Technical Notes and Correspondence

Direct Data-Driven Control of Linear Time-Varying Systems

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Abstract—Considering discrete-time linear time-varying systems with unknown dynamics, controllers guaranteeing bounded closed-loop trajectories, optimal performance, and robustness to process and measurement noise are designed via convex feasibility and optimization problems involving purely data-dependent linear matrix inequalities. For the special case of periodically time-varying systems, infinite-horizon guarantees are achieved based on finite-length data sequences.

Index Terms—Data-driven control, linear time-varying (LTV) systems, linear matrix inequalities (LMIs), optimal control, robust control.

I. INTRODUCTION

Direct data-driven control methods, which aim to control a system directly using data, without explicitly identifying a system model, have recently attracted significant interest (see, e.g., [1]). A central question in direct data-driven control is how to substitute a system model with data. For linear time-invariant (LTI) systems, a recent line of research addresses this question via Willems et al.’s fundamental lemma [2]. The result is used in [3] and [4] to replace the system model and initial conditions in the context of model predictive control with data. In [5], it is used to derive a data-driven representation of closed-loop systems under static state feedback, where the controller itself is parameterized using data only. This can be used to formulate and solve the stabilization [5], linear quadratic regulator (LQR) [5], [6], and suboptimal control [7] problems in terms of data-dependent linear matrix inequalities (LMIs). Extensions have been proposed for data from multiple datasets [8], certain classes of nonlinear systems [9], [10], [11], linear parameter-varying systems [12], and switched systems [13]. In [14], the direct data-driven control framework originally presented in [5] (for LTI systems) is extended to linear time-varying (LTV) systems. Time-varying systems arise in a variety of practical problems and LTV models emerge, for instance, when linearizing nonlinear systems around a trajectory or time-varying operating point [15].

The demand for model-free control approaches for LTV systems is apparent in the literature (see, e.g., [16], [17], [18], [19]). In this article, we provide a complete analysis of the preliminary results in [14]. Moreover, we extend the results to LTV systems affected by both measurement and process noise and provide insights for the special case of periodically time-varying systems. In contrast to the related results [12], [13], the presented data-driven methods are applicable to (linear) arbitrarily time-varying systems and do not rely on any assumptions or prior knowledge of the system structure or parameter variation. However, it is shown how such knowledge can be exploited for the special case of periodically time-varying systems. Challenges associated with direct data-driven control in the presence of noise are addressed for certain classes of control problems involving LTI systems in [5], [20], [21], [22], and [23]. Most works in this context focus on process noise only, or consider process noise and measurement noise separately. The results presented herein are inspired by [5] and [21], and can be considered as an LTV equivalent. The main difference apart from the extension to LTV systems—which itself introduces new challenges and requires a different approach to parametrize unknown systems—is that we incorporate both measurement and process noise in a single formulation and study the behavior of the system in closed-loop with feedback on the noisy state measurements.

The rest of this article is organized as follows. In Section II, some preliminaries are provided. In Section III, we consider noise-free LTV systems and show that state feedback control laws (guaranteeing a decreasing bound on the closed-loop trajectories or solving the time-varying LQR problem) can be designed via data-dependent semidefinite programming (SDPs). The problem of designing controllers with robustness guarantees directly using noisy data is addressed in Section IV. In Section V, we specialize the results to the class of periodically time-varying systems. Finally, Section VI concludes this article.

Notation. The sets of real numbers, integers, and natural numbers are denoted by \( \mathbb{R} \), \( \mathbb{Z} \), and \( \mathbb{N} \), respectively. The zero matrix of appropriate dimension is denoted by 0 and the \( n \times n \) identity matrix by \( I_n \). Given a square matrix \( A \), Tr\( (A) \) denotes its trace, and \( A \succeq 0 \) (\( A \preceq 0 \)) denotes that \( A \) is positive definite (positive semidefinite). In matrix inequalities \*, denotes blocks (or matrices), which can be inferred by symmetry. The block diagonal stacking of matrices \( A \) and \( B \) is written as diag\( (A, B) \). Given a vector \( v \in \mathbb{R}^n \), \( \|v\| \) denotes its Euclidean norm and given a matrix \( M \in \mathbb{R}^{m \times n} \), \( \|M\|_2 \) denotes the induced 2-norm of \( M \). Given a signal \( z : \mathbb{Z} \to \mathbb{R}^n \) the sequence \( \{z(k), \ldots, z(k+T)\} \) is denoted by \( z|_{[k,k+T]} \) with \( k, T \in \mathbb{Z} \) and we denote \( |z|_k = \text{sup}\{\|z(j)\|, 0 \leq j \leq k\} \leq \infty \). The space of square-summable sequences is denoted by \( l_2 \). A function \( \gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is a class K-function if it is continuous, strictly increasing, and \( \gamma(0) = 0 \).

II. PRELIMINARIES AND PROBLEM DEFINITION

Consider a discrete-time LTV system, described by

\[
x(k+1) = A(k)x(k) + B(k)u(k) + d(k),
\]

where \( x \in \mathbb{R}^n \) is the state of the system, \( u \in \mathbb{R}^m \) is the control input, \( d \in \mathbb{R}^n \) denotes an unknown additive system noise, and \( A(k) \) and \( B(k) \) denote the unknown time-varying dynamics and input matrices of appropriate dimensions, respectively. Suppose that available state measurements \( \zeta \in \mathbb{R}^n \) are corrupted by measurement noise \( v \in \mathbb{R}^n \),
Consider the system satisfying 
\( x(k+1) = A(k)x(k) + B(k)u(k) \), then for all \( k \geq n \)
(6a)

\( \in \mathbb{R}^{d,L} \)

\( P \)
a can always be verified from the measured data. A necessary
we define
simplify to
\( m \),
+ 1) = \( n \) \( A \in \mathbb{R}^{d,L} \)
\( \rho I \)
Assumption \( N \) can
represent input-state data collected during
\( \in \mathbb{R}^{d,L} \)

\( U \)
and different \( \in \mathbb{R}^{d,L} \)

\( \in \mathbb{R}^{d,L} \)

(10)

Considering the system
(8)
(11b)
(8)
(6b)
(6b)
(5)
(8)
holds for \( k = 0, \ldots, T - 1 \). Then, the closed-loop system (5) can equivalently be represented as
(9)
where \( G(k) \in \mathbb{R}^{L \times m} \) satisfies
(10)
(8)

for \( k = 0, \ldots, T - 1 \), see [14]. Hence, the sequence of control gains \( K(k) \) is parametrized using data through the identity (10). Thus, the matrices \( G(k) \), for \( k = 0, \ldots, T - 1 \), can be seen as decision variables, which can be used for identification-free design of state feedback controllers.

Remark 2: To utilize the data-driven system representation (9), (10), it is required that (8) holds for all \( k = 0, \ldots, T - 1 \). Thus, each input-state data sequence \( j, j = 1, \ldots, L \), in the ensemble must start at different initial conditions \( x_{d,j}(0) \). If this is infeasible, a common starting point can be considered as the state at \( k = -1 \), and different inputs can be applied for each experiment to obtain different state data at \( k = 0 \).

III. DATA-DRIVEN CONTROL OF LTV SYSTEMS—THE NOISE-FREE CASE

In this section, we utilize the direct data-driven system representation (9), (10) to design feedback controllers for the unknown (noise-free) LTV system (4) via the solution of convex optimization problems involving LMI constraints.

A. Bounded Closed-Loop Trajectories

Consider the problem of controlling the LTV system (4) over a finite time horizon, with the aim of ensuring that the closed-loop trajectories remain close to the equilibrium throughout the considered horizon. A solution to this problem is provided in the following statement.

Theorem 1: Consider the system (4) and suppose an ensemble of input-state data is available to form the matrices (6a), (6b), such that the rank condition (8) holds, for \( k = 0, \ldots, T - 1 \). Any sequences of matrices \( Y(k), P(k) = P(k) \)

\[ \begin{bmatrix} P(k+1) - I_n & X(k+1)Y(k) \\ Y(k)^{T}X(k+1) & P(k) \end{bmatrix} \succeq 0, \]
(11a)

\( X(k)Y(k) = P(k), \)
(11b)

for \( k = 0, \ldots, T - 1 \), and

\[ \eta I_n \preceq P(k) \preceq \rho I_n, \]
(11c)

1In the rest of this article, we refer to each interval capturing the time-variation of interest as \( k = 0, \ldots, T \), i.e., we define \( k_j = 0 \), for \( j = 1, \ldots, L \).
2Herein, we refer to the act of data collection as “experiment,” regardless of whether the data is collected via experiments or simulations.
for \( k = 0, \ldots, T \), where \( \eta \geq 1 \) and \( \rho > \eta \) are finite constants, are such that the trajectories of the system (5), with
\[
K(k) = U(k)Y(k)P(k)^{-1},
\]
for \( k = 0, \ldots, T - 1 \), satisfy
\[
\|x(k)\| \leq \left(\frac{\eta}{\sqrt{\eta}}\right)\left(1 - \frac{1}{\rho}\right)^{\frac{k}{2}}\|x(0)\|,
\]
for \( k = 0, \ldots, T \).

**Proof:** To demonstrate the claim it is useful to consider the adjoint equation (see, e.g., [25, Ch. 3.1]) of the closed-loop system (5). Namely, consider
\[
\xi(j) = A_d(j)^T \xi(j + 1),
\]
and note that the solution to (14) starting from \( \xi(0) \) is given by
\[
\xi(j) = S_t(k, j)^T \xi(k),
\]
for \( j \leq k \), where
\[
S_t(k, j) = \begin{cases} A_d(k - 1)A_d(k - 2) \ldots A_d(j), & \text{for } j < k, \\ I_n, & \text{for } j = k, \end{cases}
\]
denotes the state transition matrix corresponding to the closed-loop system (5), and where \( A_d(k) = A(k) + B(k)K(k) \). Let \( \xi(k) \neq 0 \) and suppose we can determine a sequence of matrices \( P(k) \) satisfying condition (11c) and
\[
A_d(k)P(k)A_d(k)^T - P(k + 1) = I_n \
\]
for \( k = 0, \ldots, T - 1 \). Consider the quadratic function \( V_j := V(j, \xi(j)) = \xi(j)^T P(j)\xi(j) \), for \( j = 0, \ldots, T \). It follows from (14), (16), and (11c) that \( V_{j+1} - V_j \geq \|\xi(j + 1)^T\|^2 \geq \frac{2}{\rho} \|\xi(j)\|^2 + \|P(j+1)\xi(j + 1)\|^2 \), for \( j = 0, \ldots, T - 1 \). Thus, we have that \( \|\xi(j)\|^2 \leq (1 - \frac{1}{\rho})^{k-j}\|\xi(k)\|^2 \), which, using (15), in turn yields \( \|S_t(k, j)^T \xi(k)\|^2 \leq \frac{\rho}{\eta} (1 - \frac{1}{\rho})^{k-j} \xi(0) \|\xi(k)\|^2 \), and
\[
\|S_t(k, j)^T \xi(k)\|^2 = \|S_t(k, j)\|^2 \leq \frac{\rho}{\eta} (1 - \frac{1}{\rho})^{k-j} \xi(0) \|\xi(k)\|^2 ,
\]
for \( j = 0, \ldots, T \), \( j \leq k \leq T \). Noting that \( x(k) = S_t(k, j)x(j) \), for \( k \geq j \), (17) implies
\[
\|x(k)\|^2 = \|S_t(k, j)x(j)\|^2 \leq \|S_t(k, j)\|^2 \|x(j)\|^2 \\ \leq \frac{\rho}{\eta} (1 - \frac{1}{\rho})^{k-j} \|x(j)\|^2 ,
\]
for \( j = 0, \ldots, T \), \( j \leq k \leq T \). Letting \( j = 0 \), this yields (13). Finally, using (9), (10), defining \( Y(k) := G(k)P(k) \), and via the Schur complement, (11a) is equivalent to (16), if (11b) holds and the control gain is chosen as (12). \( \square \)

**IV. DATA-DRIVEN CONTROL OF LTV SYSTEMS—ROBUSTNESS TO NOISE**

In practice, both the measurement and/or the system dynamics may be subject to noise. In this section, we consider the problem of designing feedback controllers for the general (unknown) discrete-time LTV system (1). To this end, we start by deriving a data-driven system representation of the form (9), (10) using noise corrupted data. Namely, let \( \hat{u}_{d,t}[0, T - 1], \hat{\zeta}_{d,t}[0, T] \) denote input-output data collected during the \( j \)th experiment, for \( j = 1, \ldots, L \). The data is arranged to form the matrices
\[
Z(k) = [\hat{\zeta}_{d,1}(k), \hat{\zeta}_{d,2}(k), \ldots, \hat{\zeta}_{d,L}(k)] ,
\]
over the time horizon \( N \in \mathbb{N} \), starting from the initial condition \( x(0) = x_0 \), with \( Q_f = Q_f^T \geq 0, Q(k) = Q(k)^T \geq 0 \) and \( R(k) = R(k)^T \geq 0 \), for \( k = 0, \ldots, N - 1 \). In [14], it has been shown that this finite-horizon LQR problem can equivalently by solved via a convex programme. In the following statement, we combine this with the data-driven system representation (9), (10) to formulate the time-varying LQR problem as a data-dependent SDP.

**Theorem 2:** Consider the system (4) and suppose an ensemble of input-state data is available to form the matrices (6a), (6b), such that the rank condition (8) holds, for \( k = 0, \ldots, N - 1 \). The optimal state feedback control gain sequence \( \{K^*(0), K^*(1), \ldots, K^*(N - 1)\} \) solving the finite-horizon LQR problem with \( u^*(k) = K^*(k)x(k) \) is given by
\[
K^*(k) = U(k)H^*(k)S^*(k)^{-1},
\]
for \( k = 0, \ldots, N - 1 \), with \( H^*(k) \) and \( S^*(k) \) the solution of
\[
\min_{S, H, \rho} \text{Tr}(Q_s S(N)) \quad \text{s.t.} \quad S(0) \succeq I_n ,
\]
\[
\begin{bmatrix} S(k + 1) - I_n & X(k + 1)H(k) \\ H(k)^T X(k + 1) & S(k) \end{bmatrix} \succeq 0 ,
\]
\[
\begin{bmatrix} O(k) & R(k)^{1/2}U(k)H(k) \\ H(k)^T U(k)^{-1} R(k)^{1/2} S(k) \end{bmatrix} \succeq 0 ,
\]
for \( k = 0, \ldots, N - 1 \), where \( S = \{S(1), \ldots, S(N)\} \), \( H = \{H(0), \ldots, H(N - 1)\} \) and \( O = \{O(0), \ldots, O(N - 1)\} \).

**Proof:** The proof lies in demonstrating that (21) is equivalent to the model-based convex programme corresponding to [14, Eq. (15)]. This follows by introducing (9), (10) to the constraints, letting \( H(k) := G(k)S(k) \), and taking the Schur complement of the nonlinear inequality constraints. \( \square \)
Suppose the rank condition
\[
\text{rank } \begin{bmatrix} Z(k) \\ U(k) \end{bmatrix} = n + m, \tag{23}
\]
holds for \( k = 0, \ldots, T - 1 \). Then, the dynamics matrix of the closed-loop system (3) can equivalently be represented as
\[
A(k) + B(k)K(k) = \left( Z(k+1) + W(k) \right) G(k), \tag{24}
\]
where \( G(k) \) satisfies
\[
\begin{bmatrix} I_n \\ K(k) \end{bmatrix} = \begin{bmatrix} Z(k) \\ U(k) \end{bmatrix} G(k), \tag{25}
\]
for \( k = 0, \ldots, T - 1 \), with
\[
W(k) = A(k)V(k) - V(k+1) - D(k). \tag{26}
\]
Assuming the unknown ensemble matrix \( W(k) \), containing both system\(^4\) and noise information, satisfies a quadratic bound, controllers with trajectory boundedness and performance guarantees can be designed via data-dependent convex programmes, as detailed in the following subsections.

**A. Bounded Closed-Loop Trajectories**

To ensure the boundedness of the trajectories of the closed-loop system (3), we derive a bound (alternative to (13)), which is related to the notion of input-to-state stability.

**Lemma 1:** Suppose there exists \( P(k) = P(k)^\top \) satisfying (11c), for \( k = 0, \ldots, T \), and (16) for some \( K(k) \), for \( k = 0, \ldots, T - 1 \). The state trajectories of the system (3) satisfy
\[
\| x(k) \| \leq \sqrt{\frac{\rho}{\eta}} \left( 1 - \frac{1}{\rho} \right)^{\frac{k}{2}} \| x(0) \|
+ \gamma_1 \left( | v[k-1, k] \right) + \gamma_2 \left( | d[k-1, k] \right), \tag{27}
\]
for \( k = 0, \ldots, T \), with \( \gamma_1(\cdot), \gamma_2(\cdot) \) class \( K \)-functions.

**Proof:** The state response at time \( k \) is given by \( x(k) = S_1(k, 0)x(0) + \sum_{j=0}^{k-1} S_1(k-1, j)B(j)K(j)v(j) + d(j) \), where \( S_1(k, 0) \) is the state transition matrix corresponding to (5) as defined in Section III-A. From Theorem 1, we know that if there exist \( P(k) = P(k)^\top \), \( K(k) \) satisfying (11c), for \( k = 0, \ldots, T \), and (16), for \( k = 0, \ldots, T - 1 \), then \( \| S_1(0, 0) \| \leq \sqrt{\frac{\rho}{\eta}} (1 - \frac{1}{\rho})^{\frac{k}{2}} \), for \( k = 0, \ldots, T \). Combined with properties of the operator norm this gives (27) with
\[
\gamma_1(\| v[k-1, k] \| = b \left( \sum_{j=0}^{k-1} \sqrt{\frac{\rho}{\eta}} \left( 1 - \frac{1}{\rho} \right)^{\frac{j-1}{2}} \| K(j) \| \right) \| v[k-1, k] \|
+ \gamma_2 \left( | d[k-1, k] \right) \tag{28}
\]
where \( b \) denotes the upper bound on the singular values of \( B(\cdot) \), i.e., \( \| B(j) \| \leq b \) for \( 0 \leq j \leq k - 1 \).

With the aim of designing controllers such that (27) holds for \( k = 0, \ldots, T \) directly using noisy data, we combine the result of Lemma 1 and the system representation (24)-(26).

**Theorem 3:** Consider the system (1) and suppose an ensemble of input-output data is available to form the matrices (22), (6b), such that the rank condition (23) holds, for \( k = 0, \ldots, T - 1 \). Suppose \( W(k) \) satisfies
\[
\begin{bmatrix} I_n \\ W(k)^\top \end{bmatrix} \begin{bmatrix} Q_e(k) \\ S_e(k) \\ R_e(k) \end{bmatrix} \begin{bmatrix} I_n \\ W(k)^\top \end{bmatrix} \geq 0, \tag{29}
\]
where \( Q_e(k) \in \mathbb{R}^{n \times n}, S_e(k) \in \mathbb{R}^{n \times L}, \) and \( R_e(k) \prec 0 \in \mathbb{R}^{L \times L} \), for \( k = 0, \ldots, T - 1 \). Any sequences of matrices \( Y(k), P(k) = P(k)^\top \) satisfying
\[
\begin{bmatrix} P(k+1) - I_n - Q_e(k) & -S_e(k) & Z(k+1)Y(k) \\ -S_e(k)^\top & -R_e(k) & Y(k) \\ Y(k)^\top Z(k+1) & Y(k)^\top & P(k) \end{bmatrix} \geq 0, \tag{30a}
\]
for \( k = 0, \ldots, T - 1, \) and (11c), for \( k = 0, \ldots, T, \) where \( \gamma \geq 1 \) and \( \rho > \gamma \) are finite constants, are such that the trajectories of the system (3), with \( K(k) \) given by (12), for \( k = 0, \ldots, T - 1, \) satisfy (27), for \( k = 0, \ldots, T \).

**Proof:** By Lemma 1, (27) holds for the trajectories of (3) if there exist \( P(k) = P(k)^\top \) satisfying (11c), for \( k = 0, \ldots, T, \) and (16), for \( k = 0, \ldots, T - 1 \). Using (24)-(26), letting \( Y(k) := G(k)(P(k), \) and via the concrete version of the full block S-procedure (see [26], [27]), (16) is satisfied if (29) holds and \( P(k), Y(k) \) satisfy a quadratic matrix inequality, which can be transformed into the LMI (30a) by performing the matrix multiplication, applying the Schur complement and a congruence transformation with \( \text{diag}(I_n, L, P(k)) \). The constraint (30b) stems from the upper row block of (25). The lower row block of (25) is satisfied by \( K(k) \) as in (12).

Quantifying (27) requires knowledge of \( b \) and the upper bound on the norm of the noise vectors, \( | v[T, k] \) and \( | d[T, k] \). Similarly, condition (29) cannot be verified using data alone, since \( W(k) \) (as defined in (26)) contains information of both the unknown system dynamics matrix and the noise affecting the data samples. Hence, to verify (29), knowledge of an upper bound on \( A(k) \), for \( k = 0, \ldots, T - 1 \), and the matrices \( V(k) \), for \( k = 0, \ldots, T, \) and \( D(k) \), for \( k = 0, \ldots, T - 1, \) i.e., the ensembles of (unmeasured) samples of measurement and process noise corresponding to the measured input-output data, is required.\(^5\)

The practical relevance of the result of Theorem 3 is illustrated in [29], which proposes a data-driven controller for planar snake robot locomotion, partly based on this result.

**Remark 3:** In the absence of measurement noise (29) becomes a bound on \( D(k) \) (the ensemble of process noise samples corresponding to the measured input-output data), which is similar to the bound on the noise data introduced in [21] for LTI systems subject to process noise only. Note that in the LTV case (29) is required to hold at each time step.

**Remark 4:** The result of Theorem 3 requires (29) to hold only for the measured data used for the representation (24)-(26). Subsequently, (27) is satisfied by the trajectories of (3) for arbitrary, bounded noise inputs \( d(k), k = 0, \ldots, T - 1, \) and \( v(k), k = 0, \ldots, T. \)

**Remark 5:** The matrices \( Q_e(k), S_e(k), \) and \( R_e(k) \) in (29) are chosen by the user. This makes the quadratic bound (29) a flexible condition.

\(^4\)As in the LTI case the appearance of \( A(k) \) in (26) can be interpreted as a measure of the direction of the measurement noise, which contributes to the loss of information caused [5].

\(^5\)While we assume that the system dynamics and noise are unknown, for many practical applications it is expected that reasonable upper bounds on these quantities can be estimated [28, Ch. 8].
which contains many practical bounds as special cases, e.g., a bound on
the maximum singular value (see [21]) of $W(k)$, for $k = 0, \ldots, T - 1$. The choice $Q_{k}(k) = Z(k + 1)Z(k + 1)^{\top}, S_{k}(k) = 0,$ and $R_{k}(k) = -\gamma(k)I_{L_{k}}$, for some $\gamma(k) > 0 \in \mathbb{R}$, gives the signal-to-noise ratio condition
\[
W(k)W(k)^{\top} \leq \frac{1}{\gamma(k)}Z(k + 1)Z(k + 1)^{\top}, \tag{31}
\]
for $k = 0, \ldots, T - 1$. This condition is similar (apart from being
required to hold at each time step) to the condition presented in [5, Assumption 2] for LTI systems and represents a measure of the loss of
information caused by the noise.

\section*{B. Robust Performance}

In this subsection, we consider the problem of designing controllers of
the form (2) for the (unknown) LTV system (1), such that the closed-
loop system (3) fulfills a disturbance attenuation condition. Consider
the performance output
\[
z(k) = C(k)x(k) + D_{u}(k)u(k) + D_{d}(k)d(k), \tag{32a}
\]
\[
z_{f} = C(N)x(N), \tag{32b}
\]
for $k = 0, \ldots, N - 1$, where $z(k) \in \mathbb{R}^{q}, z_{f} \in \mathbb{R}^{q},$ and $C(N), C(k) \in \mathbb{R}^{q \times n}, D_{u}(k) \in \mathbb{R}^{q \times u},$ and $D_{d}(k) \in \mathbb{R}^{q \times r}$ are known matrices. This results in the closed-loop system
\[
x(k + 1) = A_{cl}(k)x(k) + E_{cl}(k)\bar{w}(k), \tag{33a}
\]
\[
z(k) = C_{cl}(k)x(k) + D_{cl}(k)\bar{w}(k), \tag{33b}
\]
\[
z_{f} = C_{cl}(N)x(N), \tag{33c}
\]
\[
\bar{z}(k) = z(k) + v(k), \tag{33d}
\]
for $k = 0, \ldots, N - 1$, where $A_{cl}(k) = A(k) + B(k)K(k), C_{cl}(k) = C(k) + D_{uc}(k)K(k), \bar{w}(k) := [\bar{v}(k)^{\top} d(k)^{\top}]^{\top},$ $v(k) = K(v(k)), E_{cl} = [B(k) \quad I_{m}],$ and $D_{cl} = [D_{uc}(k) \quad D_{uc}(k)K(k)].$ Regarding $\bar{w}(k) \in \mathbb{R}^{(m + n)\times 1}$ as the disturbance, consider the quadratic robust performance
criterion
\[
\sum_{k=0}^{N-1} \bar{w}(k)^{\top} \bar{w}(k) \leq 0, \tag{34}
\]
for all $\bar{w} \in \ell_{2}$, where $\varepsilon > 0$ and $Q_{p}(k) \in \mathbb{R}^{(m+n)\times(m+n)}, S_{p}(k) \in \mathbb{R}^{(m+n)\times q},$ and $R_{p}(k) \geq 0 \in \mathbb{R}^{q \times q},$ for $k = 0, \ldots, N - 1$. This is
the finite-horizon equivalent to the performance robust condition
introduced in [26] and [27] and it captures many popular robust performance
measures. For example, the choice $Q_{p}(k) = -\frac{\varepsilon}{2}I_{(m+n)}, S_{p}(k) = 0,$ and $R_{p}(k) = I_{q},$ with $\varepsilon > 0 \in \mathbb{R},$ for $k = 0, \ldots, N - 1,$ recovers the finite-horizon $H_{\infty}$-control problem for discrete LTV systems (see, e.g., [25]). Assuming the performance index is invertible, let
\[
\begin{bmatrix}
\tilde{Q}_{p}(k) & \tilde{S}_{p}(k) \\
\tilde{S}_{p}(k)^{\top} & \tilde{R}_{p}(k)
\end{bmatrix} = \begin{bmatrix}
Q_{p}(k) & S_{p}(k) \\
S_{p}(k)^{\top} & R_{p}(k)
\end{bmatrix}^{-1},
\]
and further assume $\tilde{Q}_{p}(k) < 0$. The following result provides a strategy to design controllers ensuring the trajectories of (33) satisfy (34).

For further results regarding robust performance of LTV systems see,
e.g., [25, 30].

\begin{lemma}
Suppose there exists a matrix sequence $P(k) = P(k)^{\top} > 0$ satisfying
\[
\begin{bmatrix}
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast & \ast & \ast
\end{bmatrix} \begin{bmatrix}
-\tilde{Q}_{p}(k) & \tilde{S}_{p}(k) \\
\tilde{S}_{p}(k)^{\top} & \tilde{R}_{p}(k)
\end{bmatrix} \begin{bmatrix}
A_{cl}(k)^{\top} & C_{cl}(k)^{\top} \\
E_{cl}(k)^{\top} & D_{cl}(k)^{\top}
\end{bmatrix} \begin{bmatrix}
I_{q} & 0 \\
0 & I_{q}
\end{bmatrix} \geq 0, \tag{35a}
\]
for $k = 0, \ldots, N - 1,$ and
\[
I_{q} - C(N)P(N)C(N)^{\top} \geq 0. \tag{35b}
\]
The output $z(k)$ of the closed-loop system (33) subject to the disturbance input $\bar{w}(k)$ and with initial condition $x(0) = 0$ satisfies the quadratic robust performance criterion (34).

\begin{proof}
The result follows from dissipativity arguments (see, e.g., [31]) and the dualization lemma [26, Lemma 4.9].
\end{proof}

With the aim of designing controllers, such that (34) holds directly
using noisy data, consider (24)-(26). A complication then arises due
to the fact that we consider measurement noise in addition to process
noise. Namely, $E_{cl}(k)$, through which the disturbance input $\bar{w}(k)$ enters the system (33) depends on the unknown input matrix $B(k)$. Hence, (33) cannot be represented using (24)-(26) alone. To address this, we introduce an additional data-driven representation of $B(k)$. Supposing (23) holds, $B(k)$ can be written as
\[
B(k) = \begin{bmatrix}
A(k) & B(k)
\end{bmatrix} \begin{bmatrix}
0 \\
I_{m}
\end{bmatrix} = \begin{bmatrix}
Z(k + 1) + W(k)
\end{bmatrix}M(k), \tag{36}
\]
with $M(k) \in \mathbb{R}^{L \times m}$ satisfying
\[
\begin{bmatrix}
0 \\
I_{m}
\end{bmatrix} = \begin{bmatrix}
Z(k) \\
U(k)
\end{bmatrix}M(k), \tag{37}
\]
for $k = 0, \ldots, N - 1$. Using (24)-(26) and (36), (37) the system (33) can be equivalently written as a data-dependent lower linear fractional transformation (LFT, see, e.g., [32]), namely
\[
\begin{bmatrix}
x(k + 1) \\
z(k)
\end{bmatrix} = \begin{bmatrix}
Z(k + 1)G(k) & E_{cl}(k) & I_{n} & \bar{w}(k)
\end{bmatrix} \begin{bmatrix}
x(k) \\
C_{cl}(k) & D_{cl}(k) & 0 & 0
\end{bmatrix} \begin{bmatrix}
G(k) & M(k) & 0 & 0
\end{bmatrix} \bar{z}(k), \tag{38}
\]
where $\bar{w}(k) = W(k)\bar{z}(k), E_{cl}(k) = [Z(k + 1)M(k) I_{n}],$ and $M(k) = [M(k) 0],$ together with (33c), (33d), for $k = 0, \ldots, N - 1$. Using this data-dependent system representation and the result of Lemma 2, controllers ensuring the criterion (34) holds can be designed directly using noisy data.

\begin{theorem}
Consider the system (1) and suppose an ensemble of input-output data is available to form the matrices (22), (6b), such that
the rank condition (23) holds, for $k = 0, \ldots, N - 1$. Suppose $W(k)$, as defined in (26), satisfies (29), for $k = 0, \ldots, N - 1$. Any sequences of matrices $Y(k), P(k) = P(k)^{\top}$ satisfying (39a), shown at the bottom of the next page,
\[
Z(k)Y(k) = P(k), \tag{39b}
\]

for $k = 0, \ldots, N - 1$, and (35b), are such that the trajectories of the system (33), with

$$K(k) = U(k)Y(k)P(k)^{-1},$$

(40)

for $k = 0, \ldots, N - 1$, and with initial condition $x(0) = 0$, satisfy the quadratic robust performance criterion (34).

**Proof:** By Lemma 2, (34) is satisfied for trajectories of (33) if there exists $P(k) = P(k)^{\top} > 0$ such that (35) holds, for $k = 0, \ldots, N - 1$. Consider the system representation (38) (based on (24)-(26) and (36), (37)) and let $Y(k) := G(k)P(k)$. Via the concrete version of the full block $S$-procedure (see [26], [27], (35a) is satisfied if (29) holds and $P(k)$, $Y(k)$ satisfy a quadratic matrix inequality, which can be transformed into the LMI (39a) by performing the matrix multiplication and applying the Schur complement twice. The equality constraints (39b) and (39c) stem from the upper row block of (25) and (37), respectively, while the lower row block of (25) is automatically satisfied by $K(k)$ in (40).

Theorem 4 provides a general approach to design controllers guaranteeing robust quadratic performance for unknown LTV systems, affected by both measurement and process noise, directly using noisy data.

**Remark 6:** While (36), (37) correspond to uniquely identifying the matrix $B(k)$, since the sequence $M(k)$ is determined at the same time as the control gain $K(k)$ in (40), the result of Theorem 4 is still a direct data-driven control approach (as opposed to indirect approaches involving sequential system identification and control design).

**Remark 7:** In the absence of measurement noise, (29) reduces to a bound on $D(k)$ (as discussed in Remark 3). Moreover, the closed-loop system is described by (33a)-(33c) with the disturbance defined as $w(k) := d(k)$, and hence, $E_{\omega}(k) = I$, and $D_{\omega}(k) = D_{\omega}(k)$. This removes the need to represent $B(k)$ via (36), (37). The closed-loop system can be represented directly using (24)-(26) via the LFT (38) with $E_{\omega}(k) = I$, and $M(k) = 0$ and the data-dependent feasibility problem in Theorem 4 reduces to finding sequences of matrices $Y(k)$ and $P(k) = P(k)^{\top}$ satisfying (39a) and (39b), for $k = 0, \ldots, N - 1$, and (35b). The control law guaranteeing (34) for the system (33) is given by $u(k) = K(k)x(k)$, with $K(k)$ given by (40).

**V. PERIODICALLY TIME-VARYING SYSTEMS**

Consider the special case in which the time-variation of the matrices $A(k)$ and $B(k)$ is $\phi$-periodic, for some $\phi \in \mathbb{N}$, i.e.,

$$A(k + \phi) = A(k), \quad B(k + \phi) = B(k),$$

(41)

for all $k \geq 0$. While the system matrices are assumed to be unknown, the periodic nature and period $\phi$ of the system may be known a priori. Exploiting periodicity, the requirement for an ensemble of $L$ data sequences (Assumption 1) can then be replaced by the requirement of one sufficiently long input-state data sequence capturing $L$ periods, i.e., covering the time interval $k = 0, \ldots, \phi L$. Moreover, this data sequence can be used to derive a data-driven system representation beyond the interval $k = 0, \ldots, \phi L$. These observations allow us to derive finite-horizon results based on finite-horizon data. Thus, in the following, we consider the infinite-horizon versions of the control problems considered in Section III and Section IV in the context of periodically time-varying systems.

**A. Stabilization**

Exploiting periodicity, stabilizing controllers for linear periodically time-varying systems can be designed using only a single, finite-length data sequence.

**Corollary 1:** Consider the linear periodically time-varying system (4), (41) and suppose input-state data is available to form the matrices (6a), (6b), such that the rank condition (8) holds, for $k = 0, \ldots, \phi - 1$. Any sequences of matrices $Y(k)$, $P(k) = P(k)^{\top}$ satisfying (11), for $k = 0, \ldots, \phi - 1$, where $\eta \geq 1$ and $\rho > \eta$ are finite constants, and

$$P(\phi) = P(0),$$

(42)

are such that the system (5), (41), with $K(k)$ given by (12), for $k = 0, \ldots, \phi - 1$, and $K(k + n_\phi \phi) = K(k)$, for all $n_\phi \geq 0$, is exponentially stable.

**Proof:** The closed-loop LTV system (5) is exponentially stable if and only if there exists $P(k) = P(k)^{\top}$ satisfying (11c) and (16) for some $K(k)$ for all $k \geq 0$. If the system dynamics are $\phi$-periodic the system is exponentially stable if and only if there exists a $\phi$-periodic solution $P(k)$, $K(k)$ to (16) [25, Ch. 3.1]. Hence, we only need to find $K(k)$, $P(k)$ satisfying (16) for one period, i.e., for $k = 0, \ldots, \phi$. Using (9), (10) and following steps similar as in the proof of Theorem 1, (16) is equivalent to (11), with the additional constraint (42) in place to ensure that $P(k)$ is periodic.

**Remark 8:** Corollary 1 is the infinite-horizon equivalent of Theorem 1 for periodically time-varying systems. Similarly, (noise) input-to-state stabilizing controllers can be designed using noisy data by solving (30), (11c), for $k = 0, \ldots, \phi - 1$, with the additional constraint (42), supposing (29) holds. This represents the infinite-horizon counterpart to Theorem 3.

**B. Optimal Control**

Consider system (4), and suppose we are interested in finding a stabilizing $u^*(k)$, for all $k \geq 0$, minimizing

$$J(x(0), u(\cdot)) = \sum_{k=0}^{\infty} \langle x(k)^{\top} Q(k) x(k) + u(k)^{\top} R(k) u(k) \rangle,$$

(43)

with $Q(k) = Q(k)^{\top} \succeq 0$ and $R(k) = R(k)^{\top} \succeq 0$, for all $k \geq 0$. If (41) holds and $Q(k + \phi) = Q(k)$ and $R(k + \phi) = R(k)$, then the sequence of state feedback gains $K^*(k)$, $k \geq 0$, corresponding to the solution $u^*(k)$ is also $\phi$-periodic, i.e., $K^*(k + \phi) = K^*(k)$ [25, Ch.

Such data may stem from a single experiment of length $L \phi$, or from an ensemble of $L$ experiments of length $\phi$.  

$$\begin{bmatrix} P(k + 1) - Q_\phi(k) & * & * & * \\ -S_\phi(k)^{\top} & S_\phi(k)^{\top} & * & * \\ -D_\phi(k) & D_\phi(k)^{\top} & * & * \\ (Z(k + 1)Y(k))^{\top} & (C(k)P(k) + D_\phi(k)U(k)Y(k))^{\top} & Y(k)^{\top} & 0 \\ \end{bmatrix} \succ 0 \quad (39a)$$
Similarly to the infinite-horizon case considered in Section III-B, the described infinite-horizon LQR problem can be formulated and solved via a convex programme involving LMI constraints [33]. Exploiting periodicity, this can be solved directly using a single, finite-length input-state data sequence.

**Corollary 2:** Consider the linear periodically time-varying system (4), (41) and suppose input-state data is available to form the matrices (6a), (6b), such that the rank condition (8) holds, for \( k = 0, \ldots, \phi - 1 \). Consider the cost function (43) with \( Q(k + \phi) = Q(k) \) and \( R(k + \phi) = R(k) \), for all \( k \geq 0 \). The optimal state feedback control gain sequence solving the infinite-horizon LQR problem with \( u^*(k) = K^*(k)x(k) \) is given by (20), for \( k = 0, \ldots, \phi - 1 \), and

\[
K^*(k + n_p, \phi) = K^*(k), \quad \text{for all } n_p \geq 0, \quad \text{with } H^*(k) \text{ and } S^*(k) \text{ the solution of}
\]

\[
\min_{S, H, O} \sum_{k=0}^{\phi-1} \left( \text{Tr} \left( Q(k)S(k) \right) + \text{Tr} \left( O(k) \right) \right)
\]

\[
\text{s.t. } (21b) - (21e), \quad S(\phi) = S(0), \quad (44)
\]

for \( k = 0, \ldots, \phi - 1 \), where \( S = \{S(1), \ldots, S(\phi)\} \), \( H = \{H(0), \ldots, H(\phi - 1)\} \), and \( O = \{O(0), \ldots, O(\phi - 1)\} \).

**Proof:** The infinite-horizon LQR problem can be recast as a convex programme (see [33]). Then, exploiting that the solution is a state feedback law and introducing (9), (10) yields (21) with \( QI = 0 \) and \( N \to \infty \), where \( H(k) := G(k)S(k) \). Recall that \( K^*(k) \) for the considered problem is \( \phi \)-periodic [25, Ch. 3.1]. It remains to be shown that this \( \phi \)-periodic solution can be recovered by solving (44) one over period, with the additional constraint \( S(\phi) = S(0) \). Since \( K^*(k) \) is stabilizing by construction, there exists a \( \phi \)-periodic solution \( S^*(k + \phi) = S^*(k) \) satisfying (21b) and (21c) (this can be shown using analogous arguments as in the proof of Corollary 1). Thus, the solution of the slack variable \( O^*(k) = R(k)\frac{1}{2}K^*(k)S^*(k)K^*(k)^\text{T}R(k)\frac{1}{2} \) is also \( \phi \)-periodic. Hence, constraints (21b)-(21e) are satisfied at time \( k \) and \( k + n_p, \phi, \) for all \( n_p \geq 0 \), if they are satisfied at time \( k + 1 \). Consequently, the optimal stage cost \( I^*_\phi(k) = \text{Tr}(Q(k)S^*(k)) + \text{Tr}(O^*(k)) \), satisfies \( I^*_\phi(k + n_p, \phi) = I^*_\phi(k), \) for all \( n_p \geq 0 \). Hence, the optimal cost is given by \( \sum_{k=0}^{\phi-1} I^*_\phi(k) = \lim_{n_p \to \infty} \sum_{k=0}^{\phi-1} I^*_\phi(k) \). Note that \( \sum_{k=0}^{\phi} I^*_\phi(k) \) is the optimal cost obtained by solving (44). Hence, the periodic solution to the infinite-horizon LQR problem is given by \( K^*(k) \), for \( k = 0, \ldots, \phi - 1 \), and (44), and \( K^*(k + n_p, \phi) = K^*(k) \), for all \( n_p \geq 0 \).

**C. Robust Performance**

Consider the problem of designing stabilizing controllers of the form (2), such that the closed-loop system (33a), (33b), (33d) satisfies the infinite-horizon performance criterion

\[
\sum_{k=0}^{\infty} \begin{bmatrix} \bar{w}(k) \\ z(k) \end{bmatrix}^\text{T} \begin{bmatrix} Q_p(k) & S_p(k) \\ S_p(k)^\text{T} & R_p(k) \end{bmatrix} \begin{bmatrix} \bar{w}(k) \\ z(k) \end{bmatrix} + \varepsilon \sum_{k=0}^{\infty} \bar{w}(k)^\text{T} \bar{w}(k) \leq 0,
\]

for all \( \bar{w} \in \ell_2 \), with \( \varepsilon > 0 \), \( R_p(k) \geq 0 \) and such that \( Q_p(k) < 0 \), for all \( k \geq 0 \). Suppose the system dynamics and the performance index are \( \phi \)-periodic, i.e., (41) holds and \( C(k + \phi) = C(k) \), \( D_a(k + \phi) = D_a(k) \), \( D_a(k + \phi) = D_d(k) \), \( Q_p(k + \phi) = Q_p(k) \), \( S_p(k + \phi) = S_p(k) \), \( S_p(k + \phi) = S_p(k) \), \( R_p(k + \phi) = R_p(k) \).

This problem can be solved via a data-driven convex programme using a single, finite-length data sequence.

**Corollary 3:** Consider the linear periodically time-varying system (1), (41) and suppose input-output data is available to form the matrices (22), (6b), such that the rank condition (23) holds, for \( k = 0, \ldots, \phi - 1 \). Suppose the performance index is \( \phi \)-periodic and \( W(k) \), as defined in (26), satisfies (29), for \( k = 0, \ldots, \phi - 1 \). Any sequences of matrices \( Y(k) \), \( P(k) = P(k)^\text{T} \) satisfying (39a)-(39c), for \( k = 0, \ldots, \phi - 1 \), and

\[
P(\phi) = P(0),
\]

are such that the trajectories of the system (33a), (33b), (33d), with \( K(k) \) given by (40), for \( k = 0, \ldots, \phi - 1 \), and \( K(k + n_p, \phi) = K(k) \), for \( n_p \geq 0 \), and with initial condition \( x(0) \), satisfy the quadratic robust performance criterion (45).

**Proof:** Analogous to Lemma 2, it can be shown via dissipativity arguments (see, e.g., [31]) and the dualization lemma [26, Lemma 4.9] that (45) holds, if there exist \( \phi \)-periodic sequences \( K(k) \), \( P(k) = P(k)^\text{T} \) satisfying (35a) for all \( k \geq 0 \). Stability is implied by the upper left block of (35a) and the assumption that \( Q_p(k) \) for all \( k \geq 0 \). The data-driven formulation (39), (46) follows via analogous steps to those in the proof of Theorem 4, exploiting periodicity.

**VI. CONCLUSION**

A model-free, data-driven representation of closed-loop LTV systems under state feedback has been employed to design feedback controllers ensuring that the resulting closed-loop trajectories satisfy certain boundedness, performance, and robustness criteria via the formulation of convex feasibility/optimization problems involving data-dependent LMIs. Both the noise-free case and the case in which the data and the system are affected by process and measurement noise have been considered. Special insights have also been provided for the case of periodically time-varying systems.

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