EIGENVECTOR STATISTICS OF LÉVY MATRICES

AMOL AGGARWAL, PATRICK LOPATTO, AND JAKE MARCINEK

Abstract. We analyze statistics for eigenvector entries of heavy-tailed random symmetric matrices (also called Lévy matrices) whose associated eigenvalues are sufficiently small. We show that the limiting law of any such entry is non-Gaussian, given by the product of a normal distribution with another random variable that depends on the location of the corresponding eigenvalue. Although the latter random variable is typically non-explicit, for the median eigenvector it is given by the inverse of a one-sided stable law. Moreover, we show that different entries of the same eigenvector are asymptotically independent, but that there are nontrivial correlations between eigenvectors with nearby eigenvalues. Our findings contrast sharply with the known eigenvector behavior for Wigner matrices and sparse random graphs.

Contents

1. Introduction 1
2. Results 4
3. Proofs of main results 8
4. Preliminary results 14
5. Comparison 17
6. Dynamics 27
7. Scaling limit 45
Appendix A. Proofs of results from Section 4 and Section 7 55
Appendix B. Convergence in distribution 64
Appendix C. Quantum unique ergodicity of eigenvectors 64
References 65

1. Introduction

A central tenet of random matrix theory is that the spectral properties of large random matrices should be independent of the distributions of their entries. This phenomenon is known as universality, and it originates from investigations of Wigner in the 1950s [79]. One realization of this phenomenon is the Wigner–Dyson–Mehta conjecture, which asserts the universality of local eigenvalue statistics of $N \times N$ real symmetric (or complex Hermitian) Wigner matrices in the bulk of the spectrum. Over the past decade, this conjecture has been resolved and generalized to an extensive collection of random matrix models [1, 10, 19, 35, 37, 39, 41, 44, 47–53, 56–59, 64, 75, 77, 80].

A.A. is partially supported by NSF grant NSF DMS-1664619, the NSF Graduate Research Fellowship under grant DGE-1144152, and a Harvard Merit/Graduate Society Term-time Research Fellowship.
P.L. is partially supported by NSF grant DMS-1606305, NSF grant DMS-1855509, and the NSF Graduate Research Fellowship Program under grant DGE-1144152.
Universality for the limiting distributions of eigenvector entries of Wigner matrices was proven more recently in [40] by Bourgade and Yau. There, they introduced the eigenvector moment flow, a system of stochastic differential equations which govern the evolution of the moments of eigenvector entries of a matrix under the addition of Gaussian noise. Through a careful analysis of these dynamics, they prove asymptotic normality for the eigenvector entries of Wigner matrices. Extensions of this method later enabled the analysis of eigenvector statistics for sparse and deformed Wigner matrices in [23, 38], and for other eigenvector observables in [24, 41].

The matrices considered in these works all have entries of finite variance. The question of whether universality persists for matrices with entries of infinite variance was first raised in 1994 by Cizeau and Bouchaud [46]. Such matrices are now believed to be better suited for modeling heavy-tailed phenomena in physics [31, 45, 72], finance [32–34, 42, 54, 55, 61, 62], and machine learning [65–67]. They may therefore be regarded as exemplars of a new universality class for highly correlated systems.

Concretely, [46] introduced a class of symmetric matrices called Lévy matrices, whose entries are random variables in the domain of attraction of an α-stable law, and made a series of predictions regarding their spectral and eigenvector behavior. One such prediction concerns the global eigenvalue distribution. Given a Lévy matrix \( H \), [46] conjectured that as \( N \) tends to \( \infty \), its empirical spectral distribution \( \mu_H \) should converge to a deterministic, heavy-tailed measure \( \mu_\alpha \). This was later proven by Ben Arous and Guionnet in [15]. The measure \( \mu_\alpha \) contrasts with the empirical spectral densities of the other finite variance models studied previously, which are all compactly supported.

The main predictions in [46] concerned eigenvector (de)localization and local eigenvalue statistics for Lévy matrices. The more recent predictions of Tarquini, Biroli, and Tarzia are slightly different and can be stated as follows [78]. First, for \( \alpha \in [1, 2) \), all eigenvectors of \( H \) corresponding to finite eigenvalues are completely delocalized, and these eigenvalues exhibit local statistics matching those of the Gaussian Orthogonal Ensemble (GOE) as \( N \) tends to infinity. Second, for \( \alpha \in (0, 1) \), there exists a mobility edge \( E_\alpha > 0 \) separating a regime with delocalized eigenvectors and GOE local statistics around small energies \( E \in (-E_\alpha, E_\alpha) \) from one with localized eigenvectors and Poisson local statistics around large energies \( E / \in (-E_\alpha, E_\alpha) \).

Partial forms of eigenvector localization and delocalization in the regimes listed above were proven by Bordenave and Guionnet in [29, 30]. Later, in [5], complete eigenvector delocalization and convergence to GOE local statistics were established in the two delocalized phases mentioned above. For \( \alpha \in (0, 1) \), these delocalization results were only proven for eigenvalues in a non-explicit neighborhood of 0, and extending them to the conjectured mobility edge remains open.

The transition identified by [46, 78] is also known as an Anderson transition, a concept fundamental to condensed matter physics that describes phenomena such as metal–insulator transitions [3, 13, 69]. It is widely believed to exist in the context of random Schrödinger operators [12, 67, 68], but rigorously establishing this statement has remained an important open problem in mathematical physics for decades. Lévy matrices provide one of the few examples of a random matrix ensemble for which such a transition is also believed to appear. Partly for this reason, they have been a topic of intense study for both mathematicians and physicists over the past 25 years [15, 20, 21, 26, 30, 33, 68, 73, 78].

While the above results and predictions address (de)localization of Lévy matrix eigenvectors, little is known about refined properties of their entry fluctuations. In [21], Benaych-Georges and Guionnet showed that averages of \( O(N^2) \) of these eigenvector entries converge to Gaussian processes,
after scaling by \( N^{-1/2} \). However, until now, we have not been aware of any results or predictions concerning fluctuations for individual entries of Lévy matrix eigenvectors.

In this paper we establish several such results, which in many respects contrast with their known counterparts for Wigner matrices (and all other random matrix models for which the eigenvector entry distributions have previously been identified). We establish, for almost all \( \alpha < 2 \), the following statements concerning the unit (in \( L^2 \)) Lévy eigenvectors \( \mathbf{u}_k = (u_k(1), u_k(2), \ldots, u_k(N)) \) whose associated eigenvalues \( \lambda_k \approx E \) are sufficiently small.

1. An eigenvector entry \( \sqrt{N} u_j(i) \) is not asymptotically normal: its square converges to \( N^2 \cdot \mathcal{U}_*(E) \) as \( N \) tends to \( \infty \), where \( \mathcal{N} \) is a standard normal random variable and \( \mathcal{U}_*(E) \) is an independent (non-constant and typically non-explicit) random variable that depends on \( E \).
2. Different entries of the same eigenvector are asymptotically independent.
3. Entries of different eigenvectors with the same index are not asymptotically independent:
   - if \( k_1, k_2, \ldots, k_n \in [1, N] \) and \( i \in [1, N] \) are indices such that \( \lambda_{k_i} \approx E \), then the vector \((N u_{k_1}(i)^2, N u_{k_2}(i)^2, \ldots, N u_{k_n}(i)^2)\) converges to \((N_1^2 \cdot \mathcal{U}_*(E), N_2^2 \cdot \mathcal{U}_*(E), \ldots, N_n^2 \cdot \mathcal{U}_*(E))\), where the \( N_i \) are i.i.d. standard Gaussians that are independent from \( \mathcal{U}_*(E) \).
4. The law of \( \mathcal{U}_*(0) \) is given explicitly as the inverse of a one-sided \( \frac{2}{3} \)-stable law. In particular, all asymptotic moments of the median eigenvector are also explicit.

To contextualize our results, we recall the asymptotic normality statements for Wigner eigenvectors proved in \([10]\). First, an individual eigenvector entry converges to a standard normal random variable. Second, different entries of the same eigenvector are asymptotically independent. Third, the same is true for entries of different eigenvectors with the same index. Our results show that, although the second of these phenomena persists in the Lévy case, the first and third do not. We further note that while \([5]\) showed that Lévy matrices exhibit GOE local statistics at small energy, which only differ through a rescaling factor corresponding to the eigenvalue density, the eigenvector statistics follow a family of distinct random variables that vary in the energy parameter \( E \in \mathbb{R} \). This is again unlike the Wigner eigenvectors, which display the same Gaussian statistics throughout the spectrum.

Our work confronts one of the major differences between Lévy matrices and Wigner matrices: the non-concentration of the resolvent entries. In the Wigner case, these diagonal resolvent entries converge to a deterministic quantity (the Stieltjes transform of the semicircle law) as \( N \) tends to \( \infty \). However, for Lévy matrices \( \mathbf{H} \), the diagonal entries \( G_{jj}(z) \) of the resolvent \( \mathbf{G}(z) = (\mathbf{H} - z)^{-1} \) converge for fixed \( z \in \mathbb{H} \) to nontrivial (complex) random variables \( R_u(z) \) as \( N \) tends to \( \infty \) \([28]\).

This is the mechanism that generates the non-Gaussian scaling limit for the limiting eigenvector entries. Indeed, the random variable \( \mathcal{U}_*(E) \) mentioned above is defined as a (multiple of a) weak limit of \( R_u(E + i\eta) \), as \( \eta \) tends to 0.

Our proof strategy is dynamical. First, we define a matrix \( \mathbf{X} \), which is coupled to the Lévy matrix \( \mathbf{H} \) and obtained by setting all sufficiently small (in absolute value) entries of \( \mathbf{H} \) to zero. We also introduce the Gaussian perturbation \( \mathbf{X}_t = \mathbf{X} + \sqrt{s} \mathbf{W} \), where \( \mathbf{W} \) is a GOE and \( s \ll 1 \). Under a certain choice of \( s = t \), we are able to show that the eigenvector statistics of \( \mathbf{H} \) are asymptotically the same as those of \( \mathbf{X}_t \). Second, we identify the moments of the eigenvector entries of \( \mathbf{X}_t \) in terms of entries of the resolvent matrix \( \mathbf{R}(t, z) = (\mathbf{X}_t - z)^{-1} \), where \( z = E + i\eta \) and \( \eta \) tends to 0 as \( N \) tends to \( \infty \). Third, we compute the large \( N \) limit of these resolvent entries and deduce the above claims from the behavior of the resulting scaling limits. We now describe the steps of our argument, and their associated challenges, in more detail.

1. The first step is a comparison of eigenvector statistics, which has been achieved before for Wigner matrices with entries matching in the first four moments \([60, 77]\). However, these results
do not apply to Lévy matrices, since the second moments of their entries are infinite. Instead, we use the comparison scheme introduced in [5] that conditions on the locations of the large entries of $H$ and matches moments between the small entries of $H$ and the Gaussian perturbation $\sqrt{7}W$ in $X_t = X + \sqrt{7}W$; these have all moments finite by construction. While [5] considered the comparison for Green’s function elements, we apply it to general smooth functions of the matrix entries and write the eigenvector statistics as such functions using ideas from [38,57].

2. The second step uses the eigenvector moment flow to show that moments of the eigenvector entries of $X_t$ approximate moments of the $\text{Im } R(t, z)$ entries. As in [38,40], a primary idea here is to apply the maximum principle to show under these dynamics that eigenvector moment observables equilibrate after a short period of time to a polynomial in the entries of $\text{Im } R(t, z)$. However, unlike in those previous works, the entries of $\text{Im } R(s, z)$ do not concentrate and therefore might be unstable under the dynamics. To address this, we condition on the initial data $X_0$, which one might hope renders the $R_{ijj}(s, z)$ essentially constant under the flow $X_s$ for $s \ll 1$. Unfortunately, we cannot show this directly, and in fact it appears that these resolvent entries can be unstable in the beginning of the dynamics even after this conditioning. Therefore, we run the flow for a short time $\tau$ before beginning the main analysis. This has a regularizing effect and ensures the stability of the resolvent entries of $X_s$ for $s \in [\tau, t]$. Our analysis then proceeds by running the dynamics for a further amount of time $t - \tau \gg N^{-1/2}$ to prove convergence to equilibrium, given this initial regularity.

3. The third step asymptotically equates moments of $\text{Im } R_{ii}(t, z)$ with those of $\text{Im } R_{ii}(z)$, as $\eta = \text{Im } z$ tends to 0 and $N$ tends to $\infty$ simultaneously. To analyze the former, we first use the Schur complement formula and a certain integral identity to express arbitrary moments of $\text{Im } R_{ii}(t, z)$ through the $\frac{\eta}{2}$-th moments of (real and imaginary parts of) $R_{ii}(t, z)$, as in [5,29,30]. Next, using a local law from [5], we approximate these $\frac{\eta}{2}$-th moments by corresponding ones for $R_{ii}(z)$. To analyze the moments of $\text{Im } R_{ii}(z)$, we use the same integral identity and a recursive distributional equation for $R_{ii}(z)$ from [28] to express them through $\frac{\eta}{2}$-moments of (real and imaginary parts of) $R_{ii}(z)$. We then observe the two expressions are equal as $N$ tends to $\infty$.

The remainder of this paper is organized as follows. In Section 2 we state our results in detail. In Section 3 we give a full proof outline and establish our main results, assuming several preliminary claims which are shown in the remainder of the paper. Section 4 recalls results on Lévy matrices from previous works that are required for the argument. Section 5 details the comparison part of the argument. Section 6 analyzes the eigenvector moment flow. Section 7 computes the scaling limits of the resolvent entries mentioned above. Appendix A provides some preliminary results needed in the previous sections, and Appendix B addresses convergence in distribution. In Appendix C we discuss quantum unique ergodicity (QUE) for eigenvectors of Lévy matrices, whose analogue for Wigner matrices was established in [40].

Acknowledgments. The authors thank Giulio Biroli, Jiaoyang Huang, and Horng-Tzer Yau for helpful conversations.

2. Results

2.1. Definitions. Denote the upper half plane by $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$. Set $\mathbb{R}_+ = [0, \infty)$, set $\mathbb{K} = \{z \in \mathbb{C} : \text{Re } z > 0\}$, and set $\mathbb{K}^+ = \overline{\mathbb{K} \cap \mathbb{H}}$ to be the closure of the positive quadrant of the complex plane. We also let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle and define $S^1_+ = \mathbb{K}^+ \cap S^1$. 
Fix a parameter $\alpha \in (0, 2)$, and let $\sigma > 0$ and $\beta \in [-1, 1]$ be real numbers. A random variable $Z$ is a $(\beta, \sigma)$-stable law if it has the characteristic function
\[
\mathbb{E} [e^{itZ}] = \exp \left( -\sigma^\alpha |t|^\alpha (1 - i\beta \text{sgn}(t)u) \right), \quad \text{for all } t \in \mathbb{R},
\] (2.1)
where $u = u_\alpha = \tan \left( \frac{\pi \alpha}{2} \right)$ if $\alpha \neq 1$ and $u = u_1 = -\frac{2}{\pi} \log |t|$ if $\alpha = 1$. Note $\beta = 0$ ensures that $Z$ is symmetric. The case $\beta = 1$ is known as a one-sided $\alpha$-stable law and is always positive.

We now define the entry distributions we consider in this paper. Our proofs and results should also apply to wider classes of distributions, but we will not pursue this here (see the similar remark in [13, Section 2] for more on this point).

**Definition 2.1.** Let $Z$ be a $(0, \sigma)$-stable law with
\[
\sigma = \left( \frac{\pi}{2 \sin \left( \frac{\pi \alpha}{2} \right) \Gamma(\alpha)} \right)^{1/\alpha} > 0.
\] (2.2)
Let $J$ be a symmetric random variable (not necessarily independent from $Z$) such that $\mathbb{E}[J^2] < \infty$, $Z + J$ is symmetric, and
\[
\frac{C_1}{(|t| + 1)^\alpha} \leq \mathbb{P}[|Z + J| \geq t] \leq \frac{C_2}{(|t| + 1)^\alpha}, \quad \text{for each } t \geq 0 \text{ and some constants } C_1, C_2 > 0.
\] (2.3)
Denoting $\mathfrak{z} = Z + J$, the symmetry of $J$ and the condition $\mathbb{E}[J^2] < \infty$ are equivalent to imposing a coupling between $\mathfrak{z}$ and $Z$ such that $\mathfrak{z} - Z$ is symmetric and has finite variance, respectively.

For each positive integer $N$, let $\{H_{ij}\}_{1 \leq i, j \leq N}$ be mutually independent random variables that each have the same law as $N^{-1/\alpha}(Z + J) = N^{-1/\alpha}\mathfrak{z}$. Set $H_{ij} = H_{ji}$ for each $i, j$, and define the $N \times N$ random matrix $H = H_N = \{H_{ij}\} = \{H_{i,j}^{(N)}\}$, which we call an $\alpha$-Lévy matrix.

The $N^{-1/\alpha}$ scaling of the entries $H_{ij}$ is different from the usual $N^{-1/2}$ scaling for Wigner matrices. It makes the typical row sum of $H$ of order one. The constant $\sigma$ is chosen so that our notation is consistent with previous works [15, 29, 30], but can be altered by rescaling $H$ without affecting our main results.

By [13, Theorem 1.1], the empirical spectral distribution of $H$ converges to a deterministic measure that we denote $\mu_\alpha$, which is absolutely continuous with respect to the Lebesgue measure and symmetric about 0. We denote its probability density function and Stieltjes transform by $\varphi_\alpha(x)$ and
\[
m_\alpha(z) = \int_{\mathbb{R}} \varphi_\alpha(x) \frac{dx}{x - z},
\] (2.4)
defined for $z \in \mathbb{H}$, respectively.

The Stieltjes transform $m_\alpha(z)$ may be characterized as the solution to a certain self-consistent equation [29, Section 3.1]. We note it here, although we will not need this representation for our work. For any $z \in \mathbb{H}$, define the functions $\varphi = \varphi_{\alpha,z} : \mathbb{K} \to \mathbb{C}$ and $\psi = \psi_{\alpha,z} : \mathbb{K} \to \mathbb{C}$ by
\[
\varphi_{\alpha,z}(x) = \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}_+} t^{\alpha/2 - 1} e^{itx} e^{-\Gamma(1 - \alpha/2)t^{\alpha/2}z} dt, \quad \psi_{\alpha,z}(x) = \int_{\mathbb{R}_+} e^{itx} e^{-\Gamma(1 - \alpha/2)t^{\alpha/2}z} dt, \quad (2.5)
\]
for any $x \in \mathbb{K}$. For each $z \in \mathbb{H}$, there exists a unique solution $y = y(z) \in \mathbb{K}$ to the equation $y(z) = \varphi_{\alpha,z}(y(z))$. Then, the Stieltjes transform $m_\alpha(z) : \mathbb{H} \to \mathbb{H}$ is defined by setting $m_\alpha(z) = i\psi_{\alpha,z}(y(z))).$ \footnote{By symmetric, we mean that $J$ has the same law as $-J.$}
We recall that, like any Stieltjes transform of an absolutely continuous measure, \( \text{Im } m_\alpha(z) \) extends to the real line with\(^{(2.6)}\)

\[
\lim_{\eta \to 0} \text{Im } m_\alpha(E + i\eta) = \pi \varrho_\alpha(E)
\]

for \( E \in \mathbb{R} \). It is known that \( \varrho_\alpha(x) \sim \frac{\alpha}{2x^2 + \pi} \) as \( x \to \infty \) \cite{28} Theorem 1.6.

**Definition 2.2.** The classical eigenvalue locations \( \gamma_i = \gamma_i^{(\alpha)} \) for \( \varrho_\alpha(x) \) are defined by the quantiles

\[
\gamma_i = \inf \left\{ y \in \mathbb{R} : \int_{-\infty}^{y} \varrho_\alpha(x) \, dx \geq \frac{i}{N} \right\}.
\]

Given a random matrix \( A \), it is common to study its resolvent \( (A - z)^{-1} \). Contrary to those for the Wigner model, the diagonal entries \( G_{ii}(z) \) of the resolvent \( G(z) = (H - z)^{-1} \) of a Lévy matrix do not converge to a constant value but instead converge to a nontrivial limiting distribution as \( N \) tends to infinity and \( z \in \mathbb{H} \) remains fixed. This was shown in \cite{28}, where the limit \( R_s(z) \) was identified as the resolvent of a random operator defined on a space known as the Poisson Weighted Infinite Tree \cite{11,12}, which is a weighted and directed rooted tree, evaluated at its root. We note the basic construction here and refer to \cite{28} Section 2.3 for details.

Set \( d\nu = (1/2) \, d\mu \), where \( \mu \) is the Lebesgue measure on \( \mathbb{R} \). The vertex set of the tree is given by \( V = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k \), where the root is \( \mathbb{N}^0 = \emptyset \), and the children of \( v \in \mathbb{N}^k \) are denoted \((v,1),(v,2),\ldots \in \mathbb{N}^{k+1}\). To determine the weights, let \( \{\Xi_v\}_{v \in V} \) be a collection of independent Poisson point processes with intensity measure \( \nu \) on \( \mathbb{R} \). Let \( \Xi_\varnothing = \{y_1,y_2,\ldots\} \) be ordered so that \( |y_1| \leq |y_2| \leq \cdots \), and set \( y_i \) to be the weight of edge connecting \( \varnothing \) to the vertex \( i \). This process is repeated for all vertices so that, for any \( v \in \mathbb{N}^k \), the edge between vertices \( v \) and \( (v,i) \) is weighted with the value \( y_{(v,i)} \), where \( \Xi_v = \{y_{(v,1)},y_{(v,2)},\ldots\} \) is labeled so that \( |y_{(v,1)}| \leq |y_{(v,2)}| \leq \cdots \).

Let \( \mathcal{F} \) be the (dense) subset of \( L^2(V) \) of vectors with finite support and, for any \( v \in V \), let \( \delta_v \in \mathcal{F} \) denote the unit vector supported on \( v \). Then, define the linear operator \( T : \mathcal{F} \to L^2(V) \) by setting

\[
\langle \delta_v, T \delta_w \rangle = \begin{cases} \text{sign}(y_{(v,i)}) |y_{(v,i)}|^{-1/\alpha} & \text{if } w = (v,k) \text{ for some } k, \\ \text{sign}(y_{(w,i)}) |y_{(w,i)}|^{-1/\alpha} & \text{if } v = (w,k) \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}
\]

We identify \( T \) with its closure, which is self-adjoint \cite{28} Section 2.3. It can be considered a weak limit of the matrix \( \mathcal{H} \), as \( N \) tends to \( \infty \).

**Definition 2.3.** For any \( z \in \mathbb{H} \), we define \( R_s(z) : \mathbb{H} \to \mathbb{H} \) to be the resolvent entry \( \langle \delta_\varnothing, (T - z)^{-1} \delta_\varnothing \rangle \).

For any \( z \in \mathbb{H} \), define the \( N \times N \) matrix \( G(z) = \{G_{ij}(z)\} \) by \( G(z) = (H - z)^{-1} \), which is the resolvent of \( H \). It is known from \cite{28} Section 2 that any diagonal entry \( G_{jj}(z) \) converges to \( R_s(z) \) in distribution for fixed \( z \in \mathbb{H} \), as \( N \) tends to \( \infty \).

Next, we require the following result and definition concerning the limit of \( R_s(z) \) as \( \text{Im } z \) tends to \( 0 \). The following proposition will be proved in Section 7 below.

**Proposition 2.4.** There exists a (deterministic) countable set \( \mathcal{A} \subset (0,2) \) with no accumulation points in \((0,2)\) such that the following two statements hold. First, for all \( \alpha \in (0,2) \setminus \mathcal{A} \), there exists a constant \( c = c(\alpha) > 0 \) such that, for every real number \( E \in [-c,c] \), the sequence of random variables \( \{\text{Im } R_s(E + i\eta)\}_{\eta > 0} \) is tight as \( \eta \) tends to \( 0 \). Second, for any fixed \( p \in \mathbb{N} \), all limit points \( \mathcal{R}(E) \) of this sequence under the weak topology have the same moment \( \mathbb{E}[\mathcal{R}(E)^p] \).
The set $A$ is non-explicit and originally appeared in [14] from an application of the implicit function theorem for Banach spaces to a certain self-consistent equation. In our context, $A$ will come from a local law, given by Lemma 4.2 below.

**Definition 2.5.** Let $R_*(E)$ be an arbitrary limit point (under the weak topology) as $\eta$ tends to 0 of the sequence $\{\text{Im} R_*(E + i\eta)\}_{\eta > 0}$. By Proposition 2.4 and Prokhorov’s theorem, there exists at least one. Given $R_*(E)$, define the random variable $U_*(E) = (\pi_{\theta_n}(E))^{-1}R_*(E)$.

We also need the following definition to state our results.

**Definition 2.6.** Let $w = (w_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ be a random vector and $w^{(j)} = (w_i^{(j)})_{1 \leq i \leq n}$, defined for $j \geq 1$, be a sequence of random vectors in $\mathbb{R}^n$. We say that $w^{(j)}$ converges in moments to $w$ if for every polynomial $P: \mathbb{R}^n \to \mathbb{R}$ in $n$ variables, we have

$$
\lim_{N \to \infty} E \left[ P(w^{(N)}) \right] = E[P(w)].
$$

(2.9)

**2.2. Results.** In this section, we state our results, which are proved in Section 3. Our first, Theorem 2.7, identifies the joint moments of different entries of the same eigenvector. Our second, Theorem 2.8, does this for the same entries of different eigenvectors. We let $\lambda_1(H) \leq \lambda_2(H) \leq \cdots \leq \lambda_N(H)$ denote the eigenvalues of $H$ in non-decreasing order and, for each $k \in [1, N]$, we write $u_k = (u_k(1), u_k(2), \ldots, u_k(N))$ for a unit eigenvector of $H$ corresponding to $\lambda_k(H)$.

In the theorem statements, certain index parameters (for instance $i$ and $k$) may depend on $N$. For brevity, we sometimes suppress this dependence in the notation, writing for example $u_k$ instead of $u_k(N)$. Throughout, we recall the countable set $A \subset (0, 2)$ from Proposition 2.4.

**Theorem 2.7.** For all $\alpha \in (0, 2) \setminus A$, there exists a constant $c = c(\alpha) > 0$ such that the following holds. Fix an integer $n > 0$ (independently of $N$) and index sequences $\{i_j(N)\}_{1 \leq j \leq n}$ such that for every $N$, $\{i_j(N)\}_{1 \leq j \leq n}$ are distinct integers in $[1, N]$. Further let $k = k(N) \in [1, N]$ be an index sequence such that $\lim_{N \to \infty} \gamma_k = E$ for some $E \in [-c, c]$. Then the vector

$$
(Nu_k(i_1)^2, Nu_k(i_2)^2, \ldots, Nu_k(i_n)^2)
$$

(2.10)

converges in moments to

$$
(N_1^2 \cdot U_1(E), N_2^2 \cdot U_2(E), \ldots, N_n^2 \cdot U_n(E)),
$$

(2.11)

where the $N_j$ are independent, identically distributed (i.i.d.) standard Gaussians and the $U_j(E)$ are i.i.d. random variables with law $U_*(E)$ that are independent from the $N_j$.

**Theorem 2.8.** For all $\alpha \in (0, 2) \setminus A$, there exists a constant $c = c(\alpha) > 0$ such that the following holds. Fix an integer $n > 0$ (independently of $N$) and index sequences $\{k_j(N)\}_{1 \leq j \leq n}$ such that for every $N$, $\{k_j(N)\}_{1 \leq j \leq n}$ are distinct integers in $[1, N]$ and $|k_1 - k_j| < N^{1/2}$ for each $j \in [2, n]$. Suppose that $\lim_{N \to \infty} \gamma_{k_1} = E$ for some $E \in [-c, c]$. Further let $i = i(N) \in [1, N]$ be an index sequence. Then the vector

$$
(Nu_{k_1}(i)^2, Nu_{k_2}(i)^2, \ldots, Nu_{k_n}(i)^2)
$$

(2.12)

converges in moments to

$$
(N_1^2 \cdot U_*(E), N_2^2 \cdot U_*(E), \ldots, N_n^2 \cdot U_*(E)),
$$

(2.13)

where the $N_j$ are i.i.d. standard Gaussians that are independent from $U_*(E)$. 
Theorem 2.7 shows that entries of the same eigenvector are asymptotically independent, as in the Wigner case [10, Corollary 1.3]. However, unlike in the Wigner case [10, Theorem 1.2], Theorem 2.8 indicates that entries of different eigenvectors with the same index can be asymptotically correlated. This can be seen by taking $n = 2$ and $k_2 = k_1 + 1$ in that result, in which case $N u_{k_1}(m)^2$ and $N u_{k_2}(m)^2$ are correlated through $U_*(E)$.

For almost all $E \in [-c, c]$, the random variable $U_*(E)$ is not explicit. However, as a consequence of [28, Theorem 4.3], an exception occurs at $E = 0$, where $U_*(0)$ is given by the inverse of a stable law. In this case, the $n = 1$ cases of Theorem 2.7 and Theorem 2.8 reduce to the following corollary.

**Corollary 2.9.** Retain the notation of Theorem 2.7. Choose $k$ so that $E = 0$, and set $n = 1$ and $m = i_1$. Then $N u_k(m)^2$ converges in moments to

$$\frac{1}{\Gamma(1 + \frac{2}{\alpha})} \cdot N^2 \cdot \vartheta,$$

(2.14)

where $\mathcal{N}$ is a standard Gaussian and $\vartheta$ is independent with law $S^{-1}$, where $S$ is a $(1, 1)$ $\mathbb{F}$-stable law.

The non-triviality of the random variable $\vartheta$ shows that the entries of $u_k$ are asymptotically non-Gaussian; this is again different from the eigenvector behavior in the Wigner case. It is natural to wonder whether $R_*(E)$ is non-constant for $E \neq 0$. As a consequence of the last statement of Lemma 7.8 below, for all $p \in \mathbb{N}$, the moments $E[(R_*(E))^p]$ are continuous in $E$, for $|E|$ sufficiently small. This implies that moments of $U_*(E)$ are non-constant for all $E$ in a neighborhood of 0, so the eigenvectors of $H$ corresponding to sufficiently small eigenvalues are also non-Gaussian.

It is also natural to ask whether our results hold for convergence in distribution. In the case $\alpha \in (1, 2) \setminus A$ we will address this in Appendix B through Proposition B.1 by studying the rate of growth of the moments of the limiting distribution. If $\alpha < 1$, then the moments of $U_*(E)$ grow too quickly for this to determine the law of $N^2 \cdot U_*(E)$.

Finally, we note that we consider the squared eigenvector entries $u_k(i)^2$ to avoid ambiguity in the choice of sign for $u_k(i)$, since given an eigenvalue $\lambda_k$ of a real symmetric matrix and a corresponding eigenvector $v_k$, the vector $-v_k$ is also an eigenvector. In the context of Lévy random matrices, if one chooses this sign independently with probability 1/2 for each possibility, then our methods show the above results hold with the conclusion of Theorem 2.7 replaced by the convergence in moments of $(\sqrt{N} u_k(i_1), \sqrt{N} u_k(i_2), \ldots, \sqrt{N} u_k(i_n))$ to $(N_1 \cdot U_1^{1/2}(E), N_2 \cdot U_1^{1/2}(E), \ldots, N_n \cdot U_1^{1/2}(E))$, where the $N_k$ remain i.i.d. standard Gaussians, and similarly for Theorem 2.8.

### 3. Proofs of main results

Assuming some claims proven in later parts of this paper, we will in this section establish the results stated in Section 2.2. This will proceed through the following steps.

1. We define a matrix $X$, obtained by setting the small entries of the original Lévy matrix $H$ to zero, and the Gaussian perturbation $X_s = X + sW$, where $W$ is a GOE matrix. For a specific choice of $s = t$, with $N^{-1/2} \ll t \ll 1$, we show as Theorem 3.7 that the eigenvector statistics of $H$ (corresponding to small eigenvalues) are approximated by those of $X_t$.
2. We show as Theorem 3.8 that moments of the eigenvector entries of $X_t$ (corresponding to small eigenvalues) can be identified through resolvent entries of $X_t$.
3. We compute as Theorem 3.9 the limits of these resolvent entries as $N$ and $\eta$ tend to $\infty$ and 0, respectively.
In Section 3.3 we prove Theorem 2.7, Theorem 2.8, and Corollary 2.9, given that the results enumerated above and Proposition 2.4 hold. The remaining sections of the paper verify these prerequisite results.

3.1. Notation. Throughout, we write $C$ for a large constant and $c$ for a small constant. These may depend on other constants and may change line to line, but only finitely many times, so that they remain finite. We say $X \ll Y$ if there exists a small constant $c > 0$ such that $N^c|X| \leq Y$. Constants in this paper may depend on the constant $c > 0$ implicit in the claim $X \ll Y$, but we suppress this in the notation. We write $X \lesssim Y$ if there exists $C > 0$ such that $|X| \leq CY$; we also say $X \lesssim_u Y$, or equivalently $X = O_u(Y)$, if $|X| \leq C_u Y$ for some constant $C_u > 0$ depending on a parameter $u$.

In what follows, for any function (or vector) $f$, we let $\|f\|_\infty$ denote the $L^\infty$-norm of $f$. We also denote $\text{Mat}_{N \times N}$ by the set of $N \times N$ real, symmetric matrices. Given $M \in \text{Mat}_{N \times N}$, we denote its eigenvalues by $\lambda_1(M), \lambda_2(M), \ldots, \lambda_N(M)$ in non-decreasing order. We further let $u_i(M)$ denote the unit eigenvector corresponding to the eigenvalue $\lambda_i(M)$ for each $i$. We also make the following definition.

**Definition 3.1.** We say a (sequence of) vectors $q = q(N) = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^N$ has stable support if there exists a constant $C > 0$ such that the set $\{(i, q_i) : q_i \neq 0\}$ does not change for $N > C$. We let $\text{supp } q = \{i : q_i \neq 0\}$ denote the support of $q$.

We next introduce the notion of overwhelming probability.

**Definition 3.2.** We say that a family of events $\{F(u)\}$ indexed by some parameter(s) $u \in U(N)$, where $U(N)$ is a parameter set which may depend on $N$, holds with overwhelming probability if, for any $D > 0$, there exists $N(D, U(N)) > 0$ such that for $N \geq N(D, U(N))$,\[ \inf_{u \in U(N)} \mathbb{P}(F(u)) \geq 1 - N^{-D}. \] (3.1)

Next, given $\alpha \in (0, 2)$ we may select positive real numbers $b = b(\alpha) > 0$; $\nu = \nu(\alpha) > 0$; $a = a(\alpha) > 0$; and $\rho = \rho(\alpha) > 0$ such that\[ \frac{1}{\alpha} - b > 0; \quad \frac{1}{4 - \alpha} < \nu < \frac{1}{4 - 2\alpha}; \quad (2 - \alpha)\nu < a < \frac{1}{2}; \quad 0 < \rho < \nu < \frac{1}{2}. \quad \alpha \rho < (2 - \alpha)\nu. \] (3.2)

These parameters will be fixed throughout the paper, and we will let other constants depend on them (and on $\alpha$), even when not explicitly noted. We always assume $\alpha \in (0, 2) \setminus A$, where $A$ is the set from Lemma 1.2 below (or, equivalently, the one from Proposition 2.4).

3.2. Comparison. We first recall the definition of the removed model $X$ from [5, Definition 3.2].

**Definition 3.3.** Recalling the notation of Definition 2.4, let $X = (Z + J)1_{Z + J > N^\alpha}$. We call $X$ the $b$-removal of $Z + J$. Further, let $\{X_{ij}\}_{1 \leq i \leq j \leq N}$ be mutually independent random variables that each have the same law as $N^{-1/\alpha}X$. Set $X_{ij} = X_{ji}$ for each $1 \leq j < i \leq N$, and define the $N \times N$ symmetric matrix $X = \{X_{ij}\}$. We call $X$ a $b$-removed $\alpha$-Lévy matrix.

We also recall a resampling and coupling of $X$ and $H$ that was described in [5, Section 3.3.1].

**Definition 3.4.** We define mutually independent random variables $\{a_{ij}, b_{ij}, c_{ij}, \psi_{ij}, \chi_{ij}\}_{1 \leq i \leq j \leq N}$ as follows. Let $\psi_{ij}$ and $\chi_{ij}$ denote $0 - 1$ Bernoulli random variables with distributions\[ \mathbb{P}[\psi_{ij} = 1] = \mathbb{P}[|H_{ij}| \geq N^{-\rho}], \quad \mathbb{P}[\chi_{ij} = 1] = \frac{\mathbb{P}[|H_{ij}| \in [N^{-\nu}, N^{-\rho}]}{\mathbb{P}[|H_{ij}| < N^{-\rho}]. \] (3.3)
Additionally, let $a_{ij}, b_{ij},$ and $c_{ij}$ be random variables such that
\[
\mathbb{P}(a_{ij} \in I) = \frac{\mathbb{P}(H_{ij} \in (-N^{-\nu}, N^{-\nu}) \cap I)}{\mathbb{P}(|H_{ij}| < N^{-\nu})}, \quad \mathbb{P}(c_{ij} \in I) = \frac{\mathbb{P}(H_{ij} \in ((-\infty, -N^{-\rho}) \cup [N^{-\rho}, \infty)) \cap I)}{\mathbb{P}(|H_{ij}| \geq N^{-\rho})},
\]
\[
\mathbb{P}(b_{ij} \in I) = \mathbb{P}(H_{ij} \in ((-N^{-\rho}, -N^{-\nu}] \cup [N^{-\nu}, N^{-\rho})) \cap I), \quad \mathbb{P}(|H_{ij}| \in [N^{-\nu}, N^{-\rho}])
\]
for any interval $I \subset \mathbb{R}$. For each $1 \leq j < i \leq N$, define $a_{ij} = a_{ji}$ by symmetry, and similarly for each of $b_{ij}, c_{ij}, \psi_{ij},$ and $\chi_{ij}$.

Because $a_{ij}, b_{ij}, c_{ij}, \psi_{ij},$ and $\chi_{ij}$ are mutually independent, $H_{ij}$ has the same law as
\[
(1 - \psi_{ij})(1 - \chi_{ij})a_{ij} + (1 - \psi_{ij})\chi_{ij}b_{ij} + \psi_{ij}c_{ij}
\]
and $X_{ij}$ has the same law as $(1 - \psi_{ij})\chi_{ij}b_{ij} + \psi_{ij}c_{ij}$. Therefore, although the random variables $H_{ij}1_{|H_{ij}| \geq N^{-\rho}}, H_{ij}1_{|H_{ij}| < N^{-\rho}},,$ and $H_{ij}1_{|H_{ij}| \leq N^{-\nu}}$ are correlated, this decomposition expresses their dependence through the Bernoulli random variables $\psi_{ij}$ and $\chi_{ij}$.

**Definition 3.5.** For each $1 \leq i, j \leq N$, set
\[
A_{ij} = (1 - \psi_{ij})(1 - \chi_{ij})a_{ij}, \quad B_{ij} = (1 - \psi_{ij})\chi_{ij}b_{ij}, \quad C_{ij} = \psi_{ij}c_{ij},
\]
and define the four $N \times N$ matrices $A = \{A_{ij}\}, B = \{B_{ij}\}, C = \{C_{ij}\},$ and $\Psi = \{\psi_{ij}\}$.

For the remainder of the paper we sample $H$ and $X$ by setting $H = A + B + C$ and $X = B + C$, inducing a coupling between the two matrices. We commonly refer to $\Psi$ as the label of $H$ (or of $X$). Defining $H$ and $X$ in this way ensures that their entries have the same laws as in Definition 3.3 and Definition 3.3 respectively.

For any $s \in \mathbb{R}_+$, we define the matrix $X_s \in \text{Mat}_{N \times N}$ by setting
\[
X_s = X + W_s,
\]
where $W_s = (w_{ij}(s))_{1 \leq i, j \leq N} \in \text{Mat}_{N \times N}$ and $w_{ij}$ are mutually independent Brownian motions with symmetry constraint $w_{ij} = w_{ji}$ and variance $(1 + \mathbb{1}_{i=j})N^{-1}$.

We now make a specific choice of the time $t$ to enable our comparison argument. Define $t$ by
\[
t = N\mathbb{E}\left[H_{11}^2 \mathbb{1}_{|H_{11}| < N^{-\nu}} \big| |H_{11}| < N^{-\rho}\right] = \frac{NE[H_{11}^2 \mathbb{1}_{|H_{11}| < N^{-\nu}}]}{\mathbb{P}(|H_{11}| < N^{-\rho})}.
\]
The following estimate is $[5]$ Lemma 3.5] and can be quickly deduced from $[2,3]$ and $[5,9]$.

**Lemma 3.6 (5 Lemma 3.5).** Under the choice of (3.9), we have that
\[
cN^{(\alpha-2)\nu} \leq t \leq CN^{(\alpha-2)\nu}.
\]
Observe in particular that (3.10) implies that $N^{-1/2} \ll t \ll 1$, by the third inequality in (3.2).

The next theorem is proved in Section 5 and completes the first step of the outline given in the beginning of Section 5.

**Theorem 3.7.** There exist constants $c_1, c_2 > 0$ such that the following holds. Let $t$ be as in (3.9), $P: \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial in $n$ variables, and $q \in \mathbb{R}^N$ be a unit vector with stable
support. Then there exists a constant $C = C(P, |\text{supp}\ q|) > 0$ such that, for indices $i_1, i_2, \ldots, i_n \in [(1/2 - c_1)N, (1/2 + c_1)N]$,
\[
\left| \mathbb{E}\left[P\left(\left(N(q, u_{i_k}(X_t))^2\right)_{1 \leq k \leq n}\right)\right] \right| - \mathbb{E}\left[P\left(\left(N(q, u_{i_k}(H))^2\right)_{1 \leq k \leq n}\right)\right] \leq CN^{-c_2}.
\]

(3.11)

3.3. Short-time universality. For each integer $k \in [1, N]$ and real number $s \geq 0$, abbreviate $\lambda_k(s) = \lambda_k(X_s)$, and set $\lambda(s) = (\lambda_1(s), \lambda_2(s), \ldots, \lambda_N(s))$. Further let $u_k(s) \in \mathbb{R}^N$ denote the unit eigenvector of $X_s$ associated with $\lambda_k(s)$, and set $u(s) = (u_1(s), \ldots, u_N(s))$.

Next, for any unit vector $q \in \mathbb{R}^N$ and $k \in [1, N]$, set $z_k(s) = z_k(s, q) = \sqrt{N}\langle q, u_k(s)\rangle$. For any integer $m \geq 1$; indices $i_1, i_2, \ldots, i_m \in [1, N]$; and integers $j_1, j_2, \ldots, j_m \geq 0$, define
\[
Q_{i_1, \ldots, i_m}^{j_1, \ldots, j_m}(s) = \prod_{i=1}^m z_{i_i}(s)^{2j_i} \prod_{i=1}^m a(2j_i)^{-1}
\] where $a(2j) = (2j - 1)!!$.

(3.12)

The normalization factors $a(2j)$ are chosen because they are the moments of a standard Gaussian.

To any index set $\{(i_1, j_1), \ldots, (i_m, j_m)\}$ with distinct $i_k \in [1, N]$ and positive $j_k$, we may associate the vector $\xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{R}^N$ with $\xi_{i_k} = j_k$ for $1 \leq k \leq m$ and $\xi_p = 0$ for $p \notin \{i_1, \ldots, i_m\}$. We think of $\xi$ as a particle configuration on the integers, with $j_k$ particles at site $i_k$ for all $k$ and zero particles on the sites not in $\{i_1, \ldots, i_n\}$. We call the set $\{i_1, \ldots, i_m\}$ the support of $\xi$, denoted $\text{supp}\ \xi$. Denote $N(\xi) = \sum_{j=1}^m j_k$, the total number of particles. The configuration $\xi^{ij}$ is defined as the result of moving one particle in $\xi$ from $i$ to $j$, that is, if $i \neq j$ then $\xi^{ij}_k$ equals $\xi_k + 1$, $\xi_k - 1$, or $\xi_k$ for $k = j, k = i, k \notin \{i, j\}$, respectively. Under this notation, we define an observable $F_\epsilon(\xi)$ by the expectation
\[
F_\epsilon(\xi) = \mathbb{E}[Q_{i_1, \ldots, i_m}^{j_1, \ldots, j_m}(s)].
\]

(3.13)

Now fix $\epsilon \in \mathbb{R}_{>0}$, later chosen to be sufficiently small. Recalling $a$ from (3.2) and $t$ from (3.9), define
\[
\psi = N^\epsilon, \quad \eta = N^{-a}\psi, \quad \text{so that} \quad N^{-1/2} \ll \eta \ll t,
\]

(3.14)

where the last inequality in (3.14) follows from the third bound in (3.2), (3.10), and the fact that $\epsilon$ is small. For each $s \in \mathbb{R}_{>0}$ we define the resolvent $R(s, z)$, Stieltjes transform $m_N(s, z)$ of $X_s$, and the expectation of $m_N(s, z)$ by
\[
R(s, z) = (X_s - z)^{-1}, \quad m_N(s, z) = N^{-1}TrR(s, z), \quad \tilde{m}_N(s, z) = \mathbb{E}[m_N(s, z)].
\]

(3.15)

We also define the (random) empirical spectral measure for $X_s$ by
\[
\mu_s = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i(s)},
\]

(3.16)

where $\delta_x$ is the discrete probability measure that places all its mass at $x$. Further, we define $\tilde{\mu}_s = \mathbb{E}[\mu_s]$ and observe that the Stieltjes transform of $\tilde{\mu}_s$ is $\tilde{m}_N(s, z)$. The classical eigenvalue locations for $\tilde{\mu}_s$ are given by
\[
\tilde{\gamma}_i(s) = \inf \left\{ y \in \mathbb{R} : \tilde{\mu}_s([\gamma, y]) \geq \frac{i}{N} \right\}.
\]

(3.17)

Recalling the specific choice of $t$ from (3.3), we abbreviate $\tilde{\gamma}_i = \tilde{\gamma}_i(t)$.

The following theorem is proved in Section 5 and completes the second step of the above outline. We recall from [27] the notation $\gamma_k = \gamma_k^{(N)}$. 

Theorem 3.8. Fix $m \in \mathbb{N}$, let $q \in \mathbb{R}^N$ be a unit vector with stable support, and let $t$ be the time defined in (3.9). There exist constants $c_1 > 0$, $c_2 = c_2(m) > 0$, and $C = C(m, |\text{supp } q|) > 0$ such that, if $c < c_2$, then

$$
\max_{\xi : N(\xi) = m} F_t(\xi) - E \left[ \prod_{k=1}^{N} \left( \frac{\text{Im } \langle q, R(t, \gamma_k + i \eta)q \rangle}{\text{Im } m_\alpha(\gamma_k + i \eta)} \right)^{\xi_k} \right] \leq CN^{-c_2}, \quad (3.18)
$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$.

3.4. Scaling limit. The next theorem will be proven in Section 7 and establishes the scaling limit of the quantity compared to $F_t$ on the left side of (3.18), completing step 3 of the above outline. Here, we recall $U_* (E)$ from Definition 2.5.

Theorem 3.9. There exist constants $c_1, c_2 > 0$ such that the following holds. Fix an integer $n > 0$ (independently of $N$) and index sequences $\{k_j(N)\}_{1 \leq j \leq n}$ such that for every $N$, $\{k_j(N)\}_{1 \leq j \leq n}$ are distinct integers in $[1, N]$ and $|k_i - k_j| < N^{1/2}$ for each $j \in [2, n]$. Let $q = (q_1, \ldots, q_N) \in \mathbb{R}^N$ be a unit vector with stable support; let $t$ be as in (3.9); and assume that $\lim_{N \to \infty} \gamma_{k_1} = E$, for some $E \in [-c_1, c_1]$, and $c < c_2$. Then the vector

$$
\left( \frac{\text{Im } \langle q, R(t, \gamma_{k_j} + i \eta)q \rangle}{\text{Im } m_\alpha(\gamma_{k_j} + i \eta)} \right)_{1 \leq j \leq n}
$$

converges in moments to

$$
(1, 1, \ldots, 1) \cdot \sum_{i \in \text{supp } q} q_i^2 U_* (E),
$$

where the random variables $U_* (E)$ are independent and identically distributed with law $U_* (E)$.

3.5. Proofs. In this section we establish Theorem 3.8, Theorem 3.9, and Corollary 2.4.

Proof of Theorem 3.8. By symmetry, we may suppose $i = 1$ in the theorem statement. Recalling $t$ from (3.9) and applying Theorem 3.7 with $q = e_1 = (1, 0, 0, \ldots, 0)$ gives

$$
\lim_{N \to \infty} \left| E \left[ P \left( \left( N \langle \mathbf{u}_{k_j} (H), e_1 \rangle \right)^2 \right)_{1 \leq j \leq n} \right] - E \left[ P \left( \left( N \langle \mathbf{u}_{k_j} (X_t), e_1 \rangle \right)^2 \right)_{1 \leq j \leq n} \right] \right| = 0, \quad (3.21)
$$

Next, Theorem 3.8 yields

$$
\lim_{N \to \infty} \left| E \left[ P \left( \left( N \langle \mathbf{u}_{k_j} (X_t), e_1 \rangle \right)^2 \right)_{1 \leq j \leq n} \right] - E \left[ P \left( \left( N \langle \mathbf{u}_{k_j} (X_t), e_1 \rangle \right)^2 \right)_{1 \leq j \leq n} \right] \right| = 0,
$$

where the $N_j$ are i.i.d. standard Gaussians that are independent from $\text{Im } R_{11}(t, \gamma_{k_j} + i \eta)$. Here we used (3.12) and the fact that $a(2j) = E [N_j^2]$ for a standard Gaussian $N_j$.

Now the theorem follows from (3.21), (3.22), and Theorem 3.9. \hfill \Box

Proof of Theorem 3.9. By symmetry, we may suppose that $i_j = j$ for each $j \in [1, n]$. Let

$$
v = (U_1 (E), U_2 (E), \ldots, U_n (E))
$$

be a vector of i.i.d. random variables with distribution $U_* (E)$, where $U_* (E)$ is as in Definition 2.4.
For any vector $q$ with stable support such that $q_i = 0$ for $i \notin [1, n]$, let $w \in \mathbb{R}^N$ denote the vector $w = (q_1^2, q_2^2, \ldots, q_N^2)$. Fix $m \in \mathbb{N}$, recall $a(2m) = (2m - 1)!!$ from (3.12), abbreviate $u_k = u_k(H)$, and consider the polynomial

$$Q(q_1, \ldots, q_n) = \mathbb{E} \left[ (N(q, u_k)^2)^m \right] - a(2m)\mathbb{E} \left[ (w, v)^m \right].$$

Then together Theorem 3.7, Theorem 3.8, and (the $n = 1$ case of) Theorem 3.9 imply for any unit vector $q \in \mathbb{R}^N$ with $\text{supp} q \subseteq \{1, 2, \ldots, n\}$ that

$$\lim_{N \to \infty} Q(q_1, \ldots, q_n) = 0.$$ 

(3.25)

Here we recalled (3.12) and the fact that $a(2j) = \mathbb{E} [\mathcal{N}^2]_j$ for a standard Gaussian $\mathcal{N}$. Now observe that $Q$ is a polynomial of degree $2m$ in the $q_i$, that is, there exists coefficients $B_d \in \mathbb{R}$ such that

$$Q(q_1, q_2, \ldots, q_n) = \sum_{|d| = 2m} B_d \prod_{j=1}^{n} q_{j}^{d_j},$$

where $d = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}_0^n$ is summed over all $n$-tuples of nonnegative integers with $|d| = \sum_{j=1}^{n} d_j = 2m$. Thus, since (3.25) holds for all $(q_1, q_2, \ldots, q_n)$ with $\sum_{j=1}^{n} q_j^2 = 1$, we have

$$\lim_{N \to \infty} \max_{|d| = 2m} |B_d| = 0,$$

where again $d$ ranges over all $n$-tuples of nonnegative integers summing to $2m$. In particular, fixing some $n$-tuple $(m_1, m_2, \ldots, m_n)$ of nonnegative integers summing to $m$ and taking $d = (2m_1, 2m_2, \ldots, 2m_n)$ gives

$$\frac{(2m)!}{\prod_{j=1}^{m} (2m_j)!} \lim_{N \to \infty} \mathbb{E} \left[ \prod_{j=1}^{m} (N u_k(j)^2)^{m_j} \right] = \frac{m!(2m - 1)!!}{\prod_{j=1}^{m} m_j} \mathbb{E} \left[ \prod_{j=1}^{m} v(j)^{m_j} \right],$$

(3.28)

which implies that

$$\lim_{N \to \infty} \mathbb{E} \left[ \prod_{j=1}^{m} (N u_k(j)^2)^{m_j} \right] = \mathbb{E} \left[ \prod_{j=1}^{m} a(2m_j) v(j)^{m_j} \right].$$

(3.29)

This yields the desired conclusion, since (3.29) holds for all $(m_1, m_2, \ldots, m_n) \in \mathbb{Z}_0^n$ and $\mathbb{E} [\mathcal{N}^2] = a(2j)$, for any integer $j \geq 0$, if $\mathcal{N}$ is a standard Gaussian random variable.

**Proof of Corollary 2.9.** By [28] Theorem 1.6(ii],

$$q_0(0) = \frac{1}{\pi} \Gamma \left( 1 + \frac{2}{\alpha} \right) \left( \frac{\Gamma \left( 1 - \frac{\alpha}{2} \right)}{\Gamma \left( 1 + \frac{\alpha}{2} \right)} \right)^{1/\alpha}. \quad (3.30)$$

By [28] Lemma 4.3(ii], $q_\ast(0)$ has the same law as $\Upsilon^{-1}$, where $\Upsilon$ is a one-sided $\frac{\alpha}{2}$-stable law with Laplace transform

$$\mathbb{E} \left[ \exp(-t \Upsilon) \right] = \exp \left( -t^{\alpha/2} \left( \frac{\Gamma \left( 1 + \frac{\alpha}{2} \right)}{\Gamma \left( 1 - \frac{\alpha}{2} \right)} \right)^{1/2} \right), \text{ for } t \geq 0. \quad (3.31)$$
Since a \((1, 1)\) \(\frac{\alpha}{2}\)-stable law \(S\) has Laplace transform \(\exp(-t^{\alpha/2})\), \(Y\) has the same law as
\[
\left( \frac{\Gamma\left(1 + \frac{\alpha}{2}\right)}{\Gamma\left(1 - \frac{\alpha}{2}\right)} \right)^{1/\alpha} S,
\]
so the conclusion follows from Theorem 2.7, (3.30), and the fact (see Definition 2.5) that \(U \ast (E) = \left(\pi \varrho(E)\right)^{-1} R \ast (E)\).

4. Preliminary results

In this section we collect several miscellaneous known results that will be used throughout the paper. After recalling general estimates and identities on resolvent matrices in Section 4.1, we state several facts about the density \(\varrho_\alpha\) in Section 4.2. In Section 4.3, we recall several results about the removed model \(X_s\). Finally, in Section 4.4 we recall properties of a certain matrix interpolating between \(X_0\) and \(H\).

4.1. Resolvent identities and estimates. For any invertible \(K, M \in \text{Mat}_{N \times N}\), we have
\[
K^{-1} - M^{-1} = K^{-1}(M - K)M.
\]
Next, assume \(z = E + i\eta \in \mathbb{H}\) and \(K = \{K_{ij}\} = (M - z)^{-1}\). Then, we have the bound
\[
\max_{1 \leq i,j \leq N} |K_{ij}| \leq \frac{1}{\eta},
\]
and the Ward identity
\[
\sum_{j=1}^{N} |K_{ij}|^2 = \frac{\text{Im} K_{ii}}{\eta}.
\]

4.2. The density \(\varrho_\alpha\). The following properties of the density \(\varrho_\alpha\) are proved in Appendix A; here, we recall the \(\gamma_i\) from 2.7.

**Lemma 4.1.** There exists a (deterministic) countable set \(A \subset (0, 2)\) with no accumulation points in \((0, 2)\) and constants \(C, c > 0\) such that the following statements hold for \(\alpha \in (0, 2) \setminus A\).

1. For real numbers \(E_1, E_2 \in [-c, c]\),
\[
|\varrho_\alpha(E_1) - \varrho_\alpha(E_2)| \leq C|E_1 - E_2|, \quad c \leq \varrho_\alpha(E_1) \leq C.
\]
2. For real numbers \(E_1, E_2 \in [-c, c]\), and any \(\eta > 0\), we have
\[
|\text{Im} m_\alpha(E_1 + i\eta) - \text{Im} m_\alpha(E_2 + i\eta)| \leq C|E_1 - E_2| + C\eta.
\]
3. For real numbers \(|E| < c\) and \(\eta \in (0, c]\), and any integer \(j \in \left[\left(1/2 - c\right)N, (1/2 + c)N\right]\),
\[
c \leq |\text{Im} m_\alpha(E + i\eta)| \leq C, \quad c \leq |\text{Im} m_\alpha(\gamma_j + i\eta)| \leq C.
\]
4.3. **Removed model.** In this section we recall several results concerning the resolvent $R(s, z)$ and Stieltjes transform $m_N(s, z)$ of $X_\alpha$ (recall (3.8) and (3.15)). In what follows, we recall the $i$-th eigenvalue of $X_\alpha$ is denoted by $\lambda_i(s)$ and its associated unit eigenvector is denoted by $u_i(s)$. For any constants $C, \delta > 0$, we define the two spectral domains

$$\mathcal{D}_{C, \delta} = \left\{ z = E + i\eta: |E| \leq \frac{1}{C}, N^{-1+\delta} \leq \eta \leq \frac{1}{C} \right\},$$

$$\widetilde{\mathcal{D}}_{C, \delta} = \left\{ z = E + i\eta: |E| \leq \frac{1}{C}, N^{-1/2+\delta} \leq \eta \leq \frac{1}{C} \right\}.$$  \hfill (4.7)

We also recall the free convolution of $X$ with the semicircle law is defined to be the probability measure on $\mathbb{R}$ whose Stieltjes transform satisfies the equation

$$m_{fc}(s, z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i(0) - z - sm_{fc}(s, z)}.$$ \hfill (4.9)

Basic facts about the free convolution, including its existence and uniqueness, may be found in [25]. It has a density, $g_{fc}(s, x) \, dx$, and its classical eigenvalue locations are defined for $1 \leq i \leq N$ by

$$\gamma_i(s) = \inf \left\{ y \in \mathbb{R}: \int_{-\infty}^{y} g_{fc}(s, x) \, dx \geq \frac{i}{N} \right\}.$$ \hfill (4.10)

These are random variables which depend on the initial data $(\lambda_i(0))_{i=1}^{N}$ determined by $X_0$.

The following intermediate local law for $R(s, z)$ (on scale $\eta \gg N^{-1/2+\delta}$) was essentially shown as [3 Theorem 3.5].

**Lemma 4.2** ([3 Theorem 3.5]). There exists a (deterministic) countable set $\mathcal{A} \subset (0, 2)$ with no accumulation points in $(0, 2)$ such that the following holds for $\alpha \in (0, 2) \setminus \mathcal{A}$. For any fixed real number $\delta > 0$ with $\delta < \max \left\{ \frac{1}{2(1+\alpha)(2-\alpha)}, \frac{1}{3} \right\}$, there exists a constant $C = C(\delta) > 0$ such that for $s \in [0, N^{-\delta}]$, we have with overwhelming probability that

$$\sup_{z \in \mathcal{D}_{C, \delta}} |m_N(s, z) - m_\alpha(z)| < CN^{-\alpha\delta/8}, \quad \sup_{z \in \mathcal{D}_{C, \delta}} \max_{1 \leq j \leq N} |R_{jj}(s, z)| < (\log N)^C,$$ \hfill (4.11)

where we recall $\mathcal{D}_{C, \delta}$ from (4.7).

In fact, [3 Theorem 3.5] was only stated in the case $s = 0$, but it is quickly verified that the same proof applies for arbitrary $s \in [0, N^{-\delta}]$, especially since $H + s^{1/2}W$ satisfies the conditions in Definition 2.1 for $s \in [0, N^{-\delta}]$ if $H$ does.

From now on, we always assume $\alpha \in (0, 2) \setminus \mathcal{A}$, where $\mathcal{A}$ is the set from Lemma 4.2 even when this is not noted explicitly. The next lemma provides more local estimates on $R(s, z)$ and $X_\alpha$ (on scales around $N^{-1}$), if $N^{-1/2} \ll s \ll 1$; they are consequences of Lemma 4.2 using results of [63]. Specifically, the first bound in (4.12) follows from [63] Theorem 3.3 and the second follows from the first and the first estimate in (4.11); (4.13) follows from [63] Theorem 3.5; and (4.14) follows from [63] Theorem 3.6. The hypotheses of these statements from [63] are all verified by the first bound in (4.11). The final estimate is an immediate consequence of (4.12), (4.13), and (4.14).

In the following, we recall the $\gamma_i(s)$ defined in (4.10).

**Lemma 4.3** ([63]). There exists a constant $K > 0$ such that the following holds for any real numbers $r, \delta > 0$. 


(1) Set $\mathcal{D} = D_{K, \delta}$ and $\tilde{\mathcal{D}} = \tilde{D}_{K, \delta}$, where we recall the definitions (4.7) and (4.8), respectively. With overwhelming probability, we have that

$$\sup_{s \in \mathcal{D}[N^{-1/2+\delta}, N^{-\delta}]} \sup_{z \in \mathcal{D}} \left| m_N(s, z) - m_{tc}(s, z) \right| < \frac{N^\delta}{N^{\eta}},$$

(4.12)

$$\sup_{s \in \mathcal{D}[N^{-1/2+\delta}, N^{-\delta}]} \sup_{z \in \mathcal{D}} \left| m_\alpha(z) - m_{tc}(s, z) \right| < N^{-\alpha\delta/16}.$$  

(4.13)

(2) With overwhelming probability, we have that

$$\sup_{s \in \mathcal{D}[N^{-1/2+\delta}, N^{-\delta}]} \left| \lambda_i(s) - \gamma_i(s) \right| \leq N^{-1+\delta}. $$

(4.14)

(3) For any $\varepsilon \in (0, 1)$ and $s \in [N^{-1/2+\delta}, N^{-\delta}]$, we have for sufficiently large $N$ that

$$P\left( \left| \lambda_i(s) - \lambda_{i+1}(s) \right| \leq \frac{\varepsilon}{N} \right) \leq N^\delta \varepsilon^{2-r}. $$

(4.15)

(4) For $E_1, E_2 \in [-K^{-1}, K^{-1}]$ and $\eta \in [N^{-1/2+\delta}, N^{-\delta}]$, we have

$$\left| \text{Im} \ m_N(E_1 + i\eta) - \text{Im} \ m_N(E_2 + i\eta) \right| \leq C|E_1 - E_2| + C\eta + C\eta N^{-\alpha\delta/16}$$

(4.16)

with overwhelming probability.

The following lemma provides resolvent and delocalization estimates for $X_s$. The first estimate in (4.17) below follows from [5, Proposition 3.9], whose hypotheses are verified by the second bound in (4.11). We omit the proof of the second since, given the first, it follows by standard arguments (for example, see the proof of [22, Theorem 2.10]).

**Lemma 4.4 ([5, Proposition 3.9]).** There exists a constant $K > 0$ such that the following holds. Fix real numbers $\delta > 0$ and $s \in [N^{-1/2+\delta}, N^{-\delta}]$, and a unit vector $q \in \mathbb{R}^N$ with stable support. For each index $i \in [1, N]$ such that $\gamma_i(s) < K^{-1}$, we have with overwhelming probability that

$$\sup_{z \in D} \sup_{1 \leq j, k \leq N} \left| R_{jk}(s, z) \right| < N^\delta, \quad \langle u_i(s), q \rangle^2 \leq N^{-1+\delta}. $$

(4.17)

### 4.4. Interpolating matrix.

Recalling $t$ from (39), define the interpolating matrix

$$H^\gamma = \{ H_{ij}^\gamma \} = \gamma A + X + (1 - \gamma^2)^{1/2} \frac{1}{t^{1/2}} W,$$

(4.18)

where $W$ is an independent $N \times N$ GOE matrix. Namely, it is an $N \times N$ real symmetric random matrix $W = \{ w_{ij} \}$, whose upper triangular entries $w_{ij}$ are mutually independent Gaussian random variables with variances $(1 + \mathbb{I}_{i=j})N^{-1}$.

The following lemma estimates the entries of the resolvent matrix $G^\gamma = \{ G_{ij}^\gamma(z) \} = (H^\gamma - z)^{-1}$ and provides complete eigenvector delocalization for $H^\gamma$. The first bound in (4.19) below was obtained as [5, Theorem 3.16]; given this, the second bound there follows by standard arguments (again, see the proof of [22, Theorem 2.10]).

**Lemma 4.5 ([5, Theorem 3.16]).** There exists a constant $K > 0$ such that the following holds. Fix real numbers $\delta > 0$ and $\gamma \in [0, 1]$, and abbreviate $D = D_{K, \delta}$ (recall (4.7)). Let $u_i(H^\gamma)$ be a unit eigenvector of $H^\gamma$ such that the corresponding eigenvalue $\lambda_i(H^\gamma)$ satisfies $|\lambda_i(H^\gamma)| \leq K^{-1}$. Then, with overwhelming probability we have the bounds

$$\sup_{0 \leq \gamma \leq 1} \sup_{z \in D} \sup_{1 \leq j, k \leq N} \left| G_{jk}^\gamma(z) \right| < N^\delta; \quad \| u_i(H^\gamma) \|_\infty \leq N^{-1/2+\delta}.$$ 

(4.19)
In view of the identity
\[
\eta \sum_{j=1}^{N} \left( |\lambda_j(M) - E|^2 + \eta^2 \right)^{-1} = N^{-1} \text{Im Tr}(M - z)^{-1},
\] (4.20)
which holds for any \( N \times N \) matrix \( M \) and complex number \( z = E + i \eta \in \mathbb{H} \), Lemma 4.3 and Lemma 4.5 together quickly imply the following lemma that bounds the number of eigenvalues of \( H \) or \( X_s \) in a given interval.

**Lemma 4.6.** For any real number \( \delta > 0 \), there exist constants \( K > 0 \) and \( C = C(\delta) > 0 \) such that the following holds. For any interval \( I \subseteq [-K^{-1}, K^{-1}] \) of length \( |I| \geq N^{-1+\delta} \), we have with overwhelming probability that
\[
\sup_{\gamma \in [0,1]} \left| \left\{ i : \lambda_i(H^\gamma) \in I \right\} \right| \leq C|I|N^{1+\delta};
\]
\[
\sup_{s \in \{N^{-1/2+\delta},N^{-\delta}\}} \left| \left\{ i : \lambda_i(X_s) \in I \right\} \right| \leq C|I|N.
\] (4.21)

The following result states that the \( i \)-th eigenvalue of \( H^\gamma \) and \( X_s \) is close to 0 if \( i \) is close to \( \frac{N}{2} \). Its proof will be given in Appendix A.

**Lemma 4.7.** For each real number \( c_1 > 0 \), there exists a constant \( c_2 > 0 \) such that the eigenvalues \( \lambda_i(H^\gamma) \) of \( H^\gamma \) and \( \lambda_i(X_s) \) of \( X_s \) satisfy
\[
\sup_{\gamma \in [0,1]} |\lambda_i(H^\gamma)| < c_1; \quad \sup_{s \in [0,1]} |\lambda_i(X_s)| < c_1; \quad \sup_{s \in [0,1]} |\gamma_i(s)| < c_1,
\] (4.22)
for each \( i \in \left[ (1/2 - c_2)N, (1/2 + c_2)N \right] \), with overwhelming probability.

In Appendix A we use Lemma 4.7 and Lemma 4.3 to deduce the following rigidity statements comparing the classical locations \( \hat{\gamma}_i(s) \) to the \( \gamma_i \), and the \( \hat{\gamma}_i(s) \) to the \( \gamma_i(s) \) (recall (2.7), (4.10), and (3.17)).

**Lemma 4.8.** Fix \( \delta > 0 \). There exist constants \( C, c_1 > 0 \) and \( c = c(\delta) > 0 \) such that for each \( i \in \left[ (1/2 - c_1)N, (1/2 + c_1)N \right] \), we deterministically have the bound
\[
\sup_{s \in \{N^{-1/2+\delta},N^{-\delta}\}} \left| \hat{\gamma}_i(s) - \gamma_i \right| \leq CN^{-c},
\] (4.23)
and with overwhelming probability the bound
\[
\sup_{s \in \{N^{-1/2+\delta},N^{-\delta}\}} \left| \hat{\gamma}_i(s) - \gamma_i(s) \right| \leq CN^{-1/2+\delta}.
\] (4.24)

5. Comparison

This section establishes Theorem 3.7, which compares the eigenvector statistics of \( X_s \) to those of \( H \). Section 5.1 establishes this result assuming a general comparison estimate, certain derivative bounds, and a level repulsion estimate for \( H^\gamma \). We then prove the comparison estimate in Section 5.2, the necessary derivative bounds in Section 5.3 and the level repulsion estimate in Section 5.4.
5.1. Proof of Theorem 3.7 | In this section we establish Theorem 3.7 assuming Lemma 5.1, Lemma 5.3, Lemma 5.4, and Lemma 5.5 below. In what follows, for any $\kappa \in [0, 1]$, $M = \{m_{ij}\} \in \text{Mat}_{N \times N}$, and $a, b \in [1, N]$, we define $\Theta_{\kappa}^{(a, b)} M \in \text{Mat}_{N \times N}$ as follows. Recalling $\rho$ from (3.2), if $|m_{ab}| = |m_{ba}| \geq N^{-\rho}$, then set $\Theta_{\kappa}^{(a, b)} M = M$. Otherwise, if $|m_{ab}| = |m_{ba}| < N^{-\rho}$, set $\Theta_{\kappa}^{(a, b)} M$ to be the $N \times N$ matrix whose $(i, j)$ entry is equal to $m_{ij}$ if $(i, j) \notin \{(a, b), (b, a)\}$ and is equal to $\kappa m_{ab} = \kappa m_{ba}$ otherwise. Moreover, for any differentiable function $F : \text{Mat}_{N \times N} \rightarrow \mathbb{C}$ and indices $a, b \in [1, N]$, we define $\partial_{ab} F$ to be the derivative of $F$ with respect to $m_{ab}$.

We first state the following comparison theorem between functions of $H^0 = X_i$ and $H^\gamma$ (recall (3.1)), which will be established in Section 5.2 below.

**Lemma 5.1.** There exists a constant $c > 0$ such that the following holds. Let $F : \text{Mat}_{N \times N} \rightarrow \mathbb{C}$ denote a smooth function, and suppose $K, L > 1$ are such that

$$
\max_{0 \leq j \leq 4} \sup_{0 \leq \gamma \leq 1} \max_{1 \leq a, b \leq N} \sup_{0 \leq \kappa \leq 1} \left| \partial_{ab}^{(j)} F(\Theta_{\kappa}^{(a, b)} H^\gamma) \right| \leq K
$$

holds with overwhelming probability, and

$$
\max_{0 \leq j \leq 4} \sup_{0 \leq \gamma \leq 1} \max_{1 \leq a, b \leq N} \sup_{0 \leq \kappa \leq 1} \left| \partial_{ab}^{(j)} F(\Theta_{\kappa}^{(a, b)} H^\gamma) \right| \leq L
$$

holds deterministically. Then, for any $D > 0$, there exists a constant $C = C(D) > 0$ such that

$$
\sup_{0 \leq \gamma \leq 1} \left| F(H^\gamma) - F(H^0) \right| \leq KN^{-\gamma} + CLN^{-D}.
$$

Next we require the following function, originally introduced in [76] Section 3.2], that measures how close eigenvalues of some matrix $A$ are to a given eigenvalue.

**Definition 5.2.** Let $A \in \text{Mat}_{N \times N}$. If $1 \leq i \leq N$ is such that $\lambda_i(A)$ is an eigenvalue of a matrix $A$ with multiplicity one, we define

$$
Q_i(A) = \frac{1}{N^2} \sum_{1 \leq j \leq N \neq i} \left| \lambda_j(A) - \lambda_i(A) \right|^{-2}.
$$

To deal with the case of multiplicity greater than one, we introduce a cutoff. For any $M > 0$, we fix a smooth function $f_M : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ such that there exists a constant $C > 0$ (independent of $M$ and $N$) satisfying the following two properties.

1. For any $x \in \mathbb{R}_{\geq 0}$, we have that $|f_M'(x)| + |f_M''(x)| + |f_M'''(x)| \leq C$.
2. If $x \in [0, M]$ then $|f_M(x) - x| \leq 1$, and if $x \geq M$, then $f_M(x) = M$.

The function $f_M(Q_i(A))$ is then well-defined and smooth on real symmetric matrices.

The following two lemmas control the derivatives of the $Q_i(H)$ and of the eigenvector entries of $H^\gamma$ with respect to the matrix entries of $H$; the first is an overwhelming probability bound, and the second is a deterministic bound. They will be established in Section 5.3 below.

**Lemma 5.3.** There exists a constant $c > 0$ such that the following holds. Fix real numbers $\gamma, \kappa \in [0, 1]$, a constant $\omega > 0$, and integers $i, a, b \in [1, N]$. Set $M = N^{2\omega}$, and assume that

$$
\left| \lambda_i(\Theta_{\kappa}^{(a, b)} H^\gamma) \right| < c, \quad \text{and} \quad Q_i(\Theta_{\kappa}^{(a, b)} H^\gamma) \leq M = N^{2\omega}
$$

(5.5)
both hold with overwhelming probability. Then,

$$\left| \partial_{ab}^{(k)} \left( Q_i \left( \Theta_\kappa^{(a,b)} H \right) \right) \right| \leq C N^{10k(\omega + \delta)}, \quad (5.6)$$

also holds with overwhelming probability, for any integer $0 \leq k \leq 4$.

Moreover, for any $q \in \mathbb{R}^N$, there exists a constant $C = C(\|\text{supp } q\|) > 0$ such that

$$\left| \partial_{ab}^{(k)} \left( \left\langle q, u_i \left( \Theta_\kappa^{(a,b)} H \right) \right\rangle^2 \right) \right| \leq C N^{-1 + 10k(\omega + \delta)}, \quad (5.7)$$

also holds with overwhelming probability, for any integer $0 \leq k \leq 4$.

**Lemma 5.4.** Fix real numbers $\gamma, \kappa \in [0,1]$, a constant $\omega > 0$, and integers $i,a,b \in [1,N]$; assume that $Q_i \left( \Theta_\kappa^{(a,b)} H \right) < N^{2\omega}$. Then, for any integers $k \in [0,4]$ and $1 \leq i \leq N$, we have the deterministic bounds

$$\left| \partial_{ab}^{(k)} \left( Q_i \left( \Theta_\kappa^{(a,b)} H \right) \right) \right| \leq C N^{10 + 6\omega}; \quad \left| \partial_{ab}^{(k)} \left( \left\langle q, u_i \left( \Theta_\kappa^{(a,b)} H \right) \right\rangle^2 \right) \right| \leq C N^{15 + 10\omega}. \quad (5.8)$$

Next we state a level repulsion estimate, which will be established in Section 5.4 below.

**Lemma 5.5.** There exist constants $c, \nu > 0$ such that, for any fixed index $i \in \left[ \left( 1/2 - c \right) N, \left( 1/2 + c \right) N \right]$ and real number $\gamma \in [0,1]$, we have that

$$\mathbb{P}(Q_i(H^\gamma) \geq N^\nu) \leq 2N^{-\nu/4}. \quad (5.9)$$

Given these statements, we now prove Theorem 3.7. The argument follows [38, Theorem 1.1].

**Proof of Theorem 3.7.** For brevity we consider just $n = 1$; the general case is no harder.

By Lemma 5.5 there exists some $\omega > 0$ such that, for each $\gamma \in \{0,1\}$,

$$\mathbb{P}(Q_i(H^\gamma) \geq N^\omega) \leq 2N^{-\omega/4}. \quad (5.10)$$

Denote the degree of $P$ by $m$, so that $P(x) \leq C(x^m + 1)$ for $x \geq 0$. Delocalization for $H^\gamma$, \[19\], implies that $N^\gamma \left( q, u_i \right)^2 \leq N^\delta$ with overwhelming probability for each $\delta > 0$ and $\gamma \in [0,1]$, if $N$ is sufficiently large. Therefore,

$$\mathbb{E} \left[ P \left( N^\gamma \left( q, u_i \right)^2 \right)^2 \right] \leq C N^{2m\delta}. \quad (5.11)$$
Now set $M = N^{2\omega}$, and let $g = g_M$ be a smooth function with uniformly bounded derivatives such that $0 \leq g(x) \leq 1$ for each $x \in \mathbb{R}_{>0}$; $g(x) = 1$ for $x \leq M$; and $g(x) = 0$ for $x \geq 2M$. Then,

$$
\left| \mathbb{E} \left[ P \left( N \langle \mathbf{q}, \mathbf{u}_i(\mathbf{H}^1) \rangle^2 \right) \right] - \mathbb{E} \left[ P \left( N \langle \mathbf{q}, \mathbf{u}_i(\mathbf{H}^0) \rangle^2 \right) \right] \right| \leq \left| \mathbb{E} \left[ P \left( N \langle \mathbf{q}, \mathbf{u}_i(\mathbf{H}^1) \rangle^2 \right) g(Q_i(\mathbf{H}^1)) \right] - \mathbb{E} \left[ P \left( N \langle \mathbf{q}, \mathbf{u}_i(\mathbf{H}^0) \rangle^2 \right) g(Q_i(\mathbf{H}^0)) \right] \right| + \mathbb{E} \left[ P \left( N \langle \mathbf{q}, \mathbf{u}_i(\mathbf{H}^1) \rangle^2 \right)^{1/2} P(Q_i(\mathbf{H}^1)) \right] \leq \mathbb{E} \left[ P \left( N \langle \mathbf{q}, \mathbf{u}_i(\mathbf{H}^0) \rangle^2 \right) g(Q_i(\mathbf{H}^0)) \right] + CN^{-\omega/4 + m\delta},
$$

(5.15)

where in the last estimate we applied (5.10) and $\mathcal{M}_i$. Now let us define the function $h : \text{Mat}_{N \times N} \rightarrow \mathbb{R}$ by setting

$$
h(A) = h_i(A) = P \left( N \langle \mathbf{q}, \mathbf{u}_i(A) \rangle^2 \right) g(Q_i(A)),
$$

(5.16)

for any $A \in \text{Mat}_{N \times N}$. By Lemma 5.3 Lemma 5.4 a union bound over $1 \leq i, a, b \leq N$ and $\gamma$ and $\kappa$ in an $N^{-3\omega}$-net of $[0, 1]$; and the fact that $h(A) = 0$ if $Q_i(A) \geq 2M$, we have that $h$ deterministically satisfies

$$
\sup_{0 \leq k \leq 4} \sup_{\gamma \in [0, 1]} \sup_{1 \leq a, b \leq N} \| \partial_{ab}^{(k)} h(\Theta^\kappa_{(a,b)\mathbf{H}^0}) \| \leq CN^{15 + 15m\omega},
$$

(5.17)

and with overwhelming probability satisfies

$$
\sup_{0 \leq k \leq 4} \sup_{\gamma \in [0, 1]} \sup_{1 \leq a, b \leq N} \| \partial_{ab}^{(k)} h(\Theta^\kappa_{(a,b)\mathbf{H}^0}) \| \leq CN^{20m(\omega + \delta)}.
$$

(5.18)

Therefore, upon setting $\omega$ and $\delta$ sufficiently small, Lemma 5.1 implies $|\mathbb{E}[h(\mathbf{H}^1)] - \mathbb{E}[h(\mathbf{H}^0)]| < CN^{-\epsilon}$. Inserting this into (5.15) yields

$$
\left| \mathbb{E} \left[ P \left( N \langle \mathbf{q}, \mathbf{u}_i(1) \rangle^2 \right) \right] - \mathbb{E} \left[ P \left( N \langle \mathbf{q}, \mathbf{u}_i(0) \rangle^2 \right) \right] \right| \leq CN^{-\epsilon} + CN^{-\omega/4 + m\delta}.
$$

(5.19)

The lemma follows from further imposing that $5m\delta < \omega$. \hfill \Box

5.2. Proof of Lemma 5.1. In this section we establish Lemma 5.1.

Proof of Lemma 5.1. Observe (by (4.1), for instance) that

$$
\partial_{ij} \mathbb{E}[F(\mathbf{H}^\gamma)] = \sum_{1 \leq i, j \leq N} \mathbb{E} \left[ \partial_{ij} F(\mathbf{H}^\gamma) \left( A_{ij} - \frac{\gamma^{1/2}}{(1 - \gamma^2)^{1/2}} \psi_{ij} \right) \right].
$$

(5.20)

Now, we condition on the label $\Psi$ of $\mathbf{H}$ (recall Definition 3.5) and denote the associated conditional expectation by $\mathbb{E}_\Psi$. We first consider the case $\psi_{ij} = 1$. This implies $A_{ij} = B_{ij} = 0$, and
Gaussian integration by parts (see for instance [73, Appendix A.4]) yields
\[
\mathbb{E}\Phi \left[ \partial_{ij} F(\mathbf{H}^\gamma) \left( \frac{\gamma t^{1/2}}{(1 - \gamma^2)^{1/2}} w_{ij} \right) \right] = \frac{t^2}{N} \mathbb{E}\Phi \left[ \partial_{ij} F(\mathbf{H}^\gamma) \right], \quad \text{whenever } \psi_{ij} = 1. \tag{5.21}
\]

Hoeffding’s inequality applied to the Bernoulli random variable $\psi_{ij}$, whose distribution was defined in (3.3), implies that there are likely at most $CN^{1+\alpha}$ pairs $(i, j)$ such that $\psi_{ij} = 1$. Specifically,
\[
\mathbb{P} \left[ \{(i, j) \in [1, N] \times [1, N] : \psi_{ij} = 1\} \right] < CN^{1+\alpha} \geq 1 - C \exp \left( -N^{\alpha} \right). \tag{5.22}
\]

By (5.21), the contribution of (5.21) over the complement of the event described in (5.21) is bounded by $CLN^{-D}$, for some constant $C = C(D) > 0$. This, together with (5.1), (5.21), (5.22), (5.10), and (5.24) imply that the sum of (5.21) over all $(i, j)$ such that $\psi_{ij} = 1$ or $i = j$ is at most
\[
CKtN^{-1}N^{\alpha+1} + CLN^{-D} \leq CKN^{\alpha-2(2-\alpha)\nu} + CLN^{-D} < KN^{-c} + CLN^{-D}, \tag{5.23}
\]
for some constants $c > 0$ (only dependent on the fixed parameters $\alpha$, $\rho$, and $\nu$) and $C = C(D) > 0$.

We next consider the case when $\psi_{ij} = 0$ and $i \neq j$. Then, $A_{ij} = a_{ij}(1 - \chi_{ij})$ and $B_{ij} = b_{ij}\chi_{ij}$; abbreviate $a_{ij} = a$, $b_{ij} = b$, $\chi_{ij} = \chi$, and $w_{ij} = w$. Set
\[
h = \gamma(1 - \chi) + \chi b + (1 - \gamma^2)^{1/2} t^{1/2} w. \tag{5.24}
\]

Fix $(i, j) \in [1, N]^2$ such that $\psi_{ij} = 0$, abbreviate $F^{(k)} = \partial^{(k)} F$, and abbreviate $S = \Theta_{ij}^{(k)} H$. Then a Taylor expansion yields
\[
F^{(k)}(\mathbf{H}^\gamma) = F^{(k)}(\mathbf{S}) + h F^{(k)}(\mathbf{S}) + h^2 F^{(3)}(\mathbf{S}) + h^3 F^{(4)}(\Theta_{ij}^{(k)} \mathbf{H}^\gamma), \tag{5.25}
\]
for some $k \in [0, 1]$. Hence, the $(i, j)$ term in the sum on the right side of (5.20) is equal to
\[
\mathbb{E}\Phi \left[ \left( (1 - \chi) a - \frac{\gamma t^{1/2}}{(1 - \gamma^2)^{1/2}} w \right) \left( F^{(k)}(\mathbf{S}) + h F^{(k)}(\mathbf{S}) + h^2 F^{(3)}(\mathbf{S}) + h^3 F^{(4)}(\Theta_{ij}^{(k)} \mathbf{H}^\gamma) \right) \right]. \tag{5.26}
\]

Using the mutual independence between $\mathbf{S}$, $a$, $b$, $\chi$, and $w$, and the fact that $a$, $b$, and $w$ are all symmetric, we conclude that (5.20) is equal to
\[
\mathbb{E}\Phi \left[ \left( (1 - \chi) a - \frac{\gamma t^{1/2}}{(1 - \gamma^2)^{1/2}} w \right) h F^{(k)}(\mathbf{S}) \right] + \mathbb{E}\Phi \left[ \left( (1 - \chi) a - \frac{\gamma t^{1/2}}{(1 - \gamma^2)^{1/2}} w \right) h^3 F^{(4)}(\Theta_{ij}^{(k)} \mathbf{H}^\gamma) \right]. \tag{5.27}
\]

Again using the mutual independence between $\mathbf{S}$, $a$, $b$, $\chi$, and $w$; the fact that $a$, $b$, and $w$ are all symmetric; and (5.24), we find that the first term in (5.27) is
\[
\mathbb{E}\Phi \left[ F^{(k)}(\mathbf{S}) \left( (1 - \chi) a - \frac{\gamma t^{1/2}}{(1 - \gamma^2)^{1/2}} w \right) h \right] = \gamma \mathbb{E}[F^{(k)}(\mathbf{S})] \mathbb{E}[a^2(1 - \chi) - tw^2] = 0, \tag{5.28}
\]
where the final equality follows from the choice of $t$ in (3.9).

The second term in (5.27) is bounded above by
\[
C \mathbb{E}\Phi \left[ F^{(4)} \left( \Theta_{ij}^{(k)} \mathbf{H}^\gamma \right) \right] \left( (1 - \chi) a^4 + tw^4 + \chi tw^2 b^2 \right), \tag{5.29}
\]
on the complement of the event in (5.1), this expectation is bounded by $CLN^{-D-2}$, for some constant $C = C(D) > 0$. On this event, we use (5.10); the facts that $\mathbb{E}[w^2] \leq N^{-1}$ and $\mathbb{E}[w^4] \leq N^{-2}$; and the estimates (which can be quickly deduced from Definition 5.4 see [5, Section 3.3.2] for details)
\[
\mathbb{E}\Phi \left[ (1 - \chi) a^4 \right] \leq CN^{\alpha-4} - 1, \quad \mathbb{E}\Phi \left[ \chi b^2 \right] \leq CN^{\alpha-2} - 1, \tag{5.30}
\]
to bound it by

\[ CK \left( N^{\nu(\alpha-4)} - 1 + N^{2\nu(\alpha-2)} - 2 + N^{(\rho+\nu)(\alpha-2)} - 2 \right) \leq CK N^{-2-c}, \]  

(5.31)

for some constant \( c > 0 \) (only dependent on the fixed parameters \( \alpha, \nu, \) and \( \rho \)), where we have used (3.2) in the last inequality. So, the sum of (5.29) over all \((i, j) \in [1,N]^2\) such that \( i \neq j \) and \( \psi_{ij} = 0 \) is at most

\[ KN^{-c} + CLN^{-D}. \]  

(5.32)

Now the lemma follows from the fact that the contribution to (5.29) from all terms corresponding to \((i, j)\) with \( i = j \) or \( \psi_{ij} = 1 \) is bounded by (5.23) and the fact that the contribution of all terms coming from \((i, j)\) with \( i \neq j \) and \( \psi_{ij} = 0 \) is bounded by (5.32).

5.3. Proof of Lemma 5.3 and Lemma 5.4

In this section we prove Lemma 5.3 and Lemma 5.4. We begin with the following estimate on the resolvent entries of \((3.2)\) in the last inequality. So, the sum of (5.29) over all \((i, j) \in [1,N]^2\) such that \( i \neq j \) and \( \psi_{ij} = 0 \) is at most

\[ KN^{-c} + CLN^{-D}. \]  

(5.32)

Proof of Lemma 5.3 and Lemma 5.4. In this section we prove Lemma 5.3 and Lemma 5.4. We begin with the following estimate on the resolvent entries of \( \Theta_k^{(a,b)} H^\gamma \). In the below, we recall \( D_{C,\delta} \) from (4.7).

**Lemma 5.6.** There exists a constant \( K > 0 \) such that following holds. For any \( \delta > 0 \), the bound

\[ \sup_{0 < \kappa < 1} \max_{1 \leq a,b \leq N} \sup_{0 < \gamma \leq 1} \max_{i,j \in D_{K,\delta}} \sup_{z \in D_{K,\delta}} \left| \left( \Theta_k^{(a,b)} H^\gamma - z \right)^{-1} \right|_{ij} < N^\delta \]  

(5.33)

holds with overwhelming probability. Moreover, for each \( c_1 > 0 \), there exists some \( c_2 > 0 \) such that

\[ \sup_{0 < \kappa < 1} \max_{1 \leq a,b \leq N} \sup_{0 < \gamma \leq 1} \left\| u_i \left( \Theta_k^{(a,b)} H^\gamma \right) \right\|_\infty < N^{\delta - 1/2}; \]

\[ \sup_{0 < \kappa < 1} \max_{1 \leq a,b \leq N} \sup_{0 < \gamma \leq 1} \left| \lambda_i \left( \Theta_k^{(a,b)} H^\gamma \right) \right| < c_1, \]  

(5.34)

both hold for each \((1/2 - c_2)N \leq i \leq (1/2 + c_2)N\) with overwhelming probability. Additionally, for any interval \( I \subset [\alpha, \beta] \) of length \(|I| \geq N^{-1+\delta}\),

\[ \sup_{0 \leq \kappa \leq 1} \max_{1 \leq a,b \leq N} \max_{0 < \gamma \leq 1} \left\{ \left| i \in [1,N] : \lambda_i \left( \Theta_k^{(a,b)} H^\gamma \right) \in I \right\} \leq C|I|N^{1+\delta}, \]  

(5.35)

holds with overwhelming probability.

**Proof.** The second bound in (5.34) follows from Lemma 4.7 and the Weyl interlacing inequality for eigenvalues of symmetric matrices. Furthermore, the proofs of the first bound in (5.34) and (5.35) given (5.33) follow from standard arguments (see for example the proofs of [22, Theorem 2.10] and [63, Lemma 7.4]). So, we only establish (5.33).

To that end, let \( K \) be as in Lemma 4.1 and fix indices \( a,b \in [1,N]; \) real numbers \( \kappa, \gamma \in [0,1]; \) and a complex number \( z \in D_{K,\delta}. \) Set \( E = \Theta_k^{(a,b)} H^\gamma, V = (E-z)^{-1} = \{ V_{ij} \}, \) and \( \Delta = H^\gamma - E. \) We may assume throughout this proof that \( |h_{ab}| \leq N^{-c}, \) for otherwise \( E = H^\gamma, \) and the result follows from Lemma 4.3.

For any \( M \in \mathbb{N}, \) the resolvent identity (4.1) gives

\[ V - G^\gamma = \sum_{k=0}^{M} (G^\gamma \Delta)^k G^\gamma + (G^\gamma \Delta)^{M+1} V. \]  

(5.36)

Now select \( M \) in (5.36) such that \( M \rho > 10. \) Then (4.1); Lemma 4.5; the deterministic bound (4.2); and the fact that \( \Delta \) is supported on at most two entries, each of which is bounded by \( N^{-c}, \)
implies for sufficiently small $\delta > 0$ that
\[
\max_{1 \leq i,j \leq N} |V_{ij}| \leq \max_{1 \leq i,j \leq N} |G_{ij}^\gamma| + \sum_{k=0}^{M} 2^k N^{(k+1)\delta - k\rho} + 2^{M+1} N^{(M+1)\delta - \rho - 10\eta^{-1}} \leq N^\delta;
\] (5.37)
for sufficiently large $N$, with overwhelming probability. Taking a union bound of (5.37) over all $a,b \in [1,N]$; $\kappa$ and $\gamma$ in a $N^{-10}$-net of $[0,1]$; and $z$ in a $N^{-10}$-net of $\mathcal{D}$, and also applying (4.1), then yields (5.33).

Next we require the following result that essentially provides level repulsion estimates for $X_t$.

**Lemma 5.7.** For all $\omega > 0$, there exist constants $c > 0$ (independent of $\omega$) and $C = C(\omega) > 0$ such that the following holds. Set $M = N^{2\omega}$; recall $t$ from (3.9); and fix an index $i \in [(1/2 - c)N, (1/2 + c)N]$. Then,
\[
\mathbb{E} \left[ f_M(Q_i(X_t)) \right] \leq CN^{3\omega/2}.\] (5.38)
Further fix $\delta > 0$ and, for fixed real numbers $\kappa, \gamma \in [0,1]$ and indices $1 \leq a,b \leq N$, abbreviate $\mu_j = \lambda_j(\Theta_\kappa(a,b)H^\gamma)$ for each $j \in [1,N]$. Then we have with overwhelming probability that
\[
\mathbb{P}_{\omega} \left( \{ Q_i(a,b)H^\gamma \} < M \sum_{j \neq i} \frac{1}{|\mu_j - \mu_i|} \leq N^{1+\omega+\delta} \right).\] (5.39)

**Proof.** Throughout this proof, we may assume that $\delta < \frac{\omega}{7}$. Define the sets
\[
U_0 = \left\{ j \in [1,N] \setminus \{ i \} : |\lambda_j(t) - \lambda_i(t)| \leq N^{-1+\delta/2} \right\},\] (5.40)
and
\[
U_n = \left\{ j \in [1,N] : 2^{n-1}N^{-1+\delta/2} < |\lambda_j(t) - \lambda_i(t)| \leq 2^n N^{-1+\delta/2} \right\}.
\] (5.41)
for each integer $n \geq 1$.

Now choose the $c > 0$ here with respect to the $K$ from Lemma 4.6 to satisfy $c < \frac{1}{2K}$, and define $L = \lceil \log_2(2cN^{-1-\delta/2}) \rceil$. Then Lemma 4.6 and Lemma 4.7 together imply (after further decreasing $c$ if necessary) that
\[
|U_n| \leq C2^n N^\delta.
\] (5.42)
holds with overwhelming probability, for each $n \in [0,L]$. Next, for any $\theta \in (0,1)$, also define the event
\[
E(\theta) = \left\{ \min \left\{ \lambda_i(t) - \lambda_{i-1}(t), \lambda_{i+1}(t) - \lambda_i(t) \right\} > \frac{\theta}{N} \right\}.
\] (5.43)
and let $E(\theta)^c$ denote the complement of $E(\theta)$. Then (5.42) implies with overwhelming probability that
\[
\frac{1}{N^2} \sum_{n=0}^{L} \sum_{j \in U_n} |\lambda_j(t) - \lambda_i(t)|^{-2} \leq CN^{\delta} \theta^{-2}.\] (5.44)
Further, we deterministically have that
\[
\frac{1}{N^2} \sum_{n=L+1}^{\infty} \sum_{j \in U_n} |\lambda_j(t) - \lambda_i(t)|^{-2} \leq CN^{-1}.
\] (5.45)
Then combining (5.44) and (5.45) bounds
\[
E \left[ Q_i (\Theta^{(a,b)}_\kappa X_i) 1_{E(\theta)} \right] < CN^\delta \theta^{-2}.
\] (5.46)

On $E(\theta)^c$, we use the third part of Lemma 4.3 and the fact that $|f_M(x)| < M$ holds for all $x > 0$
to deduce that
\[
E \left[ f_M \left( Q_i (\Theta^{(a,b)}_\kappa X_i) \right) 1_{E(\theta)^c} \right] < CN^{2\omega} \theta.
\] (5.47)

Then selecting $\theta = N^{-\omega/2}$, using the fact that $\delta < \frac{1}{4}$, and combining (5.46) and (5.47) yields (5.38). We omit the proof of (5.39), as it is entirely analogous, and follows from replacing the above application of Lemma 4.10 and Lemma 4.12 (to establish (5.42)) with (5.38) and the second bound in (5.34), respectively, and using the fact that $\mu_i - \mu_{i-1}, \mu_{i+1} - \mu_i \geq N^{-\omega}$ holds on the event that $Q_i (\Theta^{(a,b)}_\kappa H^\gamma) < M$.

Now we can establish the derivative bounds given by Lemma 5.3 and Lemma 5.4.

**Proof of Lemma 5.3 (Outline).** In outline, the bound (5.6) is proven by expanding $\partial^{(k)}_{ab} Q_i (H^\gamma)$
using contour integration into a sum of terms which are then bounded individually. Since the proof of (5.7) uses similar expansion, we only discuss that of (5.6) here (for the former, see the proof of [38] Proposition 4.2 for further details).

Our claim (5.6) is essentially the same as that of [57] Proposition 4.6, but there are two differences. First, one of our hypotheses is weaker: we only have complete delocalization at small energies and not throughout the spectrum. Second, our conclusion is stronger: [57] Proposition 4.6] bounded derivatives up to third order, but we here we bound fourth order derivatives. This extension to fourth order derivatives parallels the proof for the third order derivatives and requires no new ideas. Therefore, let us only show how the proof in [57] Proposition 4.6] may be modified to accommodate the fact that our delocalization estimate is weaker than the one used in that reference. In what follows, we also assume for notational convenience that $\kappa = 1$, so that $\Theta^{(a,b)}_\kappa (H^\gamma) = H^\gamma$, as the proof for general $\kappa \in [0,1]$ is entirely analogous by replacing our use of (4.19) below by Lemma 5.6.

For any vector $v$, let $v^\ast$ denote its transpose. Set $\theta_{jk} = u_j^* V u_k$, where $V = V^{(a,b)} = \{ V_{ij} \}$ is
the $N \times N$ matrix whose entries are zero except for $V_{ab} = V_{ba} = 1$. In the proof of [57] Proposition
4.6], $\partial^{(k)}_{ab} Q_i (H^\gamma)$ was expanded into a sum of certain terms using a contour integral representation
and Green’s function identities. For instance, in the expansion of $\partial^{(3)}_{ab} Q_i (H^\gamma)$ there are 13 distinct
terms, which are listed after line (4.18) in [57] Proposition 4.6]. Setting $\lambda_i = \lambda_i (H^\gamma)$, one such term is
\[
\frac{1}{N^2} \sum_{1 \leq j_1, j_2, j_3 \leq N} \theta_{j_1 j_2} \theta_{j_2 j_3} \theta_{j_3 j_1} (\lambda_i - \lambda_{j_1})^3 (\lambda_i - \lambda_{j_2}) (\lambda_i - \lambda_{j_3}).
\] (5.48)

The terms produced by expanding $\partial^{(k)}_{ab} Q_i (H^\gamma)$ are fractions with a product of $k \theta_{\alpha \beta}$ terms in the
numerator, where each of $\alpha, \beta$ may be a summation index or $i$, and a product of $k + 2$ eigenvalue differences $\lambda_i - \lambda_j$ in the denominator, where $j$ is a summation index. We call $k$ the order of such a term. The proof of [57] Proposition 4.6] shows that to prove the claim (5.6), it suffices to bound each of the the order $k$ terms appearing in its expansion by $CN^{(2k+2)\delta + (k+2)\omega}$.  

For illustrative purposes, we consider just the term \[\text{5.48}\] in the \(k = 3\) case here; other terms of the same order and the cases \(k \in \{1, 2, 4\}\) are analogous. So, let us show that

\[
\frac{1}{N^2} \sum_{1 \leq j_1, j_2, j_3 \leq N} \frac{\theta_{j_1j_2} \theta_{j_2j_3} \theta_{j_3j_1}}{(\lambda_i - \lambda_{j_1})(\lambda_i - \lambda_{j_2})(\lambda_i - \lambda_{j_3})} \leq CN^{8\delta + 5\omega}.
\] (5.49)

To prove \[\text{5.49}\], we consider various cases depending on the locations of the eigenvalues \(\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3}\). Let \(K\) be the constant from Lemma 4.5 and set \(c = (2K)^{-1}\). When \(\lambda_{j_3}, \lambda_{j_2}, \lambda_{j_1} \in [-2c, 2c]\), \[\text{4.19}\] shows the corresponding eigenvectors are completely delocalized, and the proof of [57, Proposition 4.6] requires no modification. There are three remaining cases: exactly one of the \(j_i\) is such that \(|\lambda_{j_i}| > 2c\), exactly two are, or all three are.

In the first case, suppose for example that \(|\lambda_{j_2}| > 2c\). Then, \[\text{4.19}\] implies

\[
|\theta_{j_1j_2}| \leq \left( |u_{j_1}(a)| |u_{j_2}(b)| + |u_{j_1}(b)| |u_{j_2}(a)| \right) \leq N^{1/2+\delta} \left( |u_{j_2}(a)| + |u_{j_2}(b)| \right),
\] (5.50)

\[
|\theta_{j_2j_3}| \leq \left( |u_{j_2}(a)| |u_{j_3}(b)| + |u_{j_2}(b)| |u_{j_3}(a)| \right) \leq N^{1/2+\delta} \left( |u_{j_2}(a)| + |u_{j_2}(b)| \right),
\] (5.51)

\[
|\theta_{j_3j_1}| \leq \left( |u_{j_3}(a)| |u_{j_1}(b)| + |u_{j_3}(b)| |u_{j_1}(a)| \right) \leq N^{-1+\delta},
\] (5.52)

with overwhelming probability. Inserting these bounds in \[\text{5.48}\] decouples the sum into a product of a sum over \(j_2\) and a sum over \(j_1, j_3\):

\[
\text{5.48} \leq c^{-1} N^{-4+3\delta} \left( \sum_{j_2=1}^{N} \left( |u_{j_2}(a)| + |u_{j_2}(b)| \right)^2 \right) \left( \sum_{1 \leq j_1, j_3 \leq N} \frac{1}{|\lambda_i - \lambda_{j_1}| |\lambda_i - \lambda_{j_3}|} \right).
\] (5.53)

Here we used \(|\lambda_1 - \lambda_{j_2}| > c\). The first factor is at most a constant, since the matrix of eigenvectors is orthonormal:

\[
\sum_{j_2=1}^{N} \left( |u_{j_2}(a)| + |u_{j_2}(b)| \right)^2 \leq 2 \sum_{j_2=1}^{N} \left( |u_{j_2}(a)|^2 + |u_{j_2}(b)|^2 \right) = 4.
\] (5.54)

For the second factor, the assumption \[\text{5.5}\] implies that \(|\lambda_i - \lambda_{i+1}| \geq N^{-1-\omega}\), and so \[\text{5.39}\] yields

\[
\sum_{j \neq i} |\lambda_j - \lambda_i|^{-1} \leq CN^{1+\omega+\delta}.
\] (5.55)

Further, for \(k \geq 2\), the hypothesis \[\text{5.5}\] yields

\[
\sum_{j \neq i} |\lambda_j - \lambda_i|^{-k} \leq \left( \sum_{j \neq i} |\lambda_j - \lambda_i|^{-2} \right)^{k/2} \leq CN^{k(1+\omega+\delta)}.
\] (5.56)

These inequalities imply the second factor of \[\text{5.53}\] is at most \(CN^{4+4\omega+4\delta}\), and so \[\text{5.48}\] is at most \(CN^{7\delta+4\omega}\).

In the second case, suppose for example that \(|\lambda_{j_2}| > c\) and \(|\lambda_{j_3}| > c\). Since then \(|\lambda_i - \lambda_{j_2}|\) and \( |\lambda_i - \lambda_{j_3}|\) are bounded below by \(c\), in this case \[\text{5.48}\] is bounded above by

\[
\frac{1}{c^2 N^2} \sum_{1 \leq j_1, j_2, j_3 \leq N} |\theta_{j_1j_2} \theta_{j_2j_3} \theta_{j_3j_1}| \leq \frac{1}{c^2 N^2} \sum_{1 \leq j_1, j_2, j_3 \leq N} \frac{\theta_{j_1j_2} \theta_{j_2j_3} \theta_{j_3j_1}}{(\lambda_i - \lambda_{j_1})(\lambda_i - \lambda_{j_2})(\lambda_i - \lambda_{j_3})^3}.
\] (5.57)
Proceeding as in the previous case, we find that \( (5.48) \) is bounded above by
\[
c^{-2}N^{2\delta-3} \left( \sum_{1 \leq j_2, j_3 \leq N} \left( |u_{j_2}(a)| + |u_{j_2}(b)| \right) \left( |u_{j_3}(a)| + |u_{j_3}(b)| \right) \right) \\
\times \left( \left| |u_{j_3}(a)||u_{j_2}(b)| + |u_{j_2}(b)||u_{j_3}(a)| \right| \right) \left( \sum_{j_1 \neq 1} \frac{1}{|\lambda_i - \lambda_j|} \right). \tag{5.58}
\]

By \( (5.56) \), the sum over \( j_1 \) is at most \( CN^{3+3\omega+3\delta} \). Moreover, by the orthogonality of the eigenvectors of \( H^\top \) (following \( (5.54) \)), the sum over \( j_2 \) and \( j_3 \) is bounded by 8. Thus, \( (5.48) \) is bounded above by \( CN^{5\delta+3\omega} \).

Finally, when \( |\lambda_i| > c \) for all \( \ell \), \( (5.48) \) is bounded by
\[
\frac{1}{N^2 c^5} \sum_{1 \leq j_1, j_2, j_3 \leq N} |\theta_{j_1 j_2 \theta_{j_2 j_3} j_3 j_1}| \leq \frac{1}{N^2 c^5} \sum_{1 \leq j_1, j_2, j_3 \leq N} \left( |u_{j_1}(a)||u_{j_2}(b)| + |u_{j_1}(b)||u_{j_2}(a)| \right) \\
\times \left( |u_{j_3}(a)||u_{j_2}(b)| + |u_{j_2}(b)||u_{j_3}(a)| \right) \\
\times \left( |u_{j_1}(a)||u_{j_3}(b)| + |u_{j_1}(b)||u_{j_3}(a)| \right). \tag{5.60}
\]

Again by the orthogonality of the eigenvectors of \( H^\top \) (following \( (5.54) \)), the latter is bounded by 8, and so \( (5.48) \) is bounded above by \( 8c^{-5}N^{-2} \).

This completes our demonstration of how to bound the sum \( (5.48) \) and concludes the proof. \( \square \)

**Proof of Lemma 5.5 (Outline).** We again only discuss the first bound in \( (5.8) \), as the proof of the second is similar. To that end observe, since we have assumed \( Q_i(H^\top) \leq N^{2\omega} \), we must have \( |\lambda_i - \lambda_j| \geq N^{-1-\omega} \), for each \( j \in [1, N] \setminus \{i\} \). As in the proof of Lemma 5.3 the derivative \( \partial_{ab}^{(k)} Q_i(G_{a,b}^{(a)}) \) for \( k \leq 4 \) can be expressed as a sum of a uniformly bounded number of terms similar to \( (5.48) \), in which at most four indices \( j_k \) are being summed over and in which the denominator is of degree at most six in the gaps \( \lambda_i - \lambda_j \). Thus, each such term is bounded by at most \( N^{6+6\omega} \), and so their sum is bounded by a multiple of \( N^{10+6\omega} \). This yields the first estimate in \( (5.8) \) and, as mentioned previously, the proof of the second is omitted. \( \square \)

5.4. **Proof of Lemma 5.5** Now we can establish Lemma 5.5.

**Proof of Lemma 5.5** It suffices to show that there exists some \( \omega > 0 \) such that, if \( M = N^{2\omega} \), then
\[
\mathbb{E} \left[ f_M(Q_i(H^\top)) \right] \leq 2N^{3\omega/2}, \tag{5.63}
\]
since then a Markov inequality would imply
\[
\mathbb{P} (Q_i(H^\top) \geq N^{2\omega}) \leq \mathbb{P} \left( f_M(Q_i(H^\top)) \geq N^{2\omega} \right) \leq N^{-2\omega} \mathbb{E} \left[ f_M(Q_i(H^\top)) \right] \leq 2N^{-\omega/2}, \tag{5.64}
\]
and the lemma follows after setting \( v = 2\omega \). To prove \( (5.63) \), we apply Lemma 5.4 to interpolate between \( X_i = H^0 \) and \( H^\top \), carrying the level repulsion estimate \( (5.38) \) from the former to the latter. To implement this argument, let us take \( c_1 > 0 \) sufficiently small so that \( i \in [(1/2 - c_1)N, (1/2 + c_1)N] \) implies that \( |\lambda_i(G_{a,b}^{(a)})| \) is less than the \( c_1 \) from Lemma 5.6 for each \( \gamma, \kappa \in [0, 1] \) and \( 1 \leq a, b \leq N \) with overwhelming probability; such a \( c_1 \) exists by the second bound of \( (5.34) \). Further take \( \omega, \delta > 0 \) sufficiently small so that \( 100(\omega + \delta) \) is less than the constant \( c \) from Lemma 5.1 and set \( M = N^{2\omega} \).
configuration is order by $\xi$.

Theorem 6.1

where we recall 6.1. shown in Section 6.4 and Section 6.5. we use the eigenvector moment flow to establish Theorem 3.8, assuming several results that will be properties. Section 6.2 contains continuity estimates used in the proof of Theorem 3.8. In Section 6.3 we use the eigenvector moment flow to establish Theorem 3.8 assuming several results that will be shown in Section 6.4 and Section 6.5.

6. Dynamics

This section determines the eigenvector statistics of $X_t$. Our main goal is a proof of Theorem 3.8. In Section 6.1 we recall the definition of the eigenvector moment flow from [40] and some of its properties. Section 6.2 contains continuity estimates used in the proof of Theorem 3.8. In Section 6.3 we use the eigenvector moment flow to establish Theorem 3.8 assuming several results that will be shown in Section 6.4 and Section 6.5.

6.1. Eigenvector moment flow. Recall the matrix $X_t$ from [38] and that its eigenvalues are given by $\lambda_1(s) \leq \lambda_2(s) \leq \cdots \leq \lambda_N(s)$ with associated unit eigenvectors $u_1(s), u_2(s), \ldots, u_N(s)$, respectively. By [40] Theorem 2.3], these eigenvalues and eigenvectors are governed by two stochastic differential equations (SDEs):

$$d\lambda_k(s) = \frac{db_{kk}(s)}{\sqrt{N}} + \frac{1}{N} \sum_{l \neq k} \frac{ds}{\lambda_k(s) - \lambda_l(s)}, \quad (6.1)$$

$$du_k(s) = \frac{1}{\sqrt{N}} \sum_{l \neq k} \frac{db_{kl}(s)}{\lambda_k(s) - \lambda_l(s)} u_l(s) - \frac{1}{2N} \sum_{l \neq k} \frac{ds}{(\lambda_k(s) - \lambda_l(s))^2} u_k(s), \quad (6.2)$$

where $(b_{ij}(s))_{1 \leq i \leq j \leq N}$ are mutually independent Brownian motions with variance $1 + 1 - j$. The first equation, for the eigenvalues, is called Dyson Brownian motion. The second, for the eigenvectors, is called the Dyson vector flow. Using these SDEs, we define the stochastic processes $\lambda = (\lambda(s))_{0 \leq s \leq 1}$ and $u = (u(s))_{0 \leq s \leq 1}$.

A key tool for analyzing the Dyson vector flow is the eigenvector moment flow, introduced in [40] Section 3.1], which characterizes the time evolution of the observable $f_s = f_{\lambda,s}$ defined by

$$f_s(\xi) = f_{\lambda,s}(\xi) = E[Q_{i_1,\ldots,i_m}^s(\xi) | \lambda], \quad (6.3)$$

where we recall $Q_{i_1,\ldots,i_m}^s(\xi)$ was defined in (3.12).

Theorem 6.1 ([40] Theorem 3.1]). Let $q \in \mathbb{R}^N$ be a unit vector and, for each $s \in [0, 1]$, set

$$c_{ij}(s) = N^{-1}(\lambda_i(s) - \lambda_j(s))^{-2}. \quad (6.4)$$

Then,

$$\partial_s f_s = B(s)f_s, \quad \text{where} \quad B(s)f_s(\xi) = \sum_{i \neq j} c_{ij}(s) 2\xi_i(1 + 2\xi_i)(f_s(\xi^{ij}) - f_s(\xi)). \quad (6.5)$$

Recall from Section 6.3 that we view $N$-tuples $\xi = (\xi_1, \xi_2, \ldots, \xi_N) \in \mathbb{N}^N$ as particle configurations, with $\xi_k$ particles at location $k$ for each $k \in [1, N]$; the total number of particles in this configuration is $n = N(\xi) = \sum_{j=1}^N \xi_j$. We label the locations of these particles in non-decreasing order by

$$x_1(\xi) \leq \cdots \leq x_n(\xi). \quad (6.6)$$
Given another particle configuration $\zeta$ with the same number $n$ of particles, whose locations are labeled by $(y_j(\zeta))$ in non-decreasing order, we define the distance

$$d(\xi, \zeta) = \sum_{j=1}^{n} |x_j(\xi) - y_j(\zeta)|. \tag{6.7}$$

Next, we recall $c, \eta, \psi$ from (6.14), fix $n \in \mathbb{N}$, and define the parameter

$$\ell = \ell(n) = \psi^{4n+1} N^{1+2\eta}, \text{ where } \delta = \delta(n) = 50(n + 1)c. \tag{6.8}$$

Recalling $t$ from (3.9) (which satisfies (3.10)), we also set

$$\tau = \tau(n) = t - N^{7\sigma} \psi \eta, \quad t_0 = t - \psi \eta. \tag{6.9}$$

These are chosen so that for fixed $n > 0$, recalling the choices from (6.14) (after selecting $\epsilon = c(n) > 0$ to be sufficiently small),

$$N^{-1/2} \ll \psi \eta \ll N^{7\sigma} \psi \eta \ll \tau < t_0 < t < t + \frac{\ell}{N} \ll 1. \tag{6.10}$$

Recalling the operator $B = B(s)$ from (6.5), we decompose it into the sum of a “short range” and “long range” operator, given explicitly by $B = S + L$, where $S = S_n = S_n(s)$ and $L = L_n = L_n(s)$ are defined by

$$(S f_s)(\xi) = \sum_{0 < |j-k| \leq \ell} c_{jk}(s) 2 \xi_j (1 + 2 \xi_k) (f_s(\xi^{jk}) - f_s(\xi)), \tag{6.11}$$

$$(L f_s)(\xi) = \sum_{|j-k| > \ell} c_{jk}(s) 2 \xi_j (1 + 2 \xi_k) (f_s(\xi^{jk}) - f_s(\xi)). \tag{6.12}$$

We let $U_S(s_1, s_2)$ be the semigroup associated with $B$ and likewise define $U_S(s_1, s_2)$ and $U_L(s_1, s_2)$.

We also let $F_{t_0}$ denote the $\sigma$-algebra generated by $\{X_s\}_{0 \leq s \leq t_0}$, and define

$$h_s(\xi) = h_{\gamma,s}(\xi) = E[Q^{j_1, \ldots, j_m}_{i_1, \ldots, i_m}(s) | \lambda, F_{t_0}] = E[Q^{j_1, \ldots, j_m}_{i_1, \ldots, i_m}(s) | \lambda, X_{t_0}]. \tag{6.13}$$

In the last equality, we used the Markov properties for Dyson Brownian motion and the eigenvector moment flow. Informally, $h_s(\xi)$ corresponds to a solution of the eigenvector moment flow dynamics, run for time $s - t_0$, with initial data $X_{t_0}$.

For consistency with (38) we introduce the following notation. Let $K > 1$ be such that $K^{-1}$ is less than the constant $c$ from Lemma 4.1 and $K$ is greater than those from Lemma 4.3, Lemma 4.4, Lemma 4.5 and Lemma 4.6 and define

$$r = \frac{1}{2K}; \quad D_r = \left\{ z = E + \eta N : |E| \leq r, \frac{\psi^4}{N} \leq \eta \leq r \right\}. \tag{6.14}$$

We define the function $\tilde{d} = \tilde{d}_n$ on $n$-particle configurations by

$$\tilde{d}(\xi, \zeta) = \max_{1 \leq \beta \leq n} \left\{ 1 \leq i \leq N : |\gamma_i(t_0)| \leq r, \min \{x_\beta(\xi), y_\beta(\zeta)\} \leq i \leq \max \{x_\beta(\xi), y_\beta(\zeta)\} \right\}. \tag{6.15}$$

The next lemma provides several estimates necessary to analyze the eigenvector moment flow. Its first and second parts follow from Lemma 4.3 and Lemma 4.4, respectively, together with a standard stochastic continuity argument. Its third part constitutes a special case of Corollary 3.3], whose assumptions are verified by Lemma 4.2. Its fourth part is a consequence of [38 (3.48)].
Lemma 6.2 (ES). The initial data \(X_0\) and the Dyson Brownian motion \(\{X_s\}_{0 \leq s \leq t}\) together induce a measure \(\mathcal{M}\) on the space of eigenvalue and eigenvector trajectories \((\lambda(s), u(s))\) on which the following event of trajectories holds with overwhelming probability.

1. Eigenvalue rigidity holds: \(\sup_{t_0 \leq s \leq t} |M_N(s, z) - m_N(s, z)| \leq \psi(N\eta)^{-1}\) uniformly for \(z \in D_r\) and \(\sup_{t_0 \leq s \leq t} |\lambda_i(s) - \gamma_i(s)| \leq \psi N^{-1}\) uniformly for indices \(i\) such that \(\gamma_i(s) \leq r\).

2. Delocalization holds: Conditional on \(\lambda\), for any \(q \in \mathbb{R}^N\) with stable support we have that
   \[\sup_{z \in D_r} \sup_{t_0 \leq s \leq t} |\langle q, R(s, z)q \rangle| \leq C(q)\psi; \quad \sup_{z \in D_r} \sup_{t_0 \leq s \leq t} N|\langle u_t(s), q \rangle|^2 \leq C(q)\psi,\]
   where \(C(q) > 0\) is a constant depending on \(|\text{supp } q|\).

3. Finite speed of propagation holds: Let \(n > 0\) be an integer, and abbreviate \(\ell = \ell(n)\) and \(S = S_n\). Conditional on \(\lambda\), we have the following estimate that is uniform in any function \(g: \{\xi \in \mathbb{N}^N: N^4(\xi) = n\} \rightarrow \mathbb{R}\). For a particle configuration \(\xi \in \mathbb{N}^N\) with \(N(\xi) = n\) such that \(\tilde{d}_n(\xi, \xi) \geq \psi\ell\) for each \(\xi\) in the support of \(g\), we have
   \[\sup_{t_0 \leq s \leq t} |U_S(t_0, s)g(\xi)| \leq N^\alpha e^{-c\psi t}\|g\|_\infty.\]

4. For any interval \(I \subset [-r, r]\) of length \(|I| \geq \psi N^{-1}\), we have
   \[C^{-1}|I|N \leq \left| \left\{ i: \lambda_i(s) \in I \right\} \right| \leq C|I|N,\]
   uniformly in \(s \in [t_0, t]\).

The next estimate on the short range operator \(S\) is a consequence of ES Lemma 3.5 (whose conditions are verified by Lemma 6.2 and Lemma 6.7).

Lemma 6.3 (ES Lemma 3.5). There exists a constant \(C > 0\) such that the following holds with overwhelming probability with respect to \(\mathcal{F}_{t_0}\). Fix an integer \(n > 0\), and abbreviate \(\ell = \ell(n)\) (recall (6.8)) and \(S = S_n\). There exists an event \(E\) of trajectories \((\lambda(s), u(s))_{t_0 \leq s \leq t}\) of overwhelming probability on which we have
   \[\sup_{s \in [t_0, t]} \left| U_S(t_0, s)h_{t_0} - U_S(t_0, s)h_{t_0} \right| \leq C \frac{\psi^\alpha N(t - t_0)}{\ell}\]
   for any configuration \(\xi \in \mathbb{N}^N\) such that \(N(\xi) = n\) and \(\text{supp } \xi \subset \left((1/2 - c)N - 2\psi\ell, (1/2 + c)N + 2\psi\ell\right]\).

6.2. Continuity estimates. To prepare for the proof of Theorem 3.8 we require the following continuity estimates for entries of \(R(t, z)\). We recall \(a\) from (6.2); \(t_0\) and \(\tau\) from (6.9); and \(r\) from (6.14), and define
   \[\mathcal{D} = \left\{ z = E + i\eta: |E| \leq \frac{r}{4}, N^{-a} \leq \eta \leq \frac{r}{4} \right\}.\]

Lemma 6.4. Fix an integer \(n > 0\), a real number \(\delta > 0\), and a unit vector \(q\) with stable support; set \(q = |\text{supp } q|\), and abbreviate \(\tau = \tau(n)\). Then, there exist constants \(c > 0\) (independent of \(n, \delta,\)
and \( q \) and \( C = C(\delta, q, n) > 0 \) such that, uniformly in \( t_1, t_2 \in [t_0, t] \) with \( t_1 < t_2 \), we have

\[
\sup_{z \in B} \left| \langle q, R(t_1, z)q \rangle - \langle q, R(t_2, z)q \rangle \right| \leq C \frac{\psi^4}{\sqrt{\eta}} + CN^\delta \left( \frac{t_2 - t_1}{t_2 - t} + N^{-c} \right); \tag{6.21}
\]

\[
\sup_{z_1, z_2 \in B} \left| \langle q, R(t_1, z_1)q \rangle - \langle q, R(t_1, z_2)q \rangle \right| \leq C \frac{\psi^4}{\sqrt{\eta}} + CN^\delta \left( N^{-c} + \frac{|z_1 - z_2| + \text{Im} z_1}{t_1 - t} \right), \tag{6.22}
\]

both with overwhelming probability.

**Proof of Lemma 6.4.** For \( s \geq \tau \), define

\[
r_i(s, z) = \frac{1}{\lambda_i(\tau) - z - (s - \tau)m_{ic}(s, z)}, \tag{6.23}
\]

which is similar to the terms appearing in the definition of the free convolution \( \psi \) (but, in a sense, “started” at \( \lambda(\tau) \) instead of \( \lambda(0) \)).

Now let us apply \[38\] Theorem 2.1, with the \( t \) there equal to \( s - \tau \) here and the \( H_0 \) given by \( X_{\tau} \) here; the assumptions of that theorem are verified by (4.12), (4.13), and the facts that \( \tau \gg N^{-1/2} \) and \( s - \tau \geq t_0 - \tau \gg N^{-1/2} \). For any \( s \in [t_0, t] \), this gives with overwhelming probability that

\[
\left| \langle q, R(s, z)q \rangle - \sum_{i=1}^{N} \langle u_i(\tau), q \rangle^2 r_i(s, z) \right| \leq \frac{\psi^2}{\sqrt{\eta}} \text{Im} \left( \sum_{i=1}^{N} \langle u_i(\tau), q \rangle^2 |r_i(s, z)| \right). \tag{6.24}
\]

Thus, (4.11), (4.2), and a union bound over \( s \) in an \( N^{-10} \)-net of \( [t_0, t] \) together yield that (6.24) holds with overwhelming probability, uniformly in \( s \in [t_0, t] \).

To bound the right side of (6.24), observe that (4.17), (38 (2.3)), and the exchangeability of the eigenvector entries together yield the bound

\[
\sum_{i=1}^{N} \langle u_i(\tau), q \rangle^2 r_i(s, z) \leq C(\log N)^2, \tag{6.25}
\]

uniformly for any standard basis vector \( q \in \{e_1, e_2, \ldots, e_N\} \). Therefore, (6.25) also holds for unit vectors \( q \in \mathbb{R}^N \) of stable support (where the \( C \) there now depends on \( \eta = |\text{supp } q| \), by expanding \( q \) in the standard basis and using the inequality \( (a + b)^2 \leq 2(a^2 + b^2) \) on the products \( \langle u_i, e_j \rangle \langle u_i, e_k \rangle \) that appear in the corresponding expansion of \( \langle u_i(\tau), q \rangle^2 \). Thus, (6.24) and (6.25) together imply

\[
\left| \langle q, R(s, z)q \rangle - \sum_{i=1}^{N} \langle u_i(\tau), q \rangle^2 r_i(s, z) \right| \leq C(\psi)^4 \frac{\psi^2}{\sqrt{\eta}}. \tag{6.26}
\]

We now establish (6.21) by subtracting (6.26) evaluated at \( s = t_2 \) from that equation evaluated at \( s = t_1 \). To that end, we have from (6.23) that

\[
|r_i(t_1, z) - r_i(t_2, z)| = \frac{|(t_2 - \tau)m_{ic}(t_2, z) - (t_1 - \tau)m_{ic}(t_1, z)|}{(\lambda_i(\tau) - z - (t_1 - \tau)m_{ic}(t_1, z))(\lambda_i(\tau) - z - (t_2 - \tau)m_{ic}(t_2, z))} \tag{6.27}
\]
The numerator of the right side of (6.27) is with overwhelming probability bounded by
\[
| (t_1 - \tau) m_{\mathcal{E}}(t_1, z) - (t_2 - \tau) m_{\mathcal{E}}(t_2, z) |
\leq | t_1 - t_2 | | m_{\mathcal{E}}(t_2, z) | + | t_1 - \tau | | m_{\mathcal{E}}(t_1, z) - m_{\mathcal{E}}(t_2, z) | 
\leq C (\log N)^2 (t_2 - t_1) + C (t_1 - \tau) N^{-c},
\]
(6.28)

Here, in the last inequality we used the overwhelming probability bound \( | m_{\mathcal{E}}(t_2, z) | \leq C (\log N)^2 \) (which follows from [33 (2.3)], whose hypotheses are satisfied by \( (4.11) \) with \( s = 0 \)) and also the overwhelming probability estimate
\[
| m_{\mathcal{E}}(t_1, z) - m_{\mathcal{E}}(t_2, z) | \leq | m_{\mathcal{E}}(t_1, z) - m_{a}(z) | + | m_{\mathcal{E}}(t_2, z) - m_{a}(z) | \leq CN^{-c},
\]
(6.29)
where the last inequality follows from (4.13) (with the \( \delta \) there equal to \( \frac{1}{2} (\frac{1}{2} - a) \) here).

To bound the denominator of the right side of (6.27) observe, since \( \text{Im} m_{\mathcal{E}}(t_1, z) \geq c' \) for some uniform constant \( c' > 0 \) (which is a consequence of (4.13) and (4.6)), we have with overwhelming probability that
\[
\frac{1}{(\lambda_i(\tau) - z - (t_1 - \tau) m_{\mathcal{E}}(t_1, z))(\lambda_i(\tau) - z - (t_2 - \tau) m_{\mathcal{E}}(t_2, z))} \leq C | r_i(t_1, z) | \frac{t_2 - \tau}{t_2 - t_1}.
\]
(6.30)

Altogether, we obtain with overwhelming probability that
\[
| r_i(t_1, z) - r_i(t_2, z) | \leq C \frac{| r_i(t_1, z) |}{t_2 - \tau} ((\log N)^2 (t_2 - t_1) + (t_1 - \tau) N^{-c})
\leq C | r_i(t_1, z) | \left( (\log N)^2 \frac{t_2 - t_1}{t_2 - \tau} + N^{-c} \right),
\]
(6.31)
where we used \( t_1 - \tau \leq t_2 - \tau \) in the last inequality. It follows that
\[
\left| \sum_{i=1}^{N} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 r_i(t, z) - \sum_{i=1}^{N} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 r_i(\tau, z) \right|
\leq C (\log N)^2 \left( \frac{t_2 - t_1}{t_2 - \tau} + N^{-c} \right) \sum_{i=1}^{N} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 | r_i(t_1, z) |. \]
(6.32)

To bound the right side of (6.32), observe that
\[
\sum_{i=1}^{N} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 | r_i(t, z) | = \sum_{1 \leq i \leq N \atop |\lambda_i(\tau)| < r} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 | r_i(t, z) | + \sum_{1 \leq i \leq N \atop |\lambda_i(\tau)| \geq r} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 | r_i(t, z) |. \]
(6.33)

We bound the first term on the right side of (6.33) using Lemma (4.4) which yields
\[
\sum_{1 \leq i \leq N \atop |\gamma_i(\tau)| < r} \langle \mathbf{u}_i(\tau), \mathbf{q} \rangle^2 | r_i(t, z) | \leq N^{\delta/2 - 1} \sum_{i=1}^{N} | r_i(t_1, z) | \leq N^{\delta},
\]
(6.34)
for any \( \delta > 0 \), with overwhelming probability. Here, in the last bound we used the fact that \( \sum_{i=1}^{N} | r_i(t_1, z) | \leq N (\log N)^2 \) (which is a consequence of [33 Lemma 7.5], whose assumptions are verified by (4.12) and the fact that \( t_1 - \tau \gg N^{-1/2} \)).

To bound the second term in (6.33), observe for \( |\gamma_i(\tau)| > r \) we have \( |\lambda_i(\tau)| > \frac{\pi}{r} \) (by (4.14)), so
\[
|\lambda_i(\tau) - z - (t_1 - \tau) m_{\mathcal{E}}(t_1, z) | \geq c,
\]
(6.35)
for some constant $c > 0$, where we used $t_1 - \tau \ll 1$ and $|m_{tc}(t_1, z)| < C(\log N)^2$ (again by (3.3) (2.3)). We then obtain by (6.29) that

$$\sum_{1 \leq i \leq N} |\langle u_i(\tau), q\rangle|^2 |r_i(t_1, z)| \leq C \sum_{i=1}^{N} |\langle u_i(\tau), q\rangle|^2 \leq C. \quad (6.36)$$

Now the first bound (6.31) of the lemma follows from (6.29), (6.32), (6.33), (6.34), and (6.36), after absorbing the $(\log N)^2$ prefactor into $N^\delta$ and adjusting $\delta$ appropriately. We omit the proof of the second as it is analogous, but obtained by replacing (6.27) with the bound

$$|r_i(t_1, z_1) - r_i(t_1, z_2)| \leq \frac{(t_1 - \tau) |m_{tc}(t_1, z_1) - m_{tc}(t_1, z_2)| + |z_1 - z_2|}{(\lambda_i(\tau) - z_1 - (t_1 - \tau) m_{tc}(t_1, z_1))(\lambda_i(\tau) - z_2 - (t_1 - \tau) m_{tc}(t_1, z_2))}, \quad (6.37)$$

and (6.28) with the bound

$$(t_1 - \tau) |m_{tc}(t_1, z_1) - m_{tc}(t_1, z_2)| \leq (t_1 - \tau) (|m_{tc}(t_1, z_1) - m(z_1)| + |m_{tc}(t_1, z_2) - m(z_2)|) \leq C(t_1 - \tau) (N^{-\delta} + |z_1 - z_2| + \text{Im} z_1), \quad (6.38)$$

where (6.40) follows from (4.13) and (4.5). \qed

### 6.3. Short-time relaxation.

The proof of short-time relaxation here is similar to that of (3.3) Theorem 3.6. However, certain changes are necessary, since the diagonal resolvent entries $R_{ii}(t, z)$ for the removed model $X_i$ do not converge to a deterministic quantity, unlike those of the matrix model considered in (3.3). This causes the observable $f_{\lambda, t}(\xi)$ from (3.6) to now converge to the random variable $A(q, \xi)$, which is defined as follows.

Recall $t$ and $t_0$ from (6.3) and (6.34), respectively; recall that $\{\lambda_i\}$ are the eigenvalues of $X_s$ and that $\{\gamma_j(s)\}$ are given by (4.14); fix a unit vector $q \in \mathbb{R}^N$ with stable support; and set $q = |\text{supp} q|$. For any integer $k \in [1, N]$, particle configuration $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$; and eigenvalue trajectory $\lambda$, define $A(q, k) = A_{t, \lambda}(q, k)$ and $A(q, \xi) = A_{t, \lambda}(q, \xi)$ by

$$A(q, k) = \frac{\text{Im} (q, R(s, \gamma_k + i\eta)q)}{\text{Im} m_{\alpha}(\gamma_k + i\eta)}, \quad A(q, \xi) = \prod_{k=1}^{N} A(q, k)^{\xi_k}, \quad (6.41)$$

where we have recalled $\gamma_k = \gamma_k^{(s)}$ from (2.7) and $\tilde{\gamma}_k$ from (5.17).

The initial data $X_0$ and Dyson Brownian motion $X_s$ for $0 \leq s \leq t$ together induce a measure on the space of eigenvalues and eigenvectors $(\lambda(s), u(s))_{0 \leq s \leq t}$, which we denote by $\mathcal{M}$. Let $\lambda = (\lambda(s))_{0 \leq s \leq t}$ be an eigenvalue trajectory with initial data given by a realization of the spectrum of $X_0$, and recall the observable $h_s(\xi)$ from (6.13) that is associated to an eigenvalue trajectory $\lambda$ and “starts” at time $t_0$.

Before proceeding, we first fix a small constant $c_0 = c_0(\alpha) > 0$ such that the conclusions of Lemma 4.1, Lemma 5.3, Lemma 6.2, Lemma 6.3 and Lemma 6.4 apply to any $z = E + i\eta \in D_r$ with $|E| \leq 16c_0$ and any $i \in [1, N]$ with $|\gamma_i(s)| \leq 16c_0$. By Lemma 4.7 we may choose $c_1 > 0$ such that, for any fixed real number $s \in [0, t]$ and index $i \in [1, 2 - c_1]$, we have that $|\gamma_i(s)| < c_0$ with overwhelming probability. Hence, we will fix this choice of $c_1 > 0$ in what follows and apply the five lemmas listed above without further comment.
Then, to establish Theorem 3.8 it suffices prove the following proposition.

**Proposition 6.5.** For any integer \( m \geq 0 \), there exist constants \( c_2 = c_2(m) > 0 \) and \( C = C(m,q) > 0 \) such that for \( c < c_2 \) we have

\[
\max_{\xi \in \mathbb{N}^N \colon \mathcal{N}(\xi) = m \atop \text{supp} \xi \in [(1/2-c_1)N,(1/2+c_1)N]} |h_t(\xi) - A(q,\xi)| \leq CN^{-c_2},
\]

(6.42)

with overwhelming probability with respect to \( \mathcal{M} \).

**Proof of Theorem 3.8.** Recall from (3.13) and (6.13) that \( \lambda \) is over \( I \). We assume the intervals \( \mathcal{N}(\xi) \) are disjoint for all \( i \). The reason for this assumption will be seen below, in the material immediately following (6.79).

For notational simplicity, we assume that \( |\text{supp} \xi| = 2 \). The cases where \( |\text{supp} \xi| \) is supported on one site and on more than two sites constitute straightforward modifications of the following two-site argument and will be briefly outlined in Section 6.6. Denote \( \text{supp} \xi = \{i_1,i_2\} \), with \( i_1 < i_2 \), and let \( j_1 \) and \( j_2 \) denote the number of particles in \( \xi \) at sites \( i_1 \) and \( i_2 \), respectively. Thus \( \zeta_{i_1} = j_1, \zeta_{i_2} = j_2, \) and \( j_1 + j_2 = m \).

Recalling \( d = \sigma(m) = 50(1 + m)c \) and \( \ell = \ell(m) = \psi^{4m+1}N^{1+\delta} \), from (6.8), we define a “short-range averaging parameter”

\[
\bar{d} = [\ell\psi^{5m}N^\delta],
\]

which by (3.12) and (6.9) satisfies \( \psi^2 \ell \ll \bar{d} \ll Nt_0 \) (assuming \( c = c(m) > 0 \) is sufficiently small).

For \( a \in \mathbb{R} \) and \( b \in \mathbb{N} \), we further define the interval \( I_{a,b} = I_{a,b}(\xi) \) by

\[
I_{a,b} = I_{a,1}^{(b)} \cup I_{a,2}^{(b)},
\]

(6.44)

where the intervals \( I_{a,1}^{(b)} = I_{a,1}^{(b)}(\xi) \) and \( I_{a,2}^{(b)} = I_{a,2}^{(b)}(\xi) \) are given by

\[
I_{a,1}^{(b)} = [i_1 - 10b\tilde{d} - a, i_1 + 10b\tilde{d} + a], \quad I_{a,2}^{(b)} = [i_2 - 10b\tilde{d} - a, i_2 + 10b\tilde{d} + a].
\]

(6.45)

We assume the intervals \( I_{a,1}^{(b)} \) and \( I_{a,2}^{(b)} \) are disjoint for all \( a \in [0,2\bar{d}] \) and \( b \in [0,m] \). When this is not true, the argument below is carried out analogously, but instead using a single connected interval. We describe the necessary modifications in Section 6.6.

**Definition 6.6.** For any particle configuration \( \xi \in \mathbb{N}^N \) with \( \text{supp} \xi \subset I_{\bar{d}+\psi \ell} \), we further set

\[
\lambda_{1}^{(b)}(\xi) = \sum_{i \in I_{\bar{d}+\psi \ell},1} \xi_i, \quad \lambda_{2}^{(b)}(\xi) = \sum_{i \in I_{\bar{d}+\psi \ell},2} \xi_i,
\]

(6.46)

\( \text{supp} \)

\( \\)The reason for this assumption will be seen below, in the material immediately following (6.79).
which denote the number of particles in \( \xi \) in \( I_{d+\psi^{t},1}^{(b)} \) and \( I_{d+\psi^{t},2}^{(b)} \), respectively. For any integers \( k_{1}, k_{2}, b \geq 0 \) and \( n \geq 1 \), recall \( A(q, k) \) from (6.41) and set

\[
\Omega^{(b)}(k_{1}, k_{2}) = \left\{ \xi \in \mathbb{N}^{n} : \text{supp} \xi \subset I_{d+\psi^{t}}^{(b)}, \chi_{1}^{(b)}(\xi) = k_{1}, \chi_{2}^{(b)}(\xi) = k_{2} \right\}; \tag{6.47}
\]

\[
\Omega^{(b)}(n) = \bigcup_{k_{1}+k_{2}=n} \Omega^{(b)}(k_{1}, k_{2}); \quad A(k_{1}, k_{2}) = A(q, i_{1})k_{1}A(q, i_{2})k_{2}. \tag{6.48}
\]

We also define the restricted intervals

\[
\Phi^{(b)}(k_{1}, k_{2}) = \left\{ \xi \in \mathbb{N}^{n} : \text{supp} \xi \subset I_{-\psi^{t}}^{(b)}, \sum_{i \in I_{-\psi^{t},1}^{(b)}} \xi_{i} = k_{1}, \sum_{i \in I_{-\psi^{t},2}^{(b)}} \xi_{i} = k_{2} \right\}; \tag{6.49}
\]

\[
\Phi^{(b)}(n) = \bigcup_{k_{1}+k_{2}=n} \Phi^{(b)}(k_{1}, k_{2}). \tag{6.50}
\]

The following two definitions provide certain operators on the space of functions on particle configurations and an auxiliary flow. Similar definitions appeared in [40, Section 7.2].

**Definition 6.7.** Fix integers \( k_{1}, k_{2} \geq 0 \). For integers \( a, b \geq 0 \), we define operators \( \text{Flat}_{a}^{(b)} = \text{Flat}_{a,k_{1},k_{2}}^{(b)} \) and \( \text{Av}^{(b)} = \text{Av}_{k_{1},k_{2}}^{(b)} \) on the space of functions \( f : \mathbb{N}^{n} \to \mathbb{C} \) as follows. For each particle configuration \( \xi \in \mathbb{N}^{n} \) and function \( f : \mathbb{N}^{n} \to \mathbb{C} \), set

\[
(\text{Flat}_{a}^{(b)}(f))(\xi) = \begin{cases} f(\xi), & \text{if } \text{supp} \xi \subset I_{a}^{(b)}, \\ A(k_{1}, k_{2}), & \text{otherwise}; \end{cases}
\]

\[
\text{Av}^{(b)}(f) = \tilde{d}^{-1} \sum_{a=1}^{\tilde{d}} \text{Flat}_{a}^{(b)}(f). \tag{6.52}
\]

**Definition 6.8.** Adopting the notation of Definition 6.7, we define the flow \( g_{s}(\xi) = g_{s}^{(b)}(\xi) = g_{s}^{(b)}(\xi; k_{1}, k_{2}) \) for \( s \geq t_{0} \) by

\[
\partial_{s}g_{s} = S(s)g_{s}, \quad \text{with initial data } g_{t_{0}}(\xi) = (\text{Av}^{(b)} h_{t_{0}})(\xi). \tag{6.53}
\]

For each \( s \geq t_{0} \), let \( \tilde{\xi}(s) = \tilde{\xi}(s; k_{1}, k_{2}) = \tilde{\xi}(s; k_{1}, k_{2}) \in \mathbb{N}^{n} \) denote a maximizing particle configuration for \( g_{s}^{(b)} \):

\[
g_{s}^{(b)}(\tilde{\xi}) = \max_{\xi \in \Omega^{(b)}(k_{1}, k_{2})} g_{s}^{(b)}(\xi; k_{1}, k_{2}). \tag{6.54}
\]

When there are multiple maximizers, we pick one arbitrarily (in a way such that \( \tilde{\xi}(s) \) remains piecewise constant in \( s \)).

6.4. **Proof of Proposition 6.5.** To establish Proposition 6.5, we begin with the following lemma providing bounds on \( A(q, \xi) \) (recall (6.41)).

**Lemma 6.9.** For any integer \( m \geq 0 \), there exists a constant \( C = C(m) > 0 \) such that the following holds with overwhelming probability with respect to \( \mathcal{M} \) for sufficiently small \( c = c(m) > 0 \). First, for any particle configuration \( \xi \in \mathbb{N}^{n} \) with \( m = N(\xi) \) particles such that \( \text{supp} \xi \subset [(1/2 - c_{1})N, (1/2 + c_{1})N] \), we have that

\[
|A(q, \xi)| \leq C_{\psi^{m}}. \tag{6.55}
\]
Second, for any integers \(k_1, k_2, b \geq 0\) with \(k_1 + k_2 \leq m\) and \(b \leq m + 1\), we have that
\[
\max_{b \in [0,m+1]} \max_{\xi \in O^{(b)}(k_1, k_2)} |A(q, \xi) - A(k_1, k_2)| < CN^{-3b}.
\]

Proof. Recalling the definition (6.41) of \(A\), the denominator \(\Im m_\alpha(\gamma_{ij} + i\eta)\) of each \(A(q, i_j)\) is bounded below by a uniform constant by (4.6). Moreover, the numerator of each \(A(q, i_j)\) is bounded above by \(C\psi\) with overwhelming probability by the second part of Lemma 6.2. Together these estimates yield (6.55).

To establish (6.56), observe by (6.22), (3.10) and (3.2) that there exists a constant \(c > 0\) such that for indices \(i, j\) with \(|i - N/2| < c'N\) and \(|j - N/2| < c'N\),
\[
|\tilde{\gamma}_i - \tilde{\gamma}_j| \leq |\gamma_i - \gamma_j| + |\gamma_j - \gamma_j(t)| + |\gamma_i(t) - \gamma_j(t)| \leq 2\eta + CN^{-1}|i - j|.
\]
In the last inequality we used (6.24), the fact that \(N^{\delta-1/2} \leq \eta\) for \(\delta \leq c\), and the fact that the density \(g_{\psi}(t, x)\) satisfies \(c \leq g_{\psi}(t, x) \leq C\) for \(|x| \leq C^{-1}\). The latter fact is [63, Lemma 3.2], whose hypotheses are satisfied in this case by the first inequality in (4.11) and the first inequality in (4.6).

We further observe using (6.55) that for \(k \in I^{(b)}_{d+\psi_{i,j}}\),
\[
|\tilde{\gamma}_{ij} - \tilde{\gamma}_k| \lesssim N^{-1}|i_j - k| + \eta \lesssim 2(10bd + 3d)N^{-1} + \eta \leq 30bdN^{-1} + \eta \leq 31bN^{3b}\eta,
\]
where in the last inequality we used the fact that \(\tilde{d} = \lfloor \ell\psi^{5m}N^0 \rfloor\) (recall (6.43)), where \(\ell = \psi^{4m+1}N^{1+\delta}\eta\) and \(d = 50(m + 1)c\) (recall (6.3)). Further using the facts that \(\eta \gg N^{-1/2}\) and \(t - \tau = N^{\delta}t\eta\) (recall (6.3)), it follows from (6.59), after taking \(c = c(m) > 0\) sufficiently small, that with overwhelming probability
\[
|\langle q, R(t, \tilde{\gamma}_k + i\eta)q\rangle - \langle q, R(t, \tilde{\gamma}_{ij} + i\eta)q\rangle| \lesssim_m N^{-4b}.
\]

We now note that by (1.4), \(g_\alpha(x) > c\) for a constant \(c > 0\) and all \(x\) in a neighborhood of zero that contains all \(\gamma^{(a)}_k\) such that \(k \in I^{(b)}_{d+\psi_{i,j}}\). The definition (2.7) then implies that for \(j, k \in I^{(b)}_{d+\psi_{i,j}}\),
\[
|\gamma^{(a)}_j - \gamma^{(a)}_k| \leq CN^{-1}|j - k|.
\]
Using this fact along with (1.5) implies that for \(k \in I^{(b)}_{d+\psi_{i,j}}\),
\[
|\Im m_\alpha(\gamma_{ij} + i\eta) - \Im m_\alpha(\gamma_k + i\eta)| \leq C|i_j - k| + C\eta \leq 31bN^{3b}\eta \ll 1,
\]
by the same calculation as in (6.59).

Thus, the bound (6.59) follows from (6.60), (6.64), and (6.60).

The following lemma, which we will establish in Section 6.3 below, essentially states that the difference \(g_\alpha(\xi) - A(k_1, k_2)\) is either nearly negative or its derivative is bounded by a negative multiple of itself. Here, we recall the intervals \(\Phi^{(b)}(k_1, k_2)\) from Definition 6.9.

Lemma 6.10. Fix integers \(b, n, k_1, k_2 \geq 0\) such that \(n \leq m\), \(b \leq m + 1\), and \(k_1 + k_2 = n\). If \(c = c(m) > 0\) (recall (3.14)) is chosen small enough, then there exist constants \(C = C(b, n) > 0\) and \(c = c(b, n) > 0\) such following holds with overwhelming probability with respect to \(\mathcal{M}\). Fix a realization of \(X_0\) and an associated eigenvalue trajectory \(\lambda = (\lambda(s))_{t_0 \leq s \leq t}\). There exists a countable subset
$C = C(X_0, \lambda) \subset [t_0, t]$ such that, for $s \in [t_0, t] \setminus C$, the continuous function $g_s^{(b)}(\bar{\xi}(s; k_1, k_2); k_1, k_2)$ is differentiable and satisfies either

$$g_s^{(b)}(\bar{\xi}) - A(k_1, k_2) \leq N^{-1},$$

or

$$\partial_s g_s^{(b)}(\bar{\xi}) - A(k_1, k_2) \leq \frac{C}{\eta} \left( \psi \max_{\xi} |h_s(\xi) - A(\chi_1^{(b)}(\xi), \chi_2^{(b)}(\xi))| + N^{-\delta} \right) - \frac{C}{\eta} (g_s^{(b)}(\bar{\xi}) - A(k_1, k_2)),$$

(6.63)

where the maximum in (6.63) is taken over all $\xi \in \Phi^{(b+1)}(k_1 - 1, k_2) \cup \Phi^{(b+1)}(k_1, k_2 - 1)$.

Given Lemma 6.10, we can now establish Proposition 6.5.

Proof of Proposition 6.5. First observe that, for any $n \leq m$ and $\xi \in \Phi^{(m-n+1)}(n)$, we have

$$|h_s(\xi) - g_t^{(m-n+1)}(\xi)| \leq |(U_{s}(t_0, t)h_{t_0} - U_{s}(t_0, t)h_{t_0})(\xi)| + |U_{s}(t_0, t)h_{t_0} - A\chi^{(m-n+1)}h_{t_0}(\xi)|$$

$$\lesssim_m \psi^m N^\ell^{-1}(1 - t_0) + e^{-c\psi}.$$  

(6.64)

(6.65)

Here, the first term from (6.64) follows from Lemma 6.3 where the containment $\text{supp} \xi \subset [(1/2 - c)N, (1/2 + c)N]$ holds since $s_1, s_2 \in [(1/2 - c_1)N, (1/2 + c_1)N]$. The second term in (6.64) follows from (6.63), which applies due to the facts that $h_{t_0} - A\chi^{(m-n+1)}h_{t_0}(\xi') = 0$ whenever $\text{supp} \xi' \subset I_0^{(m-n+1)}$ and that $d_\alpha(\xi, \xi') > \psi \ell$ for any $\xi \in \Phi^{(m-n+1)}(n)$ and $\text{supp} \xi' \not\subset I_0^{(m-n+1)}$.

We now define a discretization of the interval $[t_0, t]$ by

$$t_k = t_0 + km^{-1}\psi\eta, \quad \text{for} \quad 0 \leq k \leq m,$$

(6.66)

and we will show for each integer $n \in [0, m]$ that, with overwhelming probability,

$$\sup_{s \in [t_0, t]} \max_{\xi \in \Phi^{(m-n+1)}(n)} |h_s(\xi) - A(\chi_1^{(m-n+1)}(\xi), \chi_2^{(m-n+1)}(\xi))| \lesssim_n \frac{\psi^m N^\delta}{N^{\delta}}.$$  

(6.67)

Given the $n = m$ case of (6.67) and using the facts that $\xi \in \Phi^{(1)}(m)$ and $t = t_m = t$, we obtain with overwhelming probability that

$$|h_s(\xi) - A(\chi_1^{(1)}(\xi), \chi_2^{(1)}(\xi))| \lesssim_m \frac{\psi^m}{N^{\delta}}.$$  

(6.68)

Our conclusion, (6.12), then follows from the facts that $\chi_1^{(1)}(\xi) = j_1 = \xi_1$, and $\chi_2^{(1)}(\xi) = j_2 = \xi_2$; and our choices $\psi = N^\ell$ and $\delta = 50(m + 1)\ell$.

So, it suffices to prove (6.67) and therefore the two estimates

$$\sup_{s \in [t_0, t]} \max_{\xi \in \Phi^{(m-n+1)}(n)} \left( h_s(\xi) - A(\chi_1^{(m-n+1)}(\xi), \chi_2^{(m-n+1)}(\xi)) \right) \lesssim_n \frac{\psi^m}{N^{\delta}},$$

(6.69)

$$\sup_{s \in [t_0, t]} \max_{\xi \in \Phi^{(m-n+1)}(n)} \left( A(\chi_1^{(m-n+1)}(\xi), \chi_2^{(m-n+1)}(\xi)) - h_s(\xi) \right) \lesssim_n \frac{\psi^m}{N^{\delta}}.$$  

(6.70)

To do this, we induct on $n \in [0, m]$. The base case $n = 0$ is trivial, since $\xi \in \Phi^{(m+1)}(0)$ implies that $h_s(\xi) = 1 = A(\chi_1^{(m+1)}(\xi), \chi_2^{(m+1)}(\xi))$. For the induction step, we assume the induction hypothesis (6.67) holds for $n - 1$ and prove (6.69) and (6.70) for $n$.
We will in fact only establish (6.69), as the proof of (6.70) is entirely analogous (by in what follows replacing the maximizer $\xi$ of $g^{(b)}$ with the minimizer). To that end, for any two fixed integers $k_1, k_2 \geq 0$ with $k_1 + k_2 = n$, it suffices to show that

$$\sup_{s \in [t_n, t]} \left( g_s^{(m-n+1)}(\xi; k_1, k_2) - A(k_1, k_2) \right) \lesssim \frac{\psi^n}{N^\delta}$$

holds with overwhelming probability, where we have abbreviated $\xi = \tilde{\xi}^{(m-n+1)}(s; k_1, k_2)$. Indeed, given (6.71), (6.69) follows upon letting $(k_1, k_2)$ range over all pairs of integers summing to $n$; the fact that $\xi$ maximizes $g^{(m-n+1)}$ over $\Omega^{(m-n+1)}(k_1, k_2)$; the fact that $\Phi^{(m-n+1)}(k_1, k_2) \subseteq \Omega^{(m-n+1)}(k_1, k_2)$; and (6.62).

To establish (6.71), we first apply the $b = m - n + 1$ case of Lemma 6.10. Since (6.62) implies (6.71), we may assume that (6.63) holds. Then the induction hypothesis (6.67) (whose $n$ is equal to $n-1$ here); the fact that $\xi \in \Phi^{(m-n+2)}(k_1 - 1, k_2) \cup \Phi^{(m-n+2)}(k_1, k_2 - 1)$ implies $\xi \in \Omega^{(m-n+1)}(n-1)$ (since $10d > 2\psi$); and (6.63) together yield that the bound

$$\partial_s \left( g_s^{(m-n+1)}(\xi; k_1, k_2) - A(k_1, k_2) \right) \leq C(m, n) \frac{\psi^n}{N^\delta} \eta - \frac{c(m, n)}{\eta} \left( g_s^{(m-n+1)}(\xi; k_1, k_2) - A(k_1, k_2) \right),$$

(6.72)

holds for all $s \in [t_{n-1}, t] \setminus C$ (for some countable subset $C$ with overwhelming probability).

In particular, if we define $F : [t_0, t] \to \mathbb{R}$ by

$$F(s) = F_{m, n; k_1, k_2}(s) = g_s^{(m-n+1)}(\xi; k_1, k_2) - A(k_1, k_2),$$

(6.73)

then there exist constants $c = c(m, n) > 0$ and $C = C(m, n) > 0$ such that

$$\partial_s \left( F(s) - C \frac{\psi^n}{N^\delta} \right) \leq -\frac{c}{\eta} \left( F(s) - C \frac{\psi^n}{N^\delta} \right), \quad \text{for each } s \in [t_{n-1}, t] \setminus C.$$

(6.74)

Thus, integration and the fact that $t_n - t_{n-1} = \frac{\psi}{m}$ together yield for $s \in [t_n, t]$ that

$$F(s) \leq \exp \left( -\frac{c}{\eta} (s - t_{n-1}) \right) \left( F(t_{n-1}) - C \frac{\psi^n}{N^\delta} \right) + C \frac{\psi^n}{N^\delta} \leq \exp \left( -\frac{c\psi}{m} \right) F(t_{n-1}) + C \frac{\psi^n}{N^\delta}.$$

(6.75)

To bound $|F(t_{n-1})|$, observe that

$$|F(t_{n-1})| \leq \|g_s^{(m-n+1)}\|_\infty + \|A(k_1, k_2)\|_\infty \leq \|g_s^{(m-n+1)}\|_\infty + C(m) \psi^m \leq Cm N^{m/2} + C(m) \psi^m.$$

(6.76)

Here, to deduce the first inequality, we used the definition (6.73) of $F$. To deduce the second, we used the fact that $\|g_s^{(b)}\|_\infty \leq \|g_s^{(b')}\|_\infty$ whenever $s' \leq s$ (since $S$ is the generator of a Markov process) and (6.65). To deduce the third inequality, we used (6.58) and (6.63).

Then (6.73), (6.74), and (6.76) together imply (6.71), from which we deduce the proposition. □

6.5. Proof of Lemma 6.10. We first establish Lemma 6.10 assuming (6.92) below; the latter will be proven as Lemma 6.11. Throughout this section, for $s \in [t_0, t]$ we occasionally abbreviate $\{\lambda_j\}_{1 \leq j \leq N} = \{\lambda_j(s)\}_{1 \leq j \leq N}$.

Proof of Lemma 6.10. The differentiability of $g_s(\xi) = g_s^{(b)}(\xi; k_1, k_2)$ follows from the general fact that the maximum of finitely many differentiable functions on an interval $I$ is itself differentiable,
Hence, the finite speed of propagation estimate (6.17) yields
\[ g_s(\xi) - A(k_1, k_2) > N^{-1}, \] (6.77)
for otherwise \ref{6.72} would hold. In this case, we set \( \tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_N) \) and use \ref{6.11} to write
\[ \partial_s (g_s(\xi) - A(k_1, k_2)) = S(s)g_s(\xi) = \sum_{0 < |j - k| \leq \ell} c_{jk}(t)2\xi_j(1 + 2\xi_j)(g_s(\xi^{jk}) - g_s(\xi)). \] (6.78)

Now let \( \text{supp} \tilde{\xi} = \{j_1, j_2, \ldots, j_h\} \). We claim that
\[ g_s(\xi^{jk}) \leq g_s(\xi), \] for any integers \( p \in [1, h] \) and \( k \in [j_p - \ell, j_p + \ell] \). (6.79)
To see this first observe that, since \( \tilde{\xi} \) maximizes \( g_s \) over \( \Omega(k_1, k_2) = \Omega^{(b)}(k_1, k_2), \) \ref{6.74} holds if \( \xi^{jk} \in \Omega(k_1, k_2) \). So, let us assume instead that \( \xi^{jk} \notin \Omega(k_1, k_2) \), meaning that there exists some \( v \in \{1, 2\} \) such a particle originally at site \( j_p \in I^{(b)}_{d+\psi \ell, v} \) in \( \xi \) jumped out of the interval \( I^{(b)}_{d+\psi \ell, v} \). This implies that \( k \notin I^{(b)}_{d+\psi \ell, v} \), since the particle can jump at most \( \ell \) sites by \ref{6.70}, but the disjoint intervals \( I^{(b)}_{d+\psi \ell, 1} \) and \( I^{(b)}_{d+\psi \ell, 2} \) are at least \( d \gg \ell \) sites apart by hypothesis.

Then, \ref{6.51} and \ref{6.52} together yield \( (\text{Av} h_{t_0})(\xi) - A(k_1, k_2) = 0 \) unless \( \text{supp} \xi \subseteq I^{(b)}_d \). Thus, any particle configuration \( \xi \in \mathbb{N}^N \) in the support of \( \text{Av} h_{t_0} - A(k_1, k_2) \) must satisfy \( d_{\xi}(\xi^{jk}) \geq \psi \ell \). Hence, the finite speed of propagation estimate \ref{6.17} yields
\[ |g_s(\xi^{jk}) - A(k_1, k_2)| = |U_s(t_0, s)((\text{Av} h_{t_0}) - A(k_1, k_2))\xi^{jk}| \lesssim \exp \left( -\frac{c\psi^2}{2} \right) < \frac{1}{N}. \] (6.80)
which contradicts \ref{6.74}.

We now set \( z_{jp} = \lambda_{jp} + i\eta \) and use \ref{6.70}, the definition \ref{6.4} of the \( c_{ij} \) and the fact that \( \xi_j(2\xi_k + 1) \geq 1 \) when \( \xi^{jk} \neq \xi \) to bound the sum in \ref{6.78} over \( j \) by
\[ \sum_{p=1}^{h} \sum_{0 < |k - j_p| \leq \ell} \frac{g_s(\xi^{jk}) - g_s(\xi)}{N(\lambda_{jp} - \lambda_k)^2} \lesssim \frac{1}{N} \sum_{p=1}^{h} \sum_{0 < |k - j_p| \leq \ell} \frac{g_s(\xi^{jk}) - g_s(\xi)}{N(\lambda_{jp} - \lambda_k)^2 + \eta^2} \] (6.81)
\[ = \frac{1}{N\eta} \sum_{p=1}^{h} \sum_{0 < |k - j_p| \leq \ell} \left( \frac{\text{Im} g_s(\xi^{jk})}{z_{jp} - \lambda_k} - \frac{\text{Im} A(k_1, k_2)}{z_{jp} - \lambda_k} \right) \] (6.82)
\[ - \frac{1}{N\eta} (g_s(\xi) - A(k_1, k_2)) \sum_{p=1}^{h} \sum_{0 < |k - j_p| \leq \ell} \text{Im} \frac{1}{z_{jp} - \lambda_k}. \] (6.83)
Since the first bound in \ref{6.18} and the fact that \( N^{-1}\ell \gg \eta \) yields
\[ \sum_{p=1}^{h} \sum_{0 < |k - j_p| \leq \ell} \text{Im} \frac{1}{z_{jp} - \lambda_k} \geq \sum_{p=1}^{h} \sum_{0 < |k - j_p| \leq \ell} \frac{\eta}{(\lambda_{jp} - \lambda_k)^2 + \eta^2} \geq \sum_{p=1}^{h} \sum_{k : |\lambda_k - \lambda_{jp}| \leq \eta} \frac{\eta}{2\eta^2} \geq cN, \] (6.84)
we have that
\[ \sum_{p=1}^{h} \sum_{0 < |k - j_p| \leq \ell} \text{Im} \frac{1}{z_{jp} - \lambda_k} \geq -\frac{c}{\eta} (g_s(\xi) - A(k_1, k_2)). \] (6.85)
To bound (6.82), we fix \( p \in [1, h] \), recall \( B \) from (6.6), and employ the decomposition

\[
\frac{1}{N} \sum_{0 < |k - j_p| \leq \ell} \left( \text{Im} \frac{g_s(\tilde{\xi}_{j_p}^{(k)})}{z_{j_p} - \lambda_k} - \text{Im} \frac{A(k_1, k_2)}{z_{j_p} - \lambda_k} \right)
\]

(6.86)

\[
= \frac{1}{N} \text{Im} \sum_{0 < |k - j_p| \leq \ell} \frac{(U_{G}(t_0, s)A_{\ell}^{(b)} h_{t_0})(\tilde{\xi}_{j_p}^{(k)}) - (A_{\ell}^{(b)} U_{G}(t_0, s)h_{t_0})(\tilde{\xi}_{j_p}^{(k)})}{z_{j_p} - \lambda_k}
\]

(6.87)

\[+ \frac{1}{N} \text{Im} \sum_{0 < |k - j_p| \leq \ell} \frac{(A_{\ell}^{(b)} U_{G}(t_0, s)h_{t_0})(\tilde{\xi}_{j_p}^{(k)}) - (A_{\ell}^{(b)} U_{G}(t_0, s)h_{t_0})(\tilde{\xi}_{j_p}^{(k)})}{z_{j_p} - \lambda_k}
\]

(6.88)

\[+ \frac{1}{N} \text{Im} \sum_{0 < |k - j_p| \leq \ell} \frac{(A_{\ell}^{(b)} U_{G}(t_0, s)h_{t_0})(\tilde{\xi}_{j_p}^{(k)}) - (A_{\ell}^{(b)} U_{G}(t_0, s)h_{t_0})(\tilde{\xi}_{j_p}^{(k)})}{z_{j_p} - \lambda_k}
\]

(6.89)

The terms (6.87) and (6.88) may be bounded as in the content following [38] (3.64)\(^3\)

\[
\| (6.87) \| \lesssim_n \psi^\eta \frac{t - t_0}{\ell}, \quad \| (6.88) \| \lesssim_n \psi^\eta \frac{N(t - t_0)}{\ell}.
\]

(6.90)

For brevity we only prove here the second inequality in (6.90) and refer the reader to [38] for details on the first. Using Lemma 6.11 (which applies as \( \text{supp} \tilde{\xi}_{j_p}^{(k)} \subseteq [(1/2 - c)N, (1/2 + c)N] \), since \( \text{supp} \tilde{\xi} \subseteq [(1/2 - c_1)N, (1/2 + c_1)N] \) and \( |k - j_p| \leq \ell \), we find

\[
\left| (A_{\ell}^{(b)} U_{G}(t_0, s)h_{t_0})(\tilde{\xi}_{j_p}^{(k)}) - (A_{\ell}^{(b)} U_{G}(t_0, s)h_{t_0})(\tilde{\xi}_{j_p}^{(k)}) \right|
\]

\[
\leq \left| (U_{G}(t_0, s)h_{t_0} - U_{G}(t_0, s)h_{t_0})(\tilde{\xi}_{j_p}^{(k)}) \right| \lesssim_n \psi^\eta \frac{N(t - t_0)}{\ell},
\]

(6.91)

which implies the second bound in (6.90).

Next, as Lemma 6.11 below, we show that, for any fixed \( p \in [1, h] \) and \( s \in [t_0, t] \setminus C \),

\[
\| (6.89) \| \lesssim_n \psi^\eta \| h_s(\tilde{\xi} \setminus j_p) - A(\chi_1^{(b+1)}(\tilde{\xi} \setminus j_p), \chi_2^{(b+1)}(\tilde{\xi} \setminus j_p)) \|_N N^{-b}.
\]

(6.92)

Combining these bounds and using the choices of \( t_0 \) from (6.9) \( ; \ell \) from (6.8) \( ; \) and \( d \) from (6.43), we obtain for fixed \( p \in [1, h] \) and \( s \in [t_0, t] \setminus C \) that (6.86) satisfies

\[
\frac{1}{N} \sum_{0 < |k - j_p| \leq \ell} \left| (A_{\ell}^{(b)} U_{G}(t_0, s)h_{t_0})(\tilde{\xi}_{j_p}^{(k)}) - (A_{\ell}^{(b)} U_{G}(t_0, s)h_{t_0})(\tilde{\xi}_{j_p}^{(k)}) \right|
\]

\[
\lesssim_n \max_{\xi \in \Omega^{(b+1)}(k_1 - 1, k_2), \Omega^{(b+1)}(k_1, k_2 - 1)} \psi^\eta \| h_s(\tilde{\xi}) - A(\chi_1^{(b+1)}(\tilde{\xi}), \chi_2^{(b+1)}(\tilde{\xi})) \|_N N^{-b}.
\]

(6.93)

(6.94)

Summing over \( p \in [1, h] \), inserting this into (6.82), and using (6.85) then completes the proof. \( \square \)

We conclude this section by establishing (6.92).

**Lemma 6.11.** Retain the hypotheses and notation of the proof of Theorem 5.8. Then equation (6.92) holds.

\(^3\)When bounding (6.87), \( h_t^{(b)} \) plays the role of the interval \([b_1 - a, b_2 + a]\) in [38].
Proof. From complete delocalization (6.10), we have with overwhelming probability that
\[
\max_{p \in [1, h]} h_s(\xi^{p,k}) \lesssim_n \psi^n. \tag{6.95}
\]
Now, for any particle configuration \( \xi \in \mathbb{N}^N \), define \( a_\xi = a_{\xi,s} \in [0, 1] \) through the equation
\[
\text{Av}(f)(\xi) = a_\xi h_s(\xi) + (1 - a_\xi)A(k_1, k_2) \quad \text{if } h_s(\xi) \neq A(k_1, k_2), \tag{6.96}
\]
and set \( a_\xi = 0 \) if \( h_s(\xi) = A(k_1, k_2) \). Since \( U(t_0, s)h_{t_0} = h_s \), the first term of (6.89) equals
\[
\frac{1}{N} \text{Im} \sum_{0 < |k - j_p| \leq \ell} a_{\xi^{j_p,k}} h_s(\xi^{j_p,k}) + (1 - a_{\xi^{j_p,k}})A(k_1, k_2) \tag{6.97}
\]
\[
= \frac{1}{N} \text{Im} \sum_{0 < |k - j_p| \leq \ell} a_{\xi} h_s(\xi^{j_p,k}) \frac{1}{z_{j_p} - \lambda_k} + \frac{1}{N} \text{Im} \sum_{0 < |k - j_p| \leq \ell} (a_{\xi^{j_p,k}} - a_{\xi}) h_s(\xi^{j_p,k}) \frac{1}{z_{j_p} - \lambda_k} \tag{6.98}
\]
\[
= \frac{1}{N} \text{Im} \sum_{0 < |k - j_p| \leq \ell} a_{\xi} h_s(\xi^{j_p,k}) \frac{1}{z_{j_p} - \lambda_k} + O_n \left( \frac{\ell \psi^n}{d} \right). \tag{6.99}
\]
In the last line we used (6.55); (6.98): the fact that \( \text{Im} \sum_{0 < |k - j_p| \leq \ell} (z_{j_p} - \lambda_k)^{-1} \leq \text{Im} m_N(s, z_{j_p}) \leq C \), which follows from the first part of Lemma 6.2, (4.13), and (4.16); and the bound
\[
|a_\xi - a_{\xi^{j_p,k}}| \leq \frac{d(\xi, \xi^{j_p,k})}{d} \leq \frac{\ell}{d} \tag{6.100}
\]
which follows from the definition of \( a_\xi \) and the definition of the \( \text{Av} \) operator, since the sums defining \( \text{Av}(f)(\xi) \) and \( \text{Av}(f)(\xi^{j_p,k}) \) can differ in at most \( \ell \) terms. The equation (6.97) implies
\[
1 \sum_{0 < |k - j_p| \leq \ell} \frac{\eta h_s(\xi^{j_p,k})}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} - \frac{\eta A(k_1, k_2)}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} + O_n \left( \frac{\ell \psi^n}{d} \right). \tag{6.101}
\]
Through (6.18) and a dyadic decomposition analogous to the one used in the proof of Lemma 5.7 (see also the proof of 38 Lemma 3.5 for more details), one has with overwhelming probability that
\[
\frac{1}{N} \sum_{|k - j_p| > \ell} \frac{\eta h_s(\xi^{j_p,k})}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} \leq \frac{C N \eta}{\ell}, \tag{6.102}
\]
which by (6.95) implies with overwhelming probability
\[
\left| \frac{1}{N} \sum_{|k - j_p| > \ell} \frac{\eta h_s(\xi^{j_p,k})}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} \right| \leq \frac{C N \eta \psi^n}{\ell}. \tag{6.103}
\]
Also,
\[
\frac{1}{N} \sum_{k=1}^N \frac{\eta}{(\lambda_{j_p} - \lambda_k)^2 + \eta^2} = \text{Im} \sum_{k=1}^N \frac{1}{z_{j_p} - \lambda_k} = \text{Im} m_N(s, z_{j_p}). \tag{6.104}
\]
We conclude from (6.99), (6.100), (6.95), (6.101), and (6.102) that with overwhelming probability,

\[
\frac{a_\xi}{N} \sum_{k=1}^{N} \frac{\eta h_s(\xi_j k)}{(\lambda_j - \lambda_k)^2 + \eta^2} - a_\xi A(k_1, k_2) \text{Im} m_N(s, z_{j_p}) + O_n \left( \frac{\psi^n}{N\eta} + \frac{\psi^N N}{\ell} + \frac{\ell\psi^n}{d} \right),
\]

where we also used (6.55) and (6.95) to restore the term with index \( k = j_p \) in the sum, which accrues an error of size \( O(\psi^n / N\eta) \).

By (4.12) and (4.13), there exists a constant \( c > 0 \) such that, with overwhelming probability,

\[
\begin{align*}
|\text{Im} m_N(s, z_{j_p}) - \text{Im} m_\alpha(\gamma_{j_p} + i\eta)| &\leq |\text{Im} m_N(s, z_{j_p}) - \text{Im} m_\alpha(z_{j_p})| + |\text{Im} m_\alpha(z_{j_p}) - \text{Im} m_\alpha(\gamma_{j_p} + i\eta)| + O(N^{-c}) \\
&\leq |\text{Im} m_\alpha(z_{j_p}) - \text{Im} m_\alpha(\gamma_{j_p} + i\eta)| + O(N^{-c}).
\end{align*}
\]

Moreover, by rigidity estimate from from the first part of Lemma 6.2, the combination of (4.23) and (4.24), and (4.3), we have that

\[
|\text{Im} m_\alpha(z_{j_p}) - \text{Im} m_\alpha(\gamma_{j_p} + i\eta)| \leq C|\lambda_{j_p}(s) - \gamma_{j_p}| + C\eta \leq C \left( \frac{\psi}{N} + N^{-c} + \eta \right) \leq CN^{-c},
\]

Thus, (6.106) and (6.107) yield

\[
|\text{Im} m_N(s, z_{j_p}) - \text{Im} m_\alpha(\gamma_{j_p} + i\eta)| \leq CN^{-c},
\]

which, together with (6.55), yields

\[
|A(k_1, k_2) \text{Im} m_N(s, z_{j_p}) - A(k_1, k_2) \text{Im} m_\alpha(\gamma_{j_p} + i\eta)| \leq n \psi^n N^{-c}
\]

with overwhelming probability. Therefore (6.103) implies

\[
\frac{a_\xi}{N} \sum_{k=1}^{N} \frac{\eta h_s(\xi_j k)}{(\lambda_j - \lambda_k)^2 + \eta^2} - a_\xi A(k_1, k_2) \text{Im} m_\alpha(\gamma_{j_p} + i\eta)
\]

\[
\text{Im} m_N(s, z_{j_p}) - \text{Im} m_\alpha(\gamma_{j_p} + i\eta)
\]

holds with overwhelming probability, where we used that \( \psi^n N^{-c} = O(N^{-\alpha}) \) if \( \alpha \) is chosen sufficiently small (depending only on the value \( m \) from Lemma 6.10). Using the definition of \( h_s \) from (6.13) and recalling from (3.12) that \( a(2j) = (2j - \ell)! \), we find that the first term on the right side of (6.110) is equal to

\[
\sum_{k=1}^{N} \left[ \prod_{1 \leq q \leq h} \left( \frac{N(q, u_{j_p})}{a(2(\xi_q - 1)_{p=q})} \right)^{2} \left( \frac{\eta(q, u_k(s))}{a(2(\xi_k - 1)_{p=q})} \right)^{2} a(2(\xi_k)) \right] \lambda, F_{t_0}
\]

\[
= a_\xi \sum_{k=1}^{N} \left[ \prod_{1 \leq q \leq h} \left( \frac{N(q, u_{j_p})}{a(2(\xi_q - 1)_{p=q})} \right)^{2} \left( \frac{\eta(q, u_k(s))}{a(2(\xi_k - 1)_{p=q})} \right)^{2} \sum_{k=1}^{N} \frac{\eta(q, u_k(s))}{a(2(\xi_k - 1)_{p=q})} \right] \lambda, F_{t_0}
\]

\[
\leq a_\xi \sum_{k=1}^{N} \left[ \prod_{1 \leq q \leq h} \left( \frac{N(q, u_{j_p})}{a(2(\xi_q - 1)_{p=q})} \right)^{2} \left( \frac{\eta(q, u_k(s))}{a(2(\xi_k - 1)_{p=q})} \right)^{2} \sum_{k=1}^{N} \frac{\eta(q, u_k(s))}{a(2(\xi_k - 1)_{p=q})} \right] \lambda, F_{t_0}
\]

. (6.113)
We now have with overwhelming probability that
\[
\sum_{k=1}^{N} \frac{\eta N^2}{(\lambda_j - \lambda_k)^2 + \eta} = \text{Im} \left( \langle q, R(s, \lambda_j, s + i\eta) q \rangle \right)
\]
\[
= \text{Im} \left( \langle q, R(t_0, \gamma_j, s + i\eta) q \rangle + O_{n,c} \left( \frac{\psi^4}{\sqrt{N\eta}} + N^{\epsilon} \left( \frac{t-t_0}{t_0 - \tau} + N^{-\epsilon} \right) \right) + N^{\epsilon} \left( N^{-\epsilon} + \frac{\psi N^{-1} + 2\eta}{t_0 - \tau} \right) \right).
\]  
(6.116)

In the last equality, we used (6.21), (6.22), $N\eta \gg N^{1/2}$, $s \in [t_0, t]$, and the overwhelming probability estimate
\[
|\lambda_j - \gamma_j| \leq |\lambda_j - \gamma_j| + |\gamma_j - \gamma_j| \leq \frac{\psi}{N} + \eta,
\]
which follows from the rigidity estimate in the first item in Lemma 6.2 and (4.24) (with $\eta \gg N^{-1/2}$).

By (6.16) and the fact that $\psi = N^x$, this yields
\[
\leq a e^{-E} \left[ \prod_{1 \leq q \leq h} \left( \frac{N^{\epsilon}}{\eta} \right)^{\xi^{q-1}_{p=q}} \Gamma \left( \frac{t-t_0}{t_0 - \tau} + N^{-\epsilon} + \frac{\psi N^{-1} + 2\eta}{t_0 - \tau} \right) \right] + O_{n,c}. \]
(6.118)

We now recognize that the second factor inside the expectation on the right side of (6.118) is measurable with respect to $F_{t_0}$. We may therefore factor it out of the expectation and rewrite the previous bound as
\[
\leq a e^{-E} \left[ \prod_{1 \leq q \leq h} \left( \frac{N^{\epsilon}}{\eta} \right)^{\xi^{q-1}_{p=q}} \Gamma \left( \frac{t-t_0}{t_0 - \tau} + N^{-\epsilon} + \frac{\psi N^{-1} + 2\eta}{t_0 - \tau} \right) \right] + O_{n,c}. \]
(6.120)

Using again the computation (6.115) with $s = t$, and (6.95) yields
\[
\leq a e^{-E} \left[ \prod_{1 \leq q \leq h} \left( \frac{N^{\epsilon}}{\eta} \right)^{\xi^{q-1}_{p=q}} \Gamma \left( \frac{t-t_0}{t_0 - \tau} + N^{-\epsilon} + \frac{\psi N^{-1} + 2\eta}{t_0 - \tau} \right) \right] + O_{n,c}. \]
(6.122)

This, together with the definition (6.11) of $A(q, j)$ and (6.110), gives
\[
\leq a e^{-E} \left[ \prod_{1 \leq q \leq h} \left( \frac{N^{\epsilon}}{\eta} \right)^{\xi^{q-1}_{p=q}} \Gamma \left( \frac{t-t_0}{t_0 - \tau} + N^{-\epsilon} + \frac{\psi N^{-1} + 2\eta}{t_0 - \tau} \right) \right] + O_{n,c}. \]
(6.124)

\[
\leq a e^{-E} \left[ \prod_{1 \leq q \leq h} \left( \frac{N^{\epsilon}}{\eta} \right)^{\xi^{q-1}_{p=q}} \Gamma \left( \frac{t-t_0}{t_0 - \tau} + N^{-\epsilon} + \frac{\psi N^{-1} + 2\eta}{t_0 - \tau} \right) \right] + O_{n,c}. \]
(6.125)

\[
\leq a e^{-E} \left[ \prod_{1 \leq q \leq h} \left( \frac{N^{\epsilon}}{\eta} \right)^{\xi^{q-1}_{p=q}} \Gamma \left( \frac{t-t_0}{t_0 - \tau} + N^{-\epsilon} + \frac{\psi N^{-1} + 2\eta}{t_0 - \tau} \right) \right] + O_{n,c}. \]
(6.126)
Recalling that in \((6.14), (6.38), (6.9),\) and \((6.43),\) we fixed small \(\delta(m) > 0\) such that \(\delta = 50(1 + m)\epsilon\) (recall \(\psi = N\psi\)) and chose parameters so that \(N^{-1/2} \ll \eta \ll t_0 \leq t:\)

\[
\eta = N^{-\alpha} \psi, \quad \ell = \psi^{5m+1} N^{1+\eta}, \quad t_0 = \tau - \psi \eta, \quad \tau = \tau - N^{70} \psi \eta, \quad \tilde{d} = \ell \psi^{5m} N^5. \tag{6.127}
\]

Then choosing \(\epsilon\) sufficiently small, we deduce from \((6.129)\) that

\[
a_\xi \Im m_\alpha(\gamma_j + \eta) \left( A(\mathbf{q}, \mathbf{j}_p) h_s(\tilde{\xi} \setminus j_p) - A(k_1, k_2) \right) \leq a_\xi \Im m_\alpha(\gamma_j + \eta) \left( A(\mathbf{q}, \mathbf{j}_p) h_s(\tilde{\xi} \setminus j_p) - A(k_1, k_2) \right) + O_n \left( N^{-\delta} \right). \tag{6.128}
\]

To complete the argument, it suffices to show that

\[
a_\xi \Im m_\alpha(\gamma_j + \eta) \left( A(\mathbf{q}, \mathbf{j}_p) h_s(\tilde{\xi} \setminus j_p) - A(k_1, k_2) \right) \leq C \psi \left| h_s(\tilde{\xi} \setminus j_p) - A(\lambda_1^{(b+1)}(\tilde{\xi} \setminus j_p), \lambda_2^{(b+1)}(\tilde{\xi} \setminus j_p)) \right| + O_n \left( N^{-\delta} \right). \tag{6.129}
\]

We recall that, for \(j_p \in [(1/2 - c)N, (1/2 + c)N],\) there exists \(C > 0\) such that \(\left| \Im m_\alpha(\gamma_j + \eta) \right| < C,\) which holds by \((4.9).\) This, together with \((6.50),\) and the definition \((6.41)\) of \(A(\mathbf{q}, \mathbf{\xi}),\) we obtain

\[
\Im m_\alpha(\gamma_j + \eta) \left( A(\mathbf{q}, \mathbf{j}_p) h_s(\tilde{\xi} \setminus j_p) - A(k_1, k_2) \right) \leq C \psi \left| h_s(\tilde{\xi} \setminus j_p) - A(\lambda_1^{(b+1)}(\tilde{\xi} \setminus j_p), \lambda_2^{(b+1)}(\tilde{\xi} \setminus j_p)) \right| + O_n \left( N^{-\delta} \right). \tag{6.130}
\]

Combining the last line with \((6.55)\) and using the bound \(\left| \Im m_\alpha(\gamma_j + \eta) \right| < C\) again, we see

\[
\Im m_\alpha(\gamma_j + \eta) \left( A(\mathbf{q}, \mathbf{j}_p) h_s(\tilde{\xi} \setminus j_p) - A(k_1, k_2) \right) \leq C \psi \left| h_s(\tilde{\xi} \setminus j_p) - A(\lambda_1^{(b+1)}(\tilde{\xi} \setminus j_p), \lambda_2^{(b+1)}(\tilde{\xi} \setminus j_p)) \right| + O_n \left( N^{-\delta} \right). \tag{6.131}
\]

Then \((6.129)\) follows because \(|a_{\xi}| \leq 1.\)

\[\square\]

### 6.6. Outline of the proof of Theorem 3.8 in the general case.

Previously, we assumed when defining the interval \(I_n\) in \((6.44)\) that \(|\text{supp} \mathbf{\xi}| = 2.\) Consider now the general case where \(|\text{supp} \mathbf{\xi}| = n' \geq 1.\) Set \(\text{supp} \mathbf{\xi} = \{i_1, i_2, \ldots, i_{n'}\},\) with \(i_1 < i_2 < \cdots < i_{n'},\) recall \(m = N(\mathbf{\xi}),\) and define

\[
I_n^{(b)}(\mathbf{\xi}) = \bigcup_{j=1}^{n'} I_n^{(b)}(\mathbf{\xi}), \quad \text{where for all } 1 \leq j \leq n', \quad I_n^{(b)}(\mathbf{\xi}) = [i_j - 10b \tilde{d} - a, i_j + 10b \tilde{d} + a]. \tag{6.132}
\]

under the assumption that all the intervals \(I_{2 \mathbf{\nu}}^{(m)}\) are disjoint; we will describe the appropriate definition when this is not the case below. We further set, for each \(j \in [1, n']\) and particle configuration \(\mathbf{\xi} \in \mathbb{N}^N,\)

\[
\chi_j^{(b)}(\mathbf{\xi}) = \sum_{i \in I_n^{(b)}(\mathbf{\xi}, \mathbf{\nu}, j)} \xi_i. \tag{6.133}
\]
For any integer \( n' \geq 1 \) and \( n' \)-tuple \( k = (k_1, k_2, \ldots, k_n) \) of nonnegative integers, we define

\[
\Omega^{(b)}(k) = \left\{ \xi \in \mathbb{N}^N : \text{supp } \xi \subset I^{(b)}_{\chi_j}(\xi) = k_j \text{ for } j \in [1, n'] \right\};
\]

\[
\Omega^{(b)}(n) = \bigcup_{|k|=n} \Omega(k); \quad A(k) = \prod_{j=1}^{n'} A(q, i_j)^{k_j},
\]

where \(|k| = \sum_{j=1}^{n'} k_j\). We also define the restricted intervals

\[
\Phi^{(b)}(k) = \left\{ \xi \in \mathbb{N}^N : \text{supp } \xi \subset I^{(b)}_{-\psi, \ell}, \sum_{i \in I^{(b)}_{-\psi, \ell}} \xi_i = k_j \text{ for } j \in [1, n'] \right\};
\]

\[
\Phi^{(b)}(n) = \bigcup_{|k|=n} \Phi^{(b)}(k).
\]

The operator \( \text{Flat}_{\psi, k} \) is then defined as in (6.51), except with \( A(k_1, k_2) \) there replaced by \( A(k) \) here. Given this change, \( Av \) is defined as in (6.52). Additionally define the flow \( g_\psi(s) = g^{(b)}_\psi(\xi; k) \) as in (6.53), and also the maximizer \( \xi = \xi(s) = \hat{\xi}(s, k) \in \mathbb{N}^N \) by

\[
g_\psi(\hat{\xi}) = \max_{\xi \in \Omega(k)} g^{(b)}_\psi(\xi; k).
\]

Now the argument then proceeds as before. Specifically, the dichotomy in Lemma 6.10 becomes that \( g^{(b)}_\psi(\hat{\xi}) \) satisfies either

\[
g^{(b)}_\psi(\hat{\xi}) = \frac{1}{N},
\]

or

\[
\partial_s \left( g^{(b)}_\psi(\hat{\xi}) - A(k) \right) \leq \frac{C}{\eta} \left( \psi \max_{\xi} \left| h_s(\xi) - A(\chi^{(b+1)}_1(\xi), \ldots, \chi^{(b+1)}_{n'}(\xi)) \right| + N^{-b} \right) - \frac{C}{\eta} \left( g^{(b)}_\psi(\hat{\xi}) - A(k) \right),
\]

where the maximum in (6.144) is taken over \( \xi \in \bigcup_{1 \leq j \leq n'} \Phi^{(b+1)}(k_1, \ldots, k_{j-1}, 1, \ldots, k_{n'}) \). The proof of this claim is the same as the one in Section 6.5, and the proof of the main result given this claim is the same as in Section 6.4.

When the \( I^{(m)}_{2d,j} \) are not disjoint, we instead partition \( \bigcup_{j=1}^{n'} I^{(m)}_{2d,j} \) into a union of disjoint intervals \( \hat{I}^{(m)}_v \) as follows. There exist an integer \( v \in [1, n'] \) and indices \( 1 \leq j_1 < j_2 < \ldots < j_v \leq n' \) such that the intervals

\[
\hat{I}^{(m)}_v = \bigcup_{j=j_u}^{j_{u+1}-1} I^{(m)}_{2d,j}.
\]

are mutually disjoint over all \( u \in [1, v] \) (where we set \( j_{v+1} = n' + 1 \)), but such that \( I^{(m)}_{2d,j} \) is nonempty for each \( j \in [j_u, j_{u+1} - 2] \). We then can make the above definitions using instead the intervals

\[
J^{(b)}_{a,i} = [i_{j_i} - 10b\tilde{d} - a, i_{j_{i+1}} + 10b\tilde{d} + a],
\]

where \( j^{(b)}_{a,i} \) is defined as in (6.52). Additionally define the flow
which are disjoint for all $a \in [0, 2\tilde{d}]$ (since the $\tilde{T}^{(m)}_a$ are). For instance, we set $\chi^{(b)}_j(\xi) = \sum_{i \in J^{(b)}_{d+\psi,t,j}} \xi_i$, and
\[
\Omega^{(b)}(k) = \{ \xi \in \mathbb{N}^N : \text{supp} \xi \subset J^{(b)}_{d+\psi,t} \chi^{(b)}_j(\xi) = k_j \text{ for } j \in [1, v] \} ,
\]
and similarly for the other intervals and quantities.

Let us motivate this procedure by very briefly considering the case $|\text{supp} \zeta| = 2$, with $\text{supp} \zeta = \{ i_1, i_2 \}$, as in the material after (6.43). However, we now suppose that $|i_2 - i_1| \leq 20md + 2\tilde{d}$, so that the intervals defined in (6.45) are not disjoint. Then, according to the above, we instead work on the single connected interval $J^{(b)}_{a,1} = [i_1 - 10bd - a, i_2 + 10bd + a]$. Then $A\psi$ is defined as in (6.52), and we observe that Lemma 6.9 holds with (6.56) replaced by the inequality
\[
\max_{b \in [0, m+1]} \max_{\xi \in \Omega^{(m+1)}(k)} |A(q, \xi) - A(k)| < CN^{-3b},
\]
so that $A(q, \xi)$, up to a small error, does not depend on $\xi$ for $\xi \in \Omega^{(m+1)}(k)$. Additionally, it is permissible to apply the finite speed of propagation estimate as in the material immediately following (6.79), which would not be the case if we retained two disjoint but nearby or overlapping intervals and attempted the original argument (since then a particle could jump from one interval to the other). The same reasoning underlies the argument in the general case.

7. Scaling limit

In Section 4 we identified the moments of $X_t$ through entries of the resolvent $R(t, z)$. Here, we determine the scaling limit of these entries, as $N$ tends to infinity. In Section 7.1 we recall some preliminary material from previous works. In Section 7.2 we compute the scaling limits of the moments $\mathbb{E}[\text{Im} R_u(E + i\eta)^p]$ for $p \in \mathbb{N}$, as $\eta$ tends to 0, and establish Proposition 2.4 as a consequence. In Section 7.3 we compute the scaling limits of the moments $\mathbb{E}[\text{Im} R_u(E + i\eta)^p]$, as $N$ tends to $\infty$, and prove Theorem 5.3. Throughout this section, we recall $t$ from (3.9).

7.1. Order parameter for $X_t$. In this section we recall several results on the diagonal resolvent entries of $X_t$. From Definition 2.3 and the content following it, the scaling limits of these entries as $N$ tends to $\infty$ and $z \in \mathbb{H}$ is fixed are given by the random variable $R_\ast(z)$.

A key property of $R_\ast(z)$, shown in [28], is that it satisfies a “recursive distributional equation,” which may be considered as a limiting analogue of the usual Schur complement formula.

**Lemma 7.1** ([28] Theorem 4.1). Denote by $\{\xi_k\}_{k \geq 1}$ a Poisson process on $\mathbb{R}_+$ with intensity measure $(\frac{2}{\pi})x^{-\alpha/2-1} \, dx$. For any $z \in \mathbb{H}$, the random variable $R_\ast(z) : \mathbb{H} \to \mathbb{H}$ satisfies the equality in law
\[
R_\ast(z) \overset{d}{=} \left( z + \sum_{k=1}^{\infty} \xi_k R_k(z) \right)^{-1},
\]
where $(R_k(z))_{k \geq 1}$ is an i.i.d. sequence with distribution $R_\ast(z)$ independent from the process $\{\xi_k\}_{k \geq 1}$.

Next we discuss a certain order parameter, which is essentially given by the $\frac{\alpha}{2}$-th moment of (linear combinations of the imaginary and real parts of) $R_\ast$. In what follows, for any $u, h \in \mathbb{C}$, we recall from [29] Section 5.1 the inner product
\[
h.u = (\text{Re} \, u)h + (\text{Im} \, u)\bar{h}.
\]
Definition 7.2. For any \( z \in \mathbb{H} \) and \( u \in \mathbb{C} \), we define \( \gamma_z^*(u) : \mathbb{C} \to \mathbb{C} \) by
\[
\gamma_z^*(u) = \Gamma \left( 1 - \frac{\alpha}{2} \right) \mathbb{E} \left[ ( -iR_z(s), u )^{\alpha/2} \right]. \tag{7.3}
\]

The following lemma establishes a lower bound on Re \( \gamma_z^* \) and the existence of a limit for \( \gamma_z^* \) as Im \( z \) tends to 0. It will be proved in Appendix A below.

Lemma 7.3. There exists a constant \( c > 0 \) such that the following two statements hold. First, we have the uniform lower bound
\[
\inf_{z \in \mathbb{H}} \inf_{u \in \mathbb{S}^1_z} \text{Re} \, \gamma_z^*(u) > c. \tag{7.4}
\]
Second, for every real number \( E \in [-c, c] \), there exists a function \( \gamma_E^* : \mathbb{S}^1_+ \to \mathbb{C} \) such that the following holds. Let \( \{ E_j \}_{j \geq 1} \) and \( \{ \eta_j \}_{j \geq 1} \) denote sequences of real numbers such that \( \lim_{N \to \infty} E_N = E \) and \( \lim_{N \to \infty} \eta_N = 0 \). Then, denoting \( z_N = E_N + i\eta_N \), we have
\[
\lim_{N \to \infty} \sup_{u \in \mathbb{S}^1_z} | \gamma_{z_N}^*(u) - \gamma_E^*(u) | = 0. \tag{7.5}
\]

We next recall from [5 (7.37)] notation for a particular analog of \( \gamma_z^* \) for finite \( N \) that will be useful for us. For any real number \( s \geq 0 \) and index set \( \mathcal{I} \subset \mathbb{N} \cap [1, N] \), let \( \mathbf{R}(s, z) = (X_s - z)^{-1} = \{ R_{ij}(s, z) \} \) denote the resolvent of \( X_s^{(\mathcal{I})} \), which we define as the matrix \( X_s \) but whose rows and columns with indices in \( \mathcal{I} \) set to zero.

Definition 7.4. Fix a real number \( s \geq 0 \). For any index set \( \mathcal{I} \subset \mathbb{N} \cap [1, N] \) and complex number \( z \in \mathbb{H} \), the function \( \gamma_z^{(\mathcal{I})} : \mathbb{K}^+ \to \mathbb{C} \) is defined by
\[
\gamma_z^{(\mathcal{I})}(u) = \gamma_z^{(\mathcal{I})}(u) = \Gamma \left( 1 - \frac{\alpha}{2} \right) \frac{1}{|\mathcal{I}|} \sum_{1 \leq k \leq N} ( -iR_{kk}^{(\mathcal{I})}(s, z), u )^{\alpha/2} \mathbb{E} \left[ |g_k|^{\alpha} \right], \tag{7.6}
\]
where \( g = (g_1, g_2, \ldots, g_N) \) is a vector of i.i.d. standard Gaussian random variables\footnote{These Gaussian variables will be useful in \[\text{(7.31)}\] below, when applying a Hubbard–Stratonovich type transform.} independent from \( X_s \). If \( \mathcal{I} = \emptyset \), we abbreviate \( \gamma_z = \gamma_z^{(\mathcal{I})} \).

We next have the following local law stating that \( \mathbb{E}[\gamma_z(u)] \approx \gamma_z^*(u) \). It is a consequence of [5 Theorem 7.6], where the \( \Omega_z \) there is equal to \( \gamma_z^* \) here by [30 Lemma 4.4].

Lemma 7.5 ([5 Theorem 7.6], [30 Lemma 4.4]). There exist constants \( K > 0 \) and \( C = C(\delta) > 0 \) such that the following holds. Fix a real number \( \delta > 0 \) with \( \delta < \max \{ \frac{1}{2}, \frac{1}{2} \} \), and abbreviate \( \mathcal{D} = \mathcal{D}, \delta \) (recall (4.8)). Then, for any \( s \in [0, t] \), we have
\[
\sup_{z \in \mathcal{D}} \sup_{u \in \mathbb{S}^1_z} \left| \mathbb{E}[\gamma_z(u)] - \gamma_z^*(u) \right| \leq CN^{-\alpha \delta/8}, \tag{7.7}
\]
where expectation is taken with respect to both \( X_s \) and the Gaussian variables \( g_k \).

Like Lemma 4.2 [5 Theorem 7.6] was only stated in the case \( s = 0 \) in [5], but it is quickly verified that the same proof applies for arbitrary \( s \in [0, t] \), especially since \( \mathbf{H} + s^{1/2} \mathbf{W} \) satisfies the conditions in Definition 2.1, for \( s \in [0, t] \) if \( \mathbf{H} \) does.

We next have the following lemma, which can be viewed as an analog of Lemma 7.4 for finite \( N \). In what follows, we recall from Definition 2.1 that there exist random variables \( \{ Z_{ij} \}_{1 \leq i, j \leq g} \) that are
mutually independent (up to the symmetry condition $Z_{ij} = Z_{ji}$) and have the following properties. First, each $Z_{ij}$ has law $N^{-1/\alpha} Z$, where $Z$ is $\alpha$-stable; second, each $N^{1/\alpha}(H_{ij} - Z_{ij})$ is symmetric and has finite variance.

The first and second bounds in (7.9) below are consequences of [5, Proposition 7.11] and [5, Proposition 7.9], respectively.

**Lemma 7.6** ([5]). Define the $\{Z_{ij}\}$ and, for each integer $j \in [1,N]$, set

$$S_{jj} = -\left( z - \sum_{k \neq j} Z^2_{jk} R_{kk}^{(j)} \right)^{-1}. \quad (7.8)$$

Then, with overwhelming probability we have the bounds

$$\max_{1 \leq j \leq N} \mathbb{E}[|S_{jj} - R_{jj}|] \leq C(\log N)^C (N\eta^2)^{a/8}, \quad \max_{1 \leq j \leq N} \{|R_{jj}|, |S_{jj}|\} \leq C(\log N)^C. \quad (7.9)$$

We conclude this section with the following concentration estimate, which is essentially [5, Proposition 7.17]. Although it was only stated in [5] for the case when $I$ is a single index, the fact that it can be extended to all $I$ of uniformly bounded size is a quick consequence [5, Lemma 5.6].

**Lemma 7.7** ([5, Proposition 7.17]). There exists a constant $K > 0$ such that the following holds. Fix a real number $\delta > 0$, and abbreviate $\mathcal{D} = \mathcal{D}_{K,A}$ from (4.8). For every index set $I \subset [1,N]$, there exists a constant $C = C(s,|I|) > 0$ such that, with overwhelming probability, we have

$$\sup_{z \in \mathcal{D}} \sup_{a \in \mathbb{Z}^s} \left| \gamma_z^{(I)}(u) - \mathbb{E}[\gamma_z(u)] \right| \leq C(\log N)^C \frac{1}{N^{s/2} \eta^{\delta/2}}. \quad (7.10)$$

Here, the expectation is taken with respect to both $X_s$ and the Gaussian variables $g_k$.

**7.2. Tightness.** The following lemma computes the scaling limits of moments of $\mathbb{E}[R_*(E + i\eta)]$, as $\eta$ tends to 0. We recall $\gamma^*_E$ from Lemma 4.3.

**Lemma 7.8.** There exists a constant $c > 0$ such that the following holds. Fix a real number $E \in [-c,c]$, and let $\{E_N\}_{N \geq 1}$ and $\{\eta_N\}_{N \geq 1}$ be sequences of real numbers such that $\lim_{N \to \infty} E_N = E$ and $\lim_{N \to \infty} \eta_N = 0$. Then, for each $p \in \mathbb{N}$, we have that

$$\lim_{N \to \infty} \mathbb{E}\left[ (\mathbb{E}[R_*(E_N + i\eta_N)]^p) \right] = 2^{-p} \left( \mathcal{X} + \mathcal{X} + \sum_{a=1}^{p-1} \binom{p}{a} \mathcal{Y}(a) \right), \quad (7.11)$$

where $\mathcal{X} = \mathcal{X}_p$ and $\mathcal{Y}(a) = \mathcal{Y}_p(a)$ are defined by

$$\mathcal{X} = \frac{1}{\Gamma(p)} \int_{\mathbb{R}^+} t^{p-1} \exp \left( iEt - t^{\alpha/2} \gamma^*_E(1) \right) dt, \quad (7.12)$$

and

$$\mathcal{Y}(a) = \frac{1}{\Gamma(a) \Gamma(p-a)} \int_{\mathbb{R}^2} t^{a-1} s^{p-a-1} \exp \left( iE(t-s) \right) \times \exp \left( - (t^2 + s^2)^{\alpha/2} \gamma^*_E \left( \frac{t+is}{\sqrt{t^2 + s^2}} \right) \right) dt ds. \quad (7.13)$$

Thus, the left side of (7.11) exists, depends only on $E$ and $p$, and is uniformly continuous in $E$. 

Proof. For brevity, we set \( R_* = R_* (E_N + i\eta_N) \). We first express moments of \( \text{Im} R_* \) in terms of \( \gamma^*_z \) (recall Definition \((7.2)\)). To that end, we fix \( p \in \mathbb{N} \) and use the identity \( 2i \text{Im} R_* = R_* - \overline{R_*} \) to write

\[
(\text{Im} R_*)^p = (2i)^{-p}(R_* - \overline{R_*})^p = (2i)^{-p} \sum_{a=0}^{p} \binom{p}{a} R^a_* (\overline{R_*})^{p-a}.
\]

So, to establish \((7.11)\), it suffices to show for each integer \( a \in [1, p] \) that

\[
\lim_{N \to \infty} \mathbb{E} \left[ (-i R_*)^a (i \overline{R_*})^b \right] = \mathcal{X}; \quad \lim_{N \to \infty} \mathbb{E} \left[ (-i R_*)^a (i \overline{R_*})^{p-a} \right] = \mathcal{Y}(a).
\]

We only establish the second equality in \((7.15)\), as the proof of the former is entirely analogous.

To that end, let \( R_1, R_2, \ldots \) denote i.i.d. complex random variables whose laws are given by \( R_* \), and let \( \{ \xi_k \}_{k \geq 1} \) denote a Poisson point process with intensity measure \( \left( \frac{2}{\pi} \right) x^{-a/2-1} \, dx \) (independent from the \( \{ R_k \} \)), as in Lemma \((7.1)\). Then, Lemma \((7.1)\) implies

\[
E \left[ (-i R_*)^a (i \overline{R_*})^{p-a} \right] = \mathcal{Y}(a).
\]

We first express moments of \( \text{Im} R_* \) in terms of \( \gamma^*_z \) (recall \((7.2)\)).\( \) Im \( R_* \geq 0 \) holds almost surely, we have

\[
E \left[ \exp \left( - \sum_{k=1}^{\infty} \xi_k w_k \right) \right] = \exp \left( - \Gamma \left( 1 - \frac{\alpha}{2} \right) \mathbb{E} [w_1^{\alpha/2}] \right).
\]

Since \( \text{Im} R_* \geq 0 \), \((7.14)\), \((7.19)\), and \((7.24)\) together imply

\[
E \left[ (-i R_*)^a (i \overline{R_*})^{b} \right] = \frac{1}{\Gamma(a) \Gamma(b)} \int_{\mathbb{R}^+} t^{a-1} s^{b-1} \exp \left( - \Gamma \left( 1 - \frac{\alpha}{2} \right) \mathbb{E} [s^{\alpha/2}] \right) ds \times \exp(i tz - is \bar{z}) \, dt \, ds.
\]

Recalling \( \gamma^*_z \) from \((7.2)\), it follows that

\[
E \left[ (-i R_*)^a (i \overline{R_*})^{b} \right] = \frac{1}{\Gamma(a) \Gamma(b)} \int_{\mathbb{R}^+} t^{a-1} s^{b-1} \exp \left( - (t^2 + s^2)^{a/4} \gamma^*_z \left( \frac{t + is}{\sqrt{t^2 + s^2}} \right) \right) \times \exp(it z - is \bar{z}) \, dt \, ds.
\]

Next, observe by \((7.1)\) and \((7.5)\), there exists a constant \( c > 0 \) such that

\[
\sup_{z \in \mathbb{H}} \inf_{u \in \mathbb{S}^1} \text{Re} \gamma^*_z(u) > c; \quad \lim_{N \to \infty} \sup_{u \in \mathbb{S}^1} \left| \gamma^*_z(u) - \gamma^*_{z_N}(u) \right| = 0.
\]
Therefore, (7.22); the dominated convergence theorem; and the fact that
\[ \int_{\mathbb{R}_+^2} s^{a-1} t^{b-1} \exp \left( - c(s^2 + t^2)^{a/4} \right) ds \, dt < \infty, \] (7.25)
together imply for \( a \in [1, p - 1] \) that \( \lim_{N \to \infty} \mathbb{E} \left[ (-iR_*) a(iR_*)^{p-a} \right] = \mathcal{Y}(a) \); this establishes the second statement in (7.15). The proof of the first is entirely analogous and is therefore omitted. Now (7.11) follows from (7.14) and (7.15).

Similarly, to verify the uniform continuity of the left side of (7.11) in the scaling limits of the moments of the integrand of (7.12), the first bound in (7.24), (7.25), and the dominated convergence theorem.

**Remark 7.9.** The proof of Lemma 7.8 implies that \( \lim_{N \to \infty} \mathbb{E} \left[ (-iR_*) a(iR_*)^{p-a} \right] = \mathcal{X} \) if \( a = 0 \), to \( \mathcal{Y}(a) \) if \( a \in [1, p - 1] \), and to \( \mathcal{X} \) if \( a = p \).

Now we can quickly establish Proposition 2.4.

**Proof of Proposition 2.4.** Since Lemma 7.8 implies that \( \mathbb{E} \left[ (\text{Im} \, R_*(E + i\eta))^2 \right] \) is uniformly bounded in \( \eta > 0 \), the sequence \( \{ \text{Im} \, R_*(E + i\eta) \}_{\eta > 0} \) of random variables is tight. This establishes the first claim of the proposition. The second is a direct consequence of Lemma 7.8. \( \square \)

### 7.3. Scaling limit of \( A(q, \xi) \)

We begin with the limit of the numerator of \( A(q, \xi) \). To compute the scaling limits of the moments of \( A(q, \xi) \), we first show that the off-diagonal resolvent entries in the numerator of \( A(q, \xi) \) are negligible. Here, we recall the \( \hat{\gamma}_i = \hat{\gamma}_i(t) \) from (5.17).

**Lemma 7.10.** For all real numbers \( \delta > 0 \); integers \( m, n > 0 \); and unit vectors \( q = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^N \) with \( \supp q = m \), there exist constants \( \epsilon > 0 \) (independent of \( \delta \), \( m \), and \( n \)) and \( C = C(\delta, m, n) > 0 \) such that the following holds. Let \( \{k_1, k_2, \ldots, k_n\} \subset [1, N] \) denote an index sequence such that \( \max_{1 \leq j \leq n} |k_j - N/2| < cN \); let \( \text{supp} q = \{j_1, j_2, \ldots, j_m\} \); and let \( t \) be as in (5.9). Then, for \( \eta \geq N^{-1/2} \),
\[
\mathbb{E} \left[ \prod_{i=1}^n \text{Im} \left\{ q_i R(t, \hat{\gamma}_{k_i} + i\eta) q_i^\dagger \right\} \right] - \mathbb{E} \left[ \prod_{i=1}^n \text{Im} \left( \sum_{b=1}^m q_j^\dagger R_{ji} q_i \right) (t, \hat{\gamma}_{k_i} + i\eta) \right] \leq \frac{C N^\delta}{\sqrt{N} \eta}. \quad (7.26)
\]

**Proof.** First observe that
\[
\mathbb{E} \left[ \prod_{i=1}^n \text{Im} \left\{ q_i R(t, \hat{\gamma}_{k_i} + i\eta) q_i^\dagger \right\} \right] = \mathbb{E} \left[ \prod_{i=1}^n \sum_{a=1}^m \sum_{b=1}^m q_j a q_j b \text{Im} R_{ja, jb} (t, \hat{\gamma}_{k_i} + i\eta) \right] \quad (7.27)
\]
\[
= \sum_{a, b} \mathbb{E} \left[ \prod_{i=1}^n q_{ja(i)} q_{jb(i)} \text{Im} R_{ja(i), jb(i)} (t, \hat{\gamma}_{k_i} + i\eta) \right], \quad (7.28)
\]
where in the right side of (7.28), \( a = (a(1), a(2), \ldots, a(n)) \) and \( b = (b(1), b(2), \ldots, b(n)) \) are summed over all sequences of \( 1, 2, \ldots, m \) to the power \( n \).

It suffices to bound by \( C N^{\delta}(N\eta)^{-1/2} \) any summand on the right side of (7.28) for which there exists some \( i' \in [1, n] \) such that \( a(i') \neq b(i') \). To that end, observe that the second bound in (7.11) (to bound \( |R_{ja(i), jb(i)}| \leq N^{\delta/2n} \) with overwhelming probability for \( i \neq i' \)) holds; the fact that \( q_j \leq 1 \)
for each $j \in [1, N]$; and the exchangeability of the matrix entries of $X_t$ together imply that any such term is bounded by

$$N^{\delta/2} \mathbb{E}\left[|R_{12}(t, \gamma_{k_j} + i\eta)|\right]. \tag{7.29}$$

To estimate this quantity, abbreviate $R_{ij} = R_{ij}(t, \gamma_{k_j} + i\eta)$, and observe that the Ward identity (4.3) and the exchangeability of $X_t$ together imply that

$$\mathbb{E}[|R_{12}|] \leq \left(\mathbb{E}[|R_{12}|^2]\right)^{1/2} \leq \left(\mathbb{E}\left[\frac{1}{N-1} \sum_{j=1}^N |R_{ij}|^2\right]\right)^{1/2} \leq \frac{2\mathbb{E}[\text{Im} R_{11}]^{1/2}}{\sqrt{N}\eta}. \tag{7.30}$$

Using the second bound in (4.11), and the deterministic bound (1.2) on the exceptional set where the former estimate does not apply, yields $\mathbb{E}[\text{Im} R_{11}] \leq C(\log N)^C$, and so

$$\mathbb{E}[|R_{12}|] \leq \frac{CN^{\delta/2}}{\sqrt{N}\eta}. \tag{7.31}$$

Together with (7.28) and (7.29), this implies (7.24). \qed

In [28] Theorem 2.8] it was shown for fixed $\eta \in \mathbb{H}$ that the diagonal resolvent elements $G_{ii}(z)$ of the matrix $H$ are asymptotically independent. The next lemma is a version of this result (for the perturbed model $X_t$) when $\eta = \text{Im} z$ is simultaneously tending to 0. Theorem 3.9 is then deduced quickly as a consequence. Below, we recall $R_\ast(E)$ from Definition 2.6

**Proposition 7.11.** There exists a constant $c > 0$ such that the following holds. Fix integers $m, n > 0$ and a unit vector $q = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^N$ with $|\text{supp } q| = m$. Let $\text{supp } q = \{j_1, j_2, \ldots, j_m\}$, and let $t \in \mathbb{R}_{>0}$ be as in [3.2]. Fix a real number $E \in [-c, c]$, and let $\{\eta_N\}_{N \geq 1}$ and $\{E_{\ast}^{(i)}\}_{N \geq 1}$ for each integer $i \in [1, n]$ be sequences of real numbers such that

$$\lim_{N \to \infty} \eta_N = 0; \quad \eta_N \gg N^{-1/2}; \quad \lim_{N \to \infty} E_{\ast}^{(i)} = E; \quad \max_{1 \leq i \leq n} |E_{\ast}^{(i)} - E_{\ast}^{(1)}| \ll \eta_N. \tag{7.32}$$

Then, letting $\{R_{j_i}(E)^{\ast}\}_{i \in [1, m]}$ be i.i.d. random variables each with law $R_\ast(E)$, we have that

$$\lim_{N \to \infty} \mathbb{E}\left[\prod_{i=1}^m \text{Im} R_{j_i}(E)^{\ast}(t, E_{\ast}^{(i)} + i\eta_N)\right] = \mathbb{E}\left[\sum_{k=1}^m q_{j_k}^2 R_{j_k}(E)^{\ast n_k}\right]. \tag{7.33}$$

**Proof.** It suffices to show that, for any sequences of nonnegative integers $n^{(i)} = (n_1^{(i)}, n_2^{(i)}, \ldots, n_m^{(i)})$ for $1 \leq i \leq n$ with $N_k = \sum_{i=1}^n n_k^{(i)}$, we have

$$\lim_{N \to \infty} \mathbb{E}\left[\prod_{i=1}^n \prod_{k=1}^{m} \left(\text{Im} R_{j_k}(t, E_{\ast}^{(i)} + i\eta_N)\right)^{n_k^{(i)}}\right] = \mathbb{E}\left[\prod_{k=1}^m \mathbb{E}[R_\ast(E)^{\ast n_k}]\right]. \tag{7.34}$$

To ease notation, we detail the proof of (7.34) when $(m, n) = (1, 1)$ and outline it when $(m, n) = (1, 2)$ and $(m, n) = (2, 2)$, which are largely analogous. We omit the proofs in the remaining cases, since they are very similar to those of the $(m, n) \in \{(1, 2), (2, 2)\}$ cases.

To that end, first assume $(m, n) = (1, 1)$; abbreviate $j_1 = j, z = E_{\ast}(1) + i\eta_N$, and $R_{ik} = R_{ik}(t, z)$; and set $p = n_1^{(1)}$. We compute $\lim_{N \to \infty} \mathbb{E}[\text{Im} R_{jj}^p]$. As in the proof of Lemma 7.8 we use the
and (7.17), we deduce that
\[ C_p(z) = \sum_{a=0}^p (-1)^{p-a} \binom{p}{a} E[R_{jj}^a | T_{jj}] . \] (7.35)

Now, recall from Definition 2.14 that each entry of $H$ has law $N^{-1/\alpha}(Z + J)$, where $Z$ is $\alpha$-stable and $J$ has finite variance. For each $1 \leq i \leq j \leq N$, let $\{Z_{ij}\}$ denote mutually independent random variables with law $N^{-1/\alpha}Z$ such that $N^{1/\alpha}|H_{ij} - Z_{ij}|$ has uniformly bounded variance. Following (7.8), for any subset $I \subset [1,N]$ and index $j \in [1,N] \setminus I$, set $J = I \cup \{j\}$ and define
\[ S_{jj}^{(I)} = S_{jj}^{(J)}(z) = - \left( z + \sum_{k \not\in J} Z_{kk}^2 R_{jj}^{(J)} \right)^{-1} . \] (7.36)

If $I$ is empty, then we abbreviate $S_{jj} = S_{jj}^{(I)}$.

Then (7.9) and the deterministic bounds given by (1.2) and $|S_{jj}| \leq \eta^{-1}$ together imply that there exists a constant $C = C(p) > 0$ such that
\[ \mathbb{E}[[R_{jj}^a - S_{jj}^a]] + \mathbb{E}[[R_{jj}^a - S_{jj}^a]] \leq \frac{C(\log N)^C}{(\eta N^{2/\alpha}/8}, \] (7.37)
for any $a \in [0, p]$. Then, (7.37) and (7.9) together imply for any $a, b \in [0, p]$ that
\[ \mathbb{E}[[R_{jj}^a - S_{jj}^a]] + \mathbb{E}[[R_{jj}^a - S_{jj}^a]] \leq \frac{C(\log N)^C}{(\eta N^{2/\alpha}/8}. \] (7.38)

So, by (7.35), the definition (2.3) of $R_\star$, and Proposition 2.4 it suffices to show for each integer $a \in [0, p]$ that
\[ \lim_{N \to \infty} \mathbb{E}[S_{jj}^{(I)} R_{jj}^{\sigma-a}] = \lim_{\eta \to 0} \mathbb{E}[R_\star(E + \eta)^a R_{jj}^\sigma E + \eta)]^{\sigma-a}] . \] (7.39)

Recalling (7.2), (7.13), and Remark 7.4, the right side of is equal to $i^{-p}X$ if $a = 0$, to $i^{-p}(-1)^a\eta(X)$ if $p \in [1, a - 1]$, and to $i^pX$ if $a = p$. Let us only show (7.39) in the case $a \in [1, p - 1]$, as the cases $a \in \{0, p\}$ are entirely analogous.

To that end, we proceed similarly to as in the proof of Lemma 7.8. More specifically, by (7.36) and (7.41), we deduce that
\[ (-i S_{jj})^a (S_{jj})^b \int \frac{t^{a-1} s^{b-1} e^{iz \eta t + i s^2 \eta t} \sum_{k \neq i} Z_{ik}^2 (t R_{kk}^{(j)} - s R_{kk}^{(j)}) dt ds . \] (7.40)

To analyze the right side of (7.40), observe by Corollary B.2 (whose proof proceeds by first applying a type of Hubbard-Stratonovich transform to linearize the exponential in the $\{Z_{jk}\}$), and then using (7.2) to evaluate the expectation
\[ \mathbb{E}\left[ \exp\left( i \sum_{k \neq j} Z_{jk}^2 (t R_{kk}^{(j)} - s R_{kk}^{(j)}) \right) \right] = \mathbb{E}\left[ \exp\left( -\frac{2i}{N} \sum_{k \neq j} (t R_{kk}^{(j)} - s R_{kk}^{(j)}) \right) \right] , \] (7.41)
Therefore, by the definition (7.6) of $\gamma_z^{(j)}$, we find
\[
E \left[ \exp \left( -\frac{21^{\alpha/2} \sigma^\alpha}{N} \sum_{k \neq j} \left( tR_{kk}^{(j)} - sR_{kk}^{(j)} \right)^{\alpha/2} |g_k|^\alpha \right) \right] = E \left[ \exp \left( -\frac{N-1}{N} \gamma_z^{(j)}(t+is) \right) \right],
\] (7.42)
where we have used the fact (see (7.39)) that $|g_k|^\alpha = 2^{-\alpha/2} \sigma^{-\alpha} \Gamma \left( 1 - \frac{\alpha}{2} \right)$. Thus,
\[
E \left[ \exp \left( -\frac{N-1}{N} \alpha/4 \gamma_z(t+is) \right) \right] + O(N^{-\alpha}).
\] (7.43)
Using (7.10) and (7.7), the fact that $a, b, c, d \geq N + 1$, we obtain
\[
\gamma_z^{(j)}(t+is) = \exp \left( -\frac{N-1}{N} \alpha/4 \gamma_z(t+is) \right) + O(N^{-\alpha}).
\] (7.44)
Combining (7.10), (7.11), (7.12), (7.43), and (7.44) yields
\[
E \left[ (-iS_{jj})^a (iS_{jj})^b \right] = \frac{1}{\Gamma(a)\Gamma(b)} \int_{R^+} t^{a-1}s^{b-1} \exp(itz - is\bar{z}) \times \left( \exp \left( -\frac{N-1}{N} \alpha/4 \gamma_z(t+is) \right) + O(N^{-\alpha}) \right) dt ds.
\] (7.45)
By (7.24) and (7.25), this yields (7.39) when $a \in [1, p - 1]$. The cases when $a \in \{0, p\}$ are handled analogously and therefore omitted. This therefore establishes (7.34) in the case $m = 1 = n$.

Next let us outline how to establish (7.34) in the case $(m, n) = (1, 2)$. We abbreviate $z_1 = E_N^{(1)} + i\eta_N$, $z_2 = E_N^{(2)} + i\eta_N$, and $j = j_1$. Then following (7.35), it suffices to show for any integers $a, b, c, d \geq 0$ that
\[
\lim_{N \to \infty} E \left[ R_{jj}(z_1)^a R_{jj}(z_2)^b R_{jj}(z_2)^c R_{jj}(z_2)^d \right] = \lim_{N \to \infty} E \left[ R_\ast(E + i\eta)^{a+c} R_{jj}(E + i\eta)^{b+d} \right].
\] (7.50)
We write
\[
E \left[ R_{jj}(z_1)^a R_{jj}(z_2)^b R_{jj}(z_2)^c R_{jj}(z_2)^d \right] = R_{jj}(z_1)^{a+c} R_{jj}(z_1)^{b+d}
\] (7.51)
\[
+ E \left[ R_{jj}(z_1)^a R_{jj}(z_2)^b \left( R_{jj}(z_2)^c - R_{jj}(z_1)^c \right) \right]
\] (7.52)
\[
+ E \left[ R_{jj}(z_1)^a R_{jj}(z_2)^b R_{jj}(z_2)^c \left( R_{jj}(z_2)^d - R_{jj}(z_2)^d - R_{jj}(z_1)^d \right) \right].
\] (7.53)
The first term is the main one, and it was shown in the preceding case, as (7.35) and (7.39), that
\[
\lim_{N \to \infty} E \left[ R_{jj}(z_1)^{a+c} R_{jj}(z_1)^{b+d} \right] = \lim_{N \to \infty} E \left[ R_\ast(E + i\eta)^{a+c} R_{jj}(E + i\eta)^{b+d} \right].
\] (7.54)
The latter two terms are error terms, and they tend to zero asymptotically. Let us show this for (7.53), as the other term is similar.

Since $R_{jj}(z) - R_{jj}(w) = (w - z) \sum_{a=1}^{N} R_{ja}(z) R_{aj}(w)$ by (4.11), we have that

$$|\partial_z R_{jj}(z)| \leq \left| \sum_{a=1}^{N} R_{ja}(z) R_{aj}(z) \right| \leq \sum_{a=1}^{N} |R_{ja}(z)|^2 = \frac{\text{Im} R_{jj}(z)}{\eta} \leq (\log N) C \eta^{-1}$$

with overwhelming probability, where in the equality we used (4.3), and in the last bound we used (4.11). Integrating $\partial_z R_{jj}(z)^d = dR_{jj}^{d-1}\partial_z R_{jj}$ from $z_1$ to $z_2$ yields

$$|R_{jj}(z_1)^d - R_{jj}(z_2)^d| \leq d|z_1 - z_2|(\log N)^d C \eta^{-1}$$

with overwhelming probability. Therefore, with overwhelming probability, we have

$$|R_{jj}(z_1)^a \overline{R}_{jj}(z_1)^b R_{jj}(z_2)^c (\overline{R}_{jj}(z_2)^d - \overline{R}_{jj}(z_1)^d)| \ll C |z_1 - z_2|(\log N)^d C \eta^{-1} |R_{jj}(z_1)^a \overline{R}_{jj}(z_1)^b R_{jj}(z_2)^c|$$

$$\ll |z_1 - z_2| \eta^{-1} (\log N)^{(a+b+c+d)} C \ll 1,$$

where we used (4.11) in the last estimate, as well as the hypothesis (7.32) that $|E_N^{(1)} - E_N^{(2)}| \ll \eta$ to bound $|z_1 - z_2|$. On the complementary event, we use the trivial bound (4.12). Together, these show that

$$\lim_{N \to \infty} \mathbb{E} \left[ R_{jj}(z_1)^a \overline{R}_{jj}(z_1)^b R_{jj}(z_2)^c (\overline{R}_{jj}(z_2)^d - \overline{R}_{jj}(z_1)^d) \right] = 0,$$

as desired. We have therefore established that (7.62) is small; since the same holds for (7.62), (7.54) implies (7.51).

Now let us outline how to establish (7.64) in the case $(m, n) = (2, 2)$. We abbreviate $z_1 = E_N^{(1)} + i \eta$ and $z_2 = E_N^{(2)} + i \eta$, and we assume for notational convenience that $j_1 = 1$ and $j_2 = 2$. As in (7.56), it suffices to show for any integers $a, b, c, d \geq 0$ that

$$\lim_{\eta \to 0} \mathbb{E} \left[ R_{11}(z_1)^a \overline{R}_{11}(z_1)^b R_{22}(z_2)^c \overline{R}_{22}(z_2)^d \right] = \lim_{\eta \to 0} \mathbb{E} \left[ R_{*}^a (E + i \eta) \overline{R}_{*}^b (E + i \eta) \mathbb{E} \left[ R_{*}^c (E + i \eta) \overline{R}_{*}^d (E + i \eta) \right] \right].$$

In what follows, we assume that $a, b, c, d > 0$ for notational simplicity. Note that [30, Lemma 5.5] implies for each $i \neq j$ that

$$\mathbb{E} \left[ |R_{jj} - R_{jj}^{(i,j)}| \right] \leq \frac{C}{N \eta}.$$

It quickly follows from (7.53), (7.49), the deterministic bound (4.2) that

$$\lim_{N \to \infty} \mathbb{E} \left[ R_{11}(z_1)^a \overline{R}_{11}(z_1)^b R_{22}(z_2)^c \overline{R}_{22}(z_2)^d \right] = \lim_{N \to \infty} \mathbb{E} \left[ R_{11}^{(2)}(z_1)^a \overline{R}_{11}^{(2)}(z_1)^b R_{22}^{(1)}(z_2)^c \overline{R}_{22}^{(1)}(z_2)^d \right],$$

as in (7.33). As before, (7.33) and (7.42) together imply that

$$\lim_{N \to \infty} \mathbb{E} \left[ R_{11}^{(2)}(z_1)^a \overline{R}_{11}^{(2)}(z_1)^b R_{22}^{(1)}(z_2)^c \overline{R}_{22}^{(1)}(z_2)^d \right] = \lim_{\eta \to 0} \mathbb{E} \left[ S_{11}^{(2)}(z_1)^a \overline{S}_{11}^{(2)}(z_1)^b S_{22}^{(1)}(z_2)^c \overline{S}_{22}^{(1)}(z_2)^d \right],$$

and so it suffices to show that

$$\lim_{N \to \infty} \mathbb{E} \left[ S_{11}^{(2)}(z_1)^a \overline{S}_{11}^{(2)}(z_1)^b S_{22}^{(1)}(z_2)^c \overline{S}_{22}^{(1)}(z_2)^d \right] = \lim_{\eta \to 0} \mathbb{E} \left[ R_{*}^a (E + i \eta) \overline{R}_{*}^b (E + i \eta) \mathbb{E} \left[ R_{*}^c (E + i \eta) \overline{R}_{*}^d (E + i \eta) \right] \right].$$
Once again using (7.24) and (7.17), we find
\[
E \left[ \left( -iS_{11}^{(2)}(z_1) \right)^a \left( iS_{11}^{(2)}(z_1) \right)^b \left( -iS_{22}^{(1)}(z_2) \right) \left( iS_{22}^{(1)}(z_2) \right)^c \right] \tag{7.68}
\]
\[
= \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} \int_{\mathbb{R}^+} t^{a-1}s^{b-1}x^{c-1}y^{d-1} \exp (itz_1 + izz_2 - is\gamma_{12} - i\gamma_{22}) \tag{7.69}
\]
\[
\times E \left[ \left( \sum_{k \in \{1,2\}} Z_{1k}^2(tR_{1k}^{(12)} - s\gamma_{12}) \right)^e \left( \sum_{k \notin \{1,2\}} Z_{2k}^2(xR_{2k}^{(12)} - y\gamma_{12}) \right)^f \right] \, dt \, ds \, dx \, dy.
\tag{7.70}
\]

We now condition on \{h_{ij}\}_{i,j \notin \{1,2\}} which makes the two exponential terms in the previous line conditionally independent. Then by following (7.11), (7.12), (7.13), (7.14), and (7.17), we obtain
\[
E \left[ \left( -iS_{11}^{(2)}(z_1) \right)^a \left( iS_{11}^{(2)}(z_1) \right)^b \left( -iS_{22}^{(1)}(z_2) \right) \left( iS_{22}^{(1)}(z_2) \right)^c \right] \tag{7.71}
\]
\[
= \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} \int_{\mathbb{R}^+} t^{a-1}s^{b-1}x^{c-1}y^{d-1} \exp (itz_1 + izz_2 - is\gamma_{12} - i\gamma_{22}) \tag{7.72}
\]
\[
\times \left( \exp \left( -\frac{N - 2}{N} \left( t^2 + s^2 \right)^{\alpha/4} \gamma_{z_1} \left( \frac{t + is}{\sqrt{t^2 + s^2}} + (x^2 + y^2)^{\alpha/4} \gamma_{z_2} \left( \frac{x + iy}{\sqrt{x^2 + y^2}} + O(N^{-e}) \right) \right) \right) \right)
\tag{7.73}
\]
\[
+ O(N^{-10}) \right) \, dt \, ds \, dx \, dy.
\tag{7.74}
\]
Thus (7.24) (with the \(z_N\) there equal to \(z_1\) and \(z_2\) here), (7.20), the dominated convergence theorem, (7.13), and Remark 7.9 together give
\[
\lim_{N \to \infty} E \left[ \left( -iS_{11}^{(2)}(z_1) \right)^a \left( iS_{11}^{(2)}(z_1) \right)^b \left( -iS_{22}^{(1)}(z_2) \right) \left( iS_{22}^{(1)}(z_2) \right)^c \right] \tag{7.75}
\]
\[
= \frac{1}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(d)} \int_{\mathbb{R}^+} t^{a-1}s^{b-1}x^{c-1}y^{d-1} \exp (iE(t - s - x - y)) \tag{7.76}
\]
\[
\times \exp \left( -(t^2 + s^2)^{\alpha/4} \gamma_E \left( \frac{t + is}{\sqrt{t^2 + s^2}} - (x^2 + y^2)^{\alpha/4} \gamma_E \left( \frac{x + iy}{\sqrt{x^2 + y^2}} \right) \right) \right) \, dt \, ds \, dx \, dy \tag{7.77}
\]
\[
= \mathcal{Y}_{a+b} \mathcal{Y}_{c+d}(c) = \lim_{\eta \to 0} E \left[ R_x(E + i\eta)^a \mathcal{R}_x(E + i\eta)^b \right] E \left[ R_x(E + i\eta)^c \mathcal{R}_x(E + i\eta)^d \right]
\tag{7.78}
\]
from which we deduce (7.60).

**Proof of Theorem 7.9** We will apply Proposition 7.11 with the \(\eta_N\) there equal to the \(\eta = N^{-\frac{\alpha}{12}}\) here (recall (3.2)) and the \(E^{(j)}\) there equal to the \(\tilde{\gamma}_{kj}\) here. To that end, we must verify the assumptions (7.32) of that proposition. The first and second statements there follow from the fact that \(\eta = N^{-\frac{\alpha}{12}}\), that \(\alpha\) is sufficiently small, and the fact that \(a < \frac{1}{2}\) (by (3.3)). The third follows from the fact that \(\lim_{N \to \infty} \gamma_{k_j} = \tilde{E}\) and (7.33).

To verify the fourth, we must show that \(|\tilde{\gamma}_{kj} - \tilde{\gamma}_{kj}| \ll \eta\). To that end, observe since \(|k_1 - k_j| \leq \sqrt{N}\), we have for any fixed \(\delta > 0\) and \(j \in [1, n]\) that
\[
|\tilde{\gamma}_{kj} - \tilde{\gamma}_{kj}| \leq |\gamma_{k_1} - \gamma_{k_j}(t)| + |\gamma_{k_1} - \gamma_{k_j}(t)| + |\gamma_{k_1}(t) - \gamma_{k_j}(t)| \tag{7.79}
\]
\[
\lesssim N^{\delta - \frac{1}{2}} + N^{1 + 4\epsilon} |k_1 - k_j| \lesssim N^{\delta + 4\epsilon - \frac{1}{2}} \tag{7.80}
\]
for sufficiently large $N$, where we used (4.24) and (6.18). Then, since $\eta \gg N^{-1/2}$, we may choose $\delta$ and $c$ small enough that the last bound in (7.32) of Proposition 7.11 is also satisfied.

Now the theorem follows from Lemma 7.10, Proposition 7.11, the facts that $\lim_{N \to \infty} \text{Im} m_\alpha(\gamma_k + i\eta) = \pi \rho_\alpha(E)$ (as $\lim_{N \to \infty} \gamma_k = E$) and $U_*(E) = (\pi \rho_\alpha(E))^{-1} R_*(E)$ (see Definition 2.5); and (4.6).

**Appendix A. Proofs of results from Section 4 and Section 7**

In this section, we prove results from Section 4 and Section 7 which are used in the rest of the paper. We begin with the proof of Lemma 7.3, since facts derived in the course of that proof will be useful for proving the statements from Section 4.

For any $w \in \mathbb{C}$, we let $\mathcal{H}_w$ denote the space of $\mathcal{C}^1$ functions $g: \mathbb{K}^+ \to \mathbb{C}$ such that $g(\lambda u) = \lambda^w g(u)$ for each $\lambda \in \mathbb{R}_+$. Following [30], we define for any $r \in [0, 1]$ a norm on $\mathcal{H}_w$ by

$$\|g\|_r = \|g\|_\infty + \sup_{u \in \mathbb{S}^1_+} \sqrt{(i.u)^r \partial_1 g(u)^2 + (i.u)^r \partial_2 g(u)^2},$$

(A.1)

where $\partial_1 g(x + iy) = \partial_x g(x + iy)$ and $\partial_2 g(x + iy) = \partial_y g(x + iy)$, and we recall $\|g\|_\infty = \sup_{u \in \mathbb{S}^1_+} |g(u)|$. We let $\mathcal{H}_{w,r}$ denote the closure of $\mathcal{H}_w$ in $\|\cdot\|_r$, which is a Banach space.

Following [30], we define for any complex numbers $u \in \mathbb{S}^1_+$ and $h \in \mathbb{K}$, and any function $g \in \mathcal{H}_{w,r}$ the function

$$F_h(g)(u) = \int_0^{\pi/2} \left( \int_{\mathbb{K}^+} \left( e^{-r^{\alpha/2} g(e^{\theta})} - e^{-r^{\alpha/2} g(e^{\theta} + uy)} - (ryh.u) - (rh.e^\theta) \right) \times r^{\alpha/2 - 1} y^{-\alpha/2 - 1} drdy \right) \sin(2\theta)^{\alpha/2 - 1} d\theta.$$  

(A.2)

Further, for any $z \in \mathbb{H}$, the map $G_z: \mathbb{S}^1_+ \to \mathbb{C}$ is given by

$$G_z(f)(u) = \frac{\alpha}{2^{\alpha/2} \Gamma((\alpha/2)^2)} F_{-iz}(f)(iu).$$  

(A.3)

The following lemma from [30] indicates that the function $G_z$ from Definition 7.2 is a fixed point of $G_z$.

**Lemma 7.3** (Lemma 4.4). For any $z \in \mathbb{H}$ and $u \in \mathbb{S}^1$, we have $G_z(\gamma_z)(u) = G_z(\gamma_z)(u)$.

**Proof of Lemma 7.3** For the first statement, we use [30] Proposition 3.3, which shows that there exists $c > 0$ such that, uniformly in $|z| < c$,

$$\frac{\gamma_z^*(e^{i\pi/4})}{\Gamma(1 - \frac{\alpha}{2})} = 2^{\alpha/4} \mathbb{E} \left[ (\text{Im } R_*(z))^{\alpha/2} \right] > c.$$

(A.5)

We now compute, for any $u \in \mathbb{S}^1_+$,

$$\frac{\text{Re } \gamma_z^*(u)}{\Gamma(1 - \frac{\alpha}{2})} = \mathbb{E} \left[ \text{Re } (-iR_*(z)u)^{\alpha/2} \right] \geq \mathbb{E} \left[ (\text{Re } (-iR_*(z)u)^{\alpha/2} \right] \geq \mathbb{E} \left[ (\text{Im } R_*(z))^{\alpha/2} \right] \geq c.$$

(A.6)

In the first inequality, we used the fact that $\text{Re } a^r \geq (\text{Re } a)^r$ for any $a \in \mathbb{K}$ and $r \in (0, 1)$ (see [30] Lemma 5.10). The second inequality follows from $\text{Re}(a.u) \geq \text{Re } a$ for any $u \in \mathbb{S}^1_+$ and $a \in \mathbb{K}^+$. The final inequality follows from (A.5). This completes the proof of the first claim.
Set $z = E + i\eta$. We now establish convergence of the order parameter $\gamma^*_z$ as $\eta \to 0$. We note that for any $\tau > 0$, there exists $c = c(\tau) > 0$ such that
\[
\|\gamma^*_z - \gamma^*_w\|_r \leq \|\gamma^*_z - \gamma^*_0\|_r + \|\gamma^*_w - \gamma^*_0\|_r \leq \tau
\]
if $z, w \in \mathbb{H}$ satisfy $|z| < c$ and $|w| < c$. The final inequality follows from the first displayed equation in the proof of [20] Proposition 3.3.

Then (A.7) and [20] Proposition 3.4 together imply there exist constants $C, c > 0$ such that for any $w\in\mathbb{H}$ such that $|z| < c$ and $|w| < c$.

We now claim
\[
\|G_w(\gamma^*_w) - G_z(\gamma^*_w)\|_r \leq C|w - z|.
\]
In the proof of [20] Lemma 4.2, it was shown that the partial (Fréchet) derivative of $F_h(g)$ in either the real or imaginary part of $h$ has finite $\|\cdot\|_r$ norm, and the exact derivative was calculated. Further, the derivative may be bounded in the $\|\cdot\|_r$ norm by a constant $C$ using [20] (20) when $g = \gamma^*_w$, which is uniform in $z, w$ with $z, w \in \mathbb{H}$ and $|z|, |w| \leq c$. Also, it is continuous by the computation following [20] (21). By definition (A.4), the same is true for $G_h(g)$, and we obtain (A.9) by integration using the fundamental theorem of calculus. Combining this with (A.8) we obtain the Lipschitz estimate
\[
\|\gamma^*_w - \gamma^*_z\|_r \leq c|w - z|
\]
for any $r \in [0, 1]$. This estimate implies that $\lim_{\eta \to 0} \gamma^*_{E+\eta}$ exists as a function in $\mathcal{H}_{\alpha/2, r}$, which we denote by $\gamma^*_E$. \hfill \Box

Proof of Lemma 4.7 We begin with the first estimate of (4.3). By (3.4) and Theorem 4.1, we have $\text{Im } m_\alpha(z) = \text{Im } \text{Re } (z)$ for $z \in \mathbb{H}$. Recalling $\lim_{\eta \to 0} \text{Im } m_\alpha(E + i\eta) = \pi \theta_\alpha(E)$, it then suffices to show that $\lim_{\eta \to 0} \text{Im } R_\alpha(E + i\eta)$ is Lipschitz in $E$. For $c$ small enough, by (7.11), this limit is given by $(\mathcal{X} + \overline{\mathcal{X}})/2$, where
\[
\mathcal{X}(E) = \int_{\mathbb{R}^+} \exp \left(iEt - t^{\alpha/2}\gamma^*_E(1)\right) dt.
\]
Further, if $c$ is small enough, by Lemma 7.3 we have the uniform lower bound
\[
\inf_{|E| \leq c} \inf_{u \in \mathbb{S}^1} \text{Re } \gamma^*_E(u) > c'.
\]
Define
\[
F(x, y, w) = \int_{\mathbb{R}^+} \exp \left(ixt - t^{\alpha/2}(y + iw)\right) dt.
\]
By (A.10), $|\gamma^*_E(1) - \gamma^*_E(1)| \leq C|E_1 - E_2|$ for some constant $C$ and $E_1, E_2 \in [-c, c]$, if $c$ is small enough. Using this inequality, to show $\mathcal{X}(E)$ is Lipschitz in $E$, it suffices by the fundamental theorem of calculus to show the partial derivatives $\partial_E F(x, y, w)$, $\partial_y F(x, y, w)$, and $\partial_w F(x, y, w)$ are uniformly bounded by a constant when $|x| \leq c$ and $y > c'$. This follows straightforwardly after differentiating under the integral sign (which is permissible as the integrand is dominated in absolute value by $\exp(-t^{\alpha/2}y) \leq \exp(-t^{\alpha/2}c')$).

\footnote{We remark that although the constant $C$ in the bound [20] (20) depends on $\text{Re } g$ and degenerates as $\text{Re } g$ goes to zero, here $g = \gamma^*_w$ and $\inf_{u \in \mathbb{S}^1} \text{Re } g(u) > c$ for some $c > 0$, from (7.4). So we obtain the claimed bound.}
We have shown that the density $\varrho_\alpha(x)$ is continuous in a neighborhood of zero. By [28] Theorem 1.6(ii), $\varrho_\alpha(0)$ is positive and bounded. Therefore the second claim in (4.14) follows from the first after possibly decreasing $c$.

For (4.5), we first suppose $|E_1 - E_2| \leq c/10$ and $|E_1| \leq c/2$, where $c$ is the constant from the previous part of this proof. We use the definition of the Stieltjes transform to write

$$\operatorname{Im} m_\alpha(E_1 + i\eta) = \operatorname{Im} \int_R \frac{\varrho_\alpha(x) \, dx}{x - E_1 - i\eta}, \quad \operatorname{Im} m_\alpha(E_2 + i\eta) = \operatorname{Im} \int_R \frac{\varrho_\alpha(x + E_2 - E_1) \, dx}{x - E_1 - i\eta}, \quad (A.14)$$

where in the second equality we used the change of variables $x \mapsto x + E_2 - E_1$. Computing the imaginary part of the integrand directly, we have

$$|\operatorname{Im} m_\alpha(E_1 + i\eta) - \operatorname{Im} m_\alpha(E_2 + i\eta)| \leq \eta \int_R \left| \frac{\varrho_\alpha(x) - \varrho_\alpha(x + E_2 - E_1)}{(x - E_1)^2 + \eta^2} \right| \, dx \quad (A.15)$$

$$= \eta \int_{[-c/2,c/2]} \left| \frac{\varrho_\alpha(x) - \varrho_\alpha(x + E_2 - E_1)}{(x - E_1)^2 + \eta^2} \right| \, dx \quad (A.16)$$

$$+ \eta \int_{[-c/2,c/2]^c} \left| \frac{\varrho_\alpha(x) - \varrho_\alpha(x + E_2 - E_1)}{(x - E_1)^2 + \eta^2} \right| \, dx. \quad (A.17)$$

Using (4.4), $|x| \leq c/2$, and $|E_1 - E_2| \leq c/10$, we find that

$$\eta \int_{[-c/2,c/2]} \frac{1}{(x - E_1)^2 + \eta^2} \, dx \leq C|E_1 - E_2|. \quad (A.18)$$

By [28] Theorem 1.6, the density $\varrho_\alpha(x)$ is uniformly bounded. Therefore

$$\eta \int_{[-c/2,c/2]} \frac{1}{(x - E_1)^2 + \eta^2} \, dx = C\eta \int_{[-c/2,c/2]} \frac{1}{(x - E_1)^2} \, dx \quad (A.19)$$

$$\leq Cc^{-1}\eta. \quad (A.20)$$

In the last line, we used the hypothesis that $|E| \leq c/2$ to show the integral is uniformly bounded and adjusted the value of $C$. This completes the proof of (4.5) after decreasing $c$ if necessary.

We now address the first claim of (4.6). Again using the uniform boundedness of $\varrho_\alpha(x)$ over all $x \in \mathbb{R}$, we deduce

$$\operatorname{Im} m_\alpha(E + i\eta) = \int_R \frac{\eta \varrho_\alpha(x) \, dx}{\eta^2 + (E - x)^2} \leq C. \quad (A.21)$$

We also note, again using (4.4), that

$$\operatorname{Im} m_\alpha(E + i\eta) \geq \int_{(-c,c]} \frac{\eta \varrho_\alpha(x) \, dx}{\eta^2 + (E - x)^2} \geq c \int_{(-c,c]} \frac{\eta \, dx}{\eta^2 + (E - x)^2}. \quad (A.22)$$

and the latter quantity is uniformly bounded below $\eta$ tends to zero because $E \in (-c,c)$. Thus, after decreasing $c$ if necessary, we have $\operatorname{Im} m_\alpha(E + i\eta) > c$ when $\eta, |E| < c$.

The second claim of (4.6) follows after noting (using the bounds on the density $\varrho_\alpha$ given in (4.4)) that for any $c$, there exists $c' > 0$ such that $|\gamma_{\alpha}(i)| < c$ for all $i \in [(1/2 - c'), (1/2 + c')] N$.

**Proof of Lemma 4.7.** We prove only the first claim in detail. The proof of the second is analogous, and the third follows from the second by (4.14).

By a standard stochastic continuity argument, it suffices to prove the desired bound holds with overwhelming probability at fixed $\gamma$; all bounds below will be independent of $\gamma$. Let $C > 0$ be a
parameter. A straightforward calculation shows that for any $1 \leq i \leq N$, the law of the sum of the absolute value of the entries of the $i$-th row and column of the matrix $H^\top$ has a power law tail with parameter $\alpha$. This implies, by Hoeffding’s inequality, that with overwhelming probability there are at most $2C^{-\alpha}N$ such $i$ whose corresponding sum is greater than $C$. When this holds, after removing at most $2C^{-\alpha}N$ rows and columns, the largest eigenvalue is at most $C$ in absolute value, since the largest absolute value of a row of a matrix bounds the magnitude of its largest eigenvalue. Then eigenvalue interlacing [17, Lemma 7.4] implies that there are at most $4C^{-\alpha}N$ eigenvalues of $H^\top$ outside of the interval $[-C, C]$.

By [18, Theorem 1.1] (or [28, Theorem 1.2]), for any fixed compact interval $I \subset \mathbb{R}$,

$$\mathbb{E} [\mu_N(I)] \to \mu_\alpha(I)$$

(A.23)

as $N$ tends to $\infty$. Here $\mu_N = \mu_N^{(\gamma)}$ denotes the empirical spectral distribution of $H^\top$. For any $C > 0$, let $I_C = [c_1/2, C]$. Then (A.23) and the concentration estimate [29, Lemma C.1] imply that for any choice of $C$ there exists $N(C)$ such that

$$\mu_N(I_C) \leq C^{-1} + \mu_\alpha(I_C)$$

(A.24)

holds for any $N > N(C)$ with overwhelming probability. By the symmetry of $\mu_\alpha$ and the second estimate in (A.4), we have $\mu_\alpha(I_C) \leq (1/2 - \delta)$ for some $\delta = \delta(c_1) > 0$ such that $\lim_{c_1 \to 0} \delta(c_1) = 0$. Combining (A.24) with the estimate for eigenvalues lying outside $[-C, C]$, we find

$$\mu_N([c_1, \infty)) \leq C^{-1} + (1/2 - \delta) + 4C^{-1/\alpha} < 1/2 - \delta/2$$

(A.25)

for large enough $C$, with overwhelming probability. Then the $(1/2 - \delta/2)N$-th eigenvalue is less than $c_1$ with overwhelming probability. A similar argument shows that the $(1/2 + \delta/2)N$-th eigenvalue is greater than $c_1$ with overwhelming probability. This completes the proof.

Before proceeding to the proof of Lemma 4.8, we require the following preliminary lemma. We recall $m_N(s, z)$ and its expectation $\tilde{m}_N(s, z) = \mathbb{E} [\tilde{m}_N(s, z)]$ from (3.15), and $\mu_s$ from (3.16). Let $\tilde{\mu}_s = \mathbb{E} [\mu_s]$, which is symmetric about the origin. We further define the counting functions

$$n_s(E) = \frac{1}{N} |\{i : \lambda_i(s) \leq E\}| = \mu_s((-\infty, E]), \quad \tilde{n}_s(E) = \tilde{\mu}_s((-\infty, E]).$$

(A.26)

Lemma A.2. Retain the notation of Lemma 4.8. There exists a constant $c_0 > 0$ such that, for any $E \in [-c_0, c_0]$, we have with overwhelming probability that

$$\sup_{s \in [N^{-1/2 + \delta}, N^{-\delta}]} |n_s(E) - \tilde{n}_s(E)| \leq N^{\delta - 1/2}.$$  

(A.27)

Proof. This proof will largely follow the calculations of [63, Section 7.3], with some modifications to account for the fact that the spectral distributions we consider are not compactly supported.

To that end, we begin with a tail bound on the smallest eigenvalue, $\lambda_1 = \lambda_1(s)$, of $X_s$. A straightforward calculation shows that, for any $1 \leq i \leq N$, the law of the sum of the absolute value of the entries of the $i$-th row and column of the matrix $X_s$ has a power law tail with parameter $\alpha$. Thus, since the largest such sum bounds $|\lambda_1|$ above, we deduce for any $t > 1$ that

$$\mathbb{P}(\lambda_1 < -t) \leq C N t^{-\alpha}.$$  

(A.28)

Now, we must show that (A.27) holds on an event of probability at least $1 - N^{-D}$ for any fixed $D > 0$ and sufficiently large $N$. Throughout the remainder of this proof, set $B = \alpha^{-1}(D + 3)$ so, by (A.28), $\mathbb{P}(\lambda_1 > -N^B) \geq 1 - N^{-D - 2}$. Thus, we may work on the event on which $\lambda_1 > -N^B$. 
By the Helffer–Sjöstrand formula (see, for example, [17, Chapter 11]), for any smooth and compactly supported function $f : \mathbb{R} \to \mathbb{R}$,

$$f(u) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{iyf''(x)g(y) + i(f(x) + iyf'(x))g'(y)}{u - x - iy} \, dx \, dy,$$

(A.29)

where $g$ is any smooth, compactly supported function that is $1$ in a neighborhood of $0$. Set $E_1 = -N^{-4B}$ and fix some $E_2 \in [-2K^{-1}, 2K^{-1}]$, where $K$ is the constant from Lemma 4.3. Let $\eta = N^{-1/2+\delta}$, and let $f$ be a smooth function satisfying $f(E) = 0$ for $E \notin [E_1 - 1, E_2 + \eta]$ and $f(E) = 1$ for $E \in [E_1, E_2]$. We can select $f$ such that $|f(x)| \leq 1$ for all $x \in \mathbb{R}$; $|f'(x)| \leq C$ and $|f''(x)| \leq C$ for $x \in [E_1 - 1, E_1]$; and $|f'(x)| \leq C\eta^{-1}$ and $|f''(x)| \leq C\eta^{-2}$ for $x \in [E_2, E_2 + \eta]$. We also let $g(y)$ be a smooth function satisfying $g(y) = 1$ for $|y| \leq N^{10B}$; $g(y) = 0$ for $|y| > N^{10B} + 1$; we may select $g$ such that $0 \leq g(y) < 1$ and $|g'(y)| < C$ for all $y \in \mathbb{R}$.

Write $\mu_\Delta = \mu_\Delta - \hat{\mu}_\Delta$, and let $m_\Delta(z) = m_N(s, z) - \hat{m}_N(s, z)$ be the Stieltjes transform of $\mu_\Delta$. Our first goal is to prove that

$$\left| \int_{\mathbb{R}} f(E) \, d\mu_\Delta(E) \right| \leq C N^{\delta/2 - 1/2}$$

(A.30)

with probability at least $1 - N^{-D-1}$, for large enough $N$. Using (A.29), we find

$$\left| \int_{\mathbb{R}} f(E) \, d\mu_\Delta(E) \right| \leq C \left| \int_{\mathbb{R}^2} yf''(x)g(y) \, \text{Im} \, m_\Delta(x + iy) \, dx \, dy \right|$$

(A.31)

$$+ C \int_{\mathbb{R}^2} |f(x)g'(y)| \, \text{Im} \, m_\Delta(x + iy) \, dx \, dy$$

(A.32)

$$+ C \int_{\mathbb{R}^2} |yf'(x)g'(y)| \, \text{Re} \, m_\Delta(x + iy) \, dx \, dy.$$  

(A.33)

Now, since [5, (5.13)] states

$$\mathbb{P} \left[ |m_N(s, z) - \hat{m}_N(s, z)| > \frac{4 \log N}{N^{1/2} \text{Im} \, z} \right] \leq 2 \exp \left( - (\log N)^2 \right),$$

(A.34)

we have (by a standard stochastic continuity argument) with overwhelming probability that

$$\sup_{|y| \leq N^{20B}} |ym_\Delta(x + iy)| \leq 5N^{-1/2} \log N.$$  

(A.35)

Now let us bound the quantities (A.31), (A.32), and (A.33). We begin with the latter. To that end, observe that since $\supp f' \subseteq [E_1 - 1, E_1] \cup [E_2, E_2 + \eta]$; since $\supp g' \subseteq [-N^{10B} - 1, -N^{10B}] \cup [N^{10B}, N^{10B} + 1]$; since $|f'(x)| \leq C$ for $x \in [E_1 - 1, E_1]$; since $|f'(x)| \leq C\eta^{-1}$ for $x \in [E_2, E_2 + \eta]$; and since $|g'(y)| \leq C$, we have by (A.35) that $A.33 \leq C(\log N)N^{-1/2}$ with overwhelming probability.

Similarly, to bound (A.32), observe that $|f(x)| \leq 1$; that $|g'(x)| \leq C$; that $\supp f$ is contained in the interval $[-N^{4B}, N^{4B}]$ of length at most $2N^{4B}$; and that $\supp g' \subseteq [-N^{10B} - 1, -N^{10B}] \cup [N^{10B}, N^{10B} + 1]$, on which we have $|m_\Delta(z)| \leq N^{-10B} \log N$ (due to the deterministic bound $|m_\Delta(z)| < |\text{Im} \, z|^{-1}$). Together, these yield the deterministic estimate $A.32 \leq N^{-1}$. 

It therefore suffices to bound (A.31), to which end we write

\begin{align}
\text{(A.31)} & \leq \left| \int_{|x-E_1| \leq 2, |y| \leq 10} y f''(x) g(y) \Im m_\Delta(x + iy) \, dx \, dy \right| \tag{A.36} \\
& \quad + \left| \int_{|x-E_1| \leq 2, |y| > 10} y f''(x) g(y) \Im m_\Delta(x + iy) \, dx \, dy \right| \tag{A.37} \\
& \quad + \left| \int_{|x-E_2| \leq 2, |y| \leq \eta} y f''(x) g(y) \Im m_\Delta(x + iy) \, dx \, dy \right| \tag{A.38} \\
& \quad + \left| \int_{|x-E_2| \leq 2, |y| > \eta} y f''(x) g(y) \Im m_\Delta(x + iy) \, dx \, dy \right|. \tag{A.39}
\end{align}

We must bound the terms (A.36), (A.37), (A.38), and (A.39); we begin with the former. Since we restricted to the event of probability \(1 - N^{-D-2}\) on which \(\lambda_1 > -N^B\), and since \(E_1 = N^B\), the definition of \(m_N(s, z)\) shows that \(|m_N(s, x+iy)| \leq CN^{-4B} \) for \(x \in [E_1-2, E_1+2]\). Using the trivial bound (A.2) on the complementary event and then taking expectation shows that \(|\tilde{m}_N(s, x+iy)| \leq CN^{-4B} + N^{-D-2} y^{-1}\) for \(x \in [E_1-1, E_1+1]\), if \(D\) is sufficiently large. Hence, with probability \(1 - N^{-D-2}\), we have \(|m_\Delta(x+iy)| \leq N^{-4B} + N^{-D-2} y^{-1}\) for \(x \in [E_1-1, E_1+1]\). Combining this with the fact that \(|f''(x)|\) for \(x \in [E_1-1, E_1+1]\), we obtain (A.36) \(\leq N^{-1}\).

To estimate (A.37), we first integrate by parts in \(x\), using the identity \(\partial_x m_\Delta = -\partial_y \Re m_\Delta\) and the fact that \(\text{supp } f' \subseteq [E_1-1, E_1+1]\) to deduce

\begin{align}
\text{(A.37)} & = \left| \int_{|x-E_1| \leq 2, |y| > 10} y f'(x) g(y) \partial_y (\Re m_\Delta(x + iy)) \, dx \, dy \right|. \tag{A.40}
\end{align}

Integrating by parts in \(y\) and using the fact that \(\partial_y (yg(y)) = g(y) + yg'(y)\) then gives

\begin{align}
\text{(A.37)} & \leq \left| \int_{|x-E_1| \leq 2, |y| > 10} f'(x)(g(y) + yg'(y)) \Re m_\Delta(x + iy) \, dx \, dy \right| \tag{A.41} \\
& \quad + \left| \int_{|x-E_2| \leq 2} f'(x)10g(10) \Re m_\Delta(x + 10i) \, dx \right|. \tag{A.42}
\end{align}

To bound (A.42), we use (A.35) and the fact that \(|f'(x)| \leq C\) for \(x \in [E_1-2, E_1+2]\) to deduce that \(\text{(A.42)} \leq C(\log N)N^{-1/2}\) with overwhelming probability. To estimate (A.41), we again use the the facts that \(|f'(x)| \leq C\) for \(x \in [E_1-2, E_1+2]\); that \(g \subseteq [-N^{10B} - 1, N^{10B} + 1]\); that \(0 \leq g(y) \leq 1\); that \(\text{supp } g' \subseteq [N^{10B}, N^{10B} + 1]\); \(g'(y) \leq C\); and (A.35) to deduce

\begin{align}
\text{(A.41)} & \leq CN^{-1/2} \log N \int_{|y| > 10} |y|^{-1} (g(y) + yg'(y)) \, dy \tag{A.43} \\
& \leq CN^{-1/2} \log N \left( \int_{|y| > 10} |y|^{-1} \, dy + \int_{|y| > 10} g'(y) \, dy \right) \leq CN^{-1/2}(\log N)^3, \tag{A.44}
\end{align}

with overwhelming probability and for sufficiently large \(N\). Hence, (A.37) \(\leq CN^{-1/2}(\log N)^3\).
To bound \( \text{(A.38)} \), first recall that the function \( y \text{ Im } m(x + iy) \) is increasing in \( y \), for any Stieltjes transform \( m \) of a positive measure. Therefore, \( \text{(A.35)} \) implies with overwhelming probability that
\[
\sup_{y \leq \eta} y \text{ Im } m_{\Delta}(x + iy) \leq \eta \text{ Im } m_{\Delta}(x + i\eta) \leq CN^{-1/2} \log N \tag{A.45}
\]
Putting this estimate into \( \text{(A.38)} \) and using that \( |f''(x)| \) vanishes except on \( [E_2, E_2 + \eta] \), where it is at most \( C\eta^{-2} \), and that \( |g(y)| \) is 1 for \( |y| \leq \eta \), we deduce that \( \text{(A.38)} \leq C(\log N)N^{-1/2} \).

For the term \( \text{(A.39)} \), we integrate by parts as we did for \( \text{(A.31)} \) to obtain
\[
\text{(A.39)} \leq \int_{|x - E_2| \leq 2\eta} f'(x) \partial_y (g(y) + yg'(y)) \text{ Re } m_{\Delta}(x + iy) \, dx \, dy \tag{A.46}
\]
\[
\quad + \int_{|x - E_2| \leq 2\eta} f'(x) \eta g(\eta) \text{ Re } m_{\Delta}(x + i\eta) \, dx \tag{A.47}
\]
We use \( \text{(A.35)} \) to estimate \( \text{(A.46)} \leq C(\log N)^3N^{-1/2} \) and \( \text{(A.47)} \leq C(\log N)N^{-1/2} \), in the same way we bounded \( \text{(A.41)} \) (in \( \text{(A.43)} \)) and \( \text{(A.42)} \), except now we note that \( f'(x) \) vanishes off of \( x \in [E_2, E_2 + \eta] \), where it is at most \( C\eta^{-1} \). This shows \( \text{(A.39)} \leq C(\log N)^3N^{-1/2} \), and so \( \text{(A.31)} \leq C(\log N)^3N^{-1/2} \), with overwhelming probability. Combining our estimates on \( \text{(A.31)}, \text{(A.32)}, \) and \( \text{(A.33)} \), we deduce \( \text{(A.30)} \) holds with probability at least \( 1 - N^{-D} \).

We now use \( \text{(A.30)} \) to estimate the difference between the eigenvalue counting functions for the measures \( \mu_s \) and \( \hat{\mu}_s \). We recall that we are working on the set of probability at least \( 1 - N^{-D-2} \) where \( \lambda_1 > -N^B \). Therefore, recalling the definition of \( f(x) \), we see \( n_s(E_2) \leq \int_{-\infty}^{\infty} f(x) \, d\mu_s(x) \leq n_s(E_2 + \eta) \) for any \( E_2 \in [-2K, 2K] \) on this event. The case of \( \hat{n}_s(E) \) is slightly more delicate, since we must estimate the contribution to the mass of \( \hat{\mu}_s \) from eigenvalues in the interval \( (-\infty, -N^B) \). With probability at least \( 1 - N^{-D-2} \), there are no eigenvalues in this interval. On the complementary event, trivially have \( \int_{-N^B}^{-\infty} d\hat{\mu}_s(x) \leq 1 \), since \( \mu_s \) is a probability measure. Therefore,
\[
\int_{-\infty}^{-N^B} d\hat{\mu}_s(x) \leq N^{-2} \text{ by taking expectation and using the previous two observations.}
\]

Using \( \text{(A.30)} \) and the overwhelming probability estimate \( |n_s(E_2 + \eta) - n_s(E_2)| \leq C\eta \) (which follows from the second inequality of \( \text{(A.21)} \)), we deduce that, with probability at least \( 1 - CN^{-D-1} \),
\[
\hat{n}_s(E_2) - n_s(E_2) \leq \hat{n}_s(E_2 + \eta) - n_s(E_2 + \eta) + C\eta \leq \int_{-\infty}^{\infty} f(x) \, d\hat{\mu}_s(x) + CN^{-2} - \int_{-\infty}^{\infty} f(x) \, d\mu_s(x) + C\eta \tag{A.48}
\]
\[
\leq CN^{\delta/2-1/2} + C\eta \leq CN^{\delta/2-1/2}. \tag{A.49}
\]
Similarly, now using the bound \( |\hat{n}_s(E_2 + \eta) - \hat{n}_s(E_2)| \leq C\eta \) (which follows from the overwhelming probability estimate \( |n_s(E_2 + \eta) - n_s(E_2)| \leq C\eta \) after taking expectation and using the trivial bound \( |n_s(E_2 + \eta) - n_s(E_2)| \leq 1 \) on the set where this estimate does not hold), we obtain
\[
n_s(E_2) - \hat{n}_s(E_2) \leq n_s(E_2 + \eta) - \hat{n}_s(E_2 + \eta) + C\eta \tag{A.50}
\]
\[
\leq \int_{-\infty}^{\infty} f(x) \, d\mu_s(x) + CN^{-2} - \int_{-\infty}^{\infty} f(x) \, d\hat{\mu}_s(x) + C\eta \tag{A.51}
\]
\[
\leq CN^{\delta/2-1/2} + C\eta \leq CN^{\delta/2-1/2}, \tag{A.52}
\]
with probability at least \( 1 - CN^{-D-1} \). Hence, with probability at least \( 1 - N^{-D} \), we conclude \( |n_s(E) - \hat{n}_s(E)| \leq N^{\delta-1/2} \), for \( |E| \leq (2K)^{-1} \).
We are now ready for the proof of Lemma 4.8.

**Proof of Lemma 4.8** We start with the first claim. By (4.12) and (4.13), we have with overwhelming probability that
\[
\sup_{s \in [N^{-1/2+\delta}, N^{-\delta}]} \sup_{z \in \mathcal{D}} |m_\alpha(z) - m_N(s, z)| < N^{-\alpha\delta/16}.
\] (A.53)

By using (4.2) on the set where this does not hold and taking expectation, we deduce the deterministic estimate
\[
\sup_{s \in [N^{-1/2+\delta}, N^{-\delta}]} \sup_{z \in \mathcal{D}} |m_\alpha(z) - \hat{m}_N(s, z)| < CN^{-\alpha\delta/16}.
\] (A.54)

Next, we define
\[
n_\alpha(E) = \int_{-\infty}^E \vartheta_\alpha(x) \, dx.
\] (A.55)

The distribution functions $n_\alpha(E) - n_\alpha(0) = \int_0^E \vartheta_\alpha(x) \, dx$ and $\hat{n}_s(E) - \hat{n}_s(0) = \int_0^E d\hat{\mu}_s(x)$ may be compared using [47, (11.3)] (see also the proof of [47, Lemma 11.3]), which by (A.54) gives
\[
|n_\alpha(E) - \hat{n}_s(E) - n_\alpha(0) + \hat{n}_s(0)| \leq CN^{-c}
\] (A.56)

for some $c = c(\delta) > 0$ and $E \in [-c_0, c_0]$, where $c_0 > 0$ is sufficiently small.

There are two cases: $N$ is even and $N$ is odd. When $N$ is even, by the symmetry of the measures $\hat{\mu}_s$ and $\mu_\alpha$, we see that $\hat{\gamma}_{N/2}(s) = \hat{\gamma}_N(s) = 0$. We consider this case first.

We will show that for any $c_1 > 0$ sufficiently small, there exists $c_2 > 0$ such that, if $|i - N/2| \leq c_2 N$, then $|\hat{\gamma}_i(s)| < c_1$. To that end, observe that (6.18) implies, with overwhelming probability,
\[
\int_{-\infty}^\nu d\mu_s \geq cv,
\] (A.57)

for any $v \in [N^{\delta-1}, c_0]$ (after decreasing $c_0$, if necessary). Taking expectation and using that $\mu_s$ is a nonnegative measure, we find
\[
\int_0^v d\hat{\mu}_s \geq \frac{cv}{2}, \quad \text{for any } v \in [N^{\delta-1}, c_0].
\] (A.58)

We may suppose by symmetry that $i \geq N/2$, so that $\hat{\gamma}_i(s) \geq 0$ by the symmetry of $\hat{\mu}_s(x)$. By definition,
\[
\int_0^{\hat{\gamma}_i(s)} d\hat{\mu}_s = \frac{i - N/2}{N} \leq c_2.
\] (A.59)

If we choose $c_2 < c_1 c/4$, then $\hat{\gamma}_i(s) > c_1$ produces a contradiction with (A.58). By (6.14), we also have that for any $c_1 > 0$ sufficiently small, there exists $c_2 > 0$ such that, if $|i - N/2| \leq c_2 N$, then $|\gamma_i| < c_1$. We take $c_2$ small enough so that $|i - N/2| \leq c_2 N$ implies $\gamma_i, \hat{\gamma}_i(s) \in [-c_0, c_0]$, and consider just this set of indices in what follows.

We observe that, for $s > 0$, $\hat{\mu}_s$ is absolutely continuous with respect to Lebesgue measure, since each entry of $X_s$ is. Then, by the definition of $\gamma_i$ and $\hat{\gamma}_i(s)$,
\[
\int_0^{\hat{\gamma}_i(s)} \vartheta_\alpha(x) \, dx + \int_{\hat{\gamma}_i(s)}^{\gamma_i} \vartheta_\alpha(x) \, dx = \int_0^{\gamma_i} \vartheta_\alpha(x) \, dx = \int_0^{\gamma_i} \vartheta_\alpha(x) \, dx = \int_0^{\hat{\gamma}_i(s)} d\hat{\mu}_s.
\] (A.60)
Using (A.56) and the fact that \( g_\alpha(x) \) is bounded below on \([-K, K]\) by some constant \( c' \) by (4.4), we see

\[
N^{-c} \geq \left| \int_{\gamma_i(s)}^{\gamma_i} g_\alpha(x) \, dx \right| \geq c' |\gamma_i - \hat{\gamma}_i(s)|
\]

(A.61)
deterministically, which completes the proof when \( N \) is even.

When \( N \) is odd, we have \( \hat{\gamma}_{[N/2]}(s) = -\hat{\gamma}_{[N/2]}(s) \) by symmetry. If \( \hat{\gamma}_{[N/2]}(s) > N^{-1+\delta} \), then setting \( v = |\hat{\gamma}_{[N/2]}(s)| \) in (A.58) yields and also using the fact that

\[
N^{-1} = \int_{\hat{\gamma}_{[N/2]}(s)}^{\hat{\gamma}_{[N/2]}(s)} d\hat{\mu}_s = 2 \int_{\hat{\gamma}_{[N/2]}(s)}^{\hat{\gamma}_{[N/2]}(s)} d\hat{\mu}_s \geq \frac{cN^{\delta-1}}{2},
\]

(A.62)
which is a contradiction. Thus, \( |\hat{\gamma}_{[N/2]}(s)| = |\hat{\gamma}_{[N/2]}(s)| \leq CN^{-1+\delta} \). We write

\[
\int_{\hat{\gamma}_{[N/2]}(s)}^{\hat{\gamma}_{[N/2]}(s)} d\hat{\mu}_s = \int_{\hat{\gamma}_{[N/2]}(s)}^{\hat{\gamma}_{[N/2]}(s)} g_\alpha(x) \, dx + \int_{\hat{\gamma}_{[N/2]}(s)}^{\hat{\gamma}_{[N/2]}(s)} g_\alpha(x) \, dx.
\]

(A.63)
Since \( |\hat{\gamma}_{[N/2]}(s)| \leq CN^{-1+\delta} \), and by (4.4), \( |\gamma_{[N/2]}(s)| \leq CN^{-1+\delta} \), we have

\[
|\hat{\gamma}_{[N/2]}(s) - \gamma_{[N/2]}(s)| \leq CN^{-1+\delta}.
\]

(A.64)
In conjunction with (4.4), this shows

\[
\int_{\hat{\gamma}_{[N/2]}(s)}^{\hat{\gamma}_{[N/2]}(s)} d\hat{\mu}_s = \int_{\hat{\gamma}_{[N/2]}(s)}^{\hat{\gamma}_{[N/2]}(s)} g_\alpha(x) \, dx + \int_{\hat{\gamma}_{[N/2]}(s)}^{\hat{\gamma}_{[N/2]}(s)} g_\alpha(x) \, dx + O(N^{-1+\delta}).
\]

(A.65)
We may then proceed using (A.56) as before in (A.61) to complete the proof.

The proof of the second claim, (4.23), uses (A.27) and proceeds similarly, except there is no need to treat the cases of even and odd \( N \) separately. By (4.17) and the discussion following (A.59), there exists \( c_2 > 0 \) such that \( |i - N/2| \leq c_2N \) implies \( \gamma_i, \lambda_i(s) \in [-c_0, c_0] \) with overwhelming probability. We then write

\[
\int_{-\infty}^{\hat{\gamma}_i(s)} d\hat{\mu}_s = \int_{-\infty}^{\hat{\gamma}_i(s)} d\mu_s + \int_{\hat{\gamma}_i(s)}^{\lambda_i(s)} d\mu_s + \int_{\lambda_i(s)}^{\hat{\gamma}_i(s)} d\mu_s.
\]

(A.66)
Then using (A.27), with overwhelming probability we have

\[
N^{\delta/2-1/2} \geq \left| \int_{\hat{\gamma}_i(s)}^{\lambda_i(s)} d\mu_s \right|.
\]

(A.67)
Assuming to the contrary that \( |\lambda_i(s) - \hat{\gamma}_i(s)| \geq N^{\delta-1} \), we use (6.18) again to show that, with overwhelming probability,

\[
CN^{\delta/2-1/2} \geq \left| \int_{\hat{\gamma}_i(s)}^{\lambda_i(s)} d\mu_s \right| \geq c' |\lambda_i(s) - \hat{\gamma}_i(s)|.
\]

(A.68)
Thus, \( |\lambda_i(s) - \hat{\gamma}_i(s)| \leq CN^{\delta/2-1/2} \), which is a contradiction and so \( |\lambda_i(s) - \hat{\gamma}_i(s)| \geq N^{\delta-1} \). Finally, we estimate

\[
|\gamma_i(s) - \hat{\gamma}_i(s)| \leq |\gamma_i(s) - \lambda_i(s)| + |\lambda_i(s) - \hat{\gamma}_i(s)| \leq N^{-1+\delta} + CN^{-1/2+\delta} \ll N^{-1/2+\delta}
\]

(A.69)
using (4.24) to bound \( |\gamma_i(s) - \lambda_i(s)| \), which proves (4.24).
APPENDIX B. CONVERGENCE IN DISTRIBUTION

Proposition B.1. For $\alpha \in (2/3, 2) \setminus \mathcal{A}$, there is a unique limit point $\mathcal{R}_*(E)$ of the sequence of random variables $\{\text{Im } R_*(E + i\eta)\}_{\eta > 0}$ in the weak topology. For $\alpha \in (1, 2) \setminus \mathcal{A}$, the conclusions of Theorem 2.8, Theorem 2.7, and Corollary 2.9 hold in the sense of convergence in distribution.

Proof. We begin by showing that there exist constants $C > 1 > c > 0$ such that, for $z \in \mathbb{H}$ with $|z| < c$, the random variable $R_*(z)$ satisfies the tail bound

$$\mathbb{P}(\text{Im } R_*(z) > s) \leq \exp \left( -\frac{s^{\alpha/(2-\alpha)}}{C} \right). \quad (B.1)$$

To that end, let $R_1(z), R_2(z), \ldots$ denote mutually independent random variables each with law $R_*(z)$. By the Lévy–Khintchine formula (7.20),

$$\mathbb{E} \left[ \exp \left( -t \text{Im } \sum_{k=1}^{\infty} \xi_k R_k(z) \right) \right] = \exp \left( -t^{\alpha/2} \Gamma \left( 1 - \frac{\alpha}{2} \right) \mathbb{E} \left[ (\text{Im } R_*(z))^{\alpha/2} \right] \right) \leq \exp \left( -\frac{2t^{\alpha/2}}{C} \right),$$

for some $C > 0$, where we used that $\mathbb{E} \left[ (\text{Im } R_*(z))^{\alpha/2} \right] > c'$ for $z$ in a neighborhood of $0$ (see (A.5)).

We now compute, using (7.20),

$$\mathbb{P}(\text{Im } R_*(z) > Ct^{1-\alpha/2}) \leq \mathbb{P} \left( \text{Im } \sum_{k=1}^{\infty} \xi_k R_k(z) < \frac{t^{\alpha/2-1}}{C} \right) \quad (B.3)$$

$$= \mathbb{P} \left( \exp \left( -t \text{Im } \sum_{k=1}^{\infty} \xi_k R_k(z) \right) > \exp \left( -\frac{t^{\alpha/2}}{C} \right) \right). \quad (B.4)$$

In the first equality, we used $\text{Im } R_k(z) > 0$. Now applying Markov’s inequality to (B.2) and (B.3), we obtain

$$\mathbb{P}(\text{Im } R_*(z) > Ct^{1-\alpha/2}) \leq \exp \left( -\frac{t^{\alpha/2}}{C} \right). \quad (B.5)$$

Setting $s = Ct^{1-\alpha/2}$, we obtain (B.1) for a new value of $C$.

Let $\mathcal{R}_*(E)$ be the limit point in the weak topology of $\{\text{Im } R_*(E + i\eta)\}_{\eta > 0}$ from Definition 2.6. Note that since the tail bound (B.1) holds for $\text{Im } R_*(z)$ uniformly in $z$, it also holds for $\mathcal{R}_*(E)$.

Using the bound (B.1) for $\mathcal{R}_*(E)$, we see that for all $k$, the $k$-th moment of $\mathcal{R}_*(E)$ is bounded by $(Ck)^{k(2-\alpha)/\alpha}$ for some $C > 0$. Therefore, the series $\sum_{k \geq 1} \mathbb{E}[\mathcal{R}_*(E)^k]^{-1/2k}$ diverges when $\alpha \in (2/3, 2)$. By Carleman’s condition for positive random variables (the Stieltjes moment problem) [71, p. 21], this implies that $\mathcal{R}_*(E)$ is determined by its moments when $\alpha \in (2/3, 2)$. By Proposition 2.4 these moments are the same for any subsequential limit of $\{\text{Im } R_*(z)\}_{\eta > 0}$, so $\mathcal{R}_*(E)$ is only possible subsequential limit. Therefore the sequence converges in distribution to $\mathcal{R}_*(E)$.

Similar reasoning may be applied to the quantities $N^2 \cdot \mathcal{R}_*(E)$ appearing in Theorem 2.8 Theorem 2.7 and Corollary 2.9. By Stirling’s formula, the $k$-th moment of $N^2$ is bounded by $(Ck)^k$ for some $C > 0$, so the moments of $N^2 \cdot \mathcal{R}_*(E)$ are bounded by $(Ck)^{k(1+(2-\alpha)/\alpha)}$. For $\alpha \in (1, 2)$, $1 + (2-\alpha)/\alpha < 2$ and Carleman’s condition applies. This completes the proof. \hfill \Box

APPENDIX C. QUANTUM UNIQUE ERGODICITY OF EIGENVECTORS

For any $a_N : [1, N] \cap \mathbb{N} \rightarrow [-1, 1]$ we denote by $|a_N| = \{ 1 \leq i \leq N : a_N(i) \neq 0 \}$ the cardinality of the integer support of $a_N$. We define $\langle u_k, a_N u_k \rangle = \sum_{i=1}^{N} |u_k(i)|^2 a_N(i)$. 
Corollary C.1. For all $\alpha \in (0, 2) \setminus \mathcal{A}$, there exists $c = c(\alpha) > 0$ such that the following holds. Fix any index sequence $k = k(N)$ such that $\lim_{N \to \infty} \gamma_k = E$ for some $E \in \mathbb{R}$ satisfying $|E| < c$. Then for every $\delta > 0$, for any $a_N \in [1, N] \cap \mathbb{N} \to [-1, 1]$ such that $\sum_{i=1}^{\infty} a_N(i) = 0$ and $|a_N| \to \infty$,

$$\mathbb{P} \left( \frac{N}{|a_N|} \langle u_k, a_N u_k \rangle > \delta \right) \to 0. \quad \text{(C.1)}$$

Proof. Letting $m_2 = \mathbb{E}[U_4(E)]$, we compute

$$\mathbb{E} \left[ \left( \frac{N}{a_N} \langle u_k, a_N u_k \rangle \right)^2 \right] = \frac{1}{|a_N|^2} \mathbb{E} \left[ \left( \sum_{i=1}^{N} a_N(i) \langle N|u_k(i)^2 - m_2 \rangle \right)^2 \right]
\le \max_{i_1, i_2 \in [1, N]} \mathbb{E} \left( \langle N|u_k(i_1)^2 - m_2 \rangle \langle N|u_k(i_2)^2 - m_2 \rangle \right) + \frac{1}{|a_N|} \max_{i \in [1, N]} \mathbb{E} \left( \langle N|u_k(i)^2 - m_2 \rangle \right). \quad \text{(C.2)}$$

The conclusion applies after applying Markov’s inequality to the second moment computed in (C.2) and applying Theorem 2.7. The hypothesis that $|a_N| \to \infty$ ensures the second term in the second moment computation tends to zero. \qed

References

[1] R. Abou-Chacra and D. Thouless. Self-consistent theory of localization. II. Localization near the band edges. Journal of Physics C: Solid State Physics, 7(1):65, 1974.
[2] R. Abou-Chacra, D. Thouless, and P. Anderson. A self-consistent theory of localization. Journal of Physics C: Solid State Physics, 6(10):1734, 1973.
[3] E. Abrahams, editor. 50 Years of Anderson Localization. World Scientific, 2010.
[4] A. Aggarwal. Bulk universality for generalized Wigner matrices with few moments. Probability Theory and Related Fields, 173(1-2):375–432, 2019.
[5] A. Aggarwal, P. Lopatto, and H.-T. Yau. GOE statistics for Lévy matrices. arXiv preprint arXiv:1806.07363, 2018.
[6] M. Aizenman and S. Warzel. Absence of mobility edge for the Anderson random potential on tree graphs at weak disorder. Europhysics Letters, 96(3):37004, 2011.
[7] M. Aizenman and S. Warzel. Resonant delocalization for random Schrödinger operators on tree graphs. Journal of the European Mathematical Society, 15(4):1167–1222, 2013.
[8] O. Ajanki, T. Krüger, and L. Erdős. Singularities of solutions to quadratic vector equations on the complex upper half-plane. Communications on Pure and Applied Mathematics, 70(9):1672–1705, 2017.
[9] O. H. Ajanki, L. Erdős, and T. Krüger. Local spectral statistics of Gaussian matrices with correlated entries. Journal of Statistical Physics, 163(2):280–302, 2016.
[10] O. H. Ajanki, L. Erdős, and T. Krüger. Universality for general Wigner-type matrices. Probability Theory and Related Fields, 169(3-4):667–727, 2017.
[11] D. Aldous. Asymptotics in the random assignment problem. Probability Theory and Related Fields, 93(4):507–534, 1992.
[12] D. Aldous and J. M. Steele. The objective method: probabilistic combinatorial optimization and local weak convergence. In Probability on discrete structures, pages 1–72. Springer, 2004.
[13] P. W. Anderson. Absence of diffusion in certain random lattices. Physical Review, 109(5):1492, 1958.
[14] P. W. Anderson. Local moments and localized states. Reviews of Modern Physics, 50(2):191, 1978.
[15] G. B. Arous and A. Guionnet. The spectrum of heavy tailed random matrices. Communications in Mathematical Physics, 278(3):715–751, 2008.
[16] A. Auffinger, G. Ben Arous, and S. Péché. Poisson convergence for the largest eigenvalues of heavy tailed random matrices. In Annales de l’IHP Probabilités et statistiques, volume 45, pages 589–610, 2009.
[17] A. Auffinger and T. Tang. Extreme eigenvalues of sparse, heavy tailed random matrices. Stochastic Processes and their Applications, 126(11):3310–3330, 2016.
[47] L. Erdős and H.-T. Yau. A dynamical approach to random matrix theory. Courant Lecture Notes in Mathematics, 28, 2017.
[48] L. Erdős, A. Knowles, H.-T. Yau, and J. Yin. Spectral statistics of Erdős–Rényi graphs II: Eigenvalue spacing and the extreme eigenvalues. Communications in Mathematical Physics, 314(3):587–640, 2012.
[49] L. Erdős, A. Knowles, H.-T. Yau, and J. Yin. Spectral statistics of Erdős–Rényi graphs I: Local semicircle law. The Annals of Probability, 41(3B):2279–2375, 2013.
[50] L. Erdős, S. Péché, J. A. Ramírez, B. Schlein, and H.-T. Yau. Bulk universality for Wigner matrices. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 63(7):895–925, 2010.
[51] L. Erdős, B. Schlein, and H.-T. Yau. Universality of random matrices and local relaxation flow. Inventiones mathematicae, 185(1):75–119, 2011.
[52] L. Erdős, B. Schlein, H.-T. Yau, and J. Yin. The local relaxation flow approach to universality of the local statistics for random matrices. Annales de l’IHP Probabilités et statistiques, 48(1):1–46, 2012.
[53] L. Erdős, H.-T. Yau, and J. Yin. Bulk universality for generalized Wigner matrices. Probability Theory and Related Fields, 154(1-2):341–407, 2012.
[54] L. Galluccio, J.-P. Bouchaud, and M. Potters. Rational decisions, random matrices and spin glasses. Physica A: Statistical Mechanics and its Applications, 259(3-4):449–456, 1998.
[55] J. Heiny and T. Mikosch. The eigenstructure of the sample covariance matrices of high-dimensional stochastic volatility models with heavy tails. Bernoulli, 25(4B):3590–3622, 2019.
[56] J. Huang and B. Landon. Spectral statistics of sparse Erdős–Rényi graph Laplacians. arXiv preprint arXiv:1510.06390, 2015.
[57] J. Huang, B. Landon, and H.-T. Yau. Bulk universality of sparse random matrices. J. Math. Phys., 56(12):123301, 2015.
[58] K. Johansson. Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices. Communications in Mathematical Physics, 215(3):683–705, 2001.
[59] K. Johansson. Universality for certain Hermitian Wigner matrices under weak moment conditions. Annales de l’IHP Probabilités et statistiques, 48(1):47–79, 2012.
[60] A. Knowles and J. Yin. Eigenvector distribution of Wigner matrices. Probability Theory and Related Fields, 155(3-4):541–582, 2013.
[61] L. Laloux, P. Cizeau, J.-P. Bouchaud, and M. Potters. Noise dressing of financial correlation matrices. Physical Review Letters, 83(7):1467, 1999.
[62] L. Laloux, P. Cizeau, M. Potters, and J.-P. Bouchaud. Random matrix theory and financial correlations. International Journal of Theoretical and Applied Finance, 3(03):391–397, 2000.
[63] B. Landon and H.-T. Yau. Convergence of local statistics of Dyson Brownian motion. Comm. Math. Phys., 355:949–1000, Nov. 2017.
[64] J. O. Lee and J. Yin. A necessary and sufficient condition for edge universality of Wigner matrices. Duke Mathematical Journal, 163(1):117–173, 2014.
[65] M. Mahoney and C. Martin. Traditional and heavy tailed self regularization in neural network models. In International Conference on Machine Learning, pages 4284–4293, 2019.
[66] A. Soshnikov. Poisson statistics for the largest eigenvalues of Wigner random matrices with heavy tails. Electronic Communications in Probability, 9:10.1214/14-ECP26, 2004.
[67] A. Soshnikov. Mean field models for spin glasses. Volume I: Basic examples, volume 54. Springer Science & Business Media, 2010.
[75] T. Tao and V. Vu. Random matrices: universality of local eigenvalue statistics. *Acta mathematica*, 206(1):127–204, 2011.

[76] T. Tao and V. Vu. Random matrices: universality of local eigenvalue statistics up to the edge. *Acta Math.*, 206(1):127–204, 2011.

[77] T. Tao and V. Vu. Random matrices: universal properties of eigenvectors. *Random Matrices: Theory and Applications*, 1(01):1150001, 2012.

[78] E. Tarquini, G. Biroli, and M. Tarzia. Level statistics and localization transitions of Lévy matrices. *Physical Review Letters*, 116(1):010601, 2016.

[79] E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Annals of Mathematics*, pages 548–564, 1955.

[80] F. Yang and J. Yin. Random band matrices in the delocalized phase, III: Averaging fluctuations. *arXiv preprint arXiv:1807.02447*, 2018.