SAMPLING SETS FOR HARDY SPACES OF THE DISK

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Abstract. We propose two possible definitions for the notion of a sampling sequence (or set) for Hardy spaces of the disk. The first one is inspired by recent work of Bruna, Nicolau, and Øyma about interpolating sequences in the same spaces, and it yields sampling sets which do not depend on the value of \( p \) and correspond to the result proved for bounded functions \( (p = \infty) \) by Brown, Shields and Zeller. The second notion, while formally closer to the one used for weighted Bergman spaces, is shown to lead to trivial situations only, but raises a possibly interesting problem.

§1. Sampling via the Bruna-Nicolau-Øyma function.

Let \( \mathbb{D} \) be the unit disk in the complex plane. Recall that for any \( 0 < p \leq \infty \) the Hardy space \( H^p(\mathbb{D}) \) is the set of holomorphic functions \( f \) such that

\[
\|f\|_p := \left( \sup_{r<1} \int_0^{2\pi} |f(re^{i\theta})| \frac{d\theta}{2\pi} \right)^{\frac{1}{p}} < \infty,
\]

where the integral is replaced by a supremum in the case \( p = \infty \), and that for \( p \geq 1 \), \( \| \cdot \|_p \) is a norm.

In accordance with many previous works (e.g. [La], [Se1], [Se2]) we would like to say that a subset \( a \) of the unit disk is of sampling for the space \( H^p(\mathbb{D}) \) when the values of a function \( f \in H^p(\mathbb{D}) \), restricted to this set, determine the function uniquely, and when we can establish some inequalities between the \( H^p \)-norm and an appropriate norm on the space of functions on the subset \( a \). Usually, this is interesting only when the subset \( a \) is a discrete sequence of points, however, that hypothesis will not be necessary for the first part of this paper. We proceed to give a more specific definition.

The Stolz angle with vertex at \( e^{i\theta} \) is \( \Gamma_{\alpha}(e^{i\theta}) := \{ z \in \mathbb{D} : \frac{|e^{i\theta} - z|}{1 - |z|} < 1 + \alpha \} \).

The nontangential maximal function is

\[
Mf(e^{i\theta}) := \sup_{z \in \Gamma_{\alpha}(e^{i\theta})} |f(z)|.
\]
For any $0 < p \leq \infty$, for any choice of $\alpha > 0$, we have $\|Mf\|_p \leq C_{p,\alpha}\|f\|_p$, where, for functions defined on the unit circle, $\| \cdot \|_p$ stands for the usual norm in the space $L^p(\frac{d\theta}{2\pi})$ ([Du], [Ga, Theorem II.3.1, p. 57]).

Now, following Bruna-Nicolau-Øyma [Br-Ni-Øy], let

$$M_a(f)(e^{i\theta}) := M_{a,\infty}(f)(e^{i\theta}) := \sup_{\Gamma_\alpha(e^{i\theta}) \cap a} |f| \leq Mf(e^{i\theta}).$$

From the above it follows that $\|M_a(f)\|_p \leq C_{p,\alpha}\|f\|_p$. We will call the set $a$ sampling if the two norms are actually equivalent.

**Definition 1.**

We say that the set $a$ is sampling for $H^p(\mathbb{D})$ iff there exists a constant $C > 0$ such that for any $f \in H^p(\mathbb{D})$, $\|M_a(f)\|_p \geq C\|f\|_p$.

In the case where $p = \infty$, this simply says that $\sup_a |f| \geq C \sup_\mathbb{D} |f|$, and by taking powers of $f$ we see that $\sup_a |f| = \sup_\mathbb{D} |f|$. This case of the problem was solved by Brown, Shields and Zeller [Br-Sh-Ze, Th. 3, (iii)-(iv)]. The main positive result of this note is that, with the appropriate definition above, this can be extended to all $p > 0$.

**Definition.** We say that a point of the circle $e^{i\theta}$ is a nontangential limit point of the set $a$ iff it is in the closure of $\Gamma_\alpha(e^{i\theta}) \cap a$ for some $\alpha > 0$. In this case, we write $e^{i\theta} \in NT(a)$.

Denote by $\lambda_1$ the 1-dimensional Lebesgue measure on the unit circle.

**Theorem 1.**

$a$ is sampling for $H^p(\mathbb{D})$ if and only if $\lambda_1$-almost every point $e^{i\theta} \in \partial \mathbb{D}$ is a nontangential limit point of the set $a$.

**Proof.** This proof is essentially the same as in [Br-Sh-Ze], so we shall keep it brief. We say that a function $f$ defined on the disk admits a nontangential limit at the point $e^{i\theta}$ iff $\lim_{z \to e^{i\theta}, z \in \Gamma_\alpha(e^{i\theta})} f(z) =: f^*(e^{i\theta})$ exists and is finite. For any choice of $\alpha$, any $f \in H^p(\mathbb{D})$ admits a nontangential limit at almost every $e^{i\theta}$ and $\|f^*\|_p = \|f\|_p$ [Du], [Ga].

Thus for almost every $e^{i\theta} \in NT(a)$,

$$M_a(f) \geq \lim_{z \to e^{i\theta}, z \in \Gamma_\alpha(e^{i\theta}) \cap a} |f(z)| = |f^*(e^{i\theta})|,$$

so that if $\lambda_1(\partial \mathbb{D} \setminus NT(a)) = 0$, $\|M_a(f)\|_p \geq \|f^*\|_p$, q.e.d.

Conversely, if $\lambda_1(\partial \mathbb{D} \setminus NT(a)) > 0$, there is an integer $N$ and $A$, a compact set of positive measure, such that for all $e^{i\theta} \in A$, $\Gamma_\alpha(e^{i\theta}) \cap a \cap \{|z| \geq 1 - 1/N\} = \emptyset$. Consider
the outer function
\[
\omega_A(z) := \exp \left\{ -\int_0^{2\pi} \frac{z + e^{i\theta}}{z - e^{i\theta}} \left( 1 - \mathbb{1}_A(e^{i\theta}) \right) \frac{d\theta}{2\pi} \right\},
\]
where \(\mathbb{1}_A\) is the indicator function of \(A\). Then \(-\log |\omega_A|\) is the harmonic measure of the set \(\partial \mathbb{D} \setminus A\), and classical estimates (e.g. [Ga, ex. 3, p. 41]) show that for all \(z \notin \bigcup_{e^{i\theta} \in A} \Gamma_\alpha(e^{i\theta})\), \(-\log |\omega_A| \geq c_\alpha > 0\).

Consider the sequence of functions \(f_n(z) := z^n \omega_A(z)^n\): we have \(\|f_n\|_p = \|f_n^*\|_p \geq \frac{\lambda_1(A)}{2\pi} > 0\), while \(M_a(f_n)(e^{i\theta}) \leq 1\) for all \(n\), and \(\lim_{n \to \infty} M_a(f_n)(e^{i\theta}) = 0\) for all \(e^{i\theta} \in A\) because \((1 - 1/N)^n \to 0\), and for all \(e^{i\theta} \notin A\) because \(\exp(-nc_\alpha) \to 0\). So \(\|M_a(f_n)\|_p \to 0\), and the sampling inequality cannot hold.

§2. Attempt at a classical definition.

The Bruna-Nicolau-Øyma function was introduced to deal with problems of interpolation of the type studied in [Sh-Sh]. This involved a norm in \(L^p(\mu)\), where for any \(E \subset a\), \(\mu(E) := \sum_{z \in E} (1 - |z|^2)\).

We can then state a similar sampling problem:

**Definition 2.**

We say that the set \(a\) is \(H^p(\mathbb{D})\)-thick iff there exists a constant \(C > 0\) such that for any \(f \in H^p(\mathbb{D})\),
\[
\|f\|_{L^p(\mu)} = \left( \sum_{z \in a} (1 - |z|^2)|f(z)|^p \right)^{\frac{1}{p}} \geq C\|f\|_p.
\]

This says that the measure \(\mu\) is *dominating* in the sense of [Lu].

Note that the sum on the left hand side may be infinite; in fact it always will be, which makes for a rather uninteresting notion of sampling.

**Theorem 2.**

1. If \(\lambda_1(NT(a)) > 0\), then \(H^p(\mathbb{D}) \cap L^p(\mu) = \{0\}\) (i.e. if \(f \in H^p(\mathbb{D})\) and \(\sum_{z \in a}(1 - |z|^2)|f(z)|^p < \infty\), then \(f = 0\)).
2. The set \(a\) is \(H^p\)-thick if and only if \(H^p(\mathbb{D}) \cap L^p(\mu) = \{0\}\).

In the case of the Bergman spaces studied in [Se1], [Se2], the definition of sampling which is given is akin to Definition 2 (but, unlike it, is not vacuous!). One can also construct an analogue to the first definition, using a supremum on invariant balls of fixed radius rather than Stolz angles. It can be seen, using boundedness of the restriction map which is part of the definition, that the sequences under consideration in [Se1, §7] can only have a bounded number of points in each such ball, which implies that the two notions are in this case equivalent. The proof is similar to that of the following Lemma.
Lemma 1.

If $a$ is sampling for $H^p(\mathbb{D})$, then $a$ is $H^p(\mathbb{D})$-thick.

Proof of Lemma 1. Let

$$M_{a,p}(f)(e^{i\theta}) := \left( \sum_{z \in \Gamma_a(e^{i\theta}) \cap a} |f(z)|^p \right)^{\frac{1}{p}}.$$ 

Then $M_{a,p}(f)(e^{i\theta}) \geq M_{a,\infty}(f)(e^{i\theta})$ for each $\theta$. But

$$\int_0^{2\pi} M_{a,p}(f)(e^{i\theta})^p d\theta = \sum_{z \in \Gamma_a(e^{i\theta}) \cap a} |f(z)|^p \int_{\theta : z \in \Gamma_a(e^{i\theta})} d\theta,$$

and the set $I_z := \{ \theta : z \in \Gamma_a(e^{i\theta}) \}$ is an arc centered at $z/|z|$, with length a multiple of $1 - |z|$, so the last sum is commensurate to the one in Definition 2.

Proof of Theorem 2. The following proof of (1) is due to Bo Berndtsson.

Assume $f \in L^p(\mu)$. Then, by the proof of Lemma 1, $M_{a,p}(f)$ must be finite almost everywhere on $\partial \mathbb{D}$. At the points $e^{i\theta} \in NT(a)$, since the infinite sum converges,

$$\lim_{z \in \Gamma_a(e^{i\theta}) \cap a, z \to e^{i\theta}} |f(z)|^p = 0.$$ 

Suppose now that in addition $f \in H^p(\mathbb{D})$. Since $f$ admits a nontangential limit almost everywhere, that limit will be zero almost everywhere on $NT(a)$, a set of positive measure, thus $f = 0$ (see e.g. [Ga, Th. 4.1, p. 64]).

As remarked after Definition 2, if $L^p(\mu) \cap H^p(\mathbb{D}) = \{0\}$, then $a$ is $H^p$-thick in a trivial way. To prove the converse implication suppose $0 \neq f \in L^p(\mu) \cap H^p(\mathbb{D})$. We shall prove that $NT(a)$ is of full measure. This leads to a contradiction in view of (1).

Assume $NT(a)$ is not of full measure. We can then construct the same sequence $f_n$ as in the proof of Theorem 1, and we have $\lim_{n \to \infty} f_n(z) = 0$ for all $z \in a$. Let $f$ be a function as in property $(F_p)$; then the sequence $\{ff_n\}$ is dominated by $|f|$, and we can apply Lebesgue’s dominated convergence theorem in $L^p(\mu)$, where for any $E \subset a$, $\mu(E) := \sum_{z \in E} (1 - |z|^2)$. So $\lim_{n \to \infty} \sum_{z \in E} (1 - |z|^2)|ff_n|^p = 0$, while $\|ff_n\|_p \geq \int_A |f^*|^p > 0$, since $f^*$ cannot vanish on a set of positive measure.

§3. A question, and an example.

Question. Can we characterize explicitly the sets $a$ such that $L^p(\mu) \cap H^p(\mathbb{D}) \neq \{0\}$?

The sets under consideration will have to be discrete sequences. I see the property as a weaker analogue of the Blaschke property (a sequence satisfying the Blaschke property, being a zero-set for an $H^\infty$ function, automatically satisfies our condition).
The condition cannot be about the mere growth of the number of points in the sequence as it approaches the boundary, as is demonstrated by the following example: take \( \{b_n\} \) a sequence of points in the disk satisfying the Blaschke condition, and \( \{b_{n,k}, 1 \leq k \leq q_n\} \), \( q_n \) distinct points in the disk \( D(b_n, q_n^{-1}(1 - |b_n|^2)^2) \). It is easy to check that the Blaschke product with simple zeroes at each of the \( b_n \) is in \( L^p(\mu) \cap H^p(\mathbb{D}) \) for any \( p \geq 1 \), but of course \( q_n \) can grow as fast as we please.

The condition does not depend on \( p \), in keeping with the fact that zero-sets or interpolating sequences for the Hardy spaces \( H^p(\mathbb{D}) \) do not depend on \( p \).

**Lemma 2.**

There exists \( H^p(\mathbb{D}) \cap L^p(\mu) \neq \{0\} \) if and only if \( H^\infty(\mathbb{D}) \cap L^1(\mu) \neq \{0\} \).

**Proof of Lemma 2.** This uses the same ideas as the canonical factorization of functions in the Nevanlinna class. Let \( f \in H^p(\mathbb{D}) \cap L^p(\mu) \setminus \{0\} \). Let \( m(z) \) be a harmonic majorant of \( |f|^p \) [Ga, p. 50]. Since \( m(z) \geq 0 \), if we denote by \( \tilde{m} \) its harmonic conjugate, \( 1 + m(z) + i\tilde{m}(z) = e^{H(z)} \), with \( e^{\frac{1}{p}RH(z)} \geq \max(1, |f(z)|) \). Then

\[
 f_1(z) := e^{-\frac{1}{p}H(z)}f(z) \in H^\infty(\mathbb{D}), \quad \|f_1\|_\infty \leq 1,
\]

and \( |f_1(z)| \leq |f(z)| \), so \( f_1 \in L^p(\mu) \). If \( p \leq 1 \), \( |f_1(z)|^p \geq |f_1(z)| \) and we may set \( g = f_1 \). Otherwise, we write \( f_1(z) = B(z)e^{h(z)} \), where \( B \) is a Blaschke product, and set \( g(z) = B(z)|\tilde{b}|^{p|}e^{ph(z)} \); then \( |g(z)| \leq |f_1(z)|^p \), and we’re done.

The converse implication is easier.

**Proposition 3.** There exists a discrete sequence, accumulating at one point of the circle only, such that \( H^p(\mathbb{D}) \cap L^p(\mu) = \{0\} \).

**Proof.** Let \( \{p_n\} \) be chosen so that \( 2^{-n}p_n \to \infty \), and take \( \gamma_n := 2^n/2p_n \).

Set \( \ell_n := p_n^{-1} \log \gamma_n^{-1/2} \), and

\[
a_{n,k} := (1 - 2^{-n})^{\frac{1}{2}} \exp ik\ell_n, \quad 0 \leq k \leq p_n.
\]

Let \( g \) be a function as in Lemma 2; to prove that \( g = 0 \), it is enough to show that \( \lim_{r \to 1^-} \int_0^{2\pi} \log |g(re^{i\theta})|d\theta = -\infty \) [Ga, proof of Th. 4.1, p. 65]. We will prove that \( |g| \) is small enough often enough on circles of radii tending to 1.

Set

\[
s_n := \sum_{k \leq p_n} (1 - |a_{n,k}|^2)|g(a_{n,k})| = 2^{-n} \sum_{k \leq p_n} |g(a_{n,k})|.
\]

Since \( \sum_n s_n < \infty \), \( \lim_{n \to \infty} s_n = 0 \). Let

\[
 A_n = \{k : 1 \leq k \leq p_n, |g(a_{n,k})| \geq \gamma_n \}.
\]
By Chebyshev’s inequality, $\# A_n \leq 2^n s_n / \gamma_n \leq p_n / 2$, thus if we set $A_n' := \{1, \ldots, p_n\} \setminus A_n$, $\# A'_n \geq p_n / 2$.

Set $J(a_{n,k}) := ((k - \frac{1}{2})\ell_n, (k + \frac{1}{2})\ell_n)$. Then

$$\lambda_1 \left( \bigcup_{k \in A'_n} J(a_{n,k}) \right) \geq Cp_n \ell_n.$$  

Since $g \in H^\infty(D)$, $|g'(z)| \leq C (1 - |z|^2)^{-1}$, so that for $\theta \in J(a_{n,k})$,

$$|g((1 - 2^{-n})^{\frac{1}{2}} e^{i\theta}) - g(a_{n,k})| \leq C 2^n \ell_n,$$

thus when $k \in A'_n$,

$$\log |g((1 - 2^{-n})^{\frac{1}{2}} e^{i\theta})| \leq \log(\gamma_n + C 2^n \ell_n) \leq C + \log \gamma_n.$$

The integral is now estimated by $p_n \ell_n \log \gamma_n \leq -C|\log \gamma_n|^{1/2} \to -\infty$, and the arc subtended by the points $a_{n,k}$ for a fixed $n$ is of length $p_n \ell_n = |\log \gamma_n|^{-1/2} \to 0$ as $n \to \infty$, so $\{a_k\}$ only accumulates at 1.

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**References**

[Br-Ni-Øy] Bruna J. - Nicolau A. - Øyma K., *A note on interpolation in the Hardy spaces in the disc*, Proc. Amer. Math. Soc. (to appear).

[Br-Sh-Ze] Brown L., Shields A., Zeller K., *On absolutely convergent exponential sums*, Trans. Amer. Math. Soc. 96 (1960), 162-183.

[Du] Duren P., *Theory of $H^p$ Spaces*, Academic Press, New York, 1970.

[Ga] Garnett J., *Bounded analytic functions*, Academic Press, New York, 1981.

[La] Landau H.J., *Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Math. 117 (1967), 37-52.

[Lu] Luecking D., *Dominating measures for spaces of analytic functions*, Ill. J. Math. 32 (1988), 23-39.

[Se1] Seip K., *Beurling type density theorems in the unit disk*, Invent. Math. 113 (1993), 21-39.

[Se2] Seip K., *Regular sets of sampling and interpolation for weighted Bergman spaces*, Proc. Amer. Math. Soc. 117 (1993), 213-220.
Shapiro H. S. - Shields A., *On some interpolation problems for analytic functions*, Amer. J. Math. **83** (1961), 513-532.

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