Distances Induced by Barbilian’s Metrization Procedure

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Abstract
Several authors have pointed out the connection between Barbilian’s metric introduced in 1934 and the recent study of Apollonian metrics. We provide examples of various distances that can be obtained by Barbilian’s metrization procedure and we discuss the relation between this metrization procedure and important Riemannian and generalized Lagrangian metrics (in the sense presented in [1, 32]). Then we prove an extension of Barbilian’s metrization procedure.

1 Introduction: The Metrization Procedure
Barbilian’s metrization procedure was introduced in [3] and it was the subject of an inspiring correspondence between D. Barbilian and W. Blaschke [4] in 1934 and thereafter. The theory received a larger audience due to P. Kelly [30] and a major development due to D. Barbilian [5, 6, 7, 8]. Over the years, the paper [3] has been cited many times. Recent studies are due to A.F. Beardon [9], F. Gehring and K. Hag [18], as well as P. Hästö, Z. Ibragimov and other authors [19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. The geometric viewpoint is discussed in the monograph [15]. All of these works cite and have a common source in Barbilian’s paper [3]. The examples explored in the present work aim to discuss Barbilian’s metrization procedure in the context of its relations with various classes of metrics, as for example Riemann, Finsler, Lagrange or Lagrange generalized metrics (see [2, 32]).

The following construction is given by Barbilian [3] and it is the development of the idea from [3]. Consider two arbitrary sets $K$ and $J$. The function $f : K \times J \to \mathbb{R}_+$ is called an influence of the set $K$ over $J$ if for any $A, B \in J$ the ratio $g_{AB}(P) = \frac{f(P,A)}{f(P,B)}$ has a maximum $M_{AB} \in \mathbb{R}$ when $P \in K$. Note that $g_{AB} : K \to \mathbb{R}_+$. In [3] it is pointed out that if we assume the existence of $\max g_{AB}(P)$, when $P \in K$, then there also exists $m_{AB} = \min_{P \in K} g_{AB}(P) = \frac{1}{M_{AB}}$.

For example (see [5]), if $T$ is a topological space, $K$ a compact subset in $T$, and $J$ some arbitrary subset, then any function $f : K \times J \to \mathbb{R}_+$ continuous in the first argument is an influence on $J$. It is known since [3] that $d : J \times J \to \mathbb{R}_+$ given by

$$
\ln \frac{\max_{P \in K} g_{AB}(P)}{\min_{P \in K} g_{AB}(P)}
$$

(1)
is a semidistance, i.e.: (1) if $A = B$ then $d(A, B) = 0$; (2) $d$ is symmetric; (3) $d$ satisfies triangle inequality.

The influence $f : K \times J \to \mathbb{R}_+^*$ is called effective if there is no pair $(A, B) \in J \times J$ such that the ratio $g_{AB}(P) = \frac{f(P, A)}{f(P, B)}$ is constant for all $P \in K$. In [5] it is shown that if $f : K \times J \to \mathbb{R}_+^*$ is an effective influence, then (1) is a distance.

2 Examples

Example 2.1. Barbilian’s metrization procedure yields the Euclidean distance in a plane $(\pi)$ in $\mathbb{R}^3$, if we consider a plane $(\delta)$ parallel to the plane $(\pi)$ and take $J = (\pi)$, $K = (\delta)$, and the influence function $f : K \times J \to \mathbb{R}_+^*$, $f(M, A) = \exp \left[ \frac{1}{2} \| (P r \times Id)(M, A) \| \right] = e^{\frac{1}{2} \| M - A \|}$.

Example 2.2. Barbilian’s metrization procedure yields the spherical distance in a complete sphere in $\mathbb{R}^3$.

To see this, consider two concentric spheres $S_1$ and $S_2$ in $\mathbb{R}^3$, and let their common center be $O$. We take $S_1 = K$ and $S_2 = J$, and $A, B \in J$ and $M \in K$. Denote by $\{ M' \} = (OM \cap J$ and define $Pr$ the radial projection from $S_1$ to $S_2$ given by $Pr(M) = M'$. Denote by ( . ) the spherical distance, and consider the influence function $f : K \times J \to \mathbb{R}_+^*$, $f(M, A) = \exp \left[ \frac{1}{2} \| (P r \times Id)(M, A) \| \right] = e^{\frac{1}{2} \| M' - A \|}$.

Thus, Barbilian’s metrization procedure can generate Riemannian metrics. Our goal is to show that Barbilian’s metrization procedure generates, for other choices of $K, J$, and $f$, Lagrange generalized metrics not reducible to a Riemannian, Finslerian or Lagrangian metric.

To complete our discussion, we mention here the following result, needed in the remaining part of this section. This is a particular form of the result from [4], part 2, paragraph 7, and a version of the argument used in [22] in the proof of Lemma 3.5.

Lemma 2.3. Let $K$ and $J$ be two subsets of the Euclidean plane $\mathbb{R}^2$, and $K = \partial J$. Consider the influence $f(M, A) = ||MA||$, where by $||MA||$ we denote the Euclidean distance. Consider

\[ g_{AB}(M) = \frac{f(M, A)}{f(M, B)} = \frac{||MA||}{||MB||} \]

and consider the distance induced on $J$ by the Barbilian’s metrization procedure, $d^B(A, B)$. Suppose furthermore that for $M \in K$ the extrema $\max g_{AB}(M)$ and $\min g_{AB}(M)$ for any $A$ and $B$ in $J$ are attained each in an unique point in $K$. Then:

(a) For any $A \in J$ and any line $d$ passing through $A$ there exist exactly two circles tangent to $K$ and to $d$ in $A$.

(b) The metric induced by the Barbilian distance has the form

\[ ds^2 = \frac{1}{4} \left( \frac{1}{R} + \frac{1}{r} \right)^2 (dx_1^2 + dx_2^2), \]
where $R$ and $r$ are the radii of the circles described in (a).

**Example 2.4.** Let $K = \{ y = 0 \}$ be the line in the $xy$-plane. Let $J = \{(x, y)/y > 0\}$. Take the function $||MA||$ as influence. Then the associated ratio is $f(M) = \frac{||MA||}{||MB||}$. By applying Barbilian’s metrization procedure, we only need to analyze the existence of minimum and maximum for the function

$$g(x) = \frac{x^2 - 2x_0 \cdot x + x_0^2 + y_0^2}{x^2 - 2x_1 \cdot x + x_1^2 + y_1^2}.$$ 

A straightforward application of Lemma 2.3 yields, after computations

$$R = \frac{y\sqrt{m^2 + 1}}{-1 + \sqrt{m^2 + 1}}$$

and

$$r = \frac{y\sqrt{m^2 + 1}}{1 + \sqrt{m^2 + 1}}$$

that is $\frac{1}{4} \left( \frac{1}{R} + \frac{1}{r} \right)^2 = \frac{1}{y^2}$, i.e. $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$, which is the Poincaré metric on the upper half-plane.

**Example 2.5.** Consider $\mathbb{R}^2$ endowed with the Euclidean distance $||.||$. It is known from [5] that for any circle $K$ of radius $\rho$ in $\mathbb{R}^2$, and for $J$ the interior of $K$, a Barbilian’s distance is obtained in $J$ by taking the influence $f(P, A) = ||PA||$. For a given point $(x, y)$ in $J$ and for an arbitrary line of slope $m$ passing through $(x, y)$, we find

$$R = \frac{\sqrt{m^2 + 1}}{2} \cdot \frac{\rho^2 - x^2 - y^2}{\rho\sqrt{m^2 + 1} - xm + y}.$$ 

Similarly, we get,

$$r = \frac{\sqrt{m^2 + 1}}{2} \cdot \frac{\rho^2 - x^2 - y^2}{\rho\sqrt{m^2 + 1} + xm - y}.$$ 

Hence, we proved the metric relation

$$\frac{1}{4} \left( \frac{1}{R} + \frac{1}{r} \right)^2 = \frac{4\rho^2}{(\rho^2 - x^2 - y^2)^2}.$$ 

By a straightforward computation, we can easily see that the Gaussian curvature of this metric is $\kappa_g = -1$. Therefore this Riemannian metric generates the hyperbolic geometry on the disk.

For the next example, we apply Lemma 2.3 to the following.

**Proposition 2.6.** Barbilian’s metrization procedure on

$$K = \{(x, 0) \in \mathbb{R}^2/x > 0\} \cup \{(0, y) \in \mathbb{R}^2/y > 0\},$$ 

3
for the influence \( f : K \times J \rightarrow \mathbb{R}^*_+ \), given by \( f(M, A) = ||MA|| \), yields the metric that at \((x_0, y_0) \in J\) satisfies

\[
d s^2 = \left( \frac{y_0m + x_0 + (x_0 + y_0)\sqrt{m^2 + 1}}{4x_0y_0(m^2 + 1)} \right)^2 (dx^2 + dy^2), \tag{3}
\]

\[m = \frac{x_0}{y_0} \mid_{(x_0,y_0)},\] where the metric \( g \) is a generalized Lagrange metric that is not reducible to a Riemannian, Finslerian or Lagrangian metric.

**Proof:** Denote as above \( g_{AB} : K \rightarrow \mathbb{R}^*_+ \), given by

\[
g_{AB}(M) = \frac{f(M, A)}{f(M, B)} = \frac{||MA||}{||MB||}.
\]

First, we need to show that \( g_{AB} \) admits maximum and minimum. Consider the points \( A, B \in J \), \( M \in K \), and denote by \( A_1 \) the foot of perpendicular from \( A \) to the \( y \)-axis and by \( A_2 \) the foot of perpendicular from \( A \) on the \( x \)-axis. Consider the inversion centered in \( A \) and of power \( ||AA_1||^2 \). This inversion induces the correspondences \( A_1 \rightarrow A_1, A_2 \rightarrow A_2' \) (such that \( A_2' \in OA_2 \) and \( ||AA_2|| \cdot ||AA_2'|| = ||AA_1||^2 \)), \( O \rightarrow O' \) (such that \( O' \in AO \) and \( A_1O' \perp AO \)), \( B \rightarrow B' \) such that \( B' \in AB \) and \( ||AB|| \cdot ||AB'|| = ||AA_1||^2 \). The positive part of the \( y \)-axis is transformed in the arc of circle \( C_1 \) of endpoints \( A \) and \( O' \), and it is part of the circle of diameter \( AA_1' \); more precisely is the arc that contains the point \( A_1 \). The positive part of the \( x \)-axis is transformed in the arc of circle \( C_2 \) of endpoints \( A \) and \( O' \), and it is part of the circle of diameter \( AA_2' \), more precisely the arc that contains the point \( A_2 \). The inverse of a point \( M \in K \) is part of the union of the two arcs described above. Keeping in mind that

\[
||B'M'|| = ||AA_1||^2 \cdot \frac{||BM||}{||AM|| \cdot ||AB||} = \frac{||AA_1||^2}{||AB||} \cdot \frac{||BM||}{||AM||}, \tag{4}
\]

we get that \( ||B'M'|| \) is maximum whenever \( \frac{||AM||}{||BM||} \) is minimum. Denote by \( M'_1 \) the point on \( C_1 \cup C_2 \) for which is attained the maximum of the Euclidean distance \( ||B'M'|| \). The ray \( AM_1' \) intersects \( K \) in \( M_1' \) for which

\[
m = \frac{||AM_1||}{||BM_1||} = \min_{M \in K} \frac{||AM||}{||BM||}.
\]

From \( g \) we deduce also that there exists a point \( M'_2 \) for which \( ||B'M'_2|| \) is the minimum for \( ||B'M'|| \), when \( M' \in C_1 \cup C_2 \). The inverse of \( M'_2 \) is \( M_2 \), obtained at the intersection between \( AM_2' \) and \( K \) and it has the property

\[
M = \frac{||AM_2||}{||BM_2||} = \max_{M \in K} \frac{||AM||}{||BM||}.
\]

This allows us to conclude that the formula \( d^B(A, B) = \ln \frac{M}{m} \) produces a Barbilian distance in \( J \). Now we obtain the coefficients of the metric from Lemma
Consider the arbitrary point \(A(x_0, y_0) \in J\) and the line \((d)\) of equation \(y - y_0 = m(x - x_0)\). By Lemma 2.3 there exist the circles \(\Gamma_1\) and \(\Gamma_2\) tangent to the line \((d)\) in \(A\) and tangent to \(K\). Denote by \(O_1(x_1, y_1)\) the center of the circle \(\Gamma_1\) and by \(O_2(x_2, y_2)\) the center of the circle \(\Gamma_2\). To determine the rays of the two circles described in Lemma 2.3 (a) we have the conditions

\[
y_1 - y_0 = -\frac{1}{m}(x_1 - x_0), \quad x_1^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2, \tag{5}
\]

with \(x_0 > x_1\), and

\[
y_2 - y_0 = -\frac{1}{m}(x_2 - x_0), \quad y_2^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2, \tag{6}
\]

for \(y_0 > y_1\). From (5) and (6), respectively, we obtain:

\[
R_1 = x_1 = \frac{x_0\sqrt{m^2 + 1}}{m + \sqrt{m^2 + 1}}, \quad R_2 = y_2 = \frac{y_0\sqrt{m^2 + 1}}{1 + \sqrt{m^2 + 1}}. \tag{7}
\]

Therefore, by applying Lemma 2.3 the metric is expressed as in (3). For the directions \(m = \frac{\dot{y}}{\dot{x}}\) with \(\dot{x} > 0\), the metric has the coefficients

\[
g_{11} = g_{22} = \frac{\dot{x}(y \cdot \dot{y} + x \cdot \dot{x} + (x + y)\sqrt{\dot{x}^2 + \dot{y}^2})^2}{4xy(\dot{x}^2 + \dot{y}^2)}, \quad g_{12} = g_{21} = 0. \tag{8}
\]

This metric (see [31, 32]) is a generalized Lagrange metric, since the tensor expressed above is a \(d\)-tensor. To see this, remark that the metric is 0–homogeneous, and \(\det g = (g_{11})^2\), therefore it is positive definite. According to section 2.2 from [32], the metric is reducible to a Lagrangian metric if and only if the Cartan tensor \(C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x_k}\) is totally symmetric (see [32, section 4.1, Theorem 1.1.). The condition of symmetry reduces for the metric (8) to

\[
\frac{\partial g_{11}}{\partial \dot{y}} = \frac{\partial g_{12}}{\partial \dot{x}}.
\]

However, \(\frac{\partial g_{12}}{\partial \dot{x}} = 0\) and \(\frac{\partial g_{11}}{\partial \dot{y}} \neq 0\), which proves that the Cartan tensor is not totally symmetric. Therefore, the metric (8) is not reducible to a Lagrangian metric. If the metric is not reducible to a Lagrangian metric, it is not reducible to either a Finslerian metric or a Riemannian metric.

### 3 An Extension of Barbilian’s Metrization Procedure

Now we present an extension of Barbilian’s metrization procedure. Our motivation to produce this extension is the fact that in the case when \(K\) is a circle in the plane and \(J\) is its interior, if we remove one point \(L\) from \(K\), we can not apply the classical Barbilian’s metrization procedure considering the influence
of $K - \{L\}$ over $J$. Suppose that $K$ and $J$ are arbitrary sets and that they satisfy the general extremum requirement, that is for any $A$ and $B$ in $J$ it exists $\sup_{Q \in K} g_{AB}(Q) < \infty$, when $Q \in K$. As we have seen in the case of maximum, if there exists $\sup_{P \in K} g_{AB}(P) < \infty$ then there exists $\inf_{P \in K} g_{AB}(P)$ and it equals $[\sup_{P \in K} g_{BA}(P)]^{-1}$. We have the following (see also [17], p.10).

**Theorem 3.1.** Suppose that $g$ satisfies the general extremum requirement. Then the function $d^* : J \times J \to \mathbb{R}_+$ given by

$$d^*(A, B) = \ln \frac{\sup_{P \in K} g_{AB}(P)}{\inf_{P \in K} g_{AB}(P)}$$

is a semidistance on $J$.

**Proof:** We need to prove that: $d^*(A, B) + d^*(B, C) \geq d^*(A, C)$. Then it is sufficient to show:

$$\frac{\sup_{P \in K} g_{AB}(P)}{\inf_{P \in K} g_{AB}(P)} \cdot \frac{\sup_{Q \in K} g_{BC}(Q)}{\inf_{Q \in K} g_{BC}(Q)} \geq \frac{\sup_{R \in K} g_{AC}(R)}{\inf_{R \in K} g_{AC}(R)}.$$

Denote by $\alpha$ the left hand side term in the inequality above and remark that

$$\alpha \geq \frac{g_{AB}(P)}{g_{AB}(Q)} \cdot \frac{g_{BC}(Q)}{g_{BC}(P)} = \frac{\frac{f(P, A)}{f(P, B)}}{\frac{f(Q, A)}{f(Q, B)}} = \frac{g_{AC}(P)}{g_{AC}(Q)}, \quad \forall P, Q \in K.$$

This means $\alpha \cdot g_{AC}(Q) \geq g_{AC}(P)$, for all $P, Q \in K$. Therefore,

$$\alpha \cdot g_{AC}(Q) \geq \sup_{P \in K} g_{AC}(P), \quad \forall Q \in K,$$

which yields

$$\alpha \cdot \inf_{Q \in K} g_{AC}(Q) \geq \sup_{P \in K} g_{AC}(P).$$

We obtain

$$\alpha \geq \frac{\sup_{R \in K} g_{AC}(R)}{\inf_{R \in K} g_{AC}(R)}.$$

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