An approximation theorem for non-decreasing functions on compact posets

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Abstract

In this short note we prove a theorem of the Stone-Weierstrass sort for subsets of the cone of non-decreasing continuous functions on compact partially ordered sets.

1 Introduction

The classic book [1] contains a theorem which states that given a compact set $M$ and a separating semi-vector lattice $S$ of continuous real-valued functions on $M$ which contains the constants, there is one and only one way of making $M$ a compact ordered space so that $S$ becomes the set of all non-decreasing continuous real-valued functions on $M$. This theorem has been used in [2] to give a putative definition of noncommutative compact ordered sets. However, infimum and supremum turned out to be quite difficult to handle in the noncommutative setting. A different kind of density theorem was thus needed. Since a “continuous non-decreasing” functional calculus was available in the noncommutative context, it was natural to look for a theorem which would replace stability under infimum and supremum with stability under continuous non-decreasing functions.

Let us introduce some vocabulary in order to be more precise. Let $S$ be a subset of the set $C(M, \mathbb{R})$ of continuous real-valued functions on some space $M$, and $H : \mathbb{R} \to \mathbb{R}$ be a function. We will say that $H$ operates on $S$ is $H \circ f \in S$ for each $f \in S$.

As remarked in [3], the real version of the classical Stone-Weierstrass theorem can be rephrased in terms of operating functions.

**Theorem 1.1** (Stone-Weierstrass) Let $S$ be a non-empty subset of $C(X, \mathbb{R})$, with $X$ a compact Hausdorff space. If

1. $S$ is stable by sum,

2. the affine functions from $\mathbb{R}$ to $\mathbb{R}$ operate on $S$, 

then $S$ is dense in $C(X, \mathbb{R})$. 


3. the function \( t \mapsto t^2 \) operates on \( S \),

then \( S \) is dense in \( C(X, \mathbb{R}) \) for the uniform norm.

The second hypothesis is a way to say that \( S \) is a cone (hence a vector space thanks to first hypothesis) which contains the constant functions.

In fact it is proved in [4] that one can replace \( t \mapsto t^2 \) in the third hypothesis by any continuous non-affine function.

It is a theorem of this kind that we prove in this note, but in the same category (compact ordered sets and non-decreasing continuous functions) as the theorem of Nachbin stated above.

2 Preliminaries

Let \( M \) be a topological set equipped with a partial order \( \preceq \). We let \( I(M) \) denote the set of all continuous non-decreasing functions from \( M \) to \( \mathbb{R} \), where \( \mathbb{R} \) has the natural topology and the natural ordering, which we write \( \leq \), as usual. The elements of \( I(M) \) are sometimes called continuous isotonies.

Let \( S \) be a subset of \( I(M) \). We define the relation \( \preceq_S \) by

\[
x \preceq_S y \iff \forall f \in S, f(x) \leq f(y)
\]  

It is obvious that \( \preceq_S \) is a preorder, which we call the preorder generated by \( S \). This preorder will be a partial order relation if, and only if, \( S \) separates the points of \( M \).

We say that \( S \) generates \( \preceq \) iff \( \preceq_S = \preceq \). This is the case if, and only if, \( S \) satisfies

\[
\forall a, b \in M, \ a \nprec b \implies \exists f \in S, \ f(a) > f(b)
\]

Since \( a \nprec b \Rightarrow a \nprec b \) or \( b \nprec a \), we see that if \( S \) generates \( \preceq \), it necessarily separates the points of \( M \).

Note that it is not guaranteed that for any poset there exists such an \( S \) generating the order. When there is one, then \( I(M) \) itself will also generate the order. Posets with the property that \( I(M) \) generates the order are called completely separated ordered sets. When \( M \) is compact and Hausdorff, complete separation is equivalent to the relation \( \preceq \) being closed in \( M \times M \) (see [I], p. 114).

Let \( A \) be a set of functions from \( \mathbb{R} \) to \( \mathbb{R} \). We will say that \( A \) operates on \( S \) iff

\[
\forall H \in A, \forall f \in S, \ H \circ f \in S
\]
3 Statement and proof of the theorem

**Theorem 3.1** Let $\langle M, \preceq \rangle$ be a compact Hausdorff partially ordered set. Let $A$ be the set of continuous non-decreasing piecewise linear functions from $\mathbb{R}$ to $\mathbb{R}$. Let $S$ be a non empty subset of $I(M)$. If
1. $S$ is stable by sum,
2. $A$ operates on $S$,
3. $S$ generates $\preceq$.
then $S$ is dense in $I(M)$ for the uniform norm.

Before proving the theorem, a few comments are in order.
- First of all, the theorem is true but empty if $M$ is not completely separated, since no $S$ can satisfy the hypotheses in this case.
- The hypothesis that $S$ is not empty is redundant if $M$ has at least two elements, by 3.
- Finally, let us remark that 2 entails that $S$ is in fact a convex cone which contains the constant functions.

To prove the theorem we need two lemmas.

**Lemma 3.2** Let $x, y \in M$ be such that $y \not\preceq x$. Then $\exists f_{x,y} \in S$ such that $0 \leq f_{x,y} \leq 1$, $f_{x,y}(x) = 0$ and $f_{x,y}(y) = 1$.

**Proof:** Since $S$ generates $\preceq$, there exists $f \in S$ such that $f(x) < f(y)$. Let $H \in A$ be such that $H(t) = 0$ for $t \leq f(x)$, $H$ is affine on the segment $[f(x), f(y)]$, and $H(t) = 1$ for $t \geq f(y)$. Then $f_{x,y} := H \circ f$ meets the requirements of the lemma.

**Lemma 3.3** Let $K, L$ be two compact subsets of $M$ such that $\forall x \in K, \forall y \in L, y \not\preceq x$. Then $\exists f_{K,L} \in S$ such that $0 \leq f_{K,L} \leq 1$, $f = 0$ on $K$ and $f = 1$ on $L$.

**Proof:** For all $x \in K$ and $y \in L$, we find a $f_{x,y} \in S$ as in lemma 3.2. We fix a $y \in L$ and let $x$ vary in $K$. Since $f_{x,y}$ is continuous, there exists an open neighbourhood $V_x$ of $x$ such that $f_{x,y}(V_x) \subset [0;1/4[$. By compactness of $K$, there exists $V_1, \ldots, V_k$ corresponding to $x_1, \ldots, x_k$ such that $K \subset V_1 \cup \ldots \cup V_k$.

Now we define $g_y := \frac{1}{k} \sum_i f_{x_i,y}$. We have $g_y \in S$ since $S$ is a convex cone (see the last remark below the theorem). It is clear that $g_y(y) = 1$ and that for all $x \in K$, $0 \leq g_y(x) \leq \frac{1}{k}(k-1+1/4) = 1- \frac{1}{4k} < 1$. We then choose $H \in A$ such that $H(t) = 0$ for $t \leq 1- \frac{1}{4k}$ and $H(t) = 1$ for $t \geq 1$. We set $f_{K,y} := H \circ g_y$. We thus have $f_{K,y} \in S$, $f_{K,y} = 0$ on $K$ and $f_{K,y}(y) = 1$.

Using the continuity of $f_{K,y}$, we find an open neighbourhood $W_y$ of $y$ such that $f_{K,y}(W_y) \subset [3/4;1]$. Since we can do this for every $y \in$...
L, and since L is compact, we can find functions \( f_{K,y_j} \), \( j = 1, \ldots, l \), and open sets \( W_1, \ldots, W_l \) of the above kind such that \( L \subset W_1 \cup \ldots \cup W_l \).

We then define \( g = \frac{1}{l} \sum_j f_{K,y_j} \). We have \( g \in S \), and \( g(K) = \{0\} \).

Moreover, for all \( z \in L, 1 \geq g(z) \geq \frac{1}{l^j} > 0 \). We then choose a function \( G \in A \) such that \( G(t) = 1 \) for \( t \geq 3/4l \) and \( G(t) = 0 \) for \( t \leq 0 \). Now the function \( f_{K,L} := G \circ g \) has the desired properties. 

We can now prove the theorem.

**Proof:** Let \( f \in I(M) \). We will show that, for all \( n \in \mathbb{N}^* \) there exists \( F \in S \) such that \( \|f - F\|_\infty \leq \frac{1}{n} \).

If \( f \) is constant then the result is obvious. Else, let \( m \) be the infimum of \( f \) and \( M \) be its supremum. Let \( \hat{f} = \frac{1}{M - m}(f - m.1) \).

Using the fact that \( S \) is a convex cone, we can work with \( \hat{f} \) instead of \( f \). Hence, we can suppose that \( f(M) = [0;1] \) without loss of generality.

We set \( K_i = f^{-1}([0;\frac{1}{i}]) \), and \( L_i = f^{-1}([\frac{1}{i};1]) \) for each \( i \in \{0;\ldots; n-1\} \). Since \( f \) is continuous and \( M \) is compact, the sets \( K_i \) and \( L_i \) are both closed, hence compact.

For each \( i \) we use lemma 3.3 to find \( f_i \in S \) such that \( f_i(K_i) = \{0\} \) and \( f_i(L_i) = \{1\} \).

We then consider the function \( F = \frac{1}{n} \sum_{i=0}^{n-1} f_i \). We clearly have \( F \in S \).

Let \( m \in M \). Suppose \( \frac{1}{n} < f(m) < \frac{i+1}{n} \) for some \( j \in \{0;\ldots; n-1\} \). We thus have \( m \in K_i \) for \( j < i < n \) and \( m \in L_i \) for \( i < j \).

Hence \( F(m) = \frac{1}{n} \sum_{i=0}^{n-1} f_i(m) = \frac{1}{n}(j + f_j(m)) \in [\frac{1}{n}; \frac{j+1}{n}] \). Thus \( |f(m) - F(m)| \leq \frac{1}{n} \).

Now suppose \( f(m) = \frac{1}{n} \), with \( j \in \{0;\ldots; n\} \). We have \( m \in K_i \) for \( i \geq j \) and \( m \in L_i \) for \( i < j \). Thus \( F(m) \leq \frac{1}{n} \sum_{i=0}^{j-1} f_i(m) = \frac{1}{n} \).

We see that \( |f(m) - F(m)| = 0 \) in this case.

Hence we have shown that \( |f(m) - F(m)| \leq \frac{1}{n} \) for all \( m \in M \), thus proving the theorem.

To conclude, let us remark that the set \( A \) in the theorem can be replaced by any subset of \( I(M) \) with the following property: for any two reals \( a < b \), there exists \( f \in A \) such that \( f = 0 \) on \( ]-\infty; a] \), and \( f = 1 \) on \( ]b; +\infty[ \). For example one can take \( A = I(M) \) itself, or the set of non-decreasing \( C^\infty \) functions.

**References**

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