On the special form of integral convolution type inequality due to Walter and Weckesser

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Dedicated to Professor Karol Baron on his 70th birthday.

Abstract. Walter and Weckesser’s result (Aequationes Math 46: 212–219, 1993), extending the Bushell–Okrasiński convolution type inequality (Bushell and Okrasiński in J Lond Math Soc (2) 41: 503–510, 1990), gave some general conditions on the functions $k: [0, d) \to \mathbb{R}$ and $g: [0, \infty) \to \mathbb{R}$ under which, for every increasing function $f: [0, d) \to [0, \infty)$, the inequality

$$\int_0^x k(x - s) g(f(s)) \, ds \leq g\left(\int_0^x f(s) \, ds\right), \quad x \in (0, d),$$

is satisfied. Applying the result on a simultaneous system of functional inequalities, we prove that if $d > 1$, then, in general, both $k$ and $g$ must be power functions.

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1. Introduction

Inequality

$$\int_0^x (x - s)^{p-1} f(s) \, ds \leq \left(\int_0^x f(s)^{1/p} \, ds\right)^p, \quad x \in [0, d],$$

(1.1)

where $d \in (0, 1]$, $p \geq 1$ and $f: [0, d] \to [0, \infty)$ is an arbitrary continuous increasing (throughout the paper we use terms increasing and decreasing in the weak sense) function, is called the classical Bushell–Okrasiński inequality (the non-classical forms of (1.1) include its generalizations to various kinds of integrals—for instance, fuzzy integrals like the Sugeno integral [8] or the
universal integral [1]). It was proved in [3] as an auxiliary result in the study of the existence of solutions of some class of Volterra integral equations and almost immediately questions about an extension of (1.1) arose [9]. The first such extension was given by Walter and Weckesser in [10], but also later many other papers were published, cointaing results that in particular lead to the Bushell–Okrasiński inequality (see [4,5,7]).

In the aforementioned article [10] Walter and Weckesser proved the following theorem:

**Theorem 1.** Let for every \( c \in (0, d] \) the function \( h_c \) be defined by

\[
h_c(y) := g(cy) - K(c)g(y), \quad y \in [0, \infty),
\]

where \( K \) is given by \( K(x) := \int_0^x k(s) \, ds \) with \( k \) being Lebesgue integrable on \([0, d)\). If one of the following two conditions is satisfied:

(i) \( f : [0, d] \to [0, \infty) \) is increasing, \( g : [0, \infty) \to \mathbb{R} \) is convex and such that for every \( c \in (0, d] \) the function \( h_c \) is nonnegative and increasing;

(ii) \( f : [0, d] \to [0, \infty) \) is decreasing, \( g : [0, \infty) \to \mathbb{R} \) is concave and such that for every \( c \in (0, d] \) the function \( h_c \) is nonnegative and decreasing,

then

\[
\int_0^x k(x-s)g(f(s)) \, ds \leq g\left(\int_0^x f(s) \, ds\right), \quad x \in (0, d].
\]

It is easy to observe that the Bushell–Okrasiński inequality (1.1) can be obtained from Theorem 1 by taking functions \( g \) and \( k \), both of which are power functions with the exponent \( p \geq 1 \) (it is worth noting that now the assumption \( d \leq 1 \) is no longer needed). Because of the importance of the Bushell–Okrasiński inequality, the natural question arises: under what additional conditions does an inequality in the Walter–Weckesser theorem reduce to the Bushell–Okrasiński inequality? In this paper we give a partial answer to this question. We show that such a reduction is enforced if, in essential, \( d > 1 \) and functions \( g \) and \( K \) are positive, provided that \( K \) also satisfies a certain inequality. The main tool that we use to prove this is a certain result on the solutions of the simultaneous system of functional inequalities stated in the next section as Theorem 2.

2. Auxiliary results

**Theorem 2.** ([6], Theorem 4) Let \( a, b, \alpha, \beta \in (0, \infty) \) be such that

\[
0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q},
\]

\[
\frac{\log \beta}{\log b} \leq \frac{\log \alpha}{\log a},
\]

\[
\text{for every } c \in (0, d] \text{ the function } h_c \text{ is nonnegative and increasing.}
\]
where \( \mathbb{Q} \) denotes the set of rational numbers. Suppose that a function \( g : (0, \infty) \to \mathbb{R} \) satisfies the system of inequalities

\[
g(ax) \leq \alpha g(x), \quad g(bx) \leq \beta g(x), \quad x > 0,
\]

\( g \) is continuous at least at one point, and \( g((0, \infty)) \subsetneq (-\infty, 0) \). Then either

\[
g(x) = 0, \quad x > 0,
\]

or

\[
p := \frac{\log \beta}{\log b} = \frac{\log \alpha}{\log a}
\]

and

\[
g(x) = g(1)x^p, \quad x > 0.
\]

The suitable result for the reversed inequalities reads as follows.

**Theorem 3.** Let \( a, b, \alpha, \beta \in (0, \infty) \) be such that

\[
0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q},
\]

\[
\frac{\log \beta}{\log b} \geq \frac{\log \alpha}{\log a}.
\]

Suppose that a function \( g : (0, \infty) \to \mathbb{R} \) satisfies the system of inequalities

\[
g(ax) \geq \alpha g(x), \quad g(bx) \geq \beta g(x), \quad x > 0,
\]

\( g \) is continuous at least at one point, and \( g((0, \infty)) \subsetneq (-\infty, 0) \). Then either

\[
g(x) = 0, \quad x > 0,
\]

or

\[
p := \frac{\log \beta}{\log b} = \frac{\log \alpha}{\log a}
\]

and

\[
g(x) = g(1)x^p, \quad x > 0.
\]

Let \( d > 0 \) be arbitrarily fixed. In order to directly use Theorems 2 and 3, we must relax some of the assumptions about the domains \( g \) and \( K \) made in Theorem 1, so in this section let \( g : (0, \infty) \to [0, \infty), k \) be a Lebesgue integrable function such that \( K : (0, d) \to (0, \infty) \), where

\[
K(x) := \int_0^x k(s) \, ds, \quad (2.1)
\]

and let a bivariable function \( h : (0, d) \times (0, \infty) \to \mathbb{R} \) be defined by the following formula

\[
h(x, y) := g(xy) - K(x)g(y), \quad x \in (0, d), \; y \in (0, \infty). \quad (2.2)
\]

Applying Theorems 2 and 3, we show now that the constant sign of \( h \) implies that \( g \) must be a power function. Namely, we have the following
Theorem 4. Let a bivariable function \( h : (0, d) \times (0, \infty) \to \mathbb{R} \) be given by (2.2), where \( g : (0, \infty) \to [0, \infty) \) is continuous at a point and \( K : (0, d) \to (0, \infty) \).

If \( 1 < d \leq \infty \), and one of the following two conditions is satisfied:

(i) the function \( h \) is nonnegative and there exist \( a, b \in (0, d) \) such that

\[
0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad \frac{\log K(b)}{\log b} \geq \frac{\log K(a)}{\log a},
\]

(ii) the function \( h \) is nonpositive and there exist \( a, b \in (0, d) \) such that

\[
0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad \frac{\log K(b)}{\log b} \leq \frac{\log K(a)}{\log a},
\]

then either

\[ g(y) = 0, \quad y > 0, \]

or \( g \) is positive,

\[ \frac{\log K(b)}{\log b} = \frac{\log K(a)}{\log a}, \]

and

\[ g(y) = g(1)y^p, \quad y \in (0, \infty), \]

where

\[ p := \frac{\log K(a)}{\log a}. \]

Proof. Assume that (i) holds true. Putting

\[ \alpha := K(a), \quad \beta := K(b) \]

and using the nonnegativity of \( h \), we notice that \( g \) satisfies the following system of inequalities

\[ g(ay) \geq \alpha g(y), \quad g(by) \geq \beta g(y), \quad y > 0. \]

Because \( g \) is continuous at a point, Theorem 3 implies the result.

In case (ii), applying Theorem 2, we argue analogously. \( \square \)

This theorem does not give any specific information about \( K \). It turns out, however, that if \( g \) is positive and \( K \) satisfies an additional condition, which is quite natural in the context of the inequality \( \frac{\log K(b)}{\log b} \geq \frac{\log K(a)}{\log a} \) in (i) or its converse in (ii), then both \( g \) and \( K \) must be power functions of the same exponent. Namely, we have the following

Theorem 5. Let a bivariable function \( h : (0, d) \times (0, \infty) \to \mathbb{R} \) be given by (2.2), where \( g : (0, \infty) \to [0, \infty) \) is continuous at a point and \( K : (0, d) \to (0, \infty) \) is continuous.

If \( 1 < d \leq \infty \), and one of the following two conditions is satisfied:
(i) the function \( h \) is nonnegative and

\[
\sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} \leq \inf \left\{ \frac{\log K(t)}{\log t} : t \in (1, d) \right\};
\]

(ii) the function \( h \) is non positive and

\[
\inf \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} \geq \sup \left\{ \frac{\log K(t)}{\log t} : t \in (1, d) \right\};
\]

then either

\( g(y) = 0, \quad y > 0, \)

or \( g \) is positive, for some real \( p \)

\( g(y) = g(1)y^p \) for all \( y \in (0, \infty) \); \( K(x) = x^p \) for all \( x \in (0, d) \),

and \( h \) is equal to zero, i.e.,

\( g(xy) = K(x)g(y), \quad x \in (0, d), \; y \in (0, \infty). \)

Proof. Assume that condition (i) holds true. For an arbitrarily fixed number \( a \in (0, 1) \), choose \( b \in (1, d) \) such that \( \log a \) and \( \log b \) are incommensurable and put

\( \alpha := K(a), \quad \beta := K(b). \)

The assumed nonnegativity of \( h \) implies that \( g \) satisfies the simultaneous system of inequalities

\( g(ay) \geq \alpha g(y), \quad g(by) \geq \beta g(y), \quad y > 0. \)

Moreover, since \( 0 < a < 1 < b < d \), we have

\[
\frac{\log \alpha}{\log a} = \frac{\log K(a)}{\log a} \leq \frac{\log K(b)}{\log b} = \frac{\log \beta}{\log b}.
\]

Since \( g \) is continuous at a point, in view of Theorem 3, either \( g \equiv 0 \) or

\( g(y) = g(1)y^p, \quad y > 0, \)

where

\( p := \frac{\log \alpha}{\log a} = \frac{\log \beta}{\log b}. \)

To find the form of \( K \) in the latter case, notice that from the definitions of \( \alpha \), \( \beta \) and \( p \), we have

\[
\frac{\log K(a)}{\log a} = \frac{\log K(b)}{\log b} = p,
\]

whence

\( K(a) = a^p \quad \text{and} \quad K(b) = b^p. \)

Since the definition of \( p \) does not depend on \( b \), and the set of the numbers \( b \) such that \( \frac{\log b}{\log a} \notin \mathbb{Q} \) is dense in \((1, d)\), the continuity of \( K \) implies that \( K(x) = x^p \) for
all \( x \in [1,d) \). Interchanging the roles of \( a \) and \( b \) in this reasoning we conclude that \( K(x) = x^p \) for all \( x \in (0,1) \). Thus we have shown that
\[
K(x) = x^p, \quad x \in (0,d).
\]
Now, for all \( x \in (0,d) \) and \( y > 0 \), making use of the definition of \( h \), we get
\[
h(x,y) = g(xy) - K(x)g(y) = q(xy)^p - x^py^p = 0,
\]
which shows that
\[
g(x) = K(x)g(y), \quad x \in (0,d), y > 0.
\]
In case (ii), the simultaneous system of inequalities is reversed, so applying Theorem 2, we can argue similarly. This completes the proof. \( \square \)

**Remark 1.** In Theorem 5 condition (i), which is equivalent to the implication:
for all \( s, t, 0 < s < 1 < t < d \),
\[
0 < s < 1 < t < d \implies \frac{\log K(t)}{\log t} \geq \frac{\log K(s)}{\log s},
\]
is satisfied, if the function
\[
[(-\infty, \log d) \setminus \{0\}] \ni t \mapsto [K(\exp t)]^{1/t}
\]
is increasing. Similarly, condition (ii), which is equivalent to the implication:
for all \( s, t, 0 < s < 1 < t < d \),
\[
0 < s < 1 < t < d \implies \frac{\log K(t)}{\log t} \leq \frac{\log K(s)}{\log s},
\]
is satisfied, if the function
\[
[(-\infty, \log d) \setminus \{0\}] \ni t \mapsto [K(\exp t)]^{1/t}
\]
is decreasing.

3. Main results

From now on, following [10], we assume that functions \( g \) and \( K \) are defined on larger intervals, i.e. on \([0,\infty)\) and \((0,d]\), respectively. Based on Theorems 4 and 5, we obtain certain refinements of Theorem 1. To be more precise, with some additional assumptions about \( K \) (the same ones as in Theorems 4 and 5, respectively) we find all admissible forms of the nonnegative function \( g \) in Theorem 1, provided that \( d > 1 \).

**Corollary 1.** Assume that \( 1 < d < \infty \). Let \( g : [0,\infty) \rightarrow [0,\infty) \) be convex, \( K : (0,\infty) \rightarrow (0,\infty) \), and for every \( c \in (0,d] \) let the function \( h_c \) defined by (1.2) be nonnegative and increasing.

(i) If there exist \( a, b \in (0,d] \) such that
\begin{align*}
0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad 1 \leq \frac{\log K(a)}{\log a} \leq \frac{\log K(b)}{\log b},
\end{align*}
then either
\begin{align*}
g \equiv 0,
\end{align*}
or \( g \) is positive on \((0, \infty)\) and
\begin{align*}
g(y) = g(1)y^{p}, \quad y \in [0, \infty),
\end{align*}
where
\begin{align*}
p = \frac{\log K(a)}{\log a} = \frac{\log K(b)}{\log b} \geq 1.
\end{align*}

(ii) If there exist \( a, b \in (0, d] \) such that
\begin{align*}
0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad 0 \leq \frac{\log K(a)}{\log a} \leq \frac{\log K(b)}{\log b} < 1,
\end{align*}
then
\begin{align*}
g \equiv 0.
\end{align*}

\textbf{Proof.} (i) The function \( g \), being a convex function, is in particular continuous in \((0, \infty)\). The nonnegativity of \( h_c \) and Theorem 4 imply that either \( g \) is zero in \((0, \infty)\) or \( g(y) = g(1)y^{p} \) for all \( y > 0 \), with \( p = \frac{\log K(a)}{\log a} = \frac{\log K(b)}{\log b} \geq 1 \). Moreover, the extension of \( g \) to the interval \([0, \infty)\) by putting \( g(0) := \alpha \), \( \alpha \geq 0 \), preserves its convexity. To finish the proof, we have to show that such extensions preserve the nonnegativity and the increasingness of \( h_c \). When \( g \equiv 0 \) on \((0, \infty)\), then, for all \( c \in (0, d] \),
\begin{align*}
h_c(0) = g(0) - K(c)g(0) = \alpha(1 - K(c)) \quad \text{and} \quad h_c(y) = 0, \quad y > 0,
\end{align*}
hence \( \alpha = 0 \), as \( h_c(0) \leq h_c(y) \) for all \( y > 0 \). We obtain the same value of \( \alpha \) for the second type of function \( g \), i.e. \( g(y) = g(1)y^{p} \) for \( y \in (0, \infty) \). In this case we have
\begin{align*}
h_c(0) = \alpha(1 - K(c)) \quad \text{and} \quad h_c(y) = g(1)y^{p}(c^{p} - K(c)), \quad c \in (0, d], \quad y > 0.
\end{align*}
As \( \lim_{y \to 0^+} h_c(y) = 0 \), the increasingness of \( h_c \) implies that \( \alpha = 0 \).

(ii) In this case a similar argument works. The difference in the thesis comes from the fact that the convexity of power functions fails when \( p \in (0, 1) \). \( \square \)

A similar argument in combination with Theorem 5 gives

\textbf{Corollary 2.} Assume that \( 1 < d < \infty \). Let \( g : [0, \infty) \to [0, \infty) \) be convex, \( K : (0, d] \to (0, \infty) \) be continuous and such that
\begin{align*}
\sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} \leq \inf \left\{ \frac{\log K(t)}{\log t} : t \in (1, d) \right\},
\end{align*}
and for every \( c \in (0, d] \) let the function \( h_c \) be nonnegative and increasing.
(i) If
\[ \sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} > 1, \]
then either
\[ g \equiv 0, \]
or \( g \) is positive on \((0, \infty)\) and for some real \( p > 1 \)
\[ g(y) = g(1)y^p \text{ for all } y \in (0, \infty), \quad \text{and} \quad K(x) = x^p \text{ for all } x \in (0, d]. \]

(ii) If
\[ \sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} < 1, \]
then
\[ g \equiv 0. \]

If \( g \) is concave, the corresponding results read as follows:

**Corollary 3.** Assume that \( 1 < d < \infty \). Let \( g : [0, \infty) \to [0, \infty) \) be concave, \( K : (0, d] \to (0, \infty) \), and for every \( c \in (0, d] \) let the function \( h_c \) be nonnegative and decreasing.

(i) If there exist \( a, b \in (0, d] \) such that
\[ 0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad 1 < \frac{\log K(a)}{\log a} \leq \frac{\log K(b)}{\log b}, \]
then
\[ g \equiv 0. \]

(ii) If there exist \( a, b \in (0, d] \) such that
\[ 0 < a < 1 < b, \quad \frac{\log b}{\log a} \notin \mathbb{Q}, \quad 0 < \frac{\log K(a)}{\log a} \leq \frac{\log K(b)}{\log b} \leq 1, \]
then either
\[ g \equiv 0, \]
or \( g \) is positive on \((0, \infty)\) and
\[ g(y) = g(1)y^p, \quad y \in [0, \infty), \]
where
\[ p = \frac{\log K(a)}{\log a} = \frac{\log K(b)}{\log b} \leq 1. \]

**Corollary 4.** Assume that \( 1 < d < \infty \). Let \( g : [0, \infty) \to [0, \infty) \) be concave, \( K : (0, d] \to (0, \infty) \) be continuous and such that
\[ \sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} \leq \inf \left\{ \frac{\log K(t)}{\log t} : t \in (1, d) \right\}, \]
and for every \( c \in (0, d] \) let the function \( h_c \) be nonnegative and decreasing.
(i) If
\[ \sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} > 1, \]
then
\[ g \equiv 0. \]

(ii) If
\[ \sup \left\{ \frac{\log K(t)}{\log t} : t \in (0, 1) \right\} < 1, \]
then either
\[ g \equiv 0, \]
or \( g \) is positive on \((0, \infty)\) and for some real \( p < 1 \)
\[ g(y) = g(1)y^p \quad \text{for all} \quad y \in (0, \infty), \quad \text{and} \quad K(x) = x^p \quad \text{for all} \quad x \in (0, d). \]

An immediate consequence of Corollary 1 is the following useful result showing that if in the Walter–Weckesser theorem \( d > 1 \) and \( k \) is taken in such a way that \( K \) is a power function, then the number of possible convex functions \( g \) is quite limited.

**Corollary 5.** Assume that \( 1 < d < \infty \). Let \( K(x) = x^p \) for \( x \in (0, d] \), \( g : [0, \infty) \to [0, \infty) \) be convex, and for every \( c \in (0, d] \) let the function \( h_c \) be nonnegative and increasing.

(i) If \( p \geq 1 \), then either
\[ g(y) = g(1)y^p, \quad g(1) > 0, \]
or
\[ g \equiv 0. \]

(ii) If \( p \in (0, 1) \), then
\[ g \equiv 0. \]

**Proof.** Choose \( a, b \) such that \( 0 < a < 1 < b \) and for which \( \log a \) and \( \log b \) are incommensurable. The form of \( K \) implies that \( \frac{\log K(a)}{\log a} = \frac{\log K(b)}{\log b} = p \), so the application of Corollary 1 ends the proof. \( \square \)

**Remark 2.** The assumption \( d > 1 \) in Corollary 5 is essential. If \( d \in (0, 1] \), then for \( K(x) = x^p \), there exist non-power convex functions \( g \), for which \( h_c \) is nonnegative and increasing. An example of such a function is
\[ g(y) = \frac{y^p}{y+1}, \]
where \( p \geq 2 \) is arbitrarily fixed. Indeed, we have
\[ h_c(y) = \frac{y^{p+1}(c^p - c^{p+1})}{(cy + 1)(y+1)}, \quad y \in [0, \infty), \quad c \in (0, d], \]
and, as \( c^p > c^{p+1} \), the function \( h_c \) is nonnegative. To check its increasingness, notice that
\[ h'_c(y) = \frac{(1-c)c^p y^p (p(y+1)(cy + 1) - cy^2 + 1)}{(cy + 1)^2(y+1)^2} \geq 0, \]
iff
\[ p(y + 1)(cy + 1) - cy^2 + 1 \geq 0, \quad y \in [0, \infty), \quad c \in (0, d]. \tag{3.1} \]

But inequality (3.1) is equivalent to
\[ (pc - c)y^2 + (p + pc)y + p + 1 \geq 0, \quad y \in [0, \infty), \quad c \in (0, d], \]

and its validity can be easily deduced from the fact that for \( y \in [0, \infty), \ c \in (0, d] \) the inequality \( y^2(pc - c) + y(p + pc) \geq 0 \) holds true.

An analogous result for concave functions follows from Corollary 3.

**Corollary 6.** Assume that \( 1 < d < \infty \). Let \( K(x) = x^p \) for \( x \in (0, d] \), \( g : [0, \infty) \to [0, \infty) \) be concave, and for every \( c \in (0, d] \) let the function \( h_c \) be nonnegative and decreasing.

(i) If \( p > 1 \), then

\[ g \equiv 0. \]

(ii) If \( p \in (0, 1] \), then either

\[ g(y) = g(1)y^p, \quad g(1) > 0, \]

or

\[ g \equiv 0. \]

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