A new analytical solving for electric polarizabilities of hydrogen-like atoms

V. F. Kharchenko
Bogolyubov Institute for Theoretical Physics,
National Academy of Sciences of Ukraine, UA - 03143, Kyiv, Ukraine
vkharchenko@bitp.kiev.ua

The direct transition-matrix approach to the description of the electric polarization of the quantum bound system of particles is used to determine the electric multipole polarizabilities of the hydrogen-like atoms. It is shown that in the case of the bound system formed by the Coulomb interaction the corresponding inhomogeneous integral equation determining an off-shell scattering function, which consistently describes virtual multiple scattering, can be solved exactly analytically for all electric multipole polarizabilities. Our method allows to reproduce the known Dalgarno-Lewis formula for electric multipole polarizabilities of the hydrogen atom in the ground state and can also be applied to determine the polarizability of the atom in excited bound states.

Keywords: Hydrogen-like atom; Electric multipole polarizability; Transition-matrix formalism; Inhomogeneous integral equation; Analytical solution

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1. Introduction

The phenomenon of polarization (deformation) under the action of an external electromagnetic field is inherent to all composite quantum systems containing one or more charged components (to molecules, atoms, nuclei, nucleons and other complexes) and is indisputably an universal property of the system.

Theoretical studies of influence of the electric and magnetic fields on structure of spectra of the simplest atoms were in the center of attention while becoming of modern quantum mechanics [1,2]. The Stark effect of the second order for hydrogen-like atoms was first calculated by Epstein [3], Wentzel [4] and Waller [5] based on the separation of the Schrödinger equation in parabolic coordinates and by using the perturbation theory.

Later on, the electric dipole polarizability of the simplest atomic systems were calculated directly using the traditional method by the formula for the energy shift in the second order of the Rayleigh-Schrödinger perturbation theory [2,6,7] in the applied electric field. As an example, employing this method, the contributions of the excited virtual states to the electric dipole polarizability of the hydrogen atom were investigated in Ref. 8. However, in general case this method is too cumbersome and nonpracticable because of the necessity of taking into account of virtual excited states, both discrete and from the continuum, that are coupled with the ground state by the perturbated interaction, especially for the systems with three and more particles.

The modification of the Rayleigh-Schrödinger perturbation theory proposed by Dalgarno and Lewis [9] (see also [10-12]) permits to avoid difficulties associated
with allowance for all possible intermediate bound and continuum states by the prior
determination of the first-order correction to the wave function of the unperturbated
state which satisfies an inhomogeneous differential equation. In the case of the
hydrogen atom Dalgarno and Lewis managed to solve analytically the differential
equation and first derive a general formula for all electric multipole polarizabilities
of the atom in the ground bound state. With the use of the O(4)-symmetry of the
energy operator of the hydrogen-like atom [13] and Dalgarno-Lewis perturbative
technique [9] purely algebraic approach to the calculation of the second-order Stark
effect for the atom has been developed in Ref. [14].

The object of this paper is to elaborate a new method of the analytical solution
of the electric multipole polarizabilities of the hydrogen-like atom on the basis of
the transition matrix approach, which has been applied by us earlier in nuclear
physics to calculate the electric dipole polarizabilities of systems with the S-wave
finite-range interaction — the deuteron [15-18] and two-cluster models of the triton
and lambda hypertriton [19,20]. In the preceding paper [21] the t-matrix formalism
was firstly applied to calculate numerically the dipole, quadrupole and octupole
polarizabilities of the hydrogen atom using the representation of the Coulomb t-
matrix with explicitly removed singularities [22].

In this paper we demonstrate that the integral equations, which appear in the
framework of the t-matrix formalism, permit in the case of the Coulomb interac-
tion exact analytical solution for all the electric multipole polarizabilities of the
hydrogen-like atom. Section 2 is devoted to description of the t-matrix formalism
of the polarization interaction of a two-partice bound complex placed in a external
electric field of a charged particle. In Section 3 the proposed approach is applied
for derivation of the main integral equation that determines the electric multipole
polarizabilities of the hydrogen-like atom. Strictly analytical solving of the ob-
tained integral equation is performed in Section 4 in the case of the ground bound
state. In Section 5 the general formula for the electric multipole polarizabilities of
the hydrogen-like atom in the ground state is derived and discussed. Section 6 is
concerned with the analytical derivation of the electric dipole polarizability of the
hydrogen-like atom in the excited 2S-state. Conclusions and outlook are presented
in Section 7.

2. Transition matrix description of polarization interaction
for a two-particle bound complex

Let us consider scattering of a two-particle complex formed from charged
particles 1 and 2 in the ground bound state by an external electric field of a charged
particle 0. The total Hamiltonian of the three-particle system is

\[ H = H^0 + V , \]

where \( H^0 = h^0_{12} + h^0_0 \) is the kinetic-energy operator of the system \( h^0_{12} \) is the operator
of the kinetic energy of relative motion of the particles 1 and 2 and \( h^0_0 \) is that of the
particle 0 and the center of mass of the particles 1 and 2) and \( V = v_{12} + v_{01} + v_{02} \)
is the potential-energy operator of the system, \( v_{01} \) and \( v_{02} \) are the potentials of the
pair (Coulomb) interaction between the particle 0 and the particles 1 and 2 of the
complex,

\[ v_{0i} = \frac{e_0 e_i}{r_{0i}}, \quad (2) \]

\( e_1 \) and \( e_2 \) are the charges of the constituents of the complex, \( e_0 \) is the charge of the particle 0 that is the source of the electric field, \( r_{0i} \) is the relative distance between the particles 0 and \( i \), \( v_{12} \) is the interaction potential that supports the formation of the bound complex.

Representing the potential \( V \) as the sum of the ’’internal” interaction potential, \( v_{12} \), and ’’external” one, \( v_0 \),

\[ V = v_{12} + v_0, \quad v_0 = v_{01} + v_{02}, \quad (3) \]

and applying the well-known Francis-Watson [23] and Feshbach [24, 25] technique (see also [26]) with the use of the projection operators \( P \equiv | \psi \rangle \langle \psi | \) and \( Q = 1 - P \), where \( \psi \) is the wave function of the two-particle complex in the bound state with the binding energy \( b \) (normalized to 1) that satisfies the Schrödinger equation

\[ (h_{12}^0 + v_{12} + b) \psi = 0, \quad (4) \]

we write the effective interaction potential between the particle 0 and the complex in the form

\[ v_{eff} = \langle \psi | R(E) | \psi \rangle. \quad (5) \]

Three-particle operator \( R(E) \) in (5) satisfies to the integral equation of the Lippmann-Schwinger type

\[ R(E) = v_0 + v_0 Q G_{12}^0(E) R(E), \quad (6) \]

in which the potential-energy operator is the ’’external” interaction potential describing the interaction between the constituent particles of the complex and the particle 0 (3), and the role of the propagator fulfils the ”truncated” Green’s operator that contains the ”internal” interaction potential \( v_{12} \),

\[ G_{12}^0(E) \equiv Q G_{12}(E), \quad G_{12}(E) = (E - H_0 - v_{12})^{-1}, \quad (7) \]

where \( E = \varepsilon - b \) is the total energy of the system, \( \varepsilon \) is the energy of the relative motion of the center of mass of the complex and the particle 0. The operator \( G_{12}(E) \) is determined by the integral equation

\[ G_{12}(E) = G^0(E) + G^0(E)v_{12}G_{12}(E). \quad (8) \]

where \( G^0(E) \) is the free three-body Green’s operator,

\[ G^0(E) = (E - H^0)^{-1}. \]

Introducing the transition operator \( T_{12}(E) \) that satisfies the Lippmann-Schwinger equation with the ”internal”potential \( v_{12} \)

\[ T_{12}(E) = v_{12} + v_{12}G^0(E)T_{12}(E), \quad (9) \]

we write the ”truncated” Green’s operator in the form

\[ G_{12}^0(E) = G^0(E) + G^0(E)T_{12}(E)G^0(E) - PG_{12}(E). \quad (10) \]
The integration in the expression for the effective potential (5) is performed over the variables of the two-particle bound system. The operator $v_{\text{eff}}$ acts only on functions of variables that describe the relative motion of the center of mass of the two-particle bound complex and the particle 0. In the general case the effective potential $v_{\text{eff}}$ is nonlocal and dependent on the energy of the relative motion of the center of mass of the two-particle bound complex and the particle 0, $\varepsilon$.

Considering in the integral equation for the operator $R$ (6) the potential of the "external" interaction between the particles of the complex and the particle 0, which is the source of the field, $v_0$ as a perturbative operator, in the second order of the expansion of $R$ in terms of $v_0$ in (6) we obtain the formula for the operator of the polarization interaction potential $v_{\text{pol}}$,

$$v_{\text{pol}} = \langle \psi | v_0 G^Q_{12}(E) v_0 | \psi \rangle$$  \hspace{1cm} (11)

where the "truncated" Green’s operator $G^Q_{12}(E)$ is determined by the expression (10).

To derive the formula for the electric multipole polarizabilities of the complex, it is necessary to consider the behaviour of the potential (11) at asymptotically large distances between the complex and the particle 0 at the kinetic energy of the relative motion of the particle 0 and the complex $\varepsilon$ much lesser than the binding energy of the complex ($\varepsilon \ll b$).

According to the uncertainty principle, in the case of large (compared to the size of the complex) distances between the particle 0 and the center of mass of the complex $\rho_0$ the variable momentum of relative motion between the particles within the complex is much greater than the variable momentum of the relative motion of the particle 0 and the center of mass of the complex. In such a case, due to the adiabatic character of motion in the expressions for operators $G^0(E)$ and $G_{12}(E)$ and for the transition operator $T_{12}$, which are contained in the formula for the "truncated" Green’s function $G^Q_{12}(E)$ (10), it is reasonably to neglect by the kinetic-energy operator $h^0_1$ as compared with the variable quantity described by the operator $h^0_{12}$, and in such a way to reduce the three-body problem to two-body one,

$$G^0(E) \equiv (E - h^0_{12} - h^0_0)^{-1} \to (E - h^0_{12})^{-1} \cdot I_0 \equiv g^0(E) \cdot I_0 ,$$

$$G_{12}(E) \equiv (E - h^0_{12} - h^0_0 - v_{12})^{-1} \to (E - h^0_{12} - v_{12})^{-1} \cdot I_0 \equiv g(E) \cdot I_0 ,$$

$$T_{12}(E) \to t(E) \cdot I_0 ,$$

$$G^Q_{12}(E) \equiv G_{12}(E) - P G_{12}(E) \to [g(E) - Pg(E)] \cdot I_0 \equiv g^Q(E) \cdot I_0 .$$

Here $I_0$ is a unit operator that acts on the functions of variables describing the position of the center of mass of the two-body particle relative to the particle 0, $g^0(E) = (E - h^0_{12})^{-1}$ and $g(E) = (E - h^0_{12} - v_{12})^{-1}$ are the free and complete two-particle Green’s operators, and $t(E)$ is the two-particle transition operators. The equations for $g$ and $t$ follow from the three-particle equations (8) and (9) becoming as

$$g(E) = g^0(E) + g^0(E)v_{12}g(E)$$  \hspace{1cm} (13)

and

$$t(E) = v_{12} + v_{12}g^0(E)g(E) .$$  \hspace{1cm} (14)
Then according to (12) the formula for the three-particle Green’s operator $G_{12}^Q$ (10) is transformed into the expression for the two-particle "truncated" operator

$$g^Q(E) = g^0(E) + g^0(E)t(E)g^0(E) - Pg(E),$$

and the formula for the polarization potential (11) under the adiabatic conditions of motion (12) and $\varepsilon = 0$ takes the form

$$v_{pol} = \langle \psi | v_0 g^Q(-b)v_0 | \psi \rangle.$$ (16)

Notice that the operator $g^Q(E)$ does not contain the pole singularity at $E = -b$ due to the mutual cancellation of the pole singularities of the operators $t(E)$ and $Pg(E)$ in the expression (15), which nearly the above point have the form

$$t(E) = | \gamma \rangle \langle \gamma | + \tilde{t}(E), \quad Pg(E) = | \psi \rangle[E - h_{12}^0 - v_{12}]^{-1}| \psi \rangle = \frac{1}{E + b} | \psi \rangle,$$ (17)

where the operator $\tilde{t}$ denotes the smooth part of the transition operator and the function $| \gamma \rangle$ is the vertex function

$$| \gamma \rangle \equiv v_{12} = [g^0(-b)]^{-1} | \psi \rangle.$$ (18)

Cancelling the pole terms in (15) at $E = -b$ with the use of (17), (18) and the identity

$$g^0(E)[g^0(-b)]^{-1} = [g^0(-b)]^{-1}g^0(E) = 1 - (E + b)g^0(E)$$ (19)

we obtain the expression for the "truncated" propagator (15) at the point $E = -b$

$$g^Q(-b) = g^0(-b) - g^0(-b) | \psi \rangle | \psi \rangle - | \psi \rangle | \psi \rangle | g^0(-b) + g^0(-b)\tilde{t}(-b)g^0(-b).$$ (20)

The potential of the "external" interaction of all particles of the complex and the charged particle 0 that generates the electric field is

$$v_0 = \frac{e_1e_0}{r_{01}} + \frac{e_2e_0}{r_{02}},$$ (21)

where

$$r_{01} = -\frac{m_2}{m_{12}}r - \rho, \quad r_{02} = \frac{m_1}{m_{12}}r - \rho, \quad r = r_{12} = r_1 - r_2,$$

$$\rho = \frac{1}{m_{12}}(m_1r_1 + m_2r_2) - r_0.$$ (22)

$r$ is the radius-vector of the particle 1 relative the particle 2, $\rho$ is the radius-vector of the center of mass of the complex relative to the charged particle 0, $m_i$ is the mass of the particle $i$, $m_{12} = m_1 + m_2$. At asymptotically large distances between the complex and the particle 0, using the expansion in power series of $(r/\rho) \ll 1$ we write the potential $v_0$ in the form

$$v_0 = \frac{e_0}{\rho} \sum_{\lambda=0}^{\infty} (-1)^\lambda \frac{M_\lambda(r, \hat{\rho})}{\rho^\lambda}.$$ (23)
Here $M_\lambda$ is the operator of the multipole moment of the charged particles of the complex (multiplication operator in coordinate space)

$$M_\lambda(r, \hat{\rho}) = \epsilon_1 r_{1c}^\lambda P_\lambda(r_{1c} \hat{\rho}) + \epsilon_2 r_{2c}^\lambda P_\lambda(r_{2c} \hat{\rho}) = \left[ \epsilon_1 \left( \frac{m_2}{m_{12}} \right)^\lambda + \epsilon_2 \left( -\frac{m_2}{m_{12}} \right)^\lambda \right] r^\lambda P_\lambda(\hat{r} \cdot \hat{\rho}),$$

(24)

$r_{ic}$ is the radius-vector of the charged particle of the complex relative to its center of mass ($i = 1, 2$),

$$r_{1c} = \frac{m_2}{m_{12}} r, \quad r_{2c} = -\frac{m_1}{m_{12}} r,$$

$P_\lambda(x)$ is the Legendre polynomial, and the unit vectors are marked with a hat, $\hat{a} = a/a$.

Substituting the asymptotical expression for $v_0$ (23) in Eq. (16) we write the polarization potential as

$$v_{\text{pol}}(\rho) = \left( \frac{e_0}{\rho} \right)^2 \sum_{\lambda, \lambda' = 0}^{\infty} \left( -\frac{1}{\rho} \right)^{\lambda + \lambda'} \langle \psi | M_\lambda g^Q(-b) M_{\lambda'} | \psi \rangle,$$

(25)

where the matrix element is determined according to Eq. (20) by the formula

$$\langle \psi | M_\lambda g^Q(-b) M_{\lambda'} | \psi \rangle = \langle \psi | M_\lambda g^0(-b) M_{\lambda'} | \psi \rangle - \langle \psi | M_\lambda g^0(-b) | M_{\lambda'} | \psi \rangle \langle \psi | M_{\lambda'} | \psi \rangle - \langle \psi | M_{\lambda} | \psi \rangle \langle \psi | g^0(-b) M_{\lambda'} | \psi \rangle + \langle \psi | M_{\lambda} g^0(-b) \tilde{t}(\tilde{t}) g^0(-b) M_{\lambda'} | \psi \rangle,$$

(26)

The general formula (26) that determines the polarization interaction potential for the quantum two-particle complex is considerably simplified when applying it to specific (atomic and nuclear) systems. If, for example, the potential of interaction between the particles of the complex $v_{12}(r)$ is invariant relative to the space reflection $r \rightarrow -r$, then the wave function of the bound complex is characterized by a definite parity. The conservation of the parity leads to nullification of the matrix elements

$$\langle \psi | M_\lambda | \psi \rangle, \quad \langle \psi | M_\lambda g^0(-b) | \psi \rangle \quad \text{and} \quad \langle \psi | g^0(-b) M_\lambda | \psi \rangle$$

(27)

(for example, for the systems with the Coulomb or nuclear interactions). In this case the wave function of the bound complex is characterized by a definite parity. The conservation of the parity leads to nullification of the matrix elements which are present in the second and third summands of the formula (26), at odd values of $\lambda$.

The formula (26) is also simplified if the interaction potential $v_{12}$ is invariant relative to spatial rotations, that is central. In such a case the wave function of the complex $\psi$ is characterized by a certain value of the orbital angular momenta $l$, and the matrix elements (27) also vanish, if the triangle condition $\Delta(l\lambda l)$ does not obey. Specifically, they vanish after integrating in angular variables for all $\lambda \neq 0$, if the total orbital moment of the complex is equal to zero, $l = 0$. 

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3. Electric multipole polarizabilities of the hydrogen-like atom

Let us apply the approach developed in the above to hydrogen-like atoms (atomic systems consisting of one electron (the particle 1) and a nucleus (the particle 2)). The interaction between the particles is described by the Coulomb potential

$$v_{12}(r) = v^C(r) = -\frac{e_1 e_2}{r},$$

(28)

where $e_1 = -e$ is the charge of the electron, $e_2 = Ze$ is the charge of the nucleus ($e$ is the elementary positive charge), $Z$ is the atomic number of the nucleus.

We describe the atomic system in the momentum space. The radius-vector operator in this case is

$$r = i\hbar\nabla_p = i\nabla_k,$$

$k$ being the wave vector. The Coulomb interaction potential

$$\langle k \mid v^C \mid k' \rangle = \int dvr^C(r) \exp\left[-i(k - k')r\right] = -\frac{4\pi Ze^2}{\mid k - k' \mid^2}$$

(29)

expanded in the spherical harmonics of the variable wave vectors $k$ and $k'$ corresponding to the orbital angular momentum $l$ and magnetic $m$ quantum numbers has the form

$$\langle k \mid v^C \mid k' \rangle = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} v_l^C(k, k')Y_{lm}(\hat{k})Y_{lm}^*(\hat{k}').$$

(30)

The wave function of the $S$-wave bound state of the atom

$$\psi(k) = 4\pi \int_0^{\infty} drr^2j_0(kr)\psi(r)$$

(31)

satisfies the integral Schrödinger equation

$$\psi = g^0(-b)v_0^C \mid \psi,$$

(32)

where $g^0(-b) = (-b - h^0)^{-1}$ is the free Green’s operator, $b$ is the binding energy of the atom, $v_0^C$ is the operator of the $S$-wave partial component of the Coulomb interaction potential in the expansion (30),

$$b = \frac{\hbar^2 \kappa^2}{2\mu_{12}}, \quad \kappa = \frac{Ze^2}{\hbar^2},$$

(33)

$\mu_{12} = m_1 m_2 / m_{12}$ is the reduced mass of the electron and the nucleus ($m_1 = m$ is the mass of the electron, $m_2$ is the mass of the nucleus).

In the following, we shall restrict our consideration in the hydrogen-like atom with the nucleus of the infinitely great mass as compared to the mass of the electron ($m_1 / m_2 \to 0$, $\mu_{12} \to 0$), neglecting the second term in the expression (24).

In the coordinate space the action of the multipole moment operator $M_\lambda$ on the the bound-state wave function of the atom is reduced to the multiplication

$$\langle r \mid M_\lambda \mid \psi \rangle = M_\lambda(r, \hat{\rho})\psi(r), \quad M_\lambda(r, \hat{\rho}) = e_1 r^\lambda P_\lambda(\hat{r} \cdot \hat{\rho}).$$

(34)
In the momentum space the formula (34) takes the form

$$\langle k \mid M_\lambda \mid \psi \rangle = (-i)^\lambda e_1 \varphi_\lambda(k) P_\lambda(\hat{k} \cdot \hat{\rho}) , \tag{35}$$

where

$$\varphi_\lambda(k) = 4\pi \int_0^\infty dr r^{\lambda+2} j_\lambda(kr) \psi(r) . \tag{36}$$

Applying the Rayleigh formula [27] for the spherical Bessel functions

$$j_\lambda(kr) = (-1)^\lambda \frac{k^\lambda}{r^\lambda} \left( \frac{d}{dk} \right)^\lambda j_0(kr) , \tag{37}$$

the function $\varphi_\lambda(k)$ can be written as

$$\varphi_\lambda(k) = (-1)^\lambda \frac{k^\lambda}{k} \left( \frac{d}{dk} \right)^\lambda \psi(k) . \tag{38}$$

From the formula (38), it follows the recurrence expression that relates the functions $\varphi_{\lambda+1}$ and $\varphi_\lambda$:

$$\varphi_{\lambda+1}(k) = \frac{\lambda}{k} \varphi_\lambda(k) - \frac{d\varphi_\lambda(k)}{dk} \text{ proceeding from } \varphi_0(k) = \psi(k) . \tag{39}$$

Using Eqs. (25), (26) and (35) we can determine the polarization potential of the interaction of the hydrogen-like atom and the particle 0. In this case the first term in the right-hand side of Eq. (26) is given by

$$< \psi \mid M_\lambda g^0(-b) M_\lambda \mid \psi > = \frac{e^2}{2\lambda + 1} \int_0^\infty \frac{k^2dk}{2\pi^2} \frac{|\varphi_\lambda(k)|^2}{\frac{k^2}{2m} + b} \delta_{\lambda\lambda'} . \tag{40}$$

The matrix elements in the second and third terms in Eq. (26) are written as

$$\langle \psi \mid M_\lambda \mid \psi \rangle = -e\delta_{\lambda 0} , \tag{41}$$

$$\langle \psi \mid g^0(-b) M_\lambda \mid \psi \rangle = \langle \psi \mid M_\lambda g^0(-b) \mid \psi \rangle = e\frac{R_1}{b} \delta_{\lambda 0} , \tag{42}$$

where

$$R_1 = b \int_0^\infty \frac{k^2dk}{2\pi^2} \frac{|\psi(k)|^2}{\frac{k^2}{2m} + b} = b \int_0^\infty \frac{k^2dk}{2\pi^2} \frac{\int <\gamma \mid k < k \mid \gamma >}{\left( \frac{k^2}{2m} + b \right)^3} \tag{43}$$

and $< k \mid \gamma >$ is the vertex function (18).

Note that the smooth part of the Coulomb transition operator (defined according to (17)) that is contained in Eq. (26) at the energy of the bound state of the complex, $\tilde{t}(-b) \equiv \tilde{t}^C(-b)$, consists of two operators — a part of the $S$-wave partial component (that corresponds to the orbital momenta $l = 0$), $\tilde{t}_0^C(-b)$, and the sum of all higher partial orbital components (with $l \geq 1$) at the same energy, $\tilde{t}_h^C(-b)$,

$$\tilde{t}^C(-b) = \tilde{t}_0^C(-b) + \tilde{t}_h^C(-b) . \tag{44}$$

As shown in the preceding work [21], the operator $\tilde{t}_0^C(E)$ at the point $E = -b$ has the factorable form

$$\tilde{t}_0^C(-b) = | \gamma > \left( \frac{R_1}{b} \right) < \gamma | . \tag{45}$$
The matrix elements of the operator \( t^C_h(-b) \) can be written in the form of the expansion over the set of the spherical functions of the angles of the momentum variables

\[
<k|t^C_h(-b)|k'> = 4\pi \sum_{l=1}^{\infty} \sum_{m=-l}^{l} t^C_{l}(k, k'; -b)Y_{lm}(\hat{k})Y_{lm}^{*}(\hat{k}').
\]  

(46)

Taking into consideration the relations (35), (41) and (44) - (46), we find the fourth term in Eq. (26) in the form of the sum of the matrix element

\[
\langle \psi | M_{\lambda}g^0(-b)t^C_h(-b)g^0(-b)M_{\lambda'} | \psi \rangle = e^2 \left( -\frac{R_1}{b} \right) \delta_{\lambda 0}\delta_{\lambda' 0}
\]  

(47)

and the matrix element

\[
\langle \psi | M_{\lambda}g^0(-b)t^C_h(-b)g^0(-b)M_{\lambda'} | \psi \rangle = (1 - \delta_{\lambda 0}) (1 - \delta_{\lambda' 0}) \delta_{\lambda \lambda'} \frac{e^2}{2\sqrt{\lambda + 1}} \int_0^{\infty} \frac{k^2dk}{2\pi^2} \int_0^{\infty} \frac{k'^2dk'}{2\pi^2} \frac{\varphi^{*}_{\lambda}(k)\varphi_{\lambda}(k')}{\left( \frac{k^2}{2m} + b \right) \left( \frac{k'^2}{2m} + b \right)} t^C_{\lambda}(k, k'; -b).
\]  

(48)

Substituting the expressions (40) – (42), (47) and (48) into Eq. (26) and taking into account that the matrix elements (47) with \( \lambda = 0, \lambda' = 1 \) and \( \lambda = 1, \lambda' = 0 \) equal to zero and the matrix elements with \( \lambda = \lambda' = 0 \) are mutually cancelled, we write the polarization potential of interaction between the particle 0 and the atom (25) in the form

\[
v_{pol}(\rho) = -\frac{e^2}{2} \sum_{\lambda=1}^{\infty} \frac{\alpha_{E\lambda}}{\rho^{2\lambda+2}},
\]  

(49)

where the coefficients

\[
\alpha_{E\lambda} = -2\langle \psi | M_{\lambda}g^0(-b)M_{\lambda} | \psi \rangle,
\]  

(50)

which represent the electric \( 2\lambda \)-pole polarizabilities of the atom are determined by the expression

\[
\alpha_{E\lambda} = \frac{2}{(2\lambda + 1)} e^2 \left\{ \int_0^{\infty} \frac{dkk^2}{2\pi^2} \frac{|\varphi_{\lambda}(k)|^2}{\frac{k^2}{2m} + b} - \int_0^{\infty} \frac{dkk^2}{2\pi^2} \int_0^{\infty} \frac{dk'k'^2}{2\pi^2} \frac{\varphi^{*}_{\lambda}(k)\varphi_{\lambda}(k')}{\left( \frac{k^2}{2m} + b \right) \left( \frac{k'^2}{2m} + b \right)} \right\}.
\]  

(51)

The formula for the electric \( 2\lambda \)-pole polarizability of the hydrogen-like atom (51) contains the function \( \varphi_{\lambda}(k) \) that, according to (38), is expressed in terms of the derivatives (up to the order \( \lambda \), inclusive) of the wave function of the bound state of the atom with respect to the relative momentum variable \( k \). For example, the corresponding functions for the dipole (\( \lambda = 1 \)), quadrupole (\( \lambda = 2 \)) and octupole (\( \lambda = 3 \)) polarizabilities have the form

\[
\varphi_1(k) = -\psi'(k),
\]

\[
\varphi_2(k) = -\frac{1}{k}\psi'(k) + \psi''(k),
\]

\[
\varphi_3(k) = -\frac{2}{k^2}\psi'(k) + \frac{3}{k}\psi''(k) - \psi'''(k).
\]  

(52)
The second term of the expression for the multipole polarizability $\alpha_{E\lambda}$ (51) describes the contribution from the virtual scattering of the particles of the system in intermediate states. It contains the partial component of the Coulomb transition matrix at the negative energy of the bound state of the atom $t^C_{\lambda}(k,k' ;-b)$ that satisfies the Lippmann-Schwinger integral equation

$$
t^C_{\lambda}(k,k' ;-b) = v^C_{\lambda}(k,k') - \int_0^\infty \frac{dk'^2}{2\pi^2} v^C_{\lambda}(k,k'') \frac{1}{k'^2 + \frac{2m}{b}} t^C_{\lambda}(k'',k' ;-b), \quad (53)
$$

where $v^C_{\lambda}(k,k')$ is the partial component of the Coulomb interaction potential.

For the Coulomb transition matrix with the negative energy $t^C_{\lambda}(k,k' ;-b)$, some representations have been derived — in the form of a sum [37], of an integral [38], with explicitly separated singularities in the variables of the transfer momentum and the energy [22] and others. They can be applied for determining $\alpha_{E\lambda}$ by the formula (51) numerically. In particular, that has been performed so in our preceding paper [21] for the dipole, quadrupole and octupole polarizabilities of the hydrogen atom using the corresponding partial components of the Coulomb transition matrix derived in Ref. [22].

In this paper, leaning upon the t-matrix formalism, we are working out a new method for rigorous analytical determination of $\alpha_{E\lambda}$ for the hydrogen-like atoms.

Applying the integral transformations with the kernels

$$
K^0_{\lambda}(k,k' ;-b) = - \frac{1}{2\pi^2} \frac{k'^2}{2m + b} v^C_{\lambda}(k,k') \quad \text{and} \quad K_{\lambda}(k,k' ;-b) = - \frac{1}{2\pi^2} \frac{k'^2}{2m + b} t^C_{\lambda}(k,k' ;-b),
$$

that transform the function $\varphi_{\lambda}(k)$, which describes $2\lambda$-pole distortion of the system, into the functions

$$
f_{\lambda}(k) = - \int_0^\infty \frac{dk'^2}{2\pi^2} v^C_{\lambda}(k,k') \frac{1}{k'^2 + \frac{2m}{b}} \varphi_{\lambda}(k') \quad (55)
$$

and

$$
\phi_{\lambda}(k) = - \int_0^\infty \frac{dk'^2}{2\pi^2} t^C_{\lambda}(k,k' ;-b) \frac{1}{k'^2 + \frac{2m}{b}} \varphi_{\lambda}(k'), \quad (56)
$$

we derive from the integral equation for t-matrix (53) the inhomogeneous integral equation for the function $\phi_{\lambda}(k)$ in the form

$$
\phi_{\lambda}(k) = f_{\lambda}(k) - \int_0^\infty \frac{dk'^2}{2\pi^2} v^C_{\lambda}(k,k') \frac{1}{k'^2 + \frac{2m}{b}} \phi_{\lambda}(k'), \quad (57)
$$

in which the free term is the function $f_{\lambda}(k)$ and the kernel is the same as the kernel of the integral equation for $t^C$-matrix (53), $K^0_{\lambda}(k,k' ;-b)$. In such a case, the formula (51) for the electric $2\lambda$-pole polarizability of the hydrogen-like atom takes the form

$$
\alpha_{E\lambda} = \frac{2}{(2\lambda + 1)\pi^2} \frac{me^2}{\hbar^2} \int_0^\infty dk k^2 \varphi^*_\lambda(k) \left[ \frac{\varphi_{\lambda}(k) + \phi_{\lambda}(k)}{k^2 + \kappa^2} \right]. \quad (58)
$$
Note, that the function $\phi_\lambda(k)$ in Eq. (58) describes the contribution into the $2^\lambda$-pole polarizability of the bound complex from multiple scattering of its constituents in the intermediate state with the orbital momentum $\lambda$ and the negative energy $-b$ (scattering completely off the energy shell). This function will be referred to as the off-shell scattering function. Below we show that the integral equation (57) determining the function $\phi_\lambda(k)$ can be solved by the rigorous analytical way.

4. Analytical solving of the integral equation for the function $\phi_\lambda(k)$ in the case of the ground bound state

In the momentum space, in accordance with Eqs. (31) and (33), the normalized wave function of the hydrogen-like atom in the ground bound state ($n = 1, \ l = 0$) is given by

$$
\psi_{10}(k) = \frac{8\sqrt{\pi\kappa^{5/2}}}{(k^2 + \kappa^2)^2}, \quad \kappa = \frac{Zme^2}{\hbar^2}. \tag{59}
$$

The partial component of the expansion of the Coulomb interaction potential (29) into the spherical functions of the orbital angular momentum (30), which is contained in the kernel of the main integral equation (57), is given by the expression

$$
v^C_\lambda(k, k') = \frac{2\pi e_1 e_2}{kk'} Q_\lambda \left(\frac{k^2 + k'^2}{2kk'}\right), \tag{60}
$$

where the function $Q_\lambda(x)$ is the Legendre function of the second kind [28]

$$
Q_\lambda(x) = \frac{1}{2} P_\lambda(x) \ln \left(\frac{x + 1}{x - 1}\right) - W_{\lambda - 1}(x), \tag{61}
$$

$$
W_{-1}(x) = 0, \quad W_{\lambda - 1}(x) = \sum_{k=1}^{\lambda} \frac{1}{k} P_{\lambda-k}(x) P_{k-1}(x).
$$

Substituting the expression for the wave function (59) into the formula (38) or (39), we obtain the function $\varphi_{10}^\lambda(k)$ in the form

$$
\varphi_{10}^\lambda(k) = 2^{\lambda+3}(\lambda + 1)! \sqrt{\pi\kappa^{5/2}} \frac{k^\lambda}{(k^2 + \kappa^2)^{\lambda+2}}, \tag{62}
$$

Using the expressions for the partial component of the interaction potential (60) and for the function $\varphi_{10}^\lambda$ (62), we find the free term of the integral equation (57) in the form

$$
f_{10}^\lambda(k) = 2^{\lambda+4}(\lambda + 1)! \sqrt{\pi\kappa^{7/2}} \frac{1}{k} \mathcal{F}_{10}^\lambda(k), \tag{63}
$$

where

$$
\mathcal{F}_{10}^\lambda(k) \equiv \int_0^\infty dk' \frac{k^{\lambda+1}}{(k'^2 + \kappa^2)^{\lambda+3}} Q_\lambda \left(\frac{k^2 + k'^2}{2kk'}\right)
$$

$$
= \frac{\pi}{4(\lambda + 1)(\lambda + 2)\kappa^3} \frac{k^{\lambda+1} [k^2 + (2\lambda + 3)\kappa^2]}{(k^2 + \kappa^2)^{\lambda+2}}. \tag{64}
$$
Substituting the integration result (64) into Eq. (63), we obtain the explicit expression for the function $f_{10}^{10}(k)$

$$f_{10}^{10}(k) = 2^{\lambda+2} \frac{\lambda!}{\lambda + 2} \sqrt{\pi} \kappa^{1/2} k^\lambda \frac{[k^2 + (2\lambda + 3)\kappa^2]}{(k^2 + \kappa^2)^{\lambda+2}}. \tag{65}$$

Taking into account Eq. (60), we write the integral equation for the function $\phi_{10}^{10}(k)$ (57) as

$$\phi_{10}^{10}(k) = f_{10}^{10}(k) + 2\kappa \pi \frac{1}{k} \int_0^\infty dk' \frac{1}{k'^2 + \kappa^2} Q_\lambda \left( \frac{k^2 + k'^2}{2kk'} \right) \phi_{10}^{10}(k'). \tag{66}$$

In view of the construction of the free term $f_{10}^{10}(k)$ (63) that is determined by the integral $F_{10}^{10}(k)$ (64), we seek for the solution of the integral equation (66) in the form of an expression that contains a specific two-term factor

$$\phi_{10}^{10}(k) = C_{\lambda} \frac{k^\lambda}{(k^2 + \kappa^2)^{\lambda+2}} \left[ A_{\lambda} k^2 + B_{\lambda} \kappa^2 \right], \tag{67}$$

where the coefficient of the expression for the free term (65) was separated out explicitly,

$$C_{\lambda} = 2^{\lambda+2} \frac{\lambda!}{\lambda + 2} \sqrt{\pi} \kappa^{1/2}, \tag{68}$$

while the coefficients $A_{\lambda}$ and $B_{\lambda}$ should be determined.

Substituting the function (67) into the equation (66), we obtain the equality

$$\frac{k^\lambda}{(k^2 + \kappa^2)^{\lambda+2}} \left[ A_{\lambda} k^2 + B_{\lambda} \kappa^2 \right] = \frac{k^\lambda \left[ k^2 + (2\lambda + 1)\kappa^2 \right]}{(k^2 + \kappa^2)^{\lambda+2}} + \frac{2\kappa \pi}{k} \left[ A_{\lambda} E_{10}^{10}(k) + B_{\lambda} \kappa^2 F_{10}^{10}(k) \right] \tag{69}$$

where the integral $E_{10}^{10}(k)$ is distinct from the integral $F_{10}^{10}(k)$ (64) only by the factor $k'^2$ in the integrand,

$$E_{10}^{10}(k) \equiv \int_0^\infty dk' \frac{k^{\lambda+3}}{(k'^2 + \kappa^2)^{\lambda+3}} Q_\lambda \left( \frac{k^2 + k'^2}{2kk'} \right) = \frac{1}{4(\lambda + 1)(\lambda + 2)\kappa} \frac{k^{\lambda+1} \left[ (2\lambda + 3)k^2 + \kappa^2 \right]}{(k^2 + \kappa^2)^{\lambda+2}}. \tag{70}$$

Taking the values of the integrals (64) and (70) and carrying out required simplifications, we write the relation (69) as

$$A_{\lambda} k^2 + B_{\lambda} \kappa^2 = \left[ k^2 + (2\lambda + 1)\kappa^2 \right] + \frac{1}{2(\lambda + 1)(\lambda + 2)} \left\{ A_{\lambda} \left[ (2\lambda + 3)k^2 + \kappa^2 \right] \right. \right. \right. \nonumber$$

$$\left. \left. \left. + B_{\lambda} \left[ k^2 + (2\lambda + 3)\kappa^2 \right] \right\} \right. \tag{71}$$

Applying the relationship (71) at two values of the variable $k$, say, at the points $k = 0$ and $k \to \infty$, we find hence the set of two coupled linear algebraic equations for the coefficients $A_{\lambda}$ and $B_{\lambda}$:
\[(2\lambda^2 + 4\lambda + 1)A_\lambda - B_\lambda = 2(\lambda + 1)(\lambda + 2)\]

\[A_\lambda - (2\lambda^2 + 4\lambda + 1)B_\lambda = -2(\lambda + 1)(\lambda + 2)(2\lambda + 3)\]  
(72)

the solution of which is

\[A_\lambda = \frac{\lambda + 2}{\lambda}, \quad B_\lambda = \frac{(2\lambda + 1)(\lambda + 2)}{\lambda}.\]  
(73)

It is worth noting that alternative equations arising with the use of other values of the variable \(k\) in the relation (71) present linear combinations of the equations (72).

Inserting the obtained values (73) into Eq. (67), we find the solution of the integral equation (66) in the form

\[\varphi_{10}\lambda(k) = 2\lambda^2(\lambda - 1)!\sqrt{\pi}\kappa^{1/2}\frac{k^\lambda}{(k^2 + \kappa^2)^{\lambda+2}} \left(k^2 + (2\lambda + 1)\kappa^2\right).\]  
(74)

5. Electric multipole polarizabilities of the hydrogen-like atom in the ground state

Substituting the expressions (62) and (74) for the functions \(\varphi_{10}\lambda(k)\) and \(\varphi_{\lambda}(k)\) into the formula (58), we obtain the final result for the electric \(2\lambda\)-pole polarizabilities of the hydrogen-like atom in the ground state

\[\alpha_{E\lambda}(10) = \frac{(\lambda + 2)(2\lambda + 1)!}{2^{2\lambda}} \frac{1}{\kappa^{2\lambda+1}}.\]  
(75)

The formula (75) is coincident with the result derived using the perturbative Dalgarno-Lewis technique [9] which reduces the problem of taking into consideration of the contributions from all excited intermediate states to solving a corresponding inhomogeneous differential equation that, as it was shown in Ref. [9], in the case of the Coulomb interaction may be performed analytically.

To analyze contributions of intermediate virtual states into the polarizabilities \(\alpha_{E\lambda}\) leaning upon our general formula (51), let us separate out the Born term from the partial Coulomb t-matrix

\[t_{C\lambda}^C(k, k'; -b) = v_{C\lambda}^C(k, k') + \Delta t_{C\lambda}^C(k, k'; -b)\]  
(76)

and taking into consideration Eqs. (55) and (56) write the polarizability \(\alpha_{E\lambda}\) as a sum of three terms,

\[\alpha_{E\lambda} = \alpha_{\lambda}^0 + \alpha_{\lambda}^{BS} + \alpha_{\lambda}^{MS},\]  
(77)

where the first term \(\alpha_{\lambda}^0\) contains the free virtual propagator

\[\alpha_{\lambda}^0 = \frac{2\kappa}{(2\lambda + 1)\pi^2} \int_0^\infty dk k^2 \frac{\varphi_{\lambda}(k)}{k^2 + \kappa^2},\]  
(78)

the term \(\alpha_{\lambda}^{BS}\) describes the Born (single) scattering in the intermediate state, and the term \(\alpha_{\lambda}^{MS}\) characterizes the rest of contributions from the multiple scattering (of the two-fold order and higher),
\[ \alpha^{BS}_\lambda = \frac{2\kappa}{(2\lambda + 1)^2 \pi^2} \int_0^\infty dk k^2 \varphi_\lambda(k) f_\lambda(k) / k^2 + \kappa^2, \]

\[ \alpha^{MS}_\lambda = \frac{2\kappa}{(2\lambda + 1)^2 \pi^2} \int_0^\infty dk k^2 \varphi_\lambda(k) [\phi_\lambda(k) - f_\lambda(k)] / k^2 + \kappa^2. \]  

(79)

Making use of (62), (65) and (74), we find for the ground state

\[ \alpha^0_\lambda(10) = \frac{(\lambda + 1)(2\lambda + 5)(2\lambda)!}{2^{2\lambda}(2\lambda + 2)\kappa^{2\lambda+1}}, \]

\[ \alpha^{BS}_\lambda(10) = \frac{(\lambda + 3)(2\lambda + 3)!}{2^{2\lambda+1}(\lambda + 1)(\lambda + 2)^2(2\lambda + 1)\kappa^{2\lambda+1}}, \]

\[ \alpha^{MS}_\lambda(10) = \frac{(2\lambda^2 + 9\lambda + 8)(2\lambda)!}{2^{2\lambda}(\lambda + 2)^2\kappa^{2\lambda+1}}. \]  

(80)

As an illustration, we show in Table 1 the values of the components \( \alpha^0_\lambda(10), \alpha^{BS}_\lambda(10) \) and \( \alpha^{MS}_\lambda(10) \) together with the corresponding electric \( 2^\lambda \)-pole polarizabilities of the hydrogen-like atom in the ground state \( \alpha_{E\lambda}(10) \) for \( \lambda = 1, 2, 3 \) and 4.

Table 1. The components \( \alpha^0_\lambda(10), \alpha^{BS}_\lambda(10) \) and \( \alpha^{MS}_\lambda(10) \) determining the electric dipole (\( \lambda = 1 \)), quadrupole (\( \lambda = 2 \)), octupole (\( \lambda = 3 \)) and hexadecapole (\( \lambda = 4 \)) polarizabilities of the hydrogen-like atom in the ground state, \( \alpha_{E\lambda}(10) \), calculated with the use of the direct t-matrix approach (in \( \kappa^{-(2\lambda+1)} \))

| \( \lambda \) | \( \alpha^0_\lambda(10) \) | \( \alpha^{BS}_\lambda(10) \) | \( \alpha^{MS}_\lambda(10) \) | \( \alpha_{E\lambda}(10) \) |
|-------------|----------------|----------------|----------------|----------------|
| 1           | \( \frac{7}{3} \) | \( \frac{10}{9} \) | \( \frac{19}{18} \) | \( \frac{9}{2} \) |
| 2           | \( \frac{81}{8} \) | \( \frac{105}{32} \) | \( \frac{51}{32} \) | 15 |
| 3           | 99 | \( \frac{243}{10} \) | \( \frac{159}{20} \) | \( \frac{525}{4} \) |
| 4           | \( \frac{6825}{8} \) | \( \frac{2695}{8} \) | \( \frac{665}{8} \) | \( \frac{8505}{4} \) |

The data of Table 1 indicate that the dipole polarizability of the hydrogen-like atom \( \alpha_{E\lambda}(10) \) in the ground state consists of two almost equal parts — of the first term containing the free propagator, \( \alpha^0_\lambda(10) = \frac{7}{3} \), and of the sum of two other terms that describe single and multiple virtual scattering in the P-wave orbital state, \( \alpha^{BS}_\lambda(10) + \alpha^{MS}_\lambda(10) = \frac{103}{6} \). In the case of the polarizabilities of the higher polarity the contribution of the term with the free propagator still escalates coming up to 67.5 %, 75.4 % and 80.2 % respectively for the quadrupole, octupole and hexadecapole polarizabilities. The contribution to the polarizability \( \alpha_{E\lambda}(10) \) from the Born term, \( \alpha^{BS}_\lambda(10) \), decreases with \( \lambda \) from 24.7 % for \( \lambda = 1 \) to 15.8 % for \( \lambda = 4 \). The contribution of the term that describes the multiple virtual scattering, \( \alpha^{MS}_\lambda(10) \), decreases with \( \lambda \) more distinctly: from the value 23.5 % for \( \lambda = 1 \) to 3.9 % for \( \lambda = 4 \).
It follows that the contribution into the polarizability \( \alpha_{E\lambda}(10) \) (77) from the first two terms — with the free propagator \( \alpha_0^\lambda(10) \) and describing the single virtual scattering \( \alpha_{BS}^\lambda(10) \) — is dominant taking the values 76.5 %, 89.4 % and 93.9 % in the case of the dipole (\( \lambda = 1 \)), quadrupole (\( \lambda = 2 \)) and octupole (\( \lambda = 3 \)) polarizabilities, respectively.

6. Electric dipole polarizability of the hydrogen-like atom in the excited 2S-state

The normalized wave function of the hydrogen-like atom in the excited bound state with \( n = 2, l = 0 \) is of the form

\[
\psi^{20}(k) = \sqrt{\frac{8\pi}{\kappa^5}} \frac{k^2 - \frac{1}{4}k^2}{\left(k^2 + \frac{1}{4}k^2\right)^3}
\]

(81)

The electric dipole polarizability of the hydrogen-like atom in the excited 2S-state is determined by the expression (58) with \( \lambda = 1 \), the function \( \varphi_1 = \varphi_1^{20} \),

\[
\varphi_1^{20}(k) = -\frac{d}{dk}\psi^{20}(k) = 4\sqrt{\frac{8\pi}{\kappa^5}} \frac{k^2 - \frac{1}{2}k^2}{\left(k^2 + \frac{1}{4}k^2\right)^3}
\]

(82)

and the off-shell scattering function \( \phi_1(k) = \phi_1^{20}(k) \). The last-mentioned function satisfies the integral equation (57) the free term of which \( f_1^{20}(k) \) is determined by the formula (55),

\[
f_1^{20}(k) = \int_0^\infty dk' k_1^{20}(k, k'; -b_{20}) \varphi_1^{20}(k')
\]

\[
= C_1^{20} \frac{k \left[ \left(k^2 - \frac{3}{4}k^2\right)^2 + \frac{7}{2}k^4 \right]}{\left(k^2 + \frac{1}{4}k^2\right)^4}, \quad C_1^{20} = -\frac{\sqrt{8\pi}}{3} \kappa^{1/2}
\]

(83)

The solution of the integral equation for the function \( \phi_1^{20}(k) \),

\[
\phi_1^{20}(k) = f_1^{20}(k) + \frac{2\kappa}{\pi k} \int_0^\infty dk' \frac{1}{k'^2 + \frac{1}{4}k^2} Q_1 \left( \frac{k^2 + k'^2}{2kk'} \right) \phi_1^{20}(k')
\]

(84)

we seek in the form of a binomial constructed from \( \left(k^2 - \frac{3}{4}k^2\right)^2 \) and \( \kappa^4 \),

\[
\phi_1^{20}(k) = \frac{C_1^{20} k}{\left(k^2 + \frac{1}{4}k^2\right)^2} \left[ \bar{A}_1 \left(k^2 - \frac{3}{4}k^2\right)^2 + \bar{B}_1 \kappa^4 \right]
\]

(85)

similarly as it was done in the case of the ground state when the integral equation (66) was analytically solved leaning upon the expression (67).

Substituting the function (85) into the equation (84) we obtain
\[
\left[ A_1 \left( k^2 - \frac{3}{4} \kappa^2 \right)^2 + B_1 \kappa^4 \right] = \left[ \left( k^2 - \frac{3}{4} \kappa^2 \right)^2 + \frac{7}{2} \kappa^4 \right] + \frac{2 \kappa}{\pi \kappa^2} \left\{ A_1 G_{1}^{20}(k) - \frac{3}{2} \bar{A}_1 \kappa^2 \mathcal{E}_{1}^{20}(k) + \left( \frac{9}{16} \bar{A}_1 + \bar{B}_1 \right) \kappa^4 \mathcal{F}_{1}^{20}(k) \right\} \left( k^2 + \frac{1}{4} \kappa^2 \right)^4 ,
\]

where the following designations for arising integrals are used

\[
G_{1}^{20}(k) \equiv \int_{0}^{\infty} dk' \frac{k'^6}{\left( k'^2 + \frac{1}{4} \kappa^2 \right)^5} Q_1 \left( \frac{k^2 + k'^2}{2kk'} \right) = \pi \frac{k^2}{96 \kappa} \left( k^2 + \frac{1}{4} \kappa^2 \right)^4 \left[ 35 k^4 + \frac{7}{2} \kappa^2 k^2 + \frac{3}{16} \kappa^4 \right] ,
\]

\[
E_{1}^{20}(k) \equiv \int_{0}^{\infty} dk' \frac{k'^4}{\left( k'^2 + \frac{1}{4} \kappa^2 \right)^5} Q_1 \left( \frac{k^2 + k'^2}{2kk'} \right) = \frac{\pi}{24 \kappa^3} \left( k^2 + \frac{1}{4} \kappa^2 \right)^4 \left[ 5 k^4 + \frac{17}{2} \kappa^2 k^2 + \frac{5}{16} \kappa^4 \right] ,
\]

\[
F_{1}^{20}(k) \equiv \int_{0}^{\infty} dk' \frac{k'^2}{\left( k'^2 + \frac{1}{4} \kappa^2 \right)^5} Q_1 \left( \frac{k^2 + k'^2}{2kk'} \right) = \frac{\pi}{6 \kappa^5} \left( k^2 + \frac{1}{4} \kappa^2 \right)^4 \left[ 3 k^4 + \frac{7}{2} \kappa^2 k^2 + \frac{35}{16} \kappa^4 \right] .
\]

Using the relations (86) and (87) at the points \( k = 0 \) and \( k \to \infty \), we deduce the set of two coupled algebraic equations for the coefficients \( \bar{A}_1 \) and \( \bar{B}_1 \) — in analogy to the derivation of the set (72) for the atom in the ground state:

\[
3 \bar{A}_1 + \frac{13}{3} \bar{B}_1 = 65 ,
\]

\[
\frac{4}{3} \bar{A}_1 - \bar{B}_1 = 1 .
\]

The solution of the set (88) is

\[
\bar{A}_1 = \frac{78}{5} , \quad \bar{B}_1 = \frac{21}{5} .
\]

In this way, the solution of the integral equation for the off-shell scattering function \( \phi_{1}^{20}(k) \) (84) takes the form

\[
\phi_{1}^{20}(k) = \frac{3}{5} C_{1}^{20} \frac{k^2}{\left( k^2 + \frac{1}{4} \kappa^2 \right)^4} \left[ 26 \left( k^2 - \frac{4 \kappa^2}{4} \right)^2 + 7 \kappa^4 \right] , \quad C_{1}^{20} = -\frac{\sqrt{8 \pi}}{3} \kappa^{1/2} .
\]

Substituting the expressions (82) and (90) for the functions \( \varphi_{1}^{20}(k) \) and \( \phi_{1}^{20}(k) \) into the formula (58) with \( \lambda = 1 \), we deduce the value of the electric dipole polarizability of the hydrogen-like atom in the excited \( 2S \)-state (with \( n = 2, \ l = 0 \))

\[
\alpha_{E1}(20) = \frac{2712}{5} \frac{1}{\kappa^3} .
\]
Analyzing the role of the scattering in the intermediate virtual state and separating out the Born term from the $P$-wave partial Coulomb transition matrix according to Eq. (76) we write the electric dipole polarizability of the hydrogen-like atom in the excited $2S$-state in the form

$$\alpha_{E1}(20) = \alpha_{0}^{1}(20) + \alpha_{BS}^{1}(20) + \alpha_{MS}^{1}(20),$$  \hspace{1cm} (92)

The values of individual components in Eq. (92) we derive using the formulas (78) and (79) and the obtained above functions $\varphi_{1}^{20}(k)$, $f_{1}^{20}(k)$ and $\phi_{1}^{20}(k)$ (Eqs. (82), (83) and (90)):

$$\alpha_{0}^{1}(20) = \frac{502}{3} \frac{1}{\kappa^{3}}, \hspace{1cm} \alpha_{BS}^{1}(20) = \frac{1205}{9} \frac{1}{\kappa^{3}}, \hspace{1cm} \alpha_{MS}^{1}(20) = \frac{10853}{45} \frac{1}{\kappa^{3}}.$$  \hspace{1cm} (93)

For comparison we show in Table 2 the values of the components $\alpha_{0}^{1}(n0)$, $\alpha_{BS}^{1}(n0)$ and $\alpha_{MS}^{1}(n0)$ and the electric dipole polarizability of the hydrogen-like atom in the ground ($n = 1, l = 0$) and excited ($n = 2, l = 0$) $S$-wave states.

Table 2. The components $\alpha_{0}^{1}(n0)$, $\alpha_{BS}^{1}(n0)$ and $\alpha_{MS}^{1}(n0)$ determining the electric dipole polarizabilities of the hydrogen-like atom $\alpha_{E1}(nl)$ in the ground ($n = 1, l = 0$) and excited ($n = 2, l = 0$) states, derived with the use of the direct t-matrix approach (in $\kappa^{-3}$)

| $n$ | $\alpha_{0}^{1}(n0)$ | $\alpha_{BS}^{1}(n0)$ | $\alpha_{MS}^{1}(n0)$ | $\alpha_{E1}(n0)$ |
|-----|-----------------|-----------------|-----------------|----------------|
| 1   | $\frac{7}{3}$   | $\frac{10}{9}$  | $\frac{19}{18}$ | $\frac{9}{2}$  |
| 2   | $\frac{502}{3}$ | $\frac{1205}{9}$| $\frac{10853}{45}$ | $\frac{2712}{5}$ |

Note that for the electric dipole polarizabilities of the atom in the excited state $\alpha_{E1}(20)$ the contribution of the terms which describe the Born and multiple scattering in virtual states, $\alpha_{BS}^{1} + \alpha_{MS}^{1}$, exceeds that of the term with the free virtual propagator, $\alpha_{0}^{1}$, as distinct from the polarizabilities of the atom in the ground state $\alpha_{E1}(10)$ for which the mentioned contributions are almost equal.

7. Conclusions and outlook

To summarize, the t-matrix approach to description of the polarization interaction of the quantum two-particle bound system (both atomic and nuclear) that arises under influence of the external electric field has been developed in this work. We show that in the case of the Coulomb system the application of the t-matrix description permits to perform the rigorous analytical solution of the corresponding Lippmann-Schwinger integral equation that determines the electric multipole polarizabilities of the hydrogen-like atom in the ground state. The obtained formula for the electric polarizability in this case agrees with the Dalgarno-Lewis result [9],
which has been derived in the framework of the modified Rayleigh-Schrödinger perturbation theory by solving analytically an inhomogeneous differential equation that describes the correction to the unperturbed state for the contributions from all the intermediate bound and continuum virtual states.

The most important advantage of the t-matrix formalism for determination of the electric polarizability consists in the fact that its application allows to avoid cumbersome calculation of contributions from a great many discrete and continuum excited virtual states. Instead, it is necessary to determine one or a few (depending on the form of the interaction) partial t-matrix components. An essential circumstance in this case is also that the t-matrix elements are contained only at the negative energy of the bound state and present therefore real functions of two variable momenta.

In the case of the simplest composite nucleus — the deuteron, when the $S$-wave partial interaction between the constituents of the complex is dominated and the contribution from the $P$-wave interaction is only slight, the electric polarizability is determined mainly by the free propagator. The corresponding calculations of the electric dipole, quadrupole and octupole polarizabilities of the deuteron have recently been performed with the application of the t-matrix formalism in Ref. [18].

The t-matrix method applied in this paper can be directly generalized to systems with more complicated interactions between the constituents. The presence of the tensor interaction can cause the anisotropic polarization properties of the system. For example, the electric dipole or quadrupole polarizabilities of the deuteron nucleus will be differed in the cases when the external electric field is directed parallel or transversely to the spin of the deuteron [31].

It is also promising to apply the t-matrix approach for analytical determination of the electric polarizabilities of the hydrogen-like atoms in excited (degenerated) states.

Another important direction of investigation on the polarization effects is the calculation of the van der Waals interactions between atoms using the t-matrix formalism. It is known [32], the direct precision measurement of the van der Waals interaction between two neutral isolated atoms in which one of electrons is in a highly excited state with large principal quantum number $n$ (in the 50–100 range)—so called Rydberg atoms. The obtained new data on fine details of the van der Waals interaction require in-depth theoretical interpretation [33].

Finally, that is of the particular importance to study the possibility of the generalization of the proposed approach for more complex three- and four-particle quantum systems, which are described the Faddeev and Faddeev-Yakubovsky integral equations [34, 35].

In nuclear physics, the electric polarizabilities of the few-body nuclei are important characteristics that contain additional independent information on the fundamental nuclear force. Unfortunately, existing value for the the electric dipole polarizability of the nucleus $^3\text{He}$ extracted from the data on the cross-sections for photodisintegration of the nucleus [36, 37] ($\alpha_{E1} = 0.15 \pm 0.02$ fm$^3$)and obtained by way of the experimental study of deviations from Rutherford scattering in elastic scattering of $^3\text{He}$ nuclei from $^{208}\text{Pb}$ at energies well below the Coulomb barrier [38] ($\alpha_{E1} = 0.25 \pm 0.04$ fm$^3$) are not to be in agreement among themselves. The application of the transition-matrix approach to the description of the deformation
properties of the three-nucleon nuclei $^3$H and $^3$He in the electric field on the basis of the Faddeev mathematical formalism will help to determine reliably the values of the dipole polarizabilities of these nuclei.

References

[1] W. Pauli, Z. Phys., 36 (1926) 336-363.
[2] E. Schrödinger, Ann. Phys. Leipzig, 80 (1926) 437-490.
[3] P. S. Epstein, Phys. Rev., 28 (1926) 695-710.
[4] G. Wentzel, Z. Phys., 38 (1926) 518-529.
[5] I. Waller, Z. Phys., 38 (1926) 635-646.
[6] J. W. S. Rayleigh, "Theory of Sound" 2nd Edition, Vol. 1, p.p. 115-118, Macmillan, London, 1894.
[7] H. A. Bethe and E. S. Salpeter, in "Handbuch der Physik," Vol. 35, Section 25β, Springer, Berlin, 1957.
[8] L. Castillejo, I. C. Percival and M. J. Seaton, Proc. Roy. Soc. A 254, No. 1277 (1960) 259-272.
[9] A. Dalgarno and J. T. Lewis, Proc. Roy. Soc. A 233, No. 1192 (1955) 70-74.
[10] C. Schwartz, Ann. Phys. NY 6(1959) 156-169.
[11] L. I. Schiff, "Quantum Mechanics" 3rd Edition, p.p. 266-267, McGraw-Hill, New York, 1968.
[12] M. A. Maize, M. A. Antonacci and F. Marsiglio, Am. J. Phys. 79(2011) 222-225.
[13] V. A. Fock, Z. Phys., 98 (1935) 145-154.
[14] H. A. Becker and K. Bleuler, Z. Naturforschung, 31a (1976) 517-523.
[15] V. F. Kharchenko, S. A. Shadchin and S. A. Permyakov, Physics Letters B, 199 (1987) 1-4.
[16] V. F. Kharchenko and S. A. Shadchin, "Three-body Theory of the Effective Interaction between a Particle and a Two-Particle Bound System," Preprint ITP-93-24E, Institute for Theoretical Physics, Kyiv, 1993.
[17] V. F. Kharchenko and S. A. Shadchin, Ukrainian Journal of Physics, 42 (1997) 912-920.
[18] V. F. Kharchenko, Int. J. Mod. Phys. E, 22 (2013) 1350031, 1-13; nucl-th/1209.2004.
[19] V. F. Kharchenko and A. V. Kharchenko, Collected Physical Papers Lvit, 7 (2008) 438-443; nucl-th/0811.2565.
[20] V. F. Kharchenko and A. V. Kharchenko, Int. J. Mod. Phys. E, 19 (2010) 225-242; nucl-th/0903.5769.
[21] V. F. Kharchenko, J. Mod. Phys., 4 (2013) 99-107; nucl-th/1208.1394.
[22] S. A. Shadchin and V. F. Kharchenko, J. Phys. B, 16 (1983) 1319-1322.
[23] N. C. Francis and K. M. Watson, Phys. Rev., 92 (1953) 291-303.
[24] H. Feshbach, Ann. Phys. NY 5(1958) 357-390.
[25] H. Feshbach, Ann. Phys. NY 19(1962) 287-313.
[26] C. J. Joachain, "Quantum Collision Theory," North-Holland, Amsterdam, 1975.
[27] M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," Dover, NY, 1965.
[28] V. F. Bratsev and E. D. Trifonov, Vest. Leningrad. Gos. Univ. 16 (1962) 36-39.
[29] J. Schwinger, J. Math. Phys. 5(1964) 1606-1608.
[30] I. S. Gradshtein and I. M. Ryzhik, "Tables of Integrals, Sums, Series and Products," Fizmatgiz, Moscow, 1962.

[31] A. V. Kharchenko, Nuclear Physics A 617 (1997) 34-44.

[32] L. Beguin, A. Vernier, R. Chicireanu, T. Lahaye and A. Browaeys, Phys. Rev. Lett., 110 (2013) 263201, 1-5.

[33] M. Weidemühler, Physics 6 (2013) 71-75.

[34] L. D. Faddeev, Soviet Physics JETP 12 (1961) 1014-1019.

[35] O. A. Yakubovsky, Soviet Journal of Nuclear Physics 5 (1967) 937-942.

[36] G. A. Rinker, Phys. Rev. A 14 (1976) 18-29.

[37] V. D. Efros, W. Leidemann and G. Orlandini, Phys. Lett. B 408 (1997) 1-6; nucl-th/9707007.

[38] F. Goeckner, L. O. Lamm and L. D. Knutson, Phys. Rev. C 43 (1991) 66-72.