The global geometry of surfaces with prescribed mean curvature in $\mathbb{R}^3$

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Abstract

We develop a global theory for complete hypersurfaces in $\mathbb{R}^{n+1}$ whose mean curvature is given as a prescribed function of its Gauss map. This theory extends the usual one of constant mean curvature hypersurfaces in $\mathbb{R}^{n+1}$, and also that of self-translating solitons of the mean curvature flow. Among other topics, we will study existence and geometric properties of compact examples, existence and classification of rotational hypersurfaces, and stability properties. For the particular case $n = 2$, we will obtain results regarding a priori height and curvature estimates, non-existence of complete stable surfaces, and classification of properly embedded surfaces with at most one end.

1 Introduction

Let $\mathcal{H}$ be a $C^1$ function on the sphere $S^n$. We will say that an immersed oriented hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ has prescribed mean curvature $\mathcal{H}$ if its mean curvature function $H_\Sigma$ is given by

$$H_\Sigma = \mathcal{H} \circ \eta,$$

where $\eta : \Sigma \to S^n$ is the Gauss map of $\Sigma$. For short, we will simply say that $\Sigma$ is an $\mathcal{H}$-hypersurface. Obviously, when $\mathcal{H}$ is constant, $\Sigma$ is a hypersurface of constant mean curvature (CMC).

Mathematics Subject Classification: 53A10, 53C42
The study of hypersurfaces in $\mathbb{R}^{n+1}$ defined by a prescribed curvature function in terms of the Gauss map goes back, at least, to the famous Christoffel and Minkowski problems for ovaloids (see e.g. [Ch]). The existence and uniqueness of ovaloids of prescribed mean curvature in $\mathbb{R}^{n+1}$ was studied among others by Alexandrov and Pogorelov in the 1950s (see [Al, Po]). However, the geometry of complete hypersurfaces of prescribed mean curvature in $\mathbb{R}^{n+1}$ remains largely unexplored. The most studied situation is, obviously, the case of complete CMC hypersurfaces in $\mathbb{R}^{n+1}$. Recently, the global geometry of self-translating solitons of the mean curvature flow (which correspond to $\mathcal{H}$-hypersurfaces for the particular choice $\mathcal{H}(x) = \langle x, e_{n+1} \rangle$) is being studied in detail, see e.g. [CSS, DDN, HS, IR, MPSS, MSS, Sm, SX].

Our objective in this paper is to develop a global theory of complete $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$, taking as a starting point the well-studied global theory of CMC hypersurfaces in $\mathbb{R}^{n+1}$. We will be specially interested in the case of complete $\mathcal{H}$-surfaces in $\mathbb{R}^3$, which is the case for which most of our main results will be obtained; for example, the ones regarding the description of properly embedded $\mathcal{H}$-surfaces of finite topology, and the geometry of stable $\mathcal{H}$-surfaces in $\mathbb{R}^3$.

The following are three trivial but fundamental properties of $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$, for a given $\mathcal{H} \in C^1(S^n)$: (1) any translation of an $\mathcal{H}$-hypersurface is also an $\mathcal{H}$-hypersurface; (2) $\mathcal{H}$-hypersurfaces are locally modeled by a quasilinear elliptic PDE when viewed as local graphs in $\mathbb{R}^{n+1}$ over each tangent hyperplane, and consequently they obey the maximum principle; and (3) any symmetry of $\mathcal{H}$ in $S^n$ induces a linear isometry of $\mathbb{R}^{n+1}$ that preserves $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$.

These three properties make many ideas of classical CMC hypersurface theory in $\mathbb{R}^{n+1}$ readily available for the case of $\mathcal{H}$-hypersurfaces, for a non-constant function $\mathcal{H} \in C^1(S^n)$. Nonetheless, the class of $\mathcal{H}$-hypersurfaces is indeed much wider and richer, and new ideas are needed for its study. For example, $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ do not come in general associated to a variational problem, and this makes many of the most useful techniques of CMC theory unavailable. Also, in contrast with CMC surfaces in $\mathbb{R}^3$, there is no holomorphic object associated to $\mathcal{H}$-surfaces in $\mathbb{R}^3$ for non-constant $\mathcal{H} \in C^1(S^2)$, and so, no clear integrable systems approach to their study. Moreover, in jumping from the CMC condition to the equation of prescribed non-constant mean curvature, one also needs to account for the loss of symmetries and isotropy of the resulting equation. For instance, one can only apply the Alexandrov reflection principle for directions with respect to which $\mathcal{H}$ is symmetric.

Some of these difficulties can already be seen in the case of rotational $\mathcal{H}$-surfaces in $\mathbb{R}^3$. In Section 3 we will present many examples of rotational $\mathcal{H}$-surfaces in $\mathbb{R}^3$ with no CMC counterpart: for instance, complete, convex $\mathcal{H}$-graphs converging to a cylinder, or properly embedded disks asymptotically wiggling around a cylinder. For the case of rotational $\mathcal{H}$-surfaces of non-trivial topology, we will show examples with a wing-like shape, or with two strictly convex ends pointing in opposite directions. Many of these examples can also be constructed so that they self-intersect. All this variety, just for the very particular class of rotational $\mathcal{H}$-surfaces in $\mathbb{R}^3$, shows that the class of $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ is indeed very large, and rich in what refers to possible examples
and geometric behaviors. It also accounts for the difficulty of obtaining classification results for the case where $\mathcal{H}$ is not constant, and justifies the need of imposing some additional regularity, positivity or symmetry properties to $\mathcal{H}$ in order to derive such classification theorems.

Despite these difficulties, the results in the present paper show that, under mild assumptions, the class of $\mathcal{H}$-surfaces in $\mathbb{R}^3$ can sometimes be given a homogeneous treatment for general classes of prescribed mean curvature functions $\mathcal{H} \in C^1(S^2)$.

In the rest of this introduction we will explain the organization of the paper, and state some of our main results.

In Section 2 we will study some basic properties of $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ and introduce some examples. In Proposition 2.6 we prove that if $\mathcal{H}(x_0) = 0$ for some $x_0 \in S^n$, then there are no compact $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$. Also, using the Alexandrov reflection principle, we prove in Proposition 2.8 that if $\mathcal{H} \in C^1(S^n)$ is invariant under $n$ independent geodesic reflections on $S^n$, then any compact embedded $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$ is diffeomorphic to $S^n$.

A fundamental theorem by B. Guan and P. Guan (see [GG]) proves that if $\mathcal{H} \in C^2(S^n)$, $\mathcal{H} > 0$, is invariant under a group of isometries of $S^n$ without fixed points, then there exists a compact strictly convex $\mathcal{H}$-sphere in $\mathbb{R}^{n+1}$, which will be called from now on the Guan-Guan $\mathcal{H}$-sphere and denoted by $S_{\mathcal{H}}$. For such a choice of $\mathcal{H}$, any other strictly convex $\mathcal{H}$-sphere in $\mathbb{R}^{n+1}$ is a translation of $\mathbb{R}^{n+1}$ (for $n = 2$, this is a classical theorem by Alexandrov [Al], see also [HW, Po]). In [GM1] Gálvez and Mira proved more generally that, for $n = 2$, any compact $\mathcal{H}$-surface of genus zero immersed in $\mathbb{R}^3$ is a translation of $S_{\mathcal{H}}$.

In Corollary 2.11 we will obtain from these results that if $\mathcal{H} \in C^2(S^2)$ is positive and invariant under a group of isometries of $S^2$ without fixed points that contains two independent reflection symmetries, then any compact embedded $\mathcal{H}$-surface in $\mathbb{R}^3$ is a translation of the Guan-Guan sphere $S_{\mathcal{H}}$.

In Section 2.4 we will construct examples of $\mathcal{H}$-hypersurfaces. In Proposition 2.12 we will give an existence theorem for $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ of the form $S \times \mathbb{R}^{n-k}$, where $S$ is a compact strictly convex hypersurface in $\mathbb{R}^{k+1}$. In Proposition 2.14 we will classify all complete $\mathcal{H}$-hypersurfaces of constant sectional curvature in $\mathbb{R}^{n+1}$. In Proposition 2.16 we will recall, following [Mar], very general conditions under which one can solve the Dirichlet problem associated to the equation of $\mathcal{H}$-graphs in $\mathbb{R}^{n+1}$.

In Section 2.5 we will prove a compactness theorem for the space of $\mathcal{H}$-surfaces in $\mathbb{R}^3$ with bounded second fundamental form, with $\mathcal{H}$ not necessarily fixed. This result (Theorem 2.17) will be key for our purposes in Sections 4 and 5.

In Section 3 we consider rotational $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$, in the case that $\mathcal{H} \in C^1(S^n)$ is rotationally symmetric, i.e. $\mathcal{H}(x) = h(\langle x, e_{n+1} \rangle)$ for some $h \in C^1([-1, 1])$. In such generality for $\mathcal{H}$, it seems hopeless to find an explicit description of such rotational $\mathcal{H}$-hypersurfaces, similar to the cases of CMC hypersurfaces, or of self-translating solitons of the mean curvature flow. Thus, we will follow a different approach. We will treat the resulting ODE as a nonlinear autonomous system and we will carry out a qualitative study of its solutions through a phase space analysis.
By using this method, we will show that any rotational $H$-sphere in $\mathbb{R}^{n+1}$ is strictly convex (Theorem 3.4). When $H$ vanishes at some point, we will describe in Section 3.2 for very general choices of $H$, a family of rotational $H$-bowls (which are entire convex graphs) and of $H$-catenoids (which resemble the usual minimal catenoids in $\mathbb{R}^{n+1}$).

In Section 3.3 we will prove a classification theorem for rotational $H$-hypersurfaces in $\mathbb{R}^{n+1}$, in the case that $H \in C^2(\mathbb{S}^n)$ is positive, rotationally symmetric and even (i.e. $H(x) = H(-x) > 0$ for every $x \in \mathbb{S}^n$). In these general conditions, we will show in Theorem 3.7 that the geometry of such rotational $H$-hypersurfaces follows the same pattern as the classical Delaunay classification of rotational hypersurfaces of non-zero constant mean curvature in $\mathbb{R}^{n+1}$. That is, all such examples are convex spheres, right circular cylinders, properly embedded surfaces of *unduloid type*, or proper, non-embedded surfaces of *nodoid type*.

In contrast with this classification theorem and the existence of bowls and catenoids, in Section 3.4 we will show that there exist many rotational $H$-surfaces in $\mathbb{R}^3$ which do not behave at all like CMC surfaces in $\mathbb{R}^3$.

In Section 4 we study the geometry of properly embedded $H$-surfaces of finite topology in $\mathbb{R}^3$, motivated by some well-known results by Meeks [Me] and Korevaar-Kusner-Solomon [KKS] for the case where $H$ is a positive constant. In [Me], Meeks proved that there are no properly embedded surfaces of positive CMC in $\mathbb{R}^3$ with finite topology and exactly one end. For that, he first obtained universal height estimates for (not necessarily compact) CMC graphs in $\mathbb{R}^3$ with planar boundary. In [KKS] it was proved that any CMC surface in $\mathbb{R}^3$ in the conditions above, but this time with two ends, is a rotational surface.

For the general case of $H$-surfaces in $\mathbb{R}^3$ with $H \in C^1(\mathbb{S}^2)$, $H > 0$, these statements are not true in general. However, our main results in Section 4 give natural additional conditions on $H$ under which Meeks’ results hold for $H$-surfaces.

Specifically, in Theorem 4.5 we will give some very general sufficient conditions on $H$ for the existence of a universal height estimate for $H$-graphs with respect to a given direction $v \in \mathbb{S}^2$ in $\mathbb{R}^3$. The way of proving these height estimates is completely different from the one used by Meeks, and relies on a previous curvature estimate for surfaces in $\mathbb{R}^3$ in the spirit of a result on CMC surfaces in Riemannian three-manifolds by Rosenberg, Sa Earp and Toubiana [RST]; see Theorem 4.2.

As corollaries to Theorem 4.5 in Section 4.4 we will provide several results about the geometry of properly embedded $H$-surfaces in $\mathbb{R}^3$ of finite topology and one end, in the case that $H > 0$ is invariant under one, two or three reflections in $\mathbb{S}^2$. For instance, in the case that $H$ is invariant with respect to three linearly independent reflections in $\mathbb{S}^2$, we will prove (Theorem 4.10) that any properly embedded $H$-surface in $\mathbb{R}^3$ of finite topology and at most one end is the Guan-Guan sphere $S_H$.

In Section 5, we study stability of $H$-hypersurfaces. As we already mentioned, except for some very particular cases, the equation describing $H$-hypersurfaces in $\mathbb{R}^{n+1}$ is not the Euler-Lagrange equation of some variational problem. Thus, the way in which we introduce the concept of stability for $H$-hypersurfaces (Definition 5.2) is not in a variational way, but in connection with the *linearized equation* of the $H$-graph equation.
In this way we obtain, for any $\mathcal{H}$-hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$, a linear stability operator of the form

$$L = \Delta + \langle \nabla \cdot X, X \rangle + q \quad \text{(1.2)}$$

for some $X \in \mathfrak{X}(\Sigma)$ and $q \in C^2(\Sigma)$. With this notion at hand, we will show that it is natural to define, in analogy with the CMC case, a stable $\mathcal{H}$-hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ as one for which there exists $u \in C^2(\Sigma)$, $u > 0$, with $L(u) \leq 0$. This definition is consistent not only with the stability notion in CMC hypersurface theory, but also with the ones in the theories of self-translating solitons of the mean curvature flow, and of marginally outer-trapped surfaces (MOTS). This stability notion also implies the non-negativity of the principal eigenvalue of the (non self-adjoint) operator $-L$. One easy consequence of this notion will be that there are no compact (without boundary) stable $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$; see Corollary 5.7.

From that point on, we will focus on stable $\mathcal{H}$-surfaces $\Sigma$ in $\mathbb{R}^3$, and will seek radius and curvature estimates for $\Sigma$. In the CMC case, Ros and Rosenberg proved in [RR] that the intrinsic distance from a point $p$ in a stable CMC surface $\Sigma$ to $\partial \Sigma$ is less than $\pi/H$, where $H > 0$ is the mean curvature of $\Sigma$. In [Ma1], Mazet improved the previous result obtaining the optimal estimate $\pi/(2H)$.

Our approach to proving this type of radius estimates for stable $\mathcal{H}$-surfaces will be different, and based on arguments introduced by Fischer-Colbrie [Fi] and López-Ros [LR] for non-negative Schrodinger operators, i.e. operators of the form (1.2) for $X = 0$. We will also make use of an argument by Galloway and Schoen [GS], which will let us bound the (non-Schrodinger) stability operator $L$ in (1.2) by a non-negative Schrodinger-type one; see Lemma 5.8.

By using these ideas, we will prove (Theorem 5.10) that there exists a uniform radius estimate for stable $\mathcal{H}$-surfaces in $\mathbb{R}^3$, in the case that $\mathcal{H} \in C^2(S^2)$ is positive and satisfies a certain additional condition, see inequality (5.15). As a trivial consequence we obtain (Corollary 5.13): if $\mathcal{H} \in C^2(S^2)$ satisfies condition (5.15), there are no complete stable $\mathcal{H}$-surfaces in $\mathbb{R}^3$.

It is important to observe here that some condition on $\mathcal{H} > 0$ is needed in order to prove a radius estimate for stable $\mathcal{H}$-surfaces. For instance, for some radially symmetric choices of $\mathcal{H} > 0$, there exist complete, non-entire, strictly convex $\mathcal{H}$-graphs in $\mathbb{R}^3$; see, e.g., the example in Figure 3.13.

Also, in Theorem 5.15 we will prove a curvature estimate, of the form

$$|\sigma(p)|d_\Sigma(p, \partial \Sigma) \leq C,$$

for stable $\mathcal{H}$-surfaces $\Sigma$ in $\mathbb{R}^3$, where $\mathcal{H} \in C^2(S^2)$ satisfies (5.15); here $\sigma$ is the second fundamental form of $\Sigma$. For the case of minimal surfaces and CMC surfaces in $\mathbb{R}^3$, such estimates were obtained by Schoen [Sc] and Berard-Hauswrith [BH], respectively.

Finally, in Section 6 we propose a list of open problems of the theory of $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ that we find of special interest. These problems refer to topics such as general properties of $\mathcal{H}$-hypersurfaces, classification of compact $\mathcal{H}$-hypersurfaces, description of properly embedded $\mathcal{H}$-surfaces of finite topology in $\mathbb{R}^3$, or stable $\mathcal{H}$-surfaces.
Acknowledgements: The authors are grateful to José M. Espinar, Francisco Martín, Joaquín Pérez, Luis A. Sánchez and Francisco Torralbo for helpful comments during the preparation of this manuscript. This work is part of the PhD thesis of the first author.

2 Basic properties of $\mathcal{H}$-hypersurfaces

2.1 Maximum and tangency principles

Locally, $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ for some given $\mathcal{H} \in C^1(S^n)$ are governed by an elliptic, second order quasilinear PDE. Specifically, let $\mathcal{H} \in C^1(S^n)$, let $\Sigma$ denote an $\mathcal{H}$-hypersurface, and take any $q \in \Sigma$. Then, if we view $\Sigma$ around $q$ as an upwards-oriented graph $x_{n+1} = u(x_1, \ldots, x_n)$ with respect to coordinates $x_i$ in $\mathbb{R}^{n+1}$ such that $\{\partial_1, \ldots, \partial_{n+1}\}$ is a positively oriented orthonormal frame of $\mathbb{R}^{n+1}$, equation (1.1) shows that the function $u$ is a solution to

$$\text{div} \left( \frac{Du}{\sqrt{1+|Du|^2}} \right) = n\mathcal{H}(Z_u), \quad Z_u := \frac{(-Du, 1)}{\sqrt{1+|Du|^2}},$$

(2.1)

where $\text{div}, D$ denote respectively the divergence and gradient operators on $\mathbb{R}^n$. In particular, $\mathcal{H}$-hypersurfaces satisfy the Hopf maximum principle (both in its interior and boundary versions), a property that we will use geometrically in the following way:

**Lemma 2.1** Given $\mathcal{H} \in C^1(S^n)$, let $\Sigma_1, \Sigma_2$ be two embedded $\mathcal{H}$-hypersurfaces, possibly with smooth boundary. Assume that one of the following two conditions holds:

1. There exists $p \in \text{int}(\Sigma_1) \cap \text{int}(\Sigma_2)$ such that $\eta_1(p) = \eta_2(p)$, where $\eta_i : \Sigma_i \to S^n$ is the unit normal of $\Sigma_i$, $i = 1, 2$.

2. There exists $p \in \partial \Sigma_1 \cap \partial \Sigma_2$ such that $\eta_1(p) = \eta_2(p)$ and $\nu_1(p) = \nu_2(p)$, where $\nu_i$ denotes the interior unit conormal of $\partial \Sigma_i$.

Assume moreover that $\Sigma_1$ lies around $p$ at one side of $\Sigma_2$. Then $\Sigma_1 = \Sigma_2$.

In the case that $\mathcal{H}$ is an odd function, i.e. $\mathcal{H}(-x) = -\mathcal{H}(x)$ for every $x \in S^n$, any $\mathcal{H}$-hypersurface is also an $\mathcal{H}$-hypersurface with the opposite orientation. Hence, the tangency principle stated in Lemma 2.1 can be formulated in a stronger way, similar to the usual tangency principle for minimal hypersurfaces. In particular, we have:

**Corollary 2.2** Let $\mathcal{H} \in C^1(S^n)$ satisfy $\mathcal{H}(-x) = -\mathcal{H}(x)$ for every $x \in S^n$, and let $\Sigma_1, \Sigma_2$ denote two immersed $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$. Assume that there exists $p \in \Sigma_1 \cap \Sigma_2$ with $T_p\Sigma_1 = T_p\Sigma_2$ and such that $\Sigma_1$ lies around $p$ at one side of $\Sigma_2$. Then $\Sigma_1 = \Sigma_2$.

It is immediate that any translation of an $\mathcal{H}$-hypersurface is again an $\mathcal{H}$-hypersurface. Moreover, the possible symmetries of the function $\mathcal{H}$ induce further invariance properties of the class of $\mathcal{H}$-hypersurfaces, as detailed in the next lemma:
Lemma 2.3 Given \( \mathcal{H} \in C^1(S^n) \), let \( \Sigma \) be an \( \mathcal{H} \)-hypersurface, and let \( \Phi \) be a linear isometry of \( \mathbb{R}^{n+1} \) such that \( \mathcal{H} \circ \Phi = \mathcal{H} \) on \( S^n \subset \mathbb{R}^{n+1} \). Then \( \Sigma' = \Phi \circ \Sigma \) is an \( \mathcal{H} \)-hypersurface in \( \mathbb{R}^{n+1} \) with respect to the orientation given by \( \eta' := d\Phi(\eta) \).

Remark 2.4 The notion of a surface of prescribed mean curvature \( \mathcal{H} \) can be generalized to the context of oriented hypersurfaces in \( n \)-dimensional Lie groups with a left invariant metric. Specifically, if \( \psi : M^n \to G^{n+1} \) is a hypersurface in such a Lie group \( G \), we can define its left invariant Gauss map \( \eta : M \to S^n = \{ v \in g : |v| = 1 \} \) by left translating to the identity element \( e \) of \( G \) the unit normal vector of \( \psi \) at every point. Thus, given \( \mathcal{H} \in C^1(S^n) \), we may define a hypersurface of prescribed mean curvature \( \mathcal{H} \) by equation (1.1). See e.g. [GM1].

2.2 A particular case: self-translating solitons of mean curvature flows

We consider next the particular case in which \( \mathcal{H} \in C^1(S^n) \) is a linear function, i.e. \( \mathcal{H}(x) = a \langle x, v \rangle + b \) for \( a, b \in \mathbb{R}, a \neq 0 \), and \( v \in S^n \). Up to a homothety and a change of Euclidean coordinates in \( \mathbb{R}^{n+1} \), we can assume that \( a = 1 \) and that \( v = e_{n+1} \), and so

\[
\mathcal{H}(x) = \langle x, e_{n+1} \rangle + b. \tag{2.2}
\]

Let \( f : \Sigma \to \mathbb{R}^{n+1} \) be an immersed oriented \( \mathcal{H} \)-hypersurface for \( \mathcal{H} \) as in (2.2), let \( \eta : \Sigma \to S^n \) be its unit normal, and consider the family of translations of \( f \) in the \( e_{n+1} \) direction, given by \( F(p,t) = f(p) + te_{n+1} \). Then, \( F(p,t) \) is a solution to the geometric flow below, which corresponds to the mean curvature flow with a constant forcing term:

\[
\left( \frac{\partial F}{\partial t} \right)^\perp = (H - b)\eta, \tag{2.3}
\]

where \( H = H(\cdot, t), \eta = \eta(\cdot, t) \) denote the mean curvature and unit normal of the hypersurface \( F(\cdot, t) : \Sigma \to \mathbb{R}^{n+1} \). In other words, \( f : \Sigma \to \mathbb{R}^{n+1} \) is a self-translating soliton of the geometric flow (2.3). Note that when \( b = 0 \), (2.3) is the usual mean curvature flow (in this paper we use the convention for \( H \) that the mean curvature of the unit sphere in \( \mathbb{R}^{n+1} \) with its inner orientation is equal to 1). The converse also holds, i.e. any self-translating soliton to (2.3) is an \( \mathcal{H} \)-hypersurface in \( \mathbb{R}^{n+1} \) for \( \mathcal{H} \) as in (2.2).

There is a second characterization for \( \mathcal{H} \)-hypersurfaces with \( \mathcal{H} \) given by (2.2). For that, recall following Gromov [Gr] (see also [BCMR]), that the weighted mean curvature \( H_\phi \) of an oriented hypersurface \( \Sigma \) in \( \mathbb{R}^{n+1} \) with respect to the density \( e^\phi \in C^1(\mathbb{R}^{n+1}) \) is given by

\[
H_\phi = nH_\Sigma - \langle \eta, D\phi \rangle, \tag{2.4}
\]

where \( H_\Sigma \) is the mean curvature of \( \Sigma \) in \( \mathbb{R}^{n+1} \), \( \eta \) is its unit normal, and \( D \) denotes the gradient in \( \mathbb{R}^{n+1} \). So, by (2.4) and the previous discussion we arrive at:

**Proposition 2.5** Let \( \Sigma \) be an immersed oriented hypersurface in \( \mathbb{R}^{n+1} \). The following three conditions are equivalent:
1. Σ is an \( \mathcal{H} \)-hypersurface, for \( \mathcal{H} \) given by (2.2).

2. Σ is a self-translating soliton of the mean curvature flow with a constant forcing term (2.3).

3. Σ has constant weighted mean curvature \( H_\phi = nb \in \mathbb{R} \) for the density \( e^\phi \) in \( \mathbb{R}^{n+1} \), where \( \phi(x) := n(x,e_{n+1}) \).

The class of \( \mathcal{H} \)-hypersurfaces in \( \mathbb{R}^{n+1} \) for \( \mathcal{H} \) of the form (2.2) is also related to the volume preserving mean curvature flow introduced by Huisken [Hu2]. Specifically, for Σ compact (maybe with non-empty boundary) in \( \mathbb{R}^{n+1} \), this flow is given, with the notation of (2.3), by

\[
\left( \frac{\partial F}{\partial t} \right) \perp = (H - \overline{H}) \eta, \tag{2.5}
\]

where

\[
\overline{H} = \overline{H}(t) = \frac{1}{V_t} \int_{\Sigma_t} H(\cdot, t) dV_t,
\]

where \( \Sigma_t = F(\cdot, t) \) and \( V_t \) is the volume of \( \Sigma_t \).

Hence, in the particular case that \( \Sigma_t = \Sigma + te_{n+1}, \overline{H}(t) \) is a constant \( b \), and any \( \mathcal{H} \)-hypersurface \( \Sigma \) with \( \mathcal{H}(x) \) given by (2.2) is a self-translating soliton for (2.5).

As explained in the introduction, there are many relevant works on self-translating solitons of the mean curvature flow (which corresponds to the case \( b = 0 \) in (2.2)). The situation when \( b \neq 0 \) in (2.2) is much less studied. Some particular references for this situation are [Es, Lo].

### 2.3 Compact \( \mathcal{H} \)-hypersurfaces in \( \mathbb{R}^{n+1} \)

**Proposition 2.6** Let \( \mathcal{H} \in C^1(S^n) \), and assume that \( \mathcal{H}(x_0) = 0 \) for some \( x_0 \in S^n \). Then, there are no compact \( \mathcal{H} \)-hypersurfaces in \( \mathbb{R}^{n+1} \).

**Proof:** Take \( \mathcal{H} \in C^1(S^n) \) with \( \mathcal{H}(x_0) = 0 \) for some \( x_0 \in S^n \), and let \( \Sigma \) denote a compact immersed \( \mathcal{H} \)-hypersurface in \( \mathbb{R}^{n+1} \). Let \( \Pi \) denote an oriented hyperplane in \( \mathbb{R}^{n+1} \) with unit normal \( x_0 \), and such that \( \Sigma \) is contained in the half-space \( \Pi^+ := \cup \{ \Pi_\lambda : \lambda > 0 \} \), where \( \Pi_\lambda := \Pi + \lambda x_0 \). Let \( \lambda_0 \) denote the smallest \( \lambda > 0 \) for which \( \Sigma \cap \Pi_\lambda \neq \emptyset \), and let \( p \in \Sigma \cap \Pi_{\lambda_0} \) be any of such first contact points. Note that \( \eta(p) = \pm x_0 \), where \( \eta : \Sigma \to S^n \) is the unit normal to \( \Sigma \).

If \( \eta(p) = x_0 \), then both \( \Sigma, \Pi_{\lambda_0} \) are \( \mathcal{H} \)-hypersurfaces and we obtain a contradiction with the maximum principle (Lemma 2.1). Hence, \( \eta(p) = -x_0 \). Consequently, the principal curvatures \( \kappa_i(p) \) of \( \Sigma \) at \( p \) are \( \leq 0 \), and hence \( \mathcal{H}(-x_0) \leq 0 \). As a matter of fact, \( \mathcal{H}(-x_0) < 0 \), for if \( \mathcal{H}(-x_0) = 0 \), then \( \Pi_{\lambda_0} \) with its opposite orientation is also an \( \mathcal{H} \)-hypersurface, and we contradict again Lemma 2.1.

Let now \( \lambda_1 \) be the largest \( \lambda > 0 \) for which \( \Sigma \cap \Pi_\lambda \neq \emptyset \), and take \( q \in \Sigma \cap \Pi_{\lambda_1} \). By previous arguments, \( \eta(q) = -x_0 \). Since \( H_\Sigma(q) < 0 \) (because \( \mathcal{H}(-x_0) < 0 \)), one of the
principal curvatures $\kappa_i(q)$ of $\Sigma$ at $q$ is negative. This is a contradiction with the fact that $\Sigma \cap \Pi_\lambda$ is empty for all $\lambda > \lambda_1$. This contradiction proves Proposition 2.6

By Proposition 2.6 we see that given $\mathcal{H} \in C^1(\mathbb{S}^n)$, a necessary condition for the existence of a compact $\mathcal{H}$-hypersurface $\Sigma$ is that $\mathcal{H} > 0$ or $\mathcal{H} < 0$ on $\mathbb{S}^n$ (up to a change of orientation, we can assume $\mathcal{H} > 0$ in these conditions). However, this is not a sufficient condition. To see this, let $\psi : \Sigma \to \mathbb{R}^{n+1}$ denote a compact immersed hypersurface of prescribed mean curvature $\mathcal{H} \in C^1(\mathbb{S}^n)$, take $v \in \mathbb{S}^n$ and denote $h = \langle \psi, v \rangle$. It is then well known that $\Delta h = nH_\Sigma \langle \eta, v \rangle$, where $\eta$ denotes the unit normal of $\Sigma$ and $\Delta$ is the Laplacian on $\Sigma$. By the divergence theorem, and since $H_\Sigma = \mathcal{H} \circ \eta$, we have

$$\int_{\Sigma} \langle \eta, v \rangle H(\eta) = 0.$$  (2.6)

From (2.6), and noting that $\int_{\Sigma} \langle \eta, v \rangle = 0$ for every $v \in \mathbb{S}^n$ if $\Sigma$ is compact, we have:

**Corollary 2.7** Let $\mathcal{H} \in C^1(\mathbb{S}^n)$ be of the form $\mathcal{H}(x) = h_0(x) + b$, where $b \in \mathbb{R}$ and $h_0 \in C^1(\mathbb{S}^n)$ is not identically zero and satisfies $h_0(x)\langle x, v \rangle \geq 0$ for every $x \in \mathbb{S}^n$ and for some $v \in \mathbb{S}^n$.

Then, there are no compact $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$.

The statement of Corollary 2.7 for the particular case $\mathcal{H}(x) = \langle x, v \rangle + b$, i.e. for hypersurfaces of constant weighted mean curvature with respect to a density $e^\phi$ with $\phi$ linear (see Proposition 2.5), is due to Rafael López, see [Lo].

Lemmas 2.1 and 2.3 allow to use the Alexandrov reflection principle in any direction in $\mathbb{R}^{n+1}$ with respect to which the function $\mathcal{H}$ is symmetric. We singularize the following consequence.

**Proposition 2.8** Let $\mathcal{H} \in C^1(\mathbb{S}^n)$ be invariant with respect to $n$ linearly independent geodesic reflections $T_1, \ldots, T_n$ of $\mathbb{S}^n$. Then, any compact embedded $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$ is diffeomorphic to $\mathbb{S}^n$. Moreover, $\Sigma$ is a symmetric bi-graph with respect to hyperplanes $\Pi_1, \ldots, \Pi_n$ parallel to the hyperplanes $\Pi_i$ of $\mathbb{R}^{n+1}$ fixed by $T_i$.

**Proof:** Let $v_1, \ldots, v_n$ be unit normal vectors to $\Pi_1, \ldots, \Pi_n$. By applying the Alexandrov reflection technique with respect to these directions, we conclude using Lemma 2.1 and Lemma 2.3 that, for each $i$, $\Sigma$ is a symmetric bi-graph with respect to some hyperplane $\Pi_i$ of $\mathbb{R}^{n+1}$ parallel to $\Pi_i$. Define now $\Gamma := \cap_{i=1}^n \Pi_i$, which is a straight line in $\mathbb{R}^{n+1}$. For definiteness, we will assume that $\Gamma$ is the $x_{n+1}$-axis, and so each $v_i$ is horizontal, i.e. $\langle v_i, e_{n+1} \rangle = 0$.

Let $P_t = \{x_{n+1} = t\} \subset \mathbb{R}^{n+1}$. By compactness of $\Sigma$, there is a smallest interval $[a, b] \subset \mathbb{R}$ such that $P_t \cap \Sigma = \emptyset$ if $t \not\in [a, b]$. Also, as $\Sigma$ is a (connected) symmetric bi-graph with respect to the horizontal directions $v_1, \ldots, v_n$, it is clear that $P_a \cap \Sigma = \{(0, a)\}$ and $P_b \cap \Sigma = \{(0, b)\}$, and that $P_t \cap \Sigma$ is non-empty and transverse for all $t \in (a, b)$.

Let $\Omega \subset \mathbb{R}^{n+1}$ be the domain bounded by $\Sigma$, and define $\Omega_t := \Omega \cap P_t$ for $t \in (a, b)$. Note that for $t < b$ sufficiently close to $b$, $\Omega_t$ is diffeomorphic to an $n$-dimensional ball.
\(B_n\). As the intersection \(P_t \cap \Sigma\) is transverse for all \(t \in (a,b)\), we deduce that \(\Omega_t\) is diffeomorphic to \(B_n\) for all \(t \in (a,b)\). This implies directly that \(\Omega\) is diffeomorphic to \(\mathbb{B}_{n+1}\) and so, that \(\Sigma\) is diffeomorphic to \(\mathbb{S}^n\).

\[\Box\]

Proposition 2.8 is an extension to \(H\)-hypersurfaces of the famous Alexandrov theorem according to which compact embedded CMC hypersurfaces in \(\mathbb{R}^{n+1}\) are round spheres. The other fundamental result for compact CMC surfaces in \(\mathbb{R}^3\) is Hopf’s theorem (i.e. CMC surfaces diffeomorphic to \(\mathbb{S}^2\) immersed in \(\mathbb{R}^3\) are round spheres). An extension of Hopf’s theorem to \(H\)-surfaces in \(\mathbb{R}^3\) follows from work by the second and third authors \([\text{GM1, GM2}]\):

**Theorem 2.9 ([GM1])** Let \(H \in C^1(\mathbb{S}^2)\), \(H > 0\), and assume that there exists a strictly convex \(H\)-sphere \(S\) in \(\mathbb{R}^3\). Then any compact immersed \(H\)-surface of genus zero is a translation of \(S\).

In \([\text{GG}]\) B. Guan and P. Guan proved the following result:

**Theorem 2.10 ([GG])** Let \(H \in C^2(\mathbb{S}^n)\) be positive and invariant under a group of isometries of \(\mathbb{S}^n\) without fixed points. Then there exists a closed strictly convex \(H\)-hypersurface in \(\mathbb{R}^{n+1}\).

In particular, if \(H \in C^2(\mathbb{S}^n)\) satisfies \(H(x) = H(-x) > 0\) for all \(x \in \mathbb{S}^n\), then there exists a closed, strictly convex \(H\)-hypersurface \(S_H\) in \(\mathbb{R}^{n+1}\). When \(n = 2\), the Guan-Guan sphere \(S_H\) is actually the unique immersed \(H\)-sphere in \(\mathbb{R}^3\), by Theorem 2.9.

By combining Proposition 2.8, Theorem 2.9 and Theorem 2.10, we then have:

**Corollary 2.11** Let \(\Sigma\) be a compact embedded \(H\)-surface in \(\mathbb{R}^3\), where \(H \in C^2(\mathbb{S}^n)\) is positive and invariant under a group of isometries of \(\mathbb{S}^2\) without fixed points that contains two independent geodesic reflections of \(\mathbb{S}^2\). Then \(\Sigma\) is a translation of the Guan-Guan sphere \(S_H\).

Examples of groups of isometries of \(\mathbb{S}^2\) in the conditions of Corollary 2.11 are those generated by reflections with respect to three linearly independent geodesics of \(\mathbb{S}^2\). In particular, the groups of isometries of \(\mathbb{S}^2\) that leave invariant a Platonic solid inscribed in \(\mathbb{S}^2\) are in the conditions of Corollary 2.11.

### 2.4 Construction of non-compact \(H\)-hypersurfaces

Let \(\Sigma\) be a hypersurface in \(\mathbb{R}^{n+1}\) of the form \(\Sigma = M \times \mathbb{R}^{n-k}\), where \(M\) is a hypersurface in \(\mathbb{R}^{k+1}\). Here, by \(\mathbb{R}^{k+1}\) we are denoting any linear subspace of \(\mathbb{R}^{n+1}\), and by \(\mathbb{R}^{n-k}\) its linear orthogonal complement. Then, it is clear that \(H_{\Sigma} = \frac{k}{n} H_M\), where \(H_{\Sigma}, H_M\) denote the mean curvatures of \(\Sigma\) in \(\mathbb{R}^{n+1}\) and of \(M\) in \(\mathbb{R}^{k+1}\), respectively. In particular, if \(H \in C^1(\mathbb{S}^n)\), and we denote \(S^k := \mathbb{S}^n \cap \mathbb{R}^{k+1}\) and \(H^* := \frac{n}{k} H_{|S^k}\), then \(M\) is an \(H^*\)-hypersurface in \(\mathbb{R}^{k+1}\) if and only if \(\Sigma\) is an \(H\)-hypersurface in \(\mathbb{R}^{n+1}\).

Then next result follows from this discussion and Theorem 2.10.
Proposition 2.12 Let $\mathcal{H} \in C^2(S^n)$, let $\mathbb{R}^{k+1}$ be a linear subspace of $\mathbb{R}^{n+1}$ with $k \in \{1, \ldots, n-1\}$, and denote $S^k := S^n \cap \mathbb{R}^{k+1}$. Assume that $\mathcal{H}^* := \frac{n}{k} \mathcal{H}|_{S^k} \in C^2(S^k)$ is positive and invariant under a group of isometries of $S^k$ without fixed points.

Then, there exists a complete $\mathcal{H}$-hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ given by $\Sigma = S \times \mathbb{R}^{n-k}$, where $S$ is a compact strictly convex hypersurface in $\mathbb{R}^{k+1}$.

As a particular case of Proposition 2.12 if $\mathcal{H} \in C^2(S^n)$ satisfies $\mathcal{H}(x) = \mathcal{H}(-x) > 0$ for every $x \in S^n$, then for any linear subspace $\mathbb{R}^{k+1}$ of $\mathbb{R}^{n+1}$ with $k \in \{1, \ldots, n-1\}$ there exists a complete $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$ diffeomorphic to $S^k \times \mathbb{R}^{n-k}$, and such that the orthogonal space $\mathbb{R}^{n-k}$ to $\mathbb{R}^{k+1}$ is everywhere tangent to $\Sigma$. For later use, we single out a particular case, which corresponds to $k = n - 1$.

Corollary 2.13 Let $\mathcal{H} \in C^2(S^n)$ satisfy $\mathcal{H}(x) = \mathcal{H}(-x) > 0$ for all $x \in S^n$. Then for any $v \in S^n$ there is a closed, $(n-1)$-dimensional strictly convex hypersurface $S_v$ of the hyperplane $v^\perp$, such that $\Sigma_v := \{S_v + tv : t \in \mathbb{R}\}$ is a complete $\mathcal{H}$-hypersurface with the topology of $S^n \times \mathbb{R}$.

We can use the construction in Proposition 2.12 in order to classify the complete $\mathcal{H}$-hypersurfaces of constant curvature in $\mathbb{R}^{n+1}$; note that when $k = 1$, the product hypersurface $\Sigma = M \times \mathbb{R}^{n-1}$ is flat.

If $\mathcal{H}$ is constant (i.e. the CMC case), then it is well known that any complete $\mathcal{H}$-hypersurfaces of constant curvature in $\mathbb{R}^{n+1}$ is a hyperplane (which corresponds to the case $\mathcal{H} = 0$), a homogeneous generalized cylinder $S^1((1/(n|\mathcal{H}|)) \times \mathbb{R}^{n-1}$, or a round sphere $S^n(1/|\mathcal{H}|)$.

Assume next that $\mathcal{H} \in C^1(S^n)$ is not constant. Then, by the classical theorems of Liebmann, Hilbert and Hartman-Nirenberg, the only complete $\mathcal{H}$-hypersurfaces of constant curvature in $\mathbb{R}^{n+1}$ are flat generalized cylinders, of the form $\Sigma = \alpha \times \mathbb{R}^{n-1}$, where $\alpha$ is a complete regular curve in a two-dimensional plane $\Pi \equiv \mathbb{R}^2 \subset \mathbb{R}^{n+1}$ (here, as usual, we denote $\mathbb{R}^{n-1} \equiv \Pi^\perp$). Moreover, it follows from the prescribed mean curvature condition (1.1) that $\alpha$ satisfies

$$\kappa_\alpha = \frac{1}{n} \mathcal{H}(\mathbf{n}), \quad (2.7)$$

where $\kappa_\alpha$, $\mathbf{n}$ denote, respectively, the geodesic curvature and unit normal of the planar curve $\alpha$.

Thus, it suffices to solve (2.7), which might be seen as an analogous of the planar Minkowski problem; note, however, that in our case $\mathcal{H}$ is not assumed to be positive, and $\alpha$ is not necessarily closed.

Let us consider then $\mathcal{H} \in C^1(S^n)$, let $\{e_1, e_2\}$ be a positively oriented orthonormal basis of $\Pi$, and define a $2\pi$-periodic function $\hat{\mathcal{H}} \in C^1(\mathbb{R})$ by

$$\hat{\mathcal{H}}(\theta) := \frac{1}{n} \mathcal{H}(-\sin \theta e_1 + \cos \theta e_2).$$

Fix $v = \cos \theta_0 e_1 + \sin \theta_0 e_2 \in \Pi$, for some $\theta_0$. If $\hat{\mathcal{H}}(\theta_0) = 0$, the straight line generated by $v$ solves (2.7), and the corresponding $\mathcal{H}$-hypersurface $\Sigma$ is a hyperplane in $\mathbb{R}^{n+1}$. 

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Suppose now that \( \hat{H}(\theta_0) \neq 0 \), and let \( I_0 \subset \mathbb{R} \) denote the largest interval containing \( \theta_0 \) where \( \hat{H} \) does not vanish. Then we may consider \( F(x) \in C^2(I_0) \) to be a primitive of \( 1/\hat{H}(x) \) in \( I_0 \), with \( F(\theta_0) = 0 \). By periodicity of \( \hat{H} \), we have \( F' \geq c > 0 \) for some \( c \). If \( \hat{H} > 0 \) everywhere, then \( I_0 = \mathbb{R} \) and we can define the inverse function \( F^{-1} \) of \( F \), which is globally defined on \( \mathbb{R} \). If \( \hat{H} = 0 \) somewhere, then \( I_0 \) is a bounded open interval \((a, b)\) in \( \mathbb{R} \), and \( F'(x) \to \infty \) as \( x \to \{a, b\} \). Thus, the same conclusion for \( F^{-1} \) holds.

If we write now \( \alpha'(s) = \cos \theta(s)e_1 + \sin \theta(s)e_2 \) for an arclength parametrization \( \alpha(s) \) of \( \alpha \), equation (2.7) is rewritten as

\[
\theta'(s) = \hat{H}(\theta(s)), \tag{2.8}
\]

that, with the initial condition \( \theta(0) = \theta_0 \), has as unique solution \( \theta(s) = F^{-1}(s) : \mathbb{R} \to I_0 \).

Thus, we arrive at the following result.

**Proposition 2.14** Let \( H \in C^1(\mathbb{S}^n) \). Fix \( v \in \mathbb{S}^n \) and a two-dimensional linear subspace \( \Pi \equiv \mathbb{R}^2 \) in \( \mathbb{R}^{n+1} \) with \( v \in \Pi \). Then, there exists a unique (up to translation) complete regular curve \( \alpha = \alpha_v \) in \( \mathbb{R}^2 \) with the following properties:

1. \( \Sigma_v, \Pi := \alpha \times \mathbb{R}^{n-1} \) is a (complete, flat) \( H \)-hypersurface in \( \mathbb{R}^{n+1} \).

2. \( v \) is tangent to \( \Sigma_v, \Pi \) at some point.

Conversely, any complete \( H \)-hypersurface of constant curvature in \( \mathbb{R}^{n+1} \) is either a round sphere (in which case \( H \) is constant), or one of the examples \( \Sigma_v, \Pi \).

One should observe that the \( H \)-hypersurface \( \Sigma_v, \Pi \) above can be explicitly constructed from the restriction of \( H \) to the geodesic \( \mathbb{S}^n \cap \Pi \) of \( \mathbb{S}^n \), following the process described just before Proposition 2.14.

Let us also note that the hypersurface \( \Sigma_v, \Pi = \alpha \times \mathbb{R}^{n-1} \) is diffeomorphic to \( \mathbb{R}^n \) or to \( S^1 \times \mathbb{R}^{n-1} \), depending on whether \( \alpha \) is a closed curve or not. By construction, a necessary condition for \( \alpha \) to be closed is that \( H \) restricted to \( \mathbb{S}^n \cap \Pi \) never vanishes. Once there, the next result follows directly from the classical solution to Minkowski problem for planar curves:

**Corollary 2.15** Given \( H \in C^1(\mathbb{S}^n) \), let \( \Sigma_v, \Pi = \alpha \times \mathbb{R}^{n-1} \) be one of the \( H \)-hypersurfaces in \( \mathbb{R}^{n-1} \) constructed in Proposition 2.14. The next two conditions are equivalent:

1. \( \Sigma_v, \Pi \) is diffeomorphic to \( S^1 \times \mathbb{R}^{n-1} \) (i.e. \( \alpha \) is closed).

2. If we denote \( S^1 := \mathbb{S}^n \cap \Pi \), then \( H(\xi) \neq 0 \) for every \( \xi \in S^1 \), and

\[
\int_{S^1} \frac{\xi}{H(\xi)} d\xi = 0.
\]

We next turn our attention to the Dirichlet problem. The next result follows from [Mar, Corollary 1], and gives general conditions for the existence of graphs with prescribed mean curvature in \( \mathbb{R}^{n+1} \).
Proposition 2.16 Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{2,\alpha}$ domain and $\varphi \in C^{2,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$. Let $H \in C^1(S^n)$, and assume that:

1. $\max_{S^n}|H| \cdot \text{vol}(\Omega) < \omega_n$, where $\omega_n$ stands for the volume of the $n$-dimensional unit ball.

2. $H_{\partial \Omega}(x) \geq \frac{n}{n-1}|H(\nu(x))|$ for all $x \in \partial \Omega$, where $\nu$ is the inner pointing unit conormal of $\partial \Omega$, viewed as a vector in $S^n \subset \mathbb{R}^{n+1}$, and $H_{\partial \Omega}$ is the mean curvature of the submanifold $\partial \Omega$ with respect to $\nu$.

Then, the Dirichlet problem

\[
\begin{align*}
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) &= nH(Z_u), & Z_u := \frac{(-Du,1)}{\sqrt{1 + |Du|^2}}, & \text{in} & \Omega, \\
u &= \varphi & \text{on} & \partial \Omega,
\end{align*}
\]

(2.9)

has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$.

Existence results for the Dirichlet problem (2.9) when $\Omega \subset \mathbb{R}^n$ is unbounded can be obtained similarly as a corollary of the main results in the recent work [JL].

2.5 A compactness theorem

We introduce next a compactness theorem for the space of $H$-surfaces in $\mathbb{R}^3$ with bounded second fundamental form (or, equivalently, with bounded curvature) that will be used in later sections. The argument in the next theorem is well-known for CMC surfaces, see e.g. Section 2 in [RST]. We sketch it here for the case where the prescribed mean curvature $H$ is not constant.

Theorem 2.17 Let $(\Sigma_n)_n$ be a sequence of $H_n$-surfaces in $\mathbb{R}^3$ for some sequence of functions $H_n \in C^k(S^2)$, $k \geq 1$, and take $p_n \in \Sigma_n$. Assume that the following conditions hold:

(i) There exists a sequence of positive numbers $r_n \to \infty$ such that the geodesic disks $D_n = D_{\Sigma_n}(p_n,r_n)$ are contained in the interior of $\Sigma_n$, i.e. $d_{\Sigma_n}(p_n,\partial \Sigma_n) \geq r_n$.

(ii) $(p_n)_n \to p$ for a certain $p \in \mathbb{R}^3$.

(iii) If $|\sigma_n|$ denotes the length of the second fundamental form of $\Sigma_n$, then there exists $C > 0$ such that $|\sigma_n|(x) \leq C$ for every $n$ and every $x \in \Sigma_n$.

(iv) $H_n \to H$ in the $C^k$ topology to some $H \in C^k(S^2)$.

Then, there exists a subsequence of $(\Sigma_n)_n$ that converges uniformly on compact sets in the $C^{k+2}$ topology to a complete, possibly non-connected, $H$-surface $\Sigma$ of bounded curvature that passes through $p$. 

Proof: By conditions (i), (iii) and by virtue of a well-known result in surface theory (see e.g. Proposition 2.3 in [RST]), there exist positive constants \( \delta, M \) that only depend on \( C \) (and not on \( n, H_n \) or \( \Sigma_n \)), such that, if \( n \) is large enough:

a) An open neighbourhood of \( p_n \) in \( D_n \subset \Sigma_n \) is the graph of a function \( u_n \) over the Euclidean disk \( D_\delta := D(0, \delta) \) of radius \( \delta \) in \( T_{p_n} \Sigma_n \).

b) The \( C^2 \) norm of \( u_n \) in \( D_\delta \) is not greater than \( M \).

Since \( \Sigma_n \) is an \( H_n \)-surface, it follows that, in adequate Euclidean coordinates \((x^n, y^n, z^n)\) with respect to which \( T_{p_n} \Sigma_n = \{ z^n = 0 \} \), each function \( u_n \) is a solution in \( D_\delta \) to the quasilinear elliptic equation

\[
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 2H_n(Z_u), \quad Z_u := \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}.
\tag{2.10}
\]

Note that (2.10) for \( u_n \) can be rewritten as a linear elliptic PDE \( L[u_n] = f_n \) where the coefficients of \( L \) depend \( C^\infty \)-smoothly on \( Du_n \), and \( f_n \) depends \( C^k \)-smoothly on \( Du_n \). By condition b) above, we have \( u_n \in C^{1,\alpha}(D_\delta) \) for all \( n \). Thus, all these coefficients are bounded in the \( C^{0,\alpha}(D_\delta) \) norm. Then, by the usual Schauder estimates (see Gilbarg-Trudinger, [GT] Chapter 6), for any \( \delta' \in (0, \delta) \) we conclude that there exists a constant \( C' \) (again independent of \( n \)) such that \( ||u_n|| \leq C' \) in the \( C^{2,\alpha}(D_{\delta'}) \) norm. In particular, all coefficients of \( L[u_n] = f_n \) are uniformly bounded in the \( C^{1,\alpha}(D_{\delta'}) \) norm. By repeating this argument we eventually obtain

\[
||u_n||_{C^{k+2,\alpha}(D_{\delta'})} \leq C'', \quad 0 < \alpha < 1,
\]

for some constant \( C'' \) independent of \( n \). In these conditions, we may apply the Arzela-Ascoli theorem, and deduce by (ii), (iv) that a subsequence of the functions \( u_n \) converge on \( D_{\delta'} \) in the \( C^{k+2} \) topology to a solution \( u \in C^{k+2}(D_{\delta'}) \) to

\[
\text{div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 2H(Z_u), \quad Z_u := \frac{(-Du, 1)}{\sqrt{1 + |Du|^2}}.
\tag{2.11}
\]

Thus, the graph of \( u \) is an \( H \)-surface in \( \mathbb{R}^3 \) that by construction passes through \( p \), and has second fundamental form bounded by \( C \).

Consider next some \( y \in D_{\delta'} \), and let \( q \) be the corresponding point in the graph of \( u \). It is then clear that there exist points \( q_n \in \mathbb{R}^3 \) in the graphs of \( u_n \), all corresponding to \( y \), and such that \( q_n \to q \). Thus, passing to a subsequence if necessary so that condition (i) is fulfilled, we can repeat the same process above, this time with respect to the points \( q_n \) and \( q \). In this way, we obtain an \( H \)-surface \( \Sigma \) in \( \mathbb{R}^3 \) that extends the graph of \( u \) over \( D_{\delta'} \).

Again by (i), it follows by a standard diagonal process that \( \Sigma \) can be extended to a complete \( H \)-surface (which will also be denoted by \( \Sigma \)), that passes through \( p \), and whose second fundamental form is bounded by \( C \). Moreover, \( \Sigma \) is by construction a limit in
the $C^{k+1}$ topology on compact sets of the sequence of surfaces $(\Sigma_n)_n$; note that some other limit connected components could also appear in this process. This completes the proof.

3 Rotational $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$

In this section we will let $\mathcal{H} \in C^1(S^n)$ be a rotationally symmetric function, i.e. $\mathcal{H}(x) = h(\langle x, v \rangle)$ for some $v \in S^n$ and some $C^1$ function $h$ on $[-1, 1]$. Up to an Euclidean change of coordinates, we will assume that $\mathcal{H}(x_1, \ldots, x_{n+1})$ only depends on $x_{n+1}$, and so

$$\mathcal{H}(x) = h(\langle x, e_{n+1} \rangle).$$

Thus, equation (1.1) for an immersed oriented hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ yields

$$H_{\Sigma} = h \circ \nu,$$

where $\nu = \langle \eta, e_{n+1} \rangle$ is the angle function of $\Sigma$.

In Section 3.1 we analyze rotational hypersurfaces of prescribed mean curvature (3.2) by means of a phase space analysis, and prove that rotational $\mathcal{H}$-spheres in $\mathbb{R}^{n+1}$ are strictly convex. In Section 3.2 we use this phase space analysis to construct, for very general choices of the rotationally symmetric function $\mathcal{H} \in C^1(S^n)$, examples of properly embedded rotational $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ with the topology of either $\mathbb{R}^{n}$ (the $h$-bowls) or $S^{n-1} \times \mathbb{R}$ (the $h$-catenoids). In Section 3.3 we give a Delaunay-type classification of rotational $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ when the prescribed mean curvature satisfies $h(y) = h(-y) > 0$ for all $y \in [-1, 1]$. In Section 3.4 we give several examples of rotational $\mathcal{H}$-surfaces in $\mathbb{R}^{3}$, for different choices of the prescribed mean curvature function, which have a different geometric behavior to the examples presented in the previous sections.

3.1 Phase space analysis of rotational $\mathcal{H}$-hypersurfaces

Let $\Sigma$ be an immersed, oriented, rotational hypersurface in $\mathbb{R}^{n+1}$, obtained as the orbit of a regular planar curve parametrized by arc-length

$$\alpha(s) = (x(s), 0, \ldots, 0, z(s)) : I \subset \mathbb{R} \to \mathbb{R}^{n+1}, \quad x(s) > 0,$$

under the action of all orientation preserving linear isometries of $\mathbb{R}^{n+1}$ that leave the $x_{n+1}$-axis pointwise fixed. A parametrization for $\Sigma$ is

$$\psi(s, p) = (x(s)p, z(s)) : I \times S^{n-1} \to \mathbb{R}^{n+1}.$$

By changing the orientation of the profile curve if necessary, the angle function of $\Sigma$ is given by $\nu = x'(s)$. There are at most two different principal curvatures on $\Sigma$, given by

$$\kappa_1 = \kappa_\alpha = x'(s)z''(s) - x''(s)z'(s), \quad \kappa_2 = \cdots = \kappa_n = \frac{z'(s)}{x(s)},$$

(3.3)
where $\kappa_\alpha$ denotes the geodesic curvature of the profile curve $\alpha(s)$.

Let now $H, h$ be in the conditions stated at the beginning of this section, related by (3.1), and let $\Sigma$ be a rotational $H$-hypersurface in $\mathbb{R}^{n+1}$. Thus, $\Sigma$ satisfies (3.2). So, from (3.3), the profile curve $\alpha(s)$ of $\Sigma$ satisfies

$$n h(x') = x' z'' - x'' z' + (n - 1) \frac{z'}{x}. \quad (3.4)$$

Noting that $x'^2 + z'^2 = 1$, we obtain from (3.4) that $x(s)$ is a solution to the autonomous second order ODE

$$x'' = (n - 1) \frac{1 - x'^2}{x} - n \varepsilon h(x') \sqrt{1 - x'^2}, \quad \varepsilon = \text{sign}(z'), \quad (3.5)$$

on every subinterval $J \subset I$ where $z'(s) \neq 0$ for all $s \in J$.

Denoting $x' = y$, (3.5) transforms into the first order autonomous system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ (n - 1) \frac{1 - y^2}{x} - n \varepsilon h(y) \sqrt{1 - y^2} \end{pmatrix} =: F(x, y). \quad (3.6)$$

The phase space of (3.6) is $\Theta_\varepsilon := (0, \infty) \times (-1, 1)$, with coordinates $(x, y)$ denoting, respectively, the distance to the rotation axis and the angle function of $\Sigma$. If $\varepsilon h(0) > 0$, there is a unique equilibrium of (3.6) in $\Theta_\varepsilon$, namely

$$e_0 := \left( \frac{n - 1}{n \varepsilon h(0)}, 0 \right). \quad (3.7)$$

This equilibrium corresponds to the case where $\Sigma$ is a right circular cylinder $S^{n-1}(r) \times \mathbb{R}$ in $\mathbb{R}^{n+1}$ of constant mean curvature $h(0)$ and vertical rulings. Otherwise, there are no equilibria in $\Theta_\varepsilon$. The orbits $(x(s), y(s))$ provide then a foliation by regular proper $C^2$ curves of $\Theta_\varepsilon$ (or of $\Theta_\varepsilon - \{e_0\}$, in case $e_0$ exists). The points in $\Theta_\varepsilon$ where $y'(s) = 0$ are those placed at the intersection of $\Theta_\varepsilon$ with the (possibly disconnected) horizontal graph given for $y \in [-1, 1]$ by

$$x = \Gamma_\varepsilon(y) = \frac{(n - 1) \sqrt{1 - y^2}}{n \varepsilon h(y)}. \quad (3.8)$$

Note that $\Gamma_\varepsilon(y)$ does not take a finite value at the zeros of $h(y)$, since $h$ is $C^1$. We will denote $\Gamma_\varepsilon := \Theta_\varepsilon \cap \{x = \Gamma_\varepsilon(y)\}$. It must be remarked that $\Gamma_\varepsilon$ might be empty; for instance, in the case $h \leq 0$ and $\varepsilon = 1$. A computation shows that the values $s \in J$ where the profile curve $\alpha(s) = (x(s), z(s))$ of $\Sigma$ has zero geodesic curvature are those where $y'(s) = 0$, i.e., those where $(x(s), y(s)) \in \Gamma_\varepsilon$.

The curve $\Gamma_\varepsilon$ and the axis $y = 0$ divide $\Theta_\varepsilon$ into connected components where both $x(s)$ and $y(s)$ are monotonous. In particular, at each of these monotonicity regions, the profile curve $\alpha(s)$ has geodesic curvature of constant sign. Specifically, by (3.3) we have at each point $\alpha(s)$, $s \in J$:

$$\text{sign}(\kappa_1) = \text{sign}(-\varepsilon y'(s)), \quad \text{sign}(\kappa_i) = \varepsilon, \quad i = 2, \ldots, n. \quad (3.9)$$
Figure 3.1: An example of phase space $\Theta_1$ for some choice of $h \geq 0$ with $h(y_0) = 0$. The point $e_0$ is the equilibrium. The curve $\Gamma_1$ has two connected components, and there exist five monotonicity regions, separated by $\Gamma_1$ and the axis $y = 0$. Each arrow indicates the monotonicity direction at each of those regions.

Also, by viewing the orbits of (3.6) as graphs $y = y(x)$ wherever possible (i.e. wherever $y \neq 0$), we have

$$y' \frac{dy}{dx} = (n - 1)\frac{1 - y^2}{x} - n\varepsilon h(y)\sqrt{1 - y^2}. \quad (3.10)$$

Thus, in each of these monotonicity regions the sign of the quantity $yy'(x)$ is constant. In particular, the behavior of the orbit of (3.6) passing through a given point $(x_0, y_0) \in \Theta_\varepsilon$ is determined by the signs of $y_0$ and $x_0 - \Gamma_\varepsilon(y_0)$ (wherever $\Gamma_\varepsilon(y_0)$ exists). We point out below some trivial particular consequences:

**Lemma 3.1** In the above conditions, for any $(x_0, y_0) \in \Theta_\varepsilon$ such that $\Gamma_\varepsilon(y_0)$ exists, the following properties hold:

1. If $x_0 > \Gamma_\varepsilon(y_0)$ (resp. $x_0 < \Gamma_\varepsilon(y_0)$) and $y_0 > 0$, then $y(x)$ is strictly decreasing (resp. increasing) at $x_0$.
2. If $x_0 > \Gamma_\varepsilon(y_0)$ (resp. $x_0 < \Gamma_\varepsilon(y_0)$) and $y_0 < 0$, then $y(x)$ is strictly increasing (resp. decreasing) at $x_0$.
3. If $y_0 = 0$, then the orbit passing through $(x_0, 0)$ is orthogonal to the $x$ axis.
4. If $x_0 = \Gamma_\varepsilon(y_0)$, then $y'(x_0) = 0$ and $y(x)$ has a local extremum at $x_0$.

In order to describe further properties of orbits in the phase space $\Theta_\varepsilon$, we first point out the following direct consequence from Proposition 2.16:

**Lemma 3.2** Let $H \in C^1(S^n)$ be in the conditions stated at the beginning of the section, and $\delta \in \{-1, 1\}$. Then, there exists a unique (up to vertical translations) $H$-hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$ that is rotational with respect to the the $x_{n+1}$-axis, and that meets this rotation axis orthogonally at some point $p \in \Sigma$, with unit normal at $p$ given by the vertical unit vector $\delta e_{n+1} \in \mathbb{R}^{n+1}$.
Proof: By Proposition 2.16, we can solve the Dirichlet problem (2.9) for the equation of upwards-oriented $H$-graphs in $\mathbb{R}^{n+1}$, on a sufficiently small ball $\Omega \subset \mathbb{R}^n$, with constant Dirichlet data $\varphi$ on the boundary. Since $H$ is rotationally invariant, the graph $\Sigma$ of the solution is a rotational $H$-hypersurface in $\mathbb{R}^{n+1}$ with unit normal $e_{n+1}$ at the origin. Since the equation in (2.9) is invariant by additive constants, the uniqueness of $\Sigma$ in these conditions is immediate from the maximum principle. The same argument can be done for downwards-oriented $H$-graphs in $\mathbb{R}^{n+1}$, what completes the proof of Lemma 3.2.

Lemma 3.2 has the following consequence for our analysis of the phase space $\Theta_{\varepsilon}$.

**Corollary 3.3** Assume that $h(\delta) \neq 0$ for some $\delta \in \{-1,1\}$, and let $\varepsilon \in \{-1,1\}$ such that $\varepsilon h(\delta) > 0$. Then, there exists a unique orbit in $\Theta_\varepsilon$ that has $(0, \delta) \in \Theta_\varepsilon$ as an endpoint. There is no such an orbit in $\Theta_{-\varepsilon}$.

**Proof:** Let $\Sigma$ be the rotational $H$-hypersurface given for $\delta$ by Lemma 3.2 (the relation between $H$ and $h$ is given by (3.1)). Let $\alpha(s) = (x(s), z(s))$ be the profile curve of $\Sigma$, defined for $s \in [0, s_0)$ or $s \in (-s_0, 0]$ depending on the orientation chosen on $\alpha$, and assume that $x(0) = z'(0) = 0$, i.e. $s = 0$ corresponds to the point $p_0$ of orthogonal intersection of $\Sigma$ with its rotation axis. Since $p_0$ is an umbilical point of $\Sigma$, all principal curvatures of $\Sigma$ at $p_0$ are equal and of the same sign as $h(\delta)$.

By (3.3), the geodesic curvature of $\alpha(s)$ at $s = 0$ is non-zero, and thus the sign of $z'(s)$, which will as usual be denoted by $\varepsilon$, is constant for $s$ small enough. It follows then again by (3.3) that $\varepsilon h(\delta) > 0$. Consequently, the profile curve $\alpha(s)$ generates an orbit in the phase space $\Theta_\varepsilon$ with $(0, \delta)$ as an endpoint. Uniqueness of this orbit follows from the uniqueness in Lemma 3.2. It is also clear from the argument that such an orbit cannot exist in $\Theta_{-\varepsilon}$, because of the condition $\varepsilon h(\delta) > 0$.

The previous corollary can be used to describe geometrically the compact, rotational $H$-hypersurfaces immersed in $\mathbb{R}^{n+1}$ that are diffeomorphic to $S^n$.

**Theorem 3.4** Let $\Sigma$ be an immersed, rotationally symmetric $H$-hypersurface in $\mathbb{R}^{n+1}$ diffeomorphic to $S^n$. Then $\Sigma$ is a strictly convex sphere.

**Proof:** Let $\alpha(s) = (x(s), z(s))$ be the profile curve of $\Sigma$, and let $h \in C^1([-1,1])$ be given by (3.1). By Proposition 2.6 and up to a change of orientation, we have $h(y) > 0$ for every $y$. So, the curve $\Gamma_{\varepsilon} := \Theta_{\varepsilon} \cap \{x = \Gamma_{\varepsilon}(y)\}$ where $\Gamma_{\varepsilon}(y)$ is given by (3.8) does not exist for $\varepsilon = -1$, and for $\varepsilon = 1$, $\Gamma := \Gamma_{\varepsilon}$ is a compact connected arc with endpoints $(0,1)$ and $(0,-1)$. Hence, we have four monotonicity regions $\Lambda_1, \ldots, \Lambda_4$ inside $\Theta_1$, with monotonicities given by Lemma 3.1, and an equilibrium $e_0$; see Figure 3.2.

By Corollary 3.3 there exists an orbit $\gamma$ in the phase space $\Theta_1$ that has $(0,1)$ as an endpoint; it corresponds to an open subset of $\Sigma$ that contains a point of $\Sigma$ with unit normal $e_{n+1}$, at which $\Sigma$ meets its rotation axis orthogonally. By the monotonicity
Figure 3.2: The phase space $\Theta_1$, showing the monotonicity direction of each region $\Lambda_i$.

properties of the regions $\Lambda_i$, the curve $\gamma$ lies in the region $\Lambda_1$ for all points near $(0, 1)$. By the regularity and compactness of $\Sigma$, the curve $\gamma$ cannot approach any point of the form $(0, y)$ with $y \in (-1, 1)$. Indeed, if $\gamma$ had such an endpoint, $\Sigma$ would be asymptotic to its rotation axis (this corresponds to $y = 0$), or it would present a non-removable isolated singularity around some point touching it rotation axis ($y \neq 0$), and these situations cannot happen. Since $\gamma$ cannot self-intersect, it becomes then clear from the monotonocity properties of the phase portrait that there are only two possible behaviors for $\gamma$:

i) If $\gamma$ enters at some moment in the region $\Lambda_3 \cup \Lambda_4$, then $\gamma$ has to converge asymptotically to the equilibrium $e_0$ of $\Theta_1$. But this means that the profile curve $\alpha(s)$ is asymptotic to a vertical straight line, i.e. $\Sigma$ is asymptotic to a cylinder, what contradicts compactness of $\Sigma$. Thus, this case is impossible.

ii) If $\gamma$ stays in $\Lambda_1 \cup \Lambda_2$, then it is a graph of the form $x = g(y) > 0$, with $y \in (y_0, 1)$ for some $y_0 \in [-1, 1)$. By compactness of $\Sigma$, we must have $y_0 = -1$. Thus, $\gamma$ can be extended to be a compact graph $x = g(y) \geq 0$ for $y \in [-1, 1]$, and it has a second endpoint at some $(x_1, -1)$ with $x_1 \geq 0$.

Next, we can repeat all the argument above, and obtain a second orbit $\sigma$ in $\Lambda_1 \cup \Lambda_2 \subset \Theta_1$, that has $(0, -1)$ as an endpoint, and which corresponds to a piece of $\Sigma$ where it intersects its rotation axis with unit normal $-e_{n+1}$. We conclude that $\sigma$ can be extended to be a graph $x = t(y) \geq 0$ for $y \in [-1, 1]$, with a second endpoint at some $(x_2, 1)$ with $x_2 \geq 0$. Since $\gamma$ and $\sigma$ cannot intersect on $\Theta_1$, the only possibility is that $x_1 = 0$ or $x_2 = 0$. Thus, by the uniqueness property of Corollary 3.3 we have $\gamma = \sigma$, which is then an orbit in $\Theta_1$ joining $(0, 1)$ with $(0, -1)$. Since, again by Corollary 3.3 there are no orbits in $\Theta_{-1}$ having any of such points as an endpoint, we conclude that $\gamma$ is the whole orbit that describes the profile curve $\alpha(s)$.

By (3.6), and since $\gamma$ stays in the region $\Lambda_1 \cup \Lambda_2$, it follows that $y'(s) < 0$ for all $s$. Consequently, by (3.9) we see that all principal curvatures of $\Sigma$ are positive at every point, and so $\Sigma$ is a strictly convex sphere in $\mathbb{R}^{n+1}$.
3.2 \( h \)-bowls and \( h \)-catenoids in \( \mathbb{R}^{n+1} \)

In this section we show examples of properly embedded rotational \( H \)-hypersurfaces in \( \mathbb{R}^{n+1} \), for very general choices of the radially symmetric function \( H \in C^1(S^n) \). As always, we let \( h \in C^1([-1, 1]) \) be given in terms of \( H \) by (3.1). First, we will construct examples of entire, strictly convex, \( H \)-graphs. In analogy with the theory of self-translating solitons of the mean curvature flow, we will call them \( h \)-bowls.

**Proposition 3.5** Let \( h \in C^1([-1, 1]) \) be given by (3.1) in terms of \( H \), and suppose that there exists \( y_0 \in [0, 1] \) (resp. \( y_0 \in [-1, 0] \)) such that \( h(y_0) = 0 \). Then there exists an upwards-oriented (resp. downwards-oriented) entire rotational \( H \)-graph \( \Sigma \) in \( \mathbb{R}^{n+1} \).

Moreover, \( \Sigma \) is either a horizontal hyperplane, or a strictly convex graph.

**Proof:** If \( h(1) = 0 \) (resp. \( h(-1) = 0 \)), \( \Sigma \) can be chosen to be an upwards-oriented (resp. downwards-oriented) horizontal hyperplane in \( \mathbb{R}^{n+1} \), and the result is trivial.

Consider next the case that \( h(1) > 0 \), and let \( y_0 \in [0, 1] \) be the largest value of \( y \) such that \( h(y) = 0 \). Since \( h(y) > 0 \) on \((y_0, 1]\) and \( h(y_0) = 0 \), the horizontal graph \( \Gamma := \Theta_1 \cap \{x = \Gamma_1(y)\} \) defined by (3.8) has a connected component given by the restriction of \( \Gamma_1(y) \) to the interval \([y_0, 1]\), and satisfies \( \Gamma_1(1) = 0 \) and \( \Gamma_1(y_0) \to \infty \).

Define now \( \Lambda \subset \Theta_1 \) by \( \Lambda = \{(x, y) \in \Theta_1 : y > y_0\} \), and let \( \Lambda^+ := \{(x, y) \in \Lambda : x > \Gamma_1(y)\} \) and \( \Lambda^- := \{(x, y) \in \Lambda : x < \Gamma_1(y)\} \). These components \( \Lambda^+ \) and \( \Lambda^- \) are the only connected components of \( \Lambda \), and they have \( \Gamma \) as their common boundary. Moreover, \( \Lambda^+ \), \( \Lambda^- \) are monotonicity regions of \( \Theta_1 \), and by Lemma 3.1 each orbit \( y = y(x) \) in \( \Lambda^+ \) (resp. \( \Lambda^- \)) satisfies that \( y'(x) < 0 \) (resp. \( y'(x) > 0 \)); see Figure 3.3.

![Figure 3.3: The phase space \( \Theta_1 \), showing the monotonicity directions of \( \Lambda^+, \Lambda^- \).](image)

Let now \( \Sigma \) be the upwards-oriented rotational \( H \)-graph in \( \mathbb{R}^{n+1} \) constructed in Lemma 3.2, and let \( \gamma \) denote the orbit in \( \Theta_1 \) associated to the profile curve \( \alpha(s) = (x(s), z(s)) \) of \( \Sigma \), which has an endpoint at \((0, 1)\); see Corollary 3.3. By the monotonicity properties explained above, \( \gamma \) lies in \( \Lambda^+ \) for points near \((0, 1)\). By the same monotonicity properties, and using item 4 of Lemma 3.1 we can conclude that \( \gamma \) is globally contained in \( \Lambda^+ \). Thus, by its monotonicity and properness, \( \gamma \) can be seen as a graph \( y = r(x) \), where \( r \in C^1([0, \infty)) \) satisfies \( r(0) = 1 \), \( r(x) > y_0 \) and \( r'(x) < 0 \) for all \( x > 0 \).
This implies that Σ is an entire rotational graph in $\mathbb{R}^{n+1}$. Since $\gamma$ is totally contained in $\Theta_\varepsilon$ for $\varepsilon = 1$, we have by (3.9) that the principal curvatures $\kappa_2, \ldots, \kappa_n$ of Σ are everywhere positive. Moreover, since $\gamma$ does not leave the monotonicity region $\Lambda^+$, we conclude from (3.6) and (3.9) that $\kappa_1$ is also everywhere positive. Thus, Σ is a strictly convex entire graph.

This concludes the proof in the case that $h(1) > 0$ and $y_0 \geq 0$. A similar argument works in the case that $h(-1) > 0$ and $y_0 \leq 0$. The remaining two cases, namely $h(1) < 0$, $y_0 \geq 0$ and $h(-1) < 0$, $y_0 \leq 0$, are reduced to the previous ones by a change of orientation. This completes the proof of Proposition 3.5.

![Figure 3.4: Profile curve and graphic of the $h$-bowl in $\mathbb{R}^3$ for the choice $h(y) = y - 1/2$.](image)

The next result shows that there exist catenoid-type rotational $H$-hypersurfaces for a large class of rotationally invariant choices of $H \in C^2(S^n)$.

**Proposition 3.6** Let $h \in C^2([-1, 1])$, $h \leq 0$, be given by (3.1) in terms of $H$, and suppose that $h(\pm 1) = 0$. Then, there exists a one-parameter family of properly embedded rotational $H$-hypersurfaces in $\mathbb{R}^{n+1}$ of strictly negative Gauss-Kronecker curvature at every point, and diffeomorphic to $S^{n-1} \times \mathbb{R}$. All of them are bi-graphs over $\mathbb{R}^n - B_n(r)$, where $B_n(r) = \{x \in \mathbb{R}^n : |x| < r\}$.

**Proof:** Let Σ be the rotational $H$-hypersurface in $\mathbb{R}^{n+1}$ generated by a unit speed curve $\alpha(s) = (x(s), z(s))$ that satisfies the initial conditions $x(0) = x_0 > 0$, $z(0) = 0$ and $z'(0) = 1$ for some $x_0$. Then, the orbit $\gamma$ of (3.6) associated to $\alpha(s)$ passes through $(x_0, 0)$ and belongs to the phase space $\Theta_1$ around that point; i.e. $\varepsilon = 1$ in (3.6).

Observe that, since $h \leq 0$, the curve $\Gamma_1 := \Theta_1 \cap \{x = \Gamma_1(y)\}$ with $\Gamma_1(y)$ given by (3.8) does not exist (i.e. $\Gamma_1$ is empty). Thus, there are two monotone regions in $\Theta_1$, given by $\Lambda^+ := \Theta_1 \cap \{y > 0\}$ and $\Lambda^- = \Theta_1 \cap \{y < 0\}$. Any orbit $y = y(x)$ in $\Lambda^+$ (resp. $\Lambda^-$) satisfies that $y'(x) > 0$ (resp. $y'(x) < 0$). We should also note that, by the condition $h(\pm 1) = 0$, and since $h \in C^2$, no orbit in $\Theta_1$ can have a limit point of the form $(x, \pm 1)$ for some $x > 0$, since this would contradict uniqueness of the solution to
the Cauchy problem for (3.6); note for this that \((x(s), y(s)) = (x, \pm 1)\) is a solution to (3.6).

Taking these properties into account, it is easy to deduce that the orbit \(\gamma\) is given as a horizontal graph \(x = r(y)\) for some \(r \in C^1([a,b])\) with \(a < 0 < b\), and so that \(r(0) = x_0\), \(r'(y) > 0\) (resp. \(r'(y) < 0\)) for all \(y \in (0,b)\) (resp. for all \(y \in (a,0)\)), and \(r(y) \to \infty\) as \(y \to \{a,b\}\). As a matter of fact, using (3.10), it is easy to show that this is possible in our conditions only if \(a = -1\), \(b = 1\). Thus, \(\Sigma\) is a bi-graph in \(\mathbb{R}^{n+1}\) over \(\Omega := \mathbb{R}^n - B(x_0)\), with the topology of \(S^{n-1} \times \mathbb{R}\). That is, \(\Sigma = \Sigma_1 \cup \Sigma_2\) where both \(\Sigma_i\) are graphs over \(\Omega\) with \(\partial \Sigma_i = \partial \Omega\), and \(\Sigma_i\) meets the \(x_{n+1} = 0\) hyperplane orthogonally along \(\partial \Sigma_i\).

By (3.6) we get \(y'(s) > 0\) for all \(s\). So, by (3.9), we have that at every \(p \in \Sigma\) the relations \(\kappa_1 < 0\) and \(\kappa_2 = \cdots = \kappa_n > 0\) hold. In particular, the Gauss-Kronecker curvature of \(\Sigma\) is negative at every point. This completes the proof.

![Figure 3.5: Profile curve and graphic of the \(h\)-catenoid in \(\mathbb{R}^3\), for \(h(y) = y^2 - 1\).](image)

We remark that these examples resemble indeed the usual catenoids in minimal hypersurface theory, and that Proposition 3.6 recovers them for the choice \(H = 0\). However, while minimal catenoids can be given an explicit parametrization, this is not the case for the \(h\)-catenoids constructed in Proposition 3.6. Moreover, these \(h\)-catenoids can have different asymptotic behaviors, i.e. different growths at infinity, depending on the choice of the prescribed mean curvature function \(h\). Understanding this asymptotic behavior could lead to proving half-space theorems for properly immersed \(H\)-hypersurfaces in \(\mathbb{R}^{n+1}\) for some rather general choices of rotationally symmetric functions \(H\), by using the classical ideas of Hoffman and Meeks in [HM] about properly immersed minimal surfaces in \(\mathbb{R}^3\).

### 3.3 Classification: a Delaunay-type theorem

Given \(H > 0\), a classical theorem by Delaunay shows that there are exactly four types of rotational hypersurfaces in \(\mathbb{R}^{n+1}\) with constant mean curvature \(H\): round spheres \(S^n\), right circular cylinders \(S^{n-1} \times \mathbb{R}\), a one-parameter family of properly embedded unduloids, and a one-parameter family of non-embedded nodoids. Both unduloids and nodoids are invariant under the discrete \(\mathbb{Z}\)-group generated by some vertical translation in \(\mathbb{R}^{n+1}\).

In this section we will extend Delaunay’s classification, and show that a similar description holds for rotational \(H\)-hypersurfaces in \(\mathbb{R}^{n+1}\), in the case that \(H \in C^2(S^n)\) is positive, rotationally invariant and even, i.e. \(H(x) = H(-x)\) for all \(x \in S^n\). In terms
of the function $h$ associated to $H$ by (3.2), this means that $h(-y) = h(y) > 0$ for all $y \in [-1, 1]$.

We should point out that, in contrast with the classical CMC case, for a prescribed function $H$ in the conditions above, the Delaunay hypersurfaces with prescribed $H$ cannot be given a general explicit description by integral formulas. We should also point out that this Delaunay-type classification is not true anymore if the prescribed mean curvature $H > 0$ is not an even function on $S^n$. In Section 3.4 we will provide examples that support this claim.

**Theorem 3.7** Let $h \in C^2([-1, 1])$ be given by (3.1) in terms of $H$, and suppose that $h(y) = h(-y) > 0$ for all $y$. Then, the following list exhausts, up to vertical translations, all existing rotational $H$-hypersurfaces in $\mathbb{R}^{n+1}$ around the $x_{n+1}$-axis:

1. The right circular cylinder $C_H := S^{n-1}(r_0) \times \mathbb{R}$, where $r_0 := (n-1)h(0)/n$.
2. The strictly convex Guan-Guan $H$-sphere $S_H \subset \mathbb{R}^{n+1}$.
3. A one-parameter family of properly embedded $h$-unduloids, $U_h$.
4. A one-parameter family of properly immersed, non-embedded, $h$-nodoids, $N_h$.

Moreover, any $h$-unduloid or $h$-nodoid is invariant by a vertical translation in $\mathbb{R}^{n+1}$, diffeomorphic to $S^{n-1} \times \mathbb{R}$, and lies in a tubular neighborhood of the $x_{n+1}$-axis in $\mathbb{R}^{n+1}$. The Gauss map image of each $h$-unduloid omits open neighborhoods of the north and south poles in $S^n$. The Gauss map image of each $h$-nodoid is $S^n$.

**Proof.** Let $(x(s), y(s))$ be any solution to (3.6). Since $h(y) = h(-y) > 0$, it follows that $(x(-s), -y(-s))$ is also a solution to (3.6). Geometrically, this means that any orbit of the phase space $\Theta_\varepsilon$, $\varepsilon = \pm 1$, is symmetric with respect to the $y = 0$ axis. The curve $\Gamma_1$ in $\Theta_1$ given by (3.8) together with $y = 0$ divides the phase space $\Theta_1$ into four monotonicity regions $\Lambda_1, \ldots, \Lambda_4$, all of them meeting at the equilibrium $e_0$ in (3.7). The curve $\Gamma_{-1}$ in $\Theta_{-1}$ does not exist, and so $\Theta_{-1}$ has only two monotonic regions: $\Lambda^+ = \Theta_{-1} \cap \{y > 0\}$ and $\Lambda^- = \Theta_{-1} \cap \{y < 0\}$. See Figure 3.6.

![Figure 3.6: The phase spaces $\Theta_1$ and $\Theta_{-1}$ for $h(y) = h(-y) > 0$.](image-url)

The description of the orbits in $\Theta_{-1}$ follows then easily from this monotonicity properties. Any such orbit is given by a horizontal $C^2$ graph $x = g(y)$, with $g(y) = g(-y) > 0$
for every $y \in (-1, 1)$, and such that $g$ restricted to $[0, 1]$ is strictly increasing. In the case that $g(y) \to \infty$ as $y \to 1$ for some orbit, the rotational $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$ described by that orbit would be a symmetric bi-graph over the exterior of an open ball in $\mathbb{R}^n$. This is impossible by the mean curvature comparison principle, since $\mathcal{H} > 0$. Thus, any orbit in $\Theta_{-1}$ has two limit endpoints of the form $(x_0, \pm 1)$ for some $x_0 > 0$. The resulting $\mathcal{H}$-hypersurface $\Sigma_{-1}$ in $\mathbb{R}^{n+1}$ associated to any such orbit is then a compact (with boundary) symmetric bi-graph over some domain $\Omega \subset \mathbb{R}^n$ of the form $\{ x \in \mathbb{R}^n : \alpha \leq |x| \leq x_0 \}$, and its boundary is given by

$$\partial \Sigma_{-1} = (S^{n-1}(x_0) \times \{a\}) \cup (S^{n-1}(x_0) \times \{b\}),$$

for some $a < b$. The $z(s)$-coordinate of the profile curve $\alpha(s)$ of $\Sigma_{-1}$ is strictly decreasing, and the unit normal to $\Sigma_{-1}$ along $\partial \Sigma_{-1} \cap \{x_{n+1} = a\}$ (resp. along $\partial \Sigma_{-1} \cap \{x_{n+1} = b\}$) is constant, and equal to $e_{n+1}$ (resp. to $-e_{n+1}$).

We describe next the orbits in $\Theta_1$ together with their associated $\mathcal{H}$-hypersurfaces. First, observe that the equilibrium $e_0 \in \Theta_1$, given by (3.7), corresponds to the cylinder $C_{\mathcal{H}} \subset \mathbb{R}^{n+1}$.

Let us analyze the structure of the orbits of $\Theta_1$ around $e_0$. Noting that $\mathfrak{h}(0) = 0$ by symmetry, we can check that the linearized system at $e_0$ associated to the nonlinear system (3.6) for $\varepsilon = 1$ is

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -n^2 \mathfrak{h}(0)^2/(n-1) & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

whose orbits are ellipses around the origin. By classical theory of nonlinear autonomous systems, this means that there are two possible configurations for the space of orbits of (3.6) near $e_0$: either all such orbits are closed curves (a center structure), or they spiral around $e_0$. However, this second possibility cannot happen, since all orbits of (3.6) are symmetric with respect to the axis $y = 0$, and $e_0$ lies in this axis. In particular, we deduce that all orbits of $\Theta_1$ stay at a positive distance from the equilibrium $e_0$. 

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{figure3.7.png}
  \caption{Left: Phase space $\Theta_1$ for the choice $\mathfrak{h}(y) = 1 + y^2$ for surfaces in $\mathbb{R}^3$. The orbit $\gamma_0$, in red, corresponds to the Guan-Guan convex sphere $S_{\mathcal{H}}$ in $\mathbb{R}^3$ (right). The green curve represents the curve $\Gamma_1$ in $\Theta_1$.}
  \end{figure}
Besides, by \([GG]\), there exists a rotational, strictly convex \(H\)-sphere \(S_H\) in \(\mathbb{R}^{n+1}\). In the phase space \(\Theta_1\), this means that there exists an orbit \(\gamma_0\) that joins the limit points \((0, 1)\) and \((0, -1)\), and that lies in the region \(\Lambda_1 \cup \Lambda_2\); see Figure 3.7. Thus, \(\gamma_0\) divides \(\Theta_1\) into two connected components: the one containing the equilibrium \(e_0\), which we will denote by \(W_0\), and the one where \(x > 0\) is unbounded (which will be denoted by \(W_\infty\)). Any orbit of \(\Theta_1\) lies entirely in one of these two open sets.

By symmetry and monotonicity, and the fact that \(\gamma_0\) is the unique orbit in \(\Theta_1\) with an endpoint of the form \((0, \pm 1)\), it is clear that any orbit in \(W_\infty\) is a symmetric horizontal graph \(x = g(y)\), with \(g(1) = g(-1) = x_0\) for some \(x_0 > 0\), and such that \(g\) is strictly increasing (resp. decreasing) in \((-1, 0]\) (resp. in \([0, 1)\)). Let \(\Sigma_1\) denote the rotational \(H\)-hypersurface in \(\mathbb{R}^{n+1}\) associated to any such orbit in \(W_\infty\), and let \(\alpha(s) = (x(s), z(s))\) be its profile curve. Note that \(z' > 0\) since \(\varepsilon = 1\). By similar arguments to the ones used for \(\Theta_{-1}\), we conclude that \(\Sigma_1\) is a compact (with boundary) symmetric bi-graph in \(\mathbb{R}^{n+1}\) over some domain in \(\mathbb{R}^n\) of the form \(\{x \in \mathbb{R}^n : x_0 \leq |x| \leq \beta\}\), and

\[
\partial \Sigma_1 = (S^{n-1}(x_0) \times \{c\}) \cup (S^{n-1}(x_0) \times \{d\}),
\]

for some \(c < d\). This time, the unit normal to \(\Sigma_1\) along \(\partial \Sigma_1 \cap \{x_{n+1} = c\}\) (resp. along \(\partial \Sigma_1 \cap \{x_{n+1} = d\}\)) is \(-e_{n+1}\) (resp. \(e_{n+1}\)).

Figure 3.8: Left: Phase portrait for the choice \(h(y) = 1 + y^2\) for surfaces in \(\mathbb{R}^3\). The green curve corresponds to \(\Gamma_1\), and the red curve to the orbit \(\gamma_0\) of the Guan-Guan \(H\)-sphere. The black curve is the orbit corresponding to the piece of the \(h\)-nodoid where \(z' > 0\), and thus lies in \(\Theta_1\). The blue curve is the orbit of the \(h\)-nodoid in the \(\Theta_{-1}\) phase space (it can intersect other orbits, since it belongs to a different phase space). Right: a picture of a section of the \(h\)-nodoid in \(\mathbb{R}^3\).

Consequently, by uniqueness of the solution to the Cauchy problem for \(H\)-graphs in \(\mathbb{R}^{n+1}\), we can deduce that, given \(x_0 > 0\), the \(H\)-hypersurfaces \(\Sigma_{-1}\) and \(\Sigma_1\) that we have constructed associated to \(x_0\) can be smoothly glued together along any of their boundary components where their unit normals coincide, to form a larger \(H\)-hypersurface. For
Figure 3.9: Left: Phase portrait $\Theta_1$ for the choice $h(y) = 1 + y^2$ for surfaces in $\mathbb{R}^3$. The green curve corresponds to $\Gamma_1$, and the red curve to the orbit $\gamma_0$ of the Guan-Guan $H$-sphere. The black curve is the orbit of an $h$-unduloid. Right: a picture of an $h$-unduloid in $\mathbb{R}^3$.

This, we should note that both $\Sigma_{-1}$ and $\Sigma_1$ are defined up to vertical translations in $\mathbb{R}^{n+1}$, and so we can assume without loss of generality in the previous construction that $a = d$ or that $b = c$ (and hence $\Sigma_1$ and $\Sigma_{-1}$ have the same Cauchy data). By iterating this process we obtain a proper, non-embedded rotational $H$-hypersurface in $\mathbb{R}^{n+1}$ diffeomorphic to $S^{n-1} \times \mathbb{R}$, invariant by a vertical translation. This proves the existence of the family of $h$-nodoids $N_h$ in $\mathbb{R}^{n+1}$.

To end the proof of Theorem 3.7, we consider an orbit $\gamma$ of $\Theta_1$ that is contained in the region $W_0$. Recall that we proved previously that $\gamma$ stays at a positive distance from the equilibrium $e_0$. As $\gamma$ is symmetric with respect to the $y = 0$ axis, if we take into account the monotonicity properties of $\Theta_1$, we see that only two possibilities can happen for $\gamma$:

i) $\gamma$ is a closed curve containing $e_0$ in its inner region, or

ii) $\gamma$ is a proper arc in $\Theta_1$ with two limit endpoints of the form $(0, y_1), (0, y_2)$, with $-1 < y_1 \leq 0 < y_2 < 1$.

Let us rule out the second case. So, assume that $\gamma$ has a limit point of the form $(0, y), \ |y| < 1$, and let $\alpha(s) = (x(s), z(s))$ denote the profile curve of its corresponding rotational $H$-hypersurface $\Sigma$. Then, $(x(s_n), x'(s_n)) \to (0, y)$ for a sequence of values $s_n$, and in particular $\alpha(s)$ approaches the rotation axis in a non-orthogonal way (since $|y| \neq 1$). So, by the monotonicity properties of the phase space, we see that a piece of $\Sigma$ is a graph $x_{n+1} = u(x_1, \ldots, x_n)$ on a punctured ball $\Omega - \{0\}$ in $\mathbb{R}^{n+1}$. Moreover, the mean curvature function of $\Sigma$, viewed as a function $H(x_1, \ldots, x_n)$ on $\Omega - \{0\}$, extends continuously to the puncture, with value $h(y)$. Hence, it is known that the graph $\Sigma$ extends smoothly to the ball $\Omega$, see e.g. \cite{LR0}. In particular, the unit normal at the puncture is vertical. This is a contradiction with $|y| < 1$. 

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Consequently, we deduce that any orbit $\gamma$ inside $W_0$ is a closed curve that contains $e_0$ inside its inner region. This implies that the profile curve $\alpha(s) = (x(s), z(s))$ of the rotational $\mathcal{H}$-hypersurface associated to any such orbit satisfies that $s$ is defined for all real values, that $z'(s) > 0$ for all $s$, and that $x(s)$ is periodic. These properties imply that $\Sigma$ is an $h$-unduloid, with all the properties asserted in the statement of the theorem (see Figure 3.9). This concludes the proof.

It is also interesting to remark that the family of $h$-unduloids for a given $h$ in the conditions of Theorem 3.7 is a continuous 1-parameter family. Similarly to what happens in the CMC case, at one extreme of the parameter domain we have a (singular) vertical chain of tangent rotational $\mathcal{H}$-spheres $S_\mathcal{H}$, and at the other extreme we have the $\mathcal{H}$-cylinder $C_\mathcal{H}$.

\[\square\]

Figure 3.10: Profile curve and picture of an $h$-unduloid in $\mathbb{R}^3$ for $h(y) = 1 - y^2$.

It is important to stress that the ideas in the proof of Theorem 3.7 can be used to classify the rotational $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ for other types of rotationally symmetric choices of $\mathcal{H}$.

For instance, assume that $\mathcal{H} \in C^2(\mathbb{S}^n)$ is rotationally symmetric, and its associated function $h \in C^2([-1, 1])$ satisfies that $h(y) = h(-y) > 0$ for every $y \in (-1, 1)$, and $h(\pm 1) = 0$. In that case it can be shown using the previous ideas (although details will be skipped here) that there are exactly four types of rotational $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ with axis $x_{n+1}$: horizontal planes, cylinders $C_\mathcal{H}$ of the form $\mathbb{S}^n(r_0) \times \mathbb{R}$, $h$-catenoids of non-vanishing mean curvature like the ones of Proposition 3.6 (with opposite orientation), and surfaces of $h$-unduloid type similar to those of Theorem 3.7. Interestingly, in this situation the family of $h$-unduloids varies between the cylinder $C_\mathcal{H}$ and a singular family of double covers of horizontal planes joined along the $x_{n+1}$-axis. See Figure 3.10.
3.4 Examples with no CMC counterpart

The results in Sections 3.2 and 3.3 show, in particular, the existence of the following types of rotational $H$-hypersurfaces in $\mathbb{R}^{n+1}$ for rather general choices of the (rotationally symmetric) prescribed mean curvature function $H \in C^1(S^n)$: planes, cylinders, bowls, catenoids, convex spheres, unduloids and nodoids. This might suggest that $H$-hypersurfaces behave roughly as CMC hypersurfaces. In this section we shall see, however, that there are many other possible types of rotational $H$-hypersurfaces in $\mathbb{R}^{n+1}$. For definiteness, we will restrict our discussion to the case $n = 2$.

So, in what follows, we will consider some specific functions $h \in C^\infty([-1,1])$, which define rotationally symmetric functions $H \in C^\infty(S^n)$ by means of (3.1), and we will construct rotational $H$-surfaces $\Sigma$ in $\mathbb{R}^3$ associated to them.

Figure 3.11: Profile curve and picture for $h(y) = y + 2$ of the rotational $H$-surface in $\mathbb{R}^3$ that meets its rotation axis with unit normal $e_3$.

We first consider the case that $\Sigma$ touches its rotation axis orthogonally. If $h$ is an even positive function on $[-1,1]$, then $\Sigma$ is a strictly convex sphere (see Theorem 3.7). However, if $h > 0$ is not even, the generic situation is that one of the orbits that start at $(0, \pm 1)$ in the phase space $\Theta_1$ ends up spiraling around the equilibrium, while the other one ends up leaving $\Theta_1$ in a finite time of its parameter across one of the boundary curves $y = \pm 1$ (and thus, it enters the other phase space $\Theta_{-1}$). In the first case, we obtain properly embedded rotational $H$-disks in $\mathbb{R}^3$ that converge asymptotically to the vertical cylinder of mean curvature $h(0)$, wiggling around it. See Figure 3.11. In the second case, we typically obtain properly immersed, non-embedded, rotational $H$-disks that do not remain at a bounded distance from their rotation axis. See Figure 3.12.

Another less common, but still possible situation, is that the orbit in $\Theta_1$ that starts at $(0, \pm 1)$ converges to the equilibrium $e_0 \in \Theta_1$ without spiraling around it. This gives examples of complete, non-entire, strictly convex rotational $H$-graphs that converge
asymptotically to a cylinder. For instance, consider the rotational surface $\Sigma$ in $\mathbb{R}^3$ around the $x_3$-axis, with profile curve $\gamma(s) = (s, 0, -\log(\cos s))$, with $s \in [0, \pi/2)$. Note that $\gamma(s)$ is the generating curve of the Grim reaper cylinder in the theory of self translating solitons of the mean curvature flow.

The curve $\gamma(s)$ is a strictly convex curve asymptotic to the vertical line $x = \pi/2$. A straightforward computation yields that $\Sigma$ has angle function given by $\nu(s) = \cos s$, and its mean curvature function can be written as a function of $\nu$, as follows: $H_\Sigma(s) = h_\gamma(\nu(s))$, where $h_\gamma \in C^1([0, 1])$ is given by

$$h_\gamma(y) = y + \frac{\sqrt{1 - y^2}}{\arccos y}.$$  \hspace{1cm} (3.14)
Since $h_\gamma(y) > 0$ for all $y \in [0, 1]$, we can extend $h_\gamma$ to be a positive $C^1$ function on $[-1, 1]$. Thus, for the prescribed mean curvature function $H_\gamma > 0$ in $S^2$ associated to such $h_\gamma$, we obtain the existence of a strictly convex rotational $H_\gamma$-graph with the topology of a disk which is asymptotic to the right circular cylinder of radius $\pi/2$.

We consider next the case that the rotational surface $\Sigma$ does not touch its rotation axis. Assume that $h \in C^1([-1, 1])$ satisfies $h(0) = 0$, with $h(y) < 0$ (resp. $h(y) > 0$) if $y < 0$ (resp. $y > 0$). Then, by analyzing the resulting phase space as in previous results, we obtain the existence of wing-like $h$-catenoids; see Figure 3.14. They have a similar shape to the well-known rotational wing-like translating solitons to the mean curvature flow (see [CSS]), which are recovered for the specific choice $h(y) = y$. All of them are properly immersed annuli in $\mathbb{R}^3$, with their two ends going both either upwards or downwards.

Consider, finally, the case that $h \in C^1([-1, 1])$ satisfies $h(0) = 0$, with $h(y) > 0$ if $y \neq 0$. In that case there exist two types of properly immersed, rotational $H$-surfaces $\Sigma$ with the topology of an annulus, but this time with one end going upwards and the other going downwards. Each of such ends can be seen as a strictly convex graph over an exterior planar domain $\Omega = \mathbb{R}^2 - D(0, r)$.

In the first type of these examples, the coordinate $z(s)$ of the profile curve of $\Sigma$ is monotonous, and so $\Sigma$ is properly embedded; see Figure 3.15. In the second type of examples, $z(s)$ is not monotonous, and the profile curve describes a nodoid-type loop along which loses embeddedness. See Figure 3.16.
Figure 3.15: A rotational embedded $\mathcal{H}$-annulus in $\mathbb{R}^3$, for the choice $h(y) = y^2(y + 2)$.

Figure 3.16: A rotational non-embedded $\mathcal{H}$-annulus in $\mathbb{R}^3$, for the choice $h(y) = y^2(y+2)$.

4 Classification of properly embedded $\mathcal{H}$-surfaces

In this section we will study properly embedded $\mathcal{H}$-surfaces of finite topology in $\mathbb{R}^3$, for $\mathcal{H} \in C^1(\mathbb{S}^2)$, $\mathcal{H} > 0$. In Section 4.1 we will recall a diameter estimate for horizontal sections of graphs with positive mean curvature by Meeks (Lemma 4.1), and we will obtain a curvature estimate for $\mathcal{H}$-surfaces away from their boundary, see Theorem 4.2 and Remark 4.3. In Section 4.2 we will provide several a priori height estimates for $\mathcal{H}$-graphs with zero boundary values over closed, not necessarily bounded, planar domains. These estimates will be used in Section 4.4 to study properly embedded $\mathcal{H}$-surfaces in $\mathbb{R}^3$. 
4.1 Curvature and horizontal diameter estimates

The next result is essentially due to Meeks [Me]:

Lemma 4.1 Let $\Sigma \subset \mathbb{R}^3$ be a graph $z = u(x,y)$ over a closed (not necessarily bounded) domain of $\mathbb{R}^2$, with zero boundary values. Assume that the mean curvature $H_\Sigma$ of $\Sigma$ satisfies $H_\Sigma > H_0$ for some $H_0 > 0$.

Then, for every $t > 2/H_0$, the diameter of each connected component of $\Sigma \cap \{|z| = t\}$ is at most $2/H_0$. In particular, all connected components of $\Sigma \cap \{|z| \geq t\}$ for $t > 2/H_0$ are compact.

Proof: The argument follows the ideas of the proof of Lemma 2.4 in [Me], so we will only give here a sketch of it, following a slightly simplified version of Meeks’ original proof, that can be found in [AEG, Theorem 4], or [EGR, Theorem 6.2].

Without loss of generality, we may assume that $u \geq 0$ and that, if $U$ denotes the unique connected component of $\mathbb{R}^3$ determined by the plane $z = 0$ and the graph $\Sigma$, the mean curvature vector of $\Sigma$ points towards $U$.

Assume that there exist $t > 2/H_0$ and a connected component of $\Sigma \cap \{z = t\}$ with a diameter greater than $2/H_0$. Let $\Omega \subset \mathbb{R}^2$ denote the domain where the graph $\Sigma$ is defined. Then, there exists a simple arc $\Gamma \subset \Omega$ such that the Euclidean distance between its extrema $p_1, p_2$ is greater than $2/H_0$, and so that $u(p) \geq t$ for all $p \in \Gamma$. Besides, there is no restriction in assuming that the Euclidean distance between any other two points of $\Gamma$ is smaller than the distance from $p_1$ to $p_2$. Up to a horizontal isometry, we can take $p_1 = (-x_0, 0), p_2 = (x_0, 0)$, with $x_0 > 1/H_0$.

In this way, the “rectangle” surface with boundary $S = \Gamma \times [0, t]$ lies entirely in $U$. Besides, $S$ divides the solid region

$$C = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq x_0, \ 0 \leq z \leq t\}$$

into two connected components $C_1, C_2$.

Hence, we can place a sphere of radius $1/H_0$ inside $C_1$, and move it continuously towards $C_2$ without leaving the interior of $C$. Consider now just the piece of sphere that passes through $S$ into $C_2$. It’s clear that this piece cannot touch $\Sigma$, by the mean curvature comparison principle. Hence, the sphere could go through $S$ completely, and end up being contained in $C_2 \cap U$. But, obviously, a sphere of radius $1/H_0$ cannot be contained in the connected component $U$, since we could then move it upwards until reaching a first contact point with $\Sigma$, and this would contradict again the mean curvature comparison principle.

The next result is a curvature estimate inspired by [RST].

Theorem 4.2 Let $\Lambda, d, \rho$ be positive constants. Then there exists $C = C(\Lambda, d, \rho) > 0$ such that the following assertion is true:
Let $\Sigma$ be any immersed oriented surface in $\mathbb{R}^3$, possibly with non-empty boundary, let $\sigma, H, \eta$ denote, respectively, its second fundamental form, its mean curvature and its unit normal, and assume that:

$$|H| + |\nabla H| \leq \Lambda \quad \text{on } \Sigma. \tag{4.1}$$

$$\eta(\Sigma) \subset S^2 \text{ omits a spherical disk of radius } \rho. \tag{4.2}$$

Then, for any $p \in \Sigma_d := \{q \in \Sigma : d_\Sigma(q, \partial \Sigma) \geq d\}$, we have

$$|\sigma(p)| \leq C.$$

**Proof:** Arguing by contradiction, assume that the statement is not true, i.e. there is a sequence $f_n : \Sigma_n \to \mathbb{R}^3$ of immersed oriented surfaces in $\mathbb{R}^3$ satisfying (4.1), (4.2), and points $p_n \in \Sigma_n$ such that $d_{\Sigma_n}(p_n, \partial \Sigma_n) \geq d$ and $|\sigma_n(p_n)| > n$ for all $n$, where $\sigma_n$ is the second fundamental form of $f_n$. Note that by rotating each $f_n(\Sigma_n)$ adequately in $\mathbb{R}^3$, we may assume that the Gaussian images of all the $f_n : \Sigma_n \to \mathbb{R}^3$ omit the open spherical disk of radius $\rho$ centered at the north pole of $S^2$.

Consider the compact intrinsic metric disk $D_n = B_{\Sigma_n}(p_n, d/2)$ in $\Sigma_n$, which by construction is at a positive distance from $\partial \Sigma_n$. Let $q_n$ be the maximum on $D_n$ of the function

$$h_n(x) = |\sigma_n(x)|d_{\Sigma_n}(x, \partial D_n)$$

Clearly, $q_n$ lies in the interior of $D_n$, as $h_n$ vanishes on $\partial D_n$. Let $\lambda_n = |\sigma_n(q_n)|$ and $r_n = d_{\Sigma_n}(q_n, \partial D_n)$. Then,

$$\lambda_n r_n = |\sigma_n(q_n)|d_{\Sigma_n}(q_n, \partial D_n).$$

(4.3)

Note that this implies that $(\lambda_n)_n \to \infty$ as $n \to \infty$. Also, note that for every $z_n \in B_{\Sigma_n}(q_n, r_n/2)$ we have

$$d_{\Sigma_n}(q_n, \partial D_n) \leq 2d_{\Sigma_n}(z_n, \partial D_n). \tag{4.4}$$

Consider now the immersed oriented surfaces $g_n : B_{\Sigma_n}(q_n, r_n/2) \to \mathbb{R}^3$ obtained by applying a rescaling of factor $\lambda_n$ to the restriction of $f_n$ to $B_{\Sigma_n}(q_n, r_n/2)$; that is, $g_n = \lambda_n f_n$ restricted to $B_{\Sigma_n}(q_n, r_n/2)$. For short, we will sometimes write $M_n$ to denote this immersed surface given by $g_n$.

By (4.4), we have the following estimate for the second fundamental form $\hat{\sigma}_n$ of $M_n$ at any point $z_n$ in $B_{\Sigma_n}(q_n, r_n/2)$:

$$|\hat{\sigma}_n(z_n)| = \frac{|\sigma_n(z_n)|}{\lambda_n} = \frac{h_n(z_n)}{\lambda_n d_{\Sigma_n}(z_n, \partial D_n)} \leq \frac{h_n(q_n)}{\lambda_n d_{\Sigma_n}(z_n, \partial D_n)} = \frac{d_{\Sigma_n}(q_n, \partial D_n)}{d_{\Sigma_n}(z_n, \partial D_n)} \leq 2. \tag{4.5}$$

In particular, the norms of the second fundamental forms of the surfaces $M_n$ are uniformly bounded. Also note that, by construction, $|\hat{\sigma}_n(q_n)| = 1$. By (4.3), the radii of $M_n$ diverge to infinity (recall that the *radius* of a compact Riemannian surface with boundary is the maximum distance of points in the surface to its boundary).
Let now \( \tilde{M}_n \) denote the translation of \( M_n \) that takes the point \( g_n(q_n) \) to the origin of \( \mathbb{R}^3 \), and let \( \xi_n \in S^2 \) denote the Gauss map image of \( M_n \) at \( q_n \). After passing to a subsequence, we may assume that \( (\xi_n)_n \to \xi \) as \( n \to \infty \), for some \( \xi \in S^2 \). By construction, the norm of the second fundamental form of \( \tilde{M}_n \) is at most 2, and it is equal to 1 at the origin.

We use next an argument similar to the one in the proof of Theorem 2.17 to show that a subsequence of the surfaces \( \tilde{M}_n \) converges uniformly on compact sets to a complete minimal surface \( M_\infty \).

First, Proposition 2.3 in [RST] ensures that there exist positive constants \( \delta_0, \mu \) (independent of \( n \)) such that, for any \( n \) large enough, we can view a neighborhood of the origin in \( \tilde{M}_n \) as a graph of a function \( u_n \) over a disk \( D_0^n \) of radius \( \delta_0 \) of its tangent plane \( T_0 \tilde{M}_n = \xi_n \perp \), and such that \( ||u_n||_{C^2(D_0^n)} \leq \mu \). Since the vectors \( \xi_n \) converge to \( \xi \) in \( S^2 \), after making if necessary \( \delta_0 \) (resp. \( \mu \)) smaller (resp. larger), and for every \( n \) large enough, we have that the same properties hold with respect to the \( \xi \) direction; that is:

a) An open neighborhood of the origin in \( \tilde{M}_n \) is the graph \( x_3 = u_n(x_1, x_2) \) of a function \( u_n \) over the Euclidean disk \( D_0 := D(0, \delta_0) \) of radius \( \delta_0 \) in \( \Pi_0 = \xi_\perp \); here \( (x_1, x_2, x_3) \) are orthonormal Euclidean coordinates centered at the origin, with \( \frac{\partial}{\partial x_3} = \xi \).

b) The \( C^2 \) norm of \( u_n \) in \( D_0 \) is at most \( \mu \).

Let \( H_n(x_1, x_2) \) denote the mean curvature function of \( \tilde{M}_n \) in these coordinates. Note that, by (4.1) and the fact that the factors \( \lambda_n \) diverge to \( \infty \), the functions \( H_n \) are uniformly bounded in the \( C^1(D_0) \) norm, and as a matter of fact they converge uniformly to zero in that norm. Also note that, since the graph of \( u_n \) has mean curvature \( H_n \), then \( u_n \) is a solution to the linear elliptic PDE for \( u \)

\[
a_{11}(Du_n)u_{11} + 2a_{12}(Du_n)u_{12} + a_{22}(Du_n)u_{22} = 2H_n(1 + |Du_n|^2)^{3/2}, \quad (4.6)
\]

where \( u_{ij} \) denotes second derivatives of \( u \) with respect to the variables \( x_i, x_j \), and the coefficients \( a_{ij} \) are smooth functions. As, by condition b) above, the functions \( u_n \) are uniformly bounded in the \( C^{1,\alpha} \) norm in \( D_0 \), we conclude that all coefficients of (4.6) are bounded in the \( C^{0,\alpha}(D_0) \) norm. By Schauder theory, the \( C^{2,\alpha} \)-norms in any \( D(0, \delta) \subset D_0 \) of the functions \( u_n \) are uniformly bounded.

Once here, we may repeat the last part of the proof in Theorem 2.17 using the Arzela-Ascoli theorem and a diagonal argument, and conclude that a subsequence of the surfaces \( \tilde{M}_n \) converges uniformly on compact sets in the \( C^2 \) topology to a complete minimal surface \( M_\infty \) of bounded curvature that passes through the origin (note that \( M_\infty \) is minimal since the mean curvatures of \( \tilde{M}_n \) converge by construction to zero). Moreover, the norm of the second fundamental form of \( M_\infty \) at the origin is equal to 1.

Also, since all the surfaces \( \tilde{M}_n \) have been obtained by translations and homotheties in \( \mathbb{R}^3 \) of the original immersions \( f_n : \Sigma_n \to \mathbb{R}^3 \), and since all the Gauss map images of the \( f_n \) omit an open spherical disk of radius \( \rho \) of the north pole in \( S^2 \), it follows that \( M_\infty \) also omits such an open disk. By a classical result of Osserman, according to which
the Gauss map image of a complete non-planar minimal surface in $\mathbb{R}^3$ is dense in $S^2$, we deduce that $M_\infty$ is a plane. This contradicts the fact that the norm of the second fundamental form of $M_\infty$ at the origin is equal to 1. This contradiction proves Theorem 4.2.

\[ \square \]

Remark 4.3 It is clear from the proof that, in Theorem 4.2, one can remove Assumption (4.1) and ask instead that $\Sigma$ is an $H$-surface for some fixed, prescribed, $H \in C^1(S^2)$. In that case, the constant $C$ only depends on $d, p$ and the $C^1$ norm of $H$ in $S^2$.

It is interesting to compare Theorem 4.2 with the family of catenoids $C_\epsilon$ in $\mathbb{R}^3$, where $\epsilon > 0$ is the necksize. When $\epsilon \to 0$, the curvature of $C_\epsilon$ blows up at its waist. Moreover, if we consider, for $d_0 > 0$ fixed, the piece $C_\epsilon(d_0)$ of $C_\epsilon$ of all points that are at a distance less than $d_0$ from the waist, then the Gaussian image in $S^2$ of $C_\epsilon(d_0)$ converges as $\epsilon \to 0$ to $S^2$ minus two antipodal points. This shows that condition (4.2) cannot be avoided in Theorem 4.2.

4.2 Height estimates for $H$-graphs

In the next definition, $\partial \Sigma$ is not assumed to be bounded, and $\Sigma$ is not compact in general.

Definition 4.4 Let $H \in C^1(S^2)$, and choose some $v \in S^2$. We will say that there exists a uniform height estimate for $H$-graphs in the $v$-direction if there exists a constant $C = C(H, v) > 0$ such that the following assertion is true:

For any graph $\Sigma$ in $\mathbb{R}^3$ of prescribed mean curvature $H$ oriented towards $v$ (i.e. $\langle \eta, v \rangle > 0$ on $\Sigma$ where $\eta$ is the unit normal of $\Sigma$), and with $\partial \Sigma$ contained in the plane $\Pi = v^\perp$, it holds that the height of any $p \in \Sigma$ over $\Pi$ is at most $C$.

Clearly, minimal graphs in $\mathbb{R}^3$ do not have a uniform height estimate (e.g., half-catenoids are counterexamples). If $H$ is a positive constant, Meeks showed in [Me] that $H$-graphs admit uniform height estimates. However, for a general $H \in C^1(S^2)$ the situation is more complicated. Indeed, our analysis in Section 3.4 showed that there exist complete, strictly convex, rotational $H$-graphs converging to a cylinder for adequate rotationally symmetric positive functions $H \in C^1(S^2)$; see Figure 3.13. The existence of such graphs shows that there are no uniform height estimates for arbitrary choices of $H \in C^1(S^2), H > 0$. We should also point out that Meeks’ proof uses that the CMC equation is invariant by reflections with respect to tilted Euclidean planes, and this is not the case anymore for a general $H \in C^1(S^2)$, not even in the rotationally symmetric case. Thus, our approach to provide uniform height estimates for $H$-graphs relies on different ideas.

Our analysis in Section 3 also indicates that it is natural to assume $H > 0$ in order to obtain a uniform height estimate for $H$-graphs, since in the case that $H$ is rotationally symmetric and vanishes somewhere, there exist entire, strictly convex $H$-graphs (and
thus, there is no uniform height estimate); see Proposition 3.5. Also note that if $\mathcal{H} > 0$, the condition $\langle \eta, v \rangle > 0$ implies by the comparison principle that $\text{int}(\Sigma) \subset \Pi^-$, where $\Pi^-$ is the connected component of $\mathbb{R}^3 - \Pi$ that contains $-v$.

Let us fix some notation. In the next theorem we will assume after choosing new coordinates $(x_1, x_2, x_3)$ that $v = e_3$. We will denote $S^2_+ = S^2 \cap \{x_3 > 0\}$, $S^1 = S^2 \cap \{x_3 = 0\}$, and $\overline{S^2_+} = S^2 \cup S^1$.

Let $\mathcal{H} \in C^1(\overline{S^2_+})$, $\mathcal{H} > 0$. By an $\mathcal{H}$-hemisphere in the $e_3$-direction we will mean a compact, strictly convex $\mathcal{H}$-surface $\Sigma_{\mathcal{H}}$ with boundary, such that $\text{int}(\Sigma_{\mathcal{H}})$ is an upwards-oriented graph $x_3 = u(x_1, x_2)$ over a $C^2$ regular convex disk in $\mathbb{R}^2$, and whose Gauss map image is $\eta(\Sigma_{\mathcal{H}}) = S^2_+$.

Given $\mathcal{H} \in C^1(\overline{S^2_+})$, recall that a necessary and sufficient condition for the existence of a closed curve $\gamma \subset \mathbb{R}^2$ such that the cylinder $\gamma \times \mathbb{R} \subset \mathbb{R}^3$ is an $\mathcal{H}$-surface (see Corollary 2.15) is that $\mathcal{H}(\xi) \neq 0$ for every $\xi \in S^1$, and

$$\int_{S^1} \frac{\xi}{\mathcal{H}(\xi)} d\xi = 0. \quad (4.7)$$

In that case, $\gamma$ is strictly convex and bounds a compact domain in $\mathbb{R}^2$, that we will denote by $\Omega_{\mathcal{H}}$.

**Theorem 4.5** Given $\mathcal{H} \in C^1(\overline{S^2_+})$, $\mathcal{H} > 0$, any of the following conditions on $\mathcal{H}$ imply that there exists a uniform height estimate for $\mathcal{H}$-graphs in the $e_3$-direction:

1. Condition (4.7) does not hold.

2. Condition (4.7) holds, and there exists a graph $\Sigma_0$ in the $e_3$-direction, oriented towards $e_3$, over a domain $\Omega \subset \mathbb{R}^2$ that contains $\Omega_{\mathcal{H}}$, and with the property that $\mathcal{H}_{\Sigma_0}(p) > \mathcal{H}(\eta(p))$ for all $p \in \Sigma_0 \cap (\Omega_{\mathcal{H}} \times \mathbb{R})$.

3. There exists an $\mathcal{H}$-hemisphere in the $e_3$-direction.

4. There is some $\mathcal{H}^* \in C^1(\overline{S^2_+})$, with $\mathcal{H} = \mathcal{H}^*$ in $S^1$ and $\mathcal{H}^* > \mathcal{H}$ in $S^2_+$, for which there exists an $\mathcal{H}^*$-hemisphere in the $e_3$-direction.

5. $\max \mathcal{H} < 2 \min \mathcal{H}|_{S^1}$.

We will prove Theorem 4.5 in Section 4.3 and devote the rest of the present Section 4.2 to discuss the sufficient conditions described in Theorem 4.5 and to deduce some corollaries from it.

The first condition in Theorem 4.5 indicates that for generic, non-symmetric choices of $\mathcal{H} \in C^1(\overline{S^2_+})$, $\mathcal{H} > 0$, there exist uniform height estimates for $\mathcal{H}$-graphs. The second condition gives, for the remaining cases of $\mathcal{H}$, a very general sufficiency property for the existence of uniform height estimates for $\mathcal{H}$-graphs, in terms of the existence of an adequate barrier $\Sigma_0$. The rest of sufficient conditions in Theorem 4.5 are obtained by applying the second condition to situations where we can construct the barrier $\Sigma_0$.

One particular consequence of Theorem 4.5 of special interest is:
Corollary 4.6 Let $\mathcal{H} \in C^1(\mathbb{S}^2)$, $\mathcal{H} > 0$, and assume that there exists a strictly convex $\mathcal{H}$-sphere $S_\mathcal{H}$ in $\mathbb{R}^3$. Then, there exist uniform height estimates for $\mathcal{H}$-graphs with respect to any direction $v \in \mathbb{S}^2$.

Proof: Simply observe that $\Sigma_{\mathcal{H},v} := \{p \in S_\mathcal{H} : \langle \eta_S(p), v \rangle \geq 0\}$ is an $\mathcal{H}$-hemisphere in the $v$-direction, and apply item 3 of Theorem 4.5 here, $\eta_S$ is the unit normal of $S_\mathcal{H}$.

The next general result is immediate from item 5 of Theorem 4.5.

Corollary 4.7 Let $\mathcal{H} \in C^1(\overline{\mathbb{S}^2_+})$, $\mathcal{H} > 0$. Choose any $H_0 > \max \mathcal{H} - 2\min \mathcal{H}|_{\mathbb{S}^1}$. Then there exists a uniform height estimate for graphs of prescribed mean curvature $\mathcal{H} + H_0$ in the $e_3$-direction.

If we impose some additional symmetry to the function $\mathcal{H}$, we can obtain from Theorem 4.5 more definite sufficient conditions for the existence of a height estimate. For instance:

Corollary 4.8 Let $\mathcal{H} \in C^2(\overline{\mathbb{S}^2_+})$, $\mathcal{H} > 0$, be rotationally symmetric, i.e. $\mathcal{H}(x) = h(\langle x, e_3 \rangle)$ for some $h \in C^2([0, 1])$. Assume that $h'(0) \leq 0$.

Then, there is a uniform height estimate for $\mathcal{H}$-graphs in the $e_3$-direction.

Proof: Assume first that $h'(0) = 0$. Then, we can extend $h$ to a positive, even, $C^2$ function on $[-1, 1]$. Hence, $\mathcal{H}$ can also be extended to a positive $C^2$ function on $\mathbb{S}^2$, also denoted $\mathcal{H}$, so that $\mathcal{H}(x) = \mathcal{H}(-x)$ for all $x \in \mathbb{S}^2$. By Theorem 2.10, there exists a strictly convex $\mathcal{H}$-sphere $S_\mathcal{H}$. Thus, the result follows from Corollary 4.6.

Assume now that $h'(0) < 0$. Then, we can construct a function $h^* \in C^2([-1, 1])$, with $h^*(t) = h^*(-t)$ for all $t$, and such that $h^*(0) = h(0)$ and $h^*(t) > h(t)$ for all $t \in (0, 1]$. Let $\mathcal{H}^* \in C^2(\mathbb{S}^2)$ be defined by $\mathcal{H}^*(x) := h^*(\langle x, e_3 \rangle)$, and note that $\mathcal{H}^*(-x) > 0$ for all $x \in \mathbb{S}^2$. Arguing as above, there exists an $\mathcal{H}^*$-hemisphere in the $e_3$-direction. Since it is clear that $\mathcal{H}^* = \mathcal{H}$ in $\mathbb{S}^1$ and $\mathcal{H}^* > \mathcal{H}$ in $\mathbb{S}^2_+$, we concluded the desired result by the fourth sufficient condition in Theorem 4.5.

Remark 4.9 The condition $h'(0) \leq 0$ in the above corollary cannot be removed altogether. Indeed, as we explained after Definition 4.4, our analysis in Section 3.4 shows that there exist rotationally symmetric functions $\mathcal{H} \in C^\infty(\mathbb{S}^2)$, $\mathcal{H} > 0$, that do not admit uniform height estimates.

For the case in which $\mathcal{H}$ is invariant under the symmetry with respect to a geodesic of $\mathbb{S}^2$, we have an estimate for compact embedded $\mathcal{H}$-surfaces, not necessarily graphs.

Corollary 4.10 Let $\mathcal{H} \in C^1(\mathbb{S}^2)$, $\mathcal{H} > 0$, satisfy:

1. $\mathcal{H}(x_1, x_2, x_3) = \mathcal{H}(x_1, x_2, -x_3)$ for all $x = (x_1, x_2, x_3) \in \mathbb{S}^2$.

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2. There exists a uniform height estimate for $\mathcal{H}$-graphs in the directions of $e_3$ and $-e_3$.

Then, there is some constant $C = C(\mathcal{H}) > 0$ such that the following assertion is true: any compact embedded $\mathcal{H}$-surface $\Sigma$ in $\mathbb{R}^3$, with $\partial \Sigma$ contained in the plane $x_3 = 0$, lies in the slab $|x_3| \leq C$ of $\mathbb{R}^3$.

Proof: It is an immediate consequence of the Alexandrov reflection principle applied to the family of horizontal planes in $\mathbb{R}^3$. Note that, due to the symmetry condition imposed on $\mathcal{H}$, we can apply this reflection method in that particular direction; see Lemma 2.3.

4.3 Proof of Theorem 4.5

We start by proving the first two items. Arguing by contradiction, assume that $\mathcal{H} \in C^1(\mathbb{S}^2)$, $\mathcal{H} > 0$, is a function for which there is no uniform height estimate for $\mathcal{H}$-graphs in the $e_3$-direction. So, there exists a sequence $(\Sigma_n)_n$ of $\mathcal{H}$-graphs with respect to the $e_3$-direction, oriented towards $e_3$, and with boundary $\partial \Sigma$ contained in $\Pi = e_3^\bot = \{x_3 = 0\}$, and points $p_n \in \Sigma_n$ such that the height of $p_n$ over $\Pi$ is greater than $n$. Note that $\Sigma_n \subset \{x_3 \leq 0\}$.

Take now $H_0 \in (0, \min \mathcal{H})$, and denote $\Sigma_n^*_n := \Sigma_n \cap \{|x_3| \geq 4/H_0\}$. By hypothesis, $p_n \in \Sigma_n^*_n$ for $n$ large enough. By Lemma 4.1, the connected component $\Sigma_n^0$ of $\Sigma_n^*$ that contains $p_n$ is compact, and contained inside a vertical solid cylinder of radius $2/H_0$.

Let $q_n \in \Sigma_n^0$ be a point of maximum height of $\Sigma_n^0$, and let $\Sigma_n^1 := \Sigma_n^0 - q_n$ denote the translation of $\Sigma_n^0$ that takes $q_n$ to the origin in $\mathbb{R}^3$. By Theorem 4.2 and Remark 4.3, the norms of the second fundamental form of the graphs $\Sigma_n^1$ are uniformly bounded by some positive constant $C > 0$ that only depends on $H_0$ and $\|\mathcal{H}\|_{C^1(\mathbb{S}^2)}$, and not on $n$. Moreover, the distances in $\mathbb{R}^3$ of the origin to $\partial \Sigma_n^1$ diverge to $\infty$. By Theorem 2.17, we deduce that, up to a subsequence, there are smooth compact sets $K_n$ of the graphs $\Sigma_n^1$, all of them containing the origin and with horizontal tangent plane at it, and that converge uniformly in the $C^2$ topology to a complete $\mathcal{H}$-surface $\Sigma_\infty$ of bounded curvature that passes through the origin.

Let us define next $\nu_\infty := \langle \eta_\infty, e_3 \rangle$, where $\eta_\infty$ is the unit normal of $\Sigma_\infty$. Note that $\eta_\infty(0) = 1$. As all graphs $\Sigma_n^1$ are oriented towards $e_3$, we deduce that $\nu_\infty \geq 0$ on $\Sigma_\infty$.

Furthermore, it follows directly from Corollary 5.3 (to be proved in Section 5) that $\nu_\infty$ is a solution to a linear elliptic equation on $\Sigma_\infty$ of the form

$$\Delta \nu_\infty + \langle X, \nabla \nu_\infty \rangle + q \nu_\infty = 0,$$

where $\Delta, \nabla$ denote the Laplacian and gradient operators on $\Sigma_\infty$, $X \in \mathfrak{X}(\Sigma_\infty)$, and $q \in C^2(\Sigma_\infty)$. By the maximum principle for (4.8), and the condition $\nu_\infty \geq 0$, we conclude that either $\nu_\infty \equiv 0$ on $\Sigma_\infty$ (which cannot happen since $\nu_\infty(0) = 1$), or $\nu_\infty > 0$ on $\Sigma_\infty$. Therefore, $\Sigma_\infty$ is a local vertical graph, i.e. for every $p \in \Sigma_\infty$ it holds that $T_p \Sigma_\infty$ is not a vertical plane in $\mathbb{R}^3$.  

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Once here, and since $\Sigma_{\infty}$ is a limit of compact pieces of the graphs $\Sigma_n$, it is clear that $\Sigma_{\infty}$ is itself a proper $\mathcal{H}$-graph in $\mathbb{R}^3$ oriented towards $e_3$. By construction, this graph has horizontal tangent plane at the origin, it has bounded second fundamental form, and lies entirely in the closed half-space $\{x_3 \geq 0\}$. Moreover, since each $\Sigma_n$ lies inside a vertical solid cylinder in $\mathbb{R}^3$ of radius $2/H_0$, we deduce that all points of $\Sigma_{\infty}$ lie at a distance in $\mathbb{R}^3$ at most $4/H_0$ from the $x_3$-axis.

Since $\Sigma_{\infty}$ is a complete proper graph, and at a distance at most $4/H_0$ from the $x_3$-axis, it is clear that for any $q_0 \in \partial \Omega$ there exists a diverging sequence of points $(a_n) \in \Sigma_{\infty}$, whose horizontal projections converge to $q_0$, and such that $\nu_{\infty}(a_n) \rightarrow 0$.

Express now $\Sigma_{\infty}$ as $x_3 = f(x_1, x_2)$ over an open bounded domain $\Omega \subset \mathbb{R}^2$. Let $L$ denote a straight line in $\mathbb{R}^2$ far away from $\Omega$, and let us start moving it towards $\Omega$ until it reaches a first contact point $q_0 \in \partial \Omega$ with the compact set $\Omega$.

Choose a diverging sequence $(a_n) \in \Sigma_{\infty}$ whose horizontal projections converge to $q_0 \in \partial \Omega$, and consider the vertical translations $\Sigma_{\infty}^* = \Sigma_{\infty} - (0,0,a^3_n)$, where $a^3_n$ denotes the third coordinate of $a_n$. Up to a subsequence we can suppose that the unit normals of $\Sigma_{\infty}$ at $a_n$ converge to a fixed vector $\eta_0 \in S^2$, which is horizontal. Again, a similar compactness argument to the ones above ensures that a subsequence of the graphs $\Sigma_{\infty}^*$ converges to a complete $\mathcal{H}$-surface $\Sigma_{\infty}^*$, that passes through $q_0 \in \partial \Omega$.

Note that, by construction, $\Sigma_{\infty}^*$ is contained in $\overline{\Omega} \times \mathbb{R}$. Moreover, it is clear from the fact that the sequence $a^3_n$ diverges to $\infty$ that $(\Omega \times \mathbb{R}) \cap \Sigma_{\infty}^* = \emptyset$ (observe that $\Sigma_{\infty}^*$ is constructed from divergent vertical translations of compact pieces of $\Sigma^*$). This implies that $\Sigma_{\infty}^*$ is contained in $\partial \Omega \times \mathbb{R}$, and that the connected component of $\Sigma_{\infty}^*$ that contains the point $(q_0,0)$ is contained in $(\partial_0 \Omega) \times \mathbb{R}$, where $\partial_0 \Omega$ denotes the connected component of $\partial \Omega$ that contains $q_0$. If we keep calling this connected component as $\Sigma_{\infty}^*$, then we have $\Sigma_{\infty}^* = \alpha \times \mathbb{R}$, where $\alpha \subset \partial_0 \Omega$ is a regular curve in $\mathbb{R}^2$ that verifies equation (2.7), since $\Sigma_{\infty}^*$ has prescribed mean curvature $\mathcal{H}$. Note that $\alpha$ is a closed curve, since $\partial \Omega$ is compact, and so $\Sigma_{\infty}^*$ is a complete flat cylinder with vertical rulings, diffeomorphic to $S^1 \times \mathbb{R}$, i.e. one of the surfaces in Corollary 2.15.

Since $\alpha \cap \Omega = \emptyset$, it follows by connectedness that $\Omega$ is contained in one of the two regions of $\mathbb{R}^2$ separated by $\alpha$. Moreover, from the way that the point $q_0 \in \partial \Omega$ was chosen, it is clear that $\Omega$ is actually contained in the inner region bounded by $\alpha$.

We are now in the conditions to prove the first two items stated in Theorem 4.3. It is important to recall here that our argument was by contradiction.

Since $\alpha$ is a closed curve, it follows from Corollary 2.15 that equation (4.7) holds. This proves item 1 of Theorem 4.3.

We next prove item 2. Assume that there exists a graph $\Sigma_0$ in the conditions of that item; that is, the domain $\Omega_0 \subset \mathbb{R}^2$ over which $\Sigma_0$ is defined contains $\alpha$, and $H_{\Sigma_0}(p) > \mathcal{H}(\eta(p))$ for all $p \in \Sigma_0 \cap (\Omega_\mathcal{H} \times \mathbb{R})$. In particular, $\overline{\Omega} \subset \Omega_0$, where $\Omega$ is the domain of the graph $\Sigma_{\infty}$, which has prescribed mean curvature $\mathcal{H}$. Recall that $\Sigma_{\infty}$ is contained in $\{x_3 \geq 0\}$. Hence, we can move $\Sigma_0$ downwards by vertical translations so that its restriction to the compact set $\overline{\Omega} \subset \Omega_0$ is contained in $\{x_3 < 0\}$, and then start moving it upwards until reaching a first contact point with $\Sigma_{\infty}$. This contradicts the mean curvature comparison principle, since $H_{\Sigma_0} > \mathcal{H} \circ \eta$ on $\Sigma_0 \cap (\Omega_\mathcal{H} \times \mathbb{R})$ by hypothesis.
This contradiction proves item 2 of Theorem 4.3

**Remark 4.11** The statement of item 2 also holds for the case that $\Sigma_0$ is an upwards-oriented $\mathcal{H}$-graph in the $e_3$-direction, defined over a domain that contains $\Omega_\mathcal{H}$ in its interior. The only difference in the proof with this new hypothesis is that the desired contradiction is reached by using the maximum principle of $\mathcal{H}$-surfaces, and not the mean curvature comparison principle.

We next prove item 4, as an application of item 2. Let $\mathcal{H}, \mathcal{H}^*$ be in the conditions of item 4, and let $\Sigma^*_\mathcal{H}$ denote an $\mathcal{H}^*$-hemisphere in the $e_3$-direction. Note that $\Sigma^*_\mathcal{H}$ can be seen as an upwards-oriented graph $x_3 = u(x_1, x_2)$ over a closed strictly convex disk $\Omega_0$ with $C^2$ regular boundary $\Gamma = \partial \Omega_0$. By item 1, we may assume that condition (4.7) holds, and in particular we can consider the closed curve $\gamma$ in $\mathbb{R}^2$ such that $\gamma \times \mathbb{R}$ is an $\mathcal{H}$-surface in $\mathbb{R}^3$ (see the comments before the statement of Theorem 4.5). If $\kappa_1^*, \kappa_2^*$ denote the (positive) principal curvatures of $\Sigma^*_\mathcal{H}$, then at any $p \in \Sigma^*_\mathcal{H}$ at which the unit normal $\eta^*(p)$ is a horizontal vector $\xi \in S^1$, we have (since $\mathcal{H} = \mathcal{H}^*$ in $S^1$)

$$\kappa_1^*(p) + \kappa_2^*(p) = 2 \mathcal{H}^*(\xi) = 2 \mathcal{H}(\xi) = \kappa_\gamma(p'),$$

where $p'$ is the point of $\gamma$ with unit normal equal to $\xi$, and $\kappa_\gamma$ denotes the (positive) geodesic curvature of $\gamma$. This implies that for any unit vector $v \in T_p \Sigma^*_\mathcal{H}$, the second fundamental form $\sigma^*$ of $\Sigma^*_\mathcal{H}$ satisfies

$$0 < \sigma^*_p(v, v) < \kappa_\gamma(p').$$

(4.9)

Since $\Sigma^*_\mathcal{H}$ is an $\mathcal{H}$-hemisphere, the points in $\partial \Sigma^*_\mathcal{H} = \{p \in \Sigma^*_\mathcal{H} : \langle \eta^*(p), e_3 \rangle = 0\}$ project regularly onto the convex curve $\Gamma$. Hence, $\gamma, \Gamma$ are two closed, strictly convex planar curves that, by (4.11), satisfy the following condition: if $n_\Gamma$ and $n_\gamma$ denote the inner unit normals of $\Gamma$, $\gamma$, then:

$$\kappa_\Gamma(p) < \kappa_\gamma(p')$$

whenever $n_\Gamma(p) = n_\gamma(p')$.

(4.11)

We will need at this point the following classical property of convex curves in $\mathbb{R}^2$, whose proof we omit:

**Fact:** If $\Gamma, \gamma$ are two closed, strictly convex regular planar curves that satisfy condition (4.11), then $\gamma$ is contained in the interior region bounded by some translation of $\Gamma$.

It follows then from the Fact above that, up to a translation of $\Sigma^*_\mathcal{H}$, the convex disk $\Omega_0$ of $\mathbb{R}^2$ over which $\Sigma^*_\mathcal{H}$ is a graph contains $\gamma = \partial \Omega_\mathcal{H}$ in its interior. Hence, we can conclude directly the existence of a uniform height estimate for $\mathcal{H}$-graphs by applying the already proved item 2 of the theorem. This proves item 4.

The proof of item 3 is analogous to the one of item 4, using the alternative formulation of item 2 that is explained in Remark 4.11 instead of item 2 itself. We omit the details.

We next prove item 5. First, observe again that, by item 1, we may assume that condition (4.7) holds, and so, we can consider again the closed planar curve $\gamma$ for which $\gamma \times \mathbb{R}$ is an $\mathcal{H}$-surface in $\mathbb{R}^3$. Denote $H_0 := \max \mathcal{H}$, and let $S^2(1/H_0)$ be the round
sphere of constant mean curvature $H_0$. From the condition on $H$ in item 5, we get $2H(\xi) > H_0$ for all $\xi \in S^1$. This implies that the geodesic curvature of $\gamma$ satisfies $\kappa_\gamma > H_0$ at every point. So, this means that, up to a translation, $\gamma$ is contained in the open disk $D(0,1/H_0)$ of $\mathbb{R}^2$. Once here, we conclude the proof by applying item 2 to the lower hemisphere $\Sigma_0$ of $S^2(1/H_0)$.

This finishes the proof of Theorem 4.5.

4.4 Properly embedded $\mathcal{H}$-surfaces with one end

The following result is due to Meeks [Me], and will play a key role in this section. See also Lemma 1.5 of [KKS].

Theorem 4.12 (Plane separation lemma) Let $\Sigma$ be a surface with boundary in $\mathbb{R}^3$, diffeomorphic to the punctured closed disk $\overline{D} - \{0\}$. Assume that $\Sigma$ is properly embedded, and that its mean curvature $H_\Sigma$ satisfies $H_\Sigma(p) \geq H_0 > 0$ for every $p \in \Sigma$, and for some $H_0$.

Let $P_1, P_2$ be two parallel planes in $\mathbb{R}^3$ at a distance greater than $2/H_0$, and let $P^+, P^-$ be the connected components of $\mathbb{R}^3 - [P_1, P_2]$, where $[P_1, P_2]$ is the open slab between both planes. Then, all the connected components of either $\Sigma \cap P^+$ or $\Sigma \cap P^-$ are compact.

In what follows, we say that a surface $\Sigma$ has finite topology if it is diffeomorphic to a compact surface (without boundary) with a finite number of points removed. If $\Sigma$ is properly embedded, each of such removed points corresponds then to an end of the surface. Any such end is of annular type, that is, the surface can be seen in a neighborhood of such punctures as a proper embedding of the punctured closed disk $\overline{D} - \{0\}$ into $\mathbb{R}^3$, and thus is in the conditions of Theorem 4.12.

In the next theorem one should recall that, by our analysis in Section 4.2, the existence of uniform height estimates for $\mathcal{H}$-graphs holds in any direction $v \in S^2$ for generic choices of $\mathcal{H} \in C^1(S^2)$, $\mathcal{H} > 0$. Thus, the second hypothesis in the theorem below, although a necessary one, is relatively weak.

Theorem 4.13 Let $\mathcal{H} \in C^1(S^2)$, $\mathcal{H} > 0$, satisfy:

(a) $\mathcal{H}$ is invariant under the reflection in $S^2$ that fixes a geodesic $S^2 \cap v^\perp$, for some $v \in S^2$.

(b) There exists a uniform height estimate for $\mathcal{H}$-graphs in the directions of $v$ and $-v$.

Then, there is some constant $D = D(\mathcal{H}, v) > 0$ such that the following assertion is true: any properly embedded $\mathcal{H}$-surface in $\mathbb{R}^3$ of finite topology and one end is contained in a slab of width at most $D$ between two planes parallel to $\Pi = v^\perp$.

Proof: Let $\Pi_1, \Pi_2$ be two planes parallel to $\Pi = v^\perp$, at a distance $2d$ greater than $2/H_0$, where $H_0 := \min \mathcal{H}$, and assume that both $\Pi_1, \Pi_2$ intersect $\Sigma$ (if such a pair of planes does not exist, then $\Sigma$ lies in a slab of $\mathbb{R}^3$ of width $2d$, and the result follows). After
a change of Euclidean coordinates, we may assume that \( v = e_3 \), that \( \Pi_1 = \{ x_3 = d \} \), and \( \Pi_2 = \{ x_3 = -d \} \) for some \( d > 1/H_0 \). Since \( \Sigma \) is properly embedded and only has one end, we can write \( \Sigma = \Sigma_0 \cup \mathcal{A} \), where \( \Sigma_0 \) is a compact surface with boundary, and \( \mathcal{A} \) is a proper embedding of \( \partial D - \{ 0 \} \) into \( \mathbb{R}^3 \). Using this decomposition and the plane separation lemma (Theorem 4.12), we deduce that either \( \Sigma \cap \{ x_3 \geq d \} \) or \( \Sigma \cap \{ x_3 \leq -d \} \) only has compact connected components. Say, for definiteness that \( \Sigma \cap \{ x_3 \geq d \} \) has this property. By Corollary 4.10, \( \Sigma \cap \{ x_3 \geq d \} \) is contained in the slab \( d \leq x_3 \leq d + C \), where \( C = C(\mathcal{H}) \) is the constant appearing in that corollary. In particular, \( \Sigma \) is contained in the half-space \( \{ x_3 \leq d + C \} \).

Consider next the planes \( x_3 = d - 2C \) and \( x_3 = -d - 2C \). By the same arguments, at least one of \( \Sigma \cap \{ x_3 \geq d - 2C \} \) or \( \Sigma \cap \{ x_3 \leq -d - 2C \} \) only has compact connected components. In case \( \Sigma \cap \{ x_3 \geq d - 2C \} \) had this property, the previous arguments show that \( \Sigma \) would lie in the half-space \( \{ x_3 \leq d - C \} \), which is not possible since \( \Sigma \) intersects by hypothesis the plane \( x_3 = d \). Thus, \( \Sigma \cap \{ x_3 \leq -d - 2C \} \) only has compact connected components, and arguing as in the previous paragraph we deduce that \( \Sigma \) is contained in the half-space \( \{ x_3 \geq -d - 3C \} \). Hence, \( \Sigma \) is contained in a slab of width \( D = 2d + 4C \) between two planes parallel to \( v^+ \). This proves Theorem 4.13.

\[ \square \]

We provide next some corollaries of this result.

**Corollary 4.14** Let \( \mathcal{H} \in C^1(\mathbb{S}^2) \), \( H > 0 \), satisfy properties (a) and (b) of Theorem 4.13 with respect to two linearly independent directions \( v, w \in \mathbb{S}^2 \). Then there is some \( \alpha = \alpha(\mathcal{H}, v, w) > 0 \) such that the following holds: any properly embedded \( \mathcal{H} \)-surface of finite topology and one end lies inside a solid cylinder \( C(v \wedge w, \alpha) \) of radius \( \alpha \) and axis orthogonal to both \( v, w \).

As a particular case of Corollary 4.14 we have:

**Corollary 4.15** Let \( \mathcal{H} \in C^1(\mathbb{S}^2) \), \( H > 0 \), be rotationally symmetric, i.e. \( \mathcal{H}(x) = h(\langle x, v \rangle) \) for some \( v \in \mathbb{S}^2 \) and some \( h \in C^1([-1, 1]) \).

Then, any properly embedded \( \mathcal{H} \)-surface in \( \mathbb{R}^3 \) of finite topology and one end lies inside a solid cylinder of \( \mathbb{R}^3 \) with axis parallel to \( v \).

**Proof.** By Theorem 4.13 (or by Corollary 4.14), it suffices to show that there exist uniform height estimates for \( \mathcal{H} \)-graphs in any direction of \( \mathbb{R}^3 \) orthogonal to \( v \). In order to prove this, we can use the argument in the proof of Theorem 6.2 of [EGR], which we sketch next.

Take \( w \in \mathbb{S}^2 \) orthogonal to \( v \). First observe that, by Lemma 4.1, in order to prove existence of uniform height estimates for \( \mathcal{H} \)-graphs in the \( w \)-direction, it suffices to do so for compact graphs \( \Sigma \subset \mathbb{R}^3 \) with \( \partial \Sigma \) contained in the plane \( \Pi = w^\perp \), and with the diameter of each connected component of \( \partial \Sigma \) being less than \( 2/H_0 \), where \( 0 < H_0 < \min_{x \in \mathcal{H}} \).

Thus, let \( \Sigma \) be an such graph, and let \( \xi \in \mathbb{S}^2 \) be another vector orthogonal to \( v \), and which makes an angle of \( \pi/4 \) with \( w \). Note that, as \( \mathcal{H} \) is rotationally symmetric, we
can apply the Alexandrov reflection principle with respect to the family of planes of $\mathbb{R}^3$ orthogonal to $\xi$. By the argument in [EGR, Theorem 6.2] using reflections in this tilted direction $\xi$, it can be shown that there exists $C = C(\mathcal{H}) > 0$ (independent of $\Sigma$) such that the distance of each $p \in \Sigma$ to the plane $\Pi$ is bounded by $C$; we omit the specific details. This proves the desired existence of height estimates, and hence, also Corollary 4.15.

For the case of three reflection symmetries for $\mathcal{H}$, we obtain a more definite classification result.

**Theorem 4.16** Let $\mathcal{H} \in C^2(\mathbb{S}^2)$, $\mathcal{H} > 0$. Assume that $\mathcal{H}$ is invariant under the isometry subgroup of $\mathbb{S}^2$ generated by three linearly independent geodesic reflections of $\mathbb{S}^2$.

Then, any properly embedded $\mathcal{H}$-surface $\Sigma_\mathcal{H}$ in $\mathbb{R}^3$ of finite topology and at most one end is the Guan-Guan sphere $S_\mathcal{H}$ associated to $\mathcal{H}$.

**Proof:** By Theorem 2.10 we know that the Guan-Guan strictly convex $\mathcal{H}$-sphere $S_\mathcal{H}$ exists. By Corollary 4.6, there exist uniform height estimates for $\mathcal{H}$-graphs in any direction $v \in \mathbb{S}^2$. Thus, by Theorem 4.13 applied to the three directions of symmetry of $\mathcal{H}$, we conclude that $\Sigma_\mathcal{H}$ lies in a compact region of $\mathbb{R}^3$. Since $\Sigma_\mathcal{H}$ is proper in $\mathbb{R}^3$, $\Sigma_\mathcal{H}$ is compact. By Corollary 2.11, $\Sigma_\mathcal{H} = S_\mathcal{H}$ (up to translation), what proves the result.

**Corollary 4.17** Let $\mathcal{H} \in C^2(\mathbb{S}^2)$, $\mathcal{H} > 0$, be rotationally symmetric and even, i.e. $\mathcal{H}(x) = h(\langle x, v \rangle)$ for some $v \in \mathbb{S}^2$ and some $h \in C^2([-1, 1])$ with $h(x) = h(-x)$.

Then, any properly embedded $\mathcal{H}$-surface in $\mathbb{R}^3$ of finite topology and at most one end is the convex rotational $\mathcal{H}$-sphere $S_\mathcal{H}$ associated to $\mathcal{H}$, with rotation axis parallel to $v$.

Note that the example in Figure 3.11 shows that Theorem 4.16 is not true if we only assume that $\mathcal{H}$ is invariant by two geodesic reflections in $\mathbb{S}^2$. Likewise, by the same example, Corollary 4.17 is not true if we do not assume that $\mathcal{H}$ is even.

**5 Stability of $\mathcal{H}$-surfaces**

**5.1 The stability operator of $\mathcal{H}$-hypersurfaces**

This section is devoted to the study of stability of $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$. We start by recalling some basic notions about stability of CMC hypersurfaces. Let $\Sigma^n$ be an immersed, oriented hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature $H \in \mathbb{R}$. We define its *stability operator* (or *Jacobi operator*) as $\mathcal{L} := \Delta + |\sigma|^2$, where $\Delta$ denotes the Laplacian of $(\Sigma, g)$ and $|\sigma|$ is the norm of the second fundamental form of $\Sigma$. As $\mathcal{L}$ is a Schrödinger operator, it is $L^2$ self-adjoint.

The stability operator $\mathcal{L}$ appears when considering the second variation of the functional $\text{Area} - nH \text{Volume}$ of which $\Sigma$ is a critical point. We say that $\Sigma$ is *strongly*
stable (or simply stable) is $-L$ is a non-negative operator, that is, $-\int_{\Sigma} f L f \geq 0$ for all $f \in C_0^\infty(\Sigma)$. By a classical criterion by Fischer-Colbrie [Fi], $\Sigma$ is stable if and only if there exists a positive function $u \in C^\infty(\Sigma)$ with $Lu \leq 0$.

A Jacobi function on $\Sigma$ is a function $u \in C^\infty(\Sigma)$ such that $Lu = 0$.

The stability operator $L$ also appears as the linearized mean curvature operator. Specifically, consider a normal variation $\eta : \Sigma \to \mathbb{S}^n$ of an immersed oriented hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$, where $\eta : \Sigma \to \mathbb{S}^n$ is the unit normal of $\Sigma$ and $f \in C_0^\infty(\Sigma)$. Then, if for each $t \in (-\varepsilon, \varepsilon)$ we denote by $H(t)$ the mean curvature function of the corresponding surface $\Sigma_t$ given by (5.1), we have

$$L f = n H'(0).$$

Let us consider next the case of $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$, for some $\mathcal{H} \in C^1(\mathbb{S}^n)$, not necessarily constant. In this general situation, there is no known variational characterization of $\mathcal{H}$-hypersurfaces similar to the one of the CMC case explained above. Still, we can define a stability operator associated to $\mathcal{H}$-hypersurfaces by considering the linearization of (1.1), just as in the above characterization of the CMC case. We do this next.

**Proposition 5.1** Let $\Sigma$ be an $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$ for some $\mathcal{H} \in C^1(\mathbb{S}^n)$, let $f \in C_0^2(\Sigma)$, and for each $t$ small enough, let $\Sigma_t$ denote the hypersurface given by (5.1), where $\eta : \Sigma \to \mathbb{S}^n$ is the unit normal of $\Sigma$. Denote $\hat{\mathcal{H}}(t) := H(t) - \mathcal{H}(\eta^t) : \Sigma \times (-\varepsilon, \varepsilon) \to \mathbb{R}$, where $H(t)$ and $\eta^t$ stand, respectively, for the mean curvature and the unit normal of $\Sigma_t$.

Then,

$$n \hat{\mathcal{H}}'(0) = L f,$$

where $X_\mathcal{H} \in \mathfrak{X}(\Sigma)$ is given for each $p \in \Sigma$ by $X_\mathcal{H}(p) := n \nabla_\mathbb{S} \mathcal{H}(\eta(p))$; here $\nabla_\mathbb{S}$ denotes the gradient in $\mathbb{S}^n$, and $\Delta, \nabla, |\sigma|$ denote, respectively, the Laplacian, gradient and norm of the second fundamental form of $\Sigma$ with its induced metric.

**Proof:** From now on we work at some fixed but arbitrary $p \in \Sigma$, which will be omitted for clarity reasons. From $\hat{\mathcal{H}}(t) = H(t) - \mathcal{H}(\eta^t)$ and (5.2), we have

$$n \hat{\mathcal{H}}'(0) = \Delta f + |\sigma|^2 f - n \frac{d}{dt}_{|t=0}(\mathcal{H}(\eta^t))$$

$$= \Delta f + |\sigma|^2 f - n \left\langle \nabla_\mathbb{S} \mathcal{H}(\eta(p)), \frac{d}{dt}_{|t=0}(\eta^t) \right\rangle.$$

Let $\{e_1, \ldots, e_n\}$ be a positively oriented orthonormal basis of principal directions in $\Sigma$ at $p$. Then, for $t \in (-\varepsilon, \varepsilon)$, this basis transforms via (5.1) to the positively oriented basis $\{e'_1, \ldots, e'_n\}$ on $\Sigma_t$ given by

$$e'_i = (1 - tf\kappa_i)e_i + tdf(e_i)\eta,$$

(5.5)
where $\kappa_i$ is the principal curvature of $\Sigma$ at $p$ associated to $e_i$. From (5.5) we get

$$
\frac{d}{dt} \bigg|_{t=0} (e_1^t \wedge \cdots \wedge e_n^t) = -n H f \eta + \sum_{i=1}^n df(e_i)e_i = -n H f \eta + \nabla f.
$$

Noting that $\eta^t = (e_1^t \wedge \cdots \wedge e_n^t)/|e_1^t \wedge \cdots \wedge e_n^t|$, we have

$$
\frac{d}{dt} \bigg|_{t=0} (\eta^t) = \nabla f.
$$

From (5.4), (5.5), (5.6) we obtain (5.3). This finishes the proof of Proposition 5.1.

Proposition 5.1 justifies the following definition:

**Definition 5.2** Let $\Sigma$ be an $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$ for some $\mathcal{H} \in C^1(S^n)$. The stability operator of $\Sigma$ is defined as the operator $L$ on $\Sigma$ given for each $f \in C^2_{0}(\Sigma)$ by

$$
L f := \Delta f + \langle X_{H}, \nabla f \rangle + |\sigma|^2 f, \quad X_{H}(p) := n\nabla S_{H}(\eta(p)).
$$

Note that when $\mathcal{H}$ is constant, this definition coincides with that of the standard stability operator of CMC hypersurfaces in $\mathbb{R}^{n+1}$ described above. When $\mathcal{H}(x) = \langle x, e_{n+1} \rangle$, this notion is also consistent with the usual definition of the stability operator of self-translating solitons of the mean curvature flow, which correspond to the previous choice of $\mathcal{H}$. See, e.g. [ES, IR, Gu, SX, Gr] for some works on self-translating solitons of the mean curvature flow in which this operator is defined.

Since the property of being an $\mathcal{H}$-hypersurface is invariant by translations of $\mathbb{R}^{n+1}$, we have:

**Corollary 5.3** Let $\Sigma$ be an $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$ for some $\mathcal{H} \in C^1(S^n)$, let $\eta : \Sigma \to S^n$ denote its unit normal, choose $a \in S^n$, and define $\nu = \langle \eta, a \rangle \in C^2(\Sigma)$. Then $L \nu = 0$, where $L$ is the stability operator (5.7) of $\Sigma$.

**Proof:** Consider the variation of $\Sigma$

$$(p, \lambda) \in \Sigma \times \mathbb{R} \mapsto p + \lambda a$$

and call $\Sigma_\lambda := \Sigma + \lambda a$. By the implicit function theorem, we can write this variation in a neighborhood of each $(p_0, 0) \in \Sigma \times \mathbb{R}$ as a normal variation of the form (5.1). Specifically, for any $p$ near $p_0$ in $\Sigma$ and any $t \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ small enough we can write

$$
p + tf(p)\eta(p) = \phi(p, t) + \lambda(t)a
$$

for some smooth function $f$ defined near $p_0$ on $\Sigma$, and where $\phi(p, t) \in \Sigma$ for all $(p, t)$, and $\lambda(t)$ is smooth with $\lambda(0) = 0$ and $\lambda'(0) \neq 0$. By taking the normal component of the derivative of (5.8) with respect to $t$ at $t = 0$ we obtain using $(\frac{\partial \phi}{\partial t}, \eta) = 0$ that

$$
f = \lambda'(0)\langle \eta, a \rangle = \lambda'(0)\nu.
$$

By Proposition 5.1, $L f = 0$, since all the surfaces $\Sigma_\lambda$ (and consequently all surfaces given by $t = \text{constant}$ in (5.8)) have the same prescribed mean curvature $\mathcal{H} \in C^1(S^n)$. Thus, by (5.9), we obtain $L \nu = 0$ as claimed.
5.2 Generalized elliptic operators and stable $\mathcal{H}$-hypersurfaces

Motivated by Definition 5.2, we introduce next a general type of linear elliptic operators on Riemannian manifolds that will be of use to us (the terminology here is taken from [Es]):

**Definition 5.4** Let $(\Sigma, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold. A generalized Schrodinger operator $L$ on $\Sigma$ is an elliptic operator of the form

$$L = \Delta + \langle X, \nabla \cdot \rangle + q,$$

(5.10)

where $q \in C^2(\Sigma)$, $X \in \mathfrak{X}(\Sigma)$, and $\Delta, \nabla$ stand for the Laplacian and gradient operators on $\Sigma$.

If $X = \nabla \phi$ for some $\phi \in C^2(\Sigma)$, we say that $L$ is a gradient Schrodinger operator. If $X = 0$, we get a standard Schrodinger operator. Note that the stability operator (5.7) for hypersurfaces of prescribed mean curvature $H \in C^1(S^n)$ is a generalized Schrodinger operator.

As explained in Section 5.1, the stability operator for CMC hypersurfaces is a Schrodinger operator, and so it is $L^2$ self-adjoint. For self-translating solitons of the mean curvature flow (this corresponds to $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ with the choice $\mathcal{H}(x) := \langle x, e_{n+1} \rangle$, see Section 2.2), the stability operator is a gradient Schrodinger operator, and so it is $L^2$ self-adjoint with respect to an adequate weighted structure. In both cases, the stability operator comes associated to the second variation of an adequate functional, since these classes of surfaces admit a variational characterization.

There also exist some special cases of (non-gradient) generalized Schrodinger operators (5.10) that appear associated to the second variation of some geometric functionals. This is the case of the stability operator of marginally outer trapped surfaces (usually called, in short, MOTS). See e.g. [AEM, AMS, AM, GS, Ca] for some relevant references on this topic.

We should observe here two key difficulties in working with the stability operator $L$ of $\mathcal{H}$-hypersurfaces in (5.7), namely: (i) that $L$ is not in general $L^2$ self-adjoint, and (ii) that $L$ does not come in general from a variational problem.

Nonetheless, the following definition is natural, as will be discussed below.

**Definition 5.5** Let $\Sigma$ be an $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$ for some $\mathcal{H} \in C^1(S^n)$, and let $L$ denote its stability operator. We say that $\Sigma$ is stable if there exists a positive function $u \in C^2(\Sigma)$ such that $Lu \leq 0$.

By the classical Fischer-Colbrie theorem [Fi], this notion agrees in the CMC case with the standard definition of stability. The definition also agrees when $\mathcal{H}(x) = \langle x, e_3 \rangle$ with the usual stability notion for self-translating solutions to the mean curvature flow, see e.g. Proposition 2 in [SX]. Also, the stability notion in Definition 5.5 is also consistent with the notion of outermost stability in MOTS theory, see e.g. [AEM, Definition 3.1].

Let us give some further motivation for Definition 5.5. Let $L$ be a generalized Schrodinger operator (5.10) in a Riemannian manifold $\Sigma$, and let $\Omega \subset \Sigma$ denote a
smooth bounded domain. Even though $L$ is not formally self-adjoint, it is well known that there exists an eigenvalue $\lambda_0$ of $-L$ in $\Omega$ (with Dirichlet conditions) with the smallest real part, that $\lambda_0$ is real, and that its corresponding eigenfunction $\psi$ (unique up to multiplication by a non-zero constant) is nowhere zero; see e.g. Section 5 in [AM]. We call $\lambda_0$ the principal eigenvalue of the operator $L$ on $\Omega$.

By the argument in Remark 5.2 of [AM], the existence of a positive function $u > 0$ on $\Sigma$ such that $Lu \leq 0$ implies that the principal value $\lambda_0$ of $-L$ is non-negative on any smooth bounded domain $\Omega \subset \Sigma$. This justifies again the chosen definition for the stability of $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$.

We provide next a geometric interpretation for the stability of $\mathcal{H}$-hypersurfaces. Let $\Sigma$ denote a stable $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$, and let $\Omega$ denote a smooth bounded domain of $\Sigma$ whose closure is contained in a larger smooth bounded domain $\Omega' \subset \Sigma$. Then, the principal eigenvalue $\lambda_0(\Omega')$ of the stability operator $-\mathcal{L}$ on $\Omega'$ is non-negative. By the monotonicity of the principal eigenvalue with respect to the inclusion of domains (see e.g. [Pa]), we deduce that $\lambda_0(\Omega) > 0$.

Let $\{\Omega_t\}_{t \in (-\varepsilon, \varepsilon)}$ be a smooth variation in $\mathbb{R}^{n+1}$ of the compact (with boundary) $\mathcal{H}$-hypersurface $\Omega_0 := \Omega$, so that the boundary is fixed by the variation, i.e. $\partial \Omega_t = \partial \Omega$ for every $t$. Assume moreover that all the hypersurfaces $\Omega_t$ are also compact $\mathcal{H}$-hypersurfaces with boundary (for the same $\mathcal{H} \in C^1(\mathbb{S}^n)$). By our arguments in Proposition 5.1, this implies the existence of a function $u \in C^2(\Omega)$ satisfying $Lu = 0$ on $\Omega$ with $u = 0$ on $\partial \Omega$. However, since $\lambda_0(\Omega) > 0$ by our previous discussion, this function $u$ cannot exist. In other words: it is not possible to deform a (strictly) stable $\mathcal{H}$-hypersurface with boundary $\mathbb{R}^{n+1}$ by fixing its boundary, and in a way that all the deformed hypersurfaces in $\mathbb{R}^{n+1}$ have the same prescribed mean curvature $\mathcal{H} \in C^1(\mathbb{S}^n)$. This justifies again the notion of stability for $\mathcal{H}$-hypersurfaces we have introduced.

By Corollary 5.3 any $\mathcal{H}$-graph in $\mathbb{R}^{n+1}$ is stable as an $\mathcal{H}$-hypersurface. More generally, we have:

**Corollary 5.6** Let $\Sigma$ be an $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$ for some $\mathcal{H} \in C^1(\mathbb{S}^n)$, let $\eta: \Sigma \rightarrow \mathbb{S}^n$ denote its unit normal, and assume that $\langle \eta, a \rangle > 0$ for some $a \in \mathbb{S}^n$. Then $\Sigma$ is stable. In particular, any $\mathcal{H}$-graph is stable.

For the case of compact $\mathcal{H}$-hypersurfaces, we have another trivial consequence:

**Corollary 5.7** There are no compact (without boundary) stable $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$.

**Proof:** Let $\Sigma$ be a compact $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$, and let $\mathcal{L}$ denote its stability operator. By Corollary 5.3 the kernel of $\mathcal{L}$ has dimension at least $n + 1$, so in particular 0 is an eigenvalue for $-\mathcal{L}$ that is not simple (we do not fix Dirichlet conditions here, as $\partial \Sigma$ is empty). Hence, the principal eigenvalue $\lambda_0(\Sigma)$ of $-\mathcal{L}$ in $\Sigma$ is negative. Thus, $\Sigma$ is not stable. \(\square\)
5.3 Radius and curvature estimates for stable $\mathcal{H}$-surfaces

The next lemma is essentially due to Galloway and Schoen [GS]:

**Lemma 5.8** Let $(\Sigma, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold, and let $L = \Delta + \langle X, \nabla \cdot \rangle + q$ denote a generalized Schrodinger operator on $\Sigma$; here $X \in \mathfrak{X}(\Sigma)$ and $q \in C^0(\Sigma)$.

Assume that there exists $u \in C^2(\Sigma)$, $u > 0$, with $Lu \leq 0$. Then the Schrodinger operator

$$L := \Delta + Q, \quad Q := q - \frac{1}{2} \text{div}(X) - \frac{|X|^2}{4}, \quad (5.11)$$

satisfies that $-L$ is non-negative, i.e. $-\int_\Sigma fL f \geq 0$ for every $f \in C^2_0(\Sigma)$.

**Proof.** The condition $Lu \leq 0$ for $u > 0$ can be rewritten as

$$\Delta u + \frac{u}{4} |X| + 2 \nabla \log u^2 - \frac{u}{4} |X|^2 - u |\nabla \log u|^2 + qu \leq 0. \quad (5.12)$$

As $u > 0$ is positive, writing $u = e^\phi$ we have from (5.12)

$$\Delta \phi + \frac{1}{4} |X| + 2 \nabla \phi^2 - \frac{|X|^2}{4} + q \leq 0.$$ 

Adding and subtracting $\frac{1}{2} \text{div} X$ we have

$$\text{div}(\nabla \phi + X/2) + |\nabla \phi + X/2|^2 - \frac{|X|^2}{4} - \frac{1}{2} \text{div} X + q \leq 0. \quad (5.13)$$

So, if we define now $Y := \nabla \phi + X/2$ and $Q := q - \frac{|X|^2}{4} - \frac{1}{2} \text{div} X$, we have from (5.13)

$$|Y|^2 + Q \leq -\text{div} Y. \quad (5.14)$$

If we consider now $f \in C^2_0(\Sigma)$, then we obtain from (5.14)

$$f^2 |Y|^2 + f^2 Q \leq - f^2 \text{div} Y = - \text{div}(f^2 Y) + 2 f \langle \nabla f, Y \rangle \leq - \text{div}(f^2 Y) + 2 |f||\nabla f||Y| \leq - \text{div}(f^2 Y) + |\nabla f|^2 + f^2 |Y|^2.$$ 

By integrating the above inequality and using that $f$ has compact support, we obtain

$$\int_\Sigma |\nabla f|^2 - Q f^2 \geq 0, \forall f \in C^2_0(\Sigma),$$

which is equivalent to $-L$ being a non-negative operator. This completes the proof. 

Let us also recall the next result for $n = 2$, implicitly proved in [LR] as a variation of the arguments introduced by Fischer-Colbrie in [Fi]; see Theorem 2.8 in [MPR].
Lemma 5.9 Let $\Sigma$ denote a Riemannian surface, and assume that the Schrödinger operator $-(\Delta - K + c)$ is non-negative for some constant $c > 0$, where $K$ denotes the Gaussian curvature of $\Sigma$. Then for every $p \in \Sigma$ we have

$$d(p, \partial \Sigma) \leq \frac{2\pi}{\sqrt{3}c},$$

i.e. the radius of any (open) geodesic ball centered at $p$ that is relatively compact in $\Sigma$ is at most $2\pi/\sqrt{3}c$.

As a consequence of Lemma 5.8 and Lemma 5.9 we can deduce a distance estimate for stable $\mathcal{H}$-surfaces in $\mathbb{R}^3$, with $\mathcal{H} \in C^2(S^2)$. In Theorem 5.10 below, $\nabla_S$, $\Delta_S$ and $\nabla^2_S$ denote, respectively, the gradient, Laplacian and Hessian operators on the unit sphere $S^2$.

We should emphasize regarding Theorem 5.10 below that, by our analysis in Section 3.4, there exist complete rotational graphs (hence, stable) of prescribed mean curvature $H$ in $\mathbb{R}^3$, for some rotationally symmetric functions $\mathcal{H} \in C^2(S^2)$, $\mathcal{H} > 0$; see the example in Figure 3.13. Thus, condition (5.15) in Theorem 5.10 cannot be eliminated altogether, or just substituted by the weaker condition $H > 0$. See Problem 18 in Section 6.

Theorem 5.10 Let $\mathcal{H} \in C^2(S^2)$ satisfy on $S^2$ the inequality

$$3\mathcal{H}^2 + \det(\nabla^2_S \mathcal{H}) + \mathcal{H}\Delta_S \mathcal{H} - |\nabla_S \mathcal{H}|^2 - \frac{1}{4}(\Delta_S \mathcal{H})^2 \geq c > 0 \quad (5.15)$$

for some constant $c > 0$. Then, for every stable $\mathcal{H}$-surface $\Sigma$ in $\mathbb{R}^3$, and for every $p \in \Sigma$, we have

$$d(p, \partial \Sigma) \leq \frac{2\pi}{\sqrt{3}c}.$$

Proof: Define

$$Q_\mathcal{H} := |\sigma|^2 - \frac{1}{2}\text{div}_\Sigma(X_\mathcal{H}) - \frac{|X_\mathcal{H}|^2}{4}, \quad X_\mathcal{H} := 2\nabla_S \mathcal{H}(\eta) \in \mathfrak{X}(\Sigma). \quad (5.16)$$

Here, as usual, $|\sigma|$ denotes the norm of the second fundamental form of $\Sigma$ and $\eta$ its unit normal. Let $\mathcal{L}$ denote the stability operator of $\Sigma$, defined in (5.7). Since $\Sigma$ is stable, it follows by Lemma 5.8 that the operator $-\mathcal{L} := -(\Delta + Q_\mathcal{H})$ is non-negative on $\Sigma$. Assume for the moment that

$$Q_\mathcal{H} \geq -K + c \quad (5.17)$$

holds, where $K$ is the Gaussian curvature of $\Sigma$. In that case, the operator $-(\Delta - K + c)$ will also be non-negative, and Theorem 5.10 will follow directly from Lemma 5.9.

Thus, it only remains to prove (5.17). To do this, we first compute $\text{div}_\Sigma(X_\mathcal{H})$. First, note that if $V \in \mathfrak{X}(\Sigma)$ and $\nabla, \nabla$ denote the Riemannian connections of $\Sigma$ and $\mathbb{R}^3$, we have

$$\langle \nabla_V X_\mathcal{H}, V \rangle = 2\langle \nabla_V (\nabla_S \mathcal{H}(\eta)), V \rangle = 2\langle \nabla (\nabla_S \mathcal{H}(\eta)), V \rangle = 2\langle \nabla \nabla_S \mathcal{H}(\eta), V, d\eta(V) \rangle. \quad (5.18)$$
Here, one should observe for the last equality that, since $\langle \eta, V \rangle = 0$, then both $V, d\eta(V)$ are tangent to $S^2$ at the point $\eta$.

Consider now at any $p \in \Sigma$ an orthonormal basis $\{e_1, e_2\}$ of principal directions of $\Sigma$, and let $\kappa_1, \kappa_2$ denote their associated principal curvatures. It follows then from (5.18) that, at $p$,

$$\text{div}_\Sigma(X_H) = \sum_{i=1}^{2} \langle \nabla_{e_i} X_H, e_i \rangle = -2 \sum_{i=1}^{2} \kappa_i \alpha_i, \quad \alpha_i := (\nabla^2_{\Sigma} H)_{\eta(p)}(e_i, e_i).$$

(5.19)

Then, by (5.16), (5.19) and the identity $|\sigma|^2 = 4H^2 - 2K$, we obtain at $p$:

$$Q_H = 4H^2 - 2K - |\nabla_{\Sigma} H(\eta)|^2 + \kappa_1 \alpha_1 + \kappa_2 \alpha_2$$

$$= 3H^2 - K + \frac{(\kappa_1 - \kappa_2)^2}{4} - |\nabla_{\Sigma} H(\eta)|^2$$

$$+ \frac{(\kappa_1 + \kappa_2)(\alpha_1 + \alpha_2)}{2} + \frac{(\kappa_1 - \kappa_2)(\alpha_1 - \alpha_2)}{2}$$

$$= 3H^2 - K - |\nabla_{\Sigma} H(\eta)|^2 + \left(\frac{\kappa_1 - \kappa_2}{2} + \frac{\alpha_1 - \alpha_2}{2}\right)^2$$

$$+ H(\alpha_1 + \alpha_2) - \left(\frac{\alpha_1 - \alpha_2}{2}\right)^2$$

(5.20)

Next, note that, by definition of $\alpha_i$ in (5.19), we have $\alpha_1 + \alpha_2 = \Delta_{\Sigma} H(\eta)$, and

$$\left(\frac{\alpha_1 - \alpha_2}{2}\right)^2 \leq \frac{(\Delta_{\Sigma} H(\eta))^2}{4} - \det(\nabla^2_{\Sigma} H)_{\eta}. $$

(5.21)

Thus, it follows from (5.20), (5.21) and the identity $H_{\Sigma} = H(\eta)$ that

$$Q_H + K \geq 3H(\eta)^2 + H(\eta)\Delta_{\Sigma} H(\eta) - \frac{(\Delta_{\Sigma} H(\eta))^2}{4} + \det(\nabla^2_{\Sigma} H)_{\eta} - |\nabla_{\Sigma} H(\eta)|^2.$$

(5.22)

Since by hypothesis, $H$ satisfies (5.15) on $S^2$, we conclude from (5.22) that $Q_H + K \geq c$, which is (5.17). Hence, the proof of Theorem 5.10 is complete.

Let us state two simple remarks that clarify the nature of condition (5.15).

**Remark 5.11** Let $H \in C^2(S^2)$. Then, there is some $H_0 > 0$ such that for any $\alpha > H_0$, the function $H^* := H + \alpha \in C^2(S^2)$ satisfies equation (5.15).

**Remark 5.12** Let $H \in C^2(S^2)$ satisfy (5.15). Then $H(x) \neq 0$ at every $x \in S^2$. Indeed, if $H(x) = 0$, equation (5.15) at $x$ turns into

$$\det(\nabla^2_{\Sigma} H)_x - \frac{(\Delta_{\Sigma} H(x))^2}{4} \geq c + |\nabla_{\Sigma} H(x)|^2 > 0,$$

which is not possible since the left hand-side of this inequality is always non-positive.
The next result follows directly from Theorem 5.10. It generalizes the well-known fact that there are no complete, stable, non-minimal CMC surfaces in $\mathbb{R}^3$.

**Corollary 5.13** Let $\mathcal{H} \in C^2(\mathbb{S}^2)$ satisfy condition (5.15). Then, there are no complete stable $\mathcal{H}$-surfaces in $\mathbb{R}^3$.

Recall that any $\mathcal{H}$-graph in $\mathbb{R}^3$ is stable. Thus, we get as an immediate consequence of Theorem 5.10:

**Corollary 5.14** Let $\mathcal{H} \in C^2(\mathbb{S}^2)$ satisfy condition (5.15). Then, for any $v \in \mathbb{S}^2$ there exist uniform height estimates for $\mathcal{H}$-graphs in $\mathbb{R}^3$ in the $v$-direction (see Definition 4.4 for this notion).

We next obtain a curvature estimate for stable $\mathcal{H}$-surfaces in $\mathbb{R}^3$, for the case that $\mathcal{H}$ satisfies condition (5.15). It generalizes to $\mathcal{H}$-surfaces the classical curvature estimate obtained by Schoen in [Sc] for minimal surfaces; see also Bérard and Hauswirth [BH] for the constant mean curvature case.

**Theorem 5.15** Let $a, c > 0$. Then, exists a constant $C = C(a,c) > 0$ such that the following statement is true: if $\mathcal{H} \in C^2(\mathbb{S}^2)$ satisfies

$$|\nabla_{\mathbb{S}} \mathcal{H}| + |\nabla_{\mathbb{S}}^2 \mathcal{H}| \leq a$$

(5.23)
on $\mathbb{S}^2$, and $\Sigma$ is any stable $\mathcal{H}$-surface in $\mathbb{R}^3$ satisfying (5.15) for the constant $c$, then for any $p \in \Sigma$ the following estimate holds:

$$|\sigma(p)|d_{\Sigma}(p, \partial \Sigma) \leq C.$$

Proof: Arguing by contradiction, assume that there exists a sequence of functions $\mathcal{H}_n \in C^2(\mathbb{S}^2)$ satisfying (5.23) for the constant $a$, a sequence $\Sigma_n$ of stable $\mathcal{H}_n$-surfaces that satisfy (5.15) for the constant $c$, and points $p_n \in \Sigma_n$ satisfying

$$|\sigma_n(p_n)|d_{\Sigma_n}(p_n, \partial \Sigma_n) > n,$$

(5.24)

where $\sigma_n$ denotes the second fundamental form of $\Sigma_n$. First, note that (5.24) implies that $|\sigma_n(p_n)|$ diverges to $\infty$, since by Theorem 5.10 we have for all $n$ a uniform upper bound $d_{\Sigma_n}(p_n, \partial \Sigma_n) \leq d$ for some $d > 0$. At this point, we can use the arguments in the proof of Theorem 4.2 with some modifications. Let $d_n := d_{\Sigma_n}(p_n, \partial \Sigma_n)$, and consider the compact intrinsic balls $D_n = \overline{B_{\Sigma_n}(p_n, d_n/2)}$. Let $q_n \in D_n$ be the maximum of the function

$$h_n(x) = |\sigma_n(x)|d_{\Sigma_n}(x, \partial D_n)$$
on $D_n$. Clearly $q_n$ is an interior point of $D_n$, as $h_n$ vanishes on $\partial D_n$. Then, if we denote $\lambda_n = |\sigma_n(q_n)|$ and $r_n = d_{\Sigma_n}(q_n, \partial D_n)$, we have by (5.24):

$$\lambda_n r_n = h_n(q_n) \geq h_n(p_n) \to \infty \quad (\text{as } n \to \infty).$$

(5.25)
Let now $H_n$ denote the maximum of the mean curvature function of $\Sigma_n$ restricted to $B_{\Sigma_n}(q_n, r_n/2)$, and define $\lambda^*_n := \max\{\lambda_n, H_n\}$. Consider the rescalings by $\lambda^*_n$ of the immersed surfaces $B_{\Sigma_n}(q_n, r_n/2) \subset \Sigma_n$, in a similar fashion to the argument in Theorem 4.2, and denote these rescalings by $M_n := \lambda^*_n B_{\Sigma_n}(q_n, r_n/2)$. Let $q_n^*$ be the point in $M_n$ that corresponds to $q_n \in \Sigma_n$, and consider the translated surfaces $M_n^* := M_n - q_n^*$ which take the points $q_n^*$ to the origin. Note that the distance in $M_n^*$ from the origin to $\partial M_n^*$ is equal to $\lambda^*_n r_n/2$, and so it diverges to $\infty$ as $n \to \infty$, by (5.25).

The surfaces $M_n^*$ have uniformly bounded second fundamental form, since for any $z_n \in B_{\Sigma_n}(q_n, r_n/2)$ we have

$$\frac{|\sigma_n(z_n)|}{\lambda^*_n} \leq \frac{|\sigma_n(z_n)|}{\lambda_n} = \frac{h_n(z_n)}{\lambda_n d_{\Sigma_n}(z_n, \partial D_n)} \leq \frac{d_{\Sigma_n}(q_n, \partial D_n)}{d_{\Sigma_n}(z_n, \partial D_n)} \leq 2,$$

where in the last inequality we have used (4.4), that also holds in the present context.

Also, each $M_n^*$ is a surface of prescribed mean curvature $H^*_n \in C^2(\mathbb{S}^2)$, where $H^*_n(x) := H_n(x)/\lambda^*_n$. By definition of $\lambda^*_n$, we have $\lambda^*_n \leq 1$ on the Gauss map image $\Omega_n \subset \mathbb{S}^2$ of $M_n^*$ in $\mathbb{S}^3$, for all $n$. Also, note that as $n \to \infty$ we have by (5.23) that

$$|\nabla_{\mathbb{S}} H^*_n| + |\nabla^2_{\mathbb{S}} H^*_n| \to 0$$

(5.26)
globally on $\mathbb{S}^2$. Consequently, a subsequence of the $H^*_n$ converges in the $C^1(\mathbb{S}^2)$ topology to a constant $H_\infty \in [0, 1]$. It follows then from Theorem 2.17 that a subsequence of the $M_n^*$ converges uniformly on compact sets in the $C^3$ topology to a complete surface $\Sigma^*$ in $\mathbb{R}^3$ of constant mean curvature $H_\infty$ that passes through the origin. We consider the connected component of $\Sigma^*$ that passes through the origin, which will still be denoted by $\Sigma^*$.

Since each $\Sigma_n$ is stable, it follows that each $M_n^*$ is a stable $H^*_n$-surface in $\mathbb{R}^3$. So, by Lemma 5.8 (see also the beginning of the proof of Theorem 5.10), we see that the Schrodinger operator $-\mathcal{L}_n := -(\Delta_n + Q_{H^*_n})$ is non-negative on $M_n^*$, here $\Delta_n$ denotes the Laplacian operator in $M_n^*$, and $Q_{H^*_n}$ is given by (5.16).

Clearly, the Schrodinger operators $\mathcal{L}_n$ converge to the Jacobi operator $\mathcal{L}_\infty$ of the CMC surface $\Sigma^*$. It follows then that $-\mathcal{L}_\infty$ is non-negative, i.e. $\Sigma^*$ is stable. As $\Sigma^*$ is also complete, then necessarily $H_\infty = 0$ and $\Sigma^*$ is a plane. In particular, the norm of the second fundamental forms of the $M_n^*$ at the origin converge to zero, which implies that $\lambda_n/\lambda^*_n \to 0$. In particular, by the definition of $\lambda^*_n$, for $n$ large enough we have $\lambda^*_n = H_n$. This implies that $H^*_n(x_n) = 1$ for some $x_n \in \mathbb{S}^2$, for each $n$ large enough. This is a contradiction with $H_\infty = 0$, which completes the proof of Theorem 5.15.

\[\square\]

6 Problem section

In this final section of the paper we collect some open problems about the geometry of $H$-hypersurfaces that we find of special interest. Most of them are motivated by well-known results from the theory of constant mean curvature surfaces in $\mathbb{R}^3$. 

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6.1 On compact $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$

1. Find necessary and/or sufficient conditions for a positive function $\mathcal{H} \in C^1(S^n)$ to be the prescribed mean curvature function of a compact $\mathcal{H}$-hypersurface $\Sigma$ in $\mathbb{R}^{n+1}$.

We know that $\mathcal{H} > 0$ is necessary (Proposition 2.6), but not sufficient. For the Minkowski problem, a necessary and sufficient condition for the existence of a closed strictly convex hypersurface $\Sigma \subset \mathbb{R}^{n+1}$ whose Gauss-Kronecker curvature is given by $K_\Sigma = K(\eta) > 0$ is that $K \in C^0(S^n)$ satisfies $\int_{S^n} x/K(x) = 0$. Such an integral condition is not known for the prescribed mean curvature case; see e.g. [GG].

2. Let $\Sigma$ be a compact embedded $\mathcal{H}$-surface in $\mathbb{R}^3$ for some $\mathcal{H} \in C^2(S^2)$. Is then $\Sigma$ diffeomorphic to $S^2$?

We know that this question has a positive answer if $\mathcal{H}$ is symmetric with respect to two different geodesics in $S^2$ (Proposition 2.8), but in the general case one cannot use the Alexandrov reflection principle. More generally, one could ask the same question in arbitrary dimension.

3. Let $\Sigma_1, \Sigma_2$ be two immersed $\mathcal{H}$-spheres in $\mathbb{R}^3$ for some $\mathcal{H} \in C^2(S^2)$. Do $\Sigma_1$ and $\Sigma_2$ differ by a translation?

4. Assume that $\mathcal{H} \in C^2(S^2)$ is rotationally symmetric, and let $\Sigma$ be an immersed $\mathcal{H}$-sphere in $\mathbb{R}^3$. Is then $\Sigma$ a sphere of revolution?

Question 3 has a positive answer provided one of the two spheres $\Sigma_1, \Sigma_2$ is strictly convex, see Theorem 2.9. Question 4 has a positive answer if the rotationally invariant function $\mathcal{H}$ is also invariant with respect to the reflection of $S^2$ that fixes the geodesic of $S^2$ orthogonal to the rotation axis; this follows from Theorem 2.9 and Theorem 2.10.

Let $\Sigma$ be a compact $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$, and let $\lambda_0(\Sigma) < 0$ denote the principal eigenvalue of $-\mathcal{L}$, where $\mathcal{L}$ is its stability operator. We say that $\Sigma$ has index one if there are no eigenvalues of $-\mathcal{L}$ whose real part lies between $\lambda_0(\Sigma)$ and 0; note that 0 is always a real eigenvalue of multiplicity at least $n + 1$ of $-\mathcal{L}$.

5. Is it true that a compact $\mathcal{H}$-hypersurface in $\mathbb{R}^{n+1}$ has index one if and only if it is a strictly convex $\mathcal{H}$-sphere?

6.2 On general properties of $\mathcal{H}$-hypersurfaces

6. For which choices of $\mathcal{H} \in C^1(S^n)$ are $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ solutions to a variational problem?

We know that for the case where $\mathcal{H}$ is linear, i.e. $\mathcal{H}(x) = a(x, v) + b$ for $a, b \in \mathbb{R}$, $v \in S^n$, $\mathcal{H}$-hypersurfaces are solutions to a variational problem; see Section 2.2. Are there more choices of $\mathcal{H}$ for which this happens?

7. Is there a (homologically invariant) flux formula for $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$?

When $\mathcal{H}$ is constant, this formula was found by Korevaar, Kusner and Solomon [KKS], and is a key tool in CMC hypersurface theory. A first case to consider would be
when $\mathcal{H} \in C^1(S^n)$ is rotationally invariant.

8. Extension of the theory of $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$ to the case where the ambient manifold is a Lie group endowed with a left invariant metric.

See Remark 2.4. The case where the ambient space is the three-dimensional Heisenberg space $\text{Nil}_3$ or the homogeneous Thurston geometry $\text{Sol}_3$ seem of special importance, since they are diffeomorphic to $\mathbb{R}^3$ and their Lie group structure is unique. The theory of constant mean curvature surfaces in homogeneous three-manifolds has experienced in the last decade a great development; see e.g. [AR1, AR2, CR, Da, DH, DM, FM1, FM2, MP, MMPR1, MMPR2] and references therein. An outline of the beginning of this theory can be found in [DHM, FM3].

9. Assuming that $\mathcal{H} \in C^2(S^n)$ is rotationally symmetric, classify all rotational $\mathcal{H}$-hypersurfaces in $\mathbb{R}^{n+1}$.

In Section 3.3 we classified all rotational $\mathcal{H}$-hypersurfaces when $\mathcal{H}$ is rotationally symmetric and $\mathcal{H}(x) = \mathcal{H}(-x) > 0$. The classification for a general, rotationally symmetric, $\mathcal{H}$ seems more involved. For instance, in the case that $\mathcal{H}$ vanishes at some points, the classification should account for the asymptotic behavior of complete rotational $\mathcal{H}$-hypersurfaces that are solutions to an exterior Dirichlet problem, and for examples such as those constructed in Section 3.2. Some cases where a classification seems of particular interest are when $\mathcal{H}$ is a linear function (see Section 2.2) or when $\mathcal{H}(x) = \mathcal{h}(\langle x, e_{n+1} \rangle)$ for some $\mathcal{h} \in C^2([-1, 1])$ with $\mathcal{h}(-u) = -\mathcal{h}(u)$ for all $u$.

10. Generalize to the case $\mathcal{H} \in C^1(S^2)$ known techniques to construct CMC surfaces in $\mathbb{R}^3$.

It seems of special interest to develop gluing constructions for $\mathcal{H}$-surfaces by inverting the Jacobi operator, or to develop a Jenkins-Serrin theory. Some important tools of CMC surface theory are clearly not suitable for the case where $\mathcal{H}$ is not constant; for instance, those using holomorphicity of the Hopf differential.

6.3 On properly embedded $\mathcal{H}$-surfaces in $\mathbb{R}^3$

11. Assume that there exists a strictly convex $\mathcal{H}$-sphere $S_{\mathcal{H}}$ in $\mathbb{R}^3$ for some $\mathcal{H} \in C^2(S^2)$. Prove that there are no properly embedded $\mathcal{H}$-surfaces in $\mathbb{R}^3$ with finite topology and one end.

We proved this result in Theorem 4.16 in the particular case that $\mathcal{H}$ is invariant with respect to three independent geodesic reflections in $S^2$. The solution to the problem above is unknown even if $\mathcal{H}$ is rotationally symmetric.

12. Assume that $\mathcal{H} \in C^2(S^2)$ is rotationally symmetric with respect to the $x_3$ axis, positive and even (i.e. $\mathcal{H}(x) = \mathcal{h}(x_3) = \mathcal{h}(-x_3) > 0$). Prove that any properly embedded $\mathcal{H}$-surface in $\mathbb{R}^3$ with finite topology and two ends is a rotational surface.

For the case where $\mathcal{H}$ is constant, this is a classical result by Korevaar, Kusner and Solomon [KKS]. See also Mazet’s paper [Ma2].
13. Assume that $H \in C^2(\mathbb{S}^2)$ is rotationally symmetric and positive. Prove that any simply connected properly embedded $H$-surface $\Sigma$ in $\mathbb{R}^3$ is a rotational surface.

If $\Sigma$ is compact, this is related to Problem 2. For the case where $\Sigma$ is not compact, there is no clear analogy of this problem with the theory of constant mean curvature surfaces in $\mathbb{R}^3$, since such examples do not exist if $H > 0$ is constant. Note that, by Theorem 4.15, any such $\Sigma$ lies inside a solid cylinder of $\mathbb{R}^3$. If $H$ is, additionally, an even function in $\mathbb{S}^2$, the result follows from Theorem 4.16.

6.4 On stable $H$-surfaces in $\mathbb{R}^3$

14. Assume that $H \in C^2(\mathbb{S}^2)$ is rotationally symmetric with respect to the $x_3$-axis, and vanishes somewhere. Let $\Sigma$ be an entire $H$-graph $z = u(x, y)$ in $\mathbb{R}^3$. Find sufficient conditions on $H$ under which $\Sigma$ must be a rotational $h$-bowl in $\mathbb{R}^3$, as constructed in Proposition 3.5.

This is a generalized version of the classical Bernstein theorem, according to which entire minimal graphs in $\mathbb{R}^3$ are planes. For the case $H(x) = \langle x, e_3 \rangle$, i.e. self-translating solitons of the mean curvature flow, Spruck and Xiao have recently proved in [SX] that any entire, mean convex $H$-graph in $\mathbb{R}^3$ is a rotational bowl.

15. Assume that $H \in C^2(\mathbb{S}^2)$ is rotationally symmetric with respect to the $x_3$-axis, positive and even. Let $S^+_H$ denote the rotational half-sphere in $\mathbb{R}^3$ with prescribed mean curvature $H$, and let $h_0$ denote its maximum height over its planar, horizontal, circular boundary. Let $\Sigma$ be any compact $H$-graph in $\mathbb{R}^3$ with $\partial \Sigma \subset \{z = 0\}$. Is it true that the maximum height that a point of $\Sigma$ can rise over the $z = 0$ plane is $h_0$?

For the case where $H$ is constant, the statement in Problem 15 is a well-known property of CMC graphs in $\mathbb{R}^3$. The statement in Problem 16 for compact, stable CMC surfaces in $\mathbb{R}^3$ is also known, and due to Mazet [Ma1].

16. Assume that $H \in C^2(\mathbb{S}^2)$ is rotationally symmetric with respect to the $x_3$-axis, positive and even. Let $S^+_H$ denote the rotational half-sphere in $\mathbb{R}^3$ with prescribed mean curvature $H$, and let $d_0$ denote the intrinsic distance from its north pole to $\partial S^+_H$. Let $\Sigma$ be any compact, stable $H$-surface in $\mathbb{R}^3$. Is it true that for any $p \in \Sigma$ we have $d_\Sigma(p, \partial \Sigma) \leq d_0$?

17. Assume that $H \in C^2(\mathbb{S}^2)$ is rotationally symmetric. Let $\Sigma$ be a compact stable $H$-disk in $\mathbb{R}^3$ whose boundary is a horizontal circle. Prove that $\Sigma$ is a rotational $H$-surface.

When $H$ is constant, this was proved by Alías, López and Palmer in [ALP].

18. Find sufficient conditions on $H \in C^2(\mathbb{S}^2)$, $H > 0$, which ensure that there do not exist complete stable $H$-surfaces in $\mathbb{R}^3$.

By Corollary 5.13, a sufficient condition for this is (5.15). Recall that some condition on $H > 0$ is needed, as for some rotationally symmetric choices of $H > 0$ there exist complete rotational $H$-graphs, which are therefore stable. For the particular case $H(x) = \langle x,
\[ a\langle x, e_3 \rangle + b, \] a better sufficient condition than \( (5.15) \) follows from work of Espinar \( [Es] \).

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The authors were partially supported by MICINN-FEDER, Grant No. MTM2016-80313-P, Junta de Andalucía Grant No. FQM325, and Programa de Apoyo a la Investigacion, Fundacion Seneca-Agencia de Ciencia y Tecnología Region de Murcia, reference 19461/PI/14.