Abstract

The classical approach to the study of dynamical systems consists in representing the dynamics of the system in the form of a "source-sink", that means identifying an attractor-repeller pair, which are attractor-repellent sets for all other trajectories of the system. If there is a way to choose this pair so that the space orbits in its complement (the characteristic space of orbits) is connected, this creates prerequisites for finding complete topological invariants of the dynamical system. It is known that such a pair always exists for arbitrary Morse-Smale diffeomorphisms given on any manifolds of dimension \( n \geq 3 \). Whereas for \( n = 2 \) the existence of a connected characteristic space has been proved only for orientation-preserving gradient-like (without heteroclinic points) diffeomorphisms defined on an orientable surface. In the present work, it is constructively shown that the violation of at least one of the above conditions (absence of heteroclinic points, orientability of a surface, orientability of a diffeomorphism) leads to the existence of Morse-Smale diffeomorphisms on surfaces that do not have a connected characteristic space of orbits.

1 Introduction and the statement of the result

Let \( f : M^n \to M^n \) be a Morse-Smale diffeomorphism defined on a closed connected \( n \)-manifold. Denote by \( \Omega_0^f, \Omega_1^f, \Omega_2^f \) the set of sinks, saddles, and sources of the diffeomorphism \( f \). For any (possibly empty) \( f \)-invariant set \( \Sigma \subset \Omega_1^f \) such that \( cl(W^u_\Sigma) \setminus W^u_\Sigma \subset \Omega_0^f \), set

\[
A_\Sigma = \Omega_0^f \cup W^u_\Sigma, \quad R_\Sigma = \Omega_2^f \cup W^s_{\Omega_1^f \setminus \Sigma}.
\]

It follows from [13] that \( A_\Sigma \) and \( R_\Sigma \) are an attractor and a repeller, which are called dual. In the monograph [12] the set

\[
V_\Sigma = M^n \setminus (A_\Sigma \cup R_\Sigma)
\]
is called the characteristic space, and the orbit space $\hat{V}_\Sigma$ of the action $f$ on $V_\Sigma$ is called the characteristic space of orbits.

There are a number of examples where a reasonable choice of a dual pair leads to a complete topological classification of some subset of Morse-Smale dynamical systems (look, for example, [1], [2], [8], [10], [3], and an overview of [9]). In most cases, finding complete topological invariants is based on the existence of a connected characteristic space of orbits for the class of systems under consideration. For example, according to [1], for any Morse-Smale 3-diffeomorphism, the characteristic space of orbits constructed for the set $\Sigma$ of saddle points with a one-dimensional unstable manifold is connected. This fact played a key role in obtaining a complete topological classification of such diffeomorphisms, obtained in [1]. According to [13], any Morse-Smale diffeomorphism defined on a manifold of dimension $n > 3$ also has a connected characteristic space of orbits. For orientation-preserving gradient-like (without heteroclinic points) diffeomorphisms on surfaces there is a result in the work [6] that the existence of a connected characteristic orbit space $\hat{V}_\Sigma$ homeomorphic to the two-dimensional torus $T^2$.

The main result of the work is the proof of the fact that the violation of at least one of the conditions (absence of heteroclinic points, orientability of the surface, orientability of the diffeomorphism) leads to the existence of Morse-Smale diffeomorphisms on the surface that do not have a connected characteristic space of orbits. Exactly, the following theorem is proved.

**Theorem 1.**

1. On any orientable surface $M^2$ there exists an orientation-changing gradient-like diffeomorphism that does not have a connected characteristic space of orbits.

2. On any non-orientable surface $M^2$ there exists a gradient-like diffeomorphism that does not have a connected characteristic space of orbits.

3. On any surface $M^2$ there exists a Morse-Smale diffeomorphism with heteroclinic points that does not have a connected characteristic space of orbits.

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2 Required information and facts

Let $M^n$ be a smooth closed orientable manifold and $f$ a diffeomorphism on $M^n$. For a diffeomorphism $f$, a point $x \in X$ is called wandering if there exists an open neighborhood $U_x$ of $x$ such that $f(U_x) \cap U_x = \emptyset$. Otherwise, the point $x$ is called non-wandering. It
is immediate from the definition that any point in the neighborhood $U_x$ of a wandering
point $x$ is wandering itself and therefore the set of wandering points is open while the set
of non-wandering points is closed.

The set of all nonwandering points of the diffeomorphism $f$ is called the non-wandering
set and usually denoted by $\Omega_f$.

The simplest examples of hyperbolic sets are primarily the hyperbolic fixed points of a
diffeomorphism, which can be classified as follows. Let $f : X \to X$ be a diffeomorphism
and $f(p) = p$. A point $p$ is hyperbolic if and only if the absolute value of each eigenvalue of
the Jacobi matrix $(\frac{\partial f}{\partial x})|_p$ is not equal to 1. If the absolute values of all the eigenvalues are
less than 1, then $p$ is called a attracting (a sink point, or sink); if the absolute values of all
the eigenvalues are greater than 1 then $p$ is called a repelling (a source point, or source).
Attracting or repelling points are called a nodes. A hyperbolic fixed point that is not a
node is called a saddle point or saddle.

If the point $p$ is a periodic point $f$ with period $\text{per}(p)$, then applying the previous
construction to the diffeomorphism $f^{\text{per}(p)}$, we obtain a classification of hyperbolic periodic
points similar to the classification of fixed hyperbolic points.

The hyperbolic structure of a periodic point $p$ leads to its stable $W_p^s = \{ x \in M^n : \lim_{k \to +\infty} d(f^{k\text{per}}(p)(x), p) \to 0 \}$ and unstable $W_p^u = \{ x \in M^n : \lim_{k \to +\infty} d(f^{-k\text{per}}(p)(x), p) \to 0 \}$
diversities that are smooth embeddings of $\mathbb{R}^{n-qp}$ and $\mathbb{R}^{qp}$ respectively. Here $qp$ is the number
of eigenvalues of the Jacobian matrix $(\frac{\partial f^{\text{per}(p)}}{\partial x})|_p$ modulo greater than 1.

For a hyperbolic fixed or periodic point $p$, the stable or unstable manifold is called the
invariant manifold of this point, the connected component of the set $W_p^n \setminus p (W_p^n \setminus p)$ is
called unstable (stable) separatrix.

A closed $f$-invariant set $A \subset M^n$ is called an attractor of a discrete dynamical system $f$
if it has a compact neighborhood $U_A$ such that $f(U_A) \subset \text{int} U_A$ and
$A = \bigcap_{k \geq 0} f^k(U_A)$. The neighborhood of $U_A$ is called captivating or isolating. Repeller is defined as an attractor for $f^{-1}$. An attractor and a repeller are called dual if the complement to the exciting
neighborhood of the attractor is the exciting neighborhood of the repeller.

A diffeomorphism $f : M^n \to M^n$ is called a Morse-Smale diffeomorphism if
1) the nonwandering set $\Omega_f$ consists of a finite number of hyperbolic orbits;
2) the manifolds $W_p^s, W_q^u$ intersect transversally for any nonwandering points $p, q$.

A Morse-Smale diffeomorphism is called gradient-like if the condition $W_{\sigma_1}^s \cap W_{\sigma_2}^u \neq \emptyset$
for different points $\sigma_1, \sigma_2 \in \Omega_f$ implies that $\dim W_{\sigma_1}^u < \dim W_{\sigma_2}^u$. In dimension $n = 2$, the
set of gradient-like diffeomorphisms coincides with the set of Morse-Smale diffeomorphisms
whose saddle separatrices do not intersect.

If $M^n$ is an orientable manifold, then the diffeomorphism $f : M^n \to M^n$ is called
orientation-preserving, if $f$ has a positive Jacobian at least at one point, otherwise the
diffeomorphism is called orientation-changing.

Let $f : M^2 \to M^2$ be a gradient-like diffeomorphism defined on a closed surface $M^2$. Let
ω be the sink point of the period \( m_\omega \) of the diffeomorphism \( f \). According to [5, Theorem 5.5], the diffeomorphism \( f^{m_\omega} \) in some neighborhood of the point \( \omega \) is topologically conjugate to the linear diffeomorphism of the plane given by the matrix \( \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \varsigma_\omega \cdot \frac{1}{2} \end{pmatrix} \), where \( \varsigma_\omega = +1 \ ( -1) \) if \( f^{m_\omega}|_{W_s} \) preserves (changes) orientation. We say that the sink \( \omega \) has a positive orientation type if \( \varsigma_\omega = +1 \) and has a negative orientation type otherwise.

Denote by \( O_\omega \) the orbit of the point \( \omega \). Let \( V_\omega = W_{s_\omega} \setminus O_\omega \). Denote by \( \tilde{V}_\omega = V_\omega / f \) the orbit space of the action of the group \( F = \{ f^k, k \in \mathbb{Z} \} \cong \mathbb{Z} \) on \( V_\omega \) and by \( p_\omega : V_\omega \to \tilde{V}_\omega \) the natural projection.

**Proposition 2.1** ([11], Утверждение 1). The manifold \( \tilde{V}_\omega \) is diffeomorphic to a two-dimensional torus if \( \varsigma_\omega = +1 \) and is diffeomorphic to a Klein bottle if \( \varsigma_\omega = -1 \). Moreover, \( \eta_\omega(\pi_1(\tilde{V}_\omega)) = m_\omega \mathbb{Z} \).

Similarly denote the orientation type \( \varsigma_\alpha \) for the periodic source \( \alpha \) of the diffeomorphism \( f \), the space of orbits \( \tilde{V}_\alpha \) and the projection of the stable separatrix of the saddle point into it.

Let \( \sigma \) be a saddle point of the period \( m_\sigma \) of the diffeomorphism \( f \). According to [5, Theorem 5.5], the diffeomorphism \( f^{m_\sigma} \) in some neighborhood of the point \( \sigma \) is topologically conjugate to the linear diffeomorphism of the plane given by the matrix \( \begin{pmatrix} \nu_\sigma \cdot \frac{1}{2} & 0 \\ 0 & \lambda_\sigma \cdot 2 \end{pmatrix} \), where \( \nu_\sigma = +1 \ ( -1) \) if \( f|_{W_s} \) preserves (changes) orientation; \( \lambda_\sigma = +1 \ ( -1) \) if \( f|_{W_u} \) preserves (changes) orientation. A pair \( \varsigma_\sigma = (\nu_\sigma, \lambda_\sigma) \) will be called the orientation type of the saddle point \( \sigma \) and denote by \( a_{\varsigma_\sigma} : \mathbb{R}^2 \to \mathbb{R}^2 \) a corresponding linear diffeomorphism. If \( \nu_\sigma > 0, \lambda_\sigma > 0 \), then the type of orientation will be called positive, and negative otherwise.

Denote by \( O_\sigma \) the orbit of the saddle point \( \sigma \) and set \( N^u_\sigma = N_{O_\sigma} \setminus W_{s_\sigma} \). Then the group \( F \) acts on \( N^u_\sigma \), generating the orbit space \( \hat{N}^u_\sigma = N^u_\sigma / f \) and the natural projection \( p^u_\sigma : N^u_\sigma \to \hat{N}^u_\sigma \) (see Figure 1 for the case \( \varsigma_\sigma = (+1, +1) \)).

**Figure 1**: An orbit space \( \hat{N}^u_\sigma \)

Denote similarly by the orbit space \( \hat{N}^s_\sigma = N^s_\sigma / f \) of the action of the group \( F \) on \( N^s_\sigma = N_{O_\sigma} \setminus W_{u_\sigma} \), the natural projection \( p^s_\sigma : N^s_\sigma \to \hat{N}^s_\sigma \) and mapping \( \eta^s_\sigma \) composed of homomorphisms into the group \( \mathbb{Z} \) from the fundamental group of each connected component of the space \( \hat{N}^s_\sigma \).
In addition, the map
\[ \hat{\psi}_\sigma = p_\sigma^*(p_\sigma^u)^{-1} : \partial \hat{N}_\sigma^u \to \partial \hat{N}_\sigma^s \]
is well defined and it will be called the rearrangement map (see Figure 2 for the case \( \varsigma = (+1, +1) \)).

\[ \varsigma = (+1, +1) \]

Figure 2: Rearrangement map

Denote by \( \Omega_0^f, \Omega_1^f, \Omega_2^f \) the set of sinks, saddles, and sources of the diffeomorphism \( f \).

For any (possibly empty) \( f \)-invariant set \( \Sigma \subset \Omega_1^f \) such that \( \text{cl}(W^u_\Sigma) \setminus W^s_\Sigma \subset \Omega_0^f \), we set
\[ A_\Sigma = \Omega_0^f \cup W^u_\Sigma, \quad R_\Sigma = \Omega_2^f \cup W^s_{\Omega_1^f \setminus \Sigma}. \]

The set
\[ V_\Sigma = M^2 \setminus (A_\Sigma \cup R_\Sigma) \]
is called the characteristic space. The factor space
\[ \hat{V}_\Sigma = V_\Sigma / f \]
is called the characteristic space of orbits. Let \( V_\Sigma^{(1)} = p_\Sigma^{-1}(\hat{V}_\Sigma^{(1)}), \ldots, V_\Sigma^{(k)} = p_\Sigma^{-1}(\hat{V}_\Sigma^{(k)}) \) and denote by \( m_1, \ldots, m_k \) the number of connected components in the sets \( V_\Sigma^{(1)}, \ldots, V_\Sigma^{(k)} \) respectively.

**Proposition 2.2** ([7], Proposition 1). Each connected component of the characteristic orbit space \( \hat{V}_\Sigma \) is homeomorphic either to a two-dimensional torus or to a Klein bottle.

**Proposition 2.3** ([4], Lemma 4.1). Let \( \Sigma' = \Sigma \cup O_\sigma \) for some saddle orbit \( O_\sigma \) and \( \hat{v}, \hat{v}' \) be the disjoint union of the connected components of the spaces \( \hat{V}_\Sigma, \hat{V}_{\Sigma'} \) which have non-empty intersection with \( \hat{N}^u_\sigma, \hat{N}^s_\sigma \), respectively. Then
\[ \hat{V}_{\Sigma'} \cong (\hat{V}_\Sigma \setminus \text{int} \hat{N}^u_\sigma) \cup_{\hat{\psi}_\sigma} \hat{N}^s_\sigma. \]

Wherein
\[ \hat{V}_{\Sigma'} \cong (\hat{V}_\Sigma \setminus \hat{v}) \cup \hat{v}' \quad (1) \]

**Corollary 2.1.** If \( \sigma \) is a saddle point with a positive orientation \( \varsigma = (+1, +1) \), then for the sets \( \hat{v}, \hat{v}' \) in the formula (1) the following features are implemented:
• $\dot{v}$ – a disjoint union of two Klein bottles, $\dot{v}$ – a disjoint union of two Klein bottles (see Figure 3(1));

• $\dot{v}$ – a torus, $\dot{v}'$ – a disjoint union of two tori (see Figure 3(2));

• $\dot{v}$ – a disjoint union of two tori and $\dot{v}'$ – a torus;

• $\dot{v}$ – a disjoint union of a torus and a Klein bottle and $\dot{v}'$ – a Klein bottle;

• $\dot{v}$ – a Klein bottle and $\dot{v}'$ – a disjoint union of a torus and a Klein bottle;

• $\dot{v}$ – a torus and $\dot{v}'$ – a torus (if $M^2$-nonorientable surface).

If $\sigma$ is a saddle point with negative orientation $\zeta_\sigma = (-1, -1)$, then the following possibilities are realized:

• $\dot{v}$ – a Klein bottle and $\dot{v}'$ – a Klein bottle;

• $\dot{v}$ – a torus and $\dot{v}'$ – a torus.

3. Construction of model diffeomorphisms

In this section, we construct several basic diffeomorphisms, the proof of the theorem will be based on them.

3.1 Gradient-like diffeomorphism $\psi_0$ on the sphere $S^2$

Define polar coordinates $(r, \varphi)$ on the plane $\mathbb{R}^2$. Denote by $\varphi(r)$ the function depicted on the graph (see Figure 4) which has the property $\varphi(r) = \varphi(\frac{1}{r})$. Also define a vector field
on the plane $\mathbb{R}^2$ using the following system of differential equations:

\[
\begin{align*}
\dot{r} &= \begin{cases} 
-r(r-1), & 0 \leq r \leq 1; \\
1-r, & r > 1;
\end{cases} \\
\dot{\varphi} &= \varphi(r) \sin 2\varphi.
\end{align*}
\]

Denote by $\chi^t$ the flow induced by this vector field, and denote by $\chi^t$ the diffeomorphism, which is the shift of the flow $\chi^t$ per unit of time. The result is a diffeomorphism that has a hyperbolic source at the origin $O$, hyperbolic saddles at points $A_1, A_3$ and hyperbolic drains at points $A_0, A_2$ (see Figure 5).

Let a diffeomorphism $\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be as follows as $\theta(r, \varphi) = (r, -\varphi)$. We define the diffeomorphism $\bar{\psi}_0 : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

\[
\bar{\psi}_0 = \theta \circ \chi.
\]

By the construction, the nonwandering set of the diffeomorphism $\bar{\psi}_0$ coincides with a nonwandering diffeomorphism set $\chi$. Consider the standard two-dimensional sphere

\[
S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}.
\]
Denote by $N(0, 0, 1)$ north pole and define a stereographic projection (see Figure 6) $\vartheta : S^2 \setminus \{N\} \to \mathbb{R}^2$ formula

$$\vartheta(x_1, x_2, x_3) = \left(\frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3}\right).$$

Define a diffeomorphism $\psi_0 : S^2 \to S^2$ by the formula

$$\psi_0(x) = \begin{cases} \vartheta^{-1} \circ \bar{\psi} \circ \vartheta(x), & x \in S^2 \setminus \{N\}, \\ N, & x = N. \end{cases}$$

By construction, $\psi_0$ is an orientation-changing gradient-like 2-sphere diffeomorphism whose nonwandering set consists of two fixed sources $\alpha_1 = N, \alpha_2 = \vartheta^{-1}(O)$ of negative orientation ($\varsigma_{\alpha_1} = \varsigma_{\alpha_2} = -1$); two fixed sinks $\omega_0 = \vartheta^{-1}(A_0), \omega_1 = \vartheta^{-1}(A_2)$ of negative orientation ($\varsigma_{\omega_0} = \varsigma_{\omega_1} = -1$) and saddle orbit $O_\sigma = \{\sigma = \vartheta^{-1}(A_1), \psi_0(\sigma) = \vartheta^{-1}(A_3)\}$ of period 2 with orientation type $\varsigma_\sigma = (+1, +1)$ (see Figure 7):

$$\Omega_{\psi_0} = \{\alpha_1, \alpha_2, \omega_0, \omega_1, \sigma, \psi_0(\sigma)\}.$$

### 3.2 Gradient-like diffeomorphism $\tilde{\psi}_1$ on the projective plane $\mathbb{R}P^2$

Consider the diffeomorphism $\psi_0 : S^2 \to S^2$ defined in 3.1 and the group $\mathbb{Z}_2 = \{+1, -1\}$ acting on the two-dimensional sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ as follows:

$$\pm 1 \cdot x = \pm x, \quad x = (x_1, x_2, x_3) \in S^2.$$

Then the orbit space $S^2/\mathbb{Z}_2$ of the action of the group $\mathbb{Z}_2$ on $S^2$ is the projective plane $\mathbb{R}P^2$. Let $p : S^2 \to \mathbb{R}P^2$ be the natural projection. Let the diffeomorphism $\tilde{\psi}_1 : \mathbb{R}P^2 \to \mathbb{R}P^2$ be defined by the formula

$$\tilde{\psi}_1(x) = p \circ \psi_0 \circ p^{-1}(x), \quad x \in \mathbb{R}P^2.$$
By construction, the nonwandering set of the constructed diffeomorphism consists of three fixed points: the source $\tilde{\alpha}$ of negative orientation ($\varsigma_{\tilde{\alpha}} = -1$), the sink $\tilde{\omega}$ of negative orientation ($\varsigma_{\tilde{\omega}} = -1$) and saddle $\tilde{\sigma}_1$ with orientation type $\varsigma_{\tilde{\sigma}_1} = (-1, -1)$ (see Figure 8):

$$\Omega_{\tilde{\psi}_1} = \{\tilde{\alpha}, \tilde{\omega}, \tilde{\sigma}_1\}.$$ 

3.3 Gradient-like diffeomorphism $\tilde{\psi}_q$ on a non-orientable surface of genus $q$

Let $S^{-}_q = \mathbb{S}^{2g}_q \# \mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$. Construct a model diffeomorphism $\tilde{\psi}_q : S^{-}_q \to S^{-}_q$. The source $\tilde{\alpha}$ and the sink $\tilde{\omega}$ of the diffeomorphism $\tilde{\psi}_1$ should be considered. Whereas they are hyperbolic, there are non-intersecting 2-discs around them $B_{\tilde{\omega}}, B_{\tilde{\alpha}}$, что $\tilde{\psi}_1(B_{\tilde{\omega}}) \subset int B_{\tilde{\omega}}, \tilde{\psi}_1^{-1}(B_{\tilde{\alpha}}) \subset int B_{\tilde{\alpha}}$. Then the connected sum of two copies of projective planes along the disks $B_{\tilde{\omega}}, B_{\tilde{\alpha}}$ is a non-orientable surface $S^{-}_2$ of genus 2 (see Figure 9). Since the
dynamics in the disk $B_\omega$ is inverse to the dynamics in the disk $B_\alpha$, a diffeomorphism $\tilde{\psi}_2$ is well defined on the surface $S_2^-$, which coincides with $\tilde{\psi}_1$ to $\mathbb{RP}^2 \setminus B_\alpha$ and to $\mathbb{RP}^2 \setminus B_\omega$. We say that the diffeomorphism $\tilde{\psi}_2$ is the connected sum of two copies of the diffeomorphism $\tilde{\psi}_1$ ($\tilde{\psi}_2 = \tilde{\psi}_1 \sharp \tilde{\psi}_1$) along the sink $\tilde{\omega}$ and source $\tilde{\alpha}$.

![Figure 9: Diffeomorphism $\tilde{\psi}_2$](image)

By induction, a diffeomorphism $\tilde{\psi}_q : S_q^- \to S_q^-$ on a nonorientable surface of genus $q \geq 2$ is constructed as a connected sum of diffeomorphisms $\tilde{\psi}_{q-1}$ and $\tilde{\psi}_1$ ($\tilde{\psi}_q = \tilde{\psi}_{q-1} \sharp \tilde{\psi}_1$) along the $\tilde{\omega}$ sink and $\tilde{\alpha}$ source. By construction, the nonwandering set of the diffeomorphism $\tilde{\psi}_q$ consists of $q + 2$ fixed points: the source $\tilde{\alpha}$ of negative orientation ($\varsigma_{\tilde{\alpha}} = -1$), the sink $\tilde{\omega}$ negative orientation ($\varsigma_{\tilde{\omega}} = -1$) and $q$ saddles $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_q$ with orientation type $\varsigma_{\tilde{\sigma}_i} = (+1, +1)$ (see Figure 9 for $q = 2$):

$$\Omega_{\tilde{\psi}_q} = \{ \tilde{\alpha}, \tilde{\omega}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_q \}.$$

### 3.4 Gradient-like diffeomorphism $\psi_1$ on a torus $\mathbb{T}^2$

We construct a diffeomorphism $\psi_1$ on the two-dimensional torus $\mathbb{T}^2$ as a cartesian product of two orientation-preserving source-sink diffeomorphisms on the circle $\mathbb{S}^1$. For this we should consider the function $\bar{F} : \mathbb{R} \to \mathbb{R}$ given by the formula:

$$\bar{F}(x) = x + \frac{1}{6\pi} \sin 2\pi x$$

(see Figure 10).

Consider the projection $\pi : \mathbb{R} \to \mathbb{S}^1$ given by the formula $\pi(x) = e^{2\pi ix}$. Since the function $\bar{F}$ is strictly monotonically increasing and satisfies the condition $\bar{F}(x + 1) = \bar{F}(x) + 1$, it admits a projection onto a circle in diffeomorphism $F : \mathbb{S}^1 \to \mathbb{S}^1$ given by

$$F(z) = \pi \bar{F} \pi^{-1}(z), \; z \in \mathbb{S}^1.$$

By construction, the diffeomorphism $F$ has a fixed hyperbolic sink and source and is an orientation-preserving source-sink diffeomorphism. Define a diffeomorphism $F_1 : \mathbb{T}^2 \to \mathbb{T}^2$.
by the formula

\[ F_1(z, w) = (F(z), F(w)), \quad z, w \in S^1. \]

Then the diffeomorphism \( F_1 \) is orientation-preserving, and its nonwandering set consists of four fixed points: a source \( \alpha \) of positive orientation \((\varsigma_{\alpha} = +1)\), a sink \( \omega \) of positive orientation \((\varsigma_{\omega} = +1)\) and two saddles \( \sigma_1, \sigma_2 \) of positive orientation type (see Figure 11):

\[ \Omega_F = \{ \alpha, \omega, \sigma_1, \sigma_2 \}. \]

Let us represent the two-dimensional torus \( \mathbb{T}^2 \) as the factor group of the group \( \mathbb{R}^2 \) with respect to the integer lattice \( \mathbb{Z}^2 : \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2. \) Consider the matrix \( A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{Z}) \) and the algebraic torus automorphism \( \hat{A} : \mathbb{T}^2 \to \mathbb{T}^2, \)

\[ \hat{A}(x, y) = (y, x) \quad (\text{mod } 1) \]
Let
\[ \psi_1 = \hat{A} \circ F_1 : \mathbb{T}^2 \to \mathbb{T}^2. \]

By construction, the diffeomorphism \( \psi_1 \) is an orientation-changing gradient-like diffeomorphism whose nonwandering set consists of a source \( \alpha \) and a sink \( \omega \) of negative orientation types \( \varsigma_\alpha = \varsigma_\omega = -1 \), as well as periodic saddle orbit \( \mathcal{O}_{\sigma_1} = \{ \sigma_1, \psi_1(\sigma_1) \} \) of period 2 and orientation type \( \varsigma_{\sigma_1} = (+1, +1) \) (see Figure 12):\n
\[ \Omega_{\psi_1} = \{ \alpha, \omega, \sigma_1, \psi_1(\sigma_1) \}. \]

\[ \text{Figure 12: Diffeomorphism } \psi_1 \]

### 3.5 Gradient-like diffeomorphism \( \psi_g \) on an orientable surface of genus \( g \)

Let \( S^+_g = \mathbb{S}^2_+ \mathbb{T}^2 \mathbb{S}^2_+ \ldots \mathbb{T}^2 \). Let us construct a model diffeomorphism \( \psi_g : S^+_g \to S^+_g \). To do this, firstly, we should construct a diffeomorphism \( \psi_2 : S^+_2 \to S^+_2 \) as a connected sum of two copies of the diffeomorphism \( \psi_1 \) (\( \psi_2 = \psi_1 \hat{A} \psi_1 \)) along the sink \( \omega \) and the source \( \alpha \) (see Fig. 13). Then, by induction, we define the diffeomorphism \( \psi_g : S^+_g \to S^+_g \) as a connected sum of diffeomorphisms \( \psi_{g-1} \) and \( \psi_1 \) (\( \psi_g = \psi_{g-1} \hat{A} \psi_1 \)) along the sink \( \omega \) and the source \( \alpha \). By construction, the nonwandering set of the diffeomorphism \( \psi_g \) consists of a fixed source \( \alpha \) and a fixed sink \( \omega \) of negative orientation types \( \varsigma_\alpha = \varsigma_\omega = -1 \), and \( g \) saddle periodic orbits \( \mathcal{O}_{\sigma_1} = \{ \sigma_1, \psi_1(\sigma_1) \}, \ldots, \mathcal{O}_{\sigma_g} = \{ \sigma_g, \psi_1(\sigma_g) \} \) of period 2 and orientation type \( \varsigma_{\sigma_1} = (+1, +1) \) (see Figure 13 for \( g = 2 \)):\n
\[ \Omega_{\psi_g} = \{ \alpha, \omega, \sigma_1, \psi_g(\sigma_1), \ldots, \sigma_g, \psi_g(\sigma_g) \}. \]
3.6 Non-gradient-like Morse-Smale diffeomorphism $\xi_0$ on the sphere $S^2$

Consider the diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ given in polar coordinates $(r, \varphi), \varphi \in \left( -\frac{\pi}{2}, \frac{3\pi}{2} \right]$ by the formula $h(r, \varphi) = \left( r^2, \varphi \right)$. Let $A_- = \{(r, \varphi) \in \mathbb{R}^2 \setminus \{0\} : \frac{3\pi}{4} \leq \varphi \leq \frac{5\pi}{4}\}, A_+ = \{(r, \varphi) \in \mathbb{R}^2 \setminus \{0\} : |\varphi| \leq \frac{\pi}{4}\}$. Let $C = \mathbb{R} \times [-2, 2]$ and define the diffeomorphisms $\eta_- : A_- \to C, \eta_+ : A_+ \to C$ by the formulas

$$\eta_-(r, \varphi) = \left( 3 - \log_2 r, 8 \left( \frac{\varphi}{\pi} - 1 \right) \right), \quad \eta_+(r, \varphi) = \left( -3 - \log_2 r, \frac{8\varphi}{\pi} \right).$$

It is directly verified that the diffeomorphism $\eta_-(\eta_+) \conjugates h$ with the diffeomorphism $g : C \to C$ defined by the formula $g(x, d) = (x + 1, d)$ and $\eta_-(2, \pi) = (2, 0) \quad \eta_+(1/2, 0) = (-2, 0))$. It is obvious that the diffeomorphism $g$ is included in the flow $g^t : C \to C$ defined by the formula

$$g^t(x, d) = (x + t, d).$$

We define the flow $\phi^t_-$ on $C$ using the formulas

$$\dot{x} = \begin{cases} 1 - \frac{1}{5}(x^2 + d^2 - 4)^2, & x^2 + d^2 \leq 4 \\ 1, & \text{else} \end{cases}$$

$$\dot{d} = \begin{cases} \frac{d}{2} \left( \sin \left( \frac{\pi}{2} (x^2 + d^2 - 3) \right) - 1 \right), & 2 < x^2 + d^2 \leq 4 \\ -d, & x^2 + d^2 \leq 2 \\ 0, & \text{else} \end{cases}$$

By construction, the flow $\phi^t_-$ coincides with the flow $g^t$ for $|x| \geq 2$. Moreover, the diffeomorphism $\phi_- = \phi^1_-$ has exactly two fixed points: the saddle $P_-(1, 0)$ and the sink $Q_-(-1, 0)$ (see Figure 14), besides both points are hyperbolic. One unstable separatrix of the $P_-$ saddle is an open interval $(-1, 1) \times \{0\}$ belonging to the $Q_-$ sink basin, and the other is a ray $(1, +\infty) \times \{0\}.$
Define the flow $\phi_t^\pm$ on $C$ by the formulas

\[
\begin{align*}
\dot{x} &= \begin{cases} 
1 - \frac{1}{5}(x^2 + d^2 - 4)^2, & x^2 + d^2 \leq 4 \\
1, & \text{else}
\end{cases} \\
\dot{d} &= \begin{cases} 
-d^2(\sin(\frac{\pi}{2}(x^2 + d^2 - 3)) - 1), & 2 < x^2 + d^2 \leq 4 \\
d, & x^2 + d^2 \leq 2 \\
0, & \text{else}
\end{cases}
\end{align*}
\]

By construction, the flow $\phi_t^\pm$ coincides with the flow $g_t$ for $|x| \geq 2$. In this case, the
diffeomorphism $\phi_+ = \phi_1^\pm$ has exactly two fixed points: the saddle $P_+(-1,0)$ and the source $Q_+(1,0)$ (see Figure 15), similarly, both points are hyperbolic. One stable separatrix of the saddle $P_+$ is the open interval $(-1,1) \times \{0\}$ belonging to the source basin $Q_+$, and the other
is the ray \((-\infty, -1) \times \{0\}\).

Define diffeomorphism \(\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2\) so that \(\tilde{f}\) coincides \(h\) outside \(A_+ \cup A_-\) and coincides \(\eta_{-1} \phi \eta_-\) and \(\eta_{+1} \phi \eta_+\) to \(A_-\) and \(A_+\) respectively.

Consider on \(\mathbb{R}^2\) the annulus \(K = \{(x_1, x_2) \in \mathbb{R}^2 : 1 \leq x_1^2 + x_2^2 \leq 4\}\). Define the function \(\nu : [1, 2] \to [1, 2]\) (see Figure 17) by the formula:

\[
\nu(t) = \begin{cases} 
1, & t = 1, \\
1 + \frac{1}{1 + \exp\left(\frac{1}{2-1}\right)} \frac{1}{(t-1)^2(t-2)^2}, & 1 < t < 2, \\
2, & t = 2. 
\end{cases}
\]

On the annulus \(K\) we can define the Dehn twist \(\bar{d} : K \to K\) formula

\[
\bar{d}(t, e^{i\phi}) = \left(t, e^{i(\phi + 2\pi\nu(t))}\right).
\]

Let \(\bar{\xi}_0 = \bar{d} \circ \tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2\) (see Figure 17).

By construction, the diffeomorphism \(\bar{\xi}_0\) coincides with \(h\) in some neighborhood of the point \(O\) and the point at infinity, therefore, it induces on \(S^2\) a Morse-Smale diffeomorphism \(\xi_0 : S^2 \to S^2\) by the formula

\[
\xi_0(x) = \begin{cases} 
\vartheta^{-1} \circ \bar{\xi}_0 \circ \vartheta(x), & x \in S^2 \setminus \{N\}, \\
N, & x = N. 
\end{cases}
\]

It follows directly from the construction that the nonwandering set of the diffeomorphism \(\xi_0\) consists of six fixed points of positive orientation: two sources \(\alpha_1 = N, \alpha_2 = \xi_0(\vartheta^{-1}(Q_+))\), two sinks \(\omega_0 = \xi_0(\vartheta^{-1}(Q_-)), \omega = S\) and two saddles \(\sigma = \xi_0(\vartheta^{-1}(P_-)), \sigma_0 = \xi_0(\vartheta^{-1}(P_+))\).
Figure 17: Diffeomorphism $\tilde{\xi}_0$

(see Figure 18):

$$\Omega_{\xi_0} = \{\alpha_1, \alpha_2, \omega_0, \omega, \sigma_0, \sigma\}.$$

Figure 18: Phase portrait of a diffeomorphism $\xi_0$

4 Proof of the main result

In this section, we will prove the theorem 1, each item in a separate lemma below.

**Lemma 4.1.** On any orientable surface $M^2$ there exists an orientation-changing gradient-like diffeomorphism that does not have a connected characteristic space of orbits.

**Proof.** To prove the lemma, consider a diffeomorphism $f_g : S_g^+ \rightarrow S_g^+$ such that $f_0 = \psi_0$ and $f_g (g > 0)$ is the connected sum of the diffeomorphism $\psi_0$ with the diffeomorphism $\psi_g$ along sink $\omega_0$ and source $\alpha$ respectively. By construction, the diffeomorphism $f_g$ changes orientation, and its nonwandering set consists of two fixed sources $\alpha_1, \alpha_2$, two fixed sinks $\omega, \omega_1$ of negative orientation ($\varsigma_{\alpha_1} = \varsigma_{\alpha_2} = \varsigma_\omega = \varsigma_{\omega_1} = -1$) and $g + 1$ saddle periodic orbits.
\[ O_{\sigma} = \{ \sigma, f_g(\sigma) \}, O_{\sigma_1} = \{ \sigma_1, f_g(\sigma_1) \}, \ldots, O_{\sigma_g} = \{ \sigma_g, f_g(\sigma_g) \} \]
of period 2 and orientation type \((+1, +1)\) (see Figure 13):

\[ \Omega_{f_g} = \{ \alpha, \omega, \sigma, f_g(\sigma_1), \sigma_1, f_g(\sigma_1), \ldots, \sigma_g, f_g(\sigma_g) \}. \]

Let us show that the diffeomorphism \(f_g\) does not have a connected characteristic space of orbits.

Indeed, by the proposition 2.1, each of the orbit spaces \(\hat{V}_\omega, \hat{V}_{\sigma_1}\) is homeomorphic to a Klein bottle. Therefore, if \(\Sigma = \emptyset\), then the characteristic orbit space \(\hat{V}_\Sigma\) is not connected and consists of two Klein bottles. Since all saddle points of the diffeomorphism \(f_g\) have a positive orientation type, then, according to the corollary 2.1, adding the orbits of such saddles to the set \(\Sigma\) does not decrease the number of connected components of the characteristic space of orbits.

\[ \text{Lemma 4.2.} \quad \text{On any non-orientable surface } M^2 \text{ there exists a gradient-like diffeomorphism that does not have a connected characteristic space of orbits.} \]

\[ \text{Proof.} \quad \text{Define a diffeomorphism } \tilde{f}_q : S_q^- \to S_q^-, q \in \mathbb{N} \text{ as a connected sum of diffeomorphisms } \psi_0 \text{ and } \tilde{\psi}_q \ (\tilde{f}_q = \psi_0 \# \tilde{\psi}_q) \text{ along the sink } \omega_0 \text{ of the diffeomorphism } \psi_0 \text{ and the source } \tilde{\alpha} \text{ of the diffeomorphism } \tilde{\psi}_q \text{ (see Fig. 20 for } q = 1). \text{ By construction, the nonwandering set } \Omega_{f_q} \text{ consists of two sources } \alpha_1, \alpha_2 \text{ and two sinks } \omega_1, \tilde{\omega} \text{ of negative orientation types } (\varsigma_{\omega_1} = \varsigma_{\tilde{\omega}} = \varsigma_{\alpha_1} = \varsigma_{\alpha_2} = -1), \text{ also of } q \text{ fixed saddles } \tilde{\sigma}_1, \ldots, \tilde{\sigma}_q \text{ of orientation type } \varsigma_{\tilde{\sigma}_i} = (-1, -1) \text{ and a saddle orbit } O_{\sigma} = \{ \sigma, \tilde{f}_q(\sigma) \} \text{ of period } 2 \text{ with a positive orientation type (see Figure 20 for } q = 1): \]

\[ \Omega_{f_q} = \{ \alpha_1, \alpha_2, \omega_1, \tilde{\omega}, \sigma, \tilde{f}_q(\sigma_1), \tilde{\sigma}_1, \ldots, \tilde{\sigma}_q \}. \]
Let us show that the diffeomorphism $\tilde{f}_q$ does not have a connected characteristic space of orbits.

By the construction and proposition 2.1, each of the orbit spaces $\hat{V}_\omega, \hat{V}_{\omega_1}$ is homeomorphic to a Klein bottle. If $\Sigma = \emptyset$, then the characteristic orbit space $\hat{V}_\Sigma$ is not connected and consists of two Klein bottles. Since all saddle points of the diffeomorphism $\tilde{f}_q$ have orientation type either $(+1, +1)$ or $(-1, -1)$, then, according to the corollary 2.1, adding the orbits of such saddles to the set $\Sigma$ does not decrease the number of connected components of the characteristic space of orbits.

**Lemma 4.3.** On any surface $M^2$ there exists a Morse-Smale diffeomorphism with heteroclinic points that does not have a connected characteristic space of orbits.

**Proof.** We construct a diffeomorphism $\xi_g : S^+_g \to S^+_g$ as a connected sum of diffeomorphisms $\xi_0$ and $\psi^2_g (\xi_g = \xi_0^g \# \psi^2_g)$ along the sink $\omega_0$ of the diffeomorphism $\xi_0$ and the source $\alpha$ of the diffeomorphism $\psi^2_g$. By construction, the diffeomorphism $\xi_g$ preserves orientation, its nonwandering set consists of points of positive orientation type: two sinks $\omega, \omega_1$ and two sources $\alpha_1, \alpha_2$ and $2 + 2g$ saddles $\sigma_0, \sigma, \sigma_1, \ldots, \sigma_{2g}$ (a special case for $g = 1$ is shown in Figure 21).

Let us show that the diffeomorphism $\xi_g$ does not have a connected characteristic space of orbits.

Let the set $\Sigma = \emptyset$, then the characteristic orbit space consists of two tori $\hat{V}_{\omega_1}$ and $\hat{V}_\omega$. Since the unstable saddle point manifolds $\sigma_1, \ldots, \sigma_{2g}$ contain the only one sink $\omega$ in their closures, then adding these saddles to the set $\Sigma$ does not reduce the number of components of the characteristic space of orbits. The unstable manifold of the saddle $\sigma_0$ contains the only one sink $\omega_1$ in its closure, then adding this saddle to the set $\Sigma$ increases the number of connected components of the characteristic space to three. The saddle $\sigma$ is above the saddle $\sigma_0$ by the Smale order, so the saddle $\sigma$ can be added to the set $\Sigma$ only together with the saddle $\sigma_0$. Whence it follows that for any set $\Sigma$ the number of connected components of the characteristic space is greater than one.

Alos, let us construct a diffeomorphism $\tilde{\xi}_q : S^-_q \to S^-_q$ as a connected sum of diffeomorphisms $\xi_0$ and $\tilde{\psi}^2_q (\zeta_q = \xi_0^q \# \tilde{\psi}^2_q)$ along the sink $\omega_0$ of the diffeomorphism $\xi_0$ and the
source \( \tilde{\alpha} \) of the diffeomorphism \( \tilde{\psi}_q^2 \) (a special case for \( q = 1 \) is shown in Figure 22). The nonwandering set of the diffeomorphism \( \zeta_q \) consists of points of positive orientation type: two sources \( \alpha_1, \alpha_2 \), two sinks \( \omega_1, \tilde{\omega} \) and \( q + 2 \) saddles \( \sigma, \sigma_0, \tilde{\sigma}_1, \ldots \tilde{\sigma}_q \).

Similar Arguments to the case of orientable surfaces prove that the diffeomorphism \( \tilde{\xi}_q \) does not have a connected characteristic space of orbits.

\[ \square \]

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