C* algebra and inverse chaos

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Abstract: If an invertible linear dynamical systems is Li-York chaotic or other chaotic, what’s about it’s inverse dynamics? what’s about it’s adjoint dynamics? With this unresolved but basic problems, this paper will give a criterion for Lebesgue operator on separable Hilbert space. Also we give a criterion for the adjoint multiplier of Cowen-Douglas functions on 2-th Hardy space. Last we give some chaos about scalars perturbation of operator and some examples of invertible bounded linear operator such that $T$ is chaotic but $T^{-1}$ is not.

Keywords: inverse, chaos, Hardy space, rooter function, Cowen-Douglas function, Spectrum, C* algebra, Lebesgue operator.

1. Introduction

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The ideas of chaos in connection with a map was introduced by Li T.Y.
and his teacher Yorke, J.A. [22], after that there are various definitions
of what it means for a map to be chaotic and there is a series of papers on
Topological Dynamics and Ergodic Theory about chaos, such as [27] [11] [17] [13] [23].

Following Topological Dynamics, Linear Dynamics is also a rapidly evolv-
ing branch of functional analysis, which was probably born in 1982 with
the Toronto Ph.D. thesis of C. Kitai [3]. It has become rather popular be-
cause of the efforts of many mathematicians, for the seminal paper [6] by
G. Godefroy and J. H. Shapiro, the notes [12] by J. H. Shapiro, the authorita-
tive survey [20] by K.-G. Grosse-Erdmann, and finally for the book [4] by F. Bayart
and E. Matheron, the book [19] by K.-G. Grosse-Erdmann and A. Peris.

For finite-dimension linear space, the authors have made a topologically
conjugate classification about Jordan blocks in [18] [24]. So for the eigenvalues
|\lambda| \neq 1, the operators of Jordan block are not Li-Yorke chaotic. With [4] we
know that a Jordan block is not supercyclic when its eigenvalues |\lambda| = 1,
and following a easy discussion it is not Li-Yorke chaotic too.

So the definition of Li-Yorke chaos should be valid only on infinite-
dimension Frechet space or Banach space such that in this paper the Hilbert
space is infinite-dimensional. Because a finite-dimensional linear operator
could be regard as a compact operator on some Banach spaces or some Hilbert
spaces, we can get the same conclusion from [25] P12 or the Theorem 7 of
[10].

For a Frechet space X, let \mathcal{L}(X) denote the set of all bounded linear
operators on $X$. Let $\mathbb{B}$ denote a Banach space and let $\mathbb{H}$ denote a Hilbert space. If $T \in \mathcal{L}(\mathbb{B})$, then define $\sigma(T) = \{ \lambda \in \mathbb{C}; T - \lambda \text{ is not invertible} \}$ and define $r_{\sigma}(T) = \sup\{ |\lambda|; \lambda \in \sigma(T) \}$.

**Definition 1.** Let $T \in \mathcal{L}(\mathbb{B})$, if there exists $x \in \mathbb{B}$ satisfies:

1. $\lim_{n \to \infty} |T^n(x)| > 0$; and
2. $\lim_{n \to \infty} \|T^n(x)\| = 0$.

Then we say that $T$ is Li-Yorke chaotic, and named $x$ is a Li-Yorke chaotic point of $T$, where $x \in \mathbb{B}, n \in \mathbb{N}$.

Define a distributional function $F^n_x(\tau) = \frac{1}{n} \sharp \{ 0 \leq i \leq n : \|T^n(x)\| < \tau \}$, where $T \in \mathcal{L}(\mathbb{B}), x \in \mathbb{B}, n \in \mathbb{N}$. And define

$$F_x(\tau) = \liminf_{n \to \infty} F^n_x(\tau); \text{ and } F^*_x(\tau) = \limsup_{n \to \infty} F^n_x(\tau).$$

**Definition 2.** Let $T \in \mathcal{L}(\mathbb{B})$, if there exists $x \in \mathbb{B}$ and

1. If $F_x(\tau) = 0$, $\exists \tau > 0$, and $F^*_x(\epsilon) = 1, \forall \epsilon > 0$, then we say that $T$ is distributional chaotic or I-distributionally chaotic.
2. If $F^*_x(\epsilon) > F_x(\tau), \forall \tau > 0$, and $F^*_x(\epsilon) = 1, \forall \epsilon > 0$, then we say that $T$ is II-distributionally chaotic.
3. If $F^*_x(\epsilon) > F_x(\tau), \forall \tau > 0$, then we say that $T$ is III-distributionally chaotic.

**Definition 3 (25).** Let $X$ is an arbitrary infinite-dimensional separable Frechet space, $T \in \mathcal{L}(X)$. If there exists a subset $X_0$ of $X$ satisfies:

1. For any $x \in X_0, \{T^n x\}_{n=1}^{\infty}$ has a subsequence converging to 0;
There is a bounded sequence \( \{a_n\}_{n=1}^{\infty} \) in \( \text{span}(X_0) \) such that the sequence \( \{T^n a_n\}_{n=1}^{\infty} \) is unbounded.

Then we say \( T \) satisfies the Li-Yorke Chaos Criterion.

**Theorem 1** ([25]). Let \( X \) is an arbitrary infinite-dimensional separable Frechet space, If \( T \in \mathcal{L}(X) \), then the following assertions are equivalent.

(i) \( T \) is Li-Yorke chaotic;

(ii) \( T \) satisfies the Li-Yorke Chaos Criterion.

**Lemma 1** ([29]). There are no hypercyclic operators on a finite-dimensional space \( X \neq 0 \).

**Example 1** ([4]P8). Let \( \phi \in \mathcal{H}^\infty(\mathbb{D}) \) and let \( M_\phi : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D}) \) be the associated multiplication operator. The adjoint multiplier \( M_\phi^* \) is hypercyclic if and only if \( \phi \) is non-constant and \( \phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset \).

**Theorem 2** ([5]). Let \( X \) is an arbitrary separable Frechet space, \( T \in \mathcal{L}(X) \). The following assertions are equivalent.

(i) \( T \) is hypercyclic.

(ii) \( T \) is topologically transitive; that is, for each pair of non-empty open sets \( U, V \subseteq X \), there exists \( n \in \mathbb{N} \) such that \( T^n(U) \cap V \neq \emptyset \).

**Theorem 3** ([4]). Let \( X \) is a topological vector space, \( T \) is a bounded linear
Let

\[
\begin{align*}
\Lambda_1(T) & \triangleq \text{span}( \bigcup_{|\lambda|=1,n \in \mathbb{N}} \ker(T - \lambda)^N \cap \text{ran}(T - \lambda)^N) ; \\
\Lambda^+(T) & \triangleq \text{span}(\Lambda_1(T) \bigcup \bigcup_{|\lambda| > 1,n \in \mathbb{N}} \ker(T - \lambda)^N) ; \\
\Lambda^-(T) & \triangleq \text{span}(\Lambda_1(T) \bigcup \bigcup_{|\lambda| < 1,n \in \mathbb{N}} \ker(T - \lambda)^N). 
\end{align*}
\]

If \( \Lambda^+(T) \) and \( \Lambda^-(T) \) are both dense in \( X \), then \( T \) is mixing.

2. From Polar Decomposition to functional calculus

The Polar Decomposition Theorem \[15\]P15 on Hilbert space is a useful theorem, especially for invertible bounded linear operator. We give some properties of \( C^* \) algebra generated by normal operator.

Let \( \mathbb{H} \) be a separable Hilbert space over \( \mathbb{C} \) and let \( X \) be a compact subset of \( \mathbb{C} \). Let \( \mathcal{C}(X) \) denote the linear space of all continuous functions on the compact space \( X \), let \( \mathcal{P}(x) \) denote the set of all polynomials on \( X \) and let \( T \) be an invertible bounded linear operator on \( \mathbb{H} \). By the Polar Decomposition Theorem \[15\]P15 we get \( T = U|T| \), where \( U \) is an unitary operator and \( |T|^2 = T^*T \). Let \( \mathcal{A}(|T|) \) denote the \( C^* \) algebra generated by the positive operator \( |T| \) and 1.

**Lemma 2.** Let \( 0 \notin X \) be a compact subset of \( \mathbb{C} \). If \( \mathcal{P}(x) \) is dense in \( \mathcal{C}(X) \), then \( \mathcal{P}(\frac{1}{x^2}) \) is also dense in \( \mathcal{C}(X) \).

**Proof.** By the property of polynomials we know that \( \mathcal{P}(\frac{1}{x^2}) \) is a algebraic closed subalgebra of \( \mathcal{C}(X) \) and we get:
(1) $1 \in \mathcal{P}(\frac{1}{x})$;

(2) $\mathcal{P}(\frac{1}{x})$ separate the points of $X$;

(3) If $p(\frac{1}{x}) \in \mathcal{P}(x)$, then $\bar{p}(\frac{1}{x}) \in \mathcal{P}(x)$.

By the Stone-Weierstrass Theorem \cite{P145} we get the conclusion.

**Lemma 3.** Let $X \subseteq \mathbb{R}_+$. If $\mathcal{P}(|x|)$ is dense in $\mathcal{C}(X)$, then $\mathcal{P}(|x|^2)$ is also dense in $\mathcal{C}(X)$.

**Proof.** For $X \subseteq \mathbb{R}_+$, $x \neq y \iff x^2 \neq y^2$. By Lemma 2 we get the conclusion.

By the GNS construction \cite{P250} for the $C^*$ algebra $\mathcal{A}(|T|)$, we get the following decomposition.

**Lemma 4.** Let $T$ be an invertible bounded linear operator on the separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$, $\mathcal{A}(|T|)$ is the complex $C^*$ algebra generated by $|T|$ and 1. There is a sequence of nonzero $\mathcal{A}(|T|)$-invariant subspace $\mathbb{H}_1, \mathbb{H}_2, \cdots$ such that:

(1) $\mathbb{H} = \mathbb{H}_1 \bigoplus \mathbb{H}_2 \bigoplus \cdots$;

(2) For every $\mathbb{H}_i$, there is a $\mathcal{A}(|T|)$-cyclic vector $\xi_i$ such that $\mathbb{H}_i = \overline{\mathcal{A}(|T|)\xi_i}$ and $|T|\mathbb{H}_i = \mathbb{H}_i = |T|^{-1}\mathbb{H}_i$.

**Proof.** By \cite{P54} we get (1), and $|T|\mathbb{H}_i \subseteq \mathbb{H}_i$, that is $\mathbb{H}_i \subseteq |T|^{-1}\mathbb{H}_i$; by Lemma 2 we get $|T|^{-1}\mathbb{H}_i \subseteq \mathbb{H}_i$. Hence we get $|T|\mathbb{H}_i = \mathbb{H}_i = |T|^{-1}\mathbb{H}_i$. □
For $\forall n \in \mathbb{N}, T^n$ is invertible when $T$ is invertible. By the Polar Decomposition Theorem $[15] P15 \ T^n = U_n |T^n|$, where $U_n$ is unitary operator and $|T^n|^2 = T^{*n}T^n$, we get the following conclusion.

**Lemma 5.** Let $T$ be an invertible bounded linear operator on the separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$, let $\mathcal{A}(|T^k|)$ be the complex $C^*$ algebra generated by $|T^k|$ and 1 and let $\mathbb{H}_i^{[T^k]} = \mathcal{A}(|T^k|) \xi_k$ be a sequence of non-zero $\mathcal{A}(|T^k|)$-invariant subspace, there is a decomposition $\mathbb{H} = \bigoplus_i \mathbb{H}_i^{[T^k]}$, $\xi_k \in \mathbb{H}_i, i, k \in \mathbb{N}$. Given a proper permutation of $\mathbb{H}_i^{[T^k]}$ and $\mathbb{H}_j^{[T(k+1)]}$, we get $T^* \mathbb{H}_i^{[T^k]} = \mathbb{H}_i^{[T(k+1)]}$ and $T^{-1} \mathbb{H}_i^{[T^k]} = \mathbb{H}_i^{[T(k+1)]}$.

**Proof.** By Lemma 3, it is enough to prove the conclusion on $\mathcal{P}(|T^k|^2) \xi_k$. For any given $\xi_k \in \mathbb{H} = \bigoplus_j \mathbb{H}_j^{[T(k+1)]}$, there is a unique $j \in \mathbb{N}$ such that $\xi_k \in \mathbb{H}_j^{[T(k+1)]}$.

(1) Because $T$ is invertible, for any given $\xi_k$, there is an unique $\eta_i \in \mathbb{H}_i^{[T(k+1)]}$ such that $\eta_i = T^{-1} \xi_k$. For $\forall p \in \mathcal{P}(|x|^2)$, we get $p(|T(k+1)|^2) \eta_i = T^* p(|T^k|^2) \xi_k$. Hence we get

$$\mathbb{H}_i^{[T(k+1)]} = \mathcal{P}(|T(k+1)|^2) \xi_k \supseteq T^* \mathcal{P}(|T^k|^2) \xi_k = T^* \mathbb{H}_i^{[T^k]}.$$

(2) Similarly, for any given $\xi_k$, there is an unique $\eta_r \in \mathbb{H}_r^{[T^k]}$ such that $\eta_r = T \xi_k$. For $\forall p \in \mathcal{P}(|x|^2)$, we get

$$p(|T^k|^2) \eta_r = p(|T^k|^2) T \xi_k = T^{* -1} p(|T(k+1)|^2) \xi_{k+1}.$$

Hence we get

$$\mathbb{H}_r^{[T^k]} = \mathcal{P}(|T^k|^2) \xi_k \supseteq T^{* -1} \mathcal{P}(|T(k+1)|^2) \xi_{k+1} = T^{* -1} \mathbb{H}_s^{[T(k+1)]}.$$
Let $i = r$, by (1)(2) we get $T^{-1}H_i^{T^{(k+1)}} \subseteq H_i^{T^k} \subseteq T^{-1}H_j^{T^{(k+1)}}$.

Fixed the order of $H_i^{T^k}$, by a proper permutation of $H_j^{T^{(k+1)}}$ we get $T^{-1}H_i^{T^k} = H_i^{T^{(k+1)}}$.

By Lemma 2 and $T$ is invertible, we get

(3) For any given $\xi_i^{k+1}$, there is an unique $\eta_i \in H_i^{T^k}$ such that $\eta_i = T^*\xi_i^{k+1}$. For $\forall p \in P(|x|^{-2})$, we get $p(|T^{(k+1)}|^{-2})\eta_i = T^{-1}p(|T^k|^{-2})\xi_i^{k+1}$. Hence we get

$$H_i^{T^{(k+1)}} = \overline{P(|T^{(k+1)}|^{-2})\xi_i^{k+1}} \supseteq T^{-1}\overline{P(|T^k|^{-2})\xi_i^{k+1}} = T^{-1}H_i^{T^k}.$$

(4) For any given $\xi_i^{k+1}$, there is an unique $\eta_i \in H_i^{T^k}$ such that $\eta_i = T^*\xi_i^{k+1}$. For $\forall p \in P(|x|^{-2})$, we get

$$T^{-1}p(|T^k|^{-2})\eta_i = T^{-1}p(|T^k|^{-2})T^{-1}p(|T^{(k+1)}|^{-2})\xi_i^{k+1} = p(|T^{(k+1)}|^{-2})\xi_i^{k+1}.$$

Hence we get

$$T^{-1}H_i^{T^k} = T^{-1}\overline{P(|T^k|^{-2})\xi_i^{k+1}} \supseteq \overline{P(|T^{(k+1)}|^{-2})\xi_i^{k+1}} = H_i^{T^{(k+1)}}.$$

By (3)(4) we get $T^{-1}H_i^{T^k} \subseteq H_i^{T^{(k+1)}} \subseteq T^{-1}H_i^{T^k}$.

That is, $T^{-1}H_i^{T^k} = H_i^{T^{(k+1)}}$. 

Let $\xi \in H$ is a $A(|T|)$-cyclic vector such that $A(|T|)\xi$ is dense in $H$. Because of $\sigma|T| \neq \emptyset$, on $C(\sigma(|T|))$ define the non-zero linear functional $\rho_{|T|}: \rho_{|T|}(f) = \langle f(|T|)\xi, \xi \rangle, \forall f \in C(\sigma(|T|))$. Then $\rho_{|T|}$ is a positive linear functional, by [30] P54 and the Riesz-Markov Theorem, on $C(\sigma(|T|))$ we
get that there exists an unique finite positive Borel measure \( \mu_{|T|} \) such that

\[
\int_{\sigma(|T|)} f(z) \, d\mu_{|T|}(z) = < f(|T|) \xi, \xi >, \quad \forall f \in C(\sigma(|T|)).
\]

**Theorem 4.** Let \( T \) be an invertible bounded linear operator on the separable Hilbert space \( \mathbb{H} \) over \( \mathbb{C} \), there is \( \xi \in \mathbb{H} \) such that \( \mathcal{A}(|T|)\xi = \mathbb{H} \). For any given \( n \in \mathbb{N} \), let \( \mathcal{A}(|T^n|) \) be the complex \( C^* \) algebra generated by \( |T^n| \) and 1 and let \( \xi_n \) be a \( \mathcal{A}(|T^n|) \)-cyclic vector such that \( \mathcal{A}(|T^n|)\xi_n = \mathbb{H} \). Then:

1. For any given \( \xi_n \), there is an unique positive linear functional

\[
\int_{\sigma(|T^n|)} f(z) \, d\mu_{|T^n|}(z) = < f(|T^n|) \xi_n, \xi_n >, \quad \forall f \in L^2(\sigma(|T^n|)).
\]

2. For any given \( \xi_n \), there is an unique finite positive complete Borel measure \( \mu_{|T^n|} \) such that \( L^2(\sigma(|T^n|)) \) is isomorphic to \( \mathbb{H} \).

**Proof.** Because \( T \) is invertible, by Lemma 5 we get that if there is a \( \mathcal{A}(|T|) \)-cyclic vector \( \xi \), then there is a \( \mathcal{A}(|T^n|) \)-cyclic vector \( \xi_n \).

(1): For any given \( \xi_n \), define the linear functional, \( \rho_{|T^n|}(f) = < f(|T^n|) \xi_n, \xi_n > \), by [30]P54 and the Riesz-Markov Theorem we get that on \( C(\sigma(|T^n|)) \) there is an unique finite positive Borel measure \( \mu_{|T^n|} \) such that

\[
\int_{\sigma(|T^n|)} f(z) \, d\mu_{|T^n|}(z) = < f(|T^n|) \xi_n, \xi_n >, \quad \forall f \in C(\sigma(|T^n|)).
\]

Moreover we can complete the Borel measure \( \mu_{|T^n|} \) on \( \sigma(|T^n|) \), also using \( \mu_{|T^n|} \) to denote the complete Borel measure, By [26] we know that the complete Borel measure is uniquely.
For $\forall f \in L^2(\sigma(|T^n|))$, because of
$$
\rho_{|T^n|}(|f|^2) = \rho_{|T^n|}(\bar{f}f) = < f(|T^n|)^* f(|T^n|) \xi_n, \xi_n > = \|f(|T^n|) \xi_n\| \geq 0.
$$
we get that $\rho_{|T^n|}$ is a positive linear functional, hence (1) is right.

(2) we know that $\mathcal{C}(\sigma(|T^n|))$ is a subspace of $L^2(\sigma(|T^n|))$ such that $\mathcal{C}(\sigma(|T^n|))$ is dense in $L^2(\sigma(|T^n|))$.

For any $f, g \in \mathcal{C}(\sigma(|T^n|))$ we get
$$
< f(|T^n|) \xi_n, g(|T^n|) \xi_n > = < g(|T^n|)^* f(|T^n|) \xi_n, \xi_n > = \rho_{|T^n|}(\bar{g}f) = \int_{\sigma(|T^n|)} f(z) \bar{g}(z) d\mu_{|T^n|}(z) = < f, g >_{L^2(\sigma(|T^n|))}.
$$

Therefor $U_0 : \mathcal{C}(\sigma(|T^n|)) \to \mathbb{H}, f(z) \to f(|T^n|) \xi_n$ is a surjection isometry from $\mathcal{C}(\sigma(|T^n|))$ to $\mathcal{A}(|T^n|) \xi_n$, also $\mathcal{C}(\sigma(|T^n|))$ and $\mathcal{A}(|T^n|) \xi_n$ is a dense subspace of $L^2(\sigma(|T^n|))$ and $\mathbb{H}$, respectively. Because of $U_0$ a closable operator, it closed extension $U : L^2(\sigma(|T^n|)) \to \mathbb{H}, f(z) \to f(|T^n|) \xi_n$ is a unitary operator. Hence for any given $\xi_n$, $U$ is the unique unitary operator induced by the unique finite positive complete Borel measure $\mu_{|T^n|}$ such that $L^2(\sigma(|T^n|))$ is isomorphic to $\mathbb{H}$.

By the Polar Decomposition Theorem [15]P15, we get $U^* T^* U = T T^*$ and $U^* |T|^{-2} U = |T|^{-2}$ when $T = U|T|$. In fact, when $T$ is invertible, we can choose an specially unitary operator such that $|T|^{-1}$ and $|T|^{-1}$ are unitary equivalent. We give the following unitary equivalent by Theorem 4.

**Theorem 5.** Let $T$ be an invertible bounded linear operator on the separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$ and let $\mathcal{A}(|T|)$ be the complex algebra generated by
$|T|$ and 1. There is $\sigma(|T|^{-1}) = \sigma(|T^{-1}|)$ and we get that $|T|^{-1}$ and $|T^{-1}|$ are unitary equivalent by the unitary operator $F_{xx^*}^{\mathbb{R}}$, more over the unitary operator $F_{xx^*}^{\mathbb{R}}$ is induced by an almost everywhere non-zero function $\sqrt{|\phi|T|}$, where $\sqrt{|\phi|T|} \in L^\infty(\sigma(|T|), \mu|T|)$. That is, $d \mu|T^{-1}| = |\phi|T| |d \mu|T|^{-1}$.

Proof. By Lemma 4 lose no generally, let $\xi|T|$ is a $|A(|T|)|$-cyclic vector such that $H_\xi = A(|T|)\xi$. 

<1> Define the function $F_{x^{-1}}: P(x) \rightarrow P(x^{-1}), F_{x^{-1}}(f(x)) = f(x^{-1})$, it is easy to find that $F_{x^{-1}}$ is linear. Because of

$$\int_{\sigma(|T|)} f(z) d\mu|T|(z) = \langle f(|T|^{-1})\xi, \xi \rangle = \int_{\sigma(|T|^{-1})} f(z) d\mu|T^{-1}|(z).$$

We get $d\mu|T^{-1}|(z) = |z|^2 d\mu|T|(z).$ Hence

$$\|F_{x^{-1}}(f(x))\|_{L^2(\sigma(|T|^{-1}), \mu|T|^{-1})}$$

$$= \int_{\sigma(|T|^{-1})} F_{x^{-1}}(f(x)) \bar{F}_{x^{-1}}(f(x)) d\mu|T^{-1}|(x)$$

$$= \int_{\sigma(|T|^{-1})} f(x^{-1}) \bar{f}(x^{-1}) d\mu|T^{-1}|(x)$$

$$= \int_{\sigma(|T|)} |z|^2 f(z) \bar{f}(z) d\mu|T|(z)$$

$$\leq \sup \sigma(|T|)^2 \int_{\sigma(|T|)} f(z) d\mu|T|'(x)$$

$$\leq \sup \sigma(|T|)^2 \|f(z)\|_{L^2(\sigma(|T|), \mu|T|)}.$$ 

So we get $\|F_{x^{-1}}\| \leq \sup \sigma(|T|)^2$, by the Banach Inverse Mapping Theorem [14]P91 we get that $F_{x^{-1}}$ is an invertible bounded linear operator from
Define the operator $F^H_{x^{-1}} : \mathcal{A}(|T|) \xi \rightarrow \mathcal{A}(|T|^{-1}) \xi$, $F(f(|T|) \xi) = f(|T|^{-1}) \xi$.

By Lemma 2 and \[30\]P55, we get that $F^H_{x^{-1}}$ is a bounded linear operator on the Hilbert space $\mathcal{A}(|T|) = \mathbb{H}$, $\|F^H_{x^{-1}}\| \leq \sup \sigma(|T|)^2$. Moreover we get $H \rightarrow \mathcal{A}(|T|^{-1}) \rightarrow \mathcal{A}(|T|)$.

(2) Define the function $F_{xx^*} : \mathcal{P}(xx^*) \rightarrow \mathcal{P}(x^*x)$, $F_{xx^*}(f(x, x^*)) = f(x^*x)$.

By [8] we get that $F_{xx^*}$ is a linear algebraic isomorphic.

For any $x \in \sigma(|T|)$, by Lemma 3 we get that $\mathcal{P}(|x|^2)$ is dense in $\mathcal{L}(|x|)$, $\mathcal{L}(|x|)$ is dense in $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1}})$. Hence $\mathcal{P}(|x|^2)$ is dense in $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1}})$.

With a similarly discussion, for any $y \in \sigma(|T|^{-1})$, we get that $\mathcal{P}(|y|^2)$ is dense in $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1}})$.

So $F_{xx^*}$ is an invertible bounded linear operator from $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1}})$ to $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1}})$. Therefore $F_{xx^*} \circ F_{x^{-1}}$ is an invertible bounded linear operator from $\mathcal{L}^2(\sigma(|T|), \mu_{|T|})$ to $\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1}})$.

Because of $\lambda \in \sigma(T^*T) \iff \| \lambda \| \in \sigma(T^*T^{-1})$, we get that $\lambda \in \sigma(|T|) \iff \| \lambda \| \in \sigma(|T|^{-1})$, that is, $\sigma(|T|) = \sigma(|T|^{-1})$.

For any $p_n \in \mathcal{P}(\sigma(|T|^{-1})) \subseteq \mathcal{A}(\sigma(|T|^{-1}))$, because of $T^*p_n(|T|^{-1}) = T^*p_n(|T|^{-1}) = \ldots$
$p_n(|T^{-1}|)T^{s-1}$, by [13]P60 we get that $P(|T|^{-1})$ and $P(|T^{-1}|)$ are unitary equivalent. Hence there is an unitary operator $U \in B(\mathbb{H})$ such that $UP(|T|^{-1}) = P(|T^{-1}|)U$, that is, $U\mathcal{A}(|T|^{-1}) = \mathcal{A}(|T^{-1}|)U$ and $U\mathcal{A}(|T|^{-1})\xi_{|T|} = \mathcal{A}(|T^{-1}|)U\xi_{|T|}$.

If let $\xi_{|T^{-1}|} = U\xi_{|T|}$, then $\xi_{|T^{-1}|}$ is a $\mathcal{A}(|T^{-1}|)$-cyclic vector and $\mathcal{A}(|T^{-1}|)\xi_{|T^{-1}|} = \mathbb{H}$.

Because of

$$\int_{\sigma(|T|^{-1})} f(z) d\mu_{|T|^{-1}}(z) = \langle f(|T|^{-1}) \xi_{|T|}, \xi_{|T|} \rangle = \int_{\sigma(|T|)} f(\frac{1}{z}) d\mu_{|T|}(z).$$

$$\int_{\sigma(|T|^{-1})} f(z) d\mu_{|T|^{-1}}(z) = \langle f(|T^{-1}|) \xi_{|T^{-1}|}, \xi_{|T^{-1}|} \rangle.$$

We get $[d\mu_{|T^{-1}|}] = [d\mu_{|T|^{-1}}]$, that is, $d\mu_{|T^{-1}|}$ and $d\mu_{|T|^{-1}}$ are mutually absolutely continuous, by [14]IX.3.6 Theorem and (1) we get that $d\mu_{|T|^{-1}} = |\phi_{|T|}(\frac{1}{z})| d\mu_{|T|^{-1}} = |z|^2 |\phi_{|T|}(z)| d\mu_{|T|}$, where $|\phi_{|T|}(z)| \neq 0, a.e.$ and $|\phi_{|T|}(z)| \in \mathcal{L}^\infty(\sigma(|T|), \mu_{|T|})$. So we get

$$\|F_{xx^*} \circ F_{x^{-1}}(f(x))\|_{\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1}})}$$

$$= \int_{\sigma(|T|^{-1})} F_{xx^*} \circ F_{x^{-1}}(f(x)) F_{xx^*} \circ F_{x^{-1}}(f(x)) d\mu_{|T|^{-1}}(x)$$

$$= \int_{\sigma(|T|^{-1})} f(x^{-1}) \tilde{f}(x^{-1}) d\mu_{|T|^{-1}}(x)$$

$$= \int_{\sigma(|T|^{-1})} |\phi_{|T|}(\frac{1}{z})| f(x^{-1}) \tilde{f}(x^{-1}) d\mu_{|T|^{-1}}(x)$$

$$= \int_{\sigma(|T|^{-1})} F_{x^{-1}}(\sqrt{\phi_{|T|}(z)} f(x)) F_{x^{-1}}(\sqrt{\phi_{|T|}(z)} \tilde{f}(x)) d\mu_{|T|^{-1}}(x)$$

$$= \|\sqrt{\phi_{|T|}(\frac{1}{z})} F_{x^{-1}}(f(x))\|_{\mathcal{L}^2(\sigma(|T|^{-1}), \mu_{|T|^{-1}})}$$

Hence $F_{xx^*}$ is an unitary operator that is induced by $Uf(\frac{1}{z}) = \sqrt{|\phi_{|T|}(\frac{1}{z})|} f(\frac{1}{z})$. 

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Define $F^H_{xx^*}: \begin{cases} \mathcal{A}(|T|^{-2})\xi_{|T|} \to \mathcal{A}(|T^{-2}|)\xi_{|T^{-1}|}, \\ F^H_{xx^*}(f(|T|^{-2})\xi_{|T|}) = f(|T^{-2}|)\xi_{|T^{-1}|}. \end{cases}$

Therefore $F^H_{xx^*}$ is an unitary operator from $\mathcal{A}(|T|^{-1})\xi_{|T|}$ to $\mathcal{A}(|T^{-1}|)\xi_{|T^{-1}|}$. By Lemma 4 and [8] we get $\mathcal{A}(|T|^{-1})\xi = \mathbb{H} = \mathcal{A}(|T^{-1}|)\xi$. That is, $F^H_{xx^*}$ is an unitary operator and we get

$$
\begin{array}{c}
\text{H} & |T^{-1}| & \text{H} \\
\downarrow F^H_{xx^*} & & \downarrow F^H_{xx^*} \\
\text{H} & |T^{-1}| & \text{H}
\end{array}
$$

So $|T|^{-1}$ and $|T^{-1}|$ are unitary equivalent by $F^H_{xx^*}$, the unitary operator $F^H_{xx^*}$ is induced by the function $\sqrt{|\phi_{|T|}(\frac{1}{x})|}$.

**Corollary 1.** Let $T$ be an invertible bounded linear operator on the separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$ and let $\mathcal{A}(|T|)$ be the complex algebra generated by $|T|$ and 1. There is $\sigma(|T|) = \sigma(|T^*|)$ and we get that $|T|$ and $|T^*|$ are unitary equivalent by the unitary operator $F^H_{xx^*}$, more over the unitary operator $F^H_{xx^*}$ is induced by an almost everywhere non-zero function $\sqrt{|\phi_{|T|}|}$, where $\sqrt{|\phi_{|T|}|} \in \mathcal{L}^\infty(\sigma(|T|), \mu_{|T|})$. That is, $d \mu_{|T^*|} = |\phi_{|T|}| d \mu_{|T|}$.

**3. The chaos between $T$ and $T^{-1}$ for Lebesgue operator**

For the example of singular integral in mathematical analysis, we know that is independent the convergence or the divergence of the weighted integral between $x$ and $x^{-1}$, however some times that indeed dependent for a special weighted function. For $T$ is an invertible bounded operator on the separable Hilbert space $\mathbb{H}$ over $\mathbb{C}$, we get $0 \notin \sigma(|T^n|) \subseteq \mathbb{R}_+$. In the view of the
singular integral in mathematical analysis and by Theorem 4 we get that
$T$ and $T^{-1}$ should not be convergence or divergence at the same time for
$T$ is an invertible bounded operator. Anyway, they should be convergence or
divergence at the same time for some special operators.

Therefor we define the Lebesgue operator and prove that $T$ and $T^{-1}$ are
Li-Yorke chaotic at the same time for $T$ is a Lebesgue operator. Then we
give an example that $T$ is a Lebesgue operator, but not is a normal operator.

Let $dx$ be the Lebesgue measure on $L^2(\mathbb{R}_+)$, by Theorem 4 we get that
$d\mu_{|T^n|}$ is the complete Borel measure and $L^2(\sigma(|T^n|))$ is a Hilbert space. If
$\exists N > 0$, for $\forall n \geq N, n, n \in \mathbb{N}, d\mu_{|T^n|}$ is absolutely continuity with respect to $dx$, by the Radon-Nikodym Theorem [14]P380 there is $f_n \in L^1(\mathbb{R}_+)$ such that
d$\mu_{|T^n|} = f_n(x) dx$.

Definition 4. Let $T$ be an invertible bounded linear operator on the separable
Hilbert space $\mathbb{H}$ over $\mathbb{C}$, moreover if $T$ satisfies the following assertions:

(1) If $\exists N > 0$, for $\forall n \geq N, n \in \mathbb{N}$

$$
\begin{cases}
  d\mu_{|T^n|} = f_n(x) dx, & f_n \in L^1(\mathbb{R}_+). \\
  x^2 f_n(x) = f_n(x^{-1}), & 0 < x \leq 1.
\end{cases}
$$

(2) If $\exists N > 0$, for $\forall n \geq N, n \in \mathbb{N}$, there is a $A(|T^n|)$-cyclic vector $\xi_n$. And
for any given $0 \neq x \in \mathbb{H}$ and for any given $0 \neq g_n(t) \in L^2(\sigma(|T^n|))$, there is
an unique $0 \neq y \in \mathbb{H}$ such that $y = g_n(|T^n|^{-1})\xi_n$ when $x = g_n(|T^n|)\xi_n$.

Then we say that $T$ is a Lebesgue operator, let $L_{Leb}(\mathbb{H})$ denote the set of
all Lebesgue operators.
**Theorem 6.** Let $T$ be a Lebesgue operator on the separable Hilbert space $H$ over $\mathbb{C}$, then $T$ is Li-Yorke chaotic if and only if $T^*-1$ is.

**Proof.** We prove the conclusion by two parts.

(1) Let $\mathbb{H}$ be $A(|T|)$-cyclic, that is, there is a vector $\xi$ such that $A(|T|)\xi = \mathbb{H}$. By Lemma 5 we get that if there is a $A(|T|)$-cyclic vector $\xi$, then there is also a $A(T_n)$-cyclic vector $\xi_n$.

Let $x_0$ be a Li-Yorke chaotic point of $T$, by Theorem 4 and the Polar Decomposition Theorem [15], P15 and by the define of Lebesgue operator, we get that for enough large $n \in \mathbb{N}$, there are $g_n(x) \in L^2(\sigma(|T_n|)), f_n(x) \in L^2(\mathbb{R}_+)$ and $y_0 \in \mathbb{H}$ such that $x_0 = g_n(|T_n|)\xi_n, y_0 = g_n(|T_n|^{-1})\xi_n$ and $d\mu_{|T_n|} = f_n(x) \, dx$.

$$\|T^nx_0\| = \langle T^nx_0, x_0 \rangle = \langle |T^n|^2 g_n(|T^n|)\xi_n, g_n(|T^n|)\xi_n \rangle = \langle g_n(|T^n|)\xi_n, \xi_n \rangle$$

$$= \int_{\sigma(|T_n|)} x^2 g_n(x) \tilde{g}(x) \, d\mu_{|T_n|}(x)$$

$$= \int_{0}^{+\infty} x^2 |g_n(x)|^2 f_n(x) \, dx$$

$$= \int_{1}^{+\infty} x^2 |g_n(x)|^2 f_n(x) \, dx + \int_{1}^{+\infty} x^2 |g_n(x)|^2 f_n(x) \, dx$$

$$= \int_{1}^{+\infty} x^2 |g_n(x)|^2 f_n(x) \, dx + \int_{0}^{1} x^{-4} |g_n(x^{-1})|^2 f_n(x^{-1}) \, dx$$

$$\Delta \int_{1}^{+\infty} x^2 |g_n(x)|^2 f_n(x) \, dx + \int_{0}^{1} x^{-2} |g_n(x^{-1})|^2 f_n(x) \, dx$$

$$= \int_{1}^{+\infty} x^{-2} |g_n(x^{-1})|^2 f_n(x) \, dx + \int_{0}^{1} x^{-2} |g_n(x^{-1})|^2 f_n(x) \, dx$$

$$= \int_{0}^{+\infty} x^{-2} |g_n(x^{-1})|^2 f_n(x) \, dx$$
= \int_{\sigma(|T^n|)} x^{-2} g_n(x^{-1}) \bar{g}_n(x^{-1}) d\mu_{|T^n|}(x)
= \langle g_n(|T^n|)|T^n|^{-2} g_n(|T^n|^{-1}) \xi_n, \xi_n \rangle
= \langle |T^n|^{-2} g_n(|T^n|^{-1}) \xi_n, g_n(|T^n|^{-1}) \xi_n \rangle
= \langle |T^n|^{-2} y_0, y_0 \rangle
= \langle T^{-n}T^{-n} y_0, y_0 \rangle
= \|T^{-n}y_0\|.

Where \( \triangleq \) following the define of \( f_n(x) \).

(2) If \( \mathbb{H} \) is not \( \mathcal{A}(|T|) \)-cyclic, by Lemma 5 we get that for \( \forall n \in \mathbb{N} \), there is a decomposition \( \mathbb{H} = \bigoplus_i \mathbb{H}_i^{[T^k]}, \xi_k \in \mathbb{H}, i, k \in \mathbb{N} \), where \( \mathbb{H}_i^{[T^k]} = \mathcal{A}(|T^k|)\xi_k \) is a sequence of \( \mathcal{A}(|T^k|) \)-invariant subspace, and do (1) for \( \mathbb{H}_i^{[T^k]} \).

By (1)(2) we get that \( T \) is Li-Yorke chaotic if and only if \( T^{-1} \) is Li-Yorke chaotic.

**Corollary 2.** Let \( T \) be a Lebesgue operator on the separable Hilbert space \( \mathbb{H} \) over \( \mathbb{C} \), then \( T \) is I-distributionally chaotic (or II-distributionally chaotic or III-distributionally chaotic) if and only if \( T^{-1} \) is I-distributionally chaotic (or II-distributionally chaotic or III-distributionally chaotic).

**Theorem 7.** There is an invertible bounded linear operator \( T \) on the separable Hilbert space \( \mathbb{H} \) over \( \mathbb{C} \), \( T \) is Lebesgue operator but not is a normal operator.

**Proof.** Let \( 0 < a < b < +\infty \), then \( \mathcal{L}^2([a, b]) \) is a separable Hilbert space over \( \mathbb{R} \), because any separable Hilbert space over \( \mathbb{R} \) can be expanded to a separable Hilbert space over \( \mathbb{C} \), it is enough to prove the conclusion on \( \mathcal{L}^2([a, b]) \). We
prove the conclusion by six parts:

(1) Let $0 < a < 1 < b < +\infty, M = \{[a, \frac{b-a}{2}], [\frac{b-a}{2}, b]\}$. Construct measure preserving transformation on $[a, b]$.

There is a Borel algebra $\xi(M)$ generated by $M$, define $\Phi : [a, b] \to [a, b]$, $\Phi([a, \frac{b-a}{2}]) = [\frac{b-a}{2}, b]$, $\Phi([\frac{b-a}{2}, b]) = [a, \frac{b-a}{2}]$. Then $\Phi$ is an invertible measure preserving transformation on the Borel algebra $\xi(M)$.

By [27]P63,$U_\Phi \neq 1$ is a unitary operator induced by $\Phi$, where $U_\Phi$ is the composition $U_\Phi h = h \circ \Phi, \forall h \in L^2([a, b])$ on $L^2([a, b])$.

(2) Define $M_x h = xh$ on $L^2([a, b])$, then $M_x$ is an invertible positive operator.

(3) For $f(x) = \frac{|\ln x|}{x}, x > 0$, define $d\mu = f(x)dx$. then $f(x)$ is continuous and $f(x) > 0, a.e.x \in [a, b]$, hence $d\mu$ that is absolutely continuous with respect to $dx$ is finite positive complete Borel measure, therefor $L^2([a, b], d\mu)$ a separable Hilbert space over $\mathbb{R}$. Moreover $L^2([a, b])$ and $L^2([a, b], d\mu)$ are unitary equivalent.

(4) Let $T = U_\Phi M_x$, we get $T^*T = U_\Phi TT^*U_\Phi$ and $U_\Phi \neq 1$. Because of $U_\Phi M_x \neq M_x U_\Phi$ and $U_\Phi M_{x^2} \neq U_\Phi M_{x^2}$, we get that $T$ is not a normal operator and $\sigma(|T|) = [a, b]$.

(5) The operator $T = U_\Phi M_x$ on $L^2([a, b])$ is corresponding to the operator $T'$ on $L^2([a, b], d\mu)$. $T'$ is invertible bounded linear operator and is not a normal operator and $\sigma(|T'|) = [a, b]$. 

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(6) From \( \int_{a}^{b} x^n f(x) \, dx = \int_{a}^{b} \frac{1}{n-1} \, dt \), let \( f_n(t) = \frac{1}{n} I_{[a^n, b^n]} f(t^{1/n}) \), we get that \( f_n(t) \) is continuous and almost everywhere positive, hence \( f_n(t) \) is a finite positive complete Borel measure.

For any \( E \subseteq \mathbb{R}_+ \) define \( I_E = 1 \) when \( x \in E \) else \( I_E = 0 \), so \( I_E \) is the identity function on \( E \).

(i) \( f_n(t^{-1}) = \frac{1}{n} I_{[a^n, b^n]} f(t^{-1/n}) \) \( \frac{1}{t - \frac{1}{n}} \) = \( \frac{1}{n} I_{[a^n, b^n]} \frac{\ln t^{1/n}}{t^{1/n}} \) \( \frac{1}{n} \)

\( = \frac{1}{n} I_{[a^n, b^n]} t \ln \frac{1}{t^n} \)

(ii) \( t^2 f_n(t) = \frac{1}{n} I_{[a^n, b^n]} f(t^{1/n}) \) \( \frac{t^2}{n-1} \) = \( \frac{1}{n} I_{[a^n, b^n]} \frac{\ln t^{1/n}}{t^{1/n}} \) \( \frac{t^2}{n-1} \)

\( = \frac{1}{n} I_{[a^n, b^n]} t \ln \frac{1}{t^n} \)

By (i)(ii) we get \( x^2 f_n(x) = f_n(x^{-1}) \).

From \( \sigma(|T^n|) = [a^n, b^n] \) and \( \int_{a^n}^{b^n} t^2 f(t^{1/n}) \frac{1}{n} \, dt = \int_{0}^{+\infty} t^2 f_n(t) \, dt \), let \( d \mu_{|T^n|} = f_n(t) \, dt \), then \( d \mu_{|T^n|} \) is the finite positive complete Borel measure.

For any given \( 0 \neq h(x) \in \mathcal{L}^2([a, b]) \) we get \( 0 \neq h(x^{-1}) \in \mathcal{L}^2([a, b]) \). \( I_{[a, b]} \) is a \( \mathcal{A}(|M^n_x|) \)-cyclic vector of the multiplication \( M^n_x = M_x^n \), and \( I_{[a^n, b^n]} \) is a \( \mathcal{A}(|T^n|) \)-cyclic vector of \( |T^n| \). By Definition 4 we get that \( T' \) is Lebesgus operator but not is a normal operator.

\[ \square \]

**Corollary 3.** There is an invertible bounded linear operator \( T \) on the separable Hilbert space \( \mathbb{H} \) over \( \mathbb{C} \), \( T \) is a Lebesgue operator and also is a positive operator.
By the cyclic representation of $C^*$ algebra and the GNS construction, also by the functional calculus of invertible bounded linear operator, we could study the operator by the integral on $\mathbb{R}$. This way neither change Li-Yorke chaotic nor the computing, but by the singular integral in mathematical analysis on the theoretical level we should find that there is a invertible bounded linear operator $T$ that is Li-Yorke chaotic but $T^{-1}$ is not Li-Yorke chaotic.

4. Cowen-Douglas function on Hardy space

For $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$, if $g$ is an complex analytic function on $\mathbb{D}$ and there is $\sup_{r < 1} \int_{-\pi}^{\pi} |g(re^{i\theta})|^2 d\theta < +\infty$, then we denote $g \in \mathcal{H}^2(\mathbb{D})$, hence $\mathcal{H}^2(\mathbb{D})$ is a Hilbert space with the norm $\|g\|_{\mathcal{H}^2}^2 = \sup_{r < 1} \int_{-\pi}^{\pi} |g(re^{i\theta})|^2 \frac{d\theta}{2\pi}$, especially $\mathcal{H}^2(\mathbb{D})$ is denoted as a Hardy space. By the completion theory of complex analytic functions, the Hardy space $\mathcal{H}^2(\mathbb{D})$ is a special Hilbert space that is relatively easy not only for theoretical but also for computing, and we should give some properties about adjoint multiplier operators on $\mathcal{H}^2(\mathbb{D})$.

Any given complex analytic function $g$ has a Taylor expansion $g(z) = \sum_{n=0}^{+\infty} a_n z^n$, so $g \in \mathcal{H}^2(\mathbb{D})$ and $\sum_{n=0}^{+\infty} a_n^2 < +\infty$ are naturally isomorphic. For $\mathbb{T} = \partial \mathbb{D}$, if $\mathcal{L}^2(\mathbb{T})$ denoted the closed span of all Taylor expansions of functions in $\mathcal{L}^2(\mathbb{T})$, then $\mathcal{H}^2(\mathbb{T})$ is a closed subspace of $\mathcal{L}^2(\mathbb{T})$. From the naturally isomorphic of $\mathcal{H}^2(\mathbb{D})$ and $\mathcal{H}^2(\mathbb{T})$ by the properties of analytic function, we denote $\mathcal{H}^2(\mathbb{T})$ also as a Hardy space.

By [14]P6 we get that any Cauchy sequence with the norm $\| \cdot \|_{\mathcal{H}^2}$ on
\( H^2(\mathbb{D}) \) is a uniformly Cauchy sequence on any closed disk in \( \mathbb{D} \), in particular, we get that the point evaluations \( f \rightarrow f(z) \) are continuous linear functional on \( H^2(\mathbb{D}) \), by the Riesz Representation Theorem\[14\]P13, for any \( g(s) \in H^2(\mathbb{D}) \), there is a unique \( f_z(s) \in H^2(\mathbb{D}) \) such that \( g(z) = \langle g(s) ; f_z(s) \rangle \), so define that \( f_z \) is a reproducing kernel at \( z \).

Let \( H^\infty(\mathbb{D}) \) denote the set of all bounded complex analytic function on \( \mathbb{D} \), for any given \( \phi \in H^\infty(\mathbb{D}) \), it is easy to get that \( \|\phi\|_\infty = \sup\{ |\phi(z)| ; |z| < 1 \} \) is a norm on \( \phi \in H^\infty(\mathbb{D}) \). for any given \( g \in H^2(\mathbb{D}) \), the multiplication operator \( M_\phi(g) = \phi g \) associated with \( \phi \) on \( H^2(\mathbb{D}) \) is a bounded linear operator, and by the norm on \( H^2(\mathbb{D}) \) we get \( \|M_\phi(g)\| \leq \|\phi\|_\infty \|g\|_{H^2} \). If we denote \( H^\infty(\mathbb{T}) \) as the closed span of all Taylor expansions of functions in \( L^\infty(\mathbb{T}) \), then \( H^2(\mathbb{T}) \) is a closed subspace of \( L^2(\mathbb{T}) \) and \( H^\infty(\mathbb{D}) \) and \( H^\infty(\mathbb{T}) \) are naturally isomorphic by the properties of complex analytic functions \[17\]P55P97 and the Dirichlet Problem \[7\]P103.

There are more properties about Hardy space in \[4\]P7, \[16\]P48, \[21\]P39, \[28\]P133 and \[30\]P106.

**Definition 5.** For a connected open subset \( \Omega \) of \( \mathbb{C} \), \( n \in \mathbb{N} \), let \( B_n(\Omega) \) denotes the set of all bounded linear operator \( T \) on \( \mathbb{H} \) that satisfies:

(a) \( \Omega \in \sigma(T) = \{ \omega \in \mathbb{C} : T - \omega \text{ not invertible} \} \);  

(b) \( \text{ran}(T - \omega) = \mathbb{H} \) for \( \omega \in \Omega \);  

(c) \( \bigvee_{\omega \in \Omega} \text{ker}(T - \omega) = \mathbb{H} \);  

(d) \( \dim \text{ker}(T - \omega) = n \) for \( \omega \in \Omega \).
If \( T \in \mathcal{B}_n(\Omega) \), then say that \( T \) is a Cowen-Douglas operator.

**Theorem 8** ([2]). For a connected open subset \( \Omega \) of \( \mathbb{C} \), \( T \in \mathcal{B}_n(\Omega) \), we get

1. If \( \Omega \cap T \neq \emptyset \), then \( T \) is Devaney chaotic.
2. If \( \Omega \cap T \neq \emptyset \), then \( T \) is distributionally chaotic.
3. If \( \Omega \cap T \neq \emptyset \), then \( T \) strong mixing.

**Definition 6** ([28] P141). Let \( \mathcal{P}(z) \) be the set of all polynomials about \( z \), where \( z \in \mathbb{T} \). Define a function \( h(z) \in \mathcal{H}^2(\mathbb{T}) \) is an outer function if \( \text{cl}[h(z)\mathcal{P}(z)] = \mathcal{H}^2(\mathbb{T}) \).

**Lemma 6** ([28] P141). A function \( h(z) \in \mathcal{H}^\infty(\mathbb{T}) \) is invertible on the Banach algebra \( \mathcal{H}^\infty(\mathbb{T}) \), if and only if \( h(z) \in \mathcal{L}^\infty(\mathbb{T}) \) and \( h(z) \) is an outer function.

**Theorem 9** ([16] P81). Let \( \mathcal{P}(z) \) be the set of all polynomials about \( z \), where \( z \in \mathbb{D} \). then \( h(z) \in \mathcal{H}^2(\mathbb{D}) \) is an outer function if and only if \( \mathcal{P}(z)h(z) = \{p(z)h(z); p \in \mathcal{P}(z)\} \) is dense in \( \mathcal{H}^2(\mathbb{D}) \).

Let \( \phi \) is a non-constant complex analytic function on \( \mathbb{D} \), for any given \( z_0 \in \mathbb{D} \), by [7] P29 we get that there exists \( \delta_{z_0} > 0 \), exists \( k_{z_0} \in \mathbb{N} \), when \( |z - z_0| < \delta_{z_0} \), there is

\[
\phi(z) - \phi(z_0) = (z - z_0)^{k_{z_0}}h_{z_0}(z)
\]

where \( h_{z_0}(z) \) is complex analytic on a neighbourhood of \( z_0 \) and \( h_{z_0}(z_0) \neq 0 \).
Definition 7. Let \( \phi \) is a non-constant complex analytic function on \( \mathbb{D} \), for any given \( z_0 \in \mathbb{D} \), there exists \( \delta_{z_0} > 0 \) such that

\[
\phi(z) - \phi(z_0)\left|\| z - z_0 \| < \delta_{z_0} \right. = p_{n_{z_0}}(z)\left|\| z - z_0 \| < \delta_{z_0} \right.,
\]

\( h_{z_0}(z) \) is complex analytic on a neighbourhood of \( z_0 \) and \( h_{z_0}(z_0) \neq 0 \), \( p_{n_{z_0}}(z) \) is a \( n_{z_0} \)-th polynomial and the \( n_{z_0} \)-th coefficient is equivalent 1. By the Analytic Continuation Theorem [7]P28, we get that there is a unique complex analytic function \( h_{z_0}(z) \) on \( \mathbb{D} \) such that \( \phi(z) - \phi(z_0) = p_{n_{z_0}}(z)h_{z_0}(z) \), then define \( h_{z_0}(z) \) is a rooter function of \( \phi \) at the point \( z_0 \). If for any given \( z_0 \in \mathbb{D} \), the rooter function \( h_{z_0}(z) \) has non-zero point but the roots of \( p_{n_{z_0}}(z) \) are all in \( \mathbb{D} \) and \( n_{z_0} \in \mathbb{N} \) is a constant on \( \mathbb{D} \) that is equivalent \( m \), then define \( \phi \) is a \( m \)-folder complex analytic function on \( \mathbb{D} \).

Definition 8. Let \( \phi(z) \in \mathcal{H}^\infty(\mathbb{D}), n \in \mathbb{N}, M_\phi \) is the multiplication by \( \phi \) on \( \mathcal{H}^2(\mathbb{D}) \). If the adjoint multiplier \( M_\phi^* \in \mathcal{B}_n(\tilde{\phi}(\mathbb{D})) \), then define \( \phi \) is a Cowen-Douglas function.

By Definition 8 we get that any constant complex analytic function is not a Cowen-Douglas function.

Theorem 10. Let \( \phi(z) \in \mathcal{H}^\infty(\mathbb{D}) \) be a \( m \)-folder complex analytic function, \( M_\phi \) is the multiplication by \( \phi \) on \( \mathcal{H}^2(\mathbb{D}) \). If for any given \( z_0 \in \mathbb{D} \), the rooter functions of \( \phi \) at \( z_0 \) is an outer function, then \( \phi \) is a Cowen-Douglas function, that is, the adjoint multiplier \( M_\phi^* \in \mathcal{B}_m(\tilde{\phi}(\mathbb{D})) \).

Proof. By the definition of \( m \)-folder complex analytic function Definition 7 we get that \( \phi \) is not a constant complex analytic function. For any given
Let \( z \in \mathbb{D} \), let \( f_z \in \mathcal{H}^2(\mathbb{D}) \) be the reproducing kernel at \( z \). We confirm that \( M_\phi^* \) is valid the conditions of Definition 5 one by one.

1. For any given \( z \in \mathbb{D} \), \( f_z \) is an eigenvector of \( M_\phi^* \) with associated eigenvalue \( \lambda = \overline{\phi(z)} \).

   Because for any \( g \in \mathcal{H}^2(\mathbb{D}) \) we get
   \[
   \langle g, M_\phi^*(f_z) \rangle_{\mathcal{H}^2} = \langle \phi(g), f_z \rangle_{\mathcal{H}^2} = \phi(z)f(z) = \langle g, \overline{\phi(z)}f_z \rangle_{\mathcal{H}^2}.
   \]

   By the Riesz Representation Theorem [14]P13 of bounded linear functional in the form of inner product on Hilbert space, we get
   \[M_\phi^*(f_z) = \overline{\phi(z)}f_z = \lambda f_z,\]
   that is, \( f_z \) is an eigenvector of \( M_\phi^* \) with associated eigenvalue \( \lambda = \overline{\phi(z)} \).

2. For any given \( \overline{\lambda} \in \phi(\mathbb{D}) \), because of \( 0 \neq \phi \in \mathcal{H}^\infty(\mathbb{D}) \), we get that the multiplication operator \( M_\phi - \overline{\lambda} \) is injection by the properties of complex analysis, hence \( \ker(M_\phi - \lambda) = 0 \). Because of \( \mathcal{H}^2(\mathbb{D}) = \ker(M_\phi - \lambda)^\perp = \text{cl}[\text{ran}(M_\phi^* - \overline{\lambda})] \), we get that \( \text{ran}(M_\phi^* - \overline{\lambda}) \) is a second category space. By [14]P305 we get that \( M_\phi^* - \overline{\lambda} \) is a closed operator, also by [14]P93 or [31]P97 we get \( \text{ran}(M_\phi^* - \overline{\lambda}) = \mathcal{H}^2(\mathbb{D}) \).

3. Suppose that \( \text{span}\{f_z; z \in \phi(\mathbb{D})\} = \text{span}\{\frac{1}{1-z}; z \in \phi(\mathbb{D})\} \) is not dense in \( \mathcal{H}^2(\mathbb{D}) \), By the definition of reproducing kernel \( f_z \) and because of \( 0 \neq \phi \in \mathcal{H}^\infty(\mathbb{D}) \), we get that there exists \( 0 \neq g \in \mathcal{H}^2(\mathbb{D}) \), for any given \( z \in \mathbb{D} \), we have

   \[
   0 = \langle g, \phi(z)f_z \rangle_{\mathcal{H}^2} = \phi(z)g(z) = \langle \phi(z)g(z), f_z \rangle_{\mathcal{H}^2}.
   \]

   So we get \( g = 0 \) by the Analytic Continuation Theorem [7]P28, that is a
contradiction for \( g \neq 0 \). Therefore we get that span\{\( f_z; z \in \phi(\mathbb{D}) \}\} is dense in \( \mathcal{H}^2(\mathbb{D}) \), that is, \( \bigvee \ker_{\lambda \in \phi(\mathbb{D})}(M_{\phi} - \lambda) = \mathcal{H}^2(\mathbb{D}) \).

(4) By Definition 7 and the conditions of this theorem, for any given \( \lambda \in \phi(\mathbb{D}) \), there exists \( z_0 \in \mathbb{D} \), exists \( m \)-th polynomial \( p_m(z) \) and outer function \( h(z) \) such that

\[
\phi(z) - \lambda = \phi(z) - \phi(z_0) = p_m(z)h(z),
\]

We give \( \dim \ker(M_{\phi(z)}^* - \lambda) = m \) by the following (i)(ii)(iii) assertions.

(i) Let the roots of \( p_m(z) \) are \( z_0, z_1, \ldots, z_{m-1} \), then there exists decomposition \( p_m(z) = (z - z_0)(z - z_1)\cdots(z - z_{m-1}) \), and denote \( p_{m,z_0z_1\cdots z_{m-1}}(z) \) is the decomposition of \( p_m(z) \) by the permutation of \( z_0, z_1, \cdots, z_{m-1} \), the following to get \( \dim \ker M_{p_{m,z_0z_1\cdots z_{m-1}}}^* = m \).

By the Taylor expansions of functions in \( \mathcal{H}^2(\mathbb{T}) \), we get there is a naturally isomorphic

\[
F_s : \mathcal{H}^2(\mathbb{D}) \to \mathcal{H}^2(\mathbb{D} - s), F_s(g(z)) \to g(z + s), s \in \mathbb{C}.
\]

It is easy to get that \( G = \{F_s; s \in \mathbb{C}\} \) is a abelian group by the composite operation \( \circ \), hence for \( 0 \leq n \leq m - 1 \),there is

\[
\begin{array}{ccc}
\mathcal{H}^2((D)) & \xrightarrow{M_{z-z_n}} & \mathcal{H}^2((D)) \\
F_{zn} \downarrow & & \downarrow F_{zn} \\
\mathcal{H}^2(\mathbb{D} - z_n) & \xrightarrow{M_{z}'} & \mathcal{H}^2(\mathbb{D} - z_n)
\end{array}
\]

Let \( T \) is the backward shift operator on the Hilbert space \( \mathcal{L}^2(\mathbb{N}) \), that is, \( T(x_1, x_2, \cdots) = (x_2, x_3, \cdots) \). With the naturally isomorphic between \( \mathcal{H}^2(\mathbb{D} - \)}
$z_n$) and $H^2(\partial(\mathbb{D} - z_n))$, $M_z^*$ is equivalent the backward shift operator $T$ on $H^2(\partial(\mathbb{D} - z_n))$, that is, $M_z^*$ is a surjection and $\dim \ker M_z^* = 1$, hence $M_{z-z_n}^*$ is a surjection and $\dim \ker M_{z-z_n}^* = 1$, where $0 \leq n \leq m - 1$.

By the composition of $F_{z-m-1} \circ F_{z-m-2} \circ \cdots \circ F_{z_0}$, $M_p^* = T^m$ on $H^2(\partial(\mathbb{D} - z_n))$, that is, $M_p^*$ is a surjection and $\dim \ker M_p^* = m$, hence $M_{p,z_0z_1\cdots z_{m-1}}^*$ is a surjection and $\dim \ker M_{p,z_0z_1\cdots z_{m-1}}^* = m$.

(ii) Because $H^\infty$ is a abelian Banach algebra, $M_{p_m}$ is independent to the permutation of 1-th factors of $p_m(z)$, that is, $M_{p_m}^*$ is independent to the 1-th factors multiplication of $p_m(z) = (z - z_0)(z - z_1)\cdots(z - z_{m-1})$.

Because $G = \{F_s; s \in \mathbb{C}\}$ is a abelian group by composition operation $\circ$, for $0 \leq n \leq m - 1$, $F_{z-m-1} \circ F_{z-m-2} \circ \cdots \circ F_{z_0}$ is independent to the permutation of composition. Hence $M_{p_m}^*$ is a surjection and

$$\dim \ker M_{p_m}^* = \dim \ker M_{p_m,z_0z_1\cdots z_{m-1}}^* = m.$$ 

(iii) By Definition 6 and Theorem 9 also by [14]P93 or [31]P97 and by [14]P305 we get that the multiplication operator $M_h$ is surjection that associated with the outer function $h$. Hence we get

$$\ker M_{h(z)}^* = (\text{ran} M_{h(z)})^\perp = (H^2(\mathbb{D}))^\perp = 0.$$ 

Because there exists decomposition $M_{p_m(z)h(z)}^* = M_{h(z)}^* M_{p_m(z)}^*$ on $H^2(\mathbb{D})$, we get

$$\dim \ker (M_0^* - \bar{\lambda}) = \dim \ker M_{p_m(z)h(z)}^* = \dim \ker M_{p_m(z)}^* = m.$$
By (1)(2)(3)(4) we get the adjoint multiplier operator $M_\phi^* \in \mathcal{B}_m(\overline{\phi(\mathbb{D})})$.

By Theorem [10] and Lemma [6] we get

**Corollary 4.** Let $\phi \in \mathcal{H}^\infty(\mathbb{D})$ is a $m$-folder complex analytic function, for any given $z_0 \in \mathbb{D}$, if the rooter function of $\phi$ at $z_0$ is invertible in the Banach algebra $\mathcal{H}^\infty(\mathbb{D})$, then $\phi$ is a Cowen-Douglas function. Especially, for any given $n \in \mathbb{D}$, if $a$ and $b$ are both non-zero complex, then $a + b z^n \in \mathcal{H}^\infty(\mathbb{D})$ is a Cowen-Douglas function.

The following gives some properties about the adjoint multiplier of Cowen-Douglas functions.

**Theorem 11.** If $\phi \in \mathcal{H}^\infty(\mathbb{D})$ is a Cowen-Douglas function, $M_\phi$ is the multiplication by $\phi$ on $\mathcal{H}^2(\mathbb{D})$, Then the following assertions are equivalent

1. $M_\phi^*$ is Devaney chaotic;
2. $M_\phi^*$ is distributionally chaotic;
3. $M_\phi^*$ is strong mixing;
4. $M_\phi^*$ is Li-Yorke chaotic;
5. $M_\phi^*$ is hypercyclic;
6. $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.
Proof. By Example 1 we get that \( M^* \) is hypercyclic if and only if \( \phi \) is non-constant and \( \phi(D) \cap T \neq \emptyset \), hence (6) is equivalent to (5).

First to get that (6) imply (1)(2)(3)(4).

Because \( \phi \in H^2(D) \) is a Cowen-Douglas function, by Definition \( M^* \in B_n(\bar{\phi}(D)) \).

By Theorem \( \emptyset \) we get that if \( \phi(D) \cap T \neq \emptyset \), then (1)(2)(3) is valid. On Banach spaces Devaney chaotic, distributionally chaotic and strong mixing imply Li-Yorke chaotic, respectively. Hence (4) is valid. Because \( \bar{\phi}(D) \cap T \neq \emptyset \) and \( \phi(D) \cap T \neq \emptyset \) are mutually equivalent, (6) imply (1)(2)(3)(4).

Then to get that (1)(2)(3)(4) imply (6). By (1)(2)(3) imply (4), respectively, it is enough to get that (4) imply (6).

If \( M^*_\phi \) is Li-Yorke chaotic, then we get that \( \phi \) is non-constant and by Theorem 3.5 we get \( \sup_{n \to +\infty} \| M^{*n} \| \to \infty \), hence \( \| M^*_\phi \| = \| M^*_\psi \| > 1 \), that is, \( \sup_{z \in D} |\phi(z)| > 1 \). Moreover, we also have \( \inf_{z \in D} |\phi(z)| < 1 \). Indeed, if we assume that \( \inf_{z \in D} |\phi(z)| \geq 1 \) then \( \frac{1}{\phi} \in H^\infty \) and \( \| M^*_1 \| = \| M^*_\phi \| \leq 1 \). Hence for any \( 0 \neq x \in H^2(D) \) we get \( \| M^{*n}x \| \geq \frac{1}{\| M^*_\phi \|} \| x \| \geq \frac{1}{\| M^*_\psi \|} \| x \| \geq \| x \| \). It is a contradiction to \( M^*_\phi \) is Li-Yorke chaotic.

Therefore that \( M^*_\phi \) is Li-Yorke chaotic imply \( \inf_{z \in D} |\phi(z)| < 1 \). By the properties of a simple connectedness argument of complex analytic functions we get \( \phi(D) \cap T \neq \emptyset \). Hence we get (1)(2)(3)(4) both imply (6).

Corollary 5. If \( \phi \in H^\infty(D) \) is an invertible Cowen-Douglas function in the Banach algebra \( H^\infty(D) \), and let \( M_\phi \) be the multiplication by \( \phi \) on \( H^2(D) \).
Then \( M^*_\phi \) is Devaney chaotic or distributionally chaotic or strong mixing or Li-Yorke chaotic if and only if \( M^s_{\phi^{-1}} \) is.

\textbf{Proof.} Because of \( T = (T^{-1})^{-1} \), it is enough to prove that \( M^*_\phi \) is Devaney chaotic or distributionally chaotic or strong mixing or Li-Yorke chaotic imply \( M^s_{\phi^{-1}} \) is.

By Definition 8 we get \( M^*_\phi \in B_n(\overline{\phi(D)}) \), with a simple computing we get \( M^s_{\phi^{-1}} \in B_n(\frac{1}{\phi}(D)) \).

If \( M^*_\phi \) is Devaney chaotic or distributionally chaotic or strong mixing or Li-Yorke chaotic, by Theorem 11 we get \( \phi(D) \cap T \neq \emptyset \), and by the properties of complex analytic functions we get \( \frac{1}{\phi}(D) \cap T \neq \emptyset \).

Because of \( M^s_{\phi^{-1}} \in B_1(\frac{1}{\phi}(D)) \) and by Theorem 11 we get \( M^s_{\phi^{-1}} \) is Devaney chaotic or distributionally chaotic or strong mixing or Li-Yorke chaotic. \( \square \)

5. The chaos of scalars perturbation of an operator

We now study some properties about scalars perturbation of an operator inspired by [10] and [11] that research some properties about the compact perturbation of scalar operator.

\textbf{Definition 9.} Let \( \lambda \in \mathbb{C}, T \in \mathcal{L}(\mathbb{H}) \). Define

(i) Let \( S_{LY}(T) \) denote the set such that \( \lambda I + T \) is Li-Yorke chaotic for every \( \lambda \in S_{LY}(T) \).
(ii) Let $S_{DC}(T)$ denote the set such that $\lambda I + T$ is distributionally chaotic for every $\lambda \in S_{DC}(T)$.

(iii) Let $S_{DV}(T)$ denote the set such that $\lambda I + T$ is Devaney chaotic for every $\lambda \in S_{DV}(T)$.

(iv) Let $S_{H}(T)$ denote the set such that $\lambda I + T$ is hypercyclic for every $\lambda \in S_{H}(T)$.

By Definition 9 we get $S_{LY}(\lambda I + T) = \lambda + S_{LY}(T)$, $S_{DC}(\lambda I + T) = \lambda + S_{DC}(T)$, $S_{DV}(\lambda I + T) = \lambda + S_{DV}(T)$ and $S_{H}(\lambda I + T) = \lambda + S_{H}(T)$.

Lemma 7. Let $T \in \mathcal{L}(\mathbb{H})$ be a normal operator, then $S_{LY}(T) = \emptyset$.

**Proof.** Because $T$ is a normal operator, $\lambda I + T$ is a normal operator, too. by [10] we get $S_{LY}(T) = \emptyset$. □

Lemma 8. There is a quasinilpotent compact operator $T \in \mathcal{L}(\mathbb{H})$ such that $S_{LY}(T) = \mathbb{T}$ is closed and $S_{LY}(T^*) = \emptyset$, where $\mathbb{T} = \partial \mathbb{D}$, $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$.

**Proof.** Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{L}^2(\mathbb{N})$ and let $T$ be a weighted backward shift operator with weight sequence $\{\omega_n = \frac{1}{n}\}_{n=1}^{+\infty}$ such that $S_{\omega}(e_0) = 0$, $S_{\omega}(e_n) = \omega_n e_{n-1}$, where $0 < |\omega_n| < M < +\infty, \forall n > 0$.

By the Spectral Radius formula [14] $r_\sigma(T) = \lim_{n \to +\infty} ||T^n||^{\frac{1}{n}}$ we get $\sigma(T) = \{0\}$, hence $\sigma(\lambda I + T) = \lambda$.

(1) If $|\lambda| < 1$, we can select $\varepsilon > 0$ such that $|\lambda| + \varepsilon < q < 1$. Then
∀0 \neq x \in \mathbb{H} we get

\lim_{n \to \infty} \|(\lambda I + T)^n x\| \leq \lim_{n \to \infty} \|(\lambda I + T)^n\| \|x\| \leq \lim_{n \to \infty} (|\lambda| + \varepsilon)^n \|x\| = 0.

(2) If |\lambda| > 1, because of \sigma((\lambda I + T)^{-1}) = \frac{1}{\lambda}, then we get

\lim_{n \to \infty} \|(\lambda I + T)^n x\| \geq \lim_{n \to \infty} \frac{1}{\|(\lambda I + T)^{-n}\|} \|x\| \geq \lim_{n \to \infty} \frac{1}{\|(\lambda I + T)^{-1}\|^n} \|x\| \geq \|x\| \neq 0.

(3) By Theorem 3 we get that if |\lambda| = 1, then \lambda I + T is mixing. Mixing imply Li-Yorke chaotic.

(4) Because of \sigma(T) = \sigma(T^*), by (1)(2) we get that if |\lambda| \neq 1, then \lambda + T^* is not Li-Yorke chaotic.

(5) If |\lambda| = 1, from the view of infinite matrix \lambda I + T^* is lower triangular matrix, then with a simple computing, for any 0 \neq x \in L^2(\mathbb{N}), we get

\lim_{n \to \infty} \|\lambda I + S^*_n x\| > 0. Hence \lambda I + T^* is not Li-Yorke chaotic.

By (1)(2)(3)(4)(5) we get that \text{SLY}(T) = \mathbb{T} is closed and \text{SLY}(T^*) = \emptyset.

Lemma 9. Let T be the backward shift operator on \mathcal{L}^2(\mathbb{N}), T(x_1, x_2, \cdots) = (x_2, x_3, \cdots). Then \text{SLY}(T) = \text{SDC}(T) = \text{SDV}(T) = \text{SH}(T) = 2\mathbb{D} \setminus \{0\}, \text{SLY}(2T) = \text{SDC}(2T) = \text{SDV}(2T) = \text{SH}(2T) = 3\mathbb{D}, Hence \text{SLY}(T) and \text{SLY}(2T) are open sets.

Proof. By [14]P209 we get \sigma(T) = c1\mathbb{D} and \sigma(2T) = c2\mathbb{D}, by Definition \sigma(T) we get \sigma(\lambda I + T) = \lambda + c1\mathbb{D}. Because of the method to prove the conclusion is similarly for T and 2T, we only to prove the conclusion for T.
By the naturally isomorphic between $\mathcal{H}^2(\mathbb{T})$ and $\mathcal{H}^2(\mathbb{D})$. Let $\mathcal{L}^2(\mathbb{N}) = \mathcal{H}^2(\mathbb{T})$, by the definition of $T$ we get $(\lambda I + T)^*$ is the multiplication operator $M_f$ by $f(z) = \bar{\lambda} + z$ on the Hardy space $\mathcal{H}^2(\mathbb{T})$. By the Dirichlet Problem [7]P103 we get that $f(z)$ is associated with the complex analytic function $\phi(z) = \bar{\lambda} + z \in \mathcal{H}^\infty(\mathbb{D})$ determined by the boundary condition $\phi(z)|_{\mathbb{T}} = f(z)$.

By Corollary 4 we get that $\phi$ is a Cowen-Douglas function. Therefor by the natural isomorphic between $\mathcal{H}^2(\mathbb{T})$ and $\mathcal{H}^2(\mathbb{D})$, $\lambda I + T$ is naturally equivalent to the operator $M^*_\phi$ on $\mathcal{H}^2(\mathbb{D})$.

By Theorem 11 we get that $M^*_\phi$ is hypercyclic or Devaney chaotic or distributionally chaotic or Li-Yorke chaotic if and only if $\phi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

Because of $\sigma(\lambda I + T) = \sigma(\bar{\lambda} I + T^*)$, we get $\sigma(\lambda I + T) = \sigma(M^*_\phi) = \sigma(M^*_\phi) \supseteq \phi(\mathbb{D})$, hence $S_{LY}(T) = S_{DC}(T) = S_{DV}(T) = S_H(T) = 2\mathbb{D} \setminus \{0\}$ is an open set.

Therefor we can get

**Corollary 6.** Let $T$ be the backward shift operator on $\mathcal{L}^2(\mathbb{N})$, $T(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$. For $\lambda \neq 0, a \neq 0, n \in \mathbb{N}$, if $\lambda + aT^n$ is a invertible bounded linear operator, then $\lambda + aT^n$ is strong mixing or Devaney chaotic or distributionally chaotic or Li-Yorke chaotic if and only if $(\lambda + aT^n)^{-1}$ is.

**Theorem 12.** There is $T \in \mathcal{L}(\mathbb{H})$, $S_{LY}(T)$ is neither open nor closed.

**Proof.** Let $T_1, T_2 \in \mathcal{L}(\mathbb{H})$, because $T_1$ or $T_2$ is Li-Yorke chaotic if and only if $T_1 \oplus T_2$ is, we get $S_{LY}(T_1 \oplus T_2) = S_{LY}(T_1) \cup S_{LY}(T_2)$.
By Lemma 8, Lemma 9 and Definition 9 we get the conclusion.

6. Examples that $T$ is chaotic but $T^{-1}$ is not

In the last we give some examples to confirm the theory giving by functional calculus on the begin that $T$ is chaotic but $T^{-1}$ is not.

**Example 2.** Let $\{e_n\}_{n\in \mathbb{N}}$ be a orthonormal basis of $L^2(\mathbb{N})$ and let $S_\omega$ be a backward shift operator on $L^2(\mathbb{N})$ with weight sequence $\omega = \{\omega_n\}_{n\geq 1}$ such that $S_\omega(e_0) = 0$, $S_\omega(e_n) = \omega_n e_{n-1}$, where $0 < |\omega_n| < M < +\infty$, $\forall n > 0$.

1. If $|\lambda| = 1$, then $\lambda I + S_\omega$ is Li-Yorke chaotic, but $\lambda I + S_\omega^*$ and $(\lambda I + S_\omega^*)^{-1}$ are not Li-Yorke chaotic.

2. Let $(\lambda I + S_\omega)^n = U_n |(\lambda I + S_\omega)^n|$ is the polar decomposition of $(\lambda I + S_\omega)^n$, $\{U_n\}_{n=1}^\infty$ is not a constant sequence.

**Proof.**

1. By Theorem 3 we get that for $|\lambda| = 1, \lambda I + S_\omega$ is mixing and mixing imply Li-Yorke chaotic, hence $\lambda I + S_\omega$ is Li-Yorke chaotic. From the view of infinite matrix, $\lambda I + S_\omega^*$ and $(\lambda I + S_\omega^*)^{-1}$ are lower triangular matrix, with a simple computing, for any $0 \neq x \in L^2(\mathbb{N})$, we get $\lim_{n \to \infty} \|\lambda I + S_\omega^* x\| > 0$, and $\lim_{n \to \infty} \|(\lambda I + S_\omega^*)^{-1} x\| > 0$. Hence $\lambda I + S_\omega^*$ and $(\lambda I + S_\omega^*)^{-1}$ are not Li-Yorke chaotic.

2. Let $(\lambda I + S_\omega)^n = U_n |(\lambda I + S_\omega)^n|$ is the polar decomposition of $(\lambda I + S_\omega)^n$. If $\{U_n\}_{n=1}^\infty$ is a constant sequence, then by Theorem 4 we get that $(\lambda I + S_\omega^*)^{-1}$ is Li-Yorke chaotic. A contradiction. Hence $\{U_n\}_{n=1}^\infty$ is not a constant sequence. □
Theorem 13 ([10]). For any $\varepsilon > 0$, there is a small compact operator $K_\varepsilon \in \mathcal{L}(\mathbb{H})$ and $\|K_\varepsilon\| < \varepsilon$ such that $I + K_\varepsilon$ is distributionally chaotic.

In [10], $I + K_\varepsilon = \bigoplus_{j=1}^{+\infty} (I_j + K_j)$ is distributionally chaotic. where $\mathbb{H} = \bigoplus_{j=1}^{+\infty} \mathbb{H}_j$, $n_j = 2m_j, \mathbb{H}_j$ is the $n_j$-dimension subspace of $\mathbb{H}$. On $\mathbb{H}_j$ define:

$$S_j = \begin{bmatrix}
0 & 2\varepsilon_j \\
& \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
0 & & & & 0
\end{bmatrix}_{n_j \times n_j}, \quad K_j = \begin{bmatrix}
-\varepsilon_j & 2\varepsilon_j \\
& \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & & & -\varepsilon_j
\end{bmatrix}_{n_j \times n_j}
$$

$$I_j + K_j = \begin{bmatrix}
1 - \varepsilon_j & 2\varepsilon_j \\
& \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & & & 1 - \varepsilon_j
\end{bmatrix}_{n_j \times n_j} = (1 - \varepsilon_j)I_j + S_j.
$$

We can construct a invertible bounded linear operator $I + K_\varepsilon$ in the same way that is Li-Yorke chaotic, but $(I + K_\varepsilon)^{-1}$ is not.

Example 3. There is a invertible bounded linear operator $I + K_\varepsilon$ on $\mathbb{H} = \mathcal{L}^2(\mathbb{N})$ such that $I + K_\varepsilon$ is Li-Yorke chaotic, but $(I + K_\varepsilon)^{-1}, (I + K_\varepsilon)^{*^{-1}}$ and $(I + K_\varepsilon)^{*}$ are not Li-Yorke chaotic.

Proof. Let $\{e_i\}_{i=1}^{\infty}$ is a orthonormal basis of $\mathbb{H} = \mathcal{L}^2(\mathbb{N})$ and Let $\mathbb{H} = \bigoplus_{j=1}^{+\infty} \mathbb{H}_j$, $j \in \mathbb{N}$, where $\mathbb{H}_j = \text{span}\{e_i\}, 1 + \frac{i(i-1)}{2} \leq i \leq \frac{i(i+1)}{2}$, $\mathbb{H}_j$ is $j$-dimension subspace of $\mathbb{H}$. For any given positive sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ such that $\varepsilon_j \to 0$ and $\sup_{j \to \infty} (1 + \varepsilon_j)^j \to +\infty$, on $\mathbb{H}_j$ define:
First to prove that \((I + K_{\varepsilon})\) is Li-Yorke chaotic.

Let \( I + K_{\varepsilon} = \bigoplus_{j=1}^{+\infty} (I + K_j) \), \( f_j = \frac{1}{\sqrt{j}}(1, \cdots, 1, 0, \cdots) \) and \( f_{j,n} = \frac{1}{\sqrt{j}}(1, \cdots, 1_n, 0, \cdots, 0) \in \mathbb{H}_j \). For any \( 1 \leq n \leq j \) we get

\[
\| (I + K_{\varepsilon})^n(f_j) \| = \| (I_j + K_j)^n(f_j) \| = \| ((1 - \varepsilon_j)I_j + S_j)^n(f_j) \| = \| \sum_{k=0}^{n} C_n^k (1 - \varepsilon_j)^k S_{j}^{n-k} f_j \| \geq \| \sum_{k=0}^{n} C_n^k (1 - \varepsilon_j)^k (2\varepsilon_j)^{n-k} (1, \cdots, 1_n, 0, \cdots, 0) \| = (1 + \varepsilon_j)^n \| f_{j,n} \|. \]

Hence we get
\[(a) \lim_{j \to \infty} \|(I + K_\varepsilon)^j (f_j)\| \geq \lim_{n \to \infty} \|f_j\|(1 + \varepsilon_j)^j = +\infty.\]

Because of \(r_\sigma(I + K_\varepsilon) < 1\), we get

\[(b) \lim_{n \to \infty} \|(I + K_\varepsilon)^n (f_j)\| = 0.\]

By \((a)(b)\) and by Definition 3 we get that \(\lambda I + K_\varepsilon\) satisfies the Li-Yorke Chaos Criterion, by Theorem 1 we get that \(\lambda I + K_\varepsilon\) is Li-Yorke chaotic.

Then to prove that \((I + K_\varepsilon)^{-1}\) is not Li-Yorke chaotic. For convenience we define \(n^m\) by induction on \(m\) for any given \(n \in \mathbb{N}\).

For any given \(j \in \mathbb{N}\), define:

1. \(j^\equiv = 1 + 2 + \cdots + j;\)

2. If defined \(j^{\equiv n}\), then define \(j^{\equiv n+1} = 1^{\equiv n} + 2^{\equiv n} + \cdots + j^{\equiv n}\).

Let \(A = \bigoplus_{j=1}^{+\infty} A_j\), where \(A_j = (I_j + K_j)^{-1}\). Because of \(A_j(I_j + K_j) = A_j(I_j + K_j) = I_j\), we get \(A(I + K_\varepsilon) = (I + K_\varepsilon)A = \bigoplus_{j=1}^{+\infty} I_j = I\). By the Banach Inverse Mapping Theorem [14]P91 we get that \(A = (I + K_\varepsilon)^{-1}\) is a bounded linear operator, that is, \(A = (I + K_\varepsilon)^{-1}\). Hence we get

\[
A_j = \frac{1}{1 - \varepsilon_j} \begin{bmatrix}
1 & -2\varepsilon_j & (-2)^{j-1}\varepsilon_j^{j-1} \\
\vdots & \ddots & \ddots \\
& \ddots & -2\varepsilon_j \\
& & 1
\end{bmatrix}_{j \times j}
\]

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\[ A_j^2 = \frac{1}{(1 - \varepsilon_j)^2} \begin{bmatrix} 1 & 2 \cdot (-2)\varepsilon_j & j \cdot (2^{j-1}\varepsilon_j^{j-1}) \\ \vdots & \vdots & \vdots \\ \vdots & 2 \cdot (-2)\varepsilon_j & 1 \end{bmatrix}_{j \times j} \]

\[ A_j^3 = \frac{1}{(1 - \varepsilon_j)^3} \begin{bmatrix} 1 & (1 + 2)(-2)\varepsilon_j & j \cdot (2^{j-1}\varepsilon_j^{j-1}) \\ \vdots & \vdots & \vdots \\ \vdots & 2 \cdot (-2)\varepsilon_j & 1 \end{bmatrix}_{j \times j} \]

For \( m \geq 3, m \in \mathbb{N}, \) if defined

\[ A_j^m = \frac{1}{(1 - \varepsilon_j)^m} \begin{bmatrix} 1 & 2^{m-2}(-2)\varepsilon_j & j \cdot 2^{m-2}(2^{j-1}\varepsilon_j^{j-1}) \\ \vdots & \vdots & \vdots \\ \vdots & 2^{m-2}(-2)\varepsilon_j & 1 \end{bmatrix}_{j \times j} \]

Then define
$A_j^{(m+1)} = AA^m = \frac{1}{(1 - \varepsilon_j)^{(m+1)}} \begin{bmatrix} 1 & 2^{m-1}(-2)\varepsilon_j & j^{m-1}(-2)^{j-1}\varepsilon_j \quad \cdots \quad \cdots \quad j^{m-1}(-2)^{j-1}\varepsilon_j \quad \cdots \quad j^{m-1}(-2)\varepsilon_j & 1 \end{bmatrix}_{j \times j}$

For any given $0 \neq x_0 = (x_1, x_2, \cdots) \in \mathbb{H}$, Let

$y_j = (x_{(1+\frac{j(j-1)}{2})}, \cdots, x_{\frac{j(j+1)}{2}});$

$y'_j = (x_{(2+\frac{j(j-1)}{2})}, \cdots, x_{\frac{j(j+1)}{2}-1});$

$z_{j,m} = x_{(1+\frac{j(j-1)}{2})} + \sum_{k=2}^{j} k^{m-2}(-2)^{k-1}\varepsilon_k.$

Following a brilliant idea of Zermelo, we shall give the conclusion by induction.

(1) If $y_1 \neq 0$, then we get

$$\lim_{n \to \infty} \|A^n x_0\| = \lim_{n \to \infty} \sum_{j=1}^{+\infty} \|A^n y_j\| \geq \lim_{n \to \infty} \|A^n y_1\| = \lim_{n \to \infty} \frac{|x_1|}{(1-\varepsilon_1)^n} = +\infty.$$  

(2) If $y_1 = 0$, but $y_2 = (x_2, x_3) \neq 0$.

(i) If $x_3 \neq 0$, by (1) we get

$$\lim_{n \to \infty} \|A^n x_0\| \geq \lim_{n \to \infty} \frac{|x_3|}{(1-\varepsilon_1)^n} \to +\infty.$$  

(ii) If $x_3 = 0$ and $\varepsilon_2 > \frac{1}{2^{1/(n+2)}}$, because of $y_2 = (x_2, x_3) \neq 0$, we get

$$\lim_{n \to \infty} \|A^n x_0\| = \lim_{n \to \infty} \sum_{j=1}^{+\infty} \|A^n y_j\| \geq \lim_{n \to \infty} \|A^n y_2\|$$
\[
= \lim_{n \to \infty} \frac{1}{(1-\varepsilon_1)^n} \sqrt{(x_2^2 + (2^n - 2)(\varepsilon_2 x_3))} \geq \lim_{n \to \infty} \frac{|x_2|}{(1-\varepsilon_1)^n} = +\infty.
\]

(3) Assume for \( k \leq m - 1 \), there is \( \lim_{n \to \infty} \|A^n x_0\| \to +\infty \) for \( A^k \) and \( y_k \neq 0 \). Then for \( k = m \) and \( y_m \neq 0 \) we get

(i) If \( x_m \neq 0 \), by (1) we get

\[
\lim_{n \to \infty} \|A^n x_0\| \geq \lim_{n \to \infty} \frac{|x_m|}{(1-\varepsilon_1)^n} = +\infty.
\]

(ii) If \( x_m = 0 \) and \( \varepsilon_m > \frac{1}{m^{2(n-1)}} \), because of \( y_m \neq 0 \), we get

\[
\lim_{n \to \infty} \|A^n x_0\| = \lim_{n \to \infty} \sum_{j=1}^{+\infty} \|A^n x_j\| \geq \lim_{n \to \infty} \|A^m y_m\|
\]

\[
= \lim_{n \to \infty} \frac{1}{(1-\varepsilon_1)^n} \sqrt{z_{m,n}^2 + \|A^{n-1} y_m\|^2}.
\]

If \( y_m \neq 0 \), by the induction hypothesis we get

\[
\lim_{n \to \infty} \|A^n x_0\| \geq \lim_{n \to \infty} \|A^{n-1} y_m\| = +\infty;
\]

If \( y_m = 0 \), because of \( y_m \neq 0 \) we get \( x_{(1+\frac{m(m-1)}{2})} \neq 0 \). By (1) we get

\[
\lim_{n \to \infty} \|A^n x_0\| \geq \lim_{n \to \infty} \frac{|x_{(1+m(m-1))}|}{(1-\varepsilon_1)^n} = +\infty.
\]

Therefore for \( k = m \) and \( y_m \neq 0 \), we get \( \lim_{n \to \infty} \|A^n x_0\| \to +\infty \), by the induction we get that for any \( m \in \mathbb{N} \) and \( y_m \neq 0 \), there is \( \lim_{n \to \infty} \|A^n x_0\| \to +\infty \).

From (1)(2)(3) and \( \mathbb{H} = \bigoplus_{j=0}^{+\infty} \mathbb{H}_j \), we get that for any given \( 0 \neq x_0 = (x_1, x_2, \ldots, ) \in \mathbb{H} \) we can find \( m \in \mathbb{N} \) such that \( y_m \neq 0 \). Hence for any given \( 0 \neq x_0 = (x_1, x_2, \ldots, ) \in \mathbb{H} \) we get \( \lim_{n \to \infty} \|A^n x_0\| = +\infty \). Therefore \( (I + K_\varepsilon)^{-1} \) is not Li-Yorke chaotic. From the view of infinite matrix, \( (I + K_\varepsilon)^* \) and

\[
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\]
\((I + K_\varepsilon)^*\) are lower triangular matrix, for any 0 \(\neq x \in \mathbb{H}\) with a simple computing we get \(\lim_{n \to \infty} \|(I + K_\varepsilon)^*\varepsilon x\| > 0\), \(\lim_{n \to \infty} \|(I + K_\varepsilon)^* - \varepsilon x\| > 0\). Hence \((I + K_\varepsilon)^*\) and \((I + K_\varepsilon)^*-1\) are not Li-Yorke chaotic.

**Corollary 7.** There is a invertible bounded linear operator \(I + K_\varepsilon\) on \(\mathbb{H} = \mathcal{L}^2(\mathbb{N})\) such that \(I + K_\varepsilon\) is distributionally chaotic, but \((I + K_\varepsilon)^{-1}, (I + K_\varepsilon)^*\) and \((I + K_\varepsilon)^*\) are not distributionally chaotic.

**Proof.** By the construction of Theorem 13, it is only to give the conclusion by induction on \(\{k_i\}_{i=1}^\infty\) as Example 3.

**Theorem 14.** There is \(T \in \mathcal{L}(\mathbb{H})\), \(S_{LY}(T) = S_{DC}(T) = \omega\) is an open arc of \(\mathbb{T} = \{||\lambda| = 1; \lambda \in \mathbb{C}\}\), and for \(\forall\lambda \in \omega\), we get that \((\lambda + T)^*, (\lambda + T)^*\) and \((\lambda + T)^{-1}\) are not Li-Yorke chaotic.

**Proof.** As Example 3 give the same \(\mathbb{H} = \bigoplus_{j=1}^{+\infty} \mathbb{H}_j, S_j\) and \(K_j\), give positive sequence \(\{\varepsilon_j\}_{j=1}^\infty\) such that \(\varepsilon_j \to 0\) and \(\sup_{j \to \infty} |i + \varepsilon_j|^j \to +\infty\), where \(i \in \mathbb{C}\).

Let \(\lambda I + K_\varepsilon = \bigoplus_{j=1}^{+\infty} (\lambda I + K_j)\), so \(\sigma(\lambda I + K_\varepsilon) = \{\lambda - \varepsilon_j; j \in \mathbb{N}\}\).

(i) If \(|\lambda| < 1\), because of \(\varepsilon_j \to 0\), we get that \(\exists N > 0\) when \(n > N\), \(|\lambda - \varepsilon_j| < 1\). With the introduction of this paper we get that Li-Yorke chaos is valid only on infinite Hilbert space. Loss no generally, for any \(j \in \mathbb{N}\), let \(|\lambda - \varepsilon_j| < 1\), so \(r_{\sigma(\lambda I + K_\varepsilon)} < 1\). Hence for any \(0 \neq x \in \mathbb{H}\) there is \(\lim_{n \to \infty} \|(I + K_\varepsilon)^n(x)\| = 0\).

(ii) If \(|\lambda| > 1\) or \(\lambda \in [\frac{\pi}{2}, \frac{3\pi}{2}]\), because of \(\varepsilon_j > 0\), for \(j \in \mathbb{N}\) we get
\[|\lambda - \varepsilon_j| > 1, \quad \frac{1}{|\lambda - \varepsilon_j|} < 1, \text{ and } \sigma(\lambda I + K_\varepsilon)^{-1} = \{\frac{1}{\lambda - \varepsilon_j}; j \in \mathbb{N}\}. \]

By Example 3, for any given \(x \neq 0\) there is \(y_m \neq 0, m \in \mathbb{N}\), hence we get

\[
\lim_{n \to \infty} ||(\lambda I + K_\varepsilon)^n x_0|| \\
\geq \lim_{n \to \infty} ||(\lambda I_m + K_m)^n y_m|| \\
\geq \lim_{n \to \infty} \frac{1}{||(\lambda I_m + K_m)^{-n}||} ||y_m|| \\
\geq \lim_{n \to \infty} \frac{1}{||(\lambda I_m + K_m)^{-1}||^n} ||y_m|| \\
\geq ||y_m|| > 0.
\]

(iii) For \(\forall \lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})\), because of \(\varepsilon_j \to 0\), there exists \(N > 0\), when \(j > N\), we get \(|\lambda - \varepsilon_j| < 1\). Let \(\mathbb{H}' = \bigoplus_{j > N} \mathbb{H}_j, (\lambda I + K_\varepsilon)' = (\lambda I + K_\varepsilon)|_{\bigoplus_{j > N} \mathbb{H}_j}, \)
then for \(f_j = \frac{1}{\sqrt{j}}(1, \cdots, 1)\) and \(f_{j,n} = \frac{1}{\sqrt{j}}(1, \cdots, 1_n, 0, \cdots, 0) \in \mathbb{H}'\), we get that when \(1 \leq n \leq j\), there is

\[
||((\lambda I + K_\varepsilon)^n)(f_j)|| \\
= ||((\lambda I_j + K_j)^n)(f_j)|| \\
= ||((\lambda - \varepsilon_j)I_j + S_j)^n(f_j)|| \\
\geq ||\sum_{k=0}^{n} C_k^n (\lambda - \varepsilon_j)^k(2\varepsilon_j)^{n-k}(1, \cdots, 1_n, 0, \cdots, 0)|| \\
= |\lambda + \varepsilon_j|^n ||f_{j,n}||.
\]

By (i)(ii), if \(|\lambda| \neq 1 \text{ or } \lambda \in [\frac{\pi}{2}, \frac{3\pi}{2}]\), \(\lambda I + K_\varepsilon\) is not Li-Yorke chaotic.

By (iii) and by the property of the triangle, if \(\lambda \in (-\frac{\pi}{2}, \frac{\pi}{2})\) and \(j > N\), we get \(|\lambda + \varepsilon_j| > |i + \varepsilon_j|\) and \(r_\sigma((\lambda I + K_\varepsilon)') < 1\). Hence we get

\[
\lim_{n \to \infty} ||(\lambda I + K_\varepsilon)'^n(f_j)|| = 0, \text{ and} \\
\overline{\lim}_{j \to \infty} ||(I + K_\varepsilon)'^j(f_j)|| \geq \lim_{j \to \infty} ||f_j|||\lambda + \varepsilon_j|^j \geq \lim_{j \to \infty} |i + \varepsilon_j|^j = +\infty.
\]

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By Definition 3 we get that $\lambda I + K_\varepsilon$ satisfies the Li-Yorke Chaos Criterion, by Theorem 1 we get that $\lambda I + K_\varepsilon$ is Li-Yorke chaotic.

Using the same proof of Example 3 we get that $\left(\lambda I + K_\varepsilon\right)^*, \left(\lambda I + K_\varepsilon\right)^{-1}$ and $(\lambda I + K_\varepsilon)^{-1}$ are not Li-Yorke chaotic.

Using Corollary 7 and Theorem 13 we get that $\lambda I + K_\varepsilon$ is distributionally chaotic, but $(\lambda I + K_\varepsilon)^*, (\lambda I + K_\varepsilon)^{-1}$ and $(\lambda I + K_\varepsilon)^{-1}$ are not Li-Yorke chaotic.

Conjecture 1. For any given $m \in \mathbb{N}$, there exists $m$-folder complex analytic function $\phi(z) \in \mathcal{H}^\infty(D)$ such that $\phi$ is not a Cowen-Douglas function.

Question 1. Gives the equivalent characterization of a $m$-folder complex analytic function; Gives the equivalent characterization of a rooter function; Gives the equivalent characterization of a Cowen-Douglas function. Gives the relations between them.

Question 2. Let $M_\phi$ is the multiplication operator of the Cowen-Douglas function $\phi(z) \in \mathcal{H}^\infty(D)$ on the Hardy space $\mathcal{H}^2(D)$, then is $M_\phi^*$ a Lebesgue operator? If not and if they have relations, gives the relations between them.

References

[1] Bermdez, Bonilla, Martinez-Gimnez and Peiris. Li-Yorke and distributionally chaotic operators. J.Math.Anal.Appl.,(373)2011:1-83-93.

[2] B.Hou, P.Cui and Y.Cao. Chaos for Cowen-Douglas operators. Pro.Amer.Math.Soc,138(2010),926-936.
[3] C. Kitai. Invariant closed sets for linear operators. Ph.D. thesis, University of Toronto, Toronto, 1982.

[4] F. Bayart and E. Matheron. Dynamics of Linear Operators, Cambridge University Press, 2009.

[5] G. D. Birkhoff. Surface transformations and their dynamical applications. Acta Mathematica, 1922: 43-1-119.

[6] G. Godefroy. Renorming of Banach spaces. In Handbook of the Geometry of Banach Spaces. volume 1, pp. 781-835, North Holland, 2003.

[7] Henri Cartan (Translated by Yu Jiarong). Theorie elementaire des fonctions analytiques d’une ou plusieurs variables complexes. Higher Education Press, Peking, 2008.

[8] Hua Loo-kang. On the automorphisms of a field. Proc. Nat. Acad. Sci. U.S.A., 1949: 35-386-389.

[9] Hou B, Liao G, Cao Y. Dynamics of shift operators. Houston Journal of Mathematics, 2012: 38(4)-1225-1239.

[10] Hou Bingzhe, Tian Geng and Shi Luoyi. Some Dynamical Properties For Linear Operators. Illinois Journal of Mathematics, Fall 2009.

[11] Iwanik, A.. Independent sets of transitive points. Dynamical Systems and Ergodic Theory. vol. 23, Banach Center publications, 1989, pp 277-282.

[12] J. H. Shapiro. Notes on the dynamics of linear operators. Available at the author’s web page 2001.

[13] John Milnor. Dynamics in one complex variable. Third Edition. Princeton University Press, 2006.
[14] John B.Conway. A Course in Functional Analysis. Second Edition, Springer-Verlag New York, 1990.

[15] John B.Conway. A Course in Operator Theory. American Mathematical Society, 2000.

[16] John B.Garnett. Bounded Analytic Functions. Revised First Edition, Springer-Verlag, 2007.

[17] J.R.Brown. Ergodic Theory and Topological Dynamics. Academic Press, New York, 1976.

[18] J.W.Robbin. Topological Conjungacy and Structural Stability for discrete Dynamical Systems. Bulletin of the American Mathematical Society, Volume 78, 1972.

[19] K.-G.Grosse-Erdmann and A.Peris Manguillot. Linear Chaos. Springer, London, 2011.

[20] K.-G.Grosse-Erdmann. Universal families and hypercyclic vectors. Bull. Amer. Math. Soc., 36(3):345-381, 1999.

[21] Kenneth Hoffman. Banach Spaces of Analytic Functions. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.

[22] Li T.Y. and Yorke J.A.. Period three implies chaos. Amer.Math.Monthly 82(1975), 985C992.

[23] M.Shub. Global Stability of Dynamical Systems. Springer-Verlag New York, 1987.

[24] N.H.Kuiper and J.W.Robbin. Topological Classification of Linear Endomorphisms. Inventiones math.19,83-106, Springer-Verlag, 1973.
[25] Nilson C. Bernardes Jr., Antonio Bonilla, Vladimr Mller and A. peiris. Li-
Yorke chaos in linear dynamics. Academy of Sciences Czech Republic,
Preprint No. 22-2012.

[26] Paul R. Halmos. Measure Theory. Springer-Verlag New York, 1974.

[27] Peter Walters. An Introduction to Ergodic Theory. Springer-Verlag New
Yorke, 1982.

[28] Ronald G. Douglas. Banach Algebra Techniques in Operator Theory. Second
Edition. Springer-Verlag New York, 1998.

[29] S. Rolewicz. On orbits of elements. Studia Math., 1969: 32-17-22.

[30] William Arveson. A Short Course on Spectral Theory. Springer Scien-
tce+Businee Media, LLC, 2002.

[31] Zhang Gongqing, Lin Yuanqu. Lecture notes of functional analysis (Volume 1).
Peking University Press, Peking, 2006.