Rigidity of Fibonacci Circle Maps with a Flat Piece and Different Critical Exponents

TANGUE NDAWA Bertuel
bertuelt@yahoo.fr
https://orcid.org/0000-0001-8995-9522
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Abstract We consider order preserving $C^3$ circle maps with a flat piece, Fibonacci rotation number, critical exponents $(\ell_1, \ell_2)$ and negative Schwarzian derivative.

This paper treat the geometry characteristic of the non-wondering (cantor (fractal)) set from a map of our class. We prove that, for $(\ell_1, \ell_2)$ in $(1, 2)^2$, the geometry of system is degenerate (double exponentially fast). As consequences, the renormalization diverges and the geometric (rigidity) class depends on the three couples $(c_u(f), c'_u(f))$, $(c_+(f), c'_+(f))$ and $(c_s(f), c'_s(f))$.

Key words: Circle map, Flat piece, Critical exponent, Geometry, Renormalization and Rigidity.

1 Introduction

We are interested at in certain class of weakly order preserving, non-injective (on an interval exactly; called flat piece) circle maps which appear naturally in the study of Cherry flows on the two dimensions torus (see [14], [19]), non-invertible circle continuous maps (see [16]) and of the dependence of the rotation interval on the parameter value for one-parameter families of circle continuous maps (see [21]). We write $S^1 = \mathbb{R}/\mathbb{Z}$ for the circle. There is the natural projection $\pi : \mathbb{R} \rightarrow S^1$. This provides a unique (up to integer translation) lift of a map $f$ of our class to a real continuous map $F$. An important characteristic of such a map is rotation number defined as follows:

$$\rho(f) := \lim_{n \to \infty} \frac{F^n(x) - x}{n} (mod 1).$$

When the rotation number $\rho(f)$ is irrational (which is more interesting in the study of dynamics, see [3] (p.19-36). Also, a map of our class is infinitely renormalizable when its rotation number is irrational, see point 1 of Remark 4.3, the denominators of the nearest rational approximants of $\rho(f)$ are defined as follows:

$$q_0 = 1, q_1 = a_0 \text{ and } q_{n+1} = a_nq_n + q_{n-1}, \quad n \geq 2$$
with
\[
\rho(f) = [a_0a_1 \cdots] := \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}} ; \quad 1 \leq a_i < \infty.
\]

We say that the rotation number \( \rho(f) \) is Fibonacci or golden mean if \( a_n = 1 \) for all \( n \in \mathbb{N} \). A circle map with a Fibonacci rotation number will simply be called a Fibonacci circle map. For technical reasons (on the asymptotic behavior of the renormalization operator) the main theorem is proved for Fibonacci circle maps with a flat piece. What prompted the title of this paper.

The purpose of this paper is to describe the rigidity (characteristic geometric) of certain weakly order preserving circle maps with a flat interval and Fibonacci rotation number. In the theory of low-dimensional dynamics, rigidity consists to study (specify) the geometry class under topological class. Being given a such map \( f \), its topological class is defined by: \( \{ h \circ f \circ h^{-1}, \, h \text{ is homeomorphism} \} \).

For circle maps with a flat interval, geometry depends on the so-called critical exponents \((\ell_1, \ell_2)\) the degrees of the singularities close to the boundary points of the flat interval; the geometry class in the symmetric case \((\ell, \ell)\) with \( \ell \in (1, 2) \) is treated in \[13\]. For a fixed irrational rotation number \( \rho \), there is exactly one topological class for circle map with a flat piece. More precisely, two continuous circle maps \( f \) and \( g \) with a flat piece and same irrational rotation number are topologically conjugate (that is, there exists a homeomorphism \( h \) such that \( h \circ f = g \circ h \), see \[14\]. \( h \) is called the topological conjugation or conjugacy between \( f \) and \( g \). Let us note that, if the choice of the topological conjugation could be arbitrary inside \( U_f \) (the flat piece of \( f \); that is, \( f(U_f) \) is a point), it still uniquely defined on \( K_f := S^1 \setminus \bigcup_{i=0}^{\infty} f^{-i}(U_f) \), the attractor of \( f \). As a consequence, the class of regularity of \( h \) on \( K_f \) is very interesting. The geometry of \( K_f \) and \( K_g \) (\( f \) and \( g \)) are closed respect to this regularity. In particular, when \( h \) is \( C^{1+\beta} \), \( \beta > 0 \) diffeomorphism, \( K_f \) and \( K_g \) have the same geometry, the geometry of two systems is rigid, it is not possible to modify it on asymptotical small scales. This result is very surprising according to the class of conjugacy. Indeed, the rigidity has been studied for circle diffeomorphisms \[24\] \[10\], critical circle homeomorphisms \[2\], \[22\], \[7\], \[1\] unimodal maps \[15\], \[4\], \[3\], circle maps with breakpoints \[11\], \[12\] for Kleinian groups \[17\]. And, in all these cases, it turned out that the conjugations are differentiable.

Before continuing to explain our results, we define our class and fix some notations.

**The class of functions:** We consider \( S^1 \) as an interval \([a, 1]\) where we identify \( a \) with 1. We consider the class \( \mathcal{L}^X \) described as follows:

We fix \( \ell_1, \ell_2 > 1 \) and denote:

- by \( \Sigma^X \) the set
  \[
  \Sigma^X = \{(x_1, x_2, x_3, x_4, s) \in \mathbb{R}^5 | x_1 < 0, \, x_3 < x_4 < 1, \, 0 < x_2, \, s < 1 \},
  \]
- for \( r \in \mathbb{N} \), by \( Diff^r([0, 1]) \) the space of \( c^r \) orientation preserving diffeomorphisms of \([0, 1]\).

The space of \( C^3 \) circle maps with a flat interval is denoted by:

\[
\mathcal{L}^X = \Sigma^X \times (Diff^3([0, 1]))^3.
\]
A point
\[ f := (x_1, x_2, x_3, x_4, s, \varphi, \varphi^l, \varphi^r) \in \mathcal{L}^X \]
is defined as follows:
\[ f : [x_1, 1] \rightarrow [x_1, 1] \text{ (by identifying } x_1 \text{ with } 1) \]
\[
f(x) = \begin{cases} 
  f_-(x) = f_1(x) = (1 - x_2)q_s \circ \varphi \left( \frac{x_1 - x}{x_1} \right) + x_2 & \text{if } x \in [x_1, 0[ \\
  f_2(x) = x_1 \left( \varphi^l \left( \frac{x_3 - x}{x_3} \right) \right) & \text{if } x \in ]0, x_3] \\
  f_3(x) = 0 & \text{if } x \in [x_3, x_4] \\
  f_4(x) = x_2 \left( \varphi^r \left( \frac{x - x_4}{1 - x_4} \right) \right) & \text{if } x \in ]x_4, 1] 
\end{cases}
\]
with
\[ q_s(x) = \frac{[(1 - s)x + s]f_2 - s f_2}{1 - s f_2}. \]

Note that the dynamics of \( f_- \) and \( f_+ \) are respectively defined by \( f^{q_0} \) and \( f^{q_1} \). These kinds of maps can be represented as follows:

![Figure 1 – A map \( f \) in \( \mathcal{L}^X \) with \( x_2 < x_3 \)](image)
The different branches of a map $f$ in $\mathcal{L}^X$ are described as follows:

Let $[a, b]$ be an interval. $A_{[a,b]} : [a, b] \rightarrow [0, 1]$ is the map defined by $x \mapsto (x - a)/(b - a)$ and $B_{[a,b]} : [a, b] \rightarrow [0, 1]$ is the map defined $x \mapsto (x - b)/(a - b)$.
**Notations.** Let $f \in \mathcal{L}^X$ with $U$ as its flat piece.

1. For every $i \in \mathbb{Z}$, instead of $f^i(U)$ we will simply write $i$. For example, $0 = U$. $|i|$ stands for the length of the interval $i$. So, $|i| = 0$ when $i > 0$. $|(i, j)|$ denotes the distance between the closest endpoints of these two intervals while $|i + (i, j)|$ stands for $|i| + |(i, j)|$.

2. Let $I$ be an interval. We write $\bar{I}$ to mean the closure of $I$.

3. The scaling ratios $\alpha_n$, $n \in \mathbb{N}$ are defined by

$$\alpha_n := \frac{|(f^{-q_n}(U), U)|}{|(f^{-q_n}(U), U)| + |f^{-q_n}(U)|} = \frac{|(-q_n, 0)|}{|(-q_n, 0)|}, \quad n \in \mathbb{N}.$$

4. For any sequence $\Gamma_n$ and for any real $d$ we have:

$$\Gamma_n^{d(l_1, l_2)} := \begin{cases} \Gamma_n^{d(l_1, l_2)} & \text{if } n \equiv 0[2] \\ \Gamma_n^{d(l_1, l_2)} & \text{if } n \equiv 1[2] \end{cases} \quad \Gamma_n^{d(l_1, l_2)} := \begin{cases} \Gamma_n^{d(l_1, l_2)} & \text{if } n \equiv 0[2] \\ \Gamma_n^{d(l_1, l_2)} & \text{if } n \equiv 1[2]. \end{cases}$$

For example, $\alpha_n^{l_1, l_2} = \alpha_n^{l_1}$. 

**Figure 3 – Branches of a map $f$ in $\mathcal{L}^X$**
5. Let $x_n$ and $y_n$ be two sequences of positive numbers. We say that $x_n$ is of the order of $y_n$ if there exists a uniform constant $k > 0$ such that, for $n$ big enough $x_n < ky_n$. We will use the notation $x_n = O(y_n)$.

**Parameters frequently used.** Let $f \in \mathcal{L}^X$ with $\ell_1, \ell_2 \in (1, 2)$ and Fibonacci rotation number.

1. The geometrical (rigidity) characteristics $(c_u(f), c_u'(f)), (c_s(f), c_s'(f))$ and $(c_+(f), c_+')(f))$ are defined in Proposition 1.2 and Lemma 4.13 Moreover,

   - $c_0'(f) = c_s'(f)$ or $c_u'(f)$, $\iota = s, u, +$. For more precision, see Proposition 4.1 or Notation 4.4.
   - $\sigma_u(f) = \min\{c_u(f), c_u'(f)\}$ and $\sigma_u(f) = \max\{c_u(f), c_u'(f)\}$.

2. For every $n \in \mathbb{N}$, $S_{i,n}, y_{i,n}, i = 1, 2, 3, 4, 5$ are defined in the subsection 4.3 respectively in (4.4) and (4.5).

3. The vectors $c_i^; i = 2, 3, 4, 5; \iota = s, u, +$ are defined in Proposition 4.8.

We consider $f \in \mathcal{L}^X$ with $\ell_1, \ell_2 \in (1, 2)$ and Fibonacci rotation number. The study of the renormalization of $f$ generates the geometrical characteristics $(c_u(f), c_u'(f)), (c_s(f), c_s'(f))$ and $(c_+(f), c_+')(f))$ of $K_f$. This study depends on the following result:

**Lemma 1.1.** Let $f$ be a $C^3$ circle map with a flat piece $U$ and Fibonacci rotation number. The scaling ratios $\alpha_\iota$ go to zero double exponentially fast when $(\ell_1, \ell_2)$ belongs to $(1, 2)^2$.

We will additionally assume in the proof of Lemma 1.1 that $f$ has a negative Schwarzian derivative. That means,

$$S f(x) := \frac{D^3 f(x)}{D f(x)} - \frac{3}{2} \left( \frac{D^2 f(x)}{D f(x)} \right)^2 < 0; \quad \forall x, \ D f(x) \neq 0.$$  

As a consequence of Lemma 1.1, the renormalization diverges with three quantitative aspects of the asymptotic divergence: an unstable part, a stable part, and a neutral part. The precise formulation is as follows:

**Proposition 1.2.** Let $(\ell_1, \ell_2) \in (1, 2)^2$. There exist $\lambda_u > 1, \lambda_s \in (0, 1)$ such that for $f$ a Fibonacci circle map with a flat piece and critical exponents $(\ell_1, \ell_2)$ the following holds. There are three geometrical invariant couples $(c_u(f), c_u'(f))$, $(c_s(f), c_s'(f))$ and $(c_+(f), c_+')(f))$ such that for all $n \in \mathbb{N}$

$$\ln \alpha_n \approx c_u^*(f) \lambda_u^n + c_u'(f) \lambda_u^n + c_+^*(f)$$

where $p_n$ is the integer part of $n$, $c_i^*(f) = c_i(f)$, $\iota = s, u, +$ if $n$ is even and $c_i^*(f) = c_i'(f)$, $\iota = s, u, +$ if $n$ is odd. That is,

$$\ln \alpha_{2p_n} \approx c_u(f) \lambda_u^{p_n} + c_s(f) \lambda_s^{p_n} + c_+^*(f)$$

and

$$\ln \alpha_{2p_n+1} \approx c_u'(f) \lambda_u^{p_n} + c_s'(f) \lambda_s^{p_n} + c_+'(f).$$

Now, we can formulate the result on rigidity as follows:
Main Theorem: Let $(\ell_1, \ell_2) \in (1, 2)^2$. There exists $\beta = \beta(\ell_1, \ell_2) \in (0, 1)$ such that if $f$ and $g$ are two circle maps with Fibonacci rotation number, critical exponents $(\ell_1, \ell_2)$ and if $h$ is the topological conjugation between $f$ and $g$ then $h$ is Hélder homeomorphism.

$h$ is a bi-lipschitz homeomorphism $\iff c^*_\alpha(f) = c^*_\alpha(g)$, $c_\alpha(f) + c'_\alpha(f) = c_\alpha(g) + c'_\alpha(g)$.

$h$ is a $C^{1+\beta}$ diffeomorphism $\iff c^*_\alpha(f) = c^*_\alpha(g)$, $c_\alpha(f) + c'_\alpha(f) = c_\alpha(g) + c'_\alpha(g)$,

$c^*_\alpha(f) = c^*_\alpha(g)$.

The rigidity is now completely described. This paper is organized as follows: in §2 and §3 we introduce the basic concepts and well-known results used in this paper. §4 is devoted to the renormalization under Lemma 1.1 the scaling ratios $\alpha_n$ go to zero double exponentially fast when $(\ell_1, \ell_2)$ belongs to $(1, 2)^2$. The asymptotic study of $\alpha_n$ does not use the main ideas of this paper thus will be postponed until the last section, §6. The main theorem is proved in §5.

2 Technical tools

Remark 2.1.

1. The sequences used in this paper are $((\ell_1, \ell_2), (\ell_1, \ell_2))$ equivalent on the parity in the following sense: Let $\eta_n$ be a sequence. $\eta_{2n}$ dependents on $(\ell_1, \ell_2)$ by a function $\Psi_{2n}$ if only if $\eta_{2n+1}$ dependents on $(\ell_2, \ell_1)$ by the same function. That is,

$$\eta_{2n} = \Psi_{2n}(\ell_1, \ell_2) \iff \eta_{2n+1} = \Psi_{2n+1}(\ell_2, \ell_1).$$

As a consequence, a statement or proof presented for $n$ even deduces by himself the case $n$ odd and vice-versa.

2. The renormalization operator $\mathcal{R}$ maps $(\ell_1, \ell_2)$ to $(\ell_2, \ell_1)$. Namely, for a fixed $f \in \mathcal{L}^X$ with critical exponents $(\ell_1, \ell_2)$, if $\mathcal{R}f$ exists, then it belongs to $\mathcal{L}^X$ with critical exponents $(\ell_2, \ell_1)$. Thus, because the asymptotic study of the operator $\mathcal{R}$ on $f$ is equivalent to that of $\mathcal{R}^2$ on $(f, \mathcal{R}(f))$, then we are interested at the operator $\mathcal{R}^2$ which preserves the critical exponents.

Fact 2.2. Let $f \in \mathcal{L}^X$ with $U$ as its flat piece. Let $l(U)$ and $r(U)$ be the left and right endpoints of $U$ respectively. There are a left-sided neighborhood $l^1$ of $l_1(U)$, a right-sided neighborhood $l^2$ of $l_2(U)$ and three positive constants $K_1, K_2, K_3$ such that the following holds:

1. Let $y \in l^1; i = 1, 2$. Then

$$K_1|l_i(U) - y|^{\ell_i} \leq |f(l_i(U)) - f(y)| \leq K_2|l_i(U) - y|^{\ell_i},$$

$$K_1|l_i(U) - y|^{\ell_i-1} \leq \frac{df}{dx}(y) \leq K_2|l_i(U) - y|^{\ell_i}.$$ 

2. Let three points with $y$ between $x$ and $z$ be arranged so that, of the three, the point $z$ is the closest to the flat piece. If the interval $(x, z)$ does intersect the flat piece $U$ (that is, $f$ is a diffeomorphism on $(x, z)$), then

$$\frac{|f(x) - f(y)|}{|f(x) - f(z)|} \leq K_3 \frac{|x - y|}{|x - z|}.$$
Remark 2.3.
1. The first part of Fact 2.2 implies that, for \(i = 1, 2\), \(f_{|U_i}(x) \approx k_i x^i\).
2. Let us note that, Fact 2.2 has also been applied in control theory, see [23].

3 Basic results

Let \(f \in \mathcal{L}^X\) with \(U\) as its flat piece.

Proposition 3.1. Let \(n \geq 1\).
- The set of “long” intervals consists of the intervals 
  \[ A_n := \{(q_{n-1} + i + 1, i + 1); 0 \leq i < q_n\}. \]
- The set of “short” intervals consists of the intervals 
  \[ B_n := \{(i + 1, i + 1 + q_n); 0 \leq i < q_n - 1\}. \]

The set \(P_n := A_n \cup B_n\) covers the circle modulo the end points and is called the \(n^{th}\) dynamical partition. The dynamic partition produced by the first \(q_{n+1} + q_n\) pre-images of \(U\) is denoted \(P^n\). It consists of 
\[ \varphi_n := \{-i; 0 \leq i \leq q_{n-1} + q_n - 1\} \]
together with the gaps between these sets. As in the case of \(P_n\) there are two kinds of gaps, “long” and “short”:
- The set of “long” intervals consists of the intervals 
  \[ A^n := \{I_i^{n-1} := (-i, -q_{n-1} - i); 0 \leq i < q_{n-1}\}. \]
- The set of “short” intervals consists of the intervals 
  \[ B^n := \{I_i^n := (-q_{n-1} - i, -i); 0 \leq i < q_{n-1}\}. \]

Proof. This comes from §1.4 in [9].

Proposition 3.2. The sequence \(|(0, q_n)|\) tends to zero uniformly and at least exponentially fast.

Proposition 3.3. If \(A\) is a pre-image of \(U\) belonging to \(P^n\) and if \(B\) is one of the gaps adjacent to \(A\), then \(|A|/|B|\) is bounded away from zero by a constant that does not depend on \(n\), \(A\) or \(B\).

The proofs of all these results can be found in [9] (proof of Proposition 1 and Proposition 2 respectively).
Cross-Ratio Inequality (CRI) The cross ratio inequality was introduced and proved by several authors, see \([21, 5, 25]\). We shall adapt the one introduced in \([21]\) and redefined in \([8]\) to our needs.

Let \(f \in \mathcal{L}^X\) with \(U\) as its flat piece and negative Schwarzian derivative. Let \(I\) and \(J\) be two intervals of finite and nonzero lengths such that \(\bar{I} \cap \bar{J} = \emptyset = (J \cup I) \cap U\) and \(J\) is on the right of \(I\). We define their cross-ratio as

\[
\text{Cr}(I, J) := \frac{|I||J|}{|[I,J]|.|(I,J)|}. 
\]

The distortion \(\mathcal{D}\text{Cr}\) of the cross-ratio by \(f\) is given by

\[
\mathcal{D}\text{Cr}(I, J; f) := \frac{\text{Cr}(f(I), f(J))}{\text{Cr}(I, J)}. 
\]

When there is no ambiguity we write \(\mathcal{D}\text{Cr}(I, J)\) instead of \(\mathcal{D}\text{Cr}(I, J; f)\).

Let \(n \in \mathbb{N}\) such that

1. \(f^n\) is a diffeomorphism on \((I, J)\);
2. Each point of the circle belongs to at most \(k\) intervals \(f^i([I, J])\).

Then,

\[
\mathcal{D}\text{Cr}(I, J; f^n) = \prod_{i=0}^{n-1} \mathcal{D}\text{Cr}(f^i(I), f^i(J); f) < 1. 
\]

Remark 3.4. Let \(f \in \mathcal{L}^X\) with \(U\) as its flat piece. Given \(n \geq 1\), and \(T\) be an interval. We have \(f^n : T \rightarrow f^n(T)\) is diffeomorphism if only if, for all \(0 \leq i \leq n - 1\), \(f^i(T) \cap U = \emptyset\).

Lemma 3.5. For all \(i = 1, \ldots, q_n - 1\), the parameter sequence

\[
\varrho_n(i) := \left|\frac{-q_n + i}{(q_n - 1 + i, -q_n + i)}\right|
\]

is bounded away from zero.

Proof. Observe that if \(0 \leq i < q_n - 2\), then \(\varrho_n(i)\) is larger than

\[
\text{Cr}\left(\left[\frac{-q_n - 2 + i}{-q_n + i}, \frac{q_n - 1 + i}{q_n + 1 + i}\right], \frac{-q_n + i}{-q_n - 1}\right)
\]

which by CRI with \(f^{q_n - 2 - i}\) is greater than

\[
\frac{|-q_n - 1|}{|(q_n, -q_n - 1)|}
\]

times a uniform constant. Moreover, the above ratio is bigger than

\[
\frac{|-q_n - 1|}{|(-q_n + 1, -q_n - 1)|}
\]

which by Proposition 3.3 go away from zero.

Now, suppose that \(q_n - 2 \leq i := q_n - 2 + j \leq q_n - 1 - 1\). Then,

\[
\varrho_n(i) \geq \frac{|-q_n - 1 + j|}{|(-q_n + 1 + j, -q_n - 1 + j)|}
\]

which by Proposition 3.3 go away from zero.

This completes the proof. \(\Box\)
A proof of the following Proposition can be found in [6] theorem:3.1 p.285.

**Proposition 3.6** (Koebe principle). Let $f \in \mathcal{L}^\infty$. For every $\zeta, \alpha > 0$ there exists a constant $\zeta(\zeta, \alpha) > 0$ such that the following holds. Let $T$ and $M \subset T$ be two intervals and let $S, D$ be the left and the right component of $T \setminus M$ and $n \in \mathbb{N}$. Suppose that:

1. $\sum_{i=0}^{n-1} f^i(T) < \zeta$,
2. $f^n : T \rightarrow f^n(T)$ is a diffeomorphism,
3. $\frac{|f^n(M)|}{|f^n(S)|} < \zeta(\zeta, \alpha)$

Then,

$$\frac{1}{\zeta(\zeta, \alpha)} \leq \frac{|A|}{|B|} \leq \frac{|f^n(A)|}{|f^n(B)|} \leq \zeta(\zeta, \alpha) \cdot \frac{|A|}{|B|}, \quad \forall A, B \text{ (intervals) } \subseteq M$$

That is,

$$\frac{1}{\zeta(\zeta, \alpha)} \cdot \frac{|A|}{|B|} \leq \frac{|f^n(A)|}{|f^n(B)|} \leq \zeta(\zeta, \alpha) \cdot \frac{|A|}{|B|}, \quad \forall A, B \text{ (intervals) } \subseteq M$$

where

$$\zeta(\zeta, \alpha) = 1 + \frac{\alpha}{\alpha} e^{C_\zeta}$$

and $C \geq 0$ only depends on $f$.

4 **Renormalization**

The key of this part is to set up for $n$ large enough a right algebraic relation between the $(n+1)$th renormalization and $n$th renormalization of the form: $R^{n+1} f = LR^n f + C_0(f)$ with $f$ a map our class with Fibonacci rotation number. The technique used requires a double change of variables on $\mathcal{S}^X$. Let us mention once again that Lemma 1.1 (the scaling ratios $\alpha_n$ go to zero double exponentially fast when $(\ell_1, \ell_2)$ belongs to $(1, 2)^2$) will be used frequently throughout this part.

4.1 **Renormalization operator**

Let $f \in \mathcal{L}^\infty$ with $[a_0, a_1, \ldots]$ as its rotation number. If $1 \leq a_0, a_1 < \infty$, then $f$ can be renormalized in the following sense: let $x_{1,1} = f_{a_1}^{-1}(x_2)/x_1$, let $h : [x_1, f_{a_1}^{-1}(x_2)) \rightarrow [x_{1,1}, 1)$ be the map $x \mapsto x/x_1$ and let $R f : [x_{1,1}, 1) \rightarrow [x_{1,1}, 1)$ be the map defined by

$$R f(x) = \begin{cases} \mathcal{R} f_-(x) = h \circ f_+ \circ h^{-1}(x) & \text{if } x \in [x_{1,1}, 0) \\ \mathcal{R} f_+(x) = h \circ f_+^n \circ f_- \circ h^{-1}(x) & \text{if } x \in (0, 1) \end{cases}$$

(4.1)

More precisely,

$$R f(x) = \begin{cases} \mathcal{R} f_-(x) = h \circ f_2 \circ h^{-1}(x) & \text{if } x \in [x_{1,1}, 0) \\ \mathcal{R} f_+(x) = h \circ f_2 \circ f_3 \circ f_4 \circ f_1 \circ h^{-1}(x) & \text{if } x \in (0, 1] \end{cases}$$

Observe that

$$R f = h \circ p R f \circ h^{-1}$$
Proof.

Let Proposition 4.1. \( f \in \mathcal{L} \) be a normalization operator, \( pR \) is the first return (Poincaré) map of \( f \) to the interval \([x_1, f_4^{a_1}(x_2)]\) and \( Rf \) is called the first renormalization of \( f \).

By \( \mathcal{L}_0^X \) we denote the subset of \( \mathcal{L}^X \) consisting of maps renormalizable. Before explaining more precisely \( Rf \), we introduce the zoom map on an interval \([a, b] \). \( Z_{[a, b]} : Diff([0, 1]) \to Diff([0, 1]) \) is the map \( \psi \mapsto A_{[\psi(a), \psi(b)]} \circ \psi \circ A_{[a, b]} \) where \( A_{[a, b]} : [a, b] \to [0, 1] \) is the map \( x \mapsto (x - a)/(b - a) \).

Now, we are ready to describe more precisely the function \( Rf \) for a fixed \( f \in \mathcal{L}_0^X \).

**Proposition 4.1.** Let \( f \in \mathcal{L}_0^X \) with \([a_0, a_1, \ldots] \) as its rotation number. If \( 1 \leq a_0, a_1 \leq \infty \), then \( Rf \) is a map of \( \mathcal{L} \). More precisely,

1. \( \rho(Rf) = [a_1, a_2, \ldots] \),
2. \( Rf := (x_1, x_2, x_3, x_4, s_1, \varphi_1, \varphi'_1, \varphi''_1) \) with

   \[
   x_{1,1} = \frac{f_4^{a_1-1}(x_2)}{x_1},
   \]

   \[
   x_{2,1} = \left( \varphi' \left( \frac{x_3 - f_4^{a_1-1}(x_2)}{x_3} \right) \right)^{t_1},
   \]

   \[
   x_{3,1} = 1 - \varphi^{-1} \circ q_1^{-1} \left( \frac{f_4^{-a_1+1}(x_4) - x_2}{1 - x_2} \right),
   \]

   \[
   x_{4,1} = 1 - \varphi^{-1} \circ q_1^{-1} \left( \frac{f_4^{-a_1+1}(x_3) - x_2}{1 - x_2} \right),
   \]

   \[
   s_1 = \varphi' \left( \frac{x_3 - f_4^{a_1-1}(x_2)}{x_3} \right),
   \]

   \[
   \varphi_1 = Z_{[1-f_4^{a_1-1}(x_2)/x_3, 1]}(\varphi'),
   \]

   \[
   \varphi'_1 = Z_{(x_2-x_1)/1-x_4, 1}(\varphi'')^{a_1-1} \circ Z_{[1-x_3, 1]}(q_1 \circ \varphi),
   \]

   \[
   \varphi''_1 = Z_{[0, x_3 - f_4^{a_1-1}(x_2)]} \left( \varphi' \circ Z_{x_2 - x_4}^{a_1-1} \circ Z_{[0, 1-x_4]}(q_1 \circ \varphi) \right).
   \]

**Proof.**

1. The point 1 is a general result on the renormalization of circle maps.
2. The graph of \( pR \) is described in the quadrant in blue dashes of the following figure.
Figure 4 – First return map of a map $f$ in $\mathcal{L}^X$

By combining (4.1), the descriptions of branches of $f$ (see, Figure 3) and the graph of $pR$ (see, Figure 4) the point 2 follows.

**Definition 4.2.** Let $f \in \mathcal{L}^X$. $f$ is infinitely renormalizable if for every $n \in \mathbb{N}$, $R^nf \in \mathcal{L}_0^X$. The class of infinitely renormalizable functions will be denoted by $W^X$.

The following observation explain more the $n$th renormalization, for every $n$ in $\mathbb{N}$.

**Remark 4.3.** Let $f$ be a map in $W^X$ with rotation number $\rho(f) = [a_0, a_1, \ldots]$ and critical exponents $(\ell_1, \ell_2)$. For every $n \in \mathbb{N}$, we have:

1. $R^nf$ belongs to $W^X$ with $\rho(R^nf) = [a_n, a_{n+1}, \ldots]$. As a consequence, $W^X$ corresponds to the subset of $\mathcal{L}^X$ with irrational rotation number.

2. The critical exponents of $R^nf$ is $(\ell_1, \ell_2)$ if $n$ is even and $(\ell_2, \ell_1)$ if $n$ is odd. For example, if $n$ is even,

$$R^n f := (x_{1,n}, x_{2,n}, x_{3,n}, x_{4,n}, s_n, \varphi_n^i, \varphi_n^j, \varphi_n^r) \in W^X$$

and

$$R^n f(x) = \begin{cases} 
R^n f_-(x) & \text{if } x \in [x_{1,n}, 0) \\
R^n f_+(x) & \text{if } x \in (0, 1]
\end{cases}$$
We claim that 
\[ \mathcal{R}^n f_-(x) = f_{1,n}(x) \]
\[ = (1 - x_{2,n})q_{n,n} \circ \varphi_n \left( \frac{x_{1,n} - x}{x_{1,n}} \right) + x_{2,n} \text{ if } x \in [x_{1,n}, 0) \]  (4.2) 
and 
\[ \mathcal{R}^n f_+(x) = \begin{cases} 
  f_{2,n} = x_{1,n} \left( \varphi_n \left( \frac{x_{3,n} - x}{x_{3,n}} \right) \right)^{\ell_1} & \text{if } x \in [0, x_{3,n}] \\
  f_{3,n} = 0 & \text{if } x \in [x_{3,n}, x_{4,n}] \\
  f_{4,n} = x_{2,n} \left( \varphi_n^{\prime} \left( \frac{x - x_{4,n}}{1 - x_{4,n}} \right) \right)^{\ell_2} & \text{if } x \in [x_{4,n}, 1]
\end{cases} \]

3. The dynamic of \( \mathcal{R}^n f \) is controlled by the couple \((q_{n-1}, q_n)\) in the following sens: \( \mathcal{R}^n f_{[x_{1,n}, 0)} \) and \( \mathcal{R}^n f_{(0,1]} \) are respectively a rescaled version of \( f^{q_{n-1}} \) and \( f^{q_n} \). Namely, the quadruplet \((x_{1,n}, x_{2,n}, x_{3,n}, x_{4,n})\) corresponds to \((q_{n}+1, q_{n+1}+1, r(-q_{n}+1), l(-q_{n}+1))\) if \( n \) is even and corresponds to \((q_{n}+1, q_{n+1}+1, l(-q_{n}+1), r(-q_{n}+1))\) if \( n \) is odd.

### 4.2 Asymptotic Distortions

This part is devoted to prove the following result.

**Proposition 4.4.** Let \( f \in \mathcal{W}^X \). For every \( n \),

\[ \text{dist}(\varphi_{n}^j) = O \left( \alpha_{n-1}^{\frac{1}{n}} \right), \quad \text{dist}(\varphi_n) = O \left( \alpha_{n-1}^{\frac{1}{n}} \right), \quad \text{dist}(\varphi_{n}^{\prime}) = O \left( \alpha_{n-1}^{\frac{1}{n}} \right) \]

**Proof.** It is based on the Koebe principle (Proposition 3.6).

**Distortion of \( \varphi_{n}^j \):**
- \( T = [-q_{n+1} + 1 + (a_n - 1)q_n, -q_n + 1] \),
- \( M = \frac{1}{1 - q_n} \),
- \( S = [-q_{n+1} + 1 + (a_n - 1)q_n, 1] \),
- \( D = \frac{-q_n}{1} \).

Observe that
\[ \varphi_{n}^j = Z_M f^{q_{n-1}}. \]

We claim that
1. The members of the family \( \{ f^i(T), 0 \leq i \leq q_n - 2 \} \) are pairwise disjoint,
2. \( f^{q_{n-1}} : T \longrightarrow f^{q_{n-1}}(T) \) is diffeomorphism,
3. \( \left| f^{q_{n-1}}(M) \right| \left| f^{q_{n-1}}(M) \right| = O \left( \alpha_{n-1}^{\frac{1}{n}} \right) \).

Points 1 and 2 come respectively from Proposition 3.1 and Remark 3.4. For point 3 we observe that by Proposition 3.3 and point 2 of Fact 2.2.

\[ \left| f^{q_{n-1}}(M) \right| \left| f^{q_{n-1}}(S) \right| \leq \frac{1}{K} \left| \frac{(-q_{n-1} + 1, 1)}{(-q_{n-1} + 1, q_n + 1)} \right| < \frac{1}{K} \left( \frac{-q_{n-1} + 1, 1}{-q_{n-1} + 1, 1} \right) \leq \frac{1}{K} \alpha_{n-1}. \]
Therefore,
\[
\frac{|f^{q_n-1}(M)|}{|f^{q_n-1}(S)|} = O\left(\frac{1}{n^{\frac{1}{2}}}\right).
\]
Thus, since
\[
\frac{|f^{q_n-1}(M)|}{|f^{q_n-1}(D)|} < \frac{|f^{q_n-1}(M)|}{|f^{q_n-1}(S)|}
\]
then by Proposition 3.6 we get the desired distortion estimate for \( \varphi_n \).

**Distortion of \( \varphi_n \):**
- \( T = [-q_{n-1} + 1, -q_n + (a_{n-1} - 1)q_n - 1 + 1] \),
- \( M = (q_n + (a_{n-1} - 1)q_n - 1 + 1, 1) \),
- \( S = [-q_{n-1} + 1, q_n + (a_{n-1} - 1)q_n - 1 + 1] \),
- \( D = (1, -q_n + (a_{n-1} - 1)q_n - 1 + 1) \).

Observe that
\[
\varphi_n = Z_M f^{q_n-1}.
\]
We claim that
1. The members of the family \( \{f^i(T), 0 \leq i \leq q_n - 2\} \) are pairwise disjoint,
2. \( f^{q_n-1} : T \to f^{q_n-1}(T) \) is diffeomorphism,
3. \( \frac{|f^{q_n-1}(M)|}{|f^{q_n-1}(S)|} = O\left(\frac{1}{n^{\frac{1}{2}}}\right) \).

The point 1 and 2 come respectively of the Proposition 3.1 and the Remark 3.4. For the point 3 observe that by the Proposition 3.3 and the point 3 of the Fact 2.2
\[
\frac{|f^{q_n-1}(M)|}{|f^{q_n-1}(D)|} = \frac{|(q_{n+1} + 1, q_n - 1 + 1)|}{|q_n + 1, -q_{n-2} + 1|} < \frac{1}{K} \frac{|(1, -q_{n-2} + 1)|}{|(1, -q_{n-2} + 1)|} < \alpha_{n-2} \frac{1}{K}.
\]
Therefore,
\[
\frac{|f^{q_n-1}(M)|}{|f^{q_n-1}(D)|} = O\left(\frac{1}{n^{\frac{1}{2}}}\right).
\]
Thus, since
\[
\frac{|f^{q_n-1}(M)|}{|f^{q_n-1}(D)|} < \frac{|f^{q_n-1}(M)|}{|f^{q_n-1}(D)|},
\]
then by the Proposition 3.6 we get the desired distortion estimate for \( \varphi_n \).

The Distortion of \( \varphi_n' \) are obtained by doing similar calculations with:

**Distortion of \( \varphi_n' \):**
- \( T = [-q_0 + 1, -q_n + 1] \),
- \( M = (q_0 + 1, q_0 + 1 + 1) \),
- \( S = q_0 + 1 \),
- \( D = ((a_{n-1} - 1)q_n + q_n - 1 + 1, -q_0 + 1) \),

Observe that
\[
\varphi_n = Z_M f^{q_n}.
\]

Proposition 4.4 shows that \( \varphi_n, \varphi_n \) and \( \varphi_n' \) go to \( Id_{[0,1]} \) when \( n \) goes to infinity. As a consequence, the asymptotic behavior of \( R^nf \) depends only on \( \Sigma_R \).
Fibonacci circle maps with a flat piece. Let \( f \in \mathcal{W}_X \) with rotation number \( \rho(f) = [a_0, a_1, \ldots] \). Let us recall that for every \( n \in \mathbb{N} \),

\[
R^nf := (x_{1,n}, x_{2,n}, x_{3,n}, x_{4,n}, s_n, \varphi_n, \varphi'_n, \varphi''_n) \in \mathcal{W}_X.
\]

By (4.1), Figure 1 and Figure 2, it is easy to prove that

\[
 x_{2,n} < x_{3,n} \iff a_n = 1.
\]

In the rest of the paper we consider \( \mathcal{W}_X^{[1]} \) the subclass of \( \mathcal{W}_X \) consisting of maps with Fibonacci rotation number.

4.3 The Asymptotic of Renormalization

The set \( \Sigma_X \) can be redefined as follows:

changes of variables \((X) \rightarrow (S)\). Let

\[
\begin{align*}
S_{1,n} &= x_{3,n} - x_{2,n}, \quad S_{2,n} = \frac{1 - x_{4,n}}{1 - x_{2,n}}, \\
S_{3,n} &= \frac{x_{3,n}}{1 - x_{4,n}}, \quad S_{4,n} = -\frac{x_{2,n}}{x_{1,n}}, \quad S_{5,n} = s_n.
\end{align*}
\]

That is,

\[
\begin{align*}
x_{1,n} &= \frac{S_{3,n}(1 - S_{1,n})S_{2,n}}{(1 + S_{3,n}(1 - S_{1,n})S_{2,n})S_{4,n}} \\
x_{2,n} &= \frac{1 + S_{3,n}(1 - S_{1,n})S_{2,n}}{S_{3,n}S_{2,n}} \\
x_{3,n} &= \frac{1 + S_{3,n}(1 - S_{1,n})S_{2,n}}{S_{2,n}} \\
x_{4,n} &= 1 - \frac{S_{2,n}}{1 + S_{3,n}(1 - S_{1})S_{2,n}}
\end{align*}
\]

\((S) \rightarrow (Y)\). Let

\[
y_{1,n} = S_{1,n}, \quad y_{2,n} = \ln S_{2,n}, \quad y_{3,n} = \ln S_{3,n}, \quad y_{4,n} = \ln S_{4,n}, \quad y_{5,n} = \ln S_{5,n}.
\]

Notation 4.5. Considering the respective changes of variables \((X) \rightarrow (S)\) and \((S) \rightarrow (Y)\), the space \( \mathcal{L}^X \) becomes successively \( \mathcal{L}^S \) and \( \mathcal{L}^Y \). It will be clear which parametrization of our space we are using. The space will then be simply denoted by \( \mathcal{L} \) instead of \( \mathcal{L}^X, \mathcal{L}^S \) or \( \mathcal{L}^Y \). Similarly, we will denote by \( \mathcal{W}^{[1]} \) the class of circle maps with a flat piece and Fibonacci rotation number.

Proposition 4.6. Let \( f \in \mathcal{W}^{[1]} \). For every \( n \in 2\mathbb{N} \),

\[
R(S_{1,n}, S_{2,n}, S_{3,n}, S_{4,n}, S_{5,n}, \varphi_n, \varphi'_n, \varphi''_n)
\]

corresponds to

\[
(S_{1,n+1}, S_{2,n+1}, S_{3,n+1}, S_{4,n+1}, S_{5,n+1}, \varphi_{n+1}, \varphi'_{n+1}, \varphi''_{n+1})
\]
Proof. It follows from equalities in $y$ depends on $\alpha_0$ to zero. As consequence, the asymptotic behavior of renormalization ultimately follows.

Observe that, 

Lemma 4.7.

Let $\phi \in W^l$ with critical exponents $(\ell_1, \ell_2)$, there exists $c_u(f), c'_u(f) < 0, c_s(f), c'_s(f), c_+(f)$ and $c'_+(f)$ such that for all $n := 2p_n \in \mathbb{N}^*$,

$$w_n(f) = c_u(f)\lambda_u^{n} E^u + c_s(f)\lambda_s^{n} E^s + c_+(f)E^+ + w_{fix} + O(c_u(f), \lambda_u, n)$$

and

$$w_{n+1}(f) = c'_u(f)\lambda_u^{n} E^u + c'_s(f)\lambda_s^{n} E^s + c'_+(f)E^+ + w_{fix} + O(c'_u(f), \lambda_u, n)$$

where $O(c, c_u(f), \lambda_u, n)$ is the vector whose components are

$$O \left( \left( ce(f)\lambda_u^{n-4} \right)^{1/2} \right)$$

$\square$

Lemma 4.7. $S_{1,n} = \frac{y_{1,n} - y_{2,n}}{x_{3,n}} = O(\alpha_{n+1})$.

Proof. Observe that,

$$S_{1,n} = \frac{|-q_{n+1} + 1, q_{n+1} + 1|}{|q_{n} + 1, 1|} = O \left( \frac{|1, q_{n+2} + 1|}{|1, q_{n} + 1|} \right) = O(\alpha_{n+1})$$

where we use Proposition 3.6 and Proposition 3.3.

Because $\alpha_n$ goes to zero, then by Lemma 4.7 it follows that $y_{1,n} = S_{1,n}$ goes to zero. As consequence, the asymptotic behavior of renormalization ultimately depends on $y_{2,n}, y_{3,n}, y_{4,n}$ and $y_{5,n}$. Let

$$w_n(f) := \begin{pmatrix} y_{2,n} \\ y_{3,n} \\ y_{4,n} \\ y_{5,n} \end{pmatrix}.$$

We have the following result.

Proposition 4.8. Let $(\ell_1, \ell_2) \in (1, 2)^2$. Then there exists $\lambda_u > 1$, $\lambda_s \in (0, 1)$, $E^u$, $E^s$, $E^+$, $w_{fix} \in \mathbb{R}^4$ such that the following holds. Given $f \in W^l[1]$ with critical exponents $(\ell_1, \ell_2)$, there exists $c_u(f), c'_u(f) < 0, c_s(f), c'_s(f), c_+(f)$ and $c'_+(f)$ such that for all $n := 2p_n \in \mathbb{N}^*$,

$$w_n(f) = c_u(f)\lambda_u^{n} E^u + c_s(f)\lambda_s^{n} E^s + c_+(f)E^+ + w_{fix} + O(c_u(f), \lambda_u, n)$$

and

$$w_{n+1}(f) = c'_u(f)\lambda_u^{n} E^u + c'_s(f)\lambda_s^{n} E^s + c'_+(f)E^+ + w_{fix} + O(c'_u(f), \lambda_u, n)$$

where $O(c, c_u(f), \lambda_u, n)$ is the vector whose components are
with
\[ \bar{7} := \max\{\ell_1, \ell_2\} \]

Also,
\[
\text{dist}(\varphi_n) = O\left( e^{\frac{c_{n}(f)\lambda}{\ell_2^n-2}} \right), \\
\text{dist}(\varphi_n^l) = O\left( e^{\frac{c_{n}(f)\lambda}{\ell_2^n-1}} \right), \\
\text{dist}(\varphi_n^u) = O\left( e^{\frac{c_{n}(f)\lambda}{\ell_2^n}} \right).
\]

The rest of the section will be devoted largely to prove this proposition. In particular, we are going to show that:

1. \[ \lambda_u = \frac{1}{2\ell_1\ell_2} \left( \ell_1 + \ell_2 + 1 + \sqrt{\ell_1^2 + (2 - 2\ell_2)\ell_1 + \ell_2^2 + 2\ell_2 + 1} \right), \]
2. \[ \lambda_s = \frac{1}{2\ell_1\ell_2} \left( \ell_1 + \ell_2 + 1 - \sqrt{\ell_1^2 + (2 - 2\ell_2)\ell_1 + \ell_2^2 + 2\ell_2 + 1} \right), \]
3. \[ E^+ = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}^t, \]
4. \[ E^n(\ell_2, \ell_1, \lambda_u - \lambda_s) = \begin{pmatrix} e_3^u(\ell_2, \ell_1, \lambda_u - \lambda_s) \\
                            e_3^u(\ell_2, \ell_1, \lambda_u - \lambda_s) \\
                            e_3^u(\ell_2, \ell_1, \lambda_u - \lambda_s) \\
                            e_2^u \end{pmatrix}, \]
5. \[ E^n(\ell_2, \ell_1, \lambda_u - \lambda_s) = \begin{pmatrix} e_3^u(\ell_2, \ell_1, \lambda_u - \lambda_s) \\
                            e_3^u(\ell_2, \ell_1, \lambda_u - \lambda_s) \\
                            e_3^u(\ell_2, \ell_1, \lambda_u - \lambda_s) \\
                            e_3^u(\ell_2, \ell_1, \lambda_u - \lambda_s) \end{pmatrix} \]

where
\[ e_3^u = e_3^u = 1, \]
\[ e_3^u(\ell_2, \ell_1, -\lambda) = \frac{(\ell_2\ell_1 - \ell_2 - 1)(\lambda + \ell_1 - \ell_2 + 1) + 2\ell_2}{(\ell_1 + \ell_1 + 1)\lambda + (\ell_1 + 1)^2 + (\ell_2 + 1)^2 - 1} \]
\[ = e_3^u(\ell_2, \ell_1, \lambda), \]
\[ e_3^u(\ell_2, \ell_1, \lambda) = \frac{(\ell_2^2 + \ell_2)\lambda + (2\ell_1\ell_2 + 1)(\ell_2 - 1)\ell_1\ell_2 - \ell_2(\ell_2 + 1)^2}{((\ell_2 - \ell_2^2)\ell_1^2 + \ell_1 + (\ell_1 + 1)^2)\lambda + D} \]
\[ = e_3^u(\ell_2, \ell_1, -\lambda) \]
and
\[ e_3^u(\ell_2, \ell_1, \lambda) = \ell_2\lambda_s = e_3^u(\ell_2, \ell_1, -\lambda) \]
with
\[ \lambda = \sqrt{\ell_1^2 + (2 - 2\ell_2)\ell_1 + \ell_2^2 + 2\ell_2 + 1} \]
and
\[ D = (\ell_2^2 - \ell_2 + (\ell_2^2 - \ell_2)\ell_1 + 1)\ell_1^2 - (\ell_2 + 2)\ell_1 - (\ell_2 + 1)^3. \]
4.3.1 Asymptotic affine behavior of renormalization

Let \( f \in W[1] \). We are going to show that \( R^{n+1}f = LR^n f + C_0(f) \), for every \( n \) large enough. We start with the following observation.

**Lemma 4.9.** Let \( f \in W[1] \). We have:

1. \( \frac{x_{2,n+1}}{x_{1,n}} = O(\alpha_{n+1}) \),
2. \( \frac{x_{3,n+1}}{x_{1,n}} = O(\alpha_{n+1}) \),
3. \( \frac{x_{1,n} - x_{4,n+1}}{x_{1,n}} = O(\alpha_n) \),
4. \( s_n = S_{5,n} = O(\alpha_n) \).

**Proof.** The two first points become directly of point 2 of **Remark 4.3**. Indeed,

\[
\frac{x_{2,n+1}}{x_{1,n}} = \frac{|(1, q_{n+2} + 1)|}{|1, q_n + 1|} < \frac{|(1, -q_{n+1} + 1)|}{|1, -q_{n+1} + 1|} < \alpha_{n+1}.
\]

\[
\frac{x_{3,n+1}}{x_{1,n}} = \frac{|(1, -q_{n+1} + 1)|}{|1, q_n + 1|} < \frac{|(1, -q_{n+1} + 1)|}{|1, -q_{n+1} + 1|} < \alpha_{n+1}.
\]

For point 3, observe that

\[
(-q_{n+1} + 1, q_n + 1) \subset (1, q_n + 1).
\]

Therefore,

\[
\frac{x_{1,n} - x_{4,n+1}}{x_{1,n}} = \frac{|(-q_{n+1} + 1, q_n + 1)|}{|1, q_n + 1|} = O \left( \frac{|(-q_{n+1} + 1, q_n + 1)|}{|q_n + 1, q_{n+1} + 1|} \right) = O(\alpha_n)
\]

where we use **Proposition 3.6** and **Proposition 3.3**.

By **Lemma 4.7** and **Proposition 4.4** we get

\[
s_n = S_{5,n} = S_{1,n-1} \frac{\varphi_{n}^{1}(S_{1,n-1})}{S_{1,n-1}} = O(\alpha_n) \left( 1 + O(\alpha_{n-1}) \right) = O(\alpha_n)
\]

and the lemma is shown. \( \square \)

**Proposition 4.10.** Let \( f \in W[1] \). We have:

1. \( S_{1,n+1} = 1 - \frac{\ell_2 S_{1,n}}{S_{2,n}} \left( 1 + O(\alpha_n^{1/2}) \right) \),
2. \( S_{2,n+1} = \frac{S_{1,n} S_{2,n} S_{3,n}}{\ell_2 S_{2,n}^2} \left( 1 + O(\alpha_n^{1/2}) \right) \),
3. \( S_{3,n+1} = \frac{S_{2,n}^{\ell_2}}{S_{1,n} S_{3,n}} \left( 1 + O(\alpha_n^{1/2}) \right) \),
4. \( S_{4,n+1} = \frac{S_{1,n}^{\ell_2}}{S_{4,n}} \left( 1 + O(\alpha_n^{1/2}) \right) \),
5. \( S_{5,n+1} = S_{1,n} \left( 1 + O(\alpha_n^{1/2}) \right) \).

The case \( n \) odd is just a permutation between \( \ell_1 \) and \( \ell_2 \) in the previous equalities, as mentioned in point 1 of **Remark 2.7**.
Proof. Observe that

\[ f^{q_n-1}(1, q_{n+1}) = (q_{n-1} + 1, q_n + 1). \]

On the other hand, by (4.2) we have

\[ R^n f_{(x_1, x_0)} = (1 - x_2, q_n \circ \varphi_n \left( \frac{x_{1,n} - x}{x_{1,n}} \right) + x_{2,n}. \]

Then by the value theorem there is \( \theta_1, \theta_2 \in (0, 1) \) such that,

\[ \left| \frac{(q_{n-1} + 1, q_n + 1)}{(1 - q_{n+1} + 1)} \right| = (1 - x_{2,n}) \varphi_n'(\theta_1)q_n'(\theta_2). \]  \hspace{1cm} (4.7)

Remark that \( q_n'(\theta_2) := D_q \varphi_n(\theta_2) = \ell_2 O(1) \) and note that by Proposition 4.4 \( \varphi_n'(\theta_1) = 1 + O(\alpha_n^{1/\ell_1}) \). Thus, by point 1 of Remark 4.3 and equality (4.7), we obtain

\[ S_{2,n} = \frac{1 - x_{4,n}}{1 - x_{2,n}} = \ell_2 x_{3,n+1}(1 + O(\alpha_n^{1/\ell_1})). \]

Therefore,

\[ x_{3,n+1} = \frac{S_{2,n}}{\ell_2} (1 + O(\alpha_n^{1/\ell_1})). \]  \hspace{1cm} (4.8)

By Proposition 4.1 and Proposition 4.10 we obtain

\[ x_{2,n+1} = (\varphi_n(S_{1,n}))^{\ell_1} = S_{1,n}^{\ell_1} \left( \frac{\varphi_n'(S_{1,n})}{\varphi_n(S_{1,n})} \right)^{\ell_1} = S_{1,n}^{\ell_1} (1 + O(\alpha_n^{1/\ell_1})). \]  \hspace{1cm} (4.9)

It follows from (4.8) and (4.9) that

\[ S_{1,n+1} = 1 - \frac{x_{2,n+1}}{x_{3,n+1}} = 1 - \frac{\ell_2 S_{1,n}^{\ell_1}}{S_{2,n}^{\ell_1}} (1 + O(\alpha_n^{1/\ell_1})). \]

By Proposition 4.1 we have

\[ (1 - x_{2,n}) q_n \circ \varphi_n(1 - x_{4,n+1}) = x_{3,n} - x_{2,n}. \]

That is,

\[ (1 - x_{4,n+1})q_n \circ \varphi_n(1 - x_{4,n+1}) \frac{x_{3,n} - x_{2,n}}{1 - x_{2,n}} = \frac{x_{3,n} - x_{2,n}}{1 - x_{2,n}}. \]

Thus, by point 3 of Lemma 4.9 and Proposition 4.4 we get

\[ \ell_2 S_{5,n}^{\ell_2 - 1}(1 + O(\alpha_n^{1/\ell_1}))(1 - x_{4,n+1}) = \frac{x_{3,n} - x_{2,n}}{1 - x_{2,n}} = S_{1,n} S_{2,n} S_{3,n}. \]

Therefore,

\[ 1 - x_{4,n+1} = \frac{S_{1,n} S_{2,n} S_{3,n}}{\ell_2 S_{5,n}^{\ell_2 - 1}} (1 + O(\alpha_n^{1/\ell_1})). \]  \hspace{1cm} (4.10)

By point 1 of Lemma 4.9 and equality (4.10), we have

\[ S_{2,n+1} = \frac{1 - x_{4,n+1}}{1 - x_{2,n+1}} = \frac{S_{1,n} S_{2,n} S_{3,n}}{\ell_2 S_{5,n}^{\ell_2 - 1}} (1 + O(\alpha_n^{1/\ell_1})). \]
From (4.8) and (4.10) we obtain
\[
S_{3,n+1} = \frac{x_{3,n+1}}{1 - x_{4,n+1}} = \frac{S_{5,n}}{S_{1,n}S_{3,n}}(1 + O(\alpha_{n-2}^{1/\ell_1})).
\]

By Proposition 4.1 and equality (4.9), it follows that
\[
S_{4,n+1} = \frac{x_{2,n+1}}{-x_{1,n+1}} = \frac{x_{2,n+1}}{S_{4,n}} = \frac{S_{1,n}^{\ell_1}}{S_{4,n}}(1 + O(\alpha_{n-2}^{1/\ell_2})).
\]

and finally, by Proposition 4.6 and Proposition 4.4 we have
\[
S_{5,n+1} = S_{1,n} \left( \psi_n^l(S_{1,n}) \right) = S_{1,n}(1 + O(\alpha_{n-2}^{1/\ell_1})).
\]

\[\square\]

Corollary 4.11. Let \( f \in W[1] \). Then
\[
\frac{\ell_2 S_{1,2n}^{\ell_1}}{S_{2,2n}} = 1 + O \left( \frac{1}{\alpha_{2n-1}} \right) \quad \text{and} \quad \frac{\ell_1 S_{1,2n+1}^{\ell_2}}{S_{2,2n+1}} = 1 + O \left( \frac{1}{\alpha_{2n}} \right).
\]

Proof. By point 1 of Proposition 4.10 and Lemma 4.7
\[
\frac{\ell_2 S_{1,2n}^{\ell_1}}{S_{2,2n}} = (1 - S_{1,n+1})(1 + O(\alpha_{n-2}^{1/\ell_2})) = 1 + O(\alpha_{n-1}^{1/\ell_1}).
\]

\[\square\]

By Proposition 4.10 and Corollary 4.11 we have the following result.

Proposition 4.12. Let \( f \in W[3] \). For \( n \in \mathbb{N} \) even, the following equality holds.
\[
w_{n+2} = L(\ell_1, \ell_2)w_{n+1} + w^*(\ell_1, \ell_2) + O(n, \ell, \alpha)
\]

where
\[
L(\ell_1, \ell_2) = \begin{pmatrix}
1 + \frac{1}{\ell_2} & 1 & 0 & 1 - \ell_2 \\
-\frac{1}{\ell_1} & -1 & 0 & \ell_2 - 1 \\
\frac{1}{\ell_1} & 0 & -1 & 0 \\
\frac{1}{\ell_1} & 0 & 0 & 0
\end{pmatrix},
\]

\[
w^*(\ell_1, \ell_2) = \begin{pmatrix}
-\left(1 + \frac{1}{\ell_1}\right) \log \ell_2 \\
\frac{1}{\ell_1} \log \ell_2 \\
-\frac{1}{\ell_1} \log \ell_2 \\
\end{pmatrix}
\]

and
\[
O(n, \ell, \alpha) = \left( O(\alpha_{n-2}^{1/\ell_1}), O(\alpha_{n-2}^{1/\ell_2}), O(\alpha_{n-2}^{1/\ell_2}), O(\alpha_{n-2}^{1/\ell_2}) \right)^t.
\]
As noted in the introduction, we study the operator $R^2$ on $(f, Rf)$ rather than $R$ on $f$. Thus, we define
\[ \mathcal{L} = \mathcal{L}_{(t_1,t_2)} = L_{(t_2,t_1)}L_{(t_1,t_2)} \]
and
\[ \mathcal{L}^+ = L_{(t_2,t_1)}^+L_{(t_1,t_2)}^+. \]
We have
\[ w_{n+2} = \mathcal{L}_{(t_1,t_2)}w_n + \mathcal{L}^+_{(t_1,t_2)}w_n + O(n, \ell, \alpha). \quad (4.11) \]
The next step is to explore equality (4.11) in order to conclude the proof of Proposition 4.8.

### 4.3.2 Asymptotic in Y-Coordinates

Let $w_{f_{ix}}$ be the fixed point of the equation $\mathcal{L}w + \mathcal{L}^+w_{(t_1,t_2)} = w$. We have the following result.

**Lemma 4.13.** Let $f \in W[1]$. Then,
\[ w_n(f) = c_u(f)\lambda_u^n E^u + c_s(f)\lambda^s E^s + c_+(f)E^+ + w_{f_{ix}} + O(n, \ell_1, \tilde{\ell}, \alpha_0); \]
\[ w_{n+1}(f) = c_u'(f)\lambda_u^n E^u + c_s'(f)\lambda^s E^s + c_+'(f)E^+ + w_{f_{ix}} + O(n, \ell_2, \tilde{\ell}, \alpha_0) \]
where $O(n, \ell_1, \tilde{\ell}, \alpha_0)$ resp $O(n, \ell_2, \tilde{\ell}, \alpha_0)$ is the vector whose components are equal to
\[ 0 \left( \alpha_0 \left( \frac{\pi}{\ell} \right)^{\nu} \right) \]
resp
\[ 0 \left( \alpha_0 \left( \frac{\pi}{\ell} \right)^{\nu} \right) \]  

**Proof.** Consider the sequence $(v_n)_{n \in \mathbb{N}}$ defined by $v_n = w_n - w_{f_{ix}}$. From equality (4.11) and Proposition 6.11 it follows that,
\[ v_{n+2} = \mathcal{L}v_n + \epsilon_n \]
where $\epsilon_n = O(n, \ell_1, \tilde{\ell}, \alpha_0)$. We obtain
\[ v_n = \mathcal{L}^p v_0 + \sum_{k=0}^{p-2} \mathcal{L}^{p-k-1} \epsilon_{2k}. \quad (4.12) \]

By expressing $v_0$ and $\epsilon_n$ in the eigenbasis, we have:
\[ v_0 = c_{u,0}E^u + c_{s,0}E^s + c_{+,0}E^+ + c_{0,0}E^0; \]
\[ \epsilon_n = c_{u,n}E^u + c_{s,n}E^s + \epsilon_{+,n}E^+ + c_{0,n}E^0. \]

We consider the following quantities
\[ C_u(f) = c_{u,0} + \sum_{k=0}^{\infty} \frac{c_{u,2k}}{\lambda_u^{2k}}, \]
\[ C_s(f) = c_{s,0} + \sum_{k=0}^{\infty} \frac{c_{s,2k}}{\lambda_s^{2k}}, \]
\[ C_+(f) = c_{+,0} + \sum_{k=0}^{\infty} \epsilon_{+,2k}. \]

By introducing $v_0$ and $\epsilon_n$ in (4.12) we get:
\[ v_n = \left( c_{u,0} + \sum_{k=0}^{p-2} \frac{c_{u,2k}}{\lambda_u^{2k+1}} \right) \lambda_u^{p_u} E^u + \left( c_{s,0} + \sum_{k=0}^{p-2} \frac{c_{s,2k}}{\lambda_s^{2k+1}} \right) \lambda_s^{p_s} E^s + \]
\[ + \left( c_{+,0} + \sum_{k=0}^{p_n-2} \epsilon_{+,2k} \right) E^+ + \epsilon_{0,n-2} E^0 \]
\[ = c_u(f) \lambda_u^{p_n} E_u + c_s(f) \lambda_s^{p_n} E^s + c_+(f) E^+ + \]
\[ + \left( \sum_{k=p_n}^{\infty} \frac{\epsilon_{u,n}}{\lambda_u^{k+1}} \right) \lambda_u^{p_n} E_u + \left( \sum_{k=p_n}^{\infty} \frac{\epsilon_{s,k}}{\lambda_s^{k+1}} \right) \lambda_s^{p_n} E^s + \]
\[ + \left( \sum_{k=p_n}^{\infty} \epsilon_{+,k} \right) E^+ + \epsilon_{0,n-2} E^0 \]
\[ = c_u(f) \lambda_u^{p_n} E_u + c_s(f) \lambda_s^{p_n} E^s + c_+(f) E^+ + O(n - 2, \ell_1, \bar{\ell}, \alpha_0). \]

In fact, the three sums are estimated by
\[ 0 \left( \alpha_0 \left( \frac{\pi}{2} \right)^{p_n-2} \right)^{\frac{1}{2}}. \]

To convince oneself of that, it suffices to remark that for \( k \) large enough, the following inequality holds:
\[ \left( \frac{\alpha_0 \left( \frac{\pi}{2} \right)^{k-1}}{\left( \frac{\pi}{2} \right)^k} \right)^{\frac{1}{2}} < \frac{1}{2} \lambda_u. \]

Therefore,
\[ \left| \sum_{k=p_n}^{\infty} \frac{\epsilon_{u,2k}}{\lambda_u^{k+1}} \right| \lambda_u^{p_n} = 0 \left( \sum_{k=p_n}^{\infty} \frac{\alpha_0 \left( \frac{\pi}{2} \right)^{k-1}}{\lambda_u^{k}} \lambda_u^{p_n-1} \right) \]
\[ = 0 \left( \sum_{k=p_n}^{\infty} \frac{\alpha_0 \left( \frac{\pi}{2} \right)^{p_n-2}}{\lambda_u^{p_n-1}} \lambda_u^{p_n} \right) \]
\[ = 0 \left( \frac{\alpha_0 \left( \frac{\pi}{2} \right)^{p_n-2}}{\lambda_u^{p_n}} \right)^{\frac{1}{2}}. \]

And by analogy, we have the other estimates:

**Notation 4.14.** We can write \( w_n \) (see Lemma 4.13) as follows
\[ w_n(f) = c_u^*(f) \lambda_u^{p_n} E_u + c_s^*(f) \lambda_s^{p_n} E^s + c_+^*(f) E^+ + w_{\text{fix}} + O^*(n, \ell_1, \bar{\ell}, \alpha_0) \]
with \( c_u^*(f) = c_u(f) \) if \( n \) is even and \( c_u^*(f) = c_u(f) \) if \( n \) is odd. Also, \( \epsilon_{u}^*(f) = c_u^*(f) \) when \( c_u^*(f) = c_u(f) \) and vice-versa. We consider the same convention for \( c_s^*(f) \) and \( c_+^*(f) \). \( O^*(n, \ell_1, \bar{\ell}, \alpha_0) \) is defined from Lemma 4.13.

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Lemma 4.15. Let \( f \in \mathcal{W}_{[1]} \). For every \( n := 2p_n \in \mathbb{N} \), we have
\[
\alpha_n = O\left(e^{c_u(f)\lambda_{2,n}^{(n)}(e_2^2 + e_3^2)}\right), \quad \alpha_{n+1} = O\left(e^{c_u'(f)\lambda_{2,n}^{(n)}(e_2^2 + e_3^2)}\right) \quad \text{and} \quad c_u(f), c_u'(f) < 0.
\]
**Proof.** We claim that \( 1 - x_{2,n} \) and \( x_{4,n} \) are comparable. Observe that:
\[
\frac{1 - x_{2,n}}{x_{4,n}} = \frac{x_{3,n} - x_{2,n}}{x_{4,n} - x_{3,n}} + 1 + \frac{1 - x_{4,n}}{x_{4,n} - x_{3,n}}
\]
and for \( n \) large enough
\[
\alpha_n = \frac{x_{3,n}}{x_{4,n}} \frac{x_{3,n} - x_{2,n}}{x_{4,n} - x_{3,n}} < \frac{x_{3,n}}{x_{4,n} - x_{3,n}} < 2\alpha_n.
\]
Therefore,
\[
\frac{1 - x_{2,n}}{x_{4,n}} = 0(1 + \alpha_n).
\]
On the other hand,
\[
\frac{1 - x_{2,n}}{x_{4,n}} > \frac{1}{x_{4,n} - x_{3,n}} + 1 > \frac{1}{2\alpha_n + 1}.
\]
And we have the result.
Now we have
\[
\alpha_n := \frac{x_{3,n}}{x_{4,n}} = S_{2,n}S_{3,n} \frac{1 - x_{2,n}}{x_{4,n}} = O(1)S_{2,n}S_{3,n}.
\]
That is,
\[
\alpha_n = O(S_{2,n}S_{3,n}). \tag{4.13}
\]
By combining equality (4.13) and Lemma 4.13, the lemma follows. \( \square \)

**Proof of Proposition 4.8** The first part of Proposition 4.8 comes from Proposition 6.11, Lemma 4.13 and Lemma 4.15 and the second part comes from Proposition 4.4 and Lemma 4.15.

**Corollary 4.16.** \( f \in \mathcal{W}_c \). Let us put \( \ell_{2k-1} := \ell_1 \) and \( \ell_{2k} := \ell_2; k \geq 1 \). If the sequence
\[
\Psi_{0,n} = \frac{S_{1,n}^{\ell_n}}{S_{2,n}}
\]
and the logarithm of sequences
\[
\Psi_{1,n} = \frac{S_{2,n}}{\ell_{n+1}(\varphi_n^{-1} \circ q_{s_n}(1 - S_{2,n}))} \left( \frac{\varphi_n(S_{1,n})}{S_{1,n}^{\ell_n}} \right)^{\ell_n},
\]
\[
\Psi_{2,n} = \frac{\ell_{n+1}s_{n,n}^{\ell_n-1} \varphi_n^{-1} \circ q_{s_n}^{-1}(S_{1,n}S_{2,n}S_{3,n})}{S_{1,n}S_{2,n}S_{3,n}} \cdot \frac{1}{1 - \varphi_n(S_{1,n})},
\]
\[
\Psi_{3,n} = \frac{S_{1,n}S_{2,n}(1 - \varphi_n^{-1} \circ q_{s_n}^{-1}(1 - S_{2,n}))}{S_{n,n}^{\ell_n+1} \varphi_n^{-1} \circ q_{s_n}^{-1}(S_{1,n}S_{2,n}S_{3,n})},
\]
and the logarithm of sequences
\[
\Phi_{1,n} = \frac{S_{2,n}}{\ell_{n+1}(\varphi_n^{-1} \circ q_{s_n}(1 - S_{2,n}))} \left( \frac{\varphi_n(S_{1,n})}{S_{1,n}^{\ell_n}} \right)^{\ell_n},
\]
\[
\Phi_{2,n} = \frac{\ell_{n+1}s_{n,n}^{\ell_n-1} \varphi_n^{-1} \circ q_{s_n}^{-1}(S_{1,n}S_{2,n}S_{3,n})}{S_{1,n}S_{2,n}S_{3,n}} \cdot \frac{1}{1 - \varphi_n(S_{1,n})},
\]
\[
\Phi_{3,n} = \frac{S_{1,n}S_{2,n}(1 - \varphi_n^{-1} \circ q_{s_n}^{-1}(1 - S_{2,n}))}{S_{n,n}^{\ell_n+1} \varphi_n^{-1} \circ q_{s_n}^{-1}(S_{1,n}S_{2,n}S_{3,n})},
\]
and
\[
\Psi_{4,n} = \frac{\varphi_n'(S_{1,n})}{S_{1,n}},
\]
are uniformly bounded, then

1. \( w_n(f) \) tends to infinity exponentially when \((\ell_1, \ell_2) \in [1, 2]^2 \setminus \{(2, 2)\} \),
2. \( w_n(f) \) is bounded if \( \lambda_n < 1 \).

**Proof.** From Proposition 4.6 we have

\[
S_{1,1} = 1 - \frac{\ell_2 S_{1}^{\ell_1}}{S_2}, \quad \left[ \frac{S_2}{\ell_2 S_{1}^{\ell_1}} \cdot \frac{(\varphi'(S_1))^{\ell_1}}{S_1^{\ell_1}} \right],
\]

\[
S_{2,1} = \frac{S_1 S_2 S_3}{\ell_2 S_{1}^{\ell_1}} \cdot \left[ \frac{S_2}{S_1 S_2 S_3} \cdot 1 - (\varphi'(S_1))^{\ell_1} \right],
\]

\[
S_{3,1} = \frac{S_3 S_{1} S_{2}}{S_{1} S_3} \cdot \left[ (S_3(1 - \varphi^{-1} \circ q_n^{-1}(1 - S_2))) \right],
\]

\[
S_{4,1} = \frac{S_{1}^{\ell_1}}{S_4} \cdot \left[ \frac{(\varphi'(S_1))^{\ell_1}}{S_1} \right],
\]

\[
\varphi = Z_{[S_1]}(\varphi'),
\]

\[
\varphi'_1 = \varphi' \circ Z_{[\varphi^{-1} \circ q_n^{-1}(1 - S_2)]}(q_n \circ \varphi),
\]

\[
\varphi'^1 = Z_{[\varphi^{-1} \circ q_n^{-1}(S_{1}, S_{2}, S_{3})]}(q_n \circ \varphi).
\]

and the result follows by Proposition 4.8.

### 4.4 Asymptotic in X-coordinates

**Lemma 4.17.** Let \((\ell_1, \ell_2) \in (1, 2)^2\) and \(f \in W_{[1]}\) with critical exponents \((\ell_1, \ell_2)\). Then for \(n := 2p_n\),

\[-x_{1,n} = e^c_n f(0)^{\lambda_n^0 (e_2^2 + e_3^2 - e_1^2)} + e_1(f) \lambda_n^0 (e_2^2 + e_3^2 - e_1^2) - c_1 + 0((e_1^c(\ell_1) \lambda_n^{0 - 4}(e_2^2 + e_3^2))^{1/\ell_2}),
\]

\[x_{2,n} = e^c_n f(0)^{\lambda_n^0 (e_2^2 + e_3^2)} + e_1(f) \lambda_n^0 (e_2^2 + e_3^2) + 0((e_1^c(\ell_1) \lambda_n^{0 - 4}(e_2^2 + e_3^2))^{1/\ell_2}),
\]

\[x_{3,n} = e^c_n f(0)^{\lambda_n^0 (e_2^2 + e_3^2)} + e_1(f) \lambda_n^0 (e_2^2 + e_3^2) + 0((e_1^c(\ell_1) \lambda_n^{0 - 4}(e_2^2 + e_3^2))^{1/\ell_2}),
\]

\[1 - x_{4,n} = e^c_n f(0)^{\lambda_n^0 (e_2^2 + e_3^2 + e_1)} + 0((e_1^c(\ell_1) \lambda_n^{0 - 4}(e_2^2 + e_3^2))^{1/\ell_2}).
\]

**Proof.** Recall that (4.4)

\[
\begin{align*}
\frac{x_{1,n}}{1 + \frac{S_{3,n}(1 - S_{1,n})S_{2,n}}{S_{3,n}S_{2,n}}} &= \frac{S_{3,n}(1 - S_{1,n})S_{2,n}}{S_{3,n}S_{2,n}}, \\
\frac{x_{2,n}}{1 + \frac{S_{3,n}(1 - S_{1,n})S_{2,n}}{S_{3,n}S_{2,n}}} &= \frac{S_{3,n}(1 - S_{1,n})S_{2,n}}{S_{3,n}S_{2,n}}, \\
\frac{x_{3,n}}{1 + \frac{S_{3,n}(1 - S_{1,n})S_{2,n}}{S_{3,n}S_{2,n}}} &= \frac{S_{3,n}(1 - S_{1,n})S_{2,n}}{S_{3,n}S_{2,n}}, \\
\frac{x_{4,n}}{1 - \frac{S_{3,n}(1 - S_{1,n})S_{2,n}}{S_{3,n}S_{2,n}}} &= \frac{S_{3,n}(1 - S_{1,n})S_{2,n}}{S_{3,n}S_{2,n}}.
\end{align*}
\]

Proposition 3.3 implies that \(S_{3,n}S_{2,n} = O(\alpha_n)\) and point 4 of Lemma 4.9 shows that \(S_{1,n} = O(\alpha_{n+1})\). Thus, by Proposition 4.8 the lemma follows.
5 Rigidity

Let $f, g \in \mathcal{W}[1]$ and let $h$ be a conjugacy between $f$ and $g$. Observe that $h(U_f) = U_g$, moreover the choice of $h$ inside $U_f$ could be arbitrary. However, $h|_{K_f}$ is uniquely defined. Being interested in the geometry of $K_f$, we will only study $h|_{K_f}$ that we will denote $h$ yet. The main question treated in this section is: When do $K_f$ and $K_f$ have the same geometry? In other words, for fixed $f$ in $\mathcal{W}[1]$, what is the geometry class of $f$ ($K_f$)? Before answering this question, we give bi-lipschitz class of $f$. This section is devoted to proving the following result:

**Theorem 1.** Let $(\ell_1, \ell_2) \in (1, 2)^2$ and let $\beta = \frac{\tau_u(f) \left(e_4^* + e_9^*\right)\left(\lambda_u - 1\right)}{c_u(f)} \in (0, 1)$. If $f, g \in \mathcal{W}[1]$ with critical exponents $(\ell_1, \ell_2)$ and $h$ is the topological conjugation between $f$ and $g$, then

- $h$ is Holder homeo;
- $h$ is a bi-lipschitz homeo $\iff c_u^*(f) = c_u^*(f), c_+(f) + c'_+(f) = c_+(g) + c'_+(g)$;
- $h$ is a $C^{1+\beta}$ diffeo $\iff c_u^*(g), c_+(f) + c'_+(f) = c_+(g) + c'_+(g)$,

$c_u^*(f) = c_u^*(g)$.

### 5.1 Preliminaries

**Proposition 5.1.** Let $(\ell_1, \ell_2) \in (1, 2)^2$. If $f, g \in \mathcal{W}[1]$ with different unstable eigenvalue. Then $h$ the topological conjugation between $f$ and $g$ is not Holder.

**Proof.** Observe that for even $n$,

$$\frac{f(U_f) - f^{n+1}(U_f)}{f^{n+1}(U_f) - f(U_f)} = \frac{x_{2,n}(f)}{x_{1,n}(f)} = S_{4,n}(f).$$

Thus, by Proposition 4.8 we obtain,

$$f(U_f) - f^{n+1}(U_f) = \prod_{k \leq n} S_{4,k}(f) \sim e^{c_u(f)\tau_u(f)\lambda_u(f) + \frac{1}{2}(c_+(f) + c'_+(f))}. \quad (5.1)$$

In the same way

$$g(U_g) - g^{n+1}(U_g) = \prod_{k \leq n} S_{4,k}(g) \sim e^{c_u(g)\tau_u(g)\lambda_u(g) + \frac{1}{2}(c_+(g) + c'_+(g))}. \quad (5.2)$$

Let us remember that $h(U_f) = U_g$. Therefore, $h(f(U_f)) = g(U_g)$. That is, $h(f^n(U_f)) = g^n(U_g)$, for all $n \in \mathbb{N}$. As a consequence, for all $\beta \in (0, 1)$

$$\lim_{n \to \infty} \frac{g(U_g) - g^{n+1}(U_g)}{f(U_f) - f^{n+1}(U_f)} = \begin{cases} 0 & \text{si } \lambda_u(f) < \lambda_u(g) \\ \infty & \text{si } \lambda_u(f) > \lambda_u(g) \end{cases}$$

and the proposition is shown. ☑️

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5.2 Bi-lipschitz homeomorphism Conjugacy

**Proposition 5.2.** Let $(\ell_1, \ell_2) \in (1, 2)^2$. If $f, g \in W_{[1]}$ with critical exponents $(\ell_1, \ell_2)$ and if $h$ is the topological conjugation between $f$ and $g$ then 

$h$ is a bi-lipschitz homeo $\iff c_u^*(f) = c_u^*(g), c_+(f) + c_+(f) = c_+(g) + c_+(g)$.

**Proof.** The proposition comes directly from (5.1) and (5.2). \(\square\)

In the following, we use the below notations. For all $n \in \mathbb{N}$,

- $A_n(f) = (f(U_f), f^{q_n+1}(U_f))$;
- $B_n(f) = (f^{-q_n+1}(U_f), f(U_f))$;
- $C_n(f) = f^{-q_n+1}(U_f)$;
- $D_n(f) = (f^{q_n-1}(U_f), f^{-q_n+1}(U_f))$.

And their iterates

- $A_n^i(f) = f^i(A_n(f))$ for $0 \leq i < q_{n-1}$;
- $B_n^i(f) = f^i(B_n(f))$ for $0 \leq i < q_n$;
- $C_n^i(f) = f^i(C_n(f))$ for $0 \leq i < q_n$;
- $D_n^i(f) = f^i(D_n(f))$ for $0 \leq i < q_n$.

Observe that for all $n \in \mathbb{N}$,

$\mathcal{P}_n = \{A_n^i(f), B_n^i(f), C_n^i(f), D_n^i(f) | 0 \leq i < q_{n-1}, 0 \leq j < q_n\}$.

Since $h(U_f) = U_g$, it follows that

- $h(A_n^i(f)) = A_n^i(g)$ for $0 \leq i < q_{n-1}$;
- $h(B_n^i(f)) = B_n^i(g)$ for $0 \leq i < q_n$;
- $h(C_n^i(f)) = C_n^i(g)$ for $0 \leq i < q_n$;
- $h(D_n^i(f)) = D_n^i(g)$ for $0 \leq i < q_n$.

Let $n \in \mathbb{N}, Dh_n : [0, 1] \rightarrow \mathbb{R}^+$ is the function defined by

$$Dh_n(x) = \frac{|h(f)|}{|I|}$$

with $I \in \mathcal{P}_n$ and $x \in I$. Observe that if $T = \cup I_i$ with $I_i \in \mathcal{P}_n$, then for all $m \geq n$,

$$h(T) = \int_T Dh_m. \tag{5.3}$$

**Lemma 5.3.** There is $C > 0$ such that for every interval $I \in \mathcal{P}_n$ the following inequality holds.

$$|I| \geq \frac{1}{C} e^{C \alpha_n \lambda_n^* e_n \lambda_n^*}$$

with $\xi_a(f) = \min\{c_u(f), c'_u(f)\}$.

**Proof.** Suppose that $I \neq C_n^i$. Let $J \in \mathcal{P}_{n-1}$ and $I \subset J$. By the construction of $\mathcal{P}_n$ from $\mathcal{P}_{n-1}$, it follows that

$$\frac{|I|}{|J|} \geq (1 + O(\alpha_n^{\frac{1}{n-2}})) \min\{1 - x_{n+1}, x_{n+1}, S_{1, n}\}$$

where we used the mean value theorem and **Proposition 4.4**.

Also, from **Corollary 4.11** Lemma 4.17 and **Proposition 4.8** we get:
\[ S_{1,n} \geq e^{\frac{\alpha_n}{2}} \lambda_n^\alpha e^\alpha f ) - K \lambda_n^\alpha; \]
\[ x_{3,n+1} \geq e^{\alpha_n(x^\alpha f ) + \alpha_n f ) - K \lambda_n^\alpha; \]
\[ 1 - x_{4,n+1} \geq e^{\alpha_n(x^\alpha f ) + \alpha_n f ) - K \lambda_n^\alpha. \]

Thus, from points 1, 2, 4 and 5 of Proposition 4.8 we obtain
\[
\frac{|I|}{|J|} \geq e^{\alpha_n(x^\alpha f ) + \alpha_n f ) - K \lambda_n^\alpha.}
\]

Therefore, if \( J \neq C_{n-1} \), we can repeat these estimates and we get
\[
|I| \geq e^{\alpha_n(x^\alpha f ) + \alpha_n f ) - K \lambda_n^\alpha.}
\]

Moreover, by Proposition 3.3 we have \( C_n > K^\prime B_n^\prime \) for some \( K' > 0 \). And the lemma follows.

\[ \textbf{Lemma 5.4.} \] There is \( C > 0 \) such that for every interval \( I \in \mathcal{P} \) the following inequality holds.
\[ |I| \leq \frac{1}{C} e^{\alpha_n(x^\alpha f ) + \alpha_n f ) - \lambda_n^\alpha.}
\]

with \( \tau_n(f) = \max\{c_n(f), c'_n(f)\} \).

\[ \textbf{Proof.} \] By mean value theorem and Lemma 4.4 we get

1. \( |A_n^\prime| = x_{2,n-1}(1 + O(\alpha_n^{-\frac{1}{2}})); \)
2. \( |B_n^\prime| = x_{3,n}(1 + O(\alpha_n^{-\frac{1}{2}})); \)
3. \( |D_n^\prime| = (1 - x_{4,n})(1 + O(\alpha_n^{-\frac{1}{2}})). \)

And by Lemma 4.17 there exists \( K > 0 \) such that

1. \( x_{2,n-1} \leq Ke^{\alpha_n(x^\alpha f ) + \alpha_n f ) - \lambda_n^\alpha.} \)
2. \( x_{3,n} \leq Ke^{\alpha_n(x^\alpha f ) + \alpha_n f ) - \lambda_n^\alpha.} \)
3. \( 1 - x_{4,n} \leq Ke^{\alpha_n(x^\alpha f ) + \alpha_n f ) - \lambda_n^\alpha.} \)

Since \( \alpha_n e_n^{\alpha_2} > \alpha_n e_n^{\alpha_1}(f) > e_n^{\alpha_2}(f) + e_n^{\alpha_1}(f), \) we have the lemma.

\[ \textbf{Lemma 5.5.} \] Let \( (\ell_1, \ell_2) \in (1, 2)^2 \). If \( f, g \in W_{[1]} \) with critical exponent \((\ell_1, \ell_2)\) such \( c_n^*(f) = c_n^*(g) \) and \( c_1^*(f) \) and \( c_1^*(g) + c_1^*(g) \) then
\[ \log \frac{Dh_{n+1}}{Dh_n} = O\left(\frac{1}{\alpha_n^{-\frac{1}{2}}} + \left|\alpha_n - \ell_1\right| \lambda_n^\alpha\right) \]

\[ \textbf{Proof.} \] From Lemma 4.15 it follows that, if \( c_n^*(f) = c_n^*(g) \), then \( \alpha_n(f) = \alpha_n(g) \). Thus, by Lemma 4.4 and Lemma 4.9 we get
\[ \frac{Dh_{n+1}|A_n^*(f)|}{Dh_n|A_n^*(f)|} = \frac{\left|A_n^*(g)\right|}{\left|A_n^*(g)\right|} = \frac{\left|A_n^*(g)\right|}{\left|A_n^*(g)\right|} = \frac{\left|B_n^*(g)\right|}{\left|B_n^*(g)\right|} = \frac{\left|B_n^*(g)\right|}{\left|B_n^*(g)\right|} = \frac{\left|B_n^*(g)\right|}{\left|B_n^*(g)\right|}. \]
where we used

\[
\frac{|A_{n+1}(g)|}{|B_n(g)|} = (1 + O(\alpha_{n-2}^{-1/2}))
\]
\[
\frac{|A_{n+1}(f)|}{|B_n(f)|} \times_2, n(g)
\]
\[
= x_{3,n}(g)
\]
\[
= x_{3,n}(f)
\]
\[
= 1 - S_{1,n}(g)
\]
\[
= 1 - S_{1,n}(f)
\]
\[
= (1 + O(\alpha_{n-2}^{-1/2}))
\]

Observe that for \(i \geq q_{n-1}\), \(B^{i}_{n+1}(f) = D^{i-q_{n-1}}_{n+1}(f)\). Therefore,

\[
\frac{Dh_{n+1}|B^{i+1}_{n+1}(f)}{Dh_{n}|B^{i+1}_{n+1}(f)} = 1.
\]

For \(i < q_{n-1}\), with a similar calculation as before, we have

\[
\frac{Dh_{n+1}|B^{i+1}_{n+1}(f)}{Dh_{n}|B^{i+1}_{n+1}(f)} = \frac{|B_{n+1}(g)|}{|A_n(g)|} \times_2, n(f)
\]
\[
= x_{3,n+1}(g)
\]
\[
= x_{3,n+1}(f)
\]
\[
= (1 + O(\alpha_{n-2}^{-1/2})) c_1(g) - c_1(f) |\lambda^n|
\]
\[
= 1 + O(\alpha_{n-2}^{-1/2}) + |c_1(g) - c_1(f) |\lambda^n|
\]

where we used Lemma 4.17.

Let us remark also that for \(i \geq q_{n-1}\), \(C_{n+1}^{i}(f) = C_{n+1}^{i-q_{n-1}}(f)\). Thus

\[
\frac{Dh_{n+1}|C^{i+1}_{n+1}(f)}{Dh_{n}|C^{i+1}_{n+1}(f)} = 1
\]

Now, when \(i < q_{n-1}\), with a similar calculation as before we obtain

\[
\frac{Dh_{n+1}|C^{i+1}_{n+1}(f)}{Dh_{n}|C^{i+1}_{n+1}(f)} = \frac{|C_{n+1}(g)|}{|A_n(g)|} \times_2, n(f)
\]
\[
= x_{1,n+1}(g) - x_{3,n+1}(g)
\]
\[
= x_{1,n+1}(f) - x_{3,n+1}(f) (1 + O(\alpha_{n-2}^{-1/2}))
\]
\[
= 1 - [x_{3,n+1}(g) + (1 - x_{4,n+1}(g))]
\]
\[
= 1 - [x_{3,n+1}(f) + (1 - x_{4,n+1}(f)) (1 + O(\alpha_{n-2}^{-1/2}))
\]
\[
= 1 - \alpha_{n+1}(g)
\]
\[
= 1 - \alpha_{n+1}(f) (1 + O(\alpha_{n-2}^{-1/2}))
\]
that implies that Lemma 5.6. Moreover, critical points given by \( \Delta_i(f) = f^i(\Delta_n(f)) \), \( i < q_n \). Observe that for \( i \geq q_n - 1 \), \( D_{n+1}^i(f) = \Delta_{n+1}^{q_n-1}(f) \). Thus, since \( S_{1,n} = O(\alpha_{n+1}) \) (see point 4 of Lemma 4.9), it follows that

\[
\frac{Dh_{n+1}|_{D_{n+1}^i(f)}}{Dh_n|_{D_{n+1}^i(f)}} = \frac{|\Delta_n(g)|}{|B_n(g)|} \left( 1 + O(\alpha_{n-2}^{\frac{1}{2}}) \right) \frac{|B_n(f)|}{|\Delta_n(f)|} \frac{\alpha_{n+1}(g)}{\alpha_{n+1}(f)} \frac{S_{1,n+1}(g)}{S_{1,n+1}(f)} \left( 1 + O(\alpha_{n-2}^{\frac{1}{2}}) \right) = 1 + O(\alpha_{n-2}^{\frac{1}{2}}).
\]

For \( i < q_n - 1 \), we obtain

\[
\frac{Dh_{n+1}|_{D_{n+1}^i(f)}}{Dh_n|_{D_{n+1}^i(f)}} = \frac{|D_{n+1}(g)|}{|D_{n+1}(f)|} \left( 1 + O(\alpha_{n-2}^{\frac{1}{2}}) \right) \frac{|A_n(f)|}{|A_n(g)|} \frac{\alpha_{n+1}(g)}{\alpha_{n+1}(f)} \left( 1 + O(\alpha_{n-2}^{\frac{1}{2}}) \right) = \frac{1 - x_{4,n+1}(g)}{1 - x_{4,n+1}(f)} \left( 1 + O(\alpha_{n-2}^{\frac{1}{2}}) \right) \left( 1 + O(\alpha_{n-2}^{\frac{1}{2}}) \right) = 1 + O(\alpha_{n-2}^{\frac{1}{2}} + |c_s(g) - c_s(f)|\lambda_n^{p_n}).
\]

This ends the proof of the lemma.  

Note that, every boundary point of an interval in \( \mathcal{P}_n \) is in the orbit of the critical points \( o(f(U)) \). So, if \( x \notin o(f(U)) \) then \( Dh_n \) is well defined. Lemma 5.5 implies that

\[
D(x) = \lim_{n \to \infty} Dh_n(x)
\]

exists. Moreover,

\[
0 < \inf_x D(x) < \sup_x D(x) < \infty. \tag{5.4}
\]

Lemma 5.6. Let \((\ell_1, \ell_2) \in (1, 2)^2\). If \( f, g \in \mathcal{W}[1] \) with critical exponents \((\ell_1, \ell_2)\) and \( c_n^*(f) = c_n^*(g) \), then

\[
D : [0, 1] \setminus o(f(U)) \to (0, \infty)
\]

is continuous. In particular, for all \( n \in \mathbb{N} \),

\[
\log \frac{D(x)}{Dh_n(x)} = O(\alpha_{n-2}^{\frac{1}{2}} + |c_s^*(g) - c_s^*(f)|\lambda_n^{p_n}).
\]
Proof. Fix $n_0 \in \mathbb{N}$, then by Lemma 5.5 we have

$$\left| \log \frac{D(x)}{Dh_{n_0}(x)} \right| \leq \sum_{k \geq 0} \log \frac{Dh_{n_0+k+1}(x)}{Dh_{n_0+k}(x)}$$

$$\leq \sum_{k \geq 0} O(\alpha_{k-2}^{1/2} + |c^*_s(g) - c^*_s(f)|\lambda_p^k)$$

$$\leq O(\alpha_{n_0}^{1/2} + |c^*_s(g) - c^*_s(f)|\lambda_p^{n_0}).$$

Therefore, if $x, y \in I \in \mathcal{P}_{n_0}$, we have

$$\left| \log \frac{D(x)}{D(y)} \right| \leq \left| \log \frac{D(x)}{Dh_{n_0}(x)} \right| + \left| \log \frac{D(y)}{Dh_{n_0}(x)} \right|$$

$$\leq O(\alpha_{n_0}^{1/2} + |c^*_s(g) - c^*_s(f)|\lambda_p^{n_0})$$

which means that $D$ is continuous. ∎

Let $T$ be an interval in $[0,1]$, then by formula (5.3),

$$|h(T)| = \lim_{n \to \infty} |h_n(T)| = \lim_{n \to \infty} \int_T Dh_n \leq \int_T \sup_x D(x) \leq \sup_x D(x)|T|. \quad (5.5)$$

Also,

$$|h(T)| = \lim_{n \to \infty} |h_n(T)| = \lim_{n \to \infty} \int_T Dh_n \geq \int_T \inf_x D(x) \geq \inf_x D(x)|T|. \quad (5.6)$$

Proposition 5.7. Let $(\ell_1, \ell_2) \in (1,2)^2$. Let $f, g \in \mathcal{W}_{[1]}$ with critical exponents $(\ell_1, \ell_2)$. Then,

$h$ is a bi-lipschitz homeo $\iff c^*_u(f) = c^*_u(g), c_+(f) + c'_+(f) = c_+(g) + c'_+(g).$

Proof. If $c^*_u(f) = c^*_u(g)$ and $c_+(f) + c'_+(f) = c_+(g) + c'_+(g)$ then by inequalities (5.5), (5.6), Lemma 5.6 and inequalities in (5.4), $h$ is a bi-lipschitz. ∎

As a consequence of Proposition 5.7, we have:

Corollary 5.8. Fix $(\ell_1, \ell_2) \in (1,2)^2$. Let $f, g \in \mathcal{W}_{[1]}$ with critical exponents $(\ell_1, \ell_2)$ and let $h$ be the topological conjugation between $f$ and $g$. If $c^*_u(f) = c^*_u(f)$ and $c_+(f) + c'_+(f) = c_+(g) + c'_+(g)$, then $K_f$ and $K_g$ have the same Hausdorff dimension.

5.3 $C^{1+\beta}$ diffeomorphism Conjugacy

The rigidity class is described as:

1. $\text{Dim}_H K_f := \inf_{s \geq 0} \left\{ H_s(K_f) := \inf_{\delta > 0} \left\{ \sum_{|U| = \delta} |U|^s, K_f \subset \cup U \mid |U| \leq \delta \right\} = 0 \right\}$. 

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Proposition 5.9. Let \((\ell_1, \ell_2) \in (1, 2)^2\) and let \(\beta = \frac{c_n(f)}{\ell_n(f)} \left( \frac{e_n^* + c_n^*}{\lambda_n - 1} \right) \in (0, 1)\). If \(f, g \in W_1\) with critical exponents \((\ell_1, \ell_2)\) and \(h\) is the topological conjugation between \(f\) and \(g\), then

\[
h \text{ is a } C^{1+\beta} \text{ diffeo } \iff c_n^*(f) = c_n^*(g), c_+^*(f) + c_+^*(f) = c_+^*(g) + c_+^*(g)
\]

and \(c_+^*(f) = c_+^*(g)\).

Proof. Suppose that \(h\) is a \(C^{1+\beta}\) diffeo, then by Proposition 5.2, \(c_n^*(f) = c_n^*(g)\) and \(c_+^*(f) = c_+^*(g)\). Thus, by Proposition 4.8 and Lemma 4.15 we get

\[
\frac{Dh(f^{q_n+1}(U_f))}{Dh(f^{q_n+1}(U_f))} = \frac{Dq^{q_n-1}(g^{q_n}(0))}{Dq^{q_n-1}(f^{q_n}(0))} = \frac{Dq(s_n(g)(0))}{Dq(s_n(g)(0))} \frac{1 - x_{2,n}(g)}{1 - x_{2,n}(g)} (1 + O(\alpha_n^{1/2}))
\]

where we used Lemma 5.4. The two above estimates imply that \(c_n^*(f) = c_n^*(g)\).

Let us suppose that \(c_n^*(f) = c_n^*(g), c_n^*(f) = c_n^*(g)\) and \(c_+^*(f) + c_+^*(f) = c_+^*(g) + c_+^*(g)\).

We are going to show first that, under conditions \(c_n^*(f) = c_n^*(g)\) and \(c_+^*(f) + c_+^*(f) = c_+^*(g) + c_+^*(g)\), \(h\) is \(C^1\) diffeo.

Since by (5.5) \(h\) is differentiable with \(Dh(x) = D(x)\), for all \(x\), then it remains to prove that \(D\) can be extended to a continuous function. This is possible, if only if, for all \(k \geq 0\)

\[
\lim_{x \to c_k^*} D(x) = \lim_{x \to c_k^*} D(x)
\]

where \(c_k = f^k(U_f)\).

Let us fix \(k \geq 0\) and let \(n \in \mathbb{N}\) big enough such that \(q_{n-1} > k\). Observe that

\[
D_-(c_k) := \lim_{x \to c_k} D(x) = \lim_{n \to \infty} \frac{|B_{2n}^k(g)|}{B_{2n}^k(f)}
\]

and

\[
D_+(c_k) := \lim_{x \to c_k} D(x) = \lim_{n \to \infty} \frac{|A_{2n}^k(g)|}{A_{2n}^k(f)}
\]

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By Proposition 4.4 and Lemma 4.17 we get
\[
\frac{D_{-}(c_{k})}{D_{+}(c_{k})} = \lim_{n \to \infty} \frac{|A_{2n}^{k}(g)|}{|B_{2n}(g)|} = \lim_{n \to \infty} \frac{|A_{2n}^{k}(f)|}{|B_{2n}(f)|} = \lim_{n \to \infty} \frac{x_{3,2n}(g)}{x_{3,2n}(f)} = \lim_{n \to \infty} e^{\lambda_{n}} = 1
\]

Now, let \(x, y \in K_{f}\) and choose \(n\) maximal such that there exists \(I \in \mathcal{P}_{n}\) and \(I\) contains \([x, y]\). Since \(B_{n+q-1} = D_{n}\), then by maximality of \(n\), \(I \neq D_{n}^{i}\). Also, as \(x, y \in K_{f}\), then \(I \neq C_{n}^{i}\). So, either \(I = A_{n}^{i}\) or \(I = B_{n}^{i}\). In the case where \(I = A_{n}^{i}\), then we can assume that \(x \in D_{n+1}^{i}\) and \(y \in B_{n+1}^{i}\). Thus, by Lemma 3.5 and by Lemma 5.3 we obtain
\[
|x - y| \geq K|A_{m}^{i}| \geq \frac{K}{C} e^{\xi_{e_{n}}^{m} e_{n}^{2} \frac{2\lambda_{n}}{\alpha_{n}}}.
\]  
(5.7)

Observe that by Lemma 5.6 and Lemma 4.15 we have
\[
\left| \log \frac{D_{h}(x)}{D_{h}(y)} \right| = O(e^{\xi_{e_{n}}^{n} e_{n}^{2} (e_{n}^{2} + e_{n}^{3})}) = O(e^{\xi_{e_{n}}^{m} e_{n}^{2} (e_{n}^{2} + e_{n}^{3})}),
\]  
(5.8)

By (5.7) and (5.8), we have
\[
\left| D_{h}(x) - D_{h}(y) \right| = O \left( e^{\lambda_{n}^{m} - 2 \xi_{e_{n}}^{m} e_{n}^{2} (e_{n}^{2} + e_{n}^{3})} \right) = O(1)
\]
for \(0 < \beta < \frac{\tau_{n}(e_{n}^{2} + e_{n}^{3})(\lambda_{n} - 1)}{2\xi_{e_{n}}^{m} e_{n}^{2} \lambda_{n}^{2}} = \frac{\tau_{n}(1 + e_{n}^{3})(\lambda_{n} - 1)}{2\xi_{e_{n}}^{m} \lambda_{n}^{2}}, \) see point 5 of Proposition 4.8.

For \(I = B_{n}^{i}\), then we can choose \((x, y)\) such that \(x \in A_{n+1}^{i}\) and \(y \in B_{n+1}^{i}\). If \(x \in (f^{n+1+1}(U_{f}), f^{n+1}(U_{f}))\), then for \(m = n + 1\), Lemma 5.5 and by Lemma 5.3 imply that
\[
|x - y| \geq K|A_{m}^{i}| \geq \frac{K}{C} e^{\xi_{e_{n}}^{m} e_{n}^{2} \frac{2\lambda_{n}}{\alpha_{n}}}.
\]  
(5.9)

Otherwise, let \(k > 0\), \(m = n + 2k\) maximal such \(x \in A_{m}^{i+n+1}\) and \(y \in B_{m}^{i+n+1}\). By Lemma 5.5 we get
\[
|x - y| \geq \frac{1}{2} \min\{|A_{m}^{i+n+1}|, |B_{m}^{i+n+1}|\} \geq \frac{1}{C} e^{\xi_{e_{n}}^{m} e_{n}^{2} \frac{2\lambda_{n}}{\alpha_{n}}}.
\]  
(5.10)

On the other hand, for all \(x, y \in B_{n}^{i}\),
\[
\left| \log \frac{D_{h}(x)}{D_{h}(y)} \right| = O(e^{\xi_{e_{n}}^{m} e_{n}^{2} (e_{n}^{2} + e_{n}^{3})}) = O(e^{\xi_{e_{n}}^{m} e_{n}^{2} (e_{n}^{2} + e_{n}^{3})})
\]  
(5.11)

where we use Lemma 5.5 and Lemma 4.15.

Thus, by (5.9) or (5.10) and (5.11) we get
\[
\left| D_{h}(x) - D_{h}(y) \right| = O \left( e^{\lambda_{n}^{m} - 2 \xi_{e_{n}}^{m} e_{n}^{2} (e_{n}^{2} + e_{n}^{3})} \right) = O(1);
\]
for \(0 < \beta < \frac{\tau_{n}(e_{n}^{2} + e_{n}^{3})(\lambda_{n} - 1)}{2\xi_{e_{n}}^{m} e_{n}^{2} \lambda_{n}^{2}} = \frac{\tau_{n}(1 + e_{n}^{3})(\lambda_{n} - 1)}{2\xi_{e_{n}}^{m} \lambda_{n}^{2}}, \) see point 5 of Proposition 4.8.

\[\Box\]
5.4 Holder Conjugacy

Proposition 5.10. Let \((\ell_1, \ell_2) \in (1, 2]^2\) and \(\beta = \frac{\tau_u(g) (e_u^0 + e_u^1)(\lambda_u - 1)}{C_u(f)} \frac{2e_u^0 \lambda_u}{2e_u^0 \lambda_u}\). If \(f, g \in \mathcal{W}_1\) with critical exponents \((\ell_1, \ell_2)\), then \(h\) the topological conjugation between \(f\) and \(g\) is \(C^\beta\).

Proof. Let \(x, y \in K_f\) and choose \(n\) maximal such that there exists \(I(f) \in \mathcal{P}_n\) and \(I(f)\) contains \([x, y]\). Since \(B^{i+q-1}_{n+1}(f) = D^n(f)\), then by maximality of \(n\), \(I(f) \neq D^n(f)\). Also, as \(x, y \in K_f\), then \(I(f) \neq C^n(f)\). So, either \(I(f) = A^n(f)\) or \(I(f) = B^n(f)\). The assumptions on \(Dh\) in the previous section are the same on \(h\) in this section. Thus, the proof is done as in the previous section where we show that \(Dh\) the derivative of \(h\) is Holder. For more details, we have:

In the case where \(I(f) = A^n(f)\), then we can assume that \(x \in D^n(f)\) and \(y \in B^n(f)\). Thus, by Lemma 3.5 and Lemma 5.3 we obtain

\[
|x - y| \geq K|A^n(f)| \geq \frac{K}{C} e^{C_u(f) \lambda^n e^n x^n} \frac{2 \lambda u}{2 \lambda u}. \tag{5.12}
\]

Also, by Lemma 5.4 we get

\[
|h(x) - h(y)| \leq |h(I(f))| = |h(I(g))| = O(e^{C_u(g) \lambda^n e^n x^n c^n}). \tag{5.13}
\]

By (5.12) and (5.13) we have

\[
\frac{|h(x) - h(y)|}{|x - y|} = O(e^{C_u(g) \lambda^n e^n x^n - 2e_u(f) e_u^1 2 \lambda u}) = O(1)
\]

for

\[
0 < \beta < \frac{\tau_u(g) (e_u^0 + e_u^1)(\lambda_u - 1)}{C_u(f)} \frac{2e_u^0 \lambda_u}{2e_u^0 \lambda_u},
\]

see point 5 of Proposition 4.8.

Now for \(I = B^n\), then we can choose \((x, y)\) such that \(x \in A^n_{m+1}\) and \(y \in B^{i+q_n-1}_{n+1}\).

If \(x \in (f^{q_n+1+i}(U_f), f^{i+1}(U_f))\) (Let us put \(m := n+1\) ) then by Lemma 3.5 and Lemma 5.3 we obtain

\[
|x - y| \geq K|A^n_{m}| \geq \frac{K}{C} e^{C_u(f) \lambda^n x^n} \frac{2 \lambda u}{2 \lambda u}. \tag{5.14}
\]

Otherwise, let \(k > 0\) \((m := n + 2k)\) maximal such \(x \in A^{i+q_n+1}(f)\) and \(y \in B^{i+q_n+1}(f)\). By Lemma 5.3 we get

\[
|x - y| \geq \frac{1}{2} \min\{|A^{i+q_n+1}_{m}(f)|, |B^{i+q_n+1}_{m}(f)|\} \geq \frac{1}{C} e^{C_u(f) \lambda^n e^n x^n} \frac{2 \lambda u}{2 \lambda u} \tag{5.15}
\]

Thus, in the both cases, by Lemma 5.5 and Lemma 4.15 we get

\[
|h(x) - h(y)| \leq |h(I(f))| = |h(I(g))| = O(e^{C_u(g) \lambda^n x^n c^n}) \tag{5.16}
\]

and by (5.14) \((5.15)\) and (5.16) we obtain

\[
\frac{|h(x) - h(y)|}{|x - y|} = O(e^{C_u(g) \lambda^n x^n - 2e_u(f) e_u^1 2 \lambda u}) = O(1)
\]

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for
\[ 0 < \beta < \frac{\zeta_u(g) (e_u^2 + e_u^3)(\lambda_u - 1)}{2e_u^2\lambda_u} = \frac{\zeta_u(g) (1 + e_u^3)(\lambda_u - 1)}{2\lambda_u}, \]

see point 5 of Proposition 4.8
This concludes the proof.

6 Proof of Lemma 1.1

We use the formalism presented in [9] where the authors find a transition between the degenerate geometry (i.e., \(\alpha_n\) goes to zero) and the bounded geometry (i.e., \(\alpha_n\) is bounded away from zero) for the critical exponents \((\ell, \ell); \ell > 1\).

These results were generalized in [18] for some \((\ell_1, \ell_2)\) belonging to \([1, \infty[\) under assumption \(k_1 = k_2\), the coefficients before the powers of \(x\) in the asymptotics of a map \(f\) our class near the ends of the flat interval (see point 1 of Remark 2.3).

More precisely, in [18] the author uses this condition in the lemma 2 (p. 658) whose the Lemma 6.7 is a reformulation for Fibonacci circle map with a flat piece. In the generic case, the condition \(k_1 = k_2\) will fail for the first return maps, so it does not hold for infinitely renormalizable maps. This is the first reason why we resume the study of the asymptotic behavior of \(\alpha_n\) in our case.

This proof can simply be adapted in [18]. Other results on the geometry of circle maps with a flat interval can be found in [8] and [20].

Let us put together sequences which are frequently used in this section.

\[ \alpha_n = \frac{|(-q_n, 0)|}{|(-q_n, 0)|}, \quad \sigma_n = \frac{|(0, q_n)|}{|(q_n-1, 0)|} \text{ and } s_n := \frac{|[-q_n, 0]|}{|0|}. \]

6.1 Preliminaries

6.1.1 Cross-Ratio Inequalities

Notation 6.1. We denote by \(\mathbb{R}_\subset^4\), the subset of \(\mathbb{R}^4\) defined by
\[ \mathbb{R}_\subset^4 := \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4, \text{ such that } x_1 < x_2 < x_3 < x_4 \}. \]

The following result comes from 1.5 Lemma in [5].

Proposition 6.2. The Cross-Ratio Inequality.
Let \(f \in \mathcal{L}^X\) with \(U\) as its flat piece. Let \((a, b, c, d) \in \mathbb{R}_\subset^4\). The cross-ratio \(\text{Poin}\) is defined by
\[ \text{Poin}(a, b, c, d) := \frac{|d-a||b-c|}{|c-a||d-b|}. \]

The distortion of the cross-ratio \(\text{Poin}\) are given by
\[ \mathcal{D}\text{Poin}(a, b, c, d) := \frac{\text{Poin}(f(a), f(b), f(c), f(d))}{\text{Poin}(a, b, c, d)}. \]

Let us consider a set of \(n + 1\) quadruples \(\{a_i, b_i, c_i, d_i\}\) with the following properties:
1. Each point of the circle belongs to at most $k$ intervals $(a_i, d_i)$.

2. The intervals $(b_i, c_i)$ do not intersect $U$.

Then

$$\prod_{i=0}^{n} \bigtriangleup \text{Poin}(a_i, b_i, c_i, d_i) \geq 1.$$ 

**Remark 6.3.** Let $I$ and $J$ be two intervals of finite and nonzero lengths such that $I \cap J = \emptyset$. We assume that, $J$ is on the right of $I$ and we put $I := [a, b]$ and $J := [c, d]$. Then

$$\text{Poin}(I, J) := \frac{|(I, J)||[I, J]|}{|[I, J]|} = \text{Poin}(a, b, c, d).$$

### 6.1.2 Basic Lemmas

**Lemma 6.4.** The ratio

$$\frac{|(2q_{n-1}, q_{n-1})|}{|(2q_{n-1}, 0)|}$$

is uniformly bounded away from zero.

The proof can be found in [9] (proof of Lemma 1.2).

**Lemma 6.5.** The sequence

$$\frac{|(1, q_{n+1})|}{|(q_{n-1} + 1, 1)|}$$

is bounded.

**Proof.** We will show that the sequence

$$\frac{|(q_{n-1} + 1, 1)|}{|(1, q_{n+1})|}$$

is uniformly bounded away from zero. Let us observe that, the previous ratio is larger than

$$\text{Cr}((q_{n-1} + 1, 1), -q_{n-1} + 1)$$

which by CRI with $f^{q_{n-1}}$ is greater than

$$\frac{|(2q_{n-1}, q_{n-1})|}{|(2q_{n-1}, 0)|}$$

times a constant. This last ratio is uniformly bounded away from zero, see Lemma 6.4.

### 6.2 A priori Bounds of $\alpha_n$

**Proposition 6.6.** Let $n \in \mathbb{N}$ and $(\ell_1, \ell_2) \in (1, 2)^2$.

For all $\alpha_n$,

$$\alpha_n^{\frac{\ell_1}{\ell_2}} < 0.55;$$
for at least every other \( \alpha_n \)

\[
\frac{\ell_1}{\alpha_n} \leq 0.3.
\]

If

\[
\frac{\ell_1}{\alpha_{n-1}} > 0.3,
\]

then either,

\[
\frac{\ell_1}{\alpha_{n-1}} < 0.44 \quad \text{or} \quad \frac{\ell_1}{\alpha_n} < 0.16.
\]

Proof. Let

\[
\gamma_{1,n} = |(-q_n, 0)|, \quad \gamma_n = \frac{\gamma_{1,n}}{\gamma_{1,n-1}},
\]

\[
\gamma_n^{(\ell_1/\ell_2)} := \frac{k_2}{k_1} \cdot \frac{\gamma_{1,n}}{\gamma_{1,n-1}} \quad \text{if } n \in 2\mathbb{Z} \quad \text{and} \quad \gamma_n^{(\ell_2/\ell_1)} := \frac{k_1}{k_2} \cdot \frac{\gamma_{1,n}}{\gamma_{1,n-1}} \quad \text{if } n \in 2\mathbb{Z} + 1
\]

where \( k_1 \) and \( k_2 \) come from Fact 2.2.

These notations simplify the formalization of the following lemma which will play an important (essential) role in the proof of Proposition 6.6.

**Lemma 6.7.** For every \( n \in \mathbb{N} \) large enough, the following inequality holds

\[
\frac{(\alpha_{n,\ell_1} + \alpha_{n-1,\ell_2})}{(1 + \gamma_n^{(\ell_1/\ell_2)}/\gamma_n^{(\ell_2/\ell_1)})}(1 + \gamma_n^{(\ell_1/\ell_2)}/\gamma_n^{(\ell_2/\ell_1)}) \leq s_n \alpha_{n-2}.
\]

**Proof.** Let \( n \in 2\mathbb{N} + 1 \) large enough. By point 1 of Remark 2.3, the left hand side of (6.1) is equal to the cross-ratio

\[
Poin((-q_n + 1, -q_{n-1} + 1)).
\]

Applying \( f^{q_{n-1}} \), by the expanding cross-ratio property, we get the inequality.

The left hand side is a function of the three variables \( \alpha_{n,\ell_1}, \alpha_{n-1,\ell_2}, \gamma_n^{(\ell_2/\ell_1)} \). Observe that the function increases monotonically with each of the first two variables. However, relatively to the third variable, the function reaches a minimum. To see this, take the logarithm of the function and check that the first derivative is equal to zero only when

\[
(\gamma_n^{(\ell_2/\ell_1)})^2 = \frac{\alpha_{n,\ell_2}}{\alpha_{n-1,\ell_2}}.
\]

By substituting this for \( \gamma_n^{(\ell_2/\ell_1)} \) we get that

\[
\left(\frac{\ell_1}{\alpha_n} + \frac{\ell_1}{\alpha_{n-1}} \right)^2 \leq s_n \alpha_{n-2}.
\]

Let be the sequence \( y'_n \) defined by

\[
y'_n = \min\{\alpha_n^{\ell_1/\ell_2}, \alpha_{n-1}^{\ell_1/\ell_2}\}.
\]

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Since $\alpha_{n-2} \leq \alpha_{n-2}^{\frac{1}{2}} \cdot \frac{1}{\alpha_{n-2}^{\frac{1}{2}}}$, Substituting the above variable into inequality \[6.2\] gives rise to a quadratic inequality $y'_n$ whose only root in the interval $(0, 1)$ is given by

\[
y'_n = \min\{\alpha_{n-1}^{\frac{1}{2}}, \alpha_{n-1}^{\frac{1}{2}}\} \leq \frac{\sqrt{s_n \alpha_{n-2}^{\frac{1}{2}}} + \sqrt{1 - s_n \alpha_{n-2}^{\frac{1}{2}}}}{1 + \sqrt{1 - s_n \alpha_{n-2}^{\frac{1}{2}}}}.
\]

(Corollary 6.10) Therefore, the sequence $\alpha_n$ is constructed.

Lemma 6.8. The function

\[h_n(z) = \frac{\sqrt{s_n z}}{1 + \sqrt{1 - s_n z}}\]

moves points to the left, $h(z) < z$, if $z \geq 0.3$ and $n$ is large enough.

Lemma 6.9. There is a subsequence of $\alpha_n$ including at least every other $\alpha_n$, such that

\[\limsup \alpha_{n-1}^{\frac{1}{2}} \leq 0.3\]

Proof. We select the subsequence.

1. The initial term: There exists $n-2 \in \mathbb{N}$, such that $\alpha_{n-2}^{\frac{1}{2}} \leq 0.3$. This comes directly from the properties of the function $h_n$ (Lemma 6.8) and inequality \[(6.3)\]

2. The next element. Suppose that $\alpha_{n-2}$ has been selected. If

\[y'_n = \min\{\alpha_{n-1}^{\frac{1}{2}}, \alpha_{n-1}^{\frac{1}{2}}\} = \alpha_{n-1}^{\frac{1}{2}} \text{ or } \alpha_{n-1}^{\frac{1}{2}} \leq 0.3,
\]

then, we select $\alpha_{n-1}$ as the next term. Otherwise, $\alpha_n$ is the next term. Thus, the sequence is constructed.

Corollary 6.10. For the whole sequence $\alpha_n$ we have

\[\limsup \alpha_n^{\frac{1}{2}} \leq (0.3)^{\frac{1}{2}}\]

Moreover, if $\alpha_{n-1}$ does belong to the subsequence $\alpha_n$ defined by Lemma 6.9 then either

\[\alpha_{n-1}^{\frac{1}{2}} < 0.44 \text{ or } \alpha_n^{\frac{1}{2}} < 0.16\]

Proof. Observe that the function

\[H : (s, t) \in \mathbb{R}^2 \mapsto F(s, t) = \frac{s + t}{1 + st}\]
is symmetric and for fixed $s$, the function $F(s, \cdot)$ reaches its minimum at zero by taking the value $s$. Therefore, for every $s, t \geq 0$, 

$$s, t \leq \frac{s + t}{1 + st}.$$ 

So,

$$\frac{\frac{\mu}{s} - \frac{s}{\alpha_n^2}, \frac{\mu + s}{\alpha_n^2}}{1 + \alpha_n^2} \leq \frac{\frac{\mu + s}{\alpha_n^2} + \frac{\mu - s}{\alpha_{n-1}^2}}{1 + \alpha_n^2}.$$ 

(6.4)

Thus, according to that $\alpha_{n-2}$ is an element of the sequence of Lemma 6.9, it follows from (6.2) that the right member of (6.4) is estimated as follows

$$\frac{\mu}{s} - \frac{s}{\alpha_n^2} + \frac{\mu + s}{\alpha_n^2} - \frac{\mu - s}{\alpha_{n-1}^2} \leq \sqrt{s_n}, \alpha_{n-2} \approx \sqrt{\alpha_{n-2}} \leq 0.3.$$ 

(6.5)

Now, suppose that $\alpha_{n-1}$ does not belong to the subsequence chosen in the proof of Lemma 6.9 then

$$\min\{\frac{\mu}{s}, \alpha_{n-1}^2\} = \frac{\mu}{s} \leq 0.3.$$ 

(6.6)

Thus, if $\frac{\mu}{s}, \alpha_{n-1}^2 \geq 0.16$, then by combining this with (6.5) and (6.6), we obtain the desired estimate.

This ends the prove of Proposition 6.6.

6.3 Recursive formula of $\alpha_n$

Proposition 6.11. Fix $\ell_1, \ell_2 > 1$. Let $n$ be an integer large enough, we have

$$\alpha_{2n}^\ell_1 \leq M_{2n}(\ell_1)\alpha_{2n-2}^2 \quad \text{and} \quad \alpha_{2n+1}^\ell_2 \leq M_{2n+1}(\ell_2)\alpha_{2n-1}^2$$

(6.7)

where

$$M_n(\ell) = s_n^2 \cdot \frac{2}{\ell} \cdot \frac{1}{1 + \sqrt{1 - \frac{2(\ell - 1)}{s_n^2}}}, \frac{1}{1 - \alpha_{n-2}} \cdot \frac{\sigma_n}{\sigma_{n-2}}.$$ 

Proof. We treat the case even $n$ and the case odd $n$ is treated in a similar way. Recall that,

$$\alpha_n = \frac{|(-q_n, 0)|}{||-q_n, 0||}$$

For every even $n$ and large enough, applying $f$ to the equality we have

$$\alpha_n^\ell_1 = \frac{|(-q_n + 1, 1)|}{||-q_n + 1, 1||}$$

which is certainly less than the cross-ratio

$$\text{Poin}(-q_n + 1, (1, -q_{n-1} + 1)).$$
Since the cross-ratio $\text{Poin}$ is expanded by $f^{q_{n-1}}$, then
\[ \alpha_n^{\ell_1} < \delta_n(1)s_n(1) \] (6.8)
with
\[ \delta_n(k) := \frac{|(-q_n + kq_{n-1}, kq_{n-1})|}{|(-q_n + kq_{n-1}, q_{n-1})|} \]
and
\[ s_n(k) := \frac{|[-q_n + kq_{n-1}, 0]|}{|(-q_n + kq_{n-1}, 0)|}. \]
By multiplying and dividing the right member of (6.8) by $\alpha_{n-2}^2$, we get.
\[ \alpha_n^{\ell_1} \leq s_n\nu_{n-2}\mu_{n-2}\alpha_{n-2}^2 \] (6.9)
with
\[ \nu_{n-2} := \frac{|[-q_{n-2}, 0]|}{|q_{n-1}, 0|} \cdot \frac{|[-q_{n-2}, 0]|}{|q_{n-2}, q_{n-1}|} \]
and
\[ \mu_{n-2} := \frac{|(-q_{n-2}, q_{n-1})|}{|(-q_{n-2}, 0)|}. \]
It remains to estimate $\nu_{n-2}$ and $\mu_{n-2}$ to end this part. For $\nu_{n-2}$, observe that
\[ |(-q_{n-2}, 0)| \leq |(q_{n-3}, 0)| \]
so that,
\[ \nu_{n-2} = \frac{1}{\sigma_{n-1}\sigma_{n-2}} \cdot \frac{1}{1 - \alpha_{n-2}} \] (6.10)

The estimate $\mu_{n-2}$ we use the following lemma (Lemma 3.1 in [9]).

Lemma 6.12. Let $\ell \in (1, 2)$. For all numbers $x > y$, we have the following inequality:
\[ \frac{x^{\ell} - y^{\ell}}{x^{\ell}} \geq \left(\frac{x - y}{x}\right) \left[\ell - \frac{\ell(\ell - 1)}{2}\left(\frac{x - y}{x}\right)\right]. \]

Now, apply $f$ into the intervals defining the ratio $\mu_{n-2}$. By Lemma 6.12 the resulting ratio is larger than
\[ \mu_{n-2}(\ell_1 - \frac{\ell_1(\ell_1 - 1)}{2}\mu_{n-2}). \]

The cross-ratio $\text{Poin}$
\[ \text{Poin} \ (-q_{n-2} + 1, (q_{n-1} + 1, 1)). \]
That is,
\[ \frac{|(-q_{n-2} + 1, q_{n-1} + 1)|}{|(-q_{n-2} + 1, q_{n-1} + 1)|} \frac{|(-q_{n-2} + 1, 1)|}{|(-q_{n-2} + 1, 1)|} \]
is larger again. Thus, by expanding cross-ratio property on $f^{q_{n-2}}$, we obtain:
\[ \mu_{n-2}(\ell_1 - \frac{\ell_1(\ell_1 - 1)}{2}\mu_{n-2}) \leq s_{n-1}\sigma_n\sigma_{n-1}. \]
By solving this quadratic inequality, we obtain
\[ \mu_{n-2} < \frac{2}{\ell_1} \cdot \frac{1}{1 + \sqrt{1 - \frac{2(\ell_1 - 1)}{\ell_1} s_{n-1}\sigma_{n-1}}} \cdot s_{n-1}\sigma_n\sigma_{n-1}. \] (6.11)
Since, \( \sigma_n\sigma_{n-1} < \alpha_{n-1} \), the first inequality in (6.7) follows by combining inequalities (6.9), (6.10) and (6.11).

\( \alpha_n \) goes to zero

Technical reformulation of Proposition 6.11

Let \( W_n \) be a sequence defined by
\[ M_n(\ell) = W_n(\ell) \frac{\sigma_n}{\sigma_{n-2}}. \]
Let
\[ M'_n(\ell) := M_n(\ell)\alpha_{n-2}^{2-\ell} \quad \text{and} \quad W'_n(\ell) := W_n(\ell)\alpha_{n-2}^{2-\ell}. \]
For every \( n \) even large enough, the recursive formula (6.7) can be written in the form:
\[ \alpha_n^{\ell_1} \leq W'_n(\ell_1) \frac{\sigma_n}{\sigma_{n-2}} \alpha_{n-2}^{\ell_1}. \]
so,
\[ \alpha_n^{\ell_1} \leq \prod_{k=2}^{n} W'_k(\ell_1) \frac{\sigma_n}{\sigma_0} \alpha_0^{\ell_1}. \]
\[ \prod_{k=2}^{n} W'_k(\ell_1) \] goes to zero.

Observe that the size of \( W'_n(\ell_1) \) is given by the study of the function
\[ W'_n(x, y, \ell_1) = \frac{1}{\ell_1 + \frac{\ell_1}{2} \sqrt{1 - \frac{2(\ell_1 - 1)}{\ell_1} x^2}} \cdot y^{\frac{1}{2}-\frac{2}{\ell_1}}. \]
The meaning of variation of \( W'_n(x, y, \ell_1) \) relative to the third variable is given by the following lemma (Lemma 3.2 in [9]).

**Lemma 6.13.** For any \( 0 < y < \frac{1}{\sqrt{e}} \), \( x \in (0, 1) \) and \( \ell_1 \in (1, 2] \) the function \( W'_n(x, y, \ell_1) \) is increasing with respect to \( \ell_1 \).

**Analyse the asymptotic size of \( W'_n(2) \).**

Since the hypotheses of Lemma 6.13 are satisfied (Proposition 6.6), it enough to verify that the convergence of \( \prod_{k=1}^{n} W'_k(2) \).
- If \( \alpha_{n-2} < (0.3)^{\ell_1} \), then \( W'(2) < W'(0.55, 0.16, 2) < 0.9 \).
- If not, then by Proposition 6.6 \( W'(2) < W'(0.3, 0.44, 2) < 0.98 \), or else, \( W'_n(2)W'_n(2) < W'(0.55, 0.16, 2)W'(0.16, 0.55, 2) < 0.85 \).

As a consequence, we have

**Corollary 6.14.** Let \( \ell_1, \ell_2 \in (1, 2) \). Then \( \alpha_n \) go to zero least double exponentially fast.

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