REAL EMBEDDING AND EQUIVARIANT ETA FORMS

BO LIU

ABSTRACT. In [BZ93], Bismut and Zhang establish a mod $\mathbb{Z}$ embedding formula of Atiyah-Patodi-Singer reduced eta invariants. In this paper, we explain the hidden mod $\mathbb{Z}$ term as a spectral flow and extend this embedding formula to the equivariant family case. In this case, the spectral flow is generalized to the equivariant chern character of some equivariant Dai-Zhang higher spectral flow.

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0. Introduction

Let $i : Y \to X$ be an embedding between two odd dimensional closed oriented spin manifolds. For any Hermitian vector bundle $\mu$ over $Y$ carrying a Hermitian connection, under a natural assumption, Bismut and Zhang [BZ93] establish a mod $\mathbb{Z}$ formula, expressing the Atiyah-Patodi-Singer reduced eta invariant of certain direct image of $\mu$ over $X$, through the reduced eta invariant of the bundle $\mu$ over $Y$, up to some geometric Chern-Simons current.
In this paper, we explain the hidden mod \( \mathbb{Z} \) term as a spectral flow in Bismut-Zhang embedding formula and extend this embedding formula to the equivariant family case. In this case, the spectral flow is generalized to the equivariant chern character of some equivariant Dai-Zhang higher spectral flow \([DZ98]\).

The main motivation of this generalization is to look for a general Grothendieck-Riemann-Roch theorem in the equivariant differential K-theory, which is already established in many important cases \([FL10, BuS09, BuS13]\). Roughly speaking, the differential K-theory is the smooth version of the arithmetic K-theory in Arakelov geometry. Our main result here is expected to play the same role in the equivariant differential K-theory of the Bunke-Schick model \([BuS09, BuS13, Liu16]\) as the Bismut-Lebeau embedding formula \([BL92]\) does in the proof of Arithmetic Grothendieck-Riemann-Roch theorem in Arakelov geometry.

In this paper, we use the language of the Clifford modules.

Let \( \pi : W \to B \) be a smooth submersion of smooth oriented manifolds with closed fibres \( Y \). Let \( TY = TW/B \) be the relative tangent bundle to the fibres \( Y \). Let \( T^H \pi W \) be a horizontal subbundle of \( TW \). Let \( g^{TY} \) be a Riemannian metric on \( TY \). Let \( C(TY) \) be the Clifford algebra bundle of \( (TY, g^{TY}) \) and \( (E, h^E) \) be a \( \mathbb{Z}_2 \)-graded self-adjoint \( C(TY) \)-module with Clifford connection \( \nabla^E \) (cf. \((1.18)\) and \((1.22)\)).

Let \( G \) be a compact Lie group which acts on \( W \) and \( B \) such that for any \( g \in G \), \( \pi \circ g = g \circ \pi \). We assume that the action of \( G \) preserves the horizontal bundle and the orientation of \( TY \) and could be lifted on \( E \) such that it is compatible with the Clifford action and the \( \mathbb{Z}_2 \)-grading. We assume that \( g^{TY}, h^E, \nabla^E \) are \( G \)-invariant.

For any \( g \in G \), if the group action is trivial on \( B \), the equivariant Bismut-Chéeger eta form \( \tilde{\eta}^g(F, A) \in \Omega^*(B, \mathbb{C})/\text{Im } d \) is defined in Definition \((1.4)\) up to exact forms with respect to the equivariant geometric family \( F = (W, E, T^H \pi W, g^{TY}, h^E, \nabla^E) \) (cf. Definition \((1.1)\) over \( B \) and a perturbation operator \( A \) (cf. Definition \((1.3)\). Remark that if \( B \) is a point and \( \dim Y \) is odd, then the equivariant eta form here degenerates to the canonical equivariant reduced eta invariant by taking a special perturbation operator \([Liu16, Remark \ 2.20]\).

Let \( i : W \to V \) be an equivariant embedding of smooth \( G \)-equivariant oriented manifolds with even codimension. Let \( \pi_V : V \to B \) be a \( G \)-equivariant submersion with closed fibres \( X \), whose restriction \( \pi_W : W \to B \) is an equivariant submersion with closed fibres \( Y \). We assume that \( G \) acts on \( B \) trivially and the normal bundle \( N_{Y/X} \) to \( Y \) in \( X \) has an equivariant Spin\(^c\) structure.

\[
\begin{array}{ccc}
Y & \longrightarrow & W \\
\downarrow \ i & \ & \downarrow \pi_W \\
X & \longrightarrow & V \\
\downarrow \pi_V & \ & \downarrow \pi_V \\
& & B.
\end{array}
\]

Let \( F_Y = (W, E_Y, T^H \pi W, g^{TY}, h^E_Y, \nabla^E_Y) \) and \( F_X = (V, E_X, T^H \pi_V V, g^{TX}, h^E_X, \nabla^E_X) \) be two equivariant geometric families over \( B \) such that \((T^H \pi W, g^{TY})\) and \((T^H \pi_V V, g^{TX})\) satisfy the totally geodesic condition \((2.11)\).

We state our main result of this paper as follows.
Theorem 0.1. Assume that the equivariant geometric families $F_Y$ and $F_X$ satisfy the fundamental assumption (2.13) and (2.15). Let $A_Y$ and $A_X$ be the perturbation operators with respect to $F_Y$ and $F_X$. Then for any compact submanifold $K$ of $B$, there exists $T_0 > 2$ such that for any $T \geq T_0$, modulo exact forms, over $K$, we have

\begin{equation}
\eta_g(F_X, A_X) = \eta_g(F_Y, A_Y) + \int_X \tilde{A}_g(TX, \nabla_{TX}^X) \gamma^X_g(F_Y, F_X) + \text{ch}_g(\text{sf}_G\{D(F_X) + A_X, D(F_X) + T\nu + A_{T,Y}\}).
\end{equation}

Here $\gamma^X_g(F_Y, F_X)$ is the equivariant Bismut-Zhang current defined in Definition 2.3. The last term in (0.1) is the equivariant chern character of the equivariant Dai-Zhang higher spectral flow which explains the $\text{mod } Z$ term in the original Bismut-Zhang embedding formula.

The proof of our main result here is highly related to the analytical localization technique developed in [BL92, B95, B97, BM04]. Thanks to the functoriality of equivariant eta forms proved in [Liu16, Liu17], we only need to prove the embedding formula when $B$ is a point and $\dim X$ is odd.

Note that in [Z05, FXZ09], the authors give another proof of the Bismut-Zhang embedding formula without using the analytical localization technique. It is interesting to ask whether there is another proof of our main result here from that line.

Our paper is organized as follows. In Section 1, we summarize the definition and the properties of equivariant eta forms in [Liu16] using the language of Clifford modules. In Section 2, we state our main result. In this section, we also discuss an application on the equivariant Atiyah-Hirzebruch direct image. In Section 3, we prove our main result in two parts. In Section 3.1, we prove Theorem 0.1 when the base space is a point using some intermediary results along the lines of [BZ93], the proof of which rely on almost identical arguments of [BZ93, B95]. In Sections 3.2, we explain how to use the functoriality to reduce Theorem 0.1 to the case in Section 3.1.

To simplify the notations, we use the Einstein summation convention in this paper.

In the whole paper, we use the superconnection formalism of Quillen [Q85].

For a fibre bundle $\pi : V \to B$, we will often use the integration of the differential forms along the oriented fibres $X$ in this paper. Since the fibres may be odd dimensional, we must make precise our sign conventions. Let $\alpha$ be a differential form on $V$ which in local coordinates is given by

\begin{equation}
\alpha = f \cdot \pi^* dx^1 \wedge \cdots \wedge \pi^* dx^q \wedge dx^{q+1} \wedge \cdots \wedge dx^n,
\end{equation}

where $\{dy^p\}$ and $\{dx^i\}$ are the local frames of $T^*B$ and $T^*X$, respectively. We set

\begin{equation}
\int_X \alpha = \int_X dy^1 \wedge \cdots \wedge dy^q \cdot \int_X f \, dx^1 \wedge \cdots \wedge dx^n.
\end{equation}
1. Equivariant eta forms

In this section, we summarize the definition and the properties of equivariant eta forms in [Lin16, Lin17] using the language of Clifford modules. Note that locally all manifolds are spin. The proofs of them are the same as in the spin case. In Section 1.1, we recall elementary results on Clifford algebras. In Section 1.2, we describe the geometry of the fibration and recall the Bismut superconnection. In Section 1.3, we define the equivariant eta form and state the anomaly formula. In Section 1.4, we explain the functoriality of the equivariant eta forms.

1.1. Clifford algebras. Let $E^n$ be an oriented Euclidean vector space, such that $\dim E^n = n$, with orthonormal basis $\{e_1\}_{1 \leq i \leq n}$. Let $C(E^n)$ be the complex Clifford algebra of $E^n$ defined by the relations
\begin{equation}
   e_i e_j + e_j e_i = -2\delta_{ij}.
\end{equation}
Sometimes, we also denote by $c(e)$ the element of $C(E^n)$ corresponding to $e \in E^n$.

If $e \in E^n$, let $e^* \in (E^n)^*$ correspond to $e$ by the scalar product of $E^n$. The exterior algebra $\Lambda(E^n)^* \otimes_R \mathbb{C}$ is a module of $C(E^n)$ defined by
\begin{equation}
   c(e)\alpha = e^* \wedge \alpha - i_\epsilon \alpha
\end{equation}
for any $\alpha \in \Lambda(E^n)^* \otimes_R \mathbb{C}$. The map $a \mapsto c(a) \cdot 1$, $a \in C(E^n)$, induces an isomorphism of vector spaces
\begin{equation}
   \sigma : C(E^n) \to \Lambda(E^n)^* \otimes_R \mathbb{C}.
\end{equation}

If $n$ is even, up to isomorphism, $C(E^n)$ has a unique irreducible module, $S(E^n)$, which is $\mathbb{Z}_2$-graded. We denote this $\mathbb{Z}_2$-grading of the spinor by $\tau$. Moreover, there are isomorphisms of $\mathbb{Z}_2$-graded algebras
\begin{equation}
   C(E^n) \simeq \text{End}(S(E^n)) \simeq S(E^n) \hat{\otimes} S(E^n)^*.
\end{equation}
We consider the group $\text{Spin}^c_n$ as a multiplicative subgroup of the group of units of $C(E^n)$. For the definition and the properties of the group $\text{Spin}^c_n$, see [LM89, Appendix D]. Note that $S(E^n)$ is also an irreducible representation of $\text{Spin}^c_n$ induced by the Clifford action.

Let $F^m$ be another oriented Euclidean vector space such that $\dim F^m = m$, with orthonormal basis $\{f_p\}_{1 \leq p \leq m}$. Then as Clifford algebras,
\begin{equation}
   c(F^m \oplus E^n) \simeq c(F^m) \hat{\otimes} c(E^n).
\end{equation}
Moreover, we have $S(F^m \oplus E^n) \simeq S(F^m) \hat{\otimes} S(E^n)$.

If one of $m$ and $n$ is even, we simply assume $m$ is even, the spinor $S(F^m \oplus E^n)$ is isomorphic to $S(F^m) \otimes S(E^n)$ with the Clifford actions
\begin{equation}
   c(f_p) \hat{\otimes} 1 \mapsto c(f_p) \otimes 1, \quad 1 \hat{\otimes} c(e_i) \mapsto \tau \otimes c(e_i).
\end{equation}

If $m$ and $n$ are both odd, let
\begin{equation}
   \Gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}.
\end{equation}
The spinor $S(F^m \oplus E^n)$ is isomorphic to $S(F^m) \otimes S(E^n) \otimes \mathbb{C}^2$ with Clifford actions
\begin{equation}
(1.8) \quad c(f_p) \otimes 1 \mapsto c(f_p) \otimes 1 \otimes \Gamma_1, \quad 1 \otimes c(e_i) \mapsto 1 \otimes c(e_i) \otimes \Gamma_2.
\end{equation}

The $\mathbb{Z}_2$-grading of $S(F^m \oplus E^n)$ under this isomorphism is
\begin{equation}
(1.9) \quad \text{Id}_{S(F^m)} \otimes \text{Id}_{S(E^n)} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{equation}

Let $\tau^A \Lambda$ be the canonical $\mathbb{Z}_2$-grading of $\Lambda(F^m)$. Then $\Lambda(F^m) \otimes S(E^n)$ could be regarded as $\Lambda(F^m) \otimes S(E^n)$ with
\begin{equation}
(1.10) \quad \alpha \otimes 1 \mapsto \alpha \otimes 1, \quad 1 \otimes c(e_i) \mapsto c(e_i) \otimes \Gamma_2.
\end{equation}

We abbreviate $c(f_p) \otimes 1, \alpha \otimes 1, 1 \otimes c(e_i)$ by $c(f_p), \alpha, c(e_i)$. Then our notation here is the same as the usual one in [B86]. The purpose of writing in this way is to explain the extension of some fibrewise operators in the following sections to the $\mathbb{Z}_2$-graded setting clearly.

1.2. Bismut superconnection. Let $\pi : W \to B$ be a smooth submersion of smooth oriented manifolds with closed fibres $Y$. We assume that $B$ is connected. Remark that $W$ here is not assumed to be connected.

Let $TY = TW/B$ be the relative tangent bundle to the fibres $Y$. Then $TY$ is orientable. Let $T^H W$ be a horizontal subbundle of $TW$ such that
\begin{equation}
(1.12) \quad TW = T^H_W \oplus TY.
\end{equation}

The splitting (1.12) gives an identification
\begin{equation}
(1.13) \quad T^H_W \cong \pi^* TB.
\end{equation}

If there is no ambiguity, we will omit the subscript $\pi$ in $T^H_W$.

Let $g^{TY}, g^{TB}$ be Riemannian metrics on $TY, TB$. We equip $TW = T^H W \oplus TY$ with the Riemannian metric
\begin{equation}
(1.14) \quad g^{TW} = \pi^* g^{TB} \oplus g^{TY}.
\end{equation}

Let $\nabla^{TW}, \nabla^{TB}$ be the Levi-Civita connections on $(W, g^{TW}), (B, g^{TB})$. Let $P^{TY}$ be the projection $P^{TY} : TW = T^H W \oplus TY \to TY$. Set
\begin{equation}
(1.15) \quad \nabla^{TY} = P^{TY} \nabla^{TW} P^{TY}.
\end{equation}

Then $\nabla^{TY}$ is a Euclidean connection on $TY$.

Let $\nabla^{TB, TY}$ be the connection on $TW = T^H W \oplus TY$ defined by
\begin{equation}
(1.16) \quad \nabla^{TB, TY} = \pi^* \nabla^{TB} \oplus \nabla^{TY}.
\end{equation}

Then $\nabla^{TB, TY}$ preserves the metric $g^{TW}$ in (1.14).

Set
\begin{equation}
(1.17) \quad S = \nabla^{TW} - \nabla^{TB, TY}.
\end{equation}
Then $S$ is a 1-form on $W$ with values in antisymmetric elements of $\text{End}(TW)$. By [B86, Theorem 1.9], we know that $\nabla^{TY}$ and the $(3,0)$-tensor $g^TY(S(\cdot),\cdot)$ only depend on $(THW,g^TY)$.

Let $C(TY)$ be the Clifford algebra bundle of $(TY,g^TY)$, whose fibre at $x \in W$ is the Clifford algebra $C(T_xY)$ of the Euclidean vector space $(T_xY,g^{T_xY})$. A $\mathbb{Z}_2$-graded self-adjoint $C(TY)$-module,

$$\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-,$$

is a $\mathbb{Z}_2$-graded vector bundle equipped with a Hermitian metric $h^\mathcal{E}$ preserving the splitting (1.18) and a fiberwise Clifford multiplication $c$ of $C(TY)$ such that the action $c$ restricted to $TY$ is skew-adjoint on $(\mathcal{E},h^\mathcal{E})$. Let $\tau^\mathcal{E}$ be the $\mathbb{Z}_2$-grading of $\mathcal{E}$ which is $\pm 1$ on $\mathcal{E}_\pm$.

For a locally oriented orthonormal basis $e_1, \cdots, e_n$ of $TY$, we define the chirality operator on $\mathcal{E}$ by

$$\Gamma = \left\{ \begin{array}{ll} \sqrt{-1} c(e_1) \cdots c(e_n), & \text{if } n \text{ is even;} \\ \text{Id}_\mathcal{E}, & \text{if } n \text{ is odd.} \end{array} \right. \quad (1.19)$$

Note that our definition here is different from [BGV, Lemma 3.17] when $n$ is odd. Then $\Gamma$ does not depend on the choice of the basis and is globally defined. We note that $\Gamma^2 = 1$ and $[\tau^\mathcal{E}, \Gamma] = 0$. Set

$$\tau^\mathcal{E}/S = \tau^\mathcal{E} \cdot \Gamma. \quad (1.20)$$

Then $(\tau^\mathcal{E}/S)^2 = 1$.

Locally, we could write

$$\mathcal{E} = \mathcal{S}_0(TY) \mathcal{\otimes} \xi_\pm,$$

where $\mathcal{S}_0(TY)$ is the spinor bundle for the (possibly non-existent) spin structure of $TY$ and $\xi_\pm$ is a $\mathbb{Z}_2$-graded vector bundle. Then $\Gamma$, $\tau^\mathcal{E}/S$ and $\tau^\mathcal{E}$ correspond to the $\mathbb{Z}_2$-gradings of $\mathcal{S}_0(TY)$, $\xi_\pm$ and $\mathcal{S}_0(TY) \mathcal{\otimes} \xi_\pm$.

Let $\nabla^\mathcal{E}$ be a Clifford connection on $\mathcal{E}$ associated with $\nabla^{TY}$, that is, $\nabla^\mathcal{E}$ preserves $h^\mathcal{E}$ and the splitting (1.18) and for any $U \in TW$, $Z \in \mathcal{C}^\infty(W,TY)$,

$$[\nabla^\mathcal{E}_U, c(Z)] = c(\nabla^{TY}_U Z). \quad (1.22)$$

Let $G$ be a compact Lie group which acts on $W$ and $B$ such that for any $g \in G$, $\pi \circ g = g \circ \pi$. We assume that the action of $G$ preserves the splitting (1.12) and the orientation of $TY$ and could be lifted on $\mathcal{E}$ such that it is compatible with the Clifford action and preserves the splitting (1.18). We assume that $g^{TY}$, $h^\mathcal{E}$, $\nabla^\mathcal{E}$ are $G$-invariant.

**Definition 1.1.** ([Compare with BuS09, Definition 2.2], Liu16, Definition 1.1])

An equivariant geometric family $\mathcal{F}$ over $B$ is a family of $G$-equivariant geometric data

$$\mathcal{F} = (W, \mathcal{E}, THW, g^TY, h^\mathcal{E}, \nabla^\mathcal{E}) \quad (1.23)$$

described as above. We call the equivariant geometric family $\mathcal{F}$ is even (resp. odd) if for any connected component of fibres, the dimension of it is even (resp. odd).
Let $D(\mathcal{F})$ be the fiberwise Dirac operator
\begin{equation}
D(\mathcal{F}) = c(e_i)\nabla^\mathcal{E}_{e_i}
\end{equation}
associated with the equivariant geometric family $\mathcal{F}$. Then the $G$-action commutes with $D(\mathcal{F})$.

For $b \in B$, let $\mathcal{E}_b$ be the set of smooth sections over $Y_b$ of $\mathcal{E}_g$. As in [BS86], we will regard $\mathcal{E}$ as an infinite dimensional fibre bundle over $B$. If $V \in TB$, let $V^H \in T^H W$ be its horizontal lift in $T^H W$ so that $\pi_* V^H = V$. For any $V \in TB$, $s \in \mathcal{C}^\infty(B, \mathcal{E}) = \mathcal{C}^\infty(W, \mathcal{E})$, by [BF86 Proposition 1.4], the connection
\begin{equation}
\nabla^\mathcal{E}_{b,s} := \nabla^\mathcal{E}_{b,s} V - \frac{1}{2}\langle S(e_i)e_i, V^H \rangle s
\end{equation}
preserves the $L^2$-product on $\mathcal{E}$. Let $\{f_p\}$ be a local orthonormal frame of $TB$ and $\{f^p\}$ be its dual. We denote by $\nabla^{\mathcal{E},a} = f^p \wedge \nabla^{\mathcal{E},a}_{f^p}$. Let $T$ be the torsion of $\nabla^{TB,\Lambda\mathcal{Y}}$. We denote by $c(T) = \frac{1}{4}\langle S(e_i)e_i, V^H \rangle s$.

From [BGV Theorem 9.51], we know that $\exp(-\mathbb{B}_u^2)$ is a smooth family of smoothing operators.

Let $P$ be a section of $\Lambda(T^*B) \otimes \text{End}(\mathcal{E})$. Set
\begin{equation}
\text{Tr}_s[P] := \text{Tr}[\tau^\mathcal{E} P].
\end{equation}
Here the trace operator on the right hand side of (1.27) only acts on $\mathcal{E}$ and takes values in $\Lambda(T^*B)$. We denote by $\text{Tr}_s^{\text{odd/even}}[P]$ the part of $\text{Tr}_s[P]$ which takes values in odd or even forms. We use the convention that if $\omega \in \Lambda(T^*B)$,
\begin{equation}
\text{Tr}_s[\omega P] = \omega \text{Tr}_s[P].
\end{equation}
It is compatible with the sign convention (0.3). Set
\begin{equation}
\tilde{\text{Tr}}[P] = \begin{cases} 
\text{Tr}_s[P], & \text{if dim } Y \text{ is even}; \\
\text{Tr}_s^{\text{odd}}[P], & \text{if dim } Y \text{ is odd}.
\end{cases}
\end{equation}

1.3. Equivariant eta forms. In this subsection, we state the definition and the anomaly formula of equivariant eta forms in the language of Clifford modules.

We assume that $G$ acts trivially on $B$.

Take $g \in G$ and set $W^g = \{x \in W : gx = x\}$. Then $W^g$ is a submanifold of $W$ and $\pi|_{W^g} : W^g \to B$ is a fibre bundle with closed fibres $Y^g$. Let $N_{W^g/W}$ denote the normal bundle of $W^g$ in $W$, then $N_{W^g/W} = TY/TY^g$. We also denote it by $N_{Y^g/Y}$.
We denote the differential of $g$ by $dg$ which gives a bundle isometry $dg : N_{Y^g/Y} \to N_{Y^g/Y}$. Since $g$ lies in a compact abelian Lie group, we know that there is an orthonormal decomposition of smooth vector bundles over $W^g$

\[(1.30) \quad TY|_{W^g} = TY^g \oplus N_{Y^g/Y} = TY^g \oplus \bigoplus_{0 < \theta \leq \pi} N(\theta),\]

where $dg|_{N(\pi)} = -\text{id}$ and for each $\theta$, $0 < \theta < \pi$, $N(\theta)$ is a complex vector bundle on which $dg$ acts by multiplication by $e^{i\theta}$. Since $g$ preserves the orientation of $TY$ and $\det(dg|_{N(\pi)}) = 1$, by the property of isometry, $\dim N(\pi)$ is even. So the normal bundle $N_{Y^g/Y}$ is even dimensional.

Observe that if $N(\pi) = 0$ or if $TY$ has a $G$-equivariant $\text{Spin}^c$ structure, then $TY^g$ is canonically oriented (cf. [BGV, Proposition 6.14], [Liu17, Proposition 2.1]). In general, $TY^g$ is not necessary oriented. For simplicity, in this paper we assume that $TY^g$ is oriented. In this case $N(\pi)$ is oriented. We fix an orientation of $TY^g$ which is induced by the orientations of $N_\theta$ and $TY$.

We remark here that if the fixed point sets are not oriented, we could also get the formulas in this paper in the sense of Berezin integral as in [BGV, Theorem 6.16].

Let $E$ be an equivariant real Euclidean vector bundle over $W$. We could get the decomposition of real vector bundles over $W^g$ in the same way as (1.30),

\[(1.31) \quad E|_{W^g} = \bigoplus_{0 \leq \theta \leq \pi} E(\theta).\]

Here we also denote $E(0)$ by $E^0$.

Let $\nabla^E$ be an equivariant Euclidean connection on $E$. Then it preserves the decomposition (1.31). Let $\nabla^{E^g}$ and $\nabla^{E(\theta)}$ be the corresponding induced connections on $E^g$ and $E(\theta)$, and let $R^{E^g}$ and $R^{E(\theta)}$ be the corresponding curvatures.

Set

\[(1.32) \quad \Lambda_g(E, \nabla^E) = \det^\frac{1}{2} \left( \frac{\sqrt{-1}}{2\pi} R^{E^g} \right) \cdot \prod_{0 < \theta \leq \pi} \left( \sqrt{-1}^{\frac{1}{2} \dim E(\theta)} \det^\frac{1}{2} \left( 1 - g \exp \left( \frac{\sqrt{-1}}{2\pi} R^{E(\theta)} \right) \right) \right)^{-1}.

Let $\text{End}_{C(TY)}(E)$ be the set of endomorphisms of $E$ commuting with the Clifford action. Then it is a vector bundle over $W$. As in [BGV, Definition 3.28], for any $a \in \text{End}_{C(TY)}(E)$, we define the relative trace $\text{Tr}^{E/S} : \text{End}_{C(TY)}(E) \to \mathbb{C}$ by

\[(1.33) \quad \text{Tr}^{E/S}[a] = \begin{cases} 2^{-n/2} \text{Tr}_s[\Gamma a], & \text{if } \dim Y \text{ is even;} \\ 2^{-(n-1)/2} \text{Tr}_s[a], & \text{if } \dim Y \text{ is odd}. \end{cases}

Let $R^E$ be the curvature of $\nabla^E$. Let

\[(1.34) \quad R^{E/S} := R^E - \frac{1}{4} (R^{TY} e_i, e_j) c(e_i)c(e_j) \in \mathcal{C}^\infty(W, \pi^* \Lambda(T^*B) \otimes \text{End}_{C(TY)}(E))

be the twisting curvature of the $C(TY)$-module $E$ as in [BGV, Proposition 3.43].
By [BGV] Lemma 6.10, along $W^g$, the action of $g \in G$ on $E$ may be identified with a section $g^E$ of $C(N_{Y^g/Y}) \otimes \text{End}_{C(TY)}(E)$. Let $\dim N_{Y^g/Y} = \ell_1$. Under the isomorphism $(\ref{1.39})$, $\sigma(g^E) \in \mathcal{C}^\infty(W^g, (\Lambda N_{Y^g/Y}^* \otimes \mathbb{R} \otimes \text{End}_{C(TY)}(E)))$. Since we assume that $N_{Y^g/Y}$ is oriented, paring with the volume form, we could get the highest degree coefficient $\sigma_{\ell_1}(g^E) \in \mathcal{C}^\infty(W^g, \text{End}_{C(TY)}(E))$ of $\sigma(g^E)$.

Then we could define the localized relative Chern character $\text{ch}_g(E/S, \nabla^E) \in \Omega^*(W^g, \mathbb{C})$ in the same way as [BGV] Definition 6.13 by

$$\text{ch}_g(E/S, \nabla^E) := \frac{2^{\ell_1/2}}{\det^{1/2}(1 - g|_{N_{Y^g/Y}})} \text{Tr}_{E/S} \left[ \sigma_{\ell_1}(g^E) \exp \left( - \frac{R^E/S}{2\pi \sqrt{-1}} \right) \right].$$

Note that if $TY$ has an equivariant spin structure, the localized relative Chern character here is just the usual equivariant Chern character.

Recall that if $B$ is compact, the equivariant $K$-group $K^0_G(B)$ is the Grothendieck group of the equivalent classes of the equivariant vector bundles over $B$. Let $\iota : B \to B \times S^1$ be a $G$-equivariant inclusion map. It is well known that if the $G$-action on $S^1$ is trivial,

$$K^1_G(B) \simeq \ker (\iota^* : K^0_G(B \times S^1) \to K^0_G(B)).$$

For $x \in K^0_G(B)$, $g \in G$, the classical equivariant Chern character map sends $x$ to $\text{ch}_g(x) \in H^{even}(B, \mathbb{C})$. By $(\ref{1.39})$, for $x \in K^1_G(B)$, we can regard $x$ as an element $x'$ in $K^0_G(B \times S^1)$. The odd equivariant Chern character map

$$\text{ch}_g : K^1_G(B) \to H^{odd}(B, \mathbb{C})$$

is defined by (cf. [Liu17] (2.52))

$$\text{ch}_g(x) := \int_{S^1} \text{ch}_g(x').$$

We adopt the sign notation in the integral as in $(\ref{1.39})$.

Furthermore, the classical construction of Atiyah-Singer assigns to each equivariant geometric family $F$ its equivariant (analytic) index $\text{ind}(D(F)) \in K^*_G(B)$.

For $\alpha \in \Omega^*(B)$, set

$$\psi_B(\alpha) = \begin{cases} 
\left( \frac{1}{2\pi \sqrt{-1}} \right)^{\frac{j}{2}} \cdot \alpha, & \text{if } j \text{ is even;} \\
\frac{1}{\sqrt{\pi}} \left( \frac{1}{2\pi \sqrt{-1}} \right)^{\frac{j-1}{2}} \cdot \alpha, & \text{if } j \text{ is odd.}
\end{cases}$$

**Theorem 1.2.** [Liu17] Theorem 2.2] For any $u > 0$ and $g \in G$, the differential form $\psi_B \text{Tr}[g \exp(-B^2)] \in \Omega^*(B, \mathbb{C})$ is closed and its cohomology class is independent of $u$. As $u \to 0$,

$$\lim_{u \to 0} \psi_B \text{Tr}[g \exp(-B^2)] = \int_{Y^g} \tilde{A}_g(TY, \nabla^{TY}) \text{ch}_g(E/S, \nabla^E).$$

If $B$ is compact, the differential form $\psi_B \text{Tr}[g \exp(-B^2)]$ represents $\text{ch}_g(\text{ind}(D(F)))$.

**Definition 1.3.** [Liu10] Definition 2.10] A perturbation operator with respect to $D(F)$, denoted by $A$, is defined to be a smooth family of $G$-equivariant bounded
self-adjoint pseudodifferential operators on \( \mathcal{E} \) along the fibres such that it commutes (resp. anti-commutes) with the \( \mathbb{Z}_2 \)-grading of \( \mathcal{E} \) when the fibres are odd (resp. even) dimensional, and \( D(\mathcal{F}) + A \) is invertible.

Remark that from [Liu16, Proposition 2.3], if \( B \) is compact and at least one component of the fibres has the non-zero dimension, then there exists a perturbation operator with respect to \( D(\mathcal{F}) \) if and only if \( \text{ind}(D(\mathcal{F})) = 0 \in K^*_G(B) \).

In the followings, we always assume that there exists a perturbation operator with respect to \( D(\mathcal{F}) \) on \( \mathcal{F} \).

For \( \alpha \in \Lambda(T^*(\mathbb{R} \times B)) \), we can expand \( \alpha \) in the form

\[
\alpha(u) = du \wedge \alpha_0(u) + \alpha_1(u), \quad \alpha_0(u), \alpha_1(u) \in \Lambda(T^*B).
\]

Set

\[
[\alpha(u)]^{du} := \alpha_0(u).
\]

Let \( \chi \in C_0^\infty(\mathbb{R}) \) be a cut-off function such that

\[
\chi(u) = \begin{cases} 0, & \text{if } u \leq 1; \\ 1, & \text{if } u \geq 2. \end{cases}
\]

Let \( A \) be a perturbation operator with respect to \( D(\mathcal{F}) \). Then \( A \) could be extended to \( 1 \hat{\otimes} A \) on \( C_0^\infty(B, \pi_\ast \Lambda(T^*B) \hat{\otimes} \mathcal{E}) \) as in (1.10). Explicitly, the extended perturbation operator \( 1 \hat{\otimes} A \) which acts along the fibres \( Y \) on \( \mathcal{E}^\infty(B, \pi_\ast \Lambda(T^*B) \hat{\otimes} \mathcal{E}) \) is considered as \( \tau^\wedge \otimes A \) on \( \mathcal{E}^\infty(B, \pi_\ast \Lambda(T^*B) \otimes \mathcal{E}) \) and \( \alpha \hat{\otimes} 1 \mapsto \alpha \otimes 1 \). Then as in (1.11), we have

\[
(\alpha \hat{\otimes} 1)(1 \hat{\otimes} A) = -(1 \hat{\otimes} A)(\alpha \hat{\otimes} 1).
\]

We usually abbreviate \( 1 \hat{\otimes} A \) by \( A \) when there is no confusion. Set

\[
B'_u = B_u + \sqrt{u} \chi(\sqrt{u})A.
\]

**Definition 1.4.** [Liu16, Definition 2.11] For any \( g \in G \), modulo exact forms, the equivariant Bismut-Cheeger eta form with perturbation operator \( A \) is defined by

\[
\tilde{\eta}_g(\mathcal{F}, A) := -\int_0^\infty \left\{ \psi_{\mathbb{R} \times B} \hat{\text{Tr}} \left[ g \exp \left( - \left( \mathbb{E}' + du + \frac{\partial}{\partial u} \right)^2 \right) \right] \right\}^{du} \in \Omega^*(B, \mathcal{C})/\text{Im } d.
\]

As in (1.11) and (1.44), we adopt the convention that \( du \) anti-commutes with \( A \) and \( c(v) \) for any \( v \in TY \).

From the discussion in [Liu16, Section 2.3], the equivariant eta form with perturbation in Definition 1.4 is well defined and does not depend on the cut-off function. Moreover, since we assume that \( Y^g \) is oriented, we have (cf. [Liu16, (2.44)])

\[
d^B \tilde{\eta}_g(\mathcal{F}, A) = \int_{Y^g} \hat{A}_g(TY, \nabla^{TY}) \text{ch}_g(\mathcal{E}/\mathcal{S}, \nabla^\mathcal{E}).
\]
Remark 1.5. After changing the variable, we have
\begin{equation}
\hat{\eta}_g(F, A) = -\int_0^\infty \left\{ \psi_B \left[ g \exp \left( -\left( B_{u^2} + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] \right\} du.
\end{equation}

We will often use this formula as the definition of the equivariant eta form.

Explicitly,
\begin{equation}
\hat{\eta}_g(F, A) = \begin{cases}
\int_0^\infty \frac{1}{\sqrt{\pi}} \psi_B T_{B_{u^2}} \left[ g \frac{\partial B_{u^2}}{\partial u} \exp(-B_{u^2}^2) \right] du \\
\int_0^\infty \frac{1}{2\sqrt{\pi} \sqrt{-1}} \psi_B T_{B_{u^2}} \left[ g \frac{\partial B_{u^2}}{\partial u} \exp(-B_{u^2}^2) \right] du
\end{cases}
\end{equation}

From [Liu16, Remark 2.20], when $B$ is a point, $\dim Y$ is odd, letting $A = P_{\ker D(F_Y)}$ be the orthogonal projection onto the kernel of $D(F_Y)$, the equivariant eta form $\hat{\eta}_g(F, A)$ is just the equivariant reduced eta invariant defined in [D78]. Note that from (1.49), if $B$ is a point and $\dim Y$ is even, we have $\hat{\eta}_g(F, A) = 0$ for any perturbation operator $A$.

Let $F = (W, \mathcal{E}, T^H W, g^{TY}, h^g, \nabla^g)$ and $F' = (W, \mathcal{E}, T^H W, g'^{TY}, h'^g, \nabla'^g)$ be two equivariant geometric families over $B$. Let
\begin{equation}
\left( \widetilde{\Lambda}_g \cdot \widetilde{c}_g \right)(\nabla^{TY}, \nabla'^{TY}, \nabla^g, \nabla'^g) \in \Omega^*(W^g, \mathbb{C})/\text{Imd}
\end{equation}
be the Chern-Simons form (cf. [MM07, Appendix B]) such that
\begin{equation}
d \left( \widetilde{\Lambda}_g \cdot \widetilde{c}_g \right)(\nabla^{TY}, \nabla'^{TY}, \nabla^g, \nabla'^g)
= \widetilde{\Lambda}_g(TY, \nabla^{TY}) \text{ch}_g(\mathcal{E}/S, \nabla'^g) - \widetilde{\Lambda}_g(TY, \nabla'^{TY}) \text{ch}_g(\mathcal{E}/S, \nabla^g).
\end{equation}

When $B$ is compact, let $\text{sf}_G\{ (D(F) + A', P'), (D(F) + A, P) \} \in \mathcal{K}_G(B)$, which we often simply denote by $\text{sf}_G\{ D(F') + A', D(F) + A \}$, be the equivariant Dai-Zhang higher spectral flow defined in [Liu16, Definition 2.5, 2.6], where $P, P'$ are the orthonormal projections onto the eigenspaces of positive eigenvalues with respect to $D(F) + A, D(F') + A'$ respectively. If $B$ is a point and $\dim Y$ is odd, it is just the canonical equivariant spectral flow.

The following anomaly formula is proved in [Liu16, Theorem 2.17] and [Liu17, Theorem 2.7].

**Theorem 1.6.** Let $A, A'$ be perturbation operators with respect to $D(F), D(F')$ respectively. For any $g \in G$, modulo exact forms, we have
(a) if $B$ is compact, then
\begin{equation}
\hat{\eta}_g(F', A') - \hat{\eta}_g(F, A) = \int_{Y^g} \left( \widetilde{\Lambda}_g \cdot \widetilde{c}_g \right)(\nabla^{TY}, \nabla'^{TY}, \nabla^g, \nabla'^g)
+ \text{ch}_g(\text{sf}_G\{ D(F') + A', D(F) + A \});
\end{equation}
(b) if $B$ is noncompact and there exists a smooth path $(F_s, A_s), s \in [0, 1]$, connecting $(F, A)$ and $(F', A')$ such that for any $s \in [0, 1], A_s$ is the perturbation operator of $D(F_s)$, then

$$\tilde{\eta}_g(F', A') - \tilde{\eta}_g(F, A) = \int_{Y_g} \left( \overline{\Lambda}_g \cdot \overline{c}_g \right) \left( \nabla^{TY}, \nabla'^{TY}, \nabla^\xi, \nabla^\xi' \right).$$

(1.52)

1.4. Functoriality. Let $\pi_M : U \to W$ be a $G$-equivariant submersion of smooth manifolds with closed oriented fibres $M$. Let $(E_M, h^{E_M})$ be a $\mathbb{Z}_2$-graded self-adjoint equivariant $C(TM)$-module. Let

$$F_M = (U, E_M, T^H_{\pi_M} U, g^{TM}, h^{E_M}, \nabla^{E_M})$$

be a $G$-equivariant geometric family over $W$. Then $\pi_Z := \pi \circ \pi_M : U \to B$ is a $G$-equivariant submersion with closed fibres $Z$, the orientation of which is the composition of the orientations of $Y$ and $M$. Then we have the diagram of submersions:

$$\begin{array}{ccc}
M & \longrightarrow & Z \\
\downarrow \pi_M & & \downarrow \pi_Z \\
Y & \longrightarrow & W & \longrightarrow & B.
\end{array}$$

Set $T^H_{\pi_M} Z := T^H_{\pi_M} U \cap TZ$. Then we have the splitting of smooth vector bundles over $U$,

$$TZ = T^H_{\pi_M} Z \oplus TM,$$

and

$$T^H_{\pi_M} Z \cong \pi_M^* TY.$$

Take the geometric data $(T^H_{\pi_M} U, g^{TZ}_{T^H})$ of $\pi_Z$ such that $T^H_{\pi_Z} U \subset T^H_{\pi_M} U$,

$$g^{TZ}_{T^H} = \pi_M^* g^{TY} \oplus \frac{1}{T^2} g^{TM}$$

and $g^{TZ} = g^{TZ}_{T^H}$. We denote the Clifford algebra bundle with respect to $g^{TZ}_{T^H}$ by $C_T(Z)$ and the corresponding 1-form in (1.17) by $S_T$.

Let $\{e_i\}, \{f_p\}$ be local orthonormal frames of $TM, TY$ with respect to $g^{TM}, g^{TY}$ respectively. Now $\{Te_i\}$ is a local orthonormal frame of $TM$ with respect to the rescaled metric $T^{-2} g^{TM}$. Let $f^H_p$ be the horizontal lift of $f_p$ with respect to (1.54). Now we define a Clifford algebra homomorphism

$$G_T : (C_T(TZ), g^{TZ}_{T^H}) \to (C(TZ), g^{TZ})$$

by $G_T(c_T(f^H_p)) = c(f^H_p)$ and $G_T(c_T(Te_i)) = c(e_i)$. Under this homomorphism,

$$E_Z := \pi_M^* E_Y \otimes E_M$$

with induced Hermitian metric $h^{E_Z}$ is a $\mathbb{Z}_2$-graded self-adjoint equivariant $C_T(TZ)$-module. .

Let

$$0^{E_Z} := \pi_M^* \nabla^{E_Y} \otimes 1 + 1 \otimes \nabla^{E_M}.$$
Then it is a Clifford connection on $E_Z$ associated with
\begin{equation}
\nabla^{TY,TM} := \pi^*_M \nabla^{TY} \otimes 1 + 1 \otimes \nabla^{TM}.
\end{equation}

Now, we denote the Levi-Civita connection on $TZ$ with respect to $\hat{g}_T^{TZ}$ by $\nabla_T^{TZ}$. Then we could calculate that
\begin{equation}
\nabla^E_Z := \nabla^E_T + \frac{1}{2} \langle S_T T e_i, f_p^H \rangle_T c_T(T e_i) c(f_p^H)
+ \frac{1}{4} \langle S_T f_p^H, f_q^H \rangle_T c(f_p^H) c(f_q^H)
\end{equation}
is a Clifford connection associated with $\nabla_T^{TZ}$, where $\langle \cdot, \cdot \rangle_T = \hat{g}_T^{TZ} \langle \cdot, \cdot \rangle$ (cf. Liu17 (4.3)). Thus we get a rescaled equivariant geometric family
\begin{equation}
F_{Z,T} := (U, E_Z, T_{\pi_Z} U, g_T^{TZ}, h^E_Z, \nabla^E_T)
\end{equation}
over $B$. We write $F_Z = F_{Z,1}$.

Let $A_M$ be a perturbation operator with respect to $D(F_M)$. Then $A_M$ could be extended to $1 \otimes A_M$ on $\mathcal{F}(U, \pi^*_Z \Lambda(T^*B) \otimes \pi^*_M \hat{E}_Y \otimes E_M)$ in the same way as $D(F_M)$.

Explicitly, if dim $Y$ is even, the extended perturbation operator $1 \otimes A_M$ which acts along the fibres $M$ on $\mathcal{F}(U, \pi^*_Z \Lambda(T^*B) \otimes \pi^*_M \hat{E}_Y \otimes E_M)$ is considered as $\tau^\Lambda \otimes \tau \otimes A_M$ on $\mathcal{F}(U, \pi^*_Z \Lambda(T^*B) \otimes \pi^*_M \hat{E}_Y \otimes E_M)$ and $\alpha \otimes 1 \otimes 1 \mapsto \alpha \otimes 1 \otimes 1$, $1 \otimes c(f_p) \otimes 1 \mapsto \tau^\Lambda \otimes c(f_p) \otimes 1$.

If dim $Y$ is odd and dim $M$ is even, $1 \otimes A_M$ is considered as $\tau^\Lambda \otimes 1 \otimes A_M$ and $\alpha \otimes 1 \otimes 1 \mapsto \alpha \otimes 1 \otimes 1$, $1 \otimes c(f_p) \otimes 1 \mapsto \tau^\Lambda \otimes c(f_p) \otimes 1$.

If dim $Y$ and dim $M$ are odd, $1 \otimes A_M$ is considered as $\tau^\Lambda \otimes 1 \otimes A_M \otimes \Gamma_2$ on $\mathcal{F}(U, \pi^*_Z \Lambda(T^*B) \otimes \pi^*_M \hat{E}_Y \otimes E_M \otimes \mathbb{C}^2)$ and $\alpha \otimes 1 \otimes 1 \mapsto \alpha \otimes 1 \otimes 1 \otimes \text{Id}$, $1 \otimes c(f_p) \otimes 1 \mapsto \tau^\Lambda \otimes c(f_p) \otimes 1 \otimes \Gamma_1$.

We abbreviate $\alpha \otimes 1 \otimes 1$, $1 \otimes c(f_p) \otimes 1$ by $\alpha$, $c(f_p)$. Then
\begin{equation}
\alpha \cdot 1 \otimes A_M = -1 \otimes A_M \cdot \alpha, \quad c(f_p) \cdot 1 \otimes A_M = -1 \otimes A_M \cdot c(f_p).
\end{equation}

In Liu16 Lemma 2.15], we prove that for any compact submanifold $K$ of $B$, there exists $T_0 > 0$ such that for $T \geq T_0$, $1 \otimes T A_M$ is a perturbation operator with respect to $D(F_{Z,T})$ over $K$.

The following theorem is the Clifford module version of Liu16 Lemma 2.16], which is related to M02 Theorem 3.1], BuM04 Theorem 5.11] and Liu17 Theorem 3.4].

**Theorem 1.7.** For any compact submanifold $K$ of $B$, there exists $T_0 > 0$ such that for $T \geq T_0$, modulo exact forms, over $K$, we have
\begin{equation}
\tilde{\eta}_g(F_{Z,T}, 1 \otimes T A_M) = \int_{Y^s} \hat{A}_g(T Y, \nabla^{TY}) \, c_T(\hat{h}_g(\hat{E}_Y/S, \nabla^{\hat{E}_Y}) \, \tilde{\eta}_g(F_M, A_M)
- \int_{Z^y} \left( \hat{A}_g \cdot \hat{h}_g \right) \left( \nabla_T^{TZ} \cdot \nabla^{TY,TM}, \nabla^E_T, \nabla^E_Z, 0 \omega^E_Z \right).
\end{equation}
2. Embedding of equivariant eta forms

In this section, we state our main result and the application in equivariant Atiyah-Hirzebruch direct image. In Section 2.1, we describe the geometry of the embedding of submersions. In Section 2.2, we explain the equivariant family version of the fundamental assumption. In Section 2.3, we introduce the equivariant Atiyah-Hirzebruch direct image. In Section 2.4, we state our main result.

2.1. Embedding of submersions. In this subsection, we introduce the embedding of submersions, the setting of which is the same as [B97, Section 1] and [BM04].

Let \( i : W \rightarrow V \) be an embedding of smooth oriented manifolds. Let \( \pi_V : V \rightarrow B \) be a submersion of smooth oriented manifolds with closed fibres \( X \), whose restriction \( \pi_W : W \rightarrow B \) is a smooth submersion with closed fibres \( Y \).

Thus, we have the diagram of maps

\[
\begin{array}{ccc}
Y & \rightarrow & W \\
\downarrow i & & \downarrow \pi_W \\
X & \rightarrow & V & \rightarrow & B
\end{array}
\]

In general, \( B, V, W \) are not connected. We simply assume that \( B \) and \( V \) are connected. For any connected component \( W_\alpha \) of \( W \), we assume that \( \dim V - \dim W_\alpha \) is even. To simplify the notations, we usually denote the connected component by \( W \) when there is no confusion.

Let \( TX = TV/B, TY = TW/B \) be the relative tangent bundles to the fibres \( X, Y \). Let \( T^H V \) be a smooth subbundle of \( TV \) such that

\( \text{(2.1)} \)

\[ TV = T^H V \oplus TX. \]

Let \( \tilde{N}_{Y/X} \) be a smooth subbundle of \( TX|_W \) such that

\( \text{(2.2)} \)

\[ TX|_W = TY \oplus \tilde{N}_{Y/X}. \]

Let \( N_{W/V} \) be the normal bundle to \( W \) in \( V \), which we usually denote by \( N_{Y/X} \). Clearly,

\( \text{(2.3)} \)

\[ T^H V \simeq \pi_V^* TB, \quad \tilde{N}_{Y/X} \simeq N_{Y/X}. \]

By \( \text{(2.1)} \) and \( \text{(2.2)} \), we get

\( \text{(2.4)} \)

\[ TV|_W = T^H V|_W \oplus TY \oplus \tilde{N}_{Y/X}. \]

By \( \text{(2.4)} \), there is a well-defined morphism

\( \text{(2.5)} \)

\[ \frac{TW}{TY} \rightarrow T^H V|_W \oplus \tilde{N}_{Y/X} \]

and this morphism maps \( TW/TY \) into a subbundle of \( TW \). Let \( T^H W \) be the subbundle of \( TW \) which is the image of \( TW/TY \) by the morphism \( \text{(2.5)} \). Clearly,

\( \text{(2.6)} \)

\[ TW = T^H W \oplus TY. \]

In general, the subbundle \( T^H W \) is not equal to \( T^H V|_W \).
Let $g^{TV}$ be a metric on $TV$. Let $g^{TW}$ be the induced metric on $TW$. Let $g^{TX}$, $g^{TY}$ be the induced metrics on $TX, TY$. Note that even if $g^{TV}$ is of the type as in (1.14), in general, $g^{TW}$ is not of this type.

We identity $N_{Y/X}$ with the orthogonal bundle $\bar{N}_{Y/X}$ to $TY$ in $TX|_W$ with respect to $g^{TX}|_W$. Let $g^{N_{Y/X}}$ be the induced metric on $N_{Y/X}$. On $W$, we have

\begin{align}
TX|_W = TY \oplus N_{Y/X}.
\end{align}

To the pairs $(T^g_{\pi_W} V, g^{TX})$ and $(T^g_{\pi_W} W, g^{TY})$, we can associate the objects that we construct in (1.14) and (1.17). In particular, $TX, TY$ are now equipped with connections $\nabla^{TX}, \nabla^{TY}$ which preserve the metrics $g^{TX}, g^{TY}$ respectively.

Let $P^{TY}, P^{N_{Y/X}}$ be the orthogonal projections $TX|_W \to TY, TX|_W \to N_{Y/X}$. By [B97, Theorem 1.9], we have

\begin{align}
\nabla^{TY} = P^{TY}\nabla^{TX}|_V.
\end{align}

Let

\begin{align}
\nabla^{N_{Y/X}} = P^{N_{Y/X}}\nabla^{TX}
\end{align}

be the connection on $N_{Y/X}$. Then $\nabla^{N_{Y/X}}$ preserves the metric $g^{N_{Y/X}}$. Put

\begin{align}
\nabla^{TY,N_{Y/X}} = \nabla^{TY} \oplus \nabla^{N_{Y/X}}
\end{align}

Then $\nabla^{TY,N_{Y/X}}$ is a Euclidean connection on $TX|_W = TY \oplus N_{Y/X}$.

Let $G$ be a compact Lie group. We assume that $W, V$ and $B$ are $G$-manifolds and the $G$-action commutes with the embedding and $\pi_V$. Obviously, the group action commutes with $\pi_W$. We assume that $G$ acts trivially on $B$. We assume that the group action preserve the splittings (2.1) and (2.6) and all metrics and connections are $G$-invariant.

Let $W^g, V^g$ be the fixed point sets of $W, V$ for $g \in G$. Then $\pi_W|_{W^g} : W^g \to B$ and $\pi_V|_{V^g} : V^g \to B$ are submersions with closed fibres $Y^g$ and $X^g$. We assume that $TY^g$ and $TX^g$ are all oriented and the orientations are compatible with those of $TY, TX$ and the normal bundles as in the arguments at the beginning of Section 1.3.

Remark 2.1. (cf. [B97, Section 7.5]) Given $G$-equivariant pair $(T^H_{\pi_W} W, g^{TY})$, we could take metrics $g^{TB}$ and $g^{TW}$ on $TB$ and $TW$ such that $g^{TW} = \pi_W^* g^{TB} \oplus g^{TY}$. Let $g^N$ be a $G$-invariant metric on $N_{Y/X}$. Let $\nabla^N$ be a $G$-invariant Euclidean connection on $(N_{Y/X}, g^N)$ and $T^H N$ be the horizontal subbundle associated with the fibration $\pi_N : N_{Y/X} \to W$ and $\nabla^N$. We take $g^{TN} = \pi_N^* g^{TW} \oplus g^N$ for $TN = T^HN \oplus N$. Since $W$ intersects $X$ orthogonally, we could take horizontal subbundle $T^H_{\pi_W} V$ over $V$ such that $T^H_{\pi_W} V|_W = T^H_{\pi_W} W$. By using the partition of unity argument, we could construct $G$-invariant metrics $g^{TX}, g^{TY}$ on $TX, TV$ such that $g^{TV} = \pi_V^* g^{TB} \oplus g^{TX}$ and $W$ is a totally geodesic submanifold of $V$. In this case, for any $b \in B$, the fibre $Y_b$ is a totally geodesic submanifold of $X_b$. It means that $\nabla^{TX|_W} = \nabla^{TY, N_{Y/X}}$. 

By Remark 2.1, in this paper, we will always assume that the pairs \((T^H_{\pi} W, g^{TY})\) and \((T^H_{\pi} V, g^{TX})\) satisfy the conditions that
\[
(2.11) \quad T^H_{\pi} V|_W = T^H_{\pi} W, \quad \nabla^{TX}|_W = \nabla^{TY, N_Y/X}.
\]

2.2. Embedding of the geometric families. In this subsection, we state our assumptions on the embedding of the geometric families, which is the equivariant family case of the assumptions in [BZ93, Section 1 b)].

Let \(\mathcal{F}_Y := (W, \mathcal{E}_Y, T^H_{\pi_Y} W, g^{TY}, h^{\mathcal{E}_Y}, \nabla^{\mathcal{E}_Y})\) and \(\mathcal{F}_X := (V, \mathcal{E}_X, T^H_{\pi_Y} V, g^{TX}, h^{\mathcal{E}_X}, \nabla^{\mathcal{E}_X})\) be two equivariant geometric families over \(B\) such that the pairs \((T^H_{\pi_Y} W, g^{TY})\) and \((T^H_{\pi_Y} V, g^{TX})\) satisfy (2.11). For simplicity, we assume that \(\tau^{\mathcal{E}_Y/S} = 1\) on \(\mathcal{E}_Y\).

Assume that \((N_{Y/X}, g^{N_{Y/X}})\) has an equivariant Spin\(^c\) structure. Then there exists an equivariant complex line bundle \(L_N\) (cf. [LM89, Appendix D]) such that \(w_2(N_{Y/X}) = c_1(L_N) \mod 2\), where \(w_2\) is the second Steifel-Whitney class and \(c_1\) is the first Chern class. Let \(S(N_{Y/X}, L_N)\) be the spinor bundle for \(L_N\) which locally may be written as
\[
(2.12) \quad S(N_{Y/X}, L_N) = S_0(N_{Y/X}) \otimes L_N^{1/2},
\]
where \(S_0(N_{Y/X})\) is the spinor bundle for the (possibly non-existent) spin structure on \(N_{Y/X}\) and \(L_N^{1/2}\) is the (possibly non-existent) square root of \(L_N\). Then the \(G\)-action on \(N_{Y/X}\) and \(L_N\) lift to \(S(N_{Y/X}, L_N)\). For simplicity, we usually simply denote the spinor bundle by \(S_N\).

Let \(h^L\) be a \(G\)-invariant Hermitian metric on \(L_N\). Let \(\nabla^L\) be a \(G\)-invariant Hermitian connection on \((L_N, h^L)\). Let \(h^{S_N}\) be the equivariant Hermitian metric on \(S_N\) induced by \(g^{N_{Y/X}}\) and \(h^L\). Let \(\nabla^{S_N}\) be the equivariant Hermitian connection on \(S_N\) induced by \(\nabla^{N_{Y/X}}\) and \(\nabla^L\).

From (1.20), the bundle \(\text{End}_{C(TX)}(\mathcal{E}_X)\) is naturally \(\mathbb{Z}_2\)-graded with respect to \(\tau^{\mathcal{E}_X/S}\). Let \(\mathcal{V}\) be a smooth self-adjoint section of \(\text{End}_{C(TX)}(\mathcal{E}_X)\) such that it exchanges this \(\mathbb{Z}_2\)-grading and commutes with the \(G\)-action. Then \(\mathcal{V}\) could be extended on \(\pi^\mathcal{T}(T^B)\otimes \mathcal{E}_X\) in the same way as the perturbation operator \(\mathcal{A}\) in (1.14).

We assume that on \(\mathcal{V}\wr W\), \(\mathcal{V}\) is invertible, and that on \(W\), \(\ker \mathcal{V}\) has locally constant nonzero dimension, so that \(\ker \mathcal{V}\) is a nonzero smooth \(\mathbb{Z}_2\)-graded \(G\)-equivariant vector subbundle of \(\mathcal{E}_X|_W\). Let \(h^{\ker \mathcal{V}}\) be the metric on \(\ker \mathcal{V}\) induced by the metric \(h^{\mathcal{E}_X}|_W\). Let \(P^{\ker \mathcal{V}}\) be the orthogonal projection operator from \(\mathcal{E}_X|_W\) to \(\ker \mathcal{V}\).

For \(y \in W\), \(U \in T_y X\), let \(\partial_U \mathcal{V}(y)\) be the derivative of \(\mathcal{V}\) with respect to \(U\) in any given smooth trivialization of \(\mathcal{E}_X\) near \(y \in W\). One then verifies that \(P^{\ker \mathcal{V}} \partial_U \mathcal{V}(y) P^{\ker \mathcal{V}}\) does not depend on the trivialization, and only depends on the image \(Z\) of \(U \in T_y X\) in \(N_{Y/X}\). From now on, we will write \(\partial_Z \mathcal{V}(y)\) instead of \(P^{\ker \mathcal{V}} \partial_U \mathcal{V}(y) P^{\ker \mathcal{V}}\). Then one verifies that \(\partial_Z \mathcal{V}(y)\) is a self-adjoint element of \(\text{End}(\ker \mathcal{V})\) and exchanges the \(\mathbb{Z}_2\)-grading.

If \(Z \in N_{Y/X}\), let \(\hat{c}(Z) \in \text{End}(S_N)\) be the transpose of \(c(Z)\) acting on \(S_N\).

Denote by \(N^*_N = N^*_{Y/X} \otimes \mathbb{R} C\). Since \(L_N \otimes L_N\) is an equivariant trivial bundle, we have \(\Lambda(N^*_N) \simeq S_N \otimes S_N^*\). We equip \(\Lambda(N^*_N) \otimes \mathcal{E}_Y\) with the induced metric \(h^A(N^*_N) \otimes \mathcal{E}_Y\).

For \(Z \in N_{Y/X}\), \(\hat{c}(Z)\) acts on \(S_N \otimes S_N^* \otimes \mathcal{E}_Y\) like \(1 \otimes \hat{c}(Z) \otimes 1\).
Fundamental assumption: Let $\pi_N : N_{Y/X} \to W$ be the projection. Over the total space $N_{Y/X}$, we have the equivariant identification

$$
(2.13) \quad \left( \pi^*_N \ker V, \pi^*_N h^{\ker V}, \hat{\partial}_Z(V)(y) \right) \cong \left( \pi^*_N (\Lambda (N_N^\ast)) \otimes E_Y, \pi^*_N h^{\Lambda (N_N^\ast)} \otimes E_Y, \sqrt{-1} c(Z) \right).
$$

Let $\nabla^{\ker V}$ be the equivariant Hermitian connection on $\ker V$,

$$
(2.14) \quad \nabla^{\ker V} = D^{\ker V} \nabla|_W D^{\ker V}.
$$

We make the assumption that under the identification (2.13),

$$
(2.15) \quad \nabla^{\ker V} = \nabla^{\Lambda (N_N^\ast)} \otimes E_Y.
$$

2.3. Atiyah-Hirzebruch direct image. In this subsection, we introduce an important example of the embedding of equivariant geometric families satisfying the fundamental assumption: the equivariant version of the Atiyah-Hirzebruch direct image [AH59, FXZ09]. We assume that the base space $B$ is compact and adopt the notations and the assumptions in Section 2.1 in this subsection.

We further assume that $TY$ and $TX$ have equivariant Spin$^c$ structures. Then there exist equivariant complex line bundles $L_Y$ and $L_X$ over $W$ and $V$ such that $w_2(TY) = c_1(L_Y) \mod 2$ and $w_2(TX) = c_1(L_X) \mod 2$. Then from the splitting (2.7), the equivariant vector bundle $N_{Y/X}$ over $W$ has an equivariant Spin$^c$ structure with associated equivariant line bundle $L_N := L_X \otimes L_Y^{-1}$. Let $h^{L_Y}$, $h^{L_X}$ be $G$-invariant Hermitian metrics on $L_Y$, $L_X$ and $\nabla^{L_Y}$, $\nabla^{L_X}$ be $G$-invariant Hermitian connections on $(L_Y, h^{L_Y})$, $(L_X, h^{L_X})$. Let $h^L_N$ and $\nabla^L_N$ be metric and connection on $L_N$ induced by $h^{L_Y}$, $h^{L_X}$ and $\nabla^{L_Y}$, $\nabla^{L_X}$. Let $S(TY, L_Y)$, $S(TX, L_X)$ and $S(N_{Y/X}, L_N)$ be the spinor bundles for $(TY, L_Y)$, $(TX, L_X)$ and $(N_{Y/X}, L_N)$, which we will simply denote by $S_Y$, $S_X$ and $S_N$. Then these spinors are $G$-equivariant vector bundles. Furthermore, $S_X|_W = S_Y \otimes S_N$. Since $\dim N_{Y/X} = \dim V - \dim W$ is even, the spinor $S_N$ is $\mathbb{Z}_2$-graded.

Let $\{W_\alpha\}_{\alpha = 1, \ldots, k}$ be the connected components of $W$. Let $(\mu, h^\mu)$ be a $G$-equivariant Hermitian vector bundle over $W$ with a $G$-invariant Hermitian connection $\nabla^\mu$. In the followings, we will describe a geometric realization of the Atiyah-Hirzebruch direct image $l!|\mu| \in K^0_c(V)$ as in [AH59, FXZ09]. We denote by $\mu_\alpha$ the restriction of $\mu$ on $W_\alpha$.

For any $r > 0$, set $N_{\alpha, r} := \{Z \in N_{Y_\alpha/X} : |Z| < r\}$. Then there is $\epsilon_0 > 0$ such that the map $(y, Z) \in N_{Y_\alpha/X} \to \exp^Y_{y}(Z)$ is a diffeomorphism of $N_{\alpha, 2\epsilon_0}$ on an open $G$-equivariant tubular neighbourhood of $W_\alpha$ in $V$ for any $\alpha$. Without confusion we will also regard $N_{\alpha, 2\epsilon_0}$ as the open $G$-equivariant tubular neighbourhood of $W$ in $V$. We choose $\epsilon_0 > 0$ small enough such that for any $1 \leq \alpha \neq \beta \leq k$, $N_{\alpha, 2\epsilon_0} \cap N_{\beta, 2\epsilon_0} = \emptyset$.

Let $\pi_{\alpha} : N_{Y_\alpha/X} \to W_\alpha$ denote the projection of the normal bundle $N_{Y_\alpha/X}$ over $W_\alpha$. For $Z \in N_{Y_\alpha/X}$, let $\hat{c}(Z) \in \text{End}(S_{N_\alpha}^*)$ be the transpose of $c(Z)$ acting on $S_{N_\alpha}$. Let $\pi_{\alpha}^*(S_{N_\alpha}^*)$ be the pull back bundle of $S_{N_\alpha}^*$ over $N_{Y_\alpha/X}$. For any $Z \in N_{Y_\alpha/X}$ with $Z \neq 0$, $\hat{c}(Z) : \pi_{\alpha}^*(S_{N_\alpha}^*)|_Z \to \pi_{\alpha}^*(S_{N_\alpha}^*)|_Z$ is an equivariant isomorphisms at $Z$. 
From the equivariant Serre-Swan theorem [S68, Proposition 2.4], there exists a $G$-equivariant Hermitian vector bundle $(E_{\alpha}, hE_{\alpha})$ such that $S_{N_{\alpha}\pm}^* \otimes \mu_\alpha + E_\alpha$ is a $G$-equivariant trivial complex vector bundle over $W_\alpha$. Then
\begin{equation}
(2.16) \quad \hat{c}(Z) \oplus \pi_*^* \text{Id}_{E_{\alpha}} : \pi_*^*(S_{N_{\alpha\pm}}^* \otimes \mu_\alpha + E_\alpha) \to \pi_*^*(S_{N_{\alpha\pm}}^* \otimes \mu_\alpha + E_\alpha)
\end{equation}
induces a $G$-equivariant isomorphism between two equivariant trivial vector bundles over $N_{\alpha,2\alpha} \setminus W_\alpha$.

By adding the equivariant trivial bundles, we could assume that for any $1 \leq \alpha \neq \beta \leq k$, $\dim(S_{N_{\alpha\pm}}^* \otimes \mu_\alpha + E_\alpha) = \dim(S_{N_{\beta\pm}}^* \otimes \mu_\beta + E_\beta)$. Clearly, $\{\pi_*^*(S_{N_{\alpha\pm}}^* \otimes \mu_\alpha + E_\alpha)|_{\partial N_{\alpha,2\alpha}}\}_{\alpha=1,\ldots,k}$ extend smoothly to two equivariant trivial complex vector bundles over $V \setminus \bigcup_{1 \leq \alpha \leq k} N_{\alpha,2\alpha}$.

In summary, what we get is a $\mathbb{Z}_2$-graded Hermitian vector bundle $(\xi, h\xi)$ such that
\begin{equation}
(2.17) \quad \xi_\pm|_{N_{\alpha,\sigma_0}} = \pi_*^*(S_{N_{\alpha\pm}}^* \otimes \mu_\alpha + E_\alpha)|_{N_{\alpha,\sigma_0}},
\end{equation}
\begin{equation}
(2.18) \quad h\xi_\pm|_{N_{\alpha,\sigma_0}} = \pi_*^* \left( hS_{N_{\alpha\pm}}^* \otimes \mu_\alpha + hE_\alpha \right)|_{N_{\alpha,\sigma_0}},
\end{equation}
where $hS_{N_{\alpha\pm}}^* \otimes \mu_\alpha$ is the equivariant Hermitian metric on $S_{N_{\alpha\pm}}^* \otimes \mu_\alpha$ induced by $g_{N_{\alpha\pm}}$, $h_{L_{N_{\alpha\pm}}}$ and $h_{\mu_\alpha}$. Let $\nabla E_{\alpha}$ be a $G$-invariant Hermitian connection on $(E_{\alpha}, hE_{\alpha})$. We can also get a $G$-invariant $\mathbb{Z}_2$-graded Hermitian connection $\nabla \xi = \nabla \xi_+ \oplus \nabla \xi_-$ on $\xi = \xi_+ \oplus \xi_-$ over $V$ such that
\begin{equation}
(2.19) \quad \nabla \xi_\pm|_{N_{\alpha,\sigma_0}} = \pi_*^* \left( \nabla S_{N_{\alpha\pm}}^* \otimes \mu_\alpha \oplus \nabla E_\alpha \right),
\end{equation}
where $\nabla S_{N_{\alpha\pm}}^* \otimes \mu_\alpha$ is the equivariant Hermitian connection on $S_{N_{\alpha\pm}}^* \otimes \mu_\alpha$ induced by $\nabla_{N_{\alpha\pm}}$, $\nabla_{L_{N_{\alpha\pm}}}$ and $\nabla_{\mu_\alpha}$.

It is easy to see that there exists an equivariant self-adjoint automorphism $V$ of $S_X \hat{\otimes} \xi$, which exchanges the $\mathbb{Z}_2$-grading of $\xi$, such that
\begin{equation}
(2.20) \quad V|_{N_{\alpha,\sigma_0}} = \text{Id}_{S_X} \otimes \left( \sqrt{-1} \hat{c}(Z) \oplus \pi_*^* \text{Id}_{E_{\alpha}} \right).
\end{equation}

From the construction above, we could see that $V$ is invertible on $V \setminus W$ and
\begin{equation}
(2.21) \quad (\ker V)|_W = S_X|_W \hat{\otimes} S_N^* \otimes \mu = S_Y \hat{\otimes} S_N \hat{\otimes} S_X^* \otimes \mu = S_Y \hat{\otimes} \Lambda(N_{\xi\pm}) \otimes \mu
\end{equation}
is an equivariant vector bundle over $W$. Let $P_{\ker V}$ be the orthogonal projection from $S_X \hat{\otimes} \xi|_W$ onto ker $V$ and $\nabla_{\ker V} = P_{\ker V} \nabla S_X \hat{\otimes} \xi|_W P_{\ker V}$. From (2.11), we have
\begin{equation}
\nabla_{\ker V} = \nabla_{S_Y} \hat{\otimes} \Lambda(N_{\xi\pm}) \otimes \mu.
\end{equation}

Here $[\xi_+] - [\xi_-] \in K_G^0(V)$ is an equivariant version of the Atiyah-Hirzebruch direct image $i_\mu$ in [AH59]. In this construction, let $\xi_Y = S_Y \otimes \mu$ and $\xi_{X,\pm} = S_X \otimes \xi_{\pm}$. Then it satisfies all assumptions in Section 2.2.

2.4. Main result. In this subsection, we state our main result.

Let $F_Y$ and $F_X$ be the equivariant geometric families satisfying the assumptions in Section 2.2.

For $T \geq 0$, let $\nabla \xi_{X,T}$ be the superconnection on $\xi_X$ by
\begin{equation}
(2.22) \quad \nabla \xi_{X,T} = \nabla \xi_X + \sqrt{T} V.
\end{equation}
Let $R^E_T$ be the twisting curvature of $\nabla^E_x:T$ as in (1.32). Let $\dim(N_{X/Y}) = \ell_1$. For $T > 0$, by (1.33) and (1.34), we have the equivariant version of [BZ93, (1.17)]:

$$
\frac{\partial}{\partial T} \text{Tr}^{E_x/S} \left[ \sigma_{\ell_2}(g^{E_x}) \exp \left( -R^E_T |_{V^S} \right) \right] = -d \text{Tr}^{E_x/S} \left[ \sigma_{\ell_2}(g^{E_x}) \frac{\nu_{|V^S}}{2\sqrt{T}} \exp \left( -R^E_T |_{V^S} \right) \right].
$$

Recall that $\psi$ is the operator defined in (1.39). The proof of the following theorem is the same as those of [B95, Theorem 6.3] and [BZ93, Theorem 1.2].

**Theorem 2.2.** For any compact set $K \subset V^g$, there exists $C > 0$, such that if $\omega \in \Omega^*(V^g)$ has support in $K$,

$$
\left| \int_{X^g} \omega \cdot \frac{2^{\ell_2/2}}{\det^{1/2}(1-g|_{N_{X/Y}})} \psi_{V^S} \text{Tr}^{E_x/S} \left[ \sigma_{\ell_2}(g^{E_x}) \exp \left( -R^E_T |_{V^S} \right) \right] - \int_{Y^g} \omega \cdot \hat{A}^{-1}(N_{Y/X}, \nabla^{N_{Y/X}}) \text{ch}_g(\mathcal{E}_Y/S, \nabla^{\mathcal{E}_Y}) \right| \leq \frac{C}{\sqrt{T}} \|\omega\|_{\mathcal{E}^1(K)},
$$

and

$$
\left| \int_{X^g} \omega \cdot \psi_{V^S} \text{Tr}^{E_x/S} \left[ \sigma_{\ell_2}(g^{E_x}) \frac{\nu_{|V^S}}{2\sqrt{T}} \exp \left( -R^E_T |_{V^S} \right) \right] \right| \leq \frac{C}{T^{\ell_2/2}} \|\omega\|_{\mathcal{E}^1(K)}.
$$

Now we could extend the Bismut-Zhang current in [BZ93, Definition 1.3] to the equivariant case.

**Definition 2.3.** The equivariant Bismut-Zhang current $\gamma^X_g(\mathcal{F}_Y, \mathcal{F}_X)$ over $V^g$ is defined by

$$
\gamma^X_g(\mathcal{F}_Y, \mathcal{F}_X) = \frac{1}{2\sqrt{\pi}\sqrt{-1}} \cdot \frac{2^{\ell_2/2}}{\det^{1/2}(1-g|_{N_{X/Y}})} \cdot \int_0^\infty \psi_{V^S} \text{Tr}^{E_x/S} \left[ \sigma_{\ell_2}(g^{E_x}) \frac{\nu_{|V^S}}{2\sqrt{T}} \exp \left( -R^E_T |_{V^S} \right) \right] \frac{dT}{2\sqrt{T}}.
$$

By Theorem 2.2, the current $\gamma^X_g(\mathcal{F}_Y, \mathcal{F}_X)$ is well-defined.

Let $\delta_{W^g}$ be the current of integration over the submanifold $W^g$ in $V^g$. By integrating (2.23) and using Theorem 2.2, we have the following equivariant extension of [BZ93, Theorem 1.4].

**Theorem 2.4.** The following equation of currents holds

$$
d\gamma^X_g(\mathcal{F}_Y, \mathcal{F}_X) = \text{ch}_g(\mathcal{E}_X/S, \nabla^{\mathcal{E}_X}) - \hat{A}^{-1}(N_{Y/X}, \nabla^{N_{Y/X}}) \text{ch}_g(\mathcal{E}_Y/S, \nabla^{\mathcal{E}_Y}) \delta_{W^g}.
$$

**Remark 2.5.** Similarly as in [BZ93], the wave front set $WF(\gamma^X_g)$ of the current $\gamma^X_g$ is included in $N_{W^g/V^g}$ and $\gamma^X_g(\mathcal{F}_Y, \mathcal{F}_X)$ is a locally integrable current.

**Proposition 2.6.** Let $\mathcal{A}_Y$ be a perturbation operator with respect to $D(\mathcal{F}_Y)$. Then there exists a family of bounded pseudodifferential operator $\mathcal{A}_{T,Y}$ on $\mathcal{F}_X$, depending continuously on $T \geq 1$, such that the norm of $\mathcal{A}_{T,Y}$ is the same as that of $\mathcal{A}_Y$ for
any $T \geq 1$ and for any compact submanifold $K$ of $B$, there exists $T_0 \geq 1$ such that $TV + A_{T,Y}$ is the perturbation operator with respect to $D(F_X)$ over $K$ for $T \geq T_0$.

Proof. Following the arguments in [BL92, Section 8, 9] and [BZ93, Section 4b)] word by word, we could construct a smooth family of equivariant isometric embeddings

$$J_{T,b} : L^2(Y_b, E_Y|_{Y_b}) \to L^2(X_b, E_X|_{X_b})$$

for $b \in B$, as in [B97, Definition 9.12].

Let $E_{T,b}$ be the image of $L^2(Y_b, E_Y|_{Y_b})$ in $L^2(X_b, E_X|_{X_b})$ by $J_{T,b}$. Let $E_{T,b}^\perp$ be the orthogonal space to $E_{T,b}$ in $L^2(X_b, E_X|_{X_b})$. Since $J_{T,b}$ is an isometric embedding, $J_{T,b} : L^2(Y_b, E_Y|_{Y_b}) \to E_T$ is invertible. We extend the domain of $J_{T,b}^{-1}$ to $L^2(X_b, E_X|_{X_b})$ such that it vanishes on $E_{T,b}^\perp$.

Let $A_{T,Y} = \{A_{T,Y,b}\}_{b \in B}$ be the family of bounded pseudodifferential operators

$$A_{T,Y,b} := J_{T,b} A_{Y,b} J_{T,b}^{-1} : L^2(X_b, E_X|_{X_b}) \to L^2(X_b, E_X|_{X_b}).$$

Then $A_{T,Y}$ is a smooth family of equivariant self-adjoint operators. From the definition of the perturbation operator $A_Y$, we see that $A_{T,Y}$ commutes (resp. anti-commutes) with the $\mathbb{Z}_2$-grading $\tau^F$ of $E_X$ when the fibres are odd (resp. even) dimensional. Since $J_{T,b}$ is isometric, the $L^2$-norm of $A_{T,Y}$ is the same as that of $A_Y$.

Since $J_T$ is continuous with respect to $T$, so is the operator $A_{T,Y}$. We only need to prove that $D(F_X) + TV + A_{T,Y}$ over $K$ is invertible for $T$ large enough.

Over a compact submanifold $K$ of $B$, the same estimates of $D(F_X) + TV$ as [BL92, Theorem 9.8, 9.10, 9.11] hold. Since $D(F_Y) + A_Y$ is invertible, the arguments in [BL92, Section 9], in which we replace $D(F_Y)$ and $D(F_X) + TV$ by $D(F_Y) + A_Y$ and $D(F_X) + TV + A_{T,Y}$, imply that there exists $T_0 \geq 1$, depending on $K$, such that for any $T \geq T_0$, $D(F_X) + TV + A_{T,Y}$ is invertible. Moreover, the absolutely value of the spectrum of $D(F_X) + TV + A_{T,Y}$ has a uniformly positive lower bound for $T \geq T_0$.

The proof of our proposition is completed. \hfill $\square$

Now we state our main result of this paper.

**Theorem 2.7.** Let $A_Y$ and $A_X$ be the perturbation operators with respect to $D(F_Y)$ and $D(F_X)$. Let $A_{T,Y}$ be the operator in Proposition 2.6 with respect to $D(F_X)$. Then for any compact submanifold $K$ of $B$, there exists $T_0 > 2$ such that for any $T \geq T_0$, modulo exact forms, over $K$, we have

$$\tilde{\eta}_g(F_X, A_X) = \tilde{\eta}_g(F_Y, A_Y) + \int_{X^{T}} \hat{\Lambda}_g(TX, \nabla^{TX}) \gamma^X_g(F_Y, F_X)$$

$$+ ch_g(D(F_X) + A_X, D(F_X) + TV + A_{T,Y}).$$

Observe that since we only need to prove (2.30) over a compact submanifold, in the proof of Theorem 2.7 we may assume that $B$ is compact.

Assume that the base space is a point and $TY$, $TX$ have equivariant Spin structure. Then there exist equivariant complex vector bundles $\mu$ and $\xi_\pm$ such that $E_Y = S_Y \otimes \mu$ and $E_{X,\pm} = S_X \otimes \xi_\pm$. The following corollary is a direct consequence of Theorem 2.7.
Corollary 2.8. There exists \( x \in R(G) \), the representation ring of \( G \), such that
\[
\eta_g(X, \xi^+) - \eta_g(X, \xi^-) = \eta_g(Y, \mu) + \int_{X^g} \tilde{A}_g(TX, \nabla^TX) \gamma_g^X(F_Y, F_X) + \chi_g(x).
\]
Here \( x \) could be written as an equivariant spectral flow, \( \chi_g(x) \) is the character of \( g \) on \( x \) and \( \eta_g \) is the equivariant reduced eta invariant.

When \( g = 1 \), Corollary 2.8 is the modification of the Bismut-Zhang embedding formula by expressing the mod \( \mathbb{Z} \) term as a spectral flow. Note that in [DZ00, Theorem 4.1], the authors give an index interpretation of the mod \( \mathbb{Z} \) term of the embedding formula when the manifolds are the boundaries. It is also interesting to find the equivariant family extension of that formula.

Corollary 2.9. Let \( X \) be an odd-dimensional closed \( G \)-equivariant Spin\(^e\) manifold. For \( g \in G \), let \( (\mu, h^\mu) \) be an equivariant Hermitian vector bundle over \( X^g \) with a \( G \)-invariant Hermitian connection \( \nabla^\mu \). Then there exist a \( \mathbb{Z}_2 \)-graded equivariant Hermitian vector bundle \( (\xi, h^\xi) \) over \( X \) with a \( G \)-invariant Hermitian connection \( \nabla^\xi \) and \( x \in R(G) \), such that
\[
\eta_g(X, \xi^+) - \eta_g(X, \xi^-) = \eta_g(X^g, \mu) + \chi_g(x).
\]

Proof. Note that \( X^g \) is naturally totally geodesic in \( X \). Take \( (\xi, h^\xi, \nabla^\xi) \) as the equivariant Atiyah-Hirzebruch direct image of \( (\mu, h^\mu, \nabla^\mu) \) as in Section 2.3. We only need to notice that \( V|_{X^g} = 0 \) in this case. It implies that \( \gamma_g^X(F_{X^g}, F_X) = 0 \). □

Remark 2.10. Note that in [FL10], the authors establish an index theorem for differential K-theory. The key analytical tool is the Bismut-Zhang embedding formula of the reduced eta invariants in [BZ93]. Using Corollary 2.8, the index theorem there could be extended to the equivariant case whenever the equivariant differential K-theory is well-defined. Using Theorem 2.7, we can also get the compatibility of the push-forward map in equivariant differential K-theory along the proper submersion and the embedding under the model of Bunck-Schick [BuS09, BuS13, Liu16]. We will study these in the subsequent paper.

3. Proof of Theorem 2.7

In this section, we prove our main result Theorem 2.7. In Section 3.1, we prove Theorem 2.7 when the base space is a point using some intermediary results along the lines of [BZ93], the proof of which rely on almost identical arguments of [B95, BZ93]. In Sections 3.2, we explain how to use the functoriality to reduce Theorem 2.7 to the case in Section 3.1.

3.1. Embedding of equivariant eta invariants. In this subsection, we will prove our main result when \( B \) is a point and \( \dim X \) is odd. Recall that in (2.11), we already assume that \( Y \) is totally geodesic in \( X \).
Theorem 3.1. Assume that $B$ is a point and $\dim X$ is odd. Then there exists $T_0 > 2$ such that for any $T \geq T_0$, we have

(3.1) $\tilde{\eta}_g(F_X, A_X) = \tilde{\eta}_g(F_Y, A_Y) + \int_{X^g} \tilde{A}_g(TX, \nabla^{TX}) \gamma_g^X(F_Y, F_X) + \text{ch}_g(s_f G \{ D(F_X) + A_X, D(F_X) + TV + A_{T,Y} \}).$

Set

(3.2) $D_{u,T} = \sqrt{u}(D(F_X) + TV + \chi((1 - \chi(T))A_X + \chi(T)A_{T,Y})),

where $\chi$ is the cut-off function defined in (1.43). Let

(3.3) $B_{u^2,T} = D_{u^2,T} + dT \wedge \frac{\partial}{\partial T} + du \wedge \frac{\partial}{\partial u}.$

Definition 3.2. We define $\beta_g = du \wedge \beta^u_g + dT \wedge \beta^T_g$ to be the part of $\pi^{-1/2} \text{Tr}_s [g \exp(-B_{u^2,T})]$ of degree one with respect to the coordinates $(T,u)$, with functions $\beta^u_g, \beta^T_g : \mathbb{R}_{+} \times \mathbb{R}_{+} \times u \rightarrow \mathbb{R}$.

From (3.3), we have

(3.4) $\beta^u_g(T,u) = -\frac{1}{\sqrt{\pi}} \text{Tr}_s \left[ g D_{u^2,T} \frac{\partial}{\partial u} \exp(-D_{u^2,T}^2) \right],$

$\beta^T_g(T,u) = -\frac{1}{\sqrt{\pi}} \text{Tr}_s \left[ g D_{u^2,T} \frac{\partial}{\partial T} \exp(-D_{u^2,T}^2) \right].$

When $0 < u < 1$, $\chi(u) = 0$. In this case,

(3.5) $\beta^u_g(T,u) = -\frac{1}{\sqrt{\pi}} \text{Tr}_s \left[ g(D(F_X) + TV) \exp(-u^2(D(F_X) + TV)^2) \right],$

$\beta^T_g(T,u) = -\frac{u}{\sqrt{\pi}} \text{Tr}_s \left[ gV \exp(-u^2(D(F_X) + TV)^2) \right].$

From (3.3), we have

(3.6) $\tilde{\eta}_g(F_X, A_X) = -\int_0^{+\infty} \beta^u_g(0,u) du.$

As in [BZ93 Theorem 3.4] (see also [Liu17 Proposition 4.2]), we have

(3.7) $\left( du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T} \right) \beta_g = 0.$

Let $T_0$ be the constant in Proposition 2.6. Take $\varepsilon, A, T_1$, $0 < \varepsilon < 1 \leq A < \infty$, $T_0 \leq T_1 < \infty$. Let $\Gamma = \Gamma_{\varepsilon, A, T_1}$ be the oriented contour in $\mathbb{R}_{+} \times \mathbb{R}_{+} \times u.$
The contour $\Gamma$ is made of four oriented pieces $\Gamma_1, \ldots, \Gamma_4$ indicated in the above picture. For $1 \leq k \leq 4$, set $I^k_0 = \int_{\Gamma_k} \beta_g$. Then by Stocks' formula and (3.7),

$$
\sum_{k=1}^{4} I^k_0 = \int_{\partial U} \beta_g g = \int_{U} \left( du \wedge \frac{\partial}{\partial u} + dT \wedge \frac{\partial}{\partial T} \right) \beta_g = 0.
$$

For any $g \in G$, set

$$
\beta^Y_g(u) = \frac{1}{\sqrt{\pi}} \text{Tr} \left[ g \exp \left( - \left( u(D(F_Y) + \chi(u)A_Y) + du \wedge \frac{\partial}{\partial u} \right)^2 \right) \right] du.
$$

Then by Definition 1.4,

$$
\tilde{\eta}_g(F_Y, A_Y) = - \int_0^{+\infty} \beta^Y_g(u) du.
$$

We now establish some estimates of $\beta_g$.

**Theorem 3.3.** i) For any $u > 0$, we have

$$
\lim_{T \to \infty} \beta^u_g(T, u) = \beta^Y_g(u).
$$

ii) For $0 < u_1 < u_2$ fixed, there exists $C > 0$ such that, for $u \in [u_1, u_2]$, $T \geq 2$, we have

$$
|\beta^u_g(T, u)| \leq C.
$$

iii) We have the following identity:

$$
\lim_{T \to +\infty} \int_{2}^{\infty} \beta^u_g(T, u) du = \int_{2}^{\infty} \beta^Y_g(u) du.
$$

**Proof.** If $P$ is an operator, let Spec$(P)$ be the spectrum of $P$. From the proof of Proposition 2.6, there exist $T_0 \geq 1, c > 0$, such that for $T \geq T_0$,

$$
\text{Spec}(D(F_X) + TV + A_{T,Y}) \cap [-c, c] = \emptyset.
$$

Recall that $E^0_T$ is the image of $J_T$ defined in (2.28). For $\delta \in [0, 1]$, we write $D(F_X) + TV + \delta A_{T,Y}$ in matrix form with respect to the splitting by $E^0_T \oplus E^0_T$,

$$
D(F_X) + TV + \delta A_{T,Y} = \begin{pmatrix}
A_{T,1} + \delta A_{T,Y} & A_{T,2} \\
A_{T,3} & A_{T,4}
\end{pmatrix}.
$$
By [BL92, Theorem 9.8] and (2.29), as $T \to +\infty$, we have

\[(3.16) \quad J_T^{-1}(A_{T,1} + \delta A_{T,Y})J_T = D(F_Y) + \delta A_Y + O\left(\frac{1}{\sqrt{T}}\right).\]

Set

\[(3.17) \quad \mathcal{T} := \{\delta \in [0, 1] : D(F_Y) + \delta A_Y \text{ is not invertible}\}.

Then $\mathcal{T}$ is a closed subset of $[0, 1]$.

We firstly assume that $\mathcal{T}$ is not empty. Fix $\delta_0 \in \mathcal{T}$. There exists $C(\delta_0) > 0$ such that

\[(3.18) \quad \text{Spec}(D(F_Y) + \delta_0 A_Y) \cap [-2C(\delta_0), 2C(\delta_0)] = \{0\}.

Since the eigenvalues are continuous with respect to $\delta$, there exists $\varepsilon > 0$ small enough, such that when $\delta \in (\delta_0 - \varepsilon, \delta_0 + \varepsilon)$,

\[(3.19) \quad \text{Spec}(D(F_Y) + \delta A_Y) \cap [-C(\delta_0), C(\delta_0)] \subset (-C(\delta_0)/4, C(\delta_0)/4)

and

\[(3.20) \quad \text{Spec}(D(F_Y) + \delta A_Y) \cap (-\infty, -C(\delta_0)] \cup [C(\delta_0), +\infty) \subset (-\infty, -7C(\delta_0)/4) \cup (7C(\delta_0)/4, +\infty).

Then following the same process in [BL92, Section 9] and [BZ93, Section 4 b]) by replacing $D(F_Y)$ and $D(F_X) + TV$ by $D(F_Y) + \delta A_Y$ and $D(F_X) + TV + \delta A_{T,Y}$, for $\alpha > 0$ fixed, when $T$ is large enough, there exists $C > 0$, such that for any $\delta \in (\delta_0 - \varepsilon, \delta_0 + \varepsilon)$,

\[
\begin{align*}
&\quad [\text{Tr}_s \left[ g(D(F_X) + TV + \delta A_{T,Y}) \exp(-\alpha(D(F_X) + TV + \delta A_{T,Y})^2) \right] - \text{Tr} \left[ g(D(F_Y) + \delta A_Y) \exp(-\alpha(D(F_Y) + \delta A_Y)^2) \right] ] \leq \frac{C}{\sqrt{T}},
\quad \text{and}

&\quad [\text{Tr}_s \left[ gA_{T,Y} \exp(-\alpha(D(F_X) + TV + \delta A_{T,Y})^2) \right] - \text{Tr} \left[ gA_Y \exp(-\alpha(D(F_Y) + \delta A_Y)^2) \right] ] \leq \frac{C}{\sqrt{T}}.
\end{align*}
\]

Since $\mathcal{T}$ is compact, there exists an open neighborhood $U$ of $\mathcal{T}$ in $[0, 1]$ such that (3.21) hold uniformly for $\delta \in U$. For $\delta \in [0, 1]\setminus U$, there is a uniformly lower positive bound of the absolute value of the spectrum of $D(F_Y) + \delta A_Y$. So the process of [BL92, Section 9] also works. It means that (3.21) hold uniformly for $\delta \in [0, 1]$. If $\mathcal{T} = \emptyset$, it means that there is a uniformly lower positive bound of the absolute value of the spectrum of $D(F_Y) + \delta A_Y$ for $\delta \in [0, 1]$. Thus (3.21) holds uniformly.

In summary, for $\alpha > 0$ fixed, when $T$ is large enough, there exists $C > 0$, such that for any $\delta \in [0, 1]$, (3.21) holds.

Therefore, from Definition 3.3 and (3.2), (3.3) and (3.9), we get the proof of Theorem 3.3 i) and ii).
For $u \geq 2$ and $T \geq T_0$, from Definition 3.2, 3.14 and 3.22, we have
\begin{equation}
\beta^u_g(T, u) = -\frac{1}{\sqrt{\pi}} \text{Tr}_s \left[ g(D(F_X) + TV + A_{T,Y}) \exp(-u^2(D(F_X) + TV + A_{T,Y})^2) \right].
\end{equation}

From [BGV, Proposition 2.37], 3.14 and 3.22, there exists $C_T > 0$, depending on $T \geq T_0$, such that for $u$ large enough,
\begin{equation}
|\beta^u_g(T, u)| \leq C_T \exp(-cu^2).
\end{equation}

From the first inequality of (3.21) for $\delta = 1$, we see that $C_T$ in (3.23) is uniformly bounded. Thus iii) follows from i) and the dominated convergence theorem.

The proof of our theorem is completed.

\textbf{Theorem 3.4.} Let $T_0$ be the constant in Proposition 2.6. When $u \to +\infty$, we have
\begin{equation}
\lim_{u \to +\infty} \int_0^{T_0} \beta^T_g(T, u) dT = \text{ch}_g\{D(F_X) + A_{X}, D(F_X) + T_0V + A_{T_0,Y}\}.
\end{equation}

and
\begin{equation}
\lim_{u \to +\infty} \int_T^\infty \beta^T_g(T, u) dT = 0.
\end{equation}

\textbf{Proof.} Set
\begin{equation}
D'_{u,T} = \sqrt{u}(D(F_X) + \chi(\sqrt{u})(TV + ((1 - \chi(T))AQ + \chi(T)AR,Y)))
\end{equation}
and
\begin{equation}
\beta^T_g(T, u)' = -\frac{1}{\sqrt{\pi}} \text{Tr}_s \left[ g\frac{\partial D'_{u,T}}{\partial T} \exp(-(D'_{u,T})^2) \right].
\end{equation}

Note that when $u > 2$,
\begin{equation}
\beta^T_g(T, u)' = \beta^T_g(T, u).
\end{equation}
The proof of the anomaly formula Theorem 1.6 (cf. [Liu16, Theorem 2.17]) show that
\begin{equation}
\lim_{u \to +\infty} \int_0^{T_0} \beta^T_g(T, u) dT = \lim_{u \to +\infty} \int_0^{T_0} \beta^T_g(T, u)' dT
= \tilde{\eta}_g(F_X, A_X) - \tilde{\eta}_g(F_X, T_0V + A_{T_0,Y})
= \text{ch}_g\{D(F_X) + A_{X}, D(F_X) + T_0V + A_{T_0,Y}\}.
\end{equation}

Since $D(F_X) + TV + A_{T,Y}$ is invertible for $T \geq T_0$, the proof of (3.25) is the same as [Liu16, Theorem 2.22]. In fact, as in [Liu17 (6.8)], for $u' > 0$ fixed, there exist $C > 0$, $T' > T_0$ and $\delta > 0$ such that for $u \geq u'$ and $T \geq T'$, we have
\begin{equation}
|\beta^T_g(T, u)| \leq C_T \exp(-cu^2).
\end{equation}
The proof of Theorem 3.4 is completed.
**Theorem 3.5.** i) For any $u \in (0, 1]$, there exist $C > 0$ and $\delta > 0$ such that, for $T$ large enough, we have

\[ |\beta_g^T(T, u)| \leq \frac{C}{T^{1+\delta}}. \tag{3.31} \]

ii) There exist $C > 0, \gamma \in (0, 1]$ such that for $u \in (0, 1]$, $0 \leq T \leq u^{-1}$,

\[ \left| u^{-1} \beta_g^T(T, u) + \frac{1}{2\sqrt{\pi} \sqrt{-1}} \cdot \frac{2^{\ell_2/2}}{\det^{1/2}(1 - g|_{N_{X^g/X}})} \int_{X^g} \hat{A}_g(TX, \nabla^{TX}) \cdot \psi_{X^g} \Phi^{(T, u)} \right| \leq C(u(1 + T))^{\gamma} \sup(T, 1). \tag{3.32} \]

iii) For any $T > 0$,

\[ \lim_{u \to 0} u^{-2} \beta_g^T(T/u^2, u) = 0. \tag{3.33} \]

iv) There exist $C > 0, \delta \in (0, 1]$ such that for $u \in (0, 1], T \geq 1$,

\[ |u^{-2} \beta_g^T(T/u^2, u)| \leq \frac{C}{T^{1+\delta}}. \tag{3.34} \]

**Proof.** It is easy to see that i) follows directly from (3.30).

Note that in this theorem, $u \in (0, 1]$. By (3.5), the perturbation operator does not appear. So the proof of ii), iii), iv) here are totally the same as that of [BZ93, Theorem 3.10-3.12] except for replacing the reference of [BL92] there by the corresponding reference of [B95].

Remark that the setting of this paper uses the language of Clifford modules, not the spin case in the references. However, there is no additional difficulty for this differences. The reason is that in each proof of Theorem 3.5, ii), iii), iv), we localize the problem first. Locally, all manifolds are spin.

The proof of Theorem 3.5 is completed. \[\square\]

Now we use the estimates in Theorem 3.3, 3.4 and 3.5 to prove Theorem 3.1.

**Proof of Theorem 3.1.** From (3.3), we know that

\[ \int A \beta_g^u(T_1, u)du - \int_0^{T_1} \beta_g^T(T, A)dT - \int_{\varepsilon}^{A} \beta_g^u(0, u)du + \int_0^{T_1} \beta_g^T(T, \varepsilon)dT = I_1 + I_2 + I_3 + I_4 = 0. \tag{3.35} \]

We take the limits $A \to +\infty, T_1 \to +\infty$ and then $\varepsilon \to 0$ in the indicated order. Let $I_j, j = 1, 2, 3, 4, k = 1, 2, 3$ denote the value of the part $I_j$ after the $k$th limit.

From Theorem 3.3 (3.10) and the dominated convergence theorem, we conclude that

\[ I_3 = -\bar{\eta}_g(\mathcal{F}_Y, \mathcal{A}_Y). \tag{3.36} \]

Furthermore, by Theorem 3.3 we get

\[ I_2 = -\text{ch}_g(s|_{\mathcal{G}}\{D(\mathcal{F}_X) + \mathcal{A}_X, D(\mathcal{F}_X) + T_0\mathcal{V} + \mathcal{A}_{\bar{\eta}_0, Y}\}). \tag{3.37} \]
From (3.6), we obtain that

\[ I_3 = \tilde{\eta}(F_X, A_X). \]  

Finally, we calculate the last part. By definition,

\[ I_4^0 = \int_0^{T_1} \beta_g^T(T, \varepsilon) dT. \]  

As \( A \to +\infty \), \( I_4^0 \) remains constant and equal to \( I_4^1 \). As \( T_1 \to +\infty \), by Theorem 3.5 i),

\[ I_4^2 = \int_{0}^{+\infty} \beta_g^T(T, \varepsilon) dT = \int_{0}^{+\infty} \varepsilon^{-1} \beta_g^T(T/\varepsilon, \varepsilon) dT. \]  

Set

\[ K_1 = \int_{0}^{1} \varepsilon^{-1} \beta_g^T(T/\varepsilon, \varepsilon) dT, \]

\[ K_2 = \int_{1}^{+\infty} \varepsilon^{-2} \beta_g^T(T/\varepsilon^2, \varepsilon) dT, \]

\[ K_3 = \int_{+\infty}^{1} \varepsilon^{-2} \beta_g^T(T/\varepsilon^2, \varepsilon) dT. \]

Clearly,

\[ I_4^2 = K_1 + K_2 + K_3. \]

To simplify the notation, we denote by

\[ D(T) := \frac{1}{\sqrt{2\pi}} \frac{2^{l_2/2}}{\sqrt{-1} \det^{1/2} (1 - g|_{N X^{g}/X})} \cdot \psi_{X^{g}} T e_{X^{g}}/S \left[ \sigma_{l_2} (g|_{X^{g}}) V|_{X^{g}} \exp \left( -R_{T_2}^{X^{g}}/S|_{X^{g}} \right) \right]. \]

Then by Definition 2.3 after changing the variable, we have

\[ \gamma_X^X (F_Y, F_X) = \int_{0}^{\infty} D(T) dT. \]

As \( \varepsilon \to 0 \), by Theorem 3.5 ii),

\[ K_1 \to - \int_{X^g} \tilde{A}_g(TX, \nabla^{TX}) \cdot \int_{0}^{1} D(T) dT. \]

We write \( K_2 \) in the form

\[ K_2 = \int_{\varepsilon}^{1} \frac{T}{\varepsilon} \left\{ \varepsilon^{-1} \beta_g^T(T/\varepsilon^2, \varepsilon) + \int_{X^g} \tilde{A}_g(TX, \nabla^{TX}) D(T/\varepsilon) \right\} \frac{dT}{T} \]

\[ - \int_{X^g} \tilde{A}_g(TX, \nabla^{TX}) \int_{1}^{\varepsilon^{-1}} D(T) dT. \]
By Theorem 3.5 ii), there exist $C > 0$, $\gamma \in (0, 1]$ such that for $0 < \varepsilon \leq T \leq 1$,
\begin{equation}
(3.47) \quad \left| \frac{T}{\varepsilon} \left\{ \varepsilon^{-1} \beta^T_g (T/\varepsilon^2, \varepsilon) + \int_{X^g} \hat{A}_g (TX, \nabla^{TX}) D(T/\varepsilon) \right\} \right| \leq C \left( \varepsilon \left( 1 + \frac{T}{\varepsilon} \right) \right)^{\gamma} \leq C (2T)^{\gamma}.
\end{equation}

Using Theorem 3.5 iii), (2.25), (3.47) and the dominated convergence theorem, as $\varepsilon \to 0$,
\begin{equation}
(3.48) \quad K_2 \to - \int_{X^g} \hat{A}_g (TX, \nabla^{TX}) \int_1^{+\infty} D(T)dT.
\end{equation}

Using Theorem 3.5 iii), iv) and the dominated convergence theorem, we see that as $\varepsilon \to 0$,
\begin{equation}
(3.49) \quad K_3 \to 0.
\end{equation}

Combining Definition 2.3, (3.42), (3.45), (3.48) and (3.49), we see that as $\varepsilon \to 0$,
\begin{equation}
(3.50) \quad I_4^3 = - \int_{X^g} \hat{A}_g (TX, \nabla^{TX}) \gamma^X_{g} (F_Y, F_X).
\end{equation}

Thus (3.1) follows from (3.35)-(3.38) and (3.50).

The proof of Theorem 3.1 is completed.

3.2. Proof of Theorem 2.7 In this subsection, we use the functoriality of the equivariant eta forms Theorem 1.7 to reduce Theorem 2.7 to the case when the base manifold is a point.

Lemma 3.6. There exist a $\mathbb{Z}_2$-graded self-adjoint $C(TB)$-module $(E_B, h^{E_B})$ and a positive integer $q \in \mathbb{Z}_+$ such that
\[ \hat{A}(TB, \nabla^{TB}) \cdot \text{ch}(E_B/S, \nabla^{E_B}) - q \]
is an exact form for any Euclidean connection $\nabla^{TB}$ and Clifford connection $\nabla^{E_B}$.

Proof. Let $(E_0, h^{E_0})$ be a $\mathbb{Z}_2$-graded self-adjoint $C(TB)$-module. Let $\nabla^{E_0}$ be a Clifford connection on $(E_0, h^{E_0})$. Then since the $G$-action is trivial on $B$, from the definition of the $\hat{A}$-genus and (1.35), there exists $m \in \mathbb{Z}_+$ such that
\begin{equation}
(3.51) \quad \hat{A}(TB, \nabla^{TB}) \cdot \text{ch}(E_0/S, \nabla^{E_0}) = m + \alpha,
\end{equation}
where $\alpha \in \Omega^{\text{even}}(B)$ is a closed form and $\deg \alpha \geq 2$. Since $\alpha$ is nilpotent,
\begin{equation}
(3.52) \quad \{ \hat{A}(TB, \nabla^{TB}) \cdot \text{ch}(E_0/S, \nabla^{E_0}) \}^{-1} = \sum_{k=0}^{\infty} (-1)^k \frac{\alpha^k}{m^{k+1}}
\end{equation}
is a closed well-defined even differential form over $B$. From the isomorphism
\begin{equation}
(3.53) \quad \text{ch} : K^0(B) \otimes \mathbb{R} \xrightarrow{\sim} H^{\text{even}}(B, \mathbb{R}),
\end{equation}
\footnote{The author thanks Prof. Xiaonan Ma for pointing out this simplification, which is related to a remark in [B97, Section 7.5].}
there exist non-zero real number \( q \in \mathbb{R} \) and virtual complex vector bundle \( E = E_+ - E_- \), such that \( q^{-1} \text{ch}([E]) = [\{\hat{A}(TB, \nabla^{TB}) \text{ch}(\mathcal{E}_0/S, \nabla^{\mathcal{E}_0})\}^{-1}] \). Let \( \nabla^E \) be a connection on \( E \). Let \( \mathcal{E}_B = \mathcal{E}_0 \otimes E \) and \( \nabla^{\mathcal{E}_B} = \nabla^{\mathcal{E}_0} \otimes 1 + 1 \otimes \nabla^E \). Then \( \text{ch}(\mathcal{E}_B/S, \nabla^{\mathcal{E}_B}) = \text{ch}(\mathcal{E}_0/S, \nabla^{\mathcal{E}_0}) \text{ch}(E, \nabla^E) \). So we have
\[
[\hat{A}(TB, \nabla^{TB}) \text{ch}(\mathcal{E}_B/S, \nabla^{\mathcal{E}_B})] = q \in H^\text{even}(B, \mathbb{R}).
\]

From (3.51), we have \( q \in \mathbb{Z}_+ \).

The proof of Lemma 3.6 is completed. \( \square \)

Let \((\mathcal{E}_B, h^{\mathcal{E}_B})\) be the \( C(TB) \)-module in Lemma 3.6. Let \( \nabla^{\mathcal{E}_B} \) be a Clifford connection on \((\mathcal{E}_B, h^{\mathcal{E}_B})\). Thus
\[
\mathcal{F}_V = (V, \pi_V^* \mathcal{E}_B \otimes \mathcal{E}_X, \pi_V^* g^{TB} \otimes g^{TX}, \pi_V^* h^{\mathcal{E}_B} \otimes h^{\mathcal{E}_X}, \pi_V^* \nabla^{\mathcal{E}_B} \otimes 1 + 1 \otimes \nabla^{\mathcal{E}_X})
\]
is an equivariant geometric family over a point in \( B \). Let
\[
\mathcal{F}_{V,t} = (V, \pi_V^* \mathcal{E}_B \otimes \mathcal{E}_X, g^{TV}_t, \pi_V^* h^{\mathcal{E}_B} \otimes h^{\mathcal{E}_X}, \nabla^{\mathcal{E}_V}_t)
\]
be the rescaled equivariant geometric family over a point constructed in the same way as in (1.62).

**Lemma 3.7.** There exist \( t_0 > 0 \) and \( T' \geq 1 \), such that for any \( t \geq t_0 \) and \( T \geq T' \), the operator \( D(\mathcal{F}_{V,t}) + tTV + t\mathcal{A}_{T,Y} \) is invertible.

**Proof.** Let \( f_1, \cdots, f_t \) be a locally orthonormal basis of \( TB \). Let \( f_p^H \) be the horizontal lift of \( f_p \) on \( T^H \). Let \( e_1, \cdots, e_n \) be a locally orthonormal basis of \( TX \). Set
\[
D_t^B = c(f_p)\nabla_{f_p}^{\mathcal{E}_B} + \frac{1}{8t} \langle [f_p^H, f_q^H], e_i \rangle c(e_i) c(f_p) c(f_q).
\]
By [Liu17] (5.6), we have
\[
D(\mathcal{F}_{V,t}) + tTV + t\mathcal{A}_{T,Y} = t(D(\mathcal{F}_X) + TV + \mathcal{A}_{T,Y}) + D_t^B.
\]

From Proposition 2.19, since \( B \) is compact, there exist \( c > 0 \) and \( T' > 0 \), such that for any \( s \in \Lambda(T^*B) \otimes \mathcal{E}_X, T \geq T' \),
\[
\| (D(\mathcal{F}_X) + TV + \mathcal{A}_{T,Y}) s \|_0^2 \geq c^2 \| s \|_0^2.
\]
Let
\[
R_{t,T} := t[D(\mathcal{F}_X) + TV + \mathcal{A}_{T,Y}, D_t^B] + D_t^B.
\]
We have
\[
(D(\mathcal{F}_{V,t}) + tTV + t\mathcal{A}_{T,Y})^2 = t^2 (D(\mathcal{F}_X) + TV + \mathcal{A}_{T,Y})^2 + R_{t,T}.
\]

Let \( | \cdot |_{T,1} \) be the norm defined in the same way as [397] Definition 9.13. In particular,
\[
\| s \|_0 \leq | s |_{T,1}.
\]
Note that the perturbation operator $A_{T,Y}$ is uniformly bounded with respect to $T \geq 1$. From the arguments in the proof of [B97, Theorem 9.14], we could obtain that there exist $C_1, C_2, C_3 > 0$, such that for $T \geq 1$, $t \geq 1$, $s \in \Lambda(T^* B) \otimes \mathcal{E}_X$,

$$
\|((D(F_X) + TV + A_{T,Y})s\|_0^2 \geq C_1 |s|_{T,1}^2 - C_2 \|s\|_0^2,
$$

(3.62)

$$
|R_t ts, s\|_0 \leq C_3 t \|s\|_0 \cdot |s|_{T,1}.
$$

Take $\alpha = c^2/(c^2 + 2C_2)$. By (3.58)–(3.62), for $T \geq T'$, $t \geq 1$, we have

$$
\|(D(F_{V,t}) + tTV + tA_{T,Y})s\|_0^2 = |(t^2 (D(F_X) + TV + A_{T,Y})s + R_{t,t} s, s)_{\alpha}|
\geq (1 - \alpha)^2 \|(D(F_X) + TV + A_{T,Y})s\|_0^2 + \alpha t^2 \|(D(F_X) + TV + A_{T,Y})s\|_0^2 - |(R_{t,t} s, s)_{\alpha}|
\geq (1 - \alpha)c^2 t^2 \|s\|_0^2 + \alpha C_1 t^2 \|s\|_0^2 - \alpha C_2 t^2 \|s\|_0 \cdot |s|_{T,1}
\geq \alpha C_2 t^2 \|s\|_0^2 + t(\alpha C_1 t - C_3)|s|_{T,1}^2.
$$

(3.63)

Take $t_0 = \max\{2C_3/\alpha C_1, 1\}$. For any $t \geq t_0$, $T \geq T'$, there exists $C > 0$, such that

$$
\|(D(F_{V,t}) + tTV + tA_{T,Y})s\|_0^2 \geq C t^2 \|s\|_0^2.
$$

(3.64)

Since $D(F_{V,t}) + tTV + tA_{T,Y}$ is self-adjoint, by (3.64), it is surjective. Thus $D(F_{V,t}) + tTV + tA_{T,Y}$ is invertible.

The proof of Lemma 3.7 is completed.

Let

$$
F_W = (W, \pi^*_W \mathcal{E}_B \otimes \mathcal{E}_Y, \pi^*_B g^{TB} \otimes g^{TY}, \pi^*_W h^{EB} \otimes h^{EY}, \pi^*_W \nabla^{EB} \otimes 1 \otimes \nabla^{EY})
$$

be the equivariant geometric family over a point in $B$. Let

$$
F_{W,t} = (W, \pi^*_W \mathcal{E}_B \otimes \mathcal{E}_Y, g_{t,t}^{TW}, \pi^*_V h^{EB} \otimes h^{EY}, \nabla^{EY}_{t,t})
$$

be the rescaled equivariant geometric family constructed in the same way as in (1.62) and (3.55).

Firstly, we assume that $B$ is closed. Let $t_0$ be the constant taking in Lemma 3.7. We may assume that when $t \geq t_0$, $D(F_{W,t}) + 1 \otimes tA_Y$ is invertible by the arguments before Theorem 1.7. By Theorem 1.7 and Lemma 3.7, we have

$$
\tilde{\eta}_g(F_{W,t_0}, 1 \otimes t_0 A_Y) = \int_B \hat{A}(TB, \nabla^{TB}) ch(\mathcal{E}_B / \mathcal{S}, \nabla^{EB}) \tilde{\eta}_g(F_Y, A_Y)
$$

$$
- \int_{W^s} \left( \hat{A}_g - \chi_{\hat{g}} \right) \left( \nabla^{TW}_{t_0}, \nabla^{TB, TY}, \nabla^{EY}_{t_0}, 0 \nabla^{EY} \right)
$$

and

$$
\tilde{\eta}_g(F_{V,t_0}, 1 \otimes (t_0 T^Y + t_0 A_{T,Y}))
$$

$$
= \int_B \hat{A}(TB, \nabla^{TB}) ch(\mathcal{E}_B / \mathcal{S}, \nabla^{EB}) \tilde{\eta}_g(F_X, t^Y + A_{T,Y})
$$

$$
- \int_{Y^s} \left( \hat{A}_g - \chi_{\hat{g}} \right) \left( \nabla^{TV}_{t_0}, \nabla^{TB, TX}, \nabla^{EY}_{t_0}, 0 \nabla^{EY} \right).
$$
Set

(3.69) \[ \Delta_B = \tilde{\eta}_g(\mathcal{F}_X, T'\mathcal{V} + A_{T', \mathcal{V}}) - \tilde{\eta}_g(\mathcal{F}_Y, A_Y) \]

\[ - \int_{X_g} (T X, \nabla^{TX} X) \gamma_g(X, F_X) \in \Omega^*(B, \mathbb{C})/\text{Im}. \]

From [BGV] (1.17), Theorem 2.4 and (1.47), , we have \( d^B \Delta_B = 0 \).

Recall that \( \mathcal{V} \in \text{End}_C(T_X)(\mathcal{E}_X) \) satisfies the fundamental assumption (2.13) with respect to \( \mathcal{A}_Y \) and \( \mathcal{F}_X \). Let \( 1 \circ \mathcal{V} \) is the extension of \( \mathcal{V} \) on \( \pi_1^t \mathcal{E}_B \otimes \mathcal{E}_X \) in the same way as \( \mathcal{A} \) in (1.63). Then \( 1 \circ \mathcal{V} \) satisfies the fundamental assumption (2.13) with respect to \( \mathcal{F}_W \) and \( \mathcal{F}_V \). Furthermore, \( 1 \circ \mathcal{V} \) satisfies the fundamental assumption (2.13) with respect to \( \mathcal{F}_W, \mathcal{V}_t \) and \( \mathcal{F}_V, \mathcal{V}_t \). Observe that \( \gamma^V \) (\( \mathcal{F}_{W,t}, \mathcal{F}_{V,t} \)) does not depend on \( t \). We also denote it by \( \gamma^V \left( \mathcal{F}_W, \mathcal{F}_V \right) \).

From Theorem 3.1 if \( \dim V \) is odd, there exists \( T_0 > 0 \) such that

(3.70) \[ \tilde{\eta}_g(\mathcal{F}_{V,t_0}, A_Y) = \tilde{\eta}_g(\mathcal{F}_{W,t_0}, 1 \circ t_0 A_Y) + \int_{V_g} \hat{\Lambda}_g(T X, \nabla^{TX}_{t_0}) \gamma^X_g(\mathcal{F}_W, \mathcal{F}_V) \]

\[ + \text{ch}_g(s (D(\mathcal{F}_{V,t_0}) + A_Y, D(\mathcal{F}_{V,Y}) + 1 \circ (T_0 t_0 \mathcal{V} + t_0 A_{t_0,Y}))). \]

We may assume that \( T_0 \geq T' \), which is determined in Lemma 3.7. By anomaly formula Theorem 1.6, we have

(3.71) \[ \tilde{\eta}_g(\mathcal{F}_{V,t_0}, 1 \circ (T_0 t_0 \mathcal{V} + t_0 A_{t_0,Y})) = \tilde{\eta}_g(\mathcal{F}_{W,t_0}, 1 \circ t_0 A_Y) \]

\[ + \int_{V_g} \hat{\Lambda}_g(T V, \nabla^{TV}_{t_0}) \gamma^V_g(\mathcal{F}_W, \mathcal{F}_V). \]

Note that if \( \dim V \) is even, (3.71) also holds, because in this case all terms in (3.71) vanish.

From the anomaly formula Theorem 1.6, Lemma 3.7 and 3.71, for \( t > t_0 \), we have

(3.72) \[ \int_{V_g} \left( \tilde{\Lambda}_g \cdot \text{ch}_g \right) (\nabla^{TV}_{t_0}, \nabla^{TV}_t, \nabla^{E_Y}_{t_0}, \nabla^{E_Y}_t) \]

\[ = \int_{W_g} \left( \tilde{\Lambda}_g \cdot \text{ch}_g \right) (\nabla^{TW}_{t_0}, \nabla^{TW}_{t}, \nabla^{E_W}_{t_0}, \nabla^{E_W}_t) \]

\[ - \int_{V_g} \hat{\Lambda}_g(T V, \nabla^{TV}_{t_0}) \gamma^V_g(\mathcal{F}_W, \mathcal{F}_V) \]

\[ + \int_{V_g} \hat{\Lambda}_g(T V, \nabla^{TV}_{t_0}) \gamma^V_g(\mathcal{F}_W, \mathcal{F}_V). \]

Note that locally the manifolds are spin. From [Liu17] Proposition 4.5 and the arguments in [Liu17] Section 5.5], we have

(3.73) \[ \lim_{t \to +\infty} \left( \tilde{\Lambda}_g \cdot \text{ch}_g \right) (\nabla^{TV}_{t_0}, \nabla^{TV}_t, \nabla^{E_Y}_{t_0}, \nabla^{E_Y}_t) \]

\[ = \left( \tilde{\Lambda}_g \cdot \text{ch}_g \right) (\nabla^{TV}_{t_0}, \nabla^{TB,TX}_{t_0}, \nabla^{E_Y}_{t_0}, 0 \nabla^{E_Y}_t) \]
and

\[
\lim_{t \to +\infty} \hat{A}_g(TV, \nabla_t TV) = \hat{A}_g(TV, \nabla^{TB} TX) = \pi_V^* \hat{A}(TB, \nabla^{TB}) \cdot \hat{A}_g(TX, \nabla^{TX}).
\]

By Definition 2.3, we have

\[
\gamma^V_g(\mathcal{F}_W, \mathcal{F}_V) = \text{ch}(\mathcal{E}_B / \mathcal{S}, \nabla^\mathcal{E}_B) \gamma^X_g(\mathcal{F}_Y, \mathcal{F}_X).
\]

From Lemma 3.6 and (3.74)-(3.75), if \( B \) is closed, we have

\[
\int_B \Delta_B = q^{-1} \cdot \int_B \hat{A}(TB, \nabla^{TB}) \cdot \text{ch}(\mathcal{E}_B / \mathcal{S}, \nabla^\mathcal{E}_B) \cdot \Delta_B = 0.
\]

In general, \( B \) is not necessary closed. Let \( K \) be a closed submanifold of \( B \). Let \( \mathcal{F}_Y|_K \) and \( \mathcal{F}_X|_K \) be the restrictions of \( \mathcal{F}_Y \) and \( \mathcal{F}_X \) on \( K \). Let \( T_0 \geq 1 \) be the constant determined in Proposition 2.6 associated with \( B \). Then \( (TV + A_{T,Y})|_K \) is the perturbation operator with respect to \( D(\mathcal{F}_X|_K) \) over \( K \) for \( T \geq T_0 \). Set

\[
\Delta_K = \tilde{\eta}_g(\mathcal{F}_X|_K, (T_0 V + A_{T_0 Y})|_K) - \tilde{\eta}_g(\mathcal{F}_Y|_K, A|_K)
+ \int_{X_K} \hat{A}_g(TX, \nabla^{TX}) \gamma^X_g(\mathcal{F}_Y|_K, \mathcal{F}_X|_K) \in \Omega^*(K, \mathbb{C})/	ext{Im}d.
\]

From Definition 1.4 and 2.3, we could see that \( \int_K \Delta_B = \int_K \Delta_K \).

On the other hand, from (3.76), we have \( \int_K \Delta_K = 0 \). So for any closed submanifold \( K \) of \( B \), we have

\[
\int_K \Delta_B = 0.
\]

From the arguments in [DR73, §21, §22], we obtain that \( \Delta_B \) is exact on \( B \). Therefore, we obtain Theorem 2.7 from the anomaly formula Theorem 1.6.

The proof of our main result is completed.

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**Department of Mathematics, East China Normal University, 500 Dongchuan Road, Shanghai, 200241 P.R. China**

*E-mail address*: boliumath@outlook.com