Uniform Approximation to the Solution of a Singly Perturbed Boundary Value Problem with an Initial Jump

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Abstract: In this study, a third-order linear integro-differential equation with a small parameter at two higher derivatives was considered. An asymptotic expansion of the solution to the boundary value problem for the considered equation is constructed by considering the phenomenon of an initial jump of the second degree zeroth order on the left end of a given segment. The asymptotics of the solution has been sought in the form of a sum of the regular part and the part of the boundary layer. The terms of the regular part are defined as solutions of integro-differential boundary value problems, in which the equations and boundary conditions contain additional terms, called the initial jumps of the integral terms and solutions. Boundary layer terms are defined as solutions of third-order differential equations with initial conditions. A theorem on the existence, uniqueness, and asymptotic representation of a solution is presented along with an asymptotic estimate of the remainder term of the asymptotics. The purpose of this study is to construct a uniform asymptotic approximation to the solution to the original boundary value problem over the entire considered segment

Keywords: initial jump; boundary layer; asymptotics; asymptotic expansion

1. Introduction

In the field of theoretical physics, perturbation theory has been applied to find methods for approximating solutions to equations containing small parameters. In singularly perturbed problems in which a small parameter is contained at the higher derivatives of differential equations, the construction of approximate solutions is achieved using the method of asymptotic expansion in the powers of small parameters. The asymptotic approximation to the solution of a singularly perturbed problem with respect to a small parameter is a function that differs a little from the exact solution within a certain region of variation of an independent variable only for a sufficiently small parameter value. The accuracy of the asymptotic approximation is determined based on the order of smallness of the values in the remainder term with respect to the small parameter. In the theory of singularly perturbed boundary value problems, a boundary layer phenomenon is observed when the solution to the perturbed problem differs significantly from the solution of a degenerate problem for arbitrarily small parameters. The behavior of the solution of a singularly perturbed system in the boundary layer is described by exponentially decaying functions [1,2]. In [1], an algorithm was formulated for constructing an asymptotic expansion of the solutions of singularly perturbed initial and boundary value problems, which were characterized by a rapid change in their solutions within the boundary layer. Under certain conditions, using boundary layer functions, it was possible to obtain an asymptotic expansion of the solution without finding the exact solution in an explicit form,

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but by solving a certain sequence of simpler problems. To do so, the solution was represented as the sum of two series in the powers of a small parameter—i.e., the regular part and the boundary layer. These series are substituted into the equation as well as the initial conditions, thereby equating the coefficients in the powers of a small parameter separately in the regular part and the boundary layer part, and a sequence of problems can be obtained to determine the terms of the expansion. The theory of a boundary layer along with its technical applications was described in [2]. Applying the theoretical basis of a boundary layer, the problems of hydrodynamics, aerodynamics, fluid, and gas mechanics can be studied. In the theory of ideal fluids, the effect of viscosity appears in a very thin layer, called the boundary layer.

Methods for constructing the asymptotics of the solution of singularly perturbed problems for differential and integral-differential equations were systematically studied in [3–5]. Based on the theory of singular perturbations, Vishik, Lyusternik [6] and Kasymov [7] noted the phenomenon of an initial jump, which occurred when the solution to a singularly perturbed problem tended to the solution to an unperturbed problem with a modified condition when the small parameter tended toward zero. As a characteristic feature of an initial jump, the derivatives of the solutions to these problems were unbounded at the initial moment when the small parameter tended toward zero. An asymptotic expansion of the solution in the powers of a small parameter is an effective method for studying singularly perturbed problems with an initial jump. Kasymov [8–10] focused on constructing the asymptotics of the solutions of singularly perturbed initial and boundary value problems with initial jumps for linear differential equations and nonlinear and hyperbolic systems of differential equations. To construct an asymptotic expansion of the solution to the problem for systems of differential equations, the boundary layer region, in which the solution of the perturbed problem was significantly different from the solution to the unperturbed problem, was divided into two parts, and the rest was a part of limited change in the solution, where the solution to the perturbed problem was already in a small neighborhood of the solution to the unperturbed problem. Methods of asymptotic expansion for singularly perturbed differential boundary value problems were considered by Nurgabyl [11,12] in cases where the boundary conditions were given in an unseparated form and were unbounded, as the small parameter tended toward zero. By studying singularly perturbed integro-differential initial and boundary value problems possessing an initial jump of both the solutions and integral terms, Daulbaev developed various approaches to the development of an asymptotic expansion [13–15]. In these studies, a nonlinear integro-differential equation of arbitrary order with initial conditions at the right point of the segment under consideration was considered, and a method was proposed for constructing an asymptotic solution using the equivalent Cauchy problem with an initial jump at the left point of the segment. Moreover, the asymptotic expansion of the solution of an undivided boundary value problem with an initial jump for singularly perturbed high-order integro-differential equations was considered, and it was determined that the solution to this problem converged to the solution of the modified degenerate problem when a small parameter tended toward zero.

Various asymptotic methods exist that significantly contributed to the development of perturbation theory. Hinch [16] considered the method of matched asymptotic expansion, the methods of strained coordinates and multiple scales. The method of matched asymptotic expansions was used to find an approximate solution of singularly perturbed differential equations by constructing different asymptotic solutions inside and outside the boundary layer of rapid changes and then, matching them together to obtain a single approximate solution. Various approximate approaches to the analysis of nonlinear systems were considered in the work of Nayfeh [17]: the method of strained coordinates, the methods of matched and composite asymptotic expansions, the method of averaging, and the method of multiple scales. These different techniques were described using examples of ordinary equations and complex partial differential equations. We considered the following studies as recent scientific approaches regarding asymptotic expansion methods under the theory of singularly perturbed problems: an investigation of asymptotic expansions of the solutions of singularly perturbed boundary value problems for third-order linear differential equations with constant coefficients [18] and second-order nonlinear differential equations [19] by reducing them to
the solution to an auxiliary Cauchy problems, the generalization of the composite asymptotic expansion method for perturbed differential equations with a singular point [20], an extension of the boundary functions method to construct uniform asymptotic expansions of the solutions of singularly perturbed equations with fractional pivot point [21], and construction asymptotic expansions of the solutions of a singularly perturbed elliptic problem with Dirichlet boundary conditions in the case when the corresponding degenerate equation had a triple root [22].

In this study, we considered the two-point boundary value problem with an initial jump for an integro-differential equation of the third order with a small parameter. Similar to other studies of such type of problems, in our study, the presence of an integral term in the equation had a significant influence on the asymptotic behavior of solutions. Issues of asymptotic convergence for an analytical solution to this problem were considered in [23]. Consequently, it was revealed that the solution of the problem possessed the phenomenon of an initial jump of the second degree zeroth order at the left point of the considered segment. It was also found that the initial jump of the integral term occurred in the unperturbed problem. Consequently, the original problem solution converged non-uniformly to the unperturbed problem solution in the neighborhood of the left point of the segment, when a small parameter tended toward zero. The purpose of this study is to construct a uniform asymptotic approximation in the powers of a small parameter to the solution to the problem under consideration.

Based on relevant studies and previous results on this problem, this study explored the following issues: the identification of the growth character of the solution and its derivatives and the phenomenon of the initial jump of second degree zeroth order at the initial point of the considered segment, an investigation of the influence of the integral term on the construction of the asymptotic expansion of the solution: the phenomenon of initial jumps in the solution and integral term, the construction of boundary value problems for determining the regular part of the asymptotic expansion and initial problems for determining a part of the boundary layer, as well as obtaining the corresponding estimates for their solutions, and establishing the uniformity of the asymptotic approximation to the solution of the original problem over the entire considered segment and estimating the remainder term with any degree of accuracy on the small parameter. The present study determined that the presence of a small parameter at the two highest derivatives and the presence of an integral term in the equations qualitatively changed the properties of solutions of singularly perturbed boundary value problems for integro-differential equations. Singularly perturbed integro-differential boundary value problems with an initial jump comprise a special class of singular perturbations theory. The study of such problems has been motivated by the theoretical generalization and development of perturbation methods to solve various boundary value problems as well as their practical significance in mathematical modeling of various physical processes in the real world.

The remainder of this paper is organized as follows. The statement of the problem and the main points based on the results of previous studies of this problem are presented in Section 2. Section 3 is devoted to the formation of an asymptotic expansion of the solution of a singularly perturbed boundary value problem with an initial jump. Finally, Section 4 provides some concluding remarks.

2. Problem formulation

The purpose of the present section is a problem formulation and statement the rationale for studying this problem.

Consider a singularly perturbed boundary value problem for an integro-differential equation with a small parameter $\varepsilon$ at higher derivatives

$$
L_{\varepsilon} y(t,\varepsilon) = \varepsilon^2 y''(t,\varepsilon) + \varepsilon A(t) y'(t,\varepsilon) + B(t) y(t,\varepsilon) + C(t) y(t,\varepsilon) =
$$

$$
=F(t) + \int_0^1 \left[ K_0(t,x) y(x,\varepsilon) + K_1(t,x) y'(x,\varepsilon) \right] dx,
$$

(1)
where \( \alpha, \beta \) and \( \gamma \) are known constants.

Suppose that the following conditions hold:

1. The functions \( A(t), B(t), C(t), F(t) \) in the segment \( 0 \leq t \leq 1 \) and \( K_0(t,x), K_1(t,x) \) in the domain \( D = \{0 \leq t \leq 1, 0 \leq x \leq 1\} \) are infinitely differentiable.
2. The roots \( \mu_1(t), \mu_2(t) \) of the equation \( \mu^2 + A(t)\mu + B(t) = 0 \) are negative.
3. The parameter \( \lambda = 1 \) for a sufficiently small \( \varepsilon \) is not an eigenvalue of the kernel (see Formula (33), [23])

\[
J(t,p,\varepsilon) = \frac{1}{\varepsilon^2} \int_0^1 [K_0(t,x)K'(x,p,\varepsilon) + K_1(t,x)K''(x,p,\varepsilon)] dx.
\]

In [23], the asymptotic convergence of the solution \( y(t,\varepsilon) \) to the problem (1), (2) was investigated and the following limit equalities were established:

\[
\lim_{\varepsilon \to 0} y^{(l)}(t,\varepsilon) = \overline{y}^{(l)}(t), \quad 0 < t \leq 1, \quad l = 0, 2,
\]

where \( \overline{y}(t) \) denotes the solution of the following corresponding modified unperturbed problem:

\[
L_0\overline{y}(t) = B(t)\overline{y}'(t) + C(t)\overline{y}(t) = F(t) + \left(\sum_{j=0}^{m} K_j(t,x)\overline{y}^{(j)}(x) dx + \Delta(t)\right),
\]

\[
\overline{y}(0) = \alpha + \Delta_0, \quad \overline{y}(1) = \gamma,
\]

where \( \Delta(t) \) and \( \Delta_0 \) are the initial jumps in the integral term and solution, respectively.

The limit transitions (3) are uniform on the segment \( 0 < t_0 \leq t \leq 1 \), where \( t_0 \) is a sufficiently small but fixed number as \( \varepsilon \to 0 \). It follows that \( \overline{y}(t) \) is not a uniform asymptotic approximation for \( y(t,\varepsilon) \). This study is devoted to the study of an algorithm for uniform asymptotic approximations to the solution and to the determination of the degree of accuracy of these approximations with respect to a small parameter. The construction of the asymptotic approximation to the solution is realized based on the results of the limit transitions, the order of the initial jumps of the solution, and the influence of the integral term of the original equation. The study of asymptotic solutions to the problem (1), (2) is presented in the following section, which includes several stages.

3. Construction of an Asymptotic Expansion of the Solution

In the present section, we describe the construction of an asymptotic expansion of the solution of the problem (1), (2). We define equations with boundary and initial conditions for determining the terms of the regular part and the part of boundary layer. We consider the accuracy of the constructed asymptotic approximation by estimating the remainder term of the asymptotics (see [7–15]).

3.1. Determination of the Regular Part and a Part of the Boundary Layer

The solution of the singularly perturbed boundary value problem (1), (2) at the point \( t = 0 \) possesses the phenomenon of an initial jump of the second degree zeroth order (see Definition 1 and Equality (47), [23]). Based on this, we conclude that the asymptotics of the solution to the problem (1), (2) should be sought in the form of the following sum:

\[
y(t,\varepsilon) = y_\varepsilon(t) + \rho_\varepsilon(\tau), \quad \tau = \frac{t}{\varepsilon},
\]
\[ t = \frac{1}{\epsilon} \] is the boundary layer independent variable, \( y_s(t) \) denotes a regular part, which is significant outside the neighborhood of a point of an initial jump \( t = 0 \), and \( \rho_s(\tau) \) denotes a part of the boundary layer, which is significant in a sufficiently small neighborhood of a point of an initial jump \( t = 0 \). Parts of asymptotics (6) can be represented through the following expansions with unknown coefficients \( y_m(t) \) and \( \rho_m(\tau) \), respectively:

\[ y_s(t) = \sum_{m=0}^{\infty} \epsilon^m y_m(t), \tag{7} \]
\[ \rho_s(\tau) = \sum_{m=0}^{\infty} \epsilon^m \rho_m(\tau). \tag{8} \]

Substituting equality (6) into Equation (1), we obtain the following expression:

\[
\epsilon^2 \left( y_s''(t) + \frac{1}{\epsilon} \rho_s'(\tau) \right) + \epsilon A(t) \left( y_s''(t) + \frac{1}{\epsilon} \rho_s'(\tau) \right) + B(t) \left( y_s'(t) + \frac{1}{\epsilon} \rho_s(\tau) \right) + C(t) \left( y_s(0) + \rho_s(0) \right) = \]
\[ = F(t) + \int_0^{\infty} \left[ K_0(t, x) y_s(x) + \rho_s \left( \frac{x}{\epsilon} \right) \right] dx + K_1(t, x) y_s'(x) dx \tag{9} \]

where the dot over the function \( \rho_s(\tau) \) implies differentiation with respect to \( \tau \).

Then, separately rewriting the terms depending on the variables \( t \) and \( \tau \), we obtain the following equation:

\[
\epsilon^2 y_s''(t) + A(t) y_s''(t) + B(t) y_s'(t) + C(t) y_s(t) +
\frac{1}{\epsilon} \rho_s'(\tau) + \frac{1}{\epsilon} A(t) \rho_s'(\tau) + \frac{1}{\epsilon} B(t) \rho_s(\tau) + C(t) \rho_s(\tau) = \]
\[ = F(t) + \int_0^{\infty} \left[ K_0(t, x) y_s(x) + K_1(t, x) y_s'(x) \right] dx + \int_0^{\infty} \left[ \epsilon K_0(t, \epsilon \eta) \rho_s(\eta) + K_1(t, \epsilon \eta) \rho_s(\eta) \right] d\eta. \tag{10} \]

The second integral in the Equation (10) is obtained from expression (9) by replacing \( \eta = \frac{x}{\epsilon} \) in the integral expression and accordingly replacing the upper limit with infinity.

Hence, we separately rewrite the terms related to the variables \( t \) and \( \tau \) in Equation (10) and obtain equations for determining the functions \( y_s(t) \) and \( \rho_s(\tau) \):

\[
\epsilon^2 y_s''(t) + A(t) y_s''(t) + B(t) y_s'(t) + C(t) y_s(t) =
\frac{1}{\epsilon} \rho_s'(\tau) + \frac{1}{\epsilon} A(t) \rho_s'(\tau) + \frac{1}{\epsilon} B(t) \rho_s(\tau) + C(t) \rho_s(\tau) =
\]
\[ = F(t) + \int_0^{\infty} \left[ K_0(t, x) y_s(x) + K_1(t, x) y_s'(x) \right] dx +
\int_0^{\infty} \left[ \epsilon K_0(t, \epsilon \eta) \rho_s(\eta) + K_1(t, \epsilon \eta) \rho_s(\eta) \right] d\eta. \tag{11} \]

Solutions to Equations (11) and (12) are sought in the form of a series of powers of a small parameter (7) and (8). Substituting these series, considering the expansions of the coefficients \( K_0(t, \epsilon \eta), K_1(t, \epsilon \eta), A(\epsilon \tau), B(\epsilon \tau), \), into the Equations (11) and (12) and into the boundary conditions (2), we obtain iterative problems. Consequently, we can determine the coefficients of the regular part and the part of the boundary layer of the asymptotics, considering the properties and additional conditions for these coefficients. Considering problems for approximations of the 1-st
order as \( m \geq 0 \), we identify that a small parameter at the highest derivatives on the left-hand side and the presence of an integral term on the right-hand side of the original equation leads to the appearance of initial jumps in solutions and integral terms.

The coefficients \( K_0(t, \eta), K_1(t, \eta), A(\varepsilon \tau), B(\varepsilon \tau), C(\varepsilon \tau) \) in Equations (11) and (12) are expanded in the form of a Taylor series for the degrees of \( \varepsilon \) at point \((t, 0)\) as follows:

\[
K_0(t, \eta) = \sum_{l=0}^{\infty} \varepsilon^l \frac{\eta^l}{l!} K_0^{(l)}(t, 0),
\]

\[
K_1(t, \eta) = \sum_{l=0}^{\infty} \varepsilon^l \frac{\eta^l}{l!} K_1^{(l)}(t, 0),
\]

\[
A(\varepsilon \tau) = \sum_{l=0}^{\infty} \varepsilon^l \frac{\tau^l}{l!} A^{(l)}(0),
\]

\[
B(\varepsilon \tau) = \sum_{l=0}^{\infty} \varepsilon^l \frac{\tau^l}{l!} B^{(l)}(0),
\]

\[
C(\varepsilon \tau) = \sum_{l=0}^{\infty} \varepsilon^l \frac{\tau^l}{l!} C^{(l)}(0).
\]

To determine the coefficients \( y_m(t), m \geq 0 \) of the regular part of the asymptotics, we substitute series (7) and (13) into Equation (11). Hence, the following integro-differential equations are obtained by equating the coefficients at the same degrees of \( \varepsilon \):

\[
\varepsilon^0: B(t)y_m''(t) + C(t)y_m'(t) = F(t) + \int_0^t K_0(t, x)y_m'(x) + K_1(t, x)y_m'(x) \, dx + \Delta_0(t),
\]

where

\[
\Delta_0(t) = \int_0^t K_0(t, 0) \rho_0(\eta) \, d\eta = -K_0(t, 0) \rho_0(0) = K_1(t, 0) \Delta_0,
\]

\[
\Delta_0 = -\rho_0(0).
\]

\[
\varepsilon^m: B(t)y_m''(t) + C(t)y_m'(t) = F_m(t) + \int_0^t K_0(t, x)y_m'(x) + K_1(t, x)y_m'(x) \, dx + \Delta_m(t), m \geq 1,
\]

where

\[
\Delta_m(t) = \int_0^t K_0(t, 0) \rho_m(\eta) \, d\eta = -K_0(t, 0) \rho_m(0) = K_1(t, 0) \Delta_m,
\]

\[
\Delta_m = -\rho_m(0), m \geq 1,
\]

\[
F_m(t) = \int_0^{t-m} \frac{\eta^m}{m!} K_0(t, 0) \rho_m(\eta) \, d\eta + \int_0^{t-m} \sum_{l=0}^{m-1} \eta^l K_1(t, 0) \rho_{m-l}(\eta) \, d\eta - y_m'(t) - A(t)y_m''(t), m \geq 1.
\]

The values of \( \Delta_m(t) \) and \( \Delta_m, m \geq 0 \) defined by equalities (18) are called the initial jumps of the integral terms and solutions, respectively.

To determine the coefficients \( \rho_m(t), m \geq 0 \) of the boundary layer part of the asymptotics, we substitute series (8) and (14) into Equation (12). Hence, equating the coefficients at the same degrees of \( \varepsilon \), we have of the following differential equations:
\[ e^\nu: \quad \bar{\rho}_n(\tau) + A(0)\dot{\rho}_n(\tau) + B(0)\rho_n(\tau) = 0, \quad (20) \]

\[ e^\nu: \quad \bar{\rho}_m(\tau) + A(0)\dot{\rho}_m(\tau) + B(0)\rho_m(\tau) = \psi_m(\tau), \quad m \geq 1, \quad (21) \]

where

\[ \psi_m(\tau) = -\sum_{l=1}^{m-1} \left[ \frac{\tau^{l+1}}{(l+1)!} (A^{(l+1)}(0)\dot{\rho}_{m-l}(\tau) + B^{(l+1)}(0)\dot{\rho}_{m-l}(\tau)) + \frac{\tau^l}{l!} C^{(l)}(0)\rho_{m-l}(\tau) \right] \quad m \geq 1. \quad (22) \]

To uniquely determine the coefficients \( y_m(t) \) and \( \rho_m(\tau) \), we substitute the equality (6) into the boundary condition (2), considering expansions (7) and (8). Hence, we have the following expressions:

\[ y_0(0) + \varepsilon y_1(0) + \cdots + \varepsilon^\nu y_\nu(0) + \cdots + \varepsilon^\nu \rho_1(0) + \cdots + \varepsilon^\nu \rho_\nu(0) + \cdots = \alpha, \]

\[ y_0'(0) + \varepsilon y_1'(0) + \cdots + \varepsilon^\nu y_\nu'(0) + \cdots + \frac{1}{\varepsilon}(\rho_1(0) + \varepsilon \rho_1(0) + \cdots + \varepsilon^\nu \rho_n(0) + \cdots) = \beta, \quad (23) \]

\[ y_0(1) + \varepsilon y_1(1) + \cdots + \varepsilon^\nu y_\nu(1) + \cdots + \rho_1(1) + \varepsilon \rho_1(1) + \cdots + \varepsilon^\nu \rho_n(1) + \cdots = \gamma. \]

Under the conditions required to decrease the coefficients \( \rho_m(\tau) \to 0, \quad m \geq 0 \) as \( \tau \to \infty \), that is,

\[ \rho_n(\infty) = 0, \quad \dot{\rho}_n(\infty) = 0, \quad \ddot{\rho}_n(\infty) = 0, \quad (24) \]

the terms \( \rho_n \left( \frac{1}{\varepsilon} \right) \to 0, \quad m \geq 0 \), as \( \varepsilon \to 0 \) (see [13–15, 18]).

Equating the coefficients for the same powers of \( \varepsilon \) on both sides of equalities (23) and taking into account the condition (24), we obtain

\[ y_0(0) + \rho_0(0) = \alpha, \quad \dot{\rho}_0(0) = 0, \quad y_0(1) = \gamma, \]

\[ y_1(0) + \rho_1(0) = 0, \quad \dot{\rho}_1(0) + \rho_1(0) = \beta, \quad y_1(1) = 0, \quad (25) \]

\[ y_n(0) + \rho_n(0) = 0, \quad \tau_n(0) + \rho_n(0) = 0, \quad y_n(1) = 0, \quad m \geq 2. \]

Hence, from equalities (25) we determine additional conditions for the coefficients \( y_m(t) \) and \( \rho_m(\tau) \).

In the zeroth approximation, to determine \( y_0(t) \), we have the following boundary conditions:

\[ y_0(0) = \alpha - \rho_0(0), \quad y_0(1) = \gamma. \quad (26) \]

For the coefficient \( \rho_0(\tau) \), the initial conditions \( \rho_0(0) \) and \( \dot{\rho}_0(0) \) are known, which are determined from problem (15), (26) and from the equalities (25), respectively. To determine the remaining initial condition for \( \rho_0(\tau) \), we lower the order of Equation (20) by integrating from \( \tau \) to \( \infty \). Then, by virtue of the condition (24), we obtain an equation

\[ \dot{\rho}_0(\tau) + A(0)\rho_0(\tau) + B(0)\rho_0(\tau) = 0. \]

Hence, for \( \tau = 0 \), we obtain the initial condition

\[ \rho_0(0) = -A(0)\rho_0(0) - B(0)\rho_0(0). \]

Thus, we have the following initial conditions for determining the coefficient \( \rho_0(\tau) \):

\[ \rho_0(0) = -\Delta_0, \quad \dot{\rho}_0(0) = 0, \quad \ddot{\rho}_0(0) = B(0)\Delta_0. \quad (27) \]

The conditions of the next coefficients are determined according to the same scheme.
Similarly, the conditions of the \( m \)-th order approximation can be determined for \( m \geq 1 \). That is, to determine the coefficients \( y_m(t), \ m \geq 1 \), we have the following boundary conditions:

\[
y_m(0) = -\rho_m(0), \ \ y_m(1) = 0. \tag{28}\n\]

For the coefficients \( \rho_m(\tau), \ m \geq 2 \), the initial conditions \( \rho_m(0) \) and \( \dot{\rho}_m(0) \) are known, which are determined from problems (17), (28). To determine the missing initial condition for \( \rho_m(\tau) \), we lower the order of Equation (21) by integrating from \( \tau \) to \( \infty \). Considering the condition (24), we obtain

\[
\ddot{\rho}_m(\tau) + A(0)\dot{\rho}_m(\tau) + B(\tau)\rho_m(\tau) = - \int_{\tau}^{\infty} \psi_m'(p) dp.
\]

Hence, for \( \tau = 0 \),

\[
\ddot{\rho}_m(0) = -A(0)\dot{\rho}_m(0) - B(0)\rho_m(0) - \int_{0}^{\infty} \psi_m'(p) dp, \ m \geq 2.
\]

That is, the coefficients \( \rho_m(\tau) \) are determined with initial conditions of the form

\[
\rho_m(0) = -\Delta_m, \ \ \dot{\rho}_m(0) = -y_{m+1}'(0),
\]
\[
\ddot{\rho}_m(0) = A(0)y_{m+1}'(0) + B(0)\Delta_m - \int_{0}^{\infty} \psi_m'(p) dp, \ m \geq 2. \tag{29}\n\]

Thus, we define the following problems to find the regular terms of the asymptotics of the solution:

\[
B(t)y_0'(t) + C(t)y_0(t) = F(t) + \int_{0}^{1} [K_0(t, x)y_0(x) + K_1(t, x)y_0'(x)] dx + \Delta_y(t), \tag{30a}\n\]

\[
y_0(0) = \alpha + \Delta_y, \ \ y_0(1) = \gamma \tag{30b}\n\]

and

\[
B(t)y_m'(t) + C(t)y_m(t) = F_m(t) + \int_{0}^{1} [K_0(t, x)y_m(x) + K_1(t, x)y_m'(x)] dx + \Delta_y(t), \ m \geq 1, \tag{31a}\n\]

\[
y_m(0) = \Delta_m, \ \ y_m(1) = 0, \ m \geq 1. \tag{31b}\n\]

The terms of the boundary layer of the asymptotics are determined from the problems

\[
\ddot{\rho}_m(\tau) + A(0)\dot{\rho}_m(\tau) + B(\tau)\rho_m(\tau) = 0, \tag{32a}\n\]

\[
\rho_m(0) = -\Delta_m, \ \ \dot{\rho}_m(0) = 0, \ \ \ddot{\rho}_m(0) = B(0)\Delta_m, \tag{32b}\n\]

and

\[
\ddot{\rho}_m(\tau) + A(0)\dot{\rho}_m(\tau) + B(0)\rho_m(\tau) = \psi_m(\tau), \ m \geq 1, \tag{33a}\n\]
\[ \rho_0 (0) = -\Delta, \quad \bar{\rho}_0 (0) = \beta - y'_0 (0), \]
\[ \bar{\rho}_1 (0) = -A(0)(\beta - y'_0 (0)) + B(0) \Delta - \int_0^\infty \psi_1 (p) dp, \]
\[ \rho_m (0) = -\Delta_m, \quad \bar{\rho}_m (0) = -y'_{m-1} (0), \]
\[ \bar{\rho}_m (0) = A(0)y'_{m-1} (0) + B(0) \Delta_m - \int_0^\infty \psi_m (p) dp, \quad m \geq 2. \]

Note that, for the solutions, \( \rho_m (\tau), \; m \geq 0 \) to the problems \((32a,b)\) and \((33a,b)\), the following exponential estimates are valid:

\[ \left| \frac{d^l}{d \tau^l} \rho_m (\tau) \right| \leq Me^{-\delta \tau}, \quad \tau \geq 0, \quad m \geq 0, \quad l = 0, 2, \]

where \( M > 0, \; \delta > 0 \) are constants independent of \( t \) and \( \varepsilon \). The proof of estimates \((34)\) is performed similar to that in \cite{7}, using the method of mathematical induction.

Thus, the equations and corresponding conditions were found for determining the coefficients of the regular part and the boundary layer part at the \( m \)-th order approximation as \( m \geq 0 \).

### 3.2. Asymptotic Approximation to the Solution

Consider the \( n \)-th partial sum of the expansions of the regular part and the part of the boundary layer of the asymptotics

\[ y_n (t, \varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m y_m (t) + \sum_{m=0}^{\infty} \varepsilon^m \rho_m (\tau), \quad \tau = \frac{t}{\varepsilon}, \; n \geq 0, \]

where the regular coefficients \( y_m (t), \; m = 0, \ldots, n \) are directly determined from problems \((30a,b)\) and \((31a,b)\), and are bounded on the segment \( 0 \leq t \leq 1 \), the coefficients \( \rho_m (\tau), \; m = 0, \ldots, n \) are directly determined from problems \((32a,b)\) and \((33a,b)\), and together with their derivatives up to the second order are functions of the boundary layer at \( \tau \geq 0 \); that is, they apply estimates \((34)\). The coefficients \( \rho_{n+1} (\tau), \rho_{n+2} (\tau) \) are uniquely determined by Equation \((33a)\) for \( m = n+1, n+2 \), subject to the following initial conditions:

\[ \rho_{n+1} (0) = 0, \quad \rho_{n+1} (0) = -y'_0 (0), \quad \rho_{n+1} (0) = A(0)y'_0 (0) - \int_0^\infty \psi_{n+1} (p) dp, \]
\[ \rho_{n+2} (0) = 0, \quad \rho_{n+2} (0) = 0, \quad \rho_{n+2} (0) = -\int_0^\infty \psi_{n+2} (p) dp. \]

The functions \( \rho_{n+1} (\tau), \rho_{n+2} (\tau) \) and \( \rho_{n+2} (\tau) \) are bounded, and the functions \( \rho_{n+1} (\tau), \rho_{n+2} (\tau) \) and \( \rho_{n+2} (\tau) \) are functions of the type of boundary layer at \( \tau \geq 0 \).

We substitute the partial sum \((35)\) into Equation \((1)\) and obtain the expression
\[ L_n \overline{y}_n (t, \varepsilon) = \sum_{m=0}^{n+1} e^m \left[ e^3 y_n^m (t) + \varepsilon A(t) y_n^m (t) + B(t) y_n^m (t) + C(t) y_n^m (t) \right] + \]
\[ + \frac{1}{\varepsilon} \sum_{m=0}^{n+1} e^m \left[ \rho_n (t) + A(\varepsilon t) \rho_n (t) + B(\varepsilon t) \rho_n (t) + C(\varepsilon t) \rho_n (t) \right] = \]
\[ = F(t) + \sum_{m=0}^{n+1} e^m \left[ K_n (t, x) y_n (x) + K_1 (t, x) y_n (x) \right] dx \]
\[ + \frac{1}{\varepsilon} \sum_{m=0}^{n+1} e^m \left( e^m K_n (t, 0) + e^m K_n (t, 0) + \ldots \right) \rho_n (\eta) d\eta + \]
\[ + \frac{1}{\varepsilon} \sum_{m=0}^{n+1} e^m \left( e^m K_n (t, 0) + e^m K_n (t, 0) + \ldots \right) \rho_n (\eta) d\eta. \]

Hence,
\[ L_n \overline{y}_n (t, \varepsilon) = \sum_{m=0}^{n+1} e^m \left[ e^3 y_n^m (t) + \varepsilon A(t) y_n^m (t) + B(t) y_n^m (t) + C(t) y_n^m (t) \right] + \]
\[ + \frac{1}{\varepsilon} \sum_{m=0}^{n+1} e^m \left[ \rho_n (t) + A(0) \rho_n (t) + B(0) \rho_n (t) + \varepsilon \rho_n (t) \right] = \]
\[ = F(t) + \sum_{m=0}^{n+1} e^m \left[ K_n (t, x) \overline{y}_n (x, \varepsilon) + K_1 (t, x) \overline{y}_n (x, \varepsilon) \right] dx + O(e^{n+1}), \]

where the functions \( F_n (t), m = 0, n + 2 \) and \( \psi_n (t), m = 0, n + 2 \) are expressed by equalities (19) and (22), respectively, at that \( F_n (t) \equiv F(t) \), \( \psi_n (t) \equiv 0 \), \( y_n (t) = 0 \), \( \rho_n (t) = 0 \), \( t > 0 \). Because the coefficients \( y_n (t), m = 0, n \) are uniquely determined from problems (30a,b) and (31a,b), and the coefficients \( \rho_n (t), m = 0, n + 2 \) are uniquely determined from problems (32a,b) and (33a,b), in Equation (38), only coefficients at \( \varepsilon \) of power of \( n + 1 \) and higher remain. It can be verified that the function \( \overline{y}_n (t, \varepsilon) \), expressed by equality (35), satisfies condition (2), therefore, it is an approximate solution to problem (1), (2) with an accuracy on the order \( O(e^{n+1}) \), that is,
\[ L_n \overline{y}_n (t, \varepsilon) = \sum_{m=0}^{n+1} e^m \left[ e^3 y_n^m (t) + \varepsilon A(t) y_n^m (t) + B(t) y_n^m (t) + C(t) y_n^m (t) \right] + \]
\[ = F(t) + \sum_{m=0}^{n+1} e^m \left[ K_n (t, x) \overline{y}_n (x, \varepsilon) + K_1 (t, x) \overline{y}_n (x, \varepsilon) \right] dx + O(e^{n+1}), \]
\[ \overline{y}_n (0, \varepsilon) = \alpha, \overline{y}_n (0, \varepsilon) = \beta, \overline{y}_n (1, \varepsilon) = \gamma. \]

3.3. Estimation of the Remainder Term of the Asymptotics

Consider the difference between the exact and approximate solutions
\[ r_n (t, \varepsilon) = y(t, \varepsilon) - \overline{y}_n (t, \varepsilon). \]

Hence,
\[ y(t, \varepsilon) = \overline{y}_n (t, \varepsilon) + r_n (t, \varepsilon). \]

The function \( r_n (t, \varepsilon) \) in the equality (42) is called the remainder term of the asymptotics of the solution.

Substituting equality (42) into Equation (1) and taking into account the solution to problem (39), (40), we have the following problem for the remainder term \( r_n (t, \varepsilon) \):
\[ e^r r_n(t, \varepsilon) + A(t)r'_n(t, \varepsilon) + B(t)r''_n(t, \varepsilon) + C(t)r^n(t, \varepsilon) = \]
\[ = \int_0^1 \left[ K_n(t, x)r_n(x, \varepsilon) + K_i(t, x)r'_n(x, \varepsilon) \right] dx + O(\varepsilon^{n+1}), \]  (43)

\[ r_n(0, \varepsilon) = 0, \quad r'_n(0, \varepsilon) = 0, \quad r^n(1, \varepsilon) = 0. \]  (44)

The asymptotic estimates of the solution to problem (43), (44) are determined in the same way as the estimates of the solution to the problem (1), (2) (see [23]), because these problems are of the same type, that is,

\[ |r_n(t, \varepsilon)| \leq Me^{\varepsilon_{n+1}}, \quad |r'_n(t, \varepsilon)| \leq Me^{\varepsilon_{n+1}}e^{-\frac{\varepsilon}{\varepsilon}}, \quad |r''_n(t, \varepsilon)| \leq Me^{\varepsilon_{n+1}} + Me^{\varepsilon_{n+1}}e^{-\frac{\varepsilon}{\varepsilon}}, \]  (45)

where \( M > 0 \) is constant, independent of \( \varepsilon \).

For the segment \( 0 \leq t \leq 1 \), the estimates for the remainder terms \( r_{n+1}(t, \varepsilon) \) and \( r_{n+2}(t, \varepsilon) \) are determined similarly to the estimates of the remainder term \( r_n(t, \varepsilon) \), that is,

\[ |r_{n+1}(t, \varepsilon)| \leq Me^{\varepsilon_{n+2}}, \quad |r'_{n+1}(t, \varepsilon)| \leq Me^{\varepsilon_{n+2}}e^{-\frac{\varepsilon}{\varepsilon}}, \quad |r''_{n+1}(t, \varepsilon)| \leq Me^{\varepsilon_{n+2}} + Me^{\varepsilon_{n+2}}e^{-\frac{\varepsilon}{\varepsilon}}, \]  (46)

and

\[ |r_{n+2}(t, \varepsilon)| \leq Me^{\varepsilon_{n+3}}, \quad |r'_{n+2}(t, \varepsilon)| \leq Me^{\varepsilon_{n+3}}e^{-\frac{\varepsilon}{\varepsilon}}, \quad |r''_{n+2}(t, \varepsilon)| \leq Me^{\varepsilon_{n+3}} + Me^{\varepsilon_{n+3}}e^{-\frac{\varepsilon}{\varepsilon}}. \]  (47)

Hence, the required estimates do not hold. Therefore, we consider the following equalities to obtain the required estimates:

\[ y^{(0)}(t, \varepsilon) = y_{n+1}^{(0)}(t, \varepsilon) + r_{n+1}^{(0)}(t, \varepsilon), \]
\[ y^{(0)}(t, \varepsilon) = y_{n+2}^{(0)}(t, \varepsilon) + r_{n+2}^{(0)}(t, \varepsilon), \]  (48)
\[ y^{(l)}(t, \varepsilon) = y_{n+1}^{(l)}(t, \varepsilon) + r_{n+1}^{(l)}(t, \varepsilon), \quad l = 0, 2. \]

The estimates (45) indicate that the following estimates,

\[ r'_n(t, \varepsilon) = O(\varepsilon^n), \quad r''_n(t, \varepsilon) = O(\varepsilon^{n-1}), \]  (49)

are valid at the point \( t = 0 \), that is, they do not correspond to the required estimates. Then consider the following equalities from equalities (48) to obtain the required estimates for \( r_{n+1}^{(l)}(t, \varepsilon) \):

\[ y'(t, \varepsilon) = y_{n+1}'(t, \varepsilon) + r_{n+1}'(t, \varepsilon), \]
\[ y'(t, \varepsilon) = y_{n+2}'(t, \varepsilon) + r_{n+2}'(t, \varepsilon). \]  (50)

Equating the right-hand sides of the equalities (50), we obtain

\[ r_{n+1}'(t, \varepsilon) = y_{n+2}'(t, \varepsilon) - y_{n+1}'(t, \varepsilon) + r_{n+1}'(t, \varepsilon), \]  (51)

where

\[ y_{n+2}'(t, \varepsilon) - y_{n+1}'(t, \varepsilon) = e^{\varepsilon_{n+1}}y_{n+1}'(t, \varepsilon) + e^{\varepsilon_{n+2}}\rho_{n+1}(t). \]  (52)

According to the estimates (46),

\[ |r_{n+1}'(t, \varepsilon)| \leq Me^{\varepsilon_{n+2}} + Me^{\varepsilon_{n+2}}e^{-\frac{\varepsilon}{\varepsilon}}. \]  (53)

Hence, we obtain the required estimate
\[ |r'_\epsilon(t, \epsilon)| \leq M\epsilon^{n+1}. \]  

Analogically, we consider
\[ y'(t, \epsilon) = \bar{y}'_\rho^n(t, \epsilon) + r'_\rho^n(t, \epsilon), \]
\[ y'(t, \epsilon) = \bar{y}'_{n+2}(t, \epsilon) + r'_{n+2}(t, \epsilon). \]  

Hence,
\[ r'_\rho^n(t, \epsilon) = \bar{y}'_{n+2}(t, \epsilon) - \bar{y}'_\rho^n(t, \epsilon) + r'_{n+2}(t, \epsilon), \]
where
\[ \bar{y}'_{n+2}(t, \epsilon) - \bar{y}'_\rho^n(t, \epsilon) = \epsilon^{n+1} \bar{y}'_{n+1}(t) + \epsilon^{n+2} \bar{y}'_{n+2}(t) + \epsilon^{n+1} \bar{\rho}'_{n+3}(\tau) + \epsilon^{n+2} \bar{\rho}'_{n+4}(\tau). \]

According to estimates (47),
\[ |r'_{n+2}(t, \epsilon)| \leq M\epsilon^{n+3} + M\epsilon^{n+1} e^{-\frac{\delta}{\epsilon}}. \]

Hence,
\[ |r'_\epsilon(t, \epsilon)| \leq M\epsilon^{n+1}. \]

Consider the following equalities:
\[ y(t, \epsilon) = \bar{y}_\rho^n(t, \epsilon) + r_\rho^n(t, \epsilon), \]
\[ y(t, \epsilon) = \bar{y}_{n+1}(t, \epsilon) + r_{n+1}(t, \epsilon), \]
\[ y(t, \epsilon) = \bar{y}_{n+2}(t, \epsilon) + r_{n+2}(t, \epsilon). \]

Equating the right-hand sides of the equalities (60), we obtain
\[ r_\rho^n(t, \epsilon) = \bar{y}'_{n+2}(t, \epsilon) - \bar{y}'_\rho^n(t, \epsilon) + r'_{n+2}(t, \epsilon). \]

Consequently,
\[ r_\rho^n(t, \epsilon) = \epsilon^{n+1} \bar{y}'_{n+1}(t) + \epsilon^{n+2} \bar{y}'_{n+2}(t) + \epsilon^{n+1} \bar{\rho}'_{n+3}(\tau) + \epsilon^{n+2} \bar{\rho}'_{n+4}(\tau) + r_{n+2}(t, \epsilon). \]

Applying the estimates (47) to equality (62), we have the following estimates:
\[ |r_\rho^n(t, \epsilon)| \leq M\epsilon^{n+1}, \quad |r'_\rho^n(t, \epsilon)| \leq M\epsilon^{n+1}, \quad |r'_{n+2}(t, \epsilon)| \leq M\epsilon^{n+1}. \]

Thus, we obtain the required estimates for the remainder term \( r_\rho^n(t, \epsilon) \).

As a result of the points described above, we obtain an approximate solution \( \bar{y}_\rho^n(t, \epsilon) \) to problem (1), (2) and an estimate of the remainder term \( r_\rho^n(t, \epsilon) \) of the asymptotics of the solution.

Then, the following theorem is valid:

**Theorem 1.** Under conditions 1–3, a unique solution \( y(t, \epsilon) \) to the boundary value problem (1), (2) for the segment \([0, 1]\) has an asymptotic representation
\[ y(t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n y_{n}(t) + \sum_{n=0}^{n+2} \epsilon^n \rho_{n}(\tau) + r_\rho^n(t, \epsilon), \]
and for the remainder term \( r_\rho^n(t, \epsilon) \) the following estimates are valid
\[ |r_\rho^n(t, \epsilon)| \leq M\epsilon^{n+1}, \quad 0 \leq t \leq 1, \quad l = \frac{1}{2}. \]
where \( M > 0 \) is constant, independent of \( t \) and \( \varepsilon \), and \( n \geq 0 \) is any natural number.

Thus, we form an asymptotic expansion of the solution with an estimation for the remainder term with any degree of accuracy on a small parameter.

4. Conclusions

In this study, we considered a two-point boundary value problem for the singularly perturbed integro-differential equation of the third order. The study revealed that the integral term in the original equation had a significant influence on the construction of the asymptotic expansion of the solution. That is, additional terms appeared in the equations and boundary conditions of the unperturbed problem, which implied the appearance of initial jumps in the integral terms and solutions. Consequently, the solution of the original singularly perturbed boundary value problem converged to the solution of the modified unperturbed problem as the small parameter tended toward zero. The asymptotic expansion of the solution was constructed in accordance with the zeroth order of the initial jump, and its uniformity over the entire considered segment was established. In conclusion, using the described algorithm, it was possible to investigate the properties and nature of changes in the asymptotic approximation to the solution of singularly perturbed boundary value problem for the third-order integro-differential equation. Further work is required to extend the results obtained to solve more high-order singularly perturbed integral-differential equations with arbitrary boundary conditions.

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