General bounded corner states in two-dimensional off-diagonal Aubry–André–Harper model with flat bands

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Abstract
We investigate the general bounded corner states in a two-dimensional off-diagonal Aubry–André–Harper square lattice model supporting flat bands. We show that for certain values of the nearest-neighbor hopping amplitudes, triply degenerate zero-energy flat bands emerge in this lattice system. Moreover, the two-dimensional off-diagonal Aubry–André–Harper model splits into isolated fragments and hosts some general bounded corner states, and the absence of the energy gap results in that these general bounded corner states are susceptible to disorder. By adding intracellular next-nearest-neighbor hoppings, two flat bands with opposite energies split off from the original triply degenerate zero-energy flat bands and some robust general bounded corner states appear in real-space energy spectrum. Our work shows a way to obtain robust general bounded corner states in the two-dimensional off-diagonal Aubry–André–Harper model by the intracellular next-nearest-neighbor hoppings.

1. Introduction
Exploring topological phases and matter in various systems is a focal point of current physical sciences [1–6]. In recent years, higher-order topological insulators (HOTIs) have become one of the most engrossing areas of research in the field of topological matter [7–13]. Different from conventional (first-order) topological insulators, a d-dimensional HOTI hosts d − 2- or lower-dimensional topological states, which is called higher-order topological states. As the simplest example of higher-order topological states, the second-order corner states have drawn more attention [14–16]. In the process of investigating higher-order topological corner states of triangular Kagome lattice, a new class of finite-energy corner sates called type-II corner states has been observed by Li et al [17]. In contrast to conventional (type-I) corner states, type-II corner states are not located at the corner points, but around the corner points. More recently, it has been shown that two-dimensional (2D) Su–Schrieffer–Heeger (SSH) model on a square lattice with intracellular next-nearest-neighbor (NNN) hopping hosts a broad class of corner states similar to type-II corner states. Significantly, the field profiles of some of the corner states are further away from the corner point than type-II corner states. In general, these new corner states are not located at the corners, but around the corners. Therefore, Xu et al refer to these new corner states as general bounded corner states [18]. These unique corner states offers possibilities for fabricating novel photonic devices [19, 20].

On the other hand, the investigation of flat bands in translationally invariant lattice systems has drawn a great deal of attention in recent decades [21–30]. The main reason why such dispersionless flat bands have kept scientists intrigued is that they can act as a good platform for the research of diverse interesting phenomena, such as Hall ferromagnetism [31, 32], high-temperature fractional quantum Hall states [33], Bose–Einstein condensates [34], and high-temperature superconductivity [35]. It is generally thought that the presence of flat bands in momentum-space energy spectrum is due to the destructive interference of electron hoppings, giving rise to compact localized states where the wave-function amplitudes remain nonzero over some lattice sites beyond which they completely vanish [36, 37]. While there are extensive
research on the distribution of wave-function amplitudes corresponding to compact localized states with flat-band energies, the real-space energy spectrum and field profiles of eigenstates for flat-band lattice systems have been rarely reported. Thus, a natural question that crosses our mind is: can we find some special states in real-space energy spectrum of flat-band lattice systems?

In this paper, we investigate the general bounded corner states based on a 2D off-diagonal Aubry–André–Harper (AAH) model supporting flat bands. The nearest-neighbor (NN) hopping amplitudes of the 2D off-diagonal AAH model are modulated by a cosine function. It is shown that for certain values of NN hopping amplitudes, the system can yield flat bands which are triply degenerate at zero energy. In addition, the 2D off-diagonal AAH model splits into isolated fragments, leading to the formation of two zero-energy conventional corner states and some finite-energy general bounded corner states. However, these general bounded corner states are not immune to disorder. By partially adding intracellular NNN hoppings, the system can yield flat bands which are triply degenerate at zero energy. In addition, the 2D off-diagonal AAH model splits into isolated fragments, leading to the formation of two zero-energy conventional corner states and some finite-energy general bounded corner states. Moreover, some of gapless states induced by the intracellular NNN hopping in real-space energy spectrum are doubly degenerate corner states, which are robust against disorder. In contrast to 2D SSH model in reference [18], the system hosts general bounded corner states even in the absence of intracellular NNN hopping. Our work reveals the existence of general bounded corner states in 2D off-diagonal AAH model and presents a way to obtain robust general bounded corner states by the intracellular NNN hoppings.

The rest of the paper is organized as follows: In section 2, we present the lattice model system and give the corresponding real-space and momentum-space Hamiltonians of the system. In section 3, we illustrate the condition for generating flat bands in momentum-space spectrum and demonstrate the general bounded corner states in corresponding real-space energy spectrum. In section 4, we explore the effect of the intracellular NNN hopping on the system in detail. In section 5, we discuss the possible experimental realization of the lattice model, using the superconducting circuit lattice. Finally, a conclusion is given in section 6.

2. Model and Hamiltonian

We consider a 2D off-diagonal AAH model with $N \times M$ lattice sites, as shown in figure 1, in which the unit cell consists of nine lattice sites. The real-space Hamiltonian of the 2D off-diagonal AAH model can be written as

$$
\hat{H}_0 = \sum_{n=1}^{N} \sum_{m=1}^{M-1} \left[ 1 + \cos \left( \frac{2\pi}{3} m + \varphi \right) + \frac{2\pi}{3} (n \mod 3 + 2) \mod 3 \right] \gamma_{n,m} q_{n,m+1}^\dagger q_{n,m+1} + \text{H.c.}
$$

where $q_{n,m}^\dagger$ ($q_{n,m}$) is the creation (annihilation) operator at lattice site $(n,m)$, $\gamma_1 = 1 + \cos(2\pi/3 + \varphi)$, $\gamma_2 = 1 + \cos(4\pi/3 + \varphi)$, and $\gamma_3 = 1 + \cos(2\pi + \varphi)$ are the amplitudes of three kinds of NN hoppings, and $J$ is the amplitude of the intracellular NNN hopping in unit cells. The Hamiltonian consists of two tunable

![Figure 1. Schematic diagram of the 2D off-diagonal AAH model on a square lattice. There are nine lattice sites (A–I) in one unit cell. Three kinds of NN hoppings $\gamma_1$, $\gamma_2$, and $\gamma_3$ are represented by red, green, and blue lines, respectively. The dotted magenta lines represent intracellular NNN hoppings in unit cells with strength $J$.](image-url)
We start with the case of parameters $\varphi$ and $J$, which can be used to control and modulate the NN hopping amplitudes and intracellular NNN hopping amplitudes. By applying the discrete Fourier transformation to the real-space Hamiltonian in equation (1), we obtain the momentum-space Hamiltonian as

$$H = \sum_{\bf k} \hat{\psi}^\dagger_{\bf k} h(\bf k) \hat{\psi}_{\bf k}, \quad (2)$$

where $\hat{\psi}^\dagger_{\bf k} \equiv (q_{kx,k_y,A}^\dagger \ q_{kx,k_y,B}^\dagger \ q_{kx,k_y,C}^\dagger \ q_{kx,k_y,D}^\dagger \ q_{kx,k_y,E}^\dagger \ q_{kx,k_y,F}^\dagger \ q_{kx,k_y,G}^\dagger \ q_{kx,k_y,H}^\dagger \ q_{kx,k_y,I}^\dagger)$, and the matrix $h(\bf k)$ is given by

$$h(\bf k) = \begin{pmatrix}
0 & \gamma_1 & \gamma_3 e^{-ik_x} & \gamma_1 & 0 & 0 & \gamma_3 e^{-ik_y} & 0 & 0 \\
\gamma_1 & 0 & \gamma_2 & J & \gamma_2 & 0 & 0 & \gamma_1 e^{-ik_y} & 0 \\
\gamma_3 e^{ik_x} & \gamma_2 & 0 & 0 & \gamma_3 e^{-ik_x} & \gamma_2 & 0 & 0 & \gamma_2 e^{-ik_y} \\
\gamma_1 & J & 0 & 0 & \gamma_2 & \gamma_3 e^{-ik_x} & \gamma_2 & 0 & 0 \\
0 & 0 & \gamma_2 & 0 & \gamma_3 e^{ik_x} & \gamma_2 & 0 & \gamma_3 & 0 \\
0 & \gamma_1 & \gamma_3 e^{ik_y} & 0 & \gamma_2 & \gamma_3 & 0 & \gamma_3 & 0 \\
0 & \gamma_1 e^{ik_y} & 0 & \gamma_2 & \gamma_3 & 0 & 0 & J & \gamma_1 \\
0 & 0 & \gamma_1 e^{ik_x} & 0 & 0 & \gamma_3 & 0 & \gamma_2 e^{ik_x} & \gamma_1 \\
0 & 0 & \gamma_2 e^{ik_y} & 0 & 0 & \gamma_1 e^{ik_x} & 0 & \gamma_2 e^{ik_x} & \gamma_1 \end{pmatrix}. \quad (3)$$

3. Flat bands and general bounded corner states

We start with the case of $J = 0$, i.e. in the absence of intracellular NNN hopping. In figures 2(a) and (b), we present the energy spectra as a function of $k_x (k_y)$ and $\varphi$ with fixed $k_y = 0 (k_x = 0)$. One can see from figure 2(b) that the values of three bands at the center of energy spectrum are close to zero when $\varphi = \pi/3$, $\pi$, $5\pi/3$, implying the presence of flat bands. Since the period of the energy spectrum with respect to $\varphi$ is $2\pi$, here we only analyze the case with $\varphi = \pi/3$. For $\varphi = \pi/3$, we obtain three flat bands which are triply degenerate at the zero energy and six dispersive bands, as shown in figure 2(c). Moreover, the three zero-energy flat bands touch the two nearest dispersive bands at two lines together when $k_y = k_x \pm \pi$, in consistent with figure 2(a) where five bands touch at zero energy when $k_x (k_y) = \pm \pi$ and $k_x (k_y) = 0$. To further investigate the characteristics of the 2D off-diagonal AAH model, the real-space energy spectrum is plotted with varying $\varphi$ in figure 2(d). As $\varphi$ approaches $\pi/3$, more and more states emerge at the energy $E_{FB} = 0$ in the spectrum. This agrees with the result in figures 2(a) and (b). To explore some special states in flat-band systems, in the following we investigate the real-space energy spectrum with $\varphi = \pi/3$.

The real-space energy spectrum for the 2D off-diagonal AAH model with $\varphi = \pi/3$ is shown in figure 3(a). While most of the states at the zero energy are bulk states, such as state-450 (the 450th eigenstate in corresponding real-space energy spectrum) in figure 3(b), there are two zero-energy

![Figure 2](image-url)
conventional corner states localized completely at the corner point, as shown in figures 3(c) and (d). The appearance of conventional corner states can be understood from the values of three NN hopping parameters. When $\phi = \pi/3$, the values of the NN hopping parameters are $\gamma_1 = 0$, $\gamma_2 = 1.5$, and $\gamma_3 = 1.5$. In this case, the 2D off-diagonal AAH model splits into fragments with sites $(n, m) = \{(1, 1), (30, 30)\}$ being two single isolated sites at the corners, as shown in figure 3(e).

In addition, the fragmentation of the system leads to many finite-energy states localized at these separate fragments, as shown in figures 4(a)–(d). The states localized at these separate fragments near the upper left corner or lower right corner are general bounded corner states of the whole square lattice, such as state-245 and state-267. Disorders, fluctuations of parameters, and other imperfections are inevitable in realistic systems. To test the robustness of the system, we consider the effect of disorder on the corner states, i.e. $\gamma_1 + \epsilon \delta$, $\gamma_2 + \epsilon \delta$, $\gamma_3 + \epsilon \delta$, where $\epsilon \in [-1, 1]$ is randomly distributed. Generally, the disorder, which acts on each hopping parameter independently, is different for each unit cell. Thus we perform 200 robustness simulations and calculate the average field profile. In figures 4(e) and (f), one can see that these general bounded corner states are not resistant to disorder. Up to now we have investigated the features of the present lattice system with only NN hopping. Next, we discuss the effect of the intracellular NNN hopping on the system.

4. Effect of the intracellular next-nearest-neighbor hopping on the system

The general bounded corner states in the previous section are sensitive to disorder. Inspired by reference [18], we devote to producing robust general bounded corner states by partially adding intracellular hoppings, i.e. adding the hopping between $B$ ($F$) site and $D$ ($H$) site in unit cells, as shown in figure 5(a) (see appendix A for more discussions). The inclusion of intracellular NNN hopping only increases the
interaction within fragments. In order not to deviate from the core issues, we check whether the system with intracellular NNN hopping is a flat-band system. For $\varphi = \pi/3$, by diagonalizing equation (3), we obtain three flat bands in momentum-space energy spectrum at energies $E_{FB} = 0$ and $\pm J$, respectively, as shown in figure 5(b). This indicates that two flat bands at $E_{FB} = \pm J$ are separated from triply degenerate zero-energy flat bands by intracellular NNN hopping $J$. One can easily adjust the energies of two flat bands split off from the original triply degenerate zero-energy flat bands by changing the value of $J$.

To explore the effect of adding intracellular NNN hopping on the system, we show the real-space energy spectrum of the 2D off-diagonal AAH model with varying intracellular NNN hopping $J$ in figure 6(a). Highly degenerate states (highlighted in red) corresponding to flat bands in figure 6(a) are not robust states that we are seeking for. It is obvious that the intracellular NNN hopping breaks the chiral symmetry of the system. Furthermore, more and more states split off from the continuum with the increase of $J$. To show the gapless states in detail, the real-space energy spectrum for the 2D off-diagonal AAH model is shown in figure 6(b) with $\varphi = \pi/3$ and $J = 2.5$. One can observe that several doubly degenerate corner states appear in the band gaps, and here we focus on four doubly degenerate corner states, state-382 and state-383, state-416 and state-417, state-586 and state-587, and state-618 and state-619 (highlighted in red).

The field profiles of four doubly degenerate corner states are shown in figure 7, which agree well with the separate fragments in figure 5(a). For each pair doubly degenerate corner states, one state is located around the upper left corner and the other state is located around the lower right corner. The field profiles of each pair of corner states are on the secondary diagonal symmetry. The average field profile of either of each pair

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Figure 4. The field profiles of several finite-energy states in (a)–(d). The field profiles of state-245 and state-267 with disorder $d = 0.15$ in (e) and (f).

Figure 5. (a) Schematic diagram of the 2D off-diagonal AAH model with $\varphi = \pi/3$ and $J \neq 0$. (b) The momentum-space energy spectrum for the 2D off-diagonal AAH model with $\varphi = \pi/3$ and $J = 2.5$. 
of corner states after 200 simulations shows that these doubly degenerate corner states are still in the band gaps, which indicates that these corner states are robust against disorder, as shown in figure 8. For doubly degenerate gapless states with finite energy, if the two states are located at different positions, the average field profile of either of them after many times of robustness simulations is usually a superposition of two states in different proportions. This is because the eigenvalues corresponding to finite-energy states under disorder no longer remain at their original values and are no longer degenerate.

These doubly degenerate gapless states with finite energy are different from gapless zero-energy states protected by chiral symmetry, the eigenvalues of which remain at zero until the disorder is strong enough to make the corresponding band gap close, such as zero-energy mode in SSH model [38]. In addition, the robustness of these finite-energy gapless states only depends on the width of the corresponding band gap. The further away the eigenvalue of finite-energy gapless state is from the adjacent states or bands, the more robust the general bounded corner state is. The boundary of the system robustness is presented in detail in appendix B.

5. Possible experimental realization of the model

Finally, we discuss the possible experimental realization of our proposed lattice model using superconducting circuit lattice. The modulation terms in equation (1) can be realized based on a 2D superconducting circuit lattice, as shown in figure 9. In this array, the NN resonators are coupled via the corresponding superconducting qubits $Q_1$, $Q_2$, or $Q_3$ while the NNN hoppings between $B$ ($F$) and $D$ ($H$) resonators in one unit cell are realized via another intermediate superconducting qubit $Q_J$. The
corresponding coupling parameters \( g_1, g_2, g_3 \), and \( g_J \) between resonators and superconducting qubits can be modulated via the magnetic flux provided by an external flux-bias line (FBL). Thus, the effective NN and NNN hoppings \( (\gamma_1, \gamma_2, \gamma_3, \text{and } J) \) between two resonators can be tuned via the assisted qubits. In the following, we take the coupling between resonator H and resonator I via the corresponding superconducting qubit \( Q_1 \) as an example to show that the effective NN hoppings \( \gamma_1 \) can be tuned.

The coupling between resonator \( H \) and resonator \( I \) via the corresponding superconducting qubit \( Q_1 \) can be described by the Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{H}_{\text{hop}}, \) with
The ground state and excited state of the qubit can be removed, leading the Hamiltonian can be written as

$$H = \hat{H}_0 + \hat{H}_{\text{hop}}.$$  

Here $H_0$ is the free energy of resonator $H$, resonator $I$, and qubit $Q_1$ with resonator frequency $\omega_H$ and Pauli operator $\sigma^z = |e\rangle\langle e| - |g\rangle\langle g|$. $H_{\text{hop}}$ express the coupling between resonator $H$ and resonator $I$ via qubit $Q_1$, in which $\sigma^z = |e\rangle\langle g| + |g\rangle\langle e|$ is the excitation of the superconducting qubit $Q_1$, with $|g\rangle$ and $|e\rangle$ being the ground state and excited state.

In the rotating frame with respect to the external driving frequency $\omega_d$ and the free energies of qubit $Q_1$, the Hamiltonian can be written as $\hat{H}' = \hat{H}_0' + \hat{H}_{\text{hop}}'$, with

$$\hat{H}_0' = \Delta_H H^d H + \Delta_I I^d I,$$

$$\hat{H}_{\text{hop}}' = \frac{g_1^2}{\Delta_1} \left[ (|e\rangle\langle e| H H^d - |g\rangle\langle g| H H^d H) + (|e\rangle\langle e| I^d I - |g\rangle\langle g| I^d I) \right]$$

$$+ \frac{g_2^2}{\Delta_2} \left[ |e\rangle\langle e| H I^d I - |g\rangle\langle g| I^d I + (|e\rangle\langle e| I H^d I - |g\rangle\langle g| I H^d I) \right],$$

where $\Delta_H = \omega_H - \omega_D(\Delta_1 = \omega_I - \omega_D)$ is the detuning of the resonator $H(I)$, $\Delta_1 = \omega_1 - \omega_D$ is the detuning of and qubit $Q_1$. If the qubit $Q_1$ are prepared in its ground state, the coupling between the resonators and qubit can be removed, leading the Hamiltonian can be written as $\hat{H}'' = \hat{H}_0'' + \hat{H}_{\text{hop}}''$ with

$$\hat{H}'_0 = \Delta_H H^H + \Delta_I I^H,$$

$$\hat{H}_{\text{hop}}'' = -\frac{g_1^2}{\Delta_1} (H^H H + I^H I) - \frac{g_2^2}{\Delta_2} (H^I I + I^H I).$$

Thus, the total effective Hamiltonian can be expressed as

$$\hat{H}_{\text{eff}} = \left( \Delta_H - \frac{g_1^2}{\Delta_1} \right) H^H + \left( \Delta_I - \frac{g_2^2}{\Delta_2} \right) I^H I - \frac{g_1^2}{\Delta_1} (H^H I + I^H H).$$

For simplicity, we set the $\Delta_H = \Delta_I = \frac{g_1^2}{\Delta_1}$ as the energy zero point, after that, the final effective Hamiltonian becomes

$$\hat{H}'_{\text{eff}} = -\frac{g_1^2}{\Delta_1} (H^H I + I^H H).$$

The equation (8) represents the effective coupling between resonator $H$ and resonator $I$ assisted by qubit $Q_1$. In the experiment, the coupling between the resonator and qubit $g_1$ or the detuning $\Delta_1$ can be tuned via the external control. Thus, the effective coupling $-\frac{g_1^2}{\Delta_1}$ between resonator $H$ and resonator $I$ can also be
modulated via the external field, which means that the modulation parameter $\varphi$ in the equation (1) can be tuned based on the superconducting circuit lattice.

6. Discussions and conclusions

We discuss the linkages between general bounded corner states and flat bands. Both the flat bands and the general bounded corner states are the results of the fragmentation of the system. Compared to the flat bands, the forming condition of the general bounded corner states is less stringent. As long as $\gamma_1$ is much smaller than $\gamma_2$ and $\gamma_3$, which leads the system is the equivalent of splitting into separate fragments, the system will host general bounded corner states. Furthermore, both the general bounded corner states are related to locality. Different from the general bounded corner states, the flat bands host compact localized states, the wave-function amplitudes of which are confined in a finite number of lattice sites beyond which the wave-function amplitudes completely vanish. The states with the flat-band energy are usually bulk states, which contain many of the same compact localized states.

In conclusion, we have investigated the momentum-space and real-space energy spectrum of the 2D off-diagonal AAH model in detail. It is shown that this lattice geometry hosts triply degenerate flat bands only with a NN hopping system, and the fragmentation of the system leads to the emergence of conventional corner states and general bounded corner states. With the inclusion of intracellular NNN hopping in the system, except for one flat band pins at zero energy, two flat bands with opposite energies separate from original triply degenerate flat bands. Our work presents a scheme for generating robust general bounded corner states with 2D off-diagonal AAH model, which provides the possibility for fabricating robust topological photonic devices.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Different selected set of intracellular NNN hoppings

The general bounded corner states in section 4 are sensitive to the disorder. Inspired by reference [18], we decide to produce robust general bounded corner states by partially adding intracellular hoppings. The upper left and lower right corner sites must be isolated in order to ensure that the general bounded corner states exist. Based on this direction, in addition to the scheme in the main text, we try other schemes: (i) adding the hopping between B ($C, E, F$) site and D ($E, G, H$) site in unit cells, as shown in figure 10(a); (ii) adding the hopping between B ($D, H, F$) site and D ($H, F, B$) site in unit cells, as shown in figure 10(b).

To explore the effects of two schemes mentioned above, we show the real-space energy spectra of the 2D off-diagonal AAH model with varying intracellular NNN hopping $J$ for different schemes in figures 11(a) and (b). To show the gapless states in detail, the real-space energy spectra for the 2D off-diagonal AAH model for different schemes are shown in figures 11(c) and (d) with $\varphi = \pi/3$ and $J = 2.5$, respectively. Obviously, both of the schemes mentioned-above are inferior to the scheme in the main text in terms of the number of induced gapless general bounded corner states.

Appendix B. The boundary of the system robustness

The aim of this appendix is to see what happens to the four doubly degenerate gapless corner states in figure 6(b) when the amounts of disorder $d$ in $\gamma_1$, $\gamma_2$, $\gamma_3$, and $J$ increase. In order to address this question, we numerically investigate the boundary of the system robustness. Figure 12 shows the corresponding results, averaged over 200 simulations, as a function of the disorder strength $d$. The boundaries of the robustness for each pair of doubly degenerate corner states are different. With the disorder strength $d$ increases, four doubly degenerate corner states (the four pairs of red lines in the gaps) touch the bulk band when the strength of disorder reaches the different amounts. For the two corner states in the lowest gap in
Figure 10. Schematic diagram of the 2D off-diagonal AAH model on a square lattice. There are nine lattice sites (A–I) in one unit cell. Three kinds of NN hoppings $\gamma_1$, $\gamma_2$, and $\gamma_3$ are represented by red, green, and blue lines, respectively. The dotted magenta lines represent intracell NNN hoppings in unit cells with strength $J$. (a) Adding hopping between B (C, E, F) site and D (E, G, H) site in unit cells. (b) Adding the hopping between B (D, H, F) site and D (H, F, B) site in unit cells.

Figure 11. The real-space energy spectrum for the 2D off-diagonal AAH model ($10 \times 10$ array of unit cells) versus varying intracellular NNN hopping $J$ when $\phi = \pi/3$. Highly degenerate states with flat-band energies are colored red. (a) Scheme (i), (b) scheme (ii). The real-space energy spectrum for the 2D off-diagonal AAH model with a $10 \times 10$ array of unit cells with $\phi = \pi/3$ and $J = 2.5$. Except for the two conventional corner states represented by a black point, the rest of the states with flat-band energies are colored red. Except for gapless general bounded corner states chosen for investigation are colored green, the states without flat-band energies are blue. (c) Scheme (i), (d) scheme (ii).

Figure 12, the corner states touch the bulk bands when $d \approx 0.2$. However, for the two corner states in the topmost gap in figure 12, the corner states touch the bulk bands when $d \approx 0.5$. For other two pairs of corner states, the corner states keep in the gap only when the strength of disorder satisfy $d < 0.15$. Thus, when the disorder strength $d$ is less than 0.15, all of four pairs of corner states cannot be destroyed by the weak disorder since they are protected by the gaps.
Figure 12. The real-space energy spectrum for a 2D off-diagonal AAH model (10 × 10 array of unit cells) versus increasing amounts of disorder $d$ in $\gamma_1, \gamma_2, \gamma_3$, and $J$ when $\varphi = \pi/3$ and $J = 2.5$. The red solid lines indicate four pairs of corner states chosen for investigation.

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References

[1] Liu F and Wakabayashi K 2017 Novel topological phase with a zero berry curvature Phys. Rev. Lett. 118 076803
[2] Leykam D, Bliokh K Y, Huang C, Chong Y D and Nori F 2017 Edge modes, degeneracies, and topological numbers in non-Hermitian systems Phys. Rev. Lett. 118 040401
[3] Lieu S 2018 Topological phases in the non-Hermitian Su–Schrieffer–Heeger model Phys. Rev. B 97 045106
[4] Yao X and Wang Z 2018 Edge states and topological invariants of non-Hermitian systems Phys. Rev. Lett. 121 086803
[5] Zeng Q B, Yang Y B and Xu J Y 2020 Topological phases in non-Hermitian Aubry–Andr–Harper models Phys. Rev. B 101 020201
[6] Wu H and An J H 2020 Floquet topological phases of non-Hermitian systems Phys. Rev. B 102 1804119
[7] Ezawa M 2018 Higher-order topological insulators and semimetals on the breathing Kagome and pyrochlore lattices Phys. Rev. Lett. 120 026801
[8] Imhof S et al 2018 Topolectrical-circuit realization of topological corner modes Nat. Phys. 14 925
[9] Li Z, Cao Y, Yan P and Wang X 2019 Higher-order topological solitonic insulators npj Comput. Mater. 5 107
[10] Xue H, Yang Y, Liu G, Gao F, Chong Y and Zhang B 2019 Realization of an acoustic third-order topological insulator Phys. Rev. Lett. 122 244301
[11] Li Z-X, Cao Y, Wang X R and Yan P 2020 Second-order topological solitonic insulator in a breathing square lattice of magnetic vortices Phys. Rev. B 101 184404
[12] Wakao H, Yoshida T, Araki H, Mizoguchi T and Hatsugai Y 2020 Higher-order topological phases in a spring-mass model on a breathing Kagome lattice Phys. Rev. B 101 094107
[13] Wu H, Wang B-Q and An J-H 2021 Floquet second-order topological insulators in non-Hermitian systems Phys. Rev. B 103 L041115
[14] Peterson C W, Benalcazar W A, Hughes T L and Bahl G A 2018 A quantized microwave quadrupole insulator with topologically protected corner states Nature 555 346
[15] Benalcazar W A, Li T and Hughes T L 2019 Quantization of fractional corner charge in Cn-symmetric higher-order topological crystalline insulators Phys. Rev. B 99 245151
[16] Mittal S, Orre V V, Zhu G, Gorlach M A, Poddubny A and Hafezi M 2019 Photonic quadrupole topological phases Nat. Photon. 13 692
[17] Li M, Zhirin D, Gorlach M, Ni X, Filonov D, Slobuzhanyuk A, Ali A and Khanikaev A B 2020 Higher-order topological states in photonic Kagome crystals with long-range interactions Nat. Photon. 14 89
[18] Xu X W, Li Y Z, Liu Z F and Chen A X 2020 General bounded corner states in the two-dimensional Su–Schrieffer–Heeger model with intracellular next-nearest-neighbor hopping Phys. Rev. A 101 063839
[19] Ota Y, Liu F, Katsumi R, Watanabe K, Wakabayashi K, Arakawa Y and Iwamoto S 2019 Photonic crystal nanocavity based on a topological corner state Optica 6 786
[20] Xie B-Y, Su G-X, Wang H-F, Su H, Shen X-P, Zhan P, Lu M-H, Wang Z-L and Chen Y-F 2019 Visualization of higher-order topological insulating phases in two-dimensional dielectric photonic crystals Phys. Rev. Lett. 122 233903
[21] Mielke A 1991 Ferromagnetism in the Hubbard model on line graphs and further considerations J. Phys. A: Math. Gen. 24 3311
[22] Nishino S, Goda M and Kusakabe K 2003 Flat bands of a tight-binding electronic system with hexagonal structure J. Phys. Soc. Japan 72 2015
[23] Nishino S and Goda M 2005 Three-dimensional flat-band models J. Phys. Soc. Japan 74 393
[24] Goda M, Nishino S and Matsuda H 2006 Inverse Anderson transition caused by flatbands Phys. Rev. Lett. 96 126401
[25] Chalker J T, Pickles T S and Shukla P 2010 Anderson localization in tight-binding models with flat bands Phys. Rev. B 82 104209
[26] Danieli C, Bodyfelt J D and Flach S 2015 Flat-band engineering of mobility edges Phys. Rev. B 91 235134
[27] Leykam D, Flach S, Bahat-Treidel O and Desyatnikov A S 2013 Flat band states: disorder and nonlinearity Phys. Rev. B 88 224203
[28] Pal B and Saha K 2018 Flat bands in fractal-like geometry Phys. Rev. B 97 195101
[29] Pal B 2018 Nontrivial topological flat bands in a diamond-octagon lattice geometry Phys. Rev. B 98 245116
[30] Bhattacharya A and Pal B 2019 Flat bands and nontrivial topological properties in an extended Lieb lattice Phys. Rev. B 100 235145
[31] Tasaki H 1992 Ferromagnetism in the Hubbard models with degenerate single-electron ground states Phys. Rev. Lett. 69 1608
[32] Maksymenko M, Honecker A, Moessner R, Richter J and Derzhko O 2012 Flat-band ferromagnetism as a Pauli-correlated percolation problem Phys. Rev. Lett. 109 096404
[33] Tang E, Mei J-W and Wen X-G 2011 High-temperature fractional quantum hall states Phys. Rev. Lett. 106 236802
[34] Huber S D and Altman E 2010 Bose condensation in flat bands Phys. Rev. B 82 184502
[35] Kauppila V J, Aikebaier F and Heikkilä T T 2016 Flat-band superconductivity in strained Dirac materials Phys. Rev. B 93 214505
[36] Khomeriki R and Flach S 2016 Landau–Zener Bloch oscillations with perturbed flat bands Phys. Rev. Lett. 116 245301
[37] Maimaiti W, Andreanov A, Park H C, Gendelman O and Flach S 2017 Compact localized states and flat-band generators in one dimension Phys. Rev. B 95 115135
[38] Lee T E 2016 Anomalous edge state in a non-Hermitian lattice Phys. Rev. Lett. 116 133903