Kim-Vu’s sandwich conjecture is true for all $d = \omega(\log^7 n)$

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Abstract

Kim and Vu made the following sandwich conjecture regarding the random $d$-regular graph $G(n,d)$: for every $d \gg \log n$, there exist $p_1 = (1 - o(1))d/n$, $p_2 = (1 + o(1))d/n$ and a coupling of three random graphs $(G^L,G,G^U)$ such that marginally $G^L \sim G(n,p_1)$, $G \sim G(n,d)$ and $G^U \sim G(n,p_2)$, and jointly, $\mathbb{P}(G^L \subseteq G \subseteq G^U) = 1 - o(1)$. Recently, Isaev, McKay and the author confirmed the sandwich conjecture for all $d \gg n/\sqrt{\log n}$, and they proved a weakened version of the sandwich conjecture for $d = O(n/\sqrt{\log n})$ where $p_2$ is roughly $(\log n) \cdot d/n$. In this paper we prove that the sandwich conjecture holds for all $d = \omega(\log^7 n)$.

1 Introduction

The binomial random graphs, introduced by Erdős and Rényi [2,3] around 1960, are the most well known, and the best studied ones among various random graph models. In the binomial random graph $G(n,p)$, the vertex set is $[n]$, and between each pair of vertices, an edge is present independently with probability $p$. On the other hand, the random regular graphs $G(n,d)$, or more generally $G(n,d)$, the uniformly random graph with degree sequence $d$, are receiving extensive attention. In the studies of random regular graphs, it was noticed that $G(n,d)$ shares many properties with $G(n,p)$ where $p \approx d/n$ when $d \gg \log n$. For instance, asymptotically, they have the same chromatic number (known for several ranges of $d$), the same diameter, the same independence number, and the same order on the spectral gap. This motivated the following sandwich conjecture, proposed by Kim and Vu [6] in 2004.

Conjecture 1 (Sandwich conjecture). For every $d \gg \log n$, there are $p_1 = (1 - o(1))d/n$, $p_2 = (1 + o(1))d/n$ and a coupling $(G^L,G,G^U)$ such that $G^L \sim G(n,p_1)$, $G^U \sim G(n,p_2)$, $G \sim G(n,d)$ and $\mathbb{P}(G^L \subseteq G \subseteq G^U) = 1 - o(1)$.

The sandwich conjecture, if is true, does not only explain the universality phenomenon between the binomial random graphs and the random regular graphs, but also gives a powerful tool for analysing the random regular graphs. Although $G(n,d)$ has been extensively studied, compared to $G(n,p)$, much less is proved. Proofs for properties of $G(n,d)$, where $d = \omega(1)$ typically involves highly technical analysis, as it is not easy to estimate subgraph probabilities in $G(n,d)$. One would use the switching method to estimate such probabilities when $d = o(n)$, and for $d = \Theta(n)$, enumeration arguments involving complex integrals are usually applied. With the sandwich conjecture confirmed, one can immediately translate properties and parameters from $G(n,p)$ to $G(n,d)$ with no effort. Many such examples have been given in [5].

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In the same paper where Kim and Vu proposed the conjecture, they proved that it is possible to couple $G(n, p)$, where $p = (1 - o(d))d/n$, with $G(n, d)$ such that asymptotically almost surely (a.a.s.) $G(n, p) \subseteq G(n, d)$, when $d \ll n^{1/3}$. However, the other side of the sandwich was not established. This result was further extended to $d = o(n)$ by Dudek, Frieze, Ruciński and Šileikis [1], with basically the same coupling scheme used by Kim and Vu, with refinement. Finally, the conjecture was confirmed to be true for all $d \gg n/\sqrt{\log n}$ by Isaev, McKay and the author [3]. For $d = O(n/\sqrt{\log n})$, they proved a weakened version of the sandwich conjecture, where $p_2$ is roughly $(\log n)d/n$ (see [4] Theorem 1.5] for a precise statement). Very recently, Klímašová, Reiher, Ruciński, and Šileikis [7] reported solving Conjecture [1] for all $d \gg (n \log n)^{3/4}$.

The contribution of this paper is a confirmation of the sandwich conjecture for all $d = \omega(\log^7 n)$.

**Theorem 2.** Conjecture [1] holds for all $d = \omega(\log^7 n)$.

**Remark 3.** Theorem [2] leaves $d = O(\log^7 n)$ the only range open for the sandwich conjecture.

**Remark 4.** We did not attempt to optimise the power on $\log^7 n$. It is possible to prove the theorem with a smaller power but we do not think it is possible to confirm Conjecture [7] for all $d \gg \log n$ with the current proof technique.

**Remark 5.** For a simpler exposition of the ideas, we only deal with $G(n, d)$ instead of the more general $G(n, \mathbf{d})$ in this paper. We also did not attempt to sharpen the error terms $o(1)$ in $p_1, p_2$ or the error $o(1)$ in the probability of $G^L \subseteq G \subseteq G^U$ in the coupling. The proof for Theorem [2] easily extends to $G(n, \mathbf{d})$ for near-regular degree sequences $\mathbf{d}$. Moreover, more precise results can be obtained with sharper errors in $p_1, p_2$, and $\mathbb{P}(G^L \subseteq G \subseteq G^U)$. These will be presented in a longer version of this paper.

**Remark 6.** With the sandwich theorem, we can immediately translate monotone or convex properties from $G(n, p)$ to $G(n, d)$. We may also translate phase transitions and graph parameters from $G(n, p)$ to $G(n, d)$. Many examples were exhibited in [3], some of which require tight sandwich on both sides in order for the translation to work. We particularly note that in [3] a false claim was made about the chromatic number of $G(n, d)$ (indeed for $G(n, \mathbf{d})$ where $\mathbf{d}$ is near-regular). The sandwiching was not tight in [3] for $d = O(n/\sqrt{\log n})$, so it was not possible to perform a direct translation from $\chi(G(n, p))$. Instead, that paper only used one side of the sandwich where $G(n, p_1) \subseteq G(n, d)$, for some $p_1 = (1 - o(1))d/n$, and bounded $\chi(G(n, d))$ by

$$\chi(G(n, p_1)) \leq \chi(G(n, d)) \leq \chi(G(n, p_1)) + \Delta(G(n, d) - G(n, p_1)) + 1.$$ 

However, the upper bound above is wrong. Indeed, a correct upper bound would be $\chi(G(n, p_1)) \cdot (\Delta(G(n, d) - G(n, p_1)) + 1)$, but that would not give an asymptotically tight upper bound. Now with Theorem [2] above, one can immediately determine $\chi(G(n, d))$ asymptotically for all $d = \omega(\log^7 n)$.

2 The coupling scheme in [5]

A procedure called **Coupling()** described below in Figure [1] was used to construct a coupling of $(G_G, G_G)$ by sequentially adding edges to these three graphs under construction (the notation $G \ll M$ in Figure [1] means that $G$ is obtained from $M$ by replacing each multiple edge with a single edge). In all but the very last few steps, $G_G \subseteq G \subseteq G_0$ is maintained, unless another procedure named **IndSample()** is called. By a careful choice for the three arguments $\mathbf{d}, \mathcal{I}$ and $\zeta$ that **Coupling()** takes, **IndSample()** is called only with a very small probability. After the last step of **Coupling()**, both $G_G$ and $G_0$ are binomial random graphs with asymptotically equal edge density...
Procedure $Coupling(d, I, \zeta)$:
Let $M_0^{(0)}$, $G^{(0)}$ and $M_0^{(0)}$ be the empty multigraphs on vertex set $[n]$. For every $1 \leq \iota \leq I$:
- Uniformly at random choose an edge $jk$ from $K_n$;
- If $jk \in G^{(\iota-1)}$ then
  - $G^{(\iota)} = G^{(\iota-1)}$;
  - $M_\zeta^{(\iota)} = M_\zeta^{(\iota-1)}$ with probability $\zeta$,
  - $M_\zeta^{(\iota)} = M_\zeta^{(\iota-1)} \cup \{jk\}$ with probability $1 - \zeta$;
- If $jk \notin G^{(\iota-1)}$, define $\eta_{jk}^{(\iota)} = 1 - \frac{\Pr(jk \in \mathcal{G}(n, d) \mid G^{(\iota-1)})}{\max_{jk \notin G^{(\iota-1)}} \Pr(jk \in \mathcal{G}(n, d) \mid G^{(\iota-1)})}$;
- If $\eta_{jk}^{(\iota)} > \zeta$ then Return $IndSample(d, M_\zeta^{(\iota-1)}, M_0^{(\iota-1)}, \iota, I, \zeta)$;
- Otherwise, generate $a \in [0, 1]$ uniformly randomly;
  - If $a \in (\zeta, 1]$ then $G^{(\iota)} = G^{(\iota-1)} \cup \{jk\}$ and $M_\zeta^{(\iota)} = M_\zeta^{(\iota-1)} \cup \{jk\}$;
  - If $a \in [\eta_{jk}^{(\iota)}, \zeta]$ then $G^{(\iota)} = G^{(\iota-1)} \cup \{jk\}$ and $M_\zeta^{(\iota)} = M_\zeta^{(\iota-1)}$;
  - If $a \in [0, \eta_{jk}^{(\iota)}]$ then $G^{(\iota)} = G^{(\iota-1)}$ and $M_\zeta^{(\iota)} = M_\zeta^{(\iota-1)}$;
For $\iota \geq I + 1$, while $G^{(\iota-1)}$ has fewer edges than $\mathcal{G}(n, d)$ repeat:
  - Pick an edge $uv \notin G^{(\iota-1)}$ with probability proportional to $Pr(uv \in \mathcal{G}(n, d) \mid G^{(\iota-1)})$, $G^{(\iota)} = G^{(\iota-1)} \cup \{uv\}$;
  - Assign $G = G^{(\iota)}$.
Return $(G_\zeta, G, G_0)$, where $G_\zeta \triangleleft M_\zeta^{(I)}$ and $G_0 \triangleleft M_0^{(I)}$.

Procedure $IndSample(d, M_\zeta, M_0, \iota, I, \zeta)$:
Let $M_\zeta^{(\iota-1)} = M_\zeta$ and $M_0^{(\iota-1)} = M_0$; and let $G$ be sampled from $\mathcal{G}(n, d)$.
For every $\iota \leq \tau \leq I$:
- Uniformly at random choose an edge $jk$ from $K_n$;
- $M_0^{(\tau)} = M_0^{(\tau-1)} \cup \{jk\}$;
- $M_\zeta^{(\tau)} = M_\zeta^{(\tau-1)}$ with probability $\zeta$.
- $M_\zeta^{(\tau)} = M_\zeta^{(\tau-1)} \cup \{jk\}$ with the remaining probability, i.e. with probability $1 - \zeta$;
Return $(G_\zeta, G, G_0)$ where $G_\zeta \triangleleft M_\zeta^{(I)}$ and $G_0 \triangleleft M_0^{(I)}$.

Figure 1: Procedures $Coupling()$ and $IndSample()$.

(see Lemma 2.2]), and $G \sim \mathcal{G}(n, d)$ (see Corollary 2.3]). Moreover, $G_\zeta \subseteq G$ if $IndSample()$ has not ever been called. But $G \not\subseteq G_0$ since in the last few steps of $Coupling()$, edges are added to $G$ but not to $G_0$.

We briefly explain what $Coupling()$ does and what roles the three arguments play. Parameter $d$ is the degree sequence of the random graph we wish to sandwich between two binomial random graphs. Only near-regular degree sequences were considered ($d$ is near-regular if all components of $d$ are asymptotically equal, and all components of $(n - 1)1 - d$ are asymptotically equal). Let $\Delta(d)$ denote the maximum component of $d$. For random $d$-regular graphs, we simply have $d = (d, d, \ldots, d)$. The parameter $I$ is a Poisson variable with mean $\mu$ whose value is determined based on $d$. Parameter $I$ controls the edge density of $G_\zeta$ and $G_0$. The parameter $\zeta$ is a small
positive real number. Parameter $\zeta$ controls the difference between $G_\zeta$ and $G_0$. Parameters $\mathcal{I}$ and $\zeta$ together control the difference between $G_\zeta$ and $G$. In each step of the first $\mathcal{I}$ steps of Coupling(), a uniformly random edge $jk$ is sampled from $K_n$. Then this edge is always added to $G_0$, and is added to $G_\zeta$ with probability $1 - \zeta$, and is added to $G$ with probability proportional to the probability that $G(n, d)$ contains $jk$ conditional on the set of edges currently having been added to $G$. If this conditional probability is sufficiently close to being equal among all $jk$ that haven’t been added to $G$ yet — more precisely, $\eta_{jk}^{(i)} \leq \zeta$ in step $i$ — then a coupling can be done so that $jk$ is added to $G$ only if $jk$ is added to $G_\zeta$. If however $\eta_{jk}^{(i)} > \zeta$ in some step then IndSample() will be called — if that happens, then $G$ will be generated independently from $G_\zeta$ and $G_0$.

After the first $\mathcal{I}$ steps, assuming that IndSample() has not been called, we have $G_\zeta \subseteq G \subseteq G_0$, and by the choice of $\mathcal{I}$ being a Poisson random variable, it was shown that both $G_\zeta$ and $G_0$ are distributed as a binomial random graph (see [4, Lemma 2.1]), and their constructions are completed.

After the first $\mathcal{I}$ steps, edges are sequentially sampled and added to $G$ according to the correct conditional edge probability until $G$ is a graph with degree sequence $d$. By its construction, $G \sim \mathcal{G}(n, d)$. Moreover, if by the end of the first $\mathcal{I}$ steps, $G$ was already close to the completion of its construction (this can be achieved by choosing appropriate $\mu$), and the number of edges that were accepted by $G$ but rejected by $G_\zeta$ is small (which is the case if $\zeta$ is small), then the edge density of $G_\zeta$ will be rather close to that of $G$, yielding a coupling of $G(n, p_1)$ and $G(n, d)$ where $p_1 = (1 - o(1))\Delta(d)/n$ (recall that $d$ is assumed near-regular so all components of $d$ are asymptotically $\Delta(d)$). However, due to the additional steps after $\mathcal{I}$ where $G$ continues its construction while the construction of $G_0$ has already been completed, $G$ cannot be a subgraph of $G_0$, and there is no way to remedy it using this scheme (see [4] Section 1.1 for more discussions about this).

Regardless of the technical work on optimising the choices for parameters $\zeta$ and $\mu$ (recall that $\mathcal{I} \sim \text{Po}(\mu)$), the key step in the proof is to show that
\[
P(jk \in \mathcal{G}(n, d) \mid G^{(i)}) \text{ are roughly equal among all } jk \notin G^{(i)}.\]

Equivalently, let $S = S^{(i)} = K_n - G^{(i)}$ and let $d^{(i)}$ be the degree sequence of $G^{(i)}$, the task is to show that
\[
P(jk \in S_t) \text{ are roughly equal among all } jk \in S,
\]
where $t = d - d^{(i)}$ and $S_t$ is a uniformly random $t$-factor of $S$. When the complement of $S$ is sparse, and $t$ is a sparse degree sequence (i.e. $\Delta(t) = o(n)$), this task can be easily done by applying a known enumeration result of McKay [8]. This yields a simple proof for the coupling of $\mathcal{G}(n, p_1)$ and $\mathcal{G}(n, d)$ such that a.a.s. $\mathcal{G}(n, p_1) \subseteq \mathcal{G}(n, d)$ when $d$ is a sparse near-regular degree sequence.

The real challenge is to couple $\mathcal{G}(n, d)$ and $\mathcal{G}(n, p_2)$ so that a.a.s. $\mathcal{G}(n, d) \subseteq \mathcal{G}(n, p_2)$ for sparse near-regular degree sequences $d$. By taking the complement, it is sufficient to couple $\mathcal{G}(n, (n - 1)1 - d)$ with $\mathcal{G}(n, 1 - p_2)$ so that a.a.s. $\mathcal{G}(n, 1 - p_2) \subseteq \mathcal{G}(n, (n - 1)1 - d)$. The same procedure Coupling() is run to couple $\mathcal{G}(n, (n - 1)1 - d)$ with $\mathcal{G}(n, 1 - p_2)$, with a different set of parameters. Again, the key step is to show that
\[
P(jk \in S_t) \text{ are roughly equal among all } jk \in S,
\]
where $t = ((n - 1)1 - d) - d^{(i)}$, and $d^{(i)}$ again denotes the degree sequence of $G^{(i)}$, the set of edges currently added to $G$. Here both $S$ and $t$ are dense at the beginning of the coupling procedure, which is hard to handle directly. It turns out more convenient to consider the complement of $S_t$ in $S$. Let $s$ be the degree sequence of $S$. Let $t' = s - t$. Then,
\[
P(jk \in S_t) = 1 - P(jk \in S_{t'}).\]
The aim is to show that \( P(jk \in S_t) \) are approximately equal among all \( jk \in S \). Here,

\[
t' = s - t = ((n - 1)1 - d') - (((n - 1)1 - d) - d') = d,
\]

which is sparse by assumption. However, \( S \) starts from a complete graph, and during the construction, it becomes sparser. The sparsity of \( S \) brings challenges to estimating \( P(jk \in S_t) \) using the switching arguments, and some pseudo-randomness of \( S \) was used for the switching argument to work. The proof is rather long and technical, and we summarise below the key ideas of the proof. This summary will be helpful for understanding our new coupling scheme to be described in the next section, and for understanding why the new coupling scheme works.

- Parameters of \( Coupling() \) are set so that \( t' = d \) is relatively sparse to \( S \) (i.e. \( \Delta(d) \ll \Delta(S) \), where \( \Delta(S) \) denotes the maximum degree of \( S \)) in every step of \( Coupling() \). There is no chance for the switching method to work when \( t' \) is dense relative to \( S \). This requirement explains why \( p_2 \gg d/n \) in the sandwich theorem in \([5, 4]\) for \( d = O(n/\sqrt{\log n}) \).

- The switchings used in the proof switches up to \( \log n \) edges (see \([4, \text{Section 8}]\) for the definition of the switchings and the switching arguments). This again is necessary when \( S \) is very sparse, e.g. when the maximum degree of \( S \) is a polynomial in \( \log n \).

- For a switching involving an unbounded number of edges, various errors build up quickly. To control these errors, pseudo-randomness of \( S \) plays a key role. The set of pseudo-random properties required for the proof to work are given in \([4, \text{Lemma 6.7}]\). Roughly speaking, the graph is near-regular, with a sufficient spectral gap, and sufficient concentration on the number of edges joining two large sets of vertices.

- Inductively, after each step of the coupling procedure (up to step \( I \)), \( G^{(i)} \) is sandwiched between two binomial random graphs. This allows to translate the pseudo-randomness from the binomial random graphs to \( S = K_n - G^{(i)} \).

After estimating the edge probabilities \( P(jk \in S_t) \) (or \( P(jk \in S_t) \)) and proving that they are approximately equal for all \( jk \in S \), there is technical work of choosing proper \( \mu \) and \( \zeta \) so that (a) the probability of calling \( IndSample() \) is small, (b) it takes a small number of steps after step \( I \) to complete the construction for \( G \). The choice of \( \zeta \) further determines \( p_1 \) so that \( G(n, p_1) \) can be embedded inside \( G(n, d) \). This embedding result is given in \([4, \text{Theorem 4.1}]\) and is presented below. The theorem discusses three regimes separately, according to the range of \( \Delta(d) \). As discussed before, the sparse case and the co-sparse case of the following embedding theorem together gives the non-tight sandwich theorem in \([4, \text{Theorem 1.5}]\), for sparse near-regular \( d \). The dense case in the theorem implies the sandwich conjecture for \( d \gg n/\sqrt{\log n} \) (see \([4, \text{Proof of Theorem 1.5}]\) for a proof).

**Theorem 7** (The embedding theorem). Let \( d = d(n) \) be a degree sequence and \( \xi = \xi(n) > 0 \) be such that \( \xi(n) = o(1) \). Denote \( \Delta = \Delta(d) \) and \( rng(d) = \Delta(d) + \Delta(-d) \). Then there exists a coupling \( (G^L, G) \) with \( G^L \sim G(n, p) \) (where \( p \) is specified below) and \( G \sim G(n, d) \) for the following three cases.

(a) **Sparse case.** Assume

\[
\text{rng}(d) \leq \xi \Delta \quad \text{and} \quad \xi n \geq \Delta \gg \xi^{-3} \log n.
\]

Then there exists \( p = (1 - O(\xi)) \Delta/n \) such that

\[
\Pr(G^L \subseteq G) = 1 - e^{-\Omega(\xi^3 \Delta)} \geq 1 - n^{-c},
\]

for any constant \( c > 0 \).
(b) **Dense case.** Assume \( n \cdot \text{rng}(d) \leq \xi \Delta(n - \Delta) \) and \( n - \Delta \gg \xi \Delta \gg n/\log n \). Then there exists \( p = (1 - O(\xi)) \Delta/n \) such that

\[
\Pr(G^L \subseteq G) = 1 - e^{-\Omega(\xi^3 \Delta)} = e^{-\omega(n/\log^3 n)}.
\]

(c) **Co-sparse case.** Assume

\[
\text{rng}(d) = O(\sigma) \quad \text{and} \quad \frac{n - \Delta}{n} \log \frac{n}{n - \Delta} = o(\sigma \xi)
\]

for some positive \( \sigma = \sigma(n) \) such that

\[
\xi n \gg n^\sigma \gg \frac{\log^3 n}{\log^2 \log n}.
\]

Then there exists \( p = 1 - O(\xi) \) such that

\[
\Pr(G^L \subseteq G) = 1 - e^{-\Omega(\xi^{1-\sigma} \log n)} \geq 1 - n^{-c},
\]

for any constant \( c > 0 \).

As mentioned in [4], proving the sandwich conjecture using the fore-mentioned coupling scheme and proof method would require the estimation of edge probabilities \( \mathbb{P}(jk \in S_k) \) where \( t \) is relatively dense to \( S \), whereas \( S \) is sparse. This requires a complex integral approach, which was already very complicated for \( S = K_n \) or dense \( S \) (See, e.g. the proof of Theorem 7(b) in [4, Section 7]). Extending the analysis to sparse \( S \) would require new development in the theory.

In this paper, we do not approach this way. Instead, we come up with a different coupling scheme, consisting of 2 rounds of running of procedure \( Coupling() \). We give the new scheme in the next Section.

### 3 The new coupling scheme

The sandwich conjecture has been shown to hold for \( d \gg n/\sqrt{\log n} \) in [5]. Thus, we only consider \( d \) such that \( d = \omega(\log^2 n) \) and \( d = O(n/\sqrt{\log n}) \). The embedding of \( G(n,p_1) \) inside \( G(n,d) \) for some \( p_1 = (1 - o(1))d/n \) is already known. Thus our goal is to embed \( G(n,d) \) inside \( G(n,p_2) \) for some \( p_2 = (1 + o(1))d/n \). Instead of embedding \( G(n,1-p_2) \) inside \( G(n,n - 1 - d) \), we will embed a subgraph \( H \) of \( G(n,d) \) into \( G(n,q_1) \), and then embed the rest of edges in \( G(n,d) \) into another binomial random graph \( G(n,q_2) \). What is rather magical out of the new coupling scheme is that, regardless of the obvious correlation between \( G(n,q_1) \) and \( H \), between \( H \) and \( G(n,d) \setminus H \), and between \( G(n,d) \setminus H \) and \( G(n,q_2) \), the two random graphs \( G(n,q_1) \) and \( G(n,q_2) \) turn out to be independent. Consequently, there is a coupling where \( G(n,d) \subseteq G(n,q_1) \cup G(n,q_2) \), and due to the independence between \( G(n,q_1) \) and \( G(n,q_2) \), \( G(n,q_1) \cup G(n,q_2) \) is a binomial random graph. Below is the description of the new coupling scheme, with the specification of several parameters postponed to later sections.

(i) There are two stages in the new coupling scheme. In the first stage, run the first \( I \) steps of \( Coupling(d,I,\zeta) \) with \( d = (d,d,\ldots,d) \), and parameters \( I \) and \( \zeta \) being set exactly the same as in the proof for Theorem 7(a) (See Section 3.1 for their values). Output \( (G_\zeta,G,G_0) \) where now \( G = G^{(I)} \). Let \( H_\zeta = G_\zeta \), \( H = G \) and \( H_0 = G_0 \). We do this notation change because we will run a modified version of \( Coupling() \) again in the second stage, and we do not want to
mix the output from the first stage with that from the second stage. As long as procedure $\text{IndSample}()$ is not called, which occurs with very small probability, we have $H_\zeta \subseteq H \subseteq H_0$ (Remember we only run the first $I$ steps of $\text{Coupling}()$). Moreover, $H_\zeta \sim \mathcal{G}(n, p_1)$ and $H_0 \sim \mathcal{G}(n, q_1)$ where $p_1, q_1 = (1 - o(1))d/n$. Call this the stage 1 of the new coupling scheme.

(ii) Let $h$ be the degree sequence of $H$. Let $t = d - h$, and let $\overline{H} = K_n - H$. Let $\overline{H}_t$ be a uniformly random $t$-factor of $\overline{H}$. Then $H \cup \overline{H}_t \sim \mathcal{G}(n, d)$.

(iii) To embed $\overline{H}_t$ inside $\mathcal{G}(n, q_2)$, it is sufficient to embed $\mathcal{G}(n, 1 - q_2)$ inside $H \cup \overline{H}_{t'}$ where $t' = ((n - 1)\mathbf{1} - h) - t$, where $(n - 1)\mathbf{1} - h$ is the degree sequence of $\overline{H}$. We will run a modified version of $\text{Coupling}()$ which, in addition to the set of parameters $t', I, \zeta$ required as in Figure 1, will take another input $H$, which is exactly the output $H$ from stage 1. This modified procedure constructs a $t'$-factor of $\overline{H}$ (instead of a $t$-factor of $K_n$). Two natural modifications that will be made along with this are the following. (a) For each edge $jk$ uniformly sampled from $K_n$ in step $\iiota$, if $jk \in H$, that is, if $jk \notin \overline{H}$, then $jk$ is not going to be added to $G^{(i)}$. It will however always be added to $M^{(i)}_0$ (denoting $G^{(i)}_\zeta \triangleq M^{(i)}_\zeta$ and $G^{(i)}_0 \triangleq M^{(i)}_0$), and with probability $1 - \zeta$ it will be added to $M^{(i)}_\zeta$, just as was in Figure 1 (b). We redefine $\eta^{(i)}_{jk}$ to be

$$\eta^{(i)}_{jk} = 1 - \frac{\mathbb{P}(jk \in \overline{H}_{t'} \mid G^{(\iota)}(\cdot))}{\max_{jk \in \overline{H} - G^{(\iota-1)}} \mathbb{P}(jk \in \overline{H}_{t'} \mid G^{(\iota-1)})},$$

for every $jk \in \overline{H} - G^{(\iota-1)}$.

With these changes, the final output $G$ will have distribution $\overline{H}_{t'}$ instead of $\mathcal{G}(n, t')$. Call this the stage 2 of the new coupling scheme. Note that $G^{(i)}_\zeta \subseteq G^{(i)} \subseteq G^{(i)}_0$ will no longer be maintained. However, the generation of $(G^{(i)}_\zeta, G^{(i)}_0)$ is independent of the triple graphs output in stage 1, since every edge in $K_n$ is added to $M^{(i)}_\zeta$ (and $M^{(i)}_0$) independently and independent of $(H_\zeta, H, H_0)$. Let $(G_\zeta, G, G_0)$ be the output of this modified version of $\text{Coupling}()$. We will show that $G_\zeta \subseteq G \cup H$ if $\text{IndSample}()$ is not called, and $G$ is distributed as $\overline{H}_{t'}$, and $G_\zeta \sim \mathcal{G}(n, 1 - q_2)$. By taking the complement, we have a coupling which embeds $\overline{H}_t$ inside $\mathcal{G}(n, q_2)$.

(iv) It follows now that $\mathcal{G}(n, d)$ can be embedded inside $\mathcal{G}(n, q_1) \cup \mathcal{G}(n, q_2)$ which, by the independence of $\mathcal{G}(n, q_1)$ and $\mathcal{G}(n, q_2)$, is again distributed as $\mathcal{G}(n, p)$ where $p = 1 - (1 - q_1)(1 - q_2) \sim q_1 + q_2$.

(v) Finally by choosing appropriate parameters for the two stages of the coupling procedure we show that $q_1 + q_2 = (1 + o(1))d/n$.

(vi) The key idea behind the success of this new coupling scheme is that, by the end of the first stage of the coupling procedure, we will have $p_1, q_1 = (1 - o(1))d/n$, which means that $\Delta(t)$ is $o(d)$. So even the embedding of $\overline{H}_t$ inside $\mathcal{G}(n, q_2)$ cannot be made “tight” — that is, $q_2n \gg \Delta(t)$ — by the techniques and analysis in [5], it is still possible to make $q_2 = o(d/n)$ and thus completing step (v) above.

3.1 Stage 1: the coupling of $(H_\zeta, H, H_0)$

We use precisely the same choices of parameters as in the proof of Theorem 7(a) in Section 6.1.1 of [4]. That is

$$\xi_1 = o(1), \quad \xi_1 n \geq d \geq \xi_1^{-3} \log n \quad (3)$$

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as $\xi$ in Theorem 7(a) (as we are working on regular $d$ we simply have $\text{rng}(d) = 0$). We will further specify $\xi_1$ later. $I \sim \textbf{Po}(\mu)$ where $\mu$ is the unique solution of

$$(1 - \xi_1) \frac{dn}{2} = p_0 = 1 - e^{-\mu/(\xi_1)}$$

and $\zeta = C\xi_1$ for some sufficiently large constant $C > 0$. The proof in [4, Section 6.1] shows the following properties of $(H_\zeta, H, H_0)$, the output of $\text{Coupling}()$ after the first $I$ steps.

**Lemma 8.** (a) A.a.s. $H_\zeta \subseteq H \subseteq H_0$;

(b) Conditioning on $m$ being the number of edges in $H$, $H$ has the same distribution of the graph obtained by first taking a uniformly random $d$-regular graph $G'$ and then keeping a uniformly random set of $m$ edges of $G'$.

(c) $H_\zeta \sim G(n, p_1)$, where $p_1 = 1 - e^{-\mu(1-\zeta)/\xi_1} = (1 - O(\xi_1))d/n$;

(d) $H_0 \sim G(n, q_1)$, where $q_1 = 1 - e^{-\mu/\xi_1} = (1 - O(\xi_1))d/n$.

**Proof.** Part (a) follows directly by the definition of $\text{Coupling}()$, and by the fact that a.a.s. $\text{Ind}()$ is not called during the first $I$ steps of $\text{Coupling}()$ (It was shown in [4, Section 6.1.2] that the probability of calling $\text{Ind}()$ in each step is $e^{-\Omega(\xi d)}$). Part (b) follows by [4, Lemma 2.2]. Parts (c,d) follow by [4, Lemma 2.1] and the estimates of $p_1$ and $q_1$ in [4, Section 6.1.2] ($p_1$ and $q_1$ were denoted by $p_\zeta$ and $q_\zeta$ respectively in [4]).

Recall that $h$ is the degree sequence of $H$ and $\overline{H} = K_n - H$. By Lemma 8 we may assume that $\overline{H}$ is a random graph sandwiched between $G(n, 1 - q_1)$ and $G(n, 1 - p_1)$. Recall that $t = d - h$. Our next and main task is to couple $\overline{H}_t$ with $G(n, q_2)$ so that $\overline{H}_t \subseteq G(n, q_2)$, and $G(n, q_2)$ is independent of $(H_\zeta, H, H_0)$. To this point, we impose another condition on $\xi_1$:

$$\xi_1 d \geq \log^4 n. \quad (4)$$

Under this condition, by Lemma 8 and a standard concentration argument, a.a.s. $t$ has the following properties:

(P.a) $\Delta(t) = \Theta(\xi_1 d)$;

(P.b) $\text{rng}(t) \leq \Delta(t)^{2/3}$, recalling that $\text{rng}(t) = \Delta(t) + \Delta(-t)$.

Property (P.a) follows directly by the facts that $H_\zeta$, $H_0$ are binomial random graphs, a.a.s. $H_\zeta \subseteq H \subseteq H_0$, and the number of edges in $H_\zeta$ and $H_0$ are both $(1 + O(\xi_1))dn/2$. The proof for (P.b) is indeed almost identical to that of [4, Lemma 6.2(b)], with minor modifications, and we include its proof below.

**Proof of (P.b).** Let $G(n, d, m)$ denote a uniformly random subgraph of $G(n, d)$ with $m$ edges. Let $m$ denote the number of edges in $H$. Then by Lemma 8(b) (see its proof in [4, Lemma 2.2]), $H \sim G(n, d, m)$. Let $\tilde{p} = (M - m)/M$ where $M = \frac{1}{2}dn$. Then by (P.a), $\tilde{p} = \Theta(\xi_1)$. Consider the random graph $\tilde{H}$ obtained by deleting every edge in $G(n, d)$ with probability $\tilde{p}$. Conditioning on $|E(\tilde{H})| = M - m$, $\tilde{H}$ has the same distribution as $H$. By the Chernoff bound, for every $i \in [n],$

$$\P\left(|t_i - \tilde{p}d| \geq \frac{1}{2}\Delta(t)^{2/3}\right) \leq 2 \exp(-\Omega((\tilde{p}d)^{1/3})) \leq \frac{\exp(-\Omega(\log^{4/3} n))}{\Omega((M - m)^{-1/2})} = o(n^{-1}),$$

where the second last equality holds by [4]. Thus, taking the union bound over all $i \in [n]$, a.a.s. $|t_i - \tilde{p}d| \leq \frac{1}{2}\Delta(t)^{2/3}$ and thus, $\text{rng}(t) \leq \Delta(t)^{2/3}$ as desired. \qed
3.2 Coupling $\overline{H}_t$ with $\mathcal{G}(n, q_2)$

Recall that $(H_\zeta, H, H_0)$ is the output of stage 1 of the coupling procedure. Recall also that $t$ has properties (P.a) and (P.b) where $t = d - h$ where $h$ is the degree sequence of $H$. We will prove the following embedding lemma.

**Lemma 9.** Let $d = \omega(\log^7 n)$ and $d = O(n/\sqrt{\log n})$. There exist $\xi_1$ satisfying (3) and (4) and $f > 0$, and a coupling $(G, H)$ such that

(a) $\xi_1 f = o(1)$;

(b) $\tilde{G} \sim \overline{H}_t$ and $\tilde{H} \sim \mathcal{G}(n, q_2)$ where $q_2 = f\xi_1 d/n$;

(c) A.a.s. $\tilde{G} \subseteq \tilde{H}$;

(d) $\tilde{H}$ is independent of $(H_\zeta, H, H_0)$.

3.3 Proofs of Theorem 2 and Lemma 9

We first prove that Lemmas 8 and 9 together imply Theorem 2. In [4] it was shown that Conjecture 1 holds for all $d \gg n/\sqrt{\log n}$. Now, assume $d = \omega(\log^7 n)$ and $d = O(n/\sqrt{\log n})$. Let $\xi_1$, $f$, and the coupling $(G, H)$ be chosen to satisfy Lemma 9(a,b,c,d). Moreover, $H \cup G$ has the same distribution as $\mathcal{G}(n, d)$ by Lemma 9(b). By Lemma 8 a.a.s. $H \subseteq H_0$ where $H_0 \sim \mathcal{G}(n, q_1)$. By Lemma 9(d), $H_0$ and $H$ are independent. Hence, $H_0 \cup H \sim \mathcal{G}(n, p)$ where

$$p = 1 - (1 - q_1)(1 - q_2) \sim q_1 + q_2 = (1 - O(\xi_1))d/n + f\xi_1 d/n = (1 + o(1))d/n,$$

as $f\xi_1 = o(1)$. Thus, we have shown now that there is a coupling of $\mathcal{G}(n, d)$ and $\mathcal{G}(n, p)$ such that a.a.s. $\mathcal{G}(n, d) \subseteq \mathcal{G}(n, p)$. By Lemma 8 there is a coupling of $\mathcal{G}(n, p_1)$, for some $p_1 = (1 - o(1))d/n$, with $\mathcal{G}(n, d)$ such that a.a.s. $\mathcal{G}(n, p_1) \subseteq \mathcal{G}(n, d)$. Then, by “stitching” these two couplings together as done in [4] Proof of Theorem 1.5], there is a coupling of $\mathcal{G}(n, p_1), \mathcal{G}(n, d)$ and $\mathcal{G}(n, p)$ such that a.a.s. $\mathcal{G}(n, p_1) \subseteq \mathcal{G}(n, d) \subseteq \mathcal{G}(n, p)$. Moreover we have $p_1, p = (1 + o(1))d/n$ as argued before. The theorem follows.

It only remains to prove Lemma 9. It follows from the following claim which is almost identical to Theorem 7(c). Recall that $h$ denotes the degree sequence of $H$, and $t = d - h$. By properties (P.a) and (P.b), $t$ is near-regular and $\Delta(t) = \Theta(\xi_1 d)$. Let

$$t' = ((n - 1)1 - h) - t = ((n - 1)1 - h) - (d - h) = (n - 1)1 - d,$$

where $(n - 1)1 - h$ is the degree sequence of $\overline{H}$. Our remaining task is to construct a coupling $(G_\zeta, G)$ where $G_\zeta \sim \mathcal{G}(n, 1 - p_2)$, $G \sim \overline{H}_t$, $G_\zeta \subseteq H \cup G$ and moreover, $G_\zeta$ is independent of $(H_\zeta, H, H_0)$, as reflected by the following lemma.

**Lemma 10.** Assume there are $\xi$ and $\sigma$ such that

$$\frac{\text{rng}(t)}{\Delta(t)} = O(\sigma), \quad \frac{\Delta(t)}{n} \log \frac{n}{\Delta(t)} = o(\sigma\xi), \quad \xi n \gg n^\sigma \gg \frac{\log^3 n}{\log^2 \log n}.$$ 

Then there exist $q = O(\xi)$ and a coupling $(G_\zeta, G)$ where $G_\zeta \sim \mathcal{G}(n, 1 - q)$ and $G \sim \overline{H}_t$ such that a.a.s. $G_\zeta \subseteq H \cup G$, and moreover, $G_\zeta$ is independent of $(H_\zeta, H, H_0)$. 

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Notice the resemblance of this lemma and Theorem 7(c). Indeed they are exactly the same if $H = \emptyset$. Indeed the proof of Lemma 10 is almost identical to that of Theorem 7(c), except for a new natural modifications where the role of $K_n$ is replaced by $\bar{H}$. We will sketch the proof and point out the main differences in the next subsection. We end this subsection by proving that Lemma 10 implies Lemma 9.

Proof of Lemma 9 Let $f$ and $\xi_1$ be specified later, and let $q_2 = f\xi_1 d/n$. By Lemma 10 and by noting that $\Delta(t) = \Theta(\xi_1 d)$ from (P.a), and $\text{rng}(t)/\Delta(t) = O((\xi_1 d)^{-1/3})$ by (P.b), it is sufficient to show that there exist $\xi_1 = o(1), f, \sigma > 0$ satisfying the following set of constraints that are required by [3], [4], Lemma 9(a) and the hypotheses in Lemma 10.

$$\begin{align*}
\xi_1 d &\geq \log^4 n \\
\xi_1 n &\geq d \geq \xi_1^{-3} \log n \\
\xi_1 f &= o(1) \\
(\xi_1 d)^{-1/3} &= O(\sigma) \\
\log \frac{n}{\xi_1 d} &= o(\sigma f) \\
f\xi_1 d &\gg n^{\sigma} \gg \frac{\log^3 n}{\log^2 \log n}.
\end{align*}$$

We discuss $d$ in two cases.

Case 1: $d = \omega(\log^7 n)$ and $d \leq (n^3 \log n)^{1/4}$. For $d$ in this range we have $(\log n / d)^{1/3} \geq d/n$. Choosing $\xi_1 = (\log n / d)^{1/3}, f = \xi_1^{-1} / \log \log n$ and $\sigma = \frac{1}{3} \log d / \log n$, one can easily verify that all conditions above are satisfied. In particular, condition $d = \omega(\log^7 n)$ is used in satisfying (10).

Case 2: $(n^3 \log n)^{1/4} < d = O(n / \sqrt{\log n})$. In this range we have $d/n > (\log n / d)^{1/3}$. Choosing $\xi_1 = d/n, f = \xi_1^{-3/4}$ and $\sigma = 3 \log \log n / \log n$ we can again verify that all the above conditions are satisfied (we use $\sigma > \sqrt{\xi_1 \log(1/\xi_1)}$ to verify (10), $\sigma > 3 \log \log n / \log n$ to verify (9) and the second inequality in (11). For the first inequality in (11), note that the logarithm of the left hand side is at least $(11/16) \log n$ by the range of $\Delta(d)$, whereas the logarithm of the right hand side is $\sigma \log n = o(\log n)$ and thus the inequality follows). This completes the proof of Lemma 9.

3.4 Proof of Lemma 10

Proof. (Sketch) We run the modified version of Coupling(), where $t' = (n - 1)1 - d$ by [5]. The other two parameters $I$ and $\zeta$ will be specified soon. Since edges $jk$ are added to $G^{(i)}_\zeta$ and $G_0^{(i)}$ with probabilities independent of $(H, H, H_0), (G, G_0)$ is independent of $(H_\zeta, H, H_0)$. Obviously, if IndSample() is not called, then in each step $0 \leq i \leq I$, the coupled triple $(G^{(i)}_\zeta, G^{(i)}, G_0^{(i)})$ have the following properties:

(P1) $G^{(i)}_\zeta \subseteq G_0^{(i)}$;

(P2) $G^{(i)}_\zeta \subseteq G^{(i)} \cup H$ (because all edges accepted by $G^{(i)}_\zeta$ but rejected by $G^{(i)}$ must be edges in $H$);

(P3) $G_\zeta = G^{(I)}_\zeta$ and $G_0 = G^{(I)}_0$ are binomial random graphs.

(P4) $(G^{(i)}_\zeta, G_0^{(i)})$ is independent of $(H_\zeta, H, H_0)$.

The parameters $\mu$ and $\zeta$ for the modified Coupling() are set the same as in [4] Section 6.3.1]. Let $\xi$ be a real number satisfying the hypotheses of Lemma 10. Set $I \sim Po(\mu)$ where $\mu$ is the solution to

$$1 - \xi = p_0 = 1 - e^{-\mu/(\zeta^2)},$$

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and set \( \zeta = C \Delta(t)/\xi n \), for some sufficiently large constant \( C > 0 \).

Let \( S = S^{(i)} = \overline{H} - G^{(i)} \) (instead of \( K_n - G^{(i)} \)), \( S' = \overline{H} - G^{(i)}_\zeta \) and \( S'' = \overline{H} - G^{(i)}_0 \). Recall that our goal is to estimate

\[
\Pr(jk \in \overline{H}_t \mid G^{(i)}) = \Pr(jk \not\in S_t) = 1 - \Pr(jk \in S_t).
\]

We briefly summarise the proof for Theorem 7(c) in [4, Section 6.3], and explain why exactly the same analysis works for Lemma 10. The proof for Theorem 7(c) proceeds by proving that

(a) Given that \( t \) is a near-regular degree sequence, with high probability, \( S \) possesses a set of pseudo-random properties listed in [4, Lemma 6.7(A2)-(A4)], and \( t \) is sufficiently sparse, quantified in [4, Lemma 6.7(A1)], compared to \( S \). The proof is done in [4, Lemma 6.8].

(b) Then [4, Lemma 6.7] says that for any \( S \) satisfying these pseudo-random properties, \( \Pr(jk \in S_t) = \Theta(\Delta(t)/\Delta(S)) \).

(c) Combining with (12) for \( \overline{H} = K_n \), we deduce that \( \Pr(jk \in G(n, t') \mid G^{(i)}) = 1 + O(\Delta(t)/\Delta(S)) \) for any \( jk \in S \).

(d) Given the setting of parameters \( \xi \) and \( \mu \), we have \( \Delta(S) = \Omega(\xi n) \) up to step \( I \) (by [4, Lemma 6.8(b2)]). Hence, \( \Pr(jk \in \overline{H}_t \mid G^{(i)}) = 1 + O(\Delta(t)/\xi n) \) for any \( jk \in S \), which is at least \( 1 - \zeta \) by the choice of \( \zeta \).

(e) This allows the coupling up to step \( I \). The conditions on \( \xi \) in the hypotheses of Theorem 7(c) are to ensure that \( t \) is sufficiently sparse (quantified in [4, Lemma 6.7(A1)]) compared to \( S \) by step \( I \). They also serve to guarantee that the probability that \( \text{IndSample()} \) is ever called is \( o(1) \).

If we can prove that in our new coupling scheme, where \( \overline{H} \neq K_n \), the graph \( S \) in each step still possesses exactly the same set of pseudo-random properties, it is a trivial task to verify that all the steps (b)–(e) follow in the same way as in [4].

In Theorem 7(c), i.e. in the case \( \overline{H} = K_n \), the pseudo-randomness of \( S \) is proved by the following two facts: (a) with high probability, the binomial random graphs whose edge density is not too small have these pseudo-random properties; (b) \( S \) is tightly sandwiched by two binomial random graphs (which follows as a corollary of Lemma [4, Lemma 6.3]), and the sandwiching allows to translate the pseudo-random properties from the binomial random graphs to \( S \).

We show that in each step of the new coupling scheme, a.a.s. \( S \) again is tightly sandwiched by two binomial random graphs, and the proof is almost identical to that in [4, Section 6.3]. The reason for that is that \( H \) itself is sandwiched tightly by two binomial random graphs \( H_\zeta \) and \( H_0 \). We skip the tedious quantitative analysis such as the bound on the difference of the edge densities of the two Erdős-Rényi random graphs (uniform random graph with a given number of edges) that sandwich \( S \), as it is the same as in [4, Lemma 6.8]. We only explain why \( S \) is sandwiched by two Erdős-Rényi random graphs. Recall that \( S = S^{(i)} = \overline{H} - G^{(i)} = K_n - (G^{(i)} \cup H) \).

By (P1) and (P2) and the assumption that \( H_\zeta \subset H \subset H_0 \), we have

\[
(G^{(i)}_\zeta \cup H_\zeta) \subset (G^{(i)} \cup H) \subset (G^{(i)}_0 \cup H_\zeta).
\]

By (P4), \( G^{(i)}_\zeta \) and \( H_\zeta \) are independent random graphs, where \( H_\zeta \) is a random binomial graph, and \( G^{(i)}_\zeta \), conditioning on the number of its edges \( m^{(i)}_\zeta \), is distributed as \( G(n, m^{(i)}_\zeta) \). It follows that
\( \left( G_\zeta^{(i)} \cup H_\zeta \right) \) is distributed as \( G(n, m_1) \) conditioning on its number of edges being \( m_1 \). Similarly, 
\( \left( G_0^{(i)} \cup H_\zeta \right) \sim G(n, m_2) \) conditioning on its number of edges being \( m_2 \). Moreover, by the tight
sandwiching \( H_\zeta \subseteq H \subseteq H_0 \) and (P1)–(P2) it can be easily shown that \( m_2 - m_1 \) is small (see the
quantitative statement in [4, Lemma 6.8] where \( S = K_n - (G^{(i)} \cup H), S' = K_n - (G_\zeta^{(i)} \cup H) \) and
\( S'' = K_n - (G_0^{(i)} \cup H) \)). This confirms that \( S \) is sandwiched tightly by two binomial random graphs.

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