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On the fold thickness of graphs

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Abstract The graph $G'$ obtained from a graph $G$ by identifying two nonadjacent vertices in $G$ having at least one common neighbor is called a 1-fold of $G$. A sequence $G_0, G_1, G_2, \ldots, G_k$ of graphs such that $G_0 = G$ and $G_i$ is a 1-fold of $G_{i-1}$ for each $i = 1, 2, \ldots, k$ is called a uniform $k$-folding of $G$ if the graphs in the sequence are all singular or all nonsingular. The fold thickness of $G$ is the largest $k$ for which there is a uniform $k$-folding of $G$. We show here that the fold thickness of a singular bipartite graph of order $n$ is $n - 3$. Furthermore, the fold thickness of a nonsingular bipartite graph is 0, i.e., every 1-fold of a nonsingular bipartite graph is singular. We also determine the fold thickness of some well-known families of graphs such as cycles, fans and some wheels. Moreover, we investigate the fold thickness of graphs obtained by performing operations on these families of graphs. Specifically, we determine the fold thickness of graphs obtained from the cartesian product of two graphs and the fold thickness of a disconnected graph whose components are all isomorphic.

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1 Introduction

The notion of folding a graph as defined by Gervacio et al. [2] was motivated by the following situation. Consider a finite number of unit bars joined together at the ends where they are free to turn. Some meter sticks are constructed with this kind of structure as shown in bottom drawing in Fig. 1. Note that this particular meter stick can be folded to look like the top drawing in Fig. 1. The meter stick with this structure can be viewed as a physical model of the path graph $P_n$. After a sequence of folding of the meter stick, it becomes a physical model of a complete graph $K_2$. Based on this observation, they defined the notion of folding a graph as follows.

Let $x$ and $y$ be nonadjacent vertices of a graph that have at least one common neighbor. Obtain a new graph $G'$ from $G$ by identifying $x$ and $y$ and reducing any resulting multiple edges to simple edges. We say that $G'$ is a 1-fold of $G$. Note that folds of a graph are idempotent homomorphisms and also known as retracts. For more details on the connection between a 1-fold and retracts, please see [6].

Example 1.1 The vertices $x$ and $y$ in the graph $G$ shown in Fig. 2 have two common neighbors, $b$ and $c$. By identifying $x$ and $y$, we obtain the graph $G'$, a 1-fold of $G$.

Consider a graph $G$ that is not isomorphic to a complete graph. Suppose that $G_1$ is a 1-fold of $G$. If $G_1$ is not a complete graph, then there should be a pair of non-adjacent vertices in $G_1$ and a graph $G_2$ which is a 1-fold of $G_1$ can be obtained. Thus, we can consider a sequence of graphs $G = G_0, G_1, G_2, \ldots, G_k$. The
largest integer $k$ for which there exists a $k$-folding is in the case where $G_k$ is a complete graph. We call this sequence a $k$-folding of $G = G_0$. An adjacency matrix can be associated with each graph in the sequence denoted by $A(G_i)$. We now say that $G_i$ is singular if the adjacency matrix of the graph is singular; otherwise, we say that the graph is non-singular. A $k$-folding of $G$ is said to be uniform if all the graphs in the sequence are singular or all of them are nonsingular. The largest integer $k$ for which there exists a uniform $k$-folding of $G$ is called the fold thickness of $G$, denoted by $\text{fold}(G)$.

**Remark** If $G_0, G_1, G_2, \ldots, G_k$ is a $k$-folding of $G$, we shall refer to $G_k$ as a $k$-fold of $G$.

**Example 1.2** The graph $G$ shown in Fig. 3 is singular and $\text{fold}(G) = 3$.

Observe that the two pendant vertices $a$ and $b$ are both adjacent to $x$ and none of them is adjacent to any other vertex. In the adjacency matrix of this graph, the rows corresponding to these vertices are identical. Thus, the determinant of the adjacency matrix is 0 and the graph is singular. We can fold this graph starting at the third pendant vertex and we can repeat the process until we obtain the graph consisting only of $a, b$ and $x$. This is still singular and by identifying $a$ and $b$, we get already a nonsingular graph. All the other graphs in the sequence are singular because of the existence of $a$ and $b$ having only $x$ as their neighbor. Thus, $\text{fold}(G) = 3$

We shall give formulas for the fold thickness of some graphs. Specifically, we shall give the fold thickness of paths, cycles, fans and wheels. Our main result gives the fold thickness of bipartite graphs. Moreover, we investigate the fold thickness of graphs obtained by performing some graphs operations. We note that, in 1979, Cooks et al. have already shown that for a given simple connected graph $G$, the largest $k$ for a $k$-folding of $G$ is $|V(G)| - \chi(G)$. This is due to the following theorem.

**Theorem 1.3** [1] Let $G$ be a simple connected graph. The smallest complete graph that $G$ folds into is the complete graph with order $\chi(G)$, where $\chi(G)$ denotes the chromatic number of $G$.

Thus, we defined a maximum folding of a graph $G$ on $n$ vertices or simply a max fold of $G$ to be a $k$-folding of $G$, where $k = n - \chi(G)$ and $\chi(G)$ denoted the chromatic number of the graph.

The readers are referred to [7,10,12] for the terminologies and concepts in graph theory not explicitly defined in this paper.
2 Preliminary results

We shall first state some known useful results and prove other results that we will need in determining the fold thickness of some graphs.

**Lemma 2.1** [11] If the independence number \( \alpha(G) \) of a graph \( G \) is greater than half the order of \( G \), then \( G \) is singular.

*Proof* Let \( G \) be a graph of order \( n \) and let \( \alpha(G) = k > n \). Let \( x_1, x_2, \ldots, x_k \) be the vertices of \( G \) and without loss of generality, assume that \( x_1, x_2, \ldots, x_k \) form an independent set. Consider the adjacency matrix \( A(G) \) of \( G \). Let \( X_1, X_2, \ldots, X_k \) be the first \( k \) rows of \( A(G) \) and let \( V \) be the subspace of \( \mathbb{R}^n \) that they span. Let \( X'_1, X'_2, \ldots, X'_k \) be the vectors in \( \mathbb{R}^{n-k} \), where \( X'_j \) is the vector whose components are the last \( n-k \) components of \( X_j \). If \( V' \) is the subspace of \( \mathbb{R}^{n-k} \) spanned by \( X'_1, X'_2, \ldots, X'_k \) then obviously \( V \) and \( V' \) are isomorphic. Since \( k > \frac{n}{2} \), it follows that \( k > n-k \) and hence the vectors \( X'_1, X'_2, \ldots, X'_k \) are linearly dependent since their number exceeds the dimension of the vector space \( \mathbb{R}^{n-k} \). Consequently, the first \( k \) rows of \( A(G) \) are linearly dependent and so \( G \) is singular. \( \square \)

If \( G \) and \( H \) are graphs, we define \( G + H \) to be the graph consisting of \( G \) and \( H \) (considered to be vertex-disjoint) and all edges of the form \( [x, y] \), where \( x \) is a vertex of \( G \) and \( y \) is a vertex of \( H \).

**Lemma 2.2** Let \( G \) be an \( r \)-regular graph of order \( n \). Then \( \det A(G + K_1) = -\frac{n}{r} \det A(G) \).

*Proof* The adjacency matrix of \( G + K_1 \) has the form

\[
A(G + K_1) = \begin{bmatrix}
A(G) & \vdots \\
1 & \ddots \\
1 & \cdots & 1 & 0
\end{bmatrix}
\]

Since \( G \) is \( r \)-regular, each column in \( A(G) \) has exactly \( r \) entries equal to 1. Let us add \(-\frac{1}{r}\) times row \( i \) to row \( n + 1 \) of \( A(G + K_1) \), for \( i = 1, 2, \ldots, n \). Then row \( n + 1 \) becomes \((0, 0, \ldots, 0, -\frac{n}{r})\). It follows that \( \det A(G + K_1) = -\frac{n}{r} \det A(G) \) by expanding it along the last row. \( \square \)

**Corollary 2.3** Let \( G \) be an \( r \)-regular graph. Then \( G + K_1 \) is singular (nonsingular) if and only if \( G \) is singular (nonsingular).

Known results on the computation of determinants of adjacency matrices, called reduction formulas, will be needed in our arguments. We shall state some of them without proof.

**Notation** For any vertex \( x \) in a graph \( G \), we shall denote by \( N(x) \) the set of all vertices \( y \) in \( G \) that are adjacent to \( x \). We shall refer to \( N(x) \) as the *neighbor set* of \( x \).

**Theorem 2.4** [5] If \( x \) and \( y \) are vertices in a graph \( G \) such that \( N(x) = N(y) \), then \( G \) is singular.

**Theorem 2.5** [5] Let \( x \) and \( y \) be vertices in a graph \( G \) such that \( N(x) \subseteq N(y) \). If \( G' \) is the graph obtained from \( G \) by deleting all the edges of the form \([y, z] \), where \( z \) is a neighbor of \( x \), then \( \det A(G) = \det A(G') \).

**Theorem 2.6** [5] Let \( x \) and \( y \) be adjacent vertices in a graph \( G \) such that \( N(x) \setminus \{y\} = N(y) \setminus \{x\} \). Then

\[
\det A(G) = -2 \det A(G \setminus \{x\}) - \det A(G \setminus \{x, y\})
\]

The theorem that follows gives an upper bound for the fold thickness of graphs.

**Theorem 2.7** For any connected graph \( G \) of order \( n \),

\[
\text{fold}(G) \leq \begin{cases} 
  n - \chi(G) & \text{if } G \text{ is nonsingular}, \\
  n - \chi(G) - 1 & \text{if } G \text{ is singular}.
\end{cases}
\]
Proof Let $G_0, G_1, G_2, \ldots, G_k$ be a uniform $k$-folding of $G$ for some positive integer $k$. Let us first consider the case $G$ is nonsingular. If $G_k$ is not the complete graph, we can find 2 non-adjacent vertices that can be identified to form a 1-fold of $G_k$. This process can be continued until there we obtain a graph $G_t$ which is complete where $t \geq k$. Clearly, the order of $G_t$ is $n - t$. Therefore, $n - t \geq \chi(G)$ and consequently, $k \leq n - \chi(G)$. Now consider the case $G$ is singular. Then $G_k$ is singular and so $k < t$. Therefore, $k \leq t - 1 \leq n - \chi(G) - 1$.

In view of the above theorem, if there exists a uniform $k$-folding of a connected graph $G$ where $k$ is equal to the upper bound in the theorem, then $k$ must already be the fold thickness of the graph. This observation will be applied in obtaining the fold thickness of some special graphs.

Consider the path of order $n$ denoted by $P_n$. Trivially, $\det A(P_1) = 0$ and $\det A(P_2) = -1$. Let $n \geq 3$ and assume that $x, y, z$ are vertices of $P_n$, where $x$ is a pendant vertex, $y$ is the unique neighbor of $x$ and $z$ is adjacent to $y$. By Theorem 2.5, we can remove the edge $[y, z]$ from $P_n$ to obtain a graph with two components $P_2$ and $P_{n-2}$. Thus, $\det A(P_n) = \det A(P_2) \det A(P_{n-2}) = -\det A(P_{n-2})$. By mathematical induction, the following theorem is established.

**Theorem 2.8** For each integer $n \geq 1$,

\[
\det A(P_n) = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{4}, \\
(-1)^{n/2} & \text{if } n \equiv \text{odd}, \\
2 & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}, \\
-4 & \text{if } n \equiv 2 \pmod{4}. 
\end{cases}
\]

**Theorem 2.9** [11] If $[1, 2, 3, 4, 5, 6]$ is an induced path of order 6 in a graph $G$ and $G'$ is the graph obtained from $G$ by deleting the vertices 2, 3, 4, 5 and joining 1 and 6 by an edge, then $\det A(G) = \det A(G')$.

By Theorem 2.9 and mathematical induction, we get the next theorem.

**Theorem 2.10** For each integer $n \geq 3$,

\[
\det A(C_n) = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{4}, \\
2 & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}, \\
-4 & \text{if } n \equiv 2 \pmod{4}. 
\end{cases}
\]

For convenience in most of our arguments, we shall denote the vertices of the cycle $C_n$ by 1, 2, …, $n$ and its edges by $[1, 2], [2, 3], \ldots, [n-1, n], [n, 1]$. The wheel of order $n + 1$ is denoted by $W_n$. It consists of the cycle $C_n$ and an additional vertex 0 that is adjacent to each of the vertices 1, 2, …, $n$ of $C_n$. Thus, $W_n = C_n + K_1$.

**Corollary 2.11** For each integer $n \geq 3$,

\[
\det A(W_n) = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{4}, \\
-n & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}, \\
2n & \text{if } n \equiv 2 \pmod{4}. 
\end{cases}
\]

**Proof** Since $C_n$ is 2-regular and $W_n = C_n + K_1$, the conclusion follows from Lemma 2.2.

\[ \square \]

3 Fold thickness of some graphs

In this section, we shall determine the fold thickness of the path $P_n$, the cycle $C_n$ and the wheel $W_n$. The cycle $C_n$ is singular if and only if $n$ is divisible by 4. In view of Corollary 2.11, the wheel $W_n$ is singular if and only if $n$ is divisible by 4.

**Theorem 3.1** The path of order $n$, denoted by $P_n$, has fold thickness given by

\[
\text{fold}(P_n) = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
\max\{0, n - 3\} & \text{if } n \text{ is odd}. 
\end{cases}
\]

\[ \square \]
Lemma 3.3 If \( n \) is even, \( P_n \) is nonsingular. Any 1-fold of \( P_n \) yields a bipartite graph of odd order. Consequently, the independence number of this bipartite graph is more than half its order. Therefore, any 1-fold of \( P_n \) is singular and \( \text{fold}(P_n) = 0 \).

Let us recall that the chromatic number of a path of order greater than 1 is 2. If \( n \) is odd, \( P_n \) is singular. Therefore, \( \text{fold}(P_n) \leq n - 3 \). If \( n = 1 \) or 3, it is easy to see that \( \text{fold}(P_n) = 0 = \max\{0, n - 3\} \). Let \( n > 3 \) and assume that the formula holds for all paths \( P_k \) with odd \( k < n \). For convenience, let the path have vertices 1, 2, 3, \ldots, \( n \) and edges \([i, i + 1], i = 1, 2, \ldots, n - 1\). Let us identify the vertices 2 and 4 to get the graph \( G \). This graph is singular since the vertices 1 and 3 have the same neighbors. Identify 1 and 3 to get \( P_{n-2} \). This is a singular graph since \( n - 2 \) is odd. By hypothesis of induction, \( \text{fold}(P_{n-2}) = \max\{0, n - 5\} \). Therefore, \( \text{fold}(P_n) = 2 + \max\{0, n - 5\} = \max\{0, n - 3\} \).

**Theorem 3.2** The cycle \( C_n \), has fold thickness given by

\[
\text{fold}(C_n) = \begin{cases} 0 & \text{if } n \equiv 2 \pmod{4}, \\ n - 3 & \text{otherwise.} \end{cases}
\]

**Proof** Note that \( \chi(C_n) = 2 \) if \( n \) is even and \( \chi(C_n) = 3 \) if \( n \) is odd. If \( n \) is odd, \( C_n \) is nonsingular. Therefore, by Theorem 2.7, \( \text{fold}(C_n) \leq n - 3 \). Obviously, \( \text{fold}(C_3) = 0 = 3 - 3 \). Let us assume that \( n \geq 5 \). Any 1-fold of \( C_n \) is isomorphic to the graph consisting of \( C_{n-2} \) plus one other vertex that is adjacent to exactly one vertex of \( C_{n-2} \). This graph is nonsingular and can be folded onto \( C_{n-2} \) which is also nonsingular. The process can be continued and so we see that we have an \((n - 3)\)-uniform folding \( G_0, G_1, G_2, \ldots, G_{n-3} \) of \( C_n \), where the last graph in the sequence is \( C_3 \). Each graph in the sequence is either an odd cycle or an odd cycle plus another vertex adjacent to exactly one vertex of the odd cycle.

If \( n \equiv 0 \pmod{4} \), the cycle \( C_n \) is singular. Consider the 4-uniform folding of \( C_n \) illustrated in Fig. 4.

Note that the last graph in the 4-folding is a cycle of order \( n - 4 \). By mathematical induction, we conclude that \( \text{fold}(C_n) = n - 3 \).

If \( n \equiv 2 \pmod{4} \), the cycle \( C_n \) is nonsingular. Any 1-fold of \( C_n \) is singular. Thus, \( \text{fold}(C_n) = 0 \).

The fan of order \( n + 1 \), denoted by \( F_n \), is the graph obtained from the path \( P_n \) by adding a new vertex and making this vertex adjacent to every vertex of \( P_n \).

**Lemma 3.3** If \( n > 4 \), \( \det(A(F_n)) = -2 \det(A(P_{n-4})) + \det(A(F_{n-4})) \).

**Proof** Let \( F_n \) be made out of the path \( P_n \) with vertices 1, 2, \ldots, \( n \) and edges \([i, i + 1], i = 1, 2, \ldots, n - 1\) plus another vertex 0 that is adjacent to each of the \( n \) vertices of \( P_n \). Apply Theorem 2.5 to the vertices \( n \) and \( n - 2 \) and call the resulting graph \( G' \). Apply the same Theorem to the vertices \( n - 4 \) and \( n - 2 \) and then to the vertices \( n - 2 \) and 0 to get the graph \( G'' \) shown in Fig. 5.

Apply Theorem 2.6 to \( G'' \) using the vertices \( n - 1 \) and \( n \). We see that \( \det(A(F_n)) = -2 \det(A(L)) - \det(A(M)) \) where \( L \) and \( M \) are the graphs shown in Fig. 6.

In the graph \( L \), note that \( N(n - 1) \subseteq N(i) \) for \( i = 1, 2, \ldots, n - 4 \). Applying Theorem 2.6 to \( L \) successively gives us a graph with three components, namely the paths \( P_{n-4}, P_2 \) and \( P_2 \). Thus, \( \det(A(F_n)) = -2 \det(A(P_{n-4})) + \det(A(F_{n-4})) \).
Lemma 3.4 For each $n \geq 1$,

$$\det A(F_n) = \begin{cases} 
-\frac{n}{2} & \text{if } n \equiv 0 \pmod{4}, \\
-1 & \text{if } n \equiv 1 \pmod{4}, \\
1 + \frac{n}{2} & \text{if } n \equiv 2 \pmod{4}, \\
0 & \text{if } n \equiv 3 \pmod{4}.
\end{cases}$$

Proof Let $n \equiv 0 \pmod{4}$. If $n = 4$, direct computation gives $\det A(F_4) = -2$. Let $n > 4$ and assume that $\det A(F_k) = -\frac{k}{2}$ for all positive $k < n$ with $k \equiv 0 \pmod{4}$. By Lemma 3.3, $\det A(F_n) = -2 \det A(F_{n-4}) + \det A(F_{n-4})$ and by Theorem 2.8 and hypothesis of induction, $\det A(F_n) = -2 - \frac{n-4}{2} = -\frac{n}{2}$. The remaining three cases are treated in a similar manner. \hfill \square

In view of Corollary 2.3, since $W_n = C_n + K_1$, then $W_n$ is singular if and only if $n$ is divisible by 4.

Consider the wheel $W_n$ where $n$ is divisible by 4. If $n = 4$, it is easy to see that $\text{fold}(W_4) = 1$. Let $n > 4$ be divisible by 4. Identify vertices 1 and 5 to form the graph $G_1$ shown in Fig. 7.

In the graph $G_1$, 2 and 4 have the same neighbors and so the graph is singular. Obtain graph $G_2$ by identifying 6 and 8. The graph $G_2$ is singular because 2 and 4 still have the same neighbors, Obtain $G_3$ by identifying 7 and 9. By the same reason, $G_3$ is singular. We continue the process until we obtain the graph $G_{n-5}$ shown in Fig. 8.

Now, all the graphs from $G_0$ to $G_{n-5}$ are singular. We can fold $G_{n-5}$ onto $G_{n-4} = W_4$ which is singular and finally, $W_4$ can be folded onto a singular graph $G_{n-3}$. This graph has only one pair of nonadjacent vertices but if this two vertices are identified we obtain a nonsingular graph. Therefore, $\text{fold}(W_n) = n - 3$. We now state this result as a theorem.
Theorem 3.5 If \( n > 0 \) is divisible by 4, then \( \text{fold}(W_n) = n - 3 \).

Three more cases remain for the fold thickness of the wheel \( W_n \), namely \( n \equiv 1, 2 \) or 3 (mod 4).

Theorem 3.6 If \( n \equiv 1 \) or 3 (mod 4), \( \text{fold}(W_n) = n - 3 \).

Proof Consider the wheel \( W_n \), where \( n \equiv 1 \) or 3 (mod 4). This wheel is nonsingular and \( \chi(W_n) = 4 \). Therefore, \( \text{fold}(W_n) \leq n - 1 \). Identify the vertices 1 and \( n - 1 \). We obtain the graph \( G_1 \) consisting of two components, namely \( P_2 \) and \( F_{n-3} \). Both components are nonsingular and hence \( G_1 \) is nonsingular. Now, we can fold \( G_1 \) onto \( G_2 = W_{n-2} \) by identifying \( n \) and \( n - 2 \). Since \( W_{n-2} \) is nonsingular, the 2-folding \( G_0, G_1, G_2 \) is uniform. Since \( n \equiv 1 \) or 3 (mod 4), then \( n - 2 \equiv 3 \) or 1 (mod 4). The process may be continued until we get the uniform \( k \)-folding \( G_0, G_1, G_2, \ldots, G_k = W_3 \). Here, \( k = n - 3 \) and so \( \text{fold}(W_n) = n - 3 \).

The case \( n \equiv 2 \) (mod 4) seems to be difficult. It is not difficult to check that \( \text{fold}(W_6) = 2 \). It can be shown in general that there is a uniform \( (n - 4) \)-folding of \( W_n \) in this case. In connection with this, we state the following conjecture.

Conjecture 3.7 \( \text{fold}(W_n) = n - 4 \) if \( n \equiv 2 \) (mod 4).

Using basically the same techniques and theorems, the next result can be obtained easily.

Theorem 3.8 The fan \( F_n \) of order \( n + 1 \) has fold thickness given by

\[
\text{fold}(F_n) = \max\{0, n - 3\}.
\]

We need another lemma before we can state and prove our main result.

Lemma 3.9 Let \( G \) be a connected bipartite graph of order greater than 3. If \( G \) is singular, then there exists a 1-fold of \( G \) that is singular.

Proof Let \( A \) and \( B \) be a partition of the vertices of \( G \) into two independent sets. Without loss of generality, assume that \( |A| \leq |B| \) and consider the following cases.

Case 1 \( |A| = |B| \). If \( G_1 \) is any one fold of \( G \), then we may assume without loss of generality that \( G_1 \) was obtained by identifying two vertices belonging to \( A \). Clearly, the independence number \( \omega(G_1) \geq |B| \) and this is more than half the order of \( G_1 \). By Lemma 2.1, \( G_1 \) is singular.

Case 2 \( |A| < |B| \). In case \( |A| = 1 \), then \( |B| \geq 3 \). Any 1-fold of \( G \) is obtained by identifying two vertices in \( B \). Since \( |B| \geq 3 \), any 1-fold of \( G \) is singular. Consider the case \( |A| \geq 2 \). Let \( x \) and \( y \) be any two vertices in \( A \). Since \( G \) is connected, there is a path in \( G \) joining \( x \) to \( y \). The second vertex of such a path must be in \( B \) and the third vertex, say \( z \), is in \( A \). Therefore, \( x \) and \( z \) in \( A \) have a common neighbor in \( B \). The 1-fold of \( G \) obtained by identifying \( x \) and \( z \) has independence number more than half its order and hence singular.

We are now ready to state and prove our main result.

Fig. 8 The graph \( G_{n-5} \) obtained after folding \( W_n \) \( n - 5 \) times.
Theorem 3.10 If $G$ is a connected bipartite graph of order $n$, then

$$\text{fold}(G) = \begin{cases} 
  n - 3 & \text{if } G \text{ is singular,} \\
  0 & \text{if } G \text{ is nonsingular.}
\end{cases}$$

Proof Let $G$ be a connected bipartite graph. If $G$ is nonsingular consider any partition of the vertices of $G$ into two independent sets $A$ and $B$, necessarily, $|A| = |B|$ for otherwise, the independence number of $G$ would be greater than half its order and this would mean that $G$ is singular. Now, if $G_1$ is any 1-fold of $G$, we may assume that $G_1$ is obtainable from $G$ by identifying two vertices in $A$. We see that $G_1$ has independence number that is greater than or equal to $|B|$ which is more than half the order of $G_1$. Therefore, $G_1$ is singular. Thus, $\text{fold}(G) = 0$.

Consider the case $G$ is singular. Then the order $n$ of $G$ must be at least 3. If $n = 3$, we have $\text{fold}(G) = 0$. If $n > 3$, by Lemma 3.9, $G$ has a singular 1-fold $G_1$. By induction, it follows that $\text{fold}(G) = n - 3$. □

Corollary 3.11 For each $m \geq 2$ and $n \geq 2$,

$$\text{fold}(K_{m,n}) = m + n - 3.$$ 

Proof We note that the complete bipartite graph $K_{v_1,v_2}$ is of order $m + n$, where $o(v_1) = m$ and $o(v_2) = n$. By Definition of a complete bipartite graph if $x, y \in v_1$ or $x, y \in v_2$ then $N(x) = N(y)$ and thus $K_{m,n}$ is always singular. From Theorem 3.10, the conclusion easily follows. □

From the above theorem, the fold thickness of a star $K_{1,n}$ is easily computed as $\text{fold}(K_{1,n}) = n - 2$ if $n \geq 2$ and $\text{fold}(K_{1,1}) = 0$ if $n = 1$.

Theorem 3.12 For $m, n$ and $p \geq 1$,

$$\text{fold}(K_p + K_{m,n}) = m + n - 3.$$ 

Proof We note that each vertex from $K_p$ is adjacent to every vertex in $K_{m,n}$, thus we can only identify vertices from $K_{m,n}$. Now, let $V_1$ and $V_2$ be a partition of the vertex set of $K_{m,n}$ such that $|V_1| = m$ and $|V_2| = n$. For any pair of vertices $x, y \in V_1$ or $V_2$, $N(x) = N(y)$. Thus, $K_p + K_{m,n}$ is singular. By definition of the sum of two graphs, every vertex in $K_{m,n}$ is adjacent to every vertex in $K_p$. Hence, we are just actually folding $K_{m,n}$ and, by Corollary 3.11, the conclusion follows. □

We now define what we mean by the cartesian product of two disjoint graphs $G$ and $H$ denoted by $G \square H$ or $G \times H$. Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$. Define $G \times H = (V, E)$, where $V = V_1 \times V_2$ and two vertices $u = (v_1, v_2)$, $v = (v_1, v_2)$ are adjacent whenever the following holds: (1) $v_1 = v_2$ and $u_2$ is adjacent to $v_2$ or (2) $v_2 = v_1$ and $u_1$ is adjacent to $v_1$. It is well known that the cartesian product of two connected graphs is connected. Moreover, if the two graphs are bipartite then the cartesian product is also a bipartite graph. For more detailed discussion on cartesian product of graphs, see [8]. Hence the following theorem follows easily from Theorem 3.10.

Theorem 3.13 Let $G$ be a bipartite graph of order $m$. Then

$$\text{fold}(P_n \times G) = \begin{cases} 
  nm - 3 & \text{if } P_n \times G \text{ is singular} \\
  0 & \text{otherwise.}
\end{cases}$$

Corollary 3.14 Let $G$ be a bipartite graph of order $n$. Then

$$\text{fold}(P_2 \times G) = \begin{cases} 
  2n - 3 & \text{if } l \text{ is an eigenvalue of } A(G) \\
  0 & \text{otherwise.}
\end{cases}$$

Proof In [4] Theorem 1.3, it was shown that $G \times K_n$ is singular if and only if $1$ or $1 - n$ is an eigenvalue of $A(G)$. Since, $P_2 = K_2$ then, $P_2 \times G$ is singular if and only if $1$ is an eigenvalue of $A(G)$. Hence, by Theorem 3.10 the conclusion follows easily. □

Corollary 3.15 For each $n \geq 2$ and for every $k \geq 1$

$$\text{fold}(P_k \times P_n) = \begin{cases} 
  kn - 3 & \text{if } \gcd(k + 1, n + 1) > 1 \\
  0 & \text{otherwise.}
\end{cases}$$
Corollary 3.16 For each $n \geq 2$ and for every $k \geq 1$,
\[
\text{fold}(C_{2k} \times P_n) = \begin{cases} 
2kn - 3 & \text{if } \gcd(2k, n+1) > 1 \\
0 & \text{otherwise}.
\end{cases}
\]

Proof In [3], it was shown that $C_m \times P_n$ is singular if and only if $\gcd(2m, 2n+2) > 2$. Thus, if $m$ is even, that is, $m = 2k$, then $C_m \times P_n$ is singular if and only if
\[
\gcd(2m, 2n+2) > 2 \Rightarrow \gcd(4k, 2n+2) > 2 \\
\Rightarrow 2 \gcd(2k, n+1) > 2 \\
\Rightarrow \gcd(2k, n+1) > 1.
\]
Hence, the conclusion easily follows from Theorem 3.10. \qed

Corollary 3.17 For each $n \geq 2$ and for every $k \geq 1$
\[
\text{fold}(C_{2n} \times C_{2m}) = 4mn - 3.
\]

Proof We first note that a cycle of even length is bipartite. And in [3], it was shown that $C_m \times C_n$ is singular if and only if either $m$ or $n$ is even, thus by Theorem 3.10 the conclusion easily follows. \qed

4 Fold thickness of a disconnected graph

In this section, we determine the fold thickness of some disconnected graphs; specifically, the fold thickness of a graph with $p$ isomorphic components denoted $pG$. Explicitly, the fold thickness of $pC_n$, $pF_n$, $pP_n$ and $pG$ are determined, where $G$ is a bipartite graph.

Theorem 4.1 Let $G$ be a bipartite graph of order $n$. Then for $p \geq 1$
\[
\text{fold}(pG) = \begin{cases} 
np - 2p - 1 & \text{if } G \text{ is singular} \\
0 & \text{if } G \text{ is nonsingular}.
\end{cases}
\]

Proof Let $G$ be a bipartite graph of order $n$. For each $p \geq 1$, $pG$ would denote $p$ copies of $G$, that is $pG$ is a bipartite graph with $p$ components where each component is isomorphic to $G$. Denote by $G_i$ the $i$th copy of the graph $G$ where $i = 1, 2, \ldots, p$. If $G$ is singular then $pG$ must also be singular. Note that the order of $pG$ is $np$ and since $G$ is a bipartite graph $\chi(G_i) = 2$ for each $i = 1, 2, 3, \ldots, p - 1$. Now to get the fold thickness of $pG$ we isolate one copy of $G$ and maximize the number of foldings of the other $p - 1$ copies of $G$. By isolating one copy of $G$, we are assured that no matter how we fold the other $p - 1$ copies of $G$ the resulting disconnected graph will still be singular. Now since the maximum fold of $G$ is $n - \chi(G)$, then
\[
\text{fold}(pG) = (p - 1)(n - \chi(G)) + \text{fold}(G) = (p - 1)(n - 2) + n - 3 = np - 2p - 1.
\]
Now suppose the $G$ is non-singular. Since $G$ is non-singular each 1-fold of $G$ is singular and hence fold$(G) = 0$. This also means that each 1-fold of $pG$ is singular; hence, fold$(pG) = 0$. \qed

We recall that the path $P_n$ is singular if $n$ is odd and non-singular if $n$ is even. By Theorem 4.1, the proof for the next corollary follows easily.

Corollary 4.2 For $p \geq 1$ and $n \geq 1$,
\[
\text{fold}(pP_n) = \begin{cases} 
np - 2p - 1 & \text{if } n \equiv 1 \pmod{4} \\
0 & \text{if } n \equiv 2 \pmod{4} \\
np - 2p - 1 & \text{if } n \equiv 0 \pmod{4} \\
p(n - 3) & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}.
\end{cases}
\]

Theorem 4.3 For $n \geq 3$,
\[
\text{fold}(pC_n) = \begin{cases} 
0 & \text{if } n \equiv 2 \pmod{4} \\
p(n - 3) & \text{if } n \equiv 0 \pmod{4} \\
np - 2p - 1 & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}.
\end{cases}
\]
Proof If \( n \) is even, then \( pC_n \) is a bipartite graph. Suppose \( n \equiv 0 \mod 4 \) then by Theorem 2.10, \( C_n \) is singular. Therefore \( pC_n \) is also singular. Hence, by Theorem 4.1, \( \text{fold}(pC_n) = np - 2p - 1 \). Now suppose that \( n \equiv 2 \mod 4 \), by Theorem 2.10 \( C_n \) is non-singular. This implies that \( pC_n \) is non-singular and, by Corollary 4.1, \( \text{fold}(pC_n) = 0 \). Consider the case where \( n \) is odd; then by Theorem 2.10 \( C_n \) nonsingular and hence, \( pC_n \) is nonsingular. By Theorem 3.2, \( \text{fold}(C_n) = n - 3 \), where \( n \) is odd, thus \( \text{fold}(pC_n) = p(n - 3) \).

**Theorem 4.4** For \( n \geq 3 \)

\[
\text{fold}(p F_n) = \begin{cases} 
np - 2p - 1 & \text{if } n \equiv 3 \mod 4 \\
p(n-3) & \text{otherwise}
\end{cases}
\]

Proof For \( n \geq 3 \). Let \( F_n \) be singular, that is, by Lemma 3.4 \( n \equiv 3 \mod 4 \). We isolate one of the \( p \) copies of \( F_n \). To get the \( \text{fold}(p F_n) \), we maximize the number of folds of the \( (p - 1)F_n \) and add the \( \text{fold}(F_n) \). Thus, we have

\[
\text{fold}(p F_n) = \text{maximum number of fold on } (p - 1)F_n + \text{fold}(F_n)
\]

\[
= (p - 1)(n + 1 - \chi(F_n)) + n - 3
\]

\[
= (p - 1)(n + 1 - 3) + n - 3
\]

\[
= (p - 1)(n - 2) + n - 3
\]

\[
= np - n - 2p + 2 + n - 3
\]

\[
= np - 2p - 1.
\]

Suppose \( F_n \) is nonsingular. By Theorem 3.8, the \( \text{fold}(F_n) = n - 3 \) for each copy of \( F_n \). Thus, \( \text{fold}(p F_n) = p(n - 3) \).

\[\square\]

5 Conclusion

We end this paper by providing some problems regarding the fold thickness of a graph.

One of the main results of the paper is to determine the fold thickness of a bipartite graph. Another result is the determination of the fold thickness of a disconnected graph with isomorphic components. One aspect of this paper relies on knowing the singularity of a graph which has been studied extensively already, which leads us to examine the following problems:

- Problem 1: Determine the value of \( \text{fold}(G) \), where \( G \) is a \( k \)-partite graph on \( n \) vertices.
- Problem 2: Determine the value of \( \text{fold}(G) \), where \( G \) is a disconnected graph with at least 2 components that are non-isomorphic.

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