Trapping wave fields in an expulsive potential by means of linear coupling

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(1) Introduction

The **nonlinear Schrödinger (NLS)** equation including a trapping (**harmonic-oscillator**) potential is a commonly known model with many physical realizations, such as waveguides for photonic and matter waves (**BEC**):

\[
i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \frac{1}{2} x^2 u + \sigma |u|^2 u = 0,
\]

with \( \sigma = +1 \) (**self-focusing**), \(-1\) (**defocusing**), or \(0\) (**the linear Schrödinger equation**). The equation is written in the notation adjusted to **optics in the spatial domain**, \(z\) being the propagation distance. In terms of BEC, it is the **Gross-Pitaevskii** equation, with \(z\) replaced by time, \(t\).
A straightforward generalization is a symmetric system of two \textit{linearly-coupled} \textbf{NLS} equations for wave fields $u$ and $v$, which models a set of two \textit{parallel waveguides ("cores") coupled by tunneling} of photons (in optics) or atoms (in \textbf{BEC}):

$$\begin{align*}
i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \lambda v - \frac{1}{2} x^2 u + \sigma |u|^2 u &= 0, \\
i \frac{\partial v}{\partial z} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \lambda u - \frac{1}{2} x^2 v + \sigma |v|^2 v &= 0,
\end{align*}$$

where real $\lambda > 0$ is the coupling constant.
In terms of **BEC** (but not in optics), it is also relevant to consider a two-dimensional (2D) version of the system, with the 2D isotropic trapping potential. The 2D system for a set of parallel BEC layers coupled by tunneling of atoms is written in polar coordinates \((r, \theta)\):

\[
\begin{align*}
\frac{\partial u}{\partial z} + \frac{1}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) u + \lambda v - \frac{1}{2} r^2 u + \sigma |u|^2 u &= 0, \\
\frac{\partial v}{\partial z} + \frac{1}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) v + \lambda u - \frac{1}{2} r^2 v + \sigma |v|^2 v &= 0.
\end{align*}
\]
The 2D system admits vortex solutions (which carry the angular momentum), in the form of

\[ \{u(r, \theta, z)\} = \exp \left( -i \mu z + i S \theta \right) \{U(r), V(r)\}, \]

where real \( \mu \) is the propagation constant (wavenumber), integer \( S \) is the vorticity (winding number), and real functions \( U(r) \) and \( V(r) \) satisfy the radial equations:

\[
\begin{align*}
\mu U + \frac{1}{2} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{S^2}{r^2} \right) U + \lambda V - \frac{1}{2} r^2 U + \sigma U^3 &= 0, \\
\mu V + \frac{1}{2} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{S^2}{r^2} \right) V + \lambda U - \frac{1}{2} r^2 V + \sigma V^3 &= 0.
\end{align*}
\]
In the case of the **self-attractive nonlinearity** ($\sigma = +1$), the **interplay** between the **intra-core self-attraction** and **inter-core linear coupling** gives rise to **spontaneous symmetry breaking**, in the **1D** and **2D** systems alike:

**Spontaneous symmetry breaking of fundamental states, vortices, and dipoles in two- and one-dimensional linearly coupled traps with cubic self-attraction**

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An example of a state with **broken symmetry**: an **asymmetric stable vortex mode** with $S = 1$ and **different amplitudes** of the two components, produced by the **2D system** with the **inter-core coupling constant** $\lambda = 0.4$. The total norm of the state is

$$N = N_u + N_v \equiv 2\pi \int_0^\infty \left[ U^2(r) + V^2(r) \right] rdr \approx 8.8.$$

The asymmetric vortices with $S = 1$ exist (i.e., the symmetry breaking takes place) at

$$N > N_{cr} \approx 0.57 + 19.06\lambda \approx 8.2 \text{ for } \lambda = 0.4.$$
A dynamical effect: *Josephson oscillations* between the linearly coupled cores, if the input is loaded into one core of the 1D system (*Symmetry* 13, 372 (2021)):

**Nonlinear Dynamics of Wave Packets in Tunnel-Coupled Harmonic-Oscillator Traps**

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Below a critical value of the norm, the system performs regular (non-chaotic) Josephson oscillations, maintaining dynamical symmetry between the two components (cores):
Above the critical norm, the system performs chaotic oscillations, which spontaneously break the dynamical symmetry between the component:
The subject of the present work: an asymmetric linearly coupled system with the trapping (harmonic-oscillator, HO) potential in one core, and an expulsive (inverted HO) potential in the other.

The results have been recently reported in:

PHYSICAL REVIEW E 105, 034213 (2022)

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Expulsive potentials (in the 1D NLS equation) were considered in optics, as they create anti-waveguiding structures, that may find various applications in all-optical signal-processing systems:

B. V. Gisin and A. A. Hardy, Stationary solutions of plane nonlinear-optical antiwaveguides, Opt. Quant. Electron 27, 565 (1995).
B. V. Gisin, A. Kaplan, and B. A. Malomed, Spontaneous symmetry breaking and switching in planar nonlinear optical antiwaveguides, Phys. Rev. E 62, 2804 (2000).
D. Bortman-Arbiv, A. D. Wilson-Gordon, and H. Friedmann, Strong parametric amplification by spatial soliton-induced cloning of transverse beam profiles in an all-optical antiwaveguide, Phys. Rev. A 63, 031801(R) (2001).
O. N. Verma and T. N. Dey, Steering, splitting, and cloning of an optical beam in a coherently driven Raman gain system, Phys. Rev. A 91, 013820 (2015).
A. Kaplan, B. V. Gisin, and B. A. Malomed, Stable propagation and all-optical switching in planar waveguide-antiwaveguide periodic structures, J. Opt. Soc. Am. B 19, 522 (2002).
Expulsive potentials were also studied in various contexts in similar BEC models:

L. D. Carr and Y. Castin, Dynamics of a matter-wave bright soliton in an expulsive potential, Phys. Rev. A 66, 063602 (2002).

L. Salasnich, Dynamics of a Bose-Einstein-condensate bright soliton in an expulsive potential, Phys. Rev. A 70, 053617 (2004).

Z. X. Liang, Z. D. Zhang, and W. M. Liu, Dynamics of a Bright Soliton in Bose-Einstein Condensates with Time-Dependent Atomic Scattering Length in an Expulsive Parabolic Potential, Phys. Rev. Lett. 94, 050402 (2005).
The system of the coupled components with *trapping* and *anti-trapping* potentials:

\[
\begin{align*}
&i \frac{\partial u}{\partial z} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \lambda v - \frac{1}{2} x^2 u + \sigma |u|^2 u = -\omega u, \\
i \frac{\partial v}{\partial z} + \frac{1}{2} \frac{\partial^2 v}{\partial x^2} + \lambda u + \frac{\kappa}{2} x^2 v + \sigma |v|^2 v = 0,
\end{align*}
\]

where \( \omega \) is a possible *mismatch* between the coupled components, and \( \kappa > 0 \) is the strength of the *expulsive* potential acting in the \( v \)-component.

**The main question**: can the linear coupling maintain *stable two-component bound (localized) states*, in spite of the obvious *delocalization effect* produced by the expulsive potential, in such the *1D* and *2D* systems?
In optics, the physical realization of such a 1D system is obvious: a planar **dual-core coupler**, with the **waveguiding** and **antiwaveguiding** structures induced by the corresponding patterns of the transversely modulated refractive index in the coupled cores.

The 2D variant of the system cannot be realized in optics, but *it is possible* (as well as the 1D case) in BEC: **trapping** and **antitrapping** optical potentials may be induced, respectively, by red- and blue-detuned laser beams focused on two **tunnel-coupled** parallel layers of BEC. The separation between the layers is expected to be a few microns, which is also **sufficient** to separate (resolve) the two optical trapping patterns.
Stationary solutions for the linearly-coupled 1D system, with propagation constant $-\mu$ (in terms of BEC, with $z$ replaced by $t$, $\mu$ is the chemical potential), are looked for as:

$$\{u(x, z), v(x, z)\} = \{U(x), V(x)\} \exp(-i\mu z),$$

with real functions $U(x)$ and $V(x)$ satisfying equations

$$(\mu + \omega)U + \frac{1}{2} \frac{d^2 U}{dx^2} + \lambda V - \frac{1}{2} x^2 U + \sigma U^3 = 0,$$

$$\mu V + \frac{1}{2} \frac{d^2 V}{dx^2} + \lambda U + \frac{\kappa}{2} x^2 V + \sigma V^3 = 0,$$

In terms of this system of equations, a mathematical problem is: does the system give rise to solutions which are localized at $|x| \to \infty$ in both components, while the potential term $\sim \kappa$ tends to expel the $v$ component?
The plan of the subsequent presentation:

(3) Exact non-generic solutions for the bound states in the 1D linear system.

(4) The variational (Rayleigh-Ritz) approximation and numerical results for generic bound states in the linear system.

(5) Coexistence of the discrete 1D bound states with the continuum of delocalized (unbound) states (the realization of “bound states in the continuum”).

(6) Nonlinear effects in the 1D system.

(7) 2D systems (linear and nonlinear): vorticity, stability, etc.

(8) Conclusion.
(3) Exceptional \((\text{codimension-1})\) exact solutions of the 1D linearized system

First of all, to confirm the existence of the bound states in the system, it is possible to find an \textit{exact} spatially-symmetric (even) solution of the \textit{linearized} \((\sigma = 0)\) coupled system, which is valid under a special condition imposed on \(\omega\) and \(\lambda\) (while \(V_0\) is an arbitrary amplitude):

\[
U(x) = (U_0 + U_2x^2) \exp \left( -\frac{x^2}{2} \right),
\]

\[
V(x) = V_0 \exp \left( -\frac{x^2}{2} \right),
\]

\[
U_0 = \frac{1 - 2\lambda^2 + \kappa}{4\lambda} V_0,
\]

\[
U_2 = -\frac{1 + \kappa}{2\lambda} V_0,
\]

\[
\mu_{\text{even}} = \frac{1}{2} \left( \lambda^2 + \frac{1}{2} \right) - \frac{\kappa}{4},
\]
This solution exists under the *restriction* imposed on the parameters (note that the *restriction* may hold for *arbitrarily large* values of strength $\kappa$ of the expulsive potential):

$$\omega_{\text{even}} = \frac{9}{4} - \frac{\lambda^2}{2} + \frac{\kappa}{4}.$$ 

Therefore it is categorized as a **codimension-1** exact solution.

The ratio of norms of the trapped and anti-trapped components in the exact solution:

$$\frac{N_u}{N_v} = \frac{\lambda^2}{4} + \frac{(1 + \kappa)^2}{8 \lambda^2}.$$ 

The trapped component ($u$) is a **dominant** one, with $N_u > N_v$, if the strength of the expulsive potential is large enough, $\kappa > 2\sqrt{2} - 1 \approx 1.83$. Otherwise, $N_u < N_v$ is possible.
It is also possible to find an **exact solution** of the linearized system for a **spatially-odd** (antisymmetric, alias **dipole**) mode, with an arbitrary amplitude $V_1$:

$$U(x) = (U_1 x + U_3 x^3) \exp\left(-\frac{x^2}{2}\right),$$

$$V(x) = V_1 x \exp\left(-\frac{x^2}{2}\right),$$

$$U_1 = (4\lambda)^{-1} \left(3 - 2\lambda^2 + 3\kappa\right) V_1,$$

$$U_3 = -(2\lambda)^{-1} (1 + \kappa) V_1,$$

$$\mu_{\text{odd}} = \frac{1}{2} \left(\lambda^2 + \frac{3}{2}\right) - \frac{3}{4} \kappa.$$  

This exact solution too exists under the **restriction** imposed on the parameters (and again, the **restriction** may hold for **arbitrarily large** values of strength $\kappa$ of the expulsive potential):

$$\omega_{\text{odd}} = \frac{11}{4} - \frac{\lambda^2}{2} + \frac{3\kappa}{4}.$$
(4) To construct generic bound eigenstates of the linear system, one can use the variational (alias Rayleigh-Ritz) approximation (VA). It is based on the integral expression for $\mu$ following from the stationary equations:

$$
\mu = -\int_{-\infty}^{+\infty} dx \left[ U \left( \omega U + \frac{1}{2} \frac{d^2 U}{dx^2} - \frac{x^2}{2} U \right) + V \left( \frac{1}{2} \frac{d^2 V}{dx^2} + \frac{\kappa x^2}{2} V \right) + 2\pi U V \right]
$$

(assuming that the total norm of the wave function is $N \equiv N_u + N_v = 1$).

The variational ansatz for the spatially even eigenstates with free parameter $\eta$ is adopted as

$$
\{U_{VA}(x), V_{VA}(x)\} = \pi^{-1/4} \{ \cos \eta, \sin \eta \} \exp(-x^2 / 2).
$$

The substitution of the ansatz in the expression for $\mu$ yields

$$
\mu_{VA} = \left( \frac{1}{2} - \omega \right) \cos^2 \eta + \left( \frac{1}{4} \right)(1 - \kappa) \sin^2 \eta - \lambda \sin(2\eta).
$$

The variational equation is $d\mu_{VA} / d\eta = 0$. It yields

$$
\mu_{VA} = \left( \frac{1}{2} - \omega - q \right) + \sqrt{q^2 + \lambda^2}, \quad q \equiv \left( \frac{1}{2} \right)[(1/4)(1 + \kappa) - \omega].
$$

In particular, in the limit of $\kappa \to \infty$ (very strong expulsive potential) the bound state persists, with $\mu_{VA} \approx 1/2 - \omega + 4\lambda^2 / \kappa$, while the amplitude of the $v$ component, which is subject to the action of the expulsive potential, is vanishing: $\eta \approx -4\lambda / \kappa$. 
Heatmaps for the VA-predicted (left) and numerically found (right) eigenvalues $\mu$ in the $(\kappa, \omega)$ plane for $\lambda = 0.1, 1, 2$ (top to bottom). The black line designates the exact solution of codimension 1.
Comparison of the variational, numerically found, and exact (codimension-1) shapes of the spatially even wave functions for $\lambda = 2$, $\omega = 1$, $\kappa = 3$. All the three solutions have $\mu = 1.5$. The VA predicts $\mu$ accurately, in spite of the discrepancy in the shape of the wave function.
The VA can be also developed for eigenvalues of the spatially odd (dipole) eigenstates, using the ansatz

\[
\{ U_{DM}^{(VA)}(x), V_{DM}^{(VA)}(x) \} = \sqrt{2\pi}^{-1/4} \{ \cos \eta, \sin \eta \} x \exp \left( -\frac{x^2}{2} \right),
\]

(53)

cf. Eq. (42), which is also subject to normalization Eq. (43). Substituting this in Eq. (44) yields

\[
\mu_{DM} = \left( \frac{3}{2} - \omega \right) \cos^2 \eta + \frac{3}{4} (1 - \kappa) \sin^2 \eta - \lambda \sin (2\eta),
\]

(54)

cf. Eq. (45). Then, the variational Eq. (46), applied to Eq. (54), produces the result

\[
\tan (2\eta) = -\lambda / q_{DM},
\]

(55)

\[
q_{DM} \equiv \frac{1}{2} \left[ \frac{3}{4} (1 + \kappa) - \omega \right],
\]

(56)

cf. Eqs. (47) and (48). The substitution of this in Eq. (54) leads to the following eigenvalue:

\[
\mu_{DM}^{(VA)} = \frac{3}{2} - \omega - q_{DM} + \sqrt{q_{DM}^2 + \lambda^2},
\]

(57)
The comparison of the variational (left) and numerical (right) results for eigenvalues $\mu$ of the dipole (spatially odd) eigenstate, for $\lambda = 0.1, 1, 2$ (top to bottom), with the black line denoting the exact solution of codimension 1:
Comparison of the variational, numerically found, and exact shapes of the \textbf{spatially odd} eigenstates for $\lambda = 2$, $\omega = 1$, $\kappa = 1/3$. All the three solutions have $\mu = 2.5$. 

![Graphs showing comparison of variational (VA), exact, and numerical solutions.](image)
(5) The same system admits a continuum of \textit{delocalized states}, at all values of $\mu$. Therefore, the localized eigenstates, existing at discrete values of $\mu$, may be categorized as \textit{bound states in the continuum} (BIC), alias \textit{embedded states}, cf. Stillinger, F.H.; Herrick, D.R. \textit{Bound states in continuum}. Phys. Rev. A \textbf{11}, 446 (1975); Kodigala, A.; Lepetit, T.; Gu, Q.; Bahari, B.; Fainman, Y.; Kante, B. \textit{Lasing action from photonic bound states in continuum}. Nature \textbf{54}, 196 (2017). Champneys, A.R.; Malomed, B.A.; Yang, J.; Kaup, D.J. \textit{“Embedded solitons”: solitary waves in resonance with the linear spectrum}. Physica D \textbf{152}, 340 (2001).
The analytical asymptotic form of the delocalized solutions, which exist at all values of $\mu$, and thus indeed form a continuum:

$$V_{\text{deloc}}(x) \xrightarrow{|x| \to \infty} V_0 |x|^{-1/2} \cos \left( \frac{\sqrt{\kappa}}{2} x^2 + \frac{\mu}{\sqrt{\kappa}} \ln(|x|) \right), \quad (60)$$

$$U_{\text{deloc}}(x) \xrightarrow{|x| \to \infty} V_0 \frac{2\lambda}{1 + \kappa} |x|^{-5/2} \cos \left( \frac{\sqrt{\kappa}}{2} x^2 + \frac{\mu}{\sqrt{\kappa}} \ln(|x|) \right), \quad (61)$$

Note the quadratic term in the phase of these expressions. The $v$ component, subject to the action of the expulsive potential, obviously dominates in this solution. These eigenstates may be considered as delocalized ones because their norm diverges (slowly) as $\int dx/|x|$ at $|x| \to \infty$. 
An example of the delocalized state, and the spatially even exact bound eigenstate existing at the same values of parameters: $\kappa = 0.5$, $\lambda = 2$, $\omega = 0.375$, and with equal eigenvalues, $\mu = 2.125$: 
(6) Effects of the cubic self-focusing and defocusing nonlinearities

Comparison of the \textit{exact} spatially-even bound state at $\sigma = 0$, $\kappa = 1$, $\lambda = 5$, $\omega = -10$, $\mu = 12.5$ (the left panel) and its \textit{numerically found counterparts} with $\sigma = +1$, $\mu = 11.97$ (center) and $\sigma = -1$, $\mu = 13.19$ (right). Bound states remain \textit{stable} in the nonlinear system:
A **stability test**: take the exact **spatially even** bound-state solution of the **linearized system** (here, with \( \kappa = 1, \lambda = 6, \omega = -15.5 \), and amplitude \( U_{\text{max}} = 0.146 \)), and use it as the **input** for simulations of the **nonlinear system** with \( \sigma = +1 \) (**self-focusing**). The result is a **robust breather**, which emits a small amount of “radiation”: 

![Graphs showing robust breather and radiation](image-url)
Similar comparison for spatially-odd solutions at $\sigma = 0, \kappa = 2.5, \lambda = 10, \omega = -45.375$ (left: the exact solution with $\mu = 48.875$) and its numerically found counterpart with $\sigma = +1, \mu = 48.363$ (right). The odd bound states are stable in the nonlinear system.
The **stability test** with the **input** taken as per the exact **spatially odd** solution of the **linearized system** with $\kappa = 0.5$, $\lambda = 5$, $\omega = -9.375$, and amplitude $U_{\text{max}} = 0.716$. A very “clean” breather is produced by the simulations, with virtually no emission of “radiation”.
On the other hand, the simulations with a much larger amplitude of the input demonstrate chaotization of the ensuing dynamics. It suppresses the effective confinement imposed by the linear coupling onto the $v$ component, which is subject to the action of the expulsive potential. This leads to the loss of the localization. An example: the simulation initiated by the same spatially odd input as before, but with the amplitude $\times 5$:
(7) Fundamental and vortex eigenstates of the two-dimensional system

A straightforward 2D extension of the linearly-coupled system (written in the polar coordinates):

\[
iu_z + \frac{1}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) u + \lambda v - \frac{1}{2} r^2 u + \sigma |u|^2 u = -\omega u,
\]

(6)

\[
i v_z + \frac{1}{2} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) v + \lambda u + \frac{1}{2} \kappa r^2 v + \sigma |v|^2 v = 0.
\]

(7)
Stationary 2D solutions for bound states with propagation constant $-\mu$ and embedded vorticity $S = 0, 1, 2, 3, \ldots$ are looked for as

$$\{u, v\} = \exp (-i\mu z + iS\theta)\{U(r), V(r)\},$$  \hspace{1cm} (8)$$

where real functions $U$ and $V$ satisfy radial equations

$$(\mu + \omega)U + \frac{1}{2} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{S^2}{r^2} \right)U + \lambda V - \frac{1}{2} r^2 U + \sigma U^3 \hspace{1cm} = 0,$$

$$= 0,$$

$$\mu V + \frac{1}{2} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{S^2}{r^2} \right)V + \lambda U + \frac{1}{2} \kappa r^2 V + \sigma V^3 = 0.$$

(10)
The linearized version of these equations admits an exact codimension-1 solution too, with any integer vorticity $S$:

\[
U(r) = \left( U_0^{(2D)} + U_2^{(2D)} r^2 \right) \exp \left( -\frac{r^2}{2} \right),
\]

\[
V(r) = V_0^{(2D)} \exp \left( -\frac{r^2}{2} \right),
\]

\[
U_0^{(2D)} = \frac{S + 1 - \lambda^2 + (S + 1) \kappa}{2\lambda} V_0^{(2D)},
\]

\[
U_2^{(2D)} = -\frac{1 + \kappa}{2\lambda} V_0^{(2D)},
\]

\[
\mu^{(2D)} = \frac{1}{2} \left[ \lambda^2 + (S + 1) (1 - \kappa) \right],
\]

This solution is valid under the following constraint imposed on parameters of the system: $\omega_{2D} = (1/2) \left[ 5 + S - \lambda^2 + (S + 1) \kappa \right]$. 
The heatmap of numerically found eigenvalues of the 2D bound states with $\sigma = 0$, $S = 0$ and $\lambda = 1$ (the exact codimension-1 solutions exist along the black line):
Similarly to the 1D setting, all bound states of the linearized 2D system may be considered as eigenmodes embedded into the continuum of delocalized states.

The asymptotic form of the 2D delocalized states at $r \to \infty$ is

$$V_{\text{deloc}}^{(2D)}(r) \underset{r \to \infty}{\approx} V_0 r^{-1} \cos \left( \frac{\sqrt{\kappa}}{2} r^2 + \frac{\mu}{\sqrt{\kappa}} \ln r \right),$$

$$U_{\text{deloc}}^{(2D)}(r) \underset{r \to \infty}{\approx} V_0 \frac{2\lambda}{1 + \kappa} r^{-3} \cos \left( \frac{\sqrt{\kappa}}{2} r^2 + \frac{\mu}{\sqrt{\kappa}} \ln r \right).$$

Note the quadratic term $\sim r^2$ in the phase of these expressions. At $r \to \infty$, their total norm slowly diverges as $\int dr/r$.

Also similar to the 1D case, the $v$ component, which is subject to the action of the expulsive potential, dominates in this solution.
The (weak) effect of the nonlinearity on the 2D state with $S = 0$ and $\kappa = 1$, $\lambda = 5$, $\omega = -10$: the exact solution with $\sigma = 0$, $\mu = 12.5$ (left), and its numerically found counterpart with $\sigma = +1$, $\mu = 12.10$ (right). The 2D bound states with $S = 0$ remain stable in the nonlinear system with either sign of $\sigma$. 
The effect of the nonlinearity on the 2D bound state with vorticity $S = 1$ and $\kappa = 0.5$, $\lambda = 10$, $\omega = -46.5$: the exact solution with $\sigma = 0$, $\mu = 24.5$ (left), and its numerically found counterpart with $\sigma = -1$, $\mu = 27.01$ (right).
The situation concerning the stability of the nonlinear 2D bound states with embedded vorticity is different, in comparison with the case of $S = 0$. As in other models [T. J. Alexander and L. Bergé, Ground states and vortices of matter-wave condensates and optical guided waves, Phys. Rev. E 65, 026611 (2002); B. A. Malomed, Vortex solitons: Old results and new perspectives, Physica D 399, 108 (2019)], the vortex is unstable against spontaneous splitting under the action of the self-focusing nonlinearity ($\sigma = +1$).

An example: the unstable evolution of the vortex state with $S = 1$ and $\kappa = 0.5$, $\lambda = 10$, $\omega = -46.5$, $U_{\text{max}} = 1$. Shown are the shapes at $z = 0$, 3.8, and 6.2.
On the other hand, the vortex eigenstates are **stable** in the nonlinear system with self-defocusing ($\sigma = -1$). An example: the **stable evolution** of the eigenstate with $S = 1$ in the system with $\kappa = 1$, $\lambda = 7$, $\omega = -20.5$, and $U_{\text{max}} = 1$. Shown are the shapes at $z = 0$ and $z = 10$. 
(8) Conclusions

(i) It is demonstrated that, quite **counter-intuitively**, bound states of the wave function subject to the action of the 1D and 2D **expulsive** parabolic potential may be supported by the linear coupling of the wave function to a mate one, which is confined by the **trapping potential**.

(ii) This finding is precisely corroborated by the exact analytical solutions of codimension 1, for both the even and odd eigenmodes in the 1D system, and for eigenstates with all values of vorticity $S$ in 2D.

(iii) Generic spatially even and odd 1D eigenstates are found by means of the variational (Rayleigh-Ritz) approximation. Along with the systematically reported numerical findings, these results corroborate the **existence of the bound states for all values of strength $\kappa$ of the expulsive potential**. In the limit of $\kappa \rightarrow \infty$, this is explained by the fact that the amplitude of the component of the bound state which is subject to the action of the expulsive potential becomes **vanishingly small**, $\sim 1/\kappa$. 
(iv) All the **bound eigenstates** coexist with the **delocalized states** which form the **continuous spectrum**, therefore the bound eigenstates may be categorized as **localized modes embedded into the continuum**.

(v) Both the self-focusing and defocusing nonlinearity produces a weak deformation of the bound states, and **does not break their stability** in the **1D** system.

(vi) In the **2D** system, bound eigenstates with **embedded vorticity** are **unstable against spontaneous splitting** under the action of the **self-focusing**, but remain **stable** in the case of **defocusing**.

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Thank you for your interest!