Transfer matrices for scalar fields on curved spaces

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Abstract

We apply Nelson's technique of constructing Euclidean fields to the case of classical scalar fields on curved spaces. It is shown how to construct a transfer matrix and, for a class of metrics, the basic spectral properties of its generator are investigated. An application concerning decoupling of non-convex disjoint region is given.

I. INTRODUCTION

We start our construction from the ideas comprised in Nelson's axioms for scalar Euclidean-Markoff quantum fields. Here, the Markoff property of certain projectors is one of the basic ingredient in defining the transfer matrix of whom generator is identified with the Hamiltonian of Wightman quantum scalar field. We found that these ideas can be used in the same way at the non-quantum level. In the case of the scalar fields on Riemannian manifolds, for an arbitrary direction, we construct a propagator by using the Markoff property. In the stationary case it becomes a semigroup which can be considered as the transfer matrix of the system and, further, it can be used in introducing a Hamiltonian. We will show that the propagator is exponentially bounded by using Agmon's results in exponential decay of solutions of second-order elliptic equations. An application concerning the decoupling (in the sense of) of two disjoint non-convex regions is given.

II. INTRODUCTORY DEFINITIONS AND RESULTS

Let us consider the Riemannian manifold $(\mathbb{R}^{n+1}, g)$ and the Laplace-Beltrami operator on it, $\Delta$. For a point in $\mathbb{R}^{n+1}$ we use the notation $(t, x)$. Let $E_m (t, x; s, y)$ be the kernel of $(\Delta + m^2)^{-1}$ on $L^2 (\mathbb{R}^{n+1}, \sqrt{g} dt dx)$. As in [2], we will not consider the additional term $\frac{1}{6} \rho$. One defines the space $N \subset \mathcal{D}' (\mathbb{R}^{n+1})$, $f \in N$ if:

$$\|f\|_N^2 = \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}^{n+1}} \tilde{f} (t, x) E_m (t, x; s, y) f (s, y) \sqrt{g (t, x)} \sqrt{g (s, y)} dtdxdsdy < \infty, \quad (1)$$
and, for each $\sigma \in R$, let $N_{\sigma} \subset D' (R^n)$ be the space: $g \in N_{\sigma}$ if
\[
\|g\|_{N_{\sigma}}^2 = \int_{R^n} \bar{g} (x) E_m (\sigma, x; \sigma, y) g (y) \sqrt{g (\sigma, x)} \sqrt{g (\sigma, y)} dxdy < \infty. \tag{2}
\]
We will consider that, as in the Euclidean case, the space $L^2 (R^n, d\mu_{\sigma}) \subset N_{\sigma}$, where $d\mu_{\sigma} (x) = \sqrt{g (\sigma, x)} dx$ and that it is dense in $N_{\sigma}$ for each $\sigma \in R$. Now, let $\hat{E}_{\sigma} : N_{\sigma} \to L^2 (R^n, d\mu_{\sigma})$ be the operator corresponding to the kernel $E_m (\sigma, x; \sigma, y)$. Then $\hat{E}_{\sigma}^{1/2}$ defines an isometry from $N_{\sigma}$ to $L^2 (R^n, d\mu_{\sigma})$ and let $(\hat{E}_{\sigma}^{1/2})^\dagger : L^2 (R^n, d\mu_{\sigma}) \to N_{\sigma}$ be its adjoint. The following are true:
\[
\hat{E}_{\sigma}^{1/2} \circ (\hat{E}_{\sigma}^{1/2})^\dagger = 1_{L^2 (R^n, d\mu_{\sigma})} \text{ and } (\hat{E}_{\sigma}^{1/2})^\dagger \circ \hat{E}_{\sigma}^{1/2} = 1_{N_{\sigma}}. \tag{3}
\]
With our assumptions, $\hat{E}_{\sigma}^{1/2} (N_{\sigma}) = L^2 (R^n, d\mu_{\sigma}) \subset N_{\sigma}$, the operator $\hat{E}_{\sigma}^{1/2}$ is bounded on $N_{\sigma}$. Moreover, one can view $(\hat{E}_{\sigma})^\dagger$ as a dense defined unbounded operator on $N_{\sigma}$, in fact, it is the inverse operator of $\hat{E}_{\sigma}$.

For $\sigma \in R$, let $j_{\sigma}$ be the operator $j_{\sigma} : N_{\sigma} \to N$, $(j_{\sigma} \psi)(t, x) = \psi (x) \delta (t - \sigma)$ and $j_{\sigma}^*$ be its adjoint. If $\Lambda$ is a closed subset of $R^{n+1}$ we denote by $N_{\Lambda}$ the subspace of $N$ which comprises all distributions with support in $\Lambda$. The orthogonal projection of $N$ in $N_{\Lambda}$ will be denoted by $e_{\Lambda}$. Following \[5\] we have:

**Proposition 1** The operators $j_{\sigma}$ are isometries and $j_{\sigma}^* j_{\sigma} = 1_{N_{\sigma}}$, $j_{\sigma} j_{\sigma}^* = e_{\sigma}$, where $e_{\sigma}$ denotes the projector corresponding to the subset of $R^{n+1}$, $t = \sigma$.

Then we define the operators:
\[
U_{\sigma, \sigma'} : N_{\sigma'} \to N_{\sigma}, U_{\sigma, \sigma'} = j_{\sigma}^* \circ j_{\sigma'}.
\tag{4}
\]

We will derive in the following that $U_{\sigma, \sigma'}$ are propagators in the sense of \[6\]. This will follow from the Markoff property of the projectors $e_{\sigma}$.

**Lemma 2** Let $A$, $B$ and $C$ be closed subsets in $R^{n+1}$ such that $C$ separates $A$ and $B$. Then $e_{A} \circ e_{C} \circ e_{B} = e_{A} \circ e_{B}$.
Solution 3 This is the consequence of the fact that $E_m$ is the kernel of a local operator. The proof is identical with that of [3].

The basics properties of $U_{\sigma,\sigma'}$ operators are stated in the following proposition.

**Proposition 4** The family of operators $U_{\sigma,\sigma'}$, $\sigma, \sigma' \in R$ has the following properties:

1) $U_{\sigma,\sigma'} \circ U_{\sigma',\sigma''} = U_{\sigma,\sigma''}$

2) $U_{\sigma,\sigma} = 1_{N_\sigma}$

3) $\|U_{\sigma,\sigma'}\| \leq 1$.

Solution 5 1) Using the Markoff property we have:

$$e_\sigma \circ e_{\sigma'} \circ e_{\sigma''} = e_\sigma e_{\sigma''} \Leftrightarrow j_\sigma \circ j^*_{\sigma'} \circ j_{\sigma''} = j_\sigma \circ j^*_{\sigma'} \circ j^*_{\sigma''}.$$  

(5)

By composition with $j_{\sigma''}$ at the right, we have

$$j_\sigma \circ (j^*_{\sigma'} \circ j_{\sigma''} - j^*_{\sigma'} \circ j_{\sigma''}) = 0.$$  

(6)

From the definition of $U_{\sigma,\sigma'}$ and since $j_\sigma$ are isometries, we conclude $U_{\sigma,\sigma'}U_{\sigma',\sigma''} = U_{\sigma,\sigma''}$.

2) It follows from proposition 1.1 and definition of $U_{\sigma,\sigma'}$.

3) Because $j^*_{\sigma'}$ and $j_{\sigma}$ are isometries, the property results immediately.

### III. EXPONENTIAL BOUNDS ON PROPAGATORS

To improve our estimates on the propagators $U_{\sigma,\sigma'}$ we need a supplementary condition on the metric $g$. We say that an application $Q : R^{n+1} \rightarrow M (n+1, n+1)$ has stable positivity if there exists $\varepsilon > 0$ such that for any application $\delta : R^{n+1} \rightarrow M (n+1, n+1)$ with $|\delta (x)^{ij}| \leq \varepsilon$ the matrices $Q (x) - \delta (x)$ are positive defined for any $x \in R^{n+1}$. The following result is a direct application of Agmon theory [4] of exponentially decay of solutions of elliptic second order operators.
Proposition 6 If the metric $g$ has stable positivity then for any $f \in N_\sigma$:

$$\int_{T_0}^{\infty} d\sigma \left\{ e^{\omega \sigma} \left[ \hat{E}_{\sigma}^{1/2} \circ U_{\sigma,\sigma'} f \right]_{N_\sigma} \right\}^2 < \infty,$$

provided $\omega < \frac{m}{\sqrt{\sup g^{11}}}$.

Solution 7 Starting from

$$\langle u, U_{\sigma,\sigma'} f \rangle_{N_\sigma} = \left\langle u, \hat{E}_{\sigma} \circ U_{\sigma,\sigma'} f \right\rangle_{L^2(R^n, d\mu)}$$

$$= \int_{R^n} u(x) \left[ \int_{R^n} E_{m}(\sigma, x; \sigma', y) f(y) d\mu(\sigma', y) \right] d\mu(\sigma)$$

for $u \in N_\sigma$ and $f \in N_{\sigma'}$, it follows that $\varphi(\sigma, x) = \left( \hat{E}_{\sigma} \circ U_{\sigma,\sigma'} f \right)(x)$ is a solution of

$$(\Delta + m^2) \varphi(\sigma, x) = 0$$

for $\sigma > \sigma'$. Let $\rho_m(\cdot, \cdot)$ denotes the distance corresponding to the metric $g_m = mg$. The metric $g$ has stable positivity so, there is an $\varepsilon \in R_+$ such that $\rho_m(\sigma_0, x_0; \sigma, x) > \frac{\varepsilon}{m} |\sigma - \sigma_0|$. For $\Omega = \{ (\sigma, x) : \sigma > T_0 \}$, $T_0 \in R_+$ and for some positive $\lambda$:

$$\int_{\Omega} |\varphi(\sigma, x)|^2 e^{-\lambda \rho_m(T_0, x_0; \sigma, x)} \sqrt{g(\sigma, x)} d\sigma d^n x$$

$$= \int_{T_0}^{\infty} d\sigma \left\langle \hat{E}_{\sigma} \circ U_{\sigma,\sigma'} f, \hat{E}_{\sigma} \circ U_{\sigma,\sigma'} f \right\rangle_{L^2(R^n, d\mu)} e^{-\lambda \rho_m(\sigma - T_0)}$$

$$< \text{ct.} \int_{T_0}^{\infty} d\sigma \left\| U_{\sigma,\sigma'} f \right\|_{N_\sigma}^2 e^{-\lambda \rho_m(\sigma - T_0)} < \infty.$$

So we are in the conditions of the main theorem of \cite{2}. It follows that:

$$\int_{\Omega} d\sigma d^n x \sqrt{g(\sigma, x)} |\varphi(\sigma, x)|^2 (m^2 - g(\nabla h(\sigma, x), \nabla h(\sigma, x))) e^{2h(\sigma, x)}$$

$$\leq \frac{2(1+2d)}{m^2} \int_{\Omega_d} |\varphi(\sigma, x)|^2 e^{2h(\sigma, x)} \sqrt{g(\sigma, x)} d\sigma$$

$$= \frac{2(1+2d)}{m^2} m^2 \int_{\Omega_d} \int_{\Omega_d} |\varphi(\sigma, x)|^2 e^{2h(\sigma, x)} \sqrt{g(\sigma, x)} d\sigma d\sigma'$$

where $d$ is a positive number and $\Omega_d = \{ (\sigma, x) : \sigma \in \Omega, \rho_m((\sigma, x), \{\infty\}) > d \}$. Here

$$\rho_m((\sigma, x), \{\infty\}) = \sup \{ \rho_m((\sigma, x), \Omega \setminus K) : K is a compact subset of \Omega \}.$$

The function $h$ is any function which satisfies the condition $g(\nabla h(\sigma, x), \nabla h(\sigma, x)) < m^2$. We choose $h(\sigma, x) = \omega \sigma$ with $\omega < \frac{m}{\sqrt{\sup g^{11}}}$. The above inequality becomes

$$\int_{T_0}^{\infty} d\sigma \left\{ e^{\omega \sigma} \left[ \hat{E}_{\sigma}^{1/2} \circ U_{\sigma,\sigma'} f \right]_{N_\sigma} \right\}^2 < \infty,$$
\[
\frac{\int_\Omega d\sigma d^n x \sqrt{g(\sigma, x)} |\varphi(\sigma, x)|^2 e^{2\omega \sigma}}{2(1+2d) m^2 - \omega^2} \int_{\Omega \setminus \Omega_d} d\sigma dx \sqrt{g(\sigma, x)} |\varphi(\sigma, x)|^2 e^{2\omega \sigma}.
\]

If for any point \((\sigma, x) \in \Omega\) there is a geodesic which starts in \((\sigma, x)\) and ends in the hyperplane \(\sigma = T_0\) then \(\Omega \setminus \Omega_d \subset \{(\tau, x) : 0 < \sigma \leq T\}\) with \(T\) sufficiently large but finite. In conclusion

\[
\int_\Omega d\sigma d^n x \sqrt{g(\sigma, x)} |\varphi(\sigma, x)|^2 e^{2\omega \sigma} = \int_{T_0}^\infty d\sigma e^{2\omega \sigma} \left\langle \hat{E}_\sigma \circ U_{\sigma,\sigma'} f, \hat{E}_\sigma \circ U_{\sigma,\sigma'} f \right\rangle_{L^2(R^n, \mu_\sigma)} < \infty,
\]

or

\[
\int_{T_0}^\infty d\sigma e^{2\omega \sigma} \left\langle \hat{E}_\sigma \circ U_{\sigma,\sigma'} f, \hat{E}_\sigma \circ U_{\sigma,\sigma'} f \right\rangle_{L^2(R^n, \mu_\sigma)} < \infty,
\]

which implies

\[
\int_{T_0}^\infty d\sigma \left\{ e^{\omega \sigma} \left\| \hat{E}^{1/2}_\sigma \circ U_{\sigma,\sigma'} f \right\|_{N_\sigma} \right\}^2 < \infty.
\]

IV. THE STATIONARY CASE

We consider in this section that there is a coordinate system such that the metric \(g\) is independent of first coordinate. In this case, the spaces \(N_\sigma\) and the operators \(\hat{E}^{1/2}_\sigma\) are identically and will be denoted by \(N_0\) and \(\hat{E}^{1/2}_0\) respectively. Thus, the operators \(U_{\sigma,\sigma'}\) are defined on the same Hilbert space and depend only on the difference \(\sigma - \sigma' : U_{\sigma,\sigma'} = U_{\sigma - \sigma'}\).

The family of operators \(\{U_\tau\}_{\tau \in R_+}\) forms a semigroup. Using the results about existence and properties of the generators of semigroups [7], we can obtain bounds directly on the transfer matrix \(U_\tau\).

Proposition 8 The semigroup \(\{U_\tau\}_{\tau \in R_+}\) is exponentially bounded: \(\|U_\tau\|_{N_0} < e^{-\tau \omega}\) provided \(\omega < \frac{m}{\sqrt{\sup g^{1/4}}}\).

Solution 9 Because we have found estimates on \(\hat{E}^{1/2}_0 \circ U_\tau\), we will consider the operators \(\hat{U}_\tau = \hat{E}^{1/2}_0 \circ U_\tau \circ \left(\hat{E}^{1/2}_0\right)^\dagger\), well defined on \(L^2(R^n, d\mu_0)\). Using the fact that \(L^2(R^n, d\mu_0)\) is
dense in $N_0$ we can extend these operators by continuity on the space $N_0$. In this way we have build the semigroup $\{\tilde{U}_\tau\}_{\tau \in \mathbb{R}_+}$ which satisfies the estimates of the precedent section:

$$\int_{T_0}^\infty d\tau \left\{ e^{\omega \tau} \left\| \tilde{U}_\tau \right\|_{N_0} \right\}^2 < \infty,$$

(17)

for some $T_0 > 0$. So $\{\tilde{U}_\tau\}_{\tau \in \mathbb{R}_+}$ is exponentially bounded and in consequence [7], if $\tilde{K}$ is its generator ($\tilde{U}_\tau = e^{-\tau \tilde{K}}$) the resolvent set of $\tilde{K}$ satisfies:

$$\{ z \in \mathbb{C} \mid \Re z \in (-\infty, \omega) \} \subset \rho \left( \tilde{K} \right).$$

(18)

If $K$ is the generator of $\{U_\tau\}_{\tau \in \mathbb{R}_+}$ then, on $\mathcal{D}(K)$ we have:

$$K = \left( \hat{E}_{0}^{1/2} \right)^\dagger \circ \tilde{K} \circ \hat{E}_{0}^{1/2}$$

(19)

by using the reciprocal formula

$$U_\tau = \left( \hat{E}_{0}^{1/2} \right)^\dagger \circ \tilde{U}_\tau \circ \hat{E}_{0}^{1/2},$$

(20)

valid on $N_0$. If the operator

$$\left( \hat{E}_{0}^{1/2} \right)^\dagger \circ (\tilde{K} - z)^{-1} \circ \hat{E}_{0}^{1/2}$$

(21)

is well defined, even on a dense subset of $N_0$, then $K - z$ is inversable. From 20 it follows that, if $(\tilde{K} - z)^{-1}$ exists, then:

$$\left( \tilde{K} - z \right)^{-1} \left( L^2(\mathbb{R}^n, d\mu_0) \right) \subset L^2(\mathbb{R}^n, d\mu_0),$$

(22)

and in consequence $\left( \hat{E}_{0}^{1/2} \right)^\dagger \circ (\tilde{K} - z)^{-1} \circ \hat{E}_{0}^{1/2}$ is well defined on the entire $N_0$. Will follow that $\rho \left( \tilde{K} \right) \subset \rho \left( K \right)$ and this ends the proof.

If the metric is symmetric at transformation $x^1 \to -x^1$, the transfer matrix generator is self-adjoint and it can be considered as the Hamiltonian of the scalar field.
V. APPLICATION

Our application is for the Euclidean case. The results concerning decoupling of different regions in quantum Euclidean fields are based primarily on estimates of \( \|e_{\Lambda_1}e_{\Lambda_2}\|_N \), where \( \Lambda_1, \Lambda_2 \) are two disjoint regions. Let us consider the two dimensional case. The most difficult case is when \( \Lambda_1, \Lambda_2 \) are not convex and there is no possibility of drawing a straight line between the two subsets. We can sharpen the existent estimates \[ \] for these cases by using the previous results. The idea is to make a change of coordinates such that for the new coordinates, lines like \( \sigma = ct. \) separate the two sets and they are as closed as possible to the boundaries of \( \Lambda_1, \Lambda_2 \). Then we can use the exponential bounds of the previous section to evaluate \( \|e_{\Lambda_1}e_{\Lambda_2}\|_N \). More precisely:

**Proposition 10** Let \( \Lambda_1, \Lambda_2 \) two regions in \( \mathbb{R}^2 \) such that the construction of the coordinates to be possible (after a rotation if necessary). Then

\[
\|e_{\Lambda_1} \circ e_{\Lambda_2}\|_N \leq e^{-m|\beta - \alpha| \min|\cos \theta|},
\]

where \( \theta \) and \( |\beta - \alpha| \) will be defined during the proof.

**Solution 11** Let \((t, x)\) denotes the original coordinates in which the metric is diagonal. Let \( \gamma : \mathbb{R} \rightarrow \mathbb{R}^2 \) be a curve which separates \( \Lambda_1, \Lambda_2 \) and \( \gamma(0) = (t = 0, x = 0) \). We define a new coordinate system \((\sigma, \xi)\) by

\[
\begin{align*}
t(\sigma, \xi) &= \sigma + \gamma^1(\xi) \\
x(\sigma, \xi) &= \gamma^2(\xi)
\end{align*}
\]

(24)

In the new coordinates, the metric is

\[
g'(\sigma, \xi) = \begin{pmatrix}
1 & \frac{d\gamma^1}{d\xi} \\
\frac{d\gamma^1}{d\xi} & \left(\frac{d\gamma^1}{d\xi}\right)^2 + \left(\frac{d\gamma^2}{d\xi}\right)^2
\end{pmatrix}
\]

(25)

so we are in the conditions of the last section. Using the Markoff property,
\[ \| e_{\Lambda_1} \circ e_{\Lambda_2} \|_N = \| e_{\Lambda_1} \circ e_{\alpha} \circ e_{\beta} \circ e_{\Lambda_2} \|_N \leq \| e_{\alpha} \circ e_{\beta} \|_N, \] 
where the lines \( \sigma = \alpha, \sigma = \beta \) separate \( \Lambda_1 \) and \( \Lambda_2 \) exactly in the order they appear in the above relation (in the sense that \( \sigma = \alpha \) separates \( \Lambda_1 \) by \( \sigma = \beta \) etc.). Further
\[ \left\| j_\alpha \circ j_\alpha^* \circ j_\beta \circ j_\beta^* \right\|_N = \left\| j_\alpha \circ U_{\alpha - \beta} \circ j_\beta^* \right\|_N = \| U_{\alpha - \beta} \|_{N_0} \cdot \] 
The element \((g')^{11}\) is given by \((g')^{11} = \frac{1}{\cos^2 \theta}\), where \(\theta\) is the angle between the tangent to the curve \(\gamma\) and the \(x\) axis. Using the bounds of the last section we have
\[ \| e_{\Lambda_1} \circ e_{\Lambda_2} \|_N \leq e^{-m|\beta - \alpha| \min|\cos \theta|}. \]

**VI. CONCLUSIONS**

Our primary goal was to define the transfer matrix for scalar fields on curved spaces and to investigate the basic spectral properties of its generator. Even though the generator is not self-adjoint in the general case, this approach allows us to investigate this problem by using at least two new tools besides the methods of Green functions. One is the perturbations of hypercontractive semigroups \([8]\) and the other is the adiabatic theorem.

Now it is straightforward to quantize the field by defining the Markoff field over the space \(N\). For the stationary, symmetric at time reflection case (static), we think that one has now all elements to construct the physical field (for example that proposed in \([4]\)) by following Nelson reconstruction method and holomorphic continuation of the transfer matrix. Note that, according to results of \([8]\), the holomorphic continuation of the transfer matrix to real time is still possible, in the stationary case without symmetry at time reflection, as long the spectrum of the generator belongs to the real axis. Of course, one has to check that the results of \([8]\) (sistematized in \([4]\)), which are the core of the reconstruction theorem, are still valid. For the general case, we think that the adiabatic theorem, especially the adiabatic reduction theory \([10]\), may play an important role in defining the physical quantum field by following Nelson's approach.
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