SOME IDENTITIES FOR THE GENERALIZED FIBONACCI POLYNOMIALS BY THE $Q(x)$ MATRIX

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Abstract. In this note, we obtain some identities for the generalized Fibonacci polynomial by using the $Q(x)$ matrix. These identities including the Cassini identity and Honsberger formula can be applied to some polynomial sequences, such as Fibonacci polynomials, Lucas polynomials, Pell polynomials, Pell-Lucas polynomials, Fermat polynomials, Fermat-Lucas polynomials, and so on.

1. Introduction

A second order polynomial sequence $F_n(x)$ is said to be the Fibonacci polynomial if for $n \geq 2$ and $x \in \mathbb{R}$,

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$$

with $F_0(x) = 0$ and $F_1(x) = 1$. The Fibonacci polynomial and other polynomials attracted a lot of attention over the last several decades (see, for instance, [3, 4, 7, 8, 9, 12]). Recently, the generalized Fibonacci polynomial is introduced and studied intensely by many authors [1, 2, 5, 6], which is a generalization of the Fibonacci polynomial. Indeed, a polynomial sequence $G_n(x)$ in [5, 6] is called the generalized Fibonacci polynomial if for $n \geq 2$,

$$G_n(x) = c(x)G_{n-1}(x) + d(x)G_{n-2}(x)$$

with initial conditions $G_0(x)$ and $G_1(x)$, where $c(x)$ and $d(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$. It should be noted that there is no unique generalization of Fibonacci polynomials. Following the similar definitions in [6], in this note, $F_n(x)$ is said to be the Fibonacci type polynomial if for $n \geq 2$,

$$F_n(x) = c(x)F_{n-1}(x) + d(x)F_{n-2}(x)$$

where $c(x)$ and $d(x)$ are fixed non-zero polynomials in $\mathbb{Q}[x]$. If for $n \geq 2$,

$$L_n(x) = q, \quad L_1(x) = b(x)$$

and

$$L_n(x) = c(x)L_{n-1}(x) + d(x)L_{n-2}(x),$$

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then the polynomial sequence $L_n(x)$ is called the *Lucas type polynomial*, where
$q \in \mathbb{R} \setminus \{0\}$ and $b(x)$ is a fixed non-zero polynomial in $\mathbb{Q}[x]$. Naturally, both $F_n(x)$ and $L_n(x)$ are the generalized Fibonacci polynomials. We note that if we assume $F_1(x) = a = 1$, then $F_n(x)$ is the Fibonacci type polynomial given in [6]. In addition, the definition of $L_n(x)$ is the same with that of Flórez et al [6] if $|q| = 1$ or 2, and $c(x) = \frac{2}{q}b(x)$. In other words, our definitions of $F_n(x)$ and $L_n(x)$ are generalizations of those in [6].

Since the investigation of identities for polynomial sequences $F_n(x)$ and $L_n(x)$ received less attention than their numerical sequences, Flórez et al [6] collected and proved many identities for both $F_n(x)$ and $L_n(x)$ by applying their Binet formulas mostly, when certain special initial conditions were satisfied for $F_n(x)$ and $L_n(x)$. These identities can be applied to Fibonacci polynomials, Lucas polynomials, Pell polynomials, Pell-Lucas polynomials, Fermat polynomials, Fermat-Lucas polynomials, Chebyshev first kind polynomials, Chebyshev second kind polynomials, Jacobsthal polynomials, Jacobsthal-Lucas polynomials, and Morgan-Voyce polynomials. Indeed, all polynomial sequences in the upper part of Table 1 below are the Fibonacci type polynomials. On the other hand, those in the lower part of Table 1 are the Lucas type polynomials. Table 1 is the rearrangement of [6] Table 1.

| Polynomial       | Initial value $G_0(x)$ | Initial value $G_1(x)$ | Recursive Formula $G_n(x) = c(x)G_{n-1}(x) + d(x)G_{n-2}(x)$ |
|------------------|------------------------|------------------------|---------------------------------------------------------------|
| Fibonacci        | 0                      | 1                      | $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$                           |
| Pell             | 0                      | 1                      | $P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)$                           |
| Fermat           | 0                      | 1                      | $\Phi_n(x) = 3x\Phi_{n-1}(x) - 2\Phi_{n-2}(x)$                |
| Chebyshev second kind | 0                 | 1                      | $U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$                           |
| Jacobsthal       | 0                      | 1                      | $J_n(x) = J_{n-2}(x) + 2xJ_{n-1}(x)$                           |
| Morgan-Voyce     | 0                      | 1                      | $B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x)$                        |
| Vieta            | 0                      | 1                      | $V_n(x) = xV_{n-1}(x) - V_{n-2}(x)$                            |
| Lucas            | 2                      | $x$                    | $I_n(x) = xI_{n-1}(x) + I_{n-2}(x)$                           |
| Pell-Lucas       | 2                      | $2x$                   | $D_n(x) = 2xD_{n-1}(x) + D_{n-2}(x)$                           |
| Pell-Lucas-prime | 1                      | $x$                    | $D'_n(x) = 2xD'_{n-1}(x) + D'_{n-2}(x)$                        |
| Chebyshev first kind | 2                | $3x$                   | $\varphi_n(x) = 3x\varphi_{n-1}(x) - 2\varphi_{n-2}(x)$      |
| Jacobsthal-Lucas | 1                      | $x$                    | $T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x)$                           |
| Morgan-Voyce     | 1                      | $1$                    | $L_n(x) = 1 + 2xL_{n-1}(x) + 2xL_{n-2}(x)$                     |
| Vieta-Lucas      | 2                      | $x + 2$                | $C_n(x) = (x+2)C_{n-1}(x) - C_{n-2}(x)$                        |
|                  |                        | $x$                    | $\nu_n(x) = x\nu_{n-1}(x) - \nu_{n-2}(x)$                     |

In the note, by using the so called the $Q(x)$ matrix of Fibonacci type polynomials rather than the Binet formulas, we will obtain some new identities or recover some well-known ones including the Cassini identity and Honsberger formula for
In Section 2, we will present the results for the Fibonacci type polynomial \( F_n(x) \). Relying on Section 2, the identities of the Lucas type polynomial \( L_n(x) \) will be demonstrated in Section 3.

2. Fibonacci type polynomials

In this section, we will provide and prove some identities for the Fibonacci type polynomial \( F_n(x) \) by applying the Fibonacci type \( Q(x) \) matrix. The original Fibonacci \( Q \) matrix was introduced by Charles H. King in his master thesis (cf. [10]), and given by

\[
Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The Fibonacci \( Q \) matrix is connected to the Fibonacci sequence \( F_n \), which is defined as below

\[
F_0 = 1, \quad F_1 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \quad \text{for} \quad n \geq 2.
\]

Indeed, it is noted in [7] that

\[
Q^n = \left( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right)^n = \begin{pmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix}.
\]

Using this relation above, some familiar identities can be obtained. For instance,

\[
\det \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \left( \det \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right)^n
\]

implies the Cassini identity

\[
F_{n+1}F_{n-1} - F_n^2 = (-1)^n.
\]

Also, using this equality \( Q^{n+m} = Q^n Q^m \), one can deduce the Honsberger formula.

In the following, we will apply some similar idea of \( Q \) matrix from the numerical cases [11] to the Fibonacci type polynomials. For \( n \geq 2 \) and \( x \in \mathbb{R} \), the Fibonacci type polynomial \( F_n(x) \) is defined by

\[
(1) \quad F_0(x) = 0, \quad F_1(x) = a \quad \text{and} \quad F_n(x) = c(x)F_{n-1}(x) + d(x)F_{n-2}(x)
\]

where \( a \in \mathbb{R} \setminus \{0\} \). Then

\[
\begin{pmatrix} F_{n+2}(x) \\ F_{n+1}(x) \end{pmatrix} = \begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1}(x) \\ F_n(x) \end{pmatrix}.
\]

Here we define the Fibonacci type \( Q(x) \) matrix by

\[
Q(x) = \begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix}.
\]
We note that if $\mathcal{F}_n(x) = P_n(x)$ is the Pell polynomial as defined in Table 1 then

$$Q(x) = \begin{pmatrix} 2x & 1 \\ 1 & 0 \end{pmatrix}$$

which appeared in [9]. In addition, we observe that

$$\left( \frac{\mathcal{F}_{n+2}(x)}{\mathcal{F}_{n+1}(x)} \right) = \left( \frac{c(x) \ d(x)}{1 \ 0} \right)^n \left( \frac{\mathcal{F}_2(x)}{\mathcal{F}_1(x)} \right) = \left( \frac{c(x) \ d(x)}{1 \ 0} \right)^n \left( \frac{ac(x)}{a} \right).$$

On the other hand,

$$\left( \frac{\mathcal{F}_{n+2}(x)}{\mathcal{F}_{n+1}(x)} \right) = \left( \frac{c(x) \mathcal{F}_{n+1}(x) + d(x) \mathcal{F}_n(x)}{c(x) \mathcal{F}_n(x) + d(x) \mathcal{F}_{n-1}(x)} \right) = \left( \frac{\frac{1}{a} \mathcal{F}_{n+1}(x)}{\frac{1}{a} \mathcal{F}_n(x)} \frac{\frac{d(x)}{a} \mathcal{F}_n(x)}{\frac{d(x)}{a} \mathcal{F}_{n-1}(x)} \right) \left( \frac{ac(x)}{a} \right).$$

Hence we have the following result.

**Theorem 2.1.** Let $\mathcal{F}_n(x)$ be the Fibonacci type polynomial as defined in Eq. (1). Then for each $n \in \mathbb{N},$

$$\left( \frac{1}{a} \mathcal{F}_{n+1}(x) \frac{d(x)}{a} \mathcal{F}_n(x) \right) = \left( \frac{c(x) \ d(x)}{1 \ 0} \right)^n = Q^n(x).$$

**Proof.** Let $n = 1$. Then

$$\left( \frac{1}{a} \mathcal{F}_2(x) \frac{d(x)}{a} \mathcal{F}_1(x) \right) = \left( \frac{c(x) \ d(x)}{1 \ 0} \right).$$

Assume the equality holds for $n = k$. Then we have

$$\left( \frac{1}{a} \mathcal{F}_{k+1}(x) \frac{d(x)}{a} \mathcal{F}_k(x) \right) = \left( \frac{c(x) \ d(x)}{1 \ 0} \right)^k.$$

If $n = k + 1,$ then

$$\left( \frac{1}{a} \mathcal{F}_{k+2}(x) \frac{d(x)}{a} \mathcal{F}_{k+1}(x) \right) = \left( \frac{c(x) \ d(x)}{1 \ 0} \right) \left( \frac{1}{a} \mathcal{F}_{k+1}(x) \frac{d(x)}{a} \mathcal{F}_k(x) \right) = \left( \frac{c(x) \ d(x)}{1 \ 0} \right)^k.$$

By induction, the result follows. \hfill $\Box$

The Cassini identity of the Fibonacci type polynomial $\mathcal{F}_n(x)$ can be obtained below by Theorem 2.1.

**Corollary 2.2.** Let $\mathcal{F}_n(x)$ be the Fibonacci type polynomial. Then for each $n \in \mathbb{N},$

$$\mathcal{F}_n^2(x) - \mathcal{F}_{n+1}(x) \mathcal{F}_{n-1}(x) = a^2(-d(x))^{n-1}.$$ 

**Proof.** By Theorem 2.1 we have

$$\det \left( \frac{1}{a} \mathcal{F}_{n+1}(x) \frac{d(x)}{a} \mathcal{F}_n(x) \right) = \left( \det \left( \frac{c(x) \ d(x)}{1 \ 0} \right) \right)^n.$$
Hence
\[ F_n^2(x) - F_{n+1}(x)F_{n-1}(x) = a^2(-d(x))^{n-1}. \]

□

**Example 2.3.** Let \( a = 1, c(x) = x, d(x) = 1 \) in Eq. (1). Then \( F_n(x) \) is the classical Fibonacci polynomial \( F_n(x) \). By Corollary 2.2, we recover the Cassini identity in [4],
\[ F_{n+1}(x)F_{n-1}(x) - F_n^2(x) = (-1)^n. \]

**Example 2.4.** Let \( F_n(x) \) be the Pell polynomial \( P_n(x) \) as defined in Table 1. By Corollary 2.2, \( P_{n+1}(x)P_{n-1}(x) - P_n^2(x) = (-1)^n \)
which is the identity (2.5) in [9].

**Example 2.5.** Let \( a = 1, c(x) = 1, d(x) = 2x \) in Eq. (1). Then \( F_n(x) = J_n(x) \) is the Jacobsthal polynomial as defined in Table 1. By Corollary 2.2, one can obtain the Cassini identity for the Jacobsthal polynomial below
\[ J_n^2(x) - J_{n+1}(x)J_{n-1}(x) = (-2x)^{n-1}. \]

By Corollary 2.2, we have the result below.

**Corollary 2.6.** Let \( F_n(x) \) be the Fibonacci type polynomial. Then for each \( n \in \mathbb{N} \),
\[ F_n^2(x) - c(x)F_n(x)F_{n-1}(x) - d(x)F_{n-1}^2(x) = a^2(-d(x))^{n-1}. \]

**Proof.** By
\[ F_n^2(x) - F_{n+1}(x)F_{n-1}(x) = a^2(-d(x))^{n-1}. \]
and
\[ F_{n+1}(x) = c(x)F_n(x) + d(x)F_{n-1}(x), \]
we have
\[ a^2(-d(x))^{n-1} = F_n^2(x) - (c(x)F_n(x) + d(x)F_{n-1}(x))F_{n-1}(x) \]
\[ = F_n^2(x) - c(x)F_n(x)F_{n-1}(x) - d(x)F_{n-1}^2(x). \]

□

By applying \( Q^{n+m}(x) = Q^n(x)Q^m(x) \), we give the Honsberger’s formula for the Fibonacci type polynomials below.
Corollary 2.7. Let $F_n(x)$ be the Fibonacci type polynomial. Then for each $n, m \in \mathbb{N}$,

$$aF_{n+m}(x) = F_n(x)F_{m+1}(x) + d(x)F_{n-1}(x)F_m(x).$$

Proof. By

$$\begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix}^{n+m} = \begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix}^m,$$

we have

$$\begin{pmatrix} \frac{1}{a}F_{n+m+1}(x) \\ \frac{1}{a}F_{n+m}(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{a}F_{n+1}(x) \\ \frac{d(x)}{a}F_n(x) \end{pmatrix} \begin{pmatrix} \frac{1}{a}F_{m+1}(x) \\ \frac{d(x)}{a}F_m(x) \end{pmatrix}.$$

Hence considering the $(2,1)$ entry of the first matrix in the equality above,

$$aF_{n+m}(x) = F_n(x)F_{m+1}(x) + d(x)F_{n-1}(x)F_m(x).$$

\[\square\]

Remark 2.8.

(i) Let $a = 1$ in Corollary 2.7. Then Corollary 2.7 is the same with the first result of [6] Proposition 1], and a generalization of [4] Proposition 5.

(ii) If $m = n - 1$ in the above corollary, then for each $n \in \mathbb{N},$

$$aF_{2n-1}(x) = F_n^2(x) + d(x)F_{n-1}^2(x)$$

which generalizes the numerical case of Fibonacci sequences.

Example 2.9. Let $a = 1, c(x) = x, d(x) = 1$ in Eq. (1). Then $F_n(x) = F_n(x)$ is the Fibonacci polynomial as defined in Table 1. By Corollary 2.7, we recover the Honsberger formula in [4] Proposition 5,

$$F_{n+m}(x) = F_n(x)F_{m+1}(x) + F_{n-1}(x)F_m(x).$$

Example 2.10. Let $a = 1, c(x) = 2x, d(x) = 1$ in Eq. (1). Then $F_n(x)$ is the Pell polynomial $P_n(x).$ By Corollary 2.7, we have

$$P_{n+m}(x) = P_n(x)P_{m+1}(x) + P_{n-1}(x)P_m(x)$$

which is the equality (3.14) in [9].

Using $Q^{n-m}(x) = Q^n(x)Q^{-m}(x)$ for $n \geq m,$ we next will prove the d’Ocagne identity for $F_n(x).$ Here we need to assume $d(x) \not= 0$ for each $x \in \mathbb{R}$ so that $Q(x)$ is invertible. Moreover, note that

$$Q^{-m}(x) = \begin{pmatrix} \frac{1}{a}F_{m+1}(x) & \frac{d(x)}{a}F_m(x) \\ \frac{1}{a}F_m(x) & \frac{d(x)}{a}F_{m-1}(x) \end{pmatrix}^{-1} = \frac{1}{(-d(x))^m} \begin{pmatrix} \frac{d(x)}{a}F_{m-1}(x) & -\frac{d(x)}{a}F_m(x) \\ -\frac{d(x)}{a}F_m(x) & \frac{1}{a}F_{m+1}(x) \end{pmatrix}$$

by Theorem 2.1 and Corollary 2.2.
Corollary 2.11. Let \( F_n(x) \) be the Fibonacci type polynomial, and let \( d(x) \neq 0 \) for each \( x \in \mathbb{R} \). Then for each \( n, m \in \mathbb{N} \) with \( n \geq m \),

\[
a(-d(x))^m F_{n-m}(x) = F_n(x)F_{m+1}(x) - F_{n+1}(x)F_m(x).
\]

Proof. By \( Q^{n-m}(x) = Q^n(x)Q^{-m}(x) \), we have

\[
\begin{pmatrix}
\frac{1}{a}F_{n+1}(x) & \frac{d(x)}{a}F_n(x) \\
\frac{1}{a}F_n(x) & \frac{d(x)}{a}F_{n-1}(x)
\end{pmatrix}
\begin{pmatrix}
\frac{d(x)}{a}F_{m+1}(x) & -\frac{d(x)}{a}F_m(x) \\
-\frac{1}{a}F_{m+1}(x) & \frac{1}{a}F_m(x)
\end{pmatrix}
\]

Hence considering the (1,2) entry of the first matrix in the equality above,

\[
a(-d(x))^m F_{n-m}(x) = F_n(x)F_{m+1}(x) - F_{n+1}(x)F_m(x).
\]

\[\square\]

Example 2.12. Let \( F_n(x) \) be the Fibonacci polynomial \( F_n(x) \) as defined in Table 1. By Corollary 2.11,

\[
(-1)^m F_{n-m}(x) = F_n(x)F_{m+1}(x) - F_{n+1}(x)F_m(x)
\]

which is the d’Ocagne identity in [4, Corollary 8], and the identity (47) of [6, Proposition 3].

We note that \( Q(x) = \begin{pmatrix} c(x) & d(x) \\ 1 & 0 \end{pmatrix} \) satisfies \( Q^2(x) = c(x)Q(x) + d(x)I \) where \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Using this equality, one can obtain the following expression of \( F_n(x) \).

Theorem 2.13. Let \( F_n(x) \) be the Fibonacci type polynomial. Then for each \( n, p \in \mathbb{N} \),

\[
F_{2n+p}(x) = \sum_{j=0}^{n} \binom{n}{j} c^j(x)d^{n-j}(x)F_{j+p}(x).
\]
Proof. Consider
\[
\begin{pmatrix}
\frac{1}{a} F_{2n+p+1}(x) & d(x) F_{2n+p}(x) \\
\frac{1}{a} F_{2n+p}(x) & d(x) a F_{2n+p-1}(x)
\end{pmatrix}
\]
\[
= Q^{2n+p}(x)
\]
\[
= Q^p(x) \left( Q^2(x) \right)^n
\]
\[
= Q^p(x) \left( c(x)Q(x) + d(x)I \right)^n
\]
\[
= Q^p(x) \left( \sum_{j=0}^{n} \binom{n}{j} c^j(x)d^{n-j}(x)Q^j(x) \right)
\]
\[
= \left( \frac{1}{a}F_{p+1}(x) \frac{d(x)}{a} F_p(x) \right) \cdot \sum_{j=0}^{n} \binom{n}{j} c^j(x)d^{n-j}(x) \left( \frac{1}{a}F_{j+1}(x) \frac{d(x)}{a} F_j(x) \right).
\]
Then by Corollary 2.7 and the (1, 2) entry of the first matrix in the above equality, we have
\[
aF_{2n+p}(x) = \sum_{j=0}^{n} \binom{n}{j} c^j(x)d^{n-j}(x) \left( F_p(x)F_{j+1}(x) + d(x)F_{p-1}(x)F_j(x) \right)
\]
\[
= a \sum_{j=0}^{n} \binom{n}{j} c^j(x)d^{n-j}(x)F_{j+p}(x).
\]
\[
\square
\]

Example 2.14. Let \( F_n(x) \) be the Fibonacci polynomial \( F_n(x) \) in which \( a = 1, c(x) = x, d(x) = 1 \) in Eq. (1). By Theorem 2.13 we have
\[
F_{2n+p}(x) = \sum_{j=0}^{n} \binom{n}{j} x^j F_{j+p}(x).
\]
Given \( n = 2 \) and \( p = 1 \), we have
\[
F_5(x) = F_1(x) + 2xF_2(x) + x^2 F_3(x).
\]
Indeed, this equality holds for \( F_1(x) = 1, F_2(x) = x, F_3(x) = x^2 + 1 \) and \( F_5(x) = x^4 + 3x^2 + 1 \).

3. Lucas type polynomials

Based on the results of Fibonacci type polynomials, some identities of Lucas type polynomials will be demonstrated in this section. Throughout this section, we assume \( L_n(x) \) and \( F_n(x) \) have the same recursive formula with \( L_0(x) = F_1(x) \), that is, for \( n \geq 2 \),
\[
F_0(x) = 0, \ F_1(x) = a \text{ and } F_n(x) = c(x)F_{n-1}(x) + d(x)F_{n-2}(x),
\]
and

\begin{equation}
L_0(x) = a, \quad L_1(x) = b(x) \quad \text{and} \quad L_n(x) = c(x)L_{n-1}(x) + d(x)L_{n-2}(x)
\end{equation}

where \( a \in \mathbb{R} \setminus \{0\} \). By applying Theorem 2.1, one can connect \( L_n(x) \) with \( F_n(x) \) below.

**Theorem 3.1.** Let \( F_n(x) \) and \( L_n(x) \) be the Fibonacci type polynomial and Lucas type polynomial respectively with \( L_0(x) = F_1(x) = a \). Then for each \( n \in \mathbb{N} \),

\[
\begin{pmatrix}
L_{n+2}(x) & L_{n+1}(x) \\
L_{n+1}(x) & L_n(x)
\end{pmatrix} = \begin{pmatrix}
\binom{b(x)c(x) + ad(x)}{b(x)d(x)} & \binom{b(x)d(x)}{a} \\
\binom{a}{b(x)} & \binom{a}{b(x)}
\end{pmatrix}
\begin{pmatrix}
F_{n+1}(x) & F_n(x) \\
F_n(x) & F_{n-1}(x)
\end{pmatrix}
\]

**Proof.** First, we will prove \( L_n(x) = \frac{b(x)}{a}F_n(x) + d(x)F_{n-1}(x) \) holds for each \( n \in \mathbb{N} \). Let \( n = 1 \). Then

\[
L_1(x) = b(x) = \frac{b(x)}{a}F_1(x) + d(x)F_0(x).
\]

Let \( n = 2 \). Then

\[
L_2(x) = b(x)c(x) + ad(x) = \frac{b(x)}{a}F_2(x) + d(x)F_1(x).
\]

Assume this equality holds for \( n = k - 1 \) and \( k \). Let \( n = k + 1 \). Then

\[
L_{k+1}(x) = c(x)L_k(x) + d(x)L_{k-1}(x)
\]
\[
= c(x) \left( \frac{b(x)}{a}F_k(x) + d(x)F_{k-1}(x) \right) + d(x) \left( \frac{b(x)}{a}F_{k-1}(x) + d(x)F_{k-2}(x) \right)
\]
\[
= \frac{b(x)}{a} \left( c(x)F_k(x) + d(x)F_{k-1}(x) \right) + d(x) \left( c(x)F_{k-1}(x) + d(x)F_{k-2}(x) \right)
\]
\[
= \frac{b(x)}{a}F_{k+1}(x) + d(x)F_k(x).
\]

By induction, \( L_n(x) = \frac{b(x)}{a}F_n(x) + d(x)F_{n-1}(x) \) holds for all \( n \in \mathbb{N} \). On the other hand, we have

\[
L_n(x) = \frac{b(x)}{a}F_n(x) + d(x)F_{n-1}(x)
\]
\[
= \frac{b(x)}{a} \left( c(x)F_{n-1}(x) + d(x)F_{n-2}(x) \right) + d(x)F_{n-1}(x)
\]
\[
= \frac{b(x)c(x) + ad(x)}{a}F_{n-1}(x) + \frac{b(x)d(x)}{a}F_{n-2}(x).
\]

One has the result by these two equalities

\[
L_n(x) = \frac{b(x)}{a}F_n(x) + d(x)F_{n-1}(x)
\]
for each $n \in \mathbb{N}$, \[ \mathcal{L}_n(x) = \frac{b(x)c(x) + ad(x)}{a} \mathcal{F}_{n-1}(x) + \frac{b(x)d(x)}{a} \mathcal{F}_{n-2}(x). \]

Next, we will demonstrate the relation between Lucas type polynomials and the Fibonacci type $Q(x)$ matrix.

**Theorem 3.2.** Let $\mathcal{L}_n(x)$ be the Lucas type polynomial. Then for each $n \in \mathbb{N}$,

\[
\begin{pmatrix}
\mathcal{L}_{n+2}(x) & d(x)\mathcal{L}_{n+1}(x) \\
\mathcal{L}_{n+1}(x) & d(x)\mathcal{L}_n(x)
\end{pmatrix} = 
\begin{pmatrix}
\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x) \\
\mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)
\end{pmatrix} Q^n(x)
\]

**Proof.** By Theorem 2.1 and Theorem 3.1 we have

\[
\begin{pmatrix}
\mathcal{L}_{n+2}(x) & d(x)\mathcal{L}_{n+1}(x) \\
\mathcal{L}_{n+1}(x) & d(x)\mathcal{L}_n(x)
\end{pmatrix} = 
\begin{pmatrix}
\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x) \\
\mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)
\end{pmatrix} Q^n(x)
\]

for each $n \in \mathbb{N}$. \[ \square \]

Using Theorem 3.2 one has the Cassini identity for the Lucas type polynomial $\mathcal{L}_n(x)$.

**Corollary 3.3.** Let $\mathcal{L}_n(x)$ be the Lucas type polynomial. Then for each $n \in \mathbb{N}$,

\[
\mathcal{L}_{n+2}(x)\mathcal{L}_n(x) - \mathcal{L}_{n+1}^2(x) = (\mathcal{L}_2(x)\mathcal{L}_0(x) - \mathcal{L}_1^2(x)) (-d(x))^n.
\]

**Proof.** By Theorem 3.2 we have

\[
\det\begin{pmatrix}
\mathcal{L}_{n+2}(x) & d(x)\mathcal{L}_{n+1}(x) \\
\mathcal{L}_{n+1}(x) & d(x)\mathcal{L}_n(x)
\end{pmatrix} = \det\begin{pmatrix}
\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x) \\
\mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)
\end{pmatrix} \left(\det \begin{pmatrix} c(x) & d(x) \end{pmatrix}\right)^n.
\]

Hence

\[
\mathcal{L}_{n+2}(x)\mathcal{L}_n(x) - \mathcal{L}_{n+1}^2(x) = (\mathcal{L}_2(x)\mathcal{L}_0(x) - \mathcal{L}_1^2(x)) (-d(x))^n.
\]

\[ \square \]
Example 3.4. Let \( a = 2, b(x) = 2x, c(x) = 2x, d(x) = 1 \) in Eq. (2). Then \( L_n(x) = D_n(x) \) is the Pell-Lucas polynomial as defined in Table I. By Corollary 3.3, the Cassini identity for the Pell-Lucas polynomial \( D_n(x) \) is given by

\[
D_{n+2}(x)D_n(x) - D_{n+1}^2(x) = (4x^2 + 4)(-1)^n.
\]

By Corollary 3.3, we have the result below.

Corollary 3.5. Let \( L_n(x) \) be the Lucas type polynomial. Then for each \( n \in \mathbb{N}, \)

\[
c(x)\mathcal{L}_{n+1}(x)\mathcal{L}_n(x) + d(x)\mathcal{L}_n^2(x) - \mathcal{L}_{n+1}^2(x) = (\mathcal{L}_2(x)\mathcal{L}_0(x) - \mathcal{L}_1^2(x))(-d(x))^n.
\]

Proof. By

\[
\mathcal{L}_{n+2}(x)\mathcal{L}_n(x) - \mathcal{L}_{n+1}^2(x) = (\mathcal{L}_2(x)\mathcal{L}_0(x) - \mathcal{L}_1^2(x))(-d(x))^n
\]

and

\[
\mathcal{L}_{n+2}(x) = c(x)\mathcal{L}_{n+1}(x) + d(x)\mathcal{L}_n(x),
\]

we have

\[
(\mathcal{L}_2(x)\mathcal{L}_0(x) - \mathcal{L}_1^2(x))(-d(x))^n = (c(x)\mathcal{L}_{n+1}(x) + d(x)\mathcal{L}_n(x))\mathcal{L}_n(x) - \mathcal{L}_{n+1}^2(x) = c(x)\mathcal{L}_{n+1}(x)\mathcal{L}_n(x) + d(x)\mathcal{L}_n^2(x) - \mathcal{L}_{n+1}^2(x).
\]

\[
\square
\]

Using \( Q^2(x) = c(x)Q(x) + d(x)I \) again, we have the expression of \( \mathcal{L}_n(x) \).

Theorem 3.6. Let \( \mathcal{L}_n(x) \) be the Lucas type polynomial. Then for each \( n, p \in \mathbb{N}, \)

\[
\mathcal{L}_{2n+p}(x) = \sum_{j=0}^{n} \binom{n}{j} c^j(x)d^{n-j}(x)\mathcal{L}_{p+j}(x).
\]
Proof. By Theorem 3.2, we have

\[
\begin{pmatrix}
\mathcal{L}_{2n+p+2}(x) & d(x)\mathcal{L}_{2n+p+1}(x) \\
\mathcal{L}_{2n+p+1}(x) & d(x)\mathcal{L}_{2n+p}(x)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x) \\
\mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)
\end{pmatrix}
Q^{2n+p}(x)
\]

\[
= \begin{pmatrix}
\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x) \\
\mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)
\end{pmatrix}
Q^p(x) (Q^2(x))^n
\]

\[
= \begin{pmatrix}
\mathcal{L}_{p+2}(x) & d(x)\mathcal{L}_{p+1}(x) \\
\mathcal{L}_{p+1}(x) & d(x)\mathcal{L}_{p}(x)
\end{pmatrix}
(c(x)Q(x) + d(x)I)^n
\]

\[
= \begin{pmatrix}
\mathcal{L}_{p+2}(x) & d(x)\mathcal{L}_{p+1}(x) \\
\mathcal{L}_{p+1}(x) & d(x)\mathcal{L}_{p}(x)
\end{pmatrix}
\left(\sum_{j=0}^{n} \binom{n}{j} c^j(x)d^{n-j}(x)Q^j(x)\right)
\]

\[
= \sum_{j=0}^{n} \binom{n}{j} c^j(x)d^{n-j}(x)\begin{pmatrix}
\mathcal{L}_{p+j+2}(x) & d(x)\mathcal{L}_{p+j+1}(x) \\
\mathcal{L}_{p+j+1}(x) & d(x)\mathcal{L}_{p+j}(x)
\end{pmatrix}
\]

By considering the \((2, 2)\) entry of the first matrix in the above equality, we have

\[
\mathcal{L}_{2n+p}(x) = \sum_{j=0}^{n} \binom{n}{j} c^j(x)d^{n-j}(x)\mathcal{L}_{p+j}(x).
\]

□

Example 3.7. Let \(\mathcal{L}_n(x)\) be the Morgan-Voyce polynomial \(C_n(x)\) in which \(a = 2, b(x) = x + 2, c(x) = x + 2, d(x) = -1\) in Eq. (2). By Theorem 3.6, we have

\[
C_{2n+p}(x) = \sum_{j=0}^{n} \binom{n}{j} (x + 2)^j (-1)^{n-j}C_{p+j}(x).
\]

Finally, we end up this note by providing an identity in which \(\mathcal{F}_n(x)\) and \(\mathcal{L}_n(x)\) are involved.

Proposition 3.8. Let \(\mathcal{F}_n(x)\) and \(\mathcal{L}_n(x)\) be the Fibonacci type polynomial and Lucas type polynomial respectively with \(\mathcal{L}_0(x) = \mathcal{F}_1(x) = a\). Then for each \(n, m \in \mathbb{N}\),

\[
a\mathcal{L}_{n+m}(x) = \mathcal{L}_{n+1}(x)\mathcal{F}_m(x) + d(x)\mathcal{L}_n(x)\mathcal{F}_{m-1}(x).
\]
By Theorem 3.2, we have
\[
\begin{pmatrix}
\mathcal{L}_{n+m+2}(x) & d(x)\mathcal{L}_{n+m+1}(x) \\
\mathcal{L}_{n+m+1}(x) & d(x)\mathcal{L}_{n+m}(x)
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x) \\
\mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)
\end{pmatrix} Q^n(x)Q^m(x)
\]
\[
= \begin{pmatrix}
\mathcal{L}_{n+2}(x) & d(x)\mathcal{L}_{n+1}(x) \\
\mathcal{L}_{n+1}(x) & d(x)\mathcal{L}_n(x)
\end{pmatrix} \left( \begin{pmatrix}
\frac{1}{a}F_{m+1}(x) & \frac{d(x)}{a}F_{m+1}(x) \\
\frac{1}{a}F_m(x) & \frac{d(x)}{a}F_{m-1}(x)
\end{pmatrix} \right) .
\]

Then by the (2, 2) entry of the first matrix in the above equality, we have
\[
a\mathcal{L}_{n+m}(x) = \mathcal{L}_{n+1}(x)F_m(x) + d(x)\mathcal{L}_n(x)F_{m-1}(x)
\]
for each \(n, m \in \mathbb{N}\). \(\square\)

**Example 3.9.** Let \(F_n(x)\) and \(\mathcal{L}_n(x)\) be the Jacobsthal polynomial \(J_n(x)\) and the Jacobsthal-Lucas polynomial \(\Lambda_n(x)\) respectively, as defined in Table II. Then \(\Lambda_0(x) = J_1(x) = 1\) which satisfies the condition in Proposition 3.8. Hence we have the following equality for \(J_n(x)\) and \(\Lambda_n(x)\):

\[
\Lambda_{n+m}(x) = \Lambda_{n+1}(x)J_m(x) + 2x\Lambda_n(x)J_{m-1}(x).
\]

**Proposition 3.10.** Let \(F_n(x)\) and \(\mathcal{L}_n(x)\) be the Fibonacci type polynomial and Lucas type polynomial respectively with \(\mathcal{L}_0(x) = F_1(x) = a\). Let \(d(x) \neq 0\) for each \(x \in \mathbb{R}\). Then for each \(n, m \in \mathbb{N}\) with \(n \geq m\),

\[
a(-d(x))^m\mathcal{L}_{n-m}(x) = \mathcal{L}_n(x)F_{m+1}(x) - \mathcal{L}_{n+1}(x)F_m(x).
\]

**Proof.** By Theorem 3.2 and \(Q^{n-m}(x) = Q^n(x)Q^{-m}(x)\), we have
\[
\begin{pmatrix}
\mathcal{L}_{n-m+2}(x) & d(x)\mathcal{L}_{n-m+1}(x) \\
\mathcal{L}_{n-m+1}(x) & d(x)\mathcal{L}_{n-m}(x)
\end{pmatrix}
\]
\[
= \begin{pmatrix}
\mathcal{L}_2(x) & d(x)\mathcal{L}_1(x) \\
\mathcal{L}_1(x) & d(x)\mathcal{L}_0(x)
\end{pmatrix} Q^n(x)Q^{-m}(x)
\]
\[
= \begin{pmatrix}
\mathcal{L}_{n+2}(x) & d(x)\mathcal{L}_{n+1}(x) \\
\mathcal{L}_{n+1}(x) & d(x)\mathcal{L}_n(x)
\end{pmatrix} \frac{1}{(-d(x))^m} \left( \begin{pmatrix}
\frac{d(x)}{a}F_{m+1}(x) & \frac{d(x)}{a}F_{m+1}(x) \\
\frac{1}{a}F_m(x) & \frac{1}{a}F_{m-1}(x)
\end{pmatrix} \right) .
\]

Then considering the (2, 2) entry of the first matrix in the above equality, we have
\[
a(-d(x))^m\mathcal{L}_{n-m}(x) = \mathcal{L}_n(x)F_{m+1}(x) - \mathcal{L}_{n+1}(x)F_m(x).
\]

\(\square\)

**Example 3.11.** Let \(F_n(x)\) and \(\mathcal{L}_n(x)\) be the Jacobsthal polynomial \(J_n(x)\) and the Jacobsthal-Lucas polynomial \(\Lambda_n(x)\) respectively. Then \(\Lambda_0(x) = J_1(x) = 1\) and

\[
(-2x)^m\Lambda_{n-m}(x) = \Lambda_n(x)J_{m+1}(x) - \Lambda_{n+1}(x)J_m(x).
\]
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