Hyperbolic Coxeter groups with Sierpiński carpet boundary

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Abstract. We give a necessary and sufficient condition for a hyperbolic Coxeter group with planar nerve to have Sierpiński curve as its Gromov boundary.

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Introduction.

In the survey [KB] concerning boundaries of hyperbolic groups the authors wrote: classifying groups which have the Sierpiński carpet as the boundary remains a challenging open problem. In this paper we deal with this problem for hyperbolic Coxeter groups. We first state (as Theorem 1 below) our main result in the framework of right angled hyperbolic Coxeter groups, as this does not require much preparations, and is particularly simple. Then, after introducing appropriate terminology, we present the main result in its full generality (as Theorem 2 below).

We call a simplicial complex unseparable if it is connected, has no separating simplex, no separating pair of nonadjacent vertices, and no separating full subcomplex isomorphic to the (simplicial) suspension of a simplex.

Theorem 1. Let \((W, S)\) be a right angled Coxeter system such that the group \(W\) is word hyperbolic and the nerve \(L\) of the system is planar, distinct from a simplex and from a triangulation of the 2-sphere \(S^2\). Then the Gromov boundary \(\partial W\) is homeomorphic to the Sierpiński curve if and only if \(L\) is unseparable.

![Figure 1. Examples of nerves of hyperbolic right-angled Coxeter groups with Sierpiński carpet boundaries.](image)

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Recall that for a right-angled Coxeter system \((W, S)\) the group \(W\) is word hyperbolic if the nerve \(L\) of this system (which is a flag simplicial complex) satisfies *no empty square* condition, i.e. any polygonal cycle of length 4 in the 1-skeleton of \(L\) has at least one diagonal (see the comment at the bottom of p. 233 in [Da]). Examples of some nerves of hyperbolic right-angled Coxeter groups with Sierpiński carpet boundaries are presented at Figure 1.

We extend the concept of unseparability to appropriately understood nerves of arbitrary (not only right-angled) Coxeter systems.

**Definition.**

1. A *labelled nerve* \(L^*\) of a Coxeter system \((W, S)\) is the nerve \(L\) of \((W, S)\) equipped with the labelling, which to any edge of \(L\) associates the corresponding entry from the Coxeter matrix of this system. More precisely, if \(e\) is an edge of \(L\) with vertices \(s, t \in S\) then the label associated to \(e\) is equal to the exponent \(m_{st}\) in the relator \((st)^{m_{st}}\) from the standard presentation for \(W\) corresponding to the system \((W, S)\).
2. A *labelled suspension* in a labelled nerve \(L^*\) is a full subcomplex \(\Lambda\) of \(L\) isomorphic to the simplicial suspension of a simplex, \(\Lambda = \{t, s\} \star \sigma\), such that any edge in \(\Lambda\) adjacent to \(t\) or \(s\) has label 2.
3. The labelled nerve \(L^*\) of a Coxeter system is *unseparable* if it is connected, has no separating simplex, no separating pair of nonadjacent vertices, and no separating labelled suspension.

**Remark.** Unseparability of the labelled nerve of \((W, S)\) is equivalent to the fact that \(W\) has no visual splitting (in the sense of Mihalik and Tschantz [MT]) along a finite or a 2-ended subgroup.

Theorem 1 is a special case of the following more general result. In its statement, the condition that \(L^*\) is a *labelled wheel* means that \(L\) is isomorphic to the simplicial cone over a triangulation of \(S^1\), and the labels at all edges adjacent to the cone vertex are equal to 2. (If \(L^*\) is a labelled wheel then \(W\) is easily seen to be a virtually surface group, so its boundary is then a circle.)

**Theorem 2.** Let \((W, S)\) be a Coxeter system such that the group \(W\) is word hyperbolic and the nerve of the system is planar, distinct from a simplex and from a triangulation of \(S^1\) or \(S^2\). Then the Gromov boundary \(\partial W\) is homeomorphic to the Sierpinski curve if and only if the labelled nerve \(L^*\) of \((W, S)\) is distinct from a labelled wheel and unseparable.

Recall that a Coxeter group is word hyperbolic iff it contains no affine special subgroup of rank \(\geq 3\), and no special subgroup being the direct sum of two infinite special subgroups (see [Da], Corollary 12.6.3 on p. 241). Figure 2 presents a sample of labelled nerves of not right-angled hyperbolic Coxeter groups with Sierpinski carpet boundaries. In this figure we use the convention that an edge with no label indicated is labelled with 2.

**Remarks.**

1. The following two conditions from the statement of Theorem 2 do not have their explicitely expressed counterparts in the statement of Theorem 1:
   - that \(L\) is not a triangulation of \(S^1\), and
• that $L^*$ is not a labelled wheel.

However, the counterparts of these conditions follow automatically from the remaining assumptions of Theorem 1. More precisely, $L$ cannot be the triangulation of $S^1$ consisting of 3 edges (the 3-cycle), or the wheel equal to the simplicial cone over the 3-cycle (the 3-wheel), because nerves of right angled Coxeter systems are flag. Furthermore, $L$ cannot be any other triangulation of $S^1$ because of the assumption that it has no separating pair of nonadjacent vertices. Similarly, $L$ cannot be any other wheel because of the assumption that it has no separating suspension. This justifies that Theorem 2 is an extension of Theorem 1.

(2) Note that Theorem 2, among others, determines all hyperbolic Coxeter groups with Gromov boundary homeomorphic to the Sierpiński curve, and whose nerves are
- planar graphs,
- triangulations of the 2-disk,
- triangulations of other planar surfaces.

It seems that triangulations of surfaces were never previously considered as potential candidates for nerves of Coxeter groups with the Sierpiński carpet boundary. Some limited results concerning planar graphs as nerves of hyperbolic Coxeter groups with carpet boundaries were obtained by N. Benakli, as well as by N. Chinen and T. Hosaka (these works seem not to be yet fully documented).

(3) It is possible that, up to product with a finite Coxeter group, planarity of the nerve is a necessary condition for any Coxeter group to have the Sierpiński carpet boundary (and even more generally, to have planar boundary). If this were the case, Theorem 2 would provide the complete necessary and sufficient condition for a hyperbolic Coxeter system to have the Sierpiński carpet boundary. However, we do not resolve this issue in the present paper.

Figure 2. Examples of labelled nerves of hyperbolic not right-angled Coxeter groups with Sierpiński carpet boundaries.
The paper is organized as follows. In Section 1 we deal with arbitrary (i.e. not necessarily hyperbolic) Coxeter groups. We state a conjecture concerning the occurrence of Sierpiński carpet as boundary for such groups, which generalizes Theorem 2. We then show various partial results related to this conjecture (which are also the steps in our proof of Theorem 2).

In Section 2 we deal with hyperbolic Coxeter groups, providing those arguments for showing Theorem 2 which rely upon the hyperbolicity assumption. We also include comments indicating why these arguments cannot be easily extended to the general (i.e. not necessarily hyperbolic) case.

The proofs in Sections 1 and 2 are relatively short, and they mainly consist of appropriate references to various so far known results.

In the appendix we present a short proof of the fact (which we use in our arguments) that boundary of a special subgroup of a Coxeter group is canonically a subspace in the boundary of this group. It seems that this folklore result, known to the experts for at least 20 years, does not yet have a proof explicitly presented in the literature of the subject.

1. A conjecture and some partial results for arbitrary Coxeter groups.

Recall that given an arbitrary Coxeter system \((W, S)\), there is canonically associated to it a \(\text{CAT}(0)\) piecewise euclidean complex \(\Sigma = \Sigma(W, S)\), called the Coxeter-Davis complex of the system. The boundary \(\partial(W, S)\) is defined as the visual boundary of this associated Coxeter-Davis complex. It is known that when the group \(W\) is word hyperbolic, the boundary \(\partial(W, S)\) coincides (up to homeomorphism) with the Gromov boundary \(\partial W\). This allows for the following generalization of Theorem 2, which we formulate as a conjecture.

1.0 Conjecture. Let \((W, S)\) be a Coxeter system with planar nerve distinct from a simplex and from a triangulation of \(S^1\) or \(S^2\). Then the boundary \(\partial(W, S)\) is homeomorphic to the Sierpinski curve if and only if the labelled nerve \(L^*\) of \((W, S)\) is distinct from a labelled wheel and unseparable.

Recall that the Sierpiński curve has the following characterization due to Whyburn [Wh]: it is the unique up to homeomorphism compact connected and locally connected metric space of topological dimension 1, which is planar (i.e. can be embedded in the 2-sphere \(S^2\)), and which has no local cut point. For short, we will denote the Sierpiński curve by \(\Pi\).

Throughout this section \((W, S)\) is a Coxeter system, \(L\) denotes its nerve, and \(L^*\) denotes its labelled nerve. We start with a lemma yielding one of the implications in the above conjecture (and thus also in Theorem 2).

1.1 Lemma. Unseparability of the labelled nerve \(L^*\) is a necessary condition for \(\partial(W, S)\) to be homeomorphic to the Sierpiński curve \(\Pi\). More precisely, it is a necessary condition for \(\partial(W, S)\) to be connected and to have no local cut points.

Proof: By Theorem 8.7.2 in [Da], if \(L\) is not connected, or has a separating simplex, then \(W\) is either 2-ended or has infinitely many ends. Consequently, \(\partial(W, S)\) is not connected, and hence not homeomorphic to \(\Pi\).
Consider the remaining case when $W$ is 1-ended. If $L$ has a separating pair of nonadjacent vertices, or $L^\bullet$ has a separating labelled suspension, then $W$ splits over a 2-ended special subgroup (induced by the corresponding separating subcomplex). Consequently, the boundary $\partial(W, S)$ has a cut pair (see remark before Theorem 2 in the introduction to [PS]), and hence it is not homeomorphic to $\Pi$. This completes the proof.

In view of the characterization of the Sierpiński curve recalled above, to prove the converse implication in Conjecture 0.1 (and in Theorem 2), it is sufficient to show that $\partial(W, S)$ satisfies the properties from this characterization. Obviously, the boundary of any Coxeter system is a compact metric space. Next three lemmas show that planarity and unseparability of the labelled nerve $L^\bullet$ implies that $\partial(W, S)$ is connected, planar and 1-dimensional, respectively.

1.2 Lemma. If $L^\bullet$ is planar, unseparable and distinct from a simplex then $W$ is 1-ended and $\partial(W, S)$ is connected.

Proof: For a CAT(0) group connectedness of its visual boundary is a consequence of its 1-endedness. Thus, in view of Theorem 8.7.2 in [Da], connectedness of $\partial(W, S)$ follows from the fact that $L$ is connected, has no separating simplex, and is distinct from a simplex.

1.3 Lemma. If the nerve $L$ is planar then the boundary $\partial(W, S)$ is also planar.

Proof: Extend $L$ to a triangulation $N$ of $S^2$ so that the following two conditions hold:

1) $L$ is a full subcomplex of $N$;
2) $N$ is flag relative to $L$, i.e. if $T$ is a set of vertices of $N$ which are pairwise connected with edges, and if $T \cap L$ spans a simplex of $L$, then $T$ spans a simplex of $N$.

(We omit an elementary argument showing that such an extension is always possible for a planar complex.) Consider the labelled complex $N^\bullet$ whose labelling coincides with the labelling of $L^\bullet$ on edges contained in $L$, and which has labels equal to 2 at all other edges of $N$. It is not hard to observe that $N^\bullet$ is then the labelled nerve of some Coxeter system, which we denote $(W_{N^\bullet}, S_{N^\bullet})$. Moreover, $W$ is then a special subgroup of $W_{N^\bullet}$, with the induced labelled nerve equal to $L^\bullet$. Consequently, by Theorem A.1 from the appendix, $\partial(W, S) \subset \partial(W_{N^\bullet}, S_{N^\bullet})$. Since the latter is homeomorphic to the sphere $S^2$ (see e.g. Theorem 3b.2 in [DJ] or Corollary 1 in [Dr]), it follows that $\partial(W, S)$ is planar.

1.4 Lemma. If $L^\bullet$ is planar, unseparable, distinct from a simplex and from a triangulation of the 2-sphere $S^2$, then $\dim \partial(W, S) = 1$.

Proof: Denote by $vcd(W)$ the virtual cohomological dimension of $W$. It follows from results of Mike Davis that
\[
vcd(W) = \max\{n : \overline{H}^{n-1}(L \setminus \sigma) \neq 0, \text{ for some simplex } \sigma \text{ of } L, \text{ or } \overline{H}^{n-1}(L) \neq 0\}\]
(see Corollary 8.5.5 in [Da]). Moreover, by the fact that $W$ acts geometrically on $\Sigma$ and is virtually torsion-free, the pair $(\Sigma \cup \partial(W, S), \partial(W, S))$ is a $\mathcal{Z}$-structure (in the sense of Bestvina described in [Be]) for any torsion-free finite index subgroup $H < W$. Since, by Theorem 1.7 of [Be] we then have $\dim \partial(W, S) = \text{cd}(H) - 1$ (where $\text{cd}(H)$ is the cohomological dimension of $H$), it follows that $\dim \partial(W, S) = vcd(W) - 1$. Since $L$ is
planar and distinct from $S^2$, we get (from the above formula of Davis) that $vcd(W) \leq 2$, and hence $\dim \partial(W, S) \leq 1$. On the other hand, by Corollary 8.5.6 in [Da], a Coxeter group $W$ is virtually free iff $\dim \partial(W, S) \leq 0$. Since under our assumptions $W$ is 1-ended (see Lemma 1.2), we conclude that $\dim \partial(W, S) = 1$, thus completing the proof.

2. Hyperbolic Coxeter groups and completion of the proof of Theorem 2.

In this section we assume that the group $W$ in a Coxeter system $(W, S)$ is word hyperbolic. Under this assumption, we deal with the conditions of local connectedness and of absence of local cutpoints in the boundary $\partial W = \partial(W, S)$. This allows to conclude the proof of Theorem 2.

2.1 Lemma. If $L^\bullet$ is planar, unseparable, distinct from a simplex, from a triangulation of $S^1$ and $S^2$, and from a labelled wheel, and if $W$ is word-hyperbolic, then the boundary $\partial W = \partial(W, S)$ is locally connected and has no local cut point.

Proof: By a result of Bestvina and Mess [BeM], Gromov boundary of any 1-ended word-hyperbolic group is locally connected. Hence, by Lemma 1.2, $\partial(W, S)$ is locally connected.

Unseparability of $L^\bullet$ means exactly that $W$ does not visually split (in the sense of the paper [MT] by Mihalik and Tschantz) over a finite or 2-ended subgroup. More precisely, this means that $W$ cannot be expressed as an essential free product of its two special subgroups, amalgamated along a finite or 2-ended special subgroup. It follows from the main result of the same paper [MT] that, under this condition, $W$ does not split along any finite or 2-ended subgroup. Then, by a result of Bowditch [B], the boundary $\partial W = \partial(W, S)$ has no local cut point, or $W$ is a cocompact Fuchsian group (i.e. a group of isometries of the hyperbolic plane acting properly and cocompactly). The latter case can be excluded as follows. By a result of Davis (see Theorem B in [D1] or Theorem 10.9.2 in [Da]), if $W$ is a cocompact Fuchsian group then its nerve is either a triangulation of $S^1$ or $W$ splits as the direct sum of a special subgroup with the nerve $S^1$, and another special subgroup, which is finite. Since these possibilities are clearly inconsistent with the assumptions of the lemma, the proof is completed.

Proof of Theorem 2: In view of the Whyburn’s characterization of the Sierpiński curve (recalled at the beginning of Section 1), Theorem 2 follows from Lemmas 1.1-1.4 and 2.1.

2.2 Remarks.

(1) Arguments as in the proof of Lemma 2.1 cannot be easily extended to cover the case of Coxeter groups that are not word-hyperbolic. For example, it is not known fully for which 1-ended Coxeter groups $W$ the boundary $\partial(W, S)$ is locally connected. Some criteria in the right angled case are provided in [MRT] and [CM]. It is quite possible that using these criteria one can show that for right angled Coxeter groups $W$, under assumptions on $L$ as in Lemma 2.1, the boundary $\partial(W, S)$ is always locally connected. Anyway, no criteria for local connectedness of $\partial(W, S)$ are known for $W$ which are neither word-hyperbolic nor right angled.

(2) If $W$ is not word-hyperbolic, nothing seems to be known about non-appearance of local cut points in the boundary $\partial(W, S)$. The main result of the paper [PS] by Papasoglu and Swenson allows, under our assumptions on the labelled link $L^\bullet$, to exclude the
appearance of cut pairs in $\partial(W, S)$. We don’t know if the methods developed in [PS] could be applied to exclude as well local cut points (even assuming that $\partial(W, S)$ is locally connected).

Appendix.

In this appendix we provide a proof of the following folklore result, which seems to have no explicitly presented proof in the literature.

A.1 Proposition. Let $(W, S)$ be any Coxeter system, and let $W_T$ be the special subgroup of $W$ corresponding to a subset $T \subset S$. Then the boundary $\partial(W_T, T)$ embeds as a subspace in the boundary $\partial(W, S)$.

Our proof of Proposition A.1 goes backwards. It consists of a sequence of reductions to some other facts of independent interest, namely Proposition A.2, Lemma A.3 and Lemma A.8. These reductions are fairly standard and well known. The final step consists of proving Lemma A.8, which we do by borrowing some arguments from Ian Leary’s paper [L] (where they were used for slightly different purposes).

Recall that to each Coxeter system $(W, S)$ there is associated its Coxeter-Davis complex $\Sigma(W, S)$. This complex is equipped with the piecewise euclidean Moussong’s metric, for which it is a CAT(0) space. The boundary $\partial(W, S)$ is by definition the visual boundary of $\Sigma(W, S)$. Recall also that the Coxeter-Davis complex $\Sigma(W_T, T)$ of a special subgroup is canonically a subcomplex in $\Sigma(W, S)$ (see Section 7.3, especially Proposition 7.3.4, in [Da]). Consequently, since the visual boundary $\partial Y$ of a convex subspace $Y$ in a CAT(0) space $X$ canonically embeds in the visual boundary $\partial X$, to prove Proposition A.1, it is sufficient to prove the following.

A.2 Proposition. Given a Coxeter system $(W, S)$, and a special subgroup $W_T < W$, the subcomplex $\Sigma(W_T, T)$ is a convex subspace in the Coxeter-Davis complex $\Sigma(W, S)$ (equipped with the Moussong’s metric).

To prove Proposition A.2 we need some preparations. Recall that a subspace $K$ of a CAT(1) space $L$ is $\pi$-convex if any geodesic in $L$ of length less than $\pi$, having both endpoints in $K$, is entirely contained in $K$. Recall also that if $X$ is a CAT(0) piecewise euclidean complex then its any vertex link is a piecewise spherical CAT(1) complex. Moreover, a connected subcomplex $Y$ in a CAT(0) piecewise euclidean complex $X$ is convex iff for any vertex $v$ of $Y$ the link of $Y$ at $v$ is a $\pi$-convex subspace in the link of $X$ at $v$. Since all vertex links in a Coxeter-Davis complex $\Sigma(W, S)$ are isometric to the nerve $L(W, S)$ of the system $(W, S)$, to prove Proposition A.2, it is sufficient to prove the following.

A.3 Lemma. Given a Coxeter system $(W, S)$, and a special subgroup $W_T < W$, the nerve $L(W_T, T)$, viewed canonically as the subcomplex in the nerve $L(W, S)$, is a $\pi$-convex subspace.

To prove Lemma A.3, we will need the following terminology and tools introduced by Gabor Moussong.

A.4 Definition. A spherical simplex $\sigma$ is of size $\geq \frac{\pi}{2}$ if its every edge has length $\geq \frac{\pi}{2}$. 

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A.5 Definition (cf. Definition I.7.1 and Lemma I.5.1 in Appendix I in [Da]). A piecewise spherical simplicial complex $L$, with all cells of size $\geq \frac{\pi}{2}$, is metrically flag if any of the following two equivalent conditions holds:

- if $v_0, \ldots, v_k$ are vertices of $L$ pairwise connected with edges, and $c_{ij} = d_L(v_i, v_j)$, then $v_0, \ldots, v_k$ span a simplex of $L$ iff the matrix $(c_{ij})$ is positive definite;
- if $v_0, \ldots, v_k$ are vertices of $L$ pairwise connected with edges, and $c_{ij} = d_L(v_i, v_j)$, then $v_0, \ldots, v_k$ span a simplex of $L$ iff $(c_{ij})$ is the matrix of edge lengths of a spherical $k$-simplex.

A.6 Lemma (cf. Lemma 12.3.1 in [Da]). For any Coxeter system $(W, S)$ its nerve $L(W, S)$, with its natural piecewise spherical structure, has all simplices of size $\geq \frac{\pi}{2}$, and is metrically flag.

A.7 Lemma (cf. Lemma I.7.4 in Appendix I in [Da]). Let $L$ be a finite piecewise spherical simplicial complex with all cells of size $\geq \frac{\pi}{2}$. Then $L$ is CAT(1) iff it is metrically flag.

In view of Lemma A.6, and since $L(W_T, T)$ is obviously a full subcomplex of $L(W, S)$, to prove Lemma A.3, it is sufficient to prove the following.

A.8 Lemma. Let $L$ be a finite piecewise spherical simplicial complex with all cells of size $\geq \frac{\pi}{2}$, and suppose it is CAT(1). Then its any full subcomplex is a $\pi$-convex subspace of $L$.

Proof: We adapt the arguments from the proof of Theorem B.7 in Appendix B of the paper [L] by Ian Leary.

Note that, since $L$ is CAT(1), it follows from Lemma A.7 that it is metrically flag. Consider the simplicial complex $L \ast_K L$ obtained by gluing the two copies of $L$ identically along the subcomplex $K$. Equip it with the piecewise spherical metric induced from the metrics of the two copies of $L$. Observe that $L \ast_K L$ has then cells of size $\geq \frac{\pi}{2}$. Moreover, since $K$ is a full subcomplex, it is easy to see that $L \ast_K L$ is metrically flag. Consequently, applying again Lemma A.7, it is CAT(1).

We claim that the metric of $L \ast_K L$ restricted to any copy of $L$ coincides with the original metric of $L$. One inequality between these metrics is obvious, because $L$ is a subcomplex of $L \ast_K L$. The other inequality follows easily from existence of the isometric involutive automorphism exchanging the two copies of $L$ in $L \ast_K L$.

Let $\gamma$ be a geodesic of length less than $\pi$ in $L$, with both endpoints in $K$. By the above claim, lift of $\gamma$ to the first copy of $L$ in $L \ast_K L$ is also a geodesic; denote it by $\gamma_1$. If $\gamma$ is not entirely contained in $K$, the image of $\gamma_1$ through the above mentioned involution exchanging the two copies of $L$ is a different geodesic in $L \ast_K L$ connecting the same endpoints. This contradicts uniqueness of geodesics of length less than $\pi$ in CAT(1) spaces, thus completing the proof.

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