Nonlinear Dynamics of the Small-World Networks—Hopf Bifurcation, Sequence of Period-Doubling Bifurcations and Chaos

Yan Liu¹,², Jia-Zhong Zhang³*, De-Jun Mu²
¹. School of Mechatronics, Northwestern Polytechnical University, Xi’an 710072, P.R.China
². School of Automation, Northwestern Polytechnical University, Xi’an 710072, P.R.China
³. School of Energy and Power Engineering, Xi’an Jiaotong University, Xi’an 710049, P.R.China
Email: jzzhang@mail.xjtu.edu.cn, Tel: +86-29-82668723, Fax: +86-29-82668723

Abstract. From viewpoint of nonlinear dynamics, the complex nonlinear dynamic phenomena of the small-world networks are studied in some details. The small-world network model, a set of evolution equations with time delay, is used to approach the nonlinear dynamics of networks, and the stability and Hopf bifurcation of the equilibrium state are investigated numerically in the vector field, and the intermittency phenomena in the networks are explained based on the analysis of Hopf bifurcation. Additionally, the ensuing period-doubling bifurcation, sequence of period-doubling bifurcation and period-3 are studied, and the existence of chaos is verified numerically.

1. Introduction
There exist a lot of complicated systems in the society and they are composed of a number of individuals, which are relevant to each other. For instance, the computer networks, and traffic system are the typical networks. There are a strongly interaction between the individuals, and a very rich variety of complex nonlinear dynamics will be presented. For such dynamical systems, there are two main methods to investigate their dynamic properties. The first one is the reconstruction of phase space. As well known, the evolution of the system is related with its component, and the information of the entire system can be provided by the evolving process. Therefore, the dynamics of system can be studied by the investigation of the component in some time interval. Indeed, the time series of the component of the system can be captured easily in the practice, such as the flux of the computer networks, the signal of the brain etc. Another method is the modeling of the complex networks. The complex networks mean there are complex topology and nonlinear dynamics in the systems, and they are graphs composed of vertex and shortcuts, which denotes individual and the coupling between the

* To whom any correspondence should be addressed.
individuals, respectively. By the modeling of the networks, the dynamic behaviors of the system can be described clearly. In this study, the nonlinear dynamics of the small-world networks will be investigated by the modeling method.

There exist four kinds of models of small-world networks, namely, regular networks, random networks, small-world networks and scale-free networks. The regular networks have regular topology, and include global coupling networks and nearest neighbor networks. However, the small-world network is generally the one between regular and stochastic networks, and consists of WS and NW networks.

In fact, the spreading of epidemic, computer virus, forest fire and information are the complicated dynamic systems and related to the daily life of human being, and much more attention has been paid to them in decades. Recently, the continuum approach is used to model the system, which is a kind of heterogeneous media between continuum and discrete media, and strong nonlinearity is included in the system. Indeed, such topic is the focus in mathematics and mechanics community recently. Hence, it is far more important to investigate the properties of the small-world networks, especially the nonlinear dynamics, such as the intermittency in the flow of network.

2. Modelling of Small-world Network

In 1998, Watts and Strogatz proposed a model of small-world network initially, which can be used to describe the transition from regular network to random one, and the model is referred as to WS model [1]. In fact, this model is obtained from rewiring each of bonds with certain probability. However, the detached portion will be rewired in the regular network by WS model, resulting in several disconnected portions and the spreading of information in the entire network will be prevented. To overcome such disadvantage, Newman and Watts improved the WS model, and a new model, namely, NW model, is presented [2]. As stated, NW model never remove the existing bonds in the evolving process, and a new bonds will be added with a certain probability. This model is much more close to the reality. In this paper, the nonlinear dynamics of the small-world network will be studied in detail based on this model.

In recent years, a number of researchers have focused their researches on the dynamic properties of the small-world networks. Newman, Watts, Moukarzel and Yang have studied the spreading and response of event in network with sparse long-range connections [2-4]. In the model presented by Newman, Watts, Moukarzel, the velocity of spreading of the influence is assumed to be constant, and the time delay and nonlinear factors are negligible. However, Yang developed the model of small-world network further, and the nonlinear factors and the time delay due to the congestion features are considered. The resulting model is more feasible for the reality. The governing equation can be constructed as the following,

\[
\frac{d^d v}{dt^d} = \xi^d + v(t-\tau) - \mu \xi v^2(t-\tau),
\]

where \( v \) is the influence range of the spreading of event, \( d \) the lattice dimension, \( \xi \) the Newman-Watts length scale, \( \mu \) the measure of nonlinear interaction, \( \tau \) the time delay due to the congestion features.

3. Nonlinear Dynamics of the Governing Equation

The nonlinear dynamics will be analyzed in detail, from viewpoint of the vector field and map, respectively. The system with lattice dimension 1 will be introduced.

3.1. Equilibrium and Its Stability, Hopf Bifurcation

For the system with lattice dimension \( d=1 \), Eq. (1) can be reduced to the following form,

\[
\frac{dv}{dt} = \xi + v(t-\tau) - \mu \xi v^2(t-\tau).
\]
In general, for such kind of nonlinear dynamic system, the steady state will lose its stability, and a periodic solution will appear, namely, Hopf bifurcation occurs, as the system parameter increases [5]. Hence, the flux of network will oscillate periodically, that is the initial state of intermittency in the networks.

For such dynamic system with time delay, the Midpoint Euler Method is used to integrate the system numerically. The time step $h$ here satisfies $0 < \frac{h}{\tau} < 1$. Then, let $\tau = (m - \delta)h + \frac{h}{2}$, where $m \in N$, $0 \leq \delta < 1$, yields,

$$v(t_{n+1}) \approx v(t_n) + \xi h + h \frac{v(t_{n-m}) + v(t_{n-m+1})}{2} - h \mu \xi (v(t_{n-m}) + v(t_{n-m+1}))^2.$$  (3)

The initial conditions, as the follows, can be given,

$$v(s) = \phi(s), \quad s \in (-\tau, 0).$$  (4)

Equation (3) can be used to simulate the influence of the spreading of event in the small-world network.

Due to the fact that the nonlinear dynamics is sensitively dependent on the parameters, especially for the system with sequence of period-doubling bifurcation, the governing equation in form of vector, as Eq.(2), will be transformed into map. In the next section, the continuous flow will be discretized via a finite different method in time, and a discrete map is obtained.

3.2. Fixed Point and Its Stability, Bifurcation and Chaos in Map

Due to the intuitive, geometrical and computational aspects of map, it is usually to reduce the study of continuous time systems (flows) to the study of an associate discrete time systems (maps). Indeed, the map provides an insightful and striking display of the dynamics of a system. The nonlinear dynamics of the system can be obtained by investigate the fixed points and their stabilities. Eq.(2) can be then reduced as the follows

$$v(t - \tau + \Delta t) - v(t - \tau) = \Delta t[\xi + v(t - \tau) - \mu \xi v^2(t - \tau)].$$  (5)

Herein, let $\Delta t = \tau$, $v_{n+1} = v(t)$, $v_n = v(t - \tau)$, then Eq.(5) can be transformed into the following form,

$$v_{n+1} - v_n = \tau(\xi + v_n - \mu \xi v_n^2),$$  (6)

namely,

$$g : v_n \mapsto v_{n+1}.$$  (7)

3.3. Fixed Point and Its Stability

Following the definition, the fixed point of map, described by Eq.(5), can be expressed as

$$\mu \xi v^2 - v - \xi = 0.$$  (8)

Then, it can be proved that there exist two fixed points, namely, $v_1 = \frac{1 + \sqrt{1 + 4 \mu \xi^2}}{2 \mu \xi}$ and $v_2 = \frac{1 - \sqrt{1 + 4 \mu \xi^2}}{2 \mu \xi}$. It is clear that $v_2 < 0$.

Further, the stabilities of fixed point can be analyzed by the following Floquent Multiplier [5],

$$Dg |_{v} = 1 - \tau \sqrt{1 + 4 \mu \xi^2}.$$  (9)
If $|Dg| > 1$, then the fixed point will lose its stability. For parameter $\mu$ and $\xi$, the critical values are $\mu \geq \frac{4 - \tau^2}{4\tau^2 \xi^2}$ and $\xi \geq \frac{\sqrt{4\mu - \tau^2 \mu}}{2\tau \mu}$, respectively.

3.4. Period-Doubling Bifurcation

In general, as the parameter increases, the fixed point of map will lose its stability, and period-doubling bifurcation will normally emanate. Period-2 solution can be determined by the follows

$$v_{n+1} = v_n + \tau (\xi + v_n - \mu \xi v_n^2), \quad v_{n+2} = v_{n+1} + \tau (\xi + v_{n+1} - \mu \xi v_{n+1}^2),$$

and

$$v_{n+2} = v_n = v^*.$$

Yields,

$$v^*(1) = -\tau - 2 + \frac{\sqrt{4\mu \tau^2 \xi^2 + \tau^2 - 4}}{2\tau \mu \xi}, \quad v^*(2) = -\tau - 2 - \frac{\sqrt{4\mu \tau^2 \xi^2 + \tau^2 - 4}}{2\tau \mu \xi}.$$}

4. Numerical Examples

4.1. Steady State and Hopf Bifurcation in the Vector Field

The parameter of the numerical example is listed as the follows: $\tau = 1$, $h = 0.01$, $m = 100$, $\delta = 0.5$, $\xi = 3$. The initial condition is $v(s) = 1$, $s \in (-\tau, 0)$. The nonlinear interaction factor $\mu$ is considered as bifurcation parameter, and the stability and bifurcation of the steady state or the equilibrium for the small-world networks governed by Eq. (2) are analyzed numerically.

Because the aim of this study is to investigate the nature of intermittency in small-world network, and due to the limit space, only some typical numerical results will be presented.

![Fig. 1(a) Time history of the system at $\mu = 0.01$](image1)

![Fig. 1(b) Phase portrait at $\mu = 0.01$](image2)

As illustrated in Fig. 1(a), the response of the autonomous system will approach a steady state after oscillation in certain time interval. This indicates that the steady state or the equilibrium of the network is stable at $\mu = 0.01$. Fig. 1(b) is the phase portrait of the system, it is clear that the system will reach a stable focus asymptotically, and the range of influence of the event is $v = 36.10339$, which can be referred to as an attractor.
As the nonlinear interaction parameter is increased further, the stable steady state of the system will become unstable, and Hopf bifurcation will occur. Then, the system will oscillate periodically, as shown in Figs. 2(a) and 2(b) for $\mu = 0.05$. It can be concluded that the system finally remains on a stable limit cycle after Hopf bifurcation as the parameter is increased, and the system will behavior as oscillation periodically.

From the numerical results above, the network will remain at a stable steady state as the nonlinear interaction parameter is at a lower value. However, as the parameter is increased further, the steady state will lose its stability, and a periodical state will appear, the system will be in the state of oscillation.

4.2. Fixed Point and Bifurcation in the Map
First, the influence of parameter $\xi$ on the nonlinear dynamics of the system will be studied and $\mu$ is kept constant at 0.04. As illustrated in Fig. 3, there is unique stable solution for $2 \leq \xi \leq 4.33$, and the system will reach it eventually. At this range, the system will keep steady and uniform. However, for $4.33 < \xi < 5.59$, a period-doubling bifurcation will take place, and period-2 fixed point will appear. By the above equations, the critical value of parameter $\xi = 4.3301$ can be verified numerically. Consequently, there is an exchange of stability of period-1 fixed points to period-2 fixed points. For the network, the influence range of the event is no long a unique value, and will oscillate between two values with period of 2 second.

As the parameter $\xi$ increases further to around 5.85, period-doubling bifurcation will take place again, and period-4 solution is emanated, that is the network will oscillate at period of 4 second. As the parameter $\xi$ increases again, a sequence of period-doubling bifurcations is spawned. At a certain $\xi$, the system routes to chaos via such sequence of period-doubling bifurcations, and the period solution or response disappears. In fact, this is the typical route to chaos, and the cascades of period-doubling bifurcation are universal property of discrete dynamical systems [6-12].
Figure 4 shows the bifurcation of the system as the nonlinear interaction factor $\mu$ is increased to $\mu=0.06$. Apparently, the first period-doubling bifurcation occurs at $\xi = 3.53$, the second one at $\xi = 4.55$, the third one at $\xi = 4.77$. In comparison with Fig.3, the bifurcation occurs at a lower parameter. Therefore, the system will be sensitive to the periodical oscillation as the nonlinear interaction factor is increasing. The reason for that is the interaction becomes stronger as parameter $\mu$ is increasing.

In the previous section, the existence and influence of time delay on the system is discussed qualitatively. Fig. 5 shows the bifurcation diagram as the time delay is increased to $\tau=0.5$. In compared with Fig. 3, the critical values of every period-doubling bifurcation are increased remarkably, that means the congestion of networks advances the periodical oscillation of the system.

![Fig. 5 Bifurcation diagram at $\tau = 0.5$](image)

![Fig. 6 Bifurcation diagram at $\xi = 3$](image)

Then, the dynamic behavior will be studied as parameter $\mu$ is varying, with $\xi = 3$, $\tau=1.0s$.

From Fig. 6, it can be seen that there exists a unique stable solution, or steady state, as the parameter $\mu$ varied in the range of $[0.04–0.083]$, the system will perform with a steady flux. As the parameter $\mu$ beyond this range, the period-doubling bifurcation will emanate, and the system will behave as oscillation with period of 2 Seconds. The parameter is increased to around 0.139, then, the second period-doubling bifurcation occurs. In this situation, the flux of the network will “jump” between four values, that is, the system will oscillate at period of 4 Seconds. The parameter is increased further to 0.152, the third period-doubling occurs with period of $2^3$ Seconds. If the parameter is increased more, the system will be in the state of chaos.

4.3. Route to Chaos via Period-Doubling Bifurcation and Period Windows

From the numerical results above, the response of the system is divided into periodic and chaotic parts in range of $\mu$-values. However, embedded in the chaotic range of $\mu$-values, there are several “windows” with periodic behavior. For instance, Fig. 7 shows the windows in the chaotic range as parameter $\mu$ is varying at $\xi = 3$, and also the inverse bifurcation is included. Fig. 8 shows the window of period-3 in the chaos range. According to the definition of chaos given by Li-York, i.e., period-3 means chaos, the system is in chaotic state.

![Fig. 7 Bifurcation diagram with $\xi = 3$](image)

![Fig. 8 Window with period-3 in chaos](image)
5. Conclusions
For complicated system, the mathematical modeling of network is an available and efficient method. By the network, the complex relationships among the individuals can be described clearly. Small-world network is one of typical network and can capture the properties of the system. As the conclusion, it can be drawn that the nonlinear interaction factor has a great influence on the dynamics of the system. The increasing of the nonlinear interaction factor will lead to the Hopf bifurcation, which results in periodical oscillation of flux for the network. In addition, as the system is reduced to a map, there exist a sequence of period-doubling bifurcation, and the flux will behave as periodical oscillation and chaotic state. In particular, the window of period-3 has been found in the chaotic state, which verifies the Li-York definition of chaos. The influences of the parameters of the system on the nonlinear dynamics have been investigated in detail, and a deep understanding of the nature of the small-world network is given.

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