No-Go Theorem in Spacetimes with Two Commuting Spacelike Killing Vectors

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Four-dimensional Riemannian spacetimes with two commuting spacelike Killing vectors are studied in Einstein’s theory of gravity, and found that no outer apparent horizons exist, provided that the dominant energy condition holds.

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I. INTRODUCTION

The final fate of a collapsing massive star, after it has exhausted its nuclear fuel, has been one of the outstanding problems in classical relativity. Despite of numerous efforts over the last three decades or so, our understanding is still mainly limited to several conjectures, such as, the cosmic censorship conjecture [1] and the hoop conjecture [2]. To the former, many counter-examples have been found [3], although it is still not clear whether those particular solutions are generic [4]. To the latter, no counter-example has been found yet in four-dimensional spacetimes, but it has been shown recently that this is no longer the case in high dimensions [5].

Due to its (mathematical) complexity, the studies of gravitational collapse have been mainly restricted to spacetimes with spherical symmetry [3]. This is a very ideal case and there have been many attempts to study the problem with less symmetry, for example, in the spacetimes with axial symmetry [6], in which only one spacelike Killing vector exists. However, analytical studies of these spacetimes seem still far beyond our reach. Therefore, the next case would be spacetimes with two spacelike Killing vectors, a subject that will be considered in this Letter.

II. SPACETIMES WITH TWO COMMUTING SPACELIKE KILLING VECTORS

Specifically, we consider a four-dimensional Riemannian spacetime \((\mathcal{M}, g)\) with a signature \(-2\), and assume that throughout the whole spacetime there exist two commuting spacelike Killing vectors, \(\xi_{(2)}\) and \(\xi_{(3)}\),

\[
[\xi_{(2)}, \xi_{(3)}] = 0. \tag{1}
\]

In order to have our results as much applicable as possible, in this Letter we shall not impose any conditions on the nature of the orbits of these Killing vectors, so they can be either open or closed. In addition, we shall also not impose any conditions in asymptotical region(s) of the spacetime, such as, asymptotical flatness. Therefore, the theorem to be given below is valid for all the spacetimes with two commuting spacelike Killing vectors, including the ones with rotation that have been rarely studied so far.

Then, it can be shown that there exist coordinates, \(x^{\mu} (\mu = 0, 1, 2, 3)\), in which we have \(\xi_{(2)} = \partial_{x^2}\) and \(\xi_{(3)} = \partial_{x^3}\), and the metric \(g\) is independent of \(x^2\) and \(x^3\). Since the two-surface \(S\) spanned by \(\xi_{(2)}\) and \(\xi_{(3)}\) is spacelike, there exist two null directions orthogonal to \(S\). Let \(n_{\pm}\) denote these directions and satisfy the condition \(g(n_+, n_-) = 1\). We assume that \(\mathcal{M}\) is orientable, and \(n_{\pm}\) are future-pointing. Because the metric coefficients are independent of \(x^2\) and \(x^3\), it can be shown that the corresponding one-forms of the null vectors \(n_{\pm}\) are proportional to gradients [7], \(n^\pm = N_{\pm}^{-1} \nabla (x_{\pm})\), where the symbols “\(\nabla\)" and \(\nabla\)" denote, respectively, the covariant dual and absolute derivative with respect to \(g\). \(N_{\pm}\) are arbitrary functions of \(x^0\) and \(x^1\), subject to \(g(n_+, n_-) = 1\). Choosing \(x_{\pm}\) as the coordinates \(x^0\) and \(x^1\), we find that

\[
n^\pm = N_{\pm}^{-1} dx_{\pm}. \tag{2}
\]

To have \(n_{\pm}\) future-pointing, we must require \(N_{\pm} > 0\). Note that such defined coordinates \(x_{\pm}\) are unique up to \(x_{\pm} = f_{\pm} (\bar{x}_{\pm})\). However, this gauge freedom does not affect the discussions to be presented below. Instead, one can use it to regularize the metric so that it is free of coordinate singularities. In the following we assume that this is the case. For the details, we refer readers to [8, 9].

On the other hand, assuming that \(m\) is a complex null vector tangent to \(S\) and satisfies the conditions \(g(m, m) = 0\), \(g(m, \bar{m}) = -1\), we find that \(z(a) = (n_+, n_-, m, \bar{m})\) \((a = 1, 2, 3, 4)\) form a null tetrad [10], \(g = n_+ \otimes n_- - m \otimes \bar{m}\), where an overbar denotes the complex conjugate. The components of these null vectors in the chosen coordinates are given by

\[
n^\mu = (0, N_+, X^2, X^3),
\]

\[
n^\epsilon = (N_-, 0, Y^2, Y^3),
\]

\[
m^\mu = (0, 0, Z^2, Z^3), \tag{3}
\]

where \(X^i, Y^i\) and \(Z^i\) \((i = 2, 3)\) are functions of \(x_{\pm}\) only. The components \(X^i\) and \(Y^i\) correspond to rotation [7], while the ones \(Z^i\) to the two degrees of polarization of gravitational waves [11].
Introducing the one-forms \( \hat{n}_\pm \) via the relations, \( \hat{n}_\pm \equiv N_\pm \nu_{\pm}^a \), from Eq. (2) we can see that \( \hat{n}_\pm \) are closed, \( d(\hat{n}_\pm) = 0 \). Then, we define the expansions in the null directions orthogonal to \( \Sigma \) by \[ \theta_\pm \equiv \nabla \cdot \hat{n}_\pm. \] (4)

On the other hand, using the commutation relations \[ 10 \], we find that the spin coefficients have the following properties,

\[
\begin{align*}
\kappa &= \nu = 0, \quad \rho = \bar{\rho}, \quad \mu = \bar{\mu}, \\
\alpha &= \bar{\alpha}, \quad \pi = \bar{\pi} = 2 \alpha, \\
\Re (\epsilon) &= -(2N_+)^{-1} D_+ N_+, \\
\Re (\gamma) &= (2N_-)^{-1} D_- N_-,
\end{align*}
\]

(5)

where \( D_\pm \equiv n_\pm \cdot \nabla \). Since \( \kappa = 0 \), the null vector \( n_+ \) defines a null geodesic congruence \[ 14 \]. Choosing \( N_+ = 1 \), from Eq. (5) we can see that \( \Re (\epsilon) = 0 \), and consequently \( n_+ \) also defines an affine parameter, say, \( \lambda_+ \), in terms of which we have \( \nabla (n_+) / \nabla \lambda_+ = 0 \). Then, the expansion \( \theta_+ \) defined by Eq. (4) gives

\[
\theta_+ \equiv \nabla \cdot \hat{n}_+ = -2 \rho|_{N_+ = 1}.
\] (6)

Replacing \( \kappa, \epsilon, \rho \) by \( -\nu, -\gamma, -\mu \) in the above discussions, we can get the geometrical properties of the null geodesic congruence defined by \( n_- \).

**Definitions \[ 8 \] [13]:** The spatial two-surface \( \Sigma \) is said trapped, marginally trapped, or untrapped, according to whether \( \theta_+ \theta_-|_\Sigma > 0 \), \( \theta_+ \theta_-|_\Sigma = 0 \), or \( \theta_+ \theta_-|_\Sigma < 0 \). Assuming that on the marginally trapped surfaces \( \Sigma \) we have \( \theta_+|_\Sigma = 0 \), then an apparent horizon is the closure \( \Sigma \) of a three-surface \( \Sigma \) foliated by the trapped surfaces \( \Sigma \) on which \( \theta_-|_\Sigma \neq 0 \). It is said outer, degenerate, or inner, according to whether \( \mathcal{L}_- \theta_+|_\Sigma < 0 \), \( \mathcal{L}_- \theta_+|_\Sigma = 0 \), or \( \mathcal{L}_- \theta_+|_\Sigma > 0 \), where \( \mathcal{L}_- \) denotes the Lie derivative along the normal direction \( a_- \). In addition, if \( \theta_-|_\Sigma > 0 \) then the apparent horizon is said future, and if \( \theta_-|_\Sigma < 0 \) it is said past.

**Black holes** are usually defined by the existence of future outer apparent horizons \[ 8 \] [12] [16]. However, in a definition given by Tipler \[ 17 \] the degenerate case was also included, as first noted by Hayward \[ 8 \].

In the following, we shall show that with a positive cosmological constant, both outer and degenerate apparent horizons do not exist in the spacetimes considered here. To this end, let us first notice that Eqs. (4.2q) and (4.2l) in \[ 10 \] now read

\[
D_- \rho = \rho (\gamma + \bar{\gamma}) - \rho \mu - \sigma \lambda - \tau \bar{\tau} - \Psi_2 - R/12,
\]

(7)

\[
0 = \rho \mu - \sigma \lambda - \Psi_2 + R/24 + \Phi_{11},
\]

(8)

where \( R \) denotes the Ricci scalar, \( \Psi_2 \) and \( \Phi_{11} \) are defined as \( \Phi_{11} \equiv -(n_\pm^a n_\mp^a + m^a \bar{m}^a) R_{\mu \nu} / 4 \), and \( \Psi_2 \equiv -C_{\alpha \beta \delta \varsigma} n_\alpha^a n_\beta^b n_\delta^c n_\varsigma^d \), where \( C_{\alpha \beta \delta \varsigma} \) is the Weyl tensor. From Eqs. (7) and (8) we find that

\[
D_- \rho = \rho (\gamma + \bar{\gamma} - 2 \mu) - \tau \bar{\tau} - (R + 8 \Phi_{11}) / 8.
\] (9)

Choosing the particular gauge \( N_+ = 1 \) and considering the fact that on the apparent horizon we have \( \theta_+|_\Sigma = 0 \), from Eqs. (9) and (10) we find that

\[
\mathcal{L}_- \theta_+|_\Sigma = 2 \tau \bar{\tau} + (R + 8 \Phi_{11}) / 4,
\]

(10)

on \( \Sigma \). Clearly, the first term in the right-hand of the above equation is always non-negative. To consider the signs of the second term, following Hawking and Ellis \[ 12 \], we shall express the components of the energy-momentum tensor \( T_{\mu \nu} \) at a given point \( p \) with respect to an orthonormal basis \( E_i (a), \quad (a = 1, 2, 3, 4) \), where

\[
E_{(4)} = \frac{n_+ + n_-}{\sqrt{2}}, \quad E_{(3)} = \frac{n_+ - n_-}{\sqrt{2}}.
\]

(11)

Then, as shown in \[ 12 \], it takes four different canonical forms, which were referred to as Type I - IV, respectively. Types III and IV don’t satisfy any of the three energy conditions (weak, dominant, and strong), and usually are not considered to represent realistic matter. For Types I and II, using the Einstein field equations, \( Ric - (R/2)g + \Lambda g = -T \), we find that

\[
(R + 8 \Phi_{11}) - 4 \Lambda = \begin{cases} 
2 (\mu - p_i), & \text{Type I}, \\
4k, & \text{Type II},
\end{cases}
\]

(12)

where \( i = 1, 2, 3 \). Note that in writing the above expressions we had considered the fact that the roles of the three spacelike vectors, \( E_i (a) \), can be exchanged.

**The dominant energy condition** requires that \( \mu \geq 0 \), \( -\mu \leq p_i \leq \mu \) \( (i = 1, 2, 3) \) for Type I fluid, and \( \nu = +1, \quad k \geq 0, \quad 0 \leq p_j \leq k \) \( (j = 1, 2) \) for Type II fluid \[ 12 \]. Then, combining Eqs. (10) and (12) we have the following:

**Theorem:** Let \((M, g)\) be a four-dimensional Riemannian spacetime to the Einstein field equations, \( Ric - (R/2)g + \Lambda g = -T \) with \( \Lambda > 0 \). Assume that throughout the spacetime there exist two commuting spacelike Killing vectors, \( \xi_2 (0), \xi_3 (0) \). Then, \((M, g)\) contains neither outer nor degenerate apparent horizons, if the dominant energy condition holds.

Note that when \( \Lambda = 0 \), the dominant energy condition only guarantees that \( R + 8 \Phi_{11} \geq 0 \), which together with Eq. (10) implies that \( \mathcal{L}_- \theta_+|_\Sigma \geq 0 \). Therefore, in this case only the existence of outer apparent horizons is excluded.

The significance of the above theorem is two-fold. First, for a stationary spacetime, a spacetime that has an additional timelike Killing vector (at least in certain region(s) of the spacetime), say, \( \xi_0 (0) \), with \( g(\xi_0 (0), \xi_0 (0)) > 0 \), then the above theorem tells us that no black hole exists, unless \( \Lambda \leq 0 \). This is consistent with the fact that so far all the black holes with different topologies rather than that of \( S^2 \) \[ 18 \] are with \( \Lambda < 0 \) \[ 10 \]. This is also in the same spirit of topological censorship \[ 23 \]. It is interesting to note that so far degenerate stationary black holes have not been found in the spacetimes considered...
here, where “degenerate” means that the future apparent horizon that defines the black hole is degenerate.

Second, in the process of gravitational collapse of a source that satisfies the dominant energy condition and has two commuting spacelike Killing vectors, the theorem tells us that black holes can never be formed, unless a negative cosmological constant is present, \( \Lambda < 0 \). In the case where \( \Lambda = 0 \), at most a “degenerate” black hole can be formed by the collapse. An example of such a dynamical “degenerate” black hole was found lately in the study of critical collapse of a cylindrically symmetric scalar field [21]. Restricting ourselves to these spacetimes, we can see that the above theorem supports the hoop conjecture [2]: Horizons form when and only when a mass \( M \) gets compacted into a region whose circumference \( C \) in every direction is \( C \leq 4\pi GM/c^2 \). It should be noted that in [22] the “asymptotical flatness” condition was imposed in the spacetimes with cylindrical symmetry, and found that no trapped surfaces can be formed in electro-vacuum case. This does not contradict with the above theorem. In fact, when this condition is relaxed, degenerate apparent horizons indeed exist in vacuum spacetimes [3]. It should be noted that the notion of asymptotical flatness in this kind of spacetimes is a very delicate issue. For the details, we would like to refer readers to [23].

To study the problem further, let us consider the case where the two Killing vectors all have closed orbits. Then, from the above theorem we can see that the collapse is more likely to form naked singularities than “degenerate” black holes for all the matter fields that satisfy the dominant energy conditions with \( \Lambda \geq 0 \). In fact, compacting the axial coordinate in the examples studied in [24] for cylindrical collapse, we can see that the resultant spacetimes can be asymptotically flat, but naked singularities may still be formed.

Spacetimes where not only the two-spaces \( S \) are compact but also all the spatial hypersurfaces are compact have been intensively studied recently [23], after the pioneering work of Gowdy [20]. This kind of spacetimes is usually divided into three different classes, according to the topologies of these spatial hypersurfaces, \( \Sigma_t \approx T^3 \approx S^1 \times S^1 \times S^1 \), \( S^2 \times S^1 \) or \( S^3 \), where \( S^n \) denotes a unit \( n \)-sphere. In the case \( \Sigma_t \approx T^3 \), one can show that the whole spacetime \((M, g)\) developed from a Cauchy data on a compact hypersurface \( \Sigma_{t=0} \) is trapped, where \( t \) is a timelike coordinate and the spacetime is foliated by \( t = \text{Const} \). In fact, in this case one can show that the following holds in the entire spacetime

\[
\theta_+ \theta_- = \mathcal{R}^{-2} e^{2f} F' + F'' > 0,
\]

where \( F_{\pm}[= F_{\pm}(x_{\pm})] \) are arbitrary functions of their indicated arguments, subject to \( F'_{+} F'_{-} > 0 \), so that the coordinates are future-pointing. A prime denotes the ordinary derivative, \( e^{f} \equiv g(n_+, n_-) \), and

\[
\mathcal{R} \equiv |\det \left( g(\xi_2, \xi_3) \right)|^{1/2} = F_+ + F_-.
\]

Thus, no apparent horizons exist in these spacetimes. In the cases \( \Sigma_t \approx S^2 \times S^1 \) or \( S^3 \), we have [25]

\[
\mathcal{R} = \sin (t) \sin (\theta),
\]

with \( t \equiv F_+ + F_- \), \( \theta \equiv F_+ - F_- \), and \( 0 \leq t, \theta \leq \pi \). Again, to have the coordinates future-pointing, we must have \( F'_{+} F'_{-} > 0 \). Then, one can show that now we have

\[
\theta_\pm = \mathcal{R}^{-1} e^{f} F'_{\pm} \sin (\theta \pm t).
\]

Thus, in this case the spacetime has several trapped regions [cf. Fig.1]. However, it can be shown that the apparent horizons that separate the trapped regions from the untrapped ones are all degenerate. As a matter of fact, from Eq. (16) we find

\[
\mathcal{L}_\pm \theta_\pm = \mathcal{R}^{-1} e^{f} \left( \mathcal{R}_f \theta_\pm - \mathcal{R}_x \theta_\pm \right) \theta_\pm,
\]

where \( f_\pm \equiv \partial f / \partial x_\pm \). Thus, at least one of the conditions \( \mathcal{L}_\pm \theta_\pm = 0 \) holds on the apparent horizons denoted by the lines \( ad \) and \( bc \) in Fig. 1, on which we have \( \theta_- \theta_+ = 0 \). Therefore, in this case all these horizons are degenerate, which is consistent with the above theorem.

III. CONCLUSIONS

In this Letter, we have studied four-dimensional spacetimes with two commuting spacelike Killing vectors and the cosmological constant \( \Lambda \). After defining trapped surfaces and apparent horizons, following Penrose [16] and FIG. 1: The spacetime for the polarized Gowdy solutions in the \((t, \theta)\)-plane with the spatial topology \( \Sigma_t \approx S^2 \times S^1 \) or \( S^3 \). In the regions \( I \) and \( I' \) the two-surfaces of constant \( t \) and \( \theta \) are untrapped (\( \theta_+ \theta_- < 0 \)), while in the regions \( II \) and \( II' \) they are trapped (\( \theta_+ \theta_- > 0 \)). The lines \( ad \) and \( bc \) are degenerate apparent horizons (\( \theta_+ \theta_- = 0 \)), \( \mathcal{L}_- \theta_+ \mathcal{L}_- \theta_- = 0 \), and the spacetime is singular in the horizontal lines \( t = 0, \pi \).
Hayward [8], we have been able to show that outer apparent horizons do not exist in such spacetimes with $\Lambda > 0$. Degenerate apparent horizons can be formed only in the cases where $\Lambda \leq 0$. These are consistent with all the results obtained so far in the studies of both stationary black holes [10, 21, 24, 27] and gravitational collapse [21, 24, 25]. In particular, it supports the hoop conjecture [2].

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