Coherent States of the $SU(N)$ groups

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Coherent states ($CS$) of the $SU(N)$ groups are constructed explicitly and their properties are investigated. They represent a nontrivial generalization of the spinning $CS$ of the $SU(2)$ group. The $CS$ are parametrized by the points of the coset space, which is, in that particular case, the projective space $CP^{N-1}$ and plays the role of the phase space of a corresponding classical mechanics. The $CS$ possess of a minimum uncertainty, they minimize an invariant dispersion of the quadratic Casimir operator. The classical limit is investigated in terms of symbols of operators. The role of the Planck constant plays $h = P^{-1}$, where $P$ is the signature of the representation. The classical limit of the so called star commutator generates the Poisson bracket in the $CP^{N-1}$ phase space. The logarithm of the modulus of the $CS$ overlapping, being in-
interpreted as a symmetric in the space, gives the Fubini-Study metric in $CP^{N-1}$. The CS constructed are useful for the quasi-classical analysis of the quantum equations of the $SU(N)$ gauge symmetric theories.
I. INTRODUCTION

As known, coherent states (CS) are widely and fruitful being utilized in different areas of the theoretical physics [1]- [5]. The CS, introduced by Schrödinger and Glauber, turned out to be orbits of the Heisenberg-Weyl group. That observation allowed one to formulate by analogy some general definition of CS for any Lie group [4]. A connection between the CS and the quantization of classical systems, in particular, systems with a curved phase space, was also established [6]. By the origin, CS are quantum states, but, at the same time, they are parametrized by the points of the phase space of a corresponding classical mechanics. Namely that circumstance makes them very convenient in the analysis of problems of the correspondence between the quantum and the classical description. All that explains the interest both to the investigation of general problems of CS theory and to the construction of CS of concrete groups. The CS of such important in physics groups as $SU(N)$ ones are built and investigated in an uniform way in the present work. The CS of the group $SU(2)$, from that family, are well known. One can point out some of the first references [8]- [12], where that states were built on the base of the well investigated structure of the $SU(2)$ matrices in the fundamental representation. Another approach to the CS construction of the $SU(2)$ group was used by Berezin [6]. That approach is connected with the utilization of the representations of the $SU(2)$ group in the space of polynomials of the powers not more that a given one. A modification of the latter method in a gauge invariant form (with extended number of variables in the coset space or phase space) allows us to build the CS for all the groups $SU(N)$ in an uniform way. We construct the CS by means of orbits of highest weights in the space of polynomials of a fixed power. The representations used are equivalent to the total symmetric irreducible unitary representations of the $SU(N)$ groups. The stationary subgroups of the highest weights, in the case of consideration, are $U(N-1)$, so that the CS are parametrized by the points of the coset space $SU(N)/U(N-1)$ which plays the role of the phase space of a corresponding classical mechanics, and at the same time it is the well known projective space $CP^{N-1}$. The logarithm of the modulus of the
CS overlapping, being interpreted as a symmetric in the space $CP^{N-1}$, generates Fubini-Study metric in the space. The $CS$ form an overcomplited system in the representation space and minimize an invariant dispersion of the quadratic Casimir operator. The classical limit is investigated in terms of the operators symbols which are constructed as the mean values in the $CS$. The role of the Planck constant plays the quantity $h = P^{-1}$, where $P$ is the signature of the representation. The classical limit of so called star commutator of symbols generates the classical Poisson bracket in the corresponding phase space. The present work is the continuation of our papers [13] - [15], where a part of results was preliminary expounded.

II. THE CONSTRUCTION OF THE CS BY MEANS OF THE REPRESENTATIONS OF THE SU($N$) GROUPS ON POLYNOMIALS

We are going to construct $CS$ of the $SU(N)$ groups as orbits in some irreducible representations of the groups, factorized with respect to stationary subgroups. First, we describe the corresponding representations.

Let $C^N$ is $N$-dimentional space of complex lines $z = (z_\mu), \mu = 1, N$, with the scalar product $(z, z')_C = \sum_\mu \bar{z}_\mu z'_\mu, \mu = 1, N$, and $\tilde{C}^N$ is the dual space of complex columns with the scalar product $(\tilde{z}, \tilde{z}')_{\tilde{C}} = \sum_\mu \bar{\tilde{z}}_\mu \tilde{z}'_\mu$. The anti-isomorphism is given by the relation $z \leftrightarrow \tilde{z} \leftrightarrow \bar{z}_\mu = \tilde{z}_\mu$. The mixed (Dirac) scalar product between elements of $C^N$ and $\tilde{C}^N$ is defined by the equation:

$$\langle z', \tilde{z} \rangle = (\tilde{z}', \tilde{z})_{\tilde{C}} = (z', \bar{z})_C = \bar{z}'_\mu \tilde{z}_\mu. \tag{1}$$

Let $g$ are matrices of the fundamental representation of the $SU(N)$ group. This representation induces irreducible representations of the group in the spaces $\Pi_P$ and $\bar{\Pi}_P$ of polynomials of a fixed power $P$ on the vectors $z$ and $\tilde{z}$ respectively:

$$T(g)\Psi_P(z) = \Psi_P(z_g), \quad z_g = zg, \quad \Psi_P \in \Pi_P,$$

$$\bar{T}(g)\Psi_P(\tilde{z}) = \Psi_P(\tilde{z}_g), \quad \tilde{z}_g = g^{-1}\tilde{z}, \quad \Psi_P(\tilde{z}) \in \bar{\Pi}_P. \tag{2}$$

The anti-isomorphism $z \leftrightarrow \tilde{z}$ induces the correspondence $\Psi_P(\tilde{z}) = \overline{\Psi_P(z)}$. 
The representation (2) is equivalent to the one on total symmetrical tensors of signature $P$. So, we will further call $P$ as the signature of the irreducible representation.

Obviously the monomials

$$\Psi_{P, \{n\}}(z) = \sqrt{\frac{P!}{n_1! \cdots n_N!}} z_1^{n_1} \cdots z_N^{n_N},$$

$$\{n\} = \{n_1, \ldots, n_N| \sum \mu n_\mu = P\},$$

form a discrete basis in $\Pi_P$, and the monomials $\Psi_{P, \{n\}}(\tilde{z}) = \overline{\Psi_{P, \{n\}}(z)}$ form a basis in $\tilde{\Pi}_P$.

The monomials obey the remarkable relation

$$\sum_{\{n\}} \Psi_{P, \{n\}}(z') \Psi_{P, \{n\}}(\tilde{z}) = \langle z', \tilde{z} \rangle^P,$$ (4)

which is group invariant on account of the invariance of the scalar product (1) under the group transformation, $\langle z'_g, \tilde{z}_g \rangle = \langle z', \tilde{z} \rangle$.

We introduce also the scalar product of two polynomials:

$$\langle \Psi_P | \Psi'_P \rangle = \int \overline{\Psi_P(z)} \Psi'_P(z) d\mu_P(\tilde{z}, z),$$

$$d\mu_P(\tilde{z}, z) = \frac{(P + N - 1)!}{(2\pi)^N P!} \delta(\sum |z_\mu|^2 - 1) \prod d\tilde{z}_\nu dz_\nu,$$

$$d\tilde{z}dz = d(|z|^2)d(\arg z).$$

Using the integral

$$\int_0^1 d\rho_1 \cdots \int_0^1 d\rho_N \delta(\sum \rho_\mu - 1) \prod_{\nu=1}^N \rho_\nu^{n_\nu} = \frac{\prod_{\nu=1}^N n_\nu!}{(\sum_{\nu=1}^N n_\nu + N - 1)!},$$

it is easy to verify that the orthonormality relation holds:

$$\langle \Psi_{P, \{n\}} | \Psi_{P, \{n'\}} \rangle = \langle P, n | P, n' \rangle = \delta_{\{n\}, \{n'\}}.$$ (6)

The completeness relation take place as well

$$\sum_{\{n\}} |P, n\rangle \langle P, n| = I_P,$$ (7)

where $|P, n\rangle$ and $\langle P, n|$ are Dirac’s denotations for the vectors $\Psi_{P, \{n\}}(z)$ and $\Psi_{P, \{n\}}(\tilde{z})$ respectively, and $I_P$ is the identical operator in the irreducible space of representation of signature $P$. 

It is convenient to introduce the operators \( a_\mu^\dagger \) and \( a^\mu \) which act on the basis vectors by formulas:

\[
a_\mu^\dagger \Psi_{P,\{n\}}(z) \rightarrow a_\mu^\dagger |P, n\rangle = \sqrt{\frac{n_\mu + 1}{P + 1}} |P + 1, \ldots, n_\mu + 1, \ldots, n_\nu - 1, \ldots, n_\mu + 1, \ldots\rangle,
\]

\[
a^\mu \Psi_{P,\{n\}}(z) = \frac{\partial}{\partial z^\mu} \Psi_{P,\{n\}}(z) \rightarrow a^\mu |P, n\rangle = \sqrt{Pn_\mu} |P - 1, \ldots, n_\mu - 1, \ldots\rangle,
\]

\[
[a^\mu, a_\nu^\dagger] = \delta_\mu^\nu, \quad [a^\mu, a^\nu] = 0.
\]

(8)

One can find the action of these operators on the left,

\[
\langle P, n | a_\mu^\dagger = \sqrt{\frac{n_\mu}{P}} \langle P - 1, \ldots, n_\mu - 1, \ldots | = \frac{1}{P} \frac{\partial}{\partial \tilde{z}^\mu} \Psi_{P,\{n\}}(\tilde{z}),
\]

\[
\langle P, n | a^\mu = \sqrt{(P + 1)(n_\mu + 1)} \langle P + 1, \ldots, n_\mu + 1, \ldots | = (P + 1) \tilde{z}^\mu \Psi_{P,\{n\}}(\tilde{z}).
\]

Their quadratic combinations \( A_\mu^\nu \) can serve as generators in each irreducible representation of signature \( P \),

\[
A_\mu^\nu = a^\mu a_\nu = z_\mu \frac{\partial}{\partial z_\nu}, \quad [A_\mu^\nu, A_\lambda^\kappa] = \delta_\kappa^\nu A_\mu^\lambda - \delta_\mu^\kappa A_\nu^\lambda,
\]

\[
A_\mu^\nu |P, n\rangle = \sqrt{n_\nu(n_\mu + 1)} |P, \ldots, n_\nu - 1, \ldots, n_\mu - 1, \ldots\rangle, \quad \mu \neq \nu,
\]

\[
A_\mu^\nu |P, n\rangle = n_\mu |P, n\rangle, \quad \sum_\mu A_\mu^\mu |P, n\rangle = P |P, n\rangle.
\]

(10)

Obviously, the \( A_\mu^\nu \) are Cartan’s generators and \((n_1, \ldots, n_N)\) the weight vector.

The independent generators \( \hat{\Gamma}_a, a = 1, N^2 - 1 \), can be expressed in terms of the operators \( A_\mu^\nu \):

\[
\hat{\Gamma}_a = (\Gamma_a)_\mu^\nu A_\nu^\mu, \quad [\hat{\Gamma}_a, \hat{\Gamma}_b] = if_{abc} \hat{\Gamma}_c,
\]

(11)

where \( \Gamma_a \) are generators in the fundamental representation, \([\Gamma_a, \Gamma_b] = if_{abc} \Gamma_c\).

The quadratic Casimir operator \( C_2 = \sum_\alpha \hat{\Gamma}_\alpha^2 \) can be only expressed via the operators \( A_\mu^\nu \) by means of the well known formula

\[
\sum_\alpha (\Gamma_a)_\mu^\nu (\Gamma_a)_\lambda^\kappa = \frac{1}{2} \delta_\kappa^\nu \delta_\lambda^\mu - \frac{1}{2N} \delta_\nu^\mu \delta_\lambda^\kappa,
\]

(12)

and evaluated in every irreducible representation explicitly.
Now we are going to construct the orbits of highest weights (of a vector of the basis (3) with the maximal length $\sqrt{\sum n^2_\mu} = P$). Let this highest weight be the state $\Psi_{P,(P,0...0)}(z) = (z^1)_P$. Then we get, in accordance with (4):

$$T(g)\Psi_{P,(P,0...0)}(z) = [z_\mu g^\mu_1]_P = \langle z, \tilde{u} \rangle_P, \quad \tilde{u}^\mu = g^\mu_1,$$  

where the vector $\tilde{u} \in \tilde{C}^N$ is the first column of the $SU(N)$ matrix in the fundamental representation.

If we interpret the representation space as a Hilbert one of quantum states, then we have to identify all the states differ each other by a constant phase. Let us turn from that point of view to the states of the orbit (14). One can notice, that the transformation $\arg \tilde{u}^\mu \to \arg \tilde{u}^\mu + \lambda$, changes all the states (4) by the constant phase $\exp(iP\lambda)$. So, one can treat the transformation as gauge one in certain sense. To select only physical different quantum states $(CS)$ from all the states of the orbit, one has to impose a gauge condition on $\tilde{u}$ which fixes the total phase of the orbit (14). Such a condition may be chosen in the form $\sum_\mu \arg \tilde{u}^\mu = 0$.

Taken into account that the quantities $\tilde{u}$ obey the condition $\sum |\tilde{u}^\mu|^2 = 1$, by the origin, as elements of the first column of the $SU(N)$ matrix, we get the explicit form of the $CS$ of the $SU(N)$ group in the space $\Pi_P$:

$$\Psi_{P,\tilde{u}}(z) = \langle z, \tilde{u} \rangle_P,$$  

$$\sum_\mu |\tilde{u}^\mu|^2 = 1, \quad \sum_\mu \arg \tilde{u}^\mu = 0.$$

In the same manner we construct the orbit of the highest weight $\Psi_{P,(P,0...0)}(\tilde{z}) = (\tilde{z}^1)_P$ in the space $\tilde{\Pi}_P$, and the corresponding $CS$ have the form:

$$\Psi_{P,u}(\tilde{z}) = \langle u, \tilde{z} \rangle_P,$$  

$$\sum_\mu |u_\mu|^2 = 1, \quad \sum_\mu \arg u_\mu = 0.$$
Obviously, \(\Psi_{P,\tilde{u}}(z) = \overline{\Psi_{P,u}(\tilde{z})}, \ z \leftrightarrow \tilde{z}, \ u \leftrightarrow \tilde{u}.\)

It is easy to see that all the elements of the discrete basis (3) with the weight vectors of the form \((n_\mu = \delta_\mu^P, \mu = 1, N)\) belong to the CS set (15) with parameters \((\tilde{u}_\mu = \delta_\mu^P, \mu = 1, N)\). The analogous statement is valid regarding to the dual basis and to the CS (17).

The quantities \(\tilde{u}\) and \(u\), which parametrize the CS (13) and (16), are elements of the coset space \(SU(N)/U(N-1)\), in accordance with the fact that the stationary subgroups of both the initial vectors from the spaces \(\Pi_P\) and \(\tilde{\Pi}_P\) are \(U(N-1)\). At the same time, the coset space is the so called projective space \(CP^{N-1}\) (we remember that the complex projective space is defined as a set of all nonzero vectors \(z\) in \(C^N\), where \(z\) and \(\lambda z, \lambda \neq 0\), are equivalent [16]). The eq.(13) or (17), are just the possible conditions which define the projective space. The coordinates \(u\) or \(\tilde{u}\) are called homogeneous ones in the \(CP^{N-1}\).

Thus, the CS constructed are parametrized by the elements of the projective space \(CP^{N-1}\), which is a symplectic manifold [16] and therefore can be considered as the phase space of a classical mechanics.

To decompose the CS in the discrete basises, one can use the scalar product (3) directly, but there exists more simple way. One can use the relation (1), on account of the right side of eq. (4) can be treated as CS (13) or (17). So, it follows from (1) :

\[
\Psi_{P,\tilde{u}}(z) = \sum_{\{n\}} \Psi_{P,\{n\}}(\tilde{u}) \Psi_{P,\{n\}}(z).
\]

That implies:

\[
\langle P, u | P, n \rangle = \Psi_{P,\{n\}}(u), \quad \langle P, n | P, u \rangle = \Psi_{P,\{n\}}(\tilde{u}),
\]

where \(|P, u\rangle\) and \(\langle P, u|\) are Dirac’s denotations for the CS \(\Psi_{P,\tilde{u}}(z)\) and \(\Psi_{P,u}(\tilde{z})\) respectively. So we come to the important for the understanding result: the discrete basises in the spaces \(\Pi_P\) and \(\tilde{\Pi}_P\) are ones in the CS representation.

The completeness relation for the CS can be extracted from the eq.(3). Using the formulas (20) in the integral (3), we get:

\[
\int \langle P, n | P, u \rangle \langle P, u | P, n^' \rangle d\mu_P(\tilde{u}, u) = \delta_{\{n\},\{n'\}}.
\]
That proves the completeness relation
\[
\int |P, u\rangle\langle P, u|d\mu_P(\bar{u}, u) = I_P .
\] (21)

III. UNCERTAINTY RELATION

The orbit of each vector of the discrete basis $|P, n\rangle$ (3) and, particularly, the CS constructed, are eigen for a nonlinear operator $C'_2$, which is defined by its action on an arbitrary vector $|\Psi\rangle$ as
\[
C'_2|\Psi\rangle = \sum_a \langle\Psi|\hat{\Gamma}_a|\Psi\rangle \hat{\Gamma}_a|\Psi\rangle .
\] (22)

First, we note that $T^{-1}(g)C'_2T(g) = C'_2$, where $T(g)$ are operators of the representation. Indeed, it follows from the relation $T^{-1}(g)\hat{\Gamma}_aT(g) = t^c_a\hat{\Gamma}_c$ and $[C_2, T(g)] = 0$, that $t^c_a$ is an orthogonal matrix, so that
\[
T^{-1}(g)C'_2T(g)|\Psi\rangle = \sum_a \langle\Psi|T^{-1}(g)\hat{\Gamma}_aT(g)|\Psi\rangle T^{-1}(g)\hat{\Gamma}_aT(g)|\Psi\rangle
= \sum_a \langle\Psi|\hat{\Gamma}_a|\Psi\rangle \hat{\Gamma}_a|\Psi\rangle = C'_2|\Psi\rangle .
\]

After that, it is easy to show, that the orbit $T(g)|P, n\rangle$ is eigen for $C'_2$. We write:
\[
C'_2T(g)|P, n\rangle = T(g)C'_2|P, n\rangle = T(g) \sum_a \langle P, n|\hat{\Gamma}_a|P, n\rangle \hat{\Gamma}_a|P, n\rangle ,
\] (23)
and use the formulas (11), (12) in the right side of (23),
\[
\sum_a \langle P, n|\hat{\Gamma}_a|P, n\rangle \hat{\Gamma}_a|P, n\rangle
= \frac{1}{2} \left[ \langle P, n|A_{\mu}^\nu|P, n\rangle A_{\mu}^\nu - \frac{1}{N} \sum_{\mu} A_{\mu}^\mu \right] |P, n\rangle = \lambda(P, n)|P, n\rangle ,
\]
\[\lambda(P, n) = \frac{1}{2} \left( \sum_{\mu} n_{\mu}^2 - P^2 / N \right) = \frac{1}{2} \sum_{\mu} (n_{\mu} - P/N)^2 .\]

The latter results in
\[
C'_2T(g)|P, n\rangle = \lambda(P, n)T(g)|P, n\rangle .
\] (24)
The eigen value $\lambda(P, n)$ attains the maximum for the highest weights, for which $\sum_\mu n_\mu^2 = P^2 = \text{max}$. The $CS |P, u\rangle$ belong to the orbit of the highest weight $\{n\} = \{P, 0, \ldots, 0\}$. So we get:

$$C'_2|P, u\rangle = \frac{P^2(N - 1)}{2N}|P, u\rangle$$

(25)

One can introduce a dispersion of the square of the length of the isospin vector $|P, u\rangle$, 

$$\Delta C_2 = \langle \Psi | \sum_a \hat{\Gamma}_a^2 | \Psi \rangle - \sum_a \langle \Psi | \hat{\Gamma}_a | \Psi \rangle^2 = \langle \Psi | C_2 - C'_2 | \Psi \rangle .$$

(26)

The dispersion serve as a measure of the uncertainty of the state $|\Psi\rangle$. Due to the properties of the operators $C_2$ and $C'_2$, it is group invariant and has the least value $P(N - 1)/2$ for the orbits of highest weights, paticularly for the $CS$ constructed, with respect to all the orbits of the discrete basis $|3\rangle$. The relative dispersion of the square of the length of the isospin vector has the value in the $CS$:

$$\Delta C_2/C_2 = \frac{N}{N + P} ,$$

(27)

and tends to zero with $h \to 0$, $h = \frac{1}{P}$. That fact indicates already, that the quantity $h$ plays here the role of the Planck constant. In the Sect.5 this analogy is traced in more details.

**IV. THE CS OVERLAPPING**

The overlapping of the CS can be evaluated in different ways. For instance, using the completeness relation (20) and formulas (19), (4), we get:

$$\langle P, u|P, v\rangle = \sum_{\{n\}} \langle P, u|P, n\rangle \langle P, n|P, v\rangle$$

$$= \sum_{\{n\}} \Psi_{P,\{n\}}(u) \Psi_{P,\{n\}}(\tilde{v}) = \langle u, \tilde{v}\rangle^P .$$

(28)

Comparing the result with eq. (14), one can write

$$\langle P, u|P, v\rangle = \Psi_{P,\tilde{v}}(u) ,$$

(29)
what confirms once again, that the spaces $\Pi_P$ and $\tilde{\Pi}_P$ are, in quantum mechanical sense, merely the spaces of abstract vectors in the CS representation.

Let $\Psi_P(u)$ be a vector $|\Psi\rangle$ in the CS representation, $\Psi_P(u) = \langle P, u|\Psi\rangle$. Then the formula take place

$$\Psi_P(u) = \int \langle P, u|P, v\rangle \Psi_P(v) d\mu_P(\bar{v}, v).$$

(30)

That means, the CS overlapping playes the role of the $\delta$- function in the CS representation.

The modulus of the CS overlapping (28) possesses the properties:

$$|\langle P, u|P, v\rangle| < 1, \lim_{P \to \infty} |\langle P, u|P, v\rangle| = 0, \text{ if } u \neq v,$$

$$|\langle P, u|P, v\rangle| = 1, \text{ only, if } u = v.$$  

(31)

That follows from the Cauchy inequality for the scalar product (1), $|\langle u, \tilde{v}\rangle| \leq \sqrt{\langle u, \tilde{u}\rangle \langle v, \tilde{v}\rangle}$, and from the conditions on the parameters of the CS, $\langle u, \tilde{u}\rangle = \langle v, \tilde{v}\rangle = 1$.

One can introduce a function $s(u, v)$ of the coordinates of two points of the projective space $CP^{N-1}$,

$$s^2(u, v) = -\ln |\langle P, u|P, v\rangle|^2 = -P \ln |\langle u, \tilde{v}\rangle|^2.$$  

(32)

The properties of the modulus of the CS overlapping (31) allows one to interprete the function as a symmetric. We remember, that a real and positive symmetric obeys only two axioms of a distance: $s(u, v) = s(v, u)$ and $s(u, v) = 0$, if and only if $u = v$, exepting the triangle axiom. For the CS of the Heisenberg-Weyl group the function $s^2(u, v) = -\ln |\langle u|v\rangle|^2 = |u - v|^2$, and gives real the square of the distance on the complex plane of the CS parameters. It turns out, that in case of consideration, the symmetric $s(u, v)$ generates the metric in the projective space $CP^{N-1}$. To demonstrate that, it is convenient to go over from the homogeneous coordinates $u_\mu$, subjected to the supplemental conditions (8), to the local independent coordinates in $CP^{N-1}$. For instance, in the domain where $u_N \neq 0$, we introduce the local coordinates $\alpha_i, i = 1, N - 1$, 

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\[ \alpha_i = \frac{u_i}{u_N}, \quad (33) \]
\[ u_i = \alpha_i u_N, \quad u_N = \frac{\exp\left(-\frac{1}{P} \sum \arg \alpha_k \right)}{\sqrt{1 + \sum |\alpha_k|^2}}. \]

In the local coordinates (33) the symmetric (32) takes the form
\[ s^2(\alpha, \beta) = -P \ln \frac{\lambda(\alpha, \bar{\beta}) \lambda(\beta, \bar{\alpha})}{\lambda(\alpha, \bar{\alpha}) \lambda(\beta, \bar{\beta})}, \quad (34) \]
where \( \lambda(\alpha, \bar{\beta}) = 1 + \sum_i \alpha_i \bar{\beta}_i \).

So, we are in position to calculate the square of the "distance" between two infinitesimal close points \( \alpha \) and \( \alpha + d\alpha \). For the \( ds^2 \), which is defined as the quadratic part of the decomposition of \( s^2(\alpha, \alpha + d\alpha) \) in the powers of \( d\alpha \), one finds:
\[ ds^2 = g_{ik} d\alpha_i d\bar{\alpha}_k, \quad g_{ik} = P \lambda^{-2}(\alpha, \bar{\alpha}) \left[ \lambda(\alpha, \bar{\alpha}) \delta_{ik} - \bar{\alpha}_i \alpha_k \right], \]
\[ g_{ik} = \frac{\partial^2 F}{\partial \alpha_i \partial \bar{\alpha}_k}, \quad F = P \ln \lambda(\alpha, \bar{\alpha}), \quad (35) \]
\[ \det \|g_{ik}\| = P^{N-1} \lambda^{-N}(\alpha, \bar{\alpha}), \quad g^{ki} = \frac{1}{P} \lambda(\alpha, \bar{\alpha}) (\delta_{ki} + \bar{\alpha}_k \alpha_i). \]

Now one can recognize in the expression (35) so called Fubini-Study metric of the complex projective space \( CP^{N-1} \) with the constant holomorphic sectional curvature \( C = 4/P \), [16].

It follows from (35), we deal with Kahlerian manifold. As known, a Kahlerian manifold is symplectic one and a classical mechanics exists on it. The Poisson bracket has the form:
\[ \{f, g\} = ig^{ki} \left( \frac{\partial f}{\partial \alpha_i} \frac{\partial g}{\partial \bar{\alpha}_k} - \frac{\partial f}{\partial \bar{\alpha}_k} \frac{\partial g}{\partial \alpha_i} \right), \quad (36) \]

In the next Sect. we show that the classical limit of the commutator of the operators symbols, connected with the \( CS \), generates namely that Poisson bracket.

**V. THE CLASSICAL LIMIT**

One of the dignity of \( CS \) is, they allow one to construct the operators symbols in a simple way, i.e. a correspondance between operators and classical functions on the phase space of a system. The reproduction of actions with operators on the symbols language is, in fact,
equivalent to the quantization problem. That approach to the quantization was developed by Berezin [6]. Our aim has more restricted character, in that Sect. we are going to investigate the conditions of the classical limit in terms of operators symbols constructed by means of the CS.

Let us turn to the so called covariant symbol [17], which is, in fact, the mean value of an operator $\hat{A}$ in the CS.

$$Q_A(u, \bar{u}) = \langle P, u | \hat{A} | P, u \rangle .$$

We also restrict ourselves with operators which are some polynomial functions on the generators, of power not more than some given one $M < P$. Such kind of operators can be written via the operators $a_{\mu}^{\dagger}, a_{\nu}$, using (10),(11), and be presented in the ”normal” form,

$$\hat{A} = \sum_{K=0}^{M} A_{\nu_1 \ldots \nu_K}^{\mu_1 \ldots \mu_K} a_{\mu_1}^{\dagger} \ldots a_{\mu_K}^{\dagger} a_{\nu_1} \ldots a_{\nu_K} .$$

It is easy to find the action of the operators $a_{\mu}^{\dagger}, a_{\nu}$ on the CS and to calculate the symbols (37),

$$a_{\mu}^{\dagger}|P, u\rangle = \frac{1}{P + 1} \frac{\partial}{\partial \bar{u}_\mu}|P + 1, u\rangle , \quad a_{\mu}|P, u\rangle = P \bar{u}_\mu|P - 1, u\rangle ,$$

$$\langle P, u | a_{\mu}^{\dagger} = u_\mu \langle P - 1, u|, \quad \langle P, u | a_{\mu} = \frac{\partial}{\partial u_\mu}\langle P + 1, u| .$$

So,

$$Q_A(u, \bar{u}) = \sum_{K=0}^{M} \frac{P!}{(P - K)!} A_{\nu_1 \ldots \nu_K}^{\mu_1 \ldots \mu_K} u_{\mu_1} \ldots u_{\mu_K} \bar{u}_{\nu_1} \ldots \bar{u}_{\nu_K} .$$

Obviously, there is a one-to-one correspondence between an operator and its covariant symbol.

In the local independent variables $\alpha$ which were defined in (33) the covariant symbol has the form

$$Q_A(\alpha, \bar{\alpha}) = \sum_{K=0}^{M} \frac{P!}{(P - K)!} \left( 1 + \sum_{i=1}^{N-1} |\alpha_i|^2 \right)^{-K} A_{\nu_1 \ldots \nu_K}^{\mu_1 \ldots \mu_K} \alpha_{\mu_1} \ldots \alpha_{\mu_K} \bar{\alpha}_{\nu_1} \ldots \bar{\alpha}_{\nu_K} ,$$

where the summation over the Greek indices runs from 1 to $N$ as before, but one has to count $\alpha_N = 1$.
In manipulations it is convenient to deal with the nondiagonal symbols

\[ Q_A(u,\bar{v}) = \frac{\langle P, u|\hat{A}|P, v \rangle}{\langle P, u|P, v \rangle} \]

\[ = \sum_{K=0}^{M} \frac{P!}{(P-K)!} \left( \sum_{\lambda}^{N} u_{\lambda} \bar{v}_{\lambda} \right)^{-K} A_{\mu_1...\mu_K}^{\mu_1...\mu_K} u_{\mu_1} \ldots u_{\mu_K} \bar{v}_{\nu_1} \ldots \bar{v}_{\nu_K} , \]

\[ Q_A(\alpha,\bar{\beta}) = \sum_{K=0}^{M} \frac{P!}{(P-K)!} \left( 1 + \sum_{i}^{N-1} \alpha_i \bar{\beta}_i \right)^{-K} A_{\mu_1...\mu_K}^{\mu_1...\mu_K} \alpha_{\mu_1} \ldots \alpha_{\mu_K} \bar{\beta}_{\nu_1} \ldots \bar{\beta}_{\nu_K} , \]

\[ \alpha_i = \frac{u_i}{u_N}, \beta_i = \frac{v_i}{v_N}, \alpha_N = \beta_N = 1 . \]

The symbols \( Q_A(\alpha,\bar{\beta}) \) are analytical functions on \( \alpha \) and \( \bar{\beta} \) separately and coincide with the covariant symbols (41) at \( \beta \to \alpha . \) These symbols namely, but not \( \langle P, \alpha|A|P, \beta \rangle \), are nondiagonal analytical continuation of the covariant symbols.

Using the completeness relation and the eq.(32) , one can find for the symbol of the product of two operators \( \hat{A}_1 \) and \( \hat{A}_2 \) :

\[ Q_{A_1A_2}(u,\bar{u}) = \int Q_{A_1}(u,\bar{v})Q_{A_2}(v,\bar{u}) e^{-s^2(u,v)} d\mu_P(\bar{v},v) . \]

Because of \( s^2(u,v) \) tends to infinity with \( P \to \infty \), if \( u \neq v \), and equals zero, if \( u = v \), one can conclude, that in that limit the domain \( v \approx u \) gives only a contribution to the integral. Thus,

\[ \lim_{h \to 0} Q_{A_1A_2}(u,\bar{u}) = Q_{A_1}(u,\bar{u})Q_{A_2}(u,\bar{u}) \int e^{-s^2(u,v)} d\mu_P(\bar{v},v) \]

\[ = Q_{A_1}(u,\bar{u})Q_{A_2}(u,\bar{u}) , \]

\[ h = \frac{1}{P} . \]

The integral in (44) equals unity because of the definition (32) and completeness relation.

If we define according to Beresin [6,17] the so called star multiplication of symbols

\[ Q_{A_1} \star Q_{A_2} = Q_{A_1A_2} , \]

then we have for the covariant symbols

\[ \lim_{h \to 0} Q_{A_1} \star Q_{A_2} = Q_{A_1Q_{A_2}} . \]

That is the first demand of the classical limit. Thus, the quantity \( h \) plays the role of the Planck constant here, as we have already notice before in Sect.3.
Now we are going to get the next term in the decomposition of the star multiplication (45) in powers of \( h \). That is more appropriate to do in the local independent coordinates \( \alpha \), because of the decomposition includes the operation of differentiation with respect to coordinates. The formula (43) in the local coordinates (33) takes the form

\[
Q_{A_1 A_2}(\alpha, \bar{\alpha}) = \int Q_{A_1}(\alpha, \bar{\beta}) Q_{A_2}(\beta, \bar{\alpha}) e^{-s^2(\alpha, \beta)} d\mu_P(\bar{\beta}, \beta) ,
\]

where \( d\mu_P(\bar{\beta}, \beta) \) is proportional to the well known G-invariant measure on \( CP^{N-1} \) (see eq.(35)). Decomposing the integrand near the point \( \beta = \alpha \), and going over to the integration over \( z = \beta - \alpha \), we get in the zero and first order in power \( h \) :

\[
Q_{A_1 A_2}(\alpha, \bar{\alpha}) = Q_{A_1}(\alpha, \bar{\alpha}) Q_{A_2}(\alpha, \bar{\alpha}) + \frac{\partial Q_{A_1}(\alpha, \bar{\alpha})}{\partial \alpha_k} \frac{\partial Q_{A_2}(\alpha, \bar{\alpha})}{\partial \bar{\alpha}_k} \times \det || g_{l, m}(\alpha, \bar{\alpha}) || \int \bar{z}_k z_i e^{-g_{a, b} z_a z_b} \prod_{j=1}^{N-1} \frac{dRe z_j dIm z_j}{\pi} ,
\]

where the matrix \( g^{ik} \) was defined in (35) and is proportional to \( h \). Taking into account the expression (36) for the Poisson bracket in the projective space \( CP^{N-1} \), we get for the star commutator of the symbols

\[
Q_{A_1} \star Q_{A_2} - Q_{A_2} \star Q_{A_1} = i\{Q_{A_1}, Q_{A_2}\} + o(h) .
\]

Thus, the second demand of the classical limit is satisfied.

VI. CONCLUSION

One has to note, that the uniform discription of the CS for all the groups \( SU(N) \), proved to be possible due to the choice of the irreducible representations of the groups in the spaces of polynomials of a fixed power, so that the coset space is parametrized by the homogeneous coordinates from \( CP^{N-1} \). If one constructs orbits, using the concrete
structure of the matrices of the group in the fundamental representation, as that was done in the majority of works devoted to the CS of the $SU(2)$ group, then the complications of the generalization of the method are connected with the increasing of the complicacy of the structure of the $SU(N)$ matrices with the growth of the number $N$. The representations in the spaces of polynomials of a power not greater then a given one, used in [7] for the $SU(2)$ group, lead at once to the parametrization of the coset space by local coordinates from $CP^{N-1}$; due to the nonsymmetrical form of the expressions in that case, the generalization to any $SU(N)$ group does not appear to be obvious. Another approach to the problem is possible in a Fock space, by means of representations of the Jordan-Schwinger type. To construct explicitly orbits here, it is necessary to disentangle rather complicated operators of the representation of the group in the Fock space. We can do that in the $SU(2)$ case, but the complicacy growth with the number $N$ essentialy.

The CS constructed are useful for the quasi-classical analysis of quantum equations of $SU(N)$ symmetrical gauge theories. With their help one can, for instance, derive the so called Wong equation and find ”quantum” corrections to the equations [14].
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