We study the large-scale inhomogeneity of the Universe based on the averaging procedure of Buchert and Ehlers. The generalized Dyer-Roeder equation for the angular diameter distance of the inhomogeneous Universe is derived and solved for different cosmological models. We make a comparison of certain cosmic observables, such as the Hubble function, angular diameter distance, cosmological correction factor of homogeneous and inhomogeneous cosmological models, which are crucial ingredients in galaxy number counts and gravitational lenses.

It seems that present cosmological data consisting of supernova searches, structure formation, gravitational lenses, etc. suggest a rather unexpected equation of state for the simple homogeneous Friedmann-Lemaître (FL) Universe with $\Omega_m \simeq 0.3$, $\Omega_{\Lambda} \simeq 0.7$. However, one should be cautious even when performing analyses of the cosmological data (for example, for supernova data see the reanalysis in [1]). Further concern is the low statistical confidence of all extracted observables. Nevertheless, this phenomenological result initiated a lot of speculative work on the origin of the positive cosmological constant.

Contrary to current investigations, there is only one derivation of the cosmological constant without any fine tuning or any obscure assumptions and this derivation is based on the Einstein-Cartan gravity [2]. When the Universe is frozen at zero temperature $T_\gamma = 0$ (spacelike infinity) and if all nonrelativistic fermionic matter is spinning matter, then owing to the same
coupling between spacetime curvature and mass density on the one side and spacetime torsion and spin density of matter on the other side, the following relations emerge: \( \Omega_m = 2, \Omega_{\Lambda} = -1 \) \[2\]. It is important to strengthen that at \( T_\gamma = 0 \) all spin densities of nonbaryonic and baryonic species act coherently to the total spin. The role of spin densities in the derivation of the primordial density fluctuation is described in Ref. \[3\]. The existence of vorticity and acceleration \[2, 3\] or nonvanishing shear \[4\] should be of fundamental importance for resolving certain cosmological problems, but the cosmological distance measures are not very much affected by relatively small deviations from the FL model \[5\]. To conclude, let us notice that it is possible to construct a nonsingular and nonanomalous gauge theory with heavy and light neutrinos as cold and hot dark matter fermionic particles respectively, but without Higgs scalars and with no asymptotic freedom in QCD \[6\].

In this paper we want to reconcile current measurements with large-scale inhomogeneous cosmic models. We assume that the Universe is inhomogeneous on large scales, thus clumpiness should not be characteristic only of small-scale structures, and that the Earth is placed in the region away from the centre of homogeneity.

Starting with the historical work of Lemaître, Tolman and Bondi (LTB) on inhomogeneous models, one can account enormous activity in this field \[7\]. However, a recent attempt \[8\] to exploit LTB models, following perturbative calculations in Ref. \[9\], was not successful at large redshifts.

The cellular structure of the Universe with a power-law distribution of matter of Ruffini et al \[10\], represents another approach to the problem of inhomogeneity.

The work of Zalaletdinov \[11\] is the first complete and consistent treatment of the averaged Einstein equations in an arbitrary Riemannian spacetime. However, even if the vorticity and acceleration do not vanish, they are small deviations (which we neglect in this paper) from the FL geometry, thus we choose the following line element and spatially averaged scalar quantites of Buchert and Ehlers \[12\]:

\[
ds^2 = dt^2 - g_{ij}dX^i dX^j,
\]

\[
\langle \Psi(t, X^i) \rangle_D \equiv \frac{1}{V_D} \int_D d^3X J \Psi(t, X^i),
\]
Introducing natural definitions

\[ a_D(t) \equiv (\frac{V_D(t)}{V_{D_0}})^{\frac{1}{3}}, \]

\[ \langle \theta \rangle_D = \frac{V_D}{V_D} = \frac{3}{a_D}, \quad \psi(t) \equiv \frac{d}{dt}\psi(t), \]

\[ M_D = \int d^3 X J\rho(t, X^i), \]

\[ \langle \rho \rangle_D = \frac{M_D}{V_{D_0}a_D^3}. \]

one can easily derive Raychaudhuri and constraint equations for averaged scalars from Einstein and conservation equations \[12, 13\]

\[ 3\frac{\ddot{a}_D}{a_D} + 4\pi G_N \frac{M_D}{V_{D_0}a_D^3} - \Lambda = Q_D, \quad (1) \]

\[ 3(\frac{\dot{a}_D}{a_D})^2 - 8\pi G_N \frac{M_D}{V_{D_0}a_D^3} + \frac{1}{2} \langle R \rangle_D - \Lambda = -\frac{1}{2} Q_D, \quad (2) \]

\[ Q_D \equiv \frac{2}{3} \langle (\theta - \langle \theta \rangle_D)^2 \rangle_D - 2 \langle \sigma^2 \rangle_D, \]

\[ \mathcal{R} = \mathcal{R}_i^i, \quad \mathcal{R}_{ij} \equiv spatial \ part \ of \ the \ Ricci \ tensor. \]

In addition, we need a condition of integrability of these equations \[12\]:

\[ (a_D^6 Q_D) + a_D^4 (a_D^2 \langle R \rangle_D) = 0. \quad (3) \]

Now we turn to the geometrical optics in order to derive the wanted observables of the inhomogeneous cosmic models.
Starting from the definition of the angular diameter distance and the focusing equation [14]:

\[ D \equiv \left( \frac{dA_S}{d\Omega} \right)^{\frac{1}{2}}, \]

\[ \frac{d^2}{dw^2} A_S^{\frac{1}{2}} + \frac{4\pi G_N}{H_0^2} (1 + z)^2 \rho A_S^{\frac{1}{2}} = 0, \tag{4} \]

\[ w \equiv H_0 \omega_0 \nu, \quad v = a f f i n e \ p a r a m e t e r. \]

We take into account that the shear of the congruence of photons vanishes for a spherically symmetric source [15]. Then, performing the Buchert-Ehlers spatial averaging and neglecting the spacetime shear in the averaged equation for the change of redshift with respect to the affine parameter:

\[ \frac{dz}{dw} = (1 + z)^{\frac{3}{2}} \frac{\dot{a}_D(z)}{a_D(0)}. \tag{5} \]

we arrive at the following equation for the averaged angular diameter distance:

\[ \frac{d^2}{dz^2} \langle D_D \rangle (dz)^2 + \frac{d\langle D_D \rangle}{dz} \frac{d^2 z}{dw^2} + \frac{4\pi G_N}{H_0^2} (1 + z)^2 \langle \rho D_D \rangle = 0. \]

Assuming the smooth space dependence of mass density and angular diameter distance, one can factorize the last term of the preceding equation as

\[ \langle \rho D_D \rangle \simeq \rho(t, \bar{X}^i) \langle D_D \rangle \simeq \frac{M_D}{V_D} \langle D_D \rangle, \quad \bar{X}^i \varepsilon D, \]

and we are left with the second-order generalized Dyer-Roeder equation with a suitable density normalization for the cosmological models:

\[ \phi(z) \frac{d^2 D(z)}{dz^2} + \chi(z) \frac{dD(z)}{dz} + \eta(z) D(z) = 0, \tag{6} \]
\[ \tilde{D} \equiv \langle D \rangle, \ \phi(z) = \left( \frac{dz}{dw} \right)^2, \ \chi(z) = \frac{d^2 z}{dw^2}, \]

\[
\frac{d^2 z}{dw^2} = \left( \frac{dz}{dw} \right)^2 \left( \frac{3}{1 + z} + \frac{1}{\dot{a}_D(z)} \frac{d \dot{a}_D(z)}{dz} \right),
\]

\[
\frac{d \dot{a}_D}{dz} = -\frac{1}{H(z)(1 + z)} \ddot{a}_D,
\]

\[ H(z) \equiv \frac{\dot{a}_D(z)}{a_D(z)}, \ a_D(z) = \frac{1}{1 + z}, \]

\[ \eta(z) = \frac{3}{2}(1 + z)^5 \Omega_m, \ \Lambda = 3H_0^2\Omega_\Lambda. \]

Under adequate initial conditions \[14\] (Adams-Bashforth integration method used)

for integration from \( z_1 \) to \( z_2 \), \( z_2 > z_1 \),

\[ \tilde{D}(z_1, z_1) = 0, \quad \frac{\tilde{D}(z_1, z)}{dz}(z = z_1) = \frac{1}{H(z_1)} \frac{1}{1 + z_1}, \]

we can calculate the averaged angular diameter distance for an arbitrary cosmological model \( \Omega_m + \Omega_\Lambda = 1 \). Needless to say, for the vanishing backreaction and curvature terms one recovers a homogeneous solution:

\[ D(z) = \frac{1}{H_0(1 + z)} \int_0^z \frac{d\zeta}{[\Omega_m(1 + \zeta)^3 + \Omega_\Lambda]^{\frac{1}{2}}}. \]

Numerical evaluations and comparisons are performed in the following manner: (1) choose some functional form for the backreaction \( Q(z) \), (2) evaluate curvature term \( R(z) \) from the integrability condition, (3) choose two inhomogeneous models with \( \Omega_m = 2, \Omega_\Lambda = -1 \) [Einstein-Cartan (EC) model] and \( \Omega_m = 1 , \Omega_\Lambda = 0 \) [Einstein-deSitter (EdS) model] and fit parameters of the backreaction and curvature to reach the input value for the Hubble constant \( H_0 = \dot{a}(z = 0)/a(z = 0) \) and to get the best fit of the \( \tilde{D}(z) \) of the inhomogeneous models to the \( D(z) \) of the homogeneous model with \( \Omega_m = 0.3 , \Omega_\Lambda = 0.7 \) for \( z \leq 0.7 \).
One can easily check that for any $0 > \alpha > -3$ (we omit index $D$)

$$Q(z) = Q_0 a^\alpha(z),$$

$$\langle R \rangle_D \equiv R(t) = -a^{-2}(t) \int_{t_0}^t d\tau a^{-4}(\tau) \frac{d}{d\tau}[a^6(\tau)Q(\tau)],$$

$$R(z) = -\frac{\alpha + 6}{\alpha + 2}[Q(z) - (\frac{a(z_0)}{a(z)})^2Q(z_0)], \quad \alpha \neq -2,$$

$$R(z) = 4Q(z) \ln \frac{1+z}{1+z_0}, \quad \alpha = -2,$$

the curvature term power-law coefficient is within the same interval as that for the backreaction. Thus, this simple functional form guarantees the homogeneity at very large scales $|Q(z)|, |R(z)| \ll H_0^2 \Omega_m a_D(z)^{-3}, z >> 1$ (it could also be worth including the Gaussian damping, if necessary).

For definite numerical comparisons of the three cosmological models we use this Ansatz:

$$Q(z) = Q_0[1 + c_0 a^{-1}(z)],$$

$$R(z) = Q_0(1 + z)^2[-3(\frac{1}{(1+z)^2} - \frac{1}{(1+z_0)^2}) + 5c_0(\frac{1}{1+z} - \frac{1}{1+z_0})].$$

In order to match the input Hubble parameter at present time $H_0$, $z_0$ has to fulfil the following equation:

$$Q(0) + R(0) = 0$$

$$\Rightarrow 2(-1 + 3c_0)(1 + z_0)^2 - 5c_0(1 + z_0) + 3 = 0.$$

Searching for the best fit values of $c_0$ and $Q_0$, we find

$$H_0 \equiv h_0 \ u, \ u \equiv 100\text{km} s^{-1} \text{Mpc}^{-1},$$

(EC) : $h_0 = 0.6, \ Q_0 = 3.789 \ u^2, \ z_0 = \sqrt{6}/2 - 1, \ c_0 = 0,$

(EdS) : $h_0 = 0.6, \ Q_0 = 1.2632 \ u^2, \ z_0 = \sqrt{5}/2 - 1, \ c_0 = 0,$

(Hom) : $h_0 = 0.6, \ \Omega_m = 0.3, \ \Omega_\Lambda = 0.7, \ Q_0 = 0.$
For $z \leq 0.5$, the angular diameter distances differ less than 5%. Figs. 1 and 2 show the Hubble flow and the angular diameter distance for the three models, while Table 1 presents the cosmological correction terms for the difference in light travel time between gravitational lens images [16]:

$$H_0 \triangle t = T f(\theta_{1,Obs}, \theta_{2,Obs}, z_d, z_s),$$

$$T = H_0 \frac{\bar{D}^2(0, z_d)(1 + z_d) z_s - z_d}{z_s z_d},$$

$$\bar{D} = \frac{\bar{D}(z_d, z_s) D(0, z_d)}{D(0, z_s)}.$$

|      | $z_d = 0.5, \ z_s = 1$ | $z_d = 0.5, \ z_s = 1.5$ | $z_d = 1, \ z_s = 1.5$ | $z_d = 1, \ z_s = 2$ |
|------|------------------------|------------------------|------------------------|------------------------|
| T(EC)| 0.7550                 | 0.6686                 | 0.4977                 | 0.4226                 |
| T(EdS)| 0.9559                | 0.9350                 | 0.8775                 | 0.8471                 |
| T(Hom)| 1.8543                | 2.1098                 | 4.7948                 | 8.9155                 |

The galaxy number count depends essentially on the cosmological model through the comoving volume:

$$\frac{dV(z)}{dz} \propto \bar{D}(z)^2.$$

We conclude with two observations: (1) For the large-scale inhomogeneity of the Universe to be established one needs clear indications that none of homogeneous models can simultaneously fit and explain all data in cosmography, structure evolution, gravitational lenses, etc. for all redshifts. At present it is difficult to make any conclusion what is the source of some recently found disparities in gravitational lenses [17] or galaxy evolution [18]. (2) If the inhomogeneity is established, then one could intend to model a backreaction function for the global fit of data, and ultimately attempt to derive it from the evolution of multicomponent imperfect fluid with photons, baryonic and nonbaryonic matter, with a necessary knowledge of all relevant masses and couplings.

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Figure 1: Solid, dashed and dot-dashed lines denote angular diameter distance $D(z)$ in $[u^{-1}]$ for (EC), (EdS) and (Hom) models, respectively.
Figure 2: Solid, dashed and dot-dashed lines denote Hubble function $H(z)$ in $[u]$ for (EC), (EdS) and (Hom) models, respectively.
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