SPECTRAL MEASURES OVER C-ALGEBRAS OF OPERATORS DEFINED IN $c_0$

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Abstract. The main goal of this work is to introduce an analogous in the non-archimedean context of the Gelfand spaces of certain Banach commutative algebras with unit. In order to do that, we study the spectrum of this algebras and we show that, under special conditions, these algebras are isometrically isomorphic to certain spaces of continuous functions defined over compacts. Such isometries preserve projections and allow to define associated measures which are known as spectral measures. We also show that each element of the algebras can be represented as an integral defined by these measures. We finish this work by showing that the studied algebras are, actually, free Banach spaces.

1. Introduction and notation

Many researchers have tried to generalize the elemental studies of Banach algebras from classical case to vectorial structures over non-archimedean fields. The first big task was to find a results similar to the Gelfand-Mazur Theorem in this context. But, this theorem failed since every field $k$ with a non-archimedean valuation is contained in another field $K$ such that its valuation is an extension of the valuation of $k$.

One of the main pioneers in the study of non-archimedean Banach algebras of linear operators and spectral theory in this context has been M. Vishik [10], especially in the class of linear operators which admit compact spectrum. We can also mention another importan pioneer, Berkovick [4], who made a deep study of this subject on his survey.

The main goal of this work is to introduce an analogous in the non-archimedean context of the Gelfand spaces of some commutative Banach algebras of linear operators with unit. In order to do that, we will study the spectrum of this algebras and will show that, under special condition, these algebras are isometrically isomorphic to a respective spaces of continuous functions defined over some compact. Such isometries will preserve idempotent elements and will allow us to define associated measures which are known as spectral measures. We will finish this work showing that each element of the commutative Banach algebra described before can be represented as an integral of some continuous function, where the integral has been defined by the spectral measure.

Throughout this paper $\mathbb{K}$ is a valued field which is complete with respect to the metric induced by the nontrivial non-archimedean valuation $|\cdot|$ and its residue class field is formally real.

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In the classical situation we can distinguish two type of normed spaces: those spaces which are separable and those which are not separable. If \( E \) is a separable normed space over \( K \), then each one-dimensional subspaces of \( E \) is homeomorphic to \( K \), so \( K \) must be separable too. Nevertheless, we know that there exist non-archimedean fields which are not separable. Thus, for non-archimedean normed spaces the concept of separability is meaningless if \( K \) is not separable. However, linearizing the notion of separability, we obtain a useful generalization of this concept. A normed space \( E \) over a non-archimedean valued field is said to be of countable type if it contains a countable subset whose linear hull is dense in \( E \). An example of a normed space of countable type is \( (c_0, \| \cdot \|_\infty) \), where \( c_0 \) is the Banach space of all sequences \( x = (a_n)_{n \in \mathbb{N}}, a_n \in K \), for which \( \lim_{n \to \infty} a_n = 0 \) and its norm is given by \( \| x \|_\infty = \sup \{ |a_n| : n \in \mathbb{N} \} \).

A non-archimedean Banach space \( E \) is said to be Free Banach space if there exists a family \( \{e_i\}_{i \in I} \) of non-null vectors of \( E \) such that any element \( x \in E \) can be written in the form of convergent sum \( x = \sum_{i \in I} x_i e_i, \ x_i \in K, \) and \( \| x \| = \sup_{i \in I} |x_i| \| e_i \| \). The family \( \{e_i\}_{i \in I} \) is called orthogonal basis of \( E \). If \( s : I \to (0, \infty) \), then an example of Free Banach space is \( c_0 (I, \mathbb{K}, s) \), the collection of all \( x = (x_i)_{i \in I} \) such that for any \( \epsilon > 0 \), the set \( \{ i \in I : |x_i| s(i) > \epsilon \} \) is, at most, finite (or, equivalently, \( \lim_{i \in I} |x_i| s(i) = 0 \), with respect to the Frechet filter on \( I \)) and \( \| x \| = \sup_{i \in I} |x_i| s(i) \).

We already know that a Free Banach space \( E \) is isometrically isomorphic to \( c_0 (I, \mathbb{K}, s) \), for some \( s : I \to (0, \infty) \). In particular if a Free Banach space is of countable type, then it is isometrically isomorphic to \( c_0 (\mathbb{N}, \mathbb{K}, s) \), for some \( s : \mathbb{N} \to (0, \infty) \). Note that if \( s(i) \in \mathbb{K} \), then \( E \) is isometrically isomorphic to \( c_0 (\mathbb{N}, \mathbb{K}) \) (or \( c_0 \) in short). For a detailed study of Free Banach spaces, in general, we refer the reader to [5] or [9]

Now, since residual class field of \( \mathbb{K} \) is formally real, the bilinear form

\[
\langle \cdot, \cdot \rangle : c_0 \times c_0 \to \mathbb{K}; \quad (x, y) = \sum_{i=1}^{\infty} x_i y_i
\]

is an inner product, \( \| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle} \) is a norm in \( c_0 \) and the supremum norm \( \| \cdot \|_\infty \) coincides with \( \| \cdot \| \), that is, \( \| \cdot \| = \| \cdot \|_\infty \) (see [7]). Therefore, to study the Free Banach spaces of countable type it is enough to study the space \( c_0 \).

If \( E \) and \( F \) are \( \mathbb{K} \)-normed spaces, then \( \mathcal{L}(E, F) \) will be the \( \mathbb{K} \)-normed space consisting of all continuous linear maps from \( E \) into \( F \). If \( F = E \), then \( \mathcal{L}(E) = \mathcal{L}(E, E) \). For any \( T \in \mathcal{L}(E, F) \), \( N(T) \) will denote its kernel and \( R(T) \) its range.

A linear operator \( T \) from \( E \) into \( F \) is said to be compact operator if \( T(B_E) \) is compactoid, where \( B_E = \{ x \in E : \| x \| \leq 1 \} \) is the unit ball of \( E \). It was proved in [9] that \( T \) is compact if and only if, for each \( \epsilon > 0 \), there exists a linear operator of finite-dimensional range \( S \) such that \( \| T - S \| \leq \epsilon \).

Since \( c_0 \) is not orthomodular, there exist operators in \( \mathcal{L}(c_0) \) which don’t admit adjoint; for example, \( T(x) = (\sum_{i=1}^{\infty} x_i) e_1, \ x = (x_i)_{i \in \mathbb{N}} \in c_0 \). We will denote by \( A_0 \) the collection of all elements of \( \mathcal{L}(c_0) \) which admit adjoint. A characterization of the elements of \( A_0 \) (see [1]) is the following:

\[
A_0 = \{ T \in \mathcal{L}(c_0) : \forall y \in c_0, \lim_{t \to \infty} (Tc_t, y) = 0 \}.
\]

Of course, \( A_0 \) is a non-commutative Banach algebra with unit.
Now, for each $a = (a_i)_{i \in \mathbb{N}} \in c_0$, the linear operator $M_a$, defined by $M_a(\cdot) = \sum_{i=1}^{\infty} a_i \langle \cdot, e_i \rangle e_i$, belongs to $A_0$; moreover,
\[
\lim_{n \to \infty} \|M_a e_n\| = \lim_{n \to \infty} \sum_{i=1}^{\infty} a_i \langle e_n, e_i \rangle e_i = \lim_{n \to \infty} |a_n| = 0,
\]
meanwhile, the identity map $I$ is also an element of $A_0$, but
\[
\lim_{n \to \infty} \|I(e_n)\| = \lim_{n \to \infty} \|e_n\| = 1.
\]

Let us denote by $A_1$ the collection of all $T \in \mathcal{L}(c_0)$ such that $\lim_{n \to \infty} Te_n = \theta$, i.e.,
\[
A_1 = \left\{ T \in \mathcal{L}(c_0) : \lim_{n \to \infty} Te_n = \theta \right\}.
\]
From the fact that
\[
|\langle Te_n, y \rangle| \leq \|Te_n\| \|y\|,
\]
we have that $A_1 \subseteq A_0$ since $I \notin A_1$.

By [5], we know that each $T \in \mathcal{L}(c_0)$ can be represented by $T = \sum_{i,j=1}^{\infty} a_{i,j} e'_j \otimes e_i$, where $\lim_{n \to \infty} a_{i,j} = 0$, for all $j \in \mathbb{N}$, $\|T\| = \sup \{\|T(e_i)\| : i \in \mathbb{N}\}$ and $T$ is compact if and only if
\[
\lim_{n \to \infty} \sup \{|a_{i,j} : i \in \mathbb{N}\} = 0.
\]
Now, since
\[
\|Te_n\| = \left\| \left( \sum_{i,j=1}^{\infty} a_{i,j} e'_j \otimes e_i \right) (e_n) \right\| = \left\| \sum_{i,j=1}^{\infty} a_{i,j} (e_n) e_i \right\|
\]
\[
= \left\| \sum_{i=1}^{\infty} a_{i,n} e_i \right\| = \sup \{|a_{i,n} : i \in \mathbb{N}\}.
\]
Therefore,
\[
T \in A_1 \iff T \in A_0 \text{ and } T \text{ is compact}
\]

Let $\{y^{(i)}\}_{i \in \mathbb{N}}$ be a sequence in $c_0$. We will say that $\{y^{(i)}\}_{i \in \mathbb{N}}$ is orthonormal if $\langle y^{(i)}, y^{(j)} \rangle = 0$, $i \neq j$, and $\|y^{(i)}\| = 1$. On the other hand, we will understand by a normal projection to any projection $P : c_0 \to c_0$ such that $\langle x, y \rangle = 0$ for any $x \in N(P)$ and $y \in R(P)$. For example, if $y \in c_0$, $y \neq 0$, is fixed, then $P(\cdot) = \frac{\langle \cdot, y \rangle}{\langle y, y \rangle} y$ is a normal projection.

The next theorem characterizes compact and self-adjoint operators. Its proof is similar to the proof given in [2], so we will omit it.

**Theorem 1.** If the linear operator $T : c_0 \to c_0$ is compact and self-adjoint, then there exists an element $\lambda = (\lambda_i)_{i \in \mathbb{N}} \in c_0$ and an orthonormal sequence $\{y^{(i)}\}_{i \in \mathbb{N}}$ in $c_0$ such that
\[
T = \sum_{i=1}^{\infty} \lambda_i P_i
\]
and $\|T\| = \|\lambda\|$, where
\[
P_i(\cdot) = \frac{\langle \cdot, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)}
\]
is the normal projection defined by $y^{(i)}$. 
Remark 1. This theorem gives us a characterization for compact and self-adjoint operators. In fact, it is not hard to see that if we take \( \lambda = (\lambda_i)_{i \in \mathbb{N}} \in c_0 \) and an orthonormal sequences \( \{y_i^{(i)}\}_{i \in \mathbb{N}} \) in \( c_0 \), the operator

\[
T = \sum_{i=1}^{\infty} \lambda_i P_i,
\]

where \( P_i \) is as in the Theorem, is compact and self-adjoint and \( \|T\| = \|\lambda\| \).

2. Algebra of operators

2.1. An algebra without unit. From now on, we will consider a fixed orthonormal sequence \( Y = \{y^{(i)}\}_{i \in \mathbb{N}} \) in \( c_0 \). We will denote by \( \mathfrak{X}_Y(c_0) \) the collection of all compact operators \( T_\lambda, \lambda \in c_0 \), where

\[
T_\lambda = \sum_{i=1}^{\infty} \lambda_i P_i
\]

As we know, the adjoint \( T_\lambda^* \) of \( T_\lambda \) is itself and \( \lim_{n \to \infty} T_\lambda(e_i) = 0 \). Obviously, the collection \( \mathfrak{X}_Y(c_0) \) is a linear space, since

\[
T_\lambda + T_\mu = T_{\lambda + \mu}; \quad \alpha T_\lambda = T_{\alpha \lambda}
\]

On the other hand, since \( c_0 \) is a commutative algebra with the operation \( \lambda \cdot \mu = (\lambda \mu_i) \), we have

\[
T_\lambda \circ T_\mu = T_{\lambda \mu} = T_\mu \circ T_\lambda.
\]

Therefore, \( \mathfrak{X}_Y(c_0) \) is a commutative algebra without unit. Even more, by the fact that \( T_\lambda = T_\mu \) implies \( \lambda = \mu \) (see [2]), the linear transformation

\[
\Lambda : c_0 \to \mathfrak{X}_Y(c_0); \quad \lambda \to \Lambda(\lambda) = T_\lambda
\]

is an isometric isomorphism of algebras.

As we know, each algebra without unit can be transformed in an algebra with unit, considering the collection \( E^+ = \mathbb{K} \oplus E \) provided with the usual linear operations and the multiplication operation defined by

\[
(\alpha, \mu) \cdot (\beta, \nu) = (\alpha \beta, \alpha \nu + \beta \mu + \mu \cdot \nu)
\]

The unit of this algebra is \((1, \theta)\). If \( E \) is, in particular, a normed space, then \( E^+ \) so is and

\[
\|(\alpha, \mu)\| = \max \{|\alpha|, \|\mu\|\}.
\]

Now, the commutative Banach algebra \( (\mathfrak{X}_Y(c_0), +, \cdot, \|\cdot\|) \) can be transformed, as above, in a commutative Banach algebra \( (\mathfrak{X}_Y(c_0)^+, +, \cdot, \|\cdot\|) \) with unit. By the fact that \( c_0 \) is isometrically isomorphic to \( \mathfrak{X}_Y(c_0) \), \( \mathfrak{X}_Y(c_0)^+ \) is isometrically isomorphic to \( c_0^+ \).

2.2. An algebra with unit. We will denote by \( \mathcal{S}_Y(c_0) \) the collection of all linear operators \( \alpha I + T_\lambda, \alpha \in \mathbb{K} \) and \( T_\lambda \in \mathfrak{X}_Y(c_0) \). \( \mathcal{S}_Y(c_0) \) is an normed space and since

\[
(\alpha_1 I + T_\mu) \circ (\alpha_2 I + T_\nu) = \alpha_1 \alpha_2 I + \alpha_1 T_\nu + \alpha_2 T_\mu + T_\mu \circ T_\nu = \alpha_1 \alpha_2 I + T_{\alpha_1 \nu + \alpha_2 \mu + \mu \nu}
\]

we conclude that \( \mathcal{S}_Y(c_0) \) is a commutative algebra with unit.

**Theorem 2.** The algebra \( \mathcal{S}_Y(c_0) \) is isometrically isomorphic to \( \mathfrak{X}_Y(c_0)^+ \). As a consequence, \( \mathcal{S}_Y(c_0) \) is a commutative Banach algebra with unit.
Proof. We define
\[ \mathcal{T}_Y(c_0^+) \rightarrow \mathcal{S}_Y(c_0^+) \]
\[ (\alpha, T_\lambda) \rightarrow \alpha I + T_\lambda \]
Since \( \alpha T_\mu + \beta T_\lambda + T_\lambda \circ T_\mu = T_{\alpha \mu + \beta \lambda + \mu \lambda} \), this transformation is an algebra homomorphism. Obviously, this homomorphism is onto; hence it is enough to prove that it is an isometry. We claim that
\[ \|\alpha I + T_\lambda\| = \|(\alpha, T_\lambda)\| = \max \{ |\alpha|, \|T_\lambda\| \} \]
If \( \alpha = 0 \) or \( \|T_\lambda\| = 0 \) or \( \|\alpha I\| \neq \|T_\lambda\| \), we are done. We only need to check when
\[ |\alpha| = \|\alpha I\| = \|T_\lambda\| \neq 0. \]
Of course,
\[ \|\alpha I + T_\lambda\| \leq \max \{ |\alpha|, \|T_\lambda\| \} \]
Now, by the compactness of \( T_\lambda \),
\[ \lim_{n \rightarrow \infty} T_\lambda(e_n) = 0. \]
Thus, there exists \( N \in \mathbb{N} \) such that
\[ n \geq N \Rightarrow \|T_\lambda(e_n)\| < |\alpha| \]
Therefore,
\[ \|\alpha I + T_\lambda\| = \sup \{\|\alpha e_n + T_\lambda(e_n)\| : n \in \mathbb{N}\} \]
\[ = \max \{|\alpha e_1 + T_\lambda(e_1)|, |\alpha e_2 + T_\lambda(e_2)|, \ldots, |\alpha e_N + T_\lambda(e_N)|, |\alpha|\} \]
\[ = |\alpha| = \max \{|\alpha|, \|T_\lambda\|\} = \|(\alpha, T_\lambda)\|. \]

Remark 2. Since \( c_0^+ \) is isometrically isomorphic to \( \mathcal{T}_Y(c_0^+) \) and, at the same time, this last is isometrically isomorphic to \( \mathcal{S}_Y(c_0) \), \( c_0^+ \) is isometrically isomorphic to \( \mathcal{S}_Y(c_0) \).

We claim that the usual norm in \( \mathcal{S}_Y(c_0) \) is power multiplicative, that is, for each \( T \in \mathcal{S}_Y(c_0) \),
\[ \|T^n\| = \|T\|^n. \]
In fact, by the remark, it is enough to study this property in \( c_0^+ \).

The norm in \( c_0^+ \) was defined by \( \|(\alpha, a)\| = \max \{|\alpha|, \|a\|\} \). On the other hand, for any \( (\alpha, a) \in c_0^+ \)
\[ (\alpha, a)^2 = (\alpha^2, 2\alpha a + a^2) \]
\[ (\alpha, a)^3 = (\alpha^2, 2\alpha a + a^2) \cdot (\alpha, a) \]
\[ = (\alpha^3, \left( \begin{array}{c} 3 \\ 1 \end{array} \right) \alpha^2 a + \left( \begin{array}{c} 3 \\ 2 \end{array} \right) \alpha a^2 + \left( \begin{array}{c} 3 \\ 3 \end{array} \right) a^3) \]
In general, for each \( k \in \mathbb{N} \),
\[ (\alpha, a)^k = \left( \alpha_k, \sum_{i=1}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) \alpha^{k-i} a^i \right) \]
Now, let us start analyzing $\| (\alpha, a)^k \|$. In general, we have

$$\| (\alpha, a)^k \| \leq \| (\alpha, a) \|^k$$

The power multiplicative property is, obviously, satisfied by the supremum norm in $c_0$, that is,

$$\| a^k \| = \| a \|^k.$$

If $\alpha = 0$ or $a = \theta$, then

$$\| (\alpha, a)^k \| = \| (\alpha, a) \|^k$$

If $|\alpha| < \| a \|$, then we have

$$\left\| \sum_{i=1}^{k-1} \left( \begin{array}{c} k \\ i \end{array} \right) \alpha^{k-i} a^i \right\| \leq \max \left\{ |\alpha|^{k-i} \| a^i \| : i = 1, \ldots, k - 1 \right\}$$

$$< \max \left\{ \| a \|^{k-i} \| a^i \| : i = 1, \ldots, k - 1 \right\} = \| a \|^k$$

Thus,

$$\left\| \sum_{i=1}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) \alpha^{k-i} a^i \right\| = \| a^k + \sum_{i=1}^{k-1} \left( \begin{array}{c} k \\ i \end{array} \right) \alpha^{k-i} a^i \| = \| a \|^k$$

Therefore,

$$\| (\alpha, a) \|^k = \max \left\{ |\alpha|^k, \| a \|^k \right\}$$

$$= \left\| (\alpha, a)^k \right\|$$

Suppose, finally, that $|\alpha| \geq \| a \|$. Then,

$$\left\| \sum_{i=1}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) \alpha^{k-i} a^i \right\| \leq \max \left\{ |\alpha|^{k-i} \| a^i \| : i = 1, \ldots, k \right\} \leq |\alpha|^k$$

which implies

$$\| (\alpha, a)^k \| = \max \left\{ |\alpha|^k, \sum_{i=1}^{k} \left( \begin{array}{c} k \\ i \end{array} \right) \alpha^{k-i} a^i \right\}$$

$$= |\alpha|^k = \max \left\{ |\alpha|^k, \| a \|^k \right\} = \| (\alpha, a) \|^k$$

**Definition 1.** An commutative Banach algebra $\mathcal{A}$ with unit is called a $C$-algebra if there exists a locally compact space $X$ such that $\mathcal{A}$ is isometrically isomorphic to $C_\infty(X)$, where $C_\infty(X)$ is the space of all continuous functions from $X$ into $\mathbb{K}$ which vanishes at infinity.

As we know $\{ e_j = (\delta_{i,j})_{i \in \mathbb{N}} : j \in \mathbb{N} \}$, where $\delta_{i,j}$ denotes the Kronecker symbol, is the canonical basis of $c_0$. Since

$$e_j^2 = e_j$$

and

$$\{ e_j : j \in \mathbb{N} \} = c_0$$

we conclude that the collection of all the idempotent elements of $c_0$ with norm less than 1 is dense in $c_0$. As a consequence, $c_0$ is a $C$-algebra (Th. 6.12 [9]).
Theorem 3. $S_Y(c_0)$ is a C-algebra.

Proof. It follows from the fact that $c_0^+$ is isometrically isomorphic to $S_Y(c_0)$.

3. The Subalgebra $\mathcal{L}_T$

Let us fix a compact and self-adjoint operator $T$; hence $I + T \in S_Y(c_0)$. We shall denote by $\mathcal{L}_T$ the closure of the algebra spanned by $\{I, T\}$, that is,

$$\mathcal{L}_T = \overline{\{I, T\}} = \left\{ \sum_{n=0}^{\infty} \alpha_n T^n : \|\alpha_n T^n\| \to 0 \right\}.$$

Clearly, $\mathcal{L}_T$ is a C-algebra since it is closed Banach subalgebra of $S_Y(c_0)$ (Cor. 6.13, [9]).

This condition guarantees that $Sp(\mathcal{L}_T)$ is compact and, for each $H \in \mathcal{L}_T$,

$$\|H\| = \sup_{i \in I} \|H(e_i)\| = \|H\|_{\text{sp}},$$

On the other hand, since the operators norm in $S_Y(c_0)$ is power multiplicative, such property is inherited by $\mathcal{L}_T$. Thus, for any $H \in \mathcal{L}_T$, we have

$$\|H^n\| = \|H\|^n.$$

Under the conditions that $\mathcal{L}_T$ is a C-algebra and $Sp(\mathcal{L}_T)$ is compact, we conclude that $\mathcal{L}_T$ is isometrically isomorphic to the space of all continuous functions $C(Sp(\mathcal{L}_T))$ provided by the supremum norm, that is, there exists an isomorphism of algebras $\Psi$ which is, at the same time, an isometry from $\mathcal{L}_T$ onto $C(Sp(\mathcal{L}_T))$.

Thus, if $H = \sum_{n=0}^{\infty} \alpha_n T^n$, then $\|H\| = \|\Psi(H)\|_{\infty}$.

Now, since $T$ is compact and self-adjoint, there exists $\lambda = (\lambda_i)_{i \in N} \in c_0$ for which

$$T = T_\lambda = \sum_{i \in N} \lambda_i P_i, \|T\| = \|\lambda\| \text{ and } T(y^{(i)}) = \lambda_i y^{(i)}; \ y^{(i)} \in Y$$

Fix $\lambda_i \neq 0$ and define the homomorphism of algebra $\phi_i : \{I, T\} \to \mathbb{K}$ by

$$\phi_i(T) = \lambda_i.$$

Let $H = \sum_{n=0}^{k} \alpha_n T^n \in \{I, T\}$.

Since

$$|\phi_i(H)| = \left| \alpha_0 + \sum_{n=1}^{k} \alpha_n \lambda_i^n \right| \leq \max \left\{ |\alpha_0|, \left\| \sum_{n=1}^{k} \alpha_n \lambda_i^n \right\| \right\}$$

$$\leq \max \left\{ |\alpha_0|, \left\| \sum_{n=1}^{k} \alpha_n \lambda_i^n \right\| \right\} = \left\| \left( \sum_{n=1}^{k} \alpha_n \lambda_i^n \right) \right\|_{c_0^+}$$

$$= \left\| \alpha_0 I + T \sum_{n=1}^{k} \alpha_n T^n \right\| = \left\| \sum_{n=0}^{k} \alpha_n T^n \right\| = \|H\|$$

we get that $\phi_i$ is continuous.

Denote by $\sigma(T) = \{\lambda_i : i \in N\} \cup \{0\}$. Since $\{I, T, T^2, \ldots\}$ also generates the closed algebra $\mathcal{L}_T$ and

$$|\alpha_n \lambda_i^n| = |\alpha_n| |\lambda_i|^n \leq |\alpha_n| \|\lambda\|^n

= |\alpha_n| \|T\|^n = |\alpha_n| \|T^n\| = \|\alpha_n T^n\| \to 0,$$

$\phi_i$ can be continuously extended to $\mathcal{L}_T$. 

From this, the following function

$$\sigma(T) \xrightarrow{\Gamma} Sp(L_T); \lambda_i \rightarrow \Gamma(\lambda_i) = \phi_i; \ 0 \rightarrow \Gamma(0) = \phi_0.$$ 

is well defined. We claim that \(\Gamma\) is bijective. Clearly, it is an injective function. To prove the surjective condition, we will start with the following result:

**Theorem 4.** If \(z \notin \sigma(T)\), then \(zI - T\) is invertible in \(S_Y(c_0)\).

**Proof.** Since \(z \notin \sigma(T)\), we have that, for \(y \in R(zI - T)\), there exists \(x \in c_0\) such that

$$\Gamma(x) = y.$$ 

Solving this equation for \(x\), we get

$$x = \frac{1}{z} y + \frac{1}{z} T x.$$ 

Applying the continuous functional \(\langle \cdot, y(k) \rangle\) in (3.1), we have

$$\langle x, y(k) \rangle = \left( \frac{1}{z} y + \frac{1}{z} \sum_{i \in N} \lambda_i \frac{\langle x, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)} \right)$$

$$= \frac{1}{z} \langle y, y^{(k)} \rangle + \frac{1}{z} \sum_{i \in N} \lambda_i \frac{\langle x, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)} - \lambda_k \langle x, y^{(k)} \rangle.$$ 

Now, solving the last equation for \(\langle x, y^{(k)} \rangle\), we obtain

$$\left( 1 - \frac{\lambda_k}{z} \right) \langle x, y^{(k)} \rangle = \frac{1}{z} \langle y, y^{(k)} \rangle$$

$$\langle x, y^{(k)} \rangle = \frac{1}{z - \lambda_k} \langle y, y^{(k)} \rangle.$$ 

Note that the sequence

$$\left( \frac{\lambda_k}{z - \lambda_k} \right)_{k \in \mathbb{N}}$$

is an element of \(c_0\); in fact, since \(z \notin \sigma(T)\), we have that \(|z| > 0\) and, therefore, for a given \(0 < \epsilon < 1\), there exists \(i_0 \in \mathbb{N}\) such that

$$i \geq i_0 \Rightarrow |\lambda_i| < \epsilon |z|.$$ 

Thus,

$$i \geq i_0 \Rightarrow \left| \frac{\lambda_i}{z - \lambda_i} \right| = \frac{|\lambda_i|}{|z|} < \epsilon.$$ 

Now, replacing in (3.1), we get

$$x = \frac{1}{z} y + \frac{1}{z} \sum_{i \in N} \frac{\lambda_i}{z - \lambda_i} \frac{\langle y, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)}$$

$$= \frac{1}{z} y + \frac{1}{z} \sum_{i \in N} \lambda_i \frac{\langle x, y^{(i)} \rangle}{z - \lambda_i} P_i(y).$$

Although \(y\) belongs to \(R(zI - T)\), the last expression holds for any \(y \in c_0\). Thus, if we denote by

$$R_z(T)(y) = \frac{1}{z} y + \frac{1}{z} \sum_{i \in N} \frac{\lambda_i}{z - \lambda_i} P_i(y),$$
then \( R_z (T)(\cdot) \in \mathcal{S}_Y (c_0) \), since \( \sum_{i \in \mathbb{N}} \frac{\lambda_i}{z - \lambda_i} P_i (\cdot) \) is a compact and self-adjoint operator.

Let us show that, effectively, \( R_z (T)(\cdot) \) is the inverse operator of \( zI - T \). Note that \( T(P_i(y)) = \lambda_i P_i(y) \). Now,

\[
[(zI - T) \circ R_z (T)](y) = (zI - T) \left( \frac{1}{z} y + \frac{1}{z} \sum_{i \in \mathbb{N}} \frac{\lambda_i}{z - \lambda_i} P_i(y) \right) \\
= y + \sum_{i \in \mathbb{N}} \frac{\lambda_i}{z - \lambda_i} P_i(y) - \frac{1}{z} \sum_{i \in \mathbb{N}} \lambda_i P_i(y) - \frac{1}{z} \sum_{i \in \mathbb{N}} \frac{\lambda_i^2}{z - \lambda_i} P_i(y) \\
= y + \sum_{i \in \mathbb{N}} \left[ \frac{\lambda_i}{z - \lambda_i} - \frac{\lambda_i}{z} \right] P_i(y) = y = I(y)
\]

In the other direction, since \( P_j \circ P_i = \begin{cases} P_i & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases} \), we have

\[
[R_z (T) \circ (zI - T)](x) = zR_z (T)(x) - R_z (T)(Tx) \\
= x + \sum_{i \in \mathbb{N}} \frac{\lambda_i}{z - \lambda_i} P_i(x) - \sum_{i \in \mathbb{N}} \lambda_i R_z (T)(P_i(x)) \\
= x + \sum_{i \in \mathbb{N}} \frac{\lambda_i}{z - \lambda_i} P_i(x) - \sum_{i \in \mathbb{N}} \lambda_i \left[ \frac{1}{z} P_i(x) + \frac{1}{z} \sum_{j \in \mathbb{N}} \frac{\lambda_j}{z - \lambda_j} P_j(P_i(x)) \right] \\
= x + \sum_{i \in \mathbb{N}} \left[ \frac{\lambda_i}{z - \lambda_i} - \frac{\lambda_i}{z} \right] P_i(x) = x = I(x)
\]

Therefore, \( R_z (T) = (zI - T)^{-1} \in \mathcal{S}_Y (c_0) \).

\( \square \)

**Corollary 1.** If \( z \notin \sigma(T) \), then \( R_z (T) \in \mathcal{L}_T \).

**Proof.** We already know that \( \mathcal{S}_Y (c_0) \) is a C-algebra with unit; hence, by Th. 6.10 in [9], we have that

\[
R_z (T) = (zI - T)^{-1} \in \mathbb{K}[zI - T] \\
= \left\{ zI - T, (zI - T)^2, (zI - T)^3, \ldots \right\}
\]

On the other hand, since \( (zI - T)^n \in \mathcal{L}_T \), for any \( n \in \mathbb{N} \), we have that \( \mathbb{K}[zI - T] \) is a subalgebra of \( \mathcal{L}_T \). This proves that \( R_z (T) \in \mathcal{L}_T \).

\( \square \)

**Corollary 2.** The function \( \sigma(T) \xrightarrow{\mathbb{F}} \text{Sp}(\mathcal{L}_T) \) is bijective.

**Proof.** If \( \phi \in \text{Sp}(\mathcal{L}_T) \), then \( \phi(T) = z \), for a \( z \in \mathbb{K} \). Suppose that \( z \notin \sigma(T) \), hence \( zI - T \) has an inverse and, for the previous theorem, \( (zI - T)^{-1} = R_z (T) \in \mathcal{L}_T \).
Since the function $\phi$ is a homomorphism of algebras with unit, we have
\[
1 = \phi(I) = \phi\left((zI - T)^{-1} \circ (zI - T)\right) = \phi\left((zI - T)^{-1}\right) \phi(zI - T),
\]
but, by the linearity of $\phi$, the factor $\phi(zI - T)$ is null, which is a contradiction. Such contradiction is coming from the fact that $z \notin \sigma(T)$.
Thus, if $\phi \in \text{Sp}(L_T)$, then there exists $\mu \in \sigma(T)$ such that $\phi = \phi_\mu$ and therefore $\Gamma$ is bijective. □

Remark 3. We have identified $\text{Sp}(L_T)$ with $\sigma(T)$ through the bijective function $\Gamma$. Note that $\Upsilon = \Gamma^{-1}$ is continuous. In fact, if $\phi_\alpha \to \phi$ in the induced topology by the product topology in $K^L_T$, then
\[
\phi_\alpha(H) \to \phi(H)
\]
for all $H \in L_T$, in particular,
\[
\phi_\alpha(T) \to \phi(T)
\]
or, equivalently,
\[
\Upsilon(\phi_\alpha) \to \Upsilon(\phi)
\]
Now, since $\Upsilon$ is bijective and continuous, $\text{Sp}(L_T)$ is compact and $\sigma(T)$ is a Hausdorff space, we conclude that $\Upsilon$ is a homeomorphism.
By these facts and by the uniqueness of $X$ (up to homeomorphism) for which $L_T \cong C(X)$, we have
\[
L_T \cong C(\sigma(T)).
\]

We claim that one of the isometric isomorphism from $L_T$ to $C(\sigma(T))$ is the Gelfand Transformation $G$. In fact, if $H \in L_T$, then
\[
G_H : \text{Sp}(L_T) \to K; \ \phi \to G_H(\phi) = \phi(H).
\]
By the fact that $\phi_i \in \text{Sp}(L_T)$ is associated to $\lambda_i \in \sigma(T)$ through $\phi_i(T) = \lambda_i$, where $i = 0, 1, 2, \ldots$ with $\lambda_0 = 0$, we have that
\[
G_H : \sigma(T) \to K; \ \lambda_i \to G_H(\lambda_i) = \sum_{n=0}^{\infty} \alpha_n \lambda_i^n.
\]
In particular, if $H = T$,
\[
G_T : \sigma(T) \to K; \ \lambda_i \to G_T(\lambda_i) = \lambda_i.
\]
Let us denote $G_T$ by $f_T$.

From this, we get
\[
G : L_T \to C(\sigma(T)); \ H \to G_H
\]

**Proposition 1.** $G$ is an isometric isomorphism algebra.

**Proof.** Clearly, $G$ is an algebra homomorphism. Since $L_T$ satisfies the condition given by Cor. 6.16, [9], we have
\[
\|G_H\|_{\infty} = \|H\|_{sp} = \|H\|.
\]
Thus, it is enough to prove that $G$ is surjective. By the fact that $G$ is an algebra homomorphism and the image of $T$ by $G$ is $f_T$, the collection $\{1, f_T, f_T^2, \ldots\}$ is the image of $\{I, T, T^2, \ldots\}$.
Now, since $\sigma(T)$ is compact, Kaplanski (Th. 5.28, [9]) guarantees that $\{1, f_T, f_T^2, \ldots\}$
is dense in $C(\sigma(T))$. Thus, if $f \in C(\sigma(T))$, then there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$ in $[\{1, ft, f^2, \ldots\}]$ such that $f = \lim_{n \to \infty} g_n$. Now, for each $n \in \mathbb{N}$, there exists $H_n \in \mathcal{L}_T$ such that

$$G_{H_n} = g_n.$$  

By the fact that $G$ is an isometry, the sequence $(H_n)$ is a Cauchy sequence in $\mathcal{L}_T$.

Let us denote by $H = \lim_{n \to \infty} H_n$. Since $G$ is continuous, we have that

$$G_H = \lim_{n \to \infty} G_{H_n} = \lim_{n \to \infty} g_n = f.$$

$\square$

4. An integral.

By the previous section, there exists an algebra isometric isomorphism $\Psi = G^{-1} : C(\sigma(T)) \to \mathcal{L}_T$. Let us denote by $\Omega(\sigma(T))$ the family of all clopen subsets of $\sigma(T)$.

For a $C \subset \sigma(T)$, $\eta_C$ denotes the characteristic function of $C$. If $C_1, C_2 \subset \sigma(T)$, then

$$\eta_{C_1} \cdot \eta_{C_2} = \eta_{C_1 \cap C_2}; \quad \eta_C^2 = \eta_C$$

$$\eta_{C_1} + \eta_{C_2} = \eta_{C_1 \cup C_2}, \quad \text{if } C_1 \cap C_2 = \emptyset.$$

Of course, $\eta_C$ is continuous if and only if $C \in \Omega(\sigma(T))$.

Now, since $\Psi$ is a homomorphism of algebras, we have

$$\Psi(\eta_C) = \Psi(\eta_C^2) = \Psi(\eta_C)^2.$$

In other words, $\Psi(\eta_C)$ is a projection, even more, if $C \in \Omega(\sigma(T)) \setminus \{\emptyset\}$, then $\Psi(\eta_C)$ is a non-null projection in $\mathcal{L}_T$.

On the other hand, we know that the linear hull of $\{\eta_C : C \in \Omega(\sigma(T))\}$ is dense in $C(\sigma(T))$; hence, for a given $\epsilon > 0$ and for $f \in C(\sigma(T))$, there exists a finite clopen partition $\{C_k : k = 1, \ldots, n\}$ of $\sigma(T)$ and a finite collection of scalars $\{\alpha_k : k = 1, \ldots, n\}$ such that

$$\left\| f - \sum_{k=1}^n \alpha_k \eta_{C_k} \right\|_\infty = \sup_{x \in \sigma(T)} \left| f(x) - \sum_{k=1}^n \alpha_k \eta_{C_k}(x) \right| < \epsilon \quad (4.1)$$

Without lost of generality, we can assume that, for each $k = 1, \ldots, n$, there exists $x_k \in \sigma(T)$ such that

$$\left\| f - \sum_{k=1}^n \alpha_k \eta_{C_k} \right\|_\infty = \sup_{x \in \sigma(T)} \left| f(x) - \sum_{k=1}^n f(x_k) \eta_{C_k}(x) \right| < \epsilon$$

Using the isometry of $\Psi$, we have

$$\left\| \Psi(f) - \sum_{k=1}^n f(x_k) \Psi(\eta_{C_k}) \right\| < \epsilon$$

If we denote by $P_{C_k}$ the corresponding projection $\Psi(\eta_{C_k})$, then

$$\left\| \Psi(f) - \sum_{k=1}^n f(x_k) P_{C_k} \right\| < \epsilon$$

This also shows that the space

$$[\{P \in \mathcal{L}_T : P^2 = P\}]$$
is dense in $L_T$.

Let us consider the following set-function:

$$m_T : \Omega (\sigma (T)) \to L_T; \quad C \to m (C) = P_C.$$ 

Thus, $m_T$ satisfies:

1. $m_T (\varnothing) = 0$
2. If $\{C_k : k = 1, \ldots, n\} \subset \Omega (\sigma (T))$ such that $C_h \cap C_k = \varnothing$ for $h \neq k$, then

$$m_T \left( \bigcup_{k=1}^n C_k \right) = \sum_{k=1}^n P_{C_k}.$$ 

In other words, $m_T$ is a vector-valued finite additive projection measures which is norm-bounded.

Take a $C \in \Omega (\sigma (T))$, $C \neq \varnothing$ and denote by $\mathcal{D}_C$ the collection of all $\alpha = \{C_1, C_2, \ldots, C_n ; x_1, x_2, \ldots, x_n\}$, where $\{C_k : k = 1, \ldots, n\}$ is a clopen partition of $C$ and $x_k \in C_k$. We define an order by $\alpha_1 \geq \alpha_2$ if and only if the clopen partition of $C$ in $\alpha_1$ is a refinement of the clopen partition of $C$ in $\alpha_2$. Thus, $\mathcal{D}_C$ is a directed set.

Now, if $f \in C (\sigma (T))$ and $\alpha = \{C_1, C_2, \ldots, C_n; x_1, x_2, \ldots, x_n\} \in \mathcal{D}_C$, then we define

$$\omega_\alpha (f, m_T, C) = \sum_{k=1}^n f (x_k) m_T (C_k) = \sum_{k=1}^n f (x_k) P_{C_k}.$$ 

For each $f \in C (\sigma (T))$ and $C \in \Omega (\sigma (T))$, the continuous function $f \eta_C$ can be approximated by a net in $C (\sigma (T))$ of type

$$\left\{ \sum_{k=1}^{(n(\alpha))} f (x_k) \eta_{C_k} \right\}_{\alpha \in \mathcal{D}_C}$$

By the isometry $\Psi$,

$$\lim_{\alpha \in \mathcal{D}_C} \omega_\alpha (f, m_T, C) = \Psi (f \eta_C)$$

Therefore, the operator $\Psi (f \eta_C)$ can be interpreted as an integral which we will denote by

$$\Psi (f \eta_C) = \int_{\sigma (T)} f \eta_C dm_T = \int_C f dm_T = \lim_{\alpha \in \mathcal{D}_C} \omega_\alpha (f, m_T, C)$$

Some particular cases are $T = \Psi (f_T)$ and $I = \Psi (1)$

$$T = \Psi (f_T) = \int_{\sigma (T)} f_T dm_T; \quad I = \Psi (1) = \int_{\sigma (T)} dm_T.$$ 

5. **Finite range self-adjoint operators**

Let $Y = \{y_1, y_2, \ldots, y_n\}$ be a fixed finite orthonormal set of $c_0$, that is, $\|y_i\| = 1$ \forall $k = 1, \ldots, n$ and $\langle y_i, y_j \rangle = 0$ \forall $i, j = 1, \ldots, n$ with $i \neq j$, and $\{\lambda_1, \ldots, \lambda_n\}$ a finite subset of $\mathbb{K} \setminus \{0\}$ such that $\lambda_i \neq \lambda_j$ \forall $i, j = 1, 2, \ldots, n$ with $i \neq j$.

We define the operator $T : c_0 \to c_0$ by

$$T(x) = \sum_{i=1}^n \lambda_i P_i (x)$$
Remark 4. From Corollary 2, \( P_i = \frac{\langle \cdot, y_i \rangle}{(y_i, y_i)} y_i \) \( \forall i = 1, \ldots, n \). Of course, \( T \) is compact and self-adjoint operator with finite range and \( \| T \| = \max \{ |\lambda_1|, \ldots, |\lambda_n| \} \). Since \( R(T) \subset c_0 \), 0 is an eigenvalue of \( T \). In that case, \( \sigma(T) = \{ \lambda_0, \lambda_1, \ldots, \lambda_n \} \) where \( \lambda_0 = 0 \).

Theorem 5.

\[ \mathcal{L}_T = \{ [I, P_1, \ldots, P_n] \} \]

Proof. By the fact that \( P_i \circ P_j = P_j \circ P_i = \theta \) \( \forall i, j = 1, \ldots, n \) with \( i \neq j \), we have that \( T^i(\cdot) = \sum_{j=1}^n \lambda_j^i P_j(\cdot) \) which implies that \( \mathcal{L}_T \subseteq \{ [I, P_1, \ldots, P_n] \} \).

Using the Van der Monde method to calculate determinants whose columns are geometric progressions, we have

\[
\begin{vmatrix}
\lambda_1 & \lambda_2 & \ldots & \lambda_n \\
\lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^n & \lambda_2^n & \ldots & \lambda_n^n
\end{vmatrix} = \left( \prod_{i=1}^n \lambda_i \right) \left( \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \right).
\]

Considering the equations system

\[
\begin{align*}
T &= \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n \\
T^2 &= \lambda_1^2 P_1 + \lambda_2^2 P_2 + \ldots + \lambda_n^2 P_n \\
&\quad \ldots \\
T^n &= \lambda_1^n P_1 + \lambda_2^n P_2 + \ldots + \lambda_n^n P_n
\end{align*}
\]

and solving it, we obtain

\[
P_k = \prod_{i=0, i \neq k}^n \left( \frac{\lambda_i I - T}{\lambda_i - \lambda_k} \right) k = 1, 2, \ldots, n.
\]

Therefore, \( P_1, \ldots, P_n \in \mathcal{L}_T \) and, as a consequences, \( \mathcal{L}_T = \{ [I, P_1, P_2, \ldots, P_n] \} \). \( \square \)

Remark 4. From Corollary 2, \( Sp(\mathcal{L}_T) \) can be identified with \( \sigma(T) \). Since \( Sp(\mathcal{L}_T) \) is finite, the induced topology by product topology in \( \mathbb{K}^{\mathcal{L}_T} \) is just the discrete topology. By the same arguments, the induced topology in \( \sigma(T) \) by the ultrametric space \( \mathbb{K} \) is also the discrete topology.

On the other hand, using the Gelfand Transformation, the Gelfand Transform of \( P_k \) is the following:

\[
G_{P_k}(\phi_j) = \phi_j(P_k) = \phi_j \left( \frac{\prod_{i=0, i \neq k}^n (\lambda_i I - T)}{\lambda_i - \lambda_k} \right)
\]

\[
= \prod_{i=0, i \neq k}^n \left( \frac{\phi_j(\lambda_i I) - \phi_j(T)}{\lambda_i - \lambda_k} \right) = \prod_{i=0, i \neq k}^n \left( \frac{\lambda_i - \lambda_k}{\lambda_i - \lambda_k} \right) = \left\{ \begin{array}{ll} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{array} \right. = \eta(\lambda_k)(\lambda_j)
\]

Open Problem: Under the conditions that \( T \) is a self-adjoint and finite-dimensional range operator, the corresponding projections \( P_i \) belongs to \( \mathcal{L}_T \). Meanwhile, if \( T \) is infinite-dimensional range operator, we don’t have information about the projections \( P_i \). Our conjecture is that these projections are not to \( \mathcal{L}_T \).
6. \((\mathcal{L}_T, \|\cdot\|)\) is a Free Banach space

From the previous sections, we can distinguished two non-archimedean norms in \(\mathcal{L}_T\); the operator norm, denoted by \(\|\cdot\|\), and the spectral norm, denoted by \(\|\cdot\|_{sp}\).

We also proved that both norms coincide on \(\mathcal{L}_T\).

Now, we know that if \(H = \sum_{n=0}^{\infty} \alpha_n T^n \in \mathcal{L}_T\), then
\[
0 = \lim_{n \to \infty} \|\alpha_n T^n\| = \lim_{n \to \infty} |\alpha_n| \|T\|^n
\]
\[
= \lim_{n \to \infty} |\alpha_n| \|\lambda\|^n = \lim_{n \to \infty} \|\alpha_n \lambda^n\|.
\]

As a consequence of this fact,
\[
\mu = (\mu_i)_{i \in \mathbb{N}} = \sum_{n=1}^{\infty} \alpha_n \lambda^n \in c_0.
\]

and then the operator
\[
\sum_{n=1}^{\infty} \alpha_n T^n = T \mu = \sum_{i=1}^{\infty} \mu_i P_i
\]
is a self-adjoint and compact.

According to this, we can define the function
\[
\|\cdot\|_{fb} : \mathcal{L}_T \to \mathbb{R}
\]
by
\[
\left\| \sum_{n=0}^{\infty} \alpha_n T^n \right\|_{fb} = \max \\{\|\alpha_n T^n\| : n \in \mathbb{N} \cup \{0\}\}
\]
and can easily prove that this function is another non–archimedean norm in \(\mathcal{L}_T\).

We claim that \(\|\cdot\|_{fb}\) and the operators norm are equal. In fact, since
\[
\left\| \sum_{n=0}^{\infty} \alpha_n T^n \right\|_{fb} = \|\alpha_0 I + T \mu\| = \max \{\|\alpha_0\|, \|T \mu\|\}
\]
\[
\leq \max \\{\|\alpha_n\| \|T^n\| : n \in \mathbb{N}\} = \left\| \sum_{n=0}^{\infty} \alpha_n T^n \right\|_{fb},
\]
we only need to prove the other inequality. First of all, note that
\[
\mathcal{L}_T = \left\{ \sum_{n=0}^{\infty} \alpha_n T^n : \lim_{n \to \infty} |\alpha_n| \rho^n = 0, \quad \rho = \|T\| \right\},
\]
therefore, \(\mathcal{L}_T\) can be considered as a space of formal power series, denoted in [9] by \(\mathbb{K}_\rho \{X\}\), where its norm is given by \(\|\sum_{n=0}^{\infty} \alpha_n X^n\| = \sup \{|\alpha_n| \rho^n : n \in \mathbb{N} \cup \{0\}\}\).

In general, the space of formal power series with its norm is a non-archimedean Banach space since it is isometrically isomorphism to \(c_0(N:s)\), where \(s : \mathbb{N} \cup \{0\} \to (0, \infty); \quad s(n) = \rho^n\). Thus,
\[
\left(\mathcal{L}_T, \|\cdot\|_{fb}\right) = \mathbb{K}_{\|T\|} \{T\} \cong c_0(N \cup \{0\} : s)
\]
and therefore, is a Free Banach space.

On the other hand, since the identity operator
\[
I : \left(\mathcal{L}_T, \|\cdot\|_{fb}\right) \to (\mathcal{L}_T, \|\cdot\|)
\]
is continuous and bijective, we have that \( I \) is a homeomorphism (Cor. 3.6, [9]). This implies that both norms are equivalent, that is, there exists \( M > 0 \) such that

\[
\|\cdot\| \leq \|\cdot\|_{fb} \leq M \|\cdot\|.
\]

We have proved previously that \( \|\cdot\| \) is power multiplicative. From the general theory (see [9]), we know that \( \mathcal{L}_T = \mathbb{K}\|T\| \{T\} \) satisfies the property

\[
\|ST\|_{fb} = \|S\|_{fb} \|T\|_{fb};
\]

hence, in particular, \( \|\cdot\|_{fb} \) is power multiplicative. Thus, for any \( n \in \mathbb{N} \), we get

\[
\|H^n\|_{fb} \leq M \|H^n\| \Rightarrow \|H\|^n \leq \|H\|^n_{fb} \leq M \|H\|^n \Rightarrow \|H\|_{fb} \leq M^{1/n} \|H\|
\]

Now, if \( n \) tends to infinity, then we obtain that

\[
\|H\| \leq \|H\|_{fb} \leq \|H\|
\]

or, equivalently, these norms are equal.

**Theorem 6.** \( \mathcal{L}_T \) provided with the operators norm \( \|\cdot\| \) is a Free Banach space.

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