Mixed solutions of monotone iterative technique for hybrid fractional differential equations

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Abstract: This paper concerns with a mathematical modelling of biological experiments, and its influence on our lives. Fractional hybrid iterative differential equations are equations that interested in mathematical model of biology. Our technique is based on the Dhage fixed point theorem. This tool describes mixed solutions by monotone iterative technique in the nonlinear analysis. This method is used to combine two solutions: lower and upper. It is shown an approximate result for the hybrid fractional differential equations iterative in the closed assembly formed by the lower and upper solutions.

Keywords: Fractional differential equation; fractional differential operator; fractional calculus; monotonous sequences; mixed solutions

AMS Mathematics Subject Classification: 26A33

1 Introduction

Calculus of fractional order power is a field of mathematical analysis (nonlinear part). It follows the traditional definition of derivatives and integrals of calculation operators in form fractional order([1],[2],[3]). Using fractional order differential operator in mathematics modeling has become more and more extended in the last years. Fractional order differential equations have been the concentrate of several studies because of their common occurrence in diverse applications in economics, biology, physics and engineering. Recently, a wealth of literature developed on the applying nonlinear differential equations of fractional order [4].

The class of fractional order differential equations is a generalization of the class of ordinary differential equations. We argue that the fractional order differential equations are more appropriate than the ordinary in mathematical modeling of biological, economic and social systems [5]. Fractional calculus is utilized in biology and medicine to explore the potential of fractional differential equations to describe and understand the biological organisms grow. Moreover, it utilized to develop the structure and functional properties of populations. Extend this concept to evaluate the changes associated with the disease hope that contribute to the understanding of the pathogenic processes of medicine [6]. Humans have learned how to employ bacteria and other microbes to making something useful, such as genetically engineered human insulin [7].

The important of the differential equations of the type hybrid implies polls number of dynamical systems dealt as special cases, ([8],[9]). Dhage, Lakshmikantham and Jadhav proved some of the major outcomes of hybrid linear differential equations of the first order and second type disturbances ([10],[11],[12]). A great a mathematical model for bacteria from growing by the iterative difference equation described. Ibrahim [13] established of the existence of an iterative fractional differential equation (Cauchy type) using the technique of nonexpansive operator. This kind is created in [14].

In this work, we discuss a mathematical model of biological experiments, and how its influence on our lives. The most prominent influence of biological organisms that is affect negative or positive in
our lives like a bacteria. Fractional hybrid iterative differential equations are equations that interested in mathematical model of biology. Our technique is based on the Dhage fixed point theorem. This tool describes mixed solutions by monotone iterative technique in the nonlinear analysis. This method is used to combine two solutions: lower and upper. It is shown an approximate result for the hybrid fractional differential equations iterative in the closed assembly formed by the lower and upper solutions.

2 Preliminaries

Recall the following preliminaries:

Definition 2.1 The derivative of fractional \( (\gamma) \) order for the function \( \phi(s) \) where \( 0 < \gamma < 1 \) is introduced by

\[
D^\gamma_a \phi(s) = \frac{d}{ds} \int_a^s (s - \beta)^{1 - \gamma} \phi(\beta)d\beta = \frac{d}{ds} I^{1-\gamma}_a \phi(s)
\]

(1)

\((\kappa - 1) < \gamma < \kappa,\)

in which \( \kappa \) is a whole number and \( \gamma \) is real number.

Definition 2.2 The integral of fractional \( (\gamma) \) order for the function \( \phi(s) \) where \( \gamma > 0 \) is introduced by

\[
I^\gamma_a \phi(s) = \int_a^s \frac{(s - \beta)^{1-\gamma}}{\Gamma(\gamma)} \phi(\beta)d\beta
\]

(2)

While \( a = 0 \), it becomes \( I^\gamma_a \phi(s) = \phi(s) * \Upsilon_\gamma(s) \), wherever \( * \) signify the convolution product

\[
\Upsilon_\gamma(s) = \frac{s^{\gamma-1}}{\Gamma(\gamma)}
\]

and \( \Upsilon_\gamma(s) = 0, \ s \leq 0 \) and \( \Upsilon_\gamma \to \delta(s)a\gamma \to 0 \) wherever \( \delta(s) \) is the delta function

Based on the Riemann-Liouville differential operator, we impose the following useful definitions:

Definition 2.3 Assume the closed period bounded interval \( I = [s_0, s_0 + a] \) in \( \mathbb{R} \) (\( \mathbb{R} \) the real line), for some \( s_0 \in \mathbb{R}, a \in \mathbb{R} \). The problem of initial value of fractional iterative hybrid differential equations (FIHDE) can be formulated as

\[
D^\alpha[v(s) - \psi(s, v(s), v(v(s)))] = \mathcal{N}(s, v(s), v(v(s))), s \in I
\]

(3)

with \( v(s_0) = v_0 \), where \( \psi, \mathcal{N} : I \times \mathbb{R} \to \mathbb{R} \) are continuous. A solution \( v \in C(I, \mathbb{R}) \) of the FIHDE (3) can be problem by

1. \( s \to v - \psi(s, v, v(v(s))) \) is a function which is continuous \( \forall v \in \mathbb{R} \), and

2. \( v \) contented the equations in (3). In which \( C(I, \mathbb{R}) \) space is of real-valued continuous functions defined on \( I \).

The definitions of the lower and upper solutions of (3) as follows:

Definition 2.4 We said that \( \iota \in C(I, \mathbb{R}) \) is a function which is a lower solution for the equation introduced on \( I \) if

1. \( s \to (\iota(s) - \psi(s, \iota(s)), \iota(\iota(s))) \), is continuous, and

2. \( D^\alpha[\iota(s) - \psi(s, v(s), v(v(s)))] \geq \mathcal{N}(s, \iota(s), \iota(\iota(s))), s \in I, \iota(s_0) \geq v_0 \).
Definition 2.5 We said that $\tau \in C(I, \mathbb{R})$ is a function which is an upper solution for the equation introduced on $I$ if

1. $s \mapsto (\tau(s) - \psi(s, \tau(s), \tau(s)))$, is continuous, and
2. $D^\alpha[\tau(s) - \psi(s, v(s), v(v(s))))] \leq \mathcal{N}(s, \tau(s), \tau(s)), s \in I, \tau(s_0) \leq v_0$.

We can build the monotonous sequence of consecutive iterations to converging towards the extremes among the lower and upper solutions of the differential equation related hybrid on $I$. We treat the case that if $\psi$ is neither non-decreasing nor non-increasing in the state of the variable $v$. If the function $\mathcal{N}$ can be separated into two components

$$\mathcal{N}(s, v, v(v))) = \mathcal{N}_1(s, v, v(v))) + \mathcal{N}_2(s, v, v(v)))$$

where $\mathcal{N}_1(s, v, v(v)))$ is a non-decreasing component while another component is not $\mathcal{N}_2(s, v, v(v)))$ increases in the state variables of $v$, then we may be constructed sequences iteration converged to solutions extremal $FIHDE$ on $I$.

Definition 2.6 Currently thought to be a initial value problem $FIHDE$

$$\begin{align*}
{D^\alpha[v(s) - \psi(s, v(s), v(v(s)))] = \mathcal{N}_1(s, v, v(v))) + \mathcal{N}_2(s, v, v(v))), s \in I,}
\end{align*}$$

$v(s_0) = v_0$ (4)

where, $\psi \in C(I \times R, R)$ and $\mathcal{N}_1, \mathcal{N}_2 \in L(I \times R, R)$.

Thus the lower and upper solutions of $FIHDE$ can be as defined as follows:

Definition 2.7 The functions $\sigma, \rho \in C(I, \mathbb{R})$ fulfill the following condition: the maps $s \rightarrow \sigma(s) - \psi(s, \sigma(s), \sigma(s)))$ and $s \rightarrow \rho(s) - \psi(s, \rho(s), \rho(\rho(s))))$ are absolute continuous on $I$. Thus the functions $(\sigma, \rho)$ are supposed to be of the kind

(a) which is mixed lower solutions and upper solutions for $FIHDE$ on $I$, as following

$$\begin{align*}
D^\alpha[\sigma(s) - \psi(s, \sigma(s), \sigma(s)))] \leq \mathcal{N}_1(s, \sigma(s), \sigma(s))), s \in I, \\
\sigma(s_0) \leq \ell_0
\end{align*}$$

(b) which is mixed lower solutions and upper solutions for $FIHDE$ on $I$, as follows

$$\begin{align*}
D^\alpha[\rho(s) - \psi(s, \rho(s), \rho(s)))] \geq \mathcal{N}_1(s, \rho(s), \rho(s))), s \in I, \\
\rho(s_0) \geq \ell_0
\end{align*}$$

Whether the sign was of equality achieves in relationships (4) and (5), hence the even of functions $(\sigma, \rho)$ set is been calling a mixed solution of kind (a) for the $FIHDE$ on $I$.

(b) which is mixed lower solutions and upper for $FIHDE$ on $I$, as follows

$$\begin{align*}
D^\alpha[\rho(s) - \psi(s, \rho(s), \rho(s)))] \geq \mathcal{N}_1(s, \rho(s), \rho(s))), s \in I, \\
\rho(s_0) \geq \ell_0
\end{align*}$$

Whether the sign was of equality achieves in relationships (7) and (8), hence the even of functions $(\sigma, \rho)$ set is been calling a mixed solution of kind (b) for the $FIHDE$ on $I$. 

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2.1 Assumptions

In the following assumptions relating to function $\psi$ is very important in the studying of Eq.(4).

(a0) The function $v \mapsto (v - \psi(s_0, v, v(v)))$ is injective in $\mathbb{R}$.

(b0) $\aleph$ is a bounded real-valued function on $I \times \mathbb{R}$.

(a1) The function $v \mapsto (v - \psi(s, v, v(v)))$ is increasing in $\mathbb{R}$ for all $s \in I$.

(a2) There is a constant $\ell > 0$ so that

$$|\psi(s, v, v(v)) - \psi(s, z, z(z))| \leq \frac{\ell |v - z|}{M + |v - z|}, \quad M > 0,$$

$\forall s \in I, v, z \in \mathbb{R}$ and $\ell \leq M$.

(b1) There is a constant $\kappa > 0$ so that $|\aleph(s, v, v(v))| \leq \kappa \forall s \in I$ and $\forall v \in \mathbb{R}$.

(b2) $\aleph_1(s, v, v(v))$ is function which is non-decreasing in $v$ function, and $\aleph_2(s, v, v(v))$ is function which is not increasing in $v$ for each $s \in I$.

(b3) $(\sigma_0, \rho_0)$ is Functions which are mixing the lower and upper solutions for $\aleph$ kind(a) on $I$ with $\sigma_0 \leq \rho_0$.

(b4) The pair is $(\sigma_0, \rho_0)$, the upper and lower mixing solutions for $\aleph$ kinds (b) on $I$ with $\sigma_0 \leq \rho_0$.

3 Main results

In this section, our purpose is to discuss the approximation outcome for $\aleph$.

Lemma 3.1 Suppose the assumptions $(a0) - (b0)$ are achieved. Then the function $v$ is a solution for Eq.(3) if and only if it must be the solution of the fractional iterative of hybrid equation integrated

$$FIHIE$$

$$v(t) = [v_0 - \psi(s_0, v_0, v(v_0))] + \psi(s, v(s), v(v(s))) + \int_0^s \aleph(\beta, v(\beta), v(v(\beta))) \frac{(s - \beta)^{\alpha - 1}}{\Gamma(\alpha)} d\beta, \quad (s \in I, v(0) = v_0).$$

Theorem 3.1 Let $\varrho$ be a closed convex and bounded subset of the Banach space $A$. Moreover, let $Q : A \to A$ and $P : \varrho \to A$ be two operators so that

(i) $Q$ is nonlinear D-contraction,

(ii) $P$ is compact and continuous,

(iii) $v = Qv + Pz$ for all $v \in \varrho \Rightarrow z \in \varrho$.

Theorem 3.2 Let the assumptions $(a1), (a2)$ and $(b1)$ be hold. Then $\aleph$ has a solution on $I$. 

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Proof. Let $A = C(I, \mathbb{R})$ be a set and $\subseteq A$, such that
\[ \varrho = \{ v \in A \| A \| \leq M \} \]
where,
\[ M = |v_0 - \psi(s_0, v_0, v(v(0)))| + \ell + \Psi_0 + \frac{\alpha}{\Gamma(\alpha + 1)} \| \xi \|_{\ell_1} \]
and $\Psi_0 = \sup_{s \in I} | \psi(s, 0, 0) |$. Obviously $\varrho$ is a convex, bounded and closed subset of the space $A$. By using the assumptions (a1) and (b1) together with the help of the Lemma 3.1, we conclude that the $FIHDE$ is tantamount to the nonlinear $FIHIE$. We define two operators $Q : A \rightarrow A$ and $P : \varrho \rightarrow A$ as follows:
\[ Qy(s) = \psi(s, v(s), v(v(s))), s \in I, \]
and
\[ Pv(s) = [v_0 - \psi(s_0, v_0, v(v_0))] + \int_0^s \mathcal{N}(\beta, v(\beta), v(v(\beta))) \frac{(s - \beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta, s \in I. \]
Consequently, the $FIHIE$ is equivalent to the operator equation
\[ Qv(s) + Pv(s) = v(s), s \in I. \]
We demonstrate that the operators $Q$ and $P$ fulfill all the conditions of Theorem 3.1. Foremost, we examine that $Q$ is a nonlinear $\Upsilon$-contraction on $Q$ with a $\Upsilon$ function $\varphi$. Let $v, z \in A$. In view of assumption (a2), we conclude that
\[ |Qv(s) - Qz(s)| = |\psi(s, v(s)) - \psi(s, z(s))| \leq \frac{\ell|v(s) - z(s)|}{M + |v(s) - z(s)|} \leq \frac{\ell|v - z|}{M + |v - z|} \]
for all $s \in I$. Take the supremum over $s$ yields
\[ \|Av - Az\| \leq \frac{\ell|v - z|}{M + |v - z|} \]
$\forall v, z \in A$. This proves that $Q$ is a nonlinear $D$-contraction $A$ with the $D$-function $\varphi$ defined by $\varphi(r) = \frac{\ell r}{M + r}$.

Next, we examine that $P$ is a continuous and compact operator on $\varrho$ into $A$. Let $\{v_t\}$ be a sequence in $\varrho$ converging to a point $v \in \varrho$, thus we have
\[
\lim_{t \rightarrow \infty} Pv_t(s) = \lim_{t \rightarrow \infty} \left[ v_0 - \psi(s_0, v_0, v(v_0)) + \int_0^s \mathcal{N}(\beta, v_t(\beta), v_t(v(\beta))) \frac{(s - \beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta \right]
\]
\[
= v_0 - \psi(s_0, v_0, v(v_0)) + \lim_{t \rightarrow \infty} \int_0^s \mathcal{N}(\beta, v_t(\beta), v_t(v(\beta))) \frac{(s - \beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta
\]
\[
= v_0 - \psi(s_0, v_0, v(v_0)) + \int_0^s \lim_{t \rightarrow \infty} \mathcal{N}(\beta, v_t(\beta), v_t(v(\beta))) \frac{(s - \beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta
\]
\[
= v_0 - \psi(s_0, v_0, v(v_0)) + \int_0^s \mathcal{N}(\beta, v(\beta), v(v(\beta))) \frac{(s - \beta)^{\alpha-1}}{\Gamma(\alpha)} d\beta = Pv(s)
\]
for all $s \in I$. Now, we proceed to prove that $\{Pv_t\}$ is equi-continuous with respect to $v$. According to [10], we attain that $P$ is a continuous operator on $\varrho$. To show that $P$ is a compact operator on $\varrho$. It suffices to examine that $\varrho$ is a regularly bounded and equi-continuous set in $A$. Let $v \in \varrho$ be arbitrary, then by the assumption (b1), we have
\[ |Pv(s)| \leq |v_0 - \psi(s_0, v_0, v(v_0))| + \int_0^s |\mathcal{N}(\beta, v(\beta), v(v(\beta))) \frac{(s - \beta)^{\alpha-1}}{\Gamma(\alpha)}| d\beta \]
\[
\leq |v_0 - \psi(s_0, v_0)| + \int_0^s \xi(\beta) \frac{(s - \beta)^{\alpha - 1}}{\Gamma(\alpha)} \, d\beta \leq |v_0 - \psi(s_0, v_0)| + \frac{a^\alpha}{\Gamma(\alpha + 1)} \|\xi\|_\ell^1
\]

for all \( s \in I \). By taking the supremum over \( t \), we obtain
\[
|Pv(s)| \leq |v_0 - \psi(s_0, v_0)| + \frac{a^\alpha}{\Gamma(\alpha + 1)} \|\xi\|_\ell^1
\]

\( \forall v_0 \in \varrho \). This proves that \( P \) is uniformly bounded on \( \varrho \).

Also let \( s_1, s_2 \in I \) with \( s_1 < s_2 \). Then for any \( v \in \varrho \), one has
\[
|Pv(s_1) - Pv(s_2)| = |\int_{s_0}^{s_1} |N(\beta, v(\beta), v(v(\beta)))| (s_1 - \beta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \, d\beta - \int_{s_0}^{s_2} |N(\beta, v(\beta), v(v(\beta)))| (s_2 - \beta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \, d\beta|
\]
\[
\leq |\int_{s_0}^{s_1} |N(\beta, v(\beta), v(v(\beta)))| (s_1 - \beta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \, d\beta - \int_{s_0}^{s_1} |N(\beta, v(\beta), v(v(\beta)))| (s_2 - \beta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \, d\beta|
\]
\[
+ \int_{s_0}^{s_1} |N(\beta, v(\beta), v(v(\beta)))| (s_2 - \beta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \, d\beta - \int_{s_0}^{s_2} |N(\beta, v(\beta), v(v(\beta)))| (s_2 - \beta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \, d\beta|
\]
\[
\leq \frac{\|\xi\|_\ell^1}{\Gamma(\alpha + 1)} \left| (s_2 - s_2)^\alpha - (s_1 - s_0)^\alpha - (s_2 - s_1)^\alpha + (s_2 - s_1)^\alpha \right|
\]

Hence, for \( \delta > 0 \), there exists a \( \epsilon > 0 \) so that
\[
|s_1 - s_2| < \epsilon \Rightarrow |Pv(s_1) - Pv(s_2)| < \delta
\]

\( \forall s_1, s_2 \in I \) and \( \forall v \in \varrho \). This examines for \( P(\varrho) \) is equi-continuous in \( A \). Presently \( P(\varrho) \) is bounded and hence it is compact by Arzel-Ascoli Theorem. Resulting, \( \varrho \) is a continuous and compact operator on \( \varrho \).

Then, we prove that assumptions (iii) of Theorem 3.1 is fulfilled. Let \( v \in A \) be fixed and \( z \in \varrho \) be arbitrary such that \( v = Qv + Pz \). In view of the assumption (a2) yields
\[
|v(s)| \leq |Qv(s)| + |Pz(s)|
\]
\[
\leq |v_0 - \psi(s_0, v_0)| + |v(s, v(s), v(v(s)))| + \int_0^s |N(\beta, v(\beta), v(v(\beta)))| (s - \beta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \, d\beta
\]
\[
\leq |v_0 - \psi(s_0, v_0)| + |v(s, v(s), v(v(s)))| + \int_0^s |N(\beta, v(\beta), v(v(\beta)))| (s - \beta)^{\alpha - 1} \frac{\Gamma(\alpha)}{\Gamma(\alpha)} \, d\beta
\]
\[
\leq |v_0 - \psi(s_0, v_0)| + \ell + \Psi_0 + \int_0^s \xi(\beta) \frac{(s - \beta)^{\alpha - 1}}{\Gamma(\alpha)} \, d\beta
\]
\[
\leq |v_0 - \psi(s_0, v_0)| + \ell + \Psi_0 + \frac{a^\alpha}{\Gamma(\alpha + 1)} \|\xi\|_\ell^1.
\]

Take the supremum over \( s \), implies
\[
\|v\| \leq |v_0 - \psi(s_0, v_0)| + \ell + \Psi_0 + \frac{a^\alpha}{\Gamma(\alpha + 1)} \|\xi\|_\ell^1 = M.
\]

Thus, \( v \in \varrho \).

Therefore, fulfilled all conditions of the Theorem 3.1 and thus the operator equation \( v = Qv + Pz \) has a solution in \( \varrho \). Resulting, the FIIHDE has a solution introduced on \( I \). This completes the proof.
\textbf{Theorem 3.3} Let \(\iota, \tau \in C(I, \mathbb{R})\) be lower and upper solutions of \(\text{FIHDE}^{(3)}\) fulfilling \(\iota(s) \leq \tau(s), s \in I\) and let the assumptions (a1) – (a2) and (b1) achieved. Then, there is a solution \(v(s)\) of (3), in the closed set \(\overline{I}\), satisfying

\[ i(s) \leq v(s) \leq \tau(s), s \in I. \]

\textbf{Proof.} Assume that \(\Theta : I \times \mathbb{R} \rightarrow \mathbb{R}\) is a function defined by

\[ \Theta(s, v, v(v)) = \max\{\iota(s), \min v(s), \tau(s)\}, \]

satisfying

\[ \tilde{\mathcal{N}}(s, v, v(v)) := \mathcal{N}(s, \Theta(s, v, v(v))). \]

Moreover, define a continuous extension of \(\mathcal{N}\) on \(I \times \mathbb{R}\) such that

\[ |\tilde{\mathcal{N}}(s, v, v(v))| = |\mathcal{N}(u, \Theta(s, v, v(v)))| \leq \kappa, s \in I, \forall v \in \mathbb{R}. \]

In view of Theorem 3.2, the \(\text{FIHDE}\)

\[ \begin{cases} D^\alpha[v(s) - \psi(s, v(s), v(v(s))] = \tilde{\mathcal{N}}(s, v, v(v)), s \in I \\ v(u_0) = v_0 \in \mathbb{R} \end{cases} \tag{14} \]

has a solution \(v\) defined on \(I\).

For any \(\delta > 0\), define

\[ \iota_\delta(s)\psi(s, \iota_\delta(s)) = (\iota(s) - \psi(s, \iota(s), \iota(s)))\delta(1 + s) \tag{15} \]

and

\[ \tau_\delta(s)\psi(s, \tau_\delta(s)) = (\tau(s) - \psi(s, \tau(s), \tau(s)))\delta(1 + s) \tag{16} \]

for \(s \in I\). In virtue of the assumptions (a1), we get

\[ \iota_\delta(s) < \iota(s), \text{ and } \tau(s) < \tau_\delta(s) \tag{17} \]

for \(s \in I\). Since

\[ \iota(s_0) \leq v_0 \leq \tau(s_0), \]

one has

\[ \iota_\delta(s_0) < v_0 < \tau_\delta(s_0). \tag{18} \]

To show that

\[ \iota_\delta(s) < v_0 < \tau_\delta(s), s \in I, \tag{19} \]

we define

\[ v(s) = v(s) - \psi(s, v(s), v(v(s))), s \in I. \]

Likewise, we consider

\[ h_\delta(s) = \iota_\delta(s) - \psi(s, \iota_\delta(s)), \]

\[ h(s) = \iota(u) - \psi(s, \iota(s), \iota(s)), \]

and

\[ T_\delta(s) = \tau_\delta(s)\psi(s, \tau_\delta(s), \tau(\tau_\delta(s))), \]

\[ T(s) = \tau(s)\psi(s, \tau(s), \tau(s)) \]

\(\forall s \in I\). If Eq. (19) is wrong, then there exists a \(s_\varepsilon \in (s_0, s_0 + \alpha]\) such that

\[ v(\varepsilon) = \tau_\delta(s_\varepsilon) \]

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and

\[ v_\delta(s) < v(s) < \tau_\delta(s), s_0 \leq s < s_\varepsilon \]

If \( v(s_\varepsilon) > \tau(s_\varepsilon) \), then \( \Theta(s_\varepsilon, v(s_\varepsilon), v(v(s_\varepsilon))) = \tau(s_\varepsilon) \). Furthermore,

\[ v(s_\varepsilon) \leq \Theta(s_\varepsilon, v(s_\varepsilon), v(v(s_\varepsilon))) \leq \tau(s_\varepsilon) \]

Now,

\[ D^\alpha T(s_\varepsilon) \geq \aleph(s_\varepsilon, \tau(s_\varepsilon), \tau(\tau(s_\varepsilon))) = \aleph(s_\varepsilon, v(s_\varepsilon), v(v(s_\varepsilon))) = D^\alpha V(s) \]

\[ \forall s \in I \]. Since \( T_\delta(u_s) > D^\alpha T(s) \), \( \forall s \in I \), we have

\[ D^\alpha T_\delta(s_\varepsilon) > D^\alpha V(s_\varepsilon) \]  \hspace{1cm} (20)

But,

\[ V(s_\varepsilon) = T_\delta(s_\varepsilon) \]

also

\[ V(s) = T_\delta(s), s_0 \leq s < s_\varepsilon, \]

means that together

\[ \frac{V(s_\varepsilon + \rho) - V(s_\varepsilon)}{\rho^\alpha} > \frac{T_\delta(s_\varepsilon + \rho) - T_\delta(s_\varepsilon)}{\rho^\alpha} \]

if \( \rho < 0 \) a small. Take the limit \( \rho \to 0 \) in the up variance yields

\[ D^\alpha V(s_\varepsilon) \geq D^\alpha T_\delta(s_\varepsilon) \]

that is a contradiction to \( 20 \). Hence,

\[ v(s) < \tau_\delta(s) \]

\[ \forall s \in I \]. Consequently

\[ v_\delta(s) < v(s) < \tau_\delta(s), s \in I. \]

Letting \( \delta \to 0 \) in the up inequality, we get

\[ v(s) \leq v(s) \leq \tau(s), s \in I. \]

This completes the proof. \( \square \)

Theorem 3.4 Let assumptions (a1) - (a2) and (b2) - (b3) achieved. Then there are the monotonous sequences \( \{\sigma_t\}, \{\rho_t\} \) such that \( \sigma_t \to \sigma \) and \( \rho_t \to \rho \) uniformly on \( I \) in which \( (\sigma, \rho) \) are mixed extremal solutions \( FIHDE \)\( \text{[4]} \) type(a) on \( I \).

Proof. Note the following a quadratic \( FIHDE \)

\[
\begin{cases}
D^\alpha[\sigma_{t+1}(s) - \psi(s, \sigma_{t+1}(s), \sigma(\sigma_{t+1}(s)))] \leq N_1(s, \sigma_t(s), \sigma(\sigma_t(s))) + N_2(s, \rho_t(s), \rho(\rho_t(s))), s \in I, \\
\sigma_{t+1}(s_0) \leq v_0
\end{cases}
\]

and

\[
\begin{cases}
D^\alpha[\rho_{t+1}(s) - \psi(s, \rho_{t+1}(s), \rho(\rho_{t+1}(s)))] \geq N_1(s, \rho_t(s), \rho(\rho_t(s))) + N_2(s, \sigma_t(s), \sigma(\sigma_t(s))), s \in I, \\
\rho_{t+1}(s_0) \geq v_0
\end{cases}
\]

for \( t \in N \).
Obviously, the equations (21) and (22) having unique solutions $\sigma_{i+1}$ and $\rho_{i+1}$ on $I$ respectively given Banach contraction mapping principle. We now want to demonstrate that

$$
\sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_i \leq \rho_i \leq \ldots \leq \rho_2 \leq \rho_1 \leq \rho_0
$$

on $I$ for $t = 0, 1, 2, \ldots$. Let $t = 0$ and set

$$
\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_0(s) - \psi(s, \sigma_0(s), \sigma(\sigma_0(s)))) \sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))
$$

for $s \in I$. Next by monotonicity of $\mathbb{R}_1$ and $\mathbb{R}_2$, we get

$$
D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] = D^\alpha[(\sigma_0(s)\psi(s, \sigma_0(s), \sigma(\sigma_0(s)))) - D^\alpha[\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))]
$$

$$
\leq \mathbb{N}_1(s_0, \sigma_0(s), \sigma(\sigma_0(s))) + \mathbb{N}_2(s, \rho_0(s), \rho(\rho_0(s))) - \mathbb{N}_1(s_0, \rho_0(s), \rho(\rho_0(s))) + \mathbb{N}_2(s, \sigma_0(s), \sigma(\sigma_0(s)))
$$

$$
= 0
$$

$\forall s \in I$ and $\Theta(s_0) = 0$. This implies that

$$
\sigma_0(s) - \psi(s, \sigma_0(s), \sigma(\sigma_0(s))) \leq \sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))),
$$

$\forall s \in I$. In view of (a1), one can get $\sigma_0(s) \leq \sigma_1(s), \forall s \in I$. Likewise it can be demonstrated which $\rho_1(s) \leq \rho_0(s)$ on $I$. Setting

$$
\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))) - (\rho_1(s) - \psi(s, \rho_1(s), \rho(\rho_1(s))))
$$

$\forall s \in I$. By monotonicity of $\mathbb{N}_1$ and $\mathbb{N}_2$, we obtain

$$
\leq \mathbb{N}_1(s_0, \sigma_0(s), \sigma(\sigma_0(s))) + \mathbb{N}_2(s, \rho_0(s), \rho(\rho_0(s))) - \mathbb{N}_1(s_0, \rho_0(s), \rho(\rho_0(s))) + \mathbb{N}_2(s, \sigma_0(s), \sigma(\sigma_0(s)))
$$

$$
\leq 0
$$

$\forall s \in I$ and $\Theta(s_0) = 0$. This leads to

$$
\sigma_1(s)\psi(s, \sigma_1(s), \sigma(\sigma_1(s))) \leq \rho_1(s) - \psi(s, \rho_1(s), \rho(\rho_1(s)))
$$

$\forall s \in I$. By (a1), we attain to

$$
\sigma_1(s) \leq \rho_1(s), \ \forall s \in I.
$$

Next, for $j \in \mathbb{N}$, yields

$$
\sigma_{j+1} \leq \sigma_j \leq \rho_j
$$

and hence

$$
\sigma_j \leq \sigma_{j+1} \leq \rho_{j+1} \leq \rho_j.
$$

Setting

$$
\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_j(s) - \psi(s, \sigma_j(s), \sigma(\sigma_j(s)))) - (\sigma_{j+1}(s) - \psi(s, \sigma_{j+1}(s), \sigma(\sigma_{j+1}(s))))
$$

Then the humdrum of $\mathbb{N}_1$ and $\mathbb{N}_2$, we receive

$$
D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] = D^\alpha[(\sigma_j(s) - \psi(s, \sigma_j(s), \sigma(\sigma_j(s)))) - D^\alpha[(\sigma_{j+1}(s) - \psi(s, \sigma_{j+1}(s), \sigma(\sigma_{j+1}(s)))]
$$

$$
\leq \mathbb{N}_1(s, \sigma_{j+1}(s), \sigma(\sigma_{j+1}(s))) + \mathbb{N}_2(s, \rho_{j-1}(s), \rho(\rho_{j-1}(s))) - \mathbb{N}_1(s, \sigma_j(s), \sigma(\sigma_j(s))) - \mathbb{N}_2(s, \sigma_j(s), \sigma(\sigma_j(s)))
$$

$$
\leq 0
$$
∀s ∈ I and Θ(s₀) = 0. This implies that

\[ σ_j - ψ(s, σ_j(s), σ(σ_j(s))) \leq σ_{j+1}(s) - ψ(s, σ_{j+1}(s), σ(σ_{j+1}(s))) \]

for every s ∈ I. Since assumption (a1) achieved, we have σ_j(s) ≤ σ_{j+1}(s), ∀s ∈ I. Likewise it can be demonstrated which ρ_{j+1}(s) ≤ ρ_j(s) on I. The same way it is assumed that the inequality

\[ σ_{j-1} \leq σ_j \leq ρ_j \leq σ_{j-1} \]

achieves on I. We are going to demonstrate that

\[ σ_j \leq σ_{j+1} \leq ρ_{j+1} \leq ρ_j \]

on I. Set

\[ Θ(s) - ψ(s, Θ(s), Θ(Θ(s))) = (σ_{j+1}(s) - ψ(s, σ_{j+1}(s), σ(σ_{j+1}(s)))) - (ρ_{j+1}(s) - ψ(s, ρ_{j+1}, ρ(ρ_{j+1}))) \]

for s ∈ I. So by monotonicity of N₁ and N₂ we get

\[ D^{α}[Θ(s) - ψ(s, Θ(s), Θ(Θ(s)))] ≡ D^{α}[(σ_{j+1}(s) - ψ(s, σ_{j+1}(s), σ(σ_{j+1}(s)))) - (ρ_{j+1}(s) - ψ(s, ρ_{j+1}, ρ(ρ_{j+1})))̃] \]

\[ ≤ N₁(s, σ_{j+1}(s), σ(σ_{j+1}(s))) + N₂(s, ρ_{j+1}(s), ρ(ρ_{j+1}))) - N₁(s, ρ_{j+1}, ρ(ρ_{j+1})) - N₂(s, σ_{j+1}(s), σ(σ_{j+1}(s))) \]

\[ ≤ 0 \]

for the whole s ∈ I and Θ(s₀) = 0. This means that

\[ σ_{j+1}(s) - ψ(s, σ_{j+1}(s), σ(σ_{j+1}(s))) \leq ρ_{j+1} - ψ(s, ρ_{j+1}, ρ(ρ_{j+1})) \]

for every s ∈ I. Since assumption (a1) achieved, we have σ_{j+1}(s) ≤ ρ_{j+1}(s), ∀s ∈ I.

Presently it is readily shown that the sequence {σ} and {ρ} are bounded uniformly and equi-continuous sequences and have therefore converge uniformly on I. As are monotonic sequences, {σₜ} and {ρₜ} converge uniformly monotonic σ and ρ on I respectively. Course, the pair (σ, ρ) is a mixed solution of these equations on I. Lastly, we establish which (σ, ρ) is a mixed solution of minimum and maximum for the equations on I. Let v whatever solution of the equations on I as σ₀(s) ≤ v(s) ≤ ρ(s) on I. Assume that for j ∈ N, σ_j(s) ≤ v(s) ≤ ρ_j(s), s ∈ I. We will demonstrate which σ_{j+1}(s) ≤ v(s) ≤ ρ_{j+1}(s), s ∈ I. adjustment

\[ Θ(s) - ψ(s, Θ(s), Θ(Θ(s))) = (σ_{j+1}(s) - ψ(s, σ_{j+1}(s), σ(σ_{j+1}(s)))) - (v(s) - ψ(s, v(s), v(v(s)))) \]

for every s ∈ I. After, for the monotony of N₁ and N₂ we get

\[ D^{α}[Θ(s) - ψ(s, Θ(s), Θ(Θ(s)))] ≡ D^{α}[(σ_{j+1}(s) - ψ(s, σ_{j+1}(s), σ(σ_{j+1}(s)))) - (v(s) - ψ(s, v(s), v(v(s)))] \]

\[ ≤ N₁(s, σ_{j+1}(s), σ(σ_{j+1}(s))) + N₂(s, ρ_{j+1}(s), ρ(ρ_{j+1}))) - N₁(s, v(s), v(v(s))) - N₂(s, v(s), v(v(s))) \]

\[ ≤ 0 \]

for the whole s ∈ I and Θ(s₀) = 0. This yields

\[ σ_{j+1}(s) - ψ(s, σ_{j+1}(s), σ(σ_{j+1}(s))) \leq v(s) - ψ(s, v(s), v(v(s))) \]

for every s ∈ I. Since assumption (a1) is valid, we get σ_{j+1}(s) ≤ v(s), ∀s ∈ I. Likewise it can be demonstrated which v(s) ≤ ρ_{j+1}(s) on I. In principle, the method of induction, σₜ ≤ v ≤ ρₜ for every s ∈ I. By taking t → ∞ limit, we get σ ≤ v ≤ ρ on I. So (σ, ρ) they are mixed type (a) extreme solutions for the equations on I, i.e.,

\[
\begin{cases}
D^{α}[σ(s) - ψ(s, σ(s), σ(σ(s)))] \leq N₁(s, σ(s), σ(σ(s))) + N₁(s, ρ(s), ρ(ρ(s)))) & , s ∈ I, \\
σ(s₀) = v₀
\end{cases}
\]

(24)
and
\[
\begin{cases}
D^\alpha [\rho(s) - \psi(s, \rho(s), \rho(\rho(s)))] \geq R_1(s, \rho(s), \rho(\rho(s))) + R_1(s, \sigma(s), \sigma(\sigma(s))),
\end{cases} s \in I,
\]
\[\rho(s_0) = v_0\]
the proof is completed. \(\square\)

**Corollary 3.1** Suppose the hypothesis of Theorem 3.4 are fulfilled. Assume that for \(i_1 \geq i_2, i_1, i_2 \in \mathcal{O}\), then
\[
R_1(s, i_1(s), i(i_1(s))) - R_1(s, i_2(s), i(i_2(s))) \leq N_1[i_1(s) - \psi(s, i_1(s), i(i_1(s))) - (i_2(s) - \psi(s, i_2(s), i(i_2(s))))], N_1 > 0,
\]
and
\[
R_2(s, i_1(s), i(i_1(s))) - R_2(s, i_2(s), i(i_2(s))) \leq N_2[i_1(s) - \psi(s, i_1(s), i(i_1(s))) - (i_2(s) - \psi(s, i_2(s), i(i_2(s))))], N_2 > 0,
\]
thus \(\sigma(s) = v(s) = \rho(s)\) on \(I\).

**Proof.** For \(\sigma \leq \rho\) on \(I\), it suffices to demonstrate that \(\rho \leq \sigma\) on \(I\). Introduce a function \(\Theta \in C(I, \mathbb{R})\)
\[
\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\rho(s) - \psi(s, \rho(s), \rho(\rho(s)))) - (\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s)))).
\]
Next, \(\Theta(s_0) = 0\) and
\[
D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] = D^\alpha[(\rho(s) - \psi(s, \rho(s), \rho(\rho(s)))) - (\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))))]
\]
\[
= R_1(s, \rho(s), \rho(\rho(s))) - R_1(s, \sigma(s), \sigma(\sigma(s))) + R_2(s, \sigma(s), \sigma(\sigma(s))) - R_2(s, \rho(s), \rho(\rho(s)))
\]
\[
\leq N_1[(\rho(s) - \psi(s, \rho(s), \rho(\rho(s))) - (\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s))))]
\]
\[
+ N_2[(\sigma(s) - \sigma(s, \sigma(s), \sigma(\sigma(s)))) - (\rho(s) - \psi(s, \rho(s), \rho(\rho(s))))]
\]
\[
= (N_1 + N_2)[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))].
\]
This demonstrates that \(\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) \leq 0\) on \(I\), demonstrating that \(\rho \leq \sigma\) on \(I\). Therefore \(\sigma = \rho = v\) on \(I\). Therefore the proof is completed. \(\square\)

**Theorem 3.5** Let us suppose that the assumption \((a1) - (a2)\) and \((b2) - (b4)\) achieved. Therefore, for any solution \(v(s)\) of (11) with \(\sigma_0 \leq v \leq \rho_0\), and we are an iteration \(\sigma_t, \rho_t\) satisfactory for \(s \in I\),
\[
\begin{cases}
\sigma_0 \leq \sigma_2 \leq \ldots \leq \sigma_t \leq v \leq \sigma_{2t+1} \leq \ldots \leq \sigma_3 \leq \sigma_1, \\
\rho_1 \leq \rho_3 \leq \ldots \leq \rho_{2t+1} \leq v \leq \rho_{2t} \leq \ldots \leq \rho_2 \leq \rho_0,
\end{cases}
\]
as long as \(\sigma_0 \leq \sigma_2\) and \(\rho_2 \leq \rho_0\) on \(I\), in which iterating is given by
\[
D^\alpha[\sigma_{2t+1}(s) - \psi(s, \sigma_{2t+1}(s), \sigma(\sigma_{2t+1}(s))) = R_1(s, \sigma_t(s), \rho(\rho_t(s))) + R_2(s, \sigma_t(s), \sigma(\sigma_t(s))), \quad s \in I,
\]
\[\sigma_{2t+1}(s_0) = v_0,\]
and
\[
D^\alpha[\rho_{2t+1}(s) - \psi(s, \rho_{2t+1}(s), \rho(\rho_{2t+1}(s))) = R_1(s, \rho_t(s), \sigma(\sigma_t(s))) + R_2(s, \rho_t(s), \rho(\rho_t(s))), \quad s \in I,
\]
\[\rho_{2t+1}(s_0) = v_0,\]
of \(t \in N\). Furthermore, the monotonous sequences \(\{\sigma_{2t}\}, \{\sigma_{2t+1}\}, \{\rho_{2t}\}, \{\rho_{2t+1}\}\) converge uniformly to \(\sigma, \rho, \sigma_0, \rho_0\), respectively, and fulfilling this assumptions:
\[
\begin{align*}
(1) \quad & D^\alpha[\sigma(s) - \psi(s, \sigma(s), \sigma(\sigma(s)))) \geq R_1(s, \rho(s), \rho(\rho(s))) + R_2(s, \sigma(s), \sigma(\sigma(s)))) \\
(2) \quad & D^\alpha[\rho(s) - \psi(s, \rho(s), \rho(\rho(s)))] \geq R_1(s, \sigma(s), \sigma(\sigma(s))) + R_2(s, \rho(s), \rho(\rho(s)))
\end{align*}
\]
Proof. By the assumptions of the theorem, we suppose that \( \sigma_0 \leq \sigma_2 \) and \( \rho_2 \leq \rho_0 \), on \( I \). We demonstrate that

\[
\begin{align*}
\sigma_0 &\leq \sigma_2 \leq v \leq \sigma_3 \leq \sigma_1, \\
\rho_1 &\leq \rho_3 \leq v \leq \rho_2 \leq \rho_0
\end{align*}
\] (29)

on \( I \). Set

\[
\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (v(s) - \psi(s, v(s), v(v(s)))) - (\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))))
\]

utilization that \( \sigma_0 \leq v \leq \rho_0 \) on \( I \), as \( v \) is any solution of (11) and the monotonous nature of functions \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \), this yields

\[
D^\alpha[\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)))] = D^\alpha[(v(s) - \psi(s, v(s), v(v(s)))) - (\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s))))] - D^\alpha[\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))]
\]

\[
= \mathcal{N}_1(s, v(s), v(v(s)))) + \mathcal{N}_2(s, v(s), v(v(s)))) - \mathcal{N}_1(s, \rho_0(s), \rho(\rho_0(s))) - \mathcal{N}_2(s, \sigma_0(s), \sigma(\sigma_0(s)))
\]

\[
\leq 0
\]

for every \( s \in I \) and \( \Theta(s_0) = 0 \). Thus, we reached the conclusion

\[
v(s) - \psi(s, v(s), v(v(s))) \leq \sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))
\]

or

\[
v(s) \leq \sigma_1(s)
\]

for every \( s \in I \). In the same way, we can show that \( \sigma_3 \leq \sigma_1, \rho_1 \leq v \) and \( \sigma_2 \leq v \), taking into account differences

\[
\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_3(s) - \psi(s, \sigma_3(s), \sigma(\sigma_3(s)))) - (\sigma_1(s) - \psi(s, \sigma_1(s), \sigma(\sigma_1(s)))),
\]

\[
\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\rho_1(s) - \psi(s, \rho_1(s), \rho(\rho_1(s)))) - (v(s) - \psi(s, v(s), v(v(s))))
\]

and

\[
\Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s))) = (\sigma_2(s) - \psi(s, \sigma_2(s), \sigma(\sigma_2(s)))) - (v(s) - \psi(s, v(s), v(v(s))))
\]

respectively. At each of these cases, we get \( \Theta(s) - \psi(s, \Theta(s), \Theta(\Theta(s)) \leq 0 \), for all \( s \in I \) and representation (29) is established. This completed prove.

Competing Interests The authors declare that they have no competing interests.

Authors' contributions All the authors jointly worked on deriving the results and approved the final manuscript.
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