ORBITAL STABILITY OF SOLITARY WAVES FOR
THE GENERALIZED CHOQUARD MODEL

VLADIMIR GEORGIEV, MIRKO TARULLI, AND GEORGE VENKOV

Abstract. We consider the generalized Choquard equation describing trapped electron gas in 3 dimensional case. The study of orbital stability of the energy minimizers (known as ground states) depends essentially in the local uniqueness of these minimizers. In equivalent way one can optimize the Gagliardo–Nirenberg inequality subject to the constraint fixing the $L^2$ norm. The uniqueness of the minimizers for the case $p = 2$, i.e. for the case of Hartree–Choquard is well known. The main difficulty for the case $p > 2$ is connected with possible lack of control on the $L^p$ norm of the minimizers.

1. Main results

The active study of the existence and qualitative behaviour of standing waves is motivated by the important question of stability/instability properties of these waves. Therefore, one has to justify the $H^1$-evolution dynamics of the corresponding Cauchy problem

\begin{equation}
\begin{aligned}
    i\partial_t u + \Delta u + I(|u|^p)|u|^{p-2}u &= 0, \\
    u(0, x) &= u_0(x)
\end{aligned}
\end{equation}

and then to approach orbital stability/instability problem. Here and below $I(f)$ is the Riesz potential defined by

\begin{equation}
    I(f)(x) = (-\Delta)^{-1}f(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)dy}{|x - y|},
\end{equation}

Key words: generalized Choquard equation, local uniqueness, ground states

AMS Subject Classifications: 37K40, 35Q55, 35Q51

The first author was supported in part by INDAM, GNAMPA - Gruppo Nazionale per l’Analisi Matematica, la Probabilita e le loro Applicazioni, by Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, by Top Global University Project, Waseda University, the Project PRA 2018 49 of University of Pisa and project “Dinamica di equazioni nonlineari dispersive”, ”Fondazione di Sardegna”, 2016.
In general the existence of ground state is studied in [2], [9], [10] and decay and scattering properties in [12]. A detailed classification result for linearized stability properties of the standing waves is obtained in [3]. Considering linearization of (1.1) around standing waves, one can apply the classification results from [3] and deduce that linearized orbital stability holds for \( p \in (5/3, 7/3) \), while linearized orbital instability is fulfilled for \( p \in [7/3, 5) \). The notion of orbital stability and the verification that the nonlinear evolution based on (1.1) is well-defined and gives orbitally stable dynamics for \( p \in (5/3, 7/3) \), depend essentially on the local uniqueness of standing waves. More precisely, the standing waves are related to the minimization problem

\[
E_\sigma = \inf_{u \in H^1, \|u\|_{L^2} = \sigma} E_p(u). 
\]

Here and below

\[
E_p(u) = \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{1}{2p} D(|u|^p, |u|^p), 
\]

where

\[
D(|u|^p, |u|^p) = \langle I(|u|^p), |u|^p \rangle_{L^2} = \|(-\Delta)^{1/2}|u|^p\|_{L^2}^2. 
\]

Any minimizer of (1.4) satisfies the Pohozaev identity

\[
\frac{\|\nabla u\|^2}{3p - 5} = \frac{D(|u|^p, |u|^p)}{2p} 
\]

and it is a solution to the Euler–Lagrange equation

\[
-\Delta u + \omega u = I(|u|^p)|u|^{p-2}u, 
\]

where \( \omega > 0 \) is the Lagrange multiplier. Then we can write the following Pohozaev normalization conditions

\[
\frac{\omega \|u\|^2}{\beta} = \frac{\|\nabla u\|^2}{\gamma} = \frac{D(|u|^p, |u|^p)}{p} = k_\sigma, 
\]

where

\[
\beta = \frac{5 - p}{2}, \quad \gamma = \frac{3p - 5}{2} = p - \beta. 
\]

We start with the following simple property.

**Lemma 1.1.** Assume \( p \in (5/3, 7/3) \) and \( u \) is a minimizer of (1.3). Then we have the following conditions:

- \( u \) satisfies the Euler–Lagrange equation (1.6) with

\[
\omega = \frac{2\beta}{\gamma - 1} \frac{E_\sigma}{\sigma}; 
\]
we have the Pohozaev normalization conditions (1.7) with

\[ k_\varepsilon = \frac{2\varepsilon_\sigma}{\gamma - 1}. \]

We introduce the space

\[ H^1_{rad} = \{ u \in H^1(\mathbb{R}^3); v(x) = v(|x|) \} \]

and state our main result, which treats the local uniqueness of minimizers \( Q \) of (1.3).

**Theorem 1.** Assume \( 2 \leq p < 7/3 \). Then one can find \( \varepsilon > 0 \) so that for any two radial positive minimizers \( Q_1, Q_2 \in H^1_{rad} \) of (1.3), satisfying

\[ \|Q_1 - Q_2\|_{H^1} \leq \varepsilon, \]

we have \( Q_1 = Q_2 \).

The classical case \( p = 2 \) has been studied in \([7]\), the approach is based on shooting method and the fact that the Riesz potential behaves like

\[ I(|u|^2)(x) = \frac{\|u\|^2_{L^2}}{4\pi|x|} + o\left(|x|^{-1}\right), \quad x \to \infty \]

so that Pohozaev normalization conditions (1.7) in this case become

\[ \frac{\omega \|u\|^2}{3} = \|\nabla u\|^2 = \frac{D(|u|^2, |u|^2)}{4}. \]

Indeed, taking any two solutions \( u_1, u_2 \), we use the previous normalization conditions and from (1.11) we deduce

\[ I(|u_1|^2)(x) - I(|u_2|^2)(x) = o\left(|x|^{-1}\right), \quad x \to \infty \]

and this gives the possibility to apply Sturm argument and follow shooting method to deduce uniqueness. If \( p \neq 2 \), then (1.11) becomes

\[ I(|u|^p)(x) = \frac{1}{4\pi} \frac{\|u\|_{L^p}^p}{|x|} + o\left(|x|^{-1}\right), \quad x \to \infty \]

and obviously we lose the control on the asymptotics of Riesz potential at infinity, since in this case the \( L^p \) norm is not presented in Pohozaev normalization conditions (1.7).

There are different methods to prove the uniqueness of positive radial minimizers of nonlinear elliptic equations with local type nonlinearities. The method of McLeod and Serin \([8]\) and the subsequent refinements due to Kwong \([6]\) are also based on Sturm oscillation argument and therefore they work effectively for local type nonlinearities. In our case the nonlinearities involve the nonlocal Riesz potential and consequently we have met essential difficulties to follow this strategy.
Alternative method to show uniqueness of minimizer for Weinstein functionals have been proposed in [1] for the case of local type nonlinearity by studying

\[ \frac{\|u\|_{L^2}^{5-p} \|\nabla u\|_{L^2}^{3p-5}}{D(|u|^p, |u|^p)}. \]

Performing the substitution of \( u \) by \( Q + \varepsilon h \) and making a Taylor expansion of the above quotient near \( \varepsilon = 0 \), one can reduce the local existence result to the proof that the operator

\[ L_+ = -\Delta + \omega - pI(Q^{p-1})Q^{p-1} - (p-1)I(Q^p)Q^{p-2}, \]

has a unique negative eigenvalue and a kernel of dimension not greater than 2. However, the lack of Sturm comparison argument for nonlocal ODE causes essential difficulties to show the non-degeneracy of \( L_+ \), i.e. to check that the kernel of \( L_+ \) on \( H^1_{rad} \) is trivial. Our approach to obtain the local uniqueness of the minimizer might allow degeneracy of \( L_+ \), but the local uniqueness is based on the appropriate analytic continuation \( K(z) \) of the function

\[ K : \varepsilon \rightarrow E_p \left( \sqrt{\sigma} \frac{Q + \varepsilon h}{\|Q + \varepsilon h\|_{L^2}} \right), \]

where \( h \in H^1_{rad} \) is a nontrivial element in the kernel of \( L_+ \). The crucial point is to show the identity \( K(z) = K(0) \) for \( z \) in the domain of analyticity of \( K(z) \) and to find a suitable curve \( z = z(R), R > 0 \) in this domain so that

\[ \lim_{R \to \infty} K(z(R)) = E_p(\sqrt{\sigma} h). \]

Another question we shall treat in this work is the characterization of the optimal constant \( C_* \) in the Gagliardo–Nirenberg inequality

\[ (1.12) \quad D(|u|^p, |u|^p) \leq C_* \|u\|_{L^2}^{5-p} \|\nabla u\|_{L^2}^{3p-5}. \]

Choosing \( C_* > 0 \) to be the best constant in this inequality, we consider the minimization problem

\[ (1.13) \quad F_\sigma = \inf_{u \in H^1, \|u\|_{L^2}^2 = \sigma} F_p(u), \]

where

\[ (1.14) \quad F_p(u) = \|u\|_{L^2}^{5-p} \|\nabla u\|_{L^2}^{3p-5} - \frac{1}{C_*} D(|u|^p, |u|^p). \]

We focus our interest to show (at least for \( 5/3 < p < 7/3 \)) that the minimizers of \( (1.13) \) are minimizers of \( (1.13) \). To give an answer to this question we start with some properties of the minimizers of \( (1.13) \). More precisely, we have the following result.
Lemma 1.2. Assume $\sigma > 0$ and $\omega > 0$, defined by

\begin{equation}
\omega^{1-\gamma} = \frac{C_*}{p} \frac{\gamma^\gamma}{\beta^{\gamma-1} \sigma^{p-1}}.
\end{equation}

If $u$ is a minimizer of $(1.13)$, then the following conditions are equivalent:

i):

\begin{equation}
\frac{\|\nabla u\|^2}{\gamma} = \frac{D(|u|^p, |u|^p)}{p};
\end{equation}

ii):

\begin{equation}
\frac{\omega \sigma}{\beta} = \frac{D(|u|^p, |u|^p)}{p};
\end{equation}

iii): $u$ is a solution to the Euler–Lagrange equation $(1.6)$.

Definition 1. We shall say that the pair $(\sigma, \omega)$ is admissible for the problem $(1.3)$ if the relation $(1.9)$ is fulfilled.

Similarly, we shall say that the pair $(\sigma, \omega)$ is admissible for $(1.13)$ if $(1.15)$ holds.

Now we are ready to give an answer to the question about the link between the two minimizers.

Theorem 2. Assume $p \in (5/3, 7/3)$. Then the following conditions are equivalent:

a): $(\sigma, \omega)$ is admissible pair for $(1.3)$ and $u$ is a minimizer of $(1.3)$;

b): $(\sigma, \omega)$ is admissible pair for $(1.13)$ and $u$ is a minimizer of $(1.13)$.

1.1. Properties of $F_\sigma$, $E_\sigma$ and the link among them. We deal first with the Proof of Lemma 1.1. Namely we have the following

Proof of Lemma 1.1. It is easy to see, by calculating the first variation of the functional $(1.3)$, that any non-negative minimizer $Q = Q_\sigma \in H^1_{\text{rad}}$ of $(1.4)$ satisfies the Euler–Lagrange equation

\begin{equation}
-\Delta Q + \omega Q = I(Q^p)Q^{p-1},
\end{equation}

where $\omega = \omega(\sigma)$ is the Lagrange multiplier. In addition we have also the classical Pohozaev relations

\begin{equation}
\|\nabla Q\|_{L^2}^2 + \omega \|Q\|_{L^2}^2 - D(|Q|^p, |Q|^p) = 0,
\end{equation}

\begin{equation}
\frac{d}{dR} \left( E_p \left( R^{3/2} Q(Rx) \right) \right) \bigg|_{R=1} = 0
\end{equation}
and

\begin{equation}
E_p(Q) = \mathcal{E}_\sigma.
\end{equation}

Combining the relations (1.19) and (1.20), and taking into account that \( \|Q\|_{L^2}^2 = \sigma \), we can represent these relations as the following system:

\begin{align*}
\|\nabla Q\|_{L^2}^2 + \omega \sigma - D(|Q|^p, |Q|^p) &= 0, \\
\|\nabla Q\|_{L^2}^2 - \frac{3p - 5}{2p} D(|Q|^p, |Q|^p) &= 0, \\
\frac{1}{2} \|\nabla Q\|_{L^2}^2 - \frac{1}{2p} D(|Q|^p, |Q|^p) &= \mathcal{E}_\sigma.
\end{align*}

(1.22)

By solving these identities and using the notations (1.8), we achieve

\[ D(|Q|^p, |Q|^p) = \frac{\sigma \omega}{\beta} p, \]
\[ \|\nabla Q\|_{L^2}^2 = \frac{\sigma \omega}{\beta} \gamma, \]
\[ \mathcal{E}_\sigma = \frac{\sigma \omega}{2\beta} (\gamma - 1). \]

(1.23)

It is clear now that \( \omega > 0 \). Then, rearranging the last identity in (1.23) above, we arrive at (1.9). Furthermore, the equality (1.10) is a straightforward consequence of the first two identities in (1.23). The proof of the Lemma is now complete. \( \square \)

2. Proof of Theorem 1

Our goal is to show the local uniqueness of the minimizer \( Q \), associated to the minimization problem

\[ \mathcal{E}_\sigma = \inf_{u \in H^1, \|u\|_{L^2}^2 = \sigma} E_p(u), \]

where \( E_p \) is defined in (1.4). The first step is to reduce the local uniqueness to the directional local uniqueness. To be more precise, any vector \( u \) on the sphere \( \|u\|_{L^2}^2 = \sigma \) close to \( Q \) can be represented as

\[ u = \sqrt{\sigma}(Q + \varepsilon h)/\|Q + \varepsilon h\|_{L^2} \]

with \( h \in \{ Q \}^\perp = \{ g \in H^1; \langle g, Q \rangle_{L^2} = 0 \} \)

and \( \|h\|_{L^2} = 1 \). Without loss of generality we can assume

\[ Q(x) + \varepsilon h(x) > 0, \]

(2.1)

provided \( \varepsilon \in I \), where \( I \) is a small interval of type \([0, a]\) with sufficiently small \( a > 0 \).
The minimizer $Q$ will be called locally unique in direction $h$, if we can find $\varepsilon_0 = \varepsilon_0(h) > 0$ and an integer $M > 1$, so that
\begin{equation}
E_p \left( \sqrt{\sigma} \frac{Q + \varepsilon h}{\|Q + \varepsilon h\|_{L^2}} \right) - E_p(Q) \gtrsim \varepsilon^M,
\end{equation}
for any $\varepsilon \in (0, \varepsilon_0]$. We shall establish the directional local uniqueness in a way that $\varepsilon_0(h) > 0$ will be a continuous function when $h$ is restricted to 2-dimensional subspace. We argue by contradiction. If the minimizer $Q$ is not unique, then we can find sequences $\varepsilon_k \to 0$, $h_k \in T_Q$ so that
\begin{equation}
Q_k = \sqrt{\sigma}(Q + \varepsilon_k h_k)/\|Q + \varepsilon_k h_k\|_{L^2}
\end{equation}
is a solution to
\begin{equation}
(\omega - \Delta)Q_k = I(|Q_k|^p)|Q_k|^{p-2}Q_k.
\end{equation}
Rewriting this equation as
\begin{equation}
Q_k = (\omega - \Delta)^{-1} I(|Q_k|^p)|Q_k|^{p-2}Q_k
\end{equation}
and taking the limit $\varepsilon_k \to 0$, we obtain
\begin{equation}
h_k - \Phi(h_k) \to 0
\end{equation}
in $L^2$ with
\begin{equation}
\Phi(h) = (\omega - \Delta)^{-1} \left( (p - 1)I(|Q|^p)Q^{p-2}h + pI(Q^{p-1}h)Q^{p-1} \right)
\end{equation}
being a compact operator in $H^1$. Then $h_k$ is convergent on $L^2$ to $h$ and satisfies $L_+(h) = 0$. Therefore, it remains to show the directional local uniqueness for $h$ in the kernel of $L_+$. Note that this kernel has dimension at most 2 due to Lemma 5.2. If for $h \in \text{Ker}L_+$ the property (2.2) is not true, then we can find decreasing sequence $\varepsilon_k \to 0$, such that
\begin{equation}
0 \leq E_p \left( \sqrt{\sigma} \frac{Q + \varepsilon_k h}{\|Q + \varepsilon_k h\|_{L^2}} \right) - E_p(Q) \lesssim \varepsilon_k^M,
\end{equation}
for any $M > 1$. However, for any smooth function $F$, such that there exists a sequence $\varepsilon_k \to 0$, with the property $|F(\varepsilon_k) - F(0)| \lesssim |\varepsilon_k|^2$, one can assert that $F'(0) = 0$. In a similar way, if there exists an integer $M > 1$ and a sequence $\varepsilon_k \to 0$, such that $|F(\varepsilon_k) - F(0)| \lesssim |\varepsilon_k|^M$, then all derivatives of $F$ up to order $M - 1$ are identically zero. Therefore, (2.3) implies that all derivatives of the function
\begin{equation}
K : \varepsilon \to E_p \left( \sqrt{\sigma} \frac{Q + \varepsilon h}{\|Q + \varepsilon h\|_{L^2}} \right)
\end{equation}
at $\varepsilon = 0$ are identically zero. We have the relation
\begin{equation}
E_p \left( \sqrt{\sigma} \frac{Q + \varepsilon h}{\|Q + \varepsilon h\|_{L^2}} \right) =
\end{equation}
\[
\frac{\sigma(\|\nabla Q\|^2 + \varepsilon^2\|\nabla h\|^2)}{2(\sigma + \varepsilon^2)} - \frac{\sigma^p D(|Q + \varepsilon h|^p, |Q + \varepsilon h|^p)}{2p(\sigma + \varepsilon^2)^p}.
\]

This function can be extended as analytic function
\[
K : z \rightarrow E_p \left( \sqrt{\sigma \frac{Q + zh}{\sqrt{\|Q\|^2_{L^2} + z^2}}} \right), \quad z \in \mathbb{C},
\]
in a small neighborhood \(|z| < \delta\). We obviously have the analyticity of
\[
z \rightarrow \frac{\sigma(\|\nabla Q\|^2 + z^2\|\nabla h\|^2)}{2(\sigma + z^2)}
\]
near \(z = 0\). More delicate is the analyticity of the map
\[
z \rightarrow D((Q + zh)^p, (Q + zh)^p).
\]
In this case, we can apply Proposition \ref{analyticity} and use the estimate
\[
|h(r)|/Q(r) \leq C.
\]
Then \(\text{Re}(1 + zh(r)/Q(r)) > 1/2\) for \(|z|\) small and the function
\[
z \rightarrow \left(1 + z \frac{h(r)Q(r)}{Q(|x|)}\right)^p
\]
is analytic near the origin, so
\[
z \rightarrow \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(1 + z \frac{h(|x|)}{Q(|x|)}\right)^p \left(1 + z \frac{h(|y|)}{Q(|y|)}\right)^p \frac{Q(x)pQ(y)pdx dy}{|x - y|}
\]
is analytic near the origin. Moreover the property (3.4) enables one to have analytic extension of
\[
z \rightarrow D((Q + zh)^p, (Q + zh)^p)
\]
in the domain \(\{|z| \leq \delta\} \cup \{\text{Im} z > \delta/3, \text{Re} z > 0\}\).

The assumption (2.3) means that all derivatives of \(K(z)\) at \(z = 0\) are identically zero, so the function \(K(z)\) is a constant
\[
(2.4) \quad K(z) = K(0).
\]

Our next step is to show that \(K(z)\) can be extended as analytic function in
\[
\Omega_\delta = \{\text{Im} z > \delta/3, \quad \text{Re} z > \delta \text{ Im} z\}.
\]
Indeed, for fixed \(x \in \mathbb{R}^3\) the estimate \(|h(|x|)| \leq CQ(|x|)|\) implies
\[
\text{Re} \left( z \frac{h(|x|)}{Q(|x|)} \right) = (\text{Re} z) \frac{h(|x|)}{Q(|x|)} \geq -C\delta,
\]
for any \(z \in \Omega_\delta\). Then choosing \(\delta > 0\) small enough, we get
\[
(2.5) \quad \text{Re} \left( 1 + z \frac{h(|x|)}{Q(|x|)} \right) > \frac{1}{2}, \quad \forall z \in \Omega_\delta.
\]
Then we can define the principal value of the argument
\[ \text{Arg} \left( 1 + z \frac{h(|x|)}{Q(|x|)} \right), \]
as well the corresponding principal value of the Log and fractional
powers, so that
\[ \left( 1 + z \frac{h(|x|)}{Q(|x|)} \right)^p \]
is analytic in \( \Omega_\delta \). It is easier to show the analyticity of \( \text{Arg}(\sigma + z^2) \) on
\( \Omega_\delta \), since \( \text{Im}(\sigma + z^2) = 2(\text{Re}z)(\text{Im}z) > 0 \). Extending in this way \( K(z) \)
as analytic function in the strip \( \Omega_\delta \), we can extend the relation \( (2.4) \) in
the whole strip \( \Omega_\delta \).

Choosing \( z(R) = R + iR\delta \) with \( R \to \infty \), we can use the relation
\[ \frac{1 + z(R)h(|x|)/Q(|x|)}{\sqrt{\sigma + z(R)^2}} \to \frac{h(|x|)}{Q(|x|)}, \]
combined with Lebesgue dominated convergence theorem to conclude
that
\[ \lim_{R \to \infty} K(z(R)) = E_p(\sqrt{\sigma} \ h). \]
The relation
\[ E_p(\sqrt{\sigma} \ h) = K(0) = E_p(Q) \]
shows that \( u(|x|) = \sqrt{\sigma} \ h \) is a minimizer of \( E_p \), satisfying the constraint condition \( \|u\|_{L^2}^2 = \sigma \). Hence the same is true for \( |u(|x|)| \) and both of
them satisfy the equation
\[-\Delta u + \omega u = I(|u|^p)|u|^{p-2}u. \]
Since \( h \) is orthogonal to \( Q \), there exists \( r_0 > 0 \), such that \( h(r_0) = u(r_0) = 0 \). Therefore, we are in position to apply Lemma \( 5.1 \) and to
conclude that \( u(r) = 0 \) for any \( r > 0 \). This is an obvious contradiction
and shows that for any \( h \in \text{Ker}L_+ \), we can find \( \varepsilon_0 = \varepsilon_0(h) > 0 \), \( \delta_0 = \delta_0(h) > 0 \) and an integer \( M > 1 \), so that \( (2.2) \) is fulfilled for any \( \varepsilon \in (0,\varepsilon_0] \).

Recalling that Lemma \( 5.2 \) guarantees that the kernel of \( L_+ \) has di-
mension at most 2. Thus, we can show that there exists uniform \( \varepsilon_0 > 0 \),
such that for any \( h \) in the kernel of \( L_+ \) the property \( (2.2) \) is fulfilled
for \( \varepsilon \in (0,\varepsilon_0] \).

The last assertion can be verified by assuming the opposite and find-
ing a sequence \( h_k \to h^* \) in the unit sphere of \( T_Q^\perp \), such that the function
\[ z \to K^*(z) = E_p\left( \sqrt{\sigma} \frac{Q + zh^*}{\|Q + zh^*\|^2_{L^2}} \right) \]
has all derivatives equal to zero at the origin. As above, the analytic extension of $K^*(z)$ in $\Omega_\delta$ shows that $h^* = 0$ and this contradiction completes the proof.

3. Characterization of Gagliardo–Nirenberg optimal constant

We start this section by the simple observation that for any $\sigma > 0$, the minimization problem

$$\mathcal{F}_\sigma = \inf_{u \in H^1, \|u\|_{L^2}^2 = \sigma} F_p(u).$$

has infimum $\mathcal{F}_\sigma = 0$.

Proof of Lemma 1.2. The Pohozaev conditions for the minimizers of (1.13) have the form

$$\frac{\|\nabla u\|^2}{\gamma} = \frac{\omega \sigma}{\beta} = \frac{D(|u|^p, |u|^p)}{p}.$$  (3.1)

The assumption that $u$ is a minimizer of (1.13) has the meaning that the Gagliardo–Nirenberg equality

$$D(|u|^p, |u|^p) = C_* \|\nabla u\|^{2\gamma} \sigma^\beta$$

holds. Moreover, for any $\sigma > 0$, the Euler–Lagrange equation for minimizers of $\mathcal{F}_\sigma$ is

$$-\Delta u + \Lambda u = I(|u|^p)|u|^{p-2}u.$$  

First we note that iii) is equivalent to (3.1) and therefore $\Lambda = \omega$. Moreover, Gagliardo–Nirenberg equality combined with (3.1) give (1.15), so $(\sigma, \omega)$ is admissible pair for (1.13) and we have iii) $\Rightarrow$ i) and ii). From Gagliardo–Nirenberg equality, (1.15) and i) imply

$$D(|u|^p, |u|^p) = C_* \|\nabla u\|^{2\gamma} \sigma^\beta = \frac{p\|\nabla u\|^2}{\gamma},$$

then

$$\|\nabla u\|^{2(1-\gamma)} = \frac{C_* \gamma}{p} \sigma^\beta.$$  

Now (1.15) can be rewritten as

$$\frac{C_* \gamma}{p} \sigma^\beta = \left(\frac{\gamma \omega \sigma}{\beta}\right)^{1-\gamma}$$

and we arrive at (3.1) so we conclude that i) $\Rightarrow$ iii). In a similar way we check ii) $\Rightarrow$ iii). This completes the proof.

□
Our next step is to connect the minimizers of $\mathcal{F}_\sigma$ with the minimization problem
\begin{equation}
\mathcal{E}_\sigma = \inf_{u \in H^1, \|u\|_{L^2}^2 = \sigma} E_p(u).
\end{equation}

**Proof of Theorem 2 a) $\implies$ b):** If $(\sigma, \omega)$ is admissible pair for (1.3), then we have (1.15).

The plan is to assume that $u$ is a minimizer of (1.3) and to prove
\begin{equation}
D(|u|^p, |u|^p) = C_* \|u\|_{L^2}^{5-p} \|\nabla u\|_{L^2}^{3p-5}.
\end{equation}
For the purpose we shall assume that
\begin{equation}
D(|u|^p, |u|^p) < C_* \|u\|_{L^2}^{5-p} \|\nabla u\|_{L^2}^{3p-5}.
\end{equation}
and we shall arrive at contradiction. From (3.4) we have the inequality
\[ E_p(u) > \frac{1}{2} \|\nabla u\|_{L^2}^2 - \frac{C_* \sigma^\beta}{2p} \|\nabla u\|_{L^2}^{2\gamma}, \]
with $\beta, \gamma$ defined in (1.8). The right hand side suggests us to consider the function
\begin{equation}
\varphi(s) = \varphi_{\sigma}(s) = \frac{s}{2} - \frac{C_* \sigma^\beta}{2p} s^\gamma, \quad s \geq 0
\end{equation}
and obviously we have then
\begin{equation}
E_p(u) > \varphi \left( \|\nabla u\|_{L^2}^2 \right) \geq \min_{s \geq 0} \varphi(s) = \varphi(s_*),
\end{equation}
with $s_*$ being the unique solution to the equation
\[ s_* = \frac{\gamma}{p} C_* \sigma^\beta s_*^{\gamma}. \]

Further we take any minimizer $v$ of (1.13) and then we know that $\|v\|_{L^2}^2 = \sigma$ and
\begin{equation}
D(|v|^p, |v|^p) = C_* \sigma^\beta \|\nabla v\|_{L^2}^{2\gamma}.
\end{equation}
Moreover, any rescaled function
\[ v_\mu(x) = \mu^{3/2} v(\mu x) \]
generated by $v$ preserves the $L^2$ norm and the Gagliardo–Nirenberg equality (3.7). Now we choose $\mu$ in such a way so that
\[ \|\nabla v_\mu\|_{L^2}^2 = \mu^2 \|\nabla v\|_{L^2}^2 = s_*.
\]
Then we have
\[ \varphi(s_*) = \varphi \left( \|\nabla v_\mu\|_{L^2}^2 \right) = E_p(v_\mu) \]
and we arrive at
\[ \mathcal{E}_\sigma = E_p(u) > E_p(v_\mu), \]
with $\|v_\mu\|_{L^2}^2 = \sigma$ and this is clearly in contradiction with the fact that $u$ is a minimizer of (1.3).

**b) $\Rightarrow$ a):** We assume that $v$ satisfies Gagliardo–Nirenberg equality (3.3), $\|v\|_{L^2}^2 = \sigma$ and we have Pohozaev normalization conditions (3.8)
\[
\frac{\omega \sigma}{\beta} = \frac{\|\nabla v\|^2}{\gamma} = \frac{D(|v|^p, |v|^p)}{p},
\]
as stated in Lemma 1.2. We shall use the properties of the function $\varphi_\sigma(s)$ defined in (3.5). As before, we choose $s_*$ to be the point of minimum of this function. Next, we choose the parameter $\mu > 0$ so that $v_\mu(x) = \mu^3/2 v(\mu x)$ satisfies
\[
\|\nabla v_\mu\|_{L^2} = s_*.
\]
Then $v_\mu$ satisfies the Gagliardo–Nirenberg equality and hence (3.9)
\[
E_p(v_\mu) = \varphi_\sigma(s_*).
\]
It is not difficult to show that (3.10)
\[
\varphi_\sigma(s_*) = \mathcal{E}_\sigma.
\]
Indeed, the identity (3.9) implies
\[
\varphi_\sigma(s_*) = E_p(v_\mu) \geq \mathcal{E}_\sigma.
\]
If we take any minimizer $u$ of (1.3) we know from the step a) $\Rightarrow$ b), that $u$ satisfies the Gagliardo–Nirenberg equality
\[
\mathcal{E}_\sigma = E_p(u) = \varphi_\sigma(\|\nabla u\|_{L^2}^2) \geq \varphi_\sigma(s_*).
\]
Therefore, we arrive at (3.10) and the identities
\[
E_p(v_\mu) = \varphi_\sigma(s_*) = \mathcal{E}_\sigma
\]
guarantee that $v_\mu$ is a minimizer of (1.3), so we can find its Lagrange multiplier $\omega(\mu)$ such that
\[
\frac{\omega(\mu) \sigma}{\beta} = \frac{\|\nabla v_\mu\|^2}{\gamma} = \frac{D(|v_\mu|^p, |v_\mu|^p)}{p}.
\]
On the other hand $v$ satisfies (3.8) and the simple rescaling relations $\|\nabla v_\mu\|^2 = \mu^2 \|\nabla v\|^2$, $D(|v_\mu|^p, |v_\mu|^p) = \mu^{2\gamma} D(|v|^p, |v|^p)$ show immediately that $\mu = 1$ and $\omega(\mu) = \omega$.

\[
\square
\]

**Corollary 3.1.** Assume $p \in (5/3, 7/3)$. If $(\sigma, \omega)$ is admissible pair for (1.3), then we have the following relation between $\mathcal{E}_\sigma < 0$ and the best Gagliardo–Nirenberg constant $C_*$
\[
C_* = \frac{p}{\gamma^\gamma} \left( \frac{2}{1 - \gamma} \right)^{1-\gamma} \frac{|\mathcal{E}_\sigma|^{1-\gamma}}{\sigma^{p-\gamma}}.
\]
4. Asymptotics at infinity

The vector \( h \in T_{Q}^{\perp} \) in the kernel of \( L_+ \) satisfies the equations (here for simplicity we take \( \omega = 1 \))
\[
(1 - \Delta)h = pBQ^{p-1} + (p - 1)AQ^{p-2}h, \\
-\Delta B = Q^{p-1}h.
\]
Note that the positive radial ground state \( Q \) satisfies the system
\[
(1 - \Delta)Q = AQ^{p-1}, \\
-\Delta A = Q^p.
\]
By using the arguments in [11], we have the following asymptotic expansions of \( Q \) and \( A \) as \( r \to \infty \)
\[
Q(r) = \frac{e^{-r}}{r} \left( c_0 + O \left( \frac{1}{r} \right) \right),
\]
\[
A(r) = \frac{1}{r} \left( d_0 + O \left( \frac{1}{r} \right) \right),
\]
with \( c_0, d_0 > 0 \).

**Proposition 4.1.** If \( h \) is a nontrivial solution to (4.1), then there exists \( r_* > 0 \), so that \( h(r) \neq 0 \) for \( r > r_* \) and the following estimate holds
\[
|h(r)| \lesssim \frac{e^{-r}}{r},
\]

**Proof.** The starting point are the following asymptotics (verified in a similar way to (4.3) and (4.4))
\[
h(r) = \frac{e^{-r}}{r} \left( c_1 + O \left( \frac{1}{r} \right) \right),
\]
\[
c_1 = \int_{\mathbb{R}^3} pB(|y|)Q^{p-1}(|y|) + (p - 1)A(|y|)Q^{p-2}(|y|)h(|y|)dy,
\]
\[
B(r) = \frac{1}{r} \left( d_1 + O \left( \frac{1}{r} \right) \right),
\]
\[
B'(r) = -\frac{1}{r^2} \left( d_1 + O \left( \frac{1}{r} \right) \right),
\]
\[
d_1 = \int_{\mathbb{R}^3} Q^{p-1}(|y|)h(|y|)dy.
\]
If \( c_1 \neq 0 \), then the assertion of the Proposition follows. If \( c_1 = 0 \), then we can show that \( d_1 = 0 \). Indeed, if \( d_1 \neq 0 \), then without loss of generality we can assume \( d_1 > 0 \), so we can find a sufficiently large \( r_0 \), so
that $B(r) > 0$ and $V(r) = A(r)Q^{p-2}(r) < 1$, for $r > r_0$. Let us assume that $h(r) = 0$ has two roots $r_2 > r_1 > r_0$ and $\min_{[r_1, r_2]} h(r) = h(\bar{r}) < 0$. Then the maximum principle for the equation

$$\tag{4.9} (1 - \Delta - V(r))h(r) = B(r)Q^{p-1}(r)$$

in the interval $[r_1, r_2]$ leads to a contradiction. Indeed, in the point $\bar{r}$ of the negative minimum of $h$ we have $\Delta h(\bar{r}) \geq 0$, then

$$(-1 - \Delta - V(\bar{r}))h(\bar{r}) \leq 0,$$

This obviously contradicts the positiveness of the right hand side in (4.9). The contradiction shows that $c_1 = d_1 = 0$. Then we can perform the substitution $h(r) = e^{-r}g(r)/r$ into (4.1) and deduce the equations

$$\tag{4.10} -g''(r) + 2g'(r) = F_1,$$

$$-B''(r) - \frac{2}{r}B'(r) = F_2,$$

with

$$F_1(r) = \text{pre} B(r)Q^{p-1}(r) + (p - 1)A(r)Q^{p-2}(r)g(r),$$

$$F_2(r) = Q^{p-1}(r)\frac{e^{-r}g(r)}{r}.$$

The asymptotic expansions (4.3), (4.4) as well the ones in (4.7) and (4.8) with $c_1 = d_1 = 0$ imply the estimates

$$|F_1(r)| \lesssim \frac{e^{-(p-2)r}}{rp-1}|B(r)| + \frac{e^{-(p-2)r}}{rp-1}|g(r)|,$$

$$|F_2(r)| \lesssim \frac{e^{-pr}}{rp}|g(r)|.$$

Integrating the equations (4.10) from $r$ to $\infty$, we find

$$g(r) \lesssim \int_{r}^{\infty} \frac{e^{-(p-2)s}}{sp-3}|B(s)| + \frac{e^{-(p-2)s}}{sp-2}|g(s)|ds,$$

$$B(r) \lesssim \int_{r}^{\infty} \frac{e^{-ps}}{sp-1}|g(s)|ds.$$

To this end we can use the following Lemma with $\psi(r) = |g(r)| + |B(r)|$.

**Lemma 4.1.** (see Lemma 4.1 in [4]) If $\varepsilon > 0$, $\psi(r) \in C(1, \infty)$ is a non negative function satisfying

$$\tag{4.11} \psi(r) \leq C, \text{ } \forall r > 1$$

and

$$\psi(r) \leq C \int_{r}^{\infty} \frac{\psi(s)ds}{s^{1+\varepsilon}}, \text{ } \forall r > 1,$$

then $\psi(r) = 0$ for $r > 1$. 

An application of this Lemma guarantees that \( h(r) = 0 \) and this contradiction completes the proof. □

5. Simple ODE lemmas

In case \( u(|x|) \) is a radial \( C^1 \)-solution of the equation
\[
(\omega - \Delta)u = V(|u|)u,
\]
with \( V(|u|)(|x|) \) being a continuous function in \( |x| > 0 \), we have the following result.

**Lemma 5.1.** If \( u \) and \( |u| \) solve (5.1), \( u \in C^1(0, \infty) \) and there exists \( r_0 > 0 \), such that \( u(r_0) = 0 \), then \( u(r) \equiv 0 \).

**Proof.** If \( u'(r_0) = 0 \), then the Cauchy problem for the ODE (5.1) implies the assertion. If \( u'(r_0) < 0 \), then \( |u(r)| \) is not differentiable in \( r_0 \). The proof is now completed. □

Next we discuss the dimension of the kernel of \( L_+ \).

**Lemma 5.2.** If \( 2 < p < 7/3 \), then we have
\[
\dim(\text{Ker} L_+) \leq 2.
\]

**Proof.** Any positive radial solution \( w \) to the equation \( L_+w = 0 \) is a solution of the ordinary differential equation
\[
-r^2 \partial_r (r^2 \partial_r w(r)) + \omega w = pI(Q^{p-1}w)Q^{p-1} + (p - 1)I(Q^p)Q^{p-2}w.
\]
Then the couple of \( w \) and \( B = I(Q^{p-1}w) \) satisfies the system of nonlinear second order differential equations
\[
\begin{align*}
\frac{d^2}{dr^2}w(r) + \frac{2}{r}w'(r) &= \omega w(r) - pBQ^{p-1} - (p - 1)I(Q^p)Q^{p-2}w, \\
\frac{d^2}{dr^2}B(r) + \frac{2}{r}B'(r) &= -Q^{p-1}w.
\end{align*}
\]
subject to initial data
\[
\begin{align*}
w(0) &= w_0 \neq 0, \quad B(0) = B_0 \neq 0, \\
w'(0) &= 0, \quad B'(0) = 0.
\end{align*}
\]
The Fuchs–Painlevé Theorem [6.1] gives the series expansions
\[
\begin{align*}
w(r) &= w_0 + \sum_{k=1}^{\infty} w_{2k}r^{2k}, \\
B(r) &= B_0 + \sum_{k=1}^{\infty} B_{2k}r^{2k},
\end{align*}
\]
where all coefficients \( w_{2k}, B_{2k}, k \geq 1 \) can be determined in a unique way by the recurrence relations in terms of the two free initial data \( w_0 \) and \( B_0 \). This completes the proof of the Lemma. □
6. Appendix: Fuchs–Painleve series expansions of ground states

The equation

\begin{equation}
- \Delta u + Eu = I(u^p)u^{p-1}
\end{equation}

(6.1)

can be rewritten as a system of nonlinear second order differential equations

\begin{equation}
Q''(r) + \frac{2}{r} Q'(r) = EQ - A(r)Q^{p-1},
\end{equation}

(6.2)

\begin{equation}
A''(r) + \frac{2}{r} A'(r) = -Q^p.
\end{equation}

(6.3)

Our goal will be to verify that imposing special initial data

\begin{equation}
Q(0) = Q_0 > 0, \quad Q'(0) = 0,
A(0) = A_0, \quad A'(0) = 0,
\end{equation}

(6.3)

we can find unique real analytic (near \( r = 0 \)) solution to this Cauchy problem. Then we can consider the following more general problem

\begin{equation}
Y''(r) + \frac{c}{r} Y'(r) = F(r, Y),
\end{equation}

(6.4)

\begin{equation}
Y(0) = Y'(0) = 0,
\end{equation}

(6.5)

where we have shifted the initial data to zero, but we assume that \( F(r, 0) \neq 0 \) may be nontrivial source term. To be more precise, here \( Y(t) \in C^2([0, 1); \mathbb{R}^3) \) is a vector-valued function, while \( F \) satisfies the assumptions

\begin{equation}
F(r, Y) \text{ is real analytic near } r = 0, Y = 0
\end{equation}

(6.5)

and

\begin{equation}
F(0, 0) \neq 0.
\end{equation}

(6.6)

As in Theorem 11.1.1 in \[3\] we can state the following Fuchs–Painleve type result

**Theorem 6.1.** If the conditions (6.5) and (6.6) are fulfilled, then the Cauchy problem (6.4) has a unique real analytic solution

\begin{equation}
Y(r) = \sum_{k=2}^{\infty} Y_k r^k
\end{equation}

near \( r = 0 \).
This result applied to the Cauchy problem \( \text{[6.2]}, \text{[6.3]} \) gives the following series expansions near \( r = 0 \)

\[
Q(r) = Q_0 + \sum_{k=1}^{\infty} Q_{2k} r^{2k}, \quad A(r) = A_0 + \sum_{k=1}^{\infty} A_{2k} r^{2k}.
\]

**References**

[1] S.M. Chang, S. Gustafson, K. Nakanishi and T. P. Tsai, Spectra of linearized operators for NLS solitary waves, *SIAM Journal on Mathematical Analysis*, 39(4) (2007), 1070–1111.

[2] H. Genev and G. Venkov, Soliton and blow-up solutions to the time-dependent Schrödinger-Hartree equation, *Discrete Contin. Dyn. Syst. Ser. S*, 5 no. 5 (2012), 903 – 923.

[3] V. Georgiev and A. Stefanov, On the classification of the spectrally stable standing waves of the Hartree problem, *Physica D: Nonlinear Phenomena*, Vol. 370, (2018), 29–39.

[4] V. Georgiev, M. Tarulli and G. Venkov, Existence and Uniqueness of Ground States for \( p \)-Choquard Model, *Nonlinear Analysis* 179 (2019), 131–145.

[5] E. Hille, *Ordinary differential equations in the complex domain*, reprint of the 1976 original. Dover Publications, Inc., Mineola, NY, 1997

[6] M.K. Kwong, Uniqueness of positive solutions of \( \Delta u - u + u^p = 0 \), *Arch. Rational Mech. Anal*. Vol. 105, no. 3 (1989), 243 – 266.

[7] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation, *Stud. Appl. Math. 57* (2) (1976/1977), 93–105.

[8] K. McLeod and J. Serrin, Uniqueness of positive radial solutions of \( \Delta u + f(u) = 0 \) in \( \mathbb{R}^N \), *Archive for Rational Mechanics and Analysis*, 99(2) (1987), 115–145.

[9] V. Moroz and J. Van Schaftingen, Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics, *J. Funct. Anal.*, 265, no. 2 (2013), 153–184.

[10] V. Moroz and J. Van Schaftingen, Existence of ground states for a class of nonlinear Choquard equations. *Trans. Amer. Math. Soc.* 367, no. 9 (2015), 6557–6579.

[11] W. Strauss. Existence of Solitary Waves in Higher Dimension, *Comm. Math. Physics*, Vol. 55 (1977), 149 – 162.

[12] M. Tarulli and G. Venkov, Decay and Scattering in energy space for the solution of weakly coupled Choquard and Hartree-Fock equations, e-print (2019): arXiv:1904.10364.
V. Georgiev, Dipartimento di Matematica, Università di Pisa Largo B. Pontecorvo 5, 56127 Pisa, Italy, and, Faculty of Science and Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan, and IMI–BAS, Acad. Georgi Bonchev Str., Block 8, 1113 Sofia, Bulgaria

E-mail address: georgiev@dm.unipi.it

M. Tarulli, Faculty of Applied Mathematics and Informatics, Technical University of Sofia, Kliment Ohridski Blvd. 8, 1000 Sofia and IMI–BAS, Acad. Georgi Bonchev Str., Block 8, 1113 Sofia, Bulgaria, Dipartimento di Matematica, Università di Pisa Largo Bruno Pontecorvo 5 I - 56127 Pisa. Italy.

E-mail address: mta@tu-sofia.bg

G. Venkov, Faculty of Applied Mathematics and Informatics, Technical University of Sofia, Kliment Ohridski Blvd. 8, 1000 Sofia, Bulgaria

E-mail address: gvenkov@tu-sofia.bg