CONDITIONS FOR OBSERVABLE BI AND TRI-SPECTRA IN TWO-FIELD SLOW-ROLL INFLATION

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We find constraints on inflationary dynamics that yield a large local bispectrum and/or trispectrum during two-field slow-roll inflation. This leads to simple relations between the non-Gaussianity parameters, simplifying the Suyama–Yamaguchi inequality and also producing a new result between the trispectrum parameters $\tau_{NL}$ and $g_{NL}$.

Keywords: non-Gaussianity; bispectrum; trispectrum.

1. Background and motivation

Observational constraints on the statistics of the primordial curvature perturbation provide a powerful test of inflation. Multi-field inflation is a well-motivated scenario that may be observationally distinguished from single-field inflation by the generation of local non-Gaussianity during the superhorizon evolution of perturbations.\(^1\)\(^2\)

In the absence of a unique inflationary model, a key task at present is to understand the predictions of classes of inflationary models, which observational data may either rule out or constrain. In this short note we derive the types of inflationary dynamics that generate a large non-Gaussianity in two-field slow-roll inflation, which represents the simplest multi-field scenario. Further details may be found in Ref. 3. Our work is an extension of earlier bispectrum work by Byrnes et al.\(^4\) which we review, simplify and then extend to the trispectrum.

The local non-Gaussianity parameters $f_{NL}$, $\tau_{NL}$ and $g_{NL}$ are defined from the three and four-point correlators of the primordial curvature perturbation on uniform density hypersurfaces as

\[
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \equiv (2\pi)^3 \delta^3(k_1 + k_2 + k_3) \frac{6}{5} f_{NL} [P_\zeta(k_1)P_\zeta(k_2) + 2 \text{ perms}],
\]

\[
\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle \equiv (2\pi)^3 \delta^3(k_1 + k_2 + k_3 + k_4) \left\{ \tau_{NL} [P_\zeta(k_{13})P_\zeta(k_3)P(k_4) + 11 \text{ perms}] + \frac{54}{25} g_{NL} [P_\zeta(k_3)P_\zeta(k_4) + 3 \text{ perms}] \right\},
\]

where $\langle \zeta_{k_1} \zeta_{k_2} \rangle \equiv (2\pi)^3 \delta^3(k_1 + k_2)P_\zeta$ defines the power spectrum and $k_{ij} = k_i + k_j$.

We consider inflation driven by two canonical and minimally coupled scalar fields $\phi$ and $\chi$, self-interacting through a potential $W(\phi, \chi)$. Slow-roll requires that the following potential slow-roll parameters are all much smaller than unity:

\[
\epsilon_i = \frac{M_{pl}^2 W_i^2}{2 W^2}, \quad \epsilon = \sum_{i=1}^2 \epsilon_i, \quad \eta_{ij} = M_{pl}^2 \frac{W_{ij}}{W}, \quad \xi_{ijk}^2 = M_{pl}^3 \sqrt{2\epsilon} \frac{W_{ijk}}{W},
\]

where $\{i,j,k\} \in \{\phi, \chi\}$ and a comma denotes partial derivatives.
Our calculations employ the $\delta N$ formalism, which allows the non-Gaussianity parameter to be determined analytically\textsuperscript{2,5} provided the potential is of either the sum-separable form $W = U(\phi) + V(\chi)$,\textsuperscript{5} or the product-separable form $W = U(\phi)V(\chi)$.\textsuperscript{6} It is useful to define $\theta$ as the angle of evolution in the $\{\phi, \chi\}$ phase space such that $\epsilon_\phi = \epsilon \cos^2 \theta$ and $\epsilon_\chi = \epsilon \sin^2 \theta$. Assuming both fields to monotonically decrease, $\theta$ is constrained as $0 \leq \theta \leq \pi/2$.

2. Analytic formulae for non-Gaussianity

We use the rotated field basis $\{\sigma, s\}$ where $ds/dt = 0$, such that $\sigma$ and $s$ respectively define the adiabatic and isocurvature directions. Labels ‘$\ast$’ denote quantities evaluated on a flat hypersurface near horizon exit; quantities without a label are evaluated on a later-time uniform density hypersurface. Following Ref. 3, we introduce $\alpha$ such that for product-separable potentials $\alpha = \theta$, whereas for sum-separable potentials $\alpha_s = \theta_s$ and subsequently $\alpha$ follows from $d\alpha/d\theta = W \sin^2 \theta/(W^* \sin^2 2\alpha)$. The parameter $\alpha$ also follows from the linear $\delta N$ expressions of Vernizzi and Wands.\textsuperscript{5}

The $f_{NL}$ parameter, after much manipulation, assumes the simple form\textsuperscript{3}

\begin{equation}
\frac{6}{5} f_{NL} \simeq f \left[ -2 \alpha_s + 2 \Omega (\eta_{ss} - \epsilon) \right] \quad \text{Sum separable,}
\end{equation}

\begin{equation}
\frac{6}{5} f_{NL} \simeq f \left[ -2 \alpha_s + 2 \eta_{ss} \right] \quad \text{Product separable,}
\end{equation}

\begin{equation}\begin{aligned}
f &= \frac{\sin^2 2\alpha}{4\Lambda^2} (\cos^2 \alpha - \cos^2 \theta)^2, \\
\Omega &= \frac{W^2 \sin^2 \theta}{W^* \sin^2 2\alpha},
\end{aligned}
\end{equation}

where $\Lambda = \cos^4 \alpha \sin^2 \theta + \sin^4 \alpha \cos^2 \theta$. Note that $f > 0$ and $0 \leq \Omega \leq 1$. We consistently use ‘$\simeq$’ such that equality holds to excellent precision if the non-Gaussianity is large enough to be observable. Eq. (3) implies that a necessary (but not sufficient) condition for $|f_{NL}| > 1$ is a fine-tuning $\theta_s \ll 1$ such that $f \gg 1$.

The $\tau_{NL}$ parameter simplifies as

\begin{equation}
\tau_{NL} \simeq C \left( \frac{6}{5} f_{NL} \right)^2, \quad C = \frac{\Lambda}{(\cos^2 \alpha - \cos^2 \theta)^2} \geq 1,
\end{equation}

which is valid for both sum and product-separable potentials. This approximate equality is a special case of the Suyama–Yamaguchi inequality.\textsuperscript{7}

The $g_{NL}$ parameter assumes the forms

\begin{equation}\begin{aligned}
\frac{27}{25} g_{NL} &\simeq \tau_{NL} \left( \frac{\eta_{ss}^* - \Omega (\eta_{ss} - \epsilon)}{\eta_{ss}^* - 2 \Omega (\eta_{ss} - \epsilon)} \right) - \frac{6}{5} f_{NL} (2\eta_{ss}^* + \Omega (\eta_{ss} - \epsilon)) - g_4 \xi_{sss}^* \frac{2}{2} f_1 f \epsilon \eta_{ss}^* \\
&\quad + g_1 \Omega^{3/2} \left[ \xi_{sss}^* - 2 \eta_{ss}^*(\eta_{ss} + \epsilon) \right] - \frac{1}{2} f_1 f \epsilon \eta_{ss}^* \\
&\quad + 4g_3 \Omega (\eta_{ss} - \epsilon) \left( \frac{W}{W^*} \cos 2\theta \eta_{ss} - \Omega \cos 2\alpha (\eta_{ss} - \epsilon) \right),
\end{aligned}
\end{equation}

\begin{equation}\begin{aligned}
\frac{27}{25} g_{NL} &\simeq \tau_{NL} \left( \frac{\eta_{ss}^* - \eta_{ss}}{\eta_{ss}^* - 2 \eta_{ss}} \right) - \frac{6}{5} f_{NL} (2\eta_{ss}^* + \eta_{ss}) - g_4 \xi_{sss}^* \frac{2}{2} + g_1 \left[ \xi_{sss}^* - 2 \eta_{ss} \eta_{ss} \right],
\end{aligned}
\end{equation}
for sum and product-separable potentials respectively, where we have used
\[
\tau_2 = \frac{\sin 2\theta^*}{\Lambda^3} (\cos^8 \alpha \sin^4 \theta^* - \sin^8 \alpha \cos^4 \theta^*), \quad g_1 = g_3 \sin 2\alpha,
\]
\[
\tau_3 = \frac{f_1}{2\Lambda^2} (\cos^8 \alpha \sin^2 \theta^* + \sin^8 \alpha \cos^2 \theta^*), \quad g_2 = g_3 \cos 2\alpha,
\]
\[
g_4 = \frac{1}{4} (\tau_3 \sin 2\theta^* \cos 2\theta^* - \tau_2), \quad g_3 = -\frac{f}{2\Lambda} (\cos^2 \alpha - \cos^2 \theta^*),
\]
and \( f_1 = \sin^2 2\theta^*/(2\Lambda) \).

**Adiabaticity:** If the dynamics reach an adiabatic limit during slow-roll inflation then our sum-separable analytic expressions simplify following the Horizon Crossing Approximation (HCA) as \( \Omega \to 0 \). Working in this limit and assuming the \( g_4 \xi_{sss}^2 \) term may be neglected, we find a new relation between \( \tau_{NL} \) and \( g_{NL} \) as
\[
27 \frac{g_{NL}}{25} \approx \tau_{NL}.
\]
(8)

We have found it very fine-tuned to generate deviations from this result in the HCA.

3. Interpretation and conclusions

By plotting the functions such as \( f \) appearing in eqs. (3)–(6), we have verified that the inflationary dynamics that give rise to a large bispectrum parameter \( f_{NL} \) are also capable of producing large values of the trispectrum parameters \( \tau_{NL} \) or \( g_{NL} \). In all cases, a necessary requirement for a large local non-Gaussianity is that the horizon crossing field velocities are dominated by one of the two fields. For quadratic potentials we find \( g_{NL} \) to be subdominant, whereas more general potentials such as inflection points have \( g_{NL} \sim \tau_{NL} \). Under the HCA, we have generated a new consistency relation (8) between the trispectrum parameters \( g_{NL} \) and \( \tau_{NL} \).

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