RESCALED PURE GREEDY ALGORITHM FOR CONVEX OPTIMIZATION

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Abstract. We suggest a new greedy strategy for convex optimization in Banach spaces and prove its convergent rates under a suitable behavior of the modulus of uniform smoothness of the objective function.

Key Words: Greedy Algorithms, Convex Optimization, Rates of Convergence.

1. Introduction

The main goal in convex optimization is the development and analysis of algorithms for solving the problem

\( \inf_{x \in \Omega} E(x), \)

where \( E \) is a given convex function and \( \Omega \) is a bounded convex subset of a Banach space \( X \). \( E \) is called the objective function and satisfies the convexity condition

\[ E(\gamma x + \delta y) \leq \gamma E(x) + \delta E(y), \quad x, y \in \Omega, \quad \gamma, \delta \geq 0, \quad \gamma + \delta = 1. \]

While the classical convex optimization deals with objective functions \( E \) defined on subsets \( \Omega \) in \( \mathbb{R}^n \) for moderate values of \( n \), see [2], some of the new applications require that the dimension \( n \) is quite large or even \( \infty \). The design of algorithms for such cases is quite challenging since typical convergent results involve \( n \), and therefore deteriorate severely with the growth of \( n \). This is the so-called curse of dimensionality. Recently, there has been an increased interest, see [12, 8, 9, 4], in developing greedy based strategies for solving (1.1) with provable convergence rate depending only on the properties of \( E \) and not on the dimension of the underlying space. These algorithms provide approximations \( \{E(x_m)\} \), \( m = 1, 2, \ldots \) to the solution of (1.1), with \( x_m \) being a linear combination of \( m \) elements from a given dictionary \( D \subset X \). A dictionary is any set \( D \) of norm one elements from \( X \) whose span is dense in \( X \). An example of a dictionary is any Shauder basis

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for $X$, or a union of several Shauder bases. The current algorithms pick the initial approximation $E(x_0)$, $x_0 = 0$, the set $\Omega$ as

$$
\Omega := \{x \in X : E(x) \leq E(0)\},
$$

since the global minimum of $E$ is attained on that set, and generate a sequence of successive approximations $E_m := E(x_m)$, $m = 1, 2, \ldots$ recursively, using the dictionary $D$. Some methods, such as the Weak Chebychev Greedy Algorithm, see [8], provide at Step $m$ an approximant $x_m$ to the point $\bar{x}$ at which $E$ attains its global minimum, determined as

$$
x_m := \arg\min_{x \in \text{span}\{\varphi_{j_1}, \ldots, \varphi_{j_m}\}} E(x),
$$

where $\varphi_{j_1}, \ldots, \varphi_{j_m}$ are suitably chosen elements from $D$. Others choose $x_m$ as

$$
x_m := \arg\min_{\omega, \lambda \in \mathbb{R}} E(\omega x_{m-1} + \lambda \varphi_m),
$$
or

$$
x_m := \arg\min_{\lambda \in [0, 1]} E((1 - \lambda)x_{m-1} + \lambda \varphi_m)
$$
for suitably chosen $\varphi_m \in D$, where $x_{m-1}$ is the previously generated point. Convergence rates for these algorithms are proved to be of order $O(m^{1-q})$, where $q$ is a parameter related to the smoothness of the objective function $E$. Note that the last two approaches are more computationally friendly, since they require solving two or one dimensional optimization problems at each step. On the other hand, some of these algorithms work only if the minimum of $E$ is attained in the convex hull of $D$, since the approximant $x_m$ is derived as a convex combination of $x_{m-1}$ and $\varphi_m$.

In this paper, we introduce a new greedy algorithm based on one dimensional optimization at each step, which does not require the solution of (1.1) to belong to the convex hull of $D$ and has a rate of convergence $O(m^{1-q})$. This algorithm is an appropriate modification of the recently introduced Rescaled Pure Greedy Algorithm (RPGA) for approximating functions in Hilbert and Banach spaces, see [7]. We call it RPGA(co). The paper is organized as follows. In Section 2, we list several definitions and known results about convex functions. In section 3, we present the RPGA(co) and prove its convergence rate. The rest of the paper describes the weak version of this algorithm.

2. Preliminaries

Let us first recall that a function $E$ is Frechet differentiable at $x \in \Omega$ if there exists a bounded linear functional, denoted by $E'(x) \in X^*$, such that

$$
\lim_{h \to 0} \frac{|E(x + h) - E(x) - \langle E'(x), h \rangle|}{\|h\|} = 0.
$$

Here we use the notation $\langle F, x \rangle := F(x)$ to denote the action of the functional $F \in X^*$ on the element $x \in X$.

The following lemmas are well known and we simply state them.
Lemma 2.1. Let $E$ be a Frechet differentiable function at each point in $\Omega$ and convex on $X$. Then, for all $x \in \Omega$ and $x' \in X$,
$$\langle E'(x), x - x' \rangle \geq E(x) - E(x').$$

Lemma 2.2. Let $E$ be a Frechet differentiable convex function, defined on a convex domain $\Omega$. Then $E$ has a global minimum at $\bar{x} \in \Omega$ if and only if $E'(\bar{x}) = 0$.

Lemma 2.3. Let $F$ be a Frechet differentiable function and $x^*$ be such that $x^* = \text{argmin}\{F(x) : x = t\varphi, t \in \mathbb{R}\}$. Then, $\langle F'(x^*), x^* \rangle = 0$.

In this paper, we consider objective functions $E$ that satisfy the following two assumptions.

- **Condition 0:** $E$ has Frechet derivative $E'(x) \in X^*$ at each point in $\Omega := \{x \in X : E(x) \leq E(0)\}$, $\Omega$ is bounded, and $\|E'(x)\| \leq M_0$, $x \in \Omega$.

- **Uniform Smoothness (US):** There are constants $0 \leq \alpha$, $M > 0$, and $1 < q \leq 2$, such that for all $x, x'$ with $\|x - x'\| \leq M$, $x \in \Omega$,
$$E(x') - E(x) - \langle E'(x), x' - x \rangle \leq \alpha\|x' - x\|^q.$$

The US condition on $E$ is closely related to a condition on the modulus of smoothness of $E$. Recall that for a convex function $E : X \to \mathbb{R}$ and a set $S \subset X$, the modulus of smoothness of $E$ on $S$ is defined by
$$\rho(E, u) := \frac{1}{2} \sup_{x \in S, \|y\|=1} \{E(x + uy) + E(x - uy) - 2E(x)\}, \quad u > 0,$$ and the modulus of uniform smoothness of $E$ on $S$ is defined by $\rho_1 := \rho_1(E, u)$
$$\rho_1 := \sup_{x \in S, \|y\|=1, \lambda \in (0, 1)} \left\{ \frac{(1 - \lambda)E(x - \lambda uy) + \lambda E(x + (1 - \lambda)uy) - E(x)}{\lambda(1 - \lambda)} \right\}.$$ These two moduli of smoothness are equivalent (see [11], page 205), as the following lemma states.

**Lemma 2.4.** Let $E$ be a convex function defined on $X$, $S \subset X$, and $\rho(E, \cdot)$ and $\rho_1(E, \cdot)$ be its modulus of smoothness and modulus of uniform smoothness, respectively. Then we have
$$4\rho(E, \frac{u}{2}) \leq \rho_1(E, u) \leq 2\rho(E, u).$$

The next lemma shows the relation between the modulus of smoothness and the US condition. The proof of the version cited here can be found in [6]. Because of this lemma, the US condition and the condition from [3, 9, 4] on the modulus of smoothness of $E$ are equivalent.
Lemma 2.5. Let $E$ be a convex function defined on a Banach space $X$ and $E$ be Frechet differentiable on a set $S \subseteq X$. The following statements are equivalent for any $q \in (1,2]$ and $M > 0$.

- There exists $\alpha > 0$ such that for any $x \in S$, $x' \in X$, $\|x - x'\| \leq M$,
  \begin{equation}
  E(x') - E(x) - \langle E'(x), x' - x \rangle \leq \alpha\|x' - x\|^q.
  \end{equation}

- There exists $\alpha_1 > 0$, such that
  \begin{equation}
  \rho(E, u, S) \leq \alpha_1 u^q, \quad 0 < u \leq M.
  \end{equation}

Next, we introduce some notation. Let $\bar{x}$ be the solution to (1.1). We denote by $\|\bar{x}\|_1$ its semi-norm with respect to the dictionary $D$, namely
\[
\|\bar{x}\|_1 := \inf \left\{ \sum_{\varphi \in D} |c_\varphi(\bar{x})| : \bar{x} = \sum_{\varphi \in D} c_\varphi(\bar{x})\varphi \right\},
\]
where the infimum is taken over all possible representations of $\bar{x}$ as a linear combination of dictionary elements. Clearly, the point $\bar{x}$ at which $E$ attains its global minimum belongs to the set
\[
\Omega := \{x : E(x) \leq E(0)\},
\]
and in what follows we will consider the minimization problem over this set. Note that this is a convex set as a level set of a convex function.

Further in the paper we will use the following lemma, proved in [6]. Other versions of this lemma have been proved in [10].

Lemma 2.6. Let $\ell > 0$, $r > 0$, $B > 0$, and $\{a_m\}_{m=1}^\infty$ and $\{r_m\}_{m=2}^\infty$ be sequences of non-negative numbers satisfying the inequalities
\[
a_1 \leq B, \quad a_{m+1} \leq a_m(1 - r_m + 1/r a_\ell), \quad m = 1,2,\ldots.
\]
Then, we have
\begin{equation}
(a_m) \leq \max\{1, r^{-1/\ell}\} r^{1/\ell}(r B^{-\ell} + \sum_{k=2}^m r_k)^{-1/\ell}, \quad m = 2,3,\ldots.
\end{equation}

3. The Rescaled Pure Greedy Algorithm for Convex Optimization

In this section, we describe our new algorithm with parameter $\mu$ and dictionary $D$.

**RPGA(co)(\mu, D):**

- **Step 0:** Define $x_0 = 0$. If $E'(x_0) = 0$, stop the algorithm and define $x_k := x_0 = \bar{x}$, $k \geq 1$.
- **Step m:** Assuming $x_{m-1}$ has been defined and $E'(x_{m-1}) \neq 0$. Choose a direction $\varphi_{jm} \in D$ such that
  \begin{equation}
  |\langle E'(x_{m-1}), \varphi_{jm} \rangle| = \sup_{\varphi \in D} |\langle E'(x_{m-1}), \varphi \rangle|.
  \end{equation}
With \( \hat{x}_m := x_{m-1} - \lambda_m \varphi_{j_m} \), where

\[
\lambda_m := \text{sgn}\{\langle E'(x_{m-1}), \varphi_{j_m} \rangle\}(\alpha \mu)^{-\frac{1}{q+1}} |\langle E'(x_{m-1}), \varphi_{j_m} \rangle|^{\frac{1}{q+1}},
\]

t_m := \arg\min_{t \in \mathbb{R}} E(t \hat{x}_m),

define the next point to be

\[
x_m = t_m \hat{x}_m.
\]

- If \( E'(x_m) = 0 \), stop the algorithm and define \( x_k = x_m = \bar{x} \), for \( k > m \).
- If \( E'(x_m) \neq 0 \), proceed to Step \( m + 1 \).

Let us observe that, because of Lemma \ref{lem:E''(x)m}, if \( E'(x_m) = 0 \) at Step \( m \), the output \( x_m \) of the algorithm is the minimizer \( \bar{x} \). Note that the algorithm requires a minimization of the objective function along the one dimensional space \( \text{span}\{\hat{x}_m\} \). This univariate optimization problem is called line search and is well studied in optimization theory, see \cite{[5]}.

If at Step \( m \) we were to use \( \hat{x}_m \) as next approximant and not \( x_m \), which is the minimizer of \( E \) along the line generated by \( \hat{x}_m \), then the algorithm would be very similar to the EGA(\( C \)) from \cite{[8]}.

The author proves a convergence rate of \( O(m^{-r}) \), for any \( r \in (0, \frac{q}{q+1}) \) for this algorithm under suitable conditions on the parameters. Note that our algorithm, which simply adds a one dimensional optimization at each step, makes it possible to achieve an optimal convergence rate of \( O(m^{1-q}) \). Observe also that, in contrast to the other greedy algorithms from \cite{[8]} that rely on one dimensional minimization at each step, this algorithm provides convergent results for all \( \bar{x} \), and not only for \( \bar{x} \) in the convex hull of the dictionary \( D \).

Notice that all outputs \( \{x_k\}_{k=1}^{\infty} \) generated by the RPGA(co)(\( \mu, D \)) are in \( \Omega \), since \( E(x_k) \leq E(0) \). The following theorem is our main convergence result.

**Theorem 3.1.** Let the convex function \( E \) satisfy Condition 0 and the US condition. Then, at Step \( k \), the RPGA(co)(\( \mu, D \)) with parameter \( \mu > \max \{1, \alpha^{-1} M_0 M^{1-q}\} \), applied to \( E \) and a dictionary \( D = \{\varphi\} \) outputs the point \( x_k \), where

\[
e_k := E(x_k) - E(\bar{x}) \leq C_1 k^{1-q}, \quad k \geq 2,
\]

with \( C_1 = C_1(q, \alpha, E, \mu) \).

**Proof.** Clearly, we have \( e_1 = E(x_1) - E(\bar{x}) \leq E(0) - E(\bar{x}) \). Next, we consider Step \( k, k = 2, 3, \ldots \) of the algorithm. Notice that \( x_k \in \Omega \), since \( E(x_k) \leq E(0) \). The definition of \( \lambda_k \) and the choice of parameter \( \mu \) assures that

\[
\|(x_{k-1} - \lambda_k \varphi_{j_k}) - x_{k-1}\| = \left( \frac{|\langle E'(x_{k-1}), \varphi_{j_k} \rangle|}{\alpha \mu} \right)^{\frac{1}{q+1}} \leq M,
\]
and therefore, applying the US condition to \((x_{k-1} - \lambda_k \varphi_{j_k})\) and \(x_{k-1}\) gives
\[
E(\hat{x}_k) = E(x_{k-1} - \lambda_k \varphi_{j_k}) \leq E(x_{k-1}) - \lambda_k \langle E'(x_{k-1}), \varphi_{j_k} \rangle + \alpha |\lambda_k|^q
\]
where we use the fact that \(\|\varphi_{j_k}\| \leq 1\). Since \(E(x_k) \leq E(\hat{x}_k)\), we derive that
\[
E(x_k) \leq E(x_{k-1}) - \frac{\mu - 1}{\mu} (\alpha \mu)^{-\frac{1}{q-1}} \|E'(x_{k-1}), \varphi_{j_k}\|^{q/(q-1)}.
\]
Next, we provide a lower bound for \(|\langle E'(x_{k-1}), \varphi_{j_k} \rangle|\). Let us fix \(\varepsilon > 0\) and choose a representation for \(\hat{x} = \sum_{\varphi \in D} c_\varphi^\varepsilon \varphi\), such that
\[
\sum_{\varphi \in D} |c_\varphi^\varepsilon| < \|\bar{x}\|_1 + \varepsilon.
\]
Since \(\langle E'(x_{k-1}), x_{k-1} \rangle = 0\), because of the choice of \(x_{k-1}\) and Lemma 2.4, we have that
\[
\langle E'(x_{k-1}), x_{k-1} - \bar{x} \rangle = -\langle E'(x_{k-1}), \bar{x} \rangle = -\sum_{\varphi} c_\varphi^\varepsilon \langle E'(x_{k-1}), \varphi \rangle
\]
\[
\leq \|\langle E'(x_{k-1}), \varphi_{j_k} \rangle \| c_\varphi^\varepsilon \|\varphi_{j_k}\| \|\bar{x}\|_1 + \varepsilon,
\]
where we have used the choice of \(\varphi_{j_k}\). We let \(\varepsilon \to 0\) and obtain the inequality
\[
\langle E'(x_{k-1}), x_{k-1} - \bar{x} \rangle \leq \|\langle E'(x_{k-1}), \varphi_{j_k} \rangle \| \|\bar{x}\|_1.
\]
On the other hand, Lemma 2.1 and (3.6) give that
\[
\|\bar{x}\|_1^{-1} e_{k-1} \leq \|\langle E'(x_{k-1}), \varphi_{j_k} \rangle \|,
\]
which is the desired estimate from below for \(|\langle E'(x_{k-1}), \varphi_{j_k} \rangle|\). We substitute the latter inequality in (3.5) and derive
\[
E(x_k) \leq E(x_{k-1}) - \frac{\mu - 1}{\mu} (\alpha \mu)^{-\frac{1}{q-1}} \|\bar{x}\|_1^{-\frac{q}{q-1}} \varepsilon \frac{q}{k-1}.
\]
Subtracting \(E(\bar{x})\) from both sides gives
\[
e_k \leq e_{k-1} \left( 1 - \frac{\mu - 1}{\mu} (\alpha \mu)^{-\frac{1}{q-1}} \|\bar{x}\|_1^{-\frac{q}{q-1}} \varepsilon \frac{q}{k-1} \right)
\]
Now we apply Lemma 2.6 for the sequence of errors \(\{e_k\}_{k=1}^\infty\) and
\[
r_k = \frac{\mu - 1}{\mu}, \quad \ell = \frac{1}{q-1} > 0, \quad B = E(0) - E(\bar{x}), \quad r = (\alpha \mu \|\bar{x}\|_1^q)^{1-\frac{1}{q-1}},
\]
and derive that
\[
e_k \leq \alpha \mu \|\bar{x}\|_1^q \left( \left( \frac{\alpha \mu \|\bar{x}\|_1^q}{E(0) - E(\bar{x})} \right)^{1-\frac{1}{q-1}} + \frac{\mu - 1}{\mu} (m - 1) \right)^{1-q},
\]
and the proof is completed. \(\square\)
Notice that we can optimize with respect to the parameter $\mu$ and select a specific value for $\mu > \max\{1, \alpha^{-1}M_0M^{1-q}\}$ that will guarantee the best convergence rate in terms of best constants.

4. The Weak Rescaled Pure Greedy Algorithm for Convex Optimization

In this section, we describe the weak version of our algorithm with weakness sequence $\{\ell_k\}$, $\ell_k \in (0, 1]$ $k = 1, 2, \ldots$, and parameter sequence $\{\mu_k\}$, $\mu_k > \max\{1, \alpha^{-1}M_0M^{1-q}\}$, $k = 1, 2, \ldots$. In the case when $\ell_k = 1$ and $\mu_k = \mu$, $k = 1, 2, \ldots$, the WRPGA(co)$\{\ell_k\}, \{\mu_k\}, D$ is the RPGA(co)$\mu, D$.

The weakness sequence allows us to have some freedom in the selection of the next direction $\varphi_{jk}$, while the parameter sequence $\{\mu_k\}$ gives more choices in how much to advance along the selected direction $\varphi_{jk}$.

WRPGA(co)$\{\ell_k\}, \{\mu_k\}, D$:

- **Step 0:** Define $x_0 = 0$. If $E'(x_0) = 0$, stop the algorithm and define $x_k := x_0 = \tilde{x}$, $k \geq 1$.
- **Step m:** Assuming $x_{m-1}$ has been defined and $E'(x_{m-1}) \neq 0$. Choose a direction $\varphi_{jm} \in D$ such that

  $$|\langle E'(x_{m-1}), \varphi_{jm} \rangle| \geq \ell_m \sup_{\varphi \in D} |\langle E'(x_{m-1}), \varphi \rangle|.$$  

  With $\hat{x}_m := x_{m-1} - \lambda_m \varphi_{jm}$, where

  $$\lambda_m := \text{sgn}\{\langle E'(x_{m-1}), \varphi_{jm} \rangle\} \left(\alpha \mu_m \right)^{-\frac{1}{\ell_m}} |\langle E'(x_{m-1}), \varphi_{jm} \rangle|^{-\frac{1}{\ell_m}},$$

  $$t_m := \arg\min_{t \in \mathbb{R}} E(t\hat{x}_m),$$

  define the next point to be

  $$x_m = t_m \hat{x}_m.$$  

- If $E'(x_m) = 0$, stop the algorithm and define $x_k = x_m = \tilde{x}$, for $k > m$.
- If $E'(x_m) \neq 0$, proceed to Step $m + 1$.

The next theorem is the main result about the convergence rate of the WRPGA(co)$\{\ell_k\}, \{\mu_k\}, D$.

**Theorem 4.1.** Let the convex function $E$ satisfy **Condition 0** and the US condition. Then, at Step $k$, the WRPGA(co)$\{\ell_k\}, \{\mu_k\}, D$, applied to $E$ and a dictionary $D = \{\varphi\}$ outputs the point $x_k$, where

$$e_k := E(x_k) - e(\tilde{x}) \leq \alpha \|x\|_1^q \left(C_1 + \sum_{j=2}^{k} (\mu_j - 1) \left(\frac{\ell_j}{\mu_j} \right)^{\frac{q}{\ell_j - 1}} \right)^{1-q},$$

with $C_1 = C_1(q, \alpha, E)$. 

Theorem 4.1. Let the convex function $E$ satisfy **Condition 0** and the US condition. Then, at Step $k$, the WRPGA(co)$\{\ell_k\}, \{\mu_k\}, D$, applied to $E$ and a dictionary $D = \{\varphi\}$ outputs the point $x_k$, where

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with $C_1 = C_1(q, \alpha, E)$.
Proof. Similarly to the proof of Theorem 4.1, we have that

\[ e_1 \leq E(0) - E(\bar{x}) \]

and for \( k \geq 2, \)

\[ E(x_k) \leq E(x_{k-1}) - \frac{\mu_k - 1}{\mu_k} (\alpha \mu_k)^{-\frac{1}{q-1}} \|E'(x_{k-1}), \varphi_{j_k}\|^{\frac{q}{q-1}}. \]  

The same way one can easily derive that

\[ \|\bar{x}\|^{-\frac{1}{q-1}} \ell_k e_{k-1} \leq |\langle E'(x_{k-1}), \varphi_{j_k}\rangle|, \]

and thus the estimate

\[ e_k \leq e_{k-1} \left( 1 - \frac{\mu_k - 1}{\mu_k} (\alpha \mu_k)^{-\frac{1}{q-1}} \ell_k^{\frac{q}{q-1}} \|\bar{x}\|^{-\frac{q}{q-1}} e_{k-1}^{-\frac{1}{q-1}} \right). \]

Now we apply Lemma 2.6 for the sequence of errors \( \{e_k\}_{k=1}^\infty \) and

\[ r_k = (\mu_k - 1) \left( \frac{\ell_k}{\mu_k} \right)^{\frac{q}{q-1}}, \quad \ell = \frac{1}{q-1} > 0, \quad B = E(0) - E(\bar{x}), \quad r = (\alpha \|\bar{x}\|)^{\frac{1}{q-1}}, \]

and derive that

\[ e_k \leq \alpha \|\bar{x}\|^q \left( \left( \frac{\alpha \|\bar{x}\|^q}{E(0) - E(\bar{x})} \right)^{\frac{1}{q-1}} + \sum_{j=2}^k (\mu_j - 1) \left( \frac{\ell_j}{\mu_j} \right)^{\frac{q}{q-1}} \right)^{1-q}, \]

and the proof is completed. \( \square \)

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