Anyons as Dirac Strings, the $A_x = 0$ Gauge

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ABSTRACT

We show how to quantize the anyon particle theory in a gauge, $A_x = 0$, where the statistical potential $\vec{A}(\vec{x})$ is a Dirac string. In this gauge, anyons obey normal statistics.
Attempts to study the non-relativistic anyon model [1] have centered on 2 methods. Either anyons are taken to be normal bosons (or fermions) interacting with a statistical potential in the coulomb gauge \((\partial^a A_a(\vec{x}) = 0)\), or anyons are taken to be free particles obeying exotic statistics, a singular gauge [1,2]. There exists a third possibility which may be more useful especially in systems having discrete translation symmetries (ex. anyonic crystals). It involves taking anyons to be bosons (or fermions) interacting with a statistical potential having the form of a Dirac string [3]. We will study this possibility which corresponds to the gauge \(A_x(\vec{x}) = 0\).

An anyon is a charged flux line in 2+1 dimensions. We start in constructing the vector potential for a unit flux line located at the origin. The associated vector potential satisfies the equation \(\vec{\partial} \times \vec{A}(\vec{x}) = \delta^2(x)\). The potential \((A_x(\vec{x}), A_y(\vec{x})) = (0, \frac{1}{2} \delta(y) \epsilon(x))\), \(\epsilon(x) = \pm 1\) for \(x > 0\), satisfies the gauge condition \(A_x(\vec{x}) = 0\) and gives the correct magnetic field. This potential has Dirac string singularities along the positive and negative \(x\)-axes and vanishes elsewhere. Thus, the \(A_x(\vec{x}) = 0\) gauge corresponds to treating anyons as pure Dirac strings. We can write the Hamiltonian for N-anyons by taking a combination of such Dirac strings. For N free non-relativistic anyons, it is given by:

\[
H_{\text{anyon}} = \frac{1}{2m} \sum_{i=1}^{N} \left[ -\partial_{x_i}^2 + (-i\partial_{y_i} + \frac{\alpha}{2} \sum_{j \neq i} \delta(y_i - y_j) \epsilon(x_i - x_j))^2 \right] \tag{1}
\]

Our gauge choice is not a singular gauge in the sense that the N-particle wavefunction of (1) satisfies normal statistics. To see this more clearly, we will relate the \(A_x = 0\) gauge to the singular gauge\(^2\) \((\vec{A}(\vec{x}) = 0)\) and to the coulomb gauge \((\partial^a A_a = 0)\) where the potential has the form \(\vec{A}(r, \theta) = \frac{1}{2\pi} \vec{\partial} \theta\) [4]. We start by showing that the statistical potential in the \(A_x = 0\) gauge can also be written as the gradient of a ”singular” gauge transformation. The transformation is singular in the sense that the gauge parameter is not a single-valued function on the plane, i.e. periodic in \(\theta\). It is a well-defined function on \(\theta\)’s covering space, i.e. \(\theta \epsilon \mathbb{R}\). This was exactly the situation for the anyon model in the Coulomb gauge where \(\vec{A}(\vec{x}) = \vec{\partial}(\frac{1}{2\pi} \theta)\). The parameter \(\theta\) is a function on the covering space.

The potential, \(A_x(\vec{x}) = 0\) and \(A_y(\vec{x}) = \frac{1}{2} \delta(y) \epsilon(x)\) can be written as a gradient of a

\(^2\) In our conventions, \(\alpha = 0\) corresponds to bosons and \(\alpha = 2\pi\) to fermions.
staircase function $\Omega(\theta)$.

\[
\vec{A}(\vec{x}) = \vec{\partial}\Omega, \quad \Omega(\theta) \equiv \frac{+m}{2} \text{ for } \theta \epsilon [2m\pi, (2m + 1)\pi), \quad mc\mathbb{Z}
\]

To see this, write the gradient in angular coordinates, $(\partial_x, \partial_y) = (\cos\theta \partial_r - \sin\theta \partial_\theta, \sin\theta \partial_r + \cos\theta \partial_\theta)$. $\Omega(\theta)$ is a multi-valued function on the plane. We can make a singular transformation, $\vec{A}' = \vec{A} - \vec{\partial}\Omega \equiv 0$, to pass from the formulation of anyons as normal particles carrying Dirac strings to the formulation as free particles obeying exotic statistics ($\alpha$ will determine the statistics). We can also combine two gauge transformations, $\vec{A}'(\vec{x}) = \vec{A}(\vec{x}) + \vec{\partial}\omega(\theta)$, with $\omega(\theta) \equiv (\frac{1}{2\pi}\theta - \Omega(\theta))$, to transform from the $A_x = 0$ gauge to the $\partial^a A_a$ gauge. This does not effect the the statistics, because the parameter $\omega(\theta)$ satisfies $\omega(\theta \pm \pi) = \omega(\theta)$. Thus, the gauge of the Hamiltonian in (1) is not a singular gauge [1,2]. In what follows, we will suppose that the N-particle wavefunction is bosonic.

We will now illustrate the $A_x = 0$ gauge formulation by finding the spectrum of 2 anyons confined to a circular box [5]. Since the Hamiltonian of (1) is free for $x \neq 0$ ($\vec{x} \equiv \frac{1}{\sqrt{2}}(\vec{x}_1 - \vec{x}_2)$), we can write the eigenfunction for the relative problem as:

\[
\begin{align*}
\Psi_{\gamma k}(r, \theta) &= \exp(i\gamma\theta)J_{|\gamma|}(kr) \quad \text{for } y > 0, \\
\Psi_{\gamma k}(r, \theta) &= A \exp(i\beta\theta)J_{|\beta|}(kr) \quad \text{for } y < 0.
\end{align*}
\]

The energy, $E = \frac{k^2}{2m}$, is fixed by the boundary condition on the disc, $\Psi_{\gamma k}(r = R, \theta) = 0$, which implies that $J_{|\gamma|}(kR) = 0$.

To determine $A$ and $\beta$, we must impose that the wavefunction is bosonic. Our choice of the statistical potential preserves the invariance of $H_{\text{anyon}}$ under the exchange of two coordinates $\vec{x}_i$ and $\vec{x}_j$. Thus, Bose statistics is implemented by requiring that $\Psi(\vec{x}) = \Psi(-\vec{x})$ or in angular coordinates that $\Psi(r, \theta) = \Psi(r, \theta + \pi)$ with $\theta \in [0, 2\pi)$. This condition determines the wavefunction for $y < 0$ in terms of the wavefunction for $y > 0$. The solutions are:

\[
\begin{align*}
\Psi_{\gamma k}(r, \theta) &= \exp(i\gamma\theta)J_{|\gamma|}(kr) \quad \text{for } y > 0, \\
\Psi_{\gamma k}(r, \theta) &= \exp(i\gamma\theta - i\tilde{\gamma}\pi)J_{|\gamma|}(kr) \quad \text{for } y < 0.
\end{align*}
\]

Where, $\tilde{\gamma} = \gamma - 2m$ with $m$ any integer. The wavefunctions (4) satisfy all the constraints of statistics. The spectrum is determined by the quantum number $\gamma$. $\gamma$ is fixed by the
equation of motion on the $x$-axis. To see how (4) solves the Hamiltonian equations on the $x$-axis, we remark that the wavefunction has phase discontinuities, $e^{-i\tilde{\gamma}\pi}$, in crossing the either the negative or positive $x$-axis in a clockwise sense. For example, in a neighborhood of the negative $x$-axis, we can write (4) as:

$$\exp\left[\frac{i\tilde{\gamma}\pi}{2}(\epsilon(y) - 1)\right]e^{i\gamma\theta} J_{|\gamma|}(kr)$$  \hspace{1cm} (5)

Substituting this form into (1), we find:

$$k^2 e^{i\gamma\theta} J_{|\gamma|}(kr) = \left[-\partial_x^2 + (-i\partial_y + [\tilde{\gamma}\pi + \frac{\alpha}{2}\delta(y)])^2\right] e^{i\gamma\theta} J_{|\gamma|}(kr)$$ \hspace{1cm} (6)

This equation only has a solution if the coefficient of the $\delta(y)$ vanishes. Thus, we arrive at a condition on $\tilde{\gamma}$,

$$\tilde{\gamma} = \frac{\alpha}{2\pi} \text{ or } \gamma = \frac{\alpha}{2\pi} + 2m.$$ \hspace{1cm} (7)

The equation of motion on the $x$-axis has determine the non-integer part of $\gamma$. This shows that the 2-anyon relative eigenfunctions, in the $A_x(\vec{x}) = 0$ gauge, are given by:

$$\Psi_{mk}(r, \theta) = \exp\left(i\frac{\alpha}{2\pi} + 2m|\theta\right)J_{|\frac{\alpha}{2\pi} + 2m|}(kr) \quad \text{for} \quad y > 0,$$

$$\Psi_{mk}(r, \theta) = \exp\left(i\frac{\alpha}{2\pi} + 2m|\theta - i\frac{\alpha}{2}\right)J_{|\frac{\alpha}{2\pi} + 2m|}(kr) \quad \text{for} \quad y < 0$$ \hspace{1cm} (7)

Here, the states have been labeled by the two traditional quantum numbers $m\epsilon\mathbb{Z}$ and $k$.

We would arrive at the same conclusions in substituting (4) in (1) on a neighborhood of the positive $x$-axis. The values of $k$ and the energy eigenvalues are given by the zeros of the Bessel function.

$$E_m(k) = \frac{1}{2m} \left(\frac{z_{sm}}{R}\right)^2 \quad k = \frac{z_{sm}}{r} \quad \text{with} \quad J_{|\frac{\alpha}{2\pi} + 2m|}(z_{sm}) \equiv 0 \quad s = 1, 2, 3 \ldots$$ \hspace{1cm} (8)

This agrees with the results of [5] in the coulomb and singular gauges.

The $A_x(\vec{x}) = 0$ gauge has free particle wavefunctions with phase discontinuities, $e^{-i\tilde{\Phi}}$ when any 2 anyons have equal $y$-coordinates. This gauge may be more useful than the coulomb gauge in problems having discrete symmetries in the $x-$ and $y-$directions, ex. crystal of anyons (or flux tubes) where one may want to expand the anyonic interaction in a unit cell by a Fourier series. These possibilities are yet to be investigated.
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