On quartics with three-divisible sets of cusps

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Abstract

We study the geometry and codes of quartic surfaces with many cusps. We apply Gröbner bases to find examples of various configurations of cusps on quartics.

1 Introduction

The main aim of this note is to study the geometry and codes of quartics $Y_4 \subset \mathbb{P}_3(\mathbb{C})$ with many cusps.

A cusp (=singularity $A_2$) on $Y_4$ is a singularity near which the surface is given in local (analytic) coordinates $x, y$ and $z$, centered at the singularity, by an equation

$$xy - z^3 = 0.$$  

Let $\pi : X_4 \to Y_4$ be the minimal desingularization introducing two $(-2)$-curves $E'_i, E''_i$ over each cusp $P_i$. If there is a way to label these curves such that the divisor class of

$$\sum_{i=1}^{n} (E'_i + 2E''_i)$$

is divisible by three, then the set $P_1, ..., P_n$ is called three-divisible ([1], [14]). Equivalently: there exists a cyclic global triple cover of $Y_4$ branched precisely over these cusps. In particular, every three-divisible set defines a vector (word) in the so-called code of the surface $Y_4$, see Sect. 2.

It is well-known that a quartic surface $Y_4 \subset \mathbb{P}_3(\mathbb{C})$ has at most eight cusps and a three-divisible set on $Y_4$ consists of six cusps ([1]). In [14] S.-L. Tan proves that every quartic with eight cusps contains a three-divisible set, but...
his method does not show which cusps on the surfaces defined in [3] form such a set. A general construction of quartics with three-divisible cusps is given in [4]. Here we use the results from [3] to give a complete picture of the geometry of quartics with three-divisible sets:

In general the six cusps lie on a twisted cubic $C_3$, which however may degenerate to three concurrent lines in special cases. The surface $Y_4$ is given by the determinantal equation
\[
\det\begin{bmatrix} S & Q_{12} \\ Q_{21} & Q_{22} - S \end{bmatrix} = 0,
\]
where the quadrics $Q_{i,j}$ generate the ideal of $C_3$. In particular, the locus of quartics with three-divisible sets is irreducible, but its germ at each point corresponding to a quartic with eight cusps is reducible.

Every three-divisible set endows $Y_4$ with two elliptic fibrations. Their fibers are cut out by the entries of the above-given matrix, and multiplication of the matrix by a constant matrix corresponds to a choice of another pair of fibers.

We compute the code of a quartic with eight cusps and find all three-divisible sets on the quartics defined in [4]. Finally, we use Gröbner bases to give examples of both configurations of cusps.

In order to render our exposition self-contained we recall some facts from [3]. For various applications of three-divisible sets we refer the reader to [14]. We work over the field of complex numbers.

## 2 Three-divisible sets and contact surfaces

Let $Y_4 \subset \mathbb{P}_3(\mathbb{C})$ be a quartic with finitely many singularities all of which are cusps (i.e. $A_2$-type double points). Let $\pi : X_4 \rightarrow Y_4$ be the minimal resolution, where $\text{sing}(Y_4) = \{P_1, \ldots, P_6\}$ and $E'_i + E''_i := \pi^{-1}(P_i)$.

**Definition** (cf. [4]) If one can order the $(-2)$-curves $E'_i, E''_i$ in such a way that there exists a divisor $L'$ which satisfies
\[
\sum_{i=1}^{6} (E'_i + 2E''_i) = 3L',
\]
then $\{P_1, \ldots, P_6\}$ is called a 3-divisible set.

Observe that if the divisor $\sum_{i=1}^{6} (E'_i + 2E''_i)$ is 3-divisible, then we can find an $L''$ such that
\[
\sum_{i=1}^{6} (2E'_i + E''_i) = 3L''.
\]
Moreover, by [6], the divisor classes $\mathcal{L}', \mathcal{L}''$ are unique.
In the sequel we assume that $Y_4$ contains a 3-divisible set $P_1, \ldots, P_6$, and $\mathcal{L}', \mathcal{L}''$ are the above-defined divisors.

We have the following lemma, which is proved in [8] under the assumption that the set $\text{sing}(Y_4)$ is 3-divisible.

**Lemma 2.1** (cf. [5, Lemma 3.2]) If $D \in |\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}'|$, then $\pi^*(D)$ is not a hyperplane section of $Y_4$.

**Proof.** Suppose to the contrary and put $C := \pi^*(D)$. Then we have

\[
\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}' = \bar{C} + \sum_{i=1}^{q} (\alpha_i' E'_i + \alpha_i'' E''_i),
\]

\[
\pi^*\mathcal{O}_{Y_4}(1) = \bar{C} + \sum_{i=1}^{q} (\beta_i' E'_i + \beta_i'' E''_i),
\]

where $\alpha_i', \alpha_i'', \beta_i', \beta_i''$ are non-negative integers and $\bar{C}$ is the proper transform of the divisor $C$. Thus

\[
\sum_{i=1}^{q} (\alpha_i' E'_i + \alpha_i'' E''_i) + \mathcal{L}' = \sum_{i=1}^{q} (\beta_i' E'_i + \beta_i'' E''_i).
\]

We intersect that divisor with $(-2)$-curves $E'_1, E''_1$ to obtain the equalities

\[-2\alpha_1' + \alpha_1'' = -2\beta_1' + \beta_1'' \quad \text{and} \quad \alpha_1' - 2\alpha_1'' - 1 = \beta_1' - 2\beta_1''.\]

Multiply the first equality by 2 and add to the other to get contradiction with the fact that $\alpha_1', \beta_1'$ are integers. \qed

We have $(\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}')^2 = 0$, so Riemann-Roch yields:

\[\chi(\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}') = 2.\]

Since the canonical system of $X_4$ is trivial, we can find divisors $D'_4 \in |\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}'|$ and $D''_4 \in |\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}''|$. Let $C'_4 := \pi_*(D'_4)$ and let $C''_4 := \pi_*(D''_4)$. We have

\[3 C'_4 \sim \pi^*\mathcal{O}_{Y_4}(3) - \sum_{i=1}^{6} (E'_i + 2E''_i),\]
so we can find a cubic $S'$ such that $S'.Y_4 = 3C_4'$. Similarly, there is a cubic $S''$ that satisfies $S''.Y_4 = 3C_4''$. Moreover, we can find a quadric $S$ such that $S.Y_4 = C_4' + C_4''$. Indeed, we have

$$
\pi_*\mathcal{O}_{X_4}((\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}') \otimes (\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}'')) \subset \mathcal{O}_{Y_4}(2) \otimes m_{P_1} \otimes \ldots \otimes m_{P_6}.
$$

Having multiplied $S', S''$, $S$ by appropriate constants, we get that $(S' \cdot S'')$ equals ($S^3$) on $Y_4$, so the polynomial $(S' \cdot S'' - S^3)$ vanishes on $Y_4$. One can prove that it does not vanish identically on $\mathbb{P}^3(\mathbb{C})$, see [4, Thm 3.1]. Let $R$ be the quadric residual to $Y_4$ in the sextic $(S' \cdot S'' - S^3)$. Then $Y_4$ is given by the equation

$$
(S' \cdot S'' - S^3)/R = 0
$$

and the cusps $P_1, \ldots, P_6$ belong to the surfaces $S', S'', S$.

In this section $S', S'', S$ and $R = R(S', S'')$ denote the surfaces constructed in this way for the quartic $Y_4$.

**Lemma 2.2** The system $|\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}'|$ has no base curve. Therefore, the cubic $S'$ induced by push-forward of a general element of this system touches the surface $Y_4$ along a smooth elliptic quartic with multiplicity three.

**Proof.** Let sing($Y_4$) = {$P_1, \ldots, P_q$}. We put $\mathcal{L} := \pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}'$. Let $\mathcal{F}$ be the free part of $\mathcal{L}$ and let $C$ be a component of a general element of $|\mathcal{F}|$. Then $C^2 \geq 0$ (see [10, p. 536]) and $\deg(\pi(C)) \leq 4$. Since $p_a(C) = 1 + \frac{1}{2} \cdot C^2$, we have

$$
p_a(\pi(C)) \geq 1.
$$

Hence $\pi(C)$ is either a quartic or a planar cubic. Thus if $L$ is a component of the base locus then either it is $\pi$-exceptional or $\pi(L)$ is a line. Moreover, at most one fixed component of $|\mathcal{L}|$ is not $\pi$-exceptional.

Case 1: Let us assume that all fixed components are $\pi$-exceptional, i.e.

$$
\mathcal{L} = \mathcal{F} + \sum_{i=1}^{q}(\alpha'_i E'_i + \alpha''_i E''_i).
$$

Then $\mathcal{F}.C \geq 0$ for every irreducible $C$. From the equalities

$$
0 = \mathcal{L}^2 = \mathcal{L}.(\mathcal{F} + \sum_{i=1}^{q}(\alpha'_i E'_i + \alpha''_i E''_i)) = \mathcal{L}.\mathcal{F} + \sum_{i=1}^{6} \alpha''_i,
$$

we obtain $\mathcal{L}.\mathcal{F} = 0$. Moreover, we have $\mathcal{L}.E'_i = 0$ for $i \leq q$ and $\mathcal{L}.E''_i = 1$ (resp. $\mathcal{L}.E''_i = 0$) for $i \leq 6$ (resp. $i > 6$), which implies

$$
0 = (\mathcal{F} + \sum_{i=1}^{q}(\alpha'_i E'_i + \alpha''_i E''_i)).\mathcal{F} \geq \sum_{i=1}^{q}((\alpha'_i)^2 + (\alpha'_i - \alpha''_i)^2 + (\alpha''_i)^2).
$$
Case 2: Let \( \mathcal{L} = \mathcal{F} + L + \sum_i^q (\alpha_i' E_i' + \alpha_i'' E_i'') \), where \( \pi(L) \) is a line. Then

\[
\mathcal{L}.L = (\pi^* \mathcal{O}_{Y_4}(1) - \frac{1}{3} \sum_1^6 (E_i' + 2E''_i)).L = 4 - \frac{1}{3} \sum_1^6 (E_i'.L + 2E''_i.L).
\]

The latter yields \( \mathcal{L}.L > 0 \). Indeed, since \( E_i'.L \leq 1 \) and \( E''_i.L \leq 1 \), the inequality \( \mathcal{L}.L \leq 0 \) implies that at least four cusps lie on \( \pi(L) \), so \( Y_4 \) is singular along \( \pi(L) \). Contradiction. We get

\[
\mathcal{L}^2 = (\mathcal{F} + L + \sum_1^q (\alpha_i' E_i' + \alpha_i'' E_i'')).\mathcal{L} = \mathcal{F}.\mathcal{L} + L.\mathcal{L} + \sum_1^6 \alpha_i'' > 0,
\]

which contradicts \( \mathcal{L}^2 = 0 \).

Thus general element of \(|\mathcal{L}|\) is irreducible and, by \( \mathcal{L}^2 = 0 \), it is smooth.

Finally, by Lemma 2.1 the curve \( \pi(C) \) is not planar, so it is a quartic. In particular \( p_\lambda(\pi(C)) = 1 \) (see [11]). If \( \pi(C) \) were singular, then its proper transform under the blow-up of cusps of \( Y_4 \) would be either singular or smooth rational. This is in conflict with the fact that \( C \) is smooth elliptic. Thus general \( S' \) meets \( Y_4 \) along a smooth elliptic quartic.

\[ \square \]

3 The equation of \( Y_4 \)

Here we find another equation of \( Y_4 \). The proof of the main proposition is preceded by two lemmas. We maintain the notation and the assumptions of Sect. 4. Moreover, in Lemmas 3.1, 3.2 we assume that

- the quadric \( S \) is smooth,
- the quartic curves \( C'_4 := S' \cap Y_4, C''_4 := S'' \cap Y_4 \) are irreducible.

The latter implies that the cubics \( S', S'' \) are irreducible.

**Lemma 3.1** There exist (possibly non-reduced) conics \( C', C'' \) such that \( S'.R = 3 C', \quad S''.R = 3 C'' \) and \( S \cap R = C' \cup C'' \).

**Proof.** We prove that every component appears in the cycle \( S'.R \) with a 3-divisible multiplicity. Let \( S_6 := (S' \cdot S'' - S^4) \). Then

\[
S'.S_6 = S'.R + S'.Y_4 = Z'_1 + Z'_2 + 3C'_4,
\]

where \( Z'_1 \) is the part of the cycle \( S'.R \) with no components in \( \text{sing}(S') \) and \( \text{supp}(Z'_2) \subset \text{sing}(S') \). Since \( S_6 \equiv -S^3 \) on \( S' \), every \( C \not\subset \text{sing}(S') \) appears in
$S_0$, $S'$ with a 3-divisible multiplicity. In particular, $\deg(Z'_1)$ is divisible by 3. If $Z'_2 = 0$, then we are done. Otherwise, $\text{supp}(Z'_2)$ is a line. Its multiplicity in the cycle $Z'_2$ is $\deg(Z'_2)$, so it is 3-divisible.

Let $C' := \text{supp}(S'.R)$. Since $\deg(C') \leq 2$, it suffices to prove that $C'$ is connected. This is obvious when $R$ is irreducible, so we can assume that $R$ is a union of planes and $C'$ consists of two lines. Then $S, S'$ intersect along the irreducible quartic $C'_4$ and the lines, so the latter form a $(1,1)$-curve. The proof for $S''$ and $C''$ is analogous.

The quadrics $S, R$ meet along the curves $C', C''$ because $S' \cdot S'' \equiv S^3$ on $R$ (if $R$ is a double plane, then consider restriction to this plane). \hfill \Box

**Lemma 3.2** The quadric $R$ is reduced and the conics $C', C''$ are its hyperplane sections. Moreover, if $R = R_1 \cdot R_2$ consists of two planes, then

$$C' = L.R,$$

(1)

where the forms $L, R_1, R_2$ are linearly independent.

**Proof.** We maintain the notation of Lemma 3.1 and prove that $R$ is reduced. Suppose that $R = x_0^2$. The sextic $S' \cdot S''$ meets $R$ along two distinct lines, say $\ell', \ell''$, because $S$ is smooth. By Lemma 3.1 we have $S'.R = 6\ell'$ and $S''.R = 6\ell''$, so we can assume that

$$S' = x_1^3 + x_0 \cdot (x_0 \cdot S'_1 + S'_2),$$

$$S'' = x_2^3 + x_0 \cdot (x_0 \cdot S''_1 + S''_2),$$

$$S = x_1 \cdot x_2 + x_0 \cdot (x_0 \cdot S_1 + S_2),$$

where $S'_2, S''_2, S_2 \in \mathbb{C}[x_1, x_2, x_3]$. Then

$$S_0 = x_0 \cdot (x_2^3 \cdot S'_2 + x_1^3 \cdot S''_2 - 3 x_1^2 \cdot x_2^2 \cdot S_2) = 0 \pmod{x_0^2},$$

so $S'_2$ (resp. $S''_2$) is divisible by $x_1^2$ (resp. $x_2^2$), and $S_2 \in \mathbb{C}[x_1, x_2]$. The latter is in conflict with the assumption that $S$ is smooth.

Suppose that $R$ is irreducible. If $R$ is a cone, then $C' \in \mathcal{O}_R(1)$. Otherwise, $S'.R = 3C'$ is a $(3,3)$-curve, so $C'$ is a hyperplane section of $R$.

Assume that $R = R_1 \cdot R_2$ consists of two planes. Then $C' = \ell_1 + \ell_2$ is a sum of lines. Suppose that the planes $R_1, R_2$ meet along the line $\ell_1$. Then $l_1 = l_2$. Indeed, assume that $l_2 \subset R_1$ and observe that if $l_2 \neq l_1$, then by Lemma 3.1 the cycle $S'.R_1 - 3l_2 - l_1$ is non-negative, so $\deg(S'.R_1) \geq 4$. This is in conflict with the irreducibility of $S'$. Thus $S', R$ meet only along $\ell_1$. Since the conics $S.R_1, S.R_2$ contain $l_1$, the cubic $S''$ meets $R$ in two distinct lines $\ell''_1, \ell''_2$, none of which coincides with $l_1$. Thus

$$S.R = 2l_1 + \ell''_1 + \ell''_2,$$

so $2l_1$ is a $(1,1)$-curve. Contradiction. \hfill \Box
Proposition 3.1 Let $Y_4$ be a quartic with a 3-divisible set $\{P_1, \ldots, P_6\}$ of cusps. Then there exist linear forms $L', L'', F', F''$ and a quadric $R$ such that $Y_4$ is given by the equation:

$$
(S' \cdot S'' - S^3)/R = 0,
$$

where the forms $L', L''$ are linearly independent,

$$
S' := (L')^3 + F' \cdot R, \quad S'' := (L'')^3 + F'' \cdot R \quad \text{and} \quad S := R + L' \cdot L''.
$$

Moreover, the inclusion $\{P_1, \ldots, P_6\} \subset S' \cap S'' \cap S$ holds.

**Proof.** We claim that $Y_4$ is given by the equation (2) with a smooth quadric $S$. Indeed, general cubic $S'$ meets $Y_4$ along a smooth irreducible quartic $C'_4$. Once we fix $S'$ and vary $S''$, the quadric $S$ varies but always contains the curve $C'_4$. There are only finitely many singular quadrics that contain $C'_4$, so the quadric $S$ is smooth for general $S', S''$.

Lemma 3.2 yields that the residual quadric $R$ is reduced. Assume that $R = R_1 \cdot R_2$ consists of two planes and choose a form $L$ which satisfies (1). Then the function $(S'/L^3)$ is constant on both planes $R_i$, so

$$
S' = \alpha' \cdot L^3 + R_1 \cdot Q'_1 = \alpha' \cdot L^3 + R_2 \cdot Q'_2 = \alpha' \cdot L^3 + R_1 \cdot R_2 \cdot F',
$$

where the last equality results from the fact that $L, R_1, R_2$ are linearly independent. Moreover, $\alpha'$ is non-zero because the cubic $S'$ is irreducible, so we can assume $\alpha' = 1$.

In the same way we find forms $L'', F''$ such that $S''$ satisfies (3).

The function $(S/(L'L''))$ is constant on $R_i$ because $S^3 \equiv (L'L'')^3$ on that planes. Therefore,

$$
S = \alpha \cdot L' \cdot L'' + \beta \cdot R \quad \text{with} \quad \alpha^3 = 1.
$$

Multiply $L', F', F''$ and $R$ by appropriate constants to obtain (3).

The proof of (3) for an irreducible quadric $R$ follows the same lines, so we leave it to the reader.

The forms $L', L''$ are linearly independent because the polynomial

$$
S' - F' \cdot S = L' \cdot ((L')^2 - F' \cdot L''),
$$

belongs to the ideal of the non-planar irreducible curve $C'_4$. $\square$
4 Configuration of cusps

Here we study the configuration of cusps in 3-divisible sets. We introduce
the following notation:

\[ Q_{1,2}(L', \ldots, F'') := L' \cdot F'' - (L'')^2, \]
\[ Q_{2,1}(L', \ldots, F'') := L'' \cdot F' - (L')^2, \]
\[ Q_{2,2}(L', \ldots, F'') := F' \cdot F'' - L' \cdot L'', \]
\[ C_3(L', \ldots, F'') := Q_{1,2} \cap Q_{2,1} \cap Q_{2,2}. \]

We omit the linear forms and write \( Q_{i,j}, \) resp. \( C_3 \) when it is not ambiguous.

Theorem 4.1 Every quartic \( Y_4 \) with a 3-divisible set is given by the equation:

\[
\det \begin{bmatrix}
S & Q_{12} \\
Q_{21} & Q_{22} - S
\end{bmatrix} = 0,
\]

(5)

where \( S \) is smooth and the 3-divisible set consists of the intersection points of the quadric \( S \) and the curve \( C_3 \). Moreover, one of the following holds:

(I) The forms \( L', L'', F', F'' \) have no common zero, i.e. \( C_3 \) is a twisted cubic.

(II) The planes \( L', L'', F', F'' \) meet in one point \( P \). Then \( C_3 \) consists of three lines that meet in \( P \). Each of the lines contains precisely two cusps. The vertex \( P \) does not lie on the quartic \( Y_4 \) and the lines are not coplanar.

Proof. Let \( P_1, \ldots, P_6 \) be the 3-divisible set on \( Y_4 \). The equation (5) can be obtained from (2) by direct computation. Furthermore, since

\[ \{P_1, \ldots, P_6\} \subset S' \cap S'' \cap S, \]

all the cusps \( P_1, \ldots, P_6 \) belong to the quadrics \( Q_{1,2}, Q_{2,1} \) (see (4)). It remains to compare multiplicities in a cusp \( P_j \):

\[ 2 = \text{mult}_{P_j}(Y_4) = \text{mult}_{P_j}(S \cdot (Q_{2,2} - S)) = 1 + \text{mult}_{P_j}(Q_{2,2} - S), \]

so \( P_j \) lies on \( Q_{2,2} \), and the cusps \( P_1, \ldots, P_6 \) belong to \( S \cap C_3 \).

If \( L', L'', F', F'' \) have no common zero, then \( C_3 \) is twisted cubic and we have \( S \cdot C_3 = \sum P_j \). This is the configuration (I).

Assume that the planes \( L', \ldots, F'' \) meet. Suppose that their intersection is a line \( l \). Then the quadrics \( Q_{i,j} \) are singular along that line, so they are sums of planes. If they met along a surface, then \( Y_4 \) would be singular along its intersection with the quadric \( S \), so it would be singular along a curve. Hence the quadrics \( Q_{i,j} \) meet precisely along the line \( l \) and the 3-divisible
cusps lie on that line. This is impossible, so the forms \( L', \ldots, F'' \) have only one common zero, say \( P \).

Since \( Q_{i,j} \) are cones with vertex \( P \), the curve \( C_3 \) is a cone, i.e. a sum of lines that pass through \( P \). The lines are not coplanar by Lemma 2.2, so there are at least three of them. If \( Q_{1,2}, Q_{2,1} \) have no common components, then their intersection is a quartic curve with the line \( L' \cap L'' \) as a component. The latter does not lie on \( Q_{2,2} \), hence \( C_3 \) consists of three lines. Otherwise, the quadrics \( Q_{1,2}, Q_{2,1} \) meet along a line and a plane. The latter intersects \( Q_{2,2} \) properly, so \( \text{deg}(C_3) = 3 \) again.

Finally, the quartic \( Y_4 \) meets \( C_3 \) in points on \( S \), i.e. in the cusps. Since there are six of them, every component of \( C_3 \) intersects \( S \) in two cusps and the vertex \( P \) does not belong to \( Y_4 \). \( \square \)

Let \( D_4 \) denote the closure of the family of quartics given by the equation

\[
S \cdot (Q_{2,2} - S) - Q_{1,2} \cdot Q_{2,1} = 0,
\]

where \( S \) is a quadric, \( L', L'', F', F'' \) are linear forms, \( Q_{i,j} := Q_{i,j}(L', \ldots, F'') \).

By Thm 4.1 every quartic with a \( 3 \)-divisible set belongs to \( D_4 \). Thus [5, Thm 2.1] implies that projective quartics with \( 3 \)-divisible sets of cusps form a dense subset of \( D_4 \).

**Corollary 4.1** The locus of quartics with \( 3 \)-divisible sets is irreducible.

It is natural to ask how to pass from the equation (5) to (2), i.e. how to find the cubics that touch \( Y_4 \) with multiplicity \( 3 \). Here we answer this question in generic case, i.e. for type (I). The answer in the other case is similar.

Let \( Y_4 \) be a quartic carrying a \( 3 \)-divisible set of type (I) with \( L' := x_0, \ldots, F'' := x_3 \) and let \( \Phi(t_0, t_1) := (t_0^2 t_1, t_0 t_1^2, t_0^3, t_1^3) \).

For a non-degenerate \((2 \times 2)\)-matrix \( \mathbf{a} := [a_{i,j}]_{i,j=0,1} \) there exists a unique automorphism \( \mathbf{a} = (L''(\mathbf{a}), L'(\mathbf{a}), F''(\mathbf{a}), F'(\mathbf{a})) \) of \( \mathbb{C}^4 \) such that \( \mathbf{a} \circ \Phi = \Phi \circ \mathbf{a} \).

Put

\[
A_1 := \begin{bmatrix} a_{0,1} & a_{0,0} \\ a_{1,1} & a_{1,0} \end{bmatrix}, \quad A_2 := \begin{bmatrix} a_{1,0} & a_{0,0} \\ a_{1,1} & a_{0,1} \end{bmatrix}
\]

Then, by direct computation, one can find a quadric \( Q(\mathbf{a}) \) such that

\[
A_1 \begin{bmatrix} Q \\ Q_{21} \\ Q_{22} - Q \end{bmatrix} A_2 = \det(\mathbf{a})^{-2} \begin{bmatrix} Q(\mathbf{a}) \\ Q_{21}(\mathbf{a}) \\ Q_{22}(\mathbf{a}) - Q(\mathbf{a}) \end{bmatrix},
\]

where \( Q_{i,j}(\mathbf{a}) := Q_{i,j}(L'(\mathbf{a}), \ldots, F''(\mathbf{a})) \). Now it suffices to use (3) to find the cubics.

The above equality shows that the intersection of a cone over \( C_3 \) (with vertex
on $C_3$) with the quartic $Y_4$ consists of two quartic curves, each of which is image under $\pi$ of a curve in $|\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}'|$, resp. $|\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}''|$. The choice of the vertex of the cone corresponds to the choice of a fiber in the elliptic fibration given by $|\pi^*\mathcal{O}_{Y_4}(1) - \mathcal{L}'|$.  

5 Quartics with eight cusps

Here we study quartics with eight cusps. In particular we find $3$-divisible sets on the quartics defined in \([2]\).

**Definition 5.1** The code of the quartic $Y_4$ is the kernel of the map

$$
\varphi : \mathbb{F}_q^3 \supseteq \sum_{1}^{q} \mu_j e_j \mapsto \sum_{1}^{q} \mu_j (E'_j - E''_j) \in \text{Pic}(X_4) \otimes \mathbb{F}_3.
$$

Observe that every vector (word) in the linear code $\ker(\varphi)$ corresponds to a $3$-divisible set of cusps on $Y_4$.

For $v \in \mathbb{F}_q^3$, one defines its weight as the number of its non-zero coordinates. A ternary $[q, d, \{r\}]$ code is a $d$-dimensional subspace $\mathbb{F}_q^3$ such that all its non-zero elements are of weight $r$. We have the Griesmer bound \([16, \text{Thm (5.2.6)}]\):

$$
q \geq \sum_{i=d-1}^{i=0} \left\lceil \frac{r}{3^i} \right\rceil \quad (6)
$$

**Theorem 5.1** If $Y_4$ is a quartic with eight cusps, then

(i) the germ of $\mathcal{D}_4$ at $Y_4$ is reducible,

(ii) the code of $Y_4$ is the $[8, 2, \{6\}]$ code that is spanned by the following words

$$
\begin{align*}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & -1 & 1 & 1
\end{align*}
$$

**Proof.** (ii) The above-defined code is the only ternary $[8, 2, \{6\}]$ code. Moreover, by (3), every $[8, p, 6]$-code is at most 2-dimensional, so it suffices to prove that $\ker(\varphi)$ is 2-dimensional. Suppose that $\dim(\ker(\varphi)) \leq 1$. Then $\dim(\text{Im}(\varphi)) = 15$ and $\dim(\text{Im}(\varphi)^\perp) = 7$. On the other hand we have

$$
E'_i \in \text{Im}(\varphi)^\perp \text{ and } (E'_i)^2 = (-2) ,
$$

so $\dim(\text{Im}(\varphi)^\perp) \geq 8$. Contradiction.

(i) Consider the projection from the set

$$
\{(x, Y) : x \in \text{sing}(Y), Y \in \mathcal{D}_4\}
$$
on the variety \( \mathcal{D}_4 \). By [3] its general fiber consists of six points, but the fiber over \( Y_4 \) has eight elements. This cannot happen when the germ of \( \mathcal{D}_4 \) at \( Y_4 \) is irreducible.

By Thm 5.1 two distinct 3-divisible sets never have five cusps in common.

**Example 5.1** (cf. [2]) Let \( k \neq 0 \). The surface \( S_k \) given by the polynomial

\[
(1 + k)^3 x_0^2 x_1^2 + 2k(1 - k^2)x_0 x_1 x_2 x_3 - (1 - k)^3 x_2^2 x_3^2 \\
+(1 - k)^2(x_0 + x_1 + x_2 + x_3)[(1 - k)x_2 x_3(x_0 + x_1) - (1 + k)x_0 x_1(x_2 + x_3)]
\]

has the obvious \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-symmetry:

\[
\psi_1 : x_0 \leftrightarrow x_1, \quad \psi_2 : x_2 \leftrightarrow x_3.
\]

The set of its cusps splits into three orbits of the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-action:

\[
(p_1) \quad P_1 := (1 : 0 : -1 : 0), \quad P_2 := (1 : 0 : 0 : -1), \\
P_3 := (0 : 1 : -1 : 0), \quad P_4 := (0 : 1 : 0 : -1),
\]

\[
(p_2) \quad P_5 := (1 : 0 : 0 : 0), \quad P_6 := (0 : 1 : 0 : 0),
\]

\[
(p_3) \quad P_7 := (0 : 0 : 1 : 0), \quad P_8 := (0 : 0 : 0 : 1).
\]

Let \( \mathcal{A} \) be a 3-divisible set on \( S_k \). Suppose that \( \mathcal{A} \) contains precisely three elements of the orbit \((p_1)\). W.l.o.g. \( P_1, P_2, P_3 \) belong to the 3-divisible set \( \mathcal{A} \). If either \( P_5 \notin \mathcal{A} \) or \( P_6 \notin \mathcal{A} \), then \( \psi_1(\mathcal{A}) \) and \( \mathcal{A} \) have five cusps in common, which is in conflict with Thm 5.1. Similar reasoning applied to \( \psi_2 \) rules out the remaining possibilities, i.e. \( P_7 \notin \mathcal{A}, P_8 \notin \mathcal{A} \).

Assume that \( \mathcal{A} \) contains the orbit \((p_1)\). As before Thm 5.1 implies that \( \mathcal{A} \) contains one of the orbits \((p_2)\), \((p_3)\). Finally, let \( \mathcal{A} \) contain the orbits \((p_2)\), \((p_3)\). By Thm 4.1 no five cusps in \( \mathcal{A} \) are coplanar, so the 3-divisible set contains either the points \( P_1, P_4 \) or \( P_2, P_3 \). To sum up, \( \mathcal{A} \) is one of the following

\[
P_1, \ldots, P_4, P_5, P_6 \\
P_1, \ldots, P_4, P_7, P_8 \\
P_5, \ldots, P_8, P_1, P_4 \\
P_5, \ldots, P_8, P_2, P_3
\]

According to Thm 5.1 there are four 3-divisible sets on \( S_k \), so all the above-given sets are 3-divisible. They are of type (II) because each of them contains four coplanar points.

## 6 Examples

In order to detect a 3-divisible set of cusps we use the following lemma.
Lemma 6.1 Let $Y_4$ be a quartic carrying a set $P_1, \ldots, P_6$ of cusps such that it has no other singularities. Let $S'$ be a cubic that touches $Y_4$ with multiplicity three, e.g. $S'.Y_4 = 3C'_4$, and $P_1, \ldots, P_6 \in S'$. If the divisor $C'_4$ is smooth in the cusps, then they form a 3-divisible set.

Proof. Let $\alpha'_i \leq \alpha''_i$ be the positive integers that satisfy

$$\pi^* S' = 3\tilde{C}'_4 + \sum_{i=1}^6 (\alpha'_i E'_i + \alpha''_i E''_i).$$

We claim that $\alpha'_i = 1$ and $\alpha''_i = 2$ for $i = 1, \ldots, 6$. Indeed, let $H \subset \mathbb{P}_3$ be a general hyperplane through the point $P_i$. In particular, we assume that $\pi^* H = \tilde{H} + E'_i + E''_i$. W.l.o.g the plane $H$ contains no components of the tangent cone of the support of the divisor $C'_4$ at $P_i$, which implies that the proper transforms $\tilde{H}$, $\tilde{C}'_4$ meet in no points on the exceptional divisors $E'_i$, $E''_i$. Moreover we have $\tilde{H} \cdot \tilde{C}'_4 = (\deg(C'_4) - \text{mult}_{P_i}(C'_4)) = 3$. It results from $(\pi^* S').E'_i = (\pi^* S').E''_i = 0$ that $(3\tilde{C}'_4).(E'_i + E''_i) = \alpha'_i + \alpha''_i$. Our claim follows from the equality

$$12 = (\pi^* S').(\pi^* H) = (3\tilde{C}'_4).(\tilde{H} + E'_i + E''_i) = 9 + \alpha'_i + \alpha''_i.$$

Finally, we have $\sum_{i=1}^6 (E'_i + 2E''_i) = 3(\pi^* O_{Y_4}(1) - \tilde{C}'_4)$. □

We give examples of both configurations of cusps (see Thm 4.1).

Example 6.1 (Configuration (I)) We put

$$S := 49x_1^2 + x_2^2 - 36x_3^2 - 14x_0^2,$$

$$R := S - x_0 \cdot x_1,$$

$$S' := x_0^3 + x_2 \cdot R,$$

$$S'' := x_1^3 + x_3 \cdot R,$$

and define the quartic $Y_4$ by the equation (2). We claim that $\text{sing}(Y_4)$ consists of the six cusps $(\pm j : j : j^3 : \pm 1)$, where $j = 1, 2, 3$, that form a 3-divisible set of type (I).

By direct computation $S'$, $S''$, $S$ meet transversally in the six points, none of which belongs to $R$, so those points are cusps on $Y_4$. If we put

$$L' := x_0, L'' := x_1, F' := x_2, F'' := x_3,$$

then $Y_4$ is given by the equation (3) and the curve $C_3$ is twisted cubic. A Gröbner basis computation (see Remark 3.1) shows that the polynomials $(Q_{ij})^4, S^4$ belong to the jacobian ideal of $Y_4$, so the only singularities of $Y_4$ are the six cusps. It remains to prove that they form a 3-divisible set.
Observe that the curves \( C'_4 := Y_4 \cap S' \), \( C''_4 := Y_4 \cap S'' \) have no common components. Indeed, we have
\[
S' \cdot S'' - S^3 = Y_4 \cdot R,
\]
so, by multiplicity count, a smooth point of \( Y_4 \) that belongs to both curves must lie on \( R \). But, the surfaces \( R, S', S'' \) meet properly because the line \( x_0 = x_1 = 0 \) does not lie on the quadric \( S \). Thus the cubic \( S' \) touches \( Y_4 \) with multiplicity 3 along \( C'_4 \). It suffices to prove that \( C'_4 \) is smooth in the cusps (see Lemma 6.1).

Let \( H \) be a general hyperplane through a fixed cusp \( P_i \) and let \( i(\cdot) \) stand for the intersection multiplicity. Then
\[
\text{mult}_{P_i}(S.Y_4) = i(S.Y_4.H; P_i) = i((S \cap H). (Y_4 \cap H); P_i) = 2,
\]
where the last equality results from the fact that the tangent cone \( C_{P_i} Y_4 \) consists of the planes \( T_{P_i} S', T_{P_i} S'' \), so the planar curves \( S \cap H, Y_4 \cap H \) have no common tangent lines at \( P_i \). The curves \( C'_4, C''_4 \) are components of the cycle \( S.Y_4 \) and pass through the point \( P_i \), so they are smooth in the cusp.

**Example 6.2** (Configuration (II)) We put
\[
L' := x_0, \quad L'' := x_1, \quad F' := x_2, \quad F'' := 6(x_1 + x_2) - 11x_0,
\]
and define a quartic \( Y_4 \) by the equation (3) with \( S := x_3^2 - x_2^2 \).

Then the curve \( C_3 \) is a sum of the lines \( l_1, l_2, l_3 \), where \( l_j : x_0 = jx_2, \ x_1 = j^2x_2 \), and \( S \) meets \( C_3 \) in the points \( (j : j^2 : 1 : \pm 1) \), where \( j = 1, 2, 3 \).

If we pass to the equation (2), then we obtain the residual quadric
\[
R : x_3^2 - x_2^2 - x_0 \cdot x_1.
\]
One can check that \( S', S'', S \) meet transversally in the six points, none of which belongs to \( R \), so those points are cusps on \( Y_4 \). A Gröbner basis computation shows that the quartic \( Y_4 \) has no extra singularities. Finally, one can imitate the proof in Example 5.1 to show that the set \( \text{sing}(Y_4) \) is 3-divisible.

We leave the details to the reader.

For the sake of completeness we explain below the way we applied Gröbner bases to count singularities of quartics.

**Remark 6.1** In order to show that a polynomial \( g \in K[x_0, x_1, x_2, x_3] \) vanishes on all singularities of a surface, we need the notion of the remainder on division of a polynomial \( g \) by a Gröbner basis \( \mathcal{B} \) of an ideal \( \mathcal{I} \subset K[x_0, x_1, x_2, x_3] \). Its definition can be found in [2, II.§6]. The only fact we need is that if the
remainder vanishes, then the polynomial $g$ belongs to the ideal $\mathcal{I}$ (see [7, II.§6.Cor. 2]). We use this fact in the following way:

Let $\mathcal{B}$ be a Gröbner basis of the jacobian ideal of the quartic $Y_4$. If we find such an integer $p$ that the remainder on division of the polynomial $g^p$ by the basis $\mathcal{B}$ vanishes, then all singularities of the surface $Y_4$ lie on the hypersurface given by $g$. The former can checked with the following Maple 6.0 commands, where $Y_i$ is defined as the partial derivative $\frac{\partial Y}{\partial x_i}$:

```maple
with(Groebner):
B := gbasis([Y0, Y1, Y2, Y3], tdeg( x0, x1, x2, x3 )):
normalf((g^p), B, tdeg( x0, x1, x2, x3 ));
```

If the output is zero, then $g^p$ belongs to the jacobian ideal of $Y_4$.

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