Exclusivity principle and unphysicality of Garg-Mermin correlation

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The question concerning the physical realizability of a probability distribution is of quite importance in Quantum foundations. Specker first pointed out that this question cannot be answered from Kolmogorov’s axioms alone. Lately, this observation of Specker has motivated simple principles (exclusivity principle/ local orthogonality principle) that can explain quantum limit regarding the possible sets of experimental probabilities in various nonlocality and contextuality experiments. We study Specker’s observation in the simplest scenario involving three inputs each with two outputs. Then using only linear constraints imposed on joint probabilities by this principle, we reveal unphysical nature of Garg-Mermin (GM) correlation. Interestingly, GM correlation was proposed to falsify the following suggestion by Fine: if the inequalities of Clauser and Horne (CH) holds, then there exists a deterministic local hidden-variable model for a spin-1/2 correlation experiment of the Einstein-Podolsky-Rosen type, even when more than two observables are involved on each side. Our result establishes that, unlike in the CH scenario, the local orthogonality principle at single copy level is not equivalent to the no-signaling condition in the GM scenario.

Introduction – The origin of probability theory, like many other branches of mathematics, lies in physical observation associated with the occurrence of different possible events/outcomes of an experiment, viz., coin tossing, rolling a dice etc. With time it becomes an increasingly large area of study in mathematics which also finds useful applications in different areas of physics– starting from the study of Brownian motion to statistical mechanics.

In this article we study a very naive question that goes back to the origin of the relation between mathematics and a physical theory, in particular, physical realizability of a mathematically well defined probability distribution. More explicitly we focus on the question whether an arbitrary probability distribution satisfying Kolmogorov’s axioms can always be realized in some physical experiment. Specker first noticed that the question of physical realizability of probability distributions can not be answered from Kolmogorov’s axioms alone [1] (see [2] for English translation). He actually pointed out that if each pair of a set of experimental outcomes are exclusive alternatives in some measurement, then their probabilities are consistent with the existence of a further measurement in which they are all exclusive. This observation was used by Wright to show that simple sets of events allow probabilities such that their sum can exceed the maximum classical value [3], and this concept is also strongly related to orthomodularity and orthocoherence [4].

In the recent past, this particular observation of Specker has gained renewed interest and different researchers have studied closely related version of it under different names:- exclusivity (E) principle [5], local orthogonality (LO) principle (when applied to nonlocality) [6], consistent exclusivity (CE) principle [7]. These principles have proved to be useful towards a more unified understanding of quantum contextuality [8, 9] and nonlocality [10–13], in particular, nonlocal scenario involving more than two parties [5–7, 14–16]. Different other information theoretic/physical principles like information causality (IC) [17], macroscopic locality (ML) [18] along with other methods [19–22] have also been introduced to point out unphysical correlations in the bipartite scenario. The LO principle, likewise IC and ML, serves the same purpose in bipartite scenario, but its domain of applicability is not limited to bipartite scenario alone as in the case of IC and ML [6]. For the single party scenario E-principle has also been successfully used to explain limited noncontextual behavior of quantum theory [5]. On the other hand, in a recent work [23] Henson has established interesting connection between this principle and a generalized version of Sorkin’s idea of ‘lack of (irreducible) third-order interference’ [24].

While studying the nonlocality scenario, the authors in [6] have proved (originally shown in [25, 26]) that the constraints imposed by LO principle on the bipartite correlation is equivalent to the no-signaling (NS) constraints and hence linear on joint probabilities. However, LO principle, while applied to several copies of a device imply non-linear constraint on the joint probability distribution and turns out to be very powerful for ruling out non-quantum correlations, viz, the canonical Popescu-Rohrlich (PR) correlation [27]. Interestingly, in this article we show that the equivalence between single copy LO and NS as established in [6, 25, 26] is restrictive in a sense. We show that there exists situation where the constraint imposed by LO at single copy level can rule out non-quantum correlations even though they satisfy
We prove the strongest no-go result in this direction by putting, but here we consider e.g., are respectively obtained when the compatible tests of the system [See Fig. 1]. By then there exists a hidden-variable model for a spin-1/2 correlation experiment of the Einstein-Podolsky-Rosen state, they did not exclude a more sophisticated experiment for the physical realization of GM correlations. We prove the strongest no-go result in this direction by showing that GM correlations cannot be realizable in any physical theory. The unphysicality of GM correlations is established using the constraints imposed by E-principle at single copy level only.

The set up – We consider the ‘black box’ framework which contemplates the probability distributions on the outputs/outcomes given some inputs/measurements and also considers correlations among outcomes of measurements in multipartite scenarios. This approach makes no assumptions on the internal working of the measurement devices (hence the name ‘black box’ framework) and also it does not consider the post measurement state of the system [See Fig. 1]. By we denote an event that the outcomes 1, 2, ..., n are respectively obtained when the compatible tests 1, 2, ..., are performed. For multipartite scenario superscripts will be used to denote different parties, e.g., , where 1's (2's) denotes inputs of Alice (Bob) with corresponding outputs 1 (2). Whenever there is no confusion the subscript will be avoided in general. Two events are called exclusive if they cannot be simultaneously true, i.e., are exclusive if for some i (at least one) 1 ≠ 2 whereas 1 = 2. Exclusivity principle places a constraint that the sum of the probabilities of pairwise exclusive events 1 cannot exceed unity, i.e., 

\[ \sum P(e_i) \leq 1. \]

Three-input two-output scenario – Consider the scenario with three inputs \( x_1, x_2, x_3 \) each with binary outputs 1, 2, 3 ∈ \{0, 1\}. Also consider that all the three bi-joint probability distributions \( P(o_i | x_i, x_j) \) exist. The tri-joint probability distribution \( P(o_1 o_2 o_3 | x_1 x_2 x_3) \) may or may not exist. We also consider the probability consistency condition, i.e.,

\[ P(o_1 | x_1) := \sum_{o_2} P(o_1 o_2 | x_1 x_2) = \sum_{o_3} P(o_1 o_3 | x_1 x_3), \]

and similar conditions for the other single marginals. These conditions are also known as no-disturbance (ND) principle [34]. General three bi-joint probability distribution box satisfying the ND principle can be expressed as in the Table-I.

We are interested in the question whether such a probability box will be realized in some physical scenario. One trivial answer to this question is that if a tri-joint probability distribution \( P(o_1 o_2 o_3 | x_1 x_2 x_3) \) exists which reproduces all the three bi-joint distributions in the above table, then the box is realizable even with a physical experiment in classical world.

However, existence of three-joint probability is not a necessary condition for physical realizability of such a probability box. For example, consider unsharp measurements on a two-level quantum system. With suitable value of unsharpness parameter one can find three observables that are pairwise jointly measurable but not triple-wise jointly measurable. In such a scenario one can obtain a probability box with no tri-joint probability but having a physical realization [35].

E-principle provides non-trivial constraints that need to be satisfied by the probability box of Table-I for having a physical realization. In the single copy level these constrains are,

\[ P(o_1 o_2 | x_1 x_2) + P(\bar{o}_2 o_3 | x_2 x_3) + P(o_1 \bar{o}_3 | x_1 x_3) \leq 1, \]
where $\tilde{o}_i$ denotes complement of the output $o_i$. Varying the outputs $\tilde{o}_1$, $\tilde{o}_2$, and $\tilde{o}_3$ we get eight different inequalities. E-principle while applies to $k \in \mathbb{Z}_+$ copy level imposes further non trivial constraint (in general nonlinear on probabilities) for $k > 1$ [6]. In this article we mostly consider the single copy scenario. Whenever a given probability box violates any of the above inequalities-(2), it become illegitimate to be a physical probability distribution. Note that whenever a probability box of Table-I is obtained from a three-joint distribution as marginal then it satisfies all the constraints imposed by the E-principle [36].

The set of three-input two-output distributions satisfying ND forms a six dimensional polytope $\mathcal{N}D$. Positivity conditions of the probabilities configure the facets of $\mathcal{N}D$. Set of probability distributions satisfying E-principle at k-level forms the set $\mathcal{E}_k$ (not convex in general), with the following set inclusion relation: $...\mathcal{E}_k+1 \subseteq \mathcal{E}_k \subseteq \mathcal{E}_k-1 \subseteq ...\mathcal{E}_1 \subseteq \mathcal{N}D$ [37]. There exist eight deterministic boxes satisfying all the above inequalities-(2), and four non deterministic boxes each violating two of the exclusivity inequalities-(2) maximally [36]. These twelve boxes (eight deterministic and four non deterministic) are the extremal points of the $\mathcal{N}D$ polytope. Note that while the algebraic value of the left hand side of the expressions in Eq.(2) can takes value up to $3/2$, the maximal value satisfying ND is limited to $3/2$. This is unlike the bipartite scenario of Clauser-Horne-Shimony-Holt (CHSH) inequalities [38], where the algebraic optimal value of the CHSH expression can be achieved by a NS correlation [27]. Though $\mathcal{E}_k$’s are in general convex sets, only $\mathcal{E}_1$ is a polytope lying strictly within $\mathcal{N}D$. The non-trivial facets of $\mathcal{E}_1$ are given by the inequalities-(2).

From the above discussion it is evident that the constraints imposed by E-principle at single copy ($k = 1$) level are linear on probabilities, while they are in general non linear for higher $k$ and hence $\mathcal{E}_k$’s are not polytopes but convex sets. Moreover, for single party scenario $\mathcal{E}_1$ is a proper subset of $\mathcal{N}D$. However, for bipartite scenario the authors in Ref.[6] have shown that E-principle (or LO principle) at single copy level and the NS principle define same set of correlations. In what follows we show that this result is restrictive in a sense. We find example of bipartite correlations that satisfy NS conditions but violate the linear constraints imposed by E-principle at single copy level. To prove this claim we use a correlation that was introduced by Garg and Mermin in a comparatively old paper [28]. Before proving our claim, let us first digress on GM correlation a bit.

**Garg-Mermin correlation** – Arthur Fine in one of his well known paper established that satisfaction of Bell/Clauser-Horne inequalities is necessary and sufficient conditions for the existence of a joint probability distribution for the four observables (two on Alice’s side and two on Bob’s side) involved in the inequalities [29]. He also suggested that the Bell/Clauser-Horne inequalities would be sufficient for more general scenario involving more than two observables on each side [29]. Garg and Mermin proved Fine’s claim invalid by providing an explicit set of pair distributions involving three observables on each side. The GM correlation $GM(c)$ satisfies the following two features: (i) Joint probability distribution $P(o_i^A, o_j^B | x_i^A x_j^B) = \{P(o_i^A, o_j^B) | o_i^A, o_j^B \in \{0, 1\}; i, j \in \{1, 2, 3\}\}$ is given in the Table-II. The marginal probability distributions for Alice and Bob are same and read as $P(0|x_i^A) = (1-c)/2$ for $\mu \in \{A, B\}$ and $i \in 1, 2, 3$. Clearly, $GM(c)$ satisfies the following two features: (i) Joint probability distribution $P(o_i^A, o_j^B | x_i^A x_j^B) = \{P(o_i^A, o_j^B) | o_i^A, o_j^B \in \{0, 1\}; i, j \in \{1, 2, 3\}\}$ does not exist, otherwise $GM(c)$ will turns out to be a local correlation [39].

According to the property (i), the most general form of the three bi-joint probability distribution on Alice’s/Bob’s side reproducing single marginals compatible with $GM(c)$ is given in the Table-III. Given three single marginal distributions $P(o_i|x_i)$, with $i = 1, 2, 3$, one can always construct a three-joint distribution $P(o_i^A, o_j^B | x_i^A x_j^B)$ reproducing the marginals. Actually the solution for three-joint distribution is not unique and one trivial example is the product distribution. However, given the three bi-joint probability distributions as in Table-III existence of a three-joint distribution reproducing the bi-joins as marginals is not guaranteed. A general problem of this kind is addressed by Farkas’s lemma [40]. For our case a straightforward calculation shows that probability box as in Table-III.

### Table II. GM correlations $GM(c)$, where $0 < c \leq 1/3$.

| in | out | $0^A0^B$ | $0^A1^B$ | $1^A0^B$ | $1^A1^B$ |
|----|-----|---------|---------|---------|---------|
| $x_1^A x_2^B$ | $(1-c)/2$ | $0$ | $0$ | $(1+c)/2$ |
| Else | $(1-3c)/6$ | $1/3$ | $1/3$ | $(1+3c)/6$ |

### Table III. Three bi-joint probability distribution, where $\mu = A, B$. Positivity conditions imply $0 \leq \alpha, \beta, \gamma \leq (1-c)/2$.

| in | out | $\alpha$ | $\beta$ | $\gamma$ |
|----|-----|---------|---------|---------|
| $x_i^A x_j^B$ | $(1-c)/2 - \alpha$ | $(1-c)/2 - \beta$ | $(1-c)/2 - \gamma$ |
| $x_2^A x_3^B$ | $\alpha$ | $\beta$ | $\gamma$ |
| $x_1^A x_3^B$ | $\gamma$ | $(1-c)/2 - \gamma$ | $(1-c)/2 - \gamma$ |
can be obtained from a three-joint probability distribution \( P(a^1_b a^2_c a^3_d | x^1_a x^2_b x^3_c) \) as marginals if the following conditions are satisfied:

\[
\begin{align*}
\alpha + \beta + \gamma & \geq \frac{1-3c}{2}; \\
\beta + \gamma & \leq \frac{1-3c}{2}; \\
\alpha + \beta & \leq \frac{1-3c}{2}; \\
\alpha + \gamma & \leq \frac{1-3c}{2}.
\end{align*}
\]

(3)

Similar conditions have also been obtained by A. Fine in [41]. According to the property (ii) of the GM correlation, the distribution in Table-I will be compatible with \( \mathcal{G}_M(c) \) only when at least one of the above four conditions is violated. Interestingly, this condition along with the E-principle is sufficient to rule out the physical realizability of \( \mathcal{G}_M(c) \) as discussed in next section.

**Unphysicality of GM correlation** — We are now in a position to show that the correlation \( \mathcal{G}_M(c) \) is an unphysical one. For this we consider the following four different sets of pairwise exclusive events:

\[
\begin{align*}
S_1 & := \{(1|x^1_1 x^2_B), (10|x^1_1 x^2_1), (00|x^1_1 x^2_A)\}, \\
S_2 & := \{(11|x^1_2 x^2_B), (10|x^1_3 x^2_B), (00|x^1_2 x^2_3)\}, \\
S_3 & := \{(11|x^1_3 x^2_B), (10|x^1_3 x^2_B), (00|x^1_3 x^2_A)\}, \\
S_4 & := \{(01|x^1_3 x^2_A), (01|x^1_2 x^2_A), (10|x^1_3 x^2_A)\}.
\end{align*}
\]

(4a, 4b, 4c, 4d)

Note that here we consider events with two inputs from the same side (each of the last events in the first three sets and all the events in set \( S_4 \)). This is not illegitimate in the GM scenario as the bi-joint distributions on each side do exist. However, considering such kind of events in the CHSH scenario while establishing unphysicality of post quantum correlation (like PR box) is illegitimate as bi-joint distribution on one side does not exist in that scenario.

E-principle while applied to first three sets of events implies the constraints \( \alpha, \beta, \gamma \leq (1-3c)/6 \) that altogether further imply

\[
\alpha + \beta + \gamma \leq (1-3c)/2.
\]

(5)

The same E-principle while applied to the set \( S_4 \), implies,

\[
\alpha + \beta + \gamma \geq (1-3c)/2.
\]

(6)

Physicality of GM correlation therefore demands \( \alpha + \beta + \gamma = (1-3c)/2 \). But it contradicts the property (ii) of \( \mathcal{G}_M(c) \) which tells that at least one of the conditions in Eq.(3) must be violated for GM correlation. Hence according to the E-principle the GM correlation cannot be a physical one. It is worth mentioning that in 1996 Horodecki and Horodecki proved that no quantum state of two spin-1/2 particles subjected to direct spin measurements generates \( \mathcal{G}_M(c) \). But, they did not exclude the possibility of physical meaning of the GM distributions with more sophisticated experiments. But, our work constitute the strongest possible no-go result in this direction as it negates any such possibility.

**Discussions** — Specker first pointed out that there are probability theories consistent with Kolmogorov’s axioms that do not satisfy the E-principle [1]. Later Cabello has brought it out in notice that he actually considered E-principle as a fundamental principle of quantum theory [16]. While implications of this principle in explaining limited behavior of nonlocality (bipartite and multipartite scenario) and contextuality in quantum theory, here we studied it in the simplest scenario involving only three inputs each with two outcomes. In this scenario, we have found all the restrictions imposed by this principle at single copy level which need to be satisfied by a general ND probability distribution to be physical. However, satisfaction of all these restrictions does not guarantees physicality of the probability box. Moreover, as shown in Ref.[37], even satisfaction of the hierarchy of restrictions imposed by this principle at multi-copy level does not guarantees physicality of a probability box. An interesting research direction for future may be finding possible connection of this research to quantum logic, in particular to the recent result of Ref.[42]. As an application of our study we show the unphysicality of GM correlation. The method we apply to do so also establishes a kind of limitation to the claim that local orthogonality principle at single copy level is equivalent to no-signaling condition [6, 25]. At this point it is worthwhile to mention that though our result negates physicality of GM correlation, it does not invalidate Garg-Mermin criticism to Fine’s suggestions. GM actually addressed a different question: given any set of pair distributions for various pairs of two-valued random variables, can there exist a single higher order distribution that returns all the pairs as marginals? Whereas Fine result established that the Bell-CH inequalities are necessary and sufficient for such a higher order distribution when there are just four pair distributions (two inputs at each side), GM construction proved insufficiency of Bell-CH inequalities when there are nine pair distributions (three inputs at each side). The unphysicality of Popescu-Rohrlich [27] correlation has initiated an interesting research direction that motivated people to introduce various new principles [43]. Unphysicality of nine pair GM distribution as established in this work may also lead to some new inquisitive results. It will be also instructive to find whether Garg-Mermin criticism of Fine’s suggestion can be established by a correlation realizable in quantum world.

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Table IV, and indeterministic ones are given in Table V.

values to each input, and 4 are indeterministic boxes

The Exclusivity principle at single copy level imposes

and no-disturbance (ND) condition,

P(o_1|x_1) := \sum_{o_2} P(o_1o_2|x_1x_2) = \sum_{o_3} P(o_1o_3|x_1x_3), \forall x_i, x_j

(8)

together contributes six independent constraints with
three constraints from each. Thus the set of these prob-
abilities forms a 6 dimensional convex polytope \mathcal{ND}
with 12 vertices. Out of 12 vertices of \mathcal{ND}, 8 are deter-
ministic boxes \{D_i | i = 1, ..., 8\} which assigns definite
values to each input, and 4 are indeterministic boxes
\{I_j | j = 1, ..., 4\}. These deterministic boxes are given in
Table IV, and indeterministic ones are given in Table V.

The Exclusivity principle at single copy level imposes
the following sets of non-trivial constraints:

P(00|x_1x_2) + P(10|x_2x_3) + P(11|x_1x_3) \leq 1, \quad (9a)
P(00|x_1x_2) + P(11|x_2x_3) + P(10|x_1x_3) \leq 1, \quad (9b)
P(01|x_1x_2) + P(00|x_2x_3) + P(11|x_1x_3) \leq 1, \quad (9c)
P(01|x_1x_2) + P(01|x_2x_3) + P(10|x_1x_3) \leq 1, \quad (9d)
P(10|x_1x_2) + P(10|x_2x_3) + P(01|x_1x_3) \leq 1, \quad (9e)
P(10|x_1x_2) + P(11|x_2x_3) + P(00|x_1x_3) \leq 1, \quad (9f)
P(11|x_1x_2) + P(00|x_2x_3) + P(01|x_1x_3) \leq 1, \quad (9g)
P(11|x_1x_2) + P(01|x_2x_3) + P(00|x_1x_3) \leq 1. \quad (9h)

All of the eight deterministic extremal points of \mathcal{ND}
satisfy all of the constraints, i.e., Eqs.(9a)-(9h). On
the other hand, each of the indeterministic boxes violates
two of these above constraints. For example the in-
deterministic box I_1 violates the constraints (9b) and

(9g). Note that the violation amount goes up to 3/2, and
this is the maximal violation of the above constraints
by any ND probability box. If we consider mixture of
indeterministic box I_1 and completely random box \mathcal{W},
i.e., pI_1 + (1-p)\mathcal{W}, then the constraints (9b) and (9g)
will be violated whenever p > 1/3.

Table IV. The 8 deterministic boxes D_i. Tri-joint probability
distributions exists for each of the boxes.

| I_1 | I_2 | I_3 | I_4 |
|-----|-----|-----|-----|
| 0   | 0   | 0   | 0   |
| 0   | p   | 0   | 1   |
| 0   | 0   | 1   | 0   |
| 0   | 0   | 0   | 1   |

Table V. The 4 indeterministic boxes I_j. No tri-joint probability
distributions exists for any of these boxes.

| I_1 | I_2 | I_3 | I_4 |
|-----|-----|-----|-----|
| 0   | 0   | 0   | 1   |
| 0   | 0   | 0   | 1   |
| 0   | 0   | 0   | 1   |
| 0   | 0   | 0   | 1   |