Extension of Summation Formulas involving Stirling series

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Abstract

This paper presents a family of rapidly convergent summation formulas for various finite sums of the form $\sum_{k=0}^{[x]} f(k)$, where $x$ is a positive real number.

1 Introduction

In this paper we will use the Euler-Maclaurin summation formula [3, 5] to obtain rapidly convergent series expansions for finite sums involving Stirling series [1]. Our key tool will be the so-called Weniger transformation [1].

For example, one of our summation formulas for the sum $\sum_{k=0}^{[x]} \sqrt{k}$, where $x \in \mathbb{R}^+$ is

$$\sum_{k=0}^{[x]} \sqrt{k} = \frac{2}{3} x^2 - \frac{1}{4\pi} \left( \frac{3}{2} \right) - \sqrt{x} B_1(\{x\}) + \sqrt{x} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} (-1)^{\frac{(2l-3)!!}{2(l+1)!}} S_k^{(1)}(l) B_{l+1}(\{x\})}{(x+1)(x+2) \cdots (x+k)}$$

$$= \frac{2}{3} x^2 - \frac{1}{4\pi} \left( \frac{3}{2} \right) + \left( \frac{1}{2} - \{x\} \right) \sqrt{x} + \left( \frac{1}{4} \{x\}^2 - \frac{1}{4} \{x\} + \frac{1}{24} \right) \sqrt{x} \frac{1}{(x+1)}$$

$$+ \left( \frac{1}{27} \{x\}^3 + \frac{3}{16} \{x\}^2 - \frac{11}{48} \{x\} + \frac{1}{27} \right) \sqrt{x} \frac{1}{(x+1)(x+2)} + \left( \frac{1}{64} \{x\}^4 + \frac{3}{32} \{x\}^3 - \frac{1}{64} \{x\} \right) \sqrt{x} \frac{1}{(x+1)(x+2)(x+3)}$$

$$+ \left( \frac{1}{128} \{x\}^5 + \frac{10}{2048} \{x\}^4 + \frac{100}{384} \{x\}^3 + \frac{29}{32} \{x\}^2 - \frac{977}{768} \{x\} + \frac{79}{320} \right) \sqrt{x} \frac{1}{(x+1)(x+2)(x+3)(x+4)} + \ldots,$$

where the $B_l(x)$‘s are the Bernoulli polynomials and $S_k^{(1)}(l)$ denotes the Stirling numbers of the first kind.

Most of the other formulas in this article have a similar shape.

Setting in the above formula for $\sum_{k=0}^{[x]} \sqrt{k}$ the variable $x := n \in \mathbb{N}$, we obtain

$$\sum_{k=0}^{n} \sqrt{k} = \frac{2}{3} n^2 + \frac{1}{2} \sqrt{n} - \frac{1}{4\pi} \left( \frac{3}{2} \right) + \sqrt{n} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} (-1)^{\frac{(2l-3)!!}{2(l+1)!}} S_k^{(1)}(l) B_{l+1}(\{x\})}{(n+1)(n+2) \cdots (n+k)}$$

$$= \frac{2}{3} n^2 + \frac{1}{2} \sqrt{n} - \frac{1}{4\pi} \left( \frac{3}{2} \right) + \frac{\sqrt{n}}{24(n+1)} + \frac{\sqrt{n}}{24(n+1)(n+2)} + \frac{53\sqrt{n}}{640(n+1)(n+2)(n+3)}$$

$$+ \frac{320(n+1)(n+2)(n+3)(n+4)}{79\sqrt{n}} + \ldots,$$

which is the corresponding formula given in our previous paper [8].
2 Definitions

As usual, we denote the floor of $x$ by $\lfloor x \rfloor$ and the fractional part of $x$ by $\{x\}$.

**Definition 1.** (Pochhammer symbol)[1]

We define the Pochhammer symbol (or rising factorial function) $(x)_k$ by

$$(x)_k := x(x+1)(x+2)(x+3)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)},$$

where $\Gamma(x)$ is the gamma function defined by

$$\Gamma(x) := \int_0^\infty e^{-t}t^{x-1}dt.$$

**Definition 2.** (Stirling numbers of the first kind)[1]

Let $k, l \in \mathbb{N}_0$ be two non-negative integers such that $k \geq l \geq 0$. We define the Stirling numbers of the first kind $S_k^{(1)}(l)$ as the connecting coefficients in the identity

$$(x)_k = (-1)^k \sum_{l=0}^{k} (-1)^l S_k^{(1)}(l)x^l,$$

where $(x)_k$ is the rising factorial function.

**Definition 3.** (Bernoulli numbers)[2]

We define the $k$-th Bernoulli number $B_k$ as the $k$-th coefficient in the generating function relation

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k \quad \forall x \in \mathbb{C} \text{ with } |x| < 2\pi.$$

**Definition 4.** (Euler numbers)[3]

We define the sequence of Euler numbers $\{E_k\}_{k=0}^{\infty}$ by the generating function identity

$$\frac{2e^x}{e^{2x} + 1} = \sum_{k=0}^{\infty} \frac{E_k}{k!} x^k.$$

**Definition 5.** (Bernoulli polynomials)[2, 3]

We define for $n \in \mathbb{N}_0$ the $n$-th Bernoulli polynomial $B_n(x)$ via the following exponential generating function as

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n \quad \forall t \in \mathbb{C} \text{ with } |t| < 2\pi.$$

Using this expression, we see that the value of the $n$-th Bernoulli polynomial $B_n(x)$ at the point $x = 0$ is

$$B_n(0) = B_n,$$

which is the $n$-th Bernoulli number.
Definition 6. (Euler polynomials)[3]
We define for \( n \in \mathbb{N}_0 \) the \( n \)-th Euler polynomial \( E_n(x) \) via the following exponential generating function as
\[
\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n.
\]
Moreover, we have that \([4]\)
\[
E_n(0) = -2(2^{n+1} - 1) \frac{B_{n+1}}{n+1} \quad \forall n \in \mathbb{N}_0.
\]

3 Extended Summation Formulas involving Stirling Series

In this section we will prove our summation formulas for various finite sums of the form \( \sum_{k=1}^{\lfloor x \rfloor} f(k) \). For this, we need the following

Lemma 7. (Extended Euler-Maclaurin summation formula)[3, 5]
Let \( f \) be an analytic function. Then for all \( x \in \mathbb{R}^+ \), we have that
\[
\sum_{k=1}^{\lfloor x \rfloor} f(k) = \int_1^x f(t) dt + \sum_{k=1}^{m} (-1)^k \frac{B_k(\{x\})}{k!} f^{(k-1)}(x) - \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(1)
\]
\[
+ \frac{(-1)^{m+1}}{m!} \int_1^x B_m(\{t\}) f^{(m)}(t) dt,
\]
where \( B_m(x) \) is the \( m \)-th Bernoulli polynomial and \( \{x\} \) denotes the fractional part of \( x \). Therefore, for many functions \( f \) we have the asymptotic expansion
\[
\sum_{k=1}^{\lfloor x \rfloor} f(k) \sim \int_1^x f(t) dt + C + \sum_{k=1}^{\infty} (-1)^k \frac{B_k(\{x\})}{k!} f^{(k-1)}(x) \quad \text{as} \ x \to \infty,
\]
for some constant \( C \in \mathbb{C} \)
and

Lemma 8. (Extended Boole summation formula)[3]
Let \( f \) be an analytic function. Then for all \( x \in \mathbb{R}^+ \), we have that
\[
\sum_{k=1}^{\lfloor x \rfloor} (-1)^{k+1} f(k) = \frac{(-1)^{x-\{x\}}}{2} \sum_{k=0}^{m} (-1)^{k+1} \frac{E_k(\{x\})}{k!} f^{(k)}(x) - \sum_{k=0}^{m} \frac{(2^{k+1} - 1)B_{k+1}}{(k+1)!} f^{(k)}(1)
\]
\[
+ \frac{(-1)^m}{2m!} \int_1^x (-1)^{t-\{t\}} E_m(\{t\}) f^{(m+1)}(t) dt,
\]
where \(E_m(x)\) is the \(m\)-th Euler polynomial and \(\{x\}\) denotes the fractional part of \(x\).

Therefore, for many functions \(f\) we have the asymptotic expansion

\[
\sum_{k=1}^{\lfloor x \rfloor} (-1)^{k+1} f(k) \sim C + \frac{(-1)^{x-\lfloor x \rfloor}}{2} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{E_k(\{x\})}{k!} f^{(k)}(x) \quad \text{as} \quad x \to \infty,
\]

for some constant \(C \in \mathbb{C}\).

From the above lemma we get by setting \(x := n \in \mathbb{N}\) the following

**Corollary 9. (Boole Summation Formula)[3]**

Let \(f\) be an analytic function. Then for all \(n \in \mathbb{N}\), we have

\[
\sum_{k=1}^{n} (-1)^{k+1} f(k) = (-1)^n \sum_{k=0}^{m} (-1)^k \frac{(2k+1) - 1}{(k+1)!} f^{(k)}(n) - \sum_{k=0}^{m} \frac{(2k+1) - 1}{(k+1)!} f^{(k)}(1)
\]

\[
+ \frac{(-1)^m}{2m!} \int_1^n (-1)^{t-(t)} E_m(\{t\}) f^{(m+1)}(t) dt,
\]

and the following asymptotic expansion

\[
\sum_{k=1}^{n} (-1)^{k+1} f(k) \sim C + (-1)^n \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1) - 1}{(k+1)!} f^{(k)}(n) \quad \text{as} \quad n \to \infty,
\]

for some constant \(C \in \mathbb{C}\).

As well as the next key result found by J. Weniger:

**Lemma 10. (Generalized Weniger transformation)[1]**

For every inverse power series \(\sum_{k=1}^{\infty} \frac{a_k(x)}{x^{k+1}}\), where \(a_k(x)\) is any function in \(x\), the following transformation formula holds

\[
\sum_{k=1}^{\infty} \frac{a_k(x)}{x^{k+1}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{(x)_{k+1}} \sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) a_l(x)
\]

\[
= \sum_{k=1}^{\infty} \frac{(-1)^k}{x(x+1)(x+2)\cdots(x+k)} \sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) a_l(x).
\]

Now, we prove as an example the following

**Theorem 11. (Extended summation formulas for the harmonic series)**

For every positive real number \(x \in \mathbb{R}^+\), we have that

\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma - \frac{B_1(\{x\})}{x} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2)\cdots(x+k)}.
\]
We also have that

\[
\sum_{k=1}^{[x]} \frac{1}{k} = \log(x) + \gamma + \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{l=1}^{k} \frac{(-1)^l S_k^{(1)}(l) B_l(\{x\})}{x(x+1)(x+2) \cdots (x+k)}.
\]

**Proof.** Applying the extended Euler-Maclaurin summation formula to the function \( f(x) := \frac{1}{x} \), we get that

\[
\sum_{k=1}^{[x]} \frac{1}{k} \sim \log(x) + \gamma - \sum_{k=1}^{\infty} \frac{B_k(\{x\})}{k x^k}.
\]

Applying now the generalized Weniger transformation to the equivalent series

\[
\sum_{k=1}^{[x]} \frac{1}{k} \sim \log(x) + \gamma - \sum_{k=2}^{\infty} \frac{B_k(\{x\})}{k x^k} - \sum_{k=1}^{\infty} \frac{B_{k+1}(\{x\})}{(k+1) x^{k+1}},
\]

we get the first claimed formula. To obtain the second expression, we apply the generalized Weniger transformation to the identity

\[
\sum_{k=1}^{[x]} \frac{1}{k} \sim \log(x) + \gamma - x \sum_{k=1}^{\infty} \frac{B_k(\{x\})}{k x^{k+1}}.
\]

\(\square\)

At this point, we want to give an overview on summation formulas obtained with this method:

1.) **Extended summation formulas for the harmonic series:**

For every positive real number \( x \in \mathbb{R}^+ \), we have that

\[
\sum_{k=1}^{[x]} \frac{1}{k} = \log(x) + \gamma - \frac{B_1(\{x\})}{x} + \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{l=1}^{k} \frac{(-1)^l S_k^{(1)}(l) B_l(\{x\})}{x(x+1)(x+2) \cdots (x+k)}
\]

and

\[
\sum_{k=1}^{[x]} \frac{1}{k} = \log(x) + \gamma + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{B_{k+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}.
\]
2.) Extended summation formulas for the partial sums of $\zeta(2)$:
For every positive real number $x \in \mathbb{R}^+$, we have that

$$
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2} = \zeta(2) - \frac{1}{x} - \frac{1}{2x^2} - \frac{B_1(\{x\})}{x^3} + \frac{1}{x} \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{l=1}^{k} \frac{(-1)^l S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}
$$

and

$$
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2} = \zeta(2) - \frac{1}{x} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}.
$$

3.) Extended summation formulas for the partial sums of $\zeta(3)$:
For every positive real number $x \in \mathbb{R}^+$, we have that

$$
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^3} = \zeta(3) - \frac{1}{2x^2} - \frac{B_1(\{x\})}{x^3} + \frac{1}{2x^2} \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{l=1}^{k} \frac{(-1)^l (l+2) S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}
$$

and

$$
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^3} = \zeta(3) - \frac{1}{2x^2} + \frac{1}{2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l (l+1) S_k^{(1)}(l) B_{l+1}(\{x\})}{x^2(x+1)(x+2) \cdots (x+k)}.
$$

4.) Extended summation formulas for the sum of the square roots:
For every positive real number $x \in \mathbb{R}^+$, we have that

$$
\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4\pi} \zeta \left( \frac{3}{2} \right) - \sqrt{x} B_1(\{x\}) + \sqrt{x} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l (2l-3)!! S_k^{(1)}(l) B_{l+1}(\{x\})}{(x+1)(x+2) \cdots (x+k)}
$$

$$
\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4\pi} \zeta \left( \frac{3}{2} \right) - \sqrt{x} B_1(\{x\}) + \frac{B_2(\{x\})}{4\sqrt{x}} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l (2l-1)!! S_k^{(1)}(l) B_{l+2}(\{x\})}{\sqrt{x}(x+1)(x+2) \cdots (x+k)}
$$

and

$$
\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{\frac{3}{2}} - \frac{1}{4\pi} \zeta \left( \frac{3}{2} \right) + \sqrt{x} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} (-1)^l (2l-5)!! S_k^{(1)}(l) B_{l+1}(\{x\})}{(x+1)(x+2) \cdots (x+k)}.
$$
5.) Extended summation formulas for the partial sums of $\zeta(-3/2)$:
For every positive real number $x \in \mathbb{R}^+$, we have that
\[
\sum_{k=0}^{\lfloor x \rfloor} k^{3/2} = \frac{2}{5} \int_{0}^{x} \zeta\left(\frac{5}{2}\right) - x^{3/2} B_1(\{x\}) + \frac{3}{2} x^{3/2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \zeta(2k+1)}{(x+1)(x+2) \cdots (x+k)} \left(\frac{1}{2^{2k+1} (k+1)} S_k^1(l) B_{l+1}(\{x\})\right),
\]
\[
\sum_{k=0}^{\lfloor x \rfloor} k^{3/2} = \frac{2}{5} \int_{0}^{x} \zeta\left(\frac{5}{2}\right) - x^{3/2} B_1(\{x\}) + \frac{3}{4} \sqrt{x} B_2(\{x\}) - \frac{3}{4} \int_{0}^{x} \zeta\left(\frac{5}{2}\right) - x^{3/2} B_1(\{x\}) + \frac{3}{2} \sqrt{x} \sum_{k=1}^{\infty} (-1)^{k+1} \zeta(2k+1) S_k^1(l) B_{l+1}(\{x\}) \left(\frac{1}{2^{2k+1} (k+1)} S_k^1(l) B_{l+1}(\{x\})\right),
\]
and
\[
\sum_{k=0}^{\lfloor x \rfloor} k^{3/2} = \frac{2}{5} \int_{0}^{x} \zeta\left(\frac{5}{2}\right) - x^{3/2} B_1(\{x\}) + \frac{3}{2} \sqrt{x} \sum_{k=1}^{\infty} (-1)^{k+1} \zeta(2k+1) S_k^1(l) B_{l+1}(\{x\}) \left(\frac{1}{2^{2k+1} (k+1)} S_k^1(l) B_{l+1}(\{x\})\right).
\]

6.) Extended summation formulas for the partial sums of $\zeta(-5/2)$:
For every positive real number $x \in \mathbb{R}^+$, we have that
\[
\sum_{k=0}^{\lfloor x \rfloor} k^{5/2} = \frac{2}{7} \int_{0}^{x} \zeta\left(\frac{7}{2}\right) - x^{5/2} B_1(\{x\}) + \frac{15}{4} \int_{0}^{x} \zeta\left(\frac{7}{2}\right) - x^{5/2} B_1(\{x\}) + \frac{15}{4} \sqrt{x} \sum_{k=1}^{\infty} (-1)^{k+1} \zeta(2k+1) S_k^1(l) B_{l+1}(\{x\}) \left(\frac{1}{2^{2k+1} (k+1)} S_k^1(l) B_{l+1}(\{x\})\right),
\]
\[
\sum_{k=0}^{\lfloor x \rfloor} k^{5/2} = \frac{2}{7} \int_{0}^{x} \zeta\left(\frac{7}{2}\right) - x^{5/2} B_1(\{x\}) + \frac{5}{4} x^{5/2} B_2(\{x\}) - \frac{5}{8} \sqrt{x} B_3(\{x\}) + \frac{5 B_4(\{x\})}{64 \sqrt{x}}
\]
\[
+ \frac{15}{4} \sum_{k=1}^{\infty} (-1)^k \zeta(2k+1) S_k^1(l) B_{l+4}(\{x\}) \left(\frac{1}{2^{2k+1} (k+4)} S_k^1(l) B_{l+4}(\{x\})\right),
\]
and
\[
\sum_{k=0}^{\lfloor x \rfloor} k^{5/2} = \frac{2}{7} \int_{0}^{x} \zeta\left(\frac{7}{2}\right) - x^{5/2} B_1(\{x\}) + \frac{15}{4} \int_{0}^{x} \zeta\left(\frac{7}{2}\right) - x^{5/2} B_1(\{x\}) + \frac{15}{4} x^{5/2} \sum_{k=1}^{\infty} (-1)^{k+1} \zeta(2k+1) S_k^1(l) B_{l+1}(\{x\}) \left(\frac{1}{2^{2k+1} (k+1)} S_k^1(l) B_{l+1}(\{x\})\right).
\]
7.) Extended summation formulas for the sum of the inverse square roots:
For every positive real number \( x \in \mathbb{R}^+ \), we have that
\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{\sqrt{k}} = 2\sqrt{x} + \zeta \left( \frac{1}{2} \right) - \frac{B_1(\{x\})}{\sqrt{x}} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(1)^{(2l-1)!!}}{(2l-1)!!} S_k^{(1)}(l) B_{l+1}(\{x\})}{\sqrt{x}(x+1)(x+2) \cdots (x+k)}
\]
and
\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{\sqrt{k}} = 2\sqrt{x} + \zeta \left( \frac{1}{2} \right) + \sqrt{x} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(1)^{(2l-3)!!}}{(2l-3)!!} S_k^{(1)}(l) B_{l+1}(\{x\})}{(x+1)(x+2) \cdots (x+k)}.
\]

8.) Extended summation formulas for the partial sums of \( \zeta(3/2) \):
For every positive real number \( x \in \mathbb{R}^+ \), we have that
\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k \sqrt{k}} = \zeta \left( \frac{3}{2} \right) - \frac{2}{\sqrt{x}} - \frac{B_1(\{x\})}{x \sqrt{x}} + \frac{2}{\sqrt{x}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(1)^{(2l+1)!!}}{(2l+1)!!} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}
\]
and
\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k \sqrt{k}} = \zeta \left( \frac{3}{2} \right) - \frac{2}{\sqrt{x}} + 2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(1)^{(2l-1)!!}}{(2l-1)!!} S_k^{(1)}(l) B_{l+1}(\{x\})}{\sqrt{x}(x+1)(x+2) \cdots (x+k)}.
\]

9.) Extended summation formulas for the partial sums of \( \zeta(5/2) \):
For every positive real number \( x \in \mathbb{R}^+ \), we have that
\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2 \sqrt{k}} = \zeta \left( \frac{5}{2} \right) - \frac{2}{3x^{3/2}} - \frac{B_1(\{x\})}{x^2 \sqrt{x}} + \frac{4}{3x^{3/2}} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(1)^{(2l+3)!!}}{(2l+3)!!} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}
\]
and
\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k^2 \sqrt{k}} = \zeta \left( \frac{5}{2} \right) - \frac{2}{3x^{3/2}} + 4 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(1)^{(2l+1)!!}}{(2l+1)!!} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}.
\]

10.) Extended Generalized Faulhaber Formulas:
For every positive real number \( x \in \mathbb{R}^+ \) and for every complex number \( m \in \mathbb{C} \setminus \{-1\} \), we have that
\[
\sum_{k=1}^{\lfloor x \rfloor} k^m = \frac{1}{m+1} x^{m+1} + \zeta(-m) + \frac{x^{m+1}}{m+1} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sum_{l=1}^{k} \frac{(m+1)!!}{(m+1)!!} S_k^{(1)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}
\]


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and
\[ \sum_{k=1}^{[x]} k^m = \frac{1}{m+1} x^{m+1} + \zeta(-m) - x^m B_1(\{x\}) + \frac{x^m}{m+1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sum_{l=1}^{k} \binom{m+1}{l} S^{(1)}_{k-l} B_{l-1}(\{x\})}{(x+1)(x+2) \cdots (x+k)}, \]

and
\[ \sum_{k=1}^{[x]} k^m = \frac{1}{m+1} x^{m+1} + \zeta(-m) + \frac{1}{m+1} \sum_{k=1}^{[m+1]} (-1)^k \binom{m+1}{k} B_k(\{x\}) x^{m-k+1} \]
\[ + (-1)^{[m+1]} \frac{x^{m-[m+1]+2}}{m+1} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \binom{m+1}{l} S^{(1)}_{k-l} B_{l-1+[m+1]-1}(\{x\})}{(x+1)(x+2) \cdots (x+k)}. \]

11.) Generalized Faulhaber Formulas:
For every complex number \( m \in \mathbb{C} \setminus \{-1\} \) and every natural number \( n \in \mathbb{N} \), we have that
\[ \sum_{k=1}^{n} k^m = \frac{1}{m+1} n^{m+1} + \zeta(-m) + \frac{n^{m+1}}{m+1} \sum_{k=1}^{\infty} (-1)^k \sum_{l=1}^{k} \binom{m+1}{l} B_{l-1} S^{(1)}_k(l) \]
\[ \frac{1}{(n+1)(n+2) \cdots (n+k)}, \]

and
\[ \sum_{k=1}^{n} k^m = \frac{1}{m+1} n^{m+1} + \zeta(-m) + \frac{1}{2} n^m + \frac{n^m}{m+1} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \binom{m+1}{l} B_{l-1} S^{(1)}_k(l)}{(n+1)(n+2) \cdots (n+k)} \]
and
\[ \sum_{k=1}^{n} k^m = \frac{1}{m+1} n^{m+1} + \zeta(-m) + \frac{1}{m+1} \sum_{k=1}^{[m+1]} (-1)^k \binom{m+1}{k} B_k n^{m-k+1} \]
\[ + (-1)^{[m+1]} \frac{n^{m-[m+1]+2}}{m+1} \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \binom{m+1}{l} B_{l-1+[m+1]-1} S^{(1)}_k(l)}{(n+1)(n+2) \cdots (n+k)}. \]

12.) Extended convergent versions of Stirling’s formula:
For every positive real number \( x \in \mathbb{R}^+ \), we have that
\[ \sum_{k=1}^{[x]} \log(k) = x \log(x) - x + \frac{1}{2} \log(2\pi) - \log(x) B_1(\{x\}) + \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \binom{-1}{l+1} S^{(1)}_l(l) B_{l+1}(\{x\})}{(x+1)(x+2) \cdots (x+k)}, \]
Extended first logarithmic summation formulas:

For every positive real number \( x \in \mathbb{R}^+ \), we have that

\[
\sum_{k=1}^{\lfloor x \rfloor} \log(k) = x \log(x) - x + \frac{1}{2} \log(2\pi) - \log(x)B_1(\{x\}) + \frac{B_2(\{x\})}{2x}
\]

and

\[
\sum_{k=1}^{\lfloor x \rfloor} \log(k) = x \log(x) - x + \frac{1}{2} \log(2\pi) - \log(x)B_1(\{x\}) + x \sum_{k=1}^{\infty} (-1)^k \sum_{l=1}^{k} \frac{(-1)^l}{l(l+1)(l+2)(l+3)} S_k^{(l)}(l) B_{l+1}(\{x\}) \]

and that

\[
\sum_{k=1}^{\lfloor x \rfloor} \log(k) = x \log(x) - x + \frac{1}{2} \log(2\pi) - \log(x)B_1(\{x\}) + \sum_{k=1}^{\infty} (-1)^k \sum_{l=2}^{k} \frac{(-1)^l}{l(l-1)} S_k^{(l)}(l) B_{l}(\{x\}) \]

\[
\frac{1}{x(x+1)(x+2) \cdots (x+k)}.
\]

13.) Extended first logarithmic summation formulas:

For every positive real number \( x \in \mathbb{R}^+ \), we have that

\[
\sum_{k=0}^{\lfloor x \rfloor} k \log(k) = \frac{1}{2} x^2 \log(x) - \frac{1}{4} x^2 + \frac{1}{12} - \zeta'(1) - x \log(x)B_1(\{x\}) + \frac{1}{2} \log(x)B_2(\{x\})
\]

and

\[
\sum_{k=0}^{\lfloor x \rfloor} k \log(k) = \frac{1}{2} x^2 \log(x) - \frac{1}{4} x^2 + \frac{1}{12} - \zeta'(1) - x \log(x)B_1(\{x\}) + \frac{1}{2} \log(x)B_2(\{x\}) - \frac{B_3(\{x\})}{6x}
\]

and

\[
\sum_{k=0}^{\lfloor x \rfloor} k \log(k) = \frac{1}{2} x^2 \log(x) - \frac{1}{4} x^2 + \frac{1}{12} - \zeta'(1) - x \log(x)B_1(\{x\}) + \frac{1}{2} \log(x)B_2(\{x\}) - \frac{B_3(\{x\})}{6x}
\]

\[
+ \sum_{k=1}^{\infty} (-1)^{k+1} \sum_{l=1}^{k} \frac{(-1)^l}{l(l+1)(l+2)(l+3)} S_k^{(l)}(l) B_{l+3}(\{x\}) \]

\[
\frac{1}{x(x+1)(x+2) \cdots (x+k)}.
\]

14.) Extended second logarithmic summation formula:

For every positive real number \( x \in \mathbb{R}^+ \), we have that

\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{\log(k)}{k} = \frac{1}{2} \log(x)^2 + \gamma_1 - \frac{\log(x)}{x} B_1(\{x\}) + \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \frac{(-1)^l}{l(l+1)!} S_k^{(l)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}
\]

\[
+ \log(x) \sum_{k=1}^{\infty} (-1)^k \frac{\sum_{l=1}^{k} \frac{1}{l(l+1)} S_k^{(l)}(l) B_{l+1}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}.
\]
15.) Extended third logarithmic summation formula:
For every positive real number \( x \in \mathbb{R}^+ \), we have that
\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{\log(k)}{k^2} = -\zeta'(2) - \frac{\log(x)}{x} - \frac{1}{x} - \frac{\log(x)}{x^2} B_1(\{x\})
\]
\[
+ \frac{1}{x} \sum_{k=1}^{\lfloor x \rfloor} (-1)^{k+1} \sum_{l=1}^{k} \frac{1}{l+1} \left( \sum_{m=0}^{\lfloor m/2 \rfloor} S^{(1)}_{m+1}(l) B_{l+1}(\{x\}) \right)
\]
\[
+ \frac{\log(x)}{x} \sum_{k=1}^{\lfloor x \rfloor} (-1)^{k} \sum_{l=1}^{k} S^{(1)}_{k}(l) B_{l+1}(\{x\})
\]
\[
= \frac{\log(x)}{x} \sum_{k=1}^{\lfloor x \rfloor} (-1)^{k} \sum_{l=1}^{k} S^{(1)}_{k}(l) B_{l+1}(\{x\})
\]

16.) Extended fourth logarithmic summation formula:
For every positive real number \( x \in \mathbb{R}^+ \), we have that
\[
\sum_{k=1}^{\lfloor x \rfloor} \log(k)^2 = x \log(x)^2 - \frac{\pi^2}{6} - \frac{\gamma^2}{2} - 2 \log(2) \log(x) + 2x + \frac{\gamma^2}{2}
\]
\[
- \frac{\log(2)^2}{2} - \log(2) \log(\pi) - \frac{\log(\pi)^2}{2} + \gamma_1
\]
\[
+ \frac{\log(x)}{x} B_2(\{x\}) + 2 \sum_{k=1}^{\lfloor x \rfloor} (-1)^{k} \sum_{l=1}^{k} (-1)^{l} S^{(1)}_{l+1}(2) S^{(1)}_{k}(l) B_{l+2}(\{x\})
\]
\[
= \frac{\gamma_1}{2} + \sum_{k=1}^{\lfloor x \rfloor} (-1)^{k} \sum_{l=1}^{k} (-1)^{l} S^{(1)}_{l+1}(2) S^{(1)}_{k}(l) B_{l+2}(\{x\})
\]
\[
= \frac{\gamma_1}{2} + \sum_{k=1}^{\lfloor x \rfloor} (-1)^{k} \sum_{l=1}^{k} (-1)^{l} S^{(1)}_{l+1}(2) S^{(1)}_{k}(l) B_{l+2}(\{x\})
\]

17.) Extended summation formula for partial sums of the Gregory-Leibniz series:
For every positive real number \( x \in \mathbb{R}^+ \), we have that
\[
\sum_{k=0}^{\lfloor x \rfloor} \frac{(-1)^{k}}{2k+1} = \frac{\pi}{4} + \frac{(-1)^{x-\{x\}}}{2} \sum_{k=0}^{\infty} \frac{(-1)^{k} \sum_{l=0}^{k} (-1)^{l} 2^{l} S^{(1)}_{k}(l) B_{l+1}(\{x\})}{(2x+1)(2x+2)(2x+3) \cdots (2x+k+1)}
\]
Setting \( x := n \in \mathbb{N} \), we get that
\[
\sum_{k=0}^{n} \frac{(-1)^{k}}{2k+1} = \frac{\pi}{4} + (-1)^{n+1} \sum_{k=0}^{\infty} \frac{(-1)^{k} \sum_{l=0}^{k} (-1)^{l} 2^{l} S^{(1)}_{k}(l) B_{l+1}(\{x\})}{(2n+1)(2n+2)(2n+3) \cdots (2n+k+1)}
\]

18.) Extended formula for the partial sums of the alternating harmonic series:
For every positive real number \( x \in \mathbb{R}^+ \), we have that
\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{(-1)^{k}+1}{k} = \log(2) + \frac{(-1)^{x-\{x\}}}{2} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \sum_{l=0}^{k} (-1)^{l} S^{(1)}_{k}(l) E_{l}(\{x\})}{x(x+1)(x+2) \cdots (x+k)}
\]
In this section, we denote by

\[ \sum_{k=1}^{n} \frac{(-1)^k+1}{k} = \log(2) + (-1)^n \sum_{k=0}^{\infty} (-1)^k \frac{\sum_{l=0}^{k} (-1)^l (2^l+1) S_k^{(1)}(l) B_{l+1}}{n(n+1)(n+2) \cdots (n+k)}. \]

4 Other Extended Summation Formulas for Finite Sums

In this section, we denote by \( \eta(s) := \sum_{k=1}^{\infty} \frac{(-1)^k+1}{k^s} \) the Dirichlet eta function.

1.) The extended alternating Faulhaber formula:

\[
\sum_{k=1}^{\lfloor x \rfloor} (-1)^k+1 k^m = \eta(-m) + \frac{(-1)^{x-(x)}}{2(m+1)} \sum_{k=0}^{m} (-1)^k (k+1) \binom{m+1}{k+1} E_k(\{x\}) x^{m-k} \quad \forall m \in \mathbb{N}_0.
\]

Setting \( x := n \in \mathbb{N} \), we get

\[
\sum_{k=1}^{n} (-1)^k+1 k^m = \eta(-m) + \frac{(-1)^n}{m+1} \sum_{k=0}^{m} (-1)^k (2^{k+1} - 1) \binom{m+1}{k+1} B_{k+1} x^{m-k} \quad \forall m \in \mathbb{N}_0.
\]

2.) The extended generalized alternating Faulhaber formula:

\[
\sum_{k=1}^{\lfloor x \rfloor} (-1)^k+1 k^m = \eta(-m) + \frac{(-1)^{x-(x)}}{2(m+1)} \sum_{k=0}^{\infty} (-1)^k+1 \sum_{l=0}^{k+1} \binom{m+1}{l+1} S_k^{(1)}(l) E_l(\{x\}) (x+1)(x+2) \cdots (x+k) \quad \forall m \in \mathbb{C}.
\]

Setting \( x := n \in \mathbb{N} \), we get

\[
\sum_{k=1}^{n} (-1)^k+1 k^m = \eta(-m) + \frac{(-1)^n}{m+1} \sum_{k=0}^{\infty} (-1)^k \sum_{l=0}^{k+1} \binom{m+1}{l+1} (2^{l+1} - 1) S_k^{(1)}(l) B_{l+1} \binom{m+1}{l+1} B_{l+1} x^{m-k} \quad \forall m \in \mathbb{C}.
\]

3.) The Geometric Summation Formula:

\[
\sum_{k=0}^{\lfloor x \rfloor} a^k = \frac{a^x}{\log(a)} + \frac{1}{1-a} + a^x \sum_{k=1}^{\infty} (-1)^k \sum_{l=1}^{\infty} \frac{\log(a)^{l-1}}{l} S_k^{(1)}(l) B_l(\{x\}) x^l \quad \forall a \neq 1.
\]

Setting \( x := n \in \mathbb{N} \), we get

\[
\sum_{k=0}^{n} a^k = \frac{a^n}{\log(a)} + \frac{1}{1-a} + a^n \sum_{k=1}^{\infty} (-1)^k \sum_{l=1}^{\infty} \frac{\log(a)^{l-1}}{l} S_k^{(1)}(l) B_l n^l \quad \forall a \neq 1.
\]
4.) The alternating Geometric Summation Formula:

\[ \sum_{k=0}^{[x]} (-1)^k a^k = \frac{1}{1 + a} + \frac{(-1)^{x-[x]} a^x}{\alpha^2} \sum_{k=0}^{\infty} (-1)^k \frac{\log(a)^l}{(x + 1)(x + 2)\cdots(x + k)} \quad \forall a \neq -1. \]

Setting \( x := n \in \mathbb{N} \), we get

\[ \sum_{k=0}^{n} (-1)^k a^k = \frac{1}{1 + a} + (-1)^{n+1} a^n \sum_{k=0}^{\infty} (-1)^k \frac{\log(a)^l}{(n + 1)(n + 2)\cdots(n + k)} \quad \forall a \neq -1. \]

5.) The Euler-Maclaurin Geometric Summation Formula:

\[ \sum_{k=0}^{[x]} a^k = \frac{a^x}{\log(a)} + \frac{1}{1 - a} + a^x \sum_{k=1}^{\infty} (-1)^k \log(a)^{k-1} k! B_k(\{x\}) \quad \text{for } \frac{1}{e^{2\pi}} < a \neq 1 < e^{2\pi}. \]

6.) The Euler-Maclaurin alternating Geometric Summation Formula:

\[ \sum_{k=0}^{[x]} (-1)^k a^k = \frac{1}{1 + a} + (-1)^{x-\{x\}} a^x (a-1) \sum_{k=0}^{\infty} (-1)^k \log(a)^{k-1} k! B_k(\{x\}) \quad \text{for } \frac{1}{e^{2\pi}} < a < e^{2\pi}. \]

7.) The Exponential Geometric Summation Formula:

\[ \sum_{k=0}^{[x]} e^k = e^x + \frac{1}{1 - e} + e^x \sum_{k=1}^{\infty} (-1)^k \frac{B_k(\{x\})}{k!} \quad \forall x \in \mathbb{R}^+. \]

8.) The Self-Counting Summation Formula:

Let \( \{a_k\}_{k=1}^{\infty} := \{1, 2, 3, 3, 4, 4, 4, 4, ...\} \) be the self-counting sequence [6, 7] defined by

\[ a_k := \left\lfloor \frac{1}{2} + \sqrt{2k} \right\rfloor. \]

Then, we have that

\[
\begin{align*}
\sum_{k=1}^{[x]} a_k &= \frac{x\sqrt{8x+1}}{3} - \frac{5\sqrt{8x+1}}{24} - \frac{\sqrt{8x+1}}{2} B_1(\{x\}) + B_1 \left( \left\{ \frac{\sqrt{8x+1}}{2} - \frac{1}{2} \right\} \right) B_1(\{x\}) \\
&\quad + \frac{1}{2} B_1 \left( \left\{ \frac{\sqrt{8x+1}}{2} - \frac{1}{2} \right\} \right) - \frac{\sqrt{8x+1}}{4} B_2 \left( \left\{ \frac{\sqrt{8x+1}}{2} - \frac{1}{2} \right\} \right) \\
&\quad + \frac{1}{6} B_3 \left( \left\{ \frac{\sqrt{8x+1}}{2} - \frac{1}{2} \right\} \right).
\end{align*}
\]
9.) Slowly convergent summation formula for the sum of the square roots:

\[
\sum_{k=0}^{\lfloor x \rfloor} \sqrt{k} = \frac{2}{3} x^{3/2} - \sqrt{x} B_1(\{x\}) - \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{\text{FresnelS}(2\sqrt{k}\sqrt{x})}{k^{3/2}}.
\]

10.) Slowly convergent summation formula for the harmonic series:

\[
\sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} = \log(x) + \gamma + 2 \sum_{k=1}^{\infty} \text{CosIntegral}(2\pi k x).
\]

5 Conclusion

This paper presents an overview on some rapidly convergent summation formulas obtained by applying the Weniger transformation [1].

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