THE TOPOLOGICAL CLASSIFICATION OF SPACES OF METRICS WITH THE UNIFORM CONVERGENCE TOPOLOGY

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Abstract. For a metrizable space $X$ of density $\kappa$, let $PM(X)$ be the space of continuous bounded pseudometrics on $X$ endowed with the uniform convergence topology. In this paper, its topology shall be classified as follows: (i) If $X$ is finite, then $PM(X)$ is homeomorphic to $\{0\}$ when $X$ is a singleton, and then $PM(X)$ is homeomorphic to $[0, 1]^{|\kappa|/2}$ when $\kappa > 1$; (ii) If $X$ is infinite and generalized compact, then $PM(X)$ is homeomorphic to the Hilbert space $\ell_2(2^{<\kappa})$ of density $2^{<\kappa}$; (iii) If $X$ is not generalized compact, then $PM(X)$ is homeomorphic to the Hilbert space $\ell_2(2^\kappa)$ of density $2^\kappa$. Furthermore, letting $M(X)$ and $AM(X)$ be the spaces of continuous bounded metrics and bounded admissible metrics on $X$ with the subspace topology of $PM(X)$ respectively, we will recognize their topological types as follows: (iv) If $X$ is infinite and compact, then $M(X)$ is homeomorphic to the Hilbert space $\ell_2(2^{\kappa})$ if $X$ is $\sigma$-compact, and moreover $AM(X)$ is also homeomorphic to the Hilbert space $\ell_2(2^{\kappa})$ if $X$ is separable locally compact.

Given a metrizable space $X$, we denote by $C(X^2)$ the space of continuous bounded real-valued functions on $X^2$ equipped with the uniform convergence topology. The space $C(X^2)$ is a Banach space with the sup-norm, and then the sup-metric $D$ is admissible on $C(X^2)$:

$$D(f, g) = \sup\{|f(x, y) - g(x, y)| \mid (x, y) \in X^2\}$$

for any $f, g \in C(X^2)$. Let $PM(X)$, $M(X)$ and $AM(X)$ be the spaces consisting of continuous bounded pseudometrics, continuous bounded metrics and bounded admissible metrics on $X$ with the subspace topology of $C(X^2)$, respectively. As is easily observed, $PM(X)$ is a convex non-negative cone, and $M(X)$ and $AM(X)$ are convex positive cones, in the linear space $C(X^2)$. Note that $M(X)$ coincides with $AM(X)$ when $X$ is compact. In this paper, we shall classify the topology of $PM(X)$ as follows:

Main Theorem. Let $X$ be a metrizable space of density $\kappa$.

(i) If $X$ is finite, then $PM(X)$ is homeomorphic to $\{0\}$ when $X$ is a singleton, and then $PM(X)$ is homeomorphic to $[0, 1]^{\kappa/2} \times [0, 1]$ when $\kappa > 1$; (ii) If $X$ is infinite and generalized compact, then $PM(X)$ is homeomorphic to the Hilbert space $\ell_2(2^{<\kappa})$ of density $2^{<\kappa}$; (iii) If $X$ is not generalized compact, then $PM(X)$ is homeomorphic to the Hilbert space $\ell_2(2^\kappa)$ of density $2^\kappa$.

In particular, in the case where $X$ is separable, $PM(X)$ is homeomorphic to the separable Hilbert space $\ell_2$ when $X$ is infinite and compact, and $PM(X)$ is homeomorphic to the Hilbert space $\ell_2(2^{\kappa})$ when $X$ is not compact.

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Furthermore, we will investigate the topological types of $M(X)$ and $AM(X)$ as follows:

**Theorem 1.** If $X$ is an infinite and compact metrizable space, then $M(X)$ \((= \text{AM}(X))\) is homeomorphic to the separable Hilbert space $\ell_2$. In the case that $X$ is not compact, $M(X)$ is homeomorphic to the Hilbert space $\ell_2(2^{\aleph_0})$ when $X$ is $\sigma$-compact, and moreover $\text{AM}(X)$ is also homeomorphic to the Hilbert space $\ell_2(2^{\aleph_0})$ when $X$ is separable locally compact.

A space $X$ is *generalized compact* if every open cover of $X$ has a subcover whose cardinality is less than the density of $X$. Recall that $\text{PM}(X)$ is closed in $C(X^2)$. If $X$ is a $\sigma$-compact metrizable space, then $M(X)$ is $G_\delta$ in $C(X^2)$, and moreover if it is a separable locally compact metrizable space, then $\text{AM}(X)$ is also $G_\delta$, refer to the proof of \cite{6} Lemma 5.1, T. Dobrowolski and H. Toruńczyk \cite{4} Theorem 2, and T. Banakh and R. Cauty \cite{2} Theorem 2 gave a sufficient condition for $G_\delta$ convex sets in Fréchet spaces to be homeomorphic to Hilbert spaces as follows:

**Theorem 2.** Let $C$ be a $G_\delta$ convex set in a Fréchet space. If the closure of $C$ is not locally compact, then $C$ is homeomorphic to a Hilbert space. Especially, if $C$ is not separable, then $C$ is homeomorphic to a non-separable Hilbert space.

Using this criterion, we will prove Main Theorem and Theorem \[\square\] It is shown that $\text{PM}(X)$ is the closure of $M(X)$ and $\text{AM}(X)$ in $C(X^2)$.

**Proposition 1.** For every metrizable space $X$, $\text{AM}(X)$ is dense in $\text{PM}(X)$.

**Proof.** For each pseudometric $d \in \text{PM}(X)$ and each positive number $\varepsilon$, we will find an admissible metric $\rho \in \text{AM}(X)$ with $D(d, \rho) \leq \varepsilon$. Fix any admissible metric $d_X \in \text{AM}(X)$ so that $d_X(x, y) \leq \varepsilon$ for all $x, y \in X$. Define a continuous function $\rho : X^2 \to \mathbb{R}$ by
\[
\rho(x, y) = \max\{d(x, y), d_X(x, y)\}
\]
for each $(x, y) \in X^2$. It is easy to show that $\rho \in \text{AM}(X)$. Moreover, for every pair $(x, y) \in X^2$, when $d(x, y) \leq \varepsilon$,
\[
|d(x, y) - \rho(x, y)| = \max\{d(x, y), d_X(x, y)\} - d(x, y) \leq \max\{d(x, y), d_X(x, y)\} \leq \varepsilon,
\]
and when $d(x, y) \geq \varepsilon$,
\[
|d(x, y) - \rho(x, y)| = \max\{d(x, y), d_X(x, y)\} - d(x, y) = d(x, y) - d(x, y) = 0.
\]
Hence $D(d, \rho) \leq \varepsilon$. We conclude that $\text{AM}(X)$ is dense in $\text{PM}(X)$. \[\square\]

We shall estimate the densities of $\text{PM}(X)$, $M(X)$ and $\text{AM}(X)$.

**Lemma 1.** Let $X$ be a metrizable space. If $X$ contains a closed discrete subset of cardinality $\kappa$, then the densities of $\text{PM}(X)$, $M(X)$ and $\text{AM}(X)$ are greater than or equal to $2^\kappa$.

**Proof.** Take a closed discrete subset $D = \{x_\gamma \in X \mid \gamma < \kappa\}$ and fix a point $x_0 \in D$. We can define a pseudometric $d_a \in \text{PM}(D)$ for each sequence $a = (a_\gamma)_{\gamma < \kappa} \in \{0, 1\}^\kappa$ by
\[
\begin{align*}
(1) & \quad d_a(x_0, x_0) = 0 \quad \text{and} \quad d_a(x_\gamma, x_0) = d_a(x_0, x_\gamma) = a_\gamma \quad \text{for any } 0 < \gamma < \kappa, \\
(2) & \quad d_a(x_\gamma, x_\lambda) = |d_a(x_\gamma, x_0) - d_a(x_\lambda, x_0)| \quad \text{for any } 0 < \gamma \leq \lambda < \kappa.
\end{align*}
\]
Note that the subset
\[
\{d_a \in \text{PM}(D) \mid a \in \{0, 1\}^\kappa\}
\]
is closed discrete in $\text{PM}(D)$. The density of $\text{PM}(D)$ is greater than or equal to $2^\kappa$, and hence so is the one of $\text{AM}(D)$ because of Proposition \[\square\] By Hausdorff’s metric extension theorem \[\square\], the restriction
\[
\text{AM}(X) \ni d \mapsto d|_{P^2} \in \text{AM}(D)
\]
is surjective, which implies that the density of $\text{AM}(X)$ is also greater than or equal to $2^\kappa$. \[\square\]
Proposition 2. Suppose that $X$ is a metrizable space of density $\kappa$. If $X$ is infinite and generalized compact, then the densities of $PM(X)$, $M(X)$ and $AM(X)$ are equal to $2^{<\kappa}$, and if $X$ is not generalized compact, then their densities are equal to $2^\kappa$.

Proof. In the case that $X$ is generalized compact, by virtue of Corollary 5 and Theorem 8 of \cite{1}, $\kappa$ has countable cofinality. Hence we can write $\kappa = \sup_n n_n \kappa_n$ with $\kappa_n < \kappa$. Then for each $n < \aleph_0$, there exists a closed discrete set in $X$ of cardinality $\geq \kappa_n$. Indeed, suppose that any closed discrete subset of $X$ is of cardinality $< \kappa_n$, so the density of $X$, which is equal to the extent, is less than or equal to $\kappa_n$. This is a contradiction. By Lemma 1, the densities of $PM(X)$, $M(X)$ and $AM(X)$ are greater than or equal to $2^{<\kappa}$. In the case that $X$ is not generalized compact, there exists a closed discrete set in $X$ of cardinality $\kappa$ due to Proposition 2.3 of \cite{3}. Using Lemma 1 again, we have that their densities are greater than or equal to $2^\kappa$. On the other hand, $C(X^2)$ is of density $2^{<\kappa}$ when $X$ is generalized compact, and $2^\kappa$ when $X$ is not so by \cite{3} Theorem 2.6. Thus the proof is complete. \Box

Now Main Theorem is shown.

Proof of Main Theorem. First, we shall prove (i). It is clear that $PM(X)$ is homeomorphic to $\{0\}$ when $X$ is a singleton. In the case where $\kappa > 1$, we show that $PM(X)$ is homeomorphic to $[0, 1]^{\kappa(\kappa - 1)/2 - 1} \times [0, 1)$. Let $X = \{x_1, \ldots, x_{\kappa}\}$. Since $d(x_i, x_i) = 0$ and $d(x_i, x_j) = d(x_j, x_i)$ for all $1 \leq i \leq j \leq \kappa$, $PM(X)$ is homeomorphic to the closed cone

$$C = \{(d(x_i, x_j))_{1 \leq i < j \leq \kappa} \mid d \in PM(X)\} \subset [0, \infty)^{\kappa(\kappa - 1)/2}.$$ 

Letting

$$H = \left\{ (z_{i,j})_{1 \leq i < j \leq \kappa} \in [0, \infty)^{\kappa(\kappa - 1)/2} \mid \sum_{1 \leq i < j \leq \kappa} z_{i,j} = 1 \right\},$$

we get that $C = [0, \infty) \cdot (C \cap H)$. As is easily observed, $C \cap H$ is compact and convex. Moreover, the dimension of $C \cap H$ is equal to $\kappa(\kappa - 1)/2 - 1$. Indeed, $H$ is of dimension $\kappa(\kappa - 1)/2 - 1$, and the $(\kappa(\kappa - 1)/2 - 1)$-dimensional convex set

$$\left\{ \sum_{1 \leq i < j \leq \kappa} (2s_{i,j}/(\kappa(\kappa - 1) - 2)) \cdot w^{i,j} \mid s_{i,j} \geq 0 \text{ and } \sum_{1 \leq i < j \leq \kappa} s_{i,j} = 1 \right\},$$

where let $w^{i,j} = (w_{k,l})_{1 \leq k < l \leq \kappa} \in [0, \infty)^{\kappa(\kappa - 1)/2}$ be defined by

$$w_{k,l} = \begin{cases} 0 & \text{if } (k, l) = (i, j), \\ 1 & \text{if } (k, l) \neq (i, j), \end{cases}$$

is contained in $C \cap H$. It follows that $C \cap H$ is homeomorphic to $[0, 1]^{\kappa(\kappa - 1)/2 - 1}$, and hence $PM(X)$ is homeomorphic to $[0, 1]^{\kappa(\kappa - 1)/2 - 1} \times [0, 1)$.

Next, we show (ii). Assume that $X$ is infinite and generalized compact. To show that $PM(X)$ is not locally compact, fix any pseudometric $d \in PM(X)$ and any neighborhood $U$ of $d$. By virtue of Arzelà-Ascoli’s Theorem, we need only to prove that $U$ is not equicontinuous, that is, there exists a point $(x, y) \in X^2$ and a positive number $\epsilon > 0$ such that for each neighborhood $V$ of $(x, y)$,

$$\rho(x', y') \notin (\rho(x, y) - \epsilon, \rho(x, y) + \epsilon)$$

for some $\rho \in U$ and some $(x', y') \in V$. Take $\epsilon > 0$ such that if $D(d, d') \leq \epsilon$, then $d' \in U$. Since $X$ is infinite and generalized compact, we can choose a point $x \in X$ all of whose neighborhoods are of density $\geq \kappa$ according to \cite{1} Lemma 11], which implies that $x$ is not isolated. Hence for
any neighborhood $V$ of $(x, x)$ in $X^2$, there is $y \in X$ such that $x \neq y$ and $(x, y) \in V$. Define an admissible metric $e \in AM(\{x, y\})$ as follows:

$$e(x, x) = 0, \ e(y, y) = 0, \ e(x, y) = e(y, x) = \epsilon.$$ 

Applying Hausdorff’s metric extension theorem [5], we can obtain an admissible metric $\tilde{e} \in AM(X)$ so that $\tilde{e}|_{\{x, y\}^2} = e$ and $\tilde{e}(u, v) \leq \epsilon$ for any $(u, v) \in X^2$. Let $\rho \in AM(X)$ be an admissible metric defined by $\rho = d + \tilde{e}$. Then for each $(u, v) \in X^2$,

$$|d(u, v) - \rho(u, v)| = |d(u, v) - (d(u, v) + \tilde{e}(u, v))| = \tilde{e}(u, v) \leq \epsilon.$$

Therefore we have $D(d, \rho) \leq \epsilon$, which implies that $\rho \in U$. Moreover,

$$\rho(x, y) = d(x, y) + \tilde{e}(x, y) \geq \tilde{e}(x, y) = \epsilon = \rho(x, x) + \epsilon.$$

Consequently, $U$ is not equicontinuous, and hence $PM(X)$ is not locally compact. According to Proposition [2], $PM(X)$ is of density $2^{<\kappa}$. It follows from Theorem [2] that $PM(X)$ is homeomorphic to $\ell_2(2^{<\kappa})$.

Combining Theorem [2] with Proposition [2] we can establish (iii). The proof is complete. $\square$

Recall that $PM(X)$ is the closure of $M(X)$ and $AM(X)$ by Proposition [1]. When $X$ is a separable metrizable space, the compactness of it coincides with the generalized compactness of it. Hence as is seem in the above proof, $PM(X)$ is not locally compact. Applying Theorem [2] we can prove Theorem [1].

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