Tunneling of a Massless Field through a 3D Gaussian Barrier.

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Abstract

We propose a method for the approximate computation of the Green function of a scalar massless field $\phi$ subjected to potential barriers of given size and shape in spacetime. This technique is applied to the case of a 3D gaussian ellipsoid-like barrier, placed on the axis between two pointlike sources of the field. Instead of the Green function we compute its temporal integral, that gives the static potential energy of the interaction of the two sources. Such interaction takes place in part by tunneling of the quanta of $\phi$ across the barrier. We evaluate numerically the correction to the potential in dependence on the barrier size and on the barrier-sources distance.

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I. Introduction.

In Quantum Field Theory it is useful in several occasions to have a general expression for the Euclidean two-point correlation function of a massless scalar field $\phi$ in the presence of potential “barriers” in spacetime of the form

$$ V(\phi(x)) = \xi J_\Omega(x) \left[ \phi^2(x) - \phi_0^2 \right]^2, $$

(1)

where $J_\Omega(x)$ is the characteristic function of the 4-region $\Omega$ where the potential has support ($J_\Omega = 1$ for $x \in \Omega$, $J_\Omega = 0$ elsewhere). The region $\Omega$ can be multiple connected, thus representing several barriers placed at different points in spacetime.

Possible applications are connected for instance to the fact that a potential of the form (1) represents a localized imaginary mass term ($m^2 < 0$) in the action of the scalar field $\phi$. Terms of this kind can be present in cosmological models with inflationary fields. It is also known that every quantum field with non-vanishing vacuum expectation value (VEV) has a global imaginary mass term in its lagrangian, which couples to the gravitational field as a cosmological term; one can show that if the VEV is not constant but depends on $x$, it becomes a local cosmological term for the gravitational field.

More generally, suppose we have a system of two interacting fields and regard one of them (or its VEV) as a fixed external source. The coupling term of the two fields becomes a local constraint for the dynamical field, a sort of external potential localized in the regions where the external field has support. It is therefore important to study the tunneling of the dynamical field through these regions, that is, its Green functions. Note that in systems like this translational invariance is generally lost.

It is easy to check that the potential $V(\phi)$ in eq. (1) implements in fact a constraint in the functional integral of the field: writing this integral as

$$ z = \int d[\phi] \exp \left[ - \int d^4 x (\partial \phi)^2 - \int d^4 x V(\phi) \right], $$

one sees that for large $\xi$ the square of the field is forced to take the value $\phi_0^2$ within the region $\Omega$.

For a characteristic function $J_\Omega(x)$ like the one specified above we say that the constraint is imposed in a “sharp” way: in spacetime the potential barrier looks like a step at the boundary of $\Omega$. Smoothing $J_\Omega$ we can obtain a smooth potential barrier. In the following we shall be more interested in this second case.
Note that the potential (1) has the shape of a double well, as long as considered only a function of the field \( \phi \), but regarded as a function of \( x \) it is positive and reminds much more a barrier.

Let us focus on the case of weak fields, such that \( \phi^4 \) can be disregarded with respect to \( \phi^2 \). If the product \( \gamma \equiv \xi \phi_0^2 \) is small, then the effect of the barriers on field correlations is small, too, and can be treated as a perturbation. One can solve the equation for the modified propagator \( G'(x_1, x_2) = \langle \phi(x_1)\phi(x_2) \rangle_V \) in closed form (see the Appendix), finding that \( G' \) is given by a double inverse Fourier transform, with the direct transform of \( J_\Omega \) evaluated at \( (p + k) \):

\[
G(x_1, x_2) = G^0(x_1, x_2) + \gamma G'(x_1, x_2);
\]
\[
G^0(x_1, x_2) = \int d^4 k \, e^{-ik(x_1-x_2)};
\]
\[
G'(x_1, x_2) = \int d^4 p \int d^4 k \, e^{ipx_1}e^{ikx_2} \frac{J_\Omega(p + k)}{k^2p^2}. \quad (2)
\]

In finite-dimensional quantum mechanics computing \( G'(x_1, x_2) \) corresponds to compute the Feynman transition amplitude, related in turn to the system’s wavefunction in the presence of barriers. In field theory the intuitive meaning of \( G'(x_1, x_2) \) is less immediate. However, we can derive from \( G'(x_1, x_2) \) a quantity with a direct physical interpretation: the static potential \( U(x_1, x_2) \) of the interaction of two pointlike sources \( q_1 \) and \( q_2 \) of the field \( \phi \) at rest. This interaction is mediated by the exchange of quanta of \( \phi \). If the barriers are placed somewhere between the sources, the interaction is clearly affected, but it still takes place – provided the product \( \gamma \) is small – with the quanta of \( \phi \) “tunneling” through the barriers (or passing over the wells, depending on the interpretation).

The leading contribution to the static potential \( U(x_1, x_2) \) is obtained from (2) as follows [2]. First one defines \( J_\Omega(x) \) as the product of a 3D function \( j_\Omega(x) \) and a function constant in time, then one integrates over \( t_1 \) and \( t_2 \), multiplies by \( q_1q_2 \) and divides by \( -T \), taking the limit for \( T \to \infty \). The result is

\[
U(x_1, x_2) = U^0(x_1, x_2) + \gamma U'(x_1, x_2) =
\]
\[
= \frac{q_1q_2}{|x_1 - x_2|} - \gamma(2\pi)^8 q_1q_2 \int d^4 p \int d^4 k \, e^{ipx_1}e^{ikx_2} \frac{\tilde{J}_\Omega(p + k)}{k^2p^2}. \quad (3)
\]

This formula is easily generalized to the case of \( N \) charges \( q_1, ..., q_N \), placed respectively at \( x_1, ..., x_N \).

A limit case of the physical situation we are considering is represented by the electrostatic potential of pointlike charges in the presence of perfect conductors. In this case the field is
exactly zero within the region $\Omega$, and $\Omega$ has sharp boundaries — thus $j_{\Omega}(x)$ is a step function and $\tilde{j}_{\Omega}(p)$ a strongly oscillating function. Eq. (3) could be applied to this case only if the parameters $\phi_0$ and $\xi$ could be chosen in such a way that $\phi_0 \to 0$ and $\xi \to \infty$, the product $\gamma = \xi \phi_0^2$ still being finite and small. We know, however, that usually in an electrostatic system the change in potential energy due to the presence of perfect conductors is not just a small correction. (It can be computed exactly, in principle, solving a classical field equation with suitable boundary conditions.)

The case of interest here is actually more subtle. In the following $j_{\Omega}(x)$ is supposed to be a smooth function and both $\phi_0$ and $\xi$ are taken to be finite. The field square has only a certain probability to be equal to $\phi_0^2$ within $\Omega$. This probability is maximum at the center of $\Omega$ and decreases towards the boundary of $\Omega$. Since $j_{\Omega}(x)$ is smooth (a gaussian function), its Fourier transform $\tilde{j}_{\Omega}(p)$ is smooth too, and the integral (3) can be computed numerically.

It is interesting to study $U'$ in dependence on the geometrical features of the barrier $\Omega$ and on the position of $q_1$ and $q_2$ with respect to it. Take, for instance, a finite size barrier (gaussian ellipsoid, see Section II) lying on the axis joining $x_1$ to $x_2$. We may expect that if one of the two charges is close to $\Omega$, then $|U'/U_0|$ is larger, decreasing if both charges are far away from $\Omega$ — or if $\Omega$ is not on their axis. This behavior is confirmed and specified by our numerical results.

The paper is organized as follows. In Section II we compute the leading correction to the static potential for a barrier with the shape of an ellipsoid. Due to the peculiar behavior of the integrand, the procedure for numerical integration is not trivial and requires some care. We describe it in detail. Results are given in Section III. They concern in particular the dependence of the correction to $U(x_1, x_2)$ on the geometrical setting (size of the barrier and its position with respect to the pointlike sources). Far from exploring all the conceivable variations and related phenomenology, the main aim of this work is to show that the general technique can be successfully applied to real cases.

II. The case of two static sources.

Let us focus now on a configuration with two static sources and one barrier only. We choose
our reference frame in such a way that the sources lie on the \( z \)-axis:

\[
\mathbf{x}_1 = (0, 0, L_1); \quad \mathbf{x}_2 = (0, 0, -L_2).
\]

The spatial shape and size of the barrier are defined by the function

\[
j_\Omega(x) = \exp \left( -\frac{x^2 + y^2}{a^2} - \frac{z^2}{b^2} \right). \tag{4}
\]

This means that the region \( \Omega \) is like an ellipsoid centered at the origin, with symmetry axis along \( O_z \), radius of the order of \( a \) and thickness of the order of \( b \). We suppose that \( a > b \), thus the ellipsoid is "squeezed" on the \( xy \)-plane. More precisely, the region \( \Omega \) itself is not sharply defined, but the surfaces where \( j_\Omega(x) \) is constant are ellipsoids. For instance, on the surface defined by

\[
\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1
\]

the function \( j_\Omega(x) \) is constant and equal to \( e^{-1} \).

The Fourier transform of (4) is

\[
\tilde{j}_\Omega(p) = \frac{\pi^{3/2} a^2 b}{2} \exp \left\{ -\frac{1}{4} \left[ -a^2(p_x^2 + p_y^2) - b^2(p_z^2) \right] \right\}. \tag{5}
\]

The charges \( q_1 \) and \( q_2 \) can be taken to be unitary and the distances \( L_1 \) and \( L_2 \) expressed as multiples of the ellipsoid radius \( a \): \( L_1 \equiv n_1 a, \ L_2 \equiv n_2 a \). From (3), (5) we obtain

\[
U'(x_1, x_2) = -(2\pi)^8 \pi^{3/2} a^2 b \int d\mathbf{p} \int d\mathbf{k} \frac{e^{ik_x n_1 a - ip_z n_2 a}}{k^2 p^2} \exp \left\{ -\frac{1}{4} a^2 (p + k)^2_{xy} - \frac{1}{4} b^2 (p_z + k_z)^2 \right\},
\]

where \( \mathbf{V}_{xy} \) denotes the component of a vector \( \mathbf{V} \) in the plane \( xy \). In the following we shall be most interested in the case with one charge far from the barrier \( (n_1 \gg 1) \), while the other charge is close to it (typically in our numerical calculations \( n_2 \) ranges between 1 and 15). Accordingly we set \( n_1^{-1} = \varepsilon, \ n_2 = n \). After rescaling \( k_z \to \varepsilon k_z \) we obtain

\[
U'(x_1, x_2) = -(2\pi)^8 a^3 b \int d\mathbf{p} \int d\mathbf{k} \frac{e^{ik_x a - ip_z n a}}{L_1 p^2(k_x^2 + k_y^2 + \varepsilon k_z^2)} \exp \left\{ -\frac{1}{4} a^2 (p + k)^2_{xy} - \frac{1}{4} b^2 (p_z + \varepsilon k_z)^2 \right\},
\]

where \( \rho = b/a \) is the ratio between the thickness \( b \) and the radius \( a \) of the ellipsoid.
Next we introduce the polar variables $\theta_k, \theta_p, \phi_k,$ and $\phi_p$. In the following $k$ and $p$ will not denote four-vectors anymore, but $|k|$ and $|p|$, respectively. The square of the component of the vector $(p + k)$ in the $xy$ plane is

$$(p + k)^2_{xy} = p^2 \sin^2 \theta_p + k^2 \sin^2 \theta_k + 2pk \sin \theta_p \sin \theta_k \cos(\phi_k - \phi_p).$$

The other components are

$$k_z = k \cos \theta_k; \quad p_z = p \cos \theta_p;$$

$$k^2_x + k^2_y = k^2_{xy} = k^2 \sin^2 \theta_k.$$

Finally, introducing the variables

$$s = \cos \theta_k, \quad t = \cos \theta_p, \quad \phi = (\phi_k - \phi_p)$$

one obtains, remembering that the integrand is even in $s, t$, the following basic formula:

$$U'(x_1, x_2) = -\frac{(2\pi)^{10}}{\sqrt{\pi}} \frac{ab}{L_1} 2\pi \int_0^{2\pi} d\phi \int_{-1}^{1} ds \int_{-1}^{1} dt \int_0^{\infty} dk \int_0^{\infty} dp \frac{\cos(ks - npt)}{1 - s^2(1 - \varepsilon^2)} \times \exp \left[\frac{-p^2(1 - t^2)}{2} - k^2(1 - s^2) - 2pk \cos \phi \sqrt{(1 - t^2)(1 - s^2)}\right]$$

$$\equiv -\frac{(2\pi)^{11}}{\sqrt{\pi}} \frac{ab}{L_1} \int_0^{2\pi} d\phi \int_{-1}^{1} ds \int_{-1}^{1} dt \int_0^{\infty} dk \int_0^{\infty} dp f(\phi, s, t, k, p; \varepsilon, \rho, n)$$

$$\equiv -\frac{(2\pi)^{11}}{\sqrt{\pi}} \frac{ab}{L_1} F(\varepsilon, \rho, n). \quad (6)$$

A. Preliminary study of the integrand.

It is important to discuss in advance the case in which $\rho$ and $\varepsilon$ take values much smaller than 1, that is, $\Omega$ is very thin and the distance of the first charge from $\Omega$ is much larger than $a$. When $t$ and $s$ approach +1 or -1, for small values of $\rho$ the integral over $k$ and $p$ converges very slowly at infinity and the factor $\cos(ks - npt)$ performs a large number of oscillations. For very small $\varepsilon$ there are many more oscillations in $k$ than in $p$. (In the limit $\rho \to 0$ the integral makes sense only as a distribution. We shall never approach this limit, however.)

Let us set, for instance, $s = 1, t = 1$ and $\phi = \pi/2$ in the argument of the exponential in (6). We obtain the exponential factors

$$\exp \left[\frac{-p^2(1 + \varepsilon k)^2}{2}\right] = \exp \left[\frac{-2\rho^2 \varepsilon kp}{2}\right]. \quad (7)$$
The first factor on the r.h.s. of (7) has a range in $p$ of the order of $\rho^{-1}$ and the second factor has a range in $k$ of the order of $(\rho \varepsilon)^{-1}$. The third factor has a range in $p$, for fixed $k$, of the order of $(\rho \sqrt{\varepsilon k})^{-1}$ and a range in $k$, for fixed $p$, of the order of $(\rho \sqrt{\varepsilon p})^{-1}$. Fortunately this latter factor is not relevant: if its range is larger than the other two ranges then it does not play any role; if it is smaller then it is sufficient to refer to the other ranges.

As soon as $s^2$ and $t^2$ go away from 1, the number of oscillations of the integrand decreases. For instance, setting $s = t = 0.98$ we obtain the exponential factors

$$\sim \exp \left[ -\left( \rho p \right)^2 - \left( \rho \varepsilon k \right)^2 - 2 \rho^2 \varepsilon k p \right] \exp \left[ -0.04 p^2 - 0.04 k^2 \right].$$

When $\rho$ is much smaller than 1 the range of this product is determined by the second exponential and does not depend on $\rho$.

It is also easy to take into account the term proportional to $\cos \phi$. After setting $\phi = \pi$ that term gives a positive contribution to the argument of the exponential; thus studying the range of the resulting expression we obtain an upper limit valid for any $\phi$.

**B. Integration domains.**

Independently of the considerations above, it is possible to plot the integrand $f(\phi, s, t, k, p; \varepsilon, \rho, n)$ for several different values of $\rho$ and $\varepsilon$ and check the ranges of the exponentials. In order to better control the oscillations of $f$, we study it in 4 different domains of the variables $s, t$:

- Domain 1: $t, s \in [0, 1 - \alpha]$;
- Domain 2: $t \in [0, 1 - \alpha]; \quad s \in [1 - \alpha, 1]$;
- Domain 3: $t \in [1 - \alpha, 1]; \quad s \in [0, 1 - \alpha]$;
- Domain 4: $t, s \in [1 - \alpha, 1 - \alpha]$.

A typical value of $\alpha$ employed in the program is $\alpha = 0.02$. The total integration domain in $s, t$ is obtained by “reflecting” each of the domains above with respect to one axis and then reflecting again the result with respect to the origin ($s \rightarrow -s, t \rightarrow -t, s, t \rightarrow -s, -t$). In each domain $i$ there is a maximum value for the variables $k$ and $p$, beyond which $f$ is equal to zero for any practical purpose. Denoting by $K_i$ and $P_i$ these ranges, Table 1 shows the results found for some considered values of $\varepsilon$ and $\rho$.

Since the integration over $k$ and $p$ is extended to wide ranges, the most reasonable technique for the numerical computation of the integral (3) appears to be a Montecarlo sampling of the integrand. The sampling algorithm evaluates the average value of $f$ in each domain, extending
the values of $k$ and $p$ up to the maximum range necessary for that domain. At the end the global average is computed, weighing each single average with the ratio between the domain volume and the total volume. Denoting by $f_i$ the average of $f$ in the domain $i$ and by $V_i$ the domain volume we have

$$F = \sum_{i=1}^{4} f_i V_i = 8\pi \left[ (1 - \alpha)^2 K_1 f_1 + \alpha (1 - \alpha) K_2 P_2 f_2 + \alpha (1 - \alpha) K_3 P_3 f_3 + \alpha^2 K_4 P_4 f_4 \right]. \quad (8)$$

III. Results of the numerical integration.

The contributions of the Domains 2, 3 and 4 to the integral $F$ (compare (8)) are found to be small with respect to the contribution of Domain 1. The fluctuations of the average of $f$ in Domains 2 and 4 (where $s^2$ approaches 1) may be very large. In order to achieve a sufficient precision these regions have been sampled with a large number of points (up to $\sim 10^{10}$). The standard routine “ran2” \footnote{3} was used for random numbers generation.

The dependence of the integral $F$ on the parameters $\varepsilon$ and $\rho$ is very weak, thus $U'$ depends on $a$, $b$ and $L_1$ mainly as $ab/L_1$ (see eq. (6)). The study of the dependence of $U'$ on $n$ is more difficult, because this dependence is entirely contained in the integral $F$ and can be evaluated only numerically. One needs to insert in the program a cycle which samples the integrand for different values of $n$, typically between 1 and 15. This is possible because the ranges $P_i$, $K_i$ do not depend on $n$.

The numerical evaluation of $F$ as a function of $n$ in the range $n = 1...15$, with $\varepsilon = 0.1$ and $\rho = 0.3$, gives the results shown in Fig. 1. With $\rho = 0.1$ and $\rho = 0.032$ one obtains very similar results, thus confirming the weak dependence on $\rho$ (Fig. 2). As expected varying $\varepsilon$ does not affect much the value of $F$ either, since the dependence on the distance $L_1$ is already factorized out of the integral (compare Fig. 3).

Figs 1, 2, 3 reveal an exponential behavior of $F(n)$ of the form

$$F(n) \sim \exp(-mn + q) + b.$$

It is also clear just from the graphs that the exponential decrease of $F$ for large $n$ leaves an asymptotic value $F = b$, with $b$ in the interval 0.1-0.2. This is an interesting behavior, as it means that the “shadow” produced by the barrier in the static field of the two sources has a long, constant tail.
A least-squares fit of the data gives the results of Table 2. Excluding from the fit the first two points \((n = 1, 2)\) we obtain better estimates for the distribution tail and for \(b\). The errors on the parameters of the fit, in particular those on \(b\), are small. They can be estimated knowing that the least-squares sum of the percentual errors \(S = \sum_n \{1 - \exp(-mn + q + b)/F(n)\}^2\) has a minimum value \(S_{min} \sim 0.05\) and that its second partial derivatives at the minimum are of the order of \(\frac{\partial^2 S}{\partial b^2} \sim 4 \cdot 10^2\), \(\frac{\partial^2 S}{\partial m^2} \sim 10^2\), \(\frac{\partial^2 S}{\partial q^2} \sim 10\).

IV. Conclusions.

Our technique for the computation of the Green function and the static potential of two pointlike sources appears to work well for weak fields, yielding reasonable results. The method is based upon a double 3D Fourier transform of the function which represents size and position in space of the potential well or barrier. This double transform is necessary, due to the lack of translational invariance of the system. Its numerical evaluation requires a preliminary analytical study and a subdivision of the integration volume in a few domains, because the range of the real exponential factors in the integrand varies considerably.

We studied the case of a smooth barrier with the form of a gaussian ellipsoid in coordinate and momentum space. For values of \(\rho\) and \(\varepsilon\) not much smaller than 1 a good precision was obtained. \((\rho\) is the ratio between the lengths \(a\) and \(b\) of the ellipsoids axes and \(\varepsilon\) is the ratio between the length of the major axis \(a\) and the distance \(L_1\) of the first source from the ellipsoid.\)

Denoting by \(n\) the distance of the second source in units of the major axis, we found that the correction to the interaction potential along the line joining the two sources and the barrier has the following form (compare eq.s (3), (6)):

\[
U = U^0 + \gamma U' = \frac{1}{|x_1 - x_2|} - \frac{(2\pi)^{11}}{\sqrt{\pi}} \frac{\gamma ab}{L_1} F(\varepsilon, \rho, n)
\]

\[
= \frac{1}{|x_1 - x_2|} \left[ 1 - \frac{(2\pi)^{11}}{\sqrt{\pi}} \gamma ab(1 + n\varepsilon) F(\varepsilon, \rho, n) \right].
\]

The function \(F\) depends very weakly on \(\rho\) and \(\varepsilon\). Its dependence on \(n\) is displayed in Fig.s 1-3 and shows an exponential decay followed by a constant tail.

The behavior summarized above is interesting in itself, being the result of a sort of “tunneling” of the scalar field through a region where it is constrained or has imaginary mass. We have seen that the local imaginary mass term affects the propagation of the field also outside
the region $\Omega$ where it has support. This feature is easily understood from the physical point of view; we gave here a method for its quantitative evaluation.

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### Appendix: Proof of the expressions for $G'$, $U'$.

We give here the proof of eq.s (2) and (3) of the main text. Expanding the square in (1) we obtain for $W[J]$

$$W[J] = \int d[\phi] \exp \left\{ - \int d^4x \left[ (\partial\phi)^2 - 2\xi \phi_0^2 J_\Omega(x) \phi^2(x) + \xi J_\Omega(x) \phi^4(x) + \xi J_\Omega(x) \phi_0^4 \right] \right\}.$$  

The last term in the square bracket is constant with respect to $\phi(x)$ and its exponential can be factorized out of the functional integral. In a first instance – for weak fields – we can disregard the $\phi^4(x)$ term. We are then led to consider a quadratic functional integral, and the “modified propagator” $G(x, y) = \langle \phi(x) \phi(y) \rangle_J$, which by definition satisfies the equation

$$\left[ \partial^2_x + \gamma J_\Omega(x) \right] G(x, y) = -(2\pi)^4 \delta^4(x - y), \quad (9)$$

where $\gamma = 2\xi \phi_0^2 > 0$. Let us focus on the case when $\phi_0^2 = 0$ inside the regions $\Omega_i$ and let us take the limit $\phi_0 \to 0$ and $\xi \to \infty$ in such a way that $\gamma$ is finite and very small, so that the term $\gamma J_\Omega(x)$ in eq. (3) constitutes only a small perturbation, compared to the kinetic term. Then we can set

$$G(x, y) = G^0(x, y) + \gamma G'(x, y),$$

where $G^0(x, y)$ is the propagator of the free scalar field, and we find immediately that $G'(x, y)$ satisfies the equation

$$\partial^2_x G'(x, y) = -J_\Omega(x) G^0(x, y). \quad (10)$$

Unlike $G^0(x, y)$, in general $G'(x, y)$ will not depend only on $(x - y)$, because the source breaks the translation invariance of the system. In order to go to momentum space it will therefore be necessary to consider the Fourier transform of $G'(x, y)$ with respect to both arguments. We define $\tilde{G}'(p, k)$ and $\tilde{J}_\Omega(p)$ as follows:

$$G'(x, y) = \int d^4 p \int d^4 k e^{ixp} e^{iky} \tilde{G}'(p, k)$$
and

\[ J_\Omega(x) = \int d^4p \, e^{ipx} \tilde{J}_\Omega(p), \quad G^0(x, y) = \int d^4k \, \frac{e^{-ik(x-y)}}{k^2}. \]

The right hand side of (10) can be rewritten as

\[ J_\Omega(x)G^0(x, y) = \int d^4p \int d^4k \, e^{ipx} \tilde{J}_\Omega(p) \frac{e^{-ik(x-y)}}{k^2} = \int d^4k \int d^4p \, e^{iky} e^{ipx} \tilde{J}_\Omega(p + k) \frac{1}{k^2} \]

and we obtain the following algebraic equation for the double Fourier transform of the first order correction to the propagator:

\[
p^2 \tilde{G}'(p, k) = \frac{\tilde{J}_\Omega(p + k)}{k^2}.
\]

Transforming back, in conclusion we find eq. (2) of the main text, namely:

\[ G'(x, y) = \int d^4p \int d^4k \, e^{ipx} e^{iky} \tilde{J}_\Omega(p + k) \frac{1}{k^2 p^2}. \] (11)

Therefore, if we know the Fourier transform of the characteristic function \( J_\Omega \) of the spacetime region where the constraint is imposed, we can in principle compute the leading order correction to the field propagator and thus to \( W[J] \).

It is known \[5\] that the vacuum-to-vacuum amplitude \( W[J] = \langle 0^+ | 0^- \rangle_J \) of a field system in the presence of an external source \( J \) is related to the logarithm of the systems’ ground state energy:

\[ E_0[J] = -T^{-1} \ln W[J], \]

where the functional integral is supposed to be suitably normalized and the source vanishes outside the temporal interval \([-T/2, +T/2]\), with \( T \) eventually approaching infinity. (We use units in which \( \hbar = c = 1 \).)

An interesting application of (11) occurs in the case when the field \( \phi(x) \) also interacts with \( N \) static pointlike sources placed at \( x_1, x_2, \ldots, x_N \). Namely, let us add a further, linear coupling term \( S_Q \) to the action of the system:

\[ S_Q = \int d^4x \, Q(x) \phi(x), \quad \text{with} \quad Q(x) = \sum_{j=1}^{N} q_j \delta^3(x - x_j). \]

The ground state energy of the system corresponds, up to a constant, to the static potential energy of the interaction of the sources through the field \( \phi \). As before, it is obtained from the functional average of the interaction term, computed keeping the constraint into account:

\[ E_0[J, Q] = U(x_1, ..., x_N) = -T^{-1} \ln \langle \exp \{-S_Q\} \rangle_J. \] (12)
Expanding (12) one finds that to leading order in the \( q_j \)s, \( U(x_1, ..., x_N) \) is given by a sum of propagators integrated on time:

\[
U(x_1, ..., x_N) = -T^{-1} \sum_{j,l=1}^{N} q_j q_l \int dt_j \int dt_l \langle \phi(t_j, x_j) \phi(t_l, x_l) \rangle_j,
\]

where \( t_j, t_l \in [-T/2, +T/2] \). Since the regions \( \Omega_i \) are infinitely elongated in the temporal direction the function \( \tilde{J}_\Omega(p + k) \) gets factorized as

\[
\tilde{J}_\Omega(p + k) = (2\pi)^4 \delta(p_0 + k_0) \tilde{J}_\Omega(p_0 + k^2).
\]

(13)

Clearly the potential is disturbed by the presence of the “barriers” \( j_\Omega(x) \). To first order in \( \gamma \) we can write

\[
U(x_1, ..., x_N) = U^0(x_1, ..., x_N) + \gamma U''(x_1, ..., x_N)
\]

and taking into account eq.s (11), (13) we find

\[
U''(x_1, ..., x_N) = -T^{-1} \sum_{j,l=1}^{N} q_j q_l \int dt_j \int dt_l G''(x_j, x_l) =
\]

\[
= -(2\pi)^4 T^{-1} \sum_{j,l=1}^{N} q_j q_l \int dt_j \int dt_l \int d^4 p \int d^4 k \frac{e^{ipx_j + ikx_l} \tilde{J}_\Omega(p + k)}{k^2 p^2} =
\]

\[
= -(2\pi)^8 T^{-1} \sum_{j,l=1}^{N} q_j q_l \int dt_j \int dt_l \int d^4 p \int d^4 k \frac{e^{i(p_0(t_j - t_l) + px_j + ikx_l)} \tilde{J}_\Omega(p + k)}{(p_0^2 + k^2)(p_0^2 + p^2)}.
\]

Changing to variables \( t = t_j - t_l \) and \( s = t_j + t_l \) and integrating we finally obtain the contribution of the perturbation to the static potential energy (eq. (3) of the main text):

\[
U''(x_1, ..., x_N) = -(2\pi)^8 \sum_{j,l=1}^{N} q_j q_l \int dp \int d^4 k e^{ipx_j + ikx_l} \tilde{J}_\Omega(p + k) \frac{p_0(p + k)}{k^2 p^2}.
\]
References

[1] S. Weinberg, Rev. Mod. Phys. 61 (1989) 1.

[2] M. Bander, Phys. Rep. 75, 205 (1981).

[3] B.P. Flannery, W.H. Press, S.A. Teukolsky, W.T. Vetterling, *Numerical Recipes in FORTRAN 77* (Cambridge University Press, Cambridge, 1992), Second Edition, p. 272.

[4] G. Modanese, Phys. Rev. D 54, 5002 (1996).

[5] K. Symanzik, Comm. Math. Phys. 16, 48 (1970). See also G. Modanese, Nucl. Phys. B 434, 697 (1995).
| $\varepsilon$ | $\rho$ | $K_1$ | $P_1$ | $K_2$ | $P_2$ | $K_3$ | $P_3$ | $K_4$ | $P_4$ |
|--------------|--------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.1          | 0.3    | 12    | 10    | 80    | 10    | 12    | 10    | 80    | 10    |
| 0.1          | 0.1    | 10    | 10    | 100   | 25    | 20    | 20    | 120   | 40    |
| 0.1          | 0.032  | 10    | 10    | 600   | 20    | 20    | 60    | 600   | 60    |
| 0.1          | 0.01   | 20    | 20    | 1500  | 30    | 20    | 200   | 1500  | 200   |
| 0.032        | 0.032  | 10    | 10    | 2000  | 15    | 20    | 70    | 1800  | 80    |
| 0.032        | 0.01   | 12    | 12    | 6000  | 15    | 15    | 180   | 6000  | 250   |
| 0.01         | 0.032  | 10    | 10    | 7500  | 15    | 10    | 70    | 7500  | 90    |
| 0.032        | 0.0032 | 15    | 15    | 18000 | 15    | 15    | 650   | 18000 | 700   |

Table 1: Integration ranges in the four domains, for some values of $\varepsilon$, $\rho$. 

Table 2: Results of the best fit $F(n) = \exp(-mn + q) + b$. 

| $\varepsilon$       | $b$  | $m$  | $q$  |
|----------------------|------|------|------|
| $0.1$ (Fig. 1)       | 0.14 | 0.29 | 0.3  |
| with $n > 2$         | 0.12 | 0.24 | 0.1  |
| $0.05$ (Fig. 3)      | 0.17 | 0.32 | 0.4  |
| with $n > 2$         | 0.16 | 0.26 | 0.1  |
FIGURE CAPTIONS.

Fig. 1 - Dependence of $F(\varepsilon, \rho, n)$ on $n$, in the range $n = 1...15$, for $\varepsilon = 0.1$ and $\rho = 0.3$. Errors are $\sim 0.01$.

Fig. 2 - Comparison of the values of $F(\varepsilon, \rho, n)$ for $\varepsilon = 0.1$ and $\rho = 0.3$ (white circles), $\rho = 0.1$ (black triangles) and $\rho = 0.032$ (black circles).

Fig. 3 - Same as in Fig. 1, for $\varepsilon = 0.05$. Errors are larger, as shown.
