SCATTERING OF STRINGY STATES IN COMPACTIFIED CLOSED BOSONIC STRING

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Abstract

We present scattering of stringy states of closed bosonic string compactified on torus $T^d$. We focus our attention on scattering of moduli and gauge bosons. These states appear when massless excitations such as graviton and antisymmetric tensor field of the uncompactified theory are dimensionally reduced to lower dimension. The toroidally compactified theory is endowed with the $T$-duality symmetry, $O(d,d)$. Therefore, it is expected that the amplitude for scattering of such states will be $T$-duality invariant. The formalism of Kawai-Llewelen-Tye is adopted and appropriately tailored to construct the vertex operators of moduli and gauge bosons. It is shown, in our approach, that N-point amplitude is $T$-duality invariant. We present illustrative examples for the four point amplitude to explicitly demonstrate the economy of our formalism when three spatial dimensions are compactified on $T^3$. It is also shown that if we construct an amplitude with a set of 'initial' backgrounds, the $T$-duality operation transforms it to an amplitude associated with another set backgrounds. We propose a modified version of KLT approach to construct vertex operators for nonabelian massless gauge bosons which appear in certain compactification schemes.

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1 Introduction

The perturbatively consistent bosonic and superstring theories live in critical dimensions 26 and 10 respectively. The five superstring theories: type IIA, type IIB, type I with $SO(32)$ gauge group, heterotic string with gauge group $SO(32)$ and heterotic string with gauge group $E_8 \otimes E_8$ offer the prospect of unifying the fundamental forces [1, 2, 3]. The heterotic string [4, 5] is the most promising candidate in the endeavors of unification programme and has accomplished several notable successes. The ingenious construction of heterotic string exploits an important attribute of closed string theory that the left moving and right moving sectors are independent so far as the evolution of the free string is concerned. Therefore, the left moving sector is a closed bosonic string with critical dimension 26 whereas the right moving sector is a 10-dimensional superstring. Of the 26 bosonic coordinates 16 are compactified on a torus, $T^{16}$, and the quantized momenta along the compact directions are required to satisfy certain constraints. Consequently, the theory defined in 10-dimensional spacetime manifests $N = 1$ supergravity and supersymmetric Yang-Mills multiplets in its massless sector. The emerging gauge group, in the compactification scheme, is either $SO(32)$ or $E_8 \otimes E_8$ depending on the boundary conditions chosen for the compact coordinates. The underlying gauge groups are elegantly unraveled once the compact bosonic coordinates are feminized and suitable boundary conditions are assigned to the resulting Weyl Majorana fermions. In order to establish connections with unified theories in four spacetime dimensions, the heterotic string theory (and for that matter other superstring theories) are to be reduced to four dimensional theories. This is achieved by compactifying six spacial dimensions. One of the universal features of the compactifications in string theory is the discovery of rich strove of symmetries. One encounters new gauge symmetries as well as global symmetries. The toroidal compactification has attracted considerable attention over decades. One of the most interesting results in this programme is the proposal of Narain [6] where he demonstrated the existence of rich symmetry structure in toroidal compactification of heterotic string and its salient features were further explored by Narain, Sar-madi and Witten [7]. One of the central features in toroidal compactification is the existence of noncompact target space duality symmetry (the T-duality). One of the simplest example is to consider toroidal compactification of a closed bosonic string on a d-dimensional torus, $T^d$. Let us consider the effective action associated with the massless states, graviton, antisymmetric tensor and dilaton of the uncompactified theory. The effective action is dimensionally reduced following the Scherk-Schwarz procedure where the background fields are assumed to be independent of the compact coordinates. In the context of the string effective action, the reduced action is expressed in a manifestly $O(d,d)$ invariant form and the generalization to heterotic string effective action is also incorporated appropriately [9].

The existence of the T-duality symmetry and the invariance of the reduced effective action under T-duality has important implications (for reviews see [10, 11, 12, 13, 14]).
For example, if a set of background correspond to a string vacuum, it is possible to generate a new set of backgrounds, satisfying the equations motion, by judiciously implementing duality transformations. In general the new set could correspond to an inequivalent string vacuum. Indeed, this prescription has been very useful to generate new backgrounds from a given set in string cosmology [18], in stringy black hole physics [19] and so on.

The scattering matrix - S-matrix - plays a very important role in string theory. S-matrix can be computed perturbatively within the first quantized framework of string theory. We associate a vertex operator with each state of the string theory. The vertex operator is constrained by imposing the requirements of conformal invariance. Therefore, it is of interests to investigate consequences of T-duality in scattering of stringy states in a compactified string theory. We ignore the winding modes throughout this investigation. These are special characteristics of compactified strings since the string can wind around compact coordinates due to its one dimensional nature. Interactions of strings, taking into effects of winding modes have interesting consequences; however, we consider the scenario where the states propagate in the noncompact spacetime dimensions.

I have proposed [20] a prescription to construct duality invariant scattering amplitudes for the massless states (moduli and gauge bosons) of compactified bosonic string. These massless states appear from the dimensional reduction of the massless sectors such as graviton and antisymmetric tensor field. The vertex operators are necessary to describe the scattering of stringy states. The duality transformation, \( O(d,d) \) generates a new set of backgrounds and their corresponding vertex operators. We evaluate the amplitude for the new backgrounds. I demonstrated how the two amplitudes, computed from the two sets of vertex operators, (which are related by duality transformation) are connected to each other. This result is analogous to the application of \( T \)-duality symmetry in the context where starting from a given initial background configuration one generates a new set of them. However, in the process of evaluating scattering amplitudes it has to be kept in mind that vertex operators are constructed in the weak field approximation. In contrast, when the \( T \)-duality transformation is implemented to generate new backgrounds, we use the M-matrix. The initial set of backgrounds is the one which satisfy equations of motion (i.e. \( \beta \)-function equation). On the other hand, when we construct the vertex operators associated with these backgrounds, we invoke the weak field approximation. The background is expanded around trivial one, for example the vertex operator for the graviton/moduli is constructed in linearized approximation to metric and corresponding equations result from requirement that it be \((1,1)\) with respect to free stress energy momentum tensor. Therefore, in my earlier formulation, it is not a straight forward implementation of \( T \)-duality as is the scenario adopted for duality transformation exploiting the solution generating technique.

Recently Hohm, Sen and Zwiebach (HSZ) have revisited the duality symmetry of heterotic string effective action [21]. They analyze which duality symmetries are realized.
in the effective field theory when the nonabelian gauge field sector is kept without truncating it to the Cartan (Abelian) sector. In this context these authors make some important observations about the attributes of the tree level S-matrix of the moduli in order to draw conclusions about some crucial features of S-matrix. Furthermore, based on their arguments, they discuss properties of the effective action. They focus their attention on the case when the background fields are independent of the compact coordinates. This is also an essential ingredient of the assumption in the present investigation and the same assumption was invoked in [20]. Furthermore, HSZ justify the reasons for focusing their consideration to the massless sector in order to achieve their goals. Therefore, the excitations of heterotic string, relevant for their purpose, are the moduli and gauge bosons. Furthermore, they provide arguments to substantiate the $T$-duality invariance of the S-matrix.

We mention, en passant, that our principal goal is to study $T$-duality invariance of S-matrix. As alluded to earlier, the intent is to construct vertex operators which will lead us to compute amplitudes of these massless states for compactified bosonic string, in contrast to those of the compactified heterotic string. We cast the amplitude, appealing to the newly proposed vertex operators, in a $T$-duality invariant form. Our explicit calculations are more efficient and economical than earlier techniques of mine [20] and it lends supports to the arguments of HSZ that S-matrix is $T$-duality invariant.

In this article we evaluate the tree level amplitudes to bring out essential features of duality symmetry in the present context and study how T-duality acts on S-matrix elements. The string perturbation theoretic corrections are not expected to affect duality transformation properties of S-matrix.

The rest of the article is organized as follows. In Section II, we recapitulate essential features of compactified closed bosonic string from the worldsheet perspective. We introduce the vertex operators associated with different levels of compactified string. Next we recall how the vertex operators could be cast in $O(d, d)$ invariant form. The vertex operators are defined in the weak field approximation in the sense that all correlation functions are evaluated using the free field OPE in the computation of S-matrix elements. In the next section, Section III, we resort to the prescription of Kawai, Llewellyn and Tye [22] for computation of S-matrix elements. We argue that duality transformation properties of the vertex operators become rather transparent in KLT formulation. Moreover, this technique is also very useful when we want to study duality properties of excited massive states. We construct N-point amplitude of moduli. These massless scalar appear when the metric and antisymmetric tensor field, massless states of the closed string, are compactified to lower dimensions. The massless Abelian gauge bosons also appear as a consequence of toroidal compactification. We also provide expressions for N-point amplitude of gauge bosons. Our principal goal is to construct $T$-duality invariant amplitudes. To this end, we resort to the proposal of Sen [23]. In this section explicit duality invariant amplitude is constructed. Moreover, we present illustrative examples of four point functions.
where a given amplitude is shown to get related to another amplitude through duality transformation. We propose a prescription to construct N-point amplitude for nonabelian gauge bosons and nonabelian massive string excitations in the spirit of KLT formalism. We discuss our results and present conclusions in Section IV.

2 Duality Symmetry of Closed String and the S-matrix

Let us consider the closed string worldsheet effective action in the presence of constant metric and antisymmetric tensor field

\[ S = -\frac{1}{2} \int d\sigma d\tau \left( \partial_\mu \hat{X}^\mu \partial_\nu \hat{X}^\nu \hat{G}_{\mu\nu}^{(0)} + \epsilon^{ab} \partial_\mu \hat{X}^\mu \partial_\nu \hat{X}^\nu \hat{B}_{\mu\nu}^{(0)} \right) \tag{1} \]

Here \( \tau, \sigma \) are the worldsheet coordinates, and \( \hat{\mu}, \hat{\nu} = 0, 1, 2, ... \) \( \hat{D} - 1 \) and \( \hat{D} \) is the number of spacetime dimensions. We have already adopted orthonormal gauge for the worldsheet metric. We have omitted \( \alpha' \) from the definition of action (1). The canonical Hamiltonian density is \[ H = \mathcal{Z}^T \mathcal{M}_0 \mathcal{Z} \tag{2} \]

where

\[ \mathcal{Z} = \begin{pmatrix} \hat{P}_\mu \\ \hat{X}^{\hat{\mu}} \end{pmatrix}, \quad \mathcal{M}_0 = \begin{pmatrix} \hat{G}^{-1} & -\hat{G}^{-1} \hat{B} \\ \hat{B} \hat{G}^{-1} & \hat{G} - \hat{B} \hat{G}^{-1} \hat{B} \end{pmatrix} \tag{3} \]

where \( \hat{P}_\mu \) is canonical conjugate momenta of \( \hat{X}^{\hat{\mu}} \) and prime stands for \( \sigma \) derivative. Note that \( \mathcal{M}_0 \) is a symmetric \( 2\hat{D} \times 2\hat{D} \) matrix. The Hamiltonian is invariant under a global \( O(\hat{D}, \hat{D}) \) symmetry transformation

\[ \mathcal{Z} \rightarrow \Omega_0 \mathcal{M}_0 \Omega_0^T, \quad \Omega_0^T \hat{\eta} \Omega_0 = \hat{\eta} \tag{4} \]

where \( \Omega_0 \in O(\hat{D}, \hat{D}) \) and \( \hat{\eta} \) is the \( O(\hat{D}, \hat{D}) \) metric, with zero diagonal elements and off diagonal elements being \( \hat{D} \times \hat{D} \) unit matrices. If we consider evolution of string in the background of its massless excitations such as \( \hat{G} \) and \( \hat{B} \), they are allowed to have \( \hat{X}^{\hat{\mu}}(X(z, \bar{z})) \) dependence and the resulting action is that of a sigma model. In toroidal compactification prescription the \( \hat{X}^{\hat{\mu}} \) is decomposed into two sets of coordinates \( \hat{X}^{\hat{\mu}} = \{ X^\mu, Y^\alpha \} \), \( \mu = 0, 1, ... \hat{D} - 1 \), and \( \alpha = D, D + 1, ... \hat{D} - 1 \) so that \( \hat{D} = D + d \); \( \{ Y^\alpha \} \) are the compactified coordinates. Furthermore, the backgrounds depend only on \( X^\mu \).

We shall summarize a few aspects which will be useful later. The general prescription of Scherk and Schwarz [8] is the essential ingredient of such compactifications. It is useful to adopt the vielbein formalism

\[ \hat{e}^A_M(\hat{X}) = \begin{pmatrix} e^\mu_\nu(X) & A^\alpha_\mu(X) E^\alpha_\mu(X) \\ 0 & E^\alpha_\mu(X) \end{pmatrix} \tag{5} \]
The well known graviton vertex operator

The last constraint is the level matching condition. Before proceeding, let us look at

The vertex operators, gauge field in sequel.

The spacetime metric is \( g_{\mu\nu} = e^{\mu}_{p}e_{\nu r} \) where the local indices \( r, s \) are raised and lowered by flat Minkowski metric. Note the appearance of Abelian gauge fields \( A^{(1)}_{\mu}, \alpha = 1, 2, \ldots d \) associated with the \( d \) isometries and \( E^{a}_{\alpha} \) are vielbein of the internal metric i.e. \( G_{\alpha\beta} = E^{a}_{\alpha}E^{b}_{\beta}\delta_{ab} \) and it transforms as a scalar under \( D \)-dimensional Lorentz transformations. Similarly, the antisymmetric tensor will be decomposed into components: \( \hat{B}_{\mu\nu}, \hat{B}_{\mu\alpha}, \hat{B}_{\alpha\beta} \). There are gauge fields \( \hat{B}_{\mu\alpha} \) and scalars \( \hat{B}_{\alpha\beta} \) in lower dimensions. We refer the interested reader to our paper [9] for details and the prescriptions for dimensional reduction in the context of string theory. It is worth noting that under the T-duality transformation the spacetime tensors remain invariant. If we were to dimensionally reduce the \( \hat{D} \) dimensional effective action then all backgrounds depend only on \( D \)-dimensional spacetime coordinates \( x^\mu \). Moreover, the M-matrix defined below, is expressed in terms of the moduli \( G \) and \( B \)

\[
M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}
\]

where \( G = G_{\alpha\beta}, \quad B = B_{\alpha\beta} \) and now \( B_{\alpha\beta} \) stands for \( \hat{B}_{\alpha\beta} \). The \( O(d, d) \) metric \( \eta \) with off diagonal \( d \times d \) unit matrix, \( 1 \) remain invariant. Under the \( O(d, d) \) transformations \( M \rightarrow \Omega M\Omega^{T} \), \( \Omega \in O(d, d) \). \( M \) is symmetric and \( M \in O(D, D) \). The gauge fields, \( A^{(1)}_{\mu} \) and \( A^{(2)}_{\mu\alpha} = \hat{B}_{\mu\alpha} + B_{\alpha\beta}A^{(1)}_{\mu\beta} \) transform as a vector under \( O(d, d) \) transformations i.e. \( A^{i}_{\mu} = \Omega^{i}_{j}A^{j}_{\mu} \), \( i, j = 1, 2, \ldots 2d \). We define \( A^{i}_{\mu} = A^{(1)}_{\mu\alpha}, \quad i = 1, 2, \ldots d \) and \( A^{i}_{\mu} = A^{(2)}_{\mu\alpha}, \quad i = d + 1, \ldots 2d \); note \( \alpha \) takes \( d \)-values. We shall need this transformation property of the gauge field in sequel.

The vertex operators, \( V(k; X(z, \bar{z})) \), in closed string theory are required two satisfy two constraints:

\[
(L_0 - 1)V = 0, \quad (\bar{L}_0 - 1)V = 0, \quad \text{and} \quad (L_0 - \bar{L}_0)V = 0
\]

The last constraint is the level matching condition. Before proceeding, let us look at the well known graviton vertex operator

\[
V_{G}(\epsilon, k, X) = \epsilon_{\mu\bar{\nu}} : e^{ik, X(z, \bar{z})} \partial X^{\mu}(z)\bar{\partial}X^{\bar{\nu}}(\bar{z}) :
\]

The above constrains (7) lead to two conditions: (i) the graviton is massless, \( k^2 = 0 \) and (ii) it is transverse \( \epsilon_{\mu\bar{\nu}}k^\mu = \epsilon_{\mu\bar{\nu}}k^\bar{\nu} = 0 \). The graviton coupling \( g_{\mu\nu}\partial X^{\mu}\bar{\partial}X^{\bar{\nu}} \) is given by (8) in the plane wave approximation where \( g_{\mu\nu} \) is expanded around flat background metric. We can adopt the same prescription for the vertex operators associated with the moduli \( G_{\alpha\beta} \) and \( B_{\alpha\beta} \). The corresponding vertex operators are

\[
V_{G} = \epsilon^{G}_{\alpha\beta} : e^{ik, X(z, \bar{z})} \partial Y^{\alpha}(z)\bar{\partial}Y^{\bar{\beta}}(\bar{z}) :
\]

Here \( \epsilon^{G}_{\alpha\beta} \), symmetric under \( \alpha \leftrightarrow \bar{\beta} \), is analog of polarization tensor carrying indices along internal direction. Moreover, the scalar propagates only in spacetime manifold and therefore, the vertex operator has no dependence on \( Y^{\alpha} \) and its (quantized) canonical momenta \( P_{\alpha} \). Thus it implies that we are not taking into account the presence
of winding modes. A similar vertex operator $V_B$ can be constructed for the other moduli, $B$, in weak field approximation. Our aim is to study T-duality invariance properties of the S-matrix for states of compactified closed bosonic string. Although transformation properties of $M$-matrix under T-duality are easy to implement (see the remarks after (6)), the transformation properties of the moduli $G$ and $B$ are rather complicated [9]. In fact $G + B$ transform as quotients under $O(d, d)$. I have proposed [20] a prescription to evaluate amplitudes in terms of $M$-matrix elements in the weak field approximation. This proposal can be implemented in the massless sector; in other words we can compute amplitudes involving the moduli and generate new amplitudes through T-duality transformations. The procedure consists of following steps. Let us start with the vertex operator (9) and examine how T-duality operation works in my scheme. (i) Just as we expand backgrounds $G_{\alpha\beta}$ and $B_{\alpha\beta}$ around trivial background, I propose such an expansion for the $M$-matrix: $M = 1 + \tilde{M}$ in weak field expansion. Since $M$ is expressed in terms of $G$ and $B$, $\tilde{M}$ will be expressed in terms of the linearized expansion of those backgrounds. Therefore, $\tilde{M}$ is constructed in terms of $\epsilon^G_{\alpha\beta}$ and $\epsilon^B_{\alpha\beta}$. However, unlike $M$ which is an element of $O(d,d)$, $\tilde{M}$ is not an $O(d,d)$ matrix. Consequently, we cannot implement T-duality transformation directly on $\tilde{M}$.

(ii) Thus my starting point was to identify the vertex operators associated with the moduli. I utilized $\epsilon^G$ and $\epsilon^B$ to construct $\tilde{M}$ and then the matrix $M = 1 + \tilde{M}$.

(iii) Now we can implement the $O(d,d)$ transformation on the constructed $M$-matrix, i.e. $M \rightarrow M' = \Omega M \Omega^T$ where $\Omega \in O(d,d)$.

(iv) The next step is to expand $M'$ as $M' = 1 + \tilde{M}'$. Finally, we can extract transformed $\epsilon^{G'}_{\alpha\beta}$ and $\epsilon^{B'}_{\alpha\beta}$ from $\tilde{M}'$.

However, for compactified string, the T-duality transformation is expected to operate for massive levels arising from compactifications which have their partners as spacetime tensors at the same excitation level of the string. Indeed, I introduced a technique to construct $O(d,d)$ invariant vertex operators arising from toroidal compactification of the closed string [29].

Let us look at a generic vertex operator [30] which appears after a $\hat{D}$-dimensional vertex operator has been compactified to $D$ dimensions.

$$\partial_+^{p} Y^{\alpha_{1}} \partial_+^{q} Y^{\alpha_{2}} \partial_+^{r} Y^{\alpha_{k}} \ldots \partial_-^{p'} Y^{\alpha_{1}'} \partial_-^{q'} Y^{\alpha_{2}'} \partial_-^{r'} Y^{\alpha_{k}'} , \quad p + q + r = p' + q' + r' = n + 1 \quad (10)$$

where $\partial_\pm = \partial_r \pm \partial_s$. Thus $(\partial_r \pm \partial_s) Y^{\alpha} = (P^{\alpha} \pm Y^{\alpha})$. The indices $\alpha, \beta, \ldots$ are raised and lowered by $\delta^{\alpha\beta}$ and $\delta_{\alpha\beta}$. The constraint $p + q + r = n + 1, p' + q' + r' = n + 1$ is required by the level matching condition. It is possible to express (10) as an $O(d,d)$ tensor by introducing projection operators [30] which convert $\partial_+ Y$ and various products of $(\partial_+)^m Y$ appearing in (10) to products of $O(d,d)$ vectors. The same argument applies to $(\partial_-)^m Y$ as well. Once (10) is converted to products of $O(d,d)$ vectors, it has to be contracted with corresponding $O(d,d)$ polarization tensor. This prescription can be implemented for first couple of massive levels. However, even the formal expression
for amplitudes involving such vertex operators is not easy to manipulate to study
T-duality invariance properties of the S-matrix elements. Therefore, it is necessary
to adopt a different strategy.
There is a proposal due to Sen [23], based on string field theory, according to which
the space of solutions of backgrounds enjoys an $O(d) \otimes O(d)$ symmetry. He noted
that the diagonal subgroup, $O(d)$, of $O(d) \otimes O(d)$ generates rotations. Furthermore,
he identifies the set of matrices
\[
\Omega_{RS} = \frac{1}{2} \begin{pmatrix}
S + R & R - S \\
R - S & S + R
\end{pmatrix}
\]  
which implement $O(d) \otimes O(d)$ transformations. Here $R$ and $S$ are matrices that belong
to $O(d) \otimes O(d)$ and $\Omega_{RS}$ is subgroup of $O(d, d)$. In the linearized approximation to
the backgrounds
\[
G_{\alpha \beta} = \delta_{\alpha \beta} + h_{\alpha \beta}, \quad \text{and} \quad B_{\alpha \beta} = b_{\alpha \beta}
\]  
the transformed linearized backgrounds are
\[
(h' + b') = S(h + b)R^T
\]  
This argument has been advanced further by Sen [28] and by Hassan and Sen [26, 27]
and the matrix defines in (11) operates on transformations of the $M$-matrix since
$\Omega_{RS} \in O(d, d)$.
In order to evaluate scattering amplitudes involving moduli $G$ and $B$, we employ the
vertex operator (9). The tree level N-point amplitude is evaluated by utilizing the
conformal field theory prescription. We have noted that two sets of Abelian gauge
fields, $A^{(1)}_{\mu}$ and $A^{(2)}_{\mu}$, $\alpha = 1, 2 ... d$ also appear in general toroidal compactification
scheme. Note that gauge fields do not appear in compactification proposed by Hassan
and Sen [26] since the backgrounds $\hat{G}$ and $\hat{B}$ are decomposed in the block diagonal
form. These two sets of gauge fields combined together transform as $O(d, d)$ vectors
as already noted. Therefore, the transformation rules for these gauge bosons, under
Sen’s $O(d) \otimes O(d)$ group, can be specified easily from the structure of $\Omega_{RS}$ matrix.
We shall exploit this information when we return to discussion of the scattering of
these gauge bosons in sequel. The N-point amplitude for the scattering of moduli is
\[
A^{(N)}_{G,B} = \int d^2z_1d^2z_2...d^2z_N < \prod_{i=1}^{N} V_i(\epsilon_i, k_i, X_i, Y_i) >
\]  
where the vertex operator $V_i$ is
\[
V_i(\epsilon_i, k_i, X_i, Y_i) = \epsilon_{\alpha_i \beta_i} : \exp[ik_i.X(z, \bar{z})] \partial Y^{\alpha_i}(z) \partial Y^{\beta_i}(\bar{z}) : 
\]  
with $k_i^2 = 0$. Where $\epsilon_{\alpha_i \beta_i}$ stands for polarization tensor of $G_{\alpha_i \beta_i}$ or $B_{\alpha_i \beta_i}$ depending on
the choice we make and then it will be symmetric or antisymmetric under $\alpha_i \leftrightarrow \beta_i$.
Note that the plane wave part, $\exp[ik_i.X_i]$, is inert under T-duality and we shall
not bring in its presence in our considerations of duality symmetry transformations.
While evaluating the above amplitude (14), we have to insert the Koba-Nielsen factor and the integration is to be carried out on $N - 3$ variables. We shall pay attention to these aspects when we compute amplitudes for specific cases. However, we remind the reader about the two correlation functions which we shall use from time to time.

\[
< \partial Y^{\alpha_i}(z_i) \partial Y^{\alpha_j}(z_j) > = -\frac{\delta_{\alpha_i\alpha_j}}{(z_i - z_j)^2}, \tag{16}
\]

and

\[
< \bar{\partial} Y^{\bar{\beta}_i}(\bar{z}_i) \bar{\partial} Y^{\bar{\beta}_j}(\bar{z}_j) > = -\frac{\delta_{\bar{\beta}_i\bar{\beta}_j}}{(\bar{z}_i - \bar{z}_j)^2} \tag{17}
\]

We have used $\alpha' = 2$ for close string throughout since all the amplitudes involve the vertex operators associated with the closed string states. Therefore, it does not appear in our computations. If we have to introduce it then the two correlation functions will be multiplied by a factor $\alpha'/2$ on the right hand sides of equations (16) and (17).

When the need will arise we shall remind where the $\alpha'$ be introduced. We draw in discussions about vertex operators of open string in the next section very briefly in the context of KLT formalism. We have suppressed the $\alpha'$ factors in this context also. Thus the products of $\epsilon_{\alpha_1\bar{\beta}_1}, \epsilon_{\alpha_2\bar{\beta}_2}, ... \epsilon_{\alpha_N\bar{\beta}_N}$ get contracted with various combinations of $\delta_{\alpha_i\alpha_j}, \delta_{\bar{\beta}_i\bar{\beta}_j}, ...$ which will come from pairwise contractions of holomorphic parts and antiholomorphic parts the the vertex operators. The plane wave contractions are like

\[
< \exp[i k_i.X_i(z_i, \bar{z}_i)] \cdot \exp[i k_j.X_j(z_j, \bar{z}_j)] > = |z_i - z_j|^{2k_i.k_j}. \tag{18}
\]

If we adopt Sen’s prescription of $O(d) \otimes O(d)$ transformations then

\[
(\epsilon^{G}_{\alpha_i\bar{\beta}_i} + \epsilon^{B}_{\alpha_i\bar{\beta}_i}) = |S(\epsilon^G + \epsilon^B)R^T|_{\alpha_i\bar{\beta}_i} \tag{19}
\]

Let us consider the amplitude for scattering of gauge bosons (we confine to amplitude for $A_{\mu_i}^{(1)\alpha_i}$ for the moment).

\[
T^{(N)}_A = \int d^2z \Pi^N \langle V^A_{i}(\epsilon_i, k_i, X_i, Y_i) > \tag{20}
\]

where

\[
V^A_{i}(\epsilon_i, k_i, X_i, Y_i) = \epsilon_{\mu_i\bar{\alpha}_i} : \exp[i k_i.X(z_i, \bar{z}_i)] \partial X_{\mu_i}(z_i) \bar{\partial} Y^{\bar{\alpha}_i}(\bar{z}_i) \tag{21}
\]

and is required to satisfy constraints: (i) $k_i^2 = 0$ and (ii) $\epsilon_{\mu_i\bar{\alpha}_i} k^{\mu_i} = 0$. Thus contracted terms in the amplitude will be multiplied by products of $\epsilon_{\mu_i\bar{\alpha}_i}$ which will be contracted by various combinations spacetime metric and internal metric.

We would like to draw attention to the fact that $(\epsilon^G + \epsilon^B) \rightarrow S(\epsilon^G + \epsilon^B)R^T$ under $O(d) \otimes O(d)$ transformations and $\epsilon_{\mu_i\bar{\alpha}_i}^{(1)} \rightarrow (S + R)\epsilon_{\mu_i\bar{\alpha}_i}^{(1)}$. In the expression for the N-point amplitude for the scattering of moduli (14) we encounter a string of the products of $\epsilon^G$ and $\epsilon^B$. Therefore, it is not yet so straightforward to demonstrate the T-duality invariance of the amplitude although Sen’s argument of implementing $O(d) \otimes O(d)$ duality is more efficient compared to my earlier proposal. It turns out that the technique introduced by Kawai, Llewellyn and Tye [22] is very useful to investigate the T-duality transformation properties of scattering amplitude in the frame work of Sen’s $O(d) \otimes O(d)$ duality group.
3 KLT Formalism and T-duality

Let us briefly recall salient features of the technique introduced by KLT [22] to calculate tree level closed string amplitudes. We shall appropriately modify their prescription for our purpose. Their principal goal was to demonstrate the relationship between closed string and open string tree level scattering amplitudes. They showed that the N-point closed string amplitude can be factorized into products of two N-point open string amplitudes with certain pre-factors. This property holds for all excited levels of string theories as long as one confines to the vertex operators associated with states lying on the leading Regge trajectory. It was observed that closed string coordinates are decomposed as sum of left and right moving sectors. The vertex operator corresponding to leading trajectory is product of equal number of holomorphic and antiholomorphic operators i.e one set is \( \Pi_{\nu=1}^{M} \partial X^{\nu}(z_{i}) \) and the other one is \( \Pi_{\nu=1}^{N} \partial X^{\nu}(\bar{z}_{i}) \); further more at each level \( M = N \) in order to satisfy the level-matching condition. The factorization property of three point and four point amplitudes was explicitly demonstrated. In order to make this article self contained, we summarize essential steps prescribed by KLT. We shall not utilize the entire mechanism of KLT; however, as we shall demonstrate, their approach brings out certain simplifications in actual computations. It is also quite relevant in the present context to investigate T-duality invariance of the S-matrix.

Let us consider below the open string vertex operators for a tachyon and a gauge boson respectively.

\[
V^{T}(k, X) =: \exp[i k.X] :, \quad \text{and} \quad V^{A} = \epsilon_{\mu} : \exp[i k.X] \partial X^{\mu} : \quad (21)
\]

The tachyon and gauge boson are required to be on-shell; moreover, the gauge boson polarization vector is transverse i.e. \( k . \epsilon = 0 \). The scattering amplitudes involving tachyons and gauge bosons are evaluated by adopting known techniques. Furthermore, we need to introduce vertex operators with each excited level of the string. KLT introduced an ingenious and elegant technique derive the scattering amplitudes for excited level through modification of the tachyon vertex operator as follows. Consider the vertex operator

\[
V^{\text{open}}_{\text{KLT}}(\epsilon, k, X) =: \exp[i k.X + i \epsilon_{\mu} \partial X^{\mu}] :
\]

Now if we expand the exponential in powers of \( \epsilon_{\mu} \), the linear term in the polarization vector reproduces gauge boson vertex operator. Moreover, if we desire to compute scattering amplitude for N gauge bosons, \( T^{(N)}_{A} \), then we compute the correlation function of products of \( V_{\text{KLT}}^{\text{open}} \) and isolate the coefficient of the product \( \Pi_{i=1}^{N} \epsilon_{\mu_{i}} \). Furthermore, the vertex operator for generic excited level of open string is

\[
V^{\text{EX}} \simeq \epsilon_{\mu_{1} \mu_{2} ... \mu_{m}} : \exp[i k.X] \partial X^{\mu_{1}} \partial X^{\mu_{2}} ... \partial X^{\mu_{m}} :
\]

The excited state momentum \( k_{\mu} \) has to be on-mass-shell and the polarization tensor, \( \epsilon_{\mu_{1} \mu_{2} ... \mu_{m}} \), is required to satisfy some transversality, tracelessness conditions as a
consequence of conformal invariance. According to KLT

$$V_{KLT}^{EX} \simeq: \exp[i(k.X + \epsilon_{\mu_1}\partial X^{\mu_1} + \epsilon_{\mu_2}\partial X^{\mu_2} + \cdots + \epsilon_{\mu_m}\partial X^{\mu_m})] :$$  \hspace{1cm} (24)

Now by expanding the exponential and keeping the multilinear term

$$\epsilon_{\mu_1}\partial X^{\mu_1}\epsilon_{\mu_2}\partial X^{\mu_2} \cdots \epsilon_{\mu_m}\partial X^{\mu_m}$$  \hspace{1cm} (25)

we recover the desired vertex operator (23).

The procedure outlined for open string is also generalized for the vertex operator of closed string states. We recall that the graviton vertex operator is

$$V_{graviton} = \epsilon_{\mu\bar{\nu}}: \exp[ik.X]\partial X^\mu\bar{\partial}X^{\bar{\nu}} :$$  \hspace{1cm} (26)

with $k^2 = 0$ and $\epsilon_{\mu\bar{\nu}}k^\mu = 0 = \epsilon_{\bar{\mu}\nu}k^\nu$. The corresponding vertex operator for antisymmetric tensor assumes a similar form, only exception being that $\epsilon_{\mu\bar{\nu}}$ is antisymmetric.

The KLT prescription for generating vertex operator the massless sector of closed string is to introduce

$$V_{closed}^{KLT} =: \exp[ik.X + i\epsilon_{\mu}\partial X^{\mu} + i\bar{\epsilon}_{\bar{\nu}}\bar{\partial}X^{\bar{\nu}}] :$$  \hspace{1cm} (27)

If we collect the coefficient of the bilinear term $\epsilon_{\mu}\bar{\epsilon}_{\bar{\nu}}$ in the expansion of the exponential (27) we note that it corresponds to graviton and antisymmetric tensor vertex if we identify the symmetric product of $\epsilon_{\mu}\bar{\epsilon}_{\bar{\nu}}$ as the graviton polarization tensor and the antisymmetric product as that of the antisymmetric tensor field. It is obvious, the vertex operators for excited closed string states can be derived by suitably generalizing (27) as was achieved for the open string case. Moreover, the constraints arising from conformal invariance such as masslessness condition for the first excited states are fulfilled. The transversality conditions on polarization tensor $\epsilon_{\mu\bar{\nu}}$ are translated to constraints on $\epsilon_{\mu}$ and $\bar{\epsilon}_{\bar{\nu}}$. Furthermore, the polarization tensors associated with excited massive states of closed string [32, 30, 31] (for both compactified and noncompact strings) are constrained by requirements of conformal invariance. Those conditions can be incorporated by imposing appropriate constraints on the set of polarization vectors $\epsilon_{\mu_i}$ and $\bar{\epsilon}_{\bar{\nu}_i}$.

Our goal is to suitably adopt the KLT approach to the vertex operators associated with states arising due to compactification of a closed string. In other words, excited states living in uncompactified $D$-dimensional theory, when compactified to lower dimension belongs to the representations of rotation group $SO(D-1)$ (for massless case it is $SO(D-2)$) whereas these states came from the representations of $SO(D-1)$ (correspondingly $(D-2)$ for massless case). Note that the moduli $G$ and $B$ transform as scalars under $D$-dimensional Lorentz transformations and spatial rotations. Therefore, for an arbitrary excited states which had Lorentz indices in $\hat{D}$-dimensions, will decompose into tensors, vectors and scalars in $D$-dimensions [30]. Consider compactification of $\hat{D}$ dimensional graviton; it decomposes into $D$-dimensional graviton,
gauge bosons and moduli. We focus the attention on the massless sector consisting of the moduli $G_{\alpha\bar{\beta}}, B_{\alpha\bar{\beta}}, A^{(1)\alpha}_\mu$ and $A^{(2)}_{\mu \alpha}$. Let us consider the KLM type vertex for the moduli

$$V_{KL}^M =: \exp[ik.X + i\epsilon_\alpha \partial Y^\alpha + i\bar{\epsilon}_{\bar{\beta}} \bar{\partial} Y^{\bar{\beta}}]:$$

Remarks: (i) The scalars propagate in the spacetime manifold and therefore, the plane wave part is of the form $\exp[ik.X]$. Thus throughout this paper, we repeat, due to the $Y$-independence of plane waves in (28) we do not take into account attributes of toroidal compactification such as presence of winding modes etc. (ii) All the states are on-shell. As stated earlier, the vertex operators of $G$ and $B$ are identified by taking symmetric and antisymmetric combination of the product $\epsilon_\alpha \bar{\epsilon}_{\bar{\beta}}$. (iii) Note that while evaluating correlation function, starting from $V_{KL}^M$ we shall have products of equal number of $\epsilon_\alpha$‘s and $\bar{\epsilon}_{\bar{\beta}}$‘s due to the level matching condition. The products of $\epsilon_\alpha$ will mutually contract themselves due the $\delta^{\alpha_i\alpha_j}$ arising from contractions of $\partial Y^\alpha(z_i)$ and $\partial Y^\alpha(z_j)$, $i \neq j$, similar contractions appear from the antiholomorphic part as well. The contraction, $< \partial Y^\alpha(z_i) \bar{\partial} Y^{\bar{\beta}}(\bar{z}_n) > = 0$, consequently there are no contraction of the indices of type $\epsilon_\alpha$ and $\bar{\epsilon}_{\bar{\beta}}$ in evaluation of amplitudes. Let us recall the $Z_2$ duality, in the phase space approach: $P \leftrightarrow Y'$ under interchange $\tau \leftrightarrow \sigma$ of worldsheet coordinates. We note that, with $P_\pm = (P \pm Y')$, $P_\pm \rightarrow \pm P_\pm$. Moreover, $O(d,d)$ vector $Z$ can be decomposed as $Z = (Z_+ + Z_-)$. Where

$$Z_+ = \frac{1}{2} \left( \begin{array}{c} P_+ \\ P_- \end{array} \right), \quad \text{and} \quad Z_- = \frac{1}{2} \left( \begin{array}{c} P_- \\ -P_- \end{array} \right)$$

Therefore, we observe that the structure of the vertex operators for leading Regge trajectories is intimately related with $O(d,d)$ symmetry from the phase space perspective.

### 3.1 Scattering of Moduli and Gauge Bosons for Compactified Stringy States

We have proposed the modified version of KLT vertex operators for the compactified closed string. We shall employ this vertex operator to evaluate N-point amplitudes for moduli $G$ and $B$ as well as for the gauge bosons arising as a consequence of toroidal compactification.

The N-point amplitude extracted from $V_{KL}^M$ will have products of $\epsilon$‘s from the holomorphic sector and products of $\bar{\epsilon}$‘s from antiholomorphic sector. A careful examination of $V_{KL}^M$ itself reveals an interesting fact. If we define $\epsilon_\alpha$ to transform as vector under, say, $O(d)_R$ and $\partial Y^\alpha$ to be also the vectors of $O(d)_R$ then the second term in the exponential of (28) is $O(d)_R$ invariant. Moreover, if I invoke similar definition for the antiholomorphic sector i.e. declare $\{\bar{\epsilon}_{\bar{\beta}}, \bar{\partial} Y^{\bar{\beta}}\}$ to transform as two sets of vectors under $O(d)_L$, then the third term in the exponential is $O(d)_L$ invariant. Therefore,
the polarizations $\epsilon_\alpha$ and $\bar{\epsilon}_\beta$ transform separately under $O(d)_R$ and $O(d)_R$ as ordained above.

Let us consider the N-point amplitude for the moduli $G_{\alpha\bar{\beta}}$ and $B_{\alpha\bar{\beta}}$. We arrive at the following general expression and then we shall discuss how to extract the N-point amplitude alluding to remarks made earlier.

$$A^{(N)}_{G,B} = \int \prod_{i=1}^N d^2 z_i \, D \, \prod_{i>j} |z_i - z_j|^{2k_i.k_j} \exp\left\{\sum_{i>j} \frac{\epsilon_i \epsilon_j}{(z_i - z_j)^2} \right\} \exp\left\{\sum_{i>j} \frac{\bar{\epsilon}_i \bar{\epsilon}_j}{(\bar{z}_i - \bar{z}_j)^2} \right\}$$  \hspace{1cm} (30)

where $D$ is the Koba-Nielsen factor:

$$D = \frac{|z_a - z_b|^2 |z_b - z_c|^2 |z_c - z_a|^2}{d^2z_a d^2z_b d^2z_c}$$  \hspace{1cm} (31)

which gauge fixes the underlying $SL(2, C)$ invariance. The variables $\{z_a, z_b, z_c\}$ and $\{\bar{z}_a, \bar{z}_b, \bar{z}_c\}$ can be chosen arbitrarily. Therefore, there are only $(N - 3)$ integrations over $\Pi d^2 z_i$. Note that $\epsilon_i \epsilon_j = \delta_\alpha^\alpha \epsilon_\alpha \epsilon_\alpha$, and similar definition applies for $\bar{\epsilon}_i \bar{\epsilon}_j$. Since we have two independent products $\Pi \epsilon_\alpha$ and $\Pi \bar{\epsilon}_\beta$, and they contract only among themselves we can make the $O(d)_R$ and $O(d)_L$ transformations on each of the products. Since these ‘polarization vectors’ contract amongst themselves, as noted earlier, evidently, the N-point amplitude is T-duality invariant in the sense described above.

The vertex operators of the gauge bosons, which arise from dimensional reduction of the metric and antisymmetric tensor fields, are given by

$$V^\alpha =: \exp\left[i k \cdot X + i \epsilon_\mu \partial X^\mu + i \bar{\epsilon}_\beta \partial Y^{\bar{\beta}}\right], \quad i = 1, 2$$  \hspace{1cm} (32)

in KLT formalism and we keep in mind that in the expansion of the exponentials we retain the bilinear $\epsilon_\mu \bar{\epsilon}_\beta$. The index of $\epsilon^{(1)}_\beta$, $\bar{\beta} = 1, 2, ... d$ and this polarization is identified with $A^{(1)}_{\mu \bar{\beta}}$. The other one $\epsilon^{(2)}_\beta$, $\bar{\beta} = 1, 2, ... d$ is polarization of gauge field $A^{(2)}_{\mu \bar{\beta}}$. In other words the third term in the exponential of (32) $\bar{\epsilon}_i \partial Y^i = \epsilon^{(1)}_\alpha \partial Y^\alpha + \epsilon^{(2)}_\alpha \partial Y^\alpha$. Now the index $\alpha = 1, 2, ... d$. Moreover, the plane wave part $e^{i.k \cdot X}$ is invariant under T-duality and so is $\epsilon_\mu \partial X^\mu$. However, the photon polarization $\epsilon_\mu \bar{\epsilon}_i$, $\mu = 0, 1, ... D - 1$, and $i = 1, 2, ... 2d$ is factorized into products of spacetime polarization vector and the internal polarization vector. Moreover, the T-duality group linearly acts on internal polarizations, $\bar{\epsilon}_\alpha$.

The N-point amplitude for vector bosons assumes the following form

$$T^{(N)}_A \simeq \int d^2 z_1 d^2 z_2 ... d^2 z_N \, D \, \prod_{i>j} |z_i - z_j|^{2k_i.k_j} \exp\left\{\sum_{i>j} \frac{\epsilon_i \epsilon_j}{(z_i - z_j)^2} - \sum_{i \neq j} \frac{\epsilon_i \bar{\epsilon}_j}{(z_i - z_j)^2} \right\} \exp\left\{\sum_{i \neq j} \frac{\bar{\epsilon}_i \bar{\epsilon}_j}{(\bar{z}_i - \bar{z}_j)^2} \right\}$$  \hspace{1cm} (33)

Here $D$ is the choice of the gauge fixing Koba-Nielsen measure. The first term in the first exponential comes from contractions of $\partial X^\mu_i(z_i) \partial X^\mu_j(z_j)$ and the second term in the first exponential is due to contraction of plane wave and $\partial X^\mu_i(z_i)$. Notice
that there is no such term in the second exponential since the plane wave part has no $Y^\alpha$ dependence. We need to pick up products of terms like $\epsilon_i, \epsilon_j$ and $\epsilon_i, k_j$ from holomorphic side and various pairwise contractions of $\bar{\epsilon}_i, \bar{\epsilon}_j$ from the contractions of the antiholomorphic side. The objects like $\epsilon_i, \epsilon_j$ and $\epsilon_i, k_j$ are inert under T-duality transformations. Therefore, T-duality will only rotate the polarization vectors $\{\epsilon_\alpha\}$. In what follows, I shall present two illustrative examples to demonstrate the operations of T-duality.

Let us consider a simple case where three spatial dimensions are compactified, i.e. $d = 3$. First we consider the four point scattering amplitude of the moduli $G_{\alpha\beta}$. The four vertex operators are given by

$$ V_i = \epsilon_{\alpha_i} \bar{\epsilon}_{\beta_i} : \exp[ik.X(z, \bar{z})] \partial Y^{\alpha_i}(z_i) \bar{\partial} Y^{\beta_i}(\bar{z}_i) : , \quad i = 1, 2, 3, 4 \quad (34) $$

Thus the four point amplitude assumes the form

$$ A^{(4)} \sim \int \Pi_i^4 \Pi_{i>j} |z_i - z_j|^{2k_i,k_j} \exp[\Sigma_{i>j} \frac{\epsilon_i \epsilon_j}{(z_i - z_j)^2}] \exp[\Sigma_{i>j} \frac{\bar{\epsilon}_i \bar{\epsilon}_j}{(\bar{z}_i - \bar{z}_j)^2}] \quad (35) $$

We fix the three Koba-Niellesen variables as $z_1 = 0$, $z_3 = 1$, $z_4 \to \infty$ and correspondingly set $\bar{z}_1, \bar{z}_2, \bar{z}_3$ to those values. Thus we are left with an integral over $z_2$ and $\bar{z}_2$. We expand the two exponentials in power series and pick up on the term which is a product $\epsilon_{\alpha_1} \epsilon_{\alpha_2} \epsilon_{\alpha_3} \epsilon_{\alpha_4}$ and $\bar{\epsilon}_{\beta_1} \bar{\epsilon}_{\beta_2} \bar{\epsilon}_{\beta_3} \bar{\epsilon}_{\beta_4}$. We identify $1-2$ as the incoming particles and $3-4$ as outgoing ones. The three Mandelstam variables are

$$ s = -(k_1 + k_2)^2, \quad t = -(k_1 + k_3)^2, \quad u = -(k_1 + k_4)^2, \quad \text{and} \quad s + t + u = 0 \quad (36) $$

The energy momentum conservation lead to the condition: $(k_1 + k_2 + k_3 + k_4) = 0$. It might be interesting to draw analogy with some familiar, well known problems in quantum mechanics. We might imagine the product $\epsilon_{\alpha_1} \epsilon_{\alpha_2} \bar{\epsilon}_{\beta_1} \bar{\epsilon}_{\beta_2}$ as the initial wave function and similarly $\epsilon_{\alpha_3} \epsilon_{\alpha_4} \bar{\epsilon}_{\beta_3} \bar{\epsilon}_{\beta_4}$ as the final state wave function so far as the products of these polarizations are concerned. These tensors are in the internal space, unlike the polarization vectors of photons which transform as vectors under Lorentz transformations. Moreover $\epsilon_{\alpha_1}$ is an $O(3)_R$ vector and $\bar{\epsilon}_{\beta_1}$ is a $O(3)_L$ vector. Therefore, the initial product tensor $\epsilon_{\alpha_1} \epsilon_{\alpha_2}$ can be decomposed into a sum of symmetric traceless second rank tensor and a scalar. The former is like a quadrupole operator in quantum mechanics when we carry out similar decomposition for operator $x^i x^j$ (in radiative transitions in atomic and nuclear physics). Thus we express

$$ \epsilon_{\alpha_1} \epsilon_{\alpha_2} = (\epsilon_{\alpha_1} \epsilon_{\alpha_2} - \frac{1}{3} \delta_{\alpha_1 \alpha_2} \epsilon_{1} \epsilon_{2}) + \frac{1}{3} \delta_{\alpha_1 \alpha_2} \epsilon_{1} \epsilon_{2} \quad (37) $$

We can carry out similar decomposition for the product $\bar{\epsilon}_{\beta_1} \bar{\epsilon}_{\beta_2}$. We would like to define 'wave functions' for the initial state and for the final state. However, in order to adopt compact notations and to bring out the tensor structures under $O(3)_R$ and $O(3)_L$ rotations let us define

$$ Q_{\alpha_1 \alpha_2}^I = (\epsilon_{\alpha_1} \epsilon_{\alpha_2} - \frac{1}{3} \delta_{\alpha_1 \alpha_2} \epsilon_{1} \epsilon_{2}), \quad \text{and} \quad S_{\alpha_1 \alpha_2}^I = \frac{1}{3} \delta_{\alpha_1 \alpha_2} \epsilon_{1} \epsilon_{2} \quad (38) $$
Correspondingly, the tensors of right mover’s polarization are expressed as

\[ Q_{\bar{\alpha}_1\bar{\alpha}_2} = (\bar{\epsilon}_{\bar{\alpha}_1}\bar{\epsilon}_{\bar{\alpha}_2} - \frac{1}{3} \delta_{\bar{\alpha}_1\bar{\alpha}_2} \bar{e}_1 \bar{e}_2), \quad \text{and} \quad \bar{S}_{\bar{\alpha}_1\bar{\alpha}_2} = \frac{1}{3} \delta_{\bar{\alpha}_1\bar{\alpha}_2} \bar{e}_1 \bar{e}_2 \]  

(39)

For the final state wave functions we define

\[ Q_{\alpha_3\alpha_4}^F = (\epsilon_{\alpha_3}\epsilon_{\alpha_4} - \frac{1}{3} \delta_{\alpha_3\alpha_4} \epsilon_3 \epsilon_4), \quad \text{and} \quad S_{\alpha_3\alpha_4}^F = \frac{1}{3} \delta_{\alpha_3\alpha_4} \epsilon_3 \epsilon_4 \]  

(40)

and

\[ \bar{Q}_{\bar{\alpha}_3\bar{\alpha}_4}^F = (\bar{\epsilon}_{\bar{\alpha}_3}\bar{\epsilon}_{\bar{\alpha}_4} - \frac{1}{3} \delta_{\bar{\alpha}_3\bar{\alpha}_4} \bar{e}_3 \bar{e}_4), \quad \text{and} \quad \bar{S}_{\bar{\alpha}_3\bar{\alpha}_4}^F = \frac{1}{3} \delta_{\bar{\alpha}_3\bar{\alpha}_4} \bar{e}_3 \bar{e}_4 \]  

(41)

The four point scattering amplitude for the moduli takes the following form after we have fixed the three Koba-Nielsen variables and therefore, left with one integration

\[ A^{(4)}(s, u) \approx= \int d^2 z_2 |z|^{2k_1,k_2} |1 - z_2|^{2k_2,k_3}(\Sigma_{a=1}^3 T_R^a)(\Sigma_{a=1}^3 \bar{T}_L^a) \]  

(42)

thus the product results in a total of nine terms (there are nine terms in the integral (42)). The terms \( T_R^1 \) and \( T_L^1 \) are

\[ T_R^1 = \frac{1}{(z_2)^2} \left( Q_{\alpha_1\alpha_2}^I + S_{\alpha_1\alpha_2}^I \right) \left( Q_{\alpha_3\alpha_4}^F + S_{\alpha_3\alpha_4}^F \right) \delta^{\alpha_1\alpha_2} \delta^{\alpha_3\alpha_4} \]  

(43)

\[ T_R^2 = \frac{1}{(z_2)^2} \left( Q_{\alpha_1\alpha_2}^I + S_{\alpha_1\alpha_2}^I \right) \left( \bar{Q}_{\bar{\alpha}_3\bar{\alpha}_4}^I + \bar{S}_{\bar{\alpha}_3\bar{\alpha}_4}^I \right) \delta^{\alpha_1\alpha_2} \delta^{\alpha_3\alpha_4} \]  

(44)

\[ T_R^3 = \frac{1}{(1 - z_2)^2} \left( Q_{\alpha_1\alpha_2}^I + S_{\alpha_1\alpha_2}^I \right) \left( Q_{\alpha_3\alpha_4}^F + S_{\alpha_3\alpha_4}^F \right) \delta^{\alpha_1\alpha_4} \delta^{\alpha_2\alpha_3} \]  

(45)

Notice the following: (i) \( T_R^1, T_R^2 \) and \( T_R^3 \) have factors of \((z_2)^{-2}, 1, (1 - z_2)^{-2}\) respectively multiplying them. (ii) Although we have the same products like \([(Q^I + S^I)_{\alpha_1\alpha_2}][(Q^F + S^F)_{\alpha_3\alpha_4}]\) in the expressions for \( T_R^1, T_R^2 \) and \( T_R^3 \) the Kronecker δ’s contracting them is different. Thus we have trace of the products of these tensors. Of course, it is natural since these products are expected to be eventually \(O(3)_R\) invariants. Since \( Q^I \) and \( Q^F \) are traceless, \( T_R^1 = (\text{Tr} S^I)(\text{Tr} S^F) \). The other two terms \( T_R^2 \) and \( T_R^3 \) have the same structure, although the former has 1 as a coefficient and latter has \((1 - z_2)^{-2}\). This term, in the expressions for \( T_L^2 \) and \( T_L^3 \), is

\[ \text{Tr} \left( Q^I Q^F + Q^I S^F + S^I Q^F + S^I S^F \right) \]  

(46)

The expressions for \( \bar{T}_L \) are

\[ \bar{T}_L^1 = \frac{1}{z_2^2} (\text{Tr} \bar{S}^I)(\text{Tr} \bar{S}^F) \]  

(47)

\[ \bar{T}_L^2 = \text{Tr} \left( \bar{Q}^I \bar{Q}^F + \bar{Q}^I \bar{S}^F + \bar{S}^I \bar{Q}^F + \bar{S}^I \bar{S}^F \right) \]  

(48)
and

$$T^3_L = \frac{1}{(1 - z_2^2) Tr \left( Q^I Q^F + Q^I S^F + S^I Q^F + S^I S^F \right) \right)} \right)$$

(49)

As expected \{T_i^4\} are \(O(d)_L\) invariant. The products \(\left(\sum_{a=1}^{3} T^3_R \right) \left(\sum_{a=1}^{3} T^3_L \right)\) have altogether nine terms. These integrals can be evaluated using the standard methods. We draw attention to the fact that \(2k_1, k_2 = -s\) and \(2k_1, k_3 = -u\) since we are considering scattering of massless particles. The integral

$$\int d^2|z_2|^2 |1 - z_2|^{-u} = \frac{2\pi \Gamma(1 - \frac{s}{2}) \Gamma(1 - \frac{s}{2}) \Gamma(-1 - \frac{s}{2} - \frac{u}{2})}{\Gamma(\frac{s}{2}) \Gamma(\frac{s}{2}) \Gamma(2 - \frac{s}{2} - \frac{u}{2})}$$

(50)

There are other terms like \(z_2^{-2}, (1 - z_2)^{-2}\) and so on, which multiply this integrand. Each of these integrations can be handled by using table integrals for gamma functions (see [22] for discussion on evaluating these type of integrals, especially the appendix).

We mention in passing that the T-duality invariance of the four point amplitude is manifest in the procedure we proposed above.

Let us consider the four point function involving gauge bosons. We could compute four gauge boson amplitude; however, in order to bring out the general features, it will suffice to consider scattering of a gauge boson \(A_{\mu i}\) from a tachyon where \(i = 1, 2, 3, 4, 5, 6\). The initial and final tachyon momenta are \(k_1\) and \(k_3\) and those of the two gauge bosons are \(k_2, k_4\). The amplitude is

$$T^{(4)} = \int \prod_{i=1}^{4} d^2 z_i \mathcal{D} \prod_{i, j} |z_i - z_j|^{2k_i, k_j} \exp \left[ \frac{\epsilon_2 \epsilon_4}{(z_2 - z_4)^2} - \sum_{i \neq j} \frac{\epsilon_i k_j}{(z_i - z_j)^2} \right] \exp \left[ \frac{\bar{\epsilon}_2 \bar{\epsilon}_4}{(z_2 - z_4)^2} \right]$$

(51)

after we expand the exponentials and collect the relevant terms. These terms are of the form \(\epsilon_2, \epsilon_4\) times \(\bar{\epsilon}_2, \bar{\epsilon}_4\) that will appear in four point amplitude since we have only two photons. \(\mathcal{D}\) is the Koba-Nielsen factor and we choose it to be the same as before and therefore, the above expression (51) consists of an integral over \(z_2\) and \(z_4\). The terms like \(\epsilon_1, \epsilon_2\) and \(\epsilon_1, \epsilon_3\) have passive transformations under T-duality. Therefore, we need pay attention to this piece. The term of interests to us is \(\bar{\epsilon}_2, \bar{\epsilon}_4\) and it is automatically \(T\)-duality invariant. However, we shall discuss how we can generate new scattering amplitudes from (51).

It is interesting to note that the four point function for gauge bosons (Abelian) can be evaluated efficiently in this formalism. We start from the general expression and the outline the prescription (amplitude is \(A^{(4)}_{\text{gauge}}\))

$$A^{(4)}_{\text{gauge}} \sim \int \prod_{i=1}^{4} d^2 z_i \mathcal{D} \prod_{i, j} |z_i - z_j|^{2k_i, k_j} \mathcal{E}_1 \mathcal{E}_2$$

(52)

where \(\mathcal{E}_1\) and \(\mathcal{E}_2\) stand for two exponentials defined below

$$\mathcal{E}_1 = \exp \left[ \sum_{i, j} \frac{\epsilon_i \epsilon_j}{(z_i - z_j)^2} - \sum_{i \neq j} \frac{\epsilon_i k_j}{(z_i - z_j)^2} \right]$$

(53)
and

$$\mathcal{E}_2 = \exp[\Sigma_{i>j} \frac{\bar{\epsilon}_i \bar{\epsilon}_j}{(z_i - z_j)^2}]$$  (54)

The four point function will have one integration left since we choose \(\{z_1, z_3, z_4\}\) to be the Koba-Nielsen variable and assign them the values as before (similarly we assign values to the other complex conjugates as before). We discuss the structure of the amplitude. We have polarizations \(\{\epsilon_{\mu_i}\}\) which are spacetime vectors and we have \(\{\bar{\epsilon}_i\}\) which are 'polarizations' coming from internal directions. The latter will mutually contract among themselves. The set \(\epsilon_{\mu_i}\) will not only contract among themselves but there will be products like \((\epsilon_i \cdot \epsilon_j)(\epsilon_m \cdot k_n)\) and so on. The essential points to note are there have to be products of four \(\epsilon_{\mu_i}\) which are contracted amongst themselves or with \(k^{\mu_j}\) keeping in mind that \(\epsilon_{\mu_i} \cdot k^{\mu_i} = 0\) (no sum over \(\mu_i\)). Such an amplitude has been computed before in several ways. We intend to present arguments which will bring out the essential features of \(A^{(4)}_{\text{gauge}}\) in this regard. Let us look at \(\mathcal{E}_2\) first. The we have to retain up to the quadratic term in the expansion of the exponential. The terms of our interests are of the form \(\bar{\epsilon}_{\alpha_i} \bar{\epsilon}_{\alpha_j} \epsilon_{\alpha_m} \epsilon_{\alpha_n}\) where \(i, j, m, n\) take values 1, 2, 3, 4 they have to be different in the products. If we go back to the 4-point amplitude for scattering of moduli and in the present case identify \(\bar{\epsilon}_{\alpha_i} \bar{\epsilon}_{\alpha_j}\) as the initial 'wave function' and \(\epsilon_{\alpha_m}\) as the final wave function then the arguments of (46) to (49) go through. Thus the T-duality invariance of \(A^{(4)}_{\text{gauge}}\) is ensured. Let us discuss the terms coming from expansion of \(\mathcal{E}_1\). Here we have to first retain up to quadratic term in the expansion and some interference terms coming from the cubic part and terms coming from quartic part. (I) We note the presence of the products of the type \((\epsilon_i \cdot \epsilon_j)(\epsilon_m \cdot \epsilon_n), i \neq j, m \neq n\) and \(i, j, m, n\) are all different. These originate from the quadratic terms in the expansion of the exponential. (II) The other type of terms are \((\epsilon_i \cdot \epsilon_j)(\epsilon_m \cdot k_l)(\epsilon_n \cdot k_s)\). Such types of terms originate from the cubic term in expansion of exponential - interference of two \(\bar{\epsilon}_i \cdot k\) with \(\epsilon_i \cdot \epsilon\) (all \(\epsilon\) indices are to be different). Note that we should have all four polarization vector’s indices are different. (III) There exist another type of term which are like \(\epsilon_i \cdot k_j\) where four of them will occur. Such types of terms will come from the fourth power in the expansion of the first exponential. In order to make this argument transparent, let us look at one combination, (there are more such terms)

$$\left(\frac{\epsilon_1 \cdot k_2}{(z_1 - z_2)} + \frac{\epsilon_2 \cdot k_1}{(z_2 - z_1)} + \frac{\epsilon_3 \cdot k_4}{(z_3 - z_4)} + \frac{\epsilon_4 \cdot k_3}{(z_4 - z_3)}\right)^2$$  (55)

There is an interference term (suppressing the difference \((z_i - z_j)\) which will all appear as products in the denominator):

$$\epsilon_1 \cdot k_2 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_4 \epsilon_4 \cdot k_3$$  (56)

apart from a numerical factor. There will be several combinations of \(\epsilon \cdot k\) in such products.
There are a few points to be made:

(i) So far as $T$-duality transformation is concerned, these are manifestly invariant.

(ii) We should bring in the $\alpha'$ dependence to see the other aspect. Notice that in $\mathcal{E}_1$ each term in multiplied by $\alpha'$ since the first one comes from correlation of pair like

$$\partial X^\mu_i \partial X^\mu_j = -\frac{\alpha'}{2} \frac{\delta^{\mu_1 \mu_2}}{(z_i - z_j)^2}$$

and the other one comes from contraction of $\partial X^\mu_i \cdot e^k_j X_j$. This also has a factor of $\alpha'$. Therefore, when we expand the exponential, we have different powers of $\alpha'$. Of course this is expected on dimensional ground. Note that this is already seen in the three point vertex of graviton [2]. There are terms which are linear in momenta with suitable metric multiplied to it (in the sense when we look at contributions of the holomorphic parts).

All these amplitudes are calculated through applications of known conformal field theory techniques. However, the KLT prescription is more economical at the tree level.

Now we present some applications of $T$-duality transformations to demonstrate how another four point function will be generated from a given one. Let us consider the case of scattering of moduli (42). We choose

$$\epsilon_\alpha = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \bar{\epsilon}_{\beta} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{(57)}$$

The corresponding moduli $G$ and $B$ are

$$G_{\alpha \bar{\beta}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B_{\alpha \bar{\beta}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{(58)}$$

Now consider a simple $T$-duality transformation where we rotate by $O(3)_R$ on the $2-3$ plane and denote this operation by $R$. The other rotation is $S$, the $O(3)_L$ and $S$ is also a rotation on $2-3$ plane. We identify $S = R^T$. We choose

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \quad \text{(59)}$$

Let the rotation angle be $\theta = \frac{\pi}{4}$ for simplicity, so that $\cos \theta = \sin \theta = \frac{1}{\sqrt{2}}$. According to our prescriptions the transformed vectors are

$$\epsilon'_\alpha = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \bar{\epsilon}'_{\beta} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{(60)}$$

and the two transformed backgrounds are

$$G' = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \end{pmatrix} \quad \text{(61)}$$

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Thus this simple example brings out the essential features of $T$-duality transformation in this approach. For a most general rotation, the $O(3)$ group is parametrized by three Euler rotation matrices. We are free to choose them for $O(3)_R$ and $O(3)_L$. Therefore, starting from very simple vectors $\varepsilon_\alpha$ and $\tilde{\varepsilon}_\beta$ as in (57), we can generate very complicated form of these vectors. Moreover, if we have the four point amplitude for these configurations, we can connect the amplitude to the one which has more complicated configurations of polarization vectors.

Let us turn our attention to the gauge boson and tachyon scattering amplitude (51). Here we shall focus on one term $\tilde{\varepsilon}_2 \tilde{\varepsilon}_4$. Recall that now $\varepsilon$ is a six component vector; three of them coming from $\tilde{\varepsilon}_(1)_{\beta}$, $\tilde{\varepsilon}_{(1)\beta}^\alpha = 1, 2, 3$, and the other three coming from $\tilde{\varepsilon}_(2)_{\beta}$, $\tilde{\varepsilon}_{(2)\beta}^\alpha = 1, 2, 3$. Let us remind ourselves that $A^{(1)}_{\mu\beta}$ and $A^{(2)}_{\mu\beta}$ transform as a doublet under $O(3, 3)$: $A_{\mu} \to \Omega A_{\mu}$. According to Sen’s prescription

$$\begin{pmatrix} A^{(1)}_{\mu} \\ A^{(2)}_{\mu} \end{pmatrix} \to \frac{1}{2} \begin{pmatrix} R + S & R - S \\ R - S & R + S \end{pmatrix} \begin{pmatrix} A^{(1)}_{\mu} \\ A^{(2)}_{\mu} \end{pmatrix}$$

(62)

Therefore, the two sets of polarizations i.e. $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$, we are dealing with, will transform according to (62). Thus if we had a configuration where the photon originated from compactification of $\hat{D}$-dimensional graviton, then through the $O(3) \otimes O(3)$ transformation we will relate this amplitude to scattering of gauge bosons which are admixtures of $A^{(1)}_{\mu\beta}$ and $A^{(2)}_{\mu\beta}$. This example illustrates the central point. If we have amplitude with four gauge bosons, we know how to extract the $T$-duality transformation part of it using the above steps. As an example, let us start with a configuration for gauge boson and tachyon scattering where we have only the gauge field $A^{(1)}_{\mu\beta}$ and we set $A^{(2)}_{\mu\beta} = 0$. Then under $O(3)_R \otimes O(3)_L$ rotations: $A^{(1)}_{\mu}\to (R + S)A^{(1)}_{\mu} \beta$ and $A^{(2)}_{\mu\beta} = (R - S)\tilde{\varepsilon}^\alpha_{\beta}A^{(1)}_{\mu\alpha}$. Thus we generate a scattering amplitude involving both the gauge fields. We can interpret this result as follows. If we start from gravi-photon (gauge boson originating from dimensional reduction of graviton in $\hat{D}$-dimensions) and tachyon amplitude then through above duality rotation it gets related to state which is admixture of gravi-photon and axi-photon (gauge boson originating from reduction of $B$-field).

### 3.2 Scattering of nonabelian Bosons in KLT Formalism

In this short subsection, we propose a generalization of the KLT approach to construct vertex operators for nonabelian states in certain compactified schemes. The most familiar example is the heterotic string. The Yang-Mills super multiplet appears in the massless sector together with the $N = 1$ supergravity multiplet. There are excited massive states [4] belonging to the representations of the chosen gauge group. Therefore, it is of interests to construct vertex operators for such states which carry the nonabelian charges. We briefly recapitulate below how the vector boson vertex operator was constructed in the bosonic coordinate representation when compact
coordinates are along the torii and the canonical momenta are quantized with further restrictions [5]. Here we present another approach which is useful to compute tree level amplitudes. We recall that there are gauge bosons in the adjoint representation of $SO(32)$ or $E_8 \otimes E_8$ in heterotic string theory in super Yang-Mills sector. These nonabelian vector states appear if the lattice is self-dual and even. In order to fulfill this requirements, the number of compactified dimensions have to be multiples of 8. One of the most attractive features of heterotic string theory is that in, $D = 10$, these are the only possible gauge groups appear since 16 of the twenty six coordinates of the bosonic sector are toroidally compactified. It is remarkable that these two gauge groups are precisely the admissible gauge groups for cancellation of anomalies discovered by Green and Schwarz [33]. The scattering of these nonabelian gauge bosons were studied in [5]. The vertex operators not only have the usual spacetime plane wave factor but also a certain term $\exp[2iP_I Y^I]$ where $P_I$ are the momenta are along compact direction, they are quantized and satisfy $(P_I)^2 = 2$. Note the appearance of $2P_I$ in vertex operator: this is generator of translation in the internal space. The emission vertex for the charged gauge field needs another factor - the operator cocycle $\hat{C}(K)$ and its action on a state of momentum $P$ gives the two cocycle $\epsilon(P, K)\hat{C}(K)|P^I > = \epsilon(P, K)|P^I >$ (63) Moreover, $\hat{C}(K)\hat{C}(L) = \epsilon(K, L)\hat{C}(K+L)$ and the two cocycle condition is $\epsilon(K, L)\epsilon(K + L, M) = \epsilon(L, M)\epsilon(K, L + M)$. It is possible to choose the cocycles such that (i) $\epsilon(K, L + M) = \epsilon(K, L)\epsilon(K, M)$. (ii) They satisfy $\epsilon(K, L)\epsilon(L, K) = (-1)^{K\cdot L}$, $\epsilon(K, 0) = -\epsilon(K, -K) = 1$, for $K^2 = L^2 = (K + L)^2 = 2$, it coincides with structure constants of the group. The three point and four point Yang-Mills amplitude have been evaluated long since [5]. The structure constants of the Yang-Mills group is identified to be $f^{K_1 K_2 K_3} = \epsilon(K_2, K_3)\delta_{K_1, K_2 + K_3}$. Indeed, Kawai,Llewellen and Tye [22] adopted this form of vertex operator and used the properties mentioned above to evaluate the gauge boson scattering amplitudes in their reformulation.

Let us consider a related string compactification of a closed bosonic string. The left moving and right moving sectors are independent and these coordinates can be compactified separately. Let us toroidally compactify 16 coordinates of left moving sector and compactify same number of coordinates in right moving sector as well. The ground state of the theory is tachyonic. The 16 compact coordinates in left and right moving sector can be fermionized to give 32 Weyl Majorana fermions in each sector. If we chose NS-NS boundary conditions for all left and right moving fermions then the gauge group is $SO(32) \otimes SO(32)$. The massless spectrum is quite interesting. This compactified string has massless graviton, antisymmetric tensor field and dilaton. In addition there are two copies of gauge bosons in the adjoint of the two $SO(32)$ groups coming from left and right movers. Moreover, the theory has massless scalars transforming as $(496, 496)$. Let us consider the vertex operators for the gauge bosons in the compactified closed
bosonic string

\[ V^{(L)} =: A_{\mu ij}^L(X)\bar{\partial}X^\mu(\bar{z})\psi^i(\bar{z})\psi^j(z) : \quad (64) \]

and the other vertex operator is

\[ V^{(R)} =: A_{\mu ij}^R(X)\partial X^\mu(z)\bar{\psi}^i(\bar{z})\psi^j(z) : \quad (65) \]

Notice that each of the gauge bosons are in the adjoint of their \( SO(32) \). In the plane wave approximation we express the two vertex operators as gauge

\[ A_{\mu ij}^L = \epsilon_\mu^a(T^a)_{ij} : \exp[ik.X]\bar{\partial}X^\mu(\bar{z})\psi^i(\bar{z})\psi^j(z) : , \quad (66) \]

and

\[ A_{\mu ij}^R = \epsilon_\mu^a(\bar{T}^a)_{ij} : \exp[ik.X]\partial X^\mu(z)\bar{\psi}^i(\bar{z})\psi^j(z) : \quad (67) \]

The two generators \( T^a \) and \( \bar{T}^a \) in the vector representation of the groups. The correlation functions for gauge boson amplitudes are evaluated through these vertex operators. Let us generalize the KLT vertex operator to the case of nonabelian gauge boson emission.

\[ V_{\text{gauge}} =: \exp[ik.X+i\epsilon_\mu\bar{\epsilon}_\nu\bar{\partial}X^\mu(\bar{z})+i\epsilon_\mu\partial X^\mu+i\epsilon^aT^a_{ij}\psi^i(\bar{z})\psi^j(z)+i\epsilon^a\bar{T}^a_{ij}\bar{\psi}^i(\bar{z})\bar{\psi}^j(z)] : \quad (68) \]

Notice the following features: (i) When we expand the exponential in powers of the 'polarizations' we shall recover the vertex operators in the massless sector. (ii) The bilinear term \( \epsilon_\mu\bar{\epsilon}_\nu \) will give the vertex operators from graviton and antisymmetric tensor as mentioned earlier. (iii) The two bilinears \( \bar{\epsilon}_\nu\epsilon^a\bar{T}^a \) and \( \epsilon_\mu\epsilon^aT^a \) will give the two vertex operators for gauge bosons once we identify \( \epsilon_\mu^a(\bar{T}^a)_{ij} \) and \( \epsilon_\mu^a(T^a)_{ij} \) with the first and second term respectively.

This generalization of KLT vertex will make calculation of scattering amplitudes quite efficient. As an example consider the three point function for gauge bosons \( A^R \). The general structure of the three point function is

\[ \epsilon_\mu^a\epsilon_\nu^b\epsilon_\kappa^c T^{\mu_1\mu_2\mu_3}_{abc} \]

Note that \( T^{\mu_1\mu_2\mu_3}_{abc} \) factorizes into a product \( f_{abc} \), the structure constant and a tensor with spacetime indices (see equation below). Since the three Koba-Nielsen variables are fixed, there is no integration to be done. This three point function was already derived in [5]. If we adopt the vertex operator proposed above (68) then tree level calculation becomes simpler. The first point to observe that polarization vector \( \epsilon_\mu^a \) factorizes as \( \epsilon_\mu^a = \epsilon_\mu\bar{\epsilon}^a \) for this case (i.e. gauge boson \( A^R \)). The second point is that we have to expand the exponentials and collect the products of the type: \( \epsilon_\mu^a\epsilon_\nu^b\epsilon_\kappa^c \). Next we compute the correlation functions of right movers and left movers. Thus the product terms are

\[ f_{abc}\left(g^{\mu_1\mu_2}k_{12}^{\mu_3}+g^{\mu_2\mu_3}k_{23}^{\mu_1}+g^{\mu_3\mu_1}k_{31}^{\mu_2}\right) \quad (70) \]
where $g^{\mu \nu}$ is the flat spacetime metric and $k_{ij}^\mu = (k_i - k_j)^\mu$. The form of (70) can be cast in a differently using energy momentum conservation condition $(k_1 + k_2 + k_3) = 0$ and that $\epsilon_\mu k^\mu = 0$ (no sum over $i$). The structure constant comes from contractions of the fermions appearing in expansion of exponential. The spacetime tensor structure, as is well known comes from various contractions involving spacetime string coordinates. We have used $\langle \tilde{\psi}^i(\bar{z}_i)\tilde{\psi}^j(\bar{z}_j) \rangle \simeq \frac{\delta^j_i}{(\bar{z}_i - \bar{z}_j)}$. We believe this prescription can be used to evaluate higher point amplitudes.

4 Summary and Conclusions

We set out to investigate $T$-duality transformation properties of scattering amplitudes. It is accepted that $T$-duality is a symmetry of compactified string theory and the S-matrix is expected to be invariant. We need to construct vertex operators in order to evaluate the amplitudes. This is achieved in the weak field approximation. For closed bosonic string, compactified on a d-dimensional torii the spectrum contains the moduli, gauge bosons, graviton, antisymmetric tensor and dilaton, in its massless sector. The moduli can be cast in an form that transforms as an adjoint under the $O(d,d)$ transformation. However, it is not very convenient to implement $O(d,d)$ transformations when the weak field expansion is made around the trivial background. We have followed Sen’s [23] argument and identified $O(d) \otimes O(d) \in O(d,d)$ as the duality group for our purpose. These choice of this duality group is very appropriate to study transformation properties of the vertex operators. Furthermore, we employed a suitably modified version the KLT [22] formalism to construct vertex operators for the moduli. This formulation has the advantage that the polarization tensors of the moduli factorize when we construct the vertex operators. Thus the $O(d) \otimes O(d)$ transformation properties of the amplitudes become simple and transparent. In particular, the N-point amplitude for moduli $G$ and $B$ can be expressed in a compact form. The Abelian gauge bosons, resulting from the toroidal compactification, transform linearly under $O(d,d)$ group. In the first place, the polarization vector of these gauge bosons are shown to factorize by adopting the KLT formulation. The N-point amplitude for moduli and gauge bosons is demonstrated to be duality invariant. The present investigation substantiated the arguments of [21] through explicit calculations that the S-matrix is indeed $T$-duality invariant.

We presented two illustrative examples. We considered a case where three string coordinates are compactified on three torus, $T^3$. First we considered the four point function of the moduli. The $O(3)_R \otimes O(3)_L$ invariance of this amplitude can be explicitly verified by adopting arguments used in quantum mechanics. The initial state, as far as the products of polarizations are concerned, are decomposed into sum of irreducible representations of $O(3)_L$ and similarly as sum of irreducible representations of $O(3)_R$. Thus the initial function (in the space of polarizations) is a director product of tensors. The same procedure can be followed for the final state wave functions.
These tensors are contracted in various ways by the metrics resulting from contractions of string coordinates in the vertex operators. Therefore, the four-point amplitude is expressed in a manifestly duality invariant form. This argument can be extended to computation of four-point functions where $d$-coordinates are compact. Thus the initial states consisting of left movers will be decomposed to sum of irreducible representations of $O(d)_L$ and similarly, the initial wave function consisting of right movers will be expressed as sum of irreducible representations of $O(d)_R$. The same procedure will be applicable to the final state wave functions as well. Therefore, all the four-point amplitudes, in the massless sector can be demonstrated to be duality invariant.

We propose vertex operators, based on KLT formulation, for scattering of nonabelian stringy states. This prescription is economical for the computation of $S$-matrix elements. We explicitly evaluate three gauge boson vertex which agrees with known results. This technique might be useful for computation of excited stringy states which carry 'color' gauge charges.

In summary, we have demonstrated that the amplitudes constructed for the massless states of compactified closed bosonic string can be expressed in manifestly $T$-duality invariant form. This is facilitated efficiently through the introduction of the vertex operators of the massless states. These vertex operators are modified versions of those introduced by KLT. Moreover, we proposed vertex operators to evaluate amplitudes for scattering of nonabelian gauge bosons. We noted earlier that such nonabelian gauge boson appear as massless states of $D = 10$ heterotic string as well as in compactification of closed bosonic string as long as the internal moments fulfill the criterion mentioned already. Moreover, there will be nonabelian excited states. The scattering of these states from gauge bosons can be evaluated by using the vertex operator introduced here and the generalization there of.

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