Invariants of Elliptic and Hyperbolic
CR-Structures of Codimension 2

V. V. Ezhov, A. V. Isaev, G. Schmalz

We reduce CR-structures on smooth elliptic and hyperbolic manifolds of CR-codimension 2 to parallelisms thus solving the problem of global equivalence for such manifolds. The parallelism that we construct is defined on a sequence of two principal bundles over the manifold, takes values in the Lie algebra of infinitesimal automorphisms of the quadric corresponding to the Levi form of the manifold, and behaves “almost” like a Cartan connection. The construction is explicit and allows us to study the properties of the parallelism as well as those of its curvature form. It also leads to a natural class of “semi-flat” manifolds for which the two bundles reduce to a single one and the parallelism turns into a true Cartan connection.

In addition, for real-analytic manifolds we describe certain local normal forms that do not require passing to bundles, but in many ways agree with the structure of the parallelism.

0 Introduction

We start with a brief overview of necessary definitions and facts from CR-geometry (see e.g. [Tu1] for a more detailed exposition).

A CR-structure on a smooth real connected manifold $M$ of dimension $m$ is a smooth distribution of subspaces in the tangent spaces $T_p^c(M) \subset T_p(M)$, $p \in M$, with operators of complex structure $J_p : T_p^c(M) \to T_p^c(M)$, $J_p^2 \equiv -\text{id}$, that depend smoothly on $p$. A manifold $M$ equipped with a CR-structure is called a CR-manifold. It follows that the number $\text{CRdim} M := \dim \mathbb{C} T_p^c(M)$ does not depend on $p$; it is called the CR-dimension of $M$. The number $\text{RCodim} M := m - 2 \text{CRdim} M$ is called the CR-codimension of $M$. CR-structures naturally arise on real submanifolds in complex manifolds. Indeed, if, for example, $M$ is a real submanifold of $\mathbb{C}^K$, then one can define the distribution $T_p^c(M)$ as follows:

$$T_p^c(M) := T_p(M) \cap iT_p(M).$$

On each $T_p^c(M)$ the operator $J_p$ is then defined as the operator of multiplication by $i$. Then $\{T_p^c(M), J_p\}_{p \in M}$ form a CR-structure on $M$, if $\dim \mathbb{C} T_p^c(M)$ is constant. This is always the case, for example, if $M$ is a real hypersurface in $\mathbb{C}^K$ (in which case $\text{RCodim} M = 1$). We say that such a CR-structure is induced by $\mathbb{C}^K$.

A mapping between two CR-manifolds $f : M_1 \to M_2$ is called a CR-mapping, if for every $p \in M_1$:

(i) $df(p)$ maps $T_p^c(M_1)$ to $T_{f(p)}^c(M_2)$, and (ii) $df(p)$ is complex linear on $T_p^c(M_1)$. Two CR-manifolds $M_1$, $M_2$ are called CR-equivalent, if there is a CR-diffeomorphism from $M_1$ onto $M_2$. Such a CR-diffeomorphism $f$ is called a CR-isomorphism.

In this paper we are interested in the equivalence problem for CR-manifolds. This problem can be viewed as a special case of the equivalence problem for G-structures. Let $G \subset GL(m, \mathbb{R})$ be a subgroup. A G-structure on an $m$-dimensional manifold $M$ is a subbundle $Q$ of the frame bundle $F(M)$ over $M$ which is a principal $G$-bundle over $M$. Two $G$-structures $Q_1$, $Q_2$ on manifolds $M_1$, $M_2$ respectively are called equivalent, if there is a diffeomorphism $f$ from $M_1$ onto $M_2$ such that the induced mapping $f_* : F(M_1) \to F(M_2)$ maps $Q_1$ onto $Q_2$. Such a diffeomorphism $f$ is called an isomorphism of the G-structures. A CR-structure on a manifold $M$ of CR-dimension $n$ and CR-codimension $k$ (so that $m = 2n + k$) is a G-structure, where $G$ is the group of linear transformations.

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of \(\mathbb{C}^n \oplus \mathbb{R}^k\) that preserve the first component and are complex linear on it. The notion of equivalence of such \(G\)-structures is then exactly that of \(CR\)-structures.

É. Cartan developed a general approach to the equivalence problem for \(G\)-structures (see [St]) that works for many important examples of \(G\)-structures. We will be looking for a solution to the equivalence problem in the spirit of Cartan’s work. Namely, we will be trying to reduce the \(CR\)-structure in consideration to an \(\{e\}\)-structure, or parallelism, where \(\{e\}\) is the one-element group. An \(\{e\}\)-structure on an \(N\)-dimensional manifold \(P\) is given by a 1-form \(\omega\) on \(P\) with values in \(\mathbb{R}^N\) such that for every \(x \in P\), \(\omega(x)\) is an isomorphism of \(T_x(P)\) onto \(\mathbb{R}^N\). The problem of local equivalence for generic parallelisms is well-understood (see [St]).

Let \(\mathcal{C}\) be a class of manifolds equipped with \(G\)-structures. We say that \(G\)-structures on manifolds from \(\mathcal{C}\) are \(s\)-reducible to parallelisms if for any \(M \in \mathcal{C}\) there is a sequence of principle bundles

\[
P^s \to \ldots \to P^2 \to P^1 \to M
\]

and a parallelism \(\omega\) on \(P^s\) such that:

(i) Any isomorphism of \(G\)-structures \(f : M_1 \to M_2\) for \(M_1, M_2 \in \mathcal{C}\) can be lifted to a diffeomorphism \(F : P^s_1 \to P^s_2\) such that \(F^*\omega_2 = \omega_1\);

(ii) Any diffeomorphism \(F : P^s_1 \to P^s_2\) such that \(F^*\omega_2 = \omega_1\), is a lift of an isomorphism of the \(G\)-structures \(f : M_1 \to M_2\), for \(M_1, M_2 \in \mathcal{C}\).

In the above definition we say that \(F\) is a lift of \(f\) if

\[
\pi_2^s \circ \ldots \circ \pi_2^s \circ F = f \circ \pi_1^s \circ \ldots \circ \pi_1^s.
\]

From now on we will concentrate on solving the equivalence problem, in the sense of reducing to parallelisms, for \(CR\)-structures of certain classes \(\mathcal{C}\) that we are now going to introduce. Let \(M\) be a \(CR\)-manifold. For every \(p \in M\) consider the complexification \(T^c_p(M) \otimes_{\mathbb{R}} \mathbb{C}\). Clearly, this complexification can be represented as the direct sum

\[
T^c_p(M) \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}_p(M) \oplus T^{(0,1)}_p(M),
\]

where

\[
T^{(1,0)}_p(M) := \{X - iJ_pX : X \in T^c_p(M)\},
\]

\[
T^{(0,1)}_p(M) := \{X + iJ_pX : X \in T^c_p(M)\}.
\]

The \(CR\)-structure on \(M\) is called integrable if for any local sections \(Z, Z'\) of the bundle \(T^{(1,0)}(M)\), the vector field \([Z, Z']\) is also a section of \(T^{(1,0)}(M)\). It is not difficult to see that if \(M \subset \mathbb{C}^K\) and the \(CR\)-structure on \(M\) is induced by \(\mathbb{C}^K\), then it is integrable. In this paper all \(CR\)-structures are assumed to be integrable.

An important characteristic of a \(CR\)-structure called the Levi form comes from taking commutators of local sections of \(T^{(1,0)}(M)\) and \(T^{(0,1)}(M)\). Let \(p \in M\), \(z, z' \in T^{(1,0)}_p(M)\), and \(Z, Z'\) be local sections of \(T^{(1,0)}(M)\) near \(p\) such that \(Z(p) = z\), \(Z'(p) = z'\). The Levi form of \(M\) at \(p\) is the Hermitian form on \(T^{(1,0)}_p(M)\) with values in \((T_p(M)/T^c_p(M)) \otimes_{\mathbb{R}} \mathbb{C}\) given by

\[
L_M(p)(z, z') := i[Z, Z'](p)(\mod T^c_p(M) \otimes_{\mathbb{R}} \mathbb{C}).
\]

The Levi form is defined uniquely up to the choice of coordinates in \((T_p(M)/T^c_p(M)) \otimes_{\mathbb{R}} \mathbb{C}\), and, for fixed \(z\) and \(z'\), its value does not depend on the choice of \(Z\) and \(Z'\).
Let $H = (H_1, \ldots, H_k) : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}^k$ be a $\mathbb{R}^k$-valued Hermitian form on $\mathbb{C}^n$. We say that $H$ is non-degenerate if:

(i) The scalar Hermitian forms $H_1, \ldots, H_k$ are linearly independent over $\mathbb{R}$;

(ii) $H(z, z') = 0$ for all $z' \in \mathbb{C}^n$ implies $z = 0$.

A CR-structure on $M$ is called Levi non-degenerate, if its Levi form at any $p \in M$ is non-degenerate. In this paper we consider only Levi-nondegenerate manifolds.

For any Hermitian form $H$ there is a corresponding standard CR-manifold $Q_H \subset \mathbb{C}^{n+k}$ of CR-dimension $n$ and CR-codimension $k$ defined as follows:

$$Q_H := \{ (z, w) : \text{Im } w = H(z, z) \},$$

where $z := (z_1, \ldots, z_n)$, $w := (w_1, \ldots, w_k)$ are coordinates in $\mathbb{C}^{n+k}$. The manifold $Q_H$ is often called the quadric associated with the form $H$. Clearly, the Levi form of $Q_H$ at any point is given by $H$.

An important tool in all the considerations below is the automorphism group of $Q_H$. Let $\text{Aut}(Q_H)$ denote the collection of all local CR-isomorphisms of $Q_H$ to itself that we call local CR-automorphisms. It turns out that, if $H$ is non-degenerate, then any $f \in \text{Aut}(Q_H)$ extends to a rational (more precisely, a matrix fractional quadratic) map of $\mathbb{C}^{n+k}$ [KT], [F], [Tu2], [ES5]. Thus, for a non-degenerate $H$, $\text{Aut}(Q_H)$ is a finite-dimensional Lie group, and we denote by $\text{Aut}_e(Q_H)$ its identity component. Note that $Q_H$ is a homogeneous manifold since the global CR-automorphisms

$$z \mapsto z + z^0, \quad w \mapsto w + w^0 + 2iH(z, z^0),$$

for $(z^0, w^0) \in Q_H$, act transitively on $Q_H$. Therefore, it is important to consider the isotropy group of a point of $Q_H$, say the origin, i.e. the group of local CR-automorphisms of $Q_H$ preserving the origin. We denote this subgroup of $\text{Aut}(Q_H)$ by $\text{Aut}_0(Q_H)$ and its identity component by $\text{Aut}_{0,e}(Q_H)$. We also introduce the group $\text{Aut}_{lin}(Q_H) \subset \text{Aut}_0(Q_H)$ of linear automorphisms of $Q_H$ and its identity component $\text{Aut}_{lin,e}(Q_H)$. All these groups are finite-dimensional Lie groups. We call a Levi non-degenerate CR-manifold $M$ weakly uniform, if for any pair of points $p, q \in M$, the groups $\text{Aut}_{lin,e}(Q_{\mathcal{L}_M(p)})$, $\text{Aut}_{lin,e}(Q_{\mathcal{L}_M(q)})$ are isomorphic, and the isomorphism extends to an isomorphism between $\text{Aut}_{0,e}(Q_{\mathcal{L}_M(p)})$ and $\text{Aut}_{0,e}(Q_{\mathcal{L}_M(q)})$.

Let $H^1, H^2$ be two $\mathbb{R}^k$-valued Hermitian forms on $\mathbb{C}^n$. We say that $H^1$ and $H^2$ are equivalent, if there exist linear transformations $A$ of $\mathbb{C}^n$ and $B$ of $\mathbb{R}^k$ such that

$$H^2(z, z) = BH^1(Az, Az)$$

for all $z \in \mathbb{C}^n$. We call a CR-manifold $M$ strongly uniform, if the forms $\mathcal{L}_M(p)$ are equivalent for all $p \in M$. If, for example, $M$ is Levi non-degenerate and $CR\text{codim}M = 1$ then $M$ is strongly uniform. Clearly, for a Levi non-degenerate $M$, strong uniformity implies weak uniformity.

Existing results on the equivalence problem for CR-structures treat two large classes of Levi-nondegenerate manifolds: (i) strongly uniform manifolds and (ii) weakly uniform manifolds, for which, in addition, the groups $\text{Aut}_0(Q_{\mathcal{L}_M(p)})$ are “sufficiently small”; in particular, $\text{Aut}_0(Q_{\mathcal{L}_M(p)}) = \text{Aut}_{lin}(Q_{\mathcal{L}_M(p)})$.

É. Cartan [Cs] solved the problem for 3-dimensional Levi-nondegenerate CR-manifolds of CR-dimension 1 (such manifolds are, of course, strongly uniform). Tanaka in 1967 obtained a solution for all Levi-nondegenerate strongly uniform manifolds [Ta], but his result became widely known
only after the Chern-Moser work in 1974 [CM] where the problem was solved independently for Levi-nondegenerate manifolds of CR-codimension 1 (see also [Ta2], [J]). We note that although Tanaka’s pioneering construction is very important and applies to very general situations (that include also geometric structures other than CR-structures), his treatment of the case of CR-codimension 1 is far less detailed and clear than that due to Chern and Moser (see [K] for a discussion of this matter). For example, Tanaka’s construction gives 3-reducibility to parallelisms, whereas Chern’s original construction gives 2-reducibility and even 1-reducibility [BS]. The structure group of the single bundle $P \to M$ that arises in Chern’s construction, is $\text{Aut}_{0,e}(Q_H)$ (or, alternatively, $\text{Aut}_0(Q_H)$), where $H$ is a Hermitian form equivalent to any of $\mathcal{L}_M(p)$, $p \in M$, and the parallelism $\omega$ takes values in the Lie algebra of infinitesimal automorphisms of $Q_H$ (we denote it by $\mathfrak{g}_H$). This algebra is the Lie algebra of the group $\text{Aut}(Q_H)$ and is well-understood (see [Sa], [B1]); in particular, $\mathfrak{g}_H$ is a graded Lie algebra: $\mathfrak{g}_H = \bigoplus_{k=-2}^2 \mathfrak{g}^k_H$, where the components $\mathfrak{g}^1_H$, $\mathfrak{g}^2_H$ are responsible for non-linear automorphisms. In Tanaka’s construction, however, the parallelism takes values in a certain maximal prolongation $\tilde{\mathfrak{g}}_H$ of $\bigoplus_{k=-2}^2 \mathfrak{g}^k_H$; the coincidence of $\tilde{\mathfrak{g}}_H$ and $\mathfrak{g}_H$ is not in general established (see Section 5 for a discussion). Further, it is shown in [CM] (see also [BS]) that the parallelism $\omega$ from Chern’s construction is in fact a Cartan connection, i.e. changes in a regular way under the action of the structure group of the bundle. Namely, if for $\eta \in \text{Aut}_{0,e}(Q_H)$, $L_\eta$ denotes the (left) action by $\eta \in \text{Aut}_{0,e}(Q_H)$ on $P$, then $L_\eta^*\omega = \text{Ad}(\eta)\omega$, where $\text{Ad}$ is the adjoint representation of $\text{Aut}_{0}(Q_H)$ on $\mathfrak{g}_H$. It is not clear from [Ta1] (even in the case of CR-codimension 1) whether the sequence of bundles $P^3 \to P^2 \to P^1 \to M$ there can be reduced to a single bundle and whether the parallelism defined on $P^3$ behaves in any sense like a Cartan connection (see, however, later work in [Ta2], [Ta3]). Being more detailed, Chern’s construction also allows one to investigate the important curvature form of $\omega$, i.e. the 2-form $\Omega := dw - \frac{1}{2}[\omega, \omega]$ and find its precise expansion. It also can be used to introduce special invariant curves on the manifold called chains that turned out to be very important in the study of real hypersurfaces in $\mathbb{C}^k$. These and other differences between Tanaka’s and Chern’s construction motivated our work.

The results in [Ta2], [Ta3], [CM], in particular, solve the equivalence problem for Levi-nondegenerate CR-manifolds of CR-codimension 1, thus we concentrate on the case of higher CR-codimensions. Certain Levi-nondegenerate weakly uniform CR-structures of codimension 2 were treated in [La], [M]. The conditions imposed on the Levi form in these papers are stronger than non-degeneracy and force the groups $\text{Aut}_0(Q_{\mathcal{L}_M(p)})$, $p \in M$, to be minimal possible; in particular, they contain only linear transformations of a special form (this of course implies that certain Levi-nondegenerate weakly uniform CR-actions by $\eta$ geometric structures other than CR-structures). Pioneering construction is very important and applies to very general situations (that include also the cases $\text{CRdim}M > 2$ and the additional condition $\text{CRdim}M > (\text{CRcodim}M)^2$ was treated in [CM]. A motivation to consider manifolds with Levi form satisfying conditions as in [M] for $\text{CRdim}M \geq 7$, [La], [CM] is that, in the considered situations, these conditions are stable, i.e., if they are satisfied at a single point $p$ of a manifold $M$, they are also satisfied on an neighbourhood of $p$. Moreover, quadrics associated with Levi forms as in [M] (for $\text{CRdim}M \geq 7$), [La] are dense in the space of all Levi non-degenerate quadrics.

In this paper we consider the case $\text{CRdim}M = \text{CRcodim}M = 2$. This is one of two exceptional cases among all CR-structures with $\text{CRcodim}M > 1$ in the following sense: typically (in fact, always except for the cases $\text{CRdim}M = \text{CRcodim}M = 2$ and $(\text{CRdim}M)^2 = \text{CRcodim}M$) for generic non-degenerate Hermitian forms, the corresponding quadrics have only linear automorphisms [M], [B2], [ES6]. In the case that we consider, however, generic quadrics always have plenty of non-linear automorphisms. Any non-degenerate Hermitian form $H = (H_1, H_2)$ on $\mathbb{C}^2$ is equivalent to one of the following:

$$
H^1(z, \bar{z}) := (|z_1|^2 + |z_2|^2, z_1 \bar{z}_2 + z_2 \bar{z}_1),
$$
$$
H^{-1}(z, \bar{z}) := (|z_1|^2 - |z_2|^2, z_1 \bar{z}_2 + z_2 \bar{z}_1),
$$
$$
H^0(z, \bar{z}) := (|z_1|^2, z_1 \bar{z}_2 + z_2 \bar{z}_1).
$$
These forms are called respectively hyperbolic, elliptic and parabolic. We will be interested in the case of strongly uniform CR-manifolds whose Levi form is either at every point equivalent to the hyperbolic form, or it is at every point equivalent to the elliptic form. We will call such manifolds hyperbolic and elliptic respectively. Clearly, the conditions of hyperbolicity and ellipticity are stable: if the Levi form of a CR-manifold $M$ at $p \in M$ is equivalent to the hyperbolic or elliptic form, then there is a neighbourhood of $p$ which is respectively a hyperbolic or elliptic manifold. Moreover, the collection of all hyperbolic and elliptic quadrics is an open dense subset in the space of all Levi non-degenerate quadrics of CR-codimension and CR-dimension 2. We denote the sets of all hyperbolic and elliptic manifolds by $C^1$ and $C^{-1}$ respectively.

The equivalence problem for hyperbolic and elliptic CR-manifolds is, of course, covered by Tanaka’s construction in [Ta1]. Therefore, our main goal is to produce, in this particular case, a construction different from that in [Ta1], such that it would be more detailed and easier to use in applications. We achieve this by following the main steps of Chern’s reduction in [CM], although a great many things will have to be treated differently. Although we study just manifolds with $CR\dim M = CR\text{codim} M = 2$, the considered case possesses a rich structure: the groups $\text{Aut}_0(Q_{H^\delta})$ are large in the sense that they contain substantial non-linear part (here $\dim (\mathfrak{q}_{H^\delta}^2 \oplus \mathfrak{g}_{H^\delta}^2) = 6$ [ES1]).

We will formulate our main result in Section 1 and discuss it in Section 3; here we list just a few features of our construction and its applications:

(i) We obtain 2-reducibility to parallelisms, i.e. sequences of two principal bundles $P^{2,\delta} \to P^{1,\delta} \to M$ for $M \in C^\delta$.

(ii) The structure groups of $P^{1,\delta}$, $P^{2,\delta}$ are simply described groups $G^{1,\delta}$, $G^{2,\delta}$, where $G^{2,\delta}$ is a subgroup of $\text{Aut}_{0,e}(Q_{H^\delta})$, whereas in Tanaka’s constructions the structure groups on each step are found as certain very special factor-groups of subgroups of $\text{Aut}_{0,e}(Q_{H^\delta})$.

(iii) The parallelism $\omega^\delta$ defined on $P^{2,\delta}$ takes values in $\mathfrak{g}_{H^\delta}$ rather than in the abstractly defined Lie algebra $\mathfrak{g}_{H^\delta}$ as in [Ta1].

(iv) There is an explicit transformation formula for $\omega^\delta$ under the action of $G^{2,\delta}$ on $P^{2,\delta}$ that shows that $\omega^\delta$ is not “too far” from being a Cartan connection. We also explicitly calculate the obstructions for $\omega^\delta$ to be a Cartan connection. The obstructions are given by two scalar CR-invariants (i.e. CR-invariant functions) on $P^{2,\delta}$, and we study manifolds for which these invariants vanish; we term such manifolds semi-flat. It turns out that the invariant theory on semi-flat manifolds is completely analogous to that in the case of CR-dimension and CR-codimension 1, if one substitutes in all the formulas scalars by matrices from a certain commutative algebra.

(v) We calculate precisely the obstruction to 1-reducibility, that is, we can say when the sequence of two bundles $P^{2,\delta} \to P^{1,\delta} \to M$ can be reduced to a single bundle with structure group $\text{Aut}_{0,e}(Q_{H^\delta})$. The obstruction is given by a single scalar CR-invariant on $P^{2,\delta}$.

(vi) We obtain exact expansions for the curvature form of $\omega^\delta$ in terms of the components of $\omega^\delta$. This allows us, for example, to describe all CR-flat manifolds in much the same way as in the case of CR-codimension 1: all such manifolds must be locally CR-equivalent to $Q_{H^\delta}$.

There is one more issue that does not seem to be tractable from Tanaka’s construction and that in fact was a starting point for our work. Namely, we are interested in finding analogues of chains for CR-manifolds of $CR$-codimension $> 1$. In the case of $CR$-codimension 1, chains arise in ...
independently in the geometric construction as well as in the construction of the analytic normal form for a defining function of a real-analytic hypersurface in $\mathbb{C}^K$. In the geometric approach chains are the projections to $M$ of the integral manifolds of a certain distribution on $P$ that consists of parallel subspaces with respect to the parallelism. In the analytic approach chains are certain curves that locally become straight lines of a special form in normal coordinates. For some classes of real-analytic $CR$-submanifolds of $\mathbb{C}^K$ of $CR$-codimension $> 1$ analogues of the Chern-Moser normal forms have been found in [Lo1], [ES2]–[ES4]. These normal forms have led to certain analogues of chains that are submanifolds of $M$ of dimension equal to $CR\text{codim}M$. However, they do not inherit all the properties of chains in the case $CR\text{codim}M = 1$. In particular, translations along such chains do not preserve all conditions of the normal forms; in other words, such chains can be regarded as proper chains only at a single point (the center of normalization). The first motivation for the present work was the fact that we did not have any reasonable explanation to this phenomenon. Our initial idea was to construct proper analogues of chains (or to understand why such construction is impossible) by using a reduction of the $CR$-structure to parallelisms rather than normal forms. Our approach to some extent clarifies the matter. Namely:

(vii) The construction leads to a certain distribution on $P^{2,\delta}$ (that we call the chain distribution) which is analogous to Chern’s distribution. However, this distribution is not in general involutive, and thus does not in general have integral manifolds. It is worth noting that the obstructions for the distribution to be integrable exactly coincide with those for the parallelism to be a Cartan connection. In particular, the distribution gives proper chains (that we call $G$-chains) on semi-flat manifolds.

Thus, the parallelism in general does not generate proper chains. However, there are many submanifolds of $P^{2,\delta}$ whose tangent space at a given point is an element of the chain distribution. Most likely, the projections of a family of such submanifolds to $M$ are the chains arising in [Lo1], [ES2] and thus are “chains at a single point”. It may be that in applications one should be content with considering the chain distribution itself without trying to integrate it, that is, with considering only “infinitesimal chains”.

The paper is organized as follows. In Section 1 we collect all necessary facts concerning the groups $\text{Aut}_{e}(Q_{H^*})$, $\text{Aut}_0(Q_{H^*})$, $\text{Aut}_{0,e}(Q_{H^*})$ and the algebra $\mathfrak{g}_{H^*}$, $\delta = \pm 1$, and formulate our main result (Theorem 1.1). We prove Theorem 1.1 in Section 2. In Section 3 we discuss some corollaries of Theorem 1.1 and applications of the construction used in its proof; in particular, we introduce semi-flat manifolds as manifolds for which the curvature form of the parallelism behaves in some sense like that in the case of $CR$-codimension 1. In the real-analytic case, we also introduce so-called matrix surfaces as submanifolds of $\mathbb{C}^4$ whose defining functions are given by power series of a special form. Matrix surfaces are examples of semi-flat manifolds, and it is very likely that semi-flat manifolds in the real-analytic case locally coincide with matrix surfaces. We conclude Section 3 by proving this statement for hyperbolic manifolds. In Section 4 we reintroduce local normal forms for defining functions of real-analytic hyperbolic and elliptic $CR$-manifolds in $\mathbb{C}^4$ that are certain interpretations of the normal forms constructed in [Lo1], [ES2]. These normal forms in many ways agree with our reduction process of $CR$-structures to parallelisms in the proof of Theorem 1.1. In particular, such normal forms independently define proper chains on matrix surfaces, and it turns out that these chains coincide with $G$-chains. We conclude the paper with Section 5 where we discuss some questions that arose during our work and that we consider important for a better understanding of high-codimensional $CR$-structures.

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the University of Adelaide. The work was completed while the first author was visiting the Centre for Mathematics and Its Applications, The Australian National University.

1 Preliminaries and Formulation of the Main Result

Before we formulate our main result, we will discuss the structure of the groups \( \text{Aut}_e(Q_{H^\delta}) \), \( \text{Aut}_0(Q_{H^\delta}) \), \( \text{Aut}_{0,e}(Q_{H^\delta}) \) and the algebra \(q_{H^\delta}, \delta = \pm 1\).

The groups \( \text{Aut}_{0,e}(Q_{H^\delta}) \) were explicitly found in [ES1] (see also [B2] for the case \( \delta = 1 \)). One of the possible interpretations of the descriptions in [ES1], [B2] is as follows. Denote by \( \mathfrak{A}^\delta \) the commutative algebra of matrices of the form

\[
\begin{pmatrix}
 a & \delta b \\
 b & a \\
\end{pmatrix}
\]

where \( a, b \in \mathbb{C} \). Let \( SU^\delta(2, 1) \) be the group of \( 3 \times 3 \) matrices \( U \) with elements from \( \mathfrak{A}^\delta \) such that

\[
U \begin{pmatrix}
0 & 0 & -\frac{i}{2}E \\
0 & E & 0 \\
\frac{i}{2}E & 0 & 0 \\
\end{pmatrix} \quad \begin{pmatrix}
0 & 0 & -\frac{i}{2}E \\
0 & E & 0 \\
\frac{i}{2}E & 0 & 0 \\
\end{pmatrix}
\]

where \( E \) is the \( 2 \times 2 \) identity matrix, and such that \( \det U = E \). Let \( \mathfrak{A}^{\delta*} \) denote the set of invertible elements in \( \mathfrak{A}^\delta \), \( \text{Re}\mathfrak{A}^\delta \) the set of elements of \( \mathfrak{A}^\delta \) with real entries, and by \( \text{Re}\mathfrak{A}^{\delta*} \) the set of invertible elements in \( \text{Re}\mathfrak{A}^\delta \). It is shown in [ES1] that any element of \( \text{Aut}_{0,e}(Q_{H^\delta}) \) can be viewed as a transformation of the form

\[
\begin{pmatrix}
 E & 0 & 0 \\
-2iA & C & 0 \\
-(R + iA^2A) & CA & C^2C \\
\end{pmatrix},
\]

(1.1)

of the “projective space” \( \mathfrak{A}^{\delta\mathbb{P}^3} := (\mathfrak{A}^\delta \oplus \mathfrak{A}^\delta \oplus \mathfrak{A}^\delta) / \mathfrak{A}^{\delta*} \), where \( A \in \mathfrak{A}^\delta, C \in \mathfrak{A}^{\delta*}, R \in \text{Re}\mathfrak{A}^\delta \). To make a matrix of the form (1.1) belong to \( SU^\delta(2, 1) \) we need to multiply it by \( \sigma \in \mathfrak{A}^{\delta*} \) such that \( \sigma C^2C = E \) and \( \sigma^3 C^2C = E \). Note that this does not change mapping (1.1) as a transformation of \( \mathfrak{A}^{\delta\mathbb{P}^3} \). It is not difficult to show that such a \( \sigma \) always exists and is unique up to multiplication by \( \lambda \in \mathfrak{A}^{\delta*} \) with \( \lambda A = E, \lambda^3 = E \). We denote the set of such \( \lambda \)'s by \( \hat{\mathfrak{A}}^\delta \). A straightforward computation gives that

\[
\hat{\mathfrak{A}}^{-1} = \{ aE : a^3 = 1 \},
\]

\[
\hat{\mathfrak{A}}^1 = \{ aE : a^3 = 1 \} \cup \left\{ a \begin{pmatrix} 1 & \pm i\sqrt{3} \\ \pm i\sqrt{3} & 1 \end{pmatrix} : a^3 = -\frac{1}{8} \right\}.
\]

Therefore, \( \text{Aut}_{0,e}(Q_{H^\delta}) \) is isomorphic to the subgroup of \( SU^\delta(2, 1) \) of matrices of the form

\[
\begin{pmatrix}
 \sigma & 0 & 0 \\
-2i\sigma A & \sigma C & 0 \\
-\sigma(R + iA^2A) & \sigma CA & \sigma C^2C \\
\end{pmatrix},
\]

(1.2)

with \( A \in \mathfrak{A}^\delta, C, \sigma \in \mathfrak{A}^{\delta*}, R \in \text{Re}\mathfrak{A}^\delta, \sigma C^2C = E, \sigma^3 C^2C = E \), factorized by the subgroup \( Z^\delta \) of matrices

\[
\lambda \begin{pmatrix} E & 0 & 0 \\
0 & E & 0 \\
0 & 0 & E \end{pmatrix},
\]

with \( \lambda \in \hat{\mathfrak{A}}^\delta \). Note that \( Z^\delta \) is a discrete subgroup of \( SU^\delta(2, 1) \).
Analogously, the group of transformations of the form (0.1) is isomorphic to the subgroup of $SU^\delta(2, 1)$ of matrices
\[
\begin{pmatrix}
  E & P & Q + iP\overline{P} \\
  0 & E & 2i\overline{P} \\
  0 & 0 & E
\end{pmatrix},
\]
with $P \in \mathfrak{a}^\delta$, $Q \in \mathbb{R} \mathfrak{a}^\delta$. Since any element of $\text{Aut}_e(Q_{H^\delta})$ is the composition of an automorphism from $\text{Aut}_{0,e}(Q_{H^\delta})$ and an automorphism of the form (0.1), it follows that $\text{Aut}_e(Q_{H^\delta})$ is isomorphic to $SU^\delta(2, 1)/Z^\delta$. Therefore, $\mathfrak{g}_{H^\delta}$ is isomorphic to $\mathfrak{su}^\delta(2, 1)$, the Lie algebra of $SU^\delta(2, 1)$, which is clearly the algebra of matrices
\[
\begin{pmatrix}
  X & Y & Z \\
  W & -2i\text{Im}X & 2i\overline{V} \\
  V & -\frac{i}{2}W & -X
\end{pmatrix},
\]
where $X, Y, W \in \mathfrak{a}^\delta$, $Z, V \in \mathbb{R} \mathfrak{a}^\delta$.

Further, the group $\text{Aut}_{0,e}(Q_{H^\delta})$ turns out to be isomorphic to the group of matrices of the form
\[
\begin{pmatrix}
  C^{-1}\overline{C}^{-1} & 0 & 0 \\
  T & \overline{C}^{-1} & 0 \\
  \overline{T} & 0 & C^{-1} \\
  S & i\overline{CT} & -i\overline{CT} E
\end{pmatrix},
\]
where $T \in \mathfrak{a}^\delta$, $C \in \mathfrak{a}^{\delta*}$, $S \in \mathbb{R} \mathfrak{a}^\delta$. The isomorphism that we denote by $\chi^\delta$ is given explicitly as follows: let an element $\eta \in \text{Aut}_{0,e}(Q_{H^\delta})$ be represented by matrix (1.2); then we set
\[
\chi^\delta(\eta) = \begin{pmatrix}
  C^{-1}\overline{C}^{-1} & 0 & 0 & 0 \\
  -2AC^{-1}\overline{C}^{-1} & C^{-1} & 0 & 0 \\
  -2AC^{-1}\overline{C}^{-1} & 0 & \overline{C}^{-1} & 0 \\
  -4RC^{-1}\overline{C}^{-1} & -2iAC^{-1} & 2iAC^{-1} & E
\end{pmatrix}.
\]
It is straightforward to check that $\chi^\delta$ is a group isomorphism.

The primary description of $\text{Aut}_{0,e}(Q_{H^\delta})$ in [ES1] was in fact given in terms of rational mappings of $\mathbb{C}^4$. In particular, it was shown that all automorphisms from $\text{Aut}_{lin,e}(Q_{H^\delta})$ have the form
\[
\begin{align*}
  z^* &= Cz, \\
  w^* &= C\overline{C}w,
\end{align*}
\]
with $C \in \mathfrak{a}^{\delta*}$. Any element of $\text{Aut}_{0,e}(Q_{H^\delta})$ is then a composition of a rational mapping $z^* = z^*(z, w)$, $w^* = w^*(z, w)$, such that $\partial z^*/\partial z(0) = E$, $\partial w^*/\partial z(0) = 0$, $\partial w^*/\partial w(0) = E$, and an automorphism of the form (1.6). It is also easy to see that the full group $\text{Aut}_0(Q_{H^\delta})$ has exactly two connected components, and that the second component is obtained by taking the compositions of mappings from $\text{Aut}_{0,e}(Q_{H^\delta})$ and the linear automorphism
\[
\begin{align*}
  z^* &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z, \\
  w^* &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} w.
\end{align*}
\]
Thus, automorphisms from $\text{Aut}_{0,e}(Q_{H^\delta})$ are characterized among all elements of $\text{Aut}_0(Q_{H^\delta})$ by the condition $\det(\partial w^*/\partial w(0)) > 0$.

We are now ready to formulate the main result of this paper. Let $G^{1,\delta}$ be the group of elements of $\text{Re}\mathfrak{a}^{\delta*}$ of the form $C\overline{C}$, $C \in \mathfrak{a}^{\delta*}$, and $G^{2,\delta}$ be the subgroup of $\text{Aut}_{0,e}(Q_{H^\delta})$ defined by the condition $C\overline{C} = E$. 
THEOREM 1.1 Let \( M \in \mathcal{C}^\delta \) be an oriented manifold. Then there are principal bundles \( P^{1,\delta}, P^{2,\delta} \)

\[
P^{2,\delta} \xrightarrow{\pi^{2,\delta}} P^{1,\delta} \xrightarrow{\pi^{1,\delta}} M
\]

with structure groups \( G^{1,\delta}, G^{2,\delta} \) respectively and a 1-form \( \omega^\delta \) on \( P^{2,\delta} \) such that at any point \( x \in P^{2,\delta} \), \( \omega^\delta(x) \) is an isomorphism between \( T_x(P^{2,\delta}) \) and \( \mathfrak{su}^\delta(2,1) \), and the following holds:

(i) Any CR-isomorphism \( f : M_1 \to M_2 \) between oriented manifolds \( M_1, M_2 \in \mathcal{C}^\delta \) that preserves orientation, can be lifted to a diffeomorphism \( F : P^{2,\delta}_1 \to P^{2,\delta}_2 \) such that \( F^*\omega^\delta_2 = \omega^\delta_1 \);

(ii) Any diffeomorphism \( F : P^{2,\delta}_1 \to P^{2,\delta}_2 \) such that \( F^*\omega^\delta_2 = \omega^\delta_1 \), is a lift of a CR-isomorphism \( f : M_1 \to M_2 \) that preserves orientation, for \( M_1, M_2 \in \mathcal{C}^\delta \).

Moreover, there exists an explicit transformation formula for \( \omega^\delta \) under the action on \( G^{2,\delta} \) on \( P^{2,\delta} \): if for \( \eta \in \text{Aut}_0(Q_H) \), \( L_\eta \) denotes the left action of \( G^{2,\delta} \) on \( P^{2,\delta} \) by \( \eta \), then \( L_\eta^\ast\omega^\delta = \text{Ad}(\eta)\omega^\delta + \ldots \), where \( \text{Ad} \) is the adjoint representation of \( \text{Aut}_0(Q_H) \) on \( \mathfrak{su}^\delta(2,1) \), and \( \ldots \) denotes an error term (see formula (2.59) below).

REMARK 1.2 The condition for the manifolds to be oriented is not important. Theorem 1.1 could be formulated for any manifold from \( \mathcal{C}^\delta \), but then the group \( G^{1,\delta} \) would have to be replaced by

\[
\tilde{G}^{1,\delta} := \left\{ C \mathcal{C} : C \in \mathfrak{a}^{\delta_+} \right\} \bigcup \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} C \mathcal{C} : C \in \mathfrak{a}^{\delta_+} \right\}.
\]

The group \( \tilde{G}^{1,\delta} \) is disconnected. Thus, the bundle \( P^{1,\delta} \) would have a disconnected fibre, and, for an oriented \( M \), the bundle \( P^{1,\delta} \) and therefore the bundle \( P^{2,\delta} \) would be disconnected. To avoid these kinds of disconnectedness, we require the manifolds to be oriented.

REMARK 1.3 Everywhere in this paper we assume manifolds to be \( C^\infty \)-smooth. However, an inspection of the proof of Theorem 1.1 below shows that it is enough to require only some sufficiently high, but finite, degree of smoothness.

2 Proof of Theorem 1.1

Let \( M \) be an oriented connected manifold from \( \mathcal{C}^\delta \). We now fix \( \delta \) and drop it in all superscripts. For any \( p \in M \) denote by \( \mathfrak{m}_p \) the set of all pairs \( (\theta^1, \theta^2) \) of real linearly independent 1-forms defined in a neighbourhood of \( p \) such that:

(i) \( T^c_q(M) = \{ X \in T_q(M) : \theta^1(q)(X) = \theta^2(q)(X) = 0 \} \) for \( q \) close to \( p \),

(ii) There exist complex 1-forms \( \omega^1, \omega^2 \) near \( p \) such that: for all \( q \) close to \( p \) they are complex linear on \( T^c_q(M) \); \( (\theta^\alpha(q), \text{Re} \omega^\alpha(q), \text{Im} \omega^\alpha(q)) \) is a coframe, and near \( p \) the following holds

\[
\begin{align*}
d\theta^1 &= i \left( \omega^1 \wedge \overline{\omega^1} + \delta \omega^2 \wedge \overline{\omega^2} \right) \pmod{\theta^\alpha}, \\
d\theta^2 &= i \left( \omega^1 \wedge \overline{\omega^2} + \omega^2 \wedge \overline{\omega^1} \right) \pmod{\theta^\alpha}.
\end{align*}
\]  

(2.1)
The integrability of the CR-structure and the type of the Levi form imply that \( \mathfrak{m}_p \neq \emptyset \) for any \( p \in M \).

We define a smooth bundle \( P^1 \to M \) as

\[
P^1 := \left\{ \mathfrak{m}_p^+ \right\}_{p \in M},
\]

where \( \mathfrak{m}_p^+ \) is the set of pairs \( y := (\theta^1(p), \theta^2(p)) \) with \( (\theta^1, \theta^2) \in \mathfrak{m}_p \) such that the orientation that they define on the cotangent space \( T^*_p(M) \) coincides with that induced on \( T^*_p(M) \) by the original orientation of \( M \). Clearly, \( P^1 \) so defined is a principal \( G^1 \)-bundle over \( M \). We introduce fibre coordinates \((a, b)\) on \( P^1 \) via the entries of \( C\bar{C} \):

\[
\begin{pmatrix} a & \delta b \\ b & a \end{pmatrix} = C\bar{C}.
\]

To construct the bundle \( P^2 \to P^1 \) we need the following technical lemma.

**Lemma 2.1** Let \((\theta^1, \theta^2) \in \mathfrak{m}_p \) be such that \((\theta^1(p), \theta^2(p)) \in \mathfrak{m}_p^+ \). Then \( \omega^1, \omega^2 \) in (2.1) can be chosen so that the following holds:

\[
d\theta^1 = i \left( \omega^1 \land \bar{\omega}^1 + \delta \omega^2 \land \bar{\omega}^2 \right) + \theta^1 \land \phi^1 + \delta \theta^2 \land \phi^2, \\
d\theta^2 = i \left( \omega^1 \land \bar{\omega}^2 + \omega^2 \land \bar{\omega}^1 \right) + \theta^1 \land \phi^2 + \theta^2 \land \phi^1 + 2\theta^1 \land \mathfrak{Re} \left( \delta r_1 \omega^1 + r_2 \omega^2 \right) + 2\theta^2 \land \mathfrak{Re} \left( r_2 \omega^1 + r_1 \omega^2 \right),
\]

where \( \phi^1, \phi^2 \) are real 1-forms and \( r_1, r_2 \) are smooth complex-valued functions near \( p \).

**Proof.** By Proposition 3.2 of [M] we can assume that \( \omega^1, \omega^2 \) are chosen in such a way that

\[
d\theta^1 = i \left( \omega^1 \land \bar{\omega}^1 + \delta \omega^2 \land \bar{\omega}^2 \right) + \theta^1 \land \phi', \\
d\theta^2 = i \left( \omega^1 \land \bar{\omega}^2 + \omega^2 \land \bar{\omega}^1 \right) + \theta^2 \land \phi',
\]

where \( \phi' \) are real 1-forms near \( p \). Since \((\theta^\alpha, \mathfrak{Re} \omega^\alpha, \mathfrak{Im} \omega^\alpha)\) gives a coframe at every point near \( p \), we have

\[
\phi' = a^\alpha_2 \omega^1 + a^\alpha_2 \omega^2 + b^\alpha_2 \theta^1, \quad \alpha = 1, 2,
\]

where \( \alpha^\alpha_2 \) are complex-valued and \( b^\alpha_2 \) are real-valued functions near \( p \).

We choose the new forms \( \omega^\alpha \) as follows:

\[
\omega^1 := \omega^1, \\
\omega^2 := \omega^2 + \frac{i\delta}{2} \left( a^2_2 - a^2_1 \right) \theta^1 + \frac{i}{2} \left( a^2_1 - a^2_1 \right) \theta^2.
\]

It is now straightforward to check that under this transformation equations (2.3) take the form (2.2).

The lemma is proved. \( \square \)

Let \( \tilde{\theta}^1, \tilde{\theta}^2 \) be the following globally defined tautological 1-forms on \( P^1 \). For \( y := (\theta^1(p), \theta^2(p)) \in P^1 \) set

\[
\tilde{\theta}^\alpha(y) = (\pi^1 \theta^\alpha)(y), \quad \alpha = 1, 2,
\]

where \( \pi^1 : P^1 \to M \) is the natural projection: \( \pi^1(y) = p \). We now define the bundle \( P^2 \) over \( P^1 \) as follows: the fibre over \( y \in P^1 \) is the collection of all coframes at \( y \) of the form \((\tilde{\theta}^\alpha(y), \mathfrak{Re} \tilde{\omega}^\alpha, \mathfrak{Im} \tilde{\omega}^\alpha, \tilde{\phi}^\alpha)\) such that:
(i) $\tilde{\omega}^\alpha = \pi^{1*}\omega^\alpha(y)$, for some complex covectors $\omega^\alpha$ at $p$ that are complex-linear on $T^*_p(M)$;
(ii) $\tilde{\phi}^\alpha$ are real covectors at $y$;

(iii) For some $r_\alpha \in \mathbb{C}$ the following holds:

\[
\begin{align*}
d\tilde{\theta}^1(y) &= i \left( \bar{\omega}^1 \wedge \omega^1 + \delta \bar{\omega}^2 \wedge \bar{\omega}^2 \right) + \tilde{\theta}^1(y) \wedge \tilde{\phi}^1 + \delta \tilde{\theta}^2(y) \wedge \tilde{\phi}^2, \\
d\tilde{\theta}^2(y) &= i \left( \bar{\omega}^1 \wedge \omega^2 + \bar{\omega}^2 \wedge \omega^1 \right) + \tilde{\theta}^1(y) \wedge \tilde{\phi}^2 + \tilde{\theta}^2(y) \wedge \tilde{\phi}^1 + 2\tilde{\theta}^2(y) \wedge \text{Re} \left( \delta r_1 \omega^1 + \delta r_2 \bar{\omega}^2 \right) + 2\tilde{\theta}^1(y) \wedge \text{Re} \left( \delta r_2 \omega^1 + \delta r_1 \bar{\omega}^2 \right).
\end{align*}
\]

The existence of such coframes follows from Lemma 2.1.

From now on we will write an element of $x \in P^2$ in the form: $x := (\tilde{\theta}^\alpha(y), \tilde{\omega}^\alpha, \bar{\omega}^\alpha, \bar{\phi}^\alpha)$. It is a routine calculation to verify that the most general linear transformation that, being applied to $x$, gives an element from the fibre of $P^2$ over $y$, is of the form (1.4):

\[
\begin{pmatrix}
E & 0 & 0 & 0 \\
T & C & 0 & 0 \\
S & iCT & C & 0
\end{pmatrix},
\]

(2.4)

where $T \in \mathfrak{X}^\delta, C \in \mathfrak{X}^{d*}$, $C\bar{C} = E$, $S \in \text{Re} \mathfrak{X}^\delta$. The group of matrices (2.4) is isomorphic to $G^2$ by the isomorphism $\chi$ defined in (1.5). Therefore, $P^2$ is a principle bundle with structure group $G^2$. We will treat the independent entries of $T, C, S$ as fibre coordinates.

We now introduce globally defined tautological 1-forms on $P^2$. Let $x = (\tilde{\theta}^\alpha(y), \tilde{\omega}^\alpha, \bar{\omega}^\alpha, \bar{\phi}^\alpha) \in P^2$. Then we set:

\[
\begin{align*}
\hat{\theta}^\alpha(x) &:= (\pi^{2*}\tilde{\theta}^\alpha)(x), \\
\hat{\omega}^\alpha(x) &:= (\pi^{2*}\tilde{\omega}^\alpha)(x), \\
\hat{\phi}^\alpha(x) &:= (\pi^{2*}\tilde{\phi}^\alpha)(x),
\end{align*}
\]

(2.5)

for $\alpha = 1, 2$, where $\pi^2 : P^2 \rightarrow P^1$ is the projection: $\pi^2(x) = y$. To simplify notation, until the end of this section we drop hats in the forms defined in (2.5). These forms satisfy the equations

\[
\begin{align*}
d\theta^1 &= i \left( \omega^1 \wedge \bar{\omega}^1 + \delta \omega^2 \wedge \bar{\omega}^2 \right) + \theta^1 \wedge \phi^1 + \delta \theta^2 \wedge \phi^2, \\
d\theta^2 &= i \left( \omega^1 \wedge \bar{\omega}^2 + \omega^2 \wedge \bar{\omega}^1 \right) + \theta^1 \wedge \phi^2 + \theta^2 \wedge \phi^1 + 2\theta^1 \wedge \text{Re} \left( \delta r_1 \omega^1 + r_2 \omega^2 \right) + 2\theta^2 \wedge \text{Re} \left( r_2 \omega^1 + r_1 \omega^2 \right),
\end{align*}
\]

(2.6)

for some uniquely determined smooth complex-valued functions $r_\alpha$ on $P^2$.

Next, it follows from the integrability of the $CR$-structure that

\[
d\omega^\alpha = \omega^\beta \wedge \phi_\beta^\alpha + \theta^\beta \wedge \mu_\beta^\alpha,
\]

(2.7)

for some locally defined 1-forms $\phi_\beta^\alpha, \mu_\beta^\alpha$. Differentiating (2.6) and plugging (2.6), (2.7) in the resulting expressions, we get

\[
\begin{align*}
i\omega^1 \wedge \bar{\omega}^1 \wedge \left( \phi_1^1 - 2\text{Re} \phi_1^1 \right) + i\omega^1 \wedge \bar{\omega}^2 \wedge \left( \delta \phi^2 - \delta \phi_1^2 - \bar{\phi}_1^2 \right) + \\
i\omega^2 \wedge \bar{\omega}^1 \wedge \left( \delta \phi^2 - \phi_2^1 - \bar{\phi}_1^2 \right) + i\delta \omega^2 \wedge \bar{\omega}^2 \left( \phi_1^2 - 2\text{Re} \phi_2^2 \right) + \\
\end{align*}
\]
\[ \theta^1 \wedge \left( -d\phi^1 + 2\text{Re} \left( i\mu^1_1 \wedge \overline{\omega^1} + i\delta \mu^1_2 \wedge \overline{\omega^2} + (r_1\omega^1 + \delta r_2^2) \wedge \phi^2 \right) \right) + \]

\[ \theta^2 \wedge \left( -d\phi^2 + 2\text{Re} \left( i\mu^2_1 \wedge \overline{\omega^1} + i\delta \mu^2_2 \wedge \overline{\omega^2} + \delta (r_2\omega^1 + r_1\omega^2) \wedge \phi^2 \right) \right) = 0, \quad (2.8.a) \]

\[ i\omega^1 \wedge \overline{\omega^1} \wedge \left( \phi^2 - 2\text{Re} \phi^2 \right) + i\omega^1 \wedge \overline{\omega^2} \wedge \left( \phi^1 - \phi^1_1 - \phi^2 \right) + \]

\[ i\omega^2 \wedge \overline{\omega^1} \wedge \left( \phi^1 - \phi^2_1 - \phi^1_2 \right) + i\omega^2 \wedge \overline{\omega^2} \left( \delta \phi^2 - 2\text{Re} \phi^2_1 \right) + \]

\[ \theta^1 \wedge \left( -d\phi^1 + 2\text{Re} \left( i\mu^1_1 \wedge \overline{\omega^1} + i\mu^1_1 \wedge \overline{\omega^2} - (r_2\omega^1 + r_1\omega^2) \wedge \phi^2 - \delta dr_1 \wedge \omega^1 - dr_2 \wedge \omega^2 - \delta r_1 (\omega^1 \wedge \phi^1_1 + \omega^2 \wedge \phi^2_1 + \theta^2 \wedge \mu^1_2) - r_2 (\omega^1 \wedge \phi^1_2 + \omega^2 \wedge \phi^2_2 + \theta^2 \wedge \mu^2_2) \right) + 4\text{Re} \left( \delta r_1 \omega^1 \wedge r_2 \omega^2 \wedge \text{Re} (r_2\omega^1 + r_1\omega^2) \right) + \]

\[ \theta^2 \wedge \left( -d\phi^2 + 2\text{Re} \left( i\mu^2_1 \wedge \overline{\omega^1} + i\mu^2_1 \wedge \overline{\omega^2} - (r_1\omega^1 + \delta r_2^2) \wedge \phi^2 - dr_2 \wedge \omega^1 - dr_1 \omega^2 - r_2 (\omega^1 \wedge \phi^1_1 + \omega^2 \wedge \phi^1_2 + \theta^1 \wedge \mu^1_1) - r_1 (\omega^1 \wedge \phi^2_1 + \omega^2 \wedge \phi^2_2 + \theta^1 \wedge \mu^2_1) \right) \right) = 0. \quad (2.8.b) \]

**Lemma 2.2** There exist \( \phi^1_2 \) satisfying (2.7) such that the following holds

\[
\begin{align*}
\text{Re} \phi^1_1 &= \frac{1}{2} \phi^1, \\
\text{Re} \phi^2_1 &= \frac{1}{2} \phi^2, \\
\text{Re} \phi^2_2 &= \frac{1}{2} \phi^1 - \text{Re} \left( \rho_1 \omega^1 + \rho_2 \omega^2 \right), \\
\text{Re} \phi^1_2 &= \frac{1}{2} \delta \phi^2 - \text{Re} \left( \rho_2 \omega^1 + \delta \rho_1 \omega^2 \right), \\
\phi^1_1 + \overline{\phi^1_2} &= \phi^1 - \rho_1 \omega^1 - \rho_2 \omega^2, \\
\delta \phi^2_1 + \overline{\phi^2_2} &= \delta \phi^2 - \rho_2 \omega^1 - \delta \rho_1 \omega^2,
\end{align*}
\]

where \( \rho_\alpha \) are locally defined smooth complex-valued functions on \( P^2 \).

**Proof.** It follows from (2.8) that

\[
\begin{align*}
\phi^1 - 2\text{Re} \phi^1 &= 2\text{Re} \left( a^1_1 \omega^1 + b^1_1 \omega^2 \right) + c^1_1 \theta^1 + d^1 \theta^2, \\
\phi^2 - 2\text{Re} \phi^2 &= 2\text{Re} \left( a^2_1 \omega^1 + b^2_1 \omega^2 \right) + c^2_1 \theta^1 + d^2 \theta^2, \\
\phi^1 - 2\text{Re} \phi^2_1 &= 2\text{Re} \left( a^1_2 \omega^1 + b^2_2 \omega^2 \right) + c^1_2 \theta^1 + d^2 \theta^2, \\
\delta \phi^2 - 2\text{Re} \phi^2_1 &= 2\text{Re} \left( a^2_1 \omega^1 + b^2_1 \omega^2 \right) + c^2_1 \theta^1 + d^2 \theta^2, \\
\phi^1 - \phi^2_2 - \overline{\phi^1_1} &= a_1^1 \omega^1 + b_1^1 \omega^2 + a_1^1 \overline{\omega^1} + b_1^1 \overline{\omega^2} + c_1^1 \theta^1 + d_1^1 \theta^2, \\
\delta \phi^2 - \delta \phi^2_2 - \overline{\phi^2_1} &= a_2^1 \omega^1 + b_2^1 \omega^2 + a_2^1 \overline{\omega^1} + b_2^1 \overline{\omega^2} + c_2^1 \theta^1 + d_1^2 \theta^2, \quad (2.10)
\end{align*}
\]

where \( a^i_\beta, b^i_\beta, a^i_\beta', b^i_\beta', a^i_\beta'', b^i_\beta'', c^i_\beta, d^i_\beta \) are complex-valued and \( c^i_\beta, d^i_\beta \) are real-valued functions satisfying the following relations

\[
a^1_1 + a^2_1 = a^1_1 + a^1_1',
\]
From now on we will assume that $\phi^\beta$ in (2.7) satisfy conditions (2.9). Then (2.8.a) implies:

$$d\phi^1 = 2\Re \left( i\mu_1^1 \wedge \overline{\omega^1} + i\delta \mu_1^2 \wedge \overline{\omega^2} + \left( r_1 \omega^1 + \delta r_2 \omega^2 \right) \wedge \phi^2 \right) + \theta^1 \wedge \psi^1 + \theta^2 \wedge \psi^2,$$

$$d\phi^2 = 2\Re \left( i\delta \mu_2^1 \wedge \overline{\omega^1} + i\mu_2^2 \wedge \overline{\omega^2} + \left( r_2 \omega^1 + r_1 \omega^2 \right) \wedge \phi^2 \right) + \theta^1 \wedge \psi^3 + \theta^2 \wedge \psi^4. \tag{2.12.a}$$

Analogously, (2.8.b) implies:

$$d\phi^1 = 2\Re \left( i\mu_1^2 \wedge \overline{\omega^1} + i\mu_1^1 \wedge \overline{\omega^2} - \left( r_1 \omega^1 + \delta r_2 \omega^2 \right) \wedge \phi^2 - \right.$$

$$\left. dr_2 \wedge \omega^1 - dr_1 \wedge \omega^2 - r_2 \left( \omega^1 \wedge \phi_1^1 + \omega^2 \wedge \phi_1^2 + \theta^1 \wedge \mu_1^1 \right) - r_1 \left( \omega^1 \wedge \phi_2^2 + \omega^2 \wedge \phi_2^1 + \theta^1 \wedge \mu_1^1 \right) \right) + \theta^1 \wedge \psi^5 + \theta^2 \wedge \psi^6, \tag{2.12.c}$$

$$d\phi^2 = 2\Re \left( i\mu_2^2 \wedge \overline{\omega^1} + i\mu_1^1 \wedge \overline{\omega^2} - \left( r_2 \omega^1 + r_1 \omega^2 \right) \wedge \phi^2 - \right.$$

$$\left. \delta dr_1 \wedge \omega^1 - dr_2 \wedge \omega^2 - \delta r_1 \left( \omega^1 \wedge \phi_1^1 + \omega^2 \wedge \phi_1^2 + \theta^2 \mu_2^1 \right) - \right.$$

$$\left. r_2 \left( \omega^1 \wedge \phi_1^2 + \omega^2 \wedge \phi_2^2 + \theta^2 \mu_2^1 \right) \right) + 4\Re \left( \delta r_1 \omega^1 + r_2 \omega^2 \right) \wedge$$

$$\left( r_2 \omega^1 + r_1 \omega^2 \right) \wedge \theta^1 \wedge \psi^7 + \theta^2 \wedge \psi^8. \tag{2.12.d}$$
In formulas (2.12) $\psi^\alpha$ are real locally defined 1-forms such that

$$\psi^2 = \delta \psi^3 + s_1 \theta^1 + s_2 \theta^2,$$
$$\psi^8 = \psi^5 + s_3 \theta^1 + s_4 \theta^2,$$

(2.13)

where $s_\alpha$ are real-valued functions.

A lengthy but elementary calculation now shows that the 1-forms $\phi^\alpha_\beta$, $\mu^\alpha_\beta$, $\psi^\alpha$ satisfying (2.7), (2.9), (2.12) are defined up to transformations of the form

$$\phi^1_\ast = \phi^1_1 + h \theta^1 + g \theta^2,$$
$$\phi^2_\ast = \phi^2_1 + \delta h' \theta^1 + \delta g' \theta^2,$$
$$\phi^2_2 = \phi^2_2 + h \theta^1 + g \theta^2,$$
$$\mu^1_\ast = \mu^1_1 + h \omega^1 + \theta^2 + p_1 \theta^1 + q_1 \theta^2,$$
$$\mu^2_\ast = \mu^2_2 + \delta h' \omega^1 + h \omega^2 + p_2 \theta^1 + q_2 \theta^2,$$
$$\psi^1 = \psi^1 + 2 \text{Re} \left( i \overline{\mu^1_1 \omega^1} + i \delta \overline{\mu^1_2 \omega^2} \right) + \sigma_1 \theta^1 + \sigma_2 \theta^2,$$
$$\psi^2 = \psi^2 + 2 \text{Re} \left( i \overline{\mu^2_1 \omega^1} + i \delta \overline{\mu^2_2 \omega^2} \right) + \sigma_2 \theta^1 + \sigma_3 \theta^2,$$
$$\psi^3 = \psi^3 + 2 \text{Re} \left( i \delta \overline{\mu^1_1 \omega^1} + i \overline{\mu^1_2 \omega^2} \right) + \sigma_4 \theta^1 + \sigma_5 \theta^2,$$
$$\psi^4 = \psi^4 + 2 \text{Re} \left( i \delta \overline{\mu^2_1 \omega^1} + i \overline{\mu^2_2 \omega^2} \right) + \sigma_5 \theta^1 + \sigma_6 \theta^2,$$
$$\psi^5 = \psi^5 + 2 \text{Re} \left( i \overline{\mu^1_1 \omega^1} + i \overline{\mu^1_2 \omega^2} \right) + \sigma_7 \theta^1 + \sigma_8 \theta^2,$$
$$\psi^6 = \psi^6 + 2 \text{Re} \left( i \overline{\mu^2_1 \omega^1} + i \overline{\mu^2_2 \omega^2} \right) + \sigma_8 - 2 \text{Re} \left( r_1 q_2 + r_2 q_1 \right) \theta^1 + \sigma_9 \theta^2,$$

(2.14)

where $g, g', h, h'$ are imaginary-valued, $\sigma_\alpha$ are real-valued, $p_\alpha$, $q_\alpha$ are complex-valued functions. The same calculation shows that $\rho_\alpha$ are chosen uniquely and therefore are globally defined on $P^2$.

We now need to introduce extra conditions that would fix the parameters in (2.14) uniquely and therefore, taken together with (2.7), (2.9) and (2.12), would fix the forms $\phi^\alpha_\beta$, $\mu^\alpha_\beta$, $\psi^\alpha$ uniquely. The first set of conditions comes from comparing two pairs of equations: (2.12.a), (2.12.c) and (2.12.b), (2.12.d). From this comparison we get:

$$\mu^2_1 = \mu^1_1 + a_1 \omega^1 + b_1 \omega^2 + c_1 \overline{\omega^1} + d_1 \overline{\omega^2} + h_1 \theta^1 + g_1 \theta^2 + 2i \overline{\sigma^1_2 \overline{\phi^2_1}} + i \overline{\sigma^1_1 \phi^2_1},$$
$$\mu^2_1 = \delta \mu^2_1 - b_2 \omega^2 + d_1 \overline{\omega^2} + d_2 \overline{\omega^2} + h_2 \theta^1 + g_2 \theta^2 + 2i \overline{\sigma^2_2 \overline{\phi^1_1}} - i \overline{\sigma^1_1 \phi^2_1} + i \overline{\sigma^1_1 \phi^2_1},$$
$$\psi^5 = \psi^1 + 2 \text{Re} \left( i \overline{\sigma^1_1 \omega^1} + i \overline{\sigma^1_2 \omega^2} \right) + s_5 \theta^1 + s_6 \theta^2 + 2 \text{Re} \left( r_2 \mu^1_1 + r_1 \mu^1_2 \right),$$
$$\psi^6 = \psi^2 + 2 \text{Re} \left( i \overline{\mu^1_1 \omega^1} + i \overline{\mu^1_2 \omega^2} \right) + s_6 \theta^1 + s_7 \theta^2,$$

(2.14)
\[
\begin{align*}
\mu_2^2 &= \mu_1^2 + a_3\omega^1 + b_3\omega^2 + c_3\omega^1 + d_3\omega^2 + h_3\theta^1 + g_3\theta^2 - \\
&\quad 2i\bar{r}_1\phi^2 + id\bar{r}_2 - id\bar{r}_1\phi_2^1 - i\bar{r}_2\phi_2^1, \\
\mu_2^1 &= \delta \mu_1^1 + a_4\omega^1 + (-\delta a_3 + i(|r_1|^2 - \delta |r_2|^2)\omega^2 + c_4\omega^1 + \\
&\quad (\delta c_3 + i(r_1^2 + \delta r_2^2))\omega^2 + \delta h_4\theta^1 + \delta g_4\theta^2 - 2i\delta r_2\phi^2 + id\bar{r}_1 - \bar{r}_1\phi_1^1 - i\delta r_2\phi_2^1, \\
\psi^7 &= \psi^3 - 2\text{Re} \left( i\bar{r}_4\omega^1 + i\bar{r}_5\omega^2 \right) + s_8\theta^1 + s_9\theta^2, \\
\psi^8 &= \psi^4 - 2\text{Re} \left( i\bar{r}_4\omega^1 + i\bar{r}_5\omega^2 \right) + s_6\theta^1 + s_10\theta^2 + \\
&\quad 2\text{Re} \left( \delta r_1\mu_1^1 + r_2\mu_2^2 \right), \quad (2.15)
\end{align*}
\]

where \(a_1, b_2\) are imaginary-valued, \(s_\alpha\) are real-valued, the rest of the functions are complex-valued, and \(\text{Re} \ b_3 = \text{Im} \ (r_2\bar{r}_3)\), \(\text{Re} \ a_4 = \delta \text{Im} \ (r_1\bar{r}_2)\).

We now choose \(g, g', h', h\) in (2.14) so that

\[
a_1^* = 0, \quad \text{Im} \ a_1^* = 0, \quad (2.16)
\]

where the functions with asterisks correspond to the forms with asterisks from (2.14). This can be achieved by setting

\[
\begin{align*}
h - \delta g' &= a_1, \\
h' - g &= ida_4. \quad (2.17)
\end{align*}
\]

Choice (2.17) uniquely fixes the functions \(a_1^*, a_4^*\) and therefore all the functions \(a_\alpha^*, b_\alpha^*, c_\alpha^*, d_\alpha^*\).

We also choose \(\sigma_\alpha\) in (2.14) so that in (2.13), (2.15) one has

\[
\begin{align*}
s_\alpha^* &= 0, \quad \alpha = 1, \ldots, 10, \quad (2.18)
\end{align*}
\]

by setting

\[
\begin{align*}
\delta \sigma_4 - \sigma_2 &= s_1, \\
\delta \sigma_5 - \sigma_3 &= s_2, \\
\sigma_7 - \sigma_{11} - 2\text{Re} \left( \delta r_1q_1 + r_2q_2 \right) &= s_3, \\
\sigma_8 - \sigma_{12} &= s_4, \\
\sigma_1 - \sigma_7 + 2\text{Re} \left( r_2p_1 + r_1p_2 \right) &= s_5, \\
\sigma_2 - \sigma_8 + 2\text{Re} \left( r_2q_1 + r_1q_2 \right) &= s_6, \\
\sigma_3 - \sigma_9 &= s_7, \\
\sigma_4 - \sigma_{10} &= s_8, \\
\sigma_5 - \sigma_{11} &= s_9, \\
\sigma_6 - \sigma_{12} + 2\text{Re} \left( \delta r_1q_3 + r_2q_4 \right) &= s_{10}. \quad (2.19)
\end{align*}
\]

To introduce further restrictions on the parameters \(g, g', h, h', p_\alpha, q_\alpha, \sigma_\alpha\) we need to differentiate equations (2.7). By doing this and using (2.6), (2.7) we get

\[
\begin{align*}
\omega^1 \wedge \left( -d\phi_1^1 + \phi_1^2 \wedge \phi_2^1 - i\mu_1^1 \wedge \omega^1 - i\mu_2^1 \wedge \omega^1 \right) + \\
\omega^2 \wedge \left( -d\phi_2^1 + \phi_2^2 \wedge \phi_1^1 + \phi_1^2 \wedge \phi_2^2 - i\mu_2^1 \wedge \omega^1 - i\delta \mu_1^1 \wedge \omega^2 \right) + \\
\theta^1 \wedge \left( -d\mu_1^1 + \mu_1^1 \wedge \phi_1^1 + \mu_2^1 \wedge \phi_2^1 - \mu_1^1 \wedge \phi^1 - \\
&\quad - \mu_2^1 \wedge \phi^2 \right).
\end{align*}
\]
\[ \mu_2^1 \wedge \left( \phi^2 + 2\text{Re} \left( \delta r_1 \omega^1 + r_2 \omega^2 \right) \right) + \]
\[ \theta_2^1 \wedge \left( -d\mu_1^1 + \mu_1^1 \wedge \phi_1^1 + \mu_2^2 \wedge \phi_2^2 - \delta \mu_1^1 \wedge \phi - \right. \]
\[ \left. \mu_2^2 \wedge \left( \phi^1 + 2\text{Re} \left( r_2 \omega^1 + r_1 \omega^2 \right) \right) \right) = 0, \]
\[ \omega^1 \wedge \left( -d\phi_1^1 + \phi_1^1 \wedge \phi_1^1 + \phi_2^2 - i\mu_1^2 \wedge \overline{\omega^1} - i\mu_2^2 \wedge \overline{\omega^2} \right) + \]
\[ \omega^2 \wedge \left( -d\phi_2^2 + \phi_1^2 \wedge \phi_1^1 - i\mu_1^2 \wedge \overline{\omega^2} - i\delta \mu_1^2 \wedge \overline{\omega^2} \right) + \]
\[ \theta_1^1 \wedge \left( -d\mu_1^2 + \mu_1^1 \wedge \phi_1^1 + \mu_1^2 \wedge \phi_2^2 - \mu_1^1 \wedge \phi - \right. \]
\[ \left. \mu_2^2 \wedge \left( \phi^2 + 2\text{Re} \left( \delta r_1 \omega^1 + r_2 \omega^2 \right) \right) \right) + \]
\[ \theta_2^2 \wedge \left( -d\mu_2^2 + \mu_1^1 \wedge \phi_1^1 + \mu_2^2 \wedge \phi_2^2 - \delta \mu_2^2 \wedge \phi^2 - \right. \]
\[ \left. \mu_2^2 \wedge \left( \phi^1 + 2\text{Re} \left( r_2 \omega^1 + r_1 \omega^2 \right) \right) \right) = 0. \]  
(2.20)

Let \( \nu^1 := \text{Im} \phi_1^1, \nu^2 := \text{Im} \phi_2^2. \) Then it is easy to see that for \( x \in \mathbb{P}^2, (\theta^o(x), \omega^o(x), \overline{\omega^o(x)}, \phi^o(x), \nu^o(x), \mu_1^o(x), \mu_2^o(x), \overline{\mu_1^o(x)}, \overline{\mu_2^o(x)}, \psi^o(x), \overline{\psi^o(x)}) \) is a coframe at \( x. \) From now on we will use the independent 1-forms \( \theta^o, \omega^o, \overline{\omega^o}, \phi^o, \nu^o, \mu_1^o, \mu_2^o, \overline{\mu_1^o}, \overline{\mu_2^o}, \psi^o, \overline{\psi^o} \) as the standard basis in which we will be writing the expansions of all differential forms that we will need in the future. Equations (2.7), (2.9), (2.15), (2.20) imply
\[ \omega^1 \wedge d\phi_1^1 + \delta \omega^2 \wedge d\phi_2^2 + A \equiv 0 \pmod{\theta^o}, \]
\[ \omega^1 \wedge d\phi_2^1 + \omega^2 \wedge d\phi_1^1 + B \equiv 0 \pmod{\theta^o}, \]  
(2.21)

where
\[ A := \omega^1 \wedge \left( \frac{1}{2}r_2 \phi^2 \wedge \overline{\omega^1} - \frac{1}{2}r_1 \phi^1 \wedge \overline{\omega^2} + \delta \left( \frac{1}{2}r_1 - \frac{5}{2}r_2 \right) \phi^1 \wedge \overline{\omega^2} + \right. \]
\[ \left. i\overline{r_1} \nu^1 \wedge \overline{\omega^2} + i\delta \left( p_1 - \overline{r_2} \right) \nu^2 \wedge \overline{\omega^2} + i\mu_1^1 \wedge \overline{\omega^1} + i \left( \delta \mu_1^2 + (b_2 - i\delta \overline{r_2} \rho_1 - i\overline{r_1} \rho_2) \omega^2 + \right. \]
\[ \left. d_1 \overline{\omega^1} - i\delta \overline{r_1} \right) \wedge \overline{\omega^2} + \omega^2 \wedge \left( \frac{1}{2}r_2 - \frac{1}{2}r_1 \right) \phi^1 \wedge \overline{\omega^1} - \delta \frac{5}{2}r_2 \phi^2 \wedge \overline{\omega^1} + \delta \frac{1}{2}r_1 \phi^1 \wedge \overline{\omega^2} + \right. \]
\[ \left. i \left( \frac{1}{2}r_1 - \overline{r_2} \right) \nu^1 \wedge \overline{\omega^1} + i \delta \left( \frac{1}{2}r_2 - 2r_1 \right) \nu^2 \wedge \overline{\omega^1} - i\delta \overline{r_1} \nu^1 \wedge \overline{\omega^2} - 2i\delta \overline{r_2} \nu^2 \wedge \overline{\omega^2} + \right. \]
\[ \left. i \left( \delta \mu_1^1 - (b_1 + \overline{r_1} \rho_2 - \overline{r_2} \rho_1) \omega^1 + (d_2 - i\delta \overline{r_1} + i\overline{r_2}) \omega^2 - i\delta \overline{r_1} + i\delta \overline{r_2} \right) \right) \wedge \overline{\omega^2} + \right. \]
\[ \left. i \left( \delta \mu_1^1 + i(\delta |\rho_1|^2 + |\rho_2|^2) \omega^1 + i\delta \overline{d_1} \right) \right) \wedge \overline{\omega^2} \right), \]
\[ B := \omega^1 \wedge \left( \frac{1}{2}r_1 \phi^2 \wedge \overline{\omega^1} - \frac{1}{2}r_2 \phi^1 \wedge \overline{\omega^2} + \right. \]
\[ \left. \left( \frac{1}{2}r_2 - \frac{5}{2}r_1 \right) \phi^2 \wedge \overline{\omega^2} + i\mu_1^1 \nu^2 \wedge \overline{\omega^1} + \right. \]
\[ \left. i\overline{r_2} \nu^1 \wedge \overline{\omega^2} + i \left( \frac{1}{2}r_2 - \overline{r_1} \right) \nu^2 \wedge \overline{\omega^1} + i\mu_1^1 \wedge \overline{\omega^1} + i \left( \mu_1^1 + b_1 \omega^2 + \right. \right. \]
\[ c_1 \omega^1 - i d \theta \] 
\[ \wedge \omega^2 \right) + \omega^2 \wedge \left( \left( \frac{1}{2} \tau_1 - \frac{1}{2} \tau_2 \right) \phi \wedge \omega_{1} - \frac{5}{2} r_{1} \phi \wedge \omega + \frac{1}{2} r_{2} \phi \wedge \omega^2 + \right. \]
\[ i \left( \left( \tau_1 - \tau_2 \right) \nu \wedge \omega^1 + \frac{5}{2} r_{1} \phi \wedge \omega_{1} - \frac{1}{2} r_{2} \nu \wedge \omega - 2 i \delta \nu \wedge \omega^2 + \right. \]
\[ \left. i \left( \mu_1 + a_1 \omega^1 + d_1 \omega^2 - i d \theta \phi \wedge \omega^2 \right) + \delta \mu_2^2 \equiv \delta \mu_1^2^2 \equiv (\mod \omega^0, \omega^0). \]
we set
\begin{align*}
p_1 - q_2 &= h_1 - \tilde{\nu}_3 = h_3 + i\tilde{\nu}_3, \\
q_1 - q_4 &= g_1 - \tilde{\nu}_4 = g_3 + i\tilde{\nu}_4, \\
\delta p_2 - q_1 &= \delta h_4 + i\tilde{\nu}_1 = h_2 - i\tilde{\nu}_1, \\
\delta q_2 - q_3 &= \delta g_4 + i\tilde{\nu}_2 = g_2 - i\tilde{\nu}_2.
\end{align*}

From now on we assume that the 1-forms \( \phi_\beta^\alpha, \mu_\beta^\alpha, \psi^\alpha \) are chosen so that (2.7), (2.9), (2.12), (2.16), (2.18), (2.26) are satisfied. It follows from (2.13), (2.15), (2.25) that this set of conditions is equivalent to (2.7), (2.9), (2.12) and
\begin{align*}
\mu_2^2 &= \mu_1^1 + \tilde{a}_1 \omega^1 + \tilde{b}_1 \omega^2 + \tilde{c}_1 \omega^3 + \tilde{d}_1 \omega^4, \\
\mu_2^1 &= \delta \mu_1^1 + \tilde{a}_2 \omega^1 + \tilde{b}_2 \omega^2 + \tilde{c}_2 \omega^3 + \tilde{d}_2 \omega^4, \\
\psi^2 &= \delta \psi^3, \\
\psi^8 &= \psi^3, \\
\psi^5 &= \psi^1 + 2\text{Re} \left( v_3 \omega^1 + v_1 \omega^2 \right) + 2\text{Re} \left( r_2 \mu_1^1 + r_1 \mu_1^2 \right), \\
\psi^6 &= \psi^2 + 2\text{Re} \left( v_4 \omega^1 + v_2 \omega^2 \right), \\
\psi^7 &= \psi^3 + 2\text{Re} \left( \delta v_1 \omega^1 + v_3 \omega^2 \right), \\
\psi^8 &= \psi^4 + 2\text{Re} \left( \delta v_2 \omega^1 + v_4 \omega^2 \right) + 2\text{Re} \left( \delta r_1 \mu_2^1 + r_2 \mu_2^2 \right),
\end{align*}

where
\begin{align*}
\tilde{a}_1 &:= -\tilde{t}_7 = a_3 + i\delta \tilde{\nu}_1 \rho_2 + i\tilde{\nu}_2 \rho_2 + i\tilde{t}_7, \\
\tilde{b}_1 &:= b_1 - \tilde{t}_8 = b_3 + i\tilde{\nu}_1 \rho_1 + i\tilde{\nu}_2 \rho_2 + i\tilde{t}_8, \\
\tilde{c}_1 &:= c_1 - i\tilde{t}_5 = c_3 + i\tilde{t}_5, \\
\tilde{d}_1 &:= d_1 - i\tilde{t}_6 = d_3 + i\tilde{t}_6, \\
\tilde{a}_2 &:= \delta \text{Im} \left( r_1 \tilde{\nu}_2 \right) + i\delta \tilde{t}_3 = -\left( b_1 + i\tilde{\nu}_1 \rho_1 + i\tilde{\nu}_2 \rho_2 + i\tilde{t}_3 \right), \\
\tilde{b}_2 &:= -\delta \text{Im} \left( r_1 \tilde{\nu}_2 \right) + i\delta \tilde{t}_3 = -\left( b_1 + i\tilde{\nu}_1 \rho_1 + i\tilde{\nu}_2 \rho_2 + i\tilde{t}_3 \right), \\
\tilde{c}_2 &:= c_4 + i\tilde{t}_1 = d_4 + i\tilde{t}_1, \\
\tilde{d}_2 &:= \delta c_3 + i\tilde{\nu}_1 = i\delta \tilde{\nu}_2 + i\tilde{t}_2 = d_2 - i\tilde{t}_2,
\end{align*}

are fixed, and therefore globally defined on \( P^2 \), functions.

Now (2.14), (2.16)-(2.19), (2.26), (2.27) give that \( \phi_\beta^\alpha, \mu_\beta^\alpha, \psi^\alpha \) are fixed up to transformations of the form
\begin{align*}
\phi_1^1 &= \phi_1^1 + h \theta^1 + g \theta^2, \\
\phi_2^1 &= \phi_1^2 + \delta g \theta^1 + h \theta^2, \\
\phi_1^2 &= \phi_2^1 + g \theta^1 + \delta h \theta^2, \\
\phi_2^2 &= \phi_2^2 + h \theta^1 + g \theta^2, \\
\mu_1^1 &= \mu_1^1 + h \omega^1 + g \omega^2 + p \theta^1 + q \theta^2, \\
\mu_2^1 &= \mu_2^1 + \delta g \omega^1 + h \omega^2 + \delta q \theta^1 + p \theta^2, \\
\mu_1^2 &= \mu_1^2 + g \omega^1 + \delta h \omega^2 + q \theta^1 + \delta p \theta^2, \\
\mu_2^2 &= \mu_2^2 + h \omega^1 + g \omega^2 + p \theta^1 + q \theta^2, \\
\psi^1 &= \psi^1 + 2\text{Re} \left( i\tilde{\nu}_1 \omega^1 + i\tilde{\nu}_2 \omega^2 \right) + s \theta^1 + t \theta^2,
\end{align*}
\[
\begin{align*}
\psi^{2*} &= \psi^2 + 2\text{Re} \left( i\bar{\eta}\omega^1 + i\bar{\eta}\omega^2 \right) + s\theta^1 + \delta\sigma\theta^2, \\
\psi^{3*} &= \psi^3 + 2\text{Re} \left( i\bar{\eta}\omega^1 + i\bar{\eta}\omega^2 \right) + s\theta^1 + \delta\sigma\theta^2, \\
\psi^{4*} &= \psi^4 + 2\text{Re} \left( i\bar{\eta}\omega^1 + i\bar{\eta}\omega^2 \right) + \sigma\theta^1 + s\theta^2, \\
\psi^{5*} &= \psi^5 + 2\text{Re} \left( i\bar{\eta}\omega^1 + i\bar{\eta}\omega^2 \right) + \\
&\quad (\sigma + 2\text{Re} (\delta r_1 q + r_2 p))\theta^1 + (s + 2\text{Re} (r_1 p + r_2 q))\theta^2, \\
\psi^{6*} &= \psi^6 + 2\text{Re} \left( i\bar{\eta}\omega^1 + i\bar{\eta}\omega^2 \right) + s\theta^1 + \delta\sigma\theta^2, \\
\psi^{7*} &= \psi^7 + 2\text{Re} \left( i\bar{\eta}\omega^1 + i\bar{\eta}\omega^2 \right) + \delta s\theta^1 + \sigma\theta^2, \\
\psi^{8*} &= \psi^8 + 2\text{Re} \left( i\bar{\eta}\omega^1 + i\bar{\eta}\omega^2 \right) + \\
&\quad (\sigma + 2\text{Re} (\delta r_1 q + r_2 p))\theta^1 + (s + 2\text{Re} (r_1 p + r_2 q))\theta^2,
\end{align*}
\] (2.30)

where \(h, g\) are imaginary-valued, \(\sigma, s\) are real-valued, \(p, q\) are complex-valued functions.

Consider the following matrix-valued 1-form

\[
\omega := \begin{pmatrix} -\frac{1}{3}\phi & -\frac{1}{3}\phi \\
-\frac{1}{3}\phi & -\frac{2}{3}\phi - \frac{1}{3}\phi \\
\frac{1}{2}\mu & 1 + \frac{1}{3}\phi + \frac{1}{3}\phi \end{pmatrix},
\] (2.31)

where

\[
\begin{align*}
\theta &= \begin{pmatrix} \theta^1 \\
\theta^2 \\
\delta\theta^2
\end{pmatrix}, & \omega &= \begin{pmatrix} \omega^1 \\
\omega^2 \\
\delta\omega^2
\end{pmatrix}, \\
\phi &= \begin{pmatrix} \phi^1 \\
\phi^2 \\
\delta\phi^2
\end{pmatrix}, & \phi' &= \begin{pmatrix} \phi^1 \\
\phi^2 \\
\phi^1
\end{pmatrix}, \\
\mu &= \begin{pmatrix} \mu^1 \\
\mu^2 \\
\mu^1
\end{pmatrix}, & \psi &= \begin{pmatrix} \psi^1 \\
\psi^2 \\
\psi^1
\end{pmatrix}.
\end{align*}
\]

It is clear from (1.3), (2.9) that the form \(\omega\) defined in (2.31) takes values in \(\mathfrak{su}(2,1)\). Also, for any point \(x \in P^2\), \(\omega(x)\) is an isomorphism between \(T_x(P^2)\) and \(\mathfrak{su}(2,1)\). However, the form \(\omega\) is defined only locally. We now need to fix the free parameters \(h, g, p, q, \sigma, s\) from (2.30) to make the choice of the corresponding forms unique. This will turn \(\omega\) into a globally defined \(\mathfrak{su}(2,1)\)-valued form on \(P^2\), and it will be the parallelism that we are looking for.

To fix the free parameters from (2.30) we consider the curvature form \(\Omega\) of \(\omega\)

\[
\Omega := d\omega - \frac{1}{2} [\omega, \omega] = d\omega - \omega \wedge \omega,
\]

which is a \(\mathfrak{su}(2,1)\)-valued 2-form. In more detail, \(\Omega\) is given by \(\Omega = (\Omega^i)_{0 \leq i, j \leq 2}\) with \(\Omega^i \in \mathfrak{g}\), and

\[
\begin{align*}
\Omega^0 &= -\frac{1}{3}d\phi - \frac{1}{3}d\phi' + i\omega \wedge \eta + \frac{1}{2}\theta \wedge \psi, \\
\Omega^1 &= -2i\text{Im} \Omega^0 + \frac{2}{3}d\phi - \frac{1}{3}d\phi' + i\eta \wedge \omega + i\mu \wedge \eta, \\
\Omega^2 &= -\Omega^0, \\
\Omega^1 &= d\omega - \omega \wedge \phi - \theta \wedge \mu, \\
\Omega^2 &= 2i\Omega^0.
\end{align*}
\]
\[\Omega_0^2 = 2 \left( d\theta - i\omega \wedge \overline{\omega} - \theta \wedge \phi \right),\]
\[\Omega_1^2 = \frac{1}{2} \left( d\mu + \mu \wedge \overline{\phi} + \frac{1}{2} \psi \wedge \omega \right),\]
\[\Omega_0^0 = -2i\Omega_1^1,\]
\[\Omega_2^0 = -\frac{1}{4} \left( d\psi - 2i\mu \wedge \overline{\pi} + \psi \wedge \phi \right). \quad (2.32)\]

Now, to fix the parameters from (2.30) we concentrate on the components \(\Omega_1^1, \Omega_2^1, \Omega_2^0\) and impose certain conditions on their expansions.

We start with \(\Omega_1^1\). It follows from (2.32) that
\[\Omega_1^1 = \left( \frac{\Phi_1^1}{\Phi_1^2}, \delta \Phi_1^2 \right),\]
where
\[\Phi_1^1 := \frac{2}{3} d\phi_1^1 - \frac{1}{3} d\phi_1^2 + i\mu_1 \wedge \omega^1 + i\delta \mu_1 \wedge \omega^2 + i\mu_1^1 \wedge \overline{\omega} + i\delta \mu_1 \wedge \overline{\omega}, \quad (2.33a)\]
\[\Phi_1^2 := \frac{2}{3} d\phi_1^2 - \frac{1}{3} d\phi_2^2 + i\mu_2 \wedge \omega^1 + i\mu_2^1 \wedge \omega^2 + i\mu_2 \wedge \overline{\omega} + i\mu_2^1 \wedge \overline{\omega}. \quad (2.33b)\]

Then (2.12), (2.21)–(2.24), (2.33) imply
\[\Phi_1^1 \equiv \frac{2}{3} \left( \frac{1}{2} (\rho_2 - r_1) \phi^2 \wedge \omega^1 + \delta \frac{1}{2} (\rho_1 - r_2) \phi^2 \wedge \omega^2 + \frac{1}{2} (\tau_1 - \tau_2) \phi^2 \wedge \overline{\omega} + \delta \frac{1}{2} (\tau_2 - \tau_1) \phi^2 \wedge \overline{\omega} - \right.
\[\left. i\rho_2 \nu^2 \wedge \omega^1 - i\delta \rho_2 \nu^2 \wedge \omega^2 \right) \quad \text{(mod } \theta^a, \text{ terms quadratic in } \omega^a, \overline{\omega}^a), \]
\[\Phi_1^2 \equiv \frac{2}{3} \left( \frac{1}{2} (\rho_1 - r_1) \phi^2 \wedge \omega^1 + \frac{1}{2} (\rho_2 - r_1) \phi^2 \wedge \omega^2 + \frac{1}{2} (\tau_1 - \tau_2) \phi^2 \wedge \overline{\omega} + \frac{1}{2} (\tau_2 - \tau_1) \phi^2 \wedge \overline{\omega} - \right.
\[\left. i\rho_2 \nu^2 \wedge \omega^1 - i\rho_2 \nu^2 \wedge \omega^2 \right) \quad \text{(mod } \theta^a, \text{ terms quadratic in } \omega^a, \overline{\omega}^a). \quad (2.34)\]

Let us consider the parts of the expansions of \(\Phi_2^0\) that are quadratic in \(\omega^\gamma, \overline{\omega}^\gamma\):
\[\Phi_2^0 = S_1^{\alpha_1} \omega^\gamma \wedge \omega^\gamma + S_2^{\alpha_1} \overline{\omega}^\gamma \wedge \overline{\omega}^\gamma + S_3^{\alpha_1} \omega^\gamma \wedge \overline{\omega}^\gamma + \ldots \quad (2.35)\]

Since \(\Phi_2^0\) are imaginary-valued, the coefficients \(S_1^{\alpha_1}, S_2^{\alpha_1}, S_3^{\alpha_1}\) are real-valued functions. It now follows from (2.6), (2.33), (2.34) that under transformation (2.30) they change as:
\[S_1^{\alpha_1} = S_1^{\alpha_1} + \frac{8}{3} h,\]
\[S_2^{\alpha_1} = S_2^{\alpha_1} + \frac{8}{3} g. \quad (2.36)\]

We now fix the parameters \(h, g\) by the conditions
\[S_1^{\alpha_1} = 0, \quad S_2^{\alpha_1} = 0, \quad (2.36)\]
i.e. we set
\[ h = \frac{3}{8} S_{111}, \]
\[ g = \frac{3}{8} S_{122}. \]

Thus, the forms $\phi^a_\beta$ are fixed and therefore are defined globally on $P^2$. It then follows that the functions $v_\alpha$, $w_\alpha$ from (2.23), (2.24), (2.28) are also fixed and defined globally on $P^2$.

From now on we assume that (2.7), (2.9), (2.12), (2.28), (2.29), (2.36) are satisfied. It then follows from (2.30) that $\mu^\alpha_\beta$, $\psi^\alpha$ are fixed up to transformations of the form
\[
\begin{align*}
\mu^1_{\alpha} &= \mu^1_{\alpha} + p_\alpha + q_\alpha, \\
\mu^2_{\alpha} &= \mu^2_{\alpha} + \delta q_\alpha + p_\alpha, \\
\mu^3_{\alpha} &= \mu^3_{\alpha} + q_\alpha + \delta p_\alpha, \\
\mu^4_{\alpha} &= \mu^4_{\alpha} + p_\alpha + q_\alpha, \\
\psi^1 &= \psi^1 + 2 \text{Re} \left( i \bar{\rho} \omega^1 + i \bar{\omega} \omega^2 \right) + \sigma^1 + s^1 + \theta^2, \\
\psi^2 &= \psi^2 + 2 \text{Re} \left( i \bar{\rho} \omega^1 + i \bar{\theta} \omega^2 \right) + s^1 + 2 \text{Re} \left( r_1 q + r_2 p \right) \theta^1, \\
\psi^3 &= \psi^3 + 2 \text{Re} \left( i \bar{\rho} \omega^1 + i \bar{\sigma} \omega^2 \right) + s^1 + \delta^1 + 2 \text{Re} \left( r_1 q + r_2 p \right) \theta^1, \\
\psi^4 &= \psi^4 + 2 \text{Re} \left( i \bar{\rho} \omega^1 + i \bar{\nu} \omega^2 \right) + s^1 + 2 \text{Re} \left( r_1 q + r_2 p \right) \theta^1, \\
\psi^5 &= \psi^5 + 2 \text{Re} \left( i \bar{\rho} \omega^1 + i \bar{\omega} \omega^2 \right) + \\
&\quad (\sigma + 2 \text{Re} \left( \delta r_1 q + r_2 p \right)) \theta^1 + (s + 2 \text{Re} \left( r_1 p + r_2 q \right)) \theta^1, \\
\psi^6 &= \psi^6 + 2 \text{Re} \left( i \bar{\rho} \omega^1 + i \bar{\omega} \omega^2 \right) + s^1 + \delta^1 + 2 \text{Re} \left( r_1 q + r_2 p \right) \theta^1, \\
\psi^7 &= \psi^7 + 2 \text{Re} \left( i \bar{\rho} \omega^1 + i \bar{\omega} \omega^2 \right) + s^1 + \delta^1 + 2 \text{Re} \left( r_1 q + r_2 p \right) \theta^1, \\
\psi^8 &= \psi^8 + 2 \text{Re} \left( i \bar{\rho} \omega^1 + i \bar{\omega} \omega^2 \right) + \\
&\quad (\sigma + 2 \text{Re} \left( \delta r_1 q + r_2 p \right)) \theta^1 + (s + 2 \text{Re} \left( r_1 p + r_2 q \right)) \theta^1, \\
\end{align*}
\]

where $\sigma, s$ are real-valued, $p, q$ are complex-valued functions.

We now consider $\Omega^1_2$. It follows from (2.32) that
\[
\Omega^1_2 = \left( \Phi^1 \delta \Phi^2 \right),
\]
where
\[
\begin{align*}
\Phi^1 &= \frac{1}{2} \left( d\mu^1_1 + \frac{1}{2} \mu^1_1 \wedge \phi^1 + \delta \frac{1}{2} \mu^2_1 \wedge \phi^2 + i \nu^1 \wedge \mu^1_1 + i \delta \nu^2 \wedge \mu^1_2 + \\
&\quad \frac{1}{2} \psi^1 \wedge \omega^1 + \delta \frac{1}{2} \psi^3 \wedge \omega^2 \right), \\
\Phi^2 &= \frac{1}{2} \left( d\mu^2_1 + \frac{1}{2} \mu^1_1 \wedge \phi^1 + \frac{1}{2} \mu^2_1 \wedge \phi^2 + i \nu^1 \wedge \mu^2_1 + i \nu^2 \wedge \mu^1_1 + \\
&\quad \frac{1}{2} \psi^3 \wedge \omega^1 + \psi^3 \wedge \omega^2 \right).
\end{align*}
\]

To get information about the expansions of $\Phi^\alpha$ we return to (2.20) and consider terms containing $\theta^\alpha$ there. Let such terms in the expansions of $\phi^1_1$, $\phi^2_1$ be
\[
\begin{align*}
d\phi^1_1 &= \lambda^1 \wedge \theta^1 + \lambda^2 \wedge \theta^2 + \ldots, \\
d\phi^2_1 &= \lambda^3 \wedge \theta^1 + \lambda^4 \wedge \theta^2 + \ldots.
\end{align*}
\]

(2.39)
for some 1-forms $\lambda^\alpha$. Then (2.7), (2.9), (2.20), (2.28) imply

$$
\begin{align*}
\theta^1 \land \Sigma^1 + \delta \theta^2 \land \Sigma^2 + U & \equiv 0 \quad \text{(mod $\theta^1 \land \theta^2$, terms quadratic in $\omega^\alpha, \overline{\omega^\alpha}$),} \\
\theta^1 \land \Sigma^2 + \theta^2 \land \Sigma^1 + V & \equiv 0 \quad \text{(mod $\theta^1 \land \theta^2$, terms quadratic in $\omega^\alpha, \overline{\omega^\alpha}$),}
\end{align*}
$$

where

$$
\begin{align*}
\Sigma^1 & := \text{d}\mu_1^1 + \frac{1}{2} \text{d}\mu_1^2 \land \phi^1 + \delta \frac{1}{2} \text{d}\mu_1^2 \land \phi^2 + \text{d}i\nu^1 \land \mu_1^1 + i \delta \nu^2 \land \mu_1^2, \\
\Sigma^2 & := \text{d}\mu_1^2 + \frac{1}{2} \text{d}\mu_1^2 \land \phi^1 + \frac{1}{2} \text{d}\mu_1^2 \land \phi^2 + \text{d}i\nu^1 \land \mu_1^2 + i \nu^2 \land \mu_1^1,
\end{align*}
$$

and

$$
\begin{align*}
U & := \theta^1 \land \left( \mu_1^2 \land \left( 2 \text{Re} \left( r_1 \omega^1 + \delta r_2 \omega^2 \right) + \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} \right) + \omega^2 \land \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \right) + \\
& \quad \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \land \phi^1 + \omega^1 \land \lambda^1 + \delta \omega^2 \land \lambda^3 \right) + \\
\theta^2 \land \left( \mu_1^1 \land \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) + 2 \delta \mu_1^2 \land \text{Re} \left( r_2 \omega^1 + r_1 \omega^2 \right) + \delta \omega^2 \land \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \right) + \\
& \quad \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \land \phi^1 + \frac{1}{2} \left( (b_2 - \delta a_1) \omega^1 + \delta (a_2 - b_1) \omega^2 + \\
& \quad \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \right) \land \phi^2 + 2 \text{d}i\nu^1 \land \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \right) + \\
& \quad i \nu^2 \land \left( (\delta a_1 - a_2) \omega^1 + \delta (b_1 - a_2) \omega^2 + \delta (c_1 + d_2) \omega^1 + \delta (c_1 + d_2) \omega^2 \right) + \\
& \quad \text{d}a_2 \land \omega^1 + \text{d}b_2 \land \omega^2 + \text{d}c_2 \land \overline{\omega^1} + \text{d}d_2 \land \overline{\omega^2} + \omega^1 \land \lambda^2 + \delta \omega^2 \land \lambda^4 \right), \\
V & := \theta^1 \land \left( 2 \mu_1^1 \land \text{Re} \left( r_1 \omega^1 + r_2 \omega^2 \right) + \mu_1^2 \land \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) + \omega^2 \land \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \right) + \\
& \quad \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \land \phi^1 + \omega^1 \land \lambda^3 + \delta \omega^2 \land \lambda^1 \right) + \\
& \quad \theta^2 \land \left( \mu_1^1 \land \left( 2 \text{Re} \left( r_2 \omega^1 + r_1 \omega^2 \right) + \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} \right) + \omega^2 \land \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \right) + \\
& \quad \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \land \phi^1 + \frac{1}{2} \left( (b_1 - \delta a_2) \omega^1 + (\delta a_1 - b_2) \omega^2 + \\
& \quad \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \right) \land \phi^2 + 2 \text{d}i\nu^1 \land \left( \frac{\text{d} \mu_1^1}{\text{d} \mu_1^2} + \frac{\text{d} \mu_1^2}{\text{d} \mu_1^2} \right) \right) + \\
& \quad i \nu^2 \land \left( (a_2 - b_1) \omega^1 + (b_2 - \delta a_1) \omega^2 + (c_2 + d_1) \omega^1 + (d_2 + \delta c_1) \omega^2 \right) + \\
& \quad \text{d}a_1 \land \omega^1 + \text{d}b_1 \land \omega^2 + \text{d}c_1 \land \overline{\omega^1} + \text{d}d_1 \land \overline{\omega^2} + \omega^1 \land \lambda^4 + \omega^2 \land \lambda^2 \right).
\end{align*}
$$

Equations (2.40) imply

$$
\begin{align*}
\theta^1 \land U & \equiv -\delta \theta^2 \land V \quad \text{(mod $\theta^1 \land \theta^2$, terms quadratic in $\omega^\alpha, \overline{\omega^\alpha}$),} \\
\theta^2 \land U & \equiv \theta^1 \land V \quad \text{(mod $\theta^1 \land \theta^2$, terms quadratic in $\omega^\alpha, \overline{\omega^\alpha}$),}
\end{align*}
$$

which together with (2.42) gives

$$
d \tilde{a}_1 \equiv \lambda^4 - \lambda^1 - r_2 \mu_1^1 + r_1 \mu_1^2 + \tilde{a}_1 \phi^1 + \frac{1}{2} \left( \tilde{b}_1 - 3 \tilde{a}_2 \right) \phi^2 +
$$
\begin{align*}
  &i(\bar{b}_1 - a_2)\nu^2 \pmod{\theta^\alpha, \omega^\alpha, \overline{\omega^\alpha}}, \\
  db_1 &\equiv \lambda^2 - \delta \lambda^3 - r_1 \mu_1^2 + \delta r_2 \mu_1^2 + \bar{b}_1 \phi^1 + \frac{1}{2}(\bar{d}a_1 - 3\bar{b}_2)\phi^2 + \\
  &i(\delta \bar{a}_1 - \bar{b}_2)\nu^2 \pmod{\theta^\alpha, \omega^\alpha, \overline{\omega^\alpha}}, \\
  dc_1 &\equiv -(\overline{\tau_2 + \overline{\rho_1}})\mu_1 + (\overline{\tau_1 + \overline{\rho_2}})\mu_1^2 + \bar{c}_1 \phi^1 + \frac{1}{2}(\bar{d}_1 - 3\bar{c}_2)\phi^2 - \\
  &2i\bar{c}_1\nu^1 - i(\bar{d}_1 + \bar{c}_2)\nu^2 \pmod{\theta^\alpha, \omega^\alpha, \overline{\omega^\alpha}}, \\
  dd_1 &\equiv -(\overline{\tau_1 + \overline{\rho_2}})\mu_1 + \delta(\overline{\tau_2 + \overline{\rho_1}})\mu_1^2 + \bar{d}_1 \phi^1 + \frac{1}{2}(\delta \bar{c}_1 - 3\bar{d}_2)\phi^2 - \\
  &2i\bar{d}_1\nu^1 - i(\delta \bar{c}_1 + \bar{d}_2)\nu^2 \pmod{\theta^\alpha, \omega^\alpha, \overline{\omega^\alpha}}, \\
  da_2 &\equiv \lambda^2 - \delta \lambda^3 + r_1 \mu_1 - \delta r_2 \mu_1^2 + \bar{a}_2 \phi^1 + \frac{1}{2}(\bar{b}_2 - 3\delta \bar{a}_1)\phi^2 + \\
  &i(\bar{b}_2 - \delta \bar{a}_1)\nu^2 \pmod{\theta^\alpha, \omega^\alpha, \overline{\omega^\alpha}}, \\
  db_2 &\equiv \delta(\lambda^4 - \lambda^1) + \delta r_2 \mu_1^2 - \delta r_1 \mu_1^2 + \bar{b}_2 \phi^1 + \frac{1}{2}(\bar{a}_2 - 3\bar{b}_1)\phi^2 + \\
  &i\delta(\bar{a}_2 - \bar{b}_1)\nu^2 \pmod{\theta^\alpha, \omega^\alpha, \overline{\omega^\alpha}}, \\
  dc_2 &\equiv (\overline{\tau_1 - \overline{\rho_2}})\mu_1 + \delta(\overline{\tau_2 - \overline{\rho_1}})\mu_1^2 + \bar{c}_2 \phi^1 + \frac{1}{2}(\bar{d}_2 - 3\delta \bar{c}_1)\phi^2 - \\
  &2i\bar{c}_2\nu^1 - i(\bar{d}_2 + \delta \bar{c}_1)\nu^2 \pmod{\theta^\alpha, \omega^\alpha, \overline{\omega^\alpha}}, \\
  dd_2 &\equiv \delta(\overline{\tau_2 - \overline{\rho_1}})\mu_1^2 + \delta(\overline{\tau_2 - \overline{\rho_1}})\mu_1^2 + \bar{d}_2 \phi^1 + \frac{1}{2}(\delta \bar{c}_2 - 3\bar{d}_1)\phi^2 - \\
  &2i\bar{d}_2\nu^1 - i\delta(\bar{d}_1 + \bar{c}_2)\nu^2 \pmod{\theta^\alpha, \omega^\alpha, \overline{\omega^\alpha}}. \quad (2.43)
\end{align*}

It now follows from (2.40), (2.42), (2.43) that

\begin{align*}
  \Sigma^1 &\equiv \left(\lambda^1 - r_1 \mu_1^2 + \bar{a}_2 \phi^2\right) \wedge \omega^1 + \left(\delta \lambda^3 - \delta r_2 \mu_1^2 + \overline{\rho_2} \mu_1 + \delta \overline{\rho_1} \mu_1^2 + \bar{b}_2 \phi^2\right) \wedge \omega^2 + \\
  &\left(-\overline{\tau_1 + \overline{\rho_2}}\mu_1^2 + \bar{c}_2 \phi^2\right) \wedge \omega^1 + \left(-\delta(\overline{\tau_2 + \overline{\rho_1}})\mu_1^2 + \bar{d}_2 \phi^2\right) \wedge \omega^2 \\
  &\pmod{\theta^\alpha, \text{terms quadratic in } \omega^\alpha, \overline{\omega^\alpha}}, \\
  \Sigma^2 &\equiv \left(\lambda^3 - \delta r_1 \mu_1^1 + \bar{a}_1 \phi^2\right) \wedge \omega^1 + \left(\lambda^1 - \delta r_2 \mu_1^1 + \delta \overline{\rho_1} \mu_1 + \delta \overline{\rho_2} \mu_1^1 + \bar{b}_1 \phi^2\right) \wedge \omega^2 + \\
  &\left(-\delta \overline{\tau_1} \mu_1^1 + \overline{\tau_1} \mu_1^2 + \bar{c}_1 \phi^2\right) \wedge \omega^1 + \left(-\delta \overline{\tau_2} \mu_1^1 + \overline{\tau_2} \mu_1^2 + \bar{d}_1 \phi^2\right) \wedge \omega^2 \\
  &\pmod{\theta^\alpha, \text{terms quadratic in } \omega^\alpha, \overline{\omega^\alpha}}. \quad (2.44)
\end{align*}

Formulas (2.38), (2.41), (2.44) give

\begin{align*}
  \Phi^1 &\equiv \frac{1}{2}\left(\left(\lambda^1 - r_1 \mu_1^2 + \bar{a}_2 \phi^2 + \frac{1}{2}\psi^1\right) \wedge \omega^1 + \left(\delta \lambda^3 - \delta r_2 \mu_1^2 + \overline{\rho_2} \mu_1 + \delta \overline{\rho_1} \mu_1 + \bar{b}_2 \phi^2 + \delta \frac{1}{2}\psi^3\right) \wedge \omega^2 + \\
  &\left(-\delta(\overline{\tau_2 + \overline{\rho_1}})\mu_1^2 + \bar{d}_2 \phi^2\right) \wedge \omega^2 \right)
\end{align*}
\( \Phi^2 \equiv \frac{1}{2} \left( (\lambda^3 - \delta r_1 \mu_1^1 + \tilde{a}_1 \phi^2 + \frac{1}{2} \psi^3) \wedge \omega^1 + (\lambda^1 - \delta r_2 \mu_1^1 + \delta \overline{\mu}_1^1 + \delta \overline{\mu}_2^1 + \tilde{b}_1 \phi^2 + \frac{1}{2} \psi^1) \wedge \omega^2 + (-\delta r_1 \overline{\mu}_1^1 - \overline{\mu}_1^2 + \tilde{c}_1 \phi^2) \wedge \overline{\omega}^1 + (-\delta r_2 \overline{\mu}_1^1 + \overline{\mu}_2^1 + \tilde{d}_1 \phi^2) \wedge \overline{\omega}^2 \right) \quad (\text{mod } \theta^a, \text{ terms quadratic in } \omega^a, \overline{\omega}^a). \) (2.45)

We will now show that the expansions of \( \Phi^a \) do not contain terms involving \( \omega^\beta \wedge \psi^\gamma \). It is clear from (2.45) that for this we only need to prove that the expansions of \( \lambda^1, \lambda^3 \) have the form

\[
\lambda^1 = -\frac{1}{2} \psi^1 + \text{terms not containing } \psi^a, \quad (2.46.a)
\]

\[
\lambda^3 = -\frac{1}{2} \psi^3 + \text{terms not containing } \psi^a. \quad (2.46.b)
\]

Let

\[
\lambda^1 = \chi_1 \psi^1 + \chi_2 \psi^3 + \text{terms not containing } \psi^a, \\
\lambda^3 = \chi_3 \psi^1 + \chi_4 \psi^3 + \text{terms not containing } \psi^a. \quad (2.47)
\]

Identities (2.41), (2.44) give

\[
d\mu_1^1 = \left( \chi_1 \psi^1 + \chi_2 \psi^3 \right) \wedge \omega^1 + \delta \left( \chi_3 \psi^1 + \chi_4 \psi^3 \right) \wedge \omega^2 + \text{terms not containing } \psi^a, \\
d\mu_1^2 = \left( \chi_3 \psi^1 + \chi_4 \psi^3 \right) \wedge \omega^1 + \left( \chi_1 \psi^1 + \chi_2 \psi^3 \right) \wedge \omega^2 + \text{terms not containing } \psi^a. \quad (2.48)
\]

We now differentiate (2.33.a) and, using (2.7), (2.9), (2.28), (2.48), in the right-hand side of the resulting equation collect terms containing \( \omega^1 \wedge \overline{\omega}^1 \wedge \psi^a \):

\[
-2 \omega^1 \wedge \overline{\omega}^1 \wedge \left( \text{Im} \chi_1 \psi^1 + \text{Im} \chi_2 \psi^3 \right). \quad (2.49)
\]

On the other hand, it follows from (2.12), (2.28), (2.33)-(2.36), (2.39), (2.47) that such terms in the left-hand side are:

\[
-2 \omega^1 \wedge \overline{\omega}^1 \wedge \left( \left( \frac{2}{3} \chi_1 + \frac{1}{3} \right) \psi^1 + \frac{2}{3} \chi_2 \psi^3 \right). \quad (2.50)
\]

Comparing (2.49), (2.50) we get

\[
\chi_1 = -\frac{1}{2}, \quad \chi_2 = 0,
\]

and (2.46.a) is proved.

Similarly, we differentiate (2.33.b) and in the right-hand side of the resulting equation collect terms containing \( \omega^2 \wedge \overline{\omega}^2 \wedge \psi^a \):

\[
-2 \delta \omega^2 \wedge \overline{\omega}^2 \wedge \left( \text{Im} \chi_3 \psi^1 + \text{Im} \chi_4 \psi^3 \right). \quad (2.51)
\]

Such terms in the left-hand side are:

\[
-2 \delta \omega^2 \wedge \overline{\omega}^2 \wedge \left( \frac{2}{3} \chi_3 \psi^1 + \frac{2}{3} \chi_4 \psi^3 \right) + \left( \frac{2}{3} \chi_4 \psi^1 + \frac{1}{3} \chi_3 \psi^3 \right). \quad (2.52)
\]
Comparing (2.51), (2.52) we obtain
\[ \chi_3 = 0, \quad \chi_4 = -\frac{1}{2}, \]
and (2.46.b) is proved.

We are now ready to fix the parameters \( p, q \) in (2.37). Let us consider the parts of the expansions of \( \Phi^\alpha \) that are quadratic in \( \omega^\beta, \omega^\gamma \):
\[ \Phi^\alpha = S_{\beta\gamma}^\alpha \omega^\beta \wedge \omega^\gamma + S_{\beta\gamma}^\alpha \omega^\beta \wedge \bar{\omega}^\gamma + S_{\beta\gamma}^\alpha \bar{\omega}^\beta \wedge \omega^\gamma + \ldots \] (2.53)

It follows from (2.6), (2.38), (2.45), (2.46) that under transformation (2.37) the coefficients \( S_{11}, S_{22} \) change as:
\[ S_{11}^* = S_{11} + \frac{3}{4} p, \]
\[ S_{22}^* = S_{22} + \frac{3}{4} q. \]

We now fix \( p, q \) by the conditions:
\[ S_{11}^* = 0, \quad S_{22}^* = 0, \] (2.54)
i.e. we set
\[ p = \frac{i}{4} S_{11}, \quad q = \frac{i}{4} S_{22}. \]

The forms \( \mu^\alpha_\beta \) are now fixed and thus are globally defined on \( \mathbb{P}^2 \).

From now on we assume that (2.7), (2.9), (2.12), (2.28), (2.29), (2.36), (2.54) are satisfied. It then follows from (2.37) that \( \psi^\alpha \) are fixed up to transformations of the form
\[ \begin{align*}
\psi_1^* &= \psi^1 + \sigma^1 + s^2 , \\
\psi_2^* &= \psi^2 + \sigma^2 , \\
\psi_3^* &= \psi^3 + \delta^1 + \sigma^2 , \\
\psi_4^* &= \psi^4 + \sigma^1 + s^2 , \\
\psi_5^* &= \psi^5 + \sigma^1 + s^2 , \\
\psi_6^* &= \psi^6 + \sigma^1 + \delta^2 , \\
\psi_7^* &= \psi^7 + \delta^1 + \sigma^2 , \\
\psi_8^* &= \psi^8 + \sigma^1 + s^2 ,
\end{align*} \] (2.55)

where \( \sigma, s \) are real-valued functions.

To fix the parameters \( \sigma, s \) in (2.55) we consider \( \Omega_2^0 \). It follows from (2.32) that
\[ \Omega_2^0 = \begin{pmatrix} \Psi^1 & \delta \Psi^2 \\ \Psi^2 & \Psi^1 \end{pmatrix}, \]
where
\[ \begin{align*}
\Psi^1 &= -\frac{1}{4} \left( d\psi^1 - 2i \left( \mu^1_1 \wedge \mu^1_1 + \delta \mu^2_1 \wedge \mu^2_1 \right) + \psi^1 \wedge \phi^1 + \delta \psi^3 \wedge \phi^2 \right), \\
\Psi^2 &= -\frac{1}{4} \left( d\psi^3 - 2i \left( \mu^2_1 \wedge \mu^1_1 + \mu^1_1 \wedge \mu^2_1 \right) + \psi^3 \wedge \phi^1 + \psi^1 \wedge \phi^2 \right). \] (2.56)
Let us consider the parts of the expansions of $\Psi^\alpha$ that are quadratic in $\omega^\beta, \overline{\omega}^\beta$:

$$\Psi^\alpha = T^\alpha_3, \omega^\beta \wedge \omega^\gamma + T^\alpha_3, \omega^\beta \wedge \overline{\omega}^\gamma + T^\alpha_3, \overline{\omega}^\beta \wedge \overline{\omega}^\gamma + \ldots$$  \hspace{1cm} (2.57)

Since $\Psi^\alpha$ are real-valued, the coefficients $T^1_{1\Pi}, T^2_{2\Pi}$ are imaginary-valued. It follows from (2.6), (2.56) that under transformation (2.55) they change as:

$$\begin{align*}
T^1_{1\Pi} &= T^1_{1\Pi} - i\frac{1}{4}\sigma, \\
T^2_{2\Pi} &= T^2_{2\Pi} - i\frac{1}{4}s.
\end{align*}$$

We now fix $\sigma, s$ by the conditions:

$$\begin{align*}
T^1_{1\Pi} &= 0, \\
T^2_{2\Pi} &= 0,
\end{align*}$$

i.e. we set

$$\begin{align*}
\sigma &= -4iT^1_{1\Pi}, \\
s &= -4iT^2_{2\Pi}.
\end{align*}$$

We have proved that the forms $\phi^\alpha_3, \mu^\alpha_3, \psi^\alpha$ are uniquely fixed by conditions (2.7), (2.9), (2.12), (2.28), (2.29), (2.36), (2.54), (2.58), and therefore we can assume that they are now defined globally on $P^2$. The form $\omega$ defined via these forms as in (2.31) is the parallelism that we needed to construct. We also note that the functions $r_\alpha$ from (2.6), $\rho_\alpha$ from (2.9), $t_\alpha, v_\alpha$ from (2.23), $u_\alpha, w_\alpha$ from (2.24), $\tilde{a}_\alpha, \tilde{b}_\alpha, \tilde{c}_\alpha, \tilde{d}_\alpha$ from (2.28) are $CR$-invariant functions, i.e. scalar invariants.

In the remainder of this section we will find a transformation formula for $\omega$ under the action of $G^2$ on $P^2$. Let $\eta \in G^2$ be given by matrix (1.2), where $CC^T = E$ and

$$C = \begin{pmatrix} C_1 & \delta C_2 \\ C_2 & C_1 \end{pmatrix}, \quad A = \begin{pmatrix} A_1 & \delta A_2 \\ A_2 & A_1 \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & \delta R_2 \\ R_2 & R_1 \end{pmatrix},$$

with $C_\alpha, A_\alpha \in C, R_\alpha \in \mathbb{R}$. Let $L_\eta$ denote the mapping of $P^2$ induced by $\chi(\eta)$ (see (1.5)).

It turns out that one can find the transformation rule for $\omega$ under $L_\eta$ in the form

$$L_\eta^* \omega = \text{Ad}(\eta) \omega + \begin{pmatrix}
-\frac{1}{3}(\Pi \overline{\omega} - \overline{\Pi} \omega + \Gamma) \\
-i(M + P + \overline{\Delta}) \\
-\frac{1}{4}(N + \overline{N} + \Theta + \Lambda) \\
\frac{1}{2}(M + P + \Delta) \\
\frac{1}{3}(\Pi \overline{\omega} - \overline{\Pi} \omega - \Gamma)
\end{pmatrix},$$

where $\text{Ad}$ is the adjoint representation of $\text{Aut}_e(Q_H)$ on $su(2,1)$ and

$$\begin{align*}
\Pi &= \begin{pmatrix} \Pi_1 & \delta \Pi_2 \\ \Pi_2 & \Pi_1 \end{pmatrix}, \\
\Gamma &= \begin{pmatrix} \Gamma_1 \theta^1 + \Gamma_2 \theta^2 & \delta(\Gamma_3 \theta^1 + \Gamma_4 \theta^2) \\ \Gamma_3 \theta^1 + \Gamma_4 \theta^2 & \Gamma_1 \theta^1 + \Gamma_2 \theta^2 \end{pmatrix}, \\
M &= \begin{pmatrix} M_1 \omega^1 + M_2 \omega^2 & \delta(M_3 \omega^1 + M_4 \omega^2) \\ M_3 \omega^1 + M_4 \omega^2 & M_1 \omega^1 + M_2 \omega^2 \end{pmatrix}, \\
\Delta &= \begin{pmatrix} \Delta_1 \theta^1 + \Delta_2 \theta^2 & \delta(\Delta_3 \theta^1 + \Delta_4 \theta^2) \\ \Delta_3 \theta^1 + \Delta_4 \theta^2 & \Delta_1 \theta^1 + \Delta_2 \theta^2 \end{pmatrix}, \\
N &= \begin{pmatrix} N_1 \omega^1 + N_2 \omega^2 & \delta(N_3 \omega^1 + N_4 \omega^2) \\ N_3 \omega^1 + N_4 \omega^2 & N_1 \omega^1 + N_2 \omega^2 \end{pmatrix}, \\
\Theta &= \begin{pmatrix} \Theta_1 \theta^1 + \Theta_2 \theta^2 & \delta(\Theta_3 \theta^1 + \Theta_4 \theta^2) \\ \Theta_3 \theta^1 + \Theta_4 \theta^2 & \Theta_1 \theta^1 + \Theta_2 \theta^2 \end{pmatrix}, \\
\Lambda &= \begin{pmatrix} \Lambda_1 \phi^2 & \delta \Lambda_2 \phi^2 \\ \Lambda_2 \phi^2 & \Lambda_1 \phi^2 \end{pmatrix},
\end{align*}$$

(2.60)
with $\Pi_\alpha, M_\alpha, P_\alpha, \Delta_\alpha, N_\alpha$ complex-valued, $\Gamma_\alpha$ imaginary-valued, and $\Theta_\alpha, \Lambda_\alpha$ real-valued functions on $\mathbb{R}^2$.

To determine the parameters in (2.60) we plug the right-hand side of (2.59) in (2.6), (2.7), (2.9), (2.12), (2.16), (2.18), (2.26), (2.36), (2.54), (2.58). The computations are elementary, but lengthy, and we only list the final results here.

Plugging in (2.6) we get

$$L_\eta^*(\begin{pmatrix} r_1^* \\ r_2^* \end{pmatrix}) := L_\eta^*(\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}) = C \left( \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \right).$$

Plugging in (2.7) gives

$$\begin{align*}
\Pi_1 &= \det \overline{CC_2 (C_1 \rho_2 + \delta C_2 \rho_1)}, \\
\Pi_2 &= \det \overline{C C_2 (C_1 \rho_1 + C_2 \rho_2)}, \\
\rho_1^* &= \overline{L^*_\eta \rho_1} = \left( C_1 |C|^2 - 2 \delta C_1 |C_2|^2 + \delta \overline{C_1 C_2^2} \right) \rho_1 + \left( 2 C_2 |C_1|^2 - \delta C_2 |C_2|^2 - \overline{C_2 C_1^2} \right) \rho_2, \\
\rho_2^* &= \overline{L^*_\eta \rho_2} = \left( 2 \delta C_2 |C_1|^2 - \delta C_2 |C_2|^2 - \delta \overline{C_2 C_1^2} \right) \rho_1 + \left( C_1 |C|^2 - 2 \delta C_1 |C_2|^2 + \delta \overline{C_1 C_2^2} \right) \rho_2, \\
P_1 &= -2 \left( \delta A_2 C_2 \rho_1^* + A_2 C_1 \rho_2^* + A_2 \overline{\rho_1^*} - A_1 \Pi_1 - \delta A_2 \Pi_2 \right), \\
P_2 &= -2 \delta \left( A_2 C_1 \rho_1^* + A_2 C_2 \rho_2^* + A_2 \overline{\rho_2^*} - A_1 \Pi_2 - A_2 \Pi_1 \right), \\
P_3 &= -2 \left( A_2 C_1 \rho_1^* + A_2 C_2 \rho_2^* + \delta A_1 \overline{\rho_1^*} - A_1 \Pi_2 - A_2 \Pi_1 \right), \\
P_4 &= -2 \left( A_2 C_2 \rho_1^* + A_2 C_1 \rho_2^* + A_1 \overline{\rho_2^*} - A_1 \Pi_1 - \delta A_2 \Pi_2 \right), \\
M_1 &= 2 \left( \delta A_2 C_2 \rho_1^* + A_2 C_1 \rho_2^* - A_2 r_1 - A_1 \Pi_1 - \delta A_2 \Pi_2 \right) + \Gamma_1 \overline{C_1} + \delta \Gamma_3 C_2, \\
M_2 &= 2 \left( \delta A_2 C_1 \rho_1^* + A_1 \rho_2^* - A_2 r_2 - \delta A_1 \Pi_2 - \delta A_2 \Pi_1 \right) + \delta \Gamma_1 C_2 + \delta \Gamma_3 C_1, \\
M_3 &= 2 \left( A_1 C_2 \rho_1^* + A_2 C_2 \rho_2^* - \delta A_1 r_1 - A_2 \Pi_1 - A_1 \Pi_2 \right) + \Gamma_1 C_2 + \Gamma_3 C_1, \\
M_4 &= 2 \left( A_1 C_1 \rho_1^* + A_2 C_1 \rho_2^* - A_1 r_2 - A_1 \Pi_1 - \delta A_2 \Pi_2 \right) + \Gamma_1 C_1 + \delta \Gamma_3 C_2,
\end{align*}$$

$$\begin{align*}
\tilde{a}_1^* &= L^*_\eta \tilde{a}_1 = \det C \left( \overline{C_1 \tilde{a}_1} - \overline{C_1 C_2 \tilde{b}_1} + \overline{C_1 C_2 \tilde{a}_2} - \overline{C_2 \tilde{b}_2} + 2 \left( \overline{A_1 C_2} + \overline{A_2 C_1} \right) r_1 - \left( A_1 \overline{C_1} + \delta A_2 \overline{C_2} \right) r_2 \right) - \Gamma_1 + \Gamma_4, \\
\tilde{b}_1^* &= L^*_\eta \tilde{b}_1 = \det C \left( -\delta \overline{C_1 C_2 \tilde{a}_1} + \overline{C_2 \tilde{b}_1} - \delta \overline{C_2 \tilde{a}_2} + \overline{C_1 C_2 \tilde{b}_2} + 2 \left( -\overline{A_1 C_1} + \delta A_2 \overline{C_2} \right) r_1 + \delta \left( \overline{A_1 C_2} + \overline{A_2 C_1} \right) r_2 \right) + \Gamma_2 - \delta \Gamma_3, \\
\tilde{c}_1^* &= L^*_\eta \tilde{c}_1 = \det C \left( |C_1|^2 \tilde{c}_1 - \overline{C_1 C_2 \tilde{d}_1} + \overline{C_2 C_1 \tilde{d}_2} - |C_2|^2 \tilde{d}_2 + 2 \left( \overline{A_1 C_2} + \overline{A_2 C_1} \right) \overline{r_1} - \left( A_1 \overline{C_1} + \delta A_2 \overline{C_2} \right) \overline{r_2} \right) - 2 \left( A_1 \overline{\rho_1^*} - A_2 \overline{\rho_2^*} \right),
\end{align*}$$
\[
\begin{align*}
d_1^* & := L_\eta^* \tilde{d}_1 = \det \overline{C} \left( -\delta \overline{C}_1 C_2 \tilde{c}_1 + |C_1|^2 \tilde{d}_1 - \delta |C_2|^2 \tilde{c}_2 + \overline{C}_2 C_1 \tilde{d}_2 + 2 \left( -(A_1 C_1 + \delta A_2 C_2) \overline{r}_1 + \delta (A_1 C_2 + A_2 C_1) \overline{r}_2 \right) \right) + 2 \left( \delta A_2 \rho_1 - A_1 \rho_2 \right), \\
\tilde{a}_2^* & := L_\eta^* \tilde{a}_2 = \det C \left( \delta \overline{C}_1 C_2 \tilde{a}_1 - \overline{\delta \overline{C}_2 b}_1 + \overline{C}_1 \tilde{a}_2 - C_1 \overline{C}_2 \tilde{b}_2 + 2 \left( (A_1 \overline{C}_1 + \delta A_2 \overline{C}_2) r_1 - \delta (A_1 \overline{C}_2 + A_2 \overline{C}_1) r_2 \right) \right) + \Gamma_2 - \delta \Gamma_3, \\
\tilde{b}_2^* & := L_\eta^* \tilde{b}_2 = \det C \left( -\overline{C}_2 \tilde{a}_1 + \delta \overline{C}_1 \overline{b}_1 - \delta \overline{C}_1 \overline{C}_2 \tilde{a}_2 + \overline{C}_1 \tilde{b}_2 + 2 \left( -\delta \overline{(A_1 \overline{C}_2 + A_2 \overline{C}_1)} r_1 + (\delta A_1 \overline{C}_1 + A_2 \overline{C}_2) r_2 \right) \right) - \delta \Gamma_1 + \delta \Gamma_4, \\
\tilde{c}_2^* & := L_\eta^* \tilde{c}_2 = \det C \left( \delta \overline{C}_2 C_1 \tilde{c}_1 - \delta |C_2|^2 \tilde{c}_1 + |C_1|^2 \tilde{c}_2 - \overline{C}_1 C_2 \tilde{d}_2 + 2 \left( (A_1 C_1 + \delta A_2 C_2) \overline{r}_1 - \delta (A_1 C_2 + A_2 C_1) \overline{r}_2 \right) \right) + 2 \left( \delta A_2 \rho_1 - A_1 \rho_2 \right), \\
\tilde{d}_2^* & := L_\eta^* \tilde{d}_2 = \det C \left( -|C_2|^2 \tilde{c}_1 + \delta \overline{C}_2 C_1 \tilde{d}_1 - \delta \overline{C}_1 \overline{C}_2 \tilde{c}_2 + |C_1|^2 \tilde{d}_2 + 2 \left( -\delta (A_1 C_2 + A_2 C_1) \overline{r}_1 + (\delta A_1 \overline{C}_1 + A_2 \overline{C}_2) \overline{r}_2 \right) \right) - 2 \delta \left( A_1 \rho_1^* - A_2 \rho_2^* \right), \\
\delta \Delta_3 - \Delta_2 & = 4 \delta \left( A_1 \overline{A}_2 - A_2 \overline{A}_1 \right) \rho_1^* + 4 \left( |A_1|^2 - \delta |A_2|^2 \right) \rho_2^* + 2 \left( A_1 \tilde{a}_1^* + A_2 \tilde{b}_1^* + \overline{A}_1 \tilde{c}_1^* + \overline{A}_2 \tilde{d}_1^* - A_1 \Gamma_2 + \delta A_2 \Gamma_1 + \delta A_1 \Gamma_3 - \delta A_2 \Gamma_4 \right), \\
\Delta_1 - \Delta_4 & = 4 \left( \left( A_1 \overline{A}_2 - A_2 \overline{A}_1 \right) \rho_1^* + 4 \left( A_1 \overline{A}_2 - A_2 \overline{A}_1 \right) \rho_2^* + 2 \left( A_1 \tilde{a}_1^* + A_2 \tilde{b}_1^* + \overline{A}_1 \tilde{c}_1^* + \overline{A}_2 \tilde{d}_1^* - A_1 \Gamma_2 + \delta A_2 \Gamma_1 - A_2 \Gamma_2 - A_1 \Gamma_3 + \delta A_2 \Gamma_4 \right) \right).
\end{align*}
\]

It follows from (2.61) that \(\delta \Delta_3 - \Delta_2\) and \(\Delta_1 - \Delta_4\) do not in fact depend on \(\Gamma_a\).

Equations (2.9) are satisfied automatically, and we now plug the right-hand side of (2.59) in (2.12). This gives

\[
\begin{align*}
\Lambda_1 & = 4 \text{Re} \left( A_1 r_1^* + \delta A_2 r_2^* \right), \\
\Lambda_2 & = 4 \text{Re} \left( A_1 r_2^* + A_2 r_1^* \right), \\
N_1 & = -8 \left( R_1 r_1 + i (C_1 \overline{A}_1 + C_1 \overline{A}_2) \text{Re} \left( A_1 r_1^* + \delta A_2 r_2^* \right) \right) + i \left( C_1 \Delta_1 + \delta C_2 \Delta_3 + 2 (A_1 \overline{r}_1 + \delta A_2 \overline{r}_3 - \overline{A}_1 M_1 - \delta \overline{A}_2 M_3) \right), \\
N_2 & = -8 \left( \delta R_2 r_2 + i (C_1 \overline{A}_1 + \delta C_2 \overline{A}_2) \text{Re} \left( A_1 r_1^* + \delta A_2 r_2^* \right) \right) + i \left( C_1 \Delta_3 + \delta C_2 \Delta_1 + 2 (A_1 \overline{r}_2 + \delta A_2 \overline{r}_4 - \overline{A}_1 M_2 - \delta \overline{A}_2 M_4) \right), \\
N_3 & = -4 \left( \delta R_1 r_1 + R_2 r_2 + 2 i (C_1 \overline{A}_1 + C_1 \overline{A}_2) \text{Re} \left( A_1 r_2^* + A_2 r_1^* \right) \right) + i \left( \delta C_1 \Delta_2 + C_2 \Delta_4 + 2 (A_1 \overline{r}_3 + A_2 \overline{r}_4 - \overline{A}_1 M_3 - \overline{A}_2 M_2 - \overline{a}_1 (C_1 \overline{A}_2 + C_2 \overline{A}_1) - \delta \overline{a}_2 (C_1 \overline{A}_1 + \delta C_2 \overline{A}_2) + \overline{c}_1 (C_1 \overline{A}_2 + C_2 \overline{A}_1) + \delta \overline{c}_2 (C_1 \overline{A}_1 + \delta C_2 \overline{A}_2) \right),
\end{align*}
\]
\[ N_4 = -4 \left( R_1 r_2 + R_2 r_1 + 2i(\overline{C_1 A_1} + \delta C_2 A_2) \text{Re} \left( A_1 r_2^* + A_2 r_1^* \right) \right) + i \left( C_1 \Delta_4 + C_2 \Delta_5 + 2 \left( A_1 \overline{P_4} + A_2 \overline{P_2} - \overline{A_1 M_4} - \overline{A_2 M_2} - \overline{b_1 (C_1 A_2 - \overline{C_2 A_1})} - \delta b_2 (C_1 A_1 + \delta C_2 A_2) + \overline{d_1 (C_1 A_2 + C_2 A_1) + \delta d_2 (C_1 A_1 + \delta C_2 A_2)} \right) \), \]

\[ \Theta_2 - \delta \Theta_3 = -16 \text{Re} \left( R_1 (A_1 r_1^* + \delta A_2 r_2^*) - \delta R_2 (A_1 r_2^* + A_2 r_1^*) \right) - 4 \text{Im} \left( A_1 (\overline{\Delta_2 - \delta \Delta_3}) + \delta A_2 (\overline{\Delta_4 - \Delta_1}) \right), \]

\[ \Theta_1 - \Theta_4 = 16 \text{Re} \left( R_1 A_1 r_2^* + A_2 r_1^* - R_2 (A_1 r_1^* + \delta A_2 r_2^*) \right) - 4 \text{Im} \left( A_1 (\overline{\Delta_1 - \Delta_4}) + A_2 (\delta \overline{\Delta_3 - \Delta_2}) \right) - 8 \text{Im} \left( A_1 (\overline{a_1^* - \delta b_2^*}) + A_2 (\overline{b_1^* - \delta a_2^*}) + \overline{A_1 (c_1^* - \delta d_2^*)} + \overline{A_2 (a_2^* - b_1^*) + A_2 (\overline{b_2^* - \delta a_1^*}) + \overline{A_2 (d_2^* - \delta c_1^*)}} \right) + 4 \text{Re} \left( A_1 G_1 + A_2 G_2 \right), \tag{2.62} \]

where \( G_1, G_2 \) are found from the following relations

\[ \overline{C_1 G_1} + \overline{C_2 G_2} = v_3 - \delta v_2 - \delta r_1 \overline{a_2} - r_2 \overline{a_1} - \delta \overline{r_1 d_2} - \overline{r_2 c_1} - 8 (R_1 r_2 - R_2 r_1) + 2i \left( -\overline{a_1 (C_1 A_1 + \delta C_2 A_2)} - \overline{a_2 (C_1 A_2 + C_2 A_1)} + \overline{c_1 (C_1 A_1 + \delta C_2 A_2)} + \overline{c_2 (C_1 A_2 + C_2 A_1)} - \overline{A_1 (C_1 a_1^* + C_2 b_1^*)} - \overline{A_2 (C_1 a_2^* + C_2 b_2^*)} - \overline{C_2 (C_2 A_2 \overline{r_1} + A_1 \overline{b_1} - \delta A_2 c_1 - A_2 \overline{d_2})} - \overline{C_1 (C_2 A_2 \overline{r_1} + A_1 \overline{b_1} - \delta A_2 c_1 - A_2 \overline{d_2})} + \overline{A_1 (A_2 a_2^* + C_1 \overline{A_1 a_1^*} + C_2 \overline{A_2 a_2^*} + C_1 \overline{A_1 d_2})} + \overline{A_2 (A_2 a_2^* + C_1 \overline{A_1 a_1^*} + C_2 \overline{A_2 a_2^*} + C_1 \overline{A_1 d_2})} + \overline{A_1 (A_2 a_2^* + C_1 \overline{A_1 a_1^*} + C_2 \overline{A_2 a_2^*} + C_1 \overline{A_1 d_2})} + \overline{A_2 (A_2 a_2^* + C_1 \overline{A_1 a_1^*} + C_2 \overline{A_2 a_2^*} + C_1 \overline{A_1 d_2})} \right), \tag{2.63} \]

Further, plugging in (2.16) gives

\[ \Gamma_1 - \Gamma_4 = 2i \text{Re} \left( (C_1 \overline{C_2 r_1} + \delta |C_2 r_2|^2) \rho_1 + (C_1 \overline{C_2 r_2} + |C_2 r_1|^2) \rho_2 \right) + 4 \text{Im} \left( A_2 r_1^* - A_1 r_2^* + C_1 \overline{C_2 a_2} \right) + 2 C_1 \overline{C_2} \text{Im} \left( r_1 \overline{r_2} \right) + 2 |C_2|^2 \left( \overline{b_2 + \delta a_1 - i |r_1|^2 + i \delta |r_2|^2} \right), \]

\[ \Gamma_2 - \delta \Gamma_3 = 2i \text{Re} \left( \delta (C_1 \overline{C_1 r_2} + |C_2 r_1|^2) \rho_1 + (C_1 \overline{C_2 r_1} + \delta |C_2 r_2|^2) \rho_2 \right) + 4 \text{Im} \left( \delta A_2 r_2^* - A_1 r_2^* + \delta C_1 \overline{C_2 a_1} \right) + 2 \delta |C_2|^2 \left( \overline{b_1 + a_2} \right). \tag{2.64} \]

Equations (2.18), (2.26) are automatically satisfied, so we now have to use (2.36), (2.54) and (2.58). The remaining part of the proof goes as follows: first, in addition to (2.64), we will obtain two more relations for \( \Gamma_\alpha \) from (2.36) and thus determine them; further, in addition to the relations for \( \Delta_\alpha \) in (2.61), conditions (2.54) will give two more relations and thus fix \( \Delta_\alpha \); finally, in addition to the relations for \( \Theta_\alpha \) in (2.62) and (2.63), we obtain two more relations from (2.58) which will determine \( \Theta_\alpha \). It is clear from (2.61), (2.62) that the choice of \( \Gamma_\alpha, \Delta_\alpha, \Theta_\alpha \) determines the rest of the parameters as well. To get these extra relations (as well as for future applications) we need to find the expansions
of $\Phi_\beta, \Phi^\alpha, \Psi^\alpha$ completely (cf. (2.34), (2.45)). The calculations turn out to be so enormous that even writing down the final formulas would be extremely lengthy; therefore we only give here an outline of the procedure that allowed us to find these expansions.

The coefficients in (2.35) can be found from (2.21) if one considers terms cubic in $\omega^\alpha, \overline{\omega}^\alpha$ there. Further, to determine terms containing $\theta^\gamma$ in the expansions of $\Phi^\alpha$, we need to find the expressions of $\lambda^\alpha \wedge \theta^3$ in (2.39). This is done analogously to the proof of (2.46): we use (2.41), (2.44) to get information about the expansions of $d\mu_1, d\mu_2^2$ in terms of $\lambda^\alpha$, differentiate (2.33) and compare appropriate terms in both sides of the resulting equations. In addition to this, to find terms containing $\omega^\gamma \wedge \theta^3$ or $\theta^1 \wedge \theta^2$ in the expansions of $\Phi^\alpha$, we use (2.20). Thus, we determine the expansions of $\Phi^\alpha$ completely, and at the same time the expansions of $\Phi^\alpha \pmod{\theta^3}$, in particular, the coefficients in (2.53); we also determine the imaginary parts of the coefficients at $\omega^\beta \wedge \theta^\gamma$ and for the terms containing $\overline{\omega}^\beta \wedge \theta^\gamma$. To find more terms in the expansions of $\Phi^\alpha$ as well as some terms in the expansions of $\Psi^\alpha$ (in particular, terms in (2.57)) we use an analogue of the above procedure where the differentiation of (2.7) (i.e. equations (2.20)) is replaced by the differentiation of (2.12) and the differentiation of (2.33) is replaced by the differentiation of (2.38).

Eventually we find all the terms in the expansions of $\Phi_\beta, \Phi^\alpha, \Psi^\alpha$ except for the coefficients at $\overline{\omega}^\gamma \wedge \theta^3$ in $\Phi^\alpha$ and coefficients at $\omega^\gamma \wedge \theta^3, \overline{\omega}^\beta \wedge \theta^3$ in $\Psi^\alpha$. All the coefficients that we have found are expressed in terms of $r_\alpha, \rho_\alpha, \tilde{a}_\alpha, \tilde{b}_\alpha, \tilde{c}_\alpha, \tilde{d}_\alpha$. An observation that is going to be important for future references is: if $r_1 \equiv 0, r_2 \equiv 0, \rho_1 \equiv 0, \rho_2 \equiv 0$, then $\tilde{a}_\alpha \equiv 0, \tilde{b}_\alpha \equiv 0, \tilde{c}_\alpha \equiv 0, \tilde{d}_\alpha \equiv 0$ and

$$\begin{align*}
\Omega^1_1 & \equiv 0, \\
\Omega^2_1 & = Q\overline{\omega} \wedge \theta, \\
\Omega^0_2 & = (P\omega + \overline{T}\overline{\omega}) \wedge \theta,
\end{align*}$$

where $Q$ and $P$ are $\mathfrak{g}$-valued functions on $P^2$.

Now that we know the expansions for $\Phi_\beta, \Phi^\alpha, \Psi^\alpha$, we can determine all the parameters in (2.60). Namely, conditions (2.36) determine $\Gamma_1, \Gamma_3$, conditions (2.54) determine $\Delta_1, \Delta_3$, conditions (2.58) determine $\Theta_1, \Theta_3$. This determines the right-hand side in (2.59) completely in terms of $r_\alpha, \rho_\alpha, a_\alpha, b_\alpha, c_\alpha, d_\alpha$.

As one can see, most of the work in the above proof came from dealing with $\phi^\beta_\alpha, \mu^\alpha_\beta, \psi^\alpha$. Since the difference $L^\eta_\omega - \text{Ad}(\eta)\omega$ does not contain these forms, we may say that the form $\omega$ on $P^2$ is reasonably close to being a Cartan connection.

The theorem is proved.

3 Corollaries and Applications

In this section we derive some corollaries from Theorem 1.1 and the construction of parallelism in Section 2.

First, we discuss the question whether the sequence of bundles $P^2 \to P^1 \to M$ actually reduces to a single principle bundle over $M$ with structure group $\text{Auto}_{\mathfrak{g}}(\mathfrak{q}^H)$.

**Proposition 3.1** If $r_1 \equiv 0, r_2 \equiv 0$, then $P^2$ is a principle $\text{Auto}_{\mathfrak{g}}(\mathfrak{q}^H)$-bundle over $M$. If, in addition, $\rho_1 \equiv 0, \rho_2 \equiv 0$, then the form $\omega$ is a Cartan connection on $P^2$.

**Proof.** Let $x \in P^2$ be $x = (\tilde{\theta}(y), \tilde{\omega}, \overline{\omega}, \tilde{\phi})$ for $y = \theta(p) \in P^1, p \in M$, where $\tilde{\theta} := (\tilde{\theta}_1, \tilde{\theta}_2)$, $
\tilde{\omega} := (\tilde{\omega}^1, \tilde{\omega}^2), \tilde{\phi} := (\tilde{\phi}^1, \tilde{\phi}^2), \tilde{\theta}(p) := (\theta^1(p), \theta^2(p)), \tilde{\theta} = \pi^1 \tilde{\theta}$ for $\tilde{\theta} \in \mathfrak{m}_\theta$, $\tilde{\omega} = \pi^1 \omega(y)$ for some
complex covectors $\omega := (\omega^1, \omega^2)$ at $p$, $\phi^\alpha$ are real covectors at $y$, $\pi^1(y) = p$, $\pi^2(x) = y$. Let an element $\eta \in \text{Aut}_{0,e}(Q_H)$ be represented by matrix (1.2). We then define $F_\eta(x)$ as follows

\[
F_\eta(x) := \left( \hat{\theta}(y'), C^{-1}\hat{\omega}^* - 2A\hat{\theta}(y'), C^{-1}\hat{\omega}^* - 2A\hat{\theta}(y'), \hat{\phi}^* - 2iC^{-1}A\hat{\omega}^* + 2iC^{-1}A\hat{\phi}^* - 4R\hat{\theta}(y') \right),
\]

where $y' = C^{-1}C^{-1}\theta(p)$ and $\hat{\omega}^*, \hat{\phi}^*$ are the pull-backs of the covectors $\omega^*, \phi^*$ respectively under the diffeomorphism $\Phi_C$ of $P^1$ locally over a neighbourhood of $p$ given by

\[
\Phi_C(D\theta(q)) = C\overline{C}D\theta(q),
\]

for $D \in G^1$.

It is now easy to check that (3.1) indeed defines an action of $\text{Aut}_{0,e}(Q_H)$ on $P^2$, provided $r_1 \equiv 0, r_2 \equiv 0$.

Further, one can derive an analogue of transformation law (2.59) for the form $\omega$ under the action of $\text{Aut}_{0,e}(Q_H)$ on $P^2$ by the procedure described in Section 2. The transformation formula has the same form (2.59), but the error terms in the right-hand side turn out to be zero due to the identical vanishing of $r_\alpha$ and $\rho_\alpha$, i.e. we get

\[
L_\eta^*\omega = \text{Ad}(\eta)\omega.
\]

The proposition is proved. \hfill \Box

**REMARK 3.2** If $r_1, r_2$ do not necessarily vanish, one can still define for any $\eta \in \text{Aut}_{0,e}(Q_H)$ the mapping $\tilde{F}_\eta$ as

\[
\tilde{F}_\eta(x) := \left( \hat{\theta}(y'), C^{-1}\hat{\omega}^* - 2A\hat{\theta}(y'), C^{-1}\hat{\omega}^* - 2A\hat{\theta}(y'), \hat{\phi}^* - 2iC^{-1}A\hat{\omega}^* + 2iC^{-1}A\hat{\phi}^* - 4R\hat{\theta}(y') + tC\overline{C}\begin{pmatrix} r_1(y') & \delta r_2(y') \\ r_2(y') & r_1(y') \end{pmatrix} \hat{\omega}^* \right),
\]

where $t$ is determined by

\[
C^{-1}C^{-1} = \begin{pmatrix} s & \delta t \\ t & s \end{pmatrix}.
\]

However, $\tilde{F}_\eta$ does not give a group action unless $r_\alpha$ identically vanish.

Next, we characterize CR-flat manifolds, i.e. manifolds for which the form $\Omega$ in (2.32) vanishes.

**PROPOSITION 3.3** The form $\Omega$ identically vanishes on $P^2$ if and only if $M$ is locally CR-equivalent to $Q_H$.

**Proof.** First, we will explicitly calculate the bundles $P^1$, $P^2$ and the forms $\omega$, $\Omega$ for $Q_H$. To do this, we identify $(z_1, z_2)$ and $(w_1, w_2)$ with the matrices

\[
Z := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{and} \quad W := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},
\]

respectively and write the equation of $Q_H$ in the form

\[
V = ZZ^*,
\]

with $W = U + iV$. Let the orientation of $Q_H$ be given by the form $\hat{\theta} := \frac{1}{2}(dU - iZdZ + iZd\overline{Z})$. For $\omega := dZ$ we have

\[
d\hat{\theta} = i\omega \wedge \overline{\omega}.
The bundle $P^1$ then consists of $D\theta$ with $D = C\overline{C}$ for $C \in \mathfrak{a}^*$. The tautological form $\tilde{\theta}$ is given by

$$\tilde{\theta} = \frac{1}{2} D \left( dU - iZdZ + iZd\overline{Z} \right),$$

and therefore $P^2$ consists of coframes of the form

$$x := (\tilde{\theta}, T\tilde{\theta} + C\tilde{\omega}, T\tilde{\omega} + C\tilde{\omega}, S\tilde{\theta} + iCT\tilde{\omega} - i\overline{C}T\tilde{\omega} - D^{-1}dD),$$

with $T, C, S$ as in (1.4), $C\overline{C} = E$, where $\tilde{\omega} := \sqrt{D}dZ$ (it is an easy exercise to prove the existence of a locally smooth operation of taking real square root on $G^1$).

Now one can check that

$$\hat{\theta} = \frac{1}{2} D \left( dU - iZdZ + iZd\overline{Z} \right),$$

$$\hat{\omega} = T\hat{\theta} + C\sqrt{D}dZ,$n
$$\hat{\phi} = S\hat{\theta} + iCT\sqrt{D}dZ - i\overline{C}T\sqrt{D}d\overline{Z} - D^{-1}dD,$n
$$\phi = \frac{1}{2} (S - 3iT\overline{T})\hat{\theta} + 2iT\hat{\omega} + iT\overline{\omega} - \overline{C}dC - \frac{1}{2} D^{-1}dD,$n
$$\mu = iT^2\hat{\theta} + \frac{1}{2} (S - iT\overline{T})\hat{\omega} - iT^2\overline{\omega} - dT + \overline{C}TdC - \frac{1}{2} TD^{-1}dD,$n
$$\psi = \frac{1}{2} (S^2 - 3T^2\overline{T}^2)\hat{\theta} + (iST + T^2\overline{T})\hat{\omega} + (-iST + T^2\overline{T})\overline{\omega} - dS - iTdT + iTd\overline{T} + 2iT\overline{C}TdC - SD^{-1}dD,$

(3.3)

and all the functions $r_\alpha, \rho_\alpha, \bar{a}_\alpha, \bar{b}_\alpha, \bar{c}_\alpha, \bar{d}_\alpha$ identically vanish. It now follows from (2.32) and (3.3) that for $\omega$ defined as in (2.31) its curvature form $\Omega \equiv 0$.

Now, if $M$ is a CR-manifold with $\Omega \equiv 0$, then $P^2$ locally can be mapped by a diffeomorphism onto a neighbourhood of identity in $\text{Aut}_e(Q_H)$ in such a way that $\omega$ transforms into the Maurer-Cartan form on $\text{Aut}_e(Q_H)$ (see [St]). Therefore, there exists a local diffeomorphism of $P^2$ to the corresponding bundle over $Q_H$ that preserves parallelism. By Theorem 1.1 this diffeomorphism is the lift of a local CR-diffeomorphism between $M$ and $Q_H$, and thus $M$ is locally CR-equivalent to $Q_H$.

The proposition is proved. □

We will now try to understand what proper analogues of chains in the case of hyperbolic and elliptic CR-manifolds should be. We define the chain distribution on $P^2$ by

$$\omega = 0, \quad \mu = 0.$$

(3.4)

This distribution is analogous to the one of Chern [CM] that, in the case of CR-codimension 1, was used to define chains on $M$ as the projections of the integral manifolds of the distribution. However, in contrast with [CM], distribution (3.4) may not be integrable. It follows from the expansions of $\Phi^\alpha, \Phi^\alpha, \Psi^\alpha$ that can be found as outlined in Section 2, that it is integrable if and only if the following conditions are satisfied (cf. Proposition 3.1):

$$r_1 \equiv r_2 \equiv 0,$n
$$p_1 \equiv p_2 \equiv 0,$$

(3.5)

(note that the functions $r_\alpha, \rho_\alpha$ are scalar CR-invariants). For manifolds satisfying conditions (3.5) one can project the integral manifolds of chain distribution (3.4) to $M$. The resulting two-dimensional submanifolds of $M$ we call $G$-chains.
It is clear from (2.23), (2.24) that conditions (3.5) are equivalent to
\[ r \equiv 0, \quad \rho \equiv 0, \]  
where \( r := r_1 r_2, \rho := \rho_1 \rho_2. \) Elliptic or hyperbolic \( CR \)-manifolds satisfying (3.6) we call semi-flat. It follows from (2.6), (2.7), (2.32) that semi-flat manifolds are characterized by the curvature conditions
\[ \Omega^1_0 \equiv 0, \quad \Omega^2_0 \equiv 0. \]  

Note that conditions (3.7) are always satisfied for the parallelism constructed in \([CM]\). For semi-flat manifolds the main formulas in Section 2 reduce to matrix forms of Chern’s formulas in the case of \( CR \)-dimension and \( CR \)-codimension 1. Namely, we have
\[
\begin{align*}
\frac{d\Theta}{\Theta} &= i\omega \wedge \overline{\omega} + \theta \wedge \phi, \\
\frac{d\omega}{\omega} &= \omega \wedge \phi + \theta \wedge \mu, \\
\text{Re } \phi &= \frac{1}{2} \phi, \\
\frac{d\phi}{\phi} &= 2\text{Re} \left( i\mu \wedge \overline{\omega} \right) + \theta \wedge \psi, \\
\Omega^1_1 &\equiv 0, \\
\Omega^2_1 &= Q \overline{\omega} \wedge \theta, \\
\Omega^0_2 &= \left( P\omega + \overline{P\omega} \right) \wedge \theta,
\end{align*}
\]
where \( Q \) and \( P \) are \( \mathfrak{a} \)-valued functions on \( P^2 \), and
\[
\begin{align*}
\mu^2_2 &= \mu^1_1, \\
\mu^2_1 &= \delta \mu^1_2, \\
\psi^2 &= \delta \psi^3, \\
\psi^4 &= \psi^1, \\
\psi^5 &= \psi^1, \\
\psi^6 &= \delta \psi^3, \\
\psi^7 &= \psi^3, \\
\psi^8 &= \psi^1.
\end{align*}
\]

It follows from the proof of Proposition 3.3 that the quadric \( Q_H \) is a semi-flat manifold. In the next proposition we describe \( G \)-chains on \( Q_H \).

**Proposition 3.4** Any \( G \)-chain on \( Q_H \) passing through the origin is the intersection of \( Q_H \) with \( Z = AW \) for some \( A \in \mathfrak{a} \).

**Proof.** Formulas (3.3) imply that distribution (3.4) in the case of \( Q_H \) is given by
\[
\begin{align*}
T\hat{\theta} + C\sqrt{D}dZ &= 0, \quad \text{(3.8.a)} \\
\frac{1}{2} \left( TS + iT^2T \right) \hat{\theta} + dT &= 0. \quad \text{(3.8.b)}
\end{align*}
\]

First we show that along the integral manifolds of (3.8) passing through points of the fibre of \( P^2 \) over the origin, \( G := C - iT\sqrt{D}Z \) is non-degenerate. To do this, we differentiate \( G\overline{G} \) and plug in the resulting expression \( dZ \) and \( dT \) found from (3.8). It then easily follows that \( d\left( G\overline{G} \right) \equiv 0 \) and thus \( \det |G|^2 \equiv \text{const.} \) Since \( \det G \neq 0 \) for \( Z = 0 \), \( G \) is non-degenerate everywhere.
Next, it follows from (3.8.a) that
\[ dZ = -\frac{1}{2} TDG^{-1} (dU + i\overline{Z}dZ + iZd\overline{Z}). \]

Therefore, to show that the projections of the integral manifolds of distribution (3.8) passing through points of the fibre of \( P^2 \) over the origin have the desired form, we need only prove that
\[ TDG^{-1} = \text{const} \ (3.9) \]
along the integral manifolds of (3.8). To prove (3.9) we differentiate \( TDG^{-1} \) and by using (3.8) conclude that \( d(TDG^{-1}) \equiv 0 \).

The proposition is proved. \( \square \)

As we have seen, semi-flat manifolds possess some of the nice properties that were observed in [CM] for manifolds of \( CR \)-codimension 1. The quadric \( Q_H \) is an example of such manifolds. Many more examples come from considering matrix surfaces in \( C^4 \), i.e. real-analytic surfaces locally near the origin given in the form
\[ V = ZZ + \sum_{k+l+m \geq 2} A_{k,l,m} Z^k \overline{Z}^l U^m, \]
where \( A_{k,l,m} \in \mathbb{A}, A_{k,l,m} = \overline{A_{l,k,m}}, A_{1,0,0} = 0 \), and the power series in the right-hand side converges in a neighbourhood of the origin. The quadric \( Q_H \) written in the form (3.2) is a matrix surface.

Hyperbolic (\( \delta = 1 \)) matrix surfaces are easily described. Indeed, the mapping
\[ z_1^* = z_1 + z_2, \]
\[ z_2^* = z_1 - z_2, \]
\[ w_1^* = w_1 + w_2, \]
\[ w_2^* = w_1 - w_2, \]
maps a hyperbolic matrix surface into a direct product of real hypersurfaces in \( C^2 \). In the next proposition we show that semi-flat hyperbolic manifolds can be characterized in a similar way.

**PROPOSITION 3.5** A semi-flat hyperbolic manifold is locally \( CR \)-equivalent to a product of 3-dimensional Levi non-degenerate \( CR \)-manifolds of codimension 1.

**Proof.** It is not hard to show that a manifold \( M \) is semi-flat if and only if near every point \( p \in M \) there exist complex 1-forms \( \omega^1, \omega^2 \) that at every point \( q \) are complex linear on \( T^c_q(M) \), real 1-forms \( \theta^1, \theta^2 \) whose common annihilator at every point is \( T^c_q(M) \), real 1-forms \( \phi^1, \phi^2 \) and complex 1-forms \( \lambda^1, \lambda^2, \mu^1, \mu^2 \) such that near \( p \) the following holds
\[
\begin{align*}
    d\theta^1 &= i \left( \omega^1 \wedge \overline{\omega}^1 + \delta \omega^2 \wedge \overline{\omega}^2 \right) + \theta^1 \wedge \phi^1 + \delta \theta^2 \wedge \phi^2, \\
    d\theta^1 &= i \left( \omega^1 \wedge \overline{\omega}^2 + \delta \omega^2 \wedge \overline{\omega}^1 \right) + \theta^1 \wedge \phi^2 + \delta \theta^2 \wedge \phi^1, \\
    d\omega^1 &= \omega^1 \wedge \lambda^1 + \delta \omega^2 \wedge \lambda^2 + \theta^1 \wedge \mu^1 + \delta \theta^2 \wedge \mu^2, \\
    d\omega^2 &= \omega^1 \wedge \lambda^2 + \delta \omega^2 \wedge \lambda^1 + \theta^1 \wedge \mu^2 + \delta \theta^2 \wedge \mu^1,
\end{align*}
\]
and \( (\theta^\alpha, \Re \omega^\alpha, \Im \omega^\alpha) \) at every point form a coframe. Then, in the case of hyperbolic manifolds, the forms
\[
\begin{align*}
    \theta^1' &= \theta^1 + \theta^2, \\
    \theta^2' &= \theta^1 - \theta^2,
\end{align*}
\]
\[ \omega^1' := \omega^1 + \omega^2, \]
\[ \omega^{2'} := \omega^1 - \omega^2, \]
\[ \phi^1' := \phi^1 + \phi^2, \]
\[ \phi^{2'} := \phi^1 - \phi^2, \]
\[ \lambda^1' := \lambda^1 + \lambda^2, \]
\[ \lambda^{2'} := \lambda^1 - \lambda^2, \]
\[ \mu^1' := \mu^1 + \mu^2, \]
\[ \mu^{2'} := \mu^1 - \mu^2. \]

satisfy

\[ d\theta^1' = i\omega^1' \wedge \overline{\omega^1'} + \theta^1' \wedge \phi^1', \]
\[ d\theta^{2'} = i\omega^{2'} \wedge \overline{\omega^{2'}} + \theta^{2'} \wedge \phi^{2'}, \]
\[ d\omega^1' = \omega^1' \wedge \lambda^1' + \theta^1' \wedge \mu^1', \]
\[ d\omega^{2'} = \omega^{2'} \wedge \lambda^{2'} + \theta^{2'} \wedge \mu^{2'}. \]

Formulas (3.12) now imply that the distribution

\[ \theta^{\alpha'} = 0, \quad \omega^{\alpha'} = 0, \]

for each \( \alpha = 1, 2 \) is integrable and thus gives a foliation of \( M \) near \( p \) by 3-dimensional Levi non-degenerate CR-manifolds of CR-dimension 1. Let \( M^1, M^2 \) be the leaves of the first and the second foliation respectively that pass through \( p \). Then \( M \) is clearly CR-equivalent near \( p \) to the product \( M^1 \times M^2 \) (see also Proposition 5.8 in [Ch]).

The proposition is proved. \( \square \)

**REMARK 3.6** If in the above proposition a semi-flat hyperbolic manifold \( M \) is in addition real-analytic, then \( M^1, M^2 \) are also real-analytic and therefore admit real-analytic CR-embeddings in \( \mathbb{C}^3 \) as hypersurfaces [AH]. Mapping the image of the point \( p \) into the origin and applying the transformation inverse to (3.11) we see that \( M \) near \( p \) is CR-equivalent to a surface of the form (3.10), and therefore real-analytic semi-flat hyperbolic manifolds are characterized locally as matrix surfaces.

## 4 Normal Forms

In this section we consider real-analytic hyperbolic and elliptic CR-manifolds that by [AH] can be assumed to be locally embedded in \( \mathbb{C}^4 \) near the origin. We denote coordinates in \( \mathbb{C}^4 \) by \( z := (z_1, z_2), w := (w_1, w_2), u := (u_1, u_2) := (\text{Re } w_1, \text{Re } w_2), v := (v_1, v_2) := (\text{Im } w_1, \text{Im } w_2). \) Let \( M \) be such a manifold and suppose that the coordinates are chosen so that \( T_0(M) \) is spanned by \( z, u \) and \( T^c_0(M) \) by \( z \). Then \( M \) is given by an equation of the form

\[ v = H^4(z, z) + F(z, \overline{z}, u), \]

where \( F \) is an \( \mathbb{R}^2 \)-valued real-analytic function such that \( F(0) = 0, dF(0) = 0, \frac{\partial^2 F}{\partial z_i \partial z_j}(0) = 0. \)

Important examples of hyperbolic and elliptic manifolds are matrix surfaces (3.10). As we noted in Section 3, transformation (3.11) maps a hyperbolic matrix surface into a direct product:

\[ v_1 = |z_1|^2 + F^1(z_1, \overline{z_1}, u_1), \]
\[ v_2 = |z_2|^2 + F^2(z_2, \overline{z_2}, u_2), \quad (4.1) \]
where $F^k$ are real-analytic functions, $F^k(0) = 0$, $dF^k(0) = 0$, $\frac{\partial^2 F^k}{\partial z_k \partial \overline{z}_k}(0) = 0$. In the elliptic case, the transformation

$$
\begin{align*}
  z_1^* &= z_1 + iz_2, \\
  z_2^* &= z_1 - iz_2, \\
  w_1^* &= w_1, \\
  w_2^* &= w_2
\end{align*}
$$

maps the surface into a surface of the form

$$
V = z_1 \overline{z}_2 + F(z_1, \overline{z}_2, U), \quad (4.3)
$$

where $U := u_1 + iu_2$, $V := v_1 + iv_2$ and $F$ is a $C$-valued analytic function, $F(0) = 0$, $dF(0) = 0$, $\frac{\partial^2 F}{\partial z_k \partial \overline{z}_k}(0) = 0$. For convenience, we will use the forms (4.1), (4.3) for matrix surfaces instead of (3.10).

Equations of hyperbolic and elliptic manifolds can be written in normal forms [Lo1], [ES2] that may be viewed as generalizations of the Chern-Moser normal forms for real-analytic Levi non-degenerate hypersurfaces in $\mathbb{C}^n$. Here we write them in a modified way as follows (the center of normalization is assumed to be at the origin).

The hyperbolic normal form:

$$
\begin{align*}
  v_1 - |z_1|^2 &= N^1(z, \overline{z}, u) := N^1_1(z_1, \overline{z}_1, u_1) + N^1_2(z, \overline{z}, u), \\
  v_2 - |z_2|^2 &= N^2(z, \overline{z}, u) := N^2_1(z_2, \overline{z}_2, u_2) + N^2_2(z, \overline{z}, u),
\end{align*}
$$

where $N^1_1$ and $N^2_2$ are in the Chern-Moser normal form, i.e.

$$
N^1_j = 2\text{Re} \left( h^j_{4\bar{\tau}}(u_j)z^4_j \bar{z}_j^2 \right) + \sum_{k, l \geq 2, \ k + l \geq 7} h^j_{k\bar{\tau}}(u_j)z^k_j \bar{z}_j^l,
$$

each monomial in $N^1_1$ contains at least one of the variables $z_2, \overline{z}_2, u_2$ and satisfies the following conditions

$$
\begin{align*}
  N^1_{2, k\bar{\tau}} &= 0, \quad k \geq 1, \\
  N^1_{2, 1\bar{\tau}} &= 0, \\
  \frac{\partial N^1_{2, k\bar{\tau}}}{\partial z_1^k} &= 0, \quad k \geq 1, \\
  \frac{\partial^2 N^1_{2, 2\bar{\tau}}}{\partial z_1^2} &= 0, \\
  \frac{\partial^3 N^1_{2, 2\bar{\tau}}}{\partial z_1^2} &= 0, \\
  \frac{\partial^4 N^1_{2, 2\bar{\tau}}}{\partial z_1^2 \partial z_2^{\overline{\tau}}} &= 0,
\end{align*}
$$

and $N^2_1$ has the same properties as $N^1_2$ above with interchanged indices 1,2.

The elliptic normal form:

$$
V = z_1 \overline{z}_2 = N(z, \overline{z}, u) := N_1(z_1, \overline{z}_2, U) + N_2(z, \overline{z}, U, \overline{U}), \quad (4.5)
$$
where the real and imaginary parts of \( N_1(\zeta, \overline{\zeta}, \tau), \zeta \in \mathbb{C}, \tau \in \mathbb{R} \), are in the Chern-Moser normal form, each monomial in \( N_2 \) contains at least one of the variables \( \overline{z_1}, z_2, \overline{\tau} \) and satisfies the conditions

\[
\begin{align*}
N_{2, k, \overline{\tau}} &= 0, \quad k \geq 1, \\
N_{2, 1, \overline{\tau}} &= 0, \\
\frac{\partial N_{2, k, \overline{\tau}}}{\partial z_2} &= 0, \quad k \geq 2, \\
\frac{\partial^2 N_{2, 2, \overline{\tau}}}{\partial z_1 \partial z_2} &= 0, \\
\frac{\partial^4 N_{2, 2, \overline{\tau}}}{\partial z_1 \partial z_2 \partial \overline{z_1} \partial \overline{z_2}} &= 0.
\end{align*}
\]

We emphasize that in representations (4.4), (4.5) all other conditions of the normal forms from \[Lo1\], \[ES2\] are satisfied automatically.

We call \( \left( \begin{array}{c} N_1^1 \\ N_2 \end{array} \right) \) and \( N_1 \) in (4.4), (4.5) respectively the matrix non-quadratic part of the normal form; \( \left( \begin{array}{c} N_1^1 \\ N_2 \end{array} \right) \) and \( N_2 \) the non-matrix part. We say that a normal form of a hyperbolic (resp. elliptic) surface \( M \) is a matrix normal form if \( N_2^1 \equiv N_2^2 \equiv 0 \) (resp. \( N_2 \equiv 0 \)).

**Proposition 4.1** Let \( M \) be a matrix surface given by a power series of the form (3.10) that converges in a neighbourhood \( \Omega \) of the origin. Then any normal form of \( M \) with center at any point of \( \Omega \) is a matrix normal form.

**Proof.** We recall \[CM\] that for a real-analytic Levi non-degenerate hypersurface in \( \mathbb{C}^2 \) given by

\[ v = |z|^2 + G(z, \overline{z}, u), \]

with \( G(0) = dG(0) = \frac{\partial^2 G}{\partial z \partial \overline{z}}(0) = 0 \), a normalizing mapping with initial data \((c, a, r)\) (see \[V\] for the definition) can be obtained in the form

\[
\begin{align*}
z &\mapsto \frac{c(z + aw + \phi(z, w))}{1 - 2iaz - (r + i|a|^2)w}, \\
w &\mapsto \frac{|c|^2(w + \psi(z, w))}{1 - 2iaz - (r + i|a|^2)w}, \\
d\phi(0) = d\psi(0) = 0,
\end{align*}
\]

where the holomorphic functions \( \phi, \psi \) are uniquely determined by \( c, a, r \) and \( G \). Analogously, for any set of initial data \((C, A, R)\) \[Lo1\], \[ES2\] we can find a mapping of the form

\[
\begin{align*}
Z &\mapsto C \left( Z + AW + \Phi(Z, W) \right) \left( E - 2i\overline{A}Z - (R + iA\overline{A})W \right)^{-1}, \\
W &\mapsto C \overline{C} \left( W + \Psi(Z, W) \right) \left( E - 2i\overline{A}Z - (R + iA\overline{A})W \right)^{-1}, \\
d\Phi(0) = d\Psi(0) = 0,
\end{align*}
\]

that transforms \( M \) into a surface given in the form (3.10) with \( A_{k, \overline{\tau}, m} = 0, A_{k, \overline{\tau}, m} = 0, A_{z, \overline{\tau}, m} = 0, A_{z, \overline{\tau}, m} = 0, A_{z, \overline{\tau}, m} = 0, A_{z, \overline{\tau}, m} = 0 \) for all \( k, m \). By transformations (3.11), (4.2) such equations are mapped into equations in the matrix normal form.

It now follows from the uniqueness of normalization with prescribed initial data (see \[Lo2\], \[ES2\]) that any normal form of \( M \) with center at the origin is a matrix normal form. Since a transformation
that maps a fixed point of Ω into the origin can be chosen in the matrix form, any normal form of
M with center at any point of Ω is also matrix.

The proposition is proved. □

Suppose M is given in normal form (4.4) or (4.5). We write the equation in a standard way as
a sum of weighted homogeneous polynomials [Lo1], [ES2]. Let ν be the smallest weight for which
there exists a nonvanishing polynomial in the non-matrix part of the equation. We define κ(0) := \frac{1}{ν}
Clearly, κ(0) = 0 if and only if M is in a matrix normal form. It follows from Proposition 4.1 that,
for a matrix surface, κ(0) = 0 for any normalization.

**PROPOSITION 4.2** The number κ(0) does not depend on the choice of normalization for any M.

**Proof.** Without loss of generality we can assume that the differential of the renormalizing
transformation at the origin is identical on \( T^1_0(M) \). We follow the scheme of normalization in [Lo1],
[ES2] and split the transformation into a matrix and non-matrix parts. The non-matrix part begins
with a term of weight \( \geq 1 \frac{1}{κ(0)} \), and the lowest weight of the new non-matrix part of the equation in
normal form remains ν.

The proposition is proved. □

Moving the center of normalization, we define a function κ on all of M. By Proposition 4.2,
κ is a holomorphic invariant. This invariant measures the “non-matricity” of the surface and is
analogous to the CR-invariant functions r, ρ on \( P^2 \) from (3.6) that measure the “non-semi-flatness”
of the manifold.

We recall that a 2-dimensional real-analytic submanifold \( Γ \subset M \) through the origin is called a
chain, if in some normal coordinates it is defined by \( \{ z = 0, v = 0 \} \) [Lo1], [ES2]. For example, such
chains on quadrics (3.2) are the intersections of the quadrics with complex planes \( Z = AW \) for \( A ∈ \mathbb{A} \)
and thus coincide with G-chains by Proposition 3.4. Chains form a holomorphically invariant family
and are defined by a mixed system of ODE’s and PDE’s. The main disadvantage of 2-dimensional
chains as opposed the one-dimensional Chern-Moser chains, is that in general translations along
2-dimensional chains spoil the normal form conditions. However, matrix manifolds are a class of
manifolds with proper chains: chains on matrix manifolds are given by a matrix generalization of
the Chern-Moser chains. More precisely, any chain on matrix surface (3.10) has the parametric form
\( Z = P(U), W = Q(U) \) with \( P, Q \) satisfying certain equations

\[
D^2 P = \Phi'(DP, P, Q, U),
D^3 Q = \Psi'(D^2 Q, DQ, Q, U),
\]

where the operator

\[
D := \left( \begin{array}{cc}
\frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} \\
\delta \frac{\partial}{\partial u_1} & \delta \frac{\partial}{\partial u_2}
\end{array} \right)
\]

is a matrix analogue of \( \frac{\partial}{\partial z} \).

In Section 3 we defined G-chains on any semi-flat manifold, in particular, on any matrix surface.
It follows from a matrix analogue of calculations on pp. 265–268 in [CM] that, in the case of matrix
surfaces, G-chains coincide with chains (note that Proposition 3.4 gives an independent proof of this
fact for the quadrics).

5 Final Remarks and Questions

We conclude the paper with a list of questions related to this work that we believe are important for
further study of the subject.
1. Let \( H \) be a non-degenerate \( \mathbb{R}^k \)-valued Hermitian form on \( \mathbb{C}^n \) and \( Q_H \) the quadric in \( \mathbb{C}^{n+k} \) associated with \( H \). It is well-known that the algebra \( g_H \) of infinitesimal automorphisms of \( Q_H \) is a graded Lie algebra: \( g_H = \bigoplus_{k=-2}^{2} g^k_H \). Is it true that \( g_H \) is isomorphic to the maximal prolongation \( \tilde{g}_H \) of \( \bigoplus_{k=-2}^{2} g^k_H \) in the sense of Tanaka (see [1a])? So far, we have not been able to find any references on this matter except for the case \( k = 1 \) and the situation considered in [Ma], and we have produced proofs that give positive answers to this question for each of \( H^1, H^{-1}, H^0 \) (here \( k = 2, n = 2 \)). Note that \( g_H \) is always isomorphic to a subalgebra of \( \tilde{g}_H \) by a mapping that preserves grading.

2. As we noted in the Introduction, every known result on the equivalence problem for \( CR \)-manifolds falls in one of the two types: strongly uniform manifolds or weakly uniform manifolds with certain generic Levi forms. It is a reasonable question whether these two groups of results treat fact non-intersecting collections of manifolds. Namely, let \( M \) be a weakly uniform connected \( CR \)-manifold, \( p \in M \), and the Levi form at \( p \) is not in general position as in [ES6]: is then \( M \) strongly uniform?

One can ask a stronger question as follows. Let \( Q_{H^1}, Q_{H^2} \) be two irreducible (i.e. not equivalent to direct products) quadrics and \( \text{Aut}_{\text{lin},e}(Q_{H^1}) \) is isomorphic to \( \text{Aut}_{\text{lin},e}(Q_{H^2}) \) in such a way that the isomorphism extends to an isomorphism between \( \text{Aut}_{0,e}(Q_{H^1}) \) and \( \text{Aut}_{0,e}(Q_{H^2}) \). Suppose that at least one of \( Q_{H^1}, Q_{H^2} \) is not in general position as in [ES6]. Is it then true that \( H^1 \) is equivalent to \( H^2 \)?

3. In Proposition 3.5 we characterized semi-flat hyperbolic manifolds. In particular, in the real-analytic case they turned out to be locally \( CR \)-equivalent to matrix surfaces. It would be interesting to obtain analogues of these facts for elliptic manifolds. In particular, is it true that a semi-flat elliptic real-analytic manifold is locally equivalent to a matrix surface?

So far, we have been able to obtain only the following result that we mention below without a proof (cf. the proof of Proposition 3.5).

**PROPOSITION 5.1** Let \( M \) be a semi-flat elliptic manifold. Then there are two foliations \( \Sigma_1, \Sigma_2 \) of \( M \) by complex curves such that for every point \( p \in M \), the complex tangent space to \( M \) at \( p \) is spanned by the tangent spaces to the leaves of \( \Sigma_1, \Sigma_2 \) at \( p \).

4. The parallelism \( \omega \) that we constructed in Theorem 1.1 is not in general a Cartan connection. Is it possible to find a parallelism that is at the same time a Cartan connection for hyperbolic and elliptic \( CR \)-manifolds and for general strongly uniform manifolds (cf. [1a])?

It can be shown that the parallelism \( \omega \) turns into a Cartan connection on \( P^2 \) with respect to the action of \( G^2 \) if and only if the manifold is semi-flat (cf. Proposition 3.1). Is it true that the existence of a \( CR \)-invariant Cartan connection on some other fibre bundle over a hyperbolic or elliptic \( CR \)-manifold \( M \) implies the semi-flatness of \( M \)?

One can introduce matrix manifolds whenever \( CR\dim(M) = CR\codim(M) \) by using matrix algebras other than \( \mathbb{R} \); can one construct Cartan connections for such manifolds?

In [Mi, Gi] \( CR \)-invariant connections (not Cartan connections) were constructed for certain weakly uniform \( CR \)-structures. In these cases the groups \( \text{Aut}_{0,e}(Q_{\mathcal{L}(M)(p)}) \) contain only linear automorphisms given by certain diagonal matrices; thus to establish that the \( \mathcal{L}(M)(p) \)-valued forms constructed in [Mi, Gi] are indeed connections, one needs to find a transformation law only with respect to a very small group. Note that although \( \omega \) from Theorem 1.1 is not a Cartan connection,
it follows from the proof that its transformation law is in fact that of a Cartan connection if one acts not by the whole group $G^2$ on $P^2$, but only by its subgroup containing linear automorphisms given by diagonal matrices as in [M], [GM].

5. Are chains as defined in Section 4 the projections of some of submanifolds of $P^2$ that are tangent to chain distribution (3.4) at some point?

6. It can be shown that if the chain distribution is integrable, then the manifold is semi-flat. Therefore, it is natural to ask the following question: suppose that, for a hyperbolic or elliptic real-analytic manifold $M$, all chains (that a priori are chains only at a single point) turn out to be chains at each point; is it then true that $M$ is locally equivalent to a matrix surface?

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