GENERIC CONTINUITY OF METRIC ENTROPY FOR VOLUME-PRESERVING DIFFEOMORPHISMS

JIAGANG YANG AND YUNHUA ZHOU

ABSTRACT. Let $M$ be a compact manifold and $\text{Diff}^1_m(M)$ be the set of $C^1$ volume-preserving diffeomorphisms of $M$. We prove that there is a residual subset $\mathcal{R} \subset \text{Diff}^1_m(M)$ such that each $f \in \mathcal{R}$ is a continuity point of the map $g \mapsto h_m(g)$ from $\text{Diff}^1_m(M)$ to $\mathbb{R}$, where $h_m(g)$ is the metric entropy of $g$ with respect to volume measure $m$.

1. INTRODUCTION

Let $M$ be a smooth compact Riemannian manifold with dimension $d$, and $m$ be a smooth volume measure on $M$. Without loss of generality, we always assume that $m(M) = 1$ in this paper. Denote by $\text{Diff}^r_m(M)$ the set of $C^r$ volume-preserving diffeomorphisms of $M$ endowed with $C^r$ topology for $r \geq 1$.

Our main result is

**Theorem 1.1.** There is a residual subset $\mathcal{R} \subset \text{Diff}^1_m(M)$ such that each $f \in \mathcal{R}$ is a continuity point of the metric entropy map

$$
\mathcal{E} : \text{Diff}^1_m(M) \to \mathbb{R},
$$

$$
g \mapsto h_m(g),
$$

where $h_m(g)$ is the metric entropy of $g$ with respect to volume measure $m$.

The study of variation of entropy mainly focuses on two issues: the continuity of topological entropies and of metric entropies. In generally, the variation of entropies is not even semicontinuous (e.g., see [7]). S. Newhouse([9]) proved that the metric entropy function

$$
\mu \mapsto h_\mu(f)
$$

**Date:** November 25, 2013.

2010 Mathematics Subject Classification. Primary 37A35; Secondary 37C20.

Key words and phrases. continuity, metric entropy, Volume-preserving.

J.Y. is partially supported by CNPq, FAPERJ, and PRONEX. Y.Z. is the corresponding author and is partially supported by Fundamental Research for Central Universities (CQDXWL2012008).
is upper semicontinuous for all \( f \in \text{Diff}^\infty(M) \). In \([8]\) (see also \([4]\)), Y. Yomdin proved that the topological entropy function
\[
f \mapsto h(f)
\]
is upper semicontinuous on \( \text{Diff}^\infty(M) \). Together with the result of A. Katok (\([5]\)), topological entropy is continuous for \( C^\infty \) systems on surface. Recently, G. Liao etc. (\([6]\)) extend the semicontinuity results of Newhouse and Yomdin to \( C^1 \) diffeomorphisms away from tangencies.

2. Preliminaries

2.1. Lyapunov exponents and dominated splitting. Given \( f \in \text{Diff}^1_m(M) \), by Oseledec Theory, there is a \( m \)-full invariant set \( O \subset M \) such that for every \( x \in O \) there exist a splitting (which is called Oseledec splitting)
\[
T_xM = E_1(x) \oplus \cdots \oplus E_k(x)
\]
and real numbers (the Lyapunov exponents at \( x \)) \( \chi_1(x,f) > \chi_2(x,f) > \cdots > \chi_k(x,f) \) satisfying \( Df(E_j(x)) = E_j(fx) \) and
\[
\lim_{n \to \pm \infty} \frac{1}{n} \ln \|Df^nv\| = \chi_j(x,f)
\]
for every \( v \in E_j(x) \setminus \{0\} \) and \( j = 1, 2, \cdots, k(x) \). In the following, by counting multiplicity, we also rewrite the Lyapunov exponents of \( m \) as
\[
\lambda_1(x,f) \geq \lambda_2(x,f) \geq \cdots \geq \lambda_d(x,f).
\]
For \( x \in O \), we denote by
\[
\xi_i(x,f) = \begin{cases} 
\lambda_i(x,f), & \text{if } \lambda_i(x,f) \geq 0; \\
0, & \text{if } \lambda_i(x,f) < 0
\end{cases}
\]
and
\[
\chi^+(x,f) = \sum \xi_i(x,f).
\]
By the definitions, it is obviously that for \( f, g \in \text{Diff}^1_m(M) \), one has
\[
(2.1) \quad \int |\chi^+(x,f) - \chi^+(x,g)|dm(x) \leq \sum \int |\lambda_i(x,f) - \lambda_i(x,g)|dm(x)
\]
For \( f \in \text{Diff}^1_m(M) \) and \( \delta > 0 \), denote by \( \mathcal{U}(f,\delta) \) the set of diffeomorphisms \( g \in \text{Diff}^1_m(M) \) such that the \( C^1 \) distance between \( g \) and \( f \) is less than \( \delta \).

Given a diffeomorphism \( f \), we say \( Df \) has a dominated splitting of index \( i \) at a point \( x \in M \) if there are a \( Df \)-invariant splitting \( T_{\text{orb}(x)}M = E \oplus F \) and a constant \( N(x) \in \mathbb{N} \) such that \( \dim(F) = i \) and
\[
\frac{\|Df^{N(x)}E(f^j(x))\|}{m(Df^{N(x)}E(f^j(x)))} < \frac{1}{2}, \quad \forall j \in \mathbb{Z}.
\]
We also denote the dominated splitting by $E \prec F$.

Let $\Lambda$ be an $f$-invariant set and $T_\Lambda M = E \oplus F$ be a $Df$-invariant splitting on $\Lambda$. We call $T_\Lambda M = E \oplus F$ be a $N$-dominated splitting, if there exists $N \in \mathbb{N}$ such that
\[
\frac{\|Df^N|_{E(y)}\|}{m(Df^N|_{E(y)})} < \frac{1}{2}, \quad \forall y \in \Lambda.
\]

Let us note that the dominated splitting has persistence property (3). That is, if $\Lambda$ is an $f$-invariant set with an $N$-dominated splitting, then there is a neighborhood $U$ of $\Lambda$ and a $C^1$-neighborhood $U$ of $f$ such that for every $g \in U$, the maximal $g$-invariant set in the closure of $U$ admits an $N$-dominated splitting, having the same dimensions of the initial dominated splitting over $K$.

2.2. $C^1$ generic properties. We recall three $C^1$ generic properties which will be used in the proof of Theorem 1.1.

The first is about the relation of Oseledec splitting and dominated splitting.

**Lemma 2.1.** (Theorem 1 of [2]) There exists a residual set $R \subset \text{Diff}_m^1(M)$ such that, for each $f \in R$ and a measurable function $N : M \to \mathbb{N}$ such that for $m$-almost every $x \in M$, the Oseledec splitting of $f$ is either trivial or is $N(x)$ dominated at $x$.

The second one is the generic continuity of the Lyapunov spectrum.

**Lemma 2.2.** (Theorem D of [1]) Fix an integer $r \geq 1$. For each $i$, the continuous points of the map
\[
\lambda_i : \text{Diff}_m^r(M) \to L^1(M) \quad f \mapsto \lambda_i(\cdot, f)
\]
form a residual subset.

The third property is the generic persistence of invariant sets. It says that if $f$ is a generic volume-preserving diffeomorphism, then its measurable invariant sets persist in a certain (measure-theoretic and topological) sense under perturbations of $f$.

**Lemma 2.3.** (Theorem C of [1]) Fix an integer $r \geq 0$. There is a residual set $R \subset \text{Diff}_m^r(M)$ such that for every $f \in R$, every $f$-invariant Borel set $\Lambda \subset M$ with positive volume, and every $\eta > 0$, if $g \in \text{Diff}_m^r(M)$ is sufficiently close to $f$ then there exists a $g$-invariant Borel set $\tilde{\Lambda}$ such that
\[
\tilde{\Lambda} \subset B_\eta(\Lambda) \text{ and } m(\tilde{\Lambda} \Delta \Lambda) < \eta,
\]
here $B_\eta(\Lambda) = \{y \in M : d(x,y) < \eta \text{ for some } x \in \Lambda\}$. 
2.3. $C^1$ Pesin entropy formula. In [10], W. Sun and X. Tian proved that the Pesin entropy formula holds for a generic $f \in \text{Diff}^1_m(M)$.

**Lemma 2.4.** (Theorem 2.5 of [10]) There exists a residual subset $\mathcal{R} \subset \text{Diff}^1_m(M)$ such that for every $f \in \mathcal{R}$, the metric entropy $h_m(f)$ satisfies Pesin’s entropy formula, i.e.,

$$h_m(f) = \int_M \chi^+(x,f) dm.$$ 

In fact, Lemma 2.4 is a corollary of Ruelle’s inequality and the following result.

**Lemma 2.5.** (Theorem 2.2 of [10]) Let $f : M \to M$ be a $C^1$ diffeomorphism on a compact Riemannian manifold with dimension $d$. Let $f$ preserve an invariant probability $\mu$ which is absolutely continuous relative to Lebesgue measure. For $\mu$-a.e. $x \in M$, denote by

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_d(x)$$

the Lyapunov exponents at $x$. Let $N(\cdot) : M \to \mathbb{N}$ be an $f$-invariant measurable function. If for $\mu$-a.e. $x \in M$, there is a $N(x)$-dominated splitting: $T_{\text{orb}(x)}M = E \prec F$, then

$$h_\mu(f) \geq \int_M \chi_F(x) dm$$

here $\chi_F(x) = \sum_{i=1}^{\dim F(x)} \lambda_i(x)$.

3. **Proof of Theorem 1.1**

**Proof of Theorem 1.1.** We will prove the Theorem by two steps. In step 1, we first prove that there is a residual subset $\mathcal{R}_1 \subset \text{Diff}^1_m(M)$ such that the entropy map $E$ is upper-semicontinuous at each $f \in \mathcal{R}_1$. In step 2, it will be proved that the set of lower-semicontinuous points of $E$ contains a residual set $\mathcal{R}_2 \subset \text{Diff}^1_m(M)$. Setting $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$, we complete the proof of Theorem 1.1.

**Step 1.** Let $\mathcal{R}_1 \subset \text{Diff}^1_m(M)$ satisfying Lemma 2.2 and Lemma 2.4. In this step, we will prove that, for any $f \in \mathcal{R}_1$,

$$\limsup_{g \to f} h_m(g) \leq h_m(f).$$

In fact, by Lemma 2.4

$$h_m(f) = \int_M \chi^+(x,f) dm, \ \forall f \in \mathcal{R}_1.$$ 

So, by the well known Ruelle’s inequality

$$h_m(g) \leq \int_M \chi^+(x,g) dm, \ \forall g \in \text{Diff}^1_m(M)$$
and (2.1), we have that for \( \forall g \in \mathcal{U}(f, \delta) \),
\[
(3.1) \quad h_m(g) - h_m(f) \leq \int_M |\chi^+(x, g) - \chi^+(x, f)| \, dm \leq \sum \int_M |\lambda_i(x, f) - \lambda_i(x, g)| \, dm.
\]

Combining with Lemma 2.2, we proved the upper-semicontinuity.

**Step 2.** Let \( \mathcal{R}_2 \subset \mathcal{R}_1 \) which satisfies Lemma 2.1 and Lemma 2.3. We will prove that the entropy map \( E \) is lower-semicontinuous at each \( f \in \mathcal{R}_2 \). That is, for any \( f \in \mathcal{R}_2 \) and \( \varepsilon > 0 \), there are positive numbers \( \delta \) and \( D \) such that
\[
(3.2) \quad h_m(g) \geq h_m(f) - D\varepsilon, \quad \forall g \in \mathcal{U}(f, \delta),
\]
here \( D \) is only dependent on \( d \) and \( D_f \).

If the Oseledec splitting of \( f \) is trivial on Lebesgue almost every point, then \( h_m(f) = 0 \) and \( h_m(g) \geq h_m(f) \) for all \( g \in \text{Diff}^1_m(M) \). This means that the metric entropy map is lower semicontinuous at \( f \). So, in the following, we always assume that the Oseledec splitting of \( f \) is not trivial.

Let
\[
M_i(f) = \{ x : \lambda_i(x, f) > 0, \lambda_{i+1}(x, f) \leq 0 \}.
\]
Then
\[
(3.3) \quad M_i(f) \cap M_j(f) = \emptyset, \quad \forall i \neq j
\]
and
\[
(3.4) \quad h_m(f) = \sum_{i=1}^{d} \int_{M_i(f)} \sum_{j=1}^{i} \lambda_j(x, f) \, dm.
\]

**Claim 1.** For any \( \varepsilon > 0 \), there is \( \delta_1 > 0 \) such that for any \( g \in \mathcal{U}(f, \delta_1) \) and \( i = 1, 2, \ldots, d \), there exists \( M'_i(f) \subset M_i(f) \) such that \( m(M_i(f) \setminus M'_i(f)) < \varepsilon \) and \( \lambda_i(x, g) > 0, \text{ m-a.e. } x \in M'_i(f) \).

**Proof of Claim 1.** For any \( \varepsilon > 0 \) and \( i = 1, 2, \ldots, n \), there is \( k(i) > 0 \) such that
\[
(3.5) \quad m(M_i(f) \setminus M_{ik(i)}(f)) < \frac{\varepsilon}{2}
\]
here
\[
M_{ik(i)}(f) = \{ x \in M_i(f) : \lambda_i(x, f) \geq \frac{1}{k(i)} \}.
\]

Let \( \varepsilon' = \min\{\varepsilon, \frac{\varepsilon}{2k(1)}, \ldots, \frac{\varepsilon}{2k(n)}\} \). By Lemma 2.2, there is \( \delta_1 > 0 \) such that for any \( g \in \mathcal{U}(f, \delta_1) \) and any \( i \),
\[
\int_M |\lambda_i(x, g) - \lambda_i(x, f)| \, dm < \varepsilon'
\]
Set
\[
M'_i(f) = \{ x \in M_{ik(i)}(f) : \lambda_i(x, g) > 0 \} \quad \text{and} \quad M''_i(f) = M_{ik(i)}(f) \setminus M'_i(f).
\]
Then we have
\[(3.6) \quad m(M''_i(f)) \leq \frac{\varepsilon}{2}.\]
In fact, if \(m(M''_i(f)) > \frac{\varepsilon}{2}\), we have
\[
\int_M |\lambda_i(x, f) - \lambda_i(x, g)| dm \geq \int_{M''_i(f)} |\lambda_i(x, f) - \lambda_i(x, g)| dm \geq \frac{1}{k(i)} m(M''_i(f)) > \varepsilon'.
\]
This is a contradiction. \(\Box\)

Claim 2. For any \(\varepsilon > 0\), there is \(\delta_2 > 0\) such that for any \(g \in U(f, \delta_2)\) and any \(i = 1, 2, \ldots, d\), there exist \(N \in \mathbb{N}\) and a \(g\)-invariant set \(\tilde{M}_i(g) \subset M\) such that
1. there is a \(N\)-dominated splitting of index \(i\);
2. \(m(\tilde{M}_i(g) \setminus \tilde{M}_i(f)) < \frac{\varepsilon}{4}\).

Proof of Claim 2. By Lemma 2.1, there is a \(N(x)\)-dominated splitting of index \(i\) at each \(x \in \tilde{M}_i(f)\), for any \(\varepsilon > 0\), there are \(N \in \mathbb{N}\) and an \(f\)-invariant subset \(\tilde{M}_i(f) \subset M_i(f)\) such that \(m(M_i(f) \setminus \tilde{M}_i(f)) < \frac{\varepsilon}{4}\) and there is a \(N\)-dominated splitting of index \(i\) at each \(x \in \tilde{M}_i(f)\).

By the persistence property of dominated splitting and Lemma 2.3, for any \(\varepsilon > 0\), there is \(\delta_2 > 0\) such that for any \(g \in U(f, \delta_2)\) there is \(g\)-invariant set \(\tilde{M}_i(g)\) closing to \(\tilde{M}_i(f)\) such that there is a \(N\)-dominated splitting of index \(i\) at each \(x \in \tilde{M}_i(g)\) for \(Dg\) and \(m(\tilde{M}_i(g) \setminus \tilde{M}_i(f)) < \frac{\varepsilon}{4}\). \(\Box\)

Set
\[
M_i^+(g) = \{x \in \tilde{M}_i(g) : \lambda_i(x, g) > 0\} \quad \text{and} \quad M^+(g) = \bigcup_{i=1}^n M_i^+(g).
\]
Then \(M_i^+(g)\) and \(M^+(g)\) are \(g\)-invariant. By (2) of Claim 2 and (3.3), for any \(g \in U(f, \delta_2)\) and any \(i \neq j\), we have
\[
m(M_i^+(g) \cap M_j^+(g)) < \varepsilon
\]
and so
\[
(3.7) \quad m(M_i^+(g) \setminus \bigcup_{j=1}^{i-1} M_j^+(g)) \geq m(M_i^+(g)) - (i - 1)\varepsilon.
\]
Furthermore, noting
\[
\tilde{M}_i(g) \cap M_i^+(g) \subset M_i^+(g) \subset \tilde{M}_i(g),
\]
by Claim 1 and Claim 2, we have
\[
m(M_i^+(g) \setminus \tilde{M}_i(f)) < 3\varepsilon, \quad \forall g \in U(f, \delta),
\]
here $\delta = \min\{\delta_1, \delta_2\}$. Since $m(\cup_{i=1}^{d} M_i(f)) = 1$, it holds that
\[
m(M^+(g)) \geq 1 - 3\varepsilon.
\]

Now, we turn to estimate $h_m(g)$ for $g \in C^1$-close to $f$.

For $g \in \mathcal{U}(f, \delta)$, we have
\[
\int_{M_i^+(g)} |\lambda_j(x, g) - \lambda_j(x, f)| dm \leq \int_M |\lambda_j(x, g) - \lambda_j(x, f)| dm < \varepsilon, \forall j = 1, 2, \ldots, n.
\]
So,
\[
(3.8) \quad \int_{M_i^+(g)} \sum_{j=1}^{i} \lambda_j(x, g) dm > \int_{M_i^+(g)} \sum_{j=1}^{i} \lambda_j(x, f) dm - i\varepsilon = \left( \int_{M_i^+(g)} \setminus M_i(f) \right) \sum_{j=1}^{i} \lambda_j(x, f) dm - i\varepsilon
\]
\[
\geq \left( \int_{M_i(f)} \setminus M_i^+(g) \right) \sum_{j=1}^{i} \lambda_j(x, f) dm - iD_f m(M_i^+(g) \setminus M_i(f)) - i\varepsilon
\]
\[
\geq \int_{M_i(f)} \sum_{j=1}^{i} \lambda_j(x, f) dm - iD_f (m(M_i^+(g) \setminus M_i(f)) + m(M_i(g) \setminus M_i^+(g))) - i\varepsilon
\]
\[
\geq \int_{M_i(f)} \sum_{j=1}^{i} \lambda_j(x, f) dm - (3iD_f + i)\varepsilon
\]
Then by $\text{(3.7), (3.8)}$ and Lemma $2.5$,
\[
h_m(g) \geq \sum_{i=1}^{d} \int_{M_i^+(g) \setminus \cup_{j=1}^{i-1} M_i^+(g)} \sum_{j=1}^{i} \lambda_j(x, g) dm
\]
\[
\geq \sum_{i=1}^{d} \left( \int_{M_i^+(g)} \sum_{j=1}^{i} \lambda_j(x, g) dm - (i - 1)iD_f \right)
\]
\[
\geq \sum_{i=1}^{d} \left( \int_{M_i(f)} \sum_{j=1}^{i} \lambda_j(x, f) dm - (4D_f + 1)i\varepsilon \right)
\]
\[
= h_m(f) - \frac{d(1+d)(4D_f+1)}{2}\varepsilon.
\]
Setting $D = 3d^2D_f + \frac{d(1+d)(4D_f+1)}{2}$, we completes the proof of (3.2).

\[\square\]

REFERENCES

1. A. Ávila, J. Bochi, Nonuniform hyperbolicity, global dominated splittings and generic properties of volume-preserving diffeomorphisms, Trans. Amer. Math. Soc., 364(2012), no. 6, 2883-2907.

2. J. Bochi, M. Viana, The Lyapunov exponents of generic volume preserving and symplectic systems, Annals of Math. 161(2005), 1423-1485.

3. C. Bonatti, L. Díaz, M. Viana: Dynamics beyond uniform hyperbolicity. Springer, (2005)
4. J. Buzzi, Intrinsic ergodicity for smooth interval maps, *Israel J. Math.*, **100**, (1997), 125-161.
5. A. Katok, Lyapounov exponents, entropy and periodic points of diffeomorphisms, *Publ. Math. IHES*, **51**(1980), 137-173.
6. L. Gang, M. Viana, J. Yang, The entropy conjecture for diffeomorphisms away from tangencies, *Journal of the European Mathematical Society*, v. **15**, (2013) p. 2043-2060.
7. M. Misiurewicz, Diffeomorphism without any measure of maximal entropy, *Bull. Acad. Pol. Sci.*, **21**(1973), 903-910.
8. Y. Yomdin, Volume growth and entropy, *Israel J. of Math.*, **57**(1987), 285-301.
9. S. Newhouse, Continuity properties of entropy, *Ann. of Math.* (1), **129**(1989), 215-235
10. W. Sun, X. Tian, Dominated splitting and Pesin’s entropy formula, *Discrete Contin. Dyn. Syst.*, **32**(2012), no. 4, 1421-1434.

**Departamento de Geometria, Instituto de Matemática e Estatística, Universidade Federal Fluminense, Niterói, 24020-140, Brazil**

*E-mail address*: yangjg@impa.br

**College of Mathematics and Statistics, Chongqing University, Chongqing, 401331, P. R. China**

*E-mail address*: zhouyh@cqu.edu.cn