Quantum Hall Effect Wave Functions as Cyclic Representations of $U_q(sl(2))$

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Abstract

Quantum Hall effect wave functions corresponding to the filling factors $1/2p+1, 2/2p+1, \cdots, 2p/2p+1, 1$, are shown to form a basis of irreducible cyclic representation of the quantum algebra $U_q(sl(2))$ at $q^{2p+1} = 1$. Thus, the wave functions $\Psi_{P/Q}$ possessing filling factors $P/Q < 1$ where $Q$ is odd and $P, Q$ are relatively prime integers are classified in terms of $U_q(sl(2))$. 

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1. Introduction:

Microscopic theory of the fractional quantum Hall effect (QHE) is not well established. Its theoretical understanding mostly is due to trial wave functions \[1\]. For filling factors \(1/m\) where \(m\) is an odd integer, trial wave functions were given by Laughlin \[2\]. Trial wave functions for the other filling factors \(\nu = P/Q < 1\), where \(P, Q\) are relatively prime integers and \(Q\) is odd, were constructed in terms of some hierarchy schemes \[3\]–\[4\] where they were obtained from a parent state which is a full filled Landau level or a Laughlin wave function. However, general properties of the QHE should be independent of the explicit form of trial wave functions, but depend on their universal features as their orthogonality.

We utilize orthogonality of QHE states for different filling factors, independent of their explicit form, to show that they can be classified as irreducible cyclic representations of \(U_q(sl(2))\) at roots of unity. In our scheme, states corresponding to filling factors possessing a common denominator are in the same representation.

Although \(U_q(sl(2))\) structures were found in the Hofstadter problem \[5\], in the Landau problem \[6\], for Laughlin wave functions \[7\] and in the QHE \[8\], the approach presented here does not have any relation to them: \(i)\) As far as we deal with flat surfaces, in all of the previous works generators of the deformed algebra were constructed in terms of magnetic translations. The construction presented here cannot be written in terms of magnetic transformations. \(ii)\) Here, wave functions possessing different filling factors which have a common denominator are treated on the same footing. However, in the other works only one state is considered and the theories were built on them without mixing different states with different parameters which correspond to filling factors in the QHE case.

First, we show explicitly that the wave functions corresponding to the filling factors \(\nu = 1/3, 2/3, 1\), can be considered as basis of cyclic irreducible representation of the quantum algebra \(U_q(sl_2)\) at \(q^3 = 1\). Then, the general case is studied. Conclusions are presented in the last section.
2. Cyclic Representation of $U_q(sl(2))$:

The deformed algebra $U_q(sl(2))$

$$[E_+, E_-] = \frac{K - K^{-1}}{q - q^{-1}},$$
$$KE_\pm K^{-1} = q^{\pm 2}E_\pm. \quad (1)$$

at roots of unity i.e. $q^{2p+1} = 1$, $p$ a positive integer, has a finite dimensional irreducible representation which has no classical finite dimensional analog. This is the cyclic representation whose dimension is $2p + 1$[9]. Cyclic means that there are no highest or lowest weight states in the spectrum. i.e. $E_+|\cdots >\neq 0$ and $E_-|\cdots >\neq 0$ for any state.

When $q^{2p+1} = 1$ irreducible cyclic representation of $U_q(sl(2))$ can be written in some basis $\{v_0, v_1, \cdots, v_{2p}\}$ as

$$Kv_m = \lambda q^{-2m}v_m,$$
$$E_+v_m = g_mv_{m+1}, \quad (2)$$
$$E_-v_m = f_mv_{m-1},$$

where $m = 0, \cdots, 2p$, and we defined $v_0 \equiv v_{2p+1}$, $v_{-1} \equiv v_{2p}$. $\lambda$, $g_m$, and $f_m$ are some complex constants which are nonzero and in the case of requesting that the representation in unitary, we should restrict their values such that

$$K^\dagger = K^{-1}; \quad E^\dagger_+ = E_+. \quad (3)$$

Although, for the purposes of this work there is no need of discussing in detail neither how unitary representations arise in the general framework nor values of Casimir operators, let us denote that there are three independent Casimir operators of $U_q(sl(2))$ at $q^{2p+1} = 1$: $K^{2p+1}$, $E_+^{2p+1}$ and $E_-^{2p+1}$.

3. Classification of $\nu = 1, \frac{1}{3}, \frac{2}{3}$ States:
When $N$ particles (electrons) move on a plane in a perpendicular magnetic field we may consider the wave functions \cite{2, 10}

\begin{align}
\psi_1(z_1, \cdots, z_N) &= \mathcal{N}_1 e^{-\frac{i}{\hbar} \sum_{k=1}^{N} |z_k|^2} \prod_{i<j}^N (z_i - z_j), \\
\psi_{1/3}(z_1, \cdots, z_N) &= \mathcal{N}_2 e^{-\frac{i}{\hbar} \sum_{k=1}^{N} |z_k|^2} \prod_{i<j}^N (z_i - z_j)^3, \\
\psi_{2/3}(z_1, \cdots, z_N) &= \mathcal{N}_3 \int d^2z_{N+1} \cdots d^2z_{N+M} e^{-\frac{i}{\hbar} \sum_{K=1}^{N+M} |z_K|^2} \\
&\quad \times \prod_{I<J}^{N+M} (\bar{z}_{N+l} - \bar{z}_{N+n})^3 \prod_{I<J}^N (z_I - z_J),
\end{align}

which possess the following values of the angular momentum $L$,

\begin{align}
L[\psi_1(z_1, \cdots, z_N)] &= \frac{N(N-1)}{2}, \\
L[\psi_{1/3}(z_1, \cdots, z_N)] &= 3 \frac{N(N-1)}{2}, \\
L[\psi_{2/3}(z_1, \cdots, z_N)] &= \frac{(N+M)(N+M-1)}{2} - 3 \frac{M(M-1)}{2}.
\end{align}

It is supposed that $N$ is large and we take $M = N/2$.

Filling factors of the $N$ particle states are given in the thermodynamical limit as

\begin{equation}
\nu \equiv \lim_{N \to \infty} \frac{N(N-1)}{2L}.
\end{equation}

Hence, filling factors of the wave functions (4)–(6) are

\begin{equation}
\nu(\psi_1) = 1, \ \nu(\psi_{1/3}) = 1/3, \ \nu(\psi_{2/3}) = 2/3.
\end{equation}

Indeed, (4) is the wave function when the lowest Landau level is fully filled and (5)–(6) are the trial wave functions which describe the QHE at the filling factors $1/3, 2/3$.

By making use of

\begin{equation}
\int d^2z e^{-|z|^2} z^m z^n = \delta_{m,n},
\end{equation}

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one can observe that the wave functions which possess different angular momentum values are orthogonal. Moreover, by choosing the normalization constants $N_a$ appropriately the wave functions (4)–(6) can be taken to satisfy ($N > 2$)

\[ (\psi_\sigma, \psi_\rho) \equiv \int d^2 z_1 \cdots d^2 z_N \overline{\psi}_\sigma(z_1, \cdots, z_N) \psi_\rho(z_1, \cdots, z_N) = \delta_{\sigma, \rho}, \]  

where $\sigma, \rho = 1, 1/3, 2/3$.

If $\hat{\nu}$ denotes the first quantized operator corresponding to the filling factor $\nu$, one can construct the physical operator

\[ \hat{k} \equiv e^{2\pi i \hat{\nu}}, \]  

which will be shown to play the main role in classifying QHE wave functions in terms of $U_q(sl(2))$ at roots of unity.

In a second quantized theory, operators corresponding to physical operators of the first quantization will be given in terms of states spanning the related field theory. Let us deal with the states, corresponding to (4)–(6),

\[ |\sigma> \equiv \int d^2 z_1 \cdots d^2 z_N e^{-\frac{i}{2} \sum_{k=1}^{N} |z_k|^2} \overline{\psi}_\sigma(z_1, \cdots, z_N) |z_1, \cdots, z_N>, \]  

where

\[ |z_1, \cdots, z_N> = \frac{1}{\sqrt{N!}} \varphi^\dagger(z_1) \cdots \varphi^\dagger(z_N) |0>. \]  

The fermionic operators $\varphi(z)$, $\varphi^\dagger(z)$ satisfy the anticommutation relation

\[ \{\varphi^\dagger(z), \varphi(z')\} = e^{z z'}. \]  

The states (12) are orthonormal:

\[ <\sigma|\rho> = \delta_{\sigma, \rho}. \]  

The second quantized operator

\[ k = e^{2\pi i}|1><1| + e^{2\pi i/3}|1/3><1/3| + e^{4\pi i/3}|2/3><2/3|. \]  

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corresponds to the first quantized physical operator (11). In terms of the vector 
\(|1>, |1/3>, |2/3>\) and the scalar product defined in (14), one can obtain the 
representation

\[
k = \begin{pmatrix}
1 & 0 & 0 \\
0 & \bar{q} & 0 \\
0 & 0 & \bar{q}^2
\end{pmatrix},
\]

(16)

where \(\bar{q} = \exp(2\pi i/3)\) i.e.

\[
\bar{q}^3 = 1.
\]

Moreover, we can construct the operators

\[
e_+ = a_1|1/3><1| + a_2|2/3><1/3| + a_3|1><2/3|, \tag{17}
\]

\[
e_- = b_1|1><1/3| + b_2|1/3><2/3| + b_3|2/3><1|, \tag{18}
\]

whose representations are

\[
e_+ = \begin{pmatrix}
0 & a_1 & 0 \\
0 & 0 & a_2 \\
a_3 & 0 & 0
\end{pmatrix}, \quad e_- = \begin{pmatrix}
0 & 0 & b_3 \\
b_1 & 0 & 0 \\
0 & b_2 & 0
\end{pmatrix}.
\]

(19)

One can show that (16) and (19) realize the \(U_q(sl(2))\) algebra

\[
[e_+, e_-] = \frac{k - k^{-1}}{\bar{q} - \bar{q}^{-1}}, \quad ke_+k^{-1} = \bar{q}^{\pm 2}e_+, \tag{20}
\]

if the coefficients satisfy

\[
a_1b_1 - a_3b_3 = 0,
\]

\[
a_2b_2 - a_1b_1 = 1,
\]

\[
a_3b_3 - a_2b_2 = -1.
\]

(21)

If one demands that the representation (16), (19) is unitary:

\[
k^{-1} = k^\dagger, \quad e_+^\dagger = e_-,
\]

(22)
the coefficients $a_l$, $b_l$ should be taken as

\[ b_1 = \bar{a}_1, \ b_2 = \bar{a}_2, \ b_3 = \bar{a}_3. \]  

Then the conditions (21) lead to

\[ |a_1|^2 = |a_3|^2 = |a_2|^2 - 1, \ a_l \neq 0. \]  

Observe that the Casimir operators

\[ k^3 = 1, \ e_+^3 = a_1 a_2 a_3 1, \ e_-^3 = b_1 b_2 b_3 1 \]

are proportional to identity.

An explicit realization is presented in terms of the trial wave functions (8)–(11). However, the construction depends only on the orthogonality of the states of the QHE for different values of the filling factors and the existence of the physical operator (11). This will be clarified in the next section.

4. The General Case:

QHE trial wave functions in the standard hierarchy scheme are given by [3]–[11]

\[
\psi_\nu(z_1, \cdots, z_{N_0}) = \int \prod_{\alpha=1}^{r} \prod_{i_\alpha=1}^{N_\alpha} [d^2 z^{(\alpha)}_{i_\alpha}] e^{-\frac{1}{2} \sum_{i} |z_i|^2} \prod_{\beta=0}^{r} \prod_{i_\beta<j_\beta}^{N_\beta} (z^{(\beta)}_{i_\beta} - z^{(\beta)}_{j_\beta}) \prod_{i_\beta+1<j_\beta=1}^{N_{\beta+1,N_\beta}} (z^{(\beta+1)}_{i_\beta+1} - z^{(\beta)}_{j_\beta}) b_{\beta,\beta+1},
\]

where $z^{(0)}_{i_0} \equiv z_i$. The measure $\prod [d^2 z^{(\alpha)}_{i_\alpha}]$ depends on $a_\beta$ and $|z^{(\beta)}_{i_\beta} - z^{(\beta)}_{j_\beta}|$, however the detailed form of it does not affect the filling factor $\nu = P/Q$. $a_0$ is an odd positive integer, $a_\alpha$ for $\alpha \neq 0$ are even integers which can be positive or negative and $b_{\beta+1,\beta} = \pm 1$, except $b_{r,r+1} = 0$. By placing the $N_0$ electrons on a spherical surface in a monopole magnetic field, one can find that filling factor of (23) is given by

\[
\nu = \frac{1}{a_0 - \frac{1}{a_1 - \cdots - \frac{1}{a_l - \cdots}}},
\]
Factors with negative powers may be replaced by complex–conjugate factors with positive powers multiplied by some exponential factors. Hence, (25) can equivalently be given as

\[
\psi_\nu(z_1, \cdots, z_{N_0}) = \int \prod_{\alpha=1}^r \left[ \prod_{i\alpha=1}^{N_\alpha} d^2 z_{i\alpha} \prod_{i\alpha < j\alpha}^{N_\alpha} |z_{i\alpha}^{(\alpha)} - z_{j\alpha}^{(\alpha)}|^{2(1-\alpha) \theta_\alpha} e^{-|q_\alpha| \sum_{i\alpha} |z_{i\alpha}^{(\alpha)}|^2} e^{-\frac{1}{2} \sum_{k=1}^{N_0} |z_k|^2} \right] \prod_{\beta=0} r N_\beta \prod_{i\beta < j\beta}^{N_\beta} (\bar{z}_{i\beta}^{(\beta)} - \bar{z}_{j\beta}^{(\beta)})^{p_\beta} \prod_{i_{\beta+1} < j_{\beta+1}}^{N_{\beta+1}} (\bar{z}_{i_{\beta+1}}^{(\beta+1)} - \bar{z}_{j_{\beta+1}}^{(\beta)}),
\]

where \( \bar{z}_{i\beta}^{(\beta)} = z_{i\beta}^{(\beta)} \) for \( \beta = \text{even} \) and \( \bar{z}_{i\beta}^{(\beta)} = \bar{z}_{i\beta}^{(\beta)} \) for \( \beta = \text{odd} \) and

\[
\theta_0 = 0, \quad \theta_r = \frac{(-1)^r}{p_{r-1} - (-1)^r \theta_{r-1}},
\]

\[
q_0 = -1, \quad q_r = (-1)^{r+1} q_{r-1} \theta_r.
\]

Now, filling factor is

\[
\nu = \frac{1}{p_0 + \frac{1}{p_1 + \cdots + \frac{1}{p_r}}},
\]

where \( p_0 \) is odd and the other \( p_i \) are even integers.

By generalizing the calculations of Laughlin given in Ref. [1] and making use of the scalar product defined in (10), one can show that \( \psi_\nu \) states are orthogonal [11].

To emphasize the second quantized character of our construction let us introduce the states

\[
|i, p>_T = \int d^2 z_1 \cdots d^2 z_{N_0} e^{-\frac{1}{2} \sum_{k=1}^{N_0} |z_k|^2} \psi_{\frac{i}{2p+1}}(z_1, \cdots, z_{N_0}) |z_1, \cdots, z_{N_0}>,
\]

where \( i = 1, \cdots, 2p+1; \ p = 1, 2, \cdots, \) so that any filling factor \( \nu = P/Q \) is considered. We used the vectors (13) with \( N \) replaced by \( N_0 \). The subscript \( T \) denotes the fact that trial wave functions are used to give an explicit realization.

The states (29) are orthonormal:

\[
_T < i, p|j, p'> = \delta_{i,j} \delta_{p,p'}.
\]

We have shown that the states \( |i, p>_T \) are orthonormal by using the explicit form of trial wave functions. However, this should be a universal feature of QHE
wave functions. Then, even if we do not know the explicit form, we can say that exact states of the QHE which we indicate with $|i, p\rangle$, should be orthonormal:

$$<i, p|j, p'\rangle = \delta_{i,j}\delta_{p,p'}.$$ (31)

Indeed, in the following we will use this universal property of QHE states without referring to any trial wave function.

To generalize the construction given in Section 3, let us deal with the states

$$|1, p\rangle, |2, p\rangle, \cdots, |2p, p\rangle, |2p + 1, p\rangle,$$ (32)

corresponding to the filling factors, respectively,

$$\nu = \frac{1}{2p+1}, \frac{2}{2p+1}, \cdots, \frac{2p}{2p+1}, 1.$$ (33)

Define the following second quantized operators acting in the space spanned by the states (32),

$$\tilde{K} = \sum_{i=1}^{2p+1} q^i|i, p\rangle\langle i, p|,$$ (34)
$$\tilde{E}_+ = \sum_{i=1}^{2p+1} a_i|i, p\rangle\langle i + 2, p|,$$ (35)
$$\tilde{E}_- = \sum_{i=1}^{2p+1} \bar{a}_i|i + 2, p\rangle\langle i, p|,$$ (36)

where

$$q^{2p+1} = 1.$$ (37)

To obtain the compact forms we adopted the definitions

$$|2p + 2, p\rangle \equiv |1, p\rangle, |2p + 3, p\rangle \equiv |2, p\rangle.$$ (38)

By using the orthonormality condition (31) one observes that inverse of $\tilde{K}$ is

$$\tilde{K}^{-1} = \sum_{i=1}^{2p+1} q^{-i}|i, p\rangle\langle i, p| = \tilde{K}^\dagger.$$ (38)
Let the coefficients $a_i$ are nonzero and satisfy

$$|a_{2p+1}|^2 - |a_{2p-1}|^2 = 0,$$
$$|a_{2p}|^2 - |a_{2p-2}|^2 = -1,$$
$$|a_{l+2}|^2 - |a_l|^2 = \frac{q^{l+2} - q^{l-2}}{q - q^{-1}},$$

where $l = -1, 0, \cdots (2p - 3)$; $a_{-1} \equiv a_{2p}$, $a_0 \equiv a_{2p+1}$. Then, in terms of the basis ($|1, p >, \cdots, |2p - 1, p >$) the operators (34)–(36) lead to a $(2p + 1)$ dimensional unitary irreducible cyclic representation of $U_q(sl(2))$ at $q$ satisfying (37).

Note that the Casimir operators are proportional to unity as before: $\tilde{K}^{2p+1} = 1$ and $\tilde{E}_+^{2p+1} = \tilde{E}_-^{2p+1} = \left(\prod_{i=1}^{2p+1} a_i\right) 1$.

5. Discussions:

It is shown that QHE wave functions can be classified as irreducible cyclic representations of $U_q(sl(2))$ at roots of unity in a very natural way. This naturalness follows from the fact that the most significant physical quantity of the QHE $\nu = P/Q$ fits very well with the integer ($m$ in (2)) characterizing irreducible cyclic representations of $U_q(sl(2))$. Obviously, any set of orthogonal states possessing a quantum number which permits a partition of unity like $\nu$,

$$\sum_{i=1}^{2p+1} \frac{\nu(|i, 2p + 1 >)}{p + 1} = 1,$$

can be classified as irreducible cyclic representation of $U_q(sl(2))$ at a root of unity.

How one can utilize the proposed classification of the QHE to calculate some physical quantities? Here, one of the most significant physical quantities is the partition function which may be obtained if the Green function in the space defined by $U_q(sl(2))$ at roots of unity with cyclic representation is available. In Ref. [13] Green function in the space defined by the q–deformed group $SU_q(2)/U(1)$ for $q$ not a root
of unity is obtained without referring to explicit forms of the representations but depending only on their general features. We hope that a similar calculation can be used in our case. Then, we can obtain Green function and in terms of that the related partition function which may give some hints about its physical interpretation which is not clear at the moment.

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