THE SL(2, C)-CHARACTER VARIETIES OF TORUS KNOTS

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Abstract. Let $G$ be the fundamental group of the complement of the torus knot of type $(m, n)$. This has a presentation $G = \langle x, y \mid x^m = y^n \rangle$. We find the geometric description of the character variety $X(G)$ of characters of representations of $G$ into $\text{SL}(2, \mathbb{C})$.

Since the foundational work of Culler and Shalen [1], the varieties of $\text{SL}(2, \mathbb{C})$-characters have been extensively studied. Given a manifold $M$, the variety of representations of $\pi_1(M)$ into $\text{SL}(2, \mathbb{C})$ and the variety of characters of such representations both contain information of the topology of $M$. This is specially interesting for 3-dimensional manifolds, where the fundamental group and the geometrical properties of the manifold are strongly related.

This can be used to study knots $K \subset S^3$, by analysing the $\text{SL}(2, \mathbb{C})$-character variety of the fundamental group of the knot complement $S^3 - K$. In this paper, we study the case of the torus knots $K_{m,n}$ of any type $(m, n)$. The case $(m, n) = (m, 2)$ was analysed in [3] and the general case was recently determined in [2] by a method different from ours.

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1. Character varieties

A representation of a group $G$ in $\text{SL}(2, \mathbb{C})$ is a homomorphism $\rho : G \to \text{SL}(2, \mathbb{C})$. Consider a finitely presented group $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_s \rangle$, and let $\rho : G \to \text{SL}(2, \mathbb{C})$ be a representation. Then $\rho$ is completely determined by the $k$-tuple $(A_1, \ldots, A_k) = (\rho(x_1), \ldots, \rho(x_k))$ subject to the relations $r_j(A_1, \ldots, A_k) = 0$, $1 \leq j \leq s$. Using the natural embedding $\text{SL}(2, \mathbb{C}) \subset \mathbb{C}^4$, we can identify the space of representations as

$$R(G) = \text{Hom}(G, \text{SL}(2, \mathbb{C}))$$

$$= \{(A_1, \ldots, A_k) \in \text{SL}(2, \mathbb{C})^k \mid r_j(A_1, \ldots, A_k) = 0, \ 1 \leq j \leq s \} \subset \mathbb{C}^{4k}.$$ 

Therefore $R(G)$ is an affine algebraic set.

We say that two representations $\rho$ and $\rho'$ are equivalent if there exists $P \in \text{SL}(2, \mathbb{C})$ such that $\rho'(g) = P^{-1}\rho(g)P$, for every $g \in G$. This produces an action of $\text{SL}(2, \mathbb{C})$ in
The purpose of this paper is to describe the character variety \( S \) shall denote as \( K \) character, and the converse is also true if \( \rho \) is irreducible [1, Prop. 1.5.2].

There is a character map \( \chi : R(G) \rightarrow \mathbb{C}^G \), \( \rho \mapsto \chi_{\rho} \), whose image

\[
X(G) = \chi(R(G))
\]

is called the character variety of \( G \). Let us give \( X(G) \) the structure of an algebraic variety. By the results of [1], there exists a collection \( g_1, \ldots, g_m \) of elements of \( G \) such that \( \chi_{\rho} \) is determined by \( \chi_{\rho}(g_1), \ldots, \chi_{\rho}(g_m) \), for any \( \rho \). Such collection gives a map

\[
\Psi : R(G) \rightarrow \mathbb{C}^g, \quad \Psi(\rho) = (\chi_{\rho}(g_1), \ldots, \chi_{\rho}(g_m)).
\]

We have a bijection \( X(G) \cong \Psi(R(G)) \). This endows \( X(G) \) with the structure of an algebraic variety. Moreover, this is independent of the chosen collection as proved in [1].

**Lemma 1.1.** The natural algebraic map \( M(G) \rightarrow X(G) \) is a bijection.

**Proof.** The map \( R(G) \rightarrow X(G) \) is algebraic and \( SL(2, \mathbb{C}) \)-invariant, hence it descends to an algebraic map \( \varphi : M(G) \rightarrow X(G) \). Let us see that \( \varphi \) is a bijection.

For \( \rho \) an irreducible representation, if \( \varphi(\rho) = \varphi(\rho') \) then \( \rho \) and \( \rho' \) are equivalent representations; so they represent the same point in \( M(G) \).

Now suppose that \( \rho \) is reducible. Consider \( e_1 \in \mathbb{C}^2 \) the common eigenvector of all \( \rho(g) \). This gives a sub-representation \( \rho' : G \rightarrow \mathbb{C}^* \) of \( \rho \). We have a quotient representation \( \rho'' = \rho / \rho' : G \rightarrow \mathbb{C}^* \), defined as the representation induced by \( \rho \) in the quotient space \( \mathbb{C}^2 / \langle e_1 \rangle \).

As characters, \( \rho'' = \rho^{-1} \). The representation \( \rho' \oplus \rho'' \) is the semisimplification of \( \rho \). It is in the closure of the \( SL(2, \mathbb{C}) \)-orbit through \( \rho \). Clearly, \( \chi_{\rho}(g) = \rho'(g) + \rho''(g)^{-1} \). Now if \( \rho \) and \( \tilde{\rho} \) are two reducible representations and \( \varphi(\rho) = \varphi(\tilde{\rho}) \), then their semisimplifications have the same character, that is

\[
\chi_{\rho}(g) = \chi_{\tilde{\rho}}(g) \Rightarrow \rho'(g) + \rho'(g)^{-1} = \tilde{\rho}'(g) + \tilde{\rho}'(g)^{-1}.
\]

Therefore \( \rho' = \tilde{\rho}' \) or \( \rho' = \tilde{\rho}'^{-1} \). In either case \( \rho \) and \( \tilde{\rho} \) represent the same point in \( M(G) \), which is actually the point represented by \( \rho' \oplus \rho'^{-1} \). \( \square \)

2. Character varieties of torus knots

Let \( T^2 = S^1 \times S^1 \) be the 2-torus and consider the standard embedding \( T^2 \subset S^3 \). Let \( m, n \) be a pair of coprime positive integers. Identifying \( T^2 \) with the quotient \( \mathbb{R}^2 / \mathbb{Z}^2 \), the image of the straight line \( y = \frac{m}{n} x \) in \( T^2 \) defines the torus knot of type \( (m, n) \), which we shall denote as \( K_{m,n} \subset S^3 \) (see [4, Chapter 3]).

For any knot \( K \subset S^3 \), we denote by \( G(K) \) the fundamental group of the exterior \( S^3 - K \) of the knot. It is known that

\[
G_{m,n} = G(K_{m,n}) \cong \langle x, y \mid x^m = y^n \rangle.
\]

The purpose of this paper is to describe the character variety \( X(G_{m,n}) \).
In [3], the character variety $X(G_{m,2})$ is computed. We want to extend the result to arbitrary $m,n$, and give an argument simpler than that of [3].

After the completion of this work, we became aware of the paper [2] where the character varieties of $X(G_{m,n})$ are determined (even without the assumption of $m,n$ being coprime). However, our method is more direct than the one presented in [2].

To start with, note that
\[ R(G_{m,n}) = \{(A, B) \in \text{SL}(2, \mathbb{C}) \mid A^m = B^n\}. \quad (2.1) \]

Therefore we shall identify a representation $\rho$ with a pair of matrices $(A, B)$ satisfying the required relation $A^m = B^n$.

We decompose the character variety
\[ X(G_{m,n}) = X_{\text{red}} \cup X_{\text{irr}}, \]
where $X_{\text{red}}$ is the subset consisting of the characters of reducible representations (which is a closed subset by [1]), and $X_{\text{irr}}$ is the closure of the subset consisting of the characters of irreducible representations.

**Proposition 2.1.** There is an isomorphism $X_{\text{red}} \cong \mathbb{C}$. The correspondence is defined by
\[ \rho = \left( A = \begin{pmatrix} t^n & 0 \\ 0 & t^{-n} \end{pmatrix}, \quad B = \begin{pmatrix} t^m & 0 \\ 0 & t^{-m} \end{pmatrix} \right) \mapsto s = t + t^{-1} \in \mathbb{C}. \]

**Proof.** By the discussion in Lemma 1.1 an element in $X_{\text{red}}$ is described as the character of a split representations $\rho = \rho' \oplus \rho''$. This means that in a suitable basis,
\[ A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}. \quad (2.2) \]
The equality $A^m = B^n$ implies $\lambda^m = \mu^n$. Therefore there is a unique $t \in \mathbb{C}$ with $t \neq 0$ such that
\[ \begin{cases} \lambda = t^n; \\ \mu = t^m. \end{cases} \]
(Here we use the coprimality of $(m,n)$.) Note that the pair $(A, B)$ is well-defined up to permuting the two vectors in the basis. This corresponds to the change $(\lambda, \mu) \mapsto (\lambda^{-1}, \mu^{-1})$, which in turn corresponds to $t \mapsto t^{-1}$. So $(A, B)$ is parametrized by $s = t + t^{-1} \in \mathbb{C}$. \qed

**Lemma 2.2.** Suppose that $\rho = (A, B) \in R(G_{m,n})$. In any of the following cases:

(a) $A^m = B^n \neq \pm \text{Id}$,
(b) $A = \pm \text{Id}$ or $B = \pm \text{Id}$,
(c) $A$ or $B$ is non-diagonalizable,
the representation $\rho$ is reducible.

**Proof.** First suppose that $A$ is diagonalizable with eigenvalues $\lambda, \lambda^{-1}$, and suppose that $\lambda^n \neq \pm 1$. Then there is a basis $e_1, e_2$ in which $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, which is well-determined up to multiplication of the basis vectors by non-zero scalars. Then
\[ B^n = A^m = \begin{pmatrix} \lambda^m & 0 \\ 0 & \lambda^{-m} \end{pmatrix} \]
is a diagonal matrix, different from $\pm \text{Id}$. Therefore $B$ must be diagonal in the same basis, $B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$, with $\lambda^m = \mu^n$. This proves the reducibility in case (a).

Now suppose that $A = \lambda \text{Id}, \lambda = \pm 1$. Then $B^n = \lambda^m \text{Id}$, so it must be that $B$ is diagonalizable. Using a basis in which $B$ is diagonal, we get the reducibility in case (b).

Finally, suppose that $A$ is not diagonalizable. Then there is a suitable basis on which $A$ takes the form $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, with $\lambda = \pm 1$. Clearly

$$B^n = A^m = \begin{pmatrix} 1 & m\lambda \\ 0 & 1 \end{pmatrix}$$

and so

$$B = \begin{pmatrix} \mu & x \\ 0 & \mu \end{pmatrix}$$

with $\mu = \pm 1$, $\mu^n = \lambda^m$ and $\mu nx = \lambda m$. In this basis, the vector $e_1$ is an eigenvector for both $A$ and $B$. Hence the representation $(A, B)$ is reducible, completing the case (c). □

**Proposition 2.3.** Let $X_{irr}^o$ be the set of irreducible characters, and $X_{irr}$ its closure. Then

$$X_{irr}^o = \{ (\lambda, \mu, r) \mid \lambda^m = \mu^n = \pm 1, \lambda \neq \pm 1, \mu \neq \pm 1, r \in \mathbb{C} - \{0,1\} \}/\mathbb{Z}_2 \times \mathbb{Z}_2,$$

$$X_{irr} = \{ (\lambda, \mu, r) \mid \lambda^m = \mu^n = \pm 1, \lambda \neq \pm 1, \mu \neq \pm 1, r \in \mathbb{C} \}/\mathbb{Z}_2 \times \mathbb{Z}_2.$$

where $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts as $(\lambda, \mu, r) \sim (\lambda^{-1}, \mu, 1 - r) \sim (\lambda, \mu^{-1}, 1 - r) \sim (\lambda^{-1}, \mu^{-1}, r)$.

**Proof.** Let $\rho = (A, B)$ be an element of $R(G_{m,n})$ which is an irreducible representation. By Lemma 2.2 $A$ is diagonalizable but not equal to $\pm \text{Id}$, and $A^m = \pm \text{Id}$. So the eigenvalues $\lambda, \lambda^{-1}$ of $A$ satisfy $\lambda^m = \pm 1$ and $\lambda \neq \pm 1$. Analogously, $B$ is diagonalizable but not equal to $\pm \text{Id}$, with eigenvalues $\mu, \mu^{-1}$, with $\mu^n = \pm 1, \mu \neq \pm 1$. Moreover,

$$\lambda^m = \mu^n.$$

We may choose a basis $\{e_1, e_2\}$ under which $A$ diagonalizes. This is well-defined up to multiplication of $e_1$ and $e_2$ by two non-zero scalars. Let $\{f_1, f_2\}$ be a basis under which $B$ diagonalizes, which is well-defined up to multiplication of $f_1, f_2$ by non-zero scalars. Then $\{[e_1], [e_2], [f_1], [f_2]\}$ are four points of the projective line $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$. Note that the pair $(A, B)$ is irreducible if and only if the four points are different.

The only invariant of four points in $\mathbb{P}^1$ is the double ratio

$$r = ([e_1] : [e_2] : [f_1] : [f_2]) \in \mathbb{P}^1 - \{0,1, \infty\} = \mathbb{C} - \{0,1\}. \quad (2.3)$$

So $(A, B)$ is parametrized, up to the action of $\text{SL}(2, \mathbb{C})$, by $(\lambda, \mu, r)$. Permuting the two basis vectors $e_1, e_2$ corresponds to $(\lambda, \mu, r) \mapsto (\lambda^{-1}, \mu, 1 - r)$, since

$$(\lambda, \mu) \mapsto (\lambda, \mu^{-1}, 1 - r).$$

Analogously, permuting the two basis vectors $f_1, f_2$ corresponds to $(\lambda, \mu) \mapsto (\lambda, \mu^{-1}, 1 - r)$. Note that this gives an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $X_{irr}^o$ is the quotient of the set of $(\lambda, \mu, r)$ as above by this action.

To describe the closure of $X_{irr}^o$, we have to allow $f_1$ to coincide with $e_1$. This corresponds to $r = 1$ (the same happens if $f_2$ coincides with $e_2$). In this case, $e_1$ is an
eigenvector of both $A$ and $B$, so the representation $(A, B)$ has the same character as its semisimplification $(A', B')$ given by

$$A' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B' = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}.$$ 

This means that the point $(\lambda, \mu, 1)$ corresponds under the identification $X_{red} \cong \mathbb{C}$ given by Proposition 2.1 to $s_1 = t_1 + t_1^{-1}$, where $t_1 \in \mathbb{C}$ satisfies

$$\begin{cases} 
\lambda = t_1^n, \\
\mu = t_1^m. 
\end{cases} \quad (2.4)$$

Also, we have to allow $f_1$ to coincide with $e_2$ (or $f_2$ to coincide with $e_1$). This corresponds to $r = 0$. The representation $(A, B)$ has semisimplification $(A', B')$ where

$$A' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad B' = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}.$$ 

So the point $(\lambda, \mu, 1)$ corresponds to $s_0 = t_0 + t_0^{-1} \in X_{red} \cong \mathbb{C}$, where $t_0 \in \mathbb{C}$ satisfies

$$\begin{cases} 
\lambda = t_0^n, \\
\mu^{-1} = t_0^m. 
\end{cases} \quad (2.5)$$

Proposition 2.3 says that $X_{irr}$ is a collection of $\frac{(m-1)(n-1)}{2}$ lines. A pair $(\lambda, \mu)$ with $\lambda^m = \pm 1$ and $\mu^n = \pm 1$ is given as

$$\lambda = e^{\pi ik/m}, \quad \mu = e^{\pi ik'/n}, \quad (2.6)$$

where $0 \leq k < 2m$, $0 \leq k' < 2n$. The condition $\lambda \neq \pm 1, \mu \neq \pm 1$ gives $k \neq 0, m, k' \neq 0, n$. Finally, the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action allows us to restrict to $0 < k < m$, $0 < k' < n$. The condition $\lambda^m = \mu^n$ means that

$$k \equiv k' \pmod{2}.$$ 

Denote by $X_{irr}^{k,k'}$ the line of $X_{irr}$ corresponding to the values of $k, k'$. Then

$$X_{irr} = \bigsqcup_{0 < k < m, 0 < k' < n \atop k \equiv k' \pmod{2}} X_{irr}^{k,k'}.$$ 

The line $X_{irr}^{k,k'}$ intersects $X_{red}$ in two points. This gives a collection of $(m-1)(n-1)$ points in $X_{red}$, which are defined as follows: under the identification $X_{red} \cong \mathbb{C}$, these are the points $s_l = t_l + t_l^{-1}$, where

$$t_l = e^{\pi il/mn},$$

and $0 < l < mn$, $m \nmid l$, $n \nmid l$. Assume that $n$ is odd (note that either $m$ or $n$ should be odd). Then from (2.4) and (2.5), the line $X_{irr}^{k,k'}$ intersects at the points $s_{l_0}, s_{l_1} \in X_{red}$ where

$$nl_0 \equiv k \pmod{m}, \quad ml_0 \equiv n - k' \pmod{n}, \quad nl_1 \equiv k \pmod{m}, \quad ml_1 \equiv k' \pmod{n}.$$ 

These two points are different since $k' \neq n - k' \pmod{n}$, as $n$ is odd.

The following is a picture of $X(G_{m,n})$.

**Figure 1**

In the case $(m, n) = (m, 2)$, this result coincides with [3, Corollary 4.2].
3. The algebraic structure of $X(G_{m,n})$

We want to give a geometric realization of $X(G_{m,n})$ which shows that the algebraic structure of this variety is that of a collection of rational lines as in Figure 1 intersecting with nodal curve singularities.

The map $R(G_{m,n}) \to \mathbb{C}^3$, $\rho = (A,B) \mapsto (\text{tr}(A), \text{tr}(B), \text{tr}(AB))$, defines a map

$$\Psi : X(G_{m,n}) \to \mathbb{C}^3.$$ 

**Theorem 3.1.** The map $\Psi$ is an isomorphism with its image $C = \Psi(X(G_{m,n}))$. $C$ is a curve consisting of $\binom{n-1}{m-1} + 1$ irreducible components, all of them smooth and isomorphic to $\mathbb{C}$. They intersect with nodal normal crossing singularities following the pattern in Figure 1.

**Proof.** Let us look first at $\Psi_0 = \Psi|_{X_{\text{red}}} : X_{\text{red}} \to \mathbb{C}^3$. For a given $\rho = (A,B) \in X_{\text{red}}$, with the shape given in Proposition 2.1, we have that

$$\Psi_0 : s = t + t^{-1} \mapsto (t^n + t^{-n}, t^m + t^{-m}, t^{n+m} + t^{-(n+m)}).$$

This map is clearly injective: the image recovers $\{(t^n, t^{-n}), (t^m, t^{-m}), (t^{n+m}, t^{-(n+m)})\}$. From this, we recover $\{(t^n, t^m), (t^{-n}, t^{-m})\}$ and hence the pair $t, t^{-1}$ (since $n,m$ are coprime).

Let us see that $\Psi_0$ is an immersion. The differential is

$$\frac{d\Psi_0}{dt} = (nt^{-n-1}(t^{2n} - 1), mt^{-m-1}(t^{2m} - 1), (n + m)t^{-n-m-1}(t^{2n+2m} - 1)).$$ (3.1)

This is non-zero at all $t \neq \pm 1$. As $\frac{ds}{dt} \neq 0$, we have $\frac{d\Psi_0}{ds} \neq (0, 0, 0)$. For $t = \pm 1$, we note that $\frac{ds}{dt} = t^{-2}(t^2 - 1)$, so

$$\frac{d\Psi_0}{ds} = \left( nt^{-n+1}\frac{t^{2n} - 1}{t^2 - 1}, mt^{-m+1}\frac{t^{2m} - 1}{t^2 - 1}, (n + m)t^{-n-m+1}\frac{t^{2n+2m} - 1}{t^2 - 1} \right),$$

which is non-zero again.

Now, consider a component of $X_{\text{irr}}$ corresponding to a pair $(\lambda, \mu)$. Take $r \in \mathbb{C}$. Fix the basis $\{e_1, e_2\}$ of $\mathbb{C}^2$ which is given as the eigenbasis of $A$. Let $\{f_1, f_2\}$ be the eigenbasis of $B$. As the double ratio $(0 : \infty : 1 : r/(r - 1)) = r$, we can take $f_1 = (1,1)$ and
\[ f_2 = (r - 1, r). \] This corresponds to the matrices:

\[
A = \begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
1 & r - 1 \\
0 & \mu^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & r - 1 \\
0 & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
\frac{r(\mu - \mu^{-1}) + \mu^{-1}}{r(\mu - \mu^{-1})} & (1 - r)(\mu - \mu^{-1}) \\
\mu - r(\mu - \mu^{-1}) & \frac{1}{r(\mu - \mu^{-1})}
\end{pmatrix}.
\]

Therefore:

\[
\Psi(A, B) = (\text{tr}(A), \text{tr}(B), \text{tr}(AB)) = (\lambda + \lambda^{-1}, \mu + \mu^{-1}, (\lambda \mu^{-1} + \lambda^{-1} \mu) + r(\lambda - \lambda^{-1})(\mu - \mu^{-1})).
\]

The image of this component is a line in \( \mathbb{C}^3 \). Its direction vector is \((0, 0, 1)\). At an intersection point with \( \Psi_0(X_{\text{red}}) \), the tangent vector to \( \Psi_0(X_{\text{red}}) \), given in (3.1), has non-zero first and second component, since \( \lambda = t^n, \mu = t^m \) and \( t \neq 0, \lambda^2 \neq 1, \mu^2 \neq 1 \).

So the intersection of these components is a transverse nodal singularity.

Finally, note that the map \( \Psi : X(G_{m,n}) \to C \) is an algebraic map, it is a bijection, and \( C \) is a nodal curve (the mildest possible type of singularities). Therefore \( \Psi \) must be an isomorphism. \( \square \)

**Corollary 3.2.** \( M(G) \cong X(G) \), for \( G = G_{m,n} \).

**Proof.** By Lemma 1.1, \( \varphi : M(G) \to X(G) \) is an algebraic map which is a bijection. As the singularities of \( X(G) \) are just transverse nodes, \( \varphi \) must be an isomorphism. \( \square \)

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