Taylor-Couette flow stability with toroidal magnetic field

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Abstract. The linear stability of the dissipative Taylor-Couette flow with imposed azimuthal magnetic field is considered. Unlike to ideal flow, the magnetic field is fixed function of radius with two parameters only: a ratio of inner to outer cylinder radii and a ratio of the magnetic field values on outer and inner cylinders. The magnetic field with boundary values ratio greater than zero and smaller than inverse radii ratio always stabilizes the flow and called stable magnetic field below. The current free magnetic field is the stable magnetic field. The unstable magnetic field destabilizes every flow if the magnetic field (or Hartmann number) exceeds some critical value. This instability survives even without rotation (for zero Reynolds number). For the stable without the magnetic field flow, the unstable modes are located into some interval of the vertical wave numbers. The interval length is zero for critical Hartmann number and increases with increasing Hartmann number. The critical Hartmann numbers and the length of the unstable vertical wave numbers interval is the same for every rotation law. There are the critical Hartmann numbers for $m=0$ sausage and $m=1$ kink modes only. The critical Hartmann numbers are smaller for kink mode and this mode is the most unstable mode like to the pinch instability case. The flow stability do not depend on the magnetic Prandtl number for $m=0$ mode. The same is true for critical Hartmann numbers for $m=0$ and $m=1$ modes. The typical value of the magnetic field destabilizing the liquid metal Taylor-Couette flow is order of 100 Gauss.

1. Introduction
The Taylor-Couette flow between concentric rotating cylinders is a classical problem of hydrodynamic and hydromagnetic stability [1,2]. According to the Rayleigh criterion the ideal flow is stable whenever the specific angular momentum increases outwards

$$\frac{d}{dR} \left( R^2 \Omega \right)^2 > 0,$$

(1)

where cylindrical system of coordinate $(R, \phi, z)$ is used, and $\Omega$ is the angular velocity.

The vertical magnetic field destabilizes every ideal flow with angular velocity decreasing outwards and the stability condition (1) changes to

$$\frac{d}{dR} \left( \frac{\Omega^2}{R} \right) > 0.$$

(2)

This magnetorotational instability (MRI) has been discovered decades ago [3], but its importance as the source of turbulence in accretion disks with differential (Keplerian) rotation was only recognized by Balbus and Hawley [4]. Their local stability analysis suggests instability regardless of the magnitude of the toroidal magnetic field. It was not a surprise that this result have been reconsidered later [5-7] in the light of the long time known Michael's necessary and sufficient condition [8].
condition says that the ideal Taylor-Couette flow stable to axisymmetric disturbances in the presence of a toroidal magnetic field, $B_{\phi}(R)$, if

$$\frac{1}{R^3} \frac{d}{dR} \left( R^2 \Omega \right)^2 - \frac{R}{\mu_0 \rho} \frac{d}{dR} \left( \frac{B_{\phi}}{R} \right)^2 > 0,$$

where $\rho$ is the density, $\mu_0 = 4\pi$ is the magnetic constant. According to (3), the Taylor-Couette flow with arbitrary angular velocity profile is unstable to axisymmetric disturbances for appropriate toroidal magnetic field value and profile. The destabilizing role of the toroidal magnetic field is also well-known as pinch instability in the plasma theory (see e.g. [9]).

The viscosity has a stabilizing effect and non-magnetized Taylor-Couette flow which is unstable due to (1) becomes really unstable only if the angular velocity of inner cylinder (or its Reynolds number) exceeds some critical value. Moreover, the MRI is very sensitive to the value of magnetic Prandtl number (ratio of viscosity to magnetic diffusivity) value, [10], and until recently did not observe at laboratory due to very small Pm values ($\sim 10^{-5}$ and smaller) of liquid metals. Recently, MRI like behaviour was observed at experiment [11]. Nevertheless, the initial non-magnetize flow was already unstable (turbulent) and a relation of these results with the laminar MRI is not clear.

Theoretical results for the viscous Taylor-Couette flow with an imposed toroidal magnetic field, [12], have demonstrated flow stabilization by the toroidal magnetic field only. Here we perform a more comprehensive study of the non-ideal Taylor-Couette flow stability in the presence of the toroidal magnetic field.

2. Basic equations
Consider a viscous electrically conducting incompressible fluid between two rotating infinite cylinders in the presence of an azimuthal magnetic field. The cylindrical system of coordinates $(R, \phi, z)$ is used. The equations govern the problem are

$$\frac{\partial u_R}{\partial t} + (u \nabla) u_R - \frac{u_R^2}{R} + \frac{1}{4\pi \rho} \left( (B \nabla) B_R - \frac{B_{\phi}^2}{R} \right) = -\frac{1}{\rho \partial R} \left( P + \frac{B^2}{8\pi} \right) + \nu \left( \Delta u_R - \frac{2}{R^2} \frac{\partial u_{\phi}}{\partial R} - \frac{u_R}{R^2} \right),$$

$$\frac{\partial u_{\phi}}{\partial t} + (u \nabla) u_{\phi} + \frac{1}{4\pi \rho} \left( (B \nabla) B_{\phi} + \frac{B_{\phi} B_R}{R} \right) = -\frac{1}{\rho \partial \phi} \left( P + \frac{B^2}{8\pi} \right) + \nu \left( \Delta u_{\phi} + \frac{2}{R^2} \frac{\partial u_R}{\partial \phi} - u_{\phi} \right),$$

$$\frac{\partial u_z}{\partial t} + (u \nabla) u_z + \frac{1}{4\pi \rho} \left( (B \nabla) B_z \right) = -\frac{1}{\rho \partial z} \left( P + \frac{B^2}{8\pi} \right) + \nu \Delta u_z,$$

$$\frac{\partial u_R}{\partial R} + \frac{u_R}{R} + \frac{1}{\rho \partial \phi} + \frac{\partial u_z}{\partial z} = 0,$$

$$\frac{\partial B_R}{\partial t} + (u \nabla) B_R - (B \nabla) u_R = \eta \left( \Delta B_R - \frac{2}{R^2} \frac{\partial B_{\phi}}{\partial \phi} - \frac{B_R}{R^2} \right),$$

$$\frac{\partial B_{\phi}}{\partial t} + (u \nabla) B_{\phi} - (B \nabla) u_{\phi} = \eta \left( \Delta B_{\phi} + \frac{2}{R^2} \frac{\partial B_R}{\partial \phi} - \frac{B_{\phi}}{R^2} \right),$$

$$\frac{\partial B_z}{\partial t} + (u \nabla) B_z - (B \nabla) u_z = \eta \Delta B_z,$$

$$\frac{\partial B_R}{\partial R} + \frac{B_R}{R} + \frac{1}{\rho \partial \phi} + \frac{\partial B_z}{\partial z} = 0,$$

(4)
where
\[ (\mathbf{uV})u_R = u_R \frac{\partial u_R}{\partial R} + u_\phi \frac{\partial u_R}{\partial \phi} + u_z \frac{\partial u_R}{\partial z}, \] (5)

and
\[ \Delta u_R = \frac{\partial^2 u_R}{\partial R^2} + \frac{1}{R} \frac{\partial u_R}{\partial R} + \frac{1}{R^2} \frac{\partial^2 u_R}{\partial \phi^2} + \frac{\partial^2 u_R}{\partial z^2}, \] (6)

\( g \) is the gravity, \( \rho \) is the density, \( \nu = \mu / \rho \) is the kinematic viscosity, \( \eta \) is the magnetic diffusivity, \( \mathbf{u} \) is the velocity, and \( \mathbf{B} \) is the magnetic field. The system (4) admits the solution
\[ U_R = U_z = B_R = B_z = 0, \quad B_\phi = a_\phi R + \frac{b_\phi}{R}, \quad U_\phi = R \Omega = a_\Omega R + \frac{b_\Omega}{R}, \] (7)

where \( a_\Omega, b_\Omega, a_\phi, b_\phi \) are constants defined by boundary conditions:
\[ a_\Omega = \Omega_{in} \mu - \frac{\tilde{\eta}_R^2}{1 - \tilde{\eta}^2}, \quad b_\Omega = \Omega_{in} R_{in} \frac{1 - \mu_\Omega}{1 - \tilde{\eta}^2}, \quad a_\phi = \frac{B_{in}}{R_{in}} \frac{\tilde{\eta}(\mu_\phi - \tilde{\eta})}{1 - \tilde{\eta}_R^2}, \quad b_\phi = \frac{B_{in} R_{in}}{1 - \tilde{\eta}_R^2}. \] (8)

\textbf{Figure 1.} The marginal stability lines for axisymmetric disturbances (m=0) at insulating cylinders for \( \tilde{\eta} = 0.5, \mu_\Omega = 0, \mu_\phi > 0.57 \) (left) and \( \mu_\phi < 0.57 \) (right). The lines are labeled by the \( \mu_\phi \) values.

where
\[ \tilde{\eta} = \frac{R_{in}}{R_{out}}, \quad \hat{\mu}_\Omega = \frac{\Omega_{out}}{\Omega_{in}}, \quad \hat{\mu}_\phi = \frac{B_{out}}{B_{in}}, \] (9)

\( R_{in} \) and \( R_{out} \) are the radii, \( \Omega_{in} \) and \( \Omega_{out} \) are the angular velocities, and \( B_{int} \) and \( B_{out} \) are the azimuthal magnetic fields of the inner and the outer cylinders, respectively.

Note, that for the viscous flow the magnetic field profile is a fixed function of the radius. The first magnetic field term at (7) corresponds to a constant vertical electric current density into the fluid. The second term is current free.

We are interested in the stability of the basic solution (7). The linear stability problem is considered. The perturbed state of the flow is described by
\[ u_R, \quad R \Omega + u_\phi, \quad u_z, \quad B_R, \quad B_\phi + b_\phi, \quad b_z. \] (10)
By developing the disturbances into normal modes, the solutions of the linearized MHD equations are considered in the form
\[ F = F(R) \exp(i(kz + m\phi + \sigma t)), \] (11)
where \( F \) is every of the velocity or magnetic field disturbances.

The dimensionless numbers of the problem are the magnetic Prandtl number, \( P_m \), Hartmann number, \( H_a \), and Reynolds number, \( R_e \),

\[ P_m = \frac{\nu}{\eta}, \quad H_a = \frac{B_{in} R_0}{(\mu_0 \rho \eta)^{0.5}}, \quad R_e = \frac{\Omega_i \omega R_0^2}{\nu}, \] (12)

Figure 2. The same as in Figure 1 but for conducting cylinders.

where \( R_0 = (R_{in} (R_{out} - R_{in}))^{0.5} \) is the length unit.

Using normal mode expansion (11), linearizing system (4) and writing it as the system of first order equations we have

\[
\begin{align*}
\frac{dP}{dR} + i \frac{m}{R} X_2 + i k X_3 + \left( k^2 + \frac{m^2}{R^2} \right) u_R + i \Re (\omega + m\Omega) u_R - 2 \Omega \Re u_\phi - i H_a \frac{m}{R} B_\phi b_R + 2 H_a \frac{B_\phi}{R} b_R = 0, \\
\frac{dX_2}{dR} - \left( k^2 + \frac{m^2}{R^2} \right) u_\phi - i \Re (\omega + m\Omega) u_\phi + 2i \frac{m}{R^2} u_R - 2 \Re \frac{1}{R} \frac{d}{dR} \left( R^2 \Omega \right) + i H_a \frac{m}{R} B_\phi b_R - i \frac{m}{R} P = 0, \\
\frac{dX_3}{dR} + \frac{X_3}{R} - \left( k^2 + \frac{m^2}{R^2} \right) u_z - i \Re (\omega + m\Omega) u_z - ik P + i H_a \frac{m}{R} B_\phi b_z = 0, \\
\frac{du_R}{dR} + \frac{u_R}{R} + i \frac{m}{R} u_\phi + i k u_z = 0, \\
\frac{du_\phi}{dR} + \frac{u_\phi}{R} - X_2 = 0, \\
\frac{du_z}{dR} - X_3 = 0,
\end{align*}
\]
\[
\begin{align*}
\frac{db_R}{dR} + \frac{b_k}{R} + i \frac{m}{R} b_\phi + i k b_z &= 0, \\
\frac{db_\phi}{dR} + \frac{b_\phi}{R} - X_4 &= 0, \\
\frac{dX_4}{dR} &= \left(k^2 + \frac{m^2}{R^2}\right)b_\phi - i \text{Re} \{\omega + m\Omega\} b_R + \text{Im} \text{Re} R \frac{d\Omega}{dR} b_R - R \frac{d}{dR} \left(\frac{B_\phi}{R}\right) u_R + i \frac{m}{R} B_\phi u_\phi = 0 \\
\frac{db_z}{dR} &= -\frac{i}{k} \left(k^2 + \frac{m^2}{R^2}\right)b_R + \text{Im} \text{Re} \left(\frac{1}{k} (\omega + m\Omega) b_R + \frac{1}{k} \frac{m}{R} X_4 - \frac{1}{k} \frac{m}{R} B_\phi u_R = 0, \right.
\end{align*}
\]

where 5-th, 6-th and 8-th equations are the definitions of \(X_2\), \(X_3\) and \(X_4\), respectively. We use \(R_0\) as unit of a length and \(R_0^{-1}\) as unit of the wave number, \(\eta/R_0\) as unit of the perturb velocity, \(\Omega_{in}\) as unit of the angular velocity and \(\omega\), and \(B_0\) as unit of the magnetic fields (basic and disturbed).

An appropriate set of ten boundary conditions is needed to solve system (13). The no-slip condition for the velocity on the walls is used, i.e.
\[
u_R = u_\phi = u_z = 0. \tag{14}
\]

The boundary conditions for the magnetic field depend on the electrical properties of the walls. The tangential currents and the radial component of the magnetic field vanish on the conducting walls, and hence
\[
\frac{dB_\phi}{dR} + \frac{B_\phi}{R} = B_R = 0. \tag{15}
\]

The boundary conditions (14) and (15) hold for both inner and outer cylinders.

The situation changes for insulating walls. The magnetic field must match the external magnetic field for vacuum. The condition \(\text{curl}_e(B) = 0\) in vacuum immediately provides
\[
B_\phi = \frac{m}{kR} B_z \tag{16}
\]

at inner and outer cylinders. From the solution of the potential equation \(\Delta \psi = 0\) (where \(B = \nabla \psi\)) one finds
\[
B_R + \frac{i B_z}{I_m(kR)} \left(\frac{m}{kR} I_m(kR) + I_{m+1}(kR)\right) = 0 \tag{17}
\]

for \(R = R_{in}\) and
\[
B_R + \frac{i B_z}{K_m(kR)} \left(\frac{m}{kR} K_m(kR) + K_{m+1}(kR)\right) = 0 \tag{18}
\]

for \(R = R_{out}\). \(I_m\) and \(K_m\) are modified Bessel function with finite limit as \(R \to 0\) and \(R \to \infty\) respectively. The same numerical method as in [10] was used.

3. Results
Using (7) - (9) for angular velocity and magnetic field, the normalized Michael's condition (3) takes the form
\[
4a^2 + 4 \frac{a\Omega b_\Omega}{R^2} + 4\epsilon \left(\frac{a_\phi b_\phi}{R^2} + \frac{b_\phi^2}{R^4}\right) > 0, \tag{19}
\]

where
\[ \alpha = \frac{V_A^2}{(R_0 \Omega_{in})^2}, \]  
(20)

and \( V_A \) is the Alfvén velocity \( V_A = B_{in}^2 / \mu_0 \rho \).

The angular velocity part (i.e. sum of the first two terms of the (17) is positive if \( \hat{\mu} \Omega > \hat{\eta}^2 \) (see e.g. [13]). The magnetic field stabilize the flow (the sum of the last two terms are positive) if

\[ 0 \leq \hat{\mu} \phi \leq \frac{1}{\hat{\eta}}. \]  
(21)

and destabilize the flow otherwise. The magnetic field will be called stable magnetic field if \( \hat{\mu} \phi \) lays into interval (21) and an unstable magnetic field otherwise.

For unstable magnetic field there is some critical value of the constant \( \alpha \) depending on \( \hat{\mu} \Omega \) for which the ideal flow becomes unstable. Let us note, that the larger value of the constant \( \alpha \) (i.e. the larger magnetic field and the slower rotation) is more preferable for instability.

\[ \text{Figure 3.} \text{ The marginal stability lines for axisymmetric disturbances (m=0) at insulating cylinders (left) and conducting ones (right) with } \hat{\eta}=0.5 \text{ and Re}=0. \text{ The lines are labeled by the } \hat{\mu} \phi \text{ values.} \]

The critical values of the Reynolds numbers above which the viscous flow becomes unstable depend on vertical wave numbers. These critical values have minimum at some wave number for fixed other parameters. This minimum value is called critical Reynolds number.

Let us first consider a case of the hydrodynamically unstable flow (i.e flow unstable even without magnetic field). Figs. 1 and 2 present the critical Reynolds number for axisymmetric disturbances as a function of the Hartmann number for insulating and conducting cylinders. The critical Reynolds number do not depend on magnetic Prandtl number. The critical Reynolds numbers increase with increasing Hartmann numbers for the stable magnetic field and decrease for large \( \text{Ha} \) for unstable magnetic field (the critical Reynolds numbers can temporarily decrease for intermediate \( \text{Ha} \) as for \( \hat{\mu} \phi = 0.5 \) on Figs. 1, 2). The critical Reynolds numbers even vanish if Hartmann number is larger than some critical Hartmann number, \( \text{Ha}_{cr} \). Figure 4 demonstrates that the unstable magnetic field with \( \text{Ha} > \text{Ha}_{cr} \) destabilizes every flow (even the flow with increasing angular velocity). Nevertheless, the stable without magnetic field flow keeps the stability in the presence of the unstable magnetic field.
with $Ha < Ha_{cr}$. Figure 3 presents the critical Hartmann numbers as function of the vertical wave numbers. Note, that for conducting boundary condition the disturbances with smallest $Ha_{cr}$ are one-dimensional (the vertical wave number $k=0$) for all $\hat{\mu}_\phi$ but $-1 < \hat{\mu}_\phi < 0$. The critical Hartmann numbers do not depend on the rotation parameter $\hat{\mu}_\Omega$. Moreover, the vertical wave numbers for which critical Reynolds numbers equal zero are also do not depend on $\hat{\mu}_\Omega$ (see Figure 4).

Figure 4 demonstrates that for unstable rotation $\hat{\mu}_\Omega < \hat{\eta}$ and unstable magnetic field with $Ha > Ha_{cr}$ the flow unstable (there is critical Reynolds number) for any wave number which does not equal 0. For stable rotation the flow unstable only for wave numbers between some minimum wave number, $k_{min}$, and maximum wave number $k_{max}$ ($k_{min} < k_{max}$). The $k_{min}$ decreases and the $k_{max}$ increases with increasing $Ha$. So, the interval between critical wave numbers is larger for larger $Ha$. For conducting boundaries $k_{min} = 0$ except for $-1 < \hat{\mu}_\phi < 0$ (see Figures 3, 4). This behavior is the same as found by Pessah and Psaltis, [7].

Figure 5 presents the critical Reynolds numbers for non-axisymmetric disturbances. The results, unlike axisymmetric case, depend on $Pm$. Depending on Hartmann number the instability is either axisymmetric or asymmetric ($m=1$). Nevertheless, the critical Hartmann number is smaller for $m=1$ mode. The critical Hartmann number do not depend on $Pm$.

### 4. Discussion

The presence of the toroidal magnetic field can strongly destabilize the Taylor-Couette flow. For non-ideal flow the magnetic field is fixed function of radius (7) and has only two parameters defined by the flow geometry ($\hat{\eta}$) and the magnetic field boundary values ($\hat{\mu}_\phi$). The flow can be only destabilized by the magnetic field with $\hat{\mu}_\phi$ out of range (21). The current free magnetic field ($a_\phi = 0$) has $\hat{\mu}_\phi = \hat{\eta}$ and stabilizes the flow only in accordance with [12].
The stable magnetic field (with $\hat{\mu}_\phi$ into interval (21)) stabilizes the unstable rotation ($\hat{\mu}_{\Omega} < \hat{\eta}^2$) and critical Reynolds numbers increase with increasing Hartmann numbers (see Figures 1,2). The unstable magnetic field with $Ha > Ha_c$ destabilizes every rotation law. The stability properties of the flow in the presence of the unstable magnetic field with $Ha < Ha_c$ depend on stability properties of the flow without the magnetic field. The stable without the magnetic field rotation ($\hat{\mu}_{\Omega} > \hat{\eta}^2$) keeps the stability. For the unstable without magnetic field rotation, the critical Reynolds numbers decrease with increasing Hartmann numbers and become 0 for $Ha=Ha_c$.

![Figure 5](image)

Figure 5. The marginal stability lines for axisymmetric (solid) and non-axisymmetric disturbances for conducting cylinders with $\hat{\eta}=0.5$, $\hat{\mu}_{\Omega}=0$, $\hat{\mu}_\phi=3$ and $m=1$ and $Pm=1$ (lower dotted); $m=1$ and $Pm=10^4$ (lower dashed); $m=2$ and $Pm=1$ (upper dotted); $m=2$ and $Pm=10^4$ (upper dashed).

The critical Hartmann numbers are smaller for conducting boundary conditions and for wider gap between cylinders. They are depend on the magnetic field parameters but, obviously, do not depend on the rotation parameter $\hat{\mu}_{\Omega}$. This means that the flow with any $\hat{\mu}_{\Omega}$ value are unstable for $Ha>Ha_c$ (see Figure 3 and 4) and rotation can not stabilize the unstable magnetic field. Every mode with non-zero wave number is unstable for the unstable without magnetic field rotation ($\hat{\mu}_{\Omega} < \hat{\eta}^2$). The modes from some interval are only unstable for the stable rotation. The interval length is 0 for $Ha=Ha_c$ and increases with increasing $Ha$.

There are the critical Hartmann number for $m=0$ axisymmetric(sausage) and $m=1$ kink modes only (see Figure 5). The critical Hartmann numbers for kink modes are a little bit smaller than for $m=0$ modes. So, we can say that kink mode is the most unstable mode in accordance with pinch stability results, [9]. The marginal stability lines for axisymmetric mode do not depend on magnetic Prandtl number. The same is true for critical Hartmann numbers for both $m=0$ and $m=1$ modes.

Finally, let us estimate the critical value of the unstable magnetic field. Taking the parameter values for liquid sodium: $\nu=0.0071$ cm$^2$/s, $\eta=810$ cm$^2$/s, $\rho=0.92$ g/cm$^3$ and typical dimension $R_{in}=10$ cm, $R_{out}=20$ cm ($\hat{\eta}=0.5$), and $Ha_c^2 \sim 10^3$ (see Figures 1, 2) we get the magnetic field value on the inner cylinder is only nearly 30 Gauss which corresponds to nearly 90 Gauss magnetic field on the outer cylinder for $\hat{\mu}_\phi=3$. The small value of the magnetic field destabilizing the flow and independence of the main results on magnetic Prandtl number makes the Taylor-Couette flow with imposed toroidal magnetic field very promising to observe the instability of the magnetize Taylor-Couette flow.
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