Consistency and lattice renormalization of the effective theory for heavy quarks

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Abstract

The effective theory describing infinite mass particles with a given velocity, has a great interest in heavy flavor physics. It has the unpleasant characteristic that the energy spectrum is unbounded from below; this fact is the source of the problems in the formulation of the euclidean theory. In this paper we present an analysis of the euclidean effective theory, that is rather complete and has positive conclusions. A proof of the consistency of the euclidean theory is presented and a technique for the evaluation of the amplitudes in perturbation theory is described. We compute also the one-loop renormalization constants of the lattice effective theory and of the heavy-heavy current that is needed for the determination of the Isgur-Wise function. A variety of effects related to the explicit breaking of the Lorentz symmetry of lattice regularization is demonstrated. The most peculiar phenomenon is that the heavy quark velocity receives a finite renormalization. Finally, we compute the lattice-continuum renormalization constant of the Isgur-Wise current. It is needed for the conversion of the values of the matrix elements computed with the lattice effective theory, to the values in the full theory.
\section{Introduction}

A test of the Standard Model can be realized in the quark mixing sector by measuring the entries of the CKM matrix and checking the unitary relations. The extraction of the values of the matrix elements from the experimental data is possible only if one is able to compute the effects of the strong interaction on the weak process. A first principle technique for the evaluation of the matrix elements of the weak hamiltonian between hadron states is lattice QCD \cite{1}. Since present lattice cut-offs are at most $2 \div 3\, GeV$, it is not possible to simulate directly the dynamics of heavy quarks. To circumvent this problem, effective theories have been constructed in which the heavy quark masses are sent to infinity \cite{2, 3, 4}.

A method to get a precise determination of $|V_{cb}|$ is offered by the analysis of the decays \cite{5}:

$$B \rightarrow D^{(*)} + l + \nu_l$$  (1)

The hadronic matrix elements for the processes (1) can be safely computed with lattice QCD by sending the beauty and the charm masses to infinity: $M_b, M_c \rightarrow \infty$. In this limit the matrix elements can be expressed in terms of a single function, the Isgur-Wise function $\xi \cite{6, 7}$:

$$\langle D, v | V_\mu(0) | B, v' \rangle = \sqrt{M_D M_B} \left( v_\mu + v'_\mu \right) \xi(v \cdot v')$$

$$\langle D^*, v, \epsilon | V_\mu(0) | B, v' \rangle = -i\sqrt{M_D M_B} \epsilon_{\mu\nu\alpha\beta} \epsilon^{\nu'} v'^\beta v^\alpha \xi(v \cdot v')$$

$$\langle D^*, v, \epsilon | A_\mu(0) | B, v' \rangle = \sqrt{M_D M_B} (\epsilon_\mu (1 + v \cdot v') - v_\mu v'_\epsilon) \xi(v \cdot v')$$  (2)

where $v'$ and $v$ denote respectively the $b$ and $c$ quark 4-velocities. $\xi(v \cdot v')$ is normalized at zero recoil, $\xi(1) = 1$.

The computation of the Isgur-Wise function with lattice QCD requires the formulation of the theory of infinite mass quarks with non zero velocity \cite{8} on an euclidean lattice.

There are also other interesting applications. The effective theory allows the computation with lattice QCD of the production rate of heavy mesons in $e^+e^-$ annihilations \cite{9}:

$$e^+e^- \rightarrow D^{(*)} + \overline{D}^{(*)}, \quad B^{(*)} + \overline{B}^{(*)}$$  (3)
For center of mass energies far away from the masses of the $c\bar{c}$ or $b\bar{b}$ resonances, the heavy quarks are produced by the electromagnetic current with velocities that are not appreciably changed by the hadronization. The approximation of neglecting recoil effects for the heavy quark dynamics is equivalent to the infinite mass limit.

Studies of the lattice effective theories have already appeared in the literature. The analytic continuation from minkowski to euclidean space has been treated in ref. [10]. Specific properties of the euclidean effective theories have been discussed in ref. [11]. These peculiarities originate from the fact that the energy spectrum of the effective theory is unbounded from below; it is the expansion of the energy-momentum relation of a heavy quark with momentum $\vec{p} = M\vec{v} + \vec{k}$ for small $\vec{k}$:

$$E = \sqrt{M^2 + \vec{p}^2} = Mv_0 + \vec{u} \cdot \vec{k} + \frac{\vec{k}^2 - (\vec{u} \cdot \vec{k})^2}{2M v_0} + \ldots$$ (4)

where $\vec{u}$ is the kinematical velocity, $\vec{u} = d\vec{x}/dt = \vec{v}/v_0$.

By removing the energy $Mv_0$ (it is a constant in a given velocity sector) and neglecting $1/M$ terms, one gets the energy-momentum relation of the effective theory,

$$\epsilon = \vec{u} \cdot \vec{k},$$ (5)

that is unbounded from below.

The presence of states with negative energy is then an intrinsic property of the effective theory and is related to the fact that one removes the energy $Mv_0$ associated to a non zero 3-momentum $M\vec{v}$. If the heavy quark picks up a residual momentum $\vec{k}$ with a component antiparallel to $M\vec{v}$, the energy decreases with respect to $Mv_0$, and one is left with negative energies in eq. (5). Nevertheless, the theory with $\vec{v} \neq 0$ is stable because it is generated with a Lorentz transformation of the static theory, and the latter has not negative energies ($\vec{u} = 0$ in eq. (3)). The states with negative energies are simply an effect of the change of reference frame and do not give rise to any instability.

The main consequences of the energy-momentum relation (5), as shown in ref. [11], are:

i) The free propagator of the heavy quark in space-time can be defined only introducing an ultraviolet cut-off $\Lambda$ on the residual momenta $\vec{k}$. 

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ii) The correlators of systems composed of the effective quark and light degrees of freedom reproduce correctly the lowest order in $1/M$ of the original theory. Considering proper observables, it is also possible to take the continuum limit $\Lambda \to \infty$. These results are suggested by the explicit computation of a heavy-light correlator in the free case.

iii) Simple Feynman rules in 4-momentum space cannot be derived.

This paper continues the analysis started in ref.[11] and is organized as follows.

In sec.2 we report a computation that includes the interaction among the effective and the light quarks, of the correlator considered in ref.[11] for the free case. The results corroborate the conclusion of ref.[11], namely that the effective theory has a sensible continuum limit and correctly reproduces the correlations of the full theory at lowest order in $1/M$.

In sec.3 it is presented a technique for computing amplitudes of the euclidean effective theory in perturbation theory. Only slight modifications with respect to ordinary field theories are needed. One still has propagators and vertices and the only difference with respect to an ordinary theory is that it is not possible to integrate any more the loops over real domains in momentum space. An $ie$ prescription is impossible, as is instead the case for the static theory [12], and one needs an additional rule for constructing contours for the energy integration.

Sec.4 deals with the lattice regularization of the effective theory. The problem of the doubling of the heavy quark species is discussed and the diagrams that are needed for the renormalization are calculated.

In sec.5 we discuss the renormalization of the lattice effective theory. A computation of the renormalization constants at full order $\alpha_S$ is presented. We compute also the renormalization constant of a current of the form $J = \overline{h_v} \Gamma h_{v'}$, where $h_v, h_{v'}$ are two heavy quark fields with 4-velocity $v$ and $v'$ respectively, and $\Gamma$ is a generic Dirac matrix. The renormalization of this operator is essential for converting the values of the Isgur-Wise function computed with lattice QCD to the values in the full theory.

In general, the full order $\alpha_S$ renormalization on the lattice shows a variety of effects related to the non-covariance of the regularization. Mass and wave function renormalization constants depend on the velocity of the heavy quark, contrary to the case of a covariant regularization (like dimensional regularization). The renormalization constant of the current $J$ does not depend only on $v \cdot v'$ (the only non trivial invariant given $v$ and $v'$), but on the
components of $v$ and $v'$ separately.

A very peculiar effect is also demonstrated, related to the fact that the effective theory contains an additional parameter with respect to the original theory, namely the heavy quark 4-velocity $v^\mu$. There is a finite renormalization of the velocity $\delta v$, that we have computed. It is absent in a covariant regularization. This effect does not spoil the effective theory of physical meaning and the original normalization of the velocity is maintained, i.e. $v_R^2 = v_B^2 = 1$, where $v_B$ is the 'bare' velocity and $v_R$ is the renormalized one $v_R = v_B + \delta v$.

In sec.6 we discuss the matching of the lattice effective theory with the original high-energy theory.

Sec.7 contains the conclusions of our analysis.

2 Consistency of the theory

In this section we present a study of the consistency of the euclidean effective theory.

We consider for simplicity a simple regularization with a cut-off $\Lambda$ on the spatial momenta. The free space-time propagator of a heavy quark in motion along the $z$ axis is given by [11] (see section 4 for a derivation):

$$iH_\Lambda(t, \vec{x}) = \int^\Lambda d^3k \frac{e^{i\vec{k} \cdot \vec{x}}}{(2\pi)^3} iH(t, \vec{k})$$

$$= \frac{\theta(t)}{v_0} \delta(x) \delta(y) \frac{1}{2\pi} e^{\Lambda(iz-ut)} - e^{-\Lambda(iz-ut)}$$

(6)

where in the last line the velocity $\vec{u}$ has been taken along the $z$ axis. $\delta_\Lambda$ is a regularized delta function.

We consider a correlator $G(t, \vec{k})$ of a system composed of an effective and a light particle. Since what matters in this context is the singularity structure of the amplitudes, let us consider for simplicity scalar particles. In the free case the correlator $G^{(0)}(t, \vec{l})$ is given by [11] (see fig.1):

$$G^{(0)}(t, \vec{l}) = \int d^3xe^{-i\vec{q} \cdot \vec{x}} iH(t, \vec{x} \mid 0) \Delta_F(0 \mid \vec{x}, t)$$

$$= \Theta(t)e^{-\vec{u} \cdot \vec{l} t} \int^\Lambda d^3k \frac{1}{(2\pi)^3 v_0 2E(m\vec{v} - \vec{k})} e^{-|\vec{u} \cdot \vec{k} + E(m\vec{v} - \vec{k})| t}$$

(7)
where $\Delta_F$ is the propagator of the light particle of mass $m$, $\vec{P}$ is the total momentum of the composite system, $\vec{q} = \vec{P} - MQ\vec{v}$ is the momentum of the meson in the effective theory and $\vec{l} = \vec{q} - m\vec{v}$ is the residual momentum. At large $|\vec{k}|$, the argument of the exponential becomes

$$\bar{u} \cdot \bar{k} + |\bar{k}|$$

and it is positive for $u < 1$. The negative energies of the effective particle are compensated by the positive energies of the light particle in the states with high virtuality. It is possible to take the continuum limit $\Lambda \to \infty$. The argument of the exponential and the prefactor in eq.(7) are the expansion of the full theory expressions, and the Green function $G^{(0)}(t,\vec{l})$ correctly describes the internal dynamics of the composite system in lowest order in $1/M$. At large times $t$ the correlator is dominated by the states with the lowest invariant mass (that eventually become the lowest bound state in the interacting theory), and it behaves like [11]:

$$G^{(0)}(t,\vec{l}) \sim e^{-(mv_0 + \bar{u} \cdot \bar{l}) t}$$

The 2-point function $G^{(0)}(t,\vec{l})$ describes then a system with an infinite mass, velocity $\bar{u}$, and residual momentum $\bar{l}$ (the energy $Mv_0$ is removed), as the result of the correct coupling of a light particle with an infinite mass particle.

We extend these results to the case of the interacting theory, considering the exchange of a single scalar massless particle. The correlator $G(t,\vec{l})$ is given at this order by:

$$G^{(1)}(t,\vec{l}) = G_a(t,\vec{l}) + G_b(t,\vec{l}) + G_c(t,\vec{l})$$

(10)

where the indices $a-c$ refer to the fig.2 at the end.

By explicit computation one derives:

$$G_a(t,\vec{l}) = \int^\Lambda d^3k d^3p \{ A(\vec{k},\vec{p}) \exp\left( -[\bar{u} \cdot \bar{k} + E(m\vec{v} + \bar{l} - \bar{k})]t \right) +$$

$$+ B(\vec{k},\vec{p}) \exp\left( -[\bar{u} \cdot (\bar{k} + \vec{p}) + E(m\vec{v} + \bar{l} - \bar{k} - \vec{p})]t \right) +$$

$$+ C(\vec{k},\vec{p}) \exp\left( -[\bar{u} \cdot (\bar{k} + \vec{p}) + E(m\vec{v} + \bar{l} - \bar{k} + |\vec{p}|)t \right) +$$

$$+ D(\vec{k},\vec{p}) \exp\left( -[\bar{u} \cdot \bar{k} + |\vec{p}| + E(m\vec{v} + \bar{l} - \bar{k} - \vec{p})]t \right) \}$$

(11)

where $\vec{p}$ is the momentum of the scalar particle exchanged and $\vec{k}$ is the momentum of the heavy particle after the interaction. The functions $A-D$
are very complicated functions of the loop momenta $\vec{k}$ and $\vec{p}$ and we do not report their expression. They are the correct expansion of the corresponding functions of the full theory.

For large loop momenta $|\vec{k}|$ and $|\vec{p}|$, the arguments of the exponentials in the square brackets of eq.(11) are positive and behave like:

$$\vec{u} \cdot |\vec{k}|, \quad \vec{u} \cdot (\vec{k} + \vec{p}) + |\vec{k} + \vec{p}|,$$

$$\vec{u} \cdot (\vec{k} + \vec{p}) + |\vec{k}| + |\vec{p}|, \quad \vec{u} \cdot |\vec{k} + \vec{p}| + |\vec{k} - \vec{p}|$$  \hspace{1cm} (12)

The mechanism of compensation of the energies works also in the interacting theory: the negative energies of the heavy 'quark' are compensated by the positive energies of the light 'quark' and/or the 'gluon'. It is then possible to take the continuum limit without encountering other divergences than those of a usual field theory. The functions $A$ and $B$ are multiplied by the exponentials with 2 energies, that depend respectively only on $\vec{k}$ and $\vec{k} + \vec{p}$. The divergences for $\Lambda \to \infty$ originate from the integration of $A$ over $\vec{p}$, and of $B$ over $\vec{k} - \vec{p}$.

At large times $t$ the correlator (11) behaves like the free one (9), apart from ultraviolet divergences in the vertex that can be factorized in the usual renormalization constants.

For the amplitudes (b) and (c) the same considerations as for the amplitude (a) hold. The explicit expression of $G_b$ is:

$$G_b(t, \vec{l}) = \int d^3 k d^3 p \{ E(\vec{k}, \vec{p}) \exp(- [\vec{u} \cdot \vec{k} + |\vec{p} - \vec{k}| - E(m \vec{v} + \vec{l} - \vec{p})] t ) +$$

$$+ [F(\vec{k}, \vec{p}) + tG(\vec{k}, \vec{p})] \exp(- [\vec{u} \cdot \vec{p} + E(m \vec{v} + \vec{l} - \vec{p})] t ) \} \hspace{1cm} (13)$$

where $\vec{p}$ is the momentum of the heavy quark before the emission of the scalar, and $\vec{k}$ the momentum of the heavy quark after the emission. The terms with $F$ and $G$ are associated respectively to the mass and wave function renormalization of the heavy quark and will be computed more easily in sec.4. $G_c$ is given by:

$$G_c(t, \vec{l}) = \int d^3 k d^3 p \{ M(\vec{k}, \vec{p}) \exp(- [\vec{u} \cdot \vec{k} + E(m \vec{v} + \vec{l} - \vec{k} - \vec{p}) + |\vec{p}|] t ) +$$

$$+ [N(\vec{k}, \vec{p}) + tP(\vec{k}, \vec{p})] \exp(- [\vec{u} \cdot \vec{k} + E(m \vec{v} + \vec{l} - \vec{k})] t ) \} \hspace{1cm} (14)$$

where $\vec{k}$ is the momentum of the heavy quark and $\vec{p}$ is the momentum of the 'gluon'.

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In lattice regularization the formulas (11-14) have obvious modifications; the energies of the particles are replaced by the energies in lattice regularization. One can easily check that the mechanism of compensation of the energies is not spoiled by lattice effects.

Even though the computation that we have presented takes into account the interaction only at the lowest order, we argue that the results shown have a general validity. Field fluctuations do not couple to the negative energies, that have a kinematical origin and do not point to any inconsistency.

3 Contour representation of the amplitudes

In this section we derive the rules for computing amplitudes of the euclidean effective theory in perturbation theory. A continuum regularization with a cut-off \( \Lambda \) on the spatial momenta is assumed for illustrative purposes; the variations for the lattice case are straightforward and will be discussed in sec.4.

As a first step, let us derive a contour representation of the euclidean effective propagator.

The correct propagator of the heavy quark \( H(t, \vec{k}) \), as a function of time \( t \) and spatial momentum \( \vec{k} \), has been derived in ref.[11]:

\[
iH(t, \vec{k}) = \frac{\Theta(t)}{v_0} e^{-\vec{u} \cdot \vec{k} t}
\]

It is forward in time, since it has to describe particle propagation only, and contains the correct energy-momentum relation (3). Because of the exponential increase with time associated with negative energy states \( \vec{u} \cdot \vec{k} < 0 \), the propagator (15) cannot be represented as the Fourier transform of a 4-momentum propagator. By allowing the euclidean energy of the heavy quark \( k_4 \) to be complex, one can write:

\[
iH(t, \vec{k}) = \int_C \frac{dk_4}{2\pi} \frac{\exp(ik_4t)}{iv_0k_4 + \vec{u} \cdot \vec{k}}
\]

where the contour \( C \) approaches the real line for \( k_4 \to \pm \infty \), is oriented in the same way, and passes below the singularity of the integrand, at \( k_4 = i\vec{u} \cdot \vec{k} \), for every sign of the energy. For positive energies \( C \) can be chosen as the
real axis, and in this case formula (16) reduces to the Fourier transform, while for negative energies $C$ has to be moved in the lower half plane. The representation (16) can also be derived by means of a Wick rotation in the complex plane of $k_0$, the energy in Minkowski space. The propagator of the heavy quark in Minkowski space is given by:

$$iH(k_0, \vec{k}) = \frac{i}{v_0 k_0 - \vec{v} \cdot \vec{k} + i\epsilon}$$

(17)

There is a pole in the lower half plane, at $k_0 = \vec{u} \cdot \vec{k} - i\epsilon$. To preserve the causal structure of the theory the Wick rotation has to be done without crossing the pole; for positive energies the pole stays in the right quadrant, and one can rotate the axes as in ordinary field theories. For negative energies the pole stays in the left quadrant, the rotation of the real axis has to be accompanied by a deformation, and this produces the contour in eq.(16).

In the static case $\vec{u} = 0$, the pole of the integrand in eq.(16) stays in the real axis and one can impose the correct analytic structure by an $i\epsilon$ prescription (14). The $i\epsilon$ of the static euclidean theory then has the meaning of a small mass or positive kinetic energy to ensure the decay of the correlations.

Let us consider now an amplitude containing a heavy quark propagator, for example, the self-energy graph of fig.4 In Minkowski space the amplitude is proportional to

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{v \cdot (p + k) + i\epsilon} \frac{1}{k^2 - \lambda^2 + i\epsilon}$$

(18)

where $p$ is the external momentum.

In the lower half-plane of $k_0$ there are the heavy quark and the gluon pole, at $k_0 = -p_0 + \vec{u} \cdot (\vec{p} + \vec{k}) - i\epsilon$, $E_\lambda - i\epsilon$, while in the upper half plane there is the 'antigluon' pole, at $k_0 = -E_\lambda + i\epsilon$, where $E_\lambda = \sqrt{\vec{k}^2 + \lambda^2}$ (see fig.3). The real line separates the poles of the particles from the poles of the antiparticles. In making the Wick rotation one has to deform the real axis in order to keep the same topology. The euclidean amplitude is then proportional to

$$\int_C \frac{d^4k}{(2\pi)^4} \frac{1}{iv_0(k_4 + p_4) + \vec{v} \cdot (\vec{p} + \vec{k})} \frac{1}{k^2 + \lambda^2}$$

(19)

where the contour $C$ divides the $k_4$-plane in two regions, one containing the gluon pole and the heavy quark pole, the other containing the 'antigluon'
pole. The integration over $k_4$ of (19) gives:

$$\frac{1}{v_0} \int \frac{d^3k}{(2\pi)^3} \frac{1}{[i\rho_4 + E_\lambda + \vec{u} \cdot (\vec{k} + \vec{p})]2E_\lambda}$$

(20)

where the integration extends now to the ordinary 3-momentum space.

By performing the integration over $k_0$ in (18) one gets:

$$\frac{i}{v_0} \int \frac{d^3k}{(2\pi)^3} \frac{1}{[-p_0 + E_\lambda + \vec{u} \cdot (\vec{k} + \vec{p})]2E_\lambda}$$

(21)

The two amplitudes are the correct continuation one of the other, i.e. they
give the same function of the external momentum $p^\mu$ if one sets $p_4 = ip_0$.

From the above example it is easy to derive the general rule for the
contour of integration $C$ of the euclidean energy $k_4$: $C$ must divide the $k_4$
plane into two connected regions, one containing only the poles of the full and
the effective particles, the other containing the antiparticles poles. To satisfy
this requirement the contour $C$ has to be deformed during the integration
over the spatial momenta.

For positive energies of the heavy quark the contour of integration of $k_4$
can be chosen as the real axis, and for negative energies the topology must
remain the same. The rule above can also be formulated in the following
way: one has to integrate $k_4$ over the real axis by assuming that the poles
of the effective particles stay always in the region they occupy for positive
energies.

4 Lattice regularization

We assume the regularization of the euclidean effective theory that has been
proposed in ref. [10], that is forward in time and symmetric in space. Considering
for simplicity a motion of the heavy quark along the $z$ axis, the action
$S$ is given by:

$$iS = -\sum_x v_0 \psi^\dagger(x) [\psi(x) - U^1_t(x)\psi(x-\vec{t})] +$$

$$-i \frac{v_0}{2} \psi^\dagger(x) [U^z(x + \vec{z})\psi(x + \vec{z}) - U^z_t(x)\psi(x - \vec{z})]$$

(22)
where $\vec{\mu}$ is a versor in the direction $\mu$, and $U_\mu(x)$ are the links related to the gauge field by: $U_\mu(x) = \exp[-igA_\mu(x - \vec{\mu}/2)]$.

Let us discuss the problem of the doubling of the heavy quark species. The energy-momentum relation of the heavy quark on the lattice is derived by computing the propagator as a function of time $t$ and the residual momentum $\vec{k}$:

$$iH(t, \vec{k}) = \int_C \frac{dk_4}{2\pi} \frac{e^{ik_4t}}{v_0(1 - e^{-ik_4}) + v_z \sin k_z} = \frac{\theta(t)}{v_0} e^{-(t+1)\ln(1+u_z \sin k_z)}$$

One has then:

$$\epsilon = 1 + u_z \sin k_z$$

(23)

The energy is zero not only at $k_z = 0$ but also at $k_z = \pi$, implying that the lattice regularization has produced a duplication of the low energy excitations. A regularization that is forward in time and in space is also affected by the doubling problem. In this case the propagator is given by:

$$iH'(k) = \frac{1}{v_0(1 - e^{-ik_4}) - iv_z(1 - e^{-ik_4})}$$

(25)

and the energy-momentum relation is:

$$\epsilon' = \ln[1 - iu_z(1 - e^{-ik_z})]$$

(26)

The energy is a complex function of $\vec{k}$ and the doubling occurs when $\epsilon'$ is purely imaginary, at:

$$\cot(k_z/2) = -u_z$$

(27)

We show now by a physical argument that the doubling has not any significant effect in the phenomenological applications of the effective theory. Consider a meson composed of a effective quark $Q$ and a light antiquark $\bar{q}$, with total momentum $\vec{P}$. We assume that the doubling has been removed for the light quark. The effective theory deals with the residual momentum $\vec{k}$ of the meson:

$$\vec{k} = \vec{P} - M_Q \vec{v}$$

(28)
where $M_Q$ is the heavy quark mass.
It holds:

$$
\vec{k} = \vec{k}_Q + \vec{k}_l
$$

(29)

where $k_Q$ and $k_l$ denote respectively the momentum of $Q$ and of the light degrees of freedom.

Since the large mass scale $M_Q$ is removed, one expects, after renormalization, $| \vec{k} |$ to be of the order of the hadronic scale $\Lambda_{QCD}$, that is much less than the lattice cut-off $1/a$.

This implies that when $(k_Q)_z = \pm \pi/a$ and the energy of the effective quark is zero, the light quark momentum $\vec{k}_l$ is very near to the ultraviolet cut-off and its kinetic energy is very large, of the order of $1/a$. The configurations in which the heavy quark has a momentum at the edge of the Brillouin zone are then suppressed because of the large energy of the heavy-light system, as it should be. The situation is similar to that of a light meson composed of a Wilson fermion $(r \neq 0)$ and a naive fermion $(r = 0)$. For small meson momenta $| \vec{P} | \ll 1/a$, the internal dynamics is described correctly, even though there is a duplication of the meson species.

The doubling has a negligible effect also in the dynamics of the transition of a heavy meson into a heavy meson (the dynamics of the Isgur-Wise form factor). Due to the change of velocity of the heavy quark after $W$ emission, the typical momentum transfer $q^\mu$ between the heavy quark and the light degrees of freedom may be greater than $\Lambda_{QCD}$. By dimensional arguments one expects $q^\mu \sim \Lambda_{QCD} v \cdot v'$. If also this scale is assumed to be much less than $1/a$, the typical momentum exchanges lie in a region in which lattice effects are negligible.

Assuming a convention for the Fourier transform according to which $\psi(x) \sim \exp(ik \cdot x)$, one derives from the action (22) the following Feynman rules:

$$
iH(k) = \frac{1}{v_0(1 - e^{-ik_4}) + v_z \sin k_z}
$$

(30)

$$
V_0 = i \ g \ v_0 \ t_a \ e^{-i(k_4 + k'_4)/2}
$$

(31)

$$
V_z = g \ v_z \ t_a \ \cos(k_z/2 + k'_z/2)
$$

$$
V_{0}^{tad} = -\frac{g^2 v_0}{2} \ t_a t_b \ e^{-ik_4}
$$

(32)
where $k$ and $k'$ denote respectively the momenta of the incoming and out-
coming heavy quark, $V_0$ and $V_z$ are the interaction vertices of the heavy quark
with a gluon with a polarization along the time or the $z$ axis. $V_{tad}$ are the
vertices of emission of two gluons, for the case of the tadpole graph ($k = k'$).

It is to note that the conventions for the sign of the Fourier trans-
form and of the velocity are not independent, if one wants to interpret $k$ as the residual
momentum of the heavy quark. If one assumes a convention ac-
(\psi(x) \sim \exp(-ik \cdot x), only the sign of $k_4$ changes in the above Feynman
rules (i.e. the sign in front of $v_z$ in eq.(22) is changed).

In usual lattice field theories every component of the loop momentum
$k_{\mu}$ is integrated in the interval $[−\pi, +\pi]$, i.e. exp(i$k_{\mu}$) is integrated along a
unitary circle. For the effective theory the integration contour of exp(i$k_4$)
has to be distorted in order to keep the poles of the effective quarks always
in the right region of the exp(i$k_4$)-plane, the region of the full particle poles.
The rule for the contour in lattice regularization is analogous to the case of
the continuum discussed in sec.3.

We apply now these rules to the calculation of the amplitudes that are
needed for the renormalization of the effective theory. The infrared diver-
gences are regulated by a fictious gluon mass $\lambda$.

The self-energy graph of fig.4 is given by:

$$A(p) = -g^2 C_F \int \frac{d^4 k}{(2\pi)^4} \frac{v_0^2 e^{-i(p_4 + k_4)} - v_z^2 \cos^2(p_z + k_z/2)}{v_0(1 - e^{-i(k_4 + p_4)}) + v_z \sin(k_z + p_z)} \frac{1}{\Delta(k)} \quad (33)$$

where the integration region is the domain $[-\pi, +\pi]^3 \times C$. $C_F = \sum t_a t_a = (N^2 - 1)/2N$ for an $SU(N)$ gauge theory, and $\Delta(k) = 2 \sum (1 - \cos k_\mu) + (a\lambda)^2$.

Since this integral has to be computed numerically, it is convenient to reduce
the integration region to a real domain. Making the contour integration
analytically, one gets:

$$A(p) = \frac{-g^2 C_F}{16\pi^2} \frac{1}{\pi} \int d^3 k \frac{1}{\sqrt{(1 + A)^2 - 1}} \times$$

$$\times \frac{v_0^2 z(k) e^{-2ip_4} - v_z^2 \cos^2(p_z + k_z/2)}{v_0(1 - z(k)e^{-ip_4}) + v_z \sin(k_z + p_z)} \quad (34)$$

where $z(k) = \int d^4 k' e^{-ik_4} \frac{1}{\sqrt{(1 + A)^2 - 1}} \times$

$$\times \frac{v_0^2 z(k') e^{-2ip_4} - v_z^2 \cos^2(p_z + k_z/2)}{v_0(1 - z(k')e^{-ip_4}) + v_z \sin(k_z + p_z)}$$

13
where $A = \sum_{i=1}^{3} (1 - \cos k_i) + \lambda^2/2$ and $z(k) = 1 + A - \sqrt{(1 + A)^2 - 1}$

The tadpole graph of fig.5 is given by:

$$T(p) = -\frac{g^2 C_F}{16\pi^2} \left( v_0 e^{-ip_4} - v_z \sin p_z \right) \frac{1}{2\pi^2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{\Lambda(k)}$$  \hspace{1cm} (35)

In this case there is no need to integrate over $k_4$ because the integrand does not contain any effective propagator and the integration region reduces to the ordinary one, $[-\pi, +\pi]^4$.

The vertex correction of the local heavy-heavy current $J(x) = \bar{h}_v(x)\Gamma h_{v'}(x)$, omitting the trivial spin structure $(1 + \hat{v}')/2 \Gamma (1 + \hat{v})/2$, is given by (see fig.6):

$$\delta V = -g^2 C_F \int \frac{d^4k}{(2\pi)^4} \frac{1}{\Lambda(k)} \times$$
$$\times \frac{v_0 v'_0 e^{-ik_4} - v_z v'_z \cos^2 (k_z/2)}{[v_0 (1 - e^{-ik_4}) + v_z \sin k_z][v'_0 (1 - e^{-ik_4}) + v'_z \sin k_z]}$$  \hspace{1cm} (36)

where we have taken the motion of the two heavy quarks along the $z$ axis, and we have set to zero the external momenta.

Integrating over $k_4$ one gets:

$$\delta V = \frac{g^2 C_F}{16\pi^2} \frac{-1}{\pi v_0 v'_0} \int \frac{d^3k}{\sqrt{(1 + A)^2 - 1}} \times$$
$$\times \frac{v_0 v'_0 z(k) - v_z v'_z \cos^2 (k_z/2)}{(1 - z(k) + u_z \sin k_z)(1 - z(k) + u'_z \sin k_z)}$$  \hspace{1cm} (37)

## 5 Lattice renormalization

In this section we describe the one-loop renormalization of the effective theory on the lattice.

The self-energy $\Sigma(k,v)$ of the heavy quark is given by the sum of the graphs considered in section 4:

$$\Sigma(k,v) = A(k,v) + T(k,v)$$  \hspace{1cm} (38)

The bare propagator is given by:

$$iH(k) = \frac{1}{v_0 (1 - e^{-ik_4}) + v_z \sin k_z + M_0 - \Sigma(k,v)}$$  \hspace{1cm} (39)
where we have inserted a bare mass term $M_0$ to compute the mass renormalization condition.

We impose on-shell renormalization conditions. Near the mass-shell the propagator looks like:

$$iH(k) = \frac{1}{(iv_0 - X)k_4 + (v_z - Y)k_z + M_0 - \sum(0) + O(k^2)}$$

(40)

where

$$X = \left(\frac{\partial \sum}{\partial k_4}\right)(0), \quad Y = \left(\frac{\partial \sum}{\partial k_z}\right)(0)$$

(41)

Because of lattice effects (see later), the vector $(X, Y)$ turns out to be not proportional to the euclidean velocity $(iv_0, v_z)$. This implies that mass and wave function renormalizations are not sufficient for a complete renormalization of the effective theory. This effect can be interpreted as a renormalization of the velocity. The velocity $v$ appearing in eq. (10) has to be identified with a 'bare' velocity $v_B$, that is modified by the field fluctuations into a 'renormalized' velocity $v_R = v_B + \delta v$. By comparing the bare propagator (40) to the expression in terms of the renormalized parameters

$$Z \frac{i(v_R)0k_4 + (v_R)z k_z + M_R + O(k^2)}{i(v_B)0k_4 + (v_B)z k_z + M_B + O(k^2)}$$

(42)

and imposing the normalization of the velocity

$$(v_R)^2 = (v_B)^2 = 1,$$

(43)

one gets, up to order first order in $\alpha_s$:

$$\delta M = -\sum(0)$$

(44)

$$\delta Z = -iv_0 X - v_z Y$$

(45)

$$\delta v_z = -iv_0 v_z X - v_0^2 Y$$

(46)

where $\delta Z = Z - 1$.

The explicit expression for the mass renormalization $\delta M$ is:

$$\delta M = \frac{g^2 C_F}{16\pi^2} \left[ \frac{1}{\pi v_0} \int d^3k \frac{v_0^2 z - v_z^2 \cos(k_z/2)}{\sqrt{(1 + A)^2 - 1}} \frac{1}{(1 - z + u_z \sin k_z)} + \frac{v_0}{4\pi^2} \int d^4k \frac{1}{\Delta_k} \right]$$

(47)
The first term comes from the amplitude (34) and has a relativistic invariant form for small $k$; it is a function of the velocity because of hard gluons. The second term originates from the tadpole graph and is then a lattice effect. It is also a function of the velocity because of the explicit factor $v_0$.

The mass renormalization $\delta M$ is a function of the velocity $u_z$. It is linearly divergent with the ultraviolet cut-off $1/a$ and can be written as:

$$\delta M = \frac{g^2 C_F}{16\pi^2} \frac{x(u)}{a}$$

The numerical values of $x(u)$ are reported in table at the end. The numerical error is at most one unit in the second decimal place. For $u = 0$ one recovers the static value already computed in ref. [13, 14]. At $\beta = 6$ the mass renormalization is about 17% of $1/a$ for $u = 0$ and decreases up to 9% at $u = 0.7$.

We note that the mass renormalization $\delta M$ in the effective theory with $\vec{v} \neq 0$ is in effect a renormalization of the residual momentum $k^\mu$ of the heavy quark. Indeed the heavy quark propagator can be written in the limit $a \to 0$ as:

$$\frac{1}{v \cdot k + \delta M} = \frac{1}{v \cdot (k - \delta M v)}$$

where $v$ is the euclidean 4-velocity, $v = (iv_0, \vec{v})$ and $v_0 = \sqrt{1 + \vec{v}^2}$.

One can restore the original form of the propagator, that has a pole at $k = 0$, by defining a renormalized residual momentum $k_R$ by means of the relation:

$$k_R = k - \delta M v$$

This effect has a very physical interpretation. The mass renormalization of the static quark is given by the energy of the Coulomb-like field surrounding the colour charge. For $\vec{v} \neq 0$, the Coulomb field moves rigidly with the source, and carries the 3-momentum $\delta M \vec{v}$ in addition to the energy $\delta M v_0$. If one wants to interpret $k$ as the fraction of the heavy quark momentum that is changed in the collisions, and that is zero in the absence of interactions with other particles, it is necessary to subtract the constant contribution from mass renormalization. In practise, it is not necessary to make the subtraction (50), because in the effective theory the energy-momentum relation is
linearized, and it does not matter if the expansion point is shifted by renormalization. The only effect of \( \delta M v \) is an additional constant decay of the propagator with time, according to
\[
iH(t, \vec{k}) \sim e^{-(\delta M/v_0 + \vec{u} \cdot \vec{k})t}
\] instead of
\[
iH(t, \vec{k}) \sim e^{-\vec{u} \cdot \vec{k} t}
\]

According to eq. (45) the expression for the renormalization constant of the field \( \delta Z \) is:
\[
\delta Z = \frac{g^2 C_F}{16\pi^2} \left[ \frac{1}{\pi} \int \frac{d^3 k}{(1 + A)^2 - 1} \frac{2v_0^2 z(k) + v_z^2 u_z \sin k_z}{1 - z(k) + u_z \sin k_z} + \right.
\]
\[
+ \left. \int \frac{d^3 k}{\sqrt{(1 + A)^2 - 1}} \frac{[v_0^2 z(k) - v_z^2 \cos^2(k_z/2)][z(k) - u_z^2 \cos k_z]}{[1 - z(k) + u_z \sin k_z]^2} \right] (53)
\]
The first term in eq. (53) is infrared finite and comes from the differentiation of the momentum-dependent vertices. The second term is infrared divergent, and the singularity is isolated with the technique introduced in ref. [15]; the remaining integral is evaluated numerically. Details are given in appendix A. One can write:
\[
Z(u) = 1 + \frac{g^2 C_F}{16\pi^2} \left[ -2 \ln(a\lambda)^2 + e(u) \right] (54)
\]
The coefficient of the logarithmic term, i.e. the anomalous dimension of the heavy quark field, is independent of the velocity. It is indeed the same in every regularization and does not depend on the velocity in a covariant regularization [19]. The finite term \( e(u_z) \) has a non trivial dependence on the velocity \( u_z \), and the numerical values are reported in the table. For \( u_z = 0 \) one recovers the static value already computed in ref. [13, 14].

The renormalization of the velocity \( \delta v_z \) is given, according to eq. (46), by:
\[
\frac{\delta v_z}{v_z} = \frac{g^2 C_F}{16\pi^2} \left[ \frac{1}{\pi} \int \frac{d^3 k}{(1 + A)^2 - 1} \frac{2v_0 z + v_z \sin k_z}{1 - z + u_z \sin k_z} + \right.
\]
\[
+ \left. \int \frac{d^3 k}{\sqrt{(1 + A)^2 - 1}} \frac{[v_0^2 z(k) - v_z^2 \cos^2(k_z/2)][z(k) - \cos k_z]}{[1 - z + u_z \sin k_z]^2} \right] (55)
\]
The tadpole graph does not contribute to the renormalization of the velocity, because the heavy quark propagator does not enter inside the loop and then it is not evaluated at large momenta. The first term in eq. (55) is a lattice effect while the second one has analog in the continuum and the integrand vanishes for small \( \vec{k} \) (\( z(k), \cos k \to 1 \) for \( \vec{k} \to 0 \)).

The velocity renormalization is a finite effect, because the infrared divergences cancel between \( X \) and \( Y \), and it can be written as:

\[
\frac{\delta v_z}{v_z} = \frac{g^2 C_F}{16\pi^2} c(u_z)
\]  

(56)

The numerical values of \( c(u_z) \) are reported in the table. At \( \beta = 6 \), formula (56) gives a positive renormalization of the velocity \( \delta v_z \) that increases from 10\% at \( u = 0.1 \) up to 18\% for \( u = 0.7 \).

Let us discuss now a method that allows in principle a non-perturbative computation of the renormalization of the velocity. We consider a specific example.

The correlator \( G_M(t, u) \) of a meson \( M \) composed of a light quark and an effective quark with kinematical velocity \( u \) behaves for \( t \to \infty \) like:

\[
G_M(t, u) \sim \exp(-\epsilon(u)t)
\]  

(57)

where \( \epsilon(u) \) is the binding energy of a meson with velocity \( u \) in the infinite mass limit. \( \epsilon \) is not a physical quantity since it contains the mass renormalization of the heavy quark \( \delta M(u) \), that is linearly divergent and has a complicate dependence on the velocity.

The correlator \( G_H(t, u) \) of an hyperion composed of light quarks and an effective quark with velocity \( u \) has a time dependence analogous to that in eq. (57), with \( \epsilon(u) \) replaced by \( \epsilon'(u) \), the hyperion binding energy. By taking the ratio of the 2-point functions, the mass renormalization contribution \( \delta M(u) \) to the binding energies cancels [17]:

\[
\frac{G_H(t, u)}{G_M(t, u)} \sim \exp[-\Delta \epsilon(u)t]
\]  

(58)

where \( \Delta \epsilon(u) = \epsilon'(u) - \epsilon(u) \) is the difference of the binding energies. \( \Delta \epsilon(u) \) is a physical quantity and, as such, satisfies the relativistic relation:

\[
\Delta \epsilon(u) = \gamma(u) \Delta \epsilon(0)
\]  

(59)
where $\gamma(u) = 1/\sqrt{1 - u^2}$ is the Lorentz factor.

By measuring $\Delta \epsilon(u)$ and $\Delta \epsilon(0)$ with numerical simulations, one can derive by means of eq.(59) the renormalized velocity $v_R$ of the heavy quark.

Let us consider now the vertex correction to the local heavy-heavy current $J = \overline{h_v} \Gamma h_{v'}$. The amplitude has already been reported in sec.4. A general parametrization is the following:

$$\delta V = \frac{g^2 C_F}{16\pi^2} \left[ 2(v \cdot v') r(v \cdot v') \ln(a\lambda)^2 + d(v, v') \right] \quad (60)$$

where $r(x) = 1/\sqrt{x^2 - 1} \ln[x + \sqrt{x^2 - 1}]$.

The logarithmic term in eq.(60), has already been computed in ref.[19]; it is a function of $v \cdot v'$, the only non trivial invariant that can be constructed with the velocities $v$ and $v'$ of the heavy quarks. The finite term $d$ is not universal and in lattice regularization depends separately on the components of $v$ and $v'$. The constant $d$ has been evaluated numerically for the case of one static quark, $u' = 0$, and one quark moving along one axis $\vec{u} = u \hat{z}$. The numerical values of $d(u)$ are reported in the table.

The one-loop matrix element of the current $J$ between heavy quark states is then given by:

$$\langle h_v | J | h_{v'} \rangle = 1 + \frac{1}{2} \delta Z(v) + \frac{1}{2} \delta Z(v') + \delta V(v, v')$$

$$= 1 + \frac{g^2 C_F}{16\pi^2} \left[ 2(v \cdot v') r(v \cdot v') - 1 \right] \ln(a\lambda)^2 + f(v, v') \right] \quad (61)$$

where we have used eqs.(54, 59) and we have defined:

$$f(v, v') = \frac{1}{2} e(v) + \frac{1}{2} e(v') + d(v, v') \quad (62)$$

In the normalization point $v \cdot v' = 1$ the anomalous dimension of $J$ vanishes due to the conservation of the effective current related to the flavor symmetry [8].

For the case of an initial static quark $\vec{u}' = 0$ and a final quark moving along the $z$ axis $\vec{u} = u \hat{z}$, the above matrix element reduces to

$$1 + \frac{g^2 C_F}{16\pi^2} \left[ \frac{1}{u} \ln \frac{1 + u}{2} - 2 \right] \ln(a\lambda)^2 + f(u) \right] \quad (63)$$
The values of \( f(u) \) are reported in the table at the end.

Let us discuss now the on-shell renormalization of the lattice effective theory in the real space \([13]\), instead of in momentum space \([14]\) as we have done up to now. The two schemes differ on the lattice and the relation between them has been clarified in \([17]\).

Near the mass-shell (i.e. at large times) the self-energy \( \Sigma(k) \) can be written as:

\[
\Sigma(k) = -\delta M + \delta Z( v_0(1 - e^{-ik_4}) + v_z \sin k_z ) + O(k^2) \tag{64}
\]

where for simplicity we have neglected the velocity renormalization \( \delta v_z \) that is not important in this context.

The bare propagator of the heavy quark on the lattice at order \( \alpha_s \), as function of time \( t \) and momentum \( \vec{k} \) is then given, at large times, by:

\[
iH(t, \vec{k}) = \int \frac{dk_4}{2\pi} e^{ik_4t} \left\{ iH(k_4, \vec{k}) + iH(k_4, \vec{k})[-\delta M + \delta Z(v_0(1 - e^{-ik_4}) + v_z \sin k_z)]iH(k_4, \vec{k}) \right\} = Z \frac{\theta(t)}{v_0} e^{-(t+1)\ln[1+u_z \sin k_z]} \left\{ 1 + \frac{-\delta M(t + 1)}{v_0(1 + u_z \sin k_z)} \right\}
\]

\[
= Z \frac{\theta(t)}{v_0} \exp\left\{ -(t + 1) \ln[1 + u_z \sin k_z + \delta M/v_0] \right\} \tag{65}
\]

where in the last line an exponentiation that is appropriate for large \( t \) has been done.

In the continuum limit \( a \to 0 \), the propagator \( \text{(65)} \) reduces to:

\[
iH(t, \vec{k}) = Z \frac{\theta(t)}{v_0} \exp\left\{ -(t + 1)(\delta M/v_0 + \vec{u} \cdot \vec{k}) \right\} \tag{66}
\]

The renormalization conditions in momentum space \( \text{(44-46)} \) imply then that the field renormalization constant \( Z \) is multiplied, for an evolution of time \( t \), by the exponential with \( t + 1 \) instead of \( t \).

The evaluation of the Isgur-Wise function on the lattice requires the computation of a 3-point function \( G \) containing two heavy quark propagators. According to eq. \( \text{(64)} \), the correlator \( G \) contains the factors \( \exp - (t + 1) \) and \( \exp - (t' + 1) \) for the times \( t \) and \( t' \) of the evolution of the two heavy quarks.
The bare propagator of the heavy quark with renormalization conditions in the real space is given, in the limit $a \to 0$, by:

$$Z' \theta(t) \frac{v}{v_0} \exp[-(\delta M'/v_0 + \vec{u} \cdot \vec{k})t]$$

(67)

Equating the expressions (66) and (67) one gets the relation between the renormalization constants in the two schemes:

$$Z' = Z - \frac{\delta M}{v_0}, \quad \delta M' = \delta M$$

(68)

There is a finite difference in the wave function renormalization constants $Z$ and $Z'$ because the mass renormalization $\delta M$ is linearly divergent, i.e. $\delta M a$ does not vanish as $a \to 0$.

The renormalization constant of the operator $J$ in the real space renormalization scheme is given by:

$$\langle h_v | J | h_{v'} \rangle = 1 + \frac{1}{2} \delta Z'(v) + \frac{1}{2} \delta Z'(v') + \delta V(v, v')$$

$$= 1 + \frac{g^2 C_F}{16\pi^2} [2(v \cdot v' r(v \cdot v') - 1) \ln(a\lambda)^2 + f'(v, v')]$$

$$= 1 + \frac{g^2 C_F}{16\pi^2} \left[ \frac{1}{u} \ln \frac{1 + u}{1 - u} - 2 \right] \ln(a\lambda)^2 + f'(u)$$

(69)

where in the last line the case $\vec{u}' = 0$ and $\vec{u} = u\vec{z}$ has been considered. The values of $f'(u)$ are reported in the table. Note that in the static limit $u = 0$, $f' = 0$, implying that the heavy-heavy current $J$ is exactly conserved on the lattice with real space renormalization conditions.

6 Matching

In this section we consider the matching of the lattice effective theory with the full theory in the continuum [18].

For a comparison of the theoretical rate of the decays (2) with the experimental one, it is necessary to convert the values of the form factors computed with the lattice effective theory, to the values in the original, 'true', theory. In general, it may be useful to regularize the full theory with a scheme different from the lattice one (and to renormalize it). This implies that the
lattice low-energy effective theory and the ’target theory’ differ both in the high energy excitations and in the regularization. The matching process is then conventionally divided in two steps:
i) Matching between the bare amplitudes of the effective theory computed with lattice regularization, and the bare (or renormalized) amplitudes computed with a continuum regularization.

ii) Matching between the amplitudes computed in the full and in the effective theory with the same regularization (or renormalization) scheme.

It is easy to see that both steps i) and ii), i.e. the whole matching process, can be done in perturbation theory.

In step i) we assume that infrared divergences are regulated in the same way. Then, the two regularizations differ only in the precise way in which they cut-off the high-momentum modes. The difference of the amplitudes in the two regularizations is related to hard parton effects, i.e. to partons with momenta of order $1/a \gg \Lambda_{QCD}$. Due to asymptotic freedom, this difference can be safely computed in perturbation theory.

In step ii) one has an effective theory that is an expansion of the full theory for momenta much less than the heavy quark mass $M_Q$. At zero external momenta, i.e. in the matching point, the loop amplitudes in the two theories differ only for virtual momenta of the order or greater than $M_Q$. Since $M_Q \gg \Lambda_{QCD}$ also in this case the difference can be computed with perturbation theory.

In this paper we consider step i); step ii) has been considered in ref. [19, 20].

The matrix element in eq. (69) is given in the $\overline{MS}$ scheme by:

$$\langle h_v | J | h_{v'} \rangle = 1 + \frac{g^2 C_F}{16\pi^2} (2 - \frac{1}{u} \ln \frac{1+u}{1-u}) \ln(\mu/\lambda)^2$$ (70)

The ratio of the $\overline{MS}$ matrix element divided by the lattice matrix element gives the factor $Z_m$ by which one has to multiply the values of the lattice simulation, to get the $\overline{MS}$ values:

$$Z_m = 1 + \frac{g^2 C_F}{16\pi^2} \left[ 2 - \frac{1}{u} \ln \frac{1+u}{1-u} \right] \ln(\mu a)^2 - f'(u)]$$ (71)

We have considered the real space renormalization scheme (77), that appears more natural for the lattice matrix element of the Isgur-Wise current.
One sees explicitly that the dependence on the gluon mass $\lambda$ cancels, implying that soft contributions cancel in the matching.

For a numerical evaluation of $Z_m$ one has to select a value for $\alpha_S$; at $\mu = 2$ GeV the lattice value is smaller than the continuum one by a factor 2.7. It is necessary to use a unique value of $\alpha_S$, otherwise the matching constant is no longer an infrared safe quantity. One has to make a guess for the higher orders of $g^2$ in $Z_m$. By using the lattice value for $\beta = 6$ and taking $\mu = 1/a$, the matching constant $Z_m$ is 1 at $u = 0$ and decreases with the velocity up to 0.95 at $u = 0.7$. With the continuum value of $\alpha_s$, $Z_m = 0.86$ at $u = 0.7$.

We note that in the general case $\vec{v} \neq 0$ and $\vec{v}' \neq 0$, the matching constant $Z_m$ depends separately on $v$ and $v'$, and not only on $v \cdot v'$. The matrix elements of $J$ computed on the lattice with different $v$ and $v'$ and the same $v \cdot v'$, must be multiplied with different matching constants $Z_m(v, v')$ to cancel the effects of the breaking of the Lorentz symmetry.

7 Conclusions

We have presented a fairly complete analysis of the effective theory for heavy quarks in euclidean space. The results are encouraging: the theory is consistent even though the energy spectrum is unbounded from below. It is stable and has a sensible continuum limit that reproduces the lowest order in $1/M$ of the full theory. It is possible to compute the amplitude of an arbitrary process of the effective theory in perturbation theory with a set of rules that include (in addition to the usual Feynman rules) a prescription for a contour integration over the energy.

Naive lattice regularization of the continuum euclidean action leads to a duplication of the heavy quark species. This phenomenon has not any significant effect in the applications to the phenomenology, and it is then not necessary to add a Wilson-like term to the heavy quark action.

The one-loop renormalization exhibits a series of phenomena that are consequences both of the non-covariance of the lattice regularization and of the fact that the effective theory selects a preferred direction in the space-time, namely the heavy quark 4-velocity.

These peculiar effects of the lattice effective theory do not pose any conceptual problem for the matching with the original relativistic theory. Effective theories describing heavy quarks with different velocities have to be
considered as different theories, due to the velocity super-selection rule that appears in the infinite mass limit \[8\]. All these effects of the lattice regularization can be removed with velocity-dependent counter-terms.

\section*{A Subtraction of the infrared singularities}

In this appendix we give the formulas that have been used for the subtraction of the infrared singularities of the loop integrals for \(a\lambda \to 0\), and for the numerical evaluation of the integrals.

The singularity is isolated by subtracting and adding back to the original integrand its expansion for small momenta. The difference is infrared finite and can be computed numerically in the limit \(a\lambda \to 0\). The term added back is simpler than the original integrand and can be integrated analytically.

In the limit of small momenta \(|ak_i| \ll 1\) the singular term in the wave function renormalization constant, eq(53), reduces to:

\[
h(\vec{k}) = \frac{1}{\pi v_0^2} \frac{1}{\sqrt{\vec{k}^2 + (a\lambda)^2}} \frac{1}{[\sqrt{\vec{k}^2 + (a\lambda)^2 + u k_z}]^2}
\] (72)

where the usual phase-space factor \(g^2 C_F/16\pi^2\) has been omitted.

The function \(h(\vec{k})\) can be easily integrated on a 3-sphere of radius \(R\):

\[
\int d^3 k \; h(\vec{k}) = 2 \ln(R/a\lambda)^2 + \ln 16 - \frac{2}{u} \ln \frac{1+u}{1-u}
\] (73)

Note that the linearized integral in eq.(73) depends on the velocity \(u\), since the domain of integration is a 3-sphere, that is not \(SO(4)\) invariant.

For the numerical computation, formula (53) is then replaced by:

\[
\frac{1}{\pi} \int \frac{d^3 k}{\sqrt{(1+A)^2-1}} \frac{2v_0^2 z(k) + v_z^2 u_z \sin k_z}{1-z(k) + u_z \sin k_z} +
\]

\[
+ \frac{1}{\pi} \int d^3 k \left\{ \frac{1}{\sqrt{(1+A)^2-1}} \frac{[v_0^2 z(k) - v_z^2 \cos^2(k_z/2)]z(k) - u_z^2 \cos k_z]}{[1-z(k) + u_z \sin k_z]^2} \right. +
\]

\[- \frac{1}{v_0^2} \frac{\theta(R-|\vec{k}|)}{|\vec{k}|(|\vec{k}|+uk_z)^2} \right\} + 2 \ln R^2 + \ln 16 - \frac{2}{u} \ln \frac{1+u}{1-u} - 2 \ln(a\lambda)^2
\] (74)
where the phase-space factor has been removed.

In the infrared limit $a k_i \to 0$, the integral of the vertex correction (37) reduces to:

$$I = -\frac{1}{\pi v_0 v'_0} \int \frac{d^3 k}{\sqrt{ k^2 + (a \lambda)^2 }} \frac{v \cdot v'}{\sqrt{ k^2 + (a \lambda)^2 + \vec{u} \cdot \vec{k} }}$$

with the usual factor removed. In the last line we have taken $\vec{u}' = 0$.

The integral above is not easy to compute analytically, and one has to make a further subtraction. $I$ has the same infrared singularity of the simpler integral

$$L = -\frac{1}{\pi} \int d^3 k \frac{1}{| \vec{k} |^2 + (a \lambda)^2} \frac{1}{| \vec{k} | + \vec{u} \cdot \vec{k} } = -\frac{1}{u} \ln \frac{1 + u}{1 - u} \ln(R/a \lambda)^2 \quad (76)$$

The vertex correction is then computed numerically by means of the following formula:

$$-\frac{1}{\pi} \int d^3 k \left\{ \frac{z(k)}{\sqrt{ (1 + A)^2 - 1 } [1 - z(k)][1 - z(k) + u_z \sin k_z]} + \frac{\theta(R - | \vec{k} |)}{| \vec{k} |^2 (| \vec{k} | + u k_z)} \right\} +$$

$$- \delta(u) - \frac{1}{u} \ln \frac{1 + u}{1 - u} \ln R^2 + \frac{1}{u} \ln \frac{1 + u}{1 - u} \ln(a \lambda)^2 \quad (77)$$

where $\delta(u)$ is a constant evaluated numerically:

$$\delta(u) = L - I =$$

$$= \frac{2}{u} \int_{0}^{\infty} \frac{dk}{1 + k^2} \left\{ \ln \frac{\sqrt{1 + k^2} + uk}{\sqrt{1 + k^2} - uk} - \ln \frac{1 + u}{1 - u} \right\} \quad (78)$$
Table

Numerical values of the renormalization constants

| u  | x(u) | e(u) | e’(u) | d(u) | c(u) | f(u) | f’(u) |
|----|------|------|-------|------|------|------|-------|
| .0 | 19.95| 24.48| 4.53  | -4.53| -    | 19.95| 0.00  |
| .1 | 19.89| 24.67| 4.88  | -4.58| 11.93| 20.00| 0.13  |
| .2 | 19.71| 25.29| 5.97  | -4.74| 12.30| 20.14| 0.51  |
| .3 | 19.37| 26.40| 7.93  | -5.05| 13.00| 20.39| 1.18  |
| .4 | 18.78| 28.19| 10.98 | -5.56| 14.07| 20.77| 2.19  |
| .5 | 17.75| 30.98| 15.61 | -6.46| 15.71| 21.27| 3.62  |
| .6 | 15.81| 35.51| 22.86 | -8.22| 18.09| 21.77| 5.48  |
| .7 | 11.17| 44.35| 36.37 | -14.4| 20.91| 19.88| 6.02  |

Acknowledgement

I wish to thank M. Crisafulli, R. Iengo, G. Martinelli, M. Masetti and M. Testa for many useful discussions.

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FIGURE CAPTIONS

Fig.1: correlator of a system composed of an effective 'quark' $Q$ (see text) and a light antiquark $\bar{q}$ in the free case.

Fig.2: the same correlator as in fig.1 with one gluon exchange.

Fig.3: Wick rotation for the self-energy graph of the effective quark $Q$ of fig.4. The crosses indicate the positions of the poles of the effective quark $Q$, of the gluon $g$ and of the antiguilon $\bar{g}$.
   a) Case of positive energy of $Q$: $\epsilon > 0$
   b) Case of $\epsilon < 0$ and $| \epsilon | < E_\lambda$, where $E_\lambda = \sqrt{k^2 + \lambda^2}$ is the gluon energy.
   c) Case of $\epsilon < 0$ and $| \epsilon | > E_\lambda$.

Fig.4: self-energy graph of order $\alpha_S$ of the heavy quark.

Fig.5: tadpole graph of order $\alpha_S$ of the heavy quark.

Fig.6: vertex correction of order $\alpha_S$ of the current $J$ describing the transition of a heavy quark with velocity $v'$ into a heavy quark with velocity $v$. 