Abstract—In the presence of additive Gaussian noise, the statistics of the nonlinear Fourier transform (NFT) of a pulse is still an open problem. In this paper, we propose a novel approach to study this problem. Our contributions are twofold: first, we extend the existing Fourier Collocation (FC) method to compute the whole discrete spectrum (eigenvalues and spectral amplitudes). We show numerically that the accuracy of FC is comparable to the state-of-the-art NFT algorithms. Second, we apply perturbation theory of linear operators to derive analytic expressions for the statistics of both the eigenvalues and the spectral amplitudes when a pulse is contaminated by additive Gaussian noise. Our analytic expressions closely match the empirical statistics obtained through simulations.

Index Terms—Nonlinear Fourier Transform, Nonlinear Frequency Multiplexing, Multi-soliton, Fourier collocation

I. INTRODUCTION

The Nonlinear Fourier Transform (NFT), or Inverse Scattering Transform [1], [2] has been proposed as an alternative for system design in an attempt to overcome the capacity peak reported in [3] of linear transmission systems over the nonlinear optical channel. The NFT consists of a continuous spectrum and a discrete spectrum, and it is known that signals with high power (i.e. more affected by nonlinearity) have more components in the discrete spectrum [2]. A comprehensive overview on the NFT and its application and challenges for optical communications is given in [4].

Communication using the NFT has been demonstrated numerically and experimentally, e.g. [4]–[9]. Despite some promising results, the effect of channel noise on the NFT is not yet well understood. The only well-studied case is the statistics of the NFT of a first-order soliton propagating along an optical fiber with distributed additive white Gaussian noise [10]–[12]. Recently, a general method has been developed in [13] to numerically compute the statistics of the spectral coefficients (not the eigenvalues) of a noisy signal. This method, however, requires knowledge of the time-domain signal.

This work builds up on our previous paper [13] and makes a twofold contribution. First, we extend the existing Fourier Collocation (FC) method [15] Sec. 2.4.3] to compute the complete discrete spectrum (eigenvalues and spectral amplitudes) of arbitrary pulses. We apply a proper windowing and truncation to overcome the ringing problem caused by non-periodic boundary conditions of NFT. Our numerical results show that our method achieves a comparable accuracy to the best existing methods when the number of time samples is low.

Second, we apply the perturbation theory of matrix eigenvalues [16] to our extended method to derive the statistics of the discrete spectrum when a pulse is contaminated by Gaussian noise. Note that the eigenvalue perturbation theory is also used in [2] Sec. IV.A] to find the variances of the eigenvalues. Our method differs from the latter in two aspects: we analyze the statistics in the frequency domain (avoiding the time-domain derivatives), and we obtain the full covariance matrix (not only the diagonal) of both the eigenvalues and the spectral amplitudes. Unlike [13], if the Jost solutions are known (as is the case for multi-solitons), our method does not require the time-domain pulse to compute the eigenvalue statistics. We show that our analytic expressions for the statistics of the discrete spectrum closely match the statistics obtained through Monte-Carlo simulations.

The paper is organized as follows. In Sec. III we briefly overview the NFT and the multi-soliton pulses. In Sec. III we describe the FC method and extend it to compute also the discrete spectral amplitudes by proper windowing and truncation. Applying a first-order perturbation method, we derive the statistics of the discrete spectrum for an arbitrary pulse in Sec. [IV] We validate in Sec. [V] both our extended FC method and our analytic expressions for the statistics of the discrete spectrum through simulations with different pulses. Sec. VI concludes the paper.

Notation: Vectors are denoted with bold lowercase letters \( \mathbf{x} \), and matrices with bold uppercase letters \( \mathbf{X} \). \( \mathbf{X}^T \) and \( \mathbf{X}^H \) are respectively the transpose and conjugate transpose of \( \mathbf{X} \). The horizontal concatenation of vectors \( \mathbf{x} \) and \( \mathbf{y} \) is denoted \( \mathbf{x} \oplus \mathbf{y} \). For a scalar \( x \) or vector \( \mathbf{x} \), \( \Re x \) or \( \Re \mathbf{x} \) is its real part and \( \Im x \) or \( \Im \mathbf{x} \) is its imaginary part.

II. SYSTEM MODEL

Consider the complex envelope \( Q(Z, T) \) of an electrical field propagating along an optical fiber, where \( Z \) is distance and \( T \) is time. The propagation is modeled by the Nonlinear Schrödinger Equation (NLSE) [17] Eq. (2.3.46):

\[
\frac{\partial Q(Z, T)}{\partial Z} = -j \frac{\beta_2}{2} \frac{\partial^2 Q(Z, T)}{\partial T^2} + j \gamma |Q(Z, T)|^2 Q(Z, T)
\]  

where \( \beta_2 \) is the chromatic dispersion parameter, and \( \gamma \) is the nonlinear coefficient. We neglect attenuation in (1) assuming that fiber loss is exactly compensated by ideal distributed amplification. By applying the following change of variables:

\[
T = T_0 t, \quad Z = 2 \frac{T_0^2}{|\beta_2|} z, \quad Q(Z, T) = \frac{1}{T_0} \sqrt{\frac{|\beta_2|}{\gamma}} q(z, t)
\]
A. Description in Nonlinear Spectrum

The NFT is calculated by solving the Zakharov-Shabat system (ZSS) \[1\], \[2\]
\[
\begin{pmatrix}
-\frac{\partial}{\partial t} & q(t) \\
q^*(t) & -\frac{\partial^2}{\partial z^2}
\end{pmatrix}
\begin{pmatrix}
v_1(t, \lambda) \\
v_2(t, \lambda)
\end{pmatrix}
= j\lambda \begin{pmatrix}
v_1(t, \lambda) \\
v_2(t, \lambda)
\end{pmatrix}
\]
with the boundary condition
\[
v(t, \lambda) \to \left(\begin{array}{c}
1 \\
0
\end{array}\right) e^{-jM\lambda}, \quad t \to -\infty
\]
where \(\lambda\) is the eigenvalue and the Jost solution \(v(t, \lambda) = (v_1(t, \lambda)v_2(t, \lambda))^T\) is the eigenvector. The spectral coefficients \(a(\lambda)\) and \(b(\lambda)\) are given by
\[
a(\lambda) = \lim_{t \to \infty} v_1(t, \lambda)e^{jM\lambda} \quad \text{(6a)}
\]
\[
b(\lambda) = \lim_{t \to \infty} v_2(t, \lambda)e^{-jM\lambda}. \quad \text{(6b)}
\]

The NFT of the signal \(q(t)\) is made up of two spectra:
- the continuous spectrum \(Q(\xi) = \frac{2v(\xi, \lambda)}{\partial\lambda a(\lambda)}\), for \(\xi \in \mathbb{R}\);
- the discrete spectrum \(Q_k = \frac{b(\lambda_k)}{a(\lambda_k)}\), for the \(K\) eigenvalues \(\{\lambda_k \in \mathbb{C}^+ : a(\lambda_k) \neq 0\}\) where \(a_\lambda = da/d\lambda\) and \(\mathbb{C}^+ = \{\lambda \in \mathbb{C} : \Re \lambda > 0\}\).

In recent works \[3\], \[9\], \[18\], the use of \(b(\lambda)\) instead of \(Q(\lambda)\) and \(b_k = b(\lambda_k)\) instead of \(Q_k\) for data modulation has been shown to achieve better results. In terms of degrees of freedom, both approaches are equivalent, as given \(\lambda_k\) and \(b(\xi)\) for \(\xi \in \mathbb{R}\), one can compute \(a(\lambda)\) for \(\Re \lambda \geq 0\) \[19\]. The usefulness of the NFT lies in the fact that, given a signal \(q(z, t)\) propagating according to the NLSE \[3\], its NFT evolves in \(z\) according to the following simple multiplicative relations:
\[
b(z, \xi) = b(0, \xi)e^{jz^2} \quad \text{(7a)}
\]
\[
\lambda_k(z) = \lambda_k(0) \quad \text{(7b)}
\]
\[
b_k(z) = b_k(0)e^{j\lambda_k^2 z}. \quad \text{(7c)}
\]

We skip the derivation in the sequel for simplicity. Plenty of numerical algorithms have been developed to compute NFT. Some are outlined in \[2\], \[4\]. As a benchmark, we use in this paper a recently proposed algorithm in \[20\] with a sixth-order Commutator-Free Quasi-Magnus (CFQM) integrator \(CE_4\). To compute \(b_k\), this algorithm is combined with the forward-backward iterations, explained in \[21\]. In fact, we replace the 2\(^{rd}\)-order trapezoidal integration of \[21\] with the CFQM to improve the performance. We call this algorithm FB-CFQM.

B. Multi-soliton Pulses

A multi-soliton is a signal that has no continuous spectrum, i.e., \(b(\xi) = 0\). From \[7\], the eigenvalues of a multi-soliton stay constant along propagation, and the spectral amplitudes evolve independently of each other. In Algorithm \[1\], we provide pseudo-code that uses the Darboux transform \[22\] to efficiently construct a time-domain multi-soliton pulse as well as its Jost solutions from the given discrete spectrum. Some other Inverse NFT algorithms are reviewed in \[4\] to generate a time-domain pulse from a nonlinear spectrum.
\[ \Omega = \begin{pmatrix} \Omega & \Gamma \\ -\Gamma^H & -\Omega \end{pmatrix}, \]

\[ L = \begin{pmatrix} \Omega & \Gamma \\ -\Gamma^H & -\Omega \end{pmatrix}, \]

where

A solution \((\lambda_k, \psi_k)\) of (10) corresponds to the Fourier coefficients of a solution \(v(t, \lambda_k)\) of the ZSS (3). Accordingly, the eigenvalues of \(L\) include the eigenvalues \(\lambda_k\) of the discrete spectrum, their complex conjugates, and \(M - 2K\) spurious eigenvalues on the real axis.

**Computation of Spectral Amplitudes \(b_k\)**

We extend the FC method to also compute the spectral amplitudes \(b_k\). We know that for any constant \(G_k\), \(G_k^{-1} \psi_k\) is still an eigenvector of \(L\). We choose the appropriate normalization constant \(G_k\) to satisfy the boundary condition (5), i.e. \(v_1(-T/2, \lambda_k) = \exp(-j\lambda_k(-T/2))\),

\[ G_k = e^{j\lambda_k \frac{T}{2}} \sum_{n=-N}^{N} \psi_{k,1}[n] e^{jn \frac{2\pi}{M} \frac{T}{2}}. \]  

Using (8), we can obtain \(b_k = v_2(T/2, \lambda_k) \exp(-j\lambda_k T/2)\) (see (9)) from the eigenvectors:

\[ b_k = \frac{e^{-j\lambda_k \frac{T}{2}}}{G_k} \sum_{n=-N}^{N} \psi_{k,2}[n] e^{jn \frac{2\pi}{M} \frac{T}{2}} \]  

The above quantities are based on the assumption that the periodic extensions of \(v_1(t, \lambda_k)\) for \(i \in \{1, 2\}\) have a bandwidth smaller than \(2N/T\). However, this assumption may not be satisfied if \(v_1(-T/2, \lambda_k) \neq v_1(T/2, \lambda_k)\). This issue may cause undesirable ripples on \(v_1(t, \lambda_k)\) around \(t = \pm T/2\) when \(v_1(t, \lambda_k)\) is obtained from (8). The ripples have a severe impact on \(G_k\) and \(b_k\). The reason is that \(G_k = v_1(t, \lambda_k) e^{j\lambda_k t}\) at \(t = -T/2\) and \(b_k\) is proportional to \(v_2(t, \lambda_k) e^{-j\lambda_k t}\) at \(t = T/2\). The exponential terms amplify the ripples further causing a wrong estimation of \(b_k\). Let us show the issue by a numerical example. Consider a 2-soliton pulse with eigenvalues \(\lambda_1 = 0.3j, \lambda_2 = 0.6j\) and \(b_1 = b_2 = \frac{1}{3}j\). The signal is truncated to \(t \in [-17.67, 17.67]\) and sampled with \(M = 365\). We computed \(v_1(t, \lambda_1) e^{j\lambda_1 t}\) and \(v_2(t, \lambda_1) e^{-j\lambda_1 t}\) using two methods: the FB-CFQM algorithm and the FC method. The results are compared in Fig. 1 showing that the FC method generates large ripples. We apply two techniques to mitigate the effect of ripples:

- **Frequency-domain windowing**: we apply windowing functions \(w_1[n]\) and \(w_2[n]\) respectively to \(\psi_{k,1}\) and \(\psi_{k,2}\), the two halves of \(\psi_k\). In our simulations, a Hann windowing function [23] gave promising results.
- **Tail truncation**: we neglect part of the tails of \(v_1(t, \lambda_k)\) and \(v_k(t, \lambda_k)\) before computing \(G_k\) and \(b_k\). This means that we replace \(T/2\) with \(T_k < T/2\) in (12), and (13).

Define the frequency-shifted windows \(u_{k,1}[n]\) and \(u_{k,2}[n]\) as

\[ u_{k,1}[n] = w_1[n] e^{-jn \frac{2\pi}{N} T_k}, \]  

\[ u_{k,2}[n] = w_2[n] e^{-jn \frac{2\pi}{N} T_k}. \]

Let \(u_{k,i} = (u_{k,1}[N], \ldots, u_{k,1}[N])^T\), with \(i \in \{1, 2\}\). Then, from (12) and (13), \(b_k\) becomes

\[ b_k = \frac{\sum_{n=-N}^{N} w_2[n] \psi_{k,2}[n] e^{jn \frac{2\pi}{N} T_k}}{\sum_{n=-N}^{N} w_1[n] \psi_{k,1}[n] e^{-jn \frac{2\pi}{N} T_k}}. \]

The choice of \(T_k\) should avoid the ripples in Fig. 1 while staying close enough to the limit value. We choose \(T_k = \min(t_{\exp}, t_{5\%})\), with:

- \(t_{\exp} = 12/3\lambda_k\) (such that \(e^{\pm j\lambda_k t}\) does not become larger than \(e^{12} \approx 1.63 \cdot 10^4\)).
- \(t_{5\%} = T/2 - 0.05T\) (5% removed tail).

This choice is heuristic, and our simulations show that it works for signals that have most of their energy inside the interval \([-t_{\exp}, t_{\exp}]\), which seems to be the case for, at least, solitons of order up to 5 with \(e^{-3\lambda_k} < |b_k| < e^{3\lambda_k}\). This heuristic is based on the condition \(|b_k| = 1\) for symmetric solitons, which have good energy confinement in time [24]. For the design of a system with signals known to have high energy outside the interval \([-t_{\exp}, t_{\exp}]\), the value of \(t_{\exp}\) can be increased at the cost of some accuracy in the corresponding \(b_k\). Note that windowing and truncation are done after solving the eigenvalue problem (10) and obtaining \(\lambda_k\). Therefore, these two techniques affect only the computation of \(b_k\).

**IV. STATISTICS OF THE DISCRETE SPECTRUM**

The NFT of a pulse changes when the pulse is distorted. In this section, we derive the second-order statistics of the discrete spectrum when a pulse is contaminated by additive Gaussian noise. The FC method enables us to use the well-established perturbation theory of linear operators to describe the desired statistics.

Consider again the signal \(q(t)\) of pulse duration \(T\) contaminated by zero-mean additive Gaussian noise \(e(t) = \sigma q(t)\) that
is wide-sense stationary (WSS), i.e., \( E[\hat{q}(t)\hat{q}(t + \tau)] = r_q(\tau) \) does not depend on \( t \). Here, \( \sigma^2 = E[(\epsilon(t))^2] \) is chosen such that \( E[(\hat{q}(t))^2] = r_q(0) = 1 \). Let us first assume that \( \sigma \hat{q}(t) \) has bandwidth \( B \) and a constant power spectral density (PSD) \( N_0 \) inside the band. This implies \( \sigma^2 = N_0 B \). If the signal is sampled at the Nyquist rate, i.e., \( B = M/T \), then we have for \( m \in \{-N, \ldots, N\} \\
\hat{q}[m] = q[m] + \sigma \hat{q}[m] 
\] (17)

where \( \sigma \hat{q}[m] \) are the zero-mean noise samples. The Fourier series coefficients of the noisy pulse are
\[
\hat{c}[n] = \frac{1}{M} \sum_{m=-N}^{N} q[m] e^{-j \frac{2\pi}{2M} mn} + \frac{\sigma}{M} \sum_{m=-N}^{N} \hat{q}[m] e^{-j \frac{2\pi}{2M} mn} 
\]
(18)

\[
= c[n] + \sigma \hat{c}[n] 
\] (19)

where the choice of \( \sigma \) implies that \( E[|\sum_n |\hat{c}[n]|^2] = 1 \). Define \( \hat{c} = (\hat{c}[-N], \ldots, \hat{c}[N])^T \). The covariance matrix of the vector \((\hat{R}_c^T, \hat{3}c^T)^T\) is
\[
\hat{R}_c = \frac{1}{2M} \hat{G} \hat{G}^H. 
\] (21)

A. Perturbation of the Discrete Spectrum

Let \( \{(\lambda_k, \hat{b}_k)\} \) denote the discrete spectrum of the noisy signal. Using the FC method, they are obtained from the solutions of the eigenvalue problem,
\[
\hat{L} \hat{\psi}_k = \hat{\lambda}_k \hat{\psi}_k 
\] (22)

where \( \hat{L} = L + \sigma \hat{L} \)
(23)

where \( L \) given in (11), corresponds to the noiseless pulse and
\[
\hat{L} = \begin{pmatrix} 0 & \hat{\Gamma} \\ -\hat{\Gamma}^H & 0 \end{pmatrix} 
\] (24)

where \( \hat{\Gamma} \in \mathbb{C}^{M \times M} \) is a Toeplitz matrix whose first column is \(-j(\hat{c}[0] \ldots \hat{c}[N])^T \) and whose first row is \(-j(\hat{c}[0] \ldots \hat{c}[N])^T \)

When \( \sigma^2 \) is relatively small, \((\lambda_k, \hat{b}_k)\) can be approximated by first-order perturbations
\[
\hat{\lambda}_k = \lambda_k + \sigma \hat{\lambda}_k + O(\sigma^2) 
\] (25)
\[
\hat{\psi}_k = \psi_k + \sigma \hat{\psi}_k + O(\sigma^2) 
\] (26)
\[
\hat{b}_k = b_k + \sigma \hat{b}_k + O(\sigma^2) 
\] (27)

The perturbation analysis of eigenvalues and eigenvectors is a mature topic. A detailed analysis is given in [16] Sections II.1 and II.2. We first define the left eigenvector \( \Phi_k \) as
\[
\Phi_k^H L = \lambda_k \Phi_k^H 
\] (28)

For our operator \( L, \Phi_k \) is equal to the flipped, conjugated version of the right eigenvector \( \psi_k \), i.e.
\[
\Phi_k = (\psi_{k,1}^T[-N], \ldots, \psi_{k,1}^T[N], \psi_{k,1}^T[N], \ldots, \psi_{k,1}^T[-N])^T 
\] (29)

The reason is that \( \Phi_k \) corresponds to the (right) eigenvector of \( \lambda_k^* \) for the signal \( -q(t) \). This can be seen by taking the conjugate transpose from both sides of (28). Moreover, if \((\lambda_k, (\psi_{k,1}^T \psi_{k,2}^T))^T\) is a solution of (4) for \( q(t) \), then \((\lambda_k^*, (\psi_{k,1}^T \psi_{k,2}^T))^T\) is a solution of (4) for \( -q(t) \). Combining these two properties concludes (29).

**Theorem 1.** Consider (23) with the condition that \( \|\sigma \hat{L}\|_F \ll \|L\|_F \) \( (\|L\|_F \) is the Frobenius norm). We define for every eigenvalue and eigenvectors \((\lambda_k, \psi_k, \Phi_k)\) of \( L \)
\[
P_k = \frac{1}{\Phi_k^H \psi_k} \Phi_k \Phi_k^H \] (30)
\[
S_k = (L - \lambda_k I - P_k)^{-1} (1 - P_k). 
\] (31)

The matrix \( P_k \) is called the eigenprojector of \( \lambda_k \) and \( S_k \) is the Drazin inverse \([23]\) of \( L - \lambda_k I \). If \( \lambda_k \) has algebraic multiplicity 1, then its first-order perturbations (25) and (26) are given by,
\[
\hat{\lambda}_k = \frac{1}{\Phi_k^H \psi_k} \Phi_k^H \hat{L} \psi_k 
\] (32)
\[
\hat{\psi}_k = -S_k \Phi_k \psi_k. 
\] (33)

**Proof.** Under the conditions that \( \|\sigma \hat{L}\|_F \ll \|L\|_F \) and \( \lambda_k \) has multiplicity 1, it is shown in [16] Ch. II, Eqs. (2.21) and (2.33)
\[
\hat{\lambda}_k = \text{tr} \left( L P_k \right) 
\] (34)

and \( P_k \) is given based on a contour integral [16] Ch. I, Eq. (5.22)] which is further simplified in [26] to (30). Equation (34) is a useful result, because it is valid for arbitrary perturbation matrices \( L \). Substituting (30) in (34) and using \( \text{tr}(L \psi_k \Phi_k^H) = \text{tr}(\Phi_k^H L \psi_k) \) results in (33). The proof of (33) is also given in [16] Ch. II, Eq. (4.22)].

Note that all eigenvalues of the discrete spectrum, \((\lambda_k)_{k=1}^K\), have multiplicity 1 satisfying the condition of Theorem 1.

The perturbation \( \hat{b}_k \) needs some further steps. From (16),
\[
\hat{b}_k = \frac{u_{k,2}^T \psi_{k,2} + \sigma u_{k,2}^T \psi_{k,1} + O(\sigma^2)}{u_{k,1}^T \psi_{k,1} + O(\sigma^2)} 
\] (35)
\[
= \frac{u_{k,2}^T \psi_{k,2} + \sigma u_{k,2}^T \psi_{k,1} + O(\sigma^2)}{u_{k,1}^T \psi_{k,1} + O(\sigma^2)} \left( 1 + \frac{u_{k,2}^T \psi_{k,2} - \sigma u_{k,2}^T \psi_{k,1}}{u_{k,1}^T \psi_{k,1} + O(\sigma^2)} \right) + O(\sigma^2). 
\]

It results \( \hat{b}_k \) in terms of given scalars \( b_k \) and \( u_{k,1}^T \psi_{k,1} \) as
\[
\hat{b}_k = \frac{1}{u_{k,1}^T \psi_{k,1}} \left( u_{k,2}^T \psi_{k,2} - b_k u_{k,1}^T \psi_{k,1} \right) 
\] (36)

Or equivalently,
\[
\hat{b}_k = \frac{1}{u_{k,1}^T \psi_{k,1}} (b_k u_{k,1}^T - u_{k,2}^T) S_k \Phi_k \psi_k 
\] (37)
Both $\tilde{\lambda}_k$ and $\tilde{b}_k$ depend on $\tilde{\lambda}_k$. From (24) we can write

$$\tilde{\lambda}_k = \left( \begin{array}{c} \hat{\psi}_{k,2}^* \\ -\hat{\lambda}_{H \psi_{k,1}} \end{array} \right)$$

(38)

The first vector $\hat{\lambda}_{H \psi_{k,1}}$ is the convolution of $-j\tilde{c}[n]$ and $\psi_{k,2}[n]$. From the commutative property of convolution, i.e., $-j\tilde{c}[n] * \psi_{k,2}[n] = -j\psi_{k,2}[n] * \tilde{c}[n]$, it can be reordered as

$$\hat{\lambda}_{H \psi_{k,1}} = -j J_{k,2} \hat{c}$$

(39)

where $J_{k,2} \in \mathbb{C}^{M \times M}$ is a Toeplitz matrix whose first column is $(\psi_{k,2}[0] \ldots \psi_{k,2}[N])^T$ and whose first row is $(\psi_{k,2}[0] \ldots \psi_{k,2}[-N])^T$. Similarly,

$$-\hat{\lambda}_{H \psi_{k,1}} = -j J_{k,1} \tilde{c}$$

(40)

where $\tilde{c} = (\tilde{c}[N], \ldots, \tilde{c}[-N])^H$ is the flipped, conjugated version of $\tilde{c}$, and $J_{k,1}$ is a Toeplitz matrix whose first column is $(\psi_{k,1}[0] \ldots \psi_{k,1}[N])^T$ and whose first row is $(\psi_{k,1}[0] \ldots \psi_{k,1}[-N])^T$. Let $\Pi$ be a permutation matrix that reverses the order of a vector. It is an anti-diagonal matrix with the anti-diagonal elements equal to one. Therefore, $\tilde{c} = \Pi \hat{c}$ and accordingly,

$$\tilde{\lambda}_k = -j \left( \begin{array}{c} J_{k,2} \hat{c} \\ J_{k,1} \Pi \hat{c} \end{array} \right)$$

(41)

$$\Sigma_k = \left( \begin{array}{cc} \mathbb{R} \hat{c} & \mathbb{I} \hat{c} \\ \mathbb{I} \hat{c} & \mathbb{I} \hat{c} \end{array} \right)$$

(42)

with

$$\Sigma_k = \left( \begin{array}{cc} -J_{k,2} & J_{k,2} \\ -J_{k,1} \Pi & -J_{k,1} \Pi \end{array} \right)$$

(43)

### B. Statistics of $\tilde{\lambda}_k$

Consider the perturbation term $\tilde{\lambda}_k$ in (32). Using (42),

$$\tilde{\lambda}_k = \frac{1}{\phi_H^* \psi_k} \phi_H^* \Sigma_k \begin{bmatrix} \mathbb{R} \tilde{c} \\ \mathbb{I} \tilde{c} \end{bmatrix} = d_k \begin{bmatrix} 1 \\ \mathbb{R} \tilde{c} \end{bmatrix}$$

(44)

From (20), we have $\phi_H = \psi_k \Pi$. We can rewrite the horizontal vector $d_k$ as

$$d_k = \frac{1}{\phi_H^* \psi_k} \phi_H^* \Sigma_k = \frac{1}{\psi_k \Pi \psi_k} \psi_k \Pi \Sigma_k.$$  

(45)

Let $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_K)^T$. We can write,

$$\hat{\lambda} = D \begin{bmatrix} \mathbb{R} \hat{c} \\ \mathbb{I} \hat{c} \end{bmatrix}$$

(46)

where $D$ is a matrix whose rows are the $d_k$. Equivalently,

$$\begin{bmatrix} \mathbb{R} \hat{\lambda} \\ \mathbb{I} \hat{\lambda} \end{bmatrix} = D \begin{bmatrix} \mathbb{R} \hat{c} \\ \mathbb{I} \hat{c} \end{bmatrix}$$

(47)

The above equation implies that $\mathbb{R} \hat{\lambda}$ and $\mathbb{I} \hat{\lambda}$ are obtained from linear combinations of $\mathbb{R} \tilde{c}$ and $\mathbb{I} \tilde{c}$. Since $(\mathbb{R} \tilde{c}^T, \mathbb{I} \tilde{c}^T)^T$ has a jointly Gaussian distribution, $(\mathbb{R} \tilde{c}^T, \mathbb{I} \tilde{c}^T)^T$ has a jointly Gaussian distribution with zero mean and covariance matrix,

$$C_\hat{\lambda} = \begin{bmatrix} \mathbb{R}D^T \\ \mathbb{I}D^T \end{bmatrix} R_{\tilde{c}} (\mathbb{R}D^T, \mathbb{I}D^T)^T$$

(48)

**Corollary 1.** Up to the first-order approximation in $\sigma$, the $\hat{\lambda}_k$ eigenvalues of the noisy signal have a joint Gaussian distribution with means $\lambda_k$ and covariance matrix $\sigma^2 C_\hat{\lambda}$.

### C. Statistics of $b_k$ coefficients

Like $\tilde{\lambda}_k$, $\tilde{b}_k$ is a linear combination of $\mathbb{R}\tilde{c}[n]$ and $\mathbb{I}\tilde{c}[n]$. To see this more clearly, define

$$h_k = \frac{1}{u_{k,1}^T \psi_{k,1}} \left(b_k u_{k,1}^T - u_{k,2}^T \right) S_k \Sigma_k.$$  

Using (37), we have simply,

$$\tilde{b}_k = h_k \begin{bmatrix} \mathbb{R} \tilde{c} \\ \mathbb{I} \tilde{c} \end{bmatrix}$$

(49)

Let $\hat{\tilde{b}} = (\hat{b}_1, \ldots, \hat{b}_K)^T$. Then we have

$$\hat{\tilde{b}} = H \begin{bmatrix} \mathbb{R} \tilde{c} \\ \mathbb{I} \tilde{c} \end{bmatrix}$$

(50)

where $H$ is a matrix whose rows are the $h_k$. Equivalently,

$$\begin{bmatrix} \mathbb{R} \tilde{b} \\ \mathbb{I} \tilde{b} \end{bmatrix} = \begin{bmatrix} \mathbb{R}H \\ \mathbb{I}H \end{bmatrix} \begin{bmatrix} \mathbb{R} \tilde{c} \\ \mathbb{I} \tilde{c} \end{bmatrix}$$

(51)

It results that $(\mathbb{R} \tilde{b}^T, \mathbb{I} \tilde{b}^T)^T$ has a jointly Gaussian distribution with zero mean and the co-variance matrix,

$$C_{\tilde{b}} \begin{bmatrix} \mathbb{R}H \\ \mathbb{I}H \end{bmatrix} R_{\tilde{c}} \begin{bmatrix} \mathbb{R} \tilde{c}^T, \mathbb{I} \tilde{c}^T \end{bmatrix}$$

(52)

**Corollary 2.** Up to the first-order approximation in $\sigma$, the spectral coefficients $b_k$ have a joint Gaussian distribution with covariance matrix $\sigma^2 C_{\tilde{b}}$. The mean of each $\tilde{b}_k$ is $b_k$.

### V. Numerical Validation

**A. Accuracy of FC in the noiseless case**

In our first round of simulations, we measured the accuracy of the computation of $\{\lambda_k, b_k\}$ using (10) and (16). We used three different pulses for the simulations:

- a 2-soliton with $\lambda_k = [0.6j, 0.3j]$ and $b_k = [1 + 0.3j, 1 - 0.3j]$
- a 5-soliton with $\lambda_k = [1.5j, 1.2j, 0.9j, 0.6j, 0.3j]$ and, respectively, $b_k = [0.8855 + 0.1109j, -1.4293 - 0.6778j, 1.0701 + 0.2486j, -0.9965 + 1.0385j, 0.3345 + 0.8551j]$.

We used the Darboux algorithm or closed-form expressions to generate the pulses, and then computed their discrete spectrum using two different methods. For our FC method, we used Hann windows $w_1[n]$ and $w_2[n]$ (see (10)).

We used a time interval with fixed length $T$, and varied the number of time samples by varying the sampling period. We used $T = 35.3$ for the 2-soliton and the 5-soliton, and $T = 24.0$ for the sech pulse. We compared the results of FB-CFQM and FC in terms of error:

$$\text{err}(\lambda_k, \hat{\lambda}_k) = \frac{1}{|\hat{\lambda}_k|} / |\lambda_k|$$

(54)

where $\hat{\lambda}_k$ is the eigenvalue obtained using FB-CFQM or FC, and $\lambda_k$ is the original eigenvalue. err($b_k, \hat{b}_k$) is defined...
similarly. The results are depicted in Fig. 2. Our FC method (dashed lines, empty markers) provides rather precise estimates of $b_k$ (especially for 2-soliton and sech pulse, resp. Fig. 2(1b) and (3b)). However, FB-CFQM (dotted lines, filled markers) has overall more precise estimates, especially when the number of samples is large.

**B. Accuracy of the FC perturbation analysis in the noisy case**

To validate our proposed closed-form expressions for the covariance matrix of the eigenvalues (Corollary 1) and spectral coefficients (Corollary 2), we simulated the same three pulses of the previous section, using $M = 365$ samples for the 2-soliton, $M = 909$ for the 5-soliton, and $M = 233$ for the sech pulse. We then added AWGN $\sigma^{\epsilon}[m]$ to the three pulses. We define the signal-to-noise ratio (SNR) as:

$$\text{SNR} = \frac{\sum_{m=-N}^{N} |q[m]|^2}{\sigma^2 \sum_{m=-N}^{N} |\hat{q}[m]|^2}. \quad (55)$$

For each value of SNR, we generated 1024 realizations of noise and computed the discrete spectrum of the noisy pulse using the FC and FB-CFQM. From this data, we estimated the covariance matrix of $\lambda_k$ and of $b_k$ for the FC and FB-CFQM case, and compared it with our analytic covariance matrices given by (48a) and (52a). We only plot the following entries

$$\sigma^2_{\lambda_k} = \sigma^2 \mathbb{E}\left[|\Re \lambda_k|^2\right] + \sigma^2 \mathbb{E}\left[|\Im \lambda_k|^2\right] \quad (56)$$

and the same for $b_k$. For the 2-soliton, we set the SNR to 10 dB and used $b_k = \frac{1}{2} \alpha \{ 1, 1 \exp(j\alpha) \}$. In the first column of Fig. 3 we plot $\sigma^2_{\lambda_k}$ and $|\Re \lambda_1 \lambda_2|$ as a function of the phase difference $\alpha$ between the spectral coefficients. As we already reported in [14], we see that the eigenvalues of 2-solitons where $b_1$ and $b_2$ have equal phase (or equivalently, $Q_d(\lambda_1)$ and $Q_d(\lambda_2)$ have opposite phase) are less robust to noise. In our simulations, we observed that the correlation between any real part and any imaginary part was negligible, i.e., $\sigma_{\Re \lambda_i \Im \lambda_k} \approx 0$ and $\sigma_{\Re b_i \Im b_k} \approx 0$ for any $i, k$. This might not be true in general.

In the second and third columns of Fig. 3 we vary the SNR and plot $\sigma^2_{\lambda_k}$ for the 5-soliton, and both $\sigma^2_{b_k}$ and $|\Re \sigma_{\lambda_1 \lambda_2}|$ for the sech pulse. We observe that our analytic expressions for the covariances (plotted as solid lines) very accurately predict the numerical covariances obtained using the FC (dashed lines with empty markers) and the FB-CFQM (dotted lines with filled markers). Only for low SNRs the numerical covariances deviate from the analytical ones, probably because of the higher-order terms neglected in our perturbation analysis. Our analytic covariances are in all cases very close to the lowest numerical ones (either FB-CFQM or FC).

**VI. CONCLUSION**

We extended the Fourier Collocation (FC) method [15 Section 2.4.3] to compute the full discrete spectrum ($\lambda_k$ and $b(\lambda_k)$) of an arbitrary pulse. We showed numerically that our extended FC method estimates the discrete spectrum rather precisely with small number of samples. In comparison to the state-of-the-art FB-CFQM algorithm, both algorithms have a comparable estimation error for small number of samples. However, the FB-CFQM estimation errors decrease monotonically in number of samples while the FC estimation errors saturates to an error floor at some number of samples.

We applied perturbation theory of linear operators [16] to our method and derived analytic expressions for the second-order statistics of the discrete spectrum (eigenvalues and spectral amplitudes) of a pulse contaminated with additive white Gaussian noise. Our simulations show that our expressions very accurately predict the numerical statistics. Our expressions, though involved, are much faster than Monte-Carlo simulations and could be used to design better NFT.
transmission systems by avoiding combinations of spectral parameters that are less robust to noise.

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