DIFFUSIVE COSMIC RAY ACCELERATION AT RELATIVISTIC SHOCK WAVES WITH MAGNETOSTATIC TURBULENCE. II. INFLUENCE OF A FINITE DOWNSTREAM MEDIUM

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ABSTRACT

The diffusive acceleration of relativistic cosmic rays at parallel shock waves with magnetostatic turbulence and a finite size of the downstream medium is investigated. For ultrarelativistic shock speeds with Lorentz factor $\Gamma_0^{\ast} \gg 1$, both the differential momentum spectrum at the shock and the volume-integrated momentum spectrum are power-law distribution functions with different spectral indices as compared to the case of an infinitely extended downstream medium. However, the spectral differences are only modest as compared to the case of nonrelativistic shocks. The behavior of the momentum spectrum of shock-accelerated particles depends sensitively on the relativistic shock wave Pelet number $G(p) = \tau_p(p)/\tau_c$, i.e., the ratio between the diffusion and convection timescales of cosmic rays to propagate from the shock position to the downstream boundary $z_0$. For large values of $G(p) \gg 1$ the free-escape boundary has no influence on the effectiveness of particle acceleration, still providing a flat momentum power-law spectrum of the accelerated particles. In the opposite case of small Pelet numbers $G(p) \ll 1$ at all momenta, the momentum spectrum at the shock steepens to the greater spectral index $\xi_0 = 3 - \delta + (3.18/\Gamma_0^{\ast})$, whereas the volume-integrated momentum spectrum flattens by the same factor $2 - \delta$ for its power-law spectral index, where $\delta$ denotes the spectral index of the downstream power spectrum of magnetostatic turbulence. This effectiveness of relativistic shocks in generating flat power-law momentum spectra irrespective of the Pelet number $G(p)$ differs completely from the behavior of nonrelativistic shocks.

Key words: acceleration of particles – cosmic rays – relativistic processes – shock waves

1. INTRODUCTION

Relativistic shock waves form during the interaction of relativistic supersonic and super-Alfvénic outflows with the ambient ionized interstellar or intergalactic medium, producing anisotropic counterstream plasma distribution functions due to shock-reflected charged particles in the upstream medium. Such outflows ultimately result from violent explosive events such as in gamma-ray burst sources, but also occur as highly collimated pulsar winds and jets of active galactic nuclei with initial bulk Lorentz factors $\Gamma_0 = (1 - (V_0/c)^2)^{-1/2} \approx 400$. There is high interest in understanding the diffusive cosmic ray acceleration at relativistic shock waves because relativistic shock sites in these powerful cosmic sources have been frequently identified as intense sources of high-energy nonthermal radiation. The radiation modeling of these sources with hard intrinsic photon power-law spectra often requires the existence of very hard momentum spectra of the radiating particles (Lefa et al. 2011). Recently, one of us (Schlickeiser 2015—hereafter referred to as Paper I) demonstrated that the diffusive acceleration of cosmic rays at parallel relativistic magnetized shock waves produces very flat power-law spectra at relativistic cosmic ray momenta with differential particle density spectra approaching $N(p) \propto p^{-1}$ for large values of the shock wave Lorentz factor $\Gamma_0^{\ast} = (1 - (U_0/c)^2)^{-1/2}$.

In Paper I, for the first time, the diffusion–convection transport equation in relativistic flows has been developed, employing the diffusion approximation for particle transport in the respective rest frames of the upstream and downstream regions of the shock wave (Webb 1985; Kirk et al. 1988). This allowed the first fully analytical calculation of cosmic ray momentum spectra accelerated at parallel relativistic magnetized shock waves. Such an analytical calculation has clear advantages compared to earlier solutions (reviewed in Kirk & Duffy 1999 and Ellison et al. 2016) based on semi-numerical eigenfunction solutions pioneered by Kirk & Schneider (1987), relativistic Monte Carlo simulations (reviewed in Ellison et al. 2016), and relativistic particle-in-cell simulations (reviewed in Sironi et al. 2015). As shown in Paper I, the relativistic analytical theory correctly reproduces the well-known results of particle acceleration at shocks with nonrelativistic speeds, so it is a suitable analytical extension to the case of shocks with arbitrarily high, even relativistic velocities. In Paper I an infinitely extended downstream medium was considered, and it was shown that the diffusive acceleration of cosmic rays is very effective in producing very flat power-law spectra at relativistic cosmic ray momenta. Here we consider the influence of a finite spatial extent of the downstream medium with a free-escape boundary condition located at $z = -z_0$. Earlier work on nonrelativistic shocks (Ostrowski & Schlickeiser 1996) has shown that such a free-escape boundary at close enough distance to the shock makes cosmic ray acceleration very inefficient. Here we demonstrate that in relativistic shocks the influence of a free-escape boundary is less severe.

2. BASIC EQUATIONS

As detailed in Paper I, the steady-state diffusion–convection transport equation for the isotropic part of the phase space density $F(z, p)$ of cosmic rays in relativistic flows with speed $U(z)$ parallel to a uniform magnetic field reads

$$\frac{\partial}{\partial z} \left[ \Gamma(UF - \Gamma \kappa_{\perp} \frac{\partial F}{\partial z}) \right] + \frac{\alpha}{p^2} \frac{\partial}{\partial p} \left[ p^2 \kappa_{\parallel} \Gamma \frac{\partial F}{\partial z} - \frac{p^4 F}{3} \right] = S(z, p)$$

(1)
with the particle injection rate \( S(z, p) \), the bulk Lorentz factor
\[ \Gamma(z) = (1 - (U^2/c^2))^{-1/2}, \]
the rate of adiabatic acceleration
\[ \alpha(z) = \frac{c^2}{U(z)} \frac{d\Gamma(z)}{dz} = \frac{d(U\Gamma)}{dz}, \tag{2} \]
and the two diffusion coefficients
\[ \kappa_{cc} = \frac{v^2 K_0}{8}, \quad \alpha \kappa_{pc} = \frac{\alpha U K_0 p}{8}. \tag{3} \]

The transport Equation (1) holds in a mixed comoving coordinate system, where the particle’s momentum coordinate \( p \) is taken in the rest frame of the streaming plasma, whereas the space coordinate \( z \) is taken in the observer’s system.

The two diffusion coefficients (3) are caused by the resonant pitch-angle scattering of cosmic rays with magnetostatic slab turbulence for symmetric pitch-angle Fokker–Planck scattering coefficients \( D_{\mu \nu}(\mu) = D_{\mu \nu}(\mu) \). In Equations (3) the common factor
\[ K_0 = K(D_0(s)) \left( \frac{p}{p_m} \right)^{2-s} \tag{4} \]
is determined by the constant length
\[ K = \frac{32}{\pi (s - 1)} \left( \frac{B_0}{\delta B} \right)^2 \lambda_{\text{max}} \simeq 10^{21} \lambda_1 \left( \frac{B_0}{\delta B} \right)^2 \text{pc} \tag{5} \]
and the maximum cosmic ray (with charge number \( q \)) momentum
\[ p_m = 1.5 \cdot 10^{17} |q| B (mG) \lambda_1 \frac{eV}{c}, \tag{6} \]
where \( s \in (1, 2) \) denotes the spectral power-law index of the adopted isospectral magnetostatic turbulence power spectrum, \( \lambda_{\text{max}} = \lambda_1 \) pc the maximum wavelength of the Alfvén waves, and
\[ D_0(s) = \frac{2}{(2 - s)(4 - s) D_1(H_c, \sigma_+, \sigma_-)}, \]
\[ D_1(H_c, \sigma_+, \sigma_-) = 4 - [\sigma_+ + \sigma_- + H_c(\sigma_+ - \sigma_-)]^2 \tag{7} \]
accounts for the cross helicity \( H_c \) and the magnetic helicities \( \sigma_{\pm} \) of forward- and backward-moving Alfvén waves.

With Equations (4) and (5) the spatial diffusion coefficient of cosmic rays is given by
\[ \kappa_{cc} = \kappa_m \beta \left( \frac{p}{p_m} \right)^{2-s}, \tag{8} \]
where
\[ \kappa_m = 10^{30} D_0(s) P \lambda_1 \left( \frac{B_0}{\delta B} \right)^2 \text{cm}^2/s \tag{9} \]
is the spatial diffusion coefficient at the maximum cosmic-ray momentum \( p_m \). Equation (9) corresponds to a scattering length of
\[ \lambda_m = \frac{3 \kappa_m}{\beta c} = 30D_0(s) \lambda_1 \left( \frac{B_0}{\delta B} \right)^2 \text{pc} \tag{10} \]
In order to justify the use of the diffusion–convection transport Equation (1) the spatial extension of the downstream medium
\[ z_0 \gg \lambda_m \tag{11} \]
should be considerably larger than the maximum cosmic ray scattering length \( \lambda_m \).

3. ACCELERATION AT RELATIVISTIC SHOCK WAVES

Adopting as in Paper I the rest frame of the shock wave as the laboratory frame and the step-like shock profile
\[ U(z) \]
\[ \begin{cases} \frac{-U_1}{-\beta_1 c} = \text{const.} & \text{for } 0 < z \leq \infty \text{ (upstream)} \\ \frac{-U_2}{-\beta_2 c} = \text{const.} & \text{for } -\infty < z < 0 \text{ (downstream)} \end{cases} \tag{12} \]
with \( U_2 < U_1 \), the rate of adiabatic acceleration (2)
\[ \alpha = \alpha_0 \delta(z), \quad \alpha_0 = -(U_1 \Gamma_1 - U_2 \Gamma_2) \tag{13} \]
is non-zero only at the position of the shock. Moreover, we assume particle injection \( S(z, p) = S(p) \delta(z) \) at the position of the shock only, and consider spatially constant flow velocities and diffusion coefficients in the upstream and downstream region. The diffusion–convection transport Equation (1) in the rest frame of the shock wave then reduces to
\[ \frac{\partial}{\partial z} \left[ \Gamma \left( UF - \Gamma \kappa_{cc} \frac{\partial F}{\partial z} \right) \right] + \alpha \kappa_{cc} \frac{\partial}{\partial p} \left[ p^2 \kappa_{pc} \frac{\partial F}{\partial z} - \frac{p^3 F}{3} \right] = S(p) \delta(z = 0). \tag{14} \]

For the upstream region \( z > 0 \) the solution of the transport Equation (14) is given by
\[ F_1(z, p) = F_0(p) \exp \left[ -\frac{U_1 z}{\Gamma_1 \kappa_{cc,1}} \right], \tag{15} \]
which approaches zero far upstream \( z \to \infty \).

In contrast to Paper I we consider here a finite size of the downstream region \( -z_0 \leq z < 0 \), allowing the free-escape of particles at \( z = -z_0 \), subject to the constraint (11), so that
\[ F_2(-z_0, p) = 0. \tag{16} \]
The solution of the steady-state transport Equation (14) in the downstream region is then given by
\[ F_2(z, p) = F_0(p) \frac{M(p)e_{\Gamma(p)} G(p)}{M(p) - 1} \tag{17} \]
where we used the continuity of the distribution function at the position of the shock \( F_1(z = 0, p) = F_2(z = 0, p) = F_0(p) \), and define the free-escape functions,
\[ M(p) = e^{\Gamma(p)} G(p) = \frac{U_2 z_0}{\Gamma_2 \kappa_{cc,2}(p)} = G_m \beta^{-1} \left( \frac{p}{p_m} \right)^{\beta-2} \tag{18} \]
and
\[ G_m = \frac{\tau_D(p_m)}{\tau_c} = \frac{U_2 z_0}{\Gamma_2 \kappa_{m,2}} = \frac{3\beta_2}{\Gamma_2 \beta^2 \lambda_{m,2}} \]
\[ \approx \frac{3\beta_2}{\Gamma_2 \beta} = \frac{3\beta_1 \sqrt{r^2 - \beta_1^2}}{r^2 \beta} = \begin{cases} \frac{3\beta_1}{r^2} & \text{for } \beta_1 \ll 1 \\ \frac{3\beta_1 r^2 - 1}{r^2} & \text{for } \Gamma_1 \gg 1 \end{cases}, \tag{19} \]
where we adopted \( z_0 \approx \lambda_m \).

3.1. Relativistic Shock Wave Peclet number

\( G(p) \) is the relativistic shock wave Peclet number (Schlickeiser et al. 1993) as is given by the ratio of the timescales of cosmic rays at momentum \( p \) to propagate from the shock position to the downstream boundary by diffusion \( \tau_D(p) = z_0^2 / \kappa_{\gamma z,2}(p) \) and by convection \( \tau_c = \Gamma_2 z_0 / U_2 \), respectively. Large values of \( G(p) \gg 1 \) indicate that cosmic ray particles are confined very well in the downstream region because of small enough spatial diffusion coefficients. In this case we expect that the particle acceleration at the shock wave will be very efficient, resulting in a rather flat momentum spectrum of the accelerated particles. In the opposite case of small values of \( G(p) \ll 1 \), the cosmic ray particles rapidly escape from the downstream region by spatial diffusion, so that only a few particles return by diffusion to the shock front, leading to a steepened momentum spectrum of accelerated particles.

We note from Equation (18) that \( G(p) \) decreases with momentum \( G(p) \propto \beta^{-1}(p/p_m)^{-\gamma} \), so it attains its minimum value \( G_m \) at \( p = p_m \) with \( G(p < p_m) > G_m \). If in the case of mono-momentum particle injection \( S(p) = \delta_0 (p - p_0) \) the relativistic shock wave Peclet number at the injection momentum \( G(p_0) \ll 1 \) is much smaller than unity, cosmic ray acceleration at all higher momenta will be very inefficient. Alternatively, if \( G_m \) itself is much larger than unity, cosmic rays at all smaller momenta will be accelerated very efficiently, and the influence of the downstream free-escape boundary is negligibly small. Our quantitative results obtained below confirm this significant influence of the relativistic shock wave Peclet number.

3.2. Momentum Spectrum of Accelerated Particles at the Shock

The particle momentum spectrum \( F_0(p) \) at the position of the shock is obtained by integrating the transport Equation (14) from \( z = -\eta \) to \( z = \eta \) and considering the limit \( \eta \to 0 \). This provides the continuity condition for the cosmic ray streaming density at the shock
\[ - \left[ \Gamma_1 U_2 F_1 + \Gamma_1 \kappa_{\gamma z,1} \frac{\partial F_1}{\partial z} \right]_0^\eta + \left[ \Gamma_2 U_2 F_2 + \Gamma_2 \kappa_{\gamma z,2} \frac{\partial F_2}{\partial z} \right]_0^\eta + \frac{\alpha_0}{3p^2} \frac{\partial}{\partial p} p^2 \left( 3\kappa_{\gamma z,1} \Gamma_1 \frac{\partial F_1}{\partial z} \right)_0^\eta - \left[ 3\kappa_{\gamma z,2} \Gamma_2 \frac{\partial F_2}{\partial z} \right]_0^\eta = p F_0(p) = S(p). \tag{20} \]

Using the up- and downstream solutions (15) and (17) we obtain
\[ \frac{U_1 \Gamma_1 - U_2 \Gamma_2}{3p^2} \frac{d}{dp} \left( p^3 + 3\kappa_{\gamma z,1} \Gamma_1 U_1 + 3\kappa_{\gamma z,2} U_2 \right) \]
\[ \times F_0(p) + \frac{\Gamma_2 U_2 F_0(p)}{1 - M^{-1}(p)} = S(p) \tag{21} \]
where we inserted \( \alpha_0 \). We note that for an infinitely extended downstream region \( z_0 \to \infty \) also \( M \to \infty \), so that Equation (21) reduces to Equation (I-41).

According to Equations (3) we obtain for the ratios \( i = 1, 2 \)
\[ \frac{3 \kappa_{\gamma z,1} U_i}{\kappa_{\gamma z,1}} = \frac{3p U_i^2}{v^2}, \tag{22} \]
so that Equation (20) becomes
\[ F_0(p) + \frac{1 - M^{-1}(p)}{\psi p^2} \frac{d}{dp} \left( p^2 T(p) F_0(p) \right) \]
\[ = \frac{(1 - M^{-1}(p)) S(p)}{\Gamma_1 U_2} \tag{23} \]
where we define the ratio
\[ \psi = \frac{3U_2 \Gamma_2}{U_1 \Gamma_1 - U_2 \Gamma_2} = \frac{3}{\Gamma_1 \sqrt{r^2 - \beta_1^2} - 1} \tag{24} \]
in terms of the flow compression ratio \( r = U_1 / U_2 = \beta_1 / \beta_2 \), and the function
\[ T(p) = p \left( 1 + \frac{3\beta_1^2}{\beta^2 (p)} + \frac{3\beta_1^2}{\beta^2 (M(p) - 1)} \right) \tag{25} \]
with \( \beta_1 = \beta / c \) and the cosmic ray particle velocity \( \beta(p) = v / c \) in units of the speed of light. Setting
\[ W(p) = p^2 T(p) F_0(p) \tag{26} \]
provides for Equation (23)
\[ T(p)[1 - M^{-1}(p)] \frac{dW(p)}{dp} + \psi W(p) \]
\[ = \psi [1 - M^{-1}(p)] p^2 S(p) T(p). \tag{27} \]
Introducing as momentum variable
\[ x(p) = \int_0^p \frac{dp'}{T(p')[1 - M^{-1}(p')]} \tag{28} \]
implying
\[ \frac{dx}{dp} = \frac{1}{T(p)[1 - M^{-1}(p)]} \tag{29} \]
yields for Equation (27)
\[ \frac{d}{dx} [W(x) e^{\psi x}] = Q(x) \tag{30} \]
with the source term

\[ Q(x) = \frac{\psi [1 - M^{-1}(p(x))] p(x)^2 S(p(x)) T(p(x)) e^{\psi x}}{\Gamma_2 U_2}. \]  

(31)

Equation (30) is solved by

\[ W(x) = e^{-\psi x} \int_0^x dx' Q(x'). \]

\[ = \frac{\psi}{\Gamma_2 U_2} \int_0^p dt t^2 S(t) \exp \left[ -\psi \int_t^p \frac{dy}{T(y)[1 - M^{-1}(y)]} \right]. \]

(32)

where we used Equations (28)–(29). According to Equation (26) we obtain

\[ F_0(p) = \frac{\psi S_0 p_0^2}{\Gamma_2 U_2 p^2 T(p)} \int_0^p dt t^2 S(t) e^{-\psi \int_t^p \frac{dy}{T(y)[1 - M^{-1}(y)]}}. \]

(33)

3.3. Momentum Injection

For a mono-momentum injection spectrum

\[ S(p) = S_0 \delta(p - p_0), \]

(34)

the solution (33) becomes

\[ F_0(p \geq p_0) = \frac{\psi S_0 p_0^2}{\Gamma_2 U_2 p^2 T(p)} e^{-\psi [p(p,p_0)]} \]

(35)

with the integral

\[ I(p, p_0) = \int_{p_0}^p \frac{dy}{T(y)[1 - M^{-1}(y)]} \]

\[ = \int_{p_0}^p dy \frac{M(y)}{y^2} \left[ 1 + \frac{3\beta_1^2}{2^2 dy} (M(y) - 1) + \frac{3\beta_1^2}{2^2 (dy)} \right]. \]

(36)

Consequently, we obtain for the differential number density of accelerated particles

\[ N_0(p \geq p_0) = 4\pi p^2 F(p \geq p_0) = \frac{4\pi S_0 \psi p_0^2 e^{-\psi [p(p,p_0)]}}{U_2 \Gamma_2} \frac{e^{\psi [p(p,p_0)]}}{T(p)} \]

(37)

at the position of the shock \( z = 0 \), and the up- and downstream number densities

\[ N_1(z > 0, p) = N_0(p \geq p_0) \exp \left[ -\frac{U_1 z}{\Gamma_1 p^2 c z} \right] \]

(38)

and

\[ N_2(z < 0, p) = N_0(p \geq p_0) \frac{M(p) - e^{-\frac{G(p)z}{c_0}}}{M(p) - 1}. \]

(39)

4. RELATIVISTIC COSMIC RAY INJECTION

Assuming that cosmic ray particles are injected at relativistic momenta \( p_0 \gg mc \), the free-escape functions (18) are

\[ M(p) = e^{G(p)}, \quad G(p) = G_m \left( \frac{p}{p_m} \right)^{1 - 2}, \]

(40)

so that the function (25) becomes

\[ T(p) \simeq \frac{1 + 3\beta_1^2 + 3\beta_1^2}{e^{\frac{G_m}{p_m}^{1 - 2}} - 1}. \]

(41)

Likewise, the integral (36) is given by

\[ I(p, p_0) \simeq \int_{p_0}^p dy \frac{e^{G_m \left( \frac{p}{p_m} \right)^{1 - 2}}}{y^2} \left[ 1 + 3\beta_1^2 + 3\beta_1^2 \right] \]

\[ = \frac{1}{(2 - s) (1 + 3\beta_1^2)} \int_{p_0}^p dy \frac{e^{G_m \left( \frac{p}{p_m} \right)^{1 - 2}}}{y^2} \frac{du}{u} J(u), \]

(42)

where we substituted \( u = G_m \left( \frac{p}{p_m} \right)^{1 - 2} \), and where we introduce the function

\[ J(u) = \frac{e^u - b}{e^u - b} = \frac{1 - be^{-u}}{1 - be^{-u}}. \]

\[ b = \frac{1 + 3(\beta_1^2 - \beta_1^2)}{1 + 3\beta_1^2} = 1 - \frac{3\beta_1^2}{1 + 3\beta_1^2} \]

\[ = 1 - \frac{3\beta_1^2}{r^2 (1 + 3\beta_1^2)} \in (0, 1). \]

(43)

We note that \( b = 11/12 = 0.917 \) for an ultrarelativistic shock \( (\beta_1 = 1) \) and a relativistic downstream medium with adiabatic index 4/3 so that \( r = 3 \) (Blandford & McKee 1976).

As \( b \) is always smaller then unity, the function \( J(u) \) has no singularity because \( e^u > 1 \). For small and large arguments the function \( J(u) \) approaches

\[ J(u) \simeq \begin{cases} \frac{1}{1 - b} & \text{for } u \ll 1 \\ 1 & \text{for } u \gg 1. \end{cases} \]

(44)
which suggests the approximation

\[ J(u) \approx J_0(u) = \frac{1 + u}{1 - b + u} = 1 + \frac{b}{1 - b + u} \tag{45} \]

with the same approximative behavior (44). In Figure 1 we calculate the ratio \( J(u)/J_0(u) \) of the function \( J(u) \) and its approximation \( J_0(u) \) as a function of \( u \) for three different values of \( b = 0.5, 0.7, 0.9 \), which is larger than 0.78 for all values of \( u \), so that the approximation is accurate within 22%.

Using the approximation (45) we obtain for the integral (42)

\[
I(p, p_0) \approx \frac{\ln \left( \frac{p}{p_0} \right)}{1 + 3\beta^2_1} + \frac{b}{(2 - s)(1 + 3\beta^2_1)} \\
\times \int_{G(p_0)}^{G(p)} \frac{du}{u(1 - b + u)} \\
= \frac{1}{1 + 3\beta^2_1} \left( \ln \left( \frac{p}{p_0} \right) + \frac{b}{(1 - b)(2 - s)} \right) \\
\times \int_{G(p_0)}^{G(p)} \frac{1 + 3(\beta^2_1 - \beta^2_s) \ln \left( \frac{p}{p_0} \right) - 3(2 - s)\beta^2_s \ln \left( \frac{p}{p_0} \right) + 1 + \frac{3\beta^2_s}{(1 + 3\beta^2_s)G(p_0)} \right) \right) \tag{46}
\]

where we have introduced \( b \) from Equation (43). Consequently, the differential number density of accelerated particles at the shock (37) becomes

\[
N_0(p \geq p_0 \gg mc) \approx \frac{4\pi S_0 \psi p_0}{U_2 \Gamma_2 \left[ 1 + 3\beta^2_1 + \frac{3\beta^2_s}{e^{\psi p_0 / \beta_1} - 1} \right]} \\
\times \left[ \frac{p}{p_0} \right]^{\frac{\psi}{1 + 3\beta^2_1}} \\
\times \left[ 1 + \frac{3\beta^2_s}{(1 + 3\beta^2_s)G(p_0)} \right]^{\psi - 2} \left[ 1 + \frac{3\beta^2_s}{(1 + 3\beta^2_s)G(p)} \right] \tag{47}
\]

We first discuss the limiting cases of the general solution (47).

### 4.1. Efficient Acceleration at All Momenta

According to our earlier discussion the acceleration is very efficient provided \( G(p_0) \gg G(p) > G_m \gg 1 \). In this case the last term in Equation (47) approaches unity, so that with \( \exp(G(p)) \approx 1 \) we obtain the power law

\[
N_0(p \geq p_0 \gg mc) \approx \frac{4\pi S_0 \psi p_0 G(p)}{U_2 \Gamma_2 \beta^2_1 \left[ 1 + 3\beta^2_1 \right] \left( \frac{p}{p_0} \right)^{-\xi}} \tag{48}
\]

with the spectral index

\[
\xi = 1 + \frac{\psi}{1 + 3\beta^2_1} = 1 + \frac{3}{(1 + \sqrt{r^2 - \beta^2_1} - 1)(1 + 3\beta^2_1)} \tag{49}
\]

where we use Equation (24). Equations (48) and (49) reproduce exactly Equations (I-90) and (I-91) in the case of symmetric Fokker–Planck coefficients \( R = 0 \). Due to the dominating Lorentz factor \( \Gamma_1 \gg 1 \) in the case of relativistic shock speeds, the spectral index (49) approaches \( \xi (\Gamma_1 \gg 1) \to 4 \). We also emphasize that in the case of efficient acceleration at all momenta, i.e., Peclét numbers \( G(p) \gg 1 \), the momentum spectrum of accelerated particles at the shock (48) is independent of the properties of the Alfvénic turbulence, especially the value of the turbulence spectral index \( s \).

In the opposite limit of nonrelativistic shock speeds \( \beta_1 \ll 1 \), corresponding to \( \Gamma_1 \approx 1 \), but still for particle injection at relativistic momenta \( p_0 \gg mc \), the spectral index (49) reduces to the classical result

\[
\xi (\beta_1 \ll 1) \approx 1 + \frac{3}{r - 1} \approx \frac{r + 2}{r - 1}, \tag{50}
\]

which is solely determined by the flow compression ratio \( r \).

### 4.2. Inefficient Acceleration at All Momenta

This limiting case applies if \( G(p) \ll G(p_0) \ll 1 \), so that we approximate \( \exp(G(p)) \ll 1 \approx G(p) \). Moreover, in this limit the last term in Equation (47) becomes

\[
\left[ 1 + \frac{3\beta^2_s}{(1 + 3\beta^2_s)G(p_0)} \right]^{\psi - 2} \left[ 1 + \frac{3\beta^2_s}{(1 + 3\beta^2_s)G(p)} \right]^{\psi \frac{1}{1 + 3\beta^2}} \approx \left[ \frac{G(p)}{G(p_0)} \right]^{\psi - 1} \left[ 1 + \frac{3\beta^2_s}{1 + 3\beta^2} \right] \tag{51}
\]

We then find for the spectrum (47)

\[
N_0(p \geq p_0 \gg mc) \approx \frac{4\pi S_0 \psi p_0 G(p)}{3U_2 \beta^2_1} \left[ \frac{p}{p_0} \right]^{\psi \frac{1}{1 + 3\beta^2}} \tag{52}
\]

with the greater power-law spectral index

\[
\xi_0 = 3 - s + \frac{\psi}{3\beta^2_1} = 3 - s + \frac{r^2}{\beta^2_1 \Gamma_1 \sqrt{r^2 - \beta^2_1} - 1} \tag{53}
\]

for \( r = 3 \) and \( \Gamma_1 \) appropriate for ultrarelativistic shocks. In this case, the momentum spectrum of accelerated particles at the shock (52) depends on the properties of the Alfvénic
turbulence: first, due to the proportionality \( N(p \geq p_c) \propto G_m \ll 1 \), the number of accelerated particles at the shock is severely reduced as compared to the case of efficient acceleration (48). Second, the power-law spectral index (53) depends on the value of turbulence spectral index \( s \). Due to the dominant Lorentz factor \( \Gamma_1 \gg 1 \) in the case of relativistic shock speeds, the spectral index (53) approaches \( \xi_0(\Gamma_1 \gg 1) \to 3 - s \), which is considerably greater than \( \xi(\Gamma_1 \gg 1) \to 1 \) for efficient acceleration. For the Kolmogorov spectral index \( s = 5/3 \) we find as the limiting value \( \xi_0(\Gamma_1 \gg 1) \to 4/3 \).

Figure 2 shows the spectral indices \( \xi \) and \( \xi_0 \) of relativistic particles accelerated at an ultrarelativistic shock with large (efficient) and small (inefficient) Peclet numbers for different values of \( s \) as a function of the shock Lorentz factor \( \Gamma_1 \). As can be seen, the difference between the power-law spectral indices for efficient and inefficient acceleration is

\[
\Delta \xi = \xi_0 - \xi = 2 - s + \frac{r^2 - \frac{3}{1 + 3\beta_1}}{\beta_1^2(\Gamma_1^2r^2 - \beta_1^2 - 1)}.
\]

In the limit of ultrarelativistic shocks with \( \Gamma_1 \gg 1 \) and \( r = 3 \) the spectral index steepens by

\[
\Delta \xi(\Gamma_1 \gg 1) \simeq 2 - s + \frac{r^2 - \frac{3}{4}}{\Gamma_1[\sqrt{r^2 - 1} - 1]}.
\]

which approaches \( 2 - s \) for \( \Gamma_1 \gg 2.92 \). In the opposite limit of nonrelativistic shock speeds \( \beta_1 \ll 1 \), corresponding to \( \Gamma_1 \simeq 1 \), the spectral index enhancement (54)

\[
\Delta \xi(\beta_1 \ll 1) = \xi_0 - \xi = 2 - s + \frac{r^2}{\beta_1^2(r - 1)}
\]

is very large due to the dominant \( \beta_1^2 \) dependence. This inefficiency of diffusive acceleration at nonrelativistic shocks with small Peclet numbers is well known (Ostrowski & Schlickeiser 1996).

However, compared to the case of nonrelativistic shock speeds, the spectral index enhancement for diffusive acceleration at relativistic shocks with small Peclet numbers is only very modest, and of the order \( 2 - s \) as indicated by Equation (55). This significant difference between diffusive acceleration at relativistic and nonrelativistic shocks is physically understandable. At nonrelativistic shocks the cosmic ray particles have to cross the shock wave many times in order to get accelerated to high momenta, as the momentum gain for a single crossing is proportional to \( \beta_1^2 \ll 1 \), which is very small. For small Peclet numbers, however, most particles leave the downstream medium quickly, without returning to the shock front, so that the acceleration becomes very inefficient, implying large spectral indices. In contrast, at relativistic shocks the momentum gain from the first crossing \( \beta_1/\beta \simeq O(1) \) is already of order unity, so that the particles do not have to return to the shock front several times in order to get accelerated very efficiently. The influence of a small Peclet number therefore is less severe than in the case of nonrelativistic shocks.

### 4.3. Intermediate Case \( G(p_0) \gg 1 \gg G(p_m) \)

In the intermediate case of efficient acceleration at small relativistic momenta \( p_0 \ll p \ll p_c \), but inefficient acceleration at \( p \ll p \ll p_m \), where \( p_c \) is determined by \( G(p) = 1 \), corresponding to \( p_c = p_mG_m^{-1/2} \), we obtain the broken power-law momentum spectrum of accelerated particles at the shock

\[
N_0(p \geq p_0 \gg mc) \simeq \frac{4\pi s_0^3p_0^2}{U_1\Gamma_1^2} \left[ \begin{array}{l}
\frac{1}{1 + 3\beta_1^2} \left( \frac{p}{p_0} \right)^{-\xi} \\
\frac{1}{3\beta_1^2} \left( \frac{p}{p_0} \right)^{2-s} \left( \frac{p}{p_0} \right)^{-\xi_0}
\end{array} \right]
\]

for \( p_0 \ll p \ll p_c \),

\[
\left[ \begin{array}{l}
\frac{1}{1 + 3\beta_1^2} \left( \frac{p_c}{p_0} \right)^{-\xi} \\
\frac{1}{3\beta_1^2} \left( \frac{p_c}{p_0} \right)^{2-s} \left( \frac{p_c}{p_0} \right)^{-\xi_0}
\end{array} \right]
\]

for \( p_c \ll p \ll p_m \).

### 4.4. Volume-integrated Momentum Spectrum of Shock-accelerated Particles

Cosmic relativistic shock systems are very far away from the solar system, so they can neither be observed in situ nor spatially resolved in the electromagnetic radiation products of the shock-accelerated particles. Therefore the volume-integrated momentum spectrum of shock-accelerated particles

\[
N_{\text{total}}(p) = \int_{-\infty}^{\infty} dz N(z, p),
\]

as indicated by Figure 2. Power-law spectral indices \( \xi \) (solid line) and \( \xi_0 \) of relativistic particles accelerated at an ultrarelativistic shock with large and small Peclet numbers, respectively, for three values of \( s = 1.1 \) (dotted curve), \( s = 1.5 \) (dashed curve), and \( s = 5/3 \) (dotted dashed curve) as a function of the shock Lorentz factor \( \Gamma_1 \). Note that in the case of efficient acceleration \( \xi \) is independent from the value of \( s \).
is of interest, as it determines the volume-integrated electromagnetic radiation spectra from synchrotron, inverse Compton, bremsstrahlung of the accelerated electrons and the pion-decay emission from inelastic nuclear interactions of the accelerated hadrons. With Equations (38) and (39) we readily obtain

\[ N_{\text{total}}(p \geq p_0, m \gg mc) = N_0(p \geq p_0, m \gg mc) \times \left[ \frac{\Gamma_{i} \kappa_{\text{ci},1}}{U_1} + \frac{z_0}{M - 1} \left( M - \frac{1}{G} - \frac{M(p)}{M(p) - 1} \right) \right] \]

\[ = N_0(p \geq p_0)z_0 \left[ \frac{1}{G(p)} - \frac{1}{G(p)} + \frac{M(p)}{M(p) - 1} \right] \]

(60)

where, in analogy to \( G(p) \) from Equation (18),

\[ G_1(p) = \frac{U_1 \Gamma_1}{\Gamma_1 \kappa_{\text{ci},1}}(p) = \frac{r^2}{\Gamma_1 \sqrt{r^2 - \beta_1^2}}G(p). \]

(61)

The volume-integrated spectrum (60) then is given by

\[ N_{\text{total}}(p \geq p_0, m \gg mc) \]

\[ = N_0(p \geq p_0)z_0 \left[ \frac{U_2 \Gamma_1}{U_1 \Gamma_1} - 1 + \frac{G(p)M(p)}{M(p) - 1} \right] \]

\[ = N_0(p \geq p_0)z_0 \left[ \frac{\Gamma_1 \sqrt{r^2 - \beta_1^2}}{r^2} - 1 + \frac{G(p)}{1 - e^{-G(p)}} \right] \]

(62)

4.4.1. Efficient Acceleration at All Momenta

For efficient acceleration at all momenta \( G(p) \gg 1 \) we insert Equation (48) to obtain in this case

\[ N_{\text{total}}(p \geq p_0) \sim N_0(p \geq p_0)z_0 \left[ 1 + \frac{\Gamma_1 \sqrt{r^2 - \beta_1^2}}{r^2G(p)} \right] \]

\[ = \frac{12\pi S_0 p_0 z_0}{\beta_1 \Gamma_1 c(1 + 3\beta_1^2)} \left[ 1 - \frac{1}{\Gamma_1 \sqrt{r^2 - \beta_1^2}} \right] \times \left[ 1 + \frac{\Gamma_1 \sqrt{r^2 - \beta_1^2}}{r^2G(p)} \right] \frac{p}{p_0}^{\xi} \]

(63)

The momentum-dependent factor

\[ V(p) = \frac{\Gamma_1 \sqrt{r^2 - \beta_1^2}}{r^2G(p)} = \frac{\Gamma_1 \sqrt{r^2 - \beta_1^2}}{r^2G_m(p/p_m)^{2/s}}, \]

(64)

being greater or smaller compared to unity, decides on a possible spectral flattening of the volume-integrated spectrum (64) with respect to the momentum spectrum at the shock. For nonrelativistic shocks \( V(p) = 1/rG(p) \ll 1 \), so that no spectral flattening occurs. For relativistic shocks we use the estimate (19) with \( \beta = 1 \) to obtain

\[ V(p) \approx \frac{1}{3} \left( \frac{p}{p_m} \right)^{2/s}, \]

(66)

which is smaller or greater than unity for \( p < p_t \) and \( p > p_t \), respectively, where

\[ p_t = \left( \frac{3}{\Gamma_1} \right)^{1/s} p_m \ll p_m. \]

(67)

Consequently, the volume-integrated spectrum (64) for efficient acceleration becomes

\[ N_{\text{total}}(p \geq p_0) \approx \frac{3\pi S_0 p_0 z_0}{\Gamma_1 c(1 - \frac{1}{\Gamma_1 \sqrt{r^2 - \beta_1^2}})} \left[ \frac{p}{p_0} \right]^{-\xi}, \]

(68)

where we inserted Equation (52). At all momenta this power-law spectrum is flattened compared to the spectrum at the shock (52) with a power-law spectral index

\[ \xi_0 + s - 2 = 1 + \frac{r^2}{\beta_1^2 \Gamma_1 \sqrt{r^2 - \beta_1^2} - 1} \]

\[ \approx \begin{cases} \frac{r^2}{\beta_1^2 (\Gamma_1 - 1)} & \text{for } \beta_1 \ll 1 \\ 1 + \frac{r^2}{\Gamma_1 (r^2 - \beta_1^2)} & \text{for } \Gamma_1 \end{cases} \]

(69)

close to unity for relativistic shocks but again very large for nonrelativistic shocks.

5. SUMMARY AND CONCLUSIONS

The diffusive acceleration of relativistic cosmic rays at parallel shock waves with magnetostatic turbulence and a finite size of the downstream medium is investigated. For a step-wise shock velocity profile the earlier derived steady-state diffusion–convection transport equation is solved. We adopt a symmetric pitch-angle scattering Fokker–Planck coefficient \( D_{\mu \nu}(-\mu) = D_{\mu \nu}(\mu) \), resulting from magnetostatic isospectral slab Alfvén waves with non-zero values of the magnetic and cross helicities, and assume that cosmic ray particles are injected at relativistic momenta. For ultrarelativistic shock speeds with Lorentz factor \( \Gamma_1 \gg 1 \), both the differential
momentum spectrum at the shock and the volume-integrated momentum spectrum are power-law distribution functions with steeper spectral indices as compared to the case of an infinitely extended downstream medium. However, the spectral steepening is less severe as compared to the case of nonrelativistic shocks.

As shown, the behavior of the momentum spectrum of shock-accelerated particles depends sensitively on the relativistic shock wave Peclet number \( G(p) = \tau_0(p)/\tau_c \) given by the ratio of the timescales of cosmic rays at momentum \( p \) to propagate from the shock position to the downstream boundary by diffusion \( \tau_0(p) = z_0^3/\nu_{\text{c}\gamma_{\text{L}}} \) and by convection \( \tau_c = \Gamma_1 z_0/U_2 \), respectively. Large values of \( G(p) \gg 1 \) indicate that cosmic ray particles are confined very well in the downstream region because of small enough spatial diffusion coefficients. In this case the cosmic ray momentum spectrum at the shock is identical to that found in Paper I. Apart from the modified spatial dependence of the downstream momentum spectrum, the free-escape boundary has no influence on the effectiveness of particle acceleration, still providing a flat momentum power-law spectrum of the accelerated particles.

In the opposite case of small values of \( G(p) \ll 1 \) at all momenta, the cosmic ray particles rapidly escape from the downstream region by spatial diffusion, and only a few of them return by diffusion to the shock front. As a result, the momentum spectrum at the shock becomes steeper with the greater spectral index \( \xi_0 = 3 - s + (3/18)\Gamma_1 \), where due to the dominating Lorentz factor \( \Gamma_1 \gg 1 \) in the case of relativistic shock speeds, the spectral index approaches \( \xi_0(\Gamma_1 \gg 1) \to 3 - s \), which is considerably greater than \( \xi(\Gamma_1 \gg 1) \to 1 \) for efficient acceleration. For the Kolmogorov spectral index \( s = 5/3 \) we find as limiting value \( \xi_0(\Gamma_1 \gg 1) \to 4/3 \). For spectral indices \( s = 1 + \epsilon, \epsilon \ll 1 \), approaching Bohm diffusion, we obtain \( \xi_0(\Gamma_1 \gg 1) \to 2 - \epsilon \).

In the intermediate case of large Peclet numbers at small but still relativistic momenta and small Peclet numbers at high momenta, the momentum spectrum of accelerated particles at the shock is a broken power-law spectrum with spectral index close to unity between \( p_0 \leq p \leq p_c \) and \( \xi_0 \) between \( p_c \leq p \leq p_m \), where the break momentum \( p_c = (3/2\Gamma_1)^{2/3}p_m \) is considerably smaller than the maximum cosmic ray momentum.

However, compared to the case of nonrelativistic shock speeds, the spectral index enhancement for diffusive acceleration at relativistic shocks with small Peclet numbers is only very modest, and at most of the order \( 2 - s \), while for nonrelativistic \( (\beta_1 \ll 1) \) shock speeds the enhancement is much larger and given by \( 2 - s + (r^2/(r - 1)^2) \). This significantly different behavior is physically understandable. At nonrelativistic shocks the cosmic ray particles have to cross the shock wave many times in order to get accelerated to high momenta, as the momentum gain for a single crossing is proportional to \( \beta_1/\beta \ll 1 \), which is very small. For small Peclet numbers, however, most particles leave the downstream medium quickly, without returning to the shock front, so that the acceleration becomes very inefficient, implying large spectral indices. In contrast, at relativistic shocks the momentum gain from the first crossing \( \beta_1/\beta \simeq O(1) \) is already of order unity, so that the particles do not have to return to the shock front several times in order to get accelerated very efficiently. The influence of a small Peclet number therefore is less severe than in the case of nonrelativistic shocks.

The relativistic shock wave Peclet number also influences the volume-integrated momentum spectrum. For small Peclet numbers \( G(p) \ll 1 \) at all momenta, the volume-integrated momentum spectrum flattens to a power law with a spectral index smaller by a factor \( 2 - s \) as compared to the spectrum of the shock. For large shock speeds \( \Gamma_1 \gg 1 \) this leads to a power-law spectral index close to unity, almost identical to the spectral index at the shock in the case of large Peclet numbers. For large Peclet numbers no spectral flattening occurs for nonrelativistic shocks at all momenta, but only at low cosmic ray momenta and for relativistic shocks at low momenta \( p < p_1 = (3/\Gamma_1)^{3/2}p_m \). However, in the case of relativistic shocks at greater momenta \( p > p_1 \) the spectral flattening by a factor \( 2 - s \) in the spectral index value occurs. This leads to a very small power-law spectral index of \( 1 - s \) for momenta above \( p_1 \), which can approach zero for slightly large than unity (Bohm diffusion).

In conclusion, our results demonstrate that diffusive particle acceleration at relativistic shock fronts generates very flat power-law momentum spectra, both for the momentum spectrum at the shock and the volume-integrated momentum spectrum. Even in the worse case of small shock wave Peclet numbers for all particle momenta, we obtain small spectral indices \( \xi_0 \approx 3 - s \) at the shock and \( \approx 1 \) for the volume-integrated spectrum. In the favorable case of large shock wave Peclet numbers for all particle momenta we obtain even smaller spectral indices \( \xi_0 \approx 1 \) at the shock and \( \approx 1 - s \) for the volume-integrated spectrum at large cosmic ray momenta \( p > p_1 \). Our result of flat power-law spectral indices close to unity of cosmic ray particles accelerated at ultrarelativistic shocks disagrees strongly with the earlier established canonical spectral index value \( \xi \in [2.23-2.30] \) from the semi-numerical eigenfunction and Monte Carlo simulation studies (for review see Kirk & Duffey 1999; Sironi et al. 2015), although more sophisticated Monte Carlo studies (Ostrowski & Bednarz 2002; Summerlin & Baring 2012) and particle-in-cell simulations (Spitkovsky 2008) indicated deviations from this canonical value depending on the mean magnetic field orientation in the shock and the nature of the MHD turbulence. It remains a challenge for future studies to unravel this significant disagreement between the spectral index values from our analytical approach and the earlier semi-numerical and numerical studies.

Our predictions on the spectral behavior of the volume-integrated momentum spectrum of accelerated particles may help to diagnose relativistic shock acceleration systems in distant space with spatially unresolved observations of the nonthermal radiation generated by the accelerated particles. The high-energy gamma-ray emission from extragalactic blazars observed by the Fermi-LAT (Abdo et al. 2009) and air Cherenkov telescopes (Hinton & Hofmann 2009) is attributed to Doppler-boosted nonthermal emission from relativistically moving emission blobs, where most likely the radiating relativistic charged particles are accelerated at relativistic shock structures forming from the interaction of the emission blobs with the ambient medium. Similar emission models are invoked for the high-energy emission from gamma-ray bursts (Baring 2011). One-zone (i.e., volume-integrated) synchrotron-self Compton and/or external Compton radiation modeling for some blazars require very flat power-law spectral indices of the injected relativistic particles: \( \xi_0 = 1.4 \) for PKS 2155-304 (Aharonian et al. 2005), \( \xi_0 = 1.5 \) for Mrk 421 (Dermer et al. 1997), and \( \xi_0 = 1.8 \) for Mrk 501
Such small values of $\xi_0$ are at odds with the canonical value $\xi_0 \in [2.23, 2.30]$, but in agreement with our analytical small spectral index values.

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REFERENCES

Abdo, A. A., Ackermann, M., Ajello, M., et al. 2009, ApJ, 707, 1310
Aharonian, F., Akhperjanian, A. G., Bazer-Bachi, A. R., et al. 2005, A&A, 442, 895
Baring, M. 2011, AdSpR, 47, 1427
Blandford, R., & McKee, C. F. 1976, PhFl, 19, 1130
Dermer, C. D., Sturmer, S. J., & Schlickeiser, R. 1997, ApJS, 109, 103
Ellison, D. C., Warren, D. C., & Bykov, A. M. 2016, MNRAS, 456, 3090
Hinton, J. A., & Hofmann, W. 2009, ARA&A, 47, 523
Katarzynski, K., Sol, H., & Kus, A. 2001, A&A, 367, 809
Kirk, J. G., & Duffy, P. 1999, J. Phys. G, 25, R163
Kirk, J. G., & Schneider, P. 1987, ApJ, 315, 425
Kirk, J. G., Schneider, P., & Schlickeiser 1988, ApJ, 328, 269
Lefa, E., Rieger, F. M., & Aharonian, F. A. 2011, ApJ, 740, 64
Ostrowski, M., & Bednarz, J. 2002, A&A, 394, 1141
Ostrowski, M., & Schlickeiser, R. 1996, SoPh, 167, 381
Schlickeiser, R. 2015, ApJ, 809, 124 (Paper I)
Schlickeiser, R., Campeanu, A., & Lerche, I. 1993, A&A, 276, 614
Sironi, L., Keshet, U., & Lemoine, M. 2015, SSRv, 191, 519
Spitkovsky, A. 2008, ApJL, 682, L5
Summerlin, E. J., & Baring, M. G. 2012, ApJ, 745, 63
Vainio, R., Virtanen, J., & Schlickeiser, R. 2003, A&A, 409, 821 (Erratum: A&A, 431, 71)
Webb, G. M. 1985, ApJ, 296, 319