A characterization of hyperbolic rational maps
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Abstract

In the early 1980’s Thurston gave a topological characterization of rational maps whose critical points have finite iterated orbits ([Th, DH1]): given a topological branched covering $F$ of the two sphere with finite critical orbits, if $F$ has no Thurston obstructions then $F$ possesses an invariant complex structure (up to isotopy), and is combinatorially equivalent to a rational map.

We extend this theory to the setting of rational maps with infinite critical orbits, assuming a certain kind of hyperbolicity. Our study includes also holomorphic dynamical systems that arise as coverings over disconnected Riemann surfaces of finite type. The obstructions we encounter are similar to those of Thurston. We give concrete criteria for verifying whether or not such obstructions exist.

Among many possible applications, these results can be used for example to construct holomorphic maps with prescribed dynamical properties; or to give a parameter description, both local and global, of bifurcations of complex dynamical systems.

Subject class [2000]: Primary 37F; Secondary 32G.

1 Introduction

Thurston’s characterization of postcritically finite rational maps is one of the major tools in complex dynamics. It enables us to produce various kind of rational maps with prescribed dynamical properties, as well as to produce combinatorial models for parameter spaces. There are many applications of Thurston’s theory. Just to mention a few, we may cite for example Douady’s proof of monotonicity of entropy for unimodel maps ([Do]), Mary Rees’ descriptions of parameter spaces (see [Re]), McMullen’s rational quotients ([Mc2]), Kiwi’s characterization of polynomial laminations (using previous work of Bielefield-Fisher-Hubbard ([BFH]) and Poirier ([Po]), etc.

One drawback of Thurston’s theorem is that it can only be applied to postcritically finite rational maps. On one hand, these maps all have a connected Julia set; on the other hand, they form a totally disconnected subset in the parameter space (except the Lattès examples). Therefore the theorem can not characterize the combinatorics of disconnected Julia sets, nor the bifurcations through continuous parameter perturbations.

Over the years, there has been several attempts to extend Thurston’s theory beyond postcritically finite maps. For example, David Brown (see [Br]), supported by previous work of Hubbard and Schleicher ([HIS]), has succeeded in extending the theory to the uni-critical polynomials with an infinite postcritical set (but always with a connected Julia set), and pushed it even further to the infinite degree case, namely the exponential maps. See also Jiang and Zhang [JZ].

We mention also a recent work of Hubbard-Schleicher-Shishikura [HSS] extending Thurston’s theorem to postcritically finite exponential maps.

In this paper, supported by previous works of Cui, Jiang and Sullivan ([CJS]), as well as unpublished manuscripts of Cui, we extend Thurston Theorem to the full setting of arbitrary non-postcritically finite hyperbolic or sub-hyperbolic rational maps. Our analysis leads naturally to the concept of repelling systems over disconnected Riemann surfaces of finitely type, and allows us to establish an analog of Thurston Theorem for these dynamical systems.
This work consists of the first step of a long program, as expos ed in [C2], to the study of deformations and bifurcations of rational maps. In a forthcoming paper ([CT]), we will extend our characterization to the setting of geometrically finite rational maps (i.e. maps with parabolic periodic points), and then give a detailed study of their relations with hyperbolic rational maps. A geometrically finite map \( g \) often sits on the boundary of several hyperbolic components, and does so in a quite subtle way: if you approach it algebraically, you may or may not get an different geometric limit, depending very much how you approach it. This subtlety makes the study for the deformation of \( g \) very difficult. However, it is relatively easy to describe combinatorially all the possible bifurcations. Then, equipped with our Thurston-like realization result, we will be able to prove easily the existence of such bifurcations. For instance we will classify all the hyperbolic components \( H \) that contain a path converging to \( g \) and that along the path the algebraic and geometric limits coincide. Conversely, given a hyperbolic component \( H \), we will apply our technique to determine all the boundary geometrically finite maps \( g \) that are path-accessible from \( H \) with the same properties.

**Statements**

All branched coverings, homeomorphisms in this paper are orientation preserving. Let \( G : \overline{C} \to \overline{C} \) be an orientation preserving branched covering with degree \( \text{deg} G \geq 2 \). Its *postcritical set* is defined to be

\[
P_G := \text{closure}\{G^n(c) \mid n > 0, c \text{ a critical point of } G\}.
\]

Denote by \( P'_G \) the accumulation set of \( P_G \).

We say that \( G \) is *postcritically finite* if every critical point has a finite orbit (i.e. \( P'_G = \emptyset \)). We say that \( G \) is a *sub-hyperbolic semi-rational map* if \( P'_G \) is finite (or empty); and in case \( P'_G \neq \emptyset \), the map \( G \) is holomorphic in a neighborhood of \( P'_G \) and every periodic point in \( P'_G \) is either attracting or super-attracting.

Two sub-hyperbolic semi-rational maps \( G_1 \) and \( G_2 \) are called *c-equivalent*, if there is a pair \((\phi, \psi)\) of homeomorphisms of \( \overline{C} \), and a neighborhood \( U_0 \) of \( P'_{G_1} \) such that:

(a) \( \phi \circ G_1 = G_2 \circ \psi \);

(b) \( \phi \) is holomorphic in \( U_0 \);

(c) the two maps \( \phi \) and \( \psi \) are equal on \( P_{G_1} \), thus on \( P_{G_1} \cup U_0 \) (by the isolated zero theorem);

(d) the two maps \( \phi \) and \( \psi \) are isotopic to each other rel \( P_{G_1} \cup U_0 \), i.e., there is a continuous map \( H : [0, 1] \times \overline{C} \to \overline{C} \) such that each \( H(t, \cdot) \) is a homeomorphism of \( \overline{C} \), \( H(0, \cdot) = \phi \), \( H(1, \cdot) = \psi \), and \( H(t, z) \equiv \phi(z) \) for any \( t \in [0, 1] \) and any \( z \in P_{G_1} \cup U_0 \).

Given a sub-hyperbolic semi-rational map \( G \), we consider the problem of whether there is a rational map c-equivalent to it.

Thurston gave a combinatorial criterion of the same problem for postcritically finite branched coverings, based on the absence of Thurston obstructions (see §3.1 and Theorem 3.2 below). We prove here:

**Theorem 1.1.** Let \( G \) be a sub-hyperbolic semi-rational map with \( P'_G \neq \emptyset \). Then \( G \) is c-equivalent to a rational map \( g \) if and only if \( G \) has no Thurston obstruction. In this case the rational map \( g \) is unique up to Möbius conjugation.
The necessity of having no Thurston obstruction, and the unicity of the rational map \( g \), are known to be true for a wider class of maps. See [Mc2] (or Theorem 3.3 below) and [C1].

Thus it remains only to prove the existence part here: i.e. to show that if \( G \) is unobstructed then it is \( c \)-equivalent to a rational map.

In the process of proving the theorem, we introduce the concept of repelling systems over disconnected Riemann surfaces of finite type, and those of constant complexity. We develop a corresponding Thurston-like theory, including the notions of \( c \)-equivalence to holomorphic models, Thurston obstructions, and then a theorem saying that such a system without obstructions is \( c \)-equivalent to a holomorphic model (see Theorems 3.5 and 5.4 for detailed statements).

The general strategy of the proof of Theorem 1.1 can be then described as follows: we define \( K_G \), its filled Julia set relative to \( \mathcal{P}'_G \), to be the set of points not attracted by the cycles in \( \mathcal{P}'_G \), i.e.

\[
K_G := \{ z \in \mathbb{C} | \bigcup_{n>0} \{G^n(z)\} \cap \mathcal{P}'_G = \emptyset \}.
\]

Step 0. We show that up to a change of representatives in the \( c \)-equivalence class of \( G \), we may assume that \( G \) is quasi-regular (Lem. 2.1).

Given now \( G : \mathbb{C} \rightarrow \mathbb{C} \) an unobstructed quasi-regular sub-hyperbolic semi-rational map.

1. there is a restriction \( G|_{L_1} : L_1 \rightarrow L_0 \) in a neighborhood of \( K_G \) which is an unobstructed repelling system (Lem. 3.6).
2. there is a sub-repelling system \( F \) of \( G|_{L_1} \) that is both unobstructed and of constant complexity (Thms. 4.1(B), 5.1)
3. This repelling system of constant complexity has no boundary obstructions nor renormalized obstructions (Lem. 5.3).
4. Any \( F \) with properties in Step 3 is \( c \)-equivalent to a holomorphic model (Thm. 5.4).
5. \( G|_{L_1} : L_1 \rightarrow L_0 \) is \( c \)-equivalent to a holomorphic model (Thm. 4.1(A)).
6. \( G : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}} \) is \( c \)-equivalent to a rational map (Prop. 2.4).

Steps 1-3 consist of detailed study of Thurston obstructions for repelling systems, as well as combinatorics of puzzle neighborhoods of \( K_G \).

Step 4 (Theorem 5.4) is the core part of this work. It is proved using Grötzsch inequalities, and, in the presence of renormalizations, Thurston’s original Theorem, together with a form of reversed Grötzsch inequality.

Steps 5-6 are standard applications of Measurable Riemann Mapping Theorem.

Steps 2-5 together lead to a Thurston-like theorem for repelling systems (see Theorem 3.5 for a precise statement), which is of independent interest.

Notice that we do not take the approach, as one might have attempted to do, of iteration in an infinite dimensional Teichmüller space.

**Remarks.** Actually the no-obstruction condition in both our theorem 1.1 and Thurston’s original one, is in general a difficult condition to verify, as there are infinitely many candidate obstructions. Therefore in order to apply them effectively, further efforts are often needed. In the postcritically finite setting, many methods have been developed to overcome this
difficulty. But in the case at hand, our result would have been left unsatisfactory if no further criteria have been given. Fortunately, what we have actually proved (see Theorems 5.4 and 9.1 below) does provide more effective criteria. More precisely we will decompose
the dynamics into several renormalization pieces that are in fact postcritically finite maps,



together with a transition matrix that records the gluing data. This decomposition is not
entirely trivial and presents some interests even for rational maps. Our proof shows then that
in order to be c-equivalent to a rational map it amounts only to check Thurston’s condition
for the renormalizations (thus back to the postcritically finite setting), and for the gluing
data, which is only one eigenvalue to calculate.

We obtain also a combination result that is very practical to use. We will show in Theorem
9.1 that for any finite collection \( f_i \) of rational maps with connected Julia set \( J_i \) (postcritically
finite or not), together with a compatible (unobstructed) gluing data \( D \), one can glue the
\( f_i \)’s on neighborhoods of \( J_i \) together following \( D \), to obtain a rational map \( g \), so that each \( f_i \)



appears as a renormalization of \( g \).

For a similar decomposition-gluing approach, we recommend [Pi]. The topology of Julia
components for hyperbolic rational maps has been well understood. See [PT].

Theorem 1.1 was already announced in [CJS], together with a sketch of the main ideas
of the proof. Numerous details there were however missing, and sometimes erroneous. The
presentation here will be totally different. In particular the concept of repelling systems
and the related Thurston-like theory are new. This will lead also to two stronger and easier to
use results: Theorems 5.4 and 9.1.

Along the proof we will provide numerous supporting diagrams and pertinent examples.

Organization.

The paper is organized as follows: In \( \text{S}2 \) we prove Step 0 and Step 6 above. We introduce
the concept of repelling system and show how it appears as a restriction near \( K_G \) of a global
map \( G \).

In \( \text{S}3 \) we first recall the definition of Thurston obstructions and state Thurston’s original
theorem. We then develop the corresponding concepts for repelling systems and state a
Thurston-like theorem in this setting (Theorem 3.5). Assuming it we prove Theorem 1.1 (we
just need to do Step 1 above).

In \( \text{S}4 \) we introduce the concepts of constant complexity repelling systems and the specific
obstructions associated to them. We state our Thurston-like theorem, Theorem 5.4 in this
setting. Assuming this we complete Steps 2-5 above and proves Theorem 8.5.

In \( \text{S}6,8 \) we give the proof of Theorem 5.4.

In the final section \( \text{S}9 \) we state Theorem 9.1.

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2 Reducing to restrictions near \( K_G \)

Let \( G : \mathbb{C} \to \mathbb{C} \) be a sub-hyperbolic semi-rational map with \( \mathcal{P}_G' \neq \emptyset \), i.e. \( G \) is a branched
covering such that the cluster set \( \mathcal{P}_G' \) of its postcritical set is finite and non-empty, that \( G \) is
holomorphic in a neighborhood of \( \mathcal{P}_G' \), and that every periodic cycle in \( \mathcal{P}_G' \) is either attracting
or superattracting.
Our objective is to show that if \( G \) has no Thurston obstruction then \( G \) is c-equivalent to a certain rational map.

### 2.1 Making the map quasi-regular

**Lemma 2.1.** Let \( G \) be a sub-hyperbolic semi-rational map with \( \mathcal{P}'_G \neq \emptyset \). Then \( G \) is c-equivalent to a quasi-regular sub-hyperbolic semi-rational map.

**Proof.** Consider \( G \) as a branched covering from \( \overline{\mathbb{C}} \) onto \( \overline{\mathbb{C}} \). There is a unique complex structure \( \mathcal{X}' \) on \( \overline{\mathbb{C}} \) such that \( G : (\overline{\mathbb{C}}, \mathcal{X}') \rightarrow \overline{\mathbb{C}} \) is holomorphic (see [DD], section 6.1.10). The uniformization theorem provides thus a conformal homeomorphism \( \xi : (\overline{\mathbb{C}}, \mathcal{X}') \rightarrow \overline{\mathbb{C}} \). Set \( R := G \circ \xi^{-1} \). Then \( R : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}} \) is a branched covering, holomorphic with respect to the standard complex structure, therefore a rational map.

Let \( U \subset \overline{\mathbb{C}} \) be a finite union of quasi-discs with mutually disjoint closures, such that \( \mathcal{P}'_G \subset U \), \( G^{-1}(U) \supset \overline{U} \) and \( G \) is holomorphic in a neighborhood of \( \overline{U} \) with respect to the standard complex structure. Then the new structure \( \mathcal{X}' \) is compatible with the chart \( G^{-1}(U) \), so \( \xi(U) \) is also a finite disjoint union of quasi-discs. Set \( L := \overline{\mathbb{C}} \setminus U \). Then there is a quasi-conformal homeomorphism \( \eta : L \rightarrow \xi(L) \) such that \( \eta = \xi \) on \( \partial L \) \( \cup \mathcal{P}_G \cap L \) and \( \eta \) is isotopic to \( \xi \) rel \( \partial L \) \( \cup \mathcal{P}_G \cap L \) (see Lemma C.2). Set \( \zeta = \eta^{-1} \circ \xi \) on \( L \) and \( \zeta = id \) on \( U \). Then \( \zeta \) is isotopic to the identity rel \( \overline{U} \) \( \cup \mathcal{P}_G \). So \( G \circ \zeta^{-1} \) is c-equivalent to \( G \). But \( G \circ \zeta^{-1} = R \circ \eta \) on \( L \), with \( \eta \) quasi-conformal and \( R \) holomorphic. One sees that \( G \circ \zeta^{-1} \) is quasi-regular in \( L \), thus quasi-regular on \( \overline{\mathbb{C}} \). \( \square \)

### 2.2 Repelling system as restriction

**Definition 1.** For two subsets \( E_1, E_2 \) of \( \overline{\mathbb{C}} \), we use the symbol \( E_1 \subset \subset E_2 \) if the closure of \( E_1 \) is contained in the interior of \( E_2 \). We use also \( E^c := \overline{\mathbb{C}} \setminus E \) to denote the complement of \( E \) in \( \overline{\mathbb{C}} \). If \( E \subset \overline{\mathbb{C}} \) is compact connected, each complementary component of \( E \) (i.e. a component of \( E^c \)) is a disc.

We will cover \( K_G \) by a suitable puzzle \( \mathcal{L} \) so that in particular \( G^{-1}(\mathcal{L}) \subset \subset \mathcal{L} \), just as in Branner-Hubbard’s study of cubic polynomials with disconnected Julia set (see [BH]). The restriction \( G|_{G^{-1}(\mathcal{L})} \), considered as a dynamical system, leads naturally to the concept of repelling systems. Such dynamical systems can be also considered as generalization of Douady-Hubbard’s polynomial-like mappings ([DH3]) in two aspects: the domain of definitions will be several components each with several boundary curves (this is necessary as we are dealing with rational maps), and the dynamics will be quasi-regular branched coverings without necessarily analyticity.

**Definition 2.** We say that \( S \subset \overline{\mathbb{C}} \) is a (quasi-circle) bordered Riemann surface if it is \( \overline{\mathbb{C}} \) minus finitely many (might be zero) open quasi-discs \( D_1 \) so that their closures \( \overline{D}_1 \) are mutually disjoint. Therefore \( \partial S \) is a disjoint union of finitely many (might be zero) quasi-circles. We say that \( \mathcal{L} = S_1 \sqcup \cdots \sqcup S_k \) is a puzzle surface, if each \( S_i \) a bordered Riemann surface, and two distinct \( S_i \) and \( S_j \) are either contained in distinct copies of the Riemann sphere, or are mutually disjoint. Each \( S_i \) is also called an \( L \)-piece.

**Definition 3.** We say that a map \( F : \mathcal{E} \rightarrow \mathcal{L} \) is a (quasi-regular) repelling system, if:

- \( \mathcal{E} \subset \subset \mathcal{L} \) are two nested puzzle surfaces and \( F : \mathcal{E} \rightarrow \mathcal{L} \) is a quasi-regular proper mapping; more precisely if \( \mathcal{L} = S_1 \sqcup \cdots \sqcup S_k \) is a puzzle surface, \( \mathcal{E} := \bigcup_{i,j \in \{1, \ldots, k\}, \delta \in \Lambda_{ij}} E_{ij\delta} \) and \( F : \mathcal{E} \rightarrow \mathcal{L} \) is a map such that,
Lemma 2.2. Let $G$ be a quasi-regular sub-hyperbolic semi-rational map with $\mathcal{P}_G' \neq \emptyset$. Then there exists a puzzle surface neighborhood $L_0$ of $K_G$ such that, setting $L_1 = G^{-1}(L_0)$,

$$L_1 \subset \subset L_0, \quad \text{and} \quad G|_{L_1} : L_1 \to L_0 \text{ is a postcritically finite repelling system.}$$

Proof. Note that for $S \subset \overline{G}$ a bordered Riemann surface, a necessary and sufficient condition for each component of $G^{-1}(S)$ to be a bordered Riemann surface is that $\partial S$ does not contain any critical value of $G$.

One can find an open set $U_0$ which is the union of a quasi-disc neighborhood for each point of $\mathcal{P}_G'$ so that these quasi-discs have disjoint closures, that $\partial U_0$ is disjoint from $\mathcal{P}_G$, that $G$ is holomorphic in a neighborhood of $\overline{U_0}$, and that $G(U_0) \subset \subset U_0$.

Set $L_0 = \overline{G} \setminus U_0$. Then $K_G \subset \subset G^{-1}(L_0) \subset \subset L_0$, $\mathcal{P}_G \cap L_0$ is finite (or empty). This $L_0$ satisfies the requirement of the lemma.

Note that one may also set $L_n = G^{-n}(L_0)$ for $n \in \mathbb{N}$, to produce a sequence of repelling systems $G|_{L_{n+1}} : L_{n+1} \to L_n$ satisfying the requirement of the lemma.

Examples.

A. Let $\mathcal{E} \subset \subset \mathcal{L}$ be two closed quasi-discs, and $F : \mathcal{E} \to \mathcal{L}$ be a holomorphic proper map. Then $F$ is a polynomial-like map in the sense of Douady-Hubbard, $K_F$ is simply the filled Julia set, and $\mathcal{P}_F$ is the postcritical set.
B. \( \mathcal{L} \) is a closed quasi-disc, \( \mathcal{E} \) is the union of finitely many disjoint closed quasi-discs contained in the interior of \( \mathcal{L} \), and \( F \) maps each \( \mathcal{E} \)-piece quasi-conformally onto the larger disc \( \mathcal{L} \). In this case \( \mathcal{P}_F = \emptyset \) and \( K_F \) is the non-escaping set of \( F \). If \( F \) is also holomorphic, the filled Julia set \( K_F \) is a Cantor set. This happens when \( F \) is \( z^2 + c \) for large \( c \) and \( \mathcal{L} \) is a disc in \( \mathbb{C} \) bounded by an equipotential such that \( 0 \in \mathcal{L} \) but \( c \not\in \mathcal{L} \).

C. By convention we may consider \( \mathcal{E} = \mathcal{L} = \overline{\mathbb{C}} \) as puzzle surfaces and a quasi-regular postcritically finite branched covering \( F : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \) as a repelling system.

More important classes of examples are provided by Lemma [6.1] by the annuli coverings (see below). See also [6.1]

**Marking.** Let \( F : \mathcal{E} \to \mathcal{L} \) be a postcritically finite repelling system. We say that it is *marked*, if it is equipped with a marked set \( \mathcal{P} \), satisfying that

\[
\mathcal{P}_F \subset \mathcal{P} \subset (\mathcal{L} \setminus \partial \mathcal{L}); \quad \# \mathcal{P} < \infty \quad \text{and} \quad F(\mathcal{P} \cap \mathcal{E}) \subset \mathcal{P}.
\]

If not explicitly mentioned, we will consider \( F \) to be marked by its postcritical set \( \mathcal{P}_F \).

Motivated by Thurston’s theory, we give the following:

**Definition 4.** We say that two marked postcritically finite repelling systems \((\mathcal{E} \xrightarrow{F} \mathcal{L}, \mathcal{P})\) and \((\mathcal{E}' \xrightarrow{R} \mathcal{L}', \mathcal{P}')\) are **c-equivalent**, if there is a pair of quasi-conformal homeomorphisms \( \Phi, \Psi : \mathcal{L} \to \mathcal{L}' \) such that

\[
\begin{cases}
\Psi(\mathcal{E}) = \mathcal{E}' \text{ and } \Psi(\mathcal{P}) = \mathcal{P}' \\
\Psi|_{\partial \mathcal{L} \cup \mathcal{P}} = \Phi|_{\partial \mathcal{L} \cup \mathcal{P}} \\
\Phi \circ F \circ \Psi^{-1}|_{\mathcal{E}'} = R
\end{cases}
\]

\( \mathcal{L} \supset \mathcal{E} \xrightarrow{\Psi} \mathcal{E}' \subset \mathcal{L}' \)

in particular \( F \downarrow \mathcal{L} \xrightarrow{\Phi} \mathcal{L}' \)

commutes. (2)

We say that \((\mathcal{E} \xrightarrow{F} \mathcal{L}, \mathcal{P})\) is c-equivalent to a **holomorphic model**, if there is a holomorphic \((\mathcal{E}' \xrightarrow{R} \mathcal{L}', \mathcal{P}')\) c-equivalent to it.

See [6.1] for examples. We have the following criterion:

**Lemma 2.3.** A marked postcritically finite repelling system \((\mathcal{E} \xrightarrow{F} \mathcal{L}, \mathcal{P})\) is c-equivalent to a holomorphic model iff there is a pair \((\Theta, \mu)\) such that:

(a) \( \Theta : \mathcal{L} \to \mathcal{L} \) is a quasi-conformal homeomorphism with \( \Theta|_{\partial \mathcal{L} \cup \mathcal{P}} = \text{id} \) and \( \Theta \) is isotopic to the identity rel \( \partial \mathcal{L} \cup \mathcal{P} \).

(b) \( \mu \) is a Beltrami differential on \( \mathcal{L} \) with \( \|\mu\|_{\infty} < 1 \) and \( (F \circ \Theta^{-1})^*(\mu) = \mu|_{\Theta(\mathcal{E})} \).

**Proof.** Assume that \((\mathcal{E} \xrightarrow{F} \mathcal{L}, \mathcal{P})\) is c-equivalent to a holomorphic \((\mathcal{E}' \xrightarrow{R} \mathcal{L}', \mathcal{P}')\). Let \((\Phi, \Psi)\) be the pair of quasi-conformal maps given by Definition 4. Set \( \Theta = \Phi^{-1} \circ \Psi \). Then \( \Theta \) satisfies the required isotopic conditions. Let \( \mu \) be the Beltrami coefficient of \( \Phi \). Then (2) means exactly that \( (F \circ \Theta^{-1})^*(\mu) = \mu|_{\Theta(\mathcal{E})} \).

Conversely assume the existence of the pair \((\Theta, \mu)\). By the Measurable Riemann Mapping Theorem, there is a quasi-conformal map \( \Phi \) defined piecewisely on \( \mathcal{L} \) with Beltrami coefficient \( \mu \). Set \( \Psi = \Phi \circ \Theta \). Then for \( \mathcal{E}' := \Psi(\mathcal{E}) \), \( \mathcal{P}' = \Psi(\mathcal{P}) \), \( \mathcal{L}' := \Phi(\mathcal{L}) \) and \( R := \Phi \circ F \circ \Psi^{-1} : \mathcal{E}' \to \mathcal{L}' \), we know that \((\mathcal{E}' \xrightarrow{R} \mathcal{L}', \mathcal{P}')\) is a holomorphic repelling system c-equivalent to \((\mathcal{E} \xrightarrow{F} \mathcal{L}, \mathcal{P})\).

The following result relates repelling systems to our main interest (Theorem [4.1]) through restriction:
Proposition 2.4. Let $G$ be a quasi-regular sub-hyperbolic semi-rational map with $\mathcal{P}_G \neq \emptyset$. If there is a puzzle surface neighborhood $\mathcal{L}$ of $K_G$ with $G^{-1}(\mathcal{L}) \subset \mathcal{L}$ and $\partial \mathcal{L} \cap \mathcal{P}_G = \emptyset$ such that $G_{G^{-1}(\mathcal{L})} : G^{-1}(\mathcal{L}) \to \mathcal{L}$, as a postcritically finite repelling system, is $c$-equivalent to a holomorphic model, then $G$ is $c$-equivalent to a rational map.

Proof. Set $E = G^{-1}(\mathcal{L})$ and $F = G|_{G^{-1}(\mathcal{L})}$. By assumption $F$ is $c$-equivalent to a holomorphic model (with marked set $\mathcal{P}_F$, which is equal to $\mathcal{P}_G \cap K_G$). By Lemma 2.3 there is a pair $(\Theta, \mu)$, with $\Theta$ a quasi-conformal map of $\mathcal{L}$ satisfying $\Theta|_{\partial \mathcal{L} \cup (\mathcal{P}_G \cap \mathcal{L})} = id$ and $\Theta$ isotopic to the identity rel $\partial \mathcal{L} \cup (\mathcal{P}_G \cap \mathcal{L})$, with $\mu$ a Beltrami differential on $\mathcal{L}$ such that $\|\mu\|_{\infty} < 1$ and $(G \circ \Theta^{-1})^* \mu = \mu|_{\Theta(E)}$.

Choose $U_0$ an open neighborhood of $\mathcal{P}_G$ disjoint from $\mathcal{L}$ so that $G^{-1}(U_0) \supset \overline{U_0}$ and $G$ is holomorphic on $G^{-1}(U_0)$. Set $L_0 = U_0^c$ and $L_n = G^{-n}(L_0)$. As $L_n$ forms a decreasing sequence of sets shrinking down to $K_G$, there is an integer $N \geq 0$ such that $L_N \subset \mathcal{L}$. So every orbit passing through $L_0 \setminus E$ stays there for at most $N + 1$ times before being trapped by $U_0$.

Extend the map $\Theta$ to a quasi-conformal map of $\overline{\mathbb{C}}$ by setting $\Theta := id$ on $\overline{\mathbb{C}} \setminus \mathcal{L}$, then $\Theta$ is quasi-conformal and isotopic to the identity rel $\mathcal{P}_G$. Set $G_1 = G \circ \Theta^{-1}$. Then $G_1$ is again quasi-regular, and is holomorphic on $\Theta(U_0) = U_0$. Clearly, each $G_1$-orbit passes through $L_0 \setminus \Theta(E)$ at most $N + 1$ times.

Extend now $\mu$ outside $\mathcal{L}$ by $\mu = 0$. Let $\Phi_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a global integrating map of this extended $\mu$. Set $G_2 := \Phi_1 \circ G_1 \circ \Phi_1^{-1}$. Then $G_2$ is again quasi-regular, and is holomorphic in the interior of $\Phi_1 \circ \Theta(E)$ and in $\Phi_1(U_0)$. Elsewhere each $G_2$-orbit passes at most $N + 1$ times.

One can now apply the Shishikura principle: we spread out the Beltrami differential $\nu_0 \equiv 0$ using iterations of $G_2$ to get an $G_2$-invariant Beltrami differential $\nu$. Note that $\nu = 0$ on $\Phi_1(U_0)$, and $\|\nu\|_{\infty} < 1$. Integrating $\nu$ by a quasi-conformal homeomorphism $\Phi_2$ (necessarily holomorphic on $\Phi_1(U_0)$), we get a new map $R := \Phi_2 \circ G_2 \circ \Phi_2^{-1}$ which is a rational model and is $c$-equivalent to $G_2$, therefore to $G$. See the following diagram.

\begin{equation*}
\begin{array}{c}
G \downarrow & \Phi_1 \circ G_1 \circ \Phi_1^{-1} & \Phi_2 \circ G_2 \circ \Phi_2^{-1} & R \\
\Theta \downarrow & \Theta & \Theta & \Theta \\
(\overline{\mathbb{C}}, E) & (\overline{\mathbb{C}}) & (\overline{\mathbb{C}}) & (\overline{\mathbb{C}})
\end{array}
\end{equation*}

\end{proof}

3 Thurston-like theory for repelling systems

We are thus interested in whether a given repelling system (for example a restriction of $G$ near $K_G$ as in Lemma 2.2) is $c$-equivalent to a holomorphic model. We will see that, similar to Thurston’s theory, the answer is yes if the map has no obstructions that are similar to Thurston’s original obstructions.

3.1 Grötzsch inequality and Thurston obstructions

Thurston obstructions are in fact closely related to the Grötzsch inequality on moduli of annuli. The best way to understand them is to start from real models.

1. Slope obstruction. Suppose we want to make a tent map $f$ on $[0, 1]$ with folding point $c$ and with $f(c) > 1$, with left slope $d_1$ and right slope $-d_2$. This is possible iff $d_1^{-1} + d_2^{-1} < 1$. More generally, suppose we have $k$ disjoint closed intervals $I_1 \sqcup \cdots \sqcup I_k$ in $\mathbb{R}$, on which we have a topological dynamical system with the following combinatorics:
For each pair $(i, j)$, there are finitely many (might be zero) intervals $I_{ij\delta}$, for $\delta$ running through some finite or empty index set $\Lambda_{ij}$ (depending on $(i, j)$), such that

(TOP) $I_{ij\delta}$ is a sub-interval of $I_i$, and the $I_{ij\delta}$’s are mutually disjoint for all possible $i, j$ and $\delta$.

(DYN) $f : I_{ij\delta} \rightarrow I_j$ is a homeomorphism for all possible $i, j$ and $\delta$.

The question we ask is: given a collection of slopes (in absolute value) $(d_{ij\delta})_{ij\delta}$, is there a collection of $(I_j, I_{ij\delta}$ and $f : I_{ij\delta} \rightarrow I_j$) such that each $f : I_{ij\delta} \rightarrow I_j$ is affine with slope (in absolute value) $d_{ij\delta}$, for every possible multi-index $(i, j, \delta)$?

Let us search at first necessary conditions. Assume that such a piece-wise affine map $f$ exists. Then (DYN) implies $|I_{ij\delta}| = \left| \frac{|I_i|}{d_{ij\delta}} \right|$ whereas (TOP) implies $\sum_{j,\delta} |I_{ij\delta}| < |I_i|$. Put these together we get

$$\sum_j \left( \sum_{\delta \in \Lambda_{ij}} \frac{1}{d_{ij\delta}} \right) |I_j| < |I_i| \; , \; i = 1, \cdots, k . \tag{3}$$

Let $D = (a_{ij})$ denotes the transition matrix defined by $a_{ij} := \sum_{\delta \in \Lambda_{ij}} 1/d_{ij\delta}$ (it is a non-negative matrix, with, by convention, $a_{ij} = 0$ if $\Lambda_{ij} = \emptyset$). Then the necessary condition (3) can be reformulated as: $Dv < v$, where $v := (|I_i|)$ is a vector with strictly positive entries.

It is quite easy to check that this necessary condition is also sufficient. Therefore the answer to the above question is: such an affine dynamical system exists if and only if the leading eigenvalue $\lambda(D)$ of $D$ from Perron-Frobenius theorem is strictly less than 1.

Once this is done, the following ‘complexification’ will become easy:

2. Grötzsch obstruction for annuli coverings. Now we may think the intervals $I_j$ are thin tubes, and the subintervals $I_{ij\delta}$ as essential sub-annuli. More precisely, let $\mathcal{A} = A_1 \sqcup \cdots \sqcup A_k$ be a puzzle surface with each $A_i$ a closed annulus. For each pair $(i, j)$, let $(A_{ij\delta})_{\delta}$ be finitely many (might be zero) sub-annuli of $A_i$ such that

(TOP) $A_{ij\delta}$ is an essential sub-annulus of $A_i$. Furthermore for each $i$ the $A_{ij\delta}$’s are mutually disjoint for all possible choices of $j$ and $\delta$.

(DYN) $f : A_{ij\delta} \rightarrow A_j$ is a quasi-regular covering of degree $d_{ij\delta}$ for all possible $i, j$ and $\delta$.

Set $\mathcal{E} = \bigsqcup_{ij\delta} A_{ij\delta}$ and consider $f : \mathcal{E} \rightarrow \mathcal{A}$ as an (un)repelling system with empty post-critical set (annuli covering). The question is: is $f : \mathcal{E} \rightarrow \mathcal{A}$ c-equivalent to a holomorphic model?

Assume that $f$ is already holomorphic. Denote by $|A_\ast|$ the modulus (rather than the length) of the interior of the annulus $A_\ast$. Then $|A_{ij\delta}| = |A_j|/d_{ij\delta}$ (due to (DYN)) and $\sum_{j,\delta} |A_{ij\delta}| < |A_i|$ (due to Grötzsch inequality and (TOP)). Therefore the leading eigenvalue $\lambda(D)$ of the transition matrix $D$ is less than 1. We have, naturally:

**Lemma 3.1.** An annuli covering $f : \mathcal{E} \rightarrow \mathcal{A}$ is c-equivalent to a holomorphic model if and only if $\lambda(D) < 1$.

**Proof.** $\Longrightarrow$. Assume $f : \mathcal{E} \rightarrow \mathcal{A}$ is c-equivalent to a holomorphic $R : \mathcal{E}' \rightarrow \mathcal{A}'$. Then the two maps have the same transition matrix $D$. By the argument above $\lambda(D) < 1$. 

$\Longleftarrow$. Since $R$ is holomorphic, $\lambda(D) < 1$. (due to Grötzsch inequality and (TOP)). Therefore the leading eigenvalue $\lambda(D)$ of the transition matrix $D$ is less than 1. We have, naturally:
This Lemma is not needed in the proof of our main result. But it helps the understanding of Thurston obstructions and its proof will shed light to our more complicated situation.

3. **Thurston obstruction** for a pair \((h, \mathcal{P})\). Let \(h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\) be a branched covering, and \(\mathcal{P} \subset \overline{\mathbb{C}}\) a closed marked set containing \(\mathcal{P}_h\) and \(h(\mathcal{P})\). For example we may take \(h\) to be a sub-hyperbolic semi-rational map \(G\) and \(\mathcal{P} = \mathcal{P}_G\).

A Jordan curve \(\gamma\) in \(\overline{\mathbb{C}} \setminus \mathcal{P}\) is said null-homotopic (resp. peripheral) within \(\overline{\mathbb{C}} \setminus \mathcal{P}\) if one of its complementary component contains zero (resp. one) point of \(\mathcal{P}\); and is said non-peripheral within \(\overline{\mathbb{C}} \setminus \mathcal{P}\) otherwise, i.e. if each of its two complementary components contains at least two points of \(\mathcal{P}\).

We say that \(\Gamma = \{\gamma_1, \ldots, \gamma_k\}\) is a multicurve of within \(\overline{\mathbb{C}} \setminus \mathcal{P}\), if each \(\gamma_i\) is a Jordan curve in \(\overline{\mathbb{C}} \setminus \mathcal{P}\) and is non-peripheral within \(\overline{\mathbb{C}} \setminus \mathcal{P}\), and the \(\gamma_j\)’s are mutually disjoint and mutually non-homotopic within \(\overline{\mathbb{C}} \setminus \mathcal{P}\).

Each multicurve \(\Gamma\) induces a \((h, \mathcal{P})\)-transition matrix \(D_\Gamma\) together with its leading eigenvalue \(\lambda(D_\Gamma)\) as follows: Let \((\gamma_{ij}\delta)_{\delta \in \Lambda_{ij}}\) denote the collection of the components of \(h^{-1}(\gamma_j)\) homotopic to \(\gamma_i\) within \(\overline{\mathbb{C}} \setminus \mathcal{P}\), with \(\Lambda_{ij}\) some finite or empty index set depending on \(ij\). Then \(h : \gamma_{ij} \to \gamma_j\) is a topological covering of a certain degree \(d_{ij}\). The transition matrix \(D_\Gamma = (a_{ij})\) is defined by \(a_{ij} = \sum_{\delta \in \Lambda_{ij}} 1/d_{ij}\delta\) (and \(a_{ij} = 0\) if \(\Lambda_{ij} = \emptyset\)).

We say that a multicurve \(\Gamma\) is \((h, \mathcal{P})\)-stable if every curve of \(h^{-1}(\gamma)\), with \(\gamma \in \Gamma\), is either null-homotopic or peripheral within \(\overline{\mathbb{C}} \setminus \mathcal{P}\), or is homotopic within \(\overline{\mathbb{C}} \setminus \mathcal{P}\) to a curve in \(\Gamma\). This implies that for any \(m > 0\), every curve of \(h^{-m}(\gamma)\), \(\gamma \in \Gamma\) is either null-homotopic or peripheral within \(\overline{\mathbb{C}} \setminus \mathcal{P}\), or is homotopic within \(\overline{\mathbb{C}} \setminus \mathcal{P}\) to a curve in \(\Gamma\).

We say that a multicurve \(\Gamma\) is a Thurston obstruction for \((h, \mathcal{P})\) if it is \((h, \mathcal{P})\)-stable and \(\lambda(D_\Gamma) \geq 1\). In the particular case \(\mathcal{P} = \mathcal{P}_h\) we say simply that \(\Gamma\) is a Thurston obstruction for \(h\).

In case that \(\mathcal{P}\) is finite (in particular \(h\) is postcritically finite) we say that two such pairs \((h, \mathcal{P})\), \((\bar{h}, \bar{\mathcal{P}})\) are \(c\)-equivalent if there is a pair of homeomorphisms \((\phi, \psi) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\) such that \(\phi(\mathcal{P}) = \bar{\mathcal{P}}\), that \(\phi\) is isotopic to \(\psi\) rel \(\mathcal{P}\), and that \(\phi \circ h \circ \psi^{-1} = \bar{h}\).

**Theorem 3.2. (Marked Thurston theorem).** Let \(h\) be a postcritically finite branched covering of \(\overline{\mathbb{C}}\) with \(\deg h \geq 2\). Assume that the signature of its orbifold is not \((2, 2, 2, 2)\), or more particularly \(\mathcal{P}_h\) contains periodic critical points or at least five points. Let \(\mathcal{P}\) be a finite marked set containing \(\mathcal{P}_h\) and \(h(\mathcal{P})\). If \((h, \mathcal{P})\) has no Thurston obstructions, then \((h, \mathcal{P})\) is \(c\)-equivalent to a unique rational map model. More precisely there are homeomorphisms \((\phi, \psi) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}\) such that \(\phi\) is isotopic to \(\psi\) rel \(\mathcal{P}\) and that \(f := \phi \circ h \circ \psi^{-1}\) is a rational map. And the conformal conjugacy class of the pair \((f, \phi(\mathcal{P}))\) is unique.

Furthermore, if \(h\) is quasi-regular, both \(\phi\) and \(\psi\) can be taken to be quasi-conformal.

Here we omit the definition of orbifold and its signature (see e.g. [DH1]). We mention only that if \(\mathcal{P}_h\) contains periodic critical points, or at least 5 points, then the signature of its orbifold is not \((2, 2, 2, 2)\). This is enough for our purpose here.

**Remark.** Our statement is slightly different than the original Thurston Theorem, where \(\mathcal{P} = \mathcal{P}_h\). But the arguments in [DH1] can be easily adapted to prove this more general form. In case \(h\) is quasi-regular, we may replace \(\phi\) by a quasi-conformal map \(\phi'\) isotopic to \(\phi\) rel \(\mathcal{P}\). This is possible since \(\mathcal{P}\) is finite (see Lemma [C2]). Lifting this isotopy will provide us a quasi-conformal map \(\psi'\) isotopic to \(\psi\) rel \(\mathcal{P}\) such that \(\phi \circ h \circ \psi^{-1} = \phi' \circ h \circ \psi'^{-1}\).
Conversely, we have the following result of McMullen ([Mc2]):

**Theorem 3.3.** Let \( f \) be a rational map with \( \deg f \geq 2 \), and let \( \mathcal{P} \) be a closed subset such that \( f(\mathcal{P}) \subset \mathcal{P} \) and \( \mathcal{P}_f \subset \mathcal{P} \). Let \( \Gamma \) be a \((f, \mathcal{P})\)-multicurve whose transition matrix is denoted by \( D \). Then \( \lambda(D) \leq 1 \). If \( \lambda(D) = 1 \), then either \( f \) is postcritically finite whose orbifold has signature \((2,2,2,2)\); or \( \Gamma \) includes a curve that is contained in the Siegel discs or Herman rings of \( f \).

Again this form is slightly stronger than McMullen’s original version. But the proof goes through without any trouble.

### 3.2 Thurston obstructions for repelling systems.

Let \((\mathcal{E} \xrightarrow{F} \mathcal{L}, \mathcal{P})\) be a marked postcritically finite repelling system, in other words \( F : \mathcal{E} \to \mathcal{L} \) is a quasi-regular branched covering among two nested puzzle surfaces, and \( \mathcal{P} \subset \mathcal{L} \) is a finite set containing \( \mathcal{P}_F \) and \( F(\mathcal{P} \cap \mathcal{E}) \). (In case \( \mathcal{L} = \mathbb{C} \) we are back to Thurston’s setting).

Two Jordan curves in \( \mathcal{L} \) are **homotopic** if they are both contained in a common \( \mathcal{L} \)-piece \( S \) and are homotopic to each other within \( S \). 

A Jordan curve \( \gamma \subset \mathcal{L} \) is said **null-homotopic** (resp. **peripheral**) within \( \mathcal{L} \) if it bounds an open disc \( D \) so that \( D \subset \mathcal{L} \) and \( D \cap \mathcal{P} = \emptyset \) (resp. \#\( D \cap \mathcal{P} = 1 \)); is said **non-peripheral** within \( \mathcal{L} \) otherwise (this is equivalent to say that \( \gamma \) is contained in \( S \) for some component \( S \) of \( \mathcal{L} \), and either \( \gamma \) bounds no disc in \( S \), or \( \gamma \) bounds a disc \( D \) in \( S \) containing at least two points of \( \mathcal{P} \)). For example if \( \gamma \) is a boundary curve of an \( \mathcal{L} \)-piece \( S \), and if \( S \) is not a closed disc, then \( \gamma \) is non-peripheral.

We say that \( \Gamma = \{ \gamma_1, \cdots, \gamma_k \} \) is a **multicurve** within \( \mathcal{L} \), if each \( \gamma_i \) is a Jordan curve in \( \mathcal{L} \) and is non-peripheral within \( \mathcal{L} \), and the \( \gamma_j \)'s are mutually disjoint and mutually non-homotopic within \( \mathcal{L} \).

Each multicurve \( \Gamma \) induces an \((F, \mathcal{P})\)-transition matrix \( W = W_\Gamma \) together with its **leading eigenvalue** \( \lambda(W_\Gamma) \) as follows: Let \( (\gamma_{ij}\delta)_{\delta \in \Lambda_{ij}} \) denote the collection of the components of \( F^{-1}(\gamma_{ij}) \) homotopic to \( \gamma_i \) within \( \mathcal{L} \), with \( \Lambda_{ij} \) some finite or empty index set depending on \( ij \). Then \( F : \gamma_{ij} \to \gamma_j \) is a topological covering of a certain degree \( d_{ij}\delta \). The transition matrix is defined by

\[
W_\Gamma = (b_{ij}), \quad b_{ij} = \sum_{\delta \in \Lambda_{ij}} 1/d_{ij}\delta
\]

(with \( b_{ij} = 0 \) if \( \Lambda_{ij} = \emptyset \)).

We say that a multicurve \( \Gamma \) within \( \mathcal{L} \) is **(F, \mathcal{P})-stable** if every curve of \( F^{-1}(\gamma) \), with \( \gamma \in \Gamma \), is either null-homotopic or peripheral within \( \mathcal{L} \), or is homotopic within \( \mathcal{L} \) to a curve in \( \Gamma \).

We say that a multicurve \( \Gamma \) within \( \mathcal{L} \) is a **Thurston obstruction for (F, \mathcal{P})** if it is **(F, \mathcal{P})-stable** and \( \lambda(W_\Gamma) \geq 1 \). See [§6.1] for examples.

The following principle will be used frequently, and is a direct consequence of the fact that \( F(\mathcal{P} \cap \mathcal{E}) \subset \mathcal{P} \) and that \( F \) is a covering over \( \mathcal{L} \):

**Basic pull-back principle.**

1. Let \( D \) be an open Jordan disc contained in \( \mathcal{L} \) with \( \partial D \cap \mathcal{P} = \emptyset \). Then every component of \( F^{-1}(D) \) is again an open disc and is contained in \( \mathcal{E} \). Each curve in \( F^{-1}(\partial D) \) is the boundary of a component of \( F^{-1}(D) \).


2. Let $A$ be an open annulus contained in $L \setminus \mathcal{P}$. Then every component of $F^{-1}(A)$ is again an open annulus and is contained in $E \setminus \mathcal{P}$.

3. Let $D$ be an open Jordan disc contained in $L$ with $\partial D \cap \mathcal{P} = \emptyset$ such that $D$ contains a unique point of $\mathcal{P}$. Then every component of $F^{-1}(D)$ is again an open disc, is contained in $E$, and contains at most one point of $\mathcal{P}$. Each curve in $F^{-1}(\partial D)$ is the boundary of a component of $F^{-1}(D)$.

The following is an easy consequence:

**Lemma 3.4.** Let $(E \xrightarrow{F} L, \mathcal{P})$ be a marked postcritically finite repelling system. For any peripheral (resp. null-homotopic) curve $\gamma \subset L \setminus \mathcal{P}$, each curve in $F^{-1}(\gamma)$ is either peripheral or null-homotopic (resp. is null-homotopic).

We will prove:

**Theorem 3.5.** (Thurston theorem for marked repelling systems). Let $(B \xrightarrow{G} M, \mathcal{Q})$ be a marked postcritically finite repelling system. We assume in addition that no $\mathcal{M}$-piece is homeomorphic to $\mathbb{C}$, in other words we require $\partial S \neq \emptyset$ for every $\mathcal{M}$-piece $S$. If $(G, \mathcal{Q})$ has no Thurston obstructions, then $(G, \mathcal{Q})$ is $c$-equivalent to a holomorphic model map.

### 3.3 Proof of Theorem 1.1 using Theorem 3.5

**Lemma 3.6.** Assume that $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a quasi-regular sub-hyperbolic semi-rational map with $\mathcal{P}'_G \neq \emptyset$ and without Thurston obstructions. Then there are puzzle surfaces $L_1, L_0$ such that

$$K_G \subset L_1 \subset L_0, \ G^{-1}(L_0) = L_1$$

and that, the restriction $G|_{L_1} : L_1 \to L_0$, marked by $\mathcal{P}_G \cap L_0$, is a postcritically finite repelling system without Thurston obstructions.

**Proof.** One can find an open set $U_0$ which is the union of a quasi-disc neighborhood for each point of $\mathcal{P}'_G$ so that these quasi-discs have disjoint closures, that $\partial U_0$ is disjoint from $\mathcal{P}_G$, that $G$ is holomorphic in a neighborhood of $U_0$, and that $G(U_0) \subset \subset U_0$. Set $L_0 = U_0^c$. Topologically $L_0$ is the sphere minus finitely many (open) holes. Set $L_1 = G^{-1}(L_0)$, $H = G|_{L_1}$ and $\mathcal{Q} := \mathcal{P}_G \cap L_0$. So $(H, \mathcal{Q})$ is a marked postcritically finite repelling system. The assumption $\mathcal{P}'_G \neq \emptyset$ implies that $\partial L_0 \neq \emptyset$.

We will show now: under the assumption that $G : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ has no Thurston obstructions, the repelling system $(H, \mathcal{Q})$ has no Thurston obstructions.

Assume at first that $L_0$ is a closed disc containing at most one point of $\mathcal{Q}$. In this case $\partial L_0$ is a single curve and is null-homotopic or peripheral within $L_0 \setminus \mathcal{Q}$. And there is no multicurve within $L_0 \setminus \mathcal{P}$ as every curve in $L_0 \setminus \mathcal{P}$ is either null-homotopic or peripheral. Consequently $(H, \mathcal{Q})$ has no Thurston obstructions.

Next assume that $L_0$ is a closed annulus disjoint from $\mathcal{Q}$. Then there is only one class of non-peripheral Jordan curves within $L_0 \setminus \mathcal{Q} = L_0$, namely that of a boundary curve $\gamma$ of $L_0$. But such a $\gamma$ is also non-peripheral within $\overline{\mathbb{C}} \setminus \mathcal{P}_G$ as each of the two discs of $\overline{\mathbb{C}} \setminus L_0$ contains points of $\mathcal{P}'_G \subset \mathcal{P}_G$. The curves in $G^{-1}(\gamma)$ are either peripheral within $L_0$, or homotopic to $\gamma$ within $L_0$. Therefore $\{\gamma\}$ is stable for both $(\overline{\mathbb{C}} \xrightarrow{G} \overline{\mathbb{C}}, \mathcal{P}_G)$ and $(H, \mathcal{Q})$ and the corresponding two transition matrices are identical. By the assumption that $(\overline{\mathbb{C}} \xrightarrow{G} \overline{\mathbb{C}}, \mathcal{P}_G)$ has no Thurston obstructions.
obstructions, the corresponding leading eigenvalue is less than 1. Therefore \( \{ \gamma \} \) is not an Thurston obstruction for \((H, Q)\). And \((H, Q)\) has no obstructions.

In the remaining case, \( L_0 \) is a bordered Riemann surface, with
\[
\#(L_0 \cap Q) + \#\{ \text{boundary curves of } L_0 \} \geq 3.
\]
In particular each of its boundary curves is non-peripheral within \( L_0 \setminus Q \).

Let now \( \Gamma \) be a multicurve within \( L_0 \setminus Q \). In other words,

a) each curve in \( \Gamma \) is non-peripheral within \( L_0 \setminus Q \),

b) the curves in \( \Gamma \) are mutually disjoint,

c) the curves in \( \Gamma \) are mutually non-homotopic within \( L_0 \setminus Q \).

We want to show that \( \Gamma \) is also a multicurve within \( \overline{\mathbb{C}} \setminus \mathcal{P}_G \), i.e. \( \Gamma \) satisfies a), b) and c) with \( L_0 \setminus Q \) replaced by \( \overline{\mathbb{C}} \setminus \mathcal{P}_G \). By a), for each curve \( \gamma \) in \( \Gamma \), either both discs of \( \overline{\mathbb{C}} \setminus \gamma \) contains a component of \( \overline{\mathbb{C}} \setminus L_0 = U_0 \) (therefore infinitely many points of \( \mathcal{P}_G \)); or one disc of \( \overline{\mathbb{C}} \setminus \gamma \) is contained in \( L_0 \) and contains at least two points of \( Q \subseteq \mathcal{P}_G \), while the other disc contains all components of \( U_0 \) (therefore infinitely many points of \( \mathcal{P}_G \)). In both cases each component of \( \overline{\mathbb{C}} \setminus \gamma \) contains at least two points of \( \mathcal{P}_G \). So \( \gamma \) is non-peripheral within \( \overline{\mathbb{C}} \setminus \mathcal{P}_G \).

By b) the curves in \( \Gamma \) are mutually disjoint.

By c), given any two curves \( \gamma, \gamma' \) of \( \Gamma \), we have \( \gamma, \gamma' \subset L_0 \setminus Q \) and the open annulus \( A(\gamma, \gamma') \) bounded by \( \gamma, \gamma' \) intersects either \( U_0 \) or \( Q \subset \mathcal{P}_G \) (or both). In the former case \( A(\gamma, \gamma') \) contains a component of \( U_0 \). Therefore in both cases \( A(\gamma, \gamma') \) intersects \( \mathcal{P}_G \). So \( \gamma, \gamma' \) are also non-homotopic within \( \overline{\mathbb{C}} \setminus \mathcal{P}_G \).

This arguments implies that \( \Gamma \) is also a multicurve within \( \overline{\mathbb{C}} \setminus \mathcal{P}_G \).

Assume now that \( \Gamma \) is a multicurve within \( L_0 \setminus Q \) and is \((H, Q)\)-stable. In other words, for any \( \gamma \in \Gamma \) and any curve \( \delta \) of \( G^{-1}(\gamma) \), either \( \delta \) bounds a disc that is contained in \( L_0 \) and that contains at most one point of \( Q = L_0 \cap \mathcal{P}_G \), or \( \delta \) is homotopic within \( L_0 \setminus Q \) to a curve \( \gamma' \) in \( \Gamma \). Thus either \( \delta \) bounds a disc that contains at most one point of \( \mathcal{P}_G \), or it is homotopic within \( \overline{\mathbb{C}} \setminus \mathcal{P}_G \) to \( \gamma' \).

This shows that \( \Gamma \) is a multicurve within \( \overline{\mathbb{C}} \setminus \mathcal{P}_G \) that is also \( (\overline{\mathbb{C}} \xrightarrow{G} \overline{\mathbb{C}}, \mathcal{P}_G) \)-stable. The two transition matrices (by \((H, Q)\) and by \((\overline{\mathbb{C}} \xrightarrow{G} \overline{\mathbb{C}}, \mathcal{P}_G)\)) are identical, therefore have the same leading eigenvalue \( \lambda \).

By the assumption that \( (\overline{\mathbb{C}} \xrightarrow{G} \overline{\mathbb{C}}, \mathcal{P}_G) \) has no Thurston obstructions, we know that \( \lambda < 1 \). So \( \Gamma \) is not a Thurston obstruction for \((H, Q)\).

Therefore \((H, Q)\) has no obstructions. \( \square \)

Assuming Theorem 3.5, we may now give the

Proof of Theorem 1.1 (the existence part). Let \( G \) be sub-hyperbolic semi-rational map with \( \mathcal{P}_G \neq \emptyset \) and without Thurston obstruction. We may assume in addition that \( G \) is globally quasi-regular, up to a change of representatives in its c-equivalence class (by Lemma 2.1).

We may then apply Lemma 3.6 to \( G \) to show that it has a restriction near \( K_G \) which is a postcritically finite repelling system without Thurston obstructions, therefore is c-equivalent to a holomorphic model by Theorem 3.5. We may then apply Proposition 2.4 to conclude that \( G \) is c-equivalent to a rational map. \( \square \)
4 Reduction to a sub-repelling system

Our main objective here is:

**Theorem 4.1.** Let \((\mathcal{B} \subseteq \mathcal{M}, \mathcal{Q})\) be a marked postcritically finite repelling system such that every \(\mathcal{M}\)-piece has a non-empty boundary.

Let \(\mathcal{E}, \mathcal{L}\) be two puzzle surface neighborhoods of \(K_G\) satisfying:

\((*)\) \(K_G \subsetneq \mathcal{E} \subsetneq \mathcal{L} \subsetneq \mathcal{M}, \mathcal{E} = G^{-1}(\mathcal{L}), \mathcal{Q} \cap \partial \mathcal{L} = \emptyset;\)

\((**)\) for every \(\mathcal{L}\)-piece \(S\), and for the \(\mathcal{M}\)-piece \(S_0\) containing \(S\) in the interior, and for the copy \(\overline{\mathcal{C}}\) of the Riemann sphere containing \(S_0\) (therefore \(S\)), every (disc-like) component of \(\overline{\mathcal{C} \setminus S}\) contains either components of \(\partial S_0\) or points of \(\mathcal{Q}\) (or both);

Let \(F = G|_{\mathcal{E}} : \mathcal{E} \to \mathcal{L}\) be the sub-repelling system marked by \(\mathcal{Q} \cap \mathcal{L}\). Then,

(A) If \((F, \mathcal{Q} \cap \mathcal{L})\) is \(c\)-equivalent to a holomorphic model so is \((G, \mathcal{Q})\).

(B) If \((G, \mathcal{Q})\) has no Thurston obstructions, so does \((F, \mathcal{Q} \cap \mathcal{L})\).

\[ G \text{ has no ob. } \quad G \sim_c \text{ hol. model } \]
\[ \downarrow \text{(B)} \quad \uparrow \text{(A)} \]
\[ F \text{ has no ob. } \quad \Rightarrow \quad F \sim_c \text{ hol. model.} \]

Once the theorem is proved the problem of \(c\)-equivalence to a holomorphic model for \((G, \mathcal{Q})\) is reduced the problem for a suitable sub-repelling system.

4.1 How to get a stable multicurve

The following criterion is very useful:

**Lemma 4.2.** A marked postcritically finite repelling system \((\mathcal{E} \xrightarrow{\mathcal{F}} \mathcal{L}, \mathcal{P})\) has a Thurston obstruction if and only if there is a multicurve \(\Gamma\) within \(\mathcal{L} \setminus \mathcal{P}\) (not necessarily \((\mathcal{F}, \mathcal{P})\)-stable) with \(\lambda(W_\Gamma) \geq 1\).

**Proof.** We will need to produce an \((\mathcal{F}, \mathcal{P})\)-stable multicurve with the leading eigenvalue of its transition matrix greater than one.

Let \(\Gamma\) be a multicurve within \(\mathcal{L} \setminus \mathcal{P}\), i.e. the curves in \(\Gamma\) are non-peripheral, mutually disjoint and mutually non-homotopic within \(\mathcal{L} \setminus \mathcal{P}\). The action of \((\mathcal{F}, \mathcal{P})\) induces a directed graph \(\Lambda_\Gamma\) as follows: the vertices are the curves in \(\Gamma\). And there is an edge directing from \(\delta\) to \(\gamma\) (maybe \(\delta = \gamma\)) if \(\delta\) is homotopic to a curve in \(F^{-1}(\gamma)\) within \(\mathcal{L} \setminus \mathcal{P}\).

If \(A \subset \Gamma\) is a sub-multicurve then \(W_A\) is the submatrix of \(W_\Gamma\) corresponding to the entries of \(A\), denoted by \(W_\Gamma|_A\), and \(\Lambda_A\) is the corresponding subgraph of \(\Lambda_\Gamma\). In this case \(\lambda(W_A) \leq \lambda(W_\Gamma)\) (by Lemma [A.3]).

We will say that a multicurve \(A\) within \(\mathcal{L} \setminus \mathcal{P}\) is irreducible if any two vertices of its graph \(\Lambda_A\) can be connected by following successively the directed edges. It is elementary that if \(\lambda(W_\Gamma) > 0\) then there is an irreducible sub-multicurve \(A \subset \Gamma\) with \(\lambda(W_A) = \lambda(W_\Gamma)\).
Each multicurve \( \Gamma \) defines a pulled-back multicurve \( \Gamma_1 \) as follows: The curves in \( \bigcup_{\gamma \in \Gamma} F^{-1}(\gamma) \) are already mutually disjoint. But some of them might be peripheral, or homotopic to another within \( \mathcal{L} \setminus \mathcal{P} \). Pick one representative in each homotopic class of the non-peripheral curves in \( F^{-1}(\Gamma) \). Together they form a new multicurve, \( \Gamma_1 \).

In general a curve in \( \Gamma_1 \) might not be homotopically disjoint from a curve in \( \Gamma \).

Saying that \( \Gamma \) is stable is equivalent to say that every curve in \( \Gamma_1 \) is homotopic to a curve in \( \Gamma \).

In case that every vertex of \( \Lambda_\Gamma \) is the departure of an edge (for example when \( \Gamma \) is irreducible), then every curve in \( \Gamma \) is homotopic to a curve in \( \Gamma_1 \). One obtains then successive pulled back multicurves \( \Gamma_2, \ldots \), such that curves in \( \Gamma_i \) are homotopic to curves in \( \Gamma_{i+1} \). As \( \mathcal{L} \setminus \mathcal{P} \) is topologically finite, there is \( N \) such that \( \#\Gamma_N = \#\Gamma_{N+1} \). Consequently \( \Gamma_N \) is \((F,\mathcal{P})\)-stable.

Assume now \( \Gamma' \) is a multicurve within \( \mathcal{L} \setminus \mathcal{P} \), not necessarily stable, such that \( \lambda(W_{\Gamma'}) \geq 1 \).

There there is an irreducible sub-multicurve \( A \subset \Gamma' \) with \( \lambda(W_A) = \lambda(W_{\Gamma'}) \). Pulling back successively \( A \) we get a multicurve \( A_N \) that is \((F,\mathcal{P})\)-stable and contains \( A \). And \( 1 \leq \lambda(W_{\Gamma'}) = \lambda(W_A) \leq \lambda(W_{A_N}) \).

\[ \Box \]

4.2 Proof of Theorem 4.1

Proof of Theorem 4.1

(A). This part can be proved similarly as Proposition 2.4. We will omit the details here.

(B). Note that Condition (**) assures a certain minimality of \( \mathcal{L} \), so that \( \mathcal{L} \) does not have trivial holes in \( \mathcal{M} \).

Set \( \mathcal{P} = Q \cap \mathcal{L} \), the marked set of \( F \).

At first we prove the following facts that will be used frequently in the sequel:

(a) Two Jordan curves \( \gamma, \gamma' \) in \( \mathcal{L} \setminus \mathcal{P} \), homotopic within \( \mathcal{L} \setminus \mathcal{P} \), are also homotopic within \( \mathcal{M} \setminus \mathcal{Q} \).

(b) For a Jordan curve \( \gamma \) in \( \mathcal{L} \setminus \mathcal{P} \), it is null-homotopic within \( \mathcal{L} \setminus \mathcal{P} \) iff it is null-homotopic within \( \mathcal{M} \setminus \mathcal{Q} \).

(c) A peripheral curve within \( \mathcal{L} \setminus \mathcal{P} \) is also peripheral within \( \mathcal{M} \setminus \mathcal{Q} \).

Proof. (a). There is an \( \mathcal{L} \)-piece \( S \) containing both \( \gamma, \gamma' \) and \( \gamma \) and \( \gamma' \) are homotopic within \( S \setminus \mathcal{P} \). But \( S \) is contained in an \( \mathcal{M} \)-piece \( S_0 \), and \( \mathcal{P} \cap S = Q \cap \mathcal{L} \cap S = Q \cap S \). So \( S \setminus \mathcal{P} \subset S_0 \setminus \mathcal{Q} \) and \( \gamma \) and \( \gamma' \) are homotopic within \( S_0 \setminus \mathcal{Q} \), therefore within \( \mathcal{M} \setminus \mathcal{Q} \).

(b). Again let \( S \) (resp. \( S_0 \)) be the \( \mathcal{L} \)-piece (resp. \( \mathcal{M} \)-piece) containing \( \gamma \).

If \( \gamma \) is null-homotopic within \( \mathcal{L} \setminus \mathcal{P} \) then it bounds a disc \( D \) contained in \( S \setminus \mathcal{P} \). But \( S \setminus \mathcal{P} \subset S_0 \setminus \mathcal{Q} \). So \( D \subset S_0 \setminus \mathcal{Q} \) and \( \gamma \) is null-homotopic within \( S_0 \setminus \mathcal{Q} \), therefore within \( \mathcal{M} \setminus \mathcal{Q} \).

Conversely if \( \gamma \) is null-homotopic within \( \mathcal{M} \setminus \mathcal{Q} \), it bounds an open disc \( D \) contained in \( S_0 \setminus \mathcal{Q} \). If \( D \) is not contained in \( S \), and as \( \gamma = \partial D \subset S \), we must have that \( D \) contains a component of \( \overline{\mathcal{C}} \setminus S \). By Condition (**) we know that \( D \) intersects \( \partial S_0 \cup \mathcal{Q} \). This contradicts the fact that \( D \subset S_0 \setminus \mathcal{Q} \). Therefore \( D \subset S \). But \( D \cap \mathcal{Q} = \emptyset \). So \( D \subset S \setminus \mathcal{Q} = S \setminus \mathcal{P} \). Therefore \( \gamma \) is also null-homotopic within \( S \setminus \mathcal{P} \), hence within \( \mathcal{L} \setminus \mathcal{P} \).
(c). By definition \( \gamma \) is peripheral within \( L \setminus P \) if it bounds a disc \( D \) that is contained in an \( L \)-piece \( S \) such that \( \#D \cap P = 1 \). In this case \( D \) is also contained in the \( M \)-piece \( S_0 \) that contains \( S \), and \( \#D \cap Q = \#D \cap (Q \cap S) = \#D \cap P = 1 \). By definition again \( \gamma \) is also peripheral within \( M \setminus Q \). 

Assume now that \( (G, Q) \) has no Thurston obstructions. We will prove that \( (F, P) \) has no obstructions either.

Let \( T \) be a multicurve within \( L \setminus P \), i.e.:

i) each curve in \( T \) is non-peripheral within \( L \setminus P \);

ii) the curves in \( T \) are mutually disjoint;

iii) the curves in \( T \) are mutually non-homotopic within \( L \setminus P \).

Change the representatives within the same homotopy classes if necessary, we may assume in addition:

iv) a curve in \( T \) is either equal to a boundary curve of \( L \), or, is disjoint from \( \partial L \) and is not homotopic to a curve in \( \partial L \) within \( L \setminus P \).

When considering homotopy within \( M \setminus Q \) (which contains \( L \setminus P \)), there are two new phenomena:

1. Some of the curves in \( T \) may now become peripheral (but never null-homotopic) within \( M \setminus Q \). (Figure 1 shows how this may happen).

2. Some of the curves in \( T \) may now become homotopic to each other within \( M \setminus Q \).

The following two parts treat each phenomenon separately:

1. **Curves in \( T \) that become peripheral within \( M \setminus Q \).**

We now consider homotopy within \( M \setminus Q \). By (b) each curve in \( T \) is non-null-homotopic within \( M \setminus Q \), is therefore either peripheral or non-peripheral within \( M \setminus Q \). We thus decompose \( T \) into \( T = Z \sqcup X \), with \( Z \) (resp. \( X \)) denoting the collection of curves in \( T \) that are peripheral (resp. non-peripheral) within \( M \setminus Q \). Denote by \( W_Z, W_X \) the \( (F, P) \)-transition matrix of \( Z \), \( X \) respectively.

**Lemma I.** The \( (F, P) \)-transition matrix \( W_T \) has the following block decomposition (where \( O \) denotes a rectangle zero-matrix): \( W_T = \begin{pmatrix} W_X & O \\ * & W_Z \end{pmatrix} \).

Proof. Let \( \gamma \in Z \) and \( \beta \) be a curve of \( F^{-1}(\gamma) = G^{-1}(\gamma) \). We just need to show that if \( \beta \) is homotopic within \( L \setminus P \) to a curve \( \gamma' \) in \( T \), then \( \gamma' \in Z \).
By definition of $Z$ the curve $\gamma$ bounds an open disc $D(\gamma)$ contained in $\mathcal{M}$ and containing a unique point of $Q$. Therefore, applying the Basic pull-back principle to $D(\gamma)$, we know that each component of $G^{-1}(D(\gamma))$ is disc-like, is contained in $\mathcal{B} = G^{-1}(\mathcal{M})$, and contains at most one point of $Q$. Let $\beta$ be a curve in $G^{-1}(\gamma)$. Then $\beta$ is the boundary of a component of $G^{-1}(D(\gamma))$. So $\beta$ is either null-homotopic or peripheral (within $\mathcal{M}\setminus Q$). On the other hand, $\beta$ is homotopic within $\mathcal{L}\setminus \mathcal{P}$ to $\gamma' \in T$ by assumption. And the homotopy can be taken within $\mathcal{M}\setminus Q$ by (a). Consequently $\gamma'$ is either null-homotopic or peripheral within $\mathcal{M}\setminus Q$.

But no curves in $T$ are null-homotopic within $\mathcal{M}\setminus Q$ (by (b) and by the definition of $T$), so both $\beta$ and $\gamma'$ are peripheral within $\mathcal{M}\setminus Q$. Therefore $\gamma' \in Z$. \hfill $\square$

**Lemma II.** Each curve of $Z$ is a boundary curve of $\mathcal{L}$.

**Proof.** Let $\gamma \in T$ that is not a boundary curve of $\mathcal{L}$. We just need to show that $\gamma$ is necessarily non-peripheral within $\mathcal{M}\setminus Q$.

There is an $\mathcal{L}$-piece $S$, an $\mathcal{M}$-piece $S_0$ and a Riemann sphere $\overline{\mathcal{C}}$ such that $\gamma$ is contained in the interior of $S$ and $S \subset S_0 \subseteq \overline{\mathcal{C}}$. Now $\overline{\mathcal{C}} \setminus \gamma$ has two disc components $D_1, D_2$.

By i) either one disc, say $D_1$, is contained in $S$ and contains at least two points of $\mathcal{P} \subset Q$, or both $D_1, D_2$ intersect $\overline{\mathcal{C}} \setminus S$.

In the former case, $D_1 \subset S \subset S_0$, so $D_2$ contains $\partial S_0$, which by assumption is non-empty. This implies that $\gamma$ is non-peripheral within $S_0 \setminus Q$.

In the latter case, as $\gamma \subset S$, each $D_i$ contains a component of $\overline{\mathcal{C}} \setminus S$. By Condition (**), each $D_i$ contains either a curve in $\partial S_0$ or points in $Q$ (or both). Assume by contradiction that $\gamma$ is peripheral within $S_0 \setminus Q$. Then one of $D_1, D_2$, denoted by $D$, contains a unique point of $Q$ and no boundary component of $S_0$. We have $D \subset S_0$. But $\gamma$ is not peripheral within $S \setminus \mathcal{P}$ by i). So $D$ is not contained in $S$. Therefore $D$ contains components of $\overline{\mathcal{C}} \setminus S$. Let $\Delta$ be one component of $\overline{\mathcal{C}} \setminus S$ contained in $D$. Then $\Delta$ is bounded by a curve $\delta$ which is also a boundary curve of $S$. By Condition (** again) $\Delta$ must intersect $\partial S_0 \cup Q$. But $\Delta \subset D \subset S_0$. So $\Delta \cap \partial S_0 = \emptyset$. On the other hand, $D \cap Q$ consists of a single point, denoted by $a$. So $a \in \Delta$. This shows that $D$ contains a unique component of $\overline{\mathcal{C}} \setminus S$. By iv) $\gamma \cap \partial S = \emptyset$, so $D$ contains a unique boundary curve of $S$, which must be $\delta$. Furthermore the annulus $A(\gamma, \delta)$ does not contain $a$. Therefore $A(\gamma, \delta) \subset S$ and

$$A(\gamma, \delta) \subset D \setminus \{a\} = D \setminus Q \subset D \setminus \mathcal{P}.$$  

So $A(\gamma, \delta) \subset S \setminus \mathcal{P}$. This means that the boundary curve $\delta$ of $S$ is homotopic to $\gamma$ within $S \setminus \mathcal{P}$. This is not possible by iv). \hfill $\square$

**Lemma III.** There is a power $q \geq 1$ such that $(W_Z)^q = O$. Therefore $\lambda(W_Z) = 0$ and $\lambda(W_X) = \lambda(W_T)$.

**Proof.** Let $\mathcal{G}$ be the union of $Z$ with the curves in $\partial \mathcal{L}$ that are peripheral within $\mathcal{L} \setminus \mathcal{P}$ (these added curves are disjoint from curves in $Z$ by i) and iv) ). By Claim II every curve in $\mathcal{G}$ is a boundary curve of $\mathcal{L}$.

By definition a curve $\gamma \in \partial \mathcal{L}$ is peripheral if it is the boundary of an $\mathcal{L}$-piece $D(\gamma)$ (therefore $D(\gamma)$ is a disc) with $\# D(\gamma) \cap \mathcal{P} = 1$. Note that $D(\gamma) \subset \mathcal{L} \subset \mathcal{M}$.

Let $\gamma \in \mathcal{G}$. Then $\gamma$ bounds a disc $D(\gamma)$ which is contained in $\mathcal{M}$ and which contains a unique point $a(\gamma)$ of $Q$.

We decompose $\mathcal{G}$ into $\mathcal{G}_p \sqcup \mathcal{G}_0$ according to $a(\gamma)$ is periodic or not.

By Basic pull-back principle, for all $k \geq 1$, the components of $G^{-k}(D(\gamma))$ are all disc-like, and each is bounded by exactly one curve of of $G^{-k}(\gamma)$. 
Assume $\gamma \in G_0$, so that $a(\gamma)$ is not periodic (as $G$ is postcritically finite, the orbit of $a(\gamma)$ is either preperiodic or eventually escapes $B$). Then there is an integer $k(\gamma) \geq 1$ such that $G^{-k(\gamma)}(a(\gamma))$ contains no points of $Q$. But $G^{-k(D(\gamma))} \cap Q \subset G^{-k(\gamma)}(a(\gamma))$. So $G^{-k(\gamma)}(D(\gamma)) \cap Q = \emptyset$. Thus the curves in $G^{-k(\gamma)}(\gamma)$ are null-homotopic within $\mathcal{M} \setminus Q$ and hence are null-homotopic within $\mathcal{L} \setminus \mathcal{P}$ by (b). Therefore for all $k \geq k(\gamma)$, the curves in $G^{-k}(\gamma)$ are all null-homotopic within $\mathcal{L} \setminus \mathcal{P}$.

There is therefore a common integer $k$, so that for every $\gamma \in G_0$, the curves in $G^{-k}(\gamma)$ are all null-homotopic within $\mathcal{L} \setminus \mathcal{P}$.

Let now $\gamma \in G_p$, i.e. with $a(\gamma)$ periodic. Set $a = a(\gamma)$. Denote by $p$ its period. This implies in particular that the orbit of $a$ does not escape $\mathcal{M}$, so $a \in K_G = K_F \subset \mathcal{L}$.

Denote by $\{\eta_1, \ldots, \eta_m\}$ the curves in $\mathcal{G}$ homotopic to $\gamma$ within $\mathcal{M} \setminus Q$, i.e. each $\eta_j$ bounds a disc $D(\eta_j)$ which is contained in $\mathcal{M}$ with $D(\eta_j) \cap Q = \{a\}$. As $\eta_i \cap \eta_j = \emptyset$ for $i \neq j$, we have either $D(\eta_i) \subset D(\eta_j)$ or $D(\eta_i) \supset D(\eta_j)$. We number the $\eta_j$’s in the increasing order, i.e. such that $D(\eta_j)$ contains $D(\eta_{j-1})$ and $\eta_{j-1}$. The smallest disc $D(\eta_1)$ must be contained in $\mathcal{L}$, since there is no curve in $\partial \mathcal{L}$ separating $a \in \mathcal{L}$ from $\eta_1 \subset \partial \mathcal{L}$. Therefore $\overline{D}(\eta_1)$ is an $\mathcal{L}$-piece, and $\eta_1$ is peripheral within $\mathcal{L} \setminus \mathcal{P}$.

Fix $j \in \{1, \ldots, m\}$. The components of $G^{-p}(D(\eta_j))$ are all disc-like, with one of them, denoted by $D(\beta_j)$, containing $a$, and the others containing a preimage of $a$ that is not periodic.

Therefore $G^{-p}(\eta_j) = F^{-p}(\eta_j)$ has a unique component $\beta_j$, which is the boundary of $D(\beta_j)$, homotopic to $\gamma$ (and to $\{a\}$) within $\mathcal{M} \setminus Q$, and the other components are either null-homotopic or homotopic within $\mathcal{L} \setminus \mathcal{P}$ to a curve in $G_0$.

**Claim.** $\overline{D}(\beta_j) \subset D(\eta_j)$ and $\overline{D}(\beta_j) \subset D(\eta_{j-1})$ for $j = 2, \ldots, m$.

Proof. At first the enlarged collection of curves $\{\eta_1, \ldots, \eta_m, \beta_1, \ldots, \beta_m\}$ are mutually disjoint. This is clearly true between the $\eta_j$’s and between de $\beta_j$’s. But $\beta_j \cap \eta_i = \emptyset$ as well, as $\eta_i \subset \partial \mathcal{L}$, $\beta_j \subset F^{-p}(\mathcal{L})$ and $F^{-p}(\mathcal{L})$ is contained in the interior of $\mathcal{L}$.

Therefore the discs $D(\eta_j), D(\beta_j)$ are nested in a certain order.

We prove now that $\beta_j \subset D(\eta_j)$ for $j = 1, \ldots, m$.

We prove it at first for $\beta_1$. Note that $\beta_1$ is the boundary of $E_1$, the component of $F^{-p}(\overline{D}(\eta_1))$ containing $a$. But $F^{-p}(\mathcal{L})$ is contained in the interior of $\mathcal{L}$, so $E_1$ is contained in the interior of an $\mathcal{L}$-piece, which must be $\overline{D}(\eta_1)$, i.e. $\beta_1 \subset D(\eta_1)$.

Assume by contradiction that there is a minimal integer $j \geq 2$ such that $\beta_j \subset D(\eta_j)$. Then $\beta_j \cap \partial \mathcal{L} = \emptyset$ (due to again, $F^{-p}(\mathcal{L}) \cap \partial \mathcal{L} = \emptyset$). The annulus $A(\eta_j, \eta_{j-1})$ is contained in $D(\eta_j)$ therefore in $\mathcal{M}$. So its inverse images by $G$ are well defined. By the Basic pull-back principle the components of $G^{-p}(A(\eta_j, \eta_{j-1}))$ are all annuli. One of them must be $A(\beta_j, \beta_{j-1})$, which contains $A(\eta_j, \eta_{j-1})$ as a sub-annulus. Therefore

$$G^m(A(\eta_j, \eta_{j-1})), \ m = 0, \ldots, p-1$$

are all contained in $\mathcal{B} = G^{-1}(\mathcal{M})$. Set $A' = \bigcup_{m=0}^{p-1} G^m(A(\eta_j, \eta_{j-1}))$. Then $G(A') \subset A'$. Trivially either $A(\eta_j, \eta_{j-1})$ is an $\mathcal{L}$-piece or there is a point $z \in A(\eta_j, \eta_{j-1}) \cap (\mathcal{B} \setminus \mathcal{L})$. The former case is not possible, as components of $F^{-p}(\mathcal{L})$ are compactly contained in $\mathcal{L}$. In the latter case, the $G$-orbit of $z$ never escapes $A'$ which is a subset of $\mathcal{B}$, so $z \in K_G$. This is again impossible as $K_G = K_F \subset \mathcal{L}$ and $z \notin \mathcal{L}$.

This proves that $\beta_j \subset D(\eta_j)$ for all $j = 1, \ldots, m$. It follows that $\overline{D}(\beta_j) \subset D(\eta_j)$.

Fix $j = 1, \ldots, m$. Denote by $S_j$ the $\mathcal{L}$-piece containing $\eta_j$ as a boundary curve. Thus $S_1 = D(\eta_1)$. We want to show now for $j \geq 2$ either $S_j \cap D(\eta_j) = \emptyset$ or $S_j = A(\eta_j, \eta_{j-1})$.
Assume $S_j \cap D(\eta_j) \neq \emptyset$. We have $S_j \subset \overline{\mathcal{D}j}\{a\}$. Then $\eta_{j-1} \subset \partial S_j$ since no curve in $\partial \mathcal{L}$ separates $\eta_j$ from $\eta_{j-1}$. So $S_j \subset \overline{\mathcal{D}(\eta_j,\eta_{j-1})} \subset \mathcal{M}\mathcal{Q}$. But $S_j$ cannot have other boundary curves due to Condition (**). Therefore $S_j = \overline{\mathcal{A}(\eta_j,\eta_{j-1})}$.

It follows $S_2 = S_3 = \overline{\mathcal{A}(\eta_2,\eta_3)}$, and more generally $S_j = S_{j+1} = \overline{\mathcal{A}(\eta_j,\eta_{j+1})}$ for any even number $j$ with $2 \leq j < m$.

Fix $j = 1, \cdots, m$. We have $\beta_j \subset F^{-p}(\mathcal{L}) \subset \mathcal{L}$. Let $S'$ be the $\mathcal{L}$-piece containing $\beta_j$. We want to show that $S'$ is one of $S_i$.

If $\beta_j \subset D(\eta_1)$ then $S' = D(\eta_1) = S_1$. Otherwise $S'$ has a boundary component $\eta$ separating $a$ from $\beta$. Therefore $\eta$ is one of $\eta_i$ and $S' = S_i$.

Let now $j$ be an even number with $2 \leq j < m$. We know that $\beta_j \subset D(\eta_j)$ and $\beta_j \subset S_i$ for some $i$. Therefore $S_i \subset \overline{D(\eta_{j-1})}$. But $\beta_j \cap \eta_{j-1} = \emptyset$. So $\beta_j \subset D(\eta_{j-1})$. Furthermore, $\overline{\mathcal{A}(\beta_j,\beta_{j+1})}$ is a component of $F^{-p}(S_j)$, so must be contained entirely in the $\mathcal{L}$-piece $S_i$. Therefore $\beta_{j+1} \subset \overline{D(\eta_{j-1})}$ and consequently $\beta_{j+1} \subset D(\eta_{j-1})$.

This ends the proof of the claim: $\overline{D(\beta_j)} \subset D(\eta_{j-1})$ for any $j \geq 2$.

Note that in each component of $G^{-p}(D(\eta_m))$ the $G^p$-preimages of the discs $D(\eta_i), D(\beta_j)$ are nested in the same order. There is therefore $q_j$, such that for any $n \geq q_j$, all curves of $G^{-np}(\eta_j)$ are either null-homotopic or homotopic within $\overline{D(\eta_j)}\{a\}$, therefore within $\mathcal{L}\mathcal{P}$, to $\eta_1$, which is a curve in $\partial \mathcal{R}$.

Combining these arguments together, we find a $q$, such that every curve in $F^{-q}(\mathcal{G})$ is either null-homotopic or homotopic within $\mathcal{L}\mathcal{P}$ to a curve in $\partial \mathcal{R}$. So $(W_Z)^q = O$.

Therefore $\lambda(W_Z) = 0$ and $\lambda(W_X) = \lambda(W_T)$. 

2. Curves in $X$ that are homotopic to each other within $\mathcal{M}\mathcal{Q}$. Now we decompose $X = T\setminus Z$ into $X_1 \cup \cdots \cup X_k$ according to the homotopy within $\mathcal{M}\mathcal{Q}$, i.e., two curves in $X$ are homotopic within $\mathcal{M}\mathcal{Q}$ if and only if they belong to some subset $X_i$. Pick one curve $\gamma_i$ in each $X_i$ and set $\Gamma := \{\gamma_1, \cdots, \gamma_k\}$. Clearly $\Gamma$ is a multicurve within $\mathcal{M}\mathcal{Q}$.

Set $D_T := (b_{ij})$ the $(G, \mathcal{Q})$-transition matrix of $\Gamma$. Set $W_X := (a_{\delta \gamma})$. By definition

$$b_{ij} = \sum_{\alpha \in \mathcal{G}_{ij}} \deg(G : \alpha \rightarrow \gamma_j) \quad \text{and} \quad a_{\delta \gamma} = \sum_{\alpha \in \mathcal{F}_{\delta \gamma}} \deg(F : \alpha \rightarrow \beta),$$

where $\mathcal{G}_{ij}$ is the collection of curves in $G^{-1}(\gamma_j)$ homotopic to $\gamma_i$ within $\mathcal{M}\mathcal{Q}$; $\mathcal{F}_{\delta \gamma}$ is the collection of curves in $F^{-1}(\beta)$ homotopic to $\delta$ within $\mathcal{L}\mathcal{P}$. We claim that,

$$\forall \ i, j \in \{1, \cdots, k\}, \ \forall \ \delta \in X_j, \ \sum_{\delta \in X_i} a_{\delta \gamma} \leq b_{ij}.$$

Assume at first $\beta = \gamma_j$. We have

$$\bigcup_{\delta \in X_i} \mathcal{F}_{\delta \gamma} = \{\eta \in F^{-1}(\gamma_j), \ \exists \delta \in X_i \text{ such that } \eta, \delta \text{ are homotopic within } \mathcal{L}\mathcal{P}\} \subset \{\eta \in F^{-1}(\gamma_j), \ \exists \delta \in X_i \text{ such that } \eta, \delta \text{ are homotopic within } \mathcal{M}\mathcal{Q}\} = \{\eta \in G^{-1}(\gamma_j), \ \exists \delta \in X_i \text{ such that } \eta, \delta \text{ are homotopic within } \mathcal{M}\mathcal{Q}\} = \{\eta \in G^{-1}(\gamma_j), \ \eta, \gamma_i \text{ are homotopic within } \mathcal{M}\mathcal{Q}\} = \mathcal{G}_{ij},$$

where the inclusion is due to (a), and the second equality is due to $F^{-1}(\gamma_j) = G^{-1}(\gamma_j)$. Therefore

$$\sum_{\delta \in X_i} a_{\delta \gamma} = \sum_{\delta \in X_i} \sum_{\alpha \in \mathcal{F}_{\delta \gamma}} \frac{1}{\deg(F : \alpha \rightarrow \gamma_j)} = \sum_{\alpha \in \bigcup_{\delta \in X_i} \mathcal{F}_{\delta \gamma}} \frac{1}{\deg(F : \alpha \rightarrow \gamma_j)}.$$
This implies the claim for $\beta = \gamma_j$.

When $\beta \neq \gamma_j$, replace $\gamma_j$ by $\beta$ in $\Gamma$. The replacement does not change the transition matrix $D_{\Gamma}$. So the claim is still true.

Applying now Corollary A.6 from linear algebra, we have $\lambda(W_X) \leq \lambda(D_{\Gamma})$.

But $\lambda(D_{\Gamma}) < 1$ as $(G, Q)$ has no Thurston obstructions. Consequently $\lambda(W_{\Gamma}) = \lambda(W_X) < 1$. So $(F, \mathcal{P})$ has no Thurston obstructions.

\section{Constant complexity under pullback}

We are now searching for repelling systems with some specific properties such that on one hand we are capable to solve their problem of c-equivalence to holomorphic models, and on the other hand they appear as sub-systems of any repelling system. This leads to an important class of repelling systems: those of constant complexity. We will prove at first that every repelling system contains a sub-system that is of constant complexity. We then introduce two particular types of obstructions for this class of maps, and state our Thurston-like theorem in this setting, Theorem 5.4: a repelling system of constant complexity without these specific obstructions is c-equivalent to a holomorphic model. The proof of Theorem 5.4 will be postponed to the next sections. We conclude the present section with a proof of Theorem 5.5 using Theorem 5.4.

\subsection{Definitions}

Let $(\mathcal{E} \xrightarrow{F} \mathcal{L}, \mathcal{P})$ be a marked postcritically finite repelling system. Let $S$ be an $\mathcal{L}$-piece. We say that $S$ is \textit{simple} if either $S$ is annular with $S \cap \mathcal{P} = \emptyset$, or $S$ is disc-like with $\# S \cap \mathcal{P} \leq 1$. Otherwise we say that $S$ is \textit{complex}, i.e. if $\# \{ \text{curves in } \partial S \} + \# (S \cap \mathcal{P}) \geq 3$.

More generally, let $E \subset \mathcal{L}$ be a bordered Riemann surface. We say that $E$ is \textit{simple} if $E$ is contained in either a closed annulus $A$ in $\mathcal{L} \setminus \mathcal{P}$; or in a closed disc $D$ in $\mathcal{L}$, such that $D$ contains zero or one point of $\mathcal{P}$ in its interior, and that $\partial D \cap \mathcal{P} = \emptyset$. Otherwise we say $E$ is \textit{complex}.

Constant complexity means roughly that under the pull-back by $F$, both the number and the (homotopic) shapes of the complex $\mathcal{L}$-pieces remain stable. More precisely:

\textbf{Definition 5.} Let $(\mathcal{E} \xrightarrow{F} \mathcal{L}, \mathcal{P})$ be a marked postcritically finite repelling system. We say that $(F, \mathcal{P})$ is of \textbf{constant complexity}, if every complex $\mathcal{L}$-piece $S$, if any, contains an $\mathcal{E}$-piece $E_S$ such that $E_S \cap \mathcal{P} = S \cap \mathcal{P}$ and that the components of $S \setminus E_S$ are either annular or disc-like (this implies in particular both $\mathcal{P}_F, \mathcal{P}$ are contained in $K_F$).

Such $E_S$ is said to be $\textit{parallel}$ to $S$. One way to obtain a parallel subsurface of $S$ is as follows: first thicken the boundary of $S$ (without touching $\mathcal{P}$) to reduce $S$ to a sub-bordered surface $E'$, then dig a few open holes compactly contained in $\text{interior}(E') \setminus \mathcal{P}$, the result is a bordered surface $E$ parallel to $S$ (see Figure 2).

For an example one may take $g(z) = z^2 - 1$. Cut off a suitable neighborhood of $\mathcal{P}_g = \{ \infty, 0, -1 \}$ to obtain a puzzle neighborhood $\mathcal{L}$ of the Julia set so that $g^{-1}(\mathcal{L}) \subset \subset \mathcal{L}$. In this case $\mathcal{L}$ has only one piece $S$, which has three boundary curves, and $E_S = g^{-1}(\mathcal{L})$ has four
boundary curves. Now \( g : g^{-1}(\mathcal{L}) \to \mathcal{L} \) is a repelling system of constant complexity. For
details and further examples, see \([6,1]\).

### 5.2 Achieving constant complexity via restriction

**Theorem 5.1.** Let \((B \xrightarrow{G} \mathcal{M}, Q)\) be a marked postcritically finite repelling system with \(\partial \mathcal{M} \neq \emptyset\). Then there are two puzzle surface neighborhoods \(\mathcal{E}, \mathcal{L}\) of \(K_G\) satisfying:

\((*)\) \(K_G \subset\subset \mathcal{E} \subset\subset \mathcal{L} \subset\subset \mathcal{M}\), \(\mathcal{E} = G^{-1}(\mathcal{L})\), \(Q \cap K_G = Q \cap \mathcal{L}\);

\((**)\) for every \(\mathcal{L}\)-piece \(S\) and for the \(\mathcal{M}\)-piece \(S_0\) containing \(S\) in the interior, and for the copy \(\overline{\mathbb{C}}\) of the Riemann sphere containing \(S_0\) (therefore \(S\)), every (disc-like) component of \(\overline{\mathbb{C}\setminus S}\) contains either components of \(\partial S_0\) or points of \(Q\) (or both);

\((***)\) the sub-repelling system \(F = G|_{\mathcal{E}} : \mathcal{E} \to \mathcal{L}\), marked by \(Q \cap K_G\), is of constant complexity.

To prove the theorem, we need the following process together with its two properties:

**Hole-filled-in process.** Let \(S_0\) be a \(\mathcal{M}\)-piece. It is contained in a Riemann sphere \(\overline{\mathbb{C}}\). Let \(E \subset S_0\) be a bordered Riemann surface. The filled-in of \(E\), denote by \(\hat{E}\), is defined to be the union of \(E\) with all components of \(E^c = \overline{\mathbb{C}\setminus E}\) disjoint from \(\partial S_0 \cup Q\). Clearly, \(\hat{E} \subset S_0\) and \(\partial \hat{E} \subset \partial E\).

**Monotonicity Property.** Let \(E_1 \subset\subset E_2\) be two nested bordered connected surfaces in \(\mathcal{M}\). Then \(\hat{E}_1 \subset\subset \hat{E}_2\).

**Proof.** This property is easier to understand from their complements. There is a \(\mathcal{M}\)-piece \(S_0\) containing both \(E_1\) and \(E_2\). Note that \(E_2^c\) is a disjoint union of discs while \((\hat{E}_2)^c\) is the union of some of them which meet \(\partial S_0 \cup Q\). Because \(E_1^c \supset\supset E_2^c\), these discs are compactly contained in \(E_1^c\), which can not be thrown away under filled-in process of \(E_1\) since they meet \(\partial S_0 \cup Q\). So \((\hat{E}_1)^c \supset\supset (\hat{E}_2)^c\) and hence \(\hat{E}_1 \subset\subset \hat{E}_2\).

**Pull-back property.** Let \(S \subset \mathcal{M}\) be a bordered Riemann surface with \(\partial S \cap Q = \emptyset\), and let \(E_1\) be a component of \(G^{-1}(S)\). Let \(\hat{E}_1\) be the component of \(G^{-1}(\hat{S})\) containing \(E_1\). Then \(E_1\) is again a bordered Riemann surface, and \(\hat{E}_1 \subset \hat{E}_1\). See the following diagram:

\[
\begin{array}{c}
\Lambda_0 \subset \hat{S} \\
\Lambda_1 \subset \hat{E}_1 = \text{fill}(E_1)
\end{array}
\]

\[
\begin{array}{c}
S \quad \uparrow G \\
\hat{E}_1 \quad \uparrow G \\
E_1
\end{array}
\]

**Proof.** Let \(\Lambda_0\) be the \(\mathcal{M}\)-piece containing \(S\) and \(\Lambda_1\) be the \(\mathcal{M}\)-piece containing \(E_1\). Assume \(\Lambda_0 \subset \overline{\mathbb{C}_0}\) and \(\Lambda_1 \subset \overline{\mathbb{C}_1}\). Denote by \(\hat{E}_i\) \((1 \leq i \leq k)\) the components of \(G^{-1}(\hat{S})\). Noticing that \(\hat{S}\) is a union of disjoint open discs, is contained in \(\Lambda_0\) and is disjoint from \(Q\). Set \(V = G^{-1}(\hat{S}) \setminus G^{-1}(S) = G^{-1}(\hat{S}\setminus S)\). Then \(V\) is also a disjoint union of discs and \(\overline{V \cap (\partial \mathcal{M} \cup Q)} = \emptyset\) (by the Basic pull-back principle). These discs are contained in \(G^{-1}(\hat{S}) = \cup \hat{E}_i\) and hence can not separate any \(\hat{E}_i\), i.e. \(\hat{E}_i \setminus V\) is also connected for \(1 \leq i \leq k\). Therefore \(E_1 = \hat{E}_1 \setminus V\). Note that each component of \(\hat{E}_1\) is the union of some components of \(V\) which are discs contained in \(\Lambda_1 \setminus Q\). They are also components of \(\Lambda_1 \setminus E_1\). By definition, \(\hat{E}_1 = E_1 \cup (\hat{E}_1 \cap V) \subset \hat{E}_1\).

**Proof of Theorem 5.1.**

**Choice of \(N'\) to stabilize the postcritical set.** Clearly there is an integer \(N_0 \geq 0\) such that for all \(n \geq N_0\), we have \(L_n \cap Q = K_G \cap Q\), in other words every critical point of \(G\) in \(L_n\) is actually in \(K_G\) and is eventually periodic. For convenience we will choose \(N' \geq \max\{1, N_0\}\).
Choice of $N''$ to stabilize the homotopy classes of the boundary curves. Note that $\partial L_n \cap \partial L_m = \emptyset$ if $n \neq m$. For any integer $m \geq 0$, we consider the homotopy classes within $M \setminus Q$ of the Jordan curves in $\bigcup_{k=0}^{m} \partial L_k$. The number of these homotopy classes is weakly increasing with respect to $m$, but is uniformly bounded from above, as $Q \cup \partial M$ has only finitely many connected components. There is therefore an integer $N'' \geq N'$ such that for any $n \geq N''$, every boundary curve of $L_n$ is either null homotopic or homotopic, within $M \setminus Q$, to a curve in $\bigcup_{k=0}^{N''-1} \partial L_k$.

Filled-in for $L_n$. For any two $L_n$-pieces, their filled-in are either disjoint or one contains another. Let $\mathcal{L}_n$ be the union of the filled-in of all the $L_n$-pieces. Then each $\mathcal{L}_n$ is the filled-in of an $L_n$-piece; for every $\mathcal{L}_n$-piece $S$, each complementary component of $S$ contains points of $\partial M \cup Q$ (i.e. $\mathcal{L}_n$ satisfies Property (**)). Note that the total number of $\mathcal{L}_n$-pieces might be less than that of $L_n$-pieces, as some $L_n$-piece might be hidden in the hole of another, thus disappears in the filled-in process.

It is easy to check from the definition that if $S$ is a $G$-complex $L_n$-piece, then $\hat{S}$ is a $G$-complex $\mathcal{L}_n$-piece.

Assume that $E$ is a component of $G^{-1}(S)$ where $S$ is an $L_n$-piece. Let $\hat{E}$ be the component of $G^{-1}(\hat{S})$ containing $E$. Then $\hat{E} \subset \hat{E} \subset \mathcal{L}_n$ by the pull-back property and the monotonicity property of filled-in, as $E$ is an $L_{n+1}$-piece and hence is contained in an $L_n$-piece. Combining with the fact that each $\mathcal{L}_n$-piece is the filled-in of an $L_n$-piece, we have $G^{-1}(\mathcal{L}_n) \subset \mathcal{L}_{n+1} \subset \subset \mathcal{L}_n$. Note that $G^{-1}(\mathcal{L}_n) \cap Q = \mathcal{L}_n \cap Q = K_G \cap Q$ for all $n \geq N'$.

Choice of $N$ to stabilize the number and the shape of the complex pieces. From now on we assume $n \geq N''$. We claim that for $k \geq 1$, each non-null-homotopic (within $M \setminus Q$) curve $\gamma$ on $G^{-k}(\partial \mathcal{L}_n)$ is homotopic to a curve on $\partial \mathcal{L}_n$ within $M \setminus Q$. Note that $\gamma \subset G^{-k}(\partial \mathcal{L}_n) \subset G^{-k}(\partial L_n) = \partial L_{n+k}$. By the stability of the homotopy classes of boundary curves, there is an integer $m$ with $m < N'' \leq n$ so that $\gamma$ is homotopic (within $M \setminus Q$) to a curve $\beta$ on $\partial L_m$. Because $L_{n+k} \subset L_n \subset L_m$, there exists a curve $\alpha$ on $\partial L_n$ so that $\alpha$ separates $\beta$ from $\gamma$. So $\alpha$ is also homotopic (within $M \setminus Q$) to $\gamma$. Let $S$ be the $L_n$-piece containing $\alpha$. The fact that $\alpha$ is not null-homotopic implies that $\hat{\alpha}$ is an $\hat{S}$-piece and $\alpha \subset \partial \hat{S}$. The claim is proved.

Let $S$ be a $G$-complex $\mathcal{L}_n$-piece. Assume that $E_1$ and $E_2$ are $G^{-1}(\mathcal{L}_n)$-pieces in $S$ and that $E_1$ is $G$-complex. Then there is a closed curve $\gamma$ on $\partial E_1$ such that $\gamma$ separates $E_1 \setminus \gamma$ from $E_2$. If $\gamma$ is null-homotopic within $M \setminus Q$, then $E_2$ is simple and $E_2 \cap Q = \emptyset$. Assume that $\gamma$ is non-null-homotopic within $M \setminus Q$. From the above claim, we have a curve $\alpha$ on $\partial \mathcal{L}_n$ such that $\alpha$ is homotopic to $\gamma$ within $M \setminus Q$. Moreover, $\alpha$ can be taken on $\partial S$ since $E_1 \subset S$. Now the closed annulus enclosed by $\gamma$ and $\alpha$, denote by $A(\gamma, \alpha)$, is disjoint from $\partial M \cap Q$. It contains either $E_1$ or $E_2$ because $\gamma$ separates $E_1 \setminus \gamma$ from $E_2$. Because $E_1$ is $G$-complex, we see that $E_2 \subset A(\gamma, \alpha)$ and hence $E_2$ is $G$-simple and $E_2 \cap Q = \emptyset$.

The above argument shows that $S$ contains at most one $G^{-1}(\mathcal{L}_n)$-piece that is $G$-complex. In case that $S$ contains a $G$-complex $G^{-1}(\mathcal{L}_n)$-piece $E_S$, other $G^{-1}(\mathcal{L}_n)$-pieces in $S$ are simple and disjoint from $\partial M \cap Q$. Combining with the fact that $G^{-1}(\mathcal{L}_n) \cap Q = \mathcal{L}_n \cap Q$, we see that $E_S \cap Q = S \cap Q$.

We can show now that each component of $E_S^c$ contains at most one component of $\partial S$. Let $D$ be a component of $E_S^c$ and $\gamma = \partial D \cap \partial E_S$. If $\gamma$ is null-homotopic within $M \setminus Q$, then $D$ contains no component of $\partial S$ since each closed curve in $\partial S$ is non-null-homotopic rel $Q$. Now assume that $\gamma$ is non-null-homotopic rel $Q$. Then there is a curve $\beta$ on $\partial S$ homotopic to $\gamma$ within $M \setminus Q$. Therefore the closed annulus $A(\gamma, \beta)$ bounded by $\gamma$ and $\beta$ is contained
in $\mathcal{M}\setminus\mathcal{Q}$.

If $\beta \subset D$, then $E_S \subset A(\gamma, \beta)$. This contradicts to the fact that $E_S$ is $G$-complex. So $\beta \subset D$. Thus $A(\gamma, \beta) \subset S$ since $S$ is the filled-in of an $L_n$-piece. Therefore no other components of $\partial S$ is contained in $D$. This implies that components of $S \setminus E_S$ are either annular or disc-like.

Let $s_n$ be the number of $G$-complex $\mathcal{L}_n$-pieces. Let $t_n$ be the number of $G$-complex $G^{-1}(\mathcal{L}_n)$-pieces. Then $t_n \leq s_n$ since each $G$-complex $\mathcal{L}_n$-piece contains at most one $G$-complex $G^{-1}(\mathcal{L}_n)$-piece. We claim that $s_{n+1} \leq t_n$.

Let $\tilde{E}_1$ and $\tilde{E}_2$ be distinct $G$-complex $\mathcal{L}_{n+1}$-pieces, where $E_1$ and $E_2$ are $L_{n+1}$-pieces. Then $E_1$ and $E_2$ are also $G$-complex by the definition. Note that $L_{n+1} \subseteq G^{-1}(\mathcal{L}_n) \subset \mathcal{L}_{n+1}$. We have two distinct $G^{-1}(\mathcal{L}_n)$-pieces $\tilde{E}_1$ and $\tilde{E}_2$ such that $E_i \subset \tilde{E}_i \subset \tilde{E}_i$ ($i = 1, 2$). Again $\tilde{E}_1$ and $\tilde{E}_2$ are also $G$-complex. So $s_{n+1} \leq t_n$.

Now we have $s_{n+1} \leq s_n$. There is therefore an integer $N \geq N''$ such that $s_n \equiv s_N$ for $n \geq N$.

Define now a new repelling system $F : \mathcal{E} \to \mathcal{L}$ to be $G|_{G^{-1}(\mathcal{L}_N)} : G^{-1}(\mathcal{L}_N) \to \mathcal{L}_N$. It is postcritically finite with $P_F = P_G \cap K_G \subset \mathcal{Q} \cap K_G$, and $K_F = K_G$. Furthermore $(F, \mathcal{Q} \cap K_G)$ is of constant complexity. 

5.3 Boundary curves and complex pieces

We now turn to the study of properties of constant complexity maps.

**Lemma 5.2.** Let $(\mathcal{E}, F, \mathcal{L}, \mathcal{P})$ be a marked postcritically finite repelling system of constant complexity.

1. For any $n \geq 0$, any curve in $F^{-n}(\partial \mathcal{L})$ is either null-homotopic or homotopic to a curve in $\partial \mathcal{L}$ within $\mathcal{L}\setminus\mathcal{P}$.

2. For any complex $\mathcal{L}$-piece $S$, there is a unique $\mathcal{E}$-piece $E_S$ parallel to $S$, and $F(E_S) =: S'$ is again a complex $\mathcal{L}$-piece.

3. $F_* : S_1 \rightarrow S_2$ if $F(E_{S_1}) = S_2$ is a well defined map from the set of complex $\mathcal{L}$-pieces into itself. Every such $\mathcal{L}$-piece is eventually periodic under $F_*$. 

4. For any complex $\mathcal{L}$-piece $S$ and any integer $m \geq 1$, there is a unique $F^{-m}(\mathcal{L})$-piece $E$ in $S$ parallel to $S$. Moreover, $F^m(E)$ is again a complex $\mathcal{L}$-piece. 

Before proving it we will decompose $L$ following its topology and its intersecting property with $\mathcal{P}$. Let $S$ be a $\mathcal{L}$-piece. We say that $S$ is of $\mathcal{A}$-type if $S \cap \mathcal{P} = \emptyset$ and $S$ has exactly two boundary curves; is of $\mathcal{O}$-type if $S \cap \mathcal{P} = \emptyset$ and $S$ has exactly one boundary curve; is of $\mathcal{R}$-type if $S \cap \mathcal{P}$ is a single point and $S$ has exactly one boundary curve; is of $\mathcal{C}$-type if $\#(S \cap \mathcal{P}) + \#\{\text{boundary curves}\} \geq 3$ (see table \[\mathcal{P}\] ). Note that an $\mathcal{L}$-piece is complex iff it is a $\mathcal{C}$-piece.

We now decompose $\mathcal{L}$ into $\mathcal{C} \sqcup \mathcal{R} \sqcup \mathcal{A} \sqcup \mathcal{O}$ with $\mathcal{C}$ the union of $\mathcal{C}$-type pieces, $\mathcal{R}$ the union of $\mathcal{R}$-type pieces, $\mathcal{A}$ the union of $\mathcal{A}$-type pieces and $\mathcal{O}$ the union of $\mathcal{O}$-type pieces. It may happen that some sets among $\mathcal{O}, \mathcal{A}, \mathcal{R}, \mathcal{C}$ are empty. See examples in \[\mathcal{P}\].
Classification of \( L \)-pieces \( S \)

| \( \cap P \backslash \text{shape} \) | 1 boundary curve (disc) | 2 boundary curves (annulus) | \( \geq 3 \) boundary curves (pants, pillowcase without corners, etc.) |
|---|---|---|---|
| \( S \cap P = \emptyset \) | \( O \)-type null-homotopic | \( A \)-type non-peripheral \( \partial S = \partial_- S \cup \partial_+ S \) | \( C \)-type non-peripheral |
| \( S \cap P \neq \emptyset \) | \# \( S \cap P = 1 \) | \# \( S \cap P > 1 \) | \( C \)-type non-peripheral |
| \( \gamma \subset \partial S \) | \( \mathcal{R} \)-type peripheral \( \mathcal{C} \)-type non-peripheral | \( \mathcal{C} \)-type non-peripheral | \( \mathcal{C} \)-type non-peripheral |

Proof of Lemma 5.2 (1). Due to the basic pull-back principle we just need to prove it for \( n = 1 \). Let \( \gamma \) be a boundary curve of \( L \). Then \( \gamma \) is a boundary curve of some \( L \)-piece \( S \).

If \( S \) is of \( O \)-type, then all components of \( F^{-1}(S) \) are discs in \( L \) and are disjoint from \( P \). Therefore all curves in \( F^{-1}(\gamma) \) are null-homotopic.

Recall that by the definition of constant complexity, each \( C \)-piece \( S' \) contains a unique complex \( E \)-piece \( E_{S'} \) and \( E_{S'} \) is parallel to \( S' \).

If \( S \) is of \( A \)-type or \( \mathcal{R} \)-type, then each component \( E \) of \( F^{-1}(S) \) is contained in

\[
\mathcal{O} \cup \mathcal{A} \cup \mathcal{R} \cup \bigcup_{S' : \mathcal{C} \text{-piece}} S' \setminus E_{S'} .
\]

But each component of \( S' \setminus E_{S'} \) for a \( \mathcal{C} \)-piece \( S' \) is either an annulus or a disc, and is contained in \( L \backslash P \). So each boundary curve of \( E, \) in particular each curve of \( F^{-1}(\gamma) \), is either null-homotopic or homotopic to a curve in \( \partial L \).

Finally if \( S \) is of \( \mathcal{C} \)-type, then a component \( E \) of \( F^{-1}(S) \) is either equal to \( E_{S'} \) for some \( \mathcal{C} \)-piece \( S' \), or is contained in \( \mathcal{E} \). In any case each boundary curve of each of \( E, \) in particular each curve of \( F^{-1}(\gamma) \), is either null-homotopic or homotopic to a curve in \( \partial L \).

(2). The existence of \( E_S \) is given by the definition of constant complexity. Its uniqueness follows from the fact that components of \( S \setminus E_S \) are annular or disc-like and are disjoint from \( P \). We know that \( S' := F(E_S) \) is again an \( L \)-piece. It must be also an \( \mathcal{C} \)-piece since each component of the \( F \)-preimage of a simple \( L \)-piece is also simple.

(3). Clearly \( F_* \) is well defined due to (2). Since the number of \( \mathcal{C} \)-pieces are finite, each of them is eventually periodic under \( F_* \).

(4). Let \( S \) be a \( \mathcal{C} \)-piece. We have seen from (3) that \( S' := F(E_S) \) is again a \( \mathcal{C} \)-piece. By definition of constant complexity, we know that \( S' = E_{S'} \cup (\sqcup_i A_i \sqcup_j D_j) \), with \( A_i \) annuli and \( D_i \) discs, and that \( (\sqcup_i A_i \sqcup_j D_j) \cap P = \emptyset \), and that one component of \( \partial A_i \) is contained in \( \partial S' \).

There are no complex \( F^{-2}(\mathcal{L}) \)-pieces in \( \sqcup_i A_i \sqcup_j D_j \). By the basic pull-back principle, \( E_S^2 := E_S \cap F^{-1}(E_{S'}) \) is connected. It is reduced from \( E_S \) after thickening the boundary and then cutting off a few disjoint holes (without touching \( P \cap E_S \)). This implies that \( E_S^2 \) is parallel to \( S \). So \( E_S^2 \) is complex. Clearly, there is no other complex \( F^{-2}(\mathcal{L}) \)-piece in \( E_S \).

Inductively, for any integer \( m \geq 1 \), there is a unique complex \( F^{-m}(\mathcal{L}) \)-piece \( E_{S'}^m \) in \( S \) and \( E_{S'}^m \) is parallel to \( S \). Moreover, \( F^m(E_{S'})^m \) is again a \( \mathcal{C} \)-piece. \( \square \)
5.4 The boundary multicurve

Let \((E \xrightarrow{F} L, \mathcal{P})\) be a marked repelling system.

We consider now the boundary curves \(\gamma\) of \(L\) that are non-peripheral (within \(L \setminus \mathcal{P}\)) (i.e. either \(\gamma\) is the boundary of a disc piece \(D\), with \(#D \cap \mathcal{P} \geq 2\), or \(\gamma\) is a boundary curve of a non-disc piece). This set might be empty, or some of the curves might be homotopic to each other (for example the two boundary curves of an annular component of \(L \setminus \mathcal{P}\)). In any case we give the

**Definition 6.** A boundary multicurve \(\mathcal{Y}\) of \((F, \mathcal{P})\) is a collection of curves in \(\partial L\) representing all the homotopy classes within \(L \setminus \mathcal{P}\) of the non-peripheral curves in \(\partial L\).

Then boundary transition matrix \(W_{\mathcal{Y}} = (a_{ij})\) is defined by

\[
a_{ij} = \sum_{\alpha} \frac{1}{\deg(F: \alpha \to \gamma_j)},
\]

where the sum is taken over all the Jordan curves (if any) of \(\alpha \subset F - 1(\gamma_j)\) that are homotopic to \(\gamma_i\) (within \(L \setminus \mathcal{P}\)).

We will say that \((F, \mathcal{P})\) has a boundary obstruction if \(\mathcal{Y} \neq \emptyset\) and \(\lambda(W_{\mathcal{Y}}) \geq 1\).

In general \(\mathcal{Y}\) is not \((F, \mathcal{P})\)-stable, or even worse, we might have \(\mathcal{Y} \neq \emptyset\) and \(W_{\mathcal{Y}}\) equal to the zero matrix.

Assume from now on that \((F, \mathcal{P})\) is also of constant complexity. Then \(\mathcal{Y}\) is \((F, \mathcal{P})\)-stable by Lemma 5.2. It can be described more explicitly using the above classification of \(L\)-pieces:

A closed curve in \(\partial L\) is null-homotopic iff it is contained in \(\partial O\), is peripheral iff it is contained in \(\partial R\). Two closed curves in \(\partial L\) are homotopic iff they are the boundary curves of an \(A\)-piece. Label by + and − the two boundary curves of each \(A\)-piece \(S\), i.e. \(\partial S = \partial_+ S \cup \partial_- S\). Let \(\partial_+ A\) and \(\partial_- A\) denote the union of the corresponding boundary curves of every \(A\)-pieces. Then the boundary multicurve \(\mathcal{Y}\) can be taken to be the collection of closed curves in \(\partial C \cup \partial_+ A\).

Polynomials with all critical points escaping to \(\infty\) provide examples, when restricted to a suitable neighborhood of the Julia set, of repelling systems with \(\mathcal{Y} = \emptyset\). For an annuli covering \(F: E \to A\) with \(A = A_1 \sqcup \cdots \sqcup A_k\) (see Lemma 3.1), the set \(\mathcal{Y}\) consists of \(k\) boundary curves of \(A\), one in each \(A_i\), and the \((F, \mathcal{P})\)-transition matrix \(W_{\mathcal{Y}}\) coincides with the transition matrix \(D\) defined in §3.1. For further examples see §6.1.

5.5 Renormalizations and renormalized obstructions

A repelling system of constant complexity has another, somewhat more important property: it admits renormalizations and they behave like postcritically finite branched coverings of \(\mathbb{C}\).

**Definition 7.** A marked postcritically finite repelling system \((E \xrightarrow{H} S, \tilde{\mathcal{P}})\) of constant complexity is of Thurston type if both \(E\) and \(S\) are connected, and

\[
\#(S \cap \tilde{\mathcal{P}}) + \#\{\text{boundary curves of } S\} \geq 3.
\]

In other words, \(S, E \subset \mathbb{C}\) are quasi-circle bordered Riemann surfaces, \(E\) is compactly contained in the interior of \(S\), the components of \(S \setminus E\) are either annular or disc-like, \(H: E \to S\) is an orientation preserving branched covering, with a finite (or empty) postcritical set \(\mathcal{P}_H\) which is contained in \(E\), the set \(\tilde{\mathcal{P}} \subset E\) is a finite (or empty) set containing both \(H(\tilde{\mathcal{P}})\) and \(\mathcal{P}_H\), and \(\#(S \cap \tilde{\mathcal{P}}) + \#\{\text{boundary curves of } S\} \geq 3\).
Again, an example can be provided by the map \( g(z) = z^2 - 1 \), with \( S \) equal to \( \overline{C} \) minus a suitable neighborhood of \( \infty, 0, -1 \), and with \( E = g^{-1}(S) \).

Let \( (E, F, \mathcal{L}, \mathcal{P}) \) be a marked postcritically finite repelling system of constant complexity. Assume \( \mathcal{C} \neq \emptyset \). By Lemma 5.2 we have a map \( F_\ast \) defined on the collection of \( \mathcal{C} \)-pieces by \( F_\ast(S_1) = S_2 \) if \( F(E_{S_1}) = S_2 \) where \( E_{S_1} \) is the unique \( \mathcal{E} \)-piece in \( S_1 \) parallel to \( S_1 \). Assume that \( S \) is an \( \mathcal{C} \)-piece that is \( p \)-periodic under \( F_\ast \). Let \( E \) be the unique \( F^{-p}(\mathcal{L}) \)-piece in \( S \) parallel to \( S \). Then \( F^p(E) = S \). Denote \( H = F^p|_E \). Then \( H : E \to S \) is a repelling system satisfying that \( \mathcal{P}_H \subset \mathcal{P}_F \cap S \subset \mathcal{P} \cap S \), and that, marked by \( \mathcal{P} \cap S \), the marked system \( (H, \mathcal{P} \cap S) \) is of Thurston type.

**Definition 8.** We will call the marked repelling system \( (E \xrightarrow{H} S, \mathcal{P} \cap S) \) a renormalization of \( (F, \mathcal{P}) \). We say that \( (F, \mathcal{P}) \) has a renormalized obstruction if it has a renormalization \( (E \xrightarrow{H} S, \mathcal{P} \cap S) \) that has a Thurston obstruction.

**Lemma 5.3.** Let \( (E, F, \mathcal{L}, \mathcal{P}) \) be a marked postcritically finite repelling system of constant complexity. If \( (F, \mathcal{P}) \) has no Thurston obstructions, then it has no boundary obstructions nor renormalized obstructions.

**Proof.** As \( (F, \mathcal{P}) \) has no Thurston obstructions, we have \( \lambda(W_T) < 1 \) for the transition matrix \( W_T \) of every multicurve \( T \) in \( \mathcal{L} \setminus \mathcal{P} \), in particular for \( T \) equal to the boundary multicurve. Therefore \( (F, \mathcal{P}) \) has no boundary obstructions.

It remains to show that any renormalization \( H : E \to S \) marked by \( \hat{\mathcal{P}} := \mathcal{P} \cap S \) has no Thurston obstructions. Assume by contradiction that \( \lambda(W_{\Gamma}) \geq 1 \), for some \( (H, \hat{\mathcal{P}}) \)-stable multicurve \( \Gamma \), with \( (H, \hat{\mathcal{P}}) \)-transition matrix \( W_{\Gamma} \).

Let \( S_0 := S, S_1, \cdots, S_{p-1} \) be the \( F_\ast \)-periodic cycle of \( S \). Let \( E_i \) be the unique \( \mathcal{E} \)-piece in \( S_i \) parallel to \( S_i \), \( i = 0, 1, \cdots, p-1 \). Set \( \Gamma_p := \Gamma \). Define inductively, for \( i = p-1, p-2, \cdots, 0 \), the multicurve \( \Gamma_i \subset F^{-i}(\Gamma_{i+1}) \cap E_i \) representing the homotopy classes (within \( S_i \setminus \mathcal{P} \)) of the non-peripheral curves in \( F^{-i}(\Gamma_{i+1}) \cap E_i \). By stability of \( \Gamma \), each curve of \( \Gamma_0 \) is homotopic to a curve in \( \Gamma_p \).

Consider \( F' : \bigcup_i E_i \to \bigcup_i S_i \) as a repelling system. Set \( \Gamma' = \Gamma_1 \cup \cdots \cup \Gamma_p \). It is a multicurve within \( \bigcup_i S_i \setminus \mathcal{P} \), therefore within \( \mathcal{L} \setminus \mathcal{P} \). Denote by \( W_{\Gamma'} \) its \( (F', \mathcal{P}) \)-transition matrix and by \( W' \) its \( (F', \mathcal{P}) \)-transition matrix. Then the \( p \)-th power \( (W')^p \) restricted to \( \Gamma_p \) is equal to \( W_{\Gamma} \). Therefore

\[
1 \leq \lambda(W_{\Gamma}) \leq \lambda((W')^p) = \lambda(W')^p.
\]

But each entry of \( W' \) is less than or equal to the corresponding entry of \( W_{\Gamma'} \). Therefore \( \lambda(W') \leq \lambda(W_{\Gamma'}) \). This implies that \( \lambda(W_{\Gamma'}) \geq 1 \). This contradicts the assumption that \( (F, \mathcal{P}) \) has no Thurston obstructions (by Lemma 4.2).

We can now state our Thurston-like result in this setting, whose proof will occupy Sections 6-8.

**Theorem 5.4.** Let \( (E, F, \mathcal{L}, \mathcal{P}) \) be a marked postcritically finite repelling system of constant complexity. Assume that \( (F, \mathcal{P}) \) has no boundary obstructions nor renormalized obstructions. Then \( (F, \mathcal{P}) \) is c-equivalent to a holomorphic model.

5.6 Proof of Theorem 3.5 using Theorems 5.4

**Proof of Theorem** Let \( (\mathcal{B}, \mathcal{G}, \mathcal{M}, \mathcal{Q}) \) be a marked postcritically finite repelling system
without Thurston obstructions. We will prove that \((G, Q)\) is c-equivalent to a holomorphic model.

As first we apply Theorem 5.1 to \((G, Q)\) to show that it has a postcritically finite repelling system restriction \(F : E \rightarrow L\) near \(K_G\) which, marked by \(\mathcal{P} := Q \cap K_G\), is of constant complexity, and satisfies the conditions in Theorem 4.1. So we may apply Theorem 4.1(B) to show that \((F, \mathcal{P})\) has no Thurston obstructions. Lemma 5.3 then leads \((F, \mathcal{P})\) to the setting of Theorem 5.4, i.e. \((F, \mathcal{P})\) is of constant complexity, and has no boundary obstructions nor renormalized obstructions. Now we may apply Theorem 5.4 to conclude that \((F, \mathcal{P})\) is c-equivalent to a holomorphic model. Finally we conclude for \((G, Q)\) using Theorem 4.1(A).

Note that it could be more difficult to check the condition of Theorem 1.1 and Theorem 3.5 namely \(G\) has no Thurston obstructions. Whereas Theorem 5.4 turns it into the problem of checking the leading eigenvalue of \(W_Y\) for a single multicurve \(Y\), and then the absence of Thurston obstructions for postcritically finite branched coverings (arising from the renormalizations), to which there is a huge literature (see for example the references in [ST]). This form is particularly suitable for combination of rational maps, i.e. starting with postcritically finite rational maps (thus already holomorphic) as the renormalizations and glue them suitably together.

6 C-equivalence to holomorphic models

From now on we concentrate on the proof of Theorem 5.4: a marked postcritically finite repelling system of constant complexity without boundary obstructions nor renormalized obstructions is c-equivalent to a holomorphic model. In this section we will prove the theorem for the non-renormalizable case. In this case only Grötzsch inequalities are needed, but not the original Thurston theorem.

Recall that from Definition 4 and Lemma 2.3 that a marked repelling system \((\mathcal{E}, \mathcal{L}, \mathcal{P})\) is c-equivalent to a holomorphic model, if there is a marked repelling system \((\mathcal{E}', \mathcal{L}', \mathcal{P}')\) with \(R\) holomorphic, and two quasi-conformal homeomorphisms \(\Theta : \mathcal{L} \rightarrow \mathcal{L}, \Phi : \mathcal{L} \rightarrow \mathcal{L}'\) with

\[
\begin{align*}
\Phi \circ \Theta(\mathcal{E}) &= \mathcal{E}', & \Phi \circ \Theta(\mathcal{P}) &= \mathcal{P}' \\
R \circ \Phi \circ \Theta|_{\mathcal{E}} &= \Phi \circ F \\
\Theta \text{ is isotopic to the identity rel } \mathcal{P} \cup \partial \mathcal{L}.
\end{align*}
\]

6.1 Examples

Example 1. \(\mathcal{L} = Q, \mathcal{E} = E_1 \cup E_2, \) with \(Q, E_1, E_2\) quasi discs. \(F : E_1 \rightarrow Q\) are quasi-conformal homeomorphisms \(\mathcal{P} = \emptyset\).
Example 2.

\[ \mathcal{L} = A, \mathcal{E} = A_1 \cup A_2, \]
with \( A, A_1, A_2 \) closed annuli.
\( F : A_i \to A \) are degree \( d_i \) quasi-regular coverings.
\( \mathcal{P} = \emptyset \).

**Lemma 6.1.** The map \( F \) in Example 1 is always c-equivalent to a holomorphic model, whereas in Example 2 it is so if and only if \( \frac{1}{d_1} + \frac{1}{d_2} < 1 \).

**Proof.** Let \( F : E_1 \cup E_2 \to Q \) be as in Example 1.

1. Construct at first the model map \( R \) by setting \( E'_i = E_i, Q' = Q \) and by choosing \( R : E'_i \to Q' \) any conformal homeomorphism.
2. Set \( \Phi = id : Q \to Q \).
3. Set \[
\begin{cases} 
\theta|_{E_i} = R^{-1} \circ F \\
\theta|_{\partial Q} = id.
\end{cases}
\]
Then \( R \circ \Phi \circ \theta|_{E_i} = \Phi \circ F \).
4. Extend \( \theta \) as a homeomorphism of \( Q \).
5. One checks easily that this \( \theta \) is isotopic to the identity rel \( \partial Q \).

Let now \( F : A_1 \cup A_2 \to A \) be a map in Example 2. Note that \( A \) has a unique multicurve \( \Gamma \), up to homotopy, with \( \Gamma \) consisting of a boundary curve of \( A \). Its \( F \)-transition matrix has only one entry, which is \( \frac{1}{d_1} + \frac{1}{d_2} \). Therefore \( F \) has a Thurston obstruction if and only if \( \frac{1}{d_1} + \frac{1}{d_2} \geq 1 \).

If \( F \) is c-equivalent to a holomorphic \( R : A'_1 \cup A'_2 \to A' \). Then by Grötzsch inequality \( \frac{1}{d_1} + \frac{1}{d_2} < 1 \).

Conversely assume \( \frac{1}{d_1} + \frac{1}{d_2} < 1 \).

a. Construct at first a round modulus \( A' \) of modulus, say \( v \). And let \( \Phi : A \to A' \) be a qc-homeomorphism.

b. Construct then two disjoint essential round submoduli \( A'_1, A'_2 \) in \( A' \) of moduli \( v/d_1 \) and \( v/d_2 \) respectively, and displaced in the same order as the \( A_i \)’s in \( A \). This is possible precisely because \( \sum \frac{1}{d_i} < 1 \). Choose \( R : A'_i \to A' \) a holomorphic covering of degree \( d_i \), matching the boundary correspondence as \( F \).

c. Pull-back \( A'_i \) by \( \Phi \).
d. Set \[
\left\{
\begin{array}{ll}
\theta|_{A_i} & = (\Phi^{-1} \circ R \circ \Phi)^{-1} \circ F : A_i \rightarrow \Phi^{-1}(A_i) \\
\theta|_{\partial A} & = \text{id}
\end{array}
\right.
\]
e. Extend \(\theta\) as a qc-homeomorphism \(A \rightarrow A\). Then \(R \circ \Phi \circ \theta|_{A_i} = \Phi \circ F\).
f. Via Dehn twist on \(A \setminus (A_1 \cup A_2)\) to modify the extension so that \(\theta(\beta) \sim \beta\).

This guarantees that \(\theta\) is isotopic to the identity rel \(\partial A\).

**Example 3.** Let \(L\) be a pair of trousers bounded by three quasi-circles \(\gamma_0, \gamma_{-1}, \gamma_*\). Let \(E \subset L\) be a surface bounded by four quasi-circles \(\beta_0, \beta_{-1}, \beta_*, \beta_1\), with \(\beta_1\) bounding a complementary disc of \(E\) that is entirely contained in \(L\), and with each other \(\beta_i\) bounding a complementary disc of \(E\) that contains the corresponding \(\gamma_i\). Let \(H : E \rightarrow L\) be a quasi-regular covering of degree 2. Again \(P_H = P = \emptyset\). And \(H\) is of Thurston type.

The boundary multicurve \(Y\) is simply \(\{\gamma_0, \gamma_{-1}, \gamma_*\}\). For example if we require \(H : \beta_* \rightarrow \gamma_*\) to be degree 2, \(H : \beta_{-1} \rightarrow \gamma_0\) of degree 1 and \(H : \beta_0 \rightarrow \gamma_{-1}\) of degree 2, then the transition matrix \(W_Y\) is
\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}
\]
with leading eigenvalue \(1/\sqrt{2}\). Such a map behaves like \(z^2 - 1\).

On the other hand, if we require instead \(H : \beta_* \rightarrow \gamma_*\) to be degree 2, \(H : \beta_{-1} \rightarrow \gamma_1\) of degree 1 and \(H : \beta_0 \rightarrow \gamma_0\) of degree 2, then the transition matrix \(W_Y\) is
\[
\begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix}
\]
with leading eigenvalue 1. This \(H\) can be constructed more explicitly as follows (suggested by X. Buff): Let \(g(z) = z^2\). Let \(L\) be \(\mathbb{C}\) minus a small round disc of radius \(\varepsilon\) around each of the three points 0, \(\infty\) and 1. Let \(E' = g^{-1}(L)\). As 1 is a repelling fixed point of \(g\), \(E'\) is not compactly contained in \(L\) and \(g : E' \rightarrow L\) is not a repelling system. Let now \(\eta : \mathcal{D}(1,2\varepsilon) \rightarrow \mathcal{D}(1,2\varepsilon)\) to be a homeomorphism fixing pointwisely the boundary and center, mapping the boundary component of \(E'\) into the interior of \(L\). Extend \(\eta\) elsewhere by identity. Set \(H = g \circ \eta^{-1} : \eta(E') \rightarrow L\).

Note that every non-peripheral curve is homotopic to one of the curves in \(Y\). So \(H\) has a Thurston obstruction if and only if \(\lambda(W_Y) \geq 1\).

**Lemma 6.2.** This \(H\) is c-equivalent to a holomorphic model if and only if the \(H\)-transition matrix of \(Y = \{\gamma_0, \gamma_{-1}, \gamma_*\}\) has a leading eigenvalue \(\lambda < 1\).

**Proof.** Mark one point in each complement component of \(L\). Denote the marked set by \(\mathcal{P}\).
Extend $H$ as a quasi-regular branched cover $\hat{H}$ of $\overline{C}$ such that the critical values are in $\mathcal{P}$ and $\hat{H}(\mathcal{P}) \subset \mathcal{P}$, and that $\hat{H}$ is holomorphic outside $L$. In particular $\hat{H}$ is postcritically finite.

Assume at first the $H$ is c-equivalent to a holomorphic model. Then by Lemma 2.3 there is a quasi-conformal automorphism $\theta : L \to L$, isotopic to the identity rel $\partial L$, and a Beltrami differential $\mu$ supported on $L$ with $\|\mu\|_{\infty} < 1$, such that $(H \circ \theta^{-1})^* \mu = \mu|_{\theta(E)}$.

Proceed now at in the proof of Proposition 2.4. Extend $\theta$ to be equal to the identity outside $L$. Set $\hat{H}_1 = \hat{H} \circ \theta^{-1}$. Extend $\mu$ outside $L$ by $\mu = 0$. Let $\phi_1 : \overline{C} \to \overline{C}$ be a global integrating map of this extended $\mu$. Set $\hat{H}_2 := \phi_1 \circ \hat{H}_1 \circ \phi_1^{-1}$. Then $\hat{H}_2$ is again quasi-regular, and is holomorphic in the interior of $\phi_1 \circ \theta(E)$ and in $\phi_1(L)$. Elsewhere each $\hat{H}_2$-orbit passes at most once. We spread out the Beltrami differential $\nu_0 \equiv 0$ using iterations of $\hat{H}_2$ to get an $\hat{H}_2$-invariant Beltrami differential $\nu$. Note that $\nu = 0$ on $\phi_1(L)$, and $\|\nu\|_{\infty} < 1$. Integrating $\nu$ by a quasi-conformal homeomorphism $\phi_2$ (necessarily holomorphic on $\phi_1(L)$), we get a new map $f := \phi_2 \circ \hat{H}_2 \circ \phi_2^{-1}$ which is a rational map and is c-equivalent to $\hat{H}_2$, therefore to $\hat{H}$. See the following diagram.

\[
\begin{array}{ccc}
\mathcal{C}, E & \xrightarrow{\theta} & \mathcal{C} \\
\hat{H} \downarrow & & \downarrow \hat{H}_1 \\
(\mathcal{C}, L) & \xrightarrow{id} & \mathcal{C}
\end{array}
\]

Set $\phi = \phi_2 \circ \phi_1$. Then $\phi(\partial L)$ is contained in the attracting basins of $f$. But $f$ is postcritically finite, each attracting cycle must be superattracting. It follows that the boundary multicurve of the repelling system $f|_{\phi \circ \theta(E)}$ has leading eigenvalue $< 1$. Therefore $\lambda(W_Y) < 1$ for our map $H$.

Conversely, assume that $\lambda(W_Y) < 1$ for $H$. The fact that $\#\mathcal{P} = 3$ implies that $(\hat{H}, \mathcal{P})$ has no Thurston obstructions. By Thurston theorem there are $\Phi = id, \Psi$ homeomorphism, and $f$ a rational map, such that $f \circ \Psi = \Phi \circ \hat{H}$, and that $\Psi$ is isotopic to $\Phi$ rel $\mathcal{P}$.

Let $v$ be a positive vector so that $W_Y \cdot v < v$. Change $\Phi$ within its isotopy class to $\phi$ so that $\phi(\gamma)$, for any $\gamma \in Y$, is an equipotential in a Fatou component $\Sigma$ of $f$. Further, the annular component of $\Sigma \setminus \phi(\gamma)$ is of modulus $v(\gamma)$. Let $\psi$ be the homeomorphism isotopic to $\Psi$ so that $f \circ \psi = \phi \circ \hat{H}$. Now $W_Y \cdot v < v$ assures that $\psi(E)$ is compactly contained in $\phi(L)$.

Use Lemma 2.3 to modify $\psi$ on $L \setminus E$ so that $\psi$ is isotopic to $\phi$ rel $\overline{C} \setminus L$, and $f \circ \psi|_E = \phi|_L \circ \hat{H}$. \hfill \Box

**Example 4.** The following $F : \bigcup E_s \to S \sqcup A$ is an unbranched repelling system of constant complexity.

\[
\begin{array}{ccc}
\gamma_3 & & \gamma_4 \\
E_{SS} & & E_{AA} \\
E_{SA1} & & E_{AA2} \\
\gamma_1 & & \gamma_2 \\
S & & A
\end{array}
\]

$L = S \sqcup A$, $S$ is a pair of pants, $A$ is an annulus

$F : E_{SS} \to S$, $E_{AS} \to S$ are coverings

$F : E_s A_j \to A$ are coverings, $\mathcal{P} = \emptyset$.

In this case the boundary multicurve $Y$ consists of the three boundary curves $\gamma_1, \gamma_2, \gamma_3$ of $S$ together with one of the two boundary curves, named by $\gamma_4$ of $A$. And $F$ has only one
Lemma 6.3. If the $F$-transition matrix $W_Y$ of $Y$ has leading eigenvalue less than 1, then $F$ is c-equivalent to a holomorphic model.

Sketch of the proof. Let $v = (v_1, v_2, v_3, v_4)$ be a vector with each $v_i > 0$ such that $W_Yv < v$. Set $H = F|_{E_{SS}}$. Now the boundary multicurve is simply $\{\gamma_1, \gamma_2, \gamma_3\}$, the its $H$-transition matrix $D_*$ is a submatrix of $W_Y$, therefore $D_*u < u$ for $u = (v_1, v_2, v_3)$. We may then, as in Example 3, construct $\phi, \psi$ making $H$ c-equivalent to a holomorphic model $f$, so that $\phi(\gamma_i)$ has the potential prescribed by $v_i$, $i = 1, 2, 3$.

But the presence of $E_{AS}$ requires further control of $\phi, \psi$.

Fix $M > 0$ a large number. Modify again $\phi, \psi$ (but not $f$) according to the vector $Mu$.

Let now $A'$ be a round annulus of modulus $Mu_4$.

Fill in the hole in $E_{AS}$ to get $\hat{E}_{AS}$ that is an essential annulus is $A$. Denote by $\alpha, \beta$ the boundary curves of $\hat{E}_{AS}$.

Assume, say, $F$ maps $\alpha$ to $\gamma_1$ with degree $d_1$, and $\beta$ to $\gamma_2$ with degree $d_2$. Then a candidate $\hat{E}'_{AS}$ has a modulus bounded by $\frac{Mu_1}{d_1} + \frac{Mu_2}{d_2} + C$ with a constant $C$ independent of $M$ (due to Lemma 6.1). Choose $M$ so that $C < M \inf_i (v_i - (W_Yv)_i)$.

Set then $E'_{SS} = f^{-1}(S') = \psi(E_{SS})$.

Now $(W_Y(Mv))_1 + C < Mu_1$ guarantees that one can insert two disjoint essential annuli $\hat{E}'_{AS}, \hat{E}'_{AA1}$ in $A'$, and insert an essential annulus $E'_{SA1}$ in the corresponding annulus of $S' \setminus E'_{SS}$, together with a model holomorphic covering $\hat{R} : E'_{SAj} \to A', E'_{*S} \to S'$ with the same degree as $F$ and the same boundary correspondence. Then the construction of $\theta$ is similar.

Details are provided in $\mathcal{S}$ \Box

6.2 A criterion for c-equivalence to holomorphic models

The following simple remark turns concept of c-equivalence into a more practical form:

Lemma 6.4. Let $(\mathcal{E}, F, \mathcal{L}, \mathcal{P})$ be a marked postcritically finite repelling system. Then $(F, \mathcal{P})$ is c-equivalent to a holomorphic model if and only if: for each $\mathcal{L}$-piece $S$, there is a pair of quasi-conformal homeomorphisms $\theta_S : S \to \hat{S}$, $\phi_S : \hat{S} \to S' \subset \mathcal{C}_{S'}$ (here we consider $\mathcal{C}_{S'}$ as a distinct copy of the Riemann sphere), such that:

(a) $\theta_S$ is the identity on $\partial S \cup \mathcal{P}$ and is isotopic to the identity rel $\partial S \cup \mathcal{P}$.

(b) For every $\mathcal{E}$-piece $E$ contained in $S$, and for $\hat{S} = F(E)$ (which is again an $\mathcal{L}$-piece), the composition $R_E := \phi_{\hat{S}} \circ F \circ \theta_{S}^{-1} \circ \phi_{S}^{-1}$ is holomorphic in the interior of $\phi_{\hat{S}} \circ \theta_{S}(E)$. See \textit{[7]}.

The proof is almost straightforward. One just need to set $\theta = \Phi^{-1} \circ \Psi$ in the definition of c-equivalence, and set $\theta_S = \theta|_S$, $\phi_{\hat{S}} = \Phi|_{\hat{S}}$ and $R_E = R|_{\phi_{\hat{S}} \circ \theta_{S}(E)}$.

Therefore to prove that $(F, \mathcal{P})$ is c-equivalent to a holomorphic model, it amounts to construct $\phi_S, \theta_S$ and $R_E$ for each $\mathcal{L}$-piece $S$ and each $\mathcal{E}$-piece $E$ in $S$ so that they satisfy (a) and (b). In practical the maps $\phi_S, R_E$ will be constructed first. One constructs then each $\theta_E = \theta_{S}|_E$, and finally glue the various $\theta_E$’s together to get $\theta_S$. See the following schema:
Order of the construction

\begin{align*}
\forall \mathcal{E}\text{-piece } E \subset S, \\
S \xrightarrow{\frac{\theta_S}{4}} E \sqcup \bigcup_{\mathcal{E}} S \xrightarrow{\frac{\theta_S}{1}} S' \subset \mathbb{C}S', \\
E \xrightarrow{\frac{\theta_S|_{E=\theta_E}}{3}} \tilde{E} \xrightarrow{\frac{\phi_{\tilde{S}}}{2}} E' \\
F \downarrow \quad \quad \downarrow 1. \quad R_E \text{ holomorphic} \\
\tilde{S} \xrightarrow{id} \tilde{S} \xrightarrow{\frac{\phi_{\tilde{S}}}{1}} S' \subset \mathbb{C}S'.
\end{align*}

## 6.3 Annuli coverings

Let now \( f : \mathcal{E} \to A \) be an annuli covering. In other words, \( A = A_1 \sqcup \cdots \sqcup A_k, \mathcal{E} = \bigcup_{ij\delta} E_{ij\delta}, \) with each \( A_i \) (and \( E_{ij\delta} \)) an annulus, and with \( E_{ij\delta} \subset A_i \) for \( i, j \in \{1, \cdots, k\} \) and \( \delta \) in some finite or empty index set \( \Lambda_{ij} \) depending on \( (i, j) \), and \( f : E_{ij\delta} \to A_j \) is a quasi-regular covering of certain degree, denoted by \( d_{ij\delta} \). Recall that the transition matrix \( D \) is defined by

\[ D = (a_{ij}), \quad a_{ij} = \sum_{\delta \in \Lambda_{ij}} \frac{1}{d_{ij\delta}}. \]

We will prove the following more concrete form of Lemma 3.1.

**Lemma 6.5.** For the annuli covering \( f : \mathcal{E} \to A \) as above, assume that there is a vector \( v = (v_1, \cdots, v_k) \) with \( v_i > 0 \) for any \( i \) such that \( Dv < v \), i.e.

\[ \forall i = 1, \cdots, k, \quad \sum_{j, \delta} v_j d_{ij\delta} < v_i. \quad (8) \]

Then \( f : \mathcal{E} \to A \) is c-equivalent to a holomorphic model \( R : \mathcal{E}' \to A' \) with \( A' = A'_1 \sqcup \cdots \sqcup A'_k \) and \( \text{mod} (A'_i) = v_i \).

Here the modulus of a closed annulus will mean the modulus of its interior as an open annulus. Now Lemma 3.1 relates \( \lambda(D) < 1 \) to the existence of such vector \( v \). And Lemma 3.1 follows.

**Proof of Lemma 6.5.**

We consider each \( A_i \) as embedded in a distinct copy \( \mathbb{C}_i \) of the Riemann sphere. Take one more copy \( \mathbb{C}'_i \) for each \( i = 1, \cdots, k \).

For each \( i \in \{1, \cdots, k\} \), we will construct a pair \( (\theta_i, \phi_i) \) such that (see (9)):

(a) \( \theta_i : A_i \to A_i \) is a quasi-conformal map, with \( \theta_i|_{\partial A_i} = id \) and with \( \theta_i \) isotopic to the identity rel \( \partial A_i \). The set \( \theta_i(E_{ij\delta}) \) is denoted by \( \bar{E}_{ij\delta} \) for each possible \( (j, \delta) \).

(b) \( \phi_i : A_i \to A'_i \) is a quasi-conformal homeomorphism. The set \( \phi_i(\bar{E}_{ij\delta}) \) is denoted by \( E'_{ij\delta} \) for each possible \( (j, \delta) \).

(c) For each multi index \( ij\delta \), the map \( R_{ij\delta} := \phi_j \circ f \circ (\phi_i \circ \theta_i)^{-1}|_{E'_{ij\delta}} \) is holomorphic in the interior of \( E'_{ij\delta} \).
\[ A_i \xrightarrow{\theta_i} A_i \xrightarrow{\phi_i} A'_i \subset \mathbb{C}'_i \]

We will follow the order of construction as indicated by (7). Once this is done we can apply Lemma \[\text{C.3}\] to conclude that \( f : \mathcal{E} \to \mathcal{A} \) is \( \mathbb{C} \)-equivalent to a holomorphic model.

1. **Definition of \((\phi_i, A'_i, E'_{ij\delta}, R_{ij\delta})\):** For every \( i \in \{1, \ldots, k\} \), choose \( A'_i \subset \mathbb{C}'_i \) a closed round annulus of modulus \( v_i \), and let \( \phi_i : A_i \to A'_i \) be a quasi-conformal homeomorphism.

   Fix an index \( i \). For every possible choice of \((j, \delta)\), the lower diagram of (9) indicates that \( R_{ij\delta} : E'_{ij\delta} \to A'_j \) is a covering isomorphic to \( \phi_j \circ f : E_{ij\delta} \to A'_j \), therefore is an annuli covering of degree \( d_{ij\delta} \). But \( R_{ij\delta} \) is supposed to be holomorphic and \( \text{mod} (A'_j) = v_j \). This imposes that \( E'_{ij\delta} \) must be a sub-annulus of \( A'_j \) with modulus \( v_j/d_{ij\delta} \).

   Choose now a closed round essential annulus \( E'_{ij\delta} \) in \( A'_i \) such that (1) \( E'_{ij\delta} \cap \partial A'_i = \emptyset \); (2) \( \text{mod} (E'_{ij\delta}) = v_j/d_{ij\delta} \) and the \( (E'_{ij\delta})'s \) are mutually disjoint for all possible indices \((j, \delta)\) (this is possible precisely because of (5)); (3) the \( (E'_{ij\delta})'s \) are displaced in \( A'_i \) in the same order as the \( (E_{ij\delta})'s \) in \( A_i \).

   Choose now \( R_{ij\delta} : E'_{ij\delta} \to A'_j \) a holomorphic covering of degree \( d_{ij\delta} \) among the two round annuli, so that it permutes the boundary curves in the same way as \( f : E_{ij\delta} \to A_j \).

   More precisely this can be done through boundary labeling: for each \( A_i \) choose a labelling by \( + \) and \( - \) for its two boundary curves. This induces a labelling by \( \pm \) on the boundary curves of each essential sub-annulus \( E_{ij\delta} \), so that \( - \partial_+ E_{ij\delta} \) separates \( \partial_- A_i \) to \( \partial_+ E_{ij\delta} \). Now use each \( \phi_i \) to transport these labellings to \( \partial A'_i \) which then induces a labelling on each \( \partial E'_{ij\delta} \). The covering \( f : E_{ij\delta} \to A_j \) maps \( - \partial_+ E_{ij\delta} \) to one of \( \partial_- A_j \). We choose \( R_{ij\delta} \) so that it sends \( - \partial_+ E'_{ij\delta} \) to \( \phi_j(f(\partial_+ E_{ij\delta})) \), the corresponding boundary component of \( A'_j \).

2. **Definition of \( \bar{E}_{ij\delta} \):** For any multi-index \((i, j, \delta)\), set \( \bar{E}_{ij\delta} := \phi_i^{-1}(E'_{ij\delta}) \) (there are a priori two ways to label its boundary curves, one as an essential sub-annulus of \( A_i \), one transported by \( \phi_i^{-1} \) of the labeling of \( \partial E'_{ij\delta} \), but these two labellings actually coincide).

3. **Definition of \( \theta_{ij\delta} \):** For any multi-index \((i, j, \delta)\), let \( \theta_{ij\delta} : E_{ij\delta} \to \bar{E}_{ij\delta} \) be a (choice of a) lift of the quasi-conformal map \( \phi_j : A_j \to A'_j \) via the two quasi-regular coverings of the same degree: \( f|_{E_{ij\delta}} \) and \( R_{ij\delta} \circ \phi_i|_{\bar{E}_{ij\delta}} \). It is a quasi-conformal map and preserves the boundary labelling.

\[
\begin{array}{ccc}
E_{ij\delta} & \xrightarrow{\theta_{ij\delta}} & \bar{E}_{ij\delta} \\
\downarrow f & & \downarrow R_{ij\delta} \\
A_j & \xrightarrow{\phi_i} & A'_j
\end{array}
\]

4. **Definition of \( \theta_i \):** Fix an index \( i \). Define \( \theta_i : A_i \to A_i \) to be a quasi-conformal map such that \( \theta_i|_{E_{ij\delta}} = \theta_{ij\delta} \) and \( \theta_i|_{\partial A_i} = \text{id} \). It exists always, because all the boundary curves are quasi-circles and all \( \theta_{ij\delta} \) are quasi-conformal maps preserving the boundary labelling (see Lemma C.2).

The map \( \theta_i \) satisfies all the required properties, except possibly the one about their homotopy type.
4'. Adjustment of the homotopy type of $\theta_i$. As the lower commuting diagram in [9] only requires information on $\theta_{ij\delta}$, we will modify each $\theta_i$ without changing its value on the $E_{ij\delta}$'s.

Fix an index $i$. Choose an arc $\beta$ connecting the two boundaries of $A_i$. Then $\theta_i(\beta)$ is again an arc in $A_i$ with the same end points. Precompose $\theta_i$ with a quasi-conformal repeated Dehn twist supported in the interior of $A_i \setminus \bigcup_{(j, \delta)} E_{ij\delta}$ if necessary we can ensure that $\theta_i(\beta)$ is homotopic to $\beta$ (rel $\partial A_i$). After this adjustment $\theta_i$ is well isotopic to the identity rel $\partial A_i$. □

6.4 The non-renormalizable case

Proof of Theorem 5.4 in case $C = R = \emptyset$ and $P = \emptyset$.

Let $F : \mathcal{E} \rightarrow \mathcal{L}$ be a repelling system. Assume that $\mathcal{P}_F = \mathcal{P} = \emptyset$ and that every $\mathcal{L}$-piece is either annular or disc-like. Then $F$ is of constant complexity and there is nothing to renormalize. Furthermore the boundary multicurve $Y$ is simply the collection of one boundary curve in each annular piece of $\mathcal{L}$. Assume now $\lambda(W_Y) < 1$. We want to prove that $F : \mathcal{E} \rightarrow \mathcal{L}$ is c-equivalent to a holomorphic model. Set

$$\mathcal{L} = A \sqcup O, \quad A = A_1 \sqcup \cdots \sqcup A_k, \quad O = O_{k+1} \sqcup \cdots \sqcup O_m$$

so that each $A_i$ is annular and each $O_j$ is disc-like.

0. The vector $v$. Choose a vector $v \in \mathbb{R}^k$ with every entry positive such that $W_Y(v) < v$.

1. Definition of $(\phi_S, S', E', R_E)$. Consider each $A \sqcup O$-piece $S$ as a subset of a distinct copy $\overline{C}_S$ of the Riemann sphere. If $S = A_i$, define as above $\phi_S : S \rightarrow S'$ to be a quasi-conformal homeomorphism so that $S'$ is an round annulus with modulus $v_i$. If $S = O_i$, set simply $\phi_S = id$ and $S' = S$.

For any $E$-piece $E$, there are two $A \sqcup O$-pieces $S$ and $\hat{S}$ (possibly the same) such that $E \subset \subset S$ and $F(E) = \hat{S}$. As $F$ is a covering, we know that $E$ is an annulus (resp. disc) if and only if $\hat{S}$ is an annulus (resp. disc).

We decompose $\mathcal{E}$ into $\mathcal{E}^A \sqcup \mathcal{E}^{0,A} \sqcup \mathcal{E}^{0,0}$ as follows:

- $\mathcal{E}^A$ consists of $E$-pieces that are essential sub-annuli in $A$. We number as above these pieces by $E_{ij\delta}$ so that $E_{ij\delta} \subset A_i$ and $F(E_{ij\delta}) = A_j$. Label by $\pm$ the boundary curves of each $A_i$, give each $\partial E_{ij\delta}$ the induced labelling.

- $\mathcal{E}^{0,A}$ consists of the remaining annular $E$-pieces. Such a piece $E$ is a closed annulus contained in some $\mathcal{L}$-piece $S$, with either $S = O_i$ or $S = A_i$. In the latter case $E$ is contained in a component $B$ of $S \setminus \bigcup_{(j, \delta)} E_{ij\delta}$, and $E$ is non-essential in $B$. See Figure 2. In any case both boundary curves of $E$ are $F$-null-homotopic. Furthermore $E$ has one complementary component $\Delta_E$ that is entirely contained in $S$. We will call $\Delta_E$ the hole of $E$. We will label $\partial E$ so that $\partial_+ E = \partial \Delta_E$. Therefore $\partial_+ E$ denotes the outer boundary of $E$.

- Finally $\mathcal{E}^{0,0}$ consists of the disc pieces of $\mathcal{E}$. Such a piece $E$ may be contained in $A$ or in $O$, but $F(E)$ is always an $O$-piece, and $F : E \rightarrow F(E)$ is a homeomorphism.

The general strategy is quite simple to explain: we shall at first construct $(E', R_E)$ for all the $E^A$-pieces as in Lemma 6.5. For the remaining $E$-pieces, as they may be nested in each others holes, we should construct $(E', R_E)$ inductively from outer pieces to inner pieces.

1.1. Construction of $(E', R_E)$ for each $E^A$-piece $E$. For all possible $(i, j, \delta)$, define $E'_{ij\delta}$ to be a round annulus in $A_i'$ exactly as in the proof of Lemma 6.5 above, in particular the various $E'_{ij\delta}$ for a given $i$ are displaced in $A_i'$ exactly in the same order as the $E_{ij\delta}$'s in $A_i$.
and mod $(E'_{ij\delta}) = v_j / \text{deg}(F : E_{ij\delta} \to A_j)$. This is where we have used the assumption $\lambda(W_Y) < 1$. Let $R_{ij\delta} : E'_{ij\delta} \to A_j'$ be a holomorphic covering of degree $\text{deg}(F : E_{ij\delta} \to A_j)$, and permuting the boundary labelling in the same way as $F : E_{ij\delta} \to A_j$.

1.2. Construction of $(E', R_E)$ for each $\mathcal{E}^{0,A}$-piece $E$. Let now $E$ be an $\mathcal{E}^{0,A}$-piece. Then $E$ is contained in a set $B$ which is either some $O_i$ or one component of $A_1 \setminus \bigcup_{j,\delta} E_{ij\delta}$ for some $A_1$. Furthermore $F$ maps $E$ onto some $A_1$ as a covering. Denote its degree by $d_E$. Denote by $B'$ the corresponding $O'$ or the corresponding component of $A_1' \setminus \bigcup_{j,\delta} E'_{ij\delta}$. We want to set $E'$ to be an inessential annulus in $B'$ with modulus $v_j / d_E$, so that there is a holomorphic covering $R_E : E' \to A_j'$ of degree $d_E$.

However one might run into some moduli difficulty if we do so randomly, as various pieces of $\mathcal{E}^{0,A}$ in $B$ may be nested in each others holes. The correct way to do this is to place $E'$ one by one from outside to inside. More precisely, numerate the $\mathcal{E}^{0,A}$-pieces in $B$ by $E_{Bj\delta}$, with $\delta$ running in some index set (depending on $j$) such that $F(E_{Bj\delta}) = A_j$ and it is a covering of degree $d_{Bj\delta}$. Define then a layer (depth) function $l(E)$ on the set of $(E_{Bj\delta})_{\delta}$ as follows: set $l(E) = 1$ if $E$ is outermost, i.e. not contained in the hole of any other $\mathcal{E}^{0,A}$-pieces. Set inductively $l(E) = m$ if $E$ is contained immediately in the hole of a $E_{Bj\delta}$ with $l(E_{Bj\delta}) = m - 1$.

Start now with an $E_{Bj\delta}$ so that $l(E_{Bj\delta}) = 1$. Choose $E'_{Bj\delta} \subset B'$ to be any round inessential annulus of modulus $v_j / d_{Bj\delta}$. Label its outer boundary curve by +. Choose then $R_{Bj\delta} : E'_{Bj\delta} \to A_j'$ a holomorphic covering of degree $d_{Bj\delta}$, so that it permutes the boundary labelling in the same way as $F : E_{Bj\delta} \to A_j$.

Construct similarly $(E', R_E)$ for every layer 1 piece in $B$, and be sure that the various $E'$'s are mutually disjoint.

Now we should construct $(E', R_E)$ for the next layer $\mathcal{E}^{0,A}$-pieces in $B$. Proceed this layer by layer. As each time we are supposed to find finitely many disjoint annuli non mutually nested of prescribed moduli in the hole of some previously constructed $E'$, the construction is always realizable.

Do this construction for every component $B$ of $\mathcal{L} \setminus \mathcal{E}^A$.

1.3. Construction of $(E', R_E)$ for each $\mathcal{E}^{0}$-piece $E$. Assume $E \subset S$ and $F(E) = \tilde{S}$. We should choose a closed quasi-disc $E'$ in $S'$ disjoint from the previously constructed pieces, together with a conformal map $R_E : E' \to \tilde{S}'$. There is no difficulty here and we omit the details.

To recapitulate we may extend the layer function $l(E)$ on all $\mathcal{E}$-pieces so that $l(E) = 0$ for $\mathcal{E}^A$-pieces and $l(E) = +\infty$ for $\mathcal{E}^0$-pieces, and then construct $(E', R_E)$ following the natural order of the layer function.

2-3. Definition of $(\tilde{E}, \theta_E)$. This is done exactly as in Lemma 6.5 by setting $\tilde{E} = \phi_{\tilde{S}}^{-1}(E')$ for $S$ the $\mathcal{L}$-piece containing $E$, and $\theta_E : E \to \tilde{E}$ as a lift of $\phi_{\tilde{S}} : \tilde{S} \to \tilde{S}'$ via $F|_E$ and $R_E \circ \phi_{\tilde{S}}|_{\tilde{E}}$, where $\tilde{S} = F(E)$.

4. Definition of $\theta_S$. Fix an $\mathcal{L}$-piece $S$. We claim that we can define a quasi-conformal map $\theta_S : S \to S$ so that $\theta_S = \theta_E$ on each $\mathcal{E}$-piece $E$ contained in $S$ and $\theta_S = \text{id}$ on $\partial S$. Clearly the extension can be chosen so that $\theta_S$ is a homeomorphism, as the $\theta_E$'s for all possible $E$ preserve the boundary labelling. But all the boundary curves are quasi-circles and all the $\theta_E$'s are quasi-conformal. One can then apply Lemma 6.2 to make the extension quasi-conformal.

4'. Adjustment of the homotopy type of $\theta_A$. Clearly $\theta_S$ for $S$ an $O$-piece is already isotopic to the identity rel $\partial S$. However for $S$ an $A$-piece one might have to precompose $\theta$
with a repeated Dehn twist supported on the interior of $S \setminus \mathcal{E}$ as in Lemma 6.5. After that $\theta_S$ is also isotopic to the identity rel $\partial S$.

\section{Holomorphic model of a renormalization cycle}

Let $(\mathcal{E} \xrightarrow{F} \mathcal{L}, \mathcal{P})$ be a marked postcritically finite repelling system of constant complexity without boundary obstructions nor renormalized obstructions. We will prove here that a renormalization cycle of $(F, \mathcal{P})$ is $c$-equivalent to a holomorphic model, which satisfies in addition some prescribed moduli properties.

We always denote by $\mathbb{D}$ the unit disc. A marked disc is a pair $(\Delta, a)$ with $\Delta$ an open hyperbolic disc in $\mathbb{C}$ and a marked point $a \in \Delta$. An equipotential $\gamma$ of $(\Delta, a)$ is a Jordan curve that is mapped to a round circle under a conformal representation $\chi : \Delta \to \mathbb{D}$ with $\chi(a) = 0$. The potential of such a $\gamma$ is defined to be $\kappa(\gamma) := \text{mod} A(\partial \Delta, \gamma)$, the modulus of the annulus between $\partial \Delta$ and $\gamma$. These notions do not depend on the choice of $\chi$. The map $\kappa$ maps the set of equipotentials bijectively onto the open interval $]0, +\infty[$. For example in the marked disc $(\mathbb{D}, 0)$, the circle $\{|z| = e^{-\tau}\}$ is an equipotential with potential $v$ (we define $\text{mod}\{r < |z| < 1\} := -\log r$).

Let $f$ be a postcritically finite rational map with non-empty Fatou set. The Julia set $J_f$ is connected and each Fatou component $\Delta$ is canonically a marked disc marked by the unique eventually periodic point $a \in \Delta$. We call $(\Delta, a)$ a marked Fatou component of $f$. The equipotentials of these marked Fatou components will be called equipotentials of $f$. Notice that equipotentials in a periodic Fatou component correspond to round circles in Böttcher coordinates. We will use $\kappa$ to denote the potential function of these marked discs.

**Marked set $\mathcal{P}_S$.** Again consider each $\mathcal{L}$-piece $S$ as being contained in a distinct copy $\overline{\mathbb{C}}_S$ of the Riemann sphere. Mark one point in each component of $\overline{\mathbb{C}}_S \setminus S$. Set $\mathcal{P}_S$ to be the union of $\mathcal{P} \cap S$ with these marked points. By definition a piece $S$ is an $\mathcal{L}$-piece if and only if $\# \mathcal{P}_S \geq 3$.

As $(F, \mathcal{P})$ is of constant complexity, there is an induced map $F_*$ on the set of $\mathcal{C}$-pieces. Let $S_1, \ldots, S_p$ be a periodic cycle of $\mathcal{C}$-pieces for $F_*$, i.e. for $i = 1, \ldots, p$ we have $F(E_i) = S_{i+1}$ (set $S_{p+1} = S_1$), where $E_i$ is the unique complex $\mathcal{E}$-piece in $S_i$. Denote by $\overline{\mathcal{C}}_i = \overline{\mathcal{C}}_{S_i}$ for simplicity.

We will prove:

**Theorem 7.1.** Denote by $D_*$ the $(F, \mathcal{P})$-transition matrix of the set of the boundary curves of $S_1 \cup \cdots \cup S_p$. Let $u > 0$ be any positive vector such that $D_* u < u$. Then there are pairs of quasi-conformal maps $(\phi_{S_i}, \psi_{S_i}) : \overline{\mathcal{C}}_{S_i} \to \overline{\mathcal{C}}_{S_{i+1}}$ and holomorphic maps $R_i : \overline{\mathcal{C}}_{S_i} \to \overline{\mathcal{C}}_{S_{i+1}}$ for $1 \leq i \leq p$ such that:

(a) $\phi_{S_i} = \psi_{S_i}$ on $\partial S_i \cup (\mathcal{P} \cap S_i)$, and $\phi_{S_i}$ is isotopic to $\psi_{S_i}$ rel $\partial S_i \cup (\mathcal{P} \cap S_i)$;

(b) $\phi_{S_{i+1}} \circ F \circ \psi_{S_i}^{-1} \big|_{\psi_{S_i}(E_i)} = R_i |_{\psi_{S_i}(E_i)}$;

(c) the return map $f_{S_i} := R_{i+1} \circ \cdots \circ R_1 \circ R_p \circ \cdots \circ R_i$ is a postcritically finite rational map whose conformal conjugacy class depends only on $F$ and $S_i$;

(d) for each $i \in \{1, \ldots, p\}$, for each Jordan curve $\gamma \subset \partial S_i$, and for $\beta_\gamma$ the curve in $\partial E_i$ homotopic to $\gamma$ within $S_i \setminus \mathcal{P}$, both $\phi_{S_i}(\gamma)$ and $\psi_{S_i}(\beta_\gamma)$ are equipotentials in the same
marked Fatou component of \( f_{S_i} \) with potentials

\[
\kappa(\phi_{S_i}(\gamma)) = u(\gamma) \quad \text{and} \quad \kappa(\psi_{S_i}(\beta_\gamma)) = \frac{u(F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})}.
\]  

(10)

Note that (a) and (b) together assert that \( R : \bigcup \psi_{S_i}(E_i) \to \bigcup \phi_{S_i}(S_i) \), marked by \( \bigcup \phi_{S_i}(\mathcal{P} \cap S_i) \), with \( R = R_i \) on \( \psi_{S_i}(E_i) \), is a holomorphic repelling system \( c \)-equivalent to \( (\bigcup E_i, E \cup S_i, \mathcal{P} \cap \bigcup S_i) \).

### 7.1 Disc-marked extension and equipotentials

For a repelling system \( F : \mathcal{E} \to \mathcal{L} \) of constant complexity, and any \( \mathcal{L} \)-piece \( S \), the above marking \( \mathcal{P}_S \) makes each complementary disc of \( S \) into a marked disc. We will use \( \kappa_S \) to denote the potential function of these complementary marked discs of \( S \). Let \( S_1 \) and \( S_2 \) be \( \mathcal{C} \)-pieces with \( F(E_1) = S_2 \) where \( E_1 \) is the unique complex \( \mathcal{E} \)-piece contained in \( S_1 \). There are many ways to extend \( F|_{E_1} \) to a branched covering. We choose the following one to rigidify the extension.

**Lemma 7.2.** Let \( S_1 \) and \( S_2 \) be \( \mathcal{C} \)-pieces with \( F(E_1) = S_2 \) where \( E_1 \) is the unique complex \( \mathcal{E} \)-piece contained in \( S_1 \). Let \( \rho \) be a positive function defined on the set of Jordan curves in \( \partial S_1 \). Then there is a quasi-regular branched covering extension \( h : \overline{\mathcal{T}}_{S_1} \to \overline{\mathcal{T}}_{S_2} \) of \( F|_{E_1} \) such that:

(a) \( h(\overline{\mathcal{T}}_{S_1} \setminus E_1) = \overline{\mathcal{T}}_{S_2} \setminus S_2 \).

(b) \( h(\mathcal{P}_{S_1}) \subset \mathcal{P}_{S_2} \) and the critical values of \( h \) are contained in \( \mathcal{P}_{S_2} \).

(c) For any Jordan curve \( \gamma \subset \partial S_1 \), \( h(\gamma) \) is an equipotential in a complementary marked disc of \( S_2 \), with potential \( \kappa_{S_2}(h(\gamma)) = \rho(\gamma) \).

(d) \( h \) is holomorphic in \( \overline{\mathcal{T}}_{S_1} \setminus S_1 \).

Such a map \( h \) will be called a **disc-marked extension** of \( F|_{E_1} \) associated to the function \( \rho \).

**Proof.** Let \( \alpha \) be a boundary component of \( E_1 \), bounding a unique complementary component \( \Delta_\alpha \) of \( E_1 \). Then \( \eta := F(\alpha) \) is a boundary curve of \( S_2 \), and bounds a unique complementary marked disc \( (\Delta_\eta, b) \) of \( S_2 \). Set \( d := \deg(F : \alpha \to \eta) \).

Note that \( \Delta_\alpha \) may contain zero or one complementary component of \( S \). In the former case, define \( h_\alpha : \Delta_\alpha \to \Delta_\eta \) to be a quasi-conformal map if \( d = 1 \), or a quasi-regular map with a unique critical value \( b \) if \( d > 1 \), such that \( h_\alpha|_\alpha = F|_\alpha \).

In the latter case \( \alpha \) is homotopic within \( S_1 \setminus \mathcal{P} \) to a unique boundary curve \( \gamma \) of \( S_1 \). Let \( \Delta_\gamma \) be the component of \( \overline{\mathcal{T}}_{S_1} \setminus S_1 \) enclosed by \( \gamma \). Then \( \Delta_\gamma \subset \Delta_\alpha \), and \( \Delta_\gamma \) together with the marked point \( a \in \Delta_\gamma \) is a complementary marked disc of \( S_1 \).

Let \( \eta_1 \) be the equipotential in the marked disc \( (\Delta_\eta, b) \) with potential \( \kappa_{S_2}(\eta_1) = \rho(\gamma) \). Denote by \( \Delta_1 \) the disc enclosed by \( \eta_1 \) and contained in \( \Delta_\eta \). Define \( h_\gamma : \Delta_\gamma \to \Delta_1 \) by \( h_\gamma(z) = \varphi^{-1} \circ (\varphi(z))^d \), where \( \varphi \) (resp. \( \varphi_1 \)) is a conformal map from the marked disc \( (\Delta_\gamma, a) \) (resp. \( (\Delta_1, b) \)) onto the unit disc \( \mathbb{D} \) with \( \varphi(a) = 0 \) (resp. \( \varphi_1(b) = 0 \)). Then there is a quasi-regular covering \( h_{\alpha \gamma} \) from \( \Delta_\alpha \setminus \overline{\Delta} \) onto \( \Delta_\eta \setminus \overline{\Delta} \) so that \( h_{\alpha \gamma}|_\alpha = F|_\alpha \) and \( h_{\alpha \gamma}|_\gamma = h_\gamma|_\gamma \). Set \( h_\alpha := h_\gamma \) on \( \Delta_\gamma \) and \( h_\alpha := h_{\alpha \gamma} \) on \( \Delta_\alpha \setminus \overline{\Delta} \). Then \( h_\alpha : \Delta_\alpha \to \Delta_\eta \) is also quasi-regular, in particular, holomorphic in \( \Delta_\gamma \).
The map \( F|_{E_1} \) together with \( h_\alpha \) for all boundary curves of \( E_1 \) forms a quasi-regular branched covering \( h : \overline{C}_{S_1} \to \overline{C}_{S_2} \). It satisfies the conditions (a)-(d).

### 7.2 Spherical holomorphic models

Consider the same marking \( \mathcal{P}_S \) (thus the function \( \kappa_S \)) for each \( \mathcal{L} \)-piece \( S \) as above.

**Lemma 7.3.** Let \( \rho \), resp. \( \sigma \), be two positives functions defined respectively on the set of Jordan curves in \( \bigcup_{i=1}^{p} \partial S_i \), resp. in \( \partial S_1 \). For \( 1 \leq i \leq p \), let \( h_i : \overline{C}_i \to \overline{C}_{i+1} \) be a disc-marked extension of \( F : E_i \to S_{i+1} \) associated to the function \( \rho \), given by Lemma 7.2.

Then there are pairs of quasi-conformal homeomorphisms \( (\Phi_i, \Psi_i) \) of \( \overline{C}_i \) onto a distinct copy \( \overline{C}_i \) of the Riemann sphere, and holomorphic maps \( R_i : \overline{C}_i \to \overline{C}_{i+1} (i = 1, \ldots, p) \), such that they satisfy the following conditions:

1. \( \Psi_i \) is isotopic to \( \Phi_i \) rel \( \mathcal{P}_S \), and \( \Phi_i \) is holomorphic on \( \overline{C}_i \backslash S_i \), \( i = 1, \ldots, p \).
2. \( R_i \equiv \Phi_{i+1} \circ h_i \circ \Phi_i^{-1} \) for \( 2 \leq i \leq p \) (with \( \Phi_{p+1} = \Phi_1 \)), and \( R_1 \equiv \Phi_2 \circ h_1 \circ \Psi_1^{-1} \) (see (13)).
3. For any \( i = 1, \ldots, p \), and \( f_i := R_{i-1} \circ \cdots \circ R_1 \circ R_p \circ \cdots \circ R_i \), we have \( f_i = \Phi_i \circ h_{i-1} \circ \cdots \circ h_1 \circ h_2 \circ \cdots \circ h_i \circ \Psi_i^{-1} \) and \( f_i \) is a postcritically finite rational map. The conformal conjugacy class of each \( f_i \) depends only on \( F \) and \( S_i \), but not on the choices of the markings, nor on the functions \( \rho \) and \( \sigma \), nor on \( h_i, \Phi_i, \Psi_i \).
4. For each Jordan curve \( \gamma \subset \partial S_1 \), the curve \( \Phi_1(\gamma) \) is an equipotential of \( f_1 \) with potential \( \kappa(\Phi_1(\gamma)) = \sigma(\gamma) \).

Consequently we have:

**Corollary 7.4.** (1) Fix \( 2 \leq i \leq p \). The map \( \Phi_i \) is holomorphic in \( \overline{C}_i \backslash S_i \). For each Jordan curve \( \gamma \subset \partial S_i \), and for \( \beta_\gamma \) the unique curve in \( \partial E_i \) homotopic to \( \gamma \) within \( S_i \setminus \mathcal{P} \), both \( \Phi_i(\beta_\gamma) \) and \( \Phi_i(\beta_\gamma) \) are equipotentials in the same marked Fatou component of \( f_i \). Their potentials are related as follows:

\[
\kappa(\Phi_i(\gamma)) = \frac{\rho(\gamma)}{\deg(F|_{\beta_\gamma})} + \kappa(\Phi_i(\beta_\gamma)), \quad \kappa(\Phi_i(\beta_\gamma)) = \frac{\kappa(\Phi_{i+1} \circ F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})}.
\] (11)

(2) For each Jordan curve \( \gamma \subset \partial S_1 \), and for \( \beta_\gamma \) the curve in \( \partial E_1 \) homotopic to \( \gamma \) within \( S_1 \setminus \mathcal{P} \), the curve \( \Psi_1(\beta_\gamma) \) is an equipotential in the marked Fatou component of \( f_1 \) that contains \( \Phi_1(\gamma) \), with potential

\[
\kappa(\Psi_1(\beta_\gamma)) = \frac{\kappa(\Phi_2 \circ F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})}.
\] (12)

See the following diagram:

\[
E_1 \xrightarrow{\Psi_1} \overline{C}_1 \xrightarrow{h_1} \overline{C}_2 \xrightarrow{h_2} \overline{C}_3 \xrightarrow{\cdots} \overline{C}_p \xrightarrow{h_p} \overline{C}_1
\]

\[
S_1 \xrightarrow{R_1} \overline{C}_1 \xrightarrow{R_2} \overline{C}_2 \xrightarrow{R_3} \overline{C}_3 \xrightarrow{\cdots} \overline{C}_p \xrightarrow{R_p} \overline{C}_1
\]

\[
E_2 \xrightarrow{\Psi_2} \overline{C}_2 \xrightarrow{\Psi_3} \overline{C}_3 \xrightarrow{\cdots} \overline{C}_p \xrightarrow{\Psi_2 \circ F_2} \overline{C}_1
\]

\[
E_3 \xrightarrow{\Psi_3} \overline{C}_3 \xrightarrow{\cdots} \overline{C}_p \xrightarrow{\Psi_3 \circ F_3} \overline{C}_1
\]

\[
E_p \xrightarrow{F_p} \overline{C}_p \xrightarrow{h_p} \overline{C}_1
\]
Proof of Lemma 7.3
Denote by \( H : E \to S_1 \) the renormalization of \( F \) relative to \( S_1 \). Set
\[
    h := h_p \circ \cdots \circ h_2 \circ h_1 : \overline{C}_1 \to \overline{C}_1.
\]
Then \( h(\mathcal{P}_{S_1}) \subset \mathcal{P}_{S_1} \) and \( \mathcal{P}_h \subset \mathcal{P}_{S_1} \). Clearly, \( (h, \mathcal{P}_{S_1}) \) is a marked extension of the renormalization \( H : E \to S_1 \).

It is easy to see that the c-equivalence class of \( (h, \mathcal{P}_{S_1}) \) does not depend on the choice of extensions.

Now the assumption that \( (F, \mathcal{P}) \) has no renormalized obstructions implies that \( (H, \mathcal{P} \cap S_1) \) as a repelling system, has no Thurston obstructions. This in turn implies that \( (h, \mathcal{P}_{S_1}) \) has an orbifold distinct from \( (2, 2, 2, 2) \) and has no Thurston obstructions, as follows: Since the marked points in \( \overline{C}_1 \setminus S_1 \) map to themselves by \( h \), they are eventually \( h \)-periodic. Let \( b \) be a periodic marked point in \( \overline{C}_1 \setminus S_1 \) with period \( k \geq 1 \). Denote by \( \Delta_b \) the component of \( \overline{C}_1 \setminus S_1 \) that contains the marked point \( b \) and \( \gamma_b := \partial \Delta_b \). Then there is unique component of \( h^{-k}(\gamma) \), denoted by \( \beta \), such that \( \beta \) is homotopic to \( \gamma \) rel \( \mathcal{P}_{S_1} \). Note that \( \gamma \) is contained in the boundary multicurve \( Y \) and \( \beta \) is a component of \( F^{-k}(\gamma) \) in \( S_1 \). Thus the assumption \( \lambda(W_Y) < 1 \) implies that
\[
    \deg(F^{k_p} : \beta \to \gamma) = \deg(h^k : \beta \to \gamma) = \deg_b h^k > 1.
\]
This implies that \( h \) has a periodic critical point (in the cycle of \( b \)). Therefore \( (h, \mathcal{P}_{S_1}) \) has an orbifold distinct from \( (2, 2, 2, 2) \). Now any multicurve within \( \overline{C} \setminus \mathcal{P}_{S_1} \) can be represented by a multicurve within \( S_1 \setminus \mathcal{P}_{S_1} = S_1 \setminus (\mathcal{P} \cap S_1) \). So its \( (h, \mathcal{P}_{S_1}) \)-transition matrix is equal to its \( (H, \mathcal{P} \cap S_1) \)-transition matrix, thus has the same leading eigenvalue, which is less than one. This implies \( (h, \mathcal{P}_{S_1}) \) has no Thurston obstructions.

Applying Thurston Theorem to get \( (\phi, \psi) \) and \( f_1 \). We can then apply Thurston Theorem 3.2 to obtain a pair of quasi-conformal maps \((\phi, \psi)\) from \( \overline{C}_1 \) onto \( \overline{C}_1 \) and a rational map \( f_1 \), whose conformal conjugacy class depends only on the c-equivalence class of \( (h, \mathcal{P}_{S_1}) \) (which depends only on \( (H, \mathcal{P} \cap S_1) \)), such that \( \psi \) is isotopic to \( \phi \) rel \( \mathcal{P}_{S_1} \) and \( f_1 = \phi \circ h \circ \psi^{-1} \). In particular \( f_1 \) does not depend on the choice of the functions \( \rho \) and \( \sigma \). Moreover, \( \mathcal{P}_{f_1} \subset \phi(\mathcal{P}_{S_1}) \).

As any periodic cycle \( b \) of marked points in \( \overline{C} \setminus S_1 \) contains a critical point of \( h \), the cycle \( (\phi(b)) \) is a superattracting periodic cycle for \( f_1 \). Consequently, for every marked point \( a \) in \( \overline{C}_1 \setminus S_1 \), \( \phi(a) \) is an eventually superattracting periodic point of \( f_1 \).

From \( (\phi, \psi) \) to \( (\Phi_1, \Psi_1) \). For every marked point \( a \) in \( \overline{C}_1 \setminus S_1 \), denote by \( \Delta_a \) the component of \( \overline{C}_1 \setminus S_1 \) that contains \( a \) and \( \gamma_a = \partial \Delta_a \). Denote by \( \eta_a \) the equipotential of the Fatou component of \( f_1 \) containing \( \phi(a) \) (with \( \phi(a) \) as a marked point), with potential \( \kappa(\eta_a) = \sigma(\gamma_a) \). Then there is a quasi-conformal map \( \Phi_1 \) in the isotopy (rel \( \mathcal{P}_{S_1} \)) class of \( \phi \) such that for every marked point \( a \) in \( \overline{C}_1 \setminus S_1 \), we have \( \Phi_1(\gamma_a) = \eta_a \) (this is because \( \gamma_a \), resp. \( \eta_a \), is peripheral around the point \( a \in \mathcal{P}_{S_1} \), resp. the point \( \phi(a) \in \phi(\mathcal{P}_{S_1}) \)). Moreover, \( \Phi_1 \) can be taken to be holomorphic on \( \bigcup_a \Delta_a = \overline{C}_1 \setminus S_1 \).

As \( \Phi_1 \) is isotopic to \( \phi \) rel \( \mathcal{P}_{S_1} \), there is a quasi-conformal map \( \Psi_1 : \overline{C}_1 \to \overline{C}_1 \) such that it is isotopic to \( \psi \) rel \( \mathcal{P}_{S_1} \) and \( \Phi_1 \circ h \circ \Psi_1^{-1} = f_1 \).

Getting (in order) \( \Phi_p, R_p, \Phi_{p-1}, R_{p-1}, \cdots, \Phi_2, R_2 \) and then \( R_1 \). This is illustrated in the following diagrams:

\[
\begin{align*}
\overline{C}_2 &\xrightarrow{h_2} \overline{C}_3 &\to \cdots &\to \overline{C}_p &\xrightarrow{h_p} \overline{C}_1 &\xrightarrow{h_1} \overline{C}_2 \\
\downarrow \Phi_2 &\quad \downarrow \Phi_3 &\quad \cdots &\quad \downarrow \Phi_p &\quad \downarrow \Phi_1 &\quad \text{and} &\quad \Psi_1 &\quad \downarrow \Phi_2 \\
\overline{C}_2 &\xrightarrow{R_2} \overline{C}_3 &\to \cdots &\to \overline{C}_p &\xrightarrow{R_p} \overline{C}_1 &\xrightarrow{R_1} \overline{C}_2
\end{align*}
\]
More precisely pull-back the complex structure of $\overline{C}_1$ to $\overline{C}_p$ by $\Phi_1 \circ h_p$, we have a quasi-conformal map $\Phi_p : \overline{C}_p \to \overline{C}_p$ such that $R_p := \Phi_1 \circ h_p \circ \Phi_p^{-1}$ is holomorphic.

As a disc-marked extension, we know that $h_p$ is holomorphic in $\overline{C}_p \setminus S_p$ whose $h_p$-image is contained in $\overline{C}_1 \setminus S_1$. Combining with the result that $\Phi_1$ is holomorphic in $\overline{C}_1 \setminus S_1$ and the equation $R_p \circ \Phi_p = \Phi_1 \circ h_p$, we see that $\Phi_p$ is holomorphic in $\overline{C}_p \setminus S_p$.

Inductively, for $i = p - 1, \ldots, 2$, we have a quasi-conformal map $\Phi_i : \overline{C}_i \to \overline{C}_i$ such that $R_i := \Phi_{i+1} \circ h_i \circ \Phi_i^{-1}$ is holomorphic and $\Phi_i$ is holomorphic in $\overline{C}_i \setminus S_i$.

Set finally $R_1 := \Phi_2 \circ h_1 \circ \Psi_1^{-1}$. Then $R_p \circ \cdots \circ R_2 \circ R_1 = f_1$. Therefore $R_1$ is also holomorphic and $\Psi_1$ is holomorphic in $\overline{C}_1 \setminus S_1$.

**Getting $\Psi_i$ and $f_i$.** As a disc-marked extension, we know that the critical values of $h_i$ is contained in $\mathcal{P}_{S_i+1}$ and $h_i(\mathcal{P}_{S_i}) \subset \mathcal{P}_{S_{i+1}}$ for $1 \leq i \leq p$. Because $\Psi_1$ is isotopic to $\Phi_1$ rel $\mathcal{P}_S$, there is a quasi-conformal map $\Psi_p : \overline{C}_p \to \overline{C}_p$ such that $\Psi_p$ is isotopic to $\Phi_p$ rel $\mathcal{P}_S$ and $\Psi_1 \circ h_p = R_p \circ \Psi_p$. Inductively, there is a quasi-conformal map $\Psi_i : \overline{C}_i \to \overline{C}_i$ for $i = p - 1, \ldots, 2$, such that $\Psi_i$ is isotopic to $\Phi_i$ rel $\mathcal{P}_S$ and $\Psi_{i+1} \circ h_i = R_i \circ \Psi_i$. Set then $f_i := R_{i-1} \circ \cdots \circ R_1 \circ R_p \circ \cdots \circ R_i$. Now we have the following commutative diagrams:

$$
\begin{array}{cccccc}
\overline{C}_i & \xrightarrow{h_i} & \cdots & \xrightarrow{h_1} & \overline{C}_1 & \xrightarrow{\Psi_1} \\
\Psi_i \downarrow & \xrightarrow{\Psi_p} & \cdots & \xrightarrow{\Psi_2} & \cdots & \xrightarrow{\Psi_1} \\
\overline{C}_i & \xrightarrow{R_i} & \cdots & \xrightarrow{R_1} & \overline{C}_1 & \xrightarrow{f_i} \\
\end{array}
$$

It is easy to see that $f_i = \Phi_i \circ h_{i-1} \circ \cdots \circ h_1 \circ h_p \cdots \circ h_i \circ \Psi_i^{-1}$ for $i \geq 2$. The above formula shows that $f_i$ is $c$-equivalent to $h_{i-1} \circ \cdots \circ h_1 \circ h_p \circ \cdots \circ h_i$, which is postcritically finite. So $f_i$ is also postcritically finite and $\mathcal{P}_{f_i} \subset \Phi_i(\mathcal{P}_{S_i})$. Clearly it is $c$-equivalent to a marked extension of the renormalization relative to $S_i$. Again its conformal conjugacy class depends only on $F$ and $S_i$.

**Proof of Corollary 7.2.** Notice that $f_{i+1} \circ R_i = R_i \circ f_i$, i.e., $R_i$ is a holomorphic (semi-)conjugacy from $f_i$ to $f_{i+1}$ (set $f_{p+1} = f_1$). It is classical that their Julia sets are related by $\mathcal{J}(f_i) = R_i^{-1}(\mathcal{J}(f_{i+1}))$. Note that the critical values of $R_i$ are contained in $\Phi_i(\mathcal{P}_{S_i+1})$, which is eventually periodic under $f_{i+1}$. We see that $R_i$ maps equipotentials of $f_i$ to equipotentials of $f_{i+1}$.

As a disc-marked extension, for each Jordan curve $\gamma \subset \partial S_p$, the curve $h_p(\gamma)$ lies on an equipotential in a complementary marked disc of $S_1$. Because each Jordan curve in $\partial \Phi_1(S_1)$ lies on an equipotential of $f_1$ and $\Phi_1$ is holomorphic in $\overline{C}_1 \setminus S_1$, the curve $h_p(\gamma)$ goes to an equipotential of $f_1$ by $\Phi_1$. This equipotential of $f_1$ is pulled back by $R_p$ to equipotentials of $f_p$. Thus $\Phi_p(\gamma)$ lies on an equipotential of $f_p$. Inductively, we have that each Jordan curve in $\partial \Phi_i(S_i)$ lies on an equipotential of $f_i$ for $i = 1, \ldots, p$.

Similarly, each curve in $\Phi_i(\partial E_i)$ lies on an equipotential of $f_i$ for $i \geq 2$ and each curve in $\Phi_1(\partial E_1)$ lies on an equipotential of $f_1$.

Fix $i \in \{1, \ldots, p\}$. For each Jordan curve $\gamma \subset \partial S_i$, and for $\beta_i$ the curve in $\partial E_i$ homotopic to $\gamma$ within $S_i \setminus \mathcal{P}$, we have that $h_i(\beta_i) = F(\beta_i)$ is a curve in $\partial S_{i+1}$. Note that $\Phi_{i+1} \circ h_i(\beta_i) = R_i \circ \Phi_i(\beta_i)$ if $i \neq 1$ (with $\Phi_{p+1} = \Phi_1$) or $\Phi_2 \circ h_1(\beta_i) = R_i \circ \Psi_1(\beta_i)$ if $i = 1$. Their potentials are related by:

$$
\kappa(\Phi_i(\beta_i)) = \frac{\kappa(\Phi_{i+1} \circ F(\beta_i))}{\deg(F|_{\beta_i})} \text{ if } i \neq 1 \quad \text{or} \quad \kappa(\Psi_1(\beta_i)) = \frac{\kappa(\Phi_2 \circ F(\beta_i))}{\deg(F|_{\beta_i})} \text{ if } i = 1.
$$

Fix now $2 \leq i \leq p$. By the construction of $h_i$ in Lemma 7.2, the curve $h_i(\gamma)$ is
an equipotential with potential $\rho(\gamma)$ in a complementary marked disc of $S_{i+1}$. We have $\mod h_i(A(\gamma, \beta_\gamma)) = \rho(\gamma)$, where $A(\gamma, \beta_\gamma)$ is the annulus between them. Notice that $\Phi_{i+1}$ is conformal in $\overline{C_{i+1} \setminus S_{i+1}}$. We also have $\mod \Phi_{i+1} \circ h_i(A(\gamma, \beta_\gamma)) = \rho(\gamma)$. From the equation $R_i \circ \Phi_i = \Phi_{i+1} \circ h_i$, we get

$$\kappa(\Phi_i(\gamma)) - \kappa(\Phi_i(\beta_\gamma)) = \mod \Phi_i(A(\gamma, \beta_\gamma)) = \frac{\rho(\gamma)}{\deg(F|_{\beta_\gamma})}.$$  

\hfill \Box

**Remark 1.** For every $i$, if we make a normalization by requiring that three given distinct points in $P_{S_i}$ (note that $\#P_{S_i} \geq 3$ since $S_i$ is a C-piece) go to $(0, -1, \infty)$ under the action of $\Phi_i$, then $f_i$ is uniquely determined, as well as the homotopy class (rel $P_{S_i}$) of $\Phi_i$.

**Remark 2.** For a $F_\ast$-periodic cycle $(S_1, \ldots, S_p)$, we have $p$ renormalizations (one for each $S_i$). Lemma [7.3] shows that none of them has Thurston obstructions if one of them has no Thurston obstructions.

### 7.3 Proof of Theorem [7.1]

Fix now the positive functions $\sigma$ and $\rho$ as follows:

$$\forall \gamma \subset \partial S_1, \quad \sigma(\gamma) := u(\gamma); \quad \forall \gamma \subset \bigcup_{i=1}^p \partial S_i, \quad \rho(\gamma) := \left( u(\gamma) - \frac{u(F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})} \right) \deg(F|_{\beta_\gamma}),$$  

where $\beta_\gamma$ is the curve in $\bigcup_{i=1}^p \partial E_i$ homotopic to $\gamma$ within $\mathcal{L} \setminus \mathcal{P}$. Note that $\rho(\gamma) > 0$ for every $\gamma$ by the assumption $D_\ast u < u$.

Let $(\Phi_1, \Psi_1, R_1, f_1, \ldots, f_p)$ be the collection of maps derived from Lemma [7.3] with the functions $\rho$ and $\sigma$ defined above. Set $f_{S_1} := f_i$.

Let $\gamma$ be a Jordan curve in $\partial S_1$. Then $\kappa(\Phi_1(\gamma)) = \sigma(\gamma) = u(\gamma)$ by Lemma [7.3] (4).

Let $\gamma$ be a Jordan curve in $\partial S_p$ and $\beta_\gamma$ be the curve in $\partial E_p$ homotopic to $\gamma$ within $S_p \setminus \mathcal{P}$. We have

$$\kappa(\Phi_p(\gamma)) = \frac{\rho(\gamma)}{\deg(F|_{\beta_\gamma})} + \frac{\kappa(\Phi_1 \circ F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})} \frac{\rho(\gamma)}{\deg(F|_{\beta_\gamma})} + \frac{u(F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})} \overset{14}{=} u(\gamma).$$

Inductively, for $i = p - 1, \ldots, 2$, we have $\kappa(\Phi_i(\gamma)) = u(\gamma)$ for any Jordan curve $\gamma \subset \partial S_i$. Therefore $\kappa(\Phi_i(\gamma)) = u(\gamma)$ for any $i$ and any $\gamma \subset \partial S_i$.

Fix any $i \in \{1, \ldots, p\}$. Let $\beta$ be a curve in $\partial E_i$ non-null-homotopic within $S_i \setminus \mathcal{P}$. By (11) and (12), we have

$$\kappa(\Phi_i(\beta)) = \frac{\kappa(\Phi_{i+1} \circ F(\beta))}{\deg(F|_{\beta})} = \frac{u(F(\beta))}{\deg(F|_{\beta})} \quad \text{if } i \neq 1 \quad \text{and} \quad \kappa(\Psi_1(\beta)) = \frac{u(F(\beta))}{\deg(F|_{\beta})} \quad \text{if } i = 1.$$

Let $\gamma$ be a Jordan curve in $\partial S_1$ and $\beta_\gamma$ be the Jordan curve in $\partial E_1$ homotopic to $\gamma$ within $S_1 \setminus \mathcal{P}$. From the above formula and the fact that $D_\ast u < u$, we deduce that $\kappa(\Psi_1(\beta_\gamma)) < \kappa(\Phi_1(\gamma))$. This implies that $\Psi_1(E_1) \subset \Phi_1(S_1)$.

For $i = 2, \ldots, p$, set $\phi_{S_i} = \psi_{S_i} = \Phi_i$. Set also $\phi_{S_1} = \Phi_1$. Obviously, (a)-(d) hold for $i \geq 2$ by the above computation. Now we want to define $\psi_{S_1}$.

Notice that $\Psi_1(E_1) \subset \Phi_1(S_1)$ and $\Psi_1$ is isotopic to $\Phi_1$ rel $\mathcal{P}_{S_1}$. This implies that for each Jordan curve $\gamma \subset \partial S_1$, both $\gamma$ and $\Psi_1^{-1} \circ \Phi_1(\gamma)$ are contained in the same disk component.
Δ of \( \overline{C_1 \setminus E_1} \), and are homotopic within \( \Delta \setminus \{a\} \), where \( a \) is the unique point of \( P_{S_1} \) in \( \Delta \). Therefore there is a quasi-conformal map \( \eta \) of \( \overline{C_1} \) isotopic to the identity rel \( P_{S_1} \) so that \( \eta|_{E_1} = id \) and \( \eta = \Phi_1^{-1} \circ \Phi_1 \) on \( \overline{C_1 \setminus S_1} \). Set \( \hat{\psi} := \Phi_1 \circ \eta \). Then \( \hat{\psi} \) is isotopic to \( \Psi_1 \) therefore to \( \Phi_1 \) rel \( P_{S_1} \), with \( \hat{\psi}|_{\overline{C_1 \setminus S_1}} = \Phi_1|_{\overline{C_1 \setminus S_1}} \) and \( \hat{\psi}|_{E_1} = \Psi_1 \). This gives already (b).

To get (a), i.e. \( \hat{\psi} \) is isotopic to \( \Phi_1 \) rel \( P_{S_1} \cup \partial S_1 \), we need to modify \( \hat{\psi} \) on the annuli of \( S_1 \setminus E_1 \).

Set \( \chi = \Phi_1^{-1} \circ \hat{\psi} \). Then by a purely topological argument (see Lemma 8.1) there is a homeomorphism \( T \) which is the identity outside \( S_1 \setminus E_1 \) such that \( \chi \circ T \) is isotopic to the identity rel \( P_{S_1} \cup (\overline{C_1 \setminus S_1}) \). Set now \( \psi_{S_1} = \hat{\psi} \circ T \). We get both (a) and (b) of the Theorem. Point (d) for \( i = 1 \) is also derived from the above computation.

\section{Proof of Theorem 5.4}

Let now \((\mathcal{E}, F, \mathcal{L}, \mathcal{P})\) be a marked repelling system of constant complexity, without boundary obstructions nor renormalized obstructions. We prove here that \((F, \mathcal{P})\) is \(c\)-equivalent to a holomorphic model. Decompose \( \mathcal{L} \) into \( O \sqcup A \sqcup R \sqcup C \) as in \([1]\).

\subsection{Choice of the positive vector}

Let \( Y \) be a boundary multicurve of \((F, \mathcal{P})\). It can be chosen to be the collection of Jordan curves in \( \partial \mathcal{C} \sqcup \partial \mathcal{A} \). By assumption, for the \((F, \mathcal{P})\)-transition matrix \( W_Y \), we have \( \lambda(W_Y) < 1 \).

Applying Lemma \([A,1]\) we have a positive vector \( v \in \mathbb{R}^Y \) so that \( W_Y v < v \), i.e. there is a positive function

\[ v : Y \to \mathbb{R}^+ \quad \text{such that} \quad (W_Y v)_\gamma = \sum_{\eta \in Y, \alpha \sim \gamma} \frac{v(\eta)}{\deg(F : \alpha \to \eta)} < v(\gamma), \tag{15} \]

where the last sum is taken over all curves \( \alpha \) in \( F^{-1}(\eta) \) that are homotopic to \( \gamma \) within \( \mathcal{L} \setminus \mathcal{P} \).

Let \( C > 0 \) be a constant to be determined later. Denote by 1 the vector whose every entry is 1. Choose \( M > 0 \) to be a large number so that \( W_Y(Mv) + C1 < Mv \), i.e.

\[ \forall \gamma \in Y, \quad \sum_{\eta \in Y, \alpha \sim \gamma} \frac{Mv(\eta)}{\deg(F : \alpha \to \eta)} + C < Mv(\gamma). \tag{16} \]

For any \( \gamma \in Y \) the quantity \( Mv(\gamma) \) will be the prescribed potential for \( \phi_S(\gamma) \), with \( S \) the \( \mathcal{L} \)-piece admitting \( \gamma \) as a boundary curve.

\subsection{Definition of \((\phi_S, \psi_S)\) and \(f_S\) for \(\mathcal{C}\)-pieces}

Assume at first that \( S_1, \ldots, S_p \) are \( \mathcal{C} \)-pieces with \( F(E_i) = S_{i+1} \) and \( S_{p+1} = S_1 \), where \( E_i \) is the unique complex \( \mathcal{E} \)-piece in \( S_i \). Set \( u := Mv|\bigcup \partial S_i \). We have \( D_* u < u \) for \( D_* \) the \((F, \mathcal{P})\)-transition matrix of the set of the boundary curves of \( S_1 \cup \ldots \cup S_p \). We construct \( \phi_{S_i}, \psi_{S_i}, R_{S_i} \) and \( f_{S_i} \) according to Theorem \([7,1]\) for \( i = 1, \ldots, p \). We do so for every periodic cycle of \( F_* \).

Assume now that \( S \) is a non-\( F_* \)-periodic \( \mathcal{C} \)-piece. Then there are \( \mathcal{C} \)-pieces \( S =: S_{-k}, S_{-k+1}, \ldots, S_0 \) \((k > 0)\) such that \( S_i \) is not \( F_* \)-periodic for \( i < 0 \) but \( S_0 \) is \( F_* \)-periodic, and \( F(E_i) = S_{i+1} \) for \( i < 0 \), where \( E_i \) is the unique complex \( \mathcal{E} \)-piece in \( S_i \).
Denote by $\overline{C}_i = \overline{C}_{S_i}$ for simplicity. As $S_0$ is $F$-periodic, we have already constructed a quasi-conformal map $\phi_{S_0} : \overline{C}_0 \to \overline{C}_0$ and a postcritically finite rational map $f_0$ on $\overline{C}_0$ such that they satisfy the conditions of Theorem 7.1 for $u = Mv$.

For $i = -1, -2, \ldots, -k$, let $h_i : \overline{C}_i \to \overline{C}_{i+1}$ be a disc-marked extension of $F : E_i \to S_{i+1}$, given by Lemma 7.2 associated to the function

$$\rho(\gamma) := \left( Mv(\gamma) - \frac{Mv(F(\beta_\gamma))}{\deg(F|_{\beta_\gamma})} \right) \deg(F|_{\beta_\gamma}),$$

where $\gamma$ is a Jordan curve in $\partial S_i$ and $\beta_\gamma$ is the curve in $\partial E_i$ homotopic to $\gamma$ within $S_i \setminus \mathcal{P}$. As before, there are quasi-conformal maps $\phi_{S_{i+1}} = \phi_i : \overline{C}_i \to \overline{C}_i$ and holomorphic maps $R_i : \overline{C}_i \to \overline{C}_{i+1}$ such that the following diagram commutes:

$$\begin{array}{cccc}
\overline{C}_{-k} & \xrightarrow{h_{-k}} & \overline{C}_{-k+1} & \xrightarrow{h_{-k+1}} & \cdots & \xrightarrow{h_{-1}} & \overline{C}_0 \\
\downarrow \phi_{-k} & & \downarrow \phi_{-k+1} & & \cdots & & \downarrow \phi_0 = \phi_{S_0} \\
\overline{C}_{-k} & \xrightarrow{R_{-k}} & \overline{C}_{-k+1} & \xrightarrow{R_{-k+1}} & \cdots & \xrightarrow{R_{-1}} & \overline{C}_0
\end{array}$$

Because $h_i(P_{S_{i+1}}) \subset P_{S_{i+1}}$ and every critical value (if exists) of $h_i$ lies on $P_{S_{i+1}}$, we have $R_i(\phi_i(P_{S_i})) \subset \phi_{i+1}(P_{S_{i+1}})$, and every critical value of $R_{-1} \circ \cdots \circ R_i$ lies on $\phi_0(P_{S_0})$.

Set $f_{S_i} := R_{-1} \circ \cdots \circ R_i$. Let $b \in \overline{C}_i \setminus S_i$ be a marked point. Then $f_{S_i} \circ \phi_i(b)$ is the center of a marked Fatou component $\Delta$ of $f_0$. The component $\Delta_{\phi_i(b)}$ of $f_{S_i}^{-1}(\Delta)$ that contains $\phi_i(b)$ is a disc. We will call $(\Delta_{\phi_i(b)}, \phi_i(b))$ a canonical marked disc.

The name 'canonical' means that up to a Möbius transformation, the configuration formed by these marked discs is uniquely determined. Note that when a disc-marked extension $h_i$ is chosen, up to a Möbius transformation, $\phi_i$ is uniquely determined by $\phi_{i-1}$. As $\phi_{i-1}$ varies in its homotopy class, $\phi_i$ varies simultaneously in its homotopy class while $R_i$ remains unchanged. On the other hand various choices of disc-marked extensions are related by quasi-conformal maps. More precisely, if $\tilde{h}_i$ is another choice of the disc-marked extension, then there is a quasi-conformal map $\xi$ of $\overline{C}_i$ isotopic to the identity rel $P_{S_i}$, such that $\tilde{h}_i = h_i \circ \xi$. Now set $\tilde{\phi}_i = \phi_i \circ \xi$, we get the same holomorphic map $R_i$. This implies that the maps $f_i$ are in dependent of the extensions (but may depend on the marking). In particular, the canonical marked discs are independent of the large number $M$ involved in the function $\rho$ (therefore involved in the extensions $h_i$).

**Lemma 8.1.** With the assumption above, for any $i = -k, \cdots, -1$, there are quasi-conformal maps $\psi_{S_i} = \phi_{S_i} : \overline{C}_{S_i} \to \overline{C}_{S_i}$ such that:

1. $R_{i} := \phi_{S_{i+1}} \circ h_i \circ \phi_{S_i}^{-1}$ is holomorphic and is independent of $M$.
2. For any marked point $b \in P_{S_i} \setminus S_i$, denote by $\gamma_b$ the component of $\partial S_i$ that separates $b$ from $S_i \setminus \gamma_b$ and by $\alpha_b$ the component of $\partial E_i$ that separates $b$ from $E_i \setminus \alpha_b$. Then both $\phi_{S_i}(\gamma_b)$ and $\phi_{S_i}(\alpha_b)$ are equipotentials in the canonical marked disc $(\Delta_{\phi_{S_i}(b)}, \phi_{S_i}(b))$ with potentials

\begin{align}
\kappa(\phi_{S_i}(\gamma_b)) = Mv(\gamma_b) \quad \text{and} \quad \kappa(\phi_{S_i}(\alpha_b)) = \kappa(\psi_{S_i}(\alpha_b)) = \frac{Mv(F(\alpha_b))}{\deg(F|_{\alpha_b})}. \quad (17)
\end{align}

**Proof.** (1) is obvious. The proof of (2) is quite easy by following the same argument as before. \qed
8.3 Definition of \((\phi_S, \psi_S)\) for other \(L\)-pieces

Define \(\phi_S = \psi_S = \text{id} \) for all \(O \cup R\)-pieces. Assume that \(S\) is an \(A\)-piece. Then it is a closed annulus and one of its boundary curve, say \(\gamma\), is contained in the boundary multicurve \(Y\). We define \(\phi_S\) to be a quasi-conformal map from \(S\) to a round annulus \(S'\) in \(\mathbb{T}_{S'}\) with modulus

\[
\mod \phi_S(S) = Mv(\gamma).
\] (18)

We will define a map \(\psi_A\) for all annular components \(A\) of \(L \setminus E^m\), including the \(A\)-pieces. For this we decompose \(E\) into \(E^m \sqcup E^2 \sqcup E^0\) as follows:

- \(E^m\) is the union of complex \(E\)-pieces;
- \(E^0\) is the union of \(E\)-pieces which are contained in a disk \(D \subset L\) with \(\#(D \cap P) \leq 1\);
- \(E^2\) is the union of \(E \setminus E^0\)-pieces which are contained in an annulus \(A \subset L\) with \(A \cap P = \emptyset\).

Clearly, the above three sets are mutually disjoint. Topologically, \(E^m \subset C\) and \(E^2 \subset C \cup A\). Dynamically, \(F^{-1}(O \cup R) \subset E^0\) and \(F^{-1}(A) \subset E^2 \cup E^0\).

See Figures 2.

![Figure 2: The \(L\)-pieces are bounded by thick curves. Light grey ones are \(E^m\)-pieces, darker-greys, for example \(E\) and \(T\), are \(E^2\)-pieces. Hatched ones are \(E^0\)-pieces (they may appear in any, necessarily disc or annular, component of \(L \\setminus (E^m \cup E^2)\), and may be nested in each others holes).](image)

1. Definition of an auxiliary map \(\varphi_E\) for \(E^2\)-pieces.

Assume that \(E\) is a \(E^2\)-piece and \(S\) is an \(L\)-piece with \(E \subset S\). Set \(\tilde{S} := F(E)\). Then both \(S\) and \(\tilde{S}\) are contained in \(C \cup A\). Decompose \(E^2\) into \(E^{(2,2)} \sqcup E^{(2,m)}\) so that \(F(E^{(2,2)}) \subset A\) and \(F(E^{(2,m)}) \subset C\).

If \(\tilde{S}\) is an \(A\)-piece, then there is a quasi-conformal map \(\varphi_E\) from \(E\) onto a closed round annulus such that \(\phi_{\tilde{S}} \circ F \circ \varphi_E^{-1}\) is holomorphic in the interior of \(\varphi_E(E)\).
Let \( \gamma \) be one of the two boundary curves in \( \partial \hat{S} \) with \( \gamma \in Y \). Then there is a Jordan curve \( \beta \) in \( \partial E \) so that \( F(\beta) = \gamma \). From (18), we have:

\[
\mod \varphi_E(E) = \frac{\mod \phi_{\hat{S}}(\hat{S})}{\deg F|_E} = \frac{Mu(F(\beta))}{\deg(F|_\beta)}.
\] (19)

Now assume \( \hat{S} \) is a \( C \)-piece. Then there is a quasi-regular branched covering \( h_E : \overline{\mathbb{T}}_S \to \overline{\mathbb{T}}_{\hat{S}} \) such that \( h_E|_E = F|_E \), \( h_E(E^c) = \hat{S}^c \) and every critical value of \( h_E \) is contained in \( \mathcal{P}_{\hat{S}} \). As before, we have a quasi-conformal map \( \varphi_E \) of \( \overline{\mathbb{T}} \) such that \( \varphi_E \circ h_E \circ \varphi_E^{-1} \) is holomorphic from \( \overline{\mathbb{T}}_{\mathcal{S}} \) to \( \overline{\mathbb{T}}_{\hat{S}} \).

\[
\begin{align*}
\overline{\mathbb{T}}_S & \supset S \supset E \xrightarrow{\varphi_E} \varphi_E(E) \subset \overline{\mathbb{T}}_{S'} \quad \alpha, \beta \xrightarrow{\varphi_E} \subset \Delta_{\alpha'}, \Delta_{\beta'} \\
h_E \downarrow \overline{\mathbb{T}}_{\hat{S}} & \supset \hat{S} \xrightarrow{\phi_{\hat{S}}} \hat{S}' \subset \overline{\mathbb{T}}_{\hat{S}'} \quad \phi_{\hat{S}} \xrightarrow{\phi_{\hat{S}}} \subset \Delta_{\alpha}, \Delta_{\beta}
\end{align*}
\]

Note that \( \partial E \) has exactly two boundary curves \( \alpha \) and \( \beta \) that are non-null-homotopic within \( S \setminus \mathcal{P} \). They are homotopic to each other within \( S \setminus \mathcal{P} \). From Theorem 7.1 and Lemma 8.1, we know that \( \phi_{\hat{S}} \circ F(\alpha) \) (resp. \( \phi_{\hat{S}} \circ F(\beta) \)) is an equipotential, in a marked disc \( (\Delta_a, a) \) (resp. \( (\Delta_b, b) \)) of the postcritically finite rational map \( f_{\hat{S}} \) when \( \hat{S} \) is \( F_* \)-periodic, or in a canonical marked disc, denoted also by \( (\Delta_a, a) \) (resp. \( (\Delta_b, b) \)) otherwise, whose potentials are

\[\kappa(\phi_{\hat{S}} \circ F(\alpha)) = Mu(F(\alpha)), \quad \kappa(\phi_{\hat{S}} \circ F(\beta)) = Mu(F(\beta)).\]

Let \( \Delta_{\alpha'} \) (resp. \( \Delta_{\beta'} \)) be the component of \( R^{-1}_E(\Delta_a) \) (resp. \( R^{-1}_E(\Delta_b) \)) that contains \( \varphi_E(\alpha) \) (resp. \( \varphi_E(\beta) \)). Then \( \Delta_{\alpha'} \) and \( \Delta_{\beta'} \) are disjoint discs since neither \( \Delta_a \setminus \{a\} \) and \( \Delta_b \setminus \{b\} \) contains critical values of \( R_E \). Set \( a' := \Delta_{\alpha'} \cap R^{-1}_E(a) \) and \( b' := \Delta_{\beta'} \cap R^{-1}_E(b) \). Then \( (\Delta_a, a') \) and \( (\Delta_b, b') \) are disjoint marked discs in \( \overline{\mathbb{T}}_{S'} \). Moreover they are independent of the choice of \( M \), because \( (\Delta_a, a) \) and \( (\Delta_b, b) \) are independent of the choice of \( M \).

Clearly, \( \varphi_E(\alpha) \) and \( \varphi_E(\beta) \) are equipotentials with potentials

\[\kappa(\varphi_E(\alpha)) = \frac{Mu(F(\alpha))}{\deg F|_{\alpha}} \quad \text{and} \quad \kappa(\varphi_E(\beta)) = \frac{Mu(F(\beta))}{\deg F|_{\beta}}.\]

Let \( A(E) = A(\alpha, \beta) \) denote the annulus bounded by \( \alpha \) and \( \beta \). Applying Lemma 13.1 there is a constant \( C(E) > 0 \) which is independent of the choice of \( M \), such that

\[\frac{Mu(F(\alpha))}{\deg F|_{\alpha}} + \frac{Mu(F(\beta))}{\deg F|_{\beta}} \leq \mod \varphi_E(\alpha, \beta) \leq \frac{Mu(F(\alpha))}{\deg F|_{\alpha}} + \frac{Mu(F(\beta))}{\deg F|_{\beta}} + C(E).\] (20)

**The constant** \( C \). The set \( \mathcal{E}^{(2,m)} \) has only finite many pieces \( E \) with \( C(E) \) independent of the choice of the number \( M \). Set \( C := \sum_E C(E) \). It is also independent of \( M \).

**2. Embedding of** \( \varphi_E(E) \) **and construction of** \( \psi_A \).

Every \( \mathcal{E}^2 \)-piece \( E \) is contained in \( \mathcal{A} \) or in an annular component of \( \mathcal{C} \setminus \mathcal{E}^m \). We will embed \( \varphi_E(E) \) into the interior of \( \mathcal{A} \cup (\mathcal{C} \setminus \mathcal{E}^m) \) so that they are mutually disjoint.

Assume that \( S \) is an \( \mathcal{A} \)-piece. Let \( \gamma \) be a boundary curve of \( S \) with \( \gamma \in Y \). From (19) and (20), we have

\[
\sum_{E \subseteq S \setminus \mathcal{E}^{(2,2)}} \mod \varphi_E(E) + \sum_{E \subseteq S \setminus \mathcal{E}^{(2,m)}} \mod \varphi_E(A(E)) \leq \sum_{\beta} \frac{Mu(F(\beta))}{\deg(F|_{\beta})} + C,
\]
where the last sum is taken over all the curves $\beta$ in $F^{-1}(\eta)$ for every $\eta \in Y$ such that $\beta$ is homotopic to $\gamma$ within $S = S \setminus \mathcal{P}$.

The right term is less than $MV(\gamma) = \text{mod}(\phi_S(S))$ by $[16]$. Therefore, as in the non-renormalizable case, one can embed holomorphically $\varphi_E(E)$ essentially into the interior of $\phi_S(S)$ for every $\mathcal{E}^2$-piece $E \subset S$ according to the original order of their non-null-homotopic boundary curves, so that they are mutually disjoint. In other words, we have a quasi-conformal map $\psi_S$ from $S$ onto $\phi_S(S)$, such that

- $\psi_S|_{\partial S} = \phi_S|_{\partial S}$ and $\psi_S$ is isotopic to $\phi_S$ rel $\partial S$;
- for every $\mathcal{E}^2$-piece $E \subset S$, $\varphi_E \circ \psi_S^{-1}$ is holomorphic in the interior of $\psi_S(E)$.

Consequently, we have

- $\phi_S \circ \varphi \circ \psi_S^{-1}$ is holomorphic in the interior of $\psi_S(E)$ for every $\mathcal{E}^2$-piece $E \subset S$ with $\hat{S} := F(E)$.

Assume now that $S$ is an $\mathcal{C}$-piece and that $A$ is an annular component of $S \setminus E_S$ where $E_S$ is the unique complex $\mathcal{E}$-piece contained in $S$. Following a similar argument as above, we have a quasi-conformal map $\psi_A$ from $A$ onto $\psi_S(A)$, such that

- $\psi_A|_{\partial A} = \psi_S|_{\partial A}$ and $\psi_A$ is isotopic to $\psi_S|_A$ rel $\partial A$;
- $\phi_S \circ F \circ \psi_A^{-1}$ is holomorphic in the interior of $\psi_A(E)$ for every $\mathcal{E}^2$-piece $E \subset A$ with $\hat{S} := F(E)$.

8.4 Definition of $\theta_S$

Define $\theta_S = \phi_S^{-1} \circ \psi_S$ for every $A$-piece $S$. If $S$ is a $\mathcal{C}$-piece, define

$$\theta_S = \left\{ \begin{array}{ll} \phi_S^{-1} \circ \psi_A & \text{on every annular component } A \text{ of } S \setminus \mathcal{E}^m; \\ \phi_S^{-1} \circ \psi_S & \text{otherwise.} \end{array} \right.$$  

Then $\theta_S|_{\partial S} = id$ and $\theta_S$ is isotopic to the identity rel $\partial S \cup (S \cap \mathcal{P})$. Moreover, for every $\mathcal{E}^2 \cup \mathcal{E}^m$-piece $E$ with $E \subset S$ and $F(E) = \hat{S}$, the map $\phi_S \circ F \circ \theta_S^{-1} \phi_S^{-1}$ is holomorphic in the interior of $\phi_S \theta_S(E)$.

Now if $\mathcal{E}^0 \cup \mathcal{O} \cup \mathcal{R} = \emptyset$, the proof of Theorem 5.4 is already completed. Otherwise one can follow the argument as in the non-renormalizable case (there is no more trouble in case $\mathcal{R} \neq \emptyset$) to modify $\theta_S$ on $S \setminus (\mathcal{E}^2 \cup \mathcal{E}^m \cup \partial S)$ with the help of a suitable layer function. This ends the proof of Theorem 5.4.

9 A combination result

A regular puzzle is by definition a subset of $\overline{\mathbb{C}}$ which is also a puzzle surface.

A regular open set is by definition the complement of a regular puzzle.

Let $U, V$ be regular open sets in $\overline{\mathbb{C}}$ with $V \subset U$. Let $G : U \rightarrow V$ be a quasi-regular branched covering. We say that $(G, U, V)$ is a locally holomorphic attracting system, if there is a finite set $\mathcal{P}' \subset U$ such that:

- $G(\mathcal{P}') = \mathcal{P}'$;
- $G$ is holomorphic in a neighborhood of $\mathcal{P}'$ and each cycle in $\mathcal{P}'$ is (super)attracting;
- for any $z \in V$ the limit set of $\{G^n(z)\}$ is contained in $\mathcal{P}'$.

Let $F : \mathcal{E} \rightarrow \mathcal{L}$ be a repelling system of constant complexity, in particular $F$ is quasi-regular. We say that $F$ has no analytization obstruction is it has no boundary obstruction, and , for each renormalization $H : E \rightarrow S$ (if any, and not necessarily postcritically finite),
either
(1) \(#P_f \cap S < \infty\) and \((H, P_f \cap S)\) as a repelling system has no Thurston obstructions; or
(2) for the integer \(p\) such that \(H = F^p|_E\), each step of the composition

\[
E \xrightarrow{F} F(E) \xrightarrow{F} F^2(E) \xrightarrow{F} \cdots \xrightarrow{F} F^{p-1}(E) \xrightarrow{F} S
\]

is holomorphic in the interior.

What we have proved in this paper can be reformulated in the following stronger form:

**Theorem 9.1.** Let \(G\) be a quasi-regular branched covering of \(\mathbb{C}\) with degree at least 2. Assume that \(\mathbb{C} = V \sqcup L\) is a splitting with \(L\) a regular puzzle such that:

(a) \(G^{-1}(V) \supset V\);
(b) \((G, G^{-1}(V), V)\) is a locally holomorphic attracting system;
(c) \(G : G^{-1}(L) \to L\) is a repelling system of constant complexity without analytization obstructions.

Let \(K\) be the union of the filled Julia set \(K_H\) of each of the holomorphic renormalizations. Then there is a rational map \(g\) and a pair of qc-homeomorphisms \(\phi, \psi\) of \(\mathbb{C}\) such that

- \(\phi \circ G = g \circ \psi\);
- \(\psi\) is isotopic to \(\phi\) rel \(P_G \cup K\);
- the Beltrami coefficient of \(\phi\) is equal to 0 almost everywhere on \(K\).

## A Non-negative matrices

For a vector \(v = (v_i) \in \mathbb{R}^n\) we write \(v > 0\) if every coordinate \(v_i\) is strictly positive.

**Lemma A.1.** Let \(D = (a_{ij})\) be a real square matrix with \(a_{ij} \geq 0\) for each entry \(a_{ij}\). Denote by \(\lambda\) its spectral radius, i.e. the maximal modulus of the eigenvalues. Then \(\lambda < 1\) iff there is a vector \(v > 0\) such that \(Dv < v\).

**Proof.** The following proof is provided by H.H. Rugh. Necessity: Assume \(v > 0\) and \(Dv < v\). Then \(Dv \leq av\) for some \(0 \leq a < 1\). Define a norm on the underlying vector space by \(\|x\| = \sum_i (v_i \cdot |x_i|)\). Then, writing \(|x|\) as the vector whose \(i\)-th entry is \(|x_i|\), we have

\[
\|tDx\| = t^\lambda v^\lambda |x| = t(Dv)|x| \leq a^\lambda v|x| = a\|x\|.
\]

Therefore, \(\lambda := \max_{\lambda' \text{ eigenvalue of } D} |\lambda'| = \max_{\lambda' \text{ eigenvalue of } tD} |\lambda'| \leq \|D\| \leq a\).

Sufficiency: Now assume \(\lambda < 1\). By continuity of the spectral radius, there is \(\epsilon > 0\) such that the spectral radius \(\lambda_{\epsilon}\) of \(D + \epsilon := (a_{ij} + \epsilon)\) satisfies \(\lambda_{\epsilon} < 1\). Now the Perron-Frobenius Theorem assures that \(\lambda_{\epsilon}\) is also an eigenvalue (called the leading eigenvalue) and it has a strictly positive eigenvector \(v > 0\). So \(Dv \leq (D + \epsilon)v = \lambda_{\epsilon}v < v\).

Note that it follows that \(\lambda\) is also an eigenvalue of \(D\) (called the leading eigenvalue). Lemma [A.1] actually gives an equivalent definition of the eigenvalues.

**Corollary A.2.** Let \(\lambda(D)\) be the leading eigenvalue of a non-negative square matrix \(D\). Then

\[
\lambda(D) = \inf\{\lambda | \exists v > 0 \text{ such that } Dv < \lambda v\}.
\]

**Corollary A.3.** Assume that \(A\) and \(B\) are non-negative \(n \times n\) matrix with \(A \leq B\) (i.e. each entry of \(A\) is less than or equal to the corresponding entry of \(B\)), then \(\lambda(A) \leq \lambda(B)\).
Proof. From Lemma A.4, we see that for any \( \lambda_0 > \lambda(B) \), there is a vector \( v > 0 \) so that \( \lambda_0^{-1}Bv < v \). Thus \( \lambda_0^{-1}Av \leq \lambda_0^{-1}Bv < v \). Again by Lemma A.4, we have \( \lambda(A) < \lambda_0 \) for any constant \( \lambda_0 > \lambda(B) \). So \( \lambda(A) \leq \lambda(B) \). \( \square \)

Let \( A \) be an \( n \times n \) matrix with a block decomposition

\[
\begin{pmatrix}
B_{11} & \cdots & B_{1k} \\
\vdots & \ddots & \vdots \\
B_{k1} & \cdots & B_{kk}
\end{pmatrix}
\]

where \( B_{ij} \) is an \( n_i \times n_j \) matrix (in particular each \( B_{ii} \) is a square matrix). We say that the block decomposition is *projected* if for each \( B_{ij} \), there is a number \( b_{ij} \) such that the summation of each column of \( B_{ij} \) is equal to \( b_{ij} \).

This property could be understood as the following: An \( n \times n \) matrix can be considered as a linear map of \( \mathbb{R}^n \) defined by the left action:

\[
\begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{pmatrix} \mapsto A \begin{pmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{pmatrix}.
\]

According to the block decomposition of \( A \), there is a corresponding decomposition of the index set \( I = \{1, \ldots, n\} \) by \( I = I_1 \sqcup \cdots \sqcup I_k \) with \( \#I_i = n_i \). Define a linear projection \( \pi : \mathbb{R}^n \to \mathbb{R}^k \) by

\[
(\pi v)_i = \sum_{\delta \in I_i} v_{\delta}.
\]

**Lemma A.4.** There is a \( k \times k \) matrix \( B \) such that \( \pi \circ A = B \circ \pi \) if and only if the block decomposition \( A = (B_{ij}) \) is projected. In this case, \( B = (b_{ij}) \).

\[
\begin{array}{ccc}
\mathbb{R}^n & \overset{A}{\to} & \mathbb{R}^n \\
\pi & \downarrow & \pi \\
\mathbb{R}^k & \overset{B}{\to} & \mathbb{R}^k
\end{array}
\]

**Proof.** Set \( A = (a_{\delta \beta}) \). For any \( v \in \mathbb{R}^n \),

\[
(\pi \circ Av)_i = \sum_{\beta \in I_j} \sum_{\delta \in I_i} a_{\delta \beta} v_{\beta}, \quad \text{and} \quad (B \circ \pi(v))_i = \sum_{\beta \in I_j} \sum_{\delta \in I_i} b_{ij} v_{\beta}.
\]

If the block decomposition is projected, then for \( \beta \in I_j \), \( \sum_{\delta \in I_i} a_{\delta \beta} = b_{ij} \). Therefore \( \pi \circ Av = B \circ \pi(v) \). Conversely, assume that \( \pi \circ A = B \circ \pi \). For \( \beta \in I_j \), let \( e_\beta \in \mathbb{R}^n \) be a vector whose \( \beta \)-entry is 1 and 0 elsewhere. Then \( (\pi \circ Ae_\beta)_i = b_{ij} \), and \( (B \circ \pi(e_\beta))_i = \sum_{\delta \in I_i} a_{\delta \beta} \). So for \( \beta \in I_j \), \( \sum_{\delta \in I_i} a_{\delta \beta} = b_{ij} \), i.e. the block decomposition is projected. \( \square \)

**Theorem A.5.** Assume that \( A \) is a non-negative square matrix with a projected block decomposition \( A = (B_{ij}) \). Set \( B = (b_{ij}) \). Then \( \lambda(A) = \lambda(B) \).

**Proof.** Let \( v \neq 0 \) be an eigenvector of \( A \) for the leading eigenvalue \( \lambda(A) \), i.e. \( Av = \lambda(A)v \). Set \( u = \pi(v) \). Then \( Bu = \pi \circ Av = \pi(\lambda(A)v) = \lambda(A)\pi(v) = \lambda(A)u \) by the above Lemma. So \( \lambda(A) \) is an eigenvalue of \( B \) and hence \( \lambda(A) \leq \lambda(B) \) since the leading eigenvalue is the maximum of the eigenvalues.
Conversely, let \( u \neq 0 \) be an eigenvector of the transpose \( B^t \) of \( B \) for the leading eigenvalue \( \lambda(B) \) (note that \( B \) and \( B^t \) have same leading eigenvalues), i.e. \( B^t u = \lambda(B) u \). Set \( v = (v_\beta) \in \mathbb{R}^n \) by \( v_\beta := u_j \) for \( \beta \in I_j \). Then for \( \delta \in I_i \),
\[
(A^t v)_\delta = \sum_j \sum_{\beta \in I_j} a_{\beta \delta} v_\beta = \sum_j b_{ji} v_j = (B^t u)_i = \lambda(B) u_i = \lambda(B) v_\delta.
\]
So \( \lambda(B) \) is an eigenvalue of \( A^t \). Therefore we again have \( \lambda(B) \leq \lambda(A) \). \( \square \)

**Corollary A.6.** Let \( A' \) be a non-negative square matrix with a block decomposition \( (B'_{ij}) \). Assume that for each \( ij \), the summation of each column of \( B'_{ij} \) is at most \( b_{ij} \). Set \( B = (b_{ij}) \). Then \( \lambda(A') \leq \lambda(B) \).

**Proof.** For each \( ij \), we just need to replace one entry of each column of \( B'_{ij} \) by a larger number so that the summation of the column becomes exactly \( b_{ij} \). Denote by \( B_{ij} \) the modified matrix. Set \( A = (B_{ij}) \). Then \( \lambda(A) = \lambda(B) \) by Theorem A.3 and \( \lambda(A') \leq \lambda(A) \) by Corollary A.3. \( \square \)

**B Reversing the Grötzsch inequality**

A *equipotential* \( \gamma \) in a marked disc \( (\Delta, a) \) is a curve mapped onto a round circle under a conformal representation \( \varphi : (\Delta, a) \to (\mathbb{D},0) \). The potential of \( \gamma \) is defined to be the modulus of \( \partial\Delta \) and \( \gamma \).

**Lemma B.1.** Let \( (D_i, z_i), \ i = 1, 2 \) be two disjoint marked hyperbolic discs. Then there is a constant \( C > 0 \) independent of \( v_1 > 0, v_2 > 0 \) such that, for the annulus \( A(v_1, v_2) \) between the equipotential in \( D_1 \) of potential \( v_1 \) and the equipotential of \( D_2 \) of potential \( v_2 \), we have
\[
 v_1 + v_2 \leq \text{mod} (A(v_1, v_2)) \leq v_1 + v_2 + C
\]

**Proof.** The left hand side is just the Grötzsch inequality.

The conformal radius of a marked disc \( (\Delta, 0) \) is defined to be the radius \( r \) if there is a conformal map \( \varphi : (\Delta, 0) \to (D(0, r), 0) \) with \( \varphi'(0) = 1 \). And the conformal radius of a marked disc \( (\Delta, \infty) \) is defined to be the conformal radius of \( (\pi(\Delta), 0) \) with \( \pi(z) = 1/z \).

Let \( \xi \) be a Möbius transformation of \( \overline{\mathbb{C}} \) with \( \xi(z_1) = 0 \) and \( \xi(z_2) = \infty \). Any two such maps differ by a multiplicative constant. So the product \( C_1 \cdot C_2 \) of the conformal radii of \( (\xi(D_1), 0) \) and \( (\xi(D_2), \infty) \) is independent of the choice of \( \xi \). Denote by \( W_i \) the component of \( A(v_1, v_2)^c \) containing \( z_i, i = 1, 2 \). By Koebe 1/4-Theorem, \( \xi(W_1) \) contains \( \{ |z| \leq C_1 r_1 / 4 \} \) and \( \xi(W_2) \) contains \( \{ |z| \geq 4 / (C_2 r_2) \} \), where \( r_i = e^{-v_i} \). Therefore
\[
\text{mod} (A(v_1, v_2)) \leq \log \left( \frac{4}{C_2 r_2} \cdot \frac{4}{C_1 r_1} \right) = \log \left( \frac{16}{C_1 C_2} \cdot \frac{1}{r_1 r_2} \right) = \log \frac{16}{C_1 C_2} + v_1 + v_2 \ . \quad \square
\]

**C Quasi-conformal extensions**

We state here several results about quasi-conformal maps that have been frequently used in the paper.

**Lemma C.1.** Let \( h : C_1 \to C_2 \) be a homeomorphism between two quasi-circles \( C_1 \) and \( C_2 \) in \( \overline{\mathbb{C}} \). If \( h \) can be extended to a quasi-conformal map on an one-side neighborhood of \( C_1 \), then \( h \) can be extended to a global quasi-conformal homeomorphism of \( \overline{\mathbb{C}} \). Moreover the extension can be chosen to be a diffeomorphism from \( \overline{\mathbb{C}} \setminus C_1 \) onto \( \overline{\mathbb{C}} \setminus C_2 \).
Lemma C.2. Let $\Omega_i \subset \overline{C}$ ($i = 1, 2$) be two open connected domains such that $\partial \Omega_i$ ($i = 1, 2$) consists of $p \geq 0$ disjoint quasi circles (we allow the case $p = 0$). Let $\mathcal{P} \subset \Omega_1$ be a finite set (may or may not be empty). Let $f : \overline{\Omega}_1 \to \overline{\Omega}_2$ be an orientation preserving homeomorphism. If, either $p = 0$, or $f|_{\partial \Omega_1}$ can be extended to a quasi-conformal map on an one-side neighborhood of each curve of $\partial \Omega_1$, then there is a quasi-conformal homeomorphism in the isotopy class of $f$ modulo $\partial \Omega_1 \cup \mathcal{P}$.

Lemma C.3. Let $h : S^1 \to S^1$ be an orientation preserving homeomorphism of the unit circle. Assume that $h$ can be extended as a quasi-conformal map $f$ on an inner neighborhood $B$ of $S^1$ (i.e. $B \supset \{1 - \varepsilon < |z| < 1\}$ for some $\varepsilon > 0$), then $h$ is quasi-symmetric.

Proof. Denote by $\mu$ the Beltrami coefficient of $f$. Denote by $\mathbb{D}$ the unit disc. Let $\nu = \mu$ on $B$ and $\nu = 0$ on $\mathbb{D} \setminus B$. By the Measurable Riemann Mapping Theorem, there is a quasi-conformal homeomorphism $g$ of $\mathbb{D}$ whose Beltrami coefficient is $\nu$. Then $g|_{S^1}$ is quasi-symmetric. On the other hand, $f \circ g^{-1}$ is holomorphic on $g(B)$. Therefore $f \circ g^{-1}|_{S^1} : S^1 \to S^1$ is real-analytic, in particular quasi-symmetric. So $h = (f \circ g^{-1}) \circ g|_{S^1}$ is also quasi-symmetric. \hfill \Box

Proof of Lemma C.3. Fix $i = 1, 2$. By definition of quasi-circles, there is a quasi-conformal homeomorphism $\phi_i$ of $\mathbb{C}$ such that $\phi_i(C_i) = S^1$. Furthermore $\phi_i$ can be chosen to be a diffeomorphism on $\mathbb{C} \setminus C_i$ as follows: Set $\Delta = \phi_i^{-1}(\mathbb{D})$. Let $\psi : \Delta \to \mathbb{D}$ be a conformal map. Then $\phi_i \circ \psi^{-1} : \mathbb{D} \to \mathbb{D}$ is a quasi-conformal homeomorphism. Thus its boundary map is quasi-symmetric. Let $\eta$ be the Beurling-Ahlfors extension of this boundary map, it is a diffeomorphism of $\mathbb{D}$. Now $\eta \circ \psi|_\Delta$ is again a diffeomorphism, whose boundary map is $\phi_i|_{S^1}$. Set $h_1 = \phi_2 \circ h \circ \phi_1^{-1}$. Then by Lemma C.3 this $h_1$ is quasi-symmetric, thus has a quasi-conformal extension to $\mathbb{C}$. Moreover its extension can be chosen to be a diffeomorphism outside $S^1$. Thus $h = \phi_2^{-1} \circ h_1 \circ \phi_1$ can be extended to a quasi-conformal homeomorphism of $\mathbb{C}$, and a diffeomorphism outside $C_1$. \hfill \Box

Proof of Lemma C.2. By Lemma C.1 we can assume that $\partial \Omega_i$ are smooth Jordan curves and that $f|_{\partial \Omega_1}$ is a diffeomorphism. Then one can find a diffeomorphism in its isotopy class rel $\partial \Omega_1 \cup \mathcal{P}$. \hfill \Box

D A lemma about isotopy

Lemma D.1. Let $(D_i, a_i), i = 1, \cdots, k, k \geq 1$ be finitely many marked Jordan discs in $S^2$, with disjoint closures. Let $P$ be a closed (or empty) set contained in $S^2 \setminus \bigcup D_i$. Assume that $h_1 : S^2 \to S^2$ is an orientation preserving homeomorphism, and $h : S^2 \times [0, 1] \to S^2$ is continuous, such that
a) $h_1|_{P \cup D_i} = id$
b1) $h(\cdot, t)$ is a homeomorphism for any $t \in [0, 1]$;
b2) $h(\cdot, 0) = id$, $h(\cdot, 1) = h_1$;
b3) $h(x, t) = x$ for any $x \in P \cup \bigcup \{a_i\}$ and any $t \in [0, 1]$.

For $i = 1, \cdots, k$, set $\gamma_i = \partial D_i$, and let $\beta_i$ be a Jordan curve disjoint from $\overline{D_i}$ so that the annuli $A_i := A(\beta_i, \gamma_i)$ have mutually disjoint closures and are disjoint from $P$. Then there is a continuous map $H : S^2 \times [0, 1] \to S^2$ such that
c1) $H(\cdot, t)$ is a homeomorphism for any $t \in [0, 1]$;
c2) $H(\cdot, 0) = id$, $H(\cdot, 1) = h_1 \circ T$ with $T = id$ outside $\bigcup A_i$;
c3) $H(x, t) = x$ for any $x \in P \cup \bigcup D_i$ and any $t \in [0, 1]$. 
Proof. Set $h_t = h(\cdot, t)$, and $E = S^2 \setminus (D_i \cup A_i)$.

For each $i$ choose a Jordan curve $\alpha_i$ in $D_i$ bounding a disc $D(\alpha_i)$ so that $a_i \in D(\alpha_i) \subset D_i$ and that $h_t(\beta_i) \cap \alpha_i = \emptyset$ for any $t \in [0, 1]$.

Define $s : S^2 \times [0, 1] \to S^2$ continuous, with each $s(\cdot, t) := s_t$ a homeomorphism of $S^2$, as follows: $s(\cdot, 0) = id$.

$s(x, t) = \begin{cases} h_t^{-1}(x) & x \in \bigcup D(\alpha_i) \\ \text{interpolation} & x \in \bigcup A(\alpha_i, \beta_i) \end{cases}$. Then $s(x, 1) = \begin{cases} x = h_t^{-1}(x) & x \in \bigcup D(\alpha_i) \\ T_0(x) & x \in \bigcup A(\alpha_i, \beta_i) \end{cases}$, $x \in E$

where $T_0$ is a certain homeomorphism of $S^2$ that is identity outside $\bigcup A(\alpha_i, \beta_i)$.

Set $\xi_t = h_t \circ s_t$. Then $\xi_0 = id$, \begin{align*}
\xi_t(x) &= \begin{cases} x \in \bigcup D(\alpha_i) \\ \text{interpolation} & x \in \bigcup A(\alpha_i, \beta_i) \end{cases} \hspace{1cm} h_t(x) \quad x \in E \\
&= h_t(x) \quad x \in P \subset E
\end{align*}

and $\xi_1(x) = h_1 \circ s_1(x) = \begin{cases} x = h_1(x) & x \in \bigcup D(\alpha_i) \\ h_1 \circ T_0(x) & x \in \bigcup A(\alpha_i, \beta_i) \end{cases}$, $h_1(x) \quad x \in E \\
&= h_1(x) \quad x \in P \subset E$

Let now $u : S^2 \to S^2$ be a homeomorphism such that $u(D_i) = D(\alpha_i)$, $u(a_i) = a_i$ for each $i$ and $u|E = id$. Define then $v : S^2 \to S^2$ be a homeomorphism such that $v|\bigcup D(\alpha_i) = u^{-1}$ and $v|_{h_1(E) \cup E} = id$. Set $\zeta_t = v \circ \xi_t \circ u$. We have $\zeta_0 = \begin{cases} id & x \notin \bigcup A_i \\ v \circ u & x \in \bigcup A_i \end{cases}$,

$\zeta_t(x) = \begin{cases} x = h_1(x) & x \in \bigcup D_i \\ \text{interpolation} & x \in \bigcup A_i \end{cases} \hspace{1cm} h_t(x) \quad x \in E \\
v \circ h_t(x) \quad x \in E \\
x = h_1(x) \quad x \in P \subset E$

$\zeta_1(x) = \begin{cases} x = h_1(x) & x \in \bigcup D_i \\ h_1(x) & x \in E \\ h_1 \circ T_0 \circ u(x) & x \in \bigcup A_i \end{cases}$, $h_1 \circ h_1(x) \quad x \in E \\
= h_1 \circ h_1 \circ T_0 \circ u \circ T_0 \circ u(x) \quad x \in \bigcup A_i = T_0(x)$

for a certain homeomorphism $T_1$ of $S^2$ with $T_1 = id$ outside $\bigcup A_i$. Set finally $H_t(x) = \zeta_t \circ \zeta_0^{-1}(x)$. It has the required properties. In particular $H_1(\cdot, 1) = \begin{cases} h_1 \quad x \notin \bigcup A_i \\ v \circ h_1 \circ T_0 \circ v^{-1} & x \in \bigcup A_i \end{cases}$, $h_1 \circ h_1 \circ T_0 \circ T_2$, with $T_2 = \zeta_0^{-1}$, and $T_1 \circ T_2 = id$ outside $\bigcup A_i$. \hfill \Box

References

[BFH] Ben Bielefield, Yuval Fisher & John H. Hubbard, The classification of critically preperiodic polynomials as dynamical systems, J. Amer. Math. Soc. 5 (1992) 721–762.

[Br] David Brown, Thurston equivalence without postcritical finiteness for a family of polynomial and exponential mappings, manuscript.

[BH] Bodil Branner & John H. Hubbard, The iteration of cubic polynomials, Part II: Patterns and parapatterns, Acta Math., 169, 1992, p. 229-325.

[CJS] Guizhen Cui, Yunting Jiang & Dennis Sullivan, On geometrically finite branched coverings-II. Realization of rational maps, in Complex dynamics and related topics, ed. Yunting Jiang and Yuefei Wang, The international press 2004, p. 15-29.

[C1] Guizhen Cui, Conjugacies between rational maps and extremal quasiconformal maps, Proc. Amer. Math. Soc.. 129, no. 7 (2001), 1949-1953.

[C2] Guizhen Cui, Dynamics of rational maps, topology, deformation and bifurcation, preprint, May 2002.
Guizhen Cui & Tan Lei, Hyperbolic-parabolic deformations of rational maps, manuscript.

Adrien Douady, Topological Entropy of Unimodal Maps, Proceedings of the NATO Adv. Study Inst. on Real and Complex Dynamical Systems, NATO ASI Series Vol. 464 (1993).

Régine & Adrien Douady, Algèbre et théories galoisiennes, Cassini, 2005.

Adrien Douady & John H. Hubbard, A proof of Thurston's topological characterization of rational functions, Acta Math., 171 (1993), 263-297.

Adrien Douady & John H. Hubbard, On the dynamics of polynomial-like mappings, Ann. Sci. École Norm. Sup. (4) 18 (1985), no. 2, 287–343.

Yunping Jiang & Gaofei Zhang, On geometrically finitebranched coverings-III. A direct proof of CJS’s theorem, in Complex dynamics and related topics, ed. Yunping Jiang and Yuefei Wang, The international press 2004, p.265-291.

Jan Kiwi, Real laminations and the topological dynamics of complex polynomials, Adv. in Math. 184 2 (2004), 207-267.

John H. Hubbard & Dierk Schleicher, The spider algorithm, in Complex dynamics: the mathematics behind the Mandelbrot and Julia sets, AMS Proceedings of the Symposia in Applied Mathematics, vol. 49, 1994.

John H. Hubbard, Dierk Schleicher & Mitsuhiro Shishikura, A topological characterization of postsingularly finite exponential maps and limits of quadratic differentials, Manuscript, 2006.

Curtis McMullen, Automorphisms of rational maps. In Holomorphic Functions and Moduli I (1988), Springer Verlag, pages 31-60.

Curtis McMullen, Complex Dynamics and Renormalization, Annals of Mathematics Studies, Princeton University Press, 1994.

Kevin Pilgrim, Combinations of complex dynamical systems, Lecture Notes in Mathematics, 1827. Springer-Verlag, Berlin, 2003.

Alfredo Poirier, On postcritically finite polynomials, part I, Critical portraits, Stony Brook IMS preprint #1993/5, arxiv.org/abs/math.DS/9305207.

Kevin Pilgrim & Tan Lei, Rational maps with disconnected Julia set, Astérisque 261 (2000), volume spécial en l’honneur d’A. Douady, pp. 349-384.

Mary Rees, Views of parameter space: Topographer and Resident, Astérisque 288, 2003.

Mitsuhiro Shishikura & Tan Lei, A family of cubic rational maps and matings of cubic polynomials, Experi. Math., Vol. 9 (2000), No. 1, pp. 29-53.

Williams Thurston, Lectures notes, Princeton University and University of Minnesota at Duluth, 1982-1983.

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