COMPLEX LIPSCHITZ STRUCTURES AND BUNDLES OF COMPLEX CLIFFORD MODULES

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Abstract. Let \((M, g)\) be a pseudo-Riemannian manifold of signature \((p, q)\). We construct mutually quasi-inverse equivalences between the groupoid of bundles of weakly-faithful complex Clifford modules on \((M, g)\) and the groupoid of reduced complex Lipschitz structures on \((M, g)\). As an application, we show that \((M, g)\) admits a bundle of irreducible complex Clifford modules if and only if it admits either a \(\text{Spin}^c(p, q)\) structure (when \(p + q\) is odd) or a \(\text{Pin}^c(p, q)\) structure (when \(p + q\) is even). When \(p - q \equiv 3, 4, 6, 7\), we compare with the classification of bundles of irreducible real Clifford modules which we obtained in previous work. The results obtained in this note form a counterpart of the classification of bundles of faithful complex Clifford modules which was previously given by T. Friedrich and A. Trautman.

CONTENTS

1. Introduction 1
2. Complexified Clifford algebras and complex Clifford groups 2
3. Complex Clifford representations and complex Lipschitz groups 4
4. Elementary complex Lipschitz groups 7
5. Complex pinor bundles and complex Lipschitz structures 11
References 19

1. INTRODUCTION

In reference [1], T. Friedrich and A. Trautman classified bundles of faithful complex Clifford modules over a connected pseudo-Riemannian manifold \((M, g)\) in terms of complex Lipschitz structures (see also [2]). In this note, we establish a general equivalence of categories between the groupoid of weakly-faithful\(^1\) complex Clifford modules over a connected pseudo-Riemannian manifold of arbitrary signature \((p, q)\) and the groupoid of complex Lipschitz structures associated to the corresponding fiberwise representation. This equivalence, which we give in Proposition 5.17 and Theorem 5.19, clarifies and extends the correspondence given in [1, Sec. 5, Theorems 2 and 3] between bundles of faithful complex Clifford modules and the corresponding complex Lipschitz structures.

The fibers of a bundle \(S\) of irreducible complex Clifford modules are faithful iff the dimension \(d = p + q\) of \(M\) is even (in which case they are “Dirac representations” in the language of [1]). In this case, the results of [1] imply that the corresponding complex Lipschitz structure is homotopy-equivalent with a \(\text{Pin}^c(p, q)\) structure. When \(d\) is odd, the fibers of a bundle \(S\) of irreducible complex Clifford modules are weakly-faithful but not faithful (they are “Pauli representations” in the language of [1, 2]). In this case, we show that the corresponding complex Lipschitz structure is homotopy-equivalent with a \(\text{Spin}^c(p, q)\) structure. In particular, this shows that an odd-dimensional pseudo-Riemannian manifold admits a bundle of irreducible complex Clifford modules if and only if it admits a \(\text{Spin}^c(p, q)\) structure, in which case any such bundle is associated to the appropriate

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\(^1\)See Definition 3.4.
Spin$^c(p, q)$ structure through the tautological representation. We also show that the classifications of Spin$^c(p, q)$ structures and of bundles of irreducible complex Clifford modules agree and hence the isomorphism classes of the latter form a torsor over $H^2(M, \mathbb{Z})$ when $(M, g)$ admits a Spin$^c(p, q)$ structure.

Reference [3] solved a similar classification problem for bundles of irreducible real Clifford modules in terms of real Lipschitz structures. The two problems are related by the faithful realification functor, which maps a bundle $S$ of complex Clifford modules to the underlying bundle of real Clifford modules. When the fibers of $S$ are complex-irreducible, they are also real irreducible exactly when $p - q \equiv 3, 4, 6, 7$, so in these cases it is natural to compare the two categories. For such signatures, we show that the realification functor is faithful and strictly surjective but not full and we describe its preimage. When $M$ is unorientable with $p - q \equiv 3, 7$, the pseudo-Riemannian manifold $(M, g)$ does not admit bundles of complex irreducible Clifford modules. However, such $(M, g)$ can admit Spin$^c(p, q)$ structures (and hence bundles of real irreducible Clifford modules) provided that the topological conditions given in [3] are satisfied.

Notations and conventions. All manifolds considered are smooth, paracompact, Hausdorff and connected. All fiber bundles and sections thereof are smooth. For a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $\text{Alg}_\mathbb{K}$ denote the category of unital associative $\mathbb{K}$-algebras and unital algebra morphisms. The unit groupoid of any category $C$ is denoted by $C^\times$; it is defined as the groupoid obtained from $C$ by keeping only the invertible morphisms.

2. Complexified Clifford algebras and complex Clifford groups

A real quadratic space is a pair $(V, h)$, where $V$ is a finite-dimensional $\mathbb{R}$-vector space and $h$ is a symmetric and non-degenerate $\mathbb{R}$-bilinear pairing defined on $V$. Given a real quadratic space $(V, h)$, let $\text{Cl}(V, h)$ denote its real Clifford algebra. By definition, the complexified Clifford algebra of $(V, h)$ is the complexification $\text{Cl}(V, h) \overset{\text{def}}{=} \text{Cl}(V, h) \otimes_{\mathbb{R}} \mathbb{C}$. There exists a natural unital isomorphism of $\mathbb{C}$-algebras:

$$\text{Cl}(V, h) \overset{\text{Alg}_\mathbb{C}}{\simeq} \text{Cl}(V_C, h_C),$$

where $V_C \overset{\text{def}}{=} V \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of $V$ and $h_C : V_C \times V_C \to \mathbb{C}$ is the $\mathbb{C}$-bilinear extension of $h : V \times V \to \mathbb{R}$.

2.1. The categories $\text{Cl}_\mathbb{C}$ and $\text{Cl}_\mathbb{R}$.

Let Quad be the category whose objects are real quadratic spaces and whose arrows are isometries between such and let Quad$^\times$ denote the unit groupoid of Quad. The Clifford algebra construction gives two functors:

- The real Clifford functor $\text{Cl} : \text{Quad} \to \text{Alg}_\mathbb{R}$, which maps a real quadratic space $(V, h)$ to its real Clifford algebra and maps an isometry $\varphi_0 : (V, h) \to (V', h')$ to the map $\text{Cl}(\varphi_0) : \text{Cl}(V, h) \to \text{Cl}(V', h')$ defined as the unique unital morphism of associative $\mathbb{R}$-algebras which satisfies the condition $\text{Cl}(\varphi_0)|_V = \varphi_0$. The image of this functor is a non-full sub-category of $\text{Alg}_\mathbb{R}$ which we denote by $\text{Cl}_\mathbb{R}$.

- The complexified Clifford functor $\text{Cl} : \text{Quad} \to \text{Alg}_\mathbb{C}$, which maps a quadratic real vector space $(V, h)$ to its complexified Clifford algebra $\text{Cl}(V, h)$ and maps an isometry $\varphi_0 : (V, h) \to (V', h')$ to the $\mathbb{C}$-linear extension $\text{Cl}(\varphi_0) \overset{\text{def}}{=} \text{Cl}(\varphi_0) \otimes_{\mathbb{R}} \mathbb{C}$ of $\text{Cl}(\varphi_0)$. The image of this functor is a non-full sub-category of $\text{Alg}_\mathbb{C}$ which we denote by $\text{Cl}_\mathbb{C}$.

The corestrictions of the functors $\text{Cl}$ and $\text{Cl}$ to their images give isomorphisms (i.e., strictly surjective equivalences) of categories between Quad and $\text{Cl}_\mathbb{R}$, respectively Quad and $\text{Cl}_\mathbb{C}$. The category Quad admits a skeleton whose objects are the standard quadratic spaces $\mathbb{R}^{p,q} \overset{\text{def}}{=} (\mathbb{R}^{p+q}, h_{p,q})$, where $h_{p,q} : \mathbb{R}^{p+q} \times \mathbb{R}^{p+q} \to \mathbb{R}$ is the standard symmetric bilinear form of signature $(p, q)$. The objects of this skeleton form a countable set indexed by the pairs $(p, q) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Accordingly, the category $\text{Cl}_\mathbb{R}$ admits a skeleton whose objects are given by the real Clifford algebras.
Finally, let \( \tilde{\pi} \) denote the "even subalgebra" of \( \Cl(V,h) \). Let \( \tilde{\tau} \) be the reversed anti-automorphism and \( \tilde{\tau} \circ \pi = \tilde{\tau} \circ \pi_\C \).

Let \( \Cl(V,h)^\times \) be the group of invertible elements of the algebra \( \Cl(V,h) \) and \( \tilde{\Ad} : \Cl(V,h)^\times \to \Aut_\C(\Cl(V,h)) \) be the twisted adjoint action:

\[
\tilde{\Ad}(x)(y) = \pi(x)y x^{-1} \quad \forall x \in \Cl(V,h)^\times \quad \forall y \in \Cl(V,h).
\]

Recall that the twisted norm of \( \Cl(V,h) \) is the map \( N : \Cl(V,h) \to \Cl(V,h) \) given by:

\[
N(x) = \tilde{\tau}_\C(x), \quad \forall x \in \Cl(V,h).
\]

Also recall the following (see [4]):

**Definition 2.2.** The complex Clifford group is the subgroup of \( \Cl(V,h)^\times \) defined through:

\[
\Gamma(V,h) \overset{\text{def}}{=} \left\{ x \in \Cl(V,h)^\times \mid \tilde{\Ad}(x)(V) = V \right\}.
\]

The complex special Clifford group is the subgroup consisting of all even elements of \( \Gamma(V,h) \):

\[
\Gamma^s(V,h) \overset{\text{def}}{=} \Gamma(V,h) \cap \Cl^+(V,h),
\]

where \( \Cl^+(V,h) \) denotes the "even subalgebra" of \( \Cl(V,h) \).

The twisted norm restricts to a group morphism \( N : \Gamma(V,h) \to \C^\times \).

**Remark 2.3.** We have \( \Gamma(V,h) \subset \G(V,h) \), where:

\[
\G(V,h) \overset{\text{def}}{=} \left\{ x \in \Cl(V,h)^\times \mid \tilde{\Ad}(x)(V_\C) = V_\C \right\}
\]

is the ordinary Clifford group of \( (V_\C,h_\C) \). Moreover, an element \( x \) of \( \G(V,h) \) belongs to \( \Gamma(V,h) \) if and only if the orthogonal transformation \( \tilde{\Ad}(x) \in O(V_\C,h_\C) \) preserves the real subspace \( V \subset V_\C \). Equivalently, if \( c : V_\C \to V_\C \) is the real structure (antilinear involution) on \( V_\C \) with real subspace (subspace of fixed points) \( V \subset V_\C \), then an element \( x \) of \( \G(V,h) \) belongs to \( \Gamma(V,h) \) if and only if \( c \circ \tilde{\Ad}(x) = \tilde{\Ad}(x) \circ c \). In contrast with \( \G(V,h) \), the complex Clifford group \( \Gamma(V,h) \) depends on the signature of the real quadratic space \( (V,h) \).

Finally, recall that the groups \( \Pin^c(V,h) \) and \( \Spin^c(V,h) \) are defined as follows [4, 5]:

\[
\Pin^c(V,h) = \Ker(|N| : \Gamma(V,h) \to \R_{>0}),
\]

\[
\Spin^c(V,h) = \Ker(|N| : \Gamma^s(V,h) \to \R_{>0}).
\]
Proposition 2.4. One has short exact sequences:

\[
1 \rightarrow \text{Pin}^c(V, h) \hookrightarrow \Gamma(V, h) \xrightarrow{|N|} \mathbb{R}_{>0} \rightarrow 1,
\]

\[
1 \rightarrow \text{Spin}^c(V, h) \hookrightarrow \Gamma^s(V, h) \xrightarrow{|N|} \mathbb{R}_{>0} \rightarrow 1.
\]

Moreover, \(\Gamma(V, h)\) is homotopy-equivalent with \(\text{Pin}^c(V, h)\) and \(\Gamma^s(V, h)\) is homotopy-equivalent with \(\text{Spin}^c(V, h)\).

Proof. Exactness of both sequences is obvious. The map:

\[
r: \Gamma(V, h) \rightarrow \text{Pin}^c(V, h),
\]

\[
r(x) \overset{\text{def.}}{=} \frac{x}{\sqrt{|N(x)|}},
\]

is a homotopy retraction of \(\Gamma(V, h)\) onto \(\text{Pin}^c(V, h)\). The proof for \(\text{Spin}^c(V, h)\) is similar, the corresponding homotopy retraction being given by \(r|_{\Gamma^s(V, h)}\). Notice the relation:

\[
\text{Ad}(r(x)) = \text{Ad}(x) \quad \forall x \in \Gamma(V, h).
\]

We have \(\Gamma(V, h) \simeq \text{Pin}^c(V, h) \cdot \mathbb{C}^\times\) and \(\Gamma^s(V, h) \simeq \text{Spin}^c(V, h) \cdot \mathbb{C}^\times\). The following theorem summarizes the key properties of \(\Gamma(V, h), \Gamma^s(V, h), \text{Pin}^c(V, h)\) and \(\text{Spin}^c(V, h)\).

Theorem 2.5. \([4]\) There exist short exact sequences:

\[
1 \rightarrow \mathbb{C}^\times \hookrightarrow \Gamma(V, h) \xrightarrow{\text{Ad}} \text{O}(V, h) \rightarrow 1,
\]

\[
1 \rightarrow U(1) \hookrightarrow \text{Pin}^c(V, h) \xrightarrow{\text{Ad}} \text{O}(V, h) \rightarrow 1,
\]

where \(U(1)\) is the subgroup of \(\text{Cl}(V, h)\) consisting of elements of the form \(1 \otimes z\) with \(|z| = 1\). Likewise, one has short exact sequences:

\[
1 \rightarrow \mathbb{C}^\times \hookrightarrow \Gamma^s(V, h) \xrightarrow{\text{Ad}} \text{SO}(V, h) \rightarrow 1,
\]

\[
1 \rightarrow U(1) \hookrightarrow \text{Spin}^c(V, h) \xrightarrow{\text{Ad}} \text{SO}(V, h) \rightarrow 1.
\]

Moreover, there exist canonical isomorphisms of groups:

\[
\text{Pin}^c(V, h) \simeq \text{Pin}(V, h) \cdot U(1) \overset{\text{def.}}{=} (\text{Pin}(V, h) \times U(1)) / \{(1, 1), (-1, -1)\}
\]

and:

\[
\text{Spin}^c(V, h) \simeq \text{Spin}(V, h) \cdot U(1) \overset{\text{def.}}{=} (\text{Spin}(V, h) \times U(1)) / \{(1, 1), (-1, -1)\}.
\]

3. Complex Clifford representations and complex Lipschitz groups

Let \((V, h)\) be a real quadratic space.

Definition 3.1. A complex Clifford representation is a unital morphism of associative \(\mathbb{R}\)-algebras \(\gamma: \text{Cl}(V, h) \rightarrow \text{End}_\mathbb{C}(S)\), where \(S\) is a finite-dimensional vector space over \(\mathbb{C}\).

A complex Clifford representation \(\gamma\) endows \(S\) with the structure of (unital) left module over \(\text{Cl}(V, h)\), while the multiplication of vectors with complex scalars endows \(S\) with a compatible structure of (unital) finite rank (and necessarily free) right module over the field of complex numbers. Conversely, any such bimodule (which we shall call a complex Clifford module) can be viewed as a complex Clifford representation.

Extending \(\gamma\) by complex linearity gives a unital morphism of associative \(\mathbb{C}\)-algebras:

\[
\gamma_\mathbb{C}: \text{Cl}(V, h) \rightarrow \text{End}_\mathbb{C}(S).
\]

Conversely, every unital morphism of associative \(\mathbb{C}\)-algebras from \(\text{Cl}(V, h)\) to \(\text{End}_\mathbb{C}(S)\) restricts to a complex representation of \(\text{Cl}(V, h)\). Any complex Clifford representation \(\gamma: \text{Cl}(V, h) \rightarrow \text{End}_\mathbb{C}(S)\) also induces a real Clifford representation \(\gamma_\mathbb{R}: \text{Cl}(V, h) \rightarrow \text{End}_\mathbb{R}(S_\mathbb{R})\) on the realification \(S_\mathbb{R}\) of
$S$ (defined as the underlying real vector space of $S$), which is an $\mathbb{R}$-vector space of dimension $\dim_{\mathbb{R}} S = 2 \dim_{\mathbb{C}} S$.

3.1. **Unbased morphisms of complex Clifford representations.** Let $\gamma : \text{Cl}(V, h) \to \text{End}_{\mathbb{C}}(S)$ and $\gamma' : \text{Cl}(V', h') \to \text{End}_{\mathbb{C}}(S')$ be two complex Clifford representations.

**Definition 3.2.** A *morphism of complex Clifford representations* from $\gamma$ to $\gamma'$ is a pair $(\varphi_0, \varphi)$ such that:

1. $\varphi_0 : V \to V'$ is an isometry from $(V, h)$ to $(V', h')$
2. $\varphi : S \to S'$ is a $\mathbb{C}$-linear map
3. $\gamma'(\text{Cl}(\varphi_0)(x)) \circ \varphi = \varphi \circ \gamma(x)$ for all $x \in \text{Cl}(V, h)$.

A morphism of complex Clifford representations is called *based* if $V' = V$ and $\varphi_0 = \text{id}_V$. A (not necessarily based) isomorphism of complex Clifford representations from $\gamma$ to itself is called an *automorphism* of $\gamma$.

In our language, a morphism of complex representations in the traditional sense corresponds to a *based* morphism. Since $\text{Cl}(V, h)$ is generated by $V$ and $\text{Cl}(V', h')$ is generated by $V'$ while the morphism $\text{Cl}(\varphi_0)$ is $\mathbb{R}$-linear, condition 3. in Definition 3.2 is equivalent with:

$$\gamma'(\varphi_0(v)) \circ \varphi = \varphi \circ \gamma(v) \; \forall v \in V,$$

which can also be written as:

$$R_{\varphi} \circ \gamma' \circ \varphi_0 = L_{\varphi} \circ \gamma|_V,$$

or

$$R_{\varphi} \circ \gamma' \circ \text{Cl}(\varphi_0) = L_{\varphi} \circ \gamma,$$

where $L_{\varphi} : \text{End}_{\mathbb{C}}(S) \to \text{Hom}_{\mathbb{C}}(S, S')$ and $R_{\varphi} : \text{End}_{\mathbb{C}}(S') \to \text{Hom}_{\mathbb{C}}(S, S')$ are defined as follows:

$$L_{\varphi}(A) \overset{\text{def}}{=} \varphi \circ A, \quad R_{\varphi}(B) \overset{\text{def}}{=} B \circ \varphi, \; \forall A \in \text{End}_{\mathbb{C}}(S), \; \forall B \in \text{End}_{\mathbb{C}}(S').$$

\[
\begin{array}{ccc}
\text{Cl}(V', h') & \overset{\gamma'}{\longrightarrow} & \text{End}_{\mathbb{C}}(S') \\
\text{Cl}(\varphi_0) & \downarrow & \text{Hom}_{\mathbb{C}}(S, S') \\
\text{Cl}(V, h) & \overset{\gamma}{\longrightarrow} & \text{End}_{\mathbb{C}}(S) \\
\end{array}
\]

With morphisms defined as above, complex Clifford representations form a category denoted $\text{ClRep}$, were compatible morphisms $(\varphi_0, \varphi)$ and $(\varphi'_0, \varphi')$ compose pairwise, that is $(\varphi'_0, \varphi') \circ (\varphi_0, \varphi) \overset{\text{def}}{=} (\varphi'_0 \circ \varphi_0, \varphi' \circ \varphi)$. The functor $\Pi : \text{ClRep} \to \text{Cl}_{\mathbb{R}}$ which takes $\gamma$ into $\text{Cl}(V, h)$ and $(\varphi_0, \varphi)$ into $\text{Cl}(\varphi_0)$ is a fibration whose fiber above $\text{Cl}(V, h)$ is the usual category $\text{Rep}_{\mathbb{C}}(\text{Cl}(V, h))$ of complex representations of $\text{Cl}(V, h)$ (whose morphisms are the based morphisms of complex representations). Isomorphisms in $\text{Rep}_{\mathbb{C}}(\text{Cl}(V, h))$ are the usual equivalences of complex representations. Hence equivalences of complex representations of real Clifford algebras coincide with based isomorphisms of $\text{ClRep}$; in particular, any isomorphism class of complex Clifford representations in the category $\text{ClRep}$ decomposes as a disjoint union of equivalence classes. A morphism $(\varphi_0, \varphi)$ is an isomorphism in $\text{ClRep}$ if and only if both $\varphi_0$ and $\varphi$ are bijective.

**Proposition 3.3.** Let $(\varphi_0, \varphi) : \gamma \to \gamma'$ be an isomorphism of complex Clifford representations. Then $(\varphi_0, \varphi)$ satisfies $\text{Ad}(\varphi)(\gamma(V)) = \gamma'(V')$ and $\gamma' \circ \varphi_0 = \text{Ad}(\varphi) \circ \gamma|_V$, where the unital isomorphism of $\mathbb{C}$-algebras $\text{Ad}(\varphi) : \text{End}_{\mathbb{C}}(S) \to \text{End}_{\mathbb{C}}(S')$ is defined through:

$$\text{Ad}(\varphi)(A) \overset{\text{def}}{=} \varphi \circ A \circ \varphi^{-1}$$

for all $A \in \text{End}_{\mathbb{C}}(S)$. 

Proof. When \((\varphi_0, \varphi)\) is an isomorphism, condition 3. in Definition 3.2 becomes:

\[ \gamma' \circ \text{Cl}(\varphi_0) = \text{Ad}(\varphi) \circ \gamma, \]

being equivalent with the condition \(\gamma' \circ \varphi_0 = \text{Ad}(\varphi) \circ \gamma|_V\), which states that \(\varphi\) implements the isometry \(\varphi_0 : (V, h) \to (V', h')\) at the level of the representation spaces. The Proposition now follows from the fact that \(\text{Cl}(\varphi_0)|_V = \varphi_0\) and \(\varphi_0(V) = V'\).

\[\square\]

3.2. Weakly-faithful complex Clifford representations.

**Definition 3.4.** A complex Clifford representation \(\gamma : \text{Cl}(V, h) \to \text{End}_C(S)\) is called weakly-faithful if the restriction \(\gamma_0 \overset{\text{def}}{=} \gamma|_V : V \to \text{End}_C(S)\) is an injective map.

**Remark 3.5.** Notice that \(\gamma|_V = \gamma|_C|_V\) and that any faithful complex Clifford representation is weakly-faithful.

Let \(\text{ClRep}_w\) denote the full sub-category of \(\text{ClRep}\) whose objects are the weakly-faithful complex Clifford representations and let \(\text{ClRep}_w^\gamma\) denote the corresponding unit groupoid. When \(\gamma\) and \(\gamma'\) are weakly-faithful and \((\varphi_0, \varphi) : \gamma \to \gamma'\) is an isomorphism of Clifford representations, Proposition 3.3 shows that \(\varphi_0\) is uniquely determined by \(\varphi\) through the relation:

\[
(3) \quad \varphi_0 = (\gamma'|_V)^{-1} \circ \text{Ad}(\varphi) \circ \gamma|_V.
\]

It is easy to see that the converse also holds, so we have:

**Proposition 3.6.** Assume that \(\gamma\) and \(\gamma'\) are weakly-faithful. Then any isomorphism \((\varphi_0, \varphi) : \gamma \overset{\sim}{\to} \gamma'\) is determined by the linear isomorphism \(\varphi : S \overset{\sim}{\to} S'\). We have \(\text{Ad}(\varphi)(\gamma(V)) = \gamma'(V')\) and \(\varphi_0\) is given by relation (3). Conversely, any linear isomorphism \(\varphi : S \to S'\) which satisfies \(\text{Ad}(\varphi)(\gamma(V)) = \gamma'(V')\) determines an isomorphism of quadratic spaces \(\varphi_0 : (V, h) \to (V', h')\) through the relation (3) and we have \((\varphi_0, \varphi) \in \text{Hom}_{\text{ClRep}_w^\gamma}(\gamma, \gamma')\).

In view of this, we denote isomorphisms of weakly-faithful complex Clifford representations only by \(\varphi\) (since \(\varphi\) determines \(\varphi_0\) in this case). From the previous proposition, we obtain:

**Corollary 3.7.** The group \(\text{Hom}_{\text{ClRep}_w^\gamma}(\gamma, \gamma')\) can be identified with the following subset of the set \(\text{Iso}_C(S, S')\) of linear isomorphisms from \(S\) to \(S'\):

\[
\text{Hom}_{\text{ClRep}_w^\gamma}(\gamma, \gamma') \equiv \{ \varphi \in \text{Iso}_C(S, S') | \text{Ad}(\varphi)(\gamma(V)) = \gamma'(V') \}, \quad \gamma, \gamma' \in \text{Ob}(\text{ClRep}_w) .
\]

3.3. Complex Lipschitz groups. When \(\gamma\) is weakly-faithful, we can identify \(V\) with its image \(W \overset{\text{def}}{=} \gamma(V)\) inside \(\text{End}_C(S)\). Equip \(W\) with the bilinear form \(\mu\) induced by \(\gamma\) from \((V, h)\), so that \((W, \mu)\) is a real quadratic space isometric with \((V, h)\).

**Definition 3.8.** The complex Lipschitz group of a weakly-faithful complex Clifford representation \(\gamma : \text{Cl}(V, h) \to \text{End}_C(S)\) is defined as the following sub-group of \(\text{Aut}_C(S)\):

\[
\mathbb{L}_\gamma \overset{\text{def}}{=} \{ \varphi \in \text{Aut}_C(S) | \text{Ad}(\varphi)(W) = W \} ,
\]

where \(W = \gamma(V)\).

Notice that \(\mathbb{L}_\gamma\) consists of \(\mathbb{C}\)-linear transformations of \(S\).

**Definition 3.9.** Given a weakly-faithful complex Clifford representation \(\gamma \in \text{Ob}(\text{ClRep}_w)\), we define the adjoint representation \(\text{Ad}_\gamma : \mathbb{L}_\gamma \to \text{O}(V, h)\) of \(\mathbb{L}_\gamma\) by:

\[
\text{Ad}_\gamma(\varphi) \overset{\text{def}}{=} (\gamma|_V)^{-1} \circ \text{Ad}(\varphi) \circ (\gamma|_V),
\]

for all \(\varphi \in \mathbb{L}_\gamma\).

The following proposition partially characterizes the image of the adjoint representation \(\text{Ad}_\gamma\), and in particular shows that \(\text{Ad}_\gamma\) is well-defined.
Proposition 3.10. Let $\gamma : \text{Cl}(V,h) \to \text{End}_{\mathbb{C}}(S)$ be a weakly-faithful complex Clifford representation and $w$ be an element of the space $W = \gamma(V)$. Then $w$ is an element of $L_\gamma$ and $\text{Ad}_\gamma(w) = -R_w$, where $R_w$ denotes the orthogonal reflection of $(V,h)$ with respect to the hyperplane orthogonal to the vector $w_0 \overset{\text{def.}}{=} (\gamma|_V)^{-1}(w) \in V$. If $d \overset{\text{def.}}{=} \dim_{\mathbb{R}} V$ is even, then we have $\text{Ad}_\gamma(L_\gamma) = \text{O}(V,h)$. If $d$ is odd, then we have $\text{SO}(V,h) \subseteq \text{Ad}_\gamma(L_\gamma)$.

Proof. Explicit computation shows that:

$$\text{Ad}_\gamma(w)(v) = w_0vw_0^{-1} = -v + 2 \frac{g(w_0,v)}{g(w_0,w_0)}w_0 = -R_w(v),$$

for every $v \in V$, where $w_0 = \gamma^{-1}(w) \in V$. The Cartan-Dieudonné theorem implies that $\text{Ad}_\gamma(L_\gamma) = \text{O}(V,h)$ for even $d$ and $\text{SO}(V,h) \subseteq \text{Ad}_\gamma(L_\gamma)$ for odd $d$. \qed

Remark 3.11. For $d$ odd, the image $\text{Ad}_\gamma(L_\gamma) \subseteq \text{O}(V,h)$ of $L_\gamma$ in $\text{O}(V,h)$ depends on the details of the particular weakly-faithful representation $\gamma$ under consideration. We will see for instance in Proposition 2.4 that if $\gamma$ is irreducible, then $\text{Ad}_\gamma(L_\gamma) = \text{SO}(V,h)$.

The following Proposition follows form Corollary 3.7, but we give a direct proof because of its importance later.

Proposition 3.12. Let $\gamma$ be a weakly-faithful complex Clifford representation. Then the Lipschitz group $L_\gamma$ is canonically isomorphic with the automorphism group $\text{Aut}_{\text{ClRep}_w}(\gamma)$ of $\gamma$ in the category $\text{ClRep}_w$ of weakly faithful complex Clifford representations. In particular, the isomorphism class of the group $L_\gamma$ depends only on the isomorphism class of $\gamma$ in that category.

Proof. Any $\varphi \in L_\gamma$ induces an invertible isometry $\varphi_0 \in \text{O}(V,h)$ through relation (3), namely:

$$\varphi_0 = (\gamma|_V)^{-1} \circ \text{Ad}(\varphi) \circ (\gamma|_V) \in \text{O}(V,h),$$

which implies:

$$\gamma \circ \text{Cl}(\varphi_0) = \text{Ad}(\varphi) \circ \gamma.$$ 

Thus $(\varphi_0, \varphi)$ is the unique automorphism of $\gamma$ in the category $\text{ClRep}_w$ whose second component equals $\varphi$. Conversely, we have $\varphi \in L_\gamma$ for any $(\varphi_0, \varphi) \in \text{Aut}_{\text{ClRep}_w}(\gamma)$ (see Proposition 3.3) and $\varphi_0$ is determined by $\varphi$ through relation (4) (see Proposition 3.6). Hence the map $F : \text{Aut}_{\text{ClRep}_w}(\gamma) \to L_\gamma$ given by $F(\varphi_0, \varphi) \overset{\text{def.}}{=} \varphi$ is an isomorphism of groups which allows us to identify $L_\gamma$ with $\text{Aut}_{\text{ClRep}_w}(\gamma)$. \qed

4. Elementary complex Lipschitz groups

Definition 4.1. The complex Lipschitz group $L_\gamma$ of an irreducible complex Clifford representation is called an elementary complex Lipschitz group.

Let $(V,h)$ be a real quadratic space of signature $(p,q)$ and dimension $d = p + q$. When $d$ is even, all irreducible complex Clifford representation $\gamma : \text{Cl}(V,h) \to \text{End}_{\mathbb{C}}(S)$ are mutually $\mathbb{C}$-equivalent and faithful (these are called Dirac representations in [1]). When $d$ is odd, there exist up to $\mathbb{C}$-equivalence exactly two complex Clifford representations $\gamma : \text{Cl}(V,h) \to \text{End}_{\mathbb{C}}(S)$, none of which is faithful (these are called Pauli representations in op. cit.). As we shall see below, both of these representations are weakly-faithful. Moreover, they are unabasedly isomorphic in the category $\text{ClRep}_w$. In particular, the category $\text{ClRep}_w$ contains a single unabased isomorphism class of irreducible complex Clifford representations of any given quadratic space $(V,h)$ (and this isomorphism class is uniquely determined by the dimension and signature of $(V,h)$).
4.1. The case when $d$ is even. In this case, there exists a single $\mathbb{C}$-equivalence class of irreducible complex Clifford representations $\gamma : \text{Cl}(V, h) \to \text{End}_{\mathbb{C}}(S)$ and any such representation is faithful and satisfies $\dim_{\mathbb{C}} S = 2^d$ (being a “Dirac representation”); in fact, the map $\gamma$ is bijective. The elementary complex Lipschitz group of such representations was determined in [1, Theorem 1]:

**Proposition 4.2.** [1] When $d$ is even, we have $L_\gamma = \text{Pin}(V, h) \cdot \mathbb{C} \cong \Gamma(V, h)$, which is homotopy-equivalent with $\text{Pin}^+(V, h)$.

4.2. The case when $d$ is odd. When $d$ is odd, an irreducible complex representation of $\text{Cl}(V, h)$ is non-faithful of dimension $\dim_{\mathbb{C}} S = 2^{\frac{d+1}{2}}$. Up to $\mathbb{C}$-equivalence, there exist two such representations $\gamma^+$ and $\gamma^-$ (called “Pauli representations” in [1]), which can be described as follows. Since $d$ is odd, the Clifford volume element $\nu \in \text{Cl}(V, h)$ determined by some fixed orientation of $V$ is central in $\text{Cl}(V, h)$ and satisfies $\nu^2 = \sigma_{p,q} \overset{\text{def}}{=} (-1)^{\frac{d-1}{2}}$. Since $\text{Cl}(V, h)$ is generated by $V$ over $\mathbb{C}$, the real volume element is also central in $\text{Cl}(V, h)$, where it can be rescaled to a complex volume element $\nu_\mathbb{C} \overset{\text{def}}{=} \lambda \nu \in \text{Cl}(V, h)$, where $\lambda$ is one of the two complex square roots of $\sigma_{p,q}$. The complex volume element is central in $\text{Cl}(V, h)$ and satisfies $\nu_\mathbb{C}^2 = 1$. Since $\nu_\mathbb{C}$ is central and equals to $+1$, we can decompose $\text{Cl}(V, h)$ as a direct sum of two-sided ideals:

$$\text{Cl}(V, h) = \frac{1}{2}(1 + \nu_\mathbb{C})\text{Cl}(V, h) \oplus \frac{1}{2}(1 - \nu_\mathbb{C})\text{Cl}(V, h),$$

which are unital $\mathbb{C}$ algebras of dimension $2^{d-1}$, with units given respectively by $\frac{1}{2}(1 + \nu_\mathbb{C})$ and $\frac{1}{2}(1 - \nu_\mathbb{C})$ (note however that they are not unital subalgebras of $\text{Cl}(V, h)$). There exist two representations $\gamma^+_\mathbb{C}$ and $\gamma^-_\mathbb{C}$ of $\text{Cl}(V, h)$ which can be realized on the same $\mathbb{C}$-vector space $S$ and are distinguished by their value on $\nu_\mathbb{C}$:

$$\gamma^+_\mathbb{C}(\nu_\mathbb{C}) = \pm \text{id}_S.$$

The kernel of $\gamma^+_\mathbb{C}$ is given by:

$$\ker \gamma^+_\mathbb{C} = \frac{1}{2}(1 + \nu_\mathbb{C})\text{Cl}(V, h).$$

Restricting $\gamma^+_\mathbb{C}$ gives unital isomorphisms of $\mathbb{C}$-algebras:

$$\gamma^+_\mathbb{C} : \frac{1}{2}(1 + \nu_\mathbb{C})\text{Cl}(V, h) \cong \text{End}_{\mathbb{C}}(S).$$

We have $\text{Cl}(V, h) = \text{Cl}(V, h) \oplus i\text{Cl}(V, h)$, where $i$ denotes the imaginary unit. The irreducible complex representations $\gamma^\pm$ are now obtained by restricting $\gamma^+_\mathbb{C}$ to $\text{Cl}(V, h)$:

$$\gamma^\pm \overset{\text{def}}{=} \gamma^+_\mathbb{C}|_{\text{Cl}(V, h)}.$$

**Proposition 4.3.** The Pauli representations $\gamma^\pm$ are weakly-faithful and isomorphic in the category $\text{ClRep}_w$.  

**Proof.** The restriction $\gamma^\pm|_V = \gamma^\pm|_V$ is injective iff $V \cap \ker \gamma^\pm = 0$. The Chevalley-Crumeyrolle-Riesz isomorphism identifies $\text{Cl}(V, h)$ with $\wedge V^\vee$. Using this identification, we have $x \nu_\mathbb{C} \in \wedge^{d-1} V^\vee$ for any vector $x \in V$. Relation (5) shows that one can have $\ker \gamma^\pm \cap V \neq 0$ only when $d = 2$, which is disallowed since $d$ is odd. This shows that we must have $\ker \gamma^\pm \cap V = 0$, which implies $\ker \gamma^\pm \cap V = 0$. This shows that $\gamma^\pm$ are weakly-faithful.

It is well-known that $\gamma^- = \gamma^+ \circ \pi$, where $\pi$ is the parity involution of $\text{Cl}(V, h)$. Since $\pi = \text{Cl}(-\text{id}_V)$ is the automorphism of $\text{Cl}(V, h)$ which is induced by minus the identity map of $V$ (which is an isometry of $V$), it follows that $\gamma^+$ and $\gamma^-$ are isomorphic in the category $\text{ClRep}_w$, being related by the involutive unbased isomorphism:

$$(-\text{id}_V, \text{id}_S) : \gamma^+ \sim \gamma^-.$$

\[\square\]

\[2\text{Notice that } \nu_\mathbb{C} \text{ belongs to } \text{Cl}(V, h) \text{ iff } \sigma_{p,q} = +1 \text{ (i.e. if and only if } p - q \equiv_8 1, 5), \text{ in which case we have } \nu_\mathbb{C} = \pm \nu.\]
Lemma 4.4. When \( d \) is odd, the restriction of \( \gamma^\pm_C \) gives a unital isomorphism of associative \( \mathbb{C} \)-algebras from \( \text{Cl}^+(V,h) \) to \( \text{End}_\mathbb{C}(S) \).

Proof. It is obvious that \( \gamma^\pm_C \) restricts to a unital morphism of algebras from \( \text{Cl}^+(V,h) \) to \( \text{End}_\mathbb{C}(S) \).
The equation \( x \nu_C = \epsilon x \) has no solutions in \( \text{Cl}^+(V,h) \) because the left-hand side is odd while the right-hand side is even; this shows that \( \gamma^\pm_C |_{\text{Cl}^+(V,h)} \) is injective. On the other hand, we have \( \dim \mathbb{C} \text{Cl}^+(V,h) = \dim \mathbb{C} \text{End}_\mathbb{C}(S) = 2^{d-1} \), so \( \gamma^\pm_C |_{\text{Cl}^+(V,h)} \) is bijective. \( \square \)

Proposition 4.5. Let \( \gamma : \text{Cl}(V,h) \rightarrow \text{End}_\mathbb{C}(S) \) be an irreducible complex Clifford representation such that \( d = \dim \mathbb{C} V \) is odd. Then \( \gamma_C \) restricts to an isomorphism between \( \Gamma^s(V,h) \simeq \text{Spin}^s(V,h) \cdot \mathbb{C}^\times \) and \( L_\gamma \). In particular, the elementary complex Lipschitz group \( L_\gamma \) is homotopy-equivalent with \( \text{Spin}^s(V,h) \).

Proof. From Lemma 4.4 we have a unital isomorphism of algebras \( \gamma_C : \text{Cl}^+(V,h) \rightarrow \text{End}_\mathbb{C}(S) \) which restricts to an isomorphism of groups \( \gamma_C : \text{Cl}(V,h)^\times \rightarrow \text{Aut}_\mathbb{C}(S) \). In turn, this further restricts to an isomorphism of groups:

\[ \gamma_C : \Gamma^s(V,h) \overset{\sim}{\rightarrow} L_\gamma. \]

The fact that \( L_\gamma \) is homotopy-equivalent to \( \text{Spin}^s(V,h) \) now follows from Proposition 2.4. \( \square \)

Since for odd \( d \) and complex irreducible \( \gamma \) the kernel of \( \text{Ad}_\gamma \) is given by the complex multiples of the identity, the previous proposition gives the short exact sequence:

\[ 1 \rightarrow \mathbb{C}^\times \rightarrow L_\gamma \overset{\text{Ad}_\gamma}{\rightarrow} \text{SO}(V,h) \rightarrow 1. \]

Remark 4.6. For odd \( d \), reference [1] determines the non-elementary complex Lipschitz group of the faithful but reducible complex Cartan representation of \( \text{Cl}(V,h) \). As shown in [1], that complex Lipschitz group is isomorphic with \( [\text{Pin}(V,h) \times (\mathbb{C}^\times)^2]/\mathbb{Z}_2 \), where the semidirect product in the numerator is determined explicitly in op. cit.

4.3. Realification of Clifford representations. Let \( (V,h) \) be a real quadratic space of signature \( (p,q) \) and dimension \( d = p + q \). For \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \), let \( \text{Rep}_\mathbb{K}(\text{Cl}(V,h)) \) denote the ordinary category of \( \mathbb{K} \)-linear representations of \( \text{Cl}(V,h) \) (whose morphisms are the based morphisms of representations). Let \( \text{Rep}_\mathbb{K}^w(\text{Cl}(V,h)) \) denote the full sub-category of \( \text{Rep}_\mathbb{K}(\text{Cl}(V,h)) \) whose objects are the weakly-faithful representations. Given a complex Clifford representation \( \gamma : \text{Cl}(V,h) \rightarrow \text{End}_\mathbb{C}(S) \), let \( S_\mathbb{R} \) be the underlying real vector space of \( S \) and \( \gamma_\mathbb{R} : \text{Cl}(V,h) \rightarrow \text{End}_\mathbb{R}(S_\mathbb{R}) \) be the realification of \( \gamma \), i.e. the real Clifford representation obtained from \( \gamma \) by forgetting the complex structure of \( S \). Given two complex Clifford representations \( \gamma_i : \text{Cl}(V,h_i) \rightarrow \text{End}_\mathbb{C}(S_i) \) (with \( i = 1, 2 \)) and a based morphism of complex representations \( \varphi : \gamma_1 \rightarrow \gamma_2 \), let \( \varphi_\mathbb{R} : \gamma_{1\mathbb{R}} \rightarrow \gamma_{2\mathbb{R}} \) be the based morphism of real representations obtained by forgetting the complex structures of \( S_1 \) and \( S_2 \).

Definition 4.7. The realification functor \( R : \text{Rep}_\mathbb{C}(\text{Cl}(V,h)) \rightarrow \text{Rep}_\mathbb{R}(\text{Cl}(V,h)) \) is the functor which maps a complex representation \( \gamma : \text{Cl}(V,h) \rightarrow \text{End}_\mathbb{C}(S) \) to the real representation \( R(\gamma) \overset{\text{def}}{=} \gamma_\mathbb{R} : \text{Cl}(V,h) \rightarrow \text{End}_\mathbb{R}(S_\mathbb{R}) \) and a based morphism \( \varphi : \gamma_1 \rightarrow \gamma_2 \) of complex representations to the based morphism of real representations \( R(\varphi) \overset{\text{def}}{=} \varphi_\mathbb{R} : \gamma_{1\mathbb{R}} \rightarrow \gamma_{2\mathbb{R}} \).

It is clear that \( R \) is faithful and that it maps \( \text{Rep}_\mathbb{C}^w(\text{Cl}(V,h)) \) to \( \text{Rep}_\mathbb{R}^w(\text{Cl}(V,h)) \).

Proposition 4.8. Let \( \gamma : \text{Cl}(V,h) \rightarrow \text{End}_\mathbb{C}(S) \) be an irreducible complex Clifford representation, where \( (V,h) \) has signature \( (p,q) \) and dimension \( d = p + q \). Then the realification \( \gamma_\mathbb{R} \) of \( \gamma \) is \( \mathbb{R} \)-irreducible iff \( p - q \equiv_8 3, 4, 6, 7 \).

Proof. Distinguish the cases:

1. When \( d \) is even. Then \( \gamma \) is a Dirac representation and we have \( \dim \mathbb{C} S = 2^d \). Thus \( \dim \mathbb{R} S_\mathbb{R} = 2^{d+1} \). Comparing with Table 4 of [6, Sec. 2.4], we see that \( \gamma_\mathbb{R} \) is irreducible iff \( p - q \equiv_8 4, 6 \) (which corresponds to the “quaternionic simple case” of op. cit.).
2. When $d$ is odd. Then we can take $\gamma$ to be one of the Pauli representations $\gamma^\pm$, both of which satisfy $\dim \mathbb{S} = 2[\frac{d}{2}]$. Thus $\dim \mathbb{S}_R = 2[\frac{d}{2}] + 1$. Comparing with Table 4 of [6, Sec. 2.4], we see that $\gamma_R$ is irreducible iff $p - q \equiv_3 3, 7$ (which corresponds to the “complex case” of op. cit.).

Let $\text{Irrep}_R(\text{Cl}(V, h))$ denote the full subcategory of $\text{Rep}_R^w(\text{Cl}(V, h))$ whose objects are the irreducible representations.

**Proposition 4.9.** When $p - q \equiv_8 3, 4, 6, 7$, the restriction $R : \text{Irrep}_C(\text{Cl}(V, h))^\times \to \text{Irrep}_R(\text{Cl}(V, h))^\times$ of the realification functor is faithful and strictly surjective but not full. Moreover:

1. When $p - q \equiv_8 3, 7$, the $R$-preimage of a real irreducible representation $\eta : \text{Cl}(V, h) \to \text{End}_R(\Sigma)$ has cardinality two and consists of the $\mathbb{C}$-inequivalent complex Pauli representations of $\text{Cl}(V, h)$ whose representation space is obtained by endowing $\Sigma$ with the complex structures $J_\pm = \pm \gamma(\nu)$, where $\nu$ is the Clifford volume form determined by some orientation of $V$.

2. When $p - q \equiv_8 4, 6$, the $R$-preimage of a real irreducible representation $\eta : \text{Cl}(V, h) \to \text{End}_R(\Sigma)$ is in bijection with the unit sphere $S^2$ and consists of the complex Dirac representations whose complex structures are the unit imaginary elements of the commutant of $\text{im}(\eta)$ (which in these cases is a quaternion algebra).

**Proof.** The fact that $R$ is faithful follows immediately from its definition. For the remaining statements, consider the cases:

1. $p - q \equiv_8 3, 7$. In this case, there exist exactly two complex structures $J_{\pm}$ on the $\mathbb{R}$-vector space $\Sigma$ which lie in the commutant of $\text{im}(\eta)$ (see [6, Section 2.7]). Namely, one has $J_{\pm} = \pm \gamma(\nu)$, where $\nu \in \text{Cl}(V, h)$ is the Clifford element defined by a fixed orientation of $V$. Since $J_{\pm}$ commute with $\eta(x)$ for all $x \in \text{Cl}(V, h)$, it follows that $\eta$ coincides with the realification of any of the complex representations $\gamma^{\pm} : \text{Cl}(V, h) \to \text{End}_C(S^\pm)$ obtained from $\eta$ by endowing $\Sigma$ with the complex structure $J_{\pm}$ to obtain a complex vector space $S^\pm$. Notice that $S^\pm$ coincides with the complex-conjugate of $S^+$, hence $\gamma^{+}$ and $\gamma^{-}$ are mutually inequivalent Pauli representations (the dimension $d = p + q$ of $V$ is odd when $p - q \equiv_8 3, 7$). We have $\Sigma = (S^\pm)_R$ and $\eta = (\gamma_{\pm})_R$, which shows that the restriction of $R$ is (strictly) surjective. Moreover, the $R$-preimage of $\eta$ consists of the two complex representations $\gamma_+$ and $\gamma_-$. Let $\eta' : \text{Cl}(V, h) \to \text{End}_R(\Sigma)$ be a real irreducible representation which is equivalent with $\gamma$ and $J_{\pm}^{\text{def}} = \pm \gamma'(\nu)$. Given a based isomorphism (equivalence of representations) $\psi : \eta \to \eta'$, we have $\psi \in \text{Hom}_R(\Sigma, \Sigma')$ and:

$$\psi \circ \eta(x) = \eta'(x) \circ \psi \quad \forall x \in \text{Cl}(V, h).$$

Since $J_{\pm}$ lies in the commutant of $\text{im}(\eta)$, this relation implies that the complex structure $\psi \circ J_{\pm} \circ \psi^{-1} \in \text{End}_R(\Sigma')$ lies in the commutant of $\text{im}(\eta')$ and hence that we must have $\psi \circ J_{\pm} \circ \psi^{-1} = \epsilon \psi J_{\pm}$, i.e. $\psi \circ J_{\pm} = \epsilon \psi J_{\pm} \circ \psi$ for some $\epsilon \in \{-1, 1\}$. This gives the disjoint union decomposition:

$$\text{Iso}_{\text{Rep}_R(\text{Cl}(V, h))}(\eta, \eta') = \text{Iso}_{\text{Rep}_R(\text{Cl}(V, h))}^+(\eta, \eta') \sqcup \text{Iso}_{\text{Rep}_R(\text{Cl}(V, h))}^-(\eta, \eta'),$$

where:

$$\text{Iso}_{\text{Rep}_R(\text{Cl}(V, h))}^\pm(\eta, \eta') \equiv \{ \psi \in \text{Iso}_{\text{Rep}_R(\text{Cl}(V, h))}(\eta, \eta') | \epsilon \psi = \pm 1 \}.$$

It is clear from the above that:

$$R(\text{Iso}_{\text{Rep}_R(\text{Cl}(V, h))}^\pm(\eta, \eta')) = \text{Iso}_{\text{Rep}_R(\text{Cl}(V, h))}^{\pm'}(\eta, \eta') \quad \forall \epsilon, \epsilon' \in \{-1, 1\}.$$  

In particular, each of the sets $\text{Iso}_{\text{Rep}_R(\text{Cl}(V, h))}^\pm(\eta, \eta')$ is non-empty and relation (6) shows that $R$ is not full.

2. $p - q \equiv_8 4, 6$. In this case, the commutant of $\text{im}(\eta)$ inside the $\mathbb{R}$-algebra $\text{End}_R(\Sigma)$ (which is called the Schur algebra of $\eta$) is a unital associative algebra isomorphic with the algebra $\mathbb{H}$ of quaternions. The set of those complex structures on $\Sigma$ which lie in the commutant of $\text{im}(\gamma)$ corresponds to the unit imaginary quaternions, being in bijection with the unit two-sphere $S^2$. 

It is clear that $\eta$ coincides with the realification of any of the complex Dirac representations obtained from $\eta$ by endowing $\Sigma$ with one of these complex structures. This immediately implies that $R$ is not full.

4.4. Comparison of complex and real elementary Lipschitz groups when $p - q \equiv_8 3, 4, 6, 7$.

For $p - q \equiv_8 3, 7$, the elementary complex Lipschitz group of an irreducible complex Clifford representation $\gamma$ is homotopy-equivalent to $\Spin^c(V, h)$, while the reduced elementary real Lipschitz group associated to $\gamma_R$ is homotopy-equivalent to $\Spin^q(V, h)$ [3]. The latter group is $\mathbb{Z}_2$-graded with even component equal to the $\Spin^c(V, h)$ subgroup consisting of those elements of $\Spin^c(V, h)$ which are $\mathbb{C}$-linear automorphisms of $S$:

$$\Spin^c(V, h) \simeq \{ a \in \Spin^q(V, h) | a \in \text{End}_\mathbb{C}(S) \}.$$ 

Unlike $\Spin^c(V, h)$, the reduced elementary real Lipschitz group $\Spin^q(V, h)$ also contains $\mathbb{R}$-linear endomorphisms of $S$ which are $\mathbb{C}$-antilinear; these form the odd component of $\Spin^c(V, h)$.

For $p - q \equiv_8 4, 6$, the elementary complex Lipschitz group of an irreducible complex Clifford representation $\gamma$ is homotopy-equivalent to $\Pin^q(V, h)$, while the reduced elementary real Lipschitz group associated to $\gamma_R$ is homotopy-equivalent to $\Pin^q(V, h)$ [3].

Remark 4.10. When $p - q \not\equiv_8 3, 4, 6, 7$, the real Clifford representation obtained by realification from an irreducible complex Clifford representation is reducible. In that case, there is no simple relation between real and complex elementary Lipschitz groups.

5. Complex pinor bundles and complex Lipschitz structures

5.1. Complex Lipschitz structures. Let $(V, h)$ be a quadratic real vector space and $\eta: \text{Cl}(V, h) \to \text{End}_\mathbb{C}(\Sigma)$ be a weakly-faithful complex Clifford representation, where $\Sigma$ is a $\mathbb{C}$-vector space. We denote by $L_\eta$ the corresponding complex Lipschitz group and by $\text{Ad}_\eta: L_\eta \to O(V, h)$ its adjoint representation.

Definition 5.1. Let $M$ be a connected manifold and $P_O$ be a principal $O(V, h)$-bundle over $M$. A complex Lipschitz structure of type $\eta$ on $P_O$ is a pair $(Q, \Lambda)$, where $Q$ is a principal $L_\eta$-bundle over $M$ and $\Lambda: Q \to P_O$ is a bundle map fitting into the following commutative diagram:

$$
\begin{array}{ccc}
Q \times L_\eta & \longrightarrow & Q \\
\Lambda \times \text{Ad}_\eta & \downarrow & \Lambda \\
P_O \times O(V, h) & \longrightarrow & P_O
\end{array}
$$

where the horizontal arrows denote the right action of the group on the corresponding bundle.

Definition 5.2. Let $(Q^1, \Lambda^1)$ and $(Q^2, \Lambda^2)$ be two complex Lipschitz structures of type $\eta$ over $P_O$. A morphism of complex Lipschitz structures from $(Q^1, \Lambda^1)$ to $(Q^2, \Lambda^2)$ is a morphism $F: Q^1 \to Q^2$ of principal $L_\eta$-bundles such that $\Lambda^2 \circ F = \Lambda^1$, i.e. such that the following diagram commutes:

$$
\begin{array}{ccc}
Q^1 & \xrightarrow{F} & Q^2 \\
\Lambda^1 & \downarrow & \Lambda^2 \\
P_O & \xrightarrow{\text{id}} & P_O
\end{array}
$$

Let $(M, g)$ be a connected pseudo-Riemannian manifold. In the following we take $P_O$ to be the orthogonal coframe bundle $P_O(M, g)$ of $(M, g)$, so $(V, h)$ is an isometric model of any of the quadratic spaces $(T^*_p M, g^*_p)$, where $p$ is a point of $M$ and $g^*_p$ is the contragradient metric induced by $g$ on $T^*_p M$. A complex Lipschitz structure on $P_O(M, g)$ be called a complex Lipschitz structure on $(M, g)$. 

Definition 5.3. Let $L_\eta(M, g)$ be the category whose objects are the complex Lipschitz structures of type $\eta$ on $(M, g)$ and whose arrows are morphisms of complex Lipschitz structures. We denote by $L_\eta(M, g)^\times$ the corresponding groupoid.

5.2. Reduced complex Lipschitz structures. Let $\eta : Cl(V, h) \to \text{End}_C(\Sigma)$ be a weakly-faithful Clifford representation and $L_\eta$ be the corresponding complex Lipschitz group.

Definition 5.4. A reduction of $L_\eta$ is a closed subgroup $L_\eta^0 \subset L_\eta$ such that there exists a surjective morphism of Lie groups $r : L_\eta \to L_\eta^0$ which satisfies the conditions:

1. $r$ is a homotopy retraction of the inclusion map $\iota : L_\eta^0 \hookrightarrow L_\eta$, i.e. we have $r \circ \iota = \text{id}_{L_\eta}$ and the map $\iota \circ r : L_\eta \to L_\eta$ is homotopic to $\text{id}_{L_\eta}$.
2. We have $\text{Ad}_\eta \circ r = \text{Ad}_\eta$, i.e. $r$ fits into the following commutative diagram:

\[
\begin{array}{ccc}
L_\eta^0 & \xrightarrow{\eta} & L_\eta \\
\downarrow{r} & & \downarrow{\eta} \\
L_\eta & \xrightarrow{\iota} & O(V, h)
\end{array}
\]

In this case, $r$ is called an equivariant homotopy retraction of $L_\eta$ onto $L_\eta^0$.

Remark 5.5. Since $r : L_\eta \to L_\eta^0$ is a retraction, the Lie group $K_\eta \overset{\text{def}}{=} \ker r = r^{-1}(1)$ is contractible (indeed, the restriction $r|_{K_\eta} : K_\eta \to 1$ is a homotopy retraction).

Example 5.6. When $\dim_{\mathbb{R}} V$ is even, the group $\text{Pin}^c(V, h)$ is a reduction of the elementary complex Lipschitz group $\text{Pin}(V, h) : \mathbb{C}^\times \simeq \Gamma(V, h)$, an equivariant homotopy retraction being given in equation (1). When $\dim_{\mathbb{R}} V$ is odd, the group $\text{Spin}^c(V, h)$ is a reduction of the elementary complex Lipschitz group $\text{Spin}(V, h) : \mathbb{C}^\times \simeq \Gamma^c(V, h)$, an equivariant homotopy retraction being given by $r|_{\text{Spin}^c(V, h)}$.

Let $L_\eta^0$ be a reduction of $L_\eta$, with inclusion map $\iota$ and equivariant retraction map $r$. Let $\text{Ad}_\eta^0 \overset{\text{def}}{=} \text{Ad}_\eta|_{L_\eta^0}$ and $K_\eta \overset{\text{def}}{=} \ker r$. A reduced complex Lipschitz structure with structure group $L_\eta^0$ is defined as in Definition 5.1, but replacing $L_\eta$ with $L_\eta^0$ and $\text{Ad}_\eta$ with $\text{Ad}_\eta^0$. A morphism of reduced complex Lipschitz structures is defined similarly to Definition 5.2. With these definitions, reduced complex Lipschitz structures with structure group $L_\eta^0$ and defined on $(M, g)$ form a category denoted $L_\eta^0(M, g)$, whose unit groupoid we denote by $L_\eta^0(M, g)^\times$.

The right-split short exact sequence of groups:

\[
1 \longrightarrow K_\eta \longrightarrow L_\eta \overset{\iota}{\longrightarrow} L_\eta^0 \longrightarrow 1
\]

induces mutually inverse isomorphisms in cohomology:

\[
H^1(M, L_\eta) \overset{r_*}{\longrightarrow} H^1(M, L_\eta^0),
\]

where we used the fact that $K_\eta$ is contractible. This shows that $r_*$ and $\iota_*$ give inverse bijections between the sets of isomorphism classes of principal $L_\eta$-bundles and principal $L_\eta^0$-bundles defined on $M$. The maps $r_*$ and $\iota_*$ are induced by the associated fiber bundle construction, which gives mutually quasi-inverse functors between the corresponding categories of principal bundles. Below, we show that these functors in turn induce mutually quasi-inverse equivalences between the categories $L_\eta^0(M, g)$ and $L_\eta^0(M, g)^\times$.

Definition 5.7. Let $(Q, \Lambda)$ be a complex Lipschitz structure of type $\eta$ over $(M, g)$. An $L_\eta^0$-reduction of $(Q, \Lambda)$ is a triple $(Q_0, \Lambda_0, I)$, where $(Q_0, \Lambda_0) \in \text{Ob}(L_\eta^0(M, g))$ and $I : Q_0 \to Q$ is an injective fiber bundle map such that the following conditions are satisfied:

1. $I$ is an $L_\eta^0$-reduction of the principal bundle $Q$. Thus $I(q_0 u_0) = I(q_0) \iota(u_0)$ for all $q_0 \in Q_0$ and all $u_0 \in L_\eta^0$. 


2. The following diagram commutes:

\[
\begin{array}{ccc}
Q_0 & \xrightarrow{I} & Q \\
\downarrow{\Lambda_0} & & \downarrow{\Lambda} \\
P_0(M, g) & \xrightarrow{\pi} & P_0(M, g)
\end{array}
\]

**Definition 5.8.** Let \((Q, \Lambda)\) be a complex Lipschitz structure of type \(\eta\) over \((M, g)\). An \(L^0_\eta\)-retraction of \((Q, \Lambda)\) along \(r\) is a triple \((Q_0, \Lambda_0, R)\), where \((Q_0, \Lambda_0) \in \text{Ob}(L^0_\eta(M, g))\) and \(R : Q \to Q_0\) is a surjective fiber bundle map such that the following conditions are satisfied:

1. \(R(qu) = R(q)r(u)\) for all \(q \in Q\) and all \(u \in L_\eta\).
2. The following diagram commutes:

\[
\begin{array}{ccc}
Q & \xrightarrow{R} & Q_0 \\
\downarrow{\Lambda} & & \downarrow{\Lambda_0} \\
P_0(M, g) & \xrightarrow{\pi} & P_0(M, g)
\end{array}
\]

**Remark 5.9.** Given an \(L^0_\eta\)-retraction \((Q_0, \Lambda_0, R)\) of \((Q, \Lambda)\) along \(r\), the map \(R\) makes \(Q\) into a principal fiber bundle over \(Q_0\) with structure group \(K_\eta = \ker r\). This principal bundle is trivial since \(K_\eta\) is contractible (which implies that the bundle has a section).

**Proposition 5.10.** Consider the functor \(I : L^0_\eta(M, g) \to L_\eta(M, g)\) defined as follows:

- For any \((Q_0, \Lambda_0) \in \text{Ob}(L^0_\eta(M, g))\), define \((Q, \Lambda) := \text{Ob}(Q_0, \Lambda_0) \in \text{Ob}(L_\eta(M, g))\) through:

\[
Q \overset{\text{def}}{=} Q_0 \times L_\eta \, ; \, \Lambda((q_0, u), \omega) \overset{\text{def}}{=} \Lambda_0(q_0) \circ \text{Ad}_\omega(u) \ \forall \ [q_0, u], \omega \in Q .
\]

- For any morphism \(F_0 : (Q_0^1, \Lambda_0^1) \to (Q_0^2, \Lambda_0^2)\) in \(L^0_\eta(M, g)\), define \(F := \text{Ob}(F_0)\) through:

\[
F([q_0, u], \omega) \overset{\text{def}}{=} [F_0(q_0), u], \omega \ \forall \ [q_0, u], \omega \in \text{Ob}(Q_0^1) .
\]

Then \((Q_0, \Lambda_0, I)\) is an \(L^0_\eta\)-reduction of \((Q, \Lambda)\), where the injective fiber bundle map \(I : Q_0 \to Q\) is given by:

\[
I(q_0) \overset{\text{def}}{=} [q_0, 1]_\omega, \ \forall q_0 \in Q_0 .
\]

Moreover, the surjective map \(\pi : Q \to Q_0\) defined through:

\[
\pi([q_0, u], \omega) = q_0r(u)
\]

satisfies \(\pi \circ I = \text{id}_{Q_0}\) and makes \((Q_0, \Lambda_0, \pi)\) into an \(L^0_\eta\)-retraction of \((Q, \Lambda)\) along \(r\).

**Proof.** The morphism \(\Lambda\) makes \(Q\) into a Lipschitz structure since \(\Lambda([q_0, u]w) = \Lambda_0(q_0) \circ \text{Ad}_w(u) = \Lambda([q_0, u]) \circ \text{Ad}_w(u)\) for all \(w \in L_\eta\). Given \([q_0, u]_\omega \in Q^1\) and \(u_0 \in L^0_\eta\), we have:

\[
F([q_0u_0, u_0^{-1}]_\omega) = [F_0(q_0u_0), u_0^{-1}]_\omega = [F_0(q_0)u_0, u_0^{-1}]_\omega = [F(q_0), u]_\omega = F([q_0, u]_\omega) ,
\]

which shows that \(F\) is well-defined. The fact that \((Q_0, \Lambda_0, I)\) is an \(L^0_\eta\)-reduction of \((Q, \Lambda)\) follows from:

\[
(\Lambda \circ I)(q_0) = \Lambda([q_0, 1]_\omega) = \Lambda_0(q_0), \quad \forall q_0 \in L^0_\eta .
\]

Any element \(q \in Q\) can be written as \(q = [q_0, x]_\omega\), where \(x \in \ker r\) and \(q_0 := \pi_0(q) \in Q_0\) are uniquely determined by \(q\). Indeed, any \(u \in L_\eta\) can be written uniquely as \(u = u_0x\) with \(u_0 \in L^0_\eta\) and \(x \in K_\eta = \ker r\), namely \(u_0 = r(u)\) and \(x = u_0^{-1}u\).

The map \(\pi\) is well-defined since:

\[
\pi([q_0u_0, u]_\omega) = q_0u_0r(u) = q_0r(u_0u) ,
\]
Then (9) implies that

\[ \Lambda_0(\pi(q_0, u_0)) = \Lambda_0(q_0(r(u))) = \Lambda_0(q_0)\text{Ad}_{\eta}(r(u)) = \Lambda_0(q_0)\text{Ad}_{\eta}(u) = \Lambda([q_0, u_0]). \]

where we used the relation \( \text{Ad}_{\eta} \circ r = \text{Ad}_{\eta} \). This shows that \((Q_0, \Lambda_0, \pi)\) is an \( L_0^\eta \)-retraction of \((Q, \Lambda)\) along \( r \). It is clear that we have \( \pi \circ I = \text{id}_{Q_0}. \]

### Proposition 5.11

Consider the functor \( R : L_0(M, g) \to L_0^0(M, g) \) defined as follows:

- For any \((Q, \Lambda) \in \text{Ob}(L_0(M, g))\), define \((Q_0, \Lambda_0) := R(Q, \Lambda) \in \text{Ob}(L_0^0(M, g))\) through:

\[
Q_0 := Q \times_r L_0^0, \quad \Lambda_0([q, u_0]) := \Lambda(q) \circ \text{Ad}_{\eta}^0(u_0) \quad \forall [q, u_0], \in Q \times_r L_0^0
\]

- For any morphism \( F : (Q^1, \Lambda^1) \to (Q^2, \Lambda^2) \) in \( L_0(M, g) \), define \( F_0 := R(F) \) through:

\[
F_0([q, u_0]) := [F(q), u_0] \quad \forall [q, u_0] \in R(Q^1).
\]

Then \((Q_0, \Lambda_0, R)\) is an \( L_0^\eta \)-retraction of \((Q, \Lambda)\) along \( r \), where the surjective fiber bundle map \( R : Q \to Q_0 \) is given by:

\[
R(q) := [q, 1]_r \quad \forall q \in Q.
\]

Moreover, there exists an injective fiber bundle map \( j : Q_0 \to Q \) which makes \((Q_0, \Lambda_0, j)\) into an \( L_0^\Lambda \)-reduction of \((Q, \Lambda)\) and satisfies \( R \circ j = \text{id}_{Q_0} \).

**Proof.** Clearly \( Q_0 \) is a principal \( L_0^0 \)-bundle. To see that \( \Lambda_0 \) is well-defined, note that for all \( u \in L_\eta \) we have:

\[
\Lambda_0([q u, r(u)^{-1} u_0]) = \Lambda(q u) \circ \text{Ad}_{\eta}^0(r(u)^{-1} u_0) = \Lambda(q) \circ \text{Ad}_{\eta}(u) \circ \text{Ad}_{\eta}(r(u))^{-1} \circ \text{Ad}_{\eta}(u_0) = \Lambda_0([q, u_0]),
\]

where we used the relation \( \text{Ad}_{\eta} \circ r = \text{Ad}_{\eta} \). The fact that \( F_0 \) is well-defined follows from direct computation by using the equivariance of \( F \). To show that \( R \) is an \( L_0^\eta \)-retraction of \((Q, \Lambda)\) along \( r \), we compute:

\[
(\Lambda_0 \circ R)(q) = \Lambda_0([q, 1]) = \Lambda(q)\text{Ad}_{\eta}^0(1) = \Lambda(q).
\]

Since \( r : L_\eta \to L_0^0 \) is surjective, any \( q_0 \in Q_0 \) can be written in the form \( q_0 = [q, 1]_r \), where \( q \in Q \) is determined by \( q_0 \) up to a transformation of the form \( q \to q x \), where \( x \in \ker r \). Hence \( R^{-1}(q) \) is a \( \ker r \)-torsor and the map \( R : Q \to Q_0 \) presents \( Q \) as a principal \( \ker r \)-bundle over \( Q_0 \). Hence the structure group of the principal bundle \( R : Q \to Q_0 \) is contractible, which implies that this principal bundle is trivial and thus has a section. We claim that we can find a section \( j : Q_0 \to Q \) of \( R \) which is \( L_\eta^0 \)-equivariant.

Let \( \text{Diff}(K_\eta) \) denote the group of diffeomorphisms of \( K_\eta \). The map \( \varphi : L_\eta \to \text{Diff}(K_\eta) \) defined through:

\[
\varphi(u)(x) := r(u)xu^{-1} \quad \forall u \in L_\eta, \forall x \in K_\eta
\]

is a morphism of groups which defines a smooth left action of the Lie group \( L_\eta \) on the underlying manifold of \( K_\eta \). Consider the fiber bundle defined through \( T \simeq Q \times_{\varphi} K_\eta \).

We next show existence\(^3\) of an \( L_0^0 \)-equivariant section of \( R \). Since \( K_\eta \) is contractible, the fiber bundle \( T \) admits a section \( s \in \Gamma(M, T) \). This section corresponds to a map \( \sigma : Q \to K_\eta \) such that \( \sigma(q u) = \varphi(u)^{-1} \sigma(q) \), i.e.:

\[
\sigma(q u) = r(u)^{-1} \sigma(q) \quad \forall q \in Q \forall u \in L_\eta.
\]

Let \( j : Q_0 \to Q \) be the map defined through:

\[
j([q, u_0]) := q \sigma(q)^{-1} u_0, \quad \forall q \in Q \forall u_0 \in L_0^0.
\]

Then (9) implies that \( j \) is well-defined, since for all \( u \in L_\eta \) we have:

\[
j([q u, u_0]) = q u \sigma(q u)^{-1} u_0 = q u \sigma(q)^{-1} \sigma(q u)^{-1} r(u) u_0 = q \sigma(q)^{-1} r(u) u_0 = j([q, r(u) u_0]).
\]

---

\(^3\)This argument is based on a suggestion of A. Moroianu.
Moreover, we have:

\[(R \circ j)([q, u_0]_r) = R(qr(q)^{-1}u_0) = R(q)r(q)^{-1}u_0 = R(q)r(q)^{-1}r(u_0) = R(q)u_0 = [q, 1]_ru_0 = [q, u_0]_r,\]

where we used the fact that \(\sigma(q)^{-1} \in K_\eta = ker r\) and the identity \(r|_{L^0_\eta} = \text{id}_{L^0_\eta}\). Thus \(R \circ j = \text{id}_{Q_0}\).

We also have:

\[j([q, u_0], v_0) = j([q, u_0v_0], r) = j([q, u_0]_r)v_0 \forall v_0 \in L^0_\eta.\]

Hence \(j\) is an \(L^0_\eta\)-equivariant section of \(R : Q \to Q_0\):

\[(10) \quad j(q_0u_0) = j(q_0)u_0 \forall q_0 \in Q_0 \forall u_0 \in L^0_\eta.\]

\[\square\]

**Theorem 5.12.** The functors \(\mathcal{I} \) and \(\mathcal{R}\) give mutually quasi inverse equivalences between the categories \(L^0_\eta(M, g)\) and \(L_\eta(M, g)\).

**Proof.** Composition of \(\mathcal{I}\) and \(\mathcal{R}\) gives functors:

\[
\mathcal{I} \circ \mathcal{R} : L_\eta(M, g) \to L_\eta(M, g) \quad \text{and} \quad \mathcal{R} \circ \mathcal{I} : L^0_\eta(M, g) \to L^0_\eta(M, g).
\]

We will construct isomorphisms of functors:

\[
\mathcal{N} : \text{id}_{L^0_\eta(M, g)} \sim \mathcal{I} \circ \mathcal{R} \quad \text{and} \quad \mathcal{K} : \text{id}_{L^0_\eta(M, g)} \sim \mathcal{R} \circ \mathcal{I}.
\]

1. **Construction of \(\mathcal{N}\).** Let \((Q, \Lambda)\) be a complex Lipschitz structure of type \(\eta\) and set \((Q_0, \Lambda_0) \overset{\text{def.}}{=} \mathcal{R}(Q, \Lambda)\). Let \(R : Q \to Q_0\) be the map which makes \((Q_0, \Lambda_0, R)\) into an \(L^0_\eta\)-retraction of \((Q, \Lambda)\) along \(r\) and let \(j : Q_0 \to Q\) be an \(L^0_\eta\)-equivariant section of \(R\) which makes \((Q_0, \Lambda_0, j)\) into an \(L^0_\eta\)-reduction of \((Q, \Lambda)\) (see Proposition 5.11). For every \(q \in Q\), let \(x_q\) be the unique element of \(L_\eta\) such that \(q = j(R(q))x_q\). For any \(u \in L_\eta\), direct computation using \(L^0_\eta\)-equivariance of \(j\) shows that \(x_{qu} = r(u)^{-1}x_q u\). Applying \(R\) to the relation \(q = j(R(q))x_q\) gives:

\[(11) \quad R(q) = (R \circ j \circ R)(q)r(x_q) = R(q)r(x_q),\]

where we used the identity \(R \circ j = \text{id}_{Q_0}\). Since the action of \(L_\eta\) on \(Q\) is free, relation (11) gives \(r(x_q) = 1\). Let \((Q', \Lambda') \overset{\text{def.}}{=} (\mathcal{I} \circ \mathcal{R})(Q, \Lambda) = \mathcal{I}(Q_0, \Lambda_0)\) and consider the bijective fiber bundle map:

\[
\mathcal{N}_{Q, \Lambda} : Q \to Q'
\]

given by \(\mathcal{N}_{Q, \Lambda}(q) \overset{\text{def.}}{=} [[q, 1]_r, x_q]_r\). For any \(u \in L_\eta\), we have:

\[
\mathcal{N}_{Q, \Lambda}(qu) = [[qu, 1]_r, r(u)^{-1}x_q u]_r = [[q, 1]_r, r(u), r(u)^{-1}x_q u]_r = [[q, 1]_r, x_q u]_r = \mathcal{N}_{Q, \Lambda}(q)u.
\]

Hence \(\mathcal{N}_{Q, \Lambda}\) is an isomorphism of principal \(L_\eta\)-bundles. Moreover, we have:

\[
\Lambda' \circ \mathcal{N}_{Q, \Lambda}(q) = \Lambda'( [[q, 1]_r, x_q]_r) = \Lambda_0([[q, 1]_r]_r) \circ \text{Ad}_q(x_q) = \Lambda_0([q, 1]_r) = \Lambda(q),
\]

where we used the fact that \(\text{Ad}_q(x_q) = \text{Ad}_q(r(x_q)) = \text{Ad}_q(1) = 1\). Hence \(\mathcal{N}_{Q, \Lambda}\) is an isomorphism of Lipschitz structures from \((Q, \Lambda)\) to \((Q', \Lambda') = (\mathcal{I} \circ \mathcal{R})(Q, \Lambda)\). Given any morphism of Lipschitz structures \(F : (Q^1, \Lambda^1) \to (Q^2, \Lambda^2)\), it is easy to see that the following diagram commutes:

\[
\begin{array}{ccc}
(I \circ \mathcal{R})(Q^1, \Lambda^1) & \overset{(I \circ \mathcal{R})(F)}{\longrightarrow} & (I \circ \mathcal{R})(Q^2, \Lambda^2) \\
\mathcal{N}_{Q^1, \Lambda^1} & & \mathcal{N}_{Q^2, \Lambda^2} \\
(Q^1, \Lambda^1) & \overset{F}{\longrightarrow} & (Q^2, \Lambda^2)
\end{array}
\]

showing that \(\mathcal{N}\) is a natural transformation.
2. Construction of $\mathcal{K}$. For any $(Q_0, \Lambda_0) \in \text{Ob}(L_0^0(M, g))$, there exists an isomorphism in $L^p_0(M, g)$:

$$\mathcal{K}_{Q_0, \Lambda_0} : (Q_0, \Lambda_0) \xrightarrow{\sim} (R \circ I)(Q_0, \Lambda_0)$$

given by $\mathcal{K}_{Q_0, \Lambda_0}(q_0) \overset{\text{def}}{=} [[q_0, 1], 1]_r = (R \circ I)(q_0)$. It is easy to see $\mathcal{K} : \text{id}_{L^p_0(M, g)} \to R \circ I$ is an invertible natural transformation.

\[ \square \]

5.3. Complex pinor bundles.

**Definition 5.13.** A complex pinor bundle over $(M, g)$ is a pair $(S, \gamma)$, where $S$ is a complex vector bundle over $M$ and $\gamma : \text{Cl}(M, g) \to \text{End}_\mathbb{C}(S)$ is a unital morphism of bundles of algebras such that $\gamma_p : \text{Cl}(T^*_p M, g^*_p) \to \text{End}_\mathbb{C}(S_p)$ is a weakly-faithful complex representation of $\text{Cl}(T^*_p M, g^*_p)$ for any $p \in M$. We say that $(S, \gamma)$ is of type $\eta$ if, for every $p \in M$, the complex Clifford representations $\eta$ and $\gamma_p$ are (unbasedly) isomorphic. We say that $(S, \gamma)$ is elementary if its type is an irreducible complex Clifford representation.

**Remark 5.14.** A complex pinor bundle is the same as a bundle of complex Clifford modules defined on $(M, g)$. We use the name “pinor” (rather than “spinor”) in order to distinguish such bundles from bundles of modules defined over the even sub-bundle of the Clifford bundle. Notice that the type $\eta$ of a complex pinor bundle $(S, \gamma)$ is well-defined up to (unbased) isomorphism of complex Clifford representations.

**Definition 5.15.** A based morphism of complex pinor bundles $F : (S, \gamma) \to (S', \gamma')$ is a based morphism $F : S \to S'$ of complex vector bundles such that:

$$L_F \circ \gamma = R_F \circ \gamma'$$

i.e. such that the fiber map $F_p : S_p \to S'_p$ at any point $p \in M$ is a based morphism of Clifford representations from $\gamma_p : \text{Cl}(T^*_p M, g^*_p) \to \text{End}_\mathbb{C}(S_p)$ to $\gamma'_p : \text{Cl}(T^*_p M, g^*_p) \to \text{End}_\mathbb{C}(S'_p)$.

Since $M$ is connected by assumption, all quadratic spaces $(T^*_p M, g^*_p)$ are mutually isometric and isometric to some model quadratic space $(V, h)$. Similarly, all fibers of $S$ are isomorphic as $\mathbb{C}$-vector spaces and hence isomorphic with some model vector space $\Sigma$. Using a common trivializing cover of $TM$ and $S$, this implies that the complex Clifford representations $\gamma_p : \text{Cl}(T^*_p M, g^*_p) \to \text{End}_\mathbb{C}(S_p)$ $(p \in M)$ are mutually isomorphic in the category $\text{ClRep}_\eta$ and hence isomorphic with some model weakly-faithful representation $\eta : \text{Cl}(V, h) \to \text{End}_\mathbb{C}(\Sigma)$, which defines the type of $(S, \gamma)$. The isomorphism class of $\eta$ in the category $\text{ClRep}_\eta$ is invariant under isomorphism of complex pinor bundles.

**Definition 5.16.** Let $\text{ClB}_\eta(M, g)$ be the category whose objects are complex pinor bundles of type $\eta$ and whose arrows are based morphisms of complex pinor bundles. We denote by $\text{ClB}_\eta(M, g)^\times$ the corresponding groupoid.

5.4. Relation between complex pinor bundles and complex Lipschitz structures. The following proposition is the analogue of [3, Proposition 6.1] for complex pinor bundles.

**Proposition 5.17.** Let $\eta \in \text{Ob}(\text{ClRep}_\eta)$. Consider the functors $Q_\eta : \text{ClB}_\eta(M, g)^\times \to L_\eta(M, g)^\times$ and $S_\eta : L_\eta(M, g)^\times \to \text{ClB}_\eta(M, g)^\times$ defined as follows:

A. Let $(S, \gamma)$ denote a complex pinor bundle of type $\eta$ on $(M, g)$. Let $Q := Q_\eta(S, \gamma)$ denote the principal bundle with structure group $L := L_\eta = \text{Aut}_{\text{ClRep}_\eta}(\eta)$, total space:

$$Q \overset{\text{def}}{=} \sqcup_{p \in M} \text{Hom}_{\text{ClRep}_\eta}(\eta, \gamma_p),$$

projection given by $\pi(q) = p$ for $q \in Q_p = \text{Hom}_{\text{ClRep}_\eta}(\eta, \gamma_p)$ and right $L$-action given by $q \cdot g \overset{\text{def}}{=} g \circ q$ for all $g \in L$. We topologize $Q$ in the obvious way. Let $\Lambda \overset{\text{def}}{=} \Lambda_\eta(S, \gamma) : Q_\eta(S, \gamma) \to \text{P}_\Omega(M, g)$ be the map defined through:

$$\Lambda_p(q) \overset{\text{def}}{=} q_0 \in \text{Hom}_{\text{Quad}^\times}(V, h, (T^*_p M, g^*_p)) = \text{P}_\Omega(M, g)_p$$

(12)
Then, \((Q, \Lambda)\) is a Lipschitz structure on \((M, g)\) relative to \(\eta\), which we call the Lipschitz structure induced by \((S, \gamma)\). A based isomorphism of complex pinor bundles \(F : (S, \gamma) \to (S', \gamma')\) of type \(\eta\) induces an isomorphism \(Q_\eta(F) : (Q_\eta(S, \gamma), \Lambda_\eta(S, \gamma)) \to (Q_\eta(S', \gamma'), \Lambda_\eta(S', \gamma'))\) of complex Lipschitz structures relative to \(\eta\), which is defined as follows (recall that \(F_p \in \text{Hom}_{\text{Cl}^0} \gamma_p, \gamma_p'\) and \((F_p)_0 = \text{id}_{T^*_p M}\):

\[
Q_\eta(F)(q) \overset{\text{def}}{=} (\text{id}_{T^*_p M}, F_p) \circ q, \quad \forall q \in Q_\eta(S, \gamma)_p = \text{Hom}_{\text{Cl}^0} \gamma_p, \gamma_p, \text{Hom}_{\text{Cl}^0} (\gamma_p, \gamma_p')。
\]

B. Let \((Q, \Lambda)\) denote a complex Lipschitz structure of type \(\eta\) on \((M, g)\). Then, the vector bundle \(S \overset{\text{def}}{=} S_\eta(Q, \Lambda) \overset{\text{def}}{=} Q \times_{\rho_\eta} \Sigma\) associated to \(Q\) through the complex tautological representation \(\rho_\eta : L \to \text{Aut}_C(\Sigma)\) of \(L\) becomes a complex pinor bundle of type \(\eta\) when equipped with the Clifford structure morphism \(\gamma \overset{\text{def}}{=} \gamma(Q, \Lambda) : \text{Cl}(M, g) \to \text{End}_C(S)\) defined as follows:

\[
\gamma_\eta(y)([q, s]) \overset{\text{def}}{=} [q, \eta(\text{Cl}(\Lambda_p(q)^{-1})(y))(s)], \quad \forall y \in \text{Cl}(T^*_p M, g^*_p),
\]

for all \(q \in Q_p\) and \(s \in \Sigma\). We call the pair \((S_\eta(Q, \Lambda), \gamma_\eta(Q, \Lambda))\) a Lipschitz structure on \((M, g)\). An isomorphism of complex Lipschitz structures \(F : (Q, \Lambda) \to (Q', \Lambda')\) relative to \(\eta\) induces a based isomorphism of complex pinor bundles \(S_\eta(F) = (S_\eta(Q, \Lambda), \gamma_\eta(Q, \Lambda)) \to (S_\eta(Q', \Lambda'), \gamma_\eta(Q', \Lambda'))\) defined as follows:

\[
S_\eta(F)_p([q, s]) = [F_p(q), s], \quad \forall q \in Q_p, \forall s \in \Sigma.
\]

**Proof.** The fact that \((12)\) is \(\text{Ad}_\eta\)-equivariant follows from the relation \((q \circ \varphi)_0 = q_0 \varphi_0 = q_0 \circ \text{Ad}_\eta(\varphi)\) for all \(\varphi \in L\), which implies that the following equation holds:

\[
\Lambda_p(q\varphi) = \Lambda_p(q) \circ \varphi_0 = \Lambda_p(q) \circ \text{Ad}_\eta(\varphi).
\]

This in turn implies that \((Q, \Lambda)\) is an elementary complex Lipschitz structure of type \(\eta\).

It remains to show that \((13)\) is well-defined. In order to do this, notice that \(\text{Ad}(\varphi \circ \eta) = \eta \circ \text{Cl}(\varphi_0)\) for any \(\varphi \in L\), which implies (using \([\text{Cl}(\Lambda_p(q))]^{-1} = \text{Cl}(\Lambda_p(q)^{-1})\) and \(\text{Ad}(\varphi^{-1}) = \text{Ad}(\varphi)^{-1}\)):

\[
\text{Ad}(\varphi^{-1}) \circ \eta \circ \text{Cl}(\Lambda_p(q)^{-1}) = \eta \circ \text{Cl}(\varphi_0^{-1} \circ \Lambda_p(q)^{-1}) = \eta \circ \text{Cl}(\Lambda_p(q) \circ \varphi_0)^{-1}\).
\]

Using relation \((15)\), this gives:

\[
\text{Ad}(\varphi^{-1}) \circ \eta \circ \text{Cl}(\Lambda_p(q)^{-1}) = \eta \circ \text{Cl}(\Lambda_p(q\varphi^{-1})^{-1}), \quad \forall \varphi \in L.
\]

Thus:

\[
\eta(\text{Cl}(\Lambda_p(q\varphi^{-1})^{-1})(x))(\varphi s) = \eta(\text{Cl}(\Lambda_p(q)^{-1}))(x)(\varphi s) = \eta(\text{Cl}(\Lambda_p(q)^{-1}))(x)(s) = \eta(\text{Cl}(\Lambda_p(q^{-1}))(x))(s),
\]

where in the last equality we used \((16)\). This shows that \((13)\) is well-defined. The fact that \(\gamma_\eta\) defined in \((13)\) is a Clifford representation is clear, as is the fact that \(P_\eta\) and \(S_\eta\) define functors. \(\square\)

**Remark 5.18.** The same construction used above for \(S_\eta\) allows us to define a functor \(S^0_\eta : L^0_\eta(M, g) \to \text{ClB}_\eta(M, g)\) which associates a complex pinor bundle to any reduced complex Lipschitz structure.

The following theorem establishes an equivalence between complex pinor bundles and complex complex Lipschitz structures defined over \((M, g)\).

**Theorem 5.19.** Let \(\eta \in \text{Ob}(\text{ClRep}_w)\). The functors \(Q_\eta\) and \(S_\eta\) are mutually quasi-inverse equivalences between the groupoids \(\text{ClB}_\eta(M, g)^\times\) and \(L_\eta(M, g)^\times\).

**Proof.** The proof is analogous to that of [3, Theorem 6.2], so we leave it to the reader. \(\square\)

**Corollary 5.20.** Let \(\eta \in \text{Ob}(\text{ClRep}_w)\) and \(L^0_\eta\) be a reduction of \(L_\eta\). Then there exists an equivalence of groupoids between \(\text{ClB}_\eta(M, g)^\times\) and \(L^0_\eta(M, g)^\times\).

**Proof.** Follows immediately from Theorems 5.12 and 5.19. \(\square\)
Proposition 5.21. Let \((Q, \Lambda)\) be a complex Lipschitz structure of type \(\eta\) and let \((Q_0, \Lambda_0, I)\) be an \(L^n_k\)-reduction of \((Q, \Lambda)\). Then the complex pinor bundles \(S\) and \(S_0\) associated to \((Q, \Lambda)\) and \((Q_0, \Lambda_0)\) are naturally isomorphic.

Proof. Let \(S = Q \times_{\rho} \Sigma\) and \(S_0 = Q_0 \times_{\rho_0} \Sigma\) denote the vector bundles associated to \(Q\) and \(Q_0\) through the tautological representations \(\rho\) and \(\rho_0 = \rho|_{L^n_k}\) of \(L_\eta\) and \(L^n_0\). Let \(\gamma\) and \(\gamma_0\) denote the structure morphisms making \((S, \gamma)\) and \((S_0, \gamma_0)\) into complex pinor bundles. A natural isomorphism \(F: S_0 \to S\) is given by:

\[
F([q_0, s]_{\rho_0}) \overset{\text{def}}{=} [I(q_0), s]_{\rho} \quad \forall q_0 \in Q_0 \quad \forall s \in \Sigma.
\]

It is clear that \(F\) is well-defined. To show that it is an isomorphism of vector bundles, let \(p\) be any point in \(M\). Since the fiber \(Q_p\) of \(Q\) at \(p\) is an \(L_\eta\)-torsor, it follows that for every \(q \in Q_p\) there exists \(q_0 \in Q^0_p\) such that \(q = I(q_0)u\) for some \(u \in L_\eta\). Writing \(u = u_0v\) with \(u_0 \overset{\text{def}}{=} r(u) \in L^0_\eta\) and \(v \overset{\text{def}}{=} u_0^{-1}u \in L_\eta\), we have \(q = I(q_0)u_0v = I(q)v\), where \(q_0 \overset{\text{def}}{=} q_0'u_0 \in Q^0_p\). Thus:

\[
[q, s]_{\rho} = [I(q_0)v, s]_{\rho} = [I(q_0), \rho(v)s]_{\rho} = F_p([q_0, \rho(v)s]_{\rho_0}).
\]

This shows that the the linear map \(F_p: S^0_{\rho} \to S_p\) is surjective and hence bijective (since \(\dim C_0 S_\rho = \text{rk}_{C_0} S = \dim_{C_0} S_\rho\)). Thus \(F\) is an isomorphism of vector bundles. For any \(x \in \text{Cl}(T^*_p M, g^p_\rho)\), any \(q_0 \in Q^0_p\) and any \(s \in \Sigma\), we have:

\[
(F_p \circ \gamma_{\rho_0,p}(x))([q_0, s]_{\rho_0}) = [I(q_0), \eta(Cl(A_{\rho}(q_0)^{-1})(x))(s)]_{\rho} = [I(q_0), \eta(Cl(\Lambda(I(q_0)^{-1})(x))(s))]_{\rho}
\]

\[
= \gamma_p(x)(([I(q_0), s]_{\rho}) = (\gamma_0(x) \circ F)([q, s]_{\rho}),
\]

where we used the relation \(\Lambda \circ I = A_0\). This shows that \((S, \gamma)\) and \((S_0, \gamma_0)\) are isomorphic as complex pinor bundles. \(\square\)

5.5. Topological obstructions for elementary complex Lipschitz structures. The topological obstructions to existence of complex Lipschitz structures associated to faithful complex Clifford representations were determined in [1]. Since for even \(d\) such representations are \(\mathbb{C}\)-irreducible, the results of op. cit. together with those of [5] give:

Theorem 5.22. [1, 5] Suppose that \(d\) is even. Then \((M, g)\) admits an elementary complex Lipschitz structure if and only if it admits a \(\text{Pin}^c(V, h)\) structure, i.e. if and only if there exists a principal \(U(1)\)-bundle \(E\) over \(M\) such that:

\[
(17) \quad w^-_2(M) + w^+_2(M) + w^-_1(M)w^+_1(M) = w_2(E).
\]

When \(d\) is odd, the results of [1] concern non-elementary complex Lipschitz structures. The following Corollary of Theorem 5.19 shows that on an odd-dimensional pseudo-Riemannian manifold any bundle of irreducible complex Clifford modules over \(\text{Cl}(M, g)\) can be understood as a complex spinor bundle associated to a \(\text{Spin}^c(V, h)\) structure on \((M, g)\) through the tautological irreducible representation of \(\text{Spin}^c(V, h)\).

Corollary 5.23. Suppose that \(d\) is odd. Then \((M, g)\) admits an elementary complex pinor bundle \((S, \gamma)\) if and only if it admits a \(\text{Spin}^c(V, h)\)-structure \((Q, \Lambda)\) such that \(S\) is a vector bundle associated to \(Q\) through the tautological representation of \(\text{Spin}^c(V, h)\). Furthermore, the groupoid of elementary complex pinor bundles is isomorphic to the groupoid of \(\text{Spin}^c(V, h)\) structures. When \((M, g)\) admits a \(\text{Spin}^c(V, h)\) structure, then isomorphism classes of elementary complex pinor bundles form a torsor over \(H^2(M, \mathbb{Z})\).

Proof. Since an irreducible complex representation is weakly faithful, the first statement in Theorem 5.19 implies that \((M, g)\) admits an elementary complex pinor bundle \((S, \gamma)\) of type \(\eta\) if and only if it admits a Lipschitz structure \((Q, \Lambda)\) of irreducible type \(\eta\). Proposition 4.5 implies that the associated Lipschitz group \(L_\eta\) is isomorphic to \(\mathbb{R}_{>0} \text{Spin}(V, h)\). By Proposition 2.4 we deduce that \(L_\eta\) is homotopy-equivalent to \(L^0_\eta = \text{Spin}^c(V, h)\) via the normalization morphism \(r\) of equation (1), a homotopy inverse of which is provided by the inclusion \(\iota : \text{Spin}^c(V, h) \hookrightarrow L_{\eta}\). Hence Theorem 5.19 applies and we conclude that the groupoid of elementary complex pinor bundles is isomorphic to the
groupoid of Spin\(^c\)(\(V, h\)) structures on \((M, g)\). In particular, every elementary complex pinor bundle is associated to a Spin\(^c\)(\(V, h\)) structure on \((M, g)\) by means of the tautological representation. The remaining statement follows since Spin\(^c\)(\(V, h\)) structures on \((M, g)\) form a torsor over \(H^2(M, \mathbb{Z})\).

**Corollary 5.24.** A pseudo-Riemannian manifold \((M, g)\) admits an elementary complex pinor bundle if and only if each of the following conditions are satisfied:

1. \(w_1(M) = 0\)
2. There exists a principal \(U(1)\)-bundle \(E\) over \(M\) such that:

\[
\tilde{w}^1(M) + w^2_+(M) = w_2(E).
\]

In particular, \(M\) must be orientable.

**Proof.** Follows immediately from Corollary 5.23 and the results of [5].

**5.6. Realification of complex pinor bundles.** The realification functor \(R: \text{Rep}^c_\mathbb{R}(\text{Cl}(V, h)) \rightarrow \text{Rep}^c_\mathbb{R}(\text{Cl}(V, h))\) of Subsection 4.3 induces a pinor bundle realification functor \(\mathcal{R}: \text{ClB}_{\eta\mathbb{R}}(M, g) \rightarrow \text{ClB}_{\eta\mathbb{R}}(M, g)\), where \(\text{ClB}_{\eta\mathbb{R}}(M, g)\) denotes the category of real pinor bundles of type \(\eta\mathbb{R}\) and based morphisms of such (see [3]). When \(p - q \equiv s 3, 4, 6, 7\), this restricts to a functor which maps elementary complex pinor bundles to elementary real pinor bundles. Since this restricted functor is determined on fibers, the results of Subsection 4.3 imply that it is faithful but not full. The restricted functor need not be essentially surjective since a real elementary pinor bundle need not admit a globally-defined complex structure which is a section of its Schur sub-bundle. However, we have:

**Proposition 5.25.** Assume that \(p - q \equiv s 3, 7\) and that \(M\) is orientable. Then \(\mathcal{R}\) restricts to a strictly surjective functor from the groupoid of elementary complex pinor bundles defined on \((M, g)\) to the groupoid of elementary real pinor bundles defined on \((M, g)\).

**Proof.** Let \(\nu \in \Omega(M)\) be the volume form of \((M, g)\) determined by some fixed orientation of \(M\). Let \((\Sigma, \rho)\) be an elementary real pinor bundle on \((M, g)\). Since \(p - q \equiv s 3, 7\), the endomorphism \(J \overset{\text{def.}}{=} \gamma(\nu) \in \Gamma(M, \text{End}(S))\) is a globally-defined complex structure on \(S\) which is a section of the Schur bundle of \(\Sigma\) (see [6]). Thus \(J_p\) lies in the commutant of the image of the real Clifford representation \(\rho_p: \text{Cl}(T^*pM, g^*_p) \rightarrow \text{End}_G(\Sigma_p)\) for any \(p \in M\). This implies that \((\Sigma, \rho)\) is the realification of the elementary complex pinor bundle obtained by endowing \(\Sigma\) with the complex structure \(J\).

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