is true: \( [F, \chi] \) is of order 3 if and only if \( \text{Tr}(F) = -1 \pmod{d} \).

**Remark.** The restriction to operations \( \in C(d)/I(d) \) is essential (because if \( [F, \chi] \) is anti-unitary its order must be even).

**Proof.** Let \( [F, \chi] \in C(d)/I(d) \), and let \( \kappa = \text{Tr}(F) \). Then, taking into account the fact that \( \text{Det}(F) = 1 \pmod{\overline{d}} \), it is straightforward to show

\[
F^2 = \kappa F - 1 \pmod{\overline{d}}
\]

implying

\[
F^3 = (\kappa^2 - 1)F - \kappa \pmod{\overline{d}} \quad (116)
\]

\[
1 + F + F^2 = (\kappa + 1)F \pmod{\overline{d}} \quad (117)
\]

Now suppose that \( \kappa = -1 \pmod{d} \). Then there are three possibilities: (a) \( d \) is odd; (b) \( d \) is even and \( \kappa = -1 \pmod{\overline{d}} \); (c) \( d \) is even and \( \kappa = -1 + d \pmod{\overline{d}} \). In case (a) or (b) we have

\[
F^3 = 1 \pmod{\overline{d}} \quad (118)
\]

\[
1 + F + F^2 = 0 \pmod{\overline{d}} \quad (119)
\]

while in case (c) we have \( \kappa^2 - 1 = d^2 - 2d = 0 \pmod{\overline{d}} \), and consequently

\[
F^3 = \begin{pmatrix} 1 + d & 0 \\ 0 & 1 + d \end{pmatrix} \pmod{\overline{d}} \quad (120)
\]

\[
1 + F + F^2 = 0 \pmod{d} \quad (121)
\]

Referring to the definition of \( K_f \) (see Theorem 1), we deduce that, in every case,

\[
(F, \chi)^3 = (F^3, (1 + F + F^2)\chi) \in K_f \quad (122)
\]

implying that \( [F, \chi]^3 = [1, 0] \). It remains to show that neither \( [F, \chi] \) nor \( [F, \chi]^2 = [1, 0] \). To see that \( [F, \chi] \neq [1, 0] \) observe that the contrary would imply \( -1 = \kappa = \text{Tr}(1) = 2 \pmod{d} \), which is not possible given that \( d \neq 3 \). Similarly, if \( [F, \chi]^2 = [1, 0] \), it would follow (taking the trace on both sides of Eq. (115)) that \( 2 = \kappa^2 - 2 = -1 \pmod{d} \), contrary to the assumption that \( d \neq 3 \). We conclude that \( [F, \chi] \) is of order 3, as claimed.

To prove the second part of the lemma suppose that \( d \) is a prime number \( \neq 3 \) and \( [F, \chi] \) is of order 3. Then \( (F^3, (1 + F + F^2)\chi) \in K_f \), implying \( F^3 = 1 \pmod{d} \).

In view of Eq. (119), this means

\[
(\kappa + 1)((\kappa - 1)F - 1) = 0 \pmod{d} \quad (123)
\]

We now proceed by reductio ad absurdum. Suppose that \( \kappa \neq -1 \pmod{d} \). Then Eq. (123) and the fact that \( d \) is prime implies

\[
(\kappa - 1)F = 1 \pmod{d} \quad (124)
\]

Taking the trace on both sides gives \( (\kappa + 1)(\kappa - 2) = 0 \pmod{d} \), implying \( \kappa = 2 \pmod{d} \). Substituting this value into Eq. (124) we deduce \( F = 1 \pmod{d} \), implying \( F^2 = 1 \pmod{\overline{d}} \) and \( F^3 = F \pmod{\overline{d}} \). So

\[
(F, 3\chi) = (F, \chi)^3 \in K_f \quad (125)
\]
implying \((F, \chi) \in K_f\). But that would mean \(|F, \chi|\) is of order 1, contrary to assumption. We conclude that \(\kappa = -1 \mod (d)\), as claimed. \hfill \Box

The result does not hold when \(d = 3\) because then the identity has Clifford trace \(= -1\). It is, however, easily verified that in dimension 3 (as in every other prime dimension) every order 3 Clifford operation has Clifford trace \(= -1\).

If \(d\) is not a prime number there may exist order 3 Clifford operations for which the Clifford trace \(\neq -1\). Consider, for example,

\[
[F, \chi] = \left[ \begin{array}{cc} 5 & 4 \\ 2 & -3 \end{array} \right] \in \mathbb{C}(6) / \mathcal{I}(6)
\]

Then \([F, \chi]\) is of order 3 yet \(\text{Tr}(F) = 2 \mod 6\).

Because these results will play an important role in the following it is convenient to introduce some terminology. We will say that an operation \([F, \chi] \in \mathbb{C}(d) / \mathcal{I}(d)\) is a canonical order 3 unitary if

(a) \(\text{Tr}(F) = -1 \mod d\).

(b) \(F\) is not the identity matrix.

Note that the second stipulation is only needed because of the possibility that \(d = 3\). If \(d \neq 3\) an operation \([F, \chi] \in \mathbb{C}(d) / \mathcal{I}(d)\) is a canonical order 3 unitary if and only if \(\text{Tr}(F) = -1 \mod d\).

6. The RBSC Vectors

For \(5 \leq d \leq 45\) RBSC \cite{17, 21} have constructed GP fiducial vectors numerically. In this section we examine the behaviour of these vectors under the action of the extended Clifford group. In particular we show that each of them is an eigenvector of a canonical order 3 Clifford unitary. This suggests

**Conjecture A:** GP fiducial vectors exist in every finite dimension. Furthermore, every such vector is an eigenvector of a canonical order 3 unitary.

Conjecture A is related to a conjecture of Zauner’s. Let

\[
[Z, 0] = \left[ \begin{array}{cc} 0 & -1 \\ 1 & -1 \end{array} \right] \in \mathbb{C}(2)
\]

It will be observed that \([Z, 0]\) is defined, and \(\in \mathbb{C}(d) / \mathcal{I}(d)\), for every dimension \(d\), and that it is canonical order 3. Zauner \cite{16} has conjectured

**Conjecture B:** In each dimension \(d\) there exists a GP fiducial vector which is an eigenvector of \([Z, 0]\).

In Section 7 we will see that RBSC’s numerical data also provides further support for Conjecture B.

Let \(|\psi_d\rangle\) be the RBSC vector in dimension \(d\). In Table 1 we list, for each value of \(d\), a unitary Clifford operation \([F_d, \chi_d]\) having \(|\psi_d\rangle\) as one of its eigenvectors. It will be seen that, in every case, \(\text{Tr}(F_d) = -1 \mod d\), implying that \([F_d, \chi_d]\) is canonical order 3. Clearly, \(|\psi_d\rangle\) is also an eigenvector of \([F_d, \chi_d]^2\). Moreover, \([F_d, \chi_d]^2\) also has Clifford trace \(= -1\). There are, however, no other Clifford operations with these properties.

In Table 1 we also list \((n_{d1}, n_{d2}, n_{d3})\), the dimensions of the three eigenspaces of \([F_d, \chi_d]\), and \(n_d\), the dimension of the particular eigenspace to which \(|\psi_d\rangle\) belongs. It will be seen that, with one exception, \(|\psi_d\rangle\) always belongs to an eigenspace of highest dimension (the exception being \(d = 17\), where \(|\psi_d\rangle\) belongs to the eigenspace of lowest dimension).
| $d$ | $F_d$ | $\chi_d$ | $(n_{d1}, n_{d2}, n_{d3})$ | $n_d$ | $d$ | $F_d$ | $\chi_d$ | $(n_{d1}, n_{d2}, n_{d3})$ | $n_d$ |
|-----|-------|--------|----------------|------|-----|-------|--------|----------------|------|
| 5   | $(-1, -1)$ | $\frac{2}{2}$ | $(1, 2, 2)$ | 2    | 26  | $(-7, -9)$ | $\frac{11}{11}$ | $(8, 9, 9)$ | 9    |
| 6   | $(-2, 3)$   | $\frac{3}{6}$ | $(1, 2, 3)$ | 3    | 27  | $(-10, 1)$ | $\frac{3}{12}$ | $(8, 9, 10)$ | 10   |
| 7   | $(-2, -2)$  | $\frac{2}{6}$ | $(2, 2, 3)$ | 3    | 28  | $(-3, 21)$ | $\frac{10}{2}$ | $(9, 9, 10)$ | 10   |
| 8   | $(-4, 3)$   | $\frac{3}{6}$ | $(2, 3, 3)$ | 3    | 29  | $(-13, 6)$ | $\frac{-10}{12}$ | $(9, 10, 10)$ | 10   |
| 9   | $(-3, 2)$   | $\frac{2}{6}$ | $(2, 3, 4)$ | 4    | 30  | $(-8, -7)$ | $\frac{11}{7}$ | $(9, 10, 11)$ | 11   |
| 10  | $(-4, -7)$  | $\frac{-2}{3}$ | $(3, 3, 4)$ | 4    | 31  | $(-9, -10)$ | $\frac{-14}{6}$ | $(10, 10, 11)$ | 11   |
| 11  | $(-5, 4)$   | $\frac{-5}{4}$ | $(3, 4, 4)$ | 4    | 32  | $(-11, -31)$ | $\frac{11}{7}$ | $(10, 11, 11)$ | 11   |
| 12  | $(-4, 11)$  | $\frac{4}{3}$ | $(3, 4, 5)$ | 5    | 33  | $(-7, -5)$ | $\frac{8}{6}$ | $(10, 11, 12)$ | 12   |
| 13  | $(-2, -2)$  | $\frac{6}{0}$ | $(4, 4, 5)$ | 5    | 34  | $(-12, 3)$ | $\frac{1}{11}$ | $(11, 11, 12)$ | 12   |
| 14  | $(-2, -3)$  | $\frac{5}{1}$ | $(4, 5, 5)$ | 5    | 35  | $(-13, -12)$ | $\frac{11}{12}$ | $(11, 12, 12)$ | 12   |
| 15  | $(-5, 1)$   | $\frac{-6}{4}$ | $(4, 5, 6)$ | 6    | 36  | $(-8, -13)$ | $\frac{0}{7}$ | $(11, 12, 13)$ | 13   |
| 16  | $(-8, 13)$  | $\frac{1}{7}$ | $(5, 5, 6)$ | 6    | 37  | $(-16, 18)$ | $\frac{-4}{15}$ | $(12, 12, 13)$ | 13   |
| 17  | $(-5, -7)$  | $\frac{6}{7}$ | $(5, 6, 6)$ | 5    | 38  | $(-6, -31)$ | $\frac{12}{10}$ | $(12, 13, 13)$ | 13   |
| 18  | $(-5, 5)$   | $\frac{9}{6}$ | $(5, 6, 7)$ | 7    | 39  | $(-17, -11)$ | $\frac{8}{15}$ | $(12, 13, 14)$ | 14   |
| 19  | $(-2, 4)$   | $\frac{-7}{4}$ | $(6, 6, 7)$ | 7    | 40  | $(-3, -13)$ | $\frac{-12}{2}$ | $(13, 13, 14)$ | 14   |
| 20  | $(-2, -3)$  | $\frac{-9}{1}$ | $(6, 7, 7)$ | 7    | 41  | $(-2, -10)$ | $\frac{19}{13}$ | $(13, 14, 14)$ | 14   |
| 21  | $(-5, -6)$  | $\frac{-6}{1}$ | $(6, 7, 8)$ | 8    | 42  | $(-15, 11)$ | $\frac{0}{15}$ | $(13, 14, 15)$ | 15   |
| 22  | $(-2, -1)$  | $\frac{8}{2}$ | $(7, 7, 8)$ | 8    | 43  | $(-11, 18)$ | $\frac{-1}{21}$ | $(14, 14, 15)$ | 15   |
| 23  | $(-11, -10)$ | $\frac{0}{-3}$ | $(7, 8, 8)$ | 8    | 44  | $(-8, -29)$ | $\frac{16}{-5}$ | $(14, 15, 15)$ | 15   |
| 24  | $(-2, -3)$  | $\frac{0}{-3}$ | $(7, 8, 9)$ | 9    | 45  | $(-20, -1)$ | $\frac{-8}{6}$ | $(14, 15, 16)$ | 16   |
| 25  | $(-6, -1)$  | $\frac{-7}{12}$ | $(8, 8, 9)$ | 9    |      |        |        |                |     |

Table 1. For each $d$ the RBSC vector $|\psi_d\rangle$ is an eigenvector of the unitary operation $[F_d, \chi_d]$. Note that in every case $\text{Tr} F_d = -1$, implying that $[F_d, \chi_d]$ is canonical order 3. $(n_{d1}, n_{d2}, n_{d3})$ are the dimensions of the three eigenspaces of $[F_d, \chi_d]$; and $n_d$ is the dimension of the eigenspace to which $|\psi_d\rangle$ belongs. Note that $n_d = \max(n_{d1}, n_{d2}, n_{d3})$, with the single exception of $d = 17$. 


We used a computer algebra package (Mathematica) to construct the table. To illustrate the method employed we give a detailed description for the case \( d = 5 \).

We begin with the observation that, if \( |\psi_5\rangle \) is an eigenvector of \([F, \chi]\), then

\[
\langle \psi_5 | \hat{D}_p | \psi_5 \rangle = e^{\frac{2\pi i}{5} \langle \chi, F \rangle} \langle \psi_5 | \hat{D}_p | \psi_5 \rangle
\]

for all \( p \). So, using the value of \( |\psi_5\rangle \) which is available on RBSC’s website [24], we look for values of \( p, q \) such that

\[
\frac{5}{2\pi} \left( \arg \left( \langle \psi_5 | \hat{D}_p | \psi_5 \rangle \right) - \arg \left( \langle \psi_5 | \hat{D}_q | \psi_5 \rangle \right) \right)
\]

is an (approximate) integer. We find that if \( p = (1, 0) \) this is only true when \( q = (1, 0), (-1, 1) \) or \((0, -1) \) (mod 5), and that if \( p = (0, 1) \) it is only true when \( q = (0, 1), (-1, 0) \) or \((1, -1) \) (mod 5). Taking account of the requirement \( \text{Det}(F) = 1 \) (mod 5) we deduce that the only candidates are (apart from the identity)

\[
[F_5, \chi_5] = \left[ \begin{array}{cc} -1 & -1 \\ 1 & 0 \end{array} \right], \left[ \begin{array}{c} 2 \\ 2 \end{array} \right]
\]

and its square, \([F_5, \chi_5]^2\). To check that \( |\psi_5\rangle \) actually is an eigenvector of \([F_5, \chi_5]\) we observe that \( F_5 \) is a prime matrix. So in view of Lemma 4, we have the following explicit formula for the \( \hat{U} \in [F_5, \chi_5] \):

\[
\hat{U} = \frac{1}{\sqrt{3}} e^{i\theta} \hat{D}(2,2) \left( \sum_{r,s=0}^{4} e^{-\frac{2\pi i}{5} s(s+2r)} |e_r\rangle \langle e_s| \right)
\]

\( e^{i\theta} \) being an arbitrary phase. Suppose we choose \( \theta = \frac{7\pi}{15} \). Then we find \( \hat{U}^3 = 1 \) and

\[
\| (\hat{U} - 1) |\psi_5\rangle \|^2 = 0
\]

to machine precision. This confirms that \( |\psi_5\rangle \) is indeed an eigenvector of \([F_5, \chi_5]\).

To calculate the dimensions of the eigenspaces define, for \( r = 0, \pm 1 \) (and with the same choice of \( \theta \)),

\[
\hat{P}_r = \frac{1}{3} \left( 1 + e^{-\frac{2\pi i}{5}} \hat{U} + e^{\frac{2\pi i}{5}} \hat{U}^2 \right)
\]

Then \( \hat{P}_r \) projects onto the eigenspace of \( \hat{U} \) with eigenvalue \( e^{\frac{2\pi i}{5}} \). We find

\[
\text{Tr}(\hat{P}_r) = \begin{cases} 1, & r = 1 \\ 2, & r = -1 \text{ or } 0 \end{cases}
\]

implying that the dimensions of the eigenspaces are 1, 2, 2, and that \( |\psi_5\rangle \) is in one of the eigenspaces with dimension 2.

In dimensions 6 to 45 the calculation goes through in essentially the same way. The calculation is, however, slightly more complicated when \( d \) is even, due to the fact that we must then require \( \text{Det} F_d = 1 \) (mod 2d). Note, also, that when \( d = 6, 21, 24, 28 \) or 36 the matrix \( F_d \) is non-prime, so we have to use the decomposition of Lemma 3.

This method also enables us to establish the full stability group of \( |\psi_d\rangle \): i.e. the set of all operations (unitary or anti-unitary) \( \in E(d)/I(d) \) of which \( |\psi_d\rangle \) is an eigenvector. It turns out that, with one exception, the stability group is the order 3 cyclic subgroup generated by \([F_d, \chi_d]\). The exception is dimension 7, where the stability group is the order 6 cyclic subgroup generated by the anti-unitary operation

\[
[A_7, \xi_7] = \left[ \left( \begin{array}{cc} 2 & -1 \\ -1 & 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \right]
\]

Note that \([A_7, \xi_7]^2 = [F_7, \chi_7]\).
7. **Zauner’s Conjecture**

In the last section we saw that RBSC’s numerical results support Conjecture A. Their results also support Conjecture B: i.e. Zauner’s conjecture, that in each dimension $d$ there exists a GP fiducial vector which is an eigenvector of $[Z, 0]$.

In fact, for each $5 \leq d \leq 45$ let $[L_d, \eta_d]$ be the operation specified in Table 2. It is easily verified that

$$[L_d, \eta_d][F_d, \chi_d][L_d, \eta_d]^{-1} = [Z, 0] \quad (136)$$

This means that if $\hat{U} \in [L_d, \eta_d]$, and if $|\psi_d\rangle$ is the RBSC vector in dimension $d$, then $\hat{U}|\psi_d\rangle$ is a GP fiducial vector which is an eigenvector of $[Z, 0]$. Conjecture B is thus confirmed numerically for every dimension $\leq 45$.

This suggests

**Conjecture C:** GP fiducial vectors exist in every finite dimension.

Furthermore, every such vector is an eigenvector of a canonical order 3 unitary which is conjugate to $[Z, 0]$.

Conjecture C is clearly stronger than Conjecture B. It also implies Conjecture A.

An operation conjugate to $[Z, 0]$ is automatically a canonical order 3 unitary. It would be interesting to know whether the converse is also true: i.e. whether every canonical order 3 unitary is conjugate to $[Z, 0]$. If that were not the case Conjecture C would be strictly stronger than Conjecture A.
Table 3. Stability groups in dimensions 2–7. In every case the stability group includes an order 3 cyclic subgroup generated by a unitary operation having Clifford trace $= -1$.

8. Dimensions 2 to 7: Vectors, Orbits and Stability Groups

In dimensions 2–7 RBSC made a numerical search, in an attempt to find the total number of GP fiducial vectors. On the assumption that their search was exhaustive we use their data to calculate, for dimensions 2–7, the number of distinct orbits under the action of the extended Clifford group. We also calculate the order of the stability group corresponding to each orbit. Our results are tabulated in Table 3. They confirm that in dimensions 2–7 every GP fiducial vector is an eigenvector of a canonical order 3 Clifford unitary (in agreement with Conjecture A). We incidentally give exact expressions for two of the GP fiducial vectors in dimension 7 (one on each of the two distinct orbits).

The calculations on which these statements are based are somewhat lengthy, and there is not the space to reproduce them here. We therefore confine ourselves to summarizing the end results, which it is straightforward (albeit tedious) to confirm with the help of (for example) Mathematica.

Dimension 2. Exact solutions in dimension 2 have been obtained by Zauner [16] and RBSC [17]. In dimension 2 the GP fiducial vectors all lie on a single orbit of the extended Clifford group. Consider the GP fiducial vector

$$|\psi_2\rangle = \sqrt{(3 + \sqrt{3})/6} |e_0\rangle + e^{i\pi/4} \sqrt{(3 - \sqrt{3})/6} |e_1\rangle$$ (137)

The stability group of $|\psi_2\rangle$ is the order 6, non-Abelian subgroup of EC(2)/I(2) generated by the unitary operation

$$[F_2, 0] = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ (138)

and the three anti-unitary operations

$$[A_2, 0] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ (139)
\[ [B_2, 0] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, (0) \] (140)

\[ [C_2, 0] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, (0) \] (141)

Note that \([B_2, 0]\) is canonical order 3. It follows from Lemmas 5 and 6 that \(|EC(2)/I(2)| = 48\). So the orbit consists of \(48 \div 6 = 8\) fiducial vectors (identifying vectors which only differ by a phase), constituting 2 distinct SIC-POVM’s (as described by RBSC).

**Dimension 3.** Exact solutions in dimension 3 have been obtained by Zauner [16] and RBSC [17]. We saw in Section 5 that dimension 3 is unusual in that it is the only dimension for which the identity operator has Clifford trace = \(-1\). It seems to be unusual in another respect also: for it is the only case presently known where the GP fiducial vectors constitute infinitely many distinct orbits of the extended Clifford group.

Consider the one parameter family of GP fiducial vectors

\[ |\psi_3(t)\rangle = \frac{1}{\sqrt{2}} (e^{-it}|e_1\rangle - e^{it}|e_2\rangle) \] (142)

The complete set of GP fiducial vectors is obtained by acting on the vectors \(|\psi_3(t)\rangle\) with elements of EC(3).

Let \(\hat{T}\) and \(\hat{J}\) be the operators defined by Eqs. (6) and (106) respectively. Then

\[ \hat{T}|\psi_3(t)\rangle = -|\psi_3(t + \frac{\pi}{3})\rangle \quad \text{and} \quad \hat{J}|\psi_3(t)\rangle = |\psi_3(-t)\rangle \] (143)

So \(|\psi_3(t)\rangle\) and \(|\psi_3(t')\rangle\) are on the same orbit if \(t' = \frac{3t}{4} \pm t\) for some integer \(n\). At the cost of rather more computational effort one can show that this condition is not only sufficient but also necessary for \(|\psi_3(t)\rangle\) and \(|\psi_3(t')\rangle\) to be on the same orbit. So for each distinct orbit there is exactly one value of \(t \in [0, \frac{\pi}{6}]\) such that \(|\psi_3(t)\rangle\) is on the orbit.

There are three kinds of orbit: a set of infinitely many generic orbits corresponding to values of \(t\) in the interior of the interval \([0, \frac{\pi}{6}]\), and two exceptional orbits corresponding to the two end points \(t = 0\) and \(\frac{\pi}{6}\).

The stability group of the exceptional vector \(|\psi_3(0)\rangle\) consists of all 48 operations of the form \([F, 0]\), where \(F\) is any element of ESL(2, \(\mathbb{Z}_3\)). The orbit thus consists of \(432 \div 48 = 9\) fiducial vectors, constituting a single SIC-POVM.

The stability group of the exceptional vector \(|\psi_3(\frac{\pi}{6})\rangle\) is the order 12 non-Abelian subgroup of EC(3)/I(3) generated by the unitary operation

\[ [F_3, \chi_3] = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, (0) \] (144)

and the anti-unitary operation

\[ [A_3, \chi_3] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, (0) \] (145)

Note that

\[ [F_3, \chi_3]^2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, (0) \] (146)

is canonical order 3. The orbit thus consists of \(432 \div 12 = 36\) fiducial vectors, constituting 4 distinct SIC-POVMs.

The stability group of a generic vector \(|\psi_3(t)\rangle\) with \(0 < t < \frac{\pi}{6}\) is the order 6 non-Abelian subgroup generated by the unitary operation

\[ [F_3, \chi_3]^2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, (0) \] (147)
and the anti-unitary operation

\[ [F_3, \chi_3] \circ [A_3, \chi_3] = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]  

(148)

The orbit thus consists of \(432 \div 6 = 72\) fiducial vectors, constituting 8 distinct SIC-POVMs.

**Dimension 4.** The vector

\[
|\psi_4\rangle = \sqrt{\frac{5 - \sqrt{5}}{40}} \left( 2 \cos \frac{\pi}{8} |e_0\rangle + i \left( e^{-\frac{i\pi}{8}} + (2 + \sqrt{5}) \frac{i}{\sqrt{2}} e^{\frac{i\pi}{8}} \right) |e_1\rangle + 2i \sin \frac{\pi}{8} |e_2\rangle + i \left( e^{-\frac{i\pi}{8}} - (2 + \sqrt{5}) \frac{i}{\sqrt{2}} e^{\frac{i\pi}{8}} \right) |e_3\rangle \right) 
\]

(149)

is a GP fiducial vector in dimension 4, as discovered by Zauner [16] and RBSC [17].

The stability group of \(|\psi_4\rangle\) is the order 6 cyclic subgroup of \(\mathrm{EC}(4)/I(4)\) generated by the anti-unitary operation

\[ [A_4, \chi_4] = \begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]  

(152)

Note that

\[ [A_4, \chi_4]^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]  

(153)

is canonical order 3 (where we used Eq. (81) to obtain the last expression on the right hand side).

It follows from Lemmas 5 and 6 that the group \(\mathrm{EC}(4)/I(4)\) is of order 1536. So the orbit generated by \(|\psi_4\rangle\) contains 1536 \(\div 6 = 256\) fiducial vectors, constituting 256 \(\div 16 = 16\) SIC-POVMs. It was shown by RBSC that there are only 16 SIC-POVMs in dimension 4. We conclude that the fiducial vectors all lie on a single orbit of the extended Clifford group.

**Dimension 5.** Let \(|\psi_5\rangle\) be RBSC’s numerical vector in dimension 5. We noted in the last section that the stability group of \(|\psi_5\rangle\) is of order 3. It follows from Lemmas 5 and 6 that the group \(\mathrm{EC}(5)/I(5)\) is of order 6000. So the orbit generated by \(|\psi_5\rangle\) contains 6000 \(\div 3 = 2000\) fiducial vectors, constituting 2000 \(\div 25 = 80\) SIC-POVMs. It was shown by RBSC that there are only 80 SIC-POVMs in dimension 5. We conclude that the fiducial vectors all lie on a single orbit of the extended Clifford group.

Note that Zauner’s analytic solution in dimension 5 (on p. 63 of his thesis [10]) can be used to give exact expressions for each of the vectors on the orbit.

---

\[ e^{-\frac{i\pi}{8}} \left( X^\psi_{1a} + \rho^3 Y^\psi_{1b} \right) \]

(150)

In RBSC’s notation it is the vector

\[
|\psi_4\rangle = r_0 |e_0\rangle + r_3 e^{i\theta_3} |e_1\rangle + r_1 e^{i\theta_1} |e_2\rangle + r_{-} e^{i\theta_{-}} |e_3\rangle 
\]

(151)

for the case \(n = j = m = 1\) and \(k = 0\) (note, however, that there is a typographical error in RBSC [17]: their expression for \(r_0\) should read \(r_0 = \sqrt{\frac{1 - 1/\sqrt{5}}{2\sqrt{2 - 2\sqrt{5}}}}\).
Dimension 6. Let $|\psi_6\rangle$ be RBSC’s numerical vector in dimension 6. We noted in the last section that the stability group of $|\psi_6\rangle$ is of order 3. It follows from Lemmas 5 and 6 that the group $EC(6)/I(6)$ is of order 10368. So the orbit generated by $|\psi_6\rangle$ contains $10368 \div 3 = 3456$ fiducial vectors, constituting $3456 \div 36 = 96$ SIC-POVMs. It was shown by RBSC that there are only 96 SIC-POVMs in dimension 5. We conclude that the fiducial vectors all lie on a single orbit of the extended Clifford group (in agreement with Grassl’s analysis, based on his exact solution in dimension 6).

Note that Grassl’s analytic solution can be used to give exact expressions for each of the vectors on the orbit.

Dimension 7. Let $|\psi_7\rangle$ be RBSC’s numerical vector in dimension 7. We noted in the last section that the stability group of $|\psi_7\rangle$ is of order 6. It follows from Lemmas 5 and 6 that the group $EC(6)/I(6)$ is of order 32928. So the orbit generated by $|\psi_7\rangle$ contains $32928 \div 6 = 5488$ fiducial vectors, constituting $5488 \div 49 = 112$ SIC-POVMs. However, it was shown by RBSC that there are 336 SIC-POVMs in dimension 7. We conclude that there must be at least one other orbit. The search for the additional orbit or orbits is facilitated by the fact that in dimension 7 there exists a canonical order 3 Clifford unitary for which the $F$ matrix is diagonal: namely

$$[F'_7, 0] = \left[ \begin{array}{cc} -3 & 0 \\ 0 & 2 \end{array} \right], \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$$

(154)

The fact that $F'_7$ is diagonal means that the $\hat{U} \in [F'_7, 0]$ are permutation matrices. Specifically

$$\hat{U} = e^{i\theta} \sum_{r=0}^{6} |e_r\rangle\langle e_r|$$

(155)

for every $\hat{U} \in [F'_7, 0]$ (where $e^{i\theta}$ is an arbitrary phase, and where we have used the decomposition described in Lemma 3). This considerably simplifies the calculations. We will also have occasion to consider the anti-unitary operation

$$[A'_7, 0] = \left[ \begin{array}{cc} 2 & 0 \\ 0 & -3 \end{array} \right], \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$$

(156)

which is a square root of $[F'_7, 0]$.

We look for eigenvectors of $[F'_7, 0]$. Let

$$l_r = \begin{cases} 1 & \text{if } r = 1, 2 \text{ or } 4 \\ -1 & \text{if } r = 3, 5 \text{ or } 6 \end{cases}$$

(157)

Also let

$$a_0 = \frac{1}{2} \left( \sqrt{\frac{1}{4 - \sqrt{2}}} + i \sqrt{\frac{4 - \sqrt{2}}{2}} \right), \quad a_1 = \frac{1}{4} \sqrt{\frac{8 - 5\sqrt{2}}{7}}, \quad a_2 = 2^{-\frac{7}{4}}$$

(158)

and

$$b_0 = \sqrt{\frac{2 + 3\sqrt{2}}{14}}, \quad b_1 = \sqrt{\frac{4 - \sqrt{2}}{28}}, \quad \theta = \cos^{-1} \left( \frac{\sqrt{\sqrt{2} + 1}}{2} \right)$$

(159)
Then define
\[ |\psi'_7\rangle = a_0 |e_0\rangle - \sum_{r=1}^{6} (a_1 + l_r a_2) |e_r\rangle \]  \hfill (160)
\[ |\psi''_7\rangle = b_0 |e_0\rangle + \sum_{r=1}^{6} b_1 e^{i l_r \theta} |e_r\rangle \]  \hfill (161)

It is readily confirmed that \(|\psi'_7\rangle\) and \(|\psi''_7\rangle\) are both GP fiducial vectors. The stability group of \(|\psi'_7\rangle\) is the order 3 subgroup generated by \([F'_7, 0]\), while the stability group of \(|\psi''_7\rangle\) is the order 6 subgroup generated by \([A'_7, 0]\). Since the stability groups are non-isomorphic the orbits generated by \(|\psi'_7\rangle\) and \(|\psi''_7\rangle\) are disjoint. The orbit generated by \(|\psi'_7\rangle\) contains \(32928 \div 3 = 10976\) fiducial vectors, constituting 10976 \(\div\) 49 = 224 SIC-POVMs. The orbit generated by \(|\psi''_7\rangle\) contains 5488 fiducial vectors, constituting a further 112 SIC-POVMs. This accounts for all 336 of the SIC-POVMs identified by RBSC. We conclude that there are no other orbits, apart from these two.

For the sake of completeness let us note that
\[ |\psi_7\rangle = \hat{U} |\psi''_7\rangle \]  \hfill (162)
where \(|\psi_7\rangle\) is RBSC’s numerical vector and \(\hat{U}\) is a unitary operator
\[ \hat{U} \in \left[ \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] \]  \hfill (163)

Finally, let us remark that \(l_r\) is the Legendre symbol (see, e.g., Nathanson \[33\] or Rose \[34\])
\[ l_r = \begin{cases} 1 & \text{if } r = 1, 4, 5, 6, 7, 9, 11, 16 \text{ or } 17 \\ -1 & \text{if } r = 2, 3, 8, 10, 12, 13, 14, 15 \text{ or } 18 \end{cases} \]  \hfill (164)

It has the important property that \(l_r = l_s l_r\) for all \(r, s \in \mathbb{Z}\).

9. A Fiducial Vector in Dimension 19

In Section 8 we saw that, except in dimension 7, each of RBSC’s numerical solutions has stability group of order 3. This might encourage one to speculate that when \(d > 7\) the stability group is always of order 3. In this section we show that there is at least one exception to that putative rule, by constructing a GP fiducial vector in dimension 19 for which the stability group has order \(\geq 18\).

The vector we construct is an eigenvector of the order 18 anti-unitary operation
\[ [A'_{19}, 0] = \left[ \begin{pmatrix} -9 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] \in EC(19)/I(19) \]  \hfill (165)
Note that
\[ [F'_{19}, 0] = [A'_{19}, 0]^6 = \left[ \begin{pmatrix} -8 & 0 \\ 0 & 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right] \]  \hfill (166)
is canonical order 3.

The construction is similar to our construction of the vector \(|\psi''_7\rangle\) in the last section. Let \(l'_r\) be the Legendre symbol
\[ l'_r = \begin{cases} 1 & \text{if } r = 1, 4, 5, 6, 7, 9, 11, 16 \text{ or } 17 \\ -1 & \text{if } r = 2, 3, 8, 10, 12, 13, 14, 15 \text{ or } 18 \end{cases} \]  \hfill (167)
and let
\[ b'_0 = \sqrt{\frac{5 + 9\sqrt{5}}{95}} \quad b'_1 = \sqrt{\frac{10 - \sqrt{5}}{190}} \quad \theta' = \cos^{-1}\left(\sqrt{\frac{\sqrt{5} - 1}{8}}\right) \]  \hfill (168)
Then define
\[ |\psi'_19\rangle = b'_0|e_0\rangle + \sum_{r=1}^{18} b'_1 e^{i\theta'_r} |e_r\rangle \] (169)

It is readily confirmed that \( |\psi'_19\rangle \) is a GP fiducial vector, and an eigenvector of \( [A'_19, 0] \).

Observe that the orbit generated by \( |\psi'_19\rangle \) is disjoint from the orbit generated by RBSC’s numerical vector \( |\psi_{19}\rangle \) (because the stability groups are non-isomorphic). It follows that there are at least two distinct orbits in dimension 19.

10. Diagonalizing the \( F \) matrix

Our construction of the exact solutions \( |\psi'_7\rangle, |\psi''_7\rangle \) and \( |\psi'_19\rangle \) in Eqs. (160), (161) and (169) was facilitated by the fact that in dimensions 7 and 19 there exist canonical order 3 unitaries for which the corresponding \( F \) matrix is diagonal. It is natural to ask in what other dimensions that is true. The theorem proved below answers that question.

We will need the following lemma:

**Lemma 8.** Let \( p \) be a prime number \( = 1 \pmod{3} \), and let \( n \) be any integer \( \geq 1 \). Then there exists an integer \( \alpha \) such that
\[ \alpha^2 + \alpha + 1 = 0 \pmod{p^n} \] (170)

**Proof.** The proof relies heavily on the theory of primitive roots, as described in (for example) Chapter 3 of Nathanson [33] or Chapter 5 of Rose [34]. Let \( \phi \) be Euler’s phi, or totient function (so for every integer \( x \geq 1 \), \( \phi(x) \) is the number of integers \( y \) in the range \( 1 \leq y < x \) which are relatively prime to \( x \)). Then there exists a single positive integer \( g \) such that for every integer \( m \geq 1 \) the multiplicative order of \( g \), considered as an element of \( \mathbb{Z}_{p^m} \), is \( \phi(p^m) = (p-1)p^{m-1} \) (see, for example, Nathanson [33], p. 93, or Rose [34], p. 91). The fact that \( p = 1 \pmod{3} \) means \( p = 3k + 1 \) for some integer \( k \geq 1 \). Define
\[ \alpha = g^{kp^{n-1}} \] (171)

It is then immediate that
\[ \alpha^3 = g^{\phi(p^n)} = 1 \pmod{p^n} \] (172)

It is also true that \( \alpha - 1 \) is relatively prime to \( p \). For suppose that were not the case. It would then follow from the definition of \( \alpha \), and the fact that \( g \) is a primitive root modulo \( p \), that
\[ kp^{n-1} = l(p-1) = 3kl \] (173)
for some integer \( l \geq 1 \). That, however, is impossible since \( p \) is not a multiple of 3.

The fact that \( \alpha - 1 \) is relatively prime to \( p \) means that there exists an integer \( \beta \) such that
\[ \beta(\alpha - 1) = 1 \pmod{p^n} \] (174)

It now follows from Eqs. (172) and (174) that
\[ \alpha^2 + \alpha + 1 = \beta(\alpha^3 - 1) = 0 \pmod{p^n} \] (175)

We are now in a position to prove our main result:

**Theorem 3.** There exists a canonical order 3 unitary \( [F, \chi] \in C(d)/I(d) \) for which the matrix \( F \) is diagonal if and only if each of the following is true
1. \( d \) has at least one prime divisor \( = 1 \pmod{3} \).
2. \( d \) has no prime divisors \( = 2 \pmod{3} \).
3. \( d \) is not divisible by 9.
Remark. So there exist canonical order 3 unitaries $[F, \chi]$ for which $F$ is diagonal in dimension $7, 13, 19, 21, 31, 37, 39, 43, 49, \ldots$

Proof. We begin by proving sufficiency. Suppose that conditions (1), (2) and (3) are all true. Then we have, for some $t \geq 1$,

$$d = 3^{n_0} p_1^{n_1} \cdots p_t^{n_t}$$  \hspace{1cm} (176)

where the $p_i$ are distinct prime numbers $\equiv 1 \pmod{3}$, where the integer $n_0 = 0$ or 1, and where the integers $n_1, \ldots, n_t$ are all $\geq 1$. It follows from Lemma 8 that there exist integers $\alpha_1, \ldots, \alpha_t$ such that

$$\alpha_i^2 + \alpha_i + 1 = 0 \pmod{p_i^{n_i}}$$  \hspace{1cm} (177)

for $i = 1, \ldots, t$. We then use the Chinese remainder theorem (see, for example, Nathanson [33] or Rose [34]) to deduce that there exists a single integer $\alpha$ such that

$$\alpha = 1 \pmod{3}$$  \hspace{1cm} (178)

and

$$\alpha = \alpha_i \pmod{p_i^{n_i}}$$  \hspace{1cm} (179)

for $i = 1, \ldots, t$. We have

$$\alpha^2 + \alpha + 1 = 0 \pmod{3}$$  \hspace{1cm} (180)

and

$$\alpha^2 + \alpha + 1 = 0 \pmod{p_i^{n_i}}$$  \hspace{1cm} (181)

for $i = 1, \ldots, t$. Consequently

$$\alpha^2 + \alpha + 1 = 0 \pmod{d}$$  \hspace{1cm} (182)

It follows that the matrix

$$F = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha - 1 \end{pmatrix}$$  \hspace{1cm} (183)

$\in$ SL(2, $\mathbb{Z}_d$) (bearing in mind that $d$ is odd). Moreover, Tr($F$) $= -1 \pmod{d}$. Since $d \neq 3$ we conclude that $[F, \chi]$ is a canonical order 3 unitary for all $\chi \in (\mathbb{Z}_d)^2$. This proves sufficiency.

To prove necessity suppose

$$F = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in$ SL(2, $\mathbb{Z}_d$)$$  \hspace{1cm} (184)

is such that $[F, \chi]$ is canonical order 3 for some $\chi \in (\mathbb{Z}_d)^2$. Then $\alpha + \delta = -1 \pmod{d}$, implying

$$\alpha^2 + \alpha + 1 = 0 \pmod{d}$$  \hspace{1cm} (185)

$$\alpha^3 = 1 \pmod{d}$$  \hspace{1cm} (186)

(in view of the fact that $\alpha \delta = 1 \pmod{d}$).

To show that $d$ has no prime divisors $\equiv 2 \pmod{3}$ assume the contrary. It would then follow from Eqs. (185) and (186) that

$$\alpha^2 + \alpha + 1 = 0 \pmod{p}$$  \hspace{1cm} (187)

$$\alpha^3 = 1 \pmod{p}$$  \hspace{1cm} (188)

for some prime number $p = 2 \pmod{3}$. Let $r$ be a primitive root of $p$ and let $k \in \mathbb{Z}$ be such that $0 \leq k < p - 1$ and $\alpha = r^k \pmod{p}$ (see, for example, Nathanson [33] or Rose [34]). Then Eq. (188) implies $r^{3k} = 1 \pmod{p}$ which, in view of the fact that $r$ is a primitive root, means $3k = l(p - 1)$ for some $l \in \mathbb{Z}$. The fact that
0 ≤ k < p − 1 implies 0 ≤ l < 3. Taking into account the fact that p − 1 is not divisible by 3 (because p = 2 (mod 3)) we deduce that l = 0. But then k = 0, implying α = 1 (mod p). In view of Eq. (185) this means 3 = 0 (mod p): which is a contradiction.

To prove that d is not divisible by 9 we again proceed by reductio ad absurdum. Suppose that d were divisible by 9. It would then follow from Eq. (185) that

\[ \alpha^2 + \alpha + 1 = 0 \pmod{9} \]

(189)

However, it is easily verified (by explicit enumeration) that this equation has no solutions.

Finally, suppose that d had no prime divisors = 1 (mod 3). In view of the results just proved it would follow that d = 3. But if d = 3, Eq. (185) implies α = 1 (mod 3). Taking into account the requirement αδ = det F = 1 (mod 3) this means δ = 1 (mod 3). But then F is the identity matrix, which contradicts the assumption that [F, χ] is a canonical order 3 unitary. We conclude that d must have at least one prime divisor = 1 (mod 3).

11. Conclusion

RBSC conclude their paper by saying “a rigorous proof of existence of SIC-POVMs in all finite dimensions seems tantalizingly close, yet remains somehow distant”. That well expresses our own perception of the matter. While working on this problem we have several times had the sense that the crucial discovery lay just round the corner, only to find that our hopes were illusory. We make our results public in the hope that they may, nevertheless, contain a few clues, which will help to take us further forward.

In particular it seems to us that significant progress would be made if it could be established whether it is in fact true that every GP fiducial vector is an eigenvector of a canonical order 3 unitary. Also, if that is the case, one would like to know exactly why it is the case.

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