GROUPS WITH FINITENESS CONDITIONS ON THE 
LOWER CENTRAL SERIES OF NON-NORMAL 
SUBGROUPS

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ABSTRACT. It is known that any locally graded group with finitely many 
derived subgroups of non-normal subgroups is finite-by-abelian. This 
result is generalized here, by proving that in a locally graded group \( G \) the 
subgroup \( \gamma_k(G) \) is finite if the set \( \{ \gamma_k(H) \mid H \not\trianglelefteq G \} \) is finite. Moreover, 
locally graded groups with finitely many \( k \)th terms of lower central series 
of infinite non-normal subgroups are also completely described.

1. INTRODUCTION

Restrictions on the derived subgroup of a group can be obtained through 
various finiteness conditions. For instance, F. de Giovanni and D.J.S. Robin-
son [6] proved that if a locally graded group \( G \) has finitely many derived 
subgroups, then its derived subgroup \( G' \) is finite, and the assumption that the 
group \( G \) is locally graded cannot be omitted as can be seen from the con-sid-
eration of Tarski groups (i.e. infinite simple groups in which any proper non-
trivial subgroup has prime order). Recall that a group \( G \) is said to be locally 
graded if each finitely generated non-trivial subgroup of \( G \) contains a proper 
subgroup of finite index; of course, all locally (soluble-by-finite) groups are 
locally graded. In [6], the authors also proved that a locally graded group 
has finitely many derived subgroups of infinite subgroups if and only if it is 
either finite-by-abelian or an irreducible Černikov group. Here a Černikov 
group is said to be irreducible if its largest divisible abelian subgroup \( D \) is 
not central and \( D \) does not contain infinite proper \( K \)-invariant subgroups 
for each subgroup \( K \) of \( G \) such that \( C_G(D) < K \). In [3], F. De Mari and F. 
de Giovanni proved that a locally graded group is finite-by-abelian provided 
it has finitely many derived subgroups of non-normal subgroups and that a 
locally graded group having finitely many derived subgroups of infinite 
non-normal subgroups is either a finite-by-abelian group or an irreducible 
Černikov group. Recently, S. Rinauro [10] proved that if \( G \) is a locally 
graded group and for some integer \( k \geq 2 \) the set \( \Gamma_k(G) = \{ \gamma_k(H) \mid H \leq G \} \) 
is finite, then the subgroup \( \gamma_k(G) \) is finite; moreover, if \( G \) is a locally graded 
group and the set \( \Gamma^\infty_k(G) = \{ \gamma_k(H) \mid H \leq G, \ H \text{ infinite} \} \) is finite, then 
either \( \gamma_k(G) \) is finite or \( G \) is an irreducible Černikov group.

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The results quoted here suggest to consider groups $G$ for which the set

$$\tilde{\Gamma}_k(G) = \{ \gamma_k(H) \mid H \not\leq G \}$$

is finite, for a given positive integer $k \geq 2$. We will refer to such a group $G$ as a $\tilde{\Gamma}_k$–group and we will prove the following result.

**Theorem A.** Let $G$ be a locally graded $\tilde{\Gamma}_k$–group. Then $\gamma_k(G)$ is finite.

A group $G$ will be called an $\tilde{\Gamma}_\infty_k$–group if the set

$$\tilde{\Gamma}_k^\infty(G) = \{ \gamma_k(H) \mid H \not\leq G, H \text{ infinite} \}$$

is finite (here again $k$ is a positive integer such that $k \geq 2$). For locally graded $\tilde{\Gamma}_k^\infty$–groups the following result will be obtained.

**Theorem B.** Let $G$ be a locally graded $\tilde{\Gamma}_k^\infty$–group. Then either $G$ is an irreducible Černikov group or $\gamma_k(G)$ is finite.

A group is metahamiltonian if all its non-normal subgroups are abelian. Such groups are involved in the consideration of groups in which the set of derived subgroups of non-normal subgroups is finite, since they are precisely the groups for which such a set does not contain non-trivial subgroups. In our consideration of $\tilde{\Gamma}_k$–groups (or $\tilde{\Gamma}_k^\infty$–groups) information is needed about groups in which the non-normal (infinite) subgroups are nilpotent of class at most $k - 1$. The behaviour of such groups will be investigated in section 2 of this paper, while section 3 will be devoted to the proof of our main theorems.

Most of our notation is standard and can be found in [11].

2. Groups whose non-normal subgroups are nilpotent

In this section groups in which each subgroup (respectively, infinite subgroup) is either normal or nilpotent of class at most $c$ (where $c$ is a fixed positive integer) are considered, such a class of groups coincides with that of all groups $G$ for which the set $\Gamma_{c-1}(G)$ (respectively, $\Gamma_{c-1}^\infty(G)$) does not contain non-trivial subgroups. The behaviour of groups with this property represents the first step in the study of groups satisfying the property $\tilde{\Gamma}_k$ or $\tilde{\Gamma}_k^\infty$, and it can also be seen in relation with [1] where (generalized) soluble groups were considered that are not locally nilpotent but in which all non-normal subgroups are locally nilpotent.

**Lemma 2.1.** Let $G$ be a locally graded group whose infinite non-normal subgroups are nilpotent. Then $G$ is soluble-by-finite and locally satisfies the maximal condition.
Proof. Let $H$ be any infinite subgroup of $\gamma_3(G)$ and assume that $H$ is not nilpotent. If $K$ is any subgroup of $G$ containing $H$, then $K$ is infinite and non-nilpotent and so it is a normal subgroup of $G$. Therefore $G/H$ is a Dedekind group and hence $\gamma_3(G)$ is contained in $H$. This proves that any proper subgroup of $\gamma_3(G)$ is either finite or nilpotent; in particular, $\gamma_3(G)$ satisfies the minimal condition on non-nilpotent subgroups and thus it is soluble-by-finite (see [5]). It follows that $G$ is soluble-by-finite and so it locally satisfies the maximal condition (see [9], Theorem A). \qed

In our argument we need the following elementary lemma, which is probably already well-known.

Lemma 2.2. Let $G$ be a locally nilpotent torsion-free group and let $H$ be a subgroup of $G$ which is nilpotent of class at most $c$. If $H$ has finite index in $G$, then also $G$ is nilpotent of class at most $c$.

Proof. Since the index $|G : H|$ is finite and $G$ is torsion-free, it follows that $Z_c(H) = H \cap Z_c(G)$ (see [8], 2.3.9). Therefore $H = Z_c(H)$ is contained in $Z_c(G)$ and hence $G/Z_c(G)$ is finite. Thus $\gamma_{c+1}(G)$ is finite (see [11] Part 1, p.113) and so even $\gamma_{c+1}(G) = \{1\}$. \qed

Lemma 2.3. Let $G$ be a locally nilpotent torsion-free group whose non-normal subgroups are nilpotent of class at most $c$. Then $G$ is nilpotent of class at most $c$.

Proof. Clearly we may suppose that $G$ is finitely generated. If $c = 1$, then all non-normal subgroups of $G$ are abelian and hence $G$ is likewise abelian (see [2], Theorem 3.4). Let now $c \geq 2$. Denote by $\mathcal{L}$ the set of all subgroups of finite index of $G$ and assume first that no subgroup of $G$ which belongs to $\mathcal{L}$ is nilpotent of class at most $c$. Then, if $H \in \mathcal{L}$, any subgroup containing $H$ is normal in $G$ and so the factor $G/H$ is a Dedekind group; in particular, $H$ contains $\gamma_3(G)$. Since any finitely generated nilpotent group is residually finite, it follows that

$$\gamma_3(G) \leq \bigcap_{H \in \mathcal{L}} H = \{1\}$$

and hence $G$ is nilpotent of class $2 \leq c$. This contradiction proves that there is a subgroup in $\mathcal{L}$ which is nilpotent of class at most $c$, and so $G$ is likewise nilpotent of class at most $c$ by Lemma 2.2. \qed

We are now in position to prove that locally graded groups whose non-normal subgroups are nilpotent of bounded class are finite-by-nilpotent.

Theorem 2.4. Let $G$ be a locally graded group whose non-normal subgroups are nilpotent of class at most $c$. Then $\gamma_{c+1}(G)$ is finite.
Proof. The group $G$ is soluble-by-finite by Lemma 2.1 and so Theorem B of [1] allows us to suppose that $G$ is locally nilpotent. Clearly we may also suppose that $G$ is not nilpotent of class $c$, so that there exists a finitely generated subgroup $E$ of $G$ which is not nilpotent of class $c$. Then $E$ is a normal subgroup of $G$ and the factor $G/E$ is a Dedekind group; thus $G'$ is a finitely generated nilpotent group. On the other hand, if $T$ is the subgroup consisting of all elements of finite order of $G$, then $G/T$ is nilpotent of class at most $c$ by Lemma 2.3, so that $\gamma_{c+1}(G)$ is periodic and so even finite. □

The next lemma follows easily from a result of D.I. Zaïćev [12], we give its proof here for the convenience of the reader (see also [4], Lemma 2.8).

Lemma 2.5. Let $G$ be a periodic soluble-by-finite group and let $H$ be a finite subgroup of $G$. If $G$ is not a Černikov group, there exists a collection $(K_i)_{i \in I}$ of infinite subgroups of $G$ such that $\bigcap_{i \in I} K_i = H$.

Proof. Since $H$ is finite, the group $G$ contains an abelian subgroup $A$ such that $A^H = A$ and $A$ does not satisfy the minimal condition on subgroups (see [12]). Then the socle $S$ of $A$ is infinite and clearly the subgroup $HS$ is residually finite, so that there exists a normal subgroup of finite index $N$ of $HS$ such that $H \cap N = \{1\}$, and a collection $(L_i)_{i \in I}$ of normal subgroups of finite index of $HS$ such that each $L_i$ is contained in $N$ and the intersection $\bigcap_{i \in I} L_i$ is trivial. Therefore each $HL_i$ is infinite and $\bigcap_{i \in I} HL_i = H$. □

Theorem 2.6. Let $G$ be a locally graded group whose infinite non-normal subgroups are nilpotent (of class at most $c$). Then either $G$ is a Černikov group or all non-normal subgroups of $G$ are nilpotent (of class at most $c$).

Proof. The group $G$ is soluble-by-finite and locally satisfies the maximal condition by Lemma 2.1. Assume that $G$ contains a finite non-normal subgroup $H$ which is not nilpotent (of class at most $c$). Let $g$ be any element of infinite order of $G$, then $\langle H, g \rangle$ is infinite and polycyclic-by-finite. If $L$ is the set of all subgroups of finite index of $\langle H, g \rangle$ containing $H$ and $K$ is any element of $L$, then $K$ is an infinite subgroup which is not nilpotent (of class at most $c$) and hence $K$ is a normal subgroup of $G$. Since a well known result due to Mal’cev yields that

$$H = \bigcap_{K \in L} K,$$

it follows that $H$ is normal in $G$. This contradiction proves that $G$ must be periodic. Since $H$ cannot be the intersection of infinite subgroups, it follows from Lemma 2.5 that $G$ is a Černikov group and the proof is completed. □
**Corollary 2.7.** Let $G$ be a locally graded group whose infinite non-normal subgroups are nilpotent of class at most $c$. Then either $G$ is a Černikov group or $\gamma_{c+1}(G)$ is finite.

**Proof.** This follows immediately from Theorem 2.6 and Theorem 2.4. \(\square\)

### 3. Proof of the main results

In this section Theorem A and Theorem B are proved. The first two lemmas show that locally graded groups with the property $\bar{\Gamma}_k^\infty$ are locally polycyclic-by-finite.

**Lemma 3.1.** Let $G$ be a locally graded $\bar{\Gamma}_k^\infty$ group. Then $G$ is soluble-by-finite.

**Proof.** We argue by induction on the number $t$ of non-trivial subgroups in the set $\bar{\Gamma}_k^\infty(G)$. If $t = 0$ the group $G$ is soluble-by-finite by Lemma 2.1, so that $t \geq 1$ and consider any infinite non-normal subgroup $H$ of $G$ such that $L = \gamma_k(H) \neq \{1\}$. Clearly $L^g = \gamma_k(H^g) \in \bar{\Gamma}_k^\infty(G)$ for every $g \in G$, so that $L$ has finitely many conjugates and hence $N = N_G(L)$ is a subgroup of finite index of $G$. Since $L \notin \bar{\Gamma}_k^\infty(L)$, the set $\bar{\Gamma}_k^\infty(L)$ contains less than $t$ non-trivial subgroups, and hence $L$ is soluble-by-finite by induction on $t$. Then the $\bar{\Gamma}_k^\infty$-group $N/L$ is likewise locally graded (see [3], Lemma 4) and hence, since also the set $\bar{\Gamma}_k^\infty(N/L)$ contains less than $t$ non-trivial subgroups, again induction on $t$ gives that $N/L$ is soluble-by-finite. Therefore $N$ is soluble-by-finite and so $G$ is likewise soluble-by-finite. \(\square\)

**Lemma 3.2.** Let $G$ be a locally graded $\bar{\Gamma}_k^\infty$ group. Then $G$ locally satisfies the maximal condition.

**Proof.** The group $G$ is soluble-by-finite by Lemma 3.1 and so, in order to prove the lemma, it can be supposed that $G$ is a finitely generated soluble group. Assume that the statement is false and choose $G$ as a counterexample with minimal derived length $d$ and such that the set $\{H_1, \ldots, H_t\}$ of all non-trivial subgroups which belong to $\bar{\Gamma}_k^\infty(G)$ has smallest order $t$; note that $t \geq 1$ by Lemma 2.1. Observe further that $d \neq 1$ and we cannot have $d > 2$, otherwise $G^{(d-2)}$ and $G/G^{(d-2)}$ would both be polycyclic, by the minimal choice of $d$, and so $G$ would be likewise polycyclic; therefore $G$ is metabelian. Since each $H_i$ has finitely many conjugates, the subgroup $N_G(H_i)$ has finite index in $G$ and hence also the subgroup $N = N_G(H_1) \cap \ldots \cap N_G(H_t)$ has likewise finite index in $G$; in particular, $N$ is not polycyclic, moreover $N$ contains each $H_i$ by the minimal choice of $t$. Since $\bar{\Gamma}_k^\infty(N/H_i)$ contains less than $t$ non-trivial subgroups, the factor $N/H_i$ is polycyclic and hence, if $H = H_1 \cap \ldots \cap H_t$, also $N/H$ is polycyclic; in particular, $H \neq \{1\}$. Let $x$ be any non-trivial element of $H$. Since $G$ is residually finite (see [11]
Part 2, Theorem 9.51), there exists a subgroup of finite index \( K \) of \( G \) such that \( x \notin K \). If \( X \) is any infinite subgroup of \( K \), then \( \gamma_k(X) \neq H_i \) for all \( i = 1, \ldots, t \), and so either \( \gamma_k(X) = \{1\} \) or \( X \) is a normal subgroup of \( G \); therefore all infinite non-normal subgroups of \( K \) are nilpotent and thus the finitely generated subgroup \( K \) is polycyclic by Lemma \( Z.1 \). It follows that \( G \) is polycyclic and this contradiction concludes the proof. \( \square \)

In what follows some lemmas are given in order to prove that the \( k \)th term of the lower central series of any locally graded \( \Gamma_k \)-group is periodic.

**Lemma 3.3.** Let \( G \) be a \( \Gamma_k \)-group and let \( A \) be a finitely generated abelian normal subgroup of \( G \). If \( A \) is torsion-free and \( g \in G \), then \( \gamma_k(\langle A, g \rangle) = \{1\} \).

**Proof.** Assume first that \( A \cap \langle g \rangle = \{1\} \). If \( \langle g \rangle \) were a normal subgroup of \( G \), \( [A, g] \) would be contained in \( A \cap \langle g \rangle = \{1\} \) and so \( [A, g] = \{1\} \); thus we may suppose that \( \langle g \rangle \) is not a normal subgroup of \( G \). Since for every infinite subset \( I \) of \( \mathbb{N} \) we have

\[
\bigcap_{n \in I} \langle A^n, g \rangle = \langle g \rangle,
\]

it follows that the subgroup \( \langle A^n, g \rangle \) is normal in \( G \) only for finitely many positive integers \( n \) and hence, since the set \( \Gamma_k^\infty(G) \) is finite, there is a positive integer \( \ell \) such that

\[
\gamma_k(\langle A^\ell, g \rangle) = \gamma_k(\langle A^{(\ell+1)^\ell}, g \rangle) = \gamma_k(\langle A^{(\ell+2)^\ell}, g \rangle) = \ldots
\]

Therefore \( \gamma_k(\langle A^\ell, g \rangle) = \gamma_k(\langle A, g \rangle)^\ell \) is a divisible subgroup of \( \langle A, g \rangle \). But \( \langle A, g \rangle \) is finitely generated and metabelian, so that it is residually finite (see [11] Part 2, Theorem 9.51) and hence \( \gamma_k(\langle A, g \rangle)^\ell = \{1\} \). Since \( \gamma_k(\langle A, g \rangle) \) is contained in \( A \), which is torsion-free, it follows that \( \gamma_k(\langle A, g \rangle) = \{1\} \).

In the general case, let

\[
A \cap \langle g \rangle = \langle g^m \rangle \neq \{1\}
\]

and put

\[
\frac{A}{A \cap \langle g \rangle} = \frac{E}{A \cap \langle g \rangle} \times \frac{F}{A \cap \langle g \rangle}
\]

where \( E/A \cap \langle g \rangle \) is finite and \( F/A \cap \langle g \rangle \) is torsion-free. Clearly, \( A \cap \langle g \rangle \) is contained in \( Z(\langle A, g \rangle) \) and the factor group \( \langle E, g \rangle \times \langle A, g \rangle \) is finite, so that \( (E, g) \) is central-by-finite. Therefore \( [E, g] \) is a finite subgroup of \( A \) (see [11] Part 1, Theorem 4.12) and so \( [E, g] = \{1\} \). Since the factor \( A/E \) is a finitely generated abelian torsion-free normal subgroup of \( \langle A, g \rangle / E \) and \( \langle gE \rangle \cap A/E = \{1\} \), the first part of this proof yields that \( \gamma_k(\langle A, g \rangle / E) = \{1\} \). Therefore \( \gamma_k(\langle A, g \rangle) \leq E \leq Z(\langle A, g \rangle) \) and so

\[
\gamma_k(\langle A, g \rangle)^m = [\gamma_k(\langle A, g \rangle), (g)]^m = [\gamma_k(\langle A, g \rangle), (g^m)].
\]

Since \( g^m \in Z(\langle A, g \rangle) \), it follows that \( \gamma_k(\langle A, g \rangle)^m = \{1\} \), thus \( \gamma_k(\langle A, g \rangle) = \{1\} \) because \( \gamma_k(\langle A, g \rangle) \) is contained in \( A \) which is torsion-free. \( \square \)
Lemma 3.4. Let \( G \) be a finitely generated locally graded \( \Gamma_k^\infty \)-group and let \( A \) be a torsion-free abelian normal subgroup of \( G \). Then \( A \) is contained in the hypercentre of \( G \).

Proof. The group \( G \) is polycyclic-by finite by Lemma 3.1 and Lemma 3.2, so that Lemma 3.3 yields that \( \gamma_k(\langle A, g \rangle) \) is trivial for any \( g \in G \). In particular, each element \( a \) of \( A \) is such that \( [a, g] = 1 \) for all \( g \in G \), so that \( A \) is contained in the hypercentre of \( G \) (see [11] Part 2, Theorem 7.21). \( \square \)

Lemma 3.5. Let \( G \) be a finitely generated locally graded \( \Gamma_k^\infty \)-group. If \( G \) has no non-trivial periodic normal subgroups, then \( G \) is nilpotent.

Proof. The group \( G \) contains a polycyclic normal subgroup of finite index by Lemma 3.1 and Lemma 3.2, so that the upper central series of any section of \( G \) becomes stationary after a finite number of steps and thus, in particular, the hypercentre \( Z(G) \) of \( G \) coincides with \( Z_n(G) \) for some positive integer \( n \). Let \( T/Z_n(G) \) be any periodic normal subgroup of \( G/Z_n(G) \). Then \( T/Z_n(G) \) is finite, so that \( \gamma_{n+1}(T) \) is finite (see [11] Part 1, p.113) and hence \( \gamma_{n+1}(T) = \{1\} \) because \( G \) does not contain non-trivial periodic normal subgroups; in particular, \( T \) is nilpotent and torsion-free. Since \( T/Z(T) \) is torsion-free (see [11] Part 1, Theorem 2.25), it follows from Lemma 3.3 and by induction on the nilpotent class of \( T \), that \( Z(T) \) is contained in \( Z_n(G) \) and that \( T/Z(T) \) is contained in the hypercentre of \( G/Z(T) \). Therefore \( T \) is contained in \( Z(G) = Z_n(G) \) and so \( G/Z_n(G) \) has no non-trivial periodic normal subgroups. Let \( K/Z_n(G) \) be a polycyclic normal subgroup of finite index of \( G/Z_n(G) \), and let \( A/Z_n(G) \) be the smallest term of the derived series of \( K/Z_n(G) \). If \( A/Z_n(G) \) were not trivial, it would be a torsion-free abelian normal subgroup of \( G/Z_n(G) \) and so it would be contained in the hypercentre of \( G/Z_n(G) \) by Lemma 3.4. Therefore \( A/Z_n(G) \) is trivial, so that \( K/Z_n(G) \) is trivial and hence \( G/Z_n(G) \) is finite. It follows that \( \gamma_{n+1}(G) \) is finite (see [11] Part 1, p.113). Since \( G \) has no non-trivial periodic normal subgroups, we obtain that \( \gamma_{n+1}(G) = \{1\} \) and the proof is completed. \( \square \)

Lemma 3.6. Let \( G \) be a finitely generated nilpotent \( \Gamma_k^\infty \)-group. If \( G \) is torsion-free, then \( \gamma_k(G) = \{1\} \).

Proof. By way of contradiction, assume that the statement is false and among all the counterexamples choose \( G \) in such a way that the set \( \Gamma_k^\infty(G) \) contains the smallest number \( t \) of non-trivial subgroups. Then \( t > 0 \) by Lemma 2.3 and hence there exists a non-normal subgroup \( H \) of \( G \) such that \( \gamma_k(H) \) contains a non-trivial element \( x \). Since \( G \) is residually finite, there exists a subgroup of finite index \( L \) of \( G \) such that \( x \notin L \). If \( X \) is any non-normal subgroup of \( L \), then \( X \) is not normal in \( G \) and \( \gamma_k(X) \neq \gamma_k(H) \), so that the set \( \Gamma_k^\infty(L) \) contains less than \( t \) non-trivial subgroups and hence \( \gamma_k(L) = \{1\} \). Therefore Lemma 2.2 yields that also \( \gamma_k(G) = \{1\} \) and this contradiction completes the proof. \( \square \)
Lemma 3.7. Let $G$ be a locally graded $\Gamma_k^\infty$-group. Then $\gamma_k(G)$ is periodic.

Proof. The group $G$ is soluble-by-finite by Lemma 3.1 and it locally satisfies the maximal condition by Lemma 3.2. Let $x$ and $y$ be elements of finite order of $G$ and let $X$ be the largest periodic normal subgroup of $\langle x, y \rangle$. Application of Lemma 3.5 yields that the factor group $\langle x, y \rangle / X$ must be trivial, so that $\langle x, y \rangle = X$ is finite. It follows that the set $T$ of all elements of finite order of $G$ is a (normal) subgroup. By Lemma 3.5 and Lemma 3.6 each finitely generated subgroup of $G / T$ is nilpotent of class at most $k - 1$, so that $G / T$ is likewise nilpotent of class at most $k - 1$. Thus $\gamma_k(G)$ is contained in $T$ and hence $\gamma_k(G)$ is periodic. □

We are now able to prove our first main result.

Proof of Theorem A. Clearly it can be assumed that $\gamma_k(G)$ is not trivial and hence $G$ contains a finitely generated subgroup $E$ such that $\gamma_k(E) \neq \{1\}$; moreover, by Theorem 2.4 it can be assumed that the set $\Gamma_k^\infty(G)$ contains $t \geq 1$ non-trivial subgroups. The group $G$ is soluble-by-finite by Lemma 3.1 and it locally satisfies the maximal condition by Lemma 3.2. Suppose that $E$ is contained in a finitely generated non-normal subgroup $F$ of $G$. Then the finite subgroup $\gamma_k(F)$ belongs to $\Gamma_k(G)$ so that, since $\Gamma_k(G)$ is finite, $\gamma_k(F)$ has finitely many conjugates and hence Dietzmann’s lemma (see \cite{11} Part 1, p.45) yields that the normal closure $N$ of $\gamma_k(F)$ in $G$ is finite. On the other hand, by induction on $t$ it follows that the subgroup $\gamma_k(G / N)$ is finite, and hence also $\gamma_k(G)$ is finite. Assume now that every finitely generated subgroup containing $E$ is normal. Then $E$ is normal and $G / E$ is a Dedekind group, so that $G' / E$ is finite and hence $G'$ is finitely generated. Then $G'$ is polycyclic-by-finite and hence its periodic subgroup $\gamma_k(G)$ is finite. □

Lemma 3.8. Let $G$ be a group and let $A$ be an abelian normal subgroup of finite index of $G$. If $A$ is the direct product of infinitely many subgroups of prime order, then there exists a collection $(B_n)_{n \in \mathbb{N}}$ of finite $G$-invariant subgroups of $A$ such that $\langle B_n \mid n \in \mathbb{N} \rangle = \bigcup_{n \in \mathbb{N}} B_n$.

Proof. Clearly each subgroup of $A$ has finitely many conjugates in $G$ so that, if $a_1$ is any non-trivial element of $A$, the subgroup $B_1 = \langle a_1 \rangle^G$ is a finite $G$-invariant subgroup of $A$. Suppose, by induction, that finite $G$-invariant subgroups $B_1, \ldots, B_n$ of $A$ have been chosen in such a way that $\langle B_1, \ldots, B_n \rangle = B_1 \times \cdots \times B_n$.

Since $\langle B_1, \ldots, B_n \rangle$ is finite and the group $G$ is residually finite, there exists a normal subgroup of finite index $N$ of $G$ such that $N \cap \langle B_1, \ldots, B_n \rangle = \{1\}$. 

Then $N \cap A$ has finite index in $A$ and hence $N \cap A$ contains a non-trivial element $a_{n+1}$. Thus $B_{n+1} = \langle a_{n+1} \rangle^G$ is a finite $G$-invariant subgroup of $N \cap A$ and
\[ \langle B_1, \ldots, B_n, B_{n+1} \rangle = B_1 \times \cdots \times B_n \times B_{n+1}, \]
so that the lemma is proved. \hfill \Box

**Lemma 3.9.** Let $G$ be a locally graded periodic $\Gamma^\infty_k$-group. Then either $G$ is a Černikov group or $\gamma_k(G)$ is finite.

**Proof.** Assume that the statement is false and let $G$ be a counterexample in which the set $\Gamma_k(G)$ has the smallest number $t$ of non-trivial subgroups. Then $t \geq 1$ by Corollary 2.7, moreover, Theorem A allows us to suppose that there exists a finite non-normal subgroup $H$ of $G$ such that $\gamma_k(H) \neq \{1\}$. Since the group $G$ is soluble-by-finite by Lemma 3.1 and it is not a Černikov group, $G$ contains an abelian subgroup $A$ such that $A^H = A$ and $A$ does not satisfy the minimal condition on subgroups (see [12]). Then the socle of $A$ is infinite and hence, by replacing $A$ by its socle, we may suppose that $A$ is the direct product of infinitely many cyclic groups of prime order. Put $K = AH$. Since $H$ is finite, also $A \cap H$ is finite and then, by replacing $A$ by a suitable $K$-invariant subgroup of finite index, we may further suppose that $A \cap H = \{1\}$. Application of Lemma 3.8 yields that there exists a collection $(B_n)_{n \in \mathbb{N}}$ of finite $K$-invariant subgroups of $A$ such that
\[ \langle B_n \mid n \in \mathbb{N} \rangle = \text{Dr} B_n. \]

Let
\[ \hat{X}_1 = \text{Dr} B_{2n} \quad \text{and} \quad \hat{X}_1 = \text{Dr} B_{2n+1}. \]
Then $\hat{X}_1$ and $\hat{X}_1$ are infinite normal subgroups of $K$ and $\hat{X}_1 \cap \hat{X}_1 = \{1\}$; so that $H = H \hat{X}_1 \cap \hat{X}_1$. Since $H$ is not normal in $G$, there exists a subgroup $L_1$ in $\{\hat{X}_1, \hat{X}_1\}$ such that $HL_1$ is not normal in $G$.

Consider now $C \in \{\hat{X}_1, \hat{X}_1\} \setminus \{L_1\}$; clearly $C$ can be written as
\[ C = \text{Dr} C_n \]
where each $C_n$ is a finite $K$-invariant subgroup of $A$ and $\langle C, L_1 \rangle = C \times L_1$. Let
\[ \hat{X}_2 = \text{Dr} C_{2n} \quad \text{and} \quad \hat{X}_2 = \text{Dr} C_{2n+1}. \]
Then $\hat{X}_2$ and $\hat{X}_2$ are infinite normal subgroups of $K$, $\hat{X}_2 \cap \hat{X}_2 = \{1\}$ and $H = H \hat{X}_2 \cap \hat{X}_2$. Since $H$ in not normal in $G$, there exists an element $L_2$ of $\{\hat{X}_2, \hat{X}_2\}$ such that $HL_2$ is not normal in $G$. Iterating this argument it is clear that $t + 1$ infinite $K$-invariant subgroups $L_1, \ldots, L_t, L_{t+1}$ can be chosen in such a way that $L_i \cap L_j = \{1\}$, $HL_i \cap HL_j = H$ and $HL_i$ is not normal in $G$ for each $i, j \in \{1, \ldots, t + 1\}$ with $i \neq j$. Since $\gamma_k(H)$ is not trivial, each $\gamma_k(HL_i)$ is likewise non-trivial and hence, since there are only $t$ non-trivial subgroups in the set $\Gamma^\infty_k(G)$, there exist $\ell, m \in \{1, \ldots, t + 1\}$,
with \( \ell \neq m \), such that \( \gamma_k(HL_\ell) = \gamma_k(HL_m) \). But \( HL_\ell \cap HL_m = H \), so that \( \gamma_k(HL_\ell) = \gamma_k(HL_m) \leq H \) and hence \( \gamma_k(HL_\ell) \) is a finite subgroup which belongs to \( \Gamma_k^\infty(G) \). Since the set \( \Gamma_k^\infty(G) \) is finite, the finite subgroup \( \gamma_k(HL_\ell) \) has finitely many conjugates and hence Dietzmann’s lemma (see [11] Part 1, p.45) yields that the normal closure \( N \) of \( \gamma_k(HL_\ell) \) in \( G \) is finite. Since \( \Gamma_k^\infty(G/N) \) contains less than \( t \) non-trivial subgroups, it follows from the minimal choice of \( t \) that \( \gamma_k(G/N) \) is finite. Thus \( \gamma_k(G) \) is finite and this contradiction concludes the proof. \( \square \)

**Lemma 3.10.** Let \( G \) be a Černikov \( \Gamma_k^\infty \)-group. If \( \gamma_k(G) \) is infinite, then \( G \) is irreducible.

**Proof.** Let \( D \) be the largest divisible subgroup of \( G \), and let \( K \) be a subgroup of \( G \) containing \( C_G(D) \). Assume that \( D \) contains an infinite proper \( K \)-invariant subgroup \( A \). Then the set \( \Gamma_k(K/A) \) is finite and so Theorem A yields that the factor group \( \gamma_k(K)A/A \) is finite. On the other hand, nilpotent groups satisfying the minimal condition are finite-by-abelian (see [11] Part 1, Theorem 3.14 and Theorem 4.12), so that the derived subgroup of \( K/\gamma_k(K)A \) is finite and hence \( K'/A/A \) is likewise finite. As the subgroup \( [D,K] \) is divisible, it follows that \( [D,K] \leq A < D \) and hence \( C_D(K) \) is infinite (see [7], Theorem G). In particular, \( Z(K) \) is infinite and Theorem A yields that \( \gamma_k(K)Z(K)/Z(K) \) is finite. Since the nilpotent group \( K/\gamma_k(K)Z(K) \) is finite-by-abelian, it follows that the factor group \( K'/Z(K)/Z(K) \) is finite. Then \( [D,K] \) is contained in \( Z(K) \), so that \( [D,K,K] = \{1\} \) and hence \( D \leq Z_2(K) \). Therefore \( K/Z_2(K) \) is finite, so that \( K \) is finite-by-nilpotent and so even finite-by-abelian. It follows that \( [D,K] = \{1\} \), so that \( K = C_G(D) \) and \( G \) is an irreducible Černikov group. \( \square \)

**Proof of Theorem B.** Let \( G \) be a counterexample with smallest number \( t \) of non-trivial subgroups which belongs to \( \Gamma_k^\infty(G) \). Then \( t \geq 1 \) by Corollary 2.7, while Lemma 3.9 and Lemma 3.10 yield that \( G \) is not periodic. The group \( G \) is soluble-by-finite by Lemma 3.11, it locally satisfies the maximal condition by Lemma 3.2 and \( \gamma_k(G) \) is periodic by Lemma 3.7. Since \( \gamma_k(G) \) is not trivial, there exists a finitely generated subgroup \( E \) of \( G \) such that \( \gamma_k(E) \neq \{1\} \) and, since \( G \) is not periodic, it can be assumed that \( E \) contains some element of infinite order. In particular, \( E \) is an infinite polycyclic-by-finite group and \( \gamma_k(E) \) is finite. Assume that \( E \) is not normal in \( G \), so that \( \gamma_k(E) \) belongs to \( \Gamma_k^\infty(G) \) and hence \( \gamma_k(E) \) has finitely many conjugates. Since \( \gamma_k(E) \) is finite, it follows from Dietzmann’s lemma (see [11] Part 1, p. 45) that the normal closure \( N \) of \( \gamma_k(E) \) in \( G \) is finite. Since the number of non-trivial subgroups in the set \( \Gamma_k^\infty(G/N) \) is less than \( t \), by the minimal choice of \( t \) it follows that \( \gamma_k(G/N) \) finite. Therefore \( \gamma_k(G) \) is likewise finite and this contradiction proves that \( E \) must be a normal subgroup of \( G \). Then the factor \( G/E \) is a \( \Gamma_k \)-group and so application of Theorem A yields that \( \gamma_k(G/E) \)
is finite. It follows that $\gamma_k(G)$ is finitely generated and so even finite. This final contradiction proves the theorem.

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