Hamiltonian Renormalization Groups

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I. Qualitative Introduction to Renormalization Groups (RG)

The RG has been introduced originally in order to handle ultraviolet divergencies in Quantum Electrodynamics. In the late sixties it has been realized by Fisher and Kadanoff (1) that critical exponents in phase transition can be computed within a RG calculation with a finite cutoff Λ. In the seventies the method has been applied to solve the Kondo problem (2,3). In the eighties it has become clear that quantum phase transition at $T = 0$ and Quantum critical behavior can be studied within the RG method. Today the RG method has become the major tool for investigation of strongly correlated systems at low temperature. In many problems we know the Hamiltonian but the Lagrangian might be complicated. Since the original RG method is based on mapping the quantum problem to Euclidean field theory the absence of a simple action might be a problem. In the last years we have developed a method based on the Hamiltonian formalism (4). Here we will present this version of RG. In particular we will use this method to solve the sine-Gordon problem in $1 + 1$ dimension which has a Kosterlitz-Thouless transition.

II. Order Parameter

The magnetization, the density, or the $U(1)$ phase represent the order parameter and characterizes the phase transition. In all these problems we are interested in the behavior at long distance. As a result many different microscopical problem might have the same long distance (low-energy) be-
behavior. We will assume that we have an ultraviolet cutoff Λ, reducing Λ to Λ' = Λ / s, s > 1 such that the measure remains invariant,

\[ T \exp \left[ -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \int d^d x h[\phi(x,t), P(x,t)] \right] \]

and we can construct the renormalized hamiltonian (h[φ, P] is the hamiltonian density, ”φ” is the order parameter, ”P” is the canonical momentum [φ(x,t), P(x',t)] = iδ(Λ(x-x'))). The long distance behavior is obtained from the existence of fixed point (a set of coupling constants which are unchanged under Λ → Λ/s)

III. Microscopic Models

We will be interested in application to strongly correlated systems \( H = \int d^d x h[\phi(x), P(x)] \), where \( h = h_0 + h_I \), \( h_0 \) is the free apart and \( h_I \) is the interacting part.

\[ h_0[\phi(x), P(x)] = \frac{v}{2}[KP^2(x) + \frac{1}{K}(\partial_x \phi)^2] \]

for \( d > 1 \), \((\partial_x \phi)^2 \equiv \sum_{i=1}^{d}(\partial_i \phi)^2\), \( x \equiv (x_1, \cdots x_d) \). The Ising model is investigated choosing \( K = 1 \) and \( h_I[\phi(x)] = \frac{1}{2}U_2 \phi^2(x) + \frac{U_4}{4!} \phi^4(x) \). Interacting electrons in 1 + 1 dimensions are described by \( d = 1 \), \( 0 < K < 2 \) with \( h_I[\phi(x)] = U \cos(\sqrt{8\pi \phi(x)}) \) with a Kosterlitz-Thouless transition phase characterized by \( U^* \) with a continuous line of fixed points \( K^* \). For \( U(s) \to 0 \) we obtain a set of Luttinger liquids characterized by \( 0 < K < 2 \).

IV Constructing RG

Our starting point is the hamiltonian density, \( h[\phi(x), P(x)] = h_0[\phi(x), P(x)] + h_I[\phi(x)] \), where \( h_0[\phi(x), P(x)] \) is given in eq. (2) and \( h_I[\phi(x)] = \sum_i U_i O_i(x) \) with \( O_i(x) \) being monomial of \( \phi(x) \). The fields \( \phi(x) \equiv \phi_\Lambda(x) \) and \( P(x) \equiv P_\Lambda(x) \) are restricted by \( \Lambda \).

The RG procedure is based on the following steps:

(a) Project out states in the momentum interval \( \Lambda/s \leq |q| < \Lambda \) using the ”Heisenberg picture”, choose \( s = \exp \ell \), with \( \ell \to 0 \) (infinitesimal). The
projection is done such that $T \exp[-\frac{i}{\hbar} \int dt \int d^d x h[\phi(x,t), P(x,t)]]$ remains invariant.

(b) In the second step the cutoff $\Lambda/s$ is restored to $\Lambda$ by increasing the unit length by a factor of $s$, $x' = x/s$, $t' = t/s$.

(c) In the last step we determine the scaling dimension of $\phi(x)$ and $P(x)$ such that $\int dt \int d^d x h_0[\phi(x,t), P(x,t)]$ remains invariant: $\phi_{\Lambda/s}(x,t) = s^\Delta \phi_{\Lambda}(x', t')$ and $P_{\Lambda/s}(x,t) = s^{\Delta-1} P_{\Lambda}(x', t')$ "$\Delta$" is the scaling dimension of $\phi(x)$. From the invariance of the free part (eq. (2)) we obtain $\Delta = \frac{1}{2}(d + 1) - 1$. Using the scaling dimension of $\phi(x)$ we find the scaling dimension of the monomials, $O_{i, \Lambda/s}(x,t) = s^\Delta O_{i, \Lambda}(x', t')$. This leads to the scaling of the coupling constants $U_i(s) = s^{d+1-\Delta_i} U_i(s = 1)$. When $d + 1 - \Delta_i > 0$, $U_i$ is relevant and flows to $\infty$ for $s \to \infty$. When $d + 1 - \Delta_i < 0$, $U_i$ is irrelevant, $U_i \to 0$. When $d + 1 - \Delta_i = 0$, $U_i$ is marginal.

In order to show how the steps (a) to (c) are implemented we will consider a model in $1 + 1$ dimension ($d = 1$).

We first separate out the fast modes from the slow ones in the field $\phi(x)$ as well as its canonical conjugate $P(x)$: $\phi(x) = \phi^<(x) + \delta \phi(x)$; $P(x) = P^<(x) + \delta P(x)$. We assumed that there is a momentum cut-off $\Lambda$. The slow parts, $\phi^<$ and $P^<$, contain those components with momenta $|q| \leq \Lambda/s$, while the fast ones, $\delta \phi$ and $\delta P$, hold the contributions from the momentum shell, $\Lambda/s < |q| \leq \Lambda$. As a result we find

$$H[\phi, P] = H[\phi^< + \delta \phi, P^< + \delta P] \equiv H[\phi^<, P^<] + \delta H[\phi^<, \delta \phi; P^<, \delta P]$$  \hspace{1cm} (3)

where

$$\delta H[\phi^<, \delta \phi; P^<, \delta P] = \int_{-\infty}^{\infty} dx \left\{ \frac{v K}{2} [\delta P(x)]^2 + \frac{v}{2K} [\partial_x \delta \phi(x)]^2 
+ h_i^1[\phi^<][\delta \phi(x)] + \frac{1}{2} h_i^2[\phi^<][\delta \phi(x)]^2 + \cdots \right\}$$  \hspace{1cm} (4)

we keep only $(\delta \phi(x))^n$, $n \leq 2$ since we work in the limit $s \to 1$.

To find out the effective hamiltonian of the slow modes, we need to integrate out the fast modes defined in the momentum shell. In the hamiltonian language we have to take the expectation value with the "fast mode" ground
state at \( t = 0 \), \(|\Psi^>(t = 0) > \equiv |GS^> >\). This corresponds to the "Heisenberg picture" with time dependent operators \( \delta \phi(x, t), \delta P(x, t) \). We can achieve this by a unitary transformation such that the linear coupling between the \( \delta \phi \) field with the slow modes \( \phi^< \) is removed. (The fields \( \phi^<(x), P^<(x) \), remain in the Schrödinger picture, only \( \delta \phi(x, t) \) and \( \delta P(x, t) \) are in the "Heisenberg picture".) This can be carried out by introducing the following unitary transformation \( U \):

\[
\delta P'(x) = U \delta P(x) U^\dagger = \delta P(x) + \Pi(x) \tag{5}
\]

\[
\delta \phi'(x) = U \delta \phi(x) U^\dagger = \delta \phi(x) + r(x) \tag{6}
\]

\( r(x) \) and \( \Pi(x) \) are chosen such that the linear term disappear from the transformed hamiltonian. Using the Heisenberg eq. of motion we have

\[
\partial_t \delta \phi(x, t) = v K \delta P(x, t)
\]

Demanding \([\delta \phi(x, t), \delta P(x', t)] = [\delta \phi'(x, t), \delta P'(x', t)]\), we obtain that, \( \Pi(x) \) and \( r(x) \) are not independent: \( \partial_t r(x, t) = v K \Pi(x, t) \). We choose \( U = \exp[\frac{i}{2} \int^\infty_{-\infty} dx [r(x) \delta P(x) - \Pi(x) \delta \phi(x)]] \). By applying the unitary transformation, we get

\[
\int dt \delta H' = \int dt \int dx \left\{ \frac{v K}{2} [\delta P(x, t)]^2 + \frac{v}{2K} [\partial_x \delta \phi(x, t)]^2 + \frac{1}{2} h''_I(\phi^<(x))[\delta \phi(x, t)]^2 + \delta \phi(x, t)[\tilde{L} r(x, t) + h'_I(\phi^<(x))] + r(x, t)[\frac{1}{2} \tilde{L} r(x, t) + h'_I(\phi^<(x)))] \right\} \tag{7}
\]

In eq. (7) we define: \( \tilde{L} = \frac{1}{vK} \partial_t^2 - \frac{v}{K} \partial_x^2 + h''_I(\phi^<(x)) \). To set the \( \delta \phi \) term zero, we need; \( \tilde{L} r(x, t) = -h'_I(\phi^<(x)) \). This gives the formal solution \( r(x, t) = -\int dy \int dt' G(x, y; t - t') h'_I(\phi^<(y, t')) \) where \( G(x, y; t - t') \) is the Green’s function of the operator \( \tilde{L}, \tilde{L} G_A(x, t; t, t') = \delta(x - y) \delta(t - t') \). Since \( \phi^<(x) \) is time independent \( r(x) \) is determined only by the spatial Green’s function, \( \tilde{G}_A(x, y) \equiv \int^\infty_{-\infty} dt' G_A(x, y; t - t') \). By substituting the solution into eq. (7) we find

\[
\int dt \delta H' = \int dt \int dx \left\{ \frac{v K}{2} [\delta P(x, t)]^2 + \frac{v}{2K} [\partial_x \delta \phi(x, t)]^2 + \frac{1}{2} h''_I(\phi^<(x))[\delta \phi(x, t)]^2 - \frac{1}{2} \int dy h'_I(\phi^<(y)) \tilde{G}_A(x, y) h'_I(\phi^<(y)) \right\} \tag{8}
\]
Eq. (8) is exact and the Green’s function $G_\Lambda(x, y; t, t')$ is restricted to the momentum shell. $G_\Lambda(q, \omega)$ is the Fourier transform of $G(x, y; t, t')$ with $\Lambda/s \leq |q| \leq \Lambda$ and $-\infty \leq \omega \leq \infty$. Since only high momenta in the shell are involved, we replace $\hat{G}_\Lambda$ by $\hat{G}_\Lambda^{(0)}$ (the free Green’s function). Now we average $\delta H'$ over the ground state of the fast modes. We use $s = \exp \ell$, $\ell \to 0$ and find:

$$\int dt \Delta H = \int dt < GS | \delta H' | GS > = \int dx \int \frac{K \ell}{4\pi} h''_I(\phi^<(x))$$

$$-\frac{1}{2} \int dy h'_I(\phi^<(x)) \hat{G}_\Lambda^{(0)}(x, y) h'_I(\phi^<(y))$$  \ (9)

V. Applications to the Sine-Gordon model

We choose $h_I(\phi^<) = U \cos(\sqrt{8\pi} \phi)$ we substitute into eq. (9) and add to eq. (9) the term $H[\phi^<, P^<]$ (see eq. (3)). As a result we find the effective hamiltonian

$$H^< = \int_{-\infty}^{\infty} dx \left\{ \frac{vK}{2} (P^<)^2 + \frac{v}{2K} [1 + \left( \frac{K}{v} \right)^2 C \frac{U^2}{\Lambda^4} \ell] (\partial_x \phi^<(x))^2 \right\}$$

$$+ U(1 - 2K\ell) \cos(\sqrt{8\pi} \phi^<(x))$$ \ (10) \ (11)

We see that the hamiltonian has the same form as the original one. The superscript ”<” indicate that the cut-off of the current model is $\Lambda/s$ and $C \approx 1$. To compare with the original system, we need to restore the momentum cut-off to $\Lambda$ by appropriately rescaling the fields and the parameters. It is advantageous to work with the dimensionless coupling constant $g$, $U = g\Lambda^2$. So by rescaling the momentum, $k \to k/s$, or in real space, $x \to xs$, $t \to ts$ and demanding that $\int dt H^<_0$ is invariant, we get the new set of parameters $\frac{v'}{K'} = \frac{v}{K}[1 + C(\frac{v}{K})^2 \frac{U^2}{\Lambda^4} \ell]$ in terms of the old ones: $v'K' = vK$ and using $s = \exp(\ell)$ with $\ell$ a positive infinitesimal, we get the $\beta$-functions:

$$\frac{d(vK)}{d\ell} = 0; \frac{d(v/K)}{d\ell} = Cg^2(v/K)^{-1}; \frac{dg}{d\ell} = 2g(1 - K)$$ \ (12)

An non-trivial fixed point is observed to locate at $K = 1$ and $g = 0$. Near the point we can define a new set of variables $x = 2(K - 1); y = \sqrt{\frac{C}{2v_0}} g$, $v_0$ is the fermi velocity before scaling $v(\ell = 0) = v_0$. The $\beta$-functions becomes
\[
\frac{dx}{dt} = -y^2; \quad \frac{dy}{dt} = -xy
\]
with the solution \( x^2(\ell) - y^2(\ell) = x^2(0) - y^2(0) \equiv \text{const}. \) The scaling equation for the speed \( v \) is not necessary since the product \( v(\ell)K(\ell) \) is a constant. The value for \( v(\ell) \) can be found trivially by
\[
v(\ell) = \frac{v(\ell=0)K(\ell=0)}{K(\ell)}.
\]

The RG flow determined by eq (12) was first reached by Kosterlitz and Thouless (5). There are two trivial fixed points, corresponding to the cases \( y \) goes to plus/minus infinity. In the fermion language the limit \( y \to +\infty \) corresponds to the infinite repulsion for particles on neighboring lattice and it is obvious at half-filling the corresponding phase is the Mott insulator. In the opposite limit \( y \to -\infty \), the corresponding phase at half-filling is a band insulator in which the unit cell of the lattice is doubled and a gap is formed at the surface of the fermi sea. There is a third phase which is described by the line \( y = 0 \) on the flow diagram. This is the Luttinger liquid, or the spin-liquid phase. The part with \( x > 0 \) is a stable liquid phase while the \( x < 0 \) part is unstable.

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