NONLINEAR VERSIONS
OF A VECTOR MAXIMAL PRINCIPLE

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Abstract. Some nonlinear extensions of the vector maximality statement established by Goepfert, Tammer and Zălinescu [Nonl. Anal., 39 (2000), 909-922] are given. Basic instruments for these are the Brezis-Browder ordering principle [Advances Math., 21 (1976), 355-364] and a (pseudometric) version of it obtained in Turinici [Demonstr. Math., 22 (1989), 213-228].

1. Introduction

Let $Y$ be a (real) separated locally convex space, and $K$, some (convex) cone of it: $\alpha K + \beta K \subseteq K$, $\forall \alpha, \beta \in R_+ := [0, \infty[$. In this case, the relation over $Y$

(a01) $(y_1, y_2 \in Y): y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$

is reflexive and transitive; hence a quasi-order; in addition, it is compatible with the linear structure of $Y$. Let $H$ be another (convex) cone of $Y$ with $K \subseteq H$; and pick some $k^0 \in K \setminus (-H)$. Further, take some complete metric space $(X, d)$; and introduce the quasi-order (on $X \times Y$)

(a02) $(x_1, y_1) \succeq (x_2, y_2)$ iff $k^0 d(x_1, x_2) \leq_K y_1 - y_2$.

Finally, take some nonempty part $A$ of $X \times Y$. For a number of both practical and theoretical reasons, it would be useful to determine sufficient conditions under which $(A, \succeq)$ has points with certain maximal properties. The basic 2000 result in the area obtained by Goepfert, Tammer and Zălinescu [9], deals with the case $H = \text{cl}(K)$ (=the closure of $K$). Precisely, assume that

(a03) $P_Y(A)$ is bounded below (modulo $K$): $\exists \tilde{y} \in Y$ with $P_Y(A) \subseteq \tilde{y} + K$

(a04) if $(x_n, y_n) \subseteq A$ is $(\succeq)$-ascending and $x_n \to x$ then $x \in P_X(A)$

and there exists $y \in A(x)$ such that $(x_n, y_n) \succeq (x, y)$, for all $n$.

[Here, for each $(x, y) \in A$, $A(x)$ (respectively, $A(y)$) stands for the $x$-section (respectively, $y$-section) of (the relation) $A$; and $P_X$, $P_Y$ are the projection operators from $X \times Y$ to $X$ and $Y$ respectively].

Theorem 1. Let the above conditions be in force. Then, for each $(x_0, y_0) \in A$ there exists $(\bar{x}, \bar{y}) \in A$ with

\begin{align*}
(x_0, y_0) \succeq (\bar{x}, \bar{y}) \ [\text{hence } y_0 \geq_K \bar{y}] \tag{1.1} \\
(\bar{x}, \bar{y}) \succeq (x', y') \in A \text{ imply } \bar{x} = x'. \tag{1.2}
\end{align*}

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This result includes the ones due to Isac [10] and Nemeth [14] (which, in turn, extend Ekeland’s variational principle [6]); and the authors’ argument is based on the Cantor intersection theorem. Further, in his 2002 paper, Turinici [18] proposed a different approach, via ordering principles related to Brezis-Browder’s [3] (cf. Section 2); and stressed that, conclusions like before are extendable to (non-topological) vector spaces $Y$ under the choice $H = \text{arch}(K)$ (=the Archimedean closure of $K$). It is our aim in this exposition to show that a further enlargement of these facts is possible (by the same techniques). This refers to the function

(a05) $\Lambda(t) = k^0 t, \ t \in R_+$ (where $k^0$ is the above one)

being no longer linear; details will be given in Section 4 (the Archimedean case) and Section 5 (the non-Archimedean case). The specific instrument of our investigations (in this last circumstance) is the concept of gauge function (developed in Section 3). Finally, in Section 6, the relationships between our statement and the recent variational principle in Bao and Mordukhovich [2] are discussed.

2. Brezis-Browder principles

(A) Let $M$ be some nonempty set. Take a quasi-order $(\leq)$ over $M$; as well as a function $\psi : M \to R_+$. Call the point $z \in M$, $(\leq, \psi)$-maximal when: $w \in M$ and $z \leq w$ imply $\psi(z) = \psi(w)$. A basic result about such points is the 1976 Brezis-Browder ordering principle [3]:

Proposition 1. Suppose that

(b01) $(M, \leq)$ is sequentially inductive:

\[
\text{each ascending sequence has an upper bound (modulo } \leq) \]

(b02) $\psi$ is $(\leq)$-decreasing ($x \leq y \implies \psi(x) \geq \psi(y)$).

Then, for each $u \in M$ there exists a $(\leq, \psi)$-maximal $v \in M$ with $u \leq v$.

This principle, including Ekeland’s [6], found some basic applications to convex and nonconvex analysis (cf. the above references). So, a discussion about its key condition (b01) would be not without profit. Let $(Z, \leq)$ be some quasi-ordered structure. Take a function $\phi : Z \to R \cup \{-\infty, \infty\}$; and let $M$ be some nonempty part of $Z$. For simplicity reasons, we let again $\phi$ stand for the restriction of $\phi$ to $M$. The following “relative” form of Proposition 1 will be useful for us.

Proposition 2. Suppose (b02) holds (modulo $\phi$), as well as

(b03) $\phi$ is inf-proper over $M$:

\[
\inf(\phi(M)) > -\infty \text{ and } \text{Dom}(\phi) := \{x \in M; \phi(x) < \infty\} \neq \emptyset
\]

(b04) $\text{Dom}(\phi)$ is sequentially inductive in $M$: each ascending sequence in $\text{Dom}(\phi)$ is bounded above in $M$ (modulo $(\leq)$).

Then, for each $u \in \text{Dom}(\phi)$ there exists $v \in \text{Dom}(\phi)$ with

i) $u \leq v$ and ii) $x \in M, v \leq x$ imply $\phi(v) = \phi(x)$.

Proof. Let $u \in \text{Dom}(\phi)$ be arbitrary fixed. Put $M(u, \leq) := \{x \in M; u \leq x\}$; and introduce the function (from $M$ to $R_+$) $\psi(x) = \phi(x) - \varphi_*, x \in M$; where $\varphi_* := \inf[\phi(M)]$. By the imposed conditions, Proposition 1 applies to $M(u, \leq)$ and $(\leq, \psi)$; wherefrom the conclusion is clear.

For the moment, Proposition 2 is a logical consequence of Proposition 1. The reciprocal is also true, by simply taking $Z = M$, $\varphi = \psi$. Hence, these two results are logically equivalent. Note that the inf-properness condition (b03) is not essential
for the conclusion above (cf. Cărja and Ursescu [4]). Moreover, \((R, \geq)\) may be substituted by a separable ordering structure \((P, \leq)\) without altering the conclusion above; see Turinici [19] for details. Further aspects were discussed in Altman [1]; see also Kang and Park [12].

\((B)\) A semi-metric version of these developments may be given along the following lines. Let \((M, \leq)\) be taken as before. By a \textit{pseudometric} over \(M\) we shall mean any map \(e : M \times M \to R_+\). If, in addition, \(e\) is \textit{reflexive} \([e(x, x) = 0, \forall x \in M]\), \textit{triangular} \([e(x, z) \leq e(x, y) + e(y, z), \forall x, y, z \in M]\) and \textit{symmetric} \([e(x, y) = e(y, x), \forall x, y \in M]\), we say that it is a \textit{semimetric} (on \(M\)). Suppose that we fixed such an object. Call \(z \in M, (\leq, e)\)-\textit{maximal}, in case: \(w \in M\) and \(z \leq w\) imply \(e(z, w) = 0\). [Note that, if (in addition) \(e\) is \textit{sufficient} \([e(x, y) = 0\) implies \(x = y\)], this property becomes: \(w \in M, z \leq w \implies z = w\) (and reads: \(z\) is strongly \((\leq)\)-\textit{maximal}). So, existence results of this type may be viewed as "metrical" versions of the Zorn-Bourbaki principle. To get such points, one may proceed as below. Call the (ascending) sequence \((x_n)\) in \(M\), \(e\)-\textit{Cauchy} when \(\forall \delta > 0, \exists n(\delta): u(\delta) \leq p \leq q \implies e(x_p, x_q) \leq \delta\); and \(e\)-\textit{asymptotic}, provided \([e(x_n, x_{n+1}) \to 0, \text{as } n \to \infty]\). Clearly, each (ascending) \(e\)-Cauchy sequence is \(e\)-asymptotic too. The reverse implication is also true when all such sequences are involved; i.e., the global conditions below are equivalent:

\begin{enumerate}[(b05)]  \item each ascending sequence is \(e\)-Cauchy  \item each ascending sequence is \(e\)-asymptotic. \end{enumerate}

By definition, either of these will be referred to as \((M, \leq)\) is \textit{regular} (modulo \(e\)). The following maximality result in Turinici [17] is available.

\textbf{Proposition 3.} Assume that \((M, \leq)\) is sequentially inductive and regular (modulo \(e\)). Then, for each \(u \in M\) there exists an \((\leq, e)\)-\textit{maximal} \(v \in M\) with \(u \leq v\).

This result includes the Brezis-Browder ordering principle [3] (Proposition [10]; to which it reduces in case \(e(x, y) = |\psi(x) - \psi(y)|\) (where \(\psi\) is the above one). The reciprocal inclusion is also true; we refer to the quoted paper for details.

\section{Conical gauge functions}

Let \(Y\) be a \textit{(real) vector space}. Take a convex cone \(L\) of \(Y\) (cf. Section 1); which, in addition, is \textit{non-degenerate} \([L \neq \{0\}]\) and \textit{proper} \([L \neq Y]\). Denote by \((\leq_L)\) its induced quasi-order (cf. (a01)); when \(L\) is understood, we indicate this as \((\leq)\), for simplicity. Further, let the map \(\Lambda : R_+ \to L\) be \textit{normal} (modulo \(L\)):

\begin{enumerate}[(c01)]  \item \(\Lambda(0) = 0\) and \(\Lambda\) is strictly increasing (modulo \(L\)):  \(\Lambda(\tau) - \Lambda(t) \in L \setminus (-L), \text{ whenever } \tau > t\)  \item \(\Lambda\) is sub-additive: \(\Lambda(t_1 + t_2) \leq \Lambda(t_1) + \Lambda(t_2), \forall t_1, t_2 \in R_+\). \end{enumerate}

Note that, as a consequence of this,

\begin{enumerate}[(c03)]  \item \(\Gamma(L; \Lambda; y) = \{s \in R_+: \Lambda(s) \leq y\}\), \(\gamma(L; \Lambda; y) = \sup\Gamma(L; \Lambda; y)\). \end{enumerate}

(By convention, \(\sup(0) = -\infty\). We therefore defined a couple of functions \(\Gamma(.) := \Gamma(L; \Lambda; .)\) and \(\gamma(.) := \gamma(L; \Lambda; .)\) from \(Y\) to \(P(R_+)\) and \(R \cup \{-\infty, \infty\}\) respectively; the latter of these will be referred to as the \textit{gauge function} attached to \((L; \Lambda)\). For the
particular case of linear normal functions [i.e., the one of (a05), with \( k^0 \in L \setminus \{-L\} \)], such objects were introduced (in the same context) by Turinici [18]; and these, in turn, appear as non-topological extensions of the locally convex ones in Goepfert, Tammer and Zălinescu [9]. The present developments may therefore be viewed as "nonlinear" extensions of the preceding ones.

i) To begin with, note that for each \( y \in L \), \( \Gamma(y) \) is hereditary \( (s \in \Gamma(y) \implies [0, s] \subseteq \Gamma(y)) \). In addition, we have \( (y \in L \iff \Gamma(y) \neq \emptyset \iff \gamma(y) \in [0, \infty]) \) (or, equivalently: \( y \notin L \iff \gamma(y) = 0 \iff \gamma(y) = -\infty \)).

ii) The gauge function is increasing \( [y_1, y_2 \in Y, y_1 \leq y_2 \implies \gamma(y_1) \leq \gamma(y_2)] \).

iii) Further, \( \gamma \) is super-additive and subtractive:

\[
\gamma(y_1 + y_2) \geq \gamma(y_1) + \gamma(y_2), \quad \text{whenever the right member exists} \quad (3.3)
\]

\[
\gamma(y_1 - y_2) \leq \gamma(y_1) - \gamma(y_2), \quad \text{if } \gamma(y_2) \text{ finite (hence } 0 \leq \gamma(y_2) < \infty) \quad (3.4)
\]

Clearly, it will suffice proving the former one. Without loss, assume that \( \gamma(y_1) > 0, \gamma(y_2) > 0 \). By definition (and the hereditary property of \( \Gamma \)) \( y_1 \geq \Lambda(t_1), y_2 \geq \Lambda(t_2) \), whenever \( 0 \leq t_1 < \gamma(y_1), 0 \leq t_2 < \gamma(y_2) \); so, combining with (c02), yields (for all such \( (t_1, t_2) \)) \( y_1 + y_2 \geq \Lambda(t_1) + \Lambda(t_2) \geq \Lambda(t_1 + t_2) \); that is, \( \gamma(y_1 + y_2) \geq t_1 + t_2 \). This, and the arbitrariness of the precise couple, ends the argument.

iv) Finally, the identity relation is available:

\[
\gamma(\Lambda(t)) = t, \quad \text{for each } t \in R_+.
\] (3.5)

In fact, let \( t \in R_+ \) be arbitrary fixed; it will suffice verifying that \( \Gamma(\Lambda(t)) = [0, t] \). Suppose not: there exists \( \tau > t \) with \( \gamma(\tau) \neq t \). By definition, \( \Lambda(\tau) \leq \Lambda(t) \); wherefrom \( \Lambda(\tau) - \Lambda(t) \in (-L) \); in contradiction to (c01). As a consequence, \( \gamma \) is proper; i.e., \( \text{Dom}(\gamma) := \{ y \in Y : \gamma(y) < \infty \} \) is nonempty. Moreover, \( \text{Dom}_L(\gamma) := \text{Dom}(\gamma) \cap L \) is nonempty too; and we have the decomposition \( \text{Dom}(\gamma) = (Y \setminus L) \cup \text{Dom}_L(\gamma) \), with (in addition) \( \gamma(Y \setminus L) = \{-\infty\} \), \( \gamma[\text{Dom}_L(\gamma)] = R_+ \); as results from (3.3). On the other hand, by super-additivity, we have the sup-translation property:

\[
\gamma(y + \Lambda(t)) \geq \gamma(y) + t, \quad \forall y \in Y, \forall t \in R_+. \quad \text{This inequality may be strict; just take } y = -\Lambda(\tau), t = \tau, \text{ for some } \tau > 0.
\]

Concerning the effectiveness of such a construction, call the function \( \psi : R_+ \rightarrow R_+ \), normal, when \( \psi(0) = 0 \) and \( \psi \) is strictly increasing (on \( R_+ \)) as well as subadditive (see above). Note that such functions exist; such as, e.g.: \( \psi(t) = t^\lambda, t \in R_+ \), for some \( \lambda \in [0, 1] \). Suppose that \( \{\psi_1, ..., \psi_m\} \) are endowed with such properties; and take some points \( \{k_1, ..., k_m\} \in L \setminus (-L) \). Then, the function

\[
(c04) \quad (\Lambda : R_+ \rightarrow L): \Lambda(t) = k_1 \psi_1(t) + ... + k_m \psi_m(t), \quad t \in R_+
\]

is a normal one, in the sense of (c01)+(c02). The obtained class of all these covers the linear one (expressed via (a05)) when (as precise) these developments reduce to the ones in Turinici [18]. Further aspects involving the locally convex (modulo \( Y \)) case (and the same linear setting) may be found in Goepfert, Riahi, Tammer and Zălinescu [8 Ch 3, Sect 10]. see also Gerth (Tammer) and Weidner [7].

4. MAIN RESULT (ARCHIMEDEAN CASE)

With these preliminaries, we may now return to the question of the introductory part. Let \( Y \) be a (real) vector space; and \( K \), some (convex) cone of it. Denote by \( (\leq_k) \) the induced quasi-order (cf. (a01)). Let \( H \) be another (convex) cone of \( Y \) with \( K \subseteq H \); and the map \( \Lambda : R_+ \rightarrow K \) be almost normal (modulo \( (K, H) \)).
to determine sufficient conditions under which \((A, \preceq)\) has points with certain maxality properties. Note that, in the linear case of \((a05)\), this problem is just the one in Turinici [18]; which (under the precise convention) \((\Lambda)\) is sub-additive (modulo \(X\))); hence a quasi-order on it. Finally, take some (nonempty) part of \((\Lambda)\) is reflexive and transitive (by the properties of \(\Lambda\); hence a quasi-order on it. 

So, it is natural asking whether similar conclusions are retainable in our "nonlinear" setting. Loosely speaking, these depend on the ambient convex cone \(H\) being or not Archimedean. So, two alternatives are open before us.

In the following, we discuss the former of these, based on \(H\) being endowed with such a property (cf. Cristescu [5], Ch 5, Sect 1):

\((d05)\) \(h, v \in H\) and \([h \tau \leq_H v, \forall \tau \in R^+\)] imply \(h \in H \cap (-H)\).

As we shall see, a positive answer is available under 

\((d06)\) each \((\succeq)\)-ascending \(e\)-Cauchy sequence \(((x_n, y_n))\) is bounded above in \(A\) (modulo \((\succeq)\)).

Here, \(e\) stands for the semi-metric on \(X \times Y\) introduced as

\((d07)\) \(e((x_1, y_1), (x_2, y_2)) = d(x_1, x_2), (x_1, y_1), (x_2, y_2) \in X \times Y\).

The (first) main result of our exposition is

**Theorem 2.** Let the assumptions \((d04)-(d07)\) hold. Then, for each starting \((x_0, y_0) \in A\) there exists \((\bar{x}, \bar{y}) \in A\) with the properties \([1.7]\) and \([1.8]\) (written for our data).

The latter of the conclusions above reads (under the precise convention)

\((\bar{x}, \bar{y}) \succeq (x', y') \in A \implies e((\bar{x}, \bar{y}), (x', y')) = 0)\).

This suggests us a possible deduction of Theorem 2 from Proposition 3. To see the effectiveness of such an approach, we need an auxiliary fact.

**Lemma 1.** Let \(((x_n, y_n))\) be an \((\succeq)\)-ascending sequence in \(A\):

\((d08)\) \(\Lambda(d(x_n, x_m)) \leq_H y_n - y_m, \text{ whenever } n \leq m\).

Then, \((x_n)\) is \(d\)-Cauchy in \(P_X(A)\); hence \(((x_n, y_n))\) is \(e\)-Cauchy in \(A\).

**Proof.** (Lemma 1) Suppose that this would be not valid; i.e. \((as d \text{ is symmetric)}\), there must be some \(\varepsilon > 0\) in such a way that

\((d09)\) for each \(n\), there exists \(m > n\) with \(d(x_n, x_m) \geq \varepsilon\).
Inductively, we may construct a subsequence \( (u_n = x_{i(n)} ) \) of \( (x_n) \) with \( d(u_n, u_{n+1}) \geq \varepsilon \), for all \( n \). This in turn yields, for the corresponding subsequence \( (v_n = y_{i(n)}) \) of \( (y_n) \), an evaluation like: \( A(\varepsilon) \leq h \ A(d(u_n, u_{n+1})) \leq k \ v_n - v_{n+1} \), for each \( n \geq 1 \).

But then, in view of \( d(04) \), one derives: \( qA(\varepsilon) \leq H \ v_1 - v_{q+1} \leq H \ v_1 - \bar{y} \), for each \( q \geq 1 \). This, along with \( d(05) \), gives \( A(\varepsilon) \in K \cap (-H) \); in contradiction with \( (1.1) \).

Hence, the working assumption \( d(09) \) cannot hold; and the claim follows. \( \square \)

**Proof. (Theorem 2)** Let \( ((x_n, y_n)) \) be an \( (\geq) \)-ascending sequence in \( A \). By Lemma \( [1] \) \( ((x_n, y_n)) \) is an \( e \)-Cauchy sequence in \( A \); which tells us that \( (A, \geq) \) is regular (modulo \( e \)). Moreover, by \( d(06) \), \( ((x_n, y_n)) \) is bounded above (modulo \( \geq \)) in \( A \); wherefrom, \( (A, \geq) \) is sequentially inductive. Summing up, Proposition \( [3] \) is applicable to \( (A, \geq, e) \); so that (from its conclusion) each \( a_0 = (x_0, y_0) \) in \( A \) is majorized (modulo \( \geq \)) by some \( (\geq, e) \)-maximal \( \bar{a} = (\bar{x}, \bar{y}) \) in \( A \). This gives the conclusions \( (1.1) + (1.2) \) we need; and completes the argument. \( \square \)

In particular, when \( A \) is taken as in \( a(05) \) (and \( d \) is complete) Theorem \( [2] \) is just the related statement in Turinici \( [13] \); which, as precise there, incorporates the (locally convex) one in Goeppert, Tammer and Zălinescu \( [9] \) (Theorem \( [1] \)). This inclusion seems to be strict; because the choice \( c(04) \) of \( \Lambda \) cannot be (completely) reduced to the linear one (appearing in all these papers). Some related aspects may be found in Tammer \( [16] \).

5. A COMPLETION (NON-ARCHIMEDEAN CASE)

Now, the key regularity assumption used in the result above is \( d(05) \). So, it is natural to discuss the alternative of this being avoided. As we shall see below, a positive answer is still available; but we must restrict the initial set \( A \) in a way imposed by the associated (to \( H \)) gauge function.

Let \( Y \) be a (real) vector space; and \( K \), some (convex) cone in it. Denote by \( (\leq_K) \) its associated quasi-order; and let \( H \) be another (convex) cone of \( Y \) with \( K \subseteq H \).

We also take a map \( \Lambda : R_+ \to K \); which is supposed to be almost normal (modulo \( (K,H) \)) in the sense of \( d(01)+(02) \). Clearly, it is also normal (modulo \( H \)); so, we may construct the gauge function \( \gamma : Y \to R \cup \{-\infty, \infty\} \) attached to \((H, \Lambda)\), under the model of \( c(03) \). Further, letting \((X,d)\) be a metric space, denote (again) by \( \geq \) the quasi-order on \( X \times Y \) introduced as in \( d(03) \); and finally, let \( A \) be some (nonempty) part of \( X \times Y \). The question to be posed is the same as in Section 4; to solve it, we list the needed conditions. The former of these is (again) \( d(04) \); which also writes

\[
\text{e}(01) \quad P_Y(A) \subseteq H \quad (\text{i.e.:} \quad \bar{y} = 0 \quad \text{in that condition}).
\]

[For, otherwise, passing to the subset \( A_0 = \{(x,y) \in X \times Y; (x, y + \bar{y}) \in A\} \), this requirement is fulfilled, via \( P_Y(A_0) = P_Y(A) - \bar{y} \). As a consequence, \( \inf[\gamma(P_Y(A))] \geq 0 \) (wherefrom \( \gamma \) is bounded below on \( P_Y(A) \)). However, the alternative \( \gamma(P_Y(A)) = \{\infty\} \) cannot be avoided; so, we must accept (as a second condition)

\[
\text{e}(02) \quad P_Y(A) \cap \text{Dom}(\gamma) \neq \emptyset \quad (\gamma(y) < \infty \quad \text{for some} \quad y \in P_Y(A)).
\]

A useful characterization of these is to be realized via the composed function \( \Phi(x,y) = \gamma(y), (x,y) \in X \times Y \) (i.e.: \( \Phi = \gamma \circ P_Y \)). Precisely, let again \( \Phi \) denote the restriction of this function to \( A \); then, \( (e(01))+(e(02)) \) may be written as

\[
\text{e}(03) \quad \inf[\Phi(A)] \geq 0 \quad \text{and} \quad \text{Dom}(\Phi) := \{(x,y) \in A; \Phi(x,y) < \infty\} \quad \text{is nonempty}.
\]

Now, the last condition to be imposed is a variant of \( a(04) \) above:
The (real) sequence ($\gamma$ sequence in $P$ hence a Cauchy one. This, added to the above, shows that (linear" product quasi-order (d03) used by us. Having these precis e, it is natural to

Theorem 3. Let the precise conditions be in force. Then, for each $(x_0, y_0) \in \text{Dom}(\Phi)$ there exists $(\bar{x}, \bar{y}) \in \text{Dom}(\Phi)$ with the properties (1.1) and

$$(\bar{x}, \bar{y}) \succeq (x', y') \in A \text{ imply } \bar{x} = x', \gamma(\bar{y}) = \gamma(y'). \quad (5.1)$$

Proof. We claim that Proposition 2 is applicable to $(Z = X \times Y, \succeq)$, $M = A$ and $\varphi = \Phi$. In fact, by the remarks above $(x_1, y_1) \succeq (x_2, y_2)$ implies $\Phi(x_1, y_1) \geq \Phi(x_2, y_2)$; i.e., $\Phi$ is $(\succeq)$-decreasing. On the other hand, (e03) is just (b03) (with $\varphi$ substituted by $\Phi$). Finally, (e04) implies (b04) (with $\varphi = \Phi$); and this will establish our claim. In fact, let $((x_n, y_n))$ be an ascending sequence in $\text{Dom}(\Phi)$; i.e.,

$$A(d(x_n, x_m)) \leq K y_n - y_m, \text{ whenever } n \leq m.$$ Combining with the subtractivity of the gauge function (cf. Section 3) yields

$$d(x_n, x_m) \leq \gamma(y_n - y_m) \leq \gamma(y_n) - \gamma(y_m), \text{ whenever } n \leq m.$$ The (real) sequence $(\gamma(y_n))$ is descending and bounded (by the choice of our data); hence a Cauchy one. This, added to the above, shows that $(x_n)$ is a $\delta$-Cauchy sequence in $P_Y(A)$; or, equivalently, that $((x_n, y_n))$ is $\epsilon$-Cauchy in $A$; wherefrom (by (e04)) the claim follows. By Proposition 2 we therefore derive that, for $(x_0, y_0) \in \text{Dom}(\Phi)$ there exists $(\bar{x}, \bar{y}) \in \text{Dom}(\Phi)$ with the properties (1.1) and

$$(\bar{x}, \bar{y}) \succeq (x', y') \in A \text{ imply } \Phi(\bar{x}, \bar{y}) = \Phi(x', y').$$

The relation in the left member of this implication yields (see the remarks above):

$$(x', y') \in \text{Dom}(\Phi), d(\bar{x}, x') \leq \gamma(\bar{y}) - \gamma(y').$$

Moreover, the relation in the right member of the same is just: $\gamma(\bar{y}) = \gamma(y')$; so that (combining these) $d(\bar{x}, x') = 0$; wherefrom $\bar{x} = x'$. This proves (5.1) as well; and concludes the argument. □

As before, when $\Lambda$ is taken as in (a05) (and $d$ is complete) Theorem 3 is nothing but the related statement in Turinici [13], obtained via similar techniques. Moreover, when $H$ is taken as in (d05), we have (cf. Section 3)

$$\text{Dom}(\gamma) = H; \text{ hence } \text{Dom}(\Phi) = A \text{ (in view of (e01))};$$

and Theorem 2 reduces to Theorem 3 above. But, in the general (nonlinear) setting, this is not true]. Further aspects (of locally convex nature) may be found in Isac and Tammer [11]; see also Rozoveanu [15]. For different structural extensions of these we refer to Khanh [13].

6. Further Aspects

Our main results (Theorem 2 and Theorem 3) were especially designed to enlarge (in two distinct manners) the product variational principle in Goepfert, Tammer and Zălinescu [3]. Unfortunately, neither of these can extend in a direct way the related variational statements in Bao and Mordukhovich [2] Theorem 3.4]; for, e.g., the quasi-boundedness assumption imposed (by the authors) upon $x \mapsto A(x)$ is weaker than the boundedness condition (b04) used here. On the other hand, the cited results cannot extend Theorem 2 or Theorem 3 because the (a02)-type product quasi-order of the authors is the linear version (cf. (a05)) of the "nonlinear" product quasi-order (d03) used by us. Having these precise, it is natural to
ask whether a common extension of all these variational principles is available; we conjecture that the answer is positive.

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