Lattice QCD at finite temperature: Evidence for calorons from the eigenvectors of the Dirac operator

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We analyze the eigenvalues and eigenvectors of the staggered Dirac operator in quenched lattice QCD in the vicinity of the deconfinement phase transition using the Lüscher-Weisz gauge action. The spectral and localization properties of the low-lying eigenmodes show characteristic differences between the $Z_3$ sectors above the critical temperature $T_c$. These findings can be interpreted in terms of calorons.

1. INTRODUCTION

The chiral phase transition is a main feature of QCD. In the instanton picture\textsuperscript{[1]} the chirally broken phase is represented by an ensemble of weakly interacting instantons and anti-instantons. Due to the fermionic quasi zero modes of the (anti-) instantons, a finite density of eigenvalues, $\rho_{\text{Dirac}}$, is generated near zero in a (large) volume $V$, and according to the Banks-Casher formula\textsuperscript{[2]}

$$\langle \bar{q}q \rangle = -\pi \rho_{\text{Dirac}}(0)/V$$

chiral symmetry is broken. In the chirally symmetric phase instantons and anti-instantons form strongly bound molecules, the eigenvalues of the Dirac operator move towards the bulk of the spectrum and the eigenvalue density vanishes near the origin. Of course one would like to verify the essential parts of this picture in QCD. Therefore we study the localization properties of the eigenmodes of the lattice Dirac operator in the neighborhood of the chiral phase transition. In a previous paper\textsuperscript{[3]} we have investigated this question with the Wilson gauge action on smaller lattices. Here we present results obtained on larger lattices with the Lüscher-Weisz gauge action.

2. PROPERTIES OF CALORONS

The temporal extension $L_t$ of the lattice is related to the temperature $T$ by $aT = 1/L_t$, where $a$ is the lattice spacing. Due to the periodic boundary condition in the time direction, an instanton becomes a chain of instantons which one calls caloron\textsuperscript{[4]}\textsuperscript{[5]}. The properties of a caloron are similar to that of an instanton: a caloron has topological charge one and a localized left-handed fermionic zero mode. However, the localization pattern of the zero mode is different. Whereas in the case of an instanton the zero mode is localized in space and time, the zero mode of a caloron is localized in space but delocalized in time.

Let us now compare the localization of the zero mode in the different sectors of the Polyakov loop $P$. We know that in the deconfined phase $P$ clusters around the values $e^{i\theta_P}$, with $\theta_P = 0$ (real sector) and $\theta_P = \pm 2\pi/3$ (complex sector)\textsuperscript{[6]}. Because the fermionic action does not share the $Z_3$ symmetry of the gauge action, the localization of the eigenmodes of the Dirac operator can depend on the Polyakov loop sectors. Indeed one finds that the zero mode $\psi_0$ of the caloron is more localized in the real sector than in the complex sector, as can be seen from\textsuperscript{[3,6]}

$$\|\psi_0\|^2 \propto \exp[-2(\pi - |\theta_P|) r T]/r^2.$$  \hfill (2)

Next we will search for caloron-like configurations concentrating on the fermionic properties of the

\textsuperscript{\*}talk given by W. Söldner at Lattice 2001, Berlin

\textsuperscript{[1]}\textsuperscript{[2]}\textsuperscript{[3]}\textsuperscript{[4]}\textsuperscript{[5]}\textsuperscript{[6]}
caloron described above.

3. LATTICE SETUP

We work in the quenched approximation with the Lüscher-Weisz gauge action

\[
S_g[U] = \beta_1 \sum_{pl} \frac{1}{3} \text{Re} \text{Tr} [1 - U_{pl}] + \beta_2 \sum_{rt} \frac{1}{3} \text{Re} \text{Tr} [1 - U_{rt}] + \beta_3 \sum_{pg} \frac{1}{3} \text{Re} \text{Tr} [1 - U_{pg}],
\]

where \( \sum_{pl}, \sum_{rt} \) and \( \sum_{pg} \) means a summation over all plaquettes, \( 2 \times 1 \) rectangles and parallelograms, respectively. The coefficients \( \beta_2 \) and \( \beta_3 \) are computed from \( \beta_1 \) via tadpole improved perturbation theory. Further we use the staggered Dirac operator

\[
D = \sum_{\mu=1}^{4} \frac{1}{2a} \alpha_\mu(x) \left[ \delta_{y,x+\mu} U_\mu(x) - \delta_{y,x-\mu} U_\mu^\dagger(y) \right],
\]

where \( \alpha_\mu(x) = (-1)^{x_1+\ldots+x_\mu-1} \). The eigenvalues of \( D \) come in pairs of \( \pm i \lambda \) with \( \lambda \) real, so we can restrict ourselves to positive \( \lambda \) in the following. In the continuum limit this action corresponds to four quark flavors. From now on we set \( a \) to 1.

We have calculated the lowest eigenvalues and eigenvectors for lattice sizes \( 12^3 \times 6, 16^3 \times 6 \) and \( 20^3 \times 6 \) below the deconfinement phase transition at \( \beta_1 = 8.10 \) (\( \beta_{\text{Wilson}} \approx 5.8 \) [6]) and above at \( \beta_1 = 8.45 \) (\( \beta_{\text{Wilson}} \approx 6.0 \) [7]) with the Arnoldi method [6]. We will present data for the largest lattice at \( \beta = 8.45 \).

4. CALORONS ON THE LATTICE

When searching for calorons on the lattice with the help of their fermionic properties, the zero mode of the Dirac operator plays the central role. Unfortunately the staggered Dirac operator has no exact zero modes, but, as we can see clearly in the left plot of Fig. 1, there are for both Polyakov loop sectors low-lying eigenmodes with a high value of \( |\langle \gamma_5 \rangle| \) and eigenmodes in the bulk of the spectrum with values of \( \langle \gamma_5 \rangle \) around zero. We find that the low-lying eigenvalues are quasi 4-fold degenerate in the sense that the number of low-lying eigenmodes is almost always 4, 8, 12, \ldots (In Fig. 1 this is illustrated by the circled eigenmodes, which belong to the same gauge field configuration. Note that the total number of low-lying eigenmodes of that configuration is 8.) So one would expect that those low-lying eigenmodes correspond to exact zero modes with definite handedness in the continuum.

Let us now look at the localization properties of these approximate zero modes. We define a gauge-invariant measure of the localization of a quark eigenmode \( \psi_\lambda(x) \) (\( \lambda \) is the Dirac eigenvalue, \( \alpha \) a color index), the inverse participation ratio

\[
I_2 = \frac{V}{\sum_x p_\lambda(x)^2},
\]

where \( V \) is the number of lattice sites and \( p_\lambda(x) \) is the gauge-invariant probability density \( p_\lambda(x) = |\psi_\lambda(x)|^2 \).
\[ \sum_{\alpha=1}^{N_c} |\psi_\alpha^\lambda(x)|^2. \] For a completely delocalized state (all \( p_\lambda(x) \) the same) one finds \( I_2 = 1 \), whereas a state localized on a single lattice site (only one non-zero \( p_\lambda(x) \)) would have \( I_2 = V \).

In the right part of Fig. 2 we plot \( I_2 \) vs. \( |\langle \gamma_5 \rangle| \). We see that the modes in the real sector are, in general, more localized than those in the complex sector. In both Polyakov loop sectors the approximate zero modes have a greater value of \( I_2 \) than the modes in the bulk spectrum, which means that they are more localized. These are exactly the properties we expect for caloron configurations.

Finally we consider the localization pattern. In Fig. 2 isosurfaces of \( p_\lambda \) and the local chirality \( p_5^\lambda(x) = \sum_{\alpha=1}^{N_c} \psi_\alpha^\lambda(x)^* \gamma_5 \psi_\alpha^\lambda(x) \) are plotted at fixed \( t \) and at fixed \( z \) for one of the encircled eigenmodes. This (typical) mode is clearly localized in space but not in time and the isosurfaces for \( p_\lambda \) and \( p_5^\lambda \) are very similar. This is again as is expected for caloron-like configurations.

5. CONCLUSIONS

Studying the localization and chirality properties of low-lying quark eigenstates in quenched lattice QCD, we could characterize semiclassical properties of the gauge field configurations without any cooling. For temperatures above \( T_c \) we found isolated modes with definite handedness. They are localized in space but not in time with characteristic differences between the Polyakov loop sectors. Thus they show essential properties of quark states associated with calorons.

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