A Highschooler’s Guide to GeV-range Electromagnetism

Shyam Wuppuluri, Satchit Chatterji, Aayush Desai, Aditya Dwarkesh, Anushree Ganesh, Ameya Kunder, Pulkit Mahotra, Roshni Sahoo, Jinal Shah, Kiranbaskar Velmurugan

R.N. Podar School (CBSE)
Jain Derasar Marg, Santacruz West, Mumbai – 400054
crypticontics@gmail.com

"Some mathematician, I believe, has said that true pleasure lies not in the discovery of truth, but in the search for it.” - Leo Tolstoy

"We must be clear that when it comes to atoms, language can be used only as in poetry. The poet, too, is not nearly so concerned with describing facts as with creating images and establishing mental connections.” - Niels Bohr

Abstract

Reconstructing influential physical theories from scratch often helps in uncovering hitherto unknown logical connections and eliciting instructive empirical checkpoints within said theory.

With this in mind, in the following article, a heuristic reconstruction of the Lorentz force equation is performed, and potentially interesting questions which come up are explored. In its most common form, the equation is written out as:

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B})$$

Only the term that includes the magnetic field $q(\vec{v} \times \vec{B})$ will be dealt with for this article. The independent parameters we use are (i) the momentum of the particles in the beam, (ii) the charge (rather, the types) of particles, either positive or negative, and (iii) the current passing through a solenoid fixed in the path of the beam. We then measure the angle by which particles get deflected while varying these three parameters and derive an empirical relationship between them.
1 Experience

The primary aspect of everything that is connected to CERN is scientific but the experience that all of us cherish contains elements of our journey which are not only scientific but also how we perceive ourselves and our lives there. This narrative forms a significant part of the beloved time we spent working on this project, a project that has given us a great deal of insight into the inner workings of the life of a scientist.

When we got to know that we had won, a feeling of disbelief washed over us, which was soon replaced by excitement that we directed towards our experiment. Our headmistress, principal and subject teachers felt a well-founded pride at our having been selected, for it was their liberal nature which allowed us to spend so much time, effort and sometimes, school classrooms and other resources on our proposal! Without their willingness to participate in this manner of their own, we would never have made it.

1.1 Before the Beam

We were all on vacation and our eyes were peeled in search of scientific endeavours when our mentor, Shyam sir, first brought to our notice, the existence of CERN’s Beamline for Schools competition.

Quickly, we formed a team to begin brainstorming, despite not really expecting to win. It was more about the journey than the destination. We began at the only place we knew to: Our school textbooks. We debated on how good an idea it would be to submit Young’s classic Double Slit Experiment or a modification of it and subsequently trawled through the internet of as-yet unperformed thought-experiments proposed by scientists, all of which we felt a deep desire to see performed; we, however, found that we had nowhere near the expertise required to execute experiments such as Popper’s experiment or Roger Penrose’s FELIX, both of which had, at a certain point of time, been considered. For a long time, ideas were flung back and forth before we finally settled on the proposal that we sent. We had just been studying classical electromagnetism at that time in school, and so our thoughts were heavily inclined in that direction. Charged particles and the Earth’s rather peculiar magnetic field occupied much of the discussions, and it finally crystallized into our proposal.

1.2 Beam me up! - Getting Ready

Safety training itself took up one full day, a day in which the importance of following protocol for our personal safety was impressed upon us. We were given a class on cryogenic safety, a simulation of an emergency situation in the LHC and a lesson on how to operate fire extinguishers and subsequently had to give online tests on computer security, radiation safety, and so on. This is something we still fondly tease each other about. We had to take multiple tests in order to pass the standards required for being eligible to get our own dosimeters, which measures the radiation dosage received by a person and is
very integral in ensuring safety in experimentation. While a few of us passed with flying colours, the others took it a tad too lightly and failed it before being helped by our mentors. The day concluded on a high, with us being handed over our safety shoes and helmets by the jolly Nelson Almeida, giving us reason to enjoy not just the beam days at CERN but also the many secondary activities that we had to perform.

We toured around Geneva, visited the Globe; were taken to the synchrocyclotron, the first particle accelerator put together, and shown an awe-inspiring 3D visualization of how it was constructed and how it functioned; were shown the antimatter factory and allowed to marvel at the mind-blowing complexity of all the detectors and accelerators that there were in it. Indeed, our first taste of real experimentation did not come from the T9. It came from the S’Cool cloud chamber workshop held by Sonia Natale, She began by teaching us about lab etiquette and then moved on to teaching us how to build a cloud chamber—you guessed it—a cloud chamber by ourselves. A cloud chamber is basically a particle detector used for visualizing the passage of ionizing radiation and typically consists of a sealed environment containing a supersaturated vapour of water or alcohol. Once we had made it, we were able to view the trajectories of the cosmic rays which keep intermittently bombarding our planet! We were in awe of having created something that would let us see the particles without any fancy equipment.

1.3 Early Beam Days

The beginning of our experiment was marked by us placing bets amongst each other regarding how long it would take: An hour or two? What could possibly prolong it to nearly two weeks?, we wondered. We all knew, in theory, of the difference between theory and practice. This difference is governed by the benignly recursive Hofstadter’s Law: it always takes a little longer than you expect, even when you take into account Hofstadter’s Law. All we knew about the Lorentz equation[8], we know from our textbooks.

Even after we entered the experimental area, things were far from smooth sailing. Not long after that, we set about recalibrating our Delay Wire Chambers. This is done in order to account for a kind of zero-error within them. Our support scientists were benevolent enough to already have done this for the Micromegas — had they not, we’d probably have spent our entire trip doing nothing but calibrations! Experimentation requires a lot more improvisation than one expects while theorizing -the Beamcats’ experiment required a scintillator to be waterproofed and dipped into a water tank. Not long after the first time that was tried, two people were back into the control room trying to dry the scintillator before it became damaged. That, perhaps, is a story for another time.

The data analysis team and our support scientists worked impressively hard trying to figure out how to calibrate the detectors and learning a lot through example Python scripts provided to us, they tested out the runs, discovered anomalies in the graphs and worked day and night to ensure that we were
acquiring the correct data. We would sometimes crash the computers and would have to reset the machines to start over again. However, our mentors helped us work efficiently by changing the commands such that a politely-worded alert would pop-up every time we typed something wrong! Eventually though, we got everything set up, and were ready to begin our formal analysis!

2 Experimental Setup

![Diagrammatic representation of Cryptic Ontic's Beamline For Schools (BL4S) setup at the T9 experimental area (not to scale). TOF0 (the first scintillator) is not shown above as it is placed quite further upstream, outside the experimental area.](image)

Our experimental setup was quite straightforward. It consisted of tracking detectors to find out where the particles hit, and thus reconstruct its trajectory, a pair of scintillators, or timing detectors\[1\]\[2\] that allowed for the speed of the particle to be measured, and an electromagnet producing a uniform magnetic field. Note that the axes are defined as: $x$ upwards in the plane, $y$ directed outside the plane, and $z$ is along the line of the beam, i.e. left-to-right in the diagram above.

2.1 T9 Beamline Characteristics

The BL4S experiments took place in a beam line (called the T9)\[1\] at CERN’s Proton Synchrotron (PS). Fast moving protons ($\approx 24\text{GeV/c}$) from the PS hit a target called the North Target or Production Target, generating particles, a stream of which is called the secondary beam. At the experimental area, the
beam will contain either positively or negatively charged particles with a well-defined momentum, from 0 to 10 GeV/c. Examples of particles used include protons, electrons, kaons and pions, and their anti-particle counterparts. Neutral particles such as photons and neutrons are not present.

2.2 Trackers

The delay-wire chambers\[1\][3] (DWCs) are tracking detectors which can give the \( x \) and \( y \) positions of where a particle hit it (hit positions). The Micromegas\[1\][4] (MMs) were slightly different, and could only supply the hit-position in one axis. Thus, simply, one could use two Micromegas placed orthogonal to each other reconstruct hit positions in two axes. In our case they were aligned to the \( x \) and \( y \) axes. In both sets of detectors, the \( z \)-axis coordinates were set as the distance of the detector face from DWC0, which were measured beforehand. Collectively, the DWCs and Micromegas are called trackers.

2.3 MDX27 Magnet

The MDX27 is an electromagnet of which we could control the amount of input electricity in terms of current, having a range of \( \approx 0A \) to \( \approx 240A \), roughly equivalent to \( \approx 0T \) to \( \approx 1T \). We substituted the strength of the field in terms of bending power (\( \beta = B \cdot l \)) during analysis. The magnet was used to create a uniform magnetic field to deflect the incoming beam of charged particles. More on the way we used the magnet is mentioned in later sections.

2.4 TOF Scintillators

In order to measure the speed of incoming particles, we used a pair of instruments called scintillators\[1\][2], which have a low response time \( \approx O(10)ns \), and thus were perfect for accurately measuring and possibly identifying particles (please see extended appendix 9.3). The Time-Of-Flight (TOF) was the difference between the times the scintillators registered a hit. Only one scintillator was placed within our experimental area, as the other was placed roughly 17 meters upstream DWC0, in order to be able to measure a time difference between the two scintillators and therefore measure its speed.

2.5 Runs

What we will be calling a run is essentially a time period wherein a set of particles hit our detectors. The particles hit more or less one at a time, but at a high frequency, so each particle’s unique track can be determined. Each run has set parameters (called run conditions), in particular, charge of the particles, momentum for the beam and amperage of the magnet. While determining the results, we take into account multiple runs to sample particles of varied parameters. Thus we may reconstruct particle tracks over several runs, and find
relations between the angle that the particle gets deflected and the aforementioned variables. This is our eventual goal - to find an overall set of criteria for particle deflection.

3 Analysis

In such a field as high-energy physics, the data must compliment the theory. Thus, a vital piece of equipment for us was code. At BL4S, the data acquisition system was borrowed from the ATLAS Experiment across the street, and the data itself was stored and parsed by CERN’s own Root framework[5]. We decided to use Root in C++, as a few of us were already well-versed with the language, but it should be noted that Root has a Python module as well[6], and was offered to us to use instead of C++.

3.1 Overview

C++ is an object-oriented language, with which classes can be used in some sense to model “real-world” objects which have attributes and functions[7]. We used this to our advantage, and created several classes, including a Beam class, classes for Magnets and Detectors, and one for Particles, which could model any of the particles we wished to work with. Properties were inherited appropriately between classes. A Run class was also created, where run conditions were specified for Run instances, such as the momentum and amperage. The purpose of this was to create a custom database system, that could find runs with specific properties, either in terms of the run conditions, or additional parameters such as maximum particles, and then load the necessary file(s). Root classes facilitated the appropriate graphing classes, which were quite easy to implement.

With the code in place, the actual analysis had a number of steps, each as important as the next. The process of the analysis has been described in detail below:

3.1.1 Calibration

As a very important first step, all detectors were calibrated. This meant carrying out procedures that gave us information about each detector’s unique “calibration constants”, used in the reconstruction process. The positional zero errors of the trackers with respect to the center of the beam path were also important to note, and were taken into account to accurately reconstruct particle paths. For the TOF scintillators, the zero-error was the recorded time difference between the hits when both instruments were positioned right next to each other. The distance between the scintillators were also measured. The magnetic field produced by the magnet when it is off was noted, due to factors such as magnetic hysteresis[10], and was taken into account during analysis.
3.1.2 Reconstruction

The data, which was stored in a `.root` binary file format, was first parsed through some code that calculated the hit-positions on the $x$ and $y$ axes from each of the DWCs and Micromegas respectively. The $z$ axis coordinates are predefined as the distance of the detector face along the beam path. This can be used to construct velocity vectors. Knowing the distance of the TOF scintillators and the time it takes a particle to traverse that distance, one can calculate the speed of the particle via

$$\text{speed} = \frac{\Delta \text{distance}}{\Delta \text{time}}.$$ 

Thus the precise 3D reconstruction of the particle’s trajectory can be traced and speed measured.

3.1.3 Graphing

Since we attained the hit-positions in the earlier step, we can now calculate the angle the particle is deflected by the magnetic field. This is simply done by subtracting the initial angle from the final angle that the particle makes, $\theta$. The mean and standard deviation of the angle for each set of runs was calculated and graphed appropriately. These were drawn with Root’s TH1D (histograms), and TGraphErrors (graphs with error bars) classes, similar to some of python’s matplotlib functions.

3.1.4 Interpretation

The point on the graphs were fit with appropriate curves, which then were interpreted as relations. The process used to determine these curves is called regression. More details on regression and how we justified fits are given in the extended appendix.

---

**Figure 2:** Selection of user-defined classes used and the interactions of their instances within the code.
3.1.5 Verification

Theoretical formulae for the angle of deflection $\theta$ based on the Lorentz force were made and curves were drawn for comparison to our experimentally obtained fits. In particular, the formula used was

$$\theta = \sin^{-1}\left(\frac{q\beta}{p}\right)$$

where $q$ is charge, $\beta = B \cdot l$ is bending power and $p$ is momentum.

For all the applicable graphs, the red fit curve is our experimental analysis, and the green curve is the expected theoretical relation.

3.2 Finding correlations

Our idea relied on finding the relation between how far the particle gets deflected laterally versus the parameters we could change, i.e. charge ($q$), momentum ($p$), and magnetic field ($B$) (viz. bending power $\beta$), and then find out how exactly they interact.

3.2.1 Charge

The T9 Beamline could provide to us either a positive or negative beam. This meant that the incoming beam could provide a beam of positive particles, such as protons and positrons, or negative particles, like negative pions and electrons. Both types comprised of singly charged particles. We see graphically,
that positive particles and negative particles get pushed in opposite directions. Thus, we can conclude:

$$\theta \propto q$$

In simple terms, oppositely charged beams deflected in opposite directions when all others properties remained the same.

### 3.2.2 Momentum

The T9 Beamline supplied to us various beam momenta, and it was suggested to us to use a variety, ranging from 3.5 GeV/c to 10 GeV/c for both positive and negative beams. Thus, by varying the momentum over several runs, a sample of results are given in figure 5. Thus we may conclude that:

$$\theta \propto \frac{1}{p}$$
Figure 5: For a particular set amperage, $\theta$ vs momentum follows a $1/x$ curve. Note that a negative momentum just implies the particles carry negative charge.

Classically speaking, one might wish to calculate velocity using $\vec{p} = m\vec{v}$, and gain a relationship between $\theta$ and $\vec{v}$. However, we are using particles moving close to the speed of light, thus relativity ensures the relationship between $\vec{p}$ and $\vec{v}$ is a little more complex. The direction of both however, remain the same. (More on this can be found in the extended appendix.)

3.2.3 Bending Power or Amperes

The magnetic field was supplied by the electromagnet MDX27, which was controlled through the amount of electricity passed through it, from $\approx 0$ to $\approx 240A$, which corresponds to roughly $\approx 0$ to $\approx 1T$. Due to factors such as the small imperfections produced in the magnetic field and the bulging of the field around the edges of the magnet itself, the actual properties of the field could not be accurately calculated. Thus we were handed bending power values ($\beta$) which were measured by engineers prior to our experiment. This could be substituted in for $B \cdot l$, i.e. field strength times length of the field. This was a useful substitution for us, and variance in $\beta$ does correspond to a proportional variance in field strength. Thus we can conclude that

$$\theta \propto \beta \quad (\therefore \theta \propto B)$$

Put simply, the stronger the bending power or magnetic field, the greater is the deflection, and hence the force.
Figure 6: Given a set momentum, $\theta$ shows a direct variance w.r.t. $\beta$ is given in units of Tesla-meter

### 3.2.4 Cross product

In order to verify our assumption of the cross product nature of force $\vec{F}$ in terms of $\vec{v}$ and $\vec{B}$, we can check the accelerations in each axis to see if they correspond to the theoretical accelerations. The way we did this, in terms of analysis, is to consider the paths before and after the magnetic field as vectors. We can then measure the change in velocities, the acceleration, in each axis.

The magnitude of velocity can be considered to remain the same before and after the magnet, as the force is centripetal in nature, and calculated using the TOF scintillators. We can thus figure out the approximate time that it is accelerating in the magnetic field:

$$t = \frac{l}{|\vec{v}| \cos \left( \frac{\theta}{2} \right)}$$

(where $l \approx$ width of magnetic field) and find out the average acceleration in each axis by simply implementing $a = v/t$
(a) Comparison of x-axis component acceleration. The means of the experimentally and theoretically calculated accelerations are quite similar.

(b) Comparison of z-axis component acceleration. Relative to the x-axis acceleration, this is quite insignificant, but is still apparently inaccurate. This can be explained on the basis of the accumulation of errors from the detectors, with the calculated acceleration’s precision $O(10^{13})$ m/s$^2$. (See Extended appendix)

Figure 7: This particular run is of 8.75 GeV/c, and with an amperage of 240 A ($\beta \approx 0.5095Tm$). Note that the theoretical calculations do take into account the initial velocity vector that each particle takes, and as such carries over some amount of experimental error and uncertainty. The y-axis is not shown as acceleration is expected only in the x− and y−axes.

Here we notice that experimental histograms have a high standard deviation value as opposed to the theoretical ones. This can be because of a few factors, most prominently scattering of particles in the air and the precision of our instruments. Thus it may or may not be reasonably expected that with more data, one would get a more confident result. For more on how the theoretical and experimental acceleration values were reached, please see the extended appendix.
3.2.5 Results

From the analysis we get the following relations:

- $\theta \propto q$
- $\theta \propto \beta$
- $\theta \propto \frac{1}{p}$

- The force acts perpendicular to both $\vec{v}$ and $\vec{B}$.

4 Concluding Derivation

In the previous sections, we have already established how $\theta$ varies with respect to $q$, $\beta$ and $p$. We can combine these as follows:

$$\theta = \frac{Kq\beta}{p}$$

Where $K$ is an arbitrary constant of proportionality, which can be determined experimentally by substituting the values of $q, \beta, \theta$ and $p$.

Note: These relationships may be generalised in the form of differential equations. (See extended appendix).

We know that

$$\lim_{x \to 0} \sin(x) = x$$

The experimental angles are in range of $0^\circ$ to $5^\circ$, thus the simple approximation of $\theta \approx \sin(\theta)$, can be used. Also

$$\sin(\theta) = \frac{l}{R}$$

Equating the two,

$$\frac{l}{R} = \frac{Kq\beta}{p}$$

Substituting $\beta = B \cdot l$ and dividing by $l$,

$$\frac{1}{R} = \frac{KqB}{p}$$

Substituting $p = mv$,

$$\frac{mv}{R} = KqB$$

Multiplying $v$ on both sides,

$$\frac{mv^2}{R} = KqvB$$
Figure 8: Deflection of particle inside the magnetic field.

\[ F = qBv \]

In terms of direction, we see that \( \hat{p} = \hat{v} \). Thus the product between \( \vec{v} \) and \( \vec{B} \) is a cross product, as the resultant is a vector that is orthogonal to both - a characteristic of cross products. Thus, in conclusion:

\[
\vec{F} = q (\vec{v} \times \vec{B})
\]

5 Discussion

Our initial idea was inspired from a flavour of Gauge system’s theory first formulated by Hermann Weyl. The idea of Weyl concerns with the very aspect of measurement which is quite central to physics. Measurement is the comparison of two different attributes, say two different lengths and Weyl suggests that when two such lengths have to be compared, the result may depend on the route pursued in passing from one place to the other. This may perhaps be due to the effect of gravitational and various other fields that are present along.

The general features of these field theories is that though these fundamental fields themselves can’t be measured, we can measure the associated attributes and infer something about the fields in general.

We had decided to use an electromagnetic field to study and observe the anomalous behaviour of muons. Now, we know that the cosmic rays hitting the earth’s atmosphere produce secondary particles that includes pions which
further decay into muons. These muons are observed to show an anomalous behaviour upon hitting the earth’s magnetic field. From our experiment we had intended to try and extrapolate anomalies in the earth’s magnetic field. However, we found that muons are scattered to a very large extent after passing through the magnet which made it difficult for us to study and quantify their behaviour.

We were thus led onto our current line of thought: A heuristic reconstruction of the mathematical quantification describing the behaviour of charged particles under the influence of a uniform magnetic field.

A striking fact comes to light when we carefully examine our reconstructive analysis: All the empirical consequences of the Lorentz force can be explicated in terms of charge, velocity and current. Within the context of our setup, future predictions can be purely made on the relations derived. We are now in a position to put forward the following question:

- Why does one need to posit an intermediate quantity known as the ‘Magnetic field’ and substitute it into the Lorentz force equation?

This leads us to the immediate conclusion that the physical quantity known as the magnetic field must necessarily have been discovered in a manner that had nothing to do with moving charges and electricity; and indeed, that is the case, for the phenomena of magnetism was first discovered via certain special rocks, in the complete absence of artificially applied flow of electrons. It hints at the fact that it is - empirically speaking, at least-wholly dispensable with for any flavour electromagnetic analysis which is based on the Lorentz force equation, and most definitely with predictions within our experimental setup.

The late physicist Murray Gell-Mann while giving a talk[16] on the beauty of physics remarked:

Three principles — the conformability of nature to herself, the applicability of the criterion of simplicity, and the ‘unreasonable effectiveness’ of certain parts of mathematics in describing physical reality — are thus consequences of the underlying law of the elementary particles and their interactions. Those three principles need not be assumed as separate metaphysical postulates. Instead, they are emergent properties of the fundamental laws of physics.

Over the countless hours of concentrated effort, our objective was to provide a visible connection between these three principles and how using basic laws of physics combined with some simple and yet elegant mathematics ties into our observation of nature so perfectly, further revealing how the fundamental relationship of electricity and magnetism, like the end of a maze, remains one - but the paths leading to it can be electrically different.
6 In Retrospect

Beam days were the best parts about our stay there. From logging, to checking the gas pressures of the detectors, to controlling current supplied to the magnets, every task was done with great enthusiasm. During shifts, we lost track of time, and before we knew it the next crew was already there. Some of us would stay back at the control room after shifts ended along with our support scientists and make a few extra runs at night in the T9. We also collaborated with the Philippines team and had learned a lot of concepts we would have otherwise not have known we enjoyed learning about.

On the penultimate day of our stay, we were given, on our request, access to the grand piano in the main hall, where Satchit, Ameya, and Anushree gave musical performances, and were joined not long after by our support scientist Gianfranco, who, as we came to know later, has a very distinguished taste in music. We then went totally off-script, and ended up having a fun jamming session which ran late into the evening, and it remains one of the highlights of our trip.

During the 14 days we spent there, there was not a single moment where any of us hadn’t a smile, except for when we knew it was the end - an end to a significant chapter in all of our lives.

6.1 Going Forward

Our time at CERN provided us new insights and skills that we could never have attained otherwise, whether it was learning how to maneuver around all the physical and mental blocks while solving real-world problems, to being able to help one another in inter-disciplinary ways, from coding to culture, from physics to public speaking, from engineering to education. When we returned, we made sure to spread awareness about STEM research in our school and others, and took hands-on workshops for middle and high school students, eager to soak up as much learning as they could.

Our team members have gone on to study in universities such as UC Berkeley, UC San Diego, University of Groningen and the Indian Institute for Science Education and Research (IISER), primarily in STEM-related fields.

The Beamline for Schools Competition continues to inspire us to take up more projects and competitions ourselves and also carry the torch forward and initiate conversation about science and collaboration in our local communities.

7 Acknowledgements

Last but not the least, the Cryptic Ontics would like to thank all those wonderful people who have made it possible for us to have this experience, from our guides Sarah and Markus, to our support scientists Gianfranco and Cristóvão, and all the other staff at CERN, including the non-science and support staff,
who added their unique human touch to the experience and helping us per aspera ad astra.

We would of course, love to thank our own school’s principal, Ms Avnita Bir, and our mentors who accompanied us, Rajeev Maurya and Shyam Wuppuluri, who have been guiding us and giving us their knowledge both before and after our trip to CERN, as well as the rest of the teaching and non-teaching staff at RN Podar School.

We also give our heartfelt gratitude to our families and friends who helped us mentally, physically, spiritually, and even technically during the course of the project. We would also like to acknowledge James Hirst, who began guiding us even before we arrived at CERN, and finally, the Beamcats, the Philippines team, for being such pleasant peers and now warm friends.

8 Extended Appendix

8.1 Regression

In theory, any curve can be fit onto any set of data points\footnote{11}. The error of the fit is calculated according to root-mean-square deviation (RMS), or the average distance between the predicted point and the actual value. If you have \( n \) unique points as a function of \( x \), you can perfectly fit an \( n - 1 \) degree polynomial onto it, i.e. with zero error. Think how any two points form a line perfectly. If the real-world relation is actually a linear fit, but you try to fit a higher degree polynomial, you will technically get a better (lower error) fit, but that may not reflect reality, or offer better predictions for new data. This idea is called overfitting\footnote{12}.

Continuing, if you fit a curve to a set of points, say the points we get from bending power \( \beta \) vs angle of deviation \( \theta \), with a higher degree polynomial (e.g. a quintic) rather than a linear curve, you’ll get a valid \( ax^5 + bx^4 + cx^3 + dx^2 + ex + f \) fit, depending on how many unique \( x \) values you have. However, the values \( a, b, c, \) and \( d \) will be extremely small, tending to zero for a perfect linear fit, whereas \( e \) and \( f \) will be much more significant. Thus, practically, you will get a straight line, or a proportional relationship. For a non-polynomial relationship, this can be approximated using the well known method of Taylor series approximation as described below.

8.2 Polynomial regression in terms of sin

In reality, \( \beta \) vs \( \theta \) follow a sinusoidal relation, but if we consider the same principle of regression while trying to fit data with the Taylor series approximation\footnote{14} of sine:

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]
Only the coefficient of the first term will have a significant value, thus approximating to a linear relationship for small values such as attained with our setup. Indeed the error in the fit for a sin or a linear function are very similar. Essentially sin (or any function) can be approximated using an arbitrary number of terms in an infinite series. In this case, if we just take the first term, we get a relative error of less than 1% for angles under \( \approx 14^\circ \) or 0.244 radians. This idea is the basis of what is commonly called the sin angle approximation for small angles.

8.3 Relativistic Momentum and Velocity

With the knowledge of the rest masses of the particles \( m \) that constituted the beam, we could then calculate the velocity of each using the relation for relativistic momentum

\[
p = \gamma mv
\]

Where \( \gamma \) is the Lorentz factor:

\[
\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

Using elementary algebra, one would reach

\[
v = \frac{pc}{\sqrt{p^2 + m^2c^2}}
\]

(a) This run is a 0.6GeV/c. It is clear that there are two peaks that line up well with the expected.

(b) At higher momenta, such as here with 10GeV/c, it gets harder to discern between particles based on TOF, and the resolution of our scintillator setup is insufficient for such a purpose.

Figure 9: Examples of TOF reconstructions for two runs. The yellow line indicates calculated positron TOF, and the red line indicated proton TOF, as calculated using special relativity.
Thus $v$ vs $p$ is non-linear at relativistic speeds. More about relativity can be seen in *Testing the Validity of the Lorentz Factor*\[13\] an article by a previous winning team of the BL4S competition.

### 8.4 Acceleration along the $z$ axis

One might also notice a significant difference between the observed acceleration in $z$ against the theoretical acceleration. But it is imperative to note how the theoretical acceleration is determined. The initial $\hat{v}_i$ (i.e. velocity along the $x$ axis) is first obtained, using detectors, whose resolution is close to about $450\mu m$, these positions obtained from the detectors are then converted to unit vectors and are thus dimensionless, and only indicate the inclination of the 3 vectors to the 3 axes i.e. $x, y, z$. Thus a variance of about $450\mu m$ can be expected there. After the unit vector is obtained it is multiplied with $V, B, q$ and divided by the $m$ which is the mass of the particle, after which we get values $O(10^{13})m/s^2$. The mean theoretical acceleration in $z$ for the particular run was $2.76 \times 10^{13}m/s^2$.

The formula for the theoretical acceleration along the $z$ axis was

$$ \vec{a}_k = \frac{qVB\hat{v}_i}{m} $$

$$ \therefore \vec{\hat{v}}_i = \frac{m\vec{a}_k}{qVB} $$

We now observe that $\hat{v}_i \approx 10^{-4}$. The resolution of the detectors is around $450\mu m= 4.5 \times 10^{-4}m$ When this variance is factored into the theoretical calculation of the acceleration, the resultant acceleration is similar to the experimental acceleration.

### 8.5 Differential Equations

We got a linear fit for the relations between $\theta$ and $\beta$ and $q$, and an inverse fit with respect to $p$.

For a linear relation between $y$ and $x$ one may express it as $y = mx + c$ where $m, c$ are constants, similarly one may express the linear relations of $\theta$ wrt $\beta$ and $q$ with constants such as:

$$ \theta = K(K_1\beta + c_1)(K_2q + c_2)\left(\frac{K_3}{p} + c_3\right) $$

$$ = KK_1K_2K_3\left(\beta + \frac{c_1}{K_1}\right)\left(q + \frac{c_2}{K_2}\right)\left(\frac{1}{p} + \frac{c_3}{K_3}\right) $$

$$ = KK_1K_2K_3\left(\beta + C_1\right)\left(q + C_2\right)\left(\frac{1}{p} + C_3\right) $$
Let \( k' = KK_1K_2K_3 \), where \( K_1 \) \( K_2 \) \( K_3 \) are the linear fit constants

\[
\theta = k' (\beta + C_1)(q + C_2) \left( \frac{1}{p} + C_3 \right)
\]

\[
\frac{\partial \theta}{\partial \beta} = k' (q + C_2) \left( \frac{1}{p} + C_3 \right) \quad \{i\}
\]

\[
\frac{\partial^2 \theta}{\partial \beta \partial q} = k' \left( \frac{1}{p} + C_3 \right) \quad \{ii\}
\]

\[
\theta = k' (\beta + C_1)(q + C_2) \left( \frac{1}{p} + C_3 \right)
\]

\[
\frac{\partial \theta}{\partial p} = -k' (q + c_2)(\beta + C_1) = -\left( \frac{\theta}{\left( \frac{1}{p} + C_3 \right)p^2} \right) \quad \{iii\}
\]

After substituting from \( \{ii\} \) in \( \{i\} \)

\[
\frac{1}{k'} \frac{\partial \theta}{\partial p} \frac{\partial^2 \theta}{\partial \beta \partial q} = \left( \frac{-\theta}{p^2} \right) \quad \{iv\}
\]

Differentiating \( \{i\} \) w.r.t. \( p \):

\[
\frac{\partial^3 \theta}{\partial p \partial q \partial \beta} = -\frac{k'}{p^2} \quad \{iv\}
\]

Substituting \( \{iv\} \) in \( \{iii\} \)

\[
\frac{1}{k'} \frac{\partial \theta}{\partial p} \frac{\partial^2 \theta}{\partial \beta \partial q} - \theta \frac{\partial^3 \theta}{\partial p \partial q \partial \beta} = 0
\]

This differential equations is in the context of our experimental setup, with relations only applying in the framework of our experiment.

**8.6 Examining the product between \( \vec{v} \) and \( \vec{B} \)**

Previously we have established a relation between the force acting on a particle and the variables tested in the scope of our experiment; thus we went one step further, to figure the nature of the product of \( \vec{v} \) and \( \vec{B} \).

**8.6.1 Calculating Experimental Acceleration**

Let us define the initial velocity vector of the particles to be \( \vec{v}_i \)

\[
\vec{v}_i = V.\hat{v}_i
\]

\[
\hat{v}_i = \frac{\{a\hat{i} + b\hat{j} + c\hat{k}\}}{\sqrt{a^2 + b^2 + c^2}}
\]
Where \( V = |\vec{v}_i| \) and \( \{a\hat{i} + b\hat{j} + c\hat{k}\} \) are the unit vectors along 3 mutually orthogonal axes, i.e \( x, y, z \) Let us define the final velocity vector of the particles to be \( \vec{v}_f \)

\[
\vec{v}_f = V\hat{v}_f
\]

\[
\vec{v}_f = \frac{\{e\hat{i} + f\hat{j} + g\hat{k}\}}{\sqrt{e^2 + f^2 + g^2}}
\]

The magnitude of the velocity \( V \) after acceleration remains the same as before, as when centripetal force is applied, it would not change the overall magnitude, just the direction of velocity. Thus,

\[
|\vec{v}_i| = |\vec{v}_f| = V
\]

We know that

\[
\vec{a} = \frac{\Delta \vec{v}}{\Delta t}
\]

Using this

\[
\vec{a}_{exp} = \frac{\vec{v}_f - \vec{v}_i}{t}
\]

Where,

\[
t = \frac{l}{V \cos(\frac{\theta}{2})}
\]

Let \( \frac{\theta}{2} = \phi \)

\[
\therefore \vec{a}_{exp} = V \cos(\phi) \frac{\vec{v}_f - \vec{v}_i}{l}
\]

\[
\vec{a}_{exp} = V \cos(\phi) \left( \frac{V\hat{v}_f - V\hat{v}_i}{l} \right)
\]

### 8.6.2 Calculating Theoretical Acceleration

We already know that the force acting on the particle is perpendicular to the velocity of the particle, due to the centripetal expression derived earlier. As for the angle with the magnetic field vector, we assume it has to be perpendicular w.r.t. the force, as if not, the \( B \) vector will have a component along the force, but since force and \( B \) do not have the same dimensions they cannot be added.

We have already defined the initial velocity vector \( \vec{v}_i \)

Where \( \vec{v}_i \)

\[
\vec{v}_i = V\hat{v}_i
\]

\[
\hat{v}_i = \frac{\{a\hat{i} + b\hat{j} + c\hat{k}\}}{\sqrt{a^2 + b^2 + c^2}}
\]

Where \( V = |\vec{v}_i| \) and \( a\hat{i} , b\hat{j} \) and \( c\hat{k} \) are the component vectors along three mutually orthogonal axes, i.e, \( x, y \) and \( z \).
Let us assume the \( \vec{B} \) to have the
\[ \vec{B} = \hat{B} \]
\[ \hat{B} = \left\{ e\hat{i} + f\hat{j} + g\hat{k} \right\} \]
\[ \frac{\sqrt{e^2 + f^2 + g^2}}{\sqrt{e^2 + f^2 + g^2}} \]

Where \( B \) = magnitude of magnetic field and \( \{e\hat{i} + f\hat{j} + g\hat{k}\} \) are the unit vectors along 3 mutually orthogonal axes, i.e \( x, y, \) and \( z \).

Let us assume the force \( \vec{F} \) to be
\[ \vec{F} = F\hat{F} \]
\[ \hat{F} = \left\{ m\hat{i} + n\hat{j} + o\hat{k} \right\} \]
\[ \frac{\sqrt{m^2 + n^2 + o^2}}{\sqrt{m^2 + n^2 + o^2}} \{i\} \]

Where \( F \) = magnitude of force and \( \{m\hat{i} + n\hat{j} + o\hat{k}\} \) are the unit vectors along 3 mutually orthogonal axes, i.e \( x, y, \) and \( z \).

Since \( \vec{F} \parallel \vec{v} \) and \( \vec{F} \parallel \vec{B} \), \( \vec{F} \) would not have any component on \( \vec{v} \) and \( \vec{B} \), thus we can equate the respective components to 0.
\[ \vec{F} \cdot \hat{v} = 0 \]
\[ \therefore am + bn + co = 0 \{i\} \]

\[ \vec{F} \cdot \hat{B} = 0 \]
\[ \therefore em + fn + go = 0 \{ii\} \]

\[ \therefore \text{Let} \quad \frac{m}{bg - fc} = \frac{-n}{ag - ec} = \frac{o}{af - be} = K' \]

(See Appendix section 9.7)

\[ \therefore m = K'(bg - fc); \quad n = -K'(ag - ec); \quad o = K'(af - be) \]

Thus after substituting in \{i\}
\[ \hat{F} = \frac{\{K'(bg - fc)i - K'(ag - ec)j + K'(af - be)k\}}{\sqrt{(K'(bg - fc))^2 + (-K'(ag - ec))^2 + (K'(ag - ec))^2}} \]

Taking out the constant \( K' \)
\[ \hat{F} = \frac{\{(bg - fc)i - (ag - ec)j + (af - be)k\}}{\sqrt{(bg - fc)^2 + (-(ag - ec))^2 + (ag - ec)^2}} \]
We already know
\[ \vec{F} = F \hat{F} ; \]
Where \( F = KqVB \)
\[ \therefore \vec{F} = \{KqVB\} \frac{((bg - fc)i - (ag - ec)j + (af - be)k)}{\sqrt{(bg - fc)^2 + (-ag - ec)^2 + (af - be)^2}} \]
Force acting upon a particle can also be defined as:
\[ \vec{F} = m \vec{a}_{th} \]
Where \( m_r = \) relativistic mass of the particle \( \gamma m_{rest} \)
Now that we have established a formula for force we can equate it to \( \vec{a}_{th} \):
\[ \therefore \vec{a}_{th} = \frac{\{KqVB\}}{m_r} \frac{((bg - fc)i - (ag - ec)j + (af - be)k)}{\sqrt{(bg - fc)^2 + (-ag - ec)^2 + (af - be)^2}} \]
We thus found that numerically that along each axis:
\[ \vec{a}_{th} = \vec{a}_{exp} \]

8.7 Linear Algebra

Since there are 3 variables and 2 equations, Cramer’s method of determining solution for the variables cannot be applied, thus a different approach is required.
Consider
\[ \begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix} \begin{bmatrix} m \\ n \\ o \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Let \( \begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix} \) = matrix A

Let \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) = matrix B

Since number of equations is less than number of variables, there are infinite solutions, but we are only interested in a single one.
Let \( K' \) be a constant.
On solving the matrix for the solutions we get
\[ \frac{m}{bg - fc} = \frac{-n}{ag - ec} = \frac{o}{af - be} = K' \]
\[ \Rightarrow m = K'(bg - fc) \]
\[ \Rightarrow n = -K'(ag - ec) \]
\[ \Rightarrow o = K'(af - be) \]
References

[1] Beamline for Schools — Beam and Detectors, CERN
   https://beamline-for-schools.web.cern.ch/sites
   /beamline-for-schools.web.cern.ch/files
   /BL4SBeam-and-detectors_2018_0.pdf

[2] Counting tubes, theory and applications, S. C. Curran, Academic Press (New
   York), 1949

[3] The Delay Wire Chamber Description A. Manarin, G Vismara

[4] Y. Giomataris, P. Rebourgeard, J.P. Robert, G. Charpak, Nucl. Instrum.
   Methods Phys. Res. A 376 (1996) 29–35, http://dx.doi.org/10.1016/0168-
   9002(96)00175-1

[5] root.cern.ch CERN.

[6] ROOT Users Guide, Python Interface CERN.

[7] Object Oriented Software Engineering, I. Jacobsen, et al., Addison-Wesley
   ACM Press (1992), pp. 43–69.

[8] I.S. Grant; W.R. Phillips; Manchester Physics (1990). Electromagnetism
   (2nd Edition). John Wiley & Sons. p. 123. ISBN 978-0-471-92712-9.

[9] Review of Particle Physics, M. Tanabashi et al. (Particle Data Group), Phys.
   Rev. D 98, 030001 (2018)

[10] Chikazumi, Sōshin (1997). Physics of ferromagnetism (2nd ed.). Oxford:
     Oxford University Press. ISBN 9780191569852. Wikipedia.

[11] Smith, Kirstine (1918). "On the Standard Deviations of Adjusted and Inter-
     polated Values of an Observed Polynomial Function and its Constants and
     the Guidance They Give Towards a Proper Choice of the Distribution of the
     Observations". Biometrika. 12 (1/2): 1–85. doi:10.2307/2331929. JSTOR
     2331929.

[12] Claeskens, G.; Hjort, N.L. (2008), Model Selection and Model Averaging,
     Cambridge University Press.

[13] H. Broomfield et al (2018) Phys. Educ. 53 055011,
     https://doi.org/10.1088/1361-6552/aaccdb

[14] Thomas, George B., Jr.; Finney, Ross L. (1996), Calculus and Analytic
     Geometry (9th ed.), Addison Wesley, ISBN 0-201-53174-7

[15] Cramer, Gabriel (1750). "Introduction à l’Analyse des lignes Courbes
     algébriques" (in French). Geneva: Europeana. pp. 656–659. Retrieved 2012-
     05-18.

[16] M. Gell-Mann, TED2007, TED Conferences LLC, March 2007.
     https://tinyurl.com/lsaeh79