EIGENVALUE ESTIMATES AND L1 ENERGY ON CLOSED MANIFOLDS

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ABSTRACT. In this paper, we study Lichnerowicz type estimate for eigenvalues of drifting Laplacian operator and L1 and L2 energy for drifting heat equation on closed manifolds with weighted measure. In some sense, this study is about the eigenvalue estimate on Ricci solitons.

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1. Introduction

In this paper we consider two kinds of problems. One is the Lichnerowicz type eigenvalue estimate for drifting Laplacian operator on closed Riemannian manifold \((M, g)\) with weighted measure \(e^{-f} dv_g\), where \(f\) is a given smooth function on \(M\). In some sense, this problem is about the eigenvalue estimate on Ricci solitons. This part is a continuation of previous studies in [14] and [15]. See also [18] for related. The other is about the L1 energy growth estimate for heat equation with drifting. We shall also study the growth of the Dirichlet energy, which can be considered as a natural extension of the eigenvalue estimate of the drifting Laplacian operator. Our results are Theorem 1 and Theorem 2 below. We use the energy method, inspired by Lichnerowicz’s estimates [1] and Qian’s work [16]. Our argument is unlike the gradient estimate through the maximum principle trick [4] for Harnack quantity (see [5], [9], [17], [10], [14], [15], and [13]). The key ingredient is again the Bochner type identity related to the drifting Laplacian operator and the Bakry-Emery-Ricci tensor (see [2] and [3]). We remark that in the recent work of [11] (see Theorem 3 there), Limoncu studies another kind of Lichnerowicz estimate for Laplacian operator.

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2. Lichnerowicz type estimate

Let us start from the proof of standard Lichnerowicz theorem through Bochner method (one may see [1] from the point of the Ricci formula). Let $L = \Delta$ on the compact Riemannian manifold of dimension $n$. Assume that $f$ is a smooth function on $M$. Recall the following Bochner formula (see [17])

$$\frac{1}{2}L|\nabla f|^2 = |D^2f|^2 + (\nabla f, \nabla Lf) + (Ric)(\nabla f, \nabla f).$$

Let $\lambda > 0$ be an eigenvalue of $-L$. Then we have a smooth function $u$ such that

$$-L u = \lambda u$$

with $\int u^2 = 1$. Then $\lambda = \int |\nabla u|^2$. Assume that $Ric \geq (n-1)k$ for some $k > 0$. Then using

$$|D^2u|^2 \geq \frac{1}{n}|\Delta u|^2 = \frac{\lambda^2}{n}u^2$$

and

$$\langle \nabla u, \nabla Lu \rangle = -\lambda |\nabla u|^2$$

we have by integration on $M$,

$$0 \geq \frac{\lambda^2}{n} \lambda^2 + (n-1)k\lambda.$$

Hence, we have

$$\lambda \geq nk.$$ 

This is the famous Lichnerowicz Theorem for eigenvalue estimate [1].

The above argument can be used to study the eigenvalue problem of the elliptic operator with drifting

$$L_f = \Delta - \nabla f \nabla$$

associated with the weighted volume form $e^{-f}dv$. Assume that

(1) 

$$-L_f u = \lambda u.$$ 

Then again $\int u^2 = 1$ and

$$\lambda = \int |\nabla u|^2.$$

Here the integral is about the weighted volume form.

Recall the following Bochner formula (see Lemma 2.1 in [7])

$$\frac{1}{2}L_f|\nabla u|^2 = |D^2u|^2 + \langle \nabla u, \nabla L_f u \rangle + (Ric + D^2f)(\nabla u, \nabla u).$$
Assume that
\[ \text{Ric} + D^2 f \geq \frac{|Df|^2}{nz} + A \]
for some \( A > 0 \) and \( z > 0 \).

Note that
\[ (a + b)^2 \geq \frac{a^2}{z + 1} - \frac{b^2}{z} \]
for any \( z > 0 \). So, we have
\[ (\Delta u)^2 = (\lambda u + \nabla f \nabla u)^2 \geq \frac{\lambda^2 u^2}{z + 1} - \frac{|\nabla f \nabla u|^2}{z}. \]

Then
\[ 0 \geq \frac{1}{n} \int \left( \frac{\lambda^2 u^2}{z + 1} - \frac{|\nabla f|^2 |\nabla u|^2}{z} \right) + \lambda \int |\nabla u|^2 + \int (\text{Ric} + D^2 f)(\nabla u, \nabla u) \]
and
\[ 0 \geq \frac{1}{n} \int \left( \frac{\lambda^2 u^2}{z + 1} \right) - \lambda \int |\nabla u|^2 + A \int |\nabla u|^2. \]

Hence,
\[ 0 \geq \frac{\lambda^2}{n(z + 1)} - \lambda^2 + A\lambda \]
and
\[ \lambda \geq \frac{n(z + 1)A}{n(z + 1) - 1}. \]

In conclusion, we get

**Theorem 1.** Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \) with assumption (2) for some \( z > 0 \) and \( A > 0 \). Let \( u \) be a smooth solution to (1) with \( \lambda > 0 \). Then we have,
\[ \lambda \geq \frac{n(z + 1)A}{n(z + 1) - 1}. \]

This result is the Lichnerowicz type eigenvalue estimate for the drifting Laplacian operator.

### 3. New Energy Bound Derived from Bochner Formula

We now study the following heat equation with drifting term
\[ \partial_t u = \Delta u + B \cdot \nabla u + cu \]
on \( M \times (0, T) \). Here \( B \) is a smooth vector field on \( M \) and we shall let \( B = \nabla f \) later, and \( c \in C^2(M) \).

We define a tensor \( \nabla B \) as in [7] that
\[ \nabla B(X, Y) = (\nabla_X B, Y). \]
We assume that there is a constant $-K$ such that
\[ (4) \quad Ric - \nabla B \geq -K \]
on $M$.

Let
\[ L^b = \Delta + B \cdot \nabla. \]
Then we have the following Bochner formula (see Lemma 2.1 in [7])
\[ \frac{1}{2} L^b |\nabla f|^2 = |D^2 f|^2 + (\nabla f, \nabla L^b f) + (Ric - \nabla B)(\nabla f, \nabla f). \]

From this, we know that
\[ (\partial_t - L^b)|\nabla u|^2 = -2|D^2 u|^2 - 2(\nabla u, \nabla (cu)) - 2(Ric - \nabla B)(\nabla u, \nabla u). \]

From now on, we take \( c = \text{constant} \). Note that by Kato's inequality,
\[ |D^2 u|^2 = |\nabla du|^2 \geq |\nabla d u|^2 \geq \frac{|\nabla |\nabla u|^2|^2}{4(|\nabla u|^2 + \epsilon)}. \]

Let \( \phi(x) \) be a monotone increasing function, i.e., \( \phi' > 0 \), and let \( H(x) = \log \phi'(x) \). It is obvious that \( H'\phi' = \phi'' \). Let \( w = \phi(|\nabla u|^2) \).

Note that
\[ (\partial_t - L^b)w = \phi' (\partial_t - L^b)|\nabla u|^2 - \phi'' |\nabla |\nabla u|^2|^2. \]
Then we have
\[ (\partial_t - L^b)w = \phi'[-2|D^2 u|^2 - 2c|\nabla u|^2 - 2(Ric - \nabla B)(\nabla u, \nabla u) - H'|\nabla u|^2|^2]. \]

Hence, by (5), we have
\[ (\partial_t - L^b)w \leq \phi'[-\left(\frac{1}{2(|\nabla u|^2 + \epsilon)} + H'\right)|\nabla |\nabla u|^2|^2 + 2(K - c)|\nabla u|^2]. \]

Assume that \( B = -\nabla f \). Integrating by part on \( M \) with respect to the weighted volume form, we have
\[ \partial_t \int_M w \leq \int_M \phi'[-\left(\frac{1}{2(|\nabla u|^2 + \epsilon)} + H'\right)|\nabla |\nabla u|^2|^2 + 2(K - c)|\nabla u|^2]. \]

We now choose \( \phi \) such that
\[ \frac{1}{2x} + H'(x) \geq 0. \]

For example, we take \( \phi(x) = x^{p/2} \) for \( x > 0 \) with \( p \geq 1 \) and \( \phi'(x) = \frac{p}{2} x^{p/2-1} \). Then we have, by sending \( \epsilon \to 0 \), that
\[ \partial_t \int_M w \leq 2(K - c) \int_M \phi' |\nabla u|^2. \]
Let \( p = 1 \), and we then have
\[
\partial_t \int_M |\nabla u| \leq (K - c) \int_M |\nabla u|,
\]
which gives us the \( L^1 \) energy bound of \(|\nabla u|\). Take \( p = 2 \) and we have
\[
\int_M |\nabla u|^2(t) \leq \exp[2(K - c)t] \int_M |\nabla u|^2(0).
\]

In conclusion we have

**Theorem 2.** Let \((M, g)\) be a compact Riemannian manifold of dimension \( n \) with \((4)\). Assume that \( f \) is a smooth function in \( M \). Let \( u \) be a smooth solution to \((3)\) with \( B = -\nabla f \) and \( c = \text{constant} \). Then we have for \( t > 0 \),
\[
\int_M |\nabla u|(t) \leq \exp[(K - c)t] \int_M |\nabla u|(0).
\]
Here the integration on \( M \) is with respect to the weighted volume form. We also have
\[
\int_M |\nabla u|^2(t) \leq \exp[2(K - c)t] \int_M |\nabla u|^2(0).
\]

We remark that in general, for \( \phi(x) = x^{p/2} \) we have the \( L^p \) control in the following way
\[
\partial_t \int_M |\nabla u|^p \leq (K - c)p \int_M |\nabla u|^p
\]
and then
\[
\int_M |\nabla u|^p(t) \leq \exp[p(K - c)t] \int_M |\nabla u|^p(0).
\]

**References**

[1] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Monogr. Math., Springer-Verlag, Berlin, 1998.

[2] D. Bakry and M. Emery, *Diffusion hypercontractivitives*, in Seminaire de Probabilites XIX, 1983/1984, 177-206, Lect. Notes in Math. 1123, Springer, Berlin, 1985.

[3] D. Bakry and Z.M. Qian, *Volume comparison theorems without Jacobi fields. Current trends in potential theory*, 115-122, Theta Ser. Adv. Math., 4, Theta, Bucharest, 2005.

[4] Ben Chow and Richard Hamilton, *Constrained and linear Harnack inequalities for parabolic equations*, Inventiones Mathematicae 129, 213-238 (1997).

[5] B. Chow, P. Lu and L. Ni, *Hamilton’s Ricci flow*, Lectures in Contemporary Mathematics 3, Science Press and American Mathematical Society, 2006.

[6] Xianzhe Dai and Li Ma, *Mass under Ricci flow*, Commun. Math. Phys., 274, 65-80 (2007).
[7] B. Gonzalez, E. Negrin, Gradient estimates for positive solutions of the Laplacian with drift, Proc. Amer. Math. Soc., 127 (1999) 619-625.

[8] R. Hamilton, The formation of Singularities in the Ricci flow, Surveys in Diff. Geom., Vol. 2, pp. 7-136, 1995.

[9] P. Li, S. T. Yau, On the parabolic kernel of the Schrödinger operator, Acta Math. 156 (1986) 153-201.

[10] X. D. Li, Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds, J. Math. Pure. Appl., 84 (2005), 1295-1361.

[11] Murat Limoncu, The Bochner technique and modification of the Ricci tensor, Ann Glob Anal Geom. (2009) 36:285-291.

[12] Jun Ling, A Lower Bound of The First Eigenvalue of a Closed Manifold with Positive Ricci Curvature, Annals of Global Analysis and Geometry, 31(4) (2007) 385-408.

[13] Li Ma, Gradient estimates for a simple elliptic equation on complete non-compact Riemannian manifolds, Journal of Functional Analysis, 241 (2006) 374-382.

[14] L. Ma, B. Y. Liu, Convex eigenfunction of a drifting Laplacian operator and the fundamental gap. Pacific J. Math. 240, 343-361 (2009)

[15] L. Ma, B. Y. Liu, Convexity of the first eigenfunction of the drifting Laplacian operator and its applications. New York J. Math. 14, 393-401 (2008)

[16] Z. M. Qian, An estimate for the vorticity of the Navier-Stokes equation, to appear in Comptes rendus - Mathematique.

[17] R. Schoen and S. T. Yau, Lectures on Differential Geometry, international Press, 1994.

[18] Changyu Xia, Universal inequality for eigenvalues of the vibration problem for a clamped plate on Riemannian manifolds, The Quarterly Journal of Mathematics, September 4, 2009. doi:10.1093/qmath/hap026

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