A coincidence theorem for holomorphic maps to $G/P$

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Dedicated to Professor Peter Zvengrowski
on the occasion of his sixty-first birthday

Abstract: The purpose of this note is to extend to an arbitrary generalized Hopf and Calabi-Eckmann manifold the following result of Kalyan Mukherjea: Let $V_n = S^{2n+1} \times S^{2n+1}$ denote a Calabi-Eckmann manifold. If $f, g : V_n \to \mathbb{P}^n$ are any two holomorphic maps, at least one of them being non-constant, then there exists a coincidence: $f(x) = g(x)$ for some $x \in V_n$. Our proof involves a coincidence theorem for holomorphic maps to complex projective varieties of the form $G/P$ where $G$ is complex simple algebraic group and $P \subset G$ is a maximal parabolic subgroup, where one of the maps is dominant.

1 Introduction

Let $G$ be a simply connected simple algebraic group over $\mathbb{C}$ and $P \subset G$ a maximal parabolic subgroup. Let $\mathcal{L}$ be the ample generator of $Pic(G/P) \cong \mathbb{Z}$. Let $E$ denote the total space of the principal $\mathbb{C}^*$ bundle associated to $\mathcal{L}^{-1}$. Let $\lambda$ be any complex number with $|\lambda| > 1$ and let $\varphi : E \to E$ denote the bundle map $e \mapsto \lambda \cdot e$, $e \in E$. The quotient space, denoted $V_{\lambda}$, is a compact complex homogeneous non-Kähler manifold. The manifold $V_{\lambda}$ (or simply $V$) is called a generalized Hopf manifold [4]. (See also §2, [10].) One has an elliptic curve bundle $q : V \to G/P$ with fibre and structure group the elliptic curve $\mathbb{T} = \mathbb{C}^*/\langle \lambda \rangle$ with periods $\{1, \tau\}$ where $\exp(2\pi \sqrt{-1}\tau) = \lambda$. (Note that $\text{Im}(\tau) \neq 0$ as $|\lambda| > 1$.) One has a diffeomorphism $V \cong S^1 \times K/L$ where $K$ is a maximal compact subgroup of $G$ and $L$ the semi simple part of the centralizer of a subgroup of $K$ isomorphic to the circle $S^1$. If we take $G/P$ to be the complex projective space $\mathbb{P}^n$, then the above construction yields the usual Hopf manifolds $S^1 \times S^{2n+1}$.

2000 A.M.S. Subject Classification:- 32H02, 54M20.
Let $G/P, G'/P'$, where both $G, G'$ are simply connected simple algebraic groups over $\mathbb{C}$ and $P, P'$, maximal parabolic subgroups of $G, G'$ respectively. Let $E \rightarrow G/P, E' \rightarrow G'/P'$ denote the principal $\mathbb{C}^*$-bundles associated to the negative ample generators of the Picard groups of $G/P, G'/P'$ respectively. The product bundle $E \times E' \rightarrow G/P \times G'/P'$ is a principal $\mathbb{C}^* \times \mathbb{C}^*$ bundle. Let $\tau$ be any complex number with $\text{Im}(\tau) \neq 0$. One has a complex analytic monomorphism $\mathbb{C} \rightarrow \mathbb{C}^* \times \mathbb{C}^*$ defined by $z \mapsto (\exp(2\pi \sqrt{-1} \tau z), \exp(2\pi \sqrt{-1} z))$. Denote the image of this group by $C_\tau$. The group $C_\tau \subset (\mathbb{C}^*)^2$ acts on $E \times E'$ as bundle automorphisms and the quotient $U = E \times E'/C_\tau$ is the total space of an elliptic curve bundle with fibre and structure group the elliptic curve $(\mathbb{C}^*)^2/C_\tau = \mathbb{T}$ with periods $\{1, \tau\}$. Up to a diffeomorphism, $U$ can be identified with the space $K'/L \times K'/L'$ where $K \subset G, K' \subset G'$ are maximal compact subgroups and $L$ and $L'$ are semi simple parts of centralizers of certain subgroups of $K$ and $K'$ isomorphic to $S^1$. The compact complex manifold $U$ is homogeneous and non-Kähler, which we call a generalized Calabi-Eckmann manifold. When $G/P = \mathbb{P}^m$, and $G'/P' = \mathbb{P}^n$, $U$ is a Calabi-Eckmann manifold $S^{2m+1} \times S^{2n+1}$ \[\mathbf{1}\]. We shall denote by $p$ the bundle projection $U \rightarrow G/P \times G'/P'$.

The manifold $U$ is an example of a simply connected compact complex homogeneous manifold. Such manifolds have been completely classified by H.-C. Wang \[\mathbf{18}\].

**Theorem 1.** We keep the above notations. (i) Let $\varphi, \psi : V \rightarrow G/P$ be any two holomorphic maps with $\varphi$ non-constant. Then there exists an $x \in V$ such that $\varphi(x) = \psi(x)$.

(ii) Assume that $\dim(G/P) \leq \dim(G'/P')$ and let $\varphi, \psi : U \rightarrow G/P$ be a holomorphic map with $\varphi$ non-constant. Then there exists an $x \in U$ such that $\varphi(x) = \psi(x)$.

The above theorem will be derived from the following

**Theorem 2.** (i) Let $M$ be any connected compact complex analytic manifold and let $\varphi, \psi : M \rightarrow G/P$ be holomorphic where $P \subset G$ is a maximal parabolic subgroup. Assume that at least one of the maps $\varphi, \psi$ is surjective. Then there exists an $x \in M$ such that $\varphi(x) = \psi(x)$.

(ii) Furthermore, if $M$ is a projective variety, or if Kähler manifold with $\dim(M) = \dim(G/P)$, and $f, g : M \rightarrow G/P$ are continuous maps homotopic to $\varphi, \psi$ respectively, then there exists an $x \in M$ such that $f(x) = g(x)$.

The special case of theorem \[\mathbf{1}\] when $U = S^{2n+1} \times S^{2n+1}$ is a Calabi-Eckmann manifold is due to K.Mukherjea \[\mathbf{12}\]. He uses D.Toledo’s approach \[\mathbf{7}\] to analytic fixed point theory. In particular he uses Borel’s computation of Dolbeault cohomology of Calabi-Eckmann manifolds and a certain ‘analytic Thom class’ $\xi_\Delta \in H^n(\mathbb{P}^n \times \mathbb{P}^n; \Omega^n) \cong H^n(\mathbb{P}^n \times \mathbb{P}^n; \mathbb{C})$ supported on the diagonal to detect coincidences. Our proof is based on the observation that any holomorphic map from $U$ (resp. $V$) to a complex projective variety factors through $G/P \times G'/P'$ (resp. $G/P$) (see lemma \[\mathbf{8}\], §3). Any non-constant holomorphic map of $G/P$ or of $G/P \times G'/P'$ into $G/P$ is shown to be dominant (lemma \[\mathbf{3}\]). Theorem \[\mathbf{1}\] is then deduced from theorem \[\mathbf{3}\]. Theorem \[\mathbf{2}\] is proved using positivity of cup products in cohomology of $G/P$ and the fact that any effective cycle is rationally equivalent to a positive linear combination of Schubert cycles. (Cf. \[\mathbf{8}\], \[\mathbf{3}\].)
2 Holomorphic maps to $G/P$

We keep the notations of §1. In particular, $G$ is a simply connected simple complex algebraic group. Let $Q \subset G$ be any parabolic subgroup (not necessarily maximal). Fix a maximal torus $T$ and a Borel subgroup $B$ containing $T$ such that $B \subset Q$. Let $W$ be the Weyl group of $G$ with respect to $T$ and $W(Q) \subset W$ that of $Q$ and let $S \subset W$ denote the set of simple reflections with respect to our choice of $B$. Recall that the $T$ fixed points of $G/Q$ are labelled by $W/W(Q)$. We shall identify $W/W(Q)$ with the set of coset representatives $W^Q \subset W$ having minimal length with respect to $S$. The $B$-orbits of these $T$ fixed points give an algebraic cell decomposition for $G/Q$. Denote by $X(w)$ the Schubert variety $X(w) \subset G/Q$ which is the $B$-orbit closure of the $T$-fixed point corresponding to $w \in W^Q$. The class of the Schubert varieties $[X(w)], w \in W^Q$, form a $\mathbb{Z}$-basis for the singular cohomology group $H^*(G/Q; \mathbb{Z})$ as well as the Chow cohomology groups $A^*(G/Q)$. Recall that $Pic(G/Q) \cong A^1(G/Q)$ which is infinite cyclic when $Q$ is a maximal parabolic subgroup.

Lemma 3. Let $P \subset G$ be a maximal parabolic subgroup and let $X$ be an irreducible complex projective variety with $Pic(X) = \mathbb{Z}$. Suppose that $\dim(X) \geq \dim(G/P)$. Then any non-constant algebraic map $\varphi : X \to G/P$ is dominant with finite fibres; in particular $\dim(G/P) = \dim(X)$.

Proof: Let $\mathcal{L}$ be a very ample line bundle over $G/P$. Let $Z = \text{Im}(\varphi)$. As $\varphi$ is non-constant, $Z$ is not a point variety and $\mathcal{L}|Z$ is very ample. Since the bundle $\mathcal{L}|Z$ is generated by global sections, it follows that $\mathcal{L}' := \varphi^*(\mathcal{L})$ is generated by its sections over $X$. Also $\mathcal{L}'$ cannot be trivial since any non-zero section of a trivial bundle is nowhere zero, whereas sections of $\mathcal{L}'$ arising as pull-back of non-zero sections of $\mathcal{L}|Z$ are non-zero sections which vanish somewhere in $X$. Since $Pic(X) \cong \mathbb{Z}$ it follows that some positive multiple of $\mathcal{L}'$ must be very ample. Hence, restricted to any fibres of $\varphi$ the bundle $\mathcal{L}' = \varphi^*(\mathcal{L})$ must be ample. This implies that the fibres of $\varphi$ must be finite. Therefore $\dim(X) \leq \dim(G/P)$. Since $\dim(X) \geq \dim(G/P)$ by hypothesis, we must actually have equality and the map $\varphi$ must be dominant. \hfill $\Box$

Remark 4. (i) The assumption that $Pic(X) = \mathbb{Z}$ is not superfluous. For example, take $X = \mathbb{P}^1 \times \mathbb{P}^n$, $n \geq 2$. Let $\varphi : X \to \mathbb{P}^3$ be the composition $\varphi_1 \circ pr_1$, where $pr_1$ is the first projection map and $\varphi_1 : \mathbb{P}^1 \to \mathbb{P}^3$ is defined by $\varphi_1(z_0 : z_1) = (z_0 : z_1 : 0 : 0)$.

(ii) Suppose that $\varphi : Z \to G/P$ is holomorphic and that $X \subset Z$ is a complex analytic subset which satisfies the hypothesis of the lemma above. If $\varphi : Z \to G/P$ is holomorphic and $\varphi|X$ is non-constant, then $\varphi$ must be dominant. On the other hand, suppose that $\pi : Z \to M$ is a complex analytic fibre bundle with fibre $X$ as in the lemma above. Suppose $\dim(X) > \dim(G/P)$, then any complex analytic map $\psi : Z \to G/P$ factors through $\pi$, i.e., $\psi = \theta \circ \pi$ for some complex analytic map $\theta : M \to G/P$ since the lemma implies that $\psi$ restricted to any fibre has to be constant.

(iii) A result of K.Paranjape and V.Srinivas $[3]$ says that if $G/P$ is not the projective space, any non-constant self morphism $\varphi : G/P \to G/P$ is an automorphism of varieties. The full group of automorphisms of $G/P$ has been determined by I.Kantor $[4]$.  

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We shall now prove theorem 2:

**Proof of theorem 2**: (i) Suppose \( \varphi : M \to G/P \) is surjective. Set \( d = \dim(G/P) \), \( m = \dim(M) \). Let \( \Delta \subset G/P \times G/P \) denote the diagonal of \( G/P \). Let \( \Gamma := \Gamma_\theta \subset G/P \times G/P \) denote the image of the map \( \theta : M \to G/P \times G/P, \theta(x) = (\varphi(x), \psi(x)), x \in M \). We need only show that \( \Gamma \cap \Delta \neq \emptyset \). Note that \( \Gamma \) is a complex analytic subspace of the projective variety \( G/P \) and hence algebraic by GAGA [13]. Also \( k := \dim(\Gamma) \geq \dim(G/P) \) since \( \varphi \) is surjective. Now, as for any effective cycle in \( G/P \times G/P \), the class \( [\Gamma] \in A_k(G/P \times G/P) \) is a *positive* linear combination of Schubert cycles in \( G/P \times G/P \). (Cf. [7]. See also [4].) Thus, \( [\Gamma] = \sum_i a_i[X(w_i)] \times [X(w_i')] \) where \( a_i \) are positive integers, \( w_i, w'_i \in W^P \) are suitable elements such that \( \dim(X(w_i)) + \dim(X(w'_i)) = \dim(\Gamma) \). Since \( \varphi \) is surjective, in the above expression for \( \Gamma \), the term \([G/P] \times [X(w)]\) must occur with positive coefficient for some \( w \in W^P \). The same arguments can be applied to the class of the diagonal \( \Delta \) in \( G/P \times G/P \) as well. In fact, it is known that \([\Delta] = \sum_{w \in W^P} [X(v)] \times [X(v'')] \) where \( x \) is the Schubert variety ‘dual’ to \( X(v) \), i.e. \( v'' = w_0.v \) where \( w_0 \in W^P \) represents the longest element of \( W/W^P \). (Cf. theorem 11.11, [1].) Hence \([\Gamma], [\Delta] = (a[G/P] \times [X(w)] + other terms) \cdot (1 \times [G/P] + other terms) = a(1 \times [X(w)]) + other terms \), where \( a > 0 \) and the coefficients of the remaining terms (with respect to the basis consisting of Schubert cocycles) in the rhs of the last equality are *non-negative* integers. Hence \([\Gamma] [\Delta] \neq 0 \) in \( A_{k-d}(G/P \times G/P) \) and so \( \Gamma \cap \Delta \neq \emptyset \).

(ii) Let \( h : M \to G/P \times G/P \) be the map \( x \mapsto (f(x), g(x)) \) for \( x \in M \). It suffices to show that \( h^*([\Delta]) \in H^{2d}(M; \mathbb{Z}) \) is non-zero, where we regard \([\Delta] \) as an element of the singular cohomology group \( H^{2d}(G/P \times G/P; \mathbb{Z}) \). Note that \( h \) is homotopic to \( \theta = (\varphi, \psi) \) and so we have \( h^*([\Delta]) = \theta^*([\Delta]) \). To complete the proof, it suffices to show that \( \theta^*([\Delta]) \neq 0 \) in \( H^*(M; \mathbb{Z}) \).

Suppose that \( M \) is Kähler and \( \dim(M) = d = \dim(G/P) \). By de Rham and Hodge theory (§15.7, [4]), we have \( H^*(M; \mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(M) \) and \( \theta^*([\Delta]) \) can be thought of as an element of \( H^{2d}((M; \mathbb{C}) \). Since \( \varphi \) is dominant, \( \dim(\Gamma) = d = \dim(M) \) and the fundamental class \( \mu_M \in H_{2d}(M; \mathbb{Z}) \) maps to \( n[\Gamma] \in H_{2d}(G/P \times G/P; \mathbb{Z}) \cong A_d(G/P \times G/P) \) for some \( n \geq 1 \). From what has been shown already the intersection product \([\Delta] \cdot [\Gamma] \neq 0 \) in the Chow ring of \( G/P \times G/P \). This implies that \([\Delta] \cap [\Gamma] \neq 0 \) in \( H_0(G/P \times G/P) \). Now we have (see ch. 5, §6, [16])

\[
\theta_* (\theta^*([\Delta])) \cap \mu_M = n[\Delta] \cap \theta_* (\mu_M) = n[\Delta] \cap [\Gamma] \neq 0.
\]

It follows that \( \theta^*([\Delta]) \neq 0 \) in \( H^*(M; \mathbb{Z}) \).

If \( M \) is a complex projective variety then, one can always find an irreducible subvariety \( Z \subset M \) with \( \dim(Z) = \dim(\Gamma) \) which maps onto \( \Gamma \). It follows that \( \theta_* ([Z]) = n[\Gamma] \in H_*(G/P \times G/P) \) for some \( n > 0 \). Using the fact that \( H_*(G/P \times G/P) \) has no torsion, proceeding just as before we conclude that \( \theta^*(\Delta) \neq 0 \).

\[\Box\]

**Remark 5.** (i) In the statement of theorem 2, the hypothesis that \( M \) be nonsingular is not necessary. Indeed, H.Hironaka [5] has shown that any irreducible complex analytic space
which is countable at infinity can be desingularized. So replacing $M$ by $\tilde{M}$ and the maps
$\varphi, \psi$ by $\tilde{\varphi} := \varphi \circ \pi, \tilde{\psi} := \psi \circ \pi$ respectively where $\pi : \tilde{M} \to M$ is a desingularization map, the
see that $\tilde{\varphi}, \tilde{\psi}$ must have a coincidence. This immediately implies that $\varphi$ and $\psi$ must have a
coincidence.

(ii) In case $M$ is not Kähler, it is not true in general that $\theta^*([\Delta]) \neq 0$ in $H^*(M)$ although $\varphi$
and $\psi$ must have a coincidence as our theorem shows. Such coincidences have been described
as “homologically invisible” by Kalyan Mukherjee [12]. He observed that when $M$ is the
Calabi-Eckmann manifold $S^{2n+1} \times S^{2n+1}$ with $n \geq 1$, for any two maps $\varphi, \psi : M \to \mathbb{P}^n$ the
homomorphism $\theta^* : H^*(\mathbb{P}^n \times \mathbb{P}^n) \to H^*(M)$ is zero in positive dimensions where $\theta = (\varphi, \psi)$.
In particular, if $f, g$ are continuous maps homotopic to holomorphic maps $\varphi, \psi$ respectively
with $\varphi$ dominant, we do not know if $f$ and $g$ must have a coincidence.

(iii) When $M$ is Kähler and $\dim(M) > \dim(G/P)$, the conclusion of the theorem is still valid
provided $[\Gamma_0] \in H_*(G/P; \mathbb{Q})$ is in the image of $\theta_* : H_*(M; \mathbb{Q}) \to H_*(G/P; \mathbb{Q})$.

Corollary 6. (i) Let $P \subset G$ be a maximal parabolic subgroup and let $f, g : G/P \to G/P$
be any two continuous maps homotopic to holomorphic maps $\varphi, \psi$ respectively where $\varphi$
is non-constant. Then $f(x) = g(x)$ for some $x \in G/P$.

(ii) Let $f, g : G/P \times G'/P' \to G/P$ be any two continuous maps which are homotopic to
holomorphic maps $\varphi, \psi$ respectively with $\varphi$ being non-constant. Assume that $\dim(G/P) \leq$
$\dim(G'/P')$. Then there exists an $x \in G/P \times G'/P'$ such that $f(x) = g(x)$.

Proof: Part (i) follows immediately from lemma 3 and theorem 2(ii). To prove (ii), suppose
$\varphi|Z$ is constant for every fibre $Z \cong G'/P'$ of the first projection $pr_1 : G/P \times G'/P' \to G/P$
map, then $\varphi$ can be factored as $\varphi_1 \circ pr_1$ where $\varphi_1 : G/P \times G'/P' \to G/P$ defined by $\varphi$.
It follows from 4 that $\varphi_1$ is dominant. Hence $\varphi$ is also dominant. Otherwise for some fibre
$Z \cong G'/P'$, $\varphi|Z$ is non-constant. By lemma 3 it follows that $\varphi|Z$ is dominant. It follows
from lemma 3 again that $\varphi|Z$ — and hence $\varphi$ — must be dominant and the corollary follows
from theorem 2.

Remark 7. It follows from the above corollary that any continuous map homotopic to a
holomorphic map has a fixed point. However, in general, the spaces $G/P$ do not have fixed
point property. For example, the Grassmannian $G_k(\mathbb{C}^n) = SL(n, \mathbb{C})/P_k$ admits a continuous
fixed point free involution whenever $n$ is even and $k$ odd or if $n = 2k$. As another example,
the complex quadric $SO(n)/P_1$ is diffeomorphic to the oriented real Grassmann manifold
$G_2(\mathbb{R}^n)$ of oriented 2-planes in $\mathbb{R}^n$. The involution that reverses the orientation on each
element of $G_2(\mathbb{R}^n)$ is obviously fixed point free. However, it is known that $G_k(\mathbb{C}^n)$ has fixed
point property (for continuous maps) when $n$ is large compared to $k$ and at most one of
$n - k, k$ is odd. See 2, 3.
3 Proof of Main Theorem

We now prove the main result of the paper, namely, theorem 1. We keep the notations of §1.

Lemma 8. Let $U, V$ be generalized Calabi-Eckmann and generalized Hopf manifolds. (See §1.) Let $Z$ be a complex projective variety. Any holomorphic maps $\varphi : U \to Z$, $\psi : V \to Z$ can be factored as $\varphi = \varphi_1 \circ p$, $\psi = \psi_1 \circ q$, where $p : U \to G/P \times G'/P'$ and $q : V \to G/P$ are projections of the principal $\mathbb{T}$-bundles.

Proof: It was shown in the proof of Theorem 3, [14], that $H^2(V; \mathbb{Z}) = H^2(K/L; \mathbb{Z}) = 0$ where $K \subset G$ is a maximal compact subgroup of $G$ and $L$ is the semi simple part of the centralizer in $K$ of a subgroup of $K$ isomorphic to $S^1$. The same argument shows that $H^2(U; \mathbb{Z}) = H^2(K/L \times K'/L'; \mathbb{Z}) = 0$. In particular the manifolds $U,V$ are not Kähler.

A theorem of Grauert and Remmert [4] says that for a compact complex homogeneous manifold $M$ of dimension $n$, the transcendence degree over $\mathbb{C}$ of the field $\mathcal{M}(M)$ of meromorphic functions on $M$ is equal to $n$ if and only if it a projective algebraic variety. In the our case $U,V$ fibre over projective varieties of dimension 1 less. It follows that $\text{tr.deg}_C((V)) \geq \text{tr.deg}_C(\mathcal{M}(X)) = \dim(G/P)$. Suppose $\psi$ is not constant along a fibre. Then there exists an open set (in the analytic topology) $N \subset G/P$ such that $\psi$ is non-constant on $q^{-1}(x)$, for any $x \in N$. Let $x \in N$. Composing with a suitable meromorphic function on $Z$ which is non-constant on $\psi(q^{-1}(x))$, we get a meromorphic function $\theta$ on $V$. We claim that $\theta$ is transcendental over $\mathcal{M}(G/P) \subset \mathcal{M}(V)$. Assume, if possible, that $\theta^k + a_1 \theta^{k-1} + \cdots + a_k = 0, a_i \in \mathcal{M}(G/P)$. By changing the $x \in N$ if necessary, we may assume that $x$ is not on the polar divisor for any $a_i$. Restricting this equation to the fibre over $x$, we see that $\theta|q^{-1}(x)$ is algebraic over $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed, we must have $\theta|p^{-1}(x) \in \mathbb{C}$. This is absurd since $\theta$ is non-constant on $q^{-1}(x)$. Hence we conclude that $\theta$ is constant along the fibres of $q$. Proof that $\varphi$ is constant along the fibres of $p$ is entirely similar.

Proof of Theorem 1: (i) By lemma 8, the maps $\varphi, \psi$ factor through the projection of the elliptic curve bundle $p : V \to G/P$. Write $\varphi = \varphi_1 \circ p$, $\psi = \psi_1 \circ p$. Now, it suffices to show that the holomorphic maps $\varphi_1$ and $\psi_1$ have a coincidence. Since $\varphi_1$ is non-constant, this is now immediate from corollary [3] (i).

(ii) Proceeding exactly as in (i), we write $\varphi = \varphi_1 \circ q$, $\psi = \psi_2 \circ q$, where $\varphi_1, \psi_1 : G/P \times G'/P' \to G/P$ are holomorphic. Note that since $\varphi_1$ is non-constant. By corollary [3] $\varphi_1$ and $\psi_1$ must have a coincidence. Hence $\varphi$ and $\psi$ must also have a coincidence. \[ \square \]

We conclude with the following observation.

Lemma 9. Let $\pi : W \to M$ be a holomorphic fibre bundle with $M$ compact connected, and fibre a complex torus $\mathbb{T}$. Suppose that $H_2(F; \mathbb{Q}) \to H_2(W; \mathbb{Q})$ is zero. Then any holomorphic map $\varphi : W \to G/Q$ is constant on the fibres of $\pi$ where $Q \subset G$ is any parabolic subgroup.
Proof: Assume that \( \varphi : W \rightarrow G/Q \) is a holomorphic map such that \( \varphi|F \) is not constant for some fibre \( F \) of the \( T \)-bundle \( \pi : W \rightarrow M \). Let \( \iota : F \subset W \) denote the inclusion map. Let \( C \subset G/Q \) be the image of \( F \). Note that \( \dim(C) = 1 = \dim(F) \). Since \( \varphi \) is holomorphic, \( C \) is an algebraic subvariety of \( G/Q \). In particular, it represents a non-zero element of \( H_2(G/Q; \mathbb{Q}) \). In fact \( C \) is rationally equivalent to a positive linear combination of certain 1-dimensional Schubert subvarieties in \( G/Q \). It follows that \( (\varphi|F)_* : H_2(F; \mathbb{Q}) \rightarrow H_2(G/Q; \mathbb{Q}) \) maps the fundamental class of \( F \) to a non-zero element of \( H_2(G/Q; \mathbb{Q}) \). On the other hand, \( (\varphi|F)_* = \varphi_* \circ \iota_* = 0 \) in dimension 2, since \( \iota_* : H_2(T; \mathbb{Q}) \rightarrow H_2(W; \mathbb{Q}) \) is zero by hypothesis. We conclude that \( \varphi \) must be constant on the fibres of \( \pi \). \( \Box \)

Remark 10. (i) Let \( \pi : W \rightarrow M \) be as in the above lemma. Let \( \varphi, \psi : W \rightarrow G/P \) be any two holomorphic maps where \( \varphi \) is non-constant and \( P \subset G \) a maximal parabolic. Let \( \varphi_1 : M \rightarrow G/P \) be such that \( \varphi = \varphi_1 \circ \pi \). Suppose \( X \subset M \) is irreducible and has the structure of a complex projective variety with \( \text{Pic}(X) = \mathbb{Z} \) and \( \dim(X) = \dim(G/P) \). If \( \varphi_1|X \) is non-constant, then, in view of the above lemma and remark 4(ii), \( \varphi_1 \) must be dominant. It follows that \( \varphi \) itself must be dominant. By theorem 2 it follows that \( \varphi \) and \( \psi \) must have a coincidence.

(ii) I do not know if theorem \( \# \) still holds if one merely assumes that \( \varphi, \psi \) are continuous maps homotopic to holomorphic maps one of which is dominant.

Acknowledgments: I am indebted to Kalyan Mukherjea for asking me the question which led to this paper and for pointing out his paper [12]. I am grateful to the referee of this paper for his/her valuable comments and suggestions for improvements. I am grateful to Issai Kantor for a copy of his paper [7] and for translating into English the relevant parts of his of paper for my benefit.

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