Geometric Thermodynamics of Schwarzschild-AdS black hole with a Cosmological Constant as State Variable

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Abstract

The thermodynamics of the Schwarzschild-AdS black hole is reformulated within the context of the recently developed formalism of geometrothermodynamics (GTD). Different choices of the metric in the equilibrium states manifold are used in order to reproduce the Hawking-Page phase transition as a divergence of the thermodynamical curvature scalar. We show that the enthalpy and total energy representations of GTD does not reproduce the transition while the entropy representation gives the expected behavior.

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I. INTRODUCTION

The thermodynamics of black holes has been studied extensively since the work of Hawking [1]. The notion of critical behavior has arisen in several contexts for black holes, ranging from the Hawking-Page [2] phase transition in anti-de-Sitter space and the pioneering work by Davies [3] on the thermodynamics of Kerr-Newman black holes, to the idea that the extremal limit of various black hole families might themselves be regarded as genuine critical points [4–6]. In most treatments of black-hole thermodynamics the cosmological constant, $\Lambda$, is treated as a fixed parameter (possibly zero) but it has been considered as a dynamical variable in [7, 8] and it has further been suggested that it is better to consider $\Lambda$ as a thermodynamic variable, [9–13]. Physically, $\Lambda$ is interpreted as a thermodynamic pressure in [14, 15], consistent with the observation in [16–18] that the conjugate thermodynamic variable is proportional to a volume.

Now, the use of geometry in statistical mechanics was pioneered by Ruppeiner [19] and Weinhold [20], who suggested that the curvature of a metric defined on the space of parameters of a statistical mechanical theory could provide information about the phase structure. However, some puzzling anomalies become apparent when these methods are applied to the study of black hole thermodynamics. A possible resolution was suggested by Quevedo’s geometrothermodynamics (GTD) whose starting point [21] was the observation that standard thermodynamics was invariant with respect to Legendre transformations, since one expects consistent results whatever starting potential one takes. The formalism of GTD indicates that phase transitions occur at those points where the thermodynamic curvature is singular, but the results of Quevedo also show that the metric structure of the phase manifold determines the type of systems that can be described by a specific thermodynamic metric. For example, an Euclidean structure describes systems with first order phase transitions, whereas a pseudo-Euclidean structure describes systems with second order phase transitions. Actually, there is no complete explanation for this result but it is clear that the phase manifold contains information about thermodynamic systems.

In this paper we apply the GTD formalism to the Schwarzschild-AdS black hole to investigate the behavior of the thermodynamical curvature. As is well known, a black hole with a positive cosmological constant has both a cosmological horizon and an event horizon. These have different Hawking temperatures associated with them in general, which compli-
cates any thermodynamical treatment. Therefore we will focus on the case of a negative cosmological constant, though many of the conclusions are applicable to the positive $\Lambda$ case. Even more, the negative $\Lambda$ case is of interest for studies on AdS/CFT correspondence and the considerations here are likely to be relevant in those studies. Trying with different thermodynamical potentials and choices of the metric in the equilibrium states manifold we will show that the entropy representation of GTD is the one that reproduces the Hawking-Page phase transition as a divergence in the thermodynamical curvature scalar and it appears to be a second order transition.

II. GEOMETROTHERMODYNAMICS IN BRIEF

The formulation of GTD is based on the use of contact geometry as a framework for thermodynamics. Consider the $(2n + 1)$-dimensional thermodynamic phase space $\mathcal{T}$ coordinatized by the thermodynamic potential $\Phi$, extensive variables $E^a$, and intensive variables $I^a$, with $a = 1, ..., n$. Let us define on $\mathcal{T}$ a non-degenerate metric $G = G(Z^A)$, with $Z^A = \{\Phi, E^a, I^a\}$, and the Gibbs 1-form $\Theta = d\Phi - \delta_{ab}I^a dE^b$, with $\delta_{ab} = \text{diag}(1, 1, ..., 1)$. If the condition $\Theta \wedge (d\Theta)^n \neq 0$ is satisfied, the set $(\mathcal{T}, \Theta, G)$ defines a contact Riemannian manifold. The Gibbs 1-form is invariant with respect to Legendre transformations, while the metric $G$ is Legendre invariant if its functional dependence on $Z^A$ does not change under a Legendre transformation. Legendre invariance guarantees that the geometric properties of $G$ do not depend on the thermodynamic potential used in its construction.

The $n$-dimensional subspace $\mathcal{E} \subset \mathcal{T}$ is called the space of equilibrium thermodynamic states if it is determined by the smooth mapping

$$\varphi : \mathcal{E} \longrightarrow \mathcal{T}$$

$$(E^a) \longrightarrow (\Phi, E^a, I^a) \tag{1}$$

with $\Phi = \Phi(E^a)$, and the condition $\varphi^*(\Theta) = 0$ is satisfied, i.e.

$$d\Phi = \delta_{ab}I^a dE^b \tag{2}$$

$$\frac{\partial \Phi}{\partial E^a} = \delta_{ab}I^b. \tag{3}$$
The first of these equations corresponds to the first law of thermodynamics, whereas the second one is usually known as the condition for thermodynamic equilibrium (the intensive thermodynamic variables are dual to the extensive ones). The mapping \( \varphi \) defined above implies that the equation \( \Phi = \Phi(E^a) \) must be explicitly given. This relation is known as the fundamental equation from which all the equations of state can be derived. Finally, the second law of thermodynamics is equivalent to the convexity condition on the thermodynamic potential,

\[
\frac{\partial^2 \Phi}{\partial E^a \partial E^b} \geq 0.
\] (4)

The thermodynamic potential satisfies the homogeneity condition \( \Phi(\lambda E^a) = \lambda^\beta \Phi(E^a) \) for constant parameters \( \lambda \) and \( \beta \). Therefore, it satisfies the Euler’s identity,

\[
\beta \Phi(E^a) = \delta_{ab} I^b E^a,
\] (5)

and using the first law of thermodynamics, we obtain the Gibbs-Duhem relation,

\[
(1 - \beta) \delta_{ab} I^a dE^b + \delta_{ab} E^a dI^b = 0.
\] (6)

We also define a non-degenerate metric structure \( g \) on \( \mathcal{E} \), that is compatible with the metric \( G \) on \( \mathcal{T} \). Now we will formulate the main statement of geometrothermodynamics. A thermodynamic system is described by a metric \( G \) which is called a thermodynamic metric [21] if it is invariant with respect to transformations which do not modify the contact structure of \( \mathcal{T} \). In particular, \( G \) must be invariant with respect to Legendre transformations in order for GTD to be able to describe thermodynamic properties in terms of geometric concepts in a manner which must be invariant with respect to changes of the thermodynamic potential. In the language of GTD, a partial Legendre transformation is written as

\[
Z^A \rightarrow \tilde{Z}^A = \left\{ \tilde{\Phi}, \tilde{E}^a, \tilde{I}^a \right\}
\] (7)

where

\[
\begin{cases}
\Phi &= \tilde{\Phi} - \delta_{kl} \tilde{E}^k \tilde{I}^l \\
E^i &= -\tilde{I}^i \\
E^j &= \tilde{E}^i \\
I^i &= \tilde{E}^i \\
I^j &= \tilde{I}^j
\end{cases}
\] (8)
with $i \cup j$ any disjoint decomposition of the set of indices $\{1, 2, ..., n\}$ and $k, l = 1, ..., i$.

As is shown in [21], a Legendre invariant metric $G$ induces a Legendre invariant metric $g$ on $\mathcal{E}$ defined by the pullback $\varphi^*$ as $g = \varphi^*(G)$. There is a vast number of metrics on $\mathcal{T}$ that satisfy the Legendre invariance condition. The results of Quevedo et al. [22–24] show that phase transitions occur at those points where the thermodynamic curvature is singular and that the metric structure of the phase manifold $\mathcal{T}$ determines the type of systems that can be described by a specific thermodynamic metric. For instance, a pseudo-Euclidean structure of the form

$$G = \Theta^2 + (\delta_{ab}E^a I^b)(\eta_{cd}dE^c dI^d)$$  \hspace{1cm} (9)

with $\eta_{cd} = \text{diag} (-1, 1, 1, ..., 1)$, is Legendre invariant (because of the invariance of the Gibbs 1-form) and induces on $\mathcal{E}$ the metric

$$g = \left( E^f \frac{\partial \Phi}{\partial E^f} \right) \left( \eta_{ab} \delta^{bc} \frac{\partial^2 \Phi}{\partial E^c \partial E^d} dE^a dE^d \right)$$  \hspace{1cm} (10)

that describes systems characterized with second order phase transitions. If the structure is an Euclidean metric on the form

$$G = \Theta^2 + (\delta_{ab}E^a I^b)(\delta_{cd}dE^c dI^d)$$  \hspace{1cm} (11)

it is also Legendre invariant and induces on $\mathcal{E}$ the metric

$$g = \left( E^c \frac{\partial \Phi}{\partial E^c} \right) \left( \frac{\partial^2 \Phi}{\partial E^a \partial E^d} dE^a dE^d \right)$$  \hspace{1cm} (12)

that describes systems with first order phase transitions.

III. THE SCHWARZSCHILD-ADS BLACK HOLE

The Einstein action with cosmological constant $\Lambda$ term is given by

$$A = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( R - 2\Lambda \right) ,$$  \hspace{1cm} (13)

and the solution representing a static black hole is given by the Kerr-AdS solution

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$$
where

\[ f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 \]  

(14)

and \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2\). The horizons are given by the condition

\[ f(r_H) = 1 - \frac{2M}{r_H} - \frac{\Lambda}{3} r_H^2 = 0 \]  

(15)

or

\[ \frac{\Lambda}{3} r_H^3 + 2M - r_H = 0 \]  

(16)

In particular, for \(\Lambda < 0\), the largest positive root located at \(r = r_+\) defines the event horizon with an area

\[ A = 4\pi r_+^2. \]  

(17)

The Smarr formula for the Schwarzschild-AdS black hole gives the relation

\[ M = \sqrt{\frac{S}{4\pi}} \left( 1 - \frac{\Lambda S}{3\pi} \right) \]  

(18)

that corresponds to the fundamental thermodynamical equation \(M = M(S, \Lambda)\) that relates the total mass \(M\) of the black hole with the extensive variables entropy \(S = \frac{A}{4}\) and cosmological constant \(\Lambda\) and from which all the thermodynamical information can be derived.

In the geometric formulation of thermodynamics we will choose \(E^a = \{S, \Lambda\}\) and the corresponding intensive variables as \(I^a = \{T, \Psi\}\), where \(T\) is the temperature and \(\Psi\) is the generalized variable conjugate to the state parameter \(\Lambda\). In this way, the coordinates that we will use in the 5-dimensional thermodynamical space \(\mathcal{T}\) are \(Z^A = \{M, S, \Lambda, T, \Psi\}\). The contact structure of \(\mathcal{T}\) is generated by the 1-form

\[ \Theta = dM - TdS - \Psi d\Lambda. \]  

(19)

No we will apply the GTD formalism to the Schwarzschild-AdS black hole using \(M\) as the thermodynamical potential.
A. Mass Representation

To obtain the induced metric in the space of equilibrium states $\mathcal{E}$ we introduce the smooth mapping

$$\varphi : \{S, \Lambda\} \mapsto \{M(S, \Lambda), S\Lambda, T(S, \Lambda), \Psi(S, \Lambda)\}$$  \hspace{1cm} (20)

along with the condition $\varphi^*(\Theta) = 0$, that corresponds to the first law $dM = TdS + \Psi d\Lambda$. This condition also gives the relation between the different variables with the use of the fundamental relation \(^{[18]}\). The Hawking temperature is evaluated as

$$T = \frac{\partial M}{\partial S} = \frac{1}{4} \frac{1}{\pi S} \left(1 - \frac{\Lambda}{\pi S}\right),$$  \hspace{1cm} (21)

and the conjugate variable to $\Lambda$ is

$$\Psi = \frac{\partial M}{\partial \Lambda} = -\frac{S}{3\pi} \sqrt{\frac{S}{4\pi}}.$$  \hspace{1cm} (22)

The variable $\Psi$ has dimensions of a volume, in fact using \(^{[17]}\) we have $\Psi = -\frac{4\pi^3}{3}$ and it can be interpreted as an effective volume excluded by the horizon, or alternatively a regularised version of the difference in the total volume of space with and without the black hole present \(^{[14–16]}\). Since the cosmological constant $\Lambda$ behaves like a pressure and its conjugate variable as a volume, the term $\Psi d\Lambda$ has dimensions of energy and is the analogue of $VdP$ in the first law. This suggests that after expanding the set of thermodynamic variables to include the cosmological constant, the mass $M$ of the AdS black hole should be interpreted as the enthalpy rather than as the total energy of the spacetime. Therefore we are working the geometrothermodynamics in the enthalpy representation.

1. Second Order Phase Transitions

Using \(^{[9]}\), $\mathcal{T}$ becomes a Riemannian manifold by defining the metric

$$G = (dM - TdS - \Psi d\Lambda)^2 + (ST + \Psi \Lambda)(-dSdT + d\Lambda d\Psi).$$  \hspace{1cm} (23)

This metric has non-zero curvature and its determinant is $\det[G] = \frac{(ST + \Psi \Lambda)^4}{16}$. To obtain the induced metric in the space of equilibrium states $\mathcal{E}$ we use equation \(^{[10]}\), obtaining
\[ g = (SM_S + \Lambda M_\Lambda) \begin{pmatrix} -M_{SS} & 0 \\ 0 & M_{AA} \end{pmatrix}, \]  

(24)

where subscripts represent partial derivative with respect to the corresponding coordinate. However, note that the determinant of this metric is null,

\[
\det [g] = -M_{SS}M_{AA} (SM_S + \Lambda M_\Lambda) = 0,
\]

(25)

because \( M_{AA} = 0 \). Therefore, this is not a suitable metric for describing the thermodynamics of the black hole.

2. First Order Phase Transitions

Using (11), we define on \( \mathcal{T} \) the euclidean metric

\[
G = (dM - TdS - \Psi d\Lambda)^2 + (ST + \Psi \Lambda) (dSdT + d\Lambda d\Psi).
\]

(26)

This metric has non-zero curvature and its determinant is again \( \det [G] = \frac{(ST + \Psi \Lambda)^4}{16} \).

Equation (12) let us define the metric structure on \( \mathcal{E} \) as

\[
g = (SM_S + \Lambda M_\Lambda) \begin{pmatrix} M_{SS} & M_{SA} \\ M_{SA} & M_{AA} \end{pmatrix}.
\]

(27)

Note that the determinant of this metric is the non-null function

\[
\det [g] = (M_{SS}M_{AA} - M^2_{SA}) (SM_S + \Lambda M_\Lambda) = -M^2_{SA} (SM_S + \Lambda M_\Lambda).
\]

(28)

B. Total Energy Representation

Since \( M \) is the enthalpy of the black hole and \( Z^A = \{\Phi, E^a, I^a\} = \{M, S, \Lambda, T, \Psi\} \), \((a = 1, 2, 3)\), one can obtain the total energy \( U = U(M, S, \Phi) \) representation by performing the partial Legendre transformation

\[
Z^A \rightarrow \tilde{Z}^A = \{\tilde{\Phi}, \tilde{E}^a, \tilde{I}^a\}
\]

(29)

with
\[
\begin{align*}
\Phi &= \tilde{\Phi} - \tilde{E}^2 \tilde{I}^2 \\
E^2 &= -\tilde{I}^2 \\
I^2 &= \tilde{E}^2,
\end{align*}
\]

(30)

that corresponds to the transformation \( U = M - \Psi \Lambda \). The Gibbs 1-form becomes

\[
\Theta = d\tilde{\Phi} - \delta_{ab} \tilde{I}^a d\tilde{E}^b = dU - TdS + \Lambda d\Psi
\]

(31)

from which the first law,

\[
dU = TdS - \Lambda d\Psi,
\]

(32)

is obtained when considering the space of equilibrium states \( \mathcal{E} \). In the total energy representation, the metric (9) becomes

\[
G^U = (dU - TdS + \Lambda d\Psi)^2 + (ST + \Psi \Lambda)(-dSdT + d\Lambda d\Psi).
\]

(33)

Equation (10) let us define a metric structure on \( \mathcal{E} \) as

\[
g^U = (SU_S + \Psi U_\Psi) \begin{pmatrix}
-U_{SS} & 0 \\
0 & U_{\Psi\Psi}
\end{pmatrix},
\]

(34)

where subscripts represent partial derivative with respect to the corresponding coordinate. The determinant of this metric is

\[
\det [g^U] = -U_{SS}U_{\Psi\Psi} (SU_S + \Psi U_\Psi).
\]

(35)

However, note that the total energy is

\[
U = M - \Psi \Lambda = \sqrt{\frac{S}{4\pi}}
\]

(36)

and therefore \( U_{\Psi\Psi} = 0 \). This imply \( \det [g^U] = 0 \) and \( g^U \) is not suitable as a thermodynamical metric.
C. Entropy Representation

In the context of GTD, it is also possible to consider the entropy representation. In this case, the fundamental equation is given as $S = S(M, \Lambda)$ and the Gibbs 1-form of the phase space can be chosen as

$$\Theta^S = dS - \frac{1}{T} dM + \frac{\Psi}{T} d\Lambda.$$  \hfill (37)

The space of equilibrium states $\mathcal{E}$ we introduce the smooth mapping

$$\varphi^S : \{M, \Lambda\} \mapsto \{S(M, \Lambda), M, \Lambda, T(M, \Lambda), \Psi(M, \Lambda)\}$$  \hfill (38)

along with with the first law $\varphi^{S*}(\Theta^S) = 0$, which gives the conditions

$$\frac{1}{T} = \frac{\partial S}{\partial M}$$  \hfill (39)

and

$$\frac{\Psi}{T} = - \frac{\partial S}{\partial \Lambda}.$$  \hfill (40)

Using (39), we define the metric

$$G^S = \left( dS - \frac{1}{T} dM + \frac{\Psi}{T} d\Lambda \right)^2 + (ST + \Psi \Lambda) (-dSdT + d\Lambda d\Psi).$$  \hfill (41)

To obtain the induced metric in the space of equilibrium states $\mathcal{E}$ we use equation (41), that gives

$$g^S = (MS_M + \Lambda S_{\Lambda}) \begin{pmatrix} -S_{MM} & 0 \\ 0 & S_{\Lambda\Lambda} \end{pmatrix},$$  \hfill (42)

This time, the determinant of the metric is

$$\det \left[ g^S \right] = -S_{MM} S_{\Lambda\Lambda} (MS_M + \Lambda S_{\Lambda}).$$  \hfill (43)

IV. PHASE TRANSITIONS AND THE CURVATURE SCALAR

Phase transitions are an interesting subject in the study of black holes thermodynamics since there is no unanimity in their definition. As is well known, ordinary thermodynamics
defines phase transitions by looking for singular points in the behavior of thermodynamical variables. Following this argument, Davis \cite{3,25} show that the divergences in the heat capacity indicate phase transitions. For example, using equation (18) we have that the heat capacity for the Schwarzschild-AdS black hole is

\[
C = T \frac{\partial S}{\partial T} = \frac{M_S}{M_{SS}},
\]

(44)

\[
C = 2S \frac{(\Lambda S - \pi)}{(\Lambda S + \pi)},
\]

(45)

One can expect that phase transitions occur at the divergences of \(C\), i.e. at \(M_{SS} = 0\). For negative \(\Lambda\) the divergence of \(C\) corresponds to the well known Hawking-Page transition. In geometrothermodynamics the apparition of phase transitions is related with the divergences of the curvature scalar \(R\) in the space of equilibrium states \(E\). If we remember that \(R\) always contains the determinant of the metric \(g\) in the denominator, we conclude that the zeros of \(\text{det} [g]\) could lead to curvature singularities (if those zeros are not canceled by the zeros of the numerator).

We have considered four different options for the metric in the case of the Schwarzschild-AdS black hole. The first metric in the enthalpy representation, (24), has no inverse since its determinant (25) is always zero because \(M (S, \Lambda)\) is linear in \(\Lambda\). Therefore, this choice of metric is not good to represent the thermodynamical system. The second choice of metric, (27), has the determinant given in equation (28), that is proportional to \(M_{SA}\). However, this term is never zero and the metric predicts no phase transitions.

Using the total energy \(U\) as thermodynamical potential we define the metric given in (34) but its determinant (35) is null because \(U (S, \Psi)\) does not depend explicitly on \(\Psi\).

Finally, we present the entropy representation in which \(S = S (M, \Lambda)\) is the thermodynamical potential. This time the metric is chosen as given by equation (42) and its determinant, (43), is proportional to \(S_{M M}\) and \(S_{\Lambda \Lambda}\). This fact makes clear the coincidence with the divergence of the heat capacity. To see that, note that the heat capacity can be written in the entropy representation as

\[
C = T \frac{\partial S}{\partial T} = \frac{M_S}{M_{SS}} = -\frac{S_M^2}{S_{MM}},
\]

(46)
so the divergences of $C$ occur at $S_{MM} = 0$. Even more, the curvature scalar $R$ for the metric $g^S$ has the denominator

$$D = (MS_M + \Lambda S_\Lambda)^3 S_{MM}^2 S_{\Lambda\Lambda}^2$$  

(47)

which makes $R$ diverge when $S_{MM} = 0$, corresponding to the Hawking-Page phase transition [2]. Since we use a metric of the form given in (10) and following [24], we conclude that the divergence in the curvature scalar corresponds to a second order phase transition.

Note that the factor $S_{\Lambda\Lambda}$ also appears in the denominator of the curvature scalar, but as can be easily seen from (18), this term gives

$$S_{\Lambda\Lambda} = \frac{\partial^2 S}{\partial \Lambda^2} = \frac{2S^3(7\pi - 5\Lambda S)}{9(\pi - \Lambda S)^3}$$  

(48)

and therefore, for $\Lambda < 0$ it never becomes zero. This fact shows that the consideration of $\Lambda$ as a thermodynamical variable does not include new phase transitions in the system.

V. CONCLUSION

Quevedo’s geometrothermodynamics is a differential geometric formalism whose objective is to describe in an invariant manner the properties of thermodynamic systems using geometric concepts. It indicates that phase transitions would occur at those points where the thermodynamic curvature $R$ is singular. However the curvature scalar depends on the choice of the thermodynamical metric and as was shown in [24] the choices given in equations (10) and (12) apparently describe the second and first order phase transitions respectively.

In this work we applied the GTD formalism to the Schwarzschild-AdS black hole, considering the cosmological constant as a new thermodynamical state variable. In this approach, the mass of the black hole is interpreted as its total enthalpy and when we apply the GTD formalism we note that none of the chosen metrics describe phase transitions. Performing a Legendre transformation to use the total energy as the thermodynamical potential, we show that the curvature scalar does not present divergences. Finally, in the entropy representation we could obtain a curvature scalar that diverges exactly at the point where the Hawking-Page phase transition occurs. Since we use a metric of the form given in (10) we conclude that this is a second order phase transition. It is also important to note that the
consideration of \( \Lambda \) as a thermodynamical variable does not include new phase transitions in the system.

From the analysis above, it is clear that the phase manifold in the GTD formalism contains information about thermodynamic systems; however, it is necessary a further exploration of the geometric properties in order to understand where is encoded this information.

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