SMOOTH LOCAL RIGIDITY FOR HYPERBOLIC TORAL AUTOMORPHISMS

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Abstract. We study the regularity of a conjugacy $H$ between a hyperbolic toral automorphism $A$ and its smooth perturbation $f$. We show that if $H$ is weakly differentiable then it is $C^{1+}\text{Hölder}$ and, if $A$ is also weakly irreducible, then $H$ is $C^\infty$. As a part of the proof, we establish results of independent interest on Hölder continuity of a measurable conjugacy between linear cocycles over a hyperbolic system. As a corollary, we improve regularity of the conjugacy to $C^\infty$ in prior local rigidity results.

1. Introduction and local rigidity results

Hyperbolic automorphisms of tori are the prime examples of hyperbolic dynamical systems. The action of a matrix $A \in SL(N, \mathbb{Z})$ on $\mathbb{R}^N$ induces an automorphism of the torus $T^N = \mathbb{R}^N/\mathbb{Z}^N$, which we denote by the same letter. An automorphism $A$ is called \textit{hyperbolic}, or \textit{Anosov}, if the matrix has no eigenvalues on the unit circle. One of the key properties of hyperbolic systems is \textit{structural stability}: any diffeomorphism $f$ of $T^N$ sufficiently $C^1$ close to such an $A$ is also hyperbolic and is topologically conjugate to $A$, that is, there exists a homeomorphism $H$ of $T^N$, called a \textit{conjugacy}, such that

\begin{equation}
A \circ H = H \circ f.
\end{equation}

Any two conjugacies differ by an affine automorphisms of $T^N$ commuting with $A$, and hence have the same regularity. Although $H$ is always bi-Hölder continuous, it is usually not even $C^1$, as there are various obstructions to smoothness. This is in sharp contrast with rigidity for actions of larger groups, where often any perturbation, or even any smooth action, is $C^\infty$ conjugate to an algebraic model.

In the classical case of a single system, the problem of establishing smoothness of the conjugacy from some weaker assumptions has been extensively studied. It is often described as rigidity, in the sense that weak equivalence implies strong equivalence.

In dimension two, definitive results were obtained in [dlL87, dlLM88, dlL92]. For hyperbolic automorphisms of $T^2$, and more generally for Anosov diffeomorphisms of $T^2$, $C^\infty$ smoothness of the conjugacy was obtained from absolute continuity of $H$ and from equality of Lyapunov exponents of $A$ and $f$ at the periodic points.

Key words: Hyperbolic toral automorphism, conjugacy, local rigidity, linear cocycle, iterative method.

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The case of higher dimensional systems is much more complicated. In particular, the problem of the exact level of regularity of $H$ is subtle: for any $k \in \mathbb{N}$ and any $N \geq 4$ there exists a reducible hyperbolic automorphism $A$ of $\mathbb{T}^N$ and its analytic perturbation $f$ such that the conjugacy is $C^k$ but is not $C^{k+1}$ \cite{dL92}. We recall that $A$ is reducible if it has a nontrivial rational invariant subspace or, equivalently, if its characteristic polynomial is reducible over $\mathbb{Q}$.

The two-dimensional results were extended in two directions. First, $C^\infty$ conjugacy was obtained for systems that are conformal on full stable and unstable subspaces under various periodic data assumptions which ensured that the perturbed system is also conformal \cite{dL02, KS03, dL04, KS09}. Second, for some classes of irreducible $A$, equality of Lyapunov exponents or similarity of the periodic data were shown to imply $C^{1+\text{Hölder}}$ smoothness of $H$ \cite{GG08, G08, GKS11, SaY19, GKS20, dW21}. Irreducibility of $A$ is necessary for these results \cite{dL92, dL02, G08}. Low smoothness of $H$ is due to the method of the proof, which establishes regularity of $H$ along natural one or two-dimensional $f$-invariant foliations of $\mathbb{T}^N$, whose leaves are typically only $C^{1+\text{Hölder}}$ smooth. Nevertheless, Gogolev conjectured in \cite{G08} that the regularity of $H$ should be close to that of $f$, and in particular if $f$ is $C^\infty$ then so is $H$. Until now, the only progress on higher regularity of $H$, outside of the conformal setting, was obtained for automorphisms of $\mathbb{T}^3$ with real spectrum in \cite{G17}. We refer to \cite{KSW22} for a more detailed account of questions and developments in the area of local rigidity.

In this paper we establish general results on bootstrap of regularity of the conjugacy. We show that for any hyperbolic automorphism $A$, if $H$ is weakly differentiable in a certain sense then it is $C^{1+\text{Hölder}}$ and, if in addition $A$ is weakly irreducible, then $H$ is $C^\infty$. We introduce and discuss the weak irreducibility property, which is weaker than irreducibility and holds for some $A$ with Jordan blocks. Our methods are different from those in the previous local rigidity results. In particular, we prove smoothness of $H$ without establishing it first along invariant foliations. For the $C^\infty$ smoothness of the conjugacy, we use a KAM type iterative scheme. This approach is novel in the setting of hyperbolic systems, and it is substantially different from previous applications of KAM, as we discuss below. As a corollary, we improve the regularity of $H$ from $C^{1+\text{Hölder}}$ to $C^\infty$ in the previous local rigidity results for the irreducible case.

Now we formulate our main results. We denote by $W^{1,q}(\mathbb{T}^N)$ the Sobolev space of $L^q$ functions with $L^q$ weak partial derivatives of first order. We note that Lipschitz functions are in $W^{1,\infty}(\mathbb{T}^N)$.

The first result holds for an arbitrary hyperbolic automorphism without any irreducibility assumption. We recall that while $H$ satisfying (1.1) is not unique, there is a unique conjugacy $C^0$ close to the identity. This is $H$ in the homotopy class of the identity with $H(p) = 0$, where $p$ is the fixed point of $f$ closest to 0.

**Theorem 1.1.** Let $A$ be a hyperbolic automorphism of $\mathbb{T}^N$ and let $f$ be a $C^{1+\text{Hölder}}$ diffeomorphism of $\mathbb{T}^N$ which is $C^1$ close to $A$. Suppose that for some conjugacy $H$ between $f$ and $A$, either $H$ or $H^{-1}$ is in $W^{1,q}(\mathbb{T}^N)$ with $q > N$. Then $H$ is a $C^{1+\text{Hölder}}$ diffeomorphism.
Moreover, for the conjugacy $H$ either theorem can be replaced with a slightly weaker one that we actually need for the proof: the differential of (1.2)

$$\|H - I\|_{C^{1+\beta}} \leq K\|f - A\|_{C^{1+\beta}}.$$  

**Remark 1.2.** The assumption of being in $W^{1,q}$ with $q > N$ in this and in the next theorem can be replaced with a slightly weaker one that we actually need for the proof: either $H^{-1}$ is in $W^{1,1}$ and its Jacoby matrix of partial derivatives is invertible and gives the differential of $H^{-1}$ for Lebesgue almost every point of $\mathbb{T}^N$, or the same holds for $H$ and $f$ preserves an absolutely continuous probability measure.

In the next theorem we obtain $C^\infty$ smoothness of the conjugacy assuming that $A$ is weakly irreducible, which we define as follows. Let $R^N = \oplus_{\rho_i} E^i$ be the splitting where $E^i$ is the sum of generalized eigenspaces of $A$ corresponding to the eigenvalues of modulus $\rho_i$, and let $\hat{E}^i = \oplus_{\rho_i \neq \rho_j} E^j$. We say that $A$ is *weakly irreducible* if each $\hat{E}^i$ contains no nonzero elements of $\mathbb{Z}^N$. This condition is weaker than irreducibility, see Section 3.3 for details.

**Theorem 1.3.** Let $A$ be a weakly irreducible hyperbolic automorphism of $\mathbb{T}^N$. Then there is a constant $\ell = \ell(A) \in \mathbb{N}$ so that for any $C^\infty$ diffeomorphism $f$ which is $C^\ell$ close to $A$ the following holds. If for some conjugacy $H$ between $f$ and $A$ either $H$ or $H^{-1}$ is in the Sobolev space $W^{1,q}(\mathbb{T}^N)$ with $q > N$, then any conjugacy between $f$ and $A$ is a $C^\infty$ diffeomorphism.

The constant $\ell = \ell(A)$ is chosen sufficiently large to satisfy the inequalities $\delta > 0$. We use these results to prove the following.

**Corollary 1.4.** Let $A : \mathbb{T}^N \to \mathbb{T}^N$ be an irreducible Anosov automorphism such that no three of its eigenvalues have the same modulus. Let $f$ be a $C^\infty$ diffeomorphism which is $C^\ell$-close to $A$ such that the derivative $D_pf^n$ is conjugate to $A^n$ whenever $p = f^n(p)$. Then $f$ is $C^\infty$ conjugate to $A$.

**Corollary 1.5.** Let $A : \mathbb{T}^N \to \mathbb{T}^N$ be an irreducible Anosov automorphism such that no three of its eigenvalues have the same modulus and there are no pairs of eigenvalues of the form $\lambda, -\lambda$ or $i\lambda, -i\lambda$, where $\lambda \in \mathbb{R}$. Let $f$ be a volume-preserving $C^\infty$ diffeomorphism of $\mathbb{T}^N$ sufficiently $C^\ell$-close to $A$. If the Lyapunov exponents of $f$ with respect to the volume are the same as the Lyapunov exponents of $A$, then $f$ is $C^\infty$ conjugate to $A$.

Now we briefly discuss our approaches. To prove Theorem 1.1 we first establish results of independent interest on Hölder continuity of a measurable conjugacy between linear cocycles over a hyperbolic system. These results are formulated and discussed in Section 2. In the proof of the theorem we apply them to the conjugacy $DH$ between the derivative...
cocycles $Df$ and $A$. The methods used yield only Hölder continuity of the conjugacy between the derivative cocycles and hence only $C^{1+\beta}$ regularity of $H$.

We note, however, that existence of some Hölder conjugacy between the derivative cocycles $Df$ and $A$ does not imply in general that $H$ is $C^1$. Indeed, if all eigenvalues of $A$ are simple with distinct moduli, then conjugacy of $D_p f^n$ and $A^n$, whenever $p = f^n(p)$, always gives Hölder conjugacy of the cocycles, but $H$ may not be $C^1$ if $A$ is reducible.

Our approach to proving Theorem 1.3 is different from prior work on this problem. We abandon the geometric arguments which use invariant foliations. Instead, we introduce a new method which combines exponential mixing of the unperturbed system with a KAM type iterative scheme. KAM methods have been extensively used to study local rigidity, primarily for elliptic systems, such as Diophantine translations of a torus. These systems are very different from the hyperbolic ones that we are considering. Closest to our setting, KAM techniques were used in [DKt10] to prove $C^\infty$ local rigidity for some $\mathbb{Z}^2$ actions by partially hyperbolic toral automorphisms. However, our approach is markedly different from the existing work in both main ingredients of the KAM method: a detailed study of the linearized conjugacy equation, and setup and convergence of the iterative process. In particular, the linearized conjugacy equation in our case is a twisted cohomological equation with the twist given by a hyperbolic matrix. In contrast to the elliptic case, this creates obstructions to solving the equation by sufficiently smooth functions. In [DKt10] the structure of $\mathbb{Z}^2$ action was used in an essential way to show vanishing of the obstructions. In our setting, we instead use the existence of a $C^{1+\beta}$ conjugacy $H$ given by Theorem 1.1. However, two difficulties arise. First, higher regularity is needed for analyzing the obstructions (see Lemma 7.1(iii)). Second, $C^{1+\beta}$ functions have slow decay of Fourier coefficients. In contrast, super-polynomial decay for $C^\infty$ functions, which yields super-exponential mixing, was crucial in obtaining suitable estimates for convergence in the KAM iteration in [DKt10]. One of the key innovations in our approach is using directional derivatives to “balance” the twist. By analyzing differentiated equations in Hölder category we are able to construct a $C^\infty$ approximate solution of the twisted cohomological equation and obtain suitable estimates. This is done in Section 7, see remarks after Theorem 7.4 for details. Relating Fourier coefficients of a function and its directional derivatives is the only place where we use weak irreducibility of $A$. The last issue is that the estimate we obtain for the approximate solution is not tame, in contrast to traditional KAM estimates [FKr09]. This creates problems in establishing convergence of the iterative procedure, which we overcome in Section 8.

The paper is structured as follows. In Section 2 we formulate our results on continuity of a measurable conjugacy between linear cocycles over a hyperbolic system, Theorems 2.1 and 2.2. These theorems are proved in Sections 4 and 5, respectively. In Section 3 we summarize basic notations and facts used throughout the paper. In Section 6 we prove Theorem 1.1. In Section 7 we obtain a result on solving a twisted cohomological equation over $A$, and in Section 8 we complete the proof of Theorem 1.3.
2. Results on Continuity of Conjugacy between Linear Cocycles

In this section we consider linear cocycles over a transitive Anosov diffeomorphism $f$ of a compact connected manifold $\mathcal{M}$. We recall that $f$ is Anosov if there exist a splitting of the tangent bundle $T\mathcal{M}$ into a direct sum of two $Df$-invariant continuous subbundles $\tilde{E}^s$ and $\tilde{E}^u$, a Riemannian metric on $\mathcal{M}$, and continuous functions $\nu$ and $\nuhat$ such that
\begin{equation}
\|Df_x(\nuhat s)\| < \nu(x) < 1 < \nuhat(x) < \|Df_x(\nu s)\|
\end{equation}
for any $x \in \mathcal{M}$ and any unit vectors $\nuhat s \in \tilde{E}^s(x)$ and $\nu s \in \tilde{E}^u(x)$. The diffeomorphism is transitive if there is a point $x \in \mathcal{M}$ with dense orbit. All known examples satisfy this property.

Let $A$ be a map from $\mathcal{M}$ to $GL(N, \mathbb{R})$. The $GL(N, \mathbb{R})$-valued cocycle over $f$ generated by $A$ is the map $A : X \times \mathbb{Z} \to GL(N, \mathbb{R})$ defined by $A(x, 0) = \text{Id}$ and for $n \in \mathbb{N}$,
\[
A(x, n) = A^n_x = A(f^{n-1}x) \circ \cdots \circ A(x) \quad \text{and} \quad A(x, -n) = A^{-n}_x = (A_{f^{-n}x})^{-1}.
\]

We say that a $GL(d, \mathbb{R})$-valued cocycle $A$ is $\beta$-Hölder continuous if there exists a constant $c$ such that
\[
d(A_x, A_y) \leq c \text{dist}(x, y)^\beta \quad \text{for all } x, y \in \mathcal{M},
\]
where the metric $d$ on $GL(N, \mathbb{R})$ is given by
\[
d(A, B) = \|A - B\| + \|A^{-1} - B^{-1}\|, \quad \text{where } \| \cdot \| \text{ is the operator norm.}
\]

More generally, we consider linear cocycles defined as follows. Let $P : E \to \mathcal{M}$ be a finite dimensional $\beta$-Hölder vector bundle over $\mathcal{M}$. A continuous linear cocycle over $f$ is a homeomorphism $A : E \to E$ such that
\[
P \circ A = f \circ P \quad \text{and} \quad A_x : E_x \to E_{fx} \quad \text{is a linear isomorphism for each } x \in \mathcal{M}.
\]

The linear cocycle $A$ is called $\beta$-Hölder if $A_x$ depends $\beta$-Hölder on $x$, with proper identification of fibers at nearby points. A detailed description of this setting is given in Section 2.2 of [KS13].

The differential of $f$ and its restrictions to invariant sub-bundles of $T\mathcal{M}$, such as $\tilde{E}^s$ and $\tilde{E}^u$, are prime examples of linear cocycles.

We say that a $\beta$-Hölder cocycle $A$ over an Anosov diffeomorphism $f$ is fiber bunched if there exist numbers $\theta < 1$ and $c$ such that for all $x \in \mathcal{M}$ and $n \in \mathbb{N}$,
\begin{equation}
\|A^n_x\| \cdot \|(A^n_x)^{-1}\| \cdot (\nu^n_x)^\beta < c \theta^n \quad \text{and} \quad \|A^{-n}_x\| \cdot \|(A^{-n}_x)^{-1}\| \cdot (\nu^{-n}_x)^\beta < c \theta^n,
\end{equation}
where $\nu^n_x = \nu(f^{n-1}x) \cdots \nu(x)$ and $\nu^{-n}_x = (\nuhat(f^{-n}x))^{-1} \cdots (\nuhat(f^{-1}x))^{-1}$.

Let $\mu$ be an ergodic $f$-invariant measure on $\mathcal{M}$. We denote by $\lambda_+(A, \mu)$ and $\lambda_-(A, \mu)$ the largest and smallest Lyapunov exponents of $A$ with respect to $\mu$ given by the Oseledets Multiplicative Ergodic Theorem. For $\mu$ almost all $x \in \mathcal{M}$, they equal the limits
\begin{equation}
\lambda_+(A, \mu) = \lim_{n \to \infty} n^{-1} \ln \|A^n_x\| \quad \text{and} \quad \lambda_-(A, \mu) = \lim_{n \to \infty} n^{-1} \ln \|(A^n_x)^{-1}\|^{-1}.
\end{equation}
We say that a cocycle $A$ has one exponent if for every $f$-periodic point $p$ the invariant measure $\mu_p$ on its orbit satisfies $\lambda_+(A, \mu_p) = \lambda_-(A, \mu_p)$. By Theorem 1.4 in [K11], this condition is equivalent to

$$\lambda_+(A, \mu) = \lambda_-(A, \mu)$$

for every ergodic $f$-invariant measure.

We note that if $A$ has one exponent, then it is fiber bunched [S15, Corollary 4.2].

For $GL(N, \mathbb{R})$ cocycles $A$ and $B$ over $f$, a (measurable or continuous) function $C : \mathcal{M} \to GL(N, \mathbb{R})$ such that

$$A_x = C(fx) B_x C(x)^{-1}$$

for all $x \in \mathcal{M}$ is called a (measurable or continuous) conjugacy or transfer map between $A$ and $B$. For linear cocycles $A, B : E \to E$ a conjugacy is defined similarly with $C(x) \in GL(E_x)$.

The question whether a measurable conjugacy between two cocycles is continuous has been studied in [PaP97, Pa99, Sch99, S13, S15]. An example in [PW01] shows that a measurable conjugacy between two fiber bunched $GL(2, \mathbb{R})$-valued cocycles is not necessarily continuous, moreover, the generators of the cocycles in this example can be chosen arbitrarily close to the identity. Continuity of a measurable conjugacy was proven for cocycles with values in a compact group [PaP97, Pa99] and, somewhat more generally for cocycles with bounded distortion [Sch99], for $GL(2, \mathbb{R})$-valued cocycles with one exponent [S13], and for $GL(N, \mathbb{R})$-valued cocycles such that one is fiber bunched and the other one is uniformly quasiconformal [S15]. The result in [S13] relied on two-dimensionality, and the uniform quasiconformality assumption in [S15] is much stronger than having one exponent. The next theorem establishes continuity of a measurable conjugacy between a fiber bunched cocycle and a cocycle with one exponent.

**Theorem 2.1.** Let $f$ be a transitive $C^{1+\text{Hölder}}$ Anosov diffeomorphism of a compact manifold $\mathcal{M}$, and let $A$ and $B$ be $\beta$-Hölder linear cocycles over $f$. Suppose that $A$ has one exponent and $B$ is fiber bunched.

Let $\mu$ be an ergodic $f$-invariant measure on $\mathcal{M}$ with full support and local product structure. Then any $\mu$-measurable conjugacy between $A$ and $B$ is $\beta$-Hölder continuous, i.e., coincides with a $\beta$-Hölder continuous conjugacy on a set of full measure.

As we mentioned above, continuity of a measurable conjugacy does not hold in general if $A$ has more than one exponent, however, we prove it in a special case of a constant $A$. Moreover, we obtain an estimate of the $\beta$-Hölder constant $K_\beta(C)$ of the conjugacy $C$ in terms of the $\beta$-Hölder constant of $B$.

**Theorem 2.2.** Let $f$ and $\mu$ be as in Theorem 2.1 and let $A$ be a constant $GL(N, \mathbb{R})$-valued cocycle over $f$. Then for any Hölder continuous $GL(N, \mathbb{R})$-valued cocycle $B$ sufficiently $C^0$ close to $A$, any $\mu$-measurable conjugacy between $A$ and $B$ is Hölder continuous.

More specifically, there exists a constant $\beta_0(A, f)$ so that the following holds. For any $0 < \beta' < \beta_0(A, f)$ there is $\delta > 0$ and $k > 0$ such that for any $0 < \beta \leq \beta'$ and any $\beta$-Hölder $GL(N, \mathbb{R})$-valued cocycle $B$ over $f$ with $\|B_x - A\|_{C^0} < \delta$, any $\mu$-measurable conjugacy $C$...
between $A$ and $B$ is $\beta$-Hölder and its $\beta$-Hölder constant satisfies

\begin{equation}
K_\beta(C) \leq k \|C\|_{C^0} K_\beta(B) \quad \text{and} \quad K_\beta(C^{-1}) \leq k \|C^{-1}\|_{C^0} K_\beta(B).
\end{equation}

The constant $\beta_0(A,f)$ is explicitly given by (5.4) in Section 5.

3. Basic notations and facts

3.1. Norms and Hölder constants. For $r \in \mathbb{N} \cup \{0\}$ we use $\|\cdot\|_{C^r}$ for the $C^r$ norm of functions with continuous derivatives up to order $r$ on $\mathbb{T}^N$.

For a $\beta$-Hölder function $g$, $0 < \beta \leq 1$, we denote its $\beta$-Hölder constant, or Hölder seminorm, by

$$K_\beta(g) = \|g\|_{C^0,\beta} \overset{\text{def}}{=} \sup \{ |g(x) - g(y)| d(x,y)^{-\beta} : x \neq y \in \mathbb{T}^N \} < \infty.$$ 

We denote by $C^{1,\beta}$ or $C^{1+\beta}$ the space of functions with $\beta$-Hölder first derivative with norm

$$\|f\|_{C^{1+\beta}} \overset{\text{def}}{=} \|f\|_{C^1} + K_\beta(Df).$$

3.2. Invariant subspaces. For $A \in GL(N,\mathbb{R})$ let $\rho_1 < \cdots < \rho_L$ be the distinct moduli of its eigenvalues and let

\begin{equation}
\mathbb{R}^N = E^1 \oplus \cdots \oplus E^L
\end{equation}

be the corresponding $A$-invariant splitting, where $E^i$ is the direct sum of generalized eigenspaces corresponding to the eigenvalues with modulus $\rho_i$. We also denote

\begin{equation}
\hat{E}^i \overset{\text{def}}{=} \oplus_{\rho_j \neq \rho_i} E^j, \quad A_i = A|_{E^i} : E^i \to E^i, \quad \text{and} \quad N_i = \dim E^i.
\end{equation}

For the Euclidean norm on $\mathbb{R}^N$ there is a constant $K_A$ such that for each $i$ we have

\begin{equation}
\|A^m_i\| \leq K_A \rho_i^m (|m| + 1)^N \quad \text{for all} \ m \in \mathbb{Z}.
\end{equation}

Also, for any $\epsilon > 0$ there is an “adapted” inner product on $\mathbb{R}^N$ such that the direct sum $\oplus E^i$ is orthogonal and for each $1 \leq i \leq L$,

\begin{equation}
(\rho_i - \epsilon)^m \leq \|A^m_i u\| \leq (\rho_i + \epsilon)^m \quad \text{for any unit vector} \ u \in E^i \text{ and any} \ m \in \mathbb{Z}.
\end{equation}

If $A$ is hyperbolic then $\rho_{i_0} < \rho_{i_0+1}$ for some $1 \leq i_0 < L$, and we define the stable and unstable subspaces of $A$ as

$$E^s \overset{\text{def}}{=} \oplus_{\rho_i < 1} E^i \quad \text{and} \quad E^u \overset{\text{def}}{=} \oplus_{\rho_i > 1} E^i.$$
3.3. Weak irreducibility. Recall that $GL(N, \mathbb{Z})$ denotes the integer matrices with determinant $\pm 1$. We say that $A \in GL(N, \mathbb{Z})$ is weakly irreducible if each $\hat{E}^i$ contains no nonzero elements of $\mathbb{Z}^N$. Irreducibility over $\mathbb{Q}$ implies weak irreducibility. Indeed, if there is a nonzero integer point $n \in \hat{E}^i$ then $\text{span}\{A^m n : m \in \mathbb{Z}\} \subset \hat{E}^i$ is a nontrivial rational invariant subspace. In fact, weak irreducibility is determined by the characteristic polynomial of $A$ as follows.

**Lemma 3.1.** A matrix $A \in GL(N, \mathbb{Z})$ is weakly irreducible if and only if there is a set $\Delta \subset \mathbb{R}$ so that for each irreducible over $\mathbb{Q}$ factor of the characteristic polynomial of $A$ the set of moduli of its roots equals $\Delta$.

**Proof.** Let $A \in GL(N, \mathbb{Z})$, let $p_A$ be its characteristic polynomial, and let $p_A = \prod_{k=1}^K p_k^{d_k}$ be its prime decomposition over $\mathbb{Q}$. Then we have the corresponding splitting $\mathbb{R}^N = \bigoplus V_k$ into rational $A$-invariant subspaces $V_k = \ker p_k^{d_k}(A)$. We also have the (non-rational) $A$-invariant splitting (3.1), and we set $\Delta = \{\rho_1, \ldots, \rho_L\}$. We will show that $A$ is weakly irreducible if and only if $\Delta$ is the set of moduli of the roots for each $p_k$.

If for some $\rho_i \in \Delta$ and $k \in \{1, \ldots, K\}$ no root of the irreducible polynomial $p_k$ has modulus $\rho_i$, then $V_k \subset \hat{E}^i$. Hence $A$ is not weakly irreducible as $V_k$ is a rational subspace and hence it contains nonzero points of $\mathbb{Z}^N$.

Conversely, suppose each $p_k$ has $\Delta$ as the set of moduli of its roots. Suppose that for some $i$ there is $0 \neq n \in (\mathbb{Z}^N \cap \hat{E}^i)$. Then for some $k$ its projection $n_k$ to $V_k$ is a nonzero rational vector. We note that $n_k \in \hat{E}^i$ as $\hat{E}^i = \bigoplus (\hat{E}^i \cap V_k)$. Then

$$W = \text{span}\{A^m n_k : m \in \mathbb{Z}\}$$

is a rational $A$-invariant subspace contained in $\hat{E}^i \cap V_k$. Then the characteristic polynomial of $A|_W$ is a power of $p_k$ and hence $W$ contains an eigenvector with eigenvalue of modulus $\rho_i \in \Delta$. Thus $W \cap E^i \neq 0$, contradicting $W \subset \hat{E}^i$. Thus $A$ is weakly irreducible. \qed

It follows from the lemma that if $A$ is irreducible or weakly irreducible then the following matrices are weakly irreducible

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & 1 \\ 0 & A \end{pmatrix}.$$ 

These matrices are not irreducible and the latter is not diagonalizable.

4. Proof of Theorem 2.1

Let $f$ be a transitive $C^{1+\text{H"older}}$ Anosov diffeomorphism of a compact manifold $\mathcal{M}$, let $E$ be a $\beta$-Hölder vector bundle over $\mathcal{M}$, and let $F : E \to E$ be a $\beta$-Hölder linear cocycle over $f$.

In Section 4.1 we recall the definition and properties of holonomies for linear cocycles, in Section 4.2 we prove a preliminary results on twisted cocycles, and in Section 4.3 we give a proof of Theorem 2.1.
4.1. Holonomies of fiber bunched linear cocycles. The notion of holonomies for linear cocycle was introduced in [BV04, V08]. Existence of holonomies was proved in [V08, ASV13] under a stronger “one-step” fiber bunching condition and then extended to bundle setting and weaker fiber bunching (2.2) in [KS13, S15].

**Proposition 4.1.** Let $F$ be a $\beta$-Hölder fiber bunched linear cocycle over $(\mathcal{M}, f)$. Then for every $x \in \mathcal{M}$ and $y \in W^s(x)$ the limit

$$H^{s}_{x,y} = H^{F,s}_{x,y} = \lim_{n \to \infty} (F^n_y)^{-1} \circ F^n_x,$$

called the stable holonomy, exists and satisfies

1. $H^{s}_{x,x} = Id$ and $H^{s}_{y,z} \circ H^{s}_{x,y} = H^{s}_{x,z}$, and hence $(H^{s}_{x,y})^{-1} = H^{s}_{y,x}$;
2. $H^{s}_{x,y} = (F^n_y)^{-1} \circ H^{s}_{f^n x, f^n y} \circ F^n_x$ for all $n \in \mathbb{N}$;
3. $\|H^{s}_{x,y} - Id\| \leq c \cdot d(x,y)^\beta$, where $c$ is independent of $x$ and $y \in W^s_{loc}(x)$.

4.2. Twisted cocycles. In this section we study the coboundary equation over $f$ twisted by a $\beta$-Hölder linear cocycle $F : E \to E$. We will use its main result, Proposition 4.3, in the inductive process in the proof of Theorem 2.1.

Let $\phi, \eta : \mathcal{M} \to E$ be sections of the bundle $E$ over $\mathcal{M}$. We consider the equation

$$\eta(x) = \phi(x) + (F_x)^{-1}(\eta(f(x))) \quad \text{equivalently} \quad \phi(x) = \eta(x) - (F_x)^{-1}(\eta(f(x))).$$

Iterating (4.2) and denoting $F^n_x = F_{f^{n-1}x} \circ \cdots \circ F_{f x} \circ F_x : E_x \to E_{f^n x}$ we obtain

$$\eta(x) = \phi(x) + (F_x)^{-1}(\eta(f(x))) = \phi(x) + (F_x)^{-1}[(\phi(f(x) + F_{f x}(\eta(f^2(x)))) = \cdots$$

$$= \phi(x) + (F_x)^{-1}(\phi(f(x)) + \cdots + (F_{f^{n-1}})^{-1}(\phi(f^{n-1}x)) + (F_{f^{n-1}})^{-1}(\eta(f^n x)).$$

Thus

$$\eta(x) = \Phi^n(x) + (F_{f^{n-1}x})^{-1}(\eta(f^n x)),$$

where

$$\Phi^n(x) = \phi(x) + (F_x)^{-1}(\phi(f(x)) + \cdots + (F_{f^{n-1}})^{-1}(\phi(f^{n-1}x)) \in E_x.$$

We say that $F$ is uniformly bounded if there exists $K$ such that $\|F^n_x\| \leq K$ for all $x \in \mathcal{M}$ and $n \in \mathbb{Z}$. A $\beta$-Hölder bounded cocycle is fiber-bunched and hence it has stable holonomies $H^s_{x,y} : E_x \to E_y$ where $y \in W^s(x)$.

**Lemma 4.2.** Suppose that $\phi$ is a $\beta$-Hölder section and that $F$ is a uniformly bounded $\beta$-Hölder cocycle. Then for any $x \in \mathcal{M}$ and $y \in W^s(x)$ the following limit exists

$$\Phi^n_{x,y} = \lim_{n \to \infty} (\Phi^n(x) - H^s_{y,x}\Phi^n(y)) = \sum_{k=0}^{\infty} [(F^n_x)^{-1}(\phi(f^k x)) - H^s_{y,x}(F^n_y)^{-1}(\phi(f^k y))]

and satisfies $\|\Phi^n_{x,y}\| \leq K'd(x,y)^\beta$ with uniform $K'$ for all $x \in \mathcal{M}$ and $y \in W^s_{loc}(x)$.
The result holds if instead of being uniformly bounded $\mathcal{F}$ satisfies the following. There exist numbers $\theta < 1$ and $L$ such that for all $x \in \mathcal{M}$ and $n \in \mathbb{N}$,

$$\|(\mathcal{F}^n_x)^{-1}\| \cdot (\nu^n_x)^\beta < L \theta^n.$$  

**Proof.** For all $x \in \mathcal{M}$ and $y \in W^s_{\text{loc}}(x)$ we have $d(f^k x, f^k y) \leq \nu^k_x d(x, y)$. As $\phi$ is $\beta$-Hölder we obtain

$$\|\phi(f^k x) - \phi(f^k y)\| \leq K_1(\nu^k_x d(x, y))^\beta,$$

and since $\mathcal{H}^{s}_{f^k y, f^k x}$ is $\beta$-Hölder close to identity by $(\mathcal{H}4)$, we have

$$\|\phi(f^k x) - \mathcal{H}^{s}_{f^k y, f^k x} \phi(f^k y)\| \leq K_2(\nu^k_x d(x, y))^\beta.$$

By uniformly boundedness of $\mathcal{F}$ we have $\|(\mathcal{F}^n_x)^{-1}\| \leq K$, and by continuity of $\phi$ we have $\sup_x \|\phi(x)\| \leq K_3$. Therefore,

$$\Phi^n(x) - \mathcal{H}^{s}_{y, x} \Phi^n(y) = \sum_{k=0}^{n-1} (\mathcal{F}^k_x)^{-1}(\phi(f^k x)) - (\mathcal{H}^{s}_{y, x} \circ (\mathcal{F}^k_y)^{-1} \circ \mathcal{H}^{s}_{f^k y, f^k x}) (\mathcal{H}^{s}_{f^k y, f^k x} \phi(f^k y))$$

Since $\mathcal{H}^{s}_{y, x} \circ (\mathcal{F}^k_y)^{-1} \circ \mathcal{H}^{s}_{f^k y, f^k x} = (\mathcal{F}^k_x)^{-1}$ by $(\mathcal{H}3)$, the $k$th term in the sum equals

$$(\mathcal{F}^k_x)^{-1}(\phi(f^k x)) - (\mathcal{F}^k_x)^{-1}(\mathcal{H}^{s}_{f^k y, f^k x} \phi(f^k y)) = (\mathcal{F}^k_x)^{-1}[\phi(f^k x) - \mathcal{H}^{s}_{f^k y, f^k x} \phi(f^k y)],$$

and we estimate

$$\|(\mathcal{F}^k_x)^{-1}[\phi(f^k x) - \mathcal{H}^{s}_{f^k y, f^k x} \phi(f^k y)]\| \leq \|(\mathcal{F}^k_x)^{-1}\| \cdot \|(\mathcal{F}^k_x)^{-1}[\phi(f^k x) - \mathcal{H}^{s}_{f^k y, f^k x} \phi(f^k y)]\| \leq \|(\mathcal{F}^k_x)^{-1}\| \cdot K_2(\nu^k_x d(x, y))^\beta \leq K K_2 \theta^k d(x, y)^\beta$$

for some $\theta < 1$.

Hence the series converges and

$$\|\Phi^n(x) - \mathcal{H}^{s}_{y, x} \Phi^n(y)\| \leq \sum_{k=0}^{n-1} K K_2 \theta^k d(x, y)^\beta \leq K' d(x, y)^\beta,$$

so the limit $\Phi^{s}_{x, y}$ satisfies $\|\Phi^{s}_{x, y}\| \leq K' d(x, y)^\beta$. \hfill \Box

**Proposition 4.3.** Let $\mathcal{F}$ be a $\beta$-Hölder uniformly bounded cocycle over an Anosov diffeomorphism $f$ (or a hyperbolic system). Let $\mu$ be an ergodic $f$-invariant measure on $\mathcal{M}$ with full support and local product structure.

Let $\phi: \mathcal{M} \to E$ be a $\beta$-Hölder section, and let $\eta: \mathcal{M} \to E$ be a $\mu$-measurable section satisfying $(\text{4.2})$. Then $\eta$ is $\beta$-Hölder and

$$\eta(x) = \mathcal{H}^{s}_{y, x} \eta(y) + \Phi^{s}_{x, y} \ 	ext{for all} \ x \in X \ \text{and} \ y \in W^s(x).$$

**Proof.** Let $x \in \mathcal{M}$ and $y \in W^s(x)$. Using equation $(\text{4.3})$ for $\eta(x)$ and $\eta(y)$ we obtain

$$\eta(x) - \mathcal{H}^{s}_{y, x} \eta(y) = \Phi^n(x) - \mathcal{H}^{s}_{y, x} \Phi^n(y) + \Delta_n,$$

where

$$\Delta_n = (\mathcal{F}^{f^n x})^{-1}(\eta(f^n x)) - \mathcal{H}^{s}_{y, x} (\mathcal{F}^{f^n y})^{-1}(\eta(f^n y)).$$

By Lemma $(\text{4.2})$ $(\Phi^n(x) - \mathcal{H}^{s}_{y, x} \Phi^n(y))$ converges to $\Phi^{s}_{x, y}$. 


Now we show that $\|\Delta_n\| \to 0$ along a subsequence for all $x, y$ in a set of full measure. First we note that by property (H3) we have $H^s_{y,x}(F_{f^{-1}x})^{-1} = (F_{f^{-1}x})^{-1} \circ H^s_{f^n y, f^n x}$. Hence

$$\Delta_n = (F_{f^{-1}x})^{-1}(\eta(f^n x) - H^s_{f^n y, f^n x}(\eta(f^n y))) = (F_{f^{-1}x})^{-1}(\Delta'_n),$$

where $\Delta'_n = \eta(f^n x) - H^s_{f^n y, f^n x}(\eta(f^n y))$. By uniform boundedness of $F$ we obtain

$$\|\Delta_n\| \leq \|(F_{f^{-1}x})^{-1}\| \cdot \|\Delta'_n\| \leq K\|\Delta'_n\|.$$  

Since the section $\eta : \mathcal{M} \to E$ is $\mu$-measurable, by Lusin’s theorem there exists a compact set $S \subset \mathcal{M}$ with $\mu(S) > 1/2$ such that $\eta$ is uniformly continuous and hence bounded on $S$. Let $Y$ be the set of points in $\mathcal{M}$ for which the frequency of visiting $S$ equals $\mu(S)$. By Birkhoff Ergodic Theorem, $\mu(Y) = 1$.

If $x, y \in Y$, there exists a subsequence $n_i \to \infty$ such that such that $f^{n_i}x, f^{n_i}y \in S$ for all $i$. Since $y \in W^s(x)$, $d(f^{n_i}x, f^{n_i}y) \to 0$ and hence $\Delta'_n \to 0$ by uniform continuity and boundedness of $\eta$ on $S$ and property (H4) of $H^s$. Thus $\Delta_n \to 0$ and we obtain that

$$\eta(x) = H^s_{y,x}(\eta(y) + \Phi^{s}_{x,y}) \text{ for all } x, y \in Y \text{ with } y \in W^s(x).$$

Since $\Phi^{s}_{x,y}$ is $\beta$-Hölder on $W^s_{loc}(x)$ by Lemma 4.2, we conclude that

$$\|\eta(x) - H^s_{y,x}(\eta(y))\| \leq K'd(x, y)^{\beta} \text{ for all } x, y \in Y \text{ with } y \in W^s(x).$$

Since $H^s_{x,y}$ is $\beta$-Hölder by property (H4), this means that $\eta$ is essentially $\beta$-Hölder along $W^s_{loc}(x)$.

Similar arguments for $y \in W^u_{loc}(x)$ show that $\eta$ is also essentially $\beta$-Hölder along $W^u_{loc}(x)$. Hence $\eta$ is $\beta$-Hölder by the local product structure of $\mu$ and of the stable and unstable manifolds.

\[\square\]

4.3. \textbf{Proof of Theorem 2.1.} For convenience, by taking inverse, we will work with a conjugacy $C$ satisfying

$$B_x = C(f(x))A_x C(x)^{-1}. \hspace{1cm} (4.4)$$

First we observe that since $\lambda_+(A, \mu) = \lambda_-(A, \mu)$ and $B$ is $\mu$-measurably conjugate to $A$, the following lemma implies that

$$\lambda_+(B, \mu) = \lambda_-(B, \mu).$$

\textbf{Lemma 4.4.} Let $\mu$ be an ergodic $f$-invariant measure. If $C$ is a $\mu$-measurable conjugacy between cocycles $A$ and $B$, then for $\mu$ a.e. $x$ and for each vector $0 \neq u \in E_x$ the forward (resp. backward) Lyapunov exponent of $u$ under $A$ equals that of $C_x(u)$ under $B$.

\textbf{Proof.} We fix a set of positive measure $Y \subseteq \mathcal{M}$ such that for some $K$ we have $\|C_x\| \leq K$ and $\|(C_x)^{-1}\| \leq K$ for all $x \in Y$. Then we choose an $f$-invariant set of full measure $X \subseteq \mathcal{M}$ such that for every $x \in X$

(i) the forward and backward Lyapunov exponents under both $A$ and $B$ exist for each non-zero vector $v \in E_x$, and
(ii) the frequency of visiting $Y$ under both forward and backward iterates of $f$ equals $\mu(Y) > 0$.

For every $x \in X$, $0 \neq u \in E_x$, and $n \in \mathbb{Z}$ we have

$$n^{-1} \ln \| \mathcal{B}_x^n(c_x(u)) \| = n^{-1} \ln \| \mathcal{C}_x^n(a_x(u)) \|.$$ 

The limit of the left hand side as $n \to \infty$ (resp. $n \to -\infty$) is the forward (resp. backward) Lyapunov exponent of $c_x(u)$ under $\mathcal{B}$. On the other hand, by the choice of $Y$, the limit of the right hand side along a subsequence $n_i \to \infty$ (resp. $n_i \to -\infty$) such that $f^{n_i}x \in Y$ equals the forward (resp. backward) Lyapunov exponent of $u$ under $A$.

We use the following results from [KS13]. In the three theorems below, $f$ is a transitive $C^{1+\text{H"older}}$ Anosov diffeomorphism, $A, B : E \to E$ are $\beta$-H"older linear cocycles over $f$, and $\mu$ is an ergodic $f$-invariant measure with full support and local product structure.

**Theorem 4.5. [KS13] Theorem 3.9** Suppose that for every $f$-periodic point $p$ the invariant measure $\mu_p$ on its orbit satisfies $\lambda_+(A, \mu_p) = \lambda_-(A, \mu_p)$. Then there exist a flag of $\beta$-H"older $A$-invariant sub-bundles

$$(4.5) \quad \{0\} = U^0 \subset U^1 \subset \ldots \subset U^{j-1} \subset U^k = E$$

and $\beta$-H"older Riemannian metrics on the quotient bundles $U^i/U^{i-1}$, $i = 1, \ldots, k$, so that for some positive $\beta$-H"older function $\phi : M \to \mathbb{R}$ the quotient-cocycles induced by the cocycle $\phi A$ on $U^i/U^{i-1}$ are isometries.

**Theorem 4.6. [KS13] Theorem 3.1 and Corollary 3.8** If $B$ is fiber bunched, then any $B$-invariant $\mu$-measurable conformal structure on $E$ coincides $\mu$-a.e. with a Hölder continuous conformal structure.

If a cocycle has more than one Lyapunov exponent, then the corresponding Lyapunov sub-bundles are invariant and measurable, but not continuous in general. For a fiber bunched cocycle with only one Lyapunov exponent, measurable invariant sub-bundles are continuous.

**Theorem 4.7. [KS13] Theorem 3.3 and Corollary 3.8** Suppose that $B$ is fiber bunched and $\lambda_+(B, \mu) = \lambda_-(B, \mu)$. Then any $\mu$-measurable $B$-invariant sub-bundle of $E$ coincides $\mu$-a.e. with a Hölder continuous one.

We consider the flag $U^i$ for $A$ given by Theorem 4.5. Denoting $U^i_x = C(x)U^i_x$ we obtain the corresponding flag of measurable $B$-invariant sub-bundles

$$\{0\} = U^0 \subset U^1 \subset \ldots \subset U^k = E.$$ 

By Theorem 4.7 we may assume that the sub-bundles $U^i$ are Hölder continuous.

The conformal structure $\sigma_1$ on $E^1$ given by the Riemannian metric in Theorem 4.5 is invariant under $A$. The push forward of $\sigma_1$ by $C$ gives a measurable $B$-invariant conformal structure $\tau_i$ on $U^i$, which is Hölder continuous by Theorem 4.6.

Similarly, we consider Hölder continuous quotient-bundles $V^i = U^i/U^{i-1}$ and $\tilde{V}^i = U^i/U^{i-1}$ over $M$ with the quotient cocycles $A^{(i)}$ and $B^{(i)}$. Since $A^{(i)}$ preserves a Hölder
continuous conformal structure $\sigma_i$ on $\tilde{V}^i$, pushing forward by $C$ we obtain a measurable conformal structure $\tau_i$ on $U^i/U^{i-1}$ invariant under $B^{(i)}$, which is Hölder continuous by Theorem 4.8. Thus we obtain a “similar structure” for $B$.

We fix a $\beta$-Hölder Riemannian metric on $E$. We denote by $V^i$ the orthogonal complement of $U^{i-1}$ in $E_i$, and we denote by $\tilde{V}^i$ the orthogonal complement of $U^i$ in $E_i$, $i = 1, \ldots, k$. Thus $U^i = V^1 \oplus \cdots \oplus V^i$ and $U^i = V^1 \oplus \cdots \oplus V^i$. All these sub-bundles are Hölder continuous but for $i > 1$ they are not invariant under $A$ and $B$, and $C$ does not necessarily map $V^i$ to $V^i$.

We denote by $P^j : E \to V^j$ the projection to the $V^j$ component in the splitting $E = V^1 \oplus \cdots \oplus V^k$ and similarly $P^j : \mathcal{E} \to V^j$.

We denote the restriction of $C$ to $V^i$ by $C^i$ and we denote by $C^{j,i}$ its $j$-component $C^{j,i} = P^j \circ C^i : V^i \to V^j$. Since $U^i_{x} = C(x)U^i_{x}$, we have $C^i : V^i \to U^i$ and thus $C^{j,i} = 0$ for $j > i$, that is $C$ has an upper triangular block structure.

We also define the corresponding blocks $A^{j,i} : V^i \to V^j$ and $B^{j,i} : V^i \to V^j$ as $A^{j,i} = P^j \circ A_{V^i}$ and similarly for $B$. The invariance of the flags also yields upper triangular block structures for $A$ and $B$: $A^{j,i} = 0 = B^{j,i}$ for $j > i$.

We will show inductively that the restriction of $C$ to $U^i$ is Hölder continuous, $i = 1, \ldots, k$. The base case $i = 1$ follows from the following result from [S15].

**Theorem 4.8.** [S15 Theorem 2.7] Let $A, B : E \to E$ be $\beta$-Hölder linear cocycles over a hyperbolic system. Suppose that $A$ uniformly quasiconformal and $B$ is fiber bunched. Let $\mu$ be an ergodic invariant measure with full support and local product structure. Then any $\mu$-measurable conjugacy between $A$ and $B$ is $\beta$-Hölder continuous, i.e. it coincides with a $\beta$-Hölder continuous conjugacy on a set of full measure.

Now we describe the inductive step. Assuming that the restriction of $C$ to $U^{i-1}$ is $\beta$-Hölder continuous we show that so is the restriction to $U^i$. Since $U^i = V^i \oplus U^{i-1}$, it suffices to show that the restriction $C^i$ of $C$ to $V^i$ is also $\beta$-Hölder continuous. We will establish this inductively for each of its components $C^{j,i}$, $j = i, \ldots, 1$.

First we observe that $C^{i,i}$ is Hölder continuous for all $i = 1, \ldots, k$. For this we identify bundles $V^i$ with $\tilde{V}^i$ and $V^i$ with $\tilde{V}^i$ via the projections. Under these identifications the cocycle $A^{i,i} : V^{i,i} \to V^{i,i}$ corresponds to the quotient cocycle $A^{(i)}$, the cocycle $B^{i,i} : V^{i,i} \to V^{i,i}$ corresponds to $B^{(i)}$, and the map $C^{i,i}$ corresponds to the quotient measurable conjugacy $C^{(i)}$ between $A^{(i)}$ and $B^{(i)}$. Since the quotient cocycles $A^{(i)}$ and $B^{(i)}$ are conformal, Theorem 4.8 shows that $C^{(i)}$ is $\beta$-Hölder continuous, and hence so is $C^{i,i}$.

Now we show that $C^{i-\ell,i}$ is $\beta$-Hölder assuming that $C^{i-j,i}$ is $\beta$-Hölder for $j = 0, 1, \ldots, \ell - 1$. Using the conjugacy equation

$$B_x \circ C_x = C_{fx} \circ A_x$$

and equating $(i - \ell, i)$ components we obtain

$$B^{i-\ell,i} \circ C^{i-\ell,i}_x + B^{i-\ell,i} \circ C^{i-\ell+1,i}_x + \cdots + B^{i-\ell,i} \circ C^{i,i}_x = C^{i-\ell,i} \circ A^{i-\ell+1,i}_x + C^{i-\ell,i} \circ A^{i-\ell+1,i}_x + \cdots + C^{i-\ell,i} \circ A^{i,i}_x$$
and hence
\[(4.6) \quad \mathcal{C}_{x}^{i-\ell,i} = (B_{x}^{i-\ell,i})^{-1} \circ C_{fx}^{i-\ell,i} \circ A_{x}^{i,i} + D_{x}\]
where
\[
D_{x} = (B_{x}^{i-\ell,i})^{-1} \circ C_{fx}^{i-\ell,i} \circ A_{x}^{i-\ell+1,i} + \cdots + C_{fx}^{i-\ell,i-1} \circ A_{x}^{i-1,i} -
- (B_{x}^{i-\ell,i})^{-1} \circ (B_{x}^{i-\ell,i-1} \circ C_{x}^{i-\ell+1,i} + \cdots + B_{x}^{i-\ell,i} \circ C_{x}^{i,i}) .
\]

We view $C_{x}^{i-\ell,i}$ and $D_{x}$ as sections of the Hölder bundle $L(V_{x}, V_{x-\ell})$ whose fiber at $x$ is the space of linear maps $L(V_{x}, V_{x-\ell})$. Thus equation (4.6) is of the form (4.2) with
\[
E = L(V_{x}, V_{x-\ell}), \quad \phi_{x} = D_{x}, \quad \eta_{x} = C_{x}^{i-\ell,i}, \quad \text{and} \quad F_{x}(\eta_{fx}) = (B_{x}^{i-\ell,i})^{-1} \circ \eta_{fx} \circ A_{x}^{i,i} .
\]

We note that $D_{x}$ is $\beta$-Hölder since we inductively know that all its terms are $\beta$-Hölder. Also $F$ is a linear cocycle on the bundle $L(V_{x}, V_{x-\ell})$ over $f^{-1}$, and it is $\beta$-Hölder since so are $B_{x}^{i-\ell,i-\ell}$ and $A^{i,i}$. Moreover, $F$ is uniformly bounded since cocycles $B_{x}^{i-\ell,i-\ell}$ and $A^{i,i}$ are conformal and their normalizations are continuously cohomologous. The latter follows since we know that $B_{x}^{i-\ell,i-\ell}$ and $A^{i,i}$ are continuously cohomologous by $C_{x}^{i-\ell,i-\ell}$ and that the normalizations of all $A^{i,i}$ are given by the same function $\phi_{x}$ from Theorem 4.5.

Hence we can apply Proposition 4.3 and conclude that $C_{x}^{i-\ell,i}$ is $\beta$-Hölder.

The argument above applies to $\ell = 1, \ldots, i - 1$ and we conclude that all $C^{i,i}, \ldots, C^{i,i}$ are Hölder. This proves that the restriction of $C$ to $U$ is Hölder and completes the inductive step. We conclude that $C$ is Hölder, completing the proof of Theorem 2.1.

5. Proof of Theorem 2.2

In this proof we will also work with a conjugacy $C$ satisfying (4.1). First, Hölder continuity of $C$ is deduced from Theorem 2.1 as follows.

Let $A \in GL(N, \mathbb{R})$ be the generator of the constant cocycle $A$. Let $\rho_{1} < \cdots < \rho_{L}$ be the distinct moduli of the eigenvalues of $A$ and let
\[(5.1) \quad \mathbb{R}^{N} = E_{1}^{1} \oplus \cdots \oplus E_{L}
\]
be the corresponding invariant splitting as in (3.1). In this section we will use the adapted norm on $\mathbb{R}^{N}$ for which we have estimates (3.4). They imply that for any $\beta > 0$ the cocycle $A_{i}$ generated by $A_{i}$ is fiber bunched if $\varepsilon$ is sufficiently small.

Let $B(x) = B_{x} : \mathcal{M} \to GL(N, \mathbb{R})$ be the generator of the cocycle $B$. If $B$ is sufficiently $C^{0}$ close to $A$, then $B$ has Hölder continuous invariant splitting $C^{0}$ close to (5.1)
\[(5.2) \quad (\rho_{i} - 2\varepsilon)^{n} \leq \|B_{x}^{n}u\| \leq (\rho_{i} + 2\varepsilon)^{n} \quad \text{for any unit vector} \quad u \in E_{i}.
\]

This is well known but also follows from Lemma 5.1 which gives explicit estimates of both Hölder exponent and Hölder constant. We conclude that all restrictions $B_{i}$ are $\beta$-Hölder and hence are fiber bunched if $\varepsilon$ is sufficiently small.
Let $\mathcal{C}$ be a measurable conjugacy between $\mathcal{A}$ and $\mathcal{B}$. We claim that $\mathcal{C}$ maps $E^i$ to $E^i$, that is $\mathcal{C}_x(E^i) = E^i_x$ for $\mu$ a.e. $x$. Indeed, by Lemma 4.4 for $\mu$ a.e. $x$ and for each unit vector $u \in E^i$ the forward and backward Lyapunov exponent of $\mathcal{C}_x(u)$ is $\ln \rho_i$. This yields that $\mathcal{C}_x(u) \in E^i$, as having a non-zero component in another $E^j$ would imply having forward or backward Lyapunov exponent under $\mathcal{B}$ different from $\ln \rho_i$ if $\epsilon$ is sufficiently small. Then $\mathcal{C}_i = \mathcal{C}|_{E^i}$ is a measurable conjugacy between fiber bunched cocycles $\mathcal{A}_i$ and $\mathcal{B}_i$. By Theorem 2.1 each $\mathcal{C}_i$ is Hölder for all $i = 1, \ldots, L$, and hence so is $\mathcal{C}$.

Now we prove the more detailed statement. We denote the Lipschitz constants of $f^{-1}$ and $f$ respectively by

\begin{equation}
\alpha_f = \sup_{x \in \mathcal{M}} \|D_x f^{-1}\| > 1 \quad \text{and} \quad \alpha'_f = \sup_{x \in \mathcal{M}} \|D_x f\| > 1.
\end{equation}

For $1 \leq i < L$ we define

$$\beta_i = \frac{\ln(\rho_{i+1}/\rho_i)}{\ln(\alpha_f)} \quad \text{and} \quad \beta'_i = \frac{\ln(\rho_{i+1}/\rho_i)}{\ln(\alpha'_f)},$$

and we choose

\begin{equation}
\beta_0 = \beta_0(A, f) = \min \{1, \beta_1, \ldots, \beta_{L-1}, \beta'_1, \ldots, \beta'_{L-1}\} > 0.
\end{equation}

Since $\mathcal{B}$ is $\beta$-Hölder with $\beta \leq \beta' < \beta_0$, Lemma 5.1 below shows that the splitting (5.2) is $\beta$-Hölder and by Lemma 5.4 so are all restrictions $\mathcal{B}_i$. Then by Theorem 2.1 each restriction $\mathcal{C}_i = \mathcal{C}|_{E^i}$ is $\beta$-Hölder and hence so is $\mathcal{C}$. Since $\mathcal{A}_i$ and $\mathcal{B}_i$ are $\beta$-fiber bunched for any sufficiently small $\epsilon$, [S15 Proposition 4.5] yields that $\beta$-Hölder $\mathcal{C}_i$ intertwines their stable holonomies, that is,

\begin{equation}
\mathcal{H}^{\mathcal{A}_i, s}_{x,y} = \mathcal{C}_i(y) \circ \mathcal{H}^{\mathcal{B}_i, s}_{x,y} \circ \mathcal{C}_i(x)^{-1} \quad \text{for all} \ x, y \in \mathcal{M} \text{ such that} y \in W^s(x).
\end{equation}

Since for the constant cocycle $\mathcal{A}_i$ the holonomies are all identity, $\mathcal{H}^{\mathcal{A}_i, s}_{x,y} = \text{Id}$, we get

$$\mathcal{C}_i(x) = \mathcal{C}_i(y) \circ \mathcal{H}^{\mathcal{B}_i, s}_{x,y}.$$ 

Thus using Lemma 5.3 we obtain that for all $y \in W^s(x)$

$$\|\mathcal{C}(x) - \mathcal{C}(y)\| = \|\mathcal{C}_i(y) \circ (\mathcal{H}^{\mathcal{B}_i, s}_{x,y} - \text{Id})\| \leq \|\mathcal{C}_i\|_{C^0} \cdot k_3 K_\beta(\mathcal{B}) \cdot d_{W^s}(x, y)^\beta.$$

Combining these estimates for all $i = 1, \ldots, L$ we conclude that all $y \in W^s(x)$

$$\|\mathcal{C}(x) - \mathcal{C}(y)\| \leq \|\mathcal{C}\|_{C^0} \cdot k_4 K_\beta(\mathcal{B}) \cdot d_{W^s}(x, y)^\beta.$$

Similarly, using the analog of Lemma 5.3 for unstable holonomies, we obtain the same estimate for $y \in W^u(y)$. Then the local product structure of stable and unstable foliations of $f$ implies that the $\beta$-Hölder constant of $\mathcal{C}$ can be estimated as

$$K_\beta(\mathcal{C}) \leq k \|\mathcal{C}\|_{C^0} K_\beta(\mathcal{B}).$$

Now, to complete the proof of the second part of the theorem, we state and prove the lemmas used in the above argument.
Lemma 5.1. For any $0 < \beta' < \beta_0$ there is $\delta > 0$ and $k_I > 0$ such that for any $0 < \beta \leq \beta'$ any $\beta$-Hölder $GL(N, \mathbb{R})$ cocycle $B$ with $\|B_x - A\|_{C^0} < \delta$ preserves $\beta$-Hölder splitting

$$
\mathbb{R}^N = \mathcal{E}_x^1 \oplus \cdots \oplus \mathcal{E}_x^L
$$

which is $C^0$ close to $E^1 \oplus \cdots \oplus E^L$ and for each $1 \leq i \leq L$ the $\beta$-Hölder constant $K_{\beta}(\mathcal{E}_i)$ of $\mathcal{E}_i$ satisfies

$$
K_{\beta}(\mathcal{E}_i) \leq k_I K_{\beta}(B)
$$

Proof. We deduce this lemma from the one below. We fix $1 \leq i < L$, and let $E^i = E^1 \oplus \cdots \oplus E^i$ and $E = E^{i+1} \oplus \cdots \oplus E^L$.

Lemma 5.3 below shows that for any $\beta' < \beta_i$ there is $\delta > 0$ and $k'_I$ such that for any $0 < \beta \leq \beta'$ any cocycle $B$ with $\|B_x - A\|_{C^0} < \delta$ preserves the bundle $\mathcal{E}$ close to $E$ with the desired estimate for $\beta$-Hölder constant. Similarly, for any $\beta' < \beta_i$ using the inverses of $A$ and $f$ we obtain that $B$ preserves a bundle $\mathcal{E}'$ close to $E'$ with a similar estimate for its $\beta$-Hölder constant. Then for each $1 \leq i \leq L$ the bundle $\mathcal{E}_i$ is defined as a suitable intersection and hence is also $C^0$ close to $E^i$ and its $\beta$-Hölder constant satisfies (5.6). \[\Box\]

Remark 5.2. Lemmas 5.1 and 5.3 do not rely on hyperbolicity of $f$ and use only that it is bi-Lipschitz.

Lemma 5.3. Let $A \in GL(N, \mathbb{R})$, let $\mathbb{R}^N = E' \oplus E$ be an $A$-invariant splitting, and let

$$
\xi' = \max \{ \| Av \| : v \in E', \| v \| = 1 \} = \| A_{E'} \| \quad \text{and} \quad \xi = \min \{ \| Av \| : v \in E, \| v \| = 1 \} = \| A^{-1}_{E} \|^{-1}.
$$

Let $\alpha_f = \sup \| Df^{-1} \| > 1$ be the Lipschitz constant of $f^{-1}$ and let $\beta' > 0$. Suppose that

$$
\xi' < \xi \quad \text{and} \quad \frac{\xi' \alpha_f^{\beta'}}{\xi} < 1, \quad \text{that is,} \quad \beta' < \frac{\ln(\xi/\xi')}{\ln \alpha_f}.
$$

Then there is $\delta > 0$ and $k'$ such that for any $0 < \beta \leq \beta'$ any $\beta$-Hölder $GL(N, \mathbb{R})$ cocycle $B$ with $\|B_x - A\|_{C^0} < \delta$ preserves a $\beta$-Hölder sub-bundle $\mathcal{E}$ which is $C^0$ close to $E$ and its $\beta$-Hölder constant $K_{\beta}(\mathcal{E})$ satisfies

$$
K_{\beta}(\mathcal{E}) \leq k' K_{\beta}(B)
$$

Proof. The argument is similar to the Hölder version the $C^r$ Section Theorem of M. Hirsch, C. Pugh, and M. Shub (see Theorem 3.8 in [HPS77]), but we give the estimate of the Hölder constant.

We consider the space $\mathcal{L} = \mathcal{L}(E, E')$ of linear operators from $E$ to $E'$ and endow it with the standard operator norm. Since $A$ preserves the splitting $E' \oplus E$ it induces the graph transform action $\hat{A}$ on $\mathcal{L}$ as follows: if $L \in \mathcal{L}$ and $G \subset \mathbb{R}^N$ is its graph then $\hat{A}(L)$ is the operator in $\mathcal{L}$ whose graph is $A(G)$. The map $\hat{A}$ is linear,

$$
\hat{A}[L] = A|_{E'} \circ L \circ (A|_E)^{-1},
$$
so we can estimate its norm as
\[ \| \hat{A} \| \leq \| A \|_E \cdot \| (A|_E)^{-1} \| \leq \xi' / \xi < 1. \]

Similarly, any linear map \( B \in GL(N, \mathbb{R}) \) sufficiently close to \( A \) induces in the same way the graph transform map \( \hat{B} \) on a unit ball \( L_1 \) in \( L \). Moreover, \( \hat{B} \) is a contraction of \( L_1 \) with Lipschitz constant \( K(\hat{B}) \) close to \( K(A) = \xi' / \xi < 1 \). Indeed, \( B \) induces an algebraic map on the Grassmannian of \((\dim E)\)-dimensional subspaces which, together with its first derivatives, depends continuously on \( B \). Also, it is easy to see that the map \( B \mapsto \hat{B} \) from a small neighborhood of \( A \) to \( C^0(L_1, L_1) \) is Lipschitz with some constant \( \hat{L} \).

Now we consider the trivial fiber bundle \( \mathcal{V} = M \times L_1 \). Then any \( \mathcal{B}_z \) which is \( C^0 \)-close to \( A \) induces graph transform maps \( \hat{B}_z : \mathcal{V}_z \to \mathcal{V}_{fx} \) and thus the bundle map \( \hat{B} : \mathcal{V} \to \mathcal{V} \) covering \( f \). We consider the space \( S \) of continuous sections of \( \mathcal{V} \) with the supremum norm, and the induced action \( F = F_{\mathcal{B}_z} \) on \( S \) defined for \( s \in S \) as \( (F_{\mathcal{B}_z}) (f) = \mathcal{B}_z (s(f)) \). If \( K^\mathcal{B} := \sup_{\mathcal{B}_z} K(\mathcal{B}_z) < 1 \) then \( F \) is a contraction on \( S \) and hence has a unique fixed point \( s_s = F s_s \). Let \( s_0(x) = 0 \in \mathcal{L} \) be the zero section, then we can write \( s_s = \lim F^n(s_0) \) and it follows that \( s_s \) is \( C^0 \)-close to \( s_0 \). Denoting the graph of \( s(x) \) by \( \mathcal{E}_x \) we obtain the unique continuous \( \mathcal{B} \)-invariant sub-bundle close to \( E \).

Now we will show that \( s_s \) is \( \beta \)-Hölder and estimate its \( \beta \)-Hölder constant. For this we will find \( M > 0 \) such that \( K_{\beta}(s_s) \leq M \) implies \( K_{\beta}(F_{\mathcal{B}_z}) \leq M \). Then \( K_{\beta}(F^n(s_0)) \leq M \) for all \( n \) and since \( s_s = \lim F^n(s_0) \) it will follow that \( K_{\beta}(s_s) \leq M \).

Fix points \( z, z' \) and let \( x = f(z), x' = f(z') \). Then for any \( \beta \)-Hölder \( s \in S \) we can estimate, as \( \| s(x) \| \leq 1 \), that
\[
\begin{align*}
\| F s(x) - F s(x') \| &= \| \hat{B}_z s(z) - \hat{B}_{z'} s(z') \| \\
&\leq \| \hat{B}_z s(z) - \hat{B}_{z'} s(z) \| + \| \hat{B}_{z'} s(z) - \hat{B}_{z'} s(z') \| \\
&\leq d_{C^0}(\hat{B}_z, \hat{B}_{z'}) + K(\hat{B}_{z'}) \| s(z) - s(z') \| \\
&\leq \hat{L} K_{\beta}(\mathcal{B}) d(z, z')^{\beta} + K^\mathcal{B} K_{\beta}(s) d(z, z')^{\beta} \\
&\leq \hat{L} K_{\beta}(\mathcal{B}) d(z, z')^{\beta} + K^\mathcal{B} K_{\beta}(s) (\alpha_f d(x, x')^{\beta}),
\end{align*}
\]
where \( \alpha_f \) is the Lipschitz constant of \( f^{-1} \) and \( \hat{L} \) is the Lipschitz constant of the map \( B \mapsto \hat{B} \) on a neighborhood of \( A \). Hence \( F_s \) is also \( \beta \)-Hölder and
\[ K_{\beta}(F_s) \leq \hat{L} \alpha_f^{\beta} K_{\beta}(\mathcal{B}) + \alpha_f^{\beta} K^\mathcal{B} K_{\beta}(s). \]

Therefore, \( K_{\beta}(s_s) \leq M \) implies \( K_{\beta}(F_{\mathcal{B}_z}) \leq M \) if we take
\[ M = (1 - K^\mathcal{B} \alpha_f^{\beta})^{-1} \hat{L} \alpha_f^{\beta} K_{\beta}(\mathcal{B}). \]
If \( \| \mathcal{B}_x - A \|_{C^0} \) is small then \( K^\mathcal{B} \) is close to \( K(\hat{A}) = \xi' / \xi \). Since \( \xi' \alpha^{\beta} / \xi < 1 \) and \( \beta \leq \beta' \) it follows that \( 1 - K^\mathcal{B} \alpha_f^{\beta} > 0 \) and is separated from 0. Then there is a constant \( k' \) which bounds \( \hat{L} a_f^{\beta} (1 - K^\mathcal{B} \alpha_f^{\beta})^{-1} \) for all \( 0 < \beta \leq \beta' \) and all \( \mathcal{B} \) with \( \| \mathcal{B}_x - A \|_{C^0} < \delta \). Hence,
\[ M \leq k' K_{\beta}(\mathcal{B}). \]
Finally, since $K_\beta(s_0) = 0$ it follows that $K_\beta(F^n(s_0)) \leq M$ for all $n$ and hence for the limit we also have $K_\beta(s_x) \leq M \leq k'K_\beta(B)$. \hfill \Box

Now we estimate the $\beta$-Hölder constants of the restricted cocycles $B_i = B|_{\mathcal{E}_i}$.

**Lemma 5.4.** For any $0 < \beta' < \beta_0$ there is $\delta > 0$ and $k_2 > 0$ such that for any $0 < \beta \leq \beta'$ and any $\beta$-Hölder cocycle $B$ with $\|B_x - A\|_{C^0} < \delta$ the $\beta$-Hölder constant of the cocycle $B_i$, $i = 1, \ldots, L$, satisfies

$$K_\beta(B_i) \leq k_2 K_\beta(B).$$

**Proof.** Denoting $B(x) = B_x$ and $B_i(x) = B_x|_{\mathcal{E}_i}$, we need to estimate the distance between $B_i(x)$ and $B_i(y)$. To do this using their difference, we fix $\beta$-Hölder identifications $I_{x,y} : \mathcal{E}_x \to \mathcal{E}_y$, say by translation from $x$ to $y$ in the trivial bundle $\mathcal{M} \times \mathbb{R}^N$ followed by an appropriate rotation. Then for a unit vector $u \in \mathcal{E}_i(x)$ we need to estimate $\|(B_i(x) - B_i(y) \circ I_{x,y})u\|$. We note that

$$\|u - I_{x,y}u\| \leq \text{dist}(\mathcal{E}_x^i, \mathcal{E}_y^i) \leq K_\beta(\mathcal{E}_i^i) d(x,y)^\beta.$$

Also, since $B(x)$ is $\beta$-Hölder have $\|B(x)u - B(y)u\| \leq K_\beta(B) d(x,y)^\beta$. Hence we obtain that for a unit vector $u \in \mathcal{E}_i^i(x)$

$$\|(B_i(x) - B_i(y) \circ I_{x,y})u\| \leq \|B(x)u - B(y)u\| + \|B(y)\| \cdot \|u - I_{x,y}u\| \
\leq K_\beta(B) d(x,y)^\beta + \|B\|_{C^0} K_\beta(\mathcal{E}_i^i) d(x,y)^\beta.$$

Since $K_\beta(\mathcal{E}_i^i) \leq k_1 K_\beta(B)$ by (5.6) and $\|B\|_{C^0} \leq \|A\| + \|B_x - A\|_{C^0} \leq \|A\| + \delta$ we conclude that

$$\|(B_i(x) - B_i(y) \circ I_{x,y})u\| \leq k_2 K_\beta(B) d(x,y)^\beta.$$

Thus $K_\beta(B_i) \leq k_2 K_\beta(B)$. \hfill \Box

In the next lemma we consider the stable holonomies of cocycles $B_i = B|_{\mathcal{E}_i}$, $i = 1, \ldots, L$.

**Lemma 5.5.** For any $0 < \beta' < \beta_0$ there is $\delta > 0$ and $k_3 > 0$ such that for any $0 < \beta \leq \beta'$ and a $\beta$-Hölder cocycle $B$ with $\|B_x - A\|_{C^0} < \delta$ the holonomies of cocycles $B_i = B|_{\mathcal{E}_i}$ satisfy

$$\|\mathcal{H}_{x,y}^s - \text{Id}\| \leq k_3 K_\beta(B) d(x,y)^\beta \text{ for any } x \in \mathcal{M} \text{ and } y \in W_{loc}^s(x).$$

**Proof.** We fix $i$ and denote $F = B_i$. The stable holonomies of $F$ are given by

$$(5.7) \quad \mathcal{H}_{x,y}^{F,s} = \lim_{n \to \infty} (F^n_y)^{-1} \circ F^n_x.$$

The existence is ensured by fiber bunching of $F$. Indeed, the contraction along $W^s$ is estimated by (2.1) as

$$d(f^n_x, f^n_y) \leq \nu^n d(x,y) \text{ for any } x \in \mathcal{M}, \ y \in W_{loc}^s(x), \ n \in \mathbb{N},$$

We also obtain from (5.2) that

$$(5.8) \quad \|F^m_x\| \cdot \|(F^m_y)^{-1}\| \leq \prod_{j=0}^{m-1} \|F_x\| \cdot \|(F_y)^{-1}\| \leq \left(\frac{\rho_1 + 2\epsilon}{\rho_1 - 2\epsilon}\right)^m = \sigma^m \text{ for all } x, y \in \mathcal{M},$$
where \( \sigma = (\rho_i + 2\epsilon)(\rho_i - 2\epsilon)^{-1} \) is close to 1 when \( \epsilon \) is small. It follows that
\[
(5.9) \quad \| F^m_x \| \cdot \|(F_y^n)^{-1}\| \cdot \nu^{m\beta} \leq \sigma^m \cdot \nu^{m\beta} = \theta^m \quad \text{for all } x, y \in M,
\]
where \( \theta = \sigma \nu^\beta < 1 \) if \( \delta \) and hence \( \epsilon \) are sufficiently small. In particular, \( F \) is fiber Cauchy) and the limit satisfies
\[
\text{Using (5.9) it follows that}
\]
\[
\text{Therefore, for every } x \in M, \text{ we have the same regularity.}
\]
\[
\text{We want to obtain a constant } c \text{ such that } \| H_{x,y}^F - Id \| \leq c d(x, y)^\beta \text{ for all } x, y \in M
\]
\[
\text{Denoting } x_m = f^m(x) \text{ and } y_m = f^m(y), \text{ we obtain}
\]
\[
(F^n_y)^{-1} \circ F^n_x = (F^n_y)^{-1} \circ ((F^n_{y_n})^{-1} \circ F^n_{x_n}) \circ F^n_x = (F^n_y)^{-1} \circ (\text{Id} + r_{n-1}) \circ F^n_x = (F^n_y)^{-1} \circ (F^n_y)^{-1} \circ (F^n_y)^{-1} \circ r_{n-1} \circ F^n_x
\]
\[
= \cdots = \text{Id} + \sum_{m=0}^{n-1} (F^n_y)^{-1} \circ r_m \circ F^n_x, \quad \text{where } r_m = (F^n_{y_m})^{-1} \circ F^n_{x_m} - \text{Id}.
\]
\[
\text{Since } F \text{ is } \beta\text{-Hölder, denoting } c' = (\rho_i - 2\epsilon)^{-1} K_\beta(F), \text{ we obtain that for every } m \geq 0
\]
\[
\| r_m \| \leq \|(F^n_{y_m})^{-1}\| \cdot \| F^n_{x_m} - F^n_{y_m} \| \leq \| F^n \| \cdot (c' K_\beta(F) d(x_m, y_m)^\beta \leq c' d(x, y)^\beta \nu^{m\beta}.
\]
\[
\text{Using (5.9) it follows that}
\]
\[
\| (F^n_y)^{-1} \circ r_m \circ F^n_x \| \leq \| (F^n_y)^{-1}\| \cdot \| F^n_x \| \cdot c' d(x, y)^\beta \nu^{m\beta} \leq \theta^m c' d(x, y)^\beta.
\]
\[
\text{Therefore, for every } n \in \mathbb{N},
\]
\[
\| \text{Id} - (F^n_y)^{-1} \circ F^n_x \| \leq \sum_{i=0}^{n-1} \| (F^n_y)^{-1} \circ r_i \circ F^i_x \| \leq c' d(x, y)^\beta \sum_{i=0}^{n-1} \theta^i \leq c d(x, y)^\beta,
\]
\[
\text{where}
\]
\[
c = \frac{c'}{1 - \theta} \leq \frac{(\rho_i - 2\epsilon)^{-1} K_\beta(F)}{1 - \sigma \nu^\beta} = k'_3 K_\beta(F) \quad \text{with } k'_3 = (\rho_i - 2\epsilon)^{-1}(1 - \sigma \nu^\beta)^{-1}.
\]
\[
\text{By (5.7) the sequence } \{(F^n_y)^{-1} \circ F^n_x\} \text{ converges to } H_{x,y}^F \text{ (in fact the estimates imply that it is Cauchy) and the limit satisfies}
\]
\[
\| H_{x,y}^F - \text{Id} \| \leq c d(x, y)^\beta \text{ for any } x \in M \text{ and } y \in W^s_{\text{loc}}(x).
\]
\[
\text{By Lemma 5.4 we have } K_\beta(F) = K_\beta(B_i) \leq k_2 K_\beta(B) \text{ and we conclude that}
\]
\[
\| H_{x,y}^F - \text{Id} \| \leq k_3 K_\beta(B) d(x, y)^\beta \text{ for any } x \in M \text{ and } y \in W^s_{\text{loc}}(x).
\]
\[
\text{This completes the proof of Lemma 5.5} \quad \square
\]

6. **Proof of Theorem 1.1**

Any two continuous conjugacies between \( f \) and \( A \) differ by an element of the centralizer of \( A \). By [Wa70, Corollary 1], any homeomorphism commuting with an ergodic, in particular hyperbolic, automorphism \( A \) is an affine automorphism, and hence all conjugacies have the same regularity.
First, using Theorem 2.2 we will show in Section 6.1 that $H$ is a $C^{1+\text{Hölder}}$ diffeomorphism, and moreover the Hölder constant of its derivative satisfies the estimate
\begin{equation}
K_\beta(DH) \leq k \|DH\|_{C^0} \|f - A\|_{C^{1+\beta}}.
\end{equation}
This part does not rely on closeness of $H$ to the identity and the estimate applies to any conjugacy $H$. Then in Section 6.2 we use (6.1) and an interpolating inequality to obtain the desired estimate (1.2) of $\|H - I\|_{C^{1+\beta}}$ for the conjugacy $C^0$ close to the identity.

6.1. Proving that $H$ is a $C^{1+\text{Hölder}}$ diffeomorphism.

First we recall some properties of a map $g \in W^{1,q}(\mathbb{R}^N, \mathbb{R}^N)$, $q > N$, which also extend to the case when $g \in W^{1,q}(\mathbb{T}^N, \mathbb{T}^N)$. It is well known that, as a consequence of Morrey’s inequality, for any such $g$ the Jacoby matrix of weak partial derivatives gives the differential $D_xg$ for almost every $x$ with respect to the Lebesgue measure $\mu$. Also, any such $g$ satisfies Lusin’s $N$-property [MM73] that $\mu(E) = 0$ implies $\mu(g(E)) = 0$, as well as Morse-Sard property [P01] that $\mu(g(C_g)) = 0$ for the set of critical points of $g$
\[
C_g = \{x \in \mathbb{T}^N : D_xg \text{ exists but is not invertible}\},
\]
see also [KK18] for sharper results and further references.

Now we assume that $H \in W^{1,q}$ with $q > N$, so that the differential $D_xH$ exists $\mu$-a.e., and for the set
\[
G_H = \{x \in \mathbb{T}^N : D_xH \text{ exists}\}
\]
and its complement $E_H = \mathbb{T}^N \setminus G_H$
we have $\mu(G_H) = 1$ and $\mu(E_H) = 0$. Further $G_H = C_H \cup R_H$ is the disjoint union of two measurable sets, the critical set $C_H$ and the regular set
\[
R_H = \{x \in \mathbb{T}^N : D_xH \text{ is invertible}\}.
\]
Since $f$ and $A$ are diffeomorphisms, it follows from the conjugacy equation $H \circ f = A \circ H$ that the sets $G_H$, $C_H$, and $R_H$ are $f$-invariant. Further, differentiating the equation on the set $G_H$ we obtain
\begin{equation}
D_{f}H \circ D_xf = A \circ D_xH.
\end{equation}
Denoting $C(x) = D_xH$ on the set $R_H$ we obtain the conjugacy equation over $f$
\begin{equation}
A = C(fx) \circ B_x \circ C(x)^{-1}
\end{equation}
for cocycles $B_x = D_xf$ and $A_x = A$.

Now we show that $\mu(R_H') = 1$ and also that $f$ preserves a measure $\tilde{\mu}$ equivalent to $\mu$. Since $\mu(E_H) = 0$, the Lusin’s $N$-property of $H$ yields $\mu(H(E_H)) = 0$. Also, we have $\mu(H(C_H)) = 0$ by the Morse-Sard property. Hence for $R_H' = H(R_H)$ we have $\mu(R_H') = 1$. Now we consider the measure $\tilde{\mu} = (H^{-1})_* (\mu)$ and note that $\tilde{\mu}(R_H) = 1$ as $\mu(R_H) = 1$. Since $H$ is a topological conjugacy between $f$ and $A$, the measure $\tilde{\mu}$ is $f$-invariant and, in fact, is the Bowen-Margulis measure of maximal entropy for $f$, since $\mu$ is that for $A$. Indeed, the Bowen-Margulis entropy by $h_{\text{top}}$ and metric entropy with respect to $\tilde{\mu}$ by $h_{\tilde{\mu}}$ we get
\[
h_{\tilde{\mu}}(f) = h_{\mu}(A) = h_{\text{top}}(A) = h_{\text{top}}(f).
\]
In particular, $\tilde{\mu}$ is ergodic with full support and local product structure. Since $C$ is a conjugacy between $B$ and $A$ on $R_H$ with $\tilde{\mu}(R_H) = 1$, by Lemma 4.4 we obtain that the Lyapunov exponents $\lambda^f_{i\tilde{\mu}}$ of $\tilde{\mu}$ for the cocycle $B = Df$ are equal to the Lyapunov exponents $\lambda^A_i$ of $A$. Hence the sum of positive Lyapunov exponents (counted with multiplicities) for $\tilde{\mu}$ equals its entropy

$$h_{\tilde{\mu}}(f) = h_{\mu}(A) = \sum_{\lambda^A_i > 0} \lambda^A_i = \sum_{\lambda^f_{i\tilde{\mu}} > 0} \lambda^f_{i\tilde{\mu}}.$$  

Thus we have equality in the Pesin-Ruelle formula, which implies that $\tilde{\mu}$ has absolutely continuous conditional measures on the unstable foliation of $f$ [Le84]. Similarly, equality of the negative Lyapunov exponents yields that $\tilde{\mu}$ has absolutely continuous conditional measures on the stable foliation of $f$. We conclude that $\tilde{\mu}$ itself is absolutely continuous. Moreover, the density $\sigma(x) = \frac{d\mu}{d\tilde{\mu}}$ is smooth and positive as a measurable solution of the coboundary equation $\sigma(f(x))\sigma(x)^{-1} = \det Df(x)$. Thus $\tilde{\mu}$ is equivalent to $\mu$, so that $\tilde{\mu}(R_H) = 1$ implies $\mu(R_H) = 1$.

Provided that $\|A - B_x\|_{C^0} = \|A - D_xf\|_{C^0} \leq \|A - f\|_{C^1} < \delta$, where $\delta > 0$ is from Theorem 2.2 we can apply this theorem with $f$ and $\tilde{\mu}$ to obtain that

$$C(x) = D_xH : \mathbb{T}^N \to GL(N, \mathbb{R})$$

coincides with a Hölder continuous function almost everywhere with respect to $\tilde{\mu}$ and hence $\mu$. Since $H \in W^{1,q}$ we conclude that $H$ is $C^{1+\text{Hölder}}$. Also, since $(D_xH)^{-1} = C(x)^{-1}$ exists and is also Hölder continuous we see that $H$ is $C^{1+\text{Hölder}}$ diffeomorphism. Further, Theorem 2.2 gives us the estimate (6.1), which we will use to obtain the desired estimate for $\|H - \text{Id}\|_{C^{1+\beta}}$ in Section 6.2. This completes the proof that $H$ is $C^{1+\text{Hölder}}$ diffeomorphism assuming that $H \in W^{1,q}$.

Now we consider the case when $\tilde{H} = H^{-1}$ is in $W^{1,q}$ and hence $D_x\tilde{H}$ exists $\mu$-a.e. We similarly define the sets $\tilde{C}_H, E_{\tilde{H}}, \tilde{C}_{\tilde{H}}$, and $R_{\tilde{H}}$, which are measurable and $A$-invariant. Hence by ergodicity of $A$ the set $R_{\tilde{H}}$ must be null or co-null for $\mu$. If $\mu(R_{\tilde{H}}) = 0$ then $\mu(\tilde{H}(R_{\tilde{H}})) = 0$ by the Lusin’s N-property of $\tilde{H}$, but this is impossible since $\mu(\tilde{H}(E_{\tilde{H}})) = 0$ by the Lusin’s N-property and $\mu(\tilde{H}(\tilde{C}_{\tilde{H}})) = 0$ by the Morse-Sard property. Hence $\mu(R_{\tilde{H}}) = 1$. Then for $R'_{\tilde{H}} = \tilde{H}(R_{\tilde{H}})$ we have $\tilde{\mu}(R'_{\tilde{H}}) = 1$, where as before $\tilde{\mu} = \tilde{H}_*(\mu)$ is the measure of maximal entropy for $f$. Now the Lusin’s N-property of $\tilde{H}$ yields that $\tilde{\mu}$ is absolutely continuous and then equivalent to $\mu$. Hence we also have $\mu(R'_{\tilde{H}}) = 1$. Since $H = \tilde{H}^{-1}$ is a homeomorphism, and $D_x\tilde{H}$ is invertible for $x \in R_{\tilde{H}}$, it follows that $D_yH = (D_x\tilde{H})^{-1}$ is the differential of $H$ for each $y = \tilde{H}(x)$ in $R'_{\tilde{H}}$.

Therefore, we can again differentiate $H \circ f = A \circ H$ to obtain (6.3) and then the conjugacy equation (6.3) with $C(x) = D_xH$ on the set $R'_{\tilde{H}}$ of full measure for both $\mu$ and $\tilde{\mu}$. Then by Theorem 2.2 applied with $f$ and $\tilde{\mu}$ we obtain that $C(x) = D_xH$ is Hölder on $\mathbb{T}^N$ and hence so is $C(y)^{-1} = D_x\tilde{H}$. Since $\tilde{H} = H^{-1}$ is in $W^{1,q}$ we conclude that $H^{-1}$ is $C^{1+\text{Hölder}}$ diffeomorphism. In this case we also get (6.1).
6.2. Estimating $\|H - I\|_{C^{1+\beta}}$. We showed that any conjugacy $H$ is a $C^{1+\text{Hölder}}$ diffeomorphism satisfying (6.1). Now we prove estimate (1.2) for the conjugacy $H$ that is $C^0$ close to the identity.

Any two conjugacies in the homotopy class of the identity differ by a composition with an affine automorphism commuting with $A$, which is translation $T_v(x) = x + v$, where $v \in \mathbb{T}^N$ is a fixed point of $A$. It is well known that if $f$ is $C^1$-close to $A$, then it has a unique fixed point $p$ which is the perturbation of 0. More precisely, there are $0 < \delta(A), r(A) < 1/5$ and $k(A)$ so that for each $f$ satisfying $\|f - A\|_{C^1} < \delta(A)$ there is a unique fixed point $p = f(p)$ with $d(p,0) < r(A)$ and it satisfies

$$d(p,0) \leq k(A)\|f - A\|_{C^0}.$$ 

Since $H$ maps fixed points of $f$ to those of $A$ we see that if $\|H - I\|_{C^0} < r(A)$ then it is in the homotopy class of the identity and satisfies $H(p) = 0$.

Replacing $f$ by $\tilde{f} = T_{-p} \circ f \circ T_p$ we can change $p$ to 0. Since for $\tilde{f}(x) = f(x + p) - p$ we have that

$$\|D\tilde{f} - A\|_{C^k} = \|Df - A\|_{C^k} \quad \text{for any} \quad k \geq 0,$$

and so only $\|f - A\|_{C^0}$ is affected by this change. Moreover, if we write $f = A + R$, then

$$\tilde{f}(x) - A(x) = A(x + p) + R(x + p) - p - A(x) = R(x + p) + A(p) - p$$

and hence

$$\|\tilde{f} - A\|_{C^0} \leq \|R\|_{C^0} + \|A(p) - p\| = \|f - A\|_{C^0} + \|A(p) - f(p)\| \leq 2\|f - A\|_{C^0}.$$ 

Thus $\|\tilde{f} - A\|_{C^{1+\beta}} \leq 2\|f - A\|_{C^{1+\beta}}$. Also, if $\tilde{H}$ is the corresponding conjugacy between $\tilde{f}$ and $A$ then $\tilde{H}(x) = \tilde{H}(x - p)$ and hence

$$\|H - \text{Id}\|_{C^{1+\beta}} \leq \|\tilde{H} - \text{Id}\|_{C^{1+\beta}} + d(p,0) \leq \|\tilde{H} - \text{Id}\|_{C^{1+\beta}} + k(A)\|f - A\|_{C^0}$$

Thus the estimate (6.1) for $\tilde{H}$ via $\tilde{f}$ would yield the corresponding estimate for $H$ via $f$. So without loss of generality we will assume that

$$f(0) = 0 \quad \text{and} \quad H(0) = 0.$$ 

Now we recall how the conjugacy equation $H \circ f = A \circ h$ can be rewritten using lifts. We denote by $\tilde{f}$ and $\tilde{H}$ the lifts of $f$ and $H$ to $\mathbb{R}^N$ satisfying $\tilde{f}(0) = 0$ and $\tilde{H}(0) = 0$ so that we have $\tilde{H} \circ \tilde{f} = A \circ \tilde{H}$ where all maps are $\mathbb{R}^N \to \mathbb{R}^N$. Since $H$ is homotopic to the identity and $f$ is homotopic to $A$ we can write

$$\tilde{H} = \text{Id} + h \quad \text{and} \quad \tilde{f} = A + R,$$

Then the commutation relation on $\mathbb{R}^N$

$$(\text{Id} + h) \circ (A + R) = A \circ (\text{Id} + h) \quad \text{yields} \quad h = A^{-1}(h \circ \tilde{f}) + A^{-1}R.$$ 

Since $h, R : \mathbb{R}^N \to \mathbb{R}^N$ are $\mathbb{Z}^N$-periodic we can view them as

$$h = H - \text{Id} : \mathbb{T}^N \to \mathbb{R}^N \quad \text{and} \quad R = f - A : \mathbb{T}^N \to \mathbb{R}^N$$
and rewrite the conjugacy equation as one for $\mathbb{R}^N$-valued functions on $\mathbb{T}^N$

\begin{equation}
(6.4) \quad h = A^{-1}(h \circ f) + A^{-1}R.
\end{equation}

Using the $A$-invariant splitting $\mathbb{R}^N = E^u \oplus E^s$ we define the projections $h_*$ and $R_*$ of $h$ and $R$ to $E^s$, where $* = s, u$, and obtain

\begin{equation}
(6.5) \quad h_* = A_*^{-1}(h_* \circ f) + A_*^{-1}R_*, \quad \text{where } A_* = A|_{E^s}.
\end{equation}

Thus $h_*$ is a fixed point of the affine operator

\begin{equation}
(6.6) \quad T_*(\psi) = A_*^{-1}(\psi \circ f) + A_*^{-1}R_*
\end{equation}

Since $\|A_u^{-1}\| < 1$, the operator $T_u$ is a contraction on the space $C^0(\mathbb{T}^d, E^u)$, and thus $h_u$ is its unique fixed point

\begin{equation}
(6.7) \quad h_u = \lim_{m \to \infty} T_u^m(0) = \sum_{m=0}^{\infty} A_u^{-m}(A_u^{-1}R_u \circ f^m).
\end{equation}

Hence

\begin{equation}
(6.8) \quad \|h_u\|_{C^0} \leq \sum_{m=0}^{\infty} \|A_u^{-1}\|^{m+1}\|R_u\|_{C^0} \leq k \|R_u\|_{C^0} \leq k \|A - f\|_{C^0}.
\end{equation}

Similarly, $h_s$ is the unique fixed point of contraction $T_s^{-1}$ and hence satisfies a similar estimate. Combining them we conclude that

\begin{equation}
(6.9) \quad \|H - \text{Id}\|_{C^0} = \|h\|_{C^0} \leq k_0\|R\|_{C^0} = k_0\|A - f\|_{C^0}.
\end{equation}

Now we estimate $\|H - \text{Id}\|_{C^{1+\beta}}$ using (6.9), (6.1), and the following elementary interpolation lemma. We note that $DH = \text{Id} + Dh$, so that $K_\beta(Dh) = K_\beta(DH)$.

**Lemma 6.1.** If $h: \mathbb{T}^N \to \mathbb{R}^N$ satisfies $K_\beta(Dh) \leq K$ then

\begin{equation}
\|Dh\|_{C^0} \leq 8 \|h\|_{C^0}^{\beta/(1+\beta)} K^{1/(1+\beta)}.
\end{equation}

**Proof.** Denote $b = \|Dh\|_{C^0}$ and choose $x \in \mathbb{T}^N$ such that $\|D_xh\| = b$. Then for some unit vectors $u, v \in \mathbb{R}^N$ we have $(D_xh)u = bv$. For $y \in \mathbb{T}^N$ let $b_y = \langle(D_yh)u, v\rangle$, so $b_x = b$. Then

\begin{equation}
|b - b_y| \leq \|(D_xh)u - (D_yh)u\| \leq K d(x, y)^\beta \leq b/2 \quad \text{if} \quad d(x, y) \leq (b/2K)^{1/\beta}
\end{equation}

and hence $b_y \geq b/2$ for such $y$. Consider $g(t) = x + tu$, with $0 \leq t \leq t_0 = (b/2K)^{1/\beta}$, and $g(t) = \langle h(y(t)), v\rangle$. Then

\begin{equation}
g'(t) = \langle(D_yh)u, v\rangle = b_{y(t)} \geq b/2,
\end{equation}

and hence by integrating we get $bt_0/2 \leq g(t_0) - g(0)$. Since $|g(t_0) - g(0)| \leq 2\|h\|_{C^0}$ we obtain $bt_0 \leq 4\|h\|_{C^0}$. Substituting $t_0 = (b/2K)^{1/\beta}$ we obtain

\begin{equation}
b(b/2K)^{1/\beta} \leq 4 \|h\|_{C^0} \Rightarrow b^{(1+\beta)/\beta} \leq 4 \|h\|_{C^0}(2K)^{1/\beta} \Rightarrow b \leq 8 \|h\|^{\beta/(1+\beta)}_{C^0} K^{1/(1+\beta)}
\end{equation}

as $4^{\beta/(1+\beta)}2^{1/(1+\beta)} < 8.$

$\square$
We denote \( a = \| h \|_{C^0}, \ b = \| Dh \|_{C^0}, \) and \( d = \| f - A \|_{C^{1+\beta}}. \) Then
\[
\| DH \|_{C^0} = \| \text{Id} + Dh \|_{C^0} \leq 1 + b,
\]
and hence (6.1) implies that
\[
(6.10) \quad K = K_\beta(Dh) = K_\beta(DH) \leq k(1 + b)d.
\]
Also, by (6.9) we have \( a = \| h \|_{C^0} \leq k_0d. \) Then Lemma 6.1 gives
\[
b \leq 8(kd)^{\beta/(1+\beta)}(k_0d)^{1/(1+\beta)} \leq k_0d(1 + b)^{\beta/(1+\beta)}.
\]
It follows that \( b \) is bounded by some \( k_2 \) if \( d \leq 1. \) Then (6.10) implies that
\[
K = K_\beta(Dh) \leq k_3d.
\]
With this \( K \) Lemma 6.1 gives
\[
b \leq 8(kd)^{\beta/(1+\beta)}(k_3d)^{1/(1+\beta)} \leq k_4d.
\]
We conclude that
\[
b = \| Dh \|_{C^0} < k_4d, \quad a = \| h \|_{C^0} \leq k_0d, \quad \text{and} \quad K_\beta(Dh) \leq k_3d;
\]
so that
\[
\| H - \text{Id} \|_{C^{1+\beta}} = \| h \|_{C^{1+\beta}} \leq k_5d = k_5 \| f - A \|_{C^{1+\beta}}.
\]
This completes the proof of Theorem 1.1.

7. Linearized Conjugacy Equation

In this section we begin the proof of Theorem 1.3, and in the next one we will complete it using an iterative process. In these sections we fix a hyperbolic matrix \( A \in SL(N, \mathbb{Z}). \) We will use \( K \) to denote any constant that depends only on \( A, \) and \( K_x \) to denote a constant that also depends on a parameter \( x. \)

7.1. Preliminaries. Set \( \tilde{A} = (A^\tau)^{-1} \) where \( A^\tau \) denotes transpose matrix. We call \( \tilde{A} \) the dual map on \( \mathbb{Z}^N. \) Since \( A \) is hyperbolic so is \( \tilde{A}, \) and we denote its stable and unstable subspaces by \( \tilde{E}^s \) and \( \tilde{E}^u. \) Thus there is \( \rho > 1 (\rho < \min\{\rho_{\text{io}+1}, \rho_{\text{io}}^{-1}\}) \) such that
\[
(7.1) \quad \| \tilde{A}^k v \| \geq K \rho^k \| v \|, \quad k \geq 0, \quad v \in \tilde{E}^u,
\]
\[
\| \tilde{A}^{-k} v \| \geq K \rho^{-k} \| v \|, \quad k \geq 0, \quad v \in \tilde{E}^s.
\]
For a subspace \( V \) of \( \mathbb{R}^N, \) we use \( \pi_V \) to denote the (orthogonal) projection to \( V. \) For any integer vector \( n \in \mathbb{Z}^N \) we write \( n_s = \pi_{\tilde{E}^s} n \) and \( n_u = \pi_{\tilde{E}^u} n. \) Since \( \tilde{A} \in SL(N, \mathbb{Z}) \) is hyperbolic, for any \( 0 \neq n \in \mathbb{Z}^N \) both \( n_s \) and \( n_u \) are nonzero and there is a unique \( k_0 = k_0(n) \in \mathbb{Z} \) such that
\[
\| \tilde{A}^k n_s \| \geq \| \tilde{A}^k n_u \| \quad \text{for all} \quad k \leq k_0 \quad \text{and}
\]
\[
\| \tilde{A}^k n_s \| < \| \tilde{A}^k n_u \| \quad \text{for all} \quad k > k_0.
\]
The corresponding element $\tilde{A}^{a(n)} n$ on the orbit of $n$ will be called \textit{minimal} and

$$M = \{ \tilde{A}^{a(n)} n : 0 \neq n \in \mathbb{Z}^N \} \subset \mathbb{Z}^N \setminus \{0\}.$$ 

For any $n \in M$ we have $\|n_s\| \geq \frac{1}{2} \|n\|$ and $\|\tilde{A} a_n\| > \frac{1}{2} \|	ilde{A} n\|.$

For a function $\theta \in L^2(\mathbb{T}^N, \mathbb{C})$ we denote its Fourier coefficients by $\hat{\theta}_n, n \in \mathbb{Z}^N,$ so that

$$\theta(x) = \sum_{n \in \mathbb{Z}^N} \hat{\theta}_n e^{2\pi i n \cdot x} \text{ in } L^2(\mathbb{T}^N).$$

We say that $\theta$ is \textit{excellent} (for $A$) if $\hat{\theta}_n = 0$ for all $n \notin M.$

To simplify our estimates, instead of the standard Sobolev spaces we will work the spaces $\mathcal{H}^s(\mathbb{T}^N), s > 0,$ defined as follows. A function $\theta \in L^2(\mathbb{T}^N)$ belongs to $\mathcal{H}^s(\mathbb{T}^N)$ if

$$\|\theta\|_s \overset{\text{def}}{=} \sup_{n \in \mathbb{Z}^N} \|\hat{\theta}_n\| |n|^{-s} + |\hat{\theta}_0| < \infty.$$ 

The following relations hold (see, for example, Section 3.1 of [dlL99]). If $\sigma > N + 1$ and $r \in \mathbb{N},$ then for any $\theta \in C^r(\mathbb{T}^N)$ and $\omega \in \mathcal{H}^{r+\sigma}$ we have $\theta \in \mathcal{H}^r$ and $\omega \in C^r(\mathbb{T}^N)$ with estimates

$$\|\theta\|_r \leq K \|\theta\|_{C^r} \quad \text{and} \quad \|\omega\|_{C^r} \leq K \|\omega\|_{r+\sigma}.$$ 

For a vector-valued function $\theta: \mathbb{T}^N \to \mathbb{C}^m$ we denote its coordinate functions by $\theta_j, j = 1, \ldots, m.$ We say that $\theta$ is in $\mathcal{H}^s(\mathbb{T}^N)$ if each $\theta_j$ is in $\mathcal{H}^s(\mathbb{T}^N)$ and set

$$\|\theta\|_s \overset{\text{def}}{=} \max_{1 \leq j \leq m} \|\theta_j\|_s, \quad \hat{\theta}_n \overset{\text{def}}{=} ((\hat{\theta}_1)_n, \ldots, (\hat{\theta}_m)_n) \quad \text{for any } n \in \mathbb{Z}^N$$

We say that $\theta$ is excellent if $\hat{\theta}_j$ is excellent for each $j.$

7.2. \textbf{Twisted cohomological equation over $A$ in high regularity.}

A crucial step in the iterative process is solving the twisted cohomological equation

$$A\omega - \omega \circ A = \theta$$

over $A$, which can be viewed as the linearized conjugacy equation. In this section we give preliminary results on solving this equation in high regularity. We start with a scalar cohomological equation over $A$ twisted by $\lambda \in \mathbb{C} \setminus \{0, 1\},$

$$\lambda \omega - \omega \circ A = \theta.$$ 

The next lemma shows that the obstructions to solving it in $C^\infty$ category are sums of Fourier coefficients of $\theta$ along the orbits of $\tilde{A}.$ Moreover, for any $C^\infty$ function $\theta$ there is a well behaved splitting $\theta = \theta^\mu + \theta^*$, where $\theta^\mu$ can be view as a projection to the space of twisted coboundaries and $\theta^*$ as the error. A similar result was proved for ergodic toral automorphisms in [DKt10] and used for establishing $C^\infty$ local rigidity of some partially hyperbolic $\mathbb{Z}^k$ actions. We prove the result for hyperbolic case to keep our exposition self-contained and get a better constant $\sigma(\lambda).$
Lemma 7.1. For a function $\theta : \mathbb{T}^N \to \mathbb{C}$ in $H^a(\mathbb{T}^N)$ and $\lambda \in \mathbb{C}\{0,1\}$ we define

$$D_\theta(n) = \sum_{i=-\infty}^{\infty} \lambda^{-(i+1)} \hat{\theta}_{\lambda^n}.$$ 

Suppose $a \geq \sigma(\lambda) = \frac{\log|\lambda|}{\log \rho} + 1$, where $\rho > 1$ is the expansion rate of $\hat{A}$ from (7.1). Then

(i) The sum $D_\theta(n)$ converges absolutely for any $n \neq 0$; moreover the function

$$\theta^* \overset{\text{def}}{=} \sum_{n \in M} D_\theta(n) e^{2\pi i n \cdot x},$$

where $M$ is from (7.2), is in $H^a(\mathbb{T}^N)$ with the estimate $\|\theta^*\|_a \leq K_{a,\lambda}\|\theta\|_a$.

(ii) If $D_\theta(n) = 0$ for any $n \neq 0$, then the equation (7.5) has a solution $\omega \in H^a(\mathbb{T}^N)$ with the estimate

$$\|\omega\|_a \leq K_{r,\lambda}\|\theta\|_a.$$

(iii) If the equation (7.5) has a solution $\omega \in H^{\sigma(\lambda)}(\mathbb{T}^N)$, then $D_\theta(n) = 0$ for any $n \neq 0$.

(iv) For $\theta^* = \theta - \theta^*$ the equation:

$$\lambda \omega - \omega \circ A = \theta^*$$

has a solution $\omega \in H^a(\mathbb{T}^N)$ with the estimate $\|\omega\|_a \leq K_{r,\lambda}\|\theta\|_a$.

Remark 7.2. We emphasize that the existence of $\theta^*$ requires a high regularity of $\theta$. In fact, for any $b \leq \sigma(\lambda)$, we have to estimate it as $\|\theta^*\|_b \leq K_{a,\lambda}\|\theta\|_{\sigma(\lambda)}$.

Proof. We define

$$D_\theta(n)_+ = \sum_{i \geq 1} \lambda^{-(i+1)} \hat{\theta}_{\lambda^n} \quad \text{and} \quad D_\theta(n)_- = -\sum_{i \leq 0} \lambda^{-(i+1)} \hat{\theta}_{\lambda^n}.$$

(i). Let $n \in M$. The inequality $\|\pi_{E^*}(n)\| \geq \frac{1}{2}\|n\|$ we obtain

$$|D_\theta(n)_-| \leq \|\theta\|_a \sum_{i \leq 0} |\lambda|^{-(i+1)} \|\hat{A} n\|^{-a} \leq \|\theta\|_a \sum_{i \leq 0} |\lambda|^{-(i+1)} \|\hat{E^*}(\hat{A} n)\|^{-a}$$

$$(7.6) \leq \|\theta\|_a C^{-a} \sum_{i \leq 0} |\lambda|^{-(i+1)} \rho^a \|\hat{E^*}(n)\|^{-a} \overset{(1)}{\leq} K_{a,\lambda}\|\theta\|_a \|n\|^{-a}.$$ 

Here in (1) convergence is guaranteed by $a > \frac{\log|\lambda|}{\log \rho}$. The sum $D_\theta(n)_+$ can be estimated similarly using the inequality $\|\pi_{E^*}(\hat{A} n)\| \geq \frac{1}{2}\|\hat{A} n\|$. Hence we get

$$\|\theta^*\|_a \leq K_{a,\lambda}\|\theta\|_a.$$

For any $z \in \mathbb{Z}^N$ and $k \in \mathbb{Z}$, we see that

$$(7.7) \quad D_\theta(\hat{A}^k z) = \lambda^k D_\theta(z).$$

This shows that $D_\theta(n)$ converges absolutely for any $n \neq 0$. 
(ii) In the dual space the equation $\lambda \omega - \omega \circ A = \theta$ has he form

$$\lambda \hat{\omega}_n - \hat{\omega}_{\hat{A}n} = \hat{\theta}_n, \quad \forall \ n \in \mathbb{Z}^N.$$  

For $n = 0$, we let $\hat{\omega}_0 = \frac{\hat{\theta}_0}{\hat{\lambda} - 1}$. For any $n \neq 0$, let $\hat{\omega}_n = D_\theta(n)$. Then $\omega = \sum_{n \in \mathbb{Z}^N} \hat{\omega}_n e^{2\pi i n}$ is a formal solution. Next, we obtain its Sobolev estimates. If $\|\pi_{\tilde{E}^+}(n)\| \geq \frac{1}{2} \|n\|$, then from (7.6) we have

$$|\hat{\omega}_n| \cdot \|n\|^\sigma \leq K_{a,\lambda}.$$  

If $\|\pi_{\tilde{E}^+}(\tilde{A}n)\| \geq \frac{1}{2} \|\tilde{A}n\|$, then the assumption $D_\theta(n) = 0$ implies that $\hat{\omega}_n = D_\theta(n)$. The arguments in (i) show that (7.8) still holds.

(iii) By (i) and (7.7) we have: for any $n \neq 0$

$$D_\theta(n) = D_{\lambda \omega - \omega \circ A}(n) = \lambda D_{\omega}(n) - D_{\omega}(\tilde{A}n) = \lambda D_{\omega}(n) - \lambda D_{\omega}(n) = 0.$$  

(iv) It is clear that $D_{\theta^*}(n) = D_{\theta - \theta^*}(n) = D_{\theta}(n) - D_{\theta^*}(n) = 0$ for any $n \neq 0$. Then the result follows from (ii). \qed

Now we extend Lemma 7.1 to the vector valued case. We consider the equation

$$A_i \omega - \omega \circ A = \theta$$

with the twist given by the restriction $A_i = A|E^i$, where $E^i$, $i = 1, \ldots, L$, is a subspace of the splitting (3.1). We note that any eigenvalue $\lambda$ of $A_i$ satisfies $|\lambda| = \rho_i$.

**Lemma 7.3.** Let $\rho > 1$ be the expansion rate for $\tilde{A}$ from (7.1) and let

$$\sigma = \max_{i=1,\ldots,L} \left( \frac{|\log \rho_i|}{\log \rho} + 1 \right) N + N + 2.$$  

Then for any $i = 1, \ldots, L$ and any $C^\infty$ map $\theta : T^N \to \mathbb{C}^N$, there is a splitting of $\theta$

$$\theta = \theta^* + \theta^\ast$$

such that the equation:

$$A_i \omega - \omega \circ A = \theta^*$$

has a $C^\infty$ solution $\omega$ with estimates

$$\|\omega\|_{C^r} \leq K_r \|\theta\|_{C^{r+\sigma}}, \quad \forall \ r \geq 0;$$

and $\theta^\ast : T^N \to \mathbb{C}^N$ is an excellent $C^\infty$ map so that for any $r \geq 0$

$$\|\theta^\ast\|_{C^r} \leq K_r \|\theta\|_{C^{r+\sigma}} \quad \text{and} \quad \|\theta^\ast\|_{r+\sigma - 2 - N} \leq K_{r+\sigma - 2 - N}.$$  

**Proof.** If $A_i$ is semisimple, then the conclusion follows directly from Lemma 7.1 as the equation (7.10) splits into finitely many equations of the type

$$\lambda_j \omega_j - \omega_j \circ A = (\theta_j)^i$$

where $\theta_j$ is a coordinate function of $\theta$ and $\lambda_j$ is the corresponding eigenvalue of $A_i$.  


If $A_i$ is not semisimple, we choose a basis in which $A_i$ is in its Jordan normal form with some nontrivial Jordan blocks. We note that the excellency of maps is preserved under the change of basis. Let $J = (J_{i,j})$ to be an $m \times m$ Jordan block of $A_i$ corresponding to an eigenvalue $\lambda$ with $|\lambda| = \rho_i$, that is, $J_{i,l} = \lambda$ for all $1 \leq l \leq m$ and $\lambda_{l,l+1} = 1$ for all $1 \leq l \leq m - 1$. Then equation \((7.10)\) splits into equations of the form
\[(7.11)\]
\[J \Omega - \Omega \circ A = \Theta^*,\]
corresponding to the Jordan blocks $J$. Each equation \((7.11)\) further splits into the following $m$ equations:
\[
\lambda \Omega_j - \Omega_j \circ A + \Omega_{j+1} = (\Theta^*)_j, \quad \text{and} \quad 
\lambda \Omega_m - \Omega_m \circ A = (\Theta^*_m) = (\Theta_m)^*,
\]
$1 \leq j \leq m - 1$. For the $m$-th equation, Lemma \(7.1\) gives the splitting
\[\Theta_m = \lambda \Omega_m - \Omega_m \circ A + (\Theta^*_m)\]
where $\Omega_m$, $(\Theta^*_m) = (\Theta_m)^*$, and $(\Theta^*_m) = \lambda \Omega_m - \Omega_m \circ A$ are $C^\infty$ functions satisfying the estimates:
\[
\max\{\|(\Theta^*_m)\|_r, \|\Omega_m\|_r\} \leq K_{r,m} \|\Theta\|_{r+\sigma(\rho_i)} , \quad \forall \, r \geq 0
\]
and $\Theta^*_m$ is excellent.

Now we proceed by induction. Fix $1 \leq k \leq m - 1$ and assume that for all $k+1 \leq j \leq m$ we already have the splitting
\[\Theta_j = \lambda \Omega_j - \Omega_j \circ A + \Omega_{j+1} + (\Theta^*)_j\]
where $\Omega_j$, $\Theta_j$, and $(\Theta^*)_j = \lambda \Omega_j - \Omega_j \circ A + \Omega_{j+1}$ are $C^\infty$ functions satisfying the estimates:
\[(7.12)\]
\[
\max\{\|\Omega_j\|_r, \|(\Theta^*)_j\|_r\} \leq K_{r,j} \|\Theta\|_{r+(m-j+1)\sigma(\rho_i)} , \quad \forall \, r \geq 0
\]
and $(\Theta^*)_j$ is excellent. By Lemma \(7.1\) we obtain the splitting
\[\Theta_k - \Omega_k + 1 = \lambda \Omega_k - \Omega_k \circ A + (\Theta_k - \Omega_k + 1)^*\]
where $\Omega_k$, $(\Theta^*)_k = (\Theta_k - \Omega_k + 1)^*$, and $(\Theta^*)_k = \lambda \Omega_k - \Omega_k \circ A + \Omega_k + 1$ are $C^\infty$ functions satisfying the estimates following from \((7.12)\):
\[
\max\{\|\Omega_k\|_r, \|(\Theta^*)_k\|_r\} \leq K_{r,k} \|\Theta\|_{r+(m-k+1)\sigma(\rho_i)} , \quad \forall \, r \geq 0
\]
and $(\Theta^*)_k$ is excellent. Let $\Omega$, $\Theta^*$ and $\Theta^*$ be maps with coordinate functions $\Omega_j$, $(\Theta^*)_j$ and $(\Theta^*)_j$, $1 \leq j \leq m$ respectively. Hence we show that there is a splitting of $\Theta$
\[\Theta = \Theta^* + \Theta^*\]
such that the equation \((7.11)\) has a $C^\infty$ solution $\Omega$ with estimates.
\[
\max\{\|\Theta^*\|_r, \|\Omega\|_r\} \leq K_r \|\Theta\|_{r+m\sigma(\rho_i)} , \quad \forall \, r \geq 0
\]
This can be repeated for all corresponding blocks of $A$. Since the maximal size of a Jordan block is bounded by $N$, we obtain estimates for the $\|\cdot\|_r$ norms of $\omega$ and $\theta^*$. This implies estimates for the $\|\cdot\|_{C^r}$ norms as well by \((7.3)\). \qed
Theorem 7.4. Let $A$ be weakly irreducible hyperbolic automorphism of $\mathbb{T}^N$. Suppose that

$$Ah - h \circ A = R + \Psi,$$

(7.13)

where maps $h, \Psi : \mathbb{T}^N \to \mathbb{R}^N$ are $C^{1+a}$ and $R : \mathbb{T}^N \to \mathbb{R}^N$ is $C^\infty$.

Then there exist $C^\infty$ maps $\omega, \Phi : \mathbb{T}^N \to \mathbb{R}^N$ satisfying the equation

$$R = A\omega - \omega \circ A + \Phi,$$

(7.14)

and the estimates

$$\|\omega\|_{Cr} \leq K_r \|R\|_{Cr+a},$$

$$\|\Phi\|_{C^0} \leq K_{l,a}(\|\Psi\|_{C^{1+a}})\frac{(2N)^{N+1}}{l^{N+1}} \left(\|R\|_{Cr+a}\right)^{2N+2}$$

for any $r \geq 0$ and $l > N + 2$, where $\sigma$ is given by (7.9).

For traditional KAM iteration scheme, the convergence requires the error $\Phi$ in solving the twisted coboundary (7.14) to be small compared with $R$. This is established by showing that $\Phi$ is tame with respect to $\Psi$, which is almost quadratically small with respect to $R$. Tameness means that the $C^r$ norm of $\Phi$ can be bounded by the $C^{r+p}$ norm of $\Psi$, where $r$ is arbitrarily large while $p$ is a constant.

One difficulty in our setting is that the estimate of $\Phi$ depends on $\Psi$ and $R$ rather than on $\Psi$ only. This results in technical issues in proving convergence of the iterative procedure, and so the traditional KAM scheme fails to work. We resolve this issue by introducing a parameter $l$ when estimating $\|\Phi\|_{C^0}$. If the parameters are well chosen, the constructed approximation behaves as if it were tame.

The main difficulty in estimating $\Phi$ in our setting is that low regularity of $h$ yields smallness of $\Psi$ only in $C^{1+\text{H"older}}$ norm, see Lemma 8.4 and equation (8.12). This does not allow us to directly estimate orbit sums of Fourier coefficients and split $R$ into a smooth coboundary $R^\circ = A\omega - \omega \circ A$ and an error term $R^* = \Phi$, see Remark 7.2. To overcome this problem we use the splitting $\mathbb{R}^N = \bigoplus E_i$ to decompose the equation (7.13) and then differentiate $i^{th}$ component along directions in $E_i$. This allows us to “balance” the twist (up to a polynomial growth of Jordan blocks) and analyze the differentiated equation using H"older regularity. This is done in the following Lemma 7.5. After that, we establish Lemma 7.6 to relate Fourier coefficients of a function and its directional derivatives. We then complete the proof of Theorem 7.4 in Section 7.5.
Now we begin the analysis of the differentiated equation \((7.13)\). For any \(1 \leq i \leq L\) and any unit vector \(u_0 \in E^i\), we consider unit vectors \(u_k\) and scalars \(a_k, k \in \mathbb{Z}\), given by

\[(7.15) \quad u_k = \frac{A_i^k u_0}{\|A_i^k u_0\|} \quad \text{and} \quad a_k = \|A_i u_k\| = \frac{\|A_i^{k+1} u_0\|}{\|A_i^k u_0\|} \quad \text{so that} \quad A_i u_k = a_k u_{k+1}.\]

We define a sequence of matrices \(P_k \in GL(N_i, \mathbb{R})\) which commute with \(A_i\) and satisfy the recursive equation

\[(7.16) \quad P_{k+1} = a_k A_i^{-1} P_k.\]

Specifically, we set

\[(7.17) \quad P_0 = \text{Id} \quad \text{and} \quad P_k = \begin{cases} a_0 \cdots a_{k-1} A_i^{-k} = \|A_i^k u_0\| A_i^{-k}, & k > 0, \\ (a_{-1} \cdots a_{-k})^{-1} A_i^{k} = \|A_i^{-k} u_0\| A_i^{k}, & k < 0. \end{cases}\]

**Lemma 7.5.** Let \(\varphi_k : \mathbb{T}^N \to \mathbb{R}^{N_i}\) be a sequence of maps in \(\mathcal{H}^a(\mathbb{T}^N), a > 0\), satisfying \(\|\varphi_k\|_a \leq b\) for all \(k \in \mathbb{Z}\), let \(P_k \in GL(N_i, \mathbb{R})\) be as in \((7.17)\), and let

\[S(n) = \sum_{k \leq 0} P_k (\tilde{\varphi}_k) \tilde{\lambda}^k n.\]

(i) For any \(n \in M\) the sum \(S(n)\) converges absolutely in \(C^N\) with the estimate

\[\|S(n)\| \leq K_a b \|n\|^{-a}.\]

(ii) If \(h_k : \mathbb{T}^N \to \mathbb{R}^{N_i}\) is another sequence in \(\mathcal{H}^a(\mathbb{T}^N)\) so that for all \(k \in \mathbb{Z}\) we have

\[\|h_k\|_a \leq c\] and

\[(7.18) \quad A_i h_k - a_k h_{k+1} \circ A = \varphi_k;\]

then \(S(n) = 0\) for every \(n \in M\).

**Proof.** (i). Since all eigenvalues of \(A_i\) have the same modulus \(\rho_i\), we have \((3.3)\), and so there exists a constant \(C\) such that all \(P_k\) satisfy the polynomial estimate

\[(7.19) \quad \|P_k\| \leq \|A_i^k\| \cdot \|A_i^{-k}\| \leq C(|k| + 1)^{2N} =: p(|k|), \quad \text{for all} \quad k \in \mathbb{Z}.\]

Let \(n \in M\). We write \(\varphi_k = (\varphi_{k,1}, \cdots, \varphi_{k,N_i})\) and set

\[S(n)_+ = \sum_{k \leq 1} P_k (\tilde{\varphi}_k) \tilde{\lambda}^k n \quad \text{and} \quad S(n)_- = \sum_{k \leq 0} P_k (\tilde{\varphi}_k) \tilde{\lambda}^k n.\]

Using the assumption \(\|\varphi_k\|_a \leq b\), estimates \((7.19)\) and \((7.1)\), and the inequality \(\|\pi_{E^*}(n)\| \geq \frac{1}{2} \|n\|\) we obtain

\[\|S(n)_-\| \leq \sum_{k \leq 0} \|P_k\| \max_{1 \leq j \leq m} |(\tilde{\varphi}_{k,j})| \tilde{\lambda}^k n | \leq \sum_{k \leq 0} \|\varphi_k\|_a \|P_k\| \|\tilde{\lambda}^k n\|^{-a}
\leq b \sum_{k \leq 0} p(|k|) \|\pi_{E^*}(\tilde{\lambda}^k n)\|^{-a} \leq b C^{-a} \sum_{k \leq 0} p(|k|) \rho^k \|\pi_{E^*}(n)\|^{-a}
\leq K_a b \|n\|^{-a}.\]
The sum $S(n)_+$ can be estimated similarly using the inequality $\|\pi_{E^u}(\hat{A}n)\| \geq \frac{1}{2}\|\hat{A}n\|$. 

(ii) Let $n \in M$. From the equation (7.18) we obtain that for any $k \in \mathbb{Z}$

$$P_k \varphi_k \circ A^k = P_k A_i h_k \circ A^k - a_k P_k h_{k+1} \circ A^{k+1}$$

Summing from $-m$ to $j$ and observing that the sum on the right is telescoping as $a_k P_k = A_i P_{k+1} = P_{k+1} A_i$ by the choice of $P_k$ in \textbf{(7.16)}, we obtain

$$\sum_{k=-m}^{j} P_k \varphi_k \circ A^k = A_i P_m h_{-m} \circ A^{-m} - a_j P_j h_{j+1} \circ A^{j+1}.$$ 

Taking Fourier coefficients and noting that $(\hat{\theta \circ A^k})_n = \hat{\hat{\theta}}_{k^n}$ we obtain

$$\sum_{k=-m}^{j} P_k(\hat{\varphi}_k)_{\hat{\theta}^k n} = A_i P_m(\hat{h}_{-m})_{\hat{\theta}^{-m} n} - a_j P_j(\hat{h}_{j+1})_{\hat{\theta}^{j+1} n}.$$ 

Since the series $\sum_{k \in \mathbb{Z}} P_k(\hat{h}_k)_{\hat{\theta}^k n}$ converges by part (i), we have $P_k(\hat{h}_k)_{\hat{\theta}^k n} \to 0$ as $k \to \pm \infty$ and hence, as $a_k$ are bounded,

$$a_j P_j(\hat{h}_{j+1})_{\hat{\theta}^{j+1} n} \to 0, \quad \text{as } j \to \infty; \quad \text{and} \quad A_i P_m(\hat{h}_m)_{\hat{\theta}^m n} \to 0, \quad \text{as } m \to -\infty.$$ 

We conclude that $S(n) = 0$. \hfill \Box

7.4. Directional derivatives. In this section we establish some estimates for Fourier coefficients of a $C^1$ function $\theta : \mathbb{T}^N \to \mathbb{R}$ via Fourier coefficients of its directional derivatives along a subspace $E^i$ of the splitting \textbf{(3.1)}. This relies on weak irreducibility of $A$.

For any $v \in \mathbb{R}^N$ with $\|v\| = 1$, we denote the directional derivative of $\theta$ along $v$ by $\theta_v$.

\textbf{Lemma 7.6.} Let $A$ be a weakly irreducible integer matrix and let $v_{i,j}$, $j = 1, \ldots, N_i$, be an orthonormal basis of a subspace $E^i$ from \textbf{(3.1)}. Then there exists a constant $K = K(A)$ such that for any $i = 1, \ldots, L$ and any $C^1$ function $\theta : \mathbb{T}^N \to \mathbb{R}$,

$$|\hat{\theta}_n| \leq K \sum_{j=1}^{N_i} |(\hat{\theta}_{v_{i,j}})_n| \cdot \|n\|^N \quad \text{for all } n \in \mathbb{Z}^N \setminus 0.$$ 

\textbf{Proof.} We denote by $\|.\|$ the standard Euclidean norm in $\mathbb{R}^N$. Since $\theta$ is $C^1$, we have

$$2\pi i (n \cdot v_{i,j}) \hat{\theta}_n = (\hat{\theta}_{v_{i,j}})_n, \quad 1 \leq j \leq N_i.$$ 

Adding over $j$ we obtain that for any $n \in \mathbb{Z}^N \setminus 0$ we have

$$|\hat{\theta}_n| = \frac{\sum_{j=1}^{N_i} |(\hat{\theta}_{v_{i,j}})_n|}{2\pi \sum_{j=1}^{N_i} |n \cdot v_{i,j}|} \leq \frac{\sum_{j=1}^{N_i} |(\hat{\theta}_{v_{i,j}})_n|}{2\pi \|\pi_{E^i} n\|}.$$
since for an orthonormal basis \( v_{i,j} \) we have \( \sum_{j=1}^{N} |n \cdot v_{i,j}| \geq \| \pi_{E^i} n \|. \) Since \( \| \pi_{E^i} n \| = d(n, (E^i)^⊥) \), to complete the proof it remains to show that \( d(n, (E^i)^⊥) \geq K' \| n \|^{-N} \).

Since \( A \) is weakly irreducible, so is the transpose \( A^T \). This follows from Lemma 3.3 which gives an equivalent condition for weak irreducibility in terms of the characteristic polynomial. We denote the splitting (3.1) for \( A^T \) by \( \mathbb{R}^N = E^1_i \oplus \cdots \oplus E^L_i \) and similarly let \( \hat{E}^i_i = \oplus_{j \neq i} E^i_j \). Then we obtain \( (E^i)^⊥ = \hat{E}^i_i \). Indeed, the polynomial

\[
p_i(x) = \prod_{|\lambda|=\rho_i} (x - \lambda)^N,\]

where the product is over all eigenvalues of \( A \) of modulus \( \rho_i \), is real and

\[
(E^i)^⊥ = (\ker p_i(A))^⊥ = \text{range}(p_i(A)^T) = \text{range}(p_i(A^T)) = \hat{E}^i_i,
\]

since \( p_i(A^T) \) is invertible on \( \hat{E}^i_i \). Now the desired inequality

\[
d(n, (E^i)^⊥) = d(n, \hat{E}^i_i) \geq K' \| n \|^{-N}
\]

follows from Katznelson’s Lemma below. We apply it to \( A^T \) with the invariant splitting \( \mathbb{R}^N = \hat{E}^i_i \oplus E^i_i \) and note that \( \hat{E}^i_i \cap \mathbb{Z}^N = \{0\} \) by weak irreducibility of \( A^T \).

\[ \square \]

Lemma 7.7 (Katznelson’s Lemma). Let \( A \) be an \( N \times N \) integer matrix. Assume that \( \mathbb{R}^N \) splits as \( \mathbb{R}^N = V_1 \oplus V_2 \) with \( V_1 \) and \( V_2 \) invariant under \( A \) and such that \( A|_{V_1} \) and \( A|_{V_2} \) have no common eigenvalues. If \( V_1 \cap \mathbb{Z}^N = \{0\} \), then there exists a constant \( K \) such that

\[
d(n, V_1) \geq K \| n \|^{-N}\]

for all \( 0 \neq n \in \mathbb{Z}^N \),

where \( \| v \| \) denotes Euclidean norm and \( d \) is Euclidean distance.

See e.g. [DK10, Lemma 4.1] for a proof.

7.5. Proof of Theorem 7.4. Using the splitting \( \mathbb{R}^N = \oplus E^i_i \) we decompose (7.13) into equations

\[
(A_i h_i - h_i \circ A = \mathcal{R}_i + \Psi_i, \quad i = 1, \ldots, L
\]

where \( h_i, \mathcal{R}_i \) and \( \Psi_i \) are coordinate maps in the of \( h, \mathcal{R} \) and \( \Psi \) respectively.

By Lemma 7.3 there is an excellent \( C^\infty \) map \( \mathcal{R}_i^* \) with estimates

\[
\| \mathcal{R}_i^* \|_{C^r} \leq K_r \| \mathcal{R}_i \|_{C^{r+\sigma}}, \quad \| \mathcal{R}_i^* \|_{r} \leq K_r \| \mathcal{R}_i \|_{r+\sigma-N-2}
\]

for any \( r \geq 0 \), such that the equation:

\[
(A_i \omega_i - \omega_i \circ A = \mathcal{R}_i + \mathcal{R}_i^*, \quad i = 1, \ldots, L
\]

has a \( C^\infty \) solution \( \omega_i \) with estimates

\[
\| \omega_i \|_{C^r} \leq K_r \| \mathcal{R}_i \|_{C^{r+\sigma}}, \quad \forall r \geq 0.
\]

Let \( \omega \) be the map with coordinate maps \( \omega_i \).
We obtain from (7.20) and (7.22) that $C^{1+a}$ maps $p_i = h_i - \omega_i$ and $\Lambda_i = -R_i^* + \Psi_i$ satisfy

$$A_i p_i - p_i \circ A = \Lambda_i.$$  

We fix $1 \leq i \leq L$ and an orthonormal basis $v_{i,j}$ of $E^i$. We fix $1 \leq j \leq N_i$ and, as in (7.15), consider unit vectors $u_0 = v_{i,j}$ and $u_k = \frac{A^k u_0}{\|A^k u_0\|}$, and let $a_k = \|A u_k\|$, $k \in \mathbb{Z}$. Taking the derivative of the previous equation in the direction of $u_k$ we obtain equations

$$A_i (p_i)_{u_k} - a_k (p_i)_{u_{k+1}} \circ A = (\Lambda_i)_{u_k}, \quad \forall k \in \mathbb{Z}.$$  

We note that for any $k \in \mathbb{Z}$ the maps $(p_i)_{u_k}$ and $(\Lambda_i)_{u_k}$ are in $C^a$ and hence in $H^a$, as we recall that for any function $g$ by (7.3) we have

$$(7.23) \quad \|g_{u_k}\|_a \leq K \|g_{u_k}\|_{C^a} \leq K_1 \|g\|_{C^{1+a}}.$$  

Now we use (ii) of Lemma 7.5 with $h_k = (p_i)_{u_k}$, $\phi_k = (\Lambda_i)_{u_k}$, and $P_k$ as defined in (7.17) to obtain that for any $n \in M$

$$\sum_{k \in \mathbb{Z}} P_k((\Psi_i)_{u_k})_{\tilde{A}^k n} - \sum_{k \in \mathbb{Z}} P_k((R_i^*)_{u_k})_{\tilde{A}^k n} = \sum_{k \in \mathbb{Z}} P_k((\Lambda_i)_{u_k})_{\tilde{A}^k n} = 0.$$  

Since $(R_i^*)_{u_k}$ is excellent, for each $k \in \mathbb{Z}$ we have

$$\sum_{k \in \mathbb{Z}} P_k((\Psi_i)_{u_k})_{\tilde{A}^k n} = \sum_{k \in \mathbb{Z}} P_k((R_i^*)_{u_k})_{\tilde{A}^k n} = ((R_i^*)_{u_0})_{n},$$

for any $n \in M$, which gives

$$(7.24) \quad \|((R_i^*)_{u_0})_n\|^{(1)} \leq K_a \max_{k \in \mathbb{Z}} \{\|(\Psi_i)_{u_k}\|_a\}\|n\|^{(2)} - a \leq K_{a,1}\|\Psi_i\|_{C^{1+a}}\|n\|^{(2)} - a.$$  

Here in (1) we use (i) of Lemma 7.5 and in (2) we use (7.23).

We conclude that for any $v_{i,j}$, $1 \leq j \leq N_i$, we have

$$(7.25) \quad \|((R_i^*)_{v_{i,j}})_n\| \leq K_a\|\Psi_i\|_{C^{1+a}}\|n\|^{(2)} - a, \quad \forall n \in M.$$  

Finally, using Lemma 7.6 and (7.24), we obtain that for any $n \in M$

$$\|(R_i^*)_n\| \leq K \sum_{j=1}^{N_i} \|((R_i^*)_{v_{i,j}})_n\|^{N} \leq K_a\|\Psi_i\|_{C^{1+a}}\|n\|^{N-a} \leq K_a\|\Psi_i\|_{C^{1+a}}\|n\|^{N}.$$  

(7.25)
Now for any \( r > N + 2 \) and any \( n \in M \) we can estimate splitting the exponent of the first term as \( \alpha \) and \( 1 - \alpha \) in the way to get the total the exponent of \( \| n \| \) be zero
\[
\| (R_i^*)_n \| \cdot \| n \|^{N+2} = \| (R_i^*)_n \| \cdot \| n \|^{N+2} \\
\leq \left( K_a \| \Psi_i \|_{C^{1+a}} \| n \|^{N} \right)^{\frac{1 - 2N}{1 + N}} \left( \| n \|^{-1} \| R_i^* \| \| n \|^{N+2} \right)^{\frac{2N+2}{1 + N}} \\
= K_a^{\frac{1 - 2N}{1 + N}} \left( \| \Psi_i \|_{C^{1+a}} \right)^{\frac{1 - 2N}{1 + N}} \left( \| R_i^* \| \| n \|^{N+2} \right)^{\frac{2N+2}{1 + N}} \\
\leq K_{la} \left( \| \Psi_i \|_{C^{1+a}} \right)^{\frac{1 - 2N}{1 + N}} \left( \| R_i \|_{C^{1+a}} \right)^{\frac{2N+2}{1 + N}}.
\]
Here in (1) we use that \( R_i^* \) is \( C^\infty \) and (1.25); in (2) we use (1.3); in (3) we use (1.21).

Then by (1.3) we get
\[
\| R_i^* \|_{C^0} \leq C \| R_i \|_{N+2} \leq K_{la} \left( \| \Psi_i \|_{C^{1+a}} \right)^{\frac{1 - 2N}{1 + N}} \left( \| R_i \|_{C^{1+a}} \right)^{\frac{2N+2}{1 + N}}.
\]
Finally, we denote by \( \Phi \) the map with coordinate maps \( R_i^* \).

8. Proof of Theorem 1.3

In this section we complete the proof of Theorem 1.3 using an iterative process. The main part is the inductive step given by Proposition 8.3. We start with a sufficiently small perturbation \( f_n \) of \( A \) which is \( C^1 \) conjugate to \( A \). We construct a smaller perturbation \( f_{n+1} \) which is smoothly conjugate to \( f_n \). The conjugacy \( H_{n+1} \) between \( f_n \) and \( f_{n+1} \) is obtained using Theorem 7.4. Then the iterative process is set up so that \( f_n \) converges to \( A \) and \( H_1 \circ \cdots \circ H_{n+1} \) converge in sufficiently high regularity.

8.1. Iterative step and error estimate.

We recall the following results, which will be used the proof of Proposition 8.3.

Lemma 8.1. [dLO98, Propositions 5.5] For any \( r \geq 1 \) there exists a constant \( M_r \) such that for any \( h, g \in C^r(M) \),
\[
\| h \circ g \|_{C^r} \leq M_r \left( 1 + \| g \|_{C^1}^{-1} \right) \left( \| h \|_{C^1} \| g \|_{C^r} + \| h \|_{C^r} \| g \|_{C^1} \right) + \| h \|_{C^0}.
\]

Lemma 8.2. [La93, Lemma AII.26] There is \( d > 0 \) such that for any \( h \in C^r(M) \), if \( \| h - I \|_{C^1} \leq d \) then \( h^{-1} \) exists with the estimate \( \| h^{-1} - I \|_{C^r} \leq K_r \| h - I \|_{C^r} \).

Proposition 8.3. Let \( A \) be a weakly irreducible Anosov automorphism of \( \mathbb{T}^N \). Let \( \beta = \frac{\beta_0}{2} \), where \( \beta_0 \) is as in Theorem 1.1. There exists \( 0 < c < \frac{1}{2} \) such that for any \( C^\infty \) perturbation \( f_n \) of \( A \) satisfying
\[
\| f_n - A \|_{C^{\sigma+2}} < c, \quad \text{where } \sigma \text{ is from Lemma 7.3}
\]
and the conjugacy equation
\[
H_n \circ f_n = A \circ H_n \quad \text{with a function } H_n \in C^1(\mathbb{T}^N) \quad \text{with } \| H_n - I \|_{C^0} \leq c
\]
the following holds. There exists $\omega_{n+1} \in C^\infty(T^N)$ so that the functions

$$(8.2) \quad \tilde{H}_{n+1} = I - \omega_{n+1}, \quad H_{n+1} = H_n \circ \tilde{H}_{n+1}, \quad f_{n+1} = \tilde{H}_{n+1}^{-1} \circ f_n \circ \tilde{H}_{n+1}$$

satisfy the new conjugacy equation

$$H_{n+1} \circ f_{n+1} = A \circ H_{n+1},$$

and we have the following estimates.

(i) For any $r \geq 0$ and any $t > 1$

$$\|\omega_{n+1}\|_{C^r} \leq K_r \min \{t^\sigma R_n\|_{C^r}, \|R_n\|_{C^{r+\alpha}}\}. \quad \text{where} \quad R_n = f_n - A.$$

(ii) For the new error $R_{n+1} = f_{n+1} - A$, we have

$$\|R_{n+1}\|_{C^0} \leq K \|R_n\|_{C^1} \|R_n\|_{C^0} + K t^{-\ell} \|R_n\|_{C^\ell}$$

$$+ K \ell (t^{-\ell+2} \|R_n\|_{C^\ell} + \|R_n\|_{C^2}^{1+\frac{\beta}{2}}) \left( \frac{t^{-2N}}{t^\alpha + 1} \right) \left( \frac{t^\alpha N + 2}{t^{N+2}} \right)$$

for any $t > 1$, $\ell \geq 0$ and $l > N + 2$; and also for any $r \geq 0$ we have

$$(8.3) \quad \|R_{n+1}\|_{C^r} \leq K_r t^\sigma \|R_n\|_{C^r} + K_r.$$

(iii) For the new conjugacy $H_{n+1}$, we have

$$(8.4) \quad \|H_{n+1} - I\|_{C^0} \leq K \|R_n\|_{C^0} + \|H_n - I\|_{C^0}$$

Proof. We denote $h_n = H_n - I$ and $R_n = f_n - A$ and, similarly to (6.4), we write the conjugacy equation (8.1) as

$$Ah_n - h_n \circ f_n = R_n$$

We can assume that $c < \delta$, where $\delta = \delta(\beta)$ is from Theorem 1.1 and that $\|H_n - I\|_{C^0} \leq c$ yields that $H$ is the conjugacy close to the identity. Then Theorem 1.1 gives the estimate

$$\|h_n\|_{C^{1+\beta}} \leq K \|R_n\|_{C^{1+\beta}}.$$

We define

$$(8.5) \quad \Omega_n = Ah_n - h_n \circ A, \quad \Theta_n = R_n - \Omega_n = h_n \circ A - h_n \circ f_n.$$
and hence
\[
\|D\Theta\|_{C^0} \leq \|A\| \|Dh(Ax) - Dh(fx)\|_{C^0} + \|Dh(fx) \circ DR(x)\|_{C^0}
\]
\[
\leq \|A\| \|Dh\|_{C^\beta} \|R\|_{C^0}^{\beta} + \|Dh\|_{C^0} \|DR\|_{C^0}
\]
\[
\leq \|A\| \|h\|_{C^{1+\beta}} \|R\|_{C^0}^{\beta} + \|h\|_{C^1} \|R\|_{C^1}.
\]
Since we also have \(\|\Theta\|_{C^0} \leq \|h\|_{C^1} \|R\|_{C^0}\), we conclude using (8.5) and \(\|R\|_{C^{1+\beta}} < 1\) that
\[
\|\Theta\|_{C^1} \leq \|A\| \|h\|_{C^{1+\beta}} \|R\|_{C^0}^{\beta} + \|h\|_{C^1} \|R\|_{C^1} \leq K \|R\|_{C^{1+\beta}}^{1+\beta}.
\]
To estimate the Hölder norm of \(D\Theta\), using equation * for any \(x, y \in \mathbb{T}^N\) we have
\[
D\Theta(x) - D\Theta(y) = (Dh(Ax) - Dh(Ay)) \circ A + Dh(fx) \circ (Df(y) - Df(x))
\]
\[
+ (Dh(fy) - Dh(fx)) \circ Df(y),
\]
and hence
\[
\|D\Theta(x) - D\Theta(y)\|
\]
\[
\leq \|A\| \|Dh(Ax) - Dh(Ay)\| + \|Dh(fx)\| \|Df(y) - Df(x)\|
\]
\[
+ \|Dh(fy) - Dh(fx)\| \|Df(y)\|
\]
\[
\leq \|A\| \|Dh\|_{C^\beta} \|Ax - Ay\|^{\beta} + \|h\|_{C^1} \|Df\|_{C^0} \|y - x\|^{\beta}
\]
\[
+ \|f\|_{C^1} \|Dh\|_{C^0} \|fx - fy\|^{\beta}
\]
\[
\leq \|A\| \|h\|_{C^{1+\beta}} \|x - y\|^{\beta} + \|h\|_{C^1} \|f\|_{C^{1+\beta}} \|y - x\|^{\beta}
\]
\[
+ \|f\|_{C^1} \|h\|_{C^{1+\beta}} \|f\|_{C^0} \|x - y\|^{\beta}.
\]
We conclude using (8.5) and \(\|f - A\|_{C^{1+\beta}} < 1\) that
\[
\|D\Theta\|_{C^0,\beta} \leq \|A\|^{1+\beta} \|h\|_{C^{1+\beta}} + \|h\|_{C^1} \|f\|_{C^{1+\beta}} + \|h\|_{C^{1+\beta}} \|f\|_{C^1}^{1+\beta} \leq K \|R\|_{C^{1+\beta}}.
\]
Therefore
\[
\|\Theta\|_{C^{1+\beta}} \leq \|\Theta\|_{C^1} + \|D\Theta\|_{C^0,\beta} \leq 2K \|R\|_{C^{1+\beta}}.
\]
Finally, we complete the proof of the lemma using an interpolation inequality
\[
\|\Theta\|_{C^{1+\beta}} \leq K \|\Theta\|_{C^1}^{\frac{1}{2}} \|\Theta\|_{C^{1+\beta}}^{\frac{1}{2}} \leq K_A \|R\|_{C^{1+\beta}}^{1+\beta}.
\]
\[\square\]
We recall that there exists a collection of smoothing operators \(s_t, t > 0\), such that for any \(s \geq s_1 \geq 0\) and \(s_2 \geq 0\), for any \(g \in C^s(\mathbb{T}^N)\) the following holds, see [DKt10] and [Ha82]:
\[
\|s_t g\|_{C^{s+s_2}} \leq K_{s,s_2} f^{s_2} \|g\|_{C^s}, \quad \text{and} \quad \|(I - s_t) g\|_{C^{s-s_1}} \leq K_{s,s_1} t^{-s_1} \|g\|_{C^s}.
\]
We write (8.6) as
\[ Ah_n - h_n \circ A = \Omega_n = R_n - \Theta_n = [s_t R_n] + [(I - s_t)R_n - \Theta_n] =: \mathcal{R} + \Psi \]
and apply Theorem 7.4 to get the new splitting and obtain the estimates:
\[ s_t R_n = A \omega_{n+1} - \omega_{n+1} \circ A + \Phi_n \]
where \( \omega_{n+1} \) and \( \Phi_n \) are \( C^\infty \) maps with the estimates:
\[
\begin{align*}
\| \omega_{n+1} \|_{C^r} &\leq K_r \| s_t(R_n) \|_{C^{r+\sigma}} \overset{(a)}{\leq} K_r \min \{ t^\sigma \| R_n \|_{C^r}, \| R_n \|_{C^{r+\sigma}} \}, \\
\| \Phi_n \|_{C^0} &\leq K_l(\| (I - s_t)R_n - \Theta_n \|_{C^{l+\frac{3}{2}}})^{\frac{l-2-N}{l+2-N}}(\| s_t R_n \|_{C^{l+\sigma}})^{\frac{2N+2}{l+2-N}} \\
&\leq K_l(\| (I - s_t)R_n \|_{C^2} + \| R_n \|_{C^2})^{\frac{l-2-N}{l+2-N}}(\| s_t R_n \|_{C^{l+\sigma}})^{\frac{2N+2}{l+2-N}} \overset{(b)}{\leq} K_l,t(t^{-l+2}\| R_n \|_{C^l} + \| R_n \|_{C^2})^{\frac{l-2-N}{l+2-N}}(t^\sigma \| R_n \|_{C^l})^{\frac{2N+2}{l+2-N}}
\end{align*}
\]
for any \( r, \ell \geq 0 \) and any \( l > N + 2 \). Here in (a) we use (8.11) and in (b) we use (8.14).

From equation (8.13) we obtain a \( C^r \) estimate for \( \Phi_n \) with \( r \geq 0 \)
\[
\| \Phi_n \|_{C^r} = \| A \omega_{n+1} - \omega_{n+1} \circ A - s_t R_n \|_{C^r} \leq K \| \omega_{n+1} \|_{C^r} + \| s_t R_n \|_{C^r} \overset{(1)}{\leq} K_r t^\sigma \| R_n \|_{C^r}.
\]
Here in (1) we use (8.11) and (8.14).

Let \( \bar{H}_{n+1} = I - \omega_{n+1} \). From (8.14) we can assume that \( \| \omega_{n+1} \|_{C^1} < \min \{ \frac{1}{2}, d \} \) (see Lemma 8.2) if \( c \) is sufficiently small. Hence \( \bar{H}_{n+1} \) is invertible. We estimate the new error
\[ R_{n+1} = f_{n+1} - A \]
by using
\[
\begin{align*}
f_{n+1} &= \bar{H}_{n+1}^{-1} \circ f_n \circ \bar{H}_{n+1} \Rightarrow \bar{H}_{n+1} \circ f_{n+1} = f_n \circ \bar{H}_{n+1} \\
&\Rightarrow (I - \omega_{n+1}) \circ f_{n+1} = f_n \circ \bar{H}_{n+1} \\
&\Rightarrow f_{n+1} = \omega_{n+1} \circ f_{n+1} + f_n \circ \bar{H}_{n+1}.
\end{align*}
\]
This gives
\[
\begin{align*}
R_{n+1} &= \omega_{n+1} \circ f_{n+1} + f_n \circ \bar{H}_{n+1} - A \\
&= \omega_{n+1} \circ f_{n+1} + (R_n + A) \circ (I - \omega_{n+1}) - A \\
&= \omega_{n+1} \circ f_{n+1} + R_n \circ (I - \omega_{n+1}) - A \circ \omega_{n+1}.
\end{align*}
\]
Hence we see that $R_{n+1}$ has three parts:

$$R_{n+1} = \frac{(\omega_{n+1} \circ f_{n+1} - \omega_{n+1} \circ A)}{\mathcal{E}_1} + \frac{(R_n \circ (I - \omega_{n+1}) - R_n)}{\mathcal{E}_2} + (\omega_{n+1} \circ A - A \circ \omega_{n+1} + R_n).$$

We note that

$$\|\mathcal{E}_1\|_{C^0} \leq \|\omega_{n+1}\|_{C^1} \|f_{n+1} - A\|_{C^0} \leq \frac{1}{2} \|R_{n+1}\|_{C^0},$$

$$\|\mathcal{E}_2\|_{C^0} \leq K \|R_n\|_{C^1} \|\omega_{n+1}\|_{C^0} \leq Kt^\sigma \|R_n\|_{C^1} \|R_n\|_{C^0};$$

and

$$\|\mathcal{E}_3\|_{C^0} = \|\Phi_n + (I - s_1)R_n\|_{C^0} \leq \|\Phi_n\|_{C^0} + \|(I - s_1)R_n\|_{C^0} \leq \|\Phi_n\|_{C^0} + 0 \leq \|\Phi_n\|_{C^0} + Kt^{-\ell} \|R_n\|_{C^\ell}$$

for any $\ell \geq 0$. Here in (0) we recall that $\|\omega_{n+1}\|_{C^1} < \frac{1}{2}$; in (1) we use (8.14); and in (2) we use (8.11).

Hence it follows that

$$\|R_{n+1}\|_{C^0} \leq \|\mathcal{E}_1\|_{C^0} + \|\mathcal{E}_2\|_{C^0} + \|\mathcal{E}_3\|_{C^0} \leq \frac{1}{2} \|R_{n+1}\|_{C^0} + \|\mathcal{E}_2\|_{C^0} + \|\mathcal{E}_3\|_{C^0},$$

which gives

$$\|R_{n+1}\|_{C^0} \leq 2\|\mathcal{E}_2\|_{C^0} + 2\|\mathcal{E}_3\|_{C^0} \leq Kt^\sigma \|R_n\|_{C^1} \|R_n\|_{C^0} + Kt^{-\ell} \|R_n\|_{C^\ell} + \|\Phi_n\|_{C^0} \leq Kt^\sigma \|R_n\|_{C^1} \|R_n\|_{C^0} + Kt^{-\ell} \|R_n\|_{C^\ell} + K_{t,\ell}(t^{-\ell+2} \|R_n\|_{C^\ell} + \|R_n\|_{C^0}^{1+\frac{\ell}{2}})^{\frac{1}{\ell+1}}(t^\sigma \|R_n\|_{C^1})^{\frac{2N+2}{\ell+1}}$$

for any $l > N + 2$. Here in (3) we use (8.13).

Now we estimate $\|R_{n+1}\|_{C^r}$. We note that

$$R_{n+1} = (I - \omega_{n+1})^{-1} \circ (R_n + A) \circ (I - \omega_{n+1}) - A = (I - \omega_{n+1})^{-1} \circ P - A.$$

By Lemma (8.1) we have

$$\|P\|_{C^r} \leq M_r \left(1 + \|I - \omega_{n+1}\|_{C^1}^{-1}\right) \cdot (\|R_n + A\|_{C^1} \|I - \omega_{n+1}\|_{C^r} + \|R_n + A\|_{C^r} \|I - \omega_{n+1}\|_{C^1} + \|R_n + A\|_{C^0} \leq K t^\sigma \|R_n\|_{C^r} + K_r, \quad \text{and} \quad \|P\|_{C^1} \leq K.$$


Here in (1) we use the fact that $\omega_{n+1}$ satisfies the estimates $\|\omega_{n+1}\|_{C^r} \leq K_r t^\sigma \|R_n\|_{C^r}$ (see (8.14)) and $\|\omega_{n+1}\|_{C^1} < \frac{1}{2}$. Using Lemma 8.2 this also implies that

$$\|(I - \omega_{n+1})^{-1}\|_{C^r} \leq K_r \|\omega_{n+1}\|_{C^r} \leq K_r t^\sigma \|R_n\|_{C^r}$$

and $\|(I - \omega_{n+1})^{-1}\|_{C^1} < 2$.

As a direct consequence of Lemma 8.1 and the above discussion we have

$$\|R_{n+1}\|_{C^r} \leq M_r (1 + \|P\|_{C^r}^{-1}) (\|(I - \omega_{n+1})^{-1}\|_{C^1} \|P\|_{C^r} + \|(I - \omega_{n+1})^{-1}\|_{C^r} \|P\|_{C^1}) + K$$

$$\leq K_r \|P\|_{C^r} + K r t^\sigma \|R_n\|_{C^r} + K$$

$$\leq K_r t^\sigma \|R_n\|_{C^r} + K r.$$

To get (8.4) we have

$$\|H_{n+1} - I\|_{C^0} = \|H_n \circ (I - \omega_{n+1}) - I\|_{C^0} \leq \|H_n \circ (I - \omega_{n+1}) - H_n\|_{C^0} + \|H_n - I\|_{C^0}$$

$$\leq \|H_n\|_{C^1} \|\omega_{n+1}\|_{C^0} + \|H_n - I\|_{C^0}$$

$$\leq K \|R_n\|_{C^r} + \|H_n - I\|_{C^0}.$$

Here in (1) we use (8.5) and (8.14).

\[\Box\]

8.2. The iteration scheme. First we note that by [dl92, Theorem 6.1] there exists $\sigma_0 = \sigma_0(A) \in \mathbb{N}$ such that if $H$ and $H^{-1}$ are $C^{\sigma_0}$ then $H$ and $H^{-1}$ are $C^\infty$.

To set up the iterative process we take $\ell$ sufficiently large so that the following holds

$$\ell \geq \max \left\{ \frac{3\sigma + 10}{1 - \frac{3}{\beta}}, \frac{24\sigma}{\beta}, 2(5 \max\{\sigma_0, \sigma\} + 1), 2(2\sigma + 5) \right\},$$

(8.16)

$$\left(1 + \frac{\beta}{2}\right) \left(1 - \frac{5}{\ell}\right) \left(\frac{\ell - 2 - N}{\ell + N}\right) - 2 \frac{2N + 2}{\ell + N} \geq 1 + \frac{\beta}{3}.$$

Now we construct $R_n, f_n, \omega_n$ and $H_n$ inductively as follows. For $n = 0$ we take

$$f_0 = f, \quad H_0 = H, \quad R_0 = f - A, \quad \omega_0 = 0, \quad \text{and define} \quad \epsilon_n = \epsilon^n$$

where $\gamma = 1 + \frac{\beta}{2}$ and $\epsilon > 0$ is sufficiently small so that the following holds

$$\|R_0\|_{C^0} \leq \epsilon_0 = \epsilon, \quad \|R_0\|_{C^\ell} \leq \epsilon_0^{-\ell}, \quad \|H_0 - I\|_{C^0} < \epsilon_0^\ell.$$

We note that $H_0 \in C^1(\mathbb{T}^N)$ by Theorem 1.1. Now we assume inductively that $H_n \in C^1(\mathbb{T}^N)$ satisfies the conjugacy equation

$$H_n \circ f_n = A \circ H_n$$

and that $H_n$ and $R_n = f_n - A$ satisfy

$$\|R_n\|_{C^0} \leq \epsilon_n, \quad \|R_n\|_{C^\ell} \leq \epsilon_n^{-\ell}, \quad \|H_n - I\|_{C^0} < \sum_{i=0}^{n-1} \epsilon_i^\ell.$$
By interpolation inequalities we have

\[(8.17) \quad \|R_n\|_{C^\sigma+2} \leq K_\ell \|R_n\|_{C^\sigma}^{\frac{\ell-2}{\ell}} \|R_n\|_{C^\sigma}^{\frac{2}{\ell}} < \epsilon_{n}^{-\frac{5+2\sigma}{\ell}} \leq \epsilon_{n}^\frac{1}{2},\]

provided \( \ell \geq 2(2\sigma+5) \). Here, and subsequently, we estimate various constants from above by \( \epsilon_{n}^{-\frac{1}{2}} \). This can be done since \( \ell \) is fixed, we can take \( \epsilon \) small enough. We also have

\[(8.18) \quad \|H_n - I\|_{C^0} < \sum_{i=0}^{n-1} \epsilon_{\ell i} \left< \epsilon_{\ell i}^{\frac{1}{2}} \right< \sum_{i=1}^{\infty} \left(\epsilon_{\ell i}^{\frac{1}{2}}\right)^i < 2\epsilon_{\ell}^{\frac{1}{2}}.\]

Then (8.17) and (8.18) allow us to use Proposition 8.3 to obtain the new iterates \( R_{n+1} \), \( f_{n+1} \), \( \omega_{n+1} \) and \( H_{n+1} \). Now we show that these iterates satisfy the inductive assumption and establish appropriate convergence.

8.3. Inductive estimates and convergence.

We use Proposition 8.3 with \( t_n = \epsilon_{n}^{-\frac{2}{\ell}} \) and \( \ell = \ell \).

1. \( C^\ell \) estimate for \( R_{n+1} \)

\[\|R_{n+1}\|_{C^\ell} \leq K_{\ell} t_n^\sigma \|R_n\|_{C^\ell} + K_{\ell} \leq K_{\ell} \epsilon_n^{-\frac{3\sigma}{\ell}} (\epsilon_n^{-1} + 1) \]

\[< \epsilon_n^{-1 - \frac{2}{\ell} - \frac{3\sigma}{\ell}} \leq \epsilon_n^{-1 - \frac{2}{\ell}} = \epsilon_{n+1},\]

provided \( \ell \geq \frac{24\sigma}{\beta} \).

2. \( C^0 \) estimate for \( R_{n+1} \)

\[\|R_{n+1}\|_{C^0} \leq K_{\ell} t_n^\sigma \|R_n\|_{C^2} + K_{\ell} t_n^{-\frac{\ell}{2}} \|R_n\|_{C^\ell} + K_{\ell} (t_n^{-\frac{\ell}{2}+2} \|R_n\|_{C^\ell} + \|R_n\|_{C^2}^{1+\frac{\ell}{2}}) \tau \epsilon_n^{-1} \leq \epsilon_{n+1} \]

\[< \epsilon_n^{-1 - \frac{2}{\ell} + \frac{3\sigma}{\ell}} \leq \epsilon_n^{-1 - \frac{2}{\ell}} = \epsilon_{n+1}.\]

Here in (a) we use interpolation inequalities:

\[(8.19) \quad \|R_n\|_{C^2} \leq C \|R_n\|_{C^0} \|R_n\|_{C^\ell} < \epsilon_n^{-\frac{2}{\ell}} ;\]

in (b) we note that

\[1 + \frac{7}{2}\frac{2}{\ell} < 2(1 - \frac{2}{\ell}) < 2 - \frac{2}{\ell} \text{ and } \frac{3\sigma}{\ell} < 1.\]
Then \( \epsilon_n^{\frac{2\sigma}{\ell}} \epsilon_n^{-1} < \epsilon_n^{-2} \) and
\[
\max\{\epsilon_n^{(1 + \frac{\sigma}{2})(1 - \frac{\ell}{2})}, \epsilon_n^{\frac{3(\ell-2)}{\ell} - 1}\} = \epsilon_n^{(1 + \frac{\sigma}{2})(1 - \frac{\ell}{2})},
\]
in (c) we use
\[
2 - \frac{2\sigma + 10}{\ell} < \epsilon_n^{\frac{1}{\ell}} \eta^{\frac{\sigma_0}{\ell}} \left( \epsilon_n^{(1 + \frac{\sigma}{2})} \right)^{\frac{\ell - 2 - N}{\ell + N}} \left( \epsilon_n^{-2} \right)^{-\frac{2N+2}{\ell + N}} < \epsilon_n^{\frac{1}{\ell}},
\]
provided
\[
2 - \frac{3\sigma + 10}{\ell} \geq 1 + \frac{\beta}{3} \cdot \left( 1 + \frac{\beta}{2} \right) (1 - \frac{5}{\ell}) \left( \frac{\ell - 2 - N}{\ell + N} \right) - 2 \frac{2N+2}{\ell + N} \geq 1 + \frac{\beta}{3}.
\]
By (8.16) and the assumption all inequalities above are satisfied.

(3) \( C^{\sigma_0} \) estimate for \( \omega_{n+1} \): By interpolation inequalities we have
\[
\| R_n \|_{C^{\sigma_0}} \leq K \| R_n \|_{C^{\sigma_0}}^{\frac{\ell - \sigma_0}{\ell}} \| R_n \|_{C^{\sigma_0}} \leq \epsilon_n^{\frac{\sigma_0}{\ell}} < \epsilon_n^{\frac{1}{\ell} - \frac{2\sigma_0 + 1}{\ell}},
\]
Hence we have
\[
\| \omega_{n+1} \|_{C^{\sigma_0}} \leq K \epsilon_n^{\frac{3\sigma}{\ell}} \| R_n \|_{C^{\sigma_0}} \leq K \epsilon_n^{\frac{3\sigma}{\ell}} \epsilon_n^{\frac{1}{\ell} - \frac{2\sigma_0 + 1}{\ell}} < \epsilon_n^{\frac{1}{\ell}},
\]
provided
\[
\frac{3\sigma}{\ell} + 1 - \frac{2\sigma_0 + 1}{\ell} > \frac{1}{2},
\]
which is satisfied for \( \ell > 2(5 \max\{\sigma_0, \sigma\} + 1) \).

(4) \( C^{0} \) estimate for \( H_{n+1} \): By (8.17) we have
\[
\| H_{n+1} - I \|_{C^{0}} \leq K \| R_n \|_{C^{\sigma}} + \| H_n - I \|_{C^{0}} < K \epsilon_n^{\frac{1}{\ell} - \frac{2\sigma_0 + 1}{\ell}} + \sum_{i=0}^{n-1} \epsilon_i^{\frac{1}{2}} \leq \epsilon_n^{\frac{1}{2}} + \sum_{i=0}^{n-1} \epsilon_i^{\frac{1}{2}} = \sum_{i=0}^{n} \epsilon_i^{\frac{1}{2}}
\]
Consequently, we have
\[
f_{n+1} = \tilde{H}_{n+1}^{-1} \circ \tilde{H}_n^{-1} \circ \cdots \circ \tilde{H}_1^{-1} \circ f \circ \tilde{H}_1 \circ \cdots \circ \tilde{H}_{n+1}
\]
where \( \tilde{H}_i = I - \omega_i, 1 \leq i \leq n + 1 \); and \( \Sigma_{n+1} = \tilde{H}_1 \circ \cdots \circ \tilde{H}_{n+1} \).

Finally, (8.20) implies that \( \Sigma_{n} \) converges in \( C^{\sigma_0} \) topology to a \( C^{\sigma_0} \) diffeomorphism \( H \), which is a conjugacy between \( f \) and \( A \). By [ALL92, Theorem 6.3] and the choice of \( \sigma_0 \) we conclude that \( H \) is a \( C^{\infty} \) diffeomorphism.

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