Nonlocal FEM Simulations of Ductile Damage with Regularized Crack Path Predictions

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Abstract. Constitutive equations of finite strain elasto-plasticity are coupled to continuum damage mechanics to simulate the initiation and propagation of cracks in ductile materials. The diffuse deterioration of material’s strength is modelled using porosity as a damage parameter. We show that the local damage model yields simulation results, pathologically dependent on the FEM mesh upon mesh refinement: (i) the structural behaviour becomes unrealistically brittle; (ii) the FEM mesh determines the crack path. To solve these problems, an integral-based approach to nonlocal damage is implemented. It enables physically sound results by introducing the linear size of the microstructure in to the model’s formulation. To prevent the effect of unrealistic diffusion of damage, the averaging operator is applied not to the porosity, but to its dual parameter, material’s continuity. Solutions of test problems for crack propagation in a compact tension specimen are presented. The convergence of FEM solutions is demonstrated for different slopes of the mesh with fine and coarse discretizations.

1. Introduction
When modelling strain-softening materials, the local approach predicts localization of plastic strain and damage in a narrow band. In FEM computations this localization zone is typically one element thick, and in SPH it occupies one inter-particle layer. Upon mesh refinement, this behaviour means convergence to non-physical cracking patterns with zero-width localization zone [2,9]. To solve this problem, two non-local approaches are used in the literature: gradient-enhanced and integral-based [1]. In the first case, the original local differential equations are enhanced by higher order derivatives. The second approach to delocalization regularizes the damage evolution by applying integral-based averaging operators.

One advantage of the integral-based approach advocated here is that it does not require higher-order finite elements with C1-approximation. Even low-order meshless discretization methods like SPH can be used [9]. Another advantage of the integral-based approach lies in simplicity of the delocalization scheme. In contrast to the gradient-based method, the issue of boundary conditions for the averaged variables does not appear. This paper demonstrates the performance of the integral-based nonlocal approach applied to the problem of ductile crack propagation. Toward that end, a general elasto-plastic framework is considered, accounting for combined geometrical and physical nonlinearities.

In continuum damage mechanics, various scalar damage parameters are implemented. In this study we employ porosity Φ and its dual variable, continuity Ψ. Classical (local) approach is the special case of the ductile damage model from [8].

In the previous work [9] a new material model was proposed and implemented; the new model accounts for isotropic hardening and non-local accumulation of ductile damage. The integral-based approach for damage delocalization resolves the problem of unrealistic abrupt loss of bearing capacity in cracked structures. Owing to correct prediction of the fracture toughness, the model describes structural ability to withstand external loads after the initiation of a crack.
We demonstrate that nonlocal damage evolution enables numerical analysis of strain localization phenomena. Owing to the regularization, the simulation results converge to a physically meaningful solution upon mesh refinement.

2. Ductile damage model on the reference configuration

We use the damage model from [9]; its backbone is the elasto-viscoplastic model of Simo and Miehe [7]. To enable the numerical treatment of the model, we transform the constitutive equations to the reference configuration. Thus, the state of the material is described by the inelastic right Cauchy-Green tensor \( \hat{C}_i \), which is active on the porous configuration, the accumulated inelastic arc-length \( s \), and the porosity \( \Phi \). The second Piola-Kirchhoff stress tensor, operating on the reference configuration, is computed through the formula

\[
\bar{T} = \frac{k(\Phi)}{10} \left( \frac{\sqrt{\det \bar{C}}}{\Phi} \right)^5 - \left( \frac{\sqrt{\det \bar{C}}}{\Phi} \right)^5 \bar{C}^{-1} + \mu(\Phi) \bar{C}^{-1} (\bar{C} \bar{C}^{-1})^D. \tag{1}
\]

Here, \( \bar{C} \) is the right Cauchy-Green tensor; \( k(\Phi) \) and \( \mu(\Phi) \) are damage-dependent bulk and shear moduli; \( \bar{C} = (\det C)^{-1/3} C \) is the unimodular part of \( C \). The first term on the right-hand side is the volumetric part of the stress, computed according to the assumption of Hartmann and Neff [3]. The second term governs the deviatoric stresses; it corresponds to the neo-Hookean material. To formulate the yield condition of the von Mises type, we introduce the Frobenius norm of the Kirchhoff stress deviator \( \bar{g} = \| (\bar{C} \bar{S})^D \| \). On the reference configuration we have

\[
\bar{g} = \Phi^{-1} \sqrt{\text{tr}(\bar{C}^T D^T)^2}.
\]

The plastic flow is governed by the ordinary differential equation (cf. [9])

\[
\frac{d}{dr} \bar{C}_i = 2 \frac{\lambda_i}{\bar{g}} \Phi^{-1} (\bar{C}^T)^D \bar{C}_i.
\]

Taking (1) into account, the evolution equation simplifies to

\[
\frac{d}{dr} \bar{C}_i = 2 \frac{\lambda_i}{\bar{g}} \Phi^{-1} \mu(\Phi) (\bar{C} \bar{C}^{-1})^D \bar{C}_i.
\]

The inelastic (plastic) strain rate is controlled by the Perzyna law:

\[
\lambda_i = \frac{1}{\eta} \left( \frac{f}{f_0} \right)^m, \quad f = \bar{g} - \sqrt{\frac{2}{3}} [K(\Phi) + R(s, \Phi)].
\]

Here, \( K(\Phi) \) is the damage dependent initial yield stress. The isotropic hardening \( R \) is a function of the Odqvist parameter \( s \):

\[
\dot{s} = \sqrt{\frac{2}{3}} \lambda_i \cdot R(s, \Phi) = \Phi^{-1} \frac{Y(\Phi)}{\tilde{\beta}_0} \left( 1 - \exp(1 - \beta_0 s) \right).
\]

To account for the damage-related deterioration of materials strength, we assume (cf. [8,9])

\[
k(\Phi) = k_0 \cdot e^{-BRR(\Phi-1)}, \quad \mu(\Phi) = \mu_0 \cdot e^{-SRR(\Phi-1)}, \quad Y(\Phi) = \gamma_0 \cdot e^{-IRR(\Phi-1)}.
\]

Material parameters \( BRR, SRR, \) and \( IRR \) govern the impact of the damage on the strength. The degradation of the yield strength is described by

\[
K = \Phi^{-1} K_0 e^{-IRR(\Phi-1)}.
\]

The hydrostatic stress component is computed according to
\[ \text{tr}(\mathbf{C}_c \mathbf{S}_{\text{ep}}) = \text{tr}(\mathbf{C}_{\text{ep}} \mathbf{F}_{\text{ep}}) = \Phi^{-1} \text{tr}(\mathbf{C}_T). \]

In the local model, the increase of porosity \( \Phi \) is due to nucleation of new pores and plasticity-induced growth of already existing:

\[ \Phi_{\text{local}}^n = A_{\text{nuc}} \Delta t_i + d_{\text{growth}} (\Phi_0 - \Phi) \Delta t_i \exp \left( \frac{\alpha}{\sqrt{2 \nu \text{tr}[(\mathbf{C}_T)^2]}} \right). \tag{2} \]

where \( A_{\text{nuc}} \) and \( d_{\text{growth}} \) are material parameters, governing nucleation and growth. Note that the growth term depends on the stress triaxiality, which is the hydrostatic stress divided by the von Mises stress. For what follows it is instructive to introduce a new parameter \( \Psi \), dual to porosity \( \Phi \); we call it continuity \[ \Psi = \exp(-\text{PCR}(\Phi - 1)), \quad \Phi = 1 - \log(\Psi)/\text{PCR}. \]

Here, \( \text{PCR} \) is a non-dimensional material parameter, responsible for porosity-continuity relation. Moreover, the constitutive equations include initial conditions and the delocalization procedure.

**Integral-based delocalization**

Following the comprehensive study \cite{1}, we implement the averaging operator:

\[ G_{\text{localized}}(x) = \int_{\text{Body}} G(y) \alpha(x,y) dy, \]

where \( \alpha(x,y) \) is the delocalization kernel. In this paper, \( \text{Body} \) is the reference configuration of the body which coincides with its initial configuration. For the predictions of the non-local model to coincide with local simulations in the uniform state, this kernel must comply with the normalization restriction

\[ \int_{\text{Body}} \alpha(x,y) dy = 1, \quad \text{for all } x \in \text{Body}. \]

In this paper we apply the following kernel \cite{1}:

\[ \alpha(x,y) = \frac{\alpha_{\infty}(\|x-y\|)}{\int_{\text{Body}} \alpha_{\infty}(\|x-z\|) dz}, \quad \alpha_{\infty}(r) = c(1 - \frac{r^2}{r_{\text{nonloc}}^2}), \quad \langle x \rangle = \max(x,0), \]

where \( r_{\text{nonloc}} \) is the internal length scale, also known as delocalization distance or interaction distance.

Following \cite{9}, the averaging operator is applied to the continuity rate \( \dot{\Psi} \). The local and nonlocal variants of the model are thermodynamically consistent. Since the delocalization is carried out on the reference configuration, the w-invariance property is lost. Other non-local models of integral type which are still w-invariant are presented in \cite{9}.

**3. Numerical procedure**

A hybrid explicit/implicit time-stepping scheme from \cite{9} is used. The constitutive equations are evaluated at each point of Gauss integration. Consider a typical time step \( t_n \rightarrow t_{n+1} \). \( \Delta t = t_{n+1} - t_n > 0 \). Suppose we know the current right Cauchy-Green tensor \( n+1 \mathbf{C} \) and all internal variables from the previous time step: \( n \mathbf{C}_i, n\Phi, nS \). We need to calculate the current stress tensor \( n+1 \mathbf{T} \) and internal variables \( n+1 \mathbf{C}_i, n+1\Phi, n+1S \). First, at the elastic predictor stage, material constants and trial stresses are calculated:

\[ \mu = \mu_0 \exp(-\text{SRR}(n\Phi - 1)), \quad k = k_0 \exp(-\text{BRR}(n\Phi - 1)), \]

\[ K = K_0 \exp(-\text{IRR}(n\Phi - 1)), \quad \gamma = \gamma_0 \exp(-\text{IRR}(n\Phi - 1)), \]
\[
\bar{T}^{\text{trial}} = \frac{k}{10} \left( (n+1) f/n \Phi - (n+1) f/n \Phi^{-5} \right) n+1 C^{-1} + \mu n+1 C^{-1} \left( n+1 \hat{C}^{-1}_i \right)^D,
\]
where \( n+1 f = \sqrt{\det(n+1 C)} \). Then the trial driving force and isotropic hardening are calculated:

\[
\dot{\theta}_i^{\text{trial}} = n \Phi^{-1} \left( \frac{1}{2} \text{tr} \left( (n+1 C \bar{T}^{\text{trial}})^D \right) \right)^2, \quad R^{\text{trial}} = n \Phi^{-1} \frac{Y}{\beta} \left( 1 - \exp(-\beta n s) \right).
\]

Viscous overstress \( f \) and plastic rain rate \( \lambda_i \) are evaluated according to

\[
f = \bar{Y}^{\text{trial}} - \sqrt{3/2} \left( K + R(n \Phi) \right), \quad \lambda_i = \frac{1}{\eta} \left( f / f_0 \right)^m, \quad (x) = \max(x, 0).
\]
With \( \lambda_i \) at hand we make the step for the Odqvist parameter and the inelastic right Cauchy-Green tensor (cf. [9])

\[
n+1 s = n_s + \Delta t \sqrt{2/3} \lambda_p, \quad n+1 C_i = n \hat{C}_i + \frac{2 \Delta t \lambda \mu}{\bar{Y}^{\text{trial}}} \left( n \Phi^{-1} \right) \frac{1}{n+1 C}, \quad \bar{A} = \left( \det A \right)^{-1/3} A.
\]

The update rule for \( \hat{C}_i \) is based on the explicit update formula, reported in [6]. Finally, at each Gauss point, we calculate the local porosity rate (2)

\[
\dot{\phi}_{i, \text{local}} = A_{\text{nuc}} \lambda_i + d_{\text{growth}} \left( n \Phi - \Phi_0 \right) \lambda_i \exp \left( \frac{3}{2} \frac{\text{tr}(n+1 C \bar{T}^{\text{trial}})}{\sqrt{\text{tr}[(n+1 C \bar{T}^{\text{trial}})^D]}} \right).
\]

Let \( N \) be the number of integration points. We denote by \( X_i \) and \( x_i \) their position vectors in the reference and current configurations, respectively. At the \( i \) th point, the local rate of porosity growth is denoted as \( \dot{\psi}_{i, \text{local}} \) (\( i = 1, 2, \ldots, N \)). Recall that the delocalization operator is applied to \( \dot{\psi}_{i, \text{local}} \), not \( \dot{\phi}_{i, \text{local}} \). In this paper, the integral-based averaging is applied in the reference configuration:

\[
\dot{\psi}_{i, \text{nonlocal}} = \sum_{j=1}^{N} \alpha_{\text{av}} \left( r(X_i, X_j) \right) \dot{\psi}_{i, \text{local}} V_j / \left( \sum_{j=1}^{N} \alpha_{\text{av}} \left( r(X_i, X_j) \right) V_j \right),
\]
where \( V_j \) is the volume related to \( X_j \) in the reference configuration. Finally, the continuity and porosity are updated:

\[
n+1 \psi_i = \max(n \psi_i + \dot{\psi}_{i, \text{nonlocal}} \cdot \Delta t, 10^{-9}), \quad n+1 \Phi_i = 1 - \frac{\ln(n+1 \psi_i)}{\text{PCR}}, \quad i = 1, 2, \ldots, N.
\]
4. Numerical results
The nonlocal damage model is implemented into a home-made FEM code. Explicit global time stepping is used in combination with low-order four-node elements with one point of Gaussian integration. Since low-order elements are prone to spurious zero-energy hour-glass modes, they are suppressed by additional penalty stiffness [10]. To demonstrate the performance of the nonlocal framework, we simulate the crack propagation through a compact sample; the sample geometry is shown in Figure 1.

The fatigue crack length is \( l_f = 5.2 \text{ mm} \). The sample is in plane stress. The boundary conditions assume a frictionless contact at the grips. The upper and lower grips are shifted in the vertical direction, such that displacement = \( 16.4975 \text{ mm} + d_{\text{max}} \left(1 - \cos\left(\pi t/(2t_0)\right)\right) \). Here, \( d_{\text{max}} = 0.25 \text{ mm} \). The implemented material parameters are summarized in Table 1. The simulations corresponding to local damage model are obtained with a vanishing delocalization distance: \( r_{\text{nonloc}} \to 0 \).

\[
egin{array}{cccccccc}
\phi_0 [-] & A_{\text{nucl}} [-] & d_{\text{growth}} [-] & BRR [-] & SRR [-] & PCR [-] & r_{\text{nonloc}} \text{ [mm]} \\
1.0 & 0.5 & 5.0 & 70.0 & 70.0 & 1.0 & 3.0 \\
\end{array}
\]

Figure 1. Geometry and dimensions of the compact tension sample. Reproduced from [9].
Following [5, 4] we study the impact of the FEM mesh. Three types of meshes are considered: with large inclination, moderate inclination, and late moderate inclination (Figure 2). Extremely coarse meshes are shown for clarity in Figure 2.

![Figure 2. Three discretizations with various mesh inclinations.](image)

Figure 2 presents the simulated force-displacement curves for various sizes of the elements and different inclinations of the mesh. When the classical local approach is implemented, numerical results exhibit extremely large scatter. Upon mesh refinement, the mechanical response converges to non-physical behaviour with vanishing fracture toughness (Figure 3, left). In contrast to that, when the newly proposed nonlocal damage model is implemented, the force-displacement curves are consistent. The simulations converge to a physically reasonable solution with a finite fracture toughness (Figure 3, right).

![Figure 3. Simulated force-displacement curves. Left: local approach. Right: non-local approach.](image)

Next, we inspect the cracking patterns. Simulations using the local model show pathological mesh dependence of the results. Although the sample geometry and applied loads are symmetric, the crack deviates from the natural symmetric path (Figure 4). The non-physical deviation of the crack path is observed both for coarse and fine meshes (Figure 4, top and bottom). Moreover, as is typical for local models, the damage localizes on the set of vanishing volume.
However, application of the nonlocal model enables consistent results, independent of the mesh inclination (Figure 5). Owing to the proper delocalization of the damage accumulation, the crack tends to propagate in the correct direction. Moreover, the width of the damage localization zone is nearly independent of the element size (Figure 5, top and bottom).

### Figure 4. Distribution of porosity according to the local damage model.

### Figure 5. Distribution of porosity according to nonlocal damage model.

### 5. Conclusion

Geometrically and physically nonlinear models of ductile damage are considered, suitable for large scale FEM simulations. The local and nonlocal integral-based models are implemented into a home-made FEM code, employing low-order finite elements. The applicability of the FEM to the solution of crack propagation problems is tested for mode-I fracture in plane stress state. The models are tested using FEM meshes with various inclinations, introduced to trigger non-physical deviation of the cracks.

The use of local ductile damage models does not allow for physically meaningful FEM simulations; the solution exhibits a pathological mesh dependence in terms of vanishing fracture toughness and...
irregular cracking patterns. In the considered crack propagation problem, the damaged material zone exhibits vanishing thickness, leading to underestimated resistance to crack propagation.

To obtain physically reasonable results, nonlocal damage model from [9] is implemented and tested. The integral-based framework is robust. In contrast to the explicit gradient approach it does not require $C^1$ finite elements. The tests show that the nonlocal framework yields consistent results. Owing to the regularization, the pathological mesh dependence is avoided.

In future studies, efficient numerical schemes and reliable calibration techniques will be developed. Moreover, the applicability of the framework to fracture under mixed-mode loading and fracture of bi-materials will be investigated.

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