On the Toda Lattice Equation with Self-Consistent Sources

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Abstract

The Toda lattice hierarchy with self-consistent sources and their Lax representation are derived. We construct a forward Darboux transformation (FDT) with arbitrary functions of time and a generalized forward Darboux transformation (GFDT) for Toda lattice with self-consistent sources (TLSCS), which can serve as a non-auto-Bäcklund transformation between TLSCS with different degrees of sources. With the help of such DT, we can construct many type of solutions to TLSCS, such as rational solution, solitons, positons, negetons, and soliton-positons, soliton-negetons, positon-negetons etc., and study properties and interactions of these solutions.

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1 Introduction

Recently, soliton equations with self-consistent sources (SESCSs) in several 1+1 and 2+1 dimensional continuous cases, which have very important applications in many fields, such as hydrodynamics, solid state physics, plasma physics, etc. have been widely studied [1–5]. For example the KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves [4]. Various methods have been used to construct their solutions, such as Inverse scattering methods [2, 3, 6, 7], Darboux transformation methods [8–11], Hirota bilinear methods [12–14] etc. Comparing with Darboux transformations (DT) for soliton equations, generalized binary Darboux transformations (GBDT) with arbitrary functions of time developed in [8–11] provides a non-auto-Bäcklund transformations between two soliton equations with different degrees of sources, and enable us to find various solutions, such as soliton, positon, negaton, soliton-positon, soliton-negaton, positon-negaton and etc.

However, in contrast with the continuous case, the integrable discrete systems with self-consistent sources and their physical applications have not been studied yet. We will present a way to construct integrable discrete systems with self-consistent sources basing on the constrained flows of the integrable discrete system [15] by regarding the latter as the stationary equation of the former. Then we derive a forward Darboux transformation (FDT) with an arbitrary function of time for the integrable discrete system with self-consistent sources by improving the method in [8–11]. We use a discrete system, the Toda lattice equation with self-consistent sources (TLSCS) to illustrate our method. The FDT with an arbitrary function of time is generated by using a linear combination of two independent eigenfunctions of the Lax pair with the combination coefficient explicitly depends on time. It serves as a non-auto-Bäcklund transformation between two TLSCS with different degrees of sources. We give a formula for multi-time repeating of such DT. We also construct the generalized FDT (GFDT) with arbitrary functions of time. The formula for GFDT is quite similar to the generalized Wronski determinant formulae [16] and the generalized Casorati determinant formulae [17, 18].

It is well known that Toda lattice equation possesses rich families of solutions including rational solution, soliton, positon, negatons and soliton-positon, soliton-negaton, positon-negaton. Soliton is a fast decaying pulse like solution without singularity [21]. It describes a wave propagating in a constant speed. Positon is an oscillating and slowly decaying solution with singularities. It leads to a trivial scattering matrix (called ”supertransparent”) when inserted as potentials in the finite-difference Schrödinger equation of the corresponding Lax pair [17]. The interaction between positon and other types of solutions is very interesting.
There is no phase shift for others during the course of collision. Negatons is another type of solutions with singularities oscillating but fast decaying and with non-trivial scattering matrix. Unlike the positon case, there is phase shift for other types of solutions during the interaction between negaton and others.

In the second part of our paper, we show that TLSCS also have such types of solutions. These solutions can be obtained easily through GFDT. Some of these solutions as well as their analytic properties are presented in our paper. Differences between solutions for TLSCS and solutions for Toda lattice equation are also studied. In particular, a new feature regarding negaton-positon (negaton-soliton) interaction is analyzed.

The paper will be organized as follows. We first review the Toda lattice hierarchy in section 2. In section 3, through the nth-constrained flow for Toda lattice hierarchy we give the Toda lattice hierarchy with self-consistent sources and its Lax representation. Based on the Darboux transformation for Toda lattice equation, we develop a method to construct the FDT and GFDT with arbitrary functions of time for TLSCS in section 4. In section 5, we construct some solutions for TLSCS by using GFDT with arbitrary functions of time. Some properties of these solutions are analyzed therein.

2 Toda Lattice Hierarchy

We first review the Toda lattice hierarchy. Assume $f = f(n,t)$ for $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. Define shift operator $E$ and difference operator $\Delta$ as follows

$$(Ef)(n,t) = f(n+1,t),$$

$$(\Delta f)(n,t) = (E - 1)f(n,t) = f(n+1,t) - f(n,t).$$

We often denote $E^k f(n,t)$ by $f^{(k)}$. The inverse of $E$ and $\Delta$ are defined as

$$(E^{-1}f)(n,t) = f(n-1,t),$$

$$(\Delta^{-1}f)(n,t) = \begin{cases} 
\sum_{i=0}^{n-1} f(i,t) & n \geq 1; \\
0 & n = 0; \\
-\sum_{i=n}^{-1} f(i,t) & n \leq -1.
\end{cases}$$

Consider the following discrete isospectral problem [17, 19]

$$L\psi = (v\psi)^{(1)} + p\psi + \psi^{(-1)} = \lambda \psi,$$  \hspace{1cm} (1)
where \( v = v(n, t) \), \( p = p(n, t) \), \( \psi = \psi(n, \lambda, t) \). This equation can be rewritten as the following \( 2 \times 2 \) matrix eigenvalue problem

\[
\Psi^{(-1)} = U(v, p, \lambda) \Psi,
\]

where

\[
U(v, p, \lambda) := \begin{pmatrix}
0 & 1 \\
-v^{(1)} & \lambda - p
\end{pmatrix}, \quad \Psi := \begin{pmatrix}
\psi^{(1)} \\
\psi
\end{pmatrix}.
\]

First consider the following *stationary zero-curvature equation* for generating function \( \Gamma \) [20],

\[
\Gamma^{(-1)} U - U \Gamma = 0,
\]

where

\[
\Gamma := \sum_{i=0}^{+\infty} \Gamma_i \lambda^{-i} = \sum_{i=0}^{+\infty} \begin{pmatrix}
a_i & b_i \\
c_i & -a_i
\end{pmatrix} \lambda^{-i},
\]

and \( a_i, b_i, c_i \) are functions of \( n \) and \( t \). The first few of these read

\[
\begin{align*}
a_0 &= \frac{1}{2}, & b_0 &= 0, & c_0 &= 0, \\
a_1 &= 0, & b_1 &= -1, & c_1 &= v^{(1)}, \\
a_2 &= v^{(1)}, & b_2 &= -p^{(1)}, & c_2 &= pv^{(1)}, \\
a_3 &= (p + p^{(1)}) v^{(1)}, & b_3 &= -(p^{(1)} + v^{(1)} + v^{(2)}), & c_3 &= v^{(1)} (p^2 + v + v^{(1)}).
\end{align*}
\]

In general

\[
\Delta a_{i+1} = p^{(1)} \Delta a_i - c_i - b_i^{(1)} v^{(2)}, \\
b_{i+1} = p^{(1)} b_i - \Delta a_i, \quad c_i = -v^{(1)} b_i^{(-1)}.
\]

Define the modification matrix

\[
\Delta_n := \text{diag}(b_{n+1} + \delta, \delta),
\]

where \( \delta \) is an arbitrary constant. Let

\[
V_n := (\lambda^n \Gamma)_+ + \Delta_n = \sum_{i=0}^{n} \Gamma_i \lambda^{n-i} + \Delta_n,
\]

and

\[
-\Psi_{\tau_n} = V_n \Psi, \quad (n \geq 1).
\]
Then the compatibility condition of (2) and (4) gives rise to the following zero-curvature representation of the Toda lattice hierarchy

\[ U_{tn} = UV_n - V_n^{(-1)}U, \quad (n \geq 1). \]  

(5)

Hamiltonian form for Toda lattice hierarchy is given by

\[
\begin{pmatrix} v \\ p \end{pmatrix}_{tn} = J \left( \frac{\delta H_n}{\delta v} \right) = J \left( \frac{a_{n+1}/v}{-b_{n+1}} \right)
\]

with the density \( H_n := -b_{n+2}/(n + 1) \) and Hamilton operator

\[
J := \begin{pmatrix} 0 & v(E^{-1} - 1) \\ (1 - E)v & 0 \end{pmatrix}.
\]

For \( n = 1 \) in (5) we have the well-known Toda lattice equation

\[
\begin{align*}
v_t &= vp^{(-1)} - vp, \\
p_t &= vv^{-1}.
\end{align*}
\]

(6)

Set \( v := \exp\left(x^{(-1)} - x\right) \), \( p := x_t \), equations (6) can be represented as

\[
x_{tt} = \exp\left(x^{(-1)} - x\right) - \exp\left(x - x^{(1)}\right).
\]

(7)

3 The Toda lattice hierarchy with self-consistent sources

First we briefly review the \( n \)th-constrained flow for Toda lattice hierarchy [15] which is defined as the following system

\[
\left( \frac{\delta H_n}{\delta v} \right) + \sum_{j=1}^{N} \left( \frac{\delta \lambda_j}{\delta v} \right) = 0
\]

(8a)

\[
L\phi_{j+} := (vp^{(1)}_{j+}) + p\phi_{j+} + \phi_{j+}^{(-1)} = \lambda_j\phi_{j+},
\]

(8b)

\[
L^*\phi_{j-} := vp^{(-1)}_{j-} + p\phi_{j-} + \phi_{j-}^{(1)} = \lambda_j\phi_{j-}, \quad j = 1, \ldots, N,
\]

(8c)

where

\[
(\delta\lambda_j/\delta v, \delta\lambda_j/\delta p)^T = \left( \phi_{j-}^{(-1)}, \phi_{j+} - \phi_{j-} \right)^T.
\]
Define
\[ \tilde{\Gamma}_n = \sum_{i=0}^{n} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i} + \sum_{j=1}^{N} \frac{1}{\lambda^{n}(\lambda - \lambda_j)} \begin{pmatrix} -v^{(1)} \phi_{j-} - \phi^{(1)}_{j+} \\ -v^{(1)} \phi_{j-} - \phi^{(1)}_{j+} \end{pmatrix} \lambda^{-j} + \sum_{j=1}^{N} \begin{pmatrix} 1 \\ \lambda - \lambda_j \end{pmatrix} \begin{pmatrix} v^{(1)} \phi_{j-} - \phi^{(1)}_{j+} \\ v^{(1)} \phi_{j-} - \phi^{(1)}_{j+} \end{pmatrix} \).

The Lax representation for (8a) is [15]
\[ \Psi^{(-1)} = U \Psi, \quad \mu \Psi = \tilde{\Gamma}_n \Psi \] (9)

Following the idea in the continuous case that by treating the constrained flows of soliton equation as the stationary equations of the soliton equation with self-consistent sources, we define Toda lattice hierarchy with \( N \) self-consistent sources as the following system
\[ \begin{pmatrix} v \\ p \end{pmatrix}_t = J \left[ \frac{\delta H_n}{\delta v} + \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta v} \phi_j - \phi^{(1)}_j \right] = J \left[ a^{(-1)}_{n+1} v + \sum_{j=1}^{N} \phi^{(-1)}_{j-} - \phi^{(1)}_{j+} \right], \] (10a)
\[ L_{\phi_{j+}} = \lambda_j \phi_{j+}, \] (10b)
\[ L_{\phi_{j-}} = \lambda_j \phi_{j-}, \quad j = 1, \ldots, N. \] (10c)

The Lax representation for (10a) can be obtained from the Lax representation (9)
\[ \Psi^{(-1)} = U \Psi, \quad -\Psi_{tn} = \left( \lambda^n \tilde{\Gamma}_n + \tilde{\Delta}_n \right) \Psi, \] (11)

where \( \tilde{\Delta}_n = \text{diag}(b_{n+1} - \sum_{j=1}^{N} \phi^{(1)}_{j-} - \phi^{(1)}_{j+} + \delta, \delta) \), with \( \delta \) an arbitrary constant.

For \( n = 1 \), we get the Toda lattice equation with \( N \) self-consistent sources (TLSCE)
\[ v_t = v \left( p^{(-1)} + \sum_{j=1}^{N} \phi^{(-1)}_{j-} \phi^{(1)}_{j+} \right) - v \left( p + \sum_{j=1}^{N} \phi_{j-} - \phi_{j+} \right), \] (12a)
\[ p_t = v \left( 1 + \sum_{j=1}^{N} \phi^{(-1)}_{j-} \phi^{(1)}_{j+} \right) - v^{(1)} \left( 1 + \sum_{j=1}^{N} \phi^{(-1)}_{j-} \phi^{(1)}_{j+} \right), \] (12b)
\[ L_{\phi_{j+}} = \lambda_j \phi_{j+}, \] (12c)
\[ L_{\phi_{j-}} = \lambda_j \phi_{j-}, \quad j = 1, \ldots, N. \] (12d)

The Lax representation for TLSCE is obtained from (11) by taking \( n = 1 \), and \( \delta = \lambda/2 \).

We prefer to rewrite it equivalently in the following scalar form
\[ L \psi = v^{(1)} \psi^{(1)} + p \psi + \psi^{(-1)} = \lambda \psi \] (13a)
\[ -\psi_t = v^{(1)} \psi^{(1)} + \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} v^{(1)} \phi_{j-} \left( \phi^{(1)}_{j+} \psi - \phi^{(1)}_{j+} \psi^{(1)} \right) \] (13b)
\[ L_{\phi_{j+}} = \lambda_j \phi_{j+}, \] (13c)
\[ L_{\phi_{j-}} = \lambda_j \phi_{j-}, \quad j = 1, \ldots, N. \] (13d)
With the substitutions
\[
v := \exp(x^{(-1)} - x), \quad p := x_t - \sum_{j=1}^{N} \phi_j \phi_{j-1},
\]
equations (12a) and (12b) can be written as
\[
x_t = \exp \left( x^{(-1)} - x \right) \left( 1 + \sum_{j=1}^{N} \phi_j \phi_{j-} \right) \]
\[
- \exp \left( x - x^{(1)} \right) \left( 1 + \sum_{j=1}^{N} \phi_j^{(1)} \phi_{j-} \right) + \sum_{j=1}^{N} (\phi_j \phi_{j-})_t.
\]

4 The Darboux Transformations for TLSCS

4.1 The forward DT with arbitrary functions of time

Based on the Darboux transformation for Toda equation in [17, 19], we can find the following theorem.

**Theorem 1.** Given the solution \( v, p, x, \phi_i \) \((i = 1, \ldots, N)\) for (12) and (15), and eigenfunction \( \psi \) for (13a) and (13b), let \( f \) and \( g \) be two independent eigenfunctions of (13a) and (13b) with \( \lambda = \mu \). Denote \( h = f + \alpha(t)g \) with the coefficient \( \alpha(t) \) being an arbitrary differentiable function of \( t \). Then the FDT is defined as follows

\[
\psi[1] = \psi - \frac{h}{h^{(1)}} \psi^{(1)},
\]
\[
v[1] = v^{(1)} \frac{h^{(1)} h^{(-1)}}{h^2},
\]
\[
p[1] = p - \frac{h}{h^{(1)}} + \frac{h^{(-1)}}{h},
\]
\[
x[1] = x^{(1)} + \frac{1}{2} \log \left( \frac{h}{h^{(1)}} \right)^2,
\]
\[
\phi_{i+}[1] = \phi_{i+} - \frac{h}{h^{(1)}} \phi_{i+}^{(1)},
\]
\[
\phi_{i-}[1] = \frac{\Delta h (h \phi_{i-}) + \kappa_i(t)}{h}, \quad i = 1, \ldots, N
\]
\[
\phi_{N+1,+}[1] = c(t) \left( f - \frac{h}{h^{(1)}} f^{(1)} \right),
\]
\[
\phi_{N+1,-}[1] = d(t) \frac{1}{h},
\]
where $c(t)$ and $d(t)$ are arbitrary fixed differentiable functions of $t$ satisfying $c(t) \cdot d(t) = -\dot{\alpha}/\alpha$, $\kappa_i(t) = \frac{1}{\mu - \lambda_i} [v^{(i)} h^{(i)} \phi_i - h \phi_i^{(i)}]|_{n=0}$. Namely $v[1], p[1], x[1], \phi_{1\pm}[1]$ ($i = 1, \ldots, N+1$), $\lambda_{N+1} = \mu$ and $\psi[1]$ give a new solution to (12) or (15) and (13) with $N+1$ self-consistent sources.

This theorem can be proved by straightforward calculation.

**Remark 1.** Theorem 1 serves as a non-auto-Bäcklund transformation between two TLSCSs with degrees of sources $N$ and $N+1$.

We give an example for obtaining solution via Theorem 1.

**Example 1 (Rational solutions).** Starting from trivial solution $v = 1, p = 0$ and vanishing sources, the Lax pair (13) becomes

$$L \psi = \psi^{(1)} + \psi^{(-1)} = \lambda \psi,$$

$$-\psi_t = \psi^{(1)}. \quad (17a)$$

$$-\psi_t = \psi^{(1)}. \quad (17b)$$

Let $\psi = \exp(kn - e^k t)$ be the solution of (17) w.r.t. $\lambda = 2 \cosh(k)$, $f = (an - at + b) \exp(-t)$, $g = \exp(-t)$ be independent solutions of (17) w.r.t. $\lambda = 2$, where $a \in \mathbb{R} - \{0\}$ and $b \in \mathbb{R}$ are arbitrary constants. Let $h = f + \alpha(t) g$ with the differentiable function $\alpha(t)$. Then we obtain the rational solution with a pair of non-vanishing sources

$$\psi_+[1] = \left[1 - e^k \frac{an + b - at + \alpha(t)}{an + a + b - at + \alpha(t)}\right] \exp(kn - e^k t),$$

$$v[1] = \frac{[an + a + b - at + \alpha(t)] [an - a + b - at + \alpha(t)]}{[an + b - at + \alpha(t)]^2},$$

$$p[1] = \frac{an - a + b - at + \alpha(t)}{an + b - at + \alpha(t)} - \frac{an + b - at + \alpha(t)}{an + a + b - at + \alpha(t)},$$

$$x[1] = \frac{1}{2} \log \left[\frac{an + b - at + \alpha(t)}{an + a + b - at + \alpha(t)}\right]^2,$$

$$\phi_{1+}[1] = -c(t) \frac{ae^{-t} \alpha(t)}{an + a + b - at + \alpha(t)},$$

$$\phi_{1-}[1] = d(t) \frac{e^t}{an + b - at + \alpha(t)},$$

where $c(t)$ and $d(t)$ satisfying $c(t) d(t) = -\dot{\alpha}/\alpha$. 


4.2 The Multi-time Repeated FDT

Theorem 2 (The multi-time repeated FDT). Given the solution \( v, p, x, \phi_i \) (\( i = 1, \ldots, N \)) for (12) and (15), and eigenfunction \( \psi \) for (13a) and (13b), let \( f_j \) and \( g_j \) be independent eigenfunctions of (13a) and (13b) w.r.t. distinct \( \mu_j \) (\( j = 1, \ldots, l \)). Let \( \alpha_j(t) \) (\( j = 1, \ldots, l \)) be arbitrary smooth functions of \( t \). Denote \( h_j = f_j + \alpha_j g_j \). Then the \( l \)-times repeated FDT is given as

\[
\psi[l] = \frac{\text{cas}(\psi, h_1, \ldots, h_l)}{\text{cas}(h_1, \ldots, h_l)^{(i)}}, \tag{18a}
\]

\[
v[l] = v^{(i)} \frac{\text{cas}(h_1, \ldots, h_l)^{(i)} \text{cas}(h_1, \ldots, h_l)^{(-1)}}{\text{cas}(h_1, \ldots, h_l)^2}, \tag{18b}
\]

\[
p[l] = p + \frac{\tilde{\text{cas}}(h_1, \ldots, h_l)^{(-1)}}{\text{cas}(h_1, \ldots, h_l)} - \frac{\tilde{\text{cas}}(h_1, \ldots, h_l)}{\text{cas}(h_1, \ldots, h_l)^{(i)}}, \tag{18c}
\]

\[
x[l] = x^{(i)} + \frac{1}{2} \log \left[ \frac{\text{cas}(h_1, \ldots, h_l)}{\text{cas}(h_1, \ldots, h_l)^{(i)}} \right]^2, \tag{18d}
\]

\[
\phi_{i+}[l] = \frac{\text{cas}(\phi_{i+}, h_1, \ldots, h_l)}{\text{cas}(h_1, \ldots, h_l)^{(i)}}, \tag{18e}
\]

\[
\phi_{i-}[l] = \frac{\text{cas}(\phi_{i-}, h_1, \ldots, h_l)}{\text{cas}(h_1, \ldots, h_l)}, \quad i = 1, \ldots, N \tag{18f}
\]

\[
\phi_{N+j,+}[l] = c_j(t) \frac{\text{cas}(f_j, h_1, \ldots, h_l)}{\text{cas}(h_1, \ldots, h_l)^{(i)}}, \tag{18g}
\]

\[
\phi_{N+j,-}[l] = d_j(t) \frac{\text{cas}(h_1, \ldots, \hat{h}_j, \ldots, h_l)}{\text{cas}(h_1, \ldots, h_l)}, \quad j = 1, \ldots, l \tag{18h}
\]

where

\[
\text{cas}(h_1, \ldots, h_l) := \det \left( \begin{array}{c}
\end{array} \right)
\]

is the Casorati determinant, and

\[
\tilde{\text{cas}}(h_1, \ldots, h_l) := \det \left( \begin{array}{c}
\end{array} \right)
\]

\[
\text{cas}(\phi_{i-}, h_1, \ldots, h_l) := \det \left[ \begin{array}{c}
S_i(h_1) \cdots S_i(h_l) \\
h_1^{(i)} \cdots h_l^{(i)} \\
\vdots & \vdots \\
h_1^{(i-1)} \cdots h_l^{(i-1)}
\end{array} \right]
\]
where
\[ S_i(h_j) := S(\phi_{i-}, h_j) = \Delta^{-1} E(\phi_{i-} h_j) + (\mu_j - \lambda_i)^{-1}[v^{(i)} h_j \phi_{i-} - h_j \phi_{i-}^{(i)}]|_{n=0}. \]

The \( \hat{h}_j \) means the removal of this term and \( c_j(t) \cdot d_j(t) = (-1)^j \hat{\alpha}_j/\alpha_j \). Functions \( \psi[l], v[l], p[l], x[l], \phi_{i\pm[l]} \) \((i = 1,\ldots,N+l)\) and \( \lambda_{N+j} = \mu_j \) \((j = 1,\ldots,l)\) satisfy \((13),(12)\) and \((15)\) with \( N \) replaced by \( N+l \).

Sketch of Proof. For the proof of formulae \((18a),(18b),(18c),(18e),(18g)\) see \[19\]. The formula \((18d)\) is proved by using \((14)\). Formulae \((18f)\) and \((18h)\) are proved by induction. The calculation is lengthy but rather straightforward. We omit it.

\[ \square \]

Remark 2. The multi-times repeated FDT \((18)\) provides a non-auto-Bäcklund transformation between two TLSCSs with degrees \( N \) and \( N+l \).

4.3 The generalized forward Darboux transformations

It is well known from \[17\] that the positon solution of the Toda lattice equations are obtained by computing the limit \( k_2 \rightarrow k_1 \) in the result of two-step DT, where \( k_{1,2} \) are parameters of eigenfunction with which DT is generated. In order to construct positon solutions for TLSCS, similar consideration can be made in our case of FDT with arbitrary functions of time. However, in our case, the arbitrary time functions \( \alpha_j(t) \) must be carefully chosen to balance the divergence of sources. Possible candidates for \( \alpha_j(t) \) are exponential functions. By using it, we get the following GFDT.

Theorem 3 (GFDT). Given the solution \( v, p, x, \phi_{i\pm}(i = 1,\ldots,N) \) for \((12)\) and \((15)\), and eigenfunction \( \psi \) for \((13a)\) and \((13b)\), let \( F_j, G_j \) \((j = 1,\ldots,l)\) be pairs of independent eigenfunctions of \((13a)\) and \((13b)\) corresponding to distinct \( \lambda_{N+j} \). Let \( f_r, g_r \) \((r = 1,\ldots,I)\) be pairs of independent eigenfunctions of \((13a)\) and \((13b)\) corresponding to \( \lambda_{N+t+r} = \lambda_{N+t+r}(\omega_r) \), where \( \lambda_{N+t+r}(\omega_r) \) is analytic function of parameter \( \omega_r \in \mathbb{C} \). Let \( \alpha_j(t) \) \((j = 1,\ldots,l)\), \( \beta_r(t) \) \((r = 1,\ldots,I)\) be arbitrary differentiable functions of \( t \). Let \( m_r \in \mathbb{N}, m_r \geq 2 \) \((r = 1,\ldots,I)\), \( m := (m_1,\ldots,m_I) \). Denote \( q_j = F_j + \alpha_j(t)G_j, h_r = f_r + g_r \) respectively. Then the following GFDT with \( l \) times of FDT with arbitrary time functions \( \alpha_j(t) \) and \( I \) times of generalized FDT of multiplicities \( m_r \) with arbitrary time functions \( \beta_r(t) \) is given by

\[ \psi[l,m] = \frac{\Delta_{l,m}(\psi)}{\Delta_{l,m}^{(1)}}, \quad (19a) \]
where $c_j(t) \cdot d_j(t) = (-1)^j \alpha_j / \alpha_j$, $|m| := \sum m_r$. Symbols are defined as follows:

\[ \Delta_{t,m}(\psi) := \text{cas}(\psi, q_1, q_2, \ldots, q_t); \]
\[ h_1, \partial_{q_1} h_1, \partial_{q_1}^2 h_1, \ldots, \partial_{q_1}^{m_1-1} h_1 + (-1)^{m_1-1} \beta_1(t) g_1; \ldots; \]
\[ h_I, \partial_{q_1} h_I, \partial_{q_1}^2 h_I, \ldots, \partial_{q_1}^{m_I-1} h_I + (-1)^{m_I-1} \beta_I(t) g_I, \]
\[ \Delta_{t,m} := \text{cas}(q_1, q_2, \ldots, q_t); \]
\[ h_1, \partial_{q_1} h_1, \partial_{q_1}^2 h_1, \ldots, \partial_{q_1}^{m_1-1} h_1 + (-1)^{m_1-1} \beta_1(t) g_1; \ldots; \]
\[ h_I, \partial_{q_1} h_I, \partial_{q_1}^2 h_I, \ldots, \partial_{q_1}^{m_I-1} h_I + (-1)^{m_I-1} \beta_I(t) g_I, \]
\[ \tilde{\Delta}_{t,m} := \tilde{\text{cas}}(q_1, q_2, \ldots, q_t); \]
\[ h_1, \partial_{q_1} h_1, \partial_{q_1}^2 h_1, \ldots, \partial_{q_1}^{m_1-1} h_1 + (-1)^{m_1-1} \beta_1(t) g_1; \ldots; \]
\[ h_I, \partial_{q_1} h_I, \partial_{q_1}^2 h_I, \ldots, \partial_{q_1}^{m_I-1} h_I + (-1)^{m_I-1} \beta_I(t) g_I, \]
\[ \Delta_{t,m}^j := \text{cas}(q_1, \ldots, q_j, \ldots, q_t); \]
\[ \Delta_{t,m}^{j+1} = \frac{\Delta_{t,m}^{j+1}}{\Delta_{t,m}^j}; \]
\[ \Delta_{t,m}^{(-1)} = \frac{\Delta_{t,m}^{(-1)}}{\Delta_{t,m}^j}; \]
\[ x[l, m] = x^{(l+|m|)} + \frac{1}{2} \log \left( \frac{\Delta_{t,m}^j}{\Delta_{t,m}^{(1)}} \right)^2; \]
\[ \phi_{i+}[l, m] = \frac{\Delta_{t,m}(\phi_{i+})}{\Delta_{t,m}^j}; \]
\[ \phi_{i-}[l, m] = \frac{\Delta_{t,m}(\phi_{i-})}{\Delta_{t,m}}; \]
\[ \phi_{N+j,+}[l, m] = c_j(t) \frac{\Delta_{t,m}(F_j)}{\Delta_{t,m}^j}; \]
\[ \phi_{N+j,-}[l, m] = d_j(t) \frac{\Delta_{t,m}^j}{\Delta_{t,m}}; \]
\[ \phi_{N+l+r,+}[l, m] = -\frac{\Delta_{t,m}(f_r)}{\Delta_{t,m}^j}; \]
\[ \phi_{N+l+r,-}[l, m] = \hat{\beta}_r \frac{\Delta_{t,m}^j}{\Delta_{t,m}^j}; \]

where $c_j(t) \cdot d_j(t) = (-1)^j \alpha_j / \alpha_j$, $|m| := \sum m_r$. Symbols are defined as follows:
We have the obvious relation \( \Omega_{s} \), where

\[
\begin{align*}
&h_1, \partial_\omega h_1, \partial^2_\omega h_1, \ldots, \partial^{m_1-1}_\omega h_1 + (-1)^{m_1-1} \beta_1(t)g_1; \ldots; \\
h_I, \partial_\omega h_I, \partial^2_\omega h_I, \ldots, \partial^{m_I-1}_\omega h_I + (-1)^{m_I-1} \beta_I(t)g_I \quad (1 \leq j \leq l),
\end{align*}
\]

\[\Delta_{t,m}^{l+r} := \text{cas}(q_1, q_2, \ldots, q_l) \]

\[
\begin{align*}
&h_1, \partial_\omega h_1, \partial^2_\omega h_1, \ldots, \partial^{m_1}_\omega h_1 + (-1)^{m_1-1} \beta_1(t)g_1; \ldots; \\
h_r, \partial_\omega h_r, \partial^2_\omega h_r, \ldots, \partial^{m_r-2}_\omega h_r; \ldots; \\
h_I, \partial_\omega h_I, \partial^2_\omega h_I, \ldots, \partial^{m_I-1}_\omega h_I + (-1)^{m_I-1} \beta_I(t)g_I \quad (1 \leq r \leq I),
\end{align*}
\]

where \( \text{cas} \) and \( \tilde{\text{cas}} \) are defined in Theorem 2.

\[\Delta_{t,m}(\phi_{i-}) := \text{det} \left( w_i(g_1), \ldots, w_i(q_l); \right) \]

\[
\begin{align*}
w_i(h_1), \partial_\omega w_i(h_1), \partial^2_\omega w_i(h_1), \ldots, \partial^{m_i}_\omega w_i(h_1) + (-1)^{m_i-1} \beta_i w_i(g_1); \ldots; \\
w_i(h_I), \partial_\omega w_i(h_I), \partial^2_\omega w_i(h_I), \ldots, \partial^{m_I}_\omega w_i(h_I) + (-1)^{m_I-1} \beta_I w_i(g_I),
\end{align*}
\]

where for any solution \( \psi \) of (13a) with eigenvalue \( \lambda \), \( w_i(\psi) \) is \( l + |m| \) dimensional column vector defined as

\[
w_i(\psi) = (S_i(\psi), \psi^{(1)}, \ldots, \psi^{(l+|m|-1)})^T,
\]

and \( S_i(\psi) \) is a scalar defined as

\[S_i(\psi) := S(\phi_{i-}, \psi) = \Delta^{-1} E(\phi_{i-}) + (\lambda - \lambda_i)^{-1} [v^{(i)} \psi^{(1)} \phi_{i-} - \phi_{i-} \psi^{(i)}]|_{n=0}.\]

Then \( \psi[l, m], v[l, m], p[l, m], x[l, m], \phi_{i\pm l}, m \) and \( \lambda_i \) \( (i = 1, \ldots, N + l + I) \) satisfy (13), (12) and (15) with \( N \) replaced by \( N + l + I \).

**Proof.** Without loss of generality, we only prove the special case \( l = 0, I = 1 \). The multiplicity \( m_1 \) is denoted by \( m \) \( (m \geq 2) \). And \( \lambda_1(\omega_1) \) is denoted by \( \lambda(\omega) \), which is analytic function of parameter \( \omega \). \( f_1, g_1, h_1 \) and \( \beta_1(t) \) are denoted by \( f, g, h \) and \( \beta(t) \) respectively.

Let \( \omega_s = \omega + \varepsilon e_s \), where \( e_s \) \( (s = 1, \ldots, m) \) are distinct complex constants, \( \varepsilon \) is a small parameter. Define \( \varrho_s(t) = \exp(\Omega_{s} b_s(t)) \) , where \( b_s(t) \) are arbitrary differentiable functions of \( t \) satisfying \( \sum_{s=1}^{m} b_s(t) = \beta(t) \), and

\[
\Omega_{s} = \frac{1}{(m-1)!} \prod_{1 \leq i < m; i \neq s} (\omega_i - \omega_s), \quad p_s = \frac{1}{(m-1)!} \prod_{1 \leq i < m; i \neq s} (e_i - e_s).
\]

We have the obvious relation \( \Omega_{s} = \varepsilon^{m-1} p_s \).
We have the following important observations. Let \( \mathbf{u}(\lambda), \mathbf{v}(\lambda) \) be two column vectors of dimension \( m \), \( \mathbf{u}(\lambda), \mathbf{v}(\lambda) \) be two column vectors of dimension \( m - 1 \), whose components are analytic functions of \( \lambda = \lambda(\omega) \). Denote \( \mathbf{u}_s = \mathbf{u}(\lambda(\omega_s)), \mathbf{v}_s = \mathbf{v}(\lambda(\omega_s)), \mathbf{u}_s = \mathbf{u}(\lambda(\omega_s)), \mathbf{v}_s = \mathbf{v}(\lambda(\omega_s)) \) then

\[
\det(\mathbf{u} + \varrho_1 \mathbf{v}_1, \ldots, \mathbf{u}_m + \varrho_m \mathbf{v}_m) = \prod_{1 \leq i < j \leq m} (e_j - e_i) \det(\mathbf{u} + \mathbf{v}, \partial_\omega(\mathbf{u} + \mathbf{v}), \\
\ldots, \partial_\omega^{m-1}(\mathbf{u} + \mathbf{v}) + (-1)^{m-1} \beta(t) \mathbf{v}) + o(\varepsilon^{\frac{1}{2}m(m-1)}),
\]

\[
\det(\mathbf{u} + \varrho_1 \mathbf{v}_1, \ldots, \mathbf{u}_r + \varrho_r \mathbf{v}_r, \ldots, \mathbf{u}_m + \varrho_m \mathbf{v}_m) = \prod_{1 \leq i < j \leq m} (e_j - e_i) \det(\mathbf{u} + \mathbf{v}, \partial_\omega(\mathbf{u} + \mathbf{v}), \ldots, \partial_\omega^{m-2}(\mathbf{u} + \mathbf{v}))
\]

\[
+ o(\varepsilon^{\frac{1}{2}(m-2)(m-1)}),
\]

These observations can be easily obtained by insert the Taylor expansion of \( \mathbf{u}_s, \mathbf{v}_s \) and \( \mathbf{u}_s, \mathbf{v}_s \) at \( \varepsilon = 0 \) into the determinant.

By applying multi-time repeated FDT (Theorem 2) with \( h(\lambda(\omega_s)) = f(\lambda(\omega_s)) + \varrho_s(t)g(\lambda(\omega_s)) \), \( c_s = -1, \partial_s = (-1)^{s-1} \partial_s/\varrho_s (s = 1, \ldots, m) \), we obtain \( \psi[m], \varphi[m], p[m], x[m], \phi_{\pm}[m] \) \( (i = 1, \ldots, N + m) \) and \( \lambda_{N+s} = \lambda(\omega_s) \) satisfying (13), (12) and (15) with \( N + m \) self-consistent sources. Since such solution contain an arbitrary parameter \( \varepsilon \), letting \( \varepsilon \to 0 \) will also give rise to solution to (13), (12) and (15) with \( N + m \) self-consistent sources. So by doing this with use of (21) and (22), we get the special case \( l = 0, m = m \) of formulae (19a), (19b), (19c), (19d), (19e) and (19f). We also obtain \( m \) self-consistent sources

\[
\phi_{N+s,+}[m] = -\frac{\text{cas}(f, h, \ldots, \partial_\omega^{m-1}h)}{\text{cas}(h, \ldots, \partial_\omega^{m-1}h + (-1)^{m-1} \beta(t)g)}, \quad s = 1, \ldots, m.
\]

\[
\phi_{N+s,-}[m] = \hat{b}_s \frac{\text{cas}(h, \ldots, \partial_\omega^{m-2}h)}{\text{cas}(h, \ldots, \partial_\omega^{m-1}h + (-1)^{m-1} \beta(t)g)}, \quad s = 1, \ldots, m.
\]

Since \( \phi_{N+s,+}[m] \)'s are equal, \( \phi_{N+s,-}[m] \) differ only in coefficients \( \hat{b}_s \) and they all corresponding to eigenvalue \( \lambda(\omega) \), it is reasonable to combine \( m \) self-consistent sources to one self-consistent source by denoting

\[
\phi_{N+1,+}[0,m] = \phi_{N+1,+}[m], \quad \phi_{N+1,-}[0,m] = \sum_{s=1}^{m} \phi_{N+s,-}[m].
\]
Thus we arrive at (19i) and (19j).

The general case can be proved by repeating above procedure.

5 Solutions of TLSCS

The GFDT technique enable us to construct various types of solutions to the TLSCS. Starting from trivial solution \( v = 1, p = 0 \) for (12) with \( N = 0 \), by choosing specific solutions of Lax pair (17) and specific \( l, I \) and \( m \) in Theorem 3, we can construct multi-solitons solutions, (multi-)positon solutions (or higher order), (multi-)negaton solutions (or higher order) and (multi-)soliton-positon solutions, (multi-)soliton-negaton solutions, (multi-)positon-negaton solutions etc..

5.1 Soliton solutions

Let \( F = \nu^n \exp(n\gamma - ne^{\gamma}t), G = \nu^n \exp(-n\gamma - ne^{-\gamma}t) \) be solutions of (17) with respect to \( \lambda = 2\nu \cosh(\gamma) \), where \( \gamma \in \mathbb{R}\{0\} \). Let \( \alpha = \exp(-2a(t)) \), where \( a(t) \) is an arbitrary differentiable functions of \( t \), \( \nu = \pm 1 \). Define

\[
q = F + \alpha G = 2\nu^n \exp(-\nu \cosh(\gamma)t - a) \cosh(U), \quad U := n\gamma - \nu \sinh(\gamma)t + a.
\]

Then using theorem 3 with \( l = 1, I = 0 \), we have the 1-soliton solution for (12) or (15) with \( N = 1 \).

\[
\begin{align*}
\psi_{\text{sol}} &= \exp(ikn - e^{ik}t) \left[ 1 - \nu e^{ik} \frac{\cosh(U)}{\cosh(U + \gamma)} \right], \\
v_{\text{sol}} &= \frac{\cosh(U - \gamma) \cosh(U + \gamma)}{\cosh^2(U)}, \\
p_{\text{sol}} &= \nu \frac{\cosh(U - \gamma)}{\cosh(U)} - \nu \frac{\cosh(U)}{\cosh(U + \gamma)}, \\
x_{\text{sol}} &= \log \left[ \frac{\cosh(U)}{\cosh(U + \gamma)} \right], \\
\phi^+_{\text{sol}} &= -\nu^n \sinh(\gamma)c(t) \frac{\exp(-\nu \cosh(\gamma)t - a)}{\cosh(U + \gamma)}, \\
\phi^-_{\text{sol}} &= \nu^n d(t) \frac{\exp(\nu \cosh(\gamma)t + a)}{2 \cosh(U)},
\end{align*}
\]

where \( c(t)d(t) = 2\dot{a} \).

The 1-soliton solution of TLCS is similar to the 1-soliton of ordinary Toda lattice equation [21]. The main difference between them is that the travelling speed of 1-soliton
solution for TLSCS can vary with $t$, for the travelling speed is $\frac{\nu \sinh \gamma - \dot{a}}{\gamma}$. The multi-soliton solutions for TLSCS (12) can be constructed similarly. We omit it.

### 5.2 Positon solutions and negaton solutions

#### 5.2.1 One-positon solution

Let $f = c_1 \exp(-t \cos \omega) \cos(n \omega - t \sin \omega)$, $g = c_2 \exp(-t \cos \omega) \sin(n \omega - t \sin \omega)$ be solutions of (17) w.r.t $\lambda = 2 \cos \omega$. Set constants $c_1 = \cos \theta$, $c_2 = \sin \theta$, where $\theta \neq \frac{n \pi}{2} (n \in \mathbb{Z})$. Define $h = f + g = \exp(-t \cos \omega) \cos(Y)$, $Y = n \omega - t \sin \omega - \theta,$

and

$$Z = \partial_\omega Y = n - t \cos \omega, \quad \eta = Z + \frac{1}{2} \beta(t) \sin(2\theta).$$

where $\beta(t)$ is an arbitrary differentiable function. Then using theorem 3 with the specification $l = 0$, $I = 1$, $m_1 = 2$, one gets one-positon solution for TLSCS (12) with $N = 1$.

$$v_{\text{pos}} = \frac{[\eta + \frac{1}{2}] \sin \omega + \frac{1}{2} \sin(2Y + 3\omega)}{[\eta + \frac{1}{2}] \sin \omega + \frac{1}{2} \sin(2Y + \omega)} \frac{[(\eta + \frac{1}{2}) \sin \omega + \frac{1}{2} \sin(2Y + \omega)]^2}{[(\eta + \frac{1}{2}) \sin \omega + \frac{1}{2} \sin(2Y + \omega)]^2}$$

(23a)

$$p_{\text{pos}} = \frac{\eta \sin(2\omega) + \sin(2Y)}{(\eta + \frac{1}{2}) \sin \omega + \frac{1}{2} \sin(2Y + \omega)} - \frac{(\eta + 1) \sin(2\omega) + \sin(2Y + 2\omega)}{(\eta + \frac{3}{2}) \sin \omega + \frac{1}{2} \sin(2Y + 3\omega)}$$

(23b)

$$x_{\text{pos}} = \frac{1}{2} \log \left[\frac{(\eta + \frac{1}{2}) \sin \omega + \frac{1}{2} \sin(2Y + \omega)}{2}\right]^2$$

(23c)

$$\phi_{+\text{pos}} = -\sin(2\theta) \frac{\exp(-t \cos \omega) \sin^2 \omega \cos(Y + \omega)}{(\eta + \frac{3}{2}) \sin \omega + \frac{1}{2} \sin(2Y + 3\omega)}$$

(23d)

$$\phi_{-\text{pos}} = -\beta \frac{\exp(t \cos \omega) \cos(Y + \omega)}{(\eta + \frac{3}{2}) \sin \omega + \frac{1}{2} \sin(2Y + \omega)}$$

(23e)

The scattering properties can be analyzed, resembling with [17]. It is noticed that the scattering matrix is an identity matrix, which is the basic feature for positon solutions.

Then we discuss the analytic properties of one-positon solution. An investigation on positon profile $x_{\text{pos}}$ shows that the one-positon profile decays to zero slowly at $\pm \infty$, oscillates and possesses singularities during their propagation. The singularities of one positon profile are determined by zeros of

$$\Delta_{\text{pos}} := (\eta + \frac{1}{2}) \sin \omega + \frac{1}{2} \sin(2Y + \omega).$$

The speed of positon profile are defined by the speed of singularities [16]. Assuming $n_0(t)$ to be one of the places where singularity occurs, differentiating by $t$ on the two side
of the equation $\Delta^{\text{pos}}(n, t) = 0$, one finds that the singularity propagation is governed by a nonlinear ODE

$$\dot{n}_0 = \frac{\sin \omega}{\omega} \left[ 1 + \frac{\omega \cos \omega - \frac{1}{2} \omega \beta \sin(2\theta) - \sin \omega}{\sin \omega + \omega \cos(2n_0 \omega - 2t \sin \omega - 2\theta + \omega)} \right].$$

The singularities of $\phi^\text{pos}_+ \text{ and } \phi^\text{pos}_-$ are determined by zeros of $(\Delta^{\text{pos}})^{(1)}$ and $\Delta^{\text{pos}}$, respectively.

### 5.2.2 Two-positon solution

Let $f_i := \cos \theta_i \exp(-t \cos \omega_i) \cos(n \omega_i - t \sin \omega_i)$, $g_i := \sin \theta_i \exp(-t \cos \omega_i) \sin(n \omega_i - t \sin \omega_i)$, $i = 1, 2$ be solutions of (17) w.r.t $\lambda_i = 2 \cos \omega_i$, $\theta_i \neq \frac{n}{2} \pi$ ($n \in \mathbb{Z}$). Define

$$h_i = f_i + g_i = \exp(-t \cos \omega_i) \cos(Y_i), \quad Y_i = n \omega_i - t \sin \omega_i - \theta_i,$$

$$Z_i = \partial_{\omega_i} Y_i = n - t \cos \omega_i, \quad \eta_i = Z_i + \frac{1}{2} \beta_i(t) \sin(2\theta_i).$$

Then using theorem 3 with $l = 0$, $I = 2$, $m = (2, 2)$, one obtains the 2-positon solution for TLCS (12),

$$x^{2p} = \frac{1}{2} \log \left[ \frac{\text{cas} \left( \cos Y_1, \eta_1 \sin Y_1, \cos Y_2, \eta_2 \sin Y_2 \right)}{\text{cas} \left( \cos Y_1, \eta_1 \sin Y_1, \cos Y_2, \eta_2 \sin Y_2 \right)^{(1)}} \right]^2,$$

$$\phi^{2p}_{1+} = -\frac{1}{2} \sin(2\theta_1) \exp(-t \cos \omega_1) \frac{\text{cas} \left( \sin Y_1, \cos Y_1, \eta_1 \sin Y_1, \cos Y_2, \eta_2 \sin Y_2 \right)}{\text{cas} \left( \cos Y_1, \eta_1 \sin Y_1, \cos Y_2, \eta_2 \sin Y_2 \right)^{(1)}},$$

$$\phi^{2p}_{1-} = -\beta_1 \exp(t \cos \omega_1) \frac{\text{cas} \left( \cos Y_1, \cos Y_2, \eta_2 \sin Y_2 \right)^{(1)}}{\text{cas} \left( \cos Y_1, \eta_1 \sin Y_1, \cos Y_2, \eta_2 \sin Y_2 \right)},$$

and $\phi^{2p}_{2\pm}$ have the similar formulae. $x^{2p}$ describes a wave profile oscillates, decreases to zero slowly as $|n| \to \infty$. $\phi^{2p}_{1\pm}$ and $\phi^{2p}_{2\pm}$ behave like $O(n^{-1})$ as $|n| \to \infty$.

The Positon-positon interaction can be analyzed as follows. Fixing $\eta_1$, assume $|\eta_2| \to \infty$ when $t \to \pm \infty$, then

$$x^{2p} \sim \frac{1}{2} \log \left[ A(\eta_1 + \frac{3}{2}) + B \sin(2Y_1 + 3\omega_1) \right]^2, \text{as } t \to \pm \infty,$$

where

$$A := 2 \sin \omega_1 \sin \omega_2 + \sin(3\omega_1) \sin \omega_2 - 2 \sin(2\omega_1) \sin(2\omega_2) + \sin \omega_1 \sin(3\omega_2),$$

$$B := \sin \omega_2 \cos(2\omega_1) + \frac{3}{2} \sin \omega_2 - \sin(2\omega_2) \cos \omega_1 + \frac{1}{2} \sin(3\omega_2).$$
Thus, we observe one-positon in terms of $\eta_1$ and $Y_1$ at $t = \pm \infty$. And there is no phase shift during the interaction. Analogously, fixing $\eta_2$ we observe one-positon at $t = \pm \infty$ in terms of $\eta_2$ and $Y_2$. Similarly, there is no phase shift in the course of interaction. Positons propagations transparently as if others were absent. This transparency of interaction is a remarkable feature of positon solutions. The N-positon solution for TLSCS (12) is obtained by taking $f_i := \cos \theta_i \exp(-t \cos \omega_i) \cos(n\omega_i - t \sin \omega_i)$, $g_i := \sin \theta_i \exp(-t \cos \omega_i) \sin(n\omega_i - t \sin \omega_i)$, $i = 1, \ldots, N$ be solutions of (17) w.r.t $\lambda_i = 2 \cos \omega_i$, $\theta_i \neq 2k\pi$ $(n \in \mathbb{Z})$, $h_i = f_i + g_i$ and use theorem 3 with $l = 0$, $I = N$, $m = (2, 2, \ldots, 2) \in \mathbb{N}^N$.

We can construct N-negaton solutions to TLSCS (12) by considering eigenfunctions $f_i := e^n \exp(n\omega_i - \epsilon_i e^{\omega_i} t + \theta_i)$, $g_i := e^n \exp(-n\omega_i - \epsilon_i e^{-\omega_i} t - \theta_i)$ of (17) with $\lambda_i = 2e_i \cosh \omega_i$, distinct $\omega_i \in \mathbb{R}$, $\theta_i \in \mathbb{R}$, $\epsilon_i = \pm 1$, $h_i = f_i + g_i$ and using theorem 3 with $l = 0$, $I = N$, $m = (2, 2, \ldots, 2) \in \mathbb{N}^N$.

The GFDT is quite general to construct not only N-soliton (positon,negaton) solutions, but also solutions of combined type.

### 5.3 Positon-negaton solution and interaction

Let

\[
\begin{align*}
    f_1 &:= \cos \theta_1 \exp(-t \cos \omega_1) \cos(n\omega_1 - t \sin \omega_1), \\
    g_1 &:= \sin \theta_1 \exp(-t \cos \omega_1) \sin(n\omega_1 - t \sin \omega_1), \\
    f_2 &:= e^n \exp(n\omega_2 - \epsilon_2 e^{\omega_2} t + \theta_2), \\
    g_2 &:= e^n \exp(-n\omega_2 - \epsilon_2 e^{-\omega_2} t - \theta_2),
\end{align*}
\]

where $\theta_1$, $\theta_2$, $\omega_1$ and $\omega_2$ are real numbers, $\theta_i \neq k\pi/2$ for any integer $k$. Define

\[
    h_1 = f_1 + g_1 = \exp(-t \cos \omega_1) \cos Y_1, \quad h_2 = f_2 + g_2 = 2e^n \exp(-t \epsilon_2 \cosh \omega_2) \cosh Y_2,
\]

where

\[
    Y_1 := n\omega_1 - t \sin \omega_1 - \theta_1, \quad Y_2 := n\omega_2 - t \epsilon_2 \sinh \omega_2 + \theta_2.
\]

Denote

\[
    \eta_1 := \partial_\omega Y_1 + \frac{1}{2} \beta_1(t) \sin 2\theta_1, \quad \eta_2 := \partial_\omega Y_2 + \frac{1}{2} \beta_2(t).
\]

Then according to theorem 3 with the specification $l = 0$, $I = 2$, $m = (2, 2)$, we obtain the positon-negaton solution for TLSCS (12).
To analyze the interaction, we always assume $\beta_1(t)$ and $\beta_2(t)$ tending to $\mp\infty$ as $n \to \pm\infty$. And all parameters, including $\omega_1$, $\omega_2$ and $\epsilon_2$ are positive. Fixing $\eta_1$, if for particular $\beta_2(t)$, $\eta_2$ increases faster than $\sinh(2Y_2)$ (i.e. $|\eta_2/\sinh(2Y_2)| \to \infty$) as $t \to \infty$, which indicates that negaton travels at a very high speed dominated by $\beta_2(t)$ then

$$x^{pm}_\omega \sim \frac{1}{2} \log \left[ \frac{\omega^3(2Y_1 + 3\omega_1) + \omega^2(\eta_1 + 1 - \Delta_8)}{\omega(2Y_1 + 3\omega_1) + \omega^2(\eta_1 + 1 - \Delta_8)} \right], \quad \text{as} \ t \to \pm\infty,$$

where

$$\Delta_8 := \sin \omega_1 \frac{2e^{3\omega_2 - \omega_1} - 2e^{-\omega_2 - \omega_1}}{2e^{3\omega_2 - \omega_1} - 2e^{-\omega_2 - \omega_1}}.$$

Thus we see the one-positon profile $(23c)$ with neither phase shift nor displacement at the two end. That is to say the negaton is transparent for positon, which is a phenomenon never observed in the ordinary Toda lattice case.

If $\eta_2$ increases slower than $\sinh(2Y_2)$ (i.e. $|\eta_2/\sinh(2Y_2)| \to \infty$) as $|t|$ increases, which implies a slowly traveling negaton in comparison with the previous case, then

$$x^{pm}_\omega \sim \frac{1}{2} \log \left[ \frac{\sin \omega_1(\eta_1 + \Delta_9) + \frac{1}{2} \sin(2Y_1 + \omega_1 - \Delta_{11})}{\sin \omega_1(\eta_1 + 1 + \Delta_9) + \frac{1}{2} \sin(2Y_1 + 3\omega_1 - \Delta_{11})} \right]^2 - 2\omega_2, \quad \text{as} \ t \to -\infty,$$

$$x^{pm}_\omega \sim \frac{1}{2} \log \left[ \frac{\sin \omega_1(\eta_1 + \Delta_{10}) + \frac{1}{2} \sin(2Y_1 + 5\omega_1 + \Delta_{11})}{\sin \omega_1(\eta_1 + 1 + \Delta_{10}) + \frac{1}{2} \sin(2Y_1 + 7\omega_1 + \Delta_{11})} \right]^2 + 2\omega_2, \quad \text{as} \ t \to +\infty,$$

where

$$\Delta_9 := \frac{1}{2} e^{-2\omega_2} \frac{(2e^{3\omega_2 - \omega_1})(2e^{i\omega_1} - 4)}{(2e^{3\omega_2 - \omega_1})(2e^{i\omega_1} - 4)},$$

$$\Delta_{10} := \frac{1}{2} e^{2\omega_2} \frac{(2e^{3\omega_2 - \omega_1})(2e^{i\omega_1} - 4)}{(2e^{3\omega_2 - \omega_1})(2e^{i\omega_1} - 4)},$$

and

$$\Delta_{11} := i \log \left( \frac{2e^{3\omega_2 - \omega_1}}{2e^{i\omega_1} - 4} \right)^2$$

are all real constants. Thus positon travels with phase shift determined by $\Delta_9$, $\Delta_{10}$ and $\Delta_{11}$, with displacement determined by $4\omega_2$ in the course of collision. This is general phenomenon caused by existence of negaton.

If we fix a coordinate frame which travelling with negaton profile, then

$$x^{pm}_\omega \sim \frac{1}{2} \log \left[ \frac{\frac{1}{2} \sinh(2Y_2 + 3\omega_2) + (\eta_2 + 3) \sinh \omega_2}{\frac{1}{2} \sinh(2Y_2 + 5\omega_2) + (\eta_2 + 4) \sinh \omega_2} \right]^2, \quad \text{as} \ t \to \pm\infty.$$

Thus, the negaton travels insensitive about the existence of positon.
Soliton-positon and soliton-negaton solutions can be obtained in the same way by using
theorem 3 with \( l = 1, \ I = 1, \ m = 2, \ q := F + \alpha G, \ h := f_1 + g_1 \) and \( q := F + \alpha G, \ h := f_2 + g_2 \)
respectively. \((f_i, g_i, i = 1, 2)\) are defined in the beginning of this section and \( F, G, \alpha \)
are defined in the section 5.1.

6 Conclusions

Based on the constrained flows of Toda lattice hierarchy, we constructed Toda lattice hierarchy
with self-consistent sources and their Lax representation.

We developed a method to construct FDT with arbitrary functions of time and the
GFDT with arbitrary functions of time, which, in contrast with the well-known Darboux
transformation for Toda lattice, provide non-auto-\(\text{B}\ddot{a}\text{cklund \ transformation} \) between two
TLSCSs with different degrees and enable us to obtain various explicit solutions to TLSCS.
Resembling the ordinary Toda lattice case, this system possesses solutions of rich families,
including solitons, positons, negatons and the solutions of combined types. A number of
solutions are listed by our method. The investigation on these solutions shows a quite
similar nature with solutions of ordinary Toda lattice. However, the new feature concerning
interactions between negaton and positon (or soliton) which is different from the ordinary
Toda lattice case is also stated. This difference is caused by the wide range of variation of
the speed of negatons in TLSCS cases. We note that variation of speed is a common feature
for continuous and discrete systems with self-consistent sources, see [6, 7] etc.

It is convinced that our approach for constructing systems with self-consistent sources
and generalized forward Darboux transformation technique are available for other discrete
systems. Some investigation will be present in the forthcoming paper.

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