An Optimal Algorithm for the Indirect Covering Subtree Problem

Joachim Spoerhase

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We consider the indirect covering subtree problem (Kim et al., 1996). The input is an edge weighted tree graph along with customers located at the nodes. Each customer is associated with a radius and a penalty. The goal is to locate a tree-shaped facility such that the sum of setup and penalty cost is minimized. The setup cost equals the sum of edge lengths taken by the facility and the penalty cost is the sum of penalties of all customers whose distance to the facility exceeds their radius. The indirect covering subtree problem generalizes the single maximum coverage location problem on trees where the facility is a node rather than a subtree. Indirect covering subtree can be solved in $O(n \log^2 n)$ time (Kim et al., 1996). A slightly faster algorithm for single maximum coverage location with a running time of $O(n \log^2 n / \log \log n)$ has been provided (Spoerhase and Wirth, 2009). We achieve time $O(n \log n)$ for indirect covering subtree thereby providing the fastest known algorithm for both problems. Our result implies also faster algorithms for competitive location problems such as $(1,X)$-medianoid and $(1,p)$-centroid on trees. We complement our result by a lower bound of $\Omega(n \log n)$ for single maximum coverage location and $(1,X)$-medianoid on a real-number RAM model showing that our algorithm is optimal in running time.

Keywords: Graph algorithm, coverage, medianoid, tree, efficient algorithm

1 Problem Definitions and Related Work

We are given a tree $T = (V, E)$ along with edge costs $c: E \rightarrow \mathbb{R}_{\geq 0}$ inducing a distance function $d: V \times V \rightarrow \mathbb{R}_{\geq 0}$. With each node $u$ we associate a non-negative penalty $\pi(u)$. Let $Y$ be a subtree of $T$. Then $c(Y)$ denotes the setup cost of $Y$ and is given by the sum $\sum_{e \in E(Y)} c(e)$ of its edge costs. A node is covered directly by $Y$ if it lies in $Y$. If some node $u$ is not covered directly it imposes the penalty $\pi(u)$ on $Y$. The direct covering subtree problem [KLTW96] asks for a subtree $Y$ such that the sum of setup cost $c(Y)$ and the total penalty $\sum_{u \in V(Y)} \pi(u)$ is minimized.
The indirect covering subtree problem goes one step further. A node $u$ is said to be covered (indirectly) if it lies within a given distance from $Y$. Again, a penalty is imposed on $Y$ if it does not cover $u$. More formally, we assign to each node $u$ some radius $\rho(u)$. The penalty imposed on $Y$ by $u$ is given as

$$p(u, Y) := \begin{cases} 0 & \text{if } d(u, Y) \leq \rho(u) \\ \pi(u) & \text{otherwise} \end{cases}$$

If $U \subseteq V$ is a set of nodes then $p(U, Y) := \sum_{u \in U} p(u, Y)$ is the penalty imposed on $Y$ by $U$. The total penalty imposed on $Y$ is given by $p(Y) := p(V, Y)$. The indirect covering subtree problem \cite{KLTW96} asks for a subtree $Y$ of $T$ such that the total cost $c(Y) + p(Y)$, given by the sum of setup and penalty cost, is minimum among all subtrees of $T$.

If we require that $Y$ be a node rather than a subtree we obtain the single maximum coverage location problem \cite{MZH83,SW09a}. It is not hard to see that single maximum coverage location is a special case of indirect covering subtree. (Scale all edge lengths and radii with a sufficiently large factor while leaving the penalties unchanged.)

1.1 Related Work and Previous Results

The multiple maximum coverage location problem allows the placement of an arbitrary set of $r$ nodes. On general graphs this problem is NP-hard \cite{MZH83} while it can be solved in time $O(rn^2)$ on trees \cite{Tam96}. This leads to an $O(n^2)$ algorithm for the single maximum coverage location problem on trees by setting $r = 1$. Kim et al. \cite{KLTW96} provide a faster algorithm running in $O(n \log^2 n)$. Their algorithm works even for the more general indirect covering subtree problem. Recently a slightly faster algorithm for single maximum coverage location with time $O(n \log^2 n / \log \log n)$ has been reported \cite{SW09a}. Finally, we remark that direct covering subtree can be solved in linear time \cite{KLTW96}.

1.2 Contribution and Outline of this Paper

In this paper we show that indirect covering subtree can be solved in $O(n \log n)$. This improves upon the previously best algorithms for this problem and single maximum coverage location on trees. Our result also implies faster algorithms for the $(1, X)$-medianoid problem and the $(1, p)$-centroid problem on trees. Specifically, we obtain an $O(n \log n)$ algorithm for $(1, X)$-medianoid and $O(n^2 \log n \log w(T))$ and $O(n^2 \log n \log w(T) \log D)$ algorithms for the discrete and absolute $(1, p)$-centroid problems on trees, respectively. Here, $w(T)$ denotes the total weight of the tree and $D$ is the maximum edge length. The previously best algorithms are slower by factor of $O(\log n / \log \log n)$ \cite{SW09a}.

Our algorithm employs the same dynamic programming framework used by Kim et al. \cite{KLTW96}. However, we improve one of their core routines by using a more sophisticated technique to subdivide trees. This technique, called two-terminal subtree subdivision (TTST), is a simplification of the recursive coarsening strategy \cite{SW09a} used for solving single maximum coverage location on a tree. The key source of our improvement is that we manage to avoid explicitly sorting the nodes according to their distances and radii.

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during the recursion, which has been necessary in the coarsening approach and also in the original algorithm of Kim et al. One further advantage of our algorithm is that it is a lot simpler than the recursive coarsening algorithm.

The two-terminal subtree technique has proved successful also for other location problems [SW10, SW09b]. I believe that there are further problem classes where it can be applied.

The paper is organized as follows. In Section 2 we briefly outline the algorithm of Kim et al. This is necessary, since our result relies on an improvement of a subroutine of that algorithm. The improved routine is then described in Section 3. In Section 4 we provide a matching lower bound on the running time needed to solve indirect covering subtree. Finally, we discuss implications on related problems such as competitive location problems in Section 5.

2 The Algorithm of Kim et al.

In the sequel we will briefly describe the algorithmic approach of Kim et al. [KLTW96] for solving the indirect covering subtree problem.

Let’s first fix some conventions and notations. We assume that the input tree $T$ is rooted at some distinguished node $s$. For technical reasons we shall adopt the convention that $s$ is the father of itself. Let $v$ be an arbitrary node. Then $f(v)$ denotes the father of $v$. We write $T_v$ for the subtree of $T$ hanging from $v$ and $T_v^+$ for the union of $T_v$ with the edge $(v, f(v))$.

Kim et al. reduce the solution of the problem to the computation of the values $p(v)$, $p(T_v, v)$ and $p(T_v, f(v))$ for all nodes $v$. They show that one can determine an optimum to the subtree location problem in linear time once these values have been precomputed for all nodes $v$.

To convince ourselves, assume that we have computed the values $p(v)$, $p(T_v, v)$ and $p(T_v, f(v))$ for all $v \in V$. Then define

$$C(v) := \min \{ c(Y) + p(T_v, Y) \mid Y \text{ is subtree of } T_v \text{ containing } v \} ,$$

and

$$C^+(v) := \min \{ c(Y) + p(T_v, Y) \mid Y \text{ is subtree of } T_v^+ \text{ containing } f(v) \} .$$

It is not hard to see that the optimum cost can now be expressed by

$$\min_{v \in V} (C(v) + p(v) - p(T_v, v)) .$$

Moreover, the $C(\cdot)$- and $C^+(\cdot)$-values can be computed in linear time by means of a simple bottom-up dynamic programming approach. To this end assume that $v$ is a leaf of $T$ then

$$C(v) = 0 \quad \text{and} \quad C^+(v) = \min \{ p(v, f(v)), c(v, f(v)) \} .$$
Otherwise, we have
\[ C(v) = \sum_{u \text{ is son of } v} C^+(u), \]
and
\[ C^+(v) = \min\{C(v) + c(v, f(v)), p(T_v, f(v))\}. \]

From this equations it follows that an optimal solution can be determined in linear time in a bottom-up fashion once the values \( p(v), p(T_v, v) \) and \( p(T_v, f(v)) \) have been computed for all \( v \in V \). Kim et al. suggest an algorithm with running time \( O(n \log^2 n) \) to compute these values.

This algorithm is based on the so-called bitree model. In this model, each (undirected) edge \((u, v)\) of the input tree is replaced with two anti-parallel, directed arcs \((u, v), (v, u)\). We call the resulting tree \( T' \) bitree of \( T \). With each arc \((u, v)\) of the bitree we associate a cost \( c_{T'}(u, v) \) representing the length of this arc. But in contrast to the edges of the input tree \( T \) we allow these costs to be negative and asymmetric. This induces a distance function \( d_{T'} : V \times V \rightarrow \mathbb{Q} \) where \( d_{T'}(u, v) \) is the length of the unique \( u-v \)-path in \( T' \). Now we define the penalty cost \( p'(u, v) \) imposed on \( v \) by \( u \) to be zero if \( d_{T'}(u, v) \leq g(u) \) and \( \pi(u) \) otherwise. We set \( p'(v) = \sum_{u \in V} p'(u, v) \).

The algorithm of Kim et al. is based on a subroutine for efficiently computing \( p'(v) \) for all nodes \( v \) on a given bitree \( T' \). By means of such a subroutine it is then possible to calculate the values \( p(v), p(T_v, v) \) and \( p(T_v, f(v)) \) for all \( v \) in the input tree \( T \). It follows from the above discussion that the knowledge of these values enables us to identify an optimal tree-shaped facility.

It remains to explain how we can employ such a subroutine to determine \( p(v), p(T_v, v) \) and \( p(T_v, f(v)) \) for all nodes \( v \) of the input tree \( T \) which, in turn, is sufficient to build an optimal tree-shaped facility.

First we describe how we can determine \( p(\cdot) \). For this purpose we simply set \( c_{T'}(u, v) := c_{T'}(v, u) := c_T(u, v) \) for all edges \((u, v)\) of the input tree \( T \). It is then immediately clear that \( p(v) = p'(v) \) for all \( v \in V \).

In order to compute \( p(T_v, v) \) for all \( v \in V \) we set \( c_{T'}(u, f(u)) := c_T(u, f(u)) \) and \( c_{T'}(f(u), u) := -\infty \) for all \( u \neq s \). This construction ensures that the penalty cost \( p'(u, v) \) is always zero if \( u \) is not a descendant of \( v \). Thus \( p(T_v, v) = p'(v) \) holds for this construction.

Finally, we wish to determine \( p(T_v, f(v)) \). To this end we introduce on each edge \((v, f(v))\) of \( T \) a new node \( f'(v) \) such that edge \((v, f'(v))\) has length \( c_T(v, f(v)) \) and edge \((f'(v), f(v))\) has length zero. This increases the number of nodes to \( 2n - 1 \). We set \( \pi(f'(v)) \) and \( g(f'(v)) \) to zero. It is easy to see that \( p(T_v, f(v)) \) in the original tree equals \( p(T_u, u) \) in the newly constructed tree where \( u := f'(v) \). Hence the problem of computing \( p(T_v, f(v)) \) for all nodes \( v \) can be reduced to the problem of computing \( p(T_v, v) \), which has been described before.

Kim et al. provide a subroutine to compute the \( p'(\cdot) \)-values on a bitree with \( n' \) nodes in \( O(n' \log^2 n') \) time which yields immediately.
The indirect covering subtree problem on a tree can be solved in $O(n \log^2 n)$.

3 An $O(n \log n)$ Algorithm

In this section we describe an algorithm for the indirect covering subtree problem with running time $O(n \log n)$.

Our algorithm uses the algorithmic framework of Kim et al. described in Section 2. Specifically, we will provide an improved routine for computing the values $p'(\cdot)$ on a given bitree in $O(n \log n)$ which can then be extended to an algorithm with the same asymptotic running time for solving indirect covering subtree.

The basic approach of the routine of Kim et al. for computing $p'(\cdot)$ is divide-and-conquer. It partitions the node set $V$ into two sets $V_1, V_2$ of bounded size such that both induce subtrees and have exactly one node (called centroid [KA75]) in common. Then it sorts the sets $V_i$ and computes, by means of a clever merge-and-scan procedure, for all $v \in V_i$ the penalties $p'(v, V_j)$ of the users in $V_j$ where $j \neq i$. Applying the routine recursively to the sub-bitrees induced by $V_1, V_2$ one can determine the $p'(v, V_i)$-values also for each $v \in V_i$. Finally, one obtains the total penalty $p'(v, V)$ of any node $v \in V$ by adding $p'(v, V_1)$ and $p'(v, V_2)$.

Our routine proceeds in a similar way but uses a more sophisticated subdivision, which allows us to avoid the explicit sorting thereby supressing the additional log-factor. Spoerhase and Wirth [SW10, SW09b] used an analogous subdivision technique for solving competitive location problems on undirected trees.

Consider the (undirected) input tree $T = (V, E)$. We may assume that $T$ has maximum degree three. Otherwise, we can split nodes of larger degree by introducing suitable zero-length edges and zero-weighted nodes. Let $T'$ be the bitree corresponding to $T$.

If $s$ and $t$ are distinct nodes then $T'_{st}$ denotes the maximal sub-bitree of $T'$ having $s$ and $t$ as leaves. Let $V_{st}$ be the node set of $T'_{st}$. We call $s$ and $t$ terminals and $T'_{st}$ two-terminal sub-bitree (TTSB).

Our algorithm divides the input bitree recursively into TTSBs. Since we are dealing with a degree-bounded bitree we can subdivide any TTSB $S$ into at most five TTSBs, called child TTSBs. Each of these child TTSBs has at most $\frac{1}{2}|S| + 1$ nodes.

Lemma 2 Let $S$ be a TTSB with maximum degree three. Then $S$ can be partitioned into at most five edge-disjoint TTSBs each of which having at most $\frac{1}{2}|S| + 1$ nodes. This subdivision can be computed in $O(|S|)$ time.

Proof. Let $S$ be a TTSB with maximum degree three and terminals $u$ and $v$. Let $m$ be the unweighted median of $S$, which can be computed in $O(|S|)$ by means of Goldman’s algorithm [Gol71]. (All node and arc weights are temporarily set to one throughout this proof.) It is a well-known fact that $m$ has the following property: Each of the connected components of $S - m$ has at most $\frac{1}{2}|S|$ nodes. Hence, if $m$ lies on path $P(u, v)$ then $S - m$ contains at most three components that form the desired subdivision (confer left part of Figure 1). If $m$ does not lie on $P(u, v)$ then consider the node $m'$ on $P(u, v)$
that is closest to $m$ (confer right part of Figure 1). Then $S - \{m, m'\}$ has at most five connected components. All of the child TTSBs obtained this way have clearly at most $\frac{1}{2}|S| + 1$ nodes.

Consider a TTSB $T_{si}$. We introduce the lists $L_{d,s}(T_{si})$ and $L_{\varnothing,s}(T_{si}')$. Both lists contain all nodes $v$ of $T_{si}$ sorted in increasing order with respect to the values $d_{T'}(s, v)$ and $\varnothing(v) - d_{T'}(v, s)$, respectively. The lists $L_{d,t}(T_{si})$ and $L_{\varnothing,t}(T_{si})$ are defined symmetrically.

The algorithm computes $p'(v, T_{si})$ for all $v \in T_{si}$ as well as the four lists $L_{d,s}(T_{si})$, $L_{d,t}(T_{si}')$, $L_{\varnothing,s}(T_{si}')$ and $L_{\varnothing,t}(T_{si}')$ for any TTSB $T_{si}'$ occurring during the recursion. We shall see that these information can be propagated inductively from child towards parent TTSBs such that we will have computed $p'(. , T_{si}') = p'(.)$ at the top of the recursion.

To this end consider an arbitrary TTSB $S = T_{si}'$ being subdivided into at most five child TTSBs $S_i$ with terminals $s_i, t_i$. Moreover assume that we have already computed $p'(. )$ and the four lists corresponding to $S_i$ for all $S_i$.

We start with computing $L_{d,s}(S)$. To this end we maintain a list $L$ which is initialized with an empty list. Now we perform the following operations for all child TTSBs $S_i$: Assume that $s_i$ is the terminal of $S_i$ closest to $s$. Then the list $L_{d,s_i}(S_i)$ contains all nodes $v \in S_i$ with associated sorting keys $d_{T'}(s_i, v)$. Now we create a copy $L'$ of this list and add the value $d_{T'}(s, s_i)$ to all sorting keys which does not affect its order. As a result $L'$ contains all nodes $v$ of $S_i$ sorted with respect to their distance $d_{T'}(s, v)$ from terminal $s$. Finally we merge $L$ with $L'$. After having carried this out for all child TTSBs $S_i$ the list $L$ equals the list $L_{d,s}(S)$ we are looking for. The list $L_{\varnothing,s}(S)$ is computed very similarly with the difference that we subtract the value $d_{T'}(s_i, s)$ from the sorting keys when handling the list $L_{\varnothing,s}(S_i)$. The respective lists for terminal $t$ are computed symmetrically. The total running time for computing the four lists associated with $S$ is $O(|S|)$ since we handle a constant number of child TTSBs.

We are now going to explain how $p'(v, S)$ can be determined for all $v \in S$. To this end assume that $v$ is contained in some $S_i$. Since we already know $p'(v, S_i)$ by the inductive hypothesis it suffices to determine $p'(v, S_j)$ for all $S_j \neq S_i$ and to add these values to $p'(v, S_i)$. Consider an arbitrary $S_j \neq S_i$ and assume that $s_i, s_j$ are the terminals of these TTSBs closest to each other. We create a copy $L'$ of list $L_{\varnothing,s_j}(S_j)$ and subtract the distance $d_{T'}(s_j, s_i)$ from all sorting keys in this list. As a result $L'$ contains all nodes $u$ of
S_j sorted with respect to the key g(u) − d_T\nu, s_i). At this point we can compute p'(v, S_j) for all v ∈ S_i by using the merge-and-scan procedure of Kim et al. To this end we merge the sorted list L' with the sorted list L_{d,s_i}(S_i) and store the result in L'. We assume that the nodes in L' are sorted in increasing order with respect to their numerical sorting keys. Ties are broken in favor of nodes in S_i. This can be achieved in linear time O(|S_i| + |S_j|).

Now recall that a node u ∈ S_j imposes a penalty π(u) on v ∈ S_i if d_T\nu, v) > g(u) or equivalently d_T\nu, v) > g(u) − d_T\nu, s_i). This is tantamount to that u precedes v in L'. Hence, in order to compute p'(v, S_j) for all v ∈ S_i it suffices to traverse L' once. In doing so, one can maintain the sum of penalties of all nodes u ∈ S_j encountered so far, which equals the penalty p'(v, S_j) whenever a node v ∈ S_i is reached.

The running time of this merge-and-scan operation is O(|S_i| + |S_j|) since the necessary sorted lists have already been computed. Thus we can compute p'(v, S) for all v ∈ S in total time O(|S|) once we know the p'-values and respective lists for all child TTSBs of S.

Note that the bottom of the recursion, that is, when T_{st}' consists merely of the pair (s, t) and (t, s) of anti-parallel arcs can trivially be handled constant time.

To sum up, this leads us to an algorithm whose running time h(|S|) can be described by the following recurrence

\[ h(|S|) = O(|S|) + \sum_{i=1}^{k} h(|S_i|), \]

where \( k \leq 5, \sum_{i=1}^{k} |S_i| = |S| + 4 \) and \( |S| \leq \frac{1}{2} |S| + 1 \). This implies that \( h(n) \) is \( O(n \log n) \).

**Theorem 3** The indirect covering subtree problem and hence also the single maximum coverage location can be solved in time \( O(n \log n) \).

\[ \square \]

### 4 A Matching Lower Bound

In this section we complement our algorithm with a lower bound \( \Omega(n \log n) \) on the running time for solving single maximum coverage location on a tree. This shows that (for certain computational models) our algorithm is optimal.

We make use of a recent result which is summarized in the following theorem.

**Theorem 4** ([BAG01]) Let \( W \subseteq \mathbb{R}^n \). If \( W \) is recognized in time \( t(n) \) on a real-number RAM that supports direct assignments, memory access, flow control, and arithmetic instructions \{+, −, ×, /\} then \( t(n) = \Omega(\log \beta(W^o)) \).

Here, \( W^o \) denotes the interior of \( W \) and \( \beta(W') \) denotes the number of connected components of some set \( W' \subseteq \mathbb{R}^n \).

To prove our lower bound we introduce a variant of the set disjointness problem. To this end let \( n \in \mathbb{N} \). The set \( W_n \subseteq \mathbb{R}^{4n} \) contains all tuples \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) such that \( x_1 < \ldots < x_n \) and \( x_i \neq y_j \) for all pairs \( i, j \). Consider a permutation \( \pi \) on the set \{1, \ldots, n\} and some tuple \( x_1 < y_{\pi(1)} < x_2 < y_{\pi(2)} < \ldots < x_n < y_{\pi(n)} \) in \( W_n \). It is easy
to see that for different permutations such tuples lie in different connected components of $W_n^*$ so $W_n^*$ contains at least $n!$ connected components. Hence any RAM of the above described type takes time $\Omega(n \log n)$ to recognize $W_n$.

We establish a linear time reduction from the problem to recognize $W_n$ to the single maximum coverage location problem on a tree with $O(n)$ nodes. To this end consider a tuple $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ for which we want to decide whether or not it is contained in $W_n$.

First we check if $x_1 < \ldots < x_n$. Then we create an edge $(u, v)$ of some length $c(u, v) > \max \{ x_i, y_i \mid i = 1, \ldots, n \}$ and choose some radius $\rho$ such that $\rho > c(u, v)$. For any $y_i$ we create two edges $(u, u_i)$ and $(v, v_i)$ of lengths $\rho - y_i$ and $y_i + \rho - c(u, v)$, respectively. Finally, we create for each $x_i$ a node $\tilde{x}_i$ on edge $(u, v)$ with distance $d(u, \tilde{x}_i) := x_i$. For each node $z$ in the node set $V := \{ u_i, v_i, \tilde{x}_i \mid i = 1, \ldots, n \} \cup \{ u, v \}$ we set $\pi(z) := 1$ and $\rho(z) := \rho$, which completes the reduction.

First suppose that we locate a facility outside the path $P(u, v)$. Assume that the facility is located at some node $u_i$. Then the distance of $u_i$ to $u$ is positive and $d(u, v_j) \geq \rho$ for any $j$. Hence, none of the nodes $v_j$ is covered by $u_i$ and the penalty cost imposed on $u_i$ must be at least $n$. The case where is the facility is placed at some node $v_i$ is treated analogously.

Now suppose for a moment that we can locate a facility anywhere at the path $P(u, v)$, that is, also at interior points of edges on $P(u, v)$. The point $x$ where the facility is located can then be identified with the distance $d(u, x)$. First, all nodes on $P(u, v)$ are covered by $x$ since $\rho > d(u, v)$. Due to our construction $x$ covers all nodes $u_i$ where $x \leq y_i$ and all nodes $v_j$ where $x \geq y_j$. Thus, the penalty imposed on $x$ is exactly $n$ if $x$ is not contained in the set $\{ y_1, \ldots, y_n \}$. If $x = y_j$ then $x$ covers both $u_j$ and $v_j$ and hence the penalty is bounded by $n - 1$. Since the facility can only be placed at nodes $\tilde{x}_j$, that is, at distances $x_i$ from $u$ we conclude that the minimum penalty cost is $n$ if the input tuple $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ lies in $W_n$ and $n - 1$ otherwise.

**Theorem 5** Any real-number RAM that complies with Theorem 4 takes at least $\Omega(n \log n)$ time to solve single maximum coverage location on a tree even for unit penalties and uniform radii. \hfill \Box

### 5 Implications for Related Problems

The variant of single maximum coverage location where the facility can be placed not only at the nodes but also at interior points of edges is called the *absolute* maximum coverage location problem. Kim et al. show [KLTW96] that a set of $O(n)$ critical points (that is a set of point which is guaranteed to to contain an optimal point) for absolute single maximum coverage location can be found in time $O(n \log n)$. We infer that also the absolute variant can be solved in $O(n \log n)$ on a tree.

Another implication of our result leads us to the realm of *competitive location*. Let a graph $G = (V, E)$ and $r, p \leq n$ be given. We assume that the graph is edge and node weighted. Let $X, Y \subseteq G$ be sets of nodes or interior points of edges. Then $w(Y \prec X)$ denotes the total weight $\sum \{ w(u) \mid u \in V \text{ and } d(u, Y) < d(u, X) \}$ of nodes that are
closer to Y than to X. Given some point set X the goal of the \((r, X)\)-medianoid problem \cite{Hak83} is to identify a set Y of \(r\) points such that \(w(Y \prec X)\) is maximized. This maximum weight is denoted by \(w_r(X)\). The goal of the \((r, p)\)-centroid problem is to find a \(p\)-element point set \(X\) such that \(w_r(X)\) is minimized.

By setting \(\rho(u) := d(u, X) - \varepsilon\) (where \(\varepsilon\) is a suitably small constant) and \(\pi(u) := w(u)\) one can easily verify that \((r, X)\)-medianoid is a special case of the multiple maximum coverage location problem with \(r\) servers. On general graphs the problem is NP-hard \cite{Hak83}. It can be solved efficiently in \(O(rn^2)\) on trees \cite{Tam96}. Our result leads to an \(O(n \log n)\) algorithm for the absolute and the discrete version of \((1, X)\)-medianoid on trees.

**Corollary 6** The discrete and the absolute \((1, X)\)-medianoid problem can be solved in \(O(n \log n)\) on trees.

It is not hard to extend the lower bound provided by Theorem 5 to \((1, X)\)-medianoid. Since the radii of the tree constructed in the reduction are uniform and hence all equal some number \(\rho\), we can furnish each node \(z\) on that tree with a pendant leaf \(z'\) at a distance \(d(z, z') = \rho + \varepsilon\). The set \(X\) contains exactly those pendant leaves. It is clear that for any node \(y\) in this enhanced tree \(T'\) the gain \(w(y \prec X)\) equals exactly \(w(T) - p(y)\) in the original tree \(T\). This implies that both instances lead to the same optimum. Thus also the algorithm for \((1, X)\)-medianoid is optimal in terms of the running time.

Now let’s turn our view to the \((r, p)\)-centroid problem. The problem is known to be \(\Sigma_2^p\)-complete on general graphs \cite{NSW07} and NP-hard even on path graphs \cite{SW09}. However, both the absolute and the discrete variant of \((1, p)\)-centroid on trees can be solved in polynomial time \(O(n^2 \log n \log w(T)/\log \log n)\) and \(O(n^2 \log^2 n \log w(T) \log D / \log \log n)\), respectively \cite{SW09}. Here \(D\) is the maximum edge length of the input tree \(T\). Those algorithms rely on \(O(n \log w(T))\) (resp. \(O(n \log w(T) \log D)\)) calls to a subroutine solving \((1, X)\)-medianoid on a tree.

The algorithm provided here allows us to solve \((1, X)\)-medianoid in \(O(n \log n)\) which yields.

**Corollary 7** The discrete and the absolute \((1, p)\)-centroid problem for trees can be solved in \(O(n^2 \log n \log w(T))\) and \(O(n^2 \log n \log w(T) \log D)\), respectively.

### 6 Concluding Remarks

We have provided an \(O(n \log n)\) algorithm for solving the indirect covering subtree problem which improves upon the previously best algorithms for this problem and single maximum coverage location on trees. We have also shown that our algorithm is optimal for certain unit-cost RAM models. Our result leads also to an optimal algorithm for \((1, X)\)-medianoid and faster algorithms for \((1, p)\)-centroid on trees.

It would be interesting to identify larger problem classes of location problems on trees where the two-terminal subtree technique can be applied. It would also be worth investigating the existence of faster algorithms on path graphs.
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