EXISTENCE OF CSCK METRICS ON SMOOTH MINIMAL MODELS

ZAKARIAS SJÖSTRÖM DYREFELT

Abstract. Given a compact Kähler manifold $X$ it is interesting to ask whether it admits a constant scalar curvature Kähler (cscK) metric in at least one Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R})$. In this short note we show that there always exist cscK metrics on compact Kähler manifolds with nef canonical bundle, thus on all smooth minimal models, and also on the blowup of any such manifold. This confirms an expectation of Jian-Shi-Song [14] and extends their main result from $K_X$ semi-ample to $K_X$ nef, with a direct proof that does not appeal to the Abundance conjecture.

1. Introduction

Let $X$ be a compact Kähler manifold of dimension $n \geq 2$, let $K_X$ be the canonical bundle and $c_1(X) := c_1(-K_X)$ the associated $(1,1)$-cohomology class. Going back to the work of Calabi [3], existence of constant scalar curvature (cscK) metrics has been a much studied question in Kähler geometry. In particular, by Yau’s celebrated theorem it has been known for decades that cscK metrics always exist on compact Kähler manifolds with ample canonical bundle. Much more recently it was shown by Jian-Shi-Song [14] that cscK metrics always exist also when $K_X$ is only semi-ample. The step from ample to semi-ample is of special interest thanks to the minimal model program, and especially in view of the Abundance conjecture. Indeed, by MMP and Abundance for surfaces, every compact Kähler surface of Kodaira dimension $\kappa \geq 0$ are birational to a surface which is minimal. In particular, it follows from [14] that all such surfaces admit cscK metrics. There are some other situations when the Abundance conjecture is known, but in general the gap $K_X$ semi-ample to $K_X$ nef is still conjecture.

In this short note we confirm the expectation of [14] that their existence result should hold in general for $-c_1(X)$ nef, with a proof not relying on the Abundance conjecture, and which is valid for arbitrary (projective or non-projective) compact Kähler manifolds in any dimension:

**Theorem 1.** Suppose that $X$ is a compact Kähler manifold with $-c_1(X)$ nef. Then for any Kähler class $[\omega]$ on $X$, there is $\epsilon_{X,[\omega]} > 0$, such that for all $0 < \epsilon < \epsilon_{X,[\omega]}$, there exists a unique cscK metric in the Kähler class $-c_1(X) + \epsilon[\omega]$. In particular, every smooth minimal model $X$ admits a cscK metric.

As a consequence of this result and Arezzo-Pacard [1] we obtain the following.

**Corollary 2.** Blowups of compact Kähler manifolds with $-c_1(X)$ nef admit cscK metrics.
A few comments on the proof are in order. Just as in [14] our proof partly relies on the work of Chen-Cheng [5], which is concerned with properness/coercivity of energy functionals on the closure $E^1$ of the space of Kähler metrics. The new ingredient is to combine the above machinery with an elementary observation about the following numerical quantity

$$\delta([\omega]) := \delta_1([\omega]) - \delta_2([\omega])$$

where

$$\delta_1([\omega]) := \sup\{\delta_1 \in \mathbb{R} : \exists C > 0, M(\varphi) \geq \delta_1 ||\varphi|| - C, \forall \varphi \in E^1\}$$

and

$$\delta_2([\omega]) := \sup\{\delta_2 \in \mathbb{R} : \exists C > 0, E_{\omega}^{\rho_\omega}(\varphi) \geq \delta_2 ||\varphi|| - C, \forall \varphi \in E^1\},$$

where $M$ is the K-energy functional, $E_{\omega}^{\rho_\omega}$ is the energy part in its Chen-Tian decomposition, see [6]. The above quantities are sometimes called stability thresholds, and they are well-defined real numbers (not equal to infinity), see [18]. It is often relevant to note also that $\delta([\omega]) \geq \alpha X([\omega]) > 0$ where the latter denotes the classical alpha invariant of Tian [20].

In order to streamline the proof of Theorem 1 we introduce the following shorthand terminology related to the J-equation and J-stability: Let $\theta$ be any smooth $(1,1)$-form on $X$ and write $[\theta] \in H^{1,1}(X, \mathbb{R})$ for the associated cohomology class. We say that $((X,[\omega]); [\theta])$ is J-coercive (or J-stable, see [4]) if and only if the $E_\theta^\omega$-functional is coercive on the space $E^1 := E^1(X, \omega)$, i.e. there are constants $\delta, C > 0$ such that

$$E_\theta^\omega(\varphi) \geq \delta d_1(0, \varphi) - C$$

for all $\varphi \in E^1$.

The strategy of proof is then based on the following observation, which in view of [18] can be interpreted as a geometric condition on the Kähler cone:

**Theorem 3.** Suppose that $-c_1(X)$ is nef. Then $X$ admits a cscK metric in the Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R})$ if and only if $((X,[\omega]); [\theta])$ is J-coercive, where

$$\theta := -c_1(X) + \delta([\omega]) [\omega].$$

Given a compact Kähler manifold with $-c_1(X)$ nef, the idea is then to show that the J-stability condition of the above theorem is always satisfied for $(1-\epsilon)(-c_1(X)) + \epsilon [\omega]$, $\epsilon \in [0, 1]$. In fact a bit more can be said, as shown by the proofs of Theorem 1 and Theorem 3 given in the next section.

**Remark 1.** By a minor alteration of the proof of Theorem 1 we can more generally obtain a twisted version of the main result: Suppose that $\eta \in H^{1,1}(X, \mathbb{R})$ is a $(1,1)$-form such that $-c_1(X) + \eta$ is nef. Then for any Kähler class $[\omega]$ on $X$, there is $\epsilon_X[\omega] > 0$ such that for all $0 < \epsilon < \epsilon_X[\omega]$, there exists a unique $\eta$-twisted cscK metric, that is, a solution to $\text{Tr}_\beta(-\text{Ric}(\omega) + \eta) = \text{constant}$, in the Kähler class $[\beta] = -c_1(X) + \eta + \epsilon [\omega]$. This follows by simply replacing $-c_1(X)$ by $-c_1(X) + \eta$ everywhere in the proof.
2. Proof

We quickly recall the notation and terminology for energy functionals that we will use, which is the standard variational setup frequently used throughout the Kähler geometry literature. To introduce our notation, let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n \geq 2\) and write \(\gamma := [\omega] \in H^{1,1}(X, \mathbb{R})\) for the associated Kähler class. Let

\[
V_\gamma := \int_X \frac{\omega^n}{n!}
\]

be the Kähler volume of \((X, \omega)\). Let \(\rho_\omega\) be the Ricci curvature form, normalized such that \([\rho_\omega] = c_1(X)\), and write \(S(\omega) := \text{Tr}_\omega \rho_\omega\) for the scalar curvature of \((X, \omega)\). Denote the automorphism group of \(X\) by \(\text{Aut}(X)\) and its connected component of the identity by \(\text{Aut}_0(X)\). Write \(C_X \subset H^{1,1}(X, \mathbb{R})\) for the cone of Kähler cohomology classes on \(X\). Let \(C_X\) be the nef cone, \(\partial C_X\) its boundary, and let \(\text{Big}_X\) be the cone of big \((1, 1)\)-classes on \(X\).

We write \((\mathcal{H}_\omega, d_1)\) for the space of Kähler potentials on \(X\) endowed with the \(L^1\)-Finsler metric \(d_1\), and denote by \((\mathcal{E}_1, d_1)\) its metric completion (see \([8, 9, 10, 11]\) and references therein). Write \(\text{PSH}(X, \omega) \cap L^\infty(X)\) for the space of bounded \(\omega\)-psh functions on \(X\).

Now consider \(\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)\). We may define well-known energy functionals

\[
I_\omega(\varphi) := \frac{1}{V_\gamma n!} \int_X \varphi (\omega^n - \omega^n_\varphi)
\]

\[
J_\omega(\varphi) = \frac{1}{V_\gamma n!} \int_X \varphi \omega^n - \frac{1}{V_\gamma (n+1)!} \int_X \varphi \sum_{j=0}^n \omega^j \wedge \omega^{n-j}_\varphi
\]

\[
E^\theta_\omega(\varphi) := \frac{1}{V_\gamma n!} \int_X \varphi \sum_{j=0}^{n-1} \theta \wedge \omega^j \wedge \omega^{n-j-1}_\varphi - \frac{1}{V_\gamma (n+1)!} \int_X \varphi \theta \sum_{j=0}^n \omega^j \wedge \omega^{n-j}_\varphi
\]

where \(\theta\) is any smooth closed \((1, 1)\)-form on \(X\) and \(\theta\) is the topological constant given by

\[
\theta := \frac{\int_X \theta \wedge \omega^{n-1}}{\int_X \omega^n/n!}.
\]

By the Chen-Tian formula \([6]\) the K-energy functional can be written as the sum of an energy/pluripotential part and an entropy part as

\[
M_\omega = E^{-\rho}_\omega + H_\omega \tag{1}
\]

where

\[
H_\omega(\varphi) := \frac{1}{V_\gamma n!} \int_X \log \left( \frac{\omega^n_\varphi}{\omega^n} \right) \omega^n_\varphi
\]

is the relative entropy of the probability measures \(\omega^n_\varphi/V_\gamma\) and \(\omega^n/V_\gamma\). In particular, it is well known that \(H_\omega(\varphi)\) is always non-negative.

For any given smooth closed \((1, 1)\)-form \(\theta\) on \(X\) we also consider the \(\theta\)-twisted K-energy functional

\[
M^\theta_\omega := M_\omega + E^\theta_\omega.
\]
As explained in [18] (and for the same reasons), in this paper it will be convenient to measure properness/coercivity of the K-energy against the functional

\[ ||\varphi|| := (I_\omega - J_\omega)(\varphi) = \frac{1}{V_\gamma (n+1)!} \int_X \varphi \sum_{j=0}^{n} \omega^j \wedge \omega^{n-j} - \frac{1}{V_\gamma n!} \int_X \omega^n \varphi \]

rather than against the Aubin J-functional or the \(d_1\)-distance.

**Definition 1.** Let \(F : \mathcal{E}^1 \to \mathbb{R}\) be any of the above considered energy functionals. We then say that \(F\) is coercive if

\[ F(\varphi) \geq \delta ||\varphi|| - C \]

for some \(\delta, C > 0\) and all \(\varphi \in \mathcal{E}^1\).

In the rest of this note we will use the notation of projective polarized manifolds \((X, L)\), while emphasizing that this is only cosmetic, and all the arguments below hold for arbitrary Kähler classes on arbitrary (possibly non-projective) compact Kähler manifolds.

**Proof of Theorem 3** The proof of Theorem 3 is a simple reformulation of the fact that existence of cscK metrics in a Kähler class \(c_1(L)\) is connected to J-stability of the triple \(((X, L); K)\) as in the introduction, where \(K := K_X\).

We will use the following necessary and sufficient existence criteria due to [21, 22] and [12]: A polarised manifold \(((X, L); K)\) is J-stable if

\[ nK.L^n - (n - 1)L - n^2 - (n - 1)K > 0 \tag{2} \]

that is this divisor is ample, and a necessary condition for J-stability of \(((X, L); K)\) is that

\[ nK.L^n - K > 0 \tag{3} \]

In dimension \(n = 2\) the condition thus becomes necessary and sufficient.

These conditions can equivalently be expressed in terms of stability thresholds, following [18]. In the notation of that paper, we can reformulate the above conditions as a double inequality

\[ nK.L^n - (n - 1)\sigma(K, L) \leq \delta_2(K)(L) \leq nK.L^n - \sigma(K, L) \tag{4} \]

where

\[ \sigma(K, L) := \inf \{ \lambda \in \mathbb{R} : K - \lambda L < 0 \} \]

and

\[ \delta_2(K)(L) := \sup \{ \delta \in \mathbb{R} : \exists C > 0, E^{\rho_\sigma}_{\omega}(\varphi) \geq \delta ||\varphi|| - C, \forall \varphi \in \mathcal{E}^1 \} \]

is the optimal coercivity constant for the energy part \(E^{\rho_\sigma}_{\omega}\) of the Mabuchi K-energy, which by [7] is positive if and only if the triple \(((X, L); K)\) is J-stable.

The relationship with the cscK equation follows by considering in the same way the optimal coercivity constant

\[ \delta_1(L) := \sup \{ \delta \in \mathbb{R} : \exists C > 0, M(\varphi) \geq \delta ||\varphi|| - C, \forall \varphi \in \mathcal{E}^1 \} \]
and then relating it to J-stability of \(((X, L); K)\), by observing that
\[
\delta_1(L) = \delta_{2, K} + (\delta_1(L) - \delta_{2, K}(L))L(L).
\]
In other words \((X, L)\) is cscK if and only if \(((X, L); \theta)\) is J-stable with respect to the smooth (not necessarily positive; stability requires positivity to make sense, but coercivity can be spoken of regardless) \((1, 1)\)-form defined by
\[
\theta := \theta_L := K + (\delta_1(L) - \delta_{2, K}(L))L.
\]
This is precisely the statement of Theorem 3. Note that \(\theta\) is not necessarily assumed positive here, since while stability requires positivity to make sense, coercivity can be defined more generally.

Proof of Theorem 1 Motivated by Theorem 3 we now study the \((1, 1)\)-form \(\theta_L\) more closely. The first thing to comment on is that we cannot expect to compute \(\theta_L\) in any easy manner, since then we would know essentially everything about the existence of cscK problem. However, it is immediate to observe the following inequality
\[
\delta_1(L) - \delta_{2, K}(L) > 0,
\]
which follows immediately because if \(E_{\omega, K}^\omega\) is coercive then also \(M\) is coercive.

Suppose now once and for all that the canonical bundle \(K := K_X\) is nef. Then by (3) it follows that \(\theta := \theta_L\) is ample. By adding \(\delta_1(L) - \delta_{2, K}(L)\) to each side of the double inequality (4), we moreover deduce a similar double inequality also for \(\delta_1(L)\):
\[
\frac{n \theta \cdot L^{n-1}}{L^n} - (n - 1)\sigma(\theta, L) \leq \delta_1(L) \leq \frac{n \theta \cdot L^{n-1}}{L^n} - \sigma(\theta, L).
\]
In order to make use of this observation, it will be enough to show that there exists an ample line bundle \(L\) on \(X\) for which the left hand side of (6) is strictly positive. This is achieved by a simple application of the methods in [18]. In case \(n = 2\) the subcone of the Kähler cone where the \(E_{\omega, K}^\omega\)-functional is coercive with respect to a given \(\theta\) is described explicitly in [18]. In higher dimension a complete description was not given, but all that is needed for the proof of our main result are the criteria (2), (3) and the following lemma, whose proof is identical to that in [18]:

Lemma 4. Let \(\Lambda, T\) be line bundles such that \(c_1(\Lambda) \in C_X\) and \(c_1(T) \in \partial C_X\), that is, \(\Lambda\) is ample and \(T\) is nef but not ample. Then \(L_t := (1 - t)T + t\Lambda, t \in [0, 1]\) satisfies
\[
\sigma(\Lambda, L_t) = \frac{1}{t}.
\]
The same result holds if \(\Lambda \in \partial C_X\), as long as \(c_1(L_t) \in C_X\) for all \(t \in (0, 1)\).

Proof. See [18, Lemma 17].

We are finally ready to prove the main result, extending [14] and replacing the hypothesis of \(K_X\) semi-ample with \(K_X\) nef, as predicted by the Abundance conjecture. We recall that while it is here presented in the terminology of projective polarized manifolds \((X, L)\), it is valid also for arbitrary compact Kähler manifolds with the same proof.
Theorem 5. (Theorem[1]) Suppose that $K_X$ is nef. Then for any ample line bundle $L$ on $X$, there is $\epsilon_{X,L} > 0$, such that for all $0 < \epsilon < \epsilon_{X,L}$, there exists a unique cscK metric in the Kähler class $-c_1(X) + \epsilon c_1(L)$. In particular, every smooth minimal model $X$ admits a cscK metric.

Proof. Suppose that $K := K_X$ is nef but not ample (if it is ample then we already know that cscK metrics exist, by Yau’s theorem). As explained above it is a consequence of [5, 7, 12, 21, 22] that cscK metrics must exist if the Kähler class

$$\theta := K + (\delta_1(L) - \delta_{2,K}(L))L$$

satisfies

$$n \frac{\theta \cdot L^{n-1}}{L^n} - (n-1)\sigma(\theta, L) > 0.$$ 

Now fix an auxiliary line bundle $T$ which is nef but not ample, in such a way that $L_t := (1-t)T + tK$ is ample for all $t \in (0, 1)$. Consider

$$R_\epsilon(t) := n \frac{(K + \epsilon L_t) \cdot L_t^{n-1}}{L_t^n} - (n-1)\sigma((K + \epsilon L_t), L_t)$$

as a function of $\epsilon \in \mathbb{R}$ and $t \in (0, 1)$. By the last part of Lemma 4 we then have

$$R_0(t) = n \frac{K \cdot L_t^{n-1}}{L_t^n} - \frac{n-1}{t},$$

so that in particular $R_0(1) = 1$. By continuity we may therefore fix $t_0 \in (0,1)$ such that $R_0(t) > 0$ for all $t \in (t_0, 1)$. Moreover, it can be easily checked that

$$\sigma((K + \epsilon L_t), L_t) = \sigma(K, L_t) + \epsilon$$

and hence also

$$R_\epsilon(t) = R(t) + \epsilon.$$

It follows that $R_\epsilon(t) > 0$ for all $t > t_0$ and all $\epsilon > 0$. Taking $\epsilon_{L_t} := \delta_1(L_t) - \delta_{2,K}(L_t)$ and noting that this quantity is always positive, it follows from Theorem 3 and 4 that $c_1(L_t)$ admits cscK metrics for all $t > t_0$. By possibly rescaling $L$ if necessary, it follows that for every ample line bundle $L$ on $X$, there is $\epsilon_{X,L} > 0$ such that for all $0 < \epsilon < \epsilon_{X,L}$, there exists a unique cscK metric in the Kähler class $-c_1(X) + \epsilon c_1(L)$. This is what we wanted to prove.

Since it is also clear that the line bundles can be replaced everywhere with arbitrary nef or Kähler $(1,1)$-cohomology classes, we obtain precisely the statement of Theorem 4 in the introduction.

Acknowledgements. We are grateful to Jian Song and Yalong Shi for helpful comments and discussions.
REFERENCES

[1] C. Arezzo and F. Pacard, Blowing up and desingularizing constant scalar curvature Kähler manifolds, Acta Math. 196 (2006), no. 2, 179-228.
[2] R. Berman and S. Boucksom and V. Guedj and A. Zeriahi, A variational approach to complex Monge-Ampère equations, Publ. Math. de l'IHES 117 (2013), 179-245.
[3] E. Calabi, Extremal Kähler metrics, Seminar on Differential Geometry, Ann. of Math. Stud., 102, Princeton Univ. Press (1982), 259-300.
[4] G. Chen, On J-equation, Preprint [arXiv:1905.10222]
[5] X.X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics, existence results, arXiv:1801.00656 (2018).
[6] X.X. Chen, On the lower bound of the Mabuchi K-energy and its application, Int. Math. Res. Not. 12 (2000), 607-623.
[7] T. Collins and G. Székelyhidi, Convergence of the J-flow on toric manifolds, J. Diff. Geom. 107 (2017) no. 1, 47-81.
[8] T. Darvas, The Mabuchi completion of the space of Kähler potentials, Amer. J. Math., 10.1353/ajm.2017.0032 (2014).
[9] T. Darvas, The Mabuchi Geometry of finite energy classes, Adv. Math. 285 (2015), 182-219.
[10] T. Darvas, Geometric pluripotential theory on Kähler manifolds, Survey article (2017).
[11] T. Darvas, Weak geodesic rays in the space of Kähler potentials and the class $E(X,\omega_0)$, J. Inst. Math. Jussieu 16 (2017), no. 4, 837-858.
[12] S.K. Donaldson, Moment maps and diffeomorphisms, Asian J. Math., 3 (1999) no. 1, pp. 1-15.
[13] S.K. Donaldson, Moment maps and diffeomorphisms, Surveys in differential geometry, 107-127, Surv. Differ. Geom., 7, Int. Press, Somerville, MA, 2000.
[14] W. Jian, Y. Shi and J. Song, A remark on constant scalar curvature Kähler metrics on minimal models, Proc. Amer. Math. Soc. 147 (2019), 3507-3513.
[15] T. Mabuchi, A functional integrating Futaki invariants, Proc. Japan Acad. 61 (1985), 119-120.
[16] T. Mabuchi, K-energy maps integrating Futaki invariants, Tohoku Math. J. 38 (1986), no. 4, 575-593.
[17] T. Mabuchi, Some symplectic geometry on compact Kähler manifolds I, Osaka J. Math. 24 (1987), 227-252.
[18] Z. Sjöström Dyrefelt, Optimal lower bounds for Donaldson’s J-functional, Preprint [arXiv:1907.01486] (2019).
[19] J. Song and B. Weinkove, On the convergence and singularities of the J-flow with applications to the Mabuchi energy, Comm. Pure Appl. Math., 61 (2008), pp. 210 - 229.
[20] G. Tian, Kähler-Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997), no. 1, 1-37.
[21] B. Weinkove, Convergence of the J-flow on Kähler surfaces, Comm. Anal. Geom. 12, no. 4 (2004), 949-965.
[22] B. Weinkove, On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy, preprint [math.DG/0309404]

The Abdus Salam International Centre for Theoretical Physics (ICTP), Str. Costiera, 11, 34151 Trieste TS, Italy.
E-mail address: zsjostro@ictp.it