SYMMETRY OF BOUND AND ANTIBOUND STATES IN THE SEMICLASSICAL LIMIT
FOR A GENERAL CLASS OF POTENTIALS

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Abstract. We consider the Schrödinger operator

\[-h^2 \partial_x^2 + V(x)\]

where \(V\) is a compactly supported potential which is positive near the endpoint of its support. We prove that the eigenvalues and the purely imaginary resonances are symmetric up to an error \(Ce^{-\delta/h}\).

1. Introduction

In this paper, we study spectral properties of the Schrödinger operator

\[P(h) = -h^2 \partial_x^2 + V(x)\]

defined for \(x\) in the half-line \((-\infty, B]\). Here \(h > 0\) is the semiclassical parameter and \(V(x)\) is a piecewise continuous real-valued potential supported in \([0, B]\).

The operator \(P(h)\) with the Neumann boundary condition at \(B\) is self-adjoint on \(L^2(-\infty, B]\); therefore, its resolvent

\[R_V(\lambda) = (P(h) - \lambda^2)^{-1}, \quad \text{Im} \lambda > 0,\]

is a bounded operator from \(L^2\) to \(H^2\) for \(\lambda^2\) not in the spectrum of \(P(h)\). This resolvent can be extended meromorphically as an operator \(L^2_{\text{comp}} \rightarrow H^2_{\text{loc}}\) to \(\lambda \in \mathbb{C}\) with isolated poles of finite rank; these poles are called resonances. (The reader is referred to [12] for details.) To each resonance \(\lambda\) corresponds a resonant state; that is, a nonzero \(u \in H^2_{\text{loc}}(-\infty, B]\) solving the equation \((P(h) - \lambda^2)u = 0\) with the Neumann boundary condition at the right endpoint and with the following outgoing condition at \(-\infty\):

\[u(x) = Ae^{-i\lambda x/h} \text{ for all } x < 0 \text{ and some constant } A.\]

(Note that for \(x < 0\), \(u\) solves the free equation \((-h^2 \partial_x^2 - \lambda^2)u = 0\), so it must be a linear combination of \(e^{\pm i\lambda x/h}\).)

For \(\text{Im } \lambda > 0\), the outgoing condition implies that \(u\) is exponentially decreasing on the negative half-line and thus \(u \in L^2\); therefore, \(\lambda\) is a pole (of the resolvent) lying in the upper half-plane if and only if \(\lambda^2\) is an eigenvalue of \(P(h)\) on \(L^2\). Since \(P(h)\) is self-adjoint, all poles in the upper half-plane have to lie on the imaginary axis. There may be poles \(\lambda\) with \(\text{Im } \lambda < 0\) and \(\text{Re } \lambda \neq 0\); however, we will restrict our attention to purely imaginary resonances:

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Definition 1. A positive number $k$ is called a \textbf{bound state} if $ik$ is a pole of the resolvent $R_V(\lambda)$, and an \textbf{antibound state} if $-ik$ is a pole.

We see from above that $k$ is an (anti)bound state if and only if there exists a nonzero solution $u$ of the problem

\begin{align}
(P(h) + k^2)u &= 0, \\ u_x|_{x=B} &= 0, \\ hu_x \pm ku|_{x=0} &= 0.
\end{align}

The plus sign in (3) corresponds to an antibound state, and the minus sign corresponds to a bound state. We will also study Neumann eigenvalues of $P(h)$ on $[0,B]$, i.e., those $k$ for which there exists a nonzero solution to (1) with boundary conditions (2) and

\begin{equation}
|k|_0 = 0.
\end{equation}

Since the space of solutions to (1) and (2) is always one dimensional, \textbf{bound states, antibound states, and Neumann eigenvalues never coincide}. However, Bündel and Zworski proved in [3] that bound and antibound states located away from zero coincide, modulo errors of order $e^{-\delta/h}$ for some $\delta > 0$, if the potential satisfies the following conditions:

\begin{align}
\exists A > 0, V_0 > 0 & : V(x) = V_0 \text{ for all } x \in [0,A], \\
\exists \varepsilon > 0 & : V(x) = 0 \text{ for all } x \in (A,A+\varepsilon).
\end{align}

In this paper, we prove a similar result with more general assumptions on the potential:

Theorem 1. Suppose that $V$ is a piecewise continuous real-valued potential supported in $[0,B]$ and satisfying the following \textbf{bump condition}:

\begin{equation}
\exists A > 0 : V(x) > 0 \text{ for all } x \in (0,A].
\end{equation}

Fix two constants $0 < c_k < C_k < \infty$. Then there exist constants $C, \delta > 0, h_0 > 0$ such that for $h < h_0$ and any $k \in [c_k,C_k]$:  

1. If $k$ is a Neumann eigenvalue, then there exist a bound state $k_+$ and an antibound state $k_-$ such that $|k - k_+| \leq Ce^{-\delta/h}$. 

2. If $k$ is a bound or an antibound state, then there exists a Neumann eigenvalue $k_0$ such that $|k - k_0| \leq Ce^{-\delta/h}$.

The bump condition (5) cannot be disposed of completely, as illustrated by the numerical experiments performed using [2]. Figure 1 shows two potentials on the whole line, each supported in $[-2,2]$, and the corresponding bound states (denoted by squares) and antibound states (denoted by circles). The vertical coordinate of each (anti)bound state on the picture corresponds to its value $k$; the horizontal coordinate corresponds to the value of $h^{-1}$ used. We see that the conclusion of the theorem does not appear to hold for the potential on the left, which does not satisfy the bump condition; at the same time, it is true for the potential on the right. Theorem 1, formulated for the half-line case, applies to these numerical experiments on the whole line since for even potentials, the set of their (anti)bound states is composed of these states for the positive half-line with Dirichlet condition and the same states for the Neumann condition; the theorem above can be applied with
Figure 1. Bound and antibound states for two spline potentials
(splinepot([0, -0.4, -1, -0.2, -1, -0.4, 0], [-2, -1.5, -1, 0, 1, 1.5, 2]) and splinepot([0, 0.2, -1, -0.2, -1, -1, 0.2, 0], [-2, 1.5, -1, 0, 1, 1.5, 2]))

Dirichlet condition in place of (2). (However, condition (1) cannot be replaced by the Dirichlet condition in the theorem.)

The study of resonances in one dimension has a long tradition going back to the origins of quantum mechanics; see for instance [8]. One of the first studies of their distribution was conducted by Regge [10]; since then, there have been many mathematical results on the topic, including [1], [4], [5], [6], [7], [9], [11], and [13]. Concerning antibound states, Hitrik has shown in [6] that for a positive compactly supported potential, there are no antibound states in the semiclassical limit. This agrees with our result since there are no bound states in this case. Simon proved in [11] that between any two bound states, there must be an odd
number of antibound states; the following corollary of this result follows almost immediately using the methods we develop to prove Theorem 1:

**Theorem 2.** Consider the half-line problem with a bounded compactly supported potential $V$ (which does not need to satisfy any positivity condition). Then for each two bound states $0 < k_1 < k_2$, the interval $(k_1, k_2)$ contains at least one antibound state. In particular, if there are $n$ bound states in some subinterval of $(0, \infty)$, then there are at least $n - 1$ antibound states in the same subinterval.

The proof of Theorem 1 works as follows: we study the evolution (in $x$) of the vectors $(u, hu_x)$ for the three solutions of (1) with initial data at $x = 0$ satisfying the conditions (3) and (4). The idea is to look at these three vectors at $x = A$. Since $V(x) + k^2 \geq 0$ on the interval $(0, A)$, the transition matrix for the considered vectors from $x = 0$ to $x = A$ will have an expanding and a contracting direction. (In fact, if we introduce rescaling $\tilde{x} = x/h$, then the behavior of the original system for small $h$ is similar to the behavior of the rescaled system for large $\tilde{x}$, and the latter will be similar to the behavior of the geodesic flow on a two-dimensional manifold of negative curvature.) It turns out that our three vectors lie in a certain angle between the expanding and the contracting directions, from which it follows that they will stay in this angle for later times (Lemma 2); what is more, their polar angles will get exponentially close to each other (Lemma 7). Finally, we can study how the polar angles of the considered vectors change with $k$ (Lemma 4): it follows (Lemma 8) that the polar angle for the solution with Neumann initial data at $x = 0$ will strictly increase in $k$ and the polar angle for the solution with the same data at $x = B$ will decrease in $k$. The proof is then completed by a perturbation argument (Lemma 5).

The detailed proofs of Theorems 1 and 2 are given in Section 3. Both are elementary and use certain properties of ordinary differential equations presented in Section 2.

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2. Preliminaries

Throughout this section, $I$ is an interval in $\mathbb{R}$, $V(x) \in L^\infty(I; \mathbb{R})$, $u(x), v(x) \in H^2(I; \mathbb{R})$, $h > 0$, and $P(h) = -h^2 \partial_x^2 + V(x)$. Any solution to the equation $P(h)u = 0$ is determined by the vector $(u, hu_x)$ at any $x$; we will sometimes view this vector in polar coordinates:

**Definition 2.** Define the **length** $L(u)$ and the **polar angle** $\theta(u)$ by the equations

\[ u = L(u) \cos \theta(u), \]
\[ hu_x = L(u) \sin \theta(u). \]

Here $\theta(u)$ lies in the circle $S^1 = \mathbb{R}/(2\pi \mathbb{Z})$.

**Lemma 1.** Define the **Wronskian** $W(u, v)$ by

\[ W(u, v) = h(uv_x - vu_x). \]

Then

\[ W(u, v) = L(u)L(v) \sin (\theta(v) - \theta(u)), \]
\[ h\partial_x W(u, v) = v \cdot P(h)u - u \cdot P(h)v. \]
Note that $W(u,v)$ is just the oriented area of the parallelogram spanned by the vectors $(u,hu_x)$ and $(v,hu_x)$. The next lemma tells us that if the vector $(u,hu_x)$ falls inside a certain angle in the plane at the initial time, then it will stay inside that angle for all later times.

**Lemma 2.** Suppose that $a^2 \leq V(x) \leq b^2$ for all $x \in I$ and some constants $a,b > 0$. Let $u$ be a solution to $P(h)u = 0$ and define

$$W_+(u) = W(u,e^{bx/h}), \quad W_-(u) = W(e^{-ax/h}, u).$$

Let $x_0$ be a point in $I$ and assume that $W_+(u), W_-(u) \geq 0$ at $x_0$. Then for $x \geq x_0$, the functions $W_\pm(u)$ are increasing in $x$ and

$$u \geq \frac{L(u)}{\sqrt{1+b^2}}. \quad \text{(8)}$$

**Proof.** We have

$$e^{-bx/h}W_+(u) = bu - hux_x, \quad e^{ax/h}W_-(u) = au + hux_x.$$ 

Therefore, $W_+(u), W_-(u) \geq 0$ yields $|hux_x| \leq bu$ and thus (8). Next,

$$P(h)e^{bx/h} = e^{bx/h}(V(x) - b^2) \leq 0,$$

$$P(h)e^{-ax/h} = e^{-ax/h}(V(x) - a^2) \geq 0.$$ 

Using (1), we see that $\partial_x W_\pm \geq 0$ as long as $u \geq 0$. It remains to prove that $u(x) \geq 0$ for $x \geq x_0$. Suppose this is false and let $x_1 = \inf\{x \geq x_0 \mid u(x) < 0\}$. Then $u$ is not identically zero; since it solves a second order linear ODE, $L(u) > 0$ everywhere. But $u \geq 0$ on $[x_0,x_1]$, so $W_\pm$ are increasing on this interval. In particular, $W_\pm \geq 0$ at $x_1$ and thus (8) holds at this point. However, by the choice of $x_1$ we have $u(x_1) = 0$, which contradicts $L(u) > 0$. \hfill \Box

In the next section, we will use the following crude estimate on how fast the solutions of an ODE can grow:

**Lemma 3.** Assume that $|V(x)| \leq M$ for $x \in I$ and that $u$ is a solution to $P(h)u = 0$. Take $x_0, x_1 \in I$; then

$$L(u)|_{x=x_1} \leq e^{(1+M)|x_0-x_1|/(2h)} \cdot L(u)|_{x=x_0}. \quad \text{(9)}$$

**Proof.** Without loss of generality we may assume that $x_1 > x_0$. We have $L(u)^2 = u^2 + (hux_x)^2$; thus

$$h\partial_x(L(u)^2) = 2hux_x(1 + V(x)) \leq (1 + M)L(u)^2,$$

and the lemma is proven by Gronwall’s inequality. \hfill \Box

**Lemma 4.** Assume that $u(x,k)$ is a family of solutions to $(P(h) + k^2)u = 0$, $x_0, x_1 \in I$, and $u(x_0,k)$ and $u_x(x_0,k)$ are independent of $k$. Let $\Theta_1(k) = \theta(u(x,k))|_{x=x_1}, \quad L_1(k) = L(u(x,k))|_{x=x_1}$. Then

$$\Theta'_1(k) = \frac{2k}{hL_1(k)^2} \int_{x_0}^{x_1} u(x,k)^2 \, dx.$$ 

**Proof.** We have $W(u, u_k)|_{x=x_1} = L_1(k)^2 \Theta'_1$. (To see that, differentiate the formulas in Definition 2 in $k$ and use the definition of the Wronskian.) Now, we differentiate the equation $(P(h) + k^2)u = 0$ in $k$ to get $(P(h) + k^2)u_k = -2ku$. It remains to apply (7) together with $W(u, u_k)|_{x=x_0} = 0$. \hfill \Box
Lemma 5. Assume that $\Phi$ is a $C^1$ map from the interval $I = [K_0, K_1]$ to the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\Phi'(k) \geq \delta > 0$ for all $k \in I$. Suppose that $\Psi : I \to S^1$ is a continuous map such that $|\Psi(k)| \leq \varepsilon < \pi$ for all $k$. Put $\nu = \varepsilon/\delta$ and $I_\nu = [K_0 + \nu, K_1 - \nu]$. Then:

1. If $k_0 \in I_\nu$ has $\Phi(k_0) = 0$, then there exists $k_1 \in I$ with $\Phi(k_1) = \Psi(k_1)$ and $|k_0 - k_1| \leq \nu$.
2. If $k_1 \in I_\nu$ has $\Phi(k_1) = \Psi(k_1)$, then there exists $k_0 \in I$ with $\Phi(k_0) = 0$ and $|k_0 - k_1| \leq \nu$.

Proof. We can lift $\Phi$ and $\Psi$ to continuous maps $\bar{\Phi}, \bar{\Psi} : I \to \mathbb{R}$; then $|\bar{\Psi}| \leq \varepsilon$ and $\Phi(k') - \Phi(k) \geq \delta(k' - k)$ for $k' \geq k$.

1. We have $\bar{\Phi}(k_0) = 2\pi m$ for some $m \in \mathbb{Z}$. Then $\bar{\Phi}(k_0 + \nu) \geq 2\pi m + \delta \nu \geq 2\pi m + \Psi(k_0 + \nu)$ and $\bar{\Phi}(k_0 - \nu) \leq 2\pi m + \Psi(k_0 - \nu)$; it remains to apply the intermediate value theorem.

2. Similar to the previous statement, we have $\bar{\Phi}(k_1) = 2\pi m + \Psi(k_1)$ for some $m \in \mathbb{Z}$ and $\bar{\Phi}(k_1 + \nu) \geq 2\pi m \geq \bar{\Phi}(k_1 - \nu)$. \hfill $\square$

Lemma 6. Assume that $\Phi$ is a $C^1$ map from some interval $I$ to the circle $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ with $\Phi'(k) > 0$ for all $k \in I$. Let $\Psi : I \to S^1$ be a continuous map such that $\Psi(k) \neq 0$ for all $k \in I$. If $k_1 < k_2$ are two roots of the equation $\Phi = 0$, then the interval $(k_1, k_2)$ contains at least one root of the equation $\Phi = \Psi$.

Proof. As in the previous lemma, lift $\Phi$ and $\Psi$ to maps $\bar{\Phi}, \bar{\Psi} : I \to \mathbb{R}$; we can make $0 < \bar{\Psi}(k) < 2\pi$ for all $k \in I$. Since $\bar{\Phi}' > 0$ everywhere, we have $\bar{\Phi}(k_j) = 2\pi m_j$, where $m_1 < m_2$ are some integers. Therefore, $\bar{\Phi}(k_1) < 2\pi m_1 + \Psi(k_1)$ and $\bar{\Phi}(k_2) > 2\pi m_1 + \Psi(k_2)$; it remains to apply the intermediate value theorem. \hfill $\square$

3. Proofs of the Theorems

We assume in this section that $0 < c'_k \leq k \leq C'_k$ for some constants $c'_k < c_k$ and $C'_k > C_k$; the constants in our estimates will depend on $c'_k$ and $C'_k$. (We need to expand the interval $[c_k, C_k]$ a little bit to be able to apply Lemma 5.)

Consider the solutions $u_\pm, u_0, u_1(x, k)$ to the equation (11) in $[0, B]$ with the initial data

\[
\begin{align*}
u_x\pm0,0,k &= 1, \quad \partial_x u_0(0, k) = 0, \quad h\partial_x u_\pm(0, k) = \pm k, \\
u_1B,k &= 1, \quad \partial_x u_1(B, k) = 0.
\end{align*}
\]

Define $\Theta_0(k), \Theta_\pm(k)$, and $\Theta_1(k)$ to be the polar angles of vectors $(u, hu_x)$ at $x = A$ for $u = u_0, u_\pm, u_1$. Then $k > 0$ is

- a Neumann eigenvalue if $u_0$ and $u_1$ are linearly dependent; that is (recalling that they solve the same second order ODE), if $2(\Theta_0(k) - \Theta_1(k)) = 0$;
- a bound state if $2(\Theta_+(k) - \Theta_1(k)) = 0$;
- an antibound state if $2(\Theta_-(k) - \Theta_1(k)) = 0$.

Here we count angles modulo $2\pi$.

To prove Theorem 1 it suffices to use Lemma 5 (for $\Phi = 2(\Theta_0 - \Theta_1)$ and $\Psi = 2(\Theta_0 - \Theta_\pm)$) together with the following two facts:

Lemma 7. For some constants $C_1$ and $\delta_1 > 0$ independent of $h$ and $k$,

\[
|2(\Theta_0(k) - \Theta_\pm(k))| \leq C_1 e^{-\delta_1/h}\text{ for all } k \in [c'_k, C'_k].
\]

Lemma 8. We have $\Theta'_0(k) - \Theta'_\pm(k) \geq 1/C_2 > 0$ for all $k \in [c'_k, C'_k]$ and some constant $C_2$ independent of $h$ and $k$. 
We first prove Lemma 7. Put $b = \max_{[0, A]} V(x)$, $k_b = \sqrt{k^2 + b}$, $\psi_0(x) = e^{-kx/h}$, $\psi_+(x) = e^{kx/h}$, and consider the Wronskians

$$W_+(u) = W(u, \psi_+), \quad W_0(u) = W(\psi_0, u).$$

These are nonnegative for $u = u_0, u_\pm$ at $x = 0$. Then by Lemma 2 all these six functions are nonnegative and increasing in $x$ for $0 \leq x \leq A$.

Our first goal is to get an exponential lower bound on the length $L(u)$ for $u = u_0, u_\pm$ at $x = A$. For $u_0$, note that by (8)

$$L(u_0) \geq \frac{W(\psi_0, u_0)}{L(\psi_0)} \geq \frac{W_0(u_0)|_{x=0}}{L(\psi_0)} \geq \frac{1}{C} e^{kx/h}.$$

The same applies to $u_+$. However, $u_-$ needs more careful analysis since $W_0(u_-) = 0$ at $x = 0$. For that, take $0 < t < 1$ and put $a = \min_{[tA, A]} V(x) > 0$, $k_a = \sqrt{k^2 + a}$, $\psi_-(x) = e^{-k_a x/h}$, and $W_-(u) = W(\psi_-, u)$. First, we have by Lemma 3

$$L(u_-) \geq e^{-(1+k^2+b)x/(2h)} \cdot L(u_-)|_{x=0}.$$

Next, $W_0(u_-) \geq 0$ and $W_+(u_-) \geq 0$, so by (8)

$$W_-(u_-) \geq (k_a - k)u_- \psi_- \geq \frac{1}{C} L(u_-) \psi_-.$$

Finally, we apply Lemma 2 on the interval $[tA, A]$ to get

$$L(u_-)|_{x=A} \geq \frac{W_-(u_-)|_{x=tA}}{L(\psi_-)|_{x=A}} \geq \frac{1}{C} e^{(k(1-t) - (1+k^2+b)t)A/h}.$$

For $t$ small enough and all $k$, $k(1-t) - (1+k^2+b)t \geq 0$, so we have

$$L(u_-)|_{x=A} \geq \frac{1}{C} > 0.$$

The next step is to use that $u_0$ and $u_\pm$ solve the same equation (1) and thus $W(u_0, u_\pm)$ is constant in $x$. Therefore, at $x = A$ we have by (6)

$$|\sin(\theta(u_\pm) - \theta(u_0))| = \frac{|W(u_0, u_\pm)|}{L(u_0)|L(u_\pm)|} \leq Ce^{-kA/h}.$$

That finishes the proof of Lemma 7.

To prove Lemma 8 first note that by Lemma 4 $\Theta_0'(k) \leq 0$ and

$$\Theta_0'(k) \geq \frac{1}{C h L(u_0)^2} \int_{x=A}^A |u_0(x, k)|^2 dx.$$

By (8), $u_0 \geq L(u_0)/C$. Also, by Lemma 3 $L(u_0) \geq e^{C(x-A)/h} L(u_0)|_{x=A}$ for $0 \leq x \leq A$; thus

$$\int_0^A |u_0(x, k)|^2 dx \geq \frac{1}{C} \int_0^A e^{C(x-A)/h} (L(u_0)^2)|_{x=A} dx \geq \frac{h}{C} L(u_0)^2|_{x=A}$$

and Lemma 8 is proven, which finishes the proof of Theorem 1.

To prove Theorem 2 let $\Phi_\pm(k) = \theta(u_\pm)|_{x=B}$; a bound state corresponds to $2\Phi_+ = 0$ and an antibound state corresponds to $2\Phi_- = 0$. Since $\theta(u_\pm)|_{x=0}$ is increasing with $k$, by an argument similar to the proof of Lemma 4 we get $\Phi_+(k) > 0$ for all $k$. Moreover, $2(\Phi_+(k) - \Phi_-(k))$ is never zero, as this would correspond to $u_+$ and $u_-$ being linearly dependent. We may now apply Lemma 5 with $\Phi = 2\Phi_+$ and $\Psi = 2(\Phi_+ - \Phi_-)$.
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