Analysis of subsystems with rooks on a chess-board representing a partial Latin square (Part 2.)*

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Abstract

A partial Latin square of order \( n \) can be represented by a 3-dimensional chess-board of size \( n \times n \times n \) with at most \( n^2 \) non-attacking rooks. In Latin squares, a subsystem and its most distant mate together have as many rooks as their capacity. That implies a simple capacity condition for the completion of partial Latin squares which is in fact the Cruse’s necessary condition for characteristic matrices.

Andersen-Hilton proved that, except for certain listed cases, a PLS of order \( n \) can be completed if it contains only \( n \) symbols. Andersen proved it for \( n + 1 \) symbols, listing the cases to be excluded. Identifying the structures of the chess-board that can be overloaded with \( n \) or \( n + 1 \) rooks, it follows that a PLS derived from a chess-board with at most \( n + 1 \) non-attacking rooks can be completed exactly if it satisfies the capacity condition.

In a layer of a Latin square, two subsystems of a remote couple are in balance. Thus, a necessary condition for completion of a layer can be formulated, the balance condition.

For an LSC, each 1-dimensional subspace of the chess-board contains exactly one rook. Consequently, for the PLSCs derived from partial Latin squares, we examine certain sets of 1-dimensional subspaces because they indicate the number of missing rooks.

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Abbreviations

|  |  |
|---|---|
| LS | Latin square |
| LSC | Latin super cube |
| d-LSC | \( d \)-dimensional Latin super cube |
| PLS | partial Latin square |
| PLSC | partial Latin super cube |
| RBC | remote brick couple |
| mRBC | minimum remote brick couple |
| RAC | remote axis couple |

*This is the second part of our consecutive articles on Latin and partial Latin squares and on bipartite graphs. Each paper holds the terminology and notation of the previous ones [5].
1 Introduction

Take a 3-dimensional chess-board $H_n^3$ with at most $n^2$ non-attacking rooks derived from a partial Latin square $P$ by composition. Place a dot in each empty cell of $H_n^3$ that has a Hamming distance of at least 2 from each rook of $P$. We consider the dots as candidates, only a dot can be replaced by a rook. When replacing a dot by a rook, then we have to delete all dots that have a Hamming distance 1 from the „new“ rook. This process is called automatic elimination. A cell is called eliminated if it has neither rooks nor dots. The structure of this chess-board with rooks and dots is called Partial Latin Super Cube, or PLSC for short. For a PLSC we always assume that the automatic elimination is done. A file of a PLSC is an eliminated file if each cell of the file is eliminated. Each file of a completion of a PLSC contains exactly one rook, consequently, a PLSC is not completable if it has an eliminated file.

Definition 1.1. The number of rooks in a PLSC $P$ is denoted by $|P|$. The PLSC $Q$ is an extension of $P$ if each rook in $P$ also is a rook in $Q$, denoted by $P \subseteq Q$. $Q$ is a completion of $P$ if $P \subseteq Q$ and $|Q| = n^2$. $P$ is called completable if such a $Q$ exists.

Definition 1.2. The RBC $(T_0, T_3)$ is stuffed, if $(T_0, T_3)$ has as many rooks as the capacity of $(T_0, T_3)$, that is $\text{cap}(T_0, T_3)$.

Definition 1.3. The RBC $(T_0, T_3)$ is called overloaded, if $(T_0, T_3)$ has more rooks than $\text{cap}(T_0, T_3)$.

The capacity function returns the number of files in an $n$-brick for degenerated RBCs, so a degenerated RBC cannot be overloaded, otherwise an $n$-brick should have more rooks than parallel files.

Definition 1.4. The closure hull of a set of cells $X$, denoted by $\text{ch}(X)$, is a brick $T$ that contains all cells of $X$ and $\text{diam}(T)$ is minimal.

Definition 1.5. A brick is perfect if it has no dots.

Remark 1.6. No rooks can be placed into a perfect brick.

We differentiate two special cases of extension.

1. When we place rooks only outside $\text{ch}(P)$:

   Definition 1.7. The PLSC $P$ is embedded in $Q$ if $Q$ is an extension of $P$ and the number of rooks in $\text{ch}(P)$ is $|P|$. If $P$ is perfect, then all extensions of $P$ contain $P$ in this embedded way.

2. When we place rooks only into the $\text{ch}(P)$:

   Definition 1.8. The PLSC $Q$ is an expansion of $P$ if $Q$ is an extension of $P$ and $\text{ch}(P) = \text{ch}(Q)$.
If \( P \) is a partial Latin square, then the conjugates of \( P \) are also partial Latin squares. A PLS \( P \) and its primary conjugates can be seen in the Figure 1.1. For example, in the proper PLSCs, the rook and cell \((3,4,5)\) in the middle square are transformed into the cell \((5,3,4)\) in the conjugate \((k,i,j)\) (left square), and into the cell \((4,5,3)\) in the conjugate \((j,k,i)\) (right square).

Sometimes we refer to a property of the cube in Euclidean space, therefore, from now on a 3-tuple \((a,b,c)\) has two meanings, on the one hand, it is a point in the Euclidean space, on the other hand, it identifies a cell. It will always be clear, which meaning we use. Let us consider \(H^3_n\) as a cube in the Euclidean space, one vertex of the cube is at the origin and the coordinate axes are \(x, y\) and \(z\). So the vertices of \(H\) in the Euclidean space are the points \((0,0,0), (0,0,n), (0,n,0), (n,0,0), (n,0,n), (0,n,n), (n,n,0)\) and \((n,n,n)\).

**Definition 1.9.** The six points \((a,b,c), (c,a,b), (b,c,a), (a,c,b), (c,b,a)\) and \((b,a,c)\) are called akin points.

If two of the coordinates are equal then we get only three distinct akin points, if all the coordinates are equal then we get only one point. Obviously, the points are equidistant from the origin in both taxicab geometry and Euclidean geometry.

## 2 Capacity Condition and its Consequences

**Definition 2.1.** An arbitrary PLSC \(P\) of order \(n\) satisfies the capacity condition if

\[
e_0 + e_3 \leq \text{cap}(T_0, T_3)
\]

for each RBC \((T_0, T_3)\) of \(P\) where \(e_0\) and \(e_3\) are the number of rooks in \(T_0\) and \(T_3\), respectively.

Due to Distribution Theorem [5] there is no completion of \(P\) if \(P\) does not satisfy the capacity condition. The same result is described by Cruse ([3, Theorem2]) based on the characteristic matrix of the Latin squares.

The PLSC \(P\) (can also be considered as a Sudoku puzzle) in the Figure 2.1 has more completions. One of them is on the right-hand side of Figure 2.1. If you put a 7 into the cell \((1,4)\) or a 3 into the cell \((4,8)\), the new partial Latin squares, shown in the Figure 2.2, are not completable because they do not satisfy the capacity condition. Performing the proper permutations of rows
and columns we get the structures shown in the Figure 2.3. The yellow brick has a height of 3, the green has a height of 6 where "height" is the number of symbol layers. The capacity of the yellow-green RBC is \((4 \times 6 \times 3 + 5 \times 3 \times 6)/9 = 18\). Some layers of the green brick have no rooks. The RBCs have 19 rooks (symbols), so they are overloaded. Therefore, the PLSCs are not completable.

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3 Incompletable PLSCs

Cruse [3] proves by induction, that the capacity condition holds, if the PLSC contains at most $n - 1$ rooks. So, we need at least $n$ rooks to produce overloaded RBCs. Here are some trivial examples in Figure 3.1 and Figure 3.2. The coordinates of the single green rook are $(k, n, n)$, $(n, n, k)$ and $(n, k, n)$ respectively, where $k \neq 1$. The capacity of the yellow-green RBCs:

$$\text{cap}(n, 1, n - 1, n - 1) = \frac{1 \cdot (n - 1)(n - 1) + (n - 1) \cdot 1 \cdot 1}{n} = n - 1$$

The yellow-green RBCs have $n$ rooks, so they are overloaded.

![Figure 3.1](image1.png)

![Figure 3.2](image2.png)

The capacity of the yellow-green RBCs:

$$\text{cap}(n, k, n - 1, 1) = \frac{k(n - 1) \cdot 1 + (n - k) \cdot 1 \cdot (n - 1)}{n} = n - 1$$

where $(1 < k < n)$.

**Remark 3.1.** The capacity of the yellow-green RBCs does not depend on $k$.

For better visualization, first we rotated the PLSCs in the Figure 3.2 by $+90$ degrees and then we represented these PLSCs in a 3-dimensional view, as shown in the Figure 3.3.
The axis of the hinge is a file and the rooks of $T_0$ eliminate $k$ dots in the axis and the rooks of $T_3$ eliminate $(n-k)$ dots in the axis, so the rooks of the RBC $(T_0, T_3)$ eliminate $n$ distinct dots in this file, where $1 \leq k \leq n - 1$. Ergo, the brown file has no dots, that means, the overloaded RBC $(T_0, T_3)$ destroys the opportunity of the completion of the brown axis.

4 RBCs of the Smallest and Second Smallest Capacity

First, we investigate the RBCs that have the smallest capacity in $H_n^3$. We put our cube into the Euclidean space. Let $H$ be a cube in the Euclidean space with vertices $(0,0,0), (0,0,n), (0,n,0), (n,0,0), (n,0,n), (0,n,n), (n,n,0)$ and $(n,n,n)$. Let $p$ be a fixed integer where $3 \leq p \leq \frac{3n}{2}$ and $a, b, c \in \{0,1,2,\ldots,n\}$ are varying integers. Let $P$ be the plane defined by the equation $p = a + b + c$. The point $(a,b,c)$ of the plane $P$ can also be considered as a vertex of a brick $T_0$ of size $a \times b \times c$. The opposite vertex of $T_0$ is at the origin. If the point $(a,b,c)$ is on the surface of $H$, then $T_0$ is not a real brick. If the point $(a,b,c)$ is inside of $H$, then $T_0$ and its remote mate $T_3$ are real bricks, the point $(a,b,c)$ is the only common point of $T_0$ and $T_3$ and the opposite vertex of $T_3$ is in the point $(n,n,n)$. A point $(a,b,c)$ is called valid, if the brick $T_3$ of size $a \times b \times c$ is a real brick. An RBC $(T_0, T_3)$ is called $mRBC$ (minimum RBC) if there is a hinge with leaves $T_0$ and $T_3$, and the axis of the hinge consists of exactly one file. If the point $(a,b,c)$ defines an $mRBC$, then the akin points of $(a,b,c)$ also define $mRBC$s. Three examples can be seen in the Figure 3.3.

Let $H'$ be a cube inside $H$ defined by the following vertices $(1,1,1), (1,1,n-1), (1,n-1,1), (n-1,1,1), (n-1,1,n-1), (1,n-1,n-1), (n-1,n-1,1)$. The point $(a,b,c)$ is valid if and only if it is in $H'$ including the surface, shown in the Figure 4.1. Ergo, we inspect the capacity function for the triples $(a,b,c)$, where $a,b,c \in \{1,2,\ldots,n-1\}$. Easy to see, that the next equation holds

$$\begin{align*}
cap(n,a,b,c) &= n^2 - (a + b + c)n + ab + bc + ca \\
&= n^2 - (a + b + c)n + \frac{(a + b + c)^2 - (a^2 + b^2 + c^2)}{2}
\end{align*}$$

(4.1)  (4.2)
We introduce a new function on the plane $P$ defined by $p = a + b + c$.

$$f_p(n, a, b, c) = n^2 - pn + \frac{p^2}{2} - \frac{(a^2 + b^2 + c^2)}{2} \tag{4.3}$$

where $a, b, c \in \{1, 2, \ldots, n - 1\}$ and $p$ is a parameter for which $3 \leq p \leq 3n - 3$.

![Figure 4.1](image1.png)

Figure 4.1.

![Figure 4.2](image2.png)

Figure 4.2.

Let $A = (a_1, b_1, c_1)$ and $B = (a_2, b_2, c_2)$ be two points, where $a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = p$, and let $K$ be the intersection of plane $P$ and the line between the points $(0, 0, 0)$ and $(n, n, n)$. $K$ is the center of the intersection of $P$ and $H^*$ as well. The sphere centered in the origin with radius $\sqrt{(a_1^2 + b_1^2 + c_1^2)}$ intersects the plane $P$ in a cycle containing point $A$. The sphere centered at the origin with radius $\sqrt{(a_2^2 + b_2^2 + c_2^2)}$ intersects $P$ in a cycle containing point $B$, as indicated in the right-hand side of Figure 4.1.
So, based on (4.3)

$$f_p(n, a_1, b_1, c_1) < f_p(n, a_2, b_2, c_2) \iff a_1^2 + b_1^2 + c_1^2 > a_2^2 + b_2^2 + c_2^2,$$

consequently, it holds that:

**Statement 4.1.** The smaller the distance of a point is from $K$, the larger the value of function $f_p$ is.

The opposite vertex of the origin in $H$ is the vertex with coordinates $(n, n, n)$. Because of the central symmetry of the cube $\text{cap}(n, a, b, c) = \text{cap}(n, n-a, n-b, n-c)$, so we can assume, that $p \leq \frac{3n}{2}$. If $3 \leq p \leq \frac{3n}{2}$, then, considering the intersection of the plane $P$ and the cube $H^*$, there are three distinct cases.

a) a point, if $p = 3$

b) a triangle, if $3 < p \leq n + 1$

c) a hexagon with vertices $(1, p - n, n - 1), (p - n, 1, n - 1), (1, n - 1, p - n), (p - n, n - 1, 1), (n - 1, 1, p - n), (n - 1, p - n, 1)$, if $n + 1 < p \leq \frac{3n}{2}$.

If $n = 2$ then $p = 3$ because of the assumption $3 \leq p \leq \frac{3n}{2}$.

**Case a)**

If $n = 2$ then the cube $H^*$ is degenerated and consists of one point $(1, 1, 1)$ and $f_3(n, 1, 1, 1) = 1 = n - 1$. If $n > 2$ then $f_p(n, 1, 1, 1) = n^2 - 3n + 3$ and because of $n^2 - 3n + 3 = (n - 2)(n - 1)$ it follows that $f_p(n, 1, 1, 1) > n - 1$.

**Case b)**

The function $f_p$ gives the minimum value in three most remote valid points $(p - 2, 1, 1), (1, p - 2, 1)$ and $(1, 1, p - 2)$. The minimum value is $f_p(n, 1, 1, p - 2) = n^2 - pn + (p - 2) + (p - 2) + 1 = n^2 - pn + 2p - 3$. If $n > 2$ and $p \leq n + 1$, then $(n - 2)(n - p + 1) \geq 0$. Consequently, if $n > 2$ and $p < n + 1$ then $f_p(n, 1, 1, p - 2) = n^2 - pn + 2p - 3 > n - 1$. Plainly, the point $(1, 1, 1)$ defines an mRBC $(T_0, T_3)$ if and only if $n = 2$ as we seen before. For $n > 2$, the distinct points $(p - 2, 1, 1), (1, p - 2, 1)$ and $(1, 1, p - 2)$ defines an mRBC $(T_0, T_3)$ if and only if $p = n + 1$. In this case the value of $f_p$ is $(n - 1)$, as depicted in the Figure 4.2.

**Case c)**

The function $f_p$ gives the minimum value in the six most remote valid points of the proper hexagon $(1, p - n, n - 1), (p - n, 1, n - 1), (1, n - 1, p - n), (p - n, n - 1, 1), (n - 1, 1, p - n), (n - 1, p - n, 1)$. The value of $f_p$ is $(n - 1)$ for these points, as depicted in the Figure 4.2.
If a triple \((a, b, c)\) is inside of the cube \(H^*\), then the point is on plane \(P\) defined by \(p = a + b + c\). If it is not a most remote point from \(K\), then \(\text{cap}(n, a, b, c) > n - 1\). Thus, we can summarize the results in the following way:

**Statement 4.2.** The minimum value of the capacity function for non-degenerated RBCs is \(n - 1\) and \(f(n, a, b, c) = n - 1\) if and only if the point \((a, b, c)\) defines an mRBC.

Consequently, the point \((a, b, c)\) defines an mRBC if and only if the point \((a, b, c)\) is on an edge of \(H^*\) that is neither adjacent to the origin nor to the vertex of \(H^*\) opposite the origin. They are the red edges on the left-hand side in Figure 4.3. If \(p = n + 1\), then the plane \(P\) intersect the cube \(H^*\) in three distinct points, \(U_x(1, 1, n - 1), U_y(1, n - 1, 1)\) and \(U_z(n - 1, 1, 1)\). If \(p = 2n - 1\), the plane \(P\) intersect the cube \(H^*\) in three points \(V_{yz}(1, n - 1, n - 1), V_{xy}(n - 1, 1, n - 1)\) and \(V_{xz}(n - 1, n - 1, 1)\). If \(n + 1 < p < 2n - 1\), then the plane \(P\) intersect the cube \(H^*\) in six points. Let \(A\) be one of the six points. The space diagonal of the proper brick \(T_0\) is between the point \(A\) and the origin. The space diagonal of the proper brick \(T_3\) is between \(A\) and the point \((n, n, n)\). If \(A\) is the point \((1, n - 1, p - n)\), then the proper bricks \(T_0\) and \(T_3\) and the proper axis of the RBC \((T_0, T_3)\) can be seen on the right-hand side of Figure 4.3. The axis of the proper hinge covers the red edge, that contains the point \((a, b, c)\). If the point \((a, b, c)\) is one of the points \(U_x, U_y, U_z, V_{xy}, V_{xz}, V_{yz}\), then there are two proper hinges defined by the point \((a, b, c)\). If we take for example the point \(U_z\), then the axis of the proper hinge covers either the edge \(U_zV_{yz}\) or \(U_zV_{xz}\).

**Remark 4.3.** The points \((p - n, 1, n - 1)\), \((n - 1, 1, p - n)\), \((n - 1, p - n, 1)\), \((p - n, n - 1, 1)\), \((1, n - 1, p - n)\) and \((1, p - n, n - 1)\) are akin points. If two of the three coordinates are equal \((p - n = n - 1)\) then we get only three distinct points \((U_x, U_y, U_z)\) or \(V_{xy}, V_{yz}, V_{xz}\) in Figure 4.3). If all the three coordinates are equal \((p - n = n - 1 = 1)\) then \(n = 2\) and \(p = 3\) so, we get only one point.

![Figure 4.3.](image-url)
vertices of the triangle \((1, 1, p - 2), (1, p - 2, 1)\) and \((p - 2, 1, 1)\), so there are no neighboring points. If \(p > 4\) then the second most remote distinct points from \(K\) are the points \((2, 1, p - 3), (1, 2, p - 3), (2, p - 3, 1), (1, p - 3, 2), (p - 3, 2, 1)\) and \((p - 3, 1, 2)\). The value of the function \(f_p\) for these points is \(f_p(n, 2, 1, p - 3) = n^2 - pn + 3p - 7\). The difference between the two values is \(f_p(n, 2, 1, p - 3) - f_p(n, 1, 1, p - 2) = p - 4\). So \(f_p(n, 2, 1, p - 3) - f_p(n, 1, 1, p - 2) = 1\) if and only if \(p = 5\), and in this case there are three distinct points. If \(n = 3\) and \(p = 5\) then \(p > 3n/2\), this case falls out, in the other case \(f_3(4, 2, 1, 2) = f_3(4, 1, 2, 2) = f_3(4, 2, 2, 1) = n\).

For \(n \geq 4\) and \(3n/2 > p > n + 1\) the proper intersection is a hexagon, as shown on the right-hand side in Figure 4.3. Let \(A = (n - 1, 1, p - n)\) be one of the vertices of the hexagon. The neighboring points of \(A\) on the edges of the hexagon are \(B = (n - 2, 1, p - n + 1)\) and \(C = (n - 1, 2, p - n - 1)\). Generally, the hexagons have three short edges of the same length and three long edges of the same length, as the example shows in the right-hand side of Figure 4.4. If \(p = n + 2\), then the point \(C = (n - 1, 2, 1)\) is the other vertex of the short edge, not a point inside. The only point inside the hexagon is the point \((2, 2, 2)\) for which \(\text{cap}(n, 2, 2, 2) = n\) if and only if \(\text{cap}(n, 2, 2, 2) - n = (n - 3)(n - 4) = 0\) ergo \(n = 3\) or \(n = 4\). The case \(n = 3\) is equivalent to the case \(p = 3\), point \((1, 1, 1)\) in case \(a)\) above. The case \(n = 4\) is a valid case because of \(\text{cap}(n, 2, 2, 2) = n\) and can be overloaded by \(n + 1\) rooks. The yellow and green bricks on the left-hand side in Figure 4.4 are \(2 \times 2 \times 2\) of size. Together they contain 5 rooks.

Let \(f_p(A)\) be the short form of \(f_p(n, n - 1, 1, p - n)\), \(f_p(B)\) be \(f_p(n, n - 2, 1, p - n + 1)\) and \(f_p(C)\) be \(f_p(n, n - 1, 2, p - n - 1)\). So, we assume, that \(3n/2 > p > n + 2\).

\[
f_p(B) = n^2 - pn + p - n + 1 + n - 2 + (p - n + 1)(n - 2) = 3n - p - 3
\]
\[
f_p(C) = n^2 - pn + 2(p - n - 1) + 2(n - 1) + (p - n - 1)(n - 1) = p - 3
\]
\[
f_p(B) - f_p(A) = 2n - p - 2
\]

Because of

\[
3n/2 > p, 3n > 2p, 2n > 2p - n = p + (p - n) > p + 2
\]

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so
\[ fp(B) > fp(A)fp(C) - fp(A) = p - (n + 2) \]
Because of \( p > n + 2 \), so
\[ fp(C) > fp(A)fp(B) - fp(C) = n - 2(p - n) \]
Because of
\[ 3n/2 \geq p, n/2 \geq p - n, n \geq 2(p - n), n - 2(p - n) \geq 0 \]
the following holds:
\[ fp(B) \geq fp(C) \]
Equality if and only if \( p = 3n/2 \). Hence, we get the minimum value of \( fp \) in the point \( C \) and
\[ fp(C) = p - 3 \]. Because of \( p > n + 2 \) the minimum value of \( p - 3 \) is \( n \) if \( p = n + 3 \). Therefore, \( A = (n - 1, 1, 3), C = (n - 1, 2, 2) \), the other end of the edge containing \( A \) and \( C \) is \( (n - 1, 3, 1) \), so this edge contains only three points and the point \( C \) is in the middle of this edge.

![Figure 4.5](image)

Assuming that \( 3 \leq p \leq 3n/2 \), all the points, that defines an RBC with capacity \( n \) are the points \((2, 1, 2), (1, 2, 2) \) and \((2, 2, 1) \) if \( n = 4 \) and the points \((n - 1, 2, 2), (2, n - 1, 2) \) and \((2, 2, n - 1) \) if \( n > 4 \). Because of the central symmetry the other points are \((2, 3, 2), (3, 2, 2) \) and \((2, 2, 3) \) if \( n = 4 \) and \((1, n - 2, n - 2), (n - 2, 1, n - 2) \) and \((n - 2, n - 2, 1) \) if \( n > 4 \). All these points define the structures shown in the Figure 4.5 or the structures derived from these structures by central symmetry.

We assume that the rooks derived from the symbols \( A, B, C, D \) are non-attacking and are not in the first layer i.e. \( A, B, C, D \neq 1 \), however the cases \( A = D \) and/or \( B = C \) are allowed.

The capacity of the yellow-green RBCs:
\[ \text{cap}(n, n - 2, n - 2, 1) = \frac{(n - 2)(n - 2) \cdot 1 + 2 \cdot 2 \cdot (n - 1)}{n} = n \]
The yellow-green RBCs have \((n + 1)\) rooks, so they are overloaded. For \( n = 4 \) the square in the middle of the Figure 4.6 is equivalent to the second square in the Figure 4.4. There are two well-known results from Andersen-Hilton [2] and Andersen [1]. They identify and list the structures.
that destroy the completability of a PLS with \( n \) or \( n + 1 \) symbols. The listed structures are the structures shown in Figure 4.5 by choosing the value of \( A, B, C, D \) in every possible ways and the left square in Figure 4.4 in case \( n = 4 \). We can add to this list the case \( n = 3 \) and the point \((1,1,1)\). Based on these results the above theorems can be restated in the following way:

**Theorem 4.4.** Assuming \( n \geq 2 \), a PLSC \( P \) of order \( n \) with at most \( (n + 1) \) rooks is completable if and only if \( P \) satisfies the capacity condition.

Generally, the capacity condition is not sufficient for completability of a PLSC. Two iconic partial Latin squares can be seen on the Figure 4.7. They are counterexamples. Cruse [3] proved that the capacity condition holds for the Cruse’s square. Using his method in the next paper (Part 3.), we prove that GW6 also satisfies the capacity condition.
5 Difference of Weights and Deficits

**Definition 5.1.** The remote mate of the axis $V$ of size $a \times b \times n$ is the axis $W$ of size $(n-a) \times (n-b) \times n$, if the edges of length $n$ of the two axes are parallel and the axes are disjoint. The pair of axes $(V, W)$ is called remote axis couple or simply RAC if $W$ is the remote mate of $V$.

The Hamming distance between the two axes of an RAC is 2. The intersection of $V$ and the layer $\sigma$ is denoted by $T_0$ and the intersection of $W$ and the layer $\sigma$ is denoted by $T_2$, as depicted in the Figure 5.1. So, $(T_0, T_2)$ is an RBC in the layer $\sigma$ for an arbitrary $\sigma \in \{1, 2, \ldots, n\}$ and $N(\sigma, V)$ denotes the number of rooks in $T_0$ and $N(\sigma, W)$ denotes the number of rooks in $T_2$.

![Figure 5.1.](image)

**Definition 5.2.** The deficit of weight of the RBC $(T_0, T_2)$ is the number of rooks needed to put into $T_0$ to get the balanced state for the RBC $(T_0, T_2)$. So the deficit of weight of $(T_0, T_2)$ is

$$dw(T_0, T_2) = dw(\sigma, V, W) = (a + b - n) - [N(\sigma, V) - N(\sigma, W)]$$

In case of the balanced state $N(\sigma, V) - N(\sigma, W) = a + b - n = Ry(T_0)$.

**Definition 5.3.** If $(T_0, T_2)$ is an RBC of a layer then $T_0$ is underweighted if $dw(T_0, T_2) > 0$ and overweighted if $dw(T_0, T_2) < 0$.

If $T_0$ is underweighted we have to put $dw(T_0, T_2)$ rooks into $T_0$ to reach the balanced state of $(T_0, T_2)$. Since we never take rooks out of a PLSC, so we need to put $|dw(T_0, T_2)|$ rooks into $T_2$ to reach the balanced state of RBC $(T_0, T_2)$ if $T_0$ is overweighted.

**Definition 5.4 (Balance condition).** The balance condition holds for a layer if any perfect brick of the layer is not underweighted.

We cannot put a rook in a perfect brick, so the Balance Condition 5.4 is a necessary condition for completion of a layer.
**Definition 5.5.** The deficit of weight of a remote axis couple (RAC) is
\[ dw(V, W) = \sum_{\sigma=1}^{n} (a + b - n) - [N(\sigma, V) - N(\sigma, W)] \]

**Definition 5.6.** Denote \( E_z(X) \) the number of files in the axis \( X \) of direction \( z \), that have no rooks.

Because of \( dw(V, W) = n(a + b - n) - \sum_{\sigma=1}^{n} N(\sigma, V) + \sum_{\sigma=1}^{n} N(\sigma, W) = [ab - \sum_{\sigma=1}^{n} N(\sigma, V)] - [(n - a)(n - b) - \sum_{\sigma=1}^{n} N(\sigma, W)] \)

therefore, \( dw(V, W) = E_z(V) - E_z(W) \). Ergo, the members of a RAC are in balance, if they have the same number of empty files of direction \( z \) (empty cells in the proper PLS). If \( V \) and \( W \) are balanced, it does not mean that \( dw(\sigma, V, W) = 0 \) for all \( \sigma \in \{1, 2, \ldots, n\} \). Obviously, \( dw(\sigma, W, V) = -dw(\sigma, V, W) \) and \( dw(\sigma, T_2, T_0) = -dw(\sigma, T_0, T_2) \).

Another evident condition for completion of a PLSC is the completablety of all layers. If a layer of a PLSC is incompletable, then the whole PLSC is incompletable as well. Therefore, we first examine the individual layers.

![Figure 5.2](image_url)

**Figure 5.2.**

**Case \( d = 2 \):**

Without loss of generality, we assume that the rows and columns have already been permuted to the form showed in the Figure 5.2, so the rooks are in the corners of the brick \( T_0 \) and \( T_2 \) and in the corners of the white bricks. Since
\[ dw(\sigma, T_0, T_2) = (a + b - n) - [N(\sigma, T_0) - N(\sigma, T_2)] = [a - N(\sigma, T_0) - (n - b) - N(\sigma, T_2)] \]
and \([a - N(\sigma, T_0)]\) is the length of the one edge of the white brick, \([n - b) - N(\sigma, T_2)]\) is the length of the other edge of the white brick in \( T_{1b} \), so the difference of the two lengths gives the
deficit of weight of the RBC \((T_0, T_2)\). Since
\[
dw(\sigma, T_0, T_2) = (a + b - n) - [N(\sigma, T_0) - N(\sigma, T_2)] = [b - N(\sigma, T_0) - [(n - a) - N(\sigma, T_2)]
\]
so, we get the same result for the white brick in \(T_{1a}\). This means that the RBC \((T_0, T_2)\) is balanced if and only if the white bricks have a square shape. Placing a rook into a white brick leaves the deficit of weight of the RBC \((T_0, T_2)\) unchanged. Even \(dw(T_0, T_2)\) can be calculated from the number of empty files of the white bricks.

\[
dw(T_0, T_2) = \frac{V(T_0) + V(T_{1b})}{n} - \frac{V(T_{1b}) + V(T_2)}{n} - (N(\sigma, T_0) - N(\sigma, T_2))
\]

\[
dw(T_0, T_2) = \frac{V(T_0) + V(T_{1b})}{n} - N(\sigma, T_0) - \frac{V(T_{1b}) + V(T_2)}{n} - N(\sigma, T_2)
\]

\[
dw(T_0, T_2) = \frac{V(T_0) + V(T_{1b})}{n} - N(\sigma, T_0) - N(\sigma, T_{1b})
\]

\[
- \frac{V(T_{1b}) + V(T_2)}{n} - N(\sigma, T_2) - N(\sigma, T_{1b})
\]

\[
dw(T_0, T_2) = E(T_0 \cup T_{1b}) - E(T_{1b} \cup T_2) \tag{5.1}
\]

In the same way we get
\[
dw(T_0, T_2) = E(T_0 \cup T_{1a}) - E(T_{1a} \cup T_2) \tag{5.2}
\]

So the RBC \((T_0, T_2)\) is balanced if and only if \(E(T_0 \cup T_{1b}) = E(T_{1b} \cup T_2)\). In this case obviously \(E(T_0 \cup T_{1a}) = E(T_{1a} \cup T_2)\) as well.

**Case \(d = 3\):**

Let us choose \(c\) arbitrary layers in \((V, W)\). After the permutation of these layers to the bottom of the axes, these layers have the z-coordinates 1, 2, \ldots, \(c\). Let \(T_0\) denote the resulting brick of size \(a \times b \times c\) in \(V\). The Hamming bricks generated by \(T_0\) can be seen in see Figure 5.3. Using the definition of the deficit of weight for a layer, we can define the deficit of weight of \((T_0, T_{2ab})\) as the sum of the deficit of weight in layer 1, 2, \ldots, \(c\) in the following way:

**Definition 5.7.** The deficit of weight of \((T_0, T_{2ab})\) is
\[
dw(T_0, T_{2ab}) = \sum_{\sigma=1}^{c} ((a + b - n) - [N(\sigma, T_0) - N(\sigma, T_{2ab})]) \tag{5.3}
\]
Because $dw(\sigma, T_0, T_2) = E(\sigma, T_0 \cup T_1b) - E(\sigma, T_{1b} \cup T_{2ab})$ for $\sigma = 1, 2, \ldots, c$. Hence,

$$dw(T_0, T_{2ab}) = E(T_0 \cup T_{1b}) - E(T_{1b} \cup T_{2ab})$$

(5.4)

In the same way we get

$$dw(T_0, T_{2ab}) = E(T_0 \cup T_{1a}) - E(T_{1a} \cup T_{2ab})$$

(5.5)

$E(T_0 \cup T_{1b})$ is the number of empty files in $T_0 \cup T_{1b}$ (parallel to edge $b$) and $E(T_{1b} \cup T_{2ab})$ is the number of empty files in $T_{1b} \cup T_{2ab}$ (parallel to edge $a$) and respectively $E(T_0 \cup T_{1a})$ is the number of empty files in $T_0 \cup T_{1a}$ (parallel to edge $a$) and $E(T_{1a} \cup T_{2ab})$ is the number of empty files in $T_{1a} \cup T_{2ab}$ (parallel to edge $b$). Therefore, $T_0$ and $T_{2ab}$ are balanced if and only if $E(T_0 \cup T_{1b}) = E(T_{1b} \cup T_{2ab})$. In this case $E(T_0 \cup T_{1a}) = E(T_{1a} \cup T_{2ab})$ as well. If you stand in the middle of the white brick $T_{1a}$ and ignore the rooks in $T_{1a}$, you should see the same number of empty files when looking through the bricks $T_0$ and $T_{2ab}$.

Let $E_x(T), E_y(T)$ and $E_z(T)$ denote the number of empty files in $T$ of directions $x, y$ and $z$ respectively. So

$$dw(T_0, T_{2ab}) = E_y(T_0) - E_x(T_{2ab}) = E_y(T_{1b}) - E_x(T_{1b})$$

and

$$dw(T_0, T_{2ab}) = E_x(T_0) - E_y(T_{2ab}) = E_x(T_{1a}) - E_y(T_{1a})$$

If $c = n, V = T_0, W = T_{2ab}, Z = T_{1a}$ and $ZZ = T_{1b}$ then

$$dw(V, W) = E_z(V) - E_z(W) = E_x(Z) - E_y(Z)$$

and

$$dw(V, W) = E_z(V) - E_z(W) = E_y(ZZ) - E_x(ZZ)$$
If you stand in the middle of the white axis \( Z \) and ignore the rooks in \( Z \), you should see the same number of empty files when looking through the axes \( V \) and \( W \). The same is true for the axes \( ZZ \), \( V \) and \( W \) respectively.

**Definition 5.8.** Let \( \text{def}(T_0, T_3) \) denote the deficit of the RBC \( (T_0, T_3) \), inclusive the degenerated cases.

\[
\text{def}(T_0, T_3) = \text{cap}(T_0, T_3) - (c_0 + c_3)
\]

The deficit tells you, how many rooks need to to be put into the bricks \( T_0 \) and \( T_3 \) combined, to reach the stuffed state. If the deficit is negative, then the RBC is overloaded. If \( c = 0 \), then

\[
\text{def}(T_0, T_3) = \text{cap}(T_0, W) - (c_{2ab} + c_3) = A_z(W) - (c_{2ab} + c_3) = E_z(W)
\]

If \( c = n \), then

\[
\text{def}(T_0, T_3) = \text{cap}(V, T_3) - (c_0 + c_1c) = A_z(V) - (c_0 + c_1c) = E_z(V)
\]

It follows from the Definition 5.7 that

\[
dw(T_0, T_{2ab}) = c(a + b - n) - (c_0 - c_{2ab})
\]

and so

\[
c_0 = c_{2ab} + c(a + b - n) - dw(T_0, T_{2ab}) \tag{5.6}
\]

On the other hand

\[
\text{def}(T_0, T_3) = \text{cap}(T_0, T_3) - (c_0 + c_3)
\]

ergo

\[
c_0 + c_3 + \text{def}(T_0, T_3) = n^2 - (a + b + c)n + ab + bc + ac
\]

that is

\[
c_0 = n^2 - (a + b + c)n + ab + bc + ac - c_3 - \text{def}(T_0, T_3) \tag{5.7}
\]

From the two right sides of the equations (5.6) and (5.7) follows

\[
c_{2ab} + c(a + b - n) - dw(T_0, T_{2ab}) = n^2 - (a + b + c)n + ab + bc + ac - c_3 - \text{def}(T_0, T_3)
\]

and

\[
n^2 - (a + b + c)n + ab + bc + ac = (n - a)(n - b) - c(n - a - b)
\]

\[
= A(W) + c(a + b - n)
\]

\[
= E(W) + c_3 + c_{2ab} + c(a + b - n)
\]

So, we get the next equation:

\[
dw(T_0, T_{2ab}) + E(W) = \text{def}(T_0, T_3) \tag{5.8}
\]
The yellow brick in the right-hand square of Figure 5.4 consists of 3 layers with the following deficit of weight:

\[
dw(1, T_0, T_{2ab}) = -2 \\
dw(2, T_0, T_{2ab}) = -3 \\
dw(3, T_0, T_{2ab}) = -2 \\
dw(T_0, T_{2ab}) = -7 \\
E(T_{2ab} \cup T_3) = E(W) = 6 \\
def(T_0, T_3) = -1
\]

We need to put 7 rooks into the brick \( T_{2ab} \) to reach the balance state. But there are only 6 empty cells in axis \( W \), because we have already put the symbol 7 into the green part of a stuffed RBC.

**Remark 5.9.** Each layer of the pair of bricks \( (T_0, T_{2ab}) \) can be balanced separately, you can put 2 or 3 rooks into the proper layer of brick \( T_{2ab} \), but it is impossible to balance them together. This is just another explanation for the incompleteness of a PLSC that has an overloaded RBC.

### 6 Capacity Condition Check and Dual Structures

Using the equations (5.4) and (5.8) we get the next equation

\[
E(T_0 \cup T_{1b}) - E(T_{1b} \cup T_{2ab}) = def(T_0, T_3) - E(W)
\]

That means

\[
def(T_0, T_3) = E(W) + E(T_0 \cup T_{1b}) - E(T_{1b} \cup T_{2ab}),
\]
and so
\[ \text{def}(T_0, T_3) = E(T_{2ab} \cup T_3) + E(T_0 \cup T_1b) - E(T_{1b} \cup T_{2ab}). \]

The same way
\[ \text{def}(T_0, T_3) = E(T_{2bc} \cup T_3) + E(T_0 \cup T_1b) - E(T_{1b} \cup T_{2bc}) \]

If we consider the hinge with axis $T_{1b} \cup T_{2bc}$ and two leaves $T_0$ and $T_3$, we get

**Theorem 6.1 (Hinge Deficit),**

\[ \text{def}(T_0 \cup T_3) = E(T_{2bc} \cup T_3) + E(T_0 \cup T_1b) - E(T_{1b} \cup T_{2bc}) \quad (6.1) \]

**Proof.**

\[ E(T_{2bc} \cup T_3) + E(T_0 \cup T_1b) - E(T_{1b} \cup T_{2bc}) \]
\[ = (n-b)(n-c) - c_{2bc} - c_3 + ac - c_0 - c_{1b} - a(n-b) + c_{1b} + c_{2bc} \]

and the right-hand side is equal to $n^2 - (a+b+c)n + (ab + bc + ca) - c_0 - c_3$. \qed

Due to this result, we got an algorithm to check whether a PLSC satisfies the capacity condition or not. Let $P$ be a PLSC of order $n$ with $|P|$ non-attacking rooks, where $|P| < n^2$.

Let $Z$ be a $z$-directional axis of $P$. Let $E(Z)$ be the number of $z$-directional empty files in $Z$, then $E(Z) = A(Z) - c_{1b} - c_{2bc}$. Let $E_x = 0$, $E_y = 0$. Since

\[ \text{dw}(T_0, T_2) = (a + b - n) - [N(\sigma, T_0) - N(\sigma, T_2)] = [a - [N(\sigma, T_0)] - [n - b - N(\sigma, T_2)] \]

we can count the number of empty files in the proper direction depending on the balance state of $(T_0, T_2)$, as depicted in the Figure 6.1.
Algorithm 6.2 (Capacity Condition Check). If \( dw(T_0, T_2) < 0 \), that is \( T_0 \) is overweighted then increment \( E_y \) by \( a - N(\sigma, T_0) \) otherwise increment \( E_x \) by \( (n - b) - N(\sigma, T_2) \) for each \( \sigma \in \{1, 2, \ldots, n\} \).

When all layers are done, then

\[
E_x = E(T_{2bc} \cup T_3) + c_{2bc} \quad E_y = E(T_0 \cup T_{1b}) + c_{1b}
\]

where the 3-dimensional \( T_0 \) consists of all layers for that \( E_y \) was incremented.

It is clear from the construction that \( E_x + E_y - A(Z) \) is the minimum deficit of all remote couples \((T_0, T_3)\) that form a hinge with axis \( Z \). If this value is not negative for all \( z \)-directional axes of \( P \) then the PLSC \( P \) satisfies the capacity condition. For any remote couple \((T_0, T_3)\) there is a \( z \)-directional axis \( Z \) so that \( Z, T_0 \) and \( T_3 \) form a hinge, consequently it is enough to check the number of empty files for \( z \)-directional axes.

Taking the advantage of the dual property of the hinge structure, we have three analogous results for volume, the number of rooks in an LSC [5] and the number of missing rooks in a PLSC. Following the method of this type of observations we can identify the next asymmetric correlations of bricks:

Corollary 6.3 (5-axes Volume).

\[
V(T_0 \cup T_3) = V(T_0 \cup T_{1c}) + V(T_{1b} \cup T_{2bc}) + V(T_{2ab} \cup T_3) - V(T_{1b} \cup T_{2ab}) - V(T_{1c} \cup T_{2bc})
\]

and dividing both sides by \( n \) we get

\[
\text{cor}(T_0, T_3) = \frac{V(T_0) + V(T_{1c})}{n} + \frac{V(T_{1b}) + V(T_{2bc})}{n} + \frac{V(T_{2ab}) + V(T_3)}{n} - \frac{V(T_{1b}) + V(T_{2ab})}{n} - \frac{V(T_{1c}) + V(T_{2bc})}{n}
\]

Corollary 6.4 (5-axes Capacity).

\[
\text{cap}(T_0, T_3) = \frac{E_z(T_0 \cup T_{1c}) + E_z(T_{1b} \cup T_{2bc}) + E_z(T_{2ab} \cup T_3)}{n} - \frac{E_z(T_{1b} \cup T_{2ab}) + E_z(T_{1c} \cup T_{2bc})}{n}
\]

Theorem 6.5 (5-axes Deficit).

\[
\text{def}(T_0, T_3) = E_z(T_0 \cup T_{1c}) + E_z(T_{1b} \cup T_{2bc}) + E_z(T_{2ab} \cup T_3)
\]

- \( E_z(T_{1b} \cup T_{2ab}) + E_z(T_{1c} \cup T_{2bc}) \)

\[
= ab - c_0 - c_{1c} + a(n - b) - c_{1b} - c_{2bc} + (n - a)(n - b) - c_{2ab} - c_3
\]

- \(- (n - b)c + c_{1b} + c_{2ab} - (n - a)c + c_{1c} - c_{2bc} \)

\[
= ab + a(n - b) + (n - a)(n - b) - (n - b)c - (n - a)c - c_0 - c_3
\]

and the right-hand side is equal to \( n^2 - (a + b + c)n + (ab + bc + ca) - c_0 - c_3. \)

\[\square\]
If $P$ is a PLSC in $H$, then the volume of $H$ is $V(H) = n^3$ and thus $\frac{V(H)}{n} = n^2$. The number of missing rooks in $P$ is $n^2 - |P|$, can be denoted by $\text{def}(H)$, so we can also identify the next symmetric correlations of bricks:

**Corollary 6.6** (3-axes Volume).

\[ V(T_0 \cup T_3) = V(H) - V(T_{1a} \cup T_{2ab}) - V(T_{1b} \cup T_{2bc}) - V(T_{1c} \cup T_{2ac}) \]

**Corollary 6.7** (3-axes Capacity).

\[ \text{cap}(T_0, T_3) = \frac{V(H)}{n} - \frac{V(T_{1a}) + V(T_{2ab})}{n} - \frac{V(T_{1b}) + V(T_{2bc})}{n} - \frac{V(T_{1c}) + V(T_{2ac})}{n} \]

**Theorem 6.8** (3-axes Deficit).

\[ \text{def}(T_0, T_3) = \text{def}(H) - E(T_{1a} \cup T_{2ab}) - E(T_{1b} \cup T_{2bc}) - E(T_{1c} \cup T_{2ac}) \]  

(6.3)

**Proof.**

\[
\begin{align*}
\text{def}(H) & = n^2 - (c_0 + c_{1a} + c_{1b} + c_{2ab} + c_{2bc} + c_{2ac} + c_3) \\
E(T_{1a} \cup T_{2ab}) & = (n - a)c + c_{1a} + c_{2ab} \\
E(T_{1b} \cup T_{2bc}) & = (n - b)a + c_{1b} + c_{2bc} \\
E(T_{1c} \cup T_{2ac}) & = (n - b)c + c_{1c} + c_{2ac}
\end{align*}
\]

Consequently,

\[ \text{def}(T_0, T_3) = n^2 - (n - a)c - (n - b)a - (n - b)c - c_0 - c_3 \]

and the right-hand side is equal to $n^2 - (a + b + c)n + (ab + bc + ca) - c_0 - c_3$. \qed

**Remark 6.9.** If $H$ has no rooks, then Theorem 6.8 is equivalent to Corollary 6.7. If you start from the empty $H$ and replace dots by rooks in $H$ one by one, then $\text{def}(H)$ decreases by one in each step, and exactly one of the four other disjoint part of the equation decreases by one as well. That is, the equation holds until you reach the position where $H$ has no dots. This is clearly not necessarily a completed LSC.

**Remark 6.10.** The equations in (6.1), (6.2) and (6.3) hold even if $\text{def}(T_0, T_3)$ is negative.

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