Biased orientation games

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Abstract

We study biased orientation games, in which the board is the complete graph $K_n$, and Maker and Breaker take turns in directing previously undirected edges of $K_n$. At the end of the game, the obtained graph is a tournament. Maker wins if the tournament has some property $\mathcal{P}$ and Breaker wins otherwise.

We provide bounds on the bias that is required for a Maker’s win and for a Breaker’s win in three different games. In the first game Maker wins if the obtained tournament has a cycle. The second game is Hamiltonicity, where Maker wins if the obtained tournament contains a Hamilton cycle. Finally, we consider the $H$-creation game, where Maker wins if the obtained tournament has a copy of some fixed graph $H$.

1 Introduction

In this work we study orientation games. The board consists of the edges of the complete graph $K_n$. In the $(p : q)$ game the two players, called Maker and Breaker, take turns in orienting (or directing) previously undirected edges. Maker starts the game and at each round, Maker directs at most $p$ edges and then Breaker directs at most $q$ edges (usually, we consider the case where $p = 1$ and $q$ is large). The game ends where all the edges are oriented, and then we obtain a tournament. Maker then wins if the tournament has some fixed property $\mathcal{P}$, and Breaker wins otherwise. Here we focus on the $1 : b$ game, which is referred to as the $b$-biased game. We stress that at each round, each player has to orient at least one edge, so the number of rounds is clearly bounded. Also, Maker (respectively, Breaker) can orient up to $p$ edges (respectively, up to $q$ edges) and hence by bias monotonicity every property $\mathcal{P}$ admits some threshold $t(n, \mathcal{P})$ so that Maker wins the $b$-biased game if $b < t(n, \mathcal{P})$ and Breaker wins the $b$-biased game if $b \geq t(n, \mathcal{P})$.

This game is a variant of the well studied classical Maker-Breaker game, which is defined by a hypergraph $(X, \mathcal{F})$ and bias $(p : q)$. In that game, at each round Maker claims $p$ elements of $X$, and Breaker claims $q$ elements of $X$. Maker wins if by the end of the game he claimed all the elements of some hyperedge $A \in \mathcal{F}$, and Breaker wins otherwise. Usually, a typical problem goes

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as follows. Given a game hypergraph $H = (X, \mathcal{F})$, $|X| = n$, determine or estimate the threshold function $t_H$ such that if $b > t_H$ then Maker wins in a $(1 : t_H)$ game, and if $b \leq t_H$ then Breaker wins in a $(1 : t_H)$ game. There has been a long line of research that studies the bias threshold of various games (see, e.g., [3, 4, 7, 9, 10, 11] and their references).

Here we study orientation games for the following three properties.

**Creating a cycle.** Maker wins if the obtained tournament contains a cycle, and Breaker wins otherwise. It is well known that a tournament contains a cycle if and only if it contains a directed triangle (cycle of length 3). This is a relatively old question which has already been studied by Alon (unpublished result) and by Bollobás and Szabó [6], and here we improve their results.

**Creating a Hamilton cycle.** Here Maker wins if the obtained tournament contains a Hamilton cycle, and Breaker wins otherwise. Recently, the second author [10] solved a long standing question and provided tight bounds on the bias threshold for Maker win in the classical Maker-Breaker Hamiltonicity game. We use a variant of his approach, together with a new application of the Gebauer-Szabó method [9] and give tight bounds in our case as well.

**Creating a copy of $H$.** Here we are given a fixed graph $H$. Maker wins if the obtained tournament contains a copy of $H$, and Breaker wins otherwise. We provide both upper and lower bounds, and give some nearly tight bounds for specific cases. We conjecture that the correct threshold is closely related to the size of the minimum feedback arc set of $H$, and provide some results that support this conjecture.

**Our results.** In this work we study the cycle game, the Hamiltonicity game and the $H$-creation game. We stress that in all these games Maker wins if the obtained tournament has the desired property, no matter who directs each edge of a winning directed subgraph. Our first theorem considers the cycle creating game. It is easy to observe that if $b \geq n - 2$ then Breaker has a winning strategy (for completeness, we give a detailed proof in Section 3). Bollobás and Szabó [6] proved that if $b = (2 - \sqrt{3})n$, Maker wins the game and conjectured that the correct threshold is $b = n - 2$.

In this work we provide a simple argument that improves their result. We have the following.

**Theorem 1** (The cycle game). For every $b \leq n/2 - 2$, Maker has a strategy guaranteeing a cycle in the $b$-biased orientation game.

The second game we consider is the Hamiltonicity game, where Maker wins if and only if the obtained tournament contains a Hamilton cycle. Here we apply old and recent techniques [7, 9, 10] to get tight bounds on the bias threshold for a win of Breaker.

**Theorem 2** (The Hamiltonicity game).

(i). If $b \geq \frac{n(1+o(1))}{\ln n}$, Breaker has a strategy to guarantee that in the $b$-biased orientation game the obtained tournament has a vertex of in-degree 0, and in particular to win the Hamiltonicity game.

(ii). If $b \leq \frac{n(1+o(1))}{\ln n}$, Maker has a strategy guaranteeing a Hamilton cycle in the $b$-biased orientation game.
In the $H$-creation game we have some partial results. We conjecture that the bias that guarantees Maker’s win depends on the minimum feedback arc set of $H$, and support this result for graphs with a small feedback arc set. We will give and discuss corresponding notion in Section 5.

## 2 Preliminaries

Let $K_n$ be the complete graph on $n$ vertices, a tournament is an orientation of $K_n$. A directed graph is called oriented if it contains nor loops neither cycles of length 2. Every oriented graph is a subgraph of a tournament. A directed graph is strongly connected if for every two vertices $u, v$ there is a directed path from $u$ to $v$ and a directed path from $v$ to $u$. All directed graphs we consider here are oriented, i.e., do not have parallel or opposite edges.

All logarithms are in base 2 unless stated otherwise.

The classical Maker and Breaker game goes as follows. Given a hypergraph $H = (V, F)$, at every round Maker occupies $p$ elements from $V$, and then Breaker occupies $q$ elements from $V$. By the end of the game, Maker wins if he occupies completely some hyperedge in $F$, and otherwise Breaker wins. The well known results of Erdős and Selfridge [8] and Beck [2] give a sufficient condition for a Maker’s win.

**Theorem 3.** Suppose that Maker and Breaker play a $(p : q)$-game on a hypergraph $H = (V, F)$. If

$$\sum_{A \in F} (q + 1) \cdot \left(\frac{1}{p}\right) < \frac{1}{q + 1},$$

Then Breaker has a winning strategy, even if Maker starts the game.

An orientation game is defined by a series of moves by Maker and Breaker. In every round, Maker orients $1 \leq m_t \leq p$ edges (usually in our settings $p = 1$) and Breaker orients $1 \leq b_t \leq q$ edges (usually in our settings $q = \omega(1)$). The game ends where all the edges are oriented, so the obtained graph is a tournament. Maker wins if the tournament has some predetermined property $\mathcal{P}$, otherwise Breaker wins.

We denote by $H_t$ the obtained oriented graph after $t$ rounds. Clearly, this graph has at most $(p + q) \cdot t$ edges.

Given a directed graph $G = (V, E)$, we write $(u, v) \in E$ if there is an edge from $u$ to $v$. Given a set $A \subseteq V$, we let

$$N^+(A) = \{u \in V \setminus A : \exists v \in A, (v, u) \in E\},$$

and

$$N^-(A) = \{u \in V \setminus A : \exists v \in A, (u, v) \in E\}.$$

A tournament $T$ on $n$ vertices is transitive if there is a bijection $\sigma : V(T) \to [n]$ such that for every edge $(u, v) \in E(T)$, $\sigma(u) < \sigma(v)$. A tournament $T = (V, E)$ is $k$-colorable if there is a partition of $V$ into $k$ sets $V_1, \ldots, V_k$ such that the induced tournament on each $V_i$ is transitive. Thus, a transitive tournament is 1-colorable.

## 3 The cycle game

In this section we prove Theorem 1. Namely, we show that in the $(n/2 - 2)$-biased game Maker can create a cycle. For the sake of completeness we also prove that in the $(n - 2)$-biased game Breaker can create an acyclic tournament.
**Breaker’s strategy.** Suppose that \( b \geq n - 2 \), we show that Breaker can block all cycles in the graph as follows. Whenever Maker orients an edge from \( u \) to \( v \), Breaker responds by orienting all edges from \( u \) to every vertex \( w \in V(K_n) \) such that the edge \( uw \) has not been oriented yet. Clearly, Breaker in his turn has to orient at most \( b \) edges from \( v \).

We proceed by proving that no cycle is created when Breaker applies this strategy. Indeed, suppose that a cycle \( C \) is created and let \((u, v)\) the first edge in \( C \) that was oriented (by either Maker or Breaker), and suppose also that \((w, u) \in C\). If Maker orients the edge from \( u \) to \( v \), by the strategy above Breaker responds by orienting the edge from \( u \) to \( w \), and thus \((w, u) \notin C\). If Breaker orients the edge from \( u \) to \( v \), he did it because Maker oriented some other edge from \( u \) to some vertex \( z \). In this case, again Breaker will also orient the edge from \( u \) to \( w \), and therefore again \((w, u) \notin C\). We conclude that no cycle is created.

**Maker’s strategy.**  Our main lemma states that Maker has a strategy so \( H_t \) contains a directed path of length \( t \) throughout the game.

**Lemma 3.1.** In the \( b \)-biased game, Maker has a strategy \( S_M \) such that for every \( t \leq n - 1 \), the graph \( H_t \) obtained after \( t \) rounds contains a directed path of length \( t \).

\[\text{Proof.}\] We prove by induction that assuming that there are no cycles in the graph, Maker can extend a longest path by one, no matter how Breaker plays. Clearly Maker can create a path of length 1 at the first round. Suppose that the longest path in \( E(H_t) \) is \( P_t = u_1, u_2, \ldots, u_r \), where \( r \geq t \). Let \( v \) be a vertex not in the path. Let \( k \) be the maximal index such that there is no edge from \( v \) to \( u_k \). This is well defined as if there is an edge from \( v \) to \( u_1 \) then \( v, u_1, \ldots, u_r \) is a longer path, contradicting the maximality of \( P_t \).

Observe first that if there is an edge in the opposite direction from \( u_k \) to \( v \) then \( u_1, \ldots, u_r \) is not a maximal path. Indeed, if \( k = r \) then \( u_1, \ldots, u_r, v \) is a longer path; Otherwise by the definition of \( k \) there is an edge from \( v \) to \( u_{k+1} \) and therefore \( u_1, \ldots, u_k, v, u_{k+1}, \ldots, r \) is a longer path, and in both cases this contradicts the maximality of \( P_t \).

Therefore Maker in his turn orients the edge from \( u_k \) to \( v \) and creates a path of length at least \( r + 1 \), and the result follows. \(\square\)

The strategy of Maker is as follows. At each round, if he can close a cycle he does so and wins. Otherwise, he increases the length of a longest directed path. We next show that after large enough number of rounds, Breaker cannot block all possible cycles.

**Proof of Theorem 1.** As long as Maker cannot orient an edge such that a cycle is created, Maker can extend a longest directed path by 1 by Lemma 3.1. After \( t \) rounds, there is a path \( P_t \) of length at least \( t \). Let \( V_t = V(P_t) \). There are \( \binom{t}{2} - t \) potential edges in \( G[V_t] \) such that orienting any of them creates a cycle.

Consider the graph \( H_{t-1} \) just before Maker starts round \( t \). There are \( \binom{t-1}{2} - (t - 1) \) edges that may close a cycle, of them at most \( (b+1)(t-1) - (t-1) \) were oriented in previous rounds. If \( (b+1)(t-1) - (t-1) < \binom{t-1}{2} - (t - 1) \) at the beginning of round \( t \) then Maker wins. Unless Maker wins before that, the game lasts at least \( \frac{\binom{n}{2}}{b+1} \) rounds, and therefore by taking \( t \geq \frac{\binom{n}{2}}{b+1} \) we get that if \( b \leq n/2 - 2 \) then Maker surely wins. \(\square\)
4 The Hamiltonicity game

In this game Maker wins if the obtained tournament contains a Hamilton cycle, and Breaker wins otherwise. We start with the following easy and well known lemma, whose proof is given here for completeness.

Lemma 4.1. Let $T$ be a strongly connected tournament. Then $T$ contains a Hamilton cycle.

Proof. Let $C = u_1, u_2, \ldots, u_r, u_1$ be a longest directed cycle in $T$. If $C$ is not a Hamilton cycle, there is a vertex $v \notin C$. Since $T$ is strongly connected, there is a path from $v$ to $C$ and a path from $C$ to $v$. Suppose first that $(u_i, v), (v, u_j) \in E(T)$ for some $1 \leq i \neq j \leq r$. Without loss of generality, assume that $j > i$. Since $T$ is a tournament, there is some index $i \leq k \leq j - 1$ such that $(u_k, v), (v, u_{k+1}) \in E(T)$ and hence we get a longer cycle $u_1, u_2, \ldots, u_k, v, u_{k+1}, \ldots, u_r, u_1$, a contradiction.

If there are no two indices $i, j$ such that $(u_i, v), (v, u_j) \in E(T)$, then all the edges between $v$ and $C$ are in the same direction. Suppose that for every $1 \leq i \leq r$, we have $(u_i, v) \in E(T)$ (the other case is similar). Since $T$ is strongly connected, there is a path $v, x_1, \ldots, x_t, u_i$ for some $1 \leq i \leq r$, where the vertices $x_1, \ldots, x_t$ are not in $C$. We therefore get a longer cycle $u_1, \ldots, u_{i-1}, v, x_1, \ldots, x_t, u_i, \ldots, u_r, u_1$, a contradiction. Therefore $T$ contains a Hamilton cycle, as claimed.  

We conclude that if Maker constructs a strongly connected graph from his own edges then he wins the game.

Breaker’s strategy. Assuming that the bias is sufficiently large, Breaker has a strategy to guarantee that the obtained tournament $T$ contains a vertex with in-degree 0. In this case clearly $T$ does not contain a Hamilton cycle. To this end, we reduce this problem to a box game, similarly to the treatment in [7].

Let $K_n$ be the complete graph on $n$ vertices, and consider the $b$-biased game, where $b \geq \frac{(1+o(1)n}{\ln n}$. Recall that $H_t$ is the oriented graph obtained after $t$ rounds. Fix a partition $V(K_n) = A \cup B$, where $A$ and $B$ are disjoint sets, $|A| = b, |B| = n - b$. Throughout the game, Breaker orients the edges from $A$ to $B$ until after some round $t$ there are two vertices $u, u' \in A$ such that for every vertex $w \in B$, both $(u, w), (u', w) \in H_t$, and the in-degree of both $u$ and $u'$ is 0. Then in the last turn he orients edges within $A$ so that either $u$ or $u'$ will have in-degree 0.

We have the following well-known result of Chvátal and Erdős [7].

Theorem 4. Suppose that there are $r$ disjoint sets (or boxes) $B_1, \ldots, B_r$, each box $B_i$ containing $k$ elements. At each round, Box-Maker claims $b$ elements and then Box-Breaker claims a single element. If

$$k \leq b \sum_{i=1}^{r} \frac{1}{i},$$

then Maker has a strategy to occupy all the elements of a single box.

Note that in each round, Box-Breaker destroys a single box, and so throughout the game Box-Maker tries to claim all elements of a single box before it is destroyed by Box-Breaker.

Here we need a variant of this theorem, for the case that Box-Maker actually has to complete two boxes.
Claim 4.2. Suppose that there are \( r \) disjoint sets, \( B_1, \ldots, B_r \), each \( B_i \) containing \( k \) elements. At each round, Box-Breaker destroys one set and then Box-Maker claims \( b \) elements. If
\[
k + b \leq b \sum_{i=1}^{k} \frac{1}{i},
\]
then Box-Maker has a strategy to occupy all the elements of two boxes.

Proof. For every box \( B_i \) we add a set \( B_i' \) of \( b \) virtual items. Consider a standard box game where the \( i \)'th box is \( B_i \cup B_i' \), and suppose that Box-Maker always claim the elements of \( B_i \) before he claims the elements of \( B_i' \), for every \( 1 \leq i \leq r \). If
\[
k + b \leq b \sum_{i=1}^{r} \frac{1}{i},
\]
then by Theorem 4 Box-Maker has a strategy to win the game. Consider the last round before Box-Maker wins, when the next move should be taken by Box-Breaker. Since Box-Breaker cannot avoid Box-Maker’s win there are at least two indices \( i \neq j \) such that all but at most \( b \) elements of boxes \( i \) and \( j \) are already claimed by Box-Maker. Therefore we conclude that there are at least two indices \( i \neq j \) such that \( B_i \) and \( B_j \) are claimed. We conclude that Box-Maker claimed all the elements of two of the original boxes, no matter what Box-Breaker did. The claim follows.  

In our setting, Maker and Breaker switch their roles. That is, we define the boxes so that Breaker will take Box-Maker’s role, and if he claims a box the obtained tournament has a vertex of in-degree 0. For every vertex \( v \in A \) we define a box \( X_v \) as \( \{vw : w \in B\} \). Note that \( |X_v| = |B| = n - b \). In every turn, Maker (that is, Box-Breaker) can destroy one box \( X_v \) by directing an edge towards \( v \), either from a vertex from \( A \) or from \( B \). On the other hand, Breaker (Box-Maker) can orient \( b \) edges from \( A \) to \( B \), which is equivalent to taking \( b \) elements from the various boxes. By Claim 4.2 if
\[
n = |X_v| + b \leq b \sum_{i=1}^{|A|} \frac{1}{i},
\]
then Breaker has a strategy to have two vertices \( u, u' \) from \( A \) for which all their incident edges that connect them to \( B \) are directed towards \( B \), and none of the edges from \( A \) enters \( u \) or \( u' \). Therefore, no matter what Maker does, Breaker can direct all the edges from either \( u \) or \( u' \), thus creating a vertex with in-degree 0 and destroying any chance for creating a Hamilton cycle. Taking \( b \geq \frac{n(1+o(1))}{\ln n} \) satisfies (4.1) and thus Breaker wins the game, and thus Item (i) in Theorem 2 follows.

Maker’s strategy. Maker’s strategy consists of two stages. His goal in the first stage is to create a graph with some expansion properties, so that all sufficiently small sets have at least one in-going edge and at least one out-going edge. To this end, he will create a graph with min in-degree and out-degree at least 3. We will show that with positive probability (and actually, with high probability) after this stage the graph has the desired expansion properties. Since the game considered is a perfect information game with no chance moves, we conclude that Maker has a deterministic strategy that guarantees these properties after the first stage. Moreover, the first stage lasts at most \( 8n \) rounds in any case.
At the second stage, Maker will ensure that for every large enough disjoint sets of vertices $A, B$ there is at least one edge from $A$ to $B$ and at least one edge from $B$ to $A$. We will show that if the he succeeds at the first stage then after the second stage we will have a strongly connected graph and hence by the end of the game Maker will win.

We say that a directed graph $G$ is $k$-expanding if the following holds.

- For every set $A$ of size at most $k$, $|N^+(A)|, |N^-(A)| > 0$.
- For every two disjoint sets $A, B$ of size at least $k$, there is an edge from $A$ to $B$ and there is an edge from $B$ to $A$.

We will show that after the first stage the obtained graph will have the first property with high probability, and after the second stage it will have the second property.

We have the following.

**Lemma 4.3.** Let $G$ be a directed graph, and suppose that $G$ is $k$-expanding for some $k$. Then $G$ is strongly connected.

**Proof.** Let $A_1, A_2, \ldots, A_t$ be the strongly connected components of $G$, and suppose that $t > 1$. Let $T$ be a graph where each $A_i$ is represented by a vertex, and there is an edge from $A_i$ to $A_j$ if and only if there is a vertex $v_i \in A_i$ and a vertex $v_j \in A_j$ such that $(v_i, v_j) \in E(G)$. It is well known that $T$ is a directed forest, and therefore contains a leaf, i.e., a set $A_i$ with no outgoing edges. If $|A_i| < k$ then since $|N^+(A_i)| > 0$ we get a contradiction. If $|A_i| > n - k$, then since $|N^-(V \setminus A_i)| > 0$ we get a contradiction. Finally, if $k \leq |A_i| \leq n - k$, then by the second property there is an edge from $A_i$ to $V \setminus A_i$. Therefore, we conclude that $t = 1$ and hence $G$ is strongly connected. □

More specifically, we will show that for $k = \frac{n}{(\ln n)^{2/3}}$, at the first stage Maker ensures that for every set $A$ of size at least $k$, $|N^+(A)|, |N^-(A)| > 0$, and at the second stage Maker ensures that for every two sets $A, B$ of size at least $k$, there is an edge from $A$ to $B$. By Lemma 4.3 and Lemma 4.1 after the second stage Maker wins.

**The first stage.** At the first stage we adapt the techniques of Gebauer and Szabó [9] in a way similar to [10] and show that if $b = \frac{(1-o(1))n}{\ln n}$ then Maker has a winning strategy. We start by reducing our game to an undirected game on the edges of a bipartite graph.

Suppose that Maker and Breaker play a biased orientation game on the edges of the complete graph $G = (V, E)$ on $n$ vertices, and let $V = \{v_1, v_2, \ldots, v_n\}$. Let $H = (V_1, V_2, E')$ be the complete bipartite graph on $2n$ vertices, where $V_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,n}\}$ and $V_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,n}\}$. Throughout the game we maintain two subgraphs, $H_M$ consisting of edges that are associated with Maker and $H_B$ consisting of edges that are associated with Breaker. Initially both graphs are empty.

If Breaker orients a previously undirected edge from $v_i$ to $v_j$ in $G$, we add the edge between $v_{2,i}$ and $v_{1,j}$ to $H_B$.

Maker, in his turn, would like to create a graph with a constant minimum degree in $H_M$. Whenever Maker, according to the strategy to be described below, wants to add some edge $(v_{1,i}, v_{2,j})$ to $H_M$ and $(v_{2,i}, v_{1,j})$ has not been taken yet, he does it and also orients $v_i$ to $v_j$ in $G$ (note that in this case the edge between $v_i$ and $v_j$ is undirected before this step). In this case we also add the edge $(v_{2,i}, v_{1,j})$ to $H_B$. If, on the other hand, $(v_{2,i}, v_{1,j}) \in E(H_B)$, then he adds $(v_{1,i}, v_{2,j})$ to
\(H_M\), and then plays another turn by taking a free edge according to his strategy, and adding the opposite edge to Breaker’s graph. Finally, if Maker takes an edge \((v_{1,i}, v_{2,i})\) then he plays another turn. Since the classical Maker-Breaker game is bias-monotone, if Maker takes more than one edge it can only help him. Also, note that edges from \(v_{1,j}\) to \(v_{2,i}\) are useless for Maker in the real game. Therefore Maker will have to construct a graph with minimum degree \(c + 1\) so that every vertex has at least \(c\) neighbors other than himself.

Observe also that \((v_{2,i}, v_{1,j}) \notin E(H_M)\), as otherwise \((v_{1,i}, v_{2,j})\) would be added to \(H_B\) in some previous step. The following proposition summarizes this reduction.

**Proposition 4.4.** If at some step \(H_M\) has minimum degree \(c + 1\) then at the same time every vertex in \(G\) has minimum in-degree and out-degree at least \(c\).

**Gebauer-Szabó proof.** In [9], Gebauer and Szabó provided a strategy for Maker (in the classical Maker-Breaker setting) to construct a spanning tree, a graph with positive minimum degree, and a connected graph with high minimum degree when \(b = \frac{(1+o(1))n}{\ln n}\). Here we summarize their method and highlight the slight differences between their strategy for the min-degree game and what we need in our case. We refer the reader to [9] for a complete proof. Their strategy is defined as follows. The goal of Maker is to construct a graph with min-degree \(c\). Throughout the game, a vertex \(v\) is dangerous if \(d_M(v) \leq c - 1\). Define the danger value of \(v\) as \(\text{dang}(v) = d_B(v) - 2b \cdot d_M(v)\). Initially, the danger of all vertices is 0. At every round, Maker takes a vertex \(v\) with maximum danger value (ties are broken arbitrarily), and then takes an arbitrary unclaimed edge incident to \(v\). The proof goes by assuming a Breaker’s win, and analyzing the change of danger value of the vertices for which Maker took incident edges in the game, and showing that the average danger value must be greater than 0. This in turn would lead to a contradiction.

In our case, our board consists of the edges of the complete bipartite graph \(K_{n,n}\) instead of the edges of the complete graph \(K_n\). Moreover, when Maker claims an edge, Breaker may get the opposite edge as well; We add this edge to the next move of Breaker. Therefore, Maker plays against Breaker that claims at most \((b + 1)\) edges in his turn. The danger of a vertex is defined only with respect to edges (and degrees) that belong to the bipartite graph \(K_{n,n}\), and hence at the beginning of the game the danger of every vertex is 0. The rest of the analysis is essentially the same as [9].

It was observed in [10] that Maker can achieve the minimum degree \(c\) at every vertex before Breaker claimed \((1 - \delta)n\) of its incident edges, for \(\delta = \frac{15}{(\ln n)^{1/4}}\). Also, if Maker claims an edge that is incident to a vertex \(v\), he chooses one of the edges randomly and uniformly among the free incident edges. Note this in this case Breaker gets only one new edge.

We will show that after the first stage, the obtained graph has typically some expanding properties. In our case after at most \(8n\) rounds, \(H_M\) has min-degree at least 4, which results in an oriented graph with the property that every vertex has in-degree and out-degree at least 3. Observe that this stage lasts at most \(8n\) moves as in every round Maker increases the degree of one of the vertices in \(K_{n,n}\) by at least one.

We conclude the description of this approach with the following proposition.

**Proposition 4.5.** Suppose that \(b = \frac{(1-\omega(1))n}{\ln n}\). Then Maker has a strategy to construct after at most \(8n\) turns a directed graph with min in-degree and min out-degree at least 4. Moreover, throughout the game, Maker chooses at each turn a vertex \(v\) according to his strategy, and picks a random incident edge out of a set of at least \(\delta n\) choices, where \(\delta = \frac{15}{(\ln n)^{1/4}}\).
Applying Gebauer-Szabó approach. Let $A$ be a set of vertices of size $O(\frac{n}{(\ln n)^{2/5}})$. We next prove that almost surely after the first stage $A$ has at least one ingoing edge and at least one outgoing edge. We start by claiming that almost surely every such set has at least one ingoing edge. Observe first that the property trivially holds for every set with a single vertex, as every vertex has in-degree at least one. Consider a fixed set $A$ of size $i$, and assume that $A$ has no ingoing edges, then all edges that enter $A$ have their other endpoint also in $A$, and there are at least $3i$ such edges. By Proposition 1.5 whenever Maker chooses a dangerous vertex $v$ from $A$, there are at least $\delta n$ unclaimed edges incident to $v$. Therefore, the probability that Maker chooses an edge between $v$ and another vertex of $A$ is at most $\frac{|A|-1}{\delta n}$. After the first stage there are $3i$ ingoing edges to vertices of $A$, hence the probability that $A$ does not have even a single ingoing edge from a vertex outside $A$ is at most $(\frac{|A|-1}{\delta n-1})^{3i}$. Therefore, by the union bound, the probability that there is set $A$ of size $i$ with no ingoing edge is at most

$$\binom{n}{i} \cdot \left(\frac{|A|-1}{\delta n-1}\right)^{3i} \leq \left(\frac{en}{i}\right)^i \cdot \left(\frac{2i}{\delta n}\right)^{3i} \leq \left(\frac{8ci^2}{\delta^3n^2}\right)^i.$$ 

By considering the two cases when $i \leq n^{1/3}$ and $i \geq n^{1/3}$ it is easy to check that for every $2 \leq i \leq \frac{n}{(\ln n)^{2/5}}$ and $\delta = \frac{15}{(\ln n)^{1/5}}$, the last expression is bounded by $o(1/n)$. Therefore by the union bound every set of size at most $\frac{n}{(\ln n)^{2/5}}$ has at least one ingoing edge, assuming that $n$ is sufficiently large. Essentially the same argument shows that almost surely every such set of that size contains at least one outgoing edge, as claimed.

Clearly, the first stage takes at most $8n$ rounds, so the total number of taken edges is at most $8n(b+1)$.

The second stage. Recall that at the second stage Maker has to connect in both directions every two disjoint sets $A, B$ of size $\frac{(1-o(1))n}{(\ln n)^{2/5}}$.

Consider a random tournament obtained from $K_n$ by directing each edge uniformly and independently of the other edges. For every two disjoint sets of vertices $A, B$, the number of edges from $A$ to $B$ is binomially distributed. Denote by $e(A,B)$ the number of edges from $A$ to $B$ and by $e(B,A)$ the number of edges from $B$ to $A$. By the Chernoff bound (see, e.g., [1]), we have

$$\Pr \left[ |e(A,B) - \frac{|A||B|}{2}| \geq \varepsilon |A||B| \right] \leq e^{-\varepsilon^2 |A||B|/2},$$ 

and similar inequality holds also for $e(B,A)$.

Therefore, if $|A| = |B| = \frac{(1-o(1))n}{(\ln n)^{2/5}}$, the probability that $e(A,B)$ or $e(B,A)$ is greater than $\frac{1}{2} \cdot |A||B|(1 + n^{-1/2+o(1)})$ is $2^{-2\varepsilon}$, and hence by the union bound for every two such sets, there are at least $\frac{1}{2} \cdot |A||B|(1 + n^{-1/2+o(1)}) \geq \frac{0.99n^2}{2(\ln n)^{4/5}}$ in each direction. Fix a tournament $T^*$ with this property.

At the second stage, Maker always directs edges that agree with $T^*$. That is, he can only direct an edge from $u$ to $v$ if $(u, v) \in E(T^*)$. For every two such sets, at most $8n(b+1) \leq \frac{12n^2}{\ln n}$ edges were directed at the first stage of the game, and hence at the beginning of the second stage at least $\frac{0.99n^2}{2(\ln n)^{4/5}}$ edges that are directed from $A$ to $B$ in $T^*$ are unclaimed.
Now Maker and Breaker switch roles. Maker clearly wins if he prevents Breaker from claiming all the edges from a set $A$ to a set $B$, where $|A| = |B| = k = \frac{n}{(\ln n)^{2/5}}$. To this end, we apply the Beck-Erdős-Selfridge criteria (Theorem 3), with $p = b = \frac{n(1+o(1))}{\ln n}$, $q = 1$, the size of each hyperedge is at least $0.99n^2 \cdot 2^{(\ln n)^4/5}$ and the total number of sets is at most $(\frac{n}{k})^2$. We have

\[
\sum_{A \in F} (q + 1) - \frac{|A|}{p} < \left( \frac{n}{k} \right)^2 \cdot (q + 1) - \frac{|A|}{p} < 2 \frac{n \ln q}{3(\ln n)^{4/5}} \leq 2 \frac{4n \log \log n}{(\ln n)^{4/5}} \cdot 2^{n(\log n)^{1/5}} \ll 1.
\]

Therefore in our case Maker wins and hence every two sets of size $k$ are connected in both ways. We conclude that at the end of the second stage Maker has a strongly connected graph, and hence by the end of the game the obtained tournament is strongly connected, and Maker wins. This proves Item (ii) in Theorem 2.

\[\Box\]

5 The $H$-creation game

In this game, a fixed oriented graph $H$ is given. Maker wins if the obtained tournament contains a copy of $H$, and Breaker wins otherwise. Note that if $H$ does not contain a directed cycle then Maker surely wins for large enough $n$, as every tournament of size $n$ contains a transitive tournament of size $\log n$, which contains $H$ as a subgraph.

Our starting point is an upper bound on the bias threshold for a general fixed graph $H$. Given a directed graph $H$, and a bijection $\sigma : V(H) \to |V(H)|$, we define the feedback arc set of $H$ with respect to $\sigma$ as

$$FAS(H, \sigma) = |\{(u, v) \in E(H) : \sigma(u) > \sigma(v)\}|.$$  

In words, this parameter measures the number of edges that are going in the wrong direction with respect to $\sigma$. Let

$$FAS(H) = \min_{\sigma} \{FAS(H, \sigma)\}.$$  

This is the minimal number of edges of $H$ that has to be deleted in order to make $H$ an acyclic graph. If, for example, $H$ is a random tournament on $t$ vertices, then it is easy to show that with high probability $FAS(H)$ is close to $t(t-1)/4$.

We have the following upper bound.

**Lemma 5.1.** Let $H$ be a graph on $t$ vertices, and let $r = FAS(H)$. Suppose that Maker and Breaker play an orientation game on $K_n$. Then if $b > c(H) \cdot n^{t/r}$ then Breaker has a strategy guaranteeing that the obtained tournament does not contain a copy of $H$, where $c(H) > 0$ depends only on $H$.

**Proof.** The proof follows by a simple application of the Beck-Erdős-Selfridge theorem (Theorem 3). Breaker will choose an arbitrary bijection $\sigma$ of the vertices, and at every turn he will direct the edges according to $\sigma$. That is, whenever he chooses to direct an edge $uv$, and $\sigma(v) > \sigma(u)$ the edge will be directed from $u$ to $v$. Hence, if Maker creates a copy of $H$, by definition he orients at least $FAS(H)$ edges in the opposite direction with respect to $\sigma$. We can thus reduce the game
to the classical Maker-Breaker game as follows. In every set of \(t\) vertices, Maker can win only if he claims at least \(r\) edges that are induced by this set, and Breaker wins if he prevents Maker from doing so. The total number of winning sets for Maker is at most \(\binom{n}{t}\) · \(\left(\frac{t}{r}\right)^t\). Therefore, if \((q + 1)^r = \Omega\left(\binom{n}{t} \cdot \left(\frac{t}{r}\right)^t\right)\) then by Theorem 3, Breaker has a winning strategy. This is the case if \(b > c(t, r) \cdot n^{t/r}\), and hence the lemma holds.

It is worth noting that following the methods of Bednarska and Luczak [4], one can prove that if \(b = O(n^{(\log n)/2})\) then Maker has a winning strategy as follows. Maker chooses at each round a random undirected edge and orients it randomly, independently of the other choices. Roughly speaking, one can show that by the end of the game the obtained graph looks random in some sense, and hence if the bias is large enough then with high probability it contains a copy of \(H\).

However, following their approach does not give sharp bounds in our case. To see this, observe that their results give a sharp bound of \(b = \Theta(\sqrt{n})\) for the triangle creation game in the classical Maker-Breaker settings, while in orientation games the correct bias for creating a cyclic triangle or even any directed cycle is \(b = \Theta(n)\), as we will see shortly.

We next generalize the result of Section 3 and show that in the case that \(H\) is a fixed cycle, Maker wins even if \(b = \Omega(n)\).

**Proposition 5.2.** For every constant \(k \geq 3\) there is a constant \(\gamma(k) > 0\) such that if \(b < \gamma(k) \cdot n\) then Maker wins the \(b\)-biased \(C_k\)-creation game.

**Proof.** We first observe that if a tournament \(T\) contains a cycle of length \(k + (k - 2)r\) for some \(r \in \mathbb{N}\) then it also contains a cycle of length \(k\). The proof of this observation is by induction. It is trivially true for \(r = 0\). Suppose this is true for all values smaller than some fixed \(r\), and let \(v_1, v_2, \ldots, v_{k+(k-2)r}, v_1\) a cycle of length \(k + (k - 2)r\). Consider the edge between \(v_k\) and \(v_1\). If the edge is directed from \(v_k\) to \(v_1\) there is a cycle of length \(k\) and we are done. Otherwise, \(v_k, v_{k+1}, \ldots, v_{k+(k-2)r}, v_1, v_k\) is a cycle of length \(k + (k - 2)(r - 1)\) and therefore by the induction hypothesis \(T\) contains a cycle of length \(k\), as required.

Therefore, in order to create a cycle \(C_k\), Maker has to create some cycle of length \(k + (k - 2)r\). By Lemma 3.3.1 at each round Maker can extend a longest directed path by 1. Maker’s strategy is to close a cycle of length \(k + (k - 2)r\) whenever it is possible, and to extend a longest directed path by 1 if it is not possible. After \(t\) rounds, there is a path of length \(k\), and we denote it by \(x_1, \ldots, x_t\). For every \(i \geq k\), the number of edges from \(x_i\) to \(x_j\), \(j < i\), that may close a cycle of length \(k + (k - 2)r\) is at least \(\frac{i}{k-2} - 2\). Hence the total number of edges that may close a cycle of length \(k + (k - 2)r\) for some \(r\) is at least

\[
\sum_{i=k}^{t} \left(\frac{i}{k-2} - 2\right) \geq \frac{(t + k)(t - k)}{2(k - 2)} - 2t.
\]

Therefore, the number of such edges is at least \(\frac{t}{k} \cdot \left(\frac{t}{k}\right)^t\) for \(t = \Omega(k^2)\), that is at least \((1/k)\)-fraction of the edges for such \(t\). Among these \(\frac{t}{k}\) edges, at most \((b + 1)t\) were oriented by either Maker or Breaker in previous rounds. Note that as long as the bias \(b\) is smaller than \(n/2\), the game lasts at least \(t = n\) rounds, and results in a path of length \(n - 1\), unless Maker wins before. Therefore if \(b < \frac{n - 1 - 2k}{2k}\) then Maker wins the game, as required. \(\square\)
Recall that a tournament $T$ is $k$-colorable if its edges can be partitioned into $k$ transitive tournaments. Berger et al. \[5\] studied the class of tournaments $H$ with the property that there is constant $c(H)$ such that every $H$-free tournament $T$ is $c(H)$-colorable. They called every such tournament $H$ a hero, and characterized the set of such tournaments. We next show that for every $k > 0$ Maker has a strategy to create a non $k$-colorable tournament as long as the bias is a sufficiently small linear function of $n$.

**Lemma 5.3.** Let $k > 0$, and suppose that $b = \frac{cn}{k \log k}$ for some sufficiently small constant $c > 0$. Then Maker has a strategy to create a non $k$-colorable tournament.

**Proof.** It is rather easy to see using a Chernoff bound (as it was done in Section 4) that a random tournament obtained by directing each edge uniformly and independently of the other choices has typically the following property. For every ordered pair of disjoint sets $A, B$ of size $n/2k$, there are $\Theta(\frac{n^2}{k^2})$ edges in each direction between $A$ and $B$. Fix a tournament $T^*$ with this property.

Define a hypergraph $H$ whose vertices are the edges of $T^*$ and whose edges are all the edges from $A$ to $B$ in $T^*$ for every ordered pair $A, B$ of size $n/2k$. Maker will win the game by orienting one edge from every hyperedge in $H$ according to $T^*$. To this end, Maker will play to prevent Breaker from orienting all the edges in some hyperedge from $H$. By the end of the game, there is an edge between every two sets of size $n/2k$ and hence the obtained tournament does not contain an acyclic set of size $n/k$, and therefore is not $k$-colorable.

There are $\left(\frac{n}{n/2k}\right)^2 \leq (2ekn)\frac{n}{k}$ choices of ordered pairs $(A, B)$, each corresponding to a hyperedge of $H$. The size of each hyperedge is $\Theta(\frac{n^2}{k^2})$. By applying the Beck-Erdős-Selfridge strategy (Theorem 3 with Maker playing role of Breaker, $p = b$ and $q = 1$), if $b = \frac{cn}{k \log k}$ then Maker has a winning strategy, as required. \qed

A simple consequence of Lemma 5.3 is the following generalization of Lemma 3.1. Berger et al. \[5\] provided a list of five minimal tournaments $H_1, H_2, \ldots, H_5$, and proved (Theorem 5.1 in \[5\]) that every non-hero tournament must contain at least one of $H_1, \ldots, H_5$ as a subtournament. For every $1 \leq i \leq 5$, one can check that $\text{FAS}(H_i) \geq 2$.

Consider any oriented graph $H$ with $\text{FAS}(H) = 1$. Let $\sigma$ be an ordering of $V(H)$ with a single edge that does not agree with $\sigma$. Let $H'$ be a tournament on $V(H)$ that contains $H$ as subgraph and is defined as follows. For every two vertices $u, v \in V(H)$, if $(u, v) \in E(H)$ we let $(u, v) \in E(H')$. If $(u, v), (v, u) \notin E(H)$, we let $(u, v) \in E(H')$ if $\sigma(v) > \sigma(u)$ and $(v, u) \in E(H')$ otherwise. We get that $\text{FAS}(H') = 1$ as well.

Clearly, it is sufficient to construct a copy of $H'$ for Maker’s win. The result of Berger et al. \[5\] can be applied only for tournaments, and hence we will use it to show that Maker can construct a copy of $H'$.

Since $\text{FAS}(H') = 1$ then $H'$ is a hero, and therefore every tournament that does not contain a copy of $H'$ is $c(H')$-colorable, where $c(H')$ is a constant that depends only on $H'$. By Lemma 5.3 if $b = \Theta(n)$ then Maker has a strategy so the obtained tournament is not $c(H')$-colorable. We therefore have the following.

**Proposition 5.4.** For every oriented graph $H$ with $\text{FAS}(H) = 1$ there is a constant $\gamma(H) > 0$ such that Maker wins the $\gamma n$-biased $H$ creation game.

We conjecture that the bias threshold that guarantees Maker’s win strongly depends on $\text{FAS}(H)$. It will be interesting to find further quantitative results in this direction.
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A Proof of Proposition 4.5

Here we provide the complete details of Gebauer-Szabó approach and show that Maker can win the min-degree game if $b = \frac{(1-o(1))n}{\ln n}$ where the base graph is the complete bipartite graph $K_{n,n}$. We give the proof for every min-degree $c$, though we need only the case $c = 4$.

We assume for simplicity that Breaker starts the game, this does not change the asymptotic threshold of this game. We say that the game ends when either all vertices have degree at least $c$ in Maker’s graph (and Maker won) or one vertex has degree at least $n - c + 1$ in Breaker’s graph (and Breaker won). With $\deg_M(v)$ and $\deg_B(v)$ we denote the degree of a vertex $v$ in Maker’s graph and in Breaker’s graph, respectively. A vertex $v$ is called dangerous if $\deg_M(v) \leq c - 1$. To establish Maker’s strategy we define the danger value of a vertex $v$ as $\text{dang}(v) := \deg_B(v) - 2b \cdot \deg_M(v)$.

Maker’s strategy $S_M$ Before his $i$th move Maker identifies a dangerous vertex $v_i$ with the largest danger value, ties are broken arbitrarily. Then, as his $i$th move Maker claims an edge incident to $v_i$. We refer to this step as “easing $v_i$”.

Observe that Maker can always make a move according to his strategy unless no vertex is dangerous (thus he won) or Breaker occupied at least $n - c + 1$ edges incident to a vertex (and Breaker won).

Also, a vertex $v_i$ was dangerous any time before Maker’s $i$th move.
Suppose, for a contradiction, that Breaker, playing with bias \( b \), has a strategy \( S_B \) to win the min-degree-c game against Maker who plays with bias 1. Let \( B_i \) and \( M_i \) denote the \( i \)th move of Breaker and Maker, respectively, in the game where they play against each other using their respective strategies \( S_B \) and \( S_M \). Let \( g \) be the length of this game, i.e., the maximum degree of Breaker’s graph becomes larger than \( n - c \) in move \( B_g \). We call this the end of the game.

For a set \( I \subseteq V \) of vertices we let \( \dang(I) \) denote the average danger value \( \frac{\sum_{v \in I} \dang(v)}{|I|} \) of the vertices of \( I \). When there is risk of confusion we add an index and write \( \dang_{B_i}(v) \) or \( \dang_{M_i}(v) \) to emphasize that we mean the danger-value of \( v \) directly before \( B_i \) or \( M_i \), respectively.

In his last move Breaker takes \( b \) edges to increase the maximum Breaker-degree of his graph to at least \( n - c \) (in fact, at least \( n - c + 1 \)). In order to be able to do that, directly before Breaker’s last move \( B_g \) there must be a dangerous vertex \( v_g \) whose Breaker-degree is at least \( n - c - b \). Thus \( \dang_{B_g}(v_g) \geq n - c - b - 2b(c - 1) \).

Recall that \( v_1, \ldots, v_{g-1} \) were defined during the game. For \( 0 \leq i \leq g - 1 \), we define the set \( I_i \) as \( I_i = \{ v_{g-i}, \ldots, v_g \} \).

The following lemma estimates the change in the average danger during Maker’s move.

**Lemma A.1.** Let \( i, 1 \leq i \leq g - 1 \),

(i) if \( I_i \neq I_{i-1} \), then \( \dang_{M_{g-i}}(I_i) - \dang_{B_{g-i+1}}(I_{i-1}) \geq 0 \).

(ii) if \( I_i = I_{i-1} \), then \( \dang_{M_{g-i}}(I_i) - \dang_{B_{g-i+1}}(I_{i-1}) \geq \frac{2b}{|I_i|} \).

**Proof:** For part (i), we have that \( v_{g-i} \notin I_{i-1} \). Since danger values do not increase during Maker’s move we have \( \dang_{M_{g-i}}(I_{i-1}) \geq \dang_{B_{g-i+1}}(I_{i-1}) \). Before \( M_{g-i} \) Maker selected to ease vertex \( v_{g-i} \) because its danger was highest among dangerous vertices. Since all vertices of \( I_{i-1} \) are dangerous before \( M_{g-i} \) we have that \( \dang(v_{g-i}) \geq \max(\dang(v_{g-i+1}), \ldots, \dang(v_g)) \), which implies \( \dang_{M_{g-i}}(I_i) \geq \dang_{M_{g-i}}(I_{i-1}) \). Combining the two inequalities establishes part (i).

For part (ii), we have that \( v_{g-i} \in I_{i-1} \). In \( M_{g-i} \) \( \deg_M(v_{g-i}) \) increases by 1 and \( \deg_M(v) \) does not decrease for any other \( v \in I_i \). Besides, the degrees in Breaker’s graph do not change during Maker’s move. So \( \dang(v_{g-i}) \) decreases by \( 2b \), whereas \( \dang(v) \) do not increase for any other vertex \( v \in I_i \). Hence \( \dang(I_i) \) decreases by at least \( \frac{2b}{|I_i|} \), which implies (ii). \( \square \)

The next lemma bounds the change of the danger value during Breaker’s moves.

**Lemma A.2.** Let \( i \) be an integer, \( 1 \leq i \leq g - 1 \).

(i) \( \dang_{B_{g-i}}(I_i) - \dang_{B_{g-i}}(I_i) \leq \frac{2b}{|I_i|} \).

(ii) \( \dang_{M_{g-i}}(I_i) - \dang_{B_{g-i}}(I_i) \leq \frac{b + |I_i| - 1 + a(i-1) - a(i)}{|I_i|} \), where \( a(i) \) denotes the number of edges spanned by \( I_i \) which Breaker took in the first \( g - i - 1 \) rounds.

**Proof:** Let \( e_{\text{double}} \) denote the number of those edges with both endpoints in \( I_i \) which are occupied by Breaker in \( B_{g-i} \). Then the increase of \( \sum_{v \in I_i} \deg_B(v) \) during \( B_{g-i} \) is at most \( b + e_{\text{double}} \). Since the degrees in Maker’s graph do not change during Breaker’s move the increase of \( \dang(I_i) \) (during \( B_{g-i} \)) is at most \( \frac{b + e_{\text{double}}}{|I_i|} \).

Part (i) is then immediate after noting that \( e_{\text{double}} \leq b \).

For (ii), we bound \( e_{\text{double}} \) more carefully. By definition, Breaker occupied \( a(i) \) edges spanned by \( I_i \) in his first \( g - i - 1 \) moves. So, all in all, Breaker occupied \( a(i) + e_{\text{double}} \) edges spanned by \( I_i \) in his first \( g - i \) moves. On the other hand, we know that among these edges exactly \( a(i - 1) \) are spanned by \( I_{i-1} \) and there are at most \( |I_i| - 1 \) edges in \( I_i \) incident to \( v_{g-i} \). Hence \( a(i) + e_{\text{double}} \leq a(i - 1) + |I_i| - 1 \), giving us \( e_{\text{double}} \leq |I_i| - 1 + a(i - 1) - a(i) \). \( \square \)
The following estimates for the change of average danger during one full round are immediate corollaries of the previous two lemmas.

**Corollary A.3.** Let \( i \) be an integer, \( 1 \leq i \leq g - 1 \).

(i) if \( I_i = I_{i-1} \), then \( \overline{\text{dang}}_{B_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \geq 0 \).

(ii) if \( I_i \neq I_{i-1} \), then \( \overline{\text{dang}}_{B_{g-i+1}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \geq -\frac{20}{I_{i-1}} \).

(iii) if \( I_i \neq I_{i-1} \), then \( \overline{\text{dang}}_{B_{g-i+1}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \geq -\frac{b_i |I_i| (i-1) - a(i)}{|I_i|} \), where \( a(i) \) denotes the number of edges spanned by \( I_i \) which Breaker took in the first \( g - i - 1 \) rounds.

Using Corollary A.3, we derive that before \( B_1 \), \( \overline{\text{dang}}(I_{g-1}) > 0 \), which contradicts the fact that at the beginning of the game every vertex has danger value 0.

Let \( k := \lceil \frac{n}{m} \rceil \). For the analysis, we split the game into two parts: The main game, and the end game which starts when \( |I_i| \leq k \).

Let \( |I_g| = r \). Let \( i_1 < \ldots < i_r-1 \) be those indices for which \( I_{i_j} \neq I_{i_j-1} \). Note that \( |I_{i_j}| = j + 1 \).

Observe that by definition \( a(i_{j-1}) \geq a(i_j - 1) \).

Recall that the danger value of \( v_g \) directly before \( B_g \) is at least \( n - c - b(2c - 1) \).

Assume first that \( k > r \).

\[
\begin{align*}
\overline{\text{dang}}_{B_1}(I_{g-1}) &= \overline{\text{dang}}_{B_g}(I_0) + \sum_{i=1}^{g-1} \left( \overline{\text{dang}}_{B_{g-i}}(I_i) - \overline{\text{dang}}_{B_{g-i+1}}(I_{i-1}) \right) \\
&\geq \overline{\text{dang}}_{B_g}(I_0) + \sum_{j=1}^{r-1} \left( \overline{\text{dang}}_{B_{g-j}}(I_{i_j}) - \overline{\text{dang}}_{B_{g-j+1}}(I_{i_j-1}) \right) \quad \text{[by Corollary A.3 (i)]} \\
&\geq \overline{\text{dang}}_{B_g}(I_0) - \sum_{j=1}^{r-1} \frac{b + j + a(i_j - 1) - a(i_j)}{j + 1} \quad \text{[by Corollary A.3 (iii)]} \\
&\geq \overline{\text{dang}}_{B_g}(I_0) - bH_r - r - \frac{a(0)}{2} + \sum_{j=2}^{r-1} \frac{a(i_j-1)}{j + 1} + \frac{a(r-1)}{r} \quad \text{[since } a(i_{j-1}) \geq a(i_j - 1) \text{]} \\
&\geq \overline{\text{dang}}_{B_g}(I_0) - bH_r - k \quad \text{[since } a(0) = 0 \text{ and } r \leq k \text{]} \\
&\geq n - c - b(2c + \ln k) - k \\
&\geq n - \frac{n}{\ln n} (2c + \ln n - \ln \ln n) - \frac{n}{\ln n} - c \quad \text{[since } b \leq \frac{n}{\ln n} \text{]} \\
&\geq \frac{n \ln \ln n}{3 \ln n} - \frac{n}{\ln n} - c \\
&> 0. \quad \text{[for large } n \text{]} \quad \text{(A.1)}
\end{align*}
\]
Assume now that $k \leq r$.

\[
\overline{\text{dan}_B(I_{g-1})} = \overline{\text{dan}_B(I_0)} + \sum_{i=1}^{g-1} \left( \overline{\text{dan}_{B_g}(I_i)} - \overline{\text{dan}_{B_{g+1}}(I_{i-1})} \right)
\]

\[
\geq \overline{\text{dan}_B(I_0)} + \sum_{j=1}^{r-1} \left( \overline{\text{dan}_{B_{g-j}}(I_{i_j})} - \overline{\text{dan}_{B_{g+1-j}}(I_{i_{j-1}})} \right) \quad \text{[by Corollary A.3(i)]}
\]

\[
= \overline{\text{dan}_B(I_0)} + \sum_{j=1}^{k-1} \left( \overline{\text{dan}_{B_{g-j}}(I_{i_j})} - \overline{\text{dan}_{B_{g+1-j}}(I_{i_{j-1}})} \right)
\]

\[
\geq \overline{\text{dan}_B(I_0)} - \sum_{j=1}^{k-1} \frac{b + j + a(i_j - 1) - a(i_j)}{j + 1} - \sum_{j=k}^{r-1} \frac{2b}{j + 1} \quad \text{[by Corollary A.3(iii) and (ii)]}
\]

\[
\geq \overline{\text{dan}_B(I_0)} - b(2H_r - H_k) - k - \frac{a(0)}{2} + \sum_{j=2}^{k-1} \frac{a(i_{j-1})}{(j+1)j} + \frac{a(i_{k-1})}{k}
\]

\[
\geq n - c - b(2c - 1 + 2H_{2n} - H_k) - k \quad \text{[since } 2n \geq r \text{ and } a(0) = 0]\]

\[
\geq n - c - \left( \frac{n}{\ln n} - \frac{n \ln \ln n}{\ln^2 n} - (2c + 3) \frac{n}{\ln^2 n} \right) (\ln n + \ln \ln n + 2c + 2) - \frac{n}{\ln n}
\]

\[
\geq \frac{n(\ln \ln n)^2}{\ln^2 n} \quad [\text{for } n \text{ large enough}]
\]

\[
> 0. \quad (A.2)
\]

Observe that in our proof we need Maker to have min-degree $c$ for every vertex $v$ before Breaker claims $(1 - \delta)n$ edges incident to $v$ (for $\delta = O(\frac{1}{(\ln n)^{3/4}})$). The same analysis essentially holds, with the following differences. Assume that Breaker wins, then before his last move the vertex $v$ has degree $(1 - \delta)n - c - 1$ (instead of $n - c - 1$). All other calculations are essentially the same by taking $b = \frac{n}{\ln n} (1 - \frac{1}{\ln \ln n})$. 

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