Necessary and sufficient conditions for regularity of interval parametric matrices

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Abstract
Matrix regularity is a key to various problems in applied mathematics. The sufficient conditions, usually used for checking regularity of interval parametric matrices, fail in case of large parameter intervals. We present necessary and sufficient conditions for regularity of interval parametric matrices in terms of boundary parametric hypersurfaces, parametric solution sets, determinants, real spectral radiiuses. The initial $n$-dimensional problem involving $K$ interval parameters is replaced by numerous problems involving $1 \leq t \leq \min\{n - 1, K\}$ interval parameters, in particular $t = 1$ is most attractive. The advantages of the proposed methodology are discussed along with its application for finding the interval hull solution to interval parametric linear system and for determining the regularity radius of an interval parametric matrix.

Keywords: interval matrix, dependent data, regularity, singularity, necessary and sufficient conditions, solution enclosure, radius of regularity. 2000 MSC: 65G40, 15A06, 15B99

1. Introduction
Contrary to the nonparametric case [1], necessary and sufficient conditions for regularity of interval parametric matrices have not been studied in much details. Regularity of the latter matrices has been studied mainly via some sufficient conditions [2], [3], [4] as part of the methods for bounding the solution set of an interval parametric linear system. Checking several
properties of interval matrices, cf. [5], like positive definiteness, \( P \)-property, stability, can be reduced to checking regularity. These properties have various other useful applications [6], [7], [8]. Some matrix properties are studied for interval parametric matrices. Stability of symmetric interval matrices is investigated in [9]. Stability in the general case of parameter dependencies is investigated in [10]. In [11] a sufficient condition for checking strong positive definiteness of an interval parametric matrix employs regularity of the matrix. In all works, regularity of an interval parametric matrix is estimated by a sufficient condition which may fail in case of large parameter intervals.

In [1] J. Rohn mentions that “regularity of interval matrices is worth further study”. This is done in the present work where we present some necessary and sufficient conditions for regularity of interval parametric matrices. This property is formulated in several equivalent forms: boundary parametric hypersurfaces, parametric solution sets, determinants, real spectral radiuses. A key approach to the presented conditions is a transformation of the initial problem depending on \( K \) interval parameters into a set of \( K2^{K-1} \) problems depending on only one interval parameter. This transformation is based on a set of one-parameter hypersurfaces, which contains the boundary of the solution set of an interval parametric linear system.

The article starts by a Notation section and some basic definitions for interval parametric matrices. The methodology and the necessary and sufficient conditions are presented in Section 3. Sections 4 and 5 discuss important applications of the regularity: (i) computing the exact interval hull of a parametric united solution set; (ii) radius of regularity of an interval parametric matrix. Numerical examples illustrate the methodology and its applications. The article ends by a Conclusion section which discusses the advantages of the presented necessary and sufficient conditions.

2. Notation

Denote by \( \mathbb{R}^{m \times n} \) the set of real \( m \times n \) matrices. Vectors are considered as one-column matrices. The inequalities are understood componentwise. The identity matrix of appropriate dimension is denoted by \( I \). The componentwise Hadamard product is denoted by \( \odot \). The vector of all ones is \( e = (1, \ldots, 1)^T \). The \( n \)-dimensional discrete cube \( Q_n := \{ y \in \mathbb{R}^n \mid \| y \| = e \} \) is the set of all \( \pm 1 \)-vectors in \( \mathbb{R}^n \) and its cardinality is \( 2^n \). For each \( y \in \mathbb{R}^n \) we denote the diagonal matrix with diagonal vector \( y \) and zero off-diagonal
elements by
\[ D_y = \text{diag}(y_1, \ldots, y_n). \]

We denote the determinant of a matrix \( A \in \mathbb{R}^{n \times n} \) by \( \det(A) \). The spectral radius of a matrix \( A \in \mathbb{R}^{n \times n} \) is denoted by \( \rho(A) \). The maximal magnitude real spectral radius (eigenvalue) of \( A \) is denoted by
\[ \rho_0(A) := \max\{|\lambda| \mid Ax = \lambda x \text{ for some } x \neq 0, \lambda \text{ real}\} \]
and we set \( \rho_0(A) = 0 \) if \( A \) has no real eigenvalue.

A real compact interval is \( a = [a^-, a^+] := \{a \in \mathbb{R} \mid a^- \leq a \leq a^+\} \) and \( \mathbb{I} \mathbb{R}^{m \times n} \) denotes the set of interval \( m \times n \) matrices. For \( a = [a^-, a^+] \), define its mid-point \( \bar{a} := (a^- + a^+)/2 \), the radius \( \hat{a} := (a^+ - a^-)/2 \). These functions are applied to interval vectors and matrices componentwise.

Let \( A(p) \) be a real \( m \times n \) parametric matrix whose elements \( a_{ij}(p), i = 1, \ldots, m, j = 1, \ldots, n \), are given functions of a number of real parameters \( p = (p_1, \ldots, p_K)^\top \).

**Definition 1.** For given real functional dependencies \( A(p) = (a_{ij}(p)) \), \( i = 1, \ldots, m, j = 1, \ldots, n \), and a given interval vector \( p \in \mathbb{I} \mathbb{R}^K \), such that \( \hat{p}_k > 0 \) for each \( 1 \leq k \leq K \), the following set of real matrices
\[ \{A(p), p\} := \{A(p) \mid \exists p \in p\} \quad (1) \]
is called an \( m \times n \) interval parametric matrix.

An interval parametric matrix is not an interval matrix and \( \{A(p), p\} \) is a short notation for a family of real matrices specified by the functional dependencies \( A(p) \) and the parameter intervals \( p \). The couple \( A(p), p \in p \), will be used as an equivalent notation for an interval parametric matrix.

In this work we consider parameter dependencies defined by affine-linear functions. Namely,
\[ A(p) = A_0 + \sum_{k=1}^{K} p_k A_k, \quad p \in p \in \mathbb{I} \mathbb{R}^K, \quad (2) \]
with prescribed numerical matrices \( A_k, k = 0, \ldots, K \) and the parameters \( p = (p_1, \ldots, p_K)^\top \) are considered to be uncertain and varying within given
non-degenerate\footnote{An interval \(a = [a^-, a^+]\) is degenerate if \(a^- = a^+\).} intervals \(p = (p_1, \ldots, p_K)^\top\). Nonlinear dependencies between interval valued parameters in interval parametric matrices are usually linearized to the form (2) and methods for the latter are applied.

Nonparametric interval matrices \(A \in \mathbb{IR}^{m \times n}\) can be considered as a special class of parametric interval matrices. Namely, \(A \in \mathbb{IR}^{m \times n}\) can be considered as a parametric interval matrix involving \(m \times n\) interval parameters \(a_{ij} \in a_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\). Thus, the following Definition 2 comprises both the parametric and nonparametric interval matrices. In this article we consider square parametric matrices.

\textbf{Definition 2.} A square interval parametric matrix \(A(p), p \in p\), is called regular if \(A(p)\) is regular for each \(p \in p\).

\(\{A(p), p\}\) is said singular if Definition 2 is not satisfied, i.e., if \(A(p)\) is singular for some \(p \in p\).

3. Necessary and sufficient conditions for regularity

Let \(K = \{1, \ldots, K\}\). Define \(K(m)\) as the set of all possible subsets of \(K\) having \(m = \min\{n - 1, K\}\) elements

\[K(m) := \{q = \{i_1, \ldots, i_m\} \mid q \subseteq K\}.\]

For a vector \(p = (p_1, \ldots, p_K)^\top \in \mathbb{IR}^K\) and a \(q = \{i_1, \ldots, i_m\}, m < K\), define \(\bar{q} := K \setminus q\) and two vectors \(p_q \in \mathbb{IR}^m, p_{\bar{q}} \in \mathbb{IR}^{K-m}\) by

\[p_q := (p_{i_1}, \ldots, p_{i_m})^\top,\]
\[p_{\bar{q}} := (p_{i_{m+1}}, \ldots, p_{i_K})^\top.\]

For completeness of the exposition, we first present \cite{12, Theorem 4.2], proven here under a more general condition. Let

\[\Sigma(A(p), b(p), p) := \{x \in \mathbb{IR}^n \mid A(p)x = b(p)\ \text{for some} \ p \in p\}\]

be the parametric united solution set of an interval parametric system \(A(p)x = b(p), p \in p \in \mathbb{IR}^K\), involving affine-linear dependencies, and \(\partial \Sigma(A(p), b(p), p)\) denote the boundary of the solution set.
Theorem 1. If $A(\hat{p})$ is nonsingular, then

$$\partial \Sigma (A(p), b(p), p) \subseteq \bigcup_{q \in \mathcal{K}(m)} \bigcup_{\varepsilon \in Q_{K-m}} x(p_q, \varepsilon \circ \hat{p}_q)|_{[-\hat{p}_q, \hat{p}_q]} \subseteq \Sigma (A(p), b, p),$$

where the parametric hypersurfaces

$$x(p_q, \varepsilon \circ \hat{p}_q) = \left( A(\hat{p}) + \sum_{i \in q} p_i A_i + \sum_{j \in K \setminus q} \varepsilon_j \hat{p}_j A_j \right)^{-1} \left( b(\hat{p}) + \sum_{i \in q} p_i b_i + \sum_{j \in K \setminus q} \varepsilon_j \hat{p}_j b_j \right)$$

are restricted to the interval vector $[-\hat{p}_q, \hat{p}_q]$.

Proof. Nonsingularity of $\hat{A}$ implies $\ker(A(p)) = \{0\}$, where the kernel (or null space) of a matrix $A(p) \in \mathbb{R}^{m \times m}$ is

$$\ker(A(p)) := \{ x \in \mathbb{R}^m \mid A(p)x = 0 \}.$$

Therefore the symbolic matrix $A(p)$ is invertible and $x(p) = (A(p))^{-1} b(p)$ has explicit representation. Then the proof goes the same way as in [12, Theorem 4.1 and Theorem 4.2].

Theorem 2. For an $n \times n$ interval parametric matrix $A(p), p \in \mathbb{R}^n$, the following conditions are equivalent:

(i) $\{A(p), p\}$ is regular,

(a) each interval parametric matrix $\{A(p_q, \varepsilon \circ \hat{p}_q), [-\hat{p}_q, \hat{p}_q]\}, q \in \mathcal{K}(m), \varepsilon \in Q_{K-m}$, of the form

$$A(p_q, \varepsilon \circ \hat{p}_q) = A(\hat{p}) + \sum_{i \in q} p_i A_i + \sum_{j \in q} \varepsilon_j \hat{p}_j A_j, \quad p_q \in [-\hat{p}_q, \hat{p}_q], \quad q = K \setminus q, \quad (3)$$

is regular,

(b) for a vector $b(p) \in \mathbb{R}^n$ and for each $q \in \mathcal{K}(m), \varepsilon \in Q_{K-m}$, defining $A(p_q, \varepsilon \circ \hat{p}_q)$ in (3) and $b(p_q, \varepsilon \circ \hat{p}_q) = b(\hat{p}) + \sum_{i \in q} p_i b_i + \sum_{j \in q} \varepsilon_j \hat{p}_j b_j$, the solution set

$$\Sigma (A(p_q, \varepsilon \circ \hat{p}_q), b(p_q, \varepsilon \circ \hat{p}_q), [-\hat{p}_q, \hat{p}_q]) := \{ A(p_q, \varepsilon \circ \hat{p}_q)x = b(p_q, \varepsilon \circ \hat{p}_q) \mid \exists p_q \in [-\hat{p}_q, \hat{p}_q] \},$$

is bounded.
Proof. (i)⇒(a) by Definition 2.
(a)⇔(b) is obvious.
(b)⇒(i) Proof by contradiction: assume that \{A(p), p\} is singular. This implies that for any \(b(p) \in \mathbb{R}^n\), some boundary hypersurfaces defining the boundary \(\partial \Sigma (A(p), b(p), p)\) are unbounded. On the other hand, (a) implies nonsingularity of \(\hat{A}\) and, by Theorem 1,
\[
\partial \Sigma (A(p), b(p), p) \subseteq \bigcup_{q \in \mathcal{K}(m)} \bigcup_{\varepsilon \in Q_{K-m}} x(p_q, \varepsilon \circ \hat{p}_q)\big|_{[-\hat{p}_q, \hat{p}_q]}.
\]
By (b), the set in the right-hand side above contains only bounded restricted parametric hypersurfaces. Thus, we have a contradiction between the assumption and (b). \(\square\)

If \(K \geq n\), Theorem 2 reduces the general regularity problem to \(\binom{K}{n-1}2^{K-n+1}\) regularity problems involving \(n-1\) interval parameters. The following theorem presents equivalent necessary and sufficient conditions for regularity of an interval parametric matrix by \(K2^{K-1}\) parametric problems involving only one interval parameter \(q \in \mathcal{K}(1)\). It should be noted, however, that with obvious modifications Theorem 3 below holds true and can be applied for any \(t, 1 \leq t \leq m\), and \(q \in \mathcal{K}(t)\).

**Theorem 3.** For an \(n \times n\) interval parametric matrix \(A(p), p \in \mathbb{I}^K\), the following conditions are equivalent:

(i) \(\{A(p), p\}\) is regular,

(ii) each interval parametric matrix \(\{A(p_k, \varepsilon \circ \hat{p}_q), [-\hat{p}_k, \hat{p}_k]\}, k \in \mathcal{K}, \varepsilon \in Q_{K-1}, \tilde{q} = \mathcal{K} \setminus \{k\}\), of the form
\[
A(p_k, \varepsilon \circ \hat{p}_q) = A(\hat{p}) + p_k A_k + \sum_{i=1, i \neq k}^K \varepsilon_i \hat{p}_i A_i, \quad p_k \in [-\hat{p}_k, \hat{p}_k], \quad (4)
\]
is regular,

(iii) for a vector \(b(p) \in \mathbb{R}^n\) and each \(k \in \mathcal{K}, \varepsilon \in Q_{K-1}\), the solution set
\[
\Sigma (A(p_k, \varepsilon \circ \hat{p}_q), b(p_k, \varepsilon \circ \hat{p}_q), [-\hat{p}_k, \hat{p}_k]) := \left\{ \left( A(\hat{p}) + p_k A_k + \sum_{i=1, i \neq k}^K \varepsilon_i \hat{p}_i A_i \right) x = b(p_k, \varepsilon \circ \hat{p}_q) \mid \exists p_k \in [-\hat{p}_k, \hat{p}_k] \right\}
\]
is bounded.
(iv) Each numerical matrix of the form

\[ A(\varepsilon^{(k)}) = A(\hat{p}) - \sum_{i=1}^{K} \varepsilon_i^{(k)} \hat{p}_i A_i, \]

where \( 0 \leq |\varepsilon_k^{(k)}| \leq 1, \varepsilon_i^{(k)} \in \{-1, 1\} \) for \( i = 1, \ldots, K, i \neq k, \) and \( k \in \mathcal{K}, \) is nonsingular,

(v) \( \det(A(p_k, \varepsilon \circ \hat{p}_q)) \neq 0 \) for each \( p_k \in [-\hat{p}_k, \hat{p}_k], k \in \mathcal{K}, \varepsilon \in \mathcal{Q}_{K-1}, \) where \( A(p_k, \varepsilon \circ \hat{p}_q) \) is defined in (4).

(vi) \( \tilde{A} \) is nonsingular and

\[ \rho_0(B(\varepsilon_k, \bar{\varepsilon})) < 1, \quad B(\varepsilon_k, \bar{\varepsilon}) := \varepsilon_k \hat{p}_k \tilde{A}^{-1} A_k + \sum_{i=1, i \neq k}^{K} \bar{\varepsilon}_i \hat{p}_i \tilde{A}^{-1} A_i, \]

for each \( k \in \mathcal{K}, 0 \leq |\varepsilon_k| \leq 1, \bar{\varepsilon} \in \mathcal{Q}_{K-1}. \)

Proof. (i)⇒(ii) by Definition 2.

(ii)⇔(iii) is obvious.

The proof (iii)⇒(i) goes similarly to the proof (b)⇒(i) in Theorem 2, since each two components of \( x(p_k, \varepsilon \circ \hat{p}_q) = A^{-1}(p_k, \varepsilon \circ \hat{p}_q)b(p_k, \varepsilon \circ \hat{p}_q) \) restricted to \([-\hat{p}_k, \hat{p}_k]\) present 2-dimensional projections of either some boundary hypersurfaces of \( \Sigma(A(p), b(p), p) \) or a subset of the solution set.

The equivalence between (ii), (iv) and (v) is obvious.

(i)⇒(vi) by contradiction. Assume that

\[ B(\varepsilon_k, \bar{\varepsilon})x = \lambda x \]

for some \( x \neq 0, |\lambda| \geq 1, k = 1, \ldots, K, 0 \leq |\varepsilon_k| \leq 1, \bar{\varepsilon} \in \mathcal{Q}_{K-1}. \) Then

\[ \left( I - \frac{1}{\lambda} B(\varepsilon_k, \bar{\varepsilon}) \right) x = 0. \]

Hence \( I - \frac{1}{\lambda} B(\varepsilon_k, \bar{\varepsilon}) \) is singular. Since \( \delta_k = \varepsilon_k / \lambda, |\delta_k| \leq 1 \) and \( \bar{\delta} = \bar{\varepsilon} / \lambda, |\bar{\delta}| \leq 1, \)

\[ I - B(\delta_k, \bar{\delta}) \in \left\{ I - \sum_{i=1}^{K} p_i \tilde{A}^{-1} A_i, [-\hat{p}, \hat{p}] \right\}. \]
The latter means that a singular matrix belongs to a set of nonsingular ones, which is a contradiction.

(vi)⇒(i) Assume that \{A(p), \mathcal{P}\} is singular and \(A = A(\hat{p})\) is nonsingular. By (i)⇔(v)

\[ \det \left( I - \hat{p}_k \hat{A}^{-1} A_k - \sum_{i=1, i \neq k}^K \varepsilon_i^{(k)} \hat{p}_i \hat{A}^{-1} A_i \right) = 0 \]  

(5)

for some \(k \in \mathcal{K}, \hat{p}_k \in [-\hat{p}_k, \hat{p}_k], \varepsilon^{(k)} \in Q_{K-1}\). This implies that the maximum magnitude real spectral radius of each matrix \(\hat{p}_k \hat{A}^{-1} A_k + \sum_{i=1, i \neq k}^K \varepsilon_i^{(k)} \hat{p}_i \hat{A}^{-1} A_i\), which satisfies (5), is equal to 1. The latter contradicts to (vi).

All the conditions of Theorem 3, except (iv) which is numerical equivalent of (ii) and (vi) which is numerical equivalent of the corresponding parametric problem, are in terms of parametric problems involving one interval parameter. By the following example we illustrate one of the advantages of considering parametric problems involving only one interval parameter (instead of more) in proving regularity/singularity of an interval parametric matrix. Some other advantages are discussed later on.

Example 1. Consider the interval parametric matrix

\[
\begin{pmatrix}
1 + p_1 & p_3 & -1 \\
p_2 & 1 + p_1 & p_3 \\
-1 & p_2 & p_1/3
\end{pmatrix}, \quad p_1 \in [-\frac{3}{4}, \frac{3}{4}],
\]

\[
p_2, p_3 \in [-\frac{1}{2}, \frac{1}{2}].
\]

The sufficient condition [4]

\[ \rho \left( \sum_{i=1}^K \hat{p}_k |A^{-1}(\hat{p})A_i| \right) < 1 \]  

(6)

for regularity of an interval parametric matrix does not hold for the considered interval parametric matrix, \(\rho \approx 1.54\). The sufficient conditions from [2] and [4] also fail.

We apply this sufficient condition for testing regularity of the interval parametric matrices in Theorem 3(ii) and prove that for each \(k = 1, 2, 3\), and each \(\varepsilon^{(k)} \in Q_2\) the corresponding interval parametric matrix (involving only one interval parameter) is regular, \(\max_{k \in \mathcal{K}, \varepsilon^{(k)} \in Q_2} \{\rho_k\} \approx 0.841\). Thus, the considered interval parametric matrix, involving three interval parameters, is regular by Theorem 3(ii).
It is advantageous that the conditions of Theorem 3 are read negated. Thus, when testing some of the equivalent conditions, we either find a singular matrix within the interval parametric one, or prove regularity of the latter. This is illustrated in the numerical examples. This double property of the necessary and sufficient regularity conditions of Theorem 3 is also illustrated by the Algorithm 1 below.

A two-fold application of Theorem 3-(iii), for verifying nonsingularity of the parametric matrix and finding the exact interval hull of a parametric solution set, is discussed in Section 4.

We do not know any method providing bounds for the determinant of an interval matrix involving general parameter dependencies. The simplest way to check condition Theorem 3-(v) is to determine if each of the univariate polynomials

$$\det \left( A(\hat{p}) + p_k A_k + \sum_{i=1, i \neq k}^{K} \varepsilon_i \hat{p}_i A_i \right), \quad k \in \mathcal{K}, \ v \in Q_{K-1},$$

has real roots in the corresponding interval $[-\hat{p}_k, \hat{p}_k]$. Furthermore, the real roots (if present) for a polynomial can be isolated in polynomial time [13]. Hence, determining the real roots, if any, we obtain singular real matrices within the interval parametric matrix.

Following the proof (vi)$\Leftrightarrow$(i) in Theorem 3 condition (vi) can be verified by finding all real solutions of the following constrained polynomial equation

$$\det \left( \lambda I - p_k A^{-1}(\hat{p}) A_k - \sum_{i=1, i \neq k}^{K} \varepsilon_i \hat{p}_i A^{-1}(\hat{p}) A_i \right) = 0,$$

$$-\hat{p}_k \leq p_k \leq \hat{p}_k, \quad 0 \leq |\lambda| \leq 1$$

for each $k \in \mathcal{K}, \ v \in Q_{K-1}$. Alternatively, one can use some interval method, if applicable, for bounding the range of the real eigenvalues of the corresponding one-parameter interval matrix in (vi), for example, [14], [15]. However, it is proven in [6] that bounding the eigenvalues is an NP-hard problem even in case of a nonparametric interval matrix.

As in the nonparametric case [16], $\rho_0$ in Theorem 3-(vi) cannot be replaced by $\rho$. This is demonstrated below in Example 2, which is parametric version of Hudak’s example [16], [17].
Example 2. Consider the interval parametric matrix
\[
\begin{pmatrix}
    p_1 & -43 & 49 \\
    -31 & p_1 & -35 \\
    25 & -35 & p_2
\end{pmatrix},
\quad p_1 \in [31, 41],
\quad p_2 \in [28, 38].
\]
The sufficient condition (5) for regularity of an interval parametric matrix is not satisfied, \( \rho \approx 1.72 \).

For each \( k = 1, 2 \) and \( \varepsilon \in \{-1, 1\} \), the corresponding polynomial (7) does not have real roots in the parameter interval, which means that Theorem 3-(v) holds true and the considered interval parametric matrix is regular.

For each \( k = 1, 2 \) and \( \varepsilon \in \{-1, 1\} \), solving the corresponding equation (8), we find the real values of \( \hat{\rho}_0 = |\lambda| \), which lie in the interval \([-1, 1]\). For each \( k = 1, 2 \) and \( \varepsilon \in \{-1, 1\} \), the obtained biggest \( \hat{\rho}_0 = |\lambda| \approx 0.968164 \).

The latter also means that Theorem 3-(vi) holds true and the considered interval parametric matrix is regular.

The proof (vi)\( \Leftrightarrow \) (i) in Theorem 3 shows that if \( \{A(p), p\} \) is singular, then it contains real singular matrices of special form. The proof also reveals a methodology (7) for finding real singular matrices within an interval parametric matrix. Solving the problems (7) is computationally simpler than solving the problems (8) since the first problem involves only one interval parameter.

Algorithm 1. Finding singular matrices within an interval parametric matrix \( \{A(p), p\} \) or proving regularity of the latter.

1. For \( k = 1, 2, \ldots, K \); \( q = K \setminus \{k\} \);
2. For \( \varepsilon^{(k)} \in Q_{K-1} \)
3. Finding the real solutions of the constrained polynomial equation
\[
\det \left( A(p_k, \varepsilon^{(k)} \circ \hat{p}_q) \right) = 0, \quad -\hat{p}_k \leq p_k \leq \hat{p}_k,
\]
where \( A(p, \varepsilon^{(k)} \circ \hat{p}_q) := A(\hat{p}) + p_k A_k + \sum_{j \in q} \varepsilon^{(k)} \hat{p}_j A_j \).

Denote
\[
\mathcal{L}(k, \varepsilon^{(k)}) := \{ \{k, \varepsilon^{(k)}, \hat{p}_k\} \mid \hat{p}_k \in [-\hat{p}_k, \hat{p}_k], \det(A(\hat{p}_k, \varepsilon^{(k)} \circ \hat{p}_q)) = 0 \}.
\]
4. If $L(k, \varepsilon^{(k)}) \neq \emptyset$, then Return $L(k, \varepsilon^{(k)})$;
    Terminate: $\{A(p), p\}$ is singular.

5. End (For of $k$)

6. End (For of $\varepsilon^{(k)}$)

7. Return $\{A(p), p\}$ is regular.

With obvious modifications, Algorithm 1 can find all real singular matrices that correspond to the negation of Theorem 3-(v). Proving singularity is not a priori exponential. One can apply some heuristics to start Algorithm 1 with a parameter, which most likely will give a singular matrix.

The following example demonstrates that, in general, proving regularity we cannot reduce neither the number $K$ of tested parameters nor the number of $\varepsilon^{(k)} \in Q_{K-1}$.

**Example 3.** Consider the interval parametric matrix

$$A(p) = \begin{pmatrix}
\frac{6}{5} + p_1 & p_3 & -1 \\
\frac{6}{5} + p_1 & -1 & 2 + p_2 \\
2 + p_2 & \frac{1}{2}p_1 & -1
\end{pmatrix}, \quad p_i \in [-1, 1], \ i = 1, 2, 3.$$

For each $k = 1, 2, 3$ and $\varepsilon^{(k)} \in Q_2$, the corresponding polynomial (7) does not have real roots in the parameter interval, except for $k = 2$ and $\varepsilon^{(2)} = (1, -1)^\top$. This means that Theorem 3-(v) does not hold true and the considered interval parametric matrix is singular.

In the exceptional case, $\det(A(p_2, \hat{p}_1, -\hat{p}_3)) = -\frac{13}{25} - \frac{22}{15}p_2 - p_2^2$ and we obtain two values for $p_2 \in [-1, 1]$ which make the determinant equal to zero. Thus, for $\tilde{p}' = (1, -13/15, -1)^\top$ and $\tilde{p}'' = (1, -3/5, -1)^\top$ we obtain explicitly two real singular matrices contained in the considered interval parametric matrix.

4. Interval hull of a parametric solution set

If $A(\hat{p})$ is nonsingular, for each $k \in K, \varepsilon \in Q_{K-1}$,

$$\Sigma(A(p_k, \varepsilon \circ \hat{p}_q), b(p_k, \varepsilon \circ \hat{p}_q), [-\hat{p}_k, \hat{p}_k]) = x(p_k, \varepsilon \circ \hat{p}_q) \big|_{-\hat{p}_k, \hat{p}_k}$$

$$= (A(p_k, \varepsilon \circ \hat{p}_q))^{-1} b(p_k, \varepsilon \circ \hat{p}_q) \big|_{-\hat{p}_k, \hat{p}_k}$$
is a piece of a one-parameter curve in $\mathbb{R}^n$, restricted to $p_k \in [-\hat{p}_k, \hat{p}_k]$, and presents either a piece of the boundary of $\Sigma(A(p), b(p), p)$ or a subset of the latter solution set. By Theorem 1 and Theorem 3-(iii) we have the following proposition.

**Proposition 1.** If $A(\hat{p}) \in \mathbb{R}^{n \times n}$ is nonsingular and $b(p)$ is a vector in $\mathbb{R}^n$, then

$$\square \Sigma(A(p), b(p), p) = \bigcup_{k \in K} \bigcup_{\varepsilon \in Q_{K-1}} \square \Sigma(A(p_k, \varepsilon \circ \hat{p}_q), b(p_k, \varepsilon \circ \hat{p}_q), [-\hat{p}_k, \hat{p}_k]),$$

where for $i = 1, \ldots, n$

$$\square \Sigma_i(A(p_k, \varepsilon \circ \hat{p}_q), b(p_k, \varepsilon \circ \hat{p}_q), [-\hat{p}_k, \hat{p}_k]) =$$

$$\left[ \inf_{p_k \in [-\hat{p}_k, \hat{p}_k]} x_i(p_k, \varepsilon \circ \hat{p}_q), \sup_{p_k \in [-\hat{p}_k, \hat{p}_k]} x_i(p_k, \varepsilon \circ \hat{p}_q) \right].$$

Specifically, by Theorem 3-(iii),

$$\inf / \sup x_i(p_k, \varepsilon \circ \hat{p}_q) = -/ + \infty$$

if and only if $\{A(p), p\}$ is singular. Thus, applying the above Proposition together with Theorem 3-(iii), we can either find the exact interval hull of an interval parametric solution set or prove that the interval parametric matrix is singular.

The traditional approach for determining the interval hull of an interval parametric linear system is to find the analytic solution of the parametric system and then to bound the ranges of the solution components in the parameter intervals. The latter problem may be very difficult in presence of several interval parameters. The methodology based on Proposition 1 replaces the second step in the traditional approach by a set of $K2^{K-1}$ range computation problems involving a single interval variable. Since each component of $x(p_k, \varepsilon \circ \hat{p}_q)$ is a rational function of one variable $p_k$, the exact extrema of $x_i(p_k, \varepsilon \circ \hat{p}_q)$ in $[-\hat{p}_k, \hat{p}_k]$ can be easily found for rational data. This approach is applicable even when interval software is not present since some available software, e.g., Mathematica® solve the latter problem. In terms of guaranteed floating point computations, the methodology based on Proposition 1 can be combined with guaranteed interval methods (for example, [18]) to reduce the number of interval variables, and thus to obtain
(i) guaranteed tight enclosure of the hull in floating point,

(ii) expanded applicability to problems with many interval parameters and/or problems with large parameter intervals.

In general, the methodology of Proposition 1 is appropriate for real-life problems involving relatively small number of interval parameters (the computing time can be reduced by distributed computations) or when the analysis is performed offline. The proposed methodology is the only option for some systems with very large parameter intervals. The present author applies this methodology for finding the exact interval hull of parametric solution sets with nonlinear boundary when constructs benchmark examples and when estimates the quality of newly designed numerical methods. Some examples of real-life applications involve analysis of manipulators in robotics \[19, 20, 21\] and guaranteed parameter set estimation for exponential sums \[22, 23\].

5. Radius of regularity

Since every interval \( p = [p^-, p^+] \) can be represented as \( p = \hat{p} + [-\hat{p}, \hat{p}] \), every interval parametric matrix \( A(p), p \in \mathbb{P} \subseteq \mathbb{R}^K \), can be equivalently represented as \( A(\hat{p} + p), p \in [-\hat{p}, \hat{p}] \).

**Definition 3.** For a square interval parametric matrix \( A(p), p \in \mathbb{P} \subseteq \mathbb{R}^K \), its regularity radius is defined by

\[
 r^*(A(p), p) := \inf \{ r \geq 0 | A(\hat{p} + rp) \text{ is singular for some } p \in \mathbb{P}, p \in [-\hat{p}, \hat{p}] \}.
\]

Specifically, \( r^*(A(p), p) = \infty \) if no real \( r \) exists such that \( \{ A(\hat{p} + rp), p \in [-\hat{p}, \hat{p}] \} \) is singular. If \( r^*(A(p), p) < \infty \), then the infimum is achieved as minimum.

Definition 3 is not restricted to interval parametric matrices involving affine linear parameter dependencies. In what follows, however, we present an explicit formula for the radius of regularity of interval parametric matrix involving affine-linear dependencies and some necessary and sufficient conditions for its infinite value.

**Theorem 4.** Let \( A(p) \) involve only affine-linear dependencies on \( p \in \mathbb{P} \subseteq \mathbb{R}^K \) and \( A(\hat{p}) \) be nonsingular. Then,

\[
 r^*(A(p), p) = \frac{1}{\max \left\{ \phi_0 \left( B(p_k, \varepsilon^{(k)}) \right) | k \in K, p_k \in [-\hat{p}_k, \hat{p}_k], \varepsilon^{(k)} \in Q_{K-1} \right\}}, \quad (9)
\]
where \( B(p_k, \varepsilon^{(k)}) = p_k (A(\hat{p}))^{-1} A_k + \sum_{i=1, i \neq k}^K \varepsilon_i^{(k)} \hat{p}_i (A(\hat{p}))^{-1} A_i \).

**Proof.** Let \( r^*(A(p), p) < \infty \). For a given \( r \geq 0 \), there exists a singular matrix within the interval parametric matrix \( \{ A(\hat{p} + rp), p \in [-\hat{p}_k, \hat{p}_k] \} \), by Theorem 3-(vi), if and only if

\[
g_0 \left( r B(p_k, \varepsilon^{(k)}) \right) \geq 1
\]

for some \( k \in K, p_k \in [-\hat{p}_k, \hat{p}_k], \varepsilon^{(k)} \in Q_{K-1} \). The latter means that

\[
r \max \left\{ g_0 \left( B(p_k, \varepsilon^{(k)}) \right) \mid k \in K, p_k \in [-\hat{p}_k, \hat{p}_k], \varepsilon^{(k)} \in Q_{K-1} \right\} \geq 1.
\]

Hence the minimum value of \( r \) is given by (9).

If \( r^*(A(p), p) = \infty \), then \( A(\hat{p} + rp) \) is nonsingular for each \( r \in \mathbb{R}, r \geq 0, p \in [-\hat{p}, \hat{p}] \), according to Definition 3. By Theorem 3-(vi) this is equivalent to \( g_0 \left( B(p_k, \varepsilon^{(k)}) \right) < 1 \) for each \( k \in K, p_k \in [-\hat{p}_k, \hat{p}_k], \varepsilon^{(k)} \in Q_{K-1} \) and each \( r \in \mathbb{R}, r \geq 0 \). Hence \( g_0 \left( B(p_k, \varepsilon^{(k)}) \right) = 0 \) for each \( k \in K, p_k \in [-\hat{p}_k, \hat{p}_k], \varepsilon^{(k)} \in Q_{K-1} \) and the convention \( 1/0 = \infty \) implies (9).

Radius of regularity for interval parametric matrices was first defined in [24], under the assumption that \( r^*(A(p), p) < \infty \). In several works, e.g., [10], [24], L. Kolev develops and applies a methodology for either finding the regularity radius of an interval parametric matrix or providing bounds for the latter. Beside various other requirements, this methodology also assumes \( r^*(A(p), p) < \infty \). We are not informed about any other work discussing a methodology for checking the condition \( r^*(A(p), p) < \infty \), respectively the condition \( r^*(A(p), p) = \infty \).

In view of Definition 3 the regularity radius can be defined equivalently as

\[
r^*(A(p), p) := \inf \{ r \geq 0 \mid \det (A(\hat{p} + rp)) = 0 \text{ for some } p \in \mathbb{R}^K, p \in [-\hat{p}, \hat{p}] \}. \tag{10}
\]

According to (10), \( r^*(A(p), p) = \infty \) if \( \det (A(\hat{p} + rp)) = 0 \) does not have real solutions for any \( r \in \mathbb{R}, r \geq 0, p \in \mathbb{R}^K, p \in [-\hat{p}, \hat{p}] \).

Applying Theorem 4 and Theorem 3 we obtain the following equivalent conditions for an infinite radius of regularity.

**Corollary 1.** An interval parametric matrix \( \{ A(p), p \in p \} \) involving affine-linear dependencies, has infinite radius of regularity \( r^* \) if and only if
(i) the constrained equation
\[
\det \left( \hat{A} - r \left( p_k A_k + \sum_{i=1, i \neq k}^{K} \varepsilon^{(k)}_i \hat{p}_i A_i \right) \right) = 0, \quad p_k \in [-\hat{p}_k, \hat{p}_k], r \geq 0,
\]
does not have real solutions for each \( k \in \mathcal{K}, \varepsilon^{(k)} \in Q_{K-1}, \)

(ii) equivalently, nonsingular \( \hat{A} = A(\hat{p}) \) and
\[
\rho_0 \left( B(p_k, \varepsilon^{(k)}) \right) = 0, \quad B(p_k, \varepsilon^{(k)}) := p_k \hat{A}^{-1} A_k + \sum_{i=1, i \neq k}^{K} \varepsilon^{(k)}_i \hat{p}_i \hat{A}^{-1} A_i,
\]
for each \( k \in \mathcal{K}, p_k \in [-\hat{p}_k, \hat{p}_k], \varepsilon^{(k)} \in Q_{K-1}. \)

Checking Corollary 1-(i), the parametric determinants involve two parameters compared to the procedure (7), respectively Algorithm 1. Checking Corollary 1-(ii), however, applies the same procedure (8) with \(|\lambda| \geq 0.\)

Checking any condition of Corollary 1 and finding all real roots of the corresponding determinant, whenever they exist, by Theorem 4 we either determine the finite regularity radius of an interval parametric matrix, or prove that the regularity radius is infinite.

**Example 4.** Determining the radius of regularity for the interval parametric matrices from Examples 1, 2 and 3, we obtain with 6 digits precision:
\[
\begin{align*}
\rho^*(A(p), p \in p, \text{Example 1}) &= 1.08209, \\
\rho^*(A(p), p \in p, \text{Example 2}) &= 1.03289, \\
\rho^*(A(p), p \in p, \text{Example 3}) &= 0.996413.
\end{align*}
\]

6. Conclusion

By Poljak and Rohn [25], see also [3], checking regularity is an NP-hard problem for the nonparametric interval matrices (note that a \( n \times n \) nonparametric interval matrix can be considered as an interval parametric matrix involving \( n^2 \) parameters). That is why, some polynomially computable sufficient conditions are used for verifying regularity. However, the sufficient regularity conditions may approximate quite rough a matrix regularity and, depending on the width of the parameter intervals, may fail proving regularity.
All of the necessary and sufficient regularity conditions, presented in Theorem 3, exhibit exponential behavior. They employ a finite set of test matrices which cardinality is $K^{2^{K-1}}$, where $K$ is the number of non-degenerate interval parameters. This makes the proposed methodology efficient for large matrices provided the number of interval parameters is small.

The methodology, presented here, for proving regularity/singularity of an interval parametric matrix is particularly useful in case of large parameter intervals for which all sufficient regularity conditions fail. This is just opposite to the requirement for narrow intervals of the most interval methods in order to provide a good quality of the result or for success. Since in case of large parameter intervals, the interval matrix is very close to a singular one, the proposed methodology may come very quickly to a singular matrix. Thus, the proposed methodology has exponential complexity only in case of regular interval parametric matrices, and could be qualified as not a priori exponential.

A key feature of the discussed methodology is that the original problem involving $K$ interval parameters is transformed to $K^{2^{K-1}}$ regularity problems involving only one interval parameter. On one side, as discussed in Example 1, this leads to an increased applicability of the easy verifiable sufficient regularity conditions. On the other hand, the one parameter problems, whose solving is required by the necessary and sufficient regularity conditions, are polynomially solvable. Most of the problems are solvable exactly in exact arithmetic. The most easily implementable criterion is Theorem 3-(v). Furthermore, the $K^{2^{K-1}}$ one parameter problems are independent of each other and, therefore, can be checked on parallel processors. The latter increases the applicability of the proposed necessary and sufficient regularity conditions.

As already mentioned, proving regularity of an interval parametric matrix implies various other properties of these matrices.

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