A TWISTED GENERALIZATION OF NOVIKOV-POISSON ALGEBRAS

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Abstract. Hom-Novikov-Poisson algebras, which are twisted generalizations of Novikov-Poisson algebras, are studied. Hom-Novikov-Poisson algebras are shown to be closed under tensor products and several kinds of twistings. Necessary and sufficient conditions are given under which Hom-Novikov-Poisson algebras give rise to Hom-Poisson algebras.

1. Introduction

A Novikov algebra has a binary operation such that the associator is left-symmetric and that the right multiplication operators commute. Novikov algebras play a major role in the studies of Hamiltonian operators and Poisson brackets of hydrodynamic type [2, 3, 4, 6, 7, 8]. The left-symmetry of the associator implies that every Novikov algebra is Lie admissible, i.e., the commutator bracket \[ [x, y] = xy - yx \] gives it a Lie algebra structure.

In [21] the author initiated the study of a twisted generalization of Novikov algebras, called Hom-Novikov algebras. A Hom-Novikov algebra \( A \) has a binary operation \( \mu \) and a linear self-map \( \alpha \), and it satisfies some \( \alpha \)-twisted versions of the defining identities of a Novikov algebra. In [21] several constructions of Hom-Novikov algebras were given and some low dimensional Hom-Novikov algebras were classified. Using some of the definitions and results in [21], a corresponding generalization of Poisson brackets of hydrodynamic type was studied in [10]. Other Hom-type algebraic structures are studied in [11, 12, 13] and the author’s papers listed in the references.

Novikov algebras, like Lie algebras, are not closed under tensor products in a non-trivial way. In order to have a satisfactory tensor theory of Novikov algebras, certain extra structures are needed. In the case of Lie algebras, the relevant structure for a tensor theory is a Poisson algebra structure. A Poisson algebra has simultaneously a Lie algebra structure and a commutative associative algebra structure, satisfying the Leibniz identity. Using Poisson algebra as a motivation, Xu in [14] defined a Novikov-Poisson algebra as a Novikov algebra that is also equipped with a commutative associative product, satisfying some compatibility conditions. Novikov-Poisson algebras are closed under tensor products [14] and some perturbations of the structure maps [15]. The relationship between Novikov-Poisson algebras and Hamiltonian super-operators was discussed in [16].

The purpose of this paper is to study Hom-Novikov-Poisson algebras, which generalize Novikov-Poisson algebras in the same way that Hom-Novikov algebras generalize Novikov algebras. In section 2 we defined Hom-Novikov-Poisson algebras and discuss some of their basic properties. It is shown that Hom-Novikov-Poisson algebras are closed under twisting by weak morphisms and that they arise from Novikov-Poisson algebras. Several examples of Hom-Novikov-Poisson algebras are given.

In section 3, it is shown that Hom-Novikov-Poisson algebras are closed under tensor products in a non-trivial way, generalizing a result in [14]. This tensor product is shown to be compatible with the twisting constructions in section 2.

In section 4, it is shown that every multiplicative Hom-Novikov-Poisson algebra can be perturbed in several ways by its own twisting map and suitable elements. These results reduce to some of those in [13] when the twisting map is the identity map.

Date: October 19, 2010.

2000 Mathematics Subject Classification. 17B63, 17D25.

Key words and phrases. Hom-Novikov-Poisson algebras, Hom-Novikov algebra, Hom-Poisson algebras.
If a Hom-Novikov-Poisson algebra gives rise to a Hom-Poisson algebra \([13, 25]\) via the commutator bracket of the Hom-Novikov product, then it is called admissible. In section 3, a necessary and sufficient condition for admissibility is given, which generalizes an observation in \([26]\). It is then shown that admissibility is preserved by the twisting constructions in section 2, the tensor products in section 3, and the perturbations in section 4.

2. Hom-Novikov-Poisson algebras

The purposes of this section are to introduce Hom-Novikov-Poisson algebras and to discuss some basic properties and examples of these objects. Before we give the definition of a Hom-Novikov-Poisson algebra, let us first fix some notations.

2.1. Notations. We work over a fixed field \(k\) of characteristic 0. For a linear self-map \(\alpha: V \to V\), denote by \(\alpha^n\) the \(n\)-fold composition of \(n\) copies of \(\alpha\), with \(\alpha^0 \equiv 1d\). If \(\mu: V^\otimes 2 \to V\) is a linear map, we often abbreviate \(\mu(x, y)\) to \(xy\) for \(x, y \in V\). Denote by \(\mu^{\text{op}}: V^\otimes 2 \to V\) the opposite map, i.e., \(\mu^{\text{op}} = \mu \tau\), where \(\tau: V^\otimes 2 \to V^\otimes 2\) interchanges the two variables.

**Definition 2.2.**

1. A **Hom-module** is a pair \((A, \alpha)\) in which \(A\) is a \(k\)-module and \(\alpha: A \to A\) is a linear self-map, called the twisting map. A **morphism** \(f: (A, \alpha_A) \to (B, \alpha_B)\) of Hom-modules is a linear map \(f: A \to B\) such that \(f \alpha_A = \alpha_B f\).

2. A **Hom-algebra** is a triple \((A, \mu, \alpha)\) in which \((A, \alpha)\) is a Hom-module and \(\mu: A^\otimes 2 \to A\) is a bilinear map. Such a Hom-algebra is **commutative** if \(\mu = \mu^{\text{op}}\). It is **multiplicative** if \(\alpha \mu = \mu \alpha \otimes 2\).

3. A **double Hom-algebra** is a quadruple \((A, \mu_1, \mu_2, \alpha)\) in which \((A, \alpha)\) is a Hom-module and each \(\mu_i: A^\otimes 2 \to A\) is a bilinear map.

4. A double Hom-algebra \((A, \mu_1, \mu_2, \alpha)\) is **multiplicative** if \(\alpha \mu_i = \mu_i \alpha^\otimes 2\) for \(i = 1, 2\).

5. A **weak morphism** \(f: A \to B\) of double Hom-algebras is a linear map such that \(f \mu_i = \mu_i f^\otimes 2\) for \(i = 1, 2\). A **morphism** \(f: A \to B\) of double Hom-algebras is a weak morphism such that \(f \alpha_A = \alpha_B f\).

**Definition 2.3.** Let \((A, \mu, \alpha)\) be a Hom-algebra. Its **Hom-associator** \([11]\) \(as_A: A^\otimes 3 \to A\) is defined as

\[
as_A = \mu(\mu \otimes \alpha - \alpha \otimes \mu),
\]

i.e.,

\[
as_A(x, y, z) = (xy)\alpha(z) - \alpha(x)(yz)
\]

for \(x, y, z \in A\).

**Definition 2.4.** Let \((A, \mu_1, \mu_2, \alpha)\) be a double Hom-algebra. Its **mixed Hom-associator** \(as_A: A^\otimes 3 \to A\) is defined as

\[
as_A = \mu_1(\mu_2 \otimes \alpha) - \mu_2(\alpha \otimes \mu_1).
\]

The Hom-associator with respect to \(\mu_i\) is denoted by \(as_{\mu_i}\), i.e.,

\[
as_{\mu_i} = \mu_i(\mu_i \otimes \alpha - \alpha \otimes \mu_i)
\]

for \(i = 1, 2\).

Let us now recall Hom-associative algebras from \([11]\); see also \([17, 18]\).

**Definition 2.5.** A **Hom-associative algebra** is a Hom-algebra \((A, \mu, \alpha)\) such that \(as_A = 0\), i.e.,

\[
(xy)\alpha(z) = \alpha(x)(yz)
\]

(2.5.1)

for all \(x, y, z \in A\).
The condition $a_s A = 0$ is called Hom-associativity. An associative algebra is a multiplicative Hom-associative algebra with $\alpha = \text{Id}$.

Next we recall the definition of a Hom-Novikov algebra from \[21\].

**Definition 2.6.** A Hom-Novikov algebra is a Hom-algebra $(A, \mu, \alpha)$ such that

\begin{align*}
& a_s A(x, y, z) = a_s A(y, x, z), \\
& (xy)\alpha(z) = (xz)\alpha(y)
\end{align*}

(2.6.1a)

for all $x, y, z \in A$.

A Novikov algebra is a multiplicative Hom-Novikov algebra with $\alpha = \text{Id}$. The condition (2.6.1a) means that the Hom-associator is left-symmetric, i.e., symmetric in the first two variables. The condition (2.6.1b) means that

\[ R_y R_{\alpha(z)} = R_{\alpha(y)} R_z \]

for all $y, z \in A$, where $R_y$ denotes right multiplication by $y$.

We can now define Hom-Novikov-Poisson algebras.

**Definition 2.7.** A Hom-Novikov-Poisson algebra is a double Hom-algebra $(A, \cdot, *, \alpha)$ such that

1. $(A, \cdot, \alpha)$ is a commutative Hom-associative algebra,
2. $(A, *, \alpha)$ is a Hom-Novikov algebra,

and that the following two compatibility conditions

\begin{align*}
& a_s A(x, y, z) = a_s A(y, x, z), \\
& (x \cdot y) * \alpha(z) = (x * z) \cdot \alpha(y)
\end{align*}

(2.7.1a)

(2.7.1b)

hold for all $x, y, z \in A$, where $a_s A$ is the mixed Hom-associator in Definition 2.4.

A Novikov-Poisson algebra \[14, 15\] is a multiplicative Hom-Novikov-Poisson algebra with $\alpha = \text{Id}$. Notice the similarity between (2.7.1) and (2.6.1). Indeed, (2.7.1a) means that the mixed Hom-associator $a_s A$ is left-symmetric. Expanding the mixed Hom-associator (Definition 2.4) in terms of $\cdot$, $*$, and $\alpha$, we can rewrite the condition (2.7.1a) as

\[(x * y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) = (y * x) \cdot \alpha(z) - \alpha(y) * (x \cdot z).\]

(2.7.2)

Likewise, (2.7.1b) means that

\[ R_y R_{\alpha(z)} = R_{\alpha(y)} R_z \]

for all $y, z \in A$, where $R_y$ (resp., $R_z$) is right multiplication by $y$ (resp., $z$) using $\cdot$ (resp., $*$).

The following observation says that there is another way to state the compatibility condition (2.7.1b) in a Hom-Novikov-Poisson algebra. It will be used many times below.

**Lemma 2.8.** Let $(A, \cdot, *, \alpha)$ be a double Hom-algebra in which $\cdot$ is commutative. Then

\[(x \cdot y) * \alpha(z) = (x * z) \cdot \alpha(y)\]

(2.8.1)

for all $x, y, z \in A$ if and only if

\[(x \cdot y) * \alpha(z) = \alpha(x) \cdot (y * z)\]

(2.8.2)

for all $x, y, z \in A$. In particular, if $A$ is a Hom-Novikov-Poisson algebra, then

\[(x \cdot y) * \alpha(z) = (x * z) \cdot \alpha(y) = \alpha(x) \cdot (y * z)\]

for all $x, y, z \in A$. 

Proof. By the commutativity of \( \cdot \) we have
\[
(x \cdot y) \ast \alpha(z) = (y \cdot x) \ast \alpha(z) \quad \text{and} \quad (y \ast z) \cdot \alpha(x) = \alpha(x) \cdot (y \ast z).
\]
Therefore, the equality
\[
(y \cdot x) \ast \alpha(z) = (y \ast z) \cdot \alpha(x)
\]
holds for all \( x, y, z \in A \) (which is equivalent to (2.8.1)) if and only if (2.8.2) holds for all \( x, y, z \in A \). □

Let us note that every non-trivial commutative Hom-associative algebra has a canonical non-trivial Hom-Novikov-Poisson algebra structure. To prove this result, we need the following preliminary observation, which will be used many times below.

**Lemma 2.9.** Let \((A, \cdot, \alpha)\) be a commutative Hom-associative algebra. Then the expressions
\[
(x \cdot y) \cdot \alpha(z) = \alpha(x) \cdot (y \cdot z)
\]
are both invariant under every permutation of \( x, y, z \in A \).

**Proof.** The expression \((x \cdot y) \cdot \alpha(z)\) is symmetric in \( x \) and \( y \) because \( \cdot \) is symmetric. Moreover, it is symmetric in \( y \) and \( z \) because
\[
(x \cdot y) \cdot \alpha(z) = (y \cdot x) \cdot \alpha(z) \quad \text{(by commutativity)}
\]
\[
= \alpha(y) \cdot (x \cdot z) \quad \text{(by Hom-associativity)}
\]
\[
= (x \cdot z) \cdot \alpha(y) \quad \text{(by commutativity).}
\]
Since the symmetric group \( S_3 \) in three letters is generated by the transpositions \((1 \ 2)\) and \((2 \ 3)\), we conclude that the expression \((x \cdot y) \cdot \alpha(z)\) is invariant under permutations of \( x, y, z \). □

**Proposition 2.10.** Let \((A, \cdot, \alpha)\) be a commutative Hom-associative algebra. Then \((A, \cdot, \ast, \alpha)\) is a Hom-Novikov-Poisson algebra.

**Proof.** Indeed, in this case the defining identities (2.6.1) of a Hom-Novikov algebra coincide with the compatibility conditions (2.7.1). The condition (2.6.1a) holds because \( a_{A} = 0 \) by Hom-associativity. The condition (2.6.1b) holds by Lemma 2.9. □

The next result says that Hom-Novikov-Poisson algebras are closed under twisting by self-weak morphisms. As we will see, this property is unique to Hom-Novikov-Poisson algebras, as Novikov-Poisson algebras are not closed under such twistings.

**Theorem 2.11.** Let \((A, \cdot, \ast, \alpha)\) be a Hom-Novikov-Poisson algebra, and let \( \beta: A \to A \) be a weak morphism. Then
\[
A_\beta = (A, \beta \cdot, \beta \ast, \beta \alpha)
\]
is also a Hom-Novikov-Poisson algebra. Moreover, if \( A \) is multiplicative and \( \beta \) is a morphism, then \( A_\beta \) is also multiplicative.

**Proof.** To see that \( A_\beta \) is a Hom-Novikov-Poisson algebra, one applies \( \beta^2 \) to (i) the Hom-associativity (2.5.1) of \((A, \cdot, \alpha)\), (ii) the conditions (2.6.1) of the Hom-Novikov algebra \((A, \ast, \alpha)\), and (iii) the compatibility conditions (2.7.1). The multiplicativity assertion follows from a direct computation. □

Let us discuss some special cases of Theorem 2.11. The following special case says that every multiplicative Hom-Novikov-Poisson algebra induces a sequence of multiplicative Hom-Novikov-Poisson algebras by twisting against its own twisting map.

**Corollary 2.12.** Let \((A, \cdot, \ast, \alpha)\) be a multiplicative Hom-Novikov-Poisson algebra. Then
\[
A^n = (A, \alpha^n \cdot, \alpha^n \ast, \alpha^{n+1})
\]
is also a multiplicative Hom-Novikov-Poisson algebra for each \( n \geq 0 \).
Proof. Since $A$ is multiplicative, $\alpha^n: A \to A$ is a morphism. By Theorem 2.11, $A_{\alpha^n} = A^n$ is a multiplicative Hom-Novikov-Poisson algebra.

The following result is the special case of Corollary 2.12 with $\cdot = 0$.

**Corollary 2.13.** Let $(A, *, \alpha)$ be a multiplicative Hom-Novikov algebra. Then

$$A^n = (A, \alpha^n *, \alpha^{n+1})$$

is also a multiplicative Hom-Novikov algebra for each $n \geq 0$.

The following result is the special case of Corollary 2.12 with $\cdot = 0$.

**Corollary 2.14.** Let $(A, \cdot, \alpha)$ be a multiplicative commutative Hom-associative algebra. Then

$$A^n = (A, \alpha^n \cdot, \alpha^{n+1})$$

is also a multiplicative commutative Hom-associative algebra for each $n \geq 0$.

The following result is the special case of Theorem 2.11.

**Corollary 2.15.** Let $(A, \cdot, \ast)$ be a Novikov-Poisson algebra and $\beta: A \to A$ be a morphism. Then

$$A_{\beta} = (A, \beta \cdot, \beta \ast, \beta)$$

is a multiplicative Hom-Novikov-Poisson algebra.

Corollary 2.15 says that multiplicative Hom-Novikov-Poisson algebras can be constructed from Novikov-Poisson algebras and their morphisms. A construction result of this form was first given by the author in [18] for $G$-Hom-associative algebras. This twisting construction highlights the fact that the category of Novikov-Poisson algebras is not closed under twisting by self-morphisms. In view of Theorem 2.11, this is a major conceptual difference between Hom-Novikov-Poisson algebras and Novikov-Poisson algebras.

The following special case of Corollary 2.15 is useful for constructing examples of Hom-Novikov-Poisson algebras.

**Corollary 2.16.** Let $(A, \mu)$ be a commutative associative algebra, $\partial: A \to A$ be a derivation, and $\alpha: A \to A$ be an algebra morphism such that $\alpha \partial = \partial \alpha$. Then $A_\alpha = (A, \cdot, \ast, \alpha)$ is a multiplicative Hom-Novikov-Poisson algebra, where

$$x \cdot y = \alpha \mu(x, y) \quad \text{and} \quad x \ast y = \alpha \mu(x, \partial y)$$

for $x, y \in A$.

Proof. It is known that $(A, \mu, \bullet)$ is a Novikov-Poisson algebra [14] (Lemma 2.1), where

$$x \bullet y = \mu(x, \partial y).$$

(That $(A, \bullet)$ is a Novikov algebra has been known since [8].) The assumptions on $\alpha$ imply that $\alpha \bullet = \bullet \alpha \circ \partial^2$, so $\alpha$ is a morphism of Novikov-Poisson algebras. The result now follows from Corollary 2.15.

The following examples illustrate Corollary 2.16.

**Example 2.17.** Starting with an example in [15], we construct infinite-dimensional multiplicative Hom-Novikov-Poisson algebras that are not Novikov-Poisson algebras.

Let the ground field be the field $\mathbb{C}$ of complex numbers. Let $J$ be either $\{0\}$ or $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\Delta$ be an additive subgroup of $\mathbb{C}$, and $A$ be the $\mathbb{C}$-module spanned by the symbols $\{x^j_a: a \in \Delta, j \in J\}$. Then $A$ is a commutative associative algebra with the multiplication

$$x^j_a \cdot x^k_b = x^{j+k}_{a+b}$$
for $a, b \in \Delta$ and $j, k \in J$. It has a multiplicative identity

$$1 = x^0_a.$$ 

The map $\partial: A \to A$ defined by

$$\partial(x^j_a) = ax^j_a + jx^{j-1}_a$$

for $a \in \Delta$ and $j \in J$ is a derivation on $(A, \cdot)$ by [13] (2.18). Therefore, by [13] Lemma 2.1, $(A, \cdot, *)$ is a Novikov-Poisson algebra, where

$$x^j_a \cdot x^k_b = x^j_a \cdot \partial(x^k_b) = bx^{j+k}_a + kx^{j+k-1}_a$$

for $a, b \in \Delta$ and $j, k \in J$. Note that if either $J = \mathbb{N}$ or $\Delta \neq \{0\}$, then $A$ is infinite-dimensional.

Let $\alpha: A \to A$ be the linear map determined by

$$\alpha(x^j_a) = e^a x^j_a$$

(2.17.1) for $a \in \Delta$ and $j \in J$. It is straightforward to check that $\alpha: A \to A$ is an algebra morphism on $(A, \cdot)$ such that $\alpha \partial = \partial \alpha$. By Corollary 2.16 there is a multiplicative Hom-Novikov-Poisson algebra

$$A_\alpha = (A, \cdot, *, \alpha),$$

(2.17.2) in which

$$x^j_a \cdot_\alpha x^k_b = e^{a+b} x^{j+k}_a, \quad x^j_a \ast_\alpha x^k_b = e^{a+b} \left( bx^{j+k}_a + kx^{j+k-1}_a \right)$$

for $a, b \in \Delta$ and $j, k \in J$.

Note that $(A, \cdot, *_\alpha)$ is not a Novikov-Poisson algebra, provided $\Delta \not\subseteq \{2\pi i : n \in \mathbb{Z}\}$. Indeed, suppose there exists $a \in \Delta$ such that $a \neq 2\pi i$ for any integer $n$. Then $e^a \neq 0, 1$. Now on the one hand we have

$$(1 \cdot_\alpha 1) \cdot_\alpha x^0_a = e^a x^0_a.$$ 

On the other hand, we have

$$1 \cdot_\alpha (1 \cdot_\alpha x^0_a) = e^{2a} x^0_a,$$

which shows that $\cdot_\alpha$ is not associative. Hence $(A, \cdot, *_\alpha)$ is not a Novikov-Poisson algebra. $\square$

**Example 2.18.** Let $(A, \cdot)$ be a commutative associative algebra and $\partial: A \to A$ be a nilpotent derivation, i.e., $\partial$ is a derivation such that $\partial^n = 0$ for some $n \geq 2$. One can check that the formal exponential map

$$\varphi = \sum_{k=0}^{n-1} \frac{\partial^k}{k!} = Id + \partial + \frac{\partial^2}{2!} + \cdots + \frac{\partial^{n-1}}{(n-1)!}$$

is a well-defined algebra automorphism on $A$ [p.26]. Moreover, $\varphi$ commutes with $\partial$ because $\varphi$ is a polynomial in $\partial$. By Corollary 2.16 there is a multiplicative Hom-Novikov-Poisson algebra

$$A_\varphi = (A, \cdot, \varphi, \varphi),$$

(2.18.1) in which

$$f \cdot_\varphi g = \varphi(f) \cdot \varphi(g) = \sum_{m=0}^{2n-2} \left( \sum_{k+j=m} \frac{(\partial^k f) \cdot (\partial^j g)}{k! j!} \right),$$

$$f \ast_\varphi g = \varphi(f) \cdot \varphi(\partial g) = \sum_{m=0}^{2n-3} \left( \sum_{k+j=m} \frac{(\partial^k f) \cdot (\partial^{j+1} g)}{k! j!} \right)$$

for $f, g \in A$. $\square$
Example 2.19. This example is a special case of Example 2.18 in which \((A, \cdot_\varphi, \ast_\varphi)\) is not a Novikov-Poisson algebra.

Let \(A\) be the truncated polynomial algebra \(k[x]/(x^N)\) for some integer \(N \geq 2\). The differential operator \(\vartheta = \frac{d}{dx}\) is a nilpotent derivation on \(A\) with \(\vartheta^N = 0\). The associated formal exponential is

\[
\varphi = \sum_{k=0}^{N-1} \frac{1}{k!} \left( \frac{d}{dx} \right)^k.
\]

As in Example 2.18, \(A_{\varphi}\) in (2.18.1) is a multiplicative Hom-Novikov-Poisson algebra. We claim that \((A, \cdot_\varphi, \ast_\varphi)\) is not a Novikov-Poisson algebra. It suffices to show that \(\cdot_\varphi\) is not associative. The element \(x \in A\) satisfies

\[
\varphi^k(x) = x + k
\]

for all \(k \geq 1\). Now on the one hand, we have

\[
(x \cdot_\varphi x) \cdot_\varphi \varphi(x) = (x + 1)^2 \cdot_\varphi (x + 1) = (x + 2)^3.
\]

On the other hand, we have

\[
x \cdot_\varphi (x \cdot_\varphi \varphi(x)) = x \cdot_\varphi ((x + 1)(x + 2)) = (x + 1)(x + 2)(x + 3).
\]

We have shown that \(\cdot_\varphi\) is not associative, so \((A, \cdot_\varphi, \ast_\varphi)\) is not a Novikov-Poisson algebra. \(\square\)

Example 2.20. Let \(A\) be the polynomial algebra \(k[x_1, \ldots, x_n]\). Fix an integer \(j \in \{1, \ldots, n\}\). The partial differential operator \(\vartheta = \frac{\partial}{\partial x_j}\) with respect to \(x_j\) is a derivation on the commutative associative algebra \(A\).

For \(1 \leq i \leq n\) let \(c_i \in k\) be arbitrary scalars, and let \(\alpha : A \to A\) be the algebra morphism determined by

\[
\alpha(x_i) = x_i + c_i.
\]

Then \(\alpha\) commutes with \(\vartheta\) because

\[
\vartheta(\alpha(x_i^k)) = \vartheta((x_i + c_i)^k)
= \delta_{i,j} k(x_i + c_i)^{k-1}
= \alpha(\delta_{i,j} k x_i^{k-1})
= \alpha(\vartheta(x_i^k)).
\]

By Corollary 2.16 there is a multiplicative Hom-Novikov-Poisson algebra \((A, \cdot, \ast, \alpha)\), in which

\[
f \cdot g = f(x_1 + c_1, \ldots, x_n + c_n) g(x_1 + c_1, \ldots, x_n + c_n),
\]

\[
f \ast g = f(x_1 + c_1, \ldots, x_n + c_n) \frac{\partial g}{\partial x_j}(x_1 + c_1, \ldots, x_n + c_n)
\]

for \(f = f(x_1, \ldots, x_n)\) and \(g = g(x_1, \ldots, x_n)\) in \(A\).

Note that, as long as \(\alpha\) is not the identity map, \((A, \cdot, \ast)\) is not a Novikov-Poisson algebra because \(\cdot\) is not associative. Indeed, suppose \(c_i \neq 0\). Then we have

\[
(1 \cdot 1) \cdot x_i = x_i + c_i,
\]

whereas

\[
1 \cdot (1 \cdot x_i) = x_i + 2c_i.
\]

This shows that \(\cdot\) is not associative, so \((A, \cdot, \ast)\) is not a Novikov-Poisson algebra. \(\square\)
3. Tensor products

The tensor product of two Novikov algebras is usually not a Novikov algebra in a non-trivial way. One reason Novikov-Poisson algebras were introduced by Xu [14] was that, unlike Novikov algebras, Novikov-Poisson algebras are closed under tensor products non-trivially ([14] Theorem 4.1). Here we show that the much larger class of Hom-Novikov-Poisson algebras is also closed under tensor products.

**Theorem 3.1.** Let \((A_i, \cdot_i, \ast_i, \alpha_i)\) be Hom-Novikov-Poisson algebras for \(i = 1, 2\), and let \(A = A_1 \otimes A_2\).

Define the operations \(\alpha: A \to A\) and \(\cdot, \ast: A^{\otimes 2} \to A\) by:

\[
\alpha = \alpha_1 \otimes \alpha_2, \\
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = (x_1 \cdot_1 y_1) \otimes (x_2 \cdot_2 y_2), \\
(x_1 \otimes x_2) \ast (y_1 \otimes y_2) = (x_1 \ast_1 y_1) \otimes (x_2 \ast_2 y_2)
\]

for \(x_i, y_i \in A_i\). Then \((A, \cdot, \ast, \alpha)\) is a Hom-Novikov-Poisson algebra. Moreover, if both \(A_i\) are multiplicative, then \(A\) is also multiplicative.

**Proof.** It is easy to see that \((A, \cdot, \alpha)\) is a commutative Hom-associative algebra and that \(A\) is multiplicative if both \(A_i\) are. We show that \((A, \ast, \alpha)\) is a Hom-Novikov algebra in Lemma 3.2 below. The compatibility conditions \(2.7.1\) for \(A\) are proved in Lemma 3.3 below. \(\square\)

**Lemma 3.2.** Under the assumptions of Theorem 3.1, \((A, \ast, \alpha)\) is a Hom-Novikov algebra.

**Proof.** To improve readability, we omit the subscripts in \(\ast_i\) and \(\alpha_i\) and write \(x \ast_i y\) as \(xy\). Let us first prove \(2.6.1\). Pick \(x = x_1 \otimes x_2, y = y_1 \otimes y_2,\) and \(z = z_1 \otimes z_2\) in \(A\). We must show that \((x \ast y) \ast \alpha(z)\) is symmetric in \(y\) and \(z\). We have:

\[
(x \ast y) \ast \alpha(z) = \{x_1 \ast_1 y_1 \otimes x_2 \ast_2 y_2 + x_1 y_1 \otimes x_2 \ast_2 y_2\} \ast \{(\alpha(z_1) \otimes \alpha(z_2))
\]

\[
= \underbrace{(x_1 \ast_1 y_1) \ast \alpha(z_1) \otimes (x_2 \ast_2 y_2) \ast \alpha(z_2)}_{a} + \underbrace{(x_1 y_1) \ast \alpha(z_1) \otimes (x_2 \ast_2 y_2) \ast \alpha(z_2)}_{b(x,y,z)}
\]

\[
+ \underbrace{(x_1 y_1) \ast \alpha(z_1) \otimes (x_2 \ast_2 y_2) \ast \alpha(z_2)}_{c(x,y,z)} + \underbrace{(x_1 \ast_1 y_1) \ast \alpha(z_1) \otimes (x_2 \ast_2 y_2) \ast \alpha(z_2)}_{d}.
\]

(3.2.1)

The terms \(a\) and \(d\) are symmetric in \(y\) and \(z\) by \(2.6.1\) in both \(A_i\) and Lemma 2.9. Moreover, we have

\[
b(x, y, z) = (x_1 y_1) \ast \alpha(y_1) \otimes (x_2 \ast_2 z_2) \ast \alpha(y_2) \text{ (by } 2.7.1)\]

\[
= c(x, z, y).
\]

This shows that \((x \ast y) \ast \alpha(z)\) is symmetric in \(y\) and \(z\), so \(2.6.1\) holds in \(A\).

For \(2.6.13\) in \(A\), we must show that the Hom-associator

\[
as_\ast(x, y, z) = (x \ast y) \ast \alpha(z) - \alpha(x) \ast (y \ast z)
\]

with respect to \(\ast\) in \(A\) is symmetric in \(x\) and \(y\). Let us compute the second term in the Hom-associator:

\[
\alpha(x) \ast (y \ast z) = (\alpha(x_1) \otimes \alpha(x_2)) \ast \{y_1 \ast_1 z_1 \otimes y_2 \ast_2 z_2 + y_1 z_1 \otimes y_2 \ast_2 z_2\}
\]

\[
= \underbrace{\alpha(x_1) \ast (y_1 \ast_1 z_1) \otimes \alpha(x_2) (y_2 \ast_2 z_2)}_{a'} + \underbrace{\alpha(x_1) \ast (y_1 \ast_1 z_1) \otimes \alpha(x_2) (y_2 \ast_2 z_2)}_{b'}
\]

\[
+ \underbrace{\alpha(x_1) \ast (y_1 z_1) \otimes \alpha(x_2) (y_2 \ast_2 z_2)}_{c'} + \underbrace{\alpha(x_1) \ast (y_1 z_1) \otimes \alpha(x_2) (y_2 \ast_2 z_2)}_{d'}.
\]

(3.2.2)
Using the notations in (3.2.1) (with $b = b(x, y, z)$ and $c = c(x, y, z)$), Hom-associativity, and Lemma 2.8, we have:

\[ a - a' = a_{*i}(x_1, y_1, z_1) \otimes (x_2y_2) \alpha(z_2), \]
\[ d - d' = (x_1y_1) \alpha(z_1) \otimes a_{*i}(x_2, y_2, z_2), \]
\[ c - b' = (x_1y_1) \cdot \alpha(z_1) \otimes a_{*A_2}(x_2, y_2, z_2), \]
\[ b - c' = a_{*A_1}(x_1, y_1, z_1) \otimes (x_2y_2) \cdot \alpha(z_2). \]  

(3.2.3)

Here $a_{*i}$ is the Hom-associator with respect to $*_i$, and $a_{*A_i}$ is the mixed Hom-associator in $A_i$ (Definition 2.4). By the commutativity of $\cdot$, (2.6.1a), and (2.7.1a) in $A_i$, it follows that $(a - a')$, $(d - d')$, $(c - b')$, and $(b - c')$ are all symmetric in $x$ and $y$. Therefore, we conclude from (3.2.1), (3.2.2), and (3.2.3) that $a_{*i}$ in $A$ is also symmetric in $x$ and $y$, thereby proving (2.6.1a) in $A$. □

Lemma 3.3. Under the assumptions of Theorem 3.1, $(A, \cdot_* \cdot, \cdot)$ satisfies the compatibility conditions (2.7.1).

Proof. Let us first prove (2.7.1b) in $A$. Using Lemma 2.8, the notations in Lemma 3.2 and (2.7.1b) in $A_i$, we have:

\[
(x * y) \cdot \alpha(z) = (x_1 \cdot y_1 \otimes x_2y_2 + x_1y_1 \otimes x_2 \cdot y_2) \cdot (\alpha(z_1) \otimes \alpha(z_2)) \\
= (x_1 \cdot y_1)\alpha(z_1) \otimes (x_2y_2)\alpha(z_2) + (x_1y_1)\alpha(z_1) \otimes (x_2 \cdot y_2)\alpha(z_2) \\
= (x_1 \cdot y_1) \cdot \alpha(z_1) \otimes (x_2y_2)\alpha(y_2) + (x_1y_1)\alpha(y_1) \otimes (x_2 \cdot y_2) \cdot \alpha(y_2) \\
= (x \cdot z) \cdot \alpha(y). \tag{3.3.1}
\]

This proves (2.7.1b) in $A$. The labels $u$ and $v$ will be used in the next paragraph.

For (2.7.1a) in $A$, we must show that the mixed Hom-associator

\[ a_{*A}(x, y, z) = (x * y) \cdot \alpha(z) - \alpha(x) \cdot (y \cdot z) \]

is symmetric in $x$ and $y$. Let us compute the second term in the mixed Hom-associator:

\[
\alpha(x) \cdot (y \cdot z) = (\alpha(x_1) \otimes \alpha(x_2)) \cdot (y_1z_1 \otimes y_2z_2) \\
= (\alpha(x_1) \cdot (y_1z_1) \otimes (x_2y_2)z_2) + (\alpha(x_1) \cdot (y_1z_1) \otimes (x_2y_2)z_2) \\
= (\alpha(x_1) \cdot (y_1z_1) \otimes (x_2y_2)z_2). \tag{3.3.2}
\]

Using Hom-associativity, we have:

\[
u - u' = a_{*A_1}(x_1, y_1, z_1) \otimes (x_2y_2)\alpha(z_2),
\]
\[
v - v' = (x_1y_1)\alpha(z_1) \otimes a_{*A_2}(x_2, y_2, z_2).
\]

It follows from the commutativity of $\cdot_i$ and (2.7.1a) in $A_i$ that both $(u - u')$ and $(v - v')$ are symmetric in $x$ and $y$. Therefore, the mixed Hom-associator

\[ a_{*A}(x, y, z) = (u - u') + (v - v') \]

is also symmetric in $x$ and $y$, thereby proving (2.7.1a) in $A$. □

Taking $\alpha_i = Id_{A_i}$ in Theorem 3.1, we recover the following result, which is Theorem 4.1 in [14].

**Corollary 3.4.** Let $(A_i, \cdot_i, \cdot_i)$ be Novikov-Poisson algebras for $i = 1, 2$, and let $A = A_1 \otimes A_2$. Define the operations $\cdot_\otimes, \cdot_* : A^{\otimes 2} \to A$ by:

\[
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = (x_1 \cdot y_1) \otimes (x_2 \cdot y_2),
\]
\[
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = (x_1 \cdot y_1) \otimes (x_2 \cdot y_2) + (x_1 \cdot y_1) \otimes (x_2 \cdot y_2)
\]

for $x_i, y_i \in A_i$. Then $(A, \cdot_\otimes, \cdot_*)$ is a Novikov-Poisson algebra.
Theorem 3.1 can be used with the results and examples in the previous section to construct new Hom-Novikov-Poisson algebras. For example, the next result is obtained by first using Corollary 2.12 and then Theorem 3.1.

**Corollary 3.5.** Let \((A_i, \cdot_i, \ast_i, \alpha_i)\) be multiplicative Hom-Novikov-Poisson algebras for \(i = 1, 2\), and let \(A = A_1 \otimes A_2\). For integers \(n, m \geq 0\), define the operations \(\alpha: A \to A\) and \(\cdot, \ast: A'^{\otimes 2} \to A\) by:

\[
\alpha = \alpha_1^{n+1} \otimes \alpha_2^{m+1},
\]

\[
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = \alpha_1^n (x_1 \cdot_1 y_1) \otimes \alpha_2^m (x_1 \cdot_2 y_2),
\]

\[
(x_1 \otimes x_2) \ast (y_1 \otimes y_2) = \alpha_1^n (x_1 \ast_1 y_1) \otimes \alpha_2^m (x_2 \ast_2 y_2) + \alpha_1^n (x_1 \cdot_1 y_1) \otimes \alpha_2^m (x_2 \cdot_2 y_2)
\]

for \(x_i, y_i \in A_i\). Then \((A, \cdot, \ast, \alpha)\) is a multiplicative Hom-Novikov-Poisson algebra.

**Proof.** Indeed, using the notations in Corollary 2.12, we have

\[
(A, \cdot, \ast, \alpha) = A_1^n \otimes A_2^m,
\]

where the tensor product is equipped with the Hom-Novikov-Poisson algebra structure in Theorem 3.1.

Observe that the tensor product in Theorem 3.1 and the operation \((-)^n\) in Corollary 2.12 commute. In other words, given any two multiplicative Hom-Novikov-Poisson algebras \(A_1\) and \(A_2\), we have

\[
(A_1 \otimes A_2)^n = A_1^n \otimes A_2^n
\]

for all \(n \geq 0\).

The following result is obtained by using Corollary 2.13 and then Theorem 3.1.

**Corollary 3.6.** Let \((A_i, \cdot_i, \ast_i, \beta_i)\) be Novikov-Poisson algebras and \(\beta_i: A_i \to A_i\) be morphisms for \(i = 1, 2\). Let \(A = A_1 \otimes A_2\). Define the operations \(\beta: A \to A\) and \(\cdot, \ast: A'^{\otimes 2} \to A\) by:

\[
\beta = \beta_1 \otimes \beta_2,
\]

\[
(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = \beta_1 (x_1 \cdot_1 y_1) \otimes \beta_2 (x_2 \cdot_2 y_2),
\]

\[
(x_1 \otimes x_2) \ast (y_1 \otimes y_2) = \beta_1 (x_1 \ast_1 y_1) \otimes \beta_2 (x_2 \ast_2 y_2) + \beta_1 (x_1 \cdot_1 y_1) \otimes \beta_2 (x_2 \cdot_2 y_2)
\]

for \(x_i, y_i \in A_i\). Then \((A, \cdot, \ast, \beta)\) is a multiplicative Hom-Novikov-Poisson algebra.

**Proof.** Indeed, in the notations of Corollary 2.13, we have

\[
(A, \cdot, \ast, \beta) = (A_1)_{\beta_1} \otimes (A_2)_{\beta_2},
\]

where the tensor product is equipped with the Hom-Novikov-Poisson algebra structure in Theorem 3.1.

Observe that the tensor product in Theorem 3.1 and the operation \((-)^\beta\) in Corollary 2.17 commute. In other words, given any two Novikov-Poisson algebras \(A_i\) and morphisms \(\beta_i: A_i \to A_i\) for \(i = 1, 2\), we have

\[
(A_1 \otimes A_2)_{\beta_1 \otimes \beta_2} = (A_1)_{\beta_1} \otimes (A_2)_{\beta_2},
\]

where both tensor products are equipped with the Hom-Novikov-Poisson algebra structures in Theorem 3.1.
4. Perturbations of Hom-Novikov-Poisson algebras

The purpose of this section is to show that certain perturbations preserve Hom-Novikov-Poisson algebra structures. We need the following preliminary observation about perturbing the structure maps in a commutative Hom-associative algebra.

**Lemma 4.1.** Let \((A, \cdot, \alpha)\) be a commutative Hom-associative algebra and \(a \in A\). Define the operation \(\odot : A^\otimes 2 \to A\) by
\[
x \odot y = a \cdot (x \cdot y)
\]
for \(x, y \in A\). Then
\[
A' = (A, \odot, \alpha^2)
\]
is also a commutative Hom-associative algebra. Moreover, if \(A\) is multiplicative and \(\alpha^2(a) = a\), then \(A'\) is also multiplicative.

**Proof.** As usual we abbreviate \(x \cdot y\) to \(xy\). The commutativity of \(\odot\) follows from that of \(\cdot\). The multiplicativity assertion is straightforward to check. To show that \(A'\) is Hom-associative, pick \(x, y, z \in A\). Then we have:
\[
(x \odot y) \odot \alpha^2(z) = a\{a(xy)a^2(z)\} = a\{\alpha(a)((xy)\alpha(z))\} \quad \text{(by Hom-associativity)}
\]
\[
= a\{\alpha(a)(\alpha(x)(yz))\} \quad \text{(by Hom-associativity)}
\]
\[
= a\{\alpha^2(x)(a(yz))\} \quad \text{(by Lemma 2.9)}
\]
\[
= \alpha^2(x) \odot (y \odot z).
\]
This shows that \(A'\) is Hom-associative. \(\square\)

The following result says that the structure maps of a multiplicative Hom-Novikov-Poisson algebra can be perturbed by a suitable element and its own twisting map.

**Theorem 4.2.** Let \((A, \cdot, \ast, \alpha)\) be a multiplicative Hom-Novikov-Poisson algebra and \(a \in A\) be an element such that \(\alpha^2(a) = a\). Then
\[
A' = (A, \odot, \ast, \alpha^2)
\]
is also a multiplicative Hom-Novikov-Poisson algebra, where
\[
x \odot y = a \cdot (x \cdot y),
\]
\[
x \ast \alpha y = \alpha(x \ast y) = \alpha(x) \ast \alpha(y)
\]
for all \(x, y \in A\).

**Proof.** By Lemma 4.1 we know that \((A, \odot, \alpha^2)\) is a multiplicative commutative Hom-associative algebra. Also, \((A, \ast, \alpha^2)\) is a multiplicative Hom-Novikov algebra by Corollary 2.13 (the \(n = 1\) case). It remains to prove the compatibility conditions (2.7.1) for \(A'\).

To prove the compatibility condition (2.7.1) (or equivalently (2.8.2)) for \(A'\), pick \(x, y, z \in A\). With \(x \cdot y\) written as \(xy\), we have:
\[
(x \odot y) \ast \alpha a^2(z) = \{\alpha(a)(\alpha(xy))\} \ast \alpha^3(z) \quad \text{(by multiplicativity)}
\]
\[
= \alpha^2(a)\{\alpha(xy) \ast \alpha^2(z)\} \quad \text{(by (2.8.2) in \(A\))}
\]
\[
= \alpha\{\alpha(x)(\alpha(y)) \ast \alpha^2(z)\} \quad \text{(by multiplicativity)}
\]
\[
= \alpha\{\alpha^2(x)(\alpha(y) \ast \alpha(z))\} \quad \text{(by (2.8.2) in \(A\))}
\]
\[
= \alpha^2(x) \odot (y \ast \alpha z).
\]
This proves that \(A'\) satisfies (2.8.2), which is equivalent to (2.7.1) by Lemma 2.8.
To prove the compatibility condition (2.7.1a) for \( A' \), we must show that the mixed Hom-associator (Definition 2.4)
\[
as_{A'}(x, y, z) = (x \ast, y) \circ \alpha^2(z) - \alpha^2(x) \ast, (y \circ z)
\] (4.2.1)
in \( A' \) is symmetric in \( x \) and \( y \). The first term in the mixed Hom-associator is:
\[
(x \ast, y) \circ \alpha^2(z) = a\{(\alpha(x) \ast, \alpha(y))\alpha^2(z)\}
\]
\[
= \{(\alpha(x) \ast, \alpha(y))\alpha^2(z)\}\alpha^2(a) \quad \text{(by commutativity)}
\]
\[
= \{\alpha^2(x) \ast, \alpha^2(y)\}(\alpha^2(z)\alpha(a)) \quad \text{(by Hom-associativity)}
\] (4.2.2)
\[
= \{\alpha^2(x) \ast, \alpha^2(y)\}(\alpha(\alpha(z)a)).
\]
The second term in the mixed Hom-associator is:
\[
\alpha^2(x) \ast, \alpha(y \circ z) = \alpha^3(x) \ast, \alpha(ayz) \]
\[
= \alpha^3(x) \ast, \{\alpha(a)(\alpha(y)\alpha(z))\}
\]
\[
= \alpha^3(x) \ast, \{\alpha^2(y)\alpha(\alpha(z)a)\} \quad \text{(by Lemma 2.9)}
\] (4.2.3)
Using (4.2.1), (4.2.2), and (4.2.3), it follows that the mixed Hom-associators of \( A' \) and \( A \) are related as follows:
\[
as_{A'}(x, y, z) = \{\alpha^2(x) \ast, \alpha^2(y)\}(\alpha(\alpha(z)a)) - \alpha^3(x) \ast, \{\alpha^2(y)\alpha(\alpha(z)a)\}
\]
\[
= as_A(\alpha^2(x), \alpha^2(y), \alpha(z)a)
\]
Since \( as_A \) is symmetric in its first two arguments (2.7.1a), we conclude that \( as_{A'}(x, y, z) \) is symmetric in \( x \) and \( y \). \( \square \)

Setting \( \alpha = Id_A \) in Theorem 4.2, we recover Lemma 2.4 in [15]:

**Corollary 4.3.** Let \((A, \ast, \ast)\) be a Novikov-Poisson algebra and \( a \in A \) be an arbitrary element. Then \((A, \circ, \ast)\) is also a Novikov-Poisson algebra, where
\[
x \circ y = a \cdot x \cdot y
\]
for all \( x, y \in A \).

The next result is a variation on the theme of Theorem 4.2. It gives a way to perturb a Hom-Novikov-Poisson algebra structure using a suitable element and its own twisting map.

**Theorem 4.4.** Let \((A, \ast, \ast, \alpha)\) be a multiplicative Hom-Novikov-Poisson algebra and \( a \in A \) be an element such that \( \alpha^2(a) = a \). Then
\[
\overline{A} = (A, \ast, \ast, \alpha^2)
\]
is also a multiplicative Hom-Novikov-Poisson algebra, where
\[
x \ast, y = \alpha(x \cdot y) = \alpha(x) \cdot \alpha(y),
\]
\[
x \times y = \alpha(x) \ast, \alpha(y) + a \cdot (x \ast, y)
\]
for all \( x, y \in A \).

**Proof.** Throughout the proof of Theorem 4.4, we abbreviate \( x \cdot y \) to \( xy \). By Corollary 2.14 (the \( n = 1 \) case) \((A, \ast, \alpha^2)\) is a multiplicative commutative Hom-associative algebra. We show that \((A, \times, \alpha^2)\) is a multiplicative Hom-Novikov algebra in Lemma 4.5 below. The compatibility conditions (2.7.1) for \( \overline{A} \) are proved in Lemma 4.6 below. \( \square \)

**Lemma 4.5.** Under the assumptions of Theorem 4.4, \((A, \times, \alpha^2)\) is a multiplicative Hom-Novikov algebra.
Proof. The multiplicativity of \((A, \times, \alpha^2)\) follows from that of \(A\) and the assumption \(\alpha^2(a) = a\). Pick arbitrary elements \(x, y, z \in A\). To check the condition (2.6.1b), we must show that the expression \((x \times y) \times \alpha^2(z)\) is symmetric in \(y\) and \(z\). Expanding this expression in terms of \(\cdot, \ast, \text{and } \alpha\), we have:

\[
(x \times y) \times \alpha^2(z) = (\alpha(x) \ast \alpha(y) + a(xy)) \times \alpha^2(z)
\]

\[
= \left(\frac{\alpha^2(x) + \alpha^2(y)}{t} + \alpha(xy) \ast \alpha(z)\right) + a\left(\alpha(x) \ast \alpha(y) \ast \alpha^2(z)\right) + a\left(\alpha(xy) \ast \alpha^2(z)\right)
\]

(4.5.1)

The term \(t\) in (4.5.1) is symmetric in \(y\) and \(z\) by (2.6.1b) in \((A, \ast, \alpha)\). By the Hom-associativity in \((A, \cdot, \alpha)\), the term \(w\) in (4.5.1) can be rewritten as

\[
w = a\{\alpha(\alpha)((\alpha(y)\alpha(z)))\},
\]

(4.5.2)

which is symmetric in \(x\), \(y\), and \(z\) by Lemma 2.9. Next we claim that

\[
u(x, y, z) = v(x, z, y)
\]

in (4.5.1). To prove this, we compute as follows:

\[
u(x, y, z) = \{\alpha(a)(\alpha(x)\alpha(y))) \ast \alpha^3(z) \text{ (by multiplicativity)}
\]

\[
= \{(\alpha(a(\alpha(y))\alpha^2(x)) \ast \alpha^3(z) \text{ (by Lemma 2.9)}
\]

\[
= \alpha(a(\alpha(y))\{\alpha^2(x) \ast \alpha^2(z)\} \text{ (by Lemma 2.8)}
\]

\[
= \{\alpha(\alpha(a))\alpha(a(x) \ast \alpha(z)) \text{ (by multiplicativity)}
\]

\[
= \alpha^2(a)\{\alpha(\alpha(y))\alpha(a(x) \ast \alpha(z))) \text{ (by Hom-associativity)}
\]

\[
= a\{\alpha(a) \ast \alpha(z))\alpha^2(y)}
\]

\[
= v(x, z, y).
\]

By (4.5.1) it follows that \((x \times y) \times \alpha^2(z)\) is symmetric in \(y\) and \(z\), thereby proving (2.6.1b) for \((A, \times, \alpha^2)\).

For (2.6.1a) we must show that the Hom-associator

\[
as_{\times}(x, y, z) = (x \times y) \times \alpha^2(z) - \alpha^2(x) \times (y \times z)
\]

in \((A, \times, \alpha^2)\) is symmetric in \(x\) and \(y\). Expanding the second term in this Hom-associator, we have:

\[
\alpha^2(x) \times (y \times z) = \alpha^2(x) \times (\alpha(y) \ast \alpha(z) + a(yz))
\]

\[
= \alpha^3(x) \ast \{\alpha^2(y) \ast \alpha(z)\} \ast \alpha(a(yz))
\]

\[
= \alpha^3(x) \ast \{\alpha^2(y) \ast \alpha(z)\} \ast \alpha(a(yz))
\]

(4.5.3)

Using the notations in (4.5.1) (with \(u = u(x, y, z)\) and (4.5.3), we have

\[
as_{\times}(x, y, z) = (t - t') + (w - w') + (u - v') + (v(x, y, z) - w'(x, y, z)).
\]

(4.5.4)

We now show that

\[
w - w' = 0 = u - v'
\]

and that both \((t - t')\) and \((v(x, y, z) - w'(x, y, z))\) are symmetric in \(x\) and \(y\).

The first summand on the right-hand side of (4.5.4) is:

\[
t - t' = as_{\times}(\alpha^2(x), \alpha^2(y), \alpha^2(z)),
\]
where \(as_*\) is the Hom-associator of \(A\) with respect to \(*\). Since \((A, *, \alpha)\) is a Hom-Novikov algebra, \(as_*\) is symmetric in its first two variables by \((2.6.1a)\), which implies that \((t - t')\) is symmetric in \(x\) and \(y\).

For the second summand on the right-hand side of \((4.5.4)\), note that by \((1.5.2)\) we have:
\[
\begin{align*}
w &= a\{\alpha(a)((xy)\alpha(z))\} \\
    &= a\{\alpha(a)(yz)\alpha(x)\} \quad \text{(by Lemma 2.9)} \\
    &= a\{\alpha^2(x)(a(yz))\} \quad \text{(by Lemma 2.9)} \\
    &= w'.
\end{align*}
\]
Therefore, we have \(w - w' = 0\).

For the third summand on the right-hand side of \((4.5.4)\), we have:
\[
\begin{align*}
u &= \{\alpha(a)(\alpha(x)\alpha(y))\} * \alpha^3(z) \\
    &= \{\alpha^2(y)\alpha(a(x))\} * \alpha^3(z) \quad \text{(by Lemma 2.9)} \\
    &= (\alpha^2(y) * \alpha^2(z))\alpha(aa(x)) \quad \text{(by (2.7.1b))} \\
    &= \{\alpha(a)a^2(x)\}\alpha(\alpha(y) * \alpha(z)) \quad \text{(by multiplicativity)} \\
    &= \alpha^2(a)\{\alpha^2(x)(\alpha(y) * \alpha(z))\} \quad \text{(by Hom-associativity)} \\
    &= u'.
\end{align*}
\]
Therefore, we have \(u - u' = 0\).

For the last summand on the right-hand side of \((4.5.4)\), first note that:
\[
\begin{align*}
v(x, y, z) &= \alpha^2(a)\{\{\alpha(x) * \alpha(y)\}\alpha^2(z)\} \\
    &= \{\alpha(a)\alpha^2(z)\}\alpha(\alpha(x) * \alpha(y)) \quad \text{(by Lemma 2.9)} \\
    &= \alpha(aa(z))\{\alpha^2(x) * \alpha^2(y)\} \quad \text{(by multiplicativity)} \quad (4.5.5) \\
    &= \{(aa(z))\alpha^2(x)\} * \alpha^3(y) \quad \text{(by Lemma 2.8)} \\
    &= \{\alpha^2(x)\alpha^2(y)\} * \alpha^2(y) * \alpha^3(x) \quad \text{(by (2.7.1b)).}
\end{align*}
\]
On the other hand, we have:
\[
\begin{align*}
u'(x, y, z) &= \alpha^3(x) * \{\alpha(a)(\alpha(y)\alpha(z))\} \\
    &= \alpha^3(x) * \{(aa(z))\alpha^2(y)\} \quad \text{(by Lemma 2.9)} \\
    &= \{\alpha^2(x) * (aa(z))\} \alpha^3(y) + \alpha(aa(z)) * (\alpha^2(x)\alpha^2(y)) \\
    & - \{(aa(z))\} \alpha^2(x)\alpha^3(y) \quad \text{(by left-symmetry of \(as_A(\alpha^2(x), aa(z), \alpha^2(y))\) \((2.7.1a)\))} \\
    &= \{\alpha^2(x)\alpha^2(y)\} * \alpha(aa(z)) + \alpha(aa(z)) * (\alpha^2(x)\alpha^2(y)) \\
    & - \alpha^3(x) * \alpha(aa(z)) - \alpha(aa(z)) * (\alpha^2(x)\alpha^2(y)),
\end{align*}
\]
Therefore, we have:
\[
\begin{align*}
v(x, y, z) - u'(x, y, z) &= v(x, y, z) + v(y, x, z) \\
    &= -\{\alpha^2(x)\alpha^2(y)\} * \alpha(aa(z)) - \alpha(aa(z)) * (\alpha^2(x)\alpha^2(y)),
\end{align*}
\]
which is symmetric in \(x\) and \(y\) because \(A\) is commutative.

In summary, the previous four paragraphs and \((4.5.4)\) show that the Hom-associator in \((A, x, \alpha^2)\) is symmetric in \(x\) and \(y\), thereby proving \((2.6.1a)\). □

**Lemma 4.6.** Under the assumptions of Theorem \(4.4\), \(A\) satisfies the compatibility conditions \((2.7.1)\).
Proof. Pick elements $x, y, z \in A$. To prove the compatibility condition (2.7.1a), or equivalently (2.8.2), for $\mathcal{A}$, we compute as follows:

\[
(x \cdot_\alpha y) \times \alpha^2(z) = (\alpha(x)\alpha(y)) \times \alpha^2(z)
\]

\[
= \{\alpha^2(x)\alpha^2(y)\} \ast \alpha^3(z) + a((\alpha(x)\alpha(y))\alpha^2(z))
\]

\[
= \alpha^3(x)\{\alpha^2(y) \ast \alpha^2(z)\} + \alpha^2(a)\{\alpha^2(x)(\alpha(y)\alpha(z))\} \quad \text{(by (2.8.2) and Hom-associativity)}
\]

\[
= \alpha^3(x)\{\alpha^2(y) \ast \alpha^2(z)\} + \alpha^3(x)\{\alpha(a)(\alpha(y)\alpha(z))\} \quad \text{(by Lemma 2.7)}
\]

\[
= \alpha^3(x)\alpha(\alpha(y) \ast \alpha(z) + a(yz)) \quad \text{(by multiplicativity)}
\]

\[
= \alpha^2(x) \cdot_\alpha (y \times z).
\]

To prove the compatibility condition (2.7.1a) for $\mathcal{A}$, we must show that the mixed Hom-associator

\[
as_\mathcal{A}(x, y, z) = (x \times y) \cdot_\alpha \alpha^2(z) - \alpha^2(x) \times (y \cdot_\alpha z)
\]

of $\mathcal{A}$ is symmetric in $x$ and $y$. The first summand in this mixed Hom-associator is:

\[
(x \times y) \cdot_\alpha \alpha^2(z) = \alpha(\alpha(x) \ast \alpha(y) + a(xy))\alpha^3(z)
\]

\[
= \{\alpha^2(x) \ast \alpha^2(y)\}\alpha^3(z) + \{\alpha(a)(\alpha(x)\alpha(y))\} \alpha^3(z).
\]

(4.6.1)

The second summand in the mixed Hom-associator is:

\[
\alpha^2(x) \times (y \cdot_\alpha z) = \alpha^2(x) \times (\alpha(y)\alpha(z))
\]

\[
= \alpha^3(x)\{\alpha^2(y)\alpha^2(z)\} + a(\alpha^2(x)(\alpha(y)\alpha(z)))
\]

\[
= \alpha^3(x)\{\alpha^2(y)\alpha^2(z)\} + \alpha^2(a)(\{\alpha(x)\alpha(y)\}\alpha^2(z)) \quad \text{(by Hom-associativity)}
\]

\[
= \alpha^3(x)\{\alpha^2(y)\alpha^2(z)\} + \{\alpha(a)(\alpha(x)\alpha(y))\} \alpha^3(z) \quad \text{(by Hom-associativity)}.
\]

(4.6.2)

Combining (4.6.1) and (4.6.2), it follows that the mixed Hom-associators of $\mathcal{A}$ and $A$ are related as follows:

\[
as_\mathcal{A}(x, y, z) = \{\alpha^2(x) \ast \alpha^2(y)\}\alpha^3(z) - \alpha^3(x) \ast (\alpha^2(y)\alpha^2(z))
\]

\[
= as_A(\alpha^2(x), \alpha^2(y), \alpha^2(z)).
\]

Since $as_A$ is left-symmetric by (2.7.1a), we conclude that $as_\mathcal{A}$ is symmetric in $x$ and $y$, as desired. □

With Lemmas 4.4 and 4.6 proved, the proof of Theorem 4.4 is complete. We now discuss some special cases of Theorem 4.4.

Setting $\alpha = Id_A$ in Theorem 4.4, we recover Lemma 2.3 in [13].

Corollary 4.7. Let $(A, \cdot, *)$ be a Novikov-Poisson algebra and $a \in A$ an arbitrary element. Then $(A, \cdot, x)$ is also a Novikov-Poisson algebra, where

\[
x \times y = x \ast y + a \cdot x \cdot y
\]

for all $x, y \in A$.

Forgetting about the Hom-associative product $\cdot_\alpha$ in Theorem 4.4, we obtain the following result, which gives a non-trivial way to construct a Hom-Novikov algebra from a Hom-Novikov-Poisson algebra.

Corollary 4.8. Let $(A, \cdot, *, \alpha)$ be a multiplicative Hom-Novikov-Poisson algebra and $a \in A$ an element such that $\alpha^2(a) = a$. Then $(A, \times, \alpha^2)$ is a multiplicative Hom-Novikov algebra, where

\[
x \times y = \alpha(x) \ast \alpha(y) + a \cdot (x \cdot y)
\]

for all $x, y \in A$.

Setting $\alpha = Id_A$ in Corollary 4.8, we obtain the following special case of Corollary 4.7.
Corollary 4.9. Let \((A, \cdot, \ast)\) be a Novikov-Poisson algebra and \(a \in A\) be an arbitrary element. Then \((A, \times)\) is a Novikov algebra, where
\[
x \times y = x \ast y + a \cdot x \cdot y
\]
for all \(x, y \in A\).

The following perturbation result is obtained by combining Theorems 4.2 and 4.4.

Corollary 4.10. Let \((A, \cdot, \ast, \alpha)\) be a multiplicative Hom-Novikov-Poisson algebra and \(a, b \in A\) be elements such that \(\alpha^2(a) = a\) and \(\alpha^4(b) = b\). Then
\[
\tilde{A} = (A, \circ, \boxtimes, \alpha^4)
\]
is also a multiplicative Hom-Novikov-Poisson algebra, where
\[
x \circ y = \alpha(b) \cdot \alpha^2(x \cdot y),
x \boxtimes y = \alpha^3(x \ast y) + a \cdot \alpha^2(x \cdot y)
\]
for all \(x, y \in A\).

Proof. By Theorem 4.4, \(A = (A, \cdot, \alpha, \times, \alpha^2)\) is a multiplicative Hom-Novikov-Poisson algebra. Now apply Theorem 4.2 to \(A\) and the element \(b \in A\), which satisfies \((\alpha^2)^2(b) = b\). We obtain a multiplicative Hom-Novikov-Poisson algebra \((\tilde{A})'\), which is \(\tilde{A}\) above. \(\square\)

Setting \(\alpha = \text{Id}_A\) in Corollary 4.10, we recover Theorem 2.5 in [15]:

Corollary 4.11. Let \((A, \cdot, \ast)\) be a Novikov-Poisson algebra and \(a, b \in A\) be arbitrary elements. Then \((A, \circ, \boxtimes)\) is also a Novikov-Poisson algebra, where
\[
x \circ y = b \cdot x \cdot y,
x \boxtimes y = x \ast y + a \cdot x \cdot y
\]
for all \(x, y \in A\).

The following result is another special case of Corollary 4.10.

Corollary 4.12. Let \(A\) be a commutative associative algebra, \(\partial: A \to A\) be a derivation, \(\alpha: A \to A\) be an algebra morphism such that \(\alpha \partial = \partial \alpha\), and \(a, b \in A\) be elements such that \(\alpha^2(a) = a\) and \(\alpha^4(b) = b\). Then \((A, \cdot, \ast, \alpha^4)\) is a multiplicative Hom-Novikov-Poisson algebra, where
\[
x \cdot y = \alpha^2(b) \alpha^4(xy),
x \ast y = \alpha^4(x \partial(y)) + \alpha(a) \alpha^4(xy)
\]
for all \(x, y \in A\).

Proof. By Corollary 4.10, \(A_\alpha = (A, \alpha \mu, \alpha \mu (\text{Id} \otimes \partial), \alpha)\) is a multiplicative Hom-Novikov-Poisson algebra, where \(\mu\) is the given commutative associative product in \(A\). Now apply Corollary 4.10 to \(A_\alpha\) and the elements \(a\) and \(b\). The result is a multiplicative Hom-Novikov-Poisson algebra \(\tilde{A}_\alpha\), whose operations are as stated above. \(\square\)

Setting \(\alpha = \text{Id}_A\) in Corollary 4.12, we recover Corollary 2.6 in [15] (see also [3]):

Corollary 4.13. Let \(A\) be a commutative associative algebra, \(\partial: A \to A\) be a derivation, and \(a, b \in A\) be arbitrary elements. Then \((A, \cdot, \ast)\) is a Novikov-Poisson algebra, where
\[
x \cdot y = bxy,
x \ast y = x \partial(y) + axy
\]
for all \(x, y \in A\).
5. From Hom-Novikov-Poisson algebras to Hom-Poisson algebras

The purpose of this section is to show how Hom-Poisson algebras arise from Hom-Novikov-Poisson algebras. A Poisson algebra is a commutative associative algebra with a Lie algebra structure that satisfies the Leibniz identity. To define a Hom-Poisson algebra, let us first recall the relevant definitions.

**Definition 5.1.** A Hom-Lie algebra \([9, 11, 17]\) is a Hom-algebra \((A, [\cdot, \cdot], \alpha)\) such that \([\cdot, \cdot]\) is anti-symmetric and that the Hom-Jacobi identity

\[
[[x, y], \alpha(z)] + [[z, x], \alpha(y)] + [[y, z], \alpha(x)] = 0
\]

holds for all \(x, y, z \in A\).

**Definition 5.2.** A Hom-Poisson algebra \([13]\) \((A, \cdot, [\cdot, \cdot], \alpha)\) consists of

1. a commutative Hom-associative algebra \((A, \cdot, \alpha)\) and
2. a Hom-Lie algebra \((A, [\cdot, \cdot], \alpha)\)

such that the Hom-Leibniz identity

\[
\alpha(x) \cdot y \cdot z = [x, y] \cdot \alpha(z) + \alpha(y) \cdot [x, z]
\]

holds for all \(x, y, z \in A\).

Hom-Poisson algebras were defined \([13]\) as Hom-type generalizations of Poisson algebras. In the context of deformations, Hom-Poisson algebras were shown in \([13]\) to be related to commutative Hom-associative algebras as Poisson algebras are related to commutative associative algebras. Further properties of (non-commutative) Hom-Poisson algebras can be found in \([25]\).

Both a Hom-Novikov-Poisson algebra and a Hom-Poisson algebra have an underlying commutative Hom-associative algebra. So it makes sense to ask whether a Hom-Poisson algebra can be constructed from a Hom-Novikov-Poisson algebra by taking the commutator bracket of the Hom-Novikov product. To answer this question, we make the following definitions.

**Definition 5.3.** Let \((A, \cdot, *, \alpha)\) be a double Hom-algebra. Its left Hom-associator \(\text{as}_A^l: A^\otimes 3 \rightarrow A\) is defined as

\[
\text{as}_A^l = * (\cdot \otimes \alpha - \alpha \otimes \cdot),
\]

or equivalently

\[
\text{as}_A^l(x, y, z) = (x \cdot y) * \alpha(z) - \alpha(x) * (y \cdot z)
\]

for \(x, y, z \in A\). The double Hom-algebra \(A\) is called left Hom-associative if \(\text{as}_A^l = 0\).

Using Lemma \([2.5]\), the left Hom-associator \(\text{as}_A^l\) in a Hom-Novikov-Poisson algebra \(A\) is equivalent to

\[
\text{as}_A^l(x, y, z) = \alpha(x) \cdot (y * z) - \alpha(x) * (y \cdot z)
\]

for all \(x, y, z \in A\).

**Definition 5.4.** Let \((A, \cdot, *, \alpha)\) be Hom-Novikov-Poisson algebra. Then \(A\) is called admissible if the double Hom-algebra

\[
A^- = (A, \cdot, [\cdot, \cdot], \alpha)
\]

is a Hom-Poisson algebra, where

\[
[x, y] = x * y - y * x
\]

for all \(x, y \in A\).

The following result gives a necessary and sufficient condition under which a Hom-Novikov-Poisson algebra is admissible. It is the Hom-type generalization of an observation in \([20]\).
Theorem 5.5. Let \((A, \cdot, *, \alpha)\) be a Hom-Novikov-Poisson algebra. Then \(A\) is admissible if and only if it is left Hom-associative.

Proof. By definition \((A, \cdot, \alpha)\) is a commutative Hom-associative algebra. Moreover, \((A, [,], \alpha)\) is a Hom-Lie algebra by Proposition 4.3 in [11]. (Equivalently, one can expand the left-hand side of the Hom-Jacobi identity in terms of \(*\) and observe that (2.6.1a) implies that the resulting sum is 0.) Therefore, we must show that \(A^-\) satisfies the Hom-Leibniz identity (5.2.1) if and only if \(as^I_A = 0\).

Pick elements \(x, y, z \in A\). As usual we abbreviate \(xy\) to \(xy\). The left-hand side of the Hom-Leibniz identity (5.2.1) for \(A^-\) is:

\[
\alpha(x), yz \equiv \alpha(x) \ast (yz) - (yz) \ast \alpha(x) = (x \ast y) \alpha(z) - (y \ast x) \alpha(z) + \underbrace{\alpha(y) \ast (xz)}_{p} - \alpha(y)(z \ast x) \quad \text{(by (2.7.2) and Lemma 2.8)} \tag{5.5.1}
\]

The right-hand side of the Hom-Leibniz identity (5.2.1) for \(A^-\) is:

\[
[x, y] = (x \ast y) \alpha(z) - (y \ast x) \alpha(z) + \alpha(y)(x \ast z) - \alpha(y)(z \ast x) = (x \ast y) \alpha(z) - (y \ast x) \alpha(z) + \underbrace{(yx) \ast \alpha(z)}_{q} - \alpha(y)(z \ast x) \quad \text{(by Lemma 2.8)} \tag{5.5.2}
\]

It follows from (5.5.1) and (5.5.2) that \(A^-\) satisfies the Hom-Leibniz identity if and only if

\[
0 = q - p = (yx) \ast \alpha(z) - \alpha(y) \ast (xz)
\]

\[
= as^I_A (y, x, z).
\]

Since \(x, y, z \in A\) are arbitrary, we conclude that \(A^-\) is a Hom-Poisson algebra if and only if \(A\) is left Hom-associative. \(\square\)

Example 5.6. Consider the multiplicative Hom-Novikov-Poisson algebra \(A_\alpha = (A, \cdot, *, \alpha)\) in Corollary 2.16. Here \(A\) is a commutative associative algebra, \(\partial: A \to A\) is a derivation, and \(\alpha: A \to A\) is an algebra morphism such that \(\alpha \partial = \partial \alpha\). The operations \(\cdot\) and \(*\) are

\[
x \cdot y = \alpha(xy) \quad \text{and} \quad x \ast y = \alpha(x \partial y)
\]

for \(x, y \in A\). Then \(A_\alpha\) is admissible if and only if

\[
\alpha^2(xy \partial z) = 0 \tag{5.6.1}
\]

for all \(x, y, z \in A\). Indeed, \(A_\alpha\) is left Hom-associative if and only if

\[
0 = \alpha(x) \ast (y \ast z) - \alpha(x) \cdot (y \ast z)
\]

\[
= \alpha\{\alpha(x) \partial(\alpha(yz))\} - \alpha\{\alpha(x) \alpha(y \partial z)\}
\]

\[
= \alpha^2(x \partial(yz) - x y \partial z)
\]

\[
= \alpha^2(xz \partial y).
\]

Therefore, \(A_\alpha\) satisfies (5.6.1) if and only if it is left Hom-associative, which by Theorem 5.5 is equivalent to \(A_\alpha\) being admissible. \(\square\)

In the rest of this section, we show that admissibility is compatible with the constructions in the previous sections. We begin with the twisting constructions in section \(\text{2}\)

Corollary 5.7. Let \((A, \cdot, *, \alpha)\) be an admissible Hom-Novikov-Poisson algebra and \(\beta: A \to A\) be a weak morphism. Then

\[
A_\beta = (A, \beta \cdot, \beta *, \beta \alpha)
\]
is also an admissible Hom-Novikov-Poisson algebra. Moreover, if \( A \) is multiplicative and \( \beta \) is a morphism, then \( A_\beta \) is also multiplicative.

**Proof.** By Theorem 2.11 \( A_\beta \) is a Hom-Novikov-Poisson algebra, and if \( A \) is multiplicative and \( \beta \) is a morphism, then \( A_\beta \) is multiplicative. The left Hom-associators in \( A \) and \( A_\beta \) are related as

\[
as_{A_\beta}^l = \beta^2 a_{A}^l.
\]

Since \( A \) is left Hom-associative by Theorem 5.5, it follows that so is \( A_\beta \). Therefore, by Theorem 5.5 again \( A_\beta \) is admissible. \( \square \)

In the context of Corollary 5.7, the Hom-Lie bracket in the Hom-Poisson algebra \( A^- \) is given by

\[
\beta(x \ast y) - \beta(y \ast x) = \beta[x, y],
\]

where \([ \cdot, \cdot ]\) is the Hom-Lie bracket in the Hom-Poisson algebra \( A^- \).

The next result is a special case of Corollary 5.7.

**Corollary 5.8.** Let \((A, \cdot, \ast, \alpha)\) be a multiplicative admissible Hom-Novikov-Poisson algebra. Then so is

\[
A^n = (A, \alpha^n \cdot, \alpha^n \ast, \alpha^{n+1})
\]

for each \( n \geq 0 \).

**Proof.** The multiplicativity of \( A \) implies that \( \alpha^n \) is a morphism. Now apply Corollary 5.7 with \( \beta = \alpha^n \). \( \square \)

Next we observe that admissibility is preserved by tensor products.

**Corollary 5.9.** Let \((A_i, \cdot_i, \ast_i, \alpha_i)\) be admissible Hom-Novikov-Poisson algebras for \( i = 1, 2 \), and let \( A = A_1 \otimes A_2 \) be the Hom-Novikov-Poisson algebra in Theorem 5.4. Then \( A \) is admissible.

**Proof.** By Theorem 5.7 we need to show that \( A \) is left Hom-associative. Pick \( x = x_1 \otimes x_2, y = y_1 \otimes y_2 \), and \( z = z_1 \otimes z_2 \) in \( A \). Then we have:

\[
\alpha(x) \cdot (y \ast z)
= \alpha(x_1) \cdot (y_1 \ast z_1) \otimes \alpha(x_2) \cdot (y_2 \ast z_2) + \alpha(x_1) \cdot (y_1 \ast z_1) \otimes \alpha(x_2) \cdot (y_2 \ast z_2)
= \alpha(x_1) \ast (y_1 \ast z_1) \otimes \alpha(x_2) \ast (y_2 \ast z_2) + \alpha(x_1) \cdot (y_1 \ast z_1) \otimes \alpha(x_2) \ast (y_2 \ast z_2)
= \alpha(x) \ast (y \ast z).
\]

Therefore, \( A \) is left Hom-associative by (5.3.1). \( \square \)

In the context of Corollary 5.9, the Hom-Lie bracket in the Hom-Poisson algebra \( A^- \) is given by:

\[
[x_1 \otimes x_2, y_1 \otimes y_2] = (x_1 \otimes x_2) \ast (y_1 \otimes y_2) - (y_1 \otimes y_2) \ast (x_1 \otimes x_2)
= x_1 \ast y_1 \otimes x_2 y_2 + x_1 y_1 \otimes x_2 \ast y_2
- y_1 \ast x_1 \otimes y_2 x_2 - y_1 x_1 \otimes y_2 \ast x_2
= [x_1, y_1] \otimes x_2 y_2 + x_1 y_1 \otimes [x_2, y_2].
\]

The last equality holds because \( \cdot_i \) is commutative, and \([x_i, y_i]\) is the Hom-Lie bracket in the Hom-Poisson algebra \( A^-_i \).

Next we observe that admissibility is preserved by the perturbations in Theorem 4.2.

**Corollary 5.10.** Let \((A, \cdot, \ast, \alpha)\) be a multiplicative admissible Hom-Novikov-Poisson algebra and \( a \in A \) be an element such that \( \alpha^2(a) = a \). Then the multiplicative Hom-Novikov-Poisson algebra

\[
A' = (A, \circ, \ast_a, \alpha^2)
\]

in Theorem 4.2 is also admissible.
Proof. By Theorem 5.5 we need to show that $A'$ is left Hom-associative. Pick $x, y, z \in A$. Recall that
\[ x \circ y = a \cdot (x \cdot y) \quad \text{and} \quad x \ast y = \alpha(x \ast y) = \alpha(x) \ast \alpha(y) \]
for $x, y \in A$. The first summand in the left Hom-associator (5.3.1) in $A'$ is:
\[
\alpha^2(x) \circ (y \ast \alpha z) = a\{\alpha^2(x)(\alpha(y) \ast \alpha(z))\}
= \alpha^2(a)\{\alpha^2(x)(\alpha(y) \ast \alpha(z))\}
= \alpha^3(x)\{\alpha(a)(\alpha(y) \ast \alpha(z))\} \quad \text{(by Lemma 2.9)}
= \alpha^3(x)\{(aa(y)) \ast \alpha^2(z)\} \quad \text{(by Lemma 2.8)}.
\]
The second summand in the left Hom-associator (5.3.1) in $A'$ is:
\[
\alpha^2(x) \ast (y \circ \alpha z) = \alpha^3(x) \ast \{\alpha(a)(\alpha(y)\alpha(z))\}
= \alpha^3(x) \ast \{(aa(y))\alpha^2(z)\} \quad \text{(by Hom-associativity)}.
\]
Therefore, the left Hom-associators (5.3.1) in $A'$ and $A$ are related as:
\[
as^l_{A'}(x, y, z) = \alpha^3(x)\{(aa(y))\alpha^2(z)\} - \alpha^3(x) \ast \{(aa(y))\alpha^2(z)\}
= as^l_A(\alpha^2(x), aa(y), \alpha^2(z)).
\]
Since $A$ is left Hom-associative by Theorem 5.5, it follows that so is $A'$.

In the context of Corollary 5.1, the Hom-Lie bracket in the Hom-Poisson algebra $(A')^-$ is given by
\[ x \ast y - y \ast x = \alpha(x \ast y - y \ast x) = \alpha[x, y], \]
where $[\cdot]$ is the Hom-Lie bracket in the Hom-Poisson algebra $A^-$. 

References

[1] E. Abe, Hopf algebras, Cambridge Tracts in Math. 74, Cambridge U. Press, Cambridge, 1977.
[2] A.A. Balinskii and S.P. Novikov, Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras, Soviet Math. Dokl. 32 (1985) 228-231.
[3] B.A. Dubrovin and S.P. Novikov, Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov-Whitham averaging method, Soviet Math. Dokl. 27 (1983) 665-669.
[4] B.A. Dubrovin and S.P. Novikov, On Poisson brackets of hydrodynamic type, Soviet Math. Dokl. 30 (1984) 651-654.
[5] V.T. Filippov, A class of simple nonassociative algebras, Mat. Zametki 45 (1989) 101-105.
[6] I.M. Gel’fand and L.A. Diki, Asymptotic behavior of the resolvent of Sturm-Liouville equations and the Lie algebras of the Korteweg-de Vries equations, Russian Math. Sur. 30 (1975) 77-113.
[7] I.M. Gel’fand and L.A. Diki, A Lie algebra structure in a formal variational calculations, Funct. Anal. Appl. 10 (1976) 16-22.
[8] I.M. Gel’fand and I.Ya. Dorfman, Hamiltonian operators and algebraic structures related to them, Funct. Anal. Appl. 13 (1979) 248-262.
[9] J.T. Hartwig, D. Larsson, and S.D. Silvestrov, Deformations of Lie algebras using $\sigma$-derivations, J. Alg. 295 (2006) 314-361.
[10] D. Hou and C. Bai, A twisted generalization of linear Poisson brackets of hydrodynamic type, J. Phys. A 43 (2010) 365205 (15pp).
[11] A. Makhlouf and S. Silvestrov, Hom-algebra structures, J. Gen. Lie Theory Appl. 2 (2008) 51-64.
[12] A. Makhlouf and S. Silvestrov, Hom-algebras and Hom-coalgebras, J. Alg. Appl. 9 (2010) 1-37.
[13] A. Makhlouf and S. Silvestrov, Notes on formal deformations of Hom-associative and Hom-Lie algebras, to appear in Forum Math., arXiv:0712.3130v1.
[14] X. Xu, On simple Novikov algebras and their irreducible modules, J. Alg. 185 (1996) 905-934.
[15] X. Xu, Novikov-Poisson algebras, J. Alg. 190 (1997) 253-279.
[16] X. Xu, Variational calculus of supervariables and related algebraic structures, J. Alg. 223 (2000) 396-437.
[17] D. Yau, Enveloping algebras of Hom-Lie algebras, J. Gen. Lie Theory Appl. 2 (2008) 95-108.
[18] D. Yau, Hom-algebras and homology, J. Lie Theory 19 (2009) 409-421.
[19] D. Yau, Hom-bialgebras and comodule Hom-algebras, Int. Elect. J. Alg. 8 (2010) 45-64.
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[20] D. Yau, The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras, J. Phys. A 42 (2009) 165202 (12pp).

[21] D. Yau, Hom-Novikov algebras, arXiv:0909.0726.

[22] D. Yau, Hom-Maltsev, Hom-alternative, and Hom-Jordan algebras, arXiv:1002.3944.

[23] D. Yau, The Hom-Yang-Baxter equation and Hom-Lie algebras, arXiv:0905.1887.

[24] D. Yau, Hom-power associative algebras, arXiv:1007.4118.

[25] D. Yau, Non-commutative Hom-Poisson algebras, preprint.

[26] Y. Zhao, C. Bai, and D. Meng, Some results on Novikov-Poisson algebras, Int. J. Theoret. Phys. 43 (2004) 519-528.

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