DIRECT CONSTRUCTION OF CODE LOOPS

GÁBOR P. NAGY

Abstract. Code loops were introduced by R. L. Griess. Griess and Hsu gave methods to construct the corresponding code loop from any given doubly even binary code; both these methods used some kind of induction. In this paper, we present a global construction of the loop, where we apply the correspondance between the concepts of Moufang loops and groups with triality.

1. Introduction

For a doubly even binary code, one can introduce three operation in a combinatorial way. These operations are related by polarization. Hsu called vector spaces over $\mathbb{F}_2$ with such operations symplectic cubic spaces. In $\mathbb{F}_2$, R. L. Griess has introduced the notion of code loops; these are Moufang loops $L$ such that $L/A$ is an elementary Abelian 2-group for some central subgroup $A$ of order 2. In code loops, the power, the commutator and the associator maps define a symplectic cubic space on $L/A$. The fact that any symplectic cubic space corresponds to a code loop was shown by Griess; he constructed the factor set of the loop extension by induction on the dimension. Hsu gave more explicit method to construct the loops as centrally twisted products. However, also Hsu’s method uses some inductive argument, since he puts together the loop from smaller parts using a clever product rule.

In this paper, we present a global construction for the loop. Therefore, we use that Moufang loops can be equivalently given by the specific group theoretical concept of groups with triality. Hence, starting from the symplectic cubic space, we first define a group $G$ and automorphisms $\sigma, \rho$ of $G$. Then, we prove that $G$ is a group with triality with respect to these automorphisms. Finally, we show that the Moufang loop $L$ corresponding to $G$ is a code loop, which gives rise to the symplectic cubic space we started with. This global approach made possible to implement our method using the GAP computer algebra package by Nagy and Vojtěchovský.

1991 Mathematics Subject Classification. 94B60, 20N05.

Key words and phrases. Doubly even binary codes, code loops, Moufang loops, groups with triality.

The author was supported by the FKFP grant 0063/2001 of the Hungarian Ministry for Education and the OTKA grants nos. F042959 and T043758.
For completeness, we mention Kitazume [8] where the author gives another explicit construction of the code loop for some special cases of doubly even codes. Also, the work of Chein and Goodaire [2] is important, where they showed the reverse implication, namely they constructed doubly even binary codes from a given symplectic cubic space. Unfortunately, this code is by far not unique; the one in the general construction has parameters $[s, n]$, where $n$ is the dimension of the symplectic cubic space and $s = O(2^{3n^3})$. Finally, we mention the paper Vojtěchovský [11], where the author generalizes the notions of symplectic cubic spaces and doubly even codes and the construction of Chein and Goodaire.

2. Preliminaries

The subspace $C \leq \mathbb{F}_2^n$ is a **doubly even binary code**, if $4|w(x)$ for all $x \in C$, where the weight $w(x)$ of $x$ is the number of nonzero coordinates of $x$. We can identify $C$ with a set of subsets of $\{1, \ldots, n\}$. In this manner, $w(x) = |x|$. Moreover, if $C$ is doubly even, we have $2|w(x \cap y)$ and the functions $\sigma : C \to \mathbb{F}_2$, $\kappa : C \times C \to \mathbb{F}_2$, $\alpha : C \times C \times C \to \mathbb{F}_2$,

\[
\sigma(x) = \frac{1}{4} w(x) \pmod{2},
\]

\[
\kappa(x, y) = \frac{1}{2} w(x \cap y) \pmod{2},
\]

\[
\alpha(x, y, z) = w(x \cap y \cap z) \pmod{2}.
\]

are well defined. Clearly, $\kappa$ and $\alpha$ are alternating and they satisfy the relations

\[
(1) \quad \sigma(x + y) = \sigma(x) + \sigma(y) + \kappa(x, y),
\]

\[
(2) \quad \kappa(x + y, z) = \kappa(x, z) + \kappa(y, z) + \alpha(x, y, z),
\]

\[
(3) \quad \alpha(x + y, z, t) = \alpha(x, z, t) + \alpha(y, z, t).
\]

With other words, $\alpha$ is trilinear and $\kappa$ and $\alpha$ is obtained from $\sigma$ and $\kappa$ by polarization, respectively.

The set $L$ together with a binary operation $(x, y) \mapsto x \cdot y = xy$ is a **quasigroup** if the equation $xy = z$ can be uniquely solved if two of the three indeterminants is given. A quasigroup with a neutral element is a **loop**. Every element $x$ of a loop $L$ determines left and right multiplication maps $L_x, R_x : L \to L$, $yL_x = xy$ and $yR_x = yx$. (As the reader can see, we write group actions on the right hand side.) The maps $R_x, L_x$ are clearly permutations of $L$. The permutation group $\text{Mlt}(L)$ generated by all left and right multiplications is called the **multiplication group** of $L$. The permutations $L_{x,y} = L_x L_y L_{y^{-1}}$, $R_{x,y} = R_y R_x R_{x^{-1}}$ are called **inner maps** of $L$.

**Moufang loops** are defined by the **Moufang identity**

\[
x(y(xz)) = ((xy)x)z.
\]
Recall that Moufang loops are diassociative, which means that every subloop generated by two elements is a subgroup. In particular, the power $x^n, n \in \mathbb{N}$, the inverse $x^{-1}$ and the commutator $[x, y] = x^{-1}y^{-1}xy$ are well-defined for every $x, y \in L$.

The associator of $x, y, z \in L$ is the element $(x, y, z) = (xy \cdot z)^{-1}(x \cdot yz)$. The center $Z(L)$ of $L$ consists of the elements $z \in L$ satisfying $[x, z] = (x, y, z) = (z, x, y) = 1$ for all $x, y \in L$. One sees easily that every subloop $A \leq Z(L)$ is a normal subloop of $L$, that is, the factor loop $L/A$ is well defined. (See Bruck [1] for basic concepts on quasigroups and Moufang loops.)

The Moufang loop $L$ is called a small Frattini Moufang loop, if $L/A$ is an elementary Abelian $p$-group for some $A \leq Z(L)$ with $|A| = p$.

By Hsu [7], for $p > 3$, any small Frattini Moufang loop is associative.

Assume $L$ to be a small Frattini Moufang 2-loop which is not an elementary Abelian 2-group. Let us identify the vector space $V$ over $\mathbb{F}_2$ with the factor loop $L/A$ and the subloop $A$ with the field $\mathbb{F}_2$ of order 2. We can introduce the following operations on $V$:

$$
\sigma(xA) = x^2, \quad \kappa(xA, yA) = [x, y], \quad \alpha(xA, yA, zA) = (x, y, z).
$$

Then, the equations (1), (2) and (3) hold for $\sigma, \alpha, \kappa$.

**Definition 2.1.** Let $V$ be a vector space over $\mathbb{F}_2$. Let $\sigma : V \to \mathbb{F}_2$, $\kappa : V \times V \to \mathbb{F}_2$ and $\sigma : V \times V \times V \to \mathbb{F}_2$ be maps satisfying (1), (2) and (3). Then, $(V, \sigma, \kappa, \alpha)$ is called a symplectic cubic space.

We mention that (1), (2) and (3) imply

$$
\begin{align*}
\kappa(x, x) &= 0, \\
\kappa(x, y) &= \kappa(y, x), \\
\alpha(x, y, z) &= \sigma(x + y + z) + \sigma(x + y) + \sigma(y + z) + \sigma(x + z) + \\
&\quad \sigma(x) + \sigma(y) + \sigma(z).
\end{align*}
$$

Hence, $\alpha$ is a trilinear alternating form on $V$.

Let $S_n$ be the symmetric group on $\{1, \ldots, n\}$. We have the following definition due to Doro [3].

**Definition 2.2.** The pair $(G, S)$ is called a group with triality, if $G$ is a group, $S \leq \text{Aut} G$, $S = \langle \sigma, \rho \mid \sigma^2 = \rho^3 = (\sigma \rho)^2 = 1 \rangle$, and for all $g \in G$ the triality identity

$$
[g, \sigma] [g, \sigma]^{\rho} [g, \sigma]^{\rho^2} = 1
$$

holds.

The following equivalent formulation of the concept of a group with triality is well known.
Lemma 2.3 (Parker). Let $G$ be a group and let $\sigma_1, \sigma_2, \sigma_3$ be involutory automorphisms of $G$. Let us denote by $C_i$ the conjugacy class $\sigma_i^G \subseteq \text{Aut}(G)$. Then, $(G, (\sigma_1, \sigma_2))$ is a group with triality if and only if $(\tau_i \tau_j)^3 = \text{id}$ for all $\tau_i \in C_i$, $\tau_j \in C_j$, $i, j \in \{1, 2, 3\}$, $i \neq j$.

The next lemma characterizes a special class of groups with triality.

Lemma 2.4. Let $G$ be a group and let $S \leq \text{Aut}(G)$ be isomorphic to the symmetric group $S_3$ on 3 elements. Let us denote by $\sigma_1, \sigma_2, \sigma_3$ the involutions of $S$ and put $H_i = C_G(\sigma_i)$ and $C_i = \sigma_i^G$. Then, if $H_i$ acts transitively on $C_j$ for some $i \neq j$, then $(G, S)$ is a group with triality. In particular, if $|G : H_j| = |H_i : H_i \cap H_j| < \infty$ for $i \neq j$, then $(G, S)$ is a group with triality.

Proof. Choose arbitrary $\tau_i = \sigma_i^g \in C_i$ and $\tau_j = \sigma_j^h \in C_j$ with $i \neq j$. By the assumption, there is $f \in H_i$ such that $\sigma_j^f = \sigma_j^{hg^{-1}}$. Then,

$$\tau_i \tau_j = (\sigma_i \sigma_j^{hg^{-1}})^g = (\sigma_i \sigma_j)^{fg},$$

which implies $(\tau_i \tau_j)^3 = ((\sigma_i \sigma_j)^3)^{fg} = \text{id}$. By Lemma 2.3, $(G, S)$ is a group with triality.

\[\Box\]

3. Constructing the group with triality

Let $V = (V, \sigma, \kappa, \alpha)$ be a symplectic cubic space. Let us choose a basis $B = \{b_1, \ldots, b_n\}$ of $V$ and denote by $\sigma_i$, $\kappa_{ij}$ and $\alpha_{ijk}$ the structure constants of $V$ with respect to $B$.

We define the group $G$ with gerenators $g_i, f_i, h_i, i \in \{1, \ldots, n\}$, $u$ and $v$ by the following relations:

(4) $g_i^2 = u^{\sigma_i}, f_i^2 = v^{\sigma_i}, h_i^2 = u^2 = v^2 = 1,$

(5) $[g_i, g_j] = u^{\kappa_{ij}}, [f_i, f_j] = v^{\kappa_{ij}},$

(6) $[g_i, f_j] = (uv)^{\alpha_{ij}} \prod_{k=1}^{n} h_k^\alpha_{ijk},$

(7) $[g_i, h_j] = u^{\delta_{ij}}, [f_i, h_j] = v^{\delta_{ij}},$

(8) $[h_i, h_j] = [g_i, u] = [f_i, u] = [h_i, u] = [g_i, v] = [f_i, v] = [h_i, v] = 1.$

Lemma 3.1. The group $G$ is well defined. Any element of $G$ is of the form $g_1^{x_1} \cdots g_n^{x_n} f_1^{y_1} \cdots f_n^{y_n} h_1^{z_1} \cdots h_n^{z_n} u^{t_1} v^{t_2}$ with $x_i, y_i, z_i, t_i \in \mathbb{Z}_2$. In particular, the order of $G$ is $2^{3n+2}$.

Proof. In order to show that $G$ is well defined, we prove the following.

(i) $E = \langle f_1, \ldots, f_n, h_1, \ldots, h_n, v \rangle$ is an extraspecial 2-group of type + and order $2^{2n+1}$.

(ii) The commutator and power relations for the $g_i$’s are consistent. In particular, modulo $\langle u \rangle$, the group $\langle g_1, \ldots, g_n \rangle$ is an elementary Abelian group of order $2^n$. 

(iii) The map $\gamma_i$ induced on $E \times \langle u \rangle$ by $g_i$ is an automorphism.
(iv) The $\gamma_i$'s induce an automorphism group $A$ of $E \times \mathbb{Z}_2$ which is an elementary Abelian 2-group.

On the one hand, in $E$, $\langle h_1, \ldots, h_n, v \rangle$ is a maximal elementary Abelian 2-group of order $2^{n+1}$. On the other hand, with \( \tilde{f}_i = f_i h_i^{\sigma_i} \prod_{k>i} h_{k}^{\kappa_{ik}} \),

we have
\[
[\tilde{f}_i, \tilde{f}_j] = [f_i, f_j] h_i^{\sigma_i} \prod_{k>i} h_k^{\kappa_{ik}}[f_j, h_j^{\sigma_j} \prod_{k>j} h_k^{\kappa_{jk}}] = v^{\kappa_{ij}} v^{\sigma_{ij}} = 1
\]

and
\[
\tilde{f}_i^2 = f_i^2 h_i^{\sigma_i} \prod_{k>i} h_k^{\kappa_{ik}} = v^{\sigma_i} = 1.
\]

This means that $\langle \tilde{f}_1, \ldots, \tilde{f}_n \rangle$ is elementary Abelian. Finally, $[\tilde{f}_i, h_j] = [f_i, h_j] = v^{\delta_{ij}}$, hence $E$ is as stated in (i).

Let $\kappa^*$ be the alternating bilinear form on $V$ with structure constants $\kappa_{ij}$ and $q$ be the quadratic form obtained by the quadratic extension of $\sigma$ with respect to $\kappa^*$. Then, $q$ determines a central extension of $V$, isomorphic to $\langle g_1, \ldots, g_n, u \rangle$. This proves (ii).

(iii) follows from the fact that $\gamma_i$ preserves the relations of $E\langle u \rangle$. Indeed, putting $f'_j = \gamma_i(f_j) = f_j (\prod_k h_k^{\alpha_{ijk}}) (uv)^{\kappa_{ij}}$, $h'_j = \gamma_i(h_j) = h_j u^{\delta_{ij}}$, we have
\[
[f'_\ell, f'_j] = [f_\ell, f_j] (\prod_k h_k^{\alpha_{\ell,jk}}) (uv)^{\kappa_{\ell,j}} = \gamma_i([f_\ell, f_j]),
\]

\[
[f'_\ell, h'_j] = [f_\ell, h_j'] (\prod_k h_k^{\alpha_{\ell,jk}}) = \gamma_i([f_\ell, h_j]),
\]

\[
(f'_j)^2 = f_j^2 (\prod_k h_k^{\alpha_{ijk}}) = f_j^2 = \gamma_i(f_j^2).
\]

To show (iv), we calculate the action of $\gamma_{i_1} \circ \gamma_{i_2}$. One obtains
\[
f_j \mapsto f_j (\prod_k h_k^{\alpha_{i_1,jk} + \alpha_{i_2,jk}}) (uv)^{\kappa_{i_1,j} + \kappa_{i_2,j}} u^{\alpha_{i_1,j} + \alpha_{i_2,j}},
\]

\[
h_j \mapsto h_j u^{\delta_{i_1,j} + \delta_{i_2,j}}.
\]

This means $\gamma_{i_1} \circ \gamma_{i_2} = \gamma_{i_2} \circ \gamma_{i_1}$ and $\gamma_i^2 = \text{id}$, hence (iv) holds. The other statements are trivial. \qed
Lemma 3.2. The maps

(9) $\sigma : g_i \leftrightarrow f_i, h_i \mapsto h_i, u \leftrightarrow v$

(10) $\rho : g_i \mapsto f_i, f_i \mapsto (g_if_i)^{-1}, h_i \mapsto h_i, u \leftrightarrow v, v \mapsto uv$

extend to automorphisms of $G$. Moreover, $\sigma^2 = \rho^3 = (\sigma\rho)^2 = \text{id}$, hence, $S = \langle \sigma, \rho \rangle$ is isomorphic to the symmetric group $S_3$ on 3 elements.

Proof. The fact that $\sigma$ is an involutionary automorphism is trivial. Also, $\rho$ preserves the relations of $G$; we only present the calculations in the two most complex cases. We first observe that $g_i$ and $f_i$ commute, therefore $(g_if_i)^2 \in Z(G)$. Moreover, $[g_i, f_j] = [f_i, g_j]$ commutes with $f_i, f_j, g_i, g_j$. Thus,

$$[\rho(f_i), \rho(f_j)] = [(g_if_i)^{-1}, (g_jf_j)^{-1}] = [g_if_i, g_jf_j] = [f_i, g_j][f_i, f_j][g_i, g_j][g_i, f_j][g_i, f_j] = [f_i, f_j][g_i, g_j] = (uv)^{\kappa_{ij}} = \rho([f_i, f_j]).$$

Similarly,

$$[\rho(g_i), \rho(f_j)] = [f_i, (g_jf_j)^{-1}] = [f_i, g_jf_j] = [f_i, f_j][f_i, g_j] = [f_i, f_j][f_i, f_j] = v^{\kappa_{ij}} \left( \prod_k h_k^{\alpha_{ijk}} \right) (uv)^{\kappa_{ij}} = \rho([g_i, f_j]).$$

Finally, the relations for $\sigma$ and $\rho$ hold, since $\sigma^2, \rho^3$ and $(\sigma\rho)^2$ leave the generators of $G$ invariant. \hfill \Box

Lemma 3.3. Let us define the subgroups

$$H_1 = \langle g_i, h_i, u \mid i = 1, \ldots, n \rangle,$$

$$H_2 = \langle f_i, h_i, v \mid i = 1, \ldots, n \rangle,$$

$$H_3 = \langle g_if_i, h_i, u \mid i = 1, \ldots, n \rangle$$

of $G$. Then, $H_3 = C_G(\sigma), H_1^\sigma = H_2, H_1^\rho = H_2, H_2^\rho = H_3$.

Proof. Clearly, $H_3 \subseteq C_G(\sigma)$. For the converse, let us write the element $a \in G$ in the form

$$a = g_1^{x_1}f_1^{y_1} \cdots g_n^{x_n}f_n^{y_n}h_1^{z_1} \cdots h_n^{z_n} u^{t_1}v^{t_2}.$$  

One immediately has that $\sigma(a) = a$ only if $x_i = y_i$ and $t_1 = t_2$, that is, $a \in H_3$. This proves $H_3 = C_G(\sigma)$, the rest is trivial. \hfill \Box
Proposition 3.4. Let \((V, \sigma, \kappa, \rho)\) be a symplectic cubic space and let us define the group \(G\) by \((1)-(5)\). Moreover, define the automorphisms \(\sigma\) and \(\rho\) by \((9)\) and \((10)\), respectively. Then, \((G, \langle \sigma, \rho \rangle)\) is a group with triality.

Proof. We have \(H_1 \cap H_2 = H_1 \cap H_3 = H_2 \cap H_3 = \langle h_q, \ldots, h_n \rangle\), hence \(|G : H_3| = |H_2 : H_2 \cap H_3| = 2^{n+1}\). By Lemma 3.3, \((G, \langle \sigma, \rho \rangle)\) is a group with triality. \(\square\)

4. Some properties of Moufang loops given by groups with triality

In the 3-net, the lines \(X = a, Y = a\) and \(XY = a^{-1}\) are permuted by \(\rho\). Indeed, let \(\sigma_1, \sigma_2, \sigma_3\) denote the Bol reflections with respect to the \(Y\)-axis \(X = 1\), \(X\)-axis \(Y = 1\) and transversal line \(XY = 1\), respectively. Then,

\[
\sigma_1 : \begin{cases}
X = a \leftrightarrow X = a^{-1}, \\
Y = a \leftrightarrow XY = a, \\
Y = a^{-1} \leftrightarrow XY = a^{-1},
\end{cases}
\]

\[
\sigma_2 : \begin{cases}
X = a \leftrightarrow XY = a, \\
Y = a \leftrightarrow Y = a^{-1}, \\
X = a^{-1} \leftrightarrow XY = a^{-1},
\end{cases}
\]

\[
\sigma_3 : \begin{cases}
X = a \leftrightarrow Y = a^{-1}, \\
Y = a \leftrightarrow X = a^{-1}, \\
XY = a \leftrightarrow XY = a^{-1},
\end{cases}
\]

and \(\rho = \sigma_2 \sigma_1\) acts as claimed.

Moreover, in the coordinate loop, \(ab = c\) holds if and only if the lines \(X = a, Y = b, XY = c\) are concurrent. Let us denote by \(\tau_a\) the Bol reflection with respect to the axis \(X = a\). Then, \(\tau_a^\rho\) and \(\tau_a^\rho^2\) are the Bol reflections with respect to the lines \(Y = a\) and \(XY = a^{-1}\), respectively. The equation \(ab = c\) holds if and only if \(\langle \tau_a, \tau_b, \tau_c \rangle \cong S_3\), that is, if and only if

\[
\tau_{c^{-1}} = (\tau_a \tau_b \tau_a)^\rho = \tau_b^{\rho a \rho}.
\]

Since \(\tau_{c^{-1}} = \tau_1 \tau_c \tau_1\) and \(\tau_1 = \sigma_1 = \sigma\), we have

\[
ab = c \iff \tau_c = \tau_b^{\rho a \rho}.
\]

Let \((G, S)\) be a group with triality, \(S = \langle \sigma, \rho | \sigma^2 = \rho^3 = (\sigma \rho)^2 = \text{id} \rangle\). As before, we denote by \(\sigma = \sigma_1, \sigma_2, \sigma_3\) the involutions of \(S\). The conjugacy class \(\sigma_i^G\) be \(C_i\). In Hall and Nagy [3], we showed how to construct a (dual) 3-net from \((G, S)\): \(C_1 \cup C_2 \cup C_3\) are the lines and \(\tau_i \in C_i, i = 1, 2, 3\) are concurrent if and only if \(\langle \tau_1, \tau_2, \tau_3 \rangle \cong S_3\).

Proposition 4.1. Let \((G, S)\) be a group with triality and use the notation \(\sigma = \sigma_1, \rho, C_1 = \sigma_1^G\) as before. Let us define the binary operation

\[
\alpha \circ \beta = \beta^{\rho \alpha \rho} = \alpha^{\rho^{-1} \beta \rho^{-1}}\sigma
\]
on $C_1$. Then,

(i) The operation is well defined, $\sigma$ is a two sided unit element. $(C_1, \circ)$ is isomorphic to the Moufang loop associated to $(G, S)$.

(ii) Put $\alpha = \sigma^g$ and $\gamma = [g, \sigma]^\rho$. Using the natural bijection between $C_1 = \sigma^G$ and the set of right cosets $X = G/C_G(\sigma)$, the right action of $\gamma$ on $X$ is equivalent with the right multiplication $R_\alpha$ of the loop. Similarly, the right action of $\gamma^\rho$ on $X$ is equivalent with the left multiplication $L_\alpha$.

(iii) Let $N$ be the largest normal subgroup of $G$, contained in $C_G(\sigma)$. Then, the multiplication group of the Moufang loop associated to $(G, S)$ is a subgroup of $G/N$.

Proof. Clearly, (11) implies (i), and (ii) implies (iii). Furthermore, on the one hand,

$$\rho^{-1} \alpha \rho^{-1} \sigma = \rho^{-1} \sigma^g \sigma \rho = [g, \sigma]^\rho = \gamma.$$ 

On the other hand,

$$\beta R_\alpha = \beta \circ \alpha = \beta \rho^{-1} \alpha \rho^{-1} \sigma = \beta \gamma.$$ 

hence (ii) follows. □

The next lemma will make possible to calculate the structure constants of small Frattini Moufang loops which are given by their groups with triality.

Lemma 4.2. Let $L$ be a Moufang loop satisfying $x^2 \in Z(L)$ for all $x \in L$. Then

(i) $[R_x, R_y] = R_{[x,y]}$,

(ii) $[R_y, L_z] = R_{y^{-1}, z} = L_{y, z^{-1}}$,

(iii) $[[R_x, L_y], R_z] = R_{(x, y, z)}$

hold for all $x, y, z \in L$.

Proof. On the one hand, by P. T. Nagy and K. Strambach [10, Theorem 1.1.6], Moufang loops satisfying $x^2 \in Z(L)$ for all $x \in L$ are conjugacy closed loops, hence $R_{y^{-1}} R_x R_y = R_{y^{-1} x y}$. On the other hand, $L/Z(L)$ has exponent 2, therefore $[x, y] \in Z(L)$ for all $x, y \in L$.

$$R_{x^{-1}} R_{y^{-1}} R_x R_y = R_{x^{-1}} R_{y^{-1}} x y = R_{x^{-1}} R_{x [x, y]} = R_{[x, y]}.$$ 

hence (i) holds. Using the Moufang identities, one obtains

$$a [R_y, L_z] = z ((z^{-1} \cdot ay^{-1}) y)$$

$$= z ((z^{-1} (ay^{-1}) \cdot z^{-1} y))$$

$$= z ((ay^{-1} \cdot z) (z^{-1} y))$$

$$= (ay^{-1} \cdot z) (z^{-1} y)$$

$$= a R_{y^{-1}, z}.$$ 

This shows (ii), since $R_{y^{-1}, z} = L_{y, z^{-1}}$ is well known from Bruck [1, Lemma VII.5.4.]. The inner map $R_{x, y} = R_x R_y R_{xy}^{-1}$ is also known to be a
pseudo-automorphism with companion \([x, y]\). Since \(L/Z(L)\) is Abelian, 
\(R_{x,y}\) turns out to be an automorphism. Moreover, 
\(z^{-1} (zR_{y,x}) \in Z(L)\) for all \(z \in L\). We have 
\[
[R_x, [R_y, L_z]] = [R_x, R_{y^{-1}, z}]
\]
\[
= R_x^{-1} R_x R_{y^{-1}, z}
\]
\[
= R_x^{-1} (x R_{y^{-1}, z})
\]
and
\[
x^{-1} (x R_{y^{-1}, z}) = (x, z^{-1}, y)^{-1}
\]
by Bruck [1, Lemma VII.5.4]. Using the fact that \(L/Z(L)\) elementary Abelian, we get (iii).

5. Structure constants of the triality group

Let \((V, \sigma, \kappa, \alpha)\) be a symplectic cubic space and let us construct the 
group with triality \((G, S)\) as in Section 3. Write \((L = \sigma^G, \circ)\) for the 
Moufang loop associated to \((G, S)\). Since the elements of \(H_3 = C_G(\sigma)\) are 
\[
\{(g_1 f_1)^{x_1} \cdots (g_n f_n)^{x_n} h_1^n \cdots h_n^n (uv)^x\}.
\]
The largest normal subgroup \(N\) of \(G\) contained in \(C_G(\sigma)\) contains the 
element \(uv\). Let \(a\) be an arbitrary element of \(G\) and let us denote by 
\(\bar{a}\) the right action of \(a\) on \(L = \sigma^G\). (This action is naturally equivalent 
with the right action on the right cosets of \(C_G(\sigma)\).) Put \(\bar{G} = G/N = \{\bar{a} \mid a \in G\}\). Clearly, \(\bar{a} = \bar{v} \in Z(\bar{G})\). Moreover, the group \(A = \langle \bar{g}_1, \ldots, \bar{g}_n, \bar{u} \rangle\) acts sharply transitively on \(L\).

As we saw in Proposition 4.1, for the element \(x = \sigma^a \in L\), one has 
\(R_x = \bar{\gamma}\) with \(\gamma = [g, \sigma]^a\) and \(L_x = \gamma \sigma^x\). Put \(s = \sigma^u\), then \(R_s \sim L_s = \bar{u}\) and \(s \in Z(L)\). For the element \(x = \sigma^{g_1} \cdots g_n \), \(R_x = \bar{\gamma}\) with 
\[
\gamma = [g_1^{x_1} \cdots g_n^{x_n}, \sigma]^a
\]
\[
= f_n^{-x_n} \cdots f_1^{-x_1} (g_1 f_1)^{-x_1} \cdots (g_n f_n)^{-x_n}
\]
\[
= g_1^{x_1} \cdots g_n^{x_n} \left( \prod_{k; i < j} h_k^{\alpha_{ij} x_i x_j} \right) a^a v^b
\]
for some \(a, b \in \mathbb{Z}_2\), that is, 
\[
R_x = \bar{\gamma} = \bar{g}_1^{x_1} \cdots \bar{g}_n^{x_n} \left( \prod_{k; i < j} \bar{h}_k^{\alpha_{ij} x_i x_j} \right) \bar{u}^z
\]
for some \(z \in \mathbb{Z}_2\). In particular, the set of right multiplications of \(L\) is 
\[
\{ \bar{g}_1^{x_1} \cdots \bar{g}_n^{x_n} \left( \prod_{k; i < j} \bar{h}_k^{\alpha_{ij} x_i x_j} \right) \bar{u}^z \mid x_1, \ldots, x_n, z \in \mathbb{Z}_2\}.
\]
A consequence of this is \(R_x^2 \in \langle \bar{u} \rangle\), or equivalently \(x^2 \in \langle s \rangle\) for all 
\(x \in L\). This means that \(L/\langle s \rangle\) is an elementary Abelian 2-group and 
\(L\) is a small Frattini Moufang loop.

We are now able to prove our main result.
Theorem 5.1. Let $(V, \sigma, \kappa, \alpha)$ be a symplectic cubic space and let us construct the group with triality $(G, S)$ using the relations (4)-(8) and (9)-(10). Let $L$ be the Moufang loop associated to $(G, S)$. Then, $L$ is a small Frattini Moufang loops and the symplectic cubic space corresponding to $L$ is $V$.

Proof. It only remained to show that the structure constants of $L$ and $V$ are the same. Let us put $x_i = \sigma g_i \in L$ and define $A = \langle g_1, \ldots, g_n, u \rangle$ as before. The set $\{g_1, \ldots, g_n\}$ is independent in $A$, that is, no proper subset of it generates $A$. This implies that the set $\{x_1, \ldots, x_n\}$ is independent in $L$. With other words, $\{x_1(s), \ldots, x_n(s)\}$ is a basis for the vector space $L/\langle s \rangle$.

Then, by Proposition 4.1(ii),
$$R_{x_i} = \bar{g}_i (\bar{g}_i \bar{f}_i) = \bar{g}_i \quad \text{and} \quad L_{x_i} = \bar{f}_i \bar{u}^{\sigma_i}.$$ By Lemma 4.2,
$$R_{x_i}^2 = \bar{g}_i^2 = \bar{u}^{\sigma_i},$$
$$R_{[x_i,x_j]} = [\bar{g}_i, \bar{g}_j] = \bar{u}^{\kappa_{ij}},$$
$$R_{(x_i,x_j,x_k)} = [[\bar{g}_i, \bar{f}_j], \bar{g}_k] = \bar{u}^{\alpha_{ijk}}.$$ This means
$$x_i^2 = s^{\sigma_i}, \quad [x_i, x_j] = s^{\kappa_{ij}}, \quad \text{and} \quad (x_i, x_j, x_k) = s^{\alpha_{ijk}},$$
hence the symplectic cubic space of $L$ is indeed $V$. □

References

[1] R. H. Bruck, A survey of binary systems, Springer, Berlin, 1958.
[2] O. Chein and E. G. Goodaire, Moufang loops with a unique non-identity commutator (associator, square). J. Algebra 130 (1990), no. 2, 369–384.
[3] S. Doro, Simple Moufang loops. Math. Proc. Cambridge Philos. Soc. 83 (1978), 377–392.
[4] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4 (2004), [http://www.gap-system.org](http://www.gap-system.org).
[5] R. L. Griess, Jr., Code loops. J. Algebra 100 (1986), no. 1, 224–234.
[6] J. I. Hall and G. P. Nagy, On Moufang 3-nets and groups with triality. Acta Sci. Math. (Szeged) 67 (2001), no. 3-4, 675–685.
[7] T. Hsu, Explicit constructions of code loops as centrally twisted products. Math. Proc. Cambridge Philos. Soc. 128 (2000), no. 2, 223–232.
[8] M. Kitazume, Code loops and even codes over $F_4$. J. Algebra 118 (1988), no. 1, 140–149.
[9] G. P. Nagy and P. Vojtěchovský. LOOPS: package for GAP, version 0.99 (2004), [http://www.math.du.edu/loops](http://www.math.du.edu/loops).
[10] P. T. Nagy and K. Strambach, Loops as invariant sections in
groups, and their geometry. Canad. J. Math. 46 (1994), no. 5,
1027–1056

[11] P. Vojtěchovský, Combinatorial aspects of code loops. Loops’99
(Prague). Comment. Math. Univ. Carolin. 41 (2000), no. 2, 429–
435.

E-mail address: nagyg@math.u-szeged.hu

Bolyai Institute, University of Szeged, Aradi vərtanūk tere 1, H-
6720 Szeged, Hungary

URL: http://www.math.u-szeged.hu/~nagyg/