Classification of finite irreducible conformal modules over a class of Lie conformal algebras of Block type

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Abstract. We classify finite irreducible conformal modules over a class of infinite Lie conformal algebras \( \mathfrak{B}(p) \) of Block type, where \( p \) is a nonzero complex number. In particular, we obtain that a finite irreducible conformal module over \( \mathfrak{B}(p) \) may be a nontrivial extension of a finite conformal module over \( \mathfrak{Vir} \) if \( p = -1 \), where \( \mathfrak{Vir} \) is a Virasoro conformal subalgebra of \( \mathfrak{B}(p) \). As a byproduct, we also obtain the classification of finite irreducible conformal modules over a series of finite Lie conformal algebras \( \mathfrak{b}(n) \) for \( n \geq 1 \).

Key words: finite conformal module; Lie conformal algebras of Block type; Virasoro conformal algebra

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1 Introduction

Lie conformal algebras, introduced by Kac [17], encode the singular part of the operator product expansion of chiral fields in conformal field theory. The theory of finite Lie conformal algebras has been greatly developed in the last two decades (e.g., [2, 7–9, 11, 17, 18, 28]). Finite simple Lie conformal algebras were classified in [9], which shows that a finite simple Lie conformal algebra is isomorphic to either the Virasoro conformal algebra or a current conformal algebra \( \text{Cur} \mathfrak{g} \) over a simple finite-dimensional Lie algebra \( \mathfrak{g} \). The theory of conformal modules and their extensions was developed in [7, 8], and the cohomology theory was developed in [2] and further in [11]. For super cases, the structure and representation theories have also been developed in recent years, see [5, 14, 15, 20] and the references therein.

However, the theory of infinite Lie conformal algebras is far from being well developed. The most important example is the general Lie conformal algebra \( \mathfrak{g}c_N \), which plays the same role in the theory of Lie conformal algebras as the general Lie algebra \( gl_N \) does in the theory of Lie algebras. The important difference between usual and conformal algebras is that \( \mathfrak{g}c_N \) is infinite. Due to these reasons, the general Lie conformal algebra \( \mathfrak{g}c_N \) and its subalgebras have been studied by many authors (e.g., [3, 4, 6, 9, 10, 21, 25, 29]). Recently, some interesting examples of infinite Lie conformal algebras associated with infinite-dimensional loop Lie algebras were constructed and studied (e.g., [12, 13, 26]).

In this paper, we focus on another class of infinite Lie conformal algebras \( \mathfrak{B}(p) \) with \( p \) being...
a nonzero complex number, where \( \mathfrak{B}(p) \) has a \( \mathbb{C}[\partial] \)-basis \( \{ L_i \mid i \in \mathbb{Z}_+ \} \) and \( \lambda \)-brackets

\[
[L_i \lambda L_j] = ((i + p) \partial + (i + j + 2p)\lambda)L_{i+j}.
\] (1.1)

We refer to \( \mathfrak{B}(p) \)'s as Lie conformal algebras of Block type due to their relations with some Lie algebras of Block type (see Remark 2.7). There are some interesting features on this class of Lie conformal algebras. Firstly, each \( \mathfrak{B}(p) \) contains a Virasoro conformal subalgebra. Set \( L = \frac{1}{p}L_0 \in \mathfrak{B}(p) \). By (1.1), we see that \( [L \lambda L] = (\partial + 2\lambda)L \).

Namely, the subalgebra

\[ \mathfrak{Vir} = \mathbb{C}[\partial]L \] (1.2)

of \( \mathfrak{B}(p) \) is exactly the Virasoro conformal algebra. Secondly, the special case \( \mathfrak{B}(1) \) has close relation with the general Lie conformal algebra \( g_{C_1} \). In fact, \( \mathfrak{B}(1) \) is a maximal subalgebra of the associated graded conformal algebra \( \text{gr } g_{C_1} \) of the filtered algebra \( g_{C_1} \) [25]. Thirdly, there are embedding relations among \( \mathfrak{B}(p) \)'s. For any integer \( n \geq 1 \), \( \mathfrak{B}(p) \) can be embedded into \( \mathfrak{B}(np) \) via \( L_i \mapsto \frac{1}{n}L'_{ni} \). Finally, \( \mathfrak{B}(-n) \) contains a series of finite Lie conformal quotient algebras (cf. (2.2))

\[ \mathfrak{b}(n) = \mathfrak{B}(-n)/\mathfrak{B}(-n)_{(n+1)} \],

including the Heisenberg-Virasoro conformal algebra \( \mathfrak{b}(1) \) and Schrödinger-Virasoro conformal algebra \( \mathfrak{b}(2) \) as the first two cases (see Subsection 2.2). Due to these observations, it seems to be interesting for us to study the Lie conformal algebra \( \mathfrak{B}(p) \) systematically.

One of the most important problems regarding infinite Lie conformal algebras is the classification of finite irreducible conformal modules (FICMs). This problem for the general Lie conformal algebra \( g_{C_N} \) was solved by Kac, Radul and Wakimoto, see also [6, 18]. In this paper, we consider this problem for \( \mathfrak{B}(p) \). Clearly, any conformal module over \( \mathfrak{Vir} \subset \mathfrak{B}(p) \) can be trivially extended to a conformal module over the whole \( \mathfrak{B}(p) \). Our main results indicate that a FICM over \( \mathfrak{B}(p) \) may be a nontrivial extension of a finite conformal module over \( \mathfrak{Vir} \) if \( p = -1 \) (see Table 1). As a byproduct, we also obtain the classification of FICMs over the finite Lie conformal algebra \( \mathfrak{b}(n) \) (see Table 2).

| \( \mathfrak{B}(p) \) | Reference |
|----------------|----------|
| \( p \neq -1 \) | \( M_{\Delta,\alpha} \) Theorems 4.1 and 5.1 |
| \( p = -1 \) | \( M_{\Delta,\alpha,\beta} \) Theorems 4.1 and 5.1 |

| \( \mathfrak{b}(n) \) | Reference |
|----------------|----------|
| \( n > 1 \) | \( M_{\Delta,\alpha} \) Corollary 5.3 |
| \( n = 1 \) | \( M_{\Delta,\alpha,\beta} \) Corollary 5.3 |

Our main results imply that \( \mathfrak{B}(p) \)'s are examples of infinite conformal algebras that have a finite module. The classification of such infinite conformal algebras (even for simple ones) is a challenging problem as Kac stated in [19]. A more general problem is to classify all simple conformal algebras of finite growth. In fact, the “bigger” \( \mathfrak{B}(p) \)'s are such examples, which can be obtained by replacing the index set \( \mathbb{Z}_+ \) of the \( \mathbb{C}[\partial] \)-base elements \( L_i \) of \( \mathfrak{B}(p) \)'s by \( \mathbb{Z} \) (see Remark 3.2).
We also would like to point out that our techniques used here may be applied to analogous problems of (both infinite and finite) Lie conformal algebras which are closely related to the Lie conformal algebra $\mathfrak{B}(p)$. This is also our motivation for writing this paper.

This paper is arranged as follows. In Section 2, we recall some definitions on Lie conformal algebras and present some related algebraic structures of $\mathfrak{B}(p)$. Then, in Section 3, we give three technical lemmas, which will gradually reduce our classification problem. In Section 4, we first consider the problem for the rank one case without the irreducibility assumption. Finally, in Section 5, we completely classify FICMs over $\mathfrak{B}(p)$ by showing that they must be free of rank one. By the last observation mentioned above, we also obtain the classification of FICMs over the finite Lie conformal algebra $\mathfrak{b}(n)$.

2 Preliminaries

2.1 Definitions

First we list some definitions that will be used. All of them are collected or reorganized from [9, 17–19].

Definition 2.1 A Lie conformal algebra $R$ is a $\mathbb{C}[[\partial]]$-module endowed with a $\mathbb{C}$-linear map $R \otimes R \to \mathbb{C}[\lambda] \otimes R$, $a \otimes b \to [a \lambda b]$ called $\lambda$-bracket, and satisfying the following axioms ($a, b, c \in R)$:

- (conformal sesquilinearity) $[\partial a \lambda b] = -\lambda[a \lambda b], \quad [a \lambda \partial b] = (\partial + \lambda)[a \lambda b]$.
- (skew-symmetry) $[a \lambda b] = -[-\lambda \partial a]$.
- (Jacobi identity) $[a \lambda [b \mu c]] = [[a \lambda b] \lambda + \mu c] + [b \mu [a \lambda c]]$.

Definition 2.2 A conformal module $M$ over a Lie conformal algebra $R$ is a $\mathbb{C}[[\partial]]$-module endowed with a $\lambda$-action $R \otimes M \to \mathbb{C}[\lambda] \otimes M$, $a \otimes v \to a \lambda b$, such that ($a, b \in R, v \in M$)

$$(\partial a) \lambda v = -\lambda a \lambda v, \quad a \lambda (\partial v) = (\partial + \lambda) a \lambda v, \quad [a \lambda b] \lambda + \mu v = a \lambda (b \mu v) - b \mu (a \lambda v).$$

A conformal $R$-module $M$ is called finite if it is finitely generated over $\mathbb{C}[[\partial]]$.

Definition 2.3 An annihilation algebra $\mathcal{A}(R)$ of a Lie conformal algebra $R$ is a Lie algebra with $\mathbb{C}$-basis $\{a(n) \mid a \in R, n \in \mathbb{Z}_+\}$ and relations

$$[a(m), b(n)] = \sum_{k \in \mathbb{Z}_+} \binom{m}{k} (a(k)b)(m+n-k), \quad (\partial a)(n) = -na(n-1),$$

where $a(k)b$ is called the $k$-product, given by $[a \lambda b] = \sum_{k \in \mathbb{Z}_+} \lambda(k)a(k)b$ with $\lambda(k) = \frac{k!}{(k+1)!}$. Furthermore, an extended annihilation algebra $\mathcal{A}(R)^e$ of $R$ is defined by $\mathcal{A}(R)^e = CT \ltimes \mathcal{A}(R)$ with

$$[T, a(n)] = -na(n-1).$$
Proposition 2.4 A conformal module $M$ over a Lie conformal algebra $R$ is the same as a module over the Lie algebra $\mathcal{A}(R)^e$ satisfying $a(n)v = 0$ for $a \in R$, $v \in M$, $n \gg 0$.

2.2 Quotient algebras of $\mathcal{B}(p)$

Start from $\mathcal{B}(p)$, one can obtain many interesting finite Lie conformal algebras. Consider, for example, the quotient algebras $\mathcal{B}(p)[n]$ with $n \in \mathbb{Z}_+$, defined by

$$\mathcal{B}(p)[n] = \mathcal{B}(p)/\mathcal{B}(p)(n+1), \quad \text{where} \quad \mathcal{B}(p)(n) = \oplus_{i \geq n} \mathbb{C}[\partial]L_i. \quad (2.1)$$

Note that $\mathcal{B}(p)[0]$ is isomorphic to the Virasoro conformal algebra $\mathcal{Vir}$. All other $\mathcal{B}(p)[n]$ with $n \geq 1$ are non-simple. The special quotients

$$b(n) = \mathcal{B}(-n)[n] = \mathcal{B}(-n)/\mathcal{B}(-n)(n+1) \quad \text{with} \quad n \geq 1, \quad (2.2)$$

seem to be particularly interesting. Let us look at the first two cases.

(Q1) Case $n = 1$. Set $L = -\bar{L}_0$ and $M = \bar{L}_1$. By (1.1) and (2.2), we see that

$$[L \lambda L] = (\partial + 2\lambda)L, \quad [L \lambda M] = (\partial + \lambda)L, \quad [M \lambda L] = \lambda L, \quad [M \lambda M] = 0.$$

Namely, $b(1) = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]M$ is exactly the Heisenberg-Virasoro conformal algebra [24], which is associated with the Heisenberg-Virasoro Lie algebra [1].

(Q2) Case $n = 2$. Set $L = -\frac{1}{2}\bar{L}_0$, $Y = \bar{L}_1$ and $M = -\bar{L}_2$. By (1.1) and (2.2), we see that (other components vanish)

$$[L \lambda L] = (\partial + 2\lambda)L, \quad [L \lambda Y] = (\partial + \frac{3}{2}\lambda)Y, \quad [L \lambda M] = (\partial + \lambda)M,$$

$$[Y \lambda L] = \frac{1}{2}(\partial + \frac{3}{2}\lambda)Y, \quad [Y \lambda Y] = (\partial + 2\lambda)M, \quad [M \lambda L] = \lambda M.$$

Namely, $b(2) = \mathbb{C}[\partial]L \oplus \mathbb{C}[\partial]Y \oplus \mathbb{C}[\partial]M$ is exactly the Schrödinger-Virasoro conformal algebra [24], which is associated with the Schrödinger-Virasoro Lie algebra [16].

2.3 Annihilation algebra of $\mathcal{B}(p)$

Denote by $\mathcal{A}(\mathcal{B}(p))$ and $\mathcal{A}(\mathcal{B}(p))^e$ the annihilation algebra and extended annihilation algebra of $\mathcal{B}(p)$, respectively. Their concrete Lie structures are as follows.

Lemma 2.5 (1) The annihilation algebra $\mathcal{A}(\mathcal{B}(p))$ is given by

$$\mathcal{A}(\mathcal{B}(p)) = \text{span}_\mathbb{C}\{L_{i,m} \mid i \in \mathbb{Z}_+, m \in \mathbb{Z}_{\geq -1}\}$$

with relations

$$[L_{i,m}, L_{j,n}] = ((j + p)(m + 1) - (i + p)(n + 1))L_{i+j,m+n}.$$
(2) The extended annihilation algebra \( \mathcal{A}(\mathfrak{B}(p))^e \) is given by
\[
\mathcal{A}(\mathfrak{B}(p))^e = \text{span}_\mathbb{C}\{L_{i,m}, T \mid i \in \mathbb{Z}_+, m \in \mathbb{Z}_{\geq -1}\}
\]
with relations as in (1) and \([T, L_{i,m}] = -(m+1)L_{i,m-1}\).

**Proof.** For \( i, j \in \mathbb{Z}_+ \), by (1.1) and Definition 2.3, we have
\[
((i+p)\partial + (i+j + 2p)\lambda)L_{i+j} = [L_i \lambda L_j] = \sum_{k \in \mathbb{Z}_+} \lambda^{(k)} L_{i(k)} L_j = L_{i(0)} L_j + \lambda L_{i(1)} L_j + \sum_{k \geq 2} \lambda^{(k)} L_{i(k)} L_j,
\]
which implies that
\[
L_{i(0)} L_j = (i+p)\partial L_{i+j}, \quad L_{i(1)} L_j = (i+j+2p)L_{i+j}, \quad L_{i(k)} L_j = 0 \quad \text{for} \quad k \geq 2.
\]
Then, for \( m, n \in \mathbb{Z}_+ \), by Definition 2.3, we have
\[
[(L_i)_{(m)}, (L_j)_{(n)}] = \sum_{k \in \mathbb{Z}_+} \binom{m}{k} (L_{i(k)} L_j)_{(m+n-k)}
\]
\[
= (L_{i(0)} L_j)_{(m+n)} + m(L_{i(1)} L_j)_{(m+n-1)}
\]
\[
= -(m+n)(i+p)(L_{i+j})_{(m+n-1)} + m(i+j+2p)(L_{i+j})_{(m+n-1)}
\]
\[
= (m(j+p) - n(i+p))(L_{i+j})_{(m+n-1)},
\]
and \([T, (L_i)_{(m)}] = -m(L_i)_{(m-1)}\). Hence, this lemma holds by setting \( L_{i,m} = (L_i)_{(m+1)} \) for \( m \geq -1 \).
\[\square\]

**Remark 2.6** Denote by \( \mathcal{A}(\mathfrak{B}(p)_{[n]}) \) and \( \mathcal{A}(\mathfrak{B}(p)_{[n]})^e \) the annihilation algebra and extended annihilation algebra of the quotient algebra \( \mathfrak{B}(p)_{[n]} \) of \( \mathfrak{B}(p) \), respectively. Then
\[
\mathcal{A}(\mathfrak{B}(p)_{[n]}) = \text{span}_\mathbb{C}\{\tilde{L}_{i,m} \mid 0 \leq i \leq n, m \in \mathbb{Z}_{\geq -1}\},
\]
\[
\mathcal{A}(\mathfrak{B}(p)_{[n]})^e = \text{span}_\mathbb{C}\{\tilde{L}_{i,m}, T \mid 0 \leq i \leq n, m \in \mathbb{Z}_{\geq -1}\},
\]
which satisfy the same relations as in Lemma 2.5 (of course, \( \tilde{L}_{i,m} \) with \( i > n \) will be viewed as zero if it technically appears). Clearly, \( \mathcal{A}(\mathfrak{B}(p)_{[n]}) \) and \( \mathcal{A}(\mathfrak{B}(p)_{[n]})^e \) can be also viewed as quotient algebras of \( \mathcal{A}(\mathfrak{B}(p)) \) and \( \mathcal{A}(\mathfrak{B}(p))^e \), respectively.

**Remark 2.7** The annihilation algebra \( \mathcal{A}(\mathfrak{B}(p)) \) has close relation with the Lie algebra \( \mathfrak{B}(q) \) of Block type studied in [22, 23, 27]. Consider a two-parameter Lie algebra \( \mathfrak{B}(p, q) \) of Block type with \( p, q \in \mathbb{C} \), where \( \mathfrak{B}(p, q) \) has a \( \mathbb{C} \)-basis \( \{L_{i,m} \mid i, m \in \mathbb{Z}\} \) and relations
\[
[L_{i,m}, L_{j,n}] = ((j+p)(m+q) - (i+p)(n+q)L_{i+j,m+n}.
\]
Then, the special case \( \mathfrak{B}(p, q) \) with \( q = 1 \) contains \( \mathcal{A}(\mathfrak{B}(p)) \) as a subalgebra, while the special case \( \mathfrak{B}(p, q) \) with \( p = 0 \) contains \( \mathfrak{B}(q) \) as a subalgebra. Hence, we refer to \( \mathfrak{B}(p) \)'s as *Lie conformal algebras of Block type*.  

\[5\]
Three technical lemmas

In this section, we shall give three lemmas for later use.

Our first lemma is the following result, which will reduce the classification of finite conformal modules over $\mathfrak{B}(p)$ to the classification of such modules over a finite conformal quotient algebra of $\mathfrak{B}(p)$. The case $p = 1$ is in fact implicit in [25]. Our proof here is a straightforward generalization of that in [25], but with a slightly different discussion (at the end of Case 1).

**Lemma 3.1** Let $M$ be a nontrivial finite conformal module over $\mathfrak{B}(p)$. Then the $\lambda$-action of $L_i \in \mathfrak{B}(p)$ on $M$ is trivial for all $i \gg 0$. In particular, a finite conformal module over $\mathfrak{B}(p)$ is simply a finite conformal module over $\mathfrak{B}(p)[n]$ for some big enough integer $n$, where $\mathfrak{B}(p)[n]$ is defined by (2.1).

**Proof.** First, recall that [7] a finite irreducible conformal module over $\mathfrak{Vir} = \mathbb{C}[\partial]L$ is isomorphic to either a free conformal module of rank one $M_{\Delta, \alpha} = \mathbb{C}[\partial]v$ with an action defined by

$$L_{\lambda}v = (\partial + \Delta \lambda + \alpha)v$$

for some $\Delta, \alpha \in \mathbb{C}$ with $\Delta \neq 0$, or a one-dimensional trivial module $\mathbb{C}c_{\alpha}$ with an action defined by $L_{\lambda}c_{\alpha} = 0, \partial c_{\alpha} = \alpha c_{\alpha}$ for some $\alpha \in \mathbb{C}$.

By regarding $M$ as a module over $\mathfrak{Vir} \subset \mathfrak{B}(p)$, we can choose a composition series

$$M = M_N \supset M_{N-1} \supset \cdots \supset M_1 \supset M_0 = 0,$$

such that for each $1 \leq k \leq N$, the composition factor $\overline{M}_k = M_k/M_{k-1}$ is either a free conformal module of rank one $M_{\Delta_k, \alpha_k}$ with $\Delta_k \neq 0$, or a one-dimensional trivial module $\mathbb{C}c_{\alpha_k}$. Denote by $\bar{v}_k$ a $\mathbb{C}[\partial]$-generator of $\overline{M}_k$ and $v_k \in M_k$ the preimage of $\bar{v}_k$. Then $\{v_k | 1 \leq k \leq N\}$ is a $\mathbb{C}[\partial]$-generating set of $M$, and the $\lambda$-action of $L_0$ on $v_k$ is a $\mathbb{C}[\partial, \lambda]$-combination of $v_1, \ldots, v_k$.

We claim that the $\lambda$-action of $L_i$ on $v_1$ is trivial for all $i \gg 0$. Namely,

$$L_i \lambda v_1 = 0 \quad \text{for all} \quad i \gg 0.$$  \hspace{1cm} (3.2)

Fix $i \gg 0$ and assume that $L_i \lambda v_1 \neq 0$. Let $k_i \geq 1$ be the largest integer such that $L_i \lambda v_1 \notin M_{k_i-1}$.

We proceed to derive a contradiction. We only need to consider the following four cases.

**Case 1:** $M_1 = M_{\Delta_1, \alpha_1}, \overline{M}_{k_i} = M_{\Delta_{k_i}, \alpha_{k_i}}$.

By assumption, we can write

$$L_i \lambda v_1 \equiv f_i(\partial, \lambda)v_{k_i} \pmod{M_{k_i-1}} \quad \text{for some} \quad 0 \neq f_i(\partial, \lambda) \in \mathbb{C}[\partial, \lambda].$$

(3.3)

Considering the action of the operator $L_0 \mu$ on (3.3), by Definition 2.2, we obtain (note that $L_0 = pL$)

$$p(\partial + \Delta_{k_i} \mu + \alpha_{k_i})f_i(\partial + \mu, \lambda) = ((i + p)\mu - p\lambda)f_i(\partial, \mu + \lambda) + p(\partial + \lambda + \Delta_{1} \mu + \alpha_{1})f_i(\partial, \lambda).$$

(3.4)
In particular, taking $\partial = 0$ in (3.4), we have
\[
f_i(\mu, \lambda) = \frac{1}{p(\Delta_{\kappa_i} \mu + \alpha_{\kappa_i})} ((i + p)\mu - p\lambda)f_i(0, \mu + \lambda) + p(\lambda + \Delta_1 \mu + \alpha_1)f_i(0, \lambda).
\] (3.5)

Taking $\partial = -\Delta_{\kappa_i} \mu - \alpha_{\kappa_i}$ and $\lambda = (1 + \frac{i}{p})\mu$ in (3.4), and then using (3.5), we obtain
\[
(1 + \frac{i}{p})(1 + \Delta_{\kappa_i} \mu + \alpha_{\kappa_i})f_i(0, (1 + \frac{i}{p} - \Delta_{\kappa_i} \mu - \alpha_{\kappa_i}) = ((1 + \frac{i}{p} - \Delta_1 \Delta_{\kappa_i})\mu - \Delta_1 \alpha_{\kappa_i} + \alpha_1)f_i(0, (1 + \frac{i}{p})\mu).
\]

Denote by $m_i$ the degree of $f_i(0, \lambda)$. Equating the coefficients of $\mu^{m_i+1}$ in the above equation, we obtain
\[
(1 + \Delta_{\kappa_i})(1 + \frac{i}{p})(1 + \frac{i}{p} - \Delta_{\kappa_i})^{m_i} = (1 + \frac{i}{p} - \Delta_1 \Delta_{\kappa_i})(1 + \frac{i}{p})^{m_i}.
\] (3.6)

Set $q(i) = 1 + \frac{i}{p}$. Then either $q(i) \gg 0$ or $q(i) \ll 0$ when $i \gg 0$. Comparing the coefficients of $q(i)^{m_i+1}$ in (3.6), one can see that it cannot hold for any $m_i \geq 0$ (note that $\Delta_{\kappa_i} \neq 0$), a contradiction.

**Case 2:** $M_1 = \mathbb{C}_{\alpha_1}$, $\overline{M}_{\kappa_i} = M_{\Delta_{\kappa_i}, \alpha_{\kappa_i}}$.

As in Case 1, we can still assume (3.3). Considering the action of the operator $L_0 \mu$ on (3.3), by Definition 2.2, we obtain
\[
p(\partial + \Delta_{\kappa_i} \mu + \alpha_{\kappa_i})f_i(\partial + \mu, \lambda) = ((i + p)\mu - p\lambda)f_i(\partial, \mu + \lambda).
\] (3.7)

Taking $\partial = \mu = 0$ in (3.7), we obtain $f_i(0, \lambda) = 0$. Using this in (3.7) with $\partial = 0$, we obtain $f_i(\mu, \lambda) = 0$, a contradiction.

**Case 3:** $M_1 = M_{\Delta_1 \alpha_1}$, $\overline{M}_{\kappa_i} = \mathbb{C}_{\alpha_{\kappa_i}}$.

In this case, since $\partial$ acts on $\tilde{v}_{\kappa_i}$ as the scalar $\alpha_{\kappa_i}$, we can write
\[
L_{i \lambda} v_1 \equiv f_i(\lambda)v_{\kappa_i} (\text{mod } M_{\kappa_i - 1}) \text{ for some } 0 \neq f_i(\lambda) \in \mathbb{C}[\lambda].
\] (3.8)

Considering the action of the operator $L_0 \mu$ on (3.8), by Definition 2.2, we obtain
\[
0 = ((i + p)\mu - p\lambda)f_i(\mu, \lambda) + p(\partial + \lambda + \Delta_1 \mu + \alpha_1)f_i(\lambda).
\] (3.9)

Equating the coefficients of $\partial$ in (3.9), we immediately obtain $f_i(\lambda) = 0$, a contradiction.

**Case 4:** $M_1 = \mathbb{C}_{\alpha_1}$, $\overline{M}_{\kappa_i} = \mathbb{C}_{\alpha_{\kappa_i}}$.

As in Case 3, one can easily obtain $f_i(\lambda) = 0$, a contradiction.

Now, start from (3.2), one can inductively show that $L_{i \lambda} v_k = 0$ for $1 \leq k \leq N$. Hence, the $\lambda$-action of $L_i$ on $M$ is trivial. This completes the proof. \qed
Remark 3.2 Clearly, $\mathcal{B}(p)$ is non-simple. If we replace the index set $\mathbb{Z}_+$ of the $\mathbb{C}[\partial]$-base elements $L_i \in \mathcal{B}(p)$ by $\mathbb{Z}$, we obtain a simple Lie conformal algebra. By Lemma 3.1 and the simplicity of this “bigger” $\mathcal{B}(p)$, one can easily show that it has no nontrivial finite conformal module. In particular, as pointed out in [25], this provides a class of examples of finitely freely generated simple Lie conformal algebras of linear growth that cannot be embedded into $\mathfrak{g}c_N$ for any $N$.

Our second lemma is a result concerning the representations of certain subquotient algebra of the annihilation algebra $\mathcal{A}(\mathcal{B}(p))$. Let $k \geq 0$ and $N \geq 0$ be two fixed integers. Define a Lie algebra $\mathcal{G}_{k,N}$ by

$$\mathcal{G}_{k,N} = \text{span}_\mathbb{C}\{J_{i,m} \mid 0 \leq i \leq k, 0 \leq m \leq N\}$$

with relations

$$[J_{i,m}, J_{j,n}] = \begin{cases} ((j + p)(m + 1) - (i + p)(n + 1))J_{i+j, m+n} & \text{if } i + j \leq k, m + n \leq N, \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 2.5, $\mathcal{G}_{k,N}$ can be viewed as a subquotient algebra of $\mathcal{A}(\mathcal{B}(p))$.

Lemma 3.3 Let $V$ be a nontrivial finite dimensional irreducible module over $\mathcal{G}_{k,N}$. Then $\dim V = 1$.

Proof. Denote by $\mathbb{Q}_+$ the set of all positive rational numbers. We divide the proof into the following two cases.

Case 1: $p \notin \mathbb{Q}_+$.

Consider the following decomposition of $\mathcal{G}_{k,N}$:

$$\mathcal{G}_{k,N} = \mathbb{C}J_{0,0} + \mathcal{K}, \quad \text{where } \mathcal{K} = \mathcal{G}_{k,N} \ominus \mathbb{C}J_{0,0}.$$

Clearly, $\mathcal{K}$ is a nilpotent ideal of $\mathcal{G}_{k,N}$. For any $J_{i,m} \in \mathcal{K}$, we have $[J_{0,0}, J_{i,m}] = (i - pm)J_{i,m}$, and $i - pm \neq 0$ since $p \notin \mathbb{Q}_+$. It follows that $\mathcal{K}$ is a completely reducible $\mathbb{C}J_{0,0}$-module with no trivial summand. By [7, Lemma 1], $\mathcal{K}$ acts trivially on $V$. Hence, $V$ can be viewed as a finite dimensional module over $\mathbb{C}J_{0,0}$, and so $\dim V = 1$.

Case 2: $p \in \mathbb{Q}_+$.

First, if $k = 0$ or $N = 0$, then this result can be proved as in Case 1.

Next, we assume that $k \geq 1$ and $N \geq 1$. Since $p \in \mathbb{Q}_+$, there exist infinitely many positive integer pairs $(i, m)$ such that $i - pm = 0$. If there are no such pairs $(i, m)$ such that $J_{i,m} \in \mathcal{G}_{k,N}$, then this result can be proved as in Case 1. Next, we assume that there exists at least one such a pair $(i, m)$ such that $J_{i,m} \in \mathcal{G}_{k,N}$. Assume that

$$i_0 = \max\{i \mid i - pm = 0 \text{ and } J_{i,m} \in \mathcal{G}_{k,N}\}.$$
Let \( m_0 = \frac{1}{p} i_0 \). Then
\[
m_0 = \max \{ m \mid i - pm = 0 \text{ and } J_{i,m} \in \mathcal{G}_{k,N} \}.
\]

We have the following three claims.

Claim 1 If \( i_0 < k \), then \( I_{k,N}^{(1)} \) acts trivially on \( V \), where \( I_{k,N}^{(1)} = \text{span}_{\mathbb{C}} \{ J_{k,m} \in \mathcal{G}_{k,N} \mid 0 \leq m \leq N \} \).

In particular, \( V \) can be viewed as a nontrivial finite dimensional irreducible module over \( \mathcal{G}_{k-1,N} \).

Assume that \( I_{k,N}^{(1)} \) acts nontrivially on \( V \). By the irreducibility of \( V \), we have
\[
V = I_{k,N}^{(1)} V, \tag{3.10}
\]

since \( I_{k,N}^{(1)} \) is an ideal of \( \mathcal{G}_{k,N} \). Note that \([J_{0,0}, J_{k,m}] = (k - pm) J_{k,m}\), and \( k - pm \neq 0 \) by the assumption \( i_0 < k \) and the maximality of \( i_0 \). Hence, each \( J_{k,m} \in I_{k,N}^{(1)} \) acts nilpotently on \( V \).

Since \( I_{k,N}^{(1)} \) is abelian, we have \( (I_{k,N}^{(1)})^n V = 0 \) for \( n \gg 0 \), which contradicts to (3.10).

Claim 2 If \( m_0 < N \), then \( I_{k,N}^{(2)} \) acts trivially on \( V \), where \( I_{k,N}^{(2)} = \text{span}_{\mathbb{C}} \{ J_{i,N} \in \mathcal{G}_{k,N} \mid 0 \leq i \leq k \} \).

In particular, \( V \) can be viewed as a nontrivial finite dimensional irreducible module over \( \mathcal{G}_{k,N-1} \).

Assume that \( I_{k,N}^{(2)} \) acts nontrivially on \( V \). As in Claim 1, by the irreducibility of \( V \), we have
\[
V = I_{k,N}^{(2)} V, \tag{3.11}
\]

since \( I_{k,N}^{(2)} \) is an ideal of \( \mathcal{G}_{k,N} \). Note that \([J_{0,0}, J_{i,N}] = (i - pN) J_{i,N}\), and \( i - pN \neq 0 \) by the assumption \( m_0 < N \) and the maximality of \( m_0 \). Hence, each \( J_{i,N} \in I_{k,N}^{(2)} \) acts nilpotently on \( V \).

Since \( I_{k,N}^{(2)} \) is abelian, we have \( (I_{k,N}^{(2)})^n V = 0 \) for \( n \gg 0 \), which contradicts to (3.11).

Claim 3 If \( i_0 = k \) and \( m_0 = N \), then \( I_{k,N}^{(3)} \) acts trivially on \( V \), where \( I_{k,N}^{(3)} = \text{span}_{\mathbb{C}} \{ J_{k,m}, J_{i,N} \in \mathcal{G}_{k,N} \mid 0 \leq m \leq N, 0 \leq i \leq k-1 \} \).

In particular, \( V \) can be viewed as a nontrivial finite dimensional irreducible module over \( \mathcal{G}_{k-1,N-1} \).

Assume that \( I_{k,N}^{(3)} \) acts nontrivially on \( V \). By the irreducibility of \( V \), we have
\[
V = I_{k,N}^{(3)} V, \tag{3.12}
\]

since \( I_{k,N}^{(3)} \) is an ideal of \( \mathcal{G}_{k,N} \). Consider the following decomposition of \( I_{k,N}^{(3)} \):
\[
I_{k,N}^{(3)} = \mathbb{C} J_{0,m_0} + \mathcal{M}, \quad \text{where} \quad \mathcal{M} = I_{k,N}^{(3)} \setminus \mathbb{C} J_{0,m_0}.
\]

As in Claims 1 and 2, one can easily show that all elements in \( \mathcal{M} \) act nilpotently on \( V \). Note that \( \mathcal{M} \) is almost abelian, except \( [J_{i_0,0}, J_{0,m_0}] = bJ_{i_0,m_0} \), where \( b = -(i_0 + p)m_0 - i_0 < 0 \). To show that \( I_k^{(3)} \) acts nilpotently on \( V \) and then derive a contradiction to (3.12), we only need to show that \( J_{i_0,0}, J_{0,m_0} \) acts trivially on \( V \). Note first that \( J_{i_0,0}, J_{0,m_0} \) must act as a scalar \( c \) on \( V \), since it is a central element in \( G_{k,N} \). Let \( d = \dim V \) and let \( Y \) be a basis of \( V \). Assume that
\[
J_{i_0,0}Y = YA, \quad J_{0,m_0}Y = YB, \quad J_{i_0,m_0}Y = YC,
\]
where \( A, B, C \) are \( d \times d \) matrices. Then, \( C = cI_d \), where \( I_d \) denotes the identity matrix of order \( d \). Applying the relation \( [J_{i_0,0}, J_{0,m_0}] = bJ_{i_0,m_0} \) to \( Y \), we obtain
\[
AB - BA = bcI_d. \tag{3.13}
\]
Considering the traces of the two sides of (3.13), we must have that \( c = 0 \) (note that \( b \neq 0 \)). Hence, \( J_{i_0,m_0} \) acts trivially on \( V \), and so Claim 3 holds.

Now, by Claims 1–3 and simultaneous induction on \( k \) and \( N \), we must have \( \dim V = 1 \). □

Our third lemma is the following result [7], which will also play an important role in our classification.

**Lemma 3.4** Let \( \mathcal{L} \) be a Lie superalgebra with a descending sequence of subspaces \( \mathcal{L} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \ldots \) and an element \( T \) satisfying \( [T, \mathcal{L}_n] = \mathcal{L}_{n-1} \) for \( n \geq 1 \). Let \( V \) be an \( \mathcal{L} \)-module and let
\[
V_n = \{ v \in V | \mathcal{L}_n v = 0 \}, \quad n \in \mathbb{Z}_+.
\]
Suppose that \( V_n \neq 0 \) for \( n \gg 0 \), and that the minimal \( N \in \mathbb{Z}_+ \) for which \( V_N \neq 0 \) is positive. Then \( \mathbb{C}[T]V_N = \mathbb{C}[T] \otimes \mathbb{C} V_N \). In particular, \( V_N \) is finite-dimensional if \( V \) is a finitely generated \( \mathbb{C}[T] \)-module.

### 4 The rank one case

In this section, we shall classify all the free conformal modules of rank one over \( \mathfrak{B}(p) \).

Let us first construct some such conformal \( \mathfrak{B}(p) \)-modules. Recall that [7] a nontrivial free conformal module of rank one over \( \mathfrak{Vir} = \mathbb{C}[\partial)L_0 \) is isomorphic to \( M_{\Delta,\alpha} = \mathbb{C}[\partial]v \) defined by (cf. (3.1) and note that \( L_0 = pL \))
\[
L_0 \lambda v = p(\partial + \Delta \lambda + \alpha)v. \tag{4.1}
\]
for some \( \Delta, \alpha \in \mathbb{C} \). Obviously, \( M_{\Delta,\alpha} \) is also a free conformal \( \mathfrak{B}(p) \)-module of rank one by extending the \( \lambda \)-actions of \( L_i \) with \( i \geq 1 \) trivially, namely
\[
L_i \lambda v = \begin{cases} 
p(\partial + \Delta \lambda + \alpha)v & \text{if } i = 0, \\
0 & \text{if } i \geq 1. \end{cases} \tag{4.2}
\]
We still denote this conformal $\mathfrak{B}(p)$-module by $M_{\Delta,\alpha}$. Clearly, $M_{\Delta,\alpha}$ is irreducible if and only if $\Delta \neq 0$. The module $M_{0,\alpha}$ contains a unique nontrivial submodule $(\partial + \alpha)M_{0,\alpha}$ isomorphic to $M_{1,\alpha}$. We are more interested in nontrivial extensions of the conformal $\mathfrak{Vir}$-module $M_{\Delta,\alpha}$. Let us consider $\mathfrak{B}(-1)$. For any $\beta \in \mathbb{C}$, by replacing the $\lambda$-actions (4.2) by

$$L_i \lambda v = \begin{cases} 
-(\partial + \Delta \lambda + \alpha)v & \text{if } i = 0, \\
\beta v & \text{if } i = 1, \\
0 & \text{if } i \geq 2,
\end{cases} \tag{4.3}$$

we obtain a free conformal $\mathfrak{B}(-1)$-module of rank one, which is a nontrivial extension of $M_{\Delta,\alpha}$ if $\beta \neq 0$. We denote this conformal $\mathfrak{B}(-1)$-module by $M_{\Delta,\alpha,\beta}$. Similarly, $M_{\Delta,\alpha,\beta}$ is irreducible if and only if $\Delta \neq 0$ or $\beta \neq 0$. The module $M_{0,\alpha,0}$ contains a unique nontrivial submodule $(\partial + \alpha)M_{0,\alpha,0}$ isomorphic to $M_{1,\alpha,0}$. Our main result in this section is as follows.

**Theorem 4.1** Let $M$ be a nontrivial free conformal module of rank one over $\mathfrak{B}(p)$.

1. If $p \neq -1$, then $M \cong M_{\Delta,\alpha}$ defined by (4.2) for some $\Delta, \alpha \in \mathbb{C}$.
2. If $p = -1$, then $M \cong M_{\Delta,\alpha,\beta}$ defined by (4.3) for some $\Delta, \alpha, \beta \in \mathbb{C}$.

Furthermore, $M_{\Delta,\alpha}$ (resp., $M_{\Delta,\alpha,\beta}$) is irreducible if and only if $\Delta \neq 0$ (resp., $\Delta \neq 0$ or $\beta \neq 0$). The module $M_{0,\alpha}$ (resp., $M_{0,\alpha,0}$) contains a unique nontrivial submodule $(\partial + \alpha)M_{0,\alpha}$ (resp., $(\partial + \alpha)M_{0,\alpha,0}$) isomorphic to $M_{1,\alpha}$ (resp., $M_{1,\alpha,0}$).

**Proof.** Write $M = \mathbb{C}[\partial]v$. First, regarding $M$ as a conformal module over $\mathfrak{Vir}$, by (4.2), we know that $L_0 \lambda v = p(\partial + \Delta \lambda + \alpha)v$ for some $\Delta, \alpha \in \mathbb{C}$. By Lemma 3.1, $L_i \lambda v = 0$ for all $i \gg 0$. Assume that $k \in \mathbb{Z}_+$ is the largest integer such that $L_k \lambda v \neq 0$.

If $k = 0$, then $M$ is simply a conformal $\mathfrak{Vir}$-module. In our notations here, $M \cong M_{\Delta,\alpha}$ if $p \neq -1$, or $M \cong M_{\Delta,\alpha,0}$ if $p = -1$.

Next, we always assume $k > 0$. By the assumption $L_k \lambda v \neq 0$, we can write $L_k \lambda v = f(\partial, \lambda)v$, where $0 \neq f(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$. By relation $[L_k L_k]_{\lambda + \mu} v = 0$, we obtain

$$f(\partial, \lambda)f(\partial + \lambda, \mu) = f(\partial, \mu)f(\partial + \mu, \lambda).$$

Comparing the coefficients of $\lambda$, we see that $f(\partial, \lambda)$ is independent of the variable $\partial$, and so we can denote $f(\lambda) = f(\partial, \lambda)$. Then, by relation $[L_0 L_k]_{\lambda + \mu} v = ((k + p)\lambda - p\mu)L_k \lambda + \mu v$, we obtain

$$(p\mu - (k + p)\lambda)f(\lambda + \mu) = p\mu f(\mu). \tag{4.4}$$

If $k \neq -p$, then $k + p \neq 0$. By (4.4) with $\mu = 0$, we immediately obtain $f(\lambda) = 0$, a contradiction.
If \( k = -p \) (note that this case can only occur when \( p \) is a negative integer), then by (4.4) we obtain \( f(\lambda + \mu) = f(\mu) \), which implies that \( f(\lambda) \) is independent of the variable \( \lambda \), and so we can denote \( \beta = f(\lambda) \). If \( p = -1 \), then \( k = 1 \), and so \( M \cong M_{\Delta,\alpha,\beta} \). If \( p \leq -2 \), then \( k \geq 2 \). Using similar arguments as in case \( k \neq -p \), one can first show that \( L_i \lambda v = 0 \) for \( 1 \leq i \leq k - 1 \). Then, by relation \( [L_1 \lambda L_k^{-1}]_{\lambda+\mu} v = 0 \), we obtain \( (\lambda + (1+p)\mu)\beta = 0 \), which implies \( \beta = 0 \), a contradiction. This completes the proof. \( \square \)

5 Classification theorem

Now, we can show that the irreducible modules appeared in Theorem 4.1 exhaust all nontrivial finite irreducible conformal modules over \( \mathfrak{B}(p) \). Namely, we have the following classification.

**Theorem 5.1** Let \( M \) be a nontrivial finite irreducible conformal module over \( \mathfrak{B}(p) \).

1. If \( p \neq -1 \), then \( M \cong M_{\Delta,\alpha} \) defined by (4.2) for some \( \Delta, \alpha \in \mathbb{C} \) with \( \Delta \neq 0 \).

2. If \( p = -1 \), then \( M \cong M_{\Delta,\alpha,\beta} \) defined by (4.3) for some \( \Delta, \alpha, \beta \in \mathbb{C} \) with \( \Delta \neq 0 \) or \( \beta \neq 0 \).

Let \( M \) be a nontrivial finite irreducible conformal module over \( \mathfrak{B}(p) \). Our basic strategy is to show that \( M \) must be free of rank one (see Lemma 5.2 below) by using the three technical lemmas given in Section 3. Then, the above result will follow from Theorem 4.1.

**Lemma 5.2** The conformal \( \mathfrak{B}(p) \)-module \( M \) must be free of rank one.

**Proof.** First, by Lemma 3.1, the \( \lambda \)-action of \( L_i \in \mathfrak{B}(p) \) on \( M \) is trivial for all \( i \gg 0 \). Assume that \( k \in \mathbb{Z}_+ \) is the largest integer such that the \( \lambda \)-action of \( L_k \) on \( M \) is nontrivial. Then \( M \) is simply a nontrivial finite irreducible conformal module over \( \mathfrak{B}(p)[k] \), where \( \mathfrak{B}(p)[k] \) is defined by (2.1).

Furthermore, by Proposition 2.4, the conformal \( \mathfrak{B}(p)[k] \)-module \( M \) can be viewed as a module over the associated extended annihilation algebra \( \mathcal{A}(\mathfrak{B}(p)[k])^e \) satisfying

\[
\hat{L}_{i,m} v = 0 \quad \text{for} \quad v \in M, \quad 0 \leq i \leq k, \quad m \gg 0.
\]  

(5.1)

For simplicity, in what follows, we denote \( \mathcal{L} = \mathcal{A}(\mathfrak{B}(p)[k])^e \). Set

\[
\mathcal{L}_n = \text{span}_\mathbb{C}\{\hat{L}_{i,m} \in \mathcal{L} \mid 0 \leq i \leq k, \, m \geq n - 1\}, \quad n \in \mathbb{Z}_+.
\]

Note that \( \mathcal{L}_0 = \mathcal{A}(\mathfrak{B}(p)[k]) \). Clearly, \( \mathcal{L} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \ldots \) and, by Remark 2.6, the element \( T \in \mathcal{L} \) satisfies \( [T, \mathcal{L}_n] = \mathcal{L}_{n-1} \) for \( n \geq 1 \). For the \( \mathcal{L} \)-module \( M \), we set

\[
M_n = \{v \in M \mid \mathcal{L}_n v = 0\}, \quad n \in \mathbb{Z}_+.
\]

By (5.1), \( M_n \neq 0 \) for \( n \gg 0 \). Assume that \( N \in \mathbb{Z}_+ \) is the smallest integer such that \( M_N \neq 0 \). 


Suppose \( N = 0 \). Take \( 0 \neq v \in M_0 \). Then \( U(\mathcal{L})v = \mathbb{C}[T]U(\mathcal{L}_0)v = \mathbb{C}[T]v \), and so \( M = \mathbb{C}[T]v \) by the irreducibility of \( M \). Since \( \mathcal{L}_0 \) is an ideal of \( \mathcal{L} \), we see that \( \mathcal{L}_0 \) acts trivially on \( M \). Hence, \( M \) is simply an irreducible \( \mathbb{C}[T] \)-module, and so \( M \) is one-dimensional. Equivalently, \( M \) is a one-dimensional trivial conformal \( \mathfrak{B}(p) \)-module, a contradiction.

Next, we always assume \( N \geq 1 \). By Remark 2.6, \( T - \frac{1}{p}\bar{L}_{0,-1} \) is a central element in \( \mathcal{L} \). Hence, \( T - \frac{1}{p}\bar{L}_{0,-1} \) acts on \( M \) as a scalar, and \( \mathcal{L}_0 \) acts irreducibly on \( M \). Furthermore, by relation \( \bar{L}_{i,-1} = \frac{1}{p}[\bar{L}_{i,0}, \bar{L}_{0,-1}] \), we see that the action of \( \mathcal{L}_0 \) is determined by \( \mathcal{L}_1 \) and \( \bar{L}_{0,-1} \) (or equivalently, determined by \( \mathcal{L}_1 \) and \( T \)). Clearly, \( M_N \) is \( \mathcal{L}_1 \)-invariant. By the irreducibility of \( M \) and Lemma 3.4, we see that \( M = \mathbb{C}[T] \otimes \mathcal{L}_N \) and \( M_N \) is a nontrivial irreducible finite-dimensional \( \mathcal{L}_1 \)-module.

If \( N = 1 \), then by the definition of \( M_1 \), we see that \( M_1 \) is a trivial \( \mathcal{L}_1 \)-module, a contradiction.

If \( N \geq 2 \), then by the definition of \( M_N \), we see that \( M_N \) can be viewed as a \( \mathcal{L}_1/\mathcal{L}_N \)-module. Note that \( \mathcal{L}_1/\mathcal{L}_N \cong \mathfrak{g}_{k,N-2} \). By Lemma 3.3, we must have that \( M_N \) is one-dimensional. Equivalently, \( M \) is free of rank one as a conformal \( \mathfrak{B}(p) \)-module. \( \square \)

At last, by Theorems 4.1 and 5.1, one can easily obtain the classification of finite irreducible conformal modules over the finite Lie conformal algebra \( \mathfrak{b}(n) \) for \( n \geq 1 \). Recall that \( \mathfrak{b}(n) \) has a \( \mathbb{C}[\partial] \)-basis \( \{ \bar{L}_i | 0 \leq i \leq n \} \) with \( \lambda \)-brackets (cf. (1.1) and (2.2))

\[
[\bar{L}_i, \bar{L}_j] = \begin{cases} (i-n)\partial + (i+j-2n)\lambda \bar{L}_{i+j} & \text{if } i+j \leq n, \\ 0 & \text{otherwise,} \end{cases}
\]

and contains a conformal subalgebra \( \mathbb{C}[\partial] \bar{L}_0 \cong \mathfrak{Vir} \), which has a free conformal module of rank one \( M_{\Delta,\alpha} \) given by (4.1) with \( p = -n \). Clearly, there is a free conformal \( \mathfrak{b}(n) \)-module of rank one \( \mathbb{C}[\partial]v \) defined by

\[
\bar{L}_i \lambda v = \begin{cases} -n(\partial + \Delta \lambda + \alpha)v & \text{if } i = 0, \\ 0 & \text{if } 1 \leq i \leq n, \end{cases}
\]  

(5.2)

where \( \Delta, \alpha \in \mathbb{C} \). We still denote this conformal \( \mathfrak{b}(n) \)-module by \( M_{\Delta,\alpha} \). In addition, consider the special case \( \mathfrak{b}(1) \). Replacing the above \( \lambda \)-actions by

\[
\bar{L}_i \lambda v = \begin{cases} -(\partial + \Delta \lambda + \alpha)v & \text{if } i = 0, \\ \beta v & \text{if } i = 1, \end{cases}
\]  

(5.3)

where \( \beta \in \mathbb{C} \), we obtain a free conformal \( \mathfrak{b}(1) \)-module of rank one, which is a nontrivial extension of the conformal \( \mathfrak{Vir} \)-module \( M_{\Delta,\alpha} \) if \( \beta \neq 0 \). We still denote this conformal \( \mathfrak{b}(1) \)-module by \( M_{\Delta,\alpha,\beta} \). Since \( \mathfrak{b}(n) \) is a quotient algebra of \( \mathfrak{Vir}(-n) \), by Theorems 4.1 and 5.1, we have

**Corollary 5.3** Let \( M \) be a nontrivial free conformal module of rank one over \( \mathfrak{b}(n) \).

(1) If \( n > 1 \), then \( M \cong M_{\Delta,\alpha} \) defined by (5.2) for some \( \Delta, \alpha \in \mathbb{C} \).
(2) If \( n = 1 \), then \( M \cong M_{\Delta,\alpha,\beta} \) defined by (5.3) for some \( \Delta, \alpha, \beta \in \mathbb{C} \).

Furthermore, \( M_{\Delta,\alpha} \) (resp., \( M_{\Delta,\alpha,\beta} \)) is irreducible if and only if \( \Delta \neq 0 \) (resp., \( \Delta \neq 0 \) or \( \beta \neq 0 \)). The module \( M_{0,\alpha} \) (resp., \( M_{0,\alpha,0} \)) contains a unique nontrivial submodule \( (\partial + \alpha)M_{0,\alpha} \) (resp., \( (\partial + \alpha)M_{0,\alpha,0} \)) isomorphic to \( M_{1,\alpha} \) (resp., \( M_{1,\alpha,0} \)). The modules \( M_{\Delta,\alpha} \) with \( \Delta \neq 0 \) and \( M_{\Delta,\alpha,\beta} \) with \( \Delta \neq 0 \) or \( \beta \neq 0 \) exhaust all nontrivial finite irreducible conformal modules over \( b(n) \).

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