BRST Quantisation of Histories Electrodynamics

Duncan Noltingk*

Blackett Laboratory
Imperial College
Prince Consort Road
London SW7 2BZ

November 4, 2018

Abstract

This paper is a continuation of earlier work where a classical history theory of pure electrodynamics was developed in which the history fields have five components. The extra component is associated with an extra constraint, thus enlarging the gauge group of histories electrodynamics. In this paper we quantise the classical theory developed previously by two methods. Firstly we quantise the reduced classical history space, to obtain a reduced quantum history theory. Secondly we quantise the classical BRST-extended history space, and use the BRST charge to define a ‘cohomological’ quantum history theory. Finally we show that the reduced history theory is isomorphic (as a history theory) to the cohomological history theory.

*e-mail d.noltingk@ic.ac.uk
1 Introduction

The history projection operator (HPO) approach to consistent histories was inspired by Isham [1], and developed by Isham and collaborators [2, 4]. An HPO theory is concerned with projection operators on the quantum history Hilbert space $\mathcal{E}$ which represent propositions about the entire *history* of the system under consideration. This should be contrasted with standard quantum logic which is concerned with propositions about the system at a particular instant of time. The quantum history theories with which this paper will be concerned are described by a pair $(\mathcal{P}(\mathcal{E}), \mathcal{D})$ where $\mathcal{P}(\mathcal{E})$ is the lattice of projection operators on the Hilbert space $\mathcal{E}$, and $\mathcal{D}$ is the space of decoherence functionals. A decoherence functional is a map $d : \mathcal{P}(\mathcal{E}) \times \mathcal{P}(\mathcal{E}) \to \mathbb{C}$ that satisfies the following conditions:

1. **Hermiticity:** $d(\alpha, \beta) = d(\beta, \alpha)^*$ for all $\alpha, \beta \in P(\mathcal{E})$.
2. **Positivity:** $d(\alpha, \alpha) \geq 0$ for all $\alpha \in P(\mathcal{E})$.
3. **Null Triviality:** $d(0, \alpha) = 0$ for all $\alpha \in P(\mathcal{E})$.
4. **Additivity:** if $\alpha \perp \beta$ then, $d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma)$, for all $\gamma \in P(\mathcal{E})$.
5. **Normalisation:** $d(1, 1) = 1$.

The off-diagonal components of the decoherence functional represent the ‘quantum interference’ between histories, while the diagonal components are interpreted as the probability that a particular history ‘occurs’.

In this paper we construct a quantum history theory of pure electrodynamics by two methods. In section 2 we quantise the reduced classical history space to obtain a reduced history theory $(\mathcal{P}(\mathcal{E}^{\text{red}}), d^{\text{red}})$. In section 3 we augment the history fields with ghost fields and quantise the extended theory to obtain a representation of the extended algebra on the BRST-extended history space $\mathcal{E}$. We then define $H^*(\Omega)$, the projection operator cohomology, and show that it has a natural lattice structure. In section 4 we define $\mathcal{D}_{gf}$, the space of gauge-fixed decoherence functionals, and show that each gauge-fixed decoherence functional induces a functional $\tilde{d}$ on $H^*(\Omega)$. Our main result is to show that the cohomological history theory $(H^*(\Omega), \tilde{d})$ is isomorphic (as a history theory) to the reduced history theory $(\mathcal{P}(\mathcal{E}^{\text{red}}), d^{\text{red}})$. 
2 Radiation Gauge Quantisation

2.1 Preliminaries

The quantum history space arises as the representation space of a certain Lie group, the history group, and the associated Lie algebra is the histories analogue of the canonical commutation relations. In the histories approach to scalar field theory proposed by Savvidou [7], there is an inequivalent representation of the history group for each Lorentzian foliation of space-time. By a Lorentzian foliation of space-time we mean a foliation in which each leaf is a space-like hyperplane. The Schrödinger picture fields satisfy the covariant history algebra

\[ \left[ \hat{\phi}_n(X), \hat{\phi}_n(X') \right] = 0 \] (1)
\[ \left[ \hat{\pi}_n(X), \hat{\pi}_n(X') \right] = 0 \] (2)
\[ \left[ \hat{\phi}_n(X), \hat{\pi}_n(X') \right] = i\hbar\delta^{(4)}(X - X'), \] (3)

where \( n \) is a future pointing time-like unit vector labelling a particular Lorentzian foliation. The fields are genuine space-time fields under the action of a representation of the Poincare group [7]. This Poincare group acts

\[ \hat{\phi}_n(X) \mapsto \hat{\phi}_{\Lambda n}(\Lambda X) \] (4)

and generates changes in the space-time foliation. Heisenberg picture fields can be defined using the time-averaged Hamiltonian which is also foliation dependent. These Heisenberg picture fields are of the form \( \hat{\phi}_n(X, s) \), and there is a second representation of the Poincare group in which the boosts act in the ‘internal’ time direction \( s \), and leave the foliation fixed.

In a previous paper [11] we considered the extension of the classical analogue of the above theory to the case of electrodynamics. It was argued that in order to preserve the two representations of the Poincare group, the history fields should have five components as opposed to the usual four. The extra component is associated with the internal time direction. It is important to note that the theory is not covariant under the action of the \( SO(3,2) \) isometry group of the space-time manifold \( \mathcal{M} \times \mathbb{R} \), but only under the internal and external \( SO(3,1) \) subgroups. It was also shown in [11] how the extra component leads to an extra constraint, and thus to an enlarged gauge group.
The theory in [11] is concerned with history configuration fields $A^n_M(X)$, and their canonical momenta $E^n_M(X)$. The index $M$ runs from 0 to 4, $X$ is a four-vector and $n$ is a future-pointing time-like unit vector which labels a particular foliation of space-time. These fields can be considered as vectors tangent to the space of embeddings of $\mathcal{M}$ into $\mathcal{N} \simeq \mathcal{M} \times \mathbb{R}$. The history fields satisfy the Poisson algebra

\begin{align}
\{A^n_M(X), A^n_N(Y)\} &= 0 \quad (5) \\
\{E^n_M(X), E^n_N(Y)\} &= 0 \quad (6) \\
\{A^n_M(X), E^n_N(Y)\} &= \delta^N_M \delta^{(4)}(X - Y). \quad (7)
\end{align}

The four-vector $n$ can be embedded in $\mathcal{N}$, resulting in a five-vector $\tilde{n}^M$ given in coordinates by $(n, 0)$. We also have the five-vector $\tilde{e}^M$ given by $(0, 1)$, and we use these vectors to decompose the fields into their temporal components

\begin{align}
A^n_t(X) := \tilde{n}^M A^n_M(X), \quad A^n_s(X) := \tilde{e}^M A^n_M(X) \quad (8)
\end{align}

and similarly for the momentum field. The $n$-spatial projection tensor is defined as

\[ n^P_M = \delta^M_N - \tilde{e}^M \tilde{e}_N - \tilde{n}^M \tilde{n}_N, \quad (9) \]

and can be used to decompose fields on the five-dimensional space-time into their ‘$n$-spatial’ components

\begin{align}
^nA^M_M(X) := n^P_M A^n_N(X). \quad (10)
\end{align}

We define ‘$n$-spatial tensors’ in a similar way, e.g.,

\begin{align}
^nF_{MN}(X) := n^P_M n^P_N F^n_{RS}(X). \quad (11)
\end{align}

The first-class constraints can now be written as

\begin{align}
E^n_s(X) &\approx 0 \quad (12) \\
E^n_t(X) &\approx 0 \quad (13) \\
^nE^M_M(X) &\approx 0, \quad (14)
\end{align}

where the longitudinal component of the electric field is defined by

\begin{align}
^nE\parallel(X) := n^\partial_M^nE^M(X). \quad (15)
\end{align}
2.2 Reduced State Space

We now augment the constraints with the histories radiation gauge conditions:

\[
\begin{align*}
A^\parallel_n(X) &= 0 \\
A^{\perp}_n(X) &= 0 \\
^nA^{\parallel}(X) &= 0
\end{align*}
\]

(16) - (18)

where

\[
^nA^{\parallel}(X) := \frac{n\partial^M nA_M(X)}{\triangle_n}
\]

(19)

and \(\triangle_n = n\partial_M n\partial^M\). The six equations (12) - (14) and (16) - (18) form a second class set of constraints and we can follow the usual procedure to find the Dirac brackets of the reduced history space \(C_n\). In terms of the transverse fields

\[
^nA^{\perp}_M(X) := nA_M(X) - n\partial_M^nA^{\parallel}(X)
\]

(20)

\[
^nE^{\perp}_M(X) := nE^M(X) - n\partial^M^nE^{\parallel}(X)
\]

(21)

they turn out to be

\[
\begin{align*}
\{nA^{\perp}_M(X), nA^{\perp}_N(Y)\}_D &= 0 \\
\{nE^{\perp}_M(X), nE^{\perp}_N(Y)\}_D &= 0 \\
\{nA^{\perp}_M(X), nE^{\perp}_N(Y)\}_D &= (nP^N_M - \triangle_n^{-1}n\partial_M n\partial^N)\delta^{(4)}(X - Y).
\end{align*}
\]

(22) - (24)

The right hand side of this algebra has become explicitly foliation dependent with the non-covariant gauge choice. In the radiation gauge the time-averaged Hamiltonian is

\[
H^0_n = \int d^4X \left( \frac{1}{2} nE^{\perp}_M nE^M_{\perp} + \frac{1}{4} nF_{MN} nF^{MN} \right),
\]

(25)

where \(F^M_{MN} = 2\partial_M A^N_N\) and \(nF_{MN}\) is the corresponding \(n\)-spatial tensor (cf. equation (11)).
2.3 Quantisation

We wish to find an irreducible representation of the commutator algebra

\[
\begin{align*}
\left[ n \hat{A}_M^\perp(X), \hat{A}_N^\perp(Y) \right] & = 0 \quad (26) \\
\left[ n \hat{E}_M^\perp(X), \hat{E}_N^\perp(Y) \right] & = 0 \quad (27) \\
\left[ n \hat{A}_M^\perp(X), \hat{E}_N^\perp(Y) \right] & = i\hbar (n P_M^n - \Delta_n^{1-n} n \partial_M n \partial_N) \delta^{(4)}(X - Y) \quad (28)
\end{align*}
\]

on a Hilbert space such that the radiation gauge quantum Hamiltonian is represented by a self-adjoint operator. The self-adjointness condition is required to select one of the infinitely many unitarily inequivalent representations of the infinite dimensional algebra (26) - (28). Such a representation exists on the bosonic Fock space \( \mathcal{E}_{\text{red}} := \mathcal{F}_B[L^2(\mathbb{R}^4)] \otimes \mathcal{F}_B[L^2(\mathbb{R}^4)] \). This space is associated with annihilation and creation operators which obey the following algebra

\[
\begin{align*}
[\hat{a}_a(X), \hat{a}_b(X')] & = 0 \quad (29) \\
\left[ \hat{a}_a(X), \hat{a}_b^\dagger(X') \right] & = \hbar \delta_{ab} \delta^{(4)}(X - X'), \quad (30)
\end{align*}
\]

for \( a = 1, 2 \). Using the Fourier transformed operators:

\[
\begin{align*}
\hat{a}_a^\dagger(K) & = \frac{1}{(2\pi)^2} \int d^4X \hat{a}_a^\dagger(X) e^{iK \cdot X} \quad (31) \\
\hat{a}_a(K) & = \frac{1}{(2\pi)^2} \int d^4X \hat{a}_a(X) e^{-iK \cdot X}, \quad (32)
\end{align*}
\]

we define field operators satisfying the algebra (26) - (28) in the following way

\[
\begin{align*}
n \hat{A}_M^\perp(X) & = \frac{1}{(2\pi)^2} \sum_{a=1}^{2} \int \frac{d^4K}{\sqrt{2\omega_n(K)}} n e_a^M(K)[\hat{a}_a(K) e^{-iK \cdot X} + \hat{a}_a^\dagger(K) e^{iK \cdot X}] \quad (33) \\
n \hat{E}_M^\perp(X) & = \frac{1}{i(2\pi)^2} \sum_{a=1}^{2} \int d^4K \sqrt{\omega_n(K)} \frac{1}{2} n e_a^M(K)[\hat{a}_a(K) e^{-iK \cdot X} - \hat{a}_a^\dagger(K) e^{iK \cdot X}] \quad (34)
\end{align*}
\]

In the above expressions, \( K \) is a four-vector representing the four-momentum of a photon and \( \omega_n(K) \) is the modulus of the \( n \)-spatial four vector \( (\delta^\mu_c - \).
\(n^\mu n_\nu)K^\nu\) with respect to the Minkowski metric. For each \(K\), we define the five-vector \(\tilde{K}\) to be the embedding of \(K\) into the five-dimensional space-time \(N\). In coordinates \((X, s)\) the vector \(\tilde{K}\) can be written as \((K, 0)\). The five-vectors \(n^a M \epsilon_a(K)\) are a pair of mutually orthogonal, \(n\)-spatial unit vectors which are in addition orthogonal to the vector \(n^M \tilde{K}^N = n^P M \tilde{K}^N\). These vectors satisfy the following completeness relation:

\[
\sum_{a=1}^{2} n^a M \epsilon_a(K) n^b M \epsilon_b(K) = n^P M \tilde{K}^P M \tilde{K}^N \omega_n(K)^2.
\]

This property ensures that the algebra defined by equations (26) - (28) is satisfied. Using the fact that the polarisation vectors are orthonormal

\[
n^a M \epsilon_a(K) n^b M \epsilon_b(K) = \delta^b_a,
\]

the normal-ordered time-averaged Hamiltonian can be written

\[
\hat{H}_n^0 = \sum_{a=1}^{2} \int d^4K \omega_n(K) \hat{a}_a^\dagger(K) \hat{a}_a(K).
\]

This Hamiltonian generates translations in internal time, and it is easy to see that

\[
e^{-is\hat{H}_n^0/\hbar} \hat{a}_a(K) e^{is\hat{H}_n^0/\hbar} = e^{is\omega_n(K)} \hat{a}_a(K).
\]

Following the argument in [4], these transformations are unitarily implementable and we conclude that \(\hat{H}_n^R\) exists as a self-adjoint operator in this representation. Therefore, for each \(n\), there exists a unitarily inequivalent representation of the radiation gauge history algebra on the Fock space \(\mathcal{E}^{red}\).

### 2.4 The Decoherence Functional

In the case when history propositions are realised as the lattice of projection operators on a Hilbert space \(\mathcal{V}\), every decoherence functional \(d\) can be written in the form [3]

\[
d(\alpha, \beta) = Tr_{\mathcal{V} \otimes \mathcal{V}} (\alpha \otimes \beta \Theta_d),
\]

where \(\Theta_d\) is an operator on \(\mathcal{V} \otimes \mathcal{V}\). In fact \(\Theta_d\) must satisfy certain conditions for \(d\) to be a decoherence functional [3]. In the case of the scalar field, the operator \(\Theta\) is dependent on the foliation, and can be written [7],

\[
\Theta_n = \langle 0 | \rho_{-\infty} | 0 \rangle (SU)_n^\dagger \otimes (SU)_n.
\]
The quantum history space for the scalar field is the bosonic Fock space $\mathcal{F}_B[L^2(\mathbb{R}^4)]$. To each operator $O$, on the base space $L^2(\mathbb{R}^4)$, there is an associated operator on $\mathcal{F}_B[L^2(\mathbb{R}^4)]$ defined by

$$\Gamma(O) = O \oplus (O \otimes O) \oplus \cdots.$$ (41)

Using this construction, the operator $(SU)_n$ can be written

$$(SU)_n = \Gamma(1 + i\sigma_n),$$ (42)

where $\sigma_n = n^\mu \partial_\mu + (-\Delta_n + m^2)^{1/2}$ is an operator on the base Hilbert space $L^2(\mathbb{R}^4)$. The operator $\sigma_n$ is related to the canonical history action $S_n$ in the following simple way

$$e^{isS_n} = \Gamma(e^{is\sigma_n}).$$ (43)

Equation (43) can be used to define the decoherence functional corresponding to a given action operator $S$ on a Fock space $\mathcal{F} = \mathcal{F}_B[\mathcal{H}]$. Firstly $S$ defines $\sigma$, the ‘generator’ of the decoherence functional, which is the operator on $\mathcal{H}$ given by

$$e^{isS} = \Gamma(e^{is\sigma}).$$ (44)

Now the decoherence functional is defined by the operator on $\mathcal{F} \otimes \mathcal{F}$ given by

$$\Theta_S \equiv \langle 0 | \rho_{-\infty} | 0 \rangle \Gamma(1 + i\sigma)^\dagger \otimes \Gamma(1 + i\sigma).$$ (45)

The reduced canonical history action of electrodynamics in the radiation gauge is

$$S_n^{red} = \int d^4X \left( n^\mu \hat{E}_\mu^M \partial_t^n \hat{A}_M^\perp - \hat{H}^0_n \right),$$ (46)

where $\partial_t^n := \hat{n}^M \partial_M$. The corresponding generator of the decoherence functional is the operator $\sigma_n^{red}$, defined on vectors of the form $f \otimes g$ in the base Hilbert space $\mathcal{H}^{red} = L^2(\mathbb{R}^4) \otimes L^2(\mathbb{R}^4)$ by

$$\sigma_n^{red}(f \otimes g) = (n^\mu \partial_\mu - \Delta_n^\perp) f \otimes (n^\mu \partial_\mu - \Delta_n^\perp) g,$$ (47)

and extended to an operator on $\mathcal{H}^{red}$ by linearity. The associated decoherence functional is denoted $d_n^{red}$. The pair $(P(\mathcal{E}^{red}), d_n^{red})$ is the reduced history theory of pure quantum electrodynamics with respect to the foliation $n$. 

7
3 BRST Cohomology

As shown in the previous section, electrodynamics can be quantised starting from the reduced state space. This is because the reduced state space has a simple structure, in particular it is a linear space. This is not the case for other constrained field theories such as Yang-Mills theory and gravity. For such theories another, more general, approach is needed. The BRST formalism [12] is a powerful approach to the quantisation of constrained systems, and can be formulated using rigorous operator methods. Motivated by these considerations we develop the BRST approach to the quantum theory of histories electrodynamics. In this section we follow closely the notation of [12].

3.1 Classical BRST Cohomology

The central idea in the BRST formalism is to extend the state space by including fermionic ‘ghost’ fields. The extended state space maintains manifest covariance and locality, unlike the reduced state space approach. The BRST charge is constructed from the ghost fields and the constraints and, in the classical case, is a functional on the extended state space. The BRST charge generates nilpotent canonical transformations on the extended state space. The physical degrees of freedom are identified with the corresponding set of cohomology classes. The idea is that the ghost fields cancel out the gauge fields in the cohomology.

We begin by briefly recalling the BRST approach to the standard classical theory of electrodynamics as given in chapter 19 of reference [12]. We have fields $E^\mu(x)$ and $A_\mu(x)$ satisfying the algebra

$$\{A_\mu(x), E^\nu(x')\} = \delta^\mu_\nu \delta^{(3)}(x - x'),$$

and a pair of constraints $E^0(x) \approx 0$ and $\partial_i E^i(x) \approx 0$. Corresponding to the first constraint we add a ghost pair $\eta(x), P(x)$ with

$$\{P(x), \eta(x')\} = -\delta^{(3)}(x - x'),$$

where the bracket is symmetric representing the fact that the ghost fields are fermionic. The Lagrange multiplier field $A_0(x)$ and its conjugate momentum are associated with an antighost field $\bar{C}(x)$ and conjugate momentum $\rho(x)$ satisfying

$$\{\rho(x), \bar{C}(x')\} = -\delta^{(3)}(x - x').$$
The BRST charge is

$$\Omega = \int d^3 x \left[ -i \rho E^0 + \eta \partial_i E^i \right], \quad (51)$$

and $\Omega$ generates nilpotent canonical transformations which are explicitly given in [12], and we denote by $\tau$, that is $\tau(F) := \{\Omega, F\}$. A functional $F$ is said to be BRST-closed if and only if

$$\tau(F) = 0, \quad (52)$$

and a functional $G$ is said to be BRST-exact if and only if

$$G = \tau(G'), \quad (53)$$

for some functional $G'$. By the nilpotency of $\tau$ closed functionals are exact, but the converse is not necessarily true, and the set of functionals which are closed but not exact is isomorphic to the set of functionals on the reduced state space. In addition [12], there is a natural Poisson algebra defined on the set of cohomology classes which is a Poisson subalgebra of the extended Poisson algebra, and is isomorphic to the Poisson algebra of the reduced classical history space.

This analysis is easy to extend to the classical history theory of electrodynamics. Corresponding to the constraint $^n E_{\parallel}(X) \approx 0$ we introduce a pair of fermionic, scalar ghost history fields $\eta_1^n(X)$ and $P_1^n(X)$ which satisfy the algebra

$$\{P_1^n(X), \eta_1^n(X')\} = -\delta^{(4)}(X - X'). \quad (54)$$

In addition we have two Lagrange multipliers in the history theory so we add two antighost fields $C_1^n(X) (a \in \{1, 2\})$, along with their conjugate momenta $\rho_1^n(X)$. These fields satisfy

$$\{\rho_1^n(X), C_1^n(X')\} = -\delta^{(4)}(X - X'). \quad (55)$$

In this way the ring of functions on the extended classical history space is given the structure of a graded Lie algebra. The history BRST charge $\Omega'_n$ is defined as

$$\Omega'_n = \int d^4 X \left[ -i \rho_1^n E^t_1 - i \rho_2^n E^s_1 + \eta_1^n E_{\parallel} \right], \quad (56)$$
and generates canonical transformations denoted by $\tau^n$. The transformations are

$$
\begin{align*}
\tau^n(nA^\parallel) &= \eta_1^n, & \tau^n(\eta_1^n) &= 0, & \tau^n(P_1^n) &= nE^\parallel, & \tau^n(nE^\parallel) &= 0, \\
\tau^n(A^n_s) &= -i\rho_1^n, & \tau^n(A^n_t) &= -i\rho_2^n, & \tau^n(\rho_1^n) &= 0, & \tau^n(\rho_2^n) &= 0, \\
\tau^n(C^n_1) &= iE^n_s, & \tau^n(C^n_2) &= iE^n_t, & \tau^n(E^n_s) &= 0, & \tau^n(E^n_t) &= 0, \\
\tau^n(nA^\perp) &= 0, & \tau^n(nE^\perp) &= 0.
\end{align*}
$$

From these transformations it is clear that $\tau^n$ is nilpotent, and thus defines a cohomology on the space of functionals on the BRST-extended classical history space. The classical history BRST cohomology, $H^*_{cl}(\Omega'_n)$, is defined to be the space of equivalence classes of BRST-closed functionals modulo BRST-exact ones. The only fields which are closed but not exact are $nA^\perp$ and $nE^\perp$, and so the cohomology classes are in bijective correspondence with functionals of the transverse fields. Thus $H^*_{cl}(\Omega'_n)$ is isomorphic to the space of functionals on the reduced classical history space.

### 3.2 Operator Quantisation

The BRST operator quantisation of standard electrodynamics proceeds by expanding the quantum fields in terms of operators which satisfy the algebra of creation and annihilation operators, thus defining a representation of the field algebra on a Fock space. Then, using the quantum BRST charge $\hat{\Omega}$, which is the operator corresponding to the functional in equation (51), a cohomology can be defined on operators on the quantum Hilbert space as follows. An operator $\hat{O}$ is defined to be $BRST$-closed if and only if

$$
[\hat{\Omega}, \hat{O}] = 0,
$$

and an operator $\hat{Q}$ is $BRST$-exact if and only if

$$
\hat{Q} = [\hat{\Omega}, \hat{W}]
$$

for some operator $\hat{W}$. Because $\hat{\Omega}$ generates nilpotent transformations and the commutator satisfies the graded Jacobi identity, a BRST-exact operator is necessarily BRST-closed. However, the converse is not true and two BRST-closed operators $\hat{O}$ and $\hat{O}'$ are defined to be BRST-equivalent if $\hat{O}' = \hat{O} + \hat{Q}$ for some BRST-exact operator $\hat{Q}$. The operator cohomology of $\hat{\Omega}$, $H^*_{op}(\hat{\Omega})$, is defined to be the set of equivalence classes of closed operators modulo this
equivalence relation. The expansion of the fields in terms of creation and annihilation operators is given in [12] and it follows that the quantum BRST charge can be written in the form
\[ \hat{\Omega} = \int d^3k \left[ \hat{c}^\dagger(k) \hat{a}(k) + \hat{a}^\dagger(k) \hat{c}(k) \right]. \] (63)

Now the non-physical field modes ‘cancel out’ the ghost modes in the cohomology or, more precisely, the operator cohomology is isomorphic to the set of operators on the reduced quantum Hilbert space. This is a consequence of a general result known as the ‘quartet mechanism’ [12] which applies to any quantum theory in which the BRST operator is the sum of terms of the form (63).

### 3.3 Quantum History Theory in Quartet Form

For the Fock space quantisation of a BRST extended theory it is necessary that the constraints come in pairs allowing the definition of creation and annihilation operators as complex linear combinations of pairs of fields. In the history theory there are three constraints which cannot be grouped in pairs for the Fock space quantisation. To proceed we include an extra Lagrange multiplier along with the associated momenta and ghosts. More precisely, we introduce a bosonic scalar field \( \hat{\lambda}_n(X) \), the Lagrange multiplier corresponding to the constraint \( \hat{E}_n(X) \approx 0 \). Its canonical momentum is denoted \( \hat{B}_n(X) \) and is constrained to vanish
\[ \hat{B}_n(X) \approx 0. \] (64)

The associated ghost pair is \( (\hat{\eta}^2_n, \hat{P}^2_n) \). Now the fields do form ‘quartets’, and the results of [12] can be applied. The detailed transformations of the fields into the ‘quartet’ form have been relegated to the appendix. The important result is that the bosonic fields can be defined in terms of six pairs of bosonic creation and annihilation operators
\[
\begin{align*}
[\hat{a}_a(K), \hat{b}^\dagger_b(K')] &= -\hbar \delta_{ab} \delta^{(4)}(K - K') \quad (65) \\
[\hat{b}_a(K), \hat{a}^\dagger_b(K')] &= -\hbar \delta_{ab} \delta^{(4)}(K - K') \quad (66) \\
[\perp \hat{a}_a(K), \perp \hat{a}^\dagger_b(K')] &= \hbar \delta_{ab} \delta^{(4)}(K - K'), \quad (67)
\end{align*}
\]

for \( a, b \in \{1, 2\} \). So we have a representation of the bosonic part of the BRST-extended history algebra on the Fock space \( \mathcal{E}^B \) which is defined as the tensor
product of six copies of the bosonic Fock space $\mathcal{F}_B[L^2(\mathbb{R}^4)]$. Similarly, the fermionic fields can be expanded in terms of four pairs of fermionic creation and annihilation operators, which satisfy the anti-commutators

\[
\begin{align*}
\left[ \hat{c}_a(K), \hat{c}^\dagger_b(K') \right] &= -\hbar \delta_{ab} \delta^{(4)}(K - K'), \\
\left[ \hat{c}_a(K), \hat{c}^\dagger_b(K') \right] &= -\hbar \delta_{ab} \delta^{(4)}(K - K'),
\end{align*}
\]

where $a, b \in \{1, 2\}$. In this way we have a representation of the fermionic part of the BRST-extended history algebra on the Fock space $\mathcal{E}_F$, which is defined as the tensor product of four copies of the fermionic Fock space $\mathcal{F}_F[L^2(\mathbb{R}^4)]$. The whole algebra is represented on the space $\mathcal{E} = \mathcal{E}_B \otimes \mathcal{E}_F$, which we call the BRST-extended quantum history space.

### 3.4 Quantum Operator Cohomology

The extended BRST charge is

\[
\hat{\Omega}_n = \int d^4X \left[ -i \hat{\rho}_a \hat{E}^s_n + \hat{\eta}_a \hat{E}_{||} - i \hat{\rho}_a \hat{E}^t_n + \hat{\eta}_a \hat{B}_n \right],
\]

and in terms of oscillators it takes the $n$-independent form

\[
\hat{\Omega} = \sum_{a=1}^{2} \int d^4K \left[ \hat{c}^\dagger_a(K) \hat{\alpha}_a(K) + \hat{\alpha}^\dagger_a(K) \hat{c}_a(K) \right].
\]

The anti-Hermitian ghost number operator $\hat{G}$ is defined in terms of oscillators as

\[
\hat{G} = \sum_a \int d^4K \left( \hat{c}^\dagger_a(K) \hat{e}_a(K) - \hat{e}_a(K) \hat{c}^\dagger_a(K) \right).
\]

Any operator $\hat{O}$ on $\mathcal{E}$ can be decomposed in components of definite ghost number

\[
\hat{O} = \sum_g \hat{O}_g, \quad [\hat{G}, \hat{O}_g] = g\hat{O}_g, \quad g \in \mathbb{Z}.
\]

It follows from these definitions that all bosonic fields are of ghost number zero, the fields $\hat{\eta}^a$ and $\hat{\rho}^a$ are of ghost number +1, and $\hat{P}^a$ and $\hat{C}^a$ are of ghost number −1. Vectors in the non-zero eigenspaces of $\hat{G}$ have an ill-defined scalar product, therefore a ghost number zero condition is often imposed.
on the physical states. However, in the Fock space quantisation the ghost number zero condition is automatically satisfied in the cohomology classes.

From equation (70) it is clear that for the case of histories electrodynamics the operators ̂η_\text{a}^n, ̂\rho_\text{a}^n, ̂E_∥^n, ̂E_\text{s}^n, ̂E_\text{t}^n, ̂B_\text{n} are closed. Similarly, the operators ̂η_\text{a}^n, ̂\rho_\text{a}^n, ̂E_∥^n, ̂E_\text{s}^n, ̂E_\text{t}^n, ̂B_\text{n} are exact; for example the smeared ghost field can be written

\[ \hat{\eta}_n^1(f) = [\hat{\Omega}, \hat{A}_∥^n(f)] \] (74)

The transverse field operators are closed but not exact.

### 3.5 Projection Operator Cohomology

In a history theory it is projection operators that appear in the decoherence functional, and operators which are not projectors lack a direct physical interpretation. In the equivalence classes of \( H^\ast(\hat{\Omega}) \), projection operators are identified with operators which are not projection operators. This identification is unnatural from the histories perspective so in this subsection we define an equivalence relation directly on the lattice of projectors. We then use this equivalence relation to define \( H^\ast(\hat{\Omega}) \), the projection operator cohomology associated with \( \hat{\Omega} \). Finally we show that \( H^\ast(\hat{\Omega}) \) can be given the structure of a lattice, and that this lattice is isomorphic to the lattice of projection operators on the reduced quantum history space.

**Definition 3.5.1** Given two closed projectors \( \alpha \) and \( \beta \), we say that ‘\( \alpha \) is an exact fine-graining of \( \beta \)’, written \( \alpha \preceq \beta \), if and only if \( \beta = \alpha + \gamma \) for some projection operator \( \gamma \) which is exact (i.e., \( \gamma = [\hat{\Omega}, \hat{Q}] \) for some operator \( \hat{Q} \)) and disjoint to \( \alpha \).

The relation \( \preceq \) is a partial order. The primitive part of \( \alpha \) is denoted \( \alpha_0 \) and is defined as the limit of exact fine-grainings of \( \alpha \). A unique \( \alpha_0 \) exists for each closed \( \alpha \) because \( \preceq \) is a partial order. If \( \alpha \) is exact then \( \alpha_0 \) is the zero projector. If \( \alpha \) is closed but not exact then \( \alpha_0 \) projects onto the spectrum of a closed but not exact field operator. For histories electrodynamics we have seen that the closed but not exact field operators are the transverse field operators, and so closed primitive projectors are in bijective correspondence with elements of \( P(\mathcal{E}^{\text{red}}) \).

**Definition 3.5.2** Two BRST-closed projection operators \( \alpha \) and \( \beta \) are said to be BRST-equivalent if and only if \( \alpha_0 = \beta_0 \).
The projection operator cohomology $H^*(\hat{\Omega})$ is defined as the space of closed projection operators modulo this equivalence relation, and elements of $H^*(\hat{\Omega})$ are identified with physical propositions. Given a primitive projection operator $\alpha_0$, the equivalence class containing $\alpha_0$ is the collection of all exact coarse-grainings of $\alpha_0$. The statement ‘$\alpha$ is an exact fine-graining of $\beta$’ is equivalent to the statement ‘$\beta$ is an exact coarse-graining of $\alpha$’. Let $[\alpha]$ denote the equivalence class containing the closed projector $\alpha$. The map $\pi : [\alpha] \mapsto \alpha_0$ is well-defined on $H^*(\hat{\Omega})$, and is in fact an isomorphism between $H^*(\hat{\Omega})$ and $P(\mathcal{E}^{red})$; the inverse is given by $\pi^{-1} : \alpha_0 \mapsto [\alpha_0]$. In order to give $H^*(\hat{\Omega})$ the structure of a lattice, we need to examine the geometry of the linear subspaces associated to closed and exact projection operators.

**Definition 3.5.3** A BRST-closed subspace $L \subset \mathcal{E}$ is a topologically closed linear subspace of $\text{Ker}(\hat{\Omega})$.

**Proposition 3.5.4** BRST-closed projection operators are in bijective correspondence with BRST-closed subspaces of $\mathcal{E}$.

**Proof:**
Each projection operator $\alpha$ is associated with a topologically closed linear subspace $L_\alpha \subset \mathcal{E}$. If $\alpha$ is a BRST-closed projection operator, i.e., $[\hat{\Omega}, \alpha] = 0$, then by writing $\alpha$ in Dirac notation as
\begin{equation}
\alpha = \sum_i |l_i\rangle\langle l_i| ,
\end{equation}
where $|l_i\rangle$ is a basis of $L_\alpha$, it follows from the independence of the basis vectors and the hermiticity of $\hat{\Omega}$ that $\hat{\Omega}|l\rangle = 0$ for all $|l\rangle \in L_\alpha$. Therefore $L_\alpha \subset \text{Ker}(\hat{\Omega})$.

Conversely, each BRST-closed subspace is associated with a BRST-closed projection operator because $\hat{\Omega}$ is self-adjoint. $\square$

In the case of histories electrodynamics BRST-closed subspaces are spanned by vectors created by the operators $\hat{a}_a^\dagger, \hat{a}_a^\dagger, \hat{c}_a^\dagger$ acting on the cyclic vacuum state.

**Definition 3.5.5** A BRST-exact subspace $M \subset \mathcal{E}$ is a topologically closed linear subspace of $\text{Im}(\hat{\Omega})$. 

14
Proposition 3.5.6  BRST-exact projection operators are in bijective correspondence with BRST-exact subspaces of $E$.

Proof:  
A BRST-exact projection operator can be written in the form $\gamma = [\hat{\Omega}, \hat{Q}]$ for some operator $\hat{Q}$. $\gamma$ is closed, and so is associated with a BRST-closed subspace $M_\gamma \subset E$. Now
\[
\gamma|m\rangle = [\hat{\Omega}, \hat{Q}]|m\rangle = \hat{\Omega}(\hat{Q}|m\rangle) \quad \forall \; |m\rangle \in M_\gamma
\]  
(76)
because $\hat{\Omega}|m\rangle = 0$. However, $\gamma|m\rangle = |m\rangle$ for any $|m\rangle \in M_\gamma$, so
\[
|m\rangle = \hat{\Omega}(\hat{Q}|m\rangle) \quad \forall \; |m\rangle \in M_\gamma,
\]  
(77)
and hence $M_\gamma \subset \text{Im}(\hat{\Omega})$.

Conversely, each BRST-exact subspace $M$ with basis $|m_i\rangle$ is associated with an exact projection operator $\gamma$
\[
\gamma = \sum_i |m_i\rangle\langle m_i| = [\hat{\Omega}, \sum_i |u_{m_i}\rangle\langle m_i|],
\]  
(78)
where $|u_{m_i}\rangle$ is any vector such that $\hat{\Omega}|u_{m_i}\rangle = |m_i\rangle$. □

In the case of histories electrodynamics BRST-exact subspaces are spanned by vectors created by the operators $\hat{a}_a^\dagger, \hat{c}_a^\dagger$ on the vacuum state.

Definition 3.5.7  A primitive subspace $R \subset E$ is a BRST-closed subspace with no BRST-exact proper subspaces. The closure of the union of all primitive subspaces is denoted $E_0$.

Proposition 3.5.8  Primitive projection operators are in bijective correspondence with primitive subspaces of $E$.

Proof:  
Let $\alpha_0$ be a primitive projector. Then the only exact fine graining of $\alpha_0$ is $\alpha_0$ itself. This implies that the linear subspace associated with $\alpha_0$ has no BRST-exact proper subspaces.

Conversely, because a primitive subspace has no BRST-exact proper subspaces it follows that the corresponding projection operator must be a limit of exact fine-grainings, and thus primitive. □
In the case of histories electrodynamics primitive subspaces are spanned by vectors created by the action of the transverse creation operators $\hat{a}_a^\dagger$ on the vacuum state.

From the above discussion it follows that $\mathcal{E}_0$ and $\text{Im}(\hat{\Omega})$ are disjoint, and that the closure of $\mathcal{E}_0 \cup \text{Im}(\hat{\Omega})$ is $\text{Ker}(\hat{\Omega})$. Therefore every exact projector is disjoint to every primitive projector, and $\text{id}_0 + \text{id}_{\text{Im}} = \text{id}_{\text{Ker}}$, where $\text{id}_0, \text{id}_{\text{Im}}$ and $\text{id}_{\text{Ker}}$ are the identity operators on $\mathcal{E}_0, \text{Im}(\hat{\Omega})$ and $\text{Ker}(\hat{\Omega})$ respectively. These results can be used to prove the following theorem.

**Theorem 3.5.9** (i) The lattice $P(\text{Ker}(\hat{\Omega}))$ induces a lattice structure on $H^*(\hat{\Omega})$ by

$$[\alpha] \land [\beta] := [\alpha \land \beta] \quad (79)$$

$$[\alpha] \lor [\beta] := [\alpha \lor \beta] \quad (80)$$

$$\lnot [\alpha] := \lnot [\alpha]. \quad (81)$$

(ii) The map $\pi$ is a lattice isomorphism of $H^*(\hat{\Omega})$ and $P(\mathcal{E}^{\text{red}})$.

**Proof:**

(i) We have to show that the definitions give the same results when evaluated on different members of an equivalence class. We define the maximal exact part of $\alpha$ to be the unique exact projector $\gamma_\alpha$ such that $\alpha = \alpha_0 + \gamma_\alpha$. Every exact subspace is orthogonal to every primitive subspace so $\alpha \land \beta = \alpha_0 \land \beta_0 + \gamma_\alpha \land \gamma_\beta$. The intersection of two exact subspaces is exact so $\gamma_\alpha \land \gamma_\beta$ is exact, and $[\alpha \land \beta] = [\alpha_0 \land \beta_0]$. In a similar way we have $[\alpha \lor \beta] = [\alpha_0 \lor \beta_0]$. Finally consider $\lnot \alpha = \text{id}_{\text{Ker}} - (\alpha_0 + \gamma_\alpha)$ which can be written $\lnot \alpha = (\text{id}_0 - \alpha_0) + (\text{id}_{\text{Im}} - \gamma_\alpha)$. Now $\text{id}_{\text{Im}} - \gamma_\alpha$ is exact, so $[\lnot \alpha] = [\text{id}_0 - \alpha_0]$. Similarly we have $\lnot \alpha_0 = (\text{id}_0 - \alpha_0) + \text{id}_{\text{Im}}$ so $[\lnot \alpha_0] = [\text{id}_0 - \alpha_0] = [\lnot \alpha]$.

(ii) It is straightforward to check that

(a) $\pi([\alpha] \land [\beta]) = \pi[\alpha] \land \pi[\beta]$.

(b) $\pi([\alpha] \lor [\beta]) = \pi[\alpha] \lor \pi[\beta]$.

(c) $\pi(\lnot [\alpha]) = \lnot \pi[\alpha]$.

where the lattice operations on the right-hand-side of the above equations are defined in $P(\mathcal{E}^{\text{red}})$. $\Box$

Thus the projection operator cohomology is isomorphic to the lattice of projection operators on the reduced history space. In order to show that the corresponding history theories are ‘the same’ we first investigate the space of decoherence functionals on the BRST-extended quantum history space.
4 Gauge-Fixed Decoherence Functionals

To each gauge-fixed action operator on $E$, there corresponds a gauge-fixed decoherence functional. The gauge-fixed decoherence functionals assign non-trivial values to propositions regarding the gauge and ghost fields. However, as we shall see, each gauge-fixed decoherence functional induces a well-defined functional on $H^*(\Omega)$ in such a way that the resulting history theory is equivalent to the reduced quantum history theory.

4.1 Radiation and Feynman Gauges

We begin by giving two explicit examples of gauge-fixed decoherence functionals. The radiation gauge corresponds to the following canonical action

$$\hat{S}_{R} = \hat{V}_{n} - \hat{H}_{0}^{0},$$

(82)

where the Louiville operator is the sum of three parts:

1) A gauge-invariant part $\hat{V}_{n}^{0}$

$$\hat{V}_{n}^{0} = \int d^{4}X \sum_{a=1}^{2}(\hat{P}_{n}^{a} \partial_{n}^{a} \hat{\eta}_{n}^{a} + \hat{\rho}_{n}^{a} \partial_{n}^{a} \hat{C}_{n}^{a}),$$

(84)

2) A ghost part

$$\hat{V}_{n}^{gh} = \int d^{4}X \sum_{a=1}^{2}(\hat{P}_{n}^{a} \partial_{n}^{a} \hat{\eta}_{n}^{a} + \hat{\rho}_{n}^{a} \partial_{n}^{a} \hat{C}_{n}^{a}),$$

(84)

3) A gauge part

$$\hat{V}_{n}^{ga} = \int d^{4}X (\hat{E}_{n}^{\perp} \partial_{n}^{\perp} \hat{A}_{n}^{\perp} + \hat{E}_{n}^{\parallel} \partial_{n}^{\parallel} \hat{A}_{n}^{\parallel} + \hat{B}_{n} \partial_{n} \hat{\lambda}_{n}).$$

(85)

The gauge-invariant Hamiltonian $\hat{H}_{0}^{0}$ is given by

$$\hat{H}_{0}^{0} = \int d^{4}X \left( \frac{1}{2} \hat{E}_{M}^{\perp} \hat{E}_{M}^{\perp} + \frac{1}{4} \hat{F}_{MN}^{\perp} \hat{F}_{MN}^{\perp} \right).$$

(86)

Using an argument similar to that in section 2, it follows that the normal ordered Hamiltonian exists as a self-adjoint operator on $E$. The Louiville operator also exists, and therefore so does the radiation gauge action $\hat{S}_{R}^{R}$.
the gauge and ghost fields vanish initially, then they vanish identically on the extrema of this action. However, note that the extrema of this action satisfy the constraints if and only if the constraints are satisfied initially.

Let \( \mathcal{H}_0 \simeq L^2(\mathbb{R}^4) \otimes L^2(\mathbb{R}^4) \) denote the base Hilbert space of the primitive Fock space \( \mathcal{E}_0 \) i.e., \( \mathcal{E}_0 = \mathcal{F}_B[\mathcal{H}_0] \). Then the generator of the decoherence functional associated with the radiation gauge action acts on vectors \( f \otimes g \in \mathcal{H}_0 \) as

\[
\sigma_n^R(f \otimes g) = (n^\mu \partial_\mu - \Delta_n^1) f \otimes (n^\mu \partial_\mu - \Delta_n^1) g,
\]

and is extended to an operator on \( \mathcal{H}_0 \) by linearity. The radiation gauge Hamiltonian commutes with the gauge and ghost fields. Therefore, as argued in [10], projectors onto the gauge and ghost fields form a canonical consistent set and the probabilities assigned to these projectors is just the probability in the initial state. This implies that if either \( \epsilon \) or \( \kappa \) are projectors onto subspaces of the orthogonal complement of \( \mathcal{E}_0 \), and the initial density matrix contains no gauge or ghost modes, then

\[
d^R(\epsilon, \kappa) = 0.
\]

This completes the definition of \( d^R \).

In the ‘Feynman gauge’ all fields satisfy the wave equation internally. Let \( \mathcal{H} \) denote the base Hilbert space of the BRST-extended quantum history space \( \mathcal{E} \). A vector in \( \mathcal{H} \) is a linear combination of homogeneous vectors of the form

\[
v = \otimes_{i=1}^{10} v_i
\]

where \( v_i \in L^2(\mathbb{R}^4) \). The generator of the Feynman gauge decoherence functional is defined on homogeneous vectors by

\[
\sigma_n^F v = \otimes_{i=1}^{10} (n^\mu \partial_\mu - \Delta_n^1) v_i,
\]

and is extended to an operator on \( \mathcal{H} \) by linearity.

4.2 Gauge Transformations

In this subsection we investigate gauge transformations. For simplicity we consider the constraint \( \hat{E}_n \approx 0 \), but analogous remarks apply to the other three constraints.
In the operator cohomology, the constraint $\hat{E}_n^t \approx 0$ is identified with operators of the form
\[
\hat{D}_n^t = \hat{E}_n^t + [\hat{\Omega}, \hat{Q}].
\] (91)
The operator $\hat{D}_n^t$ generates gauge transformations in the gauge field $\hat{A}_n^t$, and also in the ghost fields. We require that the the ghost number zero eigenspace of $\hat{G}$ is mapped into itself under gauge transformations, which implies that $[\hat{G}, \hat{D}_n^t] = 0$. As $\hat{\Omega}$ is a ghost number one operator, the operator $\hat{Q}$ must be of ghost number $-1$. In addition we choose $\hat{Q}$ to be self-adjoint, so that $\hat{D}_n^t$ generates unitary transformations.

$\hat{D}_n^t$ is exact, and can be written as $[\hat{\Omega}, \hat{G}_n]$ where $\hat{G}_n = \hat{C}_n^1 + \hat{Q}$. It follows from the Jacobi identity that
\[
[\hat{D}_n^t, \hat{O}] = [[\hat{\Omega}, \hat{G}_n], \hat{O}] = [[\hat{G}_n, \hat{O}], \hat{\Omega}],
\] (92)
if $\hat{O}$ is a closed operator. Thus closed operators are mapped into exact operators by infinitesimal gauge transformations. Under finite gauge transformations, closed operators transform as
\[
\hat{O} \mapsto \hat{U} \hat{O} \hat{U}^\dagger = \hat{O} + [\hat{\Omega}, \hat{W}].
\] (93)
where $\hat{W}$ is a ghost number $-1$ operator. Therefore, gauge transformations act trivially on the equivalence classes of the operator cohomology.

As the gauge transformations are unitary, they map projection operators onto projection operators and, because $\mathcal{E}_0$ and $\text{Im}(\hat{\Omega})$ are disjoint, exact self-adjoint operators commute with primitive projectors. Therefore the action of a gauge transformation on a BRST-closed projector, $\alpha = \alpha_0 + \gamma_\alpha$ is
\[
\alpha \mapsto \hat{U} \alpha \hat{U}^\dagger = \alpha_0 + \hat{U} \gamma_\alpha \hat{U}^\dagger
\] (94)
This shows that the gauge group acts trivially on $H^*(\hat{\Omega})$, and that primitive projectors are gauge-invariant.

There is also a natural unitary action of the gauge transformations on the space of decoherence functionals. The decoherence functional $d$ is associated with an operator on $\mathcal{E} \otimes \mathcal{E}$, denoted $\Theta_d$. Gauge transformations act on $\Theta_d$ as $\Theta_d \mapsto \Theta_{d'}$ where $\Theta_{d'} = U \otimes U \Theta_d U^\dagger \otimes U^\dagger$. As shown in [3], the operator $\Theta_{d'}$ is indeed associated with a bone fide decoherence functional $d'$. We say that the decoherence functionals $d$ and $d'$ are related by the gauge transformation $U$. 

Definition 4.2.1  The collection of all decoherence functionals related to the radiation gauge decoherence functional, $d^R$, by a gauge transformation is denoted $D_{gf}$ and called the space of gauge-fixed decoherence functionals.

5 Gauge Invariance

We now fix a particular foliation, drop the $n$-label, and use coordinates adapted to the foliation. Let $D$ denote the space of decoherence functionals associated with the quantum history space $E$. A physical symmetry of a history quantum theory (PSHQT) realised on the Hilbert space $E$ is defined in [5] as an affine one-to-one map

$$P(E) \otimes P(E) \times D \to P(E) \otimes P(E) \times D$$

$$(\alpha \otimes \beta, \Theta) \mapsto ([\alpha \otimes \beta]', \Theta').$$

(95)

(96)

that preserves the pairing between history propositions and operators associated with decoherence functionals, i.e.,

$$tr_{E\otimes E} (\alpha \otimes \beta | \Theta) = tr_{E\otimes E} ([\alpha \otimes \beta]' \Theta').$$

(97)

Schreckenberg [5] proved the following histories analogue of Wigner’s theorem:

Theorem 5.0.2 Every PSHQT can be induced by a unitary or anti-unitary operator $\hat{U}$ on $E$ in the sense that each PSHQT can be written as

$$[\alpha \otimes \beta] \mapsto \hat{U} \otimes \hat{U}[\alpha \otimes \beta]\hat{U}^\dagger \otimes \hat{U}^\dagger$$

$$\Theta \mapsto \hat{U} \otimes \hat{U}\Theta\hat{U}^\dagger \otimes \hat{U}^\dagger,$$

for some unitary or anti-unitary operator $\hat{U}$. Conversely, every transformation of the form (98),(99) for unitary or anti-unitary $\hat{U}$ induces a PSHQT.

We have seen that gauge transformations act unitarily on projection operators and on the space of decoherence functionals. Therefore gauge transformations induce a PSHQT and we can use the histories analogue of Wigner’s theorem. The following is an immediate consequence:

Proposition 5.0.3 For any gauge-fixed decoherence functional $d \in D_{gf}$, and any two primitive projectors $\alpha_0, \beta_0 \in P(E_0)$,

$$d(\alpha_0, \beta_0) = d^{red}(\alpha_0, \beta_0).$$

(100)
Proof:
Since \(d \in \mathcal{D}_{gf}\), it is related to \(d^R\) by a gauge transformation. We denote the unitary operator associated with this gauge transformation by \(\hat{U}_d\). The primitive projectors \(\alpha_0\) and \(\beta_0\) satisfy \(\hat{U}_d \alpha_0 \hat{U}_d^\dagger = \alpha_0\) and \(\hat{U}_d \beta_0 \hat{U}_d^\dagger = \beta_0\). Now equation (97) implies \(d(\alpha_0, \beta_0) = d^R(\alpha_0, \beta_0)\). Finally, from the definition of \(d^R\) it follows that \(d^R(\alpha_0, \beta_0) = d^{red}(\alpha_0, \beta_0)\). □

This shows that gauge-fixed decoherence functionals ensure that:
(i) The \emph{probabilities} assigned to gauge-invariant propositions are gauge-invariant.
(ii) The \emph{quantum interference} between gauge-invariant propositions is gauge-invariant.

We have the following lemma:

**Lemma 5.0.4** For any gauge-fixed decoherence functional \(d \in \mathcal{D}_{gf}\), any projector \(\alpha \in P(E)\), and any exact projector \(\gamma\),

\[
d(\alpha, \gamma) = 0.
\]

(101)

Proof:
We act on \(d(\alpha, \gamma)\) with \(\hat{U}_d\), the gauge transformation that maps \(d\) into \(d^R\). Under the action of \(\hat{U}_d\), \(\alpha\) and \(\gamma\) transform to \(\alpha_d := \hat{U}_d \alpha \hat{U}_d^\dagger\) and \(\gamma_d := \hat{U}_d \gamma \hat{U}_d^\dagger\). Now using equation (97) we have

\[
d(\alpha, \gamma) = d^R(\alpha_d, \gamma_d),
\]

(102)

which is equal to zero by equation (88) because \(\gamma_d\) is exact. □

**Corollary 5.0.5** Let \(\gamma\) and \(\delta\) be exact propositions. Then \(d(\gamma, \delta) = 0\) for any gauge-fixed decoherence functional \(d \in \mathcal{D}_{gf}\).

Proof: Immediate. □

This implies that there is no interference between exact projectors, and that exact propositions are assigned a probability of zero by any gauge-fixed decoherence functional. Projectors onto the spectrum of the constraint fields are exact, so a special case of this result is that any gauge-fixed decoherence functional assigns a probability of zero to any propositions that are not compatible with the constraints.
Theorem 5.0.6 Any gauge-fixed decoherence functional \( d \in \mathcal{D}_{gf} \) reduces to a well-defined functional \( \tilde{d} : H^*(\hat{\Omega}) \times H^*(\hat{\Omega}) \to \mathbb{C} \) defined by
\[
\tilde{d}([\alpha], [\beta]) := d(\alpha, \beta).
\] (103)

In addition
\[
\tilde{d}([\alpha], [\beta]) = d^{red}(\alpha_0, \beta_0),
\] (104)
for all \([\alpha], [\beta] \in H^*(\hat{\Omega})\), where \(\alpha_0\) and \(\beta_0\) are the primitive parts of \(\alpha\) and \(\beta\) respectively.

Proof:

We use the additivity axiom of the space of decoherence functionals;
\[
d(\alpha, \beta) = d(\alpha_0 + \gamma, \beta) = d(\alpha_0, \beta) + d(\gamma, \beta),
\] (105)
because \(\alpha_0\) and \(\gamma\) are disjoint. Now because \(\gamma\) is exact, lemma (5.0.4) along with the hermiticity of \(d\) implies that that \(d(\gamma, \beta) = 0\). This means that
\[
d(\alpha, \beta) = d(\alpha_0, \beta).
\] (106)

Repeating this for the other argument of \(d\), we have
\[
d(\alpha, \beta) = d(\alpha_0, \beta_0),
\] (107)
for any gauge-fixed decoherence functional \(d\). This shows that every gauge-fixed decoherence functional reduces to a well-defined functional \(\tilde{d} : H^*(\hat{\Omega}) \times H^*(\hat{\Omega}) \to \mathbb{C}\) defined by \(\tilde{d}([\alpha], [\beta]) := d(\alpha, \beta)\). Now proposition (5.0.3) proves the theorem. \(\square\)

Theorem (5.0.6) shows that the cohomological history theory \((H^*(\hat{\Omega}), \tilde{d})\) is ‘the same’ as the reduced history theory \((P(E^{red}), d^{red})\). More precisely,

Definition 5.0.7 Two quantum history theories \((P_1, \mathcal{D}_1)\) and \((P_2, \mathcal{D}_2)\) are defined to be isomorphic if there exists (i) an isomorphism of lattices \(\lambda : P_1 \to P_2\), and (ii) a bijective map \(\vartheta : \mathcal{D}_1 \to \mathcal{D}_2\), such that
\[
d(\alpha, \beta) = \vartheta(d)(\lambda(\alpha), \lambda(\beta)),
\] (108)
for all \(\alpha, \beta \in P_1\) and all \(d \in \mathcal{D}_1\).

Proposition 5.0.8 \((H^*(\hat{\Omega}), \tilde{d})\) is isomorphic to \((P(E^{red}), d^{red})\).

Proof

The map \(\pi : \alpha \mapsto \alpha_0\) provides the required lattice isomorphism between \(H^*(\hat{\Omega})\) and \(P(E^{red})\). Now define \(\vartheta\) by \(\vartheta(\tilde{d}) = d^{red}\), and equation (104) states precisely that the two history theories are isomorphic. \(\square\)
6 Conclusion

We have constructed two concrete models of history quantum electrodynamics on Fock space. Firstly we quantised the classical reduced history space by finding an inequivalent representation of the reduced history algebra on $E^{\text{red}}$ for each foliation. We then defined the decoherence functional using the canonical history action on the reduced history space. This results in the reduced history theory $(P(E^{\text{red}}), d^{\text{red}})$.

Secondly we extended the history algebra by including ghost fields, and found representations of the extended history algebra on the extended quantum history space $E$. Using the BRST charge $\Omega$, we defined $H^*(\Omega)$, the projection operator cohomology of $\Omega$, and showed that $H^*(\Omega)$ is isomorphic (as a lattice) to $P(E^{\text{red}})$. Finally we defined the space of gauge-fixed decoherence functionals and showed that, for each gauge-fixed decoherence functional $d$, $(H^*(\Omega), \tilde{d})$ is isomorphic to $(P(E^{\text{red}}), d^{\text{red}})$.

Although the construction of quantum history electrodynamics is interesting in itself, it is hoped that the results obtained here will be useful in a wider context. Given a general BRST-extended quantum history space and a nilpotent BRST charge, section 3 provides a definition of the corresponding projection operator cohomology, and shows that it is isomorphic to the lattice of projection operators on the reduced history space. In addition, the discussion of the space of gauge-fixed decoherence functionals is relevant to any gauge theory. It would be interesting to apply the histories BRST formalism developed here to mini-superspace models, or to topological quantum field theory. These examples are of particular importance in light of the recent interest in diffeomorphism invariance in history theories [8, 9].

7 Appendix

Firstly we consider the bosonic part of the BRST-extended commutator algebra:

\[
\begin{align*}
[\hat{\lambda}_n(X), \hat{B}_n(X')] &= i\hbar \delta^{(4)}(X - X') \\
[\hat{A}^a_M(X), \hat{E}^N_n(X')] &= i\hbar \delta_M^N \delta^{(4)}(X - X'),
\end{align*}
\]

(109) (110)
where all unwritten commutators vanish. We expand the configuration fields as

\[
\hat{\lambda}_n(X) = \frac{1}{(2\pi)^2} \int \frac{d^4K}{\sqrt{2\omega_n(K)}} [\hat{d}(K)e^{i\mathbf{K} \cdot \mathbf{x}} + \hat{\bar{d}}(K)e^{-i\mathbf{K} \cdot \mathbf{x}}] 
\]  
(111)

\[
\hat{A}_s^n(X) = \frac{1}{(2\pi)^2} \int \frac{d^4K}{\sqrt{2\omega_n(K)}} [\hat{d}_s(K)e^{i\mathbf{K} \cdot \mathbf{x}} + \hat{\bar{d}}_s(K)e^{-i\mathbf{K} \cdot \mathbf{x}}] 
\]  
(112)

\[
\hat{A}_t^n(X) = \frac{1}{(2\pi)^2} \int \frac{d^4K}{\sqrt{2\omega_n(K)}} [\hat{d}_t(K)e^{i\mathbf{K} \cdot \mathbf{x}} + \hat{\bar{d}}_t(K)e^{-i\mathbf{K} \cdot \mathbf{x}}] 
\]  
(113)

\[
n\hat{A}_M^n(X) = \frac{1}{(2\pi)^2} \sum_{i=1}^{3} \int \frac{d^4K}{\sqrt{2\omega_n(K)}} 
\]

\[
[\hat{d}_i(K)n^\epsilon_{M}^i(K)e^{i\mathbf{K} \cdot \mathbf{x}} + \hat{\bar{d}}_i(K)n^\epsilon_{M}^i(K)e^{-i\mathbf{K} \cdot \mathbf{x}}]. 
\]  
(114)

The five-vectors $n^\epsilon_M^a(K)$ for $a = 1, 2$ are defined as in section 2, and $n^\epsilon_M^3(K)$ is the unit vector pointing in the direction of $\mathbf{n} K$. Now the momentum fields are expanded as

\[
\hat{B}_n(X) = \frac{1}{i(2\pi)^2} \int d^4K \sqrt{\frac{\omega_n(K)}{2}} 
\]

\[
[(\hat{d}(K) + \hat{\bar{d}}(K))e^{i\mathbf{K} \cdot \mathbf{x}} - (\hat{\bar{d}}(K) + \hat{d}(K))e^{-i\mathbf{K} \cdot \mathbf{x}}] 
\]  
(116)

\[
\hat{E}_n^s(X) = \frac{1}{i(2\pi)^2} \int d^4K \sqrt{\frac{\omega_n(K)}{2}} [(\hat{d}_s(K) + \hat{\bar{d}}(K))e^{i\mathbf{K} \cdot \mathbf{x}} 
\]

\[-(\hat{\bar{d}}(K) + \hat{d}(K))e^{-i\mathbf{K} \cdot \mathbf{x}}] 
\]  
(117)

\[
\hat{E}_n^t(X) = \frac{1}{i(2\pi)^2} \int d^4K \sqrt{\frac{\omega_n(K)}{2}} [(\hat{d}_t(K) + \hat{\bar{d}}_3(K))e^{i\mathbf{K} \cdot \mathbf{x}} 
\]

\[-(\hat{\bar{d}}_3(K) + \hat{d}_t(K))e^{-i\mathbf{K} \cdot \mathbf{x}}] 
\]  
(118)

\[
n\hat{E}_n^M(X) = \frac{1}{i(2\pi)^2} \sum_{i=1}^{3} \int d^4K \sqrt{\frac{\omega_n(K)}{2}} 
\]

\[
[(\hat{d}_i(K)n^\epsilon_M^i(K) + \hat{\bar{d}}_i(K)n^\epsilon_M^i(K))e^{i\mathbf{K} \cdot \mathbf{x}} 
\]

\[-(\hat{\bar{d}}_i(K)n^\epsilon_M^i(K) + \hat{d}_i(K)n^\epsilon_M^i(K))e^{-i\mathbf{K} \cdot \mathbf{x}}]. 
\]  
(119)

Defining

\[
\hat{a}_1(K) = \frac{1}{\sqrt{2}}(\hat{d}_t(K) + \hat{\bar{d}}_3(K)) 
\]  
(120)
we obtain the commutators (65)-(67) in which \( \hat{d}_a \) has been written \( ^{\perp} \hat{a}_a \) for \( a \in \{1, 2\} \).

The fermionic part of the BRST-extended quantum history algebra is

\[
\begin{align*}
\{ \hat{\eta}^a_n(X), \hat{\mathcal{P}}^b_n(X') \} &= \hbar \delta^{ab} \delta^{(4)}(X - X') \\
[\hat{C}^a_n(X), \hat{\rho}^b_n(X')] &= \hbar \delta^{ab} \delta^{(4)}(X - X')
\end{align*}
\]

where on fermionic fields square brackets represent anti-commutators. We write the ghost fields as

\[
\begin{align*}
\hat{\eta}^1_n(X) &= -\frac{1}{(2\pi)^2} \int \frac{d^4K}{2\omega_n(K)^{3/2}} [c_1(K)e^{iK\cdot X} + c_1^\dagger(K)e^{-iK\cdot X}] \\
\hat{\mathcal{P}}^1_n(X) &= \frac{i}{(2\pi)^2} \int d^4K \omega_n(K)^{3/2} [\bar{c}_1(K)e^{iK\cdot X} + \bar{c}_1^\dagger(K)e^{-iK\cdot X}] \\
\hat{\rho}^1_n(X) &= -\frac{1}{(2\pi)^2} \int \frac{d^4K}{2\omega_n(K)^{1/2}} [c_1(K)e^{iK\cdot X} - c_1^\dagger(K)e^{-iK\cdot X}] \\
\hat{C}^1_n(X) &= \frac{1}{i(2\pi)^2} \int d^4K \omega_n(K)^{1/2} [\bar{c}_1(K)e^{iK\cdot X} + \bar{c}_1^\dagger(K)e^{-iK\cdot X}] \\
\hat{\eta}^2_n(X) &= -\frac{1}{(2\pi)^2} \int \frac{d^4K}{2\omega_n(K)^{1/2}} [c_2(K)e^{iK\cdot X} + c_2^\dagger(K)e^{-iK\cdot X}] \\
\hat{\mathcal{P}}^2_n(X) &= \frac{i}{(2\pi)^2} \int d^4K \omega_n(K)^{1/2} [\bar{c}_2(K)e^{iK\cdot X} + \bar{c}_2^\dagger(K)e^{-iK\cdot X}] \\
\hat{\rho}^2_n(X) &= -\frac{1}{(2\pi)^2} \int \frac{d^4K}{2\omega_n(K)^{1/2}} [c_2(K)e^{iK\cdot X} - c_2^\dagger(K)e^{-iK\cdot X}] \\
\hat{C}^2_n(X) &= \frac{1}{i(2\pi)^2} \int d^4K \omega_n(K)^{1/2} [\bar{c}_2(K)e^{iK\cdot X} + \bar{c}_2^\dagger(K)e^{-iK\cdot X}]
\end{align*}
\]

and the algebra (129),(130) implies the anti-commutators in equations (68), (69).
8 Acknowledgements

I would like to thank Prof. Isham for suggesting this research and for his comments on a draft of this paper. I am also grateful to Dr N. Linden and Prof. B. Hiley for finding an error in a previous version of this work. Financial support in the form of a PPARC studentship is gratefully acknowledged.

References

[1] C.J. Isham. Quantum Logic and the histories approach to quantum theory. J. Math. Phys. 35 : 2157. 1994. gr-qc/9308006.

[2] C.J. Isham and N. Linden. Continuous histories and the history group in generalised quantum theory. J. Math. Phys. 36 : 5392. 1995. gr-qc/9503063.

[3] C.J. Isham, N. Linden and S. Schreckenberg. The Classification of Decoherence Functionals: An analogue of Gleason’s Theorem. J. Math. Phys. 35 : 6360. 1994. gr-qc/9406013.

[4] C.J. Isham, N. Linden, K. Savvidou and S. Schreckenberg. Continuous Time and Consistent Histories. J. Math. Phys. 37 : 2261. 1998. quant-ph/9711031.

[5] S. Schreckenberg. Symmetry and history theory: an analogue of Wigner’s theorem. J. Math. Phys. 37 : 6086. 1996. gr-qc/9607051.

[6] K. Savvidou. The action operator in continuous-time histories. J. Math. Phys. 40: 5657. 1999.

[7] K. Savvidou. Poincare invariance for continuous-time histories. gr-qc/0104053.

[8] K. Savvidou. General relativity histories theory. Class. Quant. Grav. 18 : 3611. 2001.

[9] K. Kuchar and I. Kouletsis. Diffeomorphisms as Symplectomorphisms in history phase space: Bosonic string model. gr-qc/0108022.

[10] D. Noltingk. Consistent Histories approach to the Unruh Effect. Int. Journ. Theo. Phys. 40: 1411, (2001) gr-qc/0005063.
[11] D. Noltingk. Classical History theory of vector fields. (To appear in Journ. Math. Phys.) gr-qc/0107067.

[12] M. Henneaux and C. Teitelboim. Quantization of gauge systems. Princeton University Press. 1992.