SYNTHESIS OF FEEDBACK CONTROLLER FOR NONLINEAR CONTROL SYSTEMS WITH OPTIMAL REGION OF ATTRACTION

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ABSTRACT

The problem of computing and characterizing Region of Attraction (ROA) with its many variations have a long tradition in safety-critical systems and control theory. By virtue here comes the connections to Lyapunov functions that are considered as the centerpiece of stability theory for a non-linear dynamical systems. The agents may be imperfect because of several limitations in the sensors which ultimately restrict to fully observe the potential adversaries in the environment. Therefore while interacting with human life an autonomous robot should safely explore the outdoor environment by avoiding the largest safe region that may cause physical harm both the systems and environment. In this paper we address this problem and propose a framework of learning policies that adapt to the shape of largest safe region in the state space. At the inception the model is trained to learn an accurate safety certificate for non-linear closed loop dynamics system by constructing Lyapunov Neural Network. The current work is also an extension of the previous work of computing ROA under a fixed policy. Specifically we discuss how to design a state feedback controller by using a typical kind of performance objective function to be optimized and demonstrates our method on a simulated inverted pendulum which clearly shows that how this model can be used to resolve issues of trade-offs and extra design freedom.

Keywords Feedback control · Region Of Attraction · Neural Network · Particle Swarm Optimization · Reinforcement Learning

1 Introduction

With growing complexity of cyber-physical systems and robotics, guarantees in performing certain tasks successfully, are required while an agent interact with the environment. This requirement is an essential feature in automation. For example, multiple agents interacting with the environment should ensure to achieve the specified objectives. An ambitious goal is to design an autonomous system that would synthesize controller(s) for the system regardless of how the environment behaves. This is a fundamental problem in AI and Computer Science that has been extensively studied under different titles by control communities [1][2][3]. Modern cyber-physical systems rely heavily on the efficacy of the feedback controllers. The efficacy of a feedback controller is generally measured by its capability of keeping the system stable at an equilibrium point and making it follow a reference trajectory. Moreover in real world safety-critical systems we also need to constraint control to ensure safe-exploration by avoiding certain states from which an agent cannot recover. In other words, for safety-critical systems, another important property is the Region of Attraction (ROA) [4] which represents the region of the state-space from where if the system initiates its operation, it is guaranteed that the system will remain inside the region during its entire operation and eventually reach an equilibrium state. Though

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the feedback controller synthesis problem for stability and trajectory tracking has been widely studied, the feedback controller synthesis problem for maximizing the ROA of a nonlinear dynamical system has not received that much attention. In this paper we build work on recent techniques proposed by Berkenkamp et al [5, 6], which introduces the idea of computing the ROA of a nonlinear dynamical system. It is by combining ideas of Gaussian Process (GP) learning to approximate the model uncertainties and Lyapunov stability theory to estimate the safe operating region. The key limitation is however, given a controller it only provides an algorithm to compute its ROA, but given a nonlinear dynamical system, a radical question is how do we synthesize a feedback controller that helps the dynamical system to achieve the best possible ROA? This also leads to one of the most pivotal decision of parameters tuning while designing a controller for a dynamical systems.

Some earlier techniques involved quantization error criteria to synthesize feedback controller [7]. Off-late many of the developments happened by using deep reinforcement learning (Deep RL) methods to learn continuous control policies for dynamical systems [8]. The goal is generally to synthesize a feedback controller that has a very good trajectory tracking performance, which is captured in the form of so called LQR cost [9]. However, the controller having the best LQR cost may not achieve the best ROA for the dynamical system.

In this paper we address these limitations and makes the following contributions: We present a novel method of synthesizing an optimal controller that provides a very good tracking performance together with achieving the best possible ROA for the closed-loop system. We employ a stochastic optimization technique to solve the best ROA feedback controller synthesis problem, keeping also in mind the LQR cost as a performance index. As a specific technique, we use the Particle Swarm Optimization (PSO) [10] method to evaluate the best policies.

2 Problem

2.1 Preliminaries

In this section we review some control theoretic definitions and notations that will be used in our analysis. But before that we introduce the problem statement formally. Let us consider a nonlinear, deterministic, discrete-time dynamical system:

\[
\dot{x}_t = f(x_t, u_t)
\]

(1)

where, \(x(t) \in \mathcal{S} \subset \mathbb{R}^m\) and \(u(t) \in \mathcal{K} \subset \mathbb{R}^n\) are the states and control inputs at discrete time index \(t \in \mathbb{N}\). The system is controlled by any feedback policy \(\pi : \mathcal{S} \rightarrow \mathcal{K}\), thus the closed loop dynamics is given by \(\dot{x}_t = f(x, \pi(x))\). The policy \(\pi\) is safe to use within \(\mathcal{R}_\pi \subset \mathcal{S}\), i.e, \(\mathcal{R}_\pi\) is true largest ROA of this policy.

**Notation** We have used following symbols throughout the paper :

\[
\mathbb{R} = \text{set of all real nos.}
\]

\[
\mathbb{R}^m = m \text{ dimensional real space.}
\]

\[
\dot{x} = \frac{dx}{dt}
\]

\[
\mathcal{V}(c) = \text{ safe level set } = \{x : v(x) \leq c, \ c > 0 \in \mathbb{R}\}
\]

\[
\theta = \text{ parameter vector}
\]

\[
\mathcal{V}_\theta(c_0) = \text{ initial safe level set corresponding } v_\theta
\]

**Definition 2.1** (Cost function and LQR controller) Given a discrete time state-space model

\[
x[n + 1] = Ax[n] + Bu[n]
\]

associated quadratic cost function to be minimize

\[
J(u) = \sum_{n=1}^{\infty} \left( (x[n]^tQx[n] + u[n]^tRu[n]) \right)
\]

The solution of this control problem for linear system is given by the state-feedback law,

\[
u[n] = -Kx[n]
\]

where \(K\) is the optimal gain matrix.

**Definition 2.2** (ROA) [4] Let \(x_0\) be an asymptotically stable equilibrium point (state) then the total set of initial points (states) from which trajectories converge to \(x_0\) as \(t \rightarrow \infty\) is called the Region of Attraction.
A reliable way to estimate $R_\pi$ is by using a Lyapunov function. We have the following theorems in this concern.

**Theorem 2.1 (Lyapunov stability)** \[11\]
Suppose $f(\cdot, \pi)$ is locally Lipschitz continuous and has an equilibrium point at $x_0 = 0$ and $v : S \mapsto \mathbb{R}$ be locally Lipschitz continuous on $S$. Let, there exists a set $\Delta_0 \subseteq S$ containing 0 on which $v$ is positive-definite and $v(f(x, \pi(x))) < v(x) \forall x \in S$ then $x_0 = 0$ is an asymptotically stable equilibrium. In this case, $v$ is known as a Lyapunov function for the closed-loop dynamics $f(\cdot, \pi)$, and $\Delta_0$ is the Lyapunov decrease region for $v$.

Instead of searching a pertinent Lyapunov candidate while the computational methods often constraint to a very specific class of functions, Berkenkamp et al \[6\] proposes a technique to construct the Lyapunov candidate as an inner product function. Let, there exists a set $\Delta_0 \subseteq S$ containing 0 on which $v$ is positive-definite and $v(0) = 0$ and $v(x) > 0, \forall x \in S \sim \{0\}$. Moreover, if $v$ is Lipschitz continuous for each layer $\ell$, then $v$ is also locally Lipschitz continuous.

Following the method of estimating ROA as much of $R_\pi$ as possible, of a given policy we develop an algorithm to scrutinize the best possible policies based on the performance measure of LQR cost and ROA. We implement the following optimization technique to learn such policy through sequential iteration.

**Definition 2.3 (PSO)** It is an evolutionary computational method developed by American scholars Kennedy and Eberhart in the early 90s inspired by social behavior of fish schooling and bird flocking \[12\]. The underlying idea is to seek for an optimal solution through particles or agents, whose trajectories are adjusted by a stochastic and a deterministic component. Each particle keeps track of its coordinates in the search space and they are influenced by its local best or “best” achieved position called pbest and the group’s “best” position called gbest. It employs an objective function, also known as fitness function, to evaluate the effectiveness of each particle in an iteration, and through several iterations, it searches for the optimal solution. The iteration continues until convergence or for a pre-specified number of times.

In our methodology, the fitness function is defined to map a candidate feedback controller in the search space to the size of its ROA and its trajectory tracking performance in terms of LQR cost. The controllers are the particle in the search space and in each step, we identify the gbest particle according to the fitness function.

### 2.2 Algorithm

Initially a set of random coordinates of particles (controllers) with random positions and velocities are considered. Then the newer sets of coordinates get updated through PSO which run to obtain the global best particle (gbest) or configuration. The local best or pbest having the larger LQR cost and smaller ROA obtained locally so far, stored in the memory variable which is then followed by the searching of gbest among the set of pbest by exploration. Consequently the optimal solution achieved through iteration.

**For** each particle $i$ and each dimension $k$

Initialize position $x_{ik}$ and velocity $v_{ik}$ randomly within permissible range

**END For**

Iteration $j = 1$

**DO**

**For** each particle $i$ calculating ROA

**Input :** closed loop dynamics $f_i(\cdot, \pi(\cdot))$; parametric Lyapunov candidate $v_{i,\theta} : S \mapsto \mathbb{R}^+$; level set expansion multiplier $\lambda_i > 0$

$c_{i,0} \leftarrow \max_{x \in S} v_{i,\theta}(x)$ such that $V_\theta(c_{i,0}) \subseteq D_{v_{i,\theta}}$

Repeat

Sample a finite batch $S_b \subset V_\theta(\lambda_i G_{i,k})$

Forward-Simulate the batch with $f_i(\cdot, \pi(\cdot))$ over finite time steps

Update $\theta$ via batch SGD on $S_b$

$c_{i,k+1} \leftarrow \max_{x \in S} v_{i,\theta}(x)$, such that $V_\theta(c_{i,k+1}) \subseteq D_{v_{i,\theta}}$

Continue until convergence

**Fitness value = An weighted combination of ROA and LQR cost**
If fitness value better than $p_{best_{ik}}$ in the history
Set current fitness value as $p_{best_{ik}}$
END If
END For
Choose $gbest_k$
For each particle $i$ and each dimension $k$
$$v_{ik}(m+1) = w v_{ik}(m) + \alpha \times \text{rand}(\cdot) \times (p_{best_{ik}} - x_{ik}) + \beta \times \text{rand}(\cdot) \times (gbest_k - x_{ik})$$
$$x_{ik}(m+1) = x_{ik}(m) + v_{ik}(m+1)$$
END For
$m = m + 1$
Continue till the criteria attained.

2.3 Example

In this section we examined the effectiveness of our proposed model on real-world applications. We implement Algorithm 2.2 on the inverted pendulum benchmark with TensorFlow [13] based Python code. A three layers of 64 tanh activation units each have been used to construct the Neural Network Lyapunov candidate. The codes for computing ROA with respect to a fixed LQR policy are available at GitHub [14].

![Figure 1: Comparisons of convergence to the safe set $\mathcal{R}_p$ w.r.t the controllers $K_{LQR}, K_{max}, \text{and } K_O$](image)

The system is governed by the second order differential equation:

$$\ddot{\phi} = \frac{g}{\ell} \sin \phi - \frac{\mu}{3} \dot{\phi} + \frac{1}{3} u$$

$$\phi(0) = c$$

$$\dot{\phi}(0) = 0$$

where $I$ is the moment of inertia of pendulum, $\mu$ is a frictional coefficient, $\phi$ is an angle from the upright equilibrium position, $x := [\phi, \dot{\phi}]$ is the state matrix and $g$ is the gravitational constant. Given this system we derive three different controllers $K_{LQR}, K_{max}, K_O$, on the basis of fitness function. $K_{LQR}$ has minimal LQR cost but rather inferior ROA, $K_{max}$ has best possible ROA but maximal LQR cost and finally we obtain an optimal controller $K_O$ with moderate LQR cost and impressive ROA as well. Table 1 reflect the comparisons. We choose the number of discretized states along each dimension as 250. Following 10 outer iterations and 5 inner iterations we obtain such ROA size of each controller.

| Policy | LQR cost | ROA size | % of original Safe set |
|--------|----------|----------|------------------------|
| $K_{max}$ | 287.65 | 21,776 | 90.08 |
| $K_{LQR}$ | 245.067 | 19,721 | 81.58 |
| $K_O$ | 254.54 | 20,514 | 84.86 |

Table 1: Comparisons of ROA and LQR cost w.r.t the controllers $K_{LQR}, K_{max}, \text{and } K_O$. 


Figure 1 shows the results of training and also visualizes how $K_{\text{max}}$ characterizes $R_{\pi}$ with a level set better than both $K_{\mathcal{O}}$ (yellow ellipsoid) and $K_{\text{LQR}}$ (blue ellipsoid) candidates and almost adapts to the shape of $R_{\pi}$. Therefore on the benchmark of largest ROA $K_{\text{max}}$ is the best policy, while Table 1 also reflects its drawbacks of having worst LQR cost. Combining both the results we conclude that the feedback controller $K_{\mathcal{O}}$ has a relatively good trajectory tracking performance as well it also achieve the best possible ROA for the closed-loop system.

We have also created a Simulink model to simulate the relation between a controller and its corresponding ROA. Figure 2a-2c demonstrates the efficacy of these controllers. The angle-time plot clearly shows the failure and success rate to maintain the stability of a controller under various initial angle $c$ of the pendulum.

Figure 2: Comparison of Full Nonlinear Simulation Results.
From Figure 2b, we observe that $K_{LQR}$ fail to reach the upright equilibrium point when the pendulum is driven from $\phi(0) = c = \frac{5\pi}{12}$ while the rest two controllers are stable in this case. However, the optimal controller $K_O$ also remain unstable when driven from the position $c = \frac{\pi}{2}$ as shown in Figure 2c. Evidently $K_{max}$ is always stable throughout the experiment. In other words, these figures distinguishes the controllers on the basis of ROA, and typically shows that a certain disturbance easily drive the system out of the ROA and then fail to come back to the stable equilibrium point.

3 Related Work

In this section, we will briefly explore some of the major works that are closely related to ours. Safe-learning is an active area of research that has been drawn prominent attention from both the researchers in machine learning and control communities [15, 16, 17]. Discrete Markov Decision Process, Model Predictive Control scheme these are few areas that has considered the existence of feasible return trajectories to a safe region of the state space with high probability. Nevertheless for a non-linear dynamical system Lyapunov functions are the most convenient tools for safety certification and hence ROA estimations [18, 19, 20]. Even though searching such function analytically is not a straight forward task but can be identified efficiently via a semi definite program [21, 22], or using SOS polynomial methods [23]. Some other methods to obtain ROA includes volume over system trajectories, sampling based approaches [24] and so on.

4 Conclusion and Future Work

We have developed a novel method for synthesizing control policies for general nonlinear dynamical systems. Our work borrows insights from recent advances in estimating ROA for nonlinear dynamical system, resulting in algorithms that can learn competitive policies with continuous action spaces while minimizing the LQR cost as well as expanding the ROA, which is essential for energy-efficient, safe, and high-performance operations of life-critical and mission-critical embedded applications.

An interesting area that we want to broadly explore in future is to initiate the learning process, and updates the weights of the neural network through deep RL technique. Once the deep neural network for RL is initialized, a reward function that captures both the trajectory tracking capability and the size of the ROA will be used to improve the feedback controller. We also plan to synthesize feedback controllers for complex dynamical systems viz. quad-copters, drones etc. Our final goal would be to fly a UAV by using the feedback controllers synthesized through the proposed techniques.

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