Embedding approximately low-dimensional $\ell_2$ metrics into $\ell_1$

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Abstract

Goemans showed that any $n$ points $x_1, \ldots, x_n$ in $d$-dimensions satisfying $\ell^2_2$ triangle inequalities can be embedded into $\ell_1$, with worst-case distortion at most $\sqrt{d}$. We extend this to the case when the points are approximately low-dimensional, albeit with average distortion guarantees. More precisely, we give an $\ell^2_2$-to-$\ell_1$ embedding with average distortion at most the stable rank, $sr(M)$, of the matrix $M$ consisting of columns $\{x_i - x_j\}_{i < j}$. Average distortion embedding suffices for applications such as the SPARSEST CUT problem. Our embedding gives an approximation algorithm for the SPARSEST CUT problem on low threshold-rank graphs, where earlier work was inspired by Lasserre SDP hierarchy, and improves on a previous result of the first and third author [Deshpande and Venkat, In Proc. 17th APPROX, 2014]. Our ideas give a new perspective on $\ell_2^2$ metric, an alternate proof of Goemans’ theorem, and a simpler proof for average distortion $\sqrt{d}$. Furthermore, while the seminal result of Arora, Rao and Vazirani giving a $O(\sqrt{\log n})$ guarantee for UNIFORM SPARSEST CUT can be seen to imply Goemans’ theorem with average distortion, our work opens up the possibility of proving such a result directly via a Goemans’-like theorem.

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1 Introduction

A finite metric space consists of a pair \((X, d)\), where \(X\) is a finite set of points, and \(d : X \times X \to \mathbb{R}_{\geq 0}\) is a distance function on pairs of points in \(X\). Finite metric spaces arise naturally in combinatorial optimization (e.g., the \(\ell_1\) space in cut problems), and in practice (e.g., edit-distance between strings over some alphabet \(\Sigma\)). Since the input space may not be amenable to efficient optimization, or may not admit efficient algorithms, one looks for embeddings from these input spaces to easier spaces, while minimizing the distortion incurred. Given its importance, various aspects of such embeddings have been investigated such as dimension, distortion, efficient algorithms, and hardness results (refer to surveys [11, 17, 15] and references therein). In this paper, we provide better distortion guarantees for embedding approximately low-dimensional points in the \(\ell_2^2\)-metric into \(\ell_1\), and give applications to the SPARSEST CUT problem.

In the SPARSEST CUT problem, we are given graphs \(C, D\) on the same vertex set \(V\), with \(|V| = n\), called the cost and demand graphs, respectively. They are specified by non-negative edge weights \(c_{ij}, d_{ij} \geq 0\), for \(i < j \in [n]\) and the (non-uniform) sparsest cut problem, henceforth referred to as SPARSEST CUT, asks for a subset \(S \subseteq V\) that minimizes

\[\Phi(S) := \frac{\sum_{i,j} c_{ij} |I_S(i) - I_S(j)|}{\sum_{i,j} d_{ij} |I_S(i) - I_S(j)|},\]

where \(I_S(i)\) is the indicator function giving 1, if \(i \in S\), and 0, otherwise. We denote the optimum by \(\Phi^* := \min_{S \subseteq V} \Phi(S)\). When the demand graph is a complete graph on \(n\) vertices with uniform edge weights, the problem is then commonly referred to as the UNIFORM SPARSEST CUT problem.

The best known (unconditional) approximation guarantee for the UNIFORM SPARSEST CUT problem is \(O(\sqrt{\log n})\), due to Arora, Rao and Vazirani [4] (henceforth referred to as the ARV algorithm). Building on techniques in this work, Arora, Lee and Naor [3] give a \(O(\sqrt{\log n \log \log n})\) algorithm for non-uniform SPARSEST CUT. These results come from a semi-definite programming (SDP) relaxation to produce solutions in the \(\ell_2\)-squared metric space, i.e., a set of vectors \(\{x_i\}_{i \in V}\) in some high dimensional space that satisfy triangle inequality constraints on the squared distances in the following sense.

\[\|x_i - x_j\|_2^2 + \|x_j - x_k\|_2^2 \geq \|x_i - x_k\|_2^2 \quad \forall \ i, j, k \in [n].\]

Since the \(\ell_1\) metric lies in the non-negative cone of cut (semi-)metrics, [4] and [3] round their solutions via low-distortion embeddings of the above \(\ell_2^2\) solution into \(\ell_1\) metric. Embeddings with low average-distortion suffice for applications to the SPARSEST CUT problem.

Any \(n\) points satisfying \(\ell_2^2\) triangle inequalities make only acute angles among themselves, and therefore must lie in \(\Omega(\log n)\) dimensions (Chapter 15, [2]). However, for low threshold-rank graphs, or more generally, when the \(r\)-th smallest generalized eigenvalue of the cost and demand graphs satisfies \(\lambda_r(C, D) \gg \Phi_{SDP}\), the above SDP solution is known to be approximately low-dimensional, that is, the span of its top \(r\) eigenvecors contains nearly all of its total eigenmass (implicit in [10]). Moreover, it can be embedded into \(\ell_1\) using solutions of higher-levels of the Lasserre SDP hierarchy to obtain a PTAS-like approximation guarantee [10]. This motivates the quest for finding more efficient embeddings of low-dimensional or approximately low-dimensional \(\ell_2^2\) metrics into \(\ell_1\).

Goemans (unpublished, appears in [16]) showed that if the points satisfying \(\ell_2^2\) triangle inequalities lie in \(d\) dimensions, then they can be embedded into \(\ell_2\) (and hence into \(\ell_1\), since there is an isometry from \(\ell_2 \leftrightarrow \ell_1\) [17]) with \(\sqrt{d}\) distortion.
Theorem 1.1 (Goemans [16, Appendix B]). Let \( x_1, x_2, \ldots, x_n \in \mathbb{R}^d \) be \( n \) points satisfying \( \ell_2^2 \) triangle inequalities. Then there exists an \( \ell_2^2 \leftrightarrow \ell_2 \) embedding \( x_i \mapsto f(x_i) \) with distortion \( \sqrt{d} \), that is,

\[
\frac{1}{\sqrt{d}} \|x_i - x_j\|_2^2 \leq \|f(x_i) - f(x_j)\|_2 \leq \|x_i - x_j\|_2^2, \quad \forall i, j \in V.
\]

Comparison of Goemans and ARV: Since \( n \) points satisfying \( \ell_2^2 \) triangle inequalities must lie in \( d = \Omega(\log n) \) dimensions (Chapter 15, [2]), the ARV algorithm [4] implies an \( \ell_2^2 \leftrightarrow \ell_1 \) embedding with average distortion \( O(\sqrt{d}) \), and Arora-Lee-Naor [3] improve it to \( O(\sqrt{d}) \) worst-case distortion. In the other direction, is it possible to extend Theorem 1.1 to give ARV-like guarantees? Here are two immediate ideas that come to mind.

- Combine Theorem 1.1 with a dimension reduction to \( O(\log n) \) dimensions for \( \ell_2^2 \) metrics, similar to the Johnson-Lindenstrauss lemma for \( \ell_2 \). Such a dimension reduction for \( \ell_2^2 \) that approximately preserves all pairwise \( \ell_2^2 \) distances is ruled out by Magen and Moharrami [16], although their proof does not rule out dimension reduction for average distortion.

- Extend Theorem 1.1 to work with approximate \( \ell_2^2 \) triangle inequalities, and then combine it with the Johnson-Lindenstrauss lemma. The Johnson-Lindenstrauss lemma, when applied to points satisfying \( \ell_2^2 \) triangle inequalities, preserves their \( \ell_2^2 \) triangle inequalities only approximately. That is, the points after the Johnson-Lindenstrauss random projection satisfy

\[
\|x_i - x_j\|_2^2 + \|x_j - x_k\|_2^2 \geq (1 - O(\epsilon)) \|x_i - x_k\|_2^2, \quad \forall i, j, k \in [n].
\]

In fact, a generalization of Theorem 1.1 that accommodates approximate \( \ell_2^2 \) triangle inequalities does hold, but its only proof (due to Trevisan [18]) that we are aware of uses the technical core of the analysis of the ARV algorithm.

Here we seek a robust generalization of Goemans’ theorem that avoids the above caveats. Our generalization of Goemans’ theorem uses average distortion instead of worst-case. It is also robust in the sense that it works with approximate dimension instead of the actual dimension. Such a robust generalization opens up another possible approach to the general SPARSEST CUT problem: reduce the approximate dimension while preserving the pairwise distances on average, and then apply the robust version of Goemans’ theorem. Moreover, our definition of the approximate dimension is spectral, and our results can be easily compared to those of Guruswami-Sinop [10] on Lasserre SDP hierarchies and Kwok et al. [14] on higher order Cheeger inequalities (see Sections 1.1 and 1.2 for comparisons).

1.1 Our Results

We prove a robust version of Goemans’ theorem, when the points \( x_1, x_2, \ldots, x_n \) are only approximately low-dimensional. We quantify this approximate dimension by the stable rank of the difference matrix \( M \in \mathbb{R}^{d \times \binom{n}{2}} \) having columns \( \{x_i - x_j\}_{i<j} \). Stable rank of the difference matrix is a natural choice because (a) stable rank is a continuous proxy for rank or dimension arising naturally in many applications [6, 19], (b) the difference matrix \( M \) is invariant under any shift of origin, and (c) the difference matrix of the SDP solution for the SPARSEST CUT problem on low threshold-rank graphs indeed has low stable rank (implicit in [10]).
Definition 1.2 (Stable Rank). Given \(x_1, \ldots, x_n \in \mathbb{R}^d\), let \(M \in \mathbb{R}^{d \times n}\) be the matrix with columns \(\{x_i - x_j\}_{i < j}\). The stable rank of the points is defined as the stable rank of \(M\), given by \(sr(M) := \|M\|_F^2 / \|M\|_2^2\), where \(\|M\|_F\) and \(\|M\|_2\) are the Frobenius and spectral norm of \(M\) respectively.

Note that \(sr(M) \leq \text{rank}(M) \leq d\), when the points \(x_1, x_2, \ldots, x_n \in \mathbb{R}^d\). Our robust generalization of Goemans’ theorem is as follows.

Theorem 1.3 (Embedding almost low-dimensional vectors). Let \(x_1, x_2, \ldots, x_n \in \mathbb{R}^d\) be \(n\) points satisfying \(\ell^2_2\) triangle inequalities. Then there exists an \(\ell^2_2 \rightarrow \ell_2\) embedding \(x_i \mapsto h(x_i)\) with average distortion bounded by the stable rank of \(M\), that is,

\[
\|h(x_i) - h(x_j)\|_2 \leq \|x_i - x_j\|_2^2, \quad \forall i, j \in V,
\]

and

\[
\frac{1}{sr(M)} \sum_{i < j} \|x_i - x_j\|_2^2 \leq \sum_{i < j} \|h(x_i) - h(x_j)\|_2^2.
\]

Our proof technique gives a new perspective on \(\ell^2_2\) metric, an alternate proof of Goemans’ theorem, and a simpler proof for average distortion \(\sqrt{d}\) based on a squared-length distribution (see Section 4). Also, the result can be quantitatively compared to guarantees given by higher-order Cheeger inequalities [14]; we discuss this in more detail at the end of this section. While most known embeddings from \(\ell^2_2\) to \(\ell_1\) are Frechet embeddings, our embedding is projective (similar in spirit to [10, 8]). To obtain a truly robust version of Goemans’ theorem quantitatively, one might ask if the dependence on \(sr(M)\) in the above theorem can be improved from \(sr(M)\) to \(\sqrt{sr(M)}\).

Theorem 1.3 immediately implies an \(sr(M)\)-approximation to the uniform sparsest cut problem. In fact, with a slight modification, we obtain a similar result for the general sparsest cut problem (see theorem below).

Theorem 1.4. There is an \(r/\delta\) approximation algorithm for sparsest cut instances \(C, D\) satisfying \(\lambda_r(C, D) \geq \Phi_{SDP} / (1 - \delta)\), where \(\lambda_r(C, D)\) is the \(r\)-th smallest generalized eigenvalue of the cost and demand graphs.

The precondition on \(\lambda_r(C, D)\) is the same as in previous works [10, 8], and we improve the \(O(r/\delta^2)\)-approximation of [8] by a factor of \(1/\delta\). Our proof follows from the robust version of Goemans’ embedding into \(\ell_2\) whereas these previous works gave embeddings directly into \(\ell_1\) by either using higher levels of Lasserre explicitly [10] or using only the basic SDP solution but inspired by the properties of Lasserre vectors [8].

1.2 Related work

We recall that the best known upper bound for the worst-case distortion of embedding \(\ell^2_2 \rightarrow \ell_1\) is \(O(\sqrt{\log n \cdot \log \log n})\) [4, 3], while the best known lower bound is \((\log n)^\Omega(1)\) for worst-case distortion [7], and \(\exp(\Omega(\sqrt{\log \log n}))\) for average distortion [12]. Guarantees to sparsest cut on low threshold-rank graphs were obtained using higher levels of the Lasserre hierarchy for SDPs [5, 10]. In contrast, a previous work of the first and third author [8] showed weaker guarantees, but using just the basic SDP relaxation. Oveis Gharan and Trevisan [9] also give a rounding algorithm for the basic SDP relaxation on low-threshold rank graphs, but require a stricter pre-condition on the eigenvalues \((\lambda_r \gg \log^{2.5} r \cdot \Phi(G))\), and leverage it to give a stronger \(O(\sqrt{\log r})\)-approximation
guarantees. Their improvement comes from a new structure theorem on the SDP solutions of low threshold-rank graphs being clustered, and using the techniques in ARV for analysis.

Kwok et al. [14] showed that a better analysis of Cheeger’s inequality gives a $O(r \cdot \sqrt{d/\lambda_r})$ approximation to the sparsest cut in $d$-regular graphs. In particular, when $\lambda_r \geq ed$, this gives a $O(r/\sqrt{c})$ approximation for the UNIFORM SPARSEST CUT problem. In this regime, our result gives a slightly better approximation: Assuming $\lambda_r \geq ed$, if $\phi_{SDP} \leq ed/100n$ then $\lambda_r \geq 100\phi_{SDP}$ yielding an $O(r)$ approximation by Theorem 1.4. Otherwise, if $\phi_{SDP} \geq ed/100n$, then running a Cheeger rounding on the SDP solution would itself give a cut of sparsity $O(d\sqrt{c}/n) \leq \phi_{SDP}/\sqrt{c}$. Thus, the better of our rounding algorithm and a Cheeger rounding on the SDP solution gives a max \{O(r), 1/\sqrt{c}\}-approximation to the UNIFORM SPARSEST CUT problem.

Further, while the Kwok et al. result is tight with respect to the spectral solution, our approach allows for an improvement in terms of the dependence on $r$ to $\sqrt{r}$, since it uses the SDP relaxation rather than a spectral solution.

2 Preliminaries and Notation

Sets, Matrices, Vectors: We use $[n] = \{1, \ldots, n\}$. For a matrix $X \in \mathbb{R}^{d \times d}$, we say $X \succeq 0$ or $X$ is positive-semidefinite (psd) if $y^T X y \geq 0$ for all $y \in \mathbb{R}^d$. The Gram-matrix of a matrix $M \in \mathbb{R}^{d_1 \times d_2}$ is the matrix $M^T M$, which is psd.

Every matrix $M$ has a singular value decomposition $M = \sum_i \sigma_i u_i v_i^T = UDV^T$. Here, the matrices $U$, $V$ are Unitary, and $D$ is the diagonal matrix of the singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, in non-increasing order. When not clear from context, we denote the singular values of $M$ by $\sigma_i(M)$.

The Frobenius norm of $M$ is given by $\|M\|_F := \sqrt{\sum_i \sigma_i^2(M)} = \sqrt{\sum_{i \in [d_1], j \in [d_2]} M(i, j)^2}$. In our analysis, we will sometimes view a matrix $M$ as a collection of its columns viewed as vectors; $M = (m_j)_{j \in [d_2]}$. In this case, $\|M\|_F^2 = \sum_j \|m_j\|_2^2$. The spectral norm of $M$ is $\|M\|_2 := \sigma_1$.

Rank and Stable Rank: The rank of the matrix $M$ (denoted by rank($M$)) is the number of non-zero singular values. Recall that the stable rank of the matrix $M$, $sr(M) = \frac{\|M\|_F^2}{\sigma_1(M)}$. Note that rank($M$) $\geq$ sr($M$).

Metric spaces and embeddings: For our purposes, a (semi-)metric space $(X, d)$ consists of a finite set of points $X = \{x_1, x_2, \ldots, x_n\}$ and a distance function $d : X \times X \mapsto \mathbb{R}_{\geq 0}$ satisfying the following three conditions:

1. $d(x, x) = 0, \forall x \in X$.
2. $d(x, y) = d(y, x)$.
3. (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$.

An embedding from a metric space $(X, d)$ to a metric space $(Y, d')$ is a mapping $f : X \rightarrow Y$. The embedding is called a contraction, if

$$d'(f(x_i), f(x_j)) \leq d(x_i, x_j), \quad \forall x_i, x_j \in X.$$
For convenience, we will only deal with contractive mappings in this paper. A contractive mapping is said to have (worst-case) distortion $\Delta$, if
\[
\sup_{i,j} \frac{d(x_i, x_j)}{d'(f(x_i), f(x_j))} \leq \Delta. 
\]
It is said to have average distortion $\beta$, if
\[
\frac{\sum_{i<j} d(x_i, x_j)}{\sum_{i<j} d(f(x_i), f(x_j))} \leq \beta. 
\]
Note that a mapping with worst-case distortion $\Delta$ also has average distortion $\Delta$, but not necessarily vice-versa.

**The $\ell^2_2$ space:** A set of points $\{x_1, x_2, \ldots, x_n\} \in \mathbb{R}^d$ are said to satisfy $\ell^2_2$ triangle inequality constraints, or said to be in $\ell^2_2$ space, if it holds that
\[
\|x_i - x_j\|^2_2 + \|x_j - x_k\|^2_2 \geq \|x_i - x_k\|^2_2 \quad \forall i, j, k \in [n].
\]
These satisfy the triangle inequalities on the squares of their $\ell_2$ distances. The corresponding metric space is $(\mathcal{X}, d)$, where $d(i, j) := \|x_i - x_j\|^2_2$.

**Graphs and Laplacians:** All graphs will be defined on a vertex set $V$ of size $n$. The vertices will usually be referred to by indices $i, j, k, l \in [n]$. Given a graph with weights on pairs $W : \binom{V}{2} \mapsto \mathbb{R}^+$, the graph Laplacian matrix is defined as:
\[
L_W(i, j) := \begin{cases} -W(i, j) & \text{if } i \neq j \\ \sum_k W(i, k) & \text{if } i = j. \end{cases}
\]

**Sparseset Cut SDP:** The SDP we use for Sparsest Cut on the vertex set $V$ with costs and demands $c_{ij}, d_{kl} \geq 0$ and corresponding cost and demand graphs $C : \binom{V}{2} \mapsto \mathbb{R}^+$ and $D : \binom{V}{2} \mapsto \mathbb{R}^+$, is effectively the following:
\[
\text{SDP: } \Phi_{\text{SDP}} := \min \sum_{i<j} c_{ij} \|x_i - x_j\|^2_2 \\
\text{subject to } \begin{cases} \|x_i - x_j\|^2_2 + \|x_j - x_k\|^2_2 \geq \|x_i - x_k\|^2_2 \quad \forall i, j, k \in [n]. \\
\sum_{k<l} d_{kl} \|x_k - x_l\|^2_2 = 1. \end{cases}
\]
Note that the solution to the above SDP is in $\ell^2_2$ space.

**$\ell_1$ embeddings and cuts:** Since $\ell_1$ metrics are exactly the cone of cut-metrics, it follows from the previous discussion on embeddings, that producing an embedding of the SDP solutions $\mathcal{X} = \{x_1, \ldots, x_n\}$ in $\ell^2_2$ space to $\ell_1$ space with distortion $\alpha$ would give an $\alpha$-approximation to Sparsest Cut. Producing one with average distortion $\alpha$ would give an $\alpha$-approximation to Uniform Sparsest Cut. Furthermore, since $\ell_2$ embeds isometrically (distortion 1) into $\ell_1$, it suffices to show embeddings into $\ell_2$ for the above purposes.
Key Lemma: The following lemma about $\ell_2^2$ spaces was observed by Deshpande and Venkat [8]. We will reuse this in the rest of the paper.

Lemma 2.1 ([8, Proposition 1.3]). Let $x_1, x_2, \ldots, x_n$ be $n$ points satisfying $\ell_2^2$ triangle inequalities. Then

$$\left\langle x_i - x_j, \frac{x_k - x_l}{\|x_k - x_l\|_2} \right\rangle^2 \leq \left| \left\langle x_i - x_j, x_k - x_l \right\rangle \right| \leq \|x_i - x_j\|_2^2, \quad \forall i, j, k, l \in V.$$ 

An immediate consequence of this lemma is that we can show that a large class of naturally defined $\ell_2^2 \mapsto \ell_2^2$ embeddings are contractions.

Lemma 2.2 (Contraction). Let $x_1, x_2, \ldots, x_n$ be $n$ points satisfying $\ell_2^2$ triangle inequalities. For any probability distribution $\{p_{kl}\}_{k<l}$, let $P$ be the symmetric psd matrix defined as $P := \sum_{k<l} p_{kl} (x_k - x_l)(x_k - x_l)^T$. Then the $\ell_2^2 \mapsto \ell_2^2$ embedding given by $x_i \mapsto P^{1/2}x_i$ is a contraction, that is,

$$\|P^{1/2}(x_i - x_j)\|_2 \leq \|x_i - x_j\|_2, \quad \forall i, j \in V.$$ 

Proof. The following holds for all $i, j$:

$$\|P^{1/2}(x_i - x_j)\|_2 = \left( (x_i - x_j)^TP(x_i - x_j) \right)^{1/2} \leq \left( \sum_{k<l} p_{kl} \|x_i - x_j\|_2^4 \right)^{1/2} = \|x_i - x_j\|_2^2.$$ 

[By Lemma 2.1]

[Since $\sum_{k<l} p_{kl} = 1$]

\[\square\]

3 Embedding almost low-dimensional vectors

We now prove the robust version of Goemans’ theorem in terms of stable rank. We give two proofs, and show an application to round solutions to SPARSEST CUT on low-threshold-rank graphs. As before, given a set of points $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^d$, define their difference matrix $M \in \mathbb{R}^{d \times \binom{n}{2}}$ as the matrix with columns as $\{x_i - x_j\}_{i<j}$.

Proof of Theorem 1.3. Let $u$ and $v$ be the top left and right singular vector of $M$, respectively, and $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_d$ be the singular values of $M$. Then $Mv = \sigma_1 u$, or in other words, $\sigma_1 u = \sum_{k<l} v_{kl}(x_k - x_l)$. Now consider the embedding $x_i \mapsto h(x_i) = P^{1/2}x_i$, where the probability distribution $p_{kl} \propto |v_{kl}|$, that is

$$P = \sum_{k<l} \frac{|v_{kl}|}{\|v\|_1} (x_k - x_l)(x_k - x_l)^T.$$ 

This embedding is a contraction by Lemma 2.2. Now let’s bound its average distortion.

$$\sum_{i<j} \|h(x_i) - h(x_j)\|_2 = \sum_{i<j} \|P^{1/2}(x_i - x_j)\|_2$$
\[
= \sum_{i<j} \left( (x_i - x_j)^T P(x_i - x_j) \right)^{1/2}
= \sum_{i<j} \left( \sum_{k<l} \frac{|v_{kl}|}{||v||_1} \langle x_i - x_j, x_k - x_l \rangle \right)^{1/2}
\geq \sum_{i<j} \sum_{k<l} \frac{1}{||v||_1} |\langle x_i - x_j, x_k - x_l \rangle| \quad \text{[By Jensen’s inequality]}
\geq \sum_{i<j} \sum_{k<l} \frac{1}{||v||_1} \left| \langle x_i - x_j, \sum_{k<l} v_{kl}(x_k - x_l) \rangle \right| \quad \text{[By triangle inequality]}
\]
\[
= \frac{1}{||v||_1} \sum_{i<j} \sigma_1^2 |v_{ij}|
= \sigma_1^2 \frac{\|M\|_F^2}{\text{sr}(M)}
= \frac{1}{\text{sr}(M)} \sum_{i<j} \|x_i - x_j\|_2^2.
\]

\[ \square \]

### 3.1 An alternative proof

We can alternatively get the same guarantee as in Theorem 1.3, by giving a one-dimensional \(\ell_2\) embedding (and hence also \(\ell_1\) embedding without any extra effort) along the top singular vector of the difference matrix \(M\). This gives an interesting “spectral” algorithm that uses spectral information about the point set, akin to spectral algorithms in graphs that use the spectrum of the graph Laplacian.

**Theorem 3.1.** Let \(x_1, x_2, \ldots, x_n \in \mathbb{R}^d\) be \(n\) points satisfying \(\ell_2\) triangle inequalities with \(M\) as their difference matrix. Let \(u \in \mathbb{R}^d\) and \(v \in \mathbb{R}^{n^2}\) be its top left and right singular vectors, respectively. Then \(x_i \mapsto \frac{\sigma_1}{||v||_1} \langle x_i, u \rangle\) is an \(\ell_2 \hookrightarrow \ell_2\) embedding with average distortion bounded by the stable rank of \(M\).

**Proof.** We have \(Mv = \sigma_1 u\), or equivalently, \(\sigma_1 u = \sum_{k<l} v_{kl}(x_k - x_l)\). Our embedding is a contraction since

\[
\frac{\sigma_1}{||v||_1} \left| \langle x_i - x_j, u \rangle \right| = \frac{1}{||v||_1} \left| \langle x_i - x_j, \sum_{k<l} v_{kl}(x_k - x_l) \rangle \right|
\leq \frac{1}{||v||_1} \sum_{k<l} |v_{kl}| \left| \langle x_i - x_j, x_k - x_l \rangle \right|
\leq \frac{1}{||v||_1} \sum_{k<l} |v_{kl}| \|x_i - x_j\|_2^2 \quad \text{[By Lemma 2.1]}
= \|x_i - x_j\|_2^2.
\]
Now let’s bound the average distortion.

\[
\sum_{i<j} \frac{\sigma_1}{\|v\|_1} |\langle x_i - x_j, u \rangle| = \sum_{i<j} \frac{\sigma_1}{\|v\|_1} |\sigma_1 v_{ij}|
\]

[Since \(u^T M = \sigma_1 v^T\)]

\[
= \sigma_1^2 = \frac{\|M\|_F^2}{\text{sr}(M)}
\]

\[
= \frac{1}{\text{sr}(M)} \sum_{i<j} \|x_i - x_j\|_2^2.
\]

\[\blacksquare\]

3.2 Application to Sparsest Cut on low-threshold rank graphs

We first state a property of SDP solutions on low threshold-rank graphs, proved by Guruswami and Sinop [10] using the Von-Neumann inequality.

**Proposition 3.2** (Von-Neumann inequality [10, Theorem 3.3]). Let \(0 \leq \lambda_1 \leq \ldots \leq \lambda_m\) be the generalized eigenvalues of the Laplacian matrices of the cost and demand graphs. Let \(\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0\) be the singular vectors of the matrix \(M\) with columns \(\sqrt{d_{ij}}(x_i - x_j)\) \(\text{for } i < j\). Then

\[
\frac{\sum_{t \geq r+1} \sigma_t^2}{\sum_{t=1}^n \sigma_t^2} \leq \frac{\Phi_{\text{SDP}}}{\lambda_{r+1}}.
\]

In particular, note that on graphs where \(\lambda_r \geq \Phi_{\text{SDP}} / (1 - \delta)\), \(\sum_{i \leq r} \sigma_i^2 \geq \delta \sum_{i \leq r} \sigma_i^2\). This implies that \(\text{sr}(M) = \sum_i \sigma_i^2 / \sigma_1^2 \leq r \cdot \sum_i \sigma_i^2 / \sum_{i \leq r} \sigma_i^2 \leq r / \delta\).

We can now modify the proof of Theorem 3.1 to prove Theorem 1.4.

**Proof of Theorem 1.4.** Let \(x_1, \ldots, x_n\) be the SDP solution on given instance \(C, D\). We now let \(M\) be the matrix with columns \(\sqrt{d_{kl}}(x_k - x_l)\) \(\text{for } k < l\), and \(u, v, \sigma_1\) to be the top left singular vector, top right singular vector, and the maximum singular value respectively of \(M\). By the preceding remark, \(\text{sr}(M) \leq r / \delta\). The mapping we use is as follows

\[
x_i \mapsto \frac{1}{\sum_{kl} \sqrt{d_{kl}} v_{kl}} \langle x_i, u \rangle
\]

The proofs to show contraction and bound the distortion follow exactly as in the proof of Theorem 3.1. Note that while looking at the distortion, we need to lower bound \(\sum_{ij} d_{ij} \|g(x_i) - g(x_j)\|_2^2\).

As in Deshpande and Venkat [8], the above algorithm is a fixed polynomial time algorithm and does not grow with the threshold rank unlike the algorithm of Guruswami and Sinop [10] where they use \(r\)-levels of the Lasserre SDP hierarchy to secure the guarantee. Furthermore, the above analysis improves the guarantee of Deshpande and Venkat [8] by a factor of \(O(1/\delta)\).
4 Embedding low-dimensional vectors à la Goemans

In this section, we first view the proof of Goemans’ theorem in the framework of Lemma 2.2 by giving a probability distribution using the minimum volume enclosing ellipsoid of the difference vectors $(x_i - x_j)$’s. We then give a simpler proof, albeit for the average distortion case, based on a probability distribution arising from squared-length distribution. Via a well-known duality statement, this technique recovers Goemans’ theorem for worst-case distortion for embeddings into $\ell_1$, although non-constructively.

4.1 An alternate proof of Goemans’ theorem

Here is an adaptation of the proof from [16] re-stated in our framework. The following proof is arguably simpler and more straightforward as it works with the difference vectors instead of the original vectors and their negations.

Theorem 1.1 (Restated) (Goemans [16, Appendix B]) Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ be $n$ points satisfying $\ell_2^2$ triangle inequalities. Then there exists an $\ell_2^2 \hookrightarrow \ell_2$ embedding $x_i \mapsto f(x_i)$ with distortion $\sqrt{d}$, that is,

$$\frac{1}{\sqrt{d}} \|x_i - x_j\|_2 \leq \|f(x_i) - f(x_j)\|_2 \leq \|x_i - x_j\|_2, \quad \forall i, j \in V.$$

Proof. Consider all the difference vectors $(x_i - x_j)$’s, and let their minimum volume enclosing ellipsoid be given by $E = \{x : x^T Q x \leq 1\}$, for some psd matrix $Q \in \mathbb{R}^{d \times d}$. By John’s theorem (or Lagrangian duality for the corresponding convex program), we have $Q^{-1} = \sum_{k<l} a_{kl} (x_k - x_l)(x_k - x_l)^T$, with all $a_{kl} \geq 0$. Moreover, $a_{kl} \not= 0$ iff $(x_k - x_l)^T Q (x_k - x_l) = 1$. Notice that $d = \text{Tr}(I_d) = \text{Tr}(Q^{1/2}Q^{-1}Q^{1/2}) = \sum_{k<l} a_{kl}$. We define the embedding as

$$f(x_i) := \frac{1}{\sqrt{d}} Q^{-1/2} x_i.$$

This embedding is a contraction by Lemma 2.2. We now bound the distortion:

$$\|f(x_i) - f(x_j)\|_2 = \frac{1}{\sqrt{d}} \left\|Q^{-1/2}(x_i - x_j)\right\|_2 \geq \frac{1}{\sqrt{d}} \left\|Q^{1/2}(x_i - x_j)\right\|_2 \quad \text{[By Cauchy-Schwarz inequality]}$$

$$\geq \frac{1}{\sqrt{d}} \|x_i - x_j\|_2. \quad \text{[Since $(x_i - x_j)^T Q (x_i - x_j) \leq 1$, for all $i, j$]}$$

4.2 A simpler proof for average distortion embedding

We now give an average distortion version of Goemans’ theorem using a simple squared-length distribution on the difference vectors $(x_i - x_j)$’s in the Lemma 2.2. Interestingly, this can be modified to weighted averages and gives yet another proof of Goemans’ worst-case distortion result, although non-constructively.
**Theorem 4.1.** Let \(x_1, x_2, \ldots, x_n \in \mathbb{R}^d\) be points satisfying \(\ell_2^2\) triangle inequalities. Then there exists an \(\ell_2^2\)-to-\(\ell_2\) embedding \(x_i \mapsto g(x_i)\) with average distortion \(\sqrt{d}\), that is,

\[
\|g(x_i) - g(x_j)\|_2 \leq \|x_i - x_j\|_2^2, \quad \text{for all } i,j, \\
and \quad \frac{1}{\sqrt{d}} \sum_{i<j} \|x_i - x_j\|_2 \leq \sum_{i<j} \|g(x_i) - g(x_j)\|_2
\]

**Proof.** Let \(\{p_{kl}\}_{k<l}\) define a probability distribution with \(p_{kl} \propto \|x_k - x_l\|_2^2\). Given this distribution, let \(P\) be the symmetric psd matrix defined as \(P := \sum_{k<l} p_{kl} (x_k - x_l)(x_k - x_l)^T \in \mathbb{R}^{d \times d}\). Consider the embedding that maps \(x_i\) to \(g(x_i) := p_1^{1/2} x_i\). The embedding is a contraction by the Lemma 2.2.

Now let’s bound the average distortion. First, note that:

\[
\|g(x_i) - g(x_j)\|_2 = \|p_1^{1/2}(x_i - x_j)\|_2 \geq \frac{\|x_i - x_j\|_2^2}{\|P^{-1/2}(x_i - x_j)\|_2},
\]

where the inequality follows from the Cauchy-Schwarz inequality.

Summing over all pairs \(i, j\) and using the definition of \(p_{ij}\) we have

\[
\sum_{i<j} \|g(x_i) - g(x_j)\|_2 \geq \left( \sum_{k<l} \|x_k - x_l\|_2^2 \right) \sum_{i<j} \frac{p_{ij}}{\sqrt{(x_i - x_j)^T P^{-1}(x_i - x_j)}} \geq \left( \sum_{k<l} \|x_k - x_l\|_2^2 \right) \left( \sum_{i<j} p_{ij} \ (x_i - x_j)^T P^{-1}(x_i - x_j) \right)^{-1/2}
\]

[by Jensen’s inequality]

\[
= \left( \sum_{k<l} \|x_k - x_l\|_2^2 \right) \left( \text{Tr} \left( P^{-1/2} P P^{-1/2} \right) \right)^{-1/2}
\]

\[
= \left( \sum_{k<l} \|x_k - x_l\|_2^2 \right) \text{Tr} \left( I_d \right)^{-1/2}
\]

\[
= \frac{1}{\sqrt{d}} \sum_{i<j} \|x_i - x_j\|_2^2.
\]

We note that if \(P\) is not invertible then the same proof can be carried out using pseudo-inverse of \(P\) instead. \(\square\)

**Remark.** We note that our embedding (for the average distortion case) is based on a simple squared-length distribution and does not involve computation of minimum volume enclosing ellipsoid [13] as in the earlier proof.

Theorem 4.1 immediately gives an efficient \(\sqrt{d}\) approximation algorithm for **Uniform Sparsest Cut** when the SDP optimum solution resides in \(\mathbb{R}^d\). Furthermore, as we point out next, the same proof can be tweaked to yield a similar result for the general **Sparsest Cut** problem.

**Theorem 4.2** (**Sparsest Cut** SDP rounding in dimension \(d\)). A **Sparsest Cut instance** \(C, D\) with SDP optimum solution in \(\mathbb{R}^d\) has an integrality gap of at most \(\sqrt{d}\).
Proof. Let $x_1, \ldots, x_n$ be the optimum solution in $\mathbb{R}^d$ to the SPARSEST CUT SDP. We slightly modify the embedding given in the proof of Theorem 4.1, by choosing the $p_{ij}$'s based on the demand graph $D$. Let $P = \sum_{k<l} p_{kl} (x_k - x_l)(x_k - x_l)^T \in \mathbb{R}^{d \times d}$, where $p_{kl}$'s define a probability distribution with $p_{kl} \propto d_{kl} \|x_k - x_l\|_2^2$. We define the embedding as $x_i \mapsto g(x_i) = P^{1/2} x_i$. Lemma 2.2 shows that it is a contraction. We now need to show $\sum_{i<j} d_{ij} \|g(x_i) - g(x_j)\|_2 \geq \frac{1}{\sqrt{d}} \sum_{i<j} d_{ij} \|x_i - x_j\|_2^2$. It is easy to check that the same proof goes through without any major changes. 

By a well-known duality (cf. [17, Proposition 15.5.2 and Exercise 4]), Theorem 4.2 also implies Goemans’ worst-case distortion result (Theorem 1.1), although non-constructively.

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