EXISTENCE OF $L^p$-SOLUTIONS FOR A SEMILINEAR WAVE EQUATION WITH NON-MONOTONE NONLINEARITY

JOSÉ CAICEDO
Departmento de Matemáticas
Universidad Nacional de Colombia
Bogotá, Colombia

ALFONSO CASTRO
Department of Mathematics
Harvey Mudd College
Claremont, CA 91711, USA

RODRIGO DUQUE
Departmento de Matemáticas
Universidad Nacional de Colombia
Bogotá, Colombia

ARTURO SANJUÁN
Department of Mathematics
Universidad Distrital Francisco José de Caldas
Bogotá, Colombia

Abstract. For Dirichlet-periodic and double periodic boundary conditions, we prove the existence of solutions to a forced semilinear wave equation with large forcing terms not flat on characteristics. The nonlinearity is assumed to be non-monotone, asymptotically linear, and not resonant. We prove that the solutions are in $L^p$, ($p \geq 2$), when the forcing term is in $L^p$. This is optimal; even in the linear case there are $L^p$ forcing terms for which the solutions are only in $L^p$. Our results extend those in [9] where the forcing term is assumed to be in $L^\infty$, and are in contrast with those in [6] where the non-existence of continuous solutions is established for $C^\infty$ forcing terms flat on characteristics.

200 words.

1. Introduction and main result. Motivated by the results in [3], [16], [12], [10], [6], [9] and [7], we consider the existence of weak solutions, i.e. solutions in the sense of distributions, to the problem

\[ \square(u) + g(u) = f(x,t), \tag{1} \]

subject to the Dirichlet periodic condition

\[ u(0,t) = u(\pi,t) = 0, \quad u(x,t) = u(x,t+2\pi) \quad x \in [0,\pi], t \in \mathbb{R}, \tag{2} \]

or the $2\pi$-periodic condition

\[ u(x,t) = u(x,t+2\pi) = u(x+2\pi,t) \quad x,t \in \mathbb{R}. \tag{3} \]

2010 Mathematics Subject Classification. Primary: 35L05, 35L70; Secondary: 35P30.

Key words and phrases. Semilinear wave equation, resonance, flat on characteristic.

This work was partially supported by a grant from the Simons Foundation (♯ 245966 to Alfonso Castro).
In (1), $\Box$ denotes the D'Alembert operator $\partial_u - \partial_{xx}$. We assume $g$ to be differentiable and asymptotically linear but need not be monotone. More precisely we assume that

$$g(t) = \tau t + h(t) \quad \text{with} \quad \tau \in (0, \infty)$$

and that for some $\beta < 0$ and $a > 1$

$$|h'(u)| \leq |u|^\beta \quad \text{for} \quad |u| \geq a.$$  

Hence, there exists $M > 0$ such that

$$|h'(u)| \leq M \quad \text{for all} \quad u \in \mathbb{R}.$$  

Also, without loss of generality, we may assume that $-1 < \beta < 0$.

The spectrum of $\Box$ subject to the boundary condition (2) is given by

$$\sigma_1(\Box) = \{k^2 - j^2; k = 1, 2, \ldots, j = 0, 1, \ldots\},$$

while the spectrum of $\Box$ subject to the double-periodic condition (3) is given by

$$\sigma_2(\Box) = \{k^2 - j^2; j, k = 0, 1, \ldots\}.$$  

In both cases all the eigenvalues have finite multiplicity except for 0 which has infinite multiplicity. We denote $\Omega_1 = (0, \pi) \times (0, 2\pi)$ and $\Omega_2 = (0, 2\pi) \times (0, 2\pi)$. Also we denote by $\| \cdot \|_p$ the norm in $L^p(\Omega_1)$ or $L^p(\Omega_2)$, depending on the context.

By the arguments in [3] one sees that if $g$ is monotone then for every $f \in L^2(\Omega_1)$, the equation (1), (2) has a solution. Moreover, if $g$ and $f$ are smooth and $|g'(u)| > \epsilon$ for some $\epsilon > 0$ and all $u \in \mathbb{R}$ then such a solution is of class $C^\infty$. In the non-monotone case, [16] and [12] proved that for $f$ in a dense subset of $L^2(\Omega_1)$ the equation (1), (2) has a weak solution; however, no mechanism is provided for determining the values of $f$ for which (1) (2) has a solution. In the double-periodic case without resonance ($-\tau \notin \sigma_2(\Box)$), [6] gives a class of smooth forcing terms for which the problem has no continuous solution. In [9], a class of forcing terms in $L^\infty$ for which the double-periodic and the Dirichlet periodic problems have solutions is found.

In this paper we give a sufficient condition on the forcing term, $f \in L^p(\Omega_i)$, $i = 1, 2$, for (1)-(2) and (1)-(3) to have solutions in $L^p$.

In order to state our main result we introduce the concept of flatness on characteristics.

**Definition 1.1.** Let $J \in \{\pi, 2\pi\}$, $\phi : [0, J] \times \mathbb{R} \to \mathbb{R}$ be integrable on $[0, J] \times [0, 2\pi]$. We say that $\phi$ is not flat on characteristics if

$$\text{given} \quad \epsilon > 0 \quad \text{there exists} \quad \delta > 0 \quad \text{such that}$$

$$m(\{x \in [0, \pi]; |\phi(x, r \pm x) - \rho| < \delta\}) < \epsilon \quad \text{for all} \quad r, \rho \in \mathbb{R},$$

where $m(A)$ is the Lebesgue measure of the set $A$.

Our main result is the following.

**Theorem 1.2.** Let $-\tau \notin \sigma_1(\Box)$, $f(x, t) = cq(x, t) \in L^p(\Omega_1)$, $p \geq 2$ and $\phi$ the solution to

$$\begin{align*}
\Box \phi + \tau \phi &= g(x, t) \quad x, t \in \mathbb{R}, \\
\phi(0, t) &= \phi(t) = \phi(x, t + 2\pi).
\end{align*}$$

If $\phi$ is not flat on characteristics then there exist $c_0$ such that for $|c| \geq c_0$ the equation (1),(2) has a weak solution $u \in L^p(\Omega_1)$ (see (11)).
Remark 1. In section 5 we extend this result to the the double periodic case (1)-(3).

Theorem 1.2 is optimal; even in the linear case all we can expect is to have \(L^p\) solutions when \(f\) is in \(L^p\). Theorem 1.2 generalizes Theorem A in [10] where additional smoothness was assumed on the forcing term \(q\). Also Theorem 1.2 generalizes Theorem 1.2 in [9] where the \(q\) was assumed to be in \(L_\infty\).

The central idea for the proof of Theorem 1.2 is the estimation, in the \(L^2\) sense, of the projection into the kernel of \(\Box\) of approximate solutions to (1),(2). We achieve this using relation (25) below; similar arguments were used in [9]. Examples of functions satisfying (7) are plentiful, for instance \(q\) this using relation (25) below; similar arguments were used in [9]. Examples of functions satisfying (7) are plentiful, for instance \(q(x,t) = \sin(x + t) + \sin(t - x)\) satisfies (7). For studies on (1),(2) with \(g\) superlinear and monotone we refer the reader to [15]. For other recent results on wave equations with non-monotone nonlinearities the reader is referred to [2]. Extensions of the results in [2] using techniques introduced in [6] are found in [7].

2. Preliminaries. Let \(N\) denote the closure of the linear subspace of \(L^2(\Omega_1)\) spanned by
\[
\{\sin(kx)\cos(kt), \quad \sin(kx)\sin(kt); \quad k = 1, 2, \ldots\}. \tag{9}
\]
That is, \(N\) is the null space of the wave operator \(\Box\) subject to the boundary condition (2). We let \(H\) denote the Sobolev space of functions \(u\), \(2\pi\)-periodic in \(t\), and such that \(u\) as well as its first order partial derivatives belong to \(L^2(\Omega_1)\). The norm in \(H\) is denoted by \(\|~\|_{1,2}\). We let \(Y\) denote the subspace of \(H\) of functions \(y\) such that
\[
\int_{\Omega_1} y(x,t)v(x,t)dxdt = 0 \quad \text{for all} \quad v \in N. \tag{10}
\]
We say that \(u = y + v \in Y \oplus N\) is a weak solution of (1),(2) if
\[
\int_{\Omega_1} \{(y_t\hat{y}_t - y_x\hat{y}_x) - (g(u) - f)\hat{y} + \hat{v}\} dxdt = 0, \tag{11}
\]
for all \(\hat{y} + \hat{v} \in Y \oplus N\).

We let \(\Pi_N : L^2(\Omega_1) \to N\) and \(\Pi_{N^\perp} : L^2(\Omega_1) \to N^\perp\) denote the \(L^2(\Omega_1)\)-orthogonal projections. If \(g\) is linear, i.e. \(h = 0\), then for every \(f \in L^2(\Omega_1)\) the equation (1),(2) has a unique weak solution \(y + v\), which we denote by \((\Box + \tau I)^{-1}(f)\). Moreover, there exist a real number \(\kappa\) such that
\[
\|((\Box + \tau I)^{-1}(\Pi_Y(f)))\|_{1,2} + \|((\Box + \tau I)^{-1}(\Pi_Y(f)))\|_{C^{1/2}} \leq \kappa\|f\|_2, \tag{12}
\]

where \(C^{1/2}\) stands for the space of Hölder continuous functions of exponent 1/2 (see [11]).

On the other hand, by (5), there exists \(M_1 > 1\) such that \(|h(s)| \leq M_1\) for all \(|s| \leq a\). Therefore
\[
|h(s)| \leq M_1 + \frac{|s|^{\beta + 1}}{\beta + 1} \quad \text{for all} \quad s \in \mathbb{R}. \tag{13}
\]
Moreover, since \(0 < \beta + 1 < 1\), we have \(\lim_{|s| \to \infty} |h(s)|/|s| = 0\). Hence fore all \(\epsilon > 0\) there exists \(M_\epsilon > 0\) and \(a_\epsilon > a\) such that
\[
|h(s)| \leq \epsilon |s| \quad \text{if} \quad |s| \geq a_\epsilon, \quad \text{and} \quad |h(s)| \leq M_\epsilon + \epsilon |s|, \quad \text{for all} \quad s \in \mathbb{R}. \tag{14}
\]
Lemma 2.1. There exists $M' > 0$ and $c_1 > 0$ such that if $|c| > c_1$, $u$ is a solution to
\[
\begin{align*}
\Box u + \tau u + h(u) &= cq(x,t) + \gamma(x,t) \\
u(0,t) &= u(\pi,t) = 0 \\
u(x,t) &= u(x,t + 2\pi),
\end{align*}
\] (15)
and $\|\gamma\|_2 \leq 1$ then
\[
\|\Pi_{\mathcal{N}}(u - c\varphi)\|_{1,2} + \|\Pi_{\mathcal{N}'}(u - c\varphi)\|_{C^{1/2}} + \|\Pi_{\mathcal{N}}(u - c\varphi)\|_2 \leq M'|c|^{\beta+1}. \tag{16}
\]

Proof. Let $w = \Pi_{\mathcal{N}'}(u - c\varphi)$ and $v = \Pi_{\mathcal{N}}(u - c\varphi)$. By (13), there exists a constant $M_2 > 0$ such that
\[
\|h(u)\|_2 = M_2 + \|u\|_2^{\beta+1} \leq M_2 + (\|w\|_2 + \|v\|_2 + |c|\|\varphi\|_2)^{\beta+1}
\leq M_2 + \|w\|_2^{\beta+1} + \|v\|_2^{\beta+1} + |c|^{\beta+1}\|\varphi\|_2^{\beta+1}. \tag{17}
\]
This and (12) imply
\[
\|v\|_2 + \|w\|_{1,2} \leq k(1 + M_2 + \|w\|_2^{\beta+1} + \|v\|_2^{\beta+1} + |c|^{\beta+1}\|\varphi\|_2^{\beta+1})
\leq k(1 + M_2 + \|w\|_1^{\beta+1} + \|v\|_2^{\beta+1} + |c|^{\beta+1}\|\varphi\|_2^{\beta+1}). \tag{18}
\]
Since $\beta \in (-1,0)$, there exists $M_3 > 0$ such that $kt^{\beta+1} \leq M_3 + t/2$ for all $t \geq 0$. Hence
\[
\|v\|_2 + \|w\|_{1,2} \leq 2k(1 + M_2 + M_3) + k|c|^{\beta+1}\|\varphi\|_2^{\beta+1}
\leq M'|c|^{\beta+1}, \tag{19}
\]
with $M' = 2k\|\varphi\|_2^{\beta+1}$ and $|c| \geq c_1 = 2(1 + M_2 + M_3)/\|\varphi\|_2^{\beta+1}$, which proves the lemma. \(\Box\)

3. Proof of Theorem 1.2 for $p = 2$. Let $|c| \geq c_1$, see Lemma 2.1. From [12] (see also [16]), there exist sequences $\{\epsilon_n\}$ and $\{u_n\}$ in $L^2(\Omega_1)$ such that $\epsilon_n \to 0$ in $L^2(\Omega_1)$ and $u_n$ is a weak solution to
\[
\begin{align*}
\Box u_n + \tau u_n + h(u_n) &= f(x,t) + \epsilon_n(x,t) \\
u_n(0,t) &= u_n(\pi,t) = 0 \\
u_n(x,t) &= u_n(x,t + 2\pi),
\end{align*}
\] (20)
By (8) and (20), we have
\[
\Box(u_n - c\varphi) + \tau(u_n - c\varphi) = \epsilon_n - h(u_n). \tag{21}
\]
Let us denote $z_n = u_n - c\varphi$, $\Pi_{\mathcal{N}}(z_n) = v_n$ and $\Pi_{\mathcal{N}'}(z_n) = w_n$. Hence $z_n = v_n + w_n$ and
\[
w_n = (\Box + \tau I)^{-1}\Pi_{\mathcal{N}'}(\epsilon_n(x,t) - h(u_n)), \tag{22}
\]
and
\[
\tau v_n = \Pi_{\mathcal{N}}(\epsilon_n - h(u_n)), \tag{23}
\]
Since, by Lemma 2.1, $\{\|w_n\|_{C^{1/2}}\}$ is bounded we may assume that $\{w_n\}$ converges uniformly in $\Omega_1$. Following the arguments leading to (54) in [9], we see that $v_n$ is solution of (23) if and only if $v_n$ is solution to
\[
\tau \int_0^\pi (v_n(x, r-x) - v_n(x, r+x))dx = \int_0^\pi (\epsilon_n(x, r-x) - h(u_n(x, r-x)))dx
- \int_0^\pi (\epsilon_n(x, r+x) - h(u_n(x, r+x)))dx, \tag{24}
\]
a.e. $r \in [0, 2\pi]$. 

For each \( n = 1, 2, \ldots, u_n(x, t) = p_n(t + x) - p_n(t - x) \) with \( p_n \in L^2[0, 2\pi] \). Replacing in (24) we obtain
\[
2\pi \tau p_n(r) = \int_0^\pi \epsilon_n(x, r - x) - \epsilon_n(x, r + x)dx
- \int_0^\pi h(z_n(x, r - x) + c\varphi(x, r - x))dx
+ \int_0^\pi h(z_n(x, r + x) + c\varphi(x, r + x))dx. 
\] (25)

Let \( \Gamma_{mn}(x, r, s, c) = c\varphi(x, r + x) + z_n(x, r + x) + s(z_m(x, r + x) - z_n(x, r + x)) \)
\[
= c \left( \varphi(x, r + x) + \frac{1}{c}(z_n(x, r + x) + s(z_m(x, r + x) - z_n(x, r + x))) \right) 
\equiv c \left( \varphi(x, r + x) + \frac{1}{c} \zeta_{mn}(x, r, s) \right). 
\] (26)

Let \( D > a \) (see (5)) be such that
\[
\frac{M\tau}{128D} + \pi D^\beta < \frac{\pi\tau}{256}. 
\] (27)

Since \( \varphi \) is not flat on characteristics, for \( \epsilon = \pi\tau/256(M^2 + 1) \), there exist \( \delta \) such that \( m(\{ x \in [0, \pi]; |\varphi(x, r + x)| < \delta \}) < \epsilon \). Let
\[
A_r := A = \{ x \in [0, \pi]; |\varphi(x, r + x) - (p_n(r)/c)| < \delta \}, 
B_r := B = \{ x \in [0, \pi]; |z_n(x, r + x) - z_m(x, r + x)| \geq 128\pi D/\tau \} 
\] (28)
\[
C_r := C = \{ x \in [0, \pi]; |p_n(r + 2x)| \geq \delta|c|/2 \}.
\]

For later reference we note that, since \( \|v_n\|_2 \leq M'\|c\|^{1+\beta} \) for \( |c| \geq c_1 \), we have
\[
\int_0^\pi |p_n(r + 2x)|dx \leq K|c|^{1+\beta}, 
\] (29)
where \( K \) is a constant independent of \((n, r)\). Let
\[
c_0 = \max \left\{ c_1, \left( \frac{\pi\tau\delta}{512MK} \right)^{1/\beta}, \frac{3}{\delta} \left( \frac{\tau}{256} \right)^{1/\beta} \right\}. 
\] (30)

Let \( |c| \geq c_2 \). Now
\[
\int_A \int_0^1 |h'(\Gamma_{mn}(x, r, s, c))|dsdx \leq \int_A Mdx \leq M\epsilon < \frac{\pi\tau}{256}. 
\] (31)
Also
\[
\int_B \int_0^1 |h'(\Gamma_{mn}(x, r, s, c))|dsdx \leq \frac{M\tau}{128D} + \pi D^\beta < \frac{\pi\tau}{256}. 
\] (32)
where we have used that, for \( x \in B, \{ t; \Gamma_{mn}(x, r, t, c) \in [-D, D] \} = [0, D]|z_m(x, r - x) - z_n(x, r - x)| \).

From (29), \( m(C) \leq 2K|c|^{\beta-1} \) and
\[
\int_C \int_0^1 |h'(\Gamma_{mn}(x, r, s, c))|dsdx \leq 2KM|c|^{\beta-1}. 
\] (33)

For \( x \in [0, \pi] - (A \cup B \cup C) := E \) we have
\[
|\Gamma_{mn}(x, r, s, c)| \geq \frac{|c|\delta}{2} - D \geq \frac{|c|\delta}{3}. 
\] (34)
Similarly, which implies

\[ \int_{E} \int_{0}^{1} |h'(\Gamma_{mn}(x, r, s, c))| ds dx \leq \pi \left( \frac{|c| \delta}{3} \right)^{\beta} < \frac{\pi \tau}{256}. \]  

(35)

From (31), (32), (33), and (35) for \( |c| > c_2 \) we have

\[ \int_{0}^{\pi} \int_{0}^{1} |h'(\Gamma_{mn}(x, r, s, c))| ds dx \leq \frac{\pi \tau}{16}. \]  

(36)

Similarly

\[ \int_{0}^{\pi} \int_{0}^{1} |h'(G_{mn}(x, r, s, c))| ds dx \leq \frac{\pi \tau}{16}. \]  

(37)

where

\[ G_{mn}(x, r, s, c) = c\varphi(x, r - x) + z_n(x, r - x) + s(z_m(x, r - x) - z_n(x, r - x)). \]  

(38)

Let \( H_{mn}(x, t) = g_m(x, t) - c_n(x, t) \) and \( y_j(x, t) = z_j(x, t) + c\varphi(x, t) \). Replacing (36) and (37) in (25) yields

\[ 2\pi \tau |p_n(r) - p_m(r)| \leq \left| \int_{0}^{\pi} H_{mn}(x, r + x) - \int_{0}^{\pi} H_{mn}(x, r - x) dx \right| \]

\[ + \left| \int_{0}^{\pi} h(y_n(x, r + x)) - h(y_m(x, r + x)) dx \right| \]

\[ + \left| \int_{0}^{\pi} h(y_n(x, r - x)) - h(y_m(x, r - x)) dx \right| \]

\[ \leq \left| \int_{0}^{\pi} H_{mn}(x, r + x) - \int_{0}^{\pi} H_{mn}(x, r - x) dx \right| \]

\[ + \left| \int_{0}^{\pi} \int_{0}^{1} h'(\Gamma_{mn}(x, r, s, c)) ds |z_m(x, r + x) - z_n(x, r + x)| dx \right| \]

\[ + \left| \int_{0}^{\pi} \int_{0}^{1} h'(G_{mn}(x, r, s, c)) ds |z_m(x, r + x) - z_n(x, r + x)| dx \right| . \]

Since \( z_j(x, t) = p_j(t + x) - p_j(t - x) + w_j(x, t) \) for all \((j, x, t)\), from (36), (37) and (39) we infer

\[ \frac{15\pi \tau}{8} |p_n(r) - p_m(r)| \leq \left| \int_{0}^{\pi} H_{mn}(x, r + x) - \int_{0}^{\pi} H_{mn}(x, r - x) dx \right| \]

\[ + \left| \int_{0}^{\pi} \int_{0}^{1} h'(\Gamma_{mn}(x, r, s, c)) ds |p_m(r + 2x) - p_n(r + 2x)| dx \right| \]

\[ + \left| \int_{0}^{\pi} \int_{0}^{1} h'(\Gamma_{mn}(x, r, s, c)) ds |w_m(x, r + x) - w_n(x, r + x)| dx \right| \]

\[ + \left| \int_{0}^{\pi} \int_{0}^{1} h'(G_{mn}(x, r, s, c)) ds |p_m(r - 2x) - p_n(r - 2x)| dx \right| \]

\[ + \left| \int_{0}^{\pi} \int_{0}^{1} h'(G_{mn}(x, r, s, c)) ds |w_m(x, r - x) - w_n(x, r - x)| dx \right| . \]

Let

\[ \alpha_{mn} = \int_{0}^{2\pi} \left| \int_{0}^{\pi} H_{mn}(x, r + x) - \int_{0}^{\pi} H_{mn}(x, r - x) dx \right|^{2} dr = 0. \]  

(41)
Since \( \{\epsilon_n\}_n \) converges to zero in \( L^2(\Omega_1) \),
\[
\lim_{m,n\to+\infty} \alpha_{mn} = 0. \tag{42}
\]

Also, since \( \{\|w_j\|_{1,2}\}_j \) is bounded, we may assume without loss of generality that \( \{w_j\} \) converges uniformly in \( \Omega_1 \). Therefore
\[
\lim_{m,n\to+\infty} \left| \int_0^\pi \int_0^1 h'(\Gamma_{mn}(x,r,s,c))ds \left[ w_m(x,r + x) - w_n(x,r + x) \right] dx \right| = 0,
\]
and
\[
\lim_{m,n\to+\infty} \left| \int_0^\pi \int_0^1 h'(G_{mn}(x,r,s,c))ds \left[ w_m(x,r - x) - w_n(x,r - x) \right] dx \right| = 0,
\]
uniformly for \( r \in [0,2\pi] \). In addition, from (36) and (37),
\[
\int_0^{2\pi} \left( \int_0^\pi \int_0^1 h'(\Gamma_{mn}(x,r,s,c))ds \left[ p_m(r + 2x) - p_n(r + 2x) \right] dx \right)^2 dr \leq \frac{\pi^3 r^2}{138} \|p_m - p_n\|_2^2, \text{ and}
\]
\[
\int_0^{2\pi} \left( \int_0^\pi \int_0^1 h'(G_{mn}(x,r,s,c))ds \left[ p_m(r - 2x) - p_n(r - 2x) \right] dx \right)^2 dr \leq \frac{\pi^3 r^2}{138} \|p_m - p_n\|_2^2. \tag{44}
\]

Now from (39), (42), (43), and (44) we have
\[
\lim_{m,n\to+\infty} \|p_n - p_m\|_2 = 0. \tag{45}
\]

Hence \( \{v_n\} \) converges in \( L^2(\Omega) \) to some \( v \in N \). Let \( w \) be the weak limit in the Sobolev space \( H^1(\Omega_1) \) on \( \{w_n\} \). Therefore, for any \( \hat{y} + \hat{v} \in Y \oplus N \),
\[
\int_{\Omega_1} \{(w + \Pi_Y(c\varphi))_t \hat{y}_t - (w + \Pi_Y(c\varphi))_x \hat{y}_x\} dxdt
\]
\[
- \int_{\Omega_1} \{(\tau(v + w + c\varphi) + h(v + w + c\varphi) - f)(\hat{y} + \hat{v})\} dxdt
\]
\[
= \int_{\Omega_1} \{(w)_t \hat{y}_t - (w)_x \hat{y}_x\} - (\tau(v + w) + h(v + w + c\varphi))(\hat{y} + \hat{v})\} dxdt
\]
\[
= \lim_{n\to\infty} \int_{\Omega_1} \{(w_n)_t \hat{y}_t - (w_n)_x \hat{y}_x\} dxdt
\]
\[
- \int_{\Omega_1} \{(\tau(v_n + w_n) + h(v_n + w_n + c\varphi) - \epsilon_n)(\hat{y} + \hat{v})\} dxdt
\]
\[
= 0.
\]

Thus, \( u = v + w + c\varphi \) is a solution to (1)- (2), which proves Theorem 1.2 when \( p = 2 \).

4. Proof of Theorem 1.2 for \( p > 2 \). Next we consider the case \( q \in L^p(\Omega_1) \) with \( p > 2 \). Since \( q \in L^2(\Omega_1) \), we may write \( q = q_1 + q_2 \in N \oplus N^\perp \). Hence there exists
a 2π-periodic function \( \mu \in L^2([0, 2\pi]) \) such that \( q_1(x, t) = \mu(x + t) - \mu(t - x) \). Let us see that actually \( \mu \in L^p([0, 2\pi]) \). In fact, by the periodicity of \( \mu \) we have

\[
\int_0^\pi \int_0^{2\pi} |\mu(r + 2x) - \mu(r)|^p dx dr = \int_0^\pi \int_0^{2\pi} |q_1(x, r + x) - q_1(x, r)|^p dx dr
\]

\[
= \int_0^\pi \int_x^{x+2\pi} |q_1(x, t)|^p dt dx
\]

\[
= \int_0^\pi \int_0^{2\pi} |q_1(x, t)|^p dx dt
\]

\[
< +\infty.
\]

By Fubini’s theorem, \( \int_0^{2\pi} |\mu(r + 2x) - \mu(r)|^p dx < +\infty \), for almost every \( r \). Since \( |\mu(r)| < \infty \) for almost every \( r \in \mathbb{R} \), there exists \( r \) such that \( \int_0^{2\pi} |\mu(r + 2x)|^p dx < +\infty \). Since \( \mu \) is 2π-periodic we have \( \mu \in L^p([0, 2\pi]) \).

From the case \( p = 2 \), for \( |c| \) sufficiently large there exist \( u \in N \oplus Y \) that satisfies (1)-(2). We write \( u = v + w \) with \( v \in N \) and \( w \in Y \). By Lemma 2.1, \( w \in L_\infty(\Omega_1) \); hence \( w \in L^p(\Omega_1) \). Thus it suffices to prove that \( v \in L^p(\Omega_1) \). Since \( v \in N \), there exists a 2π-periodic function \( \psi : \mathbb{R} \to \mathbb{R}, \psi \in L^2([0, 2\pi]) \) and such that \( v(x, t) = \psi(t + x) - \psi(t - x) \). Arguing as in (25) we have that

\[
2\pi \psi(r) = 2\pi c \mu(r) + \int_0^\pi h(w(x, r + x) - \psi(r) + \psi(r + 2x)) dx
\]

\[
- \int_0^\pi h(w(x, r - x) + \psi(r) - \psi(r - 2x)) dx
\]

From (14), there exists \( K \in \mathbb{R} \) such that \( |h(s)| \leq K + (\tau/4)|s| \) for all \( s \in \mathbb{R} \).

Thus

\[
2\pi |\psi(r)| \leq 2\pi \tau |c| |\mu(r)| + \pi K + \frac{\pi \tau}{4} |\psi(r)|
\]

\[
+ \frac{\pi \tau}{4} \left( \int_0^\pi |w(x, r + x)| dx + \|\psi\|_2 2\pi^{1/2} \right)
\]

\[
+ \frac{\pi \tau}{4} |\psi(r)| + \frac{\pi \tau}{4} \left( \int_0^\pi |w(x, r + x)| dx + \|\psi\|_2 2\pi^{1/2} \right).
\]

Therefore

\[
|\psi(r)| \leq 2|c| |\mu(r)| + P(r),
\]

with \( P \in L_\infty \). Since \( \mu \in L^p([0, 2\pi]) \), \( \psi \in L^p([0, 2\pi]) \) which implies \( v \in L^p(\Omega) \) and proves the case \( 2 < p < \infty \). This completes the proof of Theorem 1.2.

5. The double-periodic case. Now we turn our attention to the equation (1) subject to the condition (3). In this case the kernel \( N \) of \( \square \) is the closure of the linear space generated by the functions

\[\{\sin(kx) \sin(kt), \cos(kx) \cos(kt), \sin(kx) \cos(kt), \cos(kx) \sin(kt), k = 0, 1, 2, \ldots\}\]

Here, \( Y \) and the weak solutions are defined as in (11) with \( \Omega_1 \) replaced by \( \Omega_2 \). Imitating the proof of Theorem 1.2 one proves the following result.

**Theorem 5.1.** Let \( -\tau \notin \sigma_2(\square), f(x, t) = cq(x, t) \in L^p(\Omega_2), p \geq 2 \) and \( \varphi \) the solution to

\[\begin{cases}
\square \varphi + \tau \varphi = q(x, t) & x, t \in \mathbb{R}, \\
\varphi(x, t) = \varphi(x, t + 2\pi) = \varphi(x + 2\pi, t).
\end{cases}\]


If $\varphi$ is not flat on characteristics then there exist $c_0$ such that for $|c| \geq c_0$ the equation (1)-(3) has a weak solution $u \in L^p(\Omega_2)$ (see (11)).

As in the proof of Theorem 1.2 we project approximate solutions to the equation (1)-(3) onto $N$ and $Y$. The only technical difference is that projection onto $N$ of the approximating solutions are now of the form $v_n(x,t) = \bar{v}_n + v_{1,n}(x+t) + v_{2,n}(t-x)$, and (24) becomes

$$
\bar{v}_n = \frac{1}{4\pi^2r} \int_{\Omega_2} (\epsilon_n - h(u_n))
$$

$$
2\pi \tau (v_{1,n}(r) + \bar{v}_n) + \int_0^{2\pi} h(u_n(x,r-x)) \, dx = \int_0^{2\pi} \epsilon_n(x,r-x) \, dx
$$

$$
2\pi \tau (v_{2,n}(r) + \bar{v}_n) + \int_0^{2\pi} h(u_n(x,r+x)) \, dx = \int_0^{2\pi} \epsilon_n(x,r+x) \, dx
$$

where $\bar{v} \in \mathbb{R}$, $v_{1,n}, v_{2,n} \in L^2[0,2\pi]$ are uniquely determined such that

$$
v_n(x,t) = \bar{v} + v_{1,n}(t+x) - v_{2,n}(t-x)
$$

and $\int_0^{2\pi} v_{1,n}(t) \, dt = \int_0^{2\pi} v_{2,n}(t) \, dt = 0$. The rest of the details are direct imitation of the case (1)-(2) and are left for reader to verify.

REFERENCES

[1] P. Bates and A. Castro, Existence and uniqueness for a variational hyperbolic system without resonance, Nonlinear Analysis TMA, 4 (1980), 1151–1156.
[2] M. Berti and L. Biasco, Forced vibrations of wave equations with non-monotone nonlinearities, Ann. Inst. H. Poincaré Anal. Non Linéaire, 23 (2006), 439–474.
[3] H. Brezis and L. Nirenberg, Characterizations of the ranges of some nonlinear operators and applications to boundary value problems, Annali della Scuola Norm. Sup. di Pisa, 5 (1978), 225–236.
[4] R. Brooks and K. Schmitt, The contraction mapping principle and some applications, Electron. J. Diff. Equns. Monograph, 90 (2009), 90 pp.
[5] J. Caicedo and A. Castro, A semilinear wave equation with derivative of nonlinearity containing multiple eigenvalues of infinite multiplicity, Contemp. Math., 208 (1997), 111–132.
[6] J. Caicedo and A. Castro, A semilinear wave equation with smooth data and no resonance having no continuous solution, Discrete and Continuous Dynamical Systems, 24 (2009), 653–658.
[7] J. Caicedo, A. Castro and R. Duque, Existence of solutions for a wave equation with non-monotone nonlinearity and a small parameter, Milan Journal of Mathematics, 79 (2011), 207–220.
[8] A. Castro, Semilinear equations with discrete spectrum, Contemporary Mathematics, 357 (2004), 1–16.
[9] A. Castro and B. Preskill, Existence of solutions for a semilinear wave equation with non-monotone nonlinearity, Discrete and Continuous Dynamical Systems, Series A, 28 (2010), 649–658.
[10] A. Castro and S. Unsurangsie, A semilinear wave equation with nonmonotone nonlinearity, Pacific J. Math., 132 (1988), 215–225.
[11] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer Verlag, 1997.
[12] H. Hofer, On the range of a wave operator with nonmonotone nonlinearity, Math. Nachr., 106 (1982), 327–340.
[13] J. Mawhin, Periodic solutions of some semilinear wave equations and systems: A survey, Chaos, Solitons and Fractals, 5 (1995), 1651–1669.
[14] P. J. McKenna, On solutions of a nonlinear wave equation when the ratio of the period to the length of the interval is irrational, Proc. Amer. Math. Soc., 93 (1985), 59–64.
[15] P. Rabinowitz, Large amplitude time periodic solutions of a semilinear wave equation, Comm. Pure Appl. Math., 37 (1984), 189–206.
[16] M. Willem, Density of the range of potential operators, Proc. Amer. Math. Soc., 83 (1981), 341–344.

Received April 2013; revised November 2013.

E-mail address: jfcaicedoc@unal.edu.co
E-mail address: castro@g.hmc.edu
E-mail address: rduqueba@unal.edu.co
E-mail address: aasanjuanc@unal.edu.co