ESTIMATE OF NUMBER OF SIMPLICIES OF TRIANGULATIONS OF LIE GROUPS

HAIBAO DUAN, WALCIAW MARZANTOWICZ, AND XUEZHI ZHAO

Abstract. We present estimates of number of simplices of given dimension of classical compact Lie groups. As in the previous work [12] the approach is a combination of an estimate of number of vertices with a use of valuation of the covering type by cohomological argument of [11] and application of the recent versions of the Lower Bound Theorem of combinatorial topology. For the case of exceptional Lie groups we made a complete calculation using the description of their cohomology rings given by the first and third author. For infinite increasing series of Lie groups of growing dimension \( d \) the rate of growth of number of simplices of highest dimension is given which extends onto the case of simplices of (fixed) codimension \( d - i \).

Keywords: minimal triangulation, covering type, compact Lie group, cup-length, Lower Bound Theorem, Manifold g-Theorem

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1. Introduction

Every smooth manifold \( X \) admits an (essentially unique) compatible piecewise linear structure, i.e a triangulation. Obviously it is finite if \( X \) is compact. But the existence theorem says nothing about the number of simplices, e.g. vertices, we need for a triangulation of \( X \). The problem to find minimal triangulation, i.e. a triangulation which has minimal number of vertices was a subject of many studies of combinatorial topology (see [2] and [17] for references). Consequently, it is important to give any estimate of the number of vertices, or more general the number of simplices of given dimension \( 0 \leq i \leq d = \text{dim} \ X \).

Definition 1.1. Let \( K \) be the complex of triangulation of a \( d \)-dimensional closed manifold, and denote by \( f_i, \ i = 0, \ldots , d \) the number of \( i \)-dimensional simplices in \( K \).

In this note we present estimates of coordinates of the vector \((f_0, f_1, \ldots , f_d)\) in the case where \( X \) is a compact Lie group. We restrict our study to classical Lie groups for which the cohomology rings have complete description. The case of five exceptional Lie groups is equipped with complete calculation of the formula based on the complete description of their cohomology rings given by the first and second author in [8]. For the remaining infinite series of classical Lie groups we describe the asymptotic in \( d \) growth of \( f_i \) the number of vertices of dimension \( i \). According to our knowledge there are no results in literature which give estimate of number of simplices of triangulation of Lie groups in general. Let us remind, that every compact Lie group has finite-sheet covering which is a product of a torus and some simply and simply-connected Lie groups, see [29 App. 1.2]. Simply and simply-connected Lie groups have four infinite series \( A_n, B_n, C_n, D_n \), and finite family of exceptional Lie groups \( G_2, F_4, E_6, E_7, E_8 \). The series \( A, B, C, \) and \( D \) correspond to the groups \( SU(n + 1), SO(2n + 1), Sp(n) \), and \( SO(2n) \) respectively.

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Dedicated to Daciberg Lima Gonçalves with best wishes of \( 2^2 \cdot 5^2 \) years.
Our approach has two factors. In [15] M. Karoubi and Ch. Weibel defined a homotopy invariant of a space \( X \) called the covering type of \( X \) and denoted \( ct(X) \). By its definition it is a lower bound for \( f_0(X) \) (cf. [15], also [11]). In paper [11] a method of estimate from below of \( ct(X) \) was presented. The main result of it provides a formula in terms of multiplicative structure of the cohomology ring \( H^*(X; R) \) in any coefficient ring \( R \). More precisely, it estimates \( ct(X) \) by the maximal weighted length of a nonzero multiple in \( \tilde{H}^*(X; R) \) (Theorem 2.1).

The second component of our approach is based on the recent much more sharper versions of the Lower Bound Theorem, shortly called LBT (see [12] for an exposition). Purly combinatorial in arguments LBT (cf. [14]) states that the number \( i \)-dimensional simplices of \( K \) grows as \( f_0 \times \) the number of \( i \)−dimensional simplices of the standard simplex \( \Delta_i \) lowered by a term \( i (d+1) \) which does not depend on \( f_0 \). Recent versions of LBT called GLBT (cf. [16], [20], [21], and [22], [23]) or \( g \)-conjecture confirmed in [1] increase the formula of LBT by adding terms which depend on the reduced Betti numbers of \( X \) (cf. Theorem 3.1). The latter not only involves the topology of \( X \) but also essentially enlarges the estimate.

The paper is organized in the following way. In the first section we derive or estimate the main formula of [11] (Theorem 2.1) for the classical Lie groups estimating the covering type of spaces in problem. Next in the second section we adapt Theorem 3.1 to the discussed spaces by substituting the estimate of \( f_0 \) of first section and the values (or estimates) of Betti numbers of studied spaces. At the end we include a notebook of the Mathematica which derives the value of main formula provided values of \( f_0 \) and \( \beta_i \) the reduced Betti numbers are known. We present the result of computation of estimates of \( f_i, 0 \leq i \leq 14 \) for the group \( G_2 \), and \( F_4 \), leaving the reader a potentiality to compute this for the remaining exceptional compact Lie groups \( E_6 \), \( E_7 \), and \( E_8 \).

2. Computations of value of covering type by use of 2.1

Our aim is derive the value of formula of Theorem 2.1 for these spaces, classes of spaces for which it can be effectively derived. To do it we restate the some facts presented already in [11], derived the value form description of cohomology rings of exceptional Lie groups given by the first and third author in [8]. We also adapt the classical results on the cohomology rings of Lie groups (for example see [10]). Finally we include the results of direct computations for some other spaces.

**Theorem 2.1 ([11, Theorem 3.5]).** If there are elements \( x_i \in H^{i_k}(X) \) with \( i_k > 0, k = 1, 2, \ldots, l \), such that \( x_{i_1} \cdot x_{i_2} \cdot \cdots \cdot x_{i_l} \neq 0 \) then

\[
ct(X) \geq l + 1 + \sum_{k=1}^{l} k i_k.
\]

Furthermore, if all \( i_k \)'s are not equal, then \( ct(X) \geq l + 2 + \sum_{k=1}^{l} k i_k \).

We begin from a general theorem which shows that for given simple-connected compact Lie group \( G \) the values which are necessary for formula 2.1 are encoded in the Cartan algebra the Weyl roots system of \( G \).

It is known [9, CH.1, §7] and [10] that the cohomology ring of a compact Lie group \( G \) with coefficients in a field \( \mathcal{R} \) of characteristic 0 is of the form

\[
H^*(G; \mathcal{R}) = \bigwedge_{\mathcal{R}} [y_{2m_1+1}, y_{2m_2+1}, \ldots, y_{2m_l+1}]
\]

where \( 0 \leq m_1 \leq m_2 \leq \cdots \leq m_l \) and the sequence \( (m_1, m_2, \ldots, m_l) \) is called "the rational type" of \( G \).
Moreover [10], it is known that if \( G \) is a simple compact Lie group then

\[
\sum_{j=1}^{l} m_j = \frac{1}{2}(d-l)
\]

where \( l = \text{rk} \ G \) is the rank of \( G \), i.e. the dimension of maximal torus \( \mathbb{T} \subset G \), and \( d = \dim G \) the dimension of \( G \).

**Theorem 2.2.** Let \( G \) be a compact simple Lie group and \((m_1, m_2, \ldots, m_l)\) its rational type. Then

\[
(*) \quad \text{ct}(G) \geq l + 1 + \sum_{j=1}^{l} j (2m_j + 1).
\]

**Corollary 2.3.** Let \( G \) be a compact simple Lie group of rank \( l \). Then

\[
\text{ct}(G) \geq \frac{(l+1)(l+2)}{2}.
\]

**Proof.** Since \( m_i \leq m_j \) if \( i < j \), and \( m_j \geq 0 \), we have

\[
\text{ct}(G) \geq l + 1 + \sum_{j=1}^{l} j (2m_j + 1) \geq l + 1 + \sum_{j=1}^{l} j = \frac{(l+1)(l+2)}{2}
\]

\[\square\]

**Remark 2.4.** A natural question about of finding the maximum and minimum of \((*)\) reduces to do the problem of estimation of sum \( \sum_{j=1}^{l} j m_j = (l+1)M_l - (M_1 + M_2 + \ldots + M_l) \), where \( M_j = m_1 + \ldots + m_j \). The above follows from the Abel identity.

Since \( \sum_{j=1}^{l} m_j = \frac{1}{2}(d-l) \), the value of \( M_l \) is fixed.

1) Note that the sum \((*)\) is monotonic with respect to each \( m_j \), consequently the minimal value is 0, for \( m_1 = \ldots = m_l = 0 \). There is not the greatest value in general, i.e. if we do not have equality \((2.2)\).

2) If \((2.2)\) holds and \( M_l \) is fixed, then the greatest value is for \( m_1 = \ldots = m_{l-1} = 0, m_l = M_l \).

3) Concerning once more the lowest value in more detail. From the monotonicity of \( m_j \) we have

\[M_j \leq |jM_{j+1}/(j+1)], j = 1, \ldots, l-1. \]

Consequently, the equality, thus the lowest value, is for \( m_l = m_{l-1} = \ldots = m_l = k = m_{k-1} + 1 = \ldots = m_1 + 1. \) Also \( m_1 = [M_l/1] \) and \( k = l + lM_l - M_l + 1. \)

Note that \( m_1 = \ldots = m_{l-1} = 0 \) and \( m_l = M_l \) implies \( l + 1 + \sum_{j=1}^{l} j (2m_j + 1) = l + 1 + l(2m_l + 1). \) This and \((*)\) give

\[
l + 1 + \sum_{j=1}^{l} j (2m_j + 1) = l + 1 + l[(d-l) + l] = 1 + l + l(d-l) + l = l + d - l^2 + 2 + l + 1
\]

**Problem 2.1.** Is there formula expressing, for a compact Lie group \( G \), the dimension \( d = \dim G \) in terms of \( l = \text{rk} \ G \) of rank of \( G \). It would be enough to know the rate of growth of \( d \) as a function of \( l \). In all known examples \( d \) is a quadratic function of \( l \).

A classical result coming form the fact \( \pi_2(G) = 0 \) and \( \pi_3(G) = \mathbb{Z} \) for any compact Lie group says (see [10] Theorem 2.6)

**Theorem 2.5.** Let \( G \) be a compact simply-connected simple Lie group. Then \( H^3(G; \mathbb{Q}) \simeq \mathbb{Z} \). This implies that \( m_1 = 1 \) and \( m_i > 1 \) for \( i > 1 \).

Consequently, \( H^*(G; \mathbb{Q}) = \Lambda[x_1, x_2m_2 + 1, \ldots, x_2m_1 + 1] \)
As a corollary we get the following estimate

**Corollary 2.6.** Let \( G \) be a compact simply-connected simple Lie group, \( T^l \subset G \) its maximal torus, and \((m_1, m_2, \ldots, m_l)\) its rational type. Then

\[
\text{ct}(G) \geq 3 + (2m_2 + 1) + \ldots + (2m_l + 1) + l + 1
\]

**Remark 2.7.** Note that estimates of Theorem 2.2 and Corollary 2.6 give only at most cubic rate of growth of \( \text{ct}(G) \) in the dimension of \( G \).

Moreover, the same argument shows that in general the estimate (2.1) of Theorem 2.1 gives only cubic rate of growth of \( \text{ct}(X) \) in \( n = \dim X \).

**Proof.** Indeed, doing very crude estimation, i.e. replacing \( l = \dim T \) by \( n = \dim G \) and each \( 2m_j + 1 \) by \( n \), or respectively \( l \) by \( n = \dim X \) and also each \( i_k \) by \( n = \dim X \) in the general case, we get the following

\[
(+)
\]

\[
l + 1 + \sum_{j=1}^l j(2m_j + 1) \leq n + 1 + \frac{n^2(n + 1)}{2} = (n + 1)(1 + \frac{1}{2}n^2)
\]

or respectively

\[
(++)
\]

\[
l + 2 + \sum_{k=1}^l k \leq n + 2 + \sum_{k=1}^n k = n + 2 + \frac{n^2(n + 1)}{2}
\]

\[\square\]

2.1. **Classical Lie groups.** The simplest compact Lie group is the torus \( T^n \), which is not simply-connected but has nice form of its cohomology algebra

**Proposition 2.8 ([11]).**

\[
\text{ct}(T^n) \geq \frac{(n+1)(n+2)}{2}
\]

**Proof.** Indeed \( H^*(T^n; \mathbb{R}) = \bigwedge[t_1, \ldots, t_n] \), where \( \deg t_j = 1 \). Thus \( \text{ct}(T^n) \geq 1+1+2+\cdots+1 \cdot n+n+1 = \frac{(n+1)(n+2)}{2} \). Of course the statement also follows from Corollary 2.3 \[\square\]

Analogous argument works for the unitary groups. But then the knowledge of degrees of generators of \( H^*(\ ; \mathbb{Z}) \) let us get better estimate than that of Corollary 2.3

**Proposition 2.9.** The covering type of unitary group is estimated as

\[
\text{ct}(U(n)) \geq \frac{1}{6}(4n^3 + 3n^2 + 5n + 12) \text{ and } \text{ct}(SU(n)) \geq \frac{1}{6}(4n^3 - 3n^2 + 5n + 6).
\]

**Proof.** The cohomology algebra \( H^*(U(n); \mathbb{Z}) \) is the exterior algebra on generators in dimensions \( 1, 3, \ldots, (2n-1) \), while \( H^*(SU(n); \mathbb{Z}) \) is the exterior algebra on generators in dimensions \( 3, 5, \ldots, (2n-1) \). Theorem 2.1 and the classical sums

\[
\sum_{j=1}^l j = \frac{l(l+1)}{2} \quad \text{and} \quad \sum_{j=1}^l j^2 = \frac{l(l+1)(2l+1)}{6}
\]

give

\[
\text{ct}(U(n)) \geq 1 + 2 \cdot 3 + 3 \cdot 5 + \ldots + n \cdot (2n-1) + (n+2) = \frac{1}{6}(4n^3 + 3n^2 + 5n + 12)
\]

and

\[
\text{ct}(SU(n)) \geq 1 \cdot 3 + 2 \cdot 5 + \ldots + (n-1) \cdot (2n+1) + (n+1) = \frac{1}{6}(4n^3 - 3n^2 + 5n + 6).
\]

\[\square\]
Proposition 2.10. The covering type of the symplectic group is estimated as
\[ \text{ct}(Sp(n)) \geq \frac{1}{6} (8n^3 + 13n^2 + 11n + 12) \]

Proof. We have \( H^*(Sp(n); \mathbb{Z}) = \bigwedge[x_3, x_7, \ldots, x_{4n-1}] \) (cf. [10]) and the statement follows by the same arguments as in Proposition 2.9 applied to the sum \( \sum_{j=1}^{n} j(4j - 1) \).

Proposition 2.11. The covering type of special orthogonal group \( SO(n) \) is estimated as
\[ \text{ct}(SO(n)) \geq \frac{1}{6} (4n^3 + 3n^2 + 5n + 12) \]

Proof. Let \( \mathbb{F}_2 \) be a field of characteristic 2, e.g. \( \mathbb{Z}_2 \) with the ring structure. Then from [10] Theorem 1.18] we have \( H^*(SO(n); \mathbb{F}_2) \) is the quotient of polynomial ring \( \mathbb{F}_2[x_1, x_3, \ldots, x_{2m-1}]/(x_i^{\alpha_i}) \), where \( m = \lfloor \frac{n}{2} \rfloor \) and \( \alpha_i \) is the smallest power of two such that \( i \alpha_i \geq n \). Since the generator of ideal \( (x_i^{\alpha_i}) \) is a multiple of variables \( x_i \) in powers which are powers of 2, the multiple \( x_1 x_3 \cdots x_{2m-1} \) is nonzero in \( \mathbb{F}_2[x_1, x_3, \ldots, x_{2m-1}]/(x_i^{\alpha_i}) \). Then the statement follows from Theorem 2.1 and the same calculation as for \( G = U(n) \) in Proposition 2.9.

2.2. Exceptional Lie groups. For the five simply connected exceptional Lie groups \( G \), the structure of the algebra \( H^*(G; \mathbb{F}_p) \) is determined by Duan and Zhao in [8] Theorem 1.11. It follows that
\[
\begin{align*}
H^*(G_2; \mathbb{F}_2) &= \mathbb{F}_2[\zeta_3]/(\zeta_3^4) \otimes \Lambda_{\mathbb{F}_2}(\zeta_5), \\
H^*(F_4; \mathbb{F}_2) &= \mathbb{F}_2[\zeta_3]/(\zeta_3^4) \otimes \Lambda_{\mathbb{F}_2}(\zeta_5, \zeta_9, \zeta_{17}, \zeta_{23}), \\
H^*(E_6; \mathbb{F}_2) &= \mathbb{F}_2[\zeta_3, \zeta_9]/(\zeta_3^4, \zeta_9^4) \otimes \Lambda_{\mathbb{F}_2}(\zeta_5, \zeta_9, \zeta_{17}, \zeta_{23}, \zeta_{27}), \\
H^*(E_7; \mathbb{F}_2) &= \mathbb{F}_2[\zeta_3, \zeta_9, \zeta_{11}]/(\zeta_3^4, \zeta_9^4, \zeta_{11}^4) \otimes \Lambda_{\mathbb{F}_2}(\zeta_5, \zeta_9, \zeta_{17}, \zeta_{23}, \zeta_{27}, \zeta_{29}).
\end{align*}
\]

Proposition 2.12. The covering type of the exceptional Lie groups \( G_2, F_4, E_6, E_7 \) and \( E_8 \) have the lower bounds 44, 259, 486, 1288 and 5870, respectively.

Consequently 44, 259, 486, 1288 and 5870 are lower estimates of \( f_0(G_2) \), \( f_0(F_4) \), \( f_0(E_6) \), \( f_0(E_7) \), and \( f_0(E_8) \) respectively.

Proof. From the cohomology algebras with coefficients in \( \mathbb{F}_2 \) stated above, we obtain following estimations:
\[
\begin{align*}
\text{ct}(G_2) & \geq (4 + 2) + 1 \times 3 + 2 \times 3 + 3 \times 3 + 4 \times 5 = 44. \\
\text{ct}(F_4) & \geq (6 + 2) + 1 \times 3 + 2 \times 3 + 3 \times 3 + 4 \times 5 + 5 \times 15 = 259. \\
\text{ct}(E_6) & \geq (8 + 2) + 1 \times 3 + 2 \times 3 + 3 \times 3 + 4 \times 5 + 5 \times 9 + 6 \times 15 + 7 \times 17 + 8 \times 23 = 486. \\
\text{ct}(E_7) & \geq (13 + 2) + 1 \times 3 + 2 \times 3 + 3 \times 3 + 4 \times 5 + 5 \times 6 + 7 \times 9 + 8 \times 9 + 9 \times 9 + 10 \times 11 + 12 \times 23 + 13 \times 27 = 1288. \\
\text{ct}(E_8) & \geq (32 + 2) + 1 \times 3 + 2 \times 3 + 3 \times 3 + 4 \times 5 + 5 \times 3 + 6 \times 6 + 7 \times 3 + 8 \times 3 + 9 \times 9 + 10 \times 3 + 11 \times 3 + 12 \times 3 + 13 \times 3 + 14 \times 3 + 15 \times 3 + 16 \times 5 + 17 \times 5 + 18 \times 5 + 19 \times 5 + 20 \times 5 + 21 \times 5 + 22 \times 5 + 23 \times 9 + 24 \times 9 + 25 \times 9 + 26 \times 15 + 27 \times 15 + 28 \times 15 + 29 \times 17 + 30 \times 23 + 31 \times 27 + 32 \times 29 = 5870.
\end{align*}
\]

Using the cohomology with other coefficients one obtains different lower bounds, among which \(F_2\) coefficient gives the best estimation.

### 2.3. Kähler or symplectic manifolds.

**Theorem 2.13.** If \(X\) is a Kähler manifold, or a closed symplectic manifold of (real) dimension \(2m\), then \(\text{ct}(X) \geq (m + 1)^2\).

**Proof.** By definition, there is a cohomology class \(\omega \in H^2(X)\) such that \(\omega^m \neq 0 \in H^{2m}(X)\) (see [3, 4.23 Theorem]). Consider \(m\) copies of \(\omega\). From Theorem 2.1, we obtain that \(\text{ct}(X) \geq m + 1 + \sum_{k=1}^{m} 2k = (m + 1)^2\). □

### 3. Estimates of number of simplices of given dimension

In this section we estimate the number of simplices of a given dimension, e.g. of facets, and of all simplices that are needed to triangulate a Lie group or flag manifold. In our approach we follow our previous work [12]. A classical tool for such an estimation is the Lower Bound Theorem of Kalai [13] and also Gromov [14] (see [12] for more information). Note that LBT is purely combinatorial and does not take into account the homology of the manifold. As in [12] we are able to obtain better estimates by using a generalized version of LBT.

First observe that the number of all simplices always increases exponentially with the dimension. In fact, even the minimal triangulation of the simplest closed manifold, the \(d\)-dimensional sphere, has \(d + 2\) vertices and \(2^{2d+2} - 2\) simplices. However, we will show that the number of simplices that are needed to triangulate Lie groups, respectively flag manifolds, of comparable dimension is several orders of magnitude bigger.

As we said LBT does not take into account the homology of the manifold, so we are led to consider stronger results. Much of the research in enumerative combinatorics of simplicial complexes has been guided by various versions of the so called \(g\)-Conjecture. These are a far-reaching generalization of the LBT. Of particular interest to us is the so called Manifold \(g\)-Conjecture (see [16, Section 4]), as it comes with a version of LBT for manifolds that takes into account the Betti numbers. Recently, Adiprasito has published a preprint [1], whose results combined with the work of Novik and Swartz [20, 22, 23, 28] imply that the Manifold \(g\)-Conjecture is indeed true.

The Strong Manifold \(g\)-Theorem comes with an even stronger version of the GLBT (cf. [12, Theorem 4.5]) which we formulate here for convenience of reader.
Theorem 3.1. Suppose $\mathbb{F}$ is a field of arbitrary characteristic. Let $M$ be a connected $d$-dimensional $\mathbb{F}$-orientable triangulated $\mathbb{F}$-homology manifold without boundary. Suppose $\beta_i$, $i = 0, \ldots, d$, are the reduced Betti numbers of $M$ with respect to $\mathbb{F}$. Then the following bounds hold:

$$f_i \geq f_0 \cdot (d+1)_i - i \cdot (d+2)_i + (d+1)_i \sum_{j=0}^{d+1} \binom{d+1}{j} \beta_j$$

$$+ \sum_{j=2}^{d+2} \binom{d+2-j}{d+1-i} \left( (d+2-j) - \binom{d+1-j}{d+1-i} \right) \beta_{j-1}$$

for $i = 0, \ldots, d - 1$

and

$$f_d \geq f_0 \cdot d - (d+2)(d-1) + \sum_{j=0}^{d-1} \binom{d}{j} \beta_j + \sum_{j=2}^{d+2} (d+2-2j) \binom{d+1}{j-1} \beta_{j-1}.$$ 

3.1. Asymptotic estimate for the classical Lie groups. We are in position to formulate a theorem which states that for the classical compact Lie groups for any triangulation the number of facets (the simplices of highest dimension of the group) growth exponentially in the rank of groups, thus also exponentially in in the dimension of these groups.

To make a use of Theorem 3.1 we should know the Betti numbers of of given space. The most common and useful for computation is a presentation of them as the coefficients of Poincare polynomial of $X$. Suppose $\beta_i$ be the dimension of $H^i(X; \mathbb{F})$ of the cohomology group in the coefficient in a field $\mathbb{F}$. The Poincare polynomial is defined as:

$$P(X)(t) := 1 + \beta_1(X) t + \beta_2(X) t^2 + \cdots + \beta_d(X) t^d + \cdots,$$

since $\beta_0(X) = 1$ if $X$ is connected. We shall use the cohomology in $\mathbb{Q}$ coefficient if no coefficient is specified.

Study of the Betti numbers of the Lie groups began even earlier than a description of their cohomology rings had been given. For a brief of its history and many positions of classical literature we refer to \[25\] and \[6\]. We have the following description of Poincare polynomial (cf. \[25\], \[6\]).

Let $G$ be a compact Lie group and $(m_1, m_2, \ldots, m_l)$ its rational type, where $l$ is the rank of $G$. Then its rational Poincare polynomial is given as

$$P(G)(t) = \prod_{i=1}^{l} (1 + t^{2m_i+1})$$

More precisely

$$P(SU(n)) = (1 + t^3)(1 + t^5) \cdots (1 + t^{2n+1}),$$

$$P(U(n)) = (1 + t)(1 + t^3)(1 + t^5) \cdots (1 + t^{2n+1}),$$

and

$$P(SO(2n+1)) = (1 + t^3)(1 + t^7) \cdots (1 + t^{4n-1}),$$

$$P(SO(2n)) = (1 + t^3)(1 + t^7) \cdots (1 + t^{4n-5})(1 + t^{2n-1}),$$

$$P(Sp(n)) = (1 + t^3)(1 + t^7) \cdots (1 + t^{4n-1}).$$

We need also an information about dimensions and ranks of the listed above groups.

Fact 3.1. For the discussed groups we have

\[In the literature, $\mathbb{F}$ is typically assumed to be infinite for technical reasons, however for the purposes of face enumeration this can be circumvented by passing to $\mathbb{F}(t)$, preserving the Betti numbers. This trick was already used for example in \[26\] Theorem 4.3.\]
Theorem 3.2. For the classical compact Lie groups $U(n)$, $SU(n)$, $SO(n)$, and $Sp(n)$ we have the following estimates of number of facets, i.e. simplices of highest dimension $d = \dim G$:

1) $f_d(U(n)) \geq \frac{1}{8}(4n^5 - 3n^4 + 5n^3 + 6n^2 + 12) + 2^n - 1,$
2) $f_d(SU(n)) \geq \frac{1}{8}(4n^5 - 9n^4 + 15n^2 + 6) + 2^n - 1,$
3) $f_d(SO(2n + 1)) \geq \frac{1}{8}(32n^5 + 64n^4 + 64k^3 + 38k^2 + 9k + 6) + 2^n - 1,$
4) $f_d(SO(2n)) \geq \frac{1}{8}(32n^5 - 16n^4 + 16n^3 - 2k^2 - 3k + 6) + 2^n - 1,$
5) $f_d(Sp(n)) \geq \frac{1}{8}(8n^5 + 15n^4 + 12n^3 + 11n^2 + 6n + 12) + 2^n - 1,$

Consequently, in all these cases the number of facets $f_d$ grows exponentially in $n$, thus in $l = \rank G$ and $d = \dim G$.

Proof. To show the statement it is enough to apply the second formula of Theorem 3.1 and do the following.

1. First substitute the values of estimates of $\beta$ stated in Propositions 2.9, 2.11, and 2.10 respectively to the formula of Theorem 3.1 as $f_0$.
2. Secondly, input to this formula the value of dimension $d$ from the table 3.1.
3. Thirdly, truncate the sum of formula of Theorem 3.1 disregarding the last term (sum) of it.
4. Next, in the term $\sum_{j=0}^{d-1} \binom{l}{i} \beta_j$ take on account only these reduced nonzero Betti numbers which correspond to the multiples $i^{2m_1+1}i^{2m_2+1} \ldots i^{2m_i+1}$ for multi-indices $(i_1, \ldots, i_s)$, $i_k \geq 1$, of lengths $1 \leq s \leq l$, where $l = \rank G$. Observe that always the upper index of summation, which is equal to $d - 1$, is greater or equal to the parameter $n$, or $n - 1$ if $G = SU(n)$. The latter is equal to $l$-the rank. Consequently $\binom{d-1}{i} \geq \binom{i}{j}$, and
\[ \sum_{j=0}^{d-1} \binom{i}{j} \beta_j \geq \sum_{j=0}^{l} \binom{l}{j} \beta_j, \]
where each $j$ corresponds to a multi-index $(i_1, \ldots, i_s)$ of length $1 \leq s \leq l$. By the above, $\beta_j \geq 1$.

Now the statement follows from the binomial formula $\sum_{j=0}^{l} \binom{l}{j} = 2^l$, since we have subtract $\binom{0}{l}$ as the zero Betti number.

Remark 3.3. Note that a modification of argument of Theorem 3.2 let us to show the exponential rate of growth of the number $f_{d - i_0}(G)$ of simplices of fixed codimension $d - i_0$ for all groups of the statement of Theorem 3.4. As in Theorem 3.2 the varying argument is $n$, which can be identified with the rank of $G$.

At the end of this subsection we turn back to the Kähler and symplectic manifolds. As a correspondent of Theorem 3.2 we have the following theorem

Theorem 3.4. If $X$ is a Kähler manifold, or a closed symplectic manifold of (real) dimension $2m$, then
\[ f_{2m}(X) \geq (2m^3 + 2m + 1) + (2^{2(m-1)} - 1) = 2m^3 + 2m + 2^{2(m-1)} \]

Proof. Since $f_0 \geq (m+1)^2$ by Proposition 3.4 and $d = 2m$ the first term of the second formula of Theorem 3.1 is equal to $2n^3 + 2n + 1$. Moreover, $\omega^j \neq 0$ in $H^{2j}(X; \mathbb{Z})$ for every $1 \leq j \leq m$ which implies that
The Poincaré polynomials

Proposition 3.6. Exceptional Lie groups but the length of outputs are too long to include them to this note.

with coefficients in a field

calculation of the numbers

for interesting algebraic varieties (cf. [24]). Consequently, for lower dimensional cases it is better to derive completely the formula of Theorem 3.7 than to estimate only.

3.2. Computations for the exceptional Lie groups. In the last section we present complete calculation of the numbers \( f_i \) for two exceptional groups \( G_2 \) and \( F_4 \), and the Poincaré polynomials for all exceptional groups. We also provide the codes sources for a notebook of Mathematica program. This and the data of Poincaré polynomials allow to derive the numbers \( f_i \) , \( 0 \leq i \leq \text{dim } G \) for the remaining exceptional Lie groups but the length of outputs are too long to include them to this note.

Proposition 3.6. The Poincaré polynomials \( P(G; F) \) of the five simply connected exceptional Lie groups \( G \) with coefficients in a field \( F \) are:

\[
P(G_2; F) = \begin{cases} 
1 - t^{12} & \text{if } F = \mathbb{F}_2, \\
1 + t^2 & \text{if } F = \mathbb{F}_3, \mathbb{F}_5, \mathbb{Q}, \\
(1 + t^3)(1 + t^11) & \text{if } F = \mathbb{F}_7, \\
(1 + t^5)(1 + t^{15})(1 + t^{23}) & \text{if } F = \mathbb{F}_2, \\
(1 + t^8 + t^{16})(1 + t^3)(1 + t^7)(1 + t^{11})(1 + t^{15}) & \text{if } F = \mathbb{F}_3, \\
(1 + t^3)(1 + t^{11})(1 + t^{15})(1 + t^{23}) & \text{if } F = \mathbb{F}_5, \mathbb{Q}, \\
(1 + t^{12})(1 + t^{15})(1 + t^{17})(1 + t^{22})(1 + t^{27}) & \text{if } F = \mathbb{F}_2, \\
(1 + t^8 + t^{16})(1 + t^3)(1 + t^7)(1 + t^9)(1 + t^{11})(1 + t^{15})(1 + t^{17}) & \text{if } F = \mathbb{F}_3, \\
(1 + t^3)(1 + t^{11})(1 + t^{15})(1 + t^{17})(1 + t^{23}) & \text{if } F = \mathbb{F}_5, \mathbb{Q}, \\
(1 + t^{12})(1 + t^{20})(1 + t^{36}) & \text{if } F = \mathbb{F}_2, \\
(1 + t^8 + t^{16})(1 + t^3)(1 + t^7)(1 + t^{11})(1 + t^{15})(1 + t^{17})(1 + t^{35}) & \text{if } F = \mathbb{F}_3, \\
(1 + t^3)(1 + t^{11})(1 + t^{15})(1 + t^{19})(1 + t^{23})(1 + t^{27})(1 + t^{35}) & \text{if } F = \mathbb{F}_5, \mathbb{Q}, \\
(1 + t^{16})(1 + t^{20})(1 + t^{36})(1 + t^{66}) & \text{if } F = \mathbb{F}_2, \\
(1 + t^8 + t^{16})(1 + t^3)(1 + t^7)(1 + t^{11})(1 + t^{15})(1 + t^{17})(1 + t^{23})(1 + t^{27})(1 + t^{29}) & \text{if } F = \mathbb{F}_3, \\
(1 + t^{22})(1 + t^{35})(1 + t^{39})(1 + t^{47}) & \text{if } F = \mathbb{F}_5, \mathbb{Q}, \\
(1 + t^{12})(1 + t^{20})(1 + t^{66}) & \text{if } F = \mathbb{F}_2, \\
(1 + t^8 + t^{16})(1 + t^3)(1 + t^7)(1 + t^{11})(1 + t^{15})(1 + t^{19}) & \text{if } F = \mathbb{F}_3, \\
(1 + t^{22})(1 + t^{35})(1 + t^{39})(1 + t^{47}) & \text{if } F = \mathbb{F}_5, \mathbb{Q}, \\
(1 + t^3)(1 + t^{15})(1 + t^{23})(1 + t^{35})(1 + t^{39})(1 + t^{47})(1 + t^{59}) & \text{if } F = \mathbb{Q}.
\end{cases}
\]

Applying Theorem 3.3, we can obtain estimations of the number \( f_i \) of simplices of dimension \( i \) of the relevant group \( G \). This task can be implemented using Mathematica with the codes noted below.

The input is the Poincaré polynomial \( P(G; F) \) of a group \( G \), while the Output is a table that lists all the estimations of \( f_i \), where \( i \) ranges from 0 to \( \text{dim } G \).
\[ d = \text{Length}[\text{CoefficientList}[p, t]] - 1 \]
\[ l_i = \text{Table}[f_0 \ast \text{Binomial}[d + 1, i] - i \ast \text{Binomial}[d + 2, i + 1] \]
\[ + \text{Binomial}[d + 1, i + 1] \ast \text{Sum}[\text{Binomial}[i, j] \ast b[j], \{j, 0, i\}] \]
\[ + \text{Sum}[(\text{Binomial}[d + 2 - j, d + 1 - i] - \text{Binomial}[j, d + 1 - i]) \ast \text{Binomial}[d + 1, j - 1] \ast b[j - 1], \]
\[ \{j, 2, (d + 2)/2\}], \]
\[ \{i, 0, d - 1\}]; \]
\[ l_i = \text{Append}[l_i, f_0 \ast d - (d + 2) \ast (d - 1) + \text{Sum}[\text{Binomial}[d, j] \ast b[j], \{j, 0, d - 1\}] \]
\[ + \text{Sum}[(d + 2 - 2 \ast j) \ast \text{Binomial}[d + 1, j - 1] \ast b[j - 1], \{j, 2, (d + 2)/2\}]]; \]
\[ l_i = l_i /. \text{Table}[b[j] \rightarrow \text{Abs}[\text{Coefficient}[p, t^j]], \{j, d\}]; \]
\[ l_i = l_i /. \{b[0] \rightarrow 0\} \]

Since the covering type \( \text{ct}(M) \) serves as an estimation of \( f_0(M) \).

**Remark 3.7.** From the Poincaré polynomial \( P(G_2; \mathbb{F}_2) \) of the group \( G_2 \) with coefficients in \( \mathbb{F}_2 \) we obtain the following estimations of the \( f_i(G_2) \)'s, where \( 1 \leq i \leq \dim G_2 = 14 \):

\[
44, 540, 3500, 16380, 60060, 180180, 460460, 1003860, 1793220,
2494492, 2582580, 1901900, 936740, 276060, 36808
\]

In comparison, the estimations coming from the Poincaré polynomials with other coefficients are much smaller.

**Remark 3.8.** In the next table we present estimations of the \( f_i \)'s for the group \( F_4 \) derived separately by use of the Poincaré polynomial with coefficients in \( \mathbb{F}_2, \mathbb{F}_3, \) or \( \mathbb{F}_5 \) respectively. To get a comparison, in each dimension \( i \) the biggest lower bounds are underlined. Note that the estimations vary according the choices of coefficients.
| i  | $F_2$   | $F_3$   | $F_5$   |
|----|---------|---------|---------|
| 0  | 259     | 259     | 259     |
| 1  | 12296   | 12296   | 12296   |
| 2  | 307294  | 307294  | 307294  |
| 3  | 5434832 | 5434832 | 5434832 |
| 4  | 75841675| 75841675| 75841675|
| 5  | 898211405| 872384240| 872384240|
| 6  | 9665099080 | 8425395160 | 8425395160 |
| 7  | 98189141960 | 70096565630 | 69056099840 |
| 8  | 936843104470 | 532679948710 | 484818522370 |
| 9  | 8207348294600 | 4019473489850 | 2942501397200 |
| 10 | 65068495431940 | 31469774114260 | 15579780596380 |
| 11 | 465207900554770 | 246812512408280 | 72908019801890 |
| 12 | 3006871679028070 | 1825555954104190 | 312895574628490 |
| 13 | 17600137873624250 | 12284217111466700 | 1357046559200900 |
| 14 | 93440294896349800 | 74314230778534600 | 6810475068598600 |
| 15 | 450286771378309757 | 403688085014794580 | 40032696051173735 |
| 16 | 1971743019982744645 | 197254627081800135 | 241874900220997225 |
| 17 | 78631823722190160 | 869199543917841515 | 1351282534317191840 |
| 18 | 2866777688396562710 | 3465935564215040210 | 6719777511823246510 |
| 19 | 9605339758806527825 | 12565150926185060840 | 29616143389087516625 |
| 20 | 29754670696222844675 | 416590123956641354875 | 116366415690518027125 |
| 21 | 85704882351332147035 | 1271376687212781173570 | 41055293101705466610 |
| 22 | 2305515484568597680 | 3594398180086388474800 | 130828024734960104800 |
| 23 | 580591666340273259140 | 946343011836668351880 | 378030092697147336640 |
| 24 | 1369174270273968084348 | 239260330203450996348 | 993359670692876306348 |
| 25 | 30203411458511162699228 | 53694335012120681194316 | 2378102456278215423648 |
| 26 | 6222295640840578835384 | 118586106264425527967208 | 5193967024820359867144 |
| 27 | 11954502608182386973476 | 233700849692810391946332 | 10360931479176120792736 |
| 28 | 214032931396835177839294 | 43949957681634358690164 | 18895182140649365763324 |
| 29 | 35711149394730846562160 | 76847135152073128466780 | 31530964549461509302080 |
| 30 | 55556068183111798392400 | 1245866340397735244442800 | 481853242200739273482800 |
Our estimations of number of all simplices, i.e. of all dimensions, which are necessary to a triangulation of the group $F_4$ are given as follows.

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