VARIETIES OF ELEMENTARY SUBALGEBRAS OF SUBMAXIMAL RANK IN TYPE A

YANG PAN

Abstract. Let $G$ be a connected simple algebraic group over an algebraically closed field $k$ of characteristic $p > 0$, and $g := \text{Lie}(G)$. We additionally assume that $G$ is standard and is of type $A_n$. Motivated by the investigation of the geometric properties of the varieties $E(r, g)$ of $r$-dimensional elementary subalgebras of a restricted Lie algebra $g$, we will show in this article the irreducible components of $E(rk_p(g) - 1, g)$ when $rk_p(g)$ is the maximal dimension of an elementary subalgebra of $g$.

Introduction

Let $(g, [p])$ be a finite dimensional restricted Lie algebra over an algebraically closed field $k$ of positive characteristic $p > 0$. The closed subset of $p$-restricted nilpotent elements

$$V(g) := \{x \in g : x^{[p]} = 0\}$$

has been studied in the modular representation theory of $(g, [p])$, aiming to understand the cohomological support variety. A Lie subalgebra $e \subset g$ is said to be elementary if it is abelian and has trivial $p$-restriction. The subset

$$E(r, g) := \{e \in \text{Gr}_r(g) : [e, e] = 0, e \in V(g)\}$$

of the Grassmannian $\text{Gr}_r(g)$ of $r$-planes in $g$ which consists of $r$-dimensional elementary subalgebras of $g$ has been expounded in [4] by Carlson, Friedlander and Pevtsova. For instance, they show us $E(r, g)$ can be endowed with a projective variety structure and it affords geometric invariant for the representations of $g$.

When concerning the geometric properties of $E(r, g)$, interest has been shown in determining its irreducible components. A prototypical example arises from $E(1, g)$, which is the projectivization of the restricted nullcone $V(g)$. When $g$ is the Lie algebra of a simple algebraic group, $V(g)$ is irreducible regardless of the characteristic $p$, so is $E(1, g)$. When $r$ equals 2, Premet in [10] shows the correspondence between the irreducible components of the nilpotent commuting variety $C^{nil}(g)$ and the distinguished nilpotent orbits of $g$ when $g$ is a reductive Lie algebra. It follows that $C^{nil}(g)$ is irreducible when $g$ is of type $A_n$, in which case the same is true of $E(2, g)$ if $p \geq n + 1$. Let

$$rk_p(g) := \max\{r \in \mathbb{N}_0 : E(r, g) \neq \emptyset\}$$

be the $p$-rank of $g$. This rank of the restricted Lie algebra of a simple algebraic group was determined earlier in [4] and in recent work by Pevtsova-Stark in [12]. Irreducible components of $E(rk_p(g), g)$ for these Lie algebras were calculated case by case and were shown in [12, Table 4].

It is the purpose of this article to give a description of the variety $E(rk_p(g) - 1, g)$ when $g$ is the Lie algebra of a connected standard simple algebraic group $G$ of type $A$. Under the standard assumption one can show that $g$ is a Lie algebra isomorphic to $s_k(u)$ such that $p$ does not divide $n$. In view of [11, Lemma 2.2], the determination of $E(rk_p(g) - 1, g)$ can be reduced to the unipotent case $E(rk_p(g) - 1, u)$ where $u = \text{Lie}(R_u(B))$ and $R_u(B) \subset B \subset G$ is the unipotent radical of a fixed Borel subgroup $B$ of
Let \( \Phi \) be an irreducible root system with positive roots \( \Phi^+ \). Recall from [12] that two positive roots commute if their sum is not a root. We define the set

\[
\text{Max}_r(\Phi) := \left\{ R \subset \Phi^+ : \alpha + \beta \notin \Phi^+, \forall \alpha, \beta \in R, |R| = r \text{ and } R \not\subseteq R' \right\},
\]

where \( R' \) is any subset of commuting positive roots.

When \( r = \text{rk}_p(\mathfrak{g}) \), we write \( \text{Max}_r(\Phi) \) as \( \text{Max}(\Phi) \) simply. By considering the set

\[
\text{Com}_r(\Phi) := \left\{ R \subset \Phi^+ : \alpha + \beta \notin \Phi^+, \forall \alpha, \beta \in R, |R| = r \right\},
\]

we find the map \( \text{LT} : \mathbb{E}(\text{rk}_p(\mathfrak{g}), u) \rightarrow \text{Max}(\Phi) \) in [12, (3.1.2)] can be defined in a generalized fashion

\[
\text{LT} : \mathbb{E}(r, u) \rightarrow \text{Com}_r(\Phi) \text{ since } \text{Com}_r(\Phi) = \text{Max}(\Phi) \text{ when } r = \text{rk}_p(\mathfrak{g}).
\]

The problem now is for any given total ordering and any element \( \epsilon \) of \( \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, u) \), it is possible to have \( \text{LT}(\epsilon) \notin \text{Max}_{\text{rk}_p(\mathfrak{g}) - 1}(\Phi) \). Let \( \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, u)_{\text{max}} \) be the subset of \( \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, u) \) consisting of maximal elementary subalgebras.

This raises the question concerning the ordering on \( \Phi^+ \), the one giving rise to the map

\[
\text{LT} : \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, u)_{\text{max}} \rightarrow \text{Max}_{\text{rk}_p(\mathfrak{g}) - 1}(\Phi).
\]

We find, for type \( A_n \), the ordering exists for \( n \) sufficiently large.

Since the set \( \text{Max}_{\text{rk}_p(\mathfrak{g}) - 1}(\Phi) \) is tractable, it will be determined within the initial step. The calculation of \( \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, u) \) then proceeds via three steps. First, we determine \( \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, u)_{\text{max}} \) as a set by using the map \( \text{LT} \). Second, we prove that the elements of \( \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, u)_{\text{max}} \) are given by the combinatorics of the root system of \( G \), which largely relies on Malcev’s linear algebraic approach. Finally, we have to utilize the result on \( \mathbb{E}(\text{rk}_p(\mathfrak{g}), u) \) to understand the elements of \( \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, u) \) which are not in \( \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, u)_{\text{max}} \). The main result of this paper is:

**Theorem.** Let \( G \) be a standard simple algebraic \( k \)-group with root system \( \Phi \) of type \( A_n \ (n \geq 5) \) and \( \mathfrak{g} := \text{Lie}(G) \). Then the irreducible components of \( \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, \mathfrak{g}) \) can be characterized as follows:

| Type | Restrictions on rank | Irreducible components |
|------|----------------------|------------------------|
| \( A_{2m+1} \) | \( m \geq 2 \) | \( \text{G.Lie}(\Phi_{m}^{\text{rad}}), \text{G.Lie}(\Phi_{m+2}^{\text{rad}}), \text{G.E}(\text{rk}_p(\mathfrak{g}) - 1, \text{Lie}(\Phi_{m+1}^{\text{rad}})) \) |
| \( A_{2m} \) | \( m \geq 3 \) | \( \text{G.E}(\text{rk}_p(\mathfrak{g}) - 1, \text{Lie}(\Phi_{m}^{\text{rad}})), \text{G.E}(\text{rk}_p(\mathfrak{g}) - 1, \text{Lie}(\Phi_{m+1}^{\text{rad}})) \) |

**Table 1. Characterization**

**Remarks.** (1). In [12] the authors have shown that \( \mathbb{E}(\text{rk}_p(\mathfrak{g}), \mathfrak{g}) \) is a finite disjoint union of partial flag varieties unless \( G \) is of type \( A_2 \), which differs from the above result.

(2). We list the results of \( \mathbb{E}(\text{rk}_p(\mathfrak{g}) - 1, \mathfrak{g}) \) for \( A_n \ (n \leq 4) \) in the following. The reference we give is the paper [14] of Warner, in which the author discuss the irreducibility of \( \mathbb{E}(r, \mathfrak{g}_n) \) in section 5.

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1. Preliminaries

1.1. Parabolic system. We assume that $G$ is a simple algebraic $\mathbb{k}$-group with irreducible root system $\Phi$. The interested reader may consult [1][2][3][13] for the theory of algebraic groups. Let $U_\alpha$ be the root subgroup corresponding to a root $\alpha$, and $B = \langle U_\alpha, T ; \alpha \in \Phi^+ \rangle$ be a Borel subgroup of $G$ containing $T$. Initially, we study the Weyl group $W$ in tandem with an irreducible root system $\Phi$. Let $\Delta := \{\alpha_1, \ldots, \alpha_n\}$ be the set of positive simple roots, and $I$ be a subset of $\Delta$. We define

$$\Phi_I := \Phi \cap \sum_{\alpha \in I} \mathbb{Z}\alpha$$

to be the parabolic subsystem of roots, and

$$W_I := \langle s_\alpha ; \alpha \in I \rangle$$

to be the standard parabolic subgroup of $W$ (see [8] for details). Then subgroups of the form $P_I := B\mathcal{W}_I B = \langle T, U_\alpha ; \alpha \in \Phi^+ \cup \Phi_I \rangle$ are called standard parabolic subgroups of $G$. The Levi decomposition $P_I = L_I \ltimes R_u(P_I)$ decomposes $P_I$ into a semi-direct product of its Levi factor $L_I$ and the unipotent radical $R_u(P_I)$, with the latter being generated by root subgroups $\{U_\alpha ; \alpha \in \Phi^+ \setminus \Phi_I^+\}$. Influenced by this, we set $S := \Delta \setminus I$ and then define

$$\Phi^\text{rad}_S = \Phi^+ \setminus \Phi_I^+$$

to be the set of positive roots that cannot be written as a linear combination of the simple roots not in $S$. If $S = \{\alpha_i\}$, then we simply write $\Phi^\text{rad}_{\{\alpha_i\}}$ instead of $\Phi^\text{rad}_S$.

1.2. Maximal subsets for type A. Suppose that $G$ is of type $A_n$. The roots of $A_n$ are the integer vectors in $\mathbb{R}^{n+1}$ of length $\sqrt{2}$ for which the coordinates sum to 0. Let $\{\epsilon_i ; 1 \leq i \leq n+1\}$ be the standard basis of $\mathbb{R}^{n+1}$. We denote by

$$\Phi = \{\epsilon_i - \epsilon_j ; i \neq j, 1 \leq i, j \leq n+1\}$$

the corresponding set of roots, and by $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ the base of $\Phi$ where $\alpha_i = \epsilon_i - \epsilon_{i+1}$. There is a bijection $\phi$ from the set of non-trivial proper subsets of $\{1, \ldots, n+1\}$ to the set of maximal subsets of commuting roots of $\Phi$ by sending $J$ to $\phi(J) := \{\epsilon_i - \epsilon_j ; i \in J, j \notin J\}$, and the condition $J < \{1, \ldots, n+1\} \setminus J$ on $J$ gives rise to a maximal subset of commuting positive roots; see [12, A.1].

Notation. Type $A_n$.

| $n$      | $\Phi^\text{odd}_{m+1,m+2}$ | $\Phi^\text{ev}_{m+1,m+2}$ |
|----------|-----------------------------|-----------------------------|
| $n = 2m + 1$, | $\phi(J) \cap \Phi^+$, for $J = \{1, \ldots, m, m+2\}$ | $\phi(J) \cap \Phi^+$, for $J = \{1, \ldots, m, m+2\}$ |
| $n = 2m$, | $\phi(J) \cap \Phi^+$, for $J = \{1, \ldots, m, m+2\}$ | $\phi(J) \cap \Phi^+$, for $J = \{1, \ldots, m, m+1\}$ |
| $n = 2m$, | $\phi(J) \cap \Phi^+$, for $J = \{1, \ldots, m-1, m+1\}$ | $\phi(J) \cap \Phi^+$, for $J = \{1, \ldots, m-1, m+1\}$ |

Table 2. small rank cases
Theorem 1.1. Keep the notations as above and set \( \mathfrak{g} := \text{Lie}(G) \). Then the elements of the set \( \text{Max}_{rk_p(\mathfrak{g})-1}(\Phi) \) are given as follows:

| Type | Restrictions on rank | \( \text{Max}_{rk_p(\mathfrak{g})-1}(\Phi) \) |
|------|----------------------|------------------------------------------------|
| \( A_{2m+1} \) | \( m \geq 0 \) | \( \Phi_{m}^{\text{rad}}, \Phi_{m+2}^{\text{rad}}, \Phi_{m+1,m+2}^{\text{odd}} \) |
| \( A_{2m} \) | \( m \geq 1 \) | \( \Phi_{m+1,m+2}^{\text{ev}}, \Phi_{m,m+1}^{\text{ev}} \) |

Table 3. Maximal subset: order \( rk_p(\mathfrak{g}) - 1 \)

Proof. Let \( M(A) \in \text{Max}_{rk_p(\mathfrak{g})-1}(\Phi) \). Notice that the maximal dimension \( rk_p(\mathfrak{g}) \) is \( (m+1)^2 \) (resp. \( m(m+1) \)) when \( n = 2m+1 \) (resp. \( n = 2m \)). If \( M(A) \) is still maximal in \( \Phi \), then \( M(A) = \phi(J) \) for certain \( J \). By letting \( |M(A)| = |\phi(J)| = |J|(n+1-|J|) \) equal \( (m+1)^2 - 1 \) when \( n = 2m+1 \) and equal \( m(m+1) - 1 \) when \( n = 2m \), we get \( |J| = m, m+2 \) for \( n = 2m+1 \), and there is no solution for \( n = 2m \). Continuing the consideration, if \( |J| = m \) or \( m+2 \) then \( M(A) \) has to be \( \Phi_{m}^{\text{rad}} \) or \( \Phi_{m+2}^{\text{rad}} \) respectively.

Alternatively, \( M(A) \) is maximal in \( \Phi^+ \) but not in \( \Phi \). Then \( M(A) \subset \phi(J) \) for some \( J \) with
\[
|\phi(J)| = rk_p(\mathfrak{g}) \quad \text{and} \quad |\phi(J) \cap \Phi^+| = rk_p(\mathfrak{g}) - 1.
\]
One gets \( M(A) \) equals \( \Phi_{m+1,m+2}^{\text{odd}} \) when \( n = 2m+1 \), and equals \( \Phi_{m+1,m+2}^{\text{ev}} \) or \( \Phi_{m,m+1}^{\text{ev}} \) when \( n = 2m \). \( \square \)

Remark. We recall the set \( \text{Max}(\Phi) \) for type \( A_n \), which is calculated by Malcev in [7]:

| Type   | Restrictions on rank | \( \text{Max}(\Phi) \) |
|--------|----------------------|-------------------------|
| \( A_{2m+1} \) | \( m \geq 0 \) | \( \Phi_{m+1}^{\text{rad}} \) |
| \( A_{2m} \) | \( m \geq 1 \) | \( \Phi_{m+1}^{\text{rad}}, \Phi_{m}^{\text{rad}} \) |

Table 4. Maximal subset: order \( rk_p(\mathfrak{g}) \)

2. Main result

Now we concentrate on \( G \) being a standard connected simple algebraic \( k \)-group of type \( A_n \) with \( \mathfrak{g} := \text{Lie}(G) \). Let \( \Phi \) be the root system of \( G \) with positive roots \( \Phi^+ \). Since \( p \) is a good prime for \( G \), we have \( [x_\alpha, x_\beta] = 0 \) if and only if \( \alpha + \beta \notin \Phi \) for \( \alpha, \beta \in \Phi \) and their associated root vectors \( x_\alpha, x_\beta \). Recall that \( x_\alpha^p = 0 \) for \( \alpha \in \Phi \), one does have an elementary subalgebra \( \text{Lie}(R) := \text{Span}_k \{ x_\alpha ; \, \alpha \in R \} \) when \( R \) is a subset of commuting roots. Let \( u \) be the Lie algebra of the unipotent radical. We will show the map
\[
\text{Lie} : \text{Max}_{rk_p(\mathfrak{g})-1}(\Phi) \longrightarrow \mathbb{E}(rk_p(\mathfrak{g}) - 1, u)_{\text{max}} ; \, R \mapsto \text{Lie}(R)
\]
is surjective up to conjugacy by \( G \). This will be done by employing the map (cf. [12, (3.1.2)])
\[
\text{LT} : \mathbb{E}(rk_p(\mathfrak{g}) - 1, u)_{\text{max}} \longrightarrow \text{Max}_{rk_p(\mathfrak{g})-1}(\Phi)
\]
according to the chosen total ordering.

2.1. Total ordering for map LT. Suppose that $G$ is of type $A_{2m+1}$. We fix the total ordering $\succeq$ by letting it be the reverse lexicographic ordering given by $\alpha_{m+1} \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{2m+1}$. We first show that the map LT is well-defined under such setting for $A_{2m+1}$.

Lemma 2.1.1. Suppose that $G$ is of type $A_{2m+1}$ $(m \geq 1)$. If $\epsilon \in \mathbb{E}(rk_p(\mathfrak{g}) - 1, u)_{\max}$, then $\operatorname{LT}(\epsilon) \in \max_{\operatorname{rk}_p(\mathfrak{g}) - 1}(\Phi)$ with respect to $\succeq$.

Proof. Assume that $\operatorname{LT}(\epsilon) \notin \max_{\operatorname{rk}_p(\mathfrak{g}) - 1}(\Phi)$, then $\operatorname{LT}(\epsilon) \subsetneq \Phi_{\operatorname{rad}}^{m+1}$ by Table 4. Since $\Phi^{+} \setminus \Phi_{\operatorname{rad}}^{m+1} > \Phi_{\operatorname{rad}}^{m+1}$, it implies that all terms of basis vectors correspond to the roots lying in $\Phi_{\operatorname{rad}}^{m+1}$. As a result, $\epsilon$ is contained in the elementary subalgebra $\operatorname{Lie}(\Phi_{\operatorname{rad}}^{m+1})$. Notice that $\dim \epsilon < \dim \operatorname{Lie}(\Phi_{\operatorname{rad}}^{m+1})$, the containment is proper which contradicts maximality. □

Now we consider the $k$-group $G$ which is of type $A_{2m}$. We choose the total ordering $\succeq$ to be the reverse lexicographic ordering given by $\alpha_{m+1} \prec \alpha_{m} \prec \alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_{2m}$. According to this choice, one can easily check that

$$\Phi^{+} \setminus (\Phi_{\operatorname{rad}}^{m} \cup \Phi_{\operatorname{rad}}^{m+1}) > \Phi_{\operatorname{rad}}^{m} \setminus \Phi_{\operatorname{rad}}^{m+1} > \Phi_{\operatorname{rad}}^{m+1} \setminus \Phi_{\operatorname{rad}}^{m} > \Phi_{\operatorname{rad}}^{m} \cap \Phi_{\operatorname{rad}}^{m+1}.$$  

Lemma 2.1.2. Suppose that $G$ is of type $A_{2m}$ with $m \geq 3$. If $\epsilon \in \mathbb{E}(rk_p(\mathfrak{g}) - 1, u)_{\max}$, then $\operatorname{LT}(\epsilon) \in \max_{\operatorname{rk}_p(\mathfrak{g}) - 1}(\Phi)$ with respect to $\succeq$.

Proof. If $\operatorname{LT}(\epsilon) \notin \max_{\operatorname{rk}_p(\mathfrak{g}) - 1}(\Phi)$, then either $\operatorname{LT}(\epsilon) \subsetneq \Phi_{\operatorname{rad}}^{m}$, or $\operatorname{LT}(\epsilon) \subsetneq \Phi_{\operatorname{rad}}^{m+1}$ according to Table 4.

• Case 1. $\operatorname{LT}(\epsilon) \subsetneq \Phi_{\operatorname{rad}}^{m+1}$. Then $\Phi_{\operatorname{rad}}^{m+1} \setminus \operatorname{LT}(\epsilon) = \{\epsilon_u - \epsilon_v\}$ for some $(u, v)$. Notice that $\Phi^{+} \setminus \Phi_{\operatorname{rad}}^{m+1}$, thus the reduced echelon form basis of $\epsilon$ is as follows

$$x_{\epsilon_i - \epsilon_j} + a_{ij} x_{\epsilon_u - \epsilon_v}, \quad a_{ij} = 0 \text{ if } i < u \text{ or } i = u, j > v$$

for $1 \leq i \leq m + 1, m + 2 \leq j \leq 2m + 1$ and $(i, j) \neq (u, v)$. Then it is readily seen that $\epsilon \subsetneq \epsilon \oplus k x_{\epsilon_u - \epsilon_v}$, and the maximality of $\epsilon$ leads to a contradiction.

• Case 2. $\Phi_{\operatorname{rad}}^{m} \setminus \operatorname{LT}(\epsilon) = \{\epsilon_u - \epsilon_v\} \subsetneq \Phi_{\operatorname{rad}}^{m} \cap \Phi_{\operatorname{rad}}^{m+1}$. Then the reduced basis of $\epsilon$ consists of elements for $1 \leq i \leq m, m + 2 \leq j \leq 2m + 1$ and $(i, j) \neq (u, v)$

$$x_{ij} = x_{\epsilon_i - \epsilon_j} + a_{ij} x_{\epsilon_u - \epsilon_v}, \quad a_{ij} = 0 \text{ if } i < u \text{ or } i = u, j > v$$

$$y_i = x_{\epsilon_i - \epsilon_m+1} + \sum_{s=m+2}^{2m+1} b_{is} x_{\epsilon_{m+1}-\epsilon_s} + d_i x_{\epsilon_u - \epsilon_v}.$$  

Now we compute

$$[y_i, y_{i'}] = \sum_{s=m+2}^{2m+1} b_{i',s} \epsilon_{m+1}-\epsilon_s x_{\epsilon_i - \epsilon_s} + \sum_{s=m+2}^{2m+1} b_{is} \epsilon_{m+1}-\epsilon_s x_{\epsilon_{i'} - \epsilon_s}.$$  

As $m \geq 3$, we may take $i \neq i'$, this gives $b_{is} = 0$ for all $i$ and $s$. As a result, we will have $\epsilon \subsetneq \epsilon \oplus k x_{\epsilon_u - \epsilon_v}$, a contradiction.

• Case 3. $\Phi_{\operatorname{rad}}^{m} \setminus \operatorname{LT}(\epsilon) = \{\epsilon_u - \epsilon_{m+1}\} \subsetneq \Phi_{\operatorname{rad}}^{m} \cap \Phi_{\operatorname{rad}}^{m+1}$. Then the reduced echelon form basis of $\epsilon$ is $x_{\epsilon_i - \epsilon_j}$ for $1 \leq i \leq m$ and $m + 2 \leq j \leq 2m + 1$ together with for $1 \leq i \leq m$ and $i \neq u$

$$y_i = x_{\epsilon_i - \epsilon_m+1} + q_i x_{\epsilon_u - \epsilon_{m+1}} + \sum_{s=m+2}^{2m+1} d_{is} x_{\epsilon_{m+1}-\epsilon_s}, \quad q_i = 0 \text{ if } i < u.$$  

If $i, i'$ are distinct and different from $u$ (which is possible as $m \geq 3$), then the coefficient of $x_{\epsilon_{i'} - \epsilon_s}$ in $[y_i, y_{i'}]$ is $N_{\epsilon_{m+1} - \epsilon_s, \epsilon_{i'} - \epsilon_{m+1}} d_{is}$, so $d_{is} = 0$ for all $i$ and $s$. Thus $\epsilon \subsetneq \epsilon \oplus k x_{\epsilon_u - \epsilon_{m+1}}$, a contradiction and we finish the proof. □
2.2. Surjectivity for map $\text{Lie}$.

**Theorem 2.2.1.** Suppose that $G$ is of type $A_{2m+1}$ with $m \geq 2$. If $\epsilon \in E(r(k_{p}(g)) - 1, u)$ satisfies $\text{LT}(\epsilon) = \Phi_{m}^{\text{rad}}$, $\Phi_{m+2}^{\text{rad}}$ or $\Phi_{m+1,m+2}^{\text{odd}}$ then $\epsilon = \text{Lie}(\Phi_{m}^{\text{rad}})$, $\text{Lie}(\Phi_{m+2}^{\text{rad}})$ or $\text{Lie}(\Phi_{m+1,m+2}^{\text{odd}})^{\text{exp(ad}(ax_{m+1}))}$ for some $a$ respectively.

**Proof.** Case 1. $\text{LT}(\epsilon) = \Phi_{m}^{\text{rad}}$. We write the reduced echelon form basis for $\epsilon$

$$x_{ij} = x_{\epsilon_{i}, -\epsilon_{j}}, 1 \leq i \leq m \text{ and } m + 2 \leq j \leq 2m + 2$$

$$y_{i} = x_{\epsilon_{i}, -\epsilon_{m+1}} + \sum_{s=1}^{i-1} \sum_{t=s+1}^{m} a_{ist}x_{\epsilon_{s}, -\epsilon_{t}} + \sum_{r=m+2}^{2m+2} b_{ir}x_{\epsilon_{m+1}, -\epsilon_{r}}, 1 \leq i \leq m$$

Let $1 \leq i \leq m$ and $2 \leq t \leq m$, the coefficient of $x_{\epsilon_{i}, -\epsilon_{j}}$ in $[y_{i}, x_{t}j]$ is $a_{ist}N_{\epsilon_{s}, -\epsilon_{t}, -\epsilon_{j}}$, this gives $a_{ist} = 0$ for all $i, s$ and $t$. If $i, j \leq m$ are distinct, then the coefficient of $x_{\epsilon_{i}, -\epsilon_{j}}$ in $[y_{i}, y_{j}]$ is $b_{ij}N_{\epsilon_{i}, -\epsilon_{j}}$. As $m \geq 2$, this gives all $b_{ij} = 0$. Therefore, we have $\epsilon = \text{Lie}(\Phi_{m}^{\text{rad}})$.

Case 2. $\text{LT}(\epsilon) = \Phi_{m+2}^{\text{rad}}$. The reduced echelon form basis of $\epsilon$ is of the form

$$x_{ij} = x_{\epsilon_{i}, -\epsilon_{j}} + \sum_{s=1}^{i-1} \sum_{t=s+1}^{m+2} a_{ij,s}x_{\epsilon_{s}, -\epsilon_{t}}, 1 \leq i \leq m + 1 \text{ and } m + 3 \leq j \leq 2m + 2$$

$$y_{j} = x_{\epsilon_{m+2}, -\epsilon_{j}} + \sum_{s=1}^{m+1} \sum_{t=s+1}^{m+2} b_{js}x_{\epsilon_{s}, -\epsilon_{t}}, m + 3 \leq j \leq 2m + 2$$

Let $m + 3 \leq j, j' \leq 2m + 2$ and $2 \leq t \leq m + 1$. If $j$ and $j'$ are distinct, then the coefficient of $x_{\epsilon_{i}, -\epsilon_{j'}}$ in $[y_{j}, x_{t}j']$ is $b_{js}N_{\epsilon_{i}, -\epsilon_{j}, -\epsilon_{j'}}$, it gives $b_{js} = 0$ for all $j, s$ and $t$. Then the coefficient of $x_{\epsilon_{i}, -\epsilon_{j}}$ in $[y_{j}, x_{t}j']$ is $b_{ij,N_{\epsilon_{i}, -\epsilon_{j}}}$, this implies $a_{ij,s} = 0$ for all $i, j, s$. It remains to consider $b_{js}(m+2)$. Let $m + 3 \leq i, j \leq 2m + 2$ are distinct, then the coefficient of $x_{\epsilon_{i}, -\epsilon_{j}}$ in $[y_{i}, y_{j}]$ is $b_{js}(m+2)N_{\epsilon_{i}, -\epsilon_{j}}$. According to this together with $m \geq 2$, we get $b_{js}(m+2) = 0$ for all $j$ and $s$. Therefore, we have $\epsilon = \text{Lie}(\Phi_{m+2}^{\text{rad}})$.

Case 3. $\text{LT}(\epsilon) = \Phi_{m+1,m+2}^{\text{odd}}$. The reduced echelon form basis of $\epsilon$ consists of

$$x_{ij} = x_{\epsilon_{i}, -\epsilon_{j}} + \sum_{s=1}^{i-1} \sum_{t=s+1}^{m} a_{ij,s}x_{\epsilon_{s}, -\epsilon_{t}}$$

$$y_{i} = x_{\epsilon_{i}, -\epsilon_{m+1}} + \sum_{s=1}^{i-1} \sum_{t=s+1}^{m} b_{ist}x_{\epsilon_{s}, -\epsilon_{t}} + \sum_{r=m+2}^{2m+2} d_{ir}x_{\epsilon_{m+1}, -\epsilon_{r}} + \sum_{r=1}^{m} k_{ir}x_{\epsilon_{r}, -\epsilon_{m+2}}$$

$$z_{j} = x_{\epsilon_{m+2}, -\epsilon_{j}} + \sum_{s=1}^{m} \sum_{t=s+1}^{m} h_{jst}x_{\epsilon_{s}, -\epsilon_{t}} + \sum_{r=m+2}^{2m+2} \ell_{jr}x_{\epsilon_{m+1}, -\epsilon_{r}} + \sum_{r=1}^{m} \xi_{jr}x_{\epsilon_{r}, -\epsilon_{m+2}}$$

where $1 \leq i \leq m$ and $m + 3 \leq j \leq 2m + 2$. By the same argument as before we deduce that $b_{ist} = h_{jst} = 0$ for all $s$ and $t$. If $i, j \leq m$ are distinct, then the coefficient of $x_{\epsilon_{i}, -\epsilon_{j}}$ in $[y_{i}, y_{j}]$ is $d_{ir}N_{\epsilon_{m+1}, -\epsilon_{r}, -\epsilon_{m+2}}$. As $m \geq 2$, this gives $d_{ir} = 0$ and the argument can also be applied to $z_{j}$ which ensures that $\xi_{jr} = 0$. $\lambda = -k_{11}N_{\epsilon_{m+1}, -\epsilon_{m+2}, -\epsilon_{1}, -\epsilon_{m+1}}$. Conjugation by $\text{exp(ad}(ax_{m+1}))$ to $\epsilon$ ensures that the image of $y_{i}$ has no term $x_{\epsilon_{i}, -\epsilon_{m+2}}$. We may assume $k_{11} = 0$. We compute the coefficient of $x_{\epsilon_{i}, -\epsilon_{j}}$ in $[y_{i}, z_{j}]$. We may assume $\ell_{jr} = 0$ for all $j$ and $r$. Then the coefficient of $x_{\epsilon_{i}, -\epsilon_{j}}$ in $[y_{i}, z_{j}]$ is $k_{ir}N_{\epsilon_{i}, -\epsilon_{j}, -\epsilon_{m+2}, -\epsilon_{m+2}}$, this gives $k_{ir} = 0$ for all $i$ and $r$, and this also applies to $[x_{ij}, z_{jr}]$ from which we can get $a_{ij,s} = 0$. As a result, we get $\epsilon = \text{Lie}(\Phi_{m+1,m+2}^{\text{odd}})^{\text{exp(ad}(ax_{m+1}))}$ where $a = -\lambda$. \qed
Theorem 2.2.2. Suppose that $G$ is of type $A_{2m}$ with $m \geq 3$. If $\epsilon \in \mathfrak{e}(rk_p(g) - 1, u)$ satisfies $\text{LT}(\epsilon) = \Phi_{m,m+1}^{ev}$ or $\Phi_{m+1,m+2}^{ev}$, then there exists some $a$ such that $\epsilon = \text{Lie}(\Phi_{m,m+1}^{ev})^{\exp(\text{ad}(ax_{m+1}))}$ or $\text{Lie}(\Phi_{m+1,m+2}^{ev})^{\exp(\text{ad}(ax_{m+1}))}$ respectively.

Proof. • Case 1. $\text{LT}(\epsilon) = \Phi_{m,m+1}^{ev}$. Then the reduced echelon form basis of $\epsilon$ is $x_{\epsilon_i - \epsilon_j}$ for $1 \leq i \leq m - 1$ and $m + 2 \leq j \leq 2m + 1$ and

$$y_j = x_{\epsilon_{m+1} - \epsilon_j} + \sum_{s=m+2}^{2m+1} a_{js} x_{\epsilon_m - \epsilon_s}$$

$$z_i = x_{\epsilon_i - \epsilon_m} + \sum_{u=1}^{i-1} \sum_{v=u+1}^{m-1} b_{iuv} x_{\epsilon_u - \epsilon_v} + \sum_{s=1}^{m} c_{is} x_{\epsilon_s - \epsilon_{m+1}} + \sum_{s=m+2}^{2m+1} d_{is} x_{\epsilon_m - \epsilon_s}$$

where $m + 2 \leq j \leq 2m + 1$ for $y_j$ and $1 \leq i < m$ for $z_i$. Let $\lambda = -a(m+2)(m+2)N_{\epsilon_m - \epsilon_{m+1} - \epsilon_m - \epsilon_{m+2}}$. Using conjugation given by $\exp(\text{ad}(\lambda x_{\epsilon_m - \epsilon_{m+1}}))$ to $\epsilon$, we have explicitly

$$\exp(\text{ad}(\lambda x_{\epsilon_m - \epsilon_{m+1}})) (y_{m+2}) = x_{\epsilon_{m+1} - \epsilon_m} + \sum_{s=m+3}^{2m+1} a_{(m+2)s} x_{\epsilon_m - \epsilon_s}$$

which allows us to assume $a_{(m+2)(m+2)} = 0$. Then we compute for $1 \leq i < m$

$$[y_{m+2}, z_i] = \sum_{s=1}^{m} c_{is} N_{\epsilon_m - \epsilon_{m+1} - \epsilon_m, \epsilon_s - \epsilon_{m+1}} x_{\epsilon_s - \epsilon_{m+2}} + \sum_{s=m+3}^{2m+1} a_{(m+2)s} N_{\epsilon_m - \epsilon_s, \epsilon_i - \epsilon_m x_{\epsilon_i - \epsilon_s}}$$

Notice that these items $x_{\epsilon_s - \epsilon_{m+2}}$ and $x_{\epsilon_i - \epsilon_s}$ are different, so $c_{is} = 0$ for all $i$ and $s$ and $a_{(m+2)s} = 0$ for all $s$. Further we can get $a_{js} = 0$ for all $j$ and $s$ by seeing $[y_j, z_1] = 0$. When $1 < v < m$, we compute the coefficient of $x_{\epsilon_v - \epsilon_{m+2}}$ in $[x_{\epsilon_v - \epsilon_{m+2}}, z_i]$, that is $N_{\epsilon_v - \epsilon_{m+2}, \epsilon_i - \epsilon_m b_{iuv}}$. This gives $b_{iuv} = 0$ for all $i, u$ and $v$. If $1 \leq i, j' < m$ are distinct, then the coefficient of $x_{\epsilon_i - \epsilon_{m+2}}$ in $[z_i, z_{j'}]$ is $N_{\epsilon_i - \epsilon_m, \epsilon_{m+1} - \epsilon_{m+2}}$. If $m \geq 3$, this gives $d_{is} = 0$ for all $i, s$, and consequently $\epsilon = \text{Lie}(\Phi_{m,m+1}^{ev})^{\exp(\text{ad}(ax_{m+1}))}$ for $a = -\lambda$.

• Case 2. $\text{LT}(\epsilon) = \Phi_{m+1,m+2}^{ev}$. We write the reduced basis for $1 \leq i \leq m$ and $m + 3 \leq j \leq 2m + 1$

$$x_{ij} = x_{\epsilon_i - \epsilon_j} + \sum_{t=1}^{i-1} a_{ijt} x_{\epsilon_t - \epsilon_{m+2}}$$

$$y_i = x_{\epsilon_i - \epsilon_{m+1}} + \sum_{s=m+2}^{2m+1} b_{is} x_{\epsilon_{m+1} - \epsilon_s} + \sum_{s=1}^{m} c_{is} x_{\epsilon_s - \epsilon_{m+2}}$$

$$z_j = x_{\epsilon_{m+2} - \epsilon_j} + \sum_{u=1}^{m} \sum_{v=u+1}^{m} d_{juv} x_{\epsilon_u - \epsilon_v} + \sum_{s=m+2}^{2m+1} f_{js} x_{\epsilon_{m+1} - \epsilon_s} + \sum_{s=1}^{m} k_{js} x_{\epsilon_s - \epsilon_{m+2}}$$

If $m + 3 \leq j, j' \leq 2m + 1$ and $j \neq j'$, choose $1 < v < m + 1$, then we compute

$$[x_{ij}, z_{j'}] = \sum_{u=1}^{v-1} d_{juv} N_{\epsilon_v - \epsilon_j, \epsilon_u - \epsilon_v x_{\epsilon_u - \epsilon_j}} + \sum_{t=1}^{i-1} a_{ijt} N_{\epsilon_t - \epsilon_{m+2}, \epsilon_{m+2} - \epsilon_j} x_{\epsilon_t - \epsilon_j}$$

As $m \geq 3$, this gives $d_{juv} = 0$ and consequently $a_{ijt} = 0$ by seeing the coefficient of $x_{\epsilon_{i} - \epsilon_{j}}$ in $[x_{ij}, z_{j'}]$. If $1 \leq i, j' \leq m$ and $i \neq j'$, then the coefficient of $x_{\epsilon_i - \epsilon_{j'}}$ in $[y_i, y_{j'}]$ is $N_{\epsilon_i - \epsilon_{m+1}, \epsilon_{m+1} - \epsilon_{j'}} b_{i j'}$, so $b_{is} = 0$ for all $i$ and $s$. Now let $\xi = -c_{11} N_{\epsilon_{m+1} - \epsilon_{m+2}, \epsilon_{m+2} - \epsilon_{m+1}}$, conjugation given by $\exp(\text{ad}(\xi x_{\epsilon_{m+1} - \epsilon_{m+2}}))$ lets
us assume that $c_{11} = 0$. Then we compute

$$[y_1, z_j] = \sum_{s=m+2}^{2m+1} f_{js} N_{\epsilon_1-\epsilon_{m+1}-\epsilon_s-\epsilon_j} + \sum_{s=2}^{m} c_{1s} N_{\epsilon_s-\epsilon_{m+2}-\epsilon_j} x_{\epsilon_s-\epsilon_j}$$

It follows that $c_{1s}$ and $f_{js}$ are zero. Further $c_{is} = 0$ for all $i$ and $s$ by computing $[y_i, z_i]$. Finally, the coefficient of $x_{\epsilon_s-\epsilon_i}$ in $[z_i, z_j]$ for $i \neq j$ is $N_{\epsilon_{m+2}-\epsilon_i-\epsilon_s-\epsilon_{m+2}} k_{js}$, so $k_{js} = 0$ for all $j$ and $s$. Now we have $e = \text{Lie}(\Phi_{m+1}^{\text{ev}})\exp(\text{ad}(ax_{m+1}))$ for $a = -\xi$ and complete the proof. □

2.3. Irreducible components.

Definition. ([12, Definition 2.10]) We say $R \subset \Phi^+$ is an ideal if $\alpha + \beta \in R$ whenever $\alpha \in R, \beta \in \Phi^+$ and $\alpha + \beta \in \Phi^+$. 

Lemma 2.3.1. Suppose that $G$ is of type $A_n$ with $n \geq 5$. Then

$$E(\text{rk}_p(g) - 1, u)_{\text{max}} \subseteq \bigcup_{R\text{ an ideal}} G.\text{Lie}(R),$$

and the ideals occurring here for each type are listed in the third column of the following Table

| Type   | Restrictions on rank | Ideal $R$                |
|--------|----------------------|--------------------------|
| $A_{2m+1}$ | $m \geq 2$          | $\Phi_m^{\text{rad}}, \Phi_{m+2}^{\text{rad}}, \Phi_{m+1}^{\text{rad}} \setminus \{\alpha_{m+1}\}$ |
| $A_{2m}$   | $m \geq 3$          | $\Phi_m^{\text{rad}} \setminus \{\alpha_m\}, \Phi_{m+1}^{\text{rad}} \setminus \{\alpha_{m+1}\}$ |

Table 5. Ideals for Lemma 2.3.1

Proof. For type $A_{2m+1}$, $\Phi_{m+1}^{\text{rad}}$ and $\Phi_{m+2}^{\text{rad}}$ both are ideals, and $\Phi_{m+1,m+2}^{\text{odd}}$ can be conjugated to $\Phi_{m+1}^{\text{rad}} \setminus \{\alpha_{m+1}\}$ by a simple reflection $s_{m+1}$. For type $A_{2m}$, $\Phi_{m,m+1}^{\text{ev}}$ is conjugate to $\Phi_m^{\text{rad}} \setminus \{\alpha_m\}$ by $s_m$, and $\Phi_{m+1,m+2}^{\text{ev}}$ is conjugate to $\Phi_{m+1}^{\text{rad}} \setminus \{\alpha_{m+1}\}$ by $s_{m+1}$. Then it is a summarization of Theorem 2.2.1 and Theorem 2.2.2. □

Corollary 2.3.2. Let $G$ be a standard simple algebraic $k$-group with root system $A_n$ ($n \geq 5$). Then

$$(*) \quad E(\text{rk}_p(g) - 1, g) = \bigcup_{R\text{ an ideal}} G.\text{Lie}(R) \cup \bigcup_{I\text{ an ideal}} G.E(\text{rk}_p(g) - 1, \text{Lie}(I))$$

is the union of irreducible closed subsets, where $R$ is taken from Table 5 and $I$ is given as follows:

| Type   | Restrictions on rank | Ideal $I$                |
|--------|----------------------|--------------------------|
| $A_{2m+1}$ | $m \geq 2$          | $\Phi_{m+1}^{\text{rad}}$ |
| $A_{2m}$   | $m \geq 3$          | $\Phi_m^{\text{rad}}, \Phi_{m+1}^{\text{rad}}$ |

Table 6. Ideals for Corollary 2.3.2
Proof. Let \( R \) be an ideal in Table 4 and \( I \) be an ideal in Table 5. We define \( X_1 := \text{Lie}(R) \), \( X_2 := \mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(I)) \) and \( Y := \mathbb{E}(\text{rk}_p(g) - 1, g) \). Since \( X_2 \) is a projective variety, it is complete, implying that \( X_2 \) is closed in \( Y \). Since \( R \) and \( I \) are ideals, it follows that \( X_i \) is stabilized by a parabolic subgroup of \( G \) for \( i \in \{1, 2\} \) respectively. By [12, Theorem 4.9] for \( X_1 \) and [6, Proposition 0.15] for \( X_2 \), we have \( G.X_i \) is closed in \( Y \), where \( i \in \{1, 2\} \).

Since \( Y \) is a \( G \)-variety, \( G.X_1 \) is irreducible as a \( G \)-orbit. Since \( \text{Lie}(I) \) is an elementary subalgebra of \( g \), it follows that \( X_2 = \text{Gr}_{\text{rk}_p(g) - 1}(\text{Lie}(I))(k) \) is the Grassmannian which is irreducible. Then \( G.X_2 \) as the image of \( X_2 \) under \( G \) is irreducible. As a result, the right hand of (\( * \)) is the union of irreducible closed subsets.

By utilizing Lemma 2.3.1 along with [12, Sect. 3.2/3.4], we have

\[
\mathbb{E}(\text{rk}_p(g) - 1, u) \subset \bigcup_{R \text{ an ideal}} G.\text{Lie}(R) \cup \bigcup_{I \text{ an ideal}} G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(I)).
\]

Therefore, we arrive at the equality of (\( * \)) according to [11, Lemma 2.2].

Lemma 2.3.3. Let \( \epsilon \) be an element of \( \mathbb{E}(r, g) \) and \( R \) be an ideal of commuting roots with \( |R| = r \). Assume that there is \( g \in G \), satisfying

\( g.\epsilon = \text{Lie}(R) \).

Then \( \text{LT}(\epsilon) \) and \( R \) are conjugate by an element of \( \mathbb{W} \).

Proof. By Bruhat decomposition of \( G \), there exist elements \( b, b' \in B \) and \( w \in \mathbb{W} \) such that \( g = bwb' \) where \( w \) is an element of \( N_G(T) \) whose image in the Weyl group \( \mathbb{W} \) is \( w \). Since \( g.\epsilon = \text{Lie}(R) \), we have \( w^{-1}.\epsilon = b^{-1}.\text{Lie}(R) \). Notice that \( R \) is an ideal, implying \( B \subset \text{Stab}_G(\text{Lie}(R)) \). Thus \( w^{-1}.\epsilon = \text{Lie}(R) \) and (\( ** \)) \( b'.\epsilon = \text{Lie}(R) \).

Observe that the action of \( U_\alpha \) on \( \epsilon \) is lower triangular with respect to \( \geq \) for \( \alpha \in \Phi^+ \). Then the equality (\( ** \)) gives \( \text{LT}(\epsilon) = w^{-1}.R \), as desired.

Theorem 2.3.4. Let \( G \) be a standard simple algebraic \( k \)-group with root system \( \Phi \) of type \( A_n \) \((n \geq 5)\). Then the irreducible components of \( \mathbb{E}(\text{rk}_p(g) - 1, g) \) can be characterised; see Table 7.

| Type | Restrictions on rank | Irreducible components |
|------|----------------------|------------------------|
| \( A_{2m+1} \) | \( m \geq 2 \) | \( G.\text{Lie}(\Phi_m^\text{rad}), G.\text{Lie}(\Phi_{m+2}^\text{rad}), G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi_{m+1}^\text{rad})) \) |
| \( A_{2m} \) | \( m \geq 3 \) | \( G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi_m^\text{rad})), G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi_{m+1}^\text{rad})) \) |

Table 7. Irreducible components for Theorem 2.3.4

Proof. By Corollary 2.3.2, it suffices to check the maximality of each irreducibly closed subset. Let \( R_v \) be an ideal of commuting roots of order \( \text{rk}_p(g) - 1 \) for \( v \in J := \{1, 2\} \). Let \( I_v \in \text{Max}(\Phi) \) be an ideal for \( v \in J \). We will apply Lemma 2.3.3 to the following three cases for \( \{u, v\} = J \):

1. \( G.\text{Lie}(R_v) \subset G.\text{Lie}(R_u) \).
2. \( G.\text{Lie}(R_v) \subset G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(I_u)) \).
3. \( G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(I_v)) \subset G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(I_u)) \).
We conclude that $R_u$ and $R_v$ are $\mathcal{W}$-conjugate from (1). In (2), we have $\text{Lie}(R_v) = g \cdot e$ for some $g \in G$ and $e \in \mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(I_u))$. Therefore $R_v$ and $\text{LT}(e)$ are conjugate by an element of $\mathcal{W}$. In (3), let $\gamma$ be the unique positive simple root in $I_v$ and $e$ be an element of $\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(I_u))$ such that $\text{Lie}(I_v \setminus \{\gamma\}) = g \cdot e$ for some $g \in G$. Then we have $I_v \setminus \{\gamma\}$ and $\text{LT}(e)$ are $\mathcal{W}$-conjugate.

Now we are in the position to classify the irreducible components for $A_n (n \geq 5)$:

- Type $A_{2m+1}$. (a) $G.\text{Lie}(\Phi^{\text{rad}}_{m+1} \setminus \{\alpha_{m+1}\})$ is not maximal because $\text{Lie}(\Phi^{\text{rad}}_{m+1} \setminus \{\alpha_{m+1}\})$ is an element of $\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi^{\text{rad}}_{m+1}))$. (b) If $G.\text{Lie}(\Phi^{\text{rad}}_{m}) \subseteq G.\text{Lie}(\Phi^{\text{rad}}_{m+2})$ or $G.\text{Lie}(\Phi^{\text{rad}}_{m}) \subseteq G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi^{\text{rad}}_{m+1}))$ then $\Phi^{\text{rad}}_{m}$ is $\mathcal{W}$-conjugate to $\Phi^{\text{rad}}_{m+2}$ or conjugate to $\text{LT}(e(\Phi^{\text{rad}}_{m}))$. Both cases are impossible when we look at [12, Lemma 2.6], this gives $G.\text{Lie}(\Phi^{\text{rad}}_{m})$ is maximal. (c) $G.\text{Lie}(\Phi^{\text{rad}}_{m+2})$ and $G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi^{\text{rad}}_{m+1}))$ are maximal by the same argument of (b).

- Type $A_{2m}$. (a) $G.\text{Lie}(\Phi^{\text{rad}}_{m} \setminus \{\alpha_{m}\})$ and $G.\text{Lie}(\Phi^{\text{rad}}_{m+1} \setminus \{\alpha_{m+1}\})$ are not maximal since they are contained in $G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(R'))$ for $R' = \Phi^{\text{rad}}_{m}, \Phi^{\text{rad}}_{m+1}$ respectively. (b) We claim $G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi^{\text{rad}}_{m}))$ and $G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi^{\text{rad}}_{m+1}))$ are maximal. Without loss of generality, we may assume that

$$G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi^{\text{rad}}_{m})) \subseteq G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi^{\text{rad}}_{m+1})).$$

Then $\Phi^{\text{rad}}_{m} \setminus \{\alpha_{m}\}$ is $\mathcal{W}$-conjugate to $\Phi^{\text{rad}}_{m+1} \setminus \{\gamma\}$ where $\gamma = \alpha_1 + \cdots + \alpha_{2m}$ is the highest root, and consequently $\Phi^{\text{rad}}_{m+1} \setminus \{\gamma\}$ are conjugate. Notice that the Weyl group of $A_{2m}$ is the permutation group $S_{2m+1}$. Let $w.\Phi^{\text{rad}}_{m} \setminus \{\gamma\} = \Phi^{\text{rad}}_{m+1} \setminus \{\gamma\}$ for some $w \in \mathcal{W}$ and $m+1 \leq j_0 < 2m+1$. We denote by $w(j_0)$ the corresponding action for $j_0$ when $w$ acts on $\{\epsilon_i - \epsilon_{j_0}\}_{1 \leq i < m+1} \subseteq \Phi^{\text{rad}}_{m} \setminus \{\gamma\}$. Then $w.\{\epsilon_i - \epsilon_{j_0}\}_{1 \leq i < m+1} \subseteq \{\epsilon_i - \epsilon_{w(j_0)}\}_{1 \leq i < m+2} \subseteq \Phi^{\text{rad}}_{m+1} \setminus \{\gamma\}$ and $w.\{\epsilon_i - \epsilon_{j_0}\}_{1 \leq i < m+1} \not\subseteq \{\epsilon_i - \epsilon_{w(j_0)}\}_{1 \leq i < m+2}$ for $m+1 \leq r < 2m+1$ with $r \neq j_0$. As $m \geq 3$, there exists $j_0$ such that $w(j_0) \neq 2m+1$. Then the equality $|\{\epsilon_i - \epsilon_{j_0}\}_{1 \leq i < m+1}| < |\{\epsilon_i - \epsilon_{w(j_0)}\}_{1 \leq i < m+2}$ shows the impossibility.

\[\square\]

**Remarks.** We would like to refer the reader to the CAU-thesis [9] for other classical types:

| Type | Restrictions on rank | Irreducible components |
|------|----------------------|------------------------|
| $B_n$ | $n \geq 5$ | $G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(S_1))$ |
| $C_n$ | $n \geq 3$ | $G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi^{\text{rad}}_{m}))$ |
| $D_n$ | $n \geq 6$ | $G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi^{\text{rad}}_{m+1})), G.\mathbb{E}(\text{rk}_p(g) - 1, \text{Lie}(\Phi^{\text{rad}}_{m+1}))$ |

**Table 8.** Irreducible components for other classical types

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School of Sciences, Zhejiang A&F University, 311300 Hangzhou, Zhejiang, China

Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Str.4, 24098 Kiel, Germany

E-mail address: ypan@outlook.de