ON THE SEMI-REGULAR MODULE AND VERTEX OPERATOR ALGEBRAS

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1. INTRODUCTION

The aim of this paper is to give a proof of a conjecture stated in a previous paper by the author ([Z1]).

Let g be a simple complex Lie algebra, 錾 be the affine Lie algebra and ꧕ be the dual Coxeter number of g. Let $A_{g,k}$ be the vertex algebroid associated to g and a complex number k, according to [GMS1], we can construct a vertex algebra $U_{A_{g,k}}$, called the enveloping algebra of $A_{g,k}$. Set $V = U_{A_{g,k}}$. It is shown in [AG] and [GMS2] that not only V is a $\hat{g}$-representation of level k, it is also a $\hat{g}$-representation of the dual level $\bar{k} = -2h^\vee - k$. Moreover the two copies of $\hat{g}$-actions commute with each other, i.e. V is a $\hat{g} \oplus \hat{g}$-representation.

When $k \notin \mathbb{Q}$, the vertex operator algebra V decomposes into
\[ \oplus_{\lambda \in P^+} V_{\lambda,k} \otimes V_{\lambda^*,\bar{k}} \]
as a $\hat{g} \oplus \hat{g}$-module (see [FS], [Z1]). Here $P^+$ is the set of dominant integral weights of g, $V_{\lambda,k}$ is the Weyl module induced from $V_{\lambda}$, the irreducible representation of g with highest weight $\lambda$, in level k, and $V_{\lambda^*,\bar{k}}$ is induced from $V_{\lambda^*}$ in the dual level $\bar{k}$. In fact the vertex operators can be constructed using intertwining operators and Knizhnik-Zamolodchikov equations (see [Z1]).

In the case where $k \in \mathbb{Q}$, the $\hat{g} \oplus \hat{g}$-module structure of V is much more complicated. In the present paper, we prove a result about the existence of canonical filtrations of V conjectured at the end of [Z1]. More precisely we will prove the following.

**Theorem 1.** Let $k \in \mathbb{Q}$, $k > -h^\vee$. The vertex operator algebra V admits an increasing (resp. a decreasing) filtration of $\hat{g} \oplus \hat{g}$-submodules with factors isomorphic to
\[ V_{\lambda,k} \otimes V_{\lambda^*,\bar{k}} \quad (\text{resp. } V_{\lambda,k}^c \otimes V_{\lambda^*,\bar{k}}), \quad \lambda \in P^+ \]
where $V_{\lambda,k}^c$ is the contragredient module of $V_{\lambda,k}$ defined by the anti-involution: $x(n) \mapsto -x(-n)$, $\xi \mapsto \xi$ of $\hat{g}$.

We need two ingredients to prove the theorem: one is the semi-regular module; the other is the regular representation of the corresponding quantum group at a root of unity.

The standard semi-regular module was first introduced by A. Voronov in [V] to treat the semi-infinite cohomology of infinite dimensional Lie algebras as a two-sided derived functor of a functor that is neither left nor right exact. It was also studied rigorously by S. M. Arkhipov. He defined the associative algebra semi-infinite cohomology in
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The semi-regular module $S_\gamma$ associated to a semi-infinite structure $\gamma$ of $\hat{g}$ (see [V]) is

the semi-infinite analogue of the universal enveloping algebra $\mathcal{U}$ of $\hat{g}$. In particular $S_\gamma$ is a $U$-bimodule, and the tensor product $S_\gamma \otimes U \mathbb{V}$ becomes a $\hat{g}_- \oplus \hat{g}_k$-representation. We will show in Section 3 that $S_\gamma \otimes U \mathbb{V}$ can be embedded into $U^*$ as a bisubmodule. In fact it is spanned by the matrix coefficients of modules from the category $\mathcal{O}_{k+h^\vee}$, defined and studied by Kazhdan and Lusztig in [KL1-4] for $k < -h^\vee$.

In the series of papers [KL1-4], Kazhdan and Lusztig defined a structure of braided category on $\mathcal{O}_{k+h^\vee}$, and constructed an equivalence between the tensor category $\mathcal{O}_{k+h^\vee}$ and the category of finite dimensional integrable representations of the quantum group with parameter $e^{i\pi/(k+h^\vee)}$ (in the simply-laced case). It motivated the author to study the structure of regular representations of the quantum group at roots of unity (see [Z2]).

One of the main results in [Z2] is that the quantum function algebra admits an increasing filtration of (bi)submodules such that the subquotients are isomorphic to the tensor products of the dual of Weyl modules $W_{\omega_0 \lambda}^* \otimes W_{\lambda}^*$ ($\omega_0$ being the longest element in the Weyl group). Translating this to the affine Lie algebra, it means that $S_\gamma \otimes U \mathbb{V}$ admits an increasing filtration of $\hat{g}_- \oplus \hat{g}_k$-submodules with factors isomorphic to $V_{\omega_0 \lambda}^* \otimes V_{\lambda}^*$. Applying the functor $\mathcal{H}om_U(S_\gamma, -)$ (see [S, Theorem 2.1]) to this filtration of $S_\gamma \otimes U \mathbb{V}$, we obtain an increasing filtration of $\hat{g}_k \oplus \hat{g}_k$-submodules of the vertex operator algebra $\mathbb{V}$ with factors described in Theorem 1. The corresponding decreasing filtration is obtained by using the non-degenerate bilinear form on $\mathbb{V}$ constructed in [Z1].

The paper is organized as follows: In Section 2, we follow [S] to recall the definition of semi-regular module $S_\gamma$ and the two functors defined with it. In Section 3, we embed $S_\gamma \otimes U \mathbb{V}$ into the dual of $U$ as a (bi)submodule. In Section 4, we prove the main theorem about the filtrations of the vertex operator algebra $\mathbb{V}$ using results of [Z2].

2. SEMI-REGULAR MODULE $S_\gamma$ AND EQUIVALENCE OF CATEGORIES

The semi-regular module of a graded Lie algebra with a semi-infinite structure was first introduced by A. Voronov in [V], where it was called the “standard semijective module”. It replaces the universal enveloping algebra (and its dual) in the semi-infinite theory, and like the universal enveloping algebra, it possesses left and right (semi)regular representations. Voronov used semijective complexes and resolutions to define the semi-infinite cohomology of infinite dimensional Lie algebras as a two-sided derived functor of a functor that is intermediate between the functors of invariants and coinvariants.

In [A2], S. M. Arkhipov generalized the classical bar duality of graded associative algebras to give an alternative construction of the semi-infinite cohomology of associative algebras. Given a graded associative algebra $A$ with a triangular decomposition, he introduced the endomorphism algebra $A^2$ of a semi-regular $A$-module $S_A$ (see [A1]). In the case where $A$ is the universal enveloping algebra of a graded Lie algebra, the algebra $A^2$ is also a universal enveloping algebra of a Lie algebra which differs from the previous one by a 1-dimensional central extension (determined by
the critical 2-cocycle). In the affine Lie algebra case, he proved that the category of all 
\( \hat{g} \)-modules with a Weyl filtration in level \( k \) is contravariantly equivalent to the
analogous category in the dual level \( \bar{k} \). This equivalence was obtained directly in \([S]\),
where W. Soergel used it to find characters of tilting modules of affine Lie algebras
and quantum groups.

Let us recall the definition of the semi-regular module from \([S, \text{Theorem } 1.3]\).

Let \( \mathfrak{g} \) be a simple complex Lie algebra. Let \( \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}_c \) be the affine Lie
algebra, where the commutator relations are given by

\[
[x(m), y(n)] = [x, y](m + n) + m\delta_{m+n,0}(x, y)c.
\]

Here \( x(n) = x \otimes t^n \) for \( x \in \mathfrak{g}, (\cdot) \) is the normalized Killing form on \( \mathfrak{g} \) and \( c \) is the
center. Define a \( \mathbb{Z} \)-grading on \( \hat{\mathfrak{g}} \) by \( \deg x(n) = n \) and \( \deg c = 0 \).

Set \( \hat{\mathfrak{g}}_{>0} = \mathfrak{g} \otimes t\mathbb{C}[t], \hat{\mathfrak{g}}_{<0} = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}], \hat{\mathfrak{g}}_0 = \mathfrak{g} \oplus \mathbb{C}_c \) and \( \hat{\mathfrak{g}}_{\geq 0} = \hat{\mathfrak{g}}_{>0} \oplus \hat{\mathfrak{g}}_0 \). Denote the
enveloping algebras of \( \hat{\mathfrak{g}}, \hat{\mathfrak{g}}_{>0}, \hat{\mathfrak{g}}_{<0} \) by \( U, B, N \). Obviously \( U, B, N \) inherit \( \mathbb{Z} \)-gradings
from the corresponding Lie algebras.

Define a character

\[
\gamma : \hat{\mathfrak{g}}_0 = \mathfrak{g} \oplus \mathbb{C}_c \to \mathbb{C}; \quad \gamma|_\mathfrak{g} = 0, \quad \gamma(c) = 2h^\vee,
\]

where \( h^\vee \) is the dual Coxeter number of \( \mathfrak{g} \). It is easy to check that \( \gamma \) is a semi-infinite
character for \( \hat{\mathfrak{g}} \) (see \([S, \text{Definition } 1.1]\)).

For any two \( \mathbb{Z} \)-graded vector spaces \( M, M' \), define the \( \mathbb{Z} \)-graded vector space
\( \mathcal{H}om_\mathbb{C}(M, M') \) with homogeneous components

\[
\mathcal{H}om_\mathbb{C}(M, M')_j = \{ f \in \mathcal{H}om_\mathbb{C}(M, M') | f(M_i) \subset M'_{i+j} \}.
\]

The graded dual \( N^\oplus = \oplus_i N_i^* \) of \( N \) is an \( N \)-bimodule via the prescriptions
\( (nf)(n_1) = f(n_1n) \) and \( (fn)(n_1) = f(nn_1) \) for any \( n, n_1 \in N, f \in N^\oplus \). We have \( N^\oplus =
\mathcal{H}om_\mathbb{C}(N, \mathbb{C}) \), if we equip \( \mathbb{C} \) with the \( \mathbb{Z} \)-grading \( \mathbb{C} = \mathbb{C}_0 \).

Consider the following sequence of isomorphisms of (\( \mathbb{Z} \)-graded) vector spaces:

\[
\mathcal{H}om_B(U, \mathbb{C}_\gamma \otimes B) \xrightarrow{\sim} \mathcal{H}om_\mathbb{C}(N, B) \xleftarrow{\sim} N^\oplus \otimes_{\mathbb{C}} B \xleftarrow{\sim} N^\oplus \otimes N U,
\]

here \( \mathbb{C}_\gamma \) is the one-dimensional representation of \( \hat{\mathfrak{g}}_{\geq 0} \) defined by the character \( \gamma : \hat{\mathfrak{g}}_0 \to \mathbb{C} \) and the surjection \( \hat{\mathfrak{g}}_{\geq 0} \to \mathfrak{g}_0 \), and \( \mathbb{C}_\gamma \otimes \mathbb{C} B \) is the tensor product of these
two representations as a left \( \hat{\mathfrak{g}}_{\geq 0} \)-module. In the leftmost term, \( U \) is considered a
\( B \)-module via left multiplication of \( B \) on \( U \), and \( \mathcal{H}om_B(U, \mathbb{C}_\gamma \otimes B) \) is made into a
(left) \( U \)-module via the right multiplication of \( U \) onto itself. The first isomorphism is
defined as the restriction to \( N \) using the identification \( \mathbb{C}_\gamma \otimes \mathbb{C} B \xrightarrow{\sim} B; 1 \otimes b \mapsto b \).

As a vector space, the semi-regular module

\[
S_\gamma = N^\oplus \otimes \mathbb{C} B.
\]

It is also a \( U \)-bimodule: the left (resp. right) \( U \)-action on \( S_\gamma \) is defined via the first
two (resp. last) isomorphisms. The semi-infinite character \( \gamma \) ensures that these two
actions commute.

**Lemma 2.1.** \( c \cdot s = s \cdot c + 2h^\vee s \) for any \( s \in S_\gamma \), where \( c \cdot s \) and \( s \cdot c \) stand for the left
and right actions of \( c \) on \( s \in S_\gamma \).

**Proof.** Easily verified. \( \Box \)
Proposition 2.2. [S, Theorem 1.3] The map $\iota : N^\otimes \hookrightarrow S_\gamma; f \mapsto f \otimes 1$ is an inclusion of $N$-bimodules. The maps $U \otimes_N N^\otimes \rightarrow S_\gamma; u \otimes f \mapsto u \cdot \iota(f)$ and $N^\otimes \otimes_U U \rightarrow S_\gamma; f \otimes u \mapsto \iota(f) \cdot u$ are bijections.

Remark 2.3. The sequence of isomorphisms

$$S_\gamma = U \otimes_N N^\otimes \cong B \otimes_C N^\otimes \cong \text{Hom}_C(N, B) \cong \text{Hom}_{B\text{-right}}(U, C_{-\gamma} \otimes B)$$

induces a right $U$-map from $S_\gamma$ to $\text{Hom}_{B\text{-right}}(U, C_{-\gamma} \otimes B)$. The right $U$-module structure of the latter is given by the left multiplication of $U$ on the first argument in $\text{Hom}$.

Let $P^+$ be the dominant integral weights of $\mathfrak{g}$ and $\lambda \in P^+$. Denote by $V_{\lambda,k} = \text{Ind}_{\mathfrak{g}_{\leq 0}}^{\mathfrak{g}} V_\lambda$ the Weyl module induced from the finite dimensional irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$ in level $k$. Let $V_{\lambda,k}^*$ be the graded dual of $V_{\lambda,k}$, on which $\hat{\mathfrak{g}}$ acts by $X f(v) = -f(X v)$ for any $X \in \hat{\mathfrak{g}}$, $f \in V_{\lambda,k}^*$ and $v \in V_{\lambda,k}$.

Let $\mathcal{M}$ (resp. $\mathcal{K}$) denote the category of all $\mathbb{Z}$-graded representations of $\hat{\mathfrak{g}}$, which are over $N$ isomorphic to finite direct sums of may-be grading shifted copies of $N$ (resp. $N^\otimes$). In fact $\mathcal{M}$ (resp. $\mathcal{K}$) consists precisely of those $\mathbb{Z}$-graded $\hat{\mathfrak{g}}$-modules, which admit a finite filtration with factors isomorphic to Weyl modules (resp. the dual of Weyl modules) (see [S, Remarks 2.4]).

Proposition 2.4. [S, Theorem 2.1] The functor $S_\gamma \otimes_U - : \mathcal{M} \rightarrow \mathcal{K}$ defines an equivalence of categories with inverse $\text{Hom}_U(S_\gamma, -)$, such that short exact sequences correspond to short exact sequences.

Proof. Note that $S_\gamma \otimes_U - \cong N^\otimes \otimes_N -$ and $\text{Hom}_U(S_\gamma, -) \cong \text{Hom}_N(N^\otimes, -)$ by Proposition 2.2. □

Proposition 2.5. Let $E$ be a $\mathbb{Z}$-graded $B$-module bounded from below, the functor $S_\gamma \otimes_U -$ maps $U \otimes_B E$ to $\text{Hom}_B(U, C_\gamma \otimes E)$.

Proof. Similar to the construction of the semi-regular module $S_\gamma$, consider the following sequence of isomorphisms of $\mathbb{Z}$-graded vector spaces:

$$S_\gamma \otimes_U (U \otimes_B E) \cong N^\otimes \otimes_C E \cong \text{Hom}_C(N, E) \cong \text{Hom}_B(U, C_\gamma \otimes E).$$

It is straightforward to check that, under these isomorphisms, the (left) $U$-module structure of $S_\gamma \otimes_U (U \otimes_B E)$ agrees with that of $\text{Hom}_B(U, C_\gamma \otimes E)$. □

Remark 2.6. In general for any $\mathbb{Z}$-graded $B$-module $E'$, the inclusion $S_\gamma \otimes_U (U \otimes_B E') \cong N^\otimes \otimes_C E' \hookrightarrow \text{Hom}_B(U, C_\gamma \otimes E')$ is a $U$-map.

Proposition 2.7. Let $F$ be a $\mathbb{Z}$-graded $B$-module bounded from above, then the functor $\text{Hom}_U(S_\gamma, -)$ maps $\text{Hom}_B(U, F)$ to $U \otimes_B (C_{-\gamma} \otimes F)$.

Proof. The isomorphism of vector spaces $U \otimes_B (C_{-\gamma} \otimes F) \cong \text{Hom}_U(S_\gamma, \text{Hom}_B(U, F))$, induced from

$$\text{Hom}_U(S_\gamma, \text{Hom}_B(U, F)) \cong \text{Hom}_N(N^\otimes, \text{Hom}_C(N, F))$$

$$\cong \text{Hom}_C(N^\otimes, F) \cong N \otimes_C F \cong U \otimes_B (C_{-\gamma} \otimes F),$$

agrees with the composition of (left) $U$-maps

$$U \otimes_B (C_{-\gamma} \otimes F) \rightarrow \text{Hom}_U(S_\gamma, S_\gamma \otimes_U (U \otimes_B (C_{-\gamma} \otimes F))) \rightarrow \text{Hom}_U(S_\gamma, \text{Hom}_B(U, F)),$$

hence it is a $U$-isomorphism. □
In particular $S_\gamma \otimes U$ — transforms Weyl modules to the dual of Weyl modules, and $\mathcal{H}om_U(\gamma, -)$ transforms the latter to the former (both with a level shift).

**Corollary 2.8.** $S_\gamma \otimes_U V_{\lambda, k} \cong V_{\lambda^*, k}^*$ and $\mathcal{H}om_U(S_\gamma, V_{\lambda, k}^*) \cong V_{\lambda^*, k}^*$, here $\lambda^*$ denotes the highest weight of $V_{\lambda}^*$.

**Proof.** Note that $U \otimes_B V_\lambda = V_{\lambda, k}$ and $\mathcal{H}om_B(U, \mathbb{C} \otimes V_\lambda) \cong V_{\lambda^*, k}^*$ if $\mathbb{C}$ acts on $V_\lambda$ as scalar multiplication by $k$. □

3. Realization of $S_\gamma \otimes_U \mathcal{V}$ inside $U^*$

Fix a complex number $k$, and let $\mathcal{V} = U\mathcal{A}_{\mathfrak{g}, k}$ be the vertex operator algebra associated to the vertex algebra $\mathcal{A}_{\mathfrak{g}, k}$ (see [AG], [GMS1, 2], [Z1]). Note that in [Z1], we used $\mathcal{V}$ to denote the vertex operator algebra for generic values of $k \notin \mathbb{Q}$, but here we adopt this notation with no restriction on $k$.

The vertex operator algebra $\mathcal{V}$ admits two commuting actions of $\hat{\mathfrak{g}}$ in dual levels $k, \bar{k} = -2\hbar^\vee - k$. It follows from Lemma 2.1 that $S_\gamma \otimes_U \mathcal{V}$, using the $\hat{\mathfrak{g}}$-module structure of $\mathcal{V}$, becomes a $\hat{\mathfrak{g}}_{-k} \otimes \hat{\mathfrak{g}}_{-\bar{k}}$-representation. Define $U(\mathfrak{h}, k) = U(\mathfrak{h})/(\mathfrak{c} - k)U(\mathfrak{h})$. Our goal is to construct an embedding of $U$-bimodules

$$\Phi : S_\gamma \otimes_U \mathcal{V} \hookrightarrow U(\mathfrak{h}, \bar{k})^*.$$  

Let $\mathbb{B} = \bigoplus_{i \leq 0} \mathbb{B}_i$ (denoted by “$B$” with opposite grading in [Z1]) be the commutative vertex subalgebra of $\mathcal{V}$ generated by $A$, where $A$ is the commutative algebra of regular functions on an affine connected algebraic group $G$ with Lie algebra $\mathfrak{g}$. Recall that $\mathbb{B}$ is closed under the actions of $U(\mathfrak{h}_{\geq 0}, k)$ and $U(\mathfrak{h}_{\geq 0}, \bar{k})$. As a $\hat{\mathfrak{g}}_k$-module, we have $\mathcal{V} \cong U \otimes_B \mathbb{B} \cong N \otimes_C \mathbb{B}$ (see e.g. [Z1, Proposition 3.16]). Since $S_\gamma \cong N^\otimes \otimes N U$ as a right $U$-module, we have

$$S_\gamma \otimes_U \mathcal{V} \cong N^\otimes \otimes_N \mathcal{V} \cong N^\otimes \otimes_C \mathbb{B}.$$  

Define a functional $\epsilon : \mathbb{B} \to \mathbb{C}$ as follows: $\epsilon|_{\mathbb{B}_0} = 0$ and its restriction to $\mathbb{B}_{0} = A$ is the evaluation of functions at identity.

Multiplication induces isomorphism of vector spaces: $N \otimes_C \mathbb{B} \cong U$, hence any $u \in U$ can be written as $u = u_{<0}u_{\geq 0}$ with $u_{<0} \in N$ and $u_{\geq 0} \in \mathbb{B}$.

Let $\gamma : U \to U; u \to \overline{u}$ be the anti-involution of $U$ determined by $-\text{Id} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$.

Define a map

$$\Phi : S \otimes_U \mathcal{V} \to U^*$$  

as follows: for any $f \in N^\otimes, b \in \mathbb{B}$,

$$\Phi(f \otimes b)(u_{<0}u_{\geq 0}) = f(u_{\geq 0})\epsilon(u_{\geq 0} \cdot b),$$

here $u_{\geq 0} \cdot b$ means the $U(\mathfrak{h}_{\geq 0}, \bar{k})$-action on $\mathbb{B}$. In fact $\Phi(f \otimes b) \in U(\hat{\mathfrak{g}}, \bar{k})^*$.

The dual space $U^*$ is a $U$-bimodule via the recipes $(u \cdot g)(u_1) = g(u_1 u)$ and $(g \cdot u)(u_1) = g(uu_1)$ for any $u, u_1 \in U, g \in U^*$.

**Theorem 3.1.** For any $u \in U$ and $f \otimes b \in N^\otimes \otimes C \mathbb{B}$ ($\cong S_\gamma \otimes_U \mathcal{V}$), we have

$$\Phi(u^l \cdot (f \otimes b)) = (\Phi(f \otimes b)) \cdot \overline{u},$$

$$\Phi(u^r \cdot (f \otimes b)) = u \cdot (\Phi(f \otimes b)),$$

here $u^l \cdot (f \otimes b), u^r \cdot (f \otimes b)$ stand for the $\hat{\mathfrak{g}}_{-k}$- and $\hat{\mathfrak{g}}_{\bar{k}}$-actions on $S_\gamma \otimes_U \mathcal{V}$ respectively.
To prove the theorem, we need some preparations. First, let

$$\Theta : S \otimes_U \mathcal{V} \to \mathcal{H} \text{om}_B(U, C \otimes \mathcal{B})$$

be the (left) $U$-map described in Remark 2.5 (taking $E' = \mathcal{B}$). Note that we regard $\mathcal{V}, \mathcal{B}$ as non-positively graded, i.e., taking the opposite of the grading defined by the conformal weights of the vertex operator algebra $\mathcal{V}$. Here $\mathcal{B}$ is regarded as a left $B$-module via the $U(\hat{\mathfrak{g}}_{\geq 0}, k)$-action on $\mathcal{B}$, and $\Theta$ is a $U(\hat{\mathfrak{g}}, -\hat{k})$-map.

Following [GMS2, Z1], let $\tau_i$ be an orthonormal basis of $\mathfrak{g}$ with respect to the normalized Killing form $(\cdot, \cdot)$. Let $C_{ijk}$ be the structure constants determined by $[\tau_i, \tau_j] = C_{ijk}\tau_k$. We identify $\mathfrak{g}$ with the tangent space to the identity of $G$. Let $\tau_i^L$ (resp. $\tau_i^R$) be the left (resp. right) invariant vector fields valued $\tau_i$ (resp. $-\tau_i$) at the identity, there exist regular functions $a^{ij} \in A$ such that $\tau_i^R = a^{ij}\tau_j^L$ and $\epsilon(a^{ij}) = -\delta_{ij}$.

**Lemma 3.2.** Let $\beta : B \to B$ be the automorphism which restricts to $\hat{\mathfrak{g}}_{\geq 0}$ as $X \mapsto \gamma(X) + X$, then for any $u_0 \geq 0$ in $B$ and $b \in \mathcal{B}$, we have $\epsilon(\beta(u_0)^l \cdot b) = \epsilon(\overline{u_0}^\tau \cdot b)$, here $\beta(u_0)^l \cdot b, \overline{u_0}^\tau \cdot b$ denote the $U(\hat{\mathfrak{g}}_{\geq 0}, k)$- and $U(\hat{\mathfrak{g}}_{\geq 0}, k)$-actions on $\mathcal{B}$.

**Proof.** By [Z1, Lemma 3.14 (10)], we have $\tau_j(n)^l \cdot b = \sum_i \sum_{p \geq 0} a_{i,j}^{ij}_{n+p} \tau_i(n+p)^r \cdot b$ for any $n \geq 0, b \in \mathcal{B}$. Since $\epsilon|_{B^1} = 0$, we have $\epsilon(\tau_j(n)^l \cdot b) = \sum_i \epsilon(a_{i,j}^{ij}_{n+1}) \tau_i(n)^r \cdot b = \sum_i (-\delta_{ij}) \epsilon(\tau_j(n)^r \cdot b) = -\epsilon(\tau_j(n)^r \cdot b)$. Since the $U(\hat{\mathfrak{g}}_{\geq 0}, k)$- and $U(\hat{\mathfrak{g}}_{\geq 0}, k)$-actions on $\mathcal{B}$ commute, for any $u_0 = \tau_j(n_1) \cdots \tau_{j_q}(n_q)$, we have $\epsilon(\beta(u_0)^l \cdot b) = \epsilon(u_0^l \cdot b) = \epsilon(-\tau_{j_1}(n_1)^r \cdot (\tau_{j_2}(n_2) \cdots \cdot (\tau_{j_q}(n_q)^r \cdot b) = \epsilon((\tau_{j_1}(n_1)^r \cdot \cdot \cdot \cdot \cdot (\tau_{j_q}(n_q)^r \cdot \cdot \cdot \cdot \cdot (\tau_{j_1}(n_1)) \cdot b) = \cdots = \epsilon((-\tau_{j_q}(n_q)) \cdot \cdots \cdot \cdot \cdot \cdot (\tau_{j_1}(n_1)) \cdot b) = \epsilon(\overline{u_0}^\tau \cdot b)$. We also have $\epsilon(\beta(u)^l \cdot b) = \epsilon((u + 2h^\tau)^l \cdot b) = \epsilon((k + 2h^\tau)b) = \epsilon(-kb) = \epsilon(\overline{c}^\tau \cdot b)$, hence the lemma is proved. □

**Proposition 3.3.** For any $f \otimes b \in N^{\otimes} \otimes_C \mathcal{B}$, we have $\Phi(f \otimes b) = \epsilon(\Theta(f \otimes b))^{-}$.

**Proof.** By the definition of $\Theta$, for any $u = u_{<0}u_0 \in U$, we have $\Theta(f \otimes b)(\overline{u_0}^\tau) = \Theta(f \otimes b)(\overline{u_0}^\tau)$. Then it follows from Lemma 3.2 that $\epsilon(\Theta(f \otimes b)(\overline{u_0}^\tau) = f(\mu_{<0})\Theta(f \otimes b)(\overline{u_0}^\tau) \cdot \overline{u}$. □

**Corollary 3.4.** For any $u \in U$ and $f \otimes b \in N^{\otimes} \otimes_C \mathcal{B}$, we have $\Phi(u^l \cdot (f \otimes b)) = (\Phi(f \otimes b)) \cdot \overline{u}$.

**Proof.** Since $\Theta$ is a (left) $U$-map, by Proposition 3.3, we have

$$\Phi(u^l \cdot (f \otimes b)) = \epsilon(\Theta(u^l \cdot (f \otimes b)))^{-} \cdot \epsilon(u \cdot \Theta(f \otimes b))^{-}$$

$$= \epsilon(\Theta(f \otimes b) \cdot r_u) = \epsilon(\Theta(f \otimes b) \cdot l_u) = \Phi(f \otimes b) \cdot \overline{u},$$

where $r_u, l_u : U \to U$ denote the right and left multiplications by $u$ and $\overline{u}$ respectively. Hence we proved one half of Theorem 3.1. □

Next we prove the other half of Theorem 3.1, which is to show that

$$\Phi(u^r \cdot (f \otimes b)) = u \cdot (\Phi(f \otimes b))$$

If $u = u_0 \geq 0 \in B$, then $u_0^{u_0^r} \cdot (f \otimes b) = f \otimes u_0^{u_0^r} \cdot b$. Hence $\Phi(f \otimes (u_0^{u_0^r} \cdot b))(u_0^{u_0^r} \cdot b) = f(u_0^{u_0^r})\epsilon(u_0^{u_0^r} \cdot u_0^{u_0^r} \cdot b) = \Phi(f \otimes b)(u_0^{u_0^r} \cdot u_0^{u_0^r} \cdot b) = u_0 \cdot (\Phi(f \otimes b))(u_0^{u_0^r} \cdot b)$, which means that $\Phi(u_0^{u_0^r} \cdot (f \otimes b)) = u_0 \cdot (\Phi(f \otimes b))$. □
To prove it holds for \( u = u_{<0} \in N \) as well, it suffices to show that \( \Phi(\tau_i(-1)^r \cdot (f \otimes b)) = \tau_i(-1) \cdot (\Phi(f \otimes b)) \) since \( \hat{g}_{<0} \) is generated by \( \hat{g}_{-1} \).

Recall that although \( \mathbb{B} \) is only closed under the action of \( U(\hat{g}_{\geq 0}, k) \), it can be equipped with a \( \hat{g}_k \)-module structure \( \tilde{\rho} : U \to \text{End}(\mathbb{B}) \) such that \( \tilde{\rho}(u_{\geq 0})b = u_{\geq 0}^r \cdot b \) for any \( u_{\geq 0} \in B \) and \( b \in \mathbb{B} \) (see [Z1, Lemma 3.29, Remark 3.30]). In addition, we have

\[
\tau_i(-1)^r \cdot (f \otimes b) = \sum_j f \cdot \tau_i(-1) \otimes (a^{ij} b) + f \otimes \tilde{\rho}(\tau_i(-1)) b
\]

(see [Z1, Lemma 3.14 (9)]). Hence for any \( u_{<0} \in N, u_0 \in U(\hat{g}_0) \) and \( u_{>0} \in U(\hat{g}_{>0}) \), we have

\[
\Phi(\tau_i(-1)^r \cdot (f \otimes b))(u_{<0}u_0u_{>0}) = \sum_j f(\tau_j(-1)u_{<0})\epsilon(u_{<0}^r \cdot u_{>0}^r \cdot (a^{ij} b)) + f(u_{<0}^r \cdot u_{>0}^r \cdot \tilde{\rho}(\tau_i(-1)) b)
\]

\[
= \sum_j f(\tau_j(-1)u_{<0})\epsilon(u_{<0}^r \cdot \tau_i(-1))^r \cdot b) + f(u_{<0}^r \cdot [u_{>0}, \tau_i(-1)]^r \cdot b) + \sum_s f(u_{<0}^r \cdot u_{<0})\epsilon(F^{i,s}(u_0)^r \cdot u_{>0}^r \cdot b)
\]

The last equality is because \( [u_{>0}^r, a_{(-1)}^{ij}] = 0 \) (see [Z1, Lemma 3.14 (4)]), and \( [u_{>0}, \tau_i(-1)] \in B, \epsilon|_{B_{>1}} = 0 \).

On the other hand, we have

\[
\tau_i(-1) \cdot (\Phi(f \otimes b))(u_{<0}u_0u_{>0}) = \Phi(f \otimes b)(u_{<0}u_0u_{>0} \tau_i(-1))
\]

\[
= \Phi(f \otimes b)(u_{<0}u_0[u_{>0}, \tau_i(-1)] + u_{<0}[u_0, \tau_i(-1)]u_{>0} + u_{<0}\tau_i(-1)u_{>0})
\]

\[
= f(u_{<0})\epsilon(u_{<0}^r \cdot [u_{>0}, \tau_i(-1)]^r \cdot b) + \sum_s f(u_{<0}^r \cdot u_{<0})\epsilon(F^{i,s}(u_0)^r \cdot u_{>0}^r \cdot b)
\]

\[
+ f(u_{<0}^r \cdot \tau_i(-1))\epsilon(u_{>0}^r \cdot b)
\]

where \( F^{i,s} : U(\hat{g}_0) \to U(\hat{g}_0) \) are maps such that \( [u_0, \tau_i(-1)] = \sum_s \tau_i(s)F^{i,s}(u_0) \) for any \( u_0 \in U(\hat{g}_0) \).

Since \( \tau_i(-1)^r(a^{ij}) = C_{kip}b^{pj} \), we have \( \tau_i(0)^r, a_{(-1)}^{ij} = C_{kip}b^{pj} \) (see [Z1, Lemma 3.14 (4)]). Compare it with the commutator \( [\tau_i(0), \tau_i(-1)] = C_{kip}\tau_p(-1) \), it follows that \( [u_0^r, a_{(-1)}^{ij}] = \sum_s a_{(-1)}^{sij}F^{i,s}(u_0)^r \). Hence we have

\[
\sum_j f(\tau_j(-1)u_{<0})\epsilon(u_{<0}^r \cdot a_{(-1)}^{sij}u_{>0}^r \cdot b)
\]

\[
= \sum_j f(\tau_j(-1)u_{<0})\epsilon(\sum_s a_{(-1)}^{sij}F^{i,s}(u_0)^r \cdot u_{>0}^r \cdot b) + \sum_j f(\tau_j(-1)u_{<0})\epsilon(a_{(-1)}^{sij}u_{>0}^r \cdot b)
\]

\[
= \sum_j f(\tau_j(-1)u_{<0})\epsilon(-F^{i,j}(u_0)^r \cdot u_{>0}^r \cdot b) + f(\tau_i(-1)u_{<0})\epsilon(-u_{>0}^r \cdot b)
\]

\[
= \sum_j f(u_{<0}\tau_j(-1))\epsilon(F^{i,j}(u_0)^r \cdot u_{>0}^r \cdot b) + f(u_{<0}\tau_i(-1))\epsilon(u_{>0}^r \cdot b),
\]

which proves that \( \Phi(\tau_i(-1)^r \cdot (f \otimes b)) = \tau_i(-1) \cdot (\Phi(f \otimes b)) \). The proof of Theorem 3.1 is now complete.
Remark 3.5. Following the notations in [Z1], let \( \{ \bar{\omega}_i \} \) be right invariant 1-forms dual to \( \{ \tau_i^R \} \), and let \( \mathcal{B}_0 \) be the linear span of elements of the form \( \partial^{(j_1)} \bar{\omega}_{i_1} \cdots \partial^{(j_n)} \bar{\omega}_{i_n} \), then \( \mathcal{B} = A \otimes \mathcal{B}_0 \). There is a non-degenerate pairing between \( U(\hat{g}_{>0}) \) and \( \mathcal{B}_0 \), defined by \( (u_{>0}, b) = \epsilon(u_{>0} \cdot \tilde{b}) \), via which \( \mathcal{B}_0 \) can be identified with \( U(\hat{g}_{>0})^\vee \), the graded dual of \( U(\hat{g}_{>0}) \). The regular functions \( A \) can be identified with the Hopf dual \( U(\mathfrak{g})^\ast \) of \( U(\mathfrak{g})^\ast \) defined by

\[
U(\mathfrak{g})^\ast \text{Hopf} = \{ \phi \in U(\mathfrak{g})^\ast \mid \text{Ker}\phi \text{ contains a two-sided ideal } J \subset U(\mathfrak{g}) \}
\]

of finite codimension).

It is not hard to see that \( \epsilon(u_{>0} \cdot u_{>0} \cdot a \tilde{b}) = \epsilon(u_{>0} \cdot a) \epsilon(u_{>0} \cdot \tilde{b}) \) for any \( u_{>0} \in U(\mathfrak{g}), u_{>0} \in U(\hat{g}_{>0}), a \in A \) and \( \tilde{b} \in \mathcal{B}_0 \). Hence

\[
S \otimes_U V \cong N^\otimes \otimes \mathcal{B} \cong N^\otimes \otimes A \otimes \mathcal{B}_0 \cong U(\hat{g}_{<0})^\otimes \otimes U(\mathfrak{g})^\ast \text{Hopf} \otimes U(\hat{g}_{>0})^\otimes \subset U(\hat{g}, \hat{k})^\ast,
\]

and \( \Phi \) is injective.

4. Filtrations of the Vertex Operator Algebra \( V \)

Fix \( k \in \mathbb{Q}, k > -h^\vee \); set \( \varkappa = k + h^\vee > 0 \). Let \( \mathcal{O}_{-\varkappa} \) be the full subcategory of the category of \( \hat{\mathfrak{g}}_k \)-modules defined by Kazhdan and Lusztig in [KL1-4]. They constructed a tensor structure on \( \mathcal{O}_{-\varkappa} \), and established an equivalence of tensor categories between \( \mathcal{O}_{-\varkappa} \) and the category of finite-dimensional integrable representations of the quantum group with quantum parameter \( q = e^{-i\pi/\varkappa} \) (in the simply-laced case).

Let \( V_{\lambda, \hat{k}} = \text{Ind}_{\hat{g}_{>0}}^{\hat{g}_k} V_\lambda \) be a Weyl module, denote the irreducible quotient of \( V_{\lambda, \hat{k}} \) by \( L_{\lambda, \hat{k}} \).

Definition 4.1. [KL1, Definition 2.15] \( \mathcal{O}_{-\varkappa} \) is the full subcategory of \( \hat{\mathfrak{g}}_k \)-modules, which admits a finite composition series with factors of the form \( L_{\lambda, \hat{k}} \) for various \( \lambda \in P^+ \).

Let us recall some basic facts about \( \mathcal{O}_{-\varkappa} \). The \( \mathbb{Z}_{>0} \)-grading on \( \hat{g}_{>0} \) induces an \( \mathbb{N} \)-grading on the enveloping algebra: \( U(\hat{g}_{>0}) = \bigoplus_{n \geq 0} U(\hat{g}_{>0})_n \). For any \( V \in \mathcal{O}_{-\varkappa} \), \( v \in V \), there exists an \( n_1 \in \mathbb{N} \) such that \( U(\hat{g}_{>0})_{n_1} \cdot v = 0 \).

A module \( \mathcal{N} \) over \( \mathfrak{g} \otimes \mathbb{C}[\hat{t}] \) is said to be a nil-module if \( \dim \mathcal{N} < \infty \) and there exists a \( n \geq 1 \) such that \( U(\hat{g}_{>0})_n \mathcal{N} = 0 \). Extend \( \mathcal{N} \) to a \( \hat{\mathfrak{g}}_{>0} \)-module by defining the action of \( \mathfrak{g} \) to be multiplication by \( \hat{k} \), and let \( \mathcal{N}_{\hat{k}} = \text{Ind}_{\hat{g}_{>0}}^{\hat{g}_k} \mathcal{N} \) be the induced module. We say that \( \mathcal{N}_{\hat{k}} \) is a generalized Weyl module.

Proposition 4.2. [KL1, Theorem 2.22] A \( \hat{\mathfrak{g}}_k \)-module \( V \) is in \( \mathcal{O}_{-\varkappa} \) if and only if \( V \) is a quotient of a generalized Weyl module.

Given \( V \in \mathcal{O}_{-\varkappa} \), let \( \hat{L}_0 : V \to V \) be the Sugawara operator defined by \( \hat{L}_0 v = -\frac{1}{\varkappa} \sum_{j \geq 0} \sum_i \tau_i(-j) \tau_i(j) v - \frac{i}{\varkappa} \sum \tau_i(0) \tau_i(0) v \), where \( \{ \tau_i \} \) is an orthonormal basis of \( \mathfrak{g} \) with respect to the normalized Killing form. Note that this operator is well defined and locally finite. Let \( V_z \) be the generalized eigenspace of \( \hat{L}_0 \) with eigenvalue \( -z \in \mathbb{C} \), we have \( V = \bigoplus_{z \in \mathbb{C}} V_z \) with \( \dim V_z < \infty \). In fact there exist \( z_1, \ldots, z_m \in \mathbb{Q} \) such that \( \{ z \in V_z \neq 0 \} \subset \{ z_1 - \mathbb{N} \} \cup \cdots \cup \{ z_m - \mathbb{N} \} \), and \( V \) becomes a \( \mathbb{Q} \)-graded \( \hat{\mathfrak{g}}_k \)-representation, i.e. \( x(n) V_z \subset V_{z+n} \) for any \( x(n) \in \hat{\mathfrak{g}} \) (see [KL1, Lemma 2.20, Proposition 2.21]).
case $V = V_{\lambda,k}$ is a Weyl module, $L_0$ acts on $V_{\lambda,k}$ semisimply. More specifically, we have $L_0|_{U(\hat{\mathfrak{g}}_{<0}) \otimes V_{\lambda}} = -\frac{(\lambda,\lambda+2\rho)}{2\pi} + n$, where $\rho$ is the half sum of positive roots.

Define the dual representation of $V$ as follows: as a vector space $V^* = \bigoplus_{z}(V_z)^*$; the $\hat{\mathfrak{g}}$-action is given by $Xf(v) = f(-Xv)$ for any $X \in \mathfrak{g}, f \in V^*, v \in V$. In particular $V^*$ is a $\hat{\mathfrak{g}}_k$-module and locally $U(\hat{\mathfrak{g}}_{<0})$-finite. In order for $V^*$ to be a graded $\hat{\mathfrak{g}}$-module as well, set $(V^*)_z = (V_z)^*$, or equivalently set $(V^*)_z$ to be the generalized $(-z)$-eigenspace of the operator $L_0' = \frac{1}{2\pi} \sum_{j > 0} \tau_j(j) + \frac{1}{2\pi} \sum_{j > 0} \tau_j(0)$ which acts on $V^*$.

The contragredient dual $V^\vee$ is isomorphic to $V^*$ as a vector space, but instead of using $-\text{Id} : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$, we use the anti-involution $x(n) \mapsto -x(-n), \mathcal{E} \mapsto \mathcal{F}$ to define the $\hat{\mathfrak{g}}$-action on $V^\vee$. Unlike $V^*$, the contragredient module $V^\vee$ is a $\hat{\mathfrak{g}}_k^*$-representation, locally $U(\hat{\mathfrak{g}}_{<0})$-finite, and in fact belongs to $O_{-\infty}$.

Given $V \in O_{-\infty}$, define a map $\phi_V : V^* \otimes V \to U(\hat{\mathfrak{g}}, \hat{\mathfrak{k}})^*$. Then $\phi_V(f \otimes v)(u) = \langle f(u) \cdot v \rangle$ for any $f \in V^*, v \in V, u \in U(\hat{\mathfrak{g}})$. It is easy to see that $\phi_V$ is a $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\infty}$-map, where the $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\infty}$-module structure of $U(\hat{\mathfrak{g}}, \hat{\mathfrak{k}})^*$ is given by $(X, 0) \cdot g = -g \cdot X$ and $(0, X) \cdot g = X \cdot g$ for any $X \in \hat{\mathfrak{g}}, g \in U(\hat{\mathfrak{g}}, \hat{\mathfrak{k}})^*$. Denote the image of $\phi_V$ by $M(V)$, which is called the matrix coefficients of $V$.

Recall the $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\infty}$-map $\Phi : S_G \otimes U\hat{\mathfrak{g}} \to U(\hat{\mathfrak{g}}, \hat{\mathfrak{k}})^*$ defined in Section 3. As pointed out in Remark 3.15, the map $\Phi$ is injective and its image, which we denote by $M^{O_{-\infty}}$, is isomorphic to $U(\hat{\mathfrak{g}}_{<0}) \otimes U(\hat{\mathfrak{g}}_{\infty}) \wedge U(\hat{\mathfrak{g}}_{>0})$. Here $U(\hat{\mathfrak{g}}_{<0}) = \bigoplus_{n \leq 0} (U(\hat{\mathfrak{g}}_{<0})_n)^*$, $U(\hat{\mathfrak{g}}_{>0}) = \bigoplus_{n > 0} (U(\hat{\mathfrak{g}}_{<0})_n)^*$ are graded duals.

**Proposition 4.3.** $M^{O_{-\infty}}$ consists of matrix coefficients of modules from the category $O_{-\infty}$, i.e. $M^{O_{-\infty}} = \sum_{V \in O_{-\infty}} M(V)$.

**Proof.** Let $V = \bigotimes V_z \in O_{-\infty}, v \in V$ and $f \in V^*$, for any $u = u_{<0} u_{>0} \in U = U(\hat{\mathfrak{g}})$, we have $\phi_V(f \otimes v)(u) = \langle f(u_{<0} u_{>0} \cdot v) \rangle = \langle u_{<0} \cdot f(u_{>0}) \cdot u_{>0} \cdot v \rangle$. Since $V \in O_{-\infty}$, there exist $n_1, n_2 \in \mathbb{N}$ such that $U(\hat{\mathfrak{g}}_{<0})_{-n_1} \cdot f = U(\hat{\mathfrak{g}}_{>0})_{n_2} \cdot v = 0$. Moreover each $V_z$ is finite-dimensional and semisimple as a $\mathfrak{g}$-module, therefore it is not hard to see that $\phi_V(f \otimes v) \in U(\hat{\mathfrak{g}}_{<0})^* \otimes U(\hat{\mathfrak{g}}_{\infty}) \wedge U(\hat{\mathfrak{g}}_{>0})^*$, i.e. $\mathcal{M}(V) \subset M^{O_{-\infty}}$.

On the other hand, let $g \in M^{O_{-\infty}}$, there exists an $n \in \mathbb{N}$ such that $U(\hat{\mathfrak{g}}_{>0})_n \cdot g = 0$. Since each $U(\hat{\mathfrak{g}}_{>0})_{n'}$ is finite-dimensional and $\mathfrak{g}$ acts on $M^{O_{-\infty}}$ locally finitely, the $\hat{\mathfrak{g}}_{>0}$-submodule generated by $g$ is a nil-module. Hence the $\mathfrak{g}$-submodule $W = U(\hat{\mathfrak{g}}) \cdot g$ generated by $g$ is a quotient of a generalized Weyl module, hence it belongs to $O_{-\infty}$. Let $\delta$ be the functional on $U^*$ defined by $\delta(g') = g'(1)$, then $\delta \in W^*$ and $g = \phi_W(\delta \otimes g) \subset M(W)$.

Define two operators $\tilde{L}_0, L_0'$ that act on $M^{O_{-\infty}}$ as follows: for any $g \in M^{O_{-\infty}}$, set $L_0 g = -\frac{1}{2\pi} \sum_{j > 0} \tau_j(j) \cdot \tau_j(0) \cdot \tau_j(0)$ and $L_0' g = \frac{1}{2\pi} \sum_{j > 0} \tau_j(j) + \frac{1}{2\pi} \sum_{j > 0} \tau_j(0) \cdot \tau_j(0)$. Let $M_{z,z}^{O_{-\infty}}$ be the subspace consisting of all $g \in M^{O_{-\infty}}$ such that $g$ is in the kernel of some power of $\tilde{L}_0 + z \text{Id}$ and the kernel of some power of $L_0' + z \text{Id}$. Then $M^{O_{-\infty}} = \bigoplus_{z'} M_{z',z}^{O_{-\infty}}$, and $\phi_V((V_z)^*)_{z'} \subset M_{z',z}^{O_{-\infty}}$ for any $V \in O_{-\infty}$. Moreover $M_{z,z}^{O_{-\infty}} \cdot x(n) \subset M_{z+n,z}^{O_{-\infty}}$ and $x(n) \cdot M_{z,z}^{O_{-\infty}} \subset M_{z,z+n}^{O_{-\infty}}$ for any $x(n) \in \mathfrak{g}$. Define a $\mathbb{Z}$-grading on $M^{O_{-\infty}}$: for any $g_1 \in (U(\hat{\mathfrak{g}}_{<0})_n)^*, \ a \in U(\hat{\mathfrak{g}}_{>0})^\text{Hopf}$, $g_2 \in (U(\hat{\mathfrak{g}}_{>0})_n)^*$, define $\deg g_1 \otimes a \otimes g_2 = -n - n'$; set $M_n^{O_{-\infty}} = \{g | \deg g = n\}$. It is not difficult to see that $M_n^{O_{-\infty}} = \bigoplus_{z+n',z} M_{z',z}^{O_{-\infty}}$.\]
Lemma 4.4. Let $V, V' \in \mathcal{O}_{-\infty}$.

1. If $V$ has a (finite) Weyl filtration with factors isomorphic to $V_{\lambda_i, k}$ for various $\lambda_i \in P^+$, then $\mathbb{M}(V) \subset \sum_i \mathbb{M}(T_{\lambda_i, k})$.

2. If $V'$ has a (finite) filtration with factors isomorphic to $V_{\mu_i, k}^c$ for various $\mu_i \in P^+$, then $\mathbb{M}(V') \subset \sum_i \mathbb{M}(T_{\mu_i, k}^c)$.

Proof. The proof is exactly the same as that of [Z2, Lemma 3.2]: we can construct an injection $V \hookrightarrow \bigoplus_i T_{\lambda_i, k}$, and a surjection $\bigoplus_i T_{\mu_i, k} \twoheadrightarrow V'$, since $\text{Ext}^1_{\mathcal{O}_{-\infty}}(V_{\lambda_i, k}, V_{\mu_i, k}^c) = 0$ (see [KL4, Proposition 27.1]). □

Corollary 4.5. $\mathbb{M}^{\mathcal{O}_{-\infty}}$ consists of the matrix coefficients of tilting modules from $\mathcal{O}_{-\infty}$, i.e. $\mathbb{M}^{\mathcal{O}_{-\infty}} = \mathbb{M}^{\mathcal{O}_{-\infty}, \text{tilting}} \mathbb{M}(V)$.

Proof. For any $V \in \mathcal{O}_{-\infty}$, choose $s$ such that $V \in \mathcal{O}_{-\infty}^s$. By [KL1, Proposition 3.9], there exists a $P$, projective in $\mathcal{O}_{-\infty}^s$ and having a (finite) Weyl filtration, such that $V$ a quotient of $P$. Hence by Lemma 4.4 (1), we have $\mathbb{M}(V) \subset \mathbb{M}(P) \subset \sum_i \mathbb{M}(T_{\lambda_i, k})$ for some $\lambda_i \in F^s$. □

Proposition 4.6. Order the dominant weights in such a way $P^+ = \{\nu_1, \ldots, \nu_i, \ldots\}$ that $\nu_i < \nu_j$ implies $i < j$. Set $\mathbb{M}_{\mathcal{O}_{-\infty}, i}^{\mathcal{O}_{-\infty}} = \sum_{j \leq i} \mathbb{M}(T_{\nu_i, k})$, then $\mathbb{M}_{\mathcal{O}_{-\infty}, i}^{\mathcal{O}_{-\infty}} \subset \cdots \subset \mathbb{M}_{\mathcal{O}_{-\infty}, i-1}^{\mathcal{O}_{-\infty}} \subset \cdots$ is an increasing filtration of $\hat{\mathfrak{g}}_{-k} \oplus \hat{\mathfrak{g}}_{k}$-submodules of $\mathbb{M}^{\mathcal{O}_{-\infty}}$ with factors $\mathbb{M}^{\mathcal{O}_{-\infty}, i}/\mathbb{M}^{\mathcal{O}_{-\infty}, i-1}$ isomorphic to $V_{\nu_i, k}^c \otimes V_{-\omega_0 \nu_i, k}$, where $\omega_0$ is the longest element in the Weyl group.

Proof. The proof is the same as that of [Z2, Theorem 3.3], using Lemma 4.4. □

Remark 4.7. The category $\mathcal{O}_{-\infty}$ is a direct sum of subcategories corresponding to the orbits of the shifted action of affine Weyl group on the weight lattice (see [KL4, Lemma 27.7]). Hence we can decompose $\mathbb{M}^{\mathcal{O}_{-\infty}}$, as a $\hat{\mathfrak{g}}_{-k} \oplus \hat{\mathfrak{g}}_{k}$-module, into summands corresponding to the orbits as well. Some summands are semisimple (see [KL4, Proposition 27.4], [Z2, Proposition 3.1]), but all have an increasing filtration of the above type.

Proposition 4.8. The vertex operator algebra $V$ is isomorphic to $\text{Hom}_U(S_\gamma, \mathbb{M}^{\mathcal{O}_{-\infty}})$ as a $\hat{\mathfrak{g}}_{k} \oplus \hat{\mathfrak{g}}_{-k}$-module.

Proof. Recall that $\mathbb{M}^{\mathcal{O}_{-\infty}} \cong S_\gamma \otimes_U V = N^\otimes \otimes \mathbb{B}$. Hence $\text{Hom}_U(S_\gamma, \mathbb{M}^{\mathcal{O}_{-\infty}}) \cong \text{Hom}_N(N^\otimes, N^\otimes \otimes \mathbb{B}) \cong \text{Hom}_C(N^\otimes, \mathbb{B}) \cong N \otimes \mathbb{B} \cong V$, the second to last isomorphism is because $\mathbb{B}$ is non-positively graded while $N^\otimes$ is non-negatively graded. Moreover the induced isomorphism $V \rightarrow \text{Hom}_U(S_\gamma, \mathbb{M}^{\mathcal{O}_{-\infty}}) \cong \text{Hom}_U(S_\gamma, S_\gamma \otimes_U V)$ is a $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{g}}$-map. □

Lemma 4.9. For any $b \in \mathbb{B}$, there exists an $i$ such that $N^\otimes \otimes b \subset \mathbb{M}^{\mathcal{O}_{-\infty}, i}$. 
Proof. For any $f \in N^\circ$ and $u_{>0} \in U(\hat{g}_{>0})$, we have $u_{>0} \cdot (f \otimes b) = f \otimes (u_{>0}^+ \cdot b)$. Let $N$ be the $U(\hat{g}_{>0}, k)$-submodule of $B$ generated by $b$, then $N$ is a nil-module and the $\hat{g}_k$-submodule $U(\hat{g}, k) \cdot (f \otimes b)$ generated by $f \otimes b$ is a quotient of the generalized Weyl module $N_k$. Hence $f \otimes b \in M(N_k)$ for any $f \in N^\circ$, hence there exists an $i$ such that $N^\circ \otimes b \subset M(\Sigma_i)$. \hfill \Box

Theorem 4.10. Set $\Sigma^i = \mathcal{H}om_U(S, \mathcal{M}_{O^\circ -x_i})$, then $\mathcal{V} = \bigcup_i \Sigma^i$ and $\Sigma^1 \subset \cdots \subset \Sigma^{i-1} \subset \Sigma^i \subset \cdots$ is an increasing filtration of $\hat{g}_k \oplus \hat{g}_k$-modules of $\mathcal{V}$ with factors $\Sigma^i / \Sigma^{i-1}$ isomorphic to $V_{-\omega \nu_i,k} \otimes V^{c}_{-\omega \nu_i,k}$.

Proof. For any $u_{<0} \otimes b \in N \otimes \mathcal{B} \cong \mathcal{V}$, let $N' \subset \mathcal{B}$ be the $U(\hat{g}_{>0}, k)$-submodule generated by $b$, then $N'$ is finite-dimensional. For any $s \in S$, we have $p(s \otimes (u_{<0} \otimes b)) \in N^\circ \otimes N'$, where $p : S \otimes \mathcal{V} \rightarrow S \otimes_U \mathcal{V}$ is the canonical projection. By Lemma 4.9 there exists an $i$ such that $p(s \otimes (u_{<0} \otimes b)) \in M_{O^\circ -x_i}$ for any $s \in S$, hence $u_{<0} \otimes b \in \mathcal{H}om_U(S, \mathcal{M}_{O^\circ -x_i}) = \Sigma^i$. This proves that $\mathcal{V} = \bigcup_i \Sigma^i$.

Note that $M_{O^\circ -x_i} = \bigoplus z^x_{i}z^x_{i-1}$ with $\dim z^x_{i}z^x_{i-1} < \infty$. Fix $z$, the exact sequence of $\hat{g}_k \oplus \hat{g}_k$-modules $0 \rightarrow M_{O^\circ -x_i} \rightarrow V^{c}_{-\omega \nu_i} \otimes V^{c}_{-\omega \nu_i} \rightarrow 0$ restricts to an exact sequence of $\hat{g}_k$-modules $0 \rightarrow \bigoplus z^x_{i}z^x_{i-1} \rightarrow \bigoplus z^x_{i}z^x_{i-1} \rightarrow V^{c}_{-\omega \nu_i} \otimes V^{c}_{-\omega \nu_i} \rightarrow 0$ since $V^{c}_{-\omega \nu_i} \otimes V^{c}_{-\omega \nu_i}$ is a quotient of the generalized Weyl algebra $U(\hat{g}_1, \hat{g}_2) \otimes \cdots \otimes U(\hat{g}_1, \hat{g}_2)$.

Remark 4.11. The decomposition of $M_{O^\circ -x_i}$ discussed in Remark 4.7 leads to a decomposition of $\mathcal{V}$, as a $\hat{g}_k \oplus \hat{g}_k$-module, into summands corresponding to the orbits of the affine Weyl group on the weight lattice. Again some summands are semisimple, but each has an increasing filtration of the above type.

Corollary 4.12. The vertex operator algebra $\mathcal{V}$ admits a decreasing filtration of $\hat{g}_k \oplus \hat{g}_k$-submodules $\mathcal{V} \supset \mathcal{V} \supset \cdots \supset \mathcal{V} \supset \cdots$ with factors $\mathcal{V} / \mathcal{V}^{i-1}$ isomorphic to $V_{-\omega \nu_i,k} \otimes V^{c}_{-\omega \nu_i,k}$, and the fact that the grading on $V^{c}_{-\omega \nu_i,k}$ is bounded from above.

Proof. Let $L_0, \bar{L}_0 : \mathcal{V} \rightarrow \mathcal{V}$ be the Sugawara operators associated to the $\hat{g}_k$- and $\hat{g}_k$-actions on $\mathcal{V}$ respectively, i.e. $L_0 = \frac{1}{2} \sum_{i > 0} \sum \tau_i(-j)\tau_j(0)$ and $\bar{L}_0 = -\frac{1}{2\tau} \sum_{i > 0} \sum \tau_i(-j)\tau_j(0)$ and $\bar{L}_0 = -\frac{1}{2\tau} \sum_{i > 0} \sum \tau_i(-j)\tau_j(0)$ and $\bar{L}_0 = -\frac{1}{2\tau} \sum_{i > 0} \sum \tau_i(-j)\tau_j(0)$. Now we regard the vertex operator algebra $\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}_n$ as non-negatively graded, then the sum $L_0 = L_0 + \bar{L}_0$ is the gradation operator, i.e. $L_0 |_{\mathcal{V}_n} = nI \mathcal{V}$ (see [Z1, Proposition 3.20, 3.24]).

Let $\mathcal{V}_{z_1,z_2}$ be the subspace consisting of $v \in \mathcal{V}$ such that $v$ is killed by some power of $L_0 - z_1I \mathcal{V}$ and some power of $\bar{L}_0 - z_2I \mathcal{V}$. It follows from Theorem 4.10 that $\mathcal{V} = \bigoplus_{z_1,z_2} \mathcal{V}_{z_1,z_2}$ with $\dim \mathcal{V}_{z_1,z_2} < \infty$.

Recall the symmetric non-degenerate bilinear form $\langle, \rangle : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$ constructed in [Z1, Proposition 3.28]. It is shown to be compatible with the vertex operator
algebra structure of $\mathbb{V}$, in particular we have $\langle x(n), \cdot \rangle = \langle \cdot, -x(-n) \rangle$ and $\langle y(n), \cdot \rangle = \langle \cdot, -\bar{y}(-n) \rangle$ for any $x(n) \in \hat{\mathfrak{g}}_k, y(n) \in \hat{\mathfrak{g}}_{\bar{k}}$. It implies that $\langle L_0, \cdot \rangle = \langle \cdot, L_0 \rangle$ and $\langle \bar{L}_0, \cdot \rangle = \langle \cdot, \bar{L}_0 \rangle$. Hence $\langle \cdot \rangle |_{V^*_{z_1, z_2} \times V^*_{z_1', z_2'}} = 0$ except when $z_1 = z_1'$ and $z_2 = z_2'$, in which case the pairing is non-degenerate.

Let $V^c = \bigoplus_{z_1, z_2} V^*_{z_1, z_2}$ be the contragredient dual of $V$, where the $\hat{\mathfrak{g}}_k$- and $\hat{\mathfrak{g}}_{\bar{k}}$-actions on $V^c$ are both defined by the anti-involution $x(n) \mapsto -x(-n); c \mapsto c$ of $\hat{\mathfrak{g}}$. Then we have $V \cong V^c$ because of the bilinear form $\langle \cdot \rangle$.

Set $\Xi_i = \{ v \in V | \langle v, \Sigma_i \rangle = 0 \}$, then $\Xi_i$ is a $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{\bar{k}}$-submodule of $V$. Moreover we have $\Xi_i \subset \Xi_{i-1}$, and $\bigcap_i \Xi_i = 0$ because $\bigcup_i \Sigma_i = V$ and $\langle \cdot \rangle$ is non-degenerate. In fact $\Xi_{i-1}/\Xi_i \cong (\Sigma_i/\Sigma_{i-1})^c \cong (V_{-\omega_0 \nu_i, k} \otimes V^c_{-\omega_0 \nu_i, \bar{k}})^c \cong V^c_{-\omega_0 \nu_i, k} \otimes V_{-\omega_0 \nu_i, \bar{k}}$. □

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