Quantum Sp(2)-antibrackets and open groups

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Abstract

The recently presented quantum antibrackets are generalized to quantum Sp(2)-antibrackets. For the class of commuting operators there are true quantum versions of the classical Sp(2)-antibrackets. For arbitrary operators we have a generalized bracket structure involving higher Sp(2)-antibrackets. It is shown that these quantum antibrackets may be obtained from generating operators involving operators in arbitrary involutions. A recently presented quantum master equation for operators, which was proposed to encode generalized quantum Maurer-Cartan equations for arbitrary open groups, is generalized to the Sp(2) formalism. In these new quantum master equations the generalized Sp(2)-brackets appear naturally.

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1 Introduction.

In [1] we introduced a quantum antibracket defined by

\[(f, g)_Q \equiv \frac{1}{2} \left( [f, [Q, g]] - [g, [Q, f]] \right) \left( (-1)^{(\varepsilon_f+1)(\varepsilon_g+1)} \right). \tag{1} \]

where \( Q \) is an odd, hermitian, and nilpotent operator \((Q^2 = 0)\). This expression satisfies all desired properties of a quantum antibracket for a class of commuting operators provided \( Q \) is such that (1) belongs to the same class of commuting operators as \( f \) and \( g \). The so-called quantum master equation in the BV quantization was then shown to have the general form

\[ Q|\phi\rangle = 0. \tag{2} \]

Remarkably enough the bracket (1) satisfies a consistent algebra even for arbitrary operators \( f \) and \( g \). However, for arbitrary operators (1) violates Leibniz’ rule, and the would-be Jacobi identities couple to higher order brackets defined in a definite way [1, 2, 3]. The 3-antibracket was explicitly given in [2]. For operators in involution it is possible to construct generating operators for (1) and all higher antibrackets [1, 2]. Furthermore, as was shown in [1, 2], there is a new type of quantum master equation which seems to encode the generalization of the Maurer-Cartan equations for arbitrary quantum open groups.

In the present paper we generalize most of the results of [1, 2] to quantum Sp(2)-antibrackets. (The classical Sp(2)-antibrackets appear in the Sp(2)-extended BV quantization [3, 4, 5].) As was already mentioned in [1] an Sp(2)-generalization of (1) involves two odd, hermitian, and nilpotent operators \( Q^a, a = 1, 2 \), which anticommute (see (5) below). The general quantum master equation (2) generalizes then to

\[ Q^a|\phi\rangle = 0, \quad a = 1, 2, \tag{3} \]

which indeed are natural equations since the Sp(2) formalism is directly related to the so-called BRST-antiBRST quantization [7]. (Nilpotent anticommuting operators and Sp(2)-antibrackets are the basic ingredients in the Sp(2)-formalism.)

In section 2 we define and give some properties of the quantum Sp(2)-antibrackets. We also give the operator form of triplectic quantization [3, 4]. In sections 3, 4 and 5 we generalize the results of [2] to the Sp(2)-case. In section 3 we define Lie equations for finite gauge transformations generated by constraints in arbitrary involutions within an Sp(2)-extended BFV-BRST scheme. In section 4 these Lie equations are used to define generating operators of the quantum Sp(2)-antibrackets for operators in arbitrary involutions. In section 5 we present quantum master equations for the integrability conditions of the Lie equations in section 3, which may be viewed as generalized quantum Maurer-Cartan equations within the Sp(2)-formalism. Finally we end the paper in section 6 where we give some formal properties of the quantum Sp(2)-antibrackets in connection with triplectic quantization. We also give the general automorphisms of the quantum master equations in [3] and section 5. In appendix A we give the defining properties of the conventional classical Sp(2)-antibrackets, and in appendix B we review the basics of the Sp(2) extended BFV-BRST scheme. It is shown how constraints in arbitrary involutions may be embedded in two nilpotent charges with an Sp(2)-relation.
2 Quantum Sp(2)-antibrackets.

The quantum Sp(2)-antibrackets are defined by

\[ (f, g)_Q^a \equiv \frac{1}{2} \left( [f, [Q^a, g]] - [g, [Q^a, f]](-1)^{(\varepsilon_f+1)(\varepsilon_g+1)} \right), \]  

where the odd operators \( Q^a \) satisfy

\[ Q^{(a}Q^{b)} \equiv Q^aQ^b + Q^bQ^a \equiv [Q^a, Q^b] = 0. \]  

These properties imply the relations

\[ [Q^{(a}, (f, g)_{Q}^{b)}] = ([Q^{(a}, f]^{b)}_{Q} + (f, [Q^{(a}, g)]^{b)}_{Q})(-1)^{\varepsilon_f+1} = [[Q^{(a}, f], [Q^{b)}, g]]. \]  

The expressions (4) satisfy all the properties of the corresponding classical Sp(2)-antibrackets as defined in the appendix A except for the Jacobi identities 4) and Leibniz’ rule 5). Instead of Leibniz’ rule we have

\[ (fg, h)_{Q}^{a} - f(g, h)_{Q}^{a} - (f, h)_{Q}^{a}g(-1)^{\varepsilon_g} = \frac{1}{2} \left( [f, h][g, Q^a](-1)^{\varepsilon_h} + [f, Q^a][g, h](-1)^{\varepsilon_g} \right), \]  

and instead of the Jacobi identities we have

\[ (f, (g, h)_{Q}^{(a}Q^{b)})(-1)^{\varepsilon_f+1} = cycle(f, g, h) = \]  

\[ = -\frac{1}{2}[(f, g, h)_{Q}^{(a}Q^{b)}(-1)^{\varepsilon_f+1}], \]  

where

\[ (f, g, h)_{Q}^{(a}Q^{b)}(-1)^{\varepsilon_f+1} \equiv \frac{1}{3} \left( [(f, g)_{Q}^{a}, h](-1)^{\varepsilon_h+\varepsilon_f+1} + cycle(f, g, h) \right) \]  

are quantum 3-antibrackets within the Sp(2) formalism. In fact, by means of generalized Jacobi identities for these 3-antibrackets we may derive 4-antibrackets and so on. (Similar classical higher Sp(2)-antibrackets were considered in [8].) These higher antibrackets are expressed in terms of the lower ones and should terminate at a certain level depending on the properties of \( Q^a \). The quantum antibrackets (3) with \( Q^a \) operators satisfying (5) determine therefore a consistent, generalized scheme involving higher antibrackets and a modified Leibniz’ rule. In sections 5 and 6, when we consider generating operators of these brackets and new kind of master equations, the above generalized scheme will appear naturally.

From (3) it is clear that Leibniz’ rule is satisfied if we restrict ourselves to the class of commuting operators \( f, g, h \). This class we denote by \( \mathcal{M} \) in the following. In terms of operators on \( \mathcal{M} \) the brackets (3) reduce to

\[ (f, g)_{Q}^{a} \equiv [f, [Q^a, g]] = [[f, Q^a], g], \quad \forall f, g \in \mathcal{M}. \]  

\[ ^{3}Eqs.(3)-(6), (10) \text{ and (11) were also given in} \]
If we, furthermore, require the vanishing of the 3-antibrackets \( \langle \rangle \) for operators on \( \mathcal{M} \), then these brackets satisfy all properties corresponding to the defining properties of the classical \( \text{Sp}(2) \)-antibrackets as given in the appendix A. This means that \( 10 \) then are the true quantum \( \text{Sp}(2) \)-antibrackets for the class of commuting operators. The vanishing of the 3-antibrackets restrict the form of the odd operators \( Q^a \) in such a way that the quantum antibrackets \( 10 \) of commuting operators also belong to the same class of commuting operators. We have

\[
[f, [g, [h, Q^a]]] = 0, \quad \forall f, g, h \in \mathcal{M} \quad \Leftrightarrow \quad (f, g)_Q^a \in \mathcal{M}, \quad \forall f, g \in \mathcal{M}.
\]  

(11)

In order to demonstrate the existence of \( Q^a \) operators satisfying \( 3 \) and \( 11 \) we give an explicit representation. Consider a triplectic manifold with Darboux coordinates \( x^\alpha \) and \( x^{*\alpha} \), \( \alpha = 1, \ldots, n \), where \( \varepsilon(x^{*\alpha}) = \varepsilon_\alpha + 1, \varepsilon_\alpha \equiv \varepsilon(x^\alpha) \) (see \( 3, 5, 6 \)). Consider then these Darboux coordinates to be coordinates on a symplectic manifold. The canonical coordinates of this symplectic manifold are then \( \{ x^\alpha, x^{*\alpha}, p_\alpha, p^{*\alpha} \} \). After quantization we choose all operators which depends on \( x^\alpha \) and \( x^{*\alpha} \) as the class of commuting operators \( \mathcal{M} \). We may then define the quantum antibracket by \( 3 \) with \( Q^a \) given by

\[
Q^a = p_\alpha p^{*\alpha}(-1)^{\varepsilon_\alpha},
\]

(12)

which obviously satisfy \( 3 \) and \( 11 \). In this case we have

\[
(f, g)_Q^a = [f, [Q^a, g]] = [f, Q^a], g = (-1)^{\varepsilon_\alpha}[f, p_\alpha][p^{*\alpha}, g] - (-1)^{\varepsilon_\alpha}[g, p_\alpha][p^{*\alpha}, f](-1)^{(\varepsilon_1 + 1)(\varepsilon_2 + 1)}.
\]

(13)

Since the nonzero canonical commutation relations are,

\[
[x^\alpha, p_\beta] = i\hbar \delta^\alpha_\beta, \quad [x^{*\alpha}, p^{*\beta}] = i\hbar \delta^{*\alpha}_\beta, p_\alpha = -i\hbar \partial_\alpha(-1)^{\varepsilon_\alpha}, \quad p^{*\alpha} = i\hbar \partial^{*\alpha}(-1)^{\varepsilon_\alpha},
\]

(14)

we find

\[
(i\hbar)^{-2}(f, g)_Q^a = f \overset{\alpha}{\partial}_a \partial^{*\alpha} \partial^{*\alpha} f - g \overset{\alpha}{\partial}_a \partial^{*\alpha} f(-1)^{(\varepsilon_1 + 1)(\varepsilon_2 + 1)}
\]

(15)

in accordance with the correspondence between quantum and classical antibrackets as given in \( 3 \), i.e. \( (i\hbar)^{-2}(f, g)_Q^a \leftrightarrow (f, g)^a \). The wave function representation \( \Delta^a \) of the \( \text{Sp}(2) \)-charges \( 12 \) follows from the equalities

\[
\langle x, x^*|Q^a|\phi \rangle = -(i\hbar)^2 \Delta^a \phi(x, x^*), \quad \phi(x, x^*) \equiv \langle x, x^*|\phi \rangle.
\]

(16)

Obviously, \( \Delta^a \) are the odd differential operators

\[
\Delta^a = (-1)^{\varepsilon_\alpha} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial x^{*\alpha}},
\]

(17)

which play a crucial role in the classical \( \text{Sp}(2) \) formalism \( 3 \).

More general brackets are obtained if we consider general coordinates \( X^A = (x^\alpha; x^{*\alpha}) \), \( \varepsilon(X^A) \equiv \varepsilon_A \). The most general form of the operators \( Q^a \) which yield \( \text{Sp}(2) \)-antibrackets satisfying all required properties are of the form

\[
Q^a = -\frac{1}{2}\rho^{-1/2} P_{A\rho} E^{ABa} P_{B\rho}^{-1/2}(-1)^{\varepsilon_B},
\]

(18)
where $E^{ABa}(X) = -E^{Baa}(X)(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}$ and $\rho(X)$ are the triplectic metric and the volume form density respectively. $P_A$ satisfies

$$[X^A, P_B] = i\hbar \delta^A_B; \quad P_A = -i\hbar \rho^{-1/2} \partial_A \circ \rho^{1/2}(-1)^{\varepsilon_A}, \quad \partial_A \equiv \partial/\partial X^A.$$  \hfill (19)

By means of (10) the quantum antibrackets (15) generalize here to (cf [5, 6])

$$(i\hbar)^{-2} (f, g)^a = f \left( \partial_A E^{ABa} \partial_B g \right),$$  \hfill (20)

where the commuting operators $f$ and $g$ are arbitrary functions of $X^A$. These are the general forms of the Sp(2)-brackets. The property (5) of $Q^a$ requires the tensor $E^{ABa}$ to satisfy the cyclic relations [5, 6]

$$E^{AD} \{ a \partial_D E^{BScb} \} (-1)^{(\varepsilon_A+1)(\varepsilon_C+1)} + \text{cycle}(A, B, C) = 0,$$  \hfill (21)

which make the antibrackets (20) satisfy the Jacobi identities. If momenta $P_A$ are allowed to enter the operators $Q^a$ more than quadratically then a nonzero contribution appears on the right-hand side of (21). This means that we then have nonzero 3-antibrackets (9).

In triplectic quantization the general master equations are (see [3])

$$Q^a_+ |\phi\rangle = 0,$$  \hfill (22)

where

$$Q^a_\pm \equiv -\frac{1}{2} \rho^{-1/2} (P_A \mp F_A) \partial_A \rho E^{ABa} (P_B \mp F_B) \rho^{-1/2}(-1)^{\varepsilon_B}.$$  \hfill (23)

$F_A$ are functions of the operators $X^A$. (This $F_A$ differ from the $F_A$ in [6] by a sign factor $(-1)^{(\varepsilon_A+1)}$.) $F_A$, $E^{ABa}$, and $\rho$ are required to be such that $Q^a_\pm$ are hermitian and

$$[Q^a_+, Q^b_+] = 0, \quad F_A E^{ABa} F_B (-1)^{\varepsilon_B} = 0.$$  \hfill (24)

Note that the antibrackets defined in terms of $Q^a_\pm$ are the same as those defined in terms of $Q^a$ in [18] for operators which are functions of $X^A$. The partition function $Z$, i.e. the path integral of the gauge fixed action, is here given by $Z = \langle \mathcal{X}|\mathcal{W}\rangle$, where $|\mathcal{W}\rangle$ is the master state and $|\mathcal{X}\rangle$ a gauge fixing state both satisfying the quantum master equations (22). $|\mathcal{W}\rangle$ and $|\mathcal{X}\rangle$ have the general form

$$|\mathcal{W}\rangle = \exp \left\{ \frac{i}{\hbar} \mathcal{W}(X) \right\} \rho^{1/2}|0\rangle_{P\pi}, \quad |\mathcal{X}\rangle = \exp \left\{ -\frac{i}{\hbar} \lambda^\dagger(X, \lambda) \right\} \rho^{1/2}|0\rangle_{P\pi},$$ \hfill (25)

where $\lambda^a$ are Lagrange multipliers and $\pi_\alpha$ their conjugate momenta. The vacuum state $|0\rangle_{P\pi}$ satisfies $P_A|0\rangle_{P\pi} = \pi_\alpha|0\rangle_{P\pi} = 0$. By means of coordinate states $|X, \lambda\rangle$ satisfying the completeness relations

$$\int |X, \lambda\rangle \rho(X) dX d\lambda |X, \lambda\rangle = 1,$$  \hfill (26)

and $\langle X, \lambda|\rho^{1/2}|0\rangle_{P\lambda} = 1$, the partition function $Z = \langle \mathcal{X}|\mathcal{W}\rangle$ becomes explicitly

$$Z = \langle \mathcal{X}|\mathcal{W}\rangle = \int \rho(X) dX d\lambda \exp \left\{ \frac{i}{\hbar} \left[ \mathcal{W}(X) + \mathcal{X}(X, \lambda) \right] \right\}$$  \hfill (27)
where \( W \) and \( X \) in the path integral denotes the master action and gauge fixing actions
\( \langle X, \lambda | W(X) \rho^{1/2} | 0 \rangle_{P, \lambda} \) and \( \langle X, \lambda | X(X, \lambda) \rho^{1/2} | 0 \rangle_{P, \lambda} \) respectively, which by (22) and (23) satisfy the quantum master equations given in [5, 6]. In fact, (27) agrees with the general partition function in [5, 6] and for its precise meaning we refer to these papers. (When (25) satisfies (2) with \( Q \) given by (see (48) in [1])
\[
Q \equiv -\frac{1}{2} \rho^{-1/2} P_A \rho^{2 AB} P_B \rho^{-1/2} (-1)^{\epsilon_B},
\]
then (27) represents the partition function for generalized BV quantization. This generalizes appendix A in [1].)

### 3 Integrating open groups within the Sp(2)-formalism.

Finite gauge transformations are obtained by integrations of Lie equations like
\[
A(\phi) \dot{\theta}_\alpha \equiv A(\phi) \partial_\alpha - (i\hbar)^{-1} [A(\phi), Y_\alpha(\phi)] = 0,
\]
where \( \partial_\alpha \) is a derivative with respect to the parameter \( \phi^\alpha \), \( \epsilon(\phi^\alpha) = \epsilon_\alpha \), and where the \( \phi \)-dependent operators \( Y_\alpha \) \( \epsilon(Y_\alpha) = \epsilon_\alpha \) should be connected to the gauge generators. Usually constraint operators in involutions like \( \theta_\alpha \) in (B.3) in appendix B generate general gauge transformations. However, if we consider a formulation which is embedded in the conventional BFV-BRST formulation, appropriate hermitian gauge generators are of the form
\[
\tilde{\theta}_\alpha \equiv [\Omega, P_\alpha] = \theta_\alpha + \{ \text{possible ghost dependent terms} \},
\]
where \( \Omega \) is the BFV-BRST charge. In [2] the above reasoning led us to consider the operator \( Y_\alpha \) in (29) to be of the general form
\[
Y_\alpha(\phi) = (i\hbar)^{-1} [\Omega, \Omega_\alpha(\phi)], \quad \epsilon(\Omega_\alpha) = \epsilon_\alpha + 1.
\]
This form of \( Y_\alpha \) implies that a BRST invariant operator remains BRST invariant when transformed according to (29). From treatments of quasigroups we found that the operator \( Y_\alpha \) should have the explicit form
\[
Y_\alpha(\phi) = \lambda_\alpha^\beta(\phi) \theta_\beta(-1)^{\epsilon_\alpha + \epsilon_\beta} + \{ \text{possible ghost dependent terms} \}, \quad \lambda_\alpha^\beta(0) = \delta_\alpha^\beta,
\]
where \( \lambda_\alpha^\beta(\phi) \) \( \epsilon(\lambda_\alpha^\beta) = \epsilon_\alpha + \epsilon_\beta \) in general are operators. For quasigroups we could also choose \( Y_\alpha(0) = \theta_\alpha \). The explicit form (32) implied that \( \Omega_\alpha \) in (31) should explicitly look like
\[
\Omega_\alpha(\phi) = \lambda_\alpha^\beta(\phi) P_\beta + \{ \text{possible ghost dependent terms} \}.
\]
In [2] a simple quantum master equation was then set up for the operators \( \Omega_\alpha \), which could be viewed as generalized quantum Maurer-Cartan equations. (See (84) in section 7.)

We shall now generalize the results of [2] to the Sp(2) scheme. First we notice that the generalization of (30) is given by
\[
\tilde{\theta}_\alpha^b \equiv [\Omega^b, P_{\alpha a}] = \theta_\alpha \delta_a^b + \{ \text{possible ghost dependent terms} \},
\]
where \( \Omega^a \) are the Sp(2)-charges in appendix B. This suggests that the generalization of (29) is

\[
A(\phi) \overset{\phi}{\partial}^b \Omega^a = A(\phi) \overset{\phi}{\partial}^a - (i\hbar)^{-1} [A(\phi), Y^b_{aa}(\phi)] = 0,
\]

(35)

where the operators \( Y^b_{aa}(\phi) \) \((\varepsilon(Y^b_{aa}) = \varepsilon_\alpha)\) are of the form

\[
Y^b_{aa}(\phi) = \lambda^\beta_{\alpha}(\phi) \theta_{\beta\alpha} \delta^b_{\alpha} (-1)^{\varepsilon_\alpha + \varepsilon_\beta} + \{ \text{possible ghost dependent terms} \},
\]

(36)

where \( \lambda^\beta_{\alpha}(\phi) \) are the same operators as in (32) satisfying \( \lambda^\beta_{\alpha}(0) = \delta^\beta_\alpha \). At least for quasi-groups it is natural to expect \( Y^b_{aa}(0) = \tilde{\theta}^b_{aa} \). The equations (39) are equivalent to (29) with \( Y_\alpha \equiv \frac{1}{2} Y^a_{aa} \) and

\[
[A(\phi), T^{ab}_\alpha(\phi)] = 0, \quad T^{ab}_\alpha(\phi) \equiv \varepsilon^{ac} Y^b_{ac}(\phi).
\]

(37)

Thus, in distinction to the case in the conventional BFV-BRST formulation we have here also to impose the boundary conditions

\[
[A(0), T^{ab}_\alpha(0)] = 0.
\]

(38)

These conditions restrict the class of operators \( A(0) \) that may be integrated in the Sp(2) scheme. From (38) we have \( T^{ab}_\alpha(0) \equiv \varepsilon^{ac} Y^b_{ac}(0) = \{ \text{possible ghost dependent terms} \} \). If we also assume that \( Y^b_{aa}(0) = \tilde{\theta}^b_{aa} \), these ghost dependent terms may be calculated from \( \Omega^a \) in appendix B. It then looks like \( A(0) \) is more and more restricted the more complicated and the higher the rank of the group. It is unrestricted for abelian theories, ghost independent for Lie group theories and so on.

The integrability of the equations (35) require the following conditions to be satisfied

\[
0 = \delta^a_\alpha \delta^b_\beta A \left( \overset{\phi}{\partial}^a \overset{\phi}{\partial}^b - \overset{\phi}{\partial}^b \overset{\phi}{\partial}^a \right) - (i\hbar)^{-1} [A, Y^c_{aa} \overset{\phi}{\partial}^c \overset{\phi}{\partial}^d - Y^d_{bb} \overset{\phi}{\partial}^d \overset{\phi}{\partial}^c \delta^b_\alpha (-1)^{\varepsilon_\alpha + \varepsilon_\beta} - (i\hbar)^{-1} [Y^c_{aa}, Y^d_{bb}]] = 0.
\]

(39)

Now since \( A \) is restricted in general we cannot conclude that the operators in the second entry of the commutator are equal to zero. The only integrability conditions independent of \( A \) are

\[
Y^{c}_{\alpha \{a} \overset{\phi}{\partial}^c_{d\}b} - Y^{d}_{\beta \{b} \overset{\phi}{\partial}^d_{e\}a} (-1)^{\varepsilon_\alpha + \varepsilon_\beta} - (i\hbar)^{-1} [Y^{c}_{\alpha \{a}, Y^{d}_{\beta \}b}] = 0.
\]

(40)

The remaining conditions in (38) are then conditions on \( A \). In fact, they are consistency conditions to (39) which ensure the properties \( (Y_\alpha \equiv \frac{1}{2} Y^a_{aa}) \)

\[
[A(\phi), [T^{ab}_\alpha(\phi), T^{cd}_\beta(\phi)]] = 0,
\]

\[
[A(\phi), T^{ab}_\alpha(\phi)] \overset{\phi}{\partial}_{\beta} = [A(\phi), T^{ab}_\alpha(\phi)] \overset{\phi}{\partial}_{\beta} - (i\hbar)^{-1} [T^{ab}_\alpha, Y_\beta] = 0.
\]

(41)

Notice also that as a consequence of (39), \( Y_\alpha \equiv \frac{1}{2} Y^a_{aa} \) satisfies the equation

\[
Y_{\alpha} \overset{\phi}{\partial}_{\beta} - Y_{\beta} \overset{\phi}{\partial}_{\alpha} (-1)^{\varepsilon_\alpha + \varepsilon_\beta} - (i\hbar)^{-1} [Y_{\alpha}, Y_{\beta}] = -\frac{1}{24} (i\hbar)^{-1} \varepsilon_{ac}[T^{ab}_\alpha, T^{cd}_\beta] \varepsilon_{bd},
\]

(42)

which together with the first equation in (41) guarantees the integrability of (29).
In analogy to \((31)\) we propose \(Y_{\alpha a}^b(\phi)\) to be of the general form
\[
Y_{\alpha a}^b(\phi) = (i\hbar)^{-1}[\Omega^b, \Omega_{\alpha a}(\phi)], \quad \varepsilon(\Omega_{\alpha a}) = \varepsilon_{\alpha} + 1, \tag{43}
\]
which makes \(Y_{\alpha a}^b\) Sp(2)-invariant in the sense
\[
[\Omega^{(\alpha}, Y_{\alpha c}^b(\phi)] = 0. \tag{44}
\]
Furthermore, we have then
\[
[\Omega^{(c}, A] \overset{\leftrightarrow}{\partial_{\alpha}} \delta^b_a = (i\hbar)^{-1}[\Omega^{(c}, A], Y_{\alpha a}^b], \tag{45}
\]
which implies that \([\Omega^a, A(\phi)] = 0\) if \([\Omega^a, A(0)] = 0\). Due to the explicit form \((46)\) of \(Y_{\alpha a}^b\), \(\Omega_{\alpha a}\) in \((43)\) should look like
\[
\Omega_{\alpha a}(\phi) = \lambda^b_{\alpha}(\phi) \mathcal{P}_{\beta a} + \{\text{possible ghost dependent terms}\}. \tag{46}
\]
If we insert the general form \((43)\) of \(Y_{aa}^b\) into the integrability conditions \((44)\), we find the following equivalent equations for \(\Omega_{\alpha a}\)
\[
\Omega_{\alpha a}(a) \overset{\leftrightarrow}{\partial_{\beta}} \delta^c_b - \Omega_{\beta b}(b) \overset{\leftrightarrow}{\partial_{\alpha}} \delta^c_a (-1)^{\varepsilon_{\alpha} \varepsilon_{\beta}} =
\[
= (i\hbar)^{-2}(\Omega_{\alpha a}(a), \Omega_{\beta b})_{\Omega} - \frac{1}{2}(i\hbar)^{-1}[\Omega_{\alpha a}, \Omega_{\beta b}], \tag{47}
\]
where \((\Omega_{\alpha a}, \Omega_{\beta b})_{\Omega}\) are the Sp(2)-antibrackets \((4)\) with \(Q^\alpha\) replaced by \(\Omega^\alpha\). \(\Omega_{\alpha a}\) is symmetric in \(a, b\), and antisymmetric in \(\alpha, \beta\). Due to the explicit form \((46)\) of \(\Omega_{\alpha a}\), these equations are generalized Maurer-Cartan equations for \(\lambda^b_{\alpha}(\phi)\) which should be equivalent to those obtained from \(\Omega_{\alpha a}\) as given in \((2)\). Now the equations \((47)\) are only integrable if \(\Omega_{\alpha a}\) and \(\Omega_{\beta b}\) commute and if simultaneously the 3-antibrackets \((3)\) for the operators \(\Omega_{\alpha a}\) are zero or equivalently if \((4)\) with \(f, g, h\) and \(Q^a\) replaced by \(\Omega_{\alpha a}\), \(\Omega_{\beta a}\), \(\Omega_{\gamma c}\), and \(\Omega^d\) are satisfied. In this case \(\Omega_{\alpha a}\) in \((47)\) is zero. If these conditions are not satisfied the integrability conditions of \((47)\) lead to equivalent first order equations of \(\Omega_{\alpha a}\) and so on. \(Y_{\alpha a}\) is then replaced by a whole set of operators, and the integrability conditions \((44)\) for \(Y_{\alpha a}^b\) are replaced by a whole set of integrability conditions. In section 6 we propose new simple quantum master equations for the operators \(\Omega_{\alpha a}\), \(\Omega_{\alpha a}\) etc.

4 Generating operators for Sp(2)-antibrackets of operators in arbitrary involutions.

In \((1)\) it was shown that quantum antibrackets for operators in arbitrary involutions may be derived from a generating, nilpotent operator \(Q(\phi)\) satisfying the Lie equations \((29)\). Here we generalize this construction to the Sp(2)-formalism. Let \(Q^a\) be nilpotent Sp(2)-charges satisfying \((3)\). We define then \(Q^a(\phi)\) with the boundary conditions \(Q^a(0) = Q^a\) and \((38)\) to be solutions to the equations \((33)\), i.e.
\[
Q^c(\phi) \overset{\leftrightarrow}{\nabla_{\alpha a}} \equiv Q^c(\phi) \overset{\leftrightarrow}{\partial_{\alpha}} \delta^b_a - (i\hbar)^{-1}[Q^c(\phi), Y_{\alpha a}^b(\phi)] = 0. \tag{48}
\]
This implies
\[
[Q^a(\phi), Q^b(\phi)] \overset{\leftrightarrow}{\partial_{\alpha}} \delta^d_a = (i\hbar)^{-1}[Q^a(\phi), Q^b(\phi)], \tag{49}
\]
which by means of the boundary conditions $Q^a(0) = Q^a$ ensures that

$$[Q^a(\phi), Q^b(\phi)] = 0.$$  \hspace{1cm} (50)

This shows that $Q^a \to Q^a(\phi)$ is a unitary transformation.

Following refs.\[1, \[2\] we define generalized quantum antibrackets in terms of $Q^c(\phi)$ according to the formula

$$(Y_{a_1a_2}^{b_1}(\phi), Y_{a_2a_3}^{b_2}(\phi), \ldots, Y_{a_na_n}^{b_n}(\phi))^{Q(\phi)}_{Q(\phi)} \equiv$$

$$\equiv -Q^c(\phi) \overset{\phi}{\partial}_{a_1} \delta_{a_1}^{b_1} \partial_{a_2} \delta_{a_2}^{b_2} \cdots \overset{\phi}{\partial}_{a_n} \delta_{a_n}^{b_n}(ih)^n(-1)^{E_n}, \quad E_n \equiv \sum_{k=0}^{\frac{n-1}{2}} \epsilon_{\alpha_{2k+1}}.$$  \hspace{1cm} (51)

Explicitly we have then \textit{e.g.}

$$(\alpha a)^{f}_{Q(\phi)} = -Q^c(\phi) \overset{\phi}{\partial}_{a} \delta_{a}^{c} \overset{\phi}{\partial}_{b} \delta_{b}^{d}(-1)^{\epsilon_{\alpha}^2} (ih)^2 =$$

$$= \frac{1}{2} \left[ [Y_{aa}^{c}(\phi), [Q^f(\phi), Y_{ab}^{d}(\phi)] - [Y_{ab}^{d}(\phi), [Q^f(\phi), Y_{aa}^{c}(\phi)]](-1)^{\epsilon_{\alpha}^2} \right] -$$

$$- \frac{1}{2} ih [Q^f(\phi), Y_{aa}^{c}(\phi) \overset{\phi}{\partial}_{b} \delta_{b}^{d} + Y_{ab}^{d}(\phi) \overset{\phi}{\partial}_{a} \delta_{a}^{c}(-1)^{\epsilon_{\alpha}^2} \right] (-1)^{\epsilon_{\alpha}^2}.$$  \hspace{1cm} (52)

Since $Y_{aa}^{b}(\phi)$ may be split as follows

$$Y_{aa}^{b}(\phi) \leftrightarrow Y_{a}(\phi) \equiv \frac{1}{2} Y_{aa}^{a}(\phi), \quad T_{a}^{ab}(\phi) \equiv \epsilon^{abc} Y_{ac}^{b}(\phi),$$  \hspace{1cm} (53)

it is easily seen that \[51\] implies that the required antibrackets are given by

$$(Y_{a_1}(\phi), Y_{a_2}(\phi), \ldots, Y_{a_n}(\phi))^{Q(\phi)} \equiv -Q^c(\phi) \overset{\phi}{\partial}_{a_1} \overset{\phi}{\partial}_{a_2} \cdots \overset{\phi}{\partial}_{a_n} (ih)^n(-1)^{E_n},$$  \hspace{1cm} (54)

and that these antibrackets are zero if any entry is $T_{a}^{ab}(\phi)$, \textit{i.e.}

$$(\ldots, T_{a}^{ab}(\phi), \ldots)^{a} = 0.$$  \hspace{1cm} (55)

Note that only $Y_{a}$ in \[53\] involves the constraint operators $\theta_{a}$. By means of the equations

$$Q^a(\phi) \overset{\phi}{\partial}_{a} = (ih)^{-1}[Q^a(\phi), Y_{a}(\phi)],$$  \hspace{1cm} (56)

which follow from \[48\], we have then from \[54\] in particular

$$(Y_{a}(\phi), Y_{\beta}(\phi))^{\alpha}_{Q(\phi)} \equiv -Q^a(\phi) \overset{\phi}{\partial}_{a} \overset{\phi}{\partial}_{\beta} (ih)^2 =$$

$$= \frac{1}{2} \left( [Y_{a}(\phi), [Q^a(\phi), Y_{\beta}(\phi)] - [Y_{\beta}(\phi), [Q^a(\phi), Y_{a}(\phi)]](-1)^{\epsilon_{\alpha}^2} \right] -$$

$$- \frac{1}{2} ih [Q^a(\phi), Y_{a}(\phi) \overset{\phi}{\partial}_{\beta} + Y_{\beta}(\phi) \overset{\phi}{\partial}_{a} (-1)^{\epsilon_{\alpha}^2} ] (-1)^{\epsilon_{\alpha}^2},$$  \hspace{1cm} (57)

and

$$(Y_{a}(\phi), Y_{\beta}(\phi), Y_{\gamma}(\phi))^{a}_{Q(\phi)}(-1)^{(\epsilon_{\alpha}+1)(\epsilon_{\gamma}+1)} \equiv$$

$$\equiv Q^a(\phi) \overset{\phi}{\partial}_{a} \overset{\phi}{\partial}_{\beta} \overset{\phi}{\partial}_{\gamma} (ih)^3(-1)^{\epsilon_{\alpha}^2} =$$

$$= \frac{1}{3} \left( [Y_{a}(\phi), Y_{\beta}(\phi)]^{a}_{Q(\phi)}, Y_{\gamma}(\phi)\right)(-1)^{(\epsilon_{\alpha}+1)(\epsilon_{\gamma}+1)} + cycle(\alpha, \beta, \gamma) +$$

$$+ \frac{1}{3} ih \left( [Q^a(\phi), Y_{a}(\phi)], Y_{\beta}(\phi) \overset{\phi}{\partial}_{\gamma} + Y_{\gamma}(\phi) \overset{\phi}{\partial}_{\beta} (-1)^{\epsilon_{\alpha}^2} \right) (-1)^{\epsilon_{\alpha}^2} + cycle(\alpha, \beta, \gamma) +$$

$$+ \frac{1}{6} (ih)^2 \left( [Q^a(\phi), (Y_{a}(\phi) \overset{\phi}{\partial}_{\beta} + Y_{\beta}(\phi) \overset{\phi}{\partial}_{a} (-1)^{\epsilon_{\alpha}^2} \overset{\phi}{\partial}_{\gamma} ] (-1)^{\epsilon_{\alpha}^2} + cycle(\alpha, \beta, \gamma) \right)$$  \hspace{1cm} (58)
These expressions deviate from (3) and (9) by \( \hbar \)-terms and \( \hbar^2 \)-terms. Such terms we also had for the ordinary quantum antibrackets generated by a nilpotent operator \( Q(\phi) \) [3]. Their interpretation is the same here. They are the price of a reparametrization independent extension of their definitions onto the space of parameters \( \phi^\alpha \). Note that (54) and (58) are second and third derivatives of scalars, which are not tensors. Thus only within a preferred coordinate frame and at least at a fixed value of the parameters \( \phi^\alpha \) can one expect the formulas (57) and (58) to reproduce the original quantum \( \text{Sp}(2) \)-antibrackets [4] and [9]. Since we expect the canonical coordinates to be the preferred ones, all antibrackets should be reproduced at \( \phi^\alpha = 0 \). The vanishing of the \( \hbar \)-terms in (57) and (58) require the same conditions, while the vanishing of the \( \hbar^2 \)-deviation imposes a new condition in (58). In the corresponding \( \phi \)-extended \( n \)-antibrackets obtained from (54) we expect to have deviations involving up to \( n - 2 \) cyclically symmetrized derivatives of the 2-antibracket deviation which in terms of canonically coordinates should vanish at \( \phi^\alpha = 0 \). The \( \phi \)-extended \( n \)-antibrackets will then exactly reproduce the original \( n \)-antibrackets at \( \phi^\alpha = 0 \). Thus, we should have

\[
(Y_{\alpha_1}(0), Y_{\alpha_2}(0), \ldots, Y_{\alpha_n}(0))^a_Q = (Y_{\alpha_1}(\phi), Y_{\alpha_2}(\phi), \ldots, Y_{\alpha_n}(\phi))^a_{Q(\phi)}|\phi=0, \tag{59}
\]

where the left-hand side are the brackets in section 2. (At \( \phi^\alpha = 0 \) we have \( Y_\alpha(0) = \theta_\alpha + \{\text{possible ghost dependent terms}\} \).) However, note that

\[
(\ldots, T^{bc}_{\alpha_k}(0), \ldots)^a_Q \neq (\ldots, T^{bc}_{\alpha_k}(\phi), \ldots)^a_{Q(\phi)}|\phi=0 = 0. \tag{60}
\]

Thus, for brackets involving the operators \( T^{ab}_\alpha \) the additional \( \hbar \)-terms do not vanish at \( \phi^\alpha = 0 \). We have e.g.

\[
(T^{ab}_\alpha, Y_\beta)^c = \frac{1}{2}[T^{ab}_\alpha, [Q^c, Y_\beta]] = -\frac{1}{2}i\hbar[Q^c \overleftrightarrow{\partial}_\beta, T^{ab}_\alpha](\overline{1})\varepsilon_\alpha(\varepsilon_\beta+1) \tag{61}
\]

from (4), while (52) yields

\[
(T^{ab}_\alpha, Y_\beta)^c = (T^{ab}_\alpha, Y_\beta)^c - \frac{1}{2}i\hbar[Q^c, T^{ab}_\alpha \overleftrightarrow{\partial}_\beta](\overline{1})\varepsilon_\alpha = \frac{1}{2}i\hbar[Q^c, T^{ab}_\alpha] \overleftrightarrow{\partial}_\beta (-1)^{\varepsilon_\alpha} = \frac{1}{2}i\hbar[Q^c, T^{ab}_\alpha] \overleftrightarrow{\partial}_\beta -(i\hbar)^{-1}[T^{ab}_\alpha, Y_\beta](\overline{1})\varepsilon_\alpha = 0, \tag{62}
\]

where the last equality follows from (59) (cf.(41)). This makes the generalized antibrackets (71) considered at \( \phi^\alpha = 0 \) to be analogues of the Dirac brackets to those in section 2.

The property (50) allows us to derive identities like

\[
[Q^a(\phi), Q^b(\phi)] \overleftrightarrow{\partial}_\alpha \overleftrightarrow{\partial}_\beta \overleftrightarrow{\partial}_\gamma (ih)^3(\overline{1})^{\varepsilon_\beta+(\varepsilon_\alpha+1)(\varepsilon_\gamma+1)} \equiv 0, \tag{63}
\]

which completely control all Jacobi identities. Eq.(63) implies in particular

\[
(Y_{\alpha_1}(\phi), Y_{\alpha_2}(\phi), Y_{\alpha_3}(\phi))^a_Q (-1)^{\varepsilon_\alpha+1}(\varepsilon_\gamma+1) + 
\text{cycle}(Y_{\alpha_1}, Y_{\alpha_2}, Y_{\alpha_3}) = -\frac{1}{2}[(Y_{\alpha_1}, Y_{\alpha_2}, Y_{\alpha_3})^a_Q, Q^b](\overline{1})^{\varepsilon_\gamma+1}, \tag{64}
\]

which is identical to (8).
5 Quantum master equations and generalized Maurer-Cartan equations in the Sp(2)-formalism.

We propose here that the operators $\Omega_{aa}$ in the integrability conditions of (35) starting with \[ \text{(47)} \] are determined by the master equations

\[
(S, S)_{\Delta}^{\Delta} = i\hbar [\Delta^a, S],
\]

where $\Delta^a$ are extended Sp(2)-charges defined by

\[
\Delta^a \equiv \Omega^a + j^a \eta^a \pi_\alpha (-1)^{\varepsilon_\alpha} = 0, \quad [\phi^\alpha, \pi_\beta] = i\hbar \delta^\alpha_\beta, \quad (66)
\]

and where $S$ is an extended ghost charge defined by

\[
S(\phi, \eta, j) \equiv G + j^a \eta^a \Omega_{aa}(\phi) + \frac{1}{4} j^b j^a \eta^\beta \eta^\alpha (-1)^{\varepsilon_\alpha} \Omega_{\alpha\beta ab}(\phi) + \\
+ \frac{1}{36} j^c j^d j^e \eta^\gamma \eta^\beta \eta^\alpha (-1)^{\varepsilon_\alpha + \varepsilon_\beta + \varepsilon_\gamma} \Omega_{\alpha\beta\gamma abc}(\phi) + \ldots \\
\ldots + \frac{1}{(n!)^2} j^{a_1} \ldots j^{a_n} \ldots \eta^{a_1} \ldots \eta^{a_n} (-1)^{\varepsilon_{a_2} + \ldots + \varepsilon_{a_{n-1}} + \varepsilon_{a_n}} \Omega_{a_1 \ldots a_{n-1} a_n}(\phi) + \ldots \quad (67)
\]

In (66) and (67) we have introduced the even Sp(2)-parameters $j^a$, and new ghost variables $\eta^a$, $\varepsilon(\eta^a) = \varepsilon_\alpha + 1$, which also are to be treated as parameters. The former parameter $\phi^\alpha$ is on the other hand turned into an operator with conjugate momentum $\pi_\alpha$. $G$ in (67) is the ghost charge operator in (3.2) and (3.4) in appendix B. Our main conjecture is that the operators $\Omega_{a_1 \ldots a_{n-1} a_n}(\phi)$ in (67) may be identified with $\Omega_{aa}, \Omega_{ab} \Omega_{ab}$ in (47) and all the $\Omega$'s in their integrability conditions in a particular manner. They satisfy the properties

\[
(S, S)_{\Delta}^{\Delta} \equiv [[S, \Delta^a], S] = [S, [\Delta^a, S]]. \quad (69)
\]

By consistency the master equations (65) require $[\Delta^a, S]$ to satisfy the same algebra as $\Omega^a$, i.e. (3.4) in appendix B, since (cf. (3.3))

\[
0 = i\hbar [\Delta^a, [\Delta^b, S]] = [\Delta^a, (S, S)_{\Delta}^{\Delta b}] = [[\Delta^a, S], [\Delta^b, S]]. \quad (70)
\]

Another property which also follows from (65) is that the master equations (65) may be written as

\[
[S, [\Delta^a, S]] = i\hbar [\Delta^a, S], \quad (71)
\]

which when compared with (3.4) in appendix B shows that $S$ indeed is an extended ghost charge and $[\Delta^a, S]$ extended Sp(2)-charges. The explicit form of $[\Delta^a, S]$ to the lowest
orders in \( \eta^\alpha \) are
\[
[S, \Delta^d] = i\hbar \Omega^d + j^a \eta^\alpha [\Omega_{\alpha a}, \Omega^d] + j^d j^a \eta^\beta \eta^\alpha \Omega_{\alpha a} \partial_\beta \frac{\hbar}{\Delta_{\beta}} i\hbar (-1)^{\varepsilon_\beta} + \frac{1}{4} j^d j^b \eta^\beta \eta^\alpha [\Omega_{\alpha ab}, \Omega^d] (-1)^{\varepsilon_\beta} + \frac{1}{4} j^d j^b \eta^\gamma \eta^\beta \eta^\alpha \Omega_{\alpha ab} \partial_\gamma \frac{\hbar}{\Delta_{\gamma}} i\hbar (-1)^{\varepsilon_\beta + \varepsilon_\gamma} + \frac{1}{36} j^c j^b \eta^\gamma \eta^\beta \eta^\alpha [\Omega_{\alpha\beta ab}, \Omega^d] (-1)^{\varepsilon_\beta + \varepsilon_\alpha + \varepsilon_\gamma} + O(\eta^4). \tag{72}
\]

Inserting (67) and (72) into the master equations (65) we find that they are satisfied identically to zeroth and first order in \( \eta^\alpha \). However, to second order in \( \eta^\alpha \) they yield exactly (47), and to third order in \( \eta^\alpha \) they yield
\[
\left( \delta^d_\alpha \partial_\alpha \Omega_{\beta\gamma bc}(-1)^{\varepsilon_\alpha \varepsilon_\gamma} + \frac{1}{2}(i\hbar)^{-2} \right) \frac{\hbar}{\Delta_{\alpha}} \frac{\hbar}{\Delta_{\beta}} \frac{\hbar}{\Delta_{\gamma}} \frac{\hbar}{\Delta_{\delta}} (-1)^{\varepsilon_\alpha \varepsilon_\gamma} + cycle(a, b, c) + cycle(\alpha, \beta, \gamma) = \left( -(i\hbar)^{-3} \Omega_{\alpha (a}, \Omega_{\beta b), \Omega_{\gamma c)}^d \Omega^d \right) (-1)^{\varepsilon_\alpha \varepsilon_\gamma} + cycle(a, b, c) - \frac{2}{3}(i\hbar)^{-1} \left[ \Omega_{\alpha\beta\gamma abc}, \Omega^d \right],
\]
where the 3-antibrackets are defined by (9) with \( Q^a \) replaced by \( \Omega^a \).

Comparing equations (73) and the integrability conditions of (47) we find exact agreement. We have also checked that the consistency conditions (70) yield exactly (40) to second order in \( \eta^\alpha \), which is consistent with (47) as they should. Similarly we have checked that (74) to third order in \( \eta^\alpha \) yield conditions which are consistent with (73), exactly like (40) are consistent with (47).

The master equations (65) yield at higher orders in \( \eta^\alpha \) equations involving still higher quantum Sp(2)-antibrackets and operators \( \Omega_{\alpha\beta\gamma abc...} \) with still more indices. We conjecture that these equations all are consistent with the integrability conditions of (73).

Notice that \( j^a \) and \( \eta^\alpha \) are parameters in (67). \( \eta^\alpha \) play the same role as in the Sp(1)-formalism in (3), i.e. they select (super)antisymmetric sector with respect to the Greek subscripts in (47), (73), etc, in order to reproduce the correct chain of Maurer-Cartan equations within the framework of the generating equations (65). The bosonic parameters \( j^a \) are necessary in order to select just the symmetric sector with respect to the Sp(2)-subscripts in all structure relations (17), (73), etc., in accordance with the assertion given in the phrase before (47).

6 Some formal properties of quantum antibrackets and quantum master equations.

In the classical Sp(2)-formalism there are first order differential operators, \( V^a \), which play a fundamental role in triplectic quantization [3, 5, 6]. The quantum analogues are odd operators \( V^a \) satisfying the properties
\[
[Q^a, V^b] = 0, \tag{74}
\]
\[ [V^{(a),(f,g)^b}_Q] = ([V^{(a),f}_Q]^b + (f,[V^{(a),g}_Q]^b)(-1)^{\varepsilon_f+1}, \quad (75) \]

\[ [V^a,V^b] = 0, \quad (76) \]

where \( Q^a \) satisfy (3). When these properties are satisfied then \( \bar{Q}^a \equiv Q^a + kV^a \) with an arbitrary constant \( k \) also satisfy (3) and may be used instead of \( Q^a \) in the definition (4) of quantum antibrackets. (Note that \( Q^a \) satisfy (74), (75), and (76) with \( V^a \) replaced by \( Q^a \).) This situation we had in (23) in section 2. A particular solution of (74)-(76) is

\[ V^a \equiv (ih)^{-1}[Q^a,H], \quad (77) \]

where \( H \) is an arbitrary even operator which satisfies the property

\[ [Q^{(a),(H,H)^b}_Q] = 0. \quad (78) \]

Consider the quantum master equations (65), i.e.

\[ (S,S)^a_\Delta = ih[\Delta^a,S], \quad [\Delta^a,\Delta^b] = 0. \quad (79) \]

If we define \( \bar{S} \) and \( \Delta^a \) by

\[ \bar{S} \equiv e^{\frac{i}{2}F}Se^{-\frac{i}{2}F}, \quad \bar{\Delta}^a \equiv e^{\frac{i}{2}F}\Delta^a e^{-\frac{i}{2}F}, \quad (80) \]

where \( F \) is an arbitrary even operator, then \( \bar{S} \) satisfy the following master equations

\[ (\bar{S},\bar{S})^a_\bar{\Delta} = ih[\bar{\Delta}^a,\bar{S}]. \quad (81) \]

If now \( F \) in (80) also satisfies the master equations (79), i.e.

\[ (F,F)^a_\Delta = ih[\Delta^a,F], \quad (82) \]

then \( \bar{\Delta}^a \) in (80) reduce to

\[ \bar{\Delta}^a = \Delta^a - (ih)^{-1}[\Delta^a,cF] = \Delta^a - cV^a, \quad (83) \]

where \( c \equiv 1 - e^{-1} \). Note that (82) implies that \( F \) satisfies condition (78) with \( H \) and \( Q^a \) replaced by \( F \) and \( \Delta^a \) respectively.

There are also transformations on \( S \) leaving \( \Delta^a \) unaffected for which the master equations are invariant. These natural automorphisms exist also for the master equation given in (2). We consider this case first. The integrability conditions of (23) expressed in terms of \( \Omega_\alpha \) in (31) were in (2) proposed to be encoded in the master equation

\[ (S,S)_\Delta = ih[\Delta,S], \quad \Delta^2 = 0, \quad (84) \]

where \( S \) is the extended ghost charge (67) without \( Sp(2) \) indices and bosonic parameters \( j \), and where \( \Delta \) is an extended BRST charge given by \( \Delta = \Omega + \eta^a \pi_\alpha (-1)^{\varepsilon_\alpha} \). The antibracket in (84) is the ordinary quantum antibracket (1). The natural automorphism of (84) is

\[ S \to S' \equiv \exp \left\{ -(ih)^{-2}[\Delta,\Psi] \right\} S \exp \left\{ (ih)^{-2}[\Delta,\Psi] \right\}, \quad (85) \]
where $\Psi$ is an arbitrary odd operator. It is easily seen that $S'$ also satisfies the master equation \[ \text{(84)}. \] For infinitesimal transformations we have
\[
\delta S = (i\hbar)^{-2}[S, [\Delta, \Psi]],
\]
\[
\delta_{21}S \equiv (\delta_2\delta_1 - \delta_1\delta_2)S = (i\hbar)^{-2}[S, [\Delta_2, \Psi]],
\]
\[
\Psi_{21} = (i\hbar)^{-2}([\Psi_2, \Psi_1]_\Delta).
\] (86)

In the Sp(2) case we have the master equations \[ \text{(79)}. \] In this case there is an automorphism under
\[
S \rightarrow S' \equiv \exp \left\{ -(i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[[\Delta^b, [\Delta^a, F]]] \right\} S \exp \left\{ (i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[[\Delta^b, [\Delta^a, F]]] \right\},
\] (87)
where $F$ is an arbitrary even operator. For infinitesimal transformations we have here
\[
\delta S = (i\hbar)^{-3}[S, \frac{1}{2}\varepsilon_{ab}[[\Delta^b, [\Delta^a, F]]]],
\]
\[
\delta_{21}S \equiv (\delta_2\delta_1 - \delta_1\delta_2)S = (i\hbar)^{-3}[S, \frac{1}{2}\varepsilon_{ab}[[\Delta^b, [\Delta^a, F]]]],
\]
\[
F_{21} = -(i\hbar)^{-3}\frac{1}{2}\varepsilon_{ab}[[\Delta^b, F_2], [\Delta^a, F_1]].
\] (88)

The above automorphisms are very similar in structure to the automorphisms of the corresponding classical master equations in the BV-quantization. Compare e.g. in the Sp(2) case \[ \text{(79)}. \] with the corresponding classical automorphism given in \[ \text{[9]}. \]

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Appendix A

Defining properties of the conventional classical Sp(2)-antibrackets.

The defining properties of the antibrackets $(f, g)^a$ for functions $f, g$ on a manifold $A$ are \[ \text{[3, 5, 6]}. \] (The complete triplectic quantization requires a $6n$ dimensional manifold $A$.)

1) Grassmann parity
\[
\varepsilon((f, g)^a) = \varepsilon_f + \varepsilon_g + 1.
\] (A.1)

2) Symmetry
\[
(f, g)^a = -(g, f)^a(-1)^{(\varepsilon_f+1)(\varepsilon_g+1)}.
\] (A.2)
3) Linearity

\[(f + g, h)^a = (f, h)^a + (g, h)^a, \quad (\varepsilon_f = \varepsilon_g).\]  
(A.3)

4) Jacobi identities

\[(f, (g, h)^{ab})^c - (f, h)^a (g, h)^b - (f, g)^a (h, h)^b = 0.\]  
(A.4)

5) Leibniz’ rule

\[(fg, h)^a = f (g, h)^a + (f, h)^a g - (a - 1) \varepsilon_f (\varepsilon_h + 1).\]  
(A.5)

6) For any odd/even parameter \(\lambda\) we have

\[(f, \lambda)^a = 0 \quad \text{any } f \in \mathcal{A}.\]  
(A.6)

The most general form of the \(\text{Sp}(2)\)-antibrackets are explicitly given by (20) [5, 6].

**Appendix B**

**Constraints in arbitrary involutions embedded in \(\text{Sp}(2)\)-charges.**

Constraints in arbitrary involutions may at least for finite number of degrees of freedom always be embedded in one single, odd, hermitian and nilpotent BFV-BRST charge \(\Omega\) provided one introduces ghost operators to the constraints [10]. It is also possible to embed the constraints in two odd, hermitian charges \(\Omega^a, (a = 1, 2)\), satisfying [11] (cf (5))

\[\Omega^{a} [a \Omega^{b}] = 0.\]  
(B.1)

This may be done in such a way that there also exists an even, hermitian ghost charge \(G\) satisfying

\[[G, \Omega^a] = i\hbar \Omega^a.\]  
(B.2)

Explicitly this may be done by means of the \(\text{Sp}(2)\)-covariant ghost operators \(C^\alpha a\) in the case \(\theta_\alpha\) are constraint operators [11]. Let \(C^\alpha a\) and their conjugate momentum operators \(P_{\alpha a}\) satisfy the properties

\[\left[ C^\alpha a, P_{\beta b} \right] = i\hbar \delta^\alpha_\beta \delta^a_b, \quad (C^\alpha a)^\dagger = C^{\alpha a}, \quad P^\dagger_{\alpha a} = - (1)^{\varepsilon_\alpha} P_{\alpha a},\]

\[\varepsilon(C^\alpha a) = \varepsilon(P_{\alpha a}) = \varepsilon_\alpha + 1, \quad \varepsilon_\alpha \equiv \varepsilon(\theta_\alpha).\]  
(B.3)

The so called new ghost operator in (B.2) is then defined by

\[G \equiv - \frac{1}{2} \left( P_{\alpha a} C^\alpha a - C^{\alpha a} P_{\alpha a} (-1)^{\varepsilon_\alpha} \right).\]  
(B.4)

The \(\text{Sp}(2)\) charges \(\Omega^a\) satisfying (B.1) and (B.2) are then of the form

\[\Omega^a = \theta_\alpha C^{\alpha a} + \frac{1}{2} C^{\beta b} C^{\alpha a} U_{\alpha \beta} \gamma \gamma \gamma (-1)^{\varepsilon_\beta + \varepsilon_\gamma + \cdots},\]  
(B.5)
where the dots denote terms which are of higher orders in the ghost momenta $P_{\alpha\alpha}$. $U_{\alpha\beta}^\gamma$ are the structure operators in the involution relations of $\theta_\alpha$, i.e.

$$[\theta_\alpha, \theta_\beta] = i\hbar U_{\alpha\beta}^\gamma \theta_\gamma.$$ (B.6)

(In general these constraint operators $\theta_\alpha$ are not hermitian.)

The complete Sp(2) formalism also involves Lagrange multipliers $\lambda_\alpha$ ($\varepsilon(\lambda_\alpha) = \varepsilon_\alpha$) and their conjugate momenta $\pi_\alpha$ ([$\lambda_\alpha$, $\pi_\beta$] = $i\hbar\delta_\alpha^\beta$) [1]. However, in distinction to standard BRST formalism, these Lagrange multipliers are considered as ghost variables in the Sp(2) case. In fact, $\lambda_\alpha$ has new ghost number two which means that $G$ in (B.4) then should be replaced by

$$G \equiv -\frac{1}{2}\left(P_{\alpha\alpha} C^{\alpha\alpha} - C^{\alpha\alpha} P_{\alpha\alpha} (-1)^{\varepsilon_\alpha}\right) - \left(\pi_\alpha \lambda_\alpha + \lambda_\alpha \pi_\alpha (-1)^{\varepsilon_\alpha}\right).$$ (B.7)

The Sp(2) charges $\Omega^a$ including the Lagrange multipliers are then explicitly

$$\Omega^a = \theta_\alpha C^{\alpha\alpha} + \frac{1}{2} C^{\alpha\beta} C^{\alpha\gamma} U_{\alpha\beta}^\gamma P_{\beta\gamma} (-1)^{\varepsilon_\beta + \varepsilon_\gamma} +$$

$$+ \varepsilon^{ab} P_{\beta\gamma} \lambda^\beta + \frac{1}{2} \lambda^\beta C^{\alpha\alpha} U_{\alpha\beta}^\gamma \pi_\gamma + \cdots,$$ (B.8)

where the dots denote terms of order square and higher in the ghost momenta $P_{\alpha\alpha}$ and/or $\pi_\alpha$. The results in sections 3-6 are valid both for (B.3) and (B.8). Note also that $\Omega^a$ may be used to construct quantum Sp(2)-antibrackets.

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