Quantum Gravity and a Time Operator in Relativistic Quantum Mechanics

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Abstract

The problem of time in the quantization of gravity arises from the fact that time in Schrödinger’s equation is a parameter. This sets time apart from the spatial coordinates, represented by operators in quantum mechanics (QM). Thus “time” in QM and “time” in General Relativity (GR) are seen as mutually incompatible notions. The introduction of a dynamical time operator in relativistic quantum mechanics (RQM), that in the Heisenberg representation is also a function of the parameter t (identified as the laboratory time), prompts to examine whether it can help to solve the disfunction referred to above. In particular, its application to the conditional interpretation of the canonical quantization approach to quantum gravity is developed.

1 Introduction

Quantization of general relativity (GR) is still an unsolved problem in physics. One of the difficulties, referred as the problem of time, arises from the fact that time in quantum mechanics (QM) is a parameter[1, 2]. The extensive experimental confirmation of the Schrödinger equation identifies this parameter as the laboratory time, thus part of the the space–time frame of reference associated to an observer. This sets time apart from the system’s physical properties (e.g., energy, momentum, position), which are represented by operators, whereas time is not[1]. On the other hand, in general relativity (GR) where matter determines the structure of spacetime, time and space acquire a dynamical nature.

1In this respect it is important to avoid, following Hilgevoord[3], the confusion between the space coordinates (x, y, z) of a system of reference and the quantum mechanical position operator \( \hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z}) \) whose expectation value gives the time evolution of the position of a system described by a certain state vector \( | \Psi(t) \rangle \). In the same way, a distinction should be made between the time coordinate in (x, y, z, t) and a “time operator \( \hat{T} \)” acting on the system state vector. The space coordinates (x, y, z) together with the parameter t constitute the background spacetime framework subject to Lorentz transformations in special relativity,
Thus "time" in QM and the "time" in GR are seen as mutually incompatible notions\[1, 5, 6, 7, 8, 9\]. Quantization of general relativity has been approached in different ways. The canonical quantization approach, to be considered here, is based on a 3+1 decomposition of spacetime, namely a foliation of three-dimension spacelike hypersurfaces and a one-dimension timelike vector that may characterize the foliation. The Dirac prescription to transform dynamical variables into operators, as used to formulate standard quantum mechanics from the Hamiltonian-Jacobi formulation of classical mechanics, is then applied\[10, 4, 5, 6, 8\]. It results, however, in the Wheeler-de Witt equation (WdW), where time is absent. The problem of time is that there is no time. The WdW equation predicts a static universe, contrary to obvious everyday experience.

To resolve this contradiction, Page and Wooters (PW)\[11, 12, 13\] advanced that a static system may describe an evolving "universe" from the point of view of an internal observer, by introducing conditional probabilities between two of the system observables, the continuum spectrum of one of them serving as the "internal time" parameter for the other. However, as in ordinary Schrödinger quantum mechanics the probability amplitudes of all dynamical variables are referred to a single time\[2\], it was soon pointed out that the chosen observable should condition not one but all other dynamical variables and, to quote\[15\]: "Evidence against the possibility of using a dynamical variable to play the role of "time" in the conditional probability interpretation is provided by the fact (proven here) that in ordinary Schrödinger quantum mechanics for a system with a Hamiltonian bounded from below, no dynamical variable can correlate monotonically with the Schrödinger parameter \( t \), and thus the role of \( t \) in the interpretation of Schrödinger quantum mechanics cannot be replaced by that of a dynamical variable". This is the well known objection to the existence of a time operator in quantum mechanics, raised by Pauli\[2\]. To be noted finally is that the non existence of a time operator seems to have been taken for granted in all up to date developments in quantum gravity\[4, 5, 6, 7, 8, 9\].

In the present paper, however, the introduction of a self-adjoint dynamical time operator in relativistic quantum mechanics (RQM)\[16\], that in the Heisenberg picture is also a function of the parameter \( t \), prompts to examine whether it can help to solve the disfunction referred to above, as well as support the conditional probability interpretation of canonical quantum gravity.

In Section 2, the proposed self-adjoint dynamical time operator in Dirac’s relativistic quantum mechanics \( (\hat{T} = \alpha \hat{\mathbf{r}}/c + \beta \tau_0) \) is presented in addition to the usual dynamical dynamical observables. It does provide a time energy uncertainty relation related to the position momentum one, as surmised by Bohr,

whereas the operators \( \hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z}) \) and, if it exists, \( \hat{T} \), are to be considered as, say, "path" dynamical space and time operators.

\[2\] Indeed, as pointed out in Ref.(13), "a consistent definition of a probability density can include only points on a space-like surface, i.e., with no possible causal connection. In the non-relativistic limit \( (c = \infty) \) all such surfaces are reduced to \( \tau = \text{const} \) planes, and the normalization applies only to the domain of space dimensions. Thus under no circumstances is the time variable on a complete equal footing as the space variables."
and circumvents Pauli’s objection by giving rise to energy changes through momentum displacements. In Section 3 the properties that the conditioning PW operator has to satisfy are reviewed and shown how they can be fulfilled by the time operator. Conclusions and further possible applications are presented finally.

2 The dynamical time operator[16]

In analogy to the Dirac free particle Hamiltonian $\hat{H}_D = c\alpha_0 \hat{p} + \beta m_0 c^2$ where $\alpha_i (i = 1, 2, 3)$ and $\beta$ are the $4 \times 4$ Dirac matrices, a dynamical “time operator” $\hat{T} = \alpha_0 \hat{r}/c + \beta \tau_0$ has been introduced. It is shown that:

A) its eigenvectors are
\[ |\tau\rangle = u_r |r\rangle \tag{1} \]
where $|r\rangle$ is the eigenvector of the position operator and $u_r$ is a four component spinor independent of the linear momentum $p$. The corresponding eigenvalues are:
\[ \tau = \pm [(r/c)^2 + \tau_0^2]^{1/2} \tag{2} \]
There is a continuous positive and a continuous negative "time branch". As $[\sigma \cdot \hat{r}/2r, \hat{T}] = 0$, where $\sigma \cdot \hat{r}/2r$ represents the spin component in the direction of $r$, there are four independent "time spinors", corresponding to two spin projections for each branch. The value of the intrinsic property $\tau_0$ is found to be $\hbar/m_0 c^2$, the de Broglie period(Appendix A).[19]

B) in the Heisenberg picture
\[ i\hbar \frac{d\hat{T}}{dt} = [\hat{T}, \hat{H}_D] = i\hbar \{ I + \beta K \} + 2\beta \{ \tau_0 \hat{H}_D - m_0 c^2 \hat{T} \} \tag{3} \]
where $K = \beta (2s \cdot l/\hbar^2 + 1)$ is a constant of motion[?]. Integrating one obtains:
\[ \hat{T}(t) = \{ 1 + \beta K \} t + \text{oscillating terms} \tag{4} \]

Thus the time operator correlates monotonically with the time parameter $t$ of the Schrödinger equation ($\tau \propto t$).

C) this operator is clearly self-adjoint and therefore can be the generator of a unitary transformation (Stone’s theorem)
\[ U_T(\varepsilon) = e^{-i\varepsilon \hat{T}/\hbar} = e^{-i\varepsilon (\alpha_0 \hat{r}/c + \beta \tau_0)/\hbar} \tag{5} \]
where $\varepsilon$ has the dimensions of energy. For infinitesimal transformations ($\delta \varepsilon << 1$), the transformed Hamiltonian $\hat{H}_D = U \hat{H}_D U^\dagger$ is approximated as:
\[ \hat{H}_D = U \hat{H}_D U^\dagger \simeq (1 - i\varepsilon \hat{T}/\hbar + ...) \hat{H}_D (1 + i\varepsilon \hat{T}/\hbar + ...) \simeq \hat{H}_D - i(\varepsilon/\hbar)[\hat{T}, \hat{H}_D] + ... \tag{6} \]
Now:
\[ [\hat{T}, \hat{H}_D] = i\hbar \{ 3I + 4s \cdot l/\hbar^2 \} + 2\beta \{ \tau_0 \hat{H}_D - m_0 c^2 \hat{T} \}. \tag{7} \]
that reduces to $[\hat{T}, \hat{H}_D] = i\hbar 3\mathbf{I}$ in the rest frame ($\mathbf{r} = 0, \mathbf{p} = 0, \mathbf{l} = 0$, $\hat{H}_D = \beta m_0c^2$ and $T = \beta \tau_0$). Otherwise there will be a transient Zitterbewgung behavior about the monotonic evolution\[17\].

Then, using $\alpha, \alpha = 3\mathbf{I}$, one has:

$$\tilde{H}_D = U\hat{H}_DU^\dagger = \hat{H}_D(\mathbf{p}) + \varepsilon \alpha, \alpha = \alpha, (\mathbf{p} + \varepsilon \alpha/c) + \beta m_0c^2 = \tilde{H}_D(\mathbf{p} + \varepsilon \alpha/c) \quad (8)$$

The unitary transformation induces a shift in momentum (Lorentz boost):

$$\delta \mathbf{p} = \{(\delta \varepsilon)/c\} \mathbf{\alpha} = \{(\delta \varepsilon)/c^2\} \varepsilon \mathbf{\alpha} \quad (9)$$

Repeated infinitesimal applications yield a momentum displacement $\mathbf{p}$ whose expectation value is

$$\langle \mathbf{p} \rangle = (\varepsilon/c^2) \mathbf{v}_{gp} = \gamma m_0 \mathbf{v}_{gp} \quad (10)$$

where $\gamma = \{1 - (v_{gp}/c)^2\}^{-1/2}$ is the Lorentz factor and $\mathbf{v}_{gp}$ the group velocity.

Clearly the shift in momentum represents a displacement in energy within the positive and the negative energy branches. In the positive branch, as $\mathbf{p}$ goes from $-\infty$ to $+\infty$ in any direction, the energy drops from $+\infty$ to a minimum $m_0c^2$ at $\mathbf{p} = 0$ and then rises again to $+\infty$. No crossing of the $2m_0c^2$ energy gap is involved. The time operator is thus the generator of a unitary transformation that corresponds to a change in energy. However, by acting on the momentum continuum space, it circumvents Pauli’s objection.

Finally it can be pointed out that, as $\hat{T}$ and $\hat{r}$ commute, the previous development applies also in the presence of any position dependent potentials, e.g., the scalar and vector electromagnetic potentials.

### 3 The dynamical time operator and the conditional probability interpretation of quantum gravity

Following Ref.5 (Kuchar), the PW conditional probability interpretation asserts that, given $\hat{B}$ and $\hat{C}$ as projection operators corresponding to observables of the system:

$$P(B \mid C) = \frac{\langle \Psi \mid \hat{C}\hat{B}\hat{C} \mid \Psi \rangle}{\langle \Psi \mid \hat{C} \mid \Psi \rangle} \quad (11)$$

represents the probability that the observation of $\hat{B}$ is subject to the observation of $\hat{C}$. It is then stated that the connection with the time problem is established by finding within the system a projection operator $\hat{C}(t)$ corresponding to the question “Does an internal clock show the time $t$?”. The operator $\hat{C}(t)$ cannot commute with the Hamiltonian as otherwise it would be constant. If $[\hat{B}, \hat{C}(t)] = 0$ the complete Hamiltonian can be written as

$$\hat{H} = (\hat{h}_B \otimes I_C) \otimes (I_B \otimes \hat{h}_C) \quad (12)$$
in the space $F_B \otimes F_C$ composed of the corresponding Hilbert spaces. Then:

$$[\hat{H}, \hat{C}(t)] = [\hat{H}_C, \hat{C}(t)] \neq 0$$  \hspace{1cm} (13)

Now, if $|\phi_0\rangle$ is a state vector such that $\hat{C}(0) = |\phi_0\rangle \langle \phi_0|$ corresponds to $t = 0$, it follows that

$$\hat{C}(t) = e^{i\hat{H}t/\hbar}\hat{C}(0)e^{-i\hat{H}t/\hbar}$$  \hspace{1cm} (14)

Then:

$$P(B|C(t)) = \frac{\langle \Psi | e^{i\hat{C}(t)/\hbar}|\phi_0\rangle \langle e^{-i\hat{H}t/\hbar}\hat{B}e^{i\hat{C}(t)/\hbar}|\phi_0\rangle \langle e^{-i\hat{H}t/\hbar}\hat{B}e^{i\hat{H}t/\hbar} |\Psi\rangle}{\langle \Psi | \langle \phi_0 | \langle \phi_0 | \Psi \rangle}$$  \hspace{1cm} (15)

where $|\psi\rangle = \langle \phi_0 | \Psi \rangle$ and:

$$\hat{B}(t) = e^{-i\hat{H}t/\hbar}\hat{B}e^{i\hat{H}t/\hbar} = e^{-i\hat{H}t/\hbar}\hat{B}e^{i\hat{H}t/\hbar}$$  \hspace{1cm} (16)

It follows then that $\hat{B}$ also satisfies:

$$i\hbar \frac{d\hat{B}}{dt} = [\hat{B}, \hat{H}_B] = [\hat{B}, \hat{H}]$$  \hspace{1cm} (17)

in spite of the fact that $|\Psi\rangle$ is a stationary state of the total system.

Note that this development is already assuming a time dependent Schrödinger equation (TDSE), without explaining the presence in it of the laboratory time $t$. This will be addressed below.

The connection of the PW conditional interpretation with the dynamical time operator defined above is as follows. Besides being a timelike operator as it is given in terms of the worldline $r(t)$, $\hat{T}$ does satisfy the following conditions:

i) it does not commute with the Hamiltonian;

ii) its spectrum is a single valued continuum in either positive or negative branch, directly proportional to the time parameter $t$;

iii) the eigenvector basis $\{|\tau\rangle = u_r |r\rangle\}$ where $\tau \propto t$ can be used to construct the normalized wave packet $|\phi_0\rangle = \int d\tau c_\tau |\tau\rangle$ such that

$$\langle \phi_0 | \hat{T} | \phi_0 \rangle = \int d\tau d\tau' c_\tau c_{\tau'} \langle \tau' | \hat{T} | \tau \rangle = \int d\tau |c_\tau|^2 \tau \propto \int dt |c_\tau|^2 t = 0$$  \hspace{1cm} (18)

iv) as $[\hat{T}, \hat{r}] = 0$, one can consider:

$$\hat{B} = |r\rangle \langle r|$$  \hspace{1cm} (19)

Then, with $\Psi(r, \tau) := \langle \Psi | \tau \rangle \langle \tau | r > < r | \tau \rangle \langle \tau | \Psi \rangle$, :

$$P(B | C) = \frac{\langle \Psi | \tau \rangle \langle \tau | r > < r | \tau \rangle \langle \tau | \Psi \rangle}{\langle \Psi | \tau > < \tau | \Psi \rangle} = \frac{|\Psi(r, \tau)|^2}{\int d\tau |\Psi(r, \tau)|^2}$$  \hspace{1cm} (20)
would be the probability density for finding the system at value \( r(t) \) at an instant \( \tau(t) \propto t \).

The eigenvectors \( |\tau, r> \) (common eigenvectors of \( r \) and \( T \) as \([T, r] = 0\)) constitute a basis. In the Schrödinger picture, they give an "intrinsic time"-space spinor representation \( \Psi(\tau, r; t) =<\tau, r | \Psi(t)> \) of the time dependent Schrödinger state vector. This is entirely analogue to the energy-momentum spinor representation \( \Phi(E, p; t) =<E, p | \Psi(t)> \) where \( |E, p> \) are the common eigenvectors of the relativistic free particle Dirac Hamiltonian \( H_D = c\alpha \cdot p + \beta m_0 c^2 \) and the momentum operator \( p \). The time dependence of \( T \) is exhibited in the Heisenberg picture and seen to correlate monotonically with the parameter \( t \) for wave packets of purely positive (or purely negative) \( \tau \) eigenstates.

Consequently \( |\Psi(\tau, r; t)|^2 \) is interpreted as the probability of finding at time \( t \) the system at position \( r \) and intrinsic time \( \tau = \pm \sqrt{(r/c)^2 + \tau_0^2} \). Normalization of \( \Psi(\tau, r; t) \) includes sum over spin but no integration over an extra dimension beyond \( r \). Then one has:

\[
P(\tau, r; t) = \frac{|\Psi(\tau, r; t)|^2}{\sum \alpha \int dr |\Psi(\tau, r; t)|^2}
\]

(21)

4 Conclusion

The introduction of a self-adjoint time operator in RQM, in addition to the usual dynamical variables, allows to consider its possible role in the "problem of time" in quantum gravity (QG). As defined, this time operator has a one to one correspondence with the timelike worldline \( r(t) \). Then to each point of its spectrum one can associate a spacelike surface that intersects the worldline at the corresponding point, thus providing a foliation of spacetime by spacelike surfaces over which one can define probability amplitudes. Furthermore it provides support to the conditional probability interpretation of PW of the canonical quantization of QG, by circumventing Pauli’s objection to the existence of such an operator, as well as providing a monotonical correlation with the time parameter in the Schrödinger equation. Consequently one can say that this operator yields an observable dynamical variable that “sets the conditions” for the other variables and defines a satisfactory notion of time.

It is interesting to note that the presence of the parameter \( t \) in the time dependent Schrödinger equation can be attributed to the monitoring that a classical environment, interacting with the microscopic system, exerts on the system. Indeed, starting from a time independent Schrödinger equation with a complete Hamiltonian (system, environment and interaction), the system is shown to satisfy a time dependent Schrödinger equation when it is disentangled.
from its classically described (dependent on the laboratory time $t$) environment. To quote: "The time dependence (- and perhaps also the space dependence, conforming together the Minkowskian spacetime laboratory reference frame) - is thus seen as an emergent property, both in QM and in QG" [20, 23]. Furthermore an intermediate subdivision can be introduced [21], that in our case allows the presence of two times, a system "internal time" constructed by the Page Wooters mechanism, and the "laboratory time" arising from the interaction with a massive classical environment. An experimental illustration of, to quote: "A static, entangled system between a clock system and the rest of the universe is perceived as evolving by internal observers that test the correlations between the two subsystems" has already been achieved [22].

The presentation in this paper is at a basic level, as the stress is on the fact that a dynamical time operator in RQM can be defined, contrary to the general view. It remains to be formulated in the usual 3+1 foliation of the spacetime with Riemann spacelike surfaces, and, as it introduces "time spinors", its possible relevance to the more advanced formulation of loop quantum gravity.

**Appendix A**

For infinitesimal transformations ($\delta \varepsilon << 1$), one can factorize the unitary operator $U_T(\varepsilon)$ generated by the time operator as follows:

$$U_T(\varepsilon) \simeq e^{-i(\delta \varepsilon)\alpha \hat{r}/c\hbar} e^{-i(\delta \varepsilon)\beta \tau_0/\hbar} = e^{-i(\delta \varepsilon)\beta \tau_0/\hbar} e^{-i(\delta \varepsilon)\alpha \hat{r}/c\hbar}$$  \hspace{1cm} (A.1)

as $[i(\delta \varepsilon)(\alpha \hat{r}/c\hbar), i(\delta \varepsilon)\beta \tau_0/\hbar] \approx (\delta \varepsilon)^2 \approx 0$ (Glauber theorem). Then the transformed Hamiltonian can be approximated as:

$$\hat{H}_D = U \hat{H}_D U^\dagger \simeq e^{i(\delta \varepsilon)\beta m c^2/\hbar} e^{i(\delta \varepsilon)\alpha \hat{r}/c\hbar} \hat{H}_D e^{-i(\delta \varepsilon)\alpha \hat{r}/c\hbar} e^{-i(\delta \varepsilon)\beta m c^2/\hbar}$$  \hspace{1cm} (A.2)

Consider first:

$$e^{i(\delta \varepsilon)\alpha \hat{r}/c\hbar} \hat{H}_D e^{-i(\delta \varepsilon)\alpha \hat{r}/c\hbar} \simeq \{I + i(\delta \varepsilon)\alpha \hat{r}/c\} \hat{H}_D \{I - i(\delta \varepsilon)\alpha \hat{r}/c\} + \ldots$$

$$\approx \hat{H}_D + i\{(\delta \varepsilon)/c\} [\alpha \hat{r}, \hat{H}_D] + \ldots$$  \hspace{1cm} (A.3)

Then using [?]:

$$[\hat{H}_D, \alpha \hat{r}] = -3\hbar c I + 2\hat{H}_D \{\alpha - c\hat{p}/H_D\} \hat{r} = -i\hbar c \alpha \alpha + 2\hat{H}_D \alpha \hat{r} - 2c\hat{p}\hat{r}$$  \hspace{1cm} (A.4)

and $\alpha \alpha = 3I$, one obtains:

$$e^{i(\delta \varepsilon)\alpha \hat{r}/c\hbar} \hat{H}_De^{-i(\delta \varepsilon)\alpha \hat{r}/c\hbar} \simeq \hat{H}_D (\hat{p} + \delta \varepsilon \alpha/c + i2(\delta \varepsilon)/c\hbar) \{\hat{H}_D \alpha \hat{r} - c\hat{p}\hat{r}\}$$  \hspace{1cm} (A.5)

Thus, the unitary transformation induces a shift in momentum:

$$\delta \hat{p} = \{\delta \varepsilon/c\} \alpha = \{(\delta \varepsilon)/c^2\} c \alpha$$  \hspace{1cm} (A.6)
as well as a Zitterbewegung behavior in the corresponding propagator $U(t) = e^{-i\hat{H}_D t/\hbar}$.

For repeated infinitessimal applications one obtains a momentum displacement $p$ whose expectation value is

$$< p > = (\varepsilon/c^2) v_{gp} = \gamma m_0 v_{gp} \quad (A.7)$$

where $\gamma = \{1 - (v_{gp}/c)^2\}^{-1/2}$ is the Lorentz factor and $v_{gp}$ the group velocity. It also induces a phase shift. Indeed:

$$\langle \Psi | \hat{H}_D | \Psi \rangle = \langle \Phi | \hat{H}_D (p + \alpha \delta \varepsilon/c) | \Phi \rangle = \langle \Phi | \hat{H}_D (p + \gamma m_0 v_{gp}) | \Phi \rangle \quad (A.8)$$

where

$$| \Phi \rangle = e^{-i(\delta \varepsilon) / \beta \tau_0 / \hbar} | \Psi \rangle \quad (A.9)$$

The phase shift is $\delta \varphi = -i(\delta \varepsilon) / \beta \tau_0 / \hbar$. For a finite transformation, its expectation value is

$$\langle \Delta \varphi \rangle = -\langle (\Delta \varepsilon) \beta \tau_0 / \hbar \rangle \quad (A.10)$$

as $< \beta > = m_0 c^2 / < H > = \pm m_0 c^2 / \varepsilon = \pm 1 / \gamma$, for a positive (negative) energy wave packet that contains both positive and negative energy free particle solutions. Thus the sign of $< \beta >$ distinguishes the positive or negative energy branch where the momentum displacement takes place. If furthermore one requires the corresponding phase shift to be equal to $2\pi$, one has to set:

$$\tau_0 = 2\pi h < \beta > \quad (A.11)$$

This is the de Broglie period. One has then:

$$\hbar/p = \hbar / \gamma m_0 v_{gp} = h c^2 / \gamma m_0 c^2 v_{gp} = (\hbar / \varepsilon)(c^2 / v_{gp}) = (1/\nu) v_{ph} \quad (A.12)$$

which is precisely the de Broglie wave length, that is, the product of the phase velocity by the period derived from the Planck relation $E = h\nu$ and the Einstein relation $E = m_0 c^2$, as originally assumed by de Broglie.[7]

As the state vector $| \Phi \rangle$ differs from the state vector $| \Psi \rangle$ by a global phase, it follows that:

$$\langle \Phi | \hat{H}_D (p + \gamma m_0 v_{gp}) | \Phi \rangle = \langle \Psi | \hat{H}_D (p + \gamma m_0 v_{gp}) | \Psi \rangle \quad (A.13)$$

Finally, it is also interesting to note the following. In the same way as above, in the case of an infinitesimal time lapse ($\delta t < 1$) the unitary operator $U(t) = e^{i\delta t (\alpha \cdot p + \beta m_0 c^2)/\hbar}$ can be approximated as:

$$U(\eta) \simeq e^{i\delta t (\alpha \cdot p) / \hbar} e^{i\delta t \beta m_0 c^2 / \hbar} \quad (A.14)$$

In configuration space this yields a displacement $\delta r = < r + \delta t c \alpha > = < r > + (\delta t) v_{gp}$ and a phase shift $\Delta \varphi = \delta t < \beta > m_0 c^2 / \hbar$. For $\delta t = \gamma \tau_0 = \gamma h / m_0 c^2$ (the boosted de Broglie period), the phase shift is

$$\Delta \varphi (\gamma h / m_0 c^2) = (\gamma h / m_0 c^2)(m_0 c^2 / < H >)m_0 c^2 / h = h / h = 2\pi \quad (A.15)$$
These results are in agreement with the fact that the Hamiltonian is actually the generator of the time development of a system described by a wave packet. The approximate treatment provides only the displacement, neglecting the dispersion of the wave packet.

References

[1] Dirac, P.A.M., "The principles of quantum mechanics", (4th ed.), Oxford, Clarendon Press, (1958)

[2] Pauli, W., “General Principles of Quantum Mechanics”, Springer Verlag, p. 63, (1980)

[3] Hilgevoord, J., "Time in quantum mechanics; a story of confusion", Studies in History and Philosophy of Modern Physics 36, pp.29-60 (2005); Hilgevoord, J. and Atkinson, D., "Time in quantum mechanics", CLARENDON PRESS-OXFORD, (2011)

[4] Anderson, E., "The problem of time in quantum gravity", [arXiv:1009.2157v3 [gr-qc]], Ann.Phys. (Berlin) 524, pp.757-786 (2012) and references therein

[5] Kuchař, K.V., "Time and Interpretations of Quantum Gravity", Int.J.Mod.Phys. D 20 (Supp.1). pp.3-86 (2011)

[6] Kiefer, C. "Concept of Time in Canonical Quantum Gravity and String Theory”, J.Phys.:Conference Series 174. 012021(2009); "Quantum Geomentry: whence, whither?", arXiv:0812.0295 (2008)

[7] Macías A. and Quevedo H., "Time Paradox in Quantum Gravity", in Quantum Gravity pp 41-60, B. Fauser, J. Tolksdorf and E. Zeidler (eds), Birkhäuser Verlag Basel / Switzerland, (2006)

[8] Isham, C.J., "Canonical Quantum Gravity and the Problem of Time", arXiv:gr-qc/9210011v1, (1992); "Prima Facie Questions in Quantum Gravity", arXiv:gr-qc/9310031v1 (1993)

[9] Ashtekar, A., "Gravity and the quantum", New J.Phys. 7, 198 (2005); "The winding road to quantum gravity”, Current Science 89, 2064-2074 (2005)

[10] DeWitt, B.S., "Quantum Theory of Gravity I. The Canonical Theory", Phys.Rev. 160, pp.1113-1148, (1967)

[11] Page, D.N. and W.K. Wootters,"Evolution without evolution: Dynamics described by stationary observables”, Phys.Rev. D 27, pp. 2885-2892 (1983); Wootters, W.K., "Time" replaced by quantum correlations”, Int.J.Theor.Phys. 23, pp.701-711 (1984)
[12] Dolby, C.E., "The Conditional Probability Interpretation of the Hamiltonian Constraint", arXiv:gr-qc/0406034v1 (2004)

[13] Giovanetti, V., S. Lloyd and L. Maccone, "Quantum time", Phys.Rev. D 92, 045033 (2015)

[14] Bauer, M., "A Time Operator in Quantum Mechanics", Ann.Phys. 150, pp.1-21, (1983)

[15] Unruh, W.G. and Wald, R.M., "Time and the interpretation of canonical quantum gravity", Phys.Rev. D 40, pp.2598-2614 (1989)

[16] Bauer, M., "A dynamical time operator in Dirac’s relativistic quantum mechanics", Int.J.Mod.Phys. A 29, 1450036 (2014)

[17] Thaller, B., The Dirac Equation, Springer-Verlag Berlin Heidelberg (1992)

[18] Greiner, W., "Relativistic Quantum Mechanics", 3ed. Springer (2000)

[19] Bauer, M., "de Broglie clock, Zitterbewegung and time in quantum mechanics", arXiv:xxxx

[20] Briggs, J.S. and Jan M. Rost, “Time dependence in quantum mechanics”, Eur.Phys.J.10, (2000); “On the Derivation of the Time-dependent Equation of Schrödinger”, Foundations of Physics 31, (2001)

[21] Briggs, J.S., S. Boonchui and S. Khemmani, "The derivation of time-dependent Schrödinger equations", J.Phys. A: Math.Theor. 40, pp.1289-1302 (2007)

[22] Moreva, E. et al., "Time from quantum entanglement: an experimental illustration", arXiv:1310.4691v1 [quant-ph] (2013); "The time as an emergent property of quantum mechanics, a synthetic description of a first experimental approach", J.Phys.: Conference Series 626, 012019 (2015)

[23] J. Butterfield and C.J. Isham, "On the Emergence of Time in Quantum Gravity", arXiv:gr-qc/9901024v1 (1999)