Adiabatic pumping in the quasi-one-dimensional triangle lattice

Michael Schulze,1,2 Dario Bercioux,2,* and Daniel F. Urban1

1Physikalisches Institut, Albert-Ludwigs-Universität, D-79104 Freiburg, Germany
2Freiburg Institute for Advanced Studies, Albert-Ludwigs-Universität, D-79104 Freiburg, Germany

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We analyze the properties of the quasi-one-dimensional triangle lattice emphasizing the occurrence of flat bands and band touching via the tuning of the lattice hopping parameters and on-site energies. The spectral properties of the infinite system will be compared with the transmission through a finite piece of the lattice with attached semi-infinite leads. Furthermore, we investigate the adiabatic pumping properties of such a system: depending on the transmission through the lattice, this results in nonzero integer charge transfers or transfers that increase linearly with the lattice size.

I. INTRODUCTION

Quantum pumping is a process, where a periodic variation of system parameters leads to a density flux of particles through the system in spite of the absence of an external voltage. The first proposal for quantum pumping was formulated by Thouless in 1983,1 and it considered particles in a periodic system described by Bloch wavefunctions. The Thouless approach is restricted to insulating systems and here the number of particles transferred in one period is always quantized to integer values.

Another pumping approach is based on elastic scattering matrices. It was formulated by Brouwer2 and Büttiker,3–6 and allowed to explain experimental results with nonzero transmission, the charge transfer but not for the pumping current. In fact, in order to fulfill the adiabatic approximation, the pumping frequency has to decrease with the number of unit-cells.

Here, we will focus on the quasi-one-dimensional triangular lattice (Fig. 1). This lattice is characterized by a basis of two lattice sites. Its band structure exhibits flat bands and band touching as a function of the lattice parameters. First we work out the conditions for the appearance of these peculiar features. Then, we relate the properties of the infinite lattice to the transmission probability of a finite piece of lattice which is connected via semi-infinite leads to some particle reservoirs. Finally, we apply Brouwer’s formalism7 for adiabatic quantum pumping (AQP) to the scattering matrix of this finite-size structure. We find that the charge transfer is finite for a pumping parameter cycle surrounding a band touching configuration, even if along the cycle the transmission is inhibited. Further, if the pumping cycle traverses configurations with nonzero transmission, the charge transfer has in average a linear dependency on $N$, the number of unit-cells of the triangle lattice (see Fig. 1) — it increases linearly with the lattice length. However, this results is valid only for the charge transfer but not for the pumping current.

II. THE TRIANGLE LATTICE

A. Spectral properties

We define the quasi-one-dimensional triangle lattice within the tight-binding formalism. It has a basis containing two lattice sites A and B with on-site energies $\epsilon_A$ and $\epsilon_B$ and hopping parameters $t_1$, $t_2$ and $t_3$ as sketched in Fig. 1. For the infinite lattice — no leads — the energy spectrum of the system can be obtained via Bloch’s theorem. The Hamiltonian in reciprocal space reads

$$\mathcal{H}(\kappa) = \left( \frac{\epsilon_A + 2t_3 \cos \kappa}{t_1 + t_2 e^{-i\kappa}} \right) \cdot \left( \frac{\epsilon_B}{t_1 + t_2 e^{i\kappa}} \right),$$

with $\kappa = k a$. The eigenvalues can be expressed with dimensionless variables as

$$\epsilon_{1,2}(\kappa) = \cos \kappa \pm \left( [\Delta \epsilon + \tau_1 \tau_2 + \cos \kappa]^2 + \tau_1^2 + \tau_2^2 \right)^{\frac{1}{2}}$$

$$- \tau_1 \tau_2 - 2 \tau_1 \tau_2 \Delta \epsilon \right)^{\frac{1}{2}}$$

FIG. 1: Tight-binding triangle lattice with semi-infinite leads. A number of unit cells ($N$) with the two sites A and B are connected with hopping parameters $t_1$, $t_2$ and $t_3$ to form the triangle lattice of length $L = N a$. 

$\epsilon_{1,2}(\kappa)$
where all parameters are expressed in units of \( t_3 \); \( \tau_1 = t_1/t_3, \epsilon(\kappa) = (E(\kappa) - (\epsilon_A + \epsilon_B)/2)/t_3 \), \( \Delta \epsilon = (\epsilon_A - \epsilon_B)/2t_3 \), we introduce here also the dimensionless coupling constant between the leads and the central system \( c = t_2^2/(t_1t_3) \). We consider the system in absence of external magnetic fields, hence, due to the conservation of time-reversal-symmetry all lattice parameters are real valued \([H^*(\kappa) = H(\kappa)]\). However, we assume the possibility of sign changes of the hopping parameters and on-site-energies. Depending on the parameters, the energy gap between the two bands, which is defined as the difference between the minimum of the upper and the maximum of the lower band, is visualized via the color scale in Fig. 2(a)–(b). Here, the spectrum (2) exhibits a one-dimensional Dirac point [Fig. 3(a)], band crossing with flat bands, and opening of gaps as visualized in Fig. 3(e) and 3(g), respectively. In order to observe a band touching or crossing the square root in (2) must vanish for a specific \( \kappa \). This is realized for either of the three configurations

\[
\begin{align*}
\tau_1 &= \tau_2 & \Delta \epsilon &= 1 \\
\tau_1 &= -\tau_2 & \Delta \epsilon &= -1 \\
\tau_1 &= \tau_2 &= 0 & |\Delta \epsilon| &\leq 1.
\end{align*}
\]

The first two cases are included in Fig. 2(a). On the other hand, case (3c) corresponds to the splitting of the lattice into a linear chain (A-sites) with a cosine band and the isolated B-sites leading to a flat band (see Fig. 3(e) and Fig. 2(b)). The parameter \( \Delta \epsilon \) is then responsible for the relative position between the flat band and the cosine band. If \(|\Delta \epsilon| < 1\), the bands cross at two points within the first Brillouin zone.

The general condition for the occurrence of a flat band is immediately obtained by (2a) and reads

\[
\tau_1^2 + \tau_2^2 - \tau_1^2 \tau_2^2 - 2\tau_1\tau_2\Delta \epsilon = 0
\]

and is visualized in Fig. 2(a)–(b) by the white dashed lines.

**B. Transport properties**

We consider a finite piece of the triangle lattice coupled to particle reservoirs via two semi-infinite one-channel leads, which are modeled via linear chains. The calculation of the elastic scattering matrix can be done via the Fisher-Lee-relation \(^{16–18,24} \) — this relates transmission and reflection amplitudes to the retarded equilibrium Green’s functions of the system. The elements of the Green’s function of the infinite system required in order to determine the transport properties are determined through the calculation of the self-energies of the leads that are added to the lattice Hamiltonian. \(^{17} \) The problem is then reduced to a finite set of algebraic equations. The details of the calculation of the scattering matrix

![FIG. 2: (Color online) Illustration of the band gap for the infinite triangle lattice. Brightness visualizes the value of the energy gap. White dashed curves correspond to the appearance of a flat band. The crosses (green) are located at band-touching (a) or band-crossing points (b). Corresponding parameters in (a): \( \tau_2 = 8; \) (b): \( \Delta \epsilon = 0.5 \).](image)

![FIG. 3: Collection of energy bands \( \epsilon(\kappa) \) of the infinite lattice, transmission probability \( T_N(\epsilon) \) and \( \alpha(\epsilon) \) for the finite open lattice \( (N = \text{number of unit-cells}). \) (a)–(d): band touching with \( \tau_1 = \tau_2 = \Delta \epsilon = 1; \) (e),(f): Band crossing with \( \tau_1 = \tau_2 = 0, \Delta \epsilon = 0.5 \); (g),(h): Splitting with \( \tau_1 = \tau_2 = 0.1, \Delta \epsilon = 0.5. \)](image)
are presented in App. A. The analytic expression for the $S$-matrix reads
\[ S_{12} = S_{21} = \frac{-2i\sigma}{B_{11} + (\beta_1 + \beta_2)B_{12} - \beta_1\beta_2B_{22}} \] (5a)
\[ S_{jj} = \frac{2i\sigma (B_{12} - \beta_1\beta_2)}{B_{11} + (\beta_1 + \beta_2)B_{12} - \beta_1\beta_2B_{22}} - 1. \] (5b)
with $j \in \{1,2\}$ are the left and the right lead, respectively. The Matrix $B$ is defined as
\[ B = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix} N^{-1}. \] (6)
Here, $N$ is the number of unit cells, c.f. Fig. 1. The variables $\alpha$, $\beta_1$, $\beta_2$ and $\sigma$ are introduced as
\[ \sigma = \frac{\epsilon + \Delta\epsilon}{\epsilon + \Delta\epsilon + \tau_1\tau_2} \]
\[ \alpha = \frac{e^2 - \Delta e^2 - \tau_2^2}{\epsilon + \Delta\epsilon + \tau_1\tau_2} \]
\[ \beta_2 = \frac{e^2 - \Delta e^2 - \tau_2^2}{\epsilon + \Delta\epsilon + \tau_1\tau_2} + i\sigma, \] (7)
where $\epsilon$ represents the energy of the incoming particles and is defined in units of $t_3$ identically to $\epsilon(\kappa)$ in (2). The wide-band-limit (energy independent lead properties) is used here for the $S$-matrix. The matrix-elements of $B$ essentially control whether the lattice is a conductor or an insulator, because it is the quantity that mainly controls the denominator of $S_{12}$ (and therefore the transmission probability $T = |S_{12}|^2$) and contains the $N$-dependency. The parameter $\alpha$ determines whether the elements of $B$ grow exponentially with $N$ or oscillate. If the definition of $\alpha$ in (7) is solved for $\epsilon$, the spectrum of the infinite lattice (2b) is recast with $\alpha$ replacing $2\cos\kappa$:
\[ \epsilon_{1,2} = \frac{\alpha}{2} \pm \sqrt{\left[\frac{\Delta\epsilon + \alpha}{2}\right]^2 + \tau_1^2 + \tau_2^2 + \tau_2\alpha}. \] (8)
One can deduce that if and only if the value of $\epsilon$ is part of the energy spectrum, there is a real valued $\kappa$ satisfying $\alpha = 2\cos\kappa$ and hence $|\alpha| \leq 2$. An elementary analysis of matrix $B$ yields that its elements grow exponentially with $N$ if $|\alpha| > 2$. With $|\alpha| < 2$, the dependency is periodic and at $|\alpha| = 2$ it is linear. Respective explicit expressions for $B$ are given in Eqs. (A15) and (A16) in the Appendix. We conclude that transmission $T = |S_{12}|^2$ through the finite lattice is suppressed exponentially with $N$ if the particle energy is not part of the spectrum of the infinite lattice. For energies that belong to the spectrum the transmission oscillates and the number of resonances is related to $N$. The transmission is visualized together with the parameter $\alpha$ as a function of the particle energy $\epsilon$ in Fig. 3. For one example of energy bands [Fig. 3(a)] the transmission is presented for $N = 3$ [Fig. 3(b)] and $N = 9$ [Fig. 3(c)]. The parameter $\alpha$, which mainly controls the transmission, is shown in Fig. 3(d). One can see that the interval $\alpha \in [-2,2]$ corresponds exactly to the spectrum. Transmission outside of this interval is suppressed as a function of $N$ [compare Fig. 3(b) and 3(c)]. Figure 3(h) illustrates the effect on the transmission when a gap opens in the energy spectrum, namely the suppression of the transmission inside the gap.

C. Adiabatic quantum pumping properties

Having the scattering matrix of the finite triangle lattice, the scattering approach to the AQP\(^2\) can be applied. By varying slowly two of the three parameters $\tau_1$, $\tau_2$ and $\Delta\epsilon$, a net current can be produced in absence of an external bias. In the adiabatic regime the current is proportional to the frequency $\omega$ of the periodic variation. The adiabatic approach is exact in the limit $\omega \rightarrow 0$. We calculate the charge transfer $Q$, which is independent of $\omega$. It is the integral of the current over one parameter cycle. Brouwer’s formula for the charge transfer in the zero temperature limit reads\(^2\)
\[ Q_j = \frac{e}{\pi} \int_A dX_1dX_2 \Im \left\{ \text{Tr} \left[ \frac{\partial S}{\partial X_1} \frac{\partial S}{\partial X_2} \right] \right\}, \] (9)
where $j$ is the lead-index and $X_1$ and $X_2$ are the pumping parameters. The trace is applied over the channels and can be omitted in the case of one-channel-leads. The scattering matrices have to be evaluated at the Fermi-energy, which will be expressed with the dimensionless scaled parameter $\epsilon_F$ in analogy to the scaling of $\epsilon(\kappa)$ in (2). The choice of the lead $j$ determines the sign of the charge transfer. It is convenient here to choose $j = R$ which fixes the result to the transfer from the left to the right side in Fig. 1.

There are three possible choices of pumping parameters: $\{\tau_1,\tau_2\}$, $\{\tau_1,\Delta\epsilon\}$ and $\{\tau_2,\Delta\epsilon\}$. Exchanging the parameters $\tau_1$ and $\tau_2$ is equivalent to the exchange of the orientation along the lattice axes. Hence, it is equivalent to a sign change of the charge transfer. It also follows that the charge transfer vanishes if $\tau_1 = \tau_2$ holds for the whole cycle. After fixing the set of pumping parameters, we study how the choice of the parameter cycle influences the charge transfer. In the following we will focus on the choices: Path $A \Rightarrow \{\tau_1,\tau_2\}$ and Path $B \Rightarrow \{\tau_1,\Delta\epsilon\}$.

For both cases the charge transfer is plotted as a function of the extreme of the pumping parameter path and is always expressed in units of the electron charge ($e = 1$).

Figure 4 shows examples where parameters Path $A$ is varied around band touching points [c.f. Fig. 2(a) and 3(a)]. Here the lower panels show the transmission probability together with the pumping parameter cycles. In the upper panels the charge transfer is plotted versus the maximum value of $\tau_1$ in the cycle. In the following we summarize the main features of the of the transferred charge $Q$ along paths $A$ and $B$:

1. Crossing of a spectral gap — If the pumping path ends in a gap of the finite system energy spectrum,
the charge transfer $Q$ is quantized to an integer value as shown in Figs. 4(a)–4(c) and 5(b). If the pumping path end in one “zero-gap” of the spectrum, thus a Dirac-like points of the energy spectrum [crosses in the lower panels of Fig. 4], the charge transfer $Q$ is also quantized but to a half–integer values so as observed also in graphene.  

2. Dependence on the number on unit cells $N$ — The charge transfer $Q$ can either be an increasing or a decreasing function of the number of unit cells $N$. The exact dependence depends on the system parameters. Figure 5(c) shows for path $B$ traversing configurations with nonzero transmission. The charge transfer for this cycle is calculated for different numbers of $N$, which effectively represents the length of the lattice, and is visualized in Fig. 5(d). On average, there is a linear increase of the charge transfer with respect to $N$.

3. Influence of the coupling to the leads $c$ — Both in the case of very weak or very strong coupling $c$ the pumped charge is quantized to a constant value. This is because in both cases the parameter paths are ending in the energy gap of the of the finite size system.

4. Crossing of a flat band — In the case of path $B$ it is possible to cross a flat band of the infinite lattice, which a Dirac-like points of the energy spectrum [crosses in the lower panels of Fig. 4] correspond to the B-site (flat band) completely decoupled from the rest of the chain (cosine band). However, if the parameter cycle surrounds this point [see Fig. 5(a)], the structure of the whole system — with nonzero coupling $\tau_1$ and $\tau_2$, e.g. of Fig. 3(g) — is responsible for the charge transfer. Figure 5(b) shows that the charge transfer becomes $Q = -2e$ for large pumping cycles that purely traverse parameter configurations where transmission

![Figure 4](image_url_4.png)

**FIG. 4:** (Color on-line) Varying parameters $\tau_1$ and $\Delta \epsilon$ — path $B$ — around two band touching points (marked with black crosses) for different Fermi-energies $\epsilon_F$ and $N$. The lower panels show the parameter cycle (blue) on the density plot of transmission $T$ (black: $T = 1$, white: $T = 0$). In the upper panels the charge transfer is plotted versus $\tau_1^{(\max)}$. Vertical dashed lines (cyan) indicate the boundary between insulating and transmitting regions. For all panels: $\tau_2 = 8$.

![Figure 5](image_url_5.png)

**FIG. 5:** Demonstration of different charge transfer results with pumping parameters $\tau_1$ and $\tau_2$ — path $A$. Panels (a) and (b): Symmetric pumping cycle with $\tau_i^{(\max)} = -\tau_i^{(\min)}$, $\Delta \epsilon = 0.5$, $\epsilon_F = 0$ and $N = 11$ around crossing point. Panels (c) and (d): Charge transfer of pumping in transmitting region increases linearly with lattice length $N$. $\Delta \epsilon = 1.8$, $\epsilon_F = 1$. Brightness of panels (a) and (c) corresponds to transmission $T$ (Black: $T = 1$, White: $T = 0$).
is inhibited. The potential benefits of crossing a flat band\textsuperscript{23} is always cancelled by the fact that a parameter path of type $\mathcal{A}$ is always crossing this flat band two times.

III. CONCLUSION

We have presented the general spectral properties of the triangular lattice. We have shown that by changing the system parameters, the spectrum shows one-dimensional Dirac cones, flat band and many other features. The transport properties of the finite length lattice are also largely affected by the system parameters.

Thus, within the scattering matrix approach to the adiabatic pumping a large set of results can be produced. It turns out that, if the parameter configurations of the cycle remains in an insulating area (no transmission), the charge transfer is quantized — either to an integer or an half–integer value. At zero transmission the Fermi-energy is positioned between the bands (without touching them). In this case, the scattering matrix only contains nonzero reflection amplitudes that vary along the parameter cycle. Although all particles are reflected in the static case, the variation in time allows for scattering processes that lead to a net charge transfer $Q$ through the system that is independent of the length of the lattice.

If the parameter path crosses areas with nonzero transmission, the charge transfer is generally a non-integer value. If transmission is allowed, the charge transfer depends sensitively on all used parameters. If particles are transmitted through the lattice, the amount energy — gained or lost — depends on $N$, the number of unit-cells of the lattice [c.f. Fig. 5(d)]. This is related to the parameter change being applied to all parts of the lattice. A perturbation affecting only one specific unit cell of the lattice does not lead to an $N$-dependency.

Note that this linear dependency on $N$ does not mean the generation of an arbitrary large current with increasing lattice size. Because of the approximations used in this model the pumping frequency $\omega$ needs to be sufficiently small. This requires $\omega$ to decrease with increasing length of the quasi-one-dimensional lattice.

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Appendix A: Calculation of the S-matrix

The Hamiltonian in position space has the form

$$H = \begin{pmatrix} H_L & T_1 & 0 \\ T_1^\dagger & H_S & T_2^\dagger \\ 0 & T_2 & H_L \end{pmatrix},$$

(A1)

where $H_L$ is the Hamiltonian of the (semi-infinite) linear chain and $H_S$ is the one of the lattice sample. $T_1$ and $T_2$ are the matrices that connect lead sites with sample sites and here have only one non-zero element. The retarded Green’s function, defined with

$$[(E + i\eta)\mathbb{1} - H] G^R(E) = \mathbb{1},$$

(A2)

can also be partitioned into sub-matrices

$$G^R(E) = \begin{pmatrix} G_{11} & G_{1S} & G_{12} \\ G_{S1} & G_S & G_{S2} \\ G_{21} & G_{2S} & G_{22} \end{pmatrix}.$$ 

(A3)

The Fisher-Lee-relation relates the Green’s function elements to the elements of the scattering matrix\textsuperscript{16–18,24}.

The leads are modeled as linear chains with dispersion relation $E_\kappa = 2t_3 \cos \kappa + \epsilon_\ell$. It is useful to consider the wide-band-limit, where $\kappa$ is fixed and the Green’s function does not depend on $E_\kappa$ or on on-site energy $\epsilon_\ell$. If $t_3 \gg E_\kappa - \epsilon_\ell$, the choice $\kappa = \frac{\pi}{2} (\cos \pi/2 = 0)$ is valid for all particle energies. The Fisher-Lee-relation for this case can be expressed with

$$S = -\mathbb{1} + 2it_3 \begin{pmatrix} (G_S)_{N_{left},N_{left}} & (G_S)_{N_{left},N_{right}} \\ (G_S)_{N_{right},N_{left}} & (G_S)_{N_{right},N_{right}} \end{pmatrix}.$$ 

(A4)

The elements of the Green’s function are restricted to the translational invariant region. Here, the edge-sites of the triangle lattice can be considered as the beginning of the translational invariant leads. Four elements of Green’s function $G_S$ are needed to calculate the scattering matrix. To determine these quantities it is useful to extract from (A2) three sub-matrix equations,

$$[(E + i\eta)\mathbb{1} - H_L] G_{1S} - T_1 G_S = 0$$

(A5)

$$[(E + i\eta)\mathbb{1} - H_S] G_S - T_2^\dagger G_{1S} = 0$$

They can be used to derive

$$G_S = [(E + i\eta)\mathbb{1} - H_S - T_1^\dagger g_{l_\ell} T_1 - T_2^\dagger g_{l_\ell} T_2]^{-1}$$

(A6)

where $g_{l_\ell} = [(E + i\eta)\mathbb{1} - H_L]^{-1}$ is the Green’s function for a single isolated lead. Its edge element in position space reads\textsuperscript{15}

$$(g_{l_\ell})_{11} = \frac{1}{t_3} e^{-i\epsilon(E)} = -\frac{i}{t_3},$$

(A7)
where the second identity holds within the wide band limit. The effective Hamiltonian in position space reads
\[(H_{\text{eff}})_{ij} = (H_S)_{ij} - (\delta_{N\text{left}}(\delta_j N_{\text{left}} + \delta_{N\text{right}}(\delta_{N\text{right}})H_3)
\]
To demonstrate its symmetry and in order to provide an appropriate calculation of $G_S$ the effective Hamiltonian is divided into sub-matrices as
\[
H_{\text{eff}} = \begin{pmatrix}
H_L & T & 0 & \cdots & 0 & 0 \\
T^\dagger & H_C & T & & & \\
0 & T^\dagger & H_C & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & T & H_C & T \\
0 & \cdots & 0 & 0 & T^\dagger & H_R
\end{pmatrix}
\]
with
\[
H_L = \begin{pmatrix}
(\epsilon_A - i\alpha t_3 & t_1 \\
t_1 & \epsilon_B \\
\end{pmatrix}, \quad H_C = \begin{pmatrix}
(\epsilon_A & t_1 \\
t_1 & \epsilon_B \\
\end{pmatrix}, \\
T = \begin{pmatrix}
t_3 \\
t_2 \\
\end{pmatrix},
\]
Now we can express the relation
\[(E + i\eta)1 - H_{\text{eff}}] G_S = 1
\]
as
\[
[(E + i\eta)1 - H_L] G_{ij} - T G_{2j} = \delta_{ij}1 \\
[(E + i\eta)1 - H_C] G_{ij} - T^\dagger G_{i-1j} - T G_{i+1j} = \delta_{ij}1 \\
[(E + i\eta)1 - H_R] G_{Nj} - T^\dagger G_{N-1j} = \delta_{ij}1
\]
with $i \in [2, N - 1]$ and $j \in [1, N]$. Equation (A9) contains twelve equations for the elements of the sub-matrices which can be solved independently for every index $j$. Only the sub-matrices $G_{11}$, $G_{1N}$, $G_{N1}$ and $G_{NN}$ are relevant for the scattering matrix. And only the (1,1)-element (A-site) is needed from the sub-matrices. By introducing $g_i = t_3(G_{ij})_{11}$, the set of relevant algebraic equations can be simplified to
\[
\beta_1 g_1 - g_2 = \delta_{ij}1 \quad (A11a)
\]
\[
\alpha g_i - g_{i+1} - g_{i-1} = 0 \quad (A11b)
\]
\[
\beta_2 g_N - g_{N-1} = \delta_{ij}1 \sigma \quad (A11c)
\]
where $\alpha$, $\sigma$, $\beta_1$ and $\beta_2$ are defined as in Eq. (7). Further, we express Eq. (A11b) as
\[
\begin{pmatrix}
g_{i+1} \\
g_{i+2}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
-1 & \alpha
\end{pmatrix} \begin{pmatrix}
g_i \\
g_{i+1}
\end{pmatrix}, \quad (A12)
\]
which allows to connect the last two elements with the first ones as
\[
\begin{pmatrix}
g_{N-1} \\
g_N
\end{pmatrix} = B \begin{pmatrix}
g_1 \\
g_2
\end{pmatrix}, \quad \text{with} \quad B = \begin{pmatrix}
0 & 1 \\
-1 & \alpha
\end{pmatrix}^{N-1} \quad (A13)
\]
From Eqs. (A11) and (A13) it follows that
\[
g_1 = \sigma \frac{\delta_{ij}1 (B_{12} - \beta_1 B_{22}) - \delta_{ij}N}{B_{11} + (\beta_1 + \beta_2)B_{12} - \beta_1 \beta_2 B_{22}} \quad (A14a)
\]
\[
g_N = \sigma \frac{\delta_{ij}N (B_{12} - \beta_1 B_{22}) - \delta_{ij}1}{B_{11} + (\beta_1 + \beta_2)B_{12} - \beta_1 \beta_2 B_{22}} \quad (A14b)
\]
which yields the scattering matrix (5).

The matrix $B$ can be evaluated analytically to read
\[
B = \frac{1}{\sin \nu} \begin{pmatrix}
-\sin((N-2)\nu) & \sin((N-1)\nu) \\
-\sin((N-1)\nu) & \sin(N\nu)
\end{pmatrix} \quad (A15)
\]
with $\cos \nu = \alpha/2$ for the case $|\alpha| \leq 2$. On the other hand, for $|\alpha| > 2$ we obtain
\[
B = \frac{1}{\sinh \nu} \begin{pmatrix}
-\sinh((N-2)\nu) & \sin((N-1)\nu) \\
-\sin((N-1)\nu) & \sinh(N\nu)
\end{pmatrix} \quad (A16)
\]
with $\cosh \nu = |\alpha/2|$.
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