Necessary conditions for optimal boundary-control of couple linear parabolic partial differential equations

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Abstract: In this research, we will study the classical continuous optimal boundary-control problem (CCOPB-CPR) determine by couple linear PDEs of parabolic type (CCOPB-CCLPDEs) in details. the existence theorem for the uniqueness state vector solution (STVSO) of couple linear parabolic PDEs (CLPPDEs) for given (fixed) continuous classical boundary-control vector (CCB-CV) is considered and proved, the existence theorem of a continuous classical-boundary optimal control vector (CCOPB-CV) associated with the CLPPDEs is developed and proved, while the Fréchet derivative (Féde) of the objective function is derived, the theorem for the existence of a unique solution of the adjoint vector equations (ADVEQ) congruous for the STVSO is considered. Lastly the necessary optimality conditions (NOPC) of the CCOPB-CPR is proved.

1. Introduction

The subject of control theory has wide applications for many real life problems in science and engineering, for examples, an electric power [1], medicine [2], economic [3], biology [4], environmental engineering [5].

In the field of mathematical sciences, optimal control problems (OPCPR) are usually considered in general either by ODEs or by PDEs so as Gerdts in 2012 [6] and Chrysssoverghi in 2010 [7] respectively. In the recent years the studied of the continuous classical OPCPR (CCOPCPR) expended to deal with more general types of PDEs as: Al-Hawasy and Al-Rawdanee in 2014 studied CCOPCPR of coupled nonlinear elliptic PDEs (CONLEPDEs)[8], Al-Hawasy and Kadhem in 2016 studied CCOPCPR of coupled nonlinear parabolic PDEs (CONLPPDEs) [9], Al-Hawasy in 2016 studied CCOPCPR of coupled nonlinear hyperbolic PDEs (CONLHPDEs) [10]. While Al-Hawasy and Naeif in 2017 studied CCOPB-CPR of CONLPPDEs [11], Al-Hawasy and Al-Qaisi in 2017 studied CCOPB-CPR of CONLEPDEs [12] and Al-Hawasy in 2016 studied CCOPB-CPR of CONLHPDEs[13].

In present work, the existence of a unique solution theorem for the STVSO of CLPPDEs for given (fixed) CCB-CV is studied, the existence theorem of a CCOPB-CV associated with a CLPPDEs is developed and proved, also the Féde is derived, the theorem for the existence of a unique solution for the ADVEQ congruous for the STVSO is studied. Lastly the NOPC of the CCOPB-CPR is developed and proved.
2. Problem Statement
Let $I = (0, T)$, $T < \infty$ and $\Omega \subset \mathbb{R}^2$ be a bounded and open region with $\Gamma = \partial \Omega$, $Q = \Omega \times I$, $\Sigma = \partial \Omega \times I$.

The following CCOPB-CPR is considered:

The state equation is assumed by the LCPPDEs with the initial and boundary conditions:

The weak form (wekfm) of (1-6) is obtained through multiplying both sides of equations (1) & (4) and (2) & (6) by $v_1 \in V_1$ and $v_2 \in V_2$ respectively, then taking the integral for both sides with respect to $(w.r.e.t.)$ the space variable, and lastly applying the generalized Greens theorem for the terms which have the $2^nd$ derivative in the L.H.N.S.of the obtained equations from (1)&(2), they get:

The following Assumptions (A):

\begin{enumerate}
  \item $f_i$ satisfies the condition $|f_i(x, t)| \leq \lambda_i(x, t), \forall i, 1 \leq 2$, where $(x, t) \in Q, \lambda_i \in L^2(Q, \mathbb{R})$.
  \item $|a_i(t, y_i, \nabla v_i)| \leq a_i \|y_i\|_1 \|v_i\|_2, |b_i(t, y_i, v_i)| \geq \beta_i \|y_i\|_0, |b_i(t, y_i, v_i)| \leq \beta_i \|y_i\|_0 \|v_i\|_0, |b_i(t, y_i, v_i)| \leq \beta_i \|y_i\|_0 \|v_i\|_0, |b_i(t, y_i, v_i)| \leq \beta_i \|y_i\|_0 \|v_i\|_0, \quad \forall i = 1, 2$.
\end{enumerate}
Lemma (3.1) The Classical Bellman-Gronwall inequality [15]:
Assume \( \boldsymbol{\alpha} \) is non-negative constant, \( y(t) \) and \( g(t) \) are non-negative, continuous function on \( 0 \leq t \leq T \) satisfying the inequality
\[
y(t) \leq \eta + \int_0^T g(s) y(s) ds.
\]
Then \( y(t) \leq \eta \exp(\int_0^T g(s) ds) \), \( \forall t \in [0, T] \).

Theorem (3.2) [14]:
Let \( A(t) \) be a continuous \( n \times n \) matrix on some interval \( \emptyset \), let \( g(t) \) be a vector with \( n \) components continuous on the same interval. Then for every \( t_0 \) in \( \emptyset \) and every constant vector \( \eta \), the initial value problem
\[
y = A(t)y + g(t), \quad y(t_0) = \eta,
\]
has a unique solution existing on the same interval \( I \).

Theorem (3.2) [16]:
Consider the assumptions (A), then for any fixed B-CV \( \bar{u} \in L^2(\Sigma) \times L^2(Q) \), the Wekfm (8-9) has a unique solution \( \bar{y} = (y_1, y_2) \), s.t.
\[
\bar{y} \in (L^2(I, V))^2, \quad \bar{y}_t = (y_{1t}, y_{2t}) \in (L^2(I, V^*))^2.
\]

Proof:
Let \( \{ \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n \} \) be the basis of \( V_n \), then the Galerkin’s method is applied here to approximate the solution \( \bar{y} \) of the Wekfm (8-9) by the approximate solution \( \bar{y}_n \) with
\[
\bar{y}_n = \sum_{j=1}^{n} c_j(t) \bar{v}_j(x),
\]
where \( \bar{v}_j = (2 - \ell) v_{1k} (\ell - 1) v_{2k} \), and \( c_j = c_{ij} \) for \( k = 1, \ldots, N, \ell = 1, 2, j = k + N(\ell - 1) \) and the coefficients functions \( c_{ij}(t) \) are unknown.

Using (10) in (8-9), give that
\[
\begin{align*}
(y_{1n}, v_1) + a_1(t, y_{1n}, v_1) + (b_1(t)y_{2n}, v_1)\Omega &= (f_1, v_1) + (u_1, v_1) \\
(y_{1n}, v_1)\Omega &= (y_0, v_1)\Omega, \quad \forall v_1 \in V_n
\end{align*}
\]
and
\[
\begin{align*}
(y_{2n}, v_2) + a_2(t, y_{2n}, v_2) + (b_2(t)y_{1n}, v_2)\Omega &= (f_2, v_2) + (u_2, v_2) \\
(y_{2n}, v_2)\Omega &= (y_0, v_2)\Omega, \quad \forall v_2 \in V_n
\end{align*}
\]
where \( y_0(x) = y_{1n}(x, 0) \in V_n \subset V \subset L^2(\Omega) \) represents the projection of \( y_0 \) in \( L^2(\Omega) \), i.e.
\[
\begin{align*}
(y_{1n}, v_1)\Omega &= (y_0, v_1)\Omega \iff \|y_{1n} - y_0\|_0 \leq \|y_0 - v_i\|_0, \forall v_i \in V_n, \forall v_i = 1, 2, 3, 4
\end{align*}
\]
Substituting (10) in (11-12) with setting \( v_1 = v_{1t} \) and \( v_2 = v_{2t} \), the Wekfm can be written as the following linear system of ordinary differential equations with its initial conditions, i.e.
\[
\begin{align*}
A_1 \bar{C}_1(t) + D_1 \bar{C}_1(t) - E_1 \bar{C}_2(t) &= b_1 \\
A_1 \bar{C}_1(0) &= b_0 \\
A_2 \bar{C}_2(t) + D_2 \bar{C}_2(t) + E_2 \bar{C}_1(t) &= b_2
\end{align*}
\]
\[ A_2 C_2(0) = b_0^2 \]
where

\[ A_1 = (a_{ij})_{n \times n}, a_{ij} = (v_{1j}, v_{2i})_{\Omega}, D_1 = (d_{ij})_{n \times n}, d_{ij} = \left[ a_1(t, v_{1j}, v_{2i}) + (b_1(t) v_{1j}, v_{2i})_{\Omega} \right], \]

\[ E_1 = (e_{ij})_{n \times n}, e_{ij} = (b(t) v_{2j}, v_{1i})_{\Omega}, C_1(t) = (c_1(t))_{n \times 1}, C_1'(t) = (c_1'(t))_{n \times 1}, \]

\[ C_1(0) = (c_1(0))_{n \times 1}, b_1 = (b_1(t))_{n \times 1}, b_2 = (b_2(t))_{n \times 1}, b_4 = (b_4(t))_{n \times 1}, \]

\[ b_0 = (b_0(t))_{n \times 1}, b_0^0 = (y_1(0), v_{2i})_{\Omega}, \]

\[ b_2 = (b_2(t))_{n \times 1}, b_2 = (b_2(t))_{n \times 1}, b_4 = (b_4(t))_{n \times 1}, \]

The norms

\[ \|y(\cdot, t)\|_{L^0(\Omega)} \]

is bounded:

Since \( y_1^0 = y_0^0(x) \in L^2(\Omega), \) there exists \( v_1^0 \in V_n \), such that \( v_1^0 \rightarrow y_1^0 \) strongly in \( L^2(\Omega) \), and since \( \|y(\cdot, t) - y_1(\cdot, t)\|_{L^0(\Omega)} \leq \|y_1(\cdot, t) - y_1^0\|_{L^0(\Omega)}, \forall v_1 \in V \) then

\[ \|y(\cdot, t) - y_1(\cdot, t)\|_{L^0(\Omega)} \leq \|y_1(\cdot, t) - y_1^0\|_{L^0(\Omega)}, \\forall v_1 \in V \]

and then \( y(\cdot, t) \rightarrow y_1(\cdot, t) \) is strongly in \( L^2(\Omega) \) and then \( \|y(\cdot, t)\|_{L^0(\Omega)} = b_1 \).

By the same way the norm \( \|H(\cdot, t)\|_{L^0(\Omega)} \) is bounded.

The norm \( \|\tilde{y}_n(\cdot, t)\|_{L^0(\Omega)} \) and \( \|\tilde{y}_n(\cdot, t)\|_{Q} \) are bounded:

Setting \( v_1 = y_{1n} \) and \( v_2 = y_{2n} \) in (1.1a) and (1.2a) respectively, integrating w.r.t. \( t \) on \([0, T]\), adding the obtain equations and using assumption (A-iii), one gets

\[ f_0^T V_{1n} \cdot y_{1n} dt + f_0^T V_{2n} \cdot y_{2n} dt + \alpha_0 f_0^T \|\tilde{y}_n\|_{Q} dt \leq f_0^T (f_1, y_{1n})_{\Omega} dt + f_0^T (u_1, y_{1n})_{\Omega} dt + f_0^T (f_2, y_{2n})_{\Omega} dt + f_0^T (u_2, y_{2n})_{\Omega} dt \]

where \( \|\tilde{y}_n(\cdot, t)\|_{L^0(\Omega)} \leq h \).
Assume the sequence of the subspaces $\{\mathcal{V}_n\}_{n=1}^\infty$ of the space $\mathcal{V}$, for which $\forall \mathbf{v} = (v_1, v_2) \in \mathcal{V}$, there exists a sequence $\{\mathbf{v}_n\}$ with $\mathbf{v}_n = (v_{1n}, v_{2n}) \in \mathcal{V}$ for all $n$, and $\mathbf{v}_n \to \mathbf{v}$ strongly in $\mathcal{V}$; $\mathbf{v}_n \to \mathbf{v}$ strongly in $(L^2(\Omega))^2$. 

Since for each $n$, with $\mathcal{V}_n \subset \mathcal{V}$, problems (11-12) has a unique solution $(y_{1n}, y_{2n})$, hence congruous to $\{\mathcal{V}_n\}_{n=1}^\infty$, there are a sequence of approximation problems like (11-12) with substituting $\mathbf{v} = \mathbf{v}_n = (v_{1n}, v_{2n})$ for $n = 1, 2, \ldots$, once get 

$$
(y_{1n}, y_{2n}) + a_2(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n}, v_{2n})_\Omega = (f_2, v_{2n})_\Omega + (u_2, v_{2n})_\Omega, \forall v_{2n} \in V_n \text{ a.e. in } I
$$

(14a) and 

$$
(y_{1n}, y_{2n}) + a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n})_\Omega = (f_1, v_{1n})_\Omega + (u_1, v_{1n})_\Omega, \forall v_{1n} \in V_n
$$

(14b)

from the above steps once got that each of $\|\mathbf{y}_n\|_{L^2(Q)}$, $\|\mathbf{y}_n\|_{L^2(I, V)}$ is bounded, therefore $\{\mathbf{y}_n\}_{n \in \mathbb{N}}$ has a subsequence "for simplicity say once more" $\{\mathbf{y}_n\}_{n \in \mathbb{N}}$ (by employing the Alaoglu’s theorem (AlaTh)) such that $\mathbf{y}_n \to \mathbf{y}$ weakly in $(L^2(Q))^2$ and $\mathbf{y}_n \to \mathbf{y}$ weakly in $(L^2(I, V))^2$.

Multiplying both sides of (14a) and (15a) by $\phi_i(t) \in C^1[0, T]$, for $i = 1, 2$ such that $\phi_i(T) = 0$, $\phi_i(0) \neq 0$, taking the integral w.r.t. $t$ from 0 to $T$ for the both sides, using integration by parts (IBPs) for the $1^{st}$ expression in the L.H.N.S. of the obtained term, one has

$$
-\int_0^T (y_{1n}, v_{1n})_\Omega \phi_1(t) dt + \int_0^T a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n})_\Omega - (b(t)y_{2n}, v_{1n})_\Omega \phi_1(t) dt = \int_0^T (f_1, v_{1n})_\Omega \phi_1(t) dt + \int_0^T (u_1, v_{1n})_\Omega \phi_1(t) dt + (y_{1n}, v_{1n})_\Omega \phi_1(0)
$$

(16)

and

$$
-\int_0^T (y_{2n}, v_{2n})_\Omega \phi_2(t) dt + \int_0^T a_2(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n}, v_{2n})_\Omega + (b(t)y_{1n}, v_{2n})_\Omega \phi_2(t) dt = \int_0^T (f_2, v_{2n})_\Omega \phi_2(t) dt + \int_0^T (u_2, v_{2n})_\Omega \phi_2(t) dt + (y_{2n}, v_{2n})_\Omega \phi_2(0)
$$

(17)

since $v_{1n} \to v_1$ strongly in $V$ and $v_{2n} \to v_2$ strongly in $L^2(\Omega)$, then $v_{1n} \phi_1 \to v_1 \phi_1$ strongly in $L^2(I, V)$ and $v_{2n} \phi_2 \to v_2 \phi_2$ strongly in $L^2(Q)$ and in $L^2(\Sigma)$, and since $y_{1n} \to y_1$ weakly in $L^2(Q)$, also since $y_{2n} \to y_2$ strongly in $L^2(\Omega)$, $\forall i = 1, 2$, then we have the following convergences to get

$$
-\int_0^T (y_{1n}, v_{1n})_\Omega \phi_1(t) dt + \int_0^T a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n})_\Omega - (b(t)y_{2n}, v_{1n})_\Omega \phi_1(t) dt = 0
$$

(18)

and

$$
-\int_0^T (y_{2n}, v_{2n})_\Omega \phi_2(t) dt + \int_0^T a_2(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n}, v_{2n})_\Omega + (b(t)y_{1n}, v_{2n})_\Omega \phi_2(t) dt = 0
$$

(19)

Case 1: Choose $\phi_i \in D[0, T]$, i.e. $\phi_i(T) = 0$, $\forall i = 1, 2$, so (18) and (19), and using IBPs for the $1^{st}$ terms in the L.H.N.S. of the obtain equality, one obtains

$$
\int_0^T (b(t)y_{2n}, v_1)_\Omega \phi_1(t) dt = \int_0^T (f_1, v_1)_\Omega \phi_1(t) dt + \int_0^T (u_1, v_1)_\Omega \phi_1(t) dt
$$

and
\[ f_0^T (y_{2t}, v_2) \varphi_2 (t) dt + \int_0^T \left[ a_2(t, y_{2t}, v_1) \varphi_2 (t) dt + \int_0^T (b_2(t)y_{2t}, v_2) \varphi_2 (t) dt + \int_0^T (b(t)y_{1t}, v_2) \varphi_2 (t) dt \right] dt + \int_0^T (u_2, v_2) \varphi_2 (t) dt \]

i.e., \((y_1, y_2)\) is a solution of the state equations (8 a)-(9 a).

Case 2: Choose \(\varphi_i \in C^1 [0, T]\), such that \(\varphi_i (T) = 0 \& \varphi_i (0) \neq 0, \forall i = 1, 2\).

Using IBPS for 1st expression in the L.H.N.S.of (20), the obtained equations is subtracted from (18) from, one obtains that

\[(y^o_{2t}, v_1)_{\Omega} = (y_1(0), v_1)_{\Omega} \Rightarrow \text{the initial condition (8 b) is hold.}\]

By the same way one will has the initial condition (9 b) holds.

The strong convergence for \(\bar{y}_n\):

By substituting \(v_1 = y_1\) and \(v_2 = y_{2n}\) in (8a) and (11a) respectively, and also substituting \(v_2 = y_2\) and \(v_2 = y_{2n}\) in (9a) and (12a) respectively, integrating the obtained four equations from 0 to \(T\), adding the 1st equation with the 3rd one, and the 2nd equation with 4th one, then using assumption (A (iii)), to get

\[ \int_0^T (f_1, y_{1n}) dt + \int_0^T (u_1, y_{1n}) dt + \int_0^T (f_2, y_{2n}) dt + \int_0^T (u_2, y_{2n}) dt \]

And

\[ \int_0^T (f_1, y_{1n}) dt + \int_0^T (u_1, y_{1n}) dt + \int_0^T (f_2, y_{2n}) dt + \int_0^T (u_2, y_{2n}) dt \]

Using Lemma(3.2) for the 1st expression in the L.H.N.S. of ((22)&(23)), they become

\[ \frac{2}{3} \| \bar{y}(T) \|_{\Omega}^2 - \frac{2}{3} \| \bar{y}(0) \|_{\Omega}^2 + \int_0^T D(t, \bar{y}_n, \bar{y}_n) dt = \int_0^T (f_1, y_{1n}) dt + \int_0^T (u_1, y_{1n}) dt + \int_0^T (f_2, y_{2n}) dt + \int_0^T (u_2, y_{2n}) dt \]

And

\[ \frac{2}{3} \| \bar{y}(T) \|_{\Omega}^2 - \frac{2}{3} \| \bar{y}(0) \|_{\Omega}^2 + \int_0^T D(t, \bar{y}_n, \bar{y}_n) dt = \int_0^T (f_1, y_{1n}) dt + \int_0^T (u_1, y_{1n}) dt + \int_0^T (f_2, y_{2n}) dt + \int_0^T (u_2, y_{2n}) dt \]

Now, consider the following inequality:

\[ \frac{2}{3} \| \bar{y}_n(T) - \bar{y}(T) \|_{\Omega}^2 - \frac{2}{3} \| \bar{y}_n(0) - \bar{y}(0) \|_{\Omega}^2 + \int_0^T D(t, \bar{y}_n - \bar{y}, \bar{y}_n - \bar{y}) dt = A_1 - A_2 - A_3 \]

Where

\[ A_1 = \frac{2}{3} \| \bar{y}_n(T) \|_{\Omega}^2 - \frac{2}{3} \| \bar{y}_n(0) \|_{\Omega}^2 + \int_0^T D(t, \bar{y}_n(T), \bar{y}_n(T)) dt \]

\[ A_2 = \frac{4}{3} \| \bar{y}_n(T), \bar{y}(T) \|_{\Omega} - \frac{4}{3} \| \bar{y}_n(0), \bar{y}(0) \|_{\Omega} + \int_0^T D(t, \bar{y}_n(T), \bar{y}(T)) dt \]

and

\[ A_3 = \frac{4}{3} \| \bar{y}(T), \bar{y}_n(T) - \bar{y}(T) \|_{\Omega} - \frac{4}{3} \| \bar{y}(0), \bar{y}_n(0) - \bar{y}(0) \|_{\Omega} + \int_0^T D(t, \bar{y}(T), \bar{y}_n(T) - \bar{y}(T)) dt \]

Since (see the above steps of the proof)

\[ \bar{y}_n(0) \to \bar{y}(0) \text{ strongly in } (L^2(\Omega))^2 \]

\[ \bar{y}_n(T) \to \bar{y}(T) \text{ strongly in } (L^2(\Omega))^2 \]

Then

\[ (\bar{y}(0), \bar{y}_n(0) - \bar{y}(0))_\Omega \to 0 \& (\bar{y}(T), \bar{y}_n(T) - \bar{y}(T))_\Omega \to 0 \]

\[ \| \bar{y}_n(T) - \bar{y}(T) \|_{\Omega} \to 0 \& \| \bar{y}_n(0) - \bar{y}(0) \|_{\Omega} \to 0 \]

and since \(\bar{y}_n \to \bar{y}\) weakly in \((L^2(\Omega))^2\), then

\[ \int_0^T D(t, \bar{y}(T), \bar{y}_n(T) - \bar{y}(T)) dt \to 0 \]
since \( y_n \to \hat{y} \) weakly in \((L^2(Q))^2\) and in \((L^2(I,V))^2\) then in \((L^2(\Sigma))^2\), one get that

\[
\int_0^t (f_1, y_{1n})_\Omega dt + \int_0^t (u_1, y_{1n})_\Gamma dt + \int_0^t (f_2, y_{2n})_\Omega dt + \int_0^t (u_2, y_{2n})_\Omega dt \to
\]

\[
\int_0^t (f_1, y_1)_\Omega dt + \int_0^t (u_1, y_{1n})_\Gamma dt + \int_0^t (f_2, y_2)_\Omega dt + \int_0^t (u_2, y_2)_\Omega dt
\]

(27f)

Now, taking \( n \to \infty \) for the both sides of (26), the following results are obtained:

1. In the L.H.N.S. of (26), the first two expressions convergence to zero (from (27d)) from

\[
\int (\tilde{f}_1, y_{1n})_\Omega dt + \int (u_1, y_{1n})_\Gamma dt + \int (\tilde{f}_2, y_{2n})_\Omega dt + \int (u_2, y_{2n})_\Omega dt
\]

from

\[
\int (f_1, y_1)_\Omega dt + \int (u_1, y_1)_\Gamma dt + \int (f_2, y_2)_\Omega dt + \int (u_2, y_2)_\Omega dt
\]

(27f)

2. Eq. (A_1) = (24)

\[
\int_0^t (f_1, y_{1n})_\Omega dt + \int_0^t (u_1, y_{1n})_\Gamma dt + \int_0^t (f_2, y_{2n})_\Omega dt + \int_0^t (u_2, y_{2n})_\Omega dt
\]

(27f)

3. Eq. (A_2) \to L.H.N.S. of (25) = \int_0^t f_1, y_1)_\Omega dt + \int_0^t (u_1, y_1)_\Gamma dt + \int_0^t (f_2, y_2)_\Omega dt + \int_0^t (u_2, y_2)_\Omega dt

4. The 1st two expressions in (A_3) convergences to zero (from (27c)), so as the last one term also tends to zero (from (27e)).

From the above convergences (1-4), the both sides of (26) give

\[
\int_0^t D(t, \tilde{y}_n - \hat{y}, \tilde{y}_n - \hat{y}) dt \to 0 , \text{ as } n \to \infty
\]

But assumption (A-iii), gives \( \int_0^T \| \tilde{y}_n - \hat{y} \|^2 dt \to 0 \Rightarrow \tilde{y}_n \to \hat{y} \) strongly in \(((L^2(I,V))^2)\).

**Uniqueness of the solution:**

Let \((y_1, y_2)\) and \((\tilde{y}_1, \tilde{y}_2)\) be two solutions of (8a) - (9a), and for the first components of the solutions \((y_1, \tilde{y}_1)\) and for the first components of the solution \(y_2\) and \(\tilde{y}_2\), we have

\[
(y_{1i}, v_1) + a_1(t, y_{1i}, v_1) + b_1(t)y_{2i} - (b(t)y_{2i}, v_1)_\Omega = (f_1, v_1)_\Omega + (u_1, v_1)_\Gamma
\]

\[
(\tilde{y}_{1i}, v_1) + a_1(t, \tilde{y}_{1i}, v_1) + b_1(t)\tilde{y}_{2i} - (b(t)\tilde{y}_{2i}, v_1)_\Omega = (f_1, v_1)_\Omega + (u_1, v_1)_\Gamma
\]

\[
(y_{2i}, v_2) + a_2(t, y_{2i}, v_2) + b_2(t)y_{2i} - (b(t)y_{2i}, v_2)_\Omega = (f_2, v_2)_\Omega + (u_2, v_2)_\Omega
\]

\[
(\tilde{y}_{2i}, v_2) + a_2(t, \tilde{y}_{2i}, v_2) + b_2(t)\tilde{y}_{2i} - (b(t)\tilde{y}_{2i}, v_2)_\Omega = (f_2, v_2)_\Omega + (u_2, v_2)_\Omega
\]

By subtracting the 2nd and the 4th equations from the 1st and the 3rd equations, then substituting \( v_i = y_i - \hat{y}_i \) for \( i = 1, 2 \) in the obtained two equations, then adding them, using Lemma (3.2) on the 1st term of the obtained equation. Finally using Assumption (A-iii) for the last term, to get

\[
\frac{1}{\pi^2} \int_0^l \frac{d}{dt} \| \tilde{y} - \hat{y} \|^2 + \frac{\alpha}{\pi^2} \| \tilde{y} - \hat{y} \|^2 \leq 0
\]

(28)

The 2nd expression of the L.H.N.S. of (28) is positive, taking the integral for both sides of (28) from 0 to t, once get

\[
\frac{1}{\pi^2} \int_0^t \| \tilde{y} - \hat{y} \|^2 dt \leq 0 \Rightarrow \| \tilde{y} - \hat{y} \|_0^2 \leq 0 \Rightarrow \| \tilde{y} - \hat{y} \|_0^2 = 0 , \forall t \in I
\]

Now, integrating equation (28) from 0 to \( T \), with using the initial conditions and the above result, one has

\[
\int_0^T \| \tilde{y} - \hat{y} \|^2 dt = 0 \Rightarrow \| \tilde{y} - \hat{y} \|^2_{L^2(I,V)} = 0 \Rightarrow \tilde{y} = \hat{y}
\]

4. Existence of a CCOPB-C

Before we state and prove the existence theorem of a OPB-C, it is necessary to deal with the following theorem and lemmas.

**Theorem (4.1):**

a-Consider all the assumptions in (A) are hold, the states \( \tilde{y} \) and \( \hat{y} + \Delta \tilde{y} \) are congruous to the controls \( \tilde{u} \) and \( \hat{u} + \Delta \tilde{u} \) respectively, where \( \tilde{u} \) and \( \Delta \tilde{u} \) are bounded in \( L^2(\Sigma) \times L^2(Q) \), then:
\[ \| \Delta \tilde{y} \|_{L^2(I; L^2(\Omega))} \leq M \| \Delta \tilde{u} \|_{\mathcal{S} \times Q}, \quad \| \Delta \tilde{y} \|_{L^2(I; V)} \leq M \| \Delta \tilde{u} \|_{\mathcal{S} \times Q}, \quad \| \Delta \tilde{y} \|_{L^2(I; V)} \leq M \| \Delta \tilde{u} \|_{\mathcal{S} \times Q} \]

b- Consider all the assumptions in (A) hold, the operator \( \tilde{u} \mapsto \tilde{y}_{\tilde{u}} \) from \( L^2(\Sigma) \times L^2(Q) \) into \( (L^2(\Sigma) \times L^2(Q))^2 \) or in to \( (L^2(I, V))^2 \) or into \( (L^2(Q))^2 \) is continuous.

**Proof:**

a- Let \( \tilde{u} = (u_1, u_2), \tilde{u} = (\tilde{u}_1, \tilde{u}_2) \in L^2(\Sigma) \times L^2(Q) \), and let \( \Delta \tilde{u} = \tilde{u} - \tilde{u} \), hence by theorem 3.2, there exist \( \tilde{y} = (y_1 = y_{u_1}, y_2 = y_{u_2}) \) and \( \tilde{y} = (\tilde{y}_{\tilde{u}_1}, \tilde{y}_{\tilde{u}_2}) \) solutions of \( (8.9) \). i.e.

\[
\begin{align*}
(y_{1t}, v_1) + a_1(t, y_1, v_1) + (b(t)y_2, v_1)_\Omega = (f_1, v_1)_\Omega + (u_1 + \Delta u_1, v_1)_\Gamma \\
(y_{2t}, v_2) + a_2(t, y_2, v_2) + (b(t)y_1, v_2)_\Omega = (f_2, v_2)_\Omega + (u_2 + \Delta u_2, v_2)_\Omega
\end{align*}
\]

(29a)

(29b)

(30a)

(30b)

Subtracting (8a&b) from (29a&b), and (9a&b) from (30a&b), then setting \( \Delta y_1 = y_1 - 1 \), and \( \Delta y_2 = y_2 - 2 \), respectively in the obtained two equations, to get

\[
\begin{align*}
(y_{1t}, v_1) + a_1(t, \Delta y_1, v_1) + (b(t)\Delta y_2, v_1)_\Omega = (\Delta u_1, v_1)_\Gamma \\
(y_{2t}, v_2) + a_2(t, \Delta y_2, v_2) + (b(t)\Delta y_1, v_2)_\Omega = (\Delta u_2, v_2)_\Omega
\end{align*}
\]

(31a)

(31b)

(32a)

(32b)

By substituting \( v_1 = \Delta y_1 \) and \( v_2 = \Delta y_2 \) into equations (31a) and (32a) respectively, using Lemma 3.2 for the 1st expression in the L.H.N.S. of each obtained equations, then adding the obtained two equations together, finally using Assumption (A-iii), to obtain

\[
\frac{1}{2} \int_0^t \| \Delta \tilde{y} \|^2 + \alpha \| \Delta \tilde{y} \|^2 \leq (\Delta u_1, \Delta y_1)_\Gamma + (\Delta u_2, \Delta y_2)_\Omega
\]

(33)

Since the 2nd term of (33) is positive, then taking the integral for both sides for \( t \) on \([0, t] \) of (33), gives

\[
\int_0^t \| \Delta \tilde{y} \|^2 + \alpha \| \Delta \tilde{y} \|^2 \leq \int_0^t (\Delta u_1, \Delta y_1)_\Gamma dt + \int_0^t (\Delta u_2, \Delta y_2)_\Omega dt
\]

Using the Cauchy Schwartz inequality, the Trace theorem of the R.H.N.S. of the above inequality, and using Lemma 3.1, gives that

\[
\| \Delta \tilde{y}(t) \|^2 \leq M \| \Delta \tilde{u} \|^2 \quad \text{for each} \quad t \in [0, T]
\]

Then

\[
\| \Delta \tilde{y} \|_{L^2(I; L^2(\Omega))} \leq M \| \Delta \tilde{u} \|_{\mathcal{S} \times Q}, \quad \text{and then} \quad \| \Delta \tilde{y} \|^2 \leq M^2 = TM^2.
\]

Using the same way which is used in the above steps for the R.H.N.S. of (33), with the upper bound of the integration \( t = T \), to get

\[
\| \Delta \tilde{y} \|_{L^2(I; V)} \leq M \| \Delta \tilde{u} \|_{\mathcal{S} \times Q}
\]

b- Let \( \Delta \tilde{u} = \tilde{u}_1 - \tilde{u}_2 \) and \( \Delta \tilde{y} = \tilde{y}_1 - \tilde{y}_2 \) where the states \( \tilde{y}_1 \) and \( \tilde{y}_2 \) are congruous to the controls \( \tilde{u}_1 \) and \( \tilde{u}_2 \), using part (a) of this theorem, to get that

\[
\| \Delta \tilde{u} \|_{L^2(I; L^2(\Omega))} \leq M \| \tilde{u}_1 - \tilde{u}_2 \|_{\mathcal{S} \times Q}
\]

Which means that \( \tilde{u} \mapsto \tilde{y} \) is a continuous Lipschitz operator from \( L^2(\Sigma) \times L^2(Q) \) to \( (L^2(\Omega), L^2(\Omega))^2 \).

The other results are obtained easily.

**Lemma (4.1) [8]:**

The cost function which is given by (7) is W.L.S.C.

**Lemma (4.2) [8]:**

The norm \( \| . \|_3 \) is strictly convex.
Theorem (4.2):
There exists a CCOPB-CV for the CCOPB problem, If the objective function (7) is coercive.

Proof:
From the coercivity of $G_0(\bar{u})$ and its non-negativity, the following minimizing sequence exists
\[
\{\bar{u}_k\} = \{(u_{1k}, u_{2k})\} \in \bar{W}_A, \forall k, \text{ such that:} \\
\lim_{n \to \infty} G_0(\bar{u}_k) = \inf_{\bar{u} \in \bar{W}_A} G_0(\bar{u}), \|\bar{u}_k\|_{\Sigma \times Q} \leq c, \forall k.
\]
By the AlaTh the sequence $\{\bar{u}_k\}$ has a subsequence say "for simplicity" $\{\bar{u}_k\}$ such that $\bar{u}_k \to \bar{u}$ weakly in $L^2(\Sigma) \times L^2(Q)$

Then by theorem (3.2), congruous to the to the sequence of the controls $\{\bar{u}_k\}$, there is a sequence of solutions $\{\bar{y}_k\}$ and and the norms $\|\bar{y}_k\|_{L^\infty(1,L^2(\Omega))}$, $\|\bar{y}_k\|_Q$ and $\|\bar{y}_k\|_{L^2(I,V)}$ are bounded, then again by the AlaTh the sequence of $\{\bar{y}_k\}$ has a subsequence say $\{\bar{y}_k\}$ such that
\[
\bar{y}_k \to \bar{y} \text{ weakly in the spaces} \left(L^\infty(1,L^2(\Omega))\right)^2, (L^2(Q))^2, \text{and in} \left(L^2(I, V)\right)^2.
\]
To show that the norm $\|\bar{y}_k\|_{L^2(I,V')}$ is bounded, the WekfM of state equations (8a) & (9a), can be rewritten as
\[
(y_{1kt}, v_1) = -a_1(t, y_{1k}, v_1) - (b_1(t)y_{1k}, v_1) + (f_1(v_1)) + (u_1)_{\Gamma} \\
(y_{2kt}, v_2) = -a_2(t, y_{2k}, v_2) - (b_2(t)y_{2k}, v_2) + (f_2(v_2)) + (u_2)_{\Omega}
\]
By adding the above two wekfm, taking the integral for both sides of the obtained wekfm w.r.t. $t$ on $[0, T]$, then taking the absolute value for both sides, and finally by Assumptions (A), one obtains
\[
\|\bar{y}_k\|_{L^2(I,V')} = \int_0^T \|\bar{y}_k\|_V dt \leq h_0, \forall \bar{y}_k \in V' \times V'
\]
Then by the AlaTh the sequence $\{\bar{y}_k\}$ has a subsequence "say" $\{\bar{y}_{k}\}$ such that
\[
\bar{y}_{k} \to \bar{y} \text{ weakly in the space} \left(L^2(I, V')\right)^2.
\]
Since for each $k$, $(y_{1k}, y_{2k})$ is a solution of the state equations, then
\[
(y_{1kt}, v_1) + a_1(t, y_{1k}, v_1) = (b_1(t)y_{1k}, v_1) + (f_1(v_1)) + (u_1)_{\Gamma} \tag{34}
\]
and
\[
(y_{2kt}, v_2) + a_2(t, y_{2k}, v_2) = (b_2(t)y_{2k}, v_2) + (f_2(v_2)) + (u_2)_{\Omega} \tag{35}
\]
Let $\varphi \in C^{1}[0, T]$, such that $\varphi(t) = 0, \forall t = 1, 2$. Now, rewriting the 1st expression in the L.H.N.S. of (34) & (35), multiplying their both sides of each obtained equality by $\varphi_1(t)$ and $\varphi_2(t)$ respectively, then taking the integral for both sides w.r.t. $t$ on $[0, T]$, using the same manner which is used in the proof of theorem (2.2) once can get that $(y_1 = y_{1u1}, y_2 = y_{1u2})$ is the solution of the state equations.

Since $G_0(\bar{u})$ is W.L.S.C. (from Lemma (4.1)), i.e. $G_0(\bar{u}) \leq \lim_{u_k \to \infty} \inf_{\bar{u}_k \in \bar{W}_A} G_0(\bar{u}_k)$

since $\bar{u}_k \to \bar{u}$ weakly in $L^2(\Sigma) \times L^2(Q)$ , then
\[
G_0(\bar{u}) \leq \lim_{u_k \to \infty} \inf_{\bar{u}_k \in \bar{W}_A} G_0(\bar{u}_k) = \lim_{u_k \to \infty} G_0(\bar{u}_k) = \inf_{\bar{u} \in \bar{W}_A} G_0(\bar{u})
\]
\[
\Rightarrow G_0(\bar{u}) = \min_{\bar{u}_k \in \bar{W}_A} G_0(\bar{u}_k) , \text{ then } \bar{u} \text{ is a CCOPB-CV .}
\]

5. The NOCP
In order to state the necessary theorem (conditions) for the CCOPB-CPR, we drive the Fède.

Theorem (5.1):
Consider the cost function $G_0(\bar{u})$ which is given by (7), and the following ADVEQ $(\tau_1, \tau_2) = (z_{1u1}, z_{2u1})$, (where $(y_1, y_2) = (y_{1u1}, y_{2u2})$) of the state equations (1-6) are given by
As in the state equations, the wekfm of the CADVEQ for each

\[ z_1(x, t) = 0 , \quad \text{in } \Omega \]  
\[ \frac{\partial z_1}{\partial n} = 0 \quad \text{on } \Sigma \]  
\[ z_2(x, t) = 0 , \quad \text{in } \Omega \]  
\[ \frac{\partial z_2}{\partial n} = 0 \quad \text{on } \Sigma \]

Then, the Fede of \( G_0 \) is given by

\[ (G_0(\vec{u}), \Delta \vec{u})_{\Sigma \times q} = (\bar{z} + \beta \vec{u}, \Delta \vec{u})_{\Sigma \times q} \]

**Proof:**

As in the state equations, the wekfm of the CADVEQ for each \( v_1, v_2 \in V \) are:

\[-z_{1t} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (a_{ij}(x, t) \frac{\partial z_1}{\partial x_j}) + b_1(x, t)z_1 + b(x, t)z_2 = (y_1 - y_{1d}), \quad \text{in } Q \]  
\[-z_{2t} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (b_{ij}(x, t) \frac{\partial z_2}{\partial x_j}) + b_2(x, t)z_2 - b(x, t)z_1 = (y_2 - y_{2d}), \quad \text{in } Q \]

Substituting

\[-\langle \vec{z}_1, \vec{u} \rangle + \langle b_1(\vec{z}_2), \vec{u} \rangle - \langle b_2(\vec{z}_2), \vec{u} \rangle = \langle \vec{y}_1 - \vec{y}_{1d}, \vec{u} \rangle \]

Now, substituting \( v_1 = z_1 \) and \( v_2 = z_2 \) in (31a) and (32a) respectively, to get

\[ \langle \Delta \vec{y}_1, \vec{u} \rangle + \langle b_1(\Delta \vec{z}_2), \vec{u} \rangle - \langle b_2(\Delta \vec{z}_2), \vec{u} \rangle = \langle \Delta \vec{y}_2, \vec{u} \rangle \]

Also, substituting \( v_1 = \Delta \vec{y}_1 \) and \( v_2 = \Delta \vec{y}_2 \) in (42) and (43) respectively, to get

\[ \langle \Delta \vec{y}_2, \vec{u} \rangle + \langle b_1(\Delta \vec{z}_2), \vec{u} \rangle - \langle b_2(\Delta \vec{z}_2), \vec{u} \rangle = \langle \Delta \vec{y}_1, \vec{u} \rangle \]

Now, integrating both sides of (44), (45), (46), (47), w.r.t. \( t \) from 0 to \( T \) using IBPs for the 1st expression of the L.H.N.S. of each obtained equality from (46) and (47), yield to

\[ J_0^T (\Delta \vec{y}_1, \vec{z}_1)dt + \int_0^T [a_1(t, \vec{y}_1, \vec{z}_1) + (b_1(t) \vec{z}_1)_{\vec{v}_1} + (b(t) \vec{z}_2)_{\vec{v}_1}]dt = \int_0^T (\Delta \vec{u}_1, \vec{z}_1)dt \]

\[ J_0^T (\Delta \vec{y}_2, \vec{z}_2)dt + \int_0^T [a_2(t, \vec{y}_2, \vec{z}_2) + (b_2(t) \vec{z}_2)_{\vec{v}_2} + (b(t) \vec{z}_1)_{\vec{v}_2}]dt = \int_0^T (\Delta \vec{u}_2, \vec{z}_2)dt \]

\[ J_0^T (\Delta \vec{y}_1, \vec{z}_1)dt + \int_0^T [a_1(t, \vec{y}_1, \vec{z}_1) + (b_1(t) \vec{y}_1 - b_2(t) \vec{y}_2)_{\vec{v}_2} + (b(t) \vec{y}_1)_{\vec{v}_2}]dt = \int_0^T (\Delta \vec{u}_1, \vec{y}_1)dt \]

\[ J_0^T (\Delta \vec{y}_2, \vec{z}_2)dt + \int_0^T [a_2(t, \vec{y}_2, \vec{z}_2) + (b_2(t) \vec{y}_2 - b_1(t) \vec{y}_1)_{\vec{v}_1} + (b(t) \vec{y}_2)_{\vec{v}_1}]dt = \int_0^T (\Delta \vec{u}_2, \vec{y}_2)dt \]

By subtracting (48) from (50) and (49) from (51), then adding the two obtained equations from to get

\[ J_0^T [(\Delta \vec{u}_1, \vec{y}_1)dt] + (\Delta \vec{u}_2, \vec{y}_2)dt = \int_0^T [(\vec{y}_1 - \vec{y}_{1d}, \Delta \vec{y}_1)_{\vec{v}_1} + (\vec{y}_2 - \vec{y}_{2d}, \Delta \vec{y}_2)_{\vec{v}_2}]dt \]

Now, adding (8a) and (31a) together and (9a) and (32a) together, one has that

\[ (\vec{y}_1 + \Delta \vec{y}_1, \vec{v}_1) + (b_1(t) \vec{y}_1 + b_2(t) \vec{y}_2, \vec{v}_1)_{\vec{v}_1} - (b(t) \vec{y}_2 + b(t) \vec{y}_1, \vec{v}_1)_{\vec{v}_1} = \left( f_1, \vec{v}_1 \right)_{\vec{v}_1} + (u_1 + \Delta u_1, v_1)_{\vec{v}_1}, \forall \vec{v}_1 \in V_1 \]

and

\[ (\vec{y}_2 + \Delta \vec{y}_2, \vec{v}_2) + (b_1(t) \vec{y}_2 + b_2(t) \vec{y}_1, \vec{v}_2)_{\vec{v}_2} - (b(t) \vec{y}_1 + b(t) \vec{y}_2, \vec{v}_2)_{\vec{v}_2} = \left( f_2, \vec{v}_2 \right)_{\vec{v}_2} + (u_2 + \Delta u_2, v_2)_{\vec{v}_2}, \forall \vec{v}_2 \in V_2 \]

Which means that, the controls \( u_1 + \Delta u_1 \) and \( u_2 + \Delta u_2 \) are given the solutions \( y_1 + \Delta y_1 \) and \( y_2 + \Delta y_2 \) of (53) and (54) respectively, hence

\[ G_0(\vec{u} + \Delta \vec{u}) - G_0(\vec{u}) = J_0^T \int_0^T (\vec{y}_1 - \vec{y}_{1d})_{\vec{v}_1} dt + \int_0^T \int_0^T \left( u_1 + \Delta u_1 \right)_{\vec{v}_1} \vec{y}_1 dt + \int_0^T \int_0^T (\vec{y}_2 - \vec{y}_{2d})_{\vec{v}_2} dt + \int_0^T \int_0^T \left( u_2 + \Delta u_2 \right)_{\vec{v}_2} \vec{y}_2 dt + \frac{1}{2} \left\| \Delta \vec{y} \right\|_0^2 + \frac{\beta}{2} \left\| \Delta \vec{u} \right\|_2^2 \]

Using (52) in the R.H.N.S. of the above equality, it becomes
\[ G_0(\bar{u} + \Delta \bar{u}) - G_0(\bar{u}) = (\bar{z} + p \bar{u}, \Delta \bar{u})_{\Sigma \times Q} + \epsilon(\Delta \bar{u}) \| \Delta \bar{u} \|_{\Sigma \times Q} \]  

where \( \epsilon(\Delta \bar{u}) \rightarrow 0 \), as \( \| \Delta \bar{u} \|_{\Sigma \times Q} \rightarrow 0 \)

from the definition of the Fède \( G_0 \), once get

\[ (G_0(\bar{u}), \Delta \bar{u})_{\Sigma \times Q} = (\bar{z} + p \bar{u}, \Delta \bar{u})_{\Sigma \times Q}. \]

**Theorem (4.2):**

The CCOPB-CV of the above problem is,

\[ G_0(\bar{u}) = \bar{z} + p \bar{u} = 0 \quad \text{with} \quad \bar{y} = \bar{y}_{\bar{u}} \quad \text{and} \quad \bar{z} = \bar{z}_{\bar{u}}. \]

**Proof:**

If \( \bar{u} \) is a CCOPB-CV, then

\[ G_0(\bar{u}) = \min_{\bar{v} \in \tilde{W}} G_0(\bar{v}) , \forall \bar{v} \in L^2(\Sigma) \times L^2(\Omega), \quad \text{i.e.} \quad G_0(\bar{u}) = 0 \]

\[ \Rightarrow \bar{z} + p \bar{u} = 0 \Rightarrow \bar{u}_1 = -\frac{z_1}{p} \quad \text{and} \quad \bar{u}_2 = -\frac{z_2}{p}. \]

The NOPC is

\[ (G_0(\bar{u}), \Delta \bar{u})_{\Sigma \times Q} \geq 0, \quad \Delta \bar{u} = \bar{w} - \bar{u} \]

\[ \Rightarrow (\bar{z} + p \bar{u}, \bar{w} - \bar{u})_{\Sigma \times Q} \geq 0 \]

\[ \Rightarrow (\bar{z} + p \bar{u}, \bar{u})_{\Sigma \times Q} \leq (\bar{z} + p \bar{u}, \bar{w})_{\Sigma \times Q} , \forall \bar{w} \in L^2(\Sigma) \times L^2(\Omega). \]

6. Conclusions

The Galerkin's method is suitable to prove the existence of a unique solution of CLPPDES when the CCBCV is fixed. The existence theorem of a CCOPB-CV for the considered CLPPDES is developed and proved. The existence and uniqueness solution of the CADVEQ associated with the considered CLPPDES is studied through derivation the Fède. The NOPC of the CCOPB-CPR is developed and proved.

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