Loop quantum cosmology
and the Wheeler-De Witt equation

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Abstract
We present some results concerning the large volume limit of loop
quantum cosmology in the flat homogeneous and isotropic case. We
derive the Wheeler-De Witt equation in this limit. Looking for the
action from which this equation can also be obtained, we then address
the problem of the modifications to be brought to the Friedman’s
equation and to the equation of motion of the scalar field, in the
classical limit.

I. Introduction.
In recent times we have witnessed an increasing interest in the applications
of the ideas of loop quantum gravity (lqg) to the problems of cosmology. This
started with a series of seminal papers by M. Bojwald [1-8] on loop quantum
cosmology, where he obtained a number of impressive results, among them
the possibility of removing, in a natural way, the presence of the cosmological
singularity.

Although importing from loop quantum gravity the techniques and ideas,
a number of issues still remain unclear. Loop quantum cosmology (lqc) begins
with a drastic reduction of the phase space of the full theory, resulting, among
other things, in contrast to the situation in the full theory, in a quantization
ambiguity in the form of a parameter which, following [9], we shall call $\mu_0$,
playing the role of regulator. The value of $\mu_0$ cannot be calculated within lqc, but has to be taken from the full theory, where it signals the fundamental discreteness of geometry, one of the most fascinating results obtained by loop quantum gravity. Other problems involve, for instance, the interpretation of the notion of quantum state and the issue of the time variable, which variable in lqc is replaced by the momentum variable controlling the evolution of the discrete quantum constraint equation (see references 10 and 11 for further comments on lqc).

Another important ambiguity arises, as a result of the reduction in the phase space of the theory. It is well known that the classical Hamiltonian of general relativity can be written as the sum of two terms which, in the homogeneous and isotropic case, become proportional to each other; the Hamiltonian reduces itself, then, to one term. Using this property, and then quantizing, a discrete equation for the evolution of the system is obtained. This is the procedure normally used in the applications [9]. However, if we separately apply, to the two terms, the same quantization procedure, with the same connection and holonomy, we obtain instead equation (20) below (see also ref. [7]). Thus, it is not indifferent the order followed to derive the evolution equation. In addition, the two equations lead to different large volume limits (section III).

An important part of the work done on lqc has turned around the study of the semi-classical behaviour of the solutions of the constraint equation, for different matter Hamiltonians, in the large volume limit. This will also be the focus of our present work. Within the context of a homogeneous and isotropic universe, assumed to be dominated by a massive scalar field, we derive a modified form of the Wheeler-De Witt equation from the large volume limit of lqc, starting from equation (20). We then address the issue of the modifications to bring to the Friedman’s equation and to the equation of motion of the scalar field, equations usually applied in lqc, by looking for the action from which, using conventional methods, we derive the above modified form of the Wheeler-De Witt equation.

II. The quantum evolution equation.
   IIa. The Hamiltonian for the scalar field.

Given the importance of scalar fields in modern cosmology, we derive in this section the Hamiltonian for the scalar field, in a flat homogeneous and isotropic universe, with the help of the expressions for the connection and holonomy given in [9].
The holonomy is

\[ h_i = \cos \frac{\mu c}{2} + 2 \sin \frac{\mu c}{2} \tau_i, \]  

(1)

where \( \mu \) defines a length along which we integrate, not a physical length, and \( c \) is the real variable containing the information on the connection. The basis vectors, defining the eigenstates of the momentum operator \( \hat{p} \), are, in the bra-ket notation,

\[ \langle c | \mu \rangle = \exp \frac{i \mu c}{2} \]  

(2)

with the operator

\[ \hat{p} = -i \frac{\gamma l_{pl}^2}{3} \frac{d}{dc} \]  

(3)

acting as

\[ \hat{p} | \mu \rangle = \frac{\gamma l_{pl}^2}{6} \mu | \mu \rangle \equiv p_\mu | \mu \rangle, \]  

(4)

\( \mu \in (-\infty, +\infty) \) and \( l_{pl}^2 = 8\pi G \). The eigenvalues \( V_\mu \) of the volume operator are obtained from the relation \( V = |p|^{3/2} \):

\[ V_\mu = \left( \frac{\gamma l_{pl}^2}{6} | \mu | \right)^{3/2}. \]  

(5)

In the expressions for \( p_\mu \) and \( V_\mu \), the constant \( \gamma \) represents the Barbero-Immirzi parameter.

We apply these expressions to the hamiltonian operator derived by Thiemann in [12], which, in the case of a real scalar field, becomes, after the appropriate calculation of the traces:

\[ \hat{H}_\phi \psi_\mu(\phi) = \frac{8m_{pl}^9}{81l_{pl}^6} (V_{\mu+\mu_0}^{1/2} - V_{\mu-\mu_0}^{1/2}) P_\phi^2 + \frac{m_{pl}^3}{l_{pl}^3} V_\mu W(\phi) \} \psi_\mu(\phi). \]  

(6)

The scalar field \( \phi \) and the momentum operator \( \hat{p}_\phi = -i\hbar d/d\phi \) have been rescaled and are dimensionless; we can check that \( \hat{H}_\phi \) has the correct dimensions of a mass. In the case of a scalar field of mass \( m \), we have the potential \( W(\phi) = \frac{1}{2} m^2 \phi^2 \) (\( m \) in planck units).
We see that the operator $\hat{H}_\phi$ naturally selects the ambiguity parameters, entering into the definition of the inverse scale factor, as $l = 3/4$ and $j = 1/2$.

Writing, as in ref. [9], the physical state as the sum

$$\langle \Psi | = \sum_\mu \psi_\mu(\phi) \langle \mu |,$$  

(7)

where $\phi$ represents the matter (scalar field of mass $m$) degrees of freedom, we derive the action of the Hamiltonian on $\psi_\mu(\phi)$. With a potential proportional to $\phi^2$, we may look for solutions in the form of linear combinations of the Hermite special functions. Taking the simplest case,

$$\psi_\mu(\phi) = \psi_\mu \exp(-\alpha_\mu \phi^2)$$  

(8)

we find that

$$\hat{H}_\phi \psi_\mu(\phi) = E_\mu \psi_\mu(\phi)$$  

(9)

where

$$E_\mu = \frac{1}{2} \sqrt{A_\mu B_\mu m}$$  

(10)

and

$$\alpha_\mu = \frac{1}{2} \sqrt{B_\mu / A_\mu m}$$  

(11)

with the expressions for $A_\mu$ and $B_\mu$ taken from (6):

$$A_\mu = \frac{4m_{pl}}{g^\mu_{\mu_o}} (V^{1/2}_{\mu + \mu_o} - V^{1/2}_{\mu - \mu_o})^6$$  

(12)

and

$$B_\mu = \frac{m_{pl}}{l^3} V_\mu.$$  

(13)

With these definitions we rewrite $\hat{H}_\phi$ in the form

$$\hat{H}_\phi = \frac{1}{2} A_\mu \dot{\phi}_\mu^2 + B_\mu \frac{1}{2} m^2 \phi^2.$$  

(14)

In section III. we shall apply these expressions to integrate the Wheeler-DeWitt equation, using numerical methods.
IIb. The discrete evolution equation obtained from the gravitational operator.

To obtain this equation we shall apply the holonomy operator (1) to the full Hamiltonian operator derived by Thiemann (see Thiemann’s [13] and also the report by A. Ashtekar and J. Lewandowski [14]), defining the regulated constraint operator:

\[
(\Psi|\hat{H}_{Re}(N) = (\Psi| \sum_{\Box} (\sqrt{\gamma} \hat{C}_{\Box}^{\mathrm{euc}}(N) - 2(1 + \gamma^2)\hat{T}_{\Box}(N)),
\]

where the Euclidean scalar constraint is

\[
\hat{C}_{\Box}^{\mathrm{euc}}(N) = -\frac{iN}{g^{3/2}h} \sum_{i,j} e^{ijk} Tr( (\hat{h}_{i\alpha j} - \hat{h}_{\alpha ij})(\hat{h}_{i\alpha j}^{\mu 0} - [\hat{h}_{i\alpha j}^{\mu 0}, \hat{V}])
\]

and the operator corresponding to the extrinsic curvature terms

\[
\hat{T}_{\Box}(N) = \frac{iN}{g^{3/2}h} e^{ijk} Tr( (\hat{h}_{i\alpha j}^{\mu 0})^{-1} [\hat{h}_{i\alpha j}^{\mu 0}, \hat{K}'][\hat{h}_{i\alpha j}^{\mu 0}]^{-1} [\hat{h}_{i\alpha j}^{\mu 0}, \hat{V}])
\]

We shall use the expansion (7) for (\Psi| and, with the help of equations (15-18), rewrite (19) as a discrete evolution equation (units \(c=1, h=1\)) (see [7], where such expression was for the first time derived):

\[
(\Psi|(|\hat{H}^{\mu 0}_{\text{grav}} + 16\pi G\hat{H}_{\phi})^{\dagger} = 0;
\]

we shall use the expansion (7) for (\Psi| and, with the help of equations (15-18), rewrite (19) as a discrete evolution equation (units \(c=1, h=1\)) (see [7], where such expression was for the first time derived):
\[-2(V_{\mu+\mu_0} - V_{\mu-\mu_0})D_{\mu}\psi_{\mu}(\phi) + (V_{\mu-7\mu_0} - V_{\mu-9\mu_0})D_{\mu-8\mu_0}\psi_{\mu-8\mu_0}(\phi)] =
\]
\[-16\pi G\hat{H}_\phi\psi_{\mu}(\phi),
\]
where the coefficients $D_{\mu}$ are complicated functions of $V_{\mu}$ given in the appendix A. This equation constrains the functions $\psi_{\mu}(\phi)$, defined in equation (7), to ensure that the $|\Psi\rangle$ belong to the space of physical states. As mentioned in the introduction, the fact that this equation is different from the corresponding equation in reference [9], shows that the operations of reducing the phase space and quantizing do not commute between them.

We used the ordering scheme adopted by Bojwald. Had we adopted a symmetrized version, we could verify that $\psi_{\mu=0}$ would not decouple from the system.

It is known that, in the case of constraint operators, we are only interested in their kernel and the operators are not required to be self-adjoint. This seems to be an advantage in quantum gravity, an open algebra category of constrained system [13], as the self-adjoint requirement may lead to quantum anomalies. On the other hand, dropping this requirement, we are left with the following awkward situation. The matter hamiltonian is required from quantum mechanics to be self-adjoint, and $\hat{H}_\phi$ indeed obeys this requirement; then, taking the adjoint of the evolution equation, we recover the same $\hat{H}_\phi$, but not the same $\hat{H}_{\text{grav}}^{(\mu_0)}$. We are left with two distinct choices, where we should have one.

However, we are here interested in the large $\mu$ behaviour of the system and it happens that both versions, symmetrized and non-symmetrized, lead to exactly the same equations; we may study this behaviour without bothering with this problem. To investigate the limit of large $\mu$, it is more convenient to study the differential equation resulting from (20) in such a limit.

Before proceeding, let us mention that, applying the methods developed in [9], section 4.3, we again find the classical limit

\[< H_{\text{grav}}^{(\mu_0)}> = \frac{6}{l_p^n^2} \epsilon_0^2 \gamma^2 \sqrt{p} + ...
\]

with $p \equiv \gamma l_p^n n_\mu_0 / 6$, $n \gg 1$, where we added contributions coming from the two terms in equation (20).

III. The large volume limit.

It is important to keep present the fact that the large volume limit, large values of $\mu$, it is not the same as the limit, usually taken, $\mu_0 \to 0$. This
can be seen, for instance, from inspection of equation (42) in [9]. It is the
behaviour in the large volume limit we are interested in. Working out this
limit in terms of \( \mu_0 / \mu \), all the volume functions appearing on the l.h.s. of
the evolution equation become proportional to \( \sqrt{\mu / n} \mu_0 \) times powers of
\( \mu_0 \). Then, adding and subtracting, after multiplication by an appropriate
constant, the terms corresponding to the first bracket (in \( \psi_{\mu \pm 4\mu_0} \) and \( \psi_{\mu} \)) of
equation (20), we can rewrite the l.h.s. in a form appropriate to this lim it:

\[
\text{l.h.s.} = \frac{1}{\gamma} - \frac{1 + \gamma^2}{\gamma^3} \frac{9}{\mu_0^2} \left[ \frac{1}{\mu_0^2} \left( \sqrt{\mu + 4\mu_0 \psi_{\mu + 4\mu_0}}(\phi) - 2\sqrt{\mu \psi_{\mu}}(\phi) + \sqrt{\mu - 4\mu_0 \psi_{\mu - 4\mu_0}}(\phi) \right) \right] - \\
\frac{-9}{4} \frac{1 + \gamma^2}{\gamma^3} \frac{1}{\mu_0} \left[ \sqrt{\mu + 8\mu_0 \psi_{\mu + 8\mu_0}}(\phi) - 4\sqrt{\mu + 4\mu_0 \psi_{\mu + 4\mu_0}}(\phi) + \\
+ 6\sqrt{\mu \psi_{\mu}}(\phi) - 4\sqrt{\mu - 4\mu_0 \psi_{\mu - 4\mu_0}}(\phi) + \sqrt{\mu - 8\mu_0 \psi_{\mu - 8\mu_0}}(\phi) \right].
\]

(22)

What we have, within the square brackets, are the algorithms for second
and fourth derivatives. The limit we are going to take is not \( \mu_0 \rightarrow 0 \), but the
limit of large \( \mu \); the replacement of the discrete expressions by derivatives is
only an approximation which becomes better and better as \( \mu \) gets larger and
larger.

Converting the combinations of \( \mu, \mu \pm 4\mu_0 \), etc. into momentum variables,
through equations (4) and (5), we finally find the equation replacing (20):

\[
\frac{2}{3} l_p^4 [f(p) \sqrt{p} \psi(p, \phi)]'' + \frac{2}{27} \gamma^2 (1 + \gamma^2) \mu_0^2 l_p^6 (\sqrt{p} \psi(p, \phi))''' \approx \\
-16 \pi G \hat{H}_{\phi} \psi(p, \phi),
\]

(23)

where the derivatives are with respect to the momentum \( p \). The term \( f(p) \)
is given by

\[
f(p) = (1 - \frac{(\mu_0 \gamma l_p^2)^2}{288} \frac{1}{p^2} + \ldots)
\]

(24)

and is usually put equal to 1, neglecting the small corrections.

We see that, had we taken the limit \( \mu_0 \rightarrow 0 \), we would have recovered the
standard form of the Wheeler-De Witt equation. This not being the case, we
find in addition an extra term involving a fourth derivative. This term comes
from the two-fold application, when deriving (20), of the \( \hat{C}^{\text{eucl}} \) operator which appears through the definition of the extrinsic curvature operator \( \hat{K}' \), in equation (18). Having begun with an hamiltonian formulation of a classical theory involving second derivatives, we end up with a quantum equation including a fourth derivative.

In the vacuum case, equation (23) has analytical solutions of the form found by Bojowald in [7], with two of them obeying the pre-classicality condition of mild variation. When the matter term is included, we have not found solutions that could be put into an analytical form. We included the matter term in the form of a scalar field, using the procedure outlined in section II.a, equations (7 - 14). We solved equation (23) by numerical methods (see figures 1-4, with \( y = \sqrt{p}\psi \)) and found that we continue to have solutions, fig. 1 for instances, whose behaviour can be made as slow as we wish, by a convenient choice of initial conditions. They thus seem to fulfill the basic requirement for pre-classicality.

IV. The classical equations of motion.

We would now like to ask which classical action, using the conventional methods, gives us back the Wheeler-De Witt equation (23), without the term in \((\sqrt{p}\psi)''''\), and with \( \hat{H}_\phi \) defined by (14). (We follow J. B. Hartle’s lectures, ref. [15]).

The restriction to a homogeneous and isotropic geometry gives the simplest class of minisuperspace models. The line element is

\[
ds^2 = -N^2(t)dt^2 + a^2(t)d\bar{x}^2,
\]

where now \( N(t) \) is the (arbitrary) lapse function and \( a(t) \) the dimensionless scale factor. We continue to take for the matter content a massive scalar field and define our effective classical action by the expression (representing, for the moment, the scalar field by \( \varphi \))

\[
S(a, \varphi) = \frac{1}{2} \int dt N \left[ \frac{1}{16\pi G} \left( -6 \frac{1}{f(a)} \frac{1}{a} \left( \frac{aa'}{N} \right)^2 + 6ka \right) + \frac{1}{A(a)} \left( \frac{\varphi'}{N} \right)^2 - \frac{1}{2} \frac{1}{B(a)} m^2 m^2 \varphi^2 \right],
\]

where \( m \) is in planck units, \( (') \equiv d/dt \) and the lagrangean is given by \( L = \delta S/\delta t; \) from now on we put \( k = 0, \) flat case. The only non-conventional elements are the functions \( A(a) \) and \( B(a), \) to be defined later, and \( f(a), \)
given by (24), now expressed as a function of the scale $a$. Introducing the momenta

$$
\pi_a = \frac{\delta L}{\delta a'} = -\frac{6}{16\pi G} a' f(a) \quad (27)
$$

$$
\pi_\phi = \frac{\delta L}{\delta \phi'} = \frac{1}{A} \phi', \quad (28)
$$

we find the constraint by varying $L$ with respect to $N$; writing the resulting equation in terms of $\pi_a$ and $\pi_\phi$ we have

$$
\frac{16\pi G}{12} a^{\frac{2}{3}} f(a) - \frac{1}{2} A(a) \pi_\phi^2 - \frac{1}{2} B(a) m_{pl}^2 \phi^2 = 0. \quad (29)
$$

(Note the positions of $A(a)$ and $f(a)$ in (26) and (29)).

We shall now replace $\phi$ by $\phi/\sqrt{6}$, followed by the usual replacements of $\pi_a$ and $\pi_\phi$ by the operators

$$
\pi_a \rightarrow -i \left( \frac{1}{16\pi G} \right)^{3/2} \frac{\partial}{\partial a}, \quad (30)
$$

$$
\pi_\phi \rightarrow -i \left( \frac{1}{16\pi G} \right) \frac{\partial}{\partial \phi}, \quad (31)
$$

where we took $\phi$ to be dimensionless. The factor ordering ambiguities, affecting the gravitational operator, will be lifted by writing it in the form suggested by lqc; then equation (29) becomes

$$
\left[ -\frac{1}{2} a \left( \frac{1}{a} \frac{\partial}{\partial a} \frac{1}{a} \frac{\partial}{\partial a} f(a) \right) + \frac{1}{2} A \frac{\partial^2}{\partial \phi^2} - \frac{1}{2} B m_{pl}^2 \phi^2 \right] \Psi(a, \phi) = 0, \quad (32)
$$

after dropping an overall constant factor $\frac{1}{6} \left( \frac{1}{16\pi G} \right)^2$.

To find the Wheeler-De Witt equation (23), all that remains is to express $a$ in terms of $p$ and introduce $A(p)$ and $B(p)$, the already known functions given by equations (12) and (13), but now expressed directly in terms of $p$ ($V = |p|^{3/2}$). Remembering that $a$ is dimensionless ($a^2$ is in fact the $\mu$ of sections II and III), we introduce the relation

$$
-\frac{1}{2} a \left( \frac{1}{a} \frac{\partial}{\partial a} \frac{1}{a} \frac{\partial}{\partial a} \right) = -2 \left( \frac{\gamma l_p^2}{6} \right) \frac{\partial^2}{\partial p^2}\quad (33)
$$

and the definitions
\[16\pi GA(p) \equiv \frac{1}{3}\left(\frac{6}{\gamma}\right)^{3/2}l_{pl}A(p)\tag{34}\]

\[16\pi GB(p) \equiv \frac{1}{3}\left(\frac{6}{\gamma}\right)^{3/2}l_{pl}B(p)\tag{35}\]

\[\Psi(p, \phi) \equiv \sqrt{p}\psi(p, \phi)\tag{36}\]

With the help of these expressions we find

\[\frac{2}{3}l_{pl}^4 \frac{\partial^2}{\partial p^2}(f(p)\sqrt{p}\psi(p, \phi)) = 16\pi G\left[\frac{1}{2}A(p)\frac{\partial^2}{\partial \phi^2} - \frac{1}{2}B(p)m^2\phi^2\right]\psi(p, \phi), \tag{37}\]

which is equation (23) without the the term \((\sqrt{p}\psi)'''\) and with \(\dot{H}_\phi\) given by (14). Thus, we may now use the action \(S(a, \varphi = \phi/\sqrt{6})\) to derive the effective classical equations of motion (see Bojowald, ref. [16], for a different way of arriving at these modified equations).

Again, varying \(L = \delta S/\delta t\) with respect to \(N\) we get the modified Friedman’s equation, after replacing \(\overline{A}(a)\) and \(\overline{B}(a)\) by \(A(a)\) and \(B(a)\):

\[\frac{1}{f(a)}\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\left(\frac{6}{\gamma}\right)^{3/2}A(a)\gamma^3\phi'\right]^2 + \frac{1}{2}\left(\frac{B(a)}{2(\gamma/6)^3}\right)m^2\phi^2\}. \tag{38}\]

We see that the form of the equation is correct. With \(A(a)\) representing, respectively, the inverse volume and the volume, apart from some constants which can be absorbed into \(\phi\) and \(m\), we have that \(a^3A(a)\) and \(B(a)/a^3\) become equal to 1 in the large volume limit; we thus recover the standard form of the equation, taking also into account that \(f(a) \approx 1\).

Variation with respect to the scalar field gives

\[\phi'' - \frac{A(a)/A(a)}{A(a)}\phi' + \left(\frac{\gamma}{6}\right)^3A(a)B(a)m^2\phi = 0. \tag{39}\]

(See also Appendix B). Equations (38) and (39) are, with small differences, the equations normally used in the classical limit, when we wish to include corrections derived from lqc.

That is, from the same effective classical action we can derive both, the modified Wheeler-De Witt equation (23) and the equations (38) and (39).
As for the term $\partial^4/\partial p^4$, we have not yet found a reasonable way of incorporating into the effective action a new term corresponding to this fourth-order derivative. At first sight, we might think that, at least in simple situations, such a term would give a correction to the Friedmann’s equation proportional to $(a'/a)^4$, but a lot of care is necessary. This would come out by adding a term to $S$ of the form $ca(a'/a)^4/N^3$, $c$ proportional to the constants multiplying $\partial^4/\partial p^4$ in (23) (in planck units, $c$ is of the order of $\gamma^2\mu_0^2$). The new momentum $\pi_a$ would become

$$\pi_a = (1 - \frac{4}{3}c_{pl}^2 \frac{(a')^2}{N^2})(-6\frac{aa'}{16\pi GN}) \simeq (-6\frac{aa'}{16\pi GN})$$

in those, and only in those, situations where the second term in the first bracket can be neglected, that is, when $(\gamma\mu_0)^2l_{pl}^2(a')^2 \ll 1$. Then, varying $L$ with respect to $N$, we find that (for simplicity, we put $f(a) = 1$)

$$\frac{16\pi G}{12} \frac{1}{a} \pi_a^2 \rightarrow \frac{16\pi G}{12} \frac{1}{a} \pi_a^2 - \frac{3}{6^4}c(16\pi G)^4 \frac{1}{a^3} \pi_a^4.$$  \hspace{1cm} (41)

After the replacement of $\pi_a$ defined in (30), we would get, as in (23), the extra fourth-order derivative term $\sqrt{p}\partial^4/\partial p^4$ (the factor $\sqrt{p}$ later cancelling out) and, in the Friedman’s equation, a new term proportional to $a^2(a'/a)^4$.

Corrections of this form to the Friedman’s equation do, sometimes, appear in the literature of brane based cosmology ([19] and references therein).

V. Summary.

In the present work, after deriving a modified form of the Wheeler-De Witt equation, as the large volume limit of the discrete evolution equation of loop quantum cosmology, we looked for an effective classical action from which this equation could be obtained, using the conventional methods. Once this action has been defined, we may also derive from it the associated classical equations, the modified Friedmann’s equation and the scalar equation of motion for the massive scalar field, which we took as our matter content. These are the equations normally used in the applications of the loop quantum cosmology to the study of inflation. Numerical simulations were done with the modified Wheeler-De Witt equation, and we found that solutions exist that comply with the requirements of pre-classicality. We also found numerical solutions to the modified equation for the scalar field (Appendix B), having, as expected, the appropriate inflationary behaviour.

Further applications will be left to a future work.
Appendix A

The functions $D_\mu$ appearing in equation (20) are given by the following expressions:

\[ D_{\mu+8\mu_0} = D'_1[(V_{\mu+6\mu_0} - V_\mu + 4\mu_0)(V_{\mu+5\mu_0} - V_\mu + \mu_0) -
(V_\mu + 4\mu_0 - V_{\mu+2\mu_0})(V_{\mu+3\mu_0} - V_{\mu-\mu_0})] \] (42)

\[ D_\mu = \frac{1}{2}D_1[(V_{\mu-2\mu_0} - V_{\mu-4\mu_0})(V_{\mu+\mu_0} - V_{\mu-3\mu_0}) -
(V_{\mu-4\mu_0} - V_{\mu-6\mu_0})(V_{\mu-\mu_0} - V_{\mu-5\mu_0})] \]

\[ + \frac{1}{2}D_2(V_{\mu+6\mu_0} - V_{\mu+4\mu_0})(V_{\mu+5\mu_0} - V_{\mu+\mu_0}) -
(V_\mu + 4\mu_0 - V_{\mu+2\mu_0})(V_{\mu+3\mu_0} - V_{\mu-\mu_0}) \] (43)

\[ D_{\mu-8\mu_0} = D'_2[(V_{\mu-2\mu_0} - V_{\mu-4\mu_0})(V_{\mu+\mu_0} - V_{\mu-3\mu_0}) -
(V_{\mu-4\mu_0} - V_{\mu-6\mu_0})(V_{\mu-\mu_0} - V_{\mu-5\mu_0})], \] (44)

where $D_1$ and $D_2$ are

\[ D_1 = [(V_{\mu+2\mu_0} - V_\mu)(V_{\mu+\mu_0} - V_{\mu-3\mu_0}) -
(V_\mu - V_{\mu-2\mu_0})(V_{\mu-\mu_0} - V_{\mu-5\mu_0})] \] (45)

\[ D_2 = [(V_{\mu+2\mu_0} - V_\mu)(V_{\mu+5\mu_0} - V_{\mu+\mu_0}) -
(V_\mu - V_{\mu-2\mu_0})(V_{\mu+3\mu_0} - V_{\mu-\mu_0})] \] (46)

and

\[ D'_1 = [(V_{\mu+10\mu_0} - V_{\mu+8\mu_0})(V_{\mu+9\mu_0} - V_{\mu+5\mu_0}) -
(V_{\mu+8\mu_0} - V_{\mu+6\mu_0})(V_{\mu+7\mu_0} - V_{\mu+3\mu_0})] \] (47)

\[ D'_2 = [(V_{\mu+6\mu_0} - V_{\mu-8\mu_0})(V_{\mu-3\mu_0} - V_{\mu-7\mu_0}) -
(V_{\mu+4\mu_0} - V_{\mu+2\mu_0})(V_{\mu+5\mu_0} - V_{\mu+\mu_0})] \] (48)

\[ + \frac{1}{2}D_2(V_{\mu+6\mu_0} - V_{\mu+4\mu_0})(V_{\mu+5\mu_0} - V_{\mu+\mu_0}) -
(V_\mu + 4\mu_0 - V_{\mu+2\mu_0})(V_{\mu+3\mu_0} - V_{\mu-\mu_0}) \] (49)
\[
- (V_{\mu-8\mu_0} - V_{\mu-10\mu_0})(V_{\mu-5\mu_0} - V_{\mu-9\mu_0})].
\] (48)

Appendix B
Let us apply a simple formal procedure, to obtain equation (39). Assume a regime where it is valid to use the commutators

\[ [\hat{\phi}, \hat{p}_\phi] = i\hbar, \]

\[ d\hat{p}_\phi/dt = \frac{1}{i\hbar}[\hat{p}_\phi, \hat{H}_\phi] \] (49)

and

\[ d\hat{\phi}/dt = \frac{1}{i\hbar}[\hat{\phi}, \hat{H}_\phi], \] (50)

followed by a discretization, through the replacement of \( dt \) by a discrete step parameter \( h \). We get the equations (\( \hat{H}_\phi \) given by (14))

\[ \phi(\mu + h) = \phi(\mu) + hA_\mu p_\phi(\mu) \] (51)

\[ p_\phi(\mu + h) = p_\phi(\mu) + hB_\mu m^2 \phi(\mu). \] (52)

A little algebra shows that we can derive the following equation for \( \phi \):

\[
\frac{1}{h^2}(\phi(\mu + 2h) - 2\phi(\mu + h) + \phi(\mu)) = \left( \frac{1}{A_\mu} A_{\mu+h} - A_\mu \right) \frac{1}{h} (\phi(\mu + h) - \phi(\mu)) -
\]

\[- A_{\mu+h} B_\mu m^2 \phi(\mu) \] (53)

which, again, suggests that, in this limit, we should replace the \( 3(\dot{a}/a) \) term by the expression within the first bracket on the r.h.s. Numerical integration of this equation, for a large variety of initial conditions, with \( h \sim 4\mu_0 \), shows that, after a period of slow increase, \( \phi \) goes through a period of very fast increase until it settles down into an almost constant value, a behaviour that is well known and is the starting point of quantum geometry inflation [17, 18]. A simple numerical example can be seen in figure 5.

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Fig. 1 - The graph shows the solution of equation (23), the initial conditions are \( y''' = y'' = y' = 0, \ y = 1 \) at \( p_0 = 100 \). We assumed a scalar field mass equal to 0.1 planck units.

Fig. 2 - The same as figure 1, except that now \( y' = 1 \) and \( y''' = y'' = y = 0 \), at \( p_0 = 100 \).
Fig. 3 - Solution of equation (23); initial conditions \(y'' = 1, y''' = y' = y = 0\), at \(p_0 = 100\).

Fig. 4 - The same as previous figures; initial conditions \(y''' = 1, y'' = y' = y = 0\), at \(p_0 = 100\).
Fig. 5 - Numerical integration of the discrete equation (53) for the scalar field. Initial conditions were $\phi(0) = 0$, $\phi'(0) = 0.1$, scalar field mass=0.1 planck units.