An Interesting Application of
the Intermediate Value Theorem:
A Simple Proof of Sharkovsky’s Theorem and
the Towers of Periodic Points

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Abstract

This note is intended primarily for college calculus students right after the introduction of the Intermediate Value Theorem, to show them how the Intermediate Value Theorem is used repeatedly and straightforwardly to prove the celebrated Sharkovsky’s theorem on the periods of coexistent periodic orbits of continuous maps on an interval. Furthermore, if the maps have a periodic orbit $P$ of odd period $\geq 3$, then we find more periodic points, in the appendix, which constitute what we call the towers of periodic points associated with $P$. These towers of periodic points have infinitely many layers. In this note, No knowledge of Dynamical Systems Theory is required.

In this revision, we add the result (Proposition 1 on page 9) that the periodic orbits of continuous unimodal maps on the interval $[0, 1]$ of least periods $\geq 2$ are nested in the sense that if $P$ and $Q$ are periodic orbits of a continuous unimodal map on $[0, 1]$ of least periods $\geq 2$ and if $\max P < \max Q$, then $[\min P, \max P] \subset [\min Q, \max Q]$.

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Throughout this note, $I$ is a compact interval, and $f : I \to I$ is a continuous map. For each integer $n \geq 1$, let $f^n$ be defined by: $f^1 = f$ and $f^n = f \circ f^{n-1}$ when $n \geq 2$. For $x_0$ in $I$, we call the set $\{x_0, f(x_0), f^2(x_0), \cdots\}$ the orbit of $x_0$ with respect to $f$ and call $x_0$ a periodic point of $f$ with least period $m$ or a period-$m$ point of $f$ if $f^m(x_0) = x_0$ and $f^i(x_0) \neq x_0$ when $0 < i < m$. If $f(x_0) = x_0$, then we call $x_0$ a fixed point of $f$.

In this note, we demonstrate how the Intermediate Value Theorem is applied repeatedly and straightforwardly to prove (1) of the following celebrated Sharkovsky’s theorem on the periods of coexistent periodic orbits of continuous maps on the interval $I$:

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**Theorem (Sharkovsky)** The Sharkovsky’s ordering of the natural numbers is as follows:

\[
3 < 5 < 7 < 9 < \cdots < 2 \cdot 3 < 2 \cdot 5 < 2 \cdot 7 < 2 \cdot 9 < \cdots < 2^2 \cdot 3 < 2^2 \cdot 5 < 2^2 \cdot 7 < 2^2 \cdot 9 < \cdots \\
< 2^i \cdot 3 < 2^i \cdot 5 < 2^i \cdot 7 < 2^i \cdot 9 < \cdots < 2^{i+1} < 2^n < \cdots < 2^3 < 2^2 < 2 < 1.
\]

Then the following three statements hold:

1. If \( f \) has a period-\(m\) point and if \( m < n \), then \( f \) also has a period-\(n\) point.
2. For each positive integer \( n \) there exists a continuous map from \( I \) into itself that has a period-\(n\) point but has no period-\(m\) point for any \( m \) with \( m < n \).
3. There exists a continuous map from \( I \) into itself that has a period-\( (2^i) \) point for \( i = 0, 1, 2, \ldots \) but has no periodic point of any other period.

To make this note self-contained, we include the following two well-known results.

**Lemma 1.** If \( f^n(x_0) = x_0 \), then the least period of \( x_0 \) with respect to \( f \) divides \( n \).

**Proof.** Let \( m \) denote the least period of \( x_0 \) with respect to \( f \) and write \( n = km + r \) with \( 0 \leq r < m \). Then \( x_0 = f^n(x_0) = f^{km+r}(x_0) = f^r(f^{km}(x_0)) = f^r(x_0) \). Since \( m \) is the smallest positive integer such that \( f^m(x_0) = x_0 \), we must have \( r = 0 \). Therefore, \( m \) divides \( n \).

**Lemma 2.** Let \( i, j, k, m, n \) and \( s \) be positive integers. Then the following statements hold:

1. If \( x_0 \) is a periodic point of \( f \) with least period \( m \), then it is a periodic point of \( f^n \) with least period \( m/(m,n) \), where \( (m,n) \) is the greatest common divisor of \( m \) and \( n \).
2. If \( x_0 \) is a periodic point of \( f^n \) with least period \( k \), then it is a periodic point of \( f \) with least period \( kn/s \), where \( s \) divides \( n \) and is relatively prime to \( k \). In particular, if \( f^{2^i-1} \) has a period-\( (2^i \cdot j) \) point for some \( i \geq 2 \) and \( j \geq 1 \), then \( f \) has a period-\( (2^i \cdot j) \) point.

**Proof.** (i) Let \( x_0 \) be a period-\( t \) point of \( f^n \). Then \( m \) divides \( nt \) since \( x_0 = (f^n)^t(x_0) = f^{nt}(x_0) \). So, \( m/(m,n) \) divides \( n/(m,n) \cdot t \). Since \( (m,n) \) and \( n/(m,n) \) are coprime, \( m/(m,n) \) divides \( t \). Furthermore, 
\[(f^n)^{(m/(m,n))}(x_0) = (f^m)^{(n/(m,n))}(x_0) = x_0.\] Thus, \( t \) divides \( m/(m,n) \). This shows that \( t = m/(m,n) \).

(ii) Since \( x_0 = (f^n)^k(x_0) = f^{kn}(x_0) \), the least period of \( x_0 \) under \( f \) is \( kn/s \) for some positive integer \( s \). By (i), \( (kn/s)/(kn/s,n) = k \). So, \( n/s = ((n/s)/k,n) \) (which is an integer) \( = ((n/s)/k,s) = (n/s)(k,s) \). This shows that \( s \) divides \( n \) and \( (s,k) = 1 \).

The version of the Intermediate Value Theorem we are going to use is the following:

**Intermediate Value Theorem** Let \( g : [a,b] \to \mathbb{R} \) be a continuous real-valued map such that \( g(a)g(b) < 0 \). Then there exists a point \( a < c < b \) such that \( g(c) = 0 \).

Following [13], we first prove the following three statements:
(a) if \( f \) has a period-\( m \) point with \( m \geq 2 \), then \( f \) has a period-2 point and a fixed point;
(b) if \( f \) has a period-\( m \) point with \( m \geq 3 \) and odd, then \( f \) has a period-(\( m + 2 \)) point; and
(c) if \( f \) has a period-\( m \) point with \( m \geq 3 \) and odd, then \( f \) has periodic points of all even periods.

Let \( P \) be a period-\( m \) orbit of \( f \) with \( m \geq 2 \) and let \( e = f^{m-1}(\min P) \).

We choose the points \( \min P \) and \( f^{m-1}(\min P) \) in the orbit \( P \) to start with.

They seem to be the right choices because they make the subsequent proofs of (a), (b) and (c) become straightforward. See also Appendix.

We now prove (a), (b) and (c)

Since \( \min P \) is a period-\( m \) point of \( f \) with \( m \geq 2 \), we have \( f(e) = \min P < e \). So, \( f(e) \neq e \).

If \( f(x) < e = f^{m-1}(\min P) \) for all \( \min P \leq x < e = f^{m-1}(\min P) \),

then, since \( \min P \leq f^i(\min P) \) for all \( i \geq 1 \), we have

\[
\text{(min } P \text{) } f^i(\min P) < e \text{ for all } i \geq 1,
\]

contradicting the fact that \( f^{m-1}(\min P) = e \). Since \( f(e) = \min P \), we have \( f([\min P, e]) \supset [\min P, e] \supset \{e\} \). Let \( v \) be any point in \([\min P, e]\) such that \( f(v) = e = f^{m-1}(\min P) \). Then

\[
f(v) - v = e - v > 0 \text{ and } f(e) - e = \min P - e < 0
\]

which imply that \( f \) has a fixed point \( z \) in \((v,e)\).

Now suppose \( m \geq 3 \). Then \( \min P < v \). Since

\[
f^2(\min P) - \min P > 0 \text{ and } f^2(v) - v = \min P - v < 0,
\]

the point \( y = \max \{ \min P \leq x \leq v : f^2(x) = x \} \) exists.

If \( f(y) = y \), then since

\[
f(y) - z = y - z < 0 \text{ and } f(v) - z = f^{m-1}(\min P) - z > 0,
\]

there is a point \( \tilde{x} \) in \((y,v)\) such that \( f(\tilde{x}) = z \). Consequently, since

\[
f^2(\tilde{x}) - \tilde{x} = z - \tilde{x} > 0 \text{ and } f^2(v) - v = \min P - v < 0,
\]

there is a point \( \tilde{y} \) in \((\tilde{x},v) \subset (y,v)\) such that \( f^2(\tilde{y}) = \tilde{y} \) which contradicts the maximality of \( y \) in \([\min P, v]\). Therefore, \( f(y) \neq y \) and \( y \) is a period-2 point of \( f \). This confirms (a).
For the proofs of (b) and (c), assume that \( m \geq 3 \) is odd. Let 
\[
z_0 = \min \{ v \leq x \leq z : f^2(x) = x \}.
\]
Then since \( f^2(v) = \min P < v \), we have
\[
f^2(x) < x < z_0 \quad \text{for all} \quad v \leq x < z_0.
\]
(\(*\))

If \( f^2(x) < z_0 \) for all \( \min P \leq x < v \), then \( f^2(x) < z_0 \) for all \( \min P \leq x < z_0 \). Consequently,
\[
(\min P \leq) \ f^{2i}(\min P) < z_0 \quad \text{for all} \quad i \geq 1,
\]
contradicting the fact that \( (f^2)^{(m-1)/2}(\min P) = e > z \geq z_0 \). So, \( f^2(\hat{x}) \geq z_0 \) for some \( \min P \leq \hat{x} < v \). Since \( f^2(\min P) \) and \( z_0 \) are periodic points of \( f \) with different periods, \( f^2(\min P) \neq z_0 \) and, since
\[
f^2(\hat{x}) - z_0 \geq 0 \quad \text{and} \quad f^2(v) - z_0 = \min P - z_0 < 0,
\]
the point
\[
d = \max \{ \min P \leq x \leq v : f^2(x) = z_0 \} \quad (> \min P \quad \text{since} \quad f^2(\min P) \neq z_0)
\]
exists and, since \( f^2(v) = \min P < x \), we have \( f^2(x) < z_0 \) for all \( d < x < v \). This, combined with the above (\(*\)), implies that
\[
f^2(x) < z_0 \quad \text{for all} \quad d < x < z_0.
\]
(\(**\))

If \( f(\bar{x}) = z \) for some \( d < x < z_0 \), then \( f^2(\bar{x}) = \bar{x} \geq z_0 \) which contradicts the fact that \( f^2(x) < z_0 \) for all \( d < x < z_0 \). Since \( f(v) = e = f^{m-1}(\min P) > z \), we have
\[
f(x) > z \quad (\geq z_0 > f^2(x)) \quad \text{for all} \quad d < x < z_0,
\]
(\(***\))

Consequently, \( f(z_0) \geq z \geq z_0 \) and \( f(d) \geq z \geq z_0 \). Now since
\[
f^2(d) - d = z_0 - d > 0 \quad \text{and} \quad f^2(v) - d = \min P - d < 0,
\]
there is a point \( u \) in \( (d, v) \) such that \( f^2(u) = d \). Let
\[
u_1 = \min \{ d \leq x \leq v : f^2(x) = d \}.
\]
In the following,

for the proof of (b), we consider the interval \([u_1, v]\) (instead of taking the period-2 point \( y \) obtained in the proof of (a) and considering the interval \([y, v]\) as in \([7]\)) and show that, for each \( n \geq 1 \), the point
\[
p_{m+2n} = \min \{ u_1 \leq x \leq v : f^{m+2n}(x) = x \}
exists and is a period- \((m + 2n)\) point of \(f\) while

for the proof of (c), we consider the interval \([d, u_1]\) and show that, for each \(n \geq 1\), the point

\[
c_{2n} = \min \{d \leq x \leq u_1 : f^{2n}(x) = x\}
\]

exists and is a period- \((2n)\) point of \(f\).

Note that \(f^3(u_1) = f(f^2(u_1)) = f(d) \geq z \geq z_0\) (by 
\((***)\)) and, for each \(j \geq 5\) and odd, \(f^j(u_1) = f^{j-4}(f^4(u_1)) = f^{j-4}(z_0) = f(z_0) \geq z \geq z_0\). Therefore,

\[
f^n(u_1) \geq z \geq z_0 \quad \text{for all odd } n \geq 3.
\]

Now since \(m \geq 3\) is odd, we have

\[
f^{m+2}(u_1) - u_1 = f(z_0) - u_1 \geq z_0 - u_1 > 0 \quad \text{and} \quad f^{m+2}(v) - v = f^m(f^2(v)) - v = \min P - v < 0.
\]

Therefore, the point

\[
p_{m+2} = \min \{u_1 \leq x \leq v : f^{m+2}(x) = x\} \quad \text{exists and} \quad f(p_{m+2}) > z \geq z_0 > p_{m+2}.
\]

Let \(k\) denote the least period of \(p_{m+2}\) with respect to \(f\). Then, \(k > 1\) and, by Lemma 1, \(k\) is odd. Now we want to show that \(k = m + 2\). Suppose (3 \(\leq\) \(k < m + 2\). Then since

\[
f^k(u_1) - v \geq z_0 - v > 0 \quad \text{and} \quad f^k(p_{m+2}) - v = p_{m+2} - v < 0,
\]

there is a point \(v_k\) in \((u_1, p_{m+2})\) such that \(f^k(v_k) = v\). Consequently, since

\[
f^{k+2}(u_1) - u_1 = f(z_0) - u_1 \geq z_0 - u_1 > 0 \quad \text{and} \quad f^{k+2}(v_k) - v_k = f^2(v) - v_k = \min P - v_k < 0,
\]

there is a point \(w_{k+2}\) in \((u_1, v_k)\) such that \(f^{k+2}(w_{k+2}) = w_{k+2}\). Similarly, since

\[
f^{k+2}(u_1) - v \geq z_0 - v > 0 \quad \text{and} \quad f^{k+2}(w_{k+2}) - v = w_{k+2} - v < 0,
\]

there is a point \(v_{k+2}\) in \((u_1, w_{k+2})\) such that \(f^{k+2}(v_{k+2}) = v\). Furthermore, since

\[
f^{k+4}(u_1) - u_1 \geq z_0 - u_1 \geq 0 \quad \text{and} \quad f^{k+4}(v_{k+2}) - v_{k+2} = f^2(v) - v_{k+2} = \min P - v_{k+2} < 0,
\]

there is a point \(w_{k+4}\) in \((u_2, v_{k+2})\) such that \(f^{k+4}(w_{k+4}) = w_{k+4}\). Inductively, there exist points

\[
u_1 < \cdots < w_{m+2} < w_m < w_{m-2} < \cdots < w_{k+4} < w_{k+2} < p_{m+2} < v
\]

such that \(f^{k+2i}(w_{k+2i}) = w_{k+2i}\) for all \(i \geq 1\). In particular, \(f^{m+2}(w_{m+2}) = w_{m+2}\) and \(u_1 < w_{m+2} < p_{m+2}\), contradicting the fact that \(p_{m+2}\) is the smallest point in \((u_1, v)\) which satisfies \(f^{m+2}(x) = x\). Therefore, \(k = m + 2\). This establishes (b).

Note that similar arguments as above show that the point

\[
p_{m+4} = \min \{u_1 \leq x \leq p_{m+2} : f^{m+4}(x) = x\}
\]


exists and is a period-$(m+4)$ point of $f$. Inductively, we obtain a sequence of points

$$u_1 < \cdots < p_{m+2i} < \cdots < p_{m+4} < p_{m+2} < v < z_0$$

such that each $p_{m+2i} = \min\{d \leq x \leq v : f^{m+2i}(x) = x\}, i \geq 1$, is a period-$(m+2i)$ point of $f$.

We now prove (c). Note that $f^{2i}(d) = z_0$ for all $i \geq 1$ and recall that we have the property

$$f^2(x) < z_0 \text{ for all } d < x < z_0.$$  

(\*)

Since

$$f^2(d) = z_0 \text{ and } u_1 = \min\{d \leq x \leq v : f^2(x) = d\} \text{ (and so, } f^2(u_1) = d),$$

we have $d < f^2(x) < z_0$ on $(d, u_1)$ and so, by (\*), $f^4(x) = (f^2)^2(x) < z_0$ on $(d, u_1)$.

Let $c_2 = \min\{d \leq x \leq u_1 : f^2(x) = x\} = \min\{d \leq x \leq v : f^2(x) = x\}$. Then $f^4([d, c_2]) = f^2(f^2([d, c_2])) \supset f^2([c_2, z_0]) \supset f^2([u_1, z_0]) \supset \{d\}$ (since $f^2(u_1) = d$).

Since

$$f^4(d) = z_0, \quad f^4(x) < z_0 \text{ on } (d, u_1) \text{ and } f^4([d, u_1]) \supset f^4([d, c_2]) \supset \{d\},$$

the point $u_2 = \min\{d \leq x \leq c_2 : (f^2)^2(x) = d\}$ (and so, $f^4(u_2) = d$) exists and we have $d < (f^2)^2(x) < z_0$ on $(d, u_2)$ and so, by (\*), $f^6(x) = (f^2)^3(x) < z_0$ on $(d, u_2)$.

Let $c_4 = \min\{d \leq x \leq u_2 : f^4(x) = x\} = \min\{d \leq x \leq v : f^4(x) = x\}$. Then $f^6([d, c_4]) = f^2(f^4([d, c_4])) \supset f^2([c_4, z_0]) \supset f^2([u_1, z_0]) \supset \{d\}$. Since

$$f^6(d) = z_0, \quad f^6(x) < z_0 \text{ on } (d, u_2) \text{ and } f^6([d, u_2]) \supset f^6([d, c_4]) \supset \{d\}$$

the point $u_3 = \min\{d \leq x \leq c_4 : (f^2)^3(x) = d\}$ exists and we have $d < (f^2)^3(x) < z_0$ on $(d, u_3)$ and so, by (\*), $f^8(x) = (f^2)^4(x) < z_0$ on $(d, u_3)$.

Let $c_6 = \min\{d \leq x \leq u_3 : f^6(x) = x\} = \min\{d \leq x \leq v : f^6(x) = x\}$. Proceeding in this manner indefinitely, we obtain points

$$d < \cdots < c_{2n} < u_n < \cdots < c_4 < u_2 < c_2 < u_1 < v < z_0$$

such that $d < (f^2)^n(x) < z_0$ on $(d, u_n)$ and $(f^2)^n(c_{2n}) = c_{2n}$ for all $n \geq 1$. Since $f(x) > z \geq z_0$ on $(d, z_0)$, we have

$$d < f^i(c_{2n}) < z_0 \leq z < f^j(c_{2n}) \text{ for all even } i \text{ and all odd } j \text{ in } [0, 2n].$$

Consequently, each $c_{2n}$ has even period $\leq 2n$ with respect to $f$. Since, for each $1 \leq k < n$, $c_{2k}$ is the smallest point in $[d, u_k] \supset [d, u_n]$ which satisfies $f^{2k}(x) = x$ and $c_{2n} < c_{2k}$, we obtain that $f^{2k}(c_{2n}) \neq c_{2n}$ for all $1 \leq k < n$. So, $c_{2n}$ is a period-$(2n)$ point of $f$. This proves (c).

Furthermore, since we have shown that, for each $1 \leq i \leq n$, $d < (f^2)^i(x) < z_0$ for all $x$ in $(d, u_i)$, we obtain that

$$d < f^{2i}(x) < z_0 \text{ for all } x \in (d, u_n) \text{ and all } 1 \leq i \leq n.$$  

(\dagger)
This fact will be used in the Appendix.

In the following, we show that, for each $n \geq 1$, the interval $[d, u_n]$ contains no periodic points of $f$ of even periods $< 2n$.

The arguments we used here will also be used repeatedly in the Appendix to show the non-existence of periodic points of $f$ of certain even periods in some intervals.

Suppose, for some $n \geq 2$, the interval $[d, u_n]$ contained a periodic point $p$ of $f$ whose least period $2k_n$ is even and $< 2n$. So, $2(k_n + 1) \leq 2n$ and $u_n \leq u_{k_n+1}$. Then since $f^{2i}(d) = z_0$ for all $i \geq 1$, we have $f^{2k_n}([d, p]) \supset [p, z_0] \supset \{v\}$ and so, there is a point $\nu$ in $[d, p]$ such that $f^{2k_n}(\nu) = v$. Therefore, $f^{2k_n+2}([d, \nu]) \supset [\min P, z_0] \supset \{d\}$. This implies the existence of a point $u_{k_n+1}^*$ in $[d, \nu]$ such that $f^{2k_n+2}(u_{k_n+1}^*) = d$. Since $d < u_{k_n+1}^* < \nu < p < u_n \leq u_{k_n+1} < v$, this contradicts the fact that $u_{k_n+1}$ is the smallest point in $[d, v]$ that satisfies $f^{2k_n+2}(x) = d$. Thus, we have shown that, for each $n \geq 1$, the interval $[d, u_n]$ contains no periodic point of $f$ whose least period is even and $< 2n$. This fact will be used below in the Appendix. In particular, for each $n \geq 1$, the point $c_{2n}$ is a period-$(2n)$ of $f$.

Note that, by combining the results in the above proofs of (b) and (c), we obtain that, if $f$ has a period-$m$ orbit $P$ with $m \geq 3$ and odd, then there exist points $v, z, z_0, d$ and, for each $n \geq 1$, a point $u_n$, a period-$(m + 2n)$ point $p_{m+2n}$ and a period-$(2n)$ point $c_{2n}$ such that

$$v = \min \{ \min P \leq x \leq f^{m-1}(\min P) : f(v) = f^{m-1}(\min P) \},$$

$z$ is a fixed point of $f$ in $[v, f^{m-1}(\min P)]$, $z_0 = \min \{ v \leq x \leq z : f^2(x) = x \}$, $d = \max \{ \min P \leq x \leq v : f^2(x) = z_0 \}$ ($> \min P$),

$$f^2(x) < z_0 \leq z < f(x) \text{ for all } d < x < z_0,$$

for each $n \geq 1$, $u_n = \min \{ d \leq x \leq v : f^{2n}(x) = d \}$ and for all $0 \leq i \leq n - 1,$

$$d < f^{2i}(c_{2n}) < z_0 \leq z < f^{2i+1}(c_{2n}),$$

and $\min P < d < \cdots < c_4 < u_2 < c_2 < u_1 < \cdots < p_{m+4} < p_{m+2} < v < z_0 < f^{m-1}(\min P)$.

We now prove (1), (2) and (3) of Sharkovsky’s theorem.

If $f$ has a period-$m$ point with $m \geq 3$ and odd, then it follows from (b) that $f$ has a period-$(m + 2)$ point and, from (c) that $f$ has periodic points of all even periods.

If $f$ has a period-$(2 \cdot m)$ point with $m \geq 3$ and odd, then, by Lemma 2(i), $f^2$ has a period-$m$ point. It follows from the above (or by (b) and (c)) that $f^2$ has a period-$(m + 2)$ point and a period-$(2 \cdot 3)$ point. If $f^2$ has a period-$(m + 2)$ point, then, by Lemma 2(ii),

$$f \text{ has either a period-} (m + 2) \text{ point or a period-} (2 \cdot (m + 2)) \text{ point.}$$

If $f$ has a period-$(m + 2)$ point, then it follows from (c) that $f$ has a period-$(2 \cdot (m + 2))$ point. In either case, $f$ has a period-$(2 \cdot (m + 2))$ point. On the other hand, if $f^2$ has a
period-(2 \cdot 3) point, then, by Lemma 2(ii), \( f \) has a period-(2\(^2\) \cdot 3) point. This shows that if \( f \) has a period-(2 \cdot m) point with \( m \geq 3 \) and odd, then \( f \) has a period-(2 \cdot (m + 2)) point and a period-(2\(^2\) \cdot 3) point.

Now if \( f \) has a period-(\( 2^k \cdot m \)) point with \( m \geq 3 \) and odd and if \( k \geq 2 \), then, by Lemma 2(i), \( f^{2k-1} \) has a period-(\( 2 \cdot m \)) point. It follows from the previous paragraph that \( f^{2k-1} \) has a period-(\( 2 \cdot (m + 2) \)) point and a period-(\( 2^2 \cdot 3 \)) point. So, by Lemma 2(ii), \( f \) has a period-(\( 2^k \cdot (m + 2) \)) point and a period-(\( 2^{k+1} \cdot 3 \)) point.

Furthermore, if \( f \) has a period-(\( 2^i \cdot m \)) point with \( m \geq 3 \) and odd and if \( i \geq 0 \), then, by Lemma 2(i), \( f^2 \) has a period-\( m \) point. For each \( \ell \geq i \), by Lemma 2(i), \( f^2\ell = (f^2)^{\ell-i} \) has a period-\( m \) point and so, by (a), \( f^{2\ell} \) has a period-2 point. This implies, by Lemma 2(ii), that \( f \) has a period-(\( 2^{\ell+1} \)) point for each \( \ell \geq i \).

Finally, if \( f \) has a period-(\( 2^k \)) point for some \( k \geq 2 \), then, by Lemma 2(i), \( f^{2k-2} \) has a period-4 point. By (a), \( f^{2k-2} \) has a period-2 point. By Lemma 2(ii), \( f \) has a period-(\( 2^{k-1} \)) point and hence, by induction, \( f \) has a period-(\( 2^j \)) point for each \( j = 1, 2, \ldots, k - 2 \). Furthermore, it follows from (a) that \( f \) has a fixed point. This completes the proof of (1).

As for the proofs of (2) and (3), there are very elegant examples in [1, 33]. Here we present different examples (see [3] for more examples). We consider the tent map \( T(x) = 1 - |2x - 1| \) and the doubly truncated tent family \( \hat{T}_{a,b}(x) \), where \( 0 < a < b < 1 \), defined on \([0, 1]\) by

\[
\hat{T}_{a,b}(x) = \begin{cases} 
  b, & \text{if } T(x) > b; \\
  T(x), & \text{if } a \leq T(x) \leq b; \\
  a, & \text{if } T(x) < a.
\end{cases}
\]

Note that the relationship between \( T(x) \) and \( \hat{T}_{a,b}(x) \) is that the periodic orbits of \( \hat{T}_{a,b}(x) \) are also periodic orbits of \( T(x) \) with the same periods and, conversely, the periodic orbits of \( T(x) \) which lie entirely in the interval \([a, b]\) are also periodic orbits of \( \hat{T}_{a,b}(x) \) with the same periods. Consequently, if \( Q_k \) is a period-\( k \) orbit of \( T(x) \), then it is also a period-\( k \) orbit of \( \hat{T}_{\min Q_k, \max Q_k}(x) \). By (1), \( \hat{T}_{\min Q_k, \max Q_k}(x) \) has a period-\( \ell \) orbit for each \( \ell \) with \( k < \ell \). In other words, the interval \([\min Q_k, \max Q_k]\) contains a period-\( \ell \) orbit of \( T(x) \) for each \( \ell \) with \( k < \ell \). Since, for each integer \( k \geq 1 \), the equation \( T^k(x) = x \) has exactly \( 2^k \) distinct solutions in \([0, 1]\), \( T(x) \) has finitely many period-\( k \) orbits. Among these finitely many period-\( k \) orbits, let

\[ P_k \]

be one with the smallest diameter \( \max P_k - \min P_k \).

For each \( x \) in \( I \), let \( \hat{T}_k(x) = \hat{T}_{a_k, b_k}(x) \), where \( a_k = \min P_k \) and \( b_k = \max P_k \). Then it is easy to see that, for each \( k \geq 1 \), \( \hat{T}_k(x) \) has exactly one period-\( k \) orbit (i.e., \( P_k \)) but has no period-\( j \) orbit for any \( j \) with \( j < k \) in the Sharkovsky ordering. This establishes (2).

Clearly, \( T(x) \) has a unique period-2 orbit, i.e., \{\frac{2}{5}, \frac{4}{5}\}. For every periodic orbit \( P \) of \( T(x) \) with least period \( \geq 3 \), it follows from (a) that \( \hat{T}_{\min P, \max P}(x) \) has a period-2 orbit. So, \( \min P \leq \frac{2}{5} < \frac{4}{5} \leq \max P \). Now let \( Q_3 \) be any period-3 orbit of \( T(x) \) of smallest diameter.
Then \([\min Q_3, \max Q_3]\) contains finitely many period-6 orbits of \(T(x)\). If \(Q_6\) is one of smallest diameter, then \([\min Q_6, \max Q_6]\) contains finitely many period-12 orbits of \(T(x)\). We choose one, say \(Q_{12}\), of smallest diameter and continue the process inductively. Let
\[
q_0 = \sup \{ \min Q_{2n,3} : n \geq 0 \} \quad \text{and} \quad q_1 = \inf \{ \max Q_{2n,3} : n \geq 0 \}
\]
Then \(q_0 \leq \frac{2}{3} < \frac{4}{5} \leq q_1\). Let \(\tilde{T}_\infty(x) = \tilde{T}_{q_0,q_1}(x)\) for all \(0 \leq x \leq 1\). If \(\tilde{T}_\infty(x)\) had a period-\((2^n \cdot m)\) orbit for some \(n \geq 0\) and some odd \(m \geq 3\), then, by (1), \(\tilde{T}_\infty(x)\) has a period-\((2^{n+1} \cdot 3)\) orbit, say \(\hat{Q}_{2^{n+1},3}\). Since \(\hat{Q}_{2^{n+1},3} \subset [q_0, q_1] \subset [\min Q_{2n+1,3}, \max Q_{2n+1,3}]\), \(Q_{2n+1,3}\) has a period-\((2^{n+1} \cdot 3)\) orbit of \(T(x)\) with diameter smaller than that of \(Q_{2n+1,3}\). This is a contradiction. So, \(\tilde{T}_\infty(x)\) has no periodic orbit of period not a power of 2. On the other hand, for each \(k \geq 0\), the map \(T(x)\) has finitely many period-\((2^k)\) orbits. If each such orbit had an exceptional point which is not in the interval \([q_0, q_1]\), then it is clear that we can find an \(n \geq 1\) such that the interval \([\min Q_{2n,3}, \max Q_{2n,3}]\) contains none of these exceptional points which implies that \([\min Q_{2n,3}, \max Q_{2n,3}]\) contains no period-\((2^k)\) orbits of \(T(x)\). Consequently, the map \(\tilde{T}_{s_k, t_k}(x)\), where \(s_n = \min Q_{2n,3}, t_n = \max Q_{2n,3}\), has no period-\((2^k)\) orbits and yet it has a period-\((2^n \cdot 3)\) orbit, i.e., \(Q_{2n,3}\). This contradicts (1). Therefore, the map \(\tilde{T}_\infty(x)\) is an example for (3).

Note that it is an easy consequence of the following result that, for each positive integer \(m\), the map \(\tilde{T}_{\min P_m, \max P_m}(x)\) defined above and the map \(T_{h(m)}(x) = \min \{ h(m), 1 - |2x - 1| \}\), where \(h(m) = \min \{ \max Q : Q\) is a period-\(m\) orbit of \(f\}\), defined in [3], although look different, have exactly one and the same period-\(m\) orbit, i.e., \(P_m\), and have, for each \(m < n \geq 2\), the same collection of period-\(n\) orbits.

**Proposition 1.** Let \(f : [0, 1] \to [0, 1]\) be a continuous map such that \(f\) is increasing on \([0, 1/2]\) and decreasing on \([1/2, 1]\). Suppose \(f\) has a periodic orbit of least period \(\geq 2\). Then the periodic orbits of \(f\) of least periods \(\geq 2\) are nested in the sense that if \(P\) and \(Q\) are periodic orbits of \(f\) of least periods \(\geq 2\) with \(\max P < \max Q\) then \([\min P, \max P] \subset [\min Q, \max Q]\).

**Proof.** Let \(P\) be a period-\(n\) orbit of \(f\) with \(n \geq 2\). Then, since \(f\) is monotonic on \([0, 1/2]\) and on \([1/2, 1]\), we have
\[
\min P \leq 1/2 \leq \max P.
\]
Let \(p\) be a point in \(P\) such that \(f(p) = \min P\). Then \(1/2 \leq p \leq \max P\). Since \(f\) is decreasing on \([1/2, 1]\), we obtain that \(\min P \leq f(\max P) \leq f(p) = \min P\). This forces \(f(\max P) = \min P\).

Now let \(P\) and \(Q\) be distinct periodic orbits of \(f\) of least periods \(\geq 2\) with \(\max P < \max Q\). Then \(1/2 \leq \max P < \max Q\). Since \(f\) is decreasing on \([1/2, 1]\), we have
\[
(\min Q =) f(\max Q) < f(\max P) (= \min P < \max P < \max Q).
\]
This establishes that \([\min P, \max P] \subset [\min Q, \max Q]\). \(\square\)

Note that Proposition 1 applies to the logistic family \(f_\alpha(x) = \alpha x(1 - x)\), the tent family \(T_\beta(x) = \beta(1 - |2x - 1|)\), the truncated tent family \(T_h(x) = \min \{ h, 1 - |2x - 1| \}\) and the above doubly truncated tent family \(\tilde{T}_{a,b}(x)\).
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Appendix.

The Towers of Periodic Points of \( f \) Associated with the Periodic Orbit \( P \)

Suppose \( f \) has a period-\( m \) orbit \( P \) with \( m \geq 3 \) and odd. In this appendix, we are going to build a tower (among other towers) by using periodic points of \( f \) which lie in the interval \([\min P, \max P]\) and call it the basic tower of periodic points of \( f \) associated with \( P \). This basic tower is divided into 5 sections which are contained respectively in the intervals:

\[ [\min P, d], \ [d, u_1], \ [u_1, v], \ [v, \bar{u}_1], \ \text{and} \ [\bar{u}_1, z_0], \]

where \( u_1 = \min \{ d \leq x \leq v : f^2(x) = d \} \) and \( \bar{u}_1 = \max \{ v \leq x \leq z_0 : f^2(x) = d \} \) (the \( ' \) corresponds to taking the maximum of a set). It is built one layer on top the other in a systematic and recursive way and consists of countably infinitely many layers. For each \( n \geq 2 \), the \( n^{th} \) layer in each section consists of countably infinitely many compartments. Each compartment consists of 3 monotonic sequences of periodic points of \( f \) and the convex hulls of these monotonic sequences are pairwise disjoint (the convex hull of a set is the smallest compact interval that contains the set). Each sequence of periodic points of \( f \) in higher layers may contain finitely many (but not all) periodic points in the lower layers. The foundation of this basic tower which is the first layer also serves to provide a simple proof of Part (1) of the celebrated Sharkovsky’s cycle coexistence theorem.

The construction of these various towers (including the basic tower) is relatively easy. The hard part is to determine the least periods of these periodic points on each layer. See Lemmas 5 & 6.

We shall follow the notations used in the main text. Furthermore, for any distinct points \( a \) and \( b \) in \( I \),

we denote \([a : b] \) as the compact interval with \( a \) and \( b \) as the endpoints.

In the following, our discussions are roughly divided into two parts depending on the relative locations of points with respect to the point \( v \):

Suppose \( a \) and \( w \) are two distinct points in \([\min P, v]\) or in \([v, z_0]\) such that \( f^n(a) \in \{ z_0, f(z_0), f(d) \} \) and \( f^n(w) = w \).

If both \( a \) and \( w \) lie in the interval \([\min P, v]\), then we have \( f^n([a : w]) \supset [w, z_0] \supset \{ v \} \). So, \( f^n(\nu) = v \) for some \( \nu \) in \([a : w]\). Consequently,

\[ f^{n+2}(a) - a \geq z_0 - a > 0 \ \text{and} \ f^{n+2}(\nu) - \nu = f^2(v) - \nu = \min P - \nu < 0. \]

Therefore, there is a point \( w^* \) between \( a \) and \( w \) such that \( f^{n+2}(w^*) = w^* \).
If both $a$ and $w$ lie in the interval $[v, z_0]$, then we may not have $f^n([a : w]) \supset \{v\}$. Fortunately, we have the fact that
\[ f^2(x) < x < z_0 \quad \text{for all} \quad v \leq x < z_0. \] (\ast)
So,
\[ f^{n+2}(a) - a \geq z_0 - a > 0 \quad \text{and} \quad f^{n+2}(w) - w = f^2(w) - w < 0. \]
Therefore, there is a point $w^*$ between $a$ and $w$ such that $f^{n+2}(w^*) = w^*$.

By repeating this process indefinitely, we obtain the following result:

**Lemma 3.** Assume that both $n$ and $k$ are positive integers such that $n - k$ is even and $\geq 0$. Let $a$ and $w$ be distinct points in one of the intervals: $[\min P, v]$ and $[v, z_0]$.

Suppose $f^k(a) \in \{z_0, f(z_0), f(d)\}$ and $f^n(w) = w$.

Then the following hold:

1. If $a < b$, then, for each $i \geq 0$, the point $p_{n+2i} = \min \{a \leq x \leq b : f^{n+2i}(x) = x\}$ exists and $a < \cdots < p_{n+6} < p_{n+4} < p_{n+2} < p_n \leq w < b$. Furthermore, if all periodic points of $f$ in $[a : b]$ of odd periods have least periods $\geq k$, then, for each $i \geq 0$, the point $p_{n+2i}$ is a period-$(n+2i)$ point of $f$;

2. If $b < a$, then, for each $i \geq 0$, the point $q_{n+2i} = \max \{a \leq x \leq b : f^{n+2i}(x) = x\}$ exists and $b < w \leq q_n < q_{n+2} < q_{n+4} < q_{n+6} < \cdots < a$. Furthermore, if all periodic points of $f$ in $[a : b]$ of odd periods have least periods $\geq k$, then, for each $i \geq 0$, the point $q_{n+2i}$ is a period-$(n+2i)$ point of $f$.

Throughout this paper, we shall call

a point $x_0$ in $[\min P, z_0]$ $d$-point if its orbit $O_f(x_0)$ contains the point $d$.

The $d$-points play fundamental role in the construction of various towers of periodic points of $f$ associated with $P$.

**Lemma 4.** For each $n \geq 1$, let
\[ u_n = \min \{d \leq x \leq v : f^{2n}(x) = d\} \quad \text{and} \quad u'_n = \max \{v \leq x \leq z_0 : f^{2n}(x) = d\}. \]

Note the relationship between the subscript $n$ of $u_n$ ($u'_n$ respectively) and the superscript $2n$ of $f^{2n}(x)$ in the definition of $u_n$ ($u'_n$ respectively). It denotes the $n^{th}$ point of the sequence $u_n > (u'_n >$ respectively) which are needed for the tower-building process to continue from the first layer to the second.

Then $d < \cdots < u_3 < u_2 < u_1 < v < u'_1 < u'_2 < u'_3 < \cdots < z_0$ and the following hold:
(1) For any \( x_0 \) in \((d, u_n)\), we have

\[
d < f^{2i}(x_0) < z_0 \quad \text{for all } 1 \leq i \leq n.
\]

Furthermore, in \([d, u_n]\), \( f \) has no periodic points of odd periods \( \leq 2n+1 \), nor has periodic points of even periods \( \leq 2n \) except period-(2n) points (we use this kind of phrasing for a reason which will become apparent in Lemmas 8 & 12 below);

(2) For any \( x_0 \) in \([\bar{u}'_n, z_0]\), we have

\[
v < f^{2n-2}(x_0) < \cdots < f^4(x_0) < f^2(x_0) < x_0 < z_0 \quad \text{and}
\]

\[
d \leq f^{2n}(x_0) < f^{2n-2}(x_0) < \cdots < f^2(x_0) < x_0 < z_0 < f^j(x) \quad \text{for all odd } 1 \leq j \leq 2n+1;
\]

(3) In \([\bar{u}'_n, z_0]\), \( f \) has no periodic points of even periods \( \leq 2n \), nor has periodic points of odd periods \( \leq 2n+1 \).

Proof. It is easy to see that \( d < \cdots < u_3 < u_2 < u_1 < v < \bar{u}'_1 < \bar{u}'_2 < \bar{u}'_3 < \cdots < z_0 \). By arguing as that (†) in the proof of (c) in the main text, we easily obtain that

\[
d < f^{2i}(x) < z_0 \quad \text{for all } x \text{ in } (d, u_n) \cup (\bar{u}'_n, z_0) \quad \text{and all } 1 \leq i \leq n.
\] (†)

This, combined with the fact

\[
f(x) > z \quad (\geq z_0 > f^2(x)) \quad \text{for all } d < x < z_0,
\] (***)

implies that \( f \) has no periodic points of odd periods \( \leq 2n+1 \) in \((d, u_n) \cup (\bar{u}'_n, z_0)\).

On the other hand, let \( x_0 \) be any point in \([\bar{u}'_n, z_0]\). Suppose \( f^{2k}(x_0) \leq v \) for some \( 1 \leq k < n \). Then, since \( f^2(z_0) = z_0 \), we obtain that \( f^{2k}([x_0, z_0]) \supset [v, z_0] \) and so, \( f^{2k+2}([x_0, z_0]) \supset [\min P, z_0] \). Let \( \bar{v}_{k+1} \) be a point in \([x_0, z_0]\) such that \( f^{2k+2}(\bar{v}_{k+1}) = \min P \) and let \( \bar{v}'_{k+1} \) be a point in \([\bar{v}_{k+1}, z_0]\) such that \( f^{2k+2}(\bar{v}'_{k+1}) = d \). Then \( \bar{v}_{k+1} < \bar{v}'_{k+1} < z_0 \). Since \( k+1 \leq n \) and \( 2(k+1) \leq 2n \), we have \( \bar{u}'_n \leq x_0 \leq \bar{v}_{k+1} < \bar{v}'_{k+1} \leq \bar{u}'_{k+1} \leq \bar{u}'_n \). This is a contradiction. So, \( f^{2i}(x) > v \) for all \( \bar{u}'_n \leq x < z_0 \) and all \( 1 \leq i \leq n-1 \). It follows from the above (†) and the following (∗)

\[
f^2(x) < x < z_0 \quad \text{for all } v \leq x < z_0
\] (∗)

that, for any \( x_0 \) in \([\bar{u}'_n, z_0]\), we have

\[
v < f^{2n-2}(x_0) < \cdots < f^4(x_0) < f^2(x_0) < x_0 < z_0 \quad \text{and,}
\]

by (∗), we have \( f^{2n}(x_0) < f^{2n-2}(x_0) \). This, combined with (†) and (*** above), implies that

\[
d \leq f^{2n}(x_0) < f^{2n-2}(x_0) < \cdots < f^2(x_0) < x_0 < z_0 < f^j(x) \quad \text{for all odd } 1 \leq j \leq 2n+1.
\]
Consequently, we have shown (2) & (3) and part of (1).

As for the rest of (1). It was already shown in the proof of (c) in the main text. We include a proof here for completeness. Suppose $f$ had a periodic point $p_{2k}^*$ of even period $2k$ with $2 \leq 2k < 2n$ in $[d,u_n]$. Then $f^{2k}([d,p_{2k}^*]) \supset [p_{2k}^*,z_0] \supset [v,z_0]$. So, $f^{2k+2}([d,p_{2k}^*]) \supset f^2([v,z_0]) \supset [\min P,z_0] \supset \{d\}$. Thus, there is a point $(d <) u_{k+1}^* (< p_{2k}^* < u_n)$ such that $f^{2k+2}(u_{k+1}^*) = d$. Since $k + 1 \leq n$, we have $u_n \leq u_{k+1} \leq u_{k+1}^* < u_n$. This is a contradiction. Therefore, $f$ has no periodic points of even periods $< 2n$ in $[d,u_n]$.

Finally, since

$$f^{2n}(d) - d = z_0 - d > 0 \text{ and } f^{2n}(u_n) - u_n = d - u_n < 0,$$

the point

$$c_{2n} = \min \{d \leq x \leq u_n : f^{2n}(x) = x\}$$

exists. Since we have just shown that $f$ has no periodic points of even periods $< 2n$ in $[d,u_n]$, The point $c_{2n}^*$ must be a period-(2n) point of $f$. Therefore, $f$ has no periodic points of even periods $\leq 2n$ except period-(2n) points in $[d,u_n]$. This proves (1) and hence Lemma 4. □

**Lemma 5.** Assume that both $a$ and $b$ are distinct points in one of the intervals: $[\min P,d]$, $[d,v]$ and $[v,z_0]$. Let $n \geq 0$ and $k \geq 0$ be integers such that $m + 2n \geq 2k + 1$.

Suppose $f^{2k+1}(a) \in \{z_0,f(z_0),f(d)\}$ and $f^{m+2n}(b) = \min P$.

Then the following hold:

1. If $a < b$, then, for each $i \geq 0$, the points

   $$p_{m+2n+2i} = \min \{a \leq x \leq b : f^{m+2n+2i}(x) = x\},$$

   $$\mu_{m,n+i} = \min \{a \leq x \leq b : f^{m+2n+2i}(x) = d\} \text{ and }$$

   $$c_{2m+2n+2i}^{(s)} = \min \{a \leq x \leq b : f^{2m+2n+2i}(x) = x\}$$

   exist (note the relationship between the subscript of $\mu_{m,n+i}$ and the superscript of $f^{m+2n+2i}$) and

   $$a < \cdots < c_{2m+2n+6}^{(s)} < c_{2m+2n+4}^{(s)} < c_{2m+2n+2}^{(s)} < c_{2m+2n}^{(s)} < b,$$

   $$a < \cdots < p_{m+2n+4} < \mu_{m,n+1} < p_{m+2n+2} < \mu_{m,n} < p_{m+2n} < b \text{ if } \min P < a < b < d,$$

   $$a < \cdots < \mu_{m,n+2} < p_{m+2n+2} < \mu_{m,n+1} < p_{m+2n} < \mu_{m,n} < b \text{ if } d \leq a < b \leq v$$

   and, if $v < a < b < z_0$, then

   $$a < \cdots < \mu_{m,n+2} < \mu_{m,n+1} < \mu_{m,n} < b,$$

   $$a < \cdots < p_{m+2n+4} < p_{m+2n+2} < p_{m+2n} \text{ and }$$

   $$p_{m+2n+2} < \mu_{m,n+i} \text{ for all } i \geq 0.$$

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Furthermore,

if \( i \geq 0 \) and \( m+2n+2i \geq 3(2k+1) \), then \( p_{m+2n+2i} \) is a period-(\( m+2n+2i \)) point of \( f \), and, if all periodic points of \( f \) in \([a, b]\) of odd periods have least periods \( \geq 2k+1 \), then,

(i) for each \( i \geq 0 \), \( p_{m+2n+2i} \) is a period-(\( m+2n+2i \)) point of \( f \);

(ii) if \( \min P < a < b < d \) or \( d \leq a < b \leq v \), then, for each \( i \geq n+1 \), \( c_{2m+2n+2i}^{(s)} \) is a period-(\( 2m+2n+2i \)) point of \( f \) and

(iii) if \( v < a < b < z_0 \), then, for each \( i \geq n + \max\{1, m + 1 - 2n\} \), \( c_{2m+2n+2i}^{(s)} \) is a period-(\( 2m+2n+2i \)) point of \( f \).

(2) If \( b < a \), then, for each \( i \geq 0 \), the points

\[
q_{m+2n+2i} = \max\{b \leq x \leq a : f^{m+2n+2i}(x) = x\},
\]

\[
\mu'_{m,n+i} = \max\{b \leq x \leq a : f^{m+2n+2i}(x) = d\}
\]

and

\[
c_{2m+2n+2i}^{(s)} = \max\{b \leq x \leq a : f^{2m+2n+2i}(x) = x\}
\]

exist (note the relationship between the subscript of \( \mu'_{m,n+i} \) and the superscript of \( f^{m+2n+2i} \)) and

\[
b < c_{m+2n}^{(s)} < c_{m+2n+2}^{(s)} < c_{m+2n+4}^{(s)} < c_{m+2n+6}^{(s)} < \cdots < a,
\]

\[
b < q_{m+2n} < \mu'_{m,n} < q_{m+2n+2} < \mu'_{m,n+1} < q_{m+2n+4} < \cdots < a \text{ if } \min P \leq b < a \leq d,
\]

\[
b < \mu'_{m,n} < q_{m+2n} < \mu'_{m,n+1} < q_{m+2n+2} < \mu'_{m,n+2} < \cdots < a \text{ if } d < b < a < v
\]

and, if \( v \leq b < a \leq z_0 \), then

\[
b < \mu'_{m,n} < \mu'_{m,n+1} < \mu'_{m,n+2} < \cdots < a,
\]

\[
b < q_{m+2n} < q_{m+2n+2} < q_{m+2n+4} < \cdots < a \text{ and }
\]

\[
\mu'_{m,n+i} < q_{m+2n+2i} \text{ for all } i \geq 0.
\]

Furthermore,

if \( i \geq 0 \) and \( m+2n+2i \geq 3(2k+1) \), then \( q_{m+2n+2i} \) is a period-(\( m+2n+2i \)) point of \( f \), and, if all periodic points of \( f \) in \([b, a]\) of odd periods have least periods \( \geq 2k+1 \), then,

(i) for each \( i \geq 0 \), \( q_{m+2n+2i} \) is a period-(\( m+2n+2i \)) point of \( f \);

(ii) if \( \min P \leq b < a \leq d \) or \( d < b < a < v \), then, for each \( i \geq n+1 \), \( c_{2m+2n+2i}^{(s)} \) is a period-(\( 2m+2n+2i \)) point of \( f \) and

(iii) if \( v \leq b < a \leq z_0 \), then, for each \( i \geq n + \max\{1, m + 1 - 2n\} \), \( c_{2m+2n+2i}^{(s)} \) is a period-(\( 2m+2n+2i \)) point of \( f \).
**Remark 1.** In the above result, the periodic points, say, $c_{2m+2n+2i}^{(s)}$'s of even periods are interspersed with the periodic points $p_{m+2n+2i}$'s of odd periods. To make things simple, we shall use these periodic points $p_{m+2n+2i}$'s of odd periods (because we apply odd iterates of $f$ to the endpoints $a$ and $b$) and ignore those periodic points $c_{2m+2n+2i}^{(s)}$ of even periods later on to build the basic tower of periodic points of $f$ associated with $P$.

**Proof.** Assume that $\min P < a < b < d$ or $d \leq a < b \leq v$. Since $f^{2k+1}(a) \in \{z_0, f(z_0), f(d)\}$ and, $m + 2n \geq 2k + 1$, we have $f^{m+2n}(a) \in \{z_0, f(z_0), f(d)\}$.

Suppose $\min P < a < b < d$.

Then $\min P < x < d$ for all $x \in [a, b]$. Since

$$f^{m+2n}(a) - a \geq z_0 - a > 0 \text{ and } f^{m+2n}(b) - b = \min P - b < 0,$$

the point $p_{m+2n} = \min \{a \leq x \leq b : f^{m+2n}(x) = x\}$ exists. Since $f^{m+2n}(a) \in \{z_0, f(z_0), f(d)\}$ and $f^{m+2n}(p_{m+2n}) = p_{m+2n}$, it follows from Lemma 3 that, for each $i \geq 0$, the point $p_{m+2n+2i} = \min \{a \leq x \leq b : f^{m+2n+2i}(x) = x\}$ exists.

Furthermore, for each $i \geq 0$, since (note that $p_{m+2n+2i} < b \leq d$)

$$f^{m+2n+2i}(a) - d \geq z_0 - d > 0 \text{ and } f^{m+2n+2i}(p_{m+2n+2i}) - d = p_{m+2n+2i} - d < 0,$$

the point $\mu_{m,n+i} = \min \{a \leq x \leq p_{m+2n+2i} : f^{m+2n+2i}(x) = d\}$ exists and is $< p_{m+2n+2i}$.

On the other hand, since $f^{m+2n+2i}([a, \mu_{m,n+i}]) \supset [d, z_0] \supset \{v\}$, there is a point $\nu_{m,n+i}$ in $[a, \mu_{m,n+i}]$ such that $f^{m+2n+2i}(\nu_{m,n+i}) = v$. Since $f^{m+2n+2i+2}(a) - a \geq z_0 - a > 0$ and $f^{m+2n+2i+2}(\nu_{m,n+i}) - \nu_{m,n+i} = f^2(v) - \nu_{m,n+i} = \min P - \nu_{m,n+i} < 0$, the point

$$p_{m+2n+2i+2} = \min \{a \leq x \leq \nu_{m,n+i} : f^{m+2n+2i+2}(x) = x\}$$

exists and is $< \nu_{m,n+i} < \mu_{m,n+i}$.

This, combined with the above, implies that

$$\min P < a < \cdots < \mu_{m,n+2} < p_{m+2n+4} < \mu_{m,n+1} < p_{m+2n+2} < \mu_{m,n} < p_{m+2n} < b < d.$$

Suppose $d \leq a < b \leq v$. Then $d \leq x \leq v$ for all $x \in [a, b]$.

Since $f^{m+2n}([a, b]) \supset [\min P, z_0] \supset \{d\}$, the point

$$\mu_{m,n} = \min \{a \leq x \leq b : f^{m+2n}(x) = d\}$$

exists.

Since $f^{m+2n}(a) - a \geq z_0 - a > 0$ and $f^{m+2n}(\mu_{m,n}) - \mu_{m,n} = d - \mu_{m,n} < 0$, the point

$$p_{m+2n} = \min \{a \leq x \leq \mu_{m,n} : f^{m+2n}(x) = x\}$$

exists and is $< \mu_{m,n}$.

Since $f^{m+2n}([a, p_{m+2n}]) \supset [p_{m+2n}, z_0] \supset \{v\}$, there is a point

$\nu_{m,n}$ in $[a, p_{m+2n}]$ such that $f^{m+2n}(\nu_{m,n}) = v$. Since $f^{m+2n+2}(a) - a \geq z_0 - d > 0$ and $f^{m+2n+2}(\nu_{m,n}) - d = f^2(v) - d = \min P - d < 0$, the point

$$\mu_{m,n+1} = \min \{a \leq x \leq \nu_{m,n} : f^{m+2n+2}(x) = d\}$$

exists and is $< \nu_{m,n} < p_{m+2n}$.

Since $f^{m+2n+2}(a) - a \geq z_0 - a > 0$ and $f^{m+2n+2}(\mu_{m,n+1}) - \mu_{m,n+1} = d - \mu_{m,n+1} < 0$, the point $p_{m+2n+2} = \min \{a \leq x \leq \mu_{m,n+1} : f^{m+2n+2}(x) = x\}$ exists and is $< \mu_{m,n+1}$.

By induction, we have

$$d \leq a < \cdots < p_{m+2n+4} < \mu_{m,n+2} < p_{m+2n+2} < \mu_{m,n+1} < p_{m+2n} < \mu_{m,n} < b < v.$$
Note that we have assumed that \( \min P < a < b < d \) or \( d < a < b \leq v \). Since \( f^{2m+2n}(a) - a \geq z_0 - a > 0 \) and \( f^{2m+2n}(b) - b = f^m(f^{m+2n}(b)) - b = f^m(\min P) - b = \min P - b < 0 \), the point \( c_{2m+2n}^{(s)} = \min \{ a \leq x \leq b : f^{2m+2n}(x) = x \} \) exists. Since \( f^{2m+2n}(a) \in \{ z_0, f(z_0) \} \) and \( f^{2m+2n}(c_{2m+2n}^{(s)}) = c_{2m+2n}^{(s)} \), it follows from Lemma 3 that, for each \( i \geq 0 \), the point \( c_{2m+2n+2i}^{(s)} = \min \{ a \leq x \leq b : f^{2m+2n+2i}(x) = x \} \) exists and \( a < \cdots < c_{2m+2n+6}^{(s)} < c_{2m+2n+4}^{(s)} < c_{2m+2n+2}^{(s)} < c_{2m+2n}^{(s)} < b \). (Note that these periodic points \( c_{2m+2n+2i}^{(s)} \)'s are interspersed with the periodic points \( p_{m+2n+2i} \)'s of odd periods. To make things simple, we shall ignore these periodic points \( c_{2m+2n+2i}^{(s)} \)'s and use only the periodic points \( p_{m+2n+2i} \)'s of odd periods later on to build the basic tower of periodic points of \( f \) associated with \( P \). We include them here for the interested readers).

Now, we want to find the least periods of \( p_{m+2n+2i} \)'s with respect to \( f \). For any periodic point \( p \) of \( f \), let \( \ell(p) \) denote the least period of \( p \) with respect to \( f \). Suppose, for some \( j \geq 0 \), \( \ell(p_{m+2n+2j}) < m + 2n + 2j \). By Lemma 1, \( \ell(p_{m+2n+2j}) \) divides \( m + 2n + 2j \) and so, is odd.

(i) If \( \ell(p_{m+2n+2j}) \geq 2k + 1 \), then, since \( \ell(p_{m+2n+2j}) \) is odd and \( f^{2k+1}(a) \in \{ z_0, f(z_0), f(d) \} \), we have

\[
f^{\ell(p_{m+2n+2j})}(a) \in \{ z_0, f(z_0), f(d) \} \quad \text{and} \quad f^{\ell(p_{m+2n+2j})}(p_{m+2n+2j}) = p_{m+2n+2j}.
\]

Since \( m + 2n + 2j > \ell(p_{m+2n+2j}) \), it follows from Lemma 3 that there is a periodic point \( w_{m+2n+2j} \) of \( f \) in \( [a, p_{m+2n+2j}] \) such that \( f^{m+2n+2j}(w_{m+2n+2j}) = w_{m+2n+2j} \). Since \( a < w_{m+2n+2j} < p_{m+2n+2j} < b \), this contradicts the minimality of \( p_{m+2n+2j} \) in \( [a, b] \). This shows that if \( j \geq 0 \) is such that \( \ell(p_{m+2n+2j}) < m + 2n + 2j \), then \( \ell(p_{m+2n+2j}) < 2k + 1 \).

(ii) Suppose \( \ell(p_{m+2n+2j}) < 2k + 1 \) and let \( r = (m + 2n + 2j)/\ell(p_{m+2n+2j}) \). By Lemma 1, \( r \geq 1 \) is odd and, since \( m + 2n + 2j > \ell(p_{m+2n+2j}) \), \( r \geq 3 \).

Suppose \( (m + 2n + 2j)/(2k + 1) \geq 3 \).

Then \( m + 2n + 2j - 2\ell(p_{m+2n+2j}) = (r - 2)\ell(p_{m+2n+2j}) \) is odd and \( \geq 3(2k + 1) - 2\ell(p_{m+2n+2j}) = (2k + 1) + 2[2k + 1 - \ell(p_{m+2n+2j})] > 2k + 1 \). Since

\[
f^{(m+2n+2j)-2\ell(p_{m+2n+2j})}(a) \in \{ z_0, f(z_0) \} \quad \text{and} \quad f^{(m+2n+2j)-2\ell(p_{m+2n+2j})}(p_{m+2n+2j}) = p_{m+2n+2j},
\]

and since \( m+2n+2j > m+n+2j - 2\ell(p_{m+2n+2j}) \), it follows from Lemma 3 that there is a periodic point \( w_{m+2n+2j} \) of \( f \) in \( [a, p_{m+2n+2j}] \) such that \( f^{m+2n+2j}(w_{m+2n+2j}) = w_{m+2n+2j} \). Since \( a < w_{m+2n+2j} < p_{m+2n+2j} < b \), this contradicts the minimality of \( p_{m+2n+2j} \) in \( [a, b] \).

By combining the above (i) and (ii), we obtain that if \( i \geq 0 \) is an integer such that \( m+2n+2i \geq 3(2k + 1) \), then \( p_{m+2n+2i} \) is a period-\((m + 2n + 2i)\) point of \( f \).

Assume that all periodic points of \( f \) in \( [a, b] \) of odd periods have least periods \( \geq 2k + 1 \). Suppose \( \ell(p_{m+2n+2j}) < m + 2n + 2j \) for some \( j \geq 0 \). Then, by hypothesis, \( \ell(p_{m+2n+2j}) \geq 2k + 1 \).
By arguing as those in (i) above, we obtain that

if all periodic points of $f$ in $[a,b]$ of odd periods have least periods $\geq 2k + 1$, then

for each $i \geq 0$, the point $p_{m+2n+2i}$ is a period-$\cdot$($\cdot$ $m+2n+2i$) point of $f$.

We now determine the least periods of $c_{2m+2n+2i}$’s under the assumption that all periodic points of $f$ in $[a,b]$ of odd periods have least periods $\geq 2k + 1$.

Recall that, we have assumed that $\min P < a < b < d$ or $d < a < b < v$ and,

for each $i \geq 0$, $c_{2m+2n+2i} = \min \{a \leq x \leq b : f^{2m+2n+2i}(x) = x\}$.

Suppose $i \geq 0$ and $m+n+i \geq 2k+2$ and let $r = (2m+2n+2i)/\ell(c_{2m+2n+2i})$.

If $r \geq 3$ is odd, then $r - 1 \geq 2$ is even. Since

$$(r-1)\ell(c_{2m+2n+2i}) = (r-1)[(2m+2n+2i)/r] \geq 2[(r-1)/r] \cdot (2k+2) \geq 2k+2,$$

we can apply Lemma 3 with

$$f^{2k+2}(a) \in \{z_0,f(z_0)\} \text{ and } f^{(r-1)\ell(c_{2m+2n+2i})}(c_{2m+2n+2i}^*) = c_{2m+2n+2i}^*.$$

If $r \geq 4$ is even, then $r - 2 \geq 2$ is even. Since

$$(r-2)\ell(c_{2m+2n+2i}) = (r-2)[(2m+2n+2i)/r] \geq 2[(r-2)/r] \cdot (2k+2) \geq 2k+2,$$

we can apply Lemma 3 with

$$f^{2k+2}(a) \in \{z_0,f(z_0)\} \text{ and } f^{(r-2)\ell(c_{2m+2n+2i})}(c_{2m+2n+2i}^*) = c_{2m+2n+2i}^*.$$

In either case, we obtain a periodic point $w_{2m+2n+2i}$ of $f$ in $[a,c_{2m+2n+2i}^*)$ such that

$$f^{2m+2n+2i}(w_{2m+2n+2i}) = w_{2m+2n+2i}^*.$$ This contradicts the minimality of $c_{2m+2n+2i}^*$ in $[a,b]$.

So, if $r > 2$, then $\ell(c_{2m+2n+2i}^*) = 2m + 2n + 2i$.

If $r = 2$ and $\ell(c_{2m+2n+2i}^*) = m + n + i \geq 2k+2$ is even, then by applying Lemma 3 with

$$f^{2k+2}(a) \in \{z_0,f(z_0)\} \text{ and } f^{\ell(c_{2m+2n+2i}^*)}(c_{2m+2n+2i}^*) = c_{2m+2n+2i}^*,$$

we obtain the same contradiction. So, in this case, $\ell(c_{2m+2n+2i}^*) = 2m + 2n + 2i$.

Consequently, if $i \geq n+1$ and $m+n+i$ is even, then $m+n+i \geq (m+2n) + 1 \geq (2k+1) + 1 \geq 2k+2$. In this case, since we have either $r > 2$ or $r = 2$, it follows from the above that $c_{2m+2n+2i}^*$ is a period-$\cdot$(2m+2n+2i) point of $f$.

On the other hand, suppose $i \geq n+1$ and $m+n+i$ is odd, then $m+n+i \geq m+2n+1 \geq 2k+2$. If $r > 2$, then it follows from the above that $c_{2m+2n+2i}^*$ is a period-$\cdot$(2m+2n+2i) point of $f$. If $r = 2$, then $\ell(c_{2m+2n+2i}^*) = m+n+i$. So, in this case, $c_{2m+2n+2i}^*$ is either a period-$\cdot$(2m+2n+2i) point of $f$ or a period-$\cdot$(m+n+i) point of $f$.  

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Suppose \( m + n + i \) is odd and \( \geq 2k + 2 \) and the point \( c_{2m+2n+2i}^* \) is a period-\((m + n + i)\) point of \( f \). Since \( m + n + i \geq 2k + 2 \) and \( f^{2k+1}(a) \in \{z_0, f(z_0), f(d)\} \), we have \( f^{m+n+i}(a) \in \{z_0, f(z_0)\} \) and, since we have assumed that \( \min P < a < b < d \) or \( d \leq a < b \leq v \), we obtain that

\[
f^{m+n+i}([a, c_{2m+2n+2i}^*]) \supset \left[ f^{m+n+i}(c_{2m+2n+2i}^*), f^{m+n+i}(a) \right] \supset [c_{2m+2n+2i}^*, z_0] \supset [v, z_0].
\]

Consequently, we have

\[
f^{m+n+i+2}([a, c_{2m+2n+2i}^*]) \supset f^2([v, z_0]) \supset [\min P, z_0].
\]

Since \( f^2([\min P, z_0]) \supset f^2([v, z_0]) \supset [\min P, z_0] \), we obtain that,

for each \( i \geq n + 1 \) such that \( m + n + i \) is odd,

\( n + i \) is even and \( \geq n + (n + 1) \geq 1 \). So, \( n + i \geq 2 \) and \( f^{2m+2n+2i}([a, c_{2m+2n+2i}^*]) \supset f^{m+n+i-2}([\min P, z_0]) \supset f^m([\min P, z_0]) \supset [\min P, z_0] \supset \{ \min P \} \).

Let \( \delta_{2m+2n+2i} \) be a point in the interval \([a, c_{2m+2n+2i}^*]\) such that \( f^{2m+2n+2i}(\delta_{2m+2n+2i}) = \min P \). It is clear that \( a < \delta_{2m+2n+2i} < c_{2m+2n+2i}^* \) such that \( f^{2m+2n+2i}(\delta_{2m+2n+2i}) = \min P \). Since \( f^{2m+2n+2i}(a) - a \geq z_0 - a > 0 \) and \( f^{2m+2n+2i}(\delta_{2m+2n+2i}) - \delta_{2m+2n+2i} = \min P - \delta_{2m+2n+2i} < 0 \), there exists a point \( w_{2m+2n+2i} \) in \([a, \delta_{2m+2n+2i}]\) such that \( f^{2m+2n+2i}(w_{2m+2n+2i}) = w_{2m+2n+2i} \). Since \( a < w_{2m+2n+2i} < \delta_{2m+2n+2i} < c_{2m+2n+2i}^* < b \) and \( f^{2m+2n+2i}(w_{2m+2n+2i}) = w_{2m+2n+2i} \), this contradicts the minimality of \( c_{2m+2n+2i}^* \) in \([a, b]\). So, the point \( c_{2m+2n+2i}^* \) can not be a period-\((m + n + i)\) point of \( f \) and must be a period-\((2m + 2n + 2i)\) point of \( f \). Therefore, we have shown that

if all periodic points of \( f \) in \([a, b]\) of odd periods have least periods \( \geq 2k + 1 \), then,

for each \( i \geq n + 1 \), \( c_{2m+2n+2i}^* \) is a period-\((2m + 2n + 2i)\) point of \( f \).

Suppose \( v < a < b < z_0 \).

In this case, we only need to show that, under the assumption that all periodic points of \( f \) in \([a, b]\) of odd periods have least periods \( \geq 2k + 1 \), for each \( i \geq n + \max \{1, m + 1 - 2n\} \), the point \( c_{2m+2n+2i}^* = \min \{a \leq x \leq b : f^{2m+2n+2i}(x) = x\} \) is a period-\((2m + 2n + 2i)\) point of \( f \). The rest statements regarding this case can be easily proved and are omitted.

By arguments similar to the case as \( \min P < a < b < d \), we obtain that

if \( m + n + i \) (\( \geq 2k + 2 \)) is even, then \( c_{2m+2n+2i}^* \) is a period-\((2m + 2n + 2i)\) point of \( f \) and

if \( m + n + i \) (\( > 2k + 2 \)) is odd, then \( c_{2m+2n+2i}^* \) is either a period-\((2m + 2n + 2i)\) point or a period-\((m + n + i)\) point of \( f \).
Now we show that,

for each \( i \geq 1 \) such that \( 2(n+i) \geq m+1 \),

the point \( c_{2^m+2n+4i} \) is a period-\( (2m+4n+4i) \) point of \( f \) by arguing as follows (note that \( m+2(n+i) = m+2n+2i \) is odd and \( = (m+2n) + 2i \geq (2k+1) + 2 > 2k+2 \)):

Recall that, for each \( i \geq 0 \), \( p_{m+2n+2i} = \min \{ a \leq x \leq b : f^{m+2n+2i}(x) = x \} \) is the smallest point in \([a, b]\) that satisfies the equation \( f^{m+2n+2i}(x) = x \).

Suppose, for some \( j \geq 1 \) such that \( 2(n+j) \geq m+1 \), \( \ell(c^*_{2^m+4n+4j}) = m+2n+2j \) (is odd).

The following arguments are different from those used in the proof of the case \( \min P < a < b < d \) above and are based on the following two facts

\[ f^2(x) < x < z_0 \text{ for all } v \leq x < z_0 \]  

\((*)\)

and

\[ f(x) > z \ (\geq z_0 > f^2(x)) \text{ for all } d < x < z_0. \]  

\((***)\)

Since \( v < a < p_{m+2n+2j} < b \), if the points \( f^2(p_{m+2n+2j}), f^4(p_{m+2n+2j}), \ldots, f^{m-1}(p_{m+2n+2j}) \) are all \( > v \), then by \((*)\) above, we have

\[ v < f^{(m-1)}(p_{m+2n+2j}) < f^{(m-3)}(p_{m+2n+2j}) < \cdots < f^2(p_{m+2n+2j}) < p_{m+2n+2j} < b < z_0. \]

But then, by \((***)\) above, \((z_0 > b > p_{m+2n+2j} = f^m(p_{m+2n+2j}) = f(f^{(m-1)}(p_{m+2n+2j})) > z \geq z_0\). This is a contradiction. Therefore, \( f^{2s}(p_{m+2n+2j}) \leq v \) for some \( 1 \leq s \leq (m-1)/2 \). Since \( f^{m+2n+2j}([a, p_{m+2n+2j}]) \supset [p_{m+2n+2j}, z_0] \), we have

\[ f^{m+2n+2j+2s+2}([a, p_{m+2n+2j}]) \supset f^{2s+2}([p_{m+2n+2j}, z_0]) \supset f^2([v, z_0]) \supset [\min P, z_0]. \]

Since \( f^2([\min P, z_0]) \supset f^2([v, z_0]) \supset [\min P, z_0] \) and \( m-1 \geq 2s \) is even, we obtain that,

\[ f^{m+2n+2j+(m+1)}([a, p_{m+2n+2j}]) = f^{m+2n+2j+2s+2+(m-1-2s)}([a, p_{m+2n+2j}]) \supset [\min P, z_0]. \]

This implies that, since \( j \geq 1 \) and \( 2(n+j) \geq m+1 \) (so, \( m+n+2j \geq (m+n) + 2j > 2k+2 \)),

\[ f^{2m+4n+4j}([a, p_{m+2n+2j}]) = f^{m+2n+2i-m-1}(f^{m+2n+2i+(m+1)}([a, p_{m+2n+2j}])) \supset f^{m+2n+2i-(m+1)}([\min P, z_0]) \supset f^m([\min P, z_0]) \supset [\min P, z_0]. \]

Let \( \delta_{2^m+4n+4j} \) be a point in \([a, p_{m+2n+2j}] \subset [a, b]\) such that \( f^{2m+4n+4j}(\delta_{2^m+4n+4j}) = \min P \). It is clear that \( a < \delta_{2^m+4n+4j} < p_{m+2n+2j} < b \). Since

\[ f^{2m+4n+4j}(a) - a \geq z_0 - a > 0 \] \( \text{and} \) \( f^{2m+4n+4j}(\delta_{2^m+4n+4j}) - \delta_{2^m+4n+4j} = \min P - \delta_{2^m+4n+4j} < 0, \)
there is a point \( w_{2m+4n+4j} \) in \([a, \delta_{2m+4n+4j}] \) \((\subset [a, b])\) so that \( f^{2m+4n+4j}(w_{2m+4n+4j}) = w_{2m+4n+4j} \). It follows from the definition of \( c^*_{2m+4n+4j} \) that \( a < c^*_{2m+4n+4j} \leq w_{2m+4n+4j} \leq \delta_{2m+4n+4j} < p_{m+2n+2j} \) < \( b \). Since \( p_{m+2n+2j} \) is the smallest point in \([a, b] \) that satisfies the equation \( f^{m+2n+2j}(x) = x \), we obtain that \( c^*_{2m+4n+4j} \) can not be a period-(\( m + 2n + 2j \)) point of \( f \). This is a contradiction. Therefore, we have shown that, for each \( i \geq 1 \) such that \( 2(n + i) = 2n + 2i \geq m + 1 \), \( c^*_{2m+4n+4i} \) is a period-(\( 2m + 4n + 4i \)) point of \( f \). In other words, we have shown that

for each even \( i \geq \max \{1, m + 1 - 2n\} \), \( c^*_{2m+4n+2i} \) is a period-(\( 2m + 4n + 2i \)) point of \( f \).

On the other hand, for each odd \( i \geq 1 \), \( m + 2n + i \) (\( \geq 2k + 2 \)) is even. It follows from the above that

for each odd \( i \geq 1 \), \( c^*_{2m+4n+2i} \) is a period-(\( 2m + 4n + 2i \)) point of \( f \).

By combining the above two results, we obtain that, for each \( i \geq \max \{1, m + 1 - 2n\} \), \( i \) even or odd, \( c^*_{2m+4n+2i} \) is a period-(\( 2m + 4n + 2i \)) point of \( f \) or, equivalently,

for each \( i \geq n + \max \{1, m + 1 - 2n\} \),

the point \( c^*_{2m+2n+2i} \) is a period-(\( 2m + 2n + 2i \)) point of \( f \).

This completes the proof of (1). (2) can be proved similarly. \( \square \)

**Lemma 6.** Assume that both \( a \) and \( b \) are distinct points in one of the intervals: \([\min P, d]\), \([d, v]\) and \([v, z_0]\). Let \( n \geq k \geq 1 \) be integers.

Suppose \( f^{2k}(a) \in \{z_0, f(z_0), f(d)\} \) and \( f^{2n}(b) = \min P \).

Then the following hold:

(1) If \( a < b \), then, for each \( i \geq 0 \), the points

\[
c_{2n+2i} = \min \{a \leq x \leq b : f^{2n+2i}(x) = x\} \quad \text{and} \quad \mu_{n+i} = \min \{a \leq x \leq b : f^{2n+2i}(x) = d\}
\]

exist (note the relationship between the subscript of \( \mu_{n+i} \) and the superscript of \( f^{2n+2i} \)) and

\[
a < \cdots < \mu_{n+2} < c_{2n+4} < \mu_{n+1} < c_{2n+2} < \mu_n < c_{2n} < b \quad \text{if} \quad \min P < a < b < d,
\]

\[
a < \cdots < c_{2n+4} < \mu_{n+2} < c_{2n+2} < \mu_{n+1} < c_{2n} < \mu_n < b \quad \text{if} \quad d \leq a < b \leq v
\]

and, if \( v < a < b < z_0 \), then

\[
a < \cdots < \mu_{n+3} < \mu_{n+2} < \mu_{n+1} < \mu_n < b,
\]

\[
a < \cdots < c_{2n+6} < c_{2n+4} < c_{2n+2} < c_{2n} \quad \text{and} \quad c_{2n+2i} < \mu_{n+i} \quad \text{for all} \quad i \geq 0.
\]
Remark 2. In the above lemma, suppose, say, \( a < b \). Since \( f^{2k}(a) \in \{z_0, f(z_0), f(d)\} \) and \( f^{2n}(b) = \min P \), we have \( f^{2k+1}(a) \in \{z_0, f(z_0)\} \) and \( f^{m+2n}(b) = f^m(\min P) = \min P \). By Lemma 5, we obtain that, for each \( i \geq 0 \) such that \( m + 2n + 2i \geq 3(2k + 1) \), the point

\[ p^*_n = \min \{a \leq x \leq b : f^{m+2n+2i}(x) = x\} \]

exists and is a period-(\( m + 2n + 2i \)) point of \( f \). These periodic points \( p^*_m \)'s of odd periods are interspersed with the periodic points of even periods obtained in the lemma. To make things simple, we choose only those periodic points of \( f \) of even periods obtained in the lemma (because we apply even iterates of \( f \) to the points \( a \) and \( b \)) to build the basic tower of periodic points of \( f \) associated with \( P \) and ignore all these periodic points \( p^*_m \)'s of odd periods.

**Proof.** The following proof is similar to that of Lemma 5. Assume that \( \min P < a < b < d \) or \( d \leq a < b \leq v \). Since \( f^{2k}(a) \in \{z_0, f(z_0), f(d)\} \) and \( 2n \geq 2k \), we have \( f^{2n}(a) \in \{z_0, f(z_0), f(d)\} \).

Suppose \( \min P < a < b < d \).
Then \( \min P < x < d \) for all \( x \) in \([a, b]\). Since \( f^{2n}(a) - a \geq z_0 - a > 0 \) and \( f^{2n}(b) - b = \min P - b < 0 \), the point \( c_{2n} = \min \{a \leq x \leq b : f^{2n}(x) = x\} \) exists. Since \( f^{2k}(a) \in \{z_0, f(z_0), f(d)\} \), \( 2n \geq 2k \) and \( f^{2n}(c_{2n}) = c_{2n} \), it follows from Lemma 3 that, for each \( i \geq 1 \), the point \( c_{2n+2i} = \min \{a \leq x \leq b : f^{2n+2i}(x) = x\} \) exists.

Furthermore, for each \( i \geq 0 \), since \( f^{2n+2i}(a) - d \geq z_0 - d > 0 \) and \( f^{2n+2i}(c_{2n+2i}) - d = c_{2n+2i} - d < 0 \), the point \( \mu_{n+i} = \min \{a \leq x \leq c_{2n+2i} : f^{2n+2i}(x) = d\} \) exists and is \( < c_{2n+2i} \).

On the other hand, since \( f^{2n+2i}(a, \mu_{n+i}) \supset [d, z_0 \supset \{v\} \), there is a point \( \nu_{n+i} \) in \([a, \mu_{n+i}] \) such that \( f^{2n+2i}(\nu_{n+i}) = v \). Since \( f^{2n+2i+2}(a) - a \geq z_0 - a > 0 \) and \( f^{2n+2i+2}(\nu_{n+i}) - \nu_{n+i} = f^2(v) - \nu_{n+i} = \min P - \nu_{n+i} < 0 \), the point

\[
c_{2n+2i+2} = \min \{a \leq x \leq \nu_{n+i} : f^{2n+2i+2}(x) = x\}
\]

exists and is \( < \nu_{n+i} < \mu_{n+i} \).

This, combined with the above, implies that

\[
\min P < a < \ldots < \mu_{n+2} < c_{2n+4} < \mu_{n+1} < c_{2n+2} < \mu_n < c_{2n} < b < d.
\]

Suppose \( d \leq a < b \leq v \). Then \( d \leq x \leq v \) for all \( x \) in \([a, b]\).

Since \( f^{2n}([a, b]) \supset [d, v] \), the point \( \mu_n = \min \{a \leq x \leq b : f^{2n}(x) = d\} \) exists.
Since \( f^{2n}(a) - a \geq z_0 - a > 0 \) and \( f^{2n}(\mu_n) = d - \mu_n < 0 \), the point

\[
c_{2n} = \min \{a \leq x \leq \mu_n : f^{2n}(x) = x\}
\]

exists and is \( < \mu_n \).

Since \( f^{2n}([a, c_{2n}]) \supset [c_{2n}, z_0 \supset \{v\} \), there is a point \( \nu_n \) in \([a, c_{2n}] \) such that

\[
f^{2n}(\nu_n) = v.
\]

Since \( f^{2n+2}(a) - a \geq z_0 - a > 0 \) and \( f^{2n+2}(\mu_{n+1}) = d - \mu_{n+1} < 0 \), the point \( \mu_{n+1} = \min \{a \leq x \leq \nu_n : f^{2n+2}(x) = x\} \) exists and is \( < \nu_n < c_{2n} \).

By induction, we have

\[
d \leq a < \ldots < c_{2n+4} < \mu_{n+2} < c_{2n+2} < \mu_{n+1} < c_{2n} < \mu_n < b \leq v.
\]

Note that we have assumed that \( \min P < a < b < d \) or \( d < a < b \leq v \). Now, we want to find the least periods of \( c_{2n+2i} \)'s with respect to \( f \). For each \( i \geq 0 \), let \( \ell(c_{2n+2i}) \) denote the least period of \( c_{2n+2i} \) with respect to \( f \).

Suppose \( i \geq 0 \) and \( n+i \geq 2k \) and let \( r = (2n+2i)/\ell(c_{2n+2i}) \).

If \( r \geq 3 \) is odd, then \( r-1 \geq 2 \) is even. Since \( (r-1)\ell(c_{2n+2i}) = (r-1)[(2n + 2i)/r] \geq 2[(r-1)/r] \cdot 2k \geq 2k \), we can apply Lemma 3 with

\[
f^{2k}(a) \in \{z_0, f(z_0), f(d)\} \quad \text{and} \quad f^{(r-1)\ell(c_{2n+2i})}(c_{2n+2i}) = c_{2n+2i}.
\]

If \( r \geq 4 \) is even, then \( r-2 \geq 2 \) is even. Since \( (r-2)\ell(c_{2n+2i}) = (r-2)[(2n + 2i)/r] \geq 2[(r-2)/r] \cdot 2k \geq 2k \), we can apply Lemma 3 with

\[
f^{2k}(a) \in \{z_0, f(z_0), f(d)\} \quad \text{and} \quad f^{(r-2)\ell(c_{2n+2i})}(c_{2n+2i}) = c_{2n+2i}.
\]
In either case, we obtain a periodic point \( w_{2n+2i} \) of \( f \) in \([a, c_{2n+2i}]\) such that \( f^{2n+2i}(w_{2n+2i}) = w_{2n+2i} \). This contradicts the minimality of \( c_{2n+2i} \) in \([a, b]\). So, if \( r > 2 \), then \( \ell(c_{2n+2i}) = 2n + 2i \).

If \( r = 2 \) and \( \ell(c_{2n+2i}) = (n + i \geq 2k) \) is even, then by applying Lemma 3 with \( f^{2k}(a) \in \{z_0, f(z_0), f(d)\} \) and \( f^{\ell(c_{2n+2i})}(c_{2n+2i}) = c_{2n+2i} \), we obtain the same contradiction. So, in this case, \( \ell(c_{2n+2i}) = 2n + 2i \).

Consequently, if \( i \geq 0 \) and \( n + i \) is an even integer such that \( n + i \geq 2k \), then since we have either \( r > 2 \) or \( r = 2 \), it follows from the above that \( c_{2n+2i} \) is a period-\((2n + 2i)\) point of \( f \).

On the other hand, suppose \( i \geq 0 \) and \( n + i \) is an odd integer such that \( n + i \geq 2k \). If \( r > 2 \), then it follows form the above that \( c_{2n+2i} \) is a period-\((2n + 2i)\) point of \( f \). If \( r = 2 \), then \( \ell(x_{2n+2i}) = n + i \). So, if \( i \geq 0 \) and \( n + i \) is an odd integer such that \( n + i \geq 2k \), then \( c_{2n+2i} \) is either a period-\((2n + 2i)\) point of \( f \) or a period-\((n + i)\) point of \( f \).

Suppose \( i \geq 0 \) and, \( n + i \) is odd and \( \geq 2k \) and \( c_{2n+2k} \) is a period-\((n + i)\) point of \( f \). Since \( n + i \geq 2k \) and \( f^{2k}(a) \in \{z_0, f(z_0), f(d)\} \), we have \( f^{(n+i)}(a) \in \{z_0, f(z_0), f(d)\} \). Since we have assumed that \( \min P < a < b < v \) or \( d \leq a < b \leq v \), we have

\[
f^{n+i}([a, c_{2n+2i}]) \supset [f^{n+i}(c_{2n+2i}), f^{n+i}(a)] \supset [c_{2n+2i}, z_0] \supset [v, z_0].
\]

Consequently, we have \( f^{n+i+2}([a, c_{2n+2i}]) \supset f^{2}([v, z_0]) \supset [\min P, z_0] \). Since \( f^{2}([\min P, z_0]) \supset f^{2}([v, z_0]) \supset [\min P, z_0] \), we obtain that,

\[
f^{m}([f^{n+i-2}([\min P, z_0])] \supset [\min P, z_0].
\]

Let \( \delta_{2n+2i} \) be a point in \([a, c_{2n+2i}]\) such that \( f^{2n+2i}(\delta_{2n+2i}) = \min P \). It is clear that \( a < \delta_{2n+2i} < c_{2n+2i} < b \). Since \( f^{2n+2i}(a) - a \geq z_0 - a > 0 \) and \( f^{2n+2i}(\delta_{2n+2i}) - \delta_{2n+2i} = \min P - \delta_{2n+2i} < 0 \), we see that there exists a point \( w_{2n+2i} \) in \([a, \delta_{2n+2i}]\) such that \( f^{2n+2i}(w_{2n+2i}) = w_{2n+2i} \). Since \( a < w_{2n+2i} < \delta_{2n+2i} < c_{2n+2i} \) and \( f^{2n+2i}(w_{2n+2i}) = w_{2n+2i} \), this contradicts the minimality of \( c_{2n+2i} \) in \([a, b]\). Therefore, the point \( c_{2n+2i} \) can not be a period-\((n + i)\) point of \( f \) and must be a period-\((2n + 2i)\) point of \( f \).

In summary, we have shown that, for each \( i \geq 1 \),

- if \( n + i \) is even and \( \geq 2k \), then the point \( c_{2n+2i} \) is a period-\((2n + 2i)\) point of \( f \);
- if \( n + i \) is odd and \( \geq \max\{2k, m+2\} \), then the point \( c_{2n+2i} \) is a period-\((2n+2i)\) point of \( f \).

Now suppose \( v < a < b < z_0 \).
In this case, let $\mu_n = \min \{ a \leq x \leq b : f^{2n}(x) = d \}$. Then $f^{2n}([a, \mu_n]) \supset [d, z_0] \supset \{ v \}$. Let $\nu_n$ be a point in $[a, \mu_n]$ such that $f^{2n}(\nu_n) = v$. Since $f^{2n+2}([a, \nu_n]) \supset [\min P, z_0] \supset \{ d \}$, the point

$$
\mu_{n+1} = \min \{ a \leq x \leq \nu_n : f^{2n+2}(x) = d \} = \min \{ a \leq x \leq b : f^{2n+2}(x) = d \}
$$

exists and is $\nu_n < \mu_n$. Inductively, for each $i \geq 2$, the point

$$
\mu_{n+i} = \min \{ a \leq x \leq \mu_{n+i-1} : f^{2n+2i}(x) = d \} = \min \{ a \leq x < b : f^{2n+2i}(x) = d \}
$$

exists and $v < a < \cdots < \mu_{n+3} < \mu_{n+2} < \mu_{n+1} < \mu_n < b < z_0$.

On the other hand, since $f^{2n}(a) - a \geq z_0 - a > 0$ and $f^{2n}(b) - b = \min P - b < 0$, the point

$$
c_{2n} = \min \{ a \leq x \leq b : f^{2n}(x) = x \}
$$

exists. Therefore, we can apply Lemma 3 with

$$
f^{2k}(a) \in \{ z_0, f(z_0), f(d) \} \quad \text{and} \quad f^{2n}(c_{2n}) = c_{2n}
$$

to obtain that, for each $i \geq 0$, the point

$$
c_{2n+2i} = \min \{ a \leq x \leq b : f^{2n+2i}(x) = x \}
$$

exists and $v < a < \cdots < c_{2n+6} < c_{2n+4} < c_{2n+2} < c_{2n} < b < z_0$.

Furthermore, for each $i \geq 0$, since

$$
f^{2n+2i}(a) - a \geq z_0 - a > 0 \quad \text{and} \quad f^{2n+2i}(\mu_{n+i}) - \mu_{n+i} = d - \mu_{n+i} < 0,
$$

we have $c_{2n+2i} = \min \{ a \leq x < b : f^{2n+2i}(x) = x \} = \min \{ a \leq x \leq \mu_{n+i} : f^{2n+2i}(x) = x \} < \mu_{n+i}$.

Now we find the least periods of $c_{2n+2i}$'s with respect to $f$. By following the arguments as those on the interval $[\min P, d]$ above, we can obtain that, for each $i \geq 0$,

if $n + i$ is even and $\geq 2k$, then $c_{2n+2i}$ is a period-$2n + 2i$ point of $f$,

if $n + i$ is odd and $\geq 2k$, then $c_{2n+2i}$ is either a period-$2n + 2i$ or a period-$(n + i)$ point of $f$.

Suppose $i \geq 0$, and $n + i$ is odd and $\geq m + 2k$ and $c_{2n+2i}$ is a period-$(n + i)$ point of $f$.

If the points $f^2(c_{2n+2i}), f^4(c_{2n+2i}), \cdots, f^{n+i-1}(c_{2n+2i})$ are all $\geq v$, then, by the fact that $f^2(x) < x$ for all $v < x < z_0$, we have

$$
d < v < f^{n+i-1}(c_{2n+2i}) < \cdots < f^4(c_{2n+2i}) < f^2(c_{2n+2i}) < c_{2n+2i} < z_0.
$$

By the fact that $f(x) > z \geq z_0$ for all $d < x < z_0$, we obtain that $c_{2n+2i} = f(f^{n+i-1}(c_{2n+2i})) > z_0 > c_{2n+2i}$ which is a contradiction. Therefore, there is an integer $s$ such that $1 \leq s \leq
(n + i - 1)/2 and \( f^{2s}(c_{2n+2i}) \leq v \). Note that \( f^2([\min P, z_0]) \supset f^2([v, z_0]) \supset [\min P, z_0] \). We have two cases to consider:

(i) If \( 2s < 2k \), and \( n + i \) is odd and \( \geq m + 2k \) \((\geq m + 2s + 2)\), then

\[
\begin{aligned}
& n + i - m \geq 2k \geq 2s + 2 \\
\Rightarrow & f^{n+i+m+2s+2}([a, c_{2n+2i}]) \supset f^{m+2s+2}([c_{2n+2i}, z_0]) \supset f^{m+2}([v, z_0]) \supset [\min P, z_0]. \\
\Rightarrow & f^{2n+2i}([a, c_{2n+2i}]) \supset f^{n+i-m-2s-2}([\min P, z_0]) \supset [\min P, z_0].
\end{aligned}
\]

(ii) If \( 2k \leq 2s \) \((\leq n + i - 1)\), then

\[
\begin{aligned}
& 2n + 2i - 2s - 2 = (n + i + 1) + [(n + i - 1) - 2s] - 2 \geq n + i - 1 \geq 2s \geq 2k > 0 \\
\Rightarrow & f^{2s+2}([a, c_{2n+2i}]) \supset f^2([f^{2s}(c_{2n+2i}), z_0]) \supset f^2([v, z_0]) \supset [\min P, z_0]. \\
\Rightarrow & f^{2n+2i}([a, c_{2n+2i}]) \supset f^{2n+2i-2s-2}([f^{2s+2}(a, c_{2n+2i})]) \supset f^{2n+2i-2s-2}([\min P, z_0]) \supset [\min P, z_0].
\end{aligned}
\]

By combining the above two results, we obtain that, for each \( i \geq 0 \) such that \( n + i \) is odd and \( \geq m + 2k \), there exists a point \( \delta_{2n+2i} \) in \([a, c_{2n+2i}]\) such that \( f^{2n+2i}(\delta_{2n+2i}) = \min P \). Since \( f^{2n+2i}(a) - a \geq z_0 - a > 0 \) and \( f^{2n+2i}(\delta_{2n+2i}) - \delta_{2n+2i} = \min P - \delta_{2n+2i} < 0 \), there is a point \( w_{2n+2i} \) in \([a, \delta_{2n+2i}]\) such that \( f^{2n+2i}(w_{2n+2i}) = w_{2n+2i} \). Since \( a < w_{2n+2i} < \delta_{2n+2i} < c_{2n+2i} < b \), this contradicts the minimality of \( c_{2n+2i} \) in \([a, b]\). Therefore, for each \( i \geq 0 \) such that \( n + i \) is odd and \( \geq m + 2k \), the point \( c_{2n+2i} \) is a period-\((2n + 2i)\) point of \( f \). Since we have known that, for each \( i \geq 0 \) such that \( n + i \) is even and \( \geq 2k \), the point \( c_{2n+2i} \) is a period-\((2n + 2i)\) point of \( f \), this proves (1). (2) can be proved similarly. \( \square \)

We now divide the interval \([\min P, z_0]\) into the 5 subintervals: \([\min P, d]\), \([d, u_1]\), \([u_1, v]\), \([v, u'_1]\) and \([u'_1, z_0]\). On each such subinterval, we can apply Lemmas 5 & 6 in various combinations to build various towers of periodic points of \( f \) associated with \( P \). As an example, we shall apply Lemmas 5 & 6 in a recursive way to build the so-called basic tower of periodic points of \( f \) associated with \( P \).

In the sequel, for the sake of clarity, we shall use the following notations:

- \( x' \) to denote points obtained by taking the maximum of a set;
- \( \bar{x}, x, \hat{x} \) and \( \bar{x} \) respectively to denote points (except endpoints) in the intervals \([\min P, d]\), \([d, u_1]\), \([u_1, v]\), \([v, \bar{u}_0]\) \((\subset [v, u'_1])\), \([\bar{u}'_1, z_0]\) respectively;
- \( \mu_{m,...} \) and \( u... \) respectively to denote \( d \)-points in \([\min P, d] \cup (u_1, v] \cup [v, \bar{u}'_1] \) and \((d, u_1] \cup [\bar{u}'_1, z_0]\) respectively;
- \( p \) and \( q \) respectively to denote periodic points of odd periods of \( f \) obtained by taking the minimum and maximum of a set respectively;
- \( c \) to denote a periodic point of even period of \( f \) obtained by taking the minimum of a set.

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§1. The first layer of the basic tower of periodic points of \( f \) associated with \( P \).

To start with, we shall find a monotonic sequence of periodic points of \( f \) (and a monotonic sequence of \( d \)-points, a \( d \)-point is a point whose orbit under \( f \) contains the point \( d \)) on each of the 3 intervals: \( [\min P, d], [d, u_1], [\bar{\hat{u}}_0, v] \subset [u_1, v] \), \( [v, \bar{u}_0] \subset [v, \bar{\hat{u}}_1] \) and \( [\bar{\hat{u}}_1, z_0] \) which comprise the first layer of the basic tower of periodic points of \( f \) associated with \( P \). Then on each of the 3 intervals: \( [\min P, d], [\bar{\hat{u}}_0, v] \subset [u_1, v] \) and \( [v, \bar{u}_0] \subset [v, \bar{\hat{u}}_1] \), where we apply odd iterates of \( f \) to the endpoints, we shall apply Lemma 5 once and Lemma 6 twice successively, and on each of the 2 intervals: \( [d, u_1] \) and \( [\bar{\hat{u}}_1, z_0] \), where we apply even iterates of \( f \) to the endpoints, we shall apply Lemma 6 once and Lemma 5 twice successively, to continue the basic tower-building process to the second and higher layers.

1.1 On the existence of periodic points of \( f \) of all odd periods \( \geq m \) in \( [\min P, d] \).

In this case, we clearly have

\[
 f(d) \in \{ z_0, f(z_0), f(d) \} \quad \text{and} \quad f^m(\min P) = \min P
\]

and trivially

all periodic points of \( f \) in \( [\min P, d] \) with odd periods have least periods \( \geq 1 \).

Therefore, we can apply Lemma 5(2) to obtain that, for each \( i \geq 0 \), the points

\[
 \tilde{q}_{m+2i} = \max \{ \min P \leq x \leq d : f^{m+2i}(x) = x \}
\]

and

\[
 \tilde{\mu}'_{m,i} = \max \{ \min P \leq x \leq d : f^{m+2i}(x) = d \}
\]

exist and \( \tilde{q}_{m+2i} \) is a period-(\( m+2i \)) point of \( f \). Furthermore, we have

\[
 \min P \leq \tilde{q}_m < \tilde{\mu}'_{m,0} < \tilde{q}_{m+2} < \tilde{\mu}'_{m,1} < \tilde{q}_{m+4} < \tilde{\mu}'_{m,2} < \tilde{q}_{m+6} < \tilde{\mu}'_{m,3} < \cdots < d.
\]

Note the relationship between the subscript of \( \tilde{\mu}'_{m,i} \) and the superscript of \( f^{m+2i} \) in the definition of \( \tilde{\mu}'_{m,i} \): We need these points \( \tilde{\mu}'_{m,i} \)'s later to continue the basic tower-building process to the next (second) layer.

**Remark 3.** Note that we can apply Lemma 5(2) with

\[
 f(d) \in \{ z_0, f(z_0), f(d) \} \quad \text{and} \quad f^m(\min P) = \min P
\]

to obtain that, for each \( i \geq 1 \), the point \( \tilde{\epsilon}^{*}_{2m+2i} = \max \{ \min P \leq x \leq d : f^{2m+2i}(x) = x \} \) exists and is a period-(\( 2m+2i \)) point of \( f \) in \([\min P, d]\) and

\[
 \min P < \tilde{\epsilon}^{*}_{2m+2} < \tilde{\epsilon}^{*}_{2m+4} < \tilde{\epsilon}^{*}_{2m+6} < \tilde{\epsilon}^{*}_{2m+8} < \cdots < d.
\]

As for the point \( \tilde{\epsilon}^{*}_{2m} = \max \{ \min P \leq x \leq d : f^{2m}(x) = x \} \), it may be a period-\( m \) point of \( f \).
For example, let $g : [0, 1] \rightarrow [0, 1]$ be the continuous map defined by putting
(i) $g(x) = x + 1/2$ and (ii) $g(x) = 2 - 2x$ and let $P = \{0, 1/2, 1\}$ be the unique
period-3 orbit of $g$. Then $\min P = 0$, $d = 1/6$, $v = 1/2$, $z = 2/3$ and $g(v) = 1.$
$g$ has exactly two period-6 orbits and they all lie in the interval $[1/6, 1]$.

In this case, $m = 3$ and $\tilde{c}_0^{2m}$ is a period-3, but not a period-6 point of $g$.

Note that, for each $i \geq 1$, the orbits $O_f(\tilde{c}_0^{2m}_i)$’s of the periodic points $\tilde{c}_0^{2m}_i$’s are disjoint
from the orbits $O_f(\tilde{c}_2i)$’s of $\tilde{c}_2i$’s (defined in the main text) and the orbits $O_f(\tilde{c}_2i)$’s of $\tilde{c}_2i$’s
(defined below) because all $O_f(\tilde{c}_2i)$’s and $O_f(\tilde{c}_2i)$’s are contained in the interval $(d, z_0)$ which
is disjoint from the interval $[\min P, d]$. However, these periodic points $\tilde{c}_0^{2m}_i$’s of $f$ of even
periods are interspersed with those periodic points $\tilde{q}_m+2i$’s of $f$ of odd periods and, to make
things simple, are not counted in the first layer of the basic tower of periodic points of $f$
associated with $P$.

**Remark 4.** On the other hand, since $f^{2m}(\min P) = \min P$ and $f^{2m}(\mu_{m,0}^{'}) = f^m(d) = f(z_0) \geq z_0$, the point $\tilde{\xi}_m = \max \{ \min P \leq x \leq \mu_{m,0}^{'} : f^{2m}(x) = d \}$ exists and $\min P < \tilde{\xi}_m < \mu_{m,0}^{'} < d$. By applying Lemma 6(2) with

$$f^{m+1}(\mu_{m,0}^{'}) = f(d) \in \{ z_0, f(z_0), f(d) \} \text{ and } f^{2m}(\min P) = \min P$$

and arguments similar to Remark 2, we obtain infinitely many more periodic points of $f$ with
even periods $> 2m$ and with odd periods $> 3m$ in the interval $[\tilde{\xi}_m, \mu_{m,0}^{'}] (\subset [\min P, \mu_{m,0}^{'}])$.

Since $f^{3m}(\min P) = \min P$ and $f^{2m+1}(\tilde{\xi}_m) = f(f^{2m}(\tilde{\xi}_m)) = f(d)$, the point $\tilde{\xi}_m = \max \{ \min P \leq x \leq \tilde{\xi}_m : f^{3m}(x) = d \}$ exists and $\min P < \tilde{\xi}_m < \tilde{\xi}_m < \mu_{m,0}^{'}, \mu_{m,0}^{'}, < d$. By applying Lemma 5(2) with

$$f^{2m+1}(\tilde{\xi}_m) = f(d) \text{ and } f^{3m}(\min P) = \min P,$$

we obtain infinitely many more periodic points of $f$ with odd periods $> 3m$ and with even
periods $> 4m$ in the interval $[\tilde{\xi}_m, \tilde{\xi}_m] (\subset [\min P, \tilde{\xi}_m])$.

Inductively, for each $k \geq 2$, the point $\tilde{\xi}_k = \max \{ \min P \leq x \leq \tilde{\xi}_k : f^{km}(x) = d \}$
exists and

$$\min P < \cdots < \tilde{\xi}_k < \cdots < \tilde{\xi}_3 < \tilde{\xi}_2 < \mu_{m,0}^{'}, < d.$$ 

By applying Lemma 5(2) or, Lemma 6(2) and Remark 2 appropriately, we obtain that, for
each $k \geq 2$, there are infinitely many more periodic points of $f$ with even periods and with
odd periods in the interval $[\tilde{\xi}_k, \tilde{\xi}_k] (\subset [\min P, \tilde{\xi}_k])$.

However, to make things simple, we choose to ignore these periodic points
of $f$ obtained in the interval $\min P, \mu_{m,0}^{'}$ as described in Remarks 3 & 4 above
and keep only the periodic points $\tilde{q}_m+2i$’s of $f$ with odd periods in the interval $[\mu_{m,0}^{'}, d]$ in the first layer and use the points $\mu_{m,i}^{'}$’s to continue the process into the
next layer of the basic tower of periodic points of $f$ associated with $P$.

**1.2 On the existence of periodic points of $f$ of all even periods $\geq 2$ in $[d, v]$**.
This is done in Part (c) in the main text. We can also argue as follows: Since we have two points \( d < v \) such that \( f^2(d) = z_0 \) and \( f^2(v) = \min P \) so that Lemma 6(1) can be applied to obtain that, for each \( n \geq 1 \), the points

\[ u_n = \min \{ d \leq x \leq v : f^{2n}(x) = d \} \quad \text{and} \quad c_{2n} = \min \{ d \leq x \leq v : f^{2n}(x) = x \} \]

exist and \( d < \cdots < c_0 < u_3 < c_4 < u_2 < c_2 < u_1 < v \). Note the relationship between the subscript of \( u_n \) and the superscript of \( f^{2n} \) in the definition of \( u_n \). We need these points \( u_n \)'s later to continue the basic tower-building process to the next (second) layer.

By Lemma 4(1), we obtain that, for each \( n \geq 1 \), the point

\[ c_{2n} = \min \{ d \leq x \leq v : f^{2n}(x) = x \} \]

is a period-(2n) point of \( f \) in \([d, u_1] (\subset [d, v])\).

**Remark 5.** On the interval \([d, u_1] \), let \( v \) be a point such that \( f^2(v) = v \). Then we have

\[ f^1(d) = f(d) \quad \text{and} \quad f^{m+2}(v) = f^m(v) = \min P. \]

On the other hand, trivially, all periodic points of \( f \) in \([d, z_0] \) of odd periods have least periods \( \geq 1 \). It follows from Lemma 5(1) that, for each \( i \geq 1 \), the point

\[ p_{m+2i} = \min \{ d \leq x \leq v : f^{m+2i}(x) = x \} \]

exists and is a period-(m+2i) point of \( f \) and \( d < \cdots < p_{m+8} < p_{m+6} < p_{m+4} < p_{m+2} < v < u_1 \). These periodic points \( p_{m+2i} \)'s of odd periods are interspersed with the periodic points \( c_{2i} \)'s of \( f \) of even periods and, to make things simple, are not counted in the first layer of the basic tower of periodic points of \( f \) associated with \( P \).

### 1.3 On the existence of periodic points of \( f \) of all odd periods \( \geq m+2 \) in \([\bar{u}', v] \) \(([u_1, v] \subset [d, v])\).

We have shown this result in the main text. Here we apply Lemma 5(1) to obtain the same result. On the interval \([u_1, v] \), we consider the point \( \bar{u}' = \max \{ u_1 \leq x \leq v : f^2(x) = d \} \) instead of the point \( u_1 \) (for a reason which will become apparent below). Then, we have

\[ f^3(\bar{u}') = f(d) \in \{ z_0, f(z_0), f(d) \} \quad \text{and} \quad f^{m+2}(v) = f^m(f^2(v)) = f^m(\min P) = \min P. \]

On the other hand, it follows from the fact

\[ f(x) > z \quad (\geq z_0 > f^2(x)) \quad \text{for all} \quad d < x < z_0, \quad (***) \]

that \( f \) has no fixed points in \((\bar{u}', v) \). So,

all periodic points of \( f \) in \([\bar{u}', v] \) with odd periods have least periods \( \geq 3 \).

By applying Lemma 5(1) with

\[ f^3(\bar{u}') = f(d) \in \{ z_0, f(z_0), f(d) \} \quad \text{and} \quad f^{m+2}(v) = \min P, \]

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we obtain that, for each $i \geq 0$, the points

$$\hat{p}_{m+2+2i} = \min \{ \hat{u}_0' \leq x \leq v : f^{m+2+2i}(x) = x \} \quad \text{and} \quad \hat{\mu}_{m,1+i} = \min \{ \hat{u}_0' \leq x \leq v : f^{m+2+2i}(x) = d \}$$

exist and $\hat{p}_{m+2+2i}$ is a period-$(m+2+2i)$ point of $f$. Furthermore, it follows from the choice of $\hat{u}_0$ that

$$f^2(x) < d \text{ for all } \hat{u}_0 < x < v.$$ 

Consequently, we have

$$u_1 \leq \hat{u}_0' < \cdots < \hat{p}_{m+6} < \hat{\mu}_{m,3} < \hat{p}_{m+4} < \hat{\mu}_{m,2} < \hat{p}_{m+2} < \hat{\mu}_{m,1} < v.$$ 

Note the relationship between the subscript of $\hat{\mu}_{m,1+i}$ and the superscript of $f^{m+2+2i}$ in the definition of $\hat{\mu}_{m,1+i}$. We need these points $\hat{\mu}_{m,1+i}$'s later to continue the basic tower-building process to the second layer.

**Remark 6.** Note that, in the interval $[\hat{u}_0, v]$ ($\subset [u_1, v]$), we have $f^3(\hat{u}_0') = f(d) \geq z_0$ and $f^{m+2}(v) = \min P$ (compare with $f^m(\min P)$ and $f(d)$ on $[\min P, d]$). As discussed in Remarks 3 & 4 above, we can apply Lemma 5(1) or, Lemma 6(1) and arguments similar to Remark 2 appropriately on the interval $[\hat{u}_0, v]$ to obtain infinitely many more periodic points of $f$ of even periods and of odd periods in $[\hat{u}_0', v]$. As before, to make things simple, we choose to ignore all of these periodic points and keep only those periodic points $\hat{p}_{m+2+2i}$'s of $f$ with odd periods (in the first layer of the basic tower) and the points $\hat{\mu}_{m,1+i}$'s to continue the process of building the basic tower of periodic points of $f$ associated with $P$ into the second layer.

### 1.4 On the existence of periodic points of $f$ of all odd periods $\geq m + 2$ in $[v, \hat{u}_0]$ ($\subset [v, \hat{u}_0'] \subset [v, z_0]$).

On the interval $[v, \hat{u}_0']$, we consider the point $\hat{u}_0 = \min \{ v \leq x \leq z_0 : f^2(x) = d \}$ ($\leq \hat{u}_1'$) instead of the point $\hat{u}_1'$ (for a reason which will become apparent below). Then, we have

$$f^3(\hat{u}_0) = f(d) \quad \text{and} \quad f^{m+2}(v) = f^m(f^2(v)) = f^m(\min P) = \min P.$$ 

On the other hand, it follows from the fact

$$f(x) > z \quad (\geq z_0 > f^2(x)) \quad \text{for all } d < x < z_0,$$

that $f$ has no fixed points in $(v, \hat{u}_0)$. So,

all periodic points of $f$ in $[v, \hat{u}_0]$ with odd periods have least periods $\geq 3$.

By applying Lemma 5(2) with

$$f^3(\hat{u}_0) = f(d) \in \{ z_0, f(z_0), f(d) \} \quad \text{and} \quad f^{m+2}(v) = \min P,$$

we obtain that, for each $i \geq 0$, the points

$$\hat{q}_{m+2+2i} = \max \{ v \leq x \leq \hat{u}_0 : f^{m+2+2i}(x) = x \}$$

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and

\[ \mu'_{m,1+i} = \max \{ v \leq x \leq \hat{u}_0 : f^{m+2+2i}(x) = d \} \]

exist and \( \hat{q}_{m+2+2i} \) is a period-\((m + 2 + 2i)\) point of \( f \). Furthermore, it follows from the choice of \( \hat{u}_0 \) that

\[ f^2(x) < d \text{ for all } v < x < \hat{u}_0. \]

Consequently, we have

\[ v < \mu'_{m,1} < \hat{q}_{m+2} < \mu'_{m,2} < \hat{q}_{m+4} < \mu'_{m,3} < \hat{q}_{m+6} < \cdots < \hat{u}_0. \]

Note the relationship between the subscript of \( \mu'_{m,1+i} \) and the superscript of \( f^{m+2+2i} \) in the definition of \( \mu'_{m,1+i} \). We need these points \( \mu'_{m,1+i} \)'s later to continue the basic tower-building process to the second layer.

**Remark 7.** Note that, on the interval \([v, \hat{u}_0]\), we have

\[ f^{m+2}(v) = \min P \text{ and } f^3(\hat{u}_0) = f(d) \geq z \geq z_0 \] (compare with \( f^m(\min P) \) and \( f(d) \) on \([\min P, d]\) and also with \( f^3(\hat{u}_0) \) and \( f^{m+2}(v) \) on \([\hat{u}_0, v] \) \((\subset [u_1, v])\)). As discussed in Remarks 3 & 4 above, by applying Lemmas 5(2) or, 6(2) and arguments similar to Remark 2 appropriately on the interval \([v, \hat{u}_0]\), we can obtain infinitely many more periodic points of \( f \) of even periods and of odd periods. However, to make things simple, we choose to ignore all of these periodic points and keep only those periodic points \( \hat{q}_{m+2+2i} \)'s of \( f \) with odd periods (in the first layer of the basic tower) and the points \( \mu'_{m,1+i} \)'s to continue the process of building the basic tower of periodic points of \( f \) associated with \( P \) into the next layer.

**1.5 On the existence of periodic points of \( f \) of all even periods \( \geq 4 \) in \([\bar{u}_1, z_0] \) \((\subset [v, z_0])\).**

In the main text, we have found points \((\min P <) d < \cdots < u_2 < u_1 < v \) such that,

\[ \text{for each } n \geq 1, \; u_n = \min \{ d \leq x \leq v : f^{2n}(x) = d \} \]

and have shown the existence of periodic points of \( f \) of all even periods in the interval \([d, u_1]\) by showing that, for each \( n \geq 1 \), the point

\[ c_{2n} = \min \{ d \leq x \leq v : f^{2n}(x) = x \} \]

is a period-\((2n)\) point of \( f \). ‘Symmetrically’, we define, as in Lemma 4, points \((v <) \hat{u}'_1 < \hat{u}'_2 < \cdots < z_0 \) by putting, for each \( n \geq 1 \),

\[ \hat{u}'_n = \max \{ v \leq x \leq z_0 : f^{2n}(x) = d \}. \]

However, to find new periodic points of \( f \) of even periods in \([v, z_0] \), we cannot 'symmetrically' define \( c'_{2n} \) as

\[ c'_{2n} = \max \{ v \leq x \leq z_0 : f^{2n}(x) = x \} \]

because in this way we only get \( z_0 \) and no new periodic points. So, we use a different strategy:
For each \( n \geq 1 \), since \( f^{2n+2}(\bar{u}'_n, \bar{u}'_{n+1}) \supset [d, z_0] \supset \{d\} \), the point \( \bar{\omega}^{*}_{n+1} = \min \{\bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{2n+2}(x) = d\} \geq \bar{u}'_n \) exists. So, \( d < f^{2n+2}(x) < z_0 \) for all \( \bar{u}'_n \leq x \leq \bar{\omega}^{*}_{n+1} \). This, combined with the following fact

\[
d < f^{2k}(x) < z_0 \quad \text{for all} \quad x \quad \text{in} \quad (d, u_n) \cup (\bar{u}'_n, z_0) \quad \text{and all} \quad 1 \leq k \leq n,
\]

implies that \( d < f^{2k}(x) < z_0 \) for all \( \bar{u}'_n \leq x \leq \bar{\omega}^{*}_{n+1} \) and all \( 1 \leq k \leq n + 1 \). Since \( f^{2n+2}(\bar{u}'_n) - \bar{u}'_n = z_0 - \bar{u}'_n > 0 \) and \( f^{2n+2}(\bar{\omega}^{*}_{n+1}) - \bar{\omega}^{*}_{n+1} = d - \bar{\omega}^{*}_{n+1} < 0 \), the point

\[
\bar{c}_{2n+2} = \min \{\bar{u}'_n \leq x \leq \bar{\omega}^{*}_{n+1} : f^{2n+2}(x) = x\} = \min \{\bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{2n+2}(x) = x\}
\]

exists. By Lemma 4(2), \( \bar{c}_{2n+2} \) is a period-(\(2n+2\)) point of \( f \) such that, for all \( \text{odd} \quad 1 \leq j \leq 2n + 1 \),

\[
d < f^{2n}(\bar{c}_{2n+2}) < f^{2n-2}(\bar{c}_{2n+2}) < \cdots < f^2(\bar{c}_{2n+2}) < \bar{c}_{2n+2} < z_0 \leq z < f^j(\bar{c}_{2n+2}).
\]

This establishes the existence of periodic points \( \bar{c}_4, \bar{c}_6, \bar{c}_8, \cdots \) of \( f \) of all even periods \( \geq 4 \) (best possible because \( f \) may have exactly 2 period-2 points which form a period-2 orbit, one in \([d,v]\) and the other is \(> z\)) in \([\bar{u}'_1, z_0]\) such that

\[
v < \bar{u}'_1 < \bar{c}_4 < \bar{u}'_2 < \bar{c}_6 < \bar{u}'_3 < \bar{c}_8 < \bar{u}'_4 < \cdots < z_0.
\]

**Remark 8.** On the interval \([\bar{u}'_1, z_0]\), let \( \bar{v} \) to be a point such that \( f^2(\bar{v}) = v \). Then we have

\[
f^1(\bar{z}_0) = f(\bar{z}_0) \in \{z_0, f(z_0), f(d)\} \quad \text{and} \quad f^{m+2}(\bar{v}) = \min P.
\]

Since, trivially, all periodic points of \( f \) in \([\bar{v}, z_0]\) of odd periods have least periods \( \geq 1 \), it follows from Lemma 5(2) that, for each \( i \geq 0 \), the point

\[
\bar{q}_{m+2+2i} = \max \{\bar{u}'_1 \leq x \leq z_0 : f^{m+2+2i}(x) = x\}
\]

exists and is a period-(\(m+2+2i\)) point of \( f \) and \( \bar{u}'_1 < \bar{v} < \bar{q}_{m+2} < \bar{q}_{m+4} < \bar{q}_{m+6} < \bar{q}_{m+8} < \cdots < z_0 \). These periodic points \( \bar{q}_{m+2+2i}' \) of \( f \) of odd periods are clearly interspersed with the periodic points \( \bar{c}_{2i+2}' \) of \( f \) of even periods and, to make things simple, are not counted in the first layer of the basic tower of periodic points of \( f \) associated with \( P \).

We can combined the results on the intervals \([d, u_1], [u_1, v]\) and \([v, \hat{u}_0] \subset [v, \hat{u}'_1]\) into one on the interval \([d, \hat{u}_0] \subset [d, \hat{u}'_1]\) as follows:

On the interval \([d, \hat{u}_0]\), we have \( f^2(d) = z_0, f^2(v) = \min P, f^2(\hat{u}_0) = d \) and

the graph of \( y = f^2(x) \) looks 'roughly' like a skew 'V' shape at points \( d, v, \hat{u}_0 \)

(meaning that the graph of \( y = f^2(x) \) passes through the 3 points \( d, z_0, (v, \min P), (\hat{u}_0, d) \) on the \( x-y \) plane with \( y \)-coordinates \( z_0, \min P, d \) respectively). Note that we choose the 3 points \( d, v, \hat{u}_0 \) instead of the 3 points \( d, v, z_0 \) for a reason which will become apparent later on.

By combining the results on the intervals \([d, u_1], [\hat{u}_0, v] \subset [u_1, v]\) and \([v, \hat{u}_0] \subset [v, \hat{u}'_1]\), we obtain that, with \( x \)-coordinates moving from point \( d \) via point \( v \) to point \( \hat{u}_0 \), for each \( i \geq 1, \)
This kind of skew \textit{V} shape phenomenon will be our basis for building recursively the basic tower of periodic points of \( f \) associated with \( P \) later on.

In summary, on each of the 5 intervals: \([\min P, d)\), \([d, u_1]\), \([\hat{u}_0', v)\) (\(\subset [u_1, v)\)), \([v, \hat{u}_0)\) (\(\subset [v, \hat{u}_1')\)), \([\hat{u}_1', z_0]\), we have obtained 2 monotonic sequences of points (one sequence of periodic points which comprises the same (first) layer of the basic tower and the other is a sequence of \( d \)-points which are needed to continue the basic tower-building process into the next (second) layer of the tower):

\[
\min P < \hat{q}_{m+2} < \hat{\mu}_{m,1} < \hat{q}_{m+4} < \hat{\mu}_{m,2} < \cdots < d < \cdots < c_4 < u_2 < c_2 < u_1 < \hat{u}_0' < \cdots
\]

\[
\hat{p}_{m+4} < \hat{\mu}_{m,2} < \hat{p}_{m+2} < \hat{\mu}_{m,1} < \hat{v} < \hat{\mu}_{m,1} < \hat{q}_{m+2} < \hat{\mu}_{m,2} < \hat{q}_{m+4} \cdots < \hat{u}_0
\]

\[
\leq \hat{u}_1' < \hat{c}_4 < \hat{u}_2' < \hat{c}_6 < \hat{u}_3' < \hat{c}_8 < \cdots < z_0
\]

such that, \( \hat{u}_0' = \max \{d \leq x \leq v : f^2(x) = d\} \) and \( \hat{u}_0 = \min \{v \leq x \leq z_0 : f^2(x) = d\} \) and, for each \( n \geq 1 \),

\[
u_n = \min \{d \leq x \leq v : f^{2n}(x) = d\} \quad \text{and} \quad \hat{\nu}_n = \max \{v \leq x \leq z_0 : f^{2n}(x) = d\},
\]

\[
d < f^{2k}(x) < z_0 \quad \text{for all} \quad 1 \leq k \leq n \quad \text{and all} \quad x \in (d, u_n) \cup (\hat{\nu}_n, z_0);
\]

\[
\hat{\mu}_{m,n-1} = \max \{\min P \leq x \leq d : f^{m+2n-2}(x) = d\},
\]

\[
\hat{\mu}_{m,n} = \min \{\hat{u}_0' \leq x \leq v : f^{m+2n}(x) = d\},
\]

\[
\hat{\mu}_{m,n}' = \max \{v \leq x \leq \hat{u}_0 : f^{m+2n}(x) = d\}
\]

and the point \( \hat{q}_{m+2n-2} = \max \{\min P \leq x \leq d : f^{m+2n-2}(x) = x\} \) is a period-\((m + 2n)\) point of \( f \), and on the interval \([d, \hat{u}_0]\) where the graph of \( y = f^2(x) \) looks like a skew \textit{V} shape on the three points \( d, v, \hat{u}_0 \), we have, with \( x \)-coordinates moving from point \( d \) via point \( v \) to point \( \hat{u}_0 \),

(a) (even periods) \( c_{2n} = \min \{d \leq x \leq v : f^{2n}(x) = x\} \) is a period-\((2n)\) point of \( f \);

(b) (odd periods) \( \hat{p}_{m+2n} = \min \{\hat{u}_0' \leq x \leq v : f^{m+2n}(x) = x\} \) is a period-\((m + 2n)\) point of \( f \);

(c) (odd periods) \( \hat{q}_{m+2n} = \max \{v \leq x \leq \hat{u}_0 : f^{m+2n}(x) = x\} \) is a period-\((m + 2n)\) point of \( f \),

and the point \( \hat{c}_{2n+2} = \min \{\hat{u}_n' \leq x \leq \hat{u}_{n+1}' : f^{2n+2}(x) = x\} \) is a period-\((2n + 2)\) point of \( f \).
Figure 1: The first layer of the basic tower of periodic points of $f$ associated with $P$, where $v$ is a point in $[\min P, f^{m-1}(\min P)]$ such that $f(v) = f^{m-1}(\min P)$; $z_0 = \min \{ v \leq x \leq f^{m-1}(\min P) : f^2(x) = x \}$; $d = \max \{ \min P \leq x \leq v : f^2(x) = z_0 \}$; and, for each $n \geq 1$,
\[
\begin{align*}
\tilde{q}_{m+2n} &= \max \{ \min P \leq x \leq d : f^{m+2n}(x) = x \}; \\
\tilde{\mu}_{m,n}' &= \max \{ \min P \leq x \leq d : f^{m+2n}(x) = d \}; \\
u_n &= \min \{ d \leq x \leq v : f^{2n}(x) = d \}; \\
c_{2n} &= \min \{ d \leq x \leq v : f^{2n}(x) = x \}; \\
\hat{u}_0' &= \max \{ d \leq x \leq v : f^2(x) = d \}; \\
\tilde{\mu}_{m,n} &= \min \{ \hat{u}_0' \leq x \leq v : f^{m+2n}(x) = d \}; \\
\tilde{p}_{m+2n} &= \min \{ \hat{u}_0' \leq x \leq v : f^{m+2n}(x) = x \}; \\
\hat{u}_0 &= \min \{ v \leq x \leq z_0 : f^2(x) = d \}; \\
\tilde{\mu}_{m,n}' &= \max \{ v \leq x \leq \hat{u}_0 : f^{m+2n}(x) = d \}; \\
\tilde{q}_{m+2n} &= \max \{ v \leq x \leq \hat{u}_0 : f^{m+2n}(x) = x \}; \\
\bar{u}_n &= \max \{ v \leq x \leq z_0 : f^{2n}(x) = d \}; \\
\bar{c}_{2n+2} &= \min \{ \bar{u}_n' \leq x \leq z_0 : f^{2n+2}(x) = x \}.
\end{align*}
\]
§2. The second layer of the basic tower of periodic points of \( f \) associated with \( P \).

For each \( n \geq 0 \), let (note the relationship between the subscript of \( \tilde{\mu}'_{m,n} \) and the superscript of \( f^{m+2n} \) in the definition of \( \tilde{\mu}'_{m,n} \))

\[
\tilde{\mu}'_{m,n} = \max \{ \min P \leq x \leq d : f^{m+2n}(x) = d \}.
\]

Let \( \tilde{\nu}_{m,n} \) be a point in \([\tilde{\mu}'_{m,n},d]\) such that \( f^{m+2n}(\tilde{\nu}_{m,n}) = v \). Then it turns out that \( \tilde{\mu}'_{m,n} < \tilde{\nu}_{m,n} < \tilde{\mu}'_{m,n+1} \). Furthermore, on the interval \([\tilde{\mu}'_{m,n},\tilde{\mu}'_{m,n+1}]\), we have

\[
f^{m+2n+2}(\tilde{\mu}'_{m,n}) = z_0, \quad f^{m+2n+2}(\tilde{\nu}_{m,n}) = \min P, \quad f^{m+2n+2}(\tilde{\mu}'_{m,n+1}) = d \quad \text{and}
\]

the graph of \( y = f^{m+2n+2}(x) \) looks 'roughly' like a skew 'V' shape at the 3 points \( \tilde{\mu}'_{m,n}, \tilde{\nu}_{m,n}, \tilde{\mu}'_{m,n+1} \) (meaning that the graph of \( y = f^{m+2n+2}(x) \) passes through the 3 points \((\tilde{\mu}'_{m,n},z_0)\), \((\tilde{\nu}_{m,n},\min P)\), \((\tilde{\mu}'_{m,n+1},d)\) on the \( x,y \) plane with \( y \)-coordinates \( z_0, \min P, d \) respectively) just like that of \( y = f^2(x) \) (at the points \( d, v, \hat{u}_0 \)) on the interval \([d, \hat{u}_0]\).

Now, for each \( n \geq 0 \) and all \( i \geq 1 \), let (the number 2 in the superscript \( (n,2) \) indicates the second layer)

\[
\tilde{\mu}'^{(n,2)}_{m+2n+2i} = \min \{ \tilde{\mu}'_{m,n} \leq x \leq \tilde{\nu}_{m,n} : f^{m+2n+2i}(x) = x \} \quad \text{and}
\]

\[
\tilde{\mu}_{m,n,i} = \min \{ \tilde{\mu}'_{m,n} \leq x \leq \tilde{\nu}_{m,n} : f^{m+2n+2i}(x) = d \}
\]

(note the relationship between the subscript of \( \tilde{\mu}_{m,n,i} \) and the superscript of \( f^{m+2n+2i} \) in the definition of \( \tilde{\mu}_{m,n,i} \). Here the \( n \) and \( i \) in the subscript of \( \tilde{\mu}_{m,n,i} \) indicate the \( i^{th} \) point of the sequence \( < \tilde{\mu}_{m,n,i} > \) in the \( n^{th} \) interval \([\tilde{\mu}'_{m,n},\tilde{\mu}'_{m,n+1}]\). We need these points \( \tilde{\mu}_{m,n,i} \)'s to continue the basic tower-building process into the third layer). By arguing as those in the proof of (c) in the main text, we obtain that

\[
\tilde{\mu}'_{m,n} < \cdots < \tilde{\mu}_{m,n,3} < \tilde{\mu}'^{(n,2)}_{m+2n+2i} < \tilde{\mu}_{m,n+6} < \tilde{\mu}'_{m,n,2} < \tilde{\mu}'^{(n,2)}_{m+2n+4} < \tilde{\mu}_{m,n,1} < \tilde{\mu}'^{(n,2)}_{m+2n+2} < \tilde{\nu}_{m,n}.
\]

In the following, we shall show that, for each \( n \geq 0 \), with \( x \)-coordinates moving from point \( \tilde{\mu}'_{m,n} \) via point \( \tilde{\nu}_{m,n} \) to point \( \tilde{\mu}'_{m,n+1} \) (see Figure 2):

(a) (odd periods) the point \( \tilde{\mu}'^{(n,2)}_{m+2n+2i} = \min \{ \tilde{\mu}'_{m,n} \leq x \leq \tilde{\nu}_{m,n} : f^{m+2n+2i}(x) = x \} \) exists and is a period-\((m+2n+2i)\) point of \( f \) for each \( i \geq 1 \);

(b) (even periods) the point \( \tilde{\mu}'^{(n,2)}_{2m+2n+2i} = \min \{ \tilde{\mu}_{m,n,1} \leq x \leq \tilde{\nu}_{m,n} : f^{2m+2n+2i}(x) = x \} \) exists for each \( i \geq 1 \) and is a period-\((2m+2n+2i)\) point of \( f \) for each \( i \geq n+3 \);

(c) (even periods) the point \( \tilde{\mu}'^{(n,2)}_{2m+2n+2i} = \max \{ \tilde{\mu}_{m,n} \leq x \leq \tilde{\mu}'_{m,n+1} : f^{2m+2n+2i}(x) = x \} \) exists for each \( i \geq 1 \) and is a period-\((2m+2n+2i)\) point of \( f \) for each \( i \geq n+3 \).

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For each fixed $n \geq 0$, the collection of all these periodic points $\tilde{P}_{m+2n+2i}^{(n,2)}$, $\tilde{c}^{(n,2)}_{2m+2n+2i}$, $\tilde{c}'^{(n,2)}_{2m+2n+2i}$, $i \geq 1$ (note that the periodic point $\tilde{P}_{m+2n+2}^{(n,2)}$ is excluded so that the convex hulls of the sets $\{\tilde{P}_{m+2n+2i}^{(n,2)} : i \geq 2\}$ and $\{\tilde{c}^{(n,2)}_{2m+2n+2i} : i \geq 1\}$ are disjoint), is called a compartment of the second layer of the basic tower of periodic points of $f$ associated with $P$.

Figure 2: A compartment of the second layer of the basic tower of periodic points of $f$ associated with $P$ in the interval $[\tilde{\mu}_{m,n}, \tilde{\mu}'_{m,n+1}] \subset [\min P, \tilde{d}]$, where $\tilde{\nu}_{m,n}$ is an auxiliary point in $[\tilde{\mu}_{m,n}, \tilde{\mu}'_{m,n+1}]$ such that $f^{m+2n}(\tilde{\nu}_{m,n}) = v$ which is needed only to determine the ordering of the following points:

and, for each $i \geq 1$,

$$\tilde{\mu}_{m,n,i} = \min \{\tilde{\mu}'_{m,n} \leq x \leq \tilde{\mu}'_{m,n+1} : f^{m+2n+2i}(x) = d\};$$
$$\tilde{P}_{m+2n+6}^{(n,2)} = \min \{\tilde{P}'_{m,n} \leq x \leq \tilde{P}'_{m,n+1} : f^{m+2n+2i}(x) = x\};$$
$$\tilde{c}^{(n,2)}_{2m+2n+2i} = \min \{\tilde{\mu}_{m,n,1} \leq x \leq \tilde{\mu}_{m,n+1} : f^{2m+2n+2i}(x) = x\};$$
$$\tilde{c}'^{(n,2)}_{2m+2n+2i} = \max \{\tilde{\mu}'_{m,n} \leq x \leq \tilde{\mu}'_{m,n+1} : f^{2m+2n+2i}(x) = x\}.$$

2.1(a) On the existence of periodic points of $f$ of all odd periods $\geq m + 2n + 2$ in $[\tilde{\mu}'_{m,n}, \tilde{\nu}_{m,n}] \subset [\tilde{\mu}'_{m,n}, \tilde{\mu}'_{m,n+1}] \subset [\min P, \tilde{d}]$ for each $n \geq 0$.

We can apply Lemma 5(1). We first prove the following result (see also Lemma 11 below which is related) (Note that, for any periodic point $p$ of $f$, $\ell(p)$ denotes the least period of $p$ with respect to $f$):

Lemma 7. All periodic points of $f$ in $[\tilde{\mu}'_{m,n}, \tilde{d}]$ of odd periods have least periods $\geq m + 2n + 2$.

Proof. Suppose $f$ had a periodic point $\tilde{p}$ in $[\tilde{\mu}'_{m,n}, \tilde{d}]$ of odd period with least period $\leq m + 2n$.

(1) If $\ell(\tilde{p}) = 1$, then $f^3(\tilde{p}) = \tilde{p}$ and $f^3(d) = f(z_0)$;

(2) If $\ell(\tilde{p}) \geq 3$, then $f^{\ell(\tilde{p})}(\tilde{p}) = \tilde{p}$ and, since $\ell(\tilde{p})$ is odd and $\geq 3$, $f^{\ell(\tilde{p})}(d) = f(z_0)$.

Since $m + 2n$ is odd and $\geq \max\{3, \ell(\tilde{p})\}$, by Lemma 3, there is a periodic point $\tilde{w}_{m+2n}$ of $f$ in $[\tilde{p}, \tilde{d}]$ such that $f^{m+2n}(\tilde{w}_{m+2n}) = \tilde{w}_{m+2n}$. Since

$$f^{m+2n}(\tilde{w}_{m+2n}) - d = \tilde{w}_{m+2n} - d < 0$$
and

$$f^{m+2n}(d) - d = f(z_0) - d \geq z_0 - d > 0,$$
there is a point $\tilde{\mu}_{m,n}^{*}$ in $(\tilde{w}_{m+2n}, d]$ such that $f^{m+2n}(\tilde{\mu}_{m,n}^{*}) = d$. Since $\tilde{\mu}_{m,n}^{*} > \tilde{w}_{m+2n} > \tilde{\mu}_{m,n}'$, this contradicts the maximality of $\tilde{\mu}_{m,n}'$ in $[\min P, d]$. Therefore, we have shown that all periodic points of $f$ in $[\tilde{\mu}_{m,n}', d]$ of odd periods have least periods $\geq m + 2n + 2$. \hfill \Box

Now, since $f^{m+2n}(\tilde{\nu}_{m,n}) = v$, we apply Lemma 5(1) with

$$f^{m+2n+2}(\tilde{\mu}_{m,n}') = z_{0} \quad \text{and} \quad f^{m+2n+2}(\tilde{\nu}_{m,n}) = \min P \quad \text{and Lemma 7}$$

to obtain that, for each $n \geq 0$ and $i \geq 0$, the points

$$\tilde{\mu}_{m,n}^{(n,2)} = \min \{ \tilde{\mu}_{m,n}' : x \leq \tilde{\mu}_{m,n}' + 2i(x) = x \} \quad \text{and} \quad \tilde{\mu}_{m,n,i+1} = \min \{ \tilde{\mu}_{m,n}' : x \leq \tilde{\mu}_{m,n}' + 2i(x) = d \}$$

exist, $\tilde{\mu}_{m,n}' < \cdots < \tilde{\mu}_{m,n,3} < \tilde{\mu}_{m,n,2} < \tilde{\mu}_{m,n} < \tilde{\mu}_{m,n,1} < \tilde{\mu}_{m,n+1} < \tilde{\mu}_{m,n+2} \quad \text{and, for each} \quad m \geq 0 \quad \text{and} \quad i \geq 0$, the point $\tilde{\mu}_{m,n}'$ is a period-$(m + 2n + 2 + 2i)$ point of $f$, or, equivalently, in $[\tilde{\mu}_{m,n}', \tilde{\nu}_{m,n}] \subset [\min P, d]$, for each $n \geq 0$ and $i \geq 1$, the point $\tilde{\mu}_{m,n}'$ is a period-$(m + 2n + 2i)$ point of $f$.

\subsection*{2.1(b) On the existence of periodic points of $f$ of all even periods $\geq 2m + 4n + 6$ in $[\tilde{\mu}_{m,n,1}, \tilde{\nu}_{m,n}] \subset [\min P, d]$ for each $n \geq 0$.}

In this case, we shall apply Lemma 6(1). Since, for each $i \geq 1$, $\tilde{\mu}_{m,n,i} = \min \{ \tilde{\mu}_{m,n}' : x \leq \tilde{\nu}_{m,n} : f^{m+2n+2i}(x) = d \}$, we have $f^{m+2n+2}(\tilde{\mu}_{m,n,1}) = d$ and so,

$$f^{m+2n+3}(\tilde{\mu}_{m,n,1}) = f(d).$$

Furthermore, since $f^{m+2n}(\tilde{\nu}_{m,n}) = v$, we have $f^{m+2n+2}(\tilde{\nu}_{m,n}) = f^{2}(v) = \min P$. Consequently, we have

$$f^{2m+2n+2}(\tilde{\nu}_{m,n}) = f^{m}(\min P) = \min P. $$

Therefore, we can apply Lemma 6(1) with

$$f^{m+2n+3}(a) = f(d) \in \{ z_{0}, f(z_{0}), f(d) \} \quad \text{and} \quad f^{2m+n+1}(b) = \min P$$

to obtain that, for each $i \geq 0$, the point

$$\tilde{c}_{2(m+n+1)+2i}^{(n,2)} = \min \{ \tilde{\mu}_{m,n,1}' \leq \tilde{\nu}_{m,n} : f^{2(m+n+1)+2i}(x) = x \}$$

exists and

$$\tilde{\mu}_{m,n,1}' < \cdots < \tilde{c}_{2m+2n+6}^{(n,2)} < \tilde{c}_{2m+2n+4}^{(n,2)} < \tilde{c}_{2m+2n+2}^{(n,2)} < \tilde{\nu}_{m,n}.$$  

Furthermore, for each $i \geq 0$ such that $(m + n + 1) + i \geq \max \{ m + 2n + 3, m + 2 \} = m + 2n + 3$, i.e., for each $i \geq n + 2$, the point $\tilde{c}_{2(m+n+1)+2i}^{(n,2)}$ is a period-$(2(m + n + 1) + 2i)$ point of $f$, or, equivalently, in $[\tilde{\mu}_{m,n,1}, \tilde{\nu}_{m,n}] \subset [\min P, d]$, for each $n \geq 0$ and $i \geq n + 3$, the point $\tilde{c}_{2m+2n+2i}^{(n,2)}$ is a period-$(2m + 2n + 2i)$ point of $f$.  

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2.1(c) On the existence of periodic points of \( f \) of all even periods \( \geq 2m + 4n + 6 \) in \([\tilde{\nu}_{m,n}, \tilde{\nu}'_{m,n+1}] (\subset [\tilde{\mu}'_{m,n}, \tilde{\mu}'_{m,n+1}] \subset [\min P, d])\) for each \( n \geq 0 \).

In this case, we shall apply Lemma 6(2). By definition of \( \tilde{\mu}'_{m,n+1} \), we have
\[
f^{m+2n+3}(\tilde{\mu}'_{m,n+1}) = f(f^{m+2n+2}(\tilde{\mu}'_{m,n+1})) = f(d).
\]
On the other hand, since \( \tilde{\nu}_{m,n} \) is a point in \([\tilde{\mu}'_{m,n}, \tilde{\mu}'_{m,n+1}] \) such that \( f^{m+2n}(\tilde{\nu}_{m,n}) = v \), we have
\[
f^{m+2n+2}(\tilde{\nu}_{m,n}) = f^2(v) = \min P.
\]

Therefore, we can apply Lemma 6(2) with
\[
f^{m+2n+3}(\tilde{\mu}'_{m,n+1}) = f(d) \in \{ z_0, f(z_0), f(d) \} \quad \text{and} \quad f^{2(m+n+1)}(\tilde{\nu}_{m,n}) = \min P
\]
to obtain that, for each \( i \geq 0 \), the point
\[
c^{(n,2)}_{2(m+n+1)+2i} = \max \{ \tilde{\nu}_{m,n} \leq x \leq \tilde{\mu}'_{m,n+1} : f^{2(m+n+1)+2i}(x) = x \}
\]
exists and
\[
\tilde{\nu}_{m,n} < c^{(n,2)}_{2m+2n+2} < c^{(n,2)}_{2m+2n+4} < c^{(n,2)}_{2m+2n+6} < \cdots < \tilde{\mu}'_{m,n+1}.
\]
Furthermore, for each \( i \geq 0 \) such that \((m+n+1)+i \geq \max \{ m+2n+3, m+2 \} = m+2n+3\), i.e., for each \( i \geq n+2 \), the point \( c^{(n,2)}_{2(m+n)+2i} \) is a period-\((2(m+n+1)+2i)\) point of \( f \), or, equivalently, in \([\tilde{\nu}_{m,n}, \tilde{\mu}'_{m,n+1}] (\subset [\tilde{\mu}'_{m,n}, \tilde{\mu}'_{m,n+1}] \subset [\min P, d])\),

for each \( n \geq 0 \) and \( i \geq n+3 \), the point \( c^{(n,2)}_{2m+2n+2i} \) is a period-\((2m+2n+2i)\) point of \( f \).

For each \( n \geq 1 \), let \( u_n = \min \{ d \leq x \leq v : f^{2n}(x) = d \} \) be defined as before. Then we have \( d < \cdots < u_3 < u_2 < u_1 < v, f^2(d) = z_0 \) and \( f^{2n}(u_n) = d \). In particular, we have \( f^{2n}([d, u_n]) = [d, z_0] \supset \{ v \} \). Let \( \nu_n \) be a point in \([d, u_n]\) such that
\[
f^{2n}(\nu_n) = v.
\]

Then, since \( f^2(v) = \min P \), we obtain that \( f^{2n+2}([d, \nu_n]) \supset [\min P, z_0] \supset \{ d \} \). So, the point \( \nu_n \) happens to satisfy that \( u_{n+1} < \nu_n < u_n \). On the interval \([u_{n+1}, u_n]\), we have
\[
f^{2n+2}(u_{n+1}) = d, f^{2n+2}(\nu_n) = \min P, f^{2n+2}(u_n) = z_0 \quad \text{and}
\]
the graph of \( y = f^{2n+2}(x) \) looks ‘roughly’ like a skew ‘V’ shape at the points \( u_{n+1}, \nu_n, u_n \) (meaning that the graph of \( y = f^{2n+2}(x) \) passes through the 3 points \((u_{n+1}, d), (\nu_n, \min P), (u_n, z_0)\) on the \( x-y \) plane with \( y \)-coordinates \( d, \min P, z_0 \) respectively) just like a mirror-symmetric copy of that of \( y = f^2(x) \) at the 3 points \( d, v, \hat{u}_0 \) on the interval \([d, \hat{u}_0]\).
In the following, we show that, for each \( n \geq 1 \), there exist 4 sequences of points in \([u_{n+1}, u_n]:(\text{the 2 in the superscript } (n, 2) \text{ below indicates second layer})\)

\[
\begin{align*}
& u_{n+1} < \cdots < p_{m+2n+4}^{(n,2)} < p_{m+2n+2}^{(n,2)} < \nu_n < q_{m+2n+2}^{(n,2)} < q_{m+2n+4}^{(n,2)} < \cdots < u_{n,1}^{(n,2)} < c_{2n+2}^{(n,2)} < u_{n,2}^{(n,2)} < c_{2n+4}^{(n,2)} < \cdots < u_n
\end{align*}
\]

such that, for each \( i \geq 1 \), \( u_{n,i} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+2i}(x) = d \} \) and, from point \( u_n \) via point \( \nu_n \) to point \( u_{n+1} \) (see Figure 3):

(a) (even periods) the point \( c_{2n+2i}^{(n,2)} = \max \{ \nu_n \leq x \leq u_{n,1} : f^{2n+2i}(x) = x \} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+2i}(x) = x \} \) is a period-\((2n+2i)\) point of \( f \) for each \( i \geq 1 \) and \( i \neq n \);

(b) (odd periods) the point \( q_{m+2n+2i}^{(n,2)} = \max \{ \nu_n \leq x \leq u_{n,1}^{(n,2)} : f^{m+2n+2i}(x) = x \} = \max \{ u_{n+1} \leq x \leq u_{n,1}^{(n,2)} : f^{m+2n+2i}(x) = x \} \) is a period-\((m + 2n + 2i)\) point of \( f \) for each \( i \geq 1 \);

(c) (odd periods) the point \( p_{m+2n+2i}^{(n,2)} = \min \{ u_{n+1} \leq x \leq \nu_n : f^{m+2n+2i}(x) = x \} = \min \{ u_{n+1} \leq x \leq u_n : f^{m+2n+2i}(x) = x \} \) is a period-\((m + 2n + 2i)\) point of \( f \) for each \( i \geq 1 \).

\[
\begin{array}{ccc}
\cdots & p_{m+2n+4}^{(n,2)} & p_{m+2n+2}^{(n,2)} \\
& q_{m+2n+2}^{(n,2)} & q_{m+2n+4}^{(n,2)}
\end{array}
\]

Figure 3: A compartment of the second layer of the basic tower of periodic points of \( f \) associated with \( P \) in the interval \([u_{n+1}, u_n] \subset [d, u_1] \),

where \( \nu_n \) is an auxiliary point in \([\min P, u_n] \) such that \( f^{2n}(\nu_n) = v \)

which is needed only to determine the ordering of the following points:

and, for each \( i \geq 1 \),

\[
\begin{align*}
& u_{n,i} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+2i}(x) = d \}; \\
& c_{2n+2i}^{(n,2)} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+2i}(x) = x \}; \\
& q_{m+2n+2i}^{(n,2)} = \max \{ u_{n+1} \leq x \leq u_{n,1}^{(n,2)} : f^{m+2n+2i}(x) = x \}; \\
& p_{m+2n+2i}^{(n,2)} = \min \{ u_{n+1} \leq x \leq u_n : f^{m+2n+2i}(x) = x \}.
\end{align*}
\]

For each fixed \( n \geq 1 \), the collection of all these periodic points \( c_{2m+2n+2i}^{(n,2)}, q_{m+2n+2i}^{(n,2)}, p_{m+2n+2i}^{(n,2)} \), \( i \geq 1 \), is called a compartment of the second layer of the basic tower of periodic points of \( f \) associated with \( P \).
2.2(a) On the existence of periodic points of $f$ of all even periods $\geq 2n + 2$ (except possibly period $4n$) in $[v_n, u_n] \subset [u_{n+1}, u_n] \subset [d, v]$ for each $n \geq 1$.

We can apply Lemma 6(2) to this case. However, since, in this special case, we can determine their exact periods, we give a proof in detail. For higher layers (layer 3 and up) of periodic points, we can determine their exact periods only when the periods are large enough.

Recall that $f^{2n}(\nu_n) = v$. On $[\nu_n, u_n]$, since $f^{2n+2}(\nu_n) \supset [\min P, z_0] \supset \{d\}$, the point
\[ u_{n,1}' = \max \{ \nu_n \leq x \leq u_n : f^{2n+2}(x) = d \} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+2}(x) = d \} \]
exists and $d < f^{2n+2}(x) < z_0$ for all $u_{n,1}' < x < u_n$. This, combined with the following fact
\[ d < f^{2i}(x) < z_0 \text{ for all } 1 \leq i \leq n \text{ and all } x \in (d, u_n) \cup [u_{n}', z_0) \]
implies that
\[ d < f^{2i}(x) < z_0 \text{ for all } 1 \leq i \leq n + 1 \text{ and all } u_{n,1}' < x < u_n. \]

Let $c_{2n+2}' = \max \{ u_{n,1}' \leq x \leq u_n : f^{2n+2}(x) = x \} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+2}(x) = x \}$. Since $f^{2n+2}\left([c_{2n+2}', u_n]\right) \supset [c_{2n+2}', z_0] \supset \{v\}$, let $\nu_{n+1}$ be a point in $[c_{2n+2}', u_n]$ such that $f^{2n+2}(\nu_{n+1}) = v$. Then $f^{2n+4}\left([\nu_{n+1}, u_n]\right) \supset [\min P, z_0] \supset \{d\}$. So, the point
\[ u_{n,2}' = \max \{ \nu_{n+1} \leq x \leq u_n : f^{2n+4}(x) = d \} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+4}(x) = d \} \]
exists and $d < f^{2n+4}(x) < z_0$ for all $u_{n,2}' < x < u_n$. This, combined with (†) above, implies that
\[ d < f^{2i}(x) < z_0 \text{ for all } 1 \leq i \leq n + 2 \text{ and all } u_{n,2}' < x < u_n. \]

Let $c_{2n+4}' = \max \{ u_{n,2}' \leq x \leq u_n : f^{2n+4}(x) = x \} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+4}(x) = x \}$. Proceeding in this manner indefinitely, we obtain 2 sequences of points
\[ u_{n+1} < \nu_{n} < u_{n,1}' < c_{2n+2}' < u_{n,2}' < c_{2n+4}' < u_{n,3}' < c_{2n+6}' < \cdots < u_n \]
such that, for each $i \geq 1$,
\[ u_{n,i}' = \max \{ \nu_n \leq x \leq u_n : f^{2n+2i}(x) = d \} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+2i}(x) = d \}, \]
\[ d < f^{2j}(x) < z_0 \text{ for all } 1 \leq j \leq n + i \text{ and all } u_{n,i}' < x < u_n, \quad (\dagger\dagger) \]
and $c_{2n+2i}' = \max \{ u_{n,i}' \leq x \leq u_n : f^{2n+2i}(x) = x \} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+2i}(x) = x \}$.

Note that it follows from (†) that each $c_{2n+2i}'$ is a periodic point of $f$ with even period. To find the least period of $c_{2n+2i}'$ with respect to $f$, we shall use the following result which can be proved by using arguments similar to those at the end of the proof of (c) in the main text (cf. Lemma 4) (see also the relevant Lemma 12 below. Unfortunately, we do not have similar result on the counterpart interval $[\tilde{u}_{n}, \tilde{u}_{n+1}]$):

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Lemma 8. For each \( n \geq 1 \) and \( i \geq 1 \), on the interval \([u'_{n,i}, u_n] (\subset [u_{n+1}, u_n])\), \( f \) has no periodic points of odd periods \( \leq 2n + 2i + 1 \), nor has periodic points of even periods \( \leq 2n + 2i \) except period-(\( 2n + 2i \)) points and possibly period-(\( 2n \)) points.

Proof. Suppose \( f \) had a period-(\( 2n + 2j \)) point, say \( w_{2n+2j} \), in \([u'_{n,i}, u_n]\) for some \( 1 \leq j \leq i - 1 \). Then since \( f^2n+2j+2([w_{2n+2j}, u_n]) \supset f^2([w_{2n+2j}, z_0]) \supset f^2([v, z_0]) \supset [\text{min } P, z_0] \supset \{d\} \), there exists a point \( u'_{n,j+1} \in (w_{2n+2j}, u_n) (\subset [u'_{n,i}, u_n]) \) such that \( f^{2n+2j+2}(u'_{n,j+1}) = d \). So, \( u'_{n,i} < u'_{n,j+1} \). Since \( n + j + 1 \leq n + i \), this contradicts the fact that \( (u'_{n,j+1} \leq u'_{n,i}) \). This, combined with Lemma 4(1), implies that \( f \) has no periodic points of even periods \( \leq 2n + 2i - 2 \) except possibly period-(\( 2n \)) points. Furthermore, it follows from the above (i†) that \( f \) has no periodic points of odd periods \( \leq 2n + 2i + 1 \).

On the other hand, since \( f^{2n+2j}(u'_{n,i}) - u'_{n,i} = d - u_{n,i} < 0 \) and \( f^{2n+2j}(u_n) - u_n = f^{2i}(d) - u_n = z_0 - u_n > 0 \), the point \( u'_{n,i} = \max \{u'_{n,i} < x \leq u_n : f^{2n+2i}(x) = x\} \) exists and, since we have just shown in the previous paragraph that \( f \) has no periodic points of least (odd or even) periods \( \leq 2n + 2i - 1 \), the point \( c_{2n+2i} \) is a period-(\( 2n + 2i \)) point of \( f \) in \([u'_{n,i}, u_n]\). \( \square \)

We now use Lemma 8 to find the least period of \( c_{2n+2i}^{\prime(n,2)} \) with respect to \( f \) for each \( i \geq 1 \):

Let \( \ell(c_{2n+2i}^{\prime(n,2)}) \) denote the least period of \( c_{2n+2i}^{\prime(n,2)} \) with respect to \( f \). Suppose \( \ell(c_{2n+2i}^{\prime(n,2)}) < 2n + 2j \) for some \( j \geq 1 \). Then by Lemma 8, \( \ell(c_{2n+2i}^{\prime(n,2)}) = 2n \) and by Lemma 1, \( 2n + 2j \) is a multiple of \( 2n \). Suppose \( (2n + 2j)/(2n) = r > 2 \). Then \((r-1)(2n) > 2n \). So, \((r-1)(2n) \geq 2n + 2 \). Since \( f^{2n+2}(u_n) = z_0 = f^{(r-1)(2n)}(c_{2n+2i}^{\prime(n,2)}) = c_{2n+2i}^{\prime(n,2)} \) and \( 2n + 2j > (r-1)(2n) \), it follows from Lemma 3 that there exists a periodic point, say \( w_{2n+2j}^{*} \), of \( f \) such that \( u'_{n,j} < c_{2n+2i}^{\prime(n,2)} < w_{2n+2j}^{*} < u_n \) and \( f^{2n+2j}(w_{2n+2j}^{*}) = w_{2n+2j}^{*} \) which contradicts the maximality of \( c_{2n+2i}^{\prime(n,2)} \) in \([u'_{n,j}, u_n]\). This shows that, for each \( i \geq 1 \),

if \( (2n + 2i)/(2n) \neq 2 \), then \( c_{2n+2i}^{\prime(n,2)} \) is a period-(\( 2n + 2i \)) point of \( f \) and,

if \( 2n + 2i = 4n \), then \( c_{2n+2i}^{\prime(n,2)} \) is either a period-(\( 2n \)) or a period-(\( 4n \)) point of \( f \).

Note that the point \( c_{2n+2i}^{\prime(n,2)} \) may be a period-(\( 2n \)) point of \( f \). For example, when \( n = 1 \), let \( g : [0, 1] \rightarrow [0, 1] \) be the continuous map defined by putting (i) \( g(x) = x + 1/2 \) and (ii) \( g(x) = 2 - 2x \) and let \( P = \{0, 1/2, 1\} \) be the unique period-3 orbit of \( g \), then \( g \) has exactly 4 period-4 points (which form a period-4 orbit of \( g \)), i.e., \( 2/9 \rightarrow 13/18 \rightarrow 5/9 \rightarrow 8/9 \rightarrow 2/9 \). In this case, \( \min P = 0 \) and \( b = g^{4}(\min P) = 1 \) and

\[
d = 1/6, \quad c_{4} = 2/9, \quad u_{2} = 11/48, \quad u_{1,1,2} = 7/24, \quad c_{2} = 1/3, \quad u_{1} = 5/12, \quad v = 1/2, \quad c_{4} = 5/9, \quad z_{0} = z = 2/3.
\]

We see that, in the interval \([u_{2}, u_{1}] = [11/48, 5/12] \subset [u_{1,1}, u_{1}] \), \( g \) has only period-2 point, i.e., \( c_{4} = c_{2} = 1/3 \), but has no period-4 point.

2.2(b) On the existence of periodic points of all odd periods \( \geq m + 2n + 2 \) in \([\nu_{n}, u'_{n,1}] (\subset [u_{n+1}, u_{n}]) \subset [d, v] \) for each \( n \geq 1 \).
By definition of $u'_{n,1}$, we have $f^{2n+2}(u'_{n,1}) = d$ and so,

$$f^{2n+3}(u'_{n,1}) = f(d).$$

Since $\nu_n$ is a point in $[u_{n+1}, u_n]$ such that $f^{2n}(\nu_n) = v$, we have

$$f^{m+2n+2}(\nu_n) = f^{m+2}(v) = f^m(\min P) = \min P.$$

On the other hand, it follows from Lemma 4(1) that

all periodic points of $f$ in $[d, u_n]$ with odd periods have periods $\geq 2n + 3$.

Therefore, by applying Lemma 5(2) with

$$f^{2n+3}(u'_{n,1}) = f(d) \in \{z_0, f(z_0), f(d)\} \quad \text{and} \quad f^{m+2(n+1)}(\nu_n) = \min P,$$

we obtain that, for each $i \geq 0$, the point

$$q_{m+2(n+1)+2i}^{(n,2)} = \max \{\nu_n \leq x \leq u'_{n,1} : f^{m+2(n+1)+2i}(x) = x\}$$

$$= \max \{u_{n+1} \leq x \leq u'_{n,1} : f^{m+2(n+1)+2i}(x) = x\}$$

exists and $u_{n+1} < \nu_n < q_{m+2n+2}^{(n,2)} < q_{m+2n+4}^{(n,2)} < q_{m+2n+6}^{(n,2)} < \cdots < u'_{n,1} < u_n < v$.

Furthermore, for each $i \geq 0$, the point $q_{m+2n+2+2i}^{(n,2)}$ is a period-$(m + 2n + 2 + 2i)$ point of $f$, or, equivalently, in $[\nu_n, u'_{n,1}] (\subset [u_{n+1}, u_n]) \subset [d, v])$,

for each $n \geq 1$ and $i \geq 1$, the point $q_{m+2n+2i}^{(n,2)}$ is a period-$(m + 2n + 2i)$ point of $f$.

2.2(c) On the existence of periodic points of all odd periods $\geq m + 2n + 2$ in $[u_{n+1}, \nu_n]$ ($\subset [u_{n+1}, u_n]) \subset [d, v]$ for each $n \geq 1$.

By definition of $u_{n+1}$, we have $f^{2n+2}(u_{n+1}) = d$ and so, $f^{2n+3}(u_{n+1}) = f(d)$. Since $\nu_n$ is a point in $[u_{n+1}, u_n]$ such that $f^{2n}(\nu_n) = v$, we have $f^{m+2n+2}(\nu_n) = f^{m+2}(v) = f^m(\min P) = \min P$. On the other hand, it follows from Lemma 4(1) that

all periodic points of $f$ in $[d, u_n]$ with odd periods have periods $\geq 2n + 3$.

Therefore, by Lemma 5(1) with

$$f^{2n+3}(u_{n+1}) = f(d) \in \{z_0, f(z_0), f(d)\} \quad \text{and} \quad f^{m+2(n+1)}(\nu_n) = \min P,$$

we obtain that, for each $i \geq 0$, the point

$$p_{m+2(n+1)+2i}^{(n,2)} = \min \{u_{n+1} \leq x \leq \nu_n : f^{m+2(n+1)+2i}(x) = x\}$$

$$= \min \{u_{n+1} \leq x \leq u_n : f^{m+2(n+1)+2i}(x) = x\},$$
exists and \( d < u_{n+1} < \cdots < p_{m+2n+6}^{(n,2)} < p_{m+2n+4}^{(n,2)} < p_{m+2n+2}^{(n,2)} < \nu_n < u_n < v \). Furthermore, for each \( i \geq 0 \), the point \( p_{m+2n+2+2i}^{(n,2)} \) is a period-\((m+2n+2+2i)\) point of \( f \), or, equivalently, in \([u_{n+1}, \nu_n] (\subset [u_{n+1}, u_n]) \subset [d, v])\),

for each \( n \geq 1 \) and \( i \geq 1 \), the point \( p_{m+2n+2+2i}^{(n,2)} \) is a period-\((m+2n+2+2i)\) point of \( f \).

Recall that \( \hat{u}'_0 = \max \{ u_1 \leq x \leq v : f^2(x) = d \} \). Since \( f^{m+2}(\hat{u}'_0) = f^m(d) = f(z_0) \geq z_0 \) and \( f^{m+2}(v) = f^m(\min P) = \min P \), we have \( f^{m+2}([\hat{u}'_0, v]) \supset [\min P, z_0] \supset \{d\} \). So, the point \( \hat{\mu}_{m,1} = \min \{ \hat{u}'_0 \leq x \leq v : f^{m+2}(x) = d \} \) exists. Since \( f^{m+4}(p_{m+2n}^{(n,2)}) = f^2(f^{m+2}([\hat{u}'_0, \hat{\mu}_{m,1}])) \supset f^2([d, z_0]) \supset [\min P, z_0] \supset \{d\} \), the point \( \hat{\mu}_{m,2} = \min \{ \hat{u}'_0 \leq x \leq \hat{\mu}_{m,1} : f^{m+4}(x) = d \} = \min \{ \hat{u}'_0 \leq x \leq v : f^{m+4}(x) = d \} \) exists and is \(< \hat{\mu}_{m,1} \). Inductively, for each \( n \geq 1 \), let (note the relationship between the subscript of \( \hat{\mu}_{m,n} \) and the superscript of \( f^{m+2n} \) in the definition of \( \hat{\mu}_{m,n} \))

\[
\hat{\mu}_{m,n} = \min \{ \hat{u}'_0 \leq x \leq v : f^{m+2n}(x) = d \}.
\]

Then we have \( u_1 \leq \hat{u}'_0 < \cdots < \hat{\mu}_{m,3} < \hat{\mu}_{m,2} < \hat{\mu}_{m,1} < v \). For each \( n \geq 1 \), let \( \hat{\nu}_{m,n} \) be a point in \([\hat{\nu}'_0, \hat{\mu}_{m,n}] \) such that \( f^{m+2n}(\hat{\nu}_{m,n}) = v \). Then it turns out that \( \hat{\mu}_{m,n+1} < \hat{\nu}_{m,n} < \hat{\mu}_{m,n} \). Furthermore, on the interval \([\hat{\mu}_{m,n+1}, \hat{\mu}_{m,n}] \), we have

\[
f^{m+2n+2}(\hat{\mu}_{m,n+1}) = d, \quad f^{m+2n+2}(\hat{\nu}_{m,n}) = \min P, \quad f^{m+2n+2}(\hat{\mu}_{m,n}) = z_0 \text{ and}
\]

the graph of \( y = f^{m+2n+2}(x) \) looks 'roughly' like a skew 'V' shape at the points \( \hat{\mu}_{m,n+1}, \hat{\nu}_{m,n}, \hat{\mu}_{m,n} \) (meaning that the graph of \( y = f^{m+2n+2}(x) \) passes through the 3 points \((\hat{\mu}_{m,n+1}, d), (\hat{\nu}_{m,n}, \min P), (\hat{\mu}_{m,n}, z_0)\) on the \( x \)-y plane with \( y \)-coordinates \( d, \min P, z_0 \) respectively) just like a mirror-symmetric copy of that of \( y = f^2(x) \) (at the 3 points \( d, v, \hat{u}_0 \)) on the interval \([d, \hat{u}_0] \), where \( \hat{u}_0 = \min \{ v \leq x \leq z_0 : f^2(x) = d \} \). Now, for each \( n \geq 1 \) and all \( i \geq 1 \), let (the number 2 in the superscript \( (n, 2) \) indicates the second layer)

\[
\hat{q}_{m+2n+2i}^{(n,2)} = \max \{ \hat{\nu}_{m,n} \leq x \leq \hat{\mu}_{m,n} : f^{m+2n+2i}(x) = x \} \quad \text{and}
\]

\[
\hat{\nu}'_{m,n,i} = \max \{ \hat{\nu}_{m,n} \leq x \leq \hat{\mu}_{m,n} : f^{m+2n+2i}(x) = d \}
\]

(note the relationship between the subscript of \( \hat{\nu}'_{m,n,i} \) and the superscript of \( f^{m+2n+2i} \) in the definition of \( \hat{\nu}'_{m,n,i} \)). Here the \( n \) and \( i \) in the subscript of \( \hat{\nu}'_{m,n,i} \) indicate the \( i \)-th point of the sequence \( < \hat{\mu}'_{m,n,i} > \) in the \( n \)-th interval \([\hat{\mu}_{m,n+1}, \hat{\mu}_{m,n}] \). We need these points \( \hat{\nu}'_{m,n,i} \)'s to continue the basic tower-building process into the third layer). By arguing as before, we obtain that

\[
\hat{\mu}_{m,n+1} < \hat{\nu}_{m,n} < \hat{\mu}'_{m,n,1} < \hat{q}_{m+2n+2}^{(n,2)} < \hat{\mu}'_{m,n,2} < \hat{q}_{m+2n+4}^{(n,2)} < \hat{\mu}'_{m,n,3} < \hat{q}_{m+2n+6}^{(n,2)} < \cdots < \hat{\mu}_{m,n}.
\]

In the following, we shall show that, for each \( n \geq 1 \), with \( x \)-coordinates moving from point \( \hat{\mu}_{m,n} \) via point \( \hat{\nu}_{m,n} \) to point \( \hat{\mu}_{m,n+1} \) (see Figure 4):
(a) (odd periods) the point \( \tilde{q}_{m+2n+2i}^{(n,2)} = \max \{ \mu'_{m+1} \leq x \leq \mu_m : f^{m+2n+2i}(x) = x \} = \max \{ \mu_{m+1} \leq x \leq \mu_m : f^{m+2n+2i}(x) = x \} \) exists and is a period-(\( m + 2n + 2i \)) point of \( f \) for each \( i \geq 1 \);

(b) (even periods) the point \( \tilde{q}_{2m+2n+2i}^{(n,2)} = \max \{ \mu'_{m,n} \leq x \leq \mu_{m,n} : f^{2m+2n+2i}(x) = x \} = \max \{ \mu_{m+1} \leq x \leq \mu_m : f^{2m+2n+2i}(x) = x \} \) exists for each \( i \geq 1 \) and is a period-(\( 2m + 2n + 2i \)) point of \( f \) for each \( i \geq 1 \);

(c) (even periods) the point \( \tilde{q}_{2m+2n+2i}^{(n,2)} = \min \{ \mu_{m+1} \leq x \leq \mu_m : f^{2m+2n+2i}(x) = x \} = \min \{ \mu_{m+1} \leq x \leq \mu_m : f^{2m+2n+2i}(x) = x \} \) exists for each \( i \geq 1 \) and is a period-(\( 2m + 2n + 2i \)) point of \( f \) for each \( i \geq n + 3 \).

\[
\begin{array}{c}
\mu_{m+1} \quad \mu_{m,n} \quad \mu_m \\
\cdots \quad \cdots \quad \cdots \quad \cdots \\
\tilde{q}_{2m+2n+4}^{(n,2)} \quad \tilde{q}_{2m+2n+2}^{(n,2)} \quad \tilde{q}_{2m+2n}^{(n,2)} \quad \tilde{q}_{2m+2n+4}^{(n,2)} \\
\mu_{m,n,1} \quad \mu_{m,n,2} \quad \mu_{m,n,3} \quad \mu_{m,n,4}
\end{array}
\]

Figure 4: A compartment of the second layer of the basic tower of periodic points of \( f \) associated with \( P \) in the interval \([\tilde{u}_{0}, \tilde{v}] \subset [u_1, v] \), where \( \tilde{v}_{m,n} \) is an auxiliary point in \([\mu_{m+1}, \mu_m] \subset [u_1, v] \).

For each fixed \( n \geq 1 \), the collection of all these periodic points \( \tilde{q}_{m+2n+2i}^{(n,2)} \), \( \tilde{c}_{m+2n+2i}^{(n,2)} \), \( \tilde{q}_{2m+2n+2i}^{(n,2)} \), \( i \geq 1 \), is called a compartment of the second layer of the basic tower of periodic points of \( f \) associated with \( P \).

2.3(a) On the existence of periodic points of \( f \) of all odd periods \( \geq m + 2n + 2 \) in \([\mu'_{m+1}, \mu_m] \subset [\tilde{u}_{0}, \tilde{v}] \subset [u_1, v] \) for each \( n \geq 1 \).

We can apply Lemma 5(2) to obtain a rough result. However, in this special case, we can get a result better than that obtained by applying Lemma 5(2). So, we argue as follows:

We first prove the following result:
Lemma 9. For each \( n \geq 1 \), all periodic points of \( f \) in \([\hat{u}'_0, \check{\mu}_{m,n}] \) (\( \subset [\check{u}'_0, v] \)) of odd periods have least periods \( \geq m + 2n \). (Note that, in Lemma 7, we have \( \geq m + 2n + 2 \))

Proof. Suppose \( f \) had a periodic point \( \check{p} \) of odd period \( \ell(\check{p}) \leq m + 2n - 2 \) in \([\hat{u}'_0, \check{\mu}_{m,n}] \) (\( \subset [\check{u}'_0, v] \)).

Since \( f(x) > z \geq z_0 \) for all \( d < x < z_0 \) and since \( \ell(\check{p}) \) is odd, we have \( \ell(\check{p}) \geq 3 \). Consequently, \( f^{\ell(\check{p})}(\check{u}_0) \in \{z_0, f(z_0), f(d)\} \).

Since \( m + 2n - 2 \geq \ell(\check{p}) \geq 3 \), by applying Lemma 3 with \( f^3(\check{u}'_0) = f(d) \) and \( f^{\ell(\check{p})}(\check{p}) = \check{p} \), there is a periodic point \( \check{w}_{m+2n-2} \) of \( f \) in \([\check{u}'_0, \check{p}] \) (\( \subset [\hat{u}'_0, \check{\mu}_{m,n}] \)) such that \( f^{m+2n-2}(\check{w}_{m+2n-2}) = \check{w}_{m+2n-2} \). Since \( f^{m+2n-2}(\check{u}'_0, \check{w}_{m+2n-2}) \supset \check{w}_{m+2n-2} \supset \{v\} \), there is a point \( \check{\nu}_{m,n-1} \) in \([\check{u}'_0, \check{w}_{m+2n-2}] \) such that \( f^{m+2n-2}(\check{\nu}_{m,n-1}) = v \). Consequently, since

\[
\begin{align*}
f^{m+2n}(\check{u}'_0) - d &\geq z_0 - d > 0 \quad \text{and} \quad f^{m+2n}(\check{\nu}_{m,n-1}) - d = f^2(v) - d = \min P - d < 0, \\
\end{align*}
\]

there is a point \( \check{\mu}_{m,n}^* \) in \([\check{u}'_0, \check{\nu}_{m,n-1}] \) (\( \subset [\check{u}'_0, \check{\mu}_{m,n}] \)) such that \( f^{m+2n}(\check{\mu}_{m,n}^*) = \check{d} \). Since \( \check{u}'_0 < \check{\mu}_{m,n}^* < \check{\nu}_{m,n} < \check{\mu}_{m,n} \), this contradicts the minimality of \( \check{\mu}_{m,n} \) in \([\check{u}'_0, v] \). So, we have shown that all periodic points of \( f \) in \([\check{u}'_0, \check{\mu}_{m,n}] \) of odd periods have least periods \( \geq m + 2n \). \( \square \)

Since the point \( \check{p}_{m+2n} = \min \{\check{u}'_0 \leq x \leq v : f^{m+2n}(x) = x\} \) (defined on the first layer of the basic tower) exists and satisfies \( \check{\mu}_{m,n+1} < \check{p}_{m+2n} < \check{\mu}_{m,n} \), the point

\[
\check{q}_{m+2n} = \max \{\check{\mu}_{m,n+1} \leq x \leq \check{\mu}_{m,n} : f^{m+2n}(x) = x\}
\]

exists and is \( \geq \check{p}_{m+2n} \). Therefore, there is no period-(\( m + 2n \)) point of \( f \) in \((\check{q}_{m+2n}, \check{\mu}_{m,n}] \). This, combined with the above Lemma 9, implies that, on the interval \((\check{q}_{m+2n}, \check{\mu}_{m,n}] \),

all periodic points of \( f \) of odd periods have least periods \( \geq m + 2n + 2 \).

Since, by the choice of \( \check{u}'_0 \), we have \( f^2(x) < d \) for all \( \check{u}'_0 < x < \check{\mu}_{m,n} \). Since \( f^{m+2n+2}(\check{q}_{m+2n}) - d = f^2(\check{q}_{m+2n}) - d < 0 \) and \( f^{m+2n+2}(\check{\mu}_{m,n}) - d = z_0 - d > 0 \), the point

\[
\check{\mu}'_{m,n,1} = \max \{\check{q}_{m+2n} \leq x \leq \check{\mu}_{m,n} : f^{m+2n+2}(x) = d\} = \max \{\check{\mu}_{m,n+1} \leq x \leq \check{\mu}_{m,n} : f^{m+2n+2}(x) = d\}
\]

satisfies \( \check{q}_{m+2n} < \check{\mu}'_{m,n,1} < \check{\mu}_{m,n} \). Therefore, since \( f^{m+2n+2}(\check{\mu}'_{m,n,1}) - \check{\mu}'_{m,n,1} = d - \check{\mu}'_{m,n,1} < 0 \) and \( f^{m+2n+2}(\check{\mu}_{m,n}) - \check{\mu}_{m,n} = z_0 - \check{\mu}_{m,n} > 0 \), the point \( \check{q}_{m+2n+2} = \max \{\check{\mu}_{m,n+1} \leq x \leq \check{\mu}_{m,n} : f^{m+2n+2}(x) = x\} \) satisfies \( \check{q}_{m+2n} < \check{\mu}'_{m,n,1} < \check{q}_{m+2n+2} \) (\( < \check{q}_{m+2n+4} < \check{q}_{m+2n+6} < \cdots < \check{\mu}_{m,n} \)).

Now suppose, for some \( j \geq 1 \), \( \ell(\check{q}_{m+2n+2j}) < m + 2n + 2j \). By Lemma 1, \( \ell(\check{q}_{m+2n+2j}) \) divides \( m + 2n + 2j \) and so, is odd. It follows from the previous paragraph that \( \ell(\check{q}_{m+2n+2j}) \geq m + 2n + 2 \). Since \( f^{m+2n+2}(\check{\mu}_{m,n}) = z_0 \) and \( f(\check{q}_{m+2n+2j}) = \check{q}_{m+2n+2j} \) and since \( m + 2n + 2j > \ell(\check{q}_{m+2n+2j}) \), it follows from Lemma 3 that there is a periodic point \( \check{w}_{m+2n+2j} \) of \( f \) in \((\check{q}_{m+2n+2j}, \check{\mu}_{m,n}] \) such that \( f^{m+2n+2j}(\check{w}_{m+2n+2j}) = \check{w}_{m+2n+2j} \). Since \( \check{q}_{m+2n+2j} < \check{w}_{m+2n+2j} \), this contradicts the maximality of \( \check{q}_{m+2n+2j} \) in \([\check{\mu}_{m,n+1}, \check{\mu}_{m,n}] \). Therefore, in \([\check{\mu}'_{m,n,1}, \check{\mu}_{m,n}] \), for each \( n \geq 1 \) and \( i \geq 1 \), the point \( \check{q}_{m+2n+2i} \) is a period-(\( m + 2n + 2i \)) point of \( f \).
2.3(b) On the existence of periodic points of $f$ of all even periods $\geq 2m + 4n + 6$ in $[\tilde{\nu}_{m,n}, \tilde{u}'_{m,n,1}]$ (⊂ $[\tilde{\mu}_{m,n+1}, \tilde{\mu}_{m,n}]$ < $[\tilde{u}'_0, v]$ ⊂ $[u_1, v]$) for each $n \geq 1$.

We apply Lemma 6(2) with

$$f^{m+2n+3}(\tilde{\mu}'_{m,n,1}) = f(d) \in \{z_0, f(z_0), f(d)\} \quad \text{and} \quad f^{2(m+n+1)}(\tilde{\nu}_{m,n}) = \min P$$

to obtain that, for each $i \geq 0$, the point

$$c'^{(n,2)}_{2(m+n+1)+2i} = \max \{\tilde{\nu}_{m,n} \leq x \leq \tilde{\mu}'_{m,n,1} : f^{2(m+n+1)+2i}(x) = x\}$$

exists and $\tilde{\nu}_{m,n} < c'^{(n,2)}_{2m+2n+2} < c'^{(n,2)}_{2m+2n+4} < c'^{(n,2)}_{2m+2n+6} < \cdots < \tilde{\mu}'_{m,n,1} < \tilde{\mu}_{m,n} < v$. Furthermore, for each $i \geq 0$ such that $(m + n + 1) + i \geq \max \{m + 2n + 3, m + 2\} = m + 2n + 3$, i.e., for each $i \geq n + 2$, the point $c'^{(2)}_{2(m+n+1)+2i}$ is a period-$(2(m + n + 1) + 2i)$ point of $f$, or equivalently, in $[\tilde{\nu}_{m,n}, \tilde{u}'_{m,n,1}]$ (⊂ $[\tilde{\mu}_{m,n+1}, \tilde{\mu}_{m,n}]$ ⊂ $[\tilde{u}'_0, v]$ ⊂ $[u_1, v]$),

for each $n \geq 1$ and $i \geq n + 3$, the point $c'^{(n,2)}_{2m+2n+2i}$ is a period-$(2m + 2n + 2i)$ point of $f$.

2.3(c) On the existence of periodic points of $f$ of all even periods $\geq 2m + 4n + 6$ in $[\tilde{\mu}_{m,n+1}, \tilde{\nu}_{m,n}]$ (⊂ $[\tilde{\mu}_{m,n+1}, \tilde{\mu}_{m,n}]$ ⊂ $[\tilde{u}'_0, v]$ ⊂ $[u_1, v]$) for each $n \geq 1$.

We apply Lemma 6(1) with

$$f^{m+2n+3}(\tilde{\mu}_{m,n+1}) = f(d) \in \{z_0, f(z_0), f(d)\} \quad \text{and} \quad f^{2(m+n+1)}(\tilde{\nu}_{m,n}) = \min P$$

to obtain that, for each $i \geq 0$, the point

$$\bar{c}^{(n,2)}_{2(m+n+1)+2i} = \min \{\tilde{\mu}_{m,n+1} \leq x \leq \tilde{\nu}_{m,n} : f^{2(m+n+1)+2i}(x) = x\}$$

exists and $\tilde{\mu}_{m,n+1} < \cdots < \bar{c}^{(n,2)}_{2m+2n+6} < \bar{c}^{(n,2)}_{2m+2n+4} < \bar{c}^{(n,2)}_{2m+2n+2} < \tilde{\nu}_{m,n} < \tilde{\mu}_{m,n} < v$. Furthermore, for each $i \geq 0$ such that $(m + n + 1) + i \geq \max \{m + 2n + 3, m + 2\} = m + 2n + 3$, i.e., for each $i \geq n + 2$, the point $\bar{c}^{(n,2)}_{2(m+n+1)+2i}$ is a period-$(2(m + n + 1) + 2i)$ point of $f$, or equivalently, in $[\tilde{\mu}_{m,n+1}, \tilde{\nu}_{m,n}]$ (⊂ $[\tilde{\mu}_{m,n+1}, \tilde{\mu}_{m,n}]$ ⊂ $[\tilde{u}'_0, v]$ ⊂ $[u_1, v]$),

for each $n \geq 1$ and $i \geq n + 3$, the point $\bar{c}^{(n,2)}_{2m+2n+2i}$ is a period-$(2m + 2n + 2i)$ point of $f$.

Recall that, on $[v, z_0]$, we define $\hat{u}'_1 = \max \{v \leq x < z_0 : f^2(x) = x\}$. However, for our purpose, we shall use the point

$$\hat{u}_0 = \min \{v \leq x \leq z_0 : f^2(x) = d\} (\leq \hat{u}'_1)$$
instead of the point \( \hat{u}_1 \) and consider the interval \([v, \hat{u}_0]\) because on this interval, we have

\[
f^2(x) < d \quad \text{for all} \quad v < x < \hat{u}_0.
\]

Since \( f^{m+2}([v, \hat{u}_0]) \supset [\min P, f(z_0)] \supset [\min P, z_0] \supset \{d\} \), the point

\[
\hat{\mu}'_{m,1} = \max \{ v \leq x \leq \hat{u}_0 : f^{m+2}(x) = d \}
\]

exists. Since \( f^{m+4}([\hat{\mu}'_{m,1}, \hat{u}_0]) \supset f^2([d, z_0]) \supset f^2([v, z_0]) \supset [\min P, z_0] \supset \{d\} \), the point

\[
\hat{\mu}'_{m,2} = \max \{ \hat{\mu}'_{m,1} \leq x \leq \hat{u}_0 : f^{m+4}(x) = d \} = \max \{ v \leq x \leq \hat{u}_0 : f^{m+4}(x) = d \}
\]

exists and is \( > \hat{\mu}'_{m,1} \). Inductively, for each \( n \geq 1 \), let (note the relationship between the subscript of \( \hat{\mu}'_{m,n} \) and the superscript of \( f^{m+2n} \) in the definition of \( \hat{\mu}'_{m,n} \))

\[
\hat{\mu}'_{m,n} = \max \{ v \leq x \leq \hat{u}_0 : f^{m+2n}(x) = d \}.
\]

Then we have \( v < \hat{\mu}'_{m,1} < \hat{\mu}'_{m,2} < \hat{\mu}'_{m,3} < \cdots < \hat{u}_0 \). For each \( n \geq 1 \), let \( \hat{\nu}_{m,n} \) be a point in \([\hat{\mu}'_{m,n}, \hat{u}_0] \) such that \( f^{m+2n}(\hat{\nu}_{m,n}) = v \). Then it turns out that \( \hat{\mu}'_{m,n} < \hat{\nu}_{m,n} < \hat{\mu}'_{m,n+1} \).

Furthermore, on the interval \([\hat{\mu}'_{m,n}, \hat{\mu}'_{m,n+1}] \subset [v, \hat{u}_0] \), we have

\[
f^{m+2n+2}(\hat{\mu}'_{m,n}) = z_0, \quad f^{m+2n+2}(\hat{\nu}_{m,n}) = \min P, \quad f^{m+2n+2}(\hat{\mu}'_{m,n+1}) = d \quad \text{and}
\]

the graph of \( y = f^{m+2n+2}(x) \) looks ‘roughly’ like a skew ‘V’ shape at the points \( \hat{\mu}'_{m,n} \), \( \hat{\nu}_{m,n} \), \( \hat{\mu}'_{m,n+1} \) (meaning that the graph of \( y = f^{m+2n+2}(x) \) passes through the 3 points \( (\hat{\mu}'_{m,n}, z_0) \), \( (\hat{\nu}_{m,n}, \min P) \), \( (\hat{\mu}'_{m,n+1}, d) \) on the \( x-y \) plane with \( y \)-coordinates \( z_0 \), \( \min P \), \( d \) respectively) just like that of \( y = f^2(x) \) at the points \( d, v, \hat{u}_0 \) on the interval \([d, \hat{u}_0] \).

Now, for each \( n \geq 1 \) and all \( i \geq 1 \), let (the number 2 in the superscript \( (n, 2) \) indicates the second layer)

\[
\hat{p}'_{m,2n+2i}^{(n,2)} = \min \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{m+2n+2i}(x) = x \} \quad (< \hat{\nu}_{m,n}) \quad \text{and}
\]

\[
\hat{\mu}_{m,n,i} = \min \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{m+2n+2i}(x) = d \} \quad (< \hat{\nu}_{m,n})
\]

(note the relationship between the subscript of \( \hat{\mu}_{m,n,i} \) and the superscript of \( f^{m+2n+2i} \) in the definition of \( \hat{\mu}_{m,n,i} \). Here the \( n \) and \( i \) in the subscript of \( \hat{\mu}_{m,n,i} \) indicate the \( i \)th point of the sequence \(< \hat{\mu}_{m,n,i} \) > in the \( n \)th interval \([\hat{\mu}'_{m,n}, \hat{\mu}'_{m,n+1}] \). We need these points \( \hat{\mu}_{m,n,i} \)'s to continue the basic tower-building process into the third layer). Note that, by the choice of the point \( \hat{u}_0 \), we have \( f^2(x) < d \) for all \( v < x < \hat{u}_0 \). By arguing as before, we obtain that

\[
\hat{\mu}'_{m,n} < \cdots < \hat{p}_{m+2n+6}^{(n,2)} < \hat{\mu}_{m,n,3} < \hat{p}_{m+2n+4}^{(n,2)} < \hat{\mu}_{m,n,2} < \hat{p}_{m+2n+2}^{(n,2)} < \hat{\mu}_{m,n,1} < \hat{\nu}_{m,n} < \hat{\mu}'_{m,n+1}.
\]

In the following, we shall show that, for each \( n \geq 1 \), with \( x \)-coordinates moving from point \( \hat{\mu}'_{m,n} \) via point \( \hat{\nu}_{m,n} \) to point \( \hat{\mu}'_{m,n+1} \) (see Figure 4):
(a) (odd periods) the point \( \hat{p}_{m,n}^{(n,2)} = \min \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{m+2n+2i}(x) = x \} = \min \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{m+2n+2i}(x) = x \} \) exists and is a period-(\( m + 2n + 2i \)) point of \( f \) for each \( i \geq 1 \);

(b) (even periods) the point \( \hat{c}_{2m+2n+2i}^{(n,2)} = \min \{ \hat{\mu}_{m,n,1} \leq x \leq \hat{\nu}_{m,n} : f^{2m+2n+2i}(x) = x \} = \min \{ \hat{\mu}_{m,n,1} \leq x \leq \hat{\mu}'_{m,n+1} : f^{2m+2n+2i}(x) = x \} \) exists for each \( i \geq 1 \) and is a period-(\( 2m + 2n + 2i \)) point of \( f \) for each \( i \geq n + 3 \);

(c) (even periods) the point \( \hat{c}'_{2m+2n+2i}^{(n,2)} = \max \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{2m+2n+2i}(x) = x \} = \min \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{2m+2n+2i}(x) = x \} \) exists for each \( i \geq 1 \) and is a period-(\( 2m + 2n + 2i \)) point of \( f \) for each \( i \geq n + 3 \).

Figure 5: A compartment of the second layer of the basic tower of periodic points of \( f \) associated with \( P \) in the interval \([\hat{\mu}'_{m,n}, \hat{\mu}'_{m,n+1}] \subset [v, \hat{u}_0] \), where \( \hat{\nu}_{m,n} \) is an auxiliary point in \([\hat{\mu}'_{m,n}, \hat{\mu}'_{m,n+1}] \) such that \( f^{m+2n}(\hat{\nu}_{m,n}) = v \) which is needed only to determine the ordering of the following points:

and, for each \( i \geq 1 \),

\[
\hat{\mu}_{m,n,i} = \min \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{m+2n+2i}(x) = d \};
\]

\[
\hat{p}_{m+2n+2i}^{(n,2)} = \min \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{m+2n+2i}(x) = x \};
\]

\[
\hat{c}_{2m+2n+2i}^{(n,2)} = \min \{ \hat{\mu}_{m,n,1} \leq x \leq \hat{\mu}'_{m,n+1} : f^{2m+2n+2i}(x) = x \};
\]

\[
\hat{c}'_{2m+2n+2i}^{(n,2)} = \max \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{2m+2n+2i}(x) = x \}.
\]

For each fixed \( n \geq 1 \), the collection of all these periodic points \( \hat{p}_{m+2n+2i}^{(n,2)}, \hat{c}_{2m+2n+2i}^{(n,2)} \), \( i \geq 1 \), is called a compartment of the second layer of the basic tower of periodic points of \( f \) associated with \( P \).

2.4(a) On the existence of periodic points of \( f \) of all odd periods \( \geq m + 2n + 2 \) in \([\hat{\mu}'_{m,n}, \hat{\mu}_{m,n,1}] \subset \hat{\mu}_{m,n,1} \leq \hat{u}_0 \subset [v, z_0] \) for each \( n \geq 1 \).

We can apply Lemma 5(2) to obtain a rough result. However, in this special case, we can get a result better than that obtained by applying Lemma 5(2). So, we argue as follows:

We first prove the following result:
Lemma 10. For each \( n \geq 1 \), all periodic points of \( f \) in \([\hat{\mu}_{m,n}^*, \hat{u}_0]\) (⊂ \([v, \hat{u}_0]\)) of odd periods have least periods \( \geq m + 2n \). (Note that, in Lemma 7, we have \( \geq m + 2n + 2 \))

Proof. Suppose \( f \) had a periodic point \( \hat{p} \) of odd period \( \ell(\hat{p}) \leq m + 2n - 2 \) in \((\hat{\mu}_{m,n}^*, \hat{u}_0)\) (⊂ \([v, \hat{u}_0]\)).

Since \( f(x) > z \geq z_0 \) for all \( d < x < z_0 \) and since \( \ell(\hat{p}) \) is odd, we have \( \ell(\hat{p}) \geq 3 \). Consequently, \( f^{\ell(\hat{p})}(\hat{u}_0) \in \{z_0, f(z_0), f(d)\} \).

Since \( m + 2n - 2 \geq \ell(\hat{p}) \geq 3 \), by Lemma 3 with \( f^3(\hat{u}_0) = f(d) \) and \( f^{\ell(\hat{p})}(\hat{p}) = \hat{p} \), there is a point \( \hat{w}_{m+2n-2} \) of \( f \) in \([\hat{p}, \hat{u}_0]\) (⊂ \((\hat{\mu}_{m,n}^*, \hat{u}_0)\) \⊂ \([v, \hat{u}_0]\)) such that \( f^{m+2n-2}(\hat{w}_{m+2n-2}) = \hat{w}_{m+2n-2} \). Since \( f^2(x) < d \) for all \( v < x < \hat{u}_0 \), we have \( f^{m+2n}(\hat{w}_{m+2n-2}, \hat{u}_0) \supseteq [f^2(\hat{w}_{m+2n-2}), z_0] \supseteq [d, z_0] \supseteq \{d\} \). So, there is a point \( \hat{mu}_{m,n}\) in \([\hat{w}_{m+2n-2}, \hat{u}_0]\) (⊂ \((\hat{\mu}_{m,n}^*, \hat{u}_0)\)) such that \( f^{m+2n}(\hat{mu}_{m,n}) = d \). Since \( \hat{mu}_{m,n} < \hat{w}_{m+2n-2} < \hat{mu}_{m,n}^* < \hat{u}_0 \), this contradicts the maximality of \( \hat{mu}_{m,n}\) in \([v, \hat{u}_0]\).
Therefore, we have shown that all periodic points of \( f \) in \((\hat{\mu}_{m,n}^*, \hat{u}_0)\) of odd periods have least periods \( \geq m + 2n \).

Now suppose, for some \( j \geq 1 \), the least period \( \ell(\hat{p}_{m+2n+2j}) \) of \( \hat{p}_{m+2n+2j}^{(n,2)} \) is \(< m + 2n + 2j \). By Lemma 1, \( \ell(\hat{p}_{m+2n+2j}^{(n,2)}) \) divides \( m + 2n + 2j \) and so, is odd.

If \( \ell(\hat{p}_{m+2n+2j}^{(n,2)}) \geq m + 2n + 2 \), then, since \( f^{m+2n+2}(\hat{\mu}_{m,n}^* = z_0 \) and \( f^{\ell(\hat{p}_{m+2n+2j}^{(n,2)})}(\hat{p}_{m+2n+2j}^{(n,2)}) = \hat{p}_{m+2n+2j}^{(n,2)} \) and since \( m + 2n + 2j \geq \ell(\hat{p}_{m+2n+2j}^{(n,2)}) \), it follows from Lemma 3 that there is a periodic point \( \hat{w}_{m+2n+2j} \) of \( f \) in \([\hat{p}_{m+n}^{(n,2)}, \hat{p}_{m+2n+2j}^{(n,2)}\]) such that \( f^{m+2n+2j}(\hat{w}_{m+2n+2j}) = \hat{w}_{m+2n+2j} \).
Since \( \hat{p}_{m+n}^* < \hat{w}_{m+2n+2j} < \hat{p}_{m+n+2j}^{(n,2)} < \hat{p}_{m+n+1}^{(n,2)} \), this contradicts the maximality of \( \hat{p}_{m+2n+2j}^{(n,2)} \) in \([\hat{p}_{m+n}^{(n,2)}, \hat{p}_{m+n+1}^{(n,2)}\])$. 

If \( \ell(\hat{p}_{m+2n+2j}^{(n,2)}) = m + 2n \), then, let \( r = (m + 2n + 2j)/\ell(\hat{p}_{m+2n+2j}^{(n,2)}) \). Suppose \( r > 3 \).
Then \( f^{3\ell(\hat{p}_{m+2n+2j}^{(n,2)})}(\hat{p}_{m+2n+2j}^{(n,2)}) = \hat{p}_{m+2n+2j}^{(n,2)} \) and \( f^{3\ell(\hat{p}_{m+2n+2j}^{(n,2)})}(\hat{\mu}_{m,n}^{(n,2)}) = f^{3(m+2n)}(\hat{\mu}_{m,n}^{(n,2)}) = z_0 \). Since \( m + 2n + 2j > 3\ell(\hat{p}_{m+2n+2j}^{(n,2)}) \), it follows from Lemma 3 that there is a periodic point \( \hat{w}_{m+2n+2j}^* \) of \( f \) in \([\hat{p}_{m+n}^{(n,2)}, \hat{p}_{m+2n+2j}^{(n,2)}\]) such that \( f^{m+2n+2j}(\hat{w}_{m+2n+2j}^*) = \hat{w}_{m+2n+2j}^* \) which contradicts the minimality of \( \hat{p}_{m+2n+2j}^{(n,2)} \) in \([\hat{p}_{m+n}^{(n,2)}, \hat{p}_{m+n+1}^{(n,2)}\])$. Therefore, if \( m + 2n = \ell(\hat{p}_{m+n+2j}^{(n,2)}) < m + 2n + 2j \), then \( m + 2n + 2j = 3(m + 2n) \). That is, the point \( \hat{p}_{3(m+2n)}^{(n,2)} \) is either a period-(\( m + 2n \)) point or a period-(\( 3(m + 2n) \)) point of \( f \).

We now show that the point \( \hat{p}_{3(m+2n)}^{(n,2)} \) is actually a period-(\( 3(m + 2n) \)) point of \( f \). Since

\[ f^{m+2n}(\hat{\mu}_{m,n}^* - \hat{\mu}_{m,n}^*) = d - \hat{\mu}_{m,n}^* < 0 \quad \text{and} \quad f^{m+2n}(\hat{u}_0) - \hat{u}_0 \geq z_0 - \hat{u}_0 > 0, \]

the point \( \hat{p}_{m+2n}^* = \min\{\hat{\mu}_{m,n}^* \leq x \leq \hat{u}_0 : f^{m+2n}(x) = x \} \) exists. By the choice of \( \hat{u}_0 \), we have \( f^2(x) < d \) for all \( v < x < \hat{u}_0 \).
Since

\[ f^{m+2n+2}(\hat{p}_{m+2n}^*) - d = f^2(\hat{p}_{m+2n}^*) - d < 0 \quad \text{and} \quad f^{m+2n+2}(\hat{u}_0) - d \geq z_0 - d > 0, \]
we obtain that \( \hat{\mu}_{m,n}^* < \hat{\mu}_{m+2n} < \hat{\mu}_{m,n+1}^* \). Recall that we have \( f^2(x) < x \) for all \( v < x < z_0 \).
Lemma 3 with

Consequently, the point \( \hat{x}_m \) is the

Therefore, we have \( f^{m+2n+2}(\hat{p}_{m+2n}) - \hat{p}_{m+2n} = f^2(\hat{p}_{m+2n}) - \hat{p}_{m+2n} < 0 \) and \( f^{m+2n+2}(\hat{p}_{m,n}) - \hat{p}_{m,n} = z_0 - \hat{p}_{m,n} > 0 \). Consequently, the point \( \hat{p}_{m+2n+2} = \min \{ \hat{p}_{m,n} : f^{m+2n+2}(x) = x \} = \min \{ \hat{p}_{m,n} : f^{m+2n+2}(x) = x \} \) is \( \hat{p}_{m+2n} \). Since \( 3(m+2n) > m + 2n + 2 \), it follows from Lemma 3 with

\[
\begin{align*}
\hat{p}_{m,n} &\in \{ z_0, f(z_0), f(d) \} \\
\hat{d} &\in \{ z_0, f(z_0), f(d) \} \\
\end{align*}
\]

that \( \hat{p}_{m,n} < \cdots < \hat{p}_{3(m+2n)} < \cdots < \hat{p}_{m+2n+4} < \hat{p}_{m+2n+2} < \hat{p}_{m+2n} \). Since the point \( \hat{p}_{m+2m} \) is the smallest point in \( [\hat{p}_{m,n}, \hat{p}_{m+1,n}] \) that satisfies \( f^{m+2n}(x) = x \) and \( \hat{p}_{3(m+2n)} < \hat{p}_{m+2m} \), we see that \( \hat{p}_{3(m+2n)} \) cannot be a period-\((m + 2n)\) point of \( f \). Therefore, \( \hat{p}_{3(m+2n)} \) is a period-\((3(m + 2n))\) point of \( f \). This, combined with the above, implies that, in \( [\hat{p}_{m,n}, \hat{p}_{m+1,n}] \subset [v, \hat{u}_0] \subset [v, z_0] \),

for each \( n \geq 1 \) and \( i \geq 1 \), the point \( \hat{p}_{m+2n+2i} \) is a period-\((m + 2n + 2i)\) point of \( f \).

2.4(b) On the existence of periodic points of \( f \) of all even periods \( \geq 4m + 4n + 6 \) in \([\hat{p}_{m,n}, \hat{p}_{m+1,n}] \subset [v, \hat{u}_0] \subset [v, z_0] \) for each \( n \geq 1 \).

We apply Lemma 6(1) with

\[
\begin{align*}
f^{m+2n+3}(\hat{p}_{m,n}) &\in \{ z_0, f(z_0), f(d) \} \\
f^{2(m+n+1)}(\hat{v}_{m,n}) &\in \{ z_0, f(z_0), f(d) \} \\
\end{align*}
\]

to obtain that, for each \( i \geq 0 \), the point

\[
\hat{c}_{2(m+n+1)+2i} = \min \{ \hat{p}_{m,n} \leq \hat{v}_{m,n} : f^{2(m+n+1)+2i}(x) = x \} = \min \{ \hat{p}_{m,n} \leq \hat{v}_{m,n+1} : f^{2(m+n+1)+2i}(x) = x \}
\]

exists and \( \hat{p}_{m,n} < \hat{p}_{m,n} < \cdots < \hat{c}_{2m+2n+6} < \hat{c}_{2m+2n+4} < \hat{c}_{2m+2n+2} < \hat{v}_{m,n} < \hat{p}_{m,n+1} = \hat{u}_0 \). Furthermore, for each \( i \geq 0 \) such that \( m + n + 1 + i \geq m + (m + 2n + 3) = 2m + 2n + 3 \), i.e., for each \( i \geq n + 2 \), the point \( \hat{c}_{2(m+n+1)+2i} \) is a period-\((2(m+n+1)+2i)\) point of \( f \), or, equivalently, in \([\hat{u}_{m,n}, \hat{u}_{m+1,n}] \subset [v, \hat{u}_0] \subset [v, z_0] \),

for each \( n \geq 1 \) and \( i \geq m + n + 3 \), the point \( \hat{c}_{2m+2n+2i} \) is a period-\((2m + 2n + 2i)\) point of \( f \).

2.4(c) On the existence of periodic points of \( f \) of all even periods \( \geq 4m + 4n + 6 \) in \([\hat{p}_{m,n}, \hat{u}_{m+1,n}] \subset [v, \hat{u}_0] \subset [v, z_0] \) for each \( n \geq 1 \).

Recall that \( \hat{v}_{m,n} \) is a point in \([\hat{p}_{m,n}, \hat{p}_{m+1,n}] \subset [v, \hat{u}_0] \subset [v, z_0] \) such that \( f^{m+2n}(\hat{v}_{m,n}) = v \).

We apply Lemma 6(2) with

\[
\begin{align*}
f^{m+2n+3}(\hat{p}_{m,n+1}) &\in \{ z_0, f(z_0), f(d) \} \\
f^{2(m+n+1)}(\hat{v}_{m,n}) &\in \{ z_0, f(z_0), f(d) \} \\
\end{align*}
\]

and \( f^{2(m+n+1)}(\hat{v}_{m,n}) = \min P \)}
to obtain that, for each \( i \geq 0 \), the point
\[
\hat{c}^{(n, 2)}_{2(m+n+1) + 2i} = \max \{ \hat{\mu}_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{2(m+n+1)+2i}(x) = x \}
\]
\[
= \max \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{2(m+n+1)+2i}(x) = x \}
\]
exists and \( v < \hat{\mu}'_{m,n} < \hat{\mu}_{m,n} < \hat{c}^{(n, 2)}_{2m+2n+2} < \hat{c}^{(n, 2)}_{2m+2n+4} < \hat{c}^{(n, 2)}_{2m+2n+6} < \cdots < \hat{\mu}'_{m,n+1} < \hat{\mu}_0 \).
Furthermore, for each \( i \geq 0 \) such that \( (m + n + 1) + i \geq m + (m + 2n + 3) = 2m + 2n + 3 \), i.e., for each \( i \geq m + n + 2 \), the point \( \hat{c}^{(n, 2)}_{2(m+n+1)+2i} \) is a period-\((2(m + n + 1) + 2i)\) point of \( f \), or, equivalently, in \( [\hat{\mu}_{m,n}, \hat{\mu}'_{m,n+1}] \subset [v, \hat{\mu}_0] \subset [v, z_0] \), for each \( n \geq 1 \) and \( i \geq m + n + 3 \), the point \( \hat{c}^{(n, 2)}_{2m+2n+2i} \) is a period-\((2m+2n+2i)\) point of \( f \).

'Symmetrically', in the interval \([\bar{u}'_n, z_0]\), for each \( n \geq 1 \), we can find similar sequences on \([\bar{u}'_n, \bar{u}'_{n+1}]\) with some variations.

Let \( \hat{\mu}_0 = \min \{ v \leq x \leq z_0 : f^2(x) = d \} \) and, for each \( n \geq 1 \), let \( \bar{u}'_n = \max \{ v \leq x \leq z_0 : f^{2n}(x) = d \} \) be defined as before. Then we have \( v < \hat{\mu}_0 < \bar{u}'_1 < \bar{u}'_2 < \bar{u}'_3 < \cdots < z_0, f^2(z_0) = z_0 \) and \( f^{2n}(\bar{u}'_n) = d \). In particular, \( f^{2n}([\bar{u}'_n, z_0]) \supset [d, z_0] \supset \{ v \} \). Let \( \hat{\nu}_n \) be a point in \( [\bar{u}'_n, z_0] \) such that \( f^{2n}(\hat{\nu}_n) = v \). Then, since \( f^2(v) = \min P \), we obtain that \( f^{2n+2}([\hat{\nu}_n, z_0]) \supset [\min P, z_0] \supset \{ d \} \). So, the point \( \hat{\nu}_n \) happens to satisfy that \( \bar{u}'_n < \hat{\nu}_n < \bar{u}'_{n+1} \). On the interval \([\bar{u}'_n, \bar{u}'_{n+1}]\), we have
\[
f^{2n+2}(\bar{u}'_n) = z_0, \quad f^{2n+2}(\hat{\nu}_n) = \min P, \quad f^{2n+2}(\bar{u}'_{n+1}) = d \quad \text{and}
\]
the graph of \( y = f^{2n+2}(x) \) looks 'roughly' like a skew 'V' shape (at the 3 points \( \bar{u}'_n, \hat{\nu}_n, \bar{u}'_{n+1} \)) (meaning that the graph of \( y = f^{2n+2}(x) \) passes through the 3 points \( (\bar{u}'_n, z_0), (\hat{\nu}_n, \min P), (\bar{u}'_{n+1}, d) \)) on the \( x-y \) plane with \( y \)-coordinates \( z_0, \min P, d \) respectively) just like that of \( y = f^2(x) \) at the 3 points \( d, v, \hat{\mu}_0 \) on the interval \([d, \hat{\mu}_0] \).

In the following, we show that, for each \( n \geq 1 \), there exist 4 sequences of points in \([\bar{u}'_n, \bar{u}'_{n+1}]\) (the 2 in the superscript \((n, 2)\) below indicates second layer):
\[
\bar{u}'_n < \cdots < \bar{u}_{n,3} < \bar{u}_{n,2} < \bar{u}_{n,1} < \nu_n < \bar{u}'_n \quad \text{and}
\]
\[
\bar{u}'_n < \cdots < \bar{c}^{(n, 2)}_{2n+4} < \bar{c}^{(n, 2)}_{2n+2} < \bar{c}^{(n, 2)}_{2n+1} < \cdots < \bar{c}^{(n, 2)}_{2n+6} < \bar{c}^{(n, 2)}_{2n+4} < \bar{c}^{(n, 2)}_{2n+2} < \nu_n < \bar{v}_n
\]
\[
< q^{(n, 2)}_{m+2n+2} < q^{(n, 2)}_{m+2n+4} < q^{(n, 2)}_{m+2n+6} < \cdots < \bar{u}'_{n+1}
\]
such that, for each \( i \geq 1 \), \( \bar{u}_{n,i} = \min \{ \bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{2n+2i}(x) = d \} \) (note the relationship between the subscript of \( \bar{u}_{n,i} \) and the superscript of \( f^{2n+2i} \), we need these points \( \bar{u}_{n,i} \)’s to continue the basic tower-building process to the third layer) and, with \( x \)-coordinates moving from point \( \bar{u}'_n \) via point \( \bar{v}_n \) to point \( \bar{u}'_{n+1} \) (see Figure 6):

(a) (even periods) the point \( \bar{c}^{(n, 2)}_{2n+2i} = \min \{ \bar{u}'_n \leq x \leq \bar{v}_n : f^{2n+2i}(x) = x \} = \min \{ \bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{2n+2i}(x) = x \} \) exists and is a period-\( (2n + 2i) \) point of \( f \) for each \( i \geq 1 \);
(b) (odd periods) the point \( \bar{u}^{(n,2)}_{m+2n+2i} = \min \{ \bar{u}_{n,1} \leq x \leq \bar{v}_n : f^{m+2n+2i}(x) = x \} = \min \{ \bar{u}_{n,1} \leq x \leq \bar{u}'_{n+1} : f^{m+2n+2i}(x) = x \} \) exists and is a period-\((m + 2n + 2i)\) point of \( f \) for each \( i \geq 1 \);

(c) (odd periods) the point \( \bar{q}^{(n,2)}_{m+2n+2i} = \max \{ \bar{v}_n \leq x \leq \bar{u}'_{n+1} : f^{m+2n+2i}(x) = x \} = \max \{ \bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{m+2n+2i}(x) = x \} \) exists and is a period-\((m + 2n + 2i)\) point of \( f \) for each \( i \geq 1 \).

\[
\begin{array}{ccccccc}
\bar{u}'_{n} & \cdots & \bar{u}_{n,3} & \bar{u}_{n,2} & \bar{u}_{n,1} & \bar{v}_{n} & \bar{u}'_{n+1} \\
\cdots & \bar{c}^{(n,2)}_{2n+6} & \bar{c}^{(n,2)}_{2n+4} & \bar{c}^{(n,2)}_{2n+2} & \bar{p}^{(n,2)}_{m+2n+2} & \bar{q}^{(n,2)}_{m+2n+2} & \cdots
\end{array}
\]

Figure 6: A compartment of the second layer of the basic tower of periodic points of \( f \) associated with \( P \) in the interval \([\bar{u}_n', \bar{u}'_{n+1}] \) such that \( f^{2n}(\bar{v}_n) = v \), where \( \bar{v}_n \) is an auxiliary point in \([\bar{u}_n', \bar{u}'_{n+1}] \) such that \( f^{2n}(\bar{v}_n) = v \), which is needed only to determine the ordering of the following points:

and, for each \( i \geq 1 \),

\[
\begin{align*}
\bar{u}_{n,i} &= \min \{ \bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{2n+2i}(x) = d \}; \\
\bar{c}^{(n,2)}_{2n+2i} &= \min \{ \bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{2n+2i}(x) = x \}; \\
\bar{p}^{(n,2)}_{m+2n+2i} &= \min \{ \bar{u}_{n,1} \leq x \leq \bar{u}'_{n+1} : f^{m+2n+2i}(x) = x \}; \\
\bar{q}^{(n,2)}_{m+2n+2i} &= \max \{ \bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{m+2n+2i}(x) = x \}.
\end{align*}
\]

(The relative locations of the periodic points \( \bar{c}^{(n,2)}_{2n+2i} \)'s with respect to \( \bar{u}_{n,i} \)'s are not known.)

For each fixed \( n \geq 1 \), the collection of all these periodic points \( \bar{c}^{(n,2)}_{2n+2i}, \bar{p}^{(n,2)}_{m+2n+2i}, \bar{q}^{(n,2)}_{m+2n+2i}, i \geq 1 \), is called a compartment of the second layer of the basic tower of periodic points of \( f \) associated with \( P \).

2.5(a) On the existence of periodic points of \( f \) of all even periods \( \geq 2n + 2 \) in \([\bar{u}_n', \bar{u}'_{n+1}] \subset [\bar{u}_n', \bar{u}'_{n+1}] \subset [v, z_0] \) for each \( n \geq 1 \).

In this case, we can apply Lemma 6(1) to get a rough result. However, since we can obtain a better result, we argue as follows:

For each \( n \geq 1 \) and \( i \geq 1 \), we define \( \bar{\nu}_n, \bar{u}_{n,i}, \) and \( \bar{c}^{(n,2)}_{2n+2i} \) in \([\bar{u}_n', \bar{u}'_{n+1}] \) 'symmetrically' with respect to \( \nu_n, u_{n,i} \) and \( c^{(n,2)}_{2n+2i} \) in \([u_{n+1}, u_n] \) as follows:
Let \( \bar{u}_n \) be a point in \([\bar{u}_n', \bar{u}_{n+1}']\) such that \( f^{2n}(\bar{u}_n) = v \). By arguing as those on \([u_{n+1}, u_n]\) and since \( f^{2}(x) < x \) for all \( v < x < z_0 \), we obtain that, for each \( i \geq 1 \), the points

\[
\bar{u}_{n,i} = \min \{ \bar{u}'_n \leq x \leq \bar{u}_n : f^{2n+2i}(x) = d \} = \min \{ \bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{2n+2i}(x) = d \}
\]

and

\[
\bar{c}_{2n+2i}^{(n,2)} = \min \{ \bar{u}'_n \leq x \leq \bar{u}_{n,i} : f^{2n+2i}(x) = x \} = \min \{ \bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{2n+2i}(x) = x \}
\]

exist and it is easy to see that \( \bar{u}_n' < \cdots < \bar{u}_{n,3} < \bar{u}_{n,2} < \bar{u}_{n,1} < \bar{u}_n \) and, by combining with Lemma 4(2), we have

\[
d < f^{2j}(x) < z_0 \quad \text{for all} \quad \bar{u}_n' < x < \bar{u}_{n,i} \quad \text{for all} \quad 1 \leq j \leq n + i.
\]

Consequently, since \( f(x) > z > z_0 \) for all \( v < x < z_0 \), we obtain that

all periodic points of \( f \) in \([\bar{u}_n', \bar{u}_{n,1}]\) of odd periods have least periods \( \geq 2n + 2i + 3 \).

This fact will be used in §3(5). As for the periodic points \( \bar{c}_{2n+2i}^{(n,2)} \)'s, we note that \( f^{2n+2}(\bar{c}_{2n+2}^{(n,2)}) = \bar{c}_{2n+2}^{(n,2)} \), since \( f^{2n+2}(\bar{u}_n') = z_0 \in \{ z_0, f(z_0), f(d) \} \), it follows from Lemma 3 that

\[
\bar{u}_n' < \cdots < \bar{c}_{2n+6}^{(n,2)} < \bar{c}_{2n+4}^{(n,2)} < \bar{c}_{2n+2}^{(n,2)} < \bar{u}_{n,1} < \bar{u}_n < \bar{u}_{n,1}.
\]

To find the least periods of \( \bar{c}_{2n+2i}^{(n,2)} \)'s with respect to \( f \), our arguments are similar to those of the counterparts \( \bar{c}_{2n+2i}^{(n,2)} \)'s on \([u_{n+1}, u_n]\). Let \( \ell(\bar{c}_{2n+2i}^{(n,2)}) \) denote the least period of \( \bar{c}_{2n+2i}^{(n,2)} \) with respect to \( f \). Then, by (††) above, \( \ell(\bar{c}_{2n+2i}^{(n,2)}) \) is even and, by the following Lemma 4(3),

\[
f \text{ has no periodic points of periods } \leq 2n + 1 \text{ in } [\bar{u}_n', z_0],
\]

we obtain that \( \ell(\bar{c}_{2n+2i}^{(n,2)}) \) is even and \( \geq 2n + 2 \).

Suppose \( (2n + 2) \leq \ell(\bar{c}_{2n+2}^{(n,2)}) \) \( < 2n + 2j \) for some \( j \geq 1 \). Then since \( 2n + 2j > \ell(\bar{c}_{2n+2}^{(n,2)}) \) and \( \ell(\bar{c}_{2n+2}^{(n,2)}) \geq 2n + 2 \), it follows from Lemma 3 with \( f^{2n+2}(\bar{c}_{2n+2}^{(n,2)}) = \bar{c}_{2n+2}^{(n,2)} \) and \( f^{2n+2}(\bar{u}_n') = z_0 \) that there is a periodic point \( \bar{c}_{2n+2}^{*} \) of \( f \) such that \( \bar{u}_n' \leq \bar{c}_{2n+2}^{*} < \bar{c}_{2n+2j}^{*} < \bar{u}_{n+1} \) and \( f^{2n+2j}(\bar{c}_{2n+2}^{*}) = \bar{c}_{2n+2j}^{*} \). This contradicts the minimality of \( \bar{c}_{2n+2j}^{(n,2)} \) in \([\bar{u}_n', \bar{u}_{n,1}]\). Therefore,

\[
\ell(\bar{c}_{2n+2j}^{(n,2)}) = 2n + 2j \quad \text{and so, we have shown that, in } [\bar{u}_n', \bar{u}_{n,1}] \subset [v, z_0],
\]

for each \( n \geq 1 \) and \( i \geq 1 \), \( \bar{c}_{2n+2i}^{(n,2)} \) is a period-\((2n+2i)\) point of \( f \).

Note that, for each \( n \geq 1 \), we have \( \bar{c}_{2n+2}^{(n,2)} = \bar{c}_{2n+2} \) (these points \( \bar{c}_{2n+2} \)'s are defined in §1(5)). That is, the second layer periodic points contain some (but not all) periodic points on the first layer.

Unfortunately, here we can not determine the relative locations of the periodic points \( \bar{c}_{2n+2i}^{(n,2)} \)'s with respect to the points \( \bar{u}_{n,i} \)'s which are needed to continue the basic tower-building process to the third layer.

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2.5(b) On the existence of periodic points of \( f \) of all odd periods \( \geq m + 2n + 2 \) in \([\bar{u}_{n,1}, \bar{v}_n] \subset [v, z_0]\) for each \( n \geq 1 \).

Since \( f^{2n+2}(\bar{u}_{n,1}) = d \), we have
\[
f^{2n+3}(\bar{u}_{n,1}) = f(d).
\]
On the other hand, recall that
\[
f^{m+2(n+1)}(\bar{v}_n) = f^{m+2}(f^{2n}(\bar{v}_n)) = f^m(f^2(v)) = \min P.
\]
Furthermore, by Lemma 4(3), we have

all periodic points of \( f \) in \([\bar{u}_n, \bar{u}'_{n+1}]\) with odd periods have least periods \( \geq 2n + 3 \).

Therefore, we can apply Lemma 5(1) with
\[
f^{2n+3}(\bar{u}_{n,1}) = f(d) \in \{z_0, f(z_0), f(d)\} \quad \text{and} \quad f^{m+2(n+1)}(\bar{v}_n) = \min P
\]
to obtain that, for each \( i \geq 0 \), the point
\[
\bar{p}^{(n,2)}_{m+2(n+1)+2i} = \min \{ \bar{u}_{n,1} \leq x \leq \bar{v}_n : f^{m+2(n+1)+2i}(x) = x \}
= \min \{ \bar{u}_{n,1} \leq x \leq \bar{u}'_{n+1} : f^{m+2(n+1)+2i}(x) = x \},
\]
exists and \( \bar{u}'_n < \bar{u}_{n,1} < \cdots < \bar{p}^{(n,2)}_{m+2n+6} < \bar{p}^{(n,2)}_{m+2n+4} < \bar{p}^{(n,2)}_{m+2n+2} < \bar{v}_n < \bar{u}'_{n+1} \).

Furthermore, for each \( i \geq 0 \), the point \( \bar{p}^{(n,2)}_{m+2n+2+2i} \) is a period-\((m + 2n + 2 + 2i)\) point of \( f \), or, equivalently, in \([\bar{u}_{n,1}, \bar{v}_n] \subset [v, z_0]\),

for each \( n \geq 1 \) and \( i \geq 1 \), the point \( \bar{p}^{(n,2)}_{m+2n+2+2i} \) is a period-\((m + 2n + 2i)\) point of \( f \).

2.5(c) On the existence of periodic points of \( f \) of all odd periods \( \geq m + 2n + 2 \) in \([\bar{v}_n, \bar{u}'_{n+1}] \subset [v, z_0]\) for each \( n \geq 1 \).

Since \( f^{2n+2}(\bar{u}'_{n+1}) = d \), we have
\[
f^{2n+3}(\bar{u}'_{n+1}) = f(d).
\]
On the other hand, We have
\[
f^{m+2(n+1)}(\bar{v}_n) = f^{m+2}(f^{2n}(\bar{v}_n)) = f^m(f^2(v)) = \min P.
\]
Furthermore, by Lemma 4(3), we have

all periodic points of \( f \) in \([\bar{u}'_n, \bar{u}'_{n+1}]\) with odd periods have least periods \( \geq 2n + 3 \).

Therefore, we can apply Lemma 5(2) with
\[
f^{m+2n+3}(\bar{u}'_{n+1}) = f(d) \in \{z_0, f(z_0), f(d)\} \quad \text{and} \quad f^{m+2(n+1)}(\bar{v}_n) = \min P
\]
to obtain that, for each \( i \geq 0 \), the point

\[
\tilde{q}_{m+2n+2i}^{(n,2)} = \max \left\{ \tilde{u}_n \leq x \leq \tilde{u}'_{n+1} : f^{m+2(n+1)+2i}(x) = x \right\}
\]

exists and \( \tilde{u}'_n < \tilde{v}_n < \bar{q}_{m+2n+2}^{(n,2)} < \bar{q}_{m+2n+4}^{(n,2)} < \bar{q}_{m+2n+6}^{(n,2)} < \bar{q}_{m+2n+8}^{(n,2)} < \cdots < \tilde{u}'_{n+1} < z_0 \).

Furthermore, for each \( i \geq 0 \), the point \( \bar{q}_{m+2n+2+2i}^{(n,2)} \) is a period-\((m + 2n + 2 + 2i)\) point of \( f \), or, equivalently, in \([\tilde{u}', \tilde{u}'_{n+1}]\) \((\subset [\tilde{u}', \tilde{u}'_{n+1}] \subset [v, z_0])\),

for each \( n \geq 1 \) and \( i \geq 1 \), the point \( \bar{q}_{m+2n+2i}^{(n,2)} \) is a period-\((m + 2n + 2i)\) point of \( f \).

In summary, on the interval \([d, u_n]\), we have \( f^{2n}(d) = z_0, f^{2n}(u_n) = d \) and \( f^{2n}(\nu_n) = v \). So, \( f^{2n+2}(d) = z_0 = f^{2n+2}(u_n) \) and \( f^{2n+2}(\nu_n) = \min P \). Thus,

the graph of \( y = f^{2n+2}(x) \) looks ‘roughly’ like a skew ‘V’ shape (at the three points \( d, \nu_n, u_n \)) and the point \( \nu_n \) happens to satisfy that \( (d <) u_{n+1} < \nu_n < u_n \).

Similarly, on the interval \([\tilde{u}', z_0]\), we have \( f^{2n}((\tilde{u}')_n) = d, f^{2n}(z_0) = z_0 \) and \( f^{2n}(\tilde{v}_n) = v \). So, \( f^{2n+2}(\tilde{u}'_n) = z_0 = f^{2n+2}(z_0) \) and \( f^{2n+2}(\tilde{v}_n) = \min P \). Thus,

the graph of \( y = f^{2n+2}(x) \) looks ‘roughly’ like a skew ‘V’ shape (at the three points \( \tilde{u}'_n, \tilde{v}_n, z_0 \)) and the point \( \tilde{v}_n \) happens to satisfy that \( \tilde{u}'_n < \tilde{v}_n < \tilde{u}'_{n+1} (< z_0) \).

We have shown that there exist 4 monotonic sequences \( < u'_{n,i} >, < c'_{2n+2i}^{(n,2)} >, < q'_{m+2n+2i}^{(n,2)} > \) and \( < p'_{m+2n+2i}^{(n,2)} > \) of points in \([u_{n+1}, u_n]\), where

\( < u'_{n,i} > \) is a monotonic sequence of \( d \)-points which will be used to continue the basic tower-building process to the third layer and, \( < c'_{2n+2i}^{(n,2)} >, < q'_{m+2n+2i}^{(n,2)} > \) and \( < p'_{m+2n+2i}^{(n,2)} > \) are 3 monotonic sequences of periodic points of \( f \) whose union constitutes a compartment of the second layer of the basic tower of periodic points of \( f \)

and 4 monotonic sequences \( < \tilde{u}_n >, < \tilde{c}_{2n+2i}^{(n,2)} >, < \tilde{p}_{m+2n+2i}^{(n,2)} > \) and \( < \tilde{q}_{m+2n+2i}^{(n,2)} > \) of points in \([\tilde{u}', \tilde{u}_{n+1}]\), where

\( < \tilde{u}_n > \) is a monotonic sequence of \( d \)-points which will be used to continue the basic tower-building process to the third layer and, \( < \tilde{c}_{2n+2i}^{(n,2)} >, < \tilde{p}_{m+2n+2i}^{(n,2)} > \) and \( < \tilde{q}_{m+2n+2i}^{(n,2)} > \) are 3 monotonic sequences of periodic points of \( f \) whose union constitutes a compartment of the second layer of the basic tower of periodic points of \( f \)

such that, for each \( n \geq 1 \),

\[
u_n < \cdots < p_{m+2n+4}^{(n,2)} < p_{m+2n+2}^{(n,2)} < \nu_n < q_{m+2n+2}^{(n,2)} < q_{m+2n+4}^{(n,2)} < \cdots < u'_{n,1} < c'_{2n+2}^{(n,2)} < u'_{n,2} < c'_{2n+4}^{(n,2)} < \cdots < u_n,
\]
\[ \bar{u}'_n < \ldots < \bar{u}_{n,3} < \bar{u}_{n,2} < \bar{u}_{n,1} \text{ and} \]
\[ \bar{u}'_n < \ldots < \bar{c}_{2n+6}^{(n,2)} < \bar{c}_{2n+4}^{(n,2)} < \bar{c}_{2n+2}^{(n,2)} < \bar{u}_{n,1} < \]
\[ \ldots < \bar{p}_{m+2n+4}^{(n,2)} < \bar{p}_{m+2n+2} < \bar{v}_n < \bar{q}_{m+2n+2} < \bar{q}_{m+2n+4} < \ldots < \bar{u}'_{n+1}, \]

and, for each \( i \geq 1, \)
\[ u'_{n,i} = \max \{ u_{n+1} \leq x \leq u_n : f^{2n+2i}(x) = d \}, \]
\[ \bar{u}_{n,i} = \min \{ \bar{u}'_n \leq x \leq \bar{u}'_{n+1} : f^{2n+2i}(x) = d \}, \]
\[ d < f^{2j}(x) < z_0 \text{ for all } 1 \leq j \leq n + i \text{ and all } x \in (u'_{n,i}, u_n) \cup (\bar{u}'_{n}, \bar{u}_{n,i}) \text{ and} \]
except possibly \( c_{4n}, \) all \( c_{2n+2i} \) and \( \bar{c}_{2n+2i}, i \neq n, \) are period-(2n + 2i) points of \( f \) and \( \bar{p}_{m+2n+2i}^{(n,2)}, \bar{q}_{m+2n+2i}^{(n,2)} \) and \( \bar{q}_{m+2n+2i}^{(n,2)} \) are all period-(m + 2n + 2i) points of \( f. \)

Furthermore, on the interval \([\min P, d],\) we have found 4 monotonic sequences \( < \bar{\mu}_{m,n,i}, >, < \bar{p}_{m+2n+2i}^{(n,2)}, >, < \bar{c}_{2m+2n+2i}^{(n,2)}, > \) and \( < \bar{c}_{2m+2n+2i}^{(n,2)}, > \) of points, where \( < \bar{\mu}_{m,n,i} > \) is a monotonic sequence of \( d \)-points which will be used to continue the basic tower-building process to the third layer and, \( < \bar{p}_{m+2n+2i}^{(n,2)}, >, < \bar{c}_{2m+2n+2i}^{(n,2)}, > \) and \( < \bar{c}_{2m+2n+2i}^{(n,2)}, > \) are 3 monotonic sequences of periodic points of \( f \) whose union constitutes a compartment of the second layer of the basic tower of periodic points of \( f \)

such that, for each \( n \geq 0 \) (the point \( \bar{p}_{m+2n+2}^{(n,2)} \) is excluded),
\[ \bar{\mu}'_{m,n} < \ldots < \bar{\mu}_{m,n,3} < \bar{\mu}_{m,n,2} < \bar{\mu}_{m,n,1} \]
\[ < \bar{c}_{2m+2n+4}^{(n,2)} < \bar{c}_{2m+2n+2} < \bar{v}_n < \bar{c}_{2m+2n+2} < \bar{c}_{2m+2n+4} < \ldots < \bar{\mu}'_{m,n+1}. \]

On the interval \([\bar{u}'_0, v] \subset [u_1, v],\) we have found 4 monotonic sequences \( < \bar{\mu}_{m,n,i}, >, < \bar{q}_{m+2n+2i}^{(n,2)}, >, < \bar{c}_{2m+2n+2i}^{(n,2)}, > \) and \( < \bar{c}_{2m+2n+2i}^{(n,2)}, > \) of points, where \( < \bar{\mu}_{m,n,i} > \) is a monotonic sequence of \( d \)-points which will be used to continue the basic tower-building process to the third layer and, \( < \bar{q}_{m+2n+2i}^{(n,2)}, >, < \bar{c}_{2m+2n+2i}^{(n,2)}, > \) and \( < \bar{c}_{2m+2n+2i}^{(n,2)}, > \) are 3 monotonic sequences of periodic points of \( f \) whose union constitutes a compartment of the second layer of the basic tower of periodic points of \( f \)

such that, for each \( n \geq 1, \)
\[ \bar{\mu}_{m,n+1} < \ldots < \bar{c}_{2m+2n+4}^{(n,2)} < \bar{c}_{2m+2n+2} < \bar{v}_n < \bar{c}_{2m+2n+2} < \bar{c}_{2m+2n+4} < \ldots \]
\[ \bar{\mu}'_{m,n+1} < \bar{q}_{m+2n+2} < \bar{\mu}'_{m,n,2} < \bar{q}_{m+2n+4} < \bar{\mu}'_{m,n,3} < \bar{\mu}_{m,n}. \]
On the interval \([v, u_0] (\subset [v, u'_0])\), we have found 4 monotonic sequences
\(<\mu_{m,n,i}, <\hat{p}_{m+2n+2i}^{(n,2)}, <\hat{c}_{2m+2n+2i}^{(n,2)}\) and \(<\hat{c}'_{2m+2n+2i}^{(n,2)}> of points, where
\(<\mu_{m,n,i}> is a monotonic sequence of \(d\)-points which will be used to continue the
\(basic\ tower\)-building process to the third layer and, \(<\hat{p}_{m+2n+2i}^{(n,2)}, <\hat{c}_{2m+2n+2i}^{(n,2)}\) and
\(<\hat{c}'_{2m+2n+2i}^{(n,2)}> are 3 monotonic sequences of periodic points of \(f\) whose union
constitutes a compartment of the second layer of the \(basic\ tower\) of periodic points of \(f\)
such that, for each \(n \geq 1\),
\[
\hat{p}'_{m,n} < \cdots < \hat{p}_{m,n,3} < \hat{p}_{m+2n+4}^{(n,2)} < \hat{p}_{m+2n+2}^{(n,2)} < \hat{p}_{m,n,1} \cdots < \hat{c}_{2m+2n+4}^{(n,2)} < \hat{c}_{2m+2n+2}^{(n,2)} < \hat{c}_{2m+2n+4}^{(n,2)} < \cdots < \hat{p}'_{m,n+1}.
\]

We call the collection of all these periodic points, arranged from \(\min P\) (exclusive) to \(z_0\) (exclusive), \(excluding\ the\ periodic\ points\ \hat{p}_{m+2n+2}^{(n,2)}, \ n \geq 1\) so
that the convex hulls of all compartments on the second layer are pairwise disjoint,
\[
\hat{p}_{m+2n+2i}^{(n,2)}, \hat{c}_{2m+2n+2i}^{(n,2)}, \hat{c}'_{2m+2n+2i}, \hat{c}_{2m+2n+2i}, \ i \geq 1, \ n \geq 1, \\
p_{m+2n+2i}, q_{m+2n+2i}, c_{2m+2n+2i}, c_{2m+2n+2i}, \ i \geq 1, \ n \geq 1, \\
\hat{c}_{2m+2n+2i}, \hat{c}_{2m+2n+2i}, \hat{q}_{m+2n+2i}, \hat{q}_{m+2n+2i}, \ i \geq 1, \ n \geq 1, \\
\hat{p}_{m+2n+2i}, \hat{c}_{2m+2n+2i}, \hat{c}_{2m+2n+2i}, \hat{c}_{2m+2n+2i}, \ i \geq 1, \ n \geq 1, \\
\hat{c}_{2m+2n+2i}, \hat{p}_{m+2n+2i}, \hat{q}_{m+2n+2i}, \hat{q}_{m+2n+2i}, \ i \geq 1, \ n \geq 1,
\]
the \(second\ layer\) of the \(basic\ tower\) of periodic points of \(f\) associated with \(P\)
(see Figures 2-6).

\(\S 3.\) The third layer of the \(basic\ tower\) of periodic points of \(f\) associated with \(P\).

By the way we find the second layer of the \(basic\ tower\) of periodic points of \(f\) associated
with \(P\), we can write down the formulas for the periodic points of the third layer as follows:

3.1 A compartment of the third layer of the \(basic\ tower\) in \([\bar{\mu}_{m,n,k+1}, \bar{\mu}_{m,n,k}] (\subset [\mu'_{m,n}, \mu'_{m,n+1}] \subset [\min P, d]).\)

For each \(n \geq 0\) and \(k \geq 1\), on the interval \([\bar{\mu}_{m,n,k+1}, \bar{\mu}_{m,n,k}] (\subset [\mu'_{m,n}, \mu'_{m,n+1}] \subset [\min P, d]),\)
let \(\tilde{v}_{m,n,k}\) be a point in \([\mu'_{m,n}, \mu'_{m,n+1}]\) such that \(f^{m+2n+2k}(\tilde{v}_{m,n,k}) = v\). It turns out that
\(\mu_{m,n,k+1} < \tilde{v}_{m,n,k} < \mu_{m,n,k}\). For each \(i \geq 1\), let, with \(x\)-coordinates moving from point \(\mu_{m,n,k}\)
via point \(\tilde{v}_{m,n,k}\) to point \(\mu_{m,n,k+1}\), (the number 3 in the superscripts indicates the third layer),
\[
\mu'_{m,n,k,i} = \max \{ \bar{\mu}_{m,n,k+1} \leq x \leq \mu_{m,n,k} : f^{m+2n+2k+2i}(x) = d \},
\]
\[ \hat{q}_{m+2n+2k+2i}^{(n,k,3)} = \max \{ \hat{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = x \}; \]
\[ \hat{c}_{2m+2n+2k+2i}^{(n,k,3)} = \max \{ \hat{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}_{m,n,k,1} : f^{m+2n+2k+2i}(x) = x \}; \]
\[ \tilde{c}_{2m+2n+2k+2i}^{(n,k,3)} = \min \{ \tilde{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = x \}. \]

Figure 7: A compartment of the third layer of the basic tower of periodic points of \( f \) associated with \( P \) in the interval \([\hat{\mu}_{m,n,k+1}, \tilde{\mu}_{m,n,k}] \) \(( \subset [\hat{\mu}_{m,n}, \hat{\mu}_{m,n+1}] \subset [\min P, d])\), where \( \tilde{\nu}_{m,n,k} \) is an auxiliary point in \([\hat{\mu}_{m,n,k+1}, \tilde{\mu}_{m,n,k}] \) such that \( f^{m+2n+2k}(\tilde{\mu}_{m,n,k}) = \nu \)
which is needed only to determine the ordering of the following points:

\[ \hat{\mu}'_{m,n,k,i} = \max \{ \hat{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = d \}; \]
\[ \hat{q}_{m+2n+2k+2i}^{(n,k,3)} = \max \{ \hat{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = x \}; \]
\[ \hat{c}_{2m+2n+2k+2i}^{(n,k,3)} = \max \{ \hat{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}_{m,n,k,1} : f^{m+2n+2k+2i}(x) = x \}; \]
\[ \tilde{c}_{2m+2n+2k+2i}^{(n,k,3)} = \min \{ \tilde{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = x \}. \]

For each fixed \( n \geq 0 \) and \( k \geq 1 \), the collection of all these periodic points \( \hat{q}_{m+2n+2k+2i}^{(n,k,3)}, \hat{c}_{2m+2n+2k+2i}^{(n,k,3)}, i \geq 1 \) (note that the point \( \hat{q}_{m+2n+2k+2i}^{(n,k,3)} \) is excluded), is called a compartment of the third layer of the basic tower of periodic points of \( f \) associated with \( P \).

It is easy to see that we have

\[ \hat{\mu}_{m,n,k+1} < \cdots < \hat{c}_{2m+2n+2k+4}^{(n,k,3)} < \hat{c}_{2m+2n+2k+2}^{(n,k,3)} < \hat{\mu}_{m,n,k} < \hat{c}_{2m+2n+2k+2}^{(n,k,3)} < \hat{c}_{2m+2n+2k+4}^{(n,k,3)} < \cdots \]
\[ < \hat{\mu}'_{m,n,k,1} < \hat{q}_{m+2n+2k+4}^{(n,k,3)} < \hat{\mu}'_{m,n,k,2} < \hat{q}_{m+2n+2k+6}^{(n,k,3)} < \hat{\mu}'_{m,n,k,3} < \cdots < \hat{\mu}_{m,n,k} \]
with the point \( \hat{q}_{m+2n+2k+2}^{(n,k,3)} \) excluded so that the convex hulls of the two sets: \( \{ \hat{c}_{2m+2n+2k+2i}^{(n,k,3)} : i \geq 1 \} \) and \( \{ \hat{q}_{m+2n+2k+2j}^{(n,k,3)} : j \geq 2 \} \) are disjoint.

The following result can be viewed as a continuation of Lemma 7 from the interval \([\hat{\mu}_{m,n}, d]\) to the subinterval \([\hat{\mu}'_{m,n}, \tilde{\mu}_{m,n,k}] \) (and later to the subinterval \([\hat{\mu}'_{m,n,k,i}, \tilde{\mu}_{m,n,k}] \) where all periodic points of \( f \) of odd periods have least periods \( \geq m + 2n + 2k + 2i + 2 \) and then continue on to the higher layers of the basic tower and so on):
Lemma 11. For each $n \geq 1$ and $k \geq 1$, all periodic points of $f$ in $[\tilde{\mu}_{m,n}, \tilde{\mu}_{m,n}]$ of odd periods have least periods $\geq m + 2n + 2k + 2$.

Proof. Recall that, by Lemma 7, all periodic points of $f$ in $[\tilde{\mu}_{m,n}, \tilde{\mu}_{m,n}]$ of odd periods have least periods $\geq m + 2n + 2$. Suppose $f$ had a periodic point $\tilde{p}$ of odd period $\ell(\tilde{p}) \leq m + 2n + 2k$ in $[\tilde{\mu}_{m,n}, \tilde{\mu}_{m,n}]$. Then, since $\ell(\tilde{p})$ is odd and $\ell(\tilde{p}) \geq m + 2n + 2$, we have $f^{\ell(\tilde{p})}(\tilde{p}_m,n) = \nu_0$. Since $m + 2n + 2k \geq \ell(\tilde{p})$, it follows from Lemma 3 that there is a point $\tilde{w}_{m+2n+2k}$ in $[\tilde{\mu}_{m,n}, \tilde{\mu}_{m,n}]$ such that $f^{m+2n+2k}(\tilde{w}_{m+2n+2k}) = \tilde{w}_{m+2n+2k}$. Since $f^{m+2n+2k}(\tilde{\mu}_{m,n}) - d = \nu_0 - d > 0$ and $f^{m+2n+2k}(\tilde{w}_{m+2n+2k}) - \nu_0$, there is a point $\tilde{\mu}_{m,n}^*$ in $[\tilde{\mu}_{m,n}, \tilde{\mu}_{m,n}]$ such that $f^{m+2n+2k}(\tilde{\mu}_{m,n}^*) = \nu_0$. Since $\tilde{\mu}_{m,n} < \tilde{\mu}_{m,n}^* < \tilde{w}_{m+2n+2k} < \tilde{\mu}_{m,n}$, this contradicts the minimality of $\tilde{\mu}_{m,n}$ in $[\tilde{\mu}_{m,n}, \tilde{\mu}_{m,n}]$.

Recall that $\tilde{\nu}_{m,n}$ is a point in $[\tilde{\mu}_{m,n}, \tilde{\mu}_{m,n}]$ such that $f^{m+2n+2k}(\tilde{\nu}_{m,n}) = \nu_0$. We can apply Lemma 5(1) and Lemma 11 with $w = \nu_0$ to obtain that, for each $(\nu_0, f(\nu_0), f(d)) = \tilde{\nu}_{m,n}$ such that $f^{m+2(n+1)}(\tilde{\nu}_{m,n}) = \min P$ to obtain that,

for each $i \geq 1$, the point $\tilde{d}^{(n,k,3)}_{m+2n+2k+2i} = \max \{ \tilde{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = x \}$

exists and is a period-$m+2n+2k$ point of $f$.

As for the periods of $\tilde{d}^{(n,k,3)}_{m+2n+2k+2i}$'s and $\tilde{c}^{(n,k,3)}_{m+2n+2k+2i}$'s, we apply Lemma 6 with

\[
\begin{align*}
f^{m+2n+2k+3}(\tilde{\mu}_{m,n,k+1}) &= f(d) \quad \text{and} \quad f^{2(m+n+1)}(\tilde{\nu}_{m,n}) = \min P \quad \text{and} \\
f^{m+2n+2k+3}(\tilde{\mu}_{m,n,k+1}) &= f(d) \quad \text{and} \quad f^{2(m+n+1)}(\tilde{\nu}_{m,n}) = \min P \quad \text{respectively}
\end{align*}
\]

to obtain that, for each $(m+n+k+1)+i \geq \max \{ m+2n+2k+3, m+2 \} = m+2n+2k+3$, i.e., for each $i \geq n+k+2$, both

\[
\begin{align*}
\tilde{d}^{(n,k,3)}_{m+2n+2k+2+2i} &= \max \{ \tilde{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}_{m,n,k}, f^{m+2n+2k+2+2i}(x) = x \} \quad \text{and} \\
\tilde{c}^{(n,k,3)}_{m+2n+2k+2+2i} &= \min \{ \tilde{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}_{m,n,k}, f^{m+2n+2k+2+2i}(x) = x \}
\end{align*}
\]

exist and are period-$2m+2n+2k+2+2i$ points of $f$, or, equivalently,

for each $i \geq n+k+3$, both $\tilde{d}^{(n,k,3)}_{m+2n+2k+2i}$ and $\tilde{c}^{(n,k,3)}_{m+2n+2k+2i}$ are period-$2m+2n+2k+2i$ points of $f$.

3.2 A compartment of the third layer of the basic tower in $[u'_{n,k}, u'_{n,k+1}]$.

For each $n \geq 1$ and $k \geq 1$, we consider the interval $[u'_{n,k}, u'_{n,k+1}]$ such that $f^{m+2n+2k}(\nu_{n,k}) = \nu_0$. It turns out that $u'_{n,k} < \nu_{n,k} < u'_{n,k+1}$. 59
For each \( i \geq 1 \), let, with \( x \)-coordinates moving from point \( u'_{n,k} \) via point \( \nu_{n,k} \) to point \( u'_{n,k+1} \), (the number 3 in the superscripts indicates the third layer),

\[
\begin{align*}
\ u_{n,k,i} &= \min \{ u'_{n,k} \leq x \leq u'_{n,k+1} : f^{2n+2k+2i}(x) = d \}, \\
\ c_{2n+2k+2i}^{(n,k,3)} &= \min \{ u'_{n,k} \leq x \leq u'_{n,k+1} : f^{2n+2k+2i}(x) = x \}, \\
\ p_{m+2n+2k+2i}^{(n,k,3)} &= \min \{ u_{n,k,1} \leq x \leq u'_{n,k+1} : f^{m+2n+2k+2i}(x) = x \} \quad \text{and} \\
\ q_{m+2n+2k+2i}^{(n,k,3)} &= \max \{ u'_{n,k} \leq x \leq u'_{n,k+1} : f^{m+2n+2k+2i}(x) = x \}.
\end{align*}
\]

For each fixed \( n \geq 1 \) and \( k \geq 1 \), the collection of all these periodic points \( c_{2n+2k+2i}^{(n,k,3)} \), \( p_{m+2n+2k+2i}^{(n,k,3)} \), \( q_{m+2n+2k+2i}^{(n,k,3)} \), \( i \geq 1 \) is called a \textit{compartment} of the third layer of the \textit{basic} tower of periodic points of \( f \) associated with \( P \).

It is easy to see that we have

\[
\begin{align*}
\ u'_{n,k} < \cdots < u_{n,k,3} < c_{2n+2k+4}^{(n,k,3)} < u_{n,k,2} < c_{2n+2k+2}^{(n,k,3)} < u_{n,k,1} < \cdots < \\
\ p_{m+2n+2k+4}^{(n,k,3)} < p_{m+2n+2k+2}^{(n,k,3)} < \nu_{n,k} < q_{m+2n+2k+2}^{(n,k,3)} < q_{m+2n+2k+4}^{(n,k,3)} < \cdots < u'_{n,k+1} \quad \text{and} \\
\ d < f^{2j}(x) < z_0 \quad \text{for all} \quad 1 \leq j \leq n + k + i \quad \text{and all} \quad u'_{n,k} < x < u_{n,k,i}, \quad (\dagger)
\end{align*}
\]

and, since \( f(x) > z \geq z_0 \) for all \( d < x < z_0 \), we also have (by taking \( i = 1 \)),

all periodic points of \( f \) in \([u'_{n,k}, u_{n,k,1}]\) of odd periods have least periods \( \geq 2n + 2k + 3 \).

Since \( f^{2n+2k+2}([u'_{n,k}, u_{n,k,1}]) \supset [d, z_0] \supset \{v\} \), we let \( \nu_{n,k+1} \) be a point in \([u'_{n,k}, u_{n,k,1}]\) such that \( f^{2n+2k+2}(\nu_{n,k+1}) = v \). Then we can apply Lemma 5(1) with

\[
\begin{align*}
\ f^{2n+2k+3}(u'_{n,k}) &= f(z_0) \quad \text{and} \quad f^{m+2n+2k+4}(\nu_{n,k+1}) = \min P
\end{align*}
\]

to obtain that, for each \( i \geq 2 \) (not \( i \geq 0 \)), the point

\[
\ p_{m+2n+2k+2i} = \min \{ u'_{n,k} \leq x \leq \nu_{n,k+1} : f^{m+2n+2k+2i}(x) = x \}
\]

exists and is a period-\((m+2n+2k+2i)\) point of \( f \). However, these periodic points \( p_{m+2n+2k+2i}^{(n,k,3)} \)'s are \textit{interspersed} with the periodic points \( c_{2n+2k+2i}^{(n,k,3)} \)'s of \( f \) of even periods. To make things simple, we do not count them in the third layer of the \textit{basic} tower of periodic points of \( f \) associated with \( P \).

As for the periods of \( p_{m+2n+2k+2i}^{(n,k,3)} \)'s and \( q_{m+2n+2k+2i}^{(n,k,3)} \)'s, we apply Lemma 5 with

\[
\begin{align*}
\ f^{2n+2k+3}(u_{n,k,1}) &= f(d) \quad \text{and} \quad f^{m+2n+2k+4}(\nu_{n,k}) = \min P \quad \text{and} \\
\ f^{2n+2k+3}(u'_{n,k+1}) &= f(d) \quad \text{and} \quad f^{m+2n+2k+4}(\nu_{n,k}) = \min P \quad \text{respectively}
\end{align*}
\]
and Lemma 8 that all periodic points of $f$ in $[u_{n,k}^′, u_n]$ of odd periods have least periods $\geq 2n + 2k + 3$ to obtain that,

for each $i \geq 1$, both $p_{m+2n+2k+2i}^{(n,k,3)}$ and $q_{m+2n+2k+2i}^{(n,k,3)}$ are period-$(m+2n+2k+2i)$ points of $f$.

To find the least periods of $c^{(n,k,3)}_{2n+2k+2}$'s with respect to $f$, we shall use the following result which can be viewed as a continuation of Lemma 8 from the interval $[u_{n,k}^′, u_n]$ to the subinterval $[u_{n,k}^′, u_{n,k,i}]$ (and later to the subinterval $[u_{n,k,i}, u_{n,k,i}]$ and then continue on to the higher layers of the basic tower. Unfortunately, we do not have similar result on the counterpart interval $[\bar{u}_{n,k,i}, \bar{u}_{n,k}]$ and so on):

**Lemma 12.** For each $n \geq 1$, $k \geq 1$, $i \geq 1$, on the interval $[u_{n,k}^′, u_{n,k,i}]$ (⊂ $[u_{n,k}^′, u_{n,k,i}]$), $f$ has no periodic points of odd periods $\leq 2n + 2k + 2i + 1$, nor has periodic points of even periods $\leq 2n + 2k + 2i$ except period-$(2n + 2k + 2i)$ points and possibly period-$(2n + 2k)$ or period-$(2n)$ points.

**Proof.** Suppose $f$ has a period-$(2n + 2k + 2i)$ point, say $c^*_{2n+2k+2j}$, in $[u_{n,k}^′, u_{n,k,i}]$ for some $1 \leq j \leq i - 1$. Then since $f^{2n+2k+2j+2}(u_{n,k}^′, c^*_{2n+2k+2j}) \supset f^{2}(c^*_{2n+2k+2j+2}, z_0) \supset f^{2}(v, z_0) \supset [\min P, z_0] \supset \{d\}$, there exists a point $u^*_{n,k,i+1}$ in $[u_{n,k}^′, c^*_{2n+2k+2j}]$ (⊂ $[u_{n,k}^′, u_{n,k,i}]$) such that $f^{2n+2k+2j+2}(u^*_{n,k,i+1}) = d$. So, $u^*_{n,k,i+1} < u_{n,k,i}$. Since $n + k + j + 1 \leq n + k + i$, this contradicts the fact that $u_{n,k,i} \leq u_{n,k,j+1} \leq u^*_{n,k,j+1}$. This, combined with Lemma 4(1) and Lemma 8, implies that $f$ has no periodic points of even periods $\leq 2n + 2k + 2i - 2$ except possibly period-$(2n)$ or period-$(2n + 2k)$ points. Furthermore, since $f(x) > z_0$ for all $d < x < z_0$, it follows from the above (††) that $f$ has no periodic points of odd periods $\leq 2n + 2k + 2i + 1$.

On the other hand, since $f^{2n+2k+2i}(u_{n,k}^′) - u_{n,k}^′ = f^{2i}(d) - u^*_{n,k} = z_0 - u^*_{n,k} > 0$ and $f^{2n+2k+2i}(u_{n,k,i}) - u_{n,k,i} = d - u_{n,k,i} < 0$, the point $q_{2n+2k+2i} = \max \{u_{n,k}^′ \leq x \leq u_{n,k,i} : f^{2n+2k+2i}(x) = x\}$ exists and, since we have just shown in the previous paragraph that $f$ has no periodic points of least periods $\leq 2n + 2k + 2i - 1$, is a period-$(2n + 2k + 2i)$ point of $f$ in $[u_{n,k}^′, u_{n,k,i}]$.

We now use Lemma 12 to find the least period of $c^{(n,k,3)}_{2n+2k+2i}$ with respect to $f$:

Let $\ell(c^{(n,k,3)}_{2n+2k+2i})$ denote the least period of $c^{(n,k,3)}_{2n+2k+2i}$ with respect to $f$. Suppose $\ell(c^{(n,k,3)}_{2n+2k+2i}) < 2n + 2k + 2i$ for some $i \geq 1$. Then by Lemma 12, $\ell(c^{(n,k,3)}_{2n+2k+2i}) = 2n$ or $\ell(c^{(n,k,3)}_{2n+2k+2i}) = 2n + 2k$.

If $2n + 2k + 2i$ is not a multiple of $2n$ nor a multiple of $2n + 2k$, then $c^{(n,k,3)}_{2n+2k+2i}$ is a period-$(2n + 2k + 2i)$ point of $f$.

If $\ell(c^{(n,k,3)}_{2n+2k+2i}) = 2n + 2k$ and $(2n+2k+2i)/(2n+2k) = r > 2$, then $(r-1)(2n+2k) > 2n+2k$. So, $(r-1)(2n+2k) \geq 2n + 2k + 2$. We apply Lemma 3 with

$$f^{(r-1)(2n+2k)}(u_{n,k}^′) \in \{z_0, f(z_0)\} \text{ and } f^{(r-1)(2n+2k)}(c^{(n,k,3)}_{2n+2k+2i}) = c^{(n,k,3)}_{2n+2k+2i}$$

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to obtain a periodic point, say $c_{2n+2k+2i}^*$, of $f$ such that $u'_{n,k} < c_{2n+2k+2i}^* < c_{2n+2k+2i}(n,k,3) < u_{n,k,1}$ and $f^{2n+2k+2i}(c_{2n+2k+2i}^*) = c_{2n+2k+2i}^*$, which contradicts the minimality of $c_{2n+2k+2i}(n,k,3)$ in $[u'_{n,k}, u_{n,k,1}]$.

So, if $\ell(c_{2n+2k+2i}(n,k,3)) = 2n + 2k$ and $\ell(c_{2n+2k+2i}(n,k,3)) < 2n + 2k + 2i$, then we must have $(2n + 2k + 2i)/(2n + 2k) = 2$. Thus, $c_{4n+4k}$ is either a period-$2n+2k$ or a period-$4n+4k$ point of $f$.

If $\ell(c_{2n+2k+2i}(n,k,3)) = 2n$ and $i \geq n + 1$, then $(2n + 2k + 2i)/(2n) = s > 2$. So, $2n(s - 1) = (2n + 2k + 2i) - 2n = 2k + 2i \geq 2k + 2(n + 1) = 2n + 2k + 2$. By applying Lemma 3 with $f^{2n(s-1)}(u'_{n,k}) = \nu_0$ and $f^{2n(s-1)}(c_{2n+2k+2i}(n,k,3)) = c_{2n+2k+2i}^{(n,k,3)}$, there exists a periodic point, say $c_{2n+2k+2i}^{(n,k,3)}$, of $f$ such that $u'_{n,k} < c_{2n+2k+2i}^{(n,k,3)} < c_{2n+2k+2i}(n,k,3) < u_{n,k,1}$ and $f^{2n+2k+2i}(c_{2n+2k+2i}^{(n,k,3)}) = c_{2n+2k+2i}^{(n,k,3)}$, which contradicts the minimality of $c_{2n+2k+2i}(n,k,3)$ in $[u'_{n,k}, u_{n,k,1}]$.

If $\ell(c_{2n+2k+2i}(n,k,3)) = 2n$ and $1 \leq i \leq n$, then, there can have exactly one $i$ such that $1 \leq i \leq n$ and $2n + 2k + 2i$ is a multiple of $2n$. For such an $i$, $c_{2n+2k+2i}(n,k,3)$ is either a period-$2n$ or a period-$2n+2k+2i$ point of $f$. For all other $i$ such that $1 \leq i \leq n$ and $2n + 2k + 2i$ is not a multiple of $2n$, $c_{2n+2k+2i}(n,k,3)$ is a period-$2n + 2k + 2i$ point of $f$.

In summary, we have shown that

for $i = 1$, the unique integer in $[1, n]$ such that $2n + 2k + 2\i$ is a multiple of $2n$,
$c_{2n+2k+2i}(n,k,3)$ is either a period-$2n$ point or a period-$2n + 2k + 2i$ point of $f$,
for $i = n + k$,
$c_{2n+2k+2i}(n,k,3)$ is either a period-$2n + 2k$ point or a period-$2n + 2k + 2i$ point of $f$,
for each $i \geq 1$ and $i \n \{1, n + k\}$,
$c_{2n+2k+2i}(n,k,3)$ is a period-$2n + 2k + 2i$ point of $f$.

3.3 A compartment of the third layer of the basic tower in $[\mu'n, m, n, k] : [\mu'n, m, n, k+1] \subset [\bar{u}_0, v] \subset [u_1, v]$.

For each $n \geq 1$ and $k \geq 1$, we consider the interval $[\mu'n, m, n, k] : [\mu'n, m, n, k+1] \subset [\bar{u}_0, v] \subset [u_1, v]$. Let $\tilde{\mu}_m,n,k$ be a point in $[\mu'n, m, n, k : \mu'm, n, m, n]$ such that $f^{m+2n+2k}(\tilde{\mu}_m,n,k) = v$. It turns out that $\mu'_m,n,k < \tilde{\mu}_m,n,k < \mu'_m,n,k+1$. For each $i \geq 1$, let, with $x$-coordinates moving from point $\tilde{\mu}_m,n,k$ via point $\tilde{\mu}_m,n,k$ to point $\tilde{\mu}_m,n,k+1$:

$\tilde{\mu}_m,n,k,i = \min\{\mu'_m,n,k \leq x \leq \tilde{\mu}_m,n,k+1 : f^{m+2n+2k+2i}(x) = \tilde{\mu}_m,n,k+1\}$,

$\tilde{\mu}_{m+2n+2k+2i}(n,k,3) = \min\{\mu'_m,n,k \leq x \leq \tilde{\mu}_m,n,k+1 : f^{m+2n+2k+2i}(x) = \mu'_m,n,k+1\},$
It is easy to see that we have

$$c_{2m+2n+2k+2} = \min\{\hat{\mu}_{m,n,k,1} \leq x \leq \hat{\mu}_{m,n,k+1} : f^{2m+2n+2k+2i}(x) = x\} \quad \text{and}$$

$$\hat{c}_{2m+2n+2k+2} = \max\{\hat{\mu}_{m,n,k} \leq x \leq \hat{\mu}_{m,n,k+1} : f^{2m+2n+2k+2i}(x) = x\}.$$  

For each fixed $n \geq 1$ and $k \geq 1$, the collection of all these periodic points $\hat{c}_{2m+2n+2k+2}$, $\hat{c}_{2m+2n+2k+2}$, $\hat{c}_{2m+2n+2k+2}$, $i \geq 1$ is called a compartment of the third layer of the basic tower of periodic points of $f$ associated with $P$.

It is easy to see that we have

$$\hat{p}_{m,n,k} < \cdots < \hat{c}_{2m+2n+2k+2} < \hat{p}_{m,n,k+2} < \hat{p}_{m+2n+2k+2} < \hat{\mu}_{m,n,k+1} <$$

$$\cdots < \hat{c}_{2m+2n+2k+2} < \hat{c}_{2m+2n+2k+2} < \hat{c}_{2m+2n+2k+2} < \hat{c}_{2m+2n+2k+2} < \cdots < \hat{\mu}_{m,n,k+1},$$

We now prove the following result which can be viewed as a continuation of Lemma 9 from the interval $[\hat{u}_0, \hat{\mu}_{m,n}]$ to the subinterval $[\hat{\mu}_{m,n,k}, \hat{\mu}_{m,n}]$ (and later to the subinterval $[\hat{\mu}_{m,n,k}, \hat{\mu}_{m,n,k,i}]$ and then continue on to the higher layers of the basic tower and so on):

**Lemma 13.** For each $n \geq 1$ and $k \geq 1$, all periodic points of $f$ in $[\hat{p}_{m,n,k}, \hat{\mu}_{m,n}]$ of odd periods have least periods $m + 2n + 2k$. (Note that, in Lemma 11, we have $m + 2n + 2k + 2$).

**Proof.** Recall that, by Lemma 9, all periodic points of $f$ in $[\hat{u}_0, \hat{\mu}_{m,n}] (\subset [\hat{\mu}_{m,n+1}, \hat{\mu}_{m,n}])$ of odd periods have least periods $m + 2n$. Since

$$f^{m+2n}(\hat{u}_0) - \hat{u}_0 = f(z_0) - \hat{u}_0 \geq z_0 - \hat{u}_0 > 0 \quad \text{and} \quad f^{m+2n}(\hat{\mu}_{m,n}) - \hat{\mu}_{m,n} = d - \hat{\mu}_{m,n} < 0,$$

the point

$$\hat{q}_{m+2n} = \max\left\{\hat{u}_0 \leq x \leq \hat{\mu}_{m,n} : f^{m+2n}(x) = x\right\}$$

exists. By the choice of $\hat{u}_0$, we have $f^2(x) < d$ for all $\hat{u}_0 < x < v$. Consequently, since

$$f^{m+2n+2}(\hat{q}_{m+2n}) - d = f^2(\hat{q}_{m+2n}) - d < 0 \quad \text{and} \quad f^{m+2n+2}(\hat{\mu}_{m,n}) - d = z_0 - d > 0,$$

we obtain that the point $\hat{\mu}_{m,n,1} = \max\{\hat{u}_0 \leq x \leq \hat{\mu}_{m,n} : f^{m+2n+2}(x) = d\}$ satisfies $\hat{q}_{m+2n} < \hat{\mu}_{m,n,1} < \hat{\mu}_{m,n,2} < \hat{\mu}_{m,n,3} < \cdots < \hat{\mu}_{m,n}$. This, combined with Lemma 9 and the maximality of $\hat{q}_{m+2n}$ in $[\hat{u}_0, \hat{\mu}_{m,n}]$, implies that

all periodic points of $f$ in $[\hat{\mu}_{m,n,1}, \hat{\mu}_{m,n}]$ of odd periods have least periods $m + 2n + 2$.

In particular, suppose $f$ had a periodic point $\hat{p}$ of odd period $\ell(\hat{p}) \leq m + 2n + 2k - 2$ in $[\hat{\mu}_{m,n,k}, \hat{\mu}_{m,n}]$. Then $\ell(\hat{p}) \geq m + 2n + 2$ and since $\ell(\hat{p})$ is odd, we have $f^{\ell(\hat{p})}(\hat{\mu}_{m,n}) = z_0$. Since $f^{\ell(\hat{p})}(\hat{p}) = \hat{p}$ and $m + 2n + 2k - 2 \geq \ell(\hat{p})$, it follows from Lemma 3 that $f$ has a period-$(m + 2n + 2k - 2)$ point $\hat{w}_{m+2n+2k-2}$ in $[\hat{p}, \hat{\mu}_{m,n}] (\subset [\hat{\mu}_{m,n,k}, \hat{\mu}_{m,n}] \subset [\hat{\mu}_{m,n+1}, \hat{\mu}_{m,n}])$. By the choice of $\hat{u}_0$, we have $f^2(x) < d$ for all $\hat{u}_0 < x < v$. Since

$$f^{m+2n+2k}(\hat{w}_{m+2n+2k-2}) - d = f^2(\hat{w}_{m+2n+2k-2}) - d < 0 \quad \text{and} \quad f^{m+2n+2k}(\hat{\mu}_{m,n}) - d = z_0 - d > 0$$
there is a point \( \tilde{\mu}_{m,n,k} \) in \([\tilde{w}_{m+2n+2k-2}, \tilde{\mu}_{m,n}]\) such that \( f^{m+2n+2k}(\tilde{\mu}_{m,n,k}) = d \). Since \( \tilde{\mu}_{m,n+1} < \tilde{\mu}_{m,n,k} < \tilde{p} \leq \tilde{w}_{m+2n+2k-2} < \tilde{\mu}_{m,n,k} < \tilde{\mu}_{m,n} \), this contradicts the maximality of \( \tilde{\mu}_{m,n,k} \) in \([\tilde{\mu}_{m,n+1}, \tilde{\mu}_{m,n}]\). This completes the proof. \( \square \)

Now since
\[
f^{m+2n+2k}(\tilde{\mu}_{m,n,k}) - \tilde{\mu}_{m,n,k} = d - \tilde{\mu}_{m,n,k} < 0 \quad \text{and} \quad f^{m+2n+2k}(\tilde{\mu}_{m,n}) - \tilde{\mu}_{m,n} = z_0 - \tilde{\mu}_{m,n} > 0
\]
the point
\[
\tilde{p}_{m+2n+2k} = \min \{ \tilde{\mu}_{m,n,k} \leq x \leq \tilde{\mu}_{m,n} : f^{m+2n+2k}(x) = x \}
\]
eants. Since, by the choice of \( \tilde{u}_0 \), we have \( f^2(x) < d \) for all \( \tilde{u}_0 < x < v \), and so,
\[
f^{m+2n+2k+2}(\tilde{p}_{m+2n+2k}) - d = f^2(\tilde{p}_{m+2n+2k}) - d < 0 \quad \text{and} \quad f^{m+2n+2k+2}(\tilde{\mu}_{m,n}) - d = z_0 - d > 0.
\]
Therefore, we have
\[
(\tilde{\mu}_{m,n,k} <) \tilde{p}_{m+2n+2k} < \tilde{\mu}_{m,n,k+1} = \max \{ \tilde{\mu}_{m,n,k} \leq x \leq \tilde{\mu}_{m,n} : f^{m+2n+2k+2}(x) = d \} < \tilde{\mu}_{m,n}.
\]
This, combined with Lemma 13, implies that all periodic point of \( f \) in \([\tilde{\mu}_{m,n,k}, \tilde{p}_{m+2n+2k}]\) of odd periods have least periods \( \geq m + 2n + 2k + 2 \).

On the other hand, by the choice of \( \tilde{u}_0 \), we have \( f^2(x) < d \) for all \( \tilde{u}_0 < x < v \). Since
\[
f^{m+2n+2k+2}(\tilde{\mu}_{m,n,k}) - \tilde{\mu}_{m,n,k} = z_0 - \tilde{\mu}_{m,n,k} > 0 \quad \text{and} \quad f^{m+2n+2k+2}(\tilde{p}_{m+2n+2k}) - \tilde{p}_{m+2n+2k} = f^2(\tilde{p}_{m+2n+2k}) - \tilde{p}_{m+2n+2k} < d - \tilde{p}_{m+2n+2k} < 0
\]
the point \( \tilde{p}_{m+2n+2k+2} = \min \{ \tilde{\mu}_{m,n,k} \leq x < \tilde{p}_{m+2n+2k} : f^{m+2n+2k+2}(x) = x \} \) exists and must be a period-(\( m + 2n + 2k + 2 \)) point of \( f \).

Since \( f^{m+2n+2k+2}(\tilde{\mu}_{m,n,k}) = z_0 \) and \( f^{m+2n+2k+2}(\tilde{p}_{m+2n+2k+2}) = \tilde{p}_{m+2n+2k+2} \), and since we have just shown in the above that all periodic point of \( f \) in the interval \([\tilde{\mu}_{m,n,k}, \tilde{p}_{m+2n+2k+2}]\) (\( \supset [\tilde{\mu}_{m,n,k}, \tilde{\mu}_{m+2n+2k+2}] \)) of odd periods have least periods \( \geq m + 2n + 2k + 2 \), it follows from Lemma 3 that,

for each \( i \geq 1 \), the point \( \tilde{p}_{m+2n+2k+2i} \) is a period-(\( m + 2n + 2k + 2i \)) point of \( f \).

As for the periods of \( \tilde{c}_{2m+2n+2k+2i} \)'s and \( \tilde{c}_{2m+2n+2k+2i} \)'s, we apply Lemma 6 with
\[
f^{m+2n+2k+3}(\tilde{\mu}_{m,n,k,1}) = f(d) \quad \text{and} \quad f^{2(m+n+k+1)}(\tilde{\mu}_{m,n,k}) = \min P \quad \text{and} \quad f^{m+2n+2k+3}(\tilde{\mu}_{m,n,k+1}) = f(d) \quad \text{and} \quad f^{2(m+n+k+1)}(\tilde{\mu}_{m,n,k}) = \min P
\]
respectively to obtain that, for each \( i \geq 0 \) such that \( (m + n + k + 1) + i \geq \max \{ m + 2n + 2k + 3, m + 2 \} = m + 2n + 2k + 3 \), both \( \tilde{c}_{2m+2n+2k+2+2i} \) and \( \tilde{c}_{2m+2n+2k+2+2i} \) are period-(\( 2m + 2n + 2k + 2 + 2i \)) points of \( f \), or, equivalently,
for each \( i \geq n+k+3 \), both \( \tilde{c}_{2m+2n+2k+2i} \) and \( \tilde{c}_{2m+2n+2k+2i} \) are period-(\( 2m + 2n + 2k + 2i \)) points of \( f \).
3.4 A compartment of the third layer of the basic tower in \([\hat{\mu}_{m,n,k+1}, \hat{\mu}_{m,n,k}] \subset [v, \hat{u}_0] \subset [v, z_0]\).

For each \(n \geq 1\) and \(k \geq 1\), we consider the interval \([\hat{\mu}_{m,n,k+1}, \hat{\mu}_{m,n,k}] \subset [v, \hat{u}_0] \subset [v, z_0]\). Let \(\hat{\nu}_{m,n,k}\) be a point in \([\hat{\mu}_{m,n}, \hat{\mu}_{m,n,k}]\) such that \(f^{m+2n+2k}(\hat{\nu}_{m,n,k}) = v\). It turns out that \(\hat{\mu}_{m,n,k+1} < \hat{\nu}_{m,n,k} < \hat{\mu}_{m,n,k}\). For each \(i \geq 1\), let, with \(x\)-coordinates moving from point \(\hat{\mu}_{m,n,k}\) via point \(\hat{\nu}_{m,n,k}\) to point \(\hat{\mu}_{m,n,k+1}\),

\[
\hat{\mu}_{m,n,k,i} = \max\{\hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = d\},
\]

\[
\hat{\mu}'_{m,n,k,i} = \max\{\hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = x\},
\]

\[
\hat{q}'_{m+2n+2k+2i} = \max\{\hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k} : f^{2m+2n+2k+2i}(x) = x\}
\]

\[
\hat{c}'_{m+2n+2k+2i} = \min\{\hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k} : f^{2m+2n+2k+2i}(x) = x\}.
\]

For each fixed \(n \geq 1\) and \(k \geq 1\), the collection of all these periodic points \(\hat{q}_{m+2n+2k+2i}, \hat{c}_{m+2n+2k+2i}, \hat{c}'_{m+2n+2k+2i}, i \geq 1\) is called a compartment of the third layer of the basic tower of periodic points of \(f\) associated with \(P\).

It is easy to see that we have

\[
\hat{\mu}_{m,n,k+1} < \cdots < \hat{c}_{m+2n+2k+4} < \hat{\mu}'_{m,n,k+1} < \hat{q}_{m+2n+2k+2} < \hat{\mu}'_{m,n,k} < \hat{c}_{m+2n+2k+2} < \hat{c}'_{m+2n+2k+4} < \cdots < \hat{\mu}_{m,n,k}. \]

We now prove the following result which can be viewed as a continuation of Lemma 10 from the interval \([\hat{\mu}_{m,n}, \hat{u}_0]\) to the subinterval \([\hat{\mu}'_{m,n}, \hat{\mu}_{m,n,k}]\) (and later to the subinterval \([\hat{\mu}_{m,n,k,i}, \hat{\mu}_{m,n,k}]\) and then continue on to the higher layers of the basic tower and so on):

**Lemma 14.** For each \(n \geq 1\) and \(k \geq 1\), all periodic points of \(f\) in \([\hat{\mu}'_{m,n}, \hat{\mu}_{m,n,k}]\) of odd periods have least periods \(\geq m + 2n + 2k\). (Note that, in Lemma 11, we have \(\geq m + 2n + 2k + 2\).)

**Proof.** The proof is similar to that of Lemma 13 but different from that of Lemma 11. Recall that, by Lemma 10, all periodic points of \(f\) in \([\hat{\mu}_{m,n}, \hat{u}_0] \subset [\hat{\mu}'_{m,n}, \hat{\mu}'_{m,n+1}]\) of odd periods have least periods \(\geq m + 2n + 2k\). Since

\[
f^{m+2n}(\hat{\mu}'_{m,n}) - \hat{\mu}'_{m,n} = d - \hat{\mu}'_{m,n} < 0 \quad \text{and} \quad f^{m+2n}(\hat{u}_0) - \hat{u}_0 = f(z_0) - \hat{u}_0 > 0,
\]

the point \(\hat{\mu}_{m,n+1} = \min\{\hat{\mu}'_{m,n} \leq x \leq \hat{u}_0 : f^{m+2n}(x) = x\}\) exists. By the choice of \(\hat{u}_0\), we have \(f^2(x) < d\) for all \(v < x < \hat{u}_0\). Consequently, since

\[
f^{m+2n+2}(\hat{\mu}'_{m,n}) - \hat{\mu}'_{m,n} = d - \hat{\mu}'_{m,n} < 0 \quad \text{and} \quad f^{m+2n+2}(\hat{u}_0) - \hat{u}_0 = f^2(\hat{u}_0) - \hat{u}_0 > 0,
\]

we obtain that the point \(\hat{\mu}_{m,n+1} = \min\{\hat{\mu}'_{m,n} \leq x \leq \hat{u}_0 : f^{m+2n+2}(x) = d\}\) satisfies \(\hat{\mu}'_{m,n} < \cdots < \hat{\mu}_{m,n+1} < \hat{\mu}_{m,n+1} < \hat{\mu}_{m,n+2} < \cdots < \hat{\mu}_{m,n+1}\). This, combined with Lemma 10, implies that all periodic points of \(f\) in \([\hat{\mu}'_{m,n}, \hat{\mu}_{m,n,k}]\) of odd periods have least periods \(\geq m + 2n + 2k\).}

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Consequently, suppose $f$ had a periodic point $\hat{p}$ of odd period $\ell(\hat{p}) \leq m + 2n + 2k - 2$ in $[\hat{\mu}'_{m,n}, \hat{\mu}_{m,n,k}]$. Then $\ell(\hat{p}) \geq m + 2n + 2$ and since $\ell(\hat{p})$ is odd, we have $f^{\ell(\hat{p})}(\hat{\mu}'_{m,n}) = z_0$. Since $f^{\ell(\hat{p})}(\hat{p}) = \hat{p}$ and $m + 2n + 2k - 2 \geq \ell(\hat{p})$, it follows from Lemma 3 that $f$ has a period-$(m + 2n + 2k - 2)$ point $\hat{w}_{m+2n+2k-2}$ in $[\hat{\mu}'_{m,n}, \hat{p}] \subset [\hat{\mu}'_{m,n}, \hat{\mu}'_{m,n+1}]$. By the choice of $\hat{u}_0$, we have $f^2(x) < d$ for all $v < x < \hat{u}_0$. Since

$$f^{m+2n+2k}(\hat{w}_{m+2n+2k-2}) - d = f^2(\hat{w}_{m+2n+2k-2}) - d < 0$$

there is a point $\hat{p}^*_{m,n,k}$ in $[\hat{\mu}'_{m,n}, \hat{w}_{m+2n+2k-2}]$ such that $f^{m+2n+2k}(\hat{p}^*_{m,n,k}) = d$. Since $\hat{p}_{m,n} < \hat{p}^*_{m,n,k} < \hat{w}_{m+2n+2k-2} \leq \hat{p} < \hat{\mu}_{m,n,k} < \hat{\mu}_{m,n,k+1}$, this contradicts the minimality of $\hat{\mu}_{m,n,k}$ in $[\hat{\mu}'_{m,n}, \hat{\mu}'_{m,n+1}]$. This completes the proof.

Now since

$$f^{m+2n+2k}(\hat{\mu}'_{m,n}) - \hat{\mu}'_{m,n} = z_0 - \hat{\mu}'_{m,n} > 0$$

and $f^{m+2n+2k}(\hat{\mu}_{m,n,k}) - \hat{\mu}_{m,n,k} = d - \hat{\mu}_{m,n,k} < 0$

the point

$$\hat{q}_{m+2n+2k} = \max \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}_{m,n,k} : f^{m+2n+2k}(x) = x \}$$

exists. Since, by the choice of $\hat{u}_0$, we have $f^2(x) < d$ for all $v < x < \hat{u}_0$, and so,

$$f^{m+2n+2k+2}(\hat{\mu}'_{m,n}) - d = z_0 - d > 0$$

and $f^{m+2n+2k+2}(\hat{q}_{m+2n+2k}) - d = f^2(\hat{q}_{m+2n+2k}) - d < 0$. Therefore, we have

$$(\hat{\mu}'_{m,n} < \hat{\mu}_{m,n,k+1}) = \min \{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{m+2n+2k+2}(x) = d \} < \hat{q}_{m+2n+2k} < \hat{\mu}_{m,n,k}$$

This, combined with Lemma 14, implies that all periodic point of $f$ in $(\hat{q}_{m+2n+2k}, \hat{\mu}_{m,n,k})$ of odd periods have least periods $\geq m + 2n + 2k + 2$.

On the other hand, we have $f^2(x) < d$ for all $v < x < z_0$. Since

$$f^{m+2n+2k+2}(\hat{q}_{m+2n+2k}) - \hat{q}_{m+2n+2k} = f^2(\hat{q}_{m+2n+2k}) - \hat{q}_{m+2n+2k} < 0$$

and

$$f^{m+2n+2k+2}(\hat{\mu}_{m,n,k}) - \hat{\mu}_{m,n,k} = z_0 - \hat{\mu}_{m,n,k} > 0$$

the point $\hat{p}_{m+2n+2k+2} = \min \{ \hat{q}_{m+2n+2k} \leq x \leq \hat{\mu}_{m,n,k} : f^{m+2n+2k+2}(x) = x \} (> \hat{q}_{m+2n+2k})$ exists and must be a period-$(m + 2n + 2k + 2)$ point of $f$.

Since $f^{m+2n+2k+2}(\hat{\mu}_{m,n,k}) = z_0$ and $f^{m+2n+2k+2}(\hat{p}_{m+2n+2k+2}) = \hat{p}_{m+2n+2k+2}$, and since we have just shown in the above that all periodic point of $f$ in the interval $(\hat{q}_{m+2n+2k}, \hat{\mu}_{m,n,k})$ of odd periods have least periods $\geq m + 2n + 2k + 2$, it follows from Lemma 3 that,

for each $i \geq 1$, the point $\hat{q}_{m+2n+2k+2i}$ is a period-$(m + 2n + 2k + 2i)$ point of $f$.

As for the periods of $\hat{c}_{2m+2n+2k+2i}'$s and $\hat{\epsilon}'_{2m+2n+2k+2i}$’s, we apply Lemma 6 with

$$f^{m+2n+2k+3}(\hat{\mu}'_{m,n,k+1}) = f(d)$$

and $f^{2(m+n+k+1)}(\hat{\nu}_{m,n,k}) = \min P$ and

$$f^{m+2n+2k+3}(\hat{\mu}_{m,n,k+1}) = f(d)$$

and $f^{2(m+n+k+1)}(\hat{\nu}_{m,n,k}) = \min P$ respectively.
to obtain that, for each $i \geq 0$ such that $(m + n + k + 1) + i \geq \max \{ m + 2n + 2k + 3, m + 2 \} = m + 2n + 2k + 3$, both $c_{2m+2n+2k+2+2i}^{(n,k,3)}$ and $\hat{c}_{2m+2n+2k+2+2i}^{(n,k,3)}$ are period-(2$m + 2n + 2k + 2 + 2i$) points of $f$, or, equivalently, in $[\mu_{m,n,k+1}, \bar{\mu}_{m,n,k}](\subset [\mu_{m,n}, \bar{\mu}_{m,n+1}] \subset [v, \bar{u}_0] \subset [v, z_0])$.

for each $i \geq n+k+3$, both $c_{2m+2n+2k+2i}^{(n,k,3)}$ and $\hat{c}_{2m+2n+2k+2i}^{(n,k,3)}$ are period-(2$m + 2n + 2k + 2 + 2i$) points of $f$.

3.5 A compartment of the third layer of the basic tower in $[\bar{u}_{n,k+1}, \bar{u}_{n,k}](\subset [\bar{u}_n', \bar{u}_{n+1}'] \subset [v, z_0])$.

For each $n \geq 1$ and $k \geq 1$, we consider the interval $[\bar{u}_{n,k+1}, \bar{u}_{n,k}](\subset [\bar{u}_n', \bar{u}_{n+1}'] \subset [v, z_0])$. Let $\nu_{n,k}$ be a point in $[\bar{u}_n', \bar{u}_{n+1}']$ such that $f^{2n+2k}(\nu_{n,k}) = v$. Then the point $\nu_{n,k}$ happens to satisfy that $\bar{u}_{n,k+1} < \nu_{n,k} < \bar{u}_{n,k}$. For each $i \geq 1$, let, with $x$-coordinates moving from point $\bar{u}_{n,k}$ via point $\nu_{n,k}$ to point $\bar{u}_{n,k+1}$,

$$\bar{u}_{n,k,i} = \max\{\bar{u}_{n,k+1} \leq x \leq \bar{u}_{n,k} : f^{2n+2k+2i}(x) = d\},$$

$$\hat{c}_{2n+2k+2i}^{(n,k,3)} = \max\{\bar{u}_{n,k+1} \leq x \leq \bar{u}_{n,k} : f^{2n+2k+2i}(x) = x\},$$

$$\bar{q}_{m+2n+2k+2i}^{(n,k,3)} = \max\{\bar{u}_{n,k+1} \leq x \leq \bar{u}_{n,k,1} : f^{m+2n+2k+2i}(x) = x\} \text{ and}$$

$$\bar{p}_{m+2n+2k+2i}^{(n,k,3)} = \min\{\bar{u}_{n,k+1} \leq x \leq \bar{u}_{n,k} : f^{m+2n+2k+2i}(x) = x\}. $$

For each fixed $n \geq 1$ and $k \geq 1$, the collection of all these periodic points $c_{2n+2k+2i}^{(n,k,3)}$, $\hat{c}_{2n+2k+2i}^{(n,k,3)}$, $q_{m+2n+2k+2i}^{(n,k,3)}$, $p_{m+2n+2k+2i}^{(n,k,3)}$, $i \geq 1$ is called a compartment of the third layer of the basic tower of periodic points of $f$ associated with $P$.

It is easy to see that

$$\bar{u}_{n,k+1} < \nu_{n,k} < \bar{u}_{n,k,1} < \bar{u}_{n,k,2} < \bar{u}_{n,k,3} < \cdots < \bar{u}_{n,k} \text{ and,}$$

combined with the fact $(\dagger \dagger \dagger)$ that $d < f^{2j}(x) < z_0$ for all $\bar{u}_n' \leq x \leq \bar{u}_{n,k}$ and all $1 \leq j \leq n + k$,

$$d < f^{2j}(x) < z_0 \text{ for all } 1 \leq j \leq n + k + i \text{ and all } \bar{u}_{n,k,i}' \leq x \leq \bar{u}_{n,k}, \ (\dagger \dagger \dagger \dagger)$$

As for the periods of $c_{2n+2k+2i}^{(n,k,3)}$'s, since we do not have a result similar to Lemma 12, we can apply Lemma 6 with

$$f^{2(n+k+1)}(\bar{u}_{n,k}) = z_0 \text{ and } f^{2(n+k+1)}(\nu_{n,k}) = \min{P}.$$
However, since, in this case, we have the following fact that
\[ d < f^{2j}(x) < z_0 \] for all \( 1 \leq j \leq n + k + i \) and all \( \bar{u}_{n,k,i} \leq x \leq \bar{u}_{n,k} \),
we can get a result better than that obtained by simply applying Lemma 6. First, by Lemma 6(4), we obtain that, for each \( i \geq 1 \),
if \((n + k + 1) + i\) is even and \( \geq 2(n + k + 1) \), then \( c_{2(n + k + 1) + 2i}^{(n,k,3)} \) is a period-(\( 2(n + 2k + 2 + 2i) \)) point of \( f \),
if \((n + k + 1) + i\) is odd and \( \geq 2(n + k + 1) \), then \( c_{2(n + k + 1) + 2i}^{(n,k,3)} \) is either a period-(\( 2(n + 2k + 2 + 2i) \)) or an odd period-(\( n + k + 1 + i \)) point of \( f \).

Now it follows from the above \((\dagger \dagger \dagger)\) that \( c_{2(n + k + 1) + 2i}^{(n,k,3)} \) cannot be an odd period-(\( n + k + 1 + i \)) point of \( f \). Consequently, we obtain that, for each \( i \geq 1 \) such that \((n + k + 1) + i \geq 2(n + k + 1)\), \( c_{2(n + k + 1) + 2i}^{(n,k,3)} \) is a period-(\( 2(n + 2k + 2 + 2i) \)) point of \( f \), or, equivalently,
for each \( i \geq n + k + 2 \), \( c_{2n+2k+2i}^{(n,k,3)} \) is a period-(\( 2(n + 2k + 2 + 2i) \)) point of \( f \).

As for the periods of \( \bar{q}_{m+2n+2k+2i}^{(n,k,3)} \)'s and \( \bar{p}_{m+2n+2k+2i}^{(n,k,3)} \)'s, we apply Lemma 5 with
\[ f^{2n+k+3}(\bar{u}_{n,k,1}) = f(d) \] and \( f^{m+2(n+k+1)}(\bar{v}_{n,k}) = \min P \) and
\[ f^{2n+k+3}(\bar{u}_{n,k+1}) = f(d) \] and \( f^{m+2(n+k+1)}(\bar{v}_{n,k}) = \min P \) respectively
and the above fact \((\dagger \dagger \dagger)\) to obtain that,
for each \( i \geq 1 \), both \( \bar{q}_{m+2n+2k+2i}^{(n,k,3)} \) and \( \bar{p}_{m+2n+2k+2i}^{(n,k,3)} \) are period-(\( m + 2n + 2k + 2i \)) points of \( f \).

Note that the relative locations of the periodic points \( c_{2n+2k+2i}^{(n,k,3)} \)'s with respect to the points \( \bar{u}_{n,k,i} \)'s are not known except that \( \bar{u}_{n,k,i}^{'} < c_{2n+2k+2i}^{(n,k,3)} \) for all \( i \geq 1 \).

We call the collection of all these periodic points, from the (smallest) point \( \min P \) (exclusive) to the (largest) point \( z_0 \) (exclusive), excluding \( \bar{q}_{m+2n+2k+2i}^{(n,k,3)} \),
\[ c_{2m+2n+2k+2i}^{(n,k,3)}, c_{2m+2n+2k+2i}^{(n,k,3)}^{\prime}, \bar{q}_{m+2n+2k+2i}^{(n,k,3)}; i \geq 1, k \geq 1, n \geq 1; \]
\[ c_{2m+2n+2k+2i}^{(n,k,3)}, q_{m+2n+2k+2i}^{(n,k,3)}; i \geq 1, k \geq 1, n \geq 1; \]
\[ \hat{p}_{m+2n+2k+2i}^{(n,k,3)}, \hat{q}_{m+2n+2k+2i}^{(n,k,3)}; i \geq 1, k \geq 1, n \geq 1; \]
\[ \hat{c}_{2m+2n+2k+2i}^{(n,k,3)}, \hat{c}_{2m+2n+2k+2i}^{(n,k,3)}^{\prime}, \hat{q}_{m+2n+2k+2i}^{(n,k,3)}; i \geq 1, k \geq 1, n \geq 1; \]
the third layer of the basic tower of periodic points of \( f \) associated with \( P \).
§4. **The higher layers of the basic tower of periodic points of \( f \) associated with \( P \).**

We wrap up the results (from the first layer to the third layer of the basic tower) obtained so far and arrange them in the order from the interval \([\min P, d]\), through \([d, u_1]\), \([u_0, v]\) \((\subset [u_1, v])\), \([v, u_0]\) \((\subset [v, u_1'])\) to \([u_1', z_0]\).

**4.1 On the interval \([\min P, d]\):**

The first layer, the starting step, for each \( n \geq 0 \),

\[
\hat{\mu}_{m,n} = \max \left\{ \min P \leq x \leq d : f^{m+2n}(x) = d \right\},
\]

\[
\tilde{q}_{m+2n} = \max \left\{ \min P \leq x \leq d : f^{m+2n}(x) = x \right\}.
\]

For each \( n \geq 0 \), \( \tilde{q}_{m+2n} \) is a period-\((m + 2n)\) point of \( f \).

The second layer, for any fixed \( n \geq 0 \) and for each \( k \geq 1 \),

\[
\hat{\mu}_{m,n,k} = \min \left\{ \hat{\mu}_{m,n} \leq x \leq \hat{\mu}_{m,n+1} : f^{m+2n+2k}(x) = d \right\},
\]

\[
\hat{p}^{(n,2)}_{m+2n+2k} = \min \left\{ \hat{\mu}_{m,n} \leq x \leq \hat{\mu}_{m,n+1} : f^{m+2n+2k}(x) = x \right\},
\]

\[
\hat{c}^{(n,2)}_{2m+2n+2k} = \min \left\{ \hat{\mu}_{m,n} \leq x \leq \hat{\mu}_{m,n+1} : f^{2m+2n+2k}(x) = x \right\},
\]

\[
\hat{c}'^{(n,2)}_{2m+2n+2k} = \max \left\{ \hat{\mu}_{m,n} \leq x \leq \hat{\mu}_{m,n+1} : f^{2m+2n+2k}(x) = x \right\}.
\]

For any fixed \( n \geq 0 \), we have

for each \( k \geq 1 \), \( \hat{p}^{(n,2)}_{m+2n+2k} \) is a period-\((m + 2n + 2k)\) point of \( f \) and,

for each \( k \geq n + 3 \), both \( \hat{c}^{(n,2)}_{2m+2n+2k} \) and \( \hat{c}'^{(n,2)}_{2m+2n+2k} \) are period-\((2m + 2n + 2k)\) points of \( f \).

The third layer, for any fixed \( n \geq 0 \), \( k \geq 1 \) and for each \( i \geq 1 \),

\[
\hat{\mu}_{m,n,k,i} = \max \left\{ \hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = d \right\},
\]

\[
\hat{q}^{(n,k,3)}_{m+2n+2k+2i} = \max \left\{ \hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = x \right\},
\]

\[
\hat{c}^{(n,k,3)}_{2m+2n+2k+2i} = \max \left\{ \hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k} : f^{2m+2n+2k+2i}(x) = x \right\},
\]

\[
\hat{c}'^{(n,k,3)}_{2m+2n+2k+2i} = \min \left\{ \hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k} : f^{2m+2n+2k+2i}(x) = x \right\}.
\]

For any fixed \( n \geq 0 \) and \( k \geq 1 \), we have

for each \( i \geq 1 \), \( \hat{q}^{(n,k,3)}_{m+2n+2k+2i} \) is a period-\((m + 2n + 2k + 2i)\) point of \( f \) and,

for each \( i \geq n + k + 3 \),

both \( \hat{c}'^{(n,k,3)}_{2m+2n+2k+2i} \) and \( \hat{c}^{(n,k,3)}_{2m+2n+2k+2i} \) are period-\((2m + 2n + 2k + 2i)\) points of \( f \).
4.2 On the interval $[d, u_1]$:

The first layer, the starting step, for each $n \geq 1$,

$$u_n = \min \{d \leq x \leq v : f^{2n}(x) = d\}$$

$$c_{2n} = \min \{d \leq x \leq u_n : f^{2n}(x) = x\}.$$  

For each $n \geq 1$, $c_{2n}$ is a period-$(2n)$ point of $f$.

The second layer, for any fixed $n \geq 1$ and for each $k \geq 1$,

$$u'_{n,k} = \max \{u_{n+1} \leq x \leq u_n : f^{2n+2k}(x) = d\},$$

$$c_{2n+2k}^{(n,2)} = \max \{u_{n+1} \leq x \leq u_n : f^{2n+2k}(x) = x\},$$

$$q_{m+2n+2k}^{(n,2)} = \max \{u_{n+1} \leq x \leq u'_{n,1} : f^{m+2n+2k}(x) = x\},$$

$$p_{m+2n+2k}^{(n,2)} = \min \{u_{n+1} \leq x \leq u_n : f^{m+2n+2k}(x) = x\}.$$

For any fixed $n \geq 1$, we have

for each $k \geq 1$, $c_{2n+2k}^{(n,2)}$ is a period-$(2n + 2k)$ (except possibly period-$(4n)$) point of $f$ and,

for each $k \geq 1$, both $q_{m+2n+2k}^{(n,2)}$ and $p_{m+2n+2k}^{(n,2)}$ are period-$(m + 2n + 2k)$ points of $f$.

The third layer, for any fixed $n \geq 1$, $k \geq 1$ and for each $i \geq 1$,

$$u_{n,k,i} = \min \{u_{n,k} \leq x \leq u_{n,k+1} : f^{2n+2k+2i}(x) = d\},$$

$$c_{2n+2k+2i}^{(n,k,3)} = \min \{u_{n,k} \leq x \leq u_{n,k+1} : f^{2n+2k+2i}(x) = x\},$$

$$p_{m+2n+2k+2i}^{(n,k,3)} = \min \{u_{n,k,1} \leq x \leq u_{n,k+1} : f^{m+2n+2k+2i}(x) = x\},$$

$$q_{m+2n+2k+2i}^{(n,k,3)} = \max \{u_{n,k} \leq x \leq u_{n,k+1} : f^{m+2n+2k+2i}(x) = x\}.$$

For any fixed $n \geq 1$ and $k \geq 1$, we have

for each $i \geq 1$ such that $i \notin \{i, n + k\}$, where $i$ is the unique integer such that

$1 \leq i \leq n$ and $2n + 2k + 2i$ is a multiple of $2n$,

$c_{2n+2k+2i}^{(n,k,3)}$ is a period-$(2n + 2k + 2i)$ point of $f$ and,

for each $i \geq 1$, both $p_{m+2n+2k+2i}^{(n,k,3)}$ and $q_{m+2n+2k+2i}^{(n,k,3)}$ are period-$(m + 2n + 2k + 2i)$ points of $f$. 

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4.3 On the interval $[\hat{u}_0, v] \subset [u_1, v]$:

The first layer, the starting step, for each $n \geq 1$,

\[
\tilde{\mu}_{m,n} = \min \left\{ \hat{u}_0 \leq x \leq v : f^{m+2n}(x) = d \right\},
\]
\[
\tilde{p}_{m+2n} = \min \left\{ \hat{u}_0 \leq x \leq v : f^{m+2n}(x) = \right\}.
\]

For each $n \geq 1$, $\tilde{p}_{m+2n}$ is a period-$(m + 2n)$ point of $f$.

The second layer, for any fixed $n \geq 1$ and for each $k \geq 1$,

\[
\tilde{\mu}'_{m,n,k} = \max \left\{ \tilde{\mu}_{m,n+1} \leq x \leq \tilde{\mu}_{m,n} : f^{m+2n+2k}(x) = d \right\},
\]
\[
\tilde{q}_{m+2n+2k}^{(n,2)} = \max \left\{ \tilde{\mu}_{m,n+1} \leq x \leq \tilde{\mu}_{m,n} : f^{m+2n+2k}(x) = x \right\},
\]
\[
\tilde{c}_{2m+2n+2k}^{(n,2)} = \max \left\{ \tilde{\mu}_{m,n+1} \leq x \leq \tilde{\mu}_{m,n} : f^{2n+2n+2k}(x) = x \right\},
\]
\[
\tilde{c}_{2m+2n+2k}^{(n,2)} = \min \left\{ \tilde{\mu}_{m,n+1} \leq x \leq \tilde{\mu}_{m,n} : f^{2n+2n+2k}(x) = x \right\}.
\]

For any fixed $n \geq 1$, we have

for each $k \geq 1$, $\tilde{q}_{m+2n+2k}^{(n,2)}$ is a period-$(m + 2n + 2k)$ point of $f$ and,

for each $k \geq n + 3$, both $\tilde{c}_{2m+2n+2k}^{(n,2)}$ and $\tilde{c}_{2m+2n+2k}^{(n,2)}$ are period-$(2m + 2n + 2k)$ points of $f$.

The third layer, for any fixed $n \geq 1$, $k \geq 1$ and for each $i \geq 1$,

\[
\tilde{\mu}_{m,n,k,i} = \min \left\{ \tilde{\mu}'_{m,n,k} \leq x \leq \tilde{\mu}'_{m,n,k+1} : f^{m+2n+2k+i}(x) = d \right\},
\]
\[
\tilde{p}_{m+2n+2k+2i}^{(n,k,3)} = \min \left\{ \tilde{\mu}'_{m,n,k} \leq x \leq \tilde{\mu}'_{m,n,k+1} : f^{m+2n+2k+2i}(x) = x \right\},
\]
\[
\tilde{c}_{2m+2n+2k+2i}^{(n,k,3)} = \min \left\{ \tilde{\mu}_{m,n,k+1} \leq x \leq \tilde{\mu}'_{m,n,k+1} : f^{2m+2n+2k+2i}(x) = x \right\},
\]
\[
\tilde{c}_{2m+2n+2k+2i}^{(n,k,3)} = \max \left\{ \tilde{\mu}'_{m,n,k} \leq x \leq \tilde{\mu}'_{m,n,k+1} : f^{2m+2n+2k+2i}(x) = x \right\}.
\]

For any fixed $n \geq 1$ and $k \geq 1$, we have

for each $i \geq 1$, $\tilde{p}_{m+2n+2k+2i}^{(n,k,3)}$ is a period-$(m + 2n + 2k + 2i)$ point of $f$, and,

for each $i \geq n + k + 3$,

both $\tilde{c}_{2m+2n+2k+2i}^{(n,k,3)}$ and $\tilde{c}_{2m+2n+2k+2i}^{(n,k,3)}$ are period-$(2m + 2n + 2k + 2i)$ points of $f$. 

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4.4 On the interval $[v, \hat{u}_0] \subset [v, \bar{u}_1]$:

The first layer, the starting step, for each $n \geq 1$,

$$\hat{\mu}_{m,n} = \max \left\{ v \leq x \leq \hat{u}_0 : f^{m+2n}(x) = d \right\},$$

$$\hat{q}_{m+2n} = \max \left\{ v \leq x \leq \hat{u}_0 : f^{m+2n}(x) = x \right\}.$$  

For each $n \geq 1$, $\hat{q}_{m+2n}$ is a period-$(m + 2n)$ point of $f$.

The second layer, for any fixed $n \geq 1$ and for each $k \geq 1$,

$$\hat{\mu}_{m,n,k} = \max \left\{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{m+2n+2k}(x) = d \right\},$$

$$\hat{p}_{m+2n+2k}^{(n,2)} = \min \left\{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{m+2n+2k}(x) = x \right\},$$

$$\hat{c}_{2m+2n+2k}^{(n,2)} = \min \left\{ \hat{\mu}_{m,n,1} \leq x \leq \hat{\mu}_{m,n+1}' : f^{2m+2n+2k}(x) = x \right\},$$

$$\hat{c}_{2m+2n+2k}^{(n,2)} = \max \left\{ \hat{\mu}'_{m,n} \leq x \leq \hat{\mu}'_{m,n+1} : f^{2m+2n+2k}(x) = x \right\}.$$  

For any fixed $n \geq 1$, we have

for each $k \geq 1$, $\hat{p}_{m+2n+2k}^{(n,2)}$ is a period-$(m + 2n + 2k)$ point of $f$ and, for each $k \geq n + 3$, both $\hat{c}_{2m+2n+2k}^{(n,2)}$ and $\hat{c}_{2m+2n+2k}^{(n,2)}$ are period-$(2m + 2n + 2k)$ points of $f$.

The third layer, for any fixed $n \geq 1$, $k \geq 1$ and for each $i \geq 1$,

$$\hat{\mu}'_{m,n,k,i} = \max \left\{ \hat{\mu}_{m,n,k} \leq x \leq \hat{\mu}_{m,n,k+1} : f^{m+2n+2k+2i}(x) = d \right\},$$

$$\hat{q}_{m+2n+2k+2i}^{(n,k,3)} = \max \left\{ \hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k} : f^{m+2n+2k+2i}(x) = x \right\},$$

$$\hat{c}_{2m+2n+2k+2i}^{(n,k,3)} = \max \left\{ \hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k+1}' : f^{2m+2n+2k+2i}(x) = x \right\},$$

$$\hat{c}_{2m+2n+2k+2i}^{(n,k,3)} = \min \left\{ \hat{\mu}_{m,n,k+1} \leq x \leq \hat{\mu}_{m,n,k} : f^{2m+2n+2k+2i}(x) = x \right\}.$$  

For any fixed $n \geq 1$ and $k \geq 1$, we have

for each $i \geq 1$, $\hat{q}_{m+2n+2k+2i}^{(n,k,3)}$ is a period-$(m + 2n + 2k + 2i)$ point of $f$ and, for each $i \geq n + k + 3$,

both $\hat{c}_{2m+2n+2k+2i}^{(n,k,3)}$ and $\hat{c}_{2m+2n+2k+2i}^{(n,k,3)}$ are period-$(2m + 2n + 2k + 2i)$ points of $f$.  

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4.5 On the interval $[\bar{u}_1', z_0]$:

The first layer, the starting step, for each $n \geq 1$ (note that there is a twist in the definition of $\bar{c}_{2n+2}$),

$$\bar{u}_n' = \max \{ v \leq x \leq z_0 : f^{2n}(x) = d \},$$

$$\bar{c}_{2n+2} = \min \{ \bar{u}_n' \leq x \leq z_0 : f^{2n+2}(x) = x \}.$$

For each $n \geq 1$, $\bar{c}_{2n+2}$ is a period-(2n + 2) point of $f$.

The second layer, for any fixed $n \geq 1$ and for each $k \geq 1$,

$$\bar{u}_{n,k} = \min \{ \bar{u}_n' \leq x \leq \bar{u}_{n+1}' : f^{2n+2k}(x) = d \},$$

$$\bar{c}^{(n,2)}_{2n+2k} = \min \{ \bar{u}_n' \leq x \leq \bar{u}_{n+1}' : f^{2n+2k}(x) = x \},$$

$$\bar{p}^{(n,2)}_{m+2n+2k} = \min \{ \bar{u}_{n+1} \leq x \leq \bar{u}_{n+1}' : f^{m+2n+2k}(x) = x \},$$

$$\bar{q}^{(n,2)}_{m+2n+2k} = \max \{ \bar{u}_n' \leq x \leq \bar{u}_{n+1}' : f^{m+2n+2k}(x) = x \}.$$

For any fixed $n \geq 1$, we have

for each $k \geq 1$, $\bar{c}^{(n,2)}_{2n+2k}$ is a period-(2n + 2k) point of $f$ and,

for each $k \geq 1$, both $p^{(n,2)}_{m+2n+2k}$ and $q^{(n,2)}_{m+2n+2k}$ are period-(m + 2n + 2k) points of $f$.

The third layer, for any fixed $n \geq 1$, $k \geq 1$ and for each $i \geq 1$,

$$\bar{u}_{n,k,i}' = \max \{ \bar{u}_{n,k+1} \leq x \leq \bar{u}_{n,k} : f^{2n+2k+2i}(x) = d \},$$

$$\bar{c}^{(n,k,3)}_{2n+2k+2i} = \max \{ \bar{u}_{n,k+1} \leq x \leq \bar{u}_{n,k} : f^{2n+2k+2i}(x) = x \},$$

$$\bar{q}^{(n,k,3)}_{m+2n+2k+2i} = \max \{ \bar{u}_{n,k+1} \leq x \leq \bar{u}_{n,k+1}' : f^{m+2n+2k+2i}(x) = x \},$$

$$\bar{p}^{(n,k,3)}_{m+2n+2k+2i} = \min \{ \bar{u}_{n,k+1} \leq x \leq \bar{u}_{n,k} : f^{m+2n+2k+2i}(x) = x \}.$$

For any fixed $n \geq 1$ and $k \geq 1$, we have

for each $i \geq n + k + 2$, $\bar{c}^{(n,k,3)}_{2n+2k+2i}$ is a period-(2n + 2k + 2i) point of $f$ and,

for each $i \geq 1$, both $p^{(n,k,3)}_{m+2n+2k+2i}$ and $q^{(n,k,3)}_{m+2n+2k+2i}$ are period-(m + 2n + 2k + 2i) points of $f$.

Now a pattern emerges which allows us to write down the formulas for the periodic points of the fourth and higher layers of the basic tower of periodic points of $f$ associated with $P$ and then we can apply Lemmas 5 & 6 to determine the least periods of all except possibly finitely many of these periodic points. We leave the details to the interested readers.
§5. An example.

Let \( P \) denote a period-3 orbit of \( f \) and let \( m = 3 \). We now describe the first and the second layers of the basic tower of periodic points of \( f \) associated with \( P \) and leave the third and the higher layers of the basic tower to the interested readers. We shall follow the notations introduced before.

On the first layer,

\[
\min P \leq \tilde{q}_3 < \tilde{q}_5 < \tilde{q}_7 < \cdots < d < \cdots < c_6 < c_4 < c_2 < u_1 \leq \tilde{u}_0 < \cdots < \tilde{p}_9 < \tilde{p}_7 < \tilde{p}_5 < v < \tilde{q}_5 < \tilde{q}_7 < \tilde{q}_9 < \cdots < \tilde{u}_1 < \tilde{c}_4 < \tilde{c}_6 < \tilde{c}_8 < \cdots < z_0;
\]

On the second layer, (we deliberately leave out the points \( \tilde{p}_5^{(0,2)}, \tilde{p}_7^{(1,2)}, \tilde{p}_9^{(2,2)}, \tilde{p}_{11}^{(3,2)}, \cdots \))

\[
\begin{align*}
\mu^{t}_{m,0} < \cdots < &\tilde{p}_{11}^{(2)} < \tilde{p}_9^{(2)} < \tilde{c}_2^{(0,2)} < \cdots < \tilde{c}_2^{(0,2)} < \tilde{c}_2^{(0,2)} < \tilde{c}_2^{(0,2)} < \cdots < \\
\mu^{t}_{m,1} < \cdots < &\tilde{p}_{13}^{(2,2)} < \tilde{p}_{11}^{(2,2)} < \tilde{p}_9^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,2} < \cdots < &\tilde{p}_{15}^{(2,2)} < \tilde{p}_{13}^{(2,2)} < \tilde{p}_{11}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,3} < \cdots < &\tilde{p}_{17}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,4} < \cdots < &\tilde{c}_1^{(2,2)} = \tilde{c}_1^{(2,2)} = \tilde{c}_1^{(2,2)} = \cdots < \\
\mu^{t}_{m,5} < \cdots < &\tilde{p}_{19}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,6} < \cdots < &\tilde{p}_{21}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,7} < \cdots < &\tilde{p}_{23}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,8} < \cdots < &\tilde{p}_{25}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,9} < \cdots < &\tilde{p}_{27}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,10} < \cdots < &\tilde{p}_{29}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,11} < \cdots < &\tilde{p}_{31}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,12} < \cdots < &\tilde{p}_{33}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,13} < \cdots < &\tilde{p}_{35}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,14} < \cdots < &\tilde{p}_{37}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,15} < \cdots < &\tilde{p}_{39}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,16} < \cdots < &\tilde{p}_{41}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,17} < \cdots < &\tilde{p}_{43}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,18} < \cdots < &\tilde{p}_{45}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,19} < \cdots < &\tilde{p}_{47}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots < \\
\mu^{t}_{m,20} < \cdots < &\tilde{p}_{49}^{(2,2)} < \cdots < \tilde{c}_2^{(2,2)} < \cdots <
\end{align*}
\]

\[
\begin{align*}
\hat{u}_1 \leq &\tilde{u}_0 < \\
\hat{u}_2 ^t < \cdots < &\tilde{c}_8^{(1,2)} < \tilde{c}_6^{(1,2)} < \cdots < \tilde{p}_{11}^{(1,2)} < \tilde{p}_9^{(1,2)} < \tilde{p}_7^{(1,2)} < \tilde{p}_5^{(1,2)} < \tilde{q}_5^{(1,2)} < \tilde{q}_7^{(1,2)} < \tilde{q}_9^{(1,2)} < \cdots < \\
\hat{u}_3 ^t < \cdots < &\tilde{c}_8^{(3,2)} < \tilde{c}_6^{(3,2)} < \cdots < \tilde{p}_{15}^{(3,2)} < \tilde{p}_{13}^{(3,2)} < \tilde{p}_{11}^{(3,2)} < \tilde{q}_5^{(3,2)} < \tilde{q}_7^{(3,2)} < \tilde{q}_9^{(3,2)} < \cdots <
\end{align*}
\]

\( (z_0). \)