Weak magneto-hydrodynamic turbulence of magnetized plasma

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Weak turbulence of magneto-hydrodynamic (MHD) waves in strongly magnetized plasma is studied when the plasma pressure is less than the magnetic field pressure. In this situation the main nonlinear mechanism is the resonance scattering of fast magneto-acoustic and Alfvénic waves on slow magneto-acoustic waves. As a result, the former waves serve as the high-frequency waves with respect to the latter ones so that the total number of HF waves - an adiabatic invariant - conserves additionally. In the weak turbulence regime this invariant is shown to generate the Kolmogorov type spectrum with a constant flux of HF waves towards large-scale region. In the short-wave region another Kolmogorov spectrum can be realized with a constant energy flux. The explicit angle dependences for both types of turbulent spectra are found for the propagation angles close to the direction of a mean magnetic field.

I. INTRODUCTION

The central place in theory of turbulence occupies the notion of spectrum of turbulence – the energy distribution on scales. The problem of its finding is one of the most difficult and nowadays is far from the complete solution. For developed hydrodynamic turbulence the pioneering works of A.N.Kolmogorov and A.M.Obukhov of 1941 about self-similar nature of turbulent spectra determined development of theory of turbulence for many years. In the seventieth years mainly by the efforts of V.E.Zakharov the Kolmogorov-Obukhov ideas were fruitfully applied to the weak wave turbulence theory (for details, see monograph and also the first papers devoted to this subject). Wave turbulence in some sense occurred to be simpler than developed hydrodynamic turbulence. The main reason of such simplicity is connected with the wave dispersion when there exists such a region of wave intensities when the nonlinear interaction between waves can be considered weak in comparison with dispersive effects. If initially phases of waves are distributed randomly then the nonlinear interaction can provide a weak correlation in phases of the interacting waves. By this reason such ensemble of waves can be described in terms of pare correlation functions the Fourier spectra of which coincide, up to the multiplier, with a number of waves \( n_k \) (occupation number) with the definite wave vector \( k \). In their turn, the occupation numbers \( n_k \) obey the kinetic equations for waves. The Kolmogorov spectra in this theory arise as stationary scale-invariant solutions of the kinetic equations amnullating their collision terms. These spectra, unlike the thermodynamic equilibrium ones, can be related to the solutions with nonzero fluxes over scales. They realize a constant flux over scales of some integral of motion: energy, number of waves, etc., in the so-called inertial interval. It is important that if for developed hydrodynamic turbulence the notion of the inertial interval – the region where influence of both pumping and dissipation can be neglected – in fact represents the hypothesis of locality, for the weak turbulence theory this property can be checked explicitly.

It is necessary to note that the main mass of works devoted to the Kolmogorov spectra of weak turbulence consider isotopic media (the corresponding bibliography can be found in). Influence of anisotropy, for instance, magnetic field in plasma, has been studied in the less extent. The first such example of Kolmogorov type spectra was considered by the author in 1972 for weak turbulence of magnetized ion-acoustic waves in plasma. For this case the collision term of the kinetic equation was shown to be invariant with respect to two independent scalings along and perpendicular the magnetic field that allowed, by means of generalization of the Zakharov transformation, to construct the Kolmogorov spectra with the power dependences relative to longitudinal \( (k_z) \) and transverse \( (k_\perp) \) components of the wave vector. Later, the ideas of these papers were used for finding Kolmogorov spectra for drift waves in plasma and for the Rossby waves (see, for instance,).

The present paper is devoted to weak turbulence of magneto-hydrodynamic waves in strongly magnetized plasma when the thermal pressure of plasma \( nT \) is small compared with the magnetic field pressure \( H^2/(8\pi) \) \( (\beta = 8\pi nT/H^2 \ll 1) \). In this case, in comparison with , , , turbulent spectra are defined from solution of three linked kinetic equations for Alfvénic, fast and slow magneto-acoustic waves.

For \( \beta \ll 1 \) the main nonlinear mechanism of the MHD waves is a scattering of fast magneto-acoustic and Alfvénic waves on slow magneto-acoustic waves (Section 2). In these processes, which can be considered as a partial case of

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decays of one wave to two another waves and the oppsite process of fusion, Alfvenic and fast magneto-acoustic waves play a role of high-frequency waves against slow magneto-acoustic waves. In each act of scattering the frequency change of the former waves (we will call them A-waves) is small enough, due to smallness of the parameter $\beta$. By this reason this process is familiar to the Mandelstamm-Brillouin scattering - scattering of electromagnetic waves on acoustic phonons. According to separation in time scales - division of all waves onto HF and LF waves - the decay interaction conserves, besides the energy, the adiabatic invariant – the total number of HF waves. However, this analogy with the Mandelstamm-Brillouin scattering has not been exhausted by the said above. It turns out that the matrix element of this interaction has maximum for the maximal value of longitudinal projection of momentum transfered by A-waves to slow magneto-acoustic waves. In particular, this result can be extracted from the expression for the growth rate of decay instability for the monochromatic Alfvenic wave obtained in 1962 by A.A.Galeev and V.N.Oraevskii [22].

Remind that for the Mandelstamm-Brillouin scattering the matrix element is also proportional to square root from the value of transferred momentum, that provides maximum for the back scattering of electromagnetic waves. Because of such behavior of the scattering amplitude of A-waves it is natural to assume that a stationary distribution of waves over angles will be very anisotropic, concentrated in the $k$-space along the magnetic field direction. Under such assumptions the kinetic equations possess two additional symmetries – invariance with respect to two independent stretching along and in transverse direction to the mean magnetic field that allow one to use the transformations of the paper [8]. Due to two these symmetries of the kinetic equations it turns out that in the transparency region there are possible two scale-invariant (against longitudinal and transverse wave vectors) Kolmogorov spectra corresponding to a constant flux of energy to the short-wave region - the direct cascade and to a constant flux of A-waves towards the large-scale region - the inverse cascade. This paper is based on the old results of the author [13] published in the form of preprint in Russian and unknown by this reason abroad (that is natural). But it turns out that these results are also unknown for Russian readers. In spite of more than twenty five years history nobody has not repeated these results. It is necessary to note, however, that recently the question about MHD turbulence has been considered in another limit $\beta \gg 1$ in [14]. This limit differs significantly from that considered in the present paper. First, at $\beta \gg 1$ plasma can be considered as almost incompressible fluid and, secondly, in this limit there is no essential difference between Alfvenic and slow magneto-acoustic waves: the latter waves have the same dispersion law as Alfvenic waves differing from them by polarization only. Such degeneracy sufficiently changes the character of nonlinear interaction.

In spite of these facts in this situation two kinds of Kolmogorov spectra are possible with the same dependences on wave numbers as the obtained ones in the present paper. However, the physical motivation of existance of the two Kolmogorov spectra at $\beta \gg 1$ is different.

Plan of the paper is the following. In Section 2 we introduce canonical description of an ideal MHD following to the original paper of 1972 by V.E.Zakharov and the author [11] and their recent review [12]. By means of the Hamiltonian description in the next – third section the average equations are derived for A-waves with account of interaction with slow magneto-acoustic waves. It is shown that from the side of A-waves to slow plasma motion there acts the HF force. The potential of this force is negative. Therefore, unlike the interaction of Langmuir waves with ion-acoustic waves [28], plasma is drawn into regions of A-wave localization forming there density humps. In the same section we analyze stability for the monochromatic A-waves. Section 4 is devoted to the Kolmogorov spectra of weak MHD turbulence.

II. VARIATIONAL PRINCIPLE AND CANONICAL DESCRIPTION

Consider the equations of ideal MHD for barotropic flows of plasma when the internal energy of plasma $\varepsilon$ depends only on its density $\rho$:

$$\frac{\partial \rho}{\partial t} + \text{div} \rho \mathbf{v} = 0;$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \nabla) \mathbf{v} = -\nabla w + \frac{1}{4\pi \rho} \left[ \text{rot} \mathbf{H} \times \mathbf{H} \right];$$

$$\frac{\partial \mathbf{H}}{\partial t} = \text{rot} \left[ \mathbf{v} \times \mathbf{H} \right].$$

Here $\mathbf{v}$ is plasma velocity, $w$ enthalpy connected with pressure $p = p(\rho)$ and internal energy $\varepsilon$ by the relations:

$$dw = \frac{dp}{\rho}, \quad w = \frac{\partial}{\partial \rho} \varepsilon(\rho).$$
Let us formulate the variational principle for this system.

First of all, it should be noticed that as follows from the equations (2.1, 2.3) the vector $H/\rho$ is advected by fluid particles, by another words, each magnetic line moves together with its own particles. This is the well known fact of frozenness of magnetic field into plasma (see, e.g. [25]). The given circumstance allows one to consider both the magnetic field $H$ and the density $\rho$ playing the roles of generalized coordinate.

To formulate the variational principle we shall use the well known expression of the Lagrangian of electromagnetic field and particles (fluid) within [21]. The Lagrangian $L$ should be written in the MHD approximation. In particular, this means that one needs to neglect by the contribution from the electric field $E$ in $L$ in comparison with that from the magnetic field since $E \sim v/cH \ll 1$ (here $c$ the light speed). Secondly, we shall account Eqs. (2.1), (2.3) and $\text{div } H = 0$ as constraints. As a result, the Lagrangian can be written as follows

$$L = \frac{\rho v^2}{2} - \varepsilon(\rho) - \frac{H^2}{8\pi} + S \cdot \left( \frac{\partial H}{\partial t} - \text{rot } [v \times H] \right) + \Phi \left( \frac{\partial \rho}{\partial t} + \text{div } \rho v \right) + \psi \text{ div } H.$$

Here $S, \Phi$ and $\psi$ are unknown Lagrange multipliers depending on $r$ and $t$.

Next, using this expression of the $L$ we write down the functional of action:

$$I = \int L dt \, dr,$$

which variations relative to the functions $v, \rho$ and $H$ yield the equations:

$$\rho v = [H \times \text{rot } S] + \rho \nabla \Phi,$$  \hspace{1cm} (2.4)

$$\frac{\partial \Phi}{\partial t} + (v \nabla) \Phi - \frac{v^2}{2} + w(\rho) = 0,$$  \hspace{1cm} (2.5)

$$\frac{\partial S}{\partial t} + \frac{H}{4\pi} - [v \times \text{rot } S] + \nabla \psi = 0.$$  \hspace{1cm} (2.6)

The first equation in this system gives the change of variables: velocity $v$ is expressed in terms of new variables $S$ and $\Phi$. It is necessary to note that this change of variables is ambiguous: it is possible to add $S$ the vector $S_0$, and to $\Phi$ the scalar $\Phi_0$ satisfying the equation

$$[H \times \text{rot } S_0] + \rho \nabla \Phi_0 = 0.$$

Two other equations in this system – Eqs. (2.3) and (2.6) – represent themselves the Bernoulli equation for the potential and the equation of motion for a new vector $S$ which contains unknown potential $\psi$. The potential $\psi$, in turn, is defined by fixing gauge of the vector $S$. For example, for the Coulomb gauge ($\text{div } S = 0$), $\psi$ is determined up to arbitrary solution $\psi_0$ of the Laplace equation $\Delta \psi_0 = 0$:

$$\psi = \frac{1}{\Delta} \text{div } [v \times \text{rot } S] + \psi_0.$$

In particular, if $v \to 0$, $H \to H_0$, $\rho \to \rho_0$ at infinity on $r$ then the value $\psi_0$ is convenient to be chosen so that $S \to 0$ at $r \to \infty$. Then obviously $\psi_0 = -(H_0 \cdot r)/(4\pi)$.

Now one needs to check that the system of equations (2.4-2.6) does not contradict to the original system of MHD equations. By plugging (2.4) into the equation of motion (2.2), after simple transformations, it is possible to verify that Eq. (2.4) can be written in the form:

$$\nabla \left( \frac{\partial \Phi}{\partial t} + (v \nabla) \Phi - \frac{v^2}{2} + w(\rho) \right)$$

$$+ \left[ \frac{H}{\rho} \times \text{rot } \left\{ \frac{\partial S}{\partial t} + \frac{H}{4\pi} - [v \times \text{rot } S] \right\} \right] = 0.$$

According to (2.3, 2.6) the obtained equation satisfies identically. Thus, one can say that the new system (2.3, 2.4, 2.5, 2.6) is equivalent the MHD system. Really, due to the formula (2.4) any solution of the system
generates some solution of the original MHD system. Under assumption about uniqueness of the Cauchy problem for the systems (2.1, 2.4) and (2.1, 2.3, 2.5, 2.6) opposite statement is valid also: for any solution of the system (2.1, 2.4) it is possible to put in correspondence some class of solutions of the system (2.1, 2.3, 2.5, 2.6). For this case it is enough by using a set of quantities \( v, H, \rho \) at some moment of time \( t_0 \) one to construct various sets of quantities \( S \) and \( \Phi \), satisfying the formula (2.3) and to take them as initial conditions for the system (2.1, 2.3, 2.5, 2.6).

After that, by means of the Lagrange function, we determine the generalized momentum and construct the Hamiltonian of the system, by the standard way:

\[
\mathcal{H} = \int \left( (S \cdot H) + \Phi \rho \right) dt + \int \left\{ \frac{\rho v^2}{2} + \left( \frac{H^2}{8\pi} - \psi \right) \right. \\
\left. \div H \right\} dr,
\]

which coincides with the total energy of the system. The equations of motion (2.1, 2.3, 2.5, 2.6) is this case are nothing more than the Hamilton equations:

\[
\frac{\partial \rho}{\partial t} = \frac{\delta \mathcal{H}}{\delta \Phi}, \quad \frac{\partial \Phi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \rho}, \quad \frac{\partial H}{\partial t} = \frac{\delta \mathcal{H}}{\delta S}, \quad \frac{\partial S}{\partial t} = -\frac{\delta \mathcal{H}}{\delta H},
\]

respectively, the variables \( (\rho, \Phi) \) and \( (H, S) \) are pairs of canonically conjugated values. Change of variables (2.4) and canonical description (2.7) were introduced for ideal MHD by Zakharov and the author in 1970 in the paper [11].

The transformation (2.4) represents analog of the Clebsch representation in the ideal hydrodynamics. Respectively, the fields \( H \) and \( S \) in the formula (2.4) play the same role as that of the Clebsch variables (about Clebsch variables see [18], [26], and recent review [12]). Later, in 1982, the authors of Ref [17] came, in fact, to the same change: the velocity vector and magnetic field were represented through the scalar Clebsch variables that after simple transformations can be reduced to (2.4).

MHD flows describing by means of (2.4) as well as flows of ideal fluids parameterized by the Clebsch variables are related to the partial type of flows. For example, for such MHD flows the topological invariant of linkage of magnetic field and vorticity lines, the so-called cross helicity \( I \) is identically equal to zero.

In 1995 Vladimirov and Moffatt [15] for ideal MHD found the analog of the Weber transformations:

\[
v = u_{0k}(a) \nabla a_k + \nabla \Phi + \frac{1}{\rho} [H \times \text{rot} S].
\]

Here \( a = a(r, t) \) are Lagrangian markers of fluid particles (this mapping is inverse to \( r = r(a, t) \) which defines particle trajectory with the marker \( a \)), \( u_0(a) \) is a new Lagrange invariant.

The Weber transformation (2.8) is the general transform which, in particular, contains at \( u_0 = 0 \), the change (2.4) to what the authors of [15] did not pay attention. It is interesting to note that the equations of motion for potentials \( \Phi \) and \( S \) for the general transform (2.8) have the same form as for (2.3) and (2.6). If \( \Phi \) and \( S \) are equal to zero at \( t = 0 \), then \( u_0(a) \) is nothing more than the initial velocity. Just the first term in (2.8) provides nonzero value of the topological invariant \( I \). One should note also that this term, as for ideal hydrodynamics, is nonlinear if one proceeds expansion over small amplitudes. Recently, Ruban [19] (see also [20]) clarified the physical meaning of the vector \( S \).

Curl of the vector \( S \) can be expressed through the shift \( d \) between electron and ion (as fluid particles) in the point \( r \) at the moment of time \( t \) if initially their coordinates coincide:

\[
\text{rot} S = \frac{e}{M c} d \rho_0(r, t).
\]

Here \( M, e \) are ion mass and its charge, respectively, \( \rho_0(a) \) the initial distribution of plasma density.

Introducing the canonical variables allow by the standard way (by means of the perturbation theory against small amplitudes of waves) both to classify and investigate all nonlinear processes. To this aim one needs in the expression for the velocity (2.8) as well as in the internal energy to perform expansion in powers of the canonical variables. In the presence of the external homogeneous magnetic field \( H_0 \), in the linear approximation, one needs to keep the linear terms relative to \( \Phi \) and \( S \), neglecting by the first (nonlinear) term in (2.8). As the result, in the velocity expansion

\[
v = v_0 + v_1 + ...,\]

(2.9)
the first order term is written as

\[ v_0 = \frac{1}{\rho_0} [H_0 \times \text{rot } S] + \nabla \Phi. \]

The three independent pairs (\( \text{div } H = \text{div } S = 0 \)) of canonically conjugated quantities will correspond to three types of waves. In the linear approximation, obviously, these waves will not interact. Their dispersion laws and polarizations can be found from analysis of the quadratic (relative to the canonical variables) Hamiltonian \( H_0 \). Three-wave interaction will correspond to the cubic (relative to the canonical variables) term \( H_3 \). Its value will be defined from the quadratic (against wave amplitudes) additions to the velocity

\[ v_1 = \frac{\rho_1}{\rho_0} [H_0 \times \text{rot } S] + \frac{1}{\rho_0} [h \times \text{rot } S]. \]

In this expression we take into account only 'wave' degrees of freedom and neglect by the first term in (2.8). Here \( h \) and \( \rho_1 \) are the magnetic field \( H \) and density \( \rho \) variations from their equilibrium values \( H_0 \) and \( \rho_0 \), respectively. As a result, the Hamiltonian represents a series with respect to powers of the wave amplitudes:

\[ H = H_0 + H_3 + ..., \tag{2.10} \]

where the quadratic Hamiltonian is

\[ H_0 = \int \left\{ \frac{\rho_0 v_0^2}{2} + \frac{h^2}{8\pi} + c_s^2 \frac{\rho_1^2}{2\rho_0^2} \right\} \text{d}r, \]

and

\[ H_3 = \int \left\{ \rho_0 (v_0 \cdot v_1) + \frac{\rho_1 v_1^2}{2} + q c_s^2 \frac{\rho_1^3}{2\rho_0^3} \right\} \text{d}r \]

is the cubic Hamiltonian. In these formulas square of the sound speed \( c_s^2 \) and the dimensionless coefficient \( q \) appeared from expansion of the internal energy in powers of \( \rho_1 \):

\[ \Delta \varepsilon(\rho) = \frac{\rho \alpha c_s^2}{2} \left\{ \left( \frac{\rho_1}{\rho_0} \right)^2 + q \left( \frac{\rho_1}{\rho_0} \right)^3 + ... \right\}. \]

Let us now perform the Fourier transform for coordinates and then introduce new variables \( a_j(k) \) \((j = 1, 2, 3)\) by means of the following formulas

\[ h(k) = e_1(k) \sqrt{2\pi \omega_1} (a_1(k) + a_1^*(-k)) + \]

\[ + e_2(k) \sum_{l=2,3} \lambda_l \sqrt{2\pi \omega_l} (a_l(k) + a_l^*(-k)); \]

\[ S(k) = -ie_1(k) \frac{1}{\sqrt{8\pi \omega_1}} (a_1(k) - a_1^*(-k)) - \]

\[ -ie_2(k) \sum_{l=2,3} \lambda_l \frac{1}{\sqrt{8\pi \omega_l}} (a_l(k) - a_l^*(-k)); \]

\[ \rho_1(k) = \sum_{l=2,3} \left( \frac{\rho_0 \omega_l}{2c_s^2 \omega_1} \right)^{1/2} \mu_1 (a_k(l) + a_k^{*-l}(l)) \]

\[ \Phi(k) = -i \sum_{l=2,3} \left( \frac{c_s^2}{2\rho_0 \omega_l} \right)^{1/2} \mu_1 (a_l(k) - a_l^*(-k)). \]
Here
\[ \omega_1(k) = |(k \cdot V_A)|, \]
\[ \omega_{2,3}(k) = \frac{1}{2} \sqrt{k^2 V_A^2 + k^2 c_s^2 + 2(k \cdot V_A)kc_s \pm \sqrt{k^2 V_A^2 + k^2 c_s^2 - 2(k \cdot V_A)kc_s}} \]
are the dispersion laws of Alfvenic waves (1), fast (2) and slow (3) magneto-acoustic waves, respectively;
\[ e_1(k) = \frac{[k \times n_0][k \cdot n_0]}{|k \times n_0||k \cdot n_0|}, \quad e_2(k) = \frac{[k \times [k \times n_0]]}{k|k \times n_0|} \]
are corresponding to them the unit polarization vectors; \( n_0 = H_0/H_0 \) the unit vector along mean magnetic field; \( V_A = H_0/(4\pi \rho_0)^{1/2} \) the Alfven velocity:
\[ \lambda_2 = -\mu_3 = - \left( 1 - \frac{\omega_1^2 - k^2 c_s^2}{\omega_2^2 - k^2 c_s^2} \right)^{1/2}, \]
\[ \lambda_3 = \mu_2 = \left( 1 - \frac{\omega_1^2 - k^2 c_s^2}{\omega_3^2 - k^2 c_s^2} \right)^{-1/2}. \]
The given change to the new variables \( a_k(j) \) represents the canonical \( U - V \) transformation diagonalizing the Hamiltonian \( H_0 \):
\[ \mathcal{H}_0 = \sum_j \int \omega_j(k)a_j(k)a_j^*(k)dk. \]
In this case the amplitudes \( a_j(k) \) have the meaning of normal variables, and, respectively, the equation of motion have the standard canonical form:
\[ \frac{\partial a_j(k)}{\partial t} = -i \frac{\delta \mathcal{H}}{\delta a_j^*(k)} \]
In the linear approximation \( a_j(k) \) obey the equations:
\[ \frac{\partial a_j(k)}{\partial t} + i\omega_j(k)a_j(k) = 0, \]

namely, with time the amplitude modulus \( |a_j(k)| \) does not change, but the phase increases linearly with \( t \).

In order to find the expression for the Hamiltonian of interaction in terms of the \( a_j(k) \) variables one needs to substitute transformation (2.11) into (2.10). As a result, the Hamiltonian of interaction will represent the integro-power series relative to these variables. In the lowest order with respect to the wave amplitudes the main nonlinear process will be the resonant three-wave interaction. The corresponding Hamiltonian is equal to
\[ \mathcal{H}_{\text{int}} = \frac{1}{2} \int \sum_{lmn} V^{l mn}_{k_1, k_2} a_l^*(k)a_m(k_1)a_n(k_2) + c.c. \delta_{k-k_1-k_2}dkdkd\kappa. \quad (2.12) \]
This Hamiltonian can be obtained after substitution of the transform (2.11) into the cubic Hamiltonian \( \mathcal{H}_3 \) and subsequent extraction from there the resonant terms. The rest terms in \( \mathcal{H}_3 \) are small; they can be excluded by means of the canonical transformation (for details see the review [13]). One should note that calculation of matrix elements \( V^{l mn}_{k_1, k_2} \) in this scheme is a pure algebraic procedure requiring performance of the Fourier transform in all integrals, substitution (2.11) and forthcoming symmetrization with respect to the \( a_k(k) \) variables, for instance, in (2.12) against pairs \( (k_1, m) \) and \( (k_2, n) \).
III. AVERAGE EQUATIONS

Expressions for dispersion laws and matrix elements of interaction can be sufficiently simplified for plasma with low $\beta = 8 \pi n T / H^2$ (it is ratio of thermal plasma pressure $nT$ and magnetic field pressure $H^2 / 8 \pi$). The condition $\beta \ll 1$ means that $V_A \gg c_s$. In this limit fast magneto-acoustic waves have isotropic dispersion law $\omega_2 = k V_A$ and their phase (as well as group) velocity coincides with the value of the group velocity of Alfvénic waves. In the linear approximation the plasma velocity in Alfvénic and fast magneto-acoustic waves is given by the expression

$$v_{HF} = \frac{1}{\rho_0} \{ [H_0 \times \text{rot} S] \}. $$

The potential part of the velocity $\nabla \Phi$ occurs to be small due to smallness of the parameter $\beta$. For slow magneto-acoustic waves, on the contrary, the main contribution is given by the potential part – the velocity turns out to be directed along the magnetic field $H_0$:

$$v_s = n_0 \frac{\partial \Phi}{\partial z}, \quad \omega_3 \equiv \Omega_s = |k_z| c_s, \quad (3.1)$$

and the dispersion law for slow magneto-acoustic waves becomes strongly anisotropic:

$$\omega_3 \equiv \Omega_s = |k_z| c_s. \quad (3.2)$$

The transverse components of the velocity for these waves $[H_0 \times \text{rot} S]/\rho_0$ are compensated by $\nabla \Phi$.

If plasma is collisionless and strongly isothermal ($T_e \gg T_i$), the slow magneto-acoustic waves represent themselves magnetized ion-acoustic waves (for details see [8]). In this case in (3.1) $c_s = \sqrt{T_e/M}$.

As far as the nonlinear interaction of the MHD waves concerns, for strongly magnetized plasma the main nonlinear process is the process of scattering of the Alfvénic and fast magneto-acoustic waves on the slow magneto-acoustic waves (that can be verified directly by comparing the computing matrix elements $V^{lmn}$ for (2.12)). In this process the former waves (further we shall call them as A-waves) plays the role of the high-frequency (HF) waves relative to the latter ones (these waves shall be simply called sound or $S$-waves). This conjecture follows directly from the analysis of the resonant conditions for the given type of decay:

$$\omega_A(k) = \omega_A(k_1) + \Omega_s(k_2), \quad k = k_1 + k_2. \quad (3.3)$$

Qualitatively it is easily to understand how this interaction looks like. While propagation of the packet of A-waves the mean characteristics of plasma (its density and mean velocity) due to the action of A-waves will be slowly varied. By this reason the mean Alfvén velocity will differ from its local value by the quantity $\Delta V_A = -V_A \rho_{1s}/(2\rho_0)$ where $\rho_{1s}$ is low-frequency (LF) density variation. It results in the frequency addition of A-waves $\Delta \omega_p \approx k \Delta V_A$. Due to slow motion of plasma with the drift velocity $v_D$ the frequency of A-waves changes at $\Delta \omega_D \approx k v_D$. The ratio of these two additions, $\Delta \omega_D$ and $\Delta \omega_p$, however, occurs to be small: $\sim c_s/V_A$. Thus, the main interaction is the scattering on the LF density fluctuations. At the same time the LF plasma characteristics will be changed due to the action of the HF force induced by A-waves.

The most simple way to find the expression of the HF force is to perform average of the Hamiltonian over the HF oscillations. The result of this average is the following

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}, \quad (3.4)$$

where

$$\mathcal{H}_0 = \int \left\{ \frac{1}{2\rho_0} \langle [H_0 \times \text{rot} S]^2 \rangle + \frac{(H^2)^2}{8\pi} \right\} dr + \int \left\{ \rho_0 \frac{\partial^2}{\partial z^2} + c_s^2 \frac{\rho_{1s}^2}{2\rho_0} \right\} dr,$$

$$\mathcal{H}_{\text{int}} = - \int \frac{\rho_{1s}}{2\rho_0} \langle [H_0 \times \text{rot} S]^2 \rangle dr.$$
In accordance with (3.5) the equation of motion for the potential $\Phi_s$ takes the form:

$$\frac{\partial \Phi_s}{\partial t} + c_s^2 \frac{\rho_{1s}}{\rho_0} = \frac{(|H_0 \times \text{rot } S|^2)}{2\rho_0^2}.$$  

(3.6)

It is important to notice that the HF potential (3.5) is negative. This means that in the region of localization of A-waves the HF force will form, instead of density wells, as it is for the interaction between Langmuir and ion-acoustic waves (see [2]), the density humps.

The equations of motion are closed by the continuity equation for $\rho_{1s}$ which, in accordance with (3.1), has the form:

$$\frac{\partial \rho_{1s}}{\partial t} + \rho_0 \frac{\partial^2 \Phi_s}{\partial z^2} = 0.$$  

(3.7)

From (3.6) and (3.7) we have

$$\frac{\partial^2 \rho_{1s}}{\partial t^2} - c_s^2 \frac{\partial^2 \rho_{1s}}{\partial z^2} = -\frac{1}{2\rho_0} \frac{\partial^2}{\partial z^2} (|H_0 \times \text{rot } S|^2).$$  

(3.8)

To write the equation for A-waves one needs to make average explicitly in the Hamiltonian of interaction $H_{int}$. It corresponds to keeping in $H_{int}$ terms containing products $a_{\lambda_i}^* a_{\lambda_i}$ where the index $\lambda = 1, 2$ enumerate the HF waves:

$$H_{int} = -\int \frac{\rho_{1s}(k)}{2\rho_0} \sum_{\lambda \lambda_i} F_{kk_1}^{\lambda \lambda_i} a_{\lambda_i}^*(k) a_{\lambda_i}(k) \delta_{k-k_1} dkdkdk.$$  

Here

$$F_{kk_1}^{\lambda \lambda_i} = (\omega_\lambda(k) \omega_{\lambda_i}(k_1))^{1/2} (n_\lambda(k) \cdot n_{\lambda_i}(k_1)), \quad n_2 = \frac{k_\perp}{k_\perp}, \quad n_1 = -[n_2 n_0]$$

As the result the equations for A-waves have the form:

$$\frac{\partial a_\lambda(k)}{\partial t} + i\omega_\lambda(k) a_\lambda(k) = -i \frac{\delta H_{int}}{\delta a_\lambda^*(k)}; \quad \lambda = 1, 2.$$  

(3.9)

For isothermal collisionless plasma ($T_e \approx T_i$) slow magneto-acoustic waves are absent due to strong Landau damping on ions. Correspondingly, the decay interaction of A-waves transforms into the induced scattering on ions. In this case the equations (3.8) have to be changed by the drift kinetic equation [27] for slow variation of distribution functions of ions $f_i$ (compare with [29]):

$$\frac{\partial f_i}{\partial t} + v_i \frac{\partial f_i}{\partial z} - \frac{1}{M} \frac{\partial}{\partial z} (v_i \delta \tilde{\phi} + U) \cdot \frac{\partial f_i}{\partial v_z} = 0,$$  

(3.10)

together with the quasi-neutrality condition for slow motions ($\Omega_k = k_z c_S \ll \omega_{pi}$)

$$\delta n_i = \int f_i d\nu = \frac{n_0}{T_e} v_i \tilde{\phi} = \frac{\rho_{1s}}{M},$$  

(3.11)

where $f_0$ is the equilibrium distribution function of ions, and $\tilde{\phi}$ the LF electrostatic potential. In this case the equation of motion for A-waves retains the form of (3.10), and the density is expressed linearly through the HF potential by means of the Green function for the system (3.10)(3.11):

$$G_{\kappa \Omega} \equiv \frac{\rho_{1s}(\kappa, \Omega)}{U_{\kappa \Omega}} = -\frac{n_0 k^2}{\omega_{pi}^2} \frac{\epsilon_{e,i}}{\epsilon_e + \epsilon_i},$$  

(3.12)

Here $\rho_{1s}(\kappa, \Omega)$ and $U_{\kappa \Omega}$ are the Fourier images of the LF density and the HF potential, respectively, $\epsilon_{e,i}$ partial dielectric constants of electrons and ions which are equal:

$$\epsilon_e = \frac{1}{\kappa^2 \epsilon_e^2},$$

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\[ \epsilon_i = \frac{4\pi e^2}{M\kappa^2} \int \frac{\kappa_z (\partial f_0/\partial v_z)}{\Omega - \kappa_z v_z} \, d\mathbf{v}, \]

where \( r_\text{D}^2 = T_e/(4\pi n_0 e^2) \) is square of the Debye radius.

In non-isothermal plasma \( (T_e \gg T_i) \) the Green function (3.12) transforms into

\[ G_{\kappa \Omega} = \frac{n_0 \kappa^2}{\Omega^2 - \kappa^2 c_s^2}, \]

that coincides with the expression given by the equation (3.8).

The system of equations (3.10)-(3.12) completely describes interaction of A-waves in magnetized plasma with arbitrary ratio of ion and electron temperatures. In this case, however, the Hamiltonian \( H_0 + H_\text{int} \) is not conservative quality due to the Landau damping on ions.

IV. INSTABILITY OF MONOCHROMATIC WAVE

Let us now analyze the obtained equations. We start from study of dynamics of the narrow packet of A-waves. A qualitative understanding about this process can be obtained from stability analysis of monochromatic A-wave. In the sake of simplicity we shall restrict by consideration of stability of Alfvenic wave in the hydrodynamic limit. For collisionless plasma the latter assumes that the phase velocity of bending \( \Omega/\kappa_z \) for A-waves exceeds the thermal ion velocity \( v_{Ti} \). In this case for slow motion one can neglect the Landau damping on ions and use the equations (3.8) or (3.12). One should remember that in strongly non-isothermal plasma plasma the sound waves are the eigen oscillations but at the same time in plasma with \( T_e \approx T_i \) sound waves represent the induced oscillations of the plasma density. However, at \( \Omega/\kappa_z \gg v_{Ti} \) the hydrodynamic description can be applied in both cases.

Then it is convenient to express \( \rho_{1s} \) through the normal variables \( a_3(k) \equiv b_k \):

\[ \rho_{1s}(k) = \left( \frac{\rho_0 \Omega_k}{2c_s} \right)^{1/2} (b_k + b_k^*). \]

The equations of motion for \( b(k) \) are obtained from variation of of the full Hamiltonian \( H_0 + H_\text{int} \):

\[ \frac{\partial b_k}{\partial t} + i\Omega(k)b_k = -i \frac{\delta H_\text{int}}{\delta b_k^*}. \quad (4.1) \]

In the equations (3.9), (4.1) the solution

\[ a_\lambda(k) = \left( \frac{\rho_0 \Omega_k}{2c_s} \right)^{1/2} \delta_{\lambda 1} e^{-i\omega_0 t} \delta_{k-k_0}, \quad b_k = 0, \quad \omega_0 = \omega_1(k_0) \]

corresponds to the monochromatic Alfvenic wave. The amplitude of Alfvenic wave here is chosen by such a way so that \( |A|^2 \) coincides with the energy density of oscillations \( W \).

Linearizing of the equations (3.10)-(3.12) on the background of exact solution and assuming for perturbations:

\[ \delta a_\lambda(k) \sim e^{-i(\Omega + \omega_0)t} \delta_{k-k_0 - \kappa}, \]

\[ \delta a_\lambda^*(k) \sim e^{-i(\Omega - \omega_0)t} \delta_{k-k_0 + \kappa}, \]

for \( \Omega \) we have the following dispersion relation:

\[ \frac{W G}{4M n_0^2 \omega_0} \sum_\lambda \left\{ \frac{|F_{\lambda k_0 k_0 + \kappa}^1|^2}{\Omega + \omega_0 - \omega_\lambda(k_0 + \kappa)} + \frac{|F_{\lambda k_0 k_0 - \kappa}^1|^2}{-\Omega + \omega_0 - \omega_\lambda(k_0 - \kappa)} \right\} = 1. \quad (4.2) \]

We shall present now results of investigations of the dispersion equation (4.2) in the different cases in dependence on the energy wave density \( W \) and on the temperature ratio.

At \( T_e \gg T_i \) and sufficiently small amplitudes the decay instability takes place with generation of ion magnetized sound [22]. For this instability the eigen frequency \( \Omega \) is expressed through the matrix element of the decay interaction.
\[
Y_{\lambda}^{k_1k_2} = \left( \frac{\Omega(k_0)}{8\rho_0 c_s^2} \right)^{1/2} \lambda^{k_1k_2} F_{\lambda}^{k_1k_2}
\]  

(4.3)

and the quantity \( W \):

\[
\Omega = \frac{1}{2} [\omega_0 - \omega_\lambda(k_0 - \kappa) + \Omega(\kappa)] \pm \frac{1}{4} \left[ \omega_0 - \omega_\lambda(k_0 - \kappa) - \Omega(\kappa) \right]^2 - \frac{W}{\omega_0} |Y_{k_0, k_0 - \kappa, \kappa}|^2)^{1/2}.
\]

(4.4)

Hence it follows that the instability takes place near the resonant surface

\[
\omega_0 = \omega_\lambda(k_0 - \kappa) + \Omega(\kappa)
\]

with maximum of the growth rate

\[
\Gamma = \left[ \frac{W}{8\rho_0 c_s^2} \frac{\Omega_s}{\omega_0} |F_{k_0, k_0 - \kappa, \kappa}|^2 \right]^{1/2}.
\]

(4.6)

The growth rate width as a function of frequency occurs to be of the order of magnitude of the maximal growth rate

Because the matrix element is proportional to the frequency of slow sound the maximum value of the growth rate on the resonant surface (4.3) is attained at the maximal value of \(|\kappa_z|\). For decay on Alfvenic wave and slow sound \(\max |\kappa_z| \approx 2|k_0z|\), so that the secondary Alfvenic wave propagates in the opposite direction to the pumping Alfvenic wave. Such behavior of the decay instability is typical for the Mandelstamm-Briullien scattering, the matrix element of which is proportional to square root from the sound momentum transfersed by scattering light. For light such dependence provides the maximal back scattering.

It is not difficult to investigate the decay instability for all other possible channels of decay \(A \rightarrow A + S\). The growth rate in all these cases are of the same order of magnitude as (4.6):

\[
\Gamma \sim (\omega_0 \Omega_s W/nT)^{1/2}.
\]

This instability takes place at

\[
W/nT < \beta^{1/2}.
\]

With increase of \( W \) the decay instability is transformed. At \( W/nT > \beta^{1/2} \) in the dispersion relation (4.2) it is possible to neglect by \( \Omega_s^2 \) against \( \Omega^2 \). Then the unstable wave vectors will lie on the surface \( \omega_1(k_0) = \omega_\lambda(k_0 - \kappa) \). This instability is called as the modified decay instability [28,30]. For the interaction of Alfvenic waves and slow sound this instability has the growth rate maximal at \( \kappa_z = 2k_0z \):

\[
\Gamma \approx \frac{\sqrt{3}}{2} \omega_0 \left( \frac{W}{\rho_0 V_A^2} \right)^{1/3}
\]

(4.7)

Value of this growth rate does not depend on temperature and therefore this instability takes place at \( W/nT > 1 \) up to the values \( \beta^{-1} \) when the main approximation - adiabaticity approximation - looses its applicability: \( \Gamma \sim \omega_0 \).

For another channels the instability with growth of \( W/nT \) has the same character: at \( W/nT > \beta^{1/2} \) the growth rate is maximal in the region \( \kappa \sim k_0 \) and has the same order of magnitudes as (4.7).

The decay instability (4.4) for arbitrary channel \( A \rightarrow A + S \), as it is easily seen, relates to the convective type of instabilities. The excited waves, according to (4.3), have the group velocities strongly different from the group velocity of the pumping wave. Therefore for the wave packet with the characteristic scale \( L \) this instability will be essential only for large enough lengths \( L \) when the amplification coefficient \( G \) exceeds a value of the Coulomb logarithm \( \Lambda \):

\[
G = \Gamma L/V_A \approx \Lambda.
\]

(4.8)

For less lengths \( L \), the decay instability is not important: during propagation of perturbation of through the whole packet perturbations amplify for a small value. In this case dynamics of the packet will be defined by slow processes. Among them the most important ones are such processes for which unstable perturbations propagate together with the wave packet. If it is a decay instability, then it has to be absolute (in the frame moving together with the packet). In particular, this is one of the reasons of appearance of collapse for fast magneto-acoustic waves and of affect of sound collapse on the fine structure of collisionless shocks in plasma [23,24]. Collapse of fast magneto-acoustic waves appears due to three-wave interaction in which only fast magneto-acoustic waves take part.
V. KOLMOGOROV SPECTRA

In the previous section we considered the stability problem for narrow in \( k \)-space wave packet. In this case for decay of monochromatic wave under the resonant conditions \((1.4)\) (namely, for the maximal value of the growth rate \((1.6)\)) sum of the phases of exciting waves \( \phi_A \) and \( \phi_s \) are strongly connected with the phase of the pumping wave \( \phi_0 \):

\[
\phi_0 + \pi/2 = \phi_A + \phi_s.
\]

(It is easily to check that this phase correlation is lost with leaving the resonance \((1.3)\).) Simultaneously, a difference in phases for the pair of exciting waves with fixed \( \kappa \) remains arbitrary. Both these factors introduce to the system of interacted triads, connected with the pumping wave an element of randomness. Thus, each triad is characterized by one random phase. At the next step - at the second cascade a new random phases are added so that a memory about the pumping wave will be lost. For multiple repetition of of this process the system of waves must transform in the turbulent state when the phases of waves can be considered random. Therefore the stochastization time should be a few inverse growth rate \((4.6)\).

Such scenario of transition to turbulence seems to be sufficiently plausible. Now there are being performed a number of numerical experiments to check this hypothesis (see, for instance, \([33], [32]\)).

In the case of the weak MHD turbulence at \( \beta \ll 1 \) we have three pair correlation functions defined by the following formulas:

\[
\langle a_\lambda (k)a_{\lambda_1}^* (k_1) \rangle = N^\lambda_k \delta_{\lambda \lambda_1} \delta_k - k_1, \quad \langle b_k b^*_k \rangle = n_k \delta_k - k_1
\]

where the quantities \( N^\lambda_k, n_k \), having a meaning of the occupation numbers, satisfy the following system of kinetic equations:

\[
\dot{n}_k = 2\pi \int |V_{k_1 k_2 k}|^2 (N_{k_1} N_{k_2} - n_k N_{k_1} + n_k N_{k_2}) \delta_{k+k_1-k_2} \delta_{\Omega+\omega_1-\omega_2} dk_1 dk_2, \quad (5.1)
\]

\[
\dot{N}_k = 2\pi \int |V_{k_1 k_2 k}|^2 (N_{k_1} n_{k_2} - N_k n_{k_2} - N_k N_{k_1}) \delta_{k-k_1-k_2} \delta_{\omega-\omega_1-\omega_2} dk_1 dk_2 \quad (5.2)
\]

\[
-2\pi \int |V_{k_1 k_2 k}|^2 (N_{k_1} n_{k_2} - N_k n_{k_2} - N_k N_{k_1}) \delta_{k+k_1-k_2} \delta_{\omega_1-\omega-\omega_2} dk_1 dk_2.
\]

Here \( \omega \equiv \omega(k), \omega_1 \equiv \omega(k_1) \), and so on. In these equations (as well as below) we omit summation over \( \lambda \). In order to include it one needs to change \( dk_1 \rightarrow d\lambda_1 dk_1 \), \( N_k \rightarrow N^\lambda_k, \omega_k \rightarrow \omega_k \lambda, V_{k_1 k_2 k} \rightarrow V_{k_1 k_2 k}^\lambda \lambda_1 \), and so on.

The equations \((5.1), (5.2)\) assume weakness of the nonlinear interaction between waves. In this concrete case the most essential criterion is

\[
\Omega_s \gg 1/\tau,
\]

where \( \tau \) is the characteristic nonlinear time defined by the kinetic equations \((5.1), (5.2)\). To estimate the value of \( \tau \) one needs to take into account that in each act of decay and inverse process - merging of waves, the frequencies of \( \Lambda \)-waves change for the small value \( \Delta \omega_A = \Omega_s \ll \omega_A \), namely, the energy transfer of \( \Lambda \)-waves along spectrum has a diffusive character. Due to this fact, we have the following estimate for \( \tau \):

\[
\frac{1}{\tau} \sim \frac{\omega_A}{\rho V_A^2} W.
\]

Notice, that this value for \( \tau \) exceeds significantly the stochastization time, defined by the inverse growth rate \((4.6)\) \( \Gamma^{-1} \).

Hence we have finally the criterion:

\[
\frac{W}{\rho V_A^2} \ll \beta^{1/2}.
\]
Next, let us include into the kinetic equations (5.1), (5.2) both sources of turbulence and its dissipation. For this aim in the left hand sides of the equations we introduce new terms $\Gamma_k n_k$ and $\gamma_{k\lambda} N_{k\lambda}$, respectively. We suppose here that the pumping region ($\Gamma_k, \gamma_{k\lambda} > 0$) and dissipation region ($\Gamma_k, \gamma_{k\lambda} < 0$) are well separated in $k$-space by the intermediate region - the inertial interval, where dynamics of turbulence is defined by nonlinear interaction between waves only. In the inertial interval we shall neglect by influence of both pumping and dissipation (that is necessary to be proved) then distributions $n_k$ and $N_{k\lambda}$ do not depend on the concrete form of $\gamma_k$ and $\Gamma_k$.

We would like to remind that in the theory of developed hydrodynamic turbulence to determine turbulent spectrum - distribution of energy for velocity fluctuations it is enough to use two hypothesizes of A.N. Kolmogorov [1]. The first hypothesis about self-similarity says that the spectrum of turbulence is defined by the unique quantity $P$ - a constant flux of energy along scales (from large to small ones where dissipation due to viscosity becomes essential). The second hypothesis assumes that the interaction of fluctuations with different scales has a local character.

If one applies these hypothesizes to the given case then spectra of turbulence in the inertial interval can be found by means of the dimensional analysis. In the given situation the kinetic equations (5.1), (5.2) have two conservation laws: conservation of the total energy and the total number of HF waves. To each integral of motion there should correspond its proper Kolmogorov spectrum. So, for a constant flux of the number of HF waves $N_k$

\[
P_N = \frac{\partial}{\partial t} \sum_{\lambda} \int N_{k\lambda} dk,
\]

we have the spectrum:

\[
N_{k\lambda} \sim P_N^{1/2} k^{-4}, \quad n_k \sim P_N^{1/2} k^{-4}.
\]  

(5.3)

For a constant flux of energy

\[
P_\varepsilon = \frac{\partial}{\partial t} \int (\omega_k n_k + \sum_{\lambda} \omega_{k\lambda} N_{k\lambda}) dk,
\]

one can get the estimate:

\[
N_{k\lambda} \sim P_\varepsilon^{1/2} k^{-3/2}, \quad n_k \sim P_\varepsilon^{1/2} k^{-3/2}.
\]  

(5.4)

From conservation in the inertial range of the total number of HF waves and the energy it is easily to establish that the flux of HF particles is directed towards small $k$-region, and energy flux is directed to the short wave region.

These – sufficiently rough – estimations for the spectra (5.3), (5.4) can pretend only to the right dependence on both wave numbers and fluxes, but they don’t account the diffusive character of decays. Notice also that these estimates are significantly based on an assumption of the interaction locality.

The spectra (5.3) and (5.4) don’t account also fine properties of distribution functions – their angle dependences, namely, they are defined up to arbitrary functions of angles. To find these dependences one needs to solve the exact equations (5.1), (5.2). It turns out that solution of these equations can be found for interaction of Alfvenic and slow acoustic waves ($N_2 \equiv 0$). For this case it is convenient to represent the equations (5.1), (5.2) in the form:

\[
\dot{n}_k = - \int U_{k|k_1} T_{k_1|k_2} \, dk_1 \, dk_2,
\]  

(5.5)

\[
\dot{N}_k = \int (U_{k|k_1} T_{k_1|k_2} - U_{k_1|k_2} T_{k_1|k_2}) \, dk_1 \, dk_2,
\]  

(5.6)

where the following notations are introduced:

\[
U_{k|k_1} = 2\pi |V_{k|k_1}^{11}|^2 \delta_{k-k_1} \delta_{\omega-k_1-\omega_2},
\]

\[
T_{k|k_1} = N_{k_1} n_{k_2} - N_k n_{k_2} - N_k N_{k_1}.
\]

It is easily to see that the equations (5.5), (5.6) have the thermodynamic equilibrium solutions:

\[
N_k = \frac{N}{\omega + \mu}, \quad n_k = \frac{T}{\Omega_k}
\]

12
– the Rayleigh-Jeans distributions which annulate the collision terms.

To find the non-equilibrium distributions it should be noted that the function $U$ has the following properties. (i) $U$ is bi-homogeneous function of its arguments $k_z$ and $k_\perp$ with homogeneity degrees, respectively, equal to +1 for $k_z$ and −2 for $k_\perp$. This means that if one performs stretching of $k_z$, $k_{1z}$, $k_{2z}$ in λ times then $U$ multiplies in $\lambda^{+1}$ times: $U \rightarrow \lambda^{+1}U$. If one makes the similar transform for all $k_\perp$: $k_\perp \rightarrow \mu k_\perp$, then $U$ transforms as $U \rightarrow \mu^{-2}U$. Besides, (ii) $U$ is invariant with respect to rotation around z-axis - the direction of mean magnetic field $H_0$.

Due to these properties solution is naturally sought in the form:

$$n_k = Ak_z^\alpha k_\perp^\beta, \quad N_k = Bk_z^\alpha k_\perp^\beta. \quad (5.7)$$

Consider the stationary equation (5.6):

$$\int (U_{k_1|k_2} T_{k_1|k_2} - U_{k_1|k_2} T_{k_1|k_2}) = 0. \quad (5.8)$$

Let us make a mapping of the integration area of the second integral (which is given by its resonance conditions – δ-functions) to the integration area of the first integral. For this aim it is convenient to introduce the complex variables $\zeta = k_z + ik_y$. Then the integration area defined by the corresponding conservation laws

$$k_{z1} - k_z - k_{z2} = 0,$$

$$\zeta_1 - \zeta - \zeta_2 = 0,$$

$$\omega_1 - \omega - \Omega_2 = 0,$$

with the help of the mapping relative to all $k_z$ and $\zeta$:

$$k_z = k_z' k_z^\perp, \quad \zeta = \zeta' \zeta^\perp; \quad (5.9)$$

$$k_{z1} = k_z k_z^\perp, \quad \zeta_1 = \zeta \zeta^\perp,$$

$$k_{z2} = k_z k_z^\perp, \quad \zeta_2 = \zeta' \zeta^\perp,$$

transforms into the integration area of the first integral in (5.8). Each such transformation (separately with respect to $k_z$ and $\zeta$) represents itself the operation of inversion: for $z$-components of wave vectors against the point $k_z$, and for transverse components relative to the circle with radius $|k_\perp|$. Under this mapping the vector $k_1$, $k_1$ into $k$ and $k_2$ into $k_2$. Simultaneously all $z$-components are stretched in the $|k_z/k_{z1}|$ times, and all transverse components get the factor $|k_\perp/k_{1\perp}|$. Besides, the rotation on the angle $\arg(\zeta/\zeta_1)$ around $z$-axis takes place.

As the result, due to the properties of both $U$ and $T$,

$$U_{k_1|k_2} \rightarrow |k_z/k_{z1}|^{+1} |k_\perp/k_{1\perp}|^{-2} U_{k_1|k_2}, \quad T_{k_1|k_2} \rightarrow |k_z/k_{z1}|^{2\alpha} |k_\perp/k_{1\perp}|^{2\beta} T_{k_1|k_2},$$

the integrand in (5.8) is factorized:

$$\int U_{k, k'} T_{k, k'} \left[1 - \left(\frac{k_z}{k_{z1}}\right)^{2\alpha+4} \left(\frac{k_\perp}{k_{1\perp}}\right)^{2\beta+4}\right] dk_1 dk_2 = 0.$$

Hence it follows that, besides the thermodynamic equilibrium spectra (which annulates $T$), the following solution is possible:

$$n_k = Ak_z^{-2} k_\perp^{-2}, \quad N_k = Bk_z^{-2} k_\perp^{-2}. \quad (5.10)$$

These spectra correspond to the solution which was obtained previously from the dimensional analysis for the constant flux of HF waves $P_N$. Connection between the coefficients $A$ and $B$ in (5.10) is determined from solution of the
stationary (\(\partial/\partial t = 0\)) equation (5.1). Hence one can get that for this case total energies containing in Alfvénic waves and in slow magneto-acoustic waves are of the same order of magnitude: \(c_s A \sim V_A B\).

It is worth to note that the whole set of the transformations (5.3) forms the group \(G\). This group is direct product of two groups \(G(1)\) and \(G(2)\): \(G = G(1) \times G(2)\). The group \(G(1)\) acts in one-dimensional space \((k_z\)-space\), and \(G(2)\) in the two-dimensional one \((k_{\perp}\)-space\). These transformations allow one to factorize the collision terms. First these transformations (in the 1D case of the frequency space) were found by V.E.Zakharov [3, 4]. The generalizations of these transformation to the both 2D and 3D cases for isotropic models were introduced by A.V.Kats and V.M.Kontorovich in 1970 [23]. The transformations (5.3) were found by the author in 1972 [3]. They represent a partial type of the so-called quasi-conformal transformations.

To find another non-equilibrium solution of (5.4) it is convenient to introduce the energy density in the \(k\)-space \(\varepsilon_k = \omega_k N_k + \Omega_k n_k\). From (5.5) and (5.6) follows that this quantity obeys the equation:

\[
\frac{\partial \varepsilon_k}{\partial t} = \int \left\{ \omega_k U_{k|k_1 k_2} T_{k|k_1 k_2} - \omega_k U_{k_1|k_2 k} T_{k_1|k_2 k} - \Omega_k U_{k_2|k_1 k} T_{k_2|k_1 k} \right\} dk_1 dk_2.
\]

(5.11)

Consider stationary solution of this equation. As before, we shall seek for solution of (5.11) in the form (5.7). In this case we have three integrals, integration areas of which are defined by the appropriate \(\delta\)-functions. Therefore we shall make transformations analogous to (5.9). The transformations of the integration area of the second integral standing in (5.11) will be the same as (5.9). The transformation of the integration area of the third integral into the corresponding integration area of the first integral in (5.11) has the form:

\[
k_z = k_z' k_z'' \quad \zeta = \zeta'' \quad \zeta', \quad k_{z_1} = k_{z_1}' k_z'' \quad \zeta_1 = \zeta' \quad \zeta''
\]

(5.12)

Applying all of these transforms yields factorization of the integrand for the stationary equation (5.11):

\[
0 = \int |V_{k_1 k_2}|^2 \delta_{k - k_1 - k_2} \delta_{\omega - \omega_1 - \omega_2} T_{k|k_1 k_2} dk_1 dk_2.
\]

\[
\left\{ \omega(k) - \omega(k_1) \left( \frac{k_z}{k_{z_1}} \right)^{2\alpha+5} \left( \frac{k_z}{k_{z_2}} \right)^{2\beta+4} - \Omega(k_2) \left( \frac{k_z}{k_{z_2}} \right)^{2\alpha+5} \left( \frac{k_z}{k_{z_2}} \right)^{2\beta+4} \right\}
\]

Hence it follows that the figure bracket vanishes at \(\alpha = -5/2, \beta = -2\), namely, the solution has the form:

\[
n_k = A k_z^{-5/2} k_{z_1}^{-2}, \quad N_k = B k_z^{-5/2} k_{z_1}^{-2}.
\]

(5.13)

The obtained solution corresponds to spectra with constant energy flux \(P_z\). Connection between constants \(A\) and \(B\), as before, is found from the stationary equation (5.1). From this equation it is possible to get the previous estimation on their ratio: \(c_s A \sim V_A B\).

The found above solutions of the Kolmogorov type are related to the only channel of interaction, namely, the interaction between Alfvénic and slow magneto-acoustic waves that demanishes significantly the value of these solutions. Remind, that processes together with fast magneto-acoustic waves have growth rates of the same order of magnitudes, and therefore they can not be ignored. Fortunately, the channel (with the fast magneto-acoustic waves) can be incorporated in the considered above scheme without essential generalizations. As was pointed out in the previous section, the maximal scattering of \(A\)-waves is attained at the maximal value of \(z\)-projection of momentum transfered to slow magneto-acoustic waves while scattering of \(A\)-waves. Therefore it is natural to assume that such behavior of the scattering amplitude of \(A\)-waves should lead to strongly anisotropic distribution of waves concentrated in a narrow cone of angles along mean magnetic field: \(k_z \gg k_{\perp}\). Under this assumption it is possible to consider the dispersion
law of fast magneto-acoustic waves to be approximated by those for Alfvenic waves: \( \omega^2 \approx |k| V_A \). Another important circumstance in this case is that the matrix element of interaction is diagonal with respect to polarization \( \lambda \):

\[
V_{\lambda\lambda_1}^{k'k_2 k_1} \approx \delta_{\lambda\lambda_1} V_{11}^{k'k_2 k_1}.
\]

Thus, for almost longitudinal (along the mean magnetic field) distribution there is almost no difference between Alfvenic and fast magneto-acoustic waves. Moreover, the direct energy exchange is absent between these waves. This means that for this region of angles Kolmogorov spectra for fast magneto-acoustic waves will have the same form as those of the obtained spectra (5.10) and (5.13). In this case in the expressions (5.10) and (5.13) \( N_k \) and \( B_\lambda \) should be changed into \( N_\lambda \) and \( B_\lambda \), and the coefficient

\[
A \sim \beta^{-1/2} \frac{\sum B_\lambda^2}{\sum B_\lambda}.
\]

The spectra, obtained in this section, will have the physical meaning if the locality property will be fulfilled. This requirement of locality consists in that contributions into interaction of the waves from both the pumping region and the dissipation region have to be small. The latter leads to the requirement of convergence of integrals in the equations (5.13) and (5.10).

Convergence of integrals relative to \( k_z \) provides by the presence of two \( \delta \)-functions containing \( k_z \). As far as convergence against transverse wave vectors concerns, the integrals are logarithmically divergent. The logarithmical divergence, to our opinion, is not so serious as a possible powerful one. Appearance of divergence is connected with bi-homogeneity of the probability \( U \). If a medium would be isotropic and matrix elements \( V \) would would have the same degrees of homogeneity as for the MHD waves at \( \beta \ll 1 \) (such situation, for instance, takes place for Mandelstamm-Brillouin scattering in isotropic dielectrics) then in such a case the locality property would be valid (compare with (34)). Violation of bi-homogeneity for interaction of the Alfvenic and slow magneto-acoustic waves appears for almost transverse propagation: \( k_\perp/k_z \sim \beta^{-1/2} \) and for interaction of the fast magneto-acoustic waves for small angles \( \sim \beta^{1/2} \). By this reason cut-off of integrals in the kinetic equations should be performed on the smaller angles: \( \sim \beta^{1/2} \). Another possibility to avoid the logarithmic divergence is in seeking for solutions containing powers of logarithm from \( k_\perp \) in (5.10) and (5.13). However, this procedure does not lead to determination of the powers, but, however, provides a convergence of the integrals.

Last remark. The spectra (5.10) as well as (5.13) have the same power dependence on transverse momenta: \( n_k, N_k \sim k_\perp^{-7/2} \). Their homogeneity degree against transverse momenta is the same as for 2D \( \delta \)-function of \( k_\perp \). This means that, besides the anisotropic Kolmogorov spectra (5.10) and (5.13), the singular Kolmogorov spectra are possible:

\[
n_k = Ak_z^{-2} \delta(k_\perp), \quad N_k = Bk_z^{-2} \delta(k_\perp)
\]

and

\[
n_k = Ak_z^{-5/2} \delta(k_\perp), \quad N_k = Bk_z^{-5/2} \delta(k_\perp).
\]

Which spectra are realized indeed? Rigorous answer to this question is possible to get by stability investigation of the spectra or numerical experiment (in the latter case one has a hope on the qualitative understanding). Both these approaches require separate consideration.

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