GENERALIZED CALABI-YAU MANIFOLDS AND THE CHIRAL DE RHAM COMPLEX

REIMUNDO HELUANI AND MAXIM ZABZINE

ABSTRACT. We show that the chiral de Rham complex of a generalized Calabi-Yau manifold carries $N = 2$ supersymmetry. We discuss the corresponding topological twist for this $N = 2$ algebra. We interpret this as an algebroid version of the super-Sugawara or Kac-Todorov construction.

1. Introduction

In [23] the authors introduced a sheaf $\Omega_{M}^{ch}$ of super vertex algebras on any smooth manifold $M$, they called it the chiral de Rham complex of $M$. It was subsequently studied both in the mathematics literature (cf. [11], [1], [8] among others) and the physics literature, in connection to the $\sigma$-model with target $M$ (cf. [18], [25] among others). In the algebraic context, when the manifold $M$ has a global holomorphic volume form, it was shown in [23] that the cohomology $H^*(M, \Omega_{M}^{ch})$ of this sheaf, a super vertex algebra, carries the structure of an $N = 2$ superconformal vertex algebra. This result was further generalized in the differential setting in [1] and more recently in [14], where it was shown that, when $M$ is Calabi-Yau, $\Omega_{M}^{ch}$ carries an $N = 2$ superconformal structure associated to the complex structure, and another $N = 2$ structure associated to its symplectic structure. Moreover, these two structures combine into two commuting $N = 2$ superconformal structures on $\Omega_{M}^{ch}$.

On the other hand, it has been known for some time now in the physics literature, that for the $\sigma$-model to possess $N = 2$ supersymmetry, the target manifold ought to have the structure of a generalized complex manifold (cf. [4], [22], [26] and references therein). The aim of this article is to show that, for this supersymmetry to subsist at the quantum level, the canonical bundle of $M$ has to be holomorphically trivial. Formally we will show that given a differentiable manifold $M$, to each pair $(J, \varphi)$ where $J$ is a generalized complex structure on $M$ and $\varphi$ is a global closed pure spinor (a closed section of the canonical line bundle $U_{J}$ corresponding to $J$), we will associate an $N = 2$ superconformal structure on $\Omega_{M}^{ch}$ of central charge $c = 3 \dim_{R} M$. This structure generalizes in the Calabi-Yau case, those structures described in [1] and [14].

We can perform a topological twist by reassigning the conformal weights of the basic fermions in this theory. For example, in the Complex case, the authors of [23] declared the conformal weight of fields corresponding to holomorphic forms to be zero, while the conformal weight of fields corresponding to holomorphic vector fields is 1. We show that the twisting in the generalized complex case is a generalization of both $A$– and $B$–models. Indeed, one has to consider linear combinations of differential forms and vector fields as the basic fermions in the twisted theory. We identify the BRST cohomology of the chiral de Rham complex of $M$ with the Lie algebroid cohomology of the corresponding Dirac structure, obtaining thus...
another interpretation for the Gerstenhaber algebra structure in the Lie algebroid cohomology of a generalized Calabi-Yau manifold.

It is well known that given a simple or commutative Lie algebra $\mathfrak{g}$ with an invariant bilinear form $(,)$, one can construct an embedding of the $N = 1$ super-vertex algebra in the corresponding super-affine vertex algebra $V^k(\mathfrak{g}_{\text{super}})$ (cf. [17], [16]). Taking Zhu algebras, one recovers the construction of the cubic Dirac operator of [20] (cf. [6]). Our construction could be viewed as a groupoid generalization of this construction. Loosely speaking, Courant algebroids could be viewed as families of Lie algebras with invariant bilinear forms $(,)$. Given a courant algebroid $E$, there exists a sheaf of SUSY vertex algebras $U^{\text{ch}}(E)$ constructed in a similar way to the sheaf of twisted differential operators corresponding to a Lie algebroid. Choosing special local frames for $E$, the superfield of $U^{\text{ch}}(E)$ that generates supersymmetry is given by the same expression as in the super-Sugawara or Kac-Todorov construction of [17].

The organization of this article is as follows. In section 2 we recall the basics of vertex algebra theory and SUSY vertex algebra theory, we refer the reader to [16] for the former and [15] for the latter. In section 3 we collect some results about Lie and Courant algebroids, we briefly recall the definition of the modular class of a Lie algebroid, as well as we recall the basics of generalized complex geometry following [12]. In section 4 we recall the construction of a sheaf of vertex algebras associated to any Courant algebroid on $M$. The chiral de Rham complex of a differentiable manifold $M$ corresponds to the case when this Courant algebroid is the standard algebroid $T_M \oplus T_M^*$. In this section we follow [14], while we refer the reader to the original literature for a complete treatment [23], [11], [5]. In section 5 we construct global sections of the chiral de Rham complex of $M$ associated to any pair $(\mathcal{J}, \phi)$ as above. We state the main results in this section (see Theorems 5.5 and 5.10). The technical proofs and computations can be found in the Appendix.

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2. Preliminaries on Vertex Algebras

2.1. Vertex superalgebras. In this section, we review the definition of vertex superalgebras, as presented in [16]. Given a vector space $V$, an $\text{End}(V)$-valued field is a formal distribution of the form

$$A(z) = \sum_{n \in \mathbb{Z}} z^{-1-n}A_{(n)}, \quad A_{(n)} \in \text{End}(V),$$

such that for every $v \in V$, we have $A_{(n)}v = 0$ for large enough $n$.

**Definition 2.1.** A vertex super-algebra consists of the data of a super vector space $V$, an even vector $|0\rangle \in V$ (the vacuum vector), an even endomorphism $T$, and a parity preserving linear map $A \mapsto Y(A, z)$ from $V$ to $\text{End}(V)$-valued fields (the state-field correspondence). This data should satisfy the following set of axioms:
• Vacuum axioms:
  \[ Y(0, z) = \text{Id}, \quad Y(A, z)0 = A + O(z), \quad T0 = 0. \]

• Translation invariance:
  \[ [T, Y(A, z)] = \partial_z Y(A, z). \]

• Locality:
  \[(z - w)^n [Y(A, z), Y(B, w)] = 0, \quad n \gg 0.\]
(The notation \(O(z)\) denotes a power series in \(z\) without constant term.)

Given a vertex super-algebra \(V\) and a vector \(A \in V\), we expand the fields
\[ Y(A, z) = \sum_{j \in \mathbb{Z}} z^{-1-j} A(j), \]
and we call the endomorphisms \(A(j)\) the Fourier modes of \(Y(A, z)\). Define now the operations:
\[ [A_\lambda B] = \sum_{j \geq 0} \frac{\lambda^j}{j!} A(j) B, \quad AB = A \cdot B := A_{(-1)} B. \]
The first operation is called the \(\lambda\)-bracket and the second is called the normally ordered product. The \(\lambda\)-bracket contains all of the information about the commutators between the Fourier coefficients of fields in \(V\).

Remark 2.2. Corresponding to a given a super-vertex algebra \(V\), there exists an associative algebra \(Z(V)\) called the Zhu algebra of \(V\). Below we will give some examples of these algebras and refer the reader to the classic literature in the subject for its definition (see for example \([6]\)).

2.2. Examples. In this section we review the standard description of the \(N = 1, 2\) superconformal vertex algebras as well as the current or affine vertex algebras. We describe the Sugawara and Kac-Todorov construction. In section 2.3, the same algebras will be described in the SUSY vertex algebra formalism.

Example 2.3. The \(N = 1\) (Neveu-Schwarz) superconformal vertex algebra

The \(N = 1\) superconformal vertex algebra \(NS_c\) ([16]) of central charge \(c\) is generated by two fields: \(L(z)\), an even field of conformal weight 2, and \(G(z)\), an odd primary field of conformal weight \(\frac{3}{2}\), with the \(\lambda\)-brackets
\[ [L_\lambda L] = (T + 2\lambda)L + \frac{c\lambda^3}{12}, \]
\[ [L_\lambda G] = (T + \frac{3}{2}\lambda)G, \quad [G_\lambda G] = 2L + \frac{c\lambda^2}{3}. \]
\(L(z)\) is called the Virasoro field. The Zhu algebra \(Z(\mathcal{N}S_c)\) is the free associative superalgebra in one odd generator \(\mathbb{C}[G]\).

Example 2.4. The \(N = 2\) superconformal vertex algebra

The \(N = 2\) superconformal vertex algebra of central charge \(c\) is generated by the Virasoro field \(L(z)\) with \(\lambda\)-bracket (2.1), an even primary field \(J(z)\) of conformal weight 1, and two odd primary fields \(G^\pm(z)\) of conformal weight \(\frac{3}{2}\), with the
remaining $\lambda$-brackets [16]

\[ [J_\lambda G^\pm] = \pm G^\pm, \quad [J_\lambda J] = \frac{c}{3} \lambda, \]
\[ [G^+ \lambda G^-] = L + \frac{1}{2} T J + \lambda J + \frac{c}{6} \lambda^2, \quad [G^\pm \lambda G^\pm] = 0. \]

**Example 2.5. The Universal affine vertex algebra** Let $g$ be a simple or commutative Lie algebra with non-degenerate invariant bilinear form $(,)$. The universal affine vertex algebra $V^k(g)$, $k \in \mathbb{C}$ is generated by fields $a, b \in g$ with the following $\lambda$-bracket:

\[ [a, b] = [a, b] + \lambda(k + h^\vee)(a, b), \quad [\bar{a}, \bar{b}] = (k + h^\vee)(a, b), \quad [a, \bar{b}] = [\bar{a}, b] = [a, b]. \]

Its corresponding Zhu algebra $Z(V^k(g)) = U(g)$, the universal enveloping algebra of $g$. If $k \neq -h^\vee$, choosing dual bases $\{a^i\}$, $\{a_i\}$ for $(,)$ we can write the field

\[ L := \frac{1}{2(k + h^\vee)} a^i a_i, \]

where $h^\vee$ is the dual Coxeter number of $g$ and we sum over repeated indexes. A simple computation shows that $L$ satisfies (2.1) and it is a superconformal vector.

Taking Zhu algebras for this morphism we find an embedding $C[x] \hookrightarrow U(g)$ of a polynomial algebra in $U(g)$ mapping the generator $x$ to the Casimir element of $g$.

**Example 2.6. The super-affine vertex algebra** Let $g$ be as above, We have a super-vertex algebra generated by even fields $a, b \in g$ and corresponding odd fields $\bar{a}, \bar{b}$ with the following $\lambda$-brackets ($k \in \mathbb{C}$):

\[ [a, b] = [a, b] + \lambda(k + h^\vee)(a, b), \quad [\bar{a}, \bar{b}] = (k + h^\vee)(a, b), \]
\[ [a, \bar{b}] = [\bar{a}, b] = [a, b]. \]

Let $\{a^i\}$, $\{a_i\}$ be dual bases as above. If $k \neq -h^\vee$, introduce the following odd field (cf. [6]):

\[ G = \frac{1}{k + h^\vee} \left( a^i a_i + \frac{1}{3(k + h^\vee)} [a^i, a^j][a_i, a_j] \right). \]

Then $G$ generates the super-vertex algebra of Example 2.3. Taking Zhu algebras we obtain the construction of the cubic Dirac operator of [20] (see [6]).

**2.3. SUSY vertex algebras.** In this section we collect some results on SUSY vertex algebras from [15].

Introduce formal variables $Z = (z, \theta)$ and $W = (w, \zeta)$, where $\theta, \zeta$ are odd anti-commuting variables and $z, w$ are even commuting variables. Given an integer $j$ and $J = 0$ or $1$ we put $Z^{iJ} = z^i \theta^J$.

Let $\mathcal{H}$ be the superalgebra generated by $\chi, \lambda$ with the relations $[\chi, \chi] = -2\lambda$, where $\chi$ is odd and $\lambda$ is even and central. We will consider another set of generators $-S, -T$ for $\mathcal{H}$ where $S$ is odd, $T$ is central, and $[S, S] = 2T$. Denote $\Lambda = (\lambda, \chi)$, $\nabla = (T, S)$, $\Lambda^{iJ} = \lambda^i \chi^J$ and $\nabla^{iJ} = T^i S^J$.

Given a super vector space $V$ and a vector $a \in V$, we will denote by $(-1)^a$ its parity. Let $U$ be a vector space, a $U$-valued formal distribution is an expression of the form

\[ \sum_{j \in \mathbb{Z}}^{j \neq \infty} Z^{-1-j} w_{(j, J)} \quad w_{(j, J)} \in U. \]
The space of such distributions will be denoted by $U[[Z, Z^{-1}]]$. If $U$ is a Lie algebra we will say that two such distributions $a(Z), b(W)$ are local if

$$(z - w)^n [a(Z), b(W)] = 0 \quad n \gg 0.$$ 

The space of distributions such that only finitely many negative powers of $z$ appear (i.e. $w_j(j) = 0$ for large enough $j$) will be denoted $U((Z))$. In the case when $U = \text{End}(V)$ for another vector space $V$, we will say that a distribution $a(Z)$ is a field if $a(Z)v \in V((Z))$ for all $v \in V$.

**Definition 2.7 ([15]).** An $N_K = 1$ SUSY vertex algebra consists of the data of a vector space $V$, an even vector $|0\rangle \in V$ (the vacuum vector), an odd endomorphism $S$ (whose square is an even endomorphism we denote $T$), and a parity preserving linear map $A \mapsto Y(A, Z)$ from $V$ to $\text{End}(V)$-valued fields (the state-field correspondence). This data should satisfy the following set of axioms:

- **Vacuum axioms:**
  $$Y(|0\rangle, Z) = \text{Id}, \quad Y(A, Z)|0\rangle = A + O(Z), \quad S|0\rangle = 0.$$

- **Translation invariance:**
  $$[S, Y(A, Z)] = (\partial_\theta - \theta \partial_z)Y(A, Z),$$
  $$[T, Y(A, Z)] = \partial_z Y(A, Z).$$

- **Locality:**
  $$(z - w)^n [Y(A, Z), Y(B, W)] = 0, \quad n \gg 0.$$ 

**Remark 2.8.** Given the vacuum axiom for a SUSY vertex algebra, we will use the state field correspondence to identify a vector $A \in V$ with its corresponding superfield $Y(A, Z)$.

Given a $N_K = 1$ SUSY vertex algebra $V$ and a vector $A \in V$, we expand the fields

$$(2.2) \quad Y(A, Z) = \sum_{j \in \mathbb{Z}, J = 0,1} Z^{-j-1} A(j, j),$$

and we call the endomorphisms $A(j, j)$ the Fourier modes of $Y(A, Z)$. Define now the operations:

$$(2.3) \quad [A_B] = \sum_{j \geq 0} \frac{A(j)}{j!} A(j, j)B, \quad AB = A(-1|1)B.$$ 

The first operation is called the $\Lambda$-bracket and the second is called the normally ordered product.

**Remark 2.9.** As in the standard setting, given a SUSY VA $V$ and a vector $A \in V$, we have:

$$Y(TA, Z) = \partial_z Y(A, Z) = [T, Y(A, Z)].$$

On the other hand, the action of the derivation $S$ is described by:

$$Y(SA, Z) = (\partial_\theta + \theta \partial_z)Y(A, Z) \neq [S, Y(A, Z)].$$
The relation with the standard field formalism is as follows. Suppose that $V$ is a vertex superalgebra as defined in section 2.1, together with a homomorphism from the $N = 1$ superconformal vertex algebra in example 2.3. $V$ therefore possesses an even vector $\nu$ of conformal weight 2, and an odd vector $\tau$ of conformal weight $\frac{3}{2}$, whose associated fields

$$Y(\nu, z) = L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$

$$Y(\tau, z) = G(z) = \sum_{n \in 1/2+\mathbb{Z}} G_n z^{-n-\frac{3}{2}},$$

have the $\lambda$-brackets as in example 2.3, and where we require $G_{-1/2} = S$ and $L_{-1} = T$. We can then endow $V$ with the structure of an $N_K = 1$ SUSY vertex algebra via the state-field correspondence [16]

$$Y(A, Z) = Y^c(A, z) + \theta Y^c(G_{-1/2} A, z),$$

where we have written $Y^c$ to emphasize that this is the usual state-field (rather than state–superfield) correspondence in the sense of section 2.1.

There exist however SUSY vertex algebras without such a map from the $N = 1$ superconformal vertex algebra.

**Definition 2.10.** Let $\mathcal{H}$ be as before. An $N_K = 1$ SUSY Lie conformal algebra is a $\mathcal{H}$-module $\mathcal{R}$ with an operation $[\Lambda]: \mathcal{R} \otimes \mathcal{R} \to \mathcal{H} \otimes \mathcal{R}$ of degree 1 satisfying:

1. **Sesquilinearity**
   
   $$[S a \Lambda b] = \chi [a \Lambda b], \quad [a \Lambda S b] = -(-1)^a (S + \chi) [a \Lambda b].$$

2. **Skew-Symmetry:**
   
   $$[b \Lambda a] = (-1)^{ab} [b - \Lambda - \nabla a].$$

   Here the bracket on the right hand side is computed as follows: first compute $[b \Gamma a]$, where $\Gamma = (\gamma, \eta)$ are generators of $\mathcal{H}$ super commuting with $\Lambda$, then replace $\Gamma$ by $(-\lambda - T, -\chi - S)$.

3. **Jacobi identity:**
   
   $$[a \Lambda [b \Gamma c]] = -(-1)^a [[a \Lambda b]_{\Gamma + \Lambda c} + (-1)^{(a+1)(b+1)} [b \Gamma [a \Lambda c]]],$$

   where the first bracket on the right hand side is computed as in Skew-Symmetry and the identity is an identity in $\mathcal{H} \otimes^2 \mathcal{R}$.

Given an $N_K = 1$ SUSY VA, it is canonically an $N_K = 1$ SUSY Lie conformal algebra with the bracket defined in (2.3). Moreover, given an $N_K = 1$ Lie conformal algebra $\mathcal{R}$, there exists a unique $N_K = 1$ SUSY VA called the universal enveloping SUSY vertex algebra of $\mathcal{R}$ with the property that if $W$ is another $N_K = 1$ SUSY VA and $\varphi : \mathcal{R} \to W$ is a morphism of Lie conformal algebras, then $\varphi$ extends uniquely to a morphism $\varphi : V \to W$ of SUSY VAs. The operations (2.3) satisfy:

- **Quasi-commutativity:**
  
  $$ab - (-1)^{ab} ba = \int_{-\Lambda}^{\Lambda} [a \Lambda b] d\Lambda.$$

- **Quasi-associativity**
  
  $$(ab)c - a(bc) = \sum_{j \geq 0} a_{(-j-2|1)} b_{(j|1)} c + (-1)^{ab} \sum_{j \geq 0} b_{(-j-2|1)} a_{(j|1)} c.$$
• Quasi-Leibniz (non-commutative Wick formula)

\[ (a \Lambda bc) = (a \Lambda b) c + (-1)^{(a+1)b}[a\Lambda c] + \int_0^\Lambda [a\Lambda b]c \, d\Gamma, \]

where the integral \( \int d\Lambda = \partial\chi \int d\lambda \). In addition, the vacuum vector is a unit for the normally ordered product and the endomorphisms \( S, T \) are odd and even derivations respectively of both operations.

2.4. Examples.

**Example 2.11.** Let \( \mathcal{R} \) be the free \( \mathcal{H} \)-module generated by an odd vector \( H \). Consider the following Lie conformal algebra structure in \( \mathcal{R} \):

\[ [H \Lambda H] = (2T + \chi S + 3\lambda)H. \]

This is the Neveu-Schwarz algebra (of central charge 0). This algebra admits a central extension of the form:

\[ [H \Lambda H] = (2T + \chi S + 3\lambda)H + \frac{c}{3}\chi\lambda^2, \]

where \( c \) is any complex number. The associated universal enveloping SUSY VA is the Neveu-Schwarz algebra of central charge \( c \). If we decompose the corresponding field

\[ H(z, \theta) = G(z) + 2\theta L(z), \]

then the fields \( G(z) \) and \( L(z) \) satisfy the commutation relations of the \( N = 1 \) super vertex algebra of Example 2.3.

**Example 2.12.** The \( N = 2 \) superconformal vertex algebra is generated by 4 fields \( \text{[16], Thm. 5.9, [15], Ex. 5.9]} \). In this context it is generated by two superfields – an \( N = 1 \) vector \( H \) as in Example 2.11 and an even current \( J \), primary of conformal weight 1, that is:

\[ [H \Lambda J] = (2T + 2\lambda + \chi S)J. \]

The remaining commutation relation is

\[ [J \Lambda J] = -(H + \frac{c}{3}\lambda\chi). \]

Note that given the current \( J \) we can recover the \( N = 1 \) vector \( H \). In terms of the fields of Example 2.4, \( H, J \) decompose as

\[
\begin{align*}
J(z, \theta) &= -\sqrt{-1}J(z) - \sqrt{-1} \theta \left(G^-(z) - G^+(z)\right), \\
H(z, \theta) &= \left(G^+(z) + G^-(z)\right) + 2\theta L(z).
\end{align*}
\]

**Example 2.13** (Super currents \[16, Thm. 5.9, [15, Ex. 5.9]\]). Let \( \mathfrak{g} \) be a finite dimensional Lie algebra with non-degenerate invariant form \( (, ) \). We construct an \( N_\mathfrak{g} = 1 \) SUSY vertex algebra generated by odd superfields:

\[ [a, b] = [a, b] + \chi(k + h^\vee)(a, b), \quad a, b \in \Pi \mathfrak{g}, \quad k \in \mathbb{C}. \]

Recall [17] (see also [15, Ex. 5.9], [19] in the superfield formalism) that when \( k \neq -h^\vee \), the superfield

\[
H_0 = \frac{1}{k + h^\vee} \left((S^{a^i})a_i + \frac{1}{3(k + h^\vee)} a^i(a^j[a_i, a_j])\right),
\]
where \( \{a_i\} \) and \( \{a^i\} \) are dual bases of \( \mathfrak{g} \) with respect to \((,\)\), generates an \( N = 1 \) SUSY vertex algebra of central charge

\[
c_0 = \frac{k \dim \mathfrak{g}}{k + h^\vee} + \frac{\dim \mathfrak{g}}{2},
\]
as in Example 2.11. Here \( h^\vee \) is the dual Coxeter number of \( \mathfrak{g} \). Moreover, for each \( a \in \mathfrak{g} \), the corresponding superfield is primary of conformal weight \( \frac{1}{2} \), namely:

\[
[H_0 \Lambda a] = (2 \lambda + \chi \mathcal{S} \otimes \mathcal{S}) a.
\]

Given any \( v \in \mathfrak{g} \) we can deform the field \( H_0 \) above as

\[
H = H_0 + T v,
\]
and it is straightforward to show that this field generates the Neveu Schwarz algebra of central charge

\[
c = c_0 - 3(k + h^\vee)(v, v).
\]

With respect to this superconformal vector, the fields \( a \in \mathfrak{g} \) are no longer primary.

2.5. Manin triples and \( N = 2 \) structures. In this section we extend the \( N = 1 \) structure of Example 2.13 to an \( N = 2 \) structure when \( g = \mathfrak{h} \oplus \mathfrak{h}^\ast \) is the double of a Lie bialgebra. The construction here presented is a particular case of the construction of E. Getzler \([10]\). We include here the proofs since we will need an algebroid version of this below. Let \( \{e_i\} \) be a basis for \( \mathfrak{h} \) and let \( \{e^i\} \) be the dual basis for \( \mathfrak{h}^\ast \). We let

\[
J := \frac{i}{k + h^\vee} e^i e_i,
\]
be the even superfield of \( V^k(\mathfrak{g}_{\text{super}}) \) corresponding to the standard R-matrix. Recall that \( 2h^\vee \) is the eigenvalue of the Casimir of \( \mathfrak{g} \) in its adjoint representation. Note that the element \( v'_h := [e^i, e_i] \in \mathfrak{g} \) does not depend on the choice of basis. Decomposing \( v'_h = w + w^\ast \), where \( w \in \mathfrak{h} \) and \( w^\ast \in \mathfrak{h}^\ast \), we define the element

\[
v_h = w - w^\ast,
\]
and with a simple computation we find

\[
(v_h, v_h) = -\frac{2}{3} h^\vee \dim \mathfrak{h}.
\]

**Proposition 2.14.**

1. \( J \) satisfies \( [J \Lambda J] = -(H + \frac{c \chi \mathcal{S}}{3}) \) where \( H \) is the odd superfield given by

\[
H := \frac{1}{(k + h^\vee)^2} \left[ e^i (e^j [e_i, e_j]) + e_i (e_j [e^i, e^j]) \right] + \frac{1}{(k + h^\vee)} (e_j (Se^j) + e^j Se_j) + \frac{1}{k + h^\vee} T v_h.
\]

and \( c = 3 \dim \mathfrak{h} \).

2. The superfields \( J \) and \( H \) generate an \( N = 2 \) vertex algebra of central charge \( c = 3 \dim \mathfrak{h} \) as in Example 2.12.
Proof. Let $c^i_j$ and $c^i_{kj}$ be the structure constants of $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively in the bases $\{e_i\}, \{e^i\}$. In order to compute $[J_J]$ we start with:

\begin{equation}
[c^i_j e^j_i] = ([e_j, e^i] + \chi(k + h^\vee)\delta^i_j) e_i + e^i[e_j, e^j] + \int_0^\Lambda \eta(k + h^\vee)([e_j, e^i], e_i) d\Gamma
\end{equation}

$$
= (c^i_{jk} e_k e_i - c^i_{kj} e^j e_i + c^i_{kj} e^j e_k) + \chi(k + h^\vee) e_j + \lambda(k + h^\vee) c^i_j
\end{equation}

$$
= c^i_{jk} e_k e_i + \chi(k + h^\vee) e_j + \lambda(k + h^\vee) c^i_j.
$$

By skew-symmetry we obtain:

\begin{equation}
[e^i e^j_i] = c^i_{jk} e_k e_i - (\chi + S)(k + h^\vee) e_j - \lambda(k + h^\vee) c^i_j.
\end{equation}

Similarly we have

\begin{equation}
[e^i e^j] = c^i_{jk} e^j e^k - \chi(k + h^\vee) e^j + \lambda(k + h^\vee) c^i_j
\end{equation}

$$
\text{and by skew-symmetry we obtain:}
\begin{equation}
[e^i e^j e^j] = c^i_{jk} e^j e^j e_i + (\chi + S)(k + h^\vee) e^j + \lambda(k + h^\vee) c^i_j + \int_0^\Lambda c^i_{jk} [e^i e^j e^j] d\Gamma + \int_0^\Lambda (\eta - \chi)(k + h^\vee) ([e^i, e^j] + \eta(k + h^\vee) \dim \mathfrak{g}) d\Gamma.
\end{equation}

We can compute the integral term easily as:

$$
2\chi(k + h^\vee) c^i_j e^j + \chi(k + h^\vee) |e^i, e^j| + \lambda\chi(k + h^\vee)^2 \dim \mathfrak{g},
$$

and replacing in (2.14) we obtain:

\begin{equation}
[e^i e^j] = (c^i_{jk} e^j e^j + (\chi + S)(k + h^\vee) e^j) e_j + e^j (c^i_{jk} e_k e_i + (\chi + S)(k + h^\vee) e_j) + \lambda\chi(k + h^\vee)^2 \dim \mathfrak{g} = e^j (c^i_{jk} e^j e_i + 2(k + h^\vee) c^i_{jk} T e^j + \epsilon_i(e^j, e^j) + (k + h^\vee) (e^j Se_j + e_j Se^j) + (k + h^\vee) T[e^j, e^j] + \lambda\chi(k + h^\vee)^2 \dim \mathfrak{g}) = (k + h^\vee) (e^j Se_j + e_j Se^j) + e^j (c^i_{jk} e_i e_k + e_i(e^j, e^i)) + e^j (c^i_{jk} e^j e_i) + \lambda\chi(k + h^\vee)^2 \dim \mathfrak{g},
\end{equation}

proving (1). In order to prove (2), a simple computation shows that $H$ can be written as in (2.7) with $v = \frac{2k \dim \mathfrak{h}}{k + h^\vee}$. Therefore $H_0$ generates an $N = 1$ algebra of central charge $c_0 = \frac{2k \dim \mathfrak{h}}{k + h^\vee} + \dim \mathfrak{h}$ and each element of $\mathfrak{g}$ is a primary field of
conformal weight 1/2 with respect to $H_0$ [6]. It follows from (2.8) that the central
charge of $H$ is given by $c = 3 \dim \mathfrak{h}$. We only need to prove that $J$ is primary of
conformal weight 1. For this we compute:

$$
[H_A e^i e_i] = \left( (2T + \lambda + \chi S) e^i - \frac{\lambda}{k + h} \right) [v_b, e^i] - \lambda \chi(v_b, e^i) e_i
$$

$$
+ e^i \left( 2T + \lambda + \chi S e_i \right) e_i - \frac{\lambda}{k + h} \right) [v_b, e^i] - \lambda \chi(v_b, e_i) e_i + e^i [v_b, e_i]
$$

$$
\int_0^\Lambda (-2\gamma + \lambda - \chi \eta) ([e^i, e_i] + \eta(k + h \gamma) \dim \mathfrak{h}) - \lambda \eta(v_b, [e^i, e_i]) d\Gamma,
$$

$$
= (2T + 2\lambda + \chi S)e^i e_i - \frac{\lambda}{k + h} ([v_b, e^i] e_i + e^i [v_b, e_i])
$$

Expanding $v_b = c_{kj} e_k + c_{kj} e^k$ we obtain

$$
[v_b, e^i] e_i = c_{kj} e_k e^i e_i - c_{kj} e_{ki} e^i e_i + c_{kj} c_{ki} e^i e_i,
$$

and similarly

$$
e^i [v_b, e_i] = c_{kj} e^i e^i e_k e^i + c_{kj} c_{ki} e^i e^i e_k - c_{kj} c_{ki} e^i e^i e_k,
$$

from where we obtain

$$
[H_A J] = (2T + 2\lambda + \chi S)J - \frac{i \lambda}{(k + h)2} \left( \text{tr}_{\mathfrak{h}} \text{ad}([e^i, e^i]) e_i e_j + \text{tr}_{\mathfrak{h}} \text{ad}([e_i, e_j]) e^i e^j \right) = (2T + 2\lambda + \chi S)J.
$$

\[\square\]

3. Preliminaries on geometry

In this section we recall the basic definitions of generalized complex geometry
following [12] and [13]. We also briefly recall the notion of unimodularity for a Lie
algebroid due to Weinstein [24].

Let $M$ be a smooth manifold and denote by $T$ the tangent bundle of $M$.

**Definition 3.1.** A Courant algebroid is a vector bundle $E$ over $M$, equipped with
a nondegenerate symmetric bilinear form $\langle, \rangle$ as well as a skew-symmetric bracket
$[\cdot, \cdot]$ on $C^\infty(E)$ and with a smooth bundle map $\pi : E \to T$ called the anchor. This
induces a natural differential operator $D : C^\infty(M) \to C^\infty(E)$ as $\langle Df, A \rangle = \frac{1}{2} \pi(A) f$
for all $f \in C^\infty(M)$ and $A \in C^\infty(E)$. These structures should satisfy:

1. $\pi([A, B]) = [\pi(A), \pi(B)]$, \hspace{1em} $\forall A, B \in C^\infty(E)$.
2. The bracket $[,]$ should satisfy the following analog of the Jacobi identity.

If we define the **Jacobiator** as $\text{Jac}(A, B, C) = [[A, B], C] + [[B, C], A] + [[C, A], B]$. And the **Nijenhuis** operator

$$
\text{Nij}(A, B, C) = \frac{1}{3} ([A, B], C) + ([B, C], A) + ([C, A], B).
$$

Then the following must be satisfied:

1. $\text{Jac}(A, B, C) = D (\text{Nij}(A, B, C))$, \hspace{1em} $\forall A, B, C \in C^\infty(E)$
2. $[A, f B] = (\pi(A)f) B + f[A, B] - \langle A, B \rangle Df$, for all $A, B \in C^\infty(E)$ and
   $f \in C^\infty(M)$
3. $\pi \circ D = 0$, i.e. $\langle Df, Dg \rangle = 0$, \hspace{1em} $\forall f, g \in C^\infty(M)$. 

\[
\pi(A) \langle B, C \rangle = \langle [A, B] + D(A, B), C \rangle + \langle B, [A, C] + D(A, C) \rangle, \quad \forall A, B, C \in C^\infty(E).
\]

A Courant algebroid \( E \) is called exact if the following sequence is exact:
\[
0 \to T^* \xrightarrow{\pi^*} E \xrightarrow{\pi} T \to 0,
\]
where we use the inner product in \( E \) to identify it with its dual.

This definition extends easily to the complexified situation.

**Example 3.2.** \( E = (T \oplus T^*) \otimes \mathbb{C}, \langle \cdot, \cdot \rangle \) and \([\cdot, \cdot]\) are respectively the natural symmetric pairing and the Courant bracket defined as:
\[
\langle X + \zeta, Y + \eta \rangle = \frac{1}{2} (i_X \eta + i_Y \zeta).
\]
\[
[X + \zeta, Y + \eta] = [X, Y] + \text{Lie}_X \eta - \text{Lie}_Y \zeta - \frac{1}{2} d(i_X \eta - i_Y \zeta).
\]

From now on, we will work only with exact Courant algebroids, although some of the results (notably Prop. 4.1) hold in a more general case.

**Definition 3.3 ([12, 4.14]).** A generalized almost complex structure on a real \( 2n \)-dimensional manifold \( M \) is given by the following equivalent data:
- an endomorphism \( J \) of \( E \simeq T \oplus T^* \) which is orthogonal with respect to the inner product \( \langle \cdot, \cdot \rangle \).
- a maximal isotropic sub-bundle \( L < E \otimes \mathbb{C} \) of real index zero, i.e. \( L \cap \bar{L} = 0 \).
- a pure spinor line sub-bundle \( U < \bigwedge^* T^* \otimes \mathbb{C} \), called the canonical line bundle satisfying \( (\varphi, \bar{\varphi}) \neq 0 \) at each point \( x \in M \) for any generator \( \varphi \in U_x \).

Here \( \langle \cdot, \cdot \rangle \) is the natural inner product induced from \( \langle \cdot, \cdot \rangle \).

The fact that \( L \) is of real index zero implies
\[
E \otimes \mathbb{C} \simeq (T \oplus T^*) \otimes \mathbb{C} = L \oplus \bar{L} = L + L^*,
\]
using \( \langle \cdot, \cdot \rangle \) to identify \( L \) with \( L^* \).

**Definition 3.4 ([12, 4.18]).** A generalized almost complex structure \( \mathcal{J} \) is said to be integrable to a generalized complex structure when its \(+i\)-eigenvalue \( L < E \otimes \mathbb{C} \) is Courant involutive.

In this case, \( L \) is a Lie bi-algebroid, and \( E \otimes \mathbb{C} \) could be viewed as its Drinfeld double. Note that \( E \) acts on the sheaf of differential forms \( \bigwedge^* T^* \) via the spinor representation, and this sheaf acquires a different grading by the eigenvalues of \( \mathcal{J} \) acting via the spinor representation:
\[
\bigwedge^* T^* = U_{-n} \oplus \cdots \oplus U_n.
\]
Clifford multiplication by sections of \( \mathcal{T} \) (resp. \( L \)) increases (resp. decreases) the grading, \( U_{-n} = U_{\mathcal{J}} \) is the canonical bundle of \((M, \mathcal{J})\). Given a non-vanishing global section of \( U_{\mathcal{J}} \), we obtain an isomorphism of sheaves:
\[
\bigwedge^k \mathcal{T} \simeq U_{k-n}.
\]
(3.1)
The de Rham differential can be split as \( d = \partial + \bar{\partial} \) such that \( \partial : U_k \to U_{k-1} \) and \( \bar{\partial} : U_k \to U_{k+1} \).
**Definition 3.5.** A generalized complex manifold \((M, \mathcal{J})\) is called generalized Calabi-Yau if the bundle \(U_\mathcal{J}\) is holomorphically trivial, i.e. it admits a non-vanishing closed global section.

In this case the isomorphism (3.1) allows us to identify the complex \((U_\ast, \bar{\partial})\) with the complex computing the Lie algebroid cohomology of \(L\). Recall that given any Lie algebroid \(L\) over \(M\), we can define a differential \(d_L : C^\infty(M) \to C^\infty(L^*)\) as \((d_L f)(l) = \pi_L(l) f\), where \(l\) is a section of \(L\) and \(\pi_L\) is the anchor map of \(L\). This differential can be extended to \(\bigwedge^\ast L^\ast\) by imposing the Leibniz rule in the usual way (for \(\zeta \in C^\infty(\bigwedge^{k-1} L^\ast)\)):

\[
(3.2) \quad (d_L \zeta)(l_1, \ldots, l_k) = \sum_i (-1)^{i+1} \pi(l_i) \zeta(l_1, \ldots, \hat{l}_i, \ldots, l_k) + \sum_{i<j} (-1)^{i+j} \zeta([l_i, l_j], \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_k).
\]

The cohomologies of the complex \((\bigwedge^\ast L^\ast, d_L)\) are denoted by \(H^\ast(L)\) and are called the Lie algebroid cohomologies of \(L\) (with trivial coefficients). If \((M, \mathcal{J})\) is a generalized Calabi-Yau manifold. The isomorphism of (3.1) is an isomorphism of complexes (using \(L = L^\ast\)). Moreover, in this case, the Lie algebroids \(L\) and \(L^\ast\) are both unimodular, a notion that we now recall.

For a Lie algebroid \(L\) we have the corresponding sheaf of twisted differential operators \(U(L)\). The sheaf \(\bigwedge^{\text{top}} T^\ast\) is always a right twisted D-module, and the corresponding left \(U(L)\)-module is then the line bundle \(Q_L = \bigwedge^{\text{top}} L \otimes \bigwedge^{\text{top}} T^\ast\). Suppose for simplicity that the line bundle \(Q_L\) is trivial, for each non-vanishing section \(s\) of \(Q_L\) we can define \(\theta_s \in C^\infty L^\ast\) by

\[
\theta_s(l)s = l \cdot s,
\]

where we use the left D-module structure of \(Q_L\) on the RHS. It turns out that \(\theta\) gives rise to a well defined element of \(H^1(L, Q_L)\), the first Lie algebroid cohomology of \(L\) with coefficients in \(Q_L\) (see [7] for details).

**Definition 3.6 ([24]).** A Lie algebroid \(L\) is called unimodular if the class \(\theta \in H^1(L, Q_L)\) above constructed vanishes.

Now let \(L\) be a unimodular Lie algebroid (of rank \(k\)). Let \(U\) be open in \(M\) and choose a local frame \(\{e_i\}\) for \(L\), we obtain a section \(s = e_1 \wedge \cdots \wedge e_k\) for \(\bigwedge^{\text{top}} L\). We can choose a local volume form \(\mu \in C^\infty(\bigwedge^{\text{top}} T^\ast)\) such that the class \(\theta\) is represented by zero (we may need to shrink \(U\)). If we define the structure constants of \(L\) by \([e_i, e_j] = c_{ij}^k e_k\) we obtain the identity:

\[
(3.3) \quad \text{div}_\mu e_k = -c_{kj}^i,
\]

where we sum over repeated indexes and \(\text{div}_\mu e_k\) is the divergence of \(e_k\) with respect to the volume form \(\mu\), defined by

\[
(\text{div}_\mu e_k)\mu = \text{Lie}_{\pi_L(e_k)} \mu.
\]

We have the following Proposition (see for example [2, Theorem 10])

**Proposition 3.7.** A generalized complex manifold \(M\) is generalized Calabi-Yau if and only if \(U_{\mathcal{J}}\) is trivial and \(L\) is unimodular.
Recall from [12, Prop. 2.2.2] that we have an isomorphism
\[ U_J \otimes U_J \simeq \det L \otimes \det T^* . \]

Therefore given a generalized Calabi-Yau manifold \( M \) and a local frame \( \{ e_i \} \) for \( L \), we can choose a closed pure spinor such that the corresponding volume form \( \mu \) satisfies (3.3).

Given a generalized complex manifold \( (M, J) \) the projection \( F = \pi_T(L) \) gives rise to a smooth integrable distribution \( \Delta \) defined by \( \Delta \otimes C = F \cap \bar{F} \).

**Proposition 3.8** ([12]). Let \( (M, J) \) be a generalized complex manifold, and let \( x \) be a point in \( M \) such that \( \dim \Delta \) is locally constant at \( x \). Then there exists an open neighborhood of \( x \) in \( M \) which is expressed as a product of a complex manifold times a symplectic one.

We remark however that there exist generalized Complex and generalized Calabi-Yau manifolds with points where \( \dim \Delta \) is not locally constant.

4. Sheaves of vertex algebras

In this section we recall some results from [11] and [5] in the language of SUSY vertex algebras, following [14].

Let \( (E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi) \) be a Courant algebroid. Let \( \Pi E \) be the corresponding purely odd super vector bundle. we will abuse notation and denote by \( \langle \cdot, \cdot \rangle \) the corresponding super-skew-symmetric bilinear form, and by \( [\cdot, \cdot] \) the corresponding super-skew-symmetric degree 1 bracket on \( \Pi E \). Similarly, we obtain an odd differential operator \( D : C^\infty(M) \to C^\infty(\Pi E) \). If no confusion should arise, when \( v \) is an element of a vector space \( V \), we will denote by the same symbol \( v \) the corresponding element of \( \Pi V \), where \( \Pi \) is the parity change operator. Recall that for sections of \( E \) we have the Dorfman bracket \( \circ \) which is defined in terms of the Courant bracket and \( D \) as
\[ X \circ Y = [X, Y] + D(X, Y). \]

The following proposition from from [14] (cf. [5]) describes the construction of the chiral de Rham complex in parallel to the construction of twisted differential operators given a Lie algebroid:

**Proposition 4.1.** For each complex Courant algebroid \( E \) over a differentiable manifold \( M \), there exists a sheaf \( \mathcal{U}_{\text{ch}}(E) \) of SUSY vertex algebras on \( M \) generated by \( \text{functions } \mathcal{I} : C(M) \hookrightarrow \mathcal{U}_{\text{ch}}(E) \), and sections of \( \Pi E, \mathcal{J} : C(\Pi E) \hookrightarrow \mathcal{U}_{\text{ch}}(E) \) subject to the relations:

1. \( \mathcal{I} \) is an “embedding of algebras”, i.e. \( \mathcal{I}(1) = \{ 0 \} \), and \( \mathcal{I}(f g) = \mathcal{I}(f) \cdot \mathcal{I}(g) \), where in the RHS we use the normally ordered product in \( \mathcal{U}_{\text{ch}}(E) \).
2. \( \mathcal{J} \) imposes a compatibility condition between the Dorfman bracket in \( E \) and the Lambda bracket in \( \mathcal{U}_{\text{ch}}(E) \):
\[ [\mathcal{J}(X) \Lambda \mathcal{J}(Y)] = \mathcal{J}(X \circ Y) + 2\chi \mathcal{I}([X, Y]). \]
3. \( \mathcal{I} \) and \( \mathcal{J} \) preserve the \( \Theta \)-module structure of \( E \), i.e. \( \mathcal{J}(fX) = \mathcal{I}(f) \cdot \mathcal{J}(X) \).
4. \( D \) and \( S \) are compatible, i.e. \( Df = Si(f) \).
5. We impose the usual commutation relation
\[ [\mathcal{J}(X) \Lambda \mathcal{I}(f)] = \mathcal{I}(\pi(X)f). \]
In the particular case when $E = (T \oplus T^*) \otimes \mathbb{C}$ is the standard Courant algebroid, then $U^\text{ch}(E)$ is the chiral de Rham complex of $M$, denoted by $\Omega^\text{ch}_M$ for historical reasons.

Using this proposition, we will abuse notation and use the same symbols for sections of $E \otimes \mathbb{C}$ when they are viewed as sections of $U^\text{ch}(E)$.

5. $N = 2$ supersymmetry

In this section we state the main results of this article, we postpone their proofs for the Appendix. We also show that these supersymmetries generalize those of [1] and [14] in the Calabi-Yau and symplectic case.

Let $M$ be an orientable and differentiable manifold of Real dimension $N$. Let $T := T_M$ be its tangent bundle and $T^* := T_M^*$ its cotangent bundle, and let $E$ be an exact Courant algebroid on $M$. Let $J \in \text{End} \, E \simeq \text{End}(T \oplus T^*)$ be a generalized complex structure and let $L \leq (T \oplus T^*) \otimes \mathbb{C}$ be the corresponding Dirac structure, and let $\rho := \rho^i_j$ be the transition functions for $L$. Let \{e_i\} be a local frame for $L$ and \{e^j\} be the corresponding dual frame for $L^* \simeq \widetilde{L}$, i.e. $\langle e^i, e_j \rangle = \delta^i_j$.

Let $U := U_L \leq \wedge^N T^*$ be the associated canonical bundle. It follows from (3.4) that if $U$ is trivial, then $\det L$ and $\det L^*$ are trivial bundles. Moreover, we will fix a closed pure spinor and a corresponding volume form $\mu$ such that (3.3) holds for the frame \{e_i\} and the corresponding dual statement holds for the frame \{e^i\}.

**Lemma 5.1.** Let $M$ be a generalized Calabi-Yau manifold. Then the following defines a global section of $U^\text{ch}(E)$: (we sum over repeated indexes)

\begin{equation}
J = \frac{-1}{2} e^i e_i.
\end{equation}

**Proof.** It follows from Proposition 4.1 that under a change of coordinates, $J$ transforms as:

\begin{equation}
\frac{i}{2} \left( (\rho^*)^i_j e^j \right) (\rho^i_k e_k).
\end{equation}

To apply quasi-associativity we need to compute the $\chi$ terms in the Lambda bracket $[e^j, \rho^i_k e_k]$ and these are easily shown to be $2\chi \rho^i_k \delta^j_i$. Therefore (5.2) reads:

\begin{equation}
\frac{i}{2} (\rho^*)^i_j \left[ e^j (\rho^i_k e_k) \right] + iT \left( (\rho^*)^i_j \right) \rho^j = \frac{i}{2} (\rho^*)^i_j \left[ e^j (e_k \rho^i) \right] +
\end{equation}

\begin{equation}
+ iT \left( (\rho^*)^i_j \right) \rho^j = \frac{i}{2} (\rho^*)^i_j \left[ (e^j e_k) \rho^i \right] + iT \left( (\rho^*)^i_j \right) \rho^j =
\end{equation}

\begin{equation}
= \frac{i}{2} \left[ (\rho^*)^i_j \rho^i \left] (e^j e_k) \right. + iT \left( (\rho^*)^i_j \right) \rho^j = \frac{i}{2} e^i e_i + iT ((\rho^*)^i_j) \rho^j.
\end{equation}

Using

\begin{equation}
\frac{\partial \det \rho}{\partial x_a} = \frac{\partial \det \rho}{\partial \rho^i_j} \frac{\partial \rho^i_j}{\partial x_a} = (\det \rho)(\rho^{-1})^i_j \frac{\partial \rho^i_j}{\partial x_a},
\end{equation}

we see that this becomes:

\begin{equation}
\frac{i}{2} e^i e_i + i \det \rho \, T \, \det \rho^{-1}.
\end{equation}

The second term of this last expression can be chosen to be zero if $c_1(L) = 0$, which in turn happens if $U$ is trivial. \hfill \Box
Remark 5.2. Note that in order to construct these sections, in [1] and [14] the authors used a connection on $T_M$. This is replaced in this setting with the existence of a global section of $U$.

Lemma 5.3. Let $M$ be a generalized Calabi-Yau manifold. Define a local section of $U^{\text{ch}}(E)$ by:

\begin{equation}
H := \frac{1}{4} \left[ e^i \left( e^j [e_i, e_j] \right) + e_i \left( e_j [e_i, e^j] \right) \right] - \frac{i}{2} T J [e^i, e_i] + \frac{1}{2} \left( e_j S e^j + e^j S e_j \right).
\end{equation}

Then the following is true:

(1) $H$ defines a global section of $U^{\text{ch}}(E)$.

(2) We have the following OPE:

\[ [J_X, J] = - \left( H + \frac{c}{3} \lambda \chi \right), \quad c = 3 \dim \mathbb{R} M. \]

Proof. The proof can be found in the appendix. \qed

Remark 5.4. Note that the field $H$ has the form (2.9) with $k + h = 2$. We therefore may view this construction as an algebroid generalization of the Kac-Todorov or super-Sugawara construction.

Theorem 5.5. Let $M$ be a generalized Calabi-Yau manifold. Let $J$ and $H$ be the corresponding sections of $U^{\text{ch}}(E)$ as constructed in Lemmas 5.1 and 5.3.

(1) For a function $f \in C^\infty(M)$, the corresponding field of $U^{\text{ch}}(E)$ is primary of conformal weight 0 with respect to $H$, namely:

\[ [H_A f] = (2 T + \chi S) f. \]

(2) For a section $X \in C^\infty(E)$, the corresponding field of $U^{\text{ch}}(E)$ has conformal weight $1/2$ with respect to $H$, but it is not primary, it satisfies:

\begin{equation}
[H_A X] = (2 T + \chi S) X + \lambda \chi \text{div}_\mu X.
\end{equation}

(3) The fields $H$ and $J$ generate an $N = 2$ SUSY vertex algebra of central charge $c = 3 \dim \mathbb{R} M$.

Proof. The proof can be found in the Appendix. \qed

Definition 5.6. Given a generalized Calabi-Yau manifold $M$, we will say that $M$ has a nice (local) volume form if we can find a (local) volume form $\mu$ and (local) dual frames $\{e_i\}$, $\{e^i\}$ for $L$ and $L^*$ such that

\[ \text{div}_\mu e_i = \text{div}_\mu e^i = 0 \quad \forall i. \]

Using Prop. 3.7, this is equivalent to finding local frames such that $\sum_i [e^i, e_i] = 0$.

Remark 5.7.

(1) Calabi-Yau manifolds and symplectic manifolds admit nice volume forms. Moreover, if the generalized complex structure is regular (i.e. there is no type jump), then $M$ admits a nice volume form. Indeed, using Prop. 3.8 one can find local frames $\{e_i\}$ and $\{e^i\}$ such that all structure constant vanish. In this case, the first two terms in the field $H$ in Lemma 5.3 vanish. We do not know if every generalized Calabi-Yau manifold admits a nice local volume form.
On the other hand, there are examples\(^1\) of unimodular Lie algebroids not admitting local frames \(\{e_i\}\) with \(\text{div}_\mu e_i = 0\) \(\forall i\).

(2) The proof of Theorem 5.5 is much simpler in the regular case as \(H\) would then be quadratic (see Remark 5.7). Moreover, if \(M\) admits a nice volume form, the proof of this Theorem, while still a long computation, would be much simpler.

In the general case we have to make explicit use of unimodularity for \(L\) and \(L^*\) to find a spinor and volume form such that \((3.3)\) holds.

(3) It follows from \((5.6)\) that if the manifold \(M\) admits a local nice volume form, we can find a local frame for \(E \simeq T \oplus T^*\) consisting of primary fields.

**Example 5.8.** Let \(M\) be a complex manifold, with complex structure \(J\), then we can consider the generalized complex structure:

\[ J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}. \]

The corresponding line bundle \(U\) is the canonical bundle of \(M\), thus we are in the usual Calabi-Yau case. \(L = T_{1,0} \oplus T_{0,1}\) and choosing holomorphic coordinates we can take \(e_\alpha = \partial_\alpha, e_\bar{\alpha} = dz^\alpha\). Similarly \(e^\alpha = 2dz^\alpha, e^{\bar{\alpha}} = 2\partial_{z^{\bar{\alpha}}}\). Finally, we have a global holomorphic volume form \(\Omega\), which satisfies:

\[ \Omega \wedge \overline{\Omega} = \sqrt{\det g_{j\bar{k}}} dz^1 \ldots dz^N d\bar{z}^1 \ldots d\bar{z}^N, \]

where \(g = g_{ij}\) is the Kähler metric on \(M\). We can pick coordinates where the volume form \(\Omega \wedge \overline{\Omega}\) is constant. In this frame, the fields \(J\) and \(H\) are (note that all Courant brackets vanish):

\[ J = idz^\alpha \partial_{z^\alpha} - idz^{\bar{\alpha}} \partial_{z^{\bar{\alpha}}}, \]
\[ H = \partial_{z^\alpha} Tz^\alpha + dz^{\bar{\alpha}} S \partial_{z^{\bar{\alpha}}} + dz^\alpha \partial_{z^\alpha} + \partial_{z^{\bar{\alpha}}} Tz^{\bar{\alpha}}, \]

which are the generators of the \(N = 2\) superconformal structure of \([23]\).

**Example 5.9.** In the symplectic case we have a generalized complex structure

\[ J = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}, \]

where \(\omega\) is a symplectic form viewed as a map \(T \to T^*\). We can use Darboux’ Theorem to find coordinates \(x^i, y^i\) such that \(\omega\) takes the standard form:

\[ \omega = dx^1 \wedge dy^1 + \ldots dx^N \wedge dy^N. \]

In this coordinate system we can take \(\{\partial_{x^i} - idy^i, dx^i - i\partial_{y^i}\}\) as a frame for \(L\) and \(\{dx^i + i\partial_{y^i}, \partial_{x^i} + idy^i\}\) as the dual frame. The supersymmetry generators are now

\[ J = \partial_{x^i} \partial_{y^i} + dx^i dy^i, \]
\[ H = dx^i S \partial_{x^i} + dy^i S \partial_{y^i} + T x^i \partial_{x^i} + T y^i \partial_{y^i}, \]

and these are the generators of \([14, \text{Lem. 5.2}]\).

In general, from the frames \(\{e_i\}\) and \(\{e^i\}\), we obtain an orthonormal frame \(\{a^i\}\) for the Courant algebroid \(E \otimes \mathbb{C} \simeq (T \oplus T^*) \otimes \mathbb{C}\) as \(a^i = \frac{1}{\sqrt{2}} (e^i + e_i)\) for

\(^1\)We owe A. Weinstein for an explanation of this point.
i = 1, \ldots, N = \dim M \text{ and } a^i = \frac{1}{\sqrt{2}}(e^i - e_{i-N}) \text{ for } i = N + 1, \ldots, 2N. \text{ If the manifold } M \text{ admits a nice volume form, then in this frame, the section } H \text{ looks like}

\begin{equation}
H = \frac{1}{2} \sum_{i=1}^{N} a^i S a^i + \frac{1}{12} \sum_{i,j=1}^{N} [a^i, a^j](a^i a^j).
\end{equation}

For a general orientable manifold \( M \) with a Courant algebroid \( E \) over it, we do not know if the expression (5.7) defines a global section of \( U^{\text{ch}}(E) \).

5.1. Topological twist. As with any vertex algebra with an \( N = 2 \) superconformal structure, some of the Fourier modes of the superfields \( H \) and \( J \), when expanded as in (2.2), play a special role. The operator \( L_0 := \frac{1}{2}(H_{(1|0)} + J_{(0|1)}) \) acts diagonally on \( V \) and its eigenvalues are called the conformal weights of the corresponding states. These conformal weights are different from those we were considering before, and they correspond to the twisted theory.

Similarly the eigenvalues of \( J_0 := -iJ_{(0|1)} \) are called charge. Sections of \( L \) have conformal weight 1 and charge \( -1 \), sections of \( L^* \) have conformal weight 0 and charge 1 (this follows for example from (A.2) and (A.4) below), while functions have conformal weight 0 and charge 0. It follows easily that the conformal weight zero part of \( U^{\text{ch}}(E) \) is just \( \bigwedge^* L^* \) and the natural grading on \( \bigwedge^* L^* \) coincides with the charge gradation on \( U^{\text{ch}}(E) \).

There are also two more important Fourier modes: the BRST charge \( Q_0 \), and the homotopy operator \( G_0 \) (we maintain here the notation of [23]). In this context \( Q_0 := \frac{1}{2}(H_{(0|1)} + iJ_{(0|0)}) \) and \( G_0 := \frac{1}{2}(H_{(0|1)} - iJ_{(0|0)}) \) (these correspond to the zero modes of the fields \( \mathcal{G}^{\pm} \) appearing in (2.4)). The operator \( Q_0 \) increases the charge and squares to zero, and we consider the complex \( (U^{\text{ch}}(E), Q_0) \). The operator \( G_0 \) decreases the charge and also squares to zero. In a similar way as in [23] we have:

**Theorem 5.10.** For a generalized Calabi-Yau manifold we have a quasi-isomorphism of complexes

\begin{equation}
(U_*, \tilde{\partial}) \simeq (\bigwedge^* L^*, d_L) \hookrightarrow (U^{\text{ch}}(E), Q_0)
\end{equation}

**Proof.** We have \([G_0, Q_0] = L_0\), hence the operator \( G_0 \) is an homotopy to zero for the operator \( Q_0 \) away from conformal weight 0. We need to show that \( Q_0 \) restricted to \( \bigwedge^* L^* \) acts as \( d_L \). The question is local in \( M \), therefore we may restrict ourselves to a small neighborhood with dual frames \( \{e_i\} \) and \( \{e^i\} \) for \( L \) and \( L^* \). \( Q_0 \) being a zero mode of a field is a derivation of all products in the vertex algebra, therefore it suffices to check that it acts as \( d_L \) on functions and the sections \( e^i \). This last statement is a straightforward computation. \( \square \)

**Remark 5.11.**

1. It is well known that the BRST cohomology of a topological vertex algebra carries the structure of a Gerstenhaber algebra [21] (see also [9] and references therein). In this case we see that we recover the Gerstenhaber algebra structure in the Lie algebroid cohomology of a generalized Calabi-Yau manifold. Indeed the sheaf \( U^{\text{ch}}(E) \) itself can be viewed as a sheaf of \( G_\infty \)-algebras [9].
(2) Just as in the usual case, the zero-th Fourier modes above defined make sense on any generalized complex manifold, even if the superfields $H$ and $J$ are not well defined. We will not pursue this further in this article.

(3) It is not obvious how to diagonalize $J_0$ on $U_{\text{ch}}(E)$, for example, the local sections

$$J_i = Se_i + \frac{1}{4}c_{jk}^i e_j e_k,$$

have charge zero and conformal weight 1. The BRST charge is the residue of the field

$$Q = e^i J_i + \frac{1}{4} e^i (e^j [e_i, e_j]),$$

and note the similarity of this field with the Chevalley differential computing Lie algebra cohomology.

The notion of conformal weight does not make sense for differential forms nor vector fields. Instead, one has to consider the mixed sections $e^i$. This is particularly interesting when the generalized Calabi-Yau manifold has type jumps. If one considers a Calabi-Yau manifold as in example 5.8 then, performing the topological twist mentioned makes holomorphic forms and antiholomorphic vector fields have conformal weight 0 (the B-model). In the symplectic case (A-model) however, for a Darboux local system of coordinates as in Example 5.9 we obtain that the basic fermions of conformal weight zero are of the form $dx^i + i\partial y^i$ and $\partial x^i + idy^i$.

6. Concluding remarks

In this article we produced an embedding of the $N = 2$ super vertex algebra of central charge $c = 3 \dim M$ into the chiral de Rham complex of any generalized Calabi-Yau manifold $M$. Our approach works without modification in the twisted generalized Calabi-Yau case (i.e. when the exact Courant algebroid $E$ is not the standard one). We discussed the topological twist of this $N = 2$ algebra, leading to an identification of the BRST-cohomology of the chiral de Rham complex of $M$ with the Lie-algebroid cohomology of the associated Dirac structure.

The formulae for the generators of $N = 2$ supersymmetry look like the generators of [10]. In particular the generators for $N = 1$ supersymmetry look like the generators of the Kac-Todorov construction. Thus, we can view our results as a Courant algebroid generalization of the Kac-Todorov construction and the results in [10].

Similar results hold in the Generalized Kähler and Generalized Calabi-Yau metric cases. In particular it is possible to show that in the latter case there are two commuting sets of $N = 2$ structures. Taking BRST cohomology with respect to the left charge we obtain a sheaf with finite dimensional cohomologies (in each conformal weight) and with a surviving topological structure (given by the right $N = 2$ structure). This allows us to define the Elliptic genus of a generalized Calabi-Yau metric manifold just as in the usual case [3]. We plan to return to this matters in the future.
Appendix A. Proofs of the main results

Proof of Lemma 5.3. 1) follows from 2). The proof of (2) is similar to the proof of Prop 2.14, the major difficulty is that the “structure constants” defined by
\[ c^i_{jk} := \langle e^i, [e_j, e_k] \rangle = \langle [e^i, e_j], e_k \rangle \in C(M), \]
\[ c^j_{ik} := \langle [e^i, e^j], e_k \rangle = \langle e^i, [e^j, e_k] \rangle \in C(M), \]
are functions, therefore we need to keep track of quasi-associativity terms. Note that we have also used axiom (5) in Def. 3.1. In order to compute \([J_\Lambda J]\) we start with:
\[
\begin{align*}
[e_\Lambda J] &= \frac{i}{2} ([e_j, e^i] + 2\chi \delta^i_j) e_i + \frac{i}{2} e^i [e_j, e_i] + i \int_0^\Lambda \eta c^i_{ij} d\Gamma,
\end{align*}
\] (A.1)
By skew-symmetry we obtain:
\[
\begin{align*}
[J_\Lambda e_j] &= \frac{i}{2} c^i_{jk} (e_k e_i) - i(\chi + S) e_j - i\lambda c^i_{ij},
\end{align*}
\] (A.2)
Similarly we have
\[
\begin{align*}
[e^j_\Lambda] &= \frac{i}{2} [e^i, e^i] e_i + \frac{i}{2} e^i ([e^j, e_i] + 2\chi \delta^i_j) + i \int_0^\Lambda \eta c^i_{ij} d\Gamma,
\end{align*}
\] (A.3)
and by skew-symmetry we obtain:
\[
\begin{align*}
[J_\Lambda e^j] &= \frac{i}{2} c^i_{ik} (e^i e^k) + i(\chi + S) e^j - i\lambda c^i_{ij},
\end{align*}
\] (A.4)
From (A.2) and (A.4) we obtain:
\[
\begin{align*}
[J_\Lambda J] &= -\frac{1}{4} \left( c^i_{jk} (e^i e^k) + 2(\chi + S) e^j - 2\lambda c^j_{ij} \right) e_j + \\
&\quad \frac{1}{4} e^i \left( c^j_{ik} (e_k e_i) - 2(\chi + S) e_j - 2\lambda c^i_{ij} \right) - \frac{1}{4} \int_0^\Lambda [c^j_{ik} (e^i e^k)] e_j d\Gamma - \\
&\quad - \frac{1}{2} \int_0^\Lambda (\eta - \chi) ([e^j, e_j] + 2\eta \dim M) d\Gamma.
\end{align*}
\] (A.5)
We can compute the integral term easily as:
\[
\lambda c^i_{ij} e^j - \frac{1}{2} \lambda [e^j, e_j] - \lambda \chi \dim M,
\]
and replacing in (A.5) using quasi-associativity we obtain:

\[
[A J] = \frac{1}{4} (e^i (e^k [e_i, e_k]) + e_i (e_k [e^i, e^k])) - \frac{1}{4} (e_j S e_j + e_j S e_j) \\
- \frac{1}{2} (\lambda + T)[e_j, e_j] + \frac{1}{2} \lambda (c_i^j e_j + c_j^i e_i) - \lambda \chi \dim M + T(c_i^j e_j) \\
= - \frac{1}{4} (e^i (e^k [e_i, e_k]) + e_i (e_k [e^i, e^k])) - \frac{1}{2} (e_j S e_j + e_j S e_j) + \\
+ \frac{1}{2} T(c_i^j e_j + c_j^i e_i) - \lambda \chi \dim M
\]

\[\Box\]

Proof of Theorem 5.5. 1) is a straightforward computation, we leave it as an exercise for the reader. We first prove 2) for the sections \(e_k\). For this we need:

\[
[e_k \chi e^i S e_i + e_i S e_i] = (c_i^j e_j - c_j^i e_j) S e_i + 2 \chi S e_k + (c_i^k e_j) S e_i + e_i (S + \chi) (c_i^k e_j) + \\
+ e_i (S + \chi) (c_i^k e_j - c_k^j e_j + 2 \delta_k^j \chi) + \\
\int_0^\Lambda [c_i^j e_j - c_k^j e_j S e_i] d\Gamma + \int_0^\Lambda [c_i^k e_j S e_i] d\Gamma.
\]

The constant term in (A.7) is given by

\[
[e^i (S e_k^i) e_j + e_i (S e_k e_j - S e_k e_j)] = (c_i^j e_j - c_k^j e_j + c_k^j e_j) S e_i + \\
(T e_i) c_i^j e_j - (T e_i)^2 c_i^j + (T e_i)^2 c_i^j + (T e_i)^2 c_i^j e_i + \\
(T c_i^j e_j - c_i^j e_j) S e_i - (T e_i) c_i^j e_j + (T e_i)^2 c_i^j e_i - (T e_i) e_i - (T c_i^j) c_i^j e_i + \\
(T e_i) (c_i^j - c_i^j) + (T e_i) c_i^j + (T c_i^j) c_i^j = \\
\frac{1}{2} e^i (c_i^j (e^i e_j + e_i e_i)) + e_i (e_i e_i) - (c_i^j e_j - c_i^j e_j) e_i = \\
(T e_i) (c_i^j e_j - c_i^j) + (T e_i)^2 c_i^j e_i + (2 T c_{i j}) e_i c_j e_i + (T e_i)^2 c_i^j e_i + \\
+ \frac{1}{2} c_i^j e_i e_i e_i + \frac{1}{2} c_i^j e_i e_i e_i + e^i T (c_i^j e_i) - \frac{1}{2} c_i^j e_i e_i e_i + \frac{1}{2} c_i^j e_i e_i e_i - \\
\frac{1}{2} c_i^j e_i e_i e_i + e_i (T c_i^j) - \frac{1}{2} c_i^j e_i e_i e_i - e_i (T c_i^j) = \\
\frac{1}{2} (c_i^j e_j - c_j^i e_j) + \frac{1}{2} (T c_i^j e_j) e_i + (T e_i) (c_i^j e_j - c_i^j) + (T e_i) (c_i^j e_j - c_i^j) + \\
(T e_i) (c_i^j e_j + T c_i^j e_j) e_i + (2 T c_i^j e_i) c_i^j + (T c_i^j e_i) + T c_i^j e_i e_i + T c_i^j e_i e_i + \]

where for a function \(f\), we use the notation

\[f := \pi(e_i) f, \quad f^i := \pi(e^i) f.\]

The \(\chi\) term in A.7 is simply:

\[
2 S e_k e^i (c_i^j e_j - e_i (c_i^j e_j - c_i^j e_j)).
\]
To compute the \( \lambda \) term we need to evaluate the integral terms in (A.7). For this we compute:

\[
\int_{0}^{\Lambda} [c_k^i e_j - c_k^j e_i] \gamma S e_i d\Gamma = \int_{0}^{\Lambda} (S + \eta) [e_k, e^i \gamma e_i] d\Gamma = \lambda [e_k, e^i], e_i] + \lambda S c_{ki}^i,
\]

and,

\[
\int_{0}^{\Lambda} [e_k^i e_j^i \gamma S e_i] d\Gamma = \int_{0}^{\Lambda} (S + \eta) [e_k, e_i] e^i d\Gamma = \lambda [e_k, e_i], e^i] - \lambda S c_{ki}^i,
\]

from where the \( \lambda \) term of (A.7) is just

\[
2\lambda e_k + [e_k, e^i], e_i] + [e_k, e_i], e^i].
\]

We also need to compute

\[
[e_k^i e_j^i (e^j, e_i)] = [e_k, e^i] e_j^i (e^j, e_i) + 2 \chi e_j [e_k, e^i] + e^i \left([e_k, e^j] e_i + 2 \chi [e_i, e_k] + e_j [e_k, e_i] + 2 \lambda [e_k, e] [e_i, e^i] + 2 \lambda c_{kj}^j - 2 \lambda e_j [e_k, e^i], e_i, e_j]\right) =
\]

\[
[e_k, e^i] (e^j, e_i) + e^i ([e_k, e^j] e_i) + e^i ([e_k, e] e^i) + 4 \chi e_k [e_i, e^i] + \lambda \left(4 [e_k, e^j], e_i, e_j]\right).
\]

Similarly:

\[
[e_k^i e_j^i (e^i, e^i)] = [e_k, e^i] e_j^i (e^i, e^i) + e_i ([e_k, e_j] e^i, e^j) + e_j (2 \chi + S) c_{kj}^j + 2 \lambda [e_k, e], e^i, e^i)] - 2 \chi e_j ([e_k, e^i], e_i, e^i]) =
\]

\[
[e_k, e_i] (e^i, e^i) + e_i ([e_k, e_j] e^i, e^j) + e_i ([e_k, e_j] e^i, e^j)) + e_i (e_j [e_k, e^i, e^j]) + e_i (e_j S c_{kj}^j) + 2 \chi e_j (e^i, e^i) + 4 \lambda [e_k, e], [e^i], e^i)] e_i.
\]

Finally we need

\[
[e_k^i - i T \gamma [e^i, e_i]] = (\lambda + T) [e_k^i c_{kj}^j + c_{ki}^j e_j] =
\]

\[
(\lambda + T) \left(c_{ki}^j e_j + c_{kj}^i e_i - c_{kj}^i c_{ki}^j e_j + c_{ki}^j c_{kj}^i e_i + 2 \chi c_{ki}^i\right).
\]

It follows from (A.9), (A.13), (A.14) and (A.15) that the \( \chi \) term of \( [e_k^i H] \) is given by

\[
S e_k - \frac{1}{2} e^i (c_{ki}^j e_j) - \frac{1}{2} c_{ki}^j (e^i e_j) + \frac{1}{2} e^i (c_{kj}^j e_j) + e^i (c_{ki}^j e_j) +
\]

\[
\frac{1}{2} e_i (e_j c_{kj}^j) + T c_{ki}^j = S e_k + \frac{1}{2} e^i (c_{ki}^j e_j) + \frac{1}{2} e_i (c_{kj}^j e^j) + T c_{ki}^j = S e_k + T c_{ki}^j.
\]
We have for the $\lambda$ term:

\begin{equation}
(A.17) \quad e_k + \frac{1}{2}[\{e_k, e^i\}, e_i] + \frac{1}{2}[\{e_k, e^j\}, e^j] + \langle\{e_k, e^i\}, [e_i, e_j]\rangle e^j + \frac{1}{2} c^i_j c^j_i e^j + \frac{1}{2} c^i_j c^j_i e^j
\end{equation}

\begin{equation}
\langle\{e_k, e_j\}, [e^i, e^j]\rangle e_i + \frac{1}{2} \left( c^i_j c^j_i e^j + c^i_j c^j_i e^j - c^j_i c^i_j e^j + c^j_i c^i_j e^j + c^j_i c^i_j e^j + c^j_i c^i_j e^j \right) = e_k + \frac{1}{2} c^i_j c^j_i e^j + \frac{1}{2} c^i_j c^j_i e^j
\end{equation}

\begin{equation}
\frac{1}{2} \left( c^i_j c^j_i e^j + c^i_j c^j_i e^j - c^j_i c^i_j e^j + c^j_i c^i_j e^j + c^j_i c^i_j e^j + c^j_i c^i_j e^j \right) = e_k - \frac{1}{2} c^i_j c^j_i e^j - \frac{1}{2} c^i_j c^j_i e^j
\end{equation}

\begin{equation}
\frac{1}{2} \left( c^i_j c^j_i e^j + c^i_j c^j_i e^j - c^j_i c^i_j e^j + c^j_i c^i_j e^j + c^j_i c^i_j e^j + c^j_i c^i_j e^j \right) = e_k - \frac{1}{2} c^i_j c^j_i e^j - \frac{1}{2} c^i_j c^j_i e^j
\end{equation}

To simplify this expression further, recall from (3.3) and its dual that we have
div $\mu e_k = -c^i_k$ and div $\mu e^j = -c^j_i$. A simple computation shows

\begin{equation}
(A.18) \quad \text{div}_\mu [e_k, e_j] = \text{div}_\mu (c^i_k e_i) = c^i_k e_i - c^i_k e_i = \text{div}_\mu \pi [e_k, e_j] = \text{div}_\mu \pi [e_k, e_j] = \pi (e_k) \text{div}_\mu e_j - \pi (e_j) \text{div}_\mu e_k = -c^i_j e_i + c^i_k e_k.
\end{equation}

A similar computation shows

\begin{equation}
(A.19) \quad \text{div}_\mu [e_k, e^j] = c^i_k e^i - c^i_k e^i = c^i k e^i + c^i k e^i = -c^i_j e_i + c^i_j e_i.
\end{equation}

Replacing (A.19) and (A.18) in (A.17) we obtain for the $\lambda$ term of $[e_k, H]_\lambda$:

\begin{equation}
(A.20) \quad c^i_j e^i + \frac{1}{2} c^i_j e^i = e_k + Sc^i_k.
\end{equation}

From (A.15) we find that the $\lambda \chi$ term in $[e_k, H]$ is simply $c^i_k$ and we need to compute only the constant term. For this we expand the terms in (A.13) which are cubic in the fermions using quasi-asociativity, a straightforward computation shows:

\begin{equation}
(A.21) \quad [e_k, e^i] [e^j (e_i, e_j)] = c^m_k c^i_j (e^m_i (e^j_i e_j)) - c^m_k c^i_j (e^m_i (e^j_i e_j)) + 2(Tc^k_j e^j k c^k_j e^j),
\end{equation}

\begin{equation}
(A.21) \quad e^i (e_k, e_j) [e^j (e_i, e_j)] = c^m_k c^i_j (e^i_k e^j_i e^m_j) - c^m_k c^i_j (e^i_k e^j_i e^m_j) - 2(Tc^k_j e^j_k c^k_j e^i),
\end{equation}

\begin{equation}
(A.21) \quad e^i (e_k, e_j) [e^j (e_i, e_j)] = c^m_k c^i_j (e^i_k e^j_i e^m_j) + c^m_k c^i_j (e^i_k e^j_i e^m_j),
\end{equation}

\begin{equation}
(A.21) \quad [e_k, e_j] (e^i (e_i, e_j)) = c^k j (e_k e^i_i (e_i e_j)) - 2(Tc^k_j e^j_k c^k_j e_j),
\end{equation}

\begin{equation}
(A.21) \quad e^i (e_k, e_j) [e^i (e_i, e_j)] = c^m_k c^i_j (e^i_k e^j_i e^m_j) + 2(Tc^k_j e^k_j c^k_j e_i),
\end{equation}

\begin{equation}
(A.21) \quad e^i (e_k, e_j) [e^i (e_i, e_j)] = c^m_k c^i_j (e^i_k e^j_i e^m_j) - c^m_k c^i_j (e^i_k e^j_i e^m_j) + c^m_k c^i_j (e^i_k e^j_i e^m_j) + c^m_k c^i_j (e^i_k e^j_i e^m_j) - \frac{1}{2} c^i_j (e_i e_j e_i) - \frac{1}{2} c^i_j (e_i e_j e_i),
\end{equation}

\begin{equation}
(A.21) \quad e_i (e_j e^j_k) = \frac{1}{2} c^i_j (e_i (e_j e_i)) + \frac{1}{2} c^i_j (e_i (e_j e_i)).
\end{equation}
Collecting all the terms of the constant term of $[e_k \Lambda H]$ that do not explicitly contain cubic products of sections of $T \oplus T^*$ we get:

$$
(A.22) \quad \frac{1}{2} (T c_i) (c^i_{k,j} - c^i_{j,k}) + \frac{1}{2} (T e^i) c^i_{k,i,j} + \frac{1}{2} \left( T c^i_{k,l,j} - 2 c^i_{j,l} (T c^l_{k,i}) e^l \right) + \\
\frac{1}{2} \left( 2 (T c^i_{k,i}) c^i_{j,j} - (T c^i_{j,j}) c^i_{k,j} + T c^i_{k,l,j} - T c^i_{j,l,j} \right) e_l + \frac{1}{2} T \left( c^i_{j,i,k} e^l + c^i_{j,l,k} e^l \right) - \\
\frac{1}{2} \left( T c^i_{k,kl} \right) c^i_{j,kl} e_j + \frac{1}{2} \left( T c^i_{kl,i} \right) c^i_{j,kl} e_j - \frac{1}{2} \left( T c^i_{kl,j} \right) c^i_{j,kl} e_j - \\
\frac{1}{2} T \left( \left( c^i_{j,kl} + c^i_{k,kl} + c^i_{l,kl} - 2 c^i_{k,l} e_j \right) - e_l \right) = \frac{1}{2} T \left( c^i_{k,kl} e^l \right) + \frac{1}{2} T \left( c^i_{j,kl} e_j \right) = T S c^i_{k,l},
$$

where we have used (A.18) and (A.19) in the last line. Collecting all the terms that contain cubic products of sections of $L$ we get:

$$
(A.23) \quad \frac{1}{4} c^i_{j,j} (e_i (e_l e_j)) + \frac{1}{4} c^i_{j,j} c^m_{i,k} (e_i (e_j e_m)) = \frac{1}{4} (c^i_{m,k} e^m_i - c^i_{k,i}) (e_i (e_l e_i)) = 0,
$$

which vanishes because of the Jacobi identity. Collecting all the terms that are quadratic in sections of $L$ and linear in $L^*$ we get:

$$
(A.24) \quad \left( - \frac{1}{4} c^i_{k,l} + \frac{1}{2} c^i_{k,kl} \right) (e^l (e_i e_j)) + \frac{1}{4} c^m_{i,k} (e_i (e_j e_m)) + \frac{1}{4} c^m_{j,k} (e_i (e_j e_m)) + \frac{1}{4} c^m_{i,k} (e_i (e_j e_m)) + \frac{1}{4} c^m_{j,k} (e_i (e_j e_m)) = \\
\left( - \frac{1}{4} c^i_{k,l} + \frac{1}{2} c^i_{k,kl} \right) (e^l (e_i e_j)) + \frac{1}{4} (c^m_{i,k} e^m_i - c^m_{j,k} e^m_i) = 0.
$$

Finally, collecting all the terms that are quadratic in sections of $L^*$ and linear in $L$:

$$
(A.25) \quad \frac{1}{2} c^i_{k,kl} (e^l (e_j e_j)) - \frac{1}{4} c^m_{i,k} c^m_{j,kl} (e^m (e_j e_i)) = \frac{1}{2} c^i_{k,kl} (e^l (e_j e_j)) + \\
\frac{1}{4} c^m_{j,k} (e^l (e_i e_j)) + \frac{1}{4} \left( 2 c^i_{k,kl} - c^m_{k,kl} c^m_{j,kl} - c^m_{j,kl} c^m_{k,kl} \right) (e^l (e_j e_i)) = 0,
$$

which vanishes because of the Jacobi identity.

It follows from (A.16), (A.20) and (A.22)-(A.25) that we have

$$
(A.26) \quad [e_k \Lambda H] = T S c^i_{k,l} + \chi (S e_k + T c^i_{k,l}) + \lambda (e_k + S c^i_{k,l}) + \lambda \chi c^i_{k,l}.
$$

Using skew-symmetry we obtain:

$$
(A.27) \quad [H \Lambda e_k] = (2 T + \lambda + \chi S) e_k - \lambda \chi c^i_{k,l}.
$$

Similarly we find

$$
(A.28) \quad [H \Lambda e^k] = (2 T + \lambda + \chi S) e^k - \lambda \chi c^i_{k,l}.
$$

Which proves 2) for the sections $e_k$ and $e^k$. To find 2) for a general section we just use the non-commutative Wick formula and 1).
3) We use the non-commutative Wick formula to obtain:

\[
\Lambda e^k e_k = \left( (2T + \lambda + \chi S)e^k \right) e_k - \lambda \chi e^k e_k + e^k (2T + \lambda + \chi S)e_k + \\
\lambda \chi e^k e_k + \int_0^\Lambda (2 \gamma + \lambda - \chi \eta) ([e^k, e_k] + 2 \eta \dim M) d\Gamma = \\
(2T + 2\lambda + \chi S)e^k e_k.
\]

The Theorem follows easily from (A.29) and the following simple Lemma that is interesting on its own:

**Lemma A.1.** Let \( V \) be an \( N_K = 1 \) SUSY vertex algebra. Let \( J \) and \( H \) be two fields of \( V \) satisfying

\[
[\Lambda e^k e_k] = (2T + \lambda + \chi S)e^k - \lambda \chi e^k e_k + e^k (2T + \lambda + \chi S)e_k + \\
\lambda \chi e^k e_k + \int_0^\Lambda (2 \gamma + \lambda - \chi \eta) ([e^k, e_k] + 2 \eta \dim M) d\Gamma = \\
(2T + 2\lambda + \chi S)e^k e_k.
\]

for some complex number \( c \). Then the fields \( J \) and \( H \) generate an \( N = 2 \) super vertex algebra of central charge \( c \), namely:

\[
[H \Lambda H] = (2T + \lambda + \chi S)H + \frac{c}{3} \lambda^2 \chi.
\]

**Proof.** This is a direct application of the Jacobi identity for \( N_K = 1 \) SUSY Lie conformal algebras [15]:

\[
[H \Lambda H] = -[H \Lambda [J]] = -[H \Lambda J]_\Lambda + [J \Gamma [H \Lambda]] = \\
-[J \Gamma (2T + \lambda + \chi S)J] - [J \Gamma (2T + 2\lambda + \chi S)J] = \\
+ (2 \gamma + \chi (\eta + \chi)) [J \Lambda + \gamma J] - (2T + 2(\lambda + \gamma) + \chi (S + \eta)) [J \Gamma J] = \\
-(2 \gamma + \chi (\eta + \chi)) \left( H + \frac{c}{3} (\lambda + \gamma) (\chi + \eta) \right) + (2T + 2(\lambda + \gamma) + \chi (S + \eta)) \left( H + \frac{c}{3} \gamma \eta \right) = \\
(2T + 3\lambda + \chi S)H + \frac{c}{3} \lambda^2 \chi.
\]

\( \square \)

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\[\text{Note that we cannot say that } J \text{ is a primary field of conformal weight } 1 \text{ from this equation since we do not know yet that } H \text{ is a superconformal field.}\]
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813 Evans Hall Dept. of Mathematics, University of California, Berkeley 94720

E-mail address: heluani@math.berkeley.edu

Department of Physics and Astronomy, Uppsala University, Box 803 SE-75108 Uppsala Sweden

E-mail address: Maxim.Zabzine@fysast.uu.se