Generalized (s-Parameterized) Weyl Transformation

Alex Granik*

Abstract

A general canonical transformation of mechanical operators of position and momentum is considered. It is shown that it automatically generates a parameter s which leads to a generalized (or s-parameterized) Wigner function. This allows one to derive a generalized (s-parameterized) Moyal brackets for any dimensions. In the classical limit the s-parameterized Wigner averages of the momentum and its square yield the respective classical values. Interestingly enough, in the latter case the classical Hamilton-Jacobi equation emerges as a consequence of such a transition only if there is a non-zero parameter s.

1 Introduction

The Moyal transformation in the context of the Weyl transformation [1] (the former being a particular case of the latter) was addressed by B.Leaf [2] who departing directly from quantum mechanics derived the Weyl-transform \( A_w(Q,K) \) of the quantum operator \( A(Q,K) \) and investigated the properties of such a transform. Here \( Q \) and \( K \) denote the eigenvalues of the coordinate \( q \) and momentum \( k \) operators respectively. The well-known Moyal formula for the phase-space distribution function [3] readily followed from the derived expressions [2]. However the work [2] did not yield a more general s-parameterized transformation unifying different quantization rules.

*Department of Physics, UOP, Stockton, CA 95211; E-mail: agranik@uop.edu
Usually this transformation is introduced "by hand" (cf. [4], [5], [6]). A closer inspection of Leaf’s approach shows that it allows one to naturally arrive at the generalized (in a sense of s-parameterization) Weyl-Wigner-Groenewold-Moyal transformation without a need for apriori introduction of the parameter s. This is associated with the fact that a shift of the operators \( K \) and \( Q \) \((K, Q \rightarrow K', Q')\) under the only condition that the resulting transformation to be canonical automatically generates an arbitrary parameter \( s \) entering the resulting transformation. In Ref. [2] this shift was chosen in such a way as to satisfy the canonicity by a special choice of the numerical coefficients entering the transformation \( K, Q \rightarrow K', Q' \) and ensuring the value of the parameter \( s = 0 \).

2 Generalized Weyl Transformation

Let us consider the Hilbert space of a quantum-mechanical system having \( n \) degrees of freedom. This space is spanned by the eigenkets \(|Q\rangle\) and \(|K\rangle\) of the Cartesian coordinate operator \( q(q_1, q_2, ..., q_n) \) and the conjugate momentum \( p = (\hbar/i)(\partial/\partial q) = 2\pi\hbar k(k_1, k_2, ..., k_n) \) [with the commutation relations \( q_ik_j - q_jk_i = (i/2\pi)\delta_{ij}; i, j = 1, 2, ..., n \):]

\[
k|K\rangle = (2\pi\hbar)^{-1}\frac{\hbar}{i}\frac{\partial}{\partial q}|K\rangle = K|K\rangle \tag{1}
\]

which means

\[
|K\rangle = e^{2\pi i q \cdot K}; \quad |Q\rangle = \delta(q - Q) \tag{2}
\]

The respective completeness relations are

\[
\int dQ|Q\rangle\langle Q| = 1, \quad \int dK|K\rangle\langle K| = 1 \tag{3}
\]

Employing (1) and (2) we find that the scalar product \( <Q|K\rangle\) is

\[
<Q|K\rangle = \int e^{2\pi i q \cdot K}\delta(q - Q)dq = e^{2\pi i Q \cdot K} \tag{4}
\]

We represent an arbitrary quantum-mechanical operator \( A \) in the Hilbert space using the completeness relation (3):

\[
A \equiv \int ... \int dQ'dQ''dK'dK''|Q''\rangle\langle K''|A|K'\rangle\langle K'|Q'\rangle\langle Q'| \tag{5}
\]
Now we perform a linear transformation from the variables $Q', Q'', K', K''$ to new variables $K, Q, u, v$ according to the following:

\begin{align*}
Q''_i &= Q_i + \alpha_i v_i, \quad Q'_i = Q_i + \beta_i v_i \\
K''_i &= K_i + \gamma_i u_i, \quad K'_i = K_i + \delta_i u_i; \quad i = 1, 2, \ldots, n
\end{align*}

where $\alpha_i, \beta_i, \gamma_i$ and $\delta_i$ are some constants to be determined from an additional condition. If we require this transformation to be canonical then its Jacobian must be 1 yielding the following relations:

\begin{equation}
\prod_{i=1}^{n} (\alpha_i - \beta_i)(\gamma_i - \delta_i) = 1
\end{equation}

Note that in [2] from the very beginning the coefficients are chosen in such a way as to identically satisfy the canonicity condition: $\alpha_i = \gamma_i = 1/2; \beta_i = \delta_i = -1/2$.

We rewrite identity (5) taking into account the transformation of variables (6),(7):

\begin{equation}
A \equiv \int \ldots \int dQ dK dudv \left| Q + \alpha v \right\rangle \langle Q + \alpha v | K + \gamma u \rangle \\
\times \langle K + \gamma u | A | K + \delta u \rangle \langle K + \delta u | Q + \beta v \rangle \langle Q + \beta v |
\end{equation}

where according to (4)

\begin{equation}
\langle Q + \alpha v | K + \delta u | Q + \beta v \rangle = \\
exp\{2\pi i[\sum_{i=1}^{n}(\gamma_i - \delta_i)u_i Q_i + (\alpha_i - \beta_i)v_i K_i + u_i v_i (\alpha_i \gamma_i - \beta_i \delta_i)]\}
\end{equation}

Inserting this expression into Eq.(9) we get

\begin{equation}
A \equiv \int \ldots \int dQ dK dudv \langle K + \gamma u | A | K + \delta u \rangle | Q + \alpha v \rangle \langle Q + \beta v | \\
exp\{2\pi i[\sum_{i=1}^{n}(\gamma_i - \delta_i)u_i Q_i + (\alpha_i - \beta_i)v_i K_i + u_i v_i (\alpha_i \gamma_i - \beta_i \delta_i)]\}
\end{equation}

By a simple change of variables we can incorporate $u_i v_i (\alpha_i \gamma_i - \beta_i \delta_i)$ into variables $Q_i$. To this end we represent the power of the exponent in (10)as
follows
\[(\gamma_i - \delta_i)u_i Q_i + (\alpha_i - \beta_i)v_i K_i + u_i v_i (\alpha_i \gamma_i - \beta_i \delta_i) =
\]
\[(\gamma_i - \delta_i)u_i [Q_i + v_i \frac{\alpha_i \gamma_i - \beta_i \delta_i}{\gamma_i - \delta_i}] + (\alpha_i - \beta_i) v_i K_i\]
(11)

Introducing a new variable
\[Q^o_i = Q_i + v_i \frac{\alpha_i \gamma_i - \beta_i \delta_i}{\gamma_i - \delta_i},\]
dropping the superscript \(o\), and using the fact that
\[\exp\{-2\pi k \cdot a\} |Q\rangle = |Q+a\rangle\]
we obtain the following representation of the operator \(A\):
\[A = \int \ldots \int dQ dK A_w(\gamma, \delta, Q, K) F(\alpha, \beta, Q, K)\]
(12)

where
\[A_w(\gamma, \delta, Q, K) = \int du e^{2\pi i \sum_{k=1}^{n} (\gamma_k - \delta_k) u_k} \langle K + \gamma u | A | K + \delta u \rangle,\]
(13)
\[F(\alpha, \beta, r, Q, K) = \int dv e^{-2\pi i (\alpha - \beta) v(K-K)} |Q + rv\rangle \langle Q + rv|\]
(14)

and
\[r = r_i = (r_1, r_2, \ldots, r_n) = \frac{\gamma_i}{\gamma_i - \delta_i} (\beta_i - \alpha_i)\]

The Weyl transformation is characterized by the properties of the operator \(F(\alpha, \beta, r, Q, K)\). To investigate it further we use the following representation of the projection operator \(|Q + rv\rangle \langle Q + rv|\):
\[|Q + rv\rangle \langle Q + rv| = \delta(q - Q - rv) =
\]
\[\int dwe^{2\pi i \sum w_j (q_j - Q_j)} e^{-2\pi i \sum r_j w_j v_j}\]
(15)

As a result we get from (14)
\[F(\alpha, \beta, r, Q, K) = \int dwe^{2\pi i \sum w_j (q_j - Q_j)} e^{-2\pi i \sum r_j w_j v_j} \int dv e^{-2\pi i \sum (\alpha_j - \beta_j) v_j (k_j - K_j)}\]
(16)
Because both \( q \) and \( k \) are hermitian the last expression demonstrates that 
\( F(\alpha, \beta, r, Q, K) \) is also Hermitian.

Since
\[
\left( \frac{\partial}{\partial Q_j} \right)^m \int d\mathbf{w} e^{2\pi i \sum w_j (q_j - Q_j)} = (-2\pi i)^m \int d\mathbf{w} (w_j)^m e^{2\pi i \sum w_j (q_j - Q_j)}
\]
and
\[
\left( \frac{\partial}{\partial K_j} \right)^m \int d\mathbf{v} e^{-2\pi i \sum (\alpha_j - \beta_j) v_j (k_j - K_j)} = (2\pi i)^m (\alpha_j - \beta_j)^m \int d\mathbf{v} (v_j)^m e^{-2\pi i \sum (\alpha_j - \beta_j) v_j (k_j - K_j)}
\]
Eq.(16) yields
\[
F(\alpha, \beta, r, Q, K) = e^{\frac{i}{2\pi i} \sum \frac{r_j}{\alpha_j - \beta_j} \frac{\partial}{\partial \alpha_j} \frac{\partial}{\partial \beta_j} \int d\mathbf{v} e^{-2\pi i \sum (\alpha_j - \beta_j) v_j (k_j - K_j)} \int d\mathbf{w} e^{2\pi i \sum w_j (q_j - Q_j)}}
\]
By introducing new variables \( k_j^\alpha = k_j (\alpha_j - \beta_j), K_j^\alpha = K_j (\alpha_j - \beta_j) \) the parameters \( (\alpha_j - \beta_j) \) are "absorbed" by these variables, which means that without any loss of generality we can set
\[
(\alpha_j - \beta_j) = 1.
\]
Therefore the operator \( F(\alpha, \beta, r, Q, K) \) becomes
\[
F(\mathbf{r}, Q, K) = e^{\frac{i}{2\pi i} \sum r_j \frac{\partial}{\partial \alpha_j} \frac{\partial}{\partial \beta_j} \int d\mathbf{v} e^{-2\pi i \sum v_j (k_j - K_j)} \int d\mathbf{w} e^{2\pi i \sum w_j (q_j - Q_j)}}
\]
Since
\[
\int d\mathbf{w} e^{2\pi i \sum w_j (q_j - Q_j)} = \delta (\mathbf{q} - \mathbf{Q}); \int d\mathbf{v} e^{-2\pi i \sum v_j (k_j - K_j)} = \delta (\mathbf{k} - \mathbf{K})
\]
we obtain from Eq.(18) another representation of \( F \)
\[
F(\mathbf{r}, Q, K) = e^{\frac{i}{2\pi i} \sum r_j \frac{\partial}{\partial \alpha_j} \frac{\partial}{\partial \beta_j} \delta (\mathbf{q} - \mathbf{Q}) \delta (\mathbf{k} - \mathbf{K})}
\]
The last equation allows us to find an explicit expression for the Weyl-transform of the quantum operator $A_w(\gamma, Q, K)$ which we represent as follows:

$$A_w(r, Q', K') = \int dQ dK A_w(r, Q, K) \delta(Q - Q') \delta(K - K')$$  \hspace{1cm} (20)

Using (19) we calculate $\langle Q' | F(r, Q, K) | K' \rangle$:

$$\langle Q' | F(r, Q, K) | K' \rangle = \langle Q' | K' \rangle e^{\frac{i}{2\pi} \sum r_j \sigma^2_j \sigma^2_K} \delta(Q' - Q) \delta(K' - K)$$  \hspace{1cm} (21)

where we use the following identity

$$\langle Q | Q \rangle \equiv \delta(Q' - Q) | Q' \rangle \equiv \delta(q - Q) | Q' \rangle$$

With the help of another identity $\langle Q' | K' \rangle \langle K' | Q' \rangle \equiv 1$ we get from (21)

$$\delta(Q' - Q) \delta(K' - K) = e^{-\frac{i}{2\pi} \sum r_j \sigma^2_j \sigma^2_K} \langle Q' | F(r, Q, K) | K' \rangle \langle K' | Q' \rangle$$  \hspace{1cm} (22)

Substitution of (22) into (20) yields

$$A_w(r, Q', K') = \int dQ dK A_w(r, Q, K) e^{-\frac{i}{2\pi} \sum r_j \sigma^2_j \sigma^2_K} \langle Q' F(r, Q, K) | K' \rangle \langle K' | Q' \rangle$$  \hspace{1cm} (23)

On the other hand, from (12) follows that

$$\langle Q' | A | K' \rangle \langle K' | Q' \rangle = \int dQ dK A_w(r, Q, K) \langle Q' | F(r, Q, K) | K' \rangle \langle K' | Q' \rangle.$$

Therefore

$$e^{-\frac{i}{2\pi} \sum r_j \sigma^2_j \sigma^2_K} \langle Q' | A | K' \rangle \langle K' | Q' \rangle = \int dQ dK A_w(r, Q, K) e^{-\frac{i}{2\pi} \sum r_j \sigma^2_j \sigma^2_K} \langle Q' | F(r, Q, K) | K' \rangle \langle K' | Q' \rangle$$  \hspace{1cm} (24)
Combining (23) and (24) we get the following expression for \( A_w(r, Q, K) \):

\[
A_w(r, Q, K) = e^{-i \frac{\hbar}{2\pi} \sum r_j \frac{\partial^2}{\partial Q_j \partial K_j} \langle Q' | A | K' \rangle \langle K' | Q' \rangle} \tag{25}
\]

If we replace \( K_j \rightarrow P_j/2\pi \hbar \) and take into account that for a system with \( N \) degrees of freedom \( | K \rangle = (2\pi \hbar)^{-N/2} | P \rangle \) then (25) takes the following form:

\[
A_w(r, Q', P') = (2\pi \hbar)^N e^{-i \hbar \sum r_j \frac{\partial^2}{\partial Q_j \partial P_j} \langle Q' | A | P' \rangle \langle P' | Q' \rangle} = (2\pi \hbar)^N e^{i \hbar \sum r_j \frac{\partial^2}{\partial Q_j \partial P_j} \langle Q' | P' \rangle \langle P' | A | Q' \rangle} \tag{26}
\]

To express Eq.(26) in the Schroedinger representation we consider an orthonormal set of eigenkets \( | \psi_m \rangle \) and expand the operator \( A \) in terms of these eigenkets (eigenbras)

\[
A = \sum_{mn} w_m | \psi_m \rangle w_n^* \langle \psi_n | \tag{27}
\]

where \( w_m \) are the respective coefficients of the expansion. Upon substitution of (27) into (26) we obtain the following expression

\[
A_w(r, Q, P) = e^{i \hbar \sum r_j \frac{\partial^2}{\partial Q_j \partial P_j} \langle \psi^{| (Q)} \psi^{| (P) \rangle e^{iPQ/\hbar}}} \tag{28}
\]

which is the generalization of the Moyal formula (expression (3.10) in [3]) which follows from (28) for the particular value of the parameter \( r_j = -1/2, j = 1, 2, ..., N. \)

As a next step, we find the commutator \([A, B]\) of operators \( A \) and \( B \) from the expression (16) for \( F(r, Q, K) \) where we use \( \alpha_j - \beta_j = 1 \). With the help of the Baker-Hausdorf identity for two operators whose commutator is a constant

\[
e^{A+B} e^{[A,B]} = e^{A} e^{B}
\]

we rewrite (16)

\[
F(r, Q, K) = \int dv dw e^{-2\pi i \sum r_j w_j v_j} e^{2\pi i \sum w_j (q_j - Q_j)} e^{-2\pi i \sum v_j (k_j - K_j)}
\]
\[ \int d\mathbf{w} d\mathbf{v} e^{-\pi \sum (1+2r_j) w_j v_j} e^{-2\pi i \sum (w_j Q_j - v_j K_j)} e^{2\pi i \sum (w_j q_j - v_j k - j)} \]

This expression yields

\[ F(\mathbf{r}, Q, K) \delta(Q' - Q) \delta(K' - K) = \]

\[ \int d\mathbf{w} d\mathbf{v} d\mathbf{v}' d\mathbf{w}' e^{2\pi i \sum w_j q_j - v_j k_j} e^{-2\pi i \sum (Q_j (w_j' + w/2) + K_j (v_j' - v/2))} \times e^{2\pi i \sum (Q_j' (w' - w/2) + K'(v' + v/2))} \]

(29)

Introducing new variables

\[ v'' = v' + v/2; \quad v''' = -(v - v/2); \]
\[ u'' = -(w' - w/2); \quad w''' = w' + w/2 \]

we obtain after some (rather lengthy) algebra

\[ F(\mathbf{r}, Q, K) \delta(Q' - Q) \delta(K' - K) = e^{-\left( i/4 \pi \right) \sum \left[ \frac{\partial}{\partial Q_j} \frac{\partial}{\partial K_j'} - \frac{\partial}{\partial K_j} \frac{\partial}{\partial Q_j'} \right]} \times \]

\[ e^{-\left( i/4 \pi \right) \sum (1+2r_j) \left[ \frac{\partial}{\partial Q_j} \frac{\partial}{\partial K_j'} + \frac{\partial}{\partial K_j} \frac{\partial}{\partial Q_j'} \right]} F(\mathbf{r}, Q, K) F(\mathbf{r}, Q', K') \]

Using this expression we readily obtain that

\[ F(\mathbf{r}, Q, K) F(\mathbf{r}, Q', K') = e^{\left( i/4 \pi \right) \sum \left[ \frac{\partial}{\partial Q_j} \frac{\partial}{\partial K_j'} - \frac{\partial}{\partial K_j} \frac{\partial}{\partial Q_j'} \right]} \times \]

\[ e^{\left( i/4 \pi \right) \sum (1+2r_j) \left[ \frac{\partial}{\partial Q_j} \frac{\partial}{\partial K_j'} + \frac{\partial}{\partial K_j} \frac{\partial}{\partial Q_j'} \right]} F(\mathbf{r}, Q, K) \delta(Q' - Q) \delta(K' - K) \]

Now we can write the product of two \( AB \) operators as follows

\[ AB = \int d\mathbf{Q} d\mathbf{K} d\mathbf{Q}' d\mathbf{K}' A_w(\mathbf{r}, Q, K) B_w(\mathbf{r}, Q', K') F(\mathbf{r}, Q, K) F(\mathbf{r}, Q', K') = \]

\[ \int d\mathbf{Q} d\mathbf{K} d\mathbf{Q}' d\mathbf{K}' A_w(\mathbf{r}, Q, K) B_w(\mathbf{r}, Q', K') e^{\left( i/4 \pi \right) \sum \left[ \frac{\partial}{\partial Q_j} \frac{\partial}{\partial K_j'} - \frac{\partial}{\partial K_j} \frac{\partial}{\partial Q_j'} \right]} \times \]

\[ e^{\left( i/4 \pi \right) \sum (1+2r_j) \left[ \frac{\partial}{\partial Q_j} \frac{\partial}{\partial K_j'} + \frac{\partial}{\partial K_j} \frac{\partial}{\partial Q_j'} \right]} F(\mathbf{r}, Q, K) \delta(Q' - Q) \delta(K' - K) = \]

\[ \int d\mathbf{Q} d\mathbf{K} d\mathbf{Q}' d\mathbf{K}' F(\mathbf{r}, Q, K) \delta(Q' - Q) \delta(K' - K) e^{\left( i/4 \pi \right) \sum \left[ \frac{\partial}{\partial Q_j} \frac{\partial}{\partial K_j'} - \frac{\partial}{\partial K_j} \frac{\partial}{\partial Q_j'} \right]} \times \]

\[ e^{\left( i/4 \pi \right) \sum (1+2r_j) \left[ \frac{\partial}{\partial Q_j} \frac{\partial}{\partial K_j'} + \frac{\partial}{\partial K_j} \frac{\partial}{\partial Q_j'} \right]} A_w(\mathbf{r}, Q, K) B_w(\mathbf{r}, Q', K')(30) \]
Quite similarly we obtain that $BA$ is

$$BA = \int dQ_dK_dQ'dK' A_w(r, Q, K) B_w(r, Q', K') F(r, Q, K) F(r, Q', K') =$$

$$\int dQ_dK_dQ'dK' A_w(r, Q, K) B_w(r, Q', K') \sum_{\rho} \left[ \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \rho_j} - \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \rho_j} \right] \times$$

$$e^{(i/4\pi) \sum (1+2r_j) \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \rho_j} + \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \rho_j}} F(r, Q, K) \delta(Q' - Q) \delta(K' - K) =$$

$$\int dQ_dK_dQ'dK' F(r, Q, K) \delta(Q' - Q) \delta(K' - K) e^{(i/4\pi) \sum (1+2r_j) \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \rho_j} + \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \rho_j}} A_w(r, Q, K) B_w(r, Q', K')(31)$$

Therefore the commutator $[A, B]$ is

$$[A, B] = 2i \int dQ_dK_dQ'dK' F(r, Q, K) \delta(Q' - Q) \delta(K' - K)$$

$$e^{(i/4\pi) \sum (1+2r_j) \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \rho_j} + \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \rho_j}} \sin \left[ \frac{1}{4\pi} \left[ \frac{\partial^2}{\partial Q' \partial K'} - \frac{\partial^2}{\partial Q \partial K} \right] \right] A_w(r, Q, K) B_w(r, Q', K')(32)$$

The quantum equation of motion for an operator $\rho$ is

$$-i\hbar \frac{\partial \rho}{\partial t} = [\rho, H]$$

(33)

where the operator $\rho$ is given by Eq.(12)

$$\rho = \int dQ_dK \rho_w(t, r, Q, K) F(r, Q, K)$$

(34)

Inserting (32) and (34) into (33) we obtain

$$\frac{\partial \rho_w(r, Q, P)}{\partial t} =$$

$$\frac{2}{\hbar} \int dQ' dK' \delta(Q' - Q) \delta(K' - K) e^{(i/4\pi) \sum (1+2r_j) \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \rho_j} + \frac{\partial}{\partial \rho_j} \frac{\partial}{\partial \rho_j}} \times$$

$$\sin \left[ \frac{1}{4\pi} \left[ \frac{\partial^2}{\partial Q' \partial K'} - \frac{\partial^2}{\partial Q \partial K} \right] \right] H_w(Q', K') \rho_w(t, Q, K)$$

(35)

Performing integration and replacing parameter $r$ by the following

$$r_j = -\frac{(1+s_j)}{2}, \ j = 1, 2, ..., N$$
we obtain the generalization of the Moyal bracket \[3\]:

\[
\frac{\partial \rho(r, Q, P)}{\partial t} = [H, \rho]_M^s
\]

where (after returning to the variables \(P_j = 2\pi\hbar K_j\)) the generalized (or \(s\)-parameterized) Moyal bracket is

\[
[H, \rho]_M^s = e^{-\frac{i}{\hbar} \sum s_j \left( \frac{\partial^2}{\partial Q_j \partial P_j} + \frac{\partial^2}{\partial Q_j \partial P_j'} \right)} \times 
\frac{2}{\hbar} \sin \left\{ \frac{\hbar}{2} \left[ \frac{\partial^2}{\partial Q_j \partial P_j'} - \frac{\partial^2}{\partial Q_j \partial P_j} \right] \right\} H_w(Q', P') \rho_w(t, Q, P)
\]

The same result was presented in [3]. However there the authors introduced parameter \(s\) by hand, without relating it to any transformation of the quantum states and simply treating it as a means to achieve a unified approach to different quantization rules. On the other hand, our approach (based on [2]) explicitly shows that such a parameter is a result of a linear transformation from one quantum state to another. Since the pure states are represented by rays, it is very natural to expect that the above transformation would result in the appearance of the phase, which is clearly seen in the exponential operator.

### 3 On a Physical Meaning of the \(s\)-Parameter

It is therefore interesting to investigate what role is played by this parameter in a transition to a classical case. To this end we restrict our attention to a \(1 - D\) case and consider (following Moyal) space-conditional moments \(<p^n>_w\) (the Wigner averages of the powers of \(p^n\) of the momentum):

\[
<p^n>_w = \frac{\int A_w(p, q, s)p^n dp}{\int A_w(p, q, s) dp}
\]

For the subsequent calculations we have to transform \(A_w\) (the phase-space distribution function given by Eq.25) into an integral form. For the convenience sake, we replace parameter \(s\) by \(-\sigma\) and choose the units with \(\hbar = 1\).
As a next step, we find Fourier-transform of \( A_w(p, q, \sigma) \), Eq. (28). We denote this transform by \( M(\tau, \theta, \sigma) \):

\[
M(\tau, \theta, \sigma) = \int \int e^{(\tau p + \theta q)} e^{-\frac{1-\sigma}{2} \sigma^2 \partial^2} [\Psi^*(q)\Psi(p)e^{ipq}]dpdq
\]

Integration of (38) by parts yields:

\[
M(\tau, \theta, \sigma) = \int \int \Psi^*(q) e^{i\theta(q + \frac{\tau(1-\sigma)}{2})} \Psi(p)e^{ip(\tau + r)}dpdq
\]

We introduce a new variable \( q_1 \):

\[
q = q_1 - \frac{\tau}{2}(1 - \sigma)
\]

Using \( q_1 \) in (39) we obtain:

\[
M(\tau, \theta, \sigma) = \int \left\{ e^{i\theta q_1} \Psi^*[q_1 - \frac{\tau}{2}(1 - \sigma)] \int e^{ip[q_1 + \frac{\tau}{2}(1+\sigma)]} \Psi(p)dp \right\} dq_1
\]

Since

\[
\frac{1}{\sqrt{\hbar}} \int e^{ip[q_1 + \frac{\tau}{2}(1+\sigma)]} \Psi(p)dp = \Psi[q_1 + \frac{\tau}{2}(1 + \sigma)]
\]

relation (40) takes the following form:

\[
M(\tau, \theta, \sigma) = \int \Psi^*[q_1 - \frac{\tau}{2}(1 - \sigma)] e^{i\theta q_1} \Psi[q_1 + \frac{\tau}{2}(1 + \sigma)]dq_1
\]

Inverse Fourier-transform of \( M(\tau, \theta, \sigma) \) gives us the desired integral form of the phase-space distribution (a \( s \)-parameterized Wigner function) \( A_w(p, q, \sigma) \):

\[
A_w(p, q, \sigma) = \frac{1}{2\pi} \int \Psi^*[q_1 - \frac{\tau}{2}(1 - \sigma)] e^{-irp} \Psi[q_1 + \frac{\tau}{2}(1 + \sigma)]d\tau
\]

If parameter \( \sigma = 0 \) then, as expected, Eq. (42) is reduced to an expression first given by Wigner [7]. Using Eq. (42) we find the probability density

\[
\int A_w(p, q, \sigma)dp = \Psi^*(q)\Psi(q) = |\Psi(q)|^2 \equiv \rho
\]

\(^1\)a simplified derivation of \( A_w(p, q, \sigma \) is given in the Appendix
In general, since the parameter $\sigma$ (or $s$) is complex-valued, the $\sigma$-parameterized Wigner function $A_w(p, q, \sigma)$ is also complex-valued, in contradistinction to its conventional counterpart (with $\sigma = 0$). However for the purely imaginary values of the parameter $\sigma$, the $\sigma$-parameterized Wigner function becomes real-valued again:

i) $A_w^*(p, q, \sigma) = A_w(q, p, -\sigma)$

ii) $A_w^*(p, q, \sigma) = A_w(q, p, \sigma), \Re(\sigma) = 0$

For the following we rewrite $A_w(p, q, \sigma)$ in terms of the momentum wave function $\Phi(p)$. After some algebra we obtain

$$A_w(p, q, \sigma) = \int dp' dp'' e^{-i q(p'' - p')} \Phi^*(p'') \Phi(p') \delta[p - \frac{p''(1 + \sigma) + p'(1 - \sigma)}{2}]$$

(44)

Upon substitution of (43), (44) into (37) we get

$$\rho < p^n >_w = \int A_w(p, q, \sigma) p^n dp =$$

$$\int \int dp dp' dp'' p^n e^{-i q(p'' - p')} \Phi^*(p'') \Phi(p') \delta[p - \frac{p''(1 + \sigma) + p'(1 - \sigma)}{2}] =$$

$$\int \int dp dp' p^n e^{-i q(p'' - p')} \Phi^*(p'') \Phi(p') \left[\frac{p''(1 + \sigma) + p'(1 - \sigma)}{2}\right]^n$$

(45)

By observing that

$$\frac{1 - \sigma}{i} \frac{\partial}{\partial q_2} e^{-i (q_1 p'' - q_2 p')} = (1 - \sigma)p' e^{-i (q_1 p'' - q_2 p')} |_{q_1 \to q_2} = (1 - \sigma)e^{-i q(p'' - p')}

- \frac{1 + \sigma}{i} \frac{\partial}{\partial q_1} e^{-i (q_1 p'' - q_2 p')} = (1 - \sigma)p'' e^{-i (q_1 p'' - q_2 p')} |_{q_1 \to q_2} = (1 + \sigma)e^{-i q(p' - p'')}

we rewrite (45)

$$\rho < p^n >_w = \left\{ \frac{1}{2i} \left[ (1 - \sigma) \frac{\partial}{\partial q_2} - (1 + \sigma) \frac{\partial}{\partial q_1} \right]^n \int e^{-i q_1 p'} \Phi^*(p'') dp'' \int e^{i q_2 p'} \right\} |_{q_2 \to q_1} =$$

$$\left\{ \frac{1}{2i} \left[ (1 - \sigma) \frac{\partial}{\partial q_2} - (1 + \sigma) \frac{\partial}{\partial q_1} \right]^n \Psi^*(q_1) \Psi(q) \right\} |_{q_2 \to q_1}$$

(46)
Returning to the units with $\hbar$ and using (43), we calculate two first momenta $<p>_w$ and $<p^2>_w$:

$$<p>_w = \frac{1}{\Psi^*(q)\Psi(q)} \left\{ \frac{\hbar}{2i} \left[ (1 - \sigma) \frac{\partial}{\partial q_2} - (1 + \sigma) \frac{\partial}{\partial q_1} \right] \Psi'(q_1) \Psi(q) \right\} \bigg|_{q_2 \to q_1} = \frac{\hbar}{2i} \left\{ \frac{\delta}{\delta q} \right\} \left[ \text{Ln} \left( \frac{\Psi}{\Psi^*} \right) - \sigma \text{Ln}(\Psi^* \Psi) \right]$$

(47)

and

$$<p^2>_w = -\frac{\hbar^2}{4} \left[ (1 - \sigma)^2 \frac{\Psi''}{\Psi} - 2(1 - \sigma^2) \frac{\partial \text{Ln} \Psi}{\partial q} \frac{\partial \text{Ln} \Psi^*}{\partial q} + (1 + \sigma^2) \frac{\Psi'''}{\Psi} \right]$$

(48)

Let us consider a semi-classical limit

$$\Psi(q,t) = \sqrt{\rho} e^{iS/\hbar}$$

(49)

where $S$ is the classical action. Inserting (49) into (47) we obtain:

$$\lim_{\hbar \to 0} <p>_w = \lim_{\hbar \to 0} \left\{ \nabla S - i \frac{\hbar S}{2} \nabla (\text{Ln} \rho) \right\} = \nabla S = p_{\text{classical}}$$

(50)

If we use the Schroedinger equation:

$$i\hbar \frac{\partial \text{Ln} \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V$$

then employing (49) in (48) we get the following:

$$\lim_{\hbar \to 0} <p^2>_w = m \lim_{\hbar \to 0} \left\{ -\frac{\partial S}{\partial t} - V + \frac{1}{2m} (\nabla S)^2 + \frac{\hbar^2}{8m} (\nabla \text{Ln} \rho)^2 + \sigma^2 \left[ -\frac{\partial S}{\partial t} - V - \frac{1}{2m} (\nabla S)^2 - \frac{\hbar^2}{8m} (\nabla \text{Ln} \rho)^2 + \frac{i\hbar}{2} \frac{\partial}{\partial t} \text{Ln} \left( \frac{\Psi}{\Psi^*} \right) \right] \right\} = m \left\{ -\frac{\partial S}{\partial t} - V + \frac{1}{2m} (\nabla S)^2 + \frac{\hbar^2}{8m} (\nabla \text{Ln} \rho)^2 + \sigma^2 \left[ -\frac{\partial S}{\partial t} - V - \frac{1}{2m} (\nabla S)^2 - \frac{\hbar^2}{8m} (\nabla \text{Ln} \rho)^2 \right] \right\}$$

(51)

This limit must yield the classical value of the square of the classical momentum

$$\lim_{\hbar \to 0} <p^2>_w = p_{\text{classical}}^2 = (\nabla S)^2$$

13
which is independent of the parameter $\sigma$. This is possible if the factor at $\sigma$ in (51) becomes 0, that is

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}(\nabla S)^2 + V$$

But amazingly enough this condition is nothing more than the classical Hamilton-Jacobi equation. Thus emergence of the parameter $\sigma$ in Wigner function is tied to an emergence of the classical Hamilton-Jacobi equation in transition to a classical regime.

## 4 Conclusion

We have demonstrated that $s$-parameterized Wigner function emerges as a result of a linear transformation from one quantum state to another. The respective change of the phase space coordinates accompanying such a transform must be necessarily canonical. This allows one to arrive in a natural way, without introducing ”by hand” the $s$-parameter into the transformation either by suitably chosen displacement operator or by using it as a ”missing link” between normal and anti-normal ordering operators.

A transition to a classical regime demonstrates that parameter $s$ plays an important role, ensuring the emergence of the classical Hamilton-Jacobi equation as a condition for the disappearance of this parameter in classical mechanics.

## 5 Appendix

Since the momentum representation $\Phi(p)$ and the coordinate representation $\Psi(q)$ of the wave function (for simplicity sake we consider a 1-D case) are related as follows:

$$\Phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipq'/\hbar} \Psi(q')dq'$$

the momentum probability density is

$$|\Psi(p)|^2 = \frac{1}{2\pi\hbar} \int \int dq'dq''\Psi(q')\Psi^*(q'')e^{ip(q''-q')/\hbar}$$

14
We introduce new variables $q$ and $\tau$

$$q' = q + \alpha \tau; \quad q'' = q + \beta \tau$$

where $\alpha$ and $\beta$ are constants such that the Jacobian of transformation from $q''$, $q'$ to $q$, $\tau$ is $\hbar$ which means

$$\alpha - \beta = \hbar$$

As a result, Eq. (53) yields

$$|\Phi(p)|^2 = \frac{1}{2\pi} \int dq d\tau \psi(q + \alpha \tau) \psi^*[q + (\alpha - \hbar)\tau] e^{-ip\tau} \quad (54)$$

Now we represent the arbitrary parameter $\alpha$ as

$$\alpha = \frac{\hbar(1 + s)}{2}$$

Upon substitution of this expression into (54) we obtain

$$|\Phi(p)|^2 = \frac{1}{2\pi} \int dq \int d\tau e^{-ip\tau} \psi\left[q + \frac{\hbar(1 + s)}{2}\right] \psi^\star\left[q - \frac{\hbar(1 - s)}{2}\right] = \int dq A_w(p, q, s) \quad (55)$$

where

$$A_w(p, q, s) = \frac{1}{2\pi} \int \psi\left[q + \frac{\hbar(1 + s)}{2}\right] e^{-ip\tau} \psi^\star\left[q - \frac{\hbar(1 - s)}{2}\right] d\tau \quad (56)$$

is the $s$-parameterized Wigner function found earlier, Eq. (42)

6 Acknowledgement

The author expresses his gratitude to C.McCallum for his help in preparation of this paper.
References

[1] H.Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publ., 1950

[2] B.Leaf, J.Math.Phys, 9, No.1, 65(1968)

[3] J.E.Moyal, Proc.Cambridge Phil.Soc.45, 99(1949)

[4] K.E.Cahill,R.J.Glauber,Phys.Rev.,177,1857(1969); Phys.Rev.,177,1882(1969)

[5] N.L.Balazs and B.K.Jennings,Physics Reports, 104, 347(1984)

[6] T.Dereli, A.Vercin, J.Math.Phys., 38, 5515(1997)

[7] E.Wigner, Phys.Rev., 40,749(1932)