Maximum likelihood estimator and its consistency for an \((L, 1)\) random walk in a parametric random environment\(^*\)

Hua-Ming WANG\(^†\) & Meijuan ZHANG\(^‡\)

Abstract

Consider an \((L, 1)\) random walk in an i.i.d. random environment, whose environment involves certain parameter. We get the maximum likelihood estimator (MLE) of the environment parameter which can be written as functionals of a multitype branching process with immigration in a random environment (BPIRE). Because the offspring distributions of the involved multitype BPIRE are of the linear fractional type, the limit invariant distribution of the multitype BPIRE can be computed explicitly. As a result, we get the consistency of the MLE. Our result is a generalization of Comets et al. [Stochastic Process. Appl. 2014, 124, 268-288].

Keywords: branching process; random walk; random environment; maximum likelihood estimator

MSC 2010: 62M05; 62F12; 60K37

1 Introduction

Random walks in random environments (RWRE) exhibit many surprising phenomena, thus attracting much attention recent years. Their limit behaviors, especially for one-dimensional case, have been extensively studied. We refer the readers to [12] for a survey. RWREs involve two kinds of randomness: the environments are chosen randomly according to some distribution; the particle evolves randomly in a given environment. So, from the statistical point of view, it is interesting to infer the distribution of the environment up to a

---

*Supported by Nature Science Foundation of China(Grant No. 11501008;11801596).
\(^†\)Email:hming@ahnu.edu.cn; Department of Statistics, Anhui Normal University, Wuhu 241003, China
\(^‡\)Email: zhangmeijuan1227@163.com; School of Statistics and Mathematics, Central University of Finance and Economics, Beijing 100081, China
single observation of a path of the random walk until it reaches a distant site. Adelman and Enriquez [1] dealt with very general RWRE and presented a procedure to infer the environment distribution through a system of moment equations. For the specific nearest neighbor ballistic RWRE, Comets et al. [2] provided a maximum likelihood estimator (MLE) for the parameter of the environment distribution and studied its consistency, whereas Falconnet et al. [3] studied the asymptotic normality and efficiency of the MLE. It turns out that the MLE of Comet et al. exhibits a much smaller variance than the one of Adelman and Enriquez.

In this paper, we study a non-nearest neighbor RWRE, say \((L, 1)\) RWRE, whose biggest left-oriented jumps are of size \(L\) and right-oriented jumps are always with size 1. Motivated by Comets et al. [2], our main goal is to provide a procedure to give the MLE of the environment parameter and show its consistency. We adopt the approach used in [2]. The MLE can be written as functionals of a multitype branching process with immigration in a random environment (BPIRE), which we get by decomposing the path of the RWRE. It turns out that the offspring distributions of the involved multitype BPIRE are of the linear fractional type. So we can compute explicitly the distribution of that BPIRE and its limit invariant distribution as well. Consequently, we can get the consistency of the MLE.

Notes: For general \(L \geq 2\), the construction of the MLE for \((L, 1)\) RWRE is basically the same as \((2, 1)\) RWRE, except that for \((L, 1)\) RWRE, a \((1 + 2 + ... + L)\)-type BPIRE is involved while for \((2, 1)\) RWRE, a \((1 + 2)\)-type BPIRE is needed. However, for general \((L, 1)\) RWRE, the notations are very heavy. So in the remainder of the paper, we fix \(L = 2\) to consider \((2, 1)\) RWRE.

The paper is organized as follows. We devote Section 2 to introducing the model and giving a procedure to construct the MLE. In Section 3 we construct a 3-type BPIRE by decomposing the path of \((2, 1)\) RWRE and compute explicitly its limit distribution. Finally, in Section 4 we show that the MLE we construct is consistent.

2 Model and main results

2.1 \((2,1)\) RWRE and its Preliminaries

For \(x \in \mathbb{Z}\), let \(\omega_x = (\omega_x(-2), \omega_x(-1), \omega_x(1))\) be a probability measure on \(\{x - 2, x - 1, x + 1\}\). Let \(\Omega\) be the collection of all \(\omega = (\omega_x, x \in \mathbb{Z})\). Equip \(\Omega\) with the weak topology induced by convergence of probability measures and let \(\mathcal{F}\) be the Borel \(\sigma\)-algebra. Let \(\nu_\theta\) be the law of \(\omega_0\) with \(\theta \in \Theta\) certain unknown parameter. We always assume that \(\Theta \subset \mathbb{R}^d\) for some \(d \geq 1\) is a compact set. Then \(\mathbb{P}_\theta := \nu_\theta^{\otimes \mathbb{Z}}\) is a probability measure on \((\Omega, \mathcal{F})\) which makes \(\omega = (\omega_x, x \in \mathbb{Z})\) an i.i.d. sequence. For a realization of \(\omega\), we consider a
Markov chain $\{X_t\}_{t \geq 0}$ on $\mathbb{Z}$ starting from 0, with transition probabilities

$$P_\omega(X_{t+1} = x + l | X_t = x) = \omega_x(l), \text{ for } l = -2, -1, 1$$

so that $P_\omega$ is the quenched law of the Markov chain under the environment $\omega$. Define a new probability measure $\mathbb{P}^\theta$ by

$$\mathbb{P}^\theta(\cdot) = \int P_\omega(\cdot)\mathbb{P}^\theta(d\omega),$$

which is usually called the annealed probability of $\{X_t\}$. We denote by $\mathbb{E}^\theta$, $E_\omega$ and $\mathbb{E}^\theta$ the expectation operators for $\mathbb{P}^\theta$, $P_\omega$ and $\mathbb{P}^\theta$, respectively. To state the recurrence criterion, we need the following condition.

**(C1)** Suppose that $\inf_{\theta \in \Theta} \mathbb{E}^\theta(\log \omega_0(l)) > -\infty$, for $l = -2, -1, 1$.

For $k \in \mathbb{Z}$, let $a_k = \frac{\omega_k(-1)}{\omega_k(1)}$, $b_k = \frac{\omega_k(-2)}{\omega_k(1)}$ and set

$$A_k = \begin{pmatrix} a_k & b_k & 0 \\ a_k & b_k & 1 \\ a_k & b_k & 0 \end{pmatrix}, \quad B_k = \begin{pmatrix} a_k + b_k & b_k \\ 1 & 0 \end{pmatrix}.$$  \hfill (1)

Clearly, $\{A_k\}$ and $\{B_k\}$ are two sequences of i.i.d. random matrices under $\mathbb{P}^\theta$. Then (C1) ensures an application of Oseledec’s multiplicative ergodic theorem (see [9]) to obtain the Lyapunov exponents of $\{A_k\}$ and $\{B_k\}$ under $\mathbb{P}^\theta$. Let $\gamma_A$ and $\gamma_B$ be the top Lyapunov exponents of $\{A_k\}$ and $\{B_k\}$, respectively. Then by the positivity of entries in $A_k$ and $B_k$, we have (see [9], Theorem 5) that $\mathbb{P}^\theta$-a.s., for all $i, j \in \{1, 2, 3\}$,

$$\gamma_A = \lim_{n \to \infty} \frac{1}{n} \log e_i A_0 A_1 \cdots A_{n-1} e_i^t, \quad \gamma_B = \lim_{n \to \infty} \frac{1}{n} \log e_i B_0 B_1 \cdots B_{n-1} e_i^t.$$  \hfill (2)

Here and throughout, $e_i$ is a vector with the $i$-th component 1 and all other components 0, and $v^t$ denotes the transpose of a vector $v$. Unless otherwise stated, the dimension of a vector depends on the matrix multiplication. By induction, it can be easily verified that $e_1 A_0 A_1 \cdots A_{k-1} e_2^t = e_1 B_0 B_1 \cdots B_{k-1} e_2^t$. Thus we can infer from (2) that

$$\gamma_A = \gamma_B.$$

The following recurrence criterion can be found in Letchikov [7].

**Recurrence criterion:** Under (C1), $\{X_t\}$ is transient to $+\infty$, recurrent or transient to $-\infty$ according as $\gamma_A < 0$, $\gamma_A = 0$ or $\gamma_A > 0$, respectively.

**Remark 1.** Here, $\gamma_A$, the top Lyapunov exponent of $\{A_k\}$, is used to give the recurrence criterion whereas in [7], $\gamma_B$ for $\{B_k\}$ is used. This causes no problem since we have shown that $\gamma_A = \gamma_B$. We use $\gamma_A$ here because we will see below that $\{A_k\}$ serves as the quenched offspring mean matrices for a 3-type BPIRE constructed from the path of $\{X_t\}$.
In this paper, we consider the case that \( \{X_t\} \) is transient to \(+\infty\). That is, we assume the following condition holds.

\((C2)\) Suppose that \( \gamma_A < 0 \).

Next we give a condition to ensure that \( \{X_t\} \) has a positive speed or in other words, \( \{X_t\} \) is ballistic. Let

\[ \pi(\omega) = 1 + \sum_{k=1}^{\infty} e_1 A_k \cdots A_1 (2, 1, 2)^t. \]

\((C3)\) Suppose that for any \( \theta \in \Theta \), we have \( E^\theta(\pi(\omega)) < \infty \).

**Law of large numbers:** Suppose that \((C1)-(C3)\) hold. Then for any \( \theta \in \Theta \), we have \( P_\theta \)-a.s.,

\[ \lim_{t \to \infty} X_t = \frac{1}{E^\theta(\pi(\omega))}. \]

Let \( T_1 := \inf\{t > 0 : X_t = 1\} \). Then under \((C1)-(C3)\), from Corollary below and the stationarity of the environment, we know that \( E^\theta T_1 = E^\theta(\pi(\omega)) < \infty \). Consequently, we can mimic the proof of Theorem 2.1.9 in [12] to prove the above law of large numbers by a hitting time decomposition approach.

### 2.2 Construction of an \( M \)-estimator

For \( n \geq 1 \), define

\[ T_n := \inf\{t > 0 : X_t = n\}, \]

the time the walk hits \( n \) for the first time. Under \((C2)\), \( \lim_{t \to \infty} X_t = +\infty, P^\theta\)-a.s. and hence for each \( n \geq 1 \), \( T_n < \infty \), \( P^\theta\)-a.s. For \( n \geq 1 \), an observation of the path of \( \{X_t\} \) until it reaches \( n \) is denoted by \( X_{[0, T_n]} = \{X_t : t = 0, 1, \ldots, T_n\} \).

Let \( x_{[0, t]} := (x_0, \cdots, x_t) \) be a path of length \( t \), \( v_t \) be the set of integers visited by the path \( x_{[0, t]} \) and \( V_{T_n} \) be the set of integers visited by the path \( X_{[0, T_n]} \). For \( x \in \mathbb{Z} \), define

\[ L_1(x, x_{[0, t]}) = \#\{0 \leq s < t : x_s = x, x_{s+1} = x - 1\}, \]

\[ L_2(x, x_{[0, t]}) = \#\{0 \leq s < t : x_s = x, x_{s+1} = x - 2\}, \]

\[ R(x, x_{[0, t]}) = \#\{0 \leq s < t : x_s = x, x_{s+1} = x + 1\}. \]

Here and throughout, we denote by \( \#\{\} \) the number of elements in a set \( \{\} \). Clearly, we have

\[
P_\omega(X_{[0, T_n]} = x_{[0, T_n]}) = \prod_{x \in v_{T_n}} \omega_x(1)^{R(x, x_{[0, T_n]})} \omega_x(-1)^{L_1(x, x_{[0, T_n]})} \omega_x(-2)^{L_2(x, x_{[0, T_n]})},
\]

\[
P^\theta(X_{[0, T_n]} = x_{[0, T_n]}) = \prod_{x \in v_{T_n}} \iint_{0 \leq a_1 + a_2 \leq 1} (1 - a_1 - a_2)^{R(x, x_{[0, T_n]})}
\]

4
\[ \times a_1^{L_1(x,x_{[0,t_n]})} a_2^{L_2(x,x_{[0,t_n]})} d\nu_d(a_1, a_2), \]

where \( x_{[0,t_n]} \) is the path up to time \( t_n \), the first hitting time of site \( n \).

One can define \( L_1(x,X_{[0,T_n]}), L_2(x,X_{[0,T_n]}) \) and \( R(x,X_{[0,T_n]}) \) by a similar way. Write simply \( L^n_{x,1} := L_1(x,X_{[0,T_n]}), L^n_{x,2} := L_2(x,X_{[0,T_n]}), R^n_x := R(x,X_{[0,T_n]}) \) and set

\[ L^n_x = (L^n_{x,1}, L^n_{x,2}). \]

Obviously, \( L^n_x \) counts the left-oriented jumps at \( x \) within the path \( X_{[0,T_n]} \).

Next we define an annealed log-likelihood function based on \( X_{[0,T_n]} \), a single observation of the path of \( \{X_t\} \) until it reaches a distant site \( n \). For \( n \geq 1 \), define

\[ l(n)_n := \sum_{x=0}^{n-1} \log \int \int_{0\leq a_1 + a_2 \leq 1} a_1^{L^n_{x,1}} a_2^{L^n_{x,2}} (1 - a_1 - a_2)^{R^n_x} d\nu_d(a_1, a_2) \]

\[ + \sum_{x=0, x \in \mathcal{V}_n} \log \int \int_{0\leq a_1 + a_2 \leq 1} a_1^{L^n_{x,1}} a_2^{L^n_{x,2}} (1 - a_1 - a_2)^{R^n_x} d\nu_d(a_1, a_2). \]

Under (C2), \( \lim_{t \to \infty} X_t = +\infty \), \( P^\theta \)-a.s. Thus \( \{X_t\} \) can only visit at most finite negative sites. Hence the second term on the righthand of (3) will not affect significantly the behavior of the normalized log-likelihood function, say \( \tilde{l}_n(\theta)/n \). So we keep only the first term which plays a dominant role.

If \( P^\theta \)-a.s. \( \lim_{t \to \infty} X_t = +\infty \), then for \( x \geq 0 \), we have

\[ R^n_x = L^n_{x+1,1} + L^n_{x+1,2} + L^n_{x+2,2} + 1. \]

Therefore, if we denote the first term on the righthand of (3) by \( l_n(\theta) \), then

\[ l_n(\theta) = \sum_{x=0}^{n-1} \log \int \int_{0\leq a_1 + a_2 \leq 1} a_1^{L^n_{x,1}} a_2^{L^n_{x,2}} (1 - a_1 - a_2)^{L^n_{x+1,1} + L^n_{x+1,2} + L^n_{x+2,2} + 1} d\nu_d(a_1, a_2). \]

\( l_n(\theta) \) will serve as the criterion function for deriving an \( M \)-estimator for \( \theta \). Set \( \mathbb{N} = \{0,1,2,\ldots\} \) and define a function \( \phi_\theta : \mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2 \to \mathbb{R} \) by setting

\[ \phi_\theta(y_1, y_2, y_3) = \log \int \int_{0\leq a_1 + a_2 \leq 1} a_1^{y_{1,1}} a_2^{y_{1,2}} (1 - a_1 - a_2)^{y_{2,1} + y_{2,2} + y_{3,1} + 1} d\nu_d(a_1, a_2), \]

where \( y_i = (y_{i,1}, y_{i,2}) \in \mathbb{N}^2, i = 1, 2, 3 \). Then

\[ l_n(\theta) = \sum_{x=0}^{n-1} \phi_\theta(L^n_x, L^n_{x+1}, L^n_{x+2}). \]

We need the following conditions in addition.
(C4) The map $\theta \mapsto \nu_\theta$ is continuous on $\Theta$ with respect to the weak topology.

(C5) $\forall (\theta, \theta') \in \Theta^2$, $\nu_\theta \neq \nu_{\theta'} \iff \theta \neq \theta'$.

Note that under (C4), since $\Theta$ is compact, it follows from (4) that for any $(y_1, y_2, y_3)$, $\phi_\theta(y_1, y_2, y_3)$ is continuous in $\theta$. Thus $l_n(\theta)$ is also a continuous function of $\theta$ and hence it can achieve its maximum over the compact set $\Theta$. Therefore, we can define an estimator $\hat{\theta}_n$ of $\theta$ as follows.

**Definition 1.** An estimator $\hat{\theta}_n$ of $\theta$ is defined as a measurable choice

$$\hat{\theta}_n \in \text{Argmax}_{\theta \in \Theta} l_n(\theta).$$

Since $\hat{\theta}_n$ is a maximiser of the criterion function $l_n(\theta)$, it is an $M$-estimator. Obviously, it is not necessarily unique. The criterion function $l_n(\theta)$ is not exactly log-likelihood. But the contribution of the negative sites which we dropped is not so important. So, we still call $\hat{\theta}_n$ a MLE.

Assume that the process $\{X_t\}$ is generated under the true parameter value $\theta^*$, an interior point of the parameter space $\Theta$, which we want to estimate. For simplicity, we use $P^*$ and $E^*$ rather than $P^{\theta^*}$ and $E^{\theta^*}$, respectively.

The following theorem, whose proof will be postponed to Section 4, gives the consistency of the estimator $\hat{\theta}_n$.

**Theorem 1 (Consistency).** Suppose that (C1)-(C5) hold. Then for any choice of $\hat{\theta}_n$ satisfying (6), we have $\lim_{n \to +\infty} \hat{\theta}_n = \theta^*$, in $P^*$-probability.

3 A 3-type BPIRE in the path of (2,1) RWRE

Throughout this section, we always assume that (C1) and (C2) hold, that is, $\lim_{t \to \infty} X_t = +\infty$, $P^\theta$-a.s.

3.1 Construction of BPIRE from the path of RWRE

In this section, we construct a 3-type BPIRE from the path of $\{X_t\}$. Hong and Wang [4] revealed a 2-type BPIRE from (2,1) RWRE. But that result is not sufficient to show the consistency for MLE. Here, we give a new construction, which is slightly different from [4], but with the same idea.

Fix an integer $n > 0$. Let

$$U_{i,1}^n = \# \{0 < k < T_n : X_{k-1} = i, X_k = i - 1\},$$

$$U_{i,2}^n = \# \{0 < k < T_n : X_{k-1} = i, X_k = i - 2\},$$

$$U_{i,3}^n = \# \{0 < k < T_n : X_{k-1} = i + 1, X_k = i - 1\}$$

and set

$$U_i^n := (U_{i,1}^n, U_{i,2}^n, U_{i,3}^n).$$
From the definition, we see that $U_{i,1}^n$, $U_{i,2}^n$, and $U_{i,3}^n$ count the number of steps by the walk from $i$ to $i-1$, $i$ to $i-2$, and $i+1$ to $i-1$ respectively in the time interval $[0, T_n]$.

**Figure 1:** The figure illustrates three types of jumps at $i$. Since $\lim_{t \to \infty} X_t = +\infty$, $\mathbf{P}^\theta$-a.s., there must be an upward jump from $i$ to $i+1$ which matches the corresponding downward jump. So in the figure, an upward (dashed) jump from $i$ to $i+1$ is attached to each downward jump. Thus, we treat $U_{i,l}$ an excursion rather than a jump, and such excursions constitute the individuals of a 3-type BPIRE.

**Theorem 2.** Assume (C1) and (C2). Then $U_{n}^0 \equiv 0, U_{n-1}^n, U_{n-2}^n, \ldots, U_{1}^n, U_{0}^n$ form the first $n$ generations of a 3-type BPIRE, which evolves as follows: given $\omega$, at time $n-(i+1)$, $0 \leq i \leq n-1$ (in view of the walk, time $n-(i+1)$ of BPIRE corresponds to the site $i+1$), a type-1 individual immigrates into the system, which, together with the existing individuals that constitute $U_{i+1}^n$, will give birth to a number of descendants with the law

$$P_{\omega}(U_i^n = (a, b, 0)|U_{i+1}^n = e_1) = \frac{(a+b)!}{a!b!}(\omega_i(-1))^a(\omega_i(-2))^b\omega_i(1), \quad (7)$$

$$P_{\omega}(U_i^n = (a, b, 1)|U_{i+1}^n = e_2) = \frac{(a+b)!}{a!b!}(\omega_i(-1))^a(\omega_i(-2))^b\omega_i(1), \quad (8)$$

$$P_{\omega}(U_i^n = (a, b, 0)|U_{i+1}^n = e_3) = \frac{(a+b)!}{a!b!}(\omega_i(-1))^a(\omega_i(-2))^b\omega_i(1). \quad (9)$$

Moreover, we have

$$T_n = n + \sum_{k=-\infty}^{n-1} U_k^n(2, 1, 2)^t. \quad (10)$$

**Proof.** We give here only the idea of the proof. For more details, we refer the readers to [4]. Given $\omega$, various independence for the branching process follows from the independence among different excursions, which follows from the strong Markov properties. As for the immigration law, note that for $1 \leq k \leq n$, before $\{X_t\}$ hits $k$ for the first time, there might be a number of excursions $U_{k-1,1}$ and $U_{k-1,2}$. But before time $T_k$, there is no step that reaches $k-1$ from some site above $k-1$. So we may treat such excursions $U_{k-1,1}$ and $U_{k-1,2}$ as individuals born to a type-1 individual, which immigrates into the system at site $k$. For the offspring distributions, we only explain that of a type-2 individual, which we illustrate in Figure 2.

We see from Figure 2 that given $\omega$, a type-2 individual at $i+1$ (the excursion starts from the dashed downward jump and ends up with the dashed upward...
jump) gives birth to type-3 individual at $i$ with probability 1 firstly and then to a number of type-1 and type-2 individuals. If it produces $a$ type-1 and $b$ type-2 individuals, that is, $U_{i,1} = a, U_{i,2} = b$, then during the excursion at $i+1$ considered here, the walk will firstly jump from $i$ to $i-1$ with $a$ times, jump from $i$ to $i-2$ with $b$ times and finally jump from $i$ to $i+1$. All of the possible combinations of those $a + b$ jumps equal $(a+b)!$. We thus conclude that

$$P_{\omega}(U_{i}^{n} = (a, b, 1)|U_{i+1}^{n} = e_2) = \frac{(a + b)!}{a!b!} (\omega_{i}(-1))^{a}(\omega_{i}(-2))^{b}\omega_{i}(1).$$

The offspring distribution for a type-1 (or type-3) individual is much easier and can be discussed similarly. Also, by some delicate computation, we see that the righthand of (10) counts all steps before the walk hits $n$. So (10) is true. □

From (7)-(9), we get the mean offspring matrix as follows.

**Corollary 1.** Given $\omega$, let $A_i$ be a $3 \times 3$ matrix whose $l$-th row is $E_{\omega}(U_{i}^{n}|U_{i+1}^{n} = e_l)$. Then $A_i$ coincides with the one defined in (7). Moreover

$$E_{\omega}T_n = n + \sum_{i=0}^{n-1} \sum_{k=-\infty}^{i} e_1 A_i \cdots A_{k+1} A_k (2, 1, 2)^t.$$

Next, we define a new 3-type BPIRE $\{Z_n\}_{n \geq 0}$. Firstly we specify its offspring distributions. For $1 \leq i \leq 3, k \geq 1$, let

$$\xi(i, k) = (\xi_1(i, k), \xi_2(i, k), \xi_3(i, k)),$$

which will serve as offsprings in $k$-th generation born to a type-$i$ individual in $(k-1)$-th generation, be a random vector satisfying

$$P_{\omega}(\xi(1, k) = (a, b, 0)) = \frac{(a+b)!}{a!b!} (\omega_k(-1))^{a}(\omega_k(-2))^{b}\omega_k(1),$$

$$P_{\omega}(\xi(2, k) = (a, b, 1)) = \frac{(a+b)!}{a!b!} (\omega_k(-1))^{a}(\omega_k(-2))^{b}\omega_k(1),$$

$$P_{\omega}(\xi(3, k) = (a, b, 0)) = \frac{(a+b)!}{a!b!} (\omega_k(-1))^{a}(\omega_k(-2))^{b}\omega_k(1).$$

Moreover, suppose that $\xi(i, k), i = 1, 2, 3, k \geq 1$ are mutually independent.
Set $Z_0 = 0$ and for $n \geq 1$, define recursively

$$Z_n = \sum_{j=1}^{1+Z_{n-1,1}} \xi^j(1, n) + \sum_{j=1}^{Z_{n-1,2}} \xi^j(2, n) + \sum_{j=1}^{Z_{n-1,3}} \xi^j(3, n),$$

where for fixed $i = 1, 2, 3$ and $n \geq 0$, $\xi^j(i, n), j = 1, 2, \ldots$ are independent copies of $\xi(i, n)$. Clearly, $\{Z_n\}_{n \geq 0}$ is a 3-type branching process with exactly one type-1 immigrant in each generation in a random environment. Since $(\omega_{n-1}, \ldots, \omega_0)$ and $(\omega_1, \ldots, \omega_n)$ share the same law under $P^\theta$, we have

$$\text{under } P^\theta, U^n_n \equiv 0, U^n_{n-1}, \ldots, U^n_0 \text{ has the same distribution as } Z_0, Z_1, \ldots, Z_n, \text{ the first } n \text{ generations of } \{Z_n\}. \quad (11)$$

In view of (11), we need only work with $\{Z_n\}$.

### 3.2 Criterion function $l_n(\theta)$ as functional of $\{Z_n\}$

It is easily seen that

$$U^n_{x,1} = L^n_{x,1}, \quad U^n_{x,2} = L^n_{x,2}, \quad U^n_{x,3} = U^n_{x+1,1} = L^n_{x+1,2}.$$

Thus, we have

$$P^n_x = L^n_{x+1,1} + L^n_{x+1,2} + L^n_{x+2,2} + 1$$
$$= U^n_{x+1,1} + U^n_{x+1,2} + U^n_{x+1,3} + 1 = |U^n_{x+1}| + 1.$$

Consequently,

$$l_n(\theta) = \sum_{x=0}^{n-1} \log \int \int_{0 \leq a_1 + a_2 \leq 1} a_1^{U^n_{x,1}} a_2^{U^n_{x,2}} (1 - a_1 - a_2)^{|U^n_{x+1}|+1} d\nu(a_1, a_2).$$

Let $\tilde{\phi}_\theta$ be a function from $\mathbb{N}^3 \times \mathbb{N}^3$ to $\mathbb{R}$ defined by

$$\tilde{\phi}_\theta(z_1, z_2) = \log \int \int_{0 \leq a_1 + a_2 \leq 1} a_1^{z_{1,1}} a_2^{z_{1,2}} (1 - a_1 - a_2)^{z_{2,1} + z_{2,2} + z_{3,3} + 1} d\nu(a_1, a_2),$$

where $z_i = (z_{i,1}, z_{i,2}, z_{i,3}) \in \mathbb{N}^3$, $i = 1, 2$. Then from [3], we have

$$l_n(\theta) = \sum_{x=0}^{n-1} \phi_\theta(L^n_x, L^n_{x+1}, L^n_{x+2}) = \sum_{x=0}^{n-1} \tilde{\phi}_\theta(U^n_{x+1}, U^n_x).$$

Therefore, it follows from (11) that

$$l_n(\theta) = \sum_{k=0}^{n-1} \tilde{\phi}_\theta(Z_k, Z_{k+1}) \quad (12)$$

in $P^\theta$-distribution. Thus, to characterize $l_n(\theta)$, it is enough to study the properties of $\{Z_n\}$. 

9
3.3 Quenched probability generating function of \( \{Z_n\} \)

Given \( \omega \), since the offspring distributions of \( \{Z_n\} \) are of the linear fractional type, we can compute explicitly its probability generating function (p.g.f.).

As a convention, set \( s = (s_1, s_2, s_3)^t \in [0, 1]^3 \) and \( 1 = (1, 1, 1)^t \). For \( k \geq 0 \), denote by
\[
F_k^1(s) = E_\omega(s^{Z_k}) = E_\omega(s_1^{Z_{k,1}}s_2^{Z_{k,2}}s_3^{Z_{k,3}})
\]
the p.g.f. of \( Z_k \). To give a formula for \( F_k^1(s) \), we introduce
\[
S_n^1 = e_1 \prod_{j=1}^n A_1 e_1^t, \quad S_n^2 = e_1 \prod_{j=1}^n A_2 e_2^t, \quad S_n^3 = e_1 \prod_{j=1}^n A_3 e_3^t,
\]
\[
\tilde{S}_n^1 = e_1 \prod_{j=1}^n A_1 e_1^t, \quad \tilde{S}_n^2 = e_1 \prod_{j=1}^n A_2 e_2^t, \quad \tilde{S}_n^3 = e_1 \prod_{j=1}^n A_3 e_3^t,
\]
and let
\[
S_n = 1 + S_n^1 + S_n^2 + S_n^3, \quad \tilde{S}_n = 1 + \tilde{S}_n^1 + \tilde{S}_n^2 + \tilde{S}_n^3.
\]
The following proposition is the main result of this section.

**Proposition 1.** We have
\[
F_n^1(s) = \frac{1}{S_n} \frac{1}{1 - \frac{S_n^1}{S_n}s_1 - \frac{S_n^2}{S_n}s_2 - \frac{S_n^3}{S_n}s_3},
\]
which equals
\[
\tilde{F}_n^1(s) := \frac{1}{S_n} \frac{1}{1 - \frac{\tilde{S}_n^1}{S_n}s_1 - \frac{\tilde{S}_n^2}{S_n}s_2 - \frac{\tilde{S}_n^3}{S_n}s_3}
\]
in \( \mathbb{P}^\omega \)-distribution.

To prove Proposition 1, we need several lemmas for preparation. Recall that for \( i = 1, 2, 3 \) and \( k \geq 0 \), \( \xi(i, k) = (\xi_1(i, k), \xi_2(i, k), \xi_3(i, k)) \) are offsprings in \( k \)-th generation born to a type-\( i \) individual in \( (k-1) \)-th generation. Given \( \omega \), for \( k \geq 0 \) and \( i \in \{1, 2, 3\} \) let
\[
f_k^i(s) = E_\omega(s^{\xi(i,k)}) = E_\omega(s_1^{\xi_1(i,k)}s_2^{\xi_2(i,k)}s_3^{\xi_3(i,k)})
\]
be the p.g.f. of \( \xi(i,k) \) and set
\[
f_k(s) = (f_k^1(s), f_k^2(s), f_k^3(s))^t.
\]

**Lemma 1.** For \( k \geq 1 \), we have
\[
F_k^1(s) = F_{k-1}^1(f_k(s)) f_k^1(s).
\]
Proof. For $k \geq 1$, note that

$$Z_{k,p} = \sum_{j=1}^{Z_{k-1,1}+1} \xi_p^j(1,k) + \sum_{j=1}^{Z_{k-1,2}} \xi_p^j(2,k) + \sum_{j=1}^{Z_{k-1,3}} \xi_p^j(3,k), \ p \in \{1, 2, 3\}.$$ 

Since for $1 \leq p \leq 3$ and $k \geq 1$, $\xi_p^j(m,k)$, $j = 1, 2, \cdots$ are independent copies of $\xi_p(m,k)$, we have

$$F_k^1(s) = E_{\omega} \left[ E_{\omega} \left( Z_{k,1} Z_{k,2} Z_{k,3} | Z_{k-1} \right) \right]$$

$$= E_{\omega} \left( s_1^{\xi_1(1,k)} s_2^{\xi_2(1,k)} s_3^{\xi_3(1,k)} \right) E_{\omega} \left\{ \prod_{m=1}^{3} E_{\omega} \left( s_1^{\xi_1(m,k)} s_2^{\xi_2(m,k)} s_3^{\xi_3(m,k)} \right) \right\}$$

$$= f_k^1(s) \cdot E_{\omega} \left\{ [f_k^1(s)]^{Z_{k-1,1}} [f_k^2(s)]^{Z_{k-1,2}} [f_k^3(s)]^{Z_{k-1,3}} \right\}$$

$$= f_k^1(s) F_{k-1}^1(f_k(s)).$$

The lemma is proved. \qed

**Lemma 2.** $F_n^1(s) = \prod_{k=1}^{n} f_k^1(f_{k+1}(\cdots f_n(s)(\cdots)).$

Proof. With Lemma 1 in hands, we can prove Lemma 2 by induction. \qed

**Lemma 3.** For $k \geq 1$, we have

$$f_k(s) = 1 - \frac{A_k(1-s)}{1+e_1 A_k(1-s)}.$$

Proof. Using the formula $\frac{1}{(1-x)^{k+1}} = \sum_{m=0}^{+\infty} \frac{(m+k)!}{m!} x^m$, we have

$$f_k^1(s) = f_k^3(s) = E_{\omega} \left( s_1^{\xi_1(1,k)} s_2^{\xi_2(1,k)} s_3^{\xi_3(1,k)} \right)$$

$$= \sum_{a=0}^{+\infty} \sum_{b=0}^{+\infty} \frac{(a+b)!}{a!b!} (\omega_k(-1))^a (\omega_k(-2))^b \omega_k(1) s_1^a s_2^b$$

$$= \frac{\omega_k(1)}{1 - \omega_k(-1)s_1 - \omega_k(-2)s_2}$$

and

$$f_k^2(s) = E_{\omega} \left( s_1^{\xi_1(2,k)} s_2^{\xi_2(2,k)} s_3^{\xi_3(2,k)} \right)$$

$$= \sum_{a=0}^{+\infty} \sum_{b=0}^{+\infty} \frac{(a+b)!}{a!b!} (\omega_k(-1))^a (\omega_k(-2))^b \omega_k(1) s_1^a s_2^b s_3^c$$

$$= \frac{\omega_k(1)s_3}{1 - \omega_k(-1)s_1 - \omega_k(-2)s_2}.$$
Then some direct computation yields that
\[
f_k^1(s) = f_k^3(s) = 1 - \frac{\gamma_k(1 - s)}{1 + \gamma_k(1 - s)}, \quad f_k^2(s) = 1 - \frac{(a_k, b_k, 1)(1 - s)}{1 + \gamma_k(1 - s)},
\]
where \( \gamma_k = (a_k, b_k, 0) \). The lemma follows. \( \square \)

**Lemma 4.** For \( n \geq 1 \) and \( 1 \leq k \leq n \), we have
\[
f_1(f_2(f_3(\ldots f_n(s) \cdots))) = 1 - \frac{\prod_{j=1}^{n} A_j(1 - s)}{1 + \sum_{j=k+1}^{n} e_1 \prod_{i=j}^{n} A_i(1 - s)},
\]
\[
f_k^l(f_{k+1}(\ldots f_n(s) \ldots)) = 1 - \frac{e_1 \prod_{j=k}^{n} A_j(1 - s)}{1 + \sum_{j=k}^{n} e_1 \prod_{i=j}^{n} A_i(1 - s)}.
\]

**Proof.** The lemma follows from Lemma 3 by induction. \( \square \)

**Proof of Proposition 7** Taking Lemma 2 and Lemma 4 together, we get
\[
F_n^1(s) = \frac{1 + \sum_{j=k+1}^{n} e_1 \prod_{i=j}^{n} A_i(1 - s)}{1 + \sum_{j=k}^{n} e_1 \prod_{i=j}^{n} A_i(1 - s)} = \frac{1}{1 + \sum_{j=1}^{n} e_1 \prod_{i=j}^{n} A_i(1 - s)},
\]
which implies (13). Moreover, since \((\omega_1, \omega_2, \ldots, \omega_n)\) and \((\omega_n, \omega_{n-1}, \ldots, \omega_1)\) share the same law under \( \mathbb{P}^\theta \), it follows from (13) that \( F_n^1(s) \) equals \( \tilde{F}_n^1(s) \) in \( \mathbb{P}^\theta \)-distribution.

**Proposition 2.** Both \( F_n^1(s) \) and \( \tilde{F}_n^1(s) \) converge in \( \mathbb{P}^\theta \)-distribution to a p.g.f.
\[
F^1(s) := \frac{\frac{1}{S}}{1 - \frac{S^1}{S} s_1 - \frac{S^2}{S} s_2 - \frac{S^3}{S} s_3},
\]
where
\[
\hat{S}^i := e_1(A_1 + A_1 A_2 + \cdots + A_1 A_2 \cdots A_n + \cdots) e_i^l, \quad i = 1, 2, 3,
\]
are all finite and \( \hat{S} := 1 + \hat{S}^1 + \hat{S}^2 + \hat{S}^3 \).

**Proof.** Note that by (C2), \( \gamma_A < 0 \). Moreover, from (2), we have
\[
\mathbb{P}^\theta (\exists N(\omega), \text{ s.t. } \forall n > N(\omega), \|A_0 A_1 \cdots A_n\| \leq e^{n\gamma_A/2}) = 1.
\]
Therefore \( \mathbb{P}^\theta \)-a.s., \( \hat{S}^i \) defined in (14) are all finite and
\[
\lim_{n \to \infty} \tilde{S}^i_n = \hat{S}^i, \quad i = 1, 2, 3 \text{ and } \lim_{n \to \infty} \tilde{S}_n = \hat{S}.
\]
Consequently, \( \tilde{F}_n^1(s) \) converges \( \mathbb{P}^\theta \)-a.s. and hence in \( \mathbb{P}^\theta \)-distribution to \( F^1(s) \).

Next, note that by Proposition 4 \( F_n^1(s) \) shares the same law with \( \tilde{F}_n^1(s) \). Thus, \( F_n^1(s) \) converges also in \( \mathbb{P}^\theta \)-distribution to \( F^1(s) \). \( \square \)
3.4 \{Z_n\} as a Markov chain with a limit invariant distribution

Since \{\omega_n\} is an i.i.d. sequence under \(P^\theta\), then \{Z_n\} is a homogeneous Markov chain under \(P^\theta\), whose transition kernel can also be computed explicitly. This observation of Markov property ensures an application of the ergodic theorem for Markov chain to \(\hat{\theta}(Z_n, Z_{n+1}), n \geq 0\), see Proposition 3 below.

**Proposition 3.** Under \(P^\theta\), \(\{Z_n\}_{n \geq 0}\) is a homogeneous Markov chain, whose transition kernel \(Q_\theta(x, y)\), for any \(\theta \in \Theta\), is given by

\[
Q_\theta(x, y) = \frac{(x_1 + x_2 + x_3 + y_1 + y_2)!}{y_1!y_2!(x_1 + x_2 + x_3)!} \times \int_{0 \leq a_1 + a_2 \leq 1} a_1^{y_1}a_2^{y_2}(1 - a_1 - a_2)^{x_1 + x_2 + x_3 + 1}d\nu(a_1, a_2), \quad (15)
\]

where \(x = (x_1, x_2, x_3) \in \mathbb{N}^3\), \(y = (y_1, y_2, y_3) \in \mathbb{N}^3\).

**Proof.** Let \(Z_n = (Z_{n,1}, Z_{n,2}, Z_{n,3})\) be individuals in the \(n\)-th generation conditioned on \(\{Z_{n-1} = (x_1, x_2, x_3)\}\). Then for \(p = 1, 2, 3\),

\[
Z_{n,p} = \sum_{j=1}^{x_1+1} \xi_p^j(1, n) + \sum_{j=1}^{x_2} \xi_p^j(2, n) + \sum_{j=1}^{x_3} \xi_p^j(3, n).
\]

The quenched p.g.f. of \(Z_n\), conditioned on \(\{Z_{n-1} = (x_1, x_2, x_3)\}\), equals

\[
f_{Z_n}(s) = [f_1^1(s)]^{x_1+1}[f_2^1(s)]^{x_2}[f_3^1(s)]^{x_3} = [1 - \omega_n(-1) - \omega_n(-2)]^{x_1 + x_3 + 1} \frac{[(1 - \omega_n(-1) - \omega_n(-2))s_3]^{x_2}}{[1 - \omega_n(-1)s_1 - \omega_n(-2)s_2]^{x_1 + x_2 + x_3 + 1}}.
\]

Some easy computation yields

\[
\frac{\partial^{y_1+y_2+y_3} f_{Z_n}(s)}{\partial y_1 \partial y_2 \partial y_3} \bigg|_{s=(0,0,0)} = \frac{(x_1 + x_2 + x_3 + y_1 + y_2)!y_3!}{(x_1 + x_2 + x_3)!} \times \left[\omega_n(-1)^{y_1} [\omega_n(-2)]^{y_2} [1 - \omega_n(-1) - \omega_n(-2)]^{x_1 + x_2 + x_3 + 1}\right].
\]

Therefore

\[
P_\omega(Z_n = (y_1, y_2, y_3)|Z_{n-1} = (x_1, x_2, x_3)) = \frac{(x_1 + x_2 + x_3 + y_1 + y_2)!}{y_1!y_2!(x_1 + x_2 + x_3)!} \times \left[\omega_n(-1)^{y_1} [\omega_n(-2)]^{y_2} [1 - \omega_n(-1) - \omega_n(-2)]^{x_1 + x_2 + x_3 + 1}\right],
\]

and hence transition kernel \(Q_\theta(x, y)\) coincides with the one defined in (15). □
**Theorem 3.** Assume (C1) and (C2). Then for any \( \theta \in \Theta \), the chain \( \{Z_n\}_{n \geq 0} \) is positive recurrent under \( P^\theta \) and admits a unique invariant probability measure \( \pi_\theta \) satisfying

\[
\lim_{n \to \infty} P^\theta(Z_n = (a, b, c)) = \pi_\theta(a, b, c). \tag{16}
\]

Moreover for all \((a, b, c) \in \mathbb{N}^3\),

\[
\pi_\theta(a, b, c) = \mathbb{E}^\theta \left[ \frac{(a + b + c)!}{a! b! c!} \left( \frac{\hat{S}^1}{S} \right)^a \left( \frac{\hat{S}^2}{S} \right)^b \left( \frac{\hat{S}^3}{S} \right)^c \left( 1 - \frac{\hat{S}^1}{S} - \frac{\hat{S}^2}{S} - \frac{\hat{S}^3}{S} \right) \right].
\]

**Remark 2.** As pointed out in [2], Key (12, Theorem 3.3) showed that \( Z_n \) converges in annealed law to a finite limit. Also, a construction of the limit was given in [10]. But here, since \( \{Z_n\} \) is Markovian and its offspring distributions are of the linear fractional type, we can compute explicitly the distribution of \( Z_n \) and its limit distribution as well.

**Proof.** Since for \( i = 1, 2, 3, 0 < \frac{S^i}{n} < 1 \), \( P^\theta \)-a.s., thus by Proposition [1] we see that, for any \((a, b, c) \in \mathbb{N}^3\),

\[
P_\omega(Z_n = (a, b, c)) = \frac{(a + b + c)!}{a! b! c!} \left( \frac{S^1_n}{S_n} \right)^a \left( \frac{S^2_n}{S_n} \right)^b \left( \frac{S^3_n}{S_n} \right)^c \left( 1 - \frac{S^1_n}{S_n} - \frac{S^2_n}{S_n} - \frac{S^3_n}{S_n} \right).
\]

As a result, using the fact \((\omega_1, \omega_2, ..., \omega_n)\) and \((\omega_n, \omega_{n-1}, ..., \omega_1)\) share the same law under \( P^\theta \) in the second equality, we get

\[
P^\theta(Z_n = (a, b, c)) = \mathbb{E}^\theta P_\omega(Z_n = (a, b, c)) = \mathbb{E}^\theta \left[ \frac{(a + b + c)!}{a! b! c!} \left( \frac{S^1_n}{S_n} \right)^a \left( \frac{S^2_n}{S_n} \right)^b \left( \frac{S^3_n}{S_n} \right)^c \left( 1 - \frac{S^1_n}{S_n} - \frac{S^2_n}{S_n} - \frac{S^3_n}{S_n} \right) \right].
\]

On the other hand, since \( 0 < \frac{S^i}{n} < 1, \ i = 1, 2, 3, \) and \( 0 < \frac{\hat{S}^1}{S_n} + \frac{\hat{S}^2}{S_n} + \frac{\hat{S}^3}{S_n} < 1 \), \( P^\theta \)-a.s., then by the dominated convergence theorem, we have

\[
\lim_{n \to \infty} P^\theta(Z_n = (a, b, c)) = \mathbb{E}^\theta \left[ \frac{(a + b + c)!}{a! b! c!} \left( \frac{\hat{S}^1}{S} \right)^a \left( \frac{\hat{S}^2}{S} \right)^b \left( \frac{\hat{S}^3}{S} \right)^c \left( 1 - \frac{\hat{S}^1}{S} - \frac{\hat{S}^2}{S} - \frac{\hat{S}^3}{S} \right) \right] =: \pi_\theta(a, b, c).
\]

Moreover, by Proposition [2] we have \( 0 < \frac{\hat{S}^i}{S} < 1, \ i = 1, 2, 3, \) \( P^\theta \)-a.s. Thus, by Fubini-Tonelli’s theorem, we get \( \sum_{a,b,c \geq 0} \pi_\theta(a, b, c) = 1 \). So \( \pi_\theta \) is a probability measure on \( \mathbb{N}^3 \). Due to [10], \( \pi_\theta \) is invariant. Moreover, since the kernel \( Q_\theta(x, y) \) is positive and \( \nu_\theta \) is not degenerate, the chain \( \{Z_n\} \) is irreducible. Therefore, \( \{Z_n\} \) is positive recurrent and \( \pi_\theta \) is unique. This completes the proof. \( \square \)
4 Consistency

In this section, we study the consistency of the estimator. Under certain conditions, we show that \( l_n(\theta)/n \) converges to a finite limit \( l(\theta) \), which identifies the true parameter value. Then, the consistency of the estimator can be established by some standard statistical arguments.

To begin with, let \( \{\tilde{Z}_n\}_{n \geq 0} \) be a process with transition kernel \( Q_\theta(x, y) \) and initial distribution \( \pi_\theta \). Then \( \{\tilde{Z}_n\} \) is a stationary Markov chain under \( P^* \). We have shown that under \( P^* \), \( \{Z_n\} \) is an irreducible Markov chain with transition kernel \( Q_{\theta^*}(x, y) \). We have the following ergodic theorem.

**Proposition 4.** Assume (C1) and (C2) hold. Then for all \( \theta \in \Theta \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \tilde{\phi}_\theta(Z_k, Z_{k+1}) = E^*[\tilde{\phi}_\theta(\tilde{Z}_0, \tilde{Z}_1)], \quad P^*-a.s.
\]

**Proof.** Note that \( -\infty < \tilde{\phi}_\theta(\tilde{Z}_0, \tilde{Z}_1) \leq 0 \). If \( E^*[\tilde{\phi}_\theta(\tilde{Z}_0, \tilde{Z}_1)] > -\infty \), then the proposition is the same as Proposition 2.4 in [3]. If \( E^*[\tilde{\phi}_\theta(\tilde{Z}_0, \tilde{Z}_1)] = -\infty \), then for \( M < 0 \), we have \( P^* \)-a.s.,

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \tilde{\phi}_\theta(Z_k, Z_{k+1}) \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (\tilde{\phi}_\theta(Z_k, Z_{k+1}) \lor M)
\]

\[
= E^*[\tilde{\phi}_\theta(\tilde{Z}_0, \tilde{Z}_1) \lor M].
\]

Letting \( M \to -\infty \) we get \( P^* \)-a.s.,

\[
\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \tilde{\phi}_\theta(Z_k, Z_{k+1}) \leq E^*[\tilde{\phi}_\theta(\tilde{Z}_0, \tilde{Z}_1)] = -\infty.
\]

The proposition is proved. \( \square \)

Next let

\[
l(\theta) := E^*[\tilde{\phi}_\theta(\tilde{Z}_0, \tilde{Z}_1)].
\]

**Lemma 5.** Assume (C1)-(C3) hold. Then \( l(\theta) \) is finite for any \( \theta \in \Theta \).

**Proof.** By Jensen’s inequality, for all \( x = (x_1, x_2, x_3) \in \mathbb{N}^3 \), \( y = (y_1, y_2, y_3) \in \mathbb{N}^3 \),

\[
\log \int_{0 \leq a_1, a_2 \leq 1} a_1 y_1 a_2 y_2 (1 - a_1 - a_2)^{x_1 + x_2 + x_3 + 1} \, d\nu_\theta(a_1, a_2)
\]

\[
\geq y_1 E^\theta[\log(\omega_0(-1))] + y_2 E^\theta[\log(\omega_0(-2))] + (|x| + 1) E^\theta[\log(\omega_0(1))].
\]

Thus

\[
\frac{1}{n} \sum_{k=0}^{n-1} \tilde{\phi}_\theta(Z_k, Z_{k+1}) \geq \frac{1}{n} \sum_{k=0}^{n-1} Z_{k+1} \cdot (E^\theta[\log(\omega_0(-1))], E^\theta[\log(\omega_0(-2))], 0)^t
\]

15
\begin{equation}
\frac{1}{n} \sum_{k=0}^{n-1} (|Z_k| + 1) \cdot \mathbb{E}^\theta[\log(\omega_0(1))].
\end{equation}

In view of Corollary 1 and the stationarity of the environment, we get

\begin{align*}
\mathbb{E}^*(\tilde{Z}_0) &= \lim_{n \to \infty} \mathbb{E}^*(Z_n) = \lim_{n \to \infty} \mathbb{E}^* \left( \sum_{k=1}^{n} e_1 A_k A_{k+1} \cdots A_n \right) \\
&= \lim_{n \to \infty} \mathbb{E}^* \left( \sum_{k=1}^{n} e_1 A_k A_{k-1} \cdots A_1 \right).
\end{align*}

(18)

By the ergodic theorem of Markov chain (Norris [8], Theorem 1.10.2), it follows that

\begin{align*}
\frac{1}{n} \sum_{x=0}^{n-1} \sup_{\theta \in U} \tilde{\phi}_\theta(L^n_x, L^n_{x+1}, L^n_{x+2}) &= \frac{1}{n} \sum_{k=0}^{n-1} \tilde{\Phi}_U(Z_k, Z_{k+1}) \text{ in } \mathbb{P}^*-\text{distribution.}
\end{align*}

Since \( \tilde{\Phi}_U \) is always negative, \( \mathbb{E}^* [\tilde{\Phi}_U(\tilde{Z}_0, \tilde{Z}_1)] \) is well defined. Thus by the ergodic theorem in Proposition 4 we conclude that

\begin{align*}
\lim_{n \to \infty} \frac{l_n(\theta)}{n} &= l(\theta), \text{ in } \mathbb{P}^*-\text{probability.}
\end{align*}

To sum up, we have the following theorem.

**Theorem 4.** Suppose that (C1)-(C3) hold. Then, there exists a finite deterministic limit \( l(\theta) \) such that \( \lim_{n \to +\infty} \frac{l_n(\theta)}{n} = l(\theta) \), in \( \mathbb{P}^* \)-probability.

**Remark 3.** Indeed, for the convergence in Theorem 4 we can say more. Suppose (C1)-(C3) hold. Fix an open \( U \subset \Theta \) and define \( \tilde{\Phi}_U = \sup_{\theta \in U} \tilde{\phi}_\theta \). Then

\begin{align*}
\frac{1}{n} \sum_{x=0}^{n-1} \sup_{\theta \in U} \tilde{\phi}_\theta(L^n_x, L^n_{x+1}, L^n_{x+2}) &= \frac{1}{n} \sum_{k=0}^{n-1} \tilde{\Phi}_U(Z_k, Z_{k+1}) \text{ in } \mathbb{P}^*-\text{distribution.}
\end{align*}

Since \( \tilde{\Phi}_U \) is always negative, \( \mathbb{E}^* [\tilde{\Phi}_U(\tilde{Z}_0, \tilde{Z}_1)] \) is well defined. Thus by the ergodic theorem in Proposition 4 we conclude that

\begin{align*}
\lim_{n \to +\infty} \frac{1}{n} \sum_{x=0}^{n-1} \sup_{\theta \in U} \tilde{\phi}_\theta(L^n_x, L^n_{x+1}, L^n_{x+2}) &= \mathbb{E}^* \left[ \sup_{\theta \in U} \tilde{\phi}_\theta(\tilde{Z}_0, \tilde{Z}_1) \right],
\end{align*}

in \( \mathbb{P}^* \)-probability. That is, \( l_n(\theta)/n \) converges to \( l(\theta) \) in \( \mathbb{P}^* \)-probability in a ‘locally uniform’ manner as \( n \to \infty \).

Similar to [2], we have from the following theorem that \( l(\theta) \) identifies the true value \( \theta^* \) as the unique point where it attains its maximum.
Theorem 5. Suppose that (C1)-(C5) hold. Then for any fixed $\varepsilon > 0$,

$$\sup_{\theta: \|\theta - \theta^*\| \geq \varepsilon} l(\theta) < l(\theta^*).$$

Proof of Theorem 4. Since $\Theta$ is compact, by applying some classical arguments of consistency for $M$-estimator (see [11], Section 5.2.1 therein), Theorem 4 follows from Theorem 3 and Theorem 5. \qed

References

[1] Adelman, O.; Enriquez, N. Random walks in random environment: what a single trajectory tells. Israel J. Math. 2004, 142, 205-220.

[2] Comets, F.; Falconnet, M.; Loukianov, O.; Loukianova, D.; Matias, C. Maximum likelihood estimator consistency for a ballistic random walk in a parametric random environment. Stochastic Process. Appl. 2014, 124, 268–288.

[3] Falconnet, M.; Loukianova, D.; Matlas, C. Asymptotic Normality and efficiency of the maximum Likelihood estimator for the parameter of a ballistic random walk in a random environment. Mathematical Methods of Stat. 2014, 23(1), 1-19.

[4] Hong, W. M.; Wang, H. M. Intrinsic branching structure within (L-1) random walk in random environment and its applications. Infinite Dimensional Analysis, Quantum Prob. Related Topics. 2013, 16(1), 1350006 (14 pages).

[5] Key, E. S. Limiting distributions and regeneration times for multitype branching processes with immigration in a random environment. Ann. Prob. 1987, 15(1), 344–353.

[6] Kingman J. F. C. Subadditive ergodic theory. Ann. Prob. 1973, 1(6), 883-909.

[7] Letchikov, A. V. A criterion for linear drift and the central limit theorem for one-dimensional random walks in a random environment. Russian Acad. Sci. Sb. Math. 1994, 79(1), 73-92.

[8] Norris, J. R. Markov Chain, in: Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press: Cambridge, 1998.

[9] Oseledec, V. I. Multiplicative ergodic theorem: characteristic Lyapunov exponents of dynamic systems. Trudy Moskovskogo Matemaricheskogo Obschchestva. 1968, 19, 179-210.

[10] Roitershtein, A. A note on multitype branching processes with immigration in a random environment. Ann. Prob. 2007, 35(4), 1573-1592.

[11] van der Vaart, A. W. Asymptotic Statistics; in: Cambridge Series in Statistical and Probabilistic Mathematic; Cambridge University Press: Cambridge, 1998.

[12] Zeitouni, O. Random walks in random environment; LNM 1837, J. Picard (Ed.); Springer-Verlag: Berlin Heidelberg, 2004; 189-312.