Non Gaussianity of General Multiple-Field Inflationary Models

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Abstract
Using the “δN-formalism”, We obtain the expression of the non-Gaussianity of multiple-field inflationary models with the nontrivial field-space metric. Further, we rewritten the result by using the slow-rolling approximation.

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Key words: non Gaussian, non linear parameter, nontrivial metric

1 Introduction
In modern cosmology, the inflation paradigm plays an important role. The simplest classes of inflation models predict Gaussian-distributed perturbations and a nearly scale-invariant spectrum of the primordial density perturbations [1]. This is in good agreement with cosmological observations [2]. Despite the appealing simplicity behind the central idea of inflation, it has

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proved difficult to discriminate between the large number of different models that have been developed to date [4]. However, it is believed that the deviation away from the Gaussian statistics represents a potential powerful discriminant between the competing inflationary models. At the same time, it is necessary to extend the theoretical framework beyond the leading-order effects of scale-invariant, Gaussian fluctuations, in order to understand the early universe further. So, recently, the non-Gaussianity has attracted considerable interest. (See [5] for a review.) In [6], Maldacena gave a general analysis of non-Gaussian perturbations in single field inflationary models. His result is that the bispectrum of the curvature perturbation for squeezed triangles ($k_1 \ll k_2, k_3$) is proportional to the tilt of the primordial power spectrum, and hence is small. Then it seems that only multiple-field inflationary models is like to generate significant non-Gaussian perturbations [7, 8, 9, 10, 11, 12]. In [11], Lyth and Rodriguez have shown that the non-Gaussianity of the curvature perturbation in multiple field models can be simply expressed in the so-called “$\delta N$-formalism” [13]. There the “separate universe” approach [14] has been used to define the curvature perturbation. It should be noted that this approach is valid only for perturbations on super-Hubble scales. But Lyth and Rodriguez only express $f_{NL}$ on a field space with the trivial metric, where $f_{NL}$ is the non-linear parameter. In [8], the authors have given the expression of $f_{NL}$ involving the metric of the field space, $G_{IJ}$, explicitly. But they restricted their attentions only on the metric that can be brought to the field-independent form $G_{IJ} = \delta_{IJ}$ by an appropriate choice of parametrization.

In this paper, first, the result in [11] is generalized to the case with a generic field-space metric. It is found that the generalized expression is similar to the expression obtained in [8] for a trivial field-space metric. Then this expression is rewritten in terms of slow-rolling parameters.
2 The background and the curvature perturbation

The starting point is the effective action of the simple coupling system of Einstein gravity and scalar fields with an arbitrary inflation potential $V(\phi^I)$

$$S = \int \sqrt{-g} d^4x \left[ \frac{M_p^2}{2} R - \frac{1}{2} G_{IJ} \partial_\mu \phi^I \partial^\mu \phi^J - V(\phi^I) \right],$$  \hspace{1cm} (1)

where $G_{IJ} \equiv G_{IJ}(\phi^K)$ represents the metric on the manifold parameterized by the scalar field values, the 'target space' metric, and $8\pi G = M_p^{-2}$ represents the reduced Planck mass. Units are chosen such that $c = \hbar = 1$. For the background model, the Friedmann-Robertson-Walker metric is used,

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j.$$  \hspace{1cm} (2)

Take the background scalar fields as $\phi^I(t)$. Then background equations of the scalar fields are

$$\ddot{\phi}^I + 3H \dot{\phi}^I + \Gamma^I_{JK} \dot{\phi}^J \dot{\phi}^K + G^{IJ} V_I = 0,$$  \hspace{1cm} (3)

where $\Gamma^I_{JK} = \frac{1}{2} G^{IL} (G_{KL,J} + G_{LK,I} - G_{JK,L})$, are the target space Christoffel symbols. $H = \dot{a}/a$ is the Hubble parameter, $\dot{\phi}^I = d\phi^I/dt$, $\ddot{\phi}^I = \frac{d^2\phi^I}{dt^2}$ and $V_I = \frac{\partial V}{\partial \phi^I}$, $G_{IJ,K} = \frac{\partial G_{IJ}}{\partial \phi^K}$. Basing on the Einstein equation, we get

$$\frac{\ddot{a}}{a} = -\frac{1}{3M_p^2} (G_{IJ} \dot{\phi}^I \dot{\phi}^J - V).$$  \hspace{1cm} (4)

Together with the Friedmann equation

$$H^2 = \frac{1}{3M_p^2} \left( \frac{1}{2} G_{IJ} \dot{\phi}^I \dot{\phi}^J + V \right),$$  \hspace{1cm} (5)

we get

$$\dot{H} \equiv \frac{dH}{dt} = -\frac{1}{2M_p^2} G_{IJ} \dot{\phi}^I \dot{\phi}^J.$$  \hspace{1cm} (6)

Now, let’s consider the perturbed scalar fields as $\phi^I(t) + \delta \phi^I(t, x)$, and define the curvature perturbation. Here the curvature perturbation refers
to the uniform density curvature perturbation, $\zeta$, which is still equivalent to the comoving curvature perturbation on super horizon scales in multiple field inflationary models. The curvature perturbation is defined as the difference between an initial space-flat fixed-$t$ slice and a final uniform energy density fixed-$t$ slice (see [20, 11, 15] for details),

$$\zeta(t, x) = \delta N = H\delta t,$$

where $N = \int Hdt$ is the integrated number of e-folds. Following the argument in [11], we expand the curvature perturbation to the second order,

$$\zeta \simeq N, I(t)\delta\phi^I(x) + \frac{1}{2}N, IJ(t)\delta\phi^I(x)\delta\phi^J(x),$$

where $N, I = \frac{\partial N}{\partial\phi^I}$, $N, IJ = \frac{\partial^2 N}{\partial\phi^I\partial\phi^J}$. In this equation, it is the partial differentiation, not the covariant differentiation that is used, which is the same as in [8]. This is due to the definition of the curvature perturbation.

3 the non-linear parameter, $f_{NL}$

The non-Gaussianity of the curvature is expressed in the form

$$\zeta = \zeta_g - \frac{3}{5}f_{NL}(\zeta^2 - \langle\zeta^2\rangle),$$

where $\zeta_g$ is Gaussian, with $\langle\zeta_g\rangle = 0$, and $f_{NL}$ is the non-linear parameter. For a generic cosmological perturbation, $\psi(t, x)$, we define its Fourier components as $\psi(k) = \int d^3x\psi(t, x)e^{ik\cdot x}$. Then, using Eq.(9), we get

$$\zeta(k) = \zeta_g(k) - \frac{3}{5}f_{NL}\left\{\int \frac{d^3k_1}{(2\pi)^3}[\zeta_g(k_1)\zeta_g(k - k_1)] - (2\pi)^3\delta^3(k)\langle\zeta^2_g\rangle\right\}$$

On the other hand, using Eq.(8), we get

$$\zeta(k) = N, I\delta\phi^I(k) + \frac{1}{2}N, IJ\left\{\int \frac{d^3k_1}{(2\pi)^3}[\delta\phi^I(k_1)\delta\phi^J(k - k_1)] - (2\pi)^3\delta^3(k)\langle\delta\phi^I\delta\phi^J\rangle\right\}.$$
The spectrum of $\zeta_g, P_\zeta(k)$, is defined in the standard way by

$$\langle \zeta_g(k_1)\zeta_g(k_2) \rangle = (2\pi)^3 \delta^3(k_1 + k_2) P_\zeta(k_1), \quad (12)$$

with $k \equiv |k|$. Together with Eq.(10), to first order of $f_{NL}$, the spectrum of $\zeta$ is

$$\langle \zeta(k_1)\zeta(k_2) \rangle \simeq \langle \zeta_g(k_1)\zeta_g(k_2) \rangle = (2\pi)^3 \delta^3(k_1 + k_2) P_\zeta(k_1). \quad (13)$$

For this multiple field model, by assuming the quasi exponential inflation, basing on the results in [15, 16], we may define the spectrum of the scalar fields as

$$\langle \delta\phi^I(k_1)\delta\phi^J(k_2) \rangle = (2\pi)^3 \delta^3(k_1 + k_2) P_{\delta\phi}(k_1) G^{IJ}(\phi_*), \quad (14)$$

with $\frac{k^3}{2\pi^2} P_{\delta\phi}(k) = \left( \frac{H}{2\pi} \right)^2$. The subscript, $\ast$, means the value calculated at the moment that the corresponding scale crosses out the Hubble horizon, $k = aH$. Using the equations (11) and (14), to leading order, we get the other expression of the spectrum of $\zeta$,

$$\langle \zeta(k_1)\zeta(k_2) \rangle \simeq (2\pi)^3 \delta^3(k_1 + k_2) P_{\delta\phi}(k_1) N_{,I}N_{,J} G^{IJ}. \quad (15)$$

Comparing Eq.(13) with Eq.(15), we obtain the relation between $P_\zeta(k)$ and $P_{\delta\phi}(k)$,

$$P_\zeta(k) = P_{\delta\phi}(k) N_{,I}N_{,J} G^{IJ}. \quad (16)$$

Now let’s calculate the bispectrum of $\zeta$. Using Eq.(10), to the first order of $f_{NL}$, we get

$$\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle = (2\pi)^3 \delta^3(k_1 + k_2 + k_3) \times \left\{ -\frac{6}{5} [P_\zeta(k_1)P_\zeta(k_2) + \text{cyclic}] \right\}, \quad (17)$$

where cyclic refers to the term, $P_\zeta(k_2)P_\zeta(k_3) + P_\zeta(k_3)P_\zeta(k_1)$. Above, we have used Eq.(12). On the other side, using Eq.(11), we get

$$\langle \zeta(k_1)\zeta(k_2)\zeta(k_3) \rangle \simeq (2\pi)^3 \delta^3(k_1 + k_2 + k_3) B(k_1, k_2, k_3), \quad (18)$$

with

$$B(k_1, k_2, k_3) \equiv N_{,I}N_{,J}N_{,K} L G^{IK} G^{JL} [P_{\delta\phi}(k_1)P_{\delta\phi}(k_2) + \text{cyclic}], \quad (19)$$
where cyclic refers to the term, \( P_{\delta\phi}(k_2)P_{\delta\phi}(k_3) + P_{\delta\phi}(k_3)P_{\delta\phi}(k_1) \). Here, we note that, in Eq.(18), we have ignored the contribution of the term,

\[
N_JN_JN_K \langle \delta\varphi^I(k_1)\delta\varphi^J(k_2)\delta\varphi^K(k_3) \rangle,
\]

which comes from the intrinsic non Gaussianity of \( \delta\varphi^I(k) \). We know, that for the case with the trivial target space metric, \( G_{IJ} = \delta_{IJ} \), in [17], it has been proved that the contribution of the intrinsic non Gaussianity is small enough to be neglected. In this paper, we suppose that, for nearly Gaussian perturbations, \( \delta\varphi^I \), the intrinsic non Gaussianity (20) is still small enough to be neglected in the context of slow-roll inflation, and the bispectrum of \( \zeta \) can be obtained from Eq.(18).

Comparing Eq.(17) and Eq.(18), we get the non-linear parameter as

\[
f_{NL} = -\frac{5}{6} \times \frac{G_{IM}G_{KN}N_JN_KN_{MN}}{(N_JN_JG^{IJ})^2}.
\]

This is an important result of this paper. Although this expression is the same as the second term on the right-hand side of Eq.(38) in Ref.[8], here we obtain it for general multi-field inflationary models.

## 4 \( f_{NL} \) and slow-rolling parameters

In this section, with the slow-rolling condition, we try to express \( f_{NL} \) in term of the slow-rolling parameters. So we define some parameters. The first is \( \varepsilon \) defined as

\[
\varepsilon = -\frac{\dot{H}}{H^2}.
\]

Using Eq.(6), we get

\[
\varepsilon = \frac{G_{IJ}\dot{\varphi}^I\dot{\varphi}^J}{2M_p^2H^2}.
\]

Now let’s use the slow-rolling approximation. Then Eq.(3) becomes

\[
3H\dot{\varphi}^I + G^{IJ}V_{,J} \approx 0 \Rightarrow \dot{\varphi}^I \approx -\frac{G^{IJ}V_{,J}}{3H}
\]

And Eq.(5) becomes

\[
H^2 \approx \frac{1}{3M_p^2}V.
\]
So $\varepsilon$ can rewritten approximately as

$$
\varepsilon \simeq \frac{G^{IJ} V_I V_J M_p^2}{2 V^2}.
$$

(26)

Then we define another parameter, $\varepsilon_I$, as

$$
\varepsilon_I \equiv -\frac{V_I M_p}{\sqrt{2} V}.
$$

(27)

This implies an relation, $\varepsilon = G_{IJ} \varepsilon^I \varepsilon^J$.

In order to express $f_{NL}$ by the slow-rolling parameters, we should firstly get the expression of $N_{IJ}$. From Eq.(7), we get

$$
\delta N = H \delta t = -\frac{1}{\varepsilon} d \ln H \simeq -\frac{1}{\varepsilon} d \ln \sqrt{V} = -\frac{1}{2 \varepsilon V} V_I \delta \varphi^I.
$$

(28)

Then it may be supposed that we can extract the derivation of $N$ with respect to $\varphi^I$,

$$
N_{IJ} \simeq -\frac{1}{2 \varepsilon V} V_I \frac{\varepsilon_I}{\sqrt{2} \varepsilon M_p}.
$$

(29)

However, this equation can not be applied to Eq.(21) unless the perturbations during multi-field inflation are purely adiabatic. In fact, for a general multi-field model, the entropy perturbation do exist. (See Ref.[21] for an extensive explanation.)

In order to use Eq.(29), in this section, we impose the condition: Adiabatic Perturbations. This implies that the result in this section is only applicable to the multi-field inflation during which the perturbations are purely adiabatic.

Then we get

$$
G^{IJ} N_{IJ} = \frac{1}{2 \varepsilon M_p^2}.
$$

(30)

The derivation of $\varepsilon_I$ or $\varepsilon$ with respect to $\varphi^J$ can be expressed approximately as

$$
\frac{\partial \varepsilon_I}{\partial \varphi^J} \simeq -\frac{\partial}{\partial \varphi^J} \left( \frac{V_I M_p}{\sqrt{2} V} \right) = \sqrt{2} \left( \varepsilon_I \frac{\varepsilon_J}{\sqrt{2}} - \frac{1}{2} \eta_{IJ} \right),
$$

(31)

$$
\frac{\partial \varepsilon}{\partial \varphi^J} = \frac{\partial}{\partial \varphi^J} (G^{IJ} \varepsilon_I \varepsilon_J) \simeq G^{KL}_{IJ} \varepsilon_K \varepsilon_L + \sqrt{2} \frac{1}{M_p} (2 \varepsilon_J - G^{KL} \varepsilon_K \eta_{LJ}),
$$

(32)
with $\eta_{IJ} \equiv \frac{V_{IJ} M^2}{\sqrt{2}}$ and $G^{KL, J} \equiv \frac{\partial G^{KL}}{\partial \varphi^J}$. Now it is the time to calculate $N_{IJ}$,

$$N_{IJ} = \frac{\partial}{\partial \varphi^J} \left( \frac{\varepsilon_I}{\sqrt{2 \varepsilon}} \right) \simeq \frac{1}{\varepsilon^2 M_p^2} \left\{ G^{KL} \varepsilon_K \varphi_J \varepsilon_I - \frac{G^{KL, J}}{\sqrt{2} M_p} \varepsilon_K \varphi_L \varepsilon_I \varepsilon_J - \varepsilon \varepsilon_I \varphi_J \varepsilon_J - \frac{1}{2} \varepsilon \eta_{IJ} \right\}. \quad (33)$$

Now it is easy to get

$$G^{IK} G^{JL} N_{I} N_{J} N_{K} N_{L} = - \frac{1}{2 \varepsilon^3 M_p^3} \beta, \quad (34)$$

with

$$\beta \equiv \frac{\varepsilon^2}{M_p} + \frac{G^{MN, L}}{\sqrt{2}} G^{JL} \varepsilon_M \varepsilon_N \varepsilon_J - \frac{1}{2 M_p} G^{JL} G^{MN} \varepsilon_J \varepsilon_M \eta_{NL}. \quad (35)$$

Substituting Eq.(30) and (34) into Eq.(21), we can express the nonlinear parameter in the form as

$$f_{NL} \simeq \frac{5}{3} \times \frac{\beta M_p}{\varepsilon}. \quad (36)$$

Here we emphasize again that Eq.(36) is only applicable to the multi-field models in which the entropy perturbations can be neglected.

5 Summary

In this paper, the “$\delta N$-formalism” suggested to express the non Gaussianity in [11] is generalized to the multi-field inflationary models with the non-trivial target space metric. One key step in our derivation is the equation (14). We believe that this equation is correct [15, 16]. We have rewritten the result by using the slow-rolling approximation, which is easy to be analyzed. But this result is obtained under the condition of Adiabatic Perturbations. For a general multi-field model, we should use Eq.(21) to calculate $f_{NL}$.

Additionally, in this paper we have restricted our attention to the contribution of the Gaussian part of $\delta \varphi'$ and ignored the contribution of the intrinsic non Gaussianity of $\delta \varphi'$, (20). We emphasize that in some case the term (20) should be included. This will be discussed in future work.
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