Two-dimensional hydrogen-like atom in a weak magnetic field

Radosław Szmytkowski*

Atomic and Optical Physics Division, Department of Atomic, Molecular and Optical Physics, Faculty of Applied Physics and Mathematics, Gdańsk University of Technology, ul. Gabriela Narutowicza 11/12, 80–233 Gdańsk, Poland

Abstract

We consider a non-relativistic two-dimensional (2D) hydrogen-like atom in a weak, static, uniform magnetic field perpendicular to the atomic plane. Within the framework of the Rayleigh–Schrödinger perturbation theory, using the Sturmian expansion of the generalized radial Coulomb Green function, we derive explicit analytical expressions for corrections to an arbitrary planar hydrogenic bound-state energy level, up to the fourth order in the strength of the perturbing magnetic field. In the case of the ground state, we correct an expression for the fourth-order correction to energy available in the literature.

Keywords: Two-dimensional (2D) atom; Zeeman effect; Perturbation theory; Coulomb Green function; Sturmian functions

PACS 2010: 03.65.Ge, 31.15.xp, 32.60.+i, 71.35.Ji

1 Introduction

Theoretical studies on elementary two-dimensional quantum structures in magnetic fields have been carried out for several decades [1–20]. Recent spectacular developments in single-layer materials science have given a fresh impact to such investigations [21–36]. In result of the work done so far, at the present moment we understand some aspects of magnetic-field-induced properties of 2D analogues of atoms and molecules, but our knowledge on the subject still appears to be far from being complete.

Recently, we have come across a need to know exact analytical representations for low-order perturbation-theory corrections to an arbitrary energy level of a two-dimensional analogue of a hydrogen-like atom placed in a weak and uniform magnetic field perpendicular to the atomic plane. The first-order correction may be obtained trivially for any atomic state. Exact values of the second-order corrections for states with the principal quantum numbers $1 \leq n \leq 4$ may be derived from a table provided in Ref. [4]. The third-order correction may be shown to vanish identically for any state (in fact, the same happens for all odd-order corrections other than the first-order one), while in Refs. [25,31] the fourth-order correction has been given, but for the ground level only. Approximate expressions for several higher even-order corrections to states with zero radial quantum numbers and with principal quantum numbers not exceeding six are contained in Ref. [35]. However, neither of the publications invoked above, nor any other related one we have had in hands in the course of browsing the literature, contains the general formulas we have been seeking for. This is a bit astonishing in view of the fact that for a similar problem of the planar one-electron atom placed in a weak, uniform, in-plane electric field, closed-form analytical expressions

*Email: radoslaw.szmytkowski@pg.edu.pl
for Stark–Lo Surdo corrections to energies of discrete parabolic eigenstates are known up to the sixth order in the perturbing field \[ B \]. Under the circumstances, we have derived expressions for the second- and fourth-order magnetic-field-induced corrections to an arbitrary energy level of the planar hydrogenic atom. The results of that study are presented in this work. We believe they may be of some interest, in particular because the result for the fourth-order correction to the ground state given in Ref. [25], and then repeated in Ref. [31], has been found to be incorrect.

2 Preliminaries

We consider a one-electron atom with a point-like and spinless nucleus at rest. The electron is constrained to move in a plane through the nucleus. A potential of interaction between the nucleus and the electron is being referred to the nucleus, the two-dimensional time-independent Schrödinger equation for the electron is

\[
\frac{\left[-i\hbar \nabla + eA(r)\right]^2}{2m} - \frac{Ze^2}{(4\pi\epsilon_0)r^2} \Psi(r) = E\Psi(r) \quad (r \in \mathbb{R}^2),
\]

where \( r = |r| \) and \( A(r) \) is a vector potential of the magnetic field. Equation (2.1) is to be solved, with the electron energy \( E \) chosen as an eigenvalue, subject to the constraint that the wave function \( \Psi(r) \) is single-valued and bounded for all \( r \in \mathbb{R}^2 \), including the point \( r = 0 \) and the point at infinity.

Throughout this paper, we shall be working in the symmetric gauge, in which the vector potential \( A(r) \) is

\[
A(r) = \frac{1}{2}B \times r.
\]

Then, the Schrödinger equation (2.1) may be rewritten as

\[
\left[-\frac{\hbar^2}{2m} \nabla^2 + \frac{eB}{2m} \cdot \Lambda + \frac{e^2B^2r^2}{8m} - \frac{Ze^2}{(4\pi\epsilon_0)r^2}\right] \Psi(r) = E\Psi(r),
\]

where

\[
\Lambda = -ir \times \nabla
\]

is a (dimensionless) orbital angular momentum operator for the electron. The form of the Hamiltonian operator in the Schrödinger equation (2.3) suggests one introduces the polar coordinates \( r \) and \( \varphi \), with \( 0 \leq r < \infty \) and \( 0 \leq \varphi < 2\pi \); Eq. (2.3) is then transformed into the following one:

\[
\left[-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2}\right) - \frac{iehB}{2m} \frac{\partial}{\partial \varphi} + \frac{e^2B^2r^2}{8m} - \frac{Ze^2}{(4\pi\epsilon_0)r}\right] \Psi(r, \varphi) = E\Psi(r, \varphi).
\]

The benefit from the use of the polar coordinates is that Eq. (2.5) is separable, in the sense that it possesses particular solutions of the form

\[
\Psi_{nlm}(r, \varphi) = \frac{1}{\sqrt{l!}}P_{nlm}(r)\Phi_{ml}(\varphi) \quad (l = |m|),
\]

where

\[
\Phi_{ml}(\varphi) = \frac{1}{\sqrt{2\pi^m}}e^{im\varphi} \quad (m_l \in \mathbb{Z}).
\]

Plugging Eq. (2.6) into Eq. (2.5) and exploiting Eq. (2.7) yields the radial Schrödinger equation

\[
\left[-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \left(l^2 - \frac{1}{4}\right) + \frac{ehB}{2m} + \frac{e^2B^2r^2}{8m} - \frac{Ze^2}{(4\pi\epsilon_0)r}\right] P_{nlm}(r) = E_{nlm}P_{nlm}(r),
\]

which is to be solved subject to the boundary conditions

\[
P_{nlm}(r)/\sqrt{r} \text{ bounded for } r \to 0 \text{ and for } r \to \infty.
\]
It is easy to deduce from the standard asymptotic analysis that for $B \neq 0$ the constraints displayed in Eq. (2.8b) may be replaced by the following ones:

$$P_{nlm}(r) \xrightarrow{r \to 0} 0, \quad P_{nlm}(r) \xrightarrow{r \to \infty} 0.$$  \hspace{1cm} (2.8c)

The symbol $n$ that has appeared the first time as a subscript in Eq. (2.6) is the principal quantum number defined as

$$n = n_r + l + 1,$$  \hspace{1cm} (2.9)

where $n_r \in \mathbb{N}_0$ is the radial quantum number which counts the number of nodes (zeroes) in the radial wave function.

Since the term linear in $B$ which appears in the differential operator in Eq. (2.8a) is independent of the variable $r$, it is clear that the energy eigenvalue $E_{nlm}$ may be written as

$$E_{nlm} = E_{nl} + E_{m}^{(1)},$$  \hspace{1cm} (2.10)

with

$$E_{m}^{(1)} = \frac{e \hbar B}{2m}.$$  \hspace{1cm} (2.11)

It is also evident that the radial function $P_{nlm}(r)$ does depend on $m_l$ through $l = |m|$: only:

$$P_{nlm}(r) \equiv P_{nl}(r).$$  \hspace{1cm} (2.12)

Consequently, the starting point for further considerations will be the radial eigenvalue problem

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2mr^2} \left( l^2 - \frac{1}{4} \right) - \frac{Ze^2}{(4\pi\varepsilon_0)r} + \frac{e^2 B^2 r^2}{8m} \right] P_{nl}(r) = E_{nl} P_{nl}(r),$$  \hspace{1cm} (2.13a)

$$P_{nl}(r) \xrightarrow{r \to 0} 0, \quad P_{nl}(r) \xrightarrow{r \to \infty} 0.$$  \hspace{1cm} (2.13b)

## 3 Perturbation-theory analysis

### 3.1 Basics and the zeroth-order problem

Closed-form analytical solutions to the eigenproblem (2.13) are not known. Therefore, below we shall attempt to find its approximate solutions, under the assumption that the magnetic field is weak, with the use of the Rayleigh–Schrödinger perturbation theory. To this end, we write the radial differential operator from Eq. (2.13a) as

$$H_l(r) = H_l^{(0)}(r) + H_l^{(2)}(r),$$  \hspace{1cm} (3.1)

where

$$H_l^{(0)}(r) = -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2mr^2} \left( l^2 - \frac{1}{4} \right) - \frac{Ze^2}{(4\pi\varepsilon_0)r}$$  \hspace{1cm} (3.2)

and

$$H_l^{(2)}(r) = \frac{e^2 B^2 r^2}{8m}.$$  \hspace{1cm} (3.3)

We shall treat the diamagnetic term (3.3) as a small perturbation of the radial Coulomb Hamiltonian (3.2). Since $H_l^{(2)}(r)$ is of the second order in the perturbing magnetic field, we seek solutions to the eigensystem (2.13) in the form of the perturbation series

$$E_{nl} = E_{nl}^{(0)} + E_{nl}^{(2)} + E_{nl}^{(4)} + \cdots$$  \hspace{1cm} (3.4)

and

$$P_{nl}(r) = P_{nl}^{(0)}(r) + P_{nl}^{(2)}(r) + P_{nl}^{(4)}(r) + \cdots,$$  \hspace{1cm} (3.5)
involving even-order terms only. Here $E_{nl}^{(0)}$ and $P_{nl}^{(0)}(r)$ are those solutions to the zeroth-order eigenproblem (being the radial Coulomb one)

$$[H_i^{(0)}(r) - E^{(0)}(r)]P^{(0)}(r) = 0,$$  \hfill (3.6a)

$$P^{(0)}(r) \xrightarrow{r \to 0} 0, \quad P^{(0)}(r) \text{ bounded for } r \to \infty$$  \hfill (3.6b)

(subscripts have been omitted intentionally), which correspond to the discrete part of its spectrum, consisting of the eigenvalues

$$E_{nl}^{(0)} = E_n^{(0)} = -\frac{Z^2}{2N_n^2} \left(\frac{e^2}{4\pi\epsilon_0}a_0\right),$$  \hfill (3.7)

with

$$N_n = n - \frac{1}{2} = n_r + l + \frac{1}{2},$$  \hfill (3.8)

and with

$$a_0 = \left(\frac{4\pi\epsilon_0}{\hbar^2 m}\right)^{1/2}$$  \hfill (3.9)

being the Bohr radius. Eigenfunctions associated with the eigenvalues (3.7), orthonormal in the sense of

$$\int_0^\infty \text{d}r P_{nl}^{(0)}(r) P_{n'l}^{(0)}(r) = \delta_{nn'},$$  \hfill (3.10)

are

$$P_{nl}^{(0)}(r) = \sqrt{\frac{Z(n - l - 1)!}{a_0 N_n^2(n + l - 1)!}} \left(\frac{2Zr}{N_n a_0}\right)^{l+1/2} e^{-Zr/N_n a_0} L_n^{(2l)}\left(\frac{2Zr}{N_n a_0}\right),$$  \hfill (3.11)

where $L_n^{(\alpha)}(x)$ is the generalized Laguerre polynomial \[39, Sec. 5.5\]. For integration purposes, it is frequently convenient to have these functions rewritten as

$$P_{nl}^{(0)}(r) = \sqrt{\frac{Zn_r!}{a_0 N_n^2(n_r + 2l)!}} \left(\frac{2Zr}{N_n a_0}\right)^{l+1/2} e^{-Zr/N_n a_0} L_n^{(2l)}\left(\frac{2Zr}{N_n a_0}\right).$$  \hfill (3.12)

### 3.2 The second-order corrections to Coulomb energies

For the present problem, the second-order correction to energy, $E_{nl}^{(2)}$, is given by

$$E_{nl}^{(2)} = \int_0^\infty \text{d}r P_{nl}^{(0)}(r) H^{(2)}(r) P_{nl}^{(0)}(r),$$  \hfill (3.13)

or equivalently, if use is made of Eq. (3.3), by

$$E_{nl}^{(2)} = \frac{e^2B^2}{8m} \int_0^\infty \text{d}r r^2 [P_{nl}^{(0)}(r)]^2.$$  \hfill (3.14)

Plugging Eq. (3.12) into the integrand and exploiting the integration formula

$$\int_0^\infty \text{d}x x^{\alpha+3} e^{-x} [L_n^{(\alpha)}(x)]^2 = (2k + \alpha + 1)(10k^2 + 10k + 10\alpha k + \alpha^2 + 5\alpha + 6) \frac{\Gamma(k + \alpha + 1)}{k!}$$  \hfill (3.15)

(Re $\alpha > -4$),

which may be deduced from the general expression \[10\] Eqs. (E54), (E56) and (E60)]

$$\int_0^\infty \text{d}x x^\gamma e^{-x} [L_n^{(\alpha)}(x)] L_m^{(\beta)}(x) = (-)^{k+k'} \sum_{m=0}^{\text{min}(k,k')} \frac{\Gamma(m+\gamma+1)}{m!} \frac{\Gamma(m+\gamma+1)}{k-m} \frac{\Gamma(m+\gamma+1)}{k'-m}$$  \hfill (3.16)

(Re $\gamma > -1$),
where
\[ E_{nl}^{(2)} = \frac{1}{2z} (n - \frac{1}{2})^2 (5n^2 - 5n - 3l^2 + 3) Z^{-2} \frac{B^2}{B_0^2} \frac{e^2}{(4\pi\epsilon_0)\alpha_0}. \] (3.17)

where
\[ B_0 = \frac{h}{ca_0} = \frac{m^2e^3}{(4\pi\epsilon_0)^2\hbar^2}, \] (3.18)
is the atomic unit of magnetic induction. For states with \( l = n - 1 \) (i.e., those with \( n_r = 0 \)), the expression in Eq. (3.17) simplifies to
\[ E_{n,n-1}^{(2)} = \frac{1}{2z} n \left(n + \frac{1}{2}\right) \left(n - \frac{1}{2}\right)^2 Z^{-2} \frac{B^2}{B_0^2} \frac{e^2}{(4\pi\epsilon_0)\alpha_0}. \] (3.19)

### 3.3 The fourth-order corrections to Coulomb energies

Proceeding along the standard route, one finds that for the present problem the fourth-order correction to energy, \( E_{nl}^{(4)} \), is given by
\[ E_{nl}^{(4)} = \int_0^\infty dr P_{nl}^{(0)}(r) H^{(2)}(r) P_{nl}^{(2)}(r), \] (3.20)

where the second-order correction to the radial wave function, \( P_{nl}^{(2)}(r) \), is a solution to the inhomogeneous boundary-value problem
\[ [H_0^{(0)}(r) - E_{nl}^{(0)}(r)] P_{nl}^{(2)}(r) = - [H^{(2)}(r) - E_{nl}^{(2)}(r)] P_{nl}^{(0)}(r), \] (3.21a)

\[ P_{nl}^{(2)}(r) \xrightarrow{r \to 0} 0, \quad P_{nl}^{(2)}(r) \xrightarrow{r \to \infty} 0, \] (3.21b)

subject to the further orthogonality restraint
\[ \int_0^\infty dr P_{nl}^{(0)}(r) P_{nl}^{(2)}(r) = 0. \] (3.22)

The formal solution to the problem \( \text{(3.21)} - \text{(3.22)} \) is
\[ P_{nl}^{(2)}(r) = - \int_0^\infty dr' \tilde{G}_{nl}^{(0)}(r,r') [H^{(2)}(r') - E_{nl}^{(2)}(r')] P_{nl}^{(0)}(r'), \] (3.23)

where \( \tilde{G}_{nl}^{(0)}(r,r') \) is a generalized (or reduced) radial Coulomb Green function associated with the Coulomb energy level \( E_{nl}^{(0)}. \) The latter function is defined as that particular solution to the inhomogeneous boundary-value problem
\[ [H_0^{(0)}(r) - E_{nl}^{(0)}(r)] \tilde{G}_{nl}^{(0)}(r,r') = \delta(r-r') - P_{nl}^{(0)}(r') P_{nl}^{(0)}(r'), \] (3.24a)

\[ \tilde{G}_{nl}^{(0)}(r,r') \xrightarrow{r \to 0} 0, \quad \tilde{G}_{nl}^{(0)}(r,r') \xrightarrow{r \to \infty} 0, \] (3.24b)

where \( \delta(r-r') \) is the Dirac delta function, which obeys the additional orthogonality constraint
\[ \int_0^\infty dr P_{nl}^{(0)}(r) \tilde{G}_{nl}^{(0)}(r,r') = 0. \] (3.25)

Since the zeroth-order eigenproblem \( \text{(3.0)} \) is self-adjoint, the function \( \tilde{G}_{nl}^{(0)}(r,r') \) is symmetric in its arguments:
\[ \tilde{G}_{nl}^{(0)}(r,r') = \tilde{G}_{nl}^{(0)}(r',r). \] (3.26)

When this is combined with Eq. \( \text{(3.25)} \), one deduces the formula
\[ \int_0^\infty dr' \tilde{G}_{nl}^{(0)}(r,r') P_{nl}^{(0)}(r') = 0, \] (3.27)
which allows us to simplify Eq. (3.23) to obtain
\[
P_{nl}^{(2)}(r) = - \int_0^\infty dr' \tilde{G}_{nl}^{(0)}(r, r') H^{(2)}(r') P_{nl}^{(0)}(r'). \tag{3.28}
\]
Plugging Eq. (3.28) into the right-hand side of Eq. (3.20) gives the energy correction $E_{nl}^{(4)}$ in the form
\[
E_{nl}^{(4)} = - \int_0^\infty dr \int_0^\infty dr' P_{nl}^{(0)}(r) H^{(2)}(r) \tilde{G}_{nl}^{(0)}(r, r') H^{(2)}(r') P_{nl}^{(0)}(r'). \tag{3.29}
\]
or, still more explicitly, in the form
\[
E_{nl}^{(4)} = - \left( \frac{e^2 B_2}{8m} \right)^2 \int_0^\infty dr \int_0^\infty dr' P_{nl}^{(0)}(r) r^2 \tilde{G}_{nl}^{(0)}(r, r') r^2 P_{nl}^{(0)}(r'). \tag{3.30}
\]
A representation of the generalized radial Coulomb Green function $\tilde{G}_{nl}^{(0)}(r, r')$ which is perhaps the most suitable for the use in Eq. (3.30) is the one in the form of a series expansion in the discrete radial Coulomb Sturmian basis. We shall construct it below.

The discrete radial Coulomb Sturmian functions are defined as solutions to the spectral problem
\[
\left[ \frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2(l^2 - \frac{1}{4})}{2mr^2} - \mu_{n, l}^{(0)}(E) \right] S_{n, l}^{(0)}(E, r) = 0 \quad (E < 0), \tag{3.31a}
\]
\[
S_{n, l}^{(0)}(E, r) \underset{r \to 0}{\to} 0, \quad S_{n, l}^{(0)}(E, r) \underset{r \to \infty}{\to} 0, \tag{3.31b}
\]
with $E < 0$ fixed and with the parameter $\mu_{n, l}^{(0)}(E)$ chosen as an eigenvalue. The spectrum of this problem is purely discrete, and eigenvalues are given by
\[
\mu_{n, l}^{(0)}(E) = (n_r + l + \frac{1}{2}) \frac{k a_0}{Z} \quad (n_r \in \mathbb{N}_0), \tag{3.32}
\]
where
\[
k = \sqrt{-\frac{2mE}{\hbar^2}}. \tag{3.33}
\]
Eigenfunctions, orthonormal in the sense of
\[
\int_0^\infty dr \frac{Ze^2}{(4\pi\epsilon_0) r} S_{n, l}^{(0)}(E, r) S_{n', l}^{(0)}(E, r) = \delta_{n, n'}, \tag{3.34}
\]
are
\[
S_{n, l}^{(0)}(E, r) = \sqrt{\frac{(4\pi\epsilon_0) a_0}{Ze^2(n_r + 2l)!}} (2kr)^{l+1/2} e^{-kr} L_{n_r}^{(2l)}(2kr). \tag{3.35}
\]
In contrary to the discrete Coulomb eigenfunctions (3.11), the Sturmians (3.35) form a complete set, the corresponding closure relation being
\[
\frac{Ze^2}{(4\pi\epsilon_0) r} \sum_{n_r=0}^\infty S_{n_r, l}^{(0)}(E, r) S_{n_r, l}^{(0)}(E, r') = \delta(r - r'). \tag{3.36}
\]
If the parameter $E$ coincides with the Coulomb energy eigenvalue $E_n^{(0)}$ displayed in Eq. (3.7) [we assume $n$ is related to $n_r$ and $l$ used here as in Eq. (2.4)], it is easy to see from Eqs. (3.32), (3.33), (3.35) and (3.36) that one has
\[
\mu_{n, l}^{(0)}(E_n^{(0)}) = 1 \quad (n = n_r + l + 1). \tag{3.37}
\]
Similarly, from Eqs. (3.35), (3.31), (3.1) and (3.12) one infers the relationship
\[
S_{n_r, l}^{(0)}(E_n^{(0)}, r) = \frac{N_n}{Z} \sqrt{\frac{(4\pi\epsilon_0)a_0}{e^2}} P_{nl}^{(0)}(r) \quad (n = n_r + l + 1). \tag{3.38}
\]
The radial Coulomb Green function, \( G_{l}^{(0)}(E, r, r') \), is defined to be a solution to the inhomogeneous equation

\[
[H_{l}^{(0)}(r) - E]G_{l}^{(0)}(E, r, r') = \delta(r - r') \quad (E < 0),
\]

subject to the boundary constraints

\[
G_{l}^{(0)}(E, r, r') \xrightarrow{r \to 0} 0, \quad G_{l}^{(0)}(E, r, r') \xrightarrow{r \to \infty} 0.
\]

Since the Sturmian functions (3.35) form a complete set, the Green function \( G_{l}^{(0)}(E, r, r') \) may be sought in the form of the series

\[
G_{l}^{(0)}(E, r, r') = \sum_{n=0}^{\infty} C_{n,l}^{(0)}(E, r')S_{n,l}^{(0)}(E, r).
\]

To determine the expansion coefficients \( C_{n,l}^{(0)}(E, r) \), we plug Eq. (3.40) into Eq. (3.39), multiply both sides of the resulting identity with \( S_{n,l}^{(0)}(E, r) \), then integrate with respect to \( r \) over the interval \([0, \infty)\), and apply the orthogonality relation (3.35). Upon the replacement of \( n' \) with \( n_{r} \), this yields

\[
C_{n,r}^{(0)}(E, r') = \frac{1}{\mu_{n,l}^{(0)}(E) - 1} S_{n,r}^{(0)}(E, r'),
\]

hence, we obtain the following symmetric Sturmian expansion of \( G_{l}^{(0)}(E, r, r') \):

\[
G_{l}^{(0)}(E, r, r') = \sum_{n=0}^{\infty} S_{n,r}^{(0)}(E, r)S_{n,r}^{(0)}(E, r') \frac{1}{\mu_{n,l}^{(0)}(E) - 1}.
\]

It follows from Eqs. (3.24), (3.25) and (3.39) that the generalized radial Coulomb Green function \( \tilde{G}_{nl}^{(0)}(r, r') \) may be obtained from the radial Coulomb Green function \( G_{l}^{(0)}(E, r, r') \) through the limit procedure

\[
\tilde{G}_{nl}^{(0)}(r, r') = \lim_{E \to E_{n}^{(0)}} \left[ G_{l}^{(0)}(E, r, r') - \frac{P_{nl}^{(0)}(r)P_{nl}^{(0)}(r')}{E_{n}^{(0)} - E} \right].
\]

By virtue of the de l’Hospital rule, the latter equation is equivalent to the following one:

\[
\tilde{G}_{nl}^{(0)}(r, r') = \lim_{E \to E_{n}^{(0)}} \frac{\partial}{\partial E} \left[ (E - E_{n}^{(0)})G_{l}^{(0)}(E, r, r') \right],
\]

which is particularly suitable for the construction of the Sturmian expansion of \( \tilde{G}_{nl}^{(0)}(r, r') \). Inserting the series representation (3.42) into the right-hand side of Eq. (3.44) and then making use of the relationships

\[
\frac{\partial S_{n,r}^{(0)}(E, r)}{\partial E} = \frac{r}{2E} \frac{d S_{n,r}^{(0)}(E, r)}{dr},
\]

\[
\frac{E - E_{n}^{(0)}}{\mu_{n,l}^{(0)}(E) - 1} = E_{n}^{(0)} \left[ \frac{\mu_{n,l}^{(0)}(E)}{\mu_{n,l}^{(0)}(E) + 1} \right],
\]

\[
\lim_{E \to E_{n}^{(0)}} \frac{E - E_{n}^{(0)}}{\mu_{n,l}^{(0)}(E) - 1} = 2E_{n}^{(0)} \quad (n = n_{r} + l + 1),
\]

\[
\lim_{E \to E_{n}^{(0)}} \frac{\partial}{\partial E} \frac{E - E_{n}^{(0)}}{\mu_{n,l}^{(0)}(E) - 1} = -\frac{1}{2} \quad (n = n_{r} + l + 1),
\]

\[
\mu_{n,l}^{(0)}(E_{n}^{(0)}) = \frac{n_{r}^{l} + l + \frac{1}{2}}{N_{n}^{(0)}},
\]

7
which may be easily derived from the defining Eqs. (3.32) and (3.35), one eventually arrives at the sought Sturmian expansion of the generalized radial Coulomb Green function, which is

\[
\tilde{G}_{nl}^{(0)}(r, r') = N_n \sum_{n'_l \neq n_l} \frac{S_{nl}^{(0)}(E_n^{(0)}, r) S_{nl'}^{(0)}(E_{n'}^{(0)}, r')} {n'_l - n_l} + \frac{1}{2} S_{nl}^{(0)}(E_n^{(0)}, r) S_{nl}^{(0)}(E_n^{(0)}, r')
\]

\[
+ r \frac{dS_{nl}^{(0)}(E_n^{(0)}, r)} {dr} S_{nl}^{(0)}(E_n^{(0)}, r') + S_{nl}^{(0)}(E_n^{(0)}, r) r \frac{dS_{nl}^{(0)}(E_n^{(0)}, r')} {dr'}
\]

\[\text{(n}_r = n - l - 1). \quad (3.50)\]

Once the Sturmian expansion of \(\tilde{G}_{nl}^{(0)}(r, r')\) has been found, we are ready to complete the task to find the fourth-order energy correction \(E_{nl}^{(4)}\). To this end, we insert Eq. (3.50) into Eq. (3.30) and use the relationship in Eq. (3.33), together with integrations by parts, to eliminate derivatives of Sturmian functions. This gives \(E_{nl}^{(4)}\) in the form

\[
E_{nl}^{(4)} = - \left( \frac{e^2 B^2}{8 m} \right)^2 \left\{ N_n \sum_{n'_l \neq n_l} \frac{\left[ \int_0^\infty dr \ r^2 p_{nl}^{(0)}(r) S_{nl}^{(0)}(E_n^{(0)}, r) \right]^2} {n'_l - n_l}
\]

\[\quad - \frac{5}{2} \left[ \int_0^\infty dr \ r^2 p_{nl}^{(0)}(r) S_{nl}^{(0)}(E_n^{(0)}, r) \right]^2 \} \quad (n_r = n - l - 1). \quad (3.51)\]

The integrals in Eq. (3.51) may be taken after one exploits Eqs. (3.12) and (3.35), with the use of the integration formula

\[
\int_0^\infty dx \ x^{\alpha+3} e^{-x} L_k^{(\alpha)}(x) L_{k'}^{(\alpha)}(x) = - \frac{\Gamma(k + \alpha + 1)}{(k - 3)!} \delta_{k', k - 3} + 3(2k + \alpha - 1) \frac{\Gamma(k + \alpha + 1)}{(k - 2)!} \delta_{k', k - 2}
\]

\[- 3(5k^2 + 5\alpha k + \alpha^2 + 1) \frac{\Gamma(k + \alpha + 1)}{(k - 1)!} \delta_{k', k - 1}
\]

\[+ (2k + \alpha + 1)(10k^2 + 10k + 10\alpha k + \alpha^2 + 5\alpha + 6) \frac{\Gamma(k + \alpha + 1)}{k!} \delta_{k', k}
\]

\[- 3(5k^2 + 10k + 5\alpha k + \alpha^2 + 5\alpha + 6) \frac{\Gamma(k + \alpha + 2)}{k!} \delta_{k', k + 1}
\]

\[+ 3(2k + \alpha + 3) \frac{\Gamma(k + \alpha + 3)}{k!} \delta_{k', k + 2} - \frac{\Gamma(k + \alpha + 4)}{k!} \delta_{k', k + 3} \quad (\Re \alpha > -4), \quad (3.52)\]

which generalizes the one in Eq. (3.15) and, similarly to the latter, may be derived from the general expression (3.10). Since only terms with \(n'_l\) constrained by \(1 \leq |n'_l - n_l| \leq 3\) are seen to contribute non-vanishingly to the sum in Eq. (3.51), we eventually obtain

\[
E_{nl}^{(4)} = - \frac{1}{2^{10}} \left( n - \frac{1}{2} \right)^6 \times \left( 143n^4 - 286n^3 - 90n^2 + 582n^2 + 90nl^2 - 439n - 21l^2 - 138l^2 + 159 \right)
\]

\[\times Z^{-6} B^4 \ e^2 \ \frac{B_0}{(4\pi e_0)a_0}. \quad (3.53)\]

For states with \(l = n - 1\) (i.e., those with \(n_r = 0\), Eq. (3.53) becomes

\[
E_{n,n-1}^{(4)} = - \frac{1}{2^{10}} \left( n + \frac{1}{2} \right)^6 \left( n - \frac{1}{2} \right)^6 \left( 16n^2 + 26n + 11 \right) Z^{-6} B^4 \ e^2 \ \frac{B_0}{(4\pi e_0)a_0}. \quad (3.54)\]
For the ground state \((n = 1)\), Eq. (3.54) yields
\[
E^{(4)}_{10} = -\frac{159}{65536} Z^{-4} B_0^4 \frac{e^2}{(4\pi\epsilon_0)a_0}.
\] (3.55)
This differs from the result announced in Refs. [25, Eq. (32)] and [31, Eq. (6.59)], which is
\[
E^{(4)}_{10} = -\frac{153}{65536} Z^{-4} B_0^4 \frac{e^2}{(4\pi\epsilon_0)a_0}.
\] (3.56)
The latter one is thus found to be incorrect.

4 Summary and concluding remarks

On the preceding pages, we have shown that energy levels of the planar hydrogen-like atom placed in a weak, static, uniform magnetic field of induction \(B\) perpendicular to the atomic plane may be expressed in the form
\[
E_{n\ell m} = E^{(0)}_n + E^{(1)}_{\ell m} + E^{(2)}_{nl} + E^{(4)}_{nl} + O \left( Z^{-10} (B/B_0)^6 \right),
\] (4.1)
where
\[
E^{(k)}_{n\ell m} = \varepsilon^{(k)} Z^{-2k+2} B_0^k \frac{e^2}{(4\pi\epsilon_0)a_0}.
\] (4.2)
In Eq. (4.2), \(Z\) is an electric charge of the atomic nucleus in units of the elementary charge \(e\), \(a_0\) is the Bohr radius,
\[
B_0 = \frac{m^2 e^3}{(4\pi\epsilon_0)^2 h^3} \approx 2.35 \times 10^5 \text{ T}
\] (4.3)
is the atomic unit of magnetic induction, while the dimensionless and \(Z\)-independent coefficients \(\varepsilon^{(k)}\) are given by
\[
\varepsilon^{(0)}_n = -\frac{1}{2 (n - \frac{1}{2})^2},
\] (4.4)
\[
\varepsilon^{(1)}_{\ell m} = \frac{1}{2} m,\n\] (4.5)
\[
\varepsilon^{(2)}_{nl} = \frac{1}{2^4} \left(n - \frac{1}{2}\right)^2 \left(5n^2 - 5n - 3\ell^2 + 3\right),
\] (4.6)
and
\[
\varepsilon^{(4)}_{nl} = -\frac{1}{2^{10}} \left(n - \frac{1}{2}\right)^6 \left(143n^4 - 286n^3 - 90n^2\ell^2 + 582n^2 + 90n\ell^2 - 439n - 21\ell^4 - 138\ell^2 + 159\right),
\] (4.7)
with \(n \in \mathbb{N}_+, m \in \mathbb{Z}\) and \(0 \leq \ell = |m| \leq n - 1\). Numerical values of the coefficients \(\varepsilon^{(2)}_{nl}\) and \(\varepsilon^{(4)}_{nl}\) for states with \(1 \leq n \leq 4\) are displayed in Table I.

It has to be emphasized that the formula in Eq. (4.1) is valid only if the electron spin is ignored. If this cannot be done, the Schrödinger equation (2.1) should be replaced with the planar Pauli equation
\[
\left\{ \sigma \cdot [-i\hbar \nabla + eA(r)] \right\}_{2m} \Psi(r) = Ze^2 / (4\pi\epsilon_0) r \Psi(r) = E\Psi(r) \quad (r \in \mathbb{R}^2),
\] (4.8)
where \(\sigma = (\sigma_x, \sigma_y)\) is the two-dimensional Pauli matrix vector, and \(\Psi(r)\) is a two-component Pauli spinor. Equation (4.8) may be cast into the form
\[
\left\{ \left[-i\hbar \nabla + eA(r) \right]^{2m} + \frac{eB}{m} \Sigma z = \frac{Ze^2}{(4\pi\epsilon_0) r} \right\} \Psi(r) = E\Psi(r),
\] (4.9)
Table I: Numerical values of the coefficients $\varepsilon_{nl}^{(2)}$ and $\varepsilon_{nl}^{(4)}$, defined in Eqs. (4.6) and (4.7), for $1 \leq n \leq 4$ and $0 \leq l \leq n - 1$.

| $n$ | $l$ | $\varepsilon_{nl}^{(2)}$ | $\varepsilon_{nl}^{(4)}$ |
|-----|-----|--------------------------|--------------------------|
| 1   | 0   | 3                        | 3 \times 53              |
|     | 64  | 159                      | 3 \times 53              |
| 2   | 0   | $3^2 \times 13$          | $3^6 \times 1609$        |
|     | 64  | $1172961$                | $3^6 \times 1609$        |
| 1   | 32  | $3^2 \times 5$          | $3^6 \times 5 \times 127$|
|     | 825 | $124078125$              | $3 \times 5 \times 2647$ |
| 3   | 0   | $3^2 \times 11$         | $3 \times 5 \times 17 \times 71$|
|     | 64  | $56578125$               | $3 \times 5 \times 7 \times 233$|
| 1   | 735 | $3 \times 5 \times 7^2$ | $3^2 \times 7^8 \times 59$|
|     | 16  | $728835555$              | $3 \times 5 \times 7^7 \times 59$|
| 2   | 2499| $3 \times 7^2 \times 17$| $3 \times 7 \times 991$   |
|     | 64  | $2448393339$             | $3 \times 7 \times 991$   |
| 3   | 441 | $3^2 \times 7^2$        | $3^2 \times 7 \times 53$  |
|     | 16  | $392830111$              | $3 \times 7 \times 53$    |

with

$$\Sigma_z = \frac{1}{2} \sigma_z, \quad (4.10)$$

where $\sigma_z$ is the third Pauli matrix. It is then evident that Eq. (4.9), supplemented with the regularity constraints on $\Psi(r)$ analogous to those introduced under Eq. (2.1), possesses separated eigenfunctions of the form

$$\Psi_{nlm_s}(r, \varphi) = \frac{1}{\sqrt{r}} P_{nl}(r) e^{im_s \varphi} \sqrt{2\pi} \chi_{m_s}, \quad (4.11)$$

where $P_{nl}(r)$ is the same radial function which has appeared in the preceding sections, while $\chi_{m_s}$ is the spin one-half eigenfunction obeying

$$\Sigma_z \chi_{m_s} = m_s \chi_{m_s}, \quad (m_s = \pm \frac{1}{2}), \quad (4.12)$$

and that the energy spectrum is of the form

$$E_{nlm_s} = E_n^{(0)} + E_{m_s}^{(1)} + E_{nl}^{(2)} + E_{nl}^{(4)} + O(Z^{-10}(B/B_0)^6), \quad (4.13)$$

with the terms $E_n^{(0)}$, $E_{nl}^{(2)}$ and $E_{nl}^{(4)}$ being identical to those derived before, and with

$$E_{nlm_s}^{(1)} = \frac{1}{2}(m_1 + 2m_s) \frac{B}{B_0} \frac{e^2}{4\pi\epsilon_0 a_0} \left(= (m_1 + 2m_s) \frac{e^2 B}{2m} \right), \quad (4.14)$$
References

[1] O. Akimoto, H. Hasegawa, Interband optical transitions in extremely anisotropic semiconductors. II. Coexistence of exciton and the Landau levels, J. Phys. Soc. Jpn. 22 (1967) 181

[2] L. P. Gor’kov, I. E. Dzyaloshinskii, Contribution to the theory of the Mott exciton in a strong magnetic field, Sov. Phys. JETP 26 (1968) 449

[3] M. Shinada, K. Tanaka, Interband optical transitions in extremely anisotropic semiconductors. III. Numerical studies of magneto-optical absorption, J. Phys. Soc. Jpn. 29 (1970) 1258

[4] A. H. MacDonald, D. S. Ritchie, Hydrogenic energy levels in two dimensions at arbitrary magnetic fields, Phys. Rev. B 33 (1986) 8336

[5] B. G. Adams, Application of 2-point Padé approximants to the ground state of the 2-dimensional hydrogen atom in an external magnetic field, Theor. Chim. Acta 73 (1988) 459

[6] W. Edelstein, H. N. Spector, R. Marasas, Two-dimensional excitons in magnetic fields, Phys. Rev. B 39 (1989) 7697

[7] J.-L. Zhu, Y. Cheng, J.-J. Xiong, Exact solutions for two-dimensional hydrogenic donor states in a magnetic field, Phys. Lett. A 145 (1990) 358

[8] J.-L. Zhu, Y. Cheng, J.-J. Xiong, Quantum levels and Zeeman splitting for two-dimensional hydrogenic donor states in a magnetic field, Phys. Rev. B 41 (1990) 10792

[9] P. Martin, J. J. Rodriguez-Nuñez, J. L. Marquez, Two-dimensional hydrogen-like atoms in the presence of a magnetic field: Quasifractional approximations, Phys. Rev. B 45 (1992) 8359

[10] V.-H. Le, T.-G. Nguyen, The algebraic method for two-dimensional quantum atomic systems, J. Phys. A 26 (1993) 1409

[11] H. Lehmann, N. H. March, The hydrogen atom in intense magnetic fields: Excitons in two and three dimensions, Pure Appl. Chem. 67 (1995) 457

[12] M. Taut, Two-dimensional hydrogen in a magnetic field: analytical solutions, J. Phys. A 28 (1995) 2081

[13] V. M. Villalba, R. Pino, Analytic computation of the energy levels of a two-dimensional hydrogenic donor states in a magnetic field, Phys. Scr. 58 (1998) 605

[14] V. M. Villalba, R. Pino, Analytic solution of a relativistic two-dimensional hydrogen-like atom in a constant magnetic field, Phys. Lett. A 238 (1998) 49

[15] C.-L. Ho, V. R. Khalilov, Planar Dirac electron in Coulomb and magnetic fields, Phys. Rev. A 61 (2000) 032104

[16] V. M. Villalba, R. Pino, Energy spectrum of a relativistic two-dimensional hydrogen-like atom in a constant magnetic field of arbitrary strength, Physica E 10 (2001) 561

[17] M. Robnik, V. G. Romanovski, Two-dimensional hydrogen atom in a strong magnetic field, J. Phys. A 36 (2003) 7923

[18] V. M. Villalba, R. Pino, Energy spectrum of the ground state of a two-dimensional relativistic hydrogen atom in the presence of a constant magnetic field, Mod. Phys. Lett. B 17 (2003) 1331

[19] A. Soylu, O. Bayrak, I. Boztosun, The energy eigenvalues of the two dimensional hydrogen atom in a magnetic field, Int. J. Mod. Phys. E 15 (2006) 1263
[20] A. Soylu, I. Boztosun, Accurate iterative solution of the energy eigenvalues of a two-dimensional hydrogenic donor in a magnetic field of arbitrary strength, Physica B 396 (2007) 150

[21] A. Rutkowski, A. Poszwa, Relativistic corrections for a two-dimensional hydrogen-like atom in the presence of a constant magnetic field, Phys. Scr. 79 (2009) 065010

[22] A. Poszwa, A. Rutkowski, Relativistic Paschen–Back effect for the two-dimensional H-like atoms, Acta Phys. Pol. A 117 (2010) 439

[23] M. Gadella, J. Negro, L. M. Nieto, G. P. Pronko, Two charged particles in the plane under a constant perpendicular magnetic field, Int. J. Theor. Phys. 50 (2011) 2019

[24] A. Poszwa, Relativistic two-dimensional H-like model atoms in an external magnetic field, Phys. Scr. 84 (2011) 055002

[25] N.-T. Hoang-Do, V.-H. Hoang, V.-H. Le, Analytical solutions of the Schrödinger equation for a two-dimensional exciton in magnetic field of arbitrary strength, J. Math. Phys. 54 (2013) 052105

[26] N.-T. Hoang-Do, D.-L. Pham, V.-H. Le, Exact numerical solutions of the Schrödinger equation for a two-dimensional exciton in a constant magnetic field of arbitrary strength, Physica B 423 (2013) 31

[27] A. Poszwa, Dirac electron in the two-dimensional Debye–Yukawa potential, Phys. Scr. 89 (2014) 065401

[28] M. A. Escobar, A. V. Turbiner, Two charges on plane in a magnetic field I. Quasi-equal charges and neutral quantum system at rest cases, Ann. Phys. 340 (2014) 37

[29] M. A. Escobar, A. V. Turbiner, Two charges on a plane in a magnetic field: II. Moving neutral quantum system across a magnetic field, Ann. Phys. 359 (2015) 405

[30] M. A. Escobar-Ruiz, Two charges on plane in a magnetic field: III. He$^+$ ion, Ann. Phys. 351 (2014) 714

[31] I. Feranchuk, A. Ivanov, V.-H. Le, A. Ulyanenkov, Non-perturbative description of quantum systems, Lecture Notes in Physics 894, Springer, Cham, 2015, chapter 6

[32] C. Flavio, C. Enrique, M. Pablo, C.-V. Luis, Analytic approximations to the energy eigenvalues of the quadratic Zeeman effect in two dimensions for hydrogen-like atoms, J. Phys.: Conf. Ser. 574 (2015) 012105

[33] L. Liu, Q. Hao, Planar hydrogen-like atom in inhomogeneous magnetic fields: exactly or quasi-exactly solvable models, Theor. Math. Phys. 183 (2015) 730

[34] J. S. Ardenghi, M. Gadella, J. Negro, Approximate solutions to the quantum problem of two opposite charges in a constant magnetic field, Phys. Lett. A 380 (2016) 1817

[35] N.-T. D. Hoang, D.-A. P. Nguyen, V.-H. Hoang, V.-H. Le, Highly accurate analytical energy of a two-dimensional exciton in a constant magnetic field, Physica B 495 (2016) 16

[36] D.-N. Le, N.-T. D. Hoang, V.-H. Le, Exact analytical solutions of a two-dimensional hydrogen atom in a constant magnetic field, J. Math. Phys. 58 (2017) 042102

[37] F. M. Fernández, Perturbation theory with canonical transformations, Phys. Rev. A 45 (1992) 1333

[38] B. G. Adams, Unified treatment of high-order perturbation theory for the Stark effect in a two- and three-dimensional hydrogen atom, Phys. Rev. A 46 (1992) 4060
[39] W. Magnus, F. Oberhettinger, R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, 3rd ed., Springer, Berlin, 1966

[40] R. Szmytkowski, The Dirac–Coulomb Sturmians and the series expansion of the Dirac–Coulomb Green function: application to the relativistic polarizability of the hydrogen-like atom, J. Phys. B 30 (1997) 825, erratum: J. Phys. B 30 (1997) 2747