Abstract

The objective of this paper is to present some results about viscosity subsolutions of the contact Hamilton-Jacobi equations on connected, closed manifold $M$

$$H(x, \partial_x u, u) = 0, \quad x \in M.$$  

Based on implicit variational principles introduced in [12, 14], we focus on the monotonicity of the solution semigroups on viscosity subsolutions and the positive invariance of the epigraph for viscosity subsolutions. Besides, we show a similar consequence for strict viscosity subsolutions on $M$.

Keywords. Hamilton-Jacobi equations, viscosity subsolutions, weak KAM theory

1 Introduction

Suppose $M$ is a closed (i.e., compact, without boundary) connected and smooth manifold. Let $H(x, p, u): T^*M \times \mathbb{R} \to \mathbb{R}$ be a $C^3$ function satisfying:

(H1) Positive Definiteness: $\frac{\partial^2 H}{\partial p^2}(x, p, u)$ is positive definite on $T^*M \times \mathbb{R}$;

(H2) Superlinearity: For every $(x, u) \in M \times \mathbb{R}$, $\lim_{|p| \to +\infty} \frac{H(x, p, u)}{|p|} = +\infty$;

(H3) Lipschitz Continuity: There exists a constant $\lambda > 0$ such that $\left| \frac{\partial H}{\partial u}(x, p, u) \right| \leq \lambda$ for any $(x, p, u) \in T^*M \times \mathbb{R}$;

We consider the viscosity solutions of the following first-order partial differential equation

$$H(x, \partial_x u, u) = 0, \quad x \in M.$$  

(HJ)}
Let $J^1(M, \mathbb{R})$ denote the manifold of 1-jet of functions on $M$. The standard contact form on $J^1(M, \mathbb{R})$ is the 1-form $\alpha = du - pdx$. Every $C^2$ function $H(x, u, p)$ determinates a unique vector field $X_H$ defined by the conditions

$$L_{X_H} \alpha = -\frac{\partial H}{\partial u}(x), \quad \alpha(X_H) = -H,$$

where $L_{X_H}$ denotes the Lie derivative along the contact vector field $X_H$. In Darboux coordinates, the contact vector field $X_H$ generated by $H$ is formulated by:

$$X_H : \begin{cases} \dot{x} = \frac{\partial H}{\partial p}(x, p, u), \\ \dot{p} = -\frac{\partial H}{\partial x}(x, p, u) - \frac{\partial H}{\partial u}(x, p, u) \cdot p, \\ \dot{u} = \frac{\partial H}{\partial p}(x, p, u) \cdot p - H(x, p, u). \end{cases} \quad (1.1)$$

$H$ is called a contact Hamiltonian, and $X_H$ is called a contact Hamiltonian vector field. Due to the fundamental theorems for ordinary differential equations, for each $(x, p, u) \in T^*M \times \mathbb{R}$, there exists a unique integral curve of $(1.1)$ through the point, which is denoted by $\Phi^t_H(x, p, u)$. In this paper, we need an extra condition on $\Phi^t_H$.

**(SH1)** Completeness: Every maximal integral curve of the contact Hamiltonian flow $\Phi^t_H$ has all of $\mathbb{R}$ as its domain of definition.

**Remark 1.1.** In Appendix A, we can give a common sufficient condition (A.1) to ensure (SH1) holds.

Actually, it is common to find non-trivial examples satisfying all of our assumptions. For instance,

**Example 1.2.** Consider the $C^3$-smooth Hamiltonian $H : T^*M \times \mathbb{R} \to \mathbb{R}$ defined by

$$H(x, p, u) = \frac{1}{2}(A(x)p, p) + V(x, u),$$

where $A(x)$ is positive definite and $V(x, u) : M \times \mathbb{R} \to \mathbb{R}$ satisfies $|\frac{\partial V}{\partial u}(x, u)| \leq \lambda$ for any $(x, u) \in M \times \mathbb{R}$.

In this paper, we assume that $H(x, p, u)$ satisfies (H1)-(H3) and (SH1). Let $L(x, \dot{x}, u)$ be the Legendre transform of $H(x, p, u)$, i.e.,

$$L(x, \dot{x}, u) = \sup_{p \in T^*_x M} \{p \cdot \dot{x} - H(x, p, u)\}.$$

Let us recall two semigroups of operators introduced in [13]. Define a family of nonlinear operators $\{T^-_t\}_{t \geq 0}$ from $C(M, \mathbb{R})$ to itself as follows. For each $\varphi \in C(M, \mathbb{R})$, we denote the unique continuous function on $(x, t) \in M \times [0, +\infty)$ by $(x, t) \mapsto T^-_t \varphi(x)$ satisfying that

$$T^-_t \varphi(x) = \inf_{\gamma} \left\{ \varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), T^-_\tau \varphi(\gamma(\tau))) d\tau \right\}, \quad (1.2)$$

where the infimum is taken among the absolutely continuous curves $\gamma : [0, t] \to M$ with $\gamma(t) = x$. It was also proved in [13] that $\{T^-_t\}_{t \geq 0}$ is a semigroup of operators and the function $(x, t) \mapsto T^-_t \varphi(x)$ is a viscosity solution of the evolutionary first-order partial differential equation

$$\begin{cases} \partial_t u + H(x, \partial_x u, u) = 0, & (x, t) \in M \times (0, +\infty), \\ u(x, 0) = \varphi(x), & x \in M, \end{cases} \quad (H_{1, \varphi})$$
We call \( \{T_t^-\}_{t \geq 0} \) the backward solution semigroup.

Similarly, one can define another semigroup of operators \( \{T_t^+\}_{t \geq 0} \), called the forward solution semigroup, by
\[
T_t^+ \varphi(x) = \sup_{\gamma} \left\{ \varphi(\gamma(t)) - \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), T_{\tau-t}^- \varphi(\gamma(\tau))) d\tau \right\},
\]
where the supremum is taken among the absolutely continuous curves \( \gamma : [0, t] \to M \) with \( \gamma(0) = x \).

It is well known that \( (HJ) \) admits a viscosity solution through Perron’s method following Ishii\cite{Ishii}, and the fundamental idea of the proof is to find a special subset of viscosity subsolutions. Actually, viscosity subsolutions decide most of the properties of viscosity solutions, and we wish to clarify the properties of the fundamental idea of the proof is to find a special subset of viscosity subsolutions. Actually, viscosity

\[ X \]
under the contact Hamiltonian flow
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to be specific, we can show that the viscosity subsolution is completely decided by a positive invariant set

\[ H \]
3

Weak KAM solutions of Hamilton-Jacobi equations

Theorem A. Suppose that \( \varphi(x) \in C(M, \mathbb{R}) \), the following statements are equivalent:

1. \( \varphi(x) \) is a viscosity subsolution of equation \((HJ)\), i.e. \( H(x, p, \varphi(x)) \leq 0 \) for any \( x \in M, p \in D^+ \varphi(x) \).
2. \( T_t^- \varphi(x) \geq \varphi(x) \), for any \( t \geq 0, x \in M \);
3. \( T_t^+ \varphi(x) \leq \varphi(x) \) for any \( t \geq 0, x \in M \);
4. \( \Phi_H^t(\Gamma_\varphi) \subset \Gamma_\varphi \) for any \( t \geq 0 \), where \( \Gamma_\varphi := \{(x, p, u), u \geq \varphi(x)\} \).

In \cite{Ishii}, a function \( \varphi(x) \in C(M, \mathbb{R}) \) is called a viscosity subsolution of \((HJ)\) is strict at \( x_0 \in M \) if there exists an open neighborhood \( V_{x_0} \) of \( x_0 \), and \( c_{x_0} < 0 \) such that \( u|V_{x_0} \) is a viscosity subsolution of \( H(x, dx_u) = c_{x_0} \) on \( V_{x_0} \). We define that \( \varphi(x) \) is a strict viscosity subsolution of \((HJ)\) on \( M \) if \( \varphi(x) \) is a viscosity subsolution of \((HJ)\) which is strict at each \( x \in M \). In Theorem B, we give the necessary and sufficient conditions for the strict viscosity subsolution.

Theorem B. Suppose that \( \varphi(x) \in C(M, \mathbb{R}) \), the following statements are equivalent:

1. \( \varphi(x) \) is a strict viscosity subsolution of equation \((HJ)\) on \( M \).
2. \( T_t^- \varphi(x) > \varphi(x) \) for any \( t \geq 0, x \in M \) and there exists \( c > 0 \) such that
\[
\lim_{t \to 0^+} \frac{1}{t} \left( T_t^- \varphi(x) - \varphi(x) \right) \geq c, \quad \forall x \in M \tag{1.3}
\]
3. \( T_t^+ \varphi(x) < \varphi(x) \) for any \( t \geq 0, x \in M \) and there exists \( c > 0 \) such that
\[
\lim_{t \to 0^+} \frac{1}{t} \left( T_t^+ \varphi(x) - \varphi(x) \right) \leq -c, \quad \forall x \in M \tag{1.4}
\]

Corollary C. \( T_t^- \varphi(x) > \varphi(x) \) for any \( t \geq 0, x \in M \) if and only if \( \Phi_H^t(\Gamma_\varphi) \subset \Gamma_\varphi \) for any \( t > 0 \).

The rest of this paper is organized as follows. We give some preliminaries and prove Theorem A in Section 2. The proof of Theorem B and Corollary C are given in Section 3.
2 Proof of theorem A

In this section, we first recall the definition of the viscosity solution of equation (HJt) and implicit action functions and some properties of them. Then we give the proof of theorem A.

2.1 Preliminaries

Definition 2.1 (Viscosity solutions of equation (HJt)). Let U be an open subset $U \subset M$.

(1) A function $u : U \rightarrow \mathbb{R}$ is called a viscosity subsolution of equation (HJt), if for every $C^1$ function $\phi : U \rightarrow \mathbb{R}$ and every point $x_0 \in U$ such that $u - \phi$ has a local maximum at $x_0$, we have $H(x_0, d_x \phi(x_0), u(x_0)) \leq c$;

(2) A function $u : U \rightarrow \mathbb{R}$ is called a viscosity supersolution of equation (HJt), if for every $C^1$ function $\psi : U \rightarrow \mathbb{R}$ and every point $y_0 \in U$ such that $u - \psi$ has a local minimum at $y_0$, we have $H(y_0, d_x \psi(y_0), u(y_0)) \geq c$;

(3) A function $u : U \rightarrow \mathbb{R}$ is called a viscosity solution of equation (HJt), if it is both a viscosity subsolution and a viscosity supersolution.

We recall that for any $x \in M$ and continuous function $u$, the closed convex sets

$$D^- u(x) = \left\{ p \in T_x^* M : \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\},$$

$$D^+ u(x) = \left\{ p \in T_x^* M : \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\},$$

are called the subdifferential and superdifferential of $u$ at $x$, respectively, see [2, 3, 6, 8] for more details on this notion and its relationship with viscosity solutions. For instance, it is proved in [6, Prop 3.1.7] that $p \in D^+ u(x)$ if and only if $p = D \phi(x)$ for some $C^1$ function $\phi$ such that $u - \phi$ attains a local maximum at $x$. Thus, we can present the other notion of the viscosity subsolution(supersolution) as

$$H(x, p, u) \leq 0 \quad (\geq 0), \quad \forall p \in D^+ u(x) \quad (D^- u(x)).$$

Let us recall two implicit action functions introduced in [12] [14] which can give the representation formulae for $T^-_t$ and $T^+_t$ by [14] Proposition 4.1:

$$T^-_t \varphi(x) = \inf_{y \in M} h_{y, \varphi}(x, t), \quad T^+_t \varphi(x) = \sup_{y \in M} h_{y, \varphi}(x, t), \quad (x, t) \in M \times (0, +\infty), \quad (2.1)$$

where the continuous functions

$$h_{x,0,u_0}(x, t) : M \times \mathbb{R} \times M \times (0, +\infty) \rightarrow \mathbb{R}, \quad h^{x_0, u_0}(x, t) : M \times \mathbb{R} \times M \times (0, +\infty) \rightarrow \mathbb{R}$$

$$(x_0, u_0, x, t) \mapsto h_{x,0,u_0}(x, t) \quad \quad (x_0, u_0, x, t) \mapsto h^{x_0, u_0}(x, t)$$

were introduced in Proposition 2.2 and 2.3 called forward and backward implicit action functions respectively. There are various properties of implicit action functions in [12] [13] [14].

Proposition 2.2. [12] Properties of backward implicit action function $h_{x,0,u_0}(x, t)$:
Weak KAM solutions of Hamilton-Jacobi equations

(1) (Minimality) Given \( x_0, x \in M \) and \( u_0 \in \mathbb{R} \) and \( t > 0 \), let \( S^{x,t}_{x_0,u_0} \) be the set of the solutions \( (x(s), p(s), u(s)) \) of (1.1) on \([0, t]\) with \( x(0) = x_0, x(t) = x, u(0) = u_0 \). Then

\[
h_{x_0,u_0}(x, t) := \inf \{ u(t) : (x(s), p(s), u(s)) \in S^{x,t}_{x_0,u_0} \},
\]

for any \((x, t) \in M \times (0, +\infty)\).

(2) (Monotonicity) Given \( x_0 \in M, u_1 < u_2 \in \mathbb{R} \), we have \( h_{x_0,u_1}(x, t) < h_{x_0,u_2}(x, t) \) for any \( t > 0, x \in M \).

(3) (Lipschitz continuity) The function \((x_0, u_0, x, t) \mapsto h_{x_0,u_0}(x, t)\) is locally Lipschitz continuous on \( M \times \mathbb{R} \times M \times (0, +\infty) \).

(4) (Implicit variational) For any given \( x_0 \in M \) and \( u_0 \in \mathbb{R} \),

\[
h_{x_0,u_0}(x, t) = u_0 + \sup \left\{ \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), h_{x_0,u_0}(\gamma(\tau), \tau)) \, d\tau \right\},
\]

where the supremum is taken among the Lipschitz continuous curves \( \gamma : [0, t] \to M \) and can be achieved.

Proposition 2.3. \([14]\) Properties of forward implicit action function \( h^{x_0,u_0}(x, t) \):

(1) (Maximality) Given \( x_0, x \in M \) and \( u_0 \in \mathbb{R} \) and \( t > 0 \), let \( S^{x,t}_{x_0,u_0} \) be the set of the solutions \( (x(s), p(s), u(s)) \) of (1.1) on \([0, t]\) with \( x(0) = x, x(t) = x_0, u(t) = u_0 \). Then

\[
h^{x_0,u_0}(x, t) := \sup \{ u(0) : (x(s), p(s), u(s)) \in S^{x,t}_{x_0,u_0} \},
\]

for any \((x, t) \in M \times (0, +\infty)\).

(2) (Monotonicity) Given \( x_0 \in M, u_1 < u_2 \in \mathbb{R} \), we have \( h^{x_0,u_1}(x, t) < h^{x_0,u_2}(x, t) \) for any \( t > 0, x \in M \).

(3) (Lipschitz continuity) The function \((x_0, u_0, x, t) \mapsto h^{x_0,u_0}(x, t)\) is locally Lipschitz continuous on \( M \times \mathbb{R} \times M \times (0, +\infty) \).

(4) (Implicit variational) For any given \( x_0 \in M \) and \( u_0 \in \mathbb{R} \),

\[
h^{x_0,u_0}(x, t) = u_0 - \inf_{\gamma(0)=x} \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), h^{x_0,u_0}(\gamma(\tau), t - \tau)) \, d\tau,
\]

where the infimum is taken among the Lipschitz continuous curves \( \gamma : [0, t] \to M \) and can be achieved.

The relation between \( h_{x_0,u_0}(x, t) \) and \( h^{x_0,u_0}(x, t) \) was shown as follows:

Proposition 2.4. \([14]\) Prop 3.5] (Equivalence) \( h_{x_0,u_0}(x, t) = u \iff h^{x_0,u_0}(x_0, t) = u_0 \).
2.2 Proof of Theorem A

Before turning to the proof of the main Theorem A, we will need one more preliminary.

**Lemma 2.5.** Given \( \varphi \in C(M, \mathbb{R}) \), we have

\[
\Phi^t_H(\Gamma_\varphi) \subset \Gamma_{T^-_t \varphi}, \quad t \geq 0.
\]

**Proof.** For any \((x_1, p_1, u_1) \in \Phi^t_H(\Gamma_\varphi)\), we assume that:

\[
(x_1, p_1, u_1) = \Phi^t_H(x_0, p_0, u_0), \quad u_0 \geq \varphi(x_0)
\]

Thus, by (2.1) and Proposition 2.2, it have

\[
T^-_t \varphi(x_1) = \inf_{y \in M} h_{y, \varphi(y)}(x_1, t) \leq h_{x_0, \varphi(x_0)}(x_1, t) \leq h_{x_0, u_0}(x_1, t)
\]

\[
= \inf \{ u(t) : (x(s), u(s), p(s)) \in S_{x_0, u_0} \} \leq u_1.
\]

It follows that \( \Phi^t_H(\Gamma_\varphi) \subset \Gamma_{T^-_t \varphi} \) for any \( t \geq 0 \).

\[ \square \]

We get back to the proof of Theorem A and assume that \( \varphi(x) \) is Lipschitz continuous at first.

**Proof of Theorem A:** (1) \( \Rightarrow \) (2): We claim that: If \( \varphi(x) \) is a viscosity subsolution of (HJ), then for any piecewise \( C^1 \) curve \( \gamma : [a, b] \to M \),

\[
\varphi(\gamma(b)) - \varphi(\gamma(a)) \leq \int_a^b L(\gamma(t), \dot{\gamma}(t), \varphi(\gamma(t))) dt,
\]

If \( \varphi(x) \) is differentiable at \( x_0 \), then \( H(x_0, \partial_x \varphi(x_0), \varphi(x_0)) \leq 0 \). Following the method mentioned in [7] Prop 4.2.3, we can always choose a sequence of piecewise \( C^1 \) curves \( \gamma_n : [a, b] \to M \), such that \( \varphi \) is differentiable on \( \gamma_n(t) \) for almost every \( t \in [a, b] \), \( \gamma_n(a) = \gamma(a), \gamma_n(b) = \gamma(b) \) and \( \gamma_n \) converges in the \( C^1 \) topology to \( \gamma \), then we obtain

\[
\varphi(\gamma(b)) - \varphi(\gamma(a)) = \lim_{n \to \infty} \int_a^b \frac{d\varphi(\gamma_n(t))}{dt} dt = \lim_{n \to \infty} \int_a^b \langle \partial_x \varphi(\gamma_n(t)), \dot{\gamma}_n(t) \rangle dt
\]

\[
\leq \lim_{n \to \infty} \int_a^b L(\gamma_n(t), \dot{\gamma}_n(t), \varphi(\gamma_n(t))) + H(\gamma_n(t), \dot{\gamma}_n(t), \varphi(\gamma_n(t))) dt
\]

\[
\leq \lim_{n \to \infty} \int_a^b L(\gamma_n(t), \dot{\gamma}_n(t), \varphi(\gamma_n(t))) dt
\]

\[
= \int_a^b L(\gamma(t), \dot{\gamma}(t), \varphi(\gamma(t))) dt,
\]

which completes our claim.

Then we want to show \( T^-_t \varphi \geq \varphi \) for any \( t \geq 0 \). By contradiction, we assume that there exists \( x_0 \in M \) and \( t > 0 \) such that \( T^-_t \varphi(x_0) < \varphi(x_0) \).

Let \( \gamma : [0, t] \to M \) be a minimizer of (1.2) with \( \gamma(t) = x_0 \), i.e.

\[
T^-_t \varphi(x_0) = \varphi(\gamma(0)) + \int_0^t L(\gamma(s), \dot{\gamma}(s), T^-_s \varphi(\gamma(s))) ds.
\]
Let $G(s) = \varphi(\gamma(s)) - T_s^- \varphi(\gamma(s))$, since $G(0) = 0$ and $G(t) > 0$, then there exists $0 \leq t_0 < t$ such that $G(t_0) = 0$ and $G(s) > 0, \forall s \in (t_0, t]$.

$$G(s) = \varphi(\gamma(s)) - T_s^- \varphi(\gamma(s)) = \varphi(\gamma(s)) - \varphi(\gamma(t_0)) - \int_{t_0}^{s} L(\gamma(\tau), \dot{\gamma}(\tau), T_{\tau}^- \varphi(\gamma(\tau))) \, d\tau$$

$$\leq \int_{t_0}^{s} L(\gamma(\tau), \dot{\gamma}(\tau), \varphi(\gamma(\tau))) - L(\gamma(\tau), \dot{\gamma}(\tau), T_{\tau}^- \varphi(\gamma(\tau))) \, d\tau \leq \lambda \int_{t_0}^{s} G(\tau) \, d\tau,$$

where $\lambda$ is the Lipschitz constant of $L$ with respect to $u$. From Gronwall inequality, it follows that $G(s) \leq 0$ for any $s \in [t_0, t]$. It derives a contradiction, which follows that $T_t^- \varphi \not\geq \varphi$ for any $t \geq 0$.

(2) $\Rightarrow$ (1): Since $T_t^- \varphi \geq \varphi$ for any $t \geq 0$, by (2.1), we have

$$T_t^- \varphi(y) = \inf_{x \in M} h_{x,\varphi(x)}(y, t) \geq \varphi(y).$$

For any fixed $x \in M$ and any $C^1$ curve $\xi : [0, t] \to M$ with $\xi(0) = x$ and $\xi(t) = y$, due to (2.3), we have

$$\varphi(y) \leq h_{x,\varphi(x)}(y, t) = \varphi(x) + \inf_{\gamma(0)=x} \int_{t_0}^{t} L(\gamma(\tau), \dot{\gamma}(\tau), h_{x,\varphi(x)}(\gamma(\tau), \tau)) \, d\tau,$$

$$\leq \varphi(x) + \int_{0}^{t} L(\xi(\tau), \dot{\xi}(\tau), h_{x,\varphi(x)}(\xi(\tau), \tau)) \, d\tau.$$

(2.5)

For any differentiable point $x \in M$ of $\varphi$,

$$\lim_{t \to 0^+} \frac{1}{t} [\varphi(\xi(t)) - \varphi(\xi(0))] \leq \lim_{t \to 0^+} \frac{1}{t} \int_{0}^{t} L(\xi(s), \dot{\xi}(s), h_{x,\varphi(x)}(\xi(s), s)) \, ds,$$

which leads to

$$\langle \partial_{\varphi} \varphi(x), \dot{\xi}(0) \rangle \leq L(x, \dot{\xi}(0), h_{x,\varphi(x)}(\xi(0), 0)) = L(x, \dot{\xi}(0), \varphi(x)).$$

By taking $\dot{\xi}(0) = \partial_p H(x, \varphi(x), \partial_{\varphi} \varphi(x))$, we get

$$L(x, \dot{\xi}(0), \varphi(x)) + H(x, \partial_{\varphi} \varphi(x), \varphi(x)) = \langle \partial_{\varphi} \varphi(x), \dot{\xi}(0) \rangle,$$

which implies $H(x, \partial_{\varphi} \varphi(x), \varphi(x)) \leq 0$.

Since $\varphi$ is Lipschitz on $M$, $\varphi$ is differentiable almost everywhere, then $\varphi(x) : M \to \mathbb{R}$ is an almost everywhere subsolution. As a result, it has to be a viscosity subsolution of $\{H_{\varphi}\}$, where the equivalence between almost everywhere subsolutions and viscosity subsolutions was proved in bunch of references [2] [3] [11].

(2) $\Rightarrow$ (4): From Lemma [2.5], we have $\Phi_H^t(\Gamma_\varphi) \subset \Gamma_{T_t^- \varphi}$ for any $t \geq 0$. Due to (2), it get $\Gamma_{T_t^- \varphi} \subset \Gamma_\varphi$ for any $t \geq 0$. Hence, $\Phi_H^t(\Gamma_\varphi) \subset \Gamma_\varphi$ for any $t \geq 0$.

(4) $\Rightarrow$ (2): For any given $x \in M$ and $t \geq 0$, by Lemma [2.5] there exists $p \in T_x M$ such that

$$(x, p, u) \in \Phi_H^t(\Gamma_\varphi), \text{ and } u = T_t^- \varphi(x).$$
Since \((x, p, u) \in \Phi_\mathcal{H}^s(\Gamma_\varphi) \subset \Gamma_\varphi\), it follows that \(T_t^\varphi(x) = u \geq \varphi(x)\) for any \(t \geq 0, x \in M\).

(2) \(\Rightarrow\) (3): Given \(t > 0\), due to (2.1), it gets that

\[
T_t^\varphi(y) = \inf_{x \in M} h_{x, \varphi(x)}(y, t) \geq \varphi(y) \quad \forall y \in M,
\]

which implies that \(h_{x, \varphi(x)}(y, t) \geq \varphi(y)\) for any \(x, y \in M\).

Let \(u = h_{x, \varphi(x)}(y, t)\), and by Proposition 2.4, if follows that \(h^{\gamma, s}(x, t) = \varphi(x)\). From (2) of Proposition 2.3, we have \(h^{\gamma, s}(y, x, t) \leq \varphi(x)\) for any \(x, y \in M\). Thus, by (2.1), it follows that \(T_t^\gamma(x) = \sup_{y \in M} h^{\gamma, \varphi(y)}(x, t) \leq \varphi(x)\) for any \(x \in M\).

(3) \(\Rightarrow\) (2): The proof is similar to the process above, and we omit it here. \(\square\)

**Remark 2.6.** Recall that any viscosity subsolution of \((\mathcal{H}_s)\) has to be Lipschitz continuous by [1] Lemma 2.2, and it is easy to verify that \(\varphi(x)\) is Lipschitz if it satisfies (2). As the statement above, it is sufficient to assume that \(\varphi(x)\) is continuous when we adapt it into Theorem A.

### 3 Proof of Theorem B and Corollary C

**Proof of Theorem B:** (1) \(\Rightarrow\) (2): By Remark 2.6, \(\varphi(x) \in C(M)\) is a Lipschitz strict viscosity subsolution of equation \((\mathcal{H}_s)\) and then for any \(x_0 \in M\), there exists an open neighborhood \(V_{x_0}\) of \(x_0\), and \(c_{x_0} < 0\) such that \(u|V_{x_0}\) is a viscosity subsolution of \(H(x, d_x u) = c_{x_0}\) on \(V_{x_0}\).

Due to \(M\) is compact, every covering of \(M\) contains a finite subcollection covering, then there exists \(c > 0\) such that \(\varphi(x) \in C(M)\) is a Lipschitz strict viscosity subsolution of \((\mathcal{H}_s)\) with \(H + c\).

Similar with Theorem A, we can choose a sequence of piecewise \(C^1\) curves \(\gamma_n : [a, b] \to M\), such that \(\varphi\) is differentiable on \(\gamma_n(t)\) on \([a, b]\), and \(\gamma_n\) converges in the \(C^1\) topology to \(\gamma\), then we obtain

\[
\varphi(\gamma(b)) - \varphi(\gamma(a)) = \lim_{n \to +\infty} \varphi(\gamma_n(b)) - \varphi(\gamma_n(a)) = \lim_{n \to +\infty} \int_a^b \langle d_x \varphi(\gamma_n(t)), \gamma_n(t) \rangle dt
\]

\[
\leq \lim_{n \to +\infty} \int_a^b L(\gamma_n(t), \dot{\gamma}_n(t), \varphi(\gamma_n(t))) + H(\gamma_n(t), \dot{\gamma}_n(t), \varphi(\gamma_n(t))) dt
\]

\[
\leq \lim_{n \to +\infty} \int_a^b L(\gamma_n(t), \dot{\gamma}_n(t), \varphi(\gamma_n(t))) - c dt
\]

\[
= \int_a^b L(\gamma(t), \dot{\gamma}(t), \varphi(\gamma(t))) - c dt.
\]

By Theorem A, \(\varphi(x)\) is a viscosity sub solution of equation \((\mathcal{H}_s)\) and \(T_t^\varphi \geq \varphi\) for any \(t > 0\). We claim that

\[
T_t^\varphi(x) \geq \varphi(x) + c \cdot \frac{1 - e^{-\lambda t}}{\lambda} \quad \forall x \in M, t \in [0, +\infty).
\]

(3.2)

By contradiction, we assume that there exists \(x \in M\) and \(t > 0\) such that \(T_t^\varphi(x) < \varphi(x) + c \cdot \frac{1 - e^{-\lambda t}}{\lambda}\). Due to (1.2), there exists \(\gamma : [0, t] \to M\) be a minimizer of (1.2) with \(\gamma(t) = x\), i.e.

\[
T_t^\varphi(x) = \varphi(\gamma(0)) + \int_0^t L(\gamma(\tau), \dot{\gamma}(\tau), T_{\tau}^\varphi(\gamma(\tau))) d\tau.
\]

(3.3)
Let $G(s) = \varphi(\gamma(s)) + c \cdot \frac{1 - e^{-\lambda s}}{\lambda} - T_s^{-} \varphi(\gamma(s))$, since $G(0) = 0$ and $G(t) > 0$, then there exists $0 \leq t_0 < t$ such that $G(t_0) = 0$ and $G(s) > 0$ for any $s \in (t_0, t]$. According to (3.1) and (3.3), it shows that

$$G(s) = G(s) - G(t_0)$$

$$= \varphi(\gamma(s)) + c \cdot \frac{1 - e^{-\lambda s}}{\lambda} - T_s^{-} \varphi(\gamma(s)) - \varphi(\gamma(t_0)) - c \cdot \frac{1 - e^{-\lambda t_0}}{\lambda} + T_{t_0}^{-} \varphi(\gamma(t_0))$$

$$= \varphi(\gamma(s)) - \varphi(\gamma(t_0)) - \int_{t_0}^{s} L(\gamma(\tau), \dot{\gamma}(\tau), T_{\tau}^{-} \varphi(\gamma(\tau))) \ d\tau + c \cdot \frac{e^{-\lambda t_0} - e^{-\lambda s}}{\lambda}$$

$$\leq \int_{t_0}^{s} \left[ L(\gamma(\tau), \dot{\gamma}(\tau), \varphi(\gamma(\tau))) - L(\gamma(\tau), \dot{\gamma}(\tau), T_{\tau}^{-} \varphi(\gamma(\tau))) \right] \ d\tau - c(s - t_0) + c \cdot \frac{e^{-\lambda t_0} - e^{-\lambda s}}{\lambda}$$

$$\leq \lambda \int_{t_0}^{s} \left[ T_{\tau}^{-} \varphi(\gamma(\tau)) - \varphi(\gamma(\tau)) \right] \ d\tau - c(s - t_0) + c \cdot \frac{e^{-\lambda t_0} - e^{-\lambda s}}{\lambda}$$

where $\lambda$ is the Lipschitz constant of $L$ with respect to $u$. It derives a contradiction, which follows that

$$T_{t}^{-} \varphi(x) \geq \varphi(x) + c \cdot \frac{1 - e^{-\lambda t}}{\lambda} > \varphi(x) \quad \forall x \in M, t \in (0, +\infty).$$

Moreover, we have

$$\lim_{t \to 0^+} \frac{1}{t} \left( T_{t}^{-} \varphi(x) - \varphi(x) \right) \geq \lim_{t \to 0^+} c \cdot \frac{1 - e^{-\lambda t}}{\lambda t} = c > 0, \quad \forall x \in M.$$

(2) $\Rightarrow$ (1): Due to (2.1),

$$\lim_{t \to 0^+} \frac{1}{t} \left( h_{y,\varphi(y)}(x, t) - \varphi(x) \right) \geq \lim_{t \to 0^+} \frac{1}{t} \left( T_{t}^{-} \varphi(x) - \varphi(x) \right) \geq c > 0, \quad \forall x \in M. \quad (3.4)$$

For any fixed $x \in M$ and any $C^1$ curve $\xi : [0, t] \to M$ with $\xi(0) = y$ and $\xi(t) = x$, due to (2.3), we have

$$h_{y,\varphi(y)}(x, t) = \varphi(y) + \inf_{\gamma(t) = x} \int_{0}^{t} L(\gamma(\tau), \dot{\gamma}(\tau), h_{y,\varphi(y)}(\gamma(\tau), \tau)) \ d\tau,$$

$$\leq \varphi(y) + \int_{0}^{t} L(\xi(\tau), \dot{\xi}(\tau), h_{x,\varphi(x)}(\xi(\tau), \tau)) \ d\tau.$$

For any differentiable point $x \in M$ of $\varphi$, putting it into (3.4), we obtain that

$$c + \lim_{t \to 0^+} \frac{1}{t} \left[ \varphi(\xi(t)) - \varphi(\xi(0)) \right] \leq \lim_{t \to 0^+} \frac{1}{t} \int_{0}^{t} L(\xi(s), \dot{\xi}(s), h_{x,\varphi(x)}(\xi(s), s)) \ ds,$$

which leads to

$$c + \langle \partial_x \varphi(x), \dot{\xi}(0) \rangle \leq L(\xi(0), \dot{\xi}(0), h_{x,\varphi(x)}(\xi(0), 0)) = L(x, \dot{\xi}(0), \varphi(x)).$$

By taking $\dot{\xi}(0) = \partial_x H(x, \varphi(x), \partial_x \varphi(x))$, we get

$$L(x, \dot{\xi}(0), \varphi(x)) + H(x, \partial_x \varphi(x), \varphi(x)) = \langle \partial_x \varphi(x), \dot{\xi}(0) \rangle,$$
which implies \( H(x, \partial_x \varphi(x), \varphi(x)) \leq -c. \)

(2) \( \Leftrightarrow \) (3): It is similar with (2) \( \Leftrightarrow \) (3) of Theorem A. \( \square \)

**Proof of Corollary C**: It is a strict version of Theorem A. On one hand, due to Lemma 2.5, \( \Phi_t^H(\Gamma_\varphi) \subset \Gamma_{T_t^\varphi} \).

Thus, by \( T_t^\varphi \varphi > \varphi \), we have \( \Gamma_{T_t^\varphi} \subset \hat{\Gamma}_\varphi \).

On the other hand, for any given \( x \in M \) and \( t > 0 \), by (2.1) and (2.2), there exists \( p \in T_x M \) such that \( (x, p, u) \in \Phi_t^H(\Gamma_\varphi) \) and \( u = T_t^\varphi \varphi(x) \).

Since \( (x, p, u) \in \Phi_t^H(\Gamma_\varphi) \subset \hat{\Gamma}_\varphi \), it follows that \( T_t^\varphi \varphi(x) = u > \varphi(x) \).

Hence, \( T_t^\varphi \varphi(x) > \varphi(x) \) for any \( t > 0 \) and \( x \in M \). \( \square \)

## A  Completeness of the flow

**Lemma A.1.** Suppose that \( H \) satisfies (H1)-(H3) and there exists a continuous function \( A(h) : \mathbb{R} \to \mathbb{R}^+ \) such that

\[
|p \cdot \frac{\partial H}{\partial p}(x, p, u)| \leq A(H(x, p, u))(1 + |u|), \quad \forall (x, p, u) \in T^*M \times \mathbb{R}.  \tag{A.1}
\]

Then condition (SH1) holds, i.e. the contact vector field \( \Phi_t^H \) generates a complete flow on \( T^*M \times \mathbb{R} \).

**Proof of Lemma A.1**: Denote that \( H(t) := H(\Phi_t^H(x, p, u)) = H(x(t), p(t), u(t)) \) for any \( t \in \mathbb{R} \). By (1.1), one can compute that

\[
\frac{d}{dt} H(\Phi_t^H(x, p, u)) = \left[ \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial p} \cdot \dot{p} + \frac{\partial H}{\partial u} \cdot \dot{u} \right](x(t), p(t), u(t))
\]

\[
= \left[ \frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial p} \cdot (\frac{\partial H}{\partial x} \cdot \dot{x} + \frac{\partial H}{\partial u} \cdot \dot{u}) + \frac{\partial H}{\partial p} \cdot \dot{p} - H \right](x(t), p(t), u(t))
\]

\[
= - \frac{\partial H}{\partial u}(x(t), p(t), u(t)) \cdot H(x(t), p(t), u(t)).
\]

which together with (H3), implies that \( |H(x(t), p(t), u(t))| \leq e^{\lambda|t|} |H(x(0), p(0), u(0))| \).

From (SH1), \( |p \cdot \frac{\partial H}{\partial p}| \leq A(H)(1 + |u|) \), then by (1.1) it obtain

\[
|\dot{u}| = \left| p \cdot \frac{\partial H}{\partial p} - H \right| \leq A(H)(1 + |u|) + |H| \leq A(h_0)|u| + A(h_0) + e^{\lambda|t|} |H(0)|.
\]

where \( A(h_0) = \sup_{|h| \leq e^{\lambda|t|} H(0)} A(h) \). It shows that \( |u(s)| \) is bounded on \([-t, t] \) for any \( t \geq 0 \). If there does not exist a flow \( \Phi_t^H(x, p, u) \) for \( t \in (-\infty, +\infty) \), it means that the solution \( (x(t), p(t), u(t)) \) can not be extended further for some finite \( t_0 \). It implies that \( |p(t_0)| \) would blow up to infinite for some finite \( t_0 \), which leads to the boundless of \( |H(t_0)| \) because of (H2). It contradicts that \( |H(t)| \) is bounded by finite \( t \). Therefore, the contact vector field \( \Phi_t^H \) generates a complete flow.

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