Wave attenuation to clock sojourn times.

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(Dated: November 20, 2018)

The subject of time in quantum mechanics is of perennial interest especially because there is no observable for the time taken by a particle to transmit (or reflect) from a particular region. Several methods have been proposed based on scattering phase shifts and using different quantum clocks, where the time taken is clocked by some external input or indirectly from the phase of the scattering amplitudes. In this work we give a general method for calculating conditional sojourn times based on wave attenuation. In this approach clock mechanism does not couple to the Hamiltonian of the system. For simplicity, specific case of a delta dimer is considered in detail. Our analysis re-affirms recent results based on correcting quantum clocks using optical potential methods, albeit in a much simpler way.

PACS numbers: 03.65.-w, 03.65.Xp, 42.25.Bs
Keywords: D. Electron Transport, A. Nanostructures, D. Sojourn times, D. Wave Propagation

There has been considerable interest on the question of time spent by a particle (interaction time) in a scattering region or in a given region of space[1-3]. This problem has been approached from many different points of view, but there is no clear consensus about a simple expression for this time as there is no hermitian operator associated with it (although experimentalists have claimed to measure it[4]). The prospect of nanoscale electronic devices has in recent years brought new urgency to this problem as this is directly related to the maximum attainable speed of such devices. When it comes to quantum phenomena of tunneling, the time taken by a particle to traverse the barrier is a subject of controversy till now[1,2]. In some formulations this time leads to a real quantity and in others to a complex quantity[1]. In certain cases tunneling time is considered to be ill-defined or quantum mechanics does not allow us to discuss this time[1,2,3]. Furthermore sometimes it is maintained that tunneling through a barrier takes zero time[3]. Recently, Anantha Ramakrishna and Kumar (AK)[6,7] have proposed the non unitary Optical potential as a clock to calculate the sojourn times without the clock affecting it. In this paper we examine another non-unitary clock, i.e., wave attenuation (or, stochastic absorption) to calculate the sojourn length, i.e., the total effective distance travelled by a particle in the region of interest. This sojourn length on appropriate division by the speed of the particle in the region of interest will give us the sojourn time.

Before, we go to the main body of our paper it would be appropriate to describe the relative merits and demerits of the proposals up-till now. The first proposal was by Wigner[10] and the so called Wigner delay time defined separately for reflection or transmission. These are asymptotic times and include self interference delays as well as the time spent in the barrier. This method is based on following the peak of the wave packet, and it loses its significance under strong distortion of the wave packet[1]. Moreover, there is no causal relationship between the peak of the transmitted packet and the peak of the incident packet. This is due to the fact that peak of the transmitted packet can leave the scattering region long before the peak of the incident packet has arrived. In this treatment one cannot address the time spent by a particle in a local region of scattering. However, it should be emphasized that this time is of physical importance as it is intimately connected to the dynamic admittance and other physical properties of microstructures[11]. The next proposal of dwell time[12,13] is an exact statement of the time spent in a region of space averaged over all incoming particles. The problem with this is that it does not distinguish between reflection and transmission and consequently it cannot answer the question “How much time a transmitted (alternatively, Reflected) particle spent in the scattering region?”. Some other proposals[14,15] include the oscillating barrier which considers only a particular limit, i.e., the opaque barrier limit, and others invoke a physical clock[16,17] possessing extra degrees of freedom that co-evolves with the sojourning particle. In some of these treatments the sum of the times spent in two non-overlapping intervals is non-additive and also the very clock mechanism affects the sojourn time to be clocked finitely even as the perturbing clock potential is infinitesimally small[18,19]. This raises the important question namely “Can quantum mechanical sojourn time be clocked without the clock affecting it?”. To this end AK have proposed a non-unitary clock wherein the absorption caused by an infinitesimal optical potential formally introduced over the locality of interest acts as a physical clock to count the time of sojourn in it. The problem for this non-unitary clock is that the optical potential itself introduces spurious scattering’s and this affects the time to be clocked. In a novel manner AK propose a formal procedure by which these spurious scattering’s are eliminated and the sojourn timesclocked by this optical potential counter satisfies all the necessary
conditions especially it is real, additive and distinguishes between reflection and transmission.

In this paper we introduce the wave attenuation (or stochastic absorption) method to calculate the sojourn lengths and times. In this method we damp the wave function by adding an exponential factor \(e^{-aL}\) every time we traverse the length of interest, here \(2\alpha\) represents the attenuation per unit length. This method is better than the optical potential model as it does not suffer from spurious scattering’s. The corrections introduced in case of Optical potential model to take care of spurious scattering’s will become manifestly difficult when we calculate the sojourn times for a superlattice involving numerous scatterer’s. Thus our method of wave attenuation scores over the optical potential model. Moreover, this method will be helpful in calculating delay times in presence of incoherence which have been done earlier using imaginary potentials.

In the presence of wave attenuation a wave attenuates exponentially and thus the transmission (or reflection) coefficient becomes exponential with the length endured in presence of the attenuator and this acts as a natural counter for the sojourn length. Following the procedure of AK we calculate the sojourn lengths and times in case of propagation as outlined in (Fig. 1).

![Diagram of wave propagation and attenuation](image)

**FIG. 1:** Summing the different paths, \(S_1\) and \(S_2\) denote the two scatterer’s. \(l\) is the distance between them. \(e^{ik'L}\) and \(e^{-aL}\) denote the propagation and attenuation factors in the locality of interest.

We calculate the closed form formulas for reflection and transmission coefficients as also the sojourn lengths and times in case of propagation. The amplitude for transmission and reflection can be calculated by summing the different paths as in Fig. 1. The scatterer \(S_1\) in Fig. 1 has as its elements \(r_1, r'_1, t_1\) and \(t'_1\). \(r_1\) is the reflection amplitude when a particle is reflected from the left side of the barrier while \(r'_1\) is the reflection amplitude when a particle is reflected from the right side of the barrier. \(t_1\) and \(t'_1\) are the amplitudes for transmission when a particle is transmitted from left to right of the barrier and vice-versa. Similar assignments are done for the scatterer \(S_2\).

Thus for the amplitude of transmission we have:

\[ t = t_1 t'_2 e^{ik'L} e^{-aL} + t_1 r_2 r'_1 t_2 e^{3ikL} e^{-3aL} + \ldots \]

which can be summed as

\[ t = \frac{t_1 t'_2 e^{ik'L} e^{-aL}}{1 - r_2 r'_1 e^{2ikL} e^{-2aL}} \]

and this is the transmission amplitude in presence of wave attenuation. Again for the case of reflection amplitude we have

\[ r = r_1 + \frac{t_1 r'_2 r_2 t'_1 e^{2ikL} e^{-2aL} + t_1 r'_2 r'_1 t_2 e^{6ikL} e^{-6aL} + \ldots}{1 - r_1 r_2 e^{2ikL} e^{-2aL}} \]

or

\[ r = \frac{r_1 - a r_2 e^{2ikL} e^{-2aL}}{1 - r_1 r_2 e^{2ikL} e^{-2aL}} \]

In Eq. 3, \( a = r_1 r'_1 - t_1 t'_1 \) is the determinant of the S-Matrix of the first barrier and as we are only dealing with unitary S-Matrices therefore the determinant is of unit modulus for all barriers. In these expressions \( k' \) is the wave vector in the region of interest. The transmission and reflection coefficients can be calculated by taking the square of the modulus of the expressions in Eq’s. (1) and (3).

The sojourn lengths for transmission and reflection are calculated as below. The sojourn length for transmission is given as

\[ \tau^T = \lim_{2a \to 0} \frac{\partial \ln |t|^2}{\partial (2a)} \]

and for reflection is defined as-

\[ \tau^R = \lim_{2a \to 0} \frac{\partial \ln |r|^2}{\partial (2a)} \]

The sojourn times for reflection or transmission can be calculated from the formula-

\[ \frac{\tau^T}{T} = \frac{1 - |r_1|^2 |r_2|^2}{1 - 2 \Re (r_1 r_2 e^{2ikL}) + |r_1|^2 |r_2|^2} \]

\[ \frac{\tau^R}{T} = \frac{1 - |r_1|^2 |r_2|^2}{1 - 2 \Re (r_1 r_2 e^{2ikL}) + |r_1|^2 |r_2|^2} \]
Here \( R \) represents real part of the quantity in brackets. In the above two equations the sojourn lengths have been normalized with respect to the length \( l \) of the locality of interest. Throughout the discussion the quantities are expressed in their dimensionless form.

\[
\frac{\tau^R}{l} = \frac{\rho}{l} + \frac{|r_2|^2 - |r_1|^2}{|r_1|^2 - |r_2|^2 - 2R(r_1^* r_2 e^{2i kl} + |r_2|^2)} \tag{7}
\]

and for reflection:

\[
\frac{\rho}{l} = \frac{\rho^T}{l} + \frac{|r_2|^2 - |r_1|^2}{|r_1|^2 - |r_2|^2}
\]

In the above two equations the sojourn lengths have been normalized with respect to the length \( l \) of the locality of interest. Throughout the discussion the quantities are expressed in their dimensionless form.

In Figures (3) and (4) we plot the Transmission and Reflection coefficients and the normalized sojourn times \( \tau^T \) and \( \tau^R \) for a non symmetrical delta dimer. We have normalized these sojourn times by the time taken by a particle to traverse a distance \( l \) in absence of barrier, i.e., \( \frac{ml}{\hbar k} \).

We observe that for an unsymmetrical delta-dimer the sojourn time in case of reflection goes negative for certain values of the potentials. Hence we get negative sojourn times in certain regions for the case of reflection. Even though additivity of local sojourn times in two non-overlapping regions holds here too. It has been argued by AK that this is because in case of reflection there is a partial wave corresponding to prompt reflection \( r_1 \) that never samples the region of interest, and also this prompt part leads to self interference delays which cause the sojourn time \( \tau^R \) to become negative for some values of the parameters. If one removes this prompt part as suggested by AK, i.e., \( r_{np} = r - r_1 \), and we calculate the sojourn time \( \tau^{R_{np}} \) with this prompt part removed we find it to be positive definite and given by \( \tau^{R_{np}} = \tau^T + 1 \), as \( \tau^T \) is positive definite. Removing the prompt part of course cures the problem of negative sojourn times but it causes another problem we find when \( V_2 \) is zero, \( \tau_R = 0 \), as expected because particle is not reflected after entering the region as there is no barrier to the right of the locality of interest, but \( \tau^{R_{np}} = 2 \). Even in absence of barriers when there is no reflection still we get \( \tau^{R_{np}} = 2 \), which follows trivially from the formal final expression \( \tau^{R_{np}} = \tau^T + 1 \), as \( \tau^T = 1 \) irrespective of the case whether \( V_2 = 0 \) or both \( V_1 \) and \( V_2 \) are zero. This is unexpected looking at the expression for \( r \) in Eq. 3. This leads us to the conclusion that to get a positive definite answer for the reflection time by removing prompt part requires careful thinking. In case of a symmetric barrier as is obvious from Eq’s. 6 and 7 \( \tau^T = \tau^R \), and \( \tau^R \) is independent of the fact whether particle is incident from the right or left, but \( \tau^R \) depends non-trivially on the direction of incidence of the particle.

A remarkable assertion found in the literature concerning the measurement of the time of transmission or

\[
S_j = \begin{pmatrix} r_j & t_j' \\ r_j' & t_j \end{pmatrix} = \begin{pmatrix} V_j & 2ik - V_j \2ik - V_j \2ik - V_j \end{pmatrix}
\]
reflection is \( \tau^D = T\tau^T + R\tau^R \). Herein \( \tau^T \) and \( \tau^R \) are as given above while the dwell time \( \tau^D = \frac{1}{v} \int_0^l |\psi|^2 dx \). \( \psi \) is the wavefunction in the locality of interest and \( v \) is the speed of the particle in the region of interest. In Fig. 5 we plot \( \tau^D \) and \( T\tau^T + R\tau^R \), for a symmetrical delta dimer. We find that they are inequivalent in the small \( kl \) regime [1].

For the case of rectangular barriers of height \( V \), \( r \) and \( t \)'s are the reflection and transmission amplitudes at the interfaces and \( k' = \sqrt{\frac{2m(E-V)}{\hbar^2}} \) in the propagation regime and the speed is \( |\frac{\hbar k}{m}| \). For this case we get the same result as obtained in the Ref.[8].

In conclusion, we have given a simple method of calculating sojourn times using wave attenuation method. In this method, wave attenuation (or, stochastic absorption) as we have treated cannot be incorporated in a Hamiltonian. Thus it inherently takes care of spurious scattering’s, which arise when Hamiltonians with an optical potential are used (or for every clock where perturbation due to clock mechanism couples to the Hamiltonian). Our results reaffirm those obtained by AK after taking care of the spurious contributions from the optical potential. The transmission sojourn times are always positive definite and are additive as mentioned above. Reflection sojourn times can become negative. By removal of the prompt part of reflection \( r_1 \) one gets positive reflection sojourn time but it further leads to another problem. Thus, if we insist on reflection sojourn time to be positive then it requires further careful analysis, or, reflection time itself needs closer inspection. Finally when we compare the dwell time with the conditional sojourn times weighted over appropriate reflection and transmission coefficients, we find them not to be equivalent.

Acknowledgments

The authors would like to thank Dr. S. Anantha Ramakrishna for useful discussions on this problem and one of us (AMJ) thanks Professor N. Kumar for his continual interest and discussions on this problem over the years.

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