Ground state energy of the $f = 1$ spinor Bose-Einstein condensates

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Abstract

We calculate, in the standard Bogoliubov approximation, the ground state energy of the spinor BEC with hyperfine spin $f = 1$ where the two-body repulsive hard-core and spin exchange interactions are both included. The coupling constants characterized these two competing interactions are expressed in terms of the corresponding $s$-wave scattering lengths using second-order perturbation methods. We show that the ultraviolet divergence arising in the ground state energy corrections can be exactly eliminated.

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Recently, Stamper-Kurn et al. [1] have successfully cooled $^{23}$Na atoms using an optical dipole trap and achieved BEC. In their experiment, a multi-component BEC which is characterized by the three hyperfine spin states $|f = 1, m_f = \pm 1, 0\rangle$ has been observed. This has opened an interesting possibility to explore the multi-component BEC with complicated internal spin dynamics in which not only the global $U(1)$ symmetry but also the rotational $SO(3)$ symmetry in spin space are involved [2,3].

An important feature of the spinor condensate is that, in addition to the repulsive binary hard-core collisions which give rise to the density-density interaction, atoms in the condensates can also couple to each other via the spin exchange interaction. Assuming that the interaction for each spin exchange channel is again characterized by zero-range delta-potential scattering, one thus obtains the interacting term $\hat{S} \cdot \hat{S}$ where $\hat{S}$ is the spin density operator. The competition between these two interactions thus lead to an intriguing scenario of the spin dynamics of the spinor BEC which is characterized by a complex ground state structure [2–4].

The question arise now that how these two-body interactions alter the dynamical properties of the condensates. It is known that in a weakly interacting Bose condensed system, the two-body interactions play a crucial role in determine the low temperature properties of the systems, which will modify the ground state of the many-particle systems and cause a depletion of the condensate fraction even at the zero temperature. Moreover, an divergence could possibly appear when we calculate the ground state energy in the standard Bogoliubov approximation. This divergence is due to the naive assumption of a constant matrix element of binary interaction irrespective of the relative momenta of the interacting particles. A well illustrated example is the ultraviolet divergence occurring in the calculated ground state energy of the one-component BEC where the two-body interaction is described by the repulsive hard-core collisions with a momentum-independent coupling constant [5]. To eliminate such a divergence and gain more insights into the ground state properties of the condensates, one has to calculate the $s$-wave scattering length at least to the second order in coupling constant. By expanding the coupling constant in powers of the $s$-wave scattering
length the previously mentioned divergent ground state energy can be rendered finite. This is expected since in a physically sensible theory the ground state energy must assume a finite value when expressed in terms of physically measurable quantities. In other generalized Bose condensed systems such as the spinor BEC ultraviolet divergences of the same sort could also appear. It is therefore quite essential to verify if a similar procedure could completely removed these divergences, and in this paper we address this issue in details for the \( f = 1 \) spinor BEC in the presence of a constant magnetic field.

Consider an assembly of homogeneous dilute Bose gas with hyperfine spin \( f = 1 \). The natural basis set to characterize such a system is the hyperfine spin states \( |m_f = \pm 1, 0\rangle \). However, in view of the special symmetrical forms of the \( S = 1 \) spin matrix representations, one may adopt the basis set \( \{ |x\rangle, |y\rangle, |z\rangle \} \) which is defined as the eigenstates of the \( \alpha \)-th component of the spin operator with eigenvalue 0, i.e., \( S_\alpha |\alpha\rangle = 0 (\alpha = x, y, z) \) such that the matrix elements for the \( \alpha \)-th spin component is given by \( \langle \gamma |S_\alpha|\beta\rangle = i\varepsilon_{\alpha\beta\gamma} \) where \(\varepsilon_{\alpha\beta\gamma}\) is the Levi-Civita tensor. This representation enables us to relate the matrix elements of \( S = 1 \) spin operators to those of the space rotation, allowing the order parameter to behave as a vector under spin space rotation.

For the \( f = 1 \) spinor BEC, the bosonic atomic field can be described by the multi-component field operator \( \Psi \), with components \( \psi_\alpha (r) (\alpha = x, y, z) \), and thus the density of the particle number and spin can be written as \( \hat{n} = \psi_\alpha^\dagger \psi_\alpha \), and \( \hat{S}_\alpha = \psi_\beta^\dagger S_\alpha \psi_\beta = -i\varepsilon_{\alpha\beta\gamma} \psi_\beta^\dagger \psi_\gamma \) respectively. Note that we have used the summation convention over the indices of component \( \alpha, \beta, \cdots \) throughout this paper. Now, without loss of generality, the Hamiltonian density can be constructed in the presence of a constant magnetic field \( B \) pointing to the \( z \)-direction:

\[
H = -\psi_\alpha^\dagger \frac{\nabla^2}{2m} \psi_\alpha + \frac{1}{2} g_n \hat{n}^2 + \frac{1}{2} g_s \hat{S} \cdot \hat{S} - \Omega \cdot \hat{S} \quad (\hbar = 1) \tag{1}
\]

where \( \Omega = \Omega \hat{z} = g_\mu B \) (\( g_\mu \): gyromagnetic ratio) is the Larmor frequency in a vectorial notation. Expanding \( \hat{n} \) and \( \hat{S}_\alpha \) in terms of the field operators, Eq.\( (1) \) can be expressed as
\[ H = -\frac{1}{2m} \nabla^2 \psi_\alpha + \frac{1}{2} g_1 \psi_\beta^\dagger \psi_\alpha \psi_\beta + \frac{1}{2} g_2 \psi_\beta^\dagger \psi_\beta \psi_\alpha + i \epsilon_{\alpha \beta \gamma} \Omega_\gamma \psi_\alpha^\dagger \psi_\beta \]  

(2)

where the two new coupling constants are given by \( g_1 = g_n + g_s \), \( g_2 = -g_s \). According to the recent spectroscopic experiment by Abraham et al. [6], it is conceivable in general that \( g_2 \) is comparable to \( g_1 \) in magnitude and can be either positive or negative. It is known that the positive \( g_2 \) implies the ferromagnetic coupling while the negative one implies the antiferromagnetic coupling for the spin exchange interaction.

Since the system is homogeneous, the field operator can be expanded in terms of creation and annihilation operators characterized by momentum \( k \)

\[ \psi_\alpha (r) = \frac{1}{\sqrt{V}} \sum_k a_{\alpha, k} e^{i k \cdot r}, \]  

(3)

where \( V \) denotes the volume of the system. Accordingly, the Hamiltonian in momentum space now reads as

\[ H = H_0 + H_{\text{mag}} + H_{\text{int}} \]  

(4)

where

\[ H_0 = \sum_k \epsilon_k a_{\alpha, k}^\dagger a_{\alpha, k} \]  

(5)

\[ H_{\text{mag}} = \sum_k i \epsilon_{\alpha \beta \gamma} \Omega_\gamma a_{\alpha, k}^\dagger a_{\beta, k} \]  

(6)

\[ H_{\text{int}} = \frac{g_1}{2V} \sum_{k_1 + k_2 = k_3 + k_4} a_{\beta, k_4}^\dagger a_{\alpha, k_3}^\dagger a_{\alpha, k_2} a_{\beta, k_1} \]

\[ + \frac{g_2}{2V} \sum_{k_1 + k_2 = k_3 + k_4} a_{\beta, k_4}^\dagger a_{\beta, k_3} a_{\alpha, k_2} a_{\alpha, k_1} \]  

(7)

In the ground state, most particles occupy the \( k = 0 \) states. As a result, the scattering between two nonzero-momentum states can be ignored and the interacting part of the Hamiltonian can be replaced by

\[ H_{\text{int}} \simeq \frac{g_1}{2V} \left[ a_{\beta, 0}^\dagger a_{\alpha, 0}^\dagger a_{\alpha, 0} a_{\beta, 0} + \sum_{k \neq 0} (a_{\beta, k}^\dagger a_{\alpha, -k} a_{\alpha, 0} a_{\beta, 0} + a_{\beta, 0}^\dagger a_{\alpha, 0} a_{\alpha, k} a_{\beta, -k} \right] \]
Before calculating the $s$-wave scattering lengths to the second order, a couple of remarks are in order. First of all, for the sake of simplicity we shall disregard the magnetic interaction $H_{\text{mag}}$ for a moment. Secondly, the $s$-wave scattering lengths are formally determined from the so-called $T$-matrix which can be computed perturbatively by using the diagrammatic techniques. Moreover, it is known that the energy correction due to the two-body interactions can be directly related to the matrix elements of the $T$-matrix\cite{8}. Hence, one expects that the desired $s$-wave scattering lengths can be obtained from the calculations of energy corrections. In fact, it is not hard to show that, to the second order, our results agree with those obtained by the $T$-matrix approach. However, as the standard second-order perturbation methods are only required in our paper, the calculations can be greatly simplified. With these remarks in mind we are motivated to compute the energy corrections due to $H_{\text{int}}$.

We first introduce a class of 2-particle states defined by

$$|0, 0; \varphi\rangle = \frac{1}{\sqrt{2}} \varphi^*_\alpha \varphi^*_\beta a^\dagger_{\beta,0} a^\dagger_{\alpha,0} |\text{vac}\rangle$$

(9)

where $\varphi_\alpha$ are constant parameters and $|\text{vac}\rangle$ is the Fock vacuum. Such states can be normalized by imposing the condition $\varphi^*_\alpha \varphi_\alpha = |\varphi|^2 = 1$. Quite clearly, the unperturbed energy vanishes in the presence of the state Eq.\(\text{(3)}\). It is easy to show that the first-order energy corrections due to $H_{\text{int}}$ is

$$E_{\text{int}}^{(1)} = \langle 0, 0; \varphi | H_{\text{int}} | 0, 0; \varphi \rangle = \frac{g_1}{V} |\varphi|^4 + \frac{g_2}{V} |\varphi|^2,$$

(10)

Next, we consider the second-order correction for the energy. Now, in view of the explicit form of $H_{\text{int}}$ given in Eq.\(\text{(3)}\), the only possible intermediate states are those states of two non-condensate particles carrying opposite momenta.
\[ |k, -k; \alpha, \beta \rangle \equiv a_{\alpha,k}^\dagger a_{\beta,-k}^\dagger |\text{vac}\rangle \]  

with which the unperturbed energy is given by

\[ \langle k, -k; \alpha, \beta | H_0 | k, -k; \alpha, \beta \rangle = \epsilon_k + \epsilon_{-k} = 2\epsilon_k. \]  

Thus the second-order energy correction due to \( H_{\text{int}} \) is

\[ E^{(2)}_{\text{int}} = -\frac{1}{2} \sum_{k\neq 0} \frac{|\langle 0, 0; \varphi | H_{\text{int}} | k, -k; \alpha, \beta \rangle|^2}{2\epsilon_k - 0}. \]  

The factor \( 1/2 \) in Eq.(13) is inserted in order to avoid the double counting of the momentum states. Now, we have

\[ \langle 0, 0; \varphi | H_{\text{int}} | k, -k; \alpha, \beta \rangle = \frac{\sqrt{2}}{V} \left[ g_1 \varphi_\alpha \varphi_\beta + g_2 \varphi^2 \delta_{\alpha\beta} \right], \]  

and hence

\[ E^{(2)}_{\text{int}} = -\frac{1}{V^2} \sum_{k\neq 0} \frac{|g_1 \varphi_\alpha \varphi_\beta + g_2 \varphi^2 \delta_{\alpha\beta}|^2}{2\epsilon_k} \]  

\[ = -\frac{1}{V} \left[ g_1^2 |\varphi|^4 + \left( 2g_1g_2 + 3g_2^2 \right) |\varphi^2|^2 \right] \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\epsilon_k}. \]  

Obviously, the integral in Eq.(13) diverges as \( |k| \to \infty \). Choosing \( \varphi_\alpha \) in such a way that \( |\varphi^2| = 0 \). yields

\[ g_1 = E^{(1)}_{\text{int}} V, \]  

indicating that \( g_1 \) is proportional to the first-order energy correction due to the two-particle interaction \( H_{\text{int}} \). At this order, \( g_1 \) is related to the corresponding s-wave scattering length \( a_1 \) by

\[ g_1 = \frac{4\pi a_1}{m}. \]
Hence, to the second order, $a_1$ is related to $g_1$ by the following equation

$$\frac{4\pi a_1}{m} \equiv \tilde{g}_1 = \left( E_{\text{int}}^{(1)} + E_{\text{int}}^{(2)} \right) V$$

$$= g_1 - g_1^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\epsilon_k}. \quad (18)$$

Here $\tilde{g}_1$ will be referred to as the corrected coupling constant of $g_1$. Writing the original coupling $g_1$ in terms of the corrected coupling $\tilde{g}_1$, we have at the same order

$$g_1 = \tilde{g}_1 + \tilde{g}_1^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\epsilon_k} \quad (19)$$

which is equivalent to the results demonstrated in the one-component case [5]. Next, we consider the corrections of $g_2$. Unlike $g_1$, $g_2$ cannot be isolated directly in the present formalism. Instead, we shall consider the sum $g_1 + g_2$ which is nothing but the coupling $g_n$ associated with the density-density interaction as $g_1 = g_n + g_s$, $g_2 = -g_s$. To this end, we may take $\varphi^2 = 1$ such that $g_1 + g_2 = E_{\text{int}}^{(1)} V$ and hence

$$\tilde{g}_1 + \tilde{g}_2 = \left( E_{\text{int}}^{(1)} + E_{\text{int}}^{(2)} \right) V$$

$$= g_1 + g_2 - \left( g_1^2 + 2g_1g_2 + 3g_2^2 \right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\epsilon_k} \quad (20)$$

Subtracting Eq.(18) from Eq.(20) and using Eq.(18) again, we have at this order

$$g_2 = \tilde{g}_2 + \left( 2\tilde{g}_1\tilde{g}_2 + 3\tilde{g}_2^2 \right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\epsilon_k} \quad (21)$$

Alternatively, $g_n$ and $g_s$ are related to the corresponding corrected couplings by

$$g_n = \tilde{g}_n + \left( \tilde{g}_n^2 + 2\tilde{g}_s^2 \right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\epsilon_k} \quad (22)$$

$$g_s = \tilde{g}_s + \left( 2\tilde{g}_n\tilde{g}_s - \tilde{g}_s^2 \right) \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\epsilon_k}$$

It should be noted that the corrected coupling constants, $\tilde{g}_1$ and $\tilde{g}_2$, are consistent with the one-loop corrections obtained by using the Feynman diagram techniques [7]. These results are actually unaltered in the presence of a constant magnetic field. The point is that the two-body interaction term $H_{\text{int}}$, in fact, commutes with the magnetic term $H_{\text{mag}}$. As a
consequence, despite that the magnetic interaction would, inevitably, introduce a Zeeman energy shift to each hyperfine spin state the total Zeeman energy is conserved in the two-particle scattering processes. Based on this point, one can easily check that both Eqs. (19) and (21) remain correct.

We now proceed to calculate the ground state energy with the foregoing results. In the standard Bogoliubov approximation the operators $a_{\alpha,0}$ and $a_{\alpha,0}^\dagger$ are replaced by the classical number $\Phi_\alpha \sqrt{V}$ and $\Phi_\alpha^* \sqrt{V}$ respectively, such that $|\Phi|^2 = N_0/V = n_0$ represents the density of condensate particles. Making these replacements into Eqs. (6) and (8) yields

$$H_{\text{mag}} \rightarrow iV \varepsilon_{\alpha\beta\gamma} \Omega_\gamma \Phi_\alpha^* \Phi_\beta + \sum_{\mathbf{k} \neq \mathbf{0}} i\varepsilon_{\alpha\beta\gamma} \Omega_\gamma a_{\alpha,\mathbf{k}}^\dagger a_{\beta,\mathbf{k}}$$

and

$$H_{\text{int}} \rightarrow \frac{1}{2} g_1 V |\Phi|^4 + \frac{1}{2} g_2 V |\Phi^2|^2$$

$$\quad + \frac{g_1}{2} \sum_{\mathbf{k} \neq \mathbf{0}} \left( \Phi_\alpha \Phi_\beta a_{\beta,\mathbf{k}}^\dagger a_{\alpha,-\mathbf{k}} + \Phi_\alpha^* \Phi_\beta^* a_{\alpha,\mathbf{k}} a_{\beta,-\mathbf{k}} + 2 \Phi_\alpha^* \Phi_\beta a_{\beta,\mathbf{k}}^\dagger a_{\alpha,\mathbf{k}} + 2 |\Phi|^2 a_{\alpha,\mathbf{k}}^\dagger a_{\alpha,\mathbf{k}} \right)$$

$$\quad + \frac{g_2}{2} \sum_{\mathbf{k} \neq \mathbf{0}} \left( \Phi_\beta^2 a_{\beta,\mathbf{k}}^\dagger a_{\beta,-\mathbf{k}} + \Phi_\beta^* a_{\alpha,\mathbf{k}} a_{\alpha,-\mathbf{k}} + 4 \Phi_\beta^* \Phi_\alpha a_{\beta,\mathbf{k}}^\dagger a_{\alpha,\mathbf{k}} \right)$$

Using Eqs. (23) and (24) we obtain the effective Hamiltonian

$$H_{\text{eff}} = H_{\text{con}} + H_{\text{non}}$$

where

$$H_{\text{con}} = \int d^3r \left[ i\varepsilon_{\alpha\beta\gamma} \Omega_\gamma \Phi_\alpha^* \Phi_\beta + \frac{g_1}{2} |\Phi|^4 + \frac{g_2}{2} |\Phi^2|^2 \right]$$

and

$$H_{\text{non}} = \sum_{\mathbf{k} \neq \mathbf{0}} \left( a_{\alpha,\mathbf{k}}^\dagger \mathcal{L}_{\alpha\beta} a_{\beta,\mathbf{k}} + \frac{1}{2} \mathcal{M}_{\alpha\beta}^* a_{\alpha,\mathbf{k}} a_{\beta,-\mathbf{k}} + \frac{1}{2} \mathcal{M}_{\alpha\beta} a_{\alpha,\mathbf{k}}^\dagger a_{\beta,-\mathbf{k}} \right)$$

are the Hamiltonians for the condensate and non-condensate part, respectively. Here the matrix elements are given by
\[ L_{\alpha\beta} = \epsilon_k \delta_{\alpha\beta} + i \varepsilon_{\alpha\beta\gamma} \Phi_\gamma + g_1 |\Phi|^2 \delta_{\alpha\beta} + g_1 \Phi^*_\beta \Phi_\alpha + 2g_2 \Phi^*_\alpha \Phi_\beta \]  
Note that \( H_{\text{eff}} \) is precisely the Hartree-Fock-Bogoliubov Hamiltonian in the standard Bogoliubov approximation \[7\] whose ground state structure can be determined by minimizing the integrand in Eq.\((26)\). As a result, two different ground state structures are found \[2,3\]:

\[ \Phi = \sqrt{n_0} \left( \frac{1}{\sqrt{2}}, i \sqrt{2}, 0 \right) \text{ for } n_0 g_2 > -\Omega \]  
which is referred to as the “ferromagnetic” state and

\[ \Phi = \sqrt{n_0} \left( \cos \theta, i \sin \theta, 0 \right) \text{ for } n_0 g_2 < -\Omega \]  
as the “polar” state. Here the cosine and the sine in Eq.\((30)\) are given by

\[ \cos \theta = \frac{1}{2} \left( \sqrt{1 + \frac{\Omega}{|g_2| n_0}} + \sqrt{1 - \frac{\Omega}{|g_2| n_0}} \right) \]

\[ \sin \theta = \frac{1}{2} \left( \sqrt{1 + \frac{\Omega}{|g_2| n_0}} - \sqrt{1 - \frac{\Omega}{|g_2| n_0}} \right) \]  
(31)

Since \( H_{\text{eff}} \) is quadratic in \( a_{\alpha,k}^\dagger \) and \( a_{\alpha,k} \), we can diagonalize this Hamiltonian by using the generalized Bogoliubov transformation

\[ a_{\alpha,k} = \sum_i \left[ u_{\alpha,k}^{(i)} b_{k}^{(i)} - v_{\alpha,-k}^{(i)} b_{-k}^{(i)} \right] \]  
(32)

where \( i \) is the mode index and \( |u_{\alpha,k}^{(i)}|^2 - |v_{\alpha,k}^{(i)}|^2 = 1 \). In terms of the quasiparticle creation and annihilation operators \( b_{k}^{(i)} \) and \( b_{-k}^{(i)} \), the Hamiltonian takes the following form

\[ H_{\text{eff}} = H_{\text{con}} + \sum_i \sum_{k \neq 0} E_{k}^{(i)} \left( i_{k}^{(i)} b_{k}^{(i)} - \left| v_{\alpha,k}^{(i)} \right|^2 \right) \]  
(33)

We now define the ground state which is annihilated by all \( b_{k}^{(i)} \) i.e., \( b_{k}^{(i)} \ket{\text{GND}} = 0 \), such that the ground state energy is found to be

\[ E_{\text{GND}} = V \left[ i \varepsilon_{\alpha\beta\gamma} \Phi^*_\alpha \Phi_\beta + \frac{g_1}{2} |\Phi|^4 + \frac{g_2}{2} |\Phi^2|^2 - \int \frac{d^3k}{(2\pi)^3} \sum_i E_{k}^{(i)} \left| v_{\alpha,k}^{(i)} \right|^2 \right], \]  
(34)
where the last integral indicates the energy shift due to the quasiparticle excitations. To calculate the ground state energy, one needs to know precisely the values of $E_{k}^{(i)}$ and $v_{\alpha,k}^{(i)}$ for the quasiparticle modes. This can be done by using the standard Hartree-Fock-Bogoliubov mean-field method, and we have calculated $E_{k}^{(i)}$ and $v_{\alpha,k}^{(i)}$ for the quasiparticle modes. In the following, we devote our attention to the case in which the ground state is “polar”, since for the “ferromagnetic” case the results are identical the those of the one-component scalar BEC [5] and can be obtained from the “polar” case by setting $\Omega = -n_{0}g_{2}$. As a result, we find that the low lying excitations can be described by two gapless modes $E_{k}^{(\pm)}$ and one massive mode $E_{k}^{(0)}$:

$$\begin{align*}
E_{k}^{(\pm)} &= \sqrt{\epsilon_{k} (\epsilon_{k} + 2n_{0}g^{(\pm)})}, & E_{k}^{(0)} &= \sqrt{\epsilon_{k} (\epsilon_{k} + 2n_{0}g^{(0)}) + \Omega^{2}},
\end{align*}$$

for which the corresponding nonvanishing distribution functions are given by [7]

$$\begin{align*}
\begin{pmatrix}
v_{x,k}^{(+)}
v_{y,k}^{(+)}
\end{pmatrix} &= \begin{pmatrix}
+ A^{(\pm)} \\
- B^{(\pm)}
\end{pmatrix} \beta_{k}^{(\pm)},
\begin{pmatrix}
v_{z,k}^{(+)}
v_{z,k}^{(0)}
\end{pmatrix} &= - \begin{pmatrix}
1 - \frac{\Omega^{2}}{n_{0}^{2}g_{2}^{2}}\end{pmatrix} \beta_{k}^{(0)}
\end{align*}$$

where

$$g^{(\pm)} = \frac{1}{2} \left[ g_{1} \pm \sqrt{g_{1}^{2} + 4g_{2} (g_{1} + g_{2}) \left( 1 - \frac{\Omega^{2}}{n_{0}^{2}g_{2}^{2}} \right)} \right], \quad g^{(0)} = |g_{2}|$$

$$\beta_{k}^{(i)} = \frac{n_{0}g^{(i)}}{\sqrt{2E_{k}^{(i)} (E_{k}^{(i)} + \epsilon_{k} + n_{0}g^{(i)})}} \quad (i = \pm, 0)$$

$$\begin{align*}
A^{(\pm)} &= \frac{g_{1} \Omega / |g_{2}|}{\sqrt{\eta^{(\pm)^{2}} + (g_{1} \Omega / g_{2})^{2}}}, & B^{(\pm)} &= \frac{-i\eta^{(\pm)}}{\sqrt{\eta^{(\pm)^{2}} + (g_{1} \Omega / g_{2})^{2}}}
\end{align*}$$

$$\eta^{(\pm)} = n_{0} (g_{1} - 2g^{(\pm)}) + n_{0} (g_{1} + 2g_{2}) \sqrt{1 - (\Omega / n_{0} |g_{2}|)^{2}}$$

Moreover, with the condensate wavefunctions described in Eqs.(30) and (31), we obtain the following results:

$$\begin{align*}
|\Phi|^{2} &= n_{0}, & |\Phi|^{2} &= n_{0}^{2} \left( 1 - \frac{\Omega^{2}}{n_{0}^{2}g_{2}^{2}} \right), & i\varepsilon_{\alpha,\beta,\gamma} \Omega_{\gamma} \Phi^{\alpha} \Phi_{\beta} &= -\frac{\Omega}{|g_{2}|}
\end{align*}$$
However, one sees that all the three integrals
\[ \int \frac{d^3k}{(2\pi)^3} E_k^{(i)} |v_{\alpha,k}^{(i)}|^2 \quad (i = \pm, 0) \]
are divergent when |k| → ∞. Note also that they are essentially of second-order in the coupling constants g_1 and g_2. To eliminate these ultraviolet divergences we substitute Eqs.(19) and (21) into Eq.(34). The resulting expression for the ground state energy is
\[ E_{\text{GND}} = V \left[ i\varepsilon_{\alpha\beta\gamma} \omega_\gamma \Phi_\alpha^* \Phi_\beta + \frac{g_1}{2} |\Phi|^4 + \frac{g_2}{2} |\Phi|^2 \right] + V \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\epsilon_k} \left\{ \left[ \frac{g_1^2}{2} |\Phi|^4 + \frac{1}{2} \left( 2\tilde{g}_1\tilde{g}_2 + 3\tilde{g}_2^2 \right) |\Phi|^2 \right] 
- \sum_i E_k^{(i)} \left( \tilde{g}_1, \tilde{g}_2 \right) \left| v_{\alpha,k}^{(i)} (\tilde{g}_1, \tilde{g}_2) \right|^2 \right\}. \] (42)

Note that the condensate wavefunction \( \Phi \) determined by minimizing the sum of terms in the first line of Eq.(42) has the same form as that in Eq.(41) except that the corrected coupling constants are substituted instead. Furthermore, the last term in Eq.(42) can be expanded in powers of the corrected coupling constants. Since Eq.(42) is valid only up to the second order in the corrected couplings, it suffices to substitute \( g_1 = \tilde{g}_1, g_2 = \tilde{g}_2 \) in the expressions for \( E_k^{(i)} \) and \( v_{\alpha,k}^{(i)} \), i.e., in Eqs.(35)-(40). On these grounds, we are now ready to calculate \( E_{\text{GND}} \) given by Eq.(42). First, we note that
\[ \tilde{g}^{(+)} + \tilde{g}^{(-)} = \tilde{g}_1^2 + 2\tilde{g}_1 (\tilde{g}_1 + \tilde{g}_2) \left( 1 - \frac{\Omega^2}{n_0^2 \tilde{g}_2^2} \right), \] (43)
and hence
\[ \tilde{g}_1 |\Phi|^4 + \left( 2\tilde{g}_1\tilde{g}_2 + 3\tilde{g}_2^2 \right) |\Phi|^2 = n_0^2 \left[ \tilde{g}^{(+)} + \tilde{g}^{(-)} + \left( 1 - \frac{\Omega^2}{n_0^2 \tilde{g}_2^2} \right) \tilde{g}^{(0)} \right]^2. \] (44)
The integral in Eq.(42) is then equal to
\[ \frac{1}{2} n_0^2 V \int \frac{d^3k}{(2\pi)^3} \left[ \sum_{i=\pm} \tilde{g}_{i,k}^{(i)} \left( \frac{1}{\epsilon_k} - \frac{1}{E_k^{(i)} + \epsilon_k + n_0 \tilde{g}_{i,k}^{(i)}} \right) \right] + \left( \tilde{g}_2^2 - \frac{\Omega^2}{n_0^2} \right) \left( \frac{1}{\epsilon_k} - \frac{1}{E_k^{(0)} + \epsilon_k + n_0 |\tilde{g}_2|} \right). \] (45)
Note that the first two terms are the same as that of the one-component case \[5\]

\[
\frac{1}{2} n_0^2 V \int \frac{d^3 k}{(2\pi)^3} \tilde{g}^2 \left( \frac{1}{2\epsilon_k} - \frac{1}{E_k + \epsilon_k + n_0 \tilde{g}^2} \right) = 2\pi n_0^{5/2} \frac{V}{m} \left( \frac{128}{15\sqrt{\pi}} \alpha^{(\pm)^{5/2}} \right) \tag{46}
\]

where \(a^{(\pm)} = m\tilde{g}^{(\pm)}/4\pi\) are the corresponding s-wave scattering wavelengths. The last integral can be expressed as

\[
\frac{1}{2} n_0^2 V \left( \frac{\Omega^2}{n_0^2} \right) \int \frac{d^3 k}{(2\pi)^3} \frac{E_k^{(0)} - \epsilon_k + n_0 |\tilde{g}_2|}{2\epsilon_k \left( E_k^{(0)} + \epsilon_k + n_0 |\tilde{g}_2| \right)} = 2\pi V \frac{n_0^{5/2}}{m} \left( \frac{128}{15\sqrt{\pi}} |a_2|^{5/2} \right) \left( 1 - t^2 \right) F\left(t^2\right) \tag{47}
\]

where \(t = \Omega/n_0 |\tilde{g}_2|\), and the function of integral is defined as

\[
F\left(t^2\right) = \frac{15\sqrt{2}}{32} \int_0^\infty dx \frac{1 - x^2 + \sqrt{t^2 + 2x^2 + x^4}}{1 + x^2 + \sqrt{t^2 + 2x^2 + x^4}} \text{ for } 0 \leq t^2 \leq 1,
\tag{48}
\]

which is a monotonically increasing function that cannot be analytically evaluated in general.

Finally, using Eq.(41) we get

\[
\left[ i \varepsilon_{\alpha\beta\gamma} \omega_\gamma \Phi^*_\alpha \Phi_\beta + \frac{\tilde{g}_1}{2} |\Phi|^4 + \frac{\tilde{g}_2}{2} |\Phi^2|^2 \right] V
= \left[ \tilde{g}_1 n_0^2 + \tilde{g}_2 n_0^2 \left( 1 - \frac{\Omega^2}{n_0^2 \tilde{g}_2^2} \right) - \frac{\Omega}{|\tilde{g}_2|} \right] V
= 2\pi n_0^2 V \left( a_n - t^2 a_s \right) \tag{49}
\]

and therefore the ground state energy is given by

\[
E_{\text{GND}} = \frac{2\pi n_0^2 V}{m} \left[ \left( a_n - t^2 a_s \right) + \frac{128}{15\sqrt{\pi}} n_0^{1/2} \left( a^{(\pm)^{5/2}} + a^{(-)^{5/2}} \right) \right. \\
\left. + \left( 1 - t^2 \right) F\left(t^2\right) a_s^{5/2} \right], \tag{50}
\]

where \(a_n = m\tilde{g}_n/4\pi, a_s = m\tilde{g}_s/4\pi\). The terms proportional to \(a^{(\pm)^{5/2}}\) are caused by the two gapless modes and have the same form of the phonon-like mode in the one-component BEC. The last term in Eq.(50) is due to the massive mode, which depends solely on the scattering length \(a_s\) for the spin exchange channel and is suppressed by the increasing magnetic field.
In conclusion, we have analytically calculated the ground state energy of a homogeneous spinor BEC with hyperfine spin \( f = 1 \) based on the Bogoliubov approximation. In this weakly interacting system, the two-body interactions are described by the hard-core collisions and the spin exchange interaction which are characterized by the coupling constants \( g_1 \) and \( g_2 \) respectively. Using the second-order perturbation methods, the two bare coupling constants \( g_1 \) and \( g_2 \), are expressed in terms of their corrected ones, \( \tilde{g}_1 \) and \( \tilde{g}_2 \) which are directly related to the physically measurable \( s \)-wave wavelengths for the corresponding scattering channels. It is found that the correction of \( g_1 \) has the same form as that of the one-component scalar BEC. However, the correction of \( g_2 \) is more complicated and has dependence on the corrected coupling constant \( \tilde{g}_1 \). With the corrected coupling constants, we are able to show that the ultraviolet divergence occurring in the calculation of ground state energy can be completely removed.

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