\textbf{q-Deformed Orthogonal and Pseudo-Orthogonal Algebras and Their Representations}

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\textbf{Abstract.} Deformed orthogonal and pseudo-orthogonal Lie algebras are constructed which differ from deformations of Lie algebras in terms of Cartan subalgebras and root vectors and which make it possible to construct representations by operators acting according to Gel’fand–Tsetlin-type formulas. Unitary representations of the \( q \)-deformed algebras \( U_q(\mathfrak{so}(n)) \) are found.

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1. In his Letter [1], M. Jimbo defined a \( q \)-deformation \( U_q(g) \) of any simple Lie algebra \( g \) by means of its Cartan subalgebra and root elements. M. Rosso has shown in [2] that to every integral highest weight there corresponds an irreducible finite-dimensional representation of \( U_q(g) \). For the \( q \)-deformed algebra \( U_q(\mathfrak{sl}(n, \mathbb{C})) \), finite-dimensional irreducible representations were explicitly constructed by M. Jimbo [3] through the \( q \)-analogue of the Gelfand–Tsetlin formulas. If one constructs the algebra \( U_q(\mathfrak{so}(n, \mathbb{C})) \) according to Jimbo’s formulas, then (as well as for the nondeformed case) it is impossible to derive irreducible finite-dimensional representations in terms of the formulas of the Gel’fand–Tsetlin type. In explicitly constructing representations of the Lie algebras of the orthogonal groups, one uses the generators

\[ I_{k,k-1} = E_{kk} - E_{k-1,k}, \]

where \( E_{ij} \) is the matrix with the elements \( \delta_{ij}\delta_{sr} \).

The purpose of this Letter is to propose another deformation \( U_q(\mathfrak{so}(n, \mathbb{C})) \) of the orthogonal algebras which allows one to construct the \( q \)-analogue of the Gel’fand–Tsetlin formulas for them. With the help of \( \ast \)-operations in \( U_q(\mathfrak{so}(n, \mathbb{C})) \), it is possible to introduce a compact deformed algebra \( U_q(\mathfrak{so}_r) \) and pseudo-orthogonal deformed algebras \( U_q(\mathfrak{so}_{r,s}), r + s = n \). We derive ‘unitary’ representations (that is, \( \ast \)-representations) of the deformed Lorentz algebras \( U_q(\mathfrak{so}_{r,s}) \). It turns out that, unlike the classical Lie algebra \( \mathfrak{so}(n, 1) \), for the algebras \( U_q(\mathfrak{so}_{r,s}) \) there also appears a continuous ‘unitary’ series (strange series) of representations. Under \( q \to 1 \), this series disappears (goes to infinity).

2. By the quantum algebra \( U_q(\mathfrak{so}(n, \mathbb{C})) \), \( n \geq 3 \), we shall mean the complex associative algebra generated by the elements \( I_{i,i-1}, i = 2, \ldots, n \), which obey the relations

\[ [I_{i,i-1}, I_{j,j-1}] = 0 \quad \text{if} \quad |i - j| > 1, \quad (1) \]
In the limit $q \to 1$, the algebra $U_q(\mathfrak{so}(n, \mathbb{C}))$ reduces to the universal enveloping algebra $U(\mathfrak{so}(n, \mathbb{C}))$ of the classical complex Lie algebra $\mathfrak{so}(n, \mathbb{C})$. Namely, relations (2) and (3) transform into the relations

\[ [I_{i+1,i}, [I_{i+1,i}, I_{i,i-1}]] = -I_{i,i-1}, \]
\[ [I_{i,i-1}, [I_{i,i-1}, I_{i+1,i}]] = -I_{i+1,i}, \]

which, together with relations (1), define the Lie algebra $\mathfrak{so}(n, \mathbb{C})$.

Several ways exist for introducing a $*$-structure (antilinear antiautomorphism) into $U_q(\mathfrak{so}(n, \mathbb{C}))$. The $*$-structure

\[ I_{i,i-1}^* = -I_{i,i-1}, \quad i = 2, \ldots, n, \quad (4a) \]

determines in $U_q(\mathfrak{so}(n, \mathbb{C}))$ the compact $q$-deformed algebra $U_q(\mathfrak{so}_n)$. The $*$-structure

\[ I_{i,i-1}^* = -I_{i,i-1}, \quad i \neq r + 1; \quad I_{r+1,r}^* = I_{r+1,r} \quad (4b) \]

determines in $U_q(\mathfrak{so}(n, \mathbb{C}))$ the noncompact $q$-deformed algebra $U_q(\mathfrak{so}_{r,s})$, $r + s = n$.

Let us remark that our definition of the algebra $U_q(\mathfrak{so}(n, \mathbb{C}))$ agrees (when $n = 3$) with a new definition of the quantum algebra $U_q(\mathfrak{su}_2)$ introduced by Witten [4]. More exactly, he defines $U_q(\mathfrak{su}_2)$ as the algebra $U_q(\mathfrak{so}_3)$. Podleś and Woronowicz [5] have described the Lorentz quantum group basing on the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We obtain the description of the quantum Lorentz algebra as the algebra $U_q(\mathfrak{so}_{3,1})$ (being a particular case $n = 4$ of our treatment).

The relation of our algebra $U_q(\mathfrak{so}(n, \mathbb{C}))$ with the corresponding algebra introduced by Jimbo [1], and the structure of a Hopf algebra in it, will be considered in a forthcoming article.

3. By a finite-dimensional representation of the algebra $U_q(\mathfrak{so}_n)$, we mean a homomorphism $T$ from this algebra into the algebra of linear operators acting on finite-dimensional space such that $T(a^*) = T(a)^*$, $a \in U_q(\mathfrak{so}_n)$. To describe a representation of $U_q(\mathfrak{so}_n)$, it is sufficient to give the action formulas for the operators $T(I_{i,i-1})$, $i = 2, 3, \ldots, n$, which obey relations (1)–(3) and the relations $T(I_{i,i-1})^* = -T(I_{i,i-1})$.

Finite-dimensional irreducible representations of the algebra $U_q(\mathfrak{so}_n)$ are characterized by highest weights analogous to the case of the Lie algebra $\mathfrak{so}_n$ [6], that is, by the numbers $m_{1n}, m_{2n}, \ldots, m_{kn}$, where $k$ is the integral part of $n/2$, which satisfy the same conditions as in the classical case. Namely, all $m_{in}$ are simultaneously integers or half-integers and

\[ m_{1n} \geq m_{2n} \geq \ldots \geq m_{kn} \geq 0 \quad \text{for} \quad n = 2k + 1, \]
\[ m_{1n} \geq m_{2n} \geq \ldots \geq m_{k-1,n} \geq |m_{kn}| \quad \text{for} \quad n = 2k. \]
We denote the set of the numbers \(m_{1n}, \ldots, m_{kn}\) (highest weight) by a single symbol \(m\) and the corresponding representation by \(T_m\).

In the carrier space of irreducible representation there exists the basis of the Gel’fand-Tsetlin type labelled by the patterns \(\alpha\) analogous to the case of the Lie algebra \(so_n\). The labels of the patterns \(\alpha\) satisfy 'betweenness' conditions [6]. We denote the \(k\)th row of the pattern \(\alpha\) by \(m_{1k}, m_{2k}, \ldots\).

The action of the operators \(T_m(I_{i,i-1})\) onto the basis elements \(|\alpha\rangle \equiv \alpha\) is written out by means of the \(q\)-numbers defined as follows. If \(b\) is a complex number, then the \(q\)-number \([b]\) is given by the formula
\[
[b] = \frac{q^{b/2} - q^{-b/2}}{q^{1/2} - q^{-1/2}}.
\]

Now the action of the operators \(T_m(I_{i,i-1})\) is given by the \(q\)-analogue of the Gel’fand-Tsetlin formulas
\[
T_m(I_{2p+1,2p})\alpha = \sum_{j=1}^{p} A_{2p}^j (\alpha) 2^{2p} - \sum_{j=1}^{p} A_{2p}^j (\bar{\alpha}_j^{2p}) \bar{\alpha}_j^{2p},
\]
\[
T_m(I_{2p+2,2p+1})\alpha = \sum_{j=1}^{p} B_{2p}^j (\alpha) 2^{2p+1} - \sum_{j=1}^{p} B_{2p}^j (\bar{\alpha}_j^{2p+1}) \bar{\alpha}_j^{2p+1} +
\]
\[
+ i \prod_{r=1}^{p+1} [l_{r,2p+2}] \prod_{r=1}^{p} [l_{r,2p}] \left( \prod_{r=1}^{p} [l_{r,2p+1}] [l_{r,2p+1} - 1] \right)^{-1} \alpha.
\]
where the pattern \(\alpha_j^{k}\) (correspondingly \(\bar{\alpha}_j^{k}\)) is obtained from \(\alpha\) by replacing \(m_{jk}\) by \(m_{jk} + 1\) (correspondingly, by \(m_{jk} - 1\)); \(l_{j,2p} = m_{j,2p} + p - j\), \(l_{j,2p+1} = m_{j,2p+1} + p - j + 1\) and
\[
A_{2p}^j (\alpha) = \left( \frac{[l_r'][l_r' + 1]}{[2l_r'] [2l_r' + 2]} \prod_{r \neq i} [l_r' + l_r'][l_r' - l_r'] \prod_{r=1}^{p-1} [l_r'' + l_r'] [l_r'' - l_r'] \right)^{1/2},
\]
\[
B_{2p}^j (\alpha) = \left( \frac{\prod_{r=1}^{p+1} [l_r' + l_r'][l_r' - l_r'] \prod_{r=1}^{p} [l_r' + l_r'][l_r'' - l_r']}{\prod_{r \neq i} [l_r' + l_r'][l_r' - l_r'] [l_r'' + l_r'][l_r'' - l_r'] [l_r'' - l_r' - 1]} \right)^{1/2} \cdot [l_r'']^{-1} [2l_r' + 1][2l_r' - 1].
\]

(Note that, for brevity, we have denoted \(l_{k,2p+1}\) in formula (7) by \(l_k\), \(l_{j,2p}\) by \(l_j\), and \(l_{i,2p-1}\) by \(l_i''\); and likewise, in (8) \(l_{i,2p+2}\) by \(l_i\), \(l_{i,2p+1}\) by \(l_i'\) and \(l_{j,2p}\) by \(l_j'\).)

The validity of relations (1)–(3) and (4a) for the operators (5) and (6) is proved directly by calculating matrix elements for the left and right-hand sides of these relations. We do not reproduce these calculations here because of their awkwardness.

4. By a representation \(T\) of the algebra \(U_q(so_{n-1,1})\) we shall mean a homomorphism of this algebra into the algebra of linear operators in a Hilbert space \(V\) defined on an everywhere dense subspace \(D\), such that the restriction of \(T\) onto the subalgebra \(U_q(so_{n-1})\) decomposes into a direct sum of finite-dimensional representations of this
subalgebra, any of which may occur in the decomposition with a finite multiplicity. 
Moreover, we suppose that $D$ contains (as its subspaces) the spaces of irreducible representations of $U_q(so_{n-1})$. If conditions (4b) are fulfilled for the representation operators $T(I_{i_1}, l_{i_1})$, then the representation is called unitary. Throughout the text, we require that $q$ is not equal to a root of unity.

To obtain representations of the algebra $U_q(so_{n-1,1})$, $m_{1n}$ in the highest weights $m = (m_{1n}, m_{2n}, \ldots, m_{kn})$ ($k$ is an integral part of $n/2$) of irreducible representations of $U_q(so_n)$ is replaced by a complex number $\sigma$. Then the numbers $m_{2n-1}, m_{3n-1}, \ldots, m_{s,n-1}$ ($s$ is an integral part of $(n-1)/2$) of highest weights $(m_{1n-1}, m_{2n-1}, \ldots, m_{s,n-1})$ of irreducible representations of the maximal compact subalgebra $U_q(so_{n-1})$ must satisfy the same betweenness conditions as in the case of the representations $T(m)$. Moreover, we suppose that $m_{1n-1}$ may occur in the decomposition with a finite multiplicity.

Theorem. The representation $T(c, m)$ of $U_q(so_{2p,1})$ is irreducible if $c$ is not an integer for integral $m_{2n}, m_{3n}, \ldots, m_{pn}$, $n = 2p + 1$, and is not a half-integer for half-integral $m_{2n}, m_{3n}, \ldots, m_{pn}$, or if one of the number $c$, $1 - c$ coincides with one of the numbers $l_j = m_{jn} + p - j + 1$, $j = 2, 3, \ldots, p$. The representation $T(c, m)$ of the algebra $U_q(so_{2p+1,1})$ is irreducible either if $c$ is not an integer for integral $m_{2n}, m_{3n}, \ldots, m_{p+1,n}$, $n = 2p + 2$, and is not a half-integer for half-integral $m_{2n}, m_{3n}, \ldots, m_{p+1,n}$, or if $c$ coincides with one of the numbers $l_j = m_{jn} + p - j + 1$, $j = 2, 3, \ldots, p$, or $|c| < |l_p + 1|$.

The proof of this theorem is similar to that of the corresponding theorem for the group $SO_0(n - 1, 1)$ (see, for example, [8]).
To select unitary representations out of the representations contained in the theorem, one has to verify for which of them the last condition of relation (4b) is fulfilled. Direct verification shows that for \( q = e^{ih}, h \in \mathbb{R} \), the following representations

1. \( T(c, m), c = it + \frac{1}{2}, 0 < t < 2\pi/h \), and also \( T(\frac{1}{2}, m) \) for integral \( m \) (principal unitary series);
2. \( T(c, m), \frac{1}{2} < c < s + 1 \), where \( s \) is an integer \((0 \leq s \leq p - 1)\). Moreover, \( s = 0 \) if \( l_{p-1} > 1 \), or at \( l_p = 1 \) the number \( s \) is equal to the largest number \( r \) for which \( l_{p-r+1} = r \) (supplementary series);
3. \( T(c, m), \text{Im } c = \pi/h, \text{Re } c > \frac{1}{2} \) (strange series)

are unitary for the algebra \( U_q(\mathfrak{so}_{2p,1}) \) and the representations

1. \( T(c, m), c = it, 0 \leq t < 2\pi/h \) (principal unitary series);
2. \( T(c, m), 0 < c < s \), where \( s \) is an integer \((0 \leq s \leq p)\) such that \( l_{p-r+2} = r - 1 \) for \( r = 1, 2, \ldots, s \) (supplementary series);
3. \( T(c, m), \text{Im } c = \pi/h, \text{Re } c > 0 \) (strange series)

are unitary for the algebra \( U_q(\mathfrak{so}_{2p+1,1}) \). If \( q = e^{ih}, h \in \mathbb{R} \), then unitary representations of supplementary series do not arise. In this case, the defining condition \( \text{Im } c = \pi/h \) is replaced by \( \text{Re } c = \pi/h \). If \( q = \exp(h_1 + ih_2), h_1, h_2 \in \mathbb{R} \), then the algebra \( U_q(\mathfrak{so}_{n,1}) \) possesses no nontrivial unitary representations.

Besides the representations of the algebra \( U_q(\mathfrak{so}_{n,1}) \) listed above, at \( q = e^{ih}, h \in \mathbb{R} \), there are other series of unitary representations (for example, \( q \)-analogue of the discrete series). These representations are subrepresentations of reducible representations \( T(c, m) \). Such series of representations are completely similar to corresponding series of the classical groups \( \text{SO}_0(n, 1) \) (see, for example, [7]) and are described in our preprint [9].

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