ON THE FORMAL DEGREE CONJECTURE FOR NON-SINGULAR SUPERCUSPIDAL REPRESENTATIONS

KAZUMA OHARA

Abstract. We prove the formal degree conjecture for non-singular supercuspidal representations based on Schwein’s work [Sch21] proving the formal degree conjecture for regular supercuspidal representations. The main difference between our work and Schwein’s work is that in non-singular case, the Deligne–Lusztig representations can be reducible, and the $S$-groups are not necessary abelian. Therefore, we have to compare the dimensions of irreducible constituents of the Deligne–Lusztig representations and the dimensions of irreducible representations of $S$-groups.

1. Introduction

Let $F$ be a non-archimedean local field of residue characteristic $p$, and let $G$ be a connected reductive group over $F$. For an essentially discrete series representation $\pi$ of $G(F)$, we can consider a positive constant $d(\pi)$ called the formal degree of $\pi$. In some sense, we can consider the formal degree as a generalization of the dimension of a finite-dimensional representation.

Recall that there is a conjectural correspondence called the local Langlands correspondence between the set of irreducible representations of $G(F)$, and a set of pairs $(\varphi, \rho)$, where $\varphi: WF \times SL_2(\mathbb{C}) \to \mathbb{G}$ is an $L$-parameter, and $\rho$ is an irreducible representation of the $S$-group $\pi_0(S^c_{\varphi})$, which is determined by $\varphi$. In [HII08b], Hiraga, Ichino, and Ikeda proposed a conjecture called the formal degree conjecture, which predicts that the formal degree $d(\pi)$ of $\pi$ is equal to

$$\frac{\dim(\rho)}{\left|\pi_0(S^c_{\varphi})\right|} \cdot \left|\gamma(0, \pi, \text{Ad}, \psi)\right|,$$

where $\gamma$ denotes the $\gamma$-factor defined in [HII08b], and $\pi_0(S^c_{\varphi})$ is a finite group determined by $\varphi$.

The formal degree conjecture depends itself on a conjectural correspondence, the local Langlands correspondence. Hence, in order to verify the formal degree conjecture, we must first have access to a candidate for the local Langlands correspondence. In [Kal19b], Kaletha defined and constructed a large class of supercuspidal representations which he calls regular, and gave a candidate for the local Langlands correspondence for regular supercuspidal representations. Kaletha’s construction starts with a pair $(S, \theta)$, where $S$ is a tame elliptic maximal torus, and $\theta$ is a character of $S(F)$ which satisfy some conditions. From the pair $(S, \theta)$, he constructed a generic cuspidal $G$-datum of [Yu01], and then he obtained a regular supercuspidal representation by Yu’s construction. In [Sch21], Schwein calculated the formal degrees of supercuspidal representations which are obtained by Yu’s construction and proved that the formal degree conjecture is true for the local Langlands correspondence in [Kal19b].

On the other hand, in [Kal19c], Kaletha defined and constructed a wider class of supercuspidal representations which he calls non-singular, and also gave a candidate for the local Langlands correspondence for non-singular supercuspidal representations. In this paper, we prove that the formal degree conjecture also holds for the local Langlands correspondence in [Kal19c].

Kaletha’s construction of non-singular supercuspidal representations and the local Langlands correspondence for them are very similar to the one for regular supercuspidal representations. Therefore, we can apply the arguments in [Sch21] in our case. However, both in the automorphic side and the Galois side, there are some differences. In the case of regular supercuspidal representations, the Deligne–Lusztig representations appearing in the construction are irreducible, so we can calculate the formal degrees of the regular supercuspidal representations by using the dimension
formula for Deligne–Lusztig representations. Moreover, in this case, the $S$-groups are abelian, so the factor of the dimension $\dim(\rho)$ of the irreducible representation of the $S$-group in the formal degree conjecture is equal to 1. In the non-singular case, however, the Deligne–Lusztig representations appearing in the construction are not necessary irreducible, and the $S$-groups are not necessary abelian. Kaletha constructed the bijections between the sets of irreducible constituents of the Deligne–Lusztig representations and some sets of irreducible representations of the $S$-groups, from which he constructed the local Langlands correspondence for non-singular supercuspidal representations. We compare the dimensions of an irreducible constituent of the Deligne–Lusztig representation and the corresponding representation of the $S$-group, and by using this comparison, we prove the formal degree conjecture for non-singular case.

We sketch the outline of this paper. In Section 3, we state the formal degree conjecture. In Section 4, we explain Yu’s construction of supercuspidal representations [Yu01] and the calculation of the formal degrees of these representations [Kal19c]. In Section 5, we review the definition and the construction of non-singular supercuspidal representations [Kal19c]. Here, we explain the description of the sets of irreducible constituents of the Deligne–Lusztig representations in $\mathbb{F}$. In Section 6, we introduce the notions of torally wild $L$-parameters and torally wild $L$-packet data, which are defined in [Kal19c], and explain the way to construct $L$-packets from these data. In Section 7, we explain the parametrization of the elements in the $L$-packet by irreducible representations of the $S$-group. Section 8 is our essential part. In this section, we prove a variant of [Kal19c] Proposition B.3. Using the results of this section, we calculate the formal degrees of non-singular supercuspidal representations in Section 9. Finally, in Section 10, we prove that the formal degree conjecture holds for the local Langlands correspondence for non-singular supercuspidal representations.

Acknowledgment. I am deeply grateful to my supervisor Noriyuki Abe for his enormous support and helpful advice. He checked the draft and gave me useful comments. I would also like to thank Tasho Kaletha. He answered my question about twisted Yu’s construction and sent me a preliminary copy of their paper about twisted Yu’s construction. I am supported by the FMSP program at Graduate School of Mathematical Sciences, the University of Tokyo.

2. Notation and Assumptions

Let $F$ be a non-archimedean local field of residue characteristic $p$, and let $k_F$ be its residue field. We fix an algebraic closure $\overline{F}$ of $F$ and write $F^{sep}$ for the separable closure of $F$ in $\overline{F}$. We denote by $k_F$ be the residue field of $F^{sep}$. We write $F^u$ for the maximal unramified extension of $F$. For a field extension $E$ over $F$, we write $O_E$ for its ring of integers.

We denote by $\Gamma_F$ the absolute Galois group $\text{Gal}(F^{sep}/F)$ of $F$, and by $W_F$ the Weil group of $F$. We also denote by $I_F$ the inertia subgroup of $\Gamma_F$, by $P_F$ the wild inertia subgroup of $\Gamma_F$, and by Frob the geometric Frobenius element in $\Gamma_F/I_F$. For $r \geq 0$, let $W_{F_r}$ be the $r$-th subgroup of $W_F$ in its upper numbering filtration computed with respect to the unique discrete valuation $\text{ord}_F$ on $F^\times$ with the value group $\mathbb{Z}$. For a representation $\pi$ of $W_F$, the depth of $\pi$ is defined as

$$\text{depth}(\pi) = \inf\{r \in \mathbb{R}_{\geq 0} \mid \pi(W_{F_r}) = 1\}.$$  

For a field extension $E$ over $F$, we compute the upper numbering filtration $\{W_{E_r}\}_{r \in \mathbb{R}_{\geq 0}}$ of the Weil group $W_E$ of $E$ by using the unique extension of $\text{ord}_F$ to $E^\times$. We compute the depth of a representation of $W_E$ by using this filtration. We also define the filtration $\{E_{r_s}\}_{r \in \mathbb{R}_{\geq 0}}$ of $E^\times$ as

$$E_{r_s} = \{x \in E^\times \mid \text{ord}_F(x - 1) \geq r\}.$$  

Here, we write the unique extension of $\text{ord}_F$ to $E^\times$ by the same symbol. We let $E_{r_+} = \cup_{s > r} E_{s+}$ for $r \geq 0$.

Let $G$ be a connected reductive group over $F$ that splits over a tamely ramified extension of $F$. We denote by $G_{\text{der}}$ the derived subgroup of $G$, by $Z(G)$ the center of $G$, by $Z$ the center of $G_{\text{der}}$, and by $A$ the maximal split central torus of $G$. Let $G^a$ denote the reductive group $G/A$. For a maximal torus $T$ of $G$, we also define $T^a$ be the torus $T/A$.

For a connected reductive group $H$ defined over $F$ or $k_F$, we denote by $\dim(H)$ the dimension of $H$ and denote by $\text{rank}(H)$ the absolute rank of $H$. For a maximal torus $T$ of $H$, let $X^*(T)$ be the character group of $T$, $X_*(T)$ be the cocharacter group of $T$, and $R(H, T)$ be the absolute root system of $H$ with respect to $T$. For a root $\alpha \in R(H, T)$, we denote by $\hat{\alpha}$ the corresponding coroot.
When $H$ is defined over $F$, we denote by $\widehat{H}$ and $L^\times H$ the dual group of $H$ and the $L$-group of $H$ respectively.

We assume that $p$ is an odd prime, is not a bad prime for any irreducible factor of the absolute root system of $G$ in the sense of [SS70, 4.1], dose not divide the number of connected components of $Z(G)$, and dose not divide the order of the fundamental group of $G_{\text{der}}$. Let $q$ be the cardinality of $k_F$, and let $\exp_p: \mathbb{R} \to \mathbb{R}$ be the function $t \mapsto q^t$.

For a connected reductive group $H$ over $F$, we denote by $\mathcal{B}(H, F)$ the enlarged Bruhat–Tits building of $H$ over $F$ and denote by $\mathcal{B}^\text{red}(H, F)$ the reduced building of $H$ over $F$. If $T$ is a maximal, maximally split torus of $H_E := H \times_F E$ for a field extension $E$ over $F$, then $\mathcal{A}(T, E)$ and $\mathcal{A}^\text{red}(T, E)$ denote the apartment of $T$ inside the Bruhat–Tits building $\mathcal{B}(H_E, E)$ and reduced building $\mathcal{B}^\text{red}(H_E, E)$ of $H_E$ over $E$ respectively. For any $y \in \mathcal{B}(H, F)$, we denote by $[y]$ the projection of $y$ on the reduced building and by $H(F)_{[y]}$ (resp. $H(F)_{y}$) the subgroup of $H(F)$ fixing $y$ (resp. $[y]$). For $y \in \mathcal{B}(H, F)$ and $r \in \mathbb{R}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{r^+ \mid r \in \mathbb{R}_{\geq 0}\}$, we write $H(F)_{y, r}$ for the Moy–Prasad filtration subgroup of $H(F)$ of depth $r$ [MP94, MP96]. In the case that $H$ is a torus, we omit the notion $y$ and write $H(F)$, for the Moy-Prasad filtration subgroup of $H(F)$ of depth $r$, which dose not depend on $y$. For an irreducible representation $\pi$ of $H(F)$, we denote by $\text{depth}(\pi)$ the depth of $\pi$ in the sense of [MP96, Theorem 3.5].

For a group $H$ and an $H$-module $M$, we denote by $M^H$ and $M_H$ its invariant and coinvariant respectively. If $H$ is a cyclic group generated by $\sigma \in H$, let $M^\sigma$ and $M_\sigma$ denote $M^H$ and $M_H$ respectively.

For a finite-dimensional representation $\rho$ of a group $H$, we denote by $\dim(\rho)$ the dimension of $\rho$. Suppose that $K$ is a subgroup of $H$ and $h \in H$. We denote $hKh^{-1}$ by $^hK$. If $\rho$ is a representation of $K$, let $^h\rho$ denote the representation $x \mapsto \rho(h^{-1}xh)$ of $^hK$.

Throughout the paper, we fix a prime number $\ell \neq p$ and an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$.

3. The formal degree conjecture

In this section, we explain the formal degree conjecture [HI08a, Conjecture 1.4] of Hiraga, Ichino, and Ikeda.

Let $(\pi, V)$ be an essentially discrete series representation of $G(F)$, i.e., it becomes a discrete series representation after twisting by a character of $G(F)$. We note that all supercuspidal representations of $G(F)$ are essentially discrete series representations. When $\pi$ is a discrete series representation, the formal degree of $\pi$ associated with a Haar measure $\mu$ on $G(F)/A(F)$ is defined by the positive constant $\text{deg}(\pi, \mu)$ which satisfies

$$\int_{G(F)/A(F)} (\pi(g)u, u')(\pi(g)v, v') \, d\mu(g) = \text{deg}(\pi, \mu)^{-1}(u, v)(u', v')$$

for all $u, u', v, v' \in V$, where $(\cdot, \cdot)$ denotes an invariant hermitian inner product on $V$. In general, we define the formal degree of $\pi$ as the formal degree of any discrete series representation of $G(F)$ obtained from $\pi$ by twisting by a character of $G(F)$. We fix a level-zero additive character $\psi$ of $F$, and let $\mu$ be the Haar measure on $G(F)/A(F)$ attached to $\psi$ as in [GG90]. We write $d(\pi) = \text{deg}(\pi, \mu)$.

**Remark 3.1.** In [HI08b], Hiraga, Ichino, and Ikeda use a different Haar measure to state the formal degree conjecture. However, the conjecture is modified in [HI08a], in which the Haar measure above is used.

Let $\varphi: W_F \times \text{SL}_2(\mathbb{C}) \to L^\times G$ be a discrete $L$-parameter. We write $S_{\varphi}$ for the centralizer of $\varphi(W_F \times \text{SL}_2(\mathbb{C}))$ in $\widehat{G}$. Let $(\widehat{G})_{\text{ad}}$ be the derived subgroup of $\widehat{G}$, $(\widehat{G})_{\text{der}}$ be its adjoint group, and $(\widehat{G})_{\text{sc}}$ be the simply connected cover of $(\widehat{G})_{\text{ad}}$. Let $S_{\varphi}^\text{ad}$ be the image of $S_{\varphi}$ in $(\widehat{G})_{\text{ad}}$ and $S_{\varphi}^{\text{sc}}$ be the preimage of $S_{\varphi}^\text{ad}$ in $(\widehat{G})_{\text{sc}}$. We also define $S_{\varphi}^{\text{red}}$ be the preimage of $S_{\varphi}^\text{ad}$ via the natural map $\widehat{G}^\text{ad} \to \widehat{G}$. For these groups, let $\pi_0(\varphi)$ denote the groups of connected components.

It is believed that the $L$-parameter $\varphi$ determines a finite set $\prod_\varphi(G)$ of irreducible representations of $G(F)$ called $L$-packet, whose elements are indexed by a set of irreducible representations of $\pi_0(S_{\varphi}^{\text{red}})$ [Art06]. We now state the formal degree conjecture, which depends on the conjectural correspondence above.
**Conjecture 3.2** ([H1083 Conjecture 1.4]). Let \( \varphi \) be a discrete \( L \)-parameter which corresponds to the \( L \)-packet \( \prod_p (G) \). Assume that \( \pi \in \prod_p (G) \) corresponds to the representation \( \rho \) of \( \pi_0(S^\varphi_\varphi) \). Then
\[
d(\pi) = \frac{\dim(\rho)}{|\pi_0(S^\varphi_\varphi)|} \cdot |\gamma(0, \pi, \text{Ad}, \psi)|,
\]
where \( \gamma \) is the \( \gamma \)-factor defined in [H1083 1].

4. **Yu’s Construction and Formal Degree**

In this section, we review Yu’s construction of supercuspidal representations in [Yu01] and the calculations of the formal degrees of these representations, which are done in [Sch21].

An input for Yu’s construction of supercuspidal representations of \( G(F) \) is a tuple \( \Psi = (\widehat{G}, y, \tau, \rho, \phi) \) called a generic cuspidal \( G \)-datum, where

\( \Phi = \text{ind}_{G(F)}^{G_0(F)} \rho, \phi \) is calculated in [Sch21].

From this datum, Yu constructed a pair \((\rho_{-1}, \phi)\) of a tame elliptic maximal torus \( T \) of \( G \) whose splitting field \( E \) is a tamely ramified extension of \( F \).

**Proposition 4.1** ([Sch21 Theorem A]). Let \( \Phi \) be a generic cuspidal \( G \)-datum. Then, the formal degree \( d(\pi_\Phi) \) of \( \pi_\Phi \) is equal to
\[
d(\pi_\Phi) = \frac{\dim(\rho_{-1})}{[G^0(F)[y] : G^0(F)_{y,0+}]} \exp_q \left( \frac{1}{2} \dim(G^a) + \frac{1}{2} \dim((G^0)^e[0]) + \frac{1}{2} \sum_{i=0}^{d-1} r_i (|R(G^{i+1}, T) - R(G^i, T)|) \right).
\]

5. **An Review of Non-singular Supercuspidal Representations**

In this section, we review the definition and the construction of non-singular supercuspidal representations [Kal99]. The construction of non-singular supercuspidal representations starts with a tame \( kp \)-non-singular elliptic pair \((S, \theta)\), i.e., \( S \) is a tame elliptic maximal torus of \( G \), and \( \theta \) is a character of \( S(F) \) which satisfy the conditions in [Kal99 Definition 3.4.1]. From the pair \((S, \theta)\), Kaletha constructed the tuple
\[
((G^i)_{i=0}^d, y, (r_i)_{i=0}^d, \kappa(S, \phi_{-1}), (\phi_i)_{i=0}^d)
\]
which satisfies the conditions \(D1, D2, D3, \) and \(D5\) in Section \(3\) where \(\kappa_{(S, \varphi, \theta)}\) is a possibly reducible representation of \(G_\emptyset^0(F)^w\) such that \(\rho_1\big|_{G_\emptyset^0(F)^w}\) is the inflation of a cuspidal representation of \(G_\emptyset^0(F)_{y, 0}/G_\emptyset^0(F)_{y, 0+}\). Hence, for each irreducible constituent \(\rho_1\) of \(\kappa_{(S, \varphi, \theta)}\), the tuple
\[
\left( \left( G_i^0 \right)^{\rho_1}_{i=0: y}, (r_i)_{i=0}^{\rho_1}, (\phi_i)_{i=0}^{\rho_1} \right)
\]
is a generic cuspidal \(G\)-datum. Then, we obtain a supercuspidal representation of \(G(F)\) from this datum by the construction of \(\Yu\). An irreducible supercuspidal representation of \(G(F)\) obtained in this way is called non-singular.

We explain the construction of the tuple
\[
\left( \left( G_i^0 \right)^{\rho_1}_{i=0: y}, (r_i)_{i=0}^{\rho_1}, (\phi_i)_{i=0}^{\rho_1} \right)
\]
from a tame \(k_F\)-non-singular elliptic pair \((S, \theta)\).

First, we consider a tame \(k_F\)-non-singular elliptic pair \((S, \theta)\) of depth-zero. In this case, let \(d = r_0 = 0, \varphi_0\) be trivial, and \(\phi_{-1} = \theta\).

Let \([y]\) be the unique Frobenius-fixed point in \(A^\mathrm{red}(S, F^u)\). According to \([\Kal19c, \text{Lemma 3.4.3}]\), the point \([y]\) is a vertex of \(B^\mathrm{red}(G, F)\). We chose \(y \in B(G, F)\) whose projection on \(B^\mathrm{red}(G, F)\) is \([y]\).

There exists a unique smooth integral model of \(G\) whose group of \(O_F\)-points equal to the stabilizer \(G(F)^w\) of \([y]\) in \(G(F)^w\). Let \(G_\emptyset([y])\) be the quotient of the special fiber of this model modulo its unipotent radical. Then \(G_\emptyset([y])\) is a smooth \(k_F\)-group scheme. Let \(G^\circ_\emptyset([y])\) be its neutral connected component, which is the reductive quotient of the special fiber of the parahoric group scheme of \(G\) associated to the vertex \([y]\). We have \(G_\emptyset([y])(k_F) = G(F)_\emptyset_{[y]}/G(F)_{y, 0+}\) and \(G^\circ_\emptyset([y])(k_F) = G(F)^{\emptyset}_{y, 0}/G(F)_{y, 0+}\). \([\Kal19c, \text{Section 3}]\).

We also define the corresponding \(k_F\)-group schemes \(S\) and \(S^\circ\) satisfying \(S(k_F) = S(F)_{S^{\emptyset}}\) and \(S^\circ(k_F) = S(F)_{S^{\emptyset}}\). Then, we put \(G^\circ_\emptyset = S \cdot G^\circ_\emptyset([y])\).

Then, Lang’s theorem implies that \(G_\emptyset|_{(k_F)} = S(F)G(F)_{y, 0}/G(F)_{y, 0+}\). We regard \(\theta\) as a character of \(S(k_F)\).

In \([\Kal19c]\), Kaletha extended the theory of \([\DL76]\) for disconnected groups and defined the representation \(\kappa_{(S, \theta)}\) of \(G_\emptyset([y])(k_F)\) from the pair \((S, \theta)\). Then, we obtain the representation \(\kappa_{(S, \theta)}\) of \(G(F)^w_{[y]}\) by inflating \(\kappa_{(S, \theta)}\) to \(G(F)_{y, 0}\). We explain the construction of \(\kappa_{(S, \theta)}\).

Let \(U \subset G^\circ_\emptyset([y])\) be the unipotent radical of a Borel subgroup of \(G^\circ_\emptyset([y])\) containing \(S^\circ\), and let \(Fr\) denote the Frobenius endomorphism of \(G_\emptyset([y])\). We define the corresponding Deligne–Lusztig variety
\[
Y^{G_\emptyset}_{U} = \{ gU \in G_{\emptyset}^{G_\emptyset} | g^{-1} Fr(g) \in U \cdot Fr(U) \}.
\]
The group \(G_\emptyset([y])(k_F)\) acts on \(Y^{G_\emptyset}_{U}\) by the left multiplication and \(S(k_F)\) acts on \(Y^{G_\emptyset}_{U}\) by the right multiplication, so these groups act on the \(\ell\)-adic cohomology group \(H^i_U(\mathbb{L}_U^{G_\emptyset}, \mathbb{Q}_\ell)\). We define
\[
H^i_U(\mathbb{L}_U^{G_\emptyset}, \mathbb{Q}_\ell) = \{ v \in H^i_U(\mathbb{L}_U^{G_\emptyset}, \mathbb{Q}_\ell) | v \cdot s = \theta(s)v \forall s \in S(k_F) \}.
\]
Then, \(G_\emptyset([y])(k_F)\) acts on \(H^i_U(\mathbb{L}_U^{G_\emptyset}, \mathbb{Q}_\ell)\). According to \([\Kal19c, \text{Corollary 2.6.3}]\), this component vanishes for all but one \(i\), namely \(i = d(U)\), where \(d(U)\) denotes the number of root hyperplanes separating the Weyl chambers of \(U\) and \(Fr(U)\) respectively. We define the representation \(\kappa_{G_\emptyset^0([y])}\) of \(G_\emptyset([y])(k_F)\) by the action of \(G_\emptyset([y])(k_F)\) on \(H^i_{\emptyset U}(\mathbb{L}_U^{G_\emptyset^0([y])}, \mathbb{Q}_\ell)\).

Remark 5.1. Replacing \(G_\emptyset([y])\) with \(G^\circ_\emptyset([y])\) in the construction above, we define the representation \(\kappa_{G^\circ_\emptyset([y])}\) of \(G^\circ_\emptyset([y])(k_F)\). Similarly, replacing \(G_\emptyset([y])\) with \(G^\circ_\emptyset([y])\), \(S\) with \(S^\circ\), and \(\theta\) with \(\theta^\circ := \theta |_{S^\circ}\), we define the representation \(\kappa_{G^\circ_\emptyset([y])}\) of \(G^\circ_\emptyset([y])(k_F)\). We note that \(\kappa_{G^\circ_\emptyset([y])}\) is the usual Deligne–Lusztig representation of \(G^\circ_\emptyset([y])\) arising from the pair \((S^\circ, \theta^\circ)\). According to \([\Kal19c, \text{Corollary 2.6.4}]\) and \([\Kal19c, \text{Remark 2.6.5}]\), \(\kappa_{G^\circ_\emptyset([y])}\) is isomorphic to the induced representation \(\text{Ind}_{G^\circ_\emptyset([y])}(k_F) \kappa_{G_\emptyset([y])}\) and the representation \(\kappa_{G^\circ_\emptyset([y])}\) is obtained by endowing \(\kappa_{G^\circ_\emptyset([y])}\) with a structure of \(G^\circ_\emptyset([y])(k_F)\)-representation. These results are used later in Section 9.
Lemma 5.3. We explain the description of the set $G$ representations, which is done in [Sch21, Corollary 52].

This datum, we obtain a regular supercuspidal representation $S,φ$ of the simply connected cover of the derived subgroup of $G$ with maximal torus $S$ and root system $R_{r_i}$. We set $G^{-1} = S$. According to [Kal19b] Lemma 3.6.1, the subgroups $G^i$ are twisted Levi subgroups of $G$.

We next recall the definition of a Howe factorization.

Definition 5.2 ([Kal19b, Definition 3.6.2]). A Howe factorization of $(S, θ)$ is a sequence of characters $φ = (φ_1, ..., φ_d)$, where $φ_i$ is a character of $G^i(F)$ with the following properties.

1. $θ = \prod_{i=1}^{d} φ_i |_{S(F)}$;
2. For all $0 ≤ i ≤ d$, the character $φ_i$ is trivial on the image of $G^i_{\text{red}}(F)$, where $G^i_{\text{red}}(F)$ denotes the simply connected cover of the derived subgroup of $G^i$;
3. For all $0 ≤ i < d$, $φ_i$ is $G^{i+1}$-generic of depth $r_i$ relative to $y$ in the sense of [Yu01, Section 9]. For $i = d$, $φ_d$ is trivial if $r_{d-1} = r_d$ and has depth $r_d$ otherwise. For $i = -1$, $φ_{-1}$ is trivial if $G^0 = S$ and otherwise satisfies $φ_{-1} |_{S(F)_{y^1}} = 1$.

According to [Kal19b] Proposition 3.6.7, $(S, θ)$ has a Howe factorization. We take a Howe factorization $φ$. Then, $(S, φ_{-1})$ is a tame $k_F$-non-singular elliptic pair of $G^0$, and the character $φ_{-1}$ is of depth-zero. Applying the construction above for the pair $(S, φ_{-1})$, we have the representation $κ_{(S, φ_{-1})}^\prime$ of $G^0(F)[u]$. In this way, we obtain the tuple

$$((G^i)^{d}_i = 0, y, (r_i)^d_{i=0}, κ_{(S, φ_{-1})}^\prime, (φ_i^d)_{i=0}^d)$$

from a tame $k_F$-non-singular elliptic pair $(S, θ)$.

If the pair $(S, θ)$ is regular in the sense of [Kal19b] Definition 3.7.5, the representation $κ_{(S, φ_{-1})}^\prime$ is irreducible, and the tuple

$$((G^i)^d_i = 0, y, (r_i)^d_{i=0}, κ_{(S, φ_{-1})}^\prime, (φ_i^d)_{i=0}^d)$$

is a generic cuspidal $G$-datum [Kal19b] Proposition 3.7.8. Applying the construction of Yu to this datum, we obtain a regular supercuspidal representation $π_{(S, θ)}$ and the formal degree of $π_{(S, θ)}$ is computed by combining Proposition 1.3 and the dimension formula for the Deligne–Lusztig representations, which is done in [Sch21] Corollary 52.

In our case, however, the representation $κ_{(S, φ_{-1})}^\prime$ can be reducible. Hence, in order to obtain a non-singular supercuspidal representation, we have to take an irreducible constituent of $κ_{(S, φ_{-1})}^\prime$. We explain the description of the set $[κ_{(S, φ_{-1})}^\prime]$ of irreducible constituents of $κ_{(S, φ_{-1})}^\prime$ in Section 3. We note that the set $[κ_{(S, φ_{-1})}^\prime]$ does not denote a multiset.

First, we assume that $θ$ is of depth-zero. Let $N(S, G)(F)_θ$ be the stabilizer of the pair $(S, θ)$ in $G(F)$ by the conjugate action, and $N(S, G)[u](k_F)_θ$ be the stabilizer of the pair $(S, θ)$ in $G[F][k_F]$ by the conjugate action. An element $G(F)$ that normalizes $S$ also normalizes $A^\text{red}(S, F^u)$ and acts on $A^\text{red}(S, F^u)$ Frobenius-equivalently, hence fixes the point $[y]$. Therefore, we get the inclusion map $N(S, G)(F)_θ → G(F)[u]$. Hence, we obtain a natural map $p: N(S, G)(F)_θ → N(S, G)[u](k_F)_θ$.

Let $\text{Irr}(N(S, G)(F)_θ)$ be the set of irreducible representations of $N(S, G)(F)_θ$ whose restriction to $S(F)$ is $θ$-isotypic, and let $\text{Irr}(N(S, G)[u](k_F)_θ, θ)$ be the set of irreducible representations of $N(S, G)[u](k_F)_θ$ whose restriction to $S(k_F)$ is $θ$-isotypic.

Lemma 5.3. The natural map $p: N(S, G)(F)_θ → N(S, G)[u](k_F)_θ$ induces a bijection $\text{Irr}(N(S, G)(F)_θ, θ) ↔ \text{Irr}(N(S, G)[u](k_F)_θ, θ)$. 
Proof. The image of $S(F)$ via the natural map $p : N(S,G)(F)_\theta \to N(S,G_{[y]})(k_F)_\theta$ is equal to $S(k_F)$, and [Kal19c, Lemma 3.2.2] implies that the induced map

$$N(S,G)(F)_\theta / S(F) \to N(S,G_{[y]})(k_F)_\theta / S(k_F)$$

is an isomorphism. Since the image of $p$ contains $S(k_F)$, this isomorphism implies that $p$ is a surjection. Let $ker p$ denote the kernel of $p$. Then, $p$ indexes an isomorphism

$$N(S,G)(F)_\theta / ker p \simeq N(S,G_{[y]})(k_F)_\theta,$$

and we can identify the set $\text{Irr}(N(S,G_{[y]})(k_F)_\theta , \theta)$ with the subset of $\text{Irr}(N(S,G)(F)_\theta , \theta)$ consisting of the representations whose restriction to $ker p$ is trivial.

On the other hand, since $p$ induces an isomorphism

$$N(S,G)(F)_\theta / ker p \simeq N(S,G_{[y]})(k_F)_\theta / S(k_F),$$

$ker p$ is contained in $S(F)$. Hence, $ker p$ is equal to the kernel of the restriction $p |_S(F) : S(F) \to S(k_F) = S(F)/S(F)_{0+}$, which is equal to $S(F)_{0+}$. Since $\theta$ is trivial on $S(F)_{0+}$, we conclude that every element in $\text{Irr}(N(S,G)(F)_\theta , \theta)$ is trivial on $ker p$. Therefore, we obtain a bijection

$$\text{Irr}(N(S,G)(F)_\theta , \theta) \longleftrightarrow \text{Irr}(N(S,G_{[y]})(k_F)_\theta , \theta).$$

\[\square\]

In [Kal19c, 2.7], Kaletha defined an action $R^G_{U^0}(\cdot)$ of $N(S,G_{[y]})(k_F)_\theta$ on $H^{d(U)}(Y^G_{[y]}; \mathbb{Q}_\ell)_\theta$ whose restriction to $S(k_F)$ is $\theta^{-1}$-isotypic. The action $R^G_{U^0}(\cdot)$ commutes with the action of $G_{[y]}(k_F)$ by $\kappa^G_{(S,\theta)}$. Therefore, we obtain a representation of $G_{[y]}(k_F) \times N(S,G_{[y]})(k_F)_\theta$ on $H^{d(U)}(Y^G_{[y]}; \mathbb{Q}_\ell)_\theta$.

Let $\kappa^G_{(S,\theta,\rho)}$ denote this $G_{[y]}(k_F) \times N(S,G_{[y]})(k_F)_\theta$-representation. For $\rho \in \text{Irr}(N(S,G_{[y]})(k_F)_\theta , \theta)$, we define

$$\kappa^G_{(S,\theta,\rho)} = \text{Hom}_{N(S,G_{[y]})(k_F)_\theta}(\hat{\rho}, \kappa^G_{(S,\theta)}),$$

where $\hat{\rho}$ denotes the contragredient representation of $\rho$. It is a representation of $G_{[y]}(k_F)$.

Proposition 5.4 ([Kal19c, Theorem 2.7.7]). The map $\rho \mapsto \kappa^G_{(S,\theta,\rho)}$ is a bijection

$$\text{Irr}(N(S,G_{[y]})(k_F)_\theta , \theta) \to \left[ \kappa^G_{(S,\theta)} \right],$$

where $\left[ \kappa^G_{(S,\theta)} \right]$ denotes the set of irreducible constituents in $\kappa^G_{(S,\theta)}$. Moreover, the multiplicity of $\kappa^G_{(S,\theta,\rho)}$ in $\kappa^G_{(S,\theta)}$ is equal to the dimension of $\rho$.

For $\rho \in \text{Irr}(S(G)(F)_\theta$, let $\kappa^G_{(S,\theta,\rho)}$ be the representation of $G(F)_{[y]}$ obtained by the inflation of $\kappa^G_{(S,\theta,\rho)}$. Here, we regard $\rho$ as an element of $\text{Irr}(N(S,G_{[y]})(k_F)_\theta , \theta)$ by the bijection

$$\text{Irr}(N(S,G)(F)_\theta , \theta) \longleftrightarrow \text{Irr}(N(S,G_{[y]})(k_F)_\theta , \theta).$$

of Lemma 5.3.

Corollary 5.5. The map $\rho \mapsto \kappa^G_{(S,\theta,\rho)}$ is a bijection

$$\text{Irr}(N(S,G)(F)_\theta , \theta) \to \left[ \kappa_{(S,\theta)} \right].$$

Moreover, the multiplicity of $\kappa^G_{(S,\theta,\rho)}$ in $\kappa_{(S,\theta)}$ is equal to the dimension of $\rho$.

We now consider the general case. Let $(S, \theta)$ be a tame $k_F$-non-singular elliptic pair of general depth. From the pair $(S, \theta)$, we obtain a sequences of twisted Levi subgroups $\mathcal{G}$ and a sequence of real numbers $\theta^\circ$, and we take a Howe decomposition $\phi$ of $(S, \theta)$ as above. As in the case of depth-zero, we define $N(S,G)(F)_\theta$ to be the stabilizer of the pair $(S, \theta)$ in $G(F)$ by the conjugate action. We also define the groups $N(S,G_{[y]})(F)_{\phi-1}$ and $N(S,G_{[y]})(F)_\theta$ similarly. According to [Kal19c Lemma 3.4.5], the natural inclusion $G_{[y]}(F) \to G(F)$ gives the identification

$$N(S,G_{[y]})(F)_{\phi-1} = N(S,G_{[y]})(F)_\theta = N(S,G)(F)_\theta.$$

ON THE FORMAL DEGREE CONJECTURE FOR NON-SINGULAR SUPERCUSPIDAL REPRESENTATIONS 7
Let \( \text{Irr}(N(S,G)(F)_\theta) \) be the set of irreducible representations of \( N(S,G)(F)_\theta \) whose restriction to \( S(F) \) is \( \theta \)-isotypic, and let \( \text{Irr}(N(S,G^0)(F)_{\phi,-1},\phi,-1) \) be the set of irreducible representations of \( N(S,G^0)(F)_{\phi,-1} = N(S,G)(F)_\theta \) whose restriction to \( S(F) \) is \( \phi,-1 \)-isotypic. Put \( \delta_0 = \prod_{i=0}^{d} \phi_i^{-1} \mid G^0(F) \). Then, \( \rho \mapsto \rho_{-1} := \delta_0 \otimes \rho \) is a bijection
\[
\text{Irr}(N(S,G)(F)_\theta,\theta) \longleftrightarrow \text{Irr}(N(S,G^0)(F)_{\phi,-1},\phi,-1).
\]
Therefore, Corollary 5.5 implies the following result.

**Corollary 5.6.** The map \( \rho \mapsto \kappa_i^{\epsilon}(S,\phi,-1,\rho_{-1}) \) is a bijection
\[
\text{Irr}(N(S,G)(F)_\theta,\theta) \longrightarrow [\kappa_i(S,\phi,-1)].
\]
Moreover, the multiplicity of \( \kappa_i^{\epsilon}(S,\phi,-1,\rho_{-1}) \) in \( \kappa_i(S,\phi,-1) \) is equal to the dimension of \( \rho \).

Let \( \rho \in \text{Irr}(N(S,G)(F)_\theta,\theta) \). We define the non-singular representation \( \pi_i^{S,S,\phi} \) of \( G(F) \) be the representation obtained from the generic cuspidal \( G \)-datum
\[
((G^0)^d_{i=0},y,(r_i)_{i=0}^d,\kappa_i(S,\phi,-1),\phi_i_{i=0}^d)
\]
by the twisted Yu’s construction defined in [FKS].

**Remark 5.7.** We need not to concern ourselves with the precise definition of the twisted Yu’s construction, but only need to know that the supercuspidal representation obtained by twisted Yu’s construction from a generic cuspidal \( G \)-datum
\[
\Psi = (\overrightarrow{G},y,\overrightarrow{r},\overrightarrow{\rho}_{-1},\overrightarrow{\phi})
\]
is the representation \( \text{ind}_{K^d}^{G(F)}(\rho_d \otimes e) \), where \( e \) is a sign character of \( K^d \) and \( \text{ind}_{K^d}^{G(F)}(\rho_d \otimes e) \) denotes the compactly induced representation. According to [Sch21, Lemma 18], the formal degree of the representation \( \pi_\Psi \) is equal to
\[
\frac{\dim(\rho_d)}{\text{vol}(K^d/(K^d \cap A))}
\]
and the formal degree of the representation \( \text{ind}_{K^d}^{G(F)}(\rho_d \otimes e) \) is equal to
\[
\frac{\dim(\rho_d \otimes e)}{\text{vol}(K^d/(K^d \cap A))},
\]
where \( \text{vol}(K^d/(K^d \cap A)) \) denotes the volume of \( K^d/(K^d \cap A) \) with respect to the Haar measure \( \mu \) on \( G(F)/A(F) \) defined in Section 3. Since \( \dim(\rho_d) = \dim(\rho_d \otimes e) \), the formal degree of two representations are equal.

We also define the supercuspidal representation \( \pi_i(S,\theta) \) of \( G(F) \), which is possibly reducible, as the representation obtained form the datum
\[
((G^0)^d_{i=0},x,(r_i)_{i=0}^d,\kappa_i(S,\phi,-1),\phi_i_{i=0}^d)
\]
by the twisted Yu’s construction. Then the set \( [\pi_i(S,\theta)] \) of irreducible constituents of \( \pi_i(S,\theta) \) is described as follows.

**Proposition 5.8 ([Kal19c, Corollary 3.4.7]).** The map \( \rho \mapsto \pi_i^{S,S,\phi} \) is a bijection
\[
\text{Irr}(N(S,G)(F)_\theta,\theta) \longrightarrow [\pi_i(S,\theta)].
\]
Moreover, the multiplicity of \( \pi_i^{S,S,\phi} \) in \( \pi_i(S,\theta) \) is equal to the dimension of \( \rho \).

6. **Toriy wild L-parameters and torially wild L-packet data**

In [Kal19c], Kaletha defined the notions of torially wild \( L \)-parameters and torially wild \( L \)-packet data. First, we recall the definitions of these notions.

**Definition 6.1 ([Kal19c, Definition 4.1.2]).** A discrete \( L \)-parameter
\[
\varphi : W_F \rightarrow L^G
\]
is called torally wild if the projection of \( \varphi(P_F) \) on \( \hat{G} \) is contained in a maximal torus of \( \hat{G} \).

If \( p \) does not divide the order of the Weyl group of \( G \), any discrete \( L \)-parameter \( \varphi : W_F \rightarrow L^G \) is torally wild [Kal19c, Lemma 4.1.3].
Definition 6.2 ([Kal19c] Definition 4.1.4]). A torally wild $L$-packet datum is a tuple $(S, \tilde{j}, \chi, \theta)$, where

1. $S$ is a torus of dimension equal to the absolute rank of $G$, defined over $F$ and splits over a tamely ramified extension of $F$;
2. $\tilde{j}: \tilde{S} \to \tilde{G}$ is an embedding of complex reductive groups whose $\tilde{G}$-conjugacy class is $\Gamma_F$-stable, i.e., for any $\gamma \in \Gamma_F$, the embedding $\gamma \circ \tilde{j} \circ \gamma^{-1}: \tilde{S} \to \tilde{G}$ is $\tilde{G}$-conjugate to $\tilde{j}$;
3. $\chi_0 = (\chi_{0\alpha})_{\alpha \in \mathcal{R}(G, F)}$ is tamely ramified $\chi$-data for $R(G, S^0)$, as explained below;
4. $\theta: S(F) \to \mathbb{C}^\times$ is a character, subject to the condition that $(S, \theta)$ is a tame $F$-non-singular elliptic pair in the sense of [Kal19c] Definition 3.4.1.

We explain the third point. Let $G'$ be the quasi-split inner form of $G$. As explained in [Kal19b, 5.1], the embedding $\tilde{j}$ determines a $\Gamma_F$-stable conjugacy class of embeddings $j: S \to G'$. By choosing an embedding $j: S \to G'$, and pulling back the root system $R(G', jS)$, we obtain a $\Gamma_F$-invariant root system $R(G, S) \subset X^\vee(S)$, which does not depend on the choice of $j$. Let $E$ be the splitting field of $S$. For each positive real number $r$, we define

$$R_r = \{ \alpha \in R(G, S) \mid (\theta \circ N_{E/F} \circ \alpha)(E_r^\times) = 1 \},$$

where $N_{E/F}$ denotes the norm map $S(E) \to S(F)$. We also define $R_{r+} = \cap_{s > r} R_s$ for $r \geq 0$. Let $S^0$ be the connected component of the intersection of the kernels of all elements of $R_{0+}$, and $R(G, S^0)$ be the image of $R(G, S)\backslash R_{0+}$ under the restriction map $X^\vee(S) \to X^\vee(S^0)$.

We now recall the definition of $\chi$-data (see [LS87, 2.5]). Let $R = R(G, S)$ or $R(G, S^0)$. For $\alpha \in R$, let $\Gamma_\alpha$ be the stabilizer of $\alpha$ in $\Gamma_F$ and $F_\alpha$ be the corresponding fixed subfield of $F^{\text{sep}}$. We also define $\Gamma_{\pm \alpha}$ to be the stabilizer of the set $\{\alpha, -\alpha \}$ and $F_{\pm \alpha}$ be the corresponding fixed subfield of $F^{\text{sep}}$. We say that $\alpha$ is symmetric if $F_{\alpha}/F_{\pm \alpha}$ is a quadratic extension, and asymmetric if $F_{\alpha} = F_{\pm \alpha}$.

Let $R_{\text{sym}}$ denote the set of symmetric roots $\alpha$ in $R$. For $\alpha \in R_{\text{sym}}$, we say that $\alpha$ is unramified (resp. ramified) if the extension $F_\alpha/F_{\pm \alpha}$ is unramified (resp. ramified).

A set of $\chi$-data for $R$ consists of characters $\chi_{\alpha}: F_{\alpha}^\times \to \mathbb{C}^\times$, one for each $\alpha \in R$, having the properties $\chi_{-\alpha} = \chi^{-1}_{\alpha}$, $\chi_{\alpha(\gamma)} = \chi_{\alpha} \circ \gamma^{-1}$ for each $\gamma \in \Gamma_F$, and $\chi_{\alpha} |_{F_{\pm \alpha}}$ is the non-trivial quadratic character of $F_{\pm \alpha}$ which is trivial on the image of the norm map $F_{\alpha}^\times \to F_{\pm \alpha}^\times$ for $\alpha \in R_{\text{sym}}$. A set of $\chi$-data $\chi = (\chi_{\alpha})_{\alpha \in R}$ for $R$ is called unramified (resp. tamely ramified) if each character $\chi_{\alpha}$ is unramified (resp. tamely ramified), i.e., trivial on $O_{F_{\alpha}}^\times$ (resp. trivial on $(F_{\alpha})_{0+}^\times$). We also say that a set of $\chi$-data $\chi = (\chi_{\alpha})_{\alpha \in R}$ for $R$ is minimally ramified, if $\chi_{\alpha} = 1$ for asymmetric $\alpha$, and $\chi_{\alpha}$ is tamely ramified for symmetric $\alpha$.

Kalétha defined the notion of morphisms between torally wild $L$-packet data [Kal19c] Definition 4.1.6 and gave a bijection between the set of $\tilde{G}$-conjugacy classes of torally wild $L$-parameters and the set of isomorphism classes of torally wild $L$-packet data [Kal19c, Proposition 4.1.8]. We recall the way to construct torally wild $L$-parameters from torally wild $L$-packet data briefly.

Let $(S, \tilde{j}, \chi_0, \theta)$ be a torally wild $L$-packet datum. From the $\chi$-data $\chi_0$ for $R(G, S^0)$, we obtain $\chi$-data $\chi = (\chi_{\alpha})_{\alpha} \in \mathcal{R}(G, S)$ as follows. For $\alpha \in R(G, S)\backslash R_{0+}$, we let $\chi_{\alpha} = \chi_{\alpha_0} \circ N_{F_{\alpha}/F_{\alpha_0}}$, where $\alpha_0$ denotes the image of $\alpha$ via the restriction map $X^\vee(S) \to X^\vee(S^0)$, and $N_{F_{\alpha}/F_{\alpha_0}}$ denotes the norm map $F_{\alpha}^\times \to F_{\alpha_0}^\times$. For $\alpha \in R_{0+}$, we let $\chi_{\alpha}$ be trivial if $\alpha$ is asymmetric and be the non-trivial unramified quadratic character if $\alpha$ is unramified symmetric. We note that since $(S, \theta)$ is a tame $F$-non-singular elliptic pair, the action of $I_F$ on $R_{0+}$ preserves a set of positive roots. Hence, every symmetric root $\alpha \in R_{0+}$ is unramified.

We extend $\tilde{j}$ to an $L$-embedding $l_{\tilde{j}}: L_\tilde{S} \to L\tilde{G}$ by using this $\chi$-data as in [Kal19a, 6.1] (see also [LS87, 2.6]). Then, let $\varphi = l_{\tilde{j}} \circ \varphi_S$, where $\varphi_S: W_F \to L_\tilde{S}$ is the $L$-parameter attached to the character $\theta$ via the local Langlands correspondence for tori [Yu09, Theorem 7.5]. In this way, we obtain from the tuple $(S, \tilde{j}, \chi_0, \theta)$ a torally wild $L$-parameter $\varphi$.

Finally, we explain the construction of the $L$-packet associated with a torally wild $L$-parameter $\varphi$. Let $(S, \tilde{j}, \chi_0, \theta)$ be the torally wild $L$-packet datum whose isomorphism class corresponds to the $\tilde{G}$-conjugacy class of $\varphi$. As explained in [Kal19c, 4.2], we may assume that the $\chi$-data for $R(G, S)$ obtained from $\theta$ as in [Kal19c, 3.5] is equal to the $\chi$-data for $R(G, S)$ obtained from $\chi_0$ as above. As explained in [Kal19b, 5.1], the embedding $\tilde{j}$ determines the class of admissible embeddings $j: S \to G$. Then, we define the $L$-packet $\mathbb{P}_\varphi(G)$ as the union of $[\pi_{(j, S, \theta)}]$.
where \( j : S \to G \) ranges over the \( G(F) \)-conjugacy classes of admissible embeddings defined over \( F \), and \( \theta_j \) is a character of \( jS(F) \) defined by \( \theta_j(j(s)) = \theta(s) \) for \( s \in S(F) \). The resulting \( L \)-packet \( \prod_\varphi(G) \) does not depend on the choice of torally wild \( L \)-packet datum \((\hat{S}, \hat{\chi}_0, \theta)\) which satisfies the above condition.

### 7. Endoscopy

Let \( \varphi \) be a torally wild \( L \)-parameter which corresponds to the torally wild \( L \)-packet datum \((\hat{S}, \hat{\chi}_0, \theta)\). We regard \( Z \) as a subgroup of \( S \) by using an admissible embedding \( j : S \to G \). We note that this structure does not depend on the choice of an embedding \( j \) (see Kal19b 5.1). Let \( \hat{S}^\varphi \) be the preimage of \( \hat{S}^F \) via the map \( \hat{S}/Z \to \hat{S} \). We also define the group \( S^\varphi \) be the preimage of \( S_\varphi \) via the map \( \hat{G}/Z \to \hat{G} \). For these groups, we denote by \( \pi_0(\tau) \) the groups of connected components.

In the previous section, we define the \( L \)-packet \( \prod_\varphi(G) \) associated with a torally wild \( L \)-parameter \( \varphi \). In Kal19c 4.4, 4.5, Kaletha constructed a bijection between \( \prod_\varphi(G) \) and a set of irreducible representations of \( \pi_0(S^\varphi) \) satisfying some conditions.

**Remark 7.1.** The formulation of the local Langlands correspondence in Kal19c uses the group \( \pi_0(S^\varphi) \) to parametrize the elements of \( L \)-packets. On the other hand, the formulation of the local Langlands correspondence in Art06, on which Conjecture 3.2 depends, uses the group \( \pi_0(S^\varphi_\tau) \) as explained in Section 3. However, in Kal18 4.6, Kaletha gave a dimension-preserving bijection between sets of irreducible representations of two different \( S \)-groups which is compatible with both formulations of the local Langlands correspondence, and proved that the local Langlands correspondence in Kal19c implies the one in Art06. Therefore, it is enough to prove Conjecture 3.2 for the formulation in Kal19c.

The bijection above is given by combining the bijections

\[
[p_{(j, S, \theta_j)}] \leftrightarrow \text{Irr}(\pi_0(S^\varphi_\tau), \eta)
\]

for all admissible embeddings \( j : S \to G \), where \( \eta \) is the character of \( \pi_0((\hat{S})^\varphi) \) which is determined by \( j \) as in Kal19c 4.4, and \( \text{Irr}(\pi_0(S^\varphi_\tau), \eta) \) denotes the set of irreducible representations of \( \pi_0(S^\varphi_\tau) \) whose restriction to \( \pi_0((\hat{S})^\varphi) \) contains \( \eta \). Here, we regard \( \pi_0((\hat{S})^\varphi) \) as a subgroup of \( \pi_0(S^\varphi_\tau) \) by using \( j \) (see Kal19c Corollary 4.3.4).

We now explain the bijection

\[
[p_{(j, S, \theta_j)}] \leftrightarrow \text{Irr}(\pi_0(S^\varphi_\tau), \eta)
\]

in Kal19c. First, we assume that \( \varphi \) is essentially of depth-zero in the sense of Kal19c 4.5.

Proposition 5.8 implies that the map \( \rho \mapsto \rho_{(j, S, \theta_j, \rho)} \) is a bijection

\[
(1) \quad \text{Irr}(N(jS(G)(F)_{\theta_j}, \theta_j) \leftrightarrow [p_{(j, S, \theta_j)}]).
\]

Let \( \Box_2 \) be the pushout of \( \theta_j : jS(F) \to \mathbb{C}^\times \) and the inclusion \( jS(F) \to N(jS(G)(F)_{\theta_j}). \) Then, we obtain the extension

\[
1 \to \mathbb{C}^\times \to \Box_2 \to N(jS(F)(F)_{\theta_j}) \xrightarrow{jS(F)} 1,
\]

and Kal19c Lemma C.5] implies that there exists a natural dimension-preserving bijection

\[
(2) \quad \text{Irr}(N(jS(G)(F)_{\theta_j}, \theta_j) \leftrightarrow \text{Irr}(\Box_2, \text{id}),
\]

where \( \text{Irr}(\Box_2, \text{id}) \) denotes the set of irreducible representations of \( \Box_2 \) whose restriction to \( \mathbb{C}^\times \) is id-isotypic.

On the other hand, according to Kal19c Corollary 4.3.4, there exists an exact sequence

\[
1 \to \pi_0((\hat{S})^\varphi) \to \pi_0(S^\varphi_\tau) \to \Omega(S(G)(F)_{\theta}) \to 1,
\]

where \( \Omega(S(G)) \) denotes the \( \Gamma_F \)-invariant subgroup of the automorphism group of \( S \) defined in Kal19b 5.1, and \( \Omega(S(G)(F)_{\theta}) \) denotes the stabilizer of \( \theta \) in \( \Omega(S(G)(F)) \). Let \( \pi_0(S^\varphi_\tau)_{\eta} \) be the
stabilizer of $\eta$ in $\pi_0(S^+_{\varphi})$ by the conjugate action, and $\Omega(S,G)(F)_{\theta,\eta}$ be the stabilizer of $\eta$ in $\Omega(S,G)(F)_{\theta}$ by the conjugate action. Then, we obtain an exact sequence
\[ 1 \rightarrow \pi_0(S^+_{\varphi}) \rightarrow \pi_0(S^+_{\varphi}) \rightarrow \Omega(S,G)(F)_{\theta,\eta} \rightarrow 1 \]
from the exact sequence above. We define $\Box_1$ be the pushout of $\eta$:
$\pi_0(S^+_{\varphi}) \rightarrow \Omega(S,G)(F)_{\theta,\eta} \rightarrow 1$,
and $\text{[Kal19c]}$ Lemma C.5] implies that there exists a natural bijection
\begin{equation}
\text{Irr}(\pi_0(S^+_{\varphi}), \eta) \leftrightarrow \text{Irr}(\Box_1, \text{id}),
\end{equation}
where $\text{Irr}(\Box_1, \text{id})$ denotes the set of irreducible representations of $\Box_1$ whose restriction to $\mathbb{C}^\times$ is id-isotypic. The bijection (3) is a composition of the natural dimension-preserving bijection
\[ \text{Irr}(\pi_0(S^+_{\varphi}), \eta) \leftrightarrow \text{Irr}(\Box_1, \text{id}) \]
and the bijection
\[ \text{Irr}(\pi_0(S^+_{\varphi}), \eta) \leftrightarrow \text{Irr}(\pi_0(S^+_{\varphi}), \eta) \]
given by the induced representation (see the proof of $\text{[Kal19c]}$ Lemma C.5)). Here, $\text{Irr}(\pi_0(S^+_{\varphi}), \eta)$ denotes the set of irreducible representations of $\pi_0(S^+_{\varphi})$ whose restriction to $\pi_0(S^+_{\varphi})$ is $\eta$-isotypic. Hence, for $\rho \in \text{Irr}(\pi_0(S^+_{\varphi}), \eta)$, the dimension of the representation of $\Box_1$ which corresponds to $\rho$ via the bijection (3) is
\[ \frac{\dim(\pi_0(S^+_{\varphi}))}{\dim(\pi_0(S^+_{\varphi}))} \cdot \dim(\rho). \]
Moreover, $\text{[Kal19b]}$ Lemma 3.4.10] and $\text{[Kal19c]}$ Lemma E.1] imply that $j$ gives an isomorphism
\[ \Omega(S,G)(F)_{\theta,\eta} \rightarrow N(jS,G)(F)_{\theta_j}/jS(F), \]
and $\text{[Kal19c]}$ Proposition 4.5.1] implies that the extensions
\[ 1 \rightarrow \mathbb{C}^\times \rightarrow \Box_2 \rightarrow \frac{N(jS,G)(F)_{\theta_j}}{jS(F)} \rightarrow \Omega(S,G)(F)_{\theta,\eta} \rightarrow 1 \]
\[ 1 \rightarrow \mathbb{C}^\times \rightarrow \Box_1 \rightarrow \Omega(S,G)(F)_{\theta,\eta} \rightarrow 1 \]
above are isomorphic. Therefore, there exists a dimension-preserving bijection
\begin{equation}
\text{Irr}(\Box_1, \text{id}) \leftrightarrow \text{Irr}(\Box_2, \text{id}).
\end{equation}
By combining the bijections (1), (2), (3), (4), we obtain a bijection
\[ [\pi_{(jS,\theta_j)}] \leftrightarrow [\pi_0(S^+_{\varphi}), \eta]. \]
We now consider the general case. Recall that we obtain a sequence of twisted Levi subgroups
\[ \mathcal{G} = (jS = G^{-1} \subset G^0 \subset \cdots \subset G^0 = G) \]
from the pair $(jS, \theta_j)$. As explained in $\text{[Kal19c]}$ 4.4], $\varphi$ is decomposed as $\varphi = L \cdot j_{G^0,G} \circ \varphi_{G^0}$, where $\varphi_{G^0} : W_F \rightarrow L \cdot G^0$ is the torally wild $L$-parameter of $G^0$ corresponding to the torally wild $L$-packet datum $(S, \tilde{j}, 0, \theta)$, which is essentially of depth-zero, and $L \cdot j_{G^0,G} : L \cdot G^0 \rightarrow L \cdot G$ is the extension of $G^0 \rightarrow \tilde{G}$ obtained from the $\chi$-data $\chi_0$ for $R(G, S^0)$ as in $\text{[Kal19a]}$ 6.1]. Moreover, the $L$-embedding $L \cdot j_{G^0,G}$ induces the identification $S^+_{\varphi} = S^+_{\varphi_{G^0}}$. In particular, there exists a canonical dimension-preserving bijection
\begin{equation}
\text{Irr}(\pi_0(S^+_{\varphi}), \eta) \leftrightarrow \text{Irr}(\pi_0(S^+_{\varphi}), \eta).
\end{equation}
On the other hand, regarding $(jS, \theta_j)$ as a tame $F$-non-singular elliptic pair of $G^0$, we obtain the supercuspidal representation $\pi_{(jS,\theta_j)}^{G^0}$ of $G^0(F)$. According to Proposition 5.8, we obtain a bijection
\[ \text{Irr}(N(jS,G^0)(F)_{\theta_j}, \theta_j) \leftrightarrow [\pi_{(jS,\theta_j)}^{G^0}]. \]
Since $N(jS,G^0)(F)_{\theta_j} = N(jS,G^0)(F)_{\theta_j}$, $\text{[Kal19b]}$ Lemma 3.6.5], we obtain a bijection
\begin{equation}
[\pi_{(jS,\theta_j)}^{G^0}] \leftrightarrow \text{Irr}(N(jS,G^0)(F)_{\theta_j}, \theta_j) \leftrightarrow \text{Irr}(N(jS,G^0)(F)_{\theta_j}, \theta_j) \leftrightarrow [\pi_{(jS,\theta_j)}].
\end{equation}
Combining the bijections \((5), (6)\) and the bijection
\[
[\pi^G_{(j,s),h,j}] \longmapsto \text{Irr}(\pi_0(S^+_\varphi|_{G^0}), \eta)
\]
obtained from depth-zero case, we obtain a bijection
\[
(7) \quad [\pi_{(j,S),h,j}] \longmapsto \text{Irr}(\pi_0(S^+_\varphi), \eta).
\]
for general case.

For \(\rho \in \text{Irr}(\pi_0(S^+_\varphi), \eta)\), we write \(\pi_\rho\) for the element in \([\pi_{(j,S),h,j}]\) which corresponds to \(\rho\) via the bijection \((7)\).

8. Key lemma

In this section, we prove a variant of [Kal19c, Proposition B.3] to compare the dimension of \(\rho\) and the formal degree of \(\pi_\rho\).

Contrary to the conventions of the rest of the paper, in this section only, we use the notations below. Let \(G\) be a locally profinite group, \(H \subset G\) be an open normal subgroup of finite index. We assume that \(G/H\) is abelian. Let \(N \subset G\) be a closed subgroup, write \(N_H = N \cap H\), and let \(S \subset N_H\) be an abelian open normal subgroup of \(N\) of finite index.

The group \(N\) acts on \(G\) by the conjugate action, and we can form \(G \times N\). Since \(H\) is a normal subgroup of \(G\), we can also define the subgroup \(H \times N\) of \(G \times N\). Since \(G/H\) is abelian, \(H \times N\) is a normal subgroup of \(G \times N\).

Let \(\theta\) be a smooth character of \(S\) and \(\text{Irr}(N_H, \theta)\) be the set of irreducible representations of \(N_H\) whose restriction to \(S\) is \(\theta\)-isotypic. We assume that \(N\) normalizes the character \(\theta\), and every element in \(\text{Irr}(N_H, \theta)\) is 1-dimensional. Let \(\sigma\) be a smooth of finite-length semisimple representation of \(H \times N\). We assume that for \(s \in S\), \(s^{-1} \times s\) acts on \(\sigma\) by \(\theta(s)\). We also assume that

\[
(1) \quad \text{End}_H(\sigma) = \bigoplus_{n \in N_H/S} C \cdot \sigma(n^{-1} \times n);
\]

\[
(2) \text{For each } g \in G, \text{ the representation } g^\times \theta \mid_H \text{ is isomorphic to } \sigma \mid_H \text{ if } g \in H \cdot N, \text{ and } g\cdot \theta \mid_H \text{ and } \sigma \mid_H \text{ have no common irreducible constituents otherwise;}
\]

\[
(3) \text{Every irreducible constituent of } \sigma \mid_H \text{ has the same dimension.}
\]

**Remark 8.1.** We define the representation \(\sigma'\) of \(H \times N_H\) by \(\sigma'(h, n) = \sigma(hn^{-1} \times n)\). Since \(S\) is of finite index in \(N\), so is \(N_H\), and \(\sigma'\) is semisimple. Then [Kal19c, Lemma B.1] and the first condition above imply that for every \(\bar{\theta} \in \text{Irr}(N_H, \theta)\), there exists exactly one irreducible constituent \(\tau\) of \(\sigma \mid_H\) such that the multiplicity of \(\tau \boxtimes \bar{\theta}\) in \(\sigma'\) is 1. Hence, the third condition above implies that every \(\bar{\theta} \in \text{Irr}(N_H, \theta)\) has the same the multiplicity in \(\sigma' \mid_{N_H}\).

Let \(\text{Ind}_{H \times N}^{G/N} \sigma\) be the induced representation on the space
\[
\{ f : G \times N \to \sigma \mid f(xy) = \sigma(x)f(y) \mid x \in H \times N, y \in G \times N \}.
\]

We define the representation \(I_\sigma\) of \(G \times N\) by \(I(g, n) = \left( \text{Ind}_{H \times N}^{G/N} \sigma \right)(gn^{-1} \times n)\). Since \(H\) is of finite index in \(G\), \(\text{Ind}_{H \times N}^{G/N} \sigma\) and \(I_\sigma\) are semisimple. For \(f \in \text{Ind}_{H \times N}^{G/N} \sigma\), \(g \in G, n \in N, s \in S\), we obtain
\[
\left( \text{Ind}_{H \times N}^{G/N} \sigma \right)(s^{-1} \times s)f \mid (g \times n) = f \left( (g \times n)(s^{-1} \times s) \right)
= f (gs^{-1}n^{-1} \times ns)
= f ( (ns^{-1}n^{-1} \times nsn^{-1})(g \times n) )
= \sigma(n^{-1}n^{-1} \times nsn^{-1})f(g \times n)
= \theta(nsn^{-1})f(g \times n)
= \theta(s)f(g \times n).
\]

Therefore, \((1, s)\) acts on \(I_\sigma\) by \(\theta\) for \(s \in S\).

Let \(\text{Irr}(N, \theta)\) be the set of irreducible representations of \(N\) whose restriction to \(S\) is \(\theta\)-isotypic, and \([I_\sigma \mid_G]\) be the set of irreducible constituents of the \(G\)-representation \(I_\sigma \mid_G\).
The representation $I_\sigma$ is decomposed as

$$I_\sigma = \bigoplus_{\pi \in [I_\sigma \cap \rho \in \text{Irr}(N, \theta)]} (\pi \boxtimes \rho)^{\oplus m_{\pi, \rho}},$$

where $m_{\pi, \rho}$ is the multiplicity of $\pi \boxtimes \rho$ in $I_\sigma$.

**Lemma 8.2.** (1) We have $m_{\pi, \rho} \in \{0, 1\}$, and for any $\rho \in \text{Irr}(N, \theta)$, there exists exactly one $\pi \in [I_\sigma \cap \rho \in \text{Irr}(N, \theta)]$ such that $m_{\pi, \rho} = 1$. So the condition $m_{\pi, \rho} = 1$ defines a correspondence

$$\text{Irr}(N, \theta) \longleftrightarrow [I_\sigma \cap \rho \in \text{Irr}(N, \theta)].$$

(2) For $\rho \in \text{Irr}(N, \theta)$, we write $\pi_\rho \in [I_\sigma \cap \rho \in \text{Irr}(N, \theta)]$ for the unique $G$-representation with $m_{\pi_\rho} = 1$.

Then,

$$\frac{\dim(\pi_\rho)}{\dim(\rho)} = \frac{|G/H| \cdot \dim(\sigma)}{|N/S|}.$$

**Proof.** The first claim follows from [Kal19c, Lemma B.1] and [Kal19c, Proposition B.3].

We prove the second claim. Let $\rho \in \text{Irr}(N, \theta)$. Because of the first claim, $\dim(\pi_\rho)$ is equal to the multiplicity of $\rho$ in $\langle I_\sigma \rangle \cap N$, which is equal to the dimension of

$$\text{Hom}_N(\rho, \langle I_\sigma \rangle \cap N).$$

We write $N' = \{n^{-1} \times n \mid n \in N\} \subset G \rtimes N$. By the isomorphism $N' \simeq N$ defined by $n^{-1} \times n \mapsto n$, we regard $\rho$ as a representation of $N'$. Then,

$$\text{Hom}_N(\rho, \langle I_\sigma \rangle \cap N) = \text{Hom}_N'(\rho, \langle \text{Ind}_{H \rtimes N}^{G \rtimes N} \rangle \cap N')$$

$$= \text{Hom}_N'(\rho, \bigoplus_{g \in N' \cap (G \rtimes N)/(H \rtimes N)} \text{Ind}_{N' \cap (H \rtimes N)}^{N' \cap (G \rtimes N)} \rho_g).$$

Let $C$ be a complete system of a representatives for $G/HN$. For $g \in G, h \in H, n_1, n_2 \in N$, we obtain

$$(n_1^{-1} \times n_1)(g \times 1)(h \times n_2) = ghn_1^{-1} \times n_1n_2.$$

This calculation implies that the set

$$\{g \times 1 \mid g \in C\}$$

is a complete system of a representatives for $N' \cap (G \rtimes N)/(H \rtimes N)$.

Therefore, we obtain

$$\text{Hom}_N'(\rho, \bigoplus_{g \in C} \text{Ind}_{N' \cap (H \rtimes N)}^{N' \cap (G \rtimes N)} \rho_g) = \text{Hom}_N'(\rho, \bigoplus_{g \in C} \text{Ind}_{N' \cap (H \rtimes N)}^{N' \cap (G \rtimes N)} \rho_g).$$

For $g \in C, h \in H, n \in N$, we obtain

$$(g \times 1)(h \times n)(g^{-1} \times 1) = ghng^{-1}n^{-1} \times n,$$

which is an element of $N'$ if and only if $ghng^{-1}n^{-1} = n^{-1}$, i.e., $h = n^{-1}$. Moreover, in the case $h = n^{-1}$, we obtain

$$(g \times 1)(h \times n)(g^{-1} \times 1) = (g \times 1)(n^{-1} \times n)(g^{-1} \times 1) = n^{-1} \times n.$$

Let $(N_H)'$ be the set

$$\{n^{-1} \times n \mid n \in N_H\}.$$

Then, the calculations above imply that for $g \in C$,

$$N' \cap (H \rtimes N) = (N_H)',$$

and the conjugate action of $g \times 1$ on $(N_H)'$ is trivial.

Hence, we deduce that

$$\text{Hom}_N'(\rho, \bigoplus_{g \in C} \text{Ind}_{N' \cap (H \rtimes N)}^{N' \cap (G \rtimes N)} \rho_g) = \text{Hom}_N'(\rho, \bigoplus_{g \in C} \text{Ind}_{(N_H)'} \rho_g).$$
By the isomorphism $N' \simeq N, n^{-1} \times n \mapsto n$, $(N_H)'$ maps to $N_H$. Therefore, by regarding $\sigma$ as a representation of $N_H$ by the isomorphism $N_H \simeq (N_H)' \subset H \rtimes N$, which is equal to $\sigma' |_{N_H}$ in Remark 6.1, we obtain
\[
\text{Hom}_{N'}(\rho, \bigoplus_{g \in C} \text{Ind}_{(N_H)'}^N(\sigma') \mid |_{N_H}) = \text{Hom}_N(\rho, \bigoplus_{g \in C} \text{Ind}_{N_H}^N(\sigma') \mid |_{N_H})
\]
\[
= \bigoplus_{g \in C} \text{Hom}_N(\rho, \text{Ind}_{N_H}^N(\sigma') \mid |_{N_H})
\]
\[
\simeq \bigoplus_{g \in C} \text{Hom}_{N_H}(\rho, \sigma' \mid |_{N_H}).
\]

According to Remark 8.1, there exists an integer $m$ such that $\sigma' \mid |_{N_H}$ is decomposed as
\[
\sigma' \mid |_{N_H} = \left( \bigoplus_{\tilde{\theta} \in \text{Irr}(N_H, \tilde{\theta})} \tilde{\theta} \right) \oplus m.
\]
Since every element in $\text{Irr}(N_H, \tilde{\theta})$ is 1-dimensional, we obtain
\[
m = \frac{\dim(\sigma)}{|\text{Irr}(N_H, \tilde{\theta})|} = \frac{\dim(\sigma)}{|N_H/S|}.
\]
Therefore, if we write the multiplicity of $\tilde{\theta} \in \text{Irr}(N_H, \tilde{\theta})$ in $\rho \mid |_{N_H}$ by $m_{\rho}(\tilde{\theta})$, the dimension of $\text{Hom}_N(\rho, (I_{\rho}) \mid |_N)$ is equal to
\[
|C| \cdot m \cdot \sum_{\tilde{\theta} \in \text{Irr}(N_H, \tilde{\theta})} m_{\rho}(\tilde{\theta}) = |G/HN| \cdot m \cdot \sum_{\tilde{\theta} \in \text{Irr}(N_H, \tilde{\theta})} m_{\rho}(\tilde{\theta})
\]
\[
= |G/HN| \cdot m \cdot \dim(\rho)
\]
\[
= |G/HN| \cdot \dim(\sigma) \cdot \dim(\rho)
\]
\[
= \frac{|G/H| \cdot \dim(\sigma)}{|N_H/S|}.
\]
This completes the proof. \hfill \Box

9. Formal degree of non-singular supercuspidal representations

In this section, we calculate the formal degree of non-singular supercuspidal representations.

Let $(S, \theta)$ be a tame $k_F$-non-singular elliptic pair and $\rho \in \text{Irr}(N(S, G)(F) \mid \theta)$. We calculate the formal degree of $\pi_{(S, \theta, \rho)}^\nu$ defined in Section 5. Since $\pi_{(S, \theta, \rho)}$ is constructed by twisted Yu’s construction from the generic cuspidal $G$-datum
\[
(G, \mathfrak{y}, \mathfrak{r}, \kappa_{(S, \phi_{-1}, \rho_{-1})}, \phi),
\]
Proposition 5.7 and Remark 5.7 imply that the formal degree $d(\pi_{(S, \theta, \rho)}^\nu)$ of $\pi_{(S, \theta, \rho)}^\nu$ is equal to
\[
\dim(\kappa_{(S, \phi_{-1}, \rho_{-1})}) \exp_{\|G^a(y) \cdot \phi_{-1} \mid |_{G^a(y)}} \left( \frac{1}{2} \dim(G^a) + \frac{1}{2} \dim(G^a, 0, 0, \phi_{-1}) + \frac{1}{2} \sum_{i=0}^{d-1} t_i \left( |R(G^a+1, S) - R(G^a, S)| \right) \right).
\]
Next, we calculate $\dim(\kappa_{(S, \phi_{-1}, \rho_{-1})})$ by using the result in Section 8. We apply Lemma 8.2 for,
\[
G = G^0_{[y]}(k_F) = G^0(F)_{[y]} / G^0(F)_{y, 0},
H = G^0_{[y]}(k_F) = S(F)G^0(F)_{y, 0} / G^0(F)_{y, 0},
S = S(k_F) = S(F) / S(F)_{0, 0},
N = N(S, G^0_{[y]}(k_F)_{\phi_{-1}},
N_H = N \cap H = N(S, G^0_{[y]}(k_F)_{\phi_{-1}},
\theta = (\phi_{-1})^{-1}.
\]
According to [Kal19c, Corollary 2.2.2] and [Kal19c, Proposition 2.3.3], every element in
\[ \text{Irr}(N(S, G^0_{[y]})(k_F)_{\phi^{-1}}, (\phi^{-1})^{-1}) \]
is 1-dimensional.

In the proof of [Kal19c, Theorem 2.7.7], Kaletha defined an action \( C_{G^0_{[y]}^\prime, \epsilon} \) of \( N(S, G^0_{[y]})(k_F)_{\phi^{-1}} \) on \( H^2_c(Y_{G^0_{[y]}^\prime}, \mathbb{Q}_{\ell})_{\phi^{-1}} \). The action \( C_{G^0_{[y]}^\prime, \epsilon} \) does not commute with the action of \( \kappa_{(S, \phi^{-1})} \), but instead translates it as
\[ C_{G^0_{[y]}^\prime, \epsilon}(n) \circ \kappa_{(S, \phi^{-1})}(h) = \kappa_{(S, \phi^{-1})}(n h n^{-1}) \circ C_{G^0_{[y]}^\prime, \epsilon}(n) \]
for \( n \in N(S, G^0_{[y]})(k_F)_{\phi^{-1}} \) and \( h \in G^0_{[y]}(k_F) \). Therefore, by using \( C_{G^0_{[y]}^\prime, \epsilon}(n) \), we can extend the representation \( \kappa_{(S, \phi^{-1})} \) to a representation \( \sigma \) of
\[ H \times N = G^0_{[y]}(k_F) \rtimes N(S, G^0_{[y]})(k_F)_{\phi^{-1}}. \]
Kaletha also defined an action \( C_{G^0_{[y]}^\prime, \epsilon} \) of \( N(S, G^0_{[y]})(k_F)_{\phi^{-1}} \) on \( H^2_c(Y_{G^0_{[y]}^\prime}, \mathbb{Q}_{\ell})_{\phi^{-1}} \), which satisfies
\[ C_{G^0_{[y]}^\prime, \epsilon}(n) = \kappa_{(S, \phi^{-1})}(n) \circ R_{G^0_{[y]}^\prime, \epsilon}(n) \]
for \( n \in N(S, G^0_{[y]})(k_F)_{\phi^{-1}} \subset G^0_{[y]}(k_F) \). Using this action, he extended the representation \( \kappa_{(S, \phi^{-1})} \) to a representation \( \Sigma \) of
\[ G \times N = G^0_{[y]}(k_F) \rtimes N(S, G^0_{[y]})(k_F)_{\phi^{-1}}. \]
and proved that
\[ \Sigma \simeq \text{Ind}_{G^0_{[y]}(k_F) \times N(S, G^0_{[y]})(k_F)}^{G^0_{[y]}(k_F) \times N(S, G^0_{[y]})(k_F)_{\phi^{-1}} \times N(S, G^0_{[y]})(k_F)_{\phi^{-1}} \times \sigma} (gn^{-1} \rtimes n) \]
of \( G^0_{[y]}(k_F) \times N(S, G^0_{[y]})(k_F)_{\phi^{-1}} \).

Since the restriction of \( R_{G^0_{[y]}^\prime, \epsilon} \) to \( S(k_F) \) is \( (\phi^{-1})^{-1} \)-isotypic, the action of \( s^{-1} \rtimes s \) by \( \Sigma \) is the multiple of \( (\phi^{-1})^{-1}(s) \) for \( s \in S(k_F) \). In particular, the action of \( s^{-1} \rtimes s \) by \( \Sigma \) is also the multiple of \( (\phi^{-1})^{-1}(s) \) for \( s \in S(k_F) \).

We will verify the assumptions in section 8
(1) \[ \text{End}_H(\sigma) = \bigoplus_{n \in N_N / S} C \cdot \sigma(n^{-1} \rtimes n); \]
(2) For each \( g \in G \), the representation \( g \cdot \sigma |_{H} \) is isomorphic to \( \sigma |_{H} \) if \( g \in H \cdot N \), and \( g \cdot \sigma |_{H} \) and \( \sigma |_{H} \) have no common irreducible constituents otherwise;
(3) Every irreducible constituent of \( \sigma |_{H} \) has the same dimension.

The first two conditions are verified in the proof of [Kal19c, Theorem 2.7.7]. We consider the third point. We note that \( \sigma |_{H} \) is isomorphic to \( \kappa_{(S, \phi^{-1})} \), which is obtained by endowing \( \kappa_{(S^*, \phi^{-1})} \) with a structure of \( G^0_{[y]}(k_F) \)-representation (see Remark 5.1).

Let \( N(S, G^0_{[y]})(k_F)_{\phi^{-1}} \) and \( N(S^*, G^0_{[y]})(k_F)_{\phi^{-1}} \) be the stabilizer of the pair \((S, \phi^{-1})\) and \((S^*, \phi^{-1})\) in \( G^0_{[y]}(k_F) \) by the conjugate action respectively. For \( * = N(S, G^0_{[y]})(k_F)_{\phi^{-1}} \) or \( N(S^*, G^0_{[y]})(k_F)_{\phi^{-1}} \), we define \( \text{Irr}(\ast, \phi^{-1}) \) be the set of irreducible representations of \( \ast \) whose restriction to \( S^* \) is \( \phi^{-1} \)-isotypic. According to [Kal19c, Corollary 2.2.2] and [Kal19c, Proposition 2.3.3], every element in \( \text{Irr}(N(S, G^0_{[y]})(k_F)_{\phi^{-1}}, \phi^{-1}) \) and \( \text{Irr}(N(S^*, G^0_{[y]})(k_F)_{\phi^{-1}}, \phi^{-1}) \) are 1-dimensional. It is shown in the proof of [Kal19c, Theorem 2.7.7] that the irreducible constituents of \( \kappa_{(S^*, \phi^{-1})} \) are indexed by the set \( \text{Irr}(N(S^*, G^0_{[y]})(k_F)_{\phi^{-1}}, \phi^{-1}) \), and the irreducible constituents of \( \kappa_{(S, \phi^{-1})} \) are indexed by the set \( \text{Irr}(N(S, G^0_{[y]})(k_F)_{\phi^{-1}}, \phi^{-1}) \).
We now explain the relationship between the irreducible constituents of $\kappa_{(S^r,\phi^r_{-1})}^{G^0_{[y]}}$ and the irreducible constituents of $\kappa_{(S^r,\phi^r_{-1})}^{G^0_{[y]}}$. Let $\rho \in \text{Irr}(N(S, G^0_{[y]}))(k_F)_{\phi_{-1}}$ and $E_\rho$ be the set of the representation $\rho^r$ of $N(S^r, G^0_{[y]})(k_F)_{\phi_{-1}}$ whose restriction to $N(S, G^0_{[y]})(k_F)_{\phi_{-1}}$ is equal to $\rho$.

**Lemma 9.1.** The cardinality of the set $E_\rho$ is equal to

$$\left| N(S^r, G^0_{[y]})(k_F)_{\phi^r_{-1}} / N(S, G^0_{[y]})(k_F)_{\phi_{-1}} \right|.$$

In particular, it is independent of $\rho$.

**Proof.** Since $E_\rho$ is an $N(S^r, G^0_{[y]})(k_F)_{\phi^r_{-1}} / N(S, G^0_{[y]})(k_F)_{\phi_{-1}}$-torsor, it is enough to show that $E_\rho$ is not empty. According to [Kal19c, Proposition 2.3.3], there exists an extension $\rho^r$ of $\phi_{-1}$ to $N(S^r, G^0_{[y]})(k_F)_{\phi^r_{-1}}$. Then,

$$\rho \cdot (\rho^r |_{N(S, G^0_{[y]})(k_F)_{\phi_{-1}}})^{-1}$$

is a character of $N(S, G^0_{[y]})(k_F)_{\phi_{-1}} / S(k_F)$. Since $N(S^r, G^0_{[y]})(k_F)_{\phi^r_{-1}} / S(k_F)$ is abelian [Kal19c, Corollary 2.2.2], we can extend

$$\rho \cdot (\rho^r |_{N(S, G^0_{[y]})(k_F)_{\phi_{-1}}})^{-1}$$

to a character $\chi$ of $N(S^r, G^0_{[y]})(k_F)_{\phi^r_{-1}} / S(k_F)$. Then, $\rho^r \cdot \chi$ is an element of $E_\rho$. \hfill \Box

In the proof of [Kal19c, Theorem 2.7.7], Kaletha proved that the irreducible constituent of $\kappa_{(S^r,\phi^r_{-1})}^{G^0_{[y]}}$ corresponding to $\rho$ is the direct sum of the irreducible constituents of $\kappa_{(S^r,\phi^r_{-1})}^{G^0_{[y]}}$ corresponding to $\rho^r \in E_\rho$.

On the other hand, according to [Kal19a, 2.3], the conjugate action of the $k_F$-point of the adjoint group of $G^0_{[y]}$ on the set of irreducible constituents of $\kappa_{(S^r,\phi^r_{-1})}^{G^0_{[y]}}$ is transitive. In particular, every irreducible constituent of $\kappa_{(S^r,\phi^r_{-1})}^{G^0_{[y]}}$ has the same dimension. Let $m$ denote the dimension of irreducible constituents of $\kappa_{(S^r,\phi^r_{-1})}^{G^0_{[y]}}$. Then, the argument above implies that the dimensions of irreducible constituents of $\kappa_{(S^r,\phi^r_{-1})}^{G^0_{[y]}}$ are equal to

$$m \cdot \left| N(S^r, G^0_{[y]})(k_F)_{\phi^r_{-1}} / N(S, G^0_{[y]})(k_F)_{\phi_{-1}} \right|.$$ 

In particular, the third assumption is verified.

Now, Lemma [5,2] implies the following theorem, which is a generalization of [Sch21 Corollary 52].

**Theorem 9.2.** The formal degree $d(\pi^r_{(S,\theta,\rho)})$ of $\pi^r_{(S,\theta,\rho)}$ is equal to

$$\frac{\dim(\rho) \cdot \exp_q \left( \frac{1}{2} \dim(G^a) + \frac{1}{2} \text{rank}(G^a)_{[y]} + \frac{1}{2} \sum_{i=0}^{d-1} r_i \left( \dim(G^{i+1}, S) - \dim(G^i, S) \right) \right)}{|N(S, G)(F)_{\theta} / S(F) / S^a(F)_{\theta}|}.$$ 

**Proof.** Recall that the formal degree $d(\pi^r_{(S,\theta,\rho)})$ of $\pi^r_{(S,\theta,\rho)}$ is equal to

$$\frac{\dim(\kappa_{(S,\phi,\rho_{-1})}^{G^0_{[y]}(F)_{[y]}})}{|G^{a,0}(F)_{[y]} : G^{a,0}(F)_{[y]_{\theta}}|} \exp_q \left( \frac{1}{2} \dim(G^a) + \frac{1}{2} \dim(G^{a,0})_{[y]} + \frac{1}{2} \sum_{i=0}^{d-1} r_i \left( \dim(G^{i+1}, S) - \dim(G^i, S) \right) \right).$$

According to Lemma [5,2], the dimension of $\kappa_{(S,\phi,\rho_{-1})}^{G^0_{[y]}}$ is equal to

$$\frac{\dim(\rho)}{|N(S, G^0_{[y]})(k_F)_{\phi_{-1}} / S(k_F)|} \left| C_{[y]}(k_F) / G^0_{[y]}(k_F) \right| \dim(\kappa_{(S,\theta,\rho_{-1})}^{G^0_{[y]}}).$$

According to [Kal19c, Lemma 3.2.2] and [Kal19c, Lemma 3.4.5], we obtain the isomorphism

$$N(S, G^0_{[y]})(k_F)_{\phi_{-1}} / S(k_F) \simeq N(S, G^0_{[y]})(F)_{\phi_{-1}} / S(F) \simeq N(S, G)(F)_{\theta} / S(F).$$
Moreover, [CAR85, Corollary 6.4.3] and [DL76, Corollary 7.2] imply that the dimension \( \dim(\kappa_{[\psi]}^{\mathbb{G}_{\mathcal{O}}}) \) of \( \kappa_{[\psi]}^{\mathbb{G}_{\mathcal{O}}(\theta, \gamma)} \) is equal to
\[
\frac{\exp_{\gamma}\left(\frac{1}{2}\left(\dim(G_{\mathcal{O}}) - \dim(S_{\theta})\right)\right)}{[G_{\mathcal{O}}^0(k_F) : S_{\theta}^0(k_F)]}.
\]

Therefore, we obtain
\[
\left|G_{\mathcal{O}}^0(k_F)/G_{\mathcal{O}}^0(k_F)\right| \dim(\kappa_{[\psi]}^{\mathbb{G}_{\mathcal{O}}(\theta, \gamma)}) = \frac{[G_{\mathcal{O}}^0(k_F) : S_{\theta}^0(k_F)]}{\exp_{\gamma}\left(\frac{1}{2}\left(\dim(G_{\mathcal{O}}^0) - \dim(S_{\theta})\right)\right)}.
\]
\[
= \frac{[G_{\mathcal{O}}^0(k_F) : S_{\theta}^0(k_F)]}{\exp_{\gamma}\left(\frac{1}{2}\left(\dim(G_{\mathcal{O}}^0) - \dim(S_{\theta})\right)\right)}.
\]
\[
= \frac{[G_{\mathcal{O}}^0(k_F) : S_{\theta}^0(k_F)]}{\exp_{\gamma}\left(\frac{1}{2}\left(\dim(G_{\mathcal{O}}^0) - \dim(S_{\theta})\right)\right)}.
\]
\[
= \frac{[G_{\mathcal{O}}^0(k_F) : S_{\theta}^0(k_F)]}{\exp_{\gamma}\left(\frac{1}{2}\left(\dim(G_{\mathcal{O}}^0) - \dim(S_{\theta})\right)\right)}.
\]
\[
\exp_{\gamma}\left(\frac{1}{2}\left(\dim(G_{\mathcal{O}}^0) - \dim(S_{\theta})\right)\right).
\]

Then, the claim follows from a computation.

10. Comparison

In this section, we prove that Kalesha’s construction of the local Langlands correspondence [Kal19c, for torally wild \( L \)-parameters satisfies Conjecture 3.2.

Let \( \varphi \) be a torally wild \( L \)-parameter which corresponds to the torally wild \( L \)-packet datum \((S, \hat{\chi}, \chi_{0}, \theta)\). Conjugating \( \varphi \) in \( \hat{G} \) if necessary, we may assume that the image \( \hat{T} \) of \( j; \hat{S} \to \hat{G} \) is \( \Gamma_F \)-stable. As explained in [Kal19c, 4.2], we may assume that the \( \chi \)-data for \( R(G, S) \) obtained from \( \theta \) as in [Kal19c, 3.5] is equal to the \( \chi \)-data for \( R(G, S) \) obtained from \( \chi_{0} \) as in Section 6.

We need not to concern ourselves with the precise way to obtain the \( \chi \)-data \((\chi_{0}, \alpha)\) for \( R(G, S) \) from \( \theta \), but we only need to know that each \( \chi_{0} \) is tamely ramified of finite order. In particular, the hypothesis of [Sch21] Corollary 67 is satisfied, hence [Sch21] Corollary 67 also holds in our situation.

Let \( j: S \to G \) be an admissible embedding defined over \( F \). By using \( j \), we regard \( A \) as a subgroup of \( S \), and let \( S^{*} \) denote the torus \( S/A \).

Assume that \( \pi \in [\pi_{(j, S, \theta, \rho)}] \) corresponds to \( \rho \in \text{Irr}(\pi_{0}(S_{\mathcal{F}}^{*}), \eta) \) via the bijection (7). We write \( \pi = \pi_{(j, S, \theta, \rho, \eta)} \) for some \( \rho \in \text{Irr}(N(j, S, G)(F)_{\theta, \rho_{0}}) \). By the construction of the bijection (7), we obtain
\[
\dim(\rho_{0}) = \frac{\dim(\rho)}{\pi_{0}(S_{\mathcal{F}}^{*})/\pi_{0}(S_{\mathcal{F}}^{*})}.
\]

On the other hand, Theorem [2.2] implies that the formal degree of \( \pi_{(j, S, \theta, \rho_{0})} \) is equal to
\[
\dim(\rho_{0}) \cdot \exp_{\gamma}\left(\frac{1}{2} \dim(G_{\mathcal{O}}) + \frac{1}{2} \text{rank}(G_{\mathcal{O}}^0)_{[\theta]} \right) + \frac{1}{2} \sum_{i=0}^{d-1} r_{i} \left(\left|R(G^{i+1}, j S) - R(G, j S)\right|\right)
\]
\[
\left|N(j, S, G)(F)_{\theta, j S}(F)\right| S_{\mathcal{O}}(F)/S_{\mathcal{O}}(F)_{0+i}
\]

Therefore, we conclude that the formal degree of \( \pi_{(j, S, \theta, \rho_{0})} \) is equal to
\[
\dim(\rho_{0}) \cdot \exp_{\gamma}\left(\frac{1}{2} \dim(G_{\mathcal{O}}) + \frac{1}{2} \text{rank}(G_{\mathcal{O}}^0)_{[\theta]} \right) + \frac{1}{2} \sum_{i=0}^{d-1} r_{i} \left(\left|R(G^{i+1}, j S) - R(G, j S)\right|\right)
\]
\[
\left|\pi_{0}(S_{\mathcal{F}}^{*})/\pi_{0}(S_{\mathcal{F}}^{*})\right| \frac{N(j, S, G)(F)_{\theta, j S}(F)}{S_{\mathcal{O}}(F)/S_{\mathcal{O}}(F)_{0+i}}
\]

Next, we calculate the \( \gamma \)-factor. We apply the calculation of \( \gamma(0, \pi, \Ad, \psi) \) in [Sch21, Section 4] for our situation. Recall that \( \varphi = \varphi_{S} \circ \phi \), where \( \varphi_{S}: W_{F} \to \hat{L} \) is the \( L \)-parameter for the character \( \theta \), and \( L_{j}: L_{S} \to L_{G} \) is an extension of \( \hat{j} \) defined by using the \( \chi \)-data for \( R(S, G) \) obtained from \( \chi_{0} \). Combining with the adjoint representation \( \Ad(G) \) of \( \hat{G} \) on \( \hat{G} \otimes F \), we define the representation \( \Ad \circ \varphi \) of \( W_{F} \). Here, \( \hat{g} \) and \( \tilde{g} \) are the Lie algebras of \( \hat{G} \) and its center respectively. We calculate the absolute value \( \gamma(0, \Ad \circ \varphi) \) of the \( \gamma \)-factor.
As explained in [Sch21, 4.3], the representation $\text{Ad} \circ \varphi$ decomposes as a direct sum
\[
\hat{\mathfrak{g}}/\hat{T}^F =: V = V_{\text{toral}} \oplus V_{\text{root}},
\]
where $V_{\text{toral}} = \hat{\mathfrak{g}}/\hat{T}^F$, and
\[
V_{\text{root}} = \bigoplus_{\alpha \in R(G,S)} \hat{\mathfrak{g}}_{\alpha}.
\]

Here, $\hat{\mathfrak{g}}$ denotes the Lie algebra of $\hat{T}$, and $\hat{\mathfrak{g}}_{\alpha}$ is the usual $\alpha$-eigenspace for $\hat{T}$, where $\alpha$ is interpreted as a root of $(\hat{G}, \hat{T})$ via the map $\hat{j}: \hat{S} \rightarrow \hat{T}$. Now, we explain the calculation of $|\gamma(0, V_{\text{toral}})|$ and $|\gamma(0, V_{\text{root}})|$ as in [Sch21, Section 4].

First, we explain the calculation of $|\gamma(0, V_{\text{toral}})|$. As explained in [Sch21, 4.4],
\[
V_{\text{toral}} \simeq \mathbb{C} \otimes X^*(\mathfrak{s}^u)
\]
as representations of $W_F$. Let $M = X^*(\mathfrak{s}^u)^F$ and write $M^\vee$ for the dual lattice of $M$. Then, $|\gamma(0, V_{\text{toral}})|$ is calculated as follows.

**Lemma 10.1 (Sch21, Lemma 69).** The absolute value $|\gamma(0, V_{\text{toral}})|$ is equal to
\[
\exp_q \left( \frac{1}{2} (\dim(S^u) + \text{rank}(M)) \right) \frac{|M_{\text{prob}}|}{|k_F^\propto \otimes M^\vee|_{\text{prob}}},
\]

**Remark 10.2.** In [Sch21, Section 4], Schwein assumes that the $L$-parameter $\varphi$ is regular in the sense of [Kal19b, Definition 5.2.3]. However, the argument in [Sch21, Lemma 69] does not require the regularity condition. Hence, [Sch21, Lemma 69] also holds in our situation.

Next, we explain the calculation of $|\gamma(0, V_{\text{root}})|$ [Sch21, 4.5]. Let $R(G, S)$ be the set of $\Gamma_F$-orbits in $R(G, S)$. The root summand $V_{\text{root}}$ decomposes as a direct sum
\[
V_{\text{root}} = \bigoplus_{\alpha \in R(G,S)} V_{\alpha},
\]
where
\[
V_{\alpha} = \bigoplus_{\alpha \in \Delta} V_{\alpha}.
\]
Moreover, by choosing $\alpha \in \Delta$, we can identify $V_{\alpha}$ with the induced representations $\text{Ind}_{W_F}^{W_E} \psi_\alpha$, where $W_\alpha$ denotes the stabilizer of $\alpha$ in $W_F$, and $\psi_\alpha$ denotes the character of $W_\alpha$ defined in [Sch21, 4.5].

Schwein calculated the depths of characters $\psi_\alpha$ in [Sch21, Corollary 71, Lemma 74] to obtain the value of $|\gamma(0, V_{\text{root}})|$. We will follow his arguments in our situation.

For $\alpha \in R(G, S)$, let $\Gamma_\alpha$ be the stabilizer of $\alpha$ in $\Gamma_F$ and $F_\alpha$ be the corresponding fixed subfield of $F_{\text{sep}}$. We consider the character
\[
\theta \circ N_{F_\alpha/F} \circ \hat{\alpha}: F_\alpha^\times \rightarrow \mathbb{C}^\times,
\]
where $N_{F_\alpha/F}$ denotes the norm map $S(F_\alpha) \rightarrow S(F)$. Recall that the depth of $\theta \circ N_{F_\alpha/F} \circ \hat{\alpha}$ is defined as
\[
\inf \{ r \in \mathbb{R} \mid r > 0 \} \supseteq \{ (\theta \circ N_{F_\alpha/F} \circ \hat{\alpha}) (x) \mid (F_\alpha)^\times \} = 1 \}
\]

**Lemma 10.3 (Sch21, Corollary 71).** If the depth of $\theta \circ N_{F_\alpha/F} \circ \hat{\alpha}$ is positive, then it is equal to the depth of $\psi_\alpha$.

**Proof.** We recall that [Sch21, Corollary 67] also holds in our situation. Then, the claim follows from the same argument as [Sch21, Corollary 71].

The case where the depth of $\theta \circ N_{F_\alpha/F} \circ \hat{\alpha}$ is equal to 0 is as follows.

**Lemma 10.4 (Sch21, Lemma 74).** If the depth of $\theta \circ N_{F_\alpha/F} \circ \hat{\alpha}$ is equal to 0, then the depth of $\psi_\alpha$ is equal to 0. Moreover, $\psi_\alpha$ is not an unramified character, i.e., $\psi_\alpha$ is not trivial on the inertia subgroup of $W_\alpha$. 

Proof. According to [Sch21, Lemma 72] and [Sch21, Remark 73], the claim follows when \( jS \) is maximally unramified in \( G \) in the sense of [Kal19b, Definition 3.4.2].

Moreover, according to [Kal19a, Proposition 6.9], we obtain the decomposition \( L_j = L_{jG_0,G} \circ L_{j,S,G_0} \), where \( G_0 \) is the twisted Levi subgroup of \( G \) with maximal torus \( jS \) and satisfying \( R(G_0,S) = R_0 \). \( L_{j,S,G_0} : L_{S} \to L_{G_0} \) is the extension of \( \hat{S} \to \hat{G} \) obtained from the unique minimally ramified \( \chi \)-data for \( R(G_0,S) \), and \( L_{jG_0,G} : \hat{G} \to \hat{G} \) is the extension of \( \hat{G} \to \hat{G} \) obtained from the \( \chi \)-data \( \chi_0 \) for \( R(G,S^0) \) [Kal19a, 6.1]. Since the depth of \( \theta \circ N_{F_u,F} \circ \alpha \) is equal to 0, we obtain from the definition of \( R_0 \) that \( \alpha \in R(G_0,S) \). From the construction of \( L_{jG_0,G} \), we obtain that the embedding \( \hat{G} \to \hat{G} \) is \( L_{jG_0,G} \)-equivariant, where \( \hat{G} \) denotes the Lie algebra of \( G_0 \). In this way, we reduce the case \( G = G_0 \), where \( jS \) is maximally unramified in \( G \). \( \square \)

Now, we calculate \( |\gamma(0, V_{\text{root}})| \). Recall that the \( F \)-non-singular pair \((jS, \theta_j)\) determines a sequence of twisted Levi subgroups

\[
\hat{G} = (jS = G^{-1} \subset G^0 \subset \cdots \subset G^d = G)
\]

and a sequence of real numbers \( r = (0 = r_{-1}, r_0, \ldots, r_d) \) [Kal19b, 3.6].

Lemma 10.5 ([Sch21, Lemma 76]). The absolute value \( |\gamma(0, V_{\text{root}})| \) is equal to

\[
\exp_q \left( \frac{1}{2} |R(G,jS)| + \frac{1}{2} \sum_{i=0}^d r_i (|R(G^{i+1},jS)| - |R(G^i,jS)|) \right).
\]

Proof. The claim follows fromLemma 10.3 and 10.4 as [Sch21, Lemma 76]. \( \square \)

Combining Lemma 10.5 and Lemma 10.6, we obtain the following result.

Proposition 10.6. The absolute value \( |\gamma(0, \pi, \text{Ad}, \psi)| \) is equal to

\[
\frac{|M_{\text{Frob}}|}{|k_F \otimes \mathbb{M}^{\text{Frob}}|} \exp_q \left( \frac{1}{2} \dim(G^a) + \frac{1}{2} \text{rank}(M) + \frac{1}{2} \sum_{i=0}^d r_i (|R(G^{i+1},jS)| - |R(G^i,jS)|) \right).
\]

According to [Sch21, Lemma 78], the rank of \( M = X^*(S^a)_{\text{Frob}} \) is equal to the rank of \( (G^a)^0_{[y]} \). Moreover, according to [Kal19a, Lemma 5.13, Lemma 5.17, Lemma 5.18], we see

\[
\frac{|M_{\text{Frob}}|}{|k_F \otimes \mathbb{M}^{\text{Frob}}|} = \frac{|\pi_0(\hat{S}^a)^0_{[y]}|}{|S^a(F)/S^a(F)_{0+}|},
\]

where \( \hat{S}^a \) denotes the preimage of \( \hat{S}^{\text{Frob}} \) via the map \( \hat{S}/\hat{A} \to \hat{S} \), and \( \pi_0(\hat{S}^a)^0_{[y]} \) denotes the group of connected components of \( \hat{S}^a \) (see [Sch21, Section 5]). Hence, we obtain

\[
|\gamma(0, \pi, \text{Ad}, \psi)| = \frac{|\pi_0(\hat{S}^a)| \cdot \exp_q \left( \frac{1}{2} \dim(G^a) + \frac{1}{2} \text{rank}((G^a)^0_{[y]}) + \frac{1}{2} \sum_{i=0}^{d+1} r_i (|R(G^{i+1},jS)| - |R(G^i,jS)|) \right)}{|S^a(F)/S^a(F)_{0+}|}.
\]

Therefore, to prove Conjecture 3.2, it is enough to show the following lemma.

Lemma 10.7.

\[
|\pi_0(S^a_{\text{Frob}})/\pi_0(S^a_{\text{Frob}})_{\eta}| \leq |N(jS,G)(F)_{\eta}/jS(F)| = \frac{|\pi_0(S^a_{\text{Frob}})|}{|\pi_0(\hat{S}^a)|}.
\]

Proof. According to [Kal19b, Proposition 4.3.2], \( j \) induces an exact sequence

\[
1 \to \hat{S}^{\text{Frob}} \to S^a \to \Omega(S,G)(F)_{\eta} \to jS(F) \to 1.
\]

From this exact sequence, we obtain an exact sequence

\[
1 \to \pi_0(\hat{S}^a) \to \pi_0(S^a) \to \Omega(S,G)(F)_{\eta} \to 1.
\]

Therefore, the right hand side of the lemma is equal to \( |\Omega(S,G)(F)_{\eta}| \).

On the other hand, [Kal19b, Lemma 3.4.10] and [Kal19b, Lemma E.1] imply that \( j \) gives an isomorphism

\[
\Omega(S,G)(F)_{\eta} \to N(jS,G)(F)_{\eta}/jS(F).
\]
Moreover, Corollary 4.3.4] and its pull back along the inclusion \(\Omega(S,G)(F)_{\theta,\eta} \to \Omega(S,G)(F)\) give the two exact sequences
\[
1 \to \pi_0(\hat{S}_F^+ \cap \pi_0(S_F^+)) \to \pi_0(S_F^+) \to \Omega(S,G)(F)_{\theta,\eta} \to 1,
\]
\[
1 \to \pi_0(\hat{S}_F^+) \to \pi_0(\hat{S}_F^+ \cap \pi_0(S_F^+))^{-1} \Omega(S,G)(F)_{\theta,\eta} \to 1.
\]
Hence, we obtain that
\[
\text{(LHS)} = \left| \frac{\pi_0(S_F^+)/\pi_0(S_F^+)}{N(jS,G)(F)_{\theta,\eta} / jS(F)} \right| = \left| \frac{\pi_0(S_F^+)/\pi_0(\hat{S}_F^+)}{\pi_0(\hat{S}_F^+ \cap \pi_0(S_F^+))^{-1} \Omega(S,G)(F)_{\theta,\eta}} \right| = \left| \Omega(S,G)(F)_{\theta,\eta} \right| = \left| \Omega(S,G)(F) \right| = \text{(RHS)}.
\]

\[\square\]

References

[Art06] James Arthur, A note on L-packets, Pure Appl. Math. Q. 2 (2006), no. 1, Special Issue: In honor of John H. Coates. Part 1, 199–217. MR 2217572

[CAR85] R. W. CARTER, Finite groups of lie type : conjugacy classes and complex characters, Pure Appl. Math. 44 (1985).

[DL76] P. Deligne and G. Lusztig, Representations of reductive groups over finite fields, Ann. of Math. (2) 103 (1976), no. 1, 103–161. MR 393266

[FKS] Jessica Fintzen, Tasho Kaletha, and Loren Spice, A twisted Yu construction, Harish-Chandra characters, and endoscopy, in preparation.

[GG99] Benedict H. Gross and Wee Teck Gan, Haar measure and the Artin conductor, Trans. Amer. Math. Soc. 351 (1999), no. 4, 1691–1704. MR 1458303

[HII08a] Kaoru Hiraga, Atsushi Ichino, and Tamotsu Ikeda, Correction to: “Formal degrees and adjoint \(\gamma\)-factors” [J. Amer. Math. Soc. 21 (2008), no. 1, 283–304; mr2350057], J. Amer. Math. Soc. 21 (2008), no. 4, 1211–1213. MR 2425185

[HII08b] Formal degrees and adjoint \(\gamma\)-factors, J. Amer. Math. Soc. 21 (2008), no. 1, 283–304. MR 2350057

[Kal15] Tasho Kaletha, Epipelagic L-packets and rectifying characters, Invent. Math. 202 (2015), no. 1, 1–89. MR 3402796

[Kal18] Global rigid inner forms and multiplicities of discrete automorphic representations, Invent. Math. 213 (2018), no. 1, 271–369. MR 3815567

[Kal19a] On L-embeddings and double covers of tori over local fields, arXiv e-prints (2019), arXiv:1907.05173

[Kal19b] Regular supercuspidal representations, J. Amer. Math. Soc. 32 (2019), no. 4, 1071–1170. MR 4013740

[Kal19c] Supercuspidal L-packets, arXiv e-prints (2019), arXiv:1912.05274

[LS87] R. P. Langlands and D. Shelstad, On the definition of transfer factors, Math. Ann. 278 (1987), no. 1-4, 219–271. MR 909227

[MP94] Allen Moy and Gopal Prasad, Unrefined minimal K-types for p-adic groups, Invent. Math. 116 (1994), no. 1-3, 393–408. MR 1253198

[MP96] Jacquet functors and unrefined minimal K-types, Comment. Math. Helv. 71 (1996), no. 1, 98–121. MR 1371680

[Sch21] David Schwein, Formal Degree of Regular Supercuspidals, arXiv e-prints (2021), arXiv:2104.00658

[SS70] Tonny A Springer and Robert Steinberg, Conjugacy classes, Seminar on algebraic groups and related finite groups, Springer, 1970, pp. 167–266.

[Yu01] Jiu-Kang Yu, Construction of tame supercuspidal representations, J. Amer. Math. Soc. 14 (2001), no. 3, 579–622. MR 1824988

[Yu09] On the local Langlands correspondence for tori, Ottawa lectures on admissible representations of reductive p-adic groups, Fields Inst. Monogr., vol. 26, Amer. Math. Soc., Providence, RI, 2009, pp. 177–183. MR 2508725

Graduate School of Mathematical Science, The University of Tokyo, 3-8-1 Komaba, Meguroku, Tokyo 153-8914, Japan.

Email address: kohara@ms.u-tokyo.ac.jp