Recursive Quantum Qudit Convolutional Codes
Need Not Be Catastrophic

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Abstract—Classical turbo codes efficiently approach the Shannon limit, and so bringing these over to the quantum scenario would allow for rapid transmission of quantum information. Early on in the work of defining the quantum analogue, it was shown that an efficient recursive subroutine (quantum convolutional codes) would always be catastrophic. This result may have stunted the further research into this coding scheme. In this document, we prove that this previously proven no-go theorem is no longer always true if we extend the coding scheme into qudit space with dimension some prime larger than 2. This removes a blockade in the development of quantum turbo codes and hopefully will stimulate further research in this area.

INTRODUCTION

HAVING efficient quantum encoding and decoding schemes are crucial for physical implementations of quantum computers. More importantly, they are mathematically interesting to study. Classical coding theory has seen a couple of relatively recent new encoding schemes, such as polar codes and turbo codes, both of which efficiently approach the Shannon limit for information transmission over a noisy channel [1] [2]. Here, we study the quantum version of turbo codes, primarily focusing on a result making it more challenging to implement these codes. Prior to reaching that, we must first introduce the notation that we will use here, then we move on to define quantum convolutional codes (a crucial component of turbo codes), show an important limitation for these convolutional codes (first shown in [3], then shown differently by [4]) then proceed to show that by generalizing the approach in [4] this limitation is removed in the case of higher dimensional qudits.

NOTATION

Throughout, we will use the Pauli group mod phase (which is physically irrelevant).

The action of these operators in the Pauli group can be expressed as:

\[ X[j] = (j + 1) \mod 2 \]
\[ Z[j] = (-1)^j \]

Let \( p \) be a prime greater than 2, then the extension of these operators to qudits over a space with \( p \) orthonormal basis states is given by operators \( X \) and \( Z \) such that:

\[ \omega = e^{2\pi i/p} \quad X[j] = (j + 1) \mod p \quad Z[j] = \omega^j j \]  

We call these operators generalized one-qudit Pauli operators, denoted \( P_1^p \), where \( p \) is a prime greater than 2. The group \( P_1^p \) is formed from all the possible compositions of the \( X \) and \( Z \) (both of which have order \( p \)), while the global phase is ignored. From here on out, we will only work with the generalized Pauli group, and often drop the superscript since it’s implicit. We note that these operators satisfy:

\[ (X^a Z^b)(X^c Z^d) = \omega^{bc - ad}(X^c Z^d)(X^a Z^b) \]  

or we can equivalently write this for generalized Paulis \( P,Q \) as:

\[ PQ = \omega^{c(P,Q)}QP \]

where \( c(P,Q) \) is defined implicitly above and can be recognized to be the symplectic inner product of the symplectic representations of the generalized Pauli operators when commuting the tensor products of Pauli operators.

We can extend this group over tensor products to create \( P_n \) where: \( P_n = \otimes_{i=1}^n P_1 \). Where Pauli entries are missing in an expression, it is to be assumed that the other entries are the identity \( I \). \( P_1 \) forms an algebra, and thus so does \( P_n \). Thus, with some independent generating set \( S \) of commuting \( P_n \), we form a finite closed algebra since each \( P_n \) has order \( p \), we will form an algebra \( S \) of size \( p^{|S|} \). When this algebra has an associated state or set of states who are all \( +1 \) eigenvectors of all \( S \), we call this \( S \) the stabilizer generators and \( S \) the stabilizer algebra.

We now turn our attention to the states that are stabilized by \( S \). These are the orthonormal states \( \psi \) such that \( s\psi = \psi \) for all \( s \in S \). These are nearly what we will define as our codewords, but we would like our states to be transformed into codewords via an invertible matrix mapping. Toward this end, we define our encoder as taking a \( n \) qudit state into a \( k \) qudit state followed by \( n-k \) copies of \( |0\rangle \), where \( n-k = |S| \) is the size of the minimal generating set for our algebra \( S \). We would like our code to be efficient, so we restrict our encoders to those involving Clifford operations only since this group is efficiently representable, as stated in the Gottesman-Knill theorem. We call this encoder \( U \).

Then, with this encoder, we can encode blocks of information at a time, allowing us to work with streams of information instead of needing to wait for the entire data set before proceeding. This has many obvious advantages, so we don’t bother discussing them here.

QUANTUM CONVOLUTIONAL CODES

Quantum convolutional codes serve as a crucial subroutine in quantum turbo codes, so being able to perform this operation efficiently implies being able to perform quantum turbo codes efficiently. In this section we define quantum convolutional codes. We also prove a critical limitation on
these codes after providing a myriad of lemmas as well as our main result showing that this limitation can be removed provided a certain criterion is satisfied.

**Definition 1.** A Pauli sequence is defined as:

\[ P_\infty = \otimes_{i=1}^\infty P_i = \lim_{k \to \infty} \otimes_{k=1}^k P_n \]

**Definition 2.** A quantum convolutional code is defined for a set of \( n - k \) generators and is the set of all Pauli sequences composed exclusively from the \( n - k \) Pauli's, where \( \bigotimes^n \) is also allowed in the Pauli sequences. All of these Pauli sequences must then commute with each other to be considered a quantum convolutional code.

The number of non-identity \( P_n \) prior to all identity operators in a Pauli sequence \( P_\infty \) will be denoted by \( d_l \).

**Definition 3.** A quantum convolutional encoder \( U \) carries \( m \) memory qudits, \( n - k \) ancilla qudits, and \( k \) logical qudits to \( n \) physical qudits and \( m \) memory qudits:

\[ U : |M_i, k_i, l_i\rangle \mapsto |n_i, M_{i+1}\rangle \]

where, aside from the initialization, each encoder takes the memory state returned from the prior encoder as its memory input.

This encoder should act as follows:

- **Initialization:** \( U\left( I^{\otimes m}, Z_i, I^{\otimes k}\right) = |h_{i,1}, g_{i,1}\rangle \)
- **Propagation:** \( U\left( |g_{i,k}, I^{\otimes n-k}, I^{\otimes k}\rangle = |h_{i,k+1}, g_{i,k+1}\rangle \right) \)
- **Termination:** \( U\left( |g_{i,d_l-1}, I^{\otimes n-k}, I^{\otimes k}\rangle = |h_{i,d}, I^{\otimes m}\right) \)

**Definition 4.** With a quantum convolutional encoder we associate a \( p^{2m} \) vertex directed multigraph, called a state diagram, where for this graph we associate vertices with memory states and with directed edges with from one vertex to another if there exists a pair \( (L, P) \) such that the encoder carries the pair of the first memory with an ancilla state \( S \in \{ I, Z, Z^2, ... \}^{\otimes n-k} \) to the other memory state. We then label the vertices by their transition operators and the vertices by the memory state associated to them.

**Definition 5.** A quantum convolutional encoder is catastrophic if there exists a cycle in the state diagram associated with it where all the edges have zero physical weight, but there is at least one edge with nonzero logical weight.

**Definition 6.** Within our state diagram, we call a path admissible if its first edge is not part of a zero physical weight cycle. Now consider any vertex belonging to a zero physical weight loop and any admissible path starting at this vertex that also has logical weight one. The encoder is recursive if all such paths do not contain a zero physical weight loop.

The following theorem has been shown in two very different ways. The original proof by Poulin et al. was correct, but somewhat challenging to understand and generalize. Then came a follow up paper that utilized group theory more to prove the contrapositive of the theorem. We follow many of the same steps as in the latter paper, but elevate them to this higher dimensional quantum computing space of qudits.

**Theorem 7.** Qubit recursive quantum convolutional codes are catastrophic.

Or equivalently:

**Theorem 8.** If a qubit quantum convolutional encoder is non-catastrophic, then it is non-recursive.

The second of these is proven rather elegantly and we modify the argument in order to show the following theorem:

**Theorem 9.** Recursive qudit quantum convolutional encoders need not be catastrophic.

In order to prove this theorem we will need to generalize a collection of lemmas and definitions presented in [4], as we have already begun to do.

**Definition 10.** A zero physical weight cycle is a sequence of operator triplets \( M, S, L \) such that upon following these operators we traverse a cycle in our state diagram where all the physical operators are the identity.

From here, we can segregate zero physical weight cycles into those with zero logical weight (all logical operators as identity) and those that are not.

**Definition 11.** Let \( P_0 \) be the subgroup of memory states that are part of a zero physical weight cycle.

\( P_0 \) is a subgroup as taking the composition of two memory states in it will result in a zero physical weight cycle as well by simply traversing both cycles at once.

**Definition 12.** A standard path satisfies the following:

- **Initialization:** \( U\left( |M_0, S_0, I^{\otimes k}\rangle = |P_1, M_1\rangle \right) \)
- **Propagation:** \( U\left( |M_1, S_i, I^{\otimes k}\rangle = |P_{i+1}, M_{i+1}\rangle \right) \)

where \( S_i \in \{ I, Z, Z^2, ... \}^{\otimes n-k} \), \( P_i \in P_m \), and \( M_i \in P_m \).

We can now define two specific versions of standard paths: finite and infinite.

**Definition 13.** A finite standard path satisfies the conditions of a standard path, as well as the additional constraint that:

- **Termination:** \( U\left( |M_{i-1}, S_{i-1}, I^{\otimes k}\rangle = |P_i, I^{\otimes m}\right) \)

where none of the memory states along the way are the identity (as that would define a shorter finite standard path), and we call \( t \) the time of the path.

Clearly finite standard paths form a finite sized memory subgroup on \( P_m \), as taking the composition of two finite standard paths (where the composition is over the initial memory state) must also be a finite standard path and also terminate. We call this subgroup the finite memory subgroup and denote it \( F_0 \).

**Definition 14.** An infinite standard path must satisfy the conditions for a standard path and in addition, the memory state must never be the identity and must also never be the starting memory state for a finite standard path.

The collection of infinite standard paths form a set, but not a group. This can be seen by creating a starting memory state which is a composition of a finite standard path \( M_f \) and a memory state in the infinite standard path \( M_i \). The
composition will still be in the infinite standard path collection, but \((M_f \cdot M_j) \cdot M_i\) is in the finite memory collection, so the set of infinite standard paths cannot be a group, since it is not closed under composition. However, if we quotient out \(\mathcal{F}_0\), then this collection is a subgroup. This subgroup of purely infinite standard paths will be denoted \(\mathcal{I}_0\) and it is the case that \(\mathcal{F}_0 \cup \mathcal{I}_0 = \mathcal{P}_m\) (they form the complete set of all possible standard paths).

Now we proceed to show that the centralizer of \(\mathcal{F}_0\) is equivalent to the subgroup \(\mathcal{P}_0\).

**Definition 15.** The centralizer of a subset \(S\) of a group \(G\) is:

\[
C_G(S) = \{g \in G \mid gs = sg, \quad \forall s \in S\}
\]  

**Lemma 16.** Let \(U\) be a quantum convolutional encoder, and suppose we have a pair of input and output states which satisfy:

\[
U([M^1_1, S_1, L_1]) = [P_1, M^0_{out}]
\]  

\[
U([M^2_2, S_2, L_2]) = [P_2, M^0_{out}]
\]

then the commutation relations between the inputs and the commutations relations between the outputs must be equal.\(^1\)

**Proof.** The encoder \(U\) is composed of Clifford gates from its definition. By definition, the Clifford group carries Paulis to Paulis, and in particular they preserve the commutation relation between any two Paulis.\(^6\) Extending this to the commutation relation over the entire input space, the action of \(U\) will preserve the commutation relation over the entire output space, thus proving our lemma. \(\Box\)

**Lemma 17.** In the qubit case: \(\mathcal{P}_0\) commutes with all elements of \(\mathcal{F}_0\).

**Proof.** We compute the commutator between our procedures in the definitions of \(\mathcal{P}_0\) and \(\mathcal{F}_0\). Clearly for any step in either procedure the stabilizer elements will commute with each other. In addition, since in \(\mathcal{P}_0\) all physical Paulis are the identity they must commute with whatever physical Pauli is in \(\mathcal{F}_0\). Likewise, since the logical Paulis in \(\mathcal{F}_0\) are all the identity, the logical Paulis in \(\mathcal{P}_0\) must commute.

Lastly, we check that the memory operators will commute. For analysis, we write the sequences of the memory states for arbitrary members of each of the two groups:

- \(P \in \mathcal{P}_0\):
  \[
m^P_i \rightarrow m^P_{i+1}, \ldots, m^P_{r} \rightarrow m^P_{i+1}, \ldots, m^P_{t-1} \rightarrow I
\]

- \(F \in \mathcal{F}_0\):
  \[
m^F_i \rightarrow m^F_{i+1}, \ldots, m^F_{t-1} \rightarrow I
\]

where the superscripts indicate which group these are members of. Since each memory state is a Pauli, it either commutes or anti-commutes: we can represent this by \((-1)^{|P|Q}|\) for Paulis \(P, Q\). Now, each of the intermediary states appear twice when attempting to commute the memory sequences, so we pick up a \((-1)^{2\sum_{k} \sum_{l} |m^P_k, m^F_l|}\) for these commutators. Next, we must commute \(I\) past all the outputs from \(P\), but these trivially commute. Lastly, by lemma \([16]\) the input and output commutators must be equal, we know that the commutation of \(m^F_k\) must be equivalent to that of \(I\). Thus the overall commutation phase incurred is:

\[
(-1)^{2\sum_{k} \sum_{l} |m^P_k, m^F_l|+0+0} = 1
\]  

Thus these two will always commute no matter the choices of \(F\) and \(P\). \(\Box\)

Before proceeding to proving that this lemma no longer carries over to the qudit case, we show that all of the remainder of the proof carries over fine.

This lemma says that \(\mathcal{P}_0 \subset C(\mathcal{F}_0)\). We now proceed to show that \(C(\mathcal{F}_0) \subset \mathcal{P}_0\) and thus \(\mathcal{P}_0 = \mathcal{F}_0\). Before proceeding, we make a note about the particular case of taking as input to our encode the state \([I^\otimes m, Z^a_i, I^\otimes k]\). As noted earlier, this induces a stabilizer on the physical qudits and some memory state.

**Lemma 18 (Precipitation of Quantum Convolutional Codes).** Let \(U\) be a quantum convolutional encoder. Consider the repeated action of \(U\) with the initial state \([I^\otimes m, Z^a_i, I^\otimes k]\). Assuming the logical operator remains as the identity, this state is in \(\mathcal{F}_0\) for all \(a \in [0, p]\), and thus commutes with \(\mathcal{P}_0\).

**Proof.** The possible progressions are either for the encoder to carry the state through the memory Pauli space (which is finite), and eventually reach the identity, or for the encoder to enter into some loop at step \(t\), with the loop having length \(l\). The former case is trivial, so we analyze the latter. The geometry of this second case is essentially a lasso and we will introduce increasing overlaps of memory states at the knot.

We consider initializing with \(Z^a_i\) on the ancilla qudits. This begins the state traveling along its path. If we then apply another \(Z^a_i\) at time \(l-t\), at the memory qudits we produce the next memory operator from our first \(Z^a_i\), call it \(m^l_{t-1}\) and the operator \(m^l_t\) from our new \(Z^a_i\). Then at time \(t\), these two operators will overlap and will travel together as \(m^l_{t+k}\), for \(k \geq n\). We can repeat this procedure of introducing a new \(Z^a_i\) at the analogous time step and produce \(m^l_t = I\). We call this action the precipitation of the memory state, as we have increased its concentration to the point of crashing out. Thus the state will always return to the identity. \(\Box\)

**Lemma 19.** Let \(M' \in C(\mathcal{F}_0)\). Then there is a unique \(M \in C(\mathcal{F}_0), S \in \{I, Z\}^{\otimes n-k}\), and \(L \in \mathcal{P}_k\) such that:

\[
U([M, S, L]) = [I^\otimes n, M']
\]

**Proof.** Clearly we can apply the inverse encoder to backtrack from the vertex \(M'\) to some other vertex \(M\) via an edge \((S, L)\) for some \(M, S, L\). Since \(M' \in C(\mathcal{F}_0)\), then this will commute with all outputs initialized by \([I^\otimes m, Z_i, I^\otimes k]\), since the output of these are members of \(\mathcal{F}_0\). This means that the inputs must also commute if the total thing is to commute as shown in lemma \([17]\) which means that \(S \in \{I, Z, Z^2, \ldots\}^{\otimes n-k}\). Next, since \(M \in C(\mathcal{F}_0)\), then \(L\) must also be uniquely defined to satisfy the commutation relation. \(\Box\)
Lemma 20. Let \( M \in C(F_0) \). Then there is a unique \( M' \in C(F_0) \), \( S \in \{ I, Z, Z^2, \ldots \}^{\otimes n-k} \), and \( L \in P_k \) such that the encoder performs:
\[
U(|M, S, L|) = |I^{\otimes n}, M'\rangle
\]  
(14)

Proof. Let \( F_0 \) have a generating set represented by \( \{T_1, \ldots, T_l\} \) and let \( \{U_i+1, \ldots, U_{2m}\} \) be the generating set for \( F_0 \). Then, for every \( i \in \{l+1, \ldots, 2m\} \) there exists \( P^{(i)} \) and \( T_i \) such that:
\[
U([U_i, I^{\otimes n-k}, I^{\otimes k}]) = |P^{(i)}, T_i\rangle
\]  
(15)

Observe that \( T_i \) form a generating set \( \{T_{i+1}, \ldots, T_{2m}\} \) for \( F_0 \) as if this wasn’t the case and these \( T_i \) were not independent, we could take compositions of the \( U_i \) at the input and generate these \( T_i \) at the output and have a path to the identity operator on the memory state, which contradicts the fact that the \( U_i \) form a generating set for all the elements in \( F_0 \). Thus, \( \{T_{i+1}, \ldots, T_{2m}\} \) is a generating set for \( F_0 \) as well.

Now we select \( M' \) the unique element in \( P_m \) such that it commutes with all elements in \( \{T_1, \ldots, T_l\} \) and such that \( [T_i, M'] = [U_i, M] \). This uniquely defines \( M' \) and guarantees it to be in \( C(F_0) \).

Lastly, we consider the result of apply \( U^{-1} \) (the inverse encoder) to \( |I^{\otimes n}, M'\rangle \). By lemma 19 there is an \( M'' \in C(F_0) \), \( S \in \{ I, Z, Z^2, \ldots \}^{\otimes n-k} \), and \( L \in P_k \) such that:
\[
U([M'', S, L]) = |I^{\otimes n}, M'\rangle
\]  
(16)

but by this construction \( M'' = M \) since the commutation relations uniquely determine \( M \). Thus we have constructed a unique \( M' \in C(F_0) \), \( S \in \{ I, Z, Z^2, \ldots \}^{\otimes n-k} \), and \( L \in P_k \) satisfying the lemma for all \( M \in C(F_0) \).

Lemma 21. Suppose \( M \in C(F_0) \). Then \( M \in P_0 \), such that all states along the way are also in \( C(F_0) \).

Proof. By lemma 20 we know that we may transition from the initial state to another state which will also have its memory state in \( C(F_0) \), and from there continue onwards repeatedly. Thus we know we will always remain in \( C(F_0) \), and since this is a finite memory standard path, but this procedure may continue ad nauseum and we must eventually return to the initial memory state and thus are in \( P_0 \). We know that we cannot backtrack by lemma 20 and thus we must return to the initial memory state and not get trapped within an infinite loop.

Corollary 22. \( P_0 = C(F_0) \)

Proof of Theorem 8. Consider the following weight one logical initialization:
\[
U(|I^{\otimes m}, I^{\otimes n-k}, X_i\rangle) = |h, g\rangle
\]  
(17)

For a non-catastrophic encoder, all edges in a zero physical weight cycle must have zero logical weight so that the input of the above commutes with the inputs of the zero physical weight cycles. This implies that the outputs also commute. Since all zero physical weight cycles have the identity acting on the physical qubits, the outputted memory state \( g \) must commute with all elements in \( C(P_0) \). Then by our corollary, \( g \in F_0 \), then by the fact that \( F_0 \) is a finite subgroup then there exists a finite standard path that returns to the identity memory state, thus this encoder cannot be recursive.

Now we finally prove our main theorem.

Lemma 23. Let \( p \) be a prime greater than 2, then \( P_0^p \) only commutes with all elements of \( F_0 \) if and only if \( 2 \sum_{jk} c(m_j^F, m_k^F) \mod p = 0 \).

Proof. We proceed by computing the commutator. As before, we only need to deal with the memory states, as all the other components commute trivially. For analysis, we write the sequences of the memory states for arbitrary members of each of the two groups:

- \( P \in P_0^p \):
\[
m_1^P \rightarrow m_2^P, \ldots, m_i^P \rightarrow m_{i+1}^P, \ldots, m_r^P \rightarrow m_i^P
\]  
(18)

- \( F \in F_0^p \):
\[
m_0^F \rightarrow m_1^F, \ldots, m_i^F \rightarrow m_{i+1}^F, \ldots, m_{i-1}^F \rightarrow I
\]  
(19)

where the superscripts indicate which group these members of. Now to compute the commutator, we take note of our definition for generalized Pauli commutation. For the moment, we focus on the outputs for all but the final step. Commuting these entries we pick up a \( \omega^{c(m_j^F, m_k^F)} \) when swapping the order of \( m_j^F \) and \( m_k^F \). We notice, however, that each time some memory state appears in the output it also appears on the input, thus the total commutation will involve the square of the commutation factor from before. Next, we need to commute the identity through, which trivially passes through and we denote the extra phase by adding to the power of \( \omega \) with \( \sum_k |I^F, m_k^F\rangle \) and likewise for the input \( m_k^F \). This gives us total commutation phase as:
\[
\omega^2 \sum_{jk} c(m_j^F, m_k^F) + \sum_k |I^F, m_k^F\rangle + \sum_{jk} |m_0^F, m_k^F\rangle
\]  
(20)

clearly the commutation for the sum over \( k \) is always 0, and by lemma 16 the summed commutator over \( j \) must also always be 0. Thus, \( [F, P] = 0 \) if and only if:
\[
2 \sum_{jk} c(m_j^F, m_k^F) \mod p = 0
\]  
(21)

Theorem 24. A recursive qudit quantum convolutional code, with prime space dimension \( p > 2 \), is not catastrophic so long as
\[
\sum_{jk} c(m_j^F, m_k^F) \mod p \neq 0
\]  
(22)

where \( c(\cdot, \cdot) \) is the generalized Pauli commutator recalled above.

Proof. The prior lemma provides the condition required for a single convolutional code not to commute. Then note that any recursive code will simply involve iteratively performing these small convolutional codes, thus the net recursive convolutional code will be not catastrophic so long as the condition stated is satisfied.
In short, the qubit case fails since the numerator is always divisible by 2.

By going through the entire proof and verifying its validity in all other locations when extended to qudit systems, we know that this no-go theorem will no longer apply if and only if our criterion is met. If our criterion is not met then the code will immediately become catastrophic just like the qubit case.

With this new result, a major limitation in implementing quantum turbo codes has been lifted, potentially allowing for this method of encoding information to be successfully implemented.

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