On an injectivity theorem for log-canonical pairs with analytic adjoint ideal sheaves

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Abstract. As an application of the residue functions corresponding to the lc-measures developed by the authors, the proof of the injectivity theorem on compact Kähler manifolds for plt pairs by Matsumura is improved in this article to allow multiplier ideal sheaves of plurisubharmonic functions with neat analytic singularities in the coefficients of the relevant cohomology groups. With the use of a refined version of analytic adjoint ideal sheaves, a plan towards a solution to the generalised version of Fujino’s conjecture (i.e. an injectivity theorem on compact Kähler manifolds for lc pairs with multiplier ideal sheaves) is laid down and, in addition to the result for plt pairs, a proof for lc pairs in dimension 2, which is also an improvement of Matsumura’s result, is given.

1. Introduction

The injectivity theorem was first formulated by Kollár in [24, Thm. 2.2] (originally as a means for proving the torsion-freeness of the higher direct images of the canonical sheaf under a proper morphism) in the algebraic setting which can be viewed as a generalisation of the celebrated Kodaira vanishing theorem (see, for example, [13, Cor. 5.2] or [26, Remark 4.3.8]). It was generalised to the setting on compact Kähler manifolds by Enoki in [12] using harmonic theory.

The theorem is further generalised to log-canonical (lc) pairs in the algebraic setting via the theory of (mixed) Hodge structures (see, for example, [13, §5], [16, §5] and [1]). On the transcendental side, the latest results that the authors are aware of are those of Fujino ([17]), Matsumura ([27]) and Gongyo–Matsumura ([19]), who obtain the injectivity theorem in the setting on compact Kähler manifolds for Kawamata log-terminal (klt) pairs, and that of Matsumura ([30]) for purely log-terminal (plt) pairs, which are sub-cases of lc pairs. All these results make use of $L^2$ theory. This is the starting point of the current research. Readers are referred to [17], [27] and [30] for more references on the development of the injectivity theorem. See also [28] for the development of the injectivity theorem in the relative setting. Moreover, readers are also referred to [25, Def. 2.8] for the precise definitions of the various notions of singularities, including klt, plt, dlt and lc, in algebraic geometry.

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In addition to the fact that this topic is an interesting research problem, another initiative of the study is to illustrate the use of the computation of residue functions corresponding to lc-measures studied in [6] and [4], and also the results on the analytic adjoint ideal sheaves studied in [5].

Readers who would like to skip the background and motivation of the statements in this work may go directly to Sections 1.2 and 1.3 for the main results of this article.

1.1. Fujino’s conjecture and Matsumura’s results. Let \( X \) be a compact Kähler manifold of dimension \( n \), \( D \) a reduced divisor on \( X \) with simple normal crossings (snc) (see, for example, [25, Def. 1.7]), and \( F \) a holomorphic line bundle on \( X \). Under the snc assumption on \( D \), the lc centres of \((X,D)\) are simply the irreducible components (with reduced structure) of any intersections of irreducible components of \( D \) (see, for example, [25, Def. 4.15] for the precise definition of lc centres; see also [6, Def. 1.4.1] for the authors’ attempt to generalise to the case when \( D \) may not be a divisor but the zero locus of certain multiplier ideal sheaf). Fujino’s conjecture on the injectivity theorem can be stated as follows.

Conjecture 1.1.1 ([18, Conj. 2.21]). Suppose that \( F \) is semi-positive and there exists a holomorphic section \( s \) of \( F^\otimes m \) on \( X \) for some positive integer \( m \) such that \( s \) does not vanish identically on any lc centres of \((X,D)\). Then, the multiplication map induced by \( \otimes s \),
\[
H^q(X, K_X \otimes D \otimes F) \to H^q(X, K_X \otimes D \otimes F^\otimes (m+1))
\]
is injective for every \( q \geq 0 \).

In the algebraic setting, i.e. \( X \) being a smooth projective manifold, the assumption on \( F \) is replaced by that \( F \) being semi-ample. Note that semi-ampleness implies semi-positivity (see, for example, [17, Lemma 1.6]). The conjecture is then known to hold true in the algebraic setting via the theory of mixed Hodge structures (see [16, §5 and §6]).

Let \( M \) be another holomorphic line bundle on the compact Kähler manifold \( X \) and let \( e^{-\varphi_F} \) and \( e^{-\varphi_M} \) be hermitian metrics on \( F \) and \( M \) respectively. Matsumura proves the conjecture for the case where \((X,D)\) being plt, i.e. \( D \) is a smooth subvariety consisting of disjoint irreducible components.

Theorem 1.1.2 ([30, Thm. 1.3 and Cor. 1.4]). Suppose that \((X,D)\) is plt, and suppose that \( \varphi_F \) and \( \varphi_M \) are smooth such that their curvature forms satisfy
\[
0 \leq i\partial\bar{\partial}\varphi_M \leq C i\partial\bar{\partial}\varphi_F \quad \text{on} \quad X
\]
for some constant \( C > 0 \) (so \( F \) is semi-positive in particular). Let \( s \) be a holomorphic section of \( M \) on \( X \) such that \( s \) does not vanish identically on any lc centres of \((X,D)\). Then, the multiplication map induced by \( \otimes s \),
\[
H^q(X, K_X \otimes D \otimes F) \to H^q(X, K_X \otimes D \otimes F \otimes M)
\]
is injective for every \( q \geq 0 \).

In particular, the conclusion holds true when \( M = F^\otimes m \) for some positive integer \( m \) (with \( \varphi_M := m\varphi_F \)), i.e. Conjecture 1.1.1 holds true for any plt pairs \((X,D)\).

Let \( \phi_D \) be a potential (of the curvature of the metric \( e^{-\phi_D} \)) on \( D \) (see Notation 2.1.2) induced from canonical sections of irreducible components of \( D \) (see Notation 2.1.3). The
proof of the theorem in [30] is proceeded by reducing the original question to the questions on the injectivity of the composition of maps

\[ H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\phi_D)) \xrightarrow{\iota_0} H^q(X, K_X \otimes D \otimes F) \xrightarrow{\otimes s} H^q(X, K_X \otimes D \otimes F \otimes M) \]

(more precisely, it is to check whether \( \ker \mu_0 = \ker \iota_0 \)) and the injectivity of the map

\[ \nu_1: H^q \left( D, K_X \otimes D \otimes F \otimes \Theta_X \mathcal{I} \phi_D \right) \xrightarrow{\otimes s \phi_D} H^q \left( D, K_X \otimes D \otimes F \otimes M \otimes \Theta_X \mathcal{I} \phi_D \right) \]

(see §1.3), where \( \mathcal{I} \phi_D = \mathcal{I} X(\phi_D) \) is the multiplier ideal sheaf of \( \phi_D \) on \( X \) and the map \( \iota_0 \) that \( \mu_0 \) factors through is induced by the inclusion \( \mathcal{I} \phi_D \subset \Theta_X \). Since \( (X, D) \) is plt, the map \( \nu_1 \) can be decomposed into a direct sum of homomorphisms between cohomology groups on irreducible components of \( D \). The injectivity of \( \nu_1 \) is thus a consequence of the injectivity theorem of Enoki ([12]). The main focus of [30] is to show that \( \ker \mu_0 = \ker \iota_0 \) (see [30, Thm. 1.6]).

When the potentials \( \varphi_F \) and \( \varphi_M \) are allowed to be singular, there is the following result for pseudo-effective line bundles.

**Theorem 1.1.3** ([19, Thm. 1.3]; see also [27, Thm. 1.3]). Suppose that \( D \) is any effective \( \mathbb{R} \)-divisor. Let \( \phi_D \) be a potential on \( D \) induced from canonical sections of irreducible components of \( D \) (see Notation 2.1.3). Suppose also that

\[ \varphi_F := a \varphi_M + \phi_D \]

for some number \( a > 0 \) and that \( \varphi_M \) (and thus \( \varphi_F \)) is plurisubharmonic (psh) locally everywhere in \( X \), i.e.

\[ i \partial \bar{\partial} \varphi_M \geq 0 \quad \text{and thus} \quad i \partial \bar{\partial} \varphi_F \geq 0 \quad \text{on} \quad X. \]

Let \( s \) be a non-zero holomorphic section of \( M \) on \( X \) such that \( \sup_X |s|^2 \varphi_M < \infty \). Then, the multiplication map induced by \( \otimes s \),

\[ H^q(X, K_X \otimes F \otimes \mathcal{I}(\varphi_F)) \xrightarrow{\otimes s} H^q(X, K_X \otimes F \otimes M \otimes \mathcal{I}(\varphi_F + \varphi_M)) \]

is injective for every \( q \geq 0 \), where \( \mathcal{I}(\varphi) \) is the multiplier ideal sheaf of the potential \( \varphi \).

Recall that \( (X, D) \) is a klt pair if, under the assumption that \( D \) is an snc \( \mathbb{R} \)-divisor, the coefficient of every irreducible component of \( D \) is \( < 1 \). In this case, \( \phi_D \) has only klt singularities, i.e. \( \mathcal{I}(\phi_D) = \Theta_X \). When \( \varphi_M \) is smooth, the above theorem can be viewed as a version of Conjecture 1.1.1 (with \( F^{\otimes m} \) replaced by \( M \)) for klt pairs \((X, D)\).

**1.2. The first result and strategy of proof.** The goal of this research is to prove Conjecture 1.1.1 while allowing certain multiplier ideal sheaves in the coefficients of the cohomology groups. This article is the first step in this direction. By revising the proofs in [27] and [30] into the one which is, in the authors’ point of view, more favourable to the study of lc pairs, the following generalisation of Theorem 1.1.2, or more precisely, the generalisation of [30, Thm. 1.6] (a statement on the map \( \mu_0 \) in Section 1.1), is obtained.

**Theorem 1.2.1.** Suppose that \( (X, \omega) \) is a compact Kähler manifold and \( D \) a reduced divisor with snc such that \( (X, D) \) is lc. Let \( \varphi_F \) and \( \varphi_M \) be potentials on \( F \) and \( M \) respectively such that

- \( i \partial \bar{\partial} \varphi_F \geq 0 \) and \( -C \omega \leq i \partial \bar{\partial} \varphi_M \leq C i \partial \bar{\partial} \varphi_F \) on \( X \) in the sense of currents for some constant \( C > 0 \),
- \( \varphi_F \) and \( \varphi_M \) have only neat analytic singularities,
the polar sets $P_F := \varphi_F^{-1}(-\infty)$ and $P_M := \varphi_M^{-1}(-\infty)$ both contain no lc centres of $(X, D)$, and
- both $P_F$ and $P_M$ are divisors and $P_F \cup P_M \cup D$ has only snc.

Also let $\phi_D$ be a potential defined by a canonical section of $D$. Suppose that there exists a non-trivial holomorphic section $s \in H^0(X, M)$ such that $\sup_X |s|_{\varphi_M}^2 < \infty$. Then, given the commutative diagram

$$
H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi_F + \phi_D)) \xrightarrow{\iota_0} H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi_F)) \xleftarrow{\otimes \iota_0} H^q(X, K_X \otimes D \otimes F \otimes M \otimes \mathcal{I}(\varphi_F + \varphi_M)) \, ,
$$

in which $\iota_0$ is induced from the inclusion $\mathcal{I}(\varphi_F + \phi_D) \subset \mathcal{I}(\varphi_F)$, one has $\ker \mu_0 = \ker \iota_0$ for every $q \geq 0$.

Together with the injectivity of the corresponding map $\nu_1$ in Section 1.1 (a consequence of Theorem 1.1.3), a statement slightly more general than Conjecture 1.1.1 in the plt case can be proved. See Corollary 1.3.3 for details.

Although Theorem 1.2.1 is only a slight improvement ($\varphi_F$ and $\varphi_M$ are allowed to be singular but only for neat analytic singularities) to the corresponding statement in [30, Thm. 1.6 or Thm. 3.9], a different proof from that in [30] is presented here. While both proofs follow the same spirit of arguments of Enoki in [12, §2] (in view of the Dolbeault isomorphism, consider a harmonic form $u$ representing a class in the domain cohomology group which is in $\ker \mu_0$ and also in the orthogonal complement of $\ker \iota_0$, then argue via the $L^2$ theory and Bochner–Kodaira–Nakano identity to show that $u = 0$ under the positivity assumption on $F$ and $M$), the two proofs differ in the handling of the non-integrable lc singularities in the potentials (namely, $\phi_D$).

Assume that $\varphi_F$ and $\varphi_M$ are smooth for the moment. Let $s_D$ be a canonical section of $D$ such that $\phi_D = \log |s_D|^2$ and let $\varphi_D^{\text{sm}}$ be a smooth potential on $D$. In the proof of [30, Thm. 3.9], $\phi_D$ is smoothed to

$$\phi_D^{(\varepsilon)} := \log \left( |s_D|^2 + \varepsilon \right) + \varphi_D^{\text{sm}} \in C_c^\infty(X) \, .$$

In view of the $L^2$ Dolbeault isomorphism (which is named as de Rham–Weil isomorphism in [27] and [30])\(^1\), let $[u]$ be a cohomology class in $\ker \mu_0$ and let $u$ be the harmonic $D \otimes F$-valued $(n, q)$-form with respect to the (global) $L^2$ norm $\|\cdot\|_{\varphi_F + \phi_D}$ induced from $\varphi_F + \phi_D$ which represents the class $[u]$. Then, $su = \bar{\partial}v$ for some $D \otimes F \otimes M$-valued $(n, q - 1)$-form.

\(^1\)The name “de Rham–Weil isomorphism” is used in [8, Ch. IV, §6] to mean more generally the isomorphisms between the cohomology of a sheaf and the cohomology of an acyclic resolution of the sheaf. The so named isomorphism in [27] and [30] stands for, more specifically, the isomorphisms between the Čech cohomology of a multiplier ideal sheaf and the $\bar{\partial}$-cohomology computed from the associated Dolbeault complex of locally $L^2$ forms (with respect to some $L^2$ norm with a possibly singular weight). This latter type of isomorphisms, while not named in [35, Prop. 4.6], [32, Thm. 4.13] and [17, Claim 1] when it is stated or proved, is named as “Leray isomorphism” in [34, §2]. The version of such isomorphism by Fujino in [17, Claim 1] allows the involving $L^2$ norm to be the one induced from a singular quasi-psh potential which is smooth on a Zariski open set, while the one by Matsumura in [27, Prop. 5.5] allows the quasi-psh potential to have arbitrary singularities and also allows a more flexible choice of the Kähler form ([30, Prop. 2.8]). In this paper, the version in [27] is used. Although it may be more proper to attribute the isomorphism to Fujino and Matsumura, the authors incline to name it as “$L^2$ Dolbeault isomorphism” ($L^2$ version of the Dolbeault isomorphism), which seems to be more suggestive and self-explanatory.
v which is $L^2$ in $\|\cdot\|_{\varphi_F+\varphi_M+\varphi^m_D}$ (but not clear whether it is $L^2$ in $\|\cdot\|_{\varphi_F+\varphi_M+\varphi^m_D}$). The proof is based on the inequality

$$\|su\|^2 \xrightarrow{\varepsilon \to 0^+} \|\vartheta(su)\|_{(\varepsilon)} = \left\langle \vartheta(su), v \right\rangle_{(\varepsilon)} \leq \|\vartheta'(su)\|_{(\varepsilon)} \|v\|_{(\varepsilon)},$$

where $\|\cdot\|_{(\varepsilon)} = \|\cdot\|_{\varphi_D+\varphi_F+\varphi_M}$ while $\|\cdot\|_{(\varepsilon)}$ and $\left\langle \cdot, \cdot \right\rangle_{(\varepsilon)}$ are the norm and inner product obtained after $\varphi_D$ is smoothed to $\varphi_{D_\varepsilon}$, and $\vartheta_{(\varepsilon)}$ is the corresponding formal adjoint of $\overline{\partial}$. In order to show that the right-hand-side converges to 0 as $\varepsilon \to 0^+$, the rate of divergence of the integral of $e^{-\varphi_{D_\varepsilon}}$ has to be controlled so that

$$\int_V e^{-\varphi_{D_\varepsilon}} \ d\text{vol}_V = o\left(\frac{1}{\varepsilon}\right) \quad (\text{little-o notation})$$

for any open set $V$ in $X$ as $\varepsilon \to 0^+$. When $(X, D)$ is plt, it is easy to show that

$$\int_V e^{-\varphi_{D_\varepsilon}} \ d\text{vol}_V = \mathcal{O}(\|\log \varepsilon\|) \quad (\text{Big-O notation})$$

as $\varepsilon \to 0^+$, which gives the required estimate (\ast) (see [30, Lemma 3.11, Prop. 3.12 and Prop. 3.14]). The proof of [30, Thm. 1.6] relies on this estimate.

In the proof presented in this article, instead of smoothing out the lc singularities on the potential $\varphi_D$, a sequence of smooth cut-off functions $\{\vartheta_\varepsilon\}_{\varepsilon > 0}$ vanishing identically on some neighbourhoods of $D$ and converging to the identity map on $X$ as $\varepsilon \to 0^+$ is considered such that

$$\|su\|^2 \xrightarrow{\varepsilon \to 0^+} \left\langle su, \vartheta_{(\varepsilon)}\overline{\partial}v \right\rangle = \left\langle \vartheta(su), \vartheta_{(\varepsilon)}v \right\rangle - \left\langle su, \overline{\partial}\vartheta_{(\varepsilon)} \wedge v \right\rangle,$$

where $\vartheta$ is the formal adjoint of $\overline{\partial}$ with respect to the potential $\varphi_D + \varphi_F + \varphi_M$ (which is denoted by $\vartheta_{(\varepsilon)}$ in latter sections). It can be shown that $\vartheta(su) = 0$ (see Corollary 3.2.6; also compare with the result $\|\vartheta_{(\varepsilon)}(su)\|^2_{(\varepsilon)} = \mathcal{O}(\varepsilon \log \varepsilon)$ in the plt case in [30, §3.2]), so it suffices to estimate the inner product on the far right-hand-side in order to show that $u = 0$. A fundamental trick at play is that, although $e^{-\varphi_D}$ is non-integrable around $D$, using the computation of the residue functions associated to lc-measures studied in [6] and [4] (or simply via a direct computation), one has

$$\varepsilon \int_V \frac{e^{-\varphi_D}}{|\psi_D|^2 + \varepsilon} \ d\text{vol}_V = \mathcal{O}(1) \quad \text{as } \varepsilon \to 0^+ \quad (\psi_D := \varphi_D - \varphi^m_D \leq -1)$$

[30, Lemma 3.11] holds only in the plt case. Indeed, on a neighbourhood $V$ such that $V \cap D = \{r_1 r_2 = 0\}$, where $r_1$ and $r_2$ are the radial components of the polar coordinates such that $(r_1, r_2) \in [0, 1]^2$ on $V$, one has

$$\int_V e^{-\varphi_{D_{\varepsilon}}} \ d\text{vol}_V \sim \int_{[0, 1]^2} \frac{dr_1 dr_2}{r_1^2 + r_2^2 + \varepsilon} \geq \int_{[0, 1]^2} \frac{dr_1 dr_2}{(r_1^2 + \sqrt{\varepsilon})(r_2^2 + \sqrt{\varepsilon})} = \mathcal{O}\left(\|\log \varepsilon\|^2\right).$$

Nevertheless, one can still obtain (\ast) by a simple adjustment, namely, when $(X, D)$ is lc but not plt and when $D = \sum_{i \in I} D_i$ such that each $D_i$ is irreducible and has a canonical section $s_{D_i}$ and a smooth potential $\varphi^m_{D_i}$, set

$$\varphi_{D_{\varepsilon}}(z_i) := \sum_{i \in I} \log \left(|s_{D_i}|^2 \varphi^m_{D_i} + \varepsilon\right) + \varphi^m_{D_i}.$$
on any local open set \( V \subset X \) when \( \sigma \geq \sigma_{\text{mlc}} \), where \( \sigma_{\text{mlc}} \) is the codimension of the minimal lc centres \( \text{mlc} \) of \( (X, D) \) (see Theorem 2.6.1 or [6, Prop. 2.2.1]; note also that the integral diverges when \( \sigma < \sigma_{\text{mlc}} \)). It turns out that, with a careful analysis on the properties possessed by \( u \), in order to prove Theorem 1.2.1 for any values of \( \sigma_{\text{mlc}} \geq 1 \) (i.e. no matter whether \( (X, D) \) is plt or not), it suffices to put \( |\psi_D|^{1+\varepsilon} \) into the denominators (via a suitable choice of the cut-off functions \( \theta_\varepsilon \)) of the integrand of the inner product (see Steps II, IV and V of the outline of the proof of Theorem 1.2.1 in §3.1).

In order to prove that \( u = 0 \), it is necessary to assume that \( u \) is sitting inside the orthogonal complement \( (\ker t_0)^\perp \) of \( \ker t_0 \) (note that \( \ker t_0 \neq 0 \) for \( q = 1 \), for example, when \( X \) is an elliptic curve, \( F = \mathcal{O}_X \) and \( D \) is an effective divisor of \( \deg D = 1 \) with the realisation that \( D \otimes \mathcal{I}(\phi_D) \cong \mathcal{O}_X \)). An argument of Takegoshi (see [36, Prop. 3.8] or [30, Prop. 3.13]; see also Step IV in §3.1) is needed to make use of such assumption, which requires \( u \) to be smooth around the lc locus \( D \) (see Remark 3.1.1 for details). Indeed, to compute the above inner product using the computation of residue functions in [6] and [4], \( u \) is also required to be smooth around the lc locus \( D \). Using the refined version of the hard Lefschetz theorem of Matsumura (see Theorem 2.5.1 or [30, Thm. 3.3]), this can be guaranteed when the Kähler metric \( \omega \) is smooth around \( D \). As a result, even though the metric \( e^{-\phi_D} \) on \( D \) is singular, one has to keep using a Kähler metric \( \omega \) which is incomplete on \( X \setminus D \) when making use of the (twisted) Bochner–Kodaira formula, and thus extra care is needed (see §2.4, Proposition 3.2.5 and Corollary 3.2.6).

When \( \varphi_F \) and \( \varphi_M \) are not smooth but have neat analytic singularities as in the assumption of Theorem 1.2.1, one would expect that the arguments employed in the smooth case should still hold true since the singularities on \( \varphi_F \) and \( \varphi_M \) along \( P_F \cup P_M \) and the lc locus \( D \) are “separated”. In practice, a suitably chosen complete Kähler metric \( \tilde{\omega} \) on \( X^\circ : = X \setminus (P_F \cup P_M) \) is considered. The corresponding harmonic forms \( u \) (denoted by \( \tilde{u} \) in latter sections) may not be smooth along \( P_F \cup P_M \), but their singularities can be determined (see Proposition 3.3.1) and are not interfering with the computations around the lc locus \( D \), thanks to Fubini’s theorem. The argument of Takegoshi is also adjusted to adapt to such situation (see Step IV in §3.1).

In the following sections, \( \omega \) is used to mean a fixed (smooth) Kähler form on \( X \) and \( \tilde{\omega} \) a chosen complete Kähler form on \( X^\circ \). The harmonic forms with respect to \( \omega \) and \( \tilde{\omega} \) in the same class \( [u] \) discussed above are denoted by \( u \) and \( \tilde{u} \) respectively.

1.3. Towards Fujino’s conjecture and its generalisation. Already in the proof of the injectivity theorem for plt pairs in [30] involves arguments of restriction of the relevant cohomology classes to the lc centres of \( (X, D) \). It is therefore natural to incorporate the corresponding adjoint ideal sheaves and their residue exact sequences into the potential solution of Fujino’s conjecture. The analytic adjoint ideal sheaves studied in [5] is introduced below for that purpose.

For any integer \( \sigma = 1, \ldots, n \), let \( \text{lc}_\sigma X(D) \) be the union of lc centres of \( (X, D) \) of codimension \( \sigma \) (or union of \( \sigma\)-lc centres for short) and \( \mathcal{I}_{\text{lc}_\sigma X}(D) \) be its defining ideal sheaf in \( \mathcal{O}_X \). If \( \sigma_{\text{mlc}} \) is the codimension of the mlc of \( (X, D) \), set \( \mathcal{I}_{\text{lc}_\sigma X}(D) : = \mathcal{O}_X \) for all \( \sigma > \sigma_{\text{mlc}} \). Let \( L \) denote, in this section, either the line bundle \( F \) or \( F \otimes M \) and let \( \varphi_L \) be either the potential \( \varphi_F \) or \( \varphi_F + \varphi_M \) accordingly. Notice that the family \( \{ \mathcal{I}(\varphi_L + m\psi_D) \}_{m \geq 0} \) of multiplier ideal sheaves on \( X \) has a jumping number \( m = 1 \) as seen from the assumptions on (the singularities of) \( \varphi_L \) and \( \psi_D : = \phi_D - \varphi_D^{\text{sm}} \leq -1 \) in Theorem 1.2.1. In [5], the first author introduces the following version of analytic adjoint ideal sheaves.
**Definition 1.3.1** ([5, Def. 1.2.1]). Given any integer $\sigma \geq 0$ and a family $\{J(\varphi_L + m\psi_D)\}_{m \in [0,1]}$ with a jumping number at $m = 1$, the (analytic) adjoint ideal sheaf $J_\sigma(\varphi_L; \psi_D) := J_{X,\sigma}(\varphi_L; \psi_D)$ of index $\sigma$ of $(X, \varphi_L, \psi_D)$ is the sheaf associated to the presheaf over $X$ given by

$$\bigcap_{\varepsilon > 0} J(\varphi_L + \psi_D + \log(|\psi_D|^\sigma (\log|e^\psi_D|)^{1+\varepsilon}))(V)$$

for every open set $V \subset X$. Then, its stalk at each $x \in X$ can be described as

$$J_\sigma(\varphi_L; \psi_D)_x = \left\{ f \in \mathcal{O}_{X,x} \mid \exists \text{ open set } V_x \ni x, \forall \varepsilon > 0, \frac{|f|^2 e^{-\varphi_L - \psi_D}}{|\psi_D|^\sigma (\log|e^\psi_D|)^{1+\varepsilon}} \in L^1(V_x) \right\}.$$

According to [5, Thm. 1.2.3], under the assumption that $\varphi_L$ and $\varphi_L + \psi_D$ have only neat analytic singularities with snc, one has

$$J_\sigma(\varphi_L; \psi_D) = J(\varphi_L) \cdot \mathcal{I}_{\log^{\gamma+1}}(D)$$

for all integers $\sigma \geq 0$, which fit into the chain of natural inclusions

$$J(\varphi_L + \psi_D) = J_0(\varphi_L; \psi_D) \subset J(\varphi_L; \psi_D) \subset \cdots \subset J_{\sigma_{\text{mlc}}}(\varphi_L; \psi_D) = J(\varphi_L).$$

Moreover, since $\varphi_L^{-1}(-\infty)$ contains no lc centres of $(X, D)$, the analytic adjoint ideal sheaves fit into the residue short exact sequence (eq1.3.1)

$$0 \rightarrow K_X \otimes J_{\sigma_{\text{mlc}}}(\varphi_L; \psi_D) \rightarrow K_X \otimes J_{\sigma}(\varphi_L; \psi_D) \xrightarrow{\text{Res}} K_X \otimes R_{\sigma}(\varphi_L) \rightarrow 0,$$

where $R_{\sigma}(\varphi_L)$ is a coherent sheaf supported on $\text{lc}_X(D)$ such that, on an open set $V$ with $\text{lc}_X(D) \cap V = \bigcup_{p \in l^\sigma_X} D_p$, where $D_p$’s are the $\sigma$-lc centres in $V$ indexed by $p \in l^\sigma_X$, one has

$$(\text{eq1.3.2}) \quad K_X \otimes R_{\sigma}(\varphi_L)(V) = \prod_{p \in l^\sigma_X} K_{D_p} \otimes D_p^{\otimes(-1)} \big|_{D_p} \otimes J_{D_p}(\varphi_L)(D_p)$$

(see [5, §4.2] for the precise construction of $R_{\sigma}(\varphi_L)$). For every $f \in K_X \otimes J_{\sigma}(\varphi_L; \psi_D)(V)$, the component of $\text{Res}(f)$ on $D_p$ is given by

$$R_{D_p}^\sigma \left( \frac{f}{s_D} \right),$$

where $s_D$ is the canonical section of $D$ such that $\phi_D = \log|s_D|^2$, $R_{D_p}^\sigma$ is the Poincaré residue map corresponding to the restriction from $X$ to $D_p$ (see [25, Def. 4.1 and para. 4.18]; see also Section 2.6). Readers are referred to [5] for the comparison between the analytic adjoint ideal sheaves introduced above and the version studied in [20] and [23], as well as the algebraic versions studied in [10], [22] and [11].

For the sake of convenience, for any sheaf $\mathcal{F}$ on $X$, set

$$\mathcal{H}^{n,q}(\mathcal{F}) := H^q(X, K_X \otimes D \otimes F \otimes \mathcal{F})$$

for any integer $q = 0, \ldots, n$. From the residue short exact sequence (eq1.3.1) and the multiplication map (where $\varphi_{F \otimes M} := \varphi_F + \varphi_M$)

$$K_X \otimes D \otimes F \otimes J_{\sigma}(\varphi_F; \psi_D) \xrightarrow{\otimes}_{\mathcal{F}} K_X \otimes D \otimes F \otimes M \otimes J_{\sigma}(\varphi_{F \otimes M}; \psi_D),$$
one obtains the following commutative diagram of cohomology groups:

(eq1.3.3)

\[
\begin{array}{ccc}
\cdots & \cdots & \cdots \\
H^n_q(\mathcal{J}_0(-\varphi_F;\psi_D)) & \xrightarrow{\iota_0} & H^n_q(\mathcal{J}_0(-\varphi_F;\psi_D)) \\
\downarrow & \downarrow & \downarrow \\
H^n_q(\mathcal{J}_\sigma(-\varphi_F;\psi_D)) & \xrightarrow{\iota_\sigma} & H^n_q(\mathcal{J}_\sigma(-\varphi_F;\psi_D)) \\
\downarrow & \downarrow & \downarrow \\
H^n_q(\mathcal{J}(\varphi_F)) & \xrightarrow{\mu_\sigma} & H^n_q(\mathcal{J}(\varphi_F)) \\
\downarrow & \downarrow & \downarrow \\
H^n_q(\mathcal{J}_0(-\varphi_F;\psi_D)) & \xrightarrow{\iota_0} & H^n_q(\mathcal{J}_0(-\varphi_F;\psi_D)) \\
\downarrow & \downarrow & \downarrow \\
H^n_q(M \otimes \mathcal{J}(\varphi_F)) & \xrightarrow{\nu_0} & H^n_q(M \otimes \mathcal{J}(\varphi_F)) \\
\downarrow & \downarrow & \downarrow \\
H^n_q(M \otimes \mathcal{J}(\varphi_F)) & \xrightarrow{\nu_\sigma} & H^n_q(M \otimes \mathcal{J}(\varphi_F)) \\
\downarrow & \downarrow & \downarrow \\
\cdots & \cdots & \cdots \\
\end{array}
\]

Note that the columns are all exact. The middle horizontal map on the left-hand-side is induced from the natural inclusion $\mathcal{J}_\sigma(-\varphi_F;\psi_D) \subset \mathcal{J}(\varphi_F)$, while the horizontal maps on the right-hand-side are induced from the multiplication map $\otimes s$. Each homomorphism of $\mu_\sigma$’s and $\nu_\sigma$’s is the composition of the maps on the corresponding row.

Through a simple diagram-chasing, one sees that, for each $\sigma \geq 1$, if the homomorphisms $\mu_{\sigma-1}$ and $\nu_\sigma$ satisfy $\ker \mu_{\sigma-1} = \ker \iota_{\sigma-1}$ and $\ker \nu_\sigma = \ker \tau_\sigma$ respectively, then it follows that $\ker \mu_\sigma = \ker \iota_\sigma$. One then obtains the following theorem via induction.

**Theorem 1.3.2.** If one has $\ker \mu_0 = \ker \iota_0$ and $\ker \nu_\sigma = \ker \tau_\sigma$ for $\sigma = 1, \ldots, \sigma_{\text{mlc}}$, then $\mu_{\sigma_{\text{mlc}}}$ is injective (as $\iota_{\sigma_{\text{mlc}}}$ is the identity map). In particular, Fujino’s conjecture, which concerns about the situation when $\varphi_F$ and $\varphi_M$ are smooth and $M = F^{\otimes m}$ for some integer $m \geq 1$, holds true under the given assumptions.

Suppose $(X, D)$ is plt and suppose that $\varphi_F$ and $\varphi_M$ both have only neat analytic singularities. The following corollary to Theorem 1.2.1 can then be stated and proved.

**Corollary 1.3.3.** (cf. [30, Thm. 3.16]). Suppose that $(X, D)$ is a plt pair. Suppose that $X$, $D$, $\varphi_F$, $\varphi_M$ and $s$ satisfy all the assumptions in Theorem 1.2.1. Assume further that the section $s \in H^0(X, M)$ does not vanish identically on any lc centres of $(X, D)$. Then, the multiplication map $\mu_1$, that is,

$$H^q(X, K_X \otimes D \otimes F \otimes \mathcal{J}(\varphi_F)) \xrightarrow{\otimes s} H^q(X, K_X \otimes D \otimes F \otimes M \otimes \mathcal{J}_X(\varphi_F + \varphi_M)),$$

is injective for any integer $q \geq 0$. (Put $M := F^{\otimes m}$ and $\varphi_M := m\varphi_F$ and assume that $\varphi_F$ is smooth when Fujino’s conjecture is concerned.)

**Proof.** The pair $(X, D)$ being plt means that $\sigma_{\text{mlc}} = 1$. By Theorem 1.3.2, it suffices to show that $\ker \mu_0 = \ker \iota_0$ and $\ker \nu_1 = \ker \tau_1$. The equality $\ker \mu_0 = \ker \iota_0$ is guaranteed by Theorem 1.2.1. Notice that, as $\sigma_{\text{mlc}} = 1$, the homomorphism $\tau_1$ is the identity map. The goal is therefore to prove that $\nu_1$ is injective.

Write $D = \sum_{i \in I} D_i$, where $D_i$’s are the mutually disjoint irreducible components of $D$. Then, it follows from (eq1.3.2) (with $\sigma = 1$ and $\{D_i \cap V \mid i \in I \text{ s.t. } D_i \cap V \neq \emptyset\} = \{D_p^i \mid p \in I^V\}$) that the homomorphism $\nu_1$ is reduced to
\[ \nu_1 : \bigoplus_{i \in I} H^q(D_i, K_{D_i} \otimes F|_{D_i} \otimes \mathcal{I}_{D_i}(\varphi_F|_{D_i})) \rightarrow \bigoplus_{i \in I} H^q(D_i, K_{D_i} \otimes (F \otimes M)|_{D_i} \otimes \mathcal{I}_{D_i}((\varphi_F + \varphi_M)|_{D_i})) , \]

which maps the \( i \)-th summand to the \( i \)-th summand via the multiplication \( \otimes s|_{D_i} \). Write the homomorphism on the \( i \)-th summand as \( \nu_{1,i} \), and thus \( \nu_1 = \bigoplus_{i \in I} \nu_{1,i} \). Note that \( s|_{D_i} \) is non-trivial and \( \varphi_F|_{D_i} \) is psh for each \( i \in I \). When \( M = F^{\otimes m} \) and \( \varphi_M = m\varphi_F \), each \( \nu_{1,i} \) is injective by Theorem 1.1.3 (putting \( D = 0 \) and \( a = \frac{1}{m} \) in the theorem; notice that each \( D_i \) is a compact Kähler manifold). For a more general pair \((M, \varphi_M)\) which satisfies the given assumptions in Theorem 1.2.1, following the proof of \([27, \text{Thm. 1.3}]\) or the arguments given in Section 3.1 under the current setup (i.e. \( \varphi_F \) and \( \varphi_M \) having only neat analytic singularities with snc), it is easy to see that the injectivity of \( \nu_{1,i} \) for each \( D_i \) (or, more precisely, for each pair \((D_i, 0)\)) still holds true. In any case, this implies that \( \nu_1 \) itself is injective. \( \square \)

**Remark 1.3.4.** When \((X, D)\) is an lc pair (which need not be plt) and \( \text{lc}^\sigma_X(D) = \bigcup_{\sigma \in I} \mathcal{D}_p^\sigma \), where \( \mathcal{D}_p^\sigma \)'s are the \( \sigma \)-lc centres, the description in the proof above implies that, if \( s \) does not vanish identically on any \( \sigma \)-lc centres \( \mathcal{D}_p^\sigma \), the multiplication map

\[ \mathcal{H}^{n,q}(\mathcal{R}_\sigma(\varphi_F)) \xrightarrow{\otimes s} \mathcal{H}^{n,q}(M \otimes \mathcal{R}_\sigma(\varphi_F + \varphi_M)) , \]

which can be rewritten as

\[ \bigoplus_{\sigma \in I} H^q(\mathcal{D}_p^\sigma, K_{\mathcal{D}_p^\sigma} \otimes F|_{\mathcal{D}_p^\sigma} \otimes \mathcal{I}_{\mathcal{D}_p^\sigma}(\varphi_F|_{\mathcal{D}_p^\sigma})) \rightarrow \bigoplus_{\sigma \in I} H^q(\mathcal{D}_p^\sigma, K_{\mathcal{D}_p^\sigma} \otimes (F \otimes M)|_{\mathcal{D}_p^\sigma} \otimes \mathcal{I}_{\mathcal{D}_p^\sigma}((\varphi_F + \varphi_M)|_{\mathcal{D}_p^\sigma})) , \]

according to (eq.1.3.2), in which the \( p \)-th summand is mapped to the \( p \)-th summand via \( \otimes s|_{\mathcal{D}_p^\sigma} \), is indeed injective.

Corollary 1.3.3 is reduced to Theorem 1.1.2 of Matsumura (with a slightly relaxed assumption on \( \tilde{\partial}\tilde{\partial}\varphi_M \)) when \( \varphi_F \) and \( \varphi_M \) are smooth. The corresponding statement for general lc pairs \((X, D)\) is a generalisation of Fujino’s conjecture (Conjecture 1.1.1). Theorem 1.3.2, together with Theorem 1.2.1, guarantees that such generalised conjecture is solved once it is shown that \( \ker \nu_\sigma = \ker \tau_\sigma \) for all \( \sigma = 1, \ldots, \sigma_{\text{mle}} \). Even without deeper analysis of the adjoint ideal sheaves, one can already solve the generalised version of Fujino’s conjecture when \( \dim \mathbb{C} X = 2 \). The same result for \( M = F^{\otimes m} \) with smooth \( \varphi_F \) and \( \varphi_M = m\varphi_F \) is obtained by Matsumura in \([29, \text{Thm. 1.4}]\).

**Theorem 1.3.5** (cf. \([29, \text{Thm. 1.4}]\)). Suppose that \( X \), \( D \), \( \varphi_F \), \( \varphi_M \) and \( s \) satisfy all the assumptions in Theorem 1.2.1 (so, in particular, \((X, D)\) is an lc pair which need not be plt) and suppose also that \( \dim \mathbb{C} X = 2 \). Assume further that the section \( s \in H^0(X, M) \) does not vanish identically on any lc centres of \((X, D)\). Then, the homomorphism

\[ H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi_F)) \xrightarrow{\otimes s} H^q(X, K_X \otimes D \otimes F \otimes M \otimes \mathcal{I}_X(\varphi_F + \varphi_M)) , \]

is injective for any integer \( q \geq 0 \).

**Proof.** Set \( \varphi_{F \otimes M} := \varphi_F + \varphi_M \) for convenience and let

\[ \nu_{\sigma - 1}^\prime := q_{\nu_{\sigma - 1}} : \mathcal{H}^{n,q}(\mathcal{I}_\sigma(\varphi_F; \psi_D)/\mathcal{I}_{\sigma - 1}(\varphi_F; \psi_D)) \xrightarrow{\otimes s} \mathcal{H}^{n,q}(M \otimes \mathcal{I}_{\sigma - 1}(\varphi_{F \otimes M}; \psi_D)/\mathcal{I}_\sigma(\varphi_{F \otimes M}; \psi_D)) , \]
for any integers $\sigma, \sigma'$ and $q$ such that $1 \leq \sigma \leq \sigma' \leq \sigma_{\text{mlc}}$ and $q \geq 0$. Then $\nu_\sigma = \nu_{\sigma-1}^\sigma \circ \tau_\sigma$ for all $\sigma = 1, \ldots, \sigma_{\text{mlc}}$. Moreover, the discussion in Remark 1.3.4 (or the injectivity theorem for the case where $D = 0$) implies that $\nu_{\sigma-1}^\sigma$ is injective for all $\sigma = 1, \ldots, \sigma_{\text{mlc}}$ and $q \geq 0$ (note that $\mathcal{R}(\varphi_L)$ is injective for all $\sigma = 1, \ldots, \sigma_{\text{mlc}}$ and $q \geq 0$). According to Theorem 1.3.2 and given Theorem 1.2.1, the claim in this theorem is proved when one shows that $\nu_{\sigma-1}^\sigma$ is injective for $\sigma = 1, \ldots, \sigma_{\text{mlc}}$.

When $\dim_C X = 2$, the codimension $\sigma_{\text{mlc}}$ of the mlc of $(X, D)$ can take only values $1$ or $2$. The case where $\sigma_{\text{mlc}} = 1$ is handled in Corollary 1.3.3. Assume $\sigma_{\text{mlc}} = 2$ in what follows.

It is known that $\nu_{\sigma-1}^\sigma$ is injective for $\sigma = 2$. It remains to check the injectivity of $\nu_0^2$ in view of Theorem 1.3.2. Considering the short exact sequence

$$
0 \longrightarrow \mathcal{R}_1(\varphi_L) \longrightarrow \mathcal{J}_2(\varphi_L; \psi_D) \longrightarrow \mathcal{R}_2(\varphi_L) \longrightarrow 0
$$

for $L = F$ or $F \otimes M$ (obtained from (eq1.3.1)), one obtains a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{R}_1(\varphi_F) \\
& & \downarrow \delta \\
0 & \longrightarrow & \mathcal{R}_2(\varphi_F) \\
& & \downarrow \delta' \\
& & \mathcal{R}_2(\varphi_F) \\
\end{array}
$$

(eq1.3.4)

where all columns and rows are exact.

Notice that $\mathcal{R}_2(\varphi_{F \otimes M})$ is supported on $\text{lc}_X^2(D)$, which has dimension $0$ and is thus a finite set of points in $X$. Therefore,

$$
\mathcal{H}^{2,q-1}(M \otimes \mathcal{R}_2(\varphi_{F \otimes M})) = 0 \quad \text{for } q \neq 1
$$

and a diagram-chasing shows that $\nu_0^2$ is injective for all $q \neq 1$.

To see that $\nu_0^2$ is injective for $q = 1$, notice that the map

$$
0^1 \nu_0^2 : \mathcal{H}^{2,0}(\mathcal{R}_2(\varphi_F)) \longrightarrow \mathcal{H}^{2,0}(M \otimes \mathcal{R}_2(\varphi_{F \otimes M}))
$$

is an isomorphism, as $s$ is non-zero at every point of the finite set $\text{lc}_X^2(D)$ by assumption. The surjectivity of $0^1 \nu_0^2$ makes it possible to show that $1^2 \nu_0^2$ is injective via again a diagram-chasing. This completes the proof. 

\[\square\]

Remark 1.3.6. In [29, Thm. 1.4] (in which $M = F^{\otimes m}$, $\varphi_M = m \varphi_F$ and $\varphi_F$ is smooth), the assumption on $s \in H^0(X, F^{\otimes m})$ is more relaxed than that in Theorem 1.3.5 in the sense that $s$ is required not to vanish identically only on every component of $(X, D)$. In other words, $s$ may vanish on some of the 2-lc centres of $(X, D)$. The proof in Theorem 1.3.5 can be adjusted to recover also this case. Following the proof above, one only has to verify the injectivity of $\nu_0^2$ for the case $q = 1$. (Note that $0^1 \nu_0^2$ may not be injective under the weakened assumption.) Suppose $\text{lc}_X^1(D) = \bigcup_{i \in I_1} D_i$ and $\text{lc}_X^2(D) = \bigcup_{p \in I_2} D_p$ (where each
In the residue short exact sequence though (see [20, Remark 2.17]), the case where sheaf then satisfies a residue short exact sequence similar to (eq1.3.1), thus being coherent, at least in as well as corresponding to the connecting morphism $H_{p}$ of Cao in [3]. Let $\phi$ non-trivial, one may, for example, put an extra assumption that the numerical dimension $p$ all $|D_{ip}| > 0$, thus $F^{\otimes m}|_{D_{ip}}$ as well as $F|_{D_{ip}}$ is ample on the curve $D_{ip}$ (true also for $j_{p}$ in place of $i_{p}$). The summand $H^{1}(D_{ip}, K_{D_{ip}} \otimes F|_{D_{ip}})$ in $H^{2,1}(\mathcal{R}(\phi_{F}))$ therefore vanishes, and the same holds true for the corresponding summand in $H^{2,1}(F^{\otimes m} \otimes \mathcal{R}(\phi_{F}))$. (If $\mathcal{I}_{D_{ip}}((m + 1)\phi_{F}|_{D_{ip}})$ is non-trivial, one may, for example, put an extra assumption that the numerical dimension of $(F|_{D_{ip}}, \phi_{F}|_{D_{ip}})$ should satisfy $\text{nd}(F|_{D_{ip}}, \phi_{F}|_{D_{ip}}) = 1$ and apply the vanishing theorem of Cao in [3].) Let $J_{2}$ be the subset of $I_{2}$ which contains all $p \in I_{2}$ such that $s$ does not vanish at $D_{p}$ and let $H^{2,0}(\mathcal{R}_{2}(\phi_{F}))|_{J_{2}}$ (resp. $H^{2,0}(F^{\otimes m} \otimes \mathcal{R}_{2}((m + 1)\phi_{F}))|_{J_{2}})$ be the sum of summands in $H^{2,0}(\mathcal{R}_{2}(\phi_{F}))$ (resp. $H^{2,0}(F^{\otimes m} \otimes \mathcal{R}_{2}((m + 1)\phi_{F}))$) corresponding to all $p \in J_{2}$. The vanishing result above implies that, after replacing the first row of the diagram (eq1.3.4) by

\[
H^{2,0}(\mathcal{R}_{2}(\phi_{F}))|_{J_{2}} \xrightarrow{\nu_{1}^{2}|_{J_{2}}} H^{2,0}(F^{\otimes m} \otimes \mathcal{R}_{2}((m + 1)\phi_{F}))|_{J_{2}},
\]

the two columns of the diagram are still exact. Since $\nu_{1}^{2}|_{J_{2}}$ is now an isomorphism, a diagram-chasing as in the proof of Theorem 1.3.5 then guarantees that $1_{\nu_{0}^{2}}$ is injective.

The general case will be discussed in subsequent papers.

1.4. Restrictions on the singularities of $\phi_{F}$ and $\phi_{M}$. It is natural to ask whether the above results can be generalised to the setting where (quasi-)psh potentials $\phi_{F}$ and $\phi_{M}$ with more general singularities are allowed. There are two apparent constraints on the singularities of the potentials as seen from the current exposition.

The first one comes from the refined hard Lefschetz theorem of Matsumura (see Theorem 2.5.1 or [30, Thm. 3.3]), in which $\phi_{F}$ (or $\phi_{F} + \phi_{D}$) is required to be smooth on some Zariski open set in $X$, although there is no restriction on its singularities on the complement.

The other one comes from the use of the adjoint ideal sheaves introduced in [5]. In that paper, all involving potentials are assumed to have neat analytic singularities. It is expected that the regularities of the potentials can be relaxed, although the singularities on the potentials may still not be arbitrary if one insists in the current definition of the adjoint ideal sheaves and requires them to satisfy the residue short exact sequences (eq1.3.1) with the quotient sheaves having some decent description as in (eq1.3.2).

In order to allow more general singularities on $\phi_{F}$ and $\phi_{M}$ in the injectivity theorem, one should first relax the requirements on their regularities from the adjoint ideal sheaves.

---

3In the version of adjoint ideal sheaves studied by Guenancia [20] and Dano Kim [23], the involving potential $\phi_{L}$ is assumed such that $e^{\phi_{L}}$ is locally Hölder continuous, and the corresponding adjoint ideal sheaf then satisfies a residue short exact sequence similar to (eq1.3.1), thus being coherent, at least in the case where $(X, D)$ is plt. There exists a psh potential such that their adjoint ideal sheaf does not fit in the residue short exact sequence though (see [20, Remark 2.17]).
1.5. **Organisation of the article.** This paper is organised as follows.

Preliminaries are given in Section 2. Sections 2.1 and 2.2 explain some less commonly used notations as well as the basic setup and assumptions used in this article. The \( L^2 \) Dolbeault isomorphism is stated in Section 2.3, also for the purpose of fixing notation. In view of the use of Kähler metrics which are incomplete on \( X \setminus D \) or \( X^0 \setminus D \), justification of the well-definedness of the formal adjoint of \( \overline{D} \), which has singularities along \( D \), is provided in Section 2.4. The (twisted) Bochner–Kodaira formulas are also stated there. In Section 2.5, the refinement of the hard Lefschetz theorem proved in [30, Thm. 3.3], with a minor adjustment for the present use, is stated and a sketch of proof is provided. The computation on the residue functions corresponding to \( \sigma \)-lc-measures, with relaxed regularity assumptions compared to the statements in [6] and [4], is given in full in Section 2.6.

Section 3 is devoted to the proof of Theorem 1.2.1. An outline of the proof is given in Section 3.1, which provides the essential arguments and leaves the technical details to latter sections. For the sake of clarity, the technical details under the assumption that both \( \varphi_\sigma \) and \( \varphi_M \) are smooth are first presented in Section 3.2. The necessary adjustments for the singular case are then presented in Section 3.3.

2. Preliminaries

2.1. **Notation.** In this paper, the following notations are used throughout.

**Notation 2.1.1.** Set \( i := \sqrt{-1} \).  

**Notation 2.1.2.** Each potential \( \varphi \) (of the curvature of a metric) on a holomorphic line bundle \( L \) in the following represents a collection of local functions \( \{ \varphi_\gamma \}_\gamma \) with respect to some fixed local coordinates and trivialisation of \( L \) on each open set \( V_\gamma \) in a fixed open cover \( \{ V_\gamma \}_\gamma \) of \( X \). The functions are related by the rule \( \varphi_\gamma = \varphi_\gamma' + 2 \text{Re} \, h_{\gamma \gamma'} \) on \( V_\gamma \cap V_{\gamma'} \), where \( e^{h_{\gamma \gamma'}} \) is a (holomorphic) transition function of \( L \) on \( V_\gamma \cap V_{\gamma'} \) (such that \( s_\gamma = s_\gamma' e^{h_{\gamma \gamma'}} \), where \( s_\gamma \) and \( s_\gamma' \) are the local representatives of a section \( s \) of \( L \) under the trivialisations on \( V_\gamma \) and \( V_{\gamma'} \) respectively). Inequalities between potentials is meant to be the inequalities under the trivialisations over open sets in the fixed open cover \( \{ V_\gamma \}_\gamma \).

**Notation 2.1.3.** For any prime (Cartier) divisor \( E \), let

- \( \phi_E := \log |s_E|^2 \), representing the collection \( \{ \log |s_{E, \gamma}|^2 \}_\gamma \), denote a potential (of the curvature of the metric) on the line bundle associated to \( E \) given by the collection of local representations \( \{ s_{E, \gamma} \}_\gamma \) of some canonical section \( s_E \) (thus \( \phi_E \) is uniquely defined up to an additive constant);
- \( \varphi_E^{\text{sm}} \) denote a smooth potential on the line bundle associated to \( E \);
- \( \psi_E := \phi_E - \varphi_E^{\text{sm}} \), which is a global function on \( X \), when both \( \phi_E \) and \( \varphi_E^{\text{sm}} \) are fixed.

All the above definitions are extended to any \( \mathbb{R} \)-divisor \( E \) by linearity. For notational convenience, the notations for a \( \mathbb{R} \)-divisor and its associated \( \mathbb{R} \)-line bundle are used interchangeably. The notation of a line bundle is also abused to mean its associated invertible sheaf.

**Notation 2.1.4.** For any \((n,0)\)-form (or \( K_X \)-valued section) \( f \) where \( n = \dim_{\mathbb{C}} X \), define \( |f|^2 := c_n f \wedge \overline{f} \), where \( c_n := (-1)^{(n+1)/2} (\pi \hbar)^n \). For any hermitian metric \( \omega = \sum_{\gamma} \omega_\gamma \),
\[\pi i \sum_{1 \leq j,k \leq n} h_{jk} \, dz^j \wedge \bar{d}z^k \] on \( X \), set \( d\text{vol}_{X,\omega} := \frac{\omega^n}{n!} \). When \( f \) is an \((n,q)\)-form with \( q \geq 1 \) \((n = \dim_{\mathbb{C}} X)\), its pointwise (squared) norm with respect to \( \omega \) for the \((0,q)\)-directions is written as \( |f|^2_\omega \) (which can be viewed as a non-negative \((n,n)\)-form or \(K_X \otimes \bar{K}_X\)-valued section). The same convention for the symbol \( |f|^{2n}_\omega \) applies also to the setup where \( X \) is replaced by its submanifolds. When the Hodge *-operator with respect to \( \omega \) is involved, let \( |*\omega f|^2_\omega \) denote the function-valued pointwise (squared) norm with respect to \( \omega \) (with differential forms of \(*\omega f\) in all directions being contracted).

**Notation 2.1.5.** For any two non-negative functions \( u \) and \( v \), write \( u \lesssim v \) (equivalently, \( v \gtrsim u \)) to mean that there exists some constant \( C > 0 \) such that \( u \leq Cv \), and \( u \sim v \) to mean that both \( u \lesssim v \) and \( v \gtrsim u \) hold true.

### 2.2. Basic setup.

Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \( n \) equipped with a Kähler metric \( \omega \) on \( X \). Let \( \mathcal{I}(\varphi) := \mathcal{I}_X(\varphi) \) be the multiplier ideal sheaf of the potential \( \varphi \) on \( X \) given at each \( x \in X \) by

\[
\mathcal{I}(\varphi)_x := \mathcal{I}_X(\varphi)_x := \left\{ f \in \mathcal{O}_{X,x} \left| \begin{array}{l} f \text{ is defined on a coord. neighbourhood } V_x \ni x \\ \text{and } \int_{V_x} |f|^2 e^{-\varphi} \, d\text{vol}_x < +\infty \end{array} \right. \right\}.
\]

A potential \( \varphi \) is said to have **Kawamata log-terminal (klt) singularities** on \( X \) if \( \mathcal{I}(\varphi) = \mathcal{O}_X \) on \( X \), and **log-canonical (lc) singularities** on \( X \) if \( \mathcal{I}(1 - \varepsilon \varphi) = \mathcal{O}_X \) for all \( \varepsilon > 0 \) on \( X \).

Throughout this paper, the following data are assumed:

1. \( D \) is a **reduced divisor** on \( X \) with **simple normal crossings (snc)** such that \((X, D)\) is log-smooth and log-canonical (lc), and it is endowed with a potential \( \phi_D \) defined from a canonical section of \( D \) (see Notations 2.1.2 and 2.1.3) and a smooth potential \( \varphi_D^{sm} \) such that the global function

\[
\psi_D := \varphi_D - \varphi_D^{sm} \leq -1 \quad \text{on } X;
\]

2. \((F, e^{-\varphi_F})\) and \((M, e^{-\varphi_M})\) are holomorphic line bundles on \( X \) equipped with singular hermitian metrics such that

- \( \varphi_F \) is **plurisubharmonic (psh)** and \( \varphi_M \) is **quasi-plurisubharmonic (quasi-psh)** such that the curvature of \( \varphi_M \) is dominated by some multiple of the curvature of \( F \), i.e. for some constant \( C > 0 \), one has

\[
i\partial\bar{\partial}\varphi_F \geq 0 \quad \text{and} \quad -C\omega \leq i\partial\bar{\partial}\varphi_M \leq C i\partial\bar{\partial}\varphi_F \quad \text{on } X
\]

in the sense of (1,1)-currents,

- \( \varphi_F \) and \( \varphi_M \) both have at worst **neat analytic singularities**, i.e. they are locally of the form (under the assumption that they are both quasi-psh)

\[
c \log \left( \sum_{j=1}^N |g_j|^2 \right) \mod \mathcal{C}^\infty \quad \text{for some } c \in \mathbb{R}_{>0} \text{ and } g_j \in \mathcal{O}_X,
\]

- the polar sets \( P_F := \varphi_F^{-1}(-\infty) \) and \( P_M := \varphi_M^{-1}(-\infty) \) do not contain any **lc centres of** \((X, D)\) (i.e. irreducible components of any intersections of irreducible components of \( D \) in \( X)\),

- the polar sets \( P_F \) and \( P_M \) are assumed to be divisors in \( X \) and \( P_F \cup P_M \cup D \) has only snc;
(3) $s \in H^0(X, M)$ is a non-trivial global holomorphic section of $M$ on $X$ such that
$$\sup_X |s|_{\varphi_M}^2 < \infty;$$

(4) $X^\circ := X \setminus (P_F \cup P_M)$ is a complete Kähler manifold (as $P_F \cup P_M$ is an analytic set, see [7, Thm. 1.5]) which is equipped with a complete Kähler form $\tilde{\omega}$ given by
$$\tilde{\omega} := 2\omega + \frac{1}{i\partial \bar{\partial} \log |\ell \psi_{P_F \cup P_M}|}
= 2\omega + \frac{i\partial \bar{\partial} \psi_{P_F \cup P_M}}{|\psi_{P_F \cup P_M}|(\log |\ell \psi_{P_F \cup P_M}|)^2}
+ \left(1 + \frac{2}{\log |\psi_{P_F \cup P_M}|} \frac{i\partial \bar{\partial} \psi_{P_F \cup P_M} \wedge \bar{\partial} \psi_{P_F \cup P_M}}{|\psi_{P_F \cup P_M}|^2(\log |\ell \psi_{P_F \cup P_M}|)^2}\right),$$
where $P_F \cup P_M$ is viewed as a reduced divisor and $\psi_{P_F \cup P_M} := \phi_{P_F \cup P_M} - \varphi_{P_F \cup P_M} \leq -1$ is defined as in Notation 2.1.3, therefore satisfying $i\partial \bar{\partial} \psi_{P_F \cup P_M} \geq -\omega$ on $X$, and $\ell \gg e$ is some constant such that
$$\omega + \frac{i\partial \bar{\partial} \psi_{P_F \cup P_M}}{|\psi_{P_F \cup P_M}|(\log |\ell \psi_{P_F \cup P_M}|)^2} \geq 0,$$
thus having $\tilde{\omega} \geq \omega$ and $\tilde{\omega} \geq i\partial (\log (e \log (|\ell \psi_{P_F \cup P_M}|))) \wedge \bar{\partial} (\log (e \log (|\ell \psi_{P_F \cup P_M}|)))$ on $X$. (Note also that the local potential of $\tilde{\omega}$ can be chosen to be locally bounded in $X$, i.e. for any $p \in X$, there exist a neighbourhood $U \ni p$ and a bounded function $\Phi$ on $U$ such that $\tilde{\omega} = i\partial \bar{\partial} \Phi$ on $U \setminus (P_F \cup P_M)$.)

Call an open set $V \subset X$ as an admissible open set with respect to the data $(\varphi_F, \varphi_M, \psi_D)$ (or simply $(\varphi_F, \psi)$ when the expression of $\varphi_M$ is not under concern) in the holomorphic coordinate system $(z_1, \ldots, z_n, \zeta_1, \ldots, \zeta_m)$ if $V$, sitting inside a coordinate chart with a holomorphic coordinate system $(z_1, \ldots, z_n) \text{ on which } F \text{ and } M \text{ are trivialised, is biholomorphic to a polydisc centred at the origin such that }$
$$D \cap V = \{z_1 \cdots z_{\sigma_V} = 0\} \quad \text{for some integer } \sigma_V \leq n,$$
$$\psi_D|_V = \sum_{j=1}^{\sigma_V} \log \cdot |z_j|^2 - \varphi_{D, sm}^m|_V \quad \text{and} \quad \varphi_*|_V = \sum_{k=\sigma_V+1}^n b_{*,k} \log \cdot |z_k|^2 + \beta_* \quad \text{for } \bullet = F, M,$$
where, after shrinking $V$ if necessary,
\begin{itemize}
  \item $\sup_V \log \cdot |z_j|^2 < 0$ for $j = 1, \ldots, n$,
  \item $\beta_F$ and $\beta_M$ are smooth functions such that $\sup_V \beta_* \leq 0$, 
  \item $b_{F,k}$'s and $b_{M,k}$'s are constants such that $b_{*,k} \geq 0$ for $k = \sigma_V + 1, \ldots, n$ (as $\varphi_F$ and $\varphi_M$ are both quasi-psh), and 
  \item $\sup_V \beta \cdot \partial_{z_j}^V \psi_D = 2 - \inf_V r_j \partial_{\sigma_V}^V \varphi_{D, sm}^m > 0$ for $j = 1, \ldots, \sigma_V$, where $r_j = |z_j|$ is the radial component of the polar coordinates.
\end{itemize}
Such an open set $V$ is simply called admissible if the data $(\varphi_F, \varphi_M, \psi_D)$ (or $(\varphi_F, \psi)$) are understood and some holomorphic coordinate system satisfying the above criteria is assumed. Note that such admissible open sets are the kind of open sets on which the computations in [6, §2.2] and [4, §2.2] are valid. The family of all such admissible open sets forms a basis of the topology of $X$. 


2.3. \textit{L}^2 \textit{Dolbeault isomorphism.} Put

\[ \varphi := \varphi_F + \phi_D. \]

The \textit{L}^2 Dolbeault isomorphism (see \cite[Prop. 5.5]{27} and \cite[Prop. 2.8]{30} for a proof; see also footnote 1 on page 4) asserts that cohomology classes on \( X \) with coefficients in \( K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi) \) can be represented by the \( \overline{\partial} \)-closed weighted-\textit{L}^2 \((n,q)\)-forms in the Hilbert space \( L^{n,q}_{(2)}(D \otimes F)_{\varphi,\omega} := L^{n,q}_{(2)}(X^0;D \otimes F)_{\varphi,\omega} \) \((\text{resp.} L^{n,q}_{(2)}(D \otimes F)_{\varphi,\omega} := L^{n,q}_{(2)}(X;D \otimes F)_{\varphi,\omega})\) for any \( q \geq 0 \), which is equipped with the \( \text{L}^2 \) norm

\[ \| \xi \|^2_{\varphi,\omega} := \int_{X^0} |\xi|^2_{\varphi,\omega} := \int_{X^0} |\xi|^2_{\varphi,\omega} e^{-\varphi} \quad \left( \text{resp. } \| \xi \|^2_{\varphi,\omega} := \int_X |\xi|^2_{\varphi,\omega} \right). \]

Note that, in the notation above, the subscript \( \varphi,\omega \) indicates the contraction only of the \((0,q)\)-forms in \( \xi \) via \( \omega \) and thus \( |\xi|^2_{\varphi,\omega} \) is a global real \((n,n)\)-form which can be integrated without any further need of metrics on \( K_X \) (see Notation 2.1.4; the same for the subscript \( \varphi,\omega \)).

One has the decomposition

\[ L^{n,q}_{(2)}(D \otimes F)_{\varphi,\omega} = \mathcal{H}^{n,q}_{\varphi,\omega} \oplus \left( \text{im} \overline{\partial} \right)_{\varphi,\omega} \oplus \left( \text{im} \overline{\partial} \right)_{\varphi,\omega} = \mathcal{H}^{n,q}_{\varphi,\omega} \oplus \left( \text{im} \overline{\partial} \right)_{\varphi,\omega} \oplus \left( \text{im} \overline{\partial}^* \right)_{\varphi,\omega}, \]

where \( \overline{\partial}^* \) is the Hilbert space adjoint of \( \overline{\partial} \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{\varphi,\omega} \), the spaces \( \left( \text{im} \overline{\partial} \right)_{\varphi,\omega} \) and \( \left( \text{im} \overline{\partial}^* \right)_{\varphi,\omega} \) being their closures in \( L^{n,q}_{(2)}(D \otimes F)_{\varphi,\omega} \), and \( \mathcal{H}^{n,q}_{\varphi,\omega} \) is the space of harmonic forms (with respect to \( \overline{\partial} \) and \( \overline{\partial}^* \)) in \( L^{n,q}_{(2)}(D \otimes F)_{\varphi,\omega} \). Since \( X \) is compact, both \( \left( \text{im} \overline{\partial} \right)_{\varphi,\omega} \) and \( \left( \text{im} \overline{\partial}^* \right)_{\varphi,\omega} \) are closed subspaces of \( L^{n,q}_{(2)}(D \otimes F)_{\varphi,\omega} \) (see, for example, \cite[Prop. 5.8]{27}). From the inclusion \( \mathcal{I}(\varphi_F + \phi_D) \subset \mathcal{I}(\varphi_F) \) and the \textit{L}^2 Dolbeault isomorphism, the following commutative diagram follows:

\[
\begin{array}{ccc}
H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi_F + \phi_D)) & \cong & H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi_F)) \\
\mathcal{H}^{n,q}_{\varphi,\omega} \cong \left( \ker \overline{\partial} \right)_{\varphi,\omega} \oplus \left( \text{im} \overline{\partial} \right)_{\varphi,\omega} & \cong & \left( \ker \overline{\partial} \right)_{\varphi_F + \phi_D} \oplus \left( \text{im} \overline{\partial} \right)_{\varphi_F + \phi_D,\omega} \\
\end{array}
\]

The above still holds true when \( \omega \) is replaced by \( \omega \) (see \cite[Prop. 2.8]{30}). Indeed, the \textit{L}^2 Dolbeault isomorphism implies that \( \mathcal{H}^{n,q}_{\varphi,\omega} \cong \mathcal{H}^{n,q}_{\varphi,\omega} \) although they may not be the same as subsets of \( L^{n,q}_{(2)}(D \otimes F)_{\varphi,\omega} \). This fact is not needed in this paper. It is stated here just for completeness.

2.4. Adjoint of \( \overline{\partial} \) and Bochner–Kodaira formulas. In this section, let \( L \) be a holomorphic line bundle on \( X \) which represents either \( D \otimes F \) or \( D \otimes F \otimes M \) in the main body of this article. Suppose that there are two \textit{quasi-psh} potentials \( \varphi \) and \( \varphi \) on \( L \) over \( X \) such that \( \varphi \) is smooth on \( X^0 \) and

\[ \varphi = \varphi + \psi_D \]

(so \( \varphi \) is \( \varphi_F + \varphi^\text{sm}_D \) \((\varphi_M)\) while \( \varphi \) is \( \varphi_F + \phi_D \) \((\varphi_M)\) in the main body of this article). Note that \( \varphi \) is singular on \( X^0 \).

Let \( \partial \varphi \) be the formal adjoint of \( \overline{\partial} \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{\varphi,\omega} \) \((\text{i.e. the adjoint of} \overline{\partial} \text{on compactly supported smooth forms} \mathcal{A}^{\bullet,\bullet}_X(X^0;L) \text{with respect to} \langle \cdot, \cdot \rangle_{\varphi,\omega} \)}
whose domain is extended in the sense of currents). Define the operator $\vartheta := \vartheta_\varphi$ via the formula

$$\vartheta \zeta := \vartheta_\varphi \zeta := e^{\psi_\varphi} \partial_\varphi (e^{-\psi_\varphi} \zeta) = \partial \varphi \zeta + (\partial \psi_\varphi)^{\varphi_\zeta} \zeta \quad \text{for } \zeta \in L^\bullet_\varphi (X^o; L)_{\varphi, \varphi},$$

where $(\partial \psi_\varphi)^{\varphi_\zeta}$ denotes the adjoint of $\bar{\partial} \psi_\varphi \wedge \cdot$ with respect to $\langle \cdot, \cdot \rangle_\varphi$ on $X^o$. Call $\vartheta$ as the formal adjoint of $\bar{\partial}$ with respect to $\langle \cdot, \cdot \rangle_\varphi$ (on $X^o$), even though it is a priori the adjoint of $\bar{\partial}$ only on $\mathcal{A}^{\bullet, \bullet}_\varphi (X^o \setminus D; L)$ with respect to $\langle \cdot, \cdot \rangle_\varphi$. Indeed, it is easy to see (for example, by using the smooth cut-off function $\theta_\varepsilon$ described in Section 3.1) that $\mathcal{A}^{\bullet, \bullet}_\varphi (X^o \setminus D; L)$ is also dense in $L^\bullet_\varphi (X^o; L)_{\varphi, \varphi}$. An argument for proving that $\vartheta \zeta$ is well-defined as a current on $X^o$ for all $\zeta \in L^\bullet_\varphi (X^o; L)_{\varphi, \varphi}$ and that $\vartheta$ induces a densely defined closed operator on $L^\bullet_\varphi (X^o; L)_{\varphi, \varphi}$ is sketched below for the convenience of readers.

**Lemma 2.4.1.** For every $\zeta \in L^{n,q}_{\varphi}(X^o; L)_{\varphi, \varphi}$, $\vartheta \zeta$ is a well-defined current on $X^o$. Moreover, $\vartheta : L^{n,q}_{\varphi}(X^o; L)_{\varphi, \varphi} \to L^{n,q-1}_{\varphi}(X^o; L)_{\varphi, \varphi}$ is a densely defined operator with closed graph.

**Sketch of proof.** First it is to show that $\vartheta \zeta$ is in $L^1_{\text{loc}}$ (unweighted) on $X^o$ for any $\zeta \in L^{n,q}_{\varphi}(X^o; L)_{\varphi, \varphi}$, hence a current. It suffices to check that for $(\partial \psi_\varphi)^{\varphi_\zeta} \zeta$. As $\psi_\varphi$ has only analytic singularities, it is easy to check that $|\partial \psi_\varphi|^{\varphi_\zeta}_{\varphi_\zeta} \in L^1_{\text{loc}}(X^o)$, since, on any admissible open set $V \subset X^o$, one has

$$|\partial \psi_\varphi|^{\varphi_\zeta}_{\varphi_\zeta} e^{\psi_\varphi} \sim \left| \sum_{j=1}^{\sigma_V} \frac{dz_j}{z_j} - \partial \varphi^m_\varphi \right|^2_{\varphi_\zeta} \left| z_1 \cdots z_{\sigma_V} \right|^2$$

(see [37, §2] for the treatment when $\psi_\varphi$ having general singularities). Then, for any relatively open set $V \subset X^o$, via the Cauchy–Schwarz inequality, one obtains

$$\int_V \left| (\partial \psi_\varphi)^{\varphi_\zeta} \zeta \right|_{\varphi_\zeta}^2 d \text{vol}_{X^o, \varphi} \leq \left( \int_V |\partial \psi_\varphi|^{\varphi_\zeta}_{\varphi_\zeta} e^{\psi_\varphi} d \text{vol}_{X^o, \varphi} \right)^{\frac{1}{2}} \left( \int_V \left| \zeta \right|_{\varphi_\zeta}^2 \right)^{\frac{1}{2}},$$

which implies that $(\partial \psi_\varphi)^{\varphi_\zeta} \zeta$ is in $L^1_{\text{loc}}$ on $X^o$, as desired.

For the remaining claims, $\vartheta$ is densely defined since it is well-defined on $\mathcal{A}_\varphi^{n,q}(X^o \setminus D; L)$. Let $\vartheta$ be the maximal domain of $\vartheta$ in $L^{n,q}_{\varphi}(X^o; L)_{\varphi, \varphi}$. To see that it has closed graph, take a sequence $\{\zeta_\nu\}_{\nu \in \mathbb{N}} \subset \text{Dom} \vartheta \subset L^{n,q}_{\varphi}(X^o; L)_{\varphi, \varphi}$ such that both sequences $\{\zeta_\nu\}_{\nu \in \mathbb{N}}$ and $\{\vartheta \zeta_\nu\}_{\nu \in \mathbb{N}}$ converge in their respective $L^2$ spaces. The identity

$$\langle \vartheta \zeta_\nu, \xi \rangle_{\varphi_\zeta} = \langle \zeta_\nu, \vartheta^\ast \xi \rangle_{\varphi_\zeta} \quad \text{for any } \xi \in \mathcal{A}^{n,q-1}_{X^o \setminus D}(\varphi_\zeta) \text{ and } \nu \in \mathbb{N}$$

concludes that $\lim_{\nu \to \infty} \vartheta \zeta_\nu = \vartheta (\lim_{\nu \to \infty} \zeta_\nu).$ \hfill $\square$

As $\mathcal{A}_\varphi^{n,q-1}(X^o \setminus D; L)$ is dense in $L^{n,q-1}_{\varphi}(X^o; L)_{\varphi, \varphi}$ and is contained in the domain $\text{Dom} \vartheta \subset L^{n,q-1}_{\varphi}(X^o; L)_{\varphi, \varphi}$ of $\vartheta : L^{n,q-1}_{\varphi}(X^o; L)_{\varphi, \varphi} \to L^{n,q}_{\varphi}(X^o; L)_{\varphi, \varphi}$, it follows easily from the standard argument that $\text{Dom} \vartheta^{\ast} \subset \text{Dom} \vartheta$ and $\vartheta^\ast = \vartheta$ on $\text{Dom} \vartheta^\ast$, where $\vartheta^\ast$ is the formal adjoint of $\vartheta$.
where \( \text{Dom} \vartheta \subset L^\sigma_{(2)}(X^o; L)_{\varphi, \bar{\omega}} \) is the maximal domain of the operator \( \vartheta \) and \( \text{Dom} \vartheta^* \subset L^\sigma_{(2)}(D \otimes F)_{\varphi, \bar{\omega}} \) is the domain of the Hilbert space adjoint \( \vartheta^* \) of \( \vartheta \).

The following lemma is the twisted Bochner–Kodaira formula for \( K_X \otimes L \)-valued \((0, q)\)-forms used in [6] with a slightly different choice of auxiliary functions, which can be derived from [33, §1.3] or [31, Eq. (8)]. Recall that \( \bar{\omega} \) is a Kähler metric. Let \( \nabla_\varphi = \nabla_\varphi^{(1,0)} + \nabla_\varphi^{(0,1)} \) be the covariant differential operator (with the decomposition according to \((1, 0)\)- and \((0, 1)\)-types) induced from the Chern connection. Moreover, \((i\Theta)\bar{\omega}(\zeta, \zeta)_{\varphi, \bar{\omega}}\) denotes, for any real \((1, 1)\)-form \( i\Theta \) (usually in the form of \( i\partial\bar{\partial}\varphi \)) and any \( K_X \otimes L \)-valued \((0, q)\)-form \( \zeta \), the trace of the contraction between \( \Theta \) and \( e^{-\bar{\varphi}}\xi \wedge \bar{\zeta} \) with respect to \( \bar{\omega} \) on \( X^o \) (in the convention such that \((i\Theta)\bar{\omega}(\zeta, \zeta)_{\varphi, \bar{\omega}} \geq 0 \) whenever \( i\Theta \geq 0 \)). To be more precise, in local coordinates, it is given as

\[
(i\Theta)\bar{\omega}(\zeta, \zeta)_{\varphi, \bar{\omega}} = \sum_{j,k} \sum_{J_q} \Theta_{j,k} \zeta_{J_q}^j \bar{\zeta}_{J_q}^k e^{-\varphi},
\]

where

\[
\Theta = \sum_{j,k} \Theta_{j,k} \, dz^j \wedge d\bar{z}^k, \quad \zeta = \sum_{J_q} \zeta_{J_q}^j \, dz^j,
\]

\( J_q = (j_1, j_2, \ldots, j_q) \) and \( \sum_{J_q}' := \sum_{j_1 < j_2 < \cdots < j_q} \)

and the indices in the components of \( \zeta \) are raised via \( \bar{\omega} \). Under this convention, the inequality \( i\Theta \geq -C\bar{\omega} \) for some constant \( C > 0 \) implies that \((i\Theta)\bar{\omega}(\zeta, \zeta)_{\varphi, \bar{\omega}} \geq -\pi q C |\zeta|_{\bar{\omega}}^2 \) for any \((0, q)\)-form \( \zeta \) (as \( \bar{\omega} = \pi i \sum_{j,k} h_{j,k} \, dz^j \wedge d\bar{z}^k \)).

**Lemma 2.4.2.** Set, for any \( \varepsilon > 0 \),

\[
\eta_{\varepsilon} := |\psi_D|^{1-\varepsilon} \quad \text{on} \ X.
\]

Then, for any given number \( \varepsilon > 0 \), the twisted Bochner–Kodaira formula, which is referred to as \((tBK)_{\varepsilon, \varphi, \bar{\omega}}\), becomes

\[
(tBK)_{\varepsilon, \varphi, \bar{\omega}} = \int_{X^o} |{\bar{\partial}}\zeta|_{\varphi, \bar{\omega}}^2 \eta_{\varepsilon} + \int_{X^o} |\partial\zeta|_{\varphi, \bar{\omega}}^2 \eta_{\varepsilon} + \int_{X^o} \left( i\partial\bar{\partial}\varphi + \frac{1 - \varepsilon}{|\psi_D|} \, i\partial\bar{\partial}\psi_D \right) (\bar{\omega}) (\zeta, \zeta)_{\varphi, \bar{\omega}} \eta_{\varepsilon} - 2(1-\varepsilon) \text{Re} \int_{X^o} \left< \partial\zeta, \frac{(\partial\psi_D) \bar{\omega} \wedge \zeta}{|\psi_D|} \right>_{\varphi, \bar{\omega}} \eta_{\varepsilon} + \varepsilon \int_{X^o} \frac{1 - \varepsilon}{|\psi_D|^2} |(\partial\psi_D) \bar{\omega} \wedge \zeta|_{\varphi, \bar{\omega}}^2 \eta_{\varepsilon}
\]

for any compactly supported \( K_X \otimes L \)-valued smooth \((0, q)\)-forms \( \zeta \in \mathcal{A}^0_{X^o \setminus D} (X^o \setminus D; K_X \otimes L) \) on \( X^o \setminus D \).

For the ease of reference, the untwisted Bochner–Kodaira formula is referred to as \((BK)_{\varphi, \bar{\omega}} \) (\( (tBK)_{1, \varphi, \bar{\omega}} \)), which is given by

\[
(BK)_{\varphi, \bar{\omega}} = \int_{X^o} |{\bar{\partial}}\zeta|_{\varphi, \bar{\omega}}^2 + \int_{X^o} |\partial\zeta|_{\varphi, \bar{\omega}}^2 = \int_{X^o} |{\bar{\partial}}\zeta|_{\varphi, \bar{\omega}}^2 + \int_{X^o} |(i\partial\bar{\partial}\varphi)(\zeta, \zeta)_{\varphi, \bar{\omega}}|
\]

for any \( \zeta \in \mathcal{A}^0_{X^o \setminus D} (X^o \setminus D; K_X \otimes L) \).
Proof. Notice that, as $\zeta$ is chosen to be compactly supported on $X^\circ \setminus D$, on which $\varphi$ and $\eta_\varphi = |\psi_D|^{1-e}$ are smooth, the classical Bochner–Kodaira formula in [33, §1.3] or [31, Eq. (2.2)] is applicable. From there, it follows that
\[
\int_{X^\circ} \langle \bar{\partial} \zeta, \bar{\partial} \zeta \rangle_{\varphi, \bar{\omega}} + \int_{X^\circ} \langle \partial \zeta, \partial \zeta \rangle_{\varphi, \bar{\omega}} + 2 \Re \int_{X^\circ} \langle \partial \zeta, (\partial \log \eta_\varphi) \bar{\omega} \rangle_{\varphi, \bar{\omega}} \eta_\varphi + \int_{X^\circ} \langle \partial \log \eta_\varphi \bar{\varphi}, \zeta \rangle_{\varphi, \bar{\omega}}^2 &\eta_\varphi \\
= \int_{X^\circ} \langle \nabla^{(0,1)} \zeta, \nabla^{(0,1)} \zeta \rangle_{\varphi, \bar{\omega}} + \int_{X^\circ} \langle i \partial \bar{\partial} \varphi - i \partial \bar{\partial} \log \eta_\varphi \rangle \bar{\omega} \zeta, \zeta \rangle_{\varphi, \bar{\omega}} \eta_\varphi 
\]
A direct computation with the choice of $\bar{\omega}$ or $\bar{\omega}$, which is derived from the commutator identities, let
\[
\text{eq2.4.1}
\]
Remarking 2.4.3. For the comparison with the Bochner–Kodaira–Nakano formula used in [27, Prop. 2.4] or [30, Prop. 2.5], which is derived from the commutator identities, let $\bar{\omega}$ be $\zeta$ but treated as an $L$-valued $(n, q)$-form. Using a local computation (in a normal coordinate system at a given point of $X^\circ$ such that $\bar{\omega}$ and $i\theta$ are simultaneously diagonalized and the first derivatives of the coefficients of $\bar{\omega}$ all vanish at that point; see, for example, [8, Ch. VII, (3.2)]) and noting that $D^* = - \ast \bar{\omega} \ast \bar{\omega}$, one sees that
\[
\langle i\theta \Lambda_{\bar{\omega}} \bar{\zeta}, \bar{\zeta} \rangle_{\varphi, \bar{\omega}} = \langle i\theta \bar{\omega}(\bar{\zeta}, \zeta) \rangle_{\varphi, \bar{\omega}} \quad \text{and} \quad |D^* \bar{\zeta}|^2_{\varphi, \bar{\omega}} = |\ast \bar{\omega} \ast \bar{\omega} \bar{\zeta}|^2_{\varphi, \bar{\omega}} = |\nabla^{(0,1)} \zeta|^2_{\varphi, \bar{\omega}}.
\]
The equalities on the right-hand side can be seen more transparently if one notices that, in a normal coordinate system at an arbitrary point in $X^\circ$,
\[
|\nabla^{(0,1)} \zeta|^2_{\varphi, \bar{\omega}} = \sum_j \sum_{J_q} \left( \langle \nabla^{(0,1)} \zeta \rangle_{\varphi, \bar{\omega}} \right)^2 e^{-\varphi} = \sum_j \sum_{J_q} \left| \partial_j \zeta \right|^2_{\varphi, \bar{\omega}} e^{-\varphi} \quad \text{and}
\]
\[
|D^* \bar{\zeta}|^2_{\varphi, \bar{\omega}} = \left| \ast \bar{\omega} \ast \bar{\zeta} \right|^2_{\varphi, \bar{\omega}} = \left| \partial \ast \bar{\omega} \bar{\zeta} - \left( \partial \ast \bar{\omega} \right) \bar{\zeta} \right|^2_{\varphi, \bar{\omega}} = \sum_j \sum_{I_{n-q}} \left| \partial_j \bar{\zeta} \right|^2_{I_{n-q}} e^{-\varphi}
\]
in which $I_{n-q} = (i_1, \ldots, i_{n-q})$ and $J_q = (j_1, \ldots, j_q)$ are multi-indices such that $I_{n-q}$ is complementary to $J_q$ in the sense that $\{i_1, \ldots, i_{n-q}, j_1, \ldots, j_q\} = \{1, \ldots, n\}$. Moreover, $\zeta_{J_q} = \bar{\zeta}_{I_{n-q}}$. In the rest of this article, unless stated otherwise, $\bar{\zeta}$ and $\zeta$ are identified with each other.\footnote{The authors’ preference of the notation used in Lemma 2.4.2 over the one used in [27, Prop. 2.4] and by many others is due to its better reflection of its hermitian nature of the integral of $i\theta$ and the ease of incorporating inequality on $i\theta$ in mental calculations. Moreover, the Chern connection $\nabla^{(0,1)}$ of type $(0, 1)$ is more apparently independent of $\varphi$ when compared to $D^*$, a fact used throughout the paper. Otherwise, the choice is made simply out of the habit and taste of the first author.}

The pointwise formula of the Laplacian
\[
\text{eq2.4.1}
\]
for any $\zeta \in \mathcal{A}_{X^\circ}^{0,q}(X^\circ; K_X \otimes L)$, which leads to $\langle \nabla_{\varphi, \bar{\omega}} \rangle_{\varphi, \bar{\omega}}$, is also stated here for later use. Note that $-\nabla_{\varphi}^{(1,0)}$ is the formal adjoint of $\nabla^{(0,1)}$ with respect to $\langle \cdot, \cdot \rangle_{\varphi, \bar{\omega}}$, which is given by
\[
-\left( \nabla_{\varphi}^{(1,0)} \cdot \nabla^{(0,1)} \zeta \right)_{J_q} = - \sum_j \left( \nabla_{\varphi}^{(1,0)} - (\partial \varphi) \right)^j \nabla^j \zeta_{J_q}
\]
for any $\zeta \in \mathcal{O}_X^{0,q}(X^\circ; K_X \otimes L)$ in local coordinates, in which the index $j$ is raised via $\tilde{\omega}$ and $\nabla^{(1,0)}$ is the $(1,0)$-Chern connection on sections of $K_X$ with respect to $\tilde{\omega}$.

2.5. **Refined hard Lefschetz theorem.** In [30, §3.1], Matsumura proves a refinement of the hard Lefschetz theorem with multiplier ideal sheaves, which indeed provides a preimage for each image of the surjective map

$$\omega^q \wedge \cdot : H^0(X, \Omega^{n-q}_X \otimes D \otimes F \otimes \mathcal{I}(\varphi)) \to H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi)),$$

where $\varphi$ is a psh potential on $D \otimes F$ over $X$ which has arbitrary singularities on a Zariski closed subset while being smooth on the complement. The precise statement (with a slight alteration) under the current setting is stated as follows.

**Theorem 2.5.1 ([30, Thm. 3.3]).** Under the setup and notation given in Section 2.2 and 2.3, for any harmonic $\tilde{u} \in \mathcal{H}^{n,q}_{\varphi, \omega}$, it is stated as follows.

For any chosen $\tilde{u} \in \mathcal{H}^{n,q}_{\varphi, \omega}$ and for any $\delta > 0$, there exists $\tilde{u}_\delta \in \mathcal{H}^{n,q}_{\varphi, \omega}$ which represents the same cohomology class as $\tilde{u}$ in $H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi))$ and satisfies

$$\|\ast_\omega \tilde{u}_\delta\|_{\varphi, \omega}^2 \leq \|\ast_\omega \tilde{u}_\delta\|_{\varphi, \omega}^2 = \|\tilde{u}_\delta\|_{\varphi, \omega}^2 \leq \|\tilde{u}\|_{\varphi, \omega}^2 \leq \|\tilde{u}\|_{\varphi, \omega}^2,$$

where $\ast_\omega$ is the Hodge $\ast$-operator with respect to $\tilde{\omega}$. As $\tilde{u}_\delta$ is complete on $X^\circ \setminus \varphi^{-1}(-\infty)$, the Bochner–Kodaira formula (BK) $\varphi, \tilde{\omega}$ (see Lemma 2.4.2) is valid for all $L^2$ sections in the domains of $\tilde{\mathcal{D}}$ and its adjoint $\tilde{\mathcal{D}}^*$ with respect to $\varphi$ and $\tilde{\omega}$. In particular, it can be applied to $\tilde{u}_\delta$ and, thanks to the fact that $\iota \tilde{\mathcal{D}} \iota \varphi \geq 0$, yields

$$\|\tilde{\mathcal{D}} \ast_\omega \tilde{u}_\delta\|_{\varphi, \tilde{\omega}}^2 = \|\ast_\omega \tilde{\mathcal{D}} \ast_\omega \tilde{u}_\delta\|_{\varphi, \tilde{\omega}}^2 = \|\nabla^{(1,0)}\tilde{\omega}\|_{\varphi, \tilde{\omega}}^2 = 0$$

(see Remark 2.4.3). From the inequality on the far left in (*), the fact that $e^{-\varphi} \geq 1$ locally on $X$, it follows that $\ast_\delta \tilde{u}_\delta$ is not only holomorphic on $X^\circ \setminus \varphi^{-1}(-\infty)$, but also on the whole of $X$. It also follows that the set of holomorphic $(n-q)$-forms $\{\ast_\delta \tilde{u}_\delta\}_{\delta \in (0,1)}$ is locally uniformly bounded in $X$, thus exists a subsequence $\{\ast_{\delta_n} \tilde{u}_{\delta_n}\}_{\nu \in \mathbb{N}}$ which converges locally uniformly to some holomorphic $(n-q)$-form $f$ on $X$ as $\delta_n \to 0^+$. For any chosen $\delta' > 0$, one has $\|\tilde{u}_\delta\|_{\varphi, \tilde{\omega}}^2 \leq \|\tilde{u}_\delta\|_{\varphi, \tilde{\omega}}^2$ for all $\delta$ such that $\delta \leq \delta'$. Together with the inequalities on the right side of (*), it follows that, by passing to a further subsequence if necessary, $\{\tilde{u}_{\delta_n}\}_{\nu \in \mathbb{N}}$ converges weakly to some $\tilde{u}_0$ in $L^{n,q}_2(D \otimes F)_{\varphi, \tilde{\omega}}$ as $\delta_n \to 0^+$. Via the use of Cantor’s diagonal argument, the subsequence $\{\tilde{u}_{\delta_n}\}_{\nu \in \mathbb{N}}$ can be chosen independent of $\delta' > 0$ as $\delta'$ shrinks to 0, and thus the weak limits of the sequence in $L^{n,q}_2(D \otimes F)_{\varphi, \tilde{\omega}}$ for various $\delta' > 0$ all coincide. It then follows from the property of weak

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8See [37, Thm. 2] for a related statement which allows more general singularities on $\varphi$. 

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On an injectivity theorem for lc pairs with adjoint ideal sheaves
limits $\|\tilde{u}_0\|^2_{\varphi,\varpi} \leq \lim_{\delta_\nu \to 0^+} \|\tilde{u}_{\delta_\nu}\|^2_{\varphi,\varpi}$, and Fatou’s lemma that $\|\tilde{u}_0\|^2_{\varphi,\varpi} \leq \|\tilde{u}\|^2_{\varphi,\varpi}$.

By considering the kernel of the composition of the maps

$$(\ker \partial)_{\varphi,\varpi} \subseteq (\ker \partial)_{\varphi,\varpi'} \subseteq (\ker \partial)_{\varphi,\varpi'} \sim (\ker \partial)_{\varphi,\varpi},$$

one obtains (cf. [30, Prop. 3.1])

$$(\text{im} \partial)_{\varphi,\varpi} = (\ker \partial)_{\varphi,\varpi} \cap (\text{im} \partial)_{\varphi,\varpi'} = L_{(2)}^{n,q}(D \otimes F)_{\varphi,\varpi} \cap (\text{im} \partial)_{\varphi,\varpi'}.$$

Since $\tilde{u}_{\delta_\nu} - \tilde{u} \in (\text{im} \partial)_{\varphi,\varpi}$ implies that $\tilde{u}_0 - \tilde{u} \in (\text{im} \partial)_{\varphi,\varpi'}$, it follows that $\tilde{u}_0 - \tilde{u} \in (\text{im} \partial)_{\varphi,\varpi'}$. As $\tilde{u} \in \mathcal{H}_{\varphi,\varpi} \subseteq (\text{im} \partial)_{\varphi,\varpi}$, it forces the relations

$$\|\tilde{u}_0 - \tilde{u}\|^2_{\varphi,\varpi} + \|\tilde{u}\|^2_{\varphi,\varpi} = \|\tilde{u}_0\|^2_{\varphi,\varpi} \leq \|\tilde{u}\|^2_{\varphi,\varpi},$$

and thus $\tilde{u}_0 \in L_{(2)}^{n,q}(D \otimes F)_{\varphi,\varpi}$.

It remains to check that $*\tilde{u} f = (-1)^{n-q} \tilde{u}$, which then implies $f = *\tilde{u} \tilde{u}$ and completes the proof. The fact that $*_{\delta_\nu} \tilde{u}_{\delta_\nu}$ converges locally uniformly to $\tilde{u}$ in $X^0$ implies that $|*_{\tilde{u}} *_{\delta_\nu} \tilde{u}_{\delta_\nu} - *_{\tilde{u}} f|^2_{\varpi} = |*_{\delta_\nu} \tilde{u}_{\delta_\nu} - f|^2_{\varpi}$ converges locally uniformly to 0 in $X^0$. A direct computation also shows that $|*_{\tilde{u}} *_{\delta_\nu} \tilde{u}_{\delta_\nu} - *_{\delta_\nu} *_{\delta_\nu} \tilde{u}_{\delta_\nu}|^2_{\varpi}$ converges locally uniformly to 0 in $X^0$ (see the proof of [30, Prop. 3.8]), which then implies that, for any $\delta' > 0$,

$$\left|(-1)^{n-q} \tilde{u}_{\delta_\nu} - *_{\tilde{u}} f\right|^2_{\varpi} \leq \left|(-1)^{n-q} \tilde{u}_{\delta_\nu} - *_{\tilde{u}} f\right|^2_{\varpi} \to 0 \quad \text{locally uniformly in } X^0.$$

From the weak convergence of $\tilde{u}_{\delta_\nu}$, it follows that, for any $\zeta \in \mathcal{A}^{\varphi,\varpi}_{\varpi}(X^0; D \otimes F)$ and for any $\delta' > 0$,

$$\left\langle (-1)^{n-q} \tilde{u} - *_{\tilde{u}} f, \zeta \right\rangle_{\varphi,\varpi'} = \lim_{\delta_\nu \to 0^+} \left\langle (-1)^{n-q} \tilde{u}_{\delta_\nu} - *_{\tilde{u}} f, \zeta \right\rangle_{\varphi,\varpi'} = 0.$$

Therefore, one obtains $*_{\tilde{u}} f = (-1)^{n-q} \tilde{u}$ in $L_{(2)}^{n,q}(D \otimes F)_{\varphi,\varpi'}$, and then in $L_{(2)}^{n,q}(D \otimes F)_{\varphi,\varpi}$ by Fatou’s lemma, as desired.

2.6. Residue functions and $\sigma$-lc-measures. In this section, any claims made in terms of $(F, \varphi_F)$ hold true when $(F, \varphi_F)$ is replaced by $(F \otimes M, \varphi_F + \varphi_M)$.

Let $V$ be an admissible open set in $X$ with respect to $(\varphi_F, \psi_D)$ (see Section 2.2). Let $\mathcal{S}_m$ be the group of permutations on the set of $m$ elements for any $m \in \mathbb{N}$ and set

$$\mathcal{C}_m^{\varphi_F} := \mathcal{S}_m^{\varphi_F} / \mathcal{S}_m \times \mathcal{S}_m^{\varphi_F - \sigma},$$

which is the set of choices of $\sigma$ elements in a set of $\varpi$ elements. Any element $p \in \mathcal{C}_m^{\varphi_F}$ is abused to mean a permutation on the set of integer $\{1, \ldots, \sigma\}$ such that, if $p, p' \in \mathcal{C}_m^{\varphi_F}$ and $p \neq p'$, then $p(\{1, \ldots, \sigma\}) \neq p'(\{1, \ldots, \sigma\})$ (and one also has $p(\{\sigma + 1, \ldots, \sigma\}) \neq p'(\{\sigma + 1, \ldots, \sigma\})$). Then, the set of subvarieties

$$D_p^{\varphi} := \{z_{p(1)} = z_{p(2)} = \cdots = z_{p(\sigma)} = 0\} \quad \text{for } p \in \mathcal{C}_m^{\varphi_F},$$

are precisely the set of all of the lc centres of codimension $\sigma$ (or $\sigma$-lc centres for short) of $(V, D \cap V)$, which is denoted by $\text{lc}_D^{\varphi}(D)$ (see [25, Def. 4.15] for the definition of lc centres of lc pairs; see also [5, §5.2] for $\sigma$-lc centres in a more general setting). Recall that $\mathfrak{s}_D$ is the canonical section of $D$ such that $\phi_D = \log|\mathfrak{s}_D|^2$ and that $\mathfrak{s}_D = z_1 \cdots z_{\varpi}$ on the admissible
open set \( V \). Let \((r_j, \theta_j)\) be the polar coordinate system of the \( z_j \)-plane for \( j = 1, \ldots, n \).

Define the ad hoc notations

\[
\mathcal{C}^\infty_{X*}(V) := \mathcal{C}^\infty_{X}(V) \left[ e^{\pm \sqrt{-1} \theta_1}, \ldots, e^{\pm \sqrt{-1} \theta_n} \right],
\]

which is given as a \( \mathcal{C}^\infty_{X*}(V) \)-algebra, and

\[
\mathcal{A}^{p,q}_{X*}(V) := \{(p, q)\text{-forms on } V \text{ with coefficients in } \mathcal{C}^\infty_{X*}(V)\}
\]

(these notations are considered only on open sets in some given coordinate chart, like admissible open sets, in this paper). Note that \( e^{m \sqrt{-1} \theta_j} \) for any \( m \in \mathbb{Z} \) and \( j = 1, \ldots, n \) can be viewed as a bounded function on \( V \) (smooth in the variable \( \theta_j \)) and thus integrable on \( V \). Apparently, one has \( \mathcal{A}^{p,q}_{X*}(V) \subset \mathcal{A}^{p,q}_{X}(V) \) for any integers \( p, q \geq 0 \). For any smooth \((n, q)\)-form \( f \in \mathcal{A}^{n,q}_{X*}(V; D \otimes F) \) and for any point \( x \in X \), \( f \) can be written locally on some admissible open set \( V \) centred at \( x \) as a finite sum

\[
(f|_V)_{(eq2.6.1)} = \sum_{p \in \mathcal{C}^\sigma_{X*}} \frac{dz_{p(1)} \wedge \cdots \wedge dz_{p(\sigma)} \wedge g_p \; z_{p(\sigma+1)} \cdots z_{p(\sigma_V)}}{z_{p(1)} \wedge \cdots \wedge z_{p(\sigma)} \wedge g_p \; s_D}
\]

for some integer \( \sigma \in [0, \sigma_V] \) and \( g_p \in \mathcal{C}^{n-\sigma,q}_{X*}(V) \) for each \( p \in \mathcal{C}^{\sigma}_{\sigma} \) (note that the set \( \mathcal{C}^{\sigma}_{\sigma} \), and therefore \( p \), depends on \( \sigma \)). Let \( \sigma_f := \sigma_V \) be the minimal \( \sigma \in [0, \sigma_V] \) such that \( g_p|_{\sigma_f} \neq 0 \) for some \( p \in \mathcal{C}^{\sigma_{\sigma_f}} \), i.e. \( \sigma_f \) is the codimension of the mlc of \( (V, D \cap V) \) with respect to \( f \) (see [6, Def. 2.2.5]; see also [5, Thm. 4.1.2 and Remark 4.1.3]). Let \( V = U \times W \) be the decomposition into a product of polydiscs such that \((z_1, \ldots, z_{\sigma_V}) \) and \((z_{\sigma_V+1}, \ldots, z_n)\) are coordinate systems on \( U = U^{\sigma_V} \) and \( W = W^{n-\sigma_V} \) respectively. To allow more general \( f \) for the applications in latter sections, the regularity on each \( g_p \) (\( p \in \mathcal{C}^{\sigma_{\sigma_f}} \)) is relaxed such that

\[
(eq2.6.2a) \quad \left( \text{coef. of } \langle g_p, g_p' \rangle_{\nabla^F} \right)_{U \times \{w\}} \in \mathcal{C}^\infty_{U*}(U) \quad \text{for a.e. } w \in W
\]

(thus all high order partial derivatives with respect to the radial coordinates \( r_1, \ldots, r_{\sigma_V} \) are well-defined for a.e. fixed point in \( W \)) and

\[
(eq2.6.2b) \quad \frac{\partial^\sigma}{\partial r_{\sigma_1} \cdots \partial r_{\sigma_1}} \left( \text{coef. of } \langle g_p, g_p' \rangle_{\nabla^F} \right) \in L^1_{\text{loc}}(V) \quad \text{for all } \sigma = 0, 1, \ldots, \sigma_V
\]

for all \( p, p' \in \mathcal{C}^{\sigma}_{\sigma_f} \).

With a suitable choice of orientation on the normal bundle of \( D^*_p \) in \( X \) which determines the sign on the Poincaré residue map \( \mathcal{R}_\mathcal{D}_p \) (see [25, para. 4.18]) and with a suitable extension of coefficients of the map \( \mathcal{R}_\mathcal{D}_p \), it follows that

\[
(eq2.6.3) \quad g_p|_{D^*_p} = \mathcal{R}_\mathcal{D}_p \left( \frac{f}{s_D} \right)
\]

\[\text{When } n = 1 \text{ (and } V \cap D = \{0\} \text{) and } f \text{ is smooth on } V \text{ with } f(0) = 0, \text{ one has } f = a z + b \bar{z} = \left(a + b e^{2 \sqrt{-1} \theta} \right) z, \text{ where } a \text{ and } b \text{ are smooth (1,1)-forms on } V, \text{ and the corresponding function } g_p = g \text{ is given by } \frac{dz}{z} \wedge g = a + b e^{2 \sqrt{-1} \theta}. \text{ Also notice that (for general } n \text{) if, for instance, } f = \rho h \text{ for some smooth function } \rho \text{ and holomorphic } (n-\sigma, q)\text{-form } h \text{ on } V, \text{ then the } (n-\sigma, q)\text{-forms } g_p \text{ can be chosen to be in (the ordinary) } \mathcal{A}^{n-\sigma,q}_{X*}(V).\]
for all \( p \in \mathcal{C}_\sigma^V \). Observe that each \( g_p |_{\mathcal{D}_p^\sigma} \), as a section of \( K_{\mathcal{D}_p^\sigma} \otimes (\wedge^q T_X \otimes F) \big|_{\mathcal{D}_p^\sigma} \) on \( \mathcal{D}_p^\sigma \) (cf. the discussion in [5, §4.2]), is uniquely determined by \( f \) up to the choice of canonical section \( \mathcal{A}_D \) of \( D \) and can be extended to the \( \sigma \)-lc centre \( \tilde{\mathcal{D}}^\sigma \) of \( (X, D) \) which contains \( \mathcal{D}_p^\sigma \).

Note also that \( g_p |_{\mathcal{D}_p^\sigma} = 0 \) for all \( p \in \mathcal{C}_\sigma^V \) when \( \sigma > \sigma_f \), as can be seen from (eq 2.6.1).

In [6, Prop. 2.2.1], the following theorem is proved for \((n,0)\)-forms via successive use of integration by parts. The proof for \((n,q)\)-forms is essentially the same, which is given below.

**Theorem 2.6.1** (cf. [6, Prop. 2.2.1]). Recall that \( \varphi = \varphi_F + \phi_D \). Let \( \rho \) be a compactly supported smooth function on an admissible open set \( V \subset X \) and let \( f \in L_{(n)}^1(D \otimes F)_{\varphi, \omega} \) which admits the decomposition (eq 2.6.1) with the coefficients satisfying (eq 2.6.2) on some neighbourhood of \( \mathcal{V} \). Then, for any \( \varepsilon > 0 \),

\[
\varepsilon \int_V \frac{\rho |f|^2_{\varphi, \omega}}{|\psi_D|^{\sigma + \varepsilon}} = \begin{cases} 
(1)^{\sigma_f \varepsilon} & \text{when } \sigma \geq \sigma_f , \\
\prod_{k=1}^{\sigma_f} \left( \varepsilon - j \right) & \text{when } 0 \leq \sigma < \sigma_f \text{ and } \varepsilon < \sigma_f - \sigma , \\
\infty & \text{when } \sigma_f = 0 , \\
0 & \text{when } \sigma > \sigma_f \text{ or } \sigma = \sigma_f = 0 ,
\end{cases}
\]

where \( G_{\sigma_f} \) is an \((n,n)\)-form on \( V \) independent of \( \varepsilon \) with \( L^1 \) coefficients, which contain derivatives of \( \rho |f|^2_{\varphi, \omega} \) of order at most \( \sigma_f \) in the normal directions of the lc centres \( \mathcal{D}_p^\sigma \). Moreover, when \( \sigma \geq \sigma_f \), one obtains the residue norm of \( f \) on \( \text{lc}_\sigma^V(D) \) with respect to the \( \sigma \)-lc-measure, given by

\[
\lim_{\varepsilon \to 0^+} \varepsilon \int_V \frac{\rho |f|^2_{\varphi, \omega}}{|\psi_D|^{\sigma + \varepsilon}} = \begin{cases} 
\sum_{p \in \mathcal{C}_\sigma^V} \frac{\pi^\sigma}{(\sigma - 1)^n} \int_{\mathcal{D}_p^\sigma} \rho |g_p|^2_{\varphi, \omega} & \text{when } \sigma > \sigma_f \text{ or } \sigma = \sigma_f = 0 , \\
0 & \text{when } \sigma \in [\sigma_f, \sigma_V] \cap \mathbb{N} .
\end{cases}
\]

(In particular, \( (\rho g_p)|_{\mathcal{D}_p^\sigma} \equiv 0 \) for \( \sigma > \sigma_f \), with the knowledge that the coefficients \( g_p \) in (eq 2.6.1) depend on \( \sigma \).)

**Remark 2.6.2.** As can be observed from the proof, Theorem 2.6.1 still holds true when \( \omega \) in the claim and also in the assumptions in (eq 2.6.2) is replaced by \( \tilde{\omega} \).

**Remark 2.6.3.** The results in [4, Prop. 2.2.1 and Cor. 2.3.3], which make similar claims as in Theorem 2.6.1 but for \((n,0)\)-forms \( f \) and for the residue function

\[
\mathcal{F}(\varepsilon)_\sigma := \varepsilon \int_V \frac{\rho |f|^2_{\varphi, \omega}}{|\psi_D|^{\sigma + \varepsilon}} (\log |\psi_D|)^{1 + \varepsilon}
\]

instead of \( \varepsilon \int_V \frac{\rho |f|^2_{\varphi, \omega}}{|\psi_D|^{\sigma + \varepsilon}} \), can also be extended similarly to the current setting (where \( f \) is an \((n,q)\)-form with more relaxed regularity assumptions). Following the proof of [4, Thm. 2.3.1], one can also show that \( \varepsilon \mapsto \int_V \frac{\rho |f|^2_{\varphi, \omega}}{|\psi_D|^{\sigma + \varepsilon}} \) can be continued analytically across the origin \( \varepsilon = 0 \), but it is not a priori an entire function as there are apparent poles at \( \varepsilon = - (\sigma - j) \) for \( j = 1, \ldots, \sigma_f - 1 \) and also at \( \varepsilon = - (\sigma - \sigma_f) \) when \( \sigma > \sigma_f \), as can be seen from the identity in Theorem 2.6.1. For the purpose of this paper, it suffices to consider \( \varepsilon \int_V \frac{\rho |f|^2_{\varphi, \omega}}{|\psi_D|^{\sigma + \varepsilon}} \) instead of \( \mathcal{F}(\varepsilon)_\sigma \).

**Proof.** Write

\[
g_p = \text{sgn}(p) \ dz_{p(\sigma + 1)} \wedge \cdots \wedge dz_{p(\sigma_V)} \wedge dz_{\sigma_V + 1} \wedge \cdots \wedge dz_n \wedge \tilde{g}_p = 0,
\]
where $\tilde{g}_p$ is a $(0, q)$-form and $\text{sgn}(p)$ is the sign of the permutation representing the choice $p$. It follows from the decomposition (eq 2.6.1) (with $\sigma = \sigma_f$) of $f$ that (note also Notation 2.1.4)

$$
\rho|f^2|_{\phi, \omega} = \sum_{p, p' \in \mathcal{C}_f} \left( F_{p, p'} := \sum_{p, p' \in \mathcal{C}_f} (g_p, g_{p'})_{\phi, \omega} \right) \left( \frac{\pi i}{1}, \frac{d \sigma}{d \tau} \right) \left( \prod_{j=1}^{n} \frac{\pi i}{z_j} \right) \prod_{k=\sigma_{V}+1}^{n} \frac{\pi i}{z_k} \right)
$$

and, following the notation in the proof of [4, Prop. 2.2.1], one can write accordingly

$$
\varepsilon \int_{V} \rho|f^2|_{\phi, \omega} = \sum_{p, p' \in \mathcal{C}_f} \mathcal{J}_{\phi, \omega}^{F_{p, p'}}(\varepsilon) = \sum_{p, p' \in \mathcal{C}_f} \mathcal{J}_{\phi, \omega}^{F_{p, p'}}(\varepsilon),
$$

where $\mathcal{J}_{\phi, \omega}^{F_{p, p'}}$ is the summand containing the term $F_{p, p'}$. Let $(r_j, \theta_j)$ be the polar coordinate system in the $z_j$-plane. That $(g_p, g_{p'})_{\phi, \omega}$ (more precisely, $(\tilde{g}_p, \tilde{g}_{p'})_{\phi, \omega}$) satisfying (eq 2.6.2) implies that the conditions in (eq 2.6.2) hold true with $F_{p, p'}$ in place of $(g_p, g_{p'})_{\phi, \omega}$ for any $p, p' \in \mathcal{C}_f$. Integration by parts with respect to the variables $r_1, \ldots, r_{\sigma_{V}}$ can therefore be applied to the integral $\mathcal{J}_{\phi, \omega}^{F_{p, p'}}$ without questions. The rest of the proof is proceeded as in [6, Prop. 2.2.1] or [4, Prop. 2.2.1].

First consider the case when $\sigma \geq \sigma_f$. To compute the summands in (*) with $p = p'$, and it suffices to consider only the case where $p = p' = \text{id}$, the identity permutation. Set

$$
F_0 := F_{\text{id}, \text{id}; 0} := F_{\text{id}, \text{id}}
$$

and

$$
F_j := F_{\text{id}, \text{id}; j} := \frac{\partial}{\partial r_j} \left( \frac{F_{j-1}}{r_j^2} \frac{\partial}{\partial r_j} \psi_D \right) = \frac{\partial}{\partial r_j} \left( \frac{F_{j-1}}{1 - \frac{r_j^2}{2} \frac{\partial}{\partial r_j} \psi_D} \right) \quad \text{for} \quad j = 1, \ldots, \sigma_{V}
$$

(with $F_{p, p'; 0} := F_{p, p'}$ is considered for general $p, p' \in \mathcal{C}_f$, all $r_j$ in the formula of $F_j := \left( F_{p, p'; j} \right)$ should be replaced by $r_{p(j)}$). Note that $\frac{1}{1 - \frac{r_j^2}{2} \frac{\partial}{\partial r_j} \psi_D} > 0$ and is smooth on $V$ by the choice of the admissible set $V$ (see Section 2.2). In view of Fubini’s theorem, it follows that

$$
\mathcal{J}_{\phi, \omega}^{F_0}(\varepsilon) = \varepsilon \int_{V} \frac{F_0}{\psi_D^{\sigma+\varepsilon}} \prod_{j=1}^{\sigma_f} \frac{\pi i}{z_j} \prod_{j=\sigma_f+1}^{\sigma_{V}} \frac{\pi i}{z_k} \prod_{k=\sigma_{V}+1}^{n} \frac{\pi i}{z_k} \left( \text{made implicit} \right)
$$

$$
= \varepsilon \int_{V} \frac{F_0}{\psi_D^{\sigma+\varepsilon}} \prod_{j=1}^{\sigma_f} \frac{\pi i}{z_j} \prod_{j=1}^{\sigma_{V}} \frac{d \theta_j}{2} \int_{V} \frac{1}{\psi_D^{\sigma-1+\varepsilon}} \prod_{j=2}^{\sigma_f} \frac{\pi i}{z_j} \prod_{j=2}^{\sigma_{V}+1} \frac{\pi i}{z_k} \prod_{k=\sigma_{V}+1}^{n} \frac{\pi i}{z_k} \left( \text{made implicit} \right)
$$

$$
= \frac{\varepsilon}{\sigma - 1 + \varepsilon} \int_{V} \frac{F_1}{\psi_D^{\sigma-1+\varepsilon}} \prod_{j=2}^{\sigma_f} \frac{\pi i}{z_j} \prod_{j=2}^{\sigma_{V}+1} \frac{\pi i}{z_k} \prod_{k=\sigma_{V}+1}^{n} \frac{\pi i}{z_k} \left( \text{made implicit} \right) d \theta
$$

where $\tilde{h}_p$ is a $(0, q)$-form and $\text{sgn}(p)$ is the sign of the permutation representing the choice $p$. It follows from the decomposition (eq 2.6.1) (with $\sigma = \sigma_f$) of $f$ that (note also Notation 2.1.4)
\[(**)
\begin{align*}
\int_{\mathbb{R}^n} & = \prod_{j=1}^d \frac{(-1)^{\sigma_j} \varepsilon}{|\psi_D|} \int_V \frac{F_{\sigma_j}}{|\psi_D|^{\sigma - \sigma_j + \varepsilon}} \prod_{j=1}^d dr_j \cdot d\theta.
\end{align*}
\]

The boundary terms arising from integration by parts all vanish as one has \(|F_\sigma|_{\{r_j=0\}} = 0\) and \(F_\sigma|_{\{r_j=1\}} = 0\) for all \(j = 1, \ldots, \sigma_V\) and \(\sigma = 0, \ldots, \sigma_V\). The assumption (eq 2.6.2b) guarantees that \(F_{\sigma_j}\) is integrable and so is the integral in (**). This implies that all equalities above which invokes integration by parts are valid and the integral \(\mathcal{J}_\sigma(F_0)\) is convergent. Moreover, when \(\sigma > \sigma_f\), the integral in (***) remains convergent when \(\varepsilon = 0\) and the coefficient of the integral is a multiple of \(\varepsilon\), it follows that
\[
\lim_{\varepsilon \to 0^+} \mathcal{J}_\sigma(F_0) = 0 \quad \text{when} \quad \sigma > \sigma_f.
\]

When \(\sigma = \sigma_f\), the fundamental theorem of calculus (which makes use of the assumption (eq 2.6.2a)) yields
\[
\lim_{\varepsilon \to 0^+} \mathcal{J}_\sigma(F_0) = \frac{(-1)^{\sigma}}{(\sigma - 1)!} \int_V F_\sigma \prod_{j=1}^d dr_j \cdot d\theta = \frac{(-1)^{\sigma}}{(\sigma - 1)!} \int_V \frac{\partial}{\partial r_\sigma} \frac{F_{\sigma - 1}}{|1 - \frac{r_\sigma}{2} \omega_{\psi D}|} \prod_{j=1}^d dr_j \cdot d\theta
\]
\[
= \frac{(-1)^{\sigma - 1}}{(\sigma - 1)!} \int_{\{r_\sigma=0\}} F_{\sigma - 1} \prod_{j=1}^d dr_j \cdot d\theta
\]
\[
= \cdots = \frac{1}{(\sigma - 1)!} \int_{\{r_1=\ldots=\sigma_V=0\}} F_0 \prod_{j=1}^d \rho |g_\psi|^2 d\theta.
\]

More generally, one has
\[
\lim_{\varepsilon \to 0^+} \mathcal{J}_{F_\sigma}(F_0) = \frac{\pi^\sigma}{(\sigma - 1)!} \int_{\{r_\sigma=0\}} F_\sigma \prod_{j=1}^d \rho |g_\psi|^2 d\theta.
\]

Note that the above computation is still valid if the expansion of \(\rho |f|^2 \varphi_{\psi \omega}\) in (eq 2.6.4) is made into a sum of \(p, p' \in C_{\sigma_V}^\sigma\) (instead of \(C_{\sigma_f}^\sigma\)), thus coming with a different set of \(\{g_p\}_p\) to start with. The above equation thus holds true for any \(p \in C_{\sigma_f}^\sigma\) and all integers \(\sigma\) such that \(\sigma_V \geq \sigma > \sigma_f\).

To compute the “cross terms” in (**), i.e. summands \(\mathcal{J}_{F_\sigma}(F_0)\) with \(p \neq p'\), consider the special case where \(p(j) = p'(j)\) for \(j = 1, \ldots, \sigma_f - 1\) but \(p(\sigma_f) \neq p'(\sigma_f)\). In this case, one has
\[
\mathcal{J}_{F_\sigma}(F_0) = \varepsilon \int_V \frac{F_{p,p'}}{|\psi_D|^{\sigma - \sigma_f}} \prod_{j=1}^{\sigma_f-1} \frac{\pi i d\sigma_j(z_{p,j}) \wedge d\sigma_j(z_{p',j})}{2(z_{p,j})} \wedge \prod_{j=\sigma_f+1}^n \frac{\pi i d\sigma_j(z_{p,j}) \wedge d\sigma_j(z_{p',j})}{2}.
\]

In view of Fubini’s theorem and the assumptions in (eq 2.6.2), the computation for the case where \(p = p' \in C_{\sigma_f}^\sigma\) still applies and thus (with the same notation as before and making the irrelevant variables implicit)
\[
\int_{\mathbb{R}^n} \frac{(-1)^{\sigma_f-1} \varepsilon}{|\psi_D|^{\sigma - \sigma_f + \varepsilon}} \int_{\mathbb{R}^n} \frac{F_{p,p';\sigma_f-1} \prod_{j=1}^{\sigma_f-1} dr_{p,j} \prod_{j=1}^d dr_{p',j}}{z_{p,(\sigma_f)} z_{p',(\sigma_f)}} d\theta.
\]
(one can further lower the exponent on $|\psi_D|$ by writing $\frac{F_{p,p';\sigma_f^{-1}}}{r_{p,\sigma_f}}$ as $\frac{F_{p,p';\sigma_f^{-1}}}{r_{\sigma_f}}$ and applying integration by parts with respect to the variable $r_{p,\sigma_f}$ as before if one wants to fit this in the identity in the claim). As the integral is convergent even when $\varepsilon = 0$ and the coefficient of the integral is a multiple of $\varepsilon$, one obtains

$$\lim_{\varepsilon \to 0^+} J^{F_{p,p'}}(\varepsilon)_\sigma = 0.$$ 

Note that this holds true even for $\sigma > \sigma_f - 1$. The computation for other summands $J^{F_{p,p'}}(\varepsilon)_\sigma$ with $p \neq p'$ is similar.

Combining the above results, the claims for the case $\sigma \geq \sigma_f$ are proved.

For the case $0 \leq \sigma < \sigma_f$ (which is to show divergence of the integral in question), as $\frac{1}{|\psi_D|^\sigma} \leq \frac{1}{|\psi_D|^\sigma}$ for any $\sigma' \leq \sigma$, it suffices to consider $\sigma$ such that $\sigma_f - 1 < \sigma < \sigma_f$.

The above computation shows that the “cross terms” $J^{F_{p,p'}}(\varepsilon)_\sigma$ in (eq\:2.6.4) are convergent, it remains to show that $J^{F_{p,p'}}(\varepsilon)_\sigma = +\infty$ for some $p \in C_{\sigma_f}^\infty$. Without loss of generality, assume that $F_{id,\sigma_f}|_{\sigma_f} \neq 0$ (such assumption is valid by the definition of $\sigma_f$). Consider a change of coordinates on the admissible open set $V$ which changes the radial coordinates $(r_1, \ldots, r_{\sigma_f})$ to $(|\psi_D|, q_2, \ldots, q_{\sigma_f})$, where

$$q_j := \frac{\log r_j^2}{\psi_D} = \frac{\log r_j^2}{|\psi_D|} \quad \text{for } j = 2, \ldots, \sigma_f,$$

in which each $q_j$ varies within $[0, 1]$ on $V$. Then, with the same notation as before and making the irrelevant variables implicit, one has, for $\varepsilon < \sigma_f - \sigma$,

$$J^{F_{id,\sigma_f}}(\varepsilon)_\sigma = \varepsilon \int_V \frac{F_{id,\sigma_f}}{r_1^2 \frac{\partial}{\partial r_1} \psi_D} |\psi_D|^{\sigma_f^{-1}} d|\psi_D|^\sigma \prod_{j=2}^{\sigma_f} dq_j \cdot d\theta$$

$$= \frac{\varepsilon}{\sigma_f - \sigma - \varepsilon} \int_V \frac{F_{id,\sigma_f}}{r_1^2 \frac{\partial}{\partial r_1} \psi_D} d(|\psi_D|^{\sigma_f^{-\varepsilon}}) \prod_{j=2}^{\sigma_f} dq_j \cdot d\theta.$$

Since $\frac{F_{id,\sigma_f}}{r_1^2 \frac{\partial}{\partial r_1} \psi_D} > 0$ on some open subset $V' \subset V$ such that $V' \cap D_{p,p'}^\sigma \neq \emptyset$ and $d(|\psi_D|^{\sigma_f^{-\varepsilon}})$ is not integrable on $V'$, the above integral diverges. This completes the proof for the case $0 \leq \sigma < \sigma_f$.

**Remark 2.6.4.** When $\psi_D$ is replaced by a global function $\psi \leq -1$ of the form

$$\psi|_V = \sum_{j=1}^{\sigma_V} \nu_j \log |z_j|^2 + \sum_{k=\sigma_V+1}^{n} c_k \log |z_k|^2 + \alpha$$

where $\nu_j > 0$ for $j = 1, \ldots, \sigma_V$ and $c_k \geq 0$ for $k = \sigma_V, \ldots, n$ are constants and $\alpha \in C_0^\infty(V)$ (as in [6, Prop. 2.2.1]), the claims in Theorem 2.6.1 still hold true when the residue norm is given by

$$\lim_{\varepsilon \to 0^+} \varepsilon \int_{V} \rho f^2_{\psi,\omega} = \begin{cases} 0 & \text{when } \sigma > \sigma_f \text{ or } \sigma = \sigma_f = 0, \\ \sum_{p \in C_{\sigma_f}^\infty} \frac{\pi^{\sigma}}{(\sigma - 1)! d_p^\sigma} \int_{D_p} \rho g_p^2 & \text{when } \sigma \in [\sigma_f, \sigma_V] \cap \mathbb{N}, \end{cases}$$
where $\nu_p := \prod_{j=1}^p \nu_{p(j)}$, which can be checked easily by following the proof of Theorem 2.6.1.

3. Proof of Theorem 1.2.1

3.1. Outline of the proof. An outline of the proof of Theorem 1.2.1 is given in this section, with some detailed verifications for the smooth case (i.e. $\varphi_F$ and $\varphi_M$ being smooth and thus $X = X^0$ and $\omega = \bar{\omega}$) given in Section 3.2. Extra treatments to the singular case (i.e. $\varphi_F$ and $\varphi_M$ having neat analytic singularities as described in Section 2.2) are given in Section 3.3.

Recall that $\varphi := \varphi_F + \phi_D$. Let $s_D$ be a canonical section of $D$ such that $\phi_D = \log|s_D|^2$. Let $\theta: [0, \infty) \to [0, 1]$ be a smooth non-decreasing cut-off function such that $\theta|_{[0, \frac{1}{2}]} \equiv 0$ and $\theta|_{[1, \infty)} \equiv 1$. For $\varepsilon > 0$, set $\theta_\varepsilon := \theta \circ \frac{1}{1 + |D|}$ and $\theta'_\varepsilon := \theta' \circ \frac{1}{1 + |D|}$ for convenience (where $\theta'$ is the derivative of $\theta$). Note that both $\theta_\varepsilon$ and $\theta'_\varepsilon$ have compact supports inside $X \setminus D$ for $\varepsilon > 0$. One also has $\theta_\varepsilon \nearrow 1$ pointwisely on $X \setminus D$ as $\varepsilon \searrow 0$.

**Step I:** Note that the space of harmonic forms $\mathcal{H}_{\varphi, \bar{\omega}}^{n,q}$ is isomorphic to $H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi))$ (see Section 2.3). The homomorphism

$$\iota_0: H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi_F + \phi_D)) \to H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi_F))$$

can be viewed as the homomorphism

$$\iota_0: \mathcal{H}_{\varphi, \bar{\omega}}^{n,q} \to H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(\varphi_F)).$$

Choose any $\tilde{u} \in (\ker \iota_0)^\perp \subset \ker \iota_0 \oplus (\ker \iota_0)^\perp = \mathcal{H}_{\varphi, \bar{\omega}}^{n,q}$ such that $\tilde{u} \in \ker \mu_0$, i.e. $\tilde{s}_D$ represents the zero class in $H^q(X, K_X \otimes D \otimes F \otimes M \otimes \mathcal{I}(\varphi_F + \varphi_M))$. Then $s\tilde{u} = \bar{\omega}\tilde{v}$ for some $\tilde{v} \in L_{(2)}^{n,q-1}(D \otimes F \otimes M)_{\varphi_F + \varphi_M + \phi_D, \bar{\omega}}$ which can be decomposed into $\tilde{v} = \tilde{v}(2) + v(\infty)$ such that

$$\tilde{v}(2) \in L_{(2)}^{n,q-1}(X^0; D \otimes F \otimes M)_{\varphi_F + \varphi_M + \phi_D, \bar{\omega}},$$

$$v(\infty) \in \mathcal{A}_{X}^{n,q-1}(X; D \otimes F \otimes M \otimes \mathcal{I}(\varphi_F + \varphi_M))$$

(see Lemma 3.2.1 and Remark 3.2.2). The goal is to show that $\tilde{u} = 0$, which will then imply that $\ker \mu_0 = \ker \iota_0$.

**Step II:** Since $\varphi_F + \phi_D$ is psh and $\tilde{u}$ is harmonic in the corresponding $L^2$ space, it follows from the refinement of the hard Lefschetz theorem proved in [30, Thm. 3.3] (see also Theorem 2.5.1) that $s_D^{\tilde{u}} \tilde{u}$ is holomorphic on $X^0$ (which also implies that $\bar{\omega} s_D^{\tilde{u}} \tilde{u}$ is smooth on $X^0$). One can derive from this fact and Theorem 2.6.1 (or the computation corresponding to the 1-lc-measure in [6, Prop. 2.2]) that

$$\lim_\varepsilon \to 0^+ \int_{X^0} |\bar{\partial} \psi_D \otimes \tilde{u}|_{\varphi, \bar{\omega}}^2 < +\infty$$

and thus

(eq3.1.1) $$\lim_\varepsilon \to 0^+ \varepsilon^2 \int_{X^0} |\bar{\partial} \psi_D \otimes \tilde{u}|_{\varphi, \bar{\omega}}^2 = 0$$

(see Proposition 3.2.3 for the smooth case and Proposition 3.3.2 for the singular case). Via a careful use of the Bochner–Kodaira formula (BK) $\varphi, \bar{\omega}$ on $X^0$ in Lemma 2.4.2 (in which the fact that the Kähler metric $\bar{\omega}$ being smooth along the general points of $D$ while
the involving potential $\phi_D$ being singular along $D$ is taken care of using (eq 3.1.1) and the fact that $i\partial\bar{\partial}\varphi_F \geq 0$ on $X$, one obtains

$$(\text{eq 3.1.2}) \quad \nabla^{(0,1)} \tilde{u} = 0 \quad \text{and} \quad (i\partial\bar{\partial}\varphi_F)(\tilde{u}, \tilde{u}) = 0 \quad \text{on } X^\circ,$$

(see Proposition 3.2.5, supplemented by Lemma 3.3.3 for the singular case) which consequently lead to

$$(\text{eq 3.1.3}) \quad \vartheta_{\varphi_M}(s\tilde{u}) = 0 \quad \text{and} \quad s\tilde{u} \in \text{Dom } \partial'_{\varphi_M},$$

where $\partial'_{\varphi_M}$ is the Hilbert space adjoint of $\overline{\partial}$ on the $L^2$ space $L^2_{(2)}(D \otimes F \otimes M)_{\varphi + \varphi_M, \bar{\omega}}$ and $\vartheta_{\varphi_M}$ is the corresponding formal adjoint (see Corollary 3.2.6, supplemented by Lemma 3.3.3 for the singular case).

**Step III:** Now consider

$$\|s\tilde{u}\|^2_{\varphi + \varphi_M, \bar{\omega}} = \langle \langle s\tilde{u}, \partial'_{\varphi} \rangle \rangle_{\varphi + \varphi_M, \bar{\omega}} = \langle \langle s\tilde{u}, \partial'_{\varphi} \rangle \rangle_{\varphi + \varphi_M, \bar{\omega}} + \langle \langle s\tilde{u}, \partial'_{\varphi} \rangle \rangle_{\varphi + \varphi_M, \bar{\omega}}$$

and apply integration by parts. The form $\tilde{v}(2)$ being $L^2$ with respect to the potential $\varphi_D + \phi_D + \varphi_M$, together with $\vartheta_{\varphi_M}(s\tilde{u}) = 0$ given in (eq 3.1.3), yields

$$\langle \langle s\tilde{u}, \partial'_{\varphi_M} \rangle \rangle_{\varphi + \varphi_M, \bar{\omega}} = \langle \langle \partial_{\varphi_M}(s\tilde{u}), \tilde{v}(2) \rangle \rangle_{\varphi + \varphi_M, \bar{\omega}} = 0.$$

For the inner product involving $v(\infty)$, as $v(\infty)$ is $L^2$ with respect to the potential $\varphi_F + \varphi_M$ but not necessarily to $\varphi_F + \phi_D + \varphi_M$, the cut-off function $\theta_\varepsilon$ is introduced to facilitate the integration by parts. As $\varepsilon \to 0^+$, one has

$$\langle \langle s\tilde{u}, \partial'_{\varphi_M} \rangle \rangle_{\varphi + \varphi_M, \bar{\omega}} = \langle \langle s\tilde{u}, \partial_{\varphi_M} \rangle \rangle_{\varphi + \varphi_M, \bar{\omega}} - \varepsilon \langle \langle \partial_{\varphi_M}(s\tilde{u}), \partial'_{\varphi_M} \rangle \rangle_{\varphi + \varphi_M, \bar{\omega}} = 0.$$

It suffices to show that the inner product on the far right-hand-side converges to 0 as $\varepsilon \to 0^+$, which then implies that $\|s\tilde{u}\|^2_{\varphi + \varphi_M, \bar{\omega}} = 0$, hence $s\tilde{u} = 0$ and the desired $\tilde{u} = 0$ on $X^\circ$.

**Step IV:** Applying to $\tilde{u}$ the twisted Bochner–Kodaira formula (tBK)$_{\varepsilon, \varphi, \bar{\omega}}$ in Lemma 2.4.2 with the twisting function $\eta_\varepsilon := |\psi_D|^{1-\varepsilon}$ for $\varepsilon > 0$, together with the help from (eq 3.1.1) and (eq 3.1.2), yields

$$\lim_{\varepsilon \to 0^+} \varepsilon \int_{X^\circ} |\tilde{u}|^2_{\psi_D} \partial'_{\varphi, \bar{\omega}} = \int_{X^\circ} i\partial\bar{\partial}'_{\varphi_D, \bar{\omega}}(\tilde{u}, \tilde{u})_{\varphi, \bar{\omega}}$$

(see Proposition 3.2.8, supplemented by the treatment described in Section 3.3.3 for the singular case). Moreover, as $\tilde{u} \in (\ker \sigma_0)^\perp$, one can apply the argument of Takegoshi ([36, Prop. 3.8], see also [30, Prop. 3.13]) to claim that the above expression vanishes as follows. Note that

$$\partial'_{\varphi_F + \varphi_M} \tilde{u} = (\partial'_{\varphi_F + \varphi_M} + \partial_{\varphi_F + \varphi_M} \partial'_{\varphi_M}) \tilde{u} = -\nabla_{\varphi_F + \varphi_M} \cdot \nabla_{\varphi_F + \varphi_M} \tilde{u} + i\partial\bar{\partial}(\varphi_F + \varphi_M) \Lambda_{\varphi_M} \tilde{u}$$

(pointwisely on $X^\circ$),

$$\text{(eq 2.4.1)} \quad \partial'_{\varphi_F + \varphi_M} \tilde{u} = (\partial'_{\varphi_F + \varphi_M} + \partial_{\varphi_F + \varphi_M} \partial'_{\varphi_M}) \tilde{u},$$

$$\text{(eq 3.1.2)} \quad \partial'_{\varphi_F + \varphi_M} \Lambda_{\varphi_M} \tilde{u}$$
where $\partial_{\varphi_F + \varphi_D^m}$ (resp. $-\nabla_{\varphi_F + \varphi_D^m}^{(1,0)}$) is the formal adjoint of $\overline{\partial}$ (resp. $\nabla^{(0,1)}$) with respect to the inner product $\langle \cdot, \cdot \rangle_{\varphi_F + \varphi_D^m \overline{\varphi}}$. The form $i \partial \overline{\partial} \varphi_D^m \Lambda \bar{u}$ is in $L^{n,q}_{(2)}(D \otimes F)_{\varphi, \overline{\varphi}}$ since so is $\bar{u}$ and $i \partial \overline{\partial} \varphi_D^m$ is a smooth form on $X$. This, together with the equality above, therefore shows that

$$i \partial \overline{\partial} \varphi_D^m \Lambda \bar{u} \in (\ker \overline{\partial})_{\varphi, \overline{\varphi}} = \mathcal{H}^{n,q}_{\varphi, \overline{\varphi}} \oplus (\im \overline{\partial})_{\varphi, \overline{\varphi}} \subset L^{n,q}_{(2)}(D \otimes F)_{\varphi, \overline{\varphi}}.$$

Recall that, on any admissible open set $V \subset X$ such that $V \cap D \neq \emptyset$, one has $\phi_D|_V = \log |s_D|^2|_V = \sum_{j=1}^{\sigma_V} \log |z_j|^2$ and, therefore,

$$\langle \partial \psi_D \rangle|_V = \sum_{j=1}^{\sigma_V} \frac{dz_j}{z_j} - \langle \partial \varphi_D^m \rangle|_V.$$  

Since $\varphi_F + \varphi_D^m = \varphi_F + \varphi_D - \psi_D = \varphi - \psi_D$ and since $\frac{\bar{u}}{s_D}$ is smooth on $X^o$ (knowing its singularities along $X \setminus X^o$ given by Proposition 3.3.1), it follows that

$$\partial_{\varphi_F + \varphi_D^m} \bar{u} = \partial \bar{u} - \langle \partial \psi_D \rangle \bar{u} = -\langle \partial \psi_D \rangle \bar{u} \in L^{n,q-1}_{(2)}(D \otimes F)_{\varphi_F + \varphi_D^m \overline{\varphi}}$$

and thus $i \partial \overline{\partial} \varphi_D^m \Lambda \bar{u} \in (\im \overline{\partial})_{\varphi_F + \varphi_D^m \overline{\varphi}} \subset L^{n,q}_{(2)}(D \otimes F)_{\varphi_F + \varphi_D^m \overline{\varphi}}$ (i.e. the class of $i \partial \overline{\partial} \varphi_D^m \Lambda \bar{u}$ in $H^q(X, K_X \otimes D \otimes F \otimes \mathcal{J}(\varphi))$ is mapped to 0 via $\iota_0$), which implies that

$$i \partial \overline{\partial} \varphi_D^m \Lambda \bar{u} \in \ker \iota_0 \oplus (\im \overline{\partial})_{\varphi_F + \varphi_D^m \overline{\varphi}} \subset L^{n,q}_{(2)}(D \otimes F)_{\varphi_F + \varphi_D^m \overline{\varphi}}.$$

As a result, $\bar{u} \in (\ker \iota_0)_{\overline{\varphi}}$ implies that

$$\lim_{\varepsilon \to 0^+} \varepsilon \int_{X^o} \frac{|\langle \partial \psi_D \rangle \bar{u}|^2}{|\psi_D|^2} \frac{|\bar{u}|^2}{\varphi_D^m} = \int_{X^o} \frac{i \partial \overline{\partial} \varphi_D^m \Lambda \bar{u}}{\varphi_D^m} = \left\langle i \partial \overline{\partial} \varphi_D^m \Lambda \bar{u}, \bar{u} \right\rangle_{\varphi_F, \overline{\varphi}} = 0.$$

**Remark 3.1.1.** The need to invoke the above argument of Takegoshi, or, more precisely, to make the claim $(\partial \psi_D) \bar{u} \in L^{n,q-1}_{(2)}(D \otimes F)_{\varphi_F + \varphi_D^m \overline{\varphi}}$ legitimate, is the reason of the use of the metric $\bar{\omega}$ which is smooth on $D \cap X^o$ instead of a complete metric on $X^o \setminus D$. If $\bar{\omega}$ were singular along $D$, the form $\frac{\bar{u}}{s_D}$ would have poles along $D$ even though $\frac{\bar{u}}{s_D}$ is holomorphic on $X$ (cf. Proposition 3.3.1, which describes the possible singularities of $\frac{\bar{u}}{s_D}$ along $X \setminus X^o$). The analysis to claim $(\partial \psi_D) \bar{u}$ being $L^2$ would then be complicated, if not impossible.

**Step V:** Let $V \subset X$ be an admissible open subset of $X$ (not only of $X^o$) with respect to $(\varphi_F, \varphi_M, \psi_D)$ in the holomorphic coordinate system $(z_1, \ldots, z_n)$, and let $\rho$ be any smooth cut-off function on $X$ compactly supported in $V$. Suppose that $V \cap D \neq \emptyset$. Taking into account the fact that $\frac{\bar{u}}{s_D}$ is smooth on $X^o$, one can derive from Theorem 2.6.1 (or the computation corresponding to the 1-cmassure in [6, Prop. 2.2]) that

$$\lim_{\varepsilon \to 0^+} \varepsilon \int_V \rho \left( \frac{|\partial \psi_D|}{|\psi_D|^2} \right)^2 \frac{|\bar{u}|^2}{|\varphi_F + \phi_D \overline{\varphi}|^2} \frac{1}{|\psi_D|^{1+\varepsilon}} = \pi \sum_{j=1}^{\sigma_V} \int_{D^j} \rho \left( \frac{|\partial \psi_D|}{|\psi_D|^2} \right)^2 \frac{|\bar{u}|^2}{|\psi_D|^{1+\varepsilon}} \mathcal{D}_{\varphi_F, \overline{\varphi}}^j.$$  

where $D^j := \{ z_j = 0 \}$ for $j = 1, \ldots, \sigma_V$ are the irreducible components (1-c centres) of $V \cap D$. (This computation is explained in the justification for the finiteness of this integral in Step II.) This implies that (see Remark 3.2.4)

$$\left( dz_j \bar{u} \right|_{D^j} = 0$$  

and thus

$$\left( dz_j \bar{u} \right|_{D^j} \in \mathcal{A}^{n,q-1}_{X^o} (V \cap X^o)$$
for all $j = 1, \ldots, \sigma_V$. Note that $(\cdots)|_D$ here means only the restriction on the coefficients of the $(n, q - 1)$-form. Considering the expression of $\partial \psi_D$ on $V$ (eq 3.1.4) and noticing that the above holds true for any admissible open subset $V \subseteq X$, it follows that

$$\text{(eq3.1.5')} \quad \text{coef. of } (\partial \psi_D)_{\bar{\omega}} \frac{\bar{\psi}}{s_D} \text{ are locally in } \mathcal{C}^\infty_{X, s} \text{ on } X^\circ.$$  

Now the last inner product in Step III can be considered and its limit as $\varepsilon \to 0^+$ can be evaluated. The absolute value of the inner product satisfies

$$\left| \varepsilon \left\langle (\partial \psi_D)_{\bar{\omega}} \frac{\bar{\psi}}{s_D}, \frac{\theta'_{\varepsilon} v_{(\infty)}}{|\psi_D|^{1+\varepsilon}} \right\rangle_{\varphi_F + \varphi_D + \varphi_M, \bar{\omega}} \right| \leq \varepsilon \int_{X^\circ} \left| (\partial \psi_D)_{\bar{\omega}} \frac{\bar{\psi}}{s_D}, \frac{v_{(\infty)} e^{-\frac{1}{2} \varphi_D - \frac{1}{2} \varphi_M}}{\varphi_{F, \bar{\omega}}}, \left| s \right| \varphi_M, \left| \theta'_{\varepsilon} \right| \right|_{B_D} \frac{\varphi_D}{|\varphi_D|^{1+\varepsilon}} |\psi_D|^{1+\varepsilon}. $$  

Note that both $\left| s \right| \varphi_M$ and $|\theta'_{\varepsilon}|$ are bounded uniformly in $\varepsilon$ on $X$. The regularity of $\tilde{u}$ along $X \setminus X^\circ$ (see Proposition 3.3.1 for the singular case) together with (eq 3.1.5') shows that the pointwise inner product $\left\langle \cdots, \cdots \right\rangle_{\varphi_F, \bar{\omega}}$ is integrable on $X$. Moreover, since $\left\langle \cdots, \cdots \right\rangle_{\varphi_F, \bar{\omega}}$ is locally in $\mathcal{C}^\infty_{X, s}$ on $X^\circ$ (with poles along $X \setminus X^\circ$ “independent” of those of $\psi_D$ along $D$) while $\frac{1}{|\varphi_D|^{1+\varepsilon}} |\psi_D|^{1+\varepsilon}$ is integrable on $X$ for all $\varepsilon \geq 0$, in view of Fubini’s theorem, $\frac{|\langle \cdots, \cdots \rangle_{\varphi_F, \bar{\omega}}|}{|\varphi_D|^{1+\varepsilon}}$ is also integrable on $X$ (see Proposition 3.3.4 for the singular case) and, therefore, the term on the right-hand-side above is in the order of $O(\varepsilon)$ as $\varepsilon \to 0^+$, that is,

$$\lim_{\varepsilon \to 0^+} \varepsilon \left\langle (\partial \psi_D)_{\bar{\omega}} \frac{\bar{\psi}}{s_D}, \frac{\theta'_{\varepsilon} v_{(\infty)}}{|\psi_D|^{1+\varepsilon}} \right\rangle_{\varphi_F + \varphi_M, \bar{\omega}} = 0. $$  

Therefore, as seen from Step III, this implies that $s \tilde{u} = 0$ and thus $\tilde{u} = 0$. This completes the proof.

3.2. **Proof for the case with smooth $\varphi_F$ and $\varphi_M$.** In this section, $\varphi_F$ and $\varphi_M$ are assumed to be smooth. Therefore, one has $X^\circ = X$ and $\tilde{\omega} = \omega$. For consistency, $\tilde{u}$, $\tilde{v}$ and $\bar{v}_{(2)}$ in Section 3.1 are written as $u$, $v$ and $v_{(2)}$ respectively.

The claims made in each step in the outline of the proof in Section 3.1 are justified in this section.

3.2.1. **Justification for Step I.** It suffices to provide a proof of the following lemma.

**Lemma 3.2.1.** Suppose $u \in H^0_{\varphi, \omega}$ and $s \in H^0(X, M)$ such that $su$ represents the zero class in $H^q(X, K_X \otimes D \otimes F \otimes M \otimes \mathcal{I}(\varphi_F + \varphi_M))$ under the $L^2$ Dolbeault isomorphism. Then, there exists $v \in L^{n,q-1}_{(2)}(D \otimes F \otimes M)_{\varphi_F + \varphi_M + \varphi_D, \omega}$ such that $su = \overline{\partial}^* v$ and $v = v_{(2)} + v_{(\infty)}$, where

$$v_{(2)} \in L^{n,q-1}_{(2)}(X; D \otimes F \otimes M)_{\varphi_F + \varphi_M + \varphi_D, \omega},$$

$$v_{(\infty)} \in \mathcal{A}^{n,q-1}(X; D \otimes F \otimes M \otimes \mathcal{I}(\varphi_F + \varphi_M)).$$

**Proof.** The lemma is proved via examining the $L^2$ Dolbeault isomorphism, which is explicitly described in [27]. Set $L := D \otimes F \otimes M$ for convenience.
Let $\mathcal{U} := \{U_i\}_{i \in I}$ be a finite open Stein cover of $X$ with a smooth partition of unity $\{\rho^i\}_{i \in I}$ subordinate to it. Write $U_{i_0 \cdots i_\nu} := U_{i_0} \cap U_{i_1} \cap \cdots \cap U_{i_\nu}$ for any indices $i_0, i_1, \ldots, i_\nu \in I$ as usual. Also let $\delta$ be the coboundary operator on the Čech cochains. Following the construction of the $L^2$ Dolbeault isomorphism (see [27, Prop. 5.5] and [30, Prop. 2.8]), with the given $(n, q)$-form $su$, solve the equations

$$
\bar{\partial}\{\beta_{i_0}\} = \left\{su|_{U_{i_0}}\right\} \quad \text{for} \quad \beta_{i_0} \in L^{n,q-1}_{(2)}(U_{i_0}; L)_{\varphi_F + \varphi_M + \phi_D, \omega} \quad \text{and} \\
\bar{\partial}\{\beta_{i_0 \cdots i_\nu}\} = \delta\{\beta_{i_0 \cdots i_{\nu-1}}\} \quad \text{for} \quad \beta_{i_0 \cdots i_\nu} \in L^{n,q-\nu-1}_{(2)}(U_{i_0 \cdots i_\nu}; L)_{\varphi_F + \varphi_M + \phi_D, \omega}
$$

for $\nu = 0, \ldots, q - 1$ via $L^2$ method on relatively compact Stein subsets (see, for example, [9, §4] for the procedures for smoothing out the metric before solving for the $\bar{\partial}$-equation on a relatively compact Stein subset). Note that $\delta\{\beta_{i_0 \cdots i_{\nu-1}}\} = \{\alpha_{i_0 \cdots i_{\nu-1}}\}$ has holomorphic components and is representing the same cohomology class as $su$ under the $L^2$ Dolbeault isomorphism. Using the fact that $\sum_{i \in I} \rho^i \equiv 0$ on $X$, the $(n, q)$-form $su$ can then be expressed as (under the Einstein summation convention)

$$su = \rho^{i_0} \bar{\partial}\beta_{i_0} = \bar{\partial}(\rho^{i_0} \beta_{i_0}) - \bar{\partial}\rho^{i_0} \wedge \beta_{i_0} = \bar{\partial}(\rho^{i_0} \beta_{i_0}) + \bar{\partial}\rho^{i_0} \wedge \rho^{i_1} \bar{\partial}\beta_{i_0 i_1}$$

$$= \bar{\partial}(\rho^{i_0} \beta_{i_0}) + \bar{\partial}\rho^{i_0} \wedge \bar{\partial}(\rho^{i_1} \beta_{i_0 i_1}) - \bar{\partial}\rho^{i_0} \wedge \bar{\partial}\rho^{i_1} \wedge \beta_{i_0 i_1}$$

$$= \bar{\partial}(\rho^{i_0} \beta_{i_0}) + \bar{\partial}\rho^{i_0} \wedge \bar{\partial}(\rho^{i_1} \beta_{i_0 i_1}) - \bar{\partial}\rho^{i_0} \wedge \bar{\partial}\rho^{i_1} \wedge \beta_{i_0 i_1}$$

$$= \bar{\partial}(\rho^{i_0} \beta_{i_0} - \rho^{i_0} \rho^{i_1} \beta_{i_0 i_1} + \rho^{i_1} \beta_{i_0 i_1} - \rho^{i_2} \beta_{i_0 i_1 i_2})$$

$$= \bar{\partial}(\rho^{i_0} \beta_{i_0} - \rho^{i_0} \rho^{i_1} \beta_{i_0 i_1} + \rho^{i_1} \beta_{i_0 i_1} - \rho^{i_2} \beta_{i_0 i_1 i_2})$$

$$= \bar{\partial}(\rho^{i_0} \beta_{i_0} - \rho^{i_0} \rho^{i_1} \beta_{i_0 i_1} + \rho^{i_1} \beta_{i_0 i_1} - \rho^{i_2} \beta_{i_0 i_1 i_2})$$

$$= \bar{\partial}(\rho^{i_0} \beta_{i_0} - \rho^{i_0} \rho^{i_1} \beta_{i_0 i_1} + \rho^{i_1} \beta_{i_0 i_1} - \rho^{i_2} \beta_{i_0 i_1 i_2})$$

$$= \bar{\partial}(\rho^{i_0} \beta_{i_0} - \rho^{i_0} \rho^{i_1} \beta_{i_0 i_1} + \rho^{i_1} \beta_{i_0 i_1} - \rho^{i_2} \beta_{i_0 i_1 i_2})$$

$$= \bar{\partial}(\rho^{i_0} \beta_{i_0} - \rho^{i_0} \rho^{i_1} \beta_{i_0 i_1} + \rho^{i_1} \beta_{i_0 i_1} - \rho^{i_2} \beta_{i_0 i_1 i_2})$$

Notice that, since all $\beta_{i_0 \cdots i_\nu}$'s are $L^2$ with respect to the weight $e^{-\varphi_F - \varphi_M - \phi_D}$, the $(n, q-1)$-form $v(2)$ can be chosen to be in $L^{n,q-1}_{(2)}(X; L)_{\varphi_F + \varphi_M + \phi_D, \omega}$.

As $su$ is representing the zero class in $H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi_F + \varphi_M))$, there exists a $(q-1)$-cochain $\{\gamma_{i_0 \cdots i_{q-1}}\}$ such that $\gamma_{i_0 \cdots i_{q-1}} \in K_X \otimes L \otimes \mathcal{I}(\varphi_F + \varphi_M)(U_{i_0 \cdots i_{q-1}})$ (holomorphic) for all indices $i_0, \ldots, i_{q-1} \in I$ and $\{\alpha_{i_0 \cdots i_{q-1}}\} = \delta\{\gamma_{i_0 \cdots i_{q-1}}\}$. Since

$$\bar{\partial}\rho^{i_{q-1}} \wedge \cdots \wedge \bar{\partial}\rho^{i_0} \cdot \rho^q \alpha_{i_0 \cdots i_{q-1}} = \bar{\partial}\rho^{i_{q-1}} \wedge \cdots \wedge \bar{\partial}\rho^{i_0} \cdot \rho^q \sum_{\nu=0}^{q} (-1)^{\nu} \gamma_{i_0 \cdots i_{\nu-1}}$$

$$= (-1)^q \bar{\partial}\rho^{i_{q-1}} \wedge \cdots \wedge \bar{\partial}\rho^{i_0} \cdot \gamma_{i_0 \cdots i_{q-1}}$$

$$= \bar{\partial}(((-1)^q \bar{\partial}\rho^{i_{q-1}} \wedge \cdots \wedge \bar{\partial}\rho^{i_0} \cdot \rho^{i_{q-1}} \gamma_{i_0 \cdots i_{q-1}}) =: \bar{\partial}v(\infty);$$
and since all $\rho^i$’s and $\gamma_{i_0...i_{q-1}}$’s are smooth functions, one sees that $v_\infty$ can be chosen to be smooth. One then obtains $v := v_{(2)} + v_\infty \in L^{n,q-1}_2(X; \Lambda)_{\varphi F + \varphi M + \varphi_D^{sm},\omega}$ such that $su = \partial v$ with the acclaimed properties.

Remark 3.2.2. The proof of Lemma 3.2.1 is still valid when $\varphi F$ and $\varphi M$ have neat analytic singularities as described in Section 2.2 and when $u \in \mathcal{H}^{n,q}_{\varphi,\omega}$, instead of $u \in \mathcal{H}^{n,q}_{\varphi,\omega}$ is considered in the statement. This can be seen by noticing that the local $\hat{\partial}$-equations on $U_{i_0...i_r}$ (for instance, $\hat{\partial}\{\beta_i\} = \{su\}_{U_{i_0}}$) for $\beta_i \in L^{n,q-1}_2(U_{i_0}; D \otimes F \otimes M)_{\varphi F + \varphi M + \varphi_D,\omega}$ can be solved first on $U_{i_0...i_r} \cap (X^o \setminus D)$ with $L^2$ estimate even though $\hat{\partial} \omega$ is not complete there (see [8, Ch. VIII, Thm. (6.1)] or [27, Lemma 5.4]) and the solution can then be extended to $U_{i_0...i_r}$ via the $L^2$ Riemann extension theorem (see [7, Lemma 6.9]). The form $v_{(2)}$ obtained in the conclusion should then be replaced by

$$\tilde{v}_{(2)} \in L^{n,q-1}_2(X^o; D \otimes F \otimes M)_{\varphi F + \varphi M + \varphi_D,\omega}.$$ 

3.2.2. Justification for Step II. To justify the arguments in Step II, one has to show that

1. (eq 3.1.1) holds true,
2. $u$ satisfies the Bochner–Kodaira formula (BK), which yields (eq 3.1.2), and
3. $su$ satisfies (BK), which yields (eq 3.1.3).

It follows from the refined hard Lefschetz theorem in [30, Thm. 3.3] (see also Theorem 2.5.1) that $\ast \omega u$ is holomorphic on $X$, which implies that $u = (-1)^{n-q} \ast \omega \ast u$ is smooth on $X$. Recall that $s_D$ is a canonical section of $D$ such that $\phi_D = \log |s_D|^2$. Since $|u|^2 = |s_D|^2 \text{vol}_{X,\omega}$ is $L^1$ with respect to the weight $e^{-\varphi F - \phi_D}$, it follows that $\frac{s_D u}{s_D}$, hence $\frac{u}{s_D}$, is smooth on $X$. This piece of information is sufficient to give the following proposition.

Proposition 3.2.3. With the knowledge that $\frac{u}{s_D}$ being smooth on $X$, one has

$$\lim_{\varepsilon \to 0^+} \int_X \frac{\hat{\partial} \phi_D \otimes u}{|\psi_D|^{1+\varepsilon}} = \pi \sum_{i \in i_D} \int_{D_i} \left( \hat{\partial} \phi_D \otimes \frac{u}{s_D} \right)^2_{\varphi F, \omega} < +\infty,$$

where $D = \sum_{i \in i_D} D_i$ is the decomposition of $D$ into irreducible components in $X$ and $R_D$, is the Poincaré residue map corresponding to the restriction from $X$ to $D_i$ (see [25, Def. 4.1 and para. 4.18], see also Section 2.6). In particular, the equation (eq 3.1.1) holds true, that is,

$$\lim_{\varepsilon \to 0^+} \int_X \frac{\hat{\partial} \phi_D \otimes u}{|\psi_D|^{1+\varepsilon}} = 0.$$

Proof. Let $\{V_\gamma\}_{\gamma \in \Gamma}$ be a finite open cover of $X$ such that each $V_\gamma$ is an admissible open set with respect to $\psi_D$ in the holomorphic coordinate system $(z^\gamma_1, \ldots, z^\gamma_n)$, and let $\{\psi_\gamma\}_{\gamma \in \Gamma}$ be a partition of unity subordinate to this cover. On any $V_\gamma$ such that $V_\gamma \cap D \neq \emptyset$, one has

$$(\hat{\partial} \phi_D)|_{V_\gamma} = \sum_{\gamma=1}^{\sigma_{V_\gamma}} \frac{dz^\gamma_j}{z^\gamma_j} - (\hat{\partial} \varphi_D^{sm})|_{V_\gamma},$$
and, therefore, on $V_\gamma$, 
\[
\overline{\partial}_D \psi \otimes u = \sum_{j=1}^{\sigma_{V_\gamma}} \left( dz_j - \frac{\overline{z}_j}{\sigma_{V_\gamma}} \overline{\partial}_D \psi \right) \otimes u \overline{z}_j = \sum_{j=1}^{\sigma_{V_\gamma}} \left( dz_j - \frac{\overline{z}_j}{\sigma_{V_\gamma}} \overline{\partial}_D \psi \right) \otimes u \overline{z}_j \prod_{k \neq j} z_k. 
\]

\[=: dz_j^\gamma \wedge g_j^\gamma \tag{8}\]

Compare this expansion with the one in (eq2.6.1). As $\frac{u}{\overline{s}_D}$ is smooth on $X$, it follows that $g_j^\gamma$ is in $C^\infty(X)$ on $V_\gamma$ for $j = 1, \ldots, \sigma_{V_\gamma}$ and thus satisfies the conditions in (eq2.6.2). Note that, for every $D_j^1, \gamma := \{z_j = 0\}$ where $j = 1, \ldots, \sigma_{V_\gamma}$ (using the notation in Section 2.6), there exists a unique $i \in I_D$ such that $D_j^1, \gamma = V_\gamma \cap D_i$ and 
\[
g_j^\gamma|_{D_j^1, \gamma} = R_{D_j^1, \gamma} \left( \overline{\partial}_D \psi \otimes \frac{u}{\overline{s}_D} \right) = \left. \left( R_{D_i} \left( \overline{\partial}_D \psi \otimes \frac{u}{\overline{s}_D} \right) \right) \right|_{V_\gamma \cap D_i}.
\]

It, therefore, follows from Theorem 2.6.1 (or the computation corresponding to the 1 lc-measure in [6, Prop. 2.2.1]) that
\[
\lim_{\varepsilon \to 0^+} \int_{V_\gamma} \rho_j \left| \overline{\partial}_D \psi \otimes u \right|_{\varphi, \omega}^2 = \frac{1}{\pi} \sum_{j=1}^{\sigma_{V_\gamma}} \int_{D_j^1, \gamma} \rho_j |g_j^\gamma|_{\varphi, \omega}^2 < +\infty.
\]

The claim thus follows after summing up over $\gamma \in \Gamma$, where $\Gamma$ is just a finite set (thus finiteness of the integral is guaranteed). \(\square\)

Remark 3.2.4. The computation and results in Proposition 3.2.3 still hold true when $\overline{\partial}_D \psi \otimes u$ is replaced by $(\partial D \psi)^\omega \wedge u$ or $\overline{\partial}_D \psi \wedge u$. For fear of being confused by the notation and also for the use in Step V of Section 3.1, notice that, if $g_j^\gamma|_{D_j^1, \gamma} = R_{D_j^1, \gamma} \left( (\partial D \psi)^\omega \wedge \frac{u}{\overline{s}_D} \right) = 0$ under the notation in the proof above, one actually has $(dz_j^\gamma \wedge g_j^\gamma)|_{D_j^1, \gamma} = (dz_j^\gamma)^\omega \wedge \frac{u}{\overline{s}_D} = 0$, where $(\cdots)|_{D_j^1, \gamma}$ here is the restriction only on the coefficients of the $(n, q - 1)$-form.

It can now be shown that $u$ satisfies the Bochner–Kodaira formula (BK)$_{\varphi, \omega}$ in Lemma 2.4.2 and therefore (eq3.1.2). For that purpose, let Dom $\overline{\partial}$ $\subset L^{n,q}_{(2)}(D \otimes F)_{\varphi, \omega}$ be the domain of $\overline{\partial}$ on $L^{n,q}_{(2)}(D \otimes F)_{\varphi, \omega}$ and Dom $\overline{\partial}^*$, Dom $\vartheta \subset L^{n,q}_{(2)}(D \otimes F)_{\varphi, \omega}$ be the domains of respectively the Hilbert space adjoint and the formal adjoint of $\overline{\partial}$: $L^{n,q+1}_{(2)}(D \otimes F)_{\varphi, \omega} \rightarrow L^{n,q}_{(2)}(D \otimes F)_{\varphi, \omega}$ (see Lemma 2.4.1). Note that one has Dom $\overline{\partial}^* \subset$ Dom $\vartheta$ and $\overline{\partial}^* = \vartheta$ on Dom $\overline{\partial}$ (see Section 2.4). Let again $\theta_{\varepsilon} := \vartheta \circ \frac{1}{|\overline{s}_D|^2}$ be the smooth cut-off function described in Section 3.1.

Proposition 3.2.5. Given $u \in H^{n,q}_{\varphi, \omega}$, which is smooth on $X$ and satisfies (eq3.1.1) according to Proposition 3.2.3, it follows that $u$ satisfies the Bochner–Kodaira formula (BK)$_{\varphi, \omega}$ in Lemma 2.4.2 and, consequently, (eq3.1.2), i.e.
\[
\nabla^{(0,1)}u = 0 \quad \text{and} \quad (i \partial \overline{\partial} \varphi F)^\omega (u, u)_{\omega} = 0 \quad \text{on} \ X.
\]

Proof. Since $u$ is smooth on $X$, it follows that $\vartheta_{\varepsilon} u$ satisfies (BK)$_{\varphi, \omega}$ by Lemma 2.4.2. Moreover, $u \in \ker \overline{\partial}^* \subset$ Dom $\overline{\partial}^*$ implies that $\vartheta_{\varepsilon} u = 0$. Noting the pointwise identities

\[\theta_{\varepsilon} = \frac{\sigma_{D_{\varepsilon}}}{{\sigma}_{D_{\varepsilon}}} \frac{1}{\vartheta_{\varepsilon}} (d\overline{z}_j - \frac{\overline{z}_j}{\sigma_{V_\gamma}} \overline{\partial}_D \psi) \otimes \frac{u}{\overline{s}_D} \frac{z_j}{\overline{z}_j}, \quad \text{so} \ g_j^\gamma \text{ contains no terms of the holomorphic differential form } dz_j^\gamma.\]
\[\overline{\partial}(\theta e u) = \overline{\partial} \theta \wedge u, \quad \partial(\theta e u) = -(\partial \theta \ell) \wedge u, \quad \nabla^{(0,1)}(\theta e u) = \theta e \nabla^{(0,1)} u + \overline{\partial} \theta \otimes u \quad \text{and} \]
\[|\overline{\partial} \theta \wedge u|_{\varphi, \omega}^2 + |(\partial \theta) \wedge u|_{\varphi, \omega}^2 = |\overline{\partial} \theta \otimes u|_{\varphi, \omega}^2 \quad \text{on} \ X \]
and \(i \overline{\partial} \overline{\partial} = 0\) on \(X \setminus D\) as well, \((\text{BK}) \varphi, \omega\) (with \(\theta e u\) in place of \(\zeta\)) yields
\[0 = \|\theta e \nabla^{(0,1)} u\|_{\varphi, \omega}^2 + 2 \Re \left(\theta e \nabla^{(0,1)} u, \frac{\varepsilon'_{\ell}}{|\psi D|^{1+\varepsilon}} \overline{\partial} \psi D \otimes u\right)_{\varphi, \omega} + \int_X \left(i \overline{\partial} \overline{\partial} \varphi_F\right)(\theta e u, \theta e u)_{\varphi, \omega}.
\]
Noting that \(\theta'\) is bounded uniformly in \(\varepsilon\) on \(X\), a use of the Cauchy–Schwarz inequality followed by the AM-GM inequality for any fixed constant \(\alpha \in (0, 1)\) on the inner product above yields
\[(1 - \alpha)\|\theta e \nabla^{(0,1)} u\|_{\varphi, \omega}^2 + \int_X \left(i \overline{\partial} \overline{\partial} \varphi_F\right)(\theta e u, \theta e u)_{\varphi, \omega} \leq \frac{\varepsilon^2}{\alpha} \int_X \left|\overline{\partial} \psi D \otimes u\right|_{\varphi, \omega}^2,
\]
where the constant involved in \(\leq\) is independent of \(\varepsilon\). As \(\frac{1}{|\psi D|^{1+\varepsilon}} \leq \frac{1}{|\psi D|^{1+\varepsilon}}\), Proposition 3.2.3 guarantees that the right-hand-side above has its limit equal 0 as \(\varepsilon \to 0^+\). Using the assumption that \(i \overline{\partial} \overline{\partial} \varphi_F \geq 0\), one can apply Fatou’s lemma on the left-hand-side of the inequality above and obtain
\[0 \leq (1 - \alpha)\|\nabla^{(0,1)} u\|_{\varphi, \omega}^2 + \int_X \left(i \overline{\partial} \overline{\partial} \varphi_F\right)(u, u)_{\varphi, \omega} \leq 0 .
\]
The desired equalities thus follow.

A similar argument can show that the Bochner–Kodaira formula \((\text{BK}) \varphi + \varphi_M, \omega\) is valid for \(su\) and the desired claims in (eq 3.1.3) can then be proved.

**Corollary 3.2.6.** Given \(u \in H^{n, \omega}_{\varphi, \omega}\), which satisfies the hypotheses and claims in Propositions 3.2.3 and 3.2.5, it follows that \(su\) satisfies the Bochner–Kodaira formula \((\text{BK}) \varphi + \varphi_M, \omega\) and, consequently, claims in (eq 3.1.3) hold, that is,
\[\partial \varphi_M(su) = 0 \quad \text{and} \quad su \in \text{Dom } \overline{\partial} \varphi_M,
\]
where \(\overline{\partial} \varphi_M\) and \(\partial \varphi_M\) are respectively the Hilbert space and formal adjoints of \(\overline{\partial}\) with respect to \([\cdot, \cdot]_{\varphi + \varphi_M, \omega}\).

**Proof.** Since \(su\) is smooth on \(X\), \((\text{BK}) \varphi + \varphi_M, \omega\) is valid for \(\theta e su\). Noting that \(\overline{\partial}(su) = s \overline{\partial} u = 0, \quad \nabla^{(0,1)}(su) = s \nabla^{(0,1)} u = 0\) and \(\partial \varphi_M(\theta e su) = \theta e \partial \varphi_M(su) - (\partial \theta) \wedge su\), together with the assumption \(i \overline{\partial} \overline{\partial} \varphi_M \leq C i \overline{\partial} \overline{\partial} \varphi_F\) for some constant \(C > 0\) (see Section 2.2), the argument as in Proposition 3.2.5 turns \((\text{BK}) \varphi + \varphi_M, \omega\) (with \(\theta e su\) in place of \(\zeta\)) into
\[\|\theta e \partial \varphi_M(su)\|_{\varphi + \varphi_M, \omega}^2 - 2 \Re \left(\theta e \partial \varphi_M(su), \frac{\varepsilon'_{\ell}}{|\psi D|^{1+\varepsilon}} \overline{\partial} \psi D \wedge su\right)_{\varphi + \varphi_M, \omega}
\]
\[= \int_X \left(i \overline{\partial} (\varphi_F + \varphi_M)\right)(\theta e su, \theta e su)_{\varphi + \varphi_M, \omega}.
\]
\[
\leq (1 + C) \int_X |s|^2 \varphi_M^2 \theta^2 (i \partial \partial \varphi_F)^\omega (u, u)_{\varphi, \omega} = 0.
\]

The use of the Cauchy–Schwarz and AM-GM inequalities for some constant \( \alpha \in (0, 1) \) then yields
\[
(1 - \alpha) \| \theta \partial \varphi_M (su) \|^2_{\varphi + \varphi_M, \omega} \leq \frac{\varepsilon^2}{\alpha} \int_X \| (\partial \psi_D)^\omega \|_{\varphi + \varphi_M, \omega}^2 |\psi_D|^{2 + 2 \varepsilon} \leq \sup_X |s|^2 \varphi_M^2 \frac{\varepsilon^2}{\alpha} \int_X \| (\partial \psi_D)^\omega \|_{\varphi + \varphi_M, \omega}^2 |\psi_D|^{2 + 2 \varepsilon},
\]
where the constant involved in \( \lesssim \) is from the estimate of \( \theta^\prime \) which is independent of \( \varepsilon \). Applying Fatou’s lemma on the left-hand-side and Proposition 3.2.3 (see also Remark 3.2.4) on the right-hand-side while taking the limit \( \varepsilon \to 0^+ \) gives
\[
\partial \varphi_M (su) = 0
\]
as desired.

To see that \( su \in \text{Dom} \overline{\partial}^* \) (see, for example, [8, Ch. VIII, §1] for the definition of the domain of an Hilbert space adjoint), notice that, for every \( \zeta \in \text{Dom} \overline{\partial} \subset L^n_{\omega+1} (D \otimes F \otimes M)_{\varphi + \varphi_M, \omega} \), by considering a partition of unity and the convolution with smoothing kernels on local coordinate charts, one obtains a sequence \( \{ \zeta_{\varepsilon, \nu} \}_{\varepsilon, \nu} \) of smooth \( D \otimes F \otimes M \)-valued \( (n, q - 1) \)-forms compactly supported in \( X \setminus D \) such that \( \zeta_{\varepsilon, \nu} \to \theta \zeta \) in the graph norm \( \left( \| \cdot \|^2_{\varphi + \varphi_M, \omega} + \| \overline{\partial} \cdot \|^2_{\varphi + \varphi_M, \omega} \right)^{1/2} \) of \( \overline{\partial} \) according to the lemma of Friedrichs (see [15] or [8, Ch. VIII, Thm. (3.2)]). Therefore, one sees that
\[
\langle su, \overline{\partial} \zeta \rangle_{\varphi + \varphi_M, \omega} \xrightarrow{\varepsilon \to 0^+} \langle su, \theta \partial \varphi_M, \omega \rangle_{\varphi + \varphi_M, \omega} = \langle su, \overline{\partial} \theta \zeta \rangle_{\varphi + \varphi_M, \omega} - \langle su, \overline{\partial} \theta \zeta \wedge \zeta \rangle_{\varphi + \varphi_M, \omega} \xrightarrow{\nu \to \infty} \langle su, \overline{\partial} \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} = \langle \partial \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} - \langle \overline{\partial} \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} = \langle \partial \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} - \langle \overline{\partial} \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} = \langle \partial \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} - \langle \overline{\partial} \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} = \langle \partial \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} - \langle \overline{\partial} \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} = \langle \partial \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega}.
\]
As \( \| (\partial \psi_D)^\omega \|_{\varphi + \varphi_M, \omega} su \|^2_{\varphi + \varphi_M, \omega} \leq \sup_X |s|^2 \varphi_M^2 \| (\partial \psi_D)^\omega \|_{\varphi + \varphi_M, \omega}^2 \), Proposition 3.2.3 (and Remark 3.2.4) guarantees that the inner product on the far right-hand-side converges to \( 0 \) when \( \varepsilon \to 0^+ \). Since \( \langle \partial \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} = 0 \) (or one has \( \lim_{\varepsilon \to 0^+} \lim_{\nu \to \infty} \langle \partial \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} = \lim_{\varepsilon \to 0^+} \langle \partial \varphi_M (su), \zeta_{\varepsilon, \nu} \rangle_{\varphi + \varphi_M, \omega} \) if \( \partial \varphi_M (su) \) were not 0), it can be seen that \( \text{Dom} \overline{\partial} \ni \zeta \mapsto \langle su, \overline{\partial} \zeta \rangle_{\varphi + \varphi_M, \omega} \) is a bounded linear functional, so \( su \in \text{Dom} \overline{\partial}^* \).

\textbf{Remark 3.2.7.} The argument for the claim \( su \in \text{Dom} \overline{\partial}^* \) indeed shows that
\[
\text{Dom} \overline{\partial}^* \cap \mathcal{A}_{n,q}^a (X; D \otimes F) = \text{Dom} \partial \cap \mathcal{A}_{X}^{n,q} (X; D \otimes F) \quad \text{and} \quad \text{Dom} \overline{\partial}^*_{\varphi, \omega} \cap \mathcal{A}_{X}^{n,q} (X; D \otimes F \otimes M) = \text{Dom} \partial_{\varphi, \omega} \cap \mathcal{A}_{X}^{n,q} (X; D \otimes F \otimes M).
\]

3.2.3. \textbf{Justification for Step IV.} There is no need of extra clarification for Step III. To justify Step IV in the current situation (where both \( \varphi_F \) and \( \varphi_M \) are smooth and \( \omega_{\varphi_F} \) is smooth on the whole of \( X \)), it suffices to show the following.

\[
\leq (1 + C) \int_X |s|^2 \varphi_M^2 \theta^2 (i \partial \partial \varphi_F)^\omega (u, u)_{\varphi, \omega} = 0.
\]
Proposition 3.2.8. Given \( u \in \mathcal{H}^{n,\omega}_{\varphi,\omega} \), which satisfies the hypotheses and claims in Propositions 3.2.3 and 3.2.5, it follows that \( u \) satisfies the twisted Bochner–Kodaira formula \((tBK)_{\varepsilon,\varphi,\omega} \) with \( \varepsilon > 0 \) and, consequently,

\[
\lim_{\varepsilon \to 0^+} \varepsilon \int_X \frac{\left| (\partial \psi_D, u)_{\varphi,\omega} \right|^2}{|\psi_D|^{1+\varepsilon}} = \int_X (i\partial\bar{\partial}\varphi_D^m)(u, u)_{\varphi,\omega}.
\]

Proof. The proof is similar to the one of Proposition 3.2.5. Note that \( \theta_{\varepsilon'}u \) satisfies \((tBK)_{\varepsilon',\varphi,\omega} \) for any \( \varepsilon, \varepsilon' > 0 \). Taking into account of the vanishing results \( \partial u = 0 \), \( \partial u = 0 \), \( \nabla^{(0,1)} u = 0 \) and \( (i\partial\bar{\partial}\varphi_F)(u, u)_{\omega} = 0 \) on \( X \), together with \( i\partial\bar{\partial}\psi_D = 0 \) on \( X \setminus D \), the argument as in Proposition 3.2.5 turns \((tBK)_{\varepsilon,\varphi,\omega} \) into

\[
0 = -\int_X \frac{1 - \varepsilon}{|\psi_D|} \left( i\partial\bar{\partial}\varphi_D^m \right)_{\theta_{\varepsilon'}u} \partial_{\varepsilon'}|\psi_D|^{1-\varepsilon} + 2(1 - \varepsilon) \Re \int_X \left( (\partial \psi_D)_{\varphi,\omega}^{\prime} u, (\partial \psi_D)_{\varphi,\omega}^{\prime} \theta_{\varepsilon'}u \right)_{\varphi,\omega} |\psi_D|^{1-\varepsilon} + \varepsilon \int_X \frac{1 - \varepsilon}{|\psi_D|} \left( (\partial \psi_D)_{\varphi,\omega}^{\prime} \theta_{\varepsilon'}u ight)^2_{\varphi,\omega} |\psi_D|^{1-\varepsilon}.
\]

After dividing the factor \( 1 - \varepsilon \), expanding \( \partial \theta_{\varepsilon'} \) and rearranging terms, it becomes

\[
\int_X \frac{\theta_{\varepsilon'}^2}{|\psi_D|^2} \left( i\partial\bar{\partial}\varphi_D^m \right)_{\theta_{\varepsilon'}u} = \int_X \left( \frac{\varepsilon\theta_{\varepsilon'}^2}{|\psi_D|^{1+\varepsilon}} + 2\varepsilon \theta_{\varepsilon'} \theta_{\varepsilon'} |(\partial \psi_D)_{\varphi,\omega}^{\prime} u |^2_{\varphi,\omega} \right) |\psi_D|^{1-\varepsilon}.
\]

On the left-hand-side, since \( i\partial\bar{\partial}\varphi_D^m \) is a smooth form on \( X \) and \( u = L^2 \) with respect to \( \varphi \) and \( \omega \), it follows that the limit as \( \varepsilon' \to 0^+ \) followed by \( \varepsilon \to 0^+ \) is finite and equal to \( \int_X \left( i\partial\bar{\partial}\varphi_D^m \right)_{\varphi,\omega} \). On the right-hand-side, notice that \( \frac{2\varepsilon' \theta_{\varepsilon'} \theta_{\varepsilon'}}{|\psi_D|^{1+\varepsilon}} \geq 0 \). As \( \theta_{\varepsilon'} \to 1 \) as \( \varepsilon' \to 0 \), it follows from the monotone convergence theorem that \( \varepsilon \int_X \frac{|(\partial \psi_D)_{\varphi,\omega}^{\prime} u |^2_{\varphi,\omega}}{|\psi_D|^{1+\varepsilon}} < \infty \) for any \( \varepsilon > 0 \). This in turn implies that \( \lim_{\varepsilon' \to 0^+} 2\varepsilon' \int_X \frac{\theta_{\varepsilon'} \theta_{\varepsilon'}}{|\psi_D|^{1+\varepsilon}} |(\partial \psi_D)_{\varphi,\omega}^{\prime} u |^2_{\varphi,\omega} = 0 \), as \( \theta_{\varepsilon'} \theta_{\varepsilon'} \) is bounded uniformly in \( \varepsilon' \). Therefore, the limit of the above equality as \( \varepsilon' \to 0^+ \) followed by \( \varepsilon \to 0^+ \) yields the desired result.

There is no need of further clarification for Step V in the case where \( \varphi_F \) and \( \varphi_M \) are smooth (thus \( X \setminus X^0 = \emptyset \)). The proof of Theorem 1.2.1 is therefore completed for this case.

3.3. Proof for the case with singular \( \varphi_F \) and \( \varphi_M \). In this section, \( \varphi_F \) and \( \varphi_M \) possess neat analytic singularities described as in Section 2.2. Recall that \( X^0 = X \setminus (P_F \cup P_M), \) where \( P_F = \varphi_F^{-1}(-\infty) \) and \( P_M = \varphi_M^{-1}(-\infty) \) are the polar sets, and \( \hat{\omega} \) is a complete Kähler metric on \( X^0 \) given by the formula in item (4) in Section 2.2 such that \( \hat{\omega} \geq \omega \). Write

\[
\psi_{FM} := \psi_{P_F \cup P_M} = \phi_{P_F \cup P_M} - \varphi_F^sm \phi_{P_F \cup P_M} =: \phi_{FM} - \varphi_F^sm
\]

for convenience in what follows. Note that \( \log(e \log(\psi_{FM})) \) is a smooth exhaustive function on \( X^0 \) with \( |d\log(e \log(\psi_{FM})))| \leq 1 \) on \( X^0 \) by the choice of \( \hat{\omega} \) (and the constant \( \ell \gg e \)). Let \( \chi_0 = 0 \to 0, 1 \) be a smooth non-increasing cut-off function such that \( \chi_0 \equiv 1 \) and \( \chi_0 \equiv 0 \). For every \( \nu \in \mathbb{N}, \) set \( \chi_{\nu} := \chi \circ \left( \frac{1}{\nu} \log(e \log(\psi_{FM})) \right) \) and \( \chi'_{\nu} := \chi' \circ \left( \frac{1}{\nu} \log(e \log(\psi_{FM})) \right) \) for convenience (where \( \chi' \) is the derivative of \( \chi \)). Notice
that both $\chi_{\nu}$ and $\chi'_{\nu}$ are compactly supported in $X^o$ for $\nu < +\infty$ and $\chi_{\nu} \not\to 1$ pointwisely on $X^o$ as $\nu \not\to +\infty$. As $|\chi'_{\nu}|$ is bounded uniformly in $\nu$, it follows that

(eq 3.3.1) \[ |d\chi_{\nu}^2| \lesssim \frac{1}{2\nu}, \]

in which the constant involved in $\lesssim$ is independent of $\nu$ (so $\{\chi_{\nu}\}_{\nu \in \mathbb{R}}$ is just the exhaustive sequence of cut-off functions introduced in [8, Ch. VIII, Lemma (2.4 c)]).

3.3.1. Justification for Step I. The justification for $\tilde{v} = \tilde{v}(2) + v(\infty)$ is already given by Lemma 3.2.1 together with Remark 3.2.2.

3.3.2. Justification for Step II. To justify the arguments in Step II for the case where $\varphi_F$ and $\varphi_M$ have neat analytic singularities, it suffices to supplement the proof of Proposition 3.2.3 with the clarification of the regularity of $\tilde{u} \in \mathcal{H}_{\varphi,\omega}^{n,q}$ along $X \setminus X^o = P_F \cup P_M$ which guarantees that the proof remains valid (thus (eq 3.1.1) holds true). Moreover, a simple adjustment, making use of the completeness of $\tilde{\omega}$ on $X^o$, to the proofs of Proposition 3.2.5 and Corollary 3.2.6 is made to guarantee that $\tilde{u}$ and $s\tilde{u}$ still satisfy the Bochner–Kodaira formulas (BK) $\tilde{\varphi}^{\tilde{F},\tilde{M},\tilde{\omega}}$ respectively.

The regularity of $\tilde{u}$ along $X \setminus X^o$ is controlled as follows.

**Proposition 3.3.1.** For any $\tilde{u} \in \mathcal{H}_{\varphi, \omega}^{n,q}$, one has $\mathcal{F}_{\omega}^{n,q}$ being smooth on $X^o$. Furthermore, on an admissible open set $V \subset X$ with respect to $(\varphi_F, \varphi_M, \psi_D)$ in the holomorphic coordinate system $(z_1, \ldots, z_n)$ such that $D \cap V = \{z_1 \cdots z_n = 0\}$ and

$$\phi_{FM}|_V = \sum_{j=\sigma_V+1}^{\sigma_V+\mu} \log|z_j|^2 = \sum_{k=1}^{\mu} \log|w_k|^2 \quad \text{(setting $w_k := z_{\sigma_V+k}$)}$$

for some integer $\mu = \mu_V \in [0, n - \sigma_V]$ ($\mu = 0$ when $V \cap (X \setminus X^o) = \emptyset$), one also has

$$\frac{\tilde{u}}{s_D}|_V \in \mathcal{A}_X^{n,q}(V) \left[ \frac{1}{|\psi_{FM}|}, \frac{1}{|\log|\psi_{FM}||}, \frac{1}{w_1}, \frac{1}{w_1}, \ldots, \frac{1}{w_{\mu}}, \frac{1}{w_{\mu}} \right],$$

where the right-hand-side is generated as an $\mathcal{A}_X^{n,q}(V)$-algebra.

**Proof.** Following the argument as in Section 3.2.2, the refined hard Lefschetz theorem in [30, Thm. 3.3] (see also Theorem 2.5.1) implies that $\ast\omega \tilde{u}$ is holomorphic on $X$, which, together with the fact that $u$ being $L^2$ with respect to $e^{-\phi_D}$, in turn implies that $\frac{\tilde{u}}{s_D}$ is smooth on $X^o$.

For the remaining claim, note that, as $\frac{\tilde{u}}{s_D}$ is smooth (indeed holomorphic) on $X$, it follows from the formula of the Hodge $*$-operator or

$$\frac{\tilde{u}}{s_D} \wedge \left( \frac{\ast\omega \tilde{u}}{s_D} \right) = \left( \frac{\tilde{u}}{s_D} \right)^2 = \frac{\ast\omega \tilde{u}}{s_D} d\text{vol}_X^{\ast \omega \tilde{u}}$$

that the singularities of $\frac{\tilde{u}}{s_D}$ along $X \setminus X^o$ is determined by those of $d\text{vol}_X^{\ast \omega \tilde{u}} = \frac{\omega^{n,n}}{n!}$. From the definition of $\omega$ in item (4) in Section 2.2 and the expression $\psi_{FM} = \phi_{FM} - \phi_{FM}^{sm} = \sum_{k=1}^n \log|w_k|^2 - \phi_{FM}^{sm}$, one can express $\frac{\omega^{n,n}}{n!}$ on $V$ as

$$\frac{\omega^{n,n}}{n!} = \frac{1}{n!} \left( 2\omega + \frac{i\partial \bar{\partial} \psi_{FM}}{|\log|\psi_{FM}||^2} \right)^n \leq \left( \sum_{j=1}^n \frac{dw_j}{w_j} - \partial \phi_{FM}^{sm} \right)^\wedge \leq \left( \sum_{j=1}^n \frac{dw_j}{w_j} - \partial \phi_{FM}^{sm} \right)^\wedge$$
where each \( \alpha^{p,q}_{j_1j_2...k_1k_2...} \) is a \((p',q')\)-form with continuous coefficients on \( V \) such that

\[
\alpha^{p',q'}_{j_1j_2...k_1k_2...} \in \Omega^{p',q'}(V) \left[ \frac{1}{|\psi_{FM}|} \right] \frac{1}{\log |\psi_{FM}|}.
\]

Note that \( \alpha^{1,1} \) can also be viewed as an \((1,1)\)-form with continuous coefficients on \( V \) since

\[
\bar{i} \partial \bar{\partial} \phi_{FM} - \frac{i}{|\psi_{FM}|(\log |\psi_{FM}|)^2} = 0 \quad \text{on } X \text{ as a current.}
\]

(Recall the Poincaré–Le long formula which states that \( i \partial \bar{\partial} \phi_{FM} = [P_F \cup P_M] \), the current of integration along \( P_F \cup P_M \).) Since \( \frac{\tilde{u}}{\tilde{s}_D} \) has at worst the singularities of \( \frac{\tilde{\omega}^{n,n}}{n!} \), the last claim follows by observing the formula of \( \frac{\tilde{\omega}^{n,n}}{n!} \).

Recall that \( D \) and \( X \setminus X^0 = P_F \cup P_M \) have no common components and intersect each other with snc. Recall also that \( \varphi_F \) has only neat analytic singularities along \( P_F \). One implication of Proposition 3.3.1 is that, in view of Fubini’s theorem, the derivatives of \( \frac{\tilde{u}}{\tilde{s}_D} \) in the normal directions of any components of \( D \) (more precisely, derivatives with respect to \( r_1, \ldots, r_{s_D} \) on an admissible set \( V \)) are locally \( L^2 \) with respect to \( \varphi_F \) and \( \tilde{\omega} \) in \( X \) (not only in \( X^0 \)), because \( \frac{\tilde{u}}{\tilde{s}_D} \) is also \( L^2 \) with respect to \( \varphi_F \) and \( \tilde{\omega} \) and the derivatives of \( \frac{1}{|\psi_{FM}|} \) and \( \frac{1}{\log |\psi_{FM}|} \) with respect to \( r_1, \ldots, r_{s_D} \) are more readily integrable than their primitives (note also that \( \frac{1}{w_k} \) and \( \frac{1}{\bar{w}_k} \) are simply constant functions with respect to these derivatives). The derivatives of coefficients of \( \tilde{\omega} \) and its inverse (as well as \( \frac{1}{\det \tilde{\omega}} \)) with respect to \( r_1, \ldots, r_{s_D} \) are also more readily integrable than their primitives for the same reason (as they live in the algebra generated over bounded functions on \( V \) with generators given by all \( \frac{1}{w_k} \)'s, \( \frac{1}{\bar{w}_k} \)'s, \( \frac{1}{|\psi_{FM}|} \), \( \frac{1}{\log |\psi_{FM}|} \) and the derivatives of the generators with respect to \( r_1, \ldots, r_{s_D} \)). Such argument is used in the following proposition.

**Proposition 3.3.2** (cf. Proposition 3.2.3). With \( \frac{\tilde{u}}{\tilde{s}_D} \) satisfying the conclusion in Proposition 3.3.1 on any admissible open sets \( V \subset X \), one has

\[
\lim_{\epsilon \to 0^+} \int_{X^0} \left| \frac{\partial \psi_D}{\psi_D} \otimes \bar{\partial} \frac{\tilde{u}}{\tilde{s}_D} \right|^2 \bar{\partial} \hat{\psi}_D \frac{1}{\hat{\psi}_D} = \pi \sum_{i \in I_D} \int_{D_i} \mathcal{R}_{D_i} \left( \partial \psi_D \otimes \frac{\tilde{u}}{\tilde{s}_D} \right)_{\varphi_F, \hat{\psi}}^2 < +\infty,
\]

where \( D = \sum_{i \in I_D} D_i \) is the decomposition of \( D \) into irreducible components in \( X \) and \( \mathcal{R}_{D_i} \) is the Poincaré residue map corresponding to the restriction from \( X \) to \( D_i \). The equation (eq. 3.1.1) therefore also holds true.

**Proof.** The proof is exactly the same as the proof of Proposition 3.2.3, provided that the singularities of \( \frac{\tilde{u}}{\tilde{s}_D} \) along \( X \setminus X^0 \) is taken care of.

Using the same notation as in the proof of Proposition 3.2.3, consider again a finite cover \( \{ V_\gamma \}_{\gamma \in \Gamma} \) of \( X \) by admissible open sets \( V_\gamma \) and write

\[
\bar{\partial}_D \otimes \bar{u} = \sum_{j=1}^{\sigma_{V_\gamma}} \left( dz_j \bar{\partial}_{\bar{\phi}_D} \right) \otimes \bar{u} = \sum_{j=1}^{\sigma_{V_\gamma}} \left( \bar{u} \right) \otimes \left( \bar{z}_j \bar{\partial}_{\bar{\phi}_D} \right) =: \left( d\bar{z}_j \right) \wedge g_{j}^{\gamma}
\]

on each \( V_\gamma \) for \( \gamma \in \Gamma \) (where \( g_{j}^{\gamma} \) is chosen as in footnote 8 on page 32 such that \( g_{j}^{\gamma} = \frac{\partial}{\partial z_j} \wedge (d\bar{z}_j \wedge g_{j}^{\gamma}) \)). With the conclusion of Proposition 3.3.1 and the fact that \( \| \bar{u} \|_{\varphi_F, \bar{\omega}} = \| \bar{u} \|_{\varphi_F, \bar{\omega}} < +\infty \), one can check readily (in view of Fubini’s theorem) that both conditions in (eq 2.6.2) are satisfied with \( \langle g_{j}^{\gamma}, g_{j'}^{\gamma} \rangle_{\varphi_F, \bar{\omega}} \) in place of \( \langle g_p, g_p' \rangle_{\varphi_F, \bar{\omega}} \) for any \( j, j' = 1, \ldots, \sigma_{V_\gamma} \).

Indeed, this can be seen from the following facts:

(a) it follows from the definition of \( g_{j}^{\gamma} \) that \( |d\bar{z}_j \wedge g_{j}^{\gamma}|_{\varphi_F, \bar{\omega}} \leq \| \bar{u} \|_{\varphi_F, \bar{\omega}}^2 \) (thus the coefficient of the \((n-1, n-1)\)-form \( g_{j}^{\gamma} \) is in \( L^1_{\text{loc}}(V_\gamma) \)), and, from the openness property of multiplier ideal sheaves and the fact that \( \phi_D \) is holomorphic, \( g_{j}^{\gamma} \) is also in \( L^1_{\text{loc}}(V_\gamma) \) for sufficiently small \( \varepsilon > 0 \);

(b) \( e^{-\varphi_F} \sim \prod_{k=1}^m \frac{1}{|\psi_{F|M}|} \) in the notation in Proposition 3.3.1, where \( b_k \geq 0 \) (but may not be integers);

(c) Proposition 3.3.1 implies that \( g_{j}^{\gamma} \), as well as \( g_{j}^{\gamma} \) (without \( e^{-\varphi_F} \)), is a polynomial in \( \prod_{k=1}^{m} \frac{1}{|\psi_{F|M}|} \) and \( \frac{1}{{\text{det}} \omega} \) with coefficients in \( \mathcal{A}_{X, \star}^\bullet(V_\gamma) \), which are therefore smooth in the variables \( r_1, \ldots, r_n \) on \( V_\gamma \), where \( r_j := |z_j| \), when the other variables are fixed (note that \( \frac{1}{{\text{det}} \omega} \) is involved as \( |g_{j}^{\gamma}|_{\omega} \) involves the coefficients of the inverse of \( \bar{\omega} \));

(d) the computation of \( \frac{\partial^n}{\partial \bar{\omega}^n} \) in the proof of Proposition 3.3.1 implies that \( \frac{1}{{\text{det}} \omega} = \prod_{k=1}^m \frac{1}{|w_k|} \prod_{k=1}^{\mu} \frac{1}{|\psi_{F|M}|} \log(|\psi_{F|M}|)^{\nu} B \) for some positive (nowhere zero) continuous function \( B \) on \( V_\gamma \) which is smooth in the variables \( r_1, \ldots, r_\sigma_{V_\gamma} \).

Taking into account the factorisation of \( \frac{1}{{\text{det}} \omega} \) in item (d) in the polynomial expression of \( |g_{j}^{\gamma}|_{\omega} \), it follows that \( |g_{j}^{\gamma}|_{\omega} \) is a polynomial in \( \prod_{k=1}^{m} \frac{1}{|w_k|} \), \( \log(|\psi_{F|M}|) \), and \( \frac{1}{{\text{det}} \omega} \) (or possibly their reciprocals) without modifying the coefficients, one can write

\[
|g_{j}^{\gamma}|_{\omega} = G \left| w_1 \right|^{m_1} \cdots \left| w_\mu \right|^{m_\mu} \frac{1}{\left| \psi_{F|M} \right|^{p} \log(|\psi_{F|M}|)^{\nu}},
\]

where \( m_1, \ldots, m_\mu, p, p' \in \mathbb{Z} \) are the maximal (possibly negative) integers such that the function \( G \) is a polynomial in \( \prod_{k=1}^{m_1} \frac{1}{|w_k|} \), \( \psi_{F|M} \) and \( \log(|\psi_{F|M}|) \) with coefficients in \( \mathcal{A}_{X, \star}^{n-1,n-1}(V_\gamma)[B] \) and that \( G \) is not divisible by \( \left| \psi_{F|M} \right| \) and \( \log(|\psi_{F|M}|) \) over \( \mathcal{A}_{X, \star}^{n-1,n-1}(V_\gamma)[B] \). With the fact that \( |g_{j}^{\gamma}|_{(1+\varepsilon)\varphi_F, \bar{\omega}} \) is integrable on \( V_\gamma \), one can adjust the exponents \( m_1, \ldots, m_\mu \) and modify the coefficients of \( G \) until the factor \( Q := \frac{|w_1|^{m_1} \cdots |w_\mu|^{m_\mu}}{\left| \psi_{F|M} \right|^{p} \log(|\psi_{F|M}|)^{\nu}} \) is integrable with respect to \( e^{-(1+\varepsilon)\varphi_F} \) on \( V_\gamma \) for all sufficiently small \( \varepsilon > 0 \). Indeed, if \( Qe^{-(1+\varepsilon)\varphi_F} \) is not integrable for all \( \varepsilon > 0 \), then \( G \) has zeros along some divisor in \( \psi_{F|M}(-\infty) \), say, \( \{ w_k = 0 \} \),
and can be factored into \(|w_k|m \cdot G'\), where \(G' \in \mathcal{A}^{n-1,n-1}(V_\gamma)\) and \(G' e^{\varepsilon} A_2\). The rest of the argument is the same as in the proof of Proposition 3.2.3.

Leibniz rule for differentiation. Therefore, Proposition 2.6.1 with Remark 2.6.2 can be applied. The proof of Proposition 3.2.5 and Corollary 3.2.6, which confirm the claims in (eq3.1.2) and (eq3.1.3), need adjustments only in replacing \(\partial_r \partial_s \psi\) in the proof of Proposition 3.3.4.

By a similar analysis, \(g_j^\gamma\) takes the form analogous to (\(*\)) (but the corresponding factor \(G\) is a polynomial over \((\mathcal{A}^{n,1}_\lambda \otimes \mathcal{A}^{n-1,q}_\lambda)(V_\gamma)[B]\) and satisfies the condition (which is stronger than (eq2.6.2))

\[
\left( \frac{\partial}{\partial r_{\sigma \gamma}} \right)^{\sigma \gamma} \cdots \left( \frac{\partial}{\partial r_{1}} \right)^{1} g_j^\gamma \in L^2_{\text{loc}}(V_\gamma) \ 	ext{ for } \alpha_1, \ldots, \alpha_{\sigma \gamma} \in \mathbb{N}_{\geq 0}
\]

when the discussion before the statement of this proposition (on the integrability of the derivatives of \(\frac{1}{|\psi_{FM}|_F} \log|\psi_{FM}|_F\) and \(\frac{1}{\det \omega}\), which applies also to multiples of \(|\psi_{FM}|\) and \(\log|\psi_{FM}|\)) is taken into account. The fact that \(\langle g_j^\gamma, g_j^{r,\omega} \rangle\) satisfies the conditions in (eq2.6.2) (even when \(j \neq j'\)) then follows from the Cauchy–Schwarz inequality and the Leibniz rule for differentiation. Therefore, Proposition 2.6.1 with Remark 2.6.2 can be applied. The rest of the argument is the same as in the proof of Proposition 3.2.3.

The proofs of Proposition 3.2.5 and Corollary 3.2.6, which confirm the claims in (eq3.1.2) and (eq3.1.3), need adjustments only in replacing \(\omega\) by \(\tilde{\omega}\) and \(\theta_\varepsilon\) by \(\chi_\nu \theta_\varepsilon\) (where \(\chi_\nu\) is defined at the beginning of Section 3.3). They remain valid without further changes once the following lemma is observed.

**Lemma 3.3.3.** Given \(\frac{\bar{\gamma}}{\bar{\nu}}\) satisfying the conclusion in Proposition 3.3.1, one has

\[
\lim_{\varepsilon \to 0^+} \lim_{\nu \to \infty} \int_{X_\nu} \left| \partial(\chi_\nu \theta_\varepsilon) \otimes \bar{u}^2 \right|_{\varphi, \bar{\omega}} = 0.
\]

Note that the statement with \(\partial(\chi_\nu \theta_\varepsilon) \otimes \bar{u} \) replaced by \(\bar{\partial}(\chi_\nu \theta_\varepsilon) \otimes \bar{u} \) also holds true since \(\left| \partial(\chi_\nu \theta_\varepsilon) \otimes \bar{u}^2 \right|_{\varphi, \bar{\omega}} \leq \left| \bar{\partial}(\chi_\nu \theta_\varepsilon) \otimes \bar{u}^2 \right|_{\varphi, \bar{\omega}}\).

**Proof.** Noting that \(\chi_\nu\), \(\theta_\varepsilon\) and \(\theta'_\varepsilon\) are bounded uniformly in \(\nu\) and \(\varepsilon\), and recalling the bound on \(d_\chi\nu\) (thus on \(\partial d_\chi\nu\)) in (eq3.3.1), a direct computation yields

\[
\left| \partial(\chi_\nu \theta_\varepsilon) \otimes \bar{u}^2 \right|_{\varphi, \bar{\omega}} = \left| (\theta_\varepsilon \partial \chi_\nu + \chi_\nu \partial \theta_\varepsilon) \otimes \bar{u}^2 \right|_{\varphi, \bar{\omega}} \leq 2|\theta_\varepsilon \partial \chi_\nu \otimes \bar{u}^2|_{\varphi, \bar{\omega}} + 2\varepsilon^2 |\chi_\nu \theta'_\varepsilon|_{\varphi, \bar{\omega}}^2 \frac{|\bar{\partial} \psi_{D} \otimes \bar{u}^2|_{\varphi, \bar{\omega}}}{|\psi_{D}|^{2+2\varepsilon}} \leq \frac{1}{2^{2\varepsilon}} |\bar{u}|_{\varphi, \bar{\omega}}^2 + \varepsilon^2 |\bar{\partial} \psi_{D} \otimes \bar{u}^2|_{\varphi, \bar{\omega}}^2 |\psi_{D}|^{2+2\varepsilon},
\]

where the constant involved in \(\lesssim\) is independent of \(\nu\) and \(\varepsilon\). Noting also the fact \(|\psi_{D}| \geq 1\), the claim then follows immediately from Proposition 3.3.2. □
This completes the justification for Step II.

3.3.3. **Justification for Step IV.** There is no extra clarification needed for Step III. To justify the argument in Step IV, it suffices to reprove Proposition 3.2.8 under the assumption that $\nu$ has singularities along $X \setminus X^\circ$ described as in Proposition 3.3.1. This can be achieved by replacing $\theta'_\nu$ in the proof of Proposition 3.2.8 by $\chi_\nu \theta'_\nu$ and noticing the identity

$$2 \text{Re} \int_X \langle (\partial(\chi_\nu \theta'_\nu)) \triangledown \tilde{u}, \frac{(\partial\psi_D) \triangledown \chi_\nu \theta'_\nu \tilde{u}}{|\psi_D|} \rangle_{\varphi, \tilde{\varphi}} |\psi_D|^{1-\varepsilon}$$

$$= 2 \text{Re} \int_X \left\langle (\partial(\chi_\nu) \triangledown \theta'_\nu \tilde{u}, \frac{(\partial\psi_D) \triangledown \chi_\nu \theta'_\nu \tilde{u}}{|\psi_D|} \right\rangle_{\varphi, \tilde{\varphi}} \varphi \tilde{\varphi} + \int_X \frac{2\varepsilon |\chi_\nu \theta'_\nu \tilde{u}}{|\psi_D|^{1+\varepsilon}} |(\partial\psi_D) \triangledown \tilde{u}|^2_{\varphi, \tilde{\varphi}}. $$

The first integral on the right-hand-side converges to $0$ as $\nu \to +\infty$ thanks to the bound on $\partial \chi_\nu$ in (eq 3.3.1). The desired result is obtained by following the remaining arguments in the proof of Proposition 3.2.8 and taking the limits $\nu \to +\infty, \varepsilon' \to 0^+$ and $\varepsilon \to 0^+$ in order.

3.3.4. **Justification for Step V.** The justification is done with the following proposition.

**Proposition 3.3.4.** Under the assumption $\varphi_F$ and $\varphi_M$ having neat analytic singularities with snc along $X \setminus X^\circ$, together with the regularity of $\frac{\tilde{u}}{s_D}$ along $X \setminus X^\circ$ described in Proposition 3.3.1, one has

$$\int_X \left\langle (\partial\psi_D) \triangledown \frac{\tilde{u}}{s_D}, \varphi \right\rangle_{\varphi, \tilde{\varphi}} < +\infty,$$

which also implies that

$$\int_X \left\langle (\partial\psi_D) \triangledown \frac{\tilde{u}}{s_D}, \varphi \right\rangle_{\varphi, \tilde{\varphi}} < +\infty \quad \text{for any } \varepsilon \in \mathbb{R}. $$

**Proof.** It suffices to prove both claims on an arbitrary admissible open set $V \Subset X$ with respect to $(\varphi_F, \varphi_M, \psi_D)$. Let $(z_1, \ldots, z_n)$ be a holomorphic coordinate system on $V$ such that $D \cap V = \{z_1 \cdots z_{\sigma_V} = 0\}$. Decompose $V$ into the product $U \times W$ of polydiscs $U = U^{\sigma_V}$ and $W = W^{n-\sigma_V}$ such that $(z_1, \ldots, z_{\sigma_V})$ and $(z_{\sigma_V+1}, \ldots, z_n) = (w_1, \ldots, w_{\mu}, z_{\sigma_V+\mu+1}, \ldots, z_n)$ are coordinate systems on $U$ and $W$ respectively (in the notation in Proposition 3.3.1).

First note that $\int_X |\varphi|^{2} \varphi_F \varphi_M \omega \leq \int_X |\varphi|^{2} \varphi_F \varphi_M \omega < +\infty$ according to Lemma 3.2.1 (together with Remark 3.2.2). Furthermore, by writing

$$(\partial\psi_D) \triangledown \frac{\tilde{u}}{s_D} = \sum_{j=1}^{\sigma_V} \left( dz_j - \frac{z_j \partial \psi_D}{\sigma_V} \right) \triangledown \frac{\tilde{u}}{s_D} \frac{1}{z_j} =: dz_j \wedge g_j$$

and following the analysis (as well as the notation) in the proof of Proposition 3.3.2, one has

$$|dz_j \wedge g_j|^{2}_{\varphi, \tilde{\varphi}} = G_j \frac{|w_1|^{m_1} \cdots |w_{\mu}|^{m_{\mu}}}{|\psi_F\psi_M|^p (\log |\psi_F\psi_M|)^p} e^{-\varphi_F},$$

$$=: Q_j$$
where $G_j \in \mathcal{A}^{n,n}_X[V, |\psi_{FM}|^{\pm 1}, \log |\psi_{FM}|]$ is bounded on $V$ and $Q_j e^{-\varphi_F}$ is integrable on $V$. The fact that $\left( \frac{dz_j}{z_j} \right)^{\sigma_j} = \frac{\tilde{u}}{\tilde{s}_D} \in \mathcal{A}^{n,q}_X(V \cap X^0)$ for all $j = 1, \ldots, \sigma_V$ according to (eq.3.1.5) (thus having no poles along $D \cap V \cap X^0$) implies that $G_j = |z_j|^2 G_j'$ for some $G_j' \in \mathcal{A}^{n,n}_X[V, |\psi_{FM}|^{\pm 1}, \log |\psi_{FM}|]$ such that $G_j'$ is also bounded on $V$. This implies immediately the convergence of $\int_V (\partial \psi_D)^{\omega} \frac{\tilde{u}}{\tilde{s}_D} \varphi_{F, \tilde{\omega}}$. The first claim then follows.

In view of Fubini’s theorem, the first claim implies that the function

$$U \ni x \mapsto \int_{\{x\} \times W} \left( (\partial \psi_D)^{\omega} \frac{\tilde{u}}{\tilde{s}_D}, v(\infty) e^{-\frac{1}{2} \varphi_{DM} - \frac{1}{2} \varphi_M} \right) |\varphi_{F, \tilde{\omega}}|$$

is bounded (indeed even continuous) on $U$, as the integrand above is continuous in $z_1, \ldots, z_{\sigma_V}$ (when the other variables are fixed at almost every point in $W$) and it has an estimate

$$\left| \langle \cdots, \cdots \rangle \right|_{\varphi_{F, \tilde{\omega}}} \leq \left( \left( \partial \psi_D)^{\omega} \frac{\tilde{u}}{\tilde{s}_D} \varphi_{F, \tilde{\omega}} \right) \right)^{\frac{1}{2}} \left( |\psi_{\infty}|^{2} \varphi_{DM} + \varphi_F + \varphi_M, \omega \right)^{\frac{1}{2}} \left( |\psi_{\infty}|^{2} \varphi_{DM} + \varphi_F + \varphi_M, \omega \right)^{\frac{1}{2}}$$

which can be seen easily that the right-hand-side is dominated by some $(n, n)$-form $\Phi d \text{vol}_X$ on $V$ such that $\Phi = \Phi(w_1, \ldots, w_\mu)$ depends only on the variables $w_1, \ldots, w_\mu$ and is integrable on $W$ (see, for example, [2, Thm. 10.38]). Since $\frac{1}{|s_D| |\psi_{DM}| |\psi_D|^{1+\varepsilon}}$ is integrable on $X$ for any $\varepsilon \in \mathbb{R}$ and has poles only along $D$, the second claim then follows again in view of Fubini’s theorem.

This completes the justification of Step V and thus the proof of Theorem 1.2.1.

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