A universal result on central charges in the presence of double-trace deformations

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Abstract

We study large $N$ conformal field theories perturbed by relevant double-trace deformations. Using the auxiliary field trick, or Hubbard-Stratonovich transformation, we show that in the infrared the theory flows to another CFT. The generating functionals of planar correlators in the ultraviolet and infrared CFT’s are shown to be related by a Legendre transform. Our main result is a universal expression for the difference of the scale anomalies between the ultraviolet and infrared fixed points, which is of order 1 in the large $N$ expansion. Our computations are entirely field theoretic, and the results are shown to agree with predictions from AdS/CFT. We also remark that a certain two-point function can be computed for all energy scales on both sides of the duality, with full agreement between the two and no scheme dependence.

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1 Introduction

The AdS/CFT correspondence is a duality between string or M-theory backgrounds of the form $AdS_{d+1} \times X$ and conformal field theories in $d$ dimensions [1, 2, 3]. There exist many checks of the duality based on studying explicit models; the most extensively studied one is the $\mathcal{N} = 4$ SYM theory dual to type IIB string theory in $AdS_5 \times S^5$ (for reviews, see [4, 5, 6]). One may be bold enough to argue that, whenever there exists an $AdS_{d+1} \times X$ solution of string or M-theory, then it serves as a constructive definition of a $d$-dimensional CFT, even if its conventional field theoretic formulation is lacking. The universal part of this construction is the $AdS_{d+1}$ space, while the “details” of the CFT, such as the number of supersymmetries, the global symmetries, etc., are encoded in the compact space $X$. It is particularly interesting to look for model-independent checks of the AdS/CFT duality which do not depend on $X$ explicitly. Examples of such checks include studies of finite temperature theories via black holes in AdS [7, 8], calculations of quark–antiquark potential [9, 10], and, quite recently, a study of dimensions of high-spin operators [11].

In this paper we present a new general check of the duality which concerns the change of the central charge under a flow produced by a double-trace deformation. Our check applies to all models which contain scalar fields in $AdS_{d+1}$ with

$$-\frac{d^2}{4} < m^2 L^2 \leq -\frac{d^2}{4} + 1,$$

so that both $\Delta_+$ and $\Delta_-$ dimensions are admissible for the dual operator $O$ [12] ($\Delta_{\pm}$ are the two roots of $\Delta(\Delta - d) = (mL)^2$, where $L$ is the radius of $AdS_{d+1}$). Multi-trace deformations were first studied in the context of AdS/CFT duality in [13]. The modification of the AdS boundary conditions by such operators was presented in [14, 15] and further elaborated in [16, 17, 18]. The boundary conditions were used in [19] to argue that, whenever a relevant case

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1 This might even be argued for an $AdS_{d+1}$ background of some as-yet-unknown theory of quantum gravity with a semi-classical limit in which Einstein gravity is recovered.
double-trace deformation $O^2$ is added to the action, then the theory flows from a UV fixed point where $O$ has dimension $\Delta_- < d/2$ to an IR fixed point where $O$ has dimension $\Delta_+ = d - \Delta_-$. The difference in the Weyl anomaly produced by this flow in even dimensions $d$ was calculated in [22] using AdS methods; the result is a remarkably simple universal formula which depends only on $\Delta_-$ and $d$. In this paper we rederive this formula using field theoretic methods. Our derivation serves as a new interesting check of the duality, and it also sheds light on the origin of the universality of the result of [22].

Indeed, the field theory computation doesn’t even depend on the existence of an anti-de Sitter dual—nor on supersymmetry, nor on gauge symmetry. It is a general result, for any even dimension $d$, for any UV dimension $\Delta_- \in (d/2 - 1, d/2)$ of $O$, and depending only on having some form of large $N$ expansion. Thus it has some interest in its own right, apart from its value as a check of AdS/CFT.

From the AdS point of view, the infrared CFT differs from the ultraviolet CFT only in one respect: the scalar field dual to $O$ is quantized with the conventional $\Delta_+$ boundary condition in the IR, but with the unconventional $\Delta_-$ boundary condition in the UV. This 2-fold ambiguity in quantization of scalar fields in $AdS_{d+1}$ with $m^2$ in the range $[0]$ was originally found in [23], while its relevance for the AdS/CFT correspondence was elucidated in [12]. In particular, [12] presented a prescription for calculating correlation functions of operators with dimension $\Delta_-$. At the leading order in $N$, this prescription asserts that the generating functional of correlation functions in the theory where $O$ has dimension $\Delta_-$ is related by a Legendre transform to the corresponding object in the theory where $O$ has dimension $\Delta_+$. In this paper we shed new light on this prescription by combining it with the proposal of [14]. The combined proposal then states that, in the large $N$ limit, the ultraviolet CFT is related to the infrared CFT produced by a relevant $O^2$ operator through a Legendre transform. We are able to derive this result in the field theory, using the Hubbard-Stratonovich auxiliary field. As a further step, we compute the leading correction in the $1/N$ expansion, which follows from the 1-loop diagram for the auxiliary field and scales as $N^0$. The anomalous part of this determinant exactly reproduces the AdS result of [22], in confirmation of the AdS/CFT duality. For this approach to work it is not necessary for the ultraviolet CFT to be supersymmetric; even if it is, the relevant double-trace interaction breaks the supersymmetry.

We also remark that our treatment of the double-trace operators parallels a similar treatment given in [24] for $c \leq 1$ matrix models of 2-d quantum gravity deformed by operators $O^2$. In particular, the integral over the Hubbard-Stratonovich auxiliary variable was used in [24] to compare the $O(N^0)$ (torus) corrections to the free energy in theories with two different scaling dimensions of operator $O$.

\footnote{This type of flow is well-known in $O(N)$ models in $2 < d < 4$ with fields $\phi^a$, $a = 1, \ldots, N$ transforming in the fundamental representation [13]. An AdS dual of the flow produced by “double-trace” operator $(\phi^a \phi^a)^2$ was recently proposed in terms of a higher-spin gauge theory in [20]. A $d = 4$ model where this flow may take place is the $SU(N) \times SU(N)$ superconformal gauge theory of [21] which contains relevant double-trace operators.}
2 Outline of the field theory calculation

Consider a conformal field theory with a gauge-singlet single-trace scalar operator $O$ whose dimension $\Delta$ falls in the range $(d/2 - 1, d/2)$. The lower limit of this range is the dimension of a free scalar field, and it can be shown that a lower dimension for $O$ would be inconsistent with unitarity. Let us also assume that in the undeformed conformal field theory, $\langle O(x)O(0) \rangle = 1/x^{2\Delta}$ on $\mathbb{R}^d$, and that higher point functions of $O$ are suppressed by some sort of $1/N$ factors, where we may take $N$ large. Then there is a general argument that, in the large $N$ limit, the renormalization group flow triggered by the relevant deformation $f^2 O^2$ terminates at an infrared fixed point where $O$ is again a scaling operator, but of dimension $d - \Delta$. The argument proceeds via a Hubbard-Stratonovich transformation, as follows. Consider the partition function

$$ Z_f[J] = \int D\phi e^{-\frac{1}{2} \int \phi^2 + J \phi} = \langle e^{-\frac{1}{2} \int \phi^2 + J \phi} \rangle_0 $$

where $f$ is a constant but $J$ may not be. The notation $\int D\phi$ indicates path integration over all the degrees of freedom of the conformal field theory, and $S[\phi]$ is the undeformed action. The notation $\langle \cdots \rangle_0$ indicates an expectation value in the undeformed conformal field theory (that is, with $J = f = 0$). The Hubbard-Stratonovich transformation amounts to introducing an auxiliary field $\sigma$:

$$ Z_f[J] = \sqrt{\text{det} \left( -\frac{1}{f} I \right)} \int D\sigma e^{\frac{1}{2} \int \sigma^2 + (f + J) \sigma} \langle e^{\int (f + J) \sigma} \rangle_0. $$

The assumption that higher point functions of $O$ are suppressed now enters in a crucial way:

$$ \langle e^{\int (\sigma + J) \sigma} \rangle_0 \approx e^{\frac{1}{2} \langle (f + J) \sigma \rangle^2}, $$

up to some $1/N$ corrections. The remaining $\sigma$ integral required for computing $Z_f[J]$ is now strictly Gaussian. If we now define three linear operators by the relations

$$ (\hat{G} \sigma)(x) = \int d^d \xi \sqrt{g} \langle O(x)O(\xi) \rangle_0 \sigma(\xi) $$

$$ \hat{K} = 1 + f \hat{G} \quad \hat{Q} = -\frac{1}{f} \left( \hat{K}^{-1} - 1 \right) = \frac{\hat{G}}{1 + f \hat{G}}, $$

then it is straightforward to show that

$$ Z_f[J] = \frac{1}{\sqrt{\text{det} \hat{K}}} e^{\frac{1}{2} \langle J, \hat{Q} J \rangle}. $$

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3The determinant in (3) is a formal expression, defined so that $\sqrt{\text{det} \left( -\frac{1}{f} I \right)} \int D\sigma e^{\frac{1}{2} \sigma^2} = 1$. The contour for $\sigma(x)$ should be rotated to run along the imaginary axis to ensure convergence. In Lorentzian signature, the $\sigma$ field may be introduced in such a way that the action remains real throughout, up to the usual $i\epsilon$ terms.
Thus in particular, the two-point function for $\mathcal{O}$ in the presence of the deformation is

$$
\langle \mathcal{O}(x)\mathcal{O}(0) \rangle_f = \frac{\partial^2 \log Z_f[J]}{\partial J(x)\partial J(0)} = Q(x, 0),
$$

(7)

where $Q(x, 0)$ is the position space representation of the operator $\hat{Q}$. In the dual AdS treatment the same formula for the two-point function was obtained in [16].

The three linear operators, $\hat{G}$, $\hat{K}$, and $\hat{Q}$, are diagonal in a momentum space basis for functions on $\mathbb{R}^d$. Explicitly,

$$
G(k) = \int d^d x \frac{e^{ik\cdot x}}{x^{2\Delta}} = 2^{d-2\Delta} \pi^{d/2} \frac{\Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma(\Delta)} k^{2\Delta-d}
$$

$$
K(k) = 1 + fG(k) \quad Q(k) = \frac{G(k)}{1 + fG(k)}.
$$

(8)

Expanding $Q(k)$ for small wave-numbers, we find

$$
Q(k) = \frac{1}{f} - \frac{1}{f^2 G(k)} + \frac{1}{f^3 G(k)^2} - \ldots \quad \text{for } fG(k) \gg 1
$$

$$
\langle \mathcal{O}(x)\mathcal{O}(0) \rangle_f \approx -\frac{1}{f^2 \pi^d} \Gamma(\Delta)\Gamma(d-\Delta) \frac{\Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma(\Delta - \frac{d}{2})} \frac{1}{x^{2(d-\Delta)}} \quad \text{for } x \gg f^{-\frac{1}{d-2\Delta}}.
$$

(9)

The position space expression shown comes from the $-1/f^2G(k)$ term, which is the leading non-analytic behavior of $Q(k)$ in the small $k$ limit. Using the relation $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$ and the constraint $\Delta \in (d/2 - 1, d/2)$, it is easy to show that the position space expression for $\langle \mathcal{O}(x)\mathcal{O}(0) \rangle_f$ is positive.

The power law behavior for $\langle \mathcal{O}\mathcal{O} \rangle_f$ in the infrared is *prima facie* evidence for an infrared fixed point. Furthermore, we find that the dimension of operator $\mathcal{O}$ has changed from $\Delta$ in the UV to $d-\Delta$ in the IR, in agreement with the reasoning presented in [14] on the AdS side of the duality. In fact, $\langle \mathcal{O}(x)\mathcal{O}(0) \rangle_f$ in the infrared limit and the original two-point function $\langle \mathcal{O}(x)\mathcal{O}(0) \rangle_0$ are related by the Legendre transform prescription proposed in [12].

More generally, introduction of the Hubbard-Stratonovich auxiliary field explains why the generating functionals of planar correlations functions in the infrared CFT, corresponding to operator $\mathcal{O}$ having dimension $\Delta_+$, and in the UV CFT, corresponding to dimension $\Delta_-$, are related by the Legendre transform. To demonstrate this, define $W[\sigma, h_i]$ to be the generating functional of correlators in the ultraviolet CFT:

$$
W[\sigma, h_i] = \left\langle e^{\int (\sigma \mathcal{O} + \sum h_i A_i)} \right\rangle_0
$$

(10)

for single-trace operators $A_i$. Using (3) and shifting the auxiliary field, $\tilde{\sigma} = \sigma + J$, we find

$$
Z_f[J, h_i] = \sqrt{\det \left(-\frac{1}{f} \mathbf{1}\right)} \int D\tilde{\sigma} e^{W[\tilde{\sigma}, h_i]} + \int \frac{1}{2f} (\tilde{\sigma} - J)^2.
$$

(11)
Now it is convenient to rescale \( J = f \tilde{J} \) and send \( f \) to \( \infty \), which corresponds to taking the IR limit. Discarding the term \( f \tilde{J}^2 / 2 \) which contributes only a contact term, we find

\[
Z_f[\tilde{J}, h_i] \sim \int D\tilde{\sigma} e^{W[\tilde{\sigma}, h_i]} + \int \tilde{\sigma} \tilde{J}.
\]  

(12)

This expression, which is analogous to the result of [24], may be used to generate the \( 1/N \) expansion in the infrared CFT. To pick out the planar limit, it is sufficient to find the saddle point for \( \tilde{\sigma} \), so that the IR generating functional is a Legendre transform of the UV one,

\[
\log Z_f[\tilde{J}, h_i] = W[\tilde{\sigma}, h_i] + \int \tilde{\sigma} \tilde{J}.
\]  

(13)

where

\[
\tilde{J} = -\frac{\delta W[\tilde{\sigma}, h_i]}{\delta \tilde{\sigma}}.
\]  

(14)

Conversely, the UV generating functional is a Legendre transform of the IR one, in agreement with the results of [12, 14]:

\[
W[\tilde{\sigma}, h_i] = \log Z_f[\tilde{J}, h_i] - \int \tilde{\sigma} \tilde{J}, \quad \tilde{\sigma} = \frac{\delta \log Z_f[\tilde{J}, h_i]}{\delta \tilde{J}}.
\]  

(15)

Now, we go on to consider \( \mathcal{O}(N^0) \) corrections to the leading order result contained in the factor \( \frac{1}{\sqrt{\det K}} \) which comes from fluctuations of the auxiliary field away from its classical value. In order to extract the central charge at the IR fixed point, it will be convenient to consider the theory on a sphere \( S^d_R \) of radius \( R \). Conformally mapping \( \mathbb{R}^d \) to \( S^d_R \) (via a stereographic projection, for example), one can easily show that

\[
\langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle = \frac{1}{s(x, x')^{2\Delta}},
\]  

(16)

where \( s(x, x') \) is the chordal distance between points \( x \) and \( x' \) on \( S^d_R \) is that is, \( s(x, x') = 2R\sin(\theta/2) \), where \( \theta \) is the angle between \( x \) and \( x' \). The operators \( \hat{G}, \hat{K}, \) and \( \hat{Q} \) are now diagonal in a basis of spherical harmonics on \( S^d \). An efficient way to find the eigenvalues \( g_\ell \) of \( \hat{G} \) is to expand

\[
\frac{1}{s(x, x')^{2\Delta}} = \sum_{\ell, m} g_\ell Y^*_{\ell m}(x) Y_{\ell m}(x'),
\]  

(17)

where \( \ell \) is the principal angular quantum number and \( m \) collectively denotes all the magnetic quantum numbers. The \( Y_{\ell m}(x) \) are assumed to include a factor of \( R^{-d/2} \), so that when they are squared and integrated over \( S^d_R \), the result is unity. For any \( x' \), we have

\[
g_\ell = \frac{1}{Y_{\ell m}(x')} \int d^d x \sqrt{g} \frac{1}{s(x, x')^{2\Delta}} Y_{\ell m}(x).
\]  

(18)

The eigenvalue \( g_\ell \) has no \( m \) dependence because of \( \text{SO}(d+1) \) invariance. Now we may choose \( x' \) to be the north pole of \( S^d \), \( \theta = 0 \). Using the fact that on \( S^d \), \( Y_{\ell 0}(\theta) \) is proportional to the
Gegenbauer polynomial \( C_{\ell}^{(d-1)/2}(\cos \theta) \), we have

\[
\frac{g_t}{R^{d-2\Delta}} = \frac{\text{Vol} S^{d-1}}{C_{\ell}^{(d-1)/2}(1)} \int_{-1}^{1} dz (1-z^2)^{(d-2)/2}(1-z)^{-\Delta} C_{\ell}^{(d-1)/2}(z)
\]

\[
= \frac{\text{Vol} S^{d-1}}{C_{\ell}^{(d-1)/2}(1)} (-1)^{\ell} 2^{d-1-\Delta} \Gamma(d/2) \frac{(\ell + d - 2)!}{\ell!(d-2)!} \frac{\Gamma(1-\Delta) \Gamma \left( \frac{d}{2} - \Delta \right)}{\Gamma(1-\ell-\Delta) \Gamma(d+\ell-\Delta)}
\]

\[
= (-1)^{\ell} \pi^{d/2} 2^{d-\Delta} \frac{\Gamma(1-\Delta) \Gamma \left( \frac{d}{2} - \Delta \right)}{\Gamma(1-\ell-\Delta) \Gamma(d+\ell-\Delta)} \frac{\Gamma(\ell + \Delta)}{\Gamma(\Delta)}
\]

(19)

where we have used

\[
C_{\ell}^{(d-1)/2}(1) = \frac{(\ell + d - 2)!}{\ell!(d-2)!},
\]

(20)

and

\[
\text{Vol} S^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}.
\]

(21)

The convenience of working on \( S^d \) stems from the fact that the central charge may be defined through the integrated one-point function of the trace of the stress tensor: for a conformal field theory,

\[
\left\langle \int_{S^d_R} d^d x \sqrt{g} T^\mu_\mu \right\rangle = c,
\]

(22)

where the radius \( R \) is arbitrary. The central charge so defined differs by factors of order unity from the usual definitions (where, for example, in two dimensions \( c = 1 \) for a free boson, and in four dimensions \( c = N^2/4 \) for \( \mathcal{N} = 4 \) \( U(N) \) gauge theory). For a flow between conformal fixed points, the one-point function in (22) will depend on \( R \), interpolating between \( c_{UV} \) for small \( R \) and \( c_{IR} \) for large \( R \). For CFT’s in even dimensions greater than 2, there are several distinct central charges, corresponding to different possible Lorentz-invariant counterterms of the appropriate dimension. In CFT’s with anti-de Sitter duals, at large \( N \) it follows from the considerations of \[25\] that all these central charges are related in a rigid way. This need not hold at subleading order in \( N \). Cardy has conjectured \[26\] an analog in four dimensions of Zamolodchikov’s c-theorem for the central charge appearing in (22). We believe that this is the identical central charge that was computed by supergravity methods, up to one loop, in \[22\]. At any rate, what we wish to do in the next section is to compute

\[
c_{IR} - c_{UV} = \left\langle \int_{S^d_R} d^d x \sqrt{g} T^\mu_\mu \right\rangle_{f} - \left\langle \int_{S^d_R} d^d x \sqrt{g} T^\mu_\mu \right\rangle_{0} = \frac{1}{d} R \frac{\partial}{\partial R} (W_f[R] - W_0[R])
\]

(23)

where \( W_f[R] \equiv \log Z_f[J = 0, S^d_R] \), and \( Z_f[J = 0, S^d_R] \) is the partition function of the CFT on the sphere \( S^d_R \), deformed by \( \int \frac{d^2 \mathcal{O}}{2} \).
3 Determinant calculation on $S^d$

With the result

$$g_\ell = R^{d-2}\pi^{d/2}2^{d-2}\Gamma\left(\frac{d}{2} - \Delta\right)\Gamma(\ell + \Delta)\frac{\Gamma(\Delta)}{\Gamma(d + \ell - \Delta)}$$  \hspace{1cm} (24)

in hand, we wish to compute

$$W_f[R] - W_0[R] = -\frac{1}{2} \text{tr} \log \hat{K}$$
$$= -\frac{1}{2} \sum_{\ell=0}^{\infty} M_d(\ell) \log k_\ell = -\frac{1}{2} \sum_{\ell=0}^{\infty} M_d(\ell) \log(1 + fg_\ell),$$  \hspace{1cm} (25)

where $M_d(\ell)$ is the degeneracy of states with angular momentum $\ell$ on $S^d$:

$$M_d(\ell) = \frac{(\ell + d - 2)!(2\ell + d - 1)}{\ell!(d - 1)!}.\hspace{1cm} (26)$$

(This is the dimension of the irreducible representation of $SO(d+1)$ formed as the symmetric traceless part of $\ell$ fundamental vector representations). In the limit $f R^{d-2\Delta} \to 0$ (a very small sphere, or hardly any deformation), $W_f[R] \to W_0[R]$. In the opposite limit, $f R^{d-2\Delta} \to \infty$, where we are probing the infrared properties of the theory, the central charge can be read off from the coefficient of $\log R$ in an expansion of $W_f[R]$. The reason for this is that the derivative in (23) picks out $\log R$.

Dropping an overall constant which is independent of both $R$ and $\Delta$, we find for large $R$ that

$$W_f[R] - W_0[R] = -\frac{1}{2} \sum_{\ell=0}^{\infty} M_d(\ell) \log g_\ell.\hspace{1cm} (27)$$

Because the factors $\pi^{d/2}2^{d-\Delta}\Gamma\left(\frac{d}{2} - \Delta\right)/\Gamma(\Delta)$ in $g_\ell$ do not depend on $\ell$ or $R$, they will not contribute to the scale anomaly (23). (The reader can check this explicitly once we have introduced our regulation scheme). Thus we are left with the computation of

$$W_f[R] = -\frac{1}{2} \sum_{\ell=0}^{\infty} M_d(\ell) \log \left(R^{d-2\Delta}\frac{\Gamma(\ell + \Delta)}{\Gamma(d + \ell - \Delta)}\right) = -\frac{1}{2}(V_1 + V_2),\hspace{1cm} (28)$$

where

$$V_1 = (d - 2\Delta)(\log R) \sum_{\ell=0}^{\infty} M_d(\ell), \hspace{1cm} V_2 = \sum_{\ell=0}^{\infty} M_d(\ell) \log \frac{\Gamma(\ell + \Delta)}{\Gamma(\ell + d - \Delta)}.\hspace{1cm} (29)$$

Naively, it seems that only $V_1$ has $\log R$ dependence. This would be in contradiction with the results of [22], because it would mean that the central charge would depend linearly on $d - 2\Delta$. The problem is that, because $M_d(\ell)$ is a polynomial of degree $d - 1$, all the sums in (23), (25), (27), and (29) are divergent, and some regulator is required. A regulator must refer to some energy scale $\Lambda$ which remains fixed as we differentiate with respect to $R$ in
calculating $\langle f T_{\mu}^\mu \rangle$. This introduces additional $R$ dependence beyond what is apparent in (29).

For reasons to be explained, we will settle eventually on zeta-function regularization. However, to appreciate the point about the regulator introducing $R$-dependence to a sum over $\ell$, consider the regulated sum

$$\sum_{\ell=1}^\infty \ell^\alpha e^{-\epsilon \ell} = \text{Li}_{-\alpha}(e^{-\epsilon})$$

where $\epsilon^{-1} = R \Lambda$ and $\alpha$ is real. When $\alpha = -1$, the right hand side diverges as $-\log \epsilon = \log(R\Lambda)$ in the $\epsilon \to 0$ limit. For $\alpha < -1$, the sum converges in the $\epsilon \to 0$ limit, while for $\alpha > -1$, the sum (defined now via analytic continuation) has, at most, divergences which are integer powers of $1/\epsilon$. Evidently, for $\alpha = -1$, there is logarithmic dependence on $R$ which persists in the limit that the cutoff $\Lambda$ is removed.

This can be compared directly to a zeta-function regulator, which can be motivated formally by considering

$$\sum_{\ell=1}^\infty \ell^\alpha \frac{1}{\ell^s} = \zeta(s - \alpha)$$

for real $\alpha$. Of course, the sum is again defined via analytic continuation when $\text{Re } s < \alpha + 1$. When $\alpha = -1$, there is a pole in the expression on the right hand side at $s = 0$ with residue 1. For other real $\alpha$, there is no pole at $s = 0$. Comparing with the remarks following (30), one can conclude that for any polynomial sum, extracting the residue from a zeta-function regularization gives the coefficient of $\log(R\Lambda)$. A convenient feature of the zeta-function approach is that

$$\sum_{\ell=1}^\infty \ell^\alpha (\log \ell) \frac{1}{\ell^s} = \frac{d}{d\alpha} \sum_{\ell=1}^\infty \ell^\alpha \frac{1}{\ell^s} = -\zeta'(s - \alpha),$$

which has no residue. Presumably it may be shown that $\ell^\alpha (\log \ell)$ terms can also be dropped with the simpler exponential regulator of the previous paragraph, but the argument seems less transparent.

The last manipulation we need in order to be able to compute the anomaly in any dimension is to set $x = \Delta - d/2$ and expand

$$\log \frac{\Gamma(\ell + \Delta)}{\Gamma(\ell + d/2 - \Delta)} = \log \Gamma(\ell + d/2 + x) - \log \Gamma(\ell + d/2 - x) = 2 \sum_{n=1}^{\infty} \frac{x^n}{n!} \psi^{(n-1)}(\ell + d/2)$$

where

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \log z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} z^{-2k} = \log z + \sum_{s=0}^{\infty} \zeta(-s) z^{-s-1}.$$
3.1 Vanishing of the anomaly in odd dimensions $d$

Because of the absence of appropriate generally covariant counter-terms for violations of scale invariance, it is expected that $\langle T_\mu^\mu \rangle = 0$ for a conformal theory in odd dimensions, regardless of which curved manifold the theory is defined on.

It is convenient here to define $k = \ell + (d - 1)/2$. The multiplicity can be expressed as

$$M_d(k - (d - 1)/2) = \frac{2}{(d - 1)!} (d - 3/2) \prod_{i=0}^{(d-3)/2} (k^2 - i^2).$$

(35)

Now we observe that $V_1 = (d - 2\Delta) \log R \sum_{k=1}^{\infty} M_d(k - (d - 1)/2)$

vanishes because $\zeta(-2n) = 0$ for all positive integer $n$. The reason we were able to shift the sum from $\ell = 0$ to $\infty$ to a sum from $k = 1$ to $\infty$ is that $M_d(k - (d - 1)/2)$ vanishes for $k = 1, 2, \ldots, (d - 1)/2$.

To show that $V_2$ also contributes nothing to the anomaly, we use the large $k$ expansion

$$\psi(k + 1/2) = \log k + \sum_{r=1}^{\infty} b_r k^{-2r},$$

(37)

where $b_1 = 1/24, b_2 = -7/960$, etc. Since both $M_d$ and $\psi(k + 1/2)$ contain only even powers of $k$, the sums

$$\sum_{k=1}^{\infty} M_d(k - (d - 1)/2) \psi^{(2n)}(k + 1/2)$$

(38)

contain no logarithmic divergences. Therefore, our calculation of the anomaly indeed gives a vanishing result in all odd dimensions $d$.

The calculation of [22] gave a definite non-zero result for the change in the bulk cosmological constant for odd $d$, but it is unclear how to translate this in a crisp way into field theory terms. Certainly the scale anomaly vanishes in the calculation of [25] for odd dimensions. The result of [22] may correspond to some non-divergent terms in field theory that we are not computing here.

3.2 Vanishing of terms linear in $d - 2\Delta$ in even dimensions $d$

A general feature predicted by [22] is that there is no term linear in $d - 2\Delta$ in the expansion of the difference $c_{IR} - c_{UV}$ in powers of $d - 2\Delta$. This can readily be shown in field theory, as follows.

It is convenient here to define $k = \ell + d/2$. The multiplicity $M_d(k - d/2)$ vanishes for $k = 1, 2, \ldots, d/2$, so we are free to shift a sum from $\ell = 0$ to $\infty$ to a sum from $k = 1$ to $\infty$. We may write

$$M_d(k - d/2) = \sum_{r=1}^{d-1} a_r k^r$$

(39)
for some set of coefficients $a_r$. Now, using the zeta-function regulator,

$$V_1 = (d - 2\Delta) \log R \sum_{k=1}^{\infty} M_d = (d - 2\Delta) \log R \sum_{r=1}^{d-1} a_r \zeta(-r). \quad (40)$$

On the other hand, the term in $V_2$ linear in $x = \Delta - d/2$ is

$$V_2 = (2\Delta - d) \sum_{k=1}^{\infty} M_d(k - d/2) \psi(k) = (2\Delta - d) \sum_{k=1}^{\infty} \left( \sum_{r=1}^{d-1} a_r k^r \sum_{s=0}^{\infty} \zeta(-s) k^{-s-1} \right)$$

$$= (2\Delta - d) \log(R\Lambda) \sum_{r=1}^{d-1} a_r \zeta(-r). \quad (41)$$

In the first equality we have used (34) and dropped the logarithmic term because it does not survive zeta-function regularization. In the second equality we have simply picked out the $1/k$ terms in the product of the two interior sums. Evidently, the $\log R$ dependence cancels between (39) and (41).

### 3.3. The cases $d = 2, 4, 6, \text{and} \ 8$

It remains to compare the coefficients of non-linear odd powers in $x = \Delta - d/2$ between our field theory calculation and the $AdS_{d+1}$ calculation of [22], where one finds an anomaly proportional to

$$\int_0^x \prod_{i=0}^{(d-2)/2} (x^2 - i^2). \quad (42)$$

In the field theory calculation, only $V_2$ contributes. We present the calculation for a few even values of $d$, using the same definition, $k = \ell + d/2$, as in section 3.3.

First we consider $d = 2$, where $M_2(\ell) = 2\ell + 1 = 2k - 1$, so that the necessary sum is

$$\frac{x^3}{3} \sum_{k=1}^{\infty} (2k - 1) \psi''(k) \rightarrow -\log(R\Lambda) \frac{2x^3}{3}, \quad (43)$$

where the arrow indicates that we have isolated the logarithmic term by zeta-function regularization. All higher powers of $x$ produce convergent sums and cannot contribute to the anomaly. The result proportional to $x^3$ agrees with [22] up to overall coefficient.

Now let us consider $d = 4$, where the result found by [22] is a particular sum of $(\Delta - 2)^3$ and $(\Delta - 2)^5$ behaviors. We have

$$M_4(\ell) = \frac{(\ell + 1)(\ell + 2)(3 + 2\ell)}{6} = \frac{k(2k^2 - 3k + 1)}{6}, \quad (44)$$

where $k = \ell + 2$. So the term of order $x^3$ is

$$\frac{x^3}{3} \sum_{k=1}^{\infty} k(2k^2 - 3k + 1) \left( \frac{1}{k^2} - \frac{1}{k^3} - \frac{1}{2k^4} + \ldots \right) \rightarrow \log(R\Lambda) \frac{x^3}{18}. \quad (45)$$
The term of order $x^5$ is evaluated similarly, and the total anomalous part of $V_2$ is

$$
\frac{1}{6} \log(R\Lambda) \left( \frac{x^3}{3} - \frac{x^5}{5} \right),
$$

which agrees with (42), again up to an overall normalization.

Now let us check the cases $d = 6$ and $d = 8$. Defining $k = \ell + 3$, we have

$$M_6(\ell) = \frac{2k^5 - 5k^4 + 5k^2 - 2k}{5!}.
$$

By computations similar to (43) and (45), we find that the anomalous part of $V_2$ is

$$-\frac{1}{180} \log(R\Lambda) \left( \frac{x^7}{7} - x^5 + \frac{4x^3}{3} \right),
$$

where $x = \Delta - 3$. In the $d = 8$ case, defining $k = \ell + 4$, we have

$$M_8(\ell) = \frac{(2k - 1)(k - 3)(k - 2)(k - 1)k(k + 1)(k + 2)}{7!},
$$

and the anomalous part of $V_2$ is

$$-\frac{1}{10080} \log(R\Lambda) \left( \frac{x^9}{9} - 2x^7 + \frac{49x^5}{5} - 12x^3 \right).
$$

The results (48) and (50) again agree with (42).

Although we have not given a general argument for the agreement of the field theory results with the simple AdS/CFT result (42) of [22], the cases that we have checked provide fairly convincing evidence that in all even $d$ there is full agreement.

4 A remark on the two-point function

One of the striking features of the general treatment in section 2 of the field theory computation is that the two-point correlator can be computed not just at the endpoints of the renormalization group flow, but also at intermediate energy scales: on $\mathbf{R}^d$, the key formulas (recapitulating (7) and (8)) are

$$Q(k) \equiv \int d^dx \ e^{-ik\cdot x} \langle \mathcal{O}(x) \mathcal{O}(0) \rangle_f = G(k) \frac{G(k)}{1 + fG(k)}
$$

$$G(k) = 2d - 2\Delta \bar{\ell}^{d/2} \frac{\Gamma \left( \frac{d}{2} - \Delta \right)}{\Gamma(\Delta)} k^{2\Delta - d},
$$

where the original deformation is $S \rightarrow S + \int \frac{4}{\bar{\ell}} \mathcal{O}^2$.

On the other hand, the AdS treatment of the renormalization group flow caused by the same deformation, initiated in [14] and continued in [19, 22], has also the capacity to yield...
information about intermediate energy scales. Indeed, in [16] the AdS methods were used to derive the formula for the two-point function identical to (51). Below we rederive this result in a somewhat different way. In [22] the two-point function for the scalar \( \phi \) dual to \( O \) is given for arbitrary points in the bulk. Parametrizing Euclidean AdS \( d + 1 \) as

\[
ds^2 = \frac{1}{x_0^2} (dx_0^2 + dx_1^2 + \ldots + dx_d^2),
\]

the two point function for points \((x_0, \vec{x}), (y_0, \vec{y})\) such that \( x_0 < y_0 \) is given in [22] as

\[
G_E(x, y; \tilde{f}) = -\int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik \cdot (\vec{x} - \vec{y})} (x_0 y_0)^{\frac{d}{2}} K_\nu(k_0)}{(1 + (2/k)^{2\nu} \tilde{f} \Gamma(1 + \nu)/\Gamma(1 - \nu))L^{d-1}} \times \left[ I_{-\nu}(k x_0) + \tilde{f} \frac{\Gamma(1 + \nu)}{\Gamma(1 - \nu)} \left( \frac{2}{k} \right)^{2\nu} I_\nu(k x_0) \right],
\]

where \( \nu = d/2 - \Delta \) is between 0 and 1.\(^4\) Expanding \( \phi \) near the boundary as

\[
\phi \sim \alpha(\vec{x}) x_0^{d-\Delta} + \beta(\vec{x}) x_0^\Delta,
\]

the normalization of \( \tilde{f} \) is fixed by writing the boundary condition of [14] as \( \alpha = \tilde{f} \beta \).

A correct though somewhat heuristic method to obtain the AdS/CFT prediction for the two-point function \( Q(k) \), up to an overall normalization, is to start with the \( k^0 \) Fourier component of \( G_E \), set \( x_0 = y_0 = \epsilon \), and extract the coefficient of the leading non-analytic term in \( \epsilon^2 \) as \( \epsilon \to 0 \). The result is supposed to be \( Q(k) \), up to an overall factor that may be \( \Delta \)-dependent. Dropping such factors, one obtains

\[
Q(k) \sim \frac{k^{-2\nu}}{1 + (2/k)^{2\nu} \tilde{f} \Gamma(1 + \nu)/\Gamma(1 - \nu)}.
\]

Only the term \( I_{-\nu}(k x_0) \) inside square brackets in (53) contributes to (55). Evidently, (55) is in agreement with (51) (up to the overall normalization which we have not endeavored to compute) provided we identify

\[
\tilde{f} = \frac{2\pi^{d/2}}{d - 2\Delta} \frac{\Gamma(\Delta - \frac{d}{2} + 1)}{\Gamma(\Delta)} \tilde{f}.
\]

It may be possible to normalize \( \tilde{f} \) in an independent manner and make a consistency check with (56). We presume that due care would allow us to reconcile the overall normalization as well; but since this can be studied entirely at the UV fixed point, and has only to do with the \( J J \mathcal{O} \) couplings that have been well-explored in other literature, we will not address the issue here.

The main point that the results of [16] and of this section illustrate is that the AdS/CFT and field theory computations yield information about the entire RG flow, with no scheme dependence in physical answers.

\(^4\)Note that this sign for \( \nu \) is opposite the one that often shows up in the literature, for example [3]. This is because \( \nu \) is being defined in reference to the UV scaling dimension of \( \mathcal{O} \), rather than the IR scaling dimension.
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References

[1] J. M. Maldacena, “The large \( N \) limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.* **2** (1998) 231–252, [hep-th/9711200](https://arxiv.org/abs/hep-th/9711200).

[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” *Phys. Lett.* **B428** (1998) 105–114, [hep-th/9802109](https://arxiv.org/abs/hep-th/9802109).

[3] E. Witten, “Anti-de Sitter space and holography,” *Adv. Theor. Math. Phys.* **2** (1998) 253–291, [hep-th/9802150](https://arxiv.org/abs/hep-th/9802150).

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, “Large \( N \) field theories, string theory and gravity,” *Phys. Rept.* **323** (2000) 183–386, [hep-th/9905111](https://arxiv.org/abs/hep-th/9905111).

[5] I. R. Klebanov, “TASI lectures: Introduction to the AdS/CFT correspondence,” [hep-th/0009139](https://arxiv.org/abs/hep-th/0009139).

[6] E. D’Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS/CFT correspondence,” [hep-th/0201253](https://arxiv.org/abs/hep-th/0201253).

[7] S. S. Gubser, I. R. Klebanov, and A. W. Peet, “Entropy and Temperature of Black 3-Branes,” *Phys. Rev.* **D54** (1996) 3915–3919, [hep-th/9602135](https://arxiv.org/abs/hep-th/9602135).

[8] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” *Adv. Theor. Math. Phys.* **2** (1998) 505–532, [hep-th/9803131](https://arxiv.org/abs/hep-th/9803131).

[9] J. M. Maldacena, “Wilson loops in large \( N \) field theories,” *Phys. Rev. Lett.* **80** (1998) 4859–4862, [hep-th/9803002](https://arxiv.org/abs/hep-th/9803002).

[10] S.-J. Rey and J. Yee, “Macroscopic strings as heavy quarks in large \( N \) gauge theory and anti-de Sitter supergravity,” *Eur. Phys. J.* **C22** (2001) 379–394, [hep-th/9803001](https://arxiv.org/abs/hep-th/9803001).

[11] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” [hep-th/0204051](https://arxiv.org/abs/hep-th/0204051).
[12] I. R. Klebanov and E. Witten, “AdS/CFT correspondence and symmetry breaking,” Nucl. Phys. B556 (1999) 89–114, hep-th/9905104.

[13] O. Aharony, M. Berkooz, and E. Silverstein, “Multiple-trace operators and non-local string theories,” JHEP 08 (2001) 006, hep-th/0105309.

[14] E. Witten, “Multi-trace operators, boundary conditions, and AdS/CFT correspondence,” hep-th/0112258.

[15] M. Berkooz, A. Sever, and A. Shomer, “Double-trace deformations, boundary conditions and spacetime singularities,” JHEP 05 (2002) 034, hep-th/0112264.

[16] W. Muck, “An improved correspondence formula for AdS/CFT with multi-trace operators,” Phys. Lett. B531 (2002) 301–304, hep-th/0201106.

[17] P. Minces, “Multi-trace operators and the generalized AdS/CFT prescription,” hep-th/0201172.

[18] A. Sever and A. Shomer, “A note on multi-trace deformations and AdS/CFT,” JHEP 07 (2002) 027, hep-th/0203168.

[19] K. G. Wilson and J. B. Kogut, “The Renormalization group and the epsilon expansion,” Phys. Rept. 12 (1974) 75–200.

[20] I. R. Klebanov and A. M. Polyakov, “AdS dual of the critical O(N) vector model,” Phys. Lett. B550 (2002) 213–219, hep-th/0210114.

[21] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B536 (1998) 199–218, hep-th/9807080.

[22] S. S. Gubser and I. Mitra, “Double-trace operators and one-loop vacuum energy in AdS/CFT,” hep-th/0210093.

[23] P. Breitenlohner and D. Z. Freedman, “Positive energy in Anti-de Sitter backgrounds and gauged extended supergravity,” Phys. Lett. B115 (1982) 197.

[24] I. R. Klebanov and A. Hashimoto, “Nonperturbative solution of matrix models modified by trace squared terms,” Nucl. Phys. B434 (1995) 264–282, hep-th/9409064.

[25] M. Henningson and K. Skenderis, “The holographic Weyl anomaly,” JHEP 07 (1998) 023, hep-th/9806087.

[26] J. L. Cardy, “Is there a c theorem in four-dimensions?,” Phys. Lett. B215 (1988) 749.