Abstract

We apply the dimensional regularization approach to study the poles in the partition function and the mean energy, that appear for specific, discrete q-values, in Tsallis’ statistics. We study the thermodynamic behavior there. The analysis is made in one, two, and three dimensions.

Keywords: Divergences, regularization, entropy, partition function.
1 Introduction

Divergences are an important topic in theoretical physics. Indeed, the study and elimination of divergences of a physical theory is perhaps one of the most important aspects of theoretical physics. The quintessential typical example is the attempt to quantify the gravitational field, which so far has not been achieved. Some examples of elimination of divergences can be seen in references [1, 2, 3, 4, 5].

There are several methods for eliminating divergences of a physical theory, and certainly the most well known (and used) is the one called \textit{dimensional regularization} [1, 2, 3], that consists in appealing to movable spatial dimensions for which the theory is defined as a regulator to eliminate its divergences. It is a method introduced by Bollini and Giambiagi [1] for regularizing integrals in the evaluation of Feynman diagrams. One assigns to them values that are meromorphic functions of an auxiliary complex parameter \(\nu\), called the dimension. We remind the reader that a meromorphic function is a single-valued one that is analytic in all but possibly a discrete subset of its domain, and at those singularities it must diverge like a polynomial (that is, these exceptional points must be poles and not essential singularities). Dimensional regularization casts a Feynman integral as one depending on i) the space-time dimension \(\nu\) and ii) the squared distances \((x_i - x_j)^2\) of the space-time points \(x_i\) appearing in it [6].

In an Euclidean space such integral often converges for \(Re(\nu)\) negative and large enough, and can be analytically continued from this region to a meromorphic function defined for all complex \(\nu\). In general, there will be a pole at the physi-
cal value (usually four) of \( \nu \), which requires to be cancelled by regularization to obtain physical quantities. Note that the parameter \( \nu \) appearing in dimensional regularization is a complex number not to be confused with the dimension of space (usually 4). Now, that if \( \nu \) happens to be a positive integer, then the equation for the dimensionally regularized integral is correct for a space-time of dimension \( d \). Let us point out, for instance, that the surface area of a unit \( \nu - 1 \)–sphere is \( 2\pi^{\nu/2}/\Gamma(\nu/2) \). Accordingly, in dimensional regularization one asserts that we are speaking of the surface area of a sphere in \( \nu \) dimensions, even if \( \nu \) is not an integer. Of course, there are no spheres in non-integral dimensions.

For computing a logarithmic-divergent loop integral in four dimensions, let us say, \( \int \frac{d^\nu p}{(2\pi)^\nu} \frac{1}{|p^2 + m^2|^2} \), we recast firstly the integral in some manner in which the number integration-variables become independent of \( \nu \). Afterwards, one varies \( d \) so as to consider non-integral values of the form \( \nu = 4 - \epsilon \), with the result

\[
\int_0^\infty \frac{dp}{(2\pi)^{4-\epsilon}} \frac{2^{7-\epsilon/2}}{\Gamma(2-\epsilon/2)} \frac{p^{3-\epsilon}}{|p^2 + m^2|^2} = m^{-\epsilon} \frac{2^{4+\epsilon} \pi^{\epsilon/2}}{\sin(\pi\epsilon/2)\Gamma(1-\epsilon/2)}. \tag{1.1}
\]

The idea of dimensional continuation has been quite early employed in statistical mechanics \[7\]. Later Wilson and Fisher discovered the \( \epsilon \)-expansion \[8\], and applied it to field-theoretic methods in statistical mechanics \[9, 10\].

In this work we will apply the dimensional regularization methodology to non-orthodox statistical theories that appeal to entropic forms other than the logarithmic Boltzmann’s one. Foremost amongst them is Tsallis q-statistics \[11, 12\], that has been recently shown to exhibit divergences in the partition function, the mean energy, and the entropy, for special values of Tsallis’ non-extensivity.
parameter $q$ [13] [14] [15]. It has been shown (see for instance, [16] [17]) that the Tsallis q-statistics is of great importance for dealing with some astrophysical issues involving self-gravitating systems [18]. Moreover, this statistics has proved its utility in variegated scientific fields, with several thousands publications and authors [12], so that studying its structural features is an important issue for physics, astronomy, biology, neurology, economics, etc. [11].

2 The divergences of q-statistics

As we have shown in [13], the partition function of the classical Harmonic Oscillator (HO) in $\nu$ dimensions can be written in the form

$$Z = \frac{\pi^\nu}{\Gamma(\nu)} \int_0^\infty \frac{u^{\nu-1}}{[1 + \beta(q - 1)u]^{1/q - 1}} du$$  \hspace{1cm} (2.1)

The result of integral (2.1) is, according to [22],

$$Z = \frac{\pi^\nu}{[\beta(q - 1)]^\nu} \frac{\Gamma \left( \frac{1}{q-1} - \nu \right)}{\Gamma \left( \frac{1}{q-1} \right)}$$  \hspace{1cm} (2.2)

This result is valid for $q \neq 1$ and we have selected $1 \leq q < 2$. Of course, $q = 1$ is the Boltzmann statistics instance, for which the q-exponential transforms itself into the ordinary exponential function (and the integral (2.1) is convergent).

We restrict herefrom the values of $q$ to $1 < q < 2$ and, according to (2.2), the singularities (divergences) of (2.1) are given by the poles of the $\Gamma$ function that appears in the numerator of (2.2), i.e., for

$$\frac{1}{q - 1} - \nu = -p \text{ for } p = 0, 1, 2, 3, \ldots,$$
or, equivalently, for

\[ q = \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots, \frac{\nu}{\nu - 1}, \frac{\nu + 1}{\nu} \]

In a similar way, we have for the mean energy of the HO,

\[ <U> = \frac{\pi^\nu}{\Gamma(\nu) Z} \int_0^\infty \frac{u^\nu}{[1 + \beta(q-1)u]^{1/q-1}} du \]  \hspace{1cm} (2.3)

The result of (2.3) is, using (22) once again,

\[ <U> = \frac{\nu \pi^\nu}{Z[\beta(q-1)]^{\nu+1}} \frac{\Gamma\left(\frac{1}{q-1} - \nu - 1\right)}{\Gamma\left(\frac{1}{q-1}\right)} \]  \hspace{1cm} (2.4)

where we assume that \( Z \) is the physical partition function, which has no singularities. In this case, the singularities of (2.4) are given by:

\[ \frac{1}{q-1} - \nu - 1 = -p \quad \text{for } p = 0, 1, 2, 3, \ldots, \]

or, equivalently,

\[ q = \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots, \frac{\nu + 1}{\nu}, \frac{\nu + 2}{\nu + 1}. \]

As usual [19], in terms of the so-called q-logarithms [11] \( \ln_q(x) = \frac{x^{1-q} - 1}{1-q} \), the entropy is cast in the fashion

\[ S = Z^{q-1} [\ln_q Z + \beta <U>] \]  \hspace{1cm} (2.5)

and it is finite if \( Z \) and \( <U> \) are also finite.

Our purpose here is then to derive, for the classical HO, physical thermodynamical variables \( Z, <U>, \) and \( S \), using dimensional regularization. As an illustration, we specify things for the cases of dimensions one, two, and three.
3 The one-dimensional case

In one dimension $Z$ is regular and $\langle U \rangle$ has a singularity at $q = \frac{3}{2}$. For $q \neq \frac{3}{2}$, $Z$, $\langle U \rangle$, and $S$ can be easily evaluated. The result is straightforward

$$Z = \frac{\pi}{\beta(2-q)},$$  \hspace{1cm} (3.1)

$$\langle U \rangle = \frac{1}{\beta(3-2q)},$$ \hspace{1cm} (3.2)

$$S = \left[ \frac{\pi}{\beta(2-q)} \right]^{q-1} \left\{ \ln q \left[ \frac{\pi}{\beta(2-q)} \right] + \frac{1}{3} - \frac{2q}{3} \right\}. $$ \hspace{1cm} (3.3)

According to (3.2), in the regular case, as $\langle U \rangle \geq 0$, one should have $q < \frac{3}{2}$.

When $q = \frac{3}{2}$ we have for $Z$

$$Z = \frac{2\pi}{\beta}. \hspace{1cm} (3.4)$$

Regularization is needed for $\langle U \rangle$, as a function of the dimension. One has

$$\langle U \rangle = \frac{2^{\nu+1} \nu \pi^\nu}{Z \beta^{\nu+1}} \Gamma(1 - \nu).$$ \hspace{1cm} (3.5)

This can be written as

$$\langle U \rangle = -\frac{1}{\pi Z} \left( \frac{2\pi}{\beta} \right)^{\nu+1} \Gamma(2 - \nu) + \frac{1}{\pi Z} \left( \frac{2\pi}{\beta} \right)^{2} \left( \frac{2\pi}{\beta} \right)^{\nu-1} \Gamma(1 - \nu)$$ \hspace{1cm} (3.6)

The first term of (3.6) is finite, while the second one is singular for $\nu = 1$. By recourse to a Taylor’s expansion around $\nu = 1$ of the factor depending on $\nu - 1$, that multiplies the $\Gamma$ function in the second term, we have

$$\langle U \rangle = -\frac{1}{\pi Z} \left( \frac{2\pi}{\beta} \right)^{\nu+1} \Gamma(2 - \nu) + \frac{1}{\pi Z} \left( \frac{2\pi}{\beta} \right)^{2} \times$$

$$\left[ 1 + (\nu - 1) \ln \left( \frac{2\pi}{\beta} \right) + \frac{(\nu - 1)^2}{2} \ln^2 \left( \frac{2\pi}{\beta} \right) + \cdots \right] \Gamma(1 - \nu)$$ \hspace{1cm} (3.7)
The sum of the first term plus the limit, for $\nu \to 1$, of the term multiplied by $\nu - 1$ in the expansion, gives the physical value of $<U>$

\[ <U> = -\frac{2}{\beta} \left[ 1 + \ln \left( \frac{2\pi}{\beta} \right) \right] \]  

(3.8)

From (3.4) and (3.8) we obtain the physical value for $S$:

\[ <S> = -2 \left[ 1 + \sqrt{\frac{2\pi}{\beta} \ln \left( \frac{2\pi}{\beta} \right)} \right] \]  

(3.9)

Since the mean energy must be positive, according to (3.8) the possible values of $\beta$ are restricted by the constraint $\beta > 2\pi e$, entailing $T < 1/2\pi ek_B$, with $k_B$ Boltzmann’s constant. There is an upper bound to the physical temperature, which cannot be infinite. This agrees with the considerations made in [20]: q-statistics refers to systems in thermal contact with a finite bath.

On a more conjectural fashion, one is also reminded here of the Hagedorn temperature. This is the temperature at which ordinary matter is no longer stable and would evaporate, transforming itself into quark matter, a sort of boiling point of hadronic matter. This temperature would exist on account of the fact that the accessible energy would be so high that quark-antiquark pairs would be spontaneously extracted from the vacuum. A putative system at such a high temperature is able to accommodate any amount of energy because the newly emerging quarks would provide additional degrees of freedom. The Hagedorn temperature would thus be unsurmountable [21].
4 The two-dimensional case

For two dimensions, $Z$ has a singularity at $q = \frac{3}{2}$ and $<\mathcal{U}>$ has singularities at $q = \frac{3}{2}$ and $q = \frac{4}{3}$. Save for the case of these singularities, we can evaluate their values of the main statistical quantities without the use of dimensional regularization. Thus, we obtain

$$Z = \frac{\pi^2}{\beta^2(2-q)(3-2q)}, \quad (4.1)$$

$$<\mathcal{U}> = \frac{2}{\beta(4-3q)}, \quad (4.2)$$

$$S = \left[\frac{\pi^2}{\beta^2(2-q)(3-2q)}\right]^{q-1} \left\{ \ln_q \left[\frac{\pi^2}{\beta^2(2-q)(3-2q)}\right] + \frac{2}{4-3q} \right\}. \quad (4.3)$$

According to (4.2), in the regular case $q < \frac{4}{3}$.

4.1 The $q = 3/2$ pole

For $q = \frac{3}{2}$ we must regularize $Z$ and $\mathcal{U}$. We start with $Z$. From (2.2) we have

$$Z = \left(\frac{2\pi}{\beta}\right)^\nu \Gamma(2 - \nu), \quad (4.4)$$

which can be rewritten as

$$Z = \left(\frac{2\pi}{\beta}\right)^2 \left(\frac{2\pi}{\beta}\right)^{\nu-2} \Gamma(2 - \nu). \quad (4.5)$$

With this form for $Z$, we can expand in Taylor’s series around $\nu = 2$ the factor multiplying the function $\Gamma$

$$Z = \left(\frac{2\pi}{\beta}\right)^2 \Gamma(2 - \nu) \left[ 1 + (\nu - 2) \ln \left(\frac{2\pi}{\beta}\right) \cdots \right], \quad (4.6)$$

and thus we obtain the physical value of $Z$ as

$$Z = -\frac{4\pi^2}{\beta^2} \ln \left(\frac{2\pi}{\beta}\right). \quad (4.7)$$
For $\mathcal{U}$ the situation is similar. From (2.4) we have

$$<\mathcal{U}> = \frac{\nu}{\pi} \left( \frac{2\pi}{\beta} \right)^{\nu+1} \Gamma(1-\nu),$$  \hspace{1cm} (4.8)

where $\mathcal{Z}$ is given by (4.7). Proceeding in the same way as for the partition function, we rewrite $<\mathcal{U}>$ in the fashion

$$<\mathcal{U}> = \frac{\Gamma(3-\nu)}{\beta^3} \left( \frac{2\pi}{\beta} \right)^{\nu+1} + \frac{2}{\beta^3} \left( \frac{2\pi}{\beta} \right)^3 \left( \frac{2\pi}{\beta} \right)^{\nu-2} \Gamma(2-\nu),$$  \hspace{1cm} (4.9)

and we obtain the physical value of $<\mathcal{U}>$:

$$<\mathcal{U}> = \frac{8\pi^2}{\beta^3} + \frac{16\pi^2}{\beta^3} \ln \left( \frac{2\pi}{\beta} \right),$$  \hspace{1cm} (4.10)

so that replacing $\mathcal{Z}$ by the value given in (4.7) we have

$$<\mathcal{U}> = \frac{2}{\beta(\ln \beta - \ln 2\pi) + 4}.$$  \hspace{1cm} (4.11)

From (4.11) we see that the possible values of $\beta$ are given by $\beta > 2\pi$. Now, from the physical values of $\mathcal{Z}$ and $<\mathcal{U}>$, as given by (4.7) and (4.11), respectively, and from (2.5), we find the physical value of $\mathcal{S}$ as

$$\mathcal{S} = \sqrt{\frac{4\pi^2}{\beta^2} \ln \left( \frac{\beta}{2\pi} \right) \left( \ln^2 \left( \frac{4\pi^2}{\beta^2} \ln \left( \frac{\beta}{2\pi} \right) \right) + \frac{2}{(\ln \beta - \ln 2\pi) + 4} \right)}.$$  \hspace{1cm} (4.12)

### 4.2 The $q = 4/3$ pole

For $q = \frac{4}{3}$, $\mathcal{Z}$ is finite and $<\mathcal{U}>$ has a pole. The procedure for finding their physical values is similar to that for the case $q = \frac{3}{2}$. For this reason, we only indicate the results obtained for $\mathcal{Z}$, $<\mathcal{U}>$, and $\mathcal{S}$. One finds

$$\mathcal{Z} = \frac{9\pi^2}{2\beta^2}.$$  \hspace{1cm} (4.13)
\[ <U> = \frac{6}{\beta} \left[ \ln \left( \frac{\beta}{3\pi} \right) - \frac{1}{2} \right], \quad (4.14) \]

\[ S = \left( \frac{9\pi^2}{2\beta^2} \right)^{\frac{3}{4}} \left[ \ln 2 + \frac{9\pi^2}{2\beta^2} + 6 \ln \left( \frac{\beta}{3\pi} \right) - 3 \right]. \quad (4.15) \]

From (4.14) we see that the possible values of \( \beta \) are given by the constraint \( \beta > 3\pi \sqrt{e}. \)

5 The three-dimensional case

In three dimensions, \( Z \) has poles at \( q = \frac{3}{2} \) and \( q = \frac{4}{3} \) while \( <U> \) exhibits them at \( q = \frac{3}{2}, q = \frac{4}{3}, \) and \( q = \frac{5}{4} \). Consequently, when regularity prevails we have

\[ Z = \frac{\pi^3}{\beta^3}(2-q)(3-2q)(4-3q), \quad (5.1) \]

\[ <U> = \frac{3}{\beta(5-4q)}. \quad (5.2) \]

From (5.1) and (5.2) we obtain for the entropy

\[ S = \left[ \frac{\pi^3}{\beta^3(2-q)(3-2q)(4-3q)} \right]^{q^{-1}} \left\{ \ln_q \left[ \frac{\pi^3}{\beta^3(2-q)(3-2q)(4-3q)} \right] + \frac{3}{5-4q} \right\} \quad (5.3) \]

In this case \( q \) should satisfy the condition \( q < \frac{5}{4} \) for the mean energy to be a positive quantity.

5.1 The \( q = 3/2 \) pole

For \( q = \frac{3}{2} \) we have

\[ Z = \left( \frac{2\pi}{\beta} \right)^\nu \Gamma(2 - \nu). \quad (5.4) \]
Proceeding as in the previous cases and making now the Taylor’s expansion around \( \nu = 3 \), \( Z \) acquires the appearance

\[
Z = \left( \frac{2\pi}{\beta} \right)^3 \frac{\Gamma(3 - \nu)}{2 - \nu} \left[ 1 + (\nu - 3) \ln \left( \frac{2\pi}{\beta} \right) + \cdots \right].
\]

(5.5)

From (5.5) it is easy to obtain the physical value of \( Z \) as

\[
Z = \frac{8\pi^3}{\beta^3} \ln \left( \frac{2\pi}{\beta} \right).
\]

(5.6)

In a similar vein have for \( \langle U \rangle \)

\[
\langle U \rangle = \frac{1}{\beta(\ln \beta - \ln 2\pi)} - \frac{3}{\beta} - \frac{1}{\ln \beta - \ln 2\pi - 3}.
\]

(5.7)

and from (5.6) and (5.7)

\[
\mathcal{S} = \sqrt{\frac{8\pi^3}{\beta^3} \ln \left( \frac{2\pi}{\beta} \right)} \left\{ \ln \frac{1}{2} \left[ \frac{8\pi^3}{\beta^3} \ln \left( \frac{2\pi}{\beta} \right) \right] + \frac{1}{\ln \beta - \ln 2\pi - 3} \right\},
\]

(5.8)

with \( 2\pi < \beta < 2\pi e^{1/3} \). This entails that the system exhibits positive entropy only for a small range of very high temperatures.

5.2 The \( q = 4/3 \) and \( q = 5/4 \) poles

For \( q = \frac{4}{3} \) and \( q = \frac{5}{4} \) we give only the corresponding results, since the calculations are entirely similar to those for the case \( q = \frac{3}{2} \). Thus, for \( q = \frac{4}{3} \) we have

\[
Z = \frac{27\pi^3}{2\beta^3} \ln \left( \frac{\beta}{3\pi} \right),
\]

(5.9)

\[
\langle U \rangle = \frac{3}{\beta(\ln \beta - \ln 3\pi)} - \frac{9}{\beta},
\]

(5.10)

\[
\mathcal{S} = \left\{ \frac{27\pi^3}{2\beta^3} \ln \left( \frac{\beta}{3\pi} \right) \right\}^{1/2} \left\{ \ln \frac{1}{2} \left[ \frac{27\pi^3}{2\beta^3} \ln \left( \frac{\beta}{3\pi} \right) \right] + \frac{2}{\ln \beta - \ln 3\pi - 9} \right\}.
\]

(5.11)
with $3\pi < \beta < 3\pi e^{\frac{1}{3}}$. This entails, again, that the system exhibits positive entropy only for a small range of very high temperatures.

For $q = \frac{5}{4}$:

$$
Z = \frac{32\pi^3}{3\beta^3},
$$

(5.12)

$$
< U >= \frac{12}{\beta} \ln \left( \frac{\beta}{4\pi} \right) - \frac{4}{\beta},
$$

(5.13)

$$
S = \left( \frac{32\pi^3}{3\beta^3} \right)^{\frac{1}{3}} \left[ \ln \left( \frac{32\pi^3}{3\beta^3} \right) + 12 \ln \left( \frac{\beta}{4\pi} \right) - 4 \right],
$$

(5.14)

with $\beta > 4\pi e^{\frac{1}{3}}$.

## 6 Specific Heats

We set $k \equiv k_B$. For $\nu = 1$, in the regular case we have for the specific heat $C$:

$$
C = \frac{k}{3 - 2q},
$$

(6.1)

with $q < \frac{3}{2}$.

For $\nu = 2$ one has

$$
C = \frac{2k}{4 - 3q},
$$

(6.2)

with $q < \frac{4}{3}$.

Finally, for $\nu = 3$ one ascertains that

$$
C = \frac{3k}{5 - 4q},
$$

(6.3)

with $q < \frac{5}{4}$. 

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6.1 Specific heats at the poles

For $\nu = 1; q = \frac{3}{2}$

$$C = -2k(\ln kT + \ln 2\pi + 2). \quad (6.4)$$

with $kT < \frac{1}{2\pi e}$.

For $\nu = 2; q = \frac{3}{2}$

$$C = \frac{2k}{(\ln kT + \ln 2\pi)^2} - \frac{2k}{(\ln kT + \ln 2\pi)} + 4k, \quad (6.5)$$

with $kT < \frac{1}{2\pi}$.

For $\nu = 2$ and $q = \frac{1}{2}$ things become:

$$C = -6k \left( \ln kT + \ln 3\pi + \frac{3}{2} \right), \quad (6.6)$$

with $kT < \frac{1}{3\pi \sqrt{e}}$.

For $\nu = 3; q = \frac{3}{2}$

$$C = \frac{k}{(\ln kT + \ln 2\pi)^2} - \frac{k}{(\ln kT + \ln 2\pi)} - 3k, \quad (6.7)$$

with $\frac{1}{2\pi e^{1/2}} < kT < \frac{1}{2\pi}$.

For $\nu = 3$ and $q = \frac{1}{3}$ one has

$$C = \frac{3k}{(\ln kT + \ln 3\pi)^2} - \frac{3k}{(\ln kT + \ln 3\pi)} - 9k, \quad (6.8)$$

with $\frac{1}{3\pi e^{1/2}} < kT < \frac{1}{3\pi}$

Finally, for $\nu = 3$ and $q = \frac{5}{4}$

$$C = -12k \left( \ln kT + \ln 4\pi + \frac{4}{3} \right) \quad (6.9)$$

with $kT < \frac{1}{4\pi e^{3/4}}$. 
Figs. 1, 2, and 3 plot the pole-specific heats within their allowed temperature ranges, for one, two, and three dimensions, respectively. The most distinguished feature emerges in the cases in which we deal with $< U >$ –poles for which $Z$ is regular. We see in such a case that negative specific heats arise. Such an occurrence has been associated to self-gravitational systems [18, 23]. In turn, Verlinde has associated this type of systems to an entropic force [24]. It is natural to conjecture then that such a force may appear at the energy poles. Notice also that temperature ranges are restricted. There is an $T$–upper bound that one may wish to link to the Hagedorn temperature (see above) [21]. In two and three dimensions there is also a lower bound, so that the system (at the poles) would be stable only in a limited $T$–range.

7 Discussion

In this work we have appealed to the dimensional regularization approach of Giambiaggi and Bollini [1] to study the poles in the partition function and the mean energy that appear, for specific, discrete q-values, in Tsallis’ statistics and studied the thermodynamic behavior there. The analysis was made in one, two, and three dimensions.
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Figure 1: One dimension: specific heats at the pole versus temperature $T$, plotted within the allowed temperature range.

Figure 2: Two dimensions: specific heats at the two poles versus temperature $T$, plotted within the allowed temperature ranges in the two cases.
Figure 3: Three dimensions: specific heats at the three poles versus temperature $T$. The vertical lines demarcate the allowed temperature ranges in the three cases. Dashed lines are continuations of the $C$—values outside the domains of validity.