A selective review on calibration information from similar studies based on parametric likelihood or empirical likelihood

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Abstract

In multi-center clinical trials, due to various reasons, the individual-level data are strictly restricted to be assessed publicly. Instead, the summarized information is widely available from published results. With the advance of computational technology, it has become very common in data analyses to run on hundreds or thousands of machines simultaneous, with the data distributed across those machines and no longer available in a single central location. How to effectively assemble the summarized clinical data information or information from each machine in parallel computation has become a challenging task for statisticians and computer scientists. In this paper, we selectively review some recently-developed statistical methods, including communication efficient distributed statistical inference, and renewal estimation and incremental inference, which can be regarded as the latest development of calibration information
methods in the era of big data. Even though those methods were developed in different fields and in different statistical frameworks, in principle, they are asymptotically equivalent to those well known methods developed in meta analysis. Almost no or little information is lost compared with the case when full data are available. As a general tool to integrate information, we also review the generalized method of moments and estimating equations approach by using empirical likelihood method.  

Some key words: Calibration information; Empirical likelihood; Estimating equations; Generalized method of moments; Meta analysis.

1 Introduction

Combining information from similar studies has been and will be an extremely important strategy in statistical inference. The most popular example of such methods is meta analysis, which pools the published results of multiple similar scientific studies together to produce an enhanced estimate without using the raw individual data from each study. We refer to Borenstein et al. (2009) for a comprehensive introduction of meta analysis. Due to various reasons such as privacy or capacity of computer storage in massive data inference, only summarized data rather than the original individual data are available. This poses a very challenging problem: how to conduct an efficient updated inference by making full use of the summarized data? In recent years, many methods of combining information have been developed in economic studies, machine learning, and distributed statistical inferences. The goal of this paper is to selectively review a few popular methods that are able to integrate information in different disciplines.

Utilizing external summary data or auxiliary information to make a sharper inference is an old and effective method in survey sampling. Due to restrictions such as cost effectiveness or convenience, the variable of interest $Y$ may be available for a small portion of individuals. However, the explanatory variable $X$ associated with $Y$ may readily be available. Cochran (1977) had a comprehensive discussion on the regression type estimators by adapting the summarized information from $X$. Chen and Qin (1993), Wu and Sitter (2001), and Chen et
al. (2002) used empirical likelihood (EL) to incorporate such information in finite population.

With the advance of technology, many summarized statistical results are available in public domains. For example, many aggregated demographic and socioeconomic status data are given in the US census reports. The Surveillance, Epidemiology and End Results (SEER) program of the National Cancer Institute provides the population-based cancer survival statistics, such as covariate specific survival probabilities. Imbens and Lancaster (1994) combined Micro and Macro data in economic studies through generalized method of moments (GMM). Chaudhuri, Handcock and Rendall (2008) showed that the inclusion of the population level information can reduce bias and increase efficiency of the parameter estimates in a generalized linear model setup. Wu and Thompson (2019) published an excellent monograph on combining auxiliary information in survey sampling.

In this paper, we will consider two situations. First, the summarized information was derived under the same statistical model. Second, the summarized information was derived under similar but not exactly the same statistical models. In general, combining information in the former case is easier. The later case is more delicate since one has to take the heterogeneity among different studies into considerations.

The rest of this paper proceeds as follows. In Section 2, we briefly review two simple and popular meta methods of combining similar analysis results. As a general tool of synthesizing information from summarized information, we review Owen’s (1988) EL method and Qin and Lawless (1994)’s over-identified parameter problem in Section 3. In particular, we present a new way of deriving the lower information bound for the over-identified parameter problem. Section 4 discusses enhanced inference by utilizing auxiliary information. Section 5 presents results on more flexible meta analysis where the covariate information is collected differently even in similar studies. Calibrating information from previous studies is given in Section 6. We discusses methods of using disease prevalence information for more efficient estimation in case and control studies in Section 7. The popular communication efficient distributed statistical inference in machine learning is discussed in Section 8. Renewal estimation and incremental inference is briefly presented in Section 9. Some discussions are provided in
2 Two simple combining information methods

2.1 Convex combination

Suppose that \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) are two asymptotically unbiased estimators for \( \theta \) from two independent studies, and that they satisfy \( \sqrt{n}(\hat{\theta}_i - \theta) \sim N(0, \sigma_i^2) \), \( i = 1, 2 \). The most straightforward way of combining \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) is a convex combination,

\[
\hat{\theta} = \alpha \hat{\theta}_1 + (1 - \alpha) \hat{\theta}_2, \quad 0 < \alpha < 1.
\]

The asymptotic variance of \( \hat{\theta} \) is \( \sigma^2 = \alpha^2 \sigma_1^2 + (1 - \alpha)^2 \sigma_2^2 \), which takes its minimum at \( \alpha = \sigma_2^2 / (\sigma_1^2 + \sigma_2^2) \). This suggests combining \( \hat{\theta}_1 \) and \( \hat{\theta}_2 \) by

\[
\hat{\theta} = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \hat{\theta}_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \hat{\theta}_2 = \frac{\hat{\theta}_1/\sigma_1^2 + \hat{\theta}_2/\sigma_2^2}{1/\sigma_1^2 + 1/\sigma_2^2},
\]

an inverse-variance weighting estimator. In general, \( \sigma_1^2 \) and \( \sigma_2^2 \) are unknown, we may replace them by their estimators \( \hat{\sigma}_1^2 \) and \( \hat{\sigma}_2^2 \) respectively, which leads to

\[
\hat{\theta} = \frac{\hat{\sigma}_2^2}{\hat{\sigma}_1^2 + \hat{\sigma}_2^2} \hat{\theta}_1 + \frac{\hat{\sigma}_1^2}{\hat{\sigma}_1^2 + \hat{\sigma}_2^2} \hat{\theta}_2 = \frac{\hat{\theta}_1/\hat{\sigma}_1^2 + \hat{\theta}_2/\hat{\sigma}_2^2}{1/\hat{\sigma}_1^2 + 1/\hat{\sigma}_2^2}.
\]

As an alternative method, we may use the maximum likelihood method to argue that this is the best estimator. We can treat \( \hat{\theta}_i \) as an direct observation from \( \hat{\theta}_i|\theta \sim N(\theta, \sigma_i^2) \), \( i = 1, 2 \). Then the log-likelihood is (regarding \( \sigma_1^2 \) and \( \sigma_2^2 \) as known constants)

\[
-(\hat{\theta}_1 - \theta)^2/(2\sigma_1^2) - (\hat{\theta}_2 - \theta)^2/(2\sigma_2^2).
\]

Maximizing this likelihood with respect to \( \theta \) or setting the score function to be zero, we end up with the same inverse-variance weighting estimator.
2.2 Random effect meta analysis

Dersimoni and Laird (1986) proposed a moment-based estimation method under a random effect model for meta analysis. Suppose

\[ \hat{\theta}_i | \theta_i \sim N(\theta_i, w_i^{-1}), \quad \theta_i \sim N(\theta, \tau^2), \quad i = 1, 2, \ldots, K, \]

where \( w_i^{-1} \)'s are treated as known. Unconditionally we have \( \hat{\theta}_i \sim N(\theta, w_i^{-1} + \tau^2) \). Consider the following inverse-variance weighting estimator for \( \theta \),

\[ \hat{\theta} = \frac{\sum_{i=1}^{K} \hat{\theta}_i w_i}{\sum_{i=1}^{K} w_i} \]

with variance \( \text{Var}(\hat{\theta}) = \frac{\sum_{i=1}^{K} w_i^2 (w_i^{-1} + \tau^2) / (\sum_{i=1}^{K} w_i)^2}{\sum_{i=1}^{K} w_i} \). Define

\[ Q = \sum_{i=1}^{K} w_i (\hat{\theta}_i - \hat{\theta})^2 = \sum_{i=1}^{K} w_i (\hat{\theta}_i - \theta)^2 - (\hat{\theta} - \theta)^2 \sum_{i=1}^{K} w_i. \]

Easily we can check

\[ \mathbb{E}(Q) = (K - 1) + \tau^2 \left( \sum_{i=1}^{K} w_i - \sum_{i=1}^{K} w_i^2 / \sum_{j=1}^{K} w_j \right), \]

which implies that a natural estimator of \( \tau^2 \) is

\[ \hat{\tau}^2 = \frac{Q - (K - 1)}{\sum_{i=1}^{K} w_i - \sum_{i=1}^{K} w_i^2 / \sum_{j=1}^{K} w_j}. \]

For small sample sizes, there is no guarantee that this estimator is non-negative; one may replace it by \( \max(\hat{\tau}^2, 0) \).

Alternatively, we may estimate \( \tau \) using the likelihood approach. The joint likelihood based on \( \hat{\theta}_i \)'s is

\[ \ell(\theta, \tau) = -\frac{1}{2} \sum_{i=1}^{K} (\hat{\theta}_i - \theta)^2 / \tau^2 + w_i^{-1} - \frac{1}{2} \sum_{i=1}^{K} \log(\tau^2 + w_i^{-1}). \]

Maximizing \( \ell \) with respect to \( \theta \) and \( \tau^2 \) gives their maximum likelihood estimators (MLEs).

Lin and Zeng (2007) made comparisons on the relative efficiency of using summary statistics versus individual-level data in meta-analysis. They found that in general there is no information loss by using the summarized information compared with the inference based on original individual data if they are indeed available.
3 Empirical likelihood and general estimating equations

In this section we will briefly review Owen’s (1988) EL and Qin and Lawless’ (1994) estimating equations approach since those methods have provided a general tool to assemble information from different sources.

The maximum likelihood method for regular parametric models has many optimality properties. As a result, it is one of the most popular methods in statistical inference. However, model mis-specification is a big concern since a misspecified model may lead to biased results. When the underlying distribution is multinomial, Hartley and Rao (1968) proposed a mean constrained estimator for the population total in survey sampling problems. To mimic the parametric likelihood but with robust properties, Owen (1988, 1990) proposed the EL method, which is a natural generalization of the multinomial likelihood when the number of categories is the same as the sample size. EL can be thought of as a bootstrap that does not resample, and as a likelihood without parametric assumptions (Owen, 2001).

3.1 Definition of empirical likelihood

Suppose $X_1, ..., X_n$ are $n$ independent and identically distributed (iid) observations from $X$ with the cumulative distribution distribution $F$. Without loss of generality, we assume there are no ties, i.e., any two observations are unequal to each other. Let $dF(X_i)$, $i = 1, 2, ..., n,$ be the jumps of $F(x)$ at the observed data points. The nonparametric likelihood is $L(F) = \prod_{i=1}^{n} p_i$. It is clear that if any $p_i = 0$, then $L(F) = 0$, and if $\sum_{i=1}^{n} p_i < 1$, then $L(F) < L(F_*)$, where $F_*(x) = \sum_{i=1}^{n} p_i I(X_i \leq x) / \sum_{i=1}^{n} p_i$. According to the likelihood principle (parameters with larger likelihoods are preferable), one need only consider the distribution functions $F(x)$ with $p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$.

If we maximize the log-likelihood

$$\ell(F) = \sum_{i=1}^{n} \log p_i$$

(1)
subject to the constraints
\[ \sum_{i=1}^{n} p_i = 1, \quad p_i \geq 0, \] (2)
then we end up with \( p_i = 1/n, \ i = 1, 2, ..., n \). Therefore the EL method estimates \( F \) by \( F_n(x) = \sum_{i=1}^{n} p_i I(X_i \leq x) = n^{-1} \sum_{i=1}^{n} I(X_i \leq x) \). This is the reason why the empirical distribution is called the nonparametric maximum likelihood estimator of \( F(x) \).

Suppose we are interested in constructing a confidence interval for \( \mu = \mathbb{E}(X) = \int x dF(x) \), the mean of \( X \). Since we have discretized \( F \) at each of the observed data points, the integral becomes \( \mu = \sum_{i=1}^{n} p_i X_i \). Next we maximize the log nonparametric likelihood subject to an extra constraint
\[ \sum_{i=1}^{n} p_i (X_i - \mu) = 0. \] (3)
Maximizing the log-likelihood (1) subject to the constraints (2) and (3), the Lagrange multiplier method gives the profile log-likelihood of \( \mu \),
\[ \ell_n(\mu) = -\sum_{i=1}^{n} \log\{1 + \lambda^\top(X_i - \mu)\} - n \log n, \] (4)
where \( \lambda \) is the solution to \( \sum_{i=1}^{n} (X_i - \mu)/\{1 + \lambda^\top(X_i - \mu)\} = 0 \).

We can treat \( \ell_n(\mu) \) as a parametric likelihood of \( \mu \). It is clear that based on this likelihood, the maximum EL estimator of \( \mu \) is \( \hat{\mu} = \bar{X} = n^{-1} \sum_{i=1}^{n} X_i \), which is exactly the sample mean. Define the likelihood ratio function as
\[ R_n(\mu) = 2\{\max_{\mu} \ell_n(\mu) - \ell_n(\mu)\} = 2\{\ell_n(\bar{X}) - \ell_n(\mu)\}. \]
Under the regularity conditions specified in Owen (1988, 1990), as \( n \) goes to infinity, \( R_n(\mu_0) \) converges in distribution to the chi-square distribution with \( p \) degrees of freedom, where \( p \) is the dimension of \( \mu \) and \( \mu_0 \) is the true value of \( \mu \).

### 3.2 General estimating equations

The original empirical likelihood was mainly used to make inference for linear functionals of the underlying population distribution such as the population mean (Owen, 1988, 1990).
Qin and Lawless (1994) applied this method to general estimating models, which greatly broadens its applications. Specifically, suppose the population of interest satisfies a general estimating equation

$$\mathbb{E}\{g(X, \theta)\} = 0,$$

for a $r \times 1$ vector-valued function $g$ and some $\theta$, which is a $p \times 1$ parameter to be estimated. We assume $r \geq p$ as otherwise the true parameter value of $\theta$ is undetermined.

For general estimating equations with $r > p$ or over-identified models, Hansen (1982) proposed the celebrated GMM, which has become one of the most popular methods in econometric literature. In essence, the GMM minimizes

$$\left\{ \sum_{i=1}^{n} g(X_i, \theta) \right\}^\top \Sigma^{-1} \left\{ \sum_{i=1}^{n} g(X_i, \theta) \right\}$$

with respect to $\theta$, where $\Sigma$ is the variance matrix of the estimating equation $g(X, \theta)$. If $\Sigma$ is unknown, we may replace it by the sample variance $\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} g(X_i, \hat{\theta})g^\top(X_i, \hat{\theta})$, where $\hat{\theta}$ is an initial consistent estimator of $\theta$.

Instead of GMM, Qin and Lawless (1994) used the EL to make inferences for parameters defined by a general estimating equation. For discretized $F(x)$ satisfying (2), equation (5) becomes

$$\sum_{i=1}^{n} p_i g(X_i, \theta) = 0.$$  

Maximizing the log-likelihood (1) subject to (2) and (6), we have the profile log-likelihood of $\theta$ (up to a constant),

$$\ell_n(\theta) = -\sum_{i=1}^{n} \log\{1 + \lambda^\top g(X_i, \theta)\},$$

where $\lambda$ is the Lagrange multiplier determined by $\sum_{i=1}^{n} g(X_i, \theta)/\{1 + \lambda^\top g(X_i, \theta)\} = 0$. We then estimate $\theta$ by the maximizer $\hat{\theta} = \arg\max_\theta \ell_n(\theta)$, whose limiting distribution is established in the following theorem. Hereafter we use $\nabla_\theta$ to denote the differentiation operator with respect to $\theta$. 

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Theorem 1 (Qin and Lawless (1994)) Denote \( g = g(X, \theta_0) \) and \( \nabla_{\theta^\top} g = \nabla_{\theta^\top} g(X, \theta_0) \). Suppose that (1) \( \mathbb{E}(gg^\top) \) is positive definite, (2) \( \nabla_{\theta^\top} g(X, \theta) \) is continuous in a neighbourhood of \( \theta_0 \), (3) \( \| \nabla_{\theta^\top} g(X, \theta) \| \) and \( \| g(X, \theta) \|^3 \) can be bounded by some integrable function \( G(X) \) in this neighbourhood, and (4) \( \mathbb{E}(\nabla_{\theta^\top} g) \) is of full rank. Then as \( n \to \infty \), \( \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, V) \), where \( \xrightarrow{d} \) stands for “convergence in distribution” and

\[
V = \left\{ \mathbb{E} \left( \nabla_{\theta^\top} g \right) \mathbb{E}(gg^\top)^{-1}(\mathbb{E}(\nabla_{\theta^\top} g))^{-1} \right\}.
\]

3.3 Information bound calculation

How good can we estimate \( \theta \) based on the over-identified parameter model \( \mathbb{E}\{g(X, \theta)\} = 0 \)? Is the maximum EL estimator optimal? To answer these questions, we consider an ideal situation: suppose the true underlying density \( f(x, \theta) \) is known. We can construct an enlarged parametric density model

\[
h(x, \eta, \theta) = \frac{\exp\{\eta^\top g(x, \theta)\} f(x, \theta)}{\int \exp\{\eta^\top g(t, \theta)\} f(t, \theta) dt},
\]

where we have implicitly assumed \( \int \exp\{\eta^\top g(t, \theta)\} f(t, \theta) dt < \infty \). Clearly \( h(x, 0, \theta) = f(x, \theta) \).

In other words, \( E_{\theta, \eta}\{g(X, \theta)\} = 0 \) if \( (\theta, \eta) = (\theta_0, 0) \), where \( \theta_0 \) is the true value of \( \theta \). We shall show that even if the form of \( f(x, \theta) \) is available, the MLE of \( \theta \) based on \( h(x, \eta, \theta) \) has the same asymptotic variance as the maximum EL estimator.

With the parametric model \( h \), we can estimate \( \theta \) by maximizing \( L(\theta, \eta) = \prod_{i=1}^n h(X_i, \eta, \theta) \) with respect to \( (\eta, \theta) \). Denote the resulting MLE by \( (\tilde{\eta}, \tilde{\theta}) \). We show in Section 3.4 that as \( n \to \infty \),

\[
\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N(0, V),
\]

where \( V \) is defined in (7). In general \( f(x, \theta) \) is unknown, hence we expect that the best estimator of \( \theta \) should have an asymptotic variance at least as large as \( V \). Because the maximum EL estimator of \( \theta \) of Qin and Lawless (1994) has the asymptotic variance \( V \), we conclude it achieves the lower information bound.
Remark 1  If \( g(x, \theta) \) is unbounded, we may construct a new density
\[
h(x, \theta, \eta) = \frac{\psi\{\eta^\top g(x, \theta)\} f(x, \theta)}{\int \psi\{\eta^\top g(x, \theta)\} f(x, \theta) dx},
\]
where \( \psi(x) = 2(1 + e^{-2x})^{-1} \) with \( \psi(0) = \psi''(0) = 1 \). We may go through the same exercise to get the same conclusion.

Remark 2  Back and Brown (1992) established a similar result by constructing an exponential family. In particular, they defined
\[
h(x, \theta) = \exp \{\xi^\top (\theta) g(x, \theta_0) - a(\theta)\} f_0(x),
\]
where \( f_0(x) = f(x, \theta_0) \) and \( \xi(\theta) \) is determined implicitly by the following conditions:
\[
\xi(\theta_0) = 0, \quad a(\theta_0) = 0, \quad \int \exp \{\xi^\top (\theta) g(x, \theta_0) - a(\theta)\} f_0(x) = 1,
\]
and
\[
\int g(x, \theta) \exp \{\xi^\top (\theta) g(x, \theta_0) - a(\theta)\} f_0(x) dx = 0.
\]
In Back and Brown (1992) approach, \( \xi(\theta) \) is determined implicitly by above constraint equation, while in our new approach \( \eta \) is an independent parameter.

3.4 A sketchy proof of (8)

The log-likelihood based on the enlarged model \( h(x, \eta, \theta) \) is
\[
\ell = \sum_{i=1}^{n} \{\eta^\top g(X_i, \theta) + \log f(X_i, \theta)\} - n \log \left[ \int \exp\{\eta^\top g(x, \theta)\} f(x, \theta) dx \right].
\]
The score functions evaluated at \( (\theta, \eta) = (\theta_0, 0) \) is
\[
\nabla_\eta \ell(\theta_0, 0) = \sum_{i=1}^{n} g(X_i, \theta_0), \quad \nabla_\theta \ell(\theta_0, 0) = 0, \quad \nabla_{\eta\eta} \ell(\theta_0, 0) = -n \mathbb{E}(gg^\top),
\]
and
\[
\nabla_{\eta\theta} \ell(\theta_0, 0) = \sum_{i=1}^{n} \nabla_{\theta} g(X_i, \theta_0) - n \mathbb{E}\{\nabla_{\theta} g(X, \theta_0) + g(X, \theta_0)(\nabla_{\theta} \log f(X, \theta))\}. 
\]
Under some mild assumptions such as that \( \int g(x, \theta) f(x, \theta) dx = 0 \) holds for \( \theta \) in a neighborhood of \( \theta_0 \), differentiating both of its sides with respect to \( \theta \) leads to

\[
\mathbb{E}\{\nabla_{\theta} g(X, \theta)\} + \mathbb{E}\{\nabla_{\theta} g(X, \theta) \nabla_{\theta} \log f(X, \theta)\} = 0,
\]

which means \( \nabla_{\eta_{\theta}} \ell(\theta_0, 0) = \sum_{i=1}^{n} \nabla_{\theta} g(X_i, \theta_0) \). Meanwhile if \( f(x, \theta) \) satisfies some regularity conditions, then

\[
\mathbb{E}[\nabla_{\theta} \log f(x, \theta_0) + \{\nabla_{\theta} \log f(x, \theta_0)\} \{\nabla_{\theta} \log f(x, \theta_0)\}^\top] = 0.
\]

Therefore

\[
\sqrt{n} \begin{pmatrix} \tilde{\eta} - \eta_0 \\ \tilde{\theta} - \theta_0 \end{pmatrix} = \begin{pmatrix} -\mathbb{E}(gg^\top) & \mathbb{E}(\nabla_{\theta} g) \\ \mathbb{E}(\nabla_{\theta} g^\top) & 0 \end{pmatrix}^{-1} \begin{pmatrix} n^{-1/2} \sum_{i=1}^{n} g(X_i, \theta_0) \\ 0 \end{pmatrix} + o_p(1).
\]

This together with the fact that \( n^{-1/2} \sum_{i=1}^{n} g(X_i, \theta_0) \xrightarrow{d} N(0, \mathbb{E}(g g^\top)) \) as \( n \) goes to infinity implies (8).

### 3.5 Entropy family

The enlarged parametric model satisfies

\[
\int h(x, \eta, \theta) g(x, \theta) dx = 0,
\]

only if \( \eta = 0 \). Naturally one may require \( \eta = \eta(\theta) \) to satisfy

\[
\int g(x, \theta) \exp\{\eta^\top g(x, \theta)\} f(x, \theta) dx = 0.
\]

In the construction of the enlarged parametric model \( h(x, \eta, \theta) \), it is often too restrictive to assume a known underlying parametric model \( f(x, \theta) \). We may replace the cumulative distribution function \( F(x, \theta) = \int_{-\infty}^{x} f(t, \theta) dt \) by the empirical distribution \( F_n(x) = n^{-1} \sum_{i=1}^{n} I(X_i \leq x) \). In this situation, \( \eta = \eta(\theta) \) is the solution to \( \sum_{i=1}^{n} g(x_i, \theta) \exp\{\eta^\top g(x_i, \theta)\} = 0 \)

Let \( H(x, \eta, \theta) = \int_{-\infty}^{x} h(t, \eta, \theta) dt \). For fixed parameter value \( (\eta, \theta) \), we have

\[
dH(X_i, \eta, \theta) = \exp\{\eta^\top g(X_i, \theta)\} / \left[ \sum_{j=1}^{n} \exp\{\eta^\top g(X_j, \theta)\} \right],
\]
and the likelihood becomes
\[
\prod_{i=1}^{n} dH(X_i, \eta, \theta) = \prod_{i=1}^{n} \frac{\exp\{\eta^\top(\theta) g(X_i, \theta)\}}{\sum_{j=1}^{n} \exp\{\eta^\top(\theta) g(X_j, \theta)\}}.
\]
In fact this is equivalent to the EL \( \prod_{i=1}^{n} p_i \), where \( p_i \)'s minimize the Kullback-Leibler divergence (up to a constant) or minus the exponential titling likelihood \( \sum_{i=1}^{n} p_i \log(p_i) \) subject to the constraint \( \sum_{i=1}^{n} p_i = 1, p_i \geq 0, \) and \( \sum_{i=1}^{n} p_i g(X_i, \theta) = 0 \). See Susanne (2007) for more details.

4 Enhance efficiency with auxiliary information

In this section, we discuss methods of incorporating auxiliary information to enhance estimation efficiency, which were also investigated by Qin (2000). We assume a parametric model \( f(y|x, \beta) \) for the conditional density function of \( Y \) given \( X \), and leave the marginal distribution \( G(x) \) of \( X \) un-specified. We wish to make inferences for \( \beta \) when some auxiliary information is summarized through an estimating equation
\[
\mathbb{E}\{\phi(X, \beta)\} = 0.
\]
For example, if we know the mean \( \mu \) of \( Y \), then we can construct an estimating equation
\[
\mathbb{E}(Y - \mu) = 0.
\]
We can take
\[
\phi(X, \beta) = \int (y - \mu) f(y|X, \beta) dy = \int y f(y|X, \beta) dy - \mu.
\]
Furthermore, we assume that the response \( Y \) may have missing values. Let \( D \) be the non-missingness indicator, being 1 if \( Y \) is available, and 0 otherwise. We assume a missing at random model
\[
P(D = 1|Y = y, X = x) = P(D = 1|X = x) = \pi(x),
\]
where \( \pi(x) \) depends only on \( x \). Denote the observed data by \((d_i, d_i y_i, x_i)\ (i = 1, 2, \ldots, n)\)
and \( p_i = dG(x_i) \). The likelihood is

\[
L = \prod_{i=1}^{n} \{ \pi(x_i) f(y_i|x_i, \beta) dG(x_i) \}^{d_i} \{ 1 - \pi(x_i) \} dG(x_i)^{1-d_i},
\]

\[
= \prod_{j=1}^{n} \{ \pi(x_j) \}^{d_j} \{ 1 - \pi(x_j) \}^{1-d_j} \cdot \prod_{i=1}^{n} \{ f(y_i|x_i, \beta) \}^{d_i} \cdot p_i.
\]

We can maximize this likelihood subject to the constraints

\[
\sum_{i=1}^{n} p_i = 1, \quad p_i \geq 0, \quad \sum_{i=1}^{n} p_i \phi(x_i, \beta) = 0.
\]

Since \( \prod_{j=1}^{n} \{ \pi(x_j) \}^{d_j} \{ 1 - \pi(x_j) \}^{1-d_j} \) is independent of \( \beta \), the profile hybrid empirical log-likelihood (up to a constant) is

\[
\ell(\beta) = \sum_{i=1}^{n} \left[ d_i \log f(y_i|x_i, \beta) - \log \left( 1 + \lambda^\top \phi(x_i, \beta) \right) \right], \quad (10)
\]

where \( \lambda \) is the Lagrange multiplier determined by

\[
\sum_{i=1}^{n} \frac{\phi(x_i, \beta)}{1 + \lambda^\top \phi(x_i, \beta)} = 0. \quad (11)
\]

In the special case that missing data is completely at random, i.e., \( \pi(x_i) \) is a constant, Qin (1992, 2000) established the following theorem.

**Theorem 2** Let \( \beta_0 \) be the true parameter value, \( \hat{\beta} \) be the maximum hybrid EL estimator, i.e., the maximizer of \( (10) \), and \( \hat{\lambda} \) be the corresponding Lagrange multiplier. Denote \( \phi = \phi(X, \beta_0), \nabla_\beta \phi = \nabla_\beta \phi(X, \beta_0) \), and

\[
J = -E \left\{ d_i \nabla_{\beta^\top} \log f(y_i|x_i, \beta_0) \right\} = \text{Var} \left\{ d_i \nabla_{\beta} \log f(y_i|x_i, \beta_0) \right\}.
\]

Under some regularity conditions, when \( n \) goes to infinity, we have

\[
\sqrt{n}((\hat{\beta} - \beta_0)^\top, \hat{\lambda}^\top) \overset{d}{\rightarrow} N(0, \Sigma),
\]

where \( \Sigma = \text{diag}(\Sigma_{11}, \Sigma_{22}) \) with

\[
\Sigma_{11} = \left\{ J + E(\nabla_\beta \phi^\top)(E\phi \phi^\top)^{-1}E(\nabla_\beta \phi) \right\}^{-1}, \quad (12)
\]

\[
\Sigma_{22} = \left\{ E(\nabla_\beta \phi^\top)J^{-1}E(\nabla_\beta \phi) + E(\phi \phi^\top) \right\}^{-1}.
\]
Remark 3 Imbens and Lancaster (1994) studied the same problem using GMM. In particular, they directly combined the conditional score estimating equation $\nabla_\beta \log f(y|x, \beta)$ and $\phi(x, \beta)$. Even though the first order large sample results are the same, the hybrid EL based approach is more appealing since it respects the parametric conditional likelihood and replaces only the marginal likelihood by the EL. Numerical comparison of the two methods was given in Qin (2000).

5 Combining summary information: A more flexible method in meta analysis

Meta analysis is a systematic way to combine published information. The method has become very popular since little extra cost is needed. The main restriction in meta analysis is that all studies must include the same variables in the analyses. The only allowed difference is the sample sizes. We have to discard some studies if they contain variables different from others. Summarized information is available from published results, such as census reports, national health studies, and so on.

Due to confidentiality or other reasons, we typically cannot gain access to the original data except for the summarized reports. Suppose we are interested in conducting a new study that may contain some new variables of interest, which are not available from the summarized information, for example, in genetic studies, some new bio-markers and genes are newly discovered. Below we discuss a more flexible method to combine published information and individual study data for enhanced inference. Chatterjee et al. (2016) discussed a related problem on the utilization of auxiliary information. As Han and Lawless (2016) pointed out, however, their methodology and theoretical results were already developed by Imbens and Lancaster (1994) and Qin (2000) in the absence of selection bias sampling case.

We consider two cases. (I) Sample size for the summarized information is much larger than the sample size in the new study. (II) Sample sizes from the two data sources are comparable. In Case I, we can treat the summarized information as known, i.e., the variation
in the summarized data is negligible compared to the variation in the new study. In Case II, we have to take the variation in the summarized information into consideration since it is comparable to the variation in the new study. We focus on Case I in this section and study Case II in Section 6.

5.1 Setup and solution

Suppose the summarized results are based on the statistical analysis from response $Y$ and covariate variables $X$ (though the original data are not available), and in the new study, in addition to $Y, X$, an extra covariate $Z$ is included. We are interested in fitting a parametric model $f(y|x, z, \beta)$ for the conditional density function of $Y$ given $X$ and $Z$. Let $(y_i^*, x_i^*)$, $,...,(y_N^*, x_N^*)$ be history data even though they are not available. The published information can be summarized in two ways:

(I) $\bar{h} = N^{-1} \sum_{i=1}^{N} h(y_i^*, x_i^*)$ is known, and

(II) $\gamma^*$ is the solution of an estimating equation $\sum_{i=1}^{N} h(y_i^*, x_i^*, \gamma) = 0$, where the function $h(y, x, \gamma)$ is known up to $\gamma$.

Let $(y_1, x_1, z_1), ..., (y_n, x_n, z_n)$ be observed data in the new study. The basic assumption is that $(y_i, x_i), i = 1, 2, ..., n$ and $(y_i^*, x_i^*)$ have the same distribution. To utilize the summarized information, we can define estimating functions

$$g = (g_1, g_3), \quad g_1(y, x, z) = \nabla_\beta \log f(y|x, z, \beta), \quad g_3(y, x) = h(y, x) - \bar{h},$$

in Scenario (I), and

$$g = (g_1, g_3), \quad g_1(y, x, z) = \nabla_\beta \log f(y|x, z, \beta), \quad g_3(y, x) = h(y, x, \gamma^*)$$

in Scenario (II). We consider only the situation that $n/N \rightarrow 0$. In other words, the variation in the auxiliary information is negligible.

The empirical likelihood approach amounts to maximizing $\sum_{i=1}^{n} \log p_i$ subject to the constraint

$$\sum_{i=1}^{n} p_i g(y_i, x_i, z_i, \beta) = 0, \quad p_i \geq 0, \quad \sum_{i=1}^{n} p_i = 1.$$
According to Qin and Lawless (1994), the asymptotic variance of the maximum EL estimator $\hat{\beta}$ based on estimating equations $g$ is
\[
[\mathbb{E}(\nabla_{\beta} g^\top)\{\mathbb{E}(gg^\top)\}^{-1}\mathbb{E}(\nabla_{\beta^\top} g)]^{-1},
\]
where $\nabla_{\beta} g = \partial g(y, x, z, \beta)/\partial \beta|_{\beta = \beta_0}$, $g = g(y, x, z, \beta_0)$, and $\beta_0$ is the truth of $\beta$. Denote
\[
A = \mathbb{E}(gg^\top) = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{pmatrix}, \quad A_{22,1} = A_{22} - A_{12}^\top A_{11}^{-1} A_{12}.
\]
Equivalently the asymptotic variance can be written as
\[
[\mathbb{E}(\nabla_{\beta} g_1^\top)A_{11}^{-1}\mathbb{E}(\nabla_{\beta^\top} g_1) + \mathbb{E}(\nabla_{\beta} g_1^\top)A_{11}^{-1}A_{12}A_{22,1}A_{21}A_{11}^{-1}\mathbb{E}(\nabla_{\beta^\top} g_1)]^{-1},
\]
or $(J + A_{12}A_{22,1}^{-1}A_{21})^{-1}$, where $A_{11} = J$ is the Fisher’s information matrix.

In the above approach the estimating equation $g_3 = h(y, x) - \bar{h}$ does not involve the parameter $\beta$. This method may not be efficient. As an alternative approach, we define $g_2(x, z, \beta) = \psi(x, z, \beta)$ with
\[
\psi(x, z, \beta) = \mathbb{E}\{h(Y, X)|X = x, Z = z\} - \bar{h} = \int h(y, x)f(y|x, z, \beta)dy - \bar{h}.
\]
Then $\mathbb{E}\{g_2(x, z, \beta)\} = 0$. If we combine the empirical log-likelihood based on the estimating equation $g_2$ and the log-likelihood $\sum_{i=1}^n \log f(y_i|x_i, z_i, \beta)$ as in last section (See Equation (12)), then the asymptotic variance of the resulting MLE $\hat{\beta}$ is given by
\[
\{J + \mathbb{E}(\nabla_{\beta}\psi^\top)(\mathbb{E}\psi\psi^\top)^{-1}\mathbb{E}(\nabla_{\beta^\top}\psi)\}^{-1}.
\]

## 5.2 A comparison

Given the two pairs of estimation functions, $\{g_1, g_3\}$ and $\{g_1, g_2\}$, we may wonder combining which pair leads to a better estimator if we directly compare their asymptotic variance formulae. Alternatively, we may enquire whether we should combine all three constraints
$g = (g_1, g_2, g_3)$ together. Write $g_{12} = (g_1, g_2)$, $a = \mathbb{E}\{h^\top(y, x) \nabla_\beta \log f(y|x, z, \beta)\}$, and

$$
\mathbb{E}(gg^\top) = \begin{pmatrix}
J & 0 & a \\
0 & \mathbb{E}(\psi_\beta \psi^\top) & \mathbb{E}(\psi_\beta)^2 \\
a^\top & \mathbb{E}(\psi_\beta \psi^\top) & \mathbb{E}(hh^\top)
\end{pmatrix} = \begin{pmatrix}
B_{11} & B_{12} \\
B_{12}^\top & B_{22}
\end{pmatrix},
$$

$$
B_{11} = \begin{pmatrix}
J & 0 \\
0 & \mathbb{E}(\psi_\beta)^2
\end{pmatrix}, \quad B_{12} = \begin{pmatrix}
a \\
\mathbb{E}(\psi_\beta)^2
\end{pmatrix}.
$$

Using the results in Qin and Lawless (1994) and

$$
\begin{pmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{pmatrix}^{-1} = \begin{pmatrix}
I & -B_{11}^{-1}B_{12} \\
0 & I
\end{pmatrix} \begin{pmatrix}
B_{11}^{-1} & 0 \\
0 & B_{22,1}^{-1}
\end{pmatrix} \begin{pmatrix}
I & 0 \\
-B_{21}B_{11}^{-1} & I
\end{pmatrix}
$$

with $B_{22,1} = B_{11} - B_{12}B_{11}^{-1}B_{12}$, we find that the asymptotic variance of $\hat{\beta}$ by combining the three estimating equations and $\sum_{i=1}^n \log f(y_i|x_i, z_i, \beta)$ is

$$
[J + \mathbb{E}(\nabla_\beta \psi^\top)\mathbb{E}(\psi_\beta)^{-1}\mathbb{E}(\nabla_\beta \psi) + \mathbb{E}(\nabla_\beta g_{12})B_{11}^{-1}B_{12}B_{22,1}^{-1}B_{21}B_{11}^{-1}\mathbb{E}(\nabla_\beta g_{12})]^{-1}.
$$

It can be shown that

$$
\mathbb{E}(\nabla_\beta g) = (-J, \mathbb{E}(\nabla_\beta \psi), 0), \quad \mathbb{E}(\nabla_\beta g_{12}) = (-J, a).
$$

Immediately, we have

$$
\mathbb{E}(\nabla_\beta g_{12})B_{11}^{-1}B_{12} = (-J, a) \begin{pmatrix}
J^{-1} & 0 \\
0 & \{\mathbb{E}(\psi_\beta)^{-1}\}
\end{pmatrix} \begin{pmatrix}
a \\
\mathbb{E}(\psi_\beta)^{-1}
\end{pmatrix} = 0,
$$

which implies that the asymptotic variance in the case of combining $g_1, g_2$, and $g_3$ is the same as that in the case of combining only $g_1$ and $g_2$. This indicates that taking $g_3$ into account leads to no efficiency gain in the estimation of $\beta$.

The method of combining $g_2$ and the parametric likelihood $\prod_{i=1}^n f(y_i|x_i, z_i, \beta)$ is better than that of combining $g_1, g_3$ and the parametric likelihood. To see this, recall that the asymptotical variances for the MLEs of $\beta$ of the two methods are

$$
V_1 = \{J + \mathbb{E}(\nabla_\beta \psi^\top)(\mathbb{E}(\psi_\beta)^{-1}\mathbb{E}(\nabla_\beta \psi))^{-1}\}^{-1}.
$$
and
\[ V_2 = (J + A_{12}A_{22,1}^{-1}A_{21})^{-1}. \]

It suffices to show that \( V_2 - V_1 \geq 0 \), namely \( V_2 - V_1 \) is non-negative definite.

### 5.3 Proof of \( V_2 - V_1 \geq 0 \)

For convenience, we assume that \( \mathbb{E}(h) = 0 \). Because \( \mathbb{E}(\nabla_\beta \psi^\top) = A_{12} \) and \( \psi = \mathbb{E}(h|X,Z) \), it suffices to show
\[
A_{22,1} - \mathbb{E}(\psi\psi^\top) = (A_{22} - A_{21}A_{11,1}^{-1}A_{12}) - \mathbb{E}[\{\mathbb{E}(h|X,Z)\}^\otimes 2] \geq 0. \tag{13}
\]

Let \( \mathbb{E}_* \) and \( \text{Var}_* \) denote \( \mathbb{E}(|X,Z) \) and \( \text{Var}(|X,Z) \), respectively. Because
\[
\begin{pmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{pmatrix}
= \mathbb{E}\left\{ \begin{pmatrix}
    g_1 \\
    h
\end{pmatrix}^\otimes 2 \right\} = \mathbb{E}\left\{ \text{Var}_*\left( \begin{pmatrix}
    g_1 \\
    h
\end{pmatrix} \right) \right\} + \text{Var}\left\{ \mathbb{E}_*\left( \begin{pmatrix}
    g_1 \\
    h
\end{pmatrix} \right) \right\}
\]
and \( \mathbb{E}_*(g_1) = 0 \), it follows that
\[
\begin{pmatrix}
    A_{11} & A_{12} \\
    A_{21} & A_{22}
\end{pmatrix}
\geq \text{Var}\left\{ \mathbb{E}_*\left( \begin{pmatrix}
    g_1 \\
    h
\end{pmatrix} \right) \right\} = \mathbb{E}\left( \begin{pmatrix}
    0 & 0 \\
    0 & \mathbb{E}_*(h)\mathbb{E}_*(h^\top)
\end{pmatrix} \right).
\]

Multiplying both sides by \((-A_{21}A_{11,1}^{-1}, I)^\top\) from the left and by \((-A_{21}A_{11,1}^{-1}, I)^\top\) from the right, we arrive at
\[
A_{22} - A_{21}A_{11,1}^{-1}A_{12} \geq \mathbb{E}\{\mathbb{E}_*(h)\mathbb{E}_*(h^\top)\},
\]
namely inequality (13) holds, which implies \( V_2 - V_1 \geq 0 \).

### 6 Calibrate information from previous studies

We consider calibrating information with parametric likelihood, EL (Owen, 2001), and GMM (Hansen, 1982). When only summary information from previous studies is available, these three well-known methods can be used to calibrate such summary information and to make
inference about the unknown parameters of interest. We may wonder whether there is efficiency loss in doing so compared with the inferences based on the pooled data as if they were all available. Lin and Zeng (2014) found that parametric-likelihood-based meta analysis of summarized information does not lose information compared with the analysis based on individual data. This is extremely important since individual data may involve privacy issues, whereas summarized information does not. We disclose that not only parametric likelihood, but also EL and GMM own this nice property.

6.1 Efficiency comparison

Suppose that the data \((Y_{ij}, X_{ij}) (j = 1, 2, \ldots, n_i; i = 1, 2, \ldots, K)\) are iid and satisfy one of the following assumptions:

\begin{enumerate}[(I)]
\item \(\text{pr}(Y_{ij} = y|X_{ij} = x) = f(y|x, \beta),\) or
\item \(\mathbb{E}\{g(Y, X, \beta)\} = 0\) with \(\beta_*\) being the true value of \(\beta.\)
\end{enumerate}

Assume that data are available batch by batch, and that \(n_i/n = \rho_i \in (0, 1)\) where \(n = \sum^K_{i=1} n_i.\)

For the \(r\)-th batch of data,

(a) under assumption (I), we define a parametric log-likelihood function

\[
\ell_{r,\text{PL}}(\beta) = \sum_{i=1}^{n_r} \log \{f(Y_{ri}, X_{ri}, \beta)\};
\]

(b) under assumption (II), we define an empirical log-likelihood function

\[
\ell_{r,\text{EL}}(\beta) = \sup \left\{ \sum_{i=1}^{n_r} \log (n_r p_i) : p_i \geq 0, \sum_{i=1}^{n_r} p_i = 1, \sum_{i=1}^{n_r} p_i g(Y_{ri}, X_{ri}; \beta) = 0 \right\}
\]

\[
= -\sum_{i=1}^{n_r} \log \{1 + \lambda_r^\top g(Y_{ri}, X_{ri}; \beta)\},
\]

where \(\lambda_r\) satisfies \(\sum_{i=1}^{n_r} \frac{g(Y_{ri}, X_{ri}; \beta)}{1 + \lambda_r^\top g(Y_{ri}, X_{ri}; \beta)} = 0;\)
(c) under assumption (II), we define the objective function of the GMM method (GMM
log-likelihood for short) as

\[ \ell_{r,GMM}(\beta) = - \left( \sum_{i=1}^{n_r} g(Y_{ri}, X_{ri}; \beta) \right) \top \Omega^{-1} \left( \sum_{i=1}^{n_r} g(Y_{ri}, X_{ri}; \theta) \right), \]

where \( \Omega = \text{Var}\{g(Y, X, \beta_*)) \}. In practice, the \( \beta_* \) in the expression of \( \Omega \) is generally
replaced by a consistent estimator of \( \beta \). Using the truth \( \beta_* \) of \( \beta \) does not affect the
theoretical analysis in this section.

Let \( \ell_r(\beta) = \ell_{r,PL}(\beta), \ell_{r,EL}(\beta) \) or \( \ell_{r,GMM}(\beta) \). Under certain regularity conditions, it can
be verified that for \( \beta = \beta_* + O_p(n^{-1/2}), \)

\[ \ell_r(\beta) = U_r \sqrt{n_r} (\beta - \beta_*) - \frac{n_r}{2} (\beta - \beta_*) \top V(\beta - \beta_*) + o_p(1). \] \( (14) \)

In Case (a),

\[ U_r = n_r^{-\frac{1}{2}} \sum_{i=1}^{n_r} \nabla_{\beta} \log \{ f(Y_{ri}|X_{ri}, \beta_* \}, \quad V = \text{Var}[\nabla_{\beta} \log \{ f(Y|X, \beta_* \}]. \]

In Case (b)

\[ U_r = n_r^{-\frac{1}{2}} \sum_{i=1}^{n_r} g(Y_{ri}, X_{ri}; \beta_*), \quad V = A_{12}A_{22}^{-1}A_{21}, \]

where

\[ A = \begin{pmatrix} 0 & \mathbb{E}\{\nabla_{\beta} g(Y, X; \beta_*)) \} \\ \mathbb{E}\{\nabla_{\beta} g(Y, X; \beta_*)) \} & \mathbb{E}\{g(Y, X; \beta_*))g(Y, X; \beta_*)) \} \end{pmatrix} \equiv \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \]

In Case (c),

\[ U_r = -\{\mathbb{E}\nabla_{\theta} g(Y, X; \beta_*)) \} \Omega^{-1} n_r^{-\frac{1}{2}} \sum_{i=1}^{n_r} g(Y_{ri}, X_{ri}; \theta_*), \quad V = \{\mathbb{E}\nabla_{\theta} g(Y, X; \beta_*)) \} \Omega^{-1}\{\mathbb{E}\nabla_{\theta} g(Y, X; \beta_*)) \} \]

Denote the MLE of \( \beta \) based on the \( r \)-th batch of data by \( \hat{\beta}_r = \arg \max \ell_r(\beta) \). The above
approximation implies that

\[ \sqrt{n_r}(\hat{\beta}_r - \beta_*) = V^{-1}U_r + o_p(1) \overset{d}{\rightarrow} N(\beta_*; V^{-1}). \]
When the $K$-th batch of individual data are available, we are not accessible to the individual data of the previous $K-1$ batches any longer, but only have summarized information $(\hat{\beta}_j, \hat{\Sigma}_j), j = 1, 2, \ldots, K - 1$, where $\hat{\beta}_j$ is the MLE based on the $j$th batch of data and $\hat{\Sigma}_j = V^{-1}/n_j + o(n^{-1})$. We can define an augmented log-likelihood

$$\ell_A(\beta) = \ell_K(\beta) - \frac{1}{2} \sum_{j=1}^{K-1} (\hat{\beta}_j - \beta)^\top \hat{\Sigma}_j^{-1}(\hat{\beta}_j - \beta)$$

and the corresponding MLE $\hat{\beta}_A = \arg \max \ell_A(\beta)$. For $\beta = \beta_* + O_p(n^{-1/2})$, using the approximation in (14), we have

$$\ell_A(\beta) = U_K^\top \sqrt{n_K} (\beta - \beta_*) - \frac{nK}{2} (\beta - \beta_*) V(\beta - \beta_*) - \sum_{j=1}^{K-1} n_j (\beta - \beta_*)^\top V(\beta - \beta_*) + C + o_p(1)$$

$$= n^{-1/2} \sum_{j=1}^{K} \sqrt{n_j} U_j^\top \sqrt{n} (\beta - \beta_*) - \frac{n}{2} (\beta - \beta_*)^\top V(\beta - \beta_*) + C + o_p(1),$$

where the constant $C$ is different from place to place.

For comparison, based on the pooled data, we define in Case (a) the parametric log-likelihood as

$$\ell_{PL}(\beta) = \sum_{r=1}^{K} \sum_{i=1}^{n_r} \log \{ f(Y_{ri}|X_{ri}, \beta) \},$$

define in Case (b) the empirical log-likelihood function as

$$\ell_{EL}(\beta) = \sup \{ \sum_{r=1}^{K} \sum_{i=1}^{n_r} \log (np_{ri}) : p_{ri} \geq 0, \sum_{r=1}^{K} \sum_{i=1}^{n_r} p_{ri} = 1, \sum_{r=1}^{K} \sum_{i=1}^{n_r} p_{ri} g(Y_{ri}, X_{ri}; \beta) = 0 \}$$

$$= - \sum_{r=1}^{K} \sum_{i=1}^{n_r} \log \{ 1 + \lambda^\top g(Y_{ri}, X_{ri}; \beta) \},$$

where $\lambda$ satisfies $\sum_{r=1}^{K} \sum_{i=1}^{n_r} \frac{g(Y_{ri}, X_{ri}; \beta)}{1 + \lambda^\top g(Y_{ri}, X_{ri}; \beta)} = 0$, and in Case (c), define the GMM log-likelihood as

$$\ell_{GMM}(\beta) = - \left\{ \sum_{r=1}^{K} \sum_{i=1}^{n_r} g(Y_{ri}, X_{ri}; \beta) \right\}^\top \Omega^{-1} \left\{ \sum_{r=1}^{K} \sum_{i=1}^{n_r} g(Y_{ri}, X_{ri}; \theta) \right\}. $$
Let the log-likelihood based on the pooled data be $\ell_{\text{pool}}(\beta) = \ell_{\text{PL}}(\beta)$, $\ell_{\text{EL}}(\beta)$, and $\ell_{\text{GMM}}(\beta)$ in Cases (a), (b), and (c), respectively. It can be found that

$$\ell_{\text{pooled}}(\beta) = n^{-1/2} \sum_{j=1}^{K} \sqrt{n}U_j^T \cdot \sqrt{n}(\beta - \beta^\ast) - \frac{n}{2}(\beta - \beta^\ast)^T V(\beta - \beta^\ast) + C + o_p(1),$$

where $C'$ is a constant different from $C$. Let $\hat{\beta}_{\text{pooled}} = \arg\max \ell_{\text{pooled}}(\beta)$. By comparing $\ell_{\text{pooled}}(\beta)$ and $\ell_A(\beta)$, we arrive at

$$\ell_{\text{pooled}}(\beta) = \ell_A(\beta) + C + o_p(1),$$

and

$$\sqrt{n}(\hat{\beta}_A - \beta^\ast) = \sqrt{n}(\hat{\beta}_{\text{pooled}} - \beta^\ast) + o_p(1)$$

$$= V^{-1} \cdot n^{-1/2} \sum_{j=1}^{K} \sqrt{n}U_j^T + o_p(1)$$

$$\xrightarrow{d} N(0, V^{-1}).$$

This indicates that compared with the methods, including parametric likelihood, EL, and GMM, based on all individual data, the calibration method based on the last batch of individual data and all summary results of the previous batches has no efficiency loss.

6.2 When nuisance parameters are present

If for batch $i$, we assume that the data $(Y_{ij}, X_{ij})$ ($j = 1, 2, \ldots, n_i$) satisfy either

$$\text{pr}(Y_{ij} = y | X_{ij} = x) = f(y | x, \beta, \gamma_i)$$

or

$$\mathbb{E}\{g(Y, X, \beta, \gamma_i)\} = 0,$$

where $\beta$ is common but $\gamma_i$ is a batch-specific parameter. We define $\ell_r(\beta, \gamma_i)$ in the same way as $\ell_r(\beta)$. Let $(\hat{\beta}_i, \hat{\gamma}_i)$ be the MLE of $(\beta, \gamma_i)$ based on the $i$-th batch of data, and assume that approximately

$$((\hat{\beta}_i - \beta)^T, (\hat{\gamma}_i - \gamma_i)^T)^T \sim N(0, \hat{\Sigma}_i)$$
with $\hat{\Sigma}_i = (\hat{\Sigma}_{i,rs})_{1 \leq r,s \leq 2}$.

We have two ways of combining information from previous studies. If we use all the previous summary information, we can define

$$\ell_A^{(1)}(\beta, \gamma_1, \ldots, \gamma_K) = \ell_K(\beta, \gamma_i) - \frac{1}{2} \sum_{i=1}^{K-1} ((\hat{\beta}_i - \beta)^\top, (\hat{\gamma}_i - \gamma_i)^\top) \hat{\Sigma}_i^{-1} ((\hat{\beta}_i - \beta)^\top, (\hat{\gamma}_i - \gamma_i)^\top)^\top.$$  

As $\hat{\beta}_i | \hat{\gamma}_i \sim N(\beta, \hat{\Sigma}_{i,11})$, where $\hat{\Sigma}_{i,11} = \hat{\Sigma}_{i,12} \hat{\Sigma}_{i,22}^{-1} \hat{\Sigma}_{i,21}$, if using only this summary information, we can define

$$\ell_A^{(2)}(\beta, \gamma_K) = \ell_K(\beta, \gamma_K) - \frac{1}{2} \sum_{i=1}^{K-1} (\hat{\beta}_i - \beta)^\top \hat{\Sigma}_{i,11}^{-1} (\hat{\beta}_i - \beta).$$

Below we show that the MLEs of $\beta$ based on these two likelihoods are actually equal to each other. In other words, there is no efficiency loss of estimating $\beta$ based on $\ell_A^{(2)}(\beta, \gamma_K)$ instead of $\ell_A^{(1)}(\beta, \gamma_1, \ldots, \gamma_K)$.

To see this, it suffices to show

$$\sup_{\gamma_1, \ldots, \gamma_{K-1}} \ell_A^{(1)}(\beta, \gamma_1, \ldots, \gamma_K) = \ell_A^{(2)}(\beta, \gamma_K). \quad (15)$$

Denote the inverse matrix of $\Sigma_i$ by $\Sigma_i^{-1} = (\Sigma_i^{rs})_{1 \leq r,s \leq 2}$, where

$$\Sigma_i^{11} = \Sigma_{i,11}^{-1}, \quad \Sigma_i^{21} = -\Sigma_{i,22}^{-1} \Sigma_{i,21} \Sigma_{i,11}^{-1}, \quad \Sigma_i^{12} = -\Sigma_{i,11}^{-1} \Sigma_{i,12} \Sigma_{i,22}^{-1},$$

$$\Sigma_i^{22} = \Sigma_{i,22}^{-1} + \Sigma_{i,22}^{-1} \Sigma_{i,21} \Sigma_{i,11}^{-1} \Sigma_{i,12} \Sigma_{i,22}^{-1}.$$  

It can be seen that

$$\ell_A^{(1)}(\beta, \gamma_1, \ldots, \gamma_K) = \ell_K(\beta, \gamma_K) - \frac{1}{2} \sum_{i=1}^{K-1} (\hat{\beta}_i - \beta)^\top \Sigma_i^{11} (\hat{\beta}_i - \beta)$$

$$+ \sum_{i=1}^{K-1} (\hat{\beta}_i - \beta)^\top \Sigma_i^{12} (\gamma_i - \hat{\gamma}_i) - \frac{1}{2} \sum_{i=1}^{K-1} (\gamma_i - \hat{\gamma}_i)^\top \Sigma_i^{22} (\gamma_i - \hat{\gamma}_i).$$

Setting $\partial \ell_A^{(1)}(\beta, \gamma_1, \ldots, \gamma_K) / \partial \gamma_i = 0 \ (1 \leq i \leq K - 1)$ gives

$$(\gamma_i - \hat{\gamma}_i) = (\Sigma_i^{22})^{-1} \Sigma_i^{21} (\hat{\beta}_i - \beta).$$
Putting this back in $\ell_A^{(1)}(\beta, \gamma_1, \ldots, \gamma_K)$ gives

$$
\sup_{\gamma_1, \ldots, \gamma_{K-1}} \ell_A^{(1)}(\beta, \gamma_1, \ldots, \gamma_K) = \ell_K(\beta, \gamma_K) - \frac{1}{2} \sum_{i=1}^{K-1} (\hat{\beta}_i - \beta) ^\top \left\{ \Sigma_i^{11} - \Sigma_i^{12}(\Sigma_i^{22})^{-1}\Sigma_i^{21} \right\} (\hat{\beta}_i - \beta) + C
$$

$$
= \ell_K(\beta, \gamma_K) - \frac{1}{2} \sum_{i=1}^{K-1} (\hat{\beta}_i - \beta) ^\top \Sigma_i^{-1}_{i,11,2} (\hat{\beta}_i - \beta) + C,
$$

where we have used the definition of $\Sigma_i^{11,2}$ in the last equation. We arrive at equation (15) after comparing this with the definition of $\ell_A^{(2)}(\beta, \gamma_K)$.

7 Use covariate specific disease prevalent information

As discussed in the previous section, summarized statistics from previous studies can sometimes be utilized to enhance the estimation efficiency in a current study. This is especially important in the big data era where many types of information can be found through internet. More specifically, suppose the disease prevalence is known at various levels of a known risk factor $X$. In this section we combine this type of information in a case-control biased sampling setup.

7.1 Induced estimating equations under case-control sampling

The case-control sampling is one of the most popular methods in cancer epidemiological studies. This is mainly due to the fact that it is the most convenient, economic and effective method. Especially in the study of rare diseases, one has to collect large samples in order to get a reasonable number of cases by using prospective sampling, which may not be practical. Using the case-control sampling, a pre-specified number of cases ($n_1$) and controls ($n_0$) are collected retrospectively from case and control populations separately. Typically this can be accomplished by sampling cases from hospitals, and sampling controls from the general disease free population.

For a given risk factor $X$, let $F_i(x) = \text{pr}(X \leq x|D = i)$ for $i = 0, 1$. Given $X$ in a range
\((a, b)\), the disease prevalence is

\[
\Pr(D = 1 | a < X \leq b) = \phi(a, b),
\]

where \(\phi(a, b)\) is known. Using Bayes’ formula we have

\[
\phi(a, b) = \frac{\pi \int_a^b dF_1(x)}{\Pr(a < X \leq b)}, \quad 1 - \phi(a, b) = \frac{(1 - \pi) \int_a^b dF_0(x)}{\Pr(a < X \leq b)}
\]

with \(\pi = \Pr(D = 1)\). It follows that

\[
\int_a^b dF_1(x) = \frac{1 - \pi}{\pi} \frac{\phi(a, b)}{1 - \phi(a, b)} \int_a^b dF_0(x),
\]

or

\[
\mathbb{E}_1 [I(a < X \leq b)] = \frac{1 - \pi}{\pi} \frac{\phi(a, b)}{1 - \phi(a, b)} \mathbb{E}_0 [I(a < X \leq b)],
\]

where \(\mathbb{E}_0\) and \(\mathbb{E}_1\) denote the expectation operators with respect to \(F_0\) and \(F_1\), respectively.

We assume that given covariates \(X\) and \(Y\), the underlying disease model is given by the conventional logistic regression

\[
\Pr(D = 1 | x, y) = \frac{\exp(\alpha^* + x\beta + y\gamma + yx\xi)}{1 + \exp(\alpha^* + x\beta + y\gamma + yx\xi)}.
\]

(16)

Let \(\alpha = \alpha^* - \eta\) with \(\eta = \log\{(1 - \pi)/\pi\}\). It can be shown (See Qin, 2017) that this is equivalent to the exponential tilting model

\[
f_1(x, y) = f(x, y|D = 1) = \exp(\alpha + x\beta + y\gamma + yx\xi) f_0(x, y),
\]

where \(f_0(x, y) = f(x, y|D = 0)\). As a consequence,

\[
\mathbb{E}_0 \left\{I(a < X \leq b)e^{\eta + \alpha + \beta X + \gamma Y + \xi XY}\right\} = \frac{1 - \pi}{\pi} \frac{\phi(a, b)}{1 - \phi(a, b)} \mathbb{E}_0 [I(a < X \leq b)],
\]

or

\[
\mathbb{E}_0 \left\{I(a < X \leq b)e^{\alpha + \beta X + \gamma Y + \xi XY} - \frac{\phi(a, b)}{1 - \phi(a, b)} I(a < X \leq b)\right\} = 0.
\]

(17)

Denote

\[
g_0(X, Y) = e^{\eta + \alpha + \beta X_i + \gamma Y_i + \xi X_i Y_i} - 1,
\]

and the summarized auxiliary information equations as

\[
g_i(X, Y) = I(a_{i-1} < X \leq a_i)e^{\alpha + \beta X + \gamma Y + \xi XY} - \frac{\phi(a_{i-1}, a_i)}{1 - \phi(a_{i-1}, a_i)} I(a_{i-1} < X \leq a_i)
\]

with \(i = 1, 2, ..., I\). Then \(\mathbb{E}_0 \{g(X, Y)\} = 0\), where \(g(X, Y) = (g_0(X, Y), g_1(X, Y), ..., g_I(X, Y))^\top\).
7.2 Empirical likelihood approach

The log-likelihood is

\[ \ell = \sum_{i=1}^{n} d_i (\eta + \alpha + \beta x_i + \gamma y_i + \xi x_i y_i) + \sum_{i=1}^{n} \log(p_i), \]  

(18)

where \( p_i = dF_0(x_i), i = 1, 2, \ldots, n \), and the constraints are

\[ p_i \geq 0, \quad \sum_{i=1}^{n} p_i = 1, \quad \sum_{i=1}^{n} p_i g(x_i, y_i) = 0. \]

The profile log-likelihood is

\[ \ell = \sum_{i=1}^{n} d_i (\eta + \alpha + \beta x_i + \gamma y_i + \xi x_i y_i) - \sum_{i=1}^{n} \log\left(1 + \lambda^\top g(x_i, y_i)\right), \]

where the Lagrange multiplier \( \lambda \) is determined by

\[ \sum_{i=1}^{n} \frac{g(x_i, y_i)}{1 + \lambda^\top g(x_i, y_i)} = 0. \]

Finally, the underlying parameters can be obtained by maximizing \( \ell \).

If the overall disease prevalence probability \( \pi = \text{pr}(D = 1) \) is known, then \( \eta = \log\{(1 - \pi)/\pi\} \) is known. On the other hand if it is unknown but \( I \geq 1 \), then \( \pi \) is identifiable. If \( I > 1 \), then we have an over-identified equation problem. This can be treated as a generalization of the empirical likelihood method for estimating functions (Qin and Lawless, 1994) to biased sampling problems. Qin et al. (2015) considered the case that \( \eta \) is unknown and \( I \geq 1 \).

Let \( \omega = (\eta, \alpha, \beta, \gamma, \xi, \lambda) \). Since the first estimating function \( g_0 \) corrects biased sampling in a case-control study, the remaining estimating functions \( g_1, \ldots, g_I \) are used for improving efficiency. When \( n \) goes to infinity, it can be shown that the limit of \( \lambda \) is a \((I+1)\)-dimensional vector with the first component being \( \lim_{n \to \infty} (n_1/n) \) and the rest all being zero. Qin et al. (2015) disclosed that if \( \rho = n_1/n_0 \) remains constant as \( n \to \infty \) and \( \rho \in (0,1) \), then under suitable regularity conditions \( \sqrt{n}(\hat{\omega} - \omega_0) \) is asymptotically normally distributed with mean zero. Moreover, the estimation of the logistic regression parameters \((\beta, \gamma, \xi)\) improves as the number \( I \) of estimating functions increases. This means that a richer set of auxiliary
information leads to better estimators. In practice, however, this must be balanced with the numerical difficulty of solving a larger number of equations.

It is interesting to note that, auxiliary information is primarily informative for estimating \( \beta \) and \( \xi \), but not for estimating \( \gamma \). This can be observed through the following equations

\[
\int I(a < x < b) \exp(\alpha + \beta x + \gamma y + \xi xy)dF_0(x, y) = \int I(a < x < b) \exp(\alpha + \beta x + s + \xi xs/\gamma)dF_0(x, s/\gamma).
\]

Since the underlying distribution \( F_0(x, y) \) is unspecified, we can treat \( F_0(x, s/\gamma) \) as a new underlying distribution \( F^*_0(x, s) \). With \( F^*_0 \) profiled out, the auxiliary information equation does not involve \( \gamma \) if \( \xi = 0 \). Hence, even if \( \xi \neq 0 \), the information for \( \gamma \) is minimal since \( \gamma \) and \( \xi \) cannot be separated.

### 7.3 Generalizations

In Qin et al. (2015)’s simulation studies, it looks like the maximum reduction of variance occurs for the estimation of the coefficient of \( X \). If the auxiliary information

\[ pr(D = 1|b_{j-1} < Y \leq b_j) = \psi_j, \quad j = 1, 2, \ldots, J \]

is also available, naturally we can combine them through estimating equations

\[
g_i(X, Y) = I(a_{i-1} < X \leq a_i) e^{\alpha + \beta X + \gamma Y + \xi XY} - \frac{\phi(a_{i-1}, a_i)}{1 - \phi(a_{i-1}, a_i)} I(a_{i-1} < X \leq a_i),
\]

\[
h_j(X, Y) = I(b_{j-1} < Y \leq b_j) e^{\alpha + \beta X + \gamma Y + \xi XY} - \frac{\psi(b_{j-1}, b_j)}{1 - \psi(b_{j-1}, b_j)} I(b_{j-1} < Y \leq b_j).
\]

It would be more informative if the auxiliary information \( pr(D = 1|a < X < b, c < Y < d) \) is available.

### 7.4 More on the use of auxiliary information

Under a logistic regression model, the case and control densities are linked by the exponential tilting model

\[ pr(x, y|D = 1) = pr(x, y|D = 0) \exp(\alpha + x\beta + y\gamma + \xi xy). \]  \( (19) \)
Suppose that for the general population $\mathbb{E}(X) = \mu_1$, $\mathbb{E}(Y) = \mu_2$ and $\mathbb{E}(XY) = \mu_3$, are all known, and $\pi = \text{pr}(D = 1)$ is known or can be estimated using external data. Under the exponential tilting model \cite{19}, the density $f(x, y)$ in the general population and the density $\text{pr}(x, y|D = 0)$ in the control population are linked by

$$\text{pr}(x, y) = \{\pi e^{\alpha + x\beta + y\gamma + x\xi xy} + (1 - \pi)\}\text{pr}(x, y|D = 0).$$

As a consequence

$$\mathbb{E}(X) = \mathbb{E}_0[\{\pi e^{\alpha + X\beta + Y\gamma + X\xi XY} + (1 - \pi)\}] = \mu_1,$$

where $\mathbb{E}_0$ is an expectation with respect to $\text{pr}(x, y|D = 0)$. Let $h(x, y) = (x - \mu_1, y - \mu_2, xy - \mu_3)$ with known $\mu_1, \mu_2$ and $\mu_3$. The log-likelihood under case-control data is still \cite{18}, where $p_i$’s satisfy the following constraints

$$\sum_{i=1}^{n} p_i = 1, \ p_i \geq 0, \ \sum_{i=1}^{n} p_i e^{\alpha + x_i\beta + x_i\gamma + x_i\xi} = 1,$$

$$\sum_{i=1}^{n} p_i h(x_i, y_i)\{\pi e^{\alpha + x_i\beta + x_i\gamma + x_i\xi xy} + (1 - \pi)\} = 0.$$

More generally, any information in the general population such as $\mathbb{E}[\psi(Y, X)] = 0$ can be converted to an equation for the control population,

$$\mathbb{E}_0[\{\pi e^{\alpha + X\beta + Y\gamma + X\xi XY} + (1 - \pi)\}\psi(Y, X)] = 0.$$

Therefore the results developed in Qin et al. (2015) can be applied too. Chatterjee et al. (2016)’s results for case-control data can be considered as a special case of Qin et al. (2015).

8 Communication efficient distributed inference

In the era of big data, it is commonplace for data analyses to run on hundreds or thousands of machines, with the data distributed across those machines and no longer available in a single central location. Recently the parallel and distributed inference has become popular in statistical literature both for frequentist settings and Bayesian settings. In essence the
data-parallel procedures are to break the overall dataset into subsets that are processed independently. To the extent that communication-avoiding procedures have been discussed explicitly, the focus has been on one-shot or embarrassingly-parallel approaches that only use one round of communication in which estimators or posterior samples are first obtained in parallel on local machines, are then communicated to a center node, and finally are combined to form a global estimator or approximation to the posterior distribution (Zhang et al., 2013, Lee et al., 2017, Wang and Dunson, 2015, Neiswanger et al., 2015). In the frequentist setting, most one-shot approaches rely on averaging (Zhang et al., 2013), where the global estimator is the average of the local estimators. Lee et al. (2017) extends this idea to high-dimensional sparse linear regression by combining local debiased Lasso estimates (van de Geer et al., 2014). Recent work by Duchi et al. (2015) shows that under certain conditions, these averaging estimators can attain the information-theoretic complexity lower bound for linear regression, and at least \( O(dk) \) bits must be communicated in order to attain the minimax rate of parameter estimation, where \( d \) is the dimension of the parameter and \( k \) is the number of machines. This result holds even in the sparse setting (Braverman et al., 2016).

The method of Jordan, Lee and Yang (2019) proceeds as follows. Suppose the big data consists of \( N \) observations and there are \( k \) machines. For convenience of presentation, we assume that each machine has \( n \) observations. That is, \( N = nk \). Denote the full-data likelihood by

\[
\ell_N(\theta) = \frac{1}{k} \sum_{j=1}^{k} \ell_j(\theta),
\]

where \( \ell_j(\theta) \) is the log-likelihood based on the data from the \( j \)th machine. For \( \theta \) near its target value \( \bar{\theta} \),

\[
\ell_N(\theta) = \ell_N(\bar{\theta}) + \nabla_\theta \ell_N(\theta) \bigg|_{\theta = \bar{\theta}} (\theta - \bar{\theta}) + R_N(\theta),
\]

\[
\ell_1(\theta) = \ell_1(\bar{\theta}) + \nabla_\theta \ell_1(\theta) \bigg|_{\theta = \bar{\theta}} (\theta - \bar{\theta}) + R_1(\theta),
\]

where \( R_N(\theta) \) and \( R_1(\theta) \) are remainders. Observing that \( R_N \approx R_1 \), they define a surrogate
log-likelihood

\[
\bar{\ell}(\theta) = \ell_N(\bar{\theta}) + (\theta - \bar{\theta})^T \nabla_{\theta} \ell_N(\theta) \bigg|_{\theta = \bar{\theta}} + \left\{ \ell_1(\theta) - \ell_1(\bar{\theta}) - (\theta - \bar{\theta})^T \nabla_{\theta} \ell_1(\theta) \bigg|_{\theta = \bar{\theta}} \right\}.
\]

With the constant terms ignored, the surrogate log-likelihood is

\[
\bar{\ell}(\theta) = \ell_1(\theta) + \theta^T \left\{ \nabla_{\theta} \ell_N(\theta) \bigg|_{\theta = \bar{\theta}} - \nabla_{\theta} \ell_1(\theta) \bigg|_{\theta = \bar{\theta}} \right\}.
\]

The score equation based on the surrogate likelihood is

\[
\nabla_{\theta} \bar{\ell}(\theta) = \nabla_{\theta} \ell_1(\theta) + \left\{ \nabla_{\theta} \ell_N(\theta) \bigg|_{\theta = \bar{\theta}} - \nabla_{\theta} \ell_1(\theta) \bigg|_{\theta = \bar{\theta}} \right\} = 0.
\]

Let \( \hat{\theta} \) be the solution. Expanding it at \( \theta_0 \) and using the fact that

\[
N^{-1} \{\nabla_{\theta \theta^T} \ell_1(\theta_0) - \nabla_{\theta \theta^T} \ell_N(\theta_0)\} \rightarrow 0 \quad \text{in probability.}
\]

Easily we can show that if \( \bar{\theta} - \theta_0 = O_p(N^{-1/2}) \), then

\[
(\hat{\theta} - \theta_0) = \left\{ \nabla_{\theta \theta^T} \ell_1(\theta_0) \right\}^{-1} \nabla_{\theta} \ell_N(\theta_0) + o_p(N^{-1/2}).
\]

If we let \( \hat{\theta} \) be the MLE based on \( \ell_1(\theta) \), then the surrogate log-likelihood can be simplified as

\[
\bar{\ell}(\theta) = \ell_1(\theta) + \theta^T \nabla_{\theta} \ell_N(\hat{\theta}),
\]

because \( \nabla_{\theta} \ell_1(\hat{\theta}) = 0 \).

If the dimension of \( \theta \) is high, naturally one may add a penalty function in the surrogate log-likelihood, and estimate \( \theta \) by \( \tilde{\theta} = \arg \max_{\theta \in \Theta} \{ \bar{\ell}(\theta) - \lambda \|\theta\|_1 \} \), where \( \|\theta\|_1 \) is the \( L_1 \)-norm of \( \theta \). Similarly Bayesian inference can be adapted to the surrogate likelihood as well.

Duan et al. (2019) proposed distributed algorithms which account for the heterogeneous distributions by allowing site-specific nuisance parameters. The proposed methods extend the surrogate likelihood approach (Wang et al., 2017; Jordan et al., 2019) to the heterogeneous setting by applying a novel density ratio tilting method to the efficient score function. It can be shown that asymptotically the approach in Section 6.2 on nuisance parameters is equivalent to Duan et al. (2019)’s.
9 Renewal estimation and incremental inference

Let $U(D_1, \beta) = \nabla_{\beta} M(D_1, \beta)$ be a score function of $\beta$ based on some objective function $M(D_1, \beta)$ from the first batch of data, where $M$ can either be the log-likelihood $M(D_1, \beta) = \sum_{i=1}^{n_1} \log f(y_{1i}|x_{1i}, \beta)$ or a log pseudo-likelihood.

Let $\hat{\beta}_1$ be the solution to $U(D_1, \beta) = 0$, when only the first batch of data $D_1$ are available. Let $D_2$ denote the second batch of data. If both of them are available, we let $\hat{\beta}_2$ be the solution to the pooled score equation, $U(D_1, \beta) + U(D_2, \beta) = 0$. Clearly $\hat{\beta}_2$ is the most efficient estimator of $\beta$ when $D_1$ and $D_2$ are both available.

In addition to $D_2$, if not $D_1$ but only some summary information $\hat{\beta}_1$ and $\hat{\Sigma}_1$ from it are available, how to utilize the summary information efficiently? It is not feasible to estimate $\beta$ by directly solving

$$U(\beta) \equiv U(D_1, \beta) + U(D_2, \beta) = 0,$$

which involves the individual data of the unavailable $D_1$. Luo and Song (2020) consider expanding $U(D_1, \beta)$ at $\beta = \hat{\beta}_1$, i.e.,

$$U(D_1, \beta) = U(D_1, \hat{\beta}_1) + (\beta - \hat{\beta}_1)^\top \nabla_{\beta} U(D_1, \hat{\beta}_1) + O(\Vert \beta - \hat{\beta}_1 \Vert^2)$$

for $\beta$ close to $\hat{\beta}_1$. Since $U(D_1, \hat{\beta}_1) = 0$, it follows that

$$U(\beta) = U(D_2, \beta) + (\beta - \hat{\beta}_1)^\top \nabla_{\beta} U(D_1, \hat{\beta}_1) + O(\Vert \beta - \hat{\beta}_1 \Vert^2).$$

Luo and Song (2020) propose to get an updated estimator of $\beta$ by solving

$$(\beta - \hat{\beta}_1)^\top \nabla_{\beta} U(D_1, \hat{\beta}_1) + U(D_2, \beta) = 0. \quad (20)$$

Alternatively we may understand this renewal estimation strategy in the way of Zhang et al. (2020), who propose to estimate $\beta$ by maximizing

$$\sum_{i=1}^{n_2} \log f(y_{2i}|x_{2i}, \beta) - 0.5n_1(\hat{\beta}_1 - \theta)\Sigma(\hat{\beta} - \beta)^\top$$

with $\Sigma = \mathbb{E}\left\{ \nabla_{\beta} \log f(Y|X, \beta) \nabla_{\beta}^\top \log f(Y|X, \beta) \right\}$ being the Fisher information. If both batches are available, the score for $\beta$ is

$$S(\beta) = \sum_{i=1}^{n_1} \nabla_{\beta} \log f(y_{1i}|x_{1i}, \beta) + \sum_{i=1}^{n_2} \nabla_{\beta} \log f(y_{2i}|x_{2i}, \beta). \quad (21)$$
After recording $\hat{\beta}_1$ and $\Sigma$, we do not have the raw data $(y_{1i}, x_{1i}), i = 1, 2, ..., n_1$ anymore. Because

$$\hat{\beta}_1 - \beta = -n_1^{-1} \Sigma^{-1} \sum_{i=1}^{n_1} \nabla_{\beta} \log f(y_{1i}|x_{1i}, \beta) + o_p(n_1^{-1/2}),$$

differentiating (21) with respect to $\beta$ gives

$$\sum_{i=1}^{n_2} \nabla_{\beta} \log f(y_{2i}|x_{2i}, \beta) - n_1 \Sigma(\hat{\beta}_1 - \beta)$$

$$= \sum_{i=1}^{n_1} \nabla_{\beta} \log f(y_{1i}|x_{1i}, \beta) + \sum_{i=1}^{n_2} \nabla_{\beta} \log f(y_{2i}|x_{2i}, \beta) + o_p(n^{1/2}).$$

Here we have assumed that $n_1 = O(n_2) = O(n)$. This indicates that estimating $\beta$ by maximizing (21) has no efficiency loss asymptotically compared with the MLE based on all individual data, where the latter is infeasible in the current situation.

## 10 Concluding remarks

Rapid growth in hardware technology has made the data collection much easier and more effectively. In many applications, data often arrive in streams and chunks, which leads to batch by batch data or streaming data. For example, web sites severed by widely distributed web servers may need to coordinate many distributed clickstream analyses, e.g. to track heavily accessed web pages as part of their real-time performance monitoring. Other examples include financial applications, network monitoring, security, telecommunications data management, manufacturing, and sensor networks (Babcock et al., 2002; Nguyen et al., 2020). The continuous arrival of such data in multiple, rapid, time-varying, possibly unpredictable and unbounded streams yields some fundamentally new research problems. One of the most challenging issues is how to address statistics in an online updating framework, without storage requirement for raw data.

Assembling information from difference data sources has become indispensable in big data and artificial intelligence research. Statistical tools play an essential role in updating
information. In this paper, we have made a selective review on several traditional statistical methods, such as meta analysis, calibration information methods in survey sampling, EL together with over-identified estimating equations, and GMM. We also briefly review some recently-developed statistical methods, including communication efficient distributed statistical inference and renewal estimation and incremental inference, which can be regarded as the latest development of calibration information methods in the era of big data. Even though those methods were developed in different fields and in different statistical frameworks, in principle, they are asymptotically equivalent to those well known methods developed in meta analysis. Almost no or little information is lost compared with the case when full data are available.

Due to deficiency of our knowledge, finally we have to apology for individuals whose works inadvertently have been left off in our reference lists.

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