NON-VIRTUALLY NILPOTENT GROUPS HAVE INFINITE CONJUGACY CLASS QUOTIENTS

JOSHUA FRISCH AND POOYA VAHIDI FERDOWSI

ABSTRACT. We offer in this note a self-contained proof of the fact that a finitely generated group is not virtually nilpotent if and only if it has a quotient with the infinite conjugacy class (ICC) property. This proof is a modern presentation of the original proof, by McLain (1956) and Duguid and McLain (1956).

1. INTRODUCTION

It has been long known that a finitely generated group is not virtually nilpotent if and only if it has a quotient with the infinite conjugacy class (ICC) property. This result, due to McLain [4] and Duguid and McLain [1], has recently found novel applications in the study of proximal actions and strong amenability [3] and the Poisson boundary [2]. This has motivated us to offer in this note a self-contained presentation of the original proof of this fact, which is divided between [4, Theorem 2] and [1, Theorem 2].

Let $G$ be a group. An element $g \in G$ is said to be a finite conjugacy (or FC) element if it has only finitely many conjugates in $G$. The FC-center of $G$ is the set of all FC-elements in $G$. The upper FC-series of $G$ is defined as follows

$$\{e\} = F_0 \leq F_1 \leq \cdots \leq F_\alpha \leq \cdots,$$

where $F_{\alpha+1}/F_\alpha$ is the set of all FC-elements of $G/F_\alpha$, and $F_\beta = \bigcup_{\alpha<\beta} F_\alpha$ for a limit ordinal $\beta$. This series will stabilize at some ordinal $\gamma$. $F_\gamma$ is called the hyper-FC center of $G$ and the least such $\gamma$ is called the FC-rank of $G$. If $G$ is equal to its hyper-FC center, $G$ is called hyper-FC central.

Theorem 1. For a finitely generated group $G$ the following are equivalent.

1. $G$ is virtually nilpotent.
2. $G$ is hyper-FC central.
3. $G$ has no non-trivial ICC quotients.

Date: March 15, 2018.
As a corollary, a finitely generated group is either virtually nilpotent or has an ICC quotient.

2. Proof

The following easy, but important, proposition shows that the obstruction to a group being ICC is the hyper-FC center of the group. Before we state the proposition, we define a universal ICC quotient of a group. This notion is useful to see the relation between the hyper-FC center of a group and its ICC quotients.

**Definition 2.1.** Let $G$ be a group. A universal ICC quotient of $G$, which we denote by by $\phi : G \to I$, is a quotient of $G$ onto an ICC group $I$ such that any quotient $\tau : G \to J$ of $G$ onto an ICC group $J$, lifts to a homomorphism $\rho : I \to J$ such that the following diagram commutes.

$$
\begin{array}{ccc}
G & \xrightarrow{\phi} & I \\
\downarrow{\tau} & & \downarrow{\rho} \\
J
\end{array}
$$

Now we can state the following proposition.

**Proposition 2.2.** Let $G$ be a group and let $H \triangleleft G$ be the hyper-FC center of $G$. The quotient map $\phi : G \to G/H$ is the unique, up to isomorphism, universal ICC quotient of $G$.

**Proof.** First, we show that $H$ is in the kernel of any ICC quotient $\tau : G \to J$. Let 

\[ \{e\} = F_0 \leq F_1 \leq \cdots \leq F_\alpha \leq \cdots \]

be the upper FC-series of $G$. If $H$ is not in the kernel of $\tau$, then there exists a minimum ordinal $\alpha$ such that $F_\alpha \not\subseteq \ker \tau$. Obviously $\alpha$ is not a limit ordinal. Let $h \in F_\alpha \setminus \ker \tau$. Since $hF_{\alpha-1}$ is FC in $G/F_{\alpha-1}$ and $F_{\alpha-1} \subseteq \ker \tau$ and $\tau$ is a surjective homomorphism, we get that $\tau(h)$ is FC in $J$. So, $\tau(h)$ is the identity in $J$, which is a contradiction. So, $H$ is in the kernel of any ICC quotient of $G$.

Now, we show that the quotient map $\phi : G \to G/H$ is an ICC quotient. Let $\gamma$ be the FC-rank of $G$. So $H = F_\gamma$. Note that since $F_\gamma = F_{\gamma+1}$, we know that any non-identity element of $G/H = G/F_\gamma$ has infinitely many conjugates, which shows that $G/H$ is ICC.

Thus $\phi : G \to G/H$ is a universal ICC quotient. Uniqueness follows from a standard fact about universal properties.

An immediate corollary of the above result is that hyper-FC central groups are exactly those with no non-trivial ICC quotients. **Theorem 1,**
which we prove next, gives a third equivalent condition when the group is finitely generated.

Proof of Theorem 1. The equivalence of (2) and (3) follows from the above corollary. We will show that (1) and (2) are equivalent. For that, we first show that the upper FC-series of a finitely generated hyper-FC central group stabilizes at some finite ordinal, i.e. the FC-rank is finite.

Claim 2.3. Let $G$ be a finitely generated hyper-FC central group. The upper FC-series of $G$ stabilizes at some $n \in \mathbb{N}$, i.e. its FC-rank is finite.

Proof. Let $S$ be a finite symmetric generating set for $G$. Let
\[
\{e\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_\alpha \subseteq \cdots \subseteq F_\gamma = G
\]
be the upper FC-series of $G$. We need to show that for some $n \in \mathbb{N}$ we have $S \subseteq F_n$. For that, we will define a sequence $X_0, X_1, \ldots$ of finite subsets of $G$ with the following properties:

1. $X_0 = S$.
2. If $\alpha_i$ is the least ordinal with $X_i \subseteq F_{\alpha_i}$, then either $\alpha_i = \alpha_{i-1} = 0$ or $\alpha_i = \alpha_{i-1} - 1$.

Given such a sequence, if none of the $\alpha_i$’s are 0, then $\alpha_0, \alpha_1, \ldots$ is an infinite strictly decreasing sequence of ordinals, which is a contradiction. So, some $\alpha_i$ is 0. Let $n$ be the least index with $\alpha_n = 0$. Then $\alpha_0 = n$. By the definition of $\alpha_0$, we get that $S = X_0 \subseteq F_n$. But since $S$ generates $G$, we get that $G \subseteq F_n$. So the upper FC-series stabilizes at $n$.

Now we define the sequence $X_0, X_1, \ldots$ and prove it has the properties we claimed. Let $X_0 = S$. Assume that $X_0, \ldots, X_i$ are defined. We want to define $X_{i+1}$. If $\alpha_i = 0$, then simply let $X_{i+1} = X_i$. And if $\alpha_i \neq 0$, we define $X_{i+1}$ below. First, we make a few observations.

- Note that since $\alpha_i$ is the least ordinal such that $F_{\alpha_i}$ contains the finite set $X_i$, we get that $\alpha_i$ is not a limit ordinal.
- Since $F_{\alpha_i}/F_{\alpha_{i-1}}$ is the FC-center of $G/F_{\alpha_{i-1}}$ and $X_i \subseteq F_{\alpha_i}$, we have that $xF_{\alpha_{i-1}}$ is FC in $G/F_{\alpha_{i-1}}$ for each $x \in X_i$.
- For each $x \in X_i$ and each conjugate of $xF_{\alpha_{i-1}}$, pick an element of $G$ in that conjugate, and let $Y_i$ be the union of $X_i$ and the collection of all the elements we chose. So, $X_i \subseteq Y_i \subseteq F_{\alpha_i}$ and $Y_i$ is finite.

Note that if $y \in Y_i$ and $g \in G$, then $A = (g^{-1}yg)F_{\alpha_{i-1}}$ is a conjugate of $xF_{\alpha_{i-1}}$ for some $x \in X_i$. Thus $A = zF_{\alpha_{i-1}}$ for some $z \in Y_i$, and so $z^{-1}(g^{-1}yg) \in F_{\alpha_{i-1}}$ for some $z \in Y_i$. Let
\[
X_{i+1} = \{z^{-1}(g^{-1}yg) \mid z^{-1}(g^{-1}yg) \in F_{\alpha_{i-1}}, \ g \in S, \ y, z \in Y_i\}.
\]
Note that $X_{i+1}$ is finite and $X_{i+1} \subseteq F_{\alpha_i}$. So, $\alpha_{i+1} \leq \alpha_i - 1$. To show that $\alpha_{i+1} = \alpha_i - 1$, we just need to show that $\alpha_i \leq \alpha_{i+1} + 1$, i.e. $X_i \subseteq F_{\alpha_{i+1}+1}$, which is the same as showing that $xF_{\alpha_{i+1}+1}$ is FC in $G/F_{\alpha_{i+1}}$ for all $x \in X_i$. Since $X_i \subseteq Y_i$, it suffices to show that $yF_{\alpha_{i+1}}$ is FC in $G/F_{\alpha_{i+1}}$ for all $y \in Y_i$.

Since $X_{i+1} \subseteq F_{\alpha_{i+1}}$, for any $y \in Y_i$ and $s \in S$, there exists a $z \in Y_i$ with $z^{-1} (s^{-1} ys) \in F_{\alpha_{i+1}}$. Since $S$ generates $G$ and $F_{\alpha_{i+1}}$ is normal in $G$, the same holds for any $g \in G$ replacing $s \in S$. Thus $Y_i F_{\alpha_{i+1}} \subseteq G/F_{\alpha_{i+1}}$ is closed under taking conjugates. Since $Y_i$ is finite, $yF_{\alpha_{i+1}}$ is thus FC in $G/F_{\alpha_{i+1}}$ for any $y \in Y_i$. This completes the proof. 

Now, we show that a finitely generated group has finite FC-rank if and only if it is virtually nilpotent. For $n \in \mathbb{N}$, denote the class of finitely generated hyper-FC central groups of FC-rank less than or equal to $n$ by $\mathcal{FC}_n$, and the class of finitely generated virtually nilpotent groups of rank less than or equal to $n$ by $\mathcal{VN}_n$.

First we prove a useful lemma.

**Lemma 2.4.** Let $G$ be a group, and $H$ be a finitely generated subgroup of the FC-center of $G$. The centralizer of $H$ in $G$, denoted by $C_G(H)$, has finite index in $G$.

**Proof.** Let $\{h_1, \ldots, h_n\}$ be a set of generators for $H$. Note that for each $h_i$, since it is FC in $G$, its centralizer $C_G(h_i)$ has finite index. Thus, the intersection of the centralizers $\cap_{i=1}^{n} C_G(h_i)$, which is the same as $C_G(H)$, has finite index in $G$. 

**Claim 2.5.** We have

$$\mathcal{VN}_0 \subseteq \mathcal{FC}_1 \subseteq \mathcal{VN}_1 \subseteq \mathcal{FC}_2 \subseteq \cdots \subseteq \mathcal{FC}_n \subseteq \mathcal{VN}_n \subseteq \mathcal{FC}_{n+1} \subseteq \cdots$$

**Proof.** First, we show that $\mathcal{VN}_{n-1} \subseteq \mathcal{FC}_n$ for any $n \in \mathbb{N}$. Let $G$ be a group in $\mathcal{VN}_{n-1}$ for $n \in \mathbb{N}$. Let $N \trianglelefteq G$ be a finite index normal subgroup with the upper central series

$$\{e\} = Z_0 \leq Z_1 \leq \cdots \leq Z_m = N,$$

where $m \leq n - 1$. Since $Z_1$ is the center of a normal subgroup of $G$, we get that $Z_1$ is normal in $G$. Similarly, we can show that each $Z_k$ is normal in $G$. Since $Z_k/Z_{k-1}$ is in the center of $N/Z_{k-1}$, we get that $N/Z_{k-1} \leq C_{G/Z_{k-1}}(zZ_{k-1})$ for any $z \in Z_k$, which means that $C_{G/Z_{k-1}}(zZ_{k-1})$ is of finite index in $G/Z_{k-1}$ for any $z \in Z_k$. So, $Z_k/Z_{k-1}$ is in the FC-center of $G/Z_{k-1}$. Obviously, since $G/N$ is finite, we have that $G/N$ is FC. So, we have that

$$\{e\} = Z_0 \leq Z_1 \leq \cdots \leq Z_m = N \leq G$$

is an FC-series for $G$ with length $m+1 \leq n$. So, $G$ belongs to $\mathcal{FC}_n$. 


Now, by induction on $n \in \mathbb{N}$ we show that $\mathcal{F}C_n \subseteq \mathcal{V}N_n$. Let $G$ be a group that belongs to $\mathcal{F}C_1$. By Lemma 2.4, the center of $G$ has finite index in $G$. So, $G$ is virtually abelian, which means that $G$ belongs to $\mathcal{V}N_1$. Thus $\mathcal{F}C_1 \subseteq \mathcal{V}N_1$.

Let $G$ be a group that belongs to $\mathcal{F}C_n$ for $n \geq 2$. Let
\[ \{e\} = F_0 \leq F_1 \leq \cdots \leq F_m = G \]
be the upper FC-series of $G$, where $m \leq n$. Since $G/F_1$ is in $\mathcal{F}C_{m-1}$, by the induction hypothesis we know that $G/F_1$ is virtually nilpotent of rank at most $m - 1$. So, there is a normal subgroup $N \trianglelefteq G$ with finite index such that $F_1 \leq N$ and $N/F_1$ is nilpotent of rank at most $m - 1$. We can make the following observations:

- Since $N$ has finite index in a finitely generated group, $N$ is finitely generated. Let $S$ be a finite symmetric set of generators for $N$.
- Let $N = \Gamma_0 \triangleright \Gamma_1 \triangleright \cdots \triangleright \Gamma_{m-1}$ be the first $m$ subgroups in the lower central series of $N$. Since $N/F_1$ is nilpotent of rank at most $m - 1$, we know that $\Gamma_{m-1} \leq F_1$. So, $\Gamma_{m-1}$ is FC.
- It is easy to see that $\Gamma_{m-1}$ is the least normal subgroup of $N$ that contains all the $(m-1)$-fold commutators $[s_1, s_2, \ldots, s_{m-1}]$, where $s_i$'s are elements of $S$. Note that 1) since $\Gamma_{m-1}$ is FC, we know that each of $[s_1, s_2, \ldots, s_{m-1}]$ has finitely many conjugates in $N$, and 2) since $S$ is finite, we have finitely many elements of the form $[s_1, s_2, \ldots, s_{m-1}]$. So, $\Gamma_{m-1}$ is finitely generated.

From the last two observations we know that $\Gamma_{m-1}$ is a finitely generated FC subgroup of $N$. By Lemma 2.4, we know that $C_N(\Gamma_{m-1})$ has finite index in $N$. Obviously, $C_N(\Gamma_{m-1})$ has a normal subgroup $M$ with finite index in $N$. It is clear that $Z = \Gamma_{m-1} \cap M$ is in the center of $M$. Since $N$ has finite index in $G$, we get that $M$ also has finite index in $G$.

By the second isomorphism theorem for groups, we have that
\[ M/Z = M/(\Gamma_{m-1} \cap M) \cong (M\Gamma_{m-1})/\Gamma_{m-1} \leq N/\Gamma_{m-1}. \]

But we know that $N/\Gamma_{m-1}$ is nilpotent with rank at most $m - 1$. So, $M/Z$ is nilpotent with rank at most $m - 1$. Also, $Z$ is in the center of $M$. Hence, $M$ is nilpotent with rank at most $m \leq n$. So, $G$ is virtually nilpotent with rank at most $n$. □

Now, we can show that (1) and (2) are equivalent. Let $G$ be a finitely generated group.

If $G$ is a virtually nilpotent group of rank $n$, then by Claim 2.5 we know that it is FC with FC-rank at most $n + 1$. 

If, on the other hand, $G$ is hyper-FC central, then by Claim 2.3 we know that its FC-rank is finite, say $n \in \mathbb{N}$. So, by Claim 2.5 we know that it is virtually nilpotent of rank at most $n$. □

REFERENCES

[1] AM Duguid and DH McLain, *FC-nilpotent and FC-soluble groups*, Mathematical proceedings of the Cambridge philosophical society, 1956, pp. 391–398.

[2] Joshua Frisch, Yair Hartman, Omer Tamuz, and Pooya Vahidi Ferdowsi, *Choquet-deny groups and the infinite conjugacy class property*, arXiv preprint arXiv:1802.00751 (2018).

[3] Joshua Frisch, Omer Tamuz, and Pooya Vahidi Ferdowsi, *Strong amenability and the infinite conjugacy class property*, arXiv preprint arXiv:1801.04024 (2018).

[4] DH McLain, *Remarks on the upper central series of a group*, Glasgow Mathematical Journal 3 (1956), no. 1, 38–44.

California Institute of Technology