Fermions, strings, and gauge fields in lattice spin models

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We investigate the general properties of lattice spin models with emerging fermionic excitations. We argue that fermions always come in pairs and their creation operator always has a string-like structure with the newly created particles appearing at the endpoints of the string. The physical implication of this structure is that the fermions always couple to a nontrivial gauge field. We present exactly soluble examples of this phenomenon in 2 and 3 dimensions. Our analysis is based on an algebraic formula that relates the statistics of a lattice particle to the properties of its hopping operators. This approach has the advantage that it works in any number of dimensions - unlike the flux-binding picture developed in FQH theory.

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I. INTRODUCTION

For many years, it was thought that Fermi statistics were fundamental, in the sense that one could only obtain a theory with fermionic excitations by introducing them by hand (via anti-commuting fields). Then, over the past two decades, this view began to change. A number of real world and theoretical examples showed that fermions and anyons could emerge as low energy collective modes of purely bosonic systems. The first examples along these lines were the fractional quantum Hall states. [1, 2] Usually we think of the FQH states as examples of anyonic excitations emerging from interacting fermions. However, from a purely theoretical point of view, the same effects should occur in systems of interacting bosons in a magnetic field.[3] There were also indications of emerging fermions in the slave-boson approach to spin-1/2 systems[4-10] and in the study of resonating valence bond (RVB) states.[11-14] Unfortunately, the RVB picture and the slave boson approach both rely on approximate or mean-field techniques to construct and analyze these exotic states. More recently, a number of researchers have introduced exactly soluble or quasi-exactly soluble models with emerging fermions. [14-21] These models allow for a more well-controlled analysis, albeit in very specific cases.

The mean-field approach and the exactly-soluble examples both provide clues to the structure and basic properties of bosonic models with fermionic excitations. They indicate, among other things, that fermions never appear alone in lattice spin systems. Instead they always come together with a nontrivial gauge field and the emerging fermions are associated with the deconfined phase of the gauge field.[7, 9, 10] The deconfined phases always seem to contain a new kind of order - topological order,[10, 22] and the emerging fermions and anyons are intimately related to the new order.

This was particularly apparent in the context of the slave-boson approach. In this technique, one expresses a spin-1/2 Hamiltonian in terms of fermion fields and gauge fields.[4, 5] Clearly, the presence of fermion fields does not, by itself, imply the existence of fermionic quasi-particles - the fermions appear only when the gauge field is in the deconfined phase. Ref. [6–10] constructed several deconfined phases where the fermion fields do describe well defined quasi-particles. Depending on the properties of the deconfined phases, these quasi-particles can carry fractional statistics (for the chiral spin states) [6–8] or Fermi statistics (for the $Z_2$ deconfined states).[9, 10]

Although it was less evident initially, a similar picture has emerged from the study of RVB states and the associated quantum dimer models. Ref. [11, 12] originally proposed that fermions (spinons) could emerge from a nearest neighbor dimer model on a square lattice. Later, it was realized that the fermions couple to a $Z_2$ gauge field, and the fermionic excitations only appear when the field is in the deconfined phase.[13] It turns out that the dimer liquid on square lattice with only nearest neighbor dimers is not in the deconfined phase (except at a critical point).[12, 13] However, on a triangular lattice, the dimer liquid does have a $Z_2$ deconfined phase.[14] The results in Ref. [11, 12] are valid in this case and fermionic quasi-particles do emerge in a dimer liquid on a triangular lattice.

In this paper, we attempt to clarify these observations and to put them on firmer foundations. We give a general argument which shows that emerging fermions always occur together with a nontrivial deconfined gauge field. In addition, we derive an algebraic formula that allows one to calculate the statistics of a lattice particle from the properties of its hopping operators. We feel that this formula both elucidates the fundamental meaning of statistics in a lattice system and simplifies their computation. This approach has the added advantage that it works in any number of dimensions - unlike the flux-binding picture[3] developed in FQH theory. We would like to point out that all the previously mentioned examples of emerging fermions are two dimensional models. The emerging fermions in those models are related to the flux binding picture in Ref. [3]. Our algebraic approach allows us to establish the emergence of fermions in an exactly soluble 3D bosonic model.

Our paper is organized as follows: in the first section, we give a definition of the statistics of a lattice particle.
In the second section, we derive the algebraic statistics formula discussed above. In the third section, we apply the formula to the case of fermionic (or anyonic) excitations in a lattice spin system. The formula demands that the fermions always come in pairs, and that their pair creation operator has a string-like structure with the newly created particles appearing at the ends. We show that the strings represent gauge fluctuations, and the fermions carry the corresponding gauge charge. After presenting the general argument, we devote the last two sections to exactly soluble examples of this phenomenon in 2 and 3 dimensions.

The above result suggests a very interesting picture of emerging fermions. Fermions and gauge fields appear to be two sides of the same coin: in some sense, fermions are the key to understanding both Fermi statistics and gauge fields.

II. DEFINITION OF STATISTICS

What do we mean when we say that a particle is a “boson” or a “fermion”? The most common way to define statistics is to use the algebra of creation and annihilation operators. If these operators satisfy the bosonic algebra, the corresponding particles are said to be bosons; if they satisfy the fermionic algebra, the particles are said to be fermions.

In this paper, we will use another, equivalent, definition based on the path integral formulation of quantum mechanics. Given any $m$ particle system with short-range interactions, we can write down the corresponding multi-particle action $S$. The most general action $S$ has two components:

$$S = S_{\text{loc}} + S_{\text{top}}$$

The first term, $S_{\text{loc}}$, can be any local expression. Typically, it’s just the usual classical action. The second term, $S_{\text{top}}$, is less familiar. In general, it can be any expression which only depends on the topology (e.g. homotopy class) of paths. In 3 or more dimensions, the form of $S_{\text{top}}$ is highly constrained. The amplitude for a closed path $P$ must be of the form

$$e^{iS_{\text{top}}} = (\pm 1)^n$$

where $n$ is the number of particle exchanges that occur in $P$. The statistics of the particles are defined by the sign in this expression. If this sign is positive, we say the particles are “bosons”, if the sign is negative, the particles are “fermions”. In 2 dimensions, there are other, more complicated possibilities for $S_{\text{top}}$. These particles are called “anyons.”

According to this definition, the statistics are completely determined by the topological term $S_{\text{top}}$ in the action. One way to isolate this term is to compare the amplitude for two paths which are the same locally (this will be made more precise in the next section) but differ in their global properties. In particular, we can compare the amplitude for a path which exchanges two particles with another path which is the same locally, but doesn’t exchange the two particles. The difference between the phases of the two amplitudes is then precisely $e^{iS_{\text{top}}} = (\pm 1)$, the statistical phase.

We now reformulate this (theoretical) test of statistics so that it can be applied to particles on a lattice. The prescription is as follows: we take a two particle state $|r_1, s_1\rangle$, and consider a product of hopping amplitudes along a lattice path which exchanges the particles

$$\langle r_n, s_n|H|r_{n-1}, s_{n-1}\rangle \langle r_{n-1}, s_{n-1}|H|r_{n-2}, s_{n-2}\rangle \ldots \langle r_3, s_3|H|r_2, s_2\rangle \langle r_2, s_2|H|r_1, s_1\rangle$$

Each hopping amplitude in this expression involves one of the two particles moving to a neighboring site while the other particle remains fixed. At the end, the two particles have exchanged places: $r_n = s_1, s_n = r_1$.

Now we construct another path which is the same locally, but doesn’t exchange the two particles, and take a product of hopping amplitudes along this path. The claim is that the difference between the phases of these two expressions is precisely the statistical phase of the particles. That is, the particles are bosons, fermions, or anyons depending on whether the phase difference is $+1$, $-1$, or something else.

One way to see this is to remember the derivation of the path integral formulation of quantum mechanics. According to the standard derivation, the amplitude $e^{iS}$ for a two particle path $|r(t), s(t)\rangle$ is given by a product

$$e^{iS} \propto \langle r(t_n), s(t_n)|e^{-iH\Delta t}|r(t_{n-1}), s(t_{n-1})\rangle \ldots \langle r(t_3), s(t_3)|e^{-iH\Delta t}|r(t_2), s(t_2)\rangle \langle r(t_2), s(t_2)|e^{-iH\Delta t}|r(t_1), s(t_1)\rangle$$

in the limit that $\Delta t = t_n - t_{n-1} \to 0$.

Now, in the discrete (lattice) case, the above expression can be further simplified. Since $\Delta t \to 0$, we can rewrite it as

$$e^{iS} \propto \langle r(t_n), s(t_n)|(1 - iH\Delta t)|r(t_{n-1}), s(t_{n-1})\rangle \ldots \langle r(t_3), s(t_3)|(1 - iH\Delta t)|r(t_2), s(t_2)\rangle \langle r(t_2), s(t_2)|(1 - iH\Delta t)|r(t_1), s(t_1)\rangle$$

Successive states $|r(t_i), s(t_i)\rangle, |r(t_{i+1}), s(t_{i+1})\rangle$ must either be identical, or must differ by a single particle hop. The matrix elements between identical states don’t contribute to the phase. Thus, we can drop them without affecting our result. We are left with:

$$e^{iS} \propto \langle r'_k(t'_k), s(t'_k)|(1 - iH\Delta t)|r'_k(t'_{k-1}), s(t'_{k-1})\rangle \ldots \langle r'_2(t'_2), s(t'_2)|(1 - iH\Delta t)|r'_2(t'_1), s(t'_1)\rangle$$

where the $t'_k$’s are all distinct.

Now, as we discussed earlier, the statistical phase can be obtained by comparing the phase of this amplitude with another amplitude which is the same locally but doesn’t exchange the two particles. When we make this
III. STATISTICS AND THE HOPPING OPERATOR ALGEBRA

In this section we derive a simple algebraic formula for the statistics of a particle hopping on a lattice. This formula is completely general and holds irrespective of whether the particles are fundamental or are low energy excitations of an underlying condensed matter system (e.g., quasi-particles).

We begin with a Hilbert space which describes \( n \) hard-core particles hopping on a \( d \) dimensional lattice. The states can be labeled by listing the positions of the \( n \) particles: \(|i_1, i_2, \ldots, i_n\rangle\). The particles are identical so the states \(|i_1, i_2, \ldots, i_n\rangle\) do not depend on the order of \( i_1, i_2, \ldots, i_n \). For example,

\[
|i_1, i_2, \ldots, i_n\rangle = |i_2, i_1, \ldots, i_n\rangle
\]

A typical Hamiltonian for this system is of the form

\[
H = \sum_{<ij>} (t_{ij} + t_{ji})
\]

where \( t_{ij} \) are “hopping operators” with the property that

\[
t_{ij} |i_1, i_2, \ldots, i_{n-1}\rangle \propto |i_1, i_2, \ldots, i_{n-1}\rangle
\]

We assume that the hopping is local, i.e.

\[
[t_{ij}, t_{kl}] = 0
\]

if \( i, j, k, l \) are all different.

Our goal is to compute the statistics of the particles described by this hopping Hamiltonian \( H \). In the following, we show that the statistical angle can be derived from the simple algebraic properties of the hopping operators. More precisely, we show that the particles obey statistics \( e^{i\theta} \) if

\[
t_{il}t_{ki}t_{ij} = e^{i\theta}t_{ij}t_{ki}t_{il}
\]

for any three hopping operators \( t_{ij}, t_{ki}, t_{il} \), where \( j, k, l \) are (distinct) neighbors of \( i \) (ordered in the clockwise direction in the case of 2 dimensions). The orientation convention in 2 dimensions is necessary for the anyonic case.

The simplest examples where we can apply this formula are the cases of noninteracting hard-core bosons or fermions. In these cases the \( n \) particle Hamiltonian can be written as

\[
H = -t \sum_{<ij>} (c_i^\dagger c_j + c_j^\dagger c_i)
\]

where the \( c_i \)'s are the boson or fermion annihilation operators. The hopping operators are just \( t_{ij} = -tc_i^\dagger c_j \). A little algebra confirms that the boson and fermion hopping operators do indeed satisfy Eq. (4) with \( e^{i\theta} = +1 \) and \( -1 \), respectively.

We now give a general derivation of the formula. We begin with the state \(|i, j, \ldots\rangle\) which contains particles at sites \( i, j \) and other particles far away.

Imagine that we exchange the two particles at \( i, j \) using an appropriate product of hopping operators. One expression for the final swapped state is

\[
|i, j, \text{swapped}\rangle = (t_{jj'})(t_{j'p})(t_{pq})(t_{q\ldots})(t_{nn'})
\]

Here, we’ve grouped the hopping operators into 5 terms. Each of these terms moves a particle along some path on the lattice. The cumulative effect of all 5 terms is to exchange the particles (see Fig. 1a).

Similarly, we can construct a path which doesn’t swap the two particles. One expression for the final unswapped state is

\[
|i, j, \text{unswapped}\rangle = (t_{jj'})(t_{j'j})(t_{j\ldots})(t_{nn'})
\]

Once again, we’ve grouped the hopping operators into 5 terms. In this case, the cumulative effect of these terms is simply to move the two particles along independent loops without any swapping taking place (see Fig. 1b).
comes from gauge fields or other Berry phases contributes equally to the swapped and unswapped states. The phase difference between the swapped and unswapped states is therefore exactly equal to the statistical phase of the particles.

This intuitive argument can be made rigorous using the discussion in the previous section. Indeed, it’s not hard to see that the two space-time paths traced out by the swapped and unswapped states are exactly the same locally - they only differ by a reordering. Furthermore, the phase difference between the swapped and unswapped states can be written as a phase difference between two expressions of the form (1). Thus, taking the phase difference between the swapped and unswapped states is completely equivalent to the procedure derived in the previous section.

We now compute this phase difference. We use the assumed algebraic relation

\[ t'_{ij} t_{nj} = e^{i\theta} t_{j'i} t_{nj} t_{j'p} \]

Applying this relation to the swapped state and reordering the hopping operators using locality, we find

\[ |i, j, \text{swapped}\rangle = e^{i\theta} |i, j, \text{unswapped}\rangle \]

The phase difference is \( e^{i\theta} \), so the particles obey statistics \( e^{i\theta} \), as claimed. A similar formula can be derived for the relative statistics of distinguishable particles. This formula, together with its derivation, is given in the appendix. (See Eq. (A1)).

### IV. FERMIONS AND STRINGS

In this section we consider the properties of a lattice spin system with fermionic or anyonic excitations. We argue that the excitations are always created in pairs, and the creation operator for a pair of particles has a string-like structure, with the new particles located at the ends. One interpretation of this is that fermions never appear alone - they always come with some kind of gauge field.

We now make these statements more precise. Suppose we have a lattice spin system with fermionic excitations (the anyonic case is completely analogous). The total Hilbert space of the lattice spin model is a direct product of local Hilbert spaces \( \mathcal{H}_I \), each associated with an individual lattice site \( I \). We expect that the total Hilbert space contains a low energy subspace which is spanned by fermion states. The corresponding low energy effective Hamiltonian is given by

\[ H_{\text{eff}} = \mathcal{P} \mathcal{H} \mathcal{P} \]

where \( \mathcal{P} \) is a projection operator onto the \( n \) fermion subspace. Typically, \( H_{\text{eff}} \) can be written as a sum

\[ H_{\text{eff}} = \sum_{<ij>} (t_{ij} + t_{ji}) \]

where the \( t_{ij} \) are hopping operators. We expect that the \( t_{ij} \) are local in the underlying spin degrees of freedom.

That is, they act only within the local Hilbert spaces near sites \( i, j \). [30] We would like to point out that the lattice on which the fermion hops (labeled \( i \)) may not be the same as the lattice formed by the local Hilbert spaces (labeled \( I \)).

Now, consider the string-like product of hopping operators

\[ W_{ir} = t_{ij} t_{jk} t_{kl} \ldots t_{pq} t_{qr} \]

This operator destroys a fermion at site \( r \) and creates one at site \( i \):

\[ W_{ir} |r, \ldots\rangle \propto |i, \ldots\rangle \]

where \( |r, \ldots\rangle, |i, \ldots\rangle \) are low-energy states with fermions at \( r, i \), (and possibly other fermions far away). Furthermore, this operator clearly has a string-like structure: it is made up of a string-like product of operators, each of which is local in the underlying spin degrees of freedom. The only issue is that this string operator might be trivial. That is, \( W_{ir} \) might act trivially on all the intermediate spins at \( j, k, \ldots, q \), and only have a nontrivial effect near \( i \) and \( r \). In that case, \( W_{ir} \) isn’t really a string at all.

This is a legitimate concern, since the string operator typically is trivial in the case of bosonic quasi-particles. Consider, for example, a lattice of noninteracting spins in a magnetic field: \( H = -B \sum_i \sigma_i^3 \). The ground state has \( \sigma_i^3 = 1 \) for all \( i \). The excitations are obtained by flipping one spin: \( \sigma_j^3 = -1 \) for some \( j \). If we perturb the Hamiltonian by a term \( t \sum_{<ij>} \sigma_i^1 \sigma_j^1 \), these excitations acquire dynamics. The corresponding hopping operators are then \( t_{ij} = 1/2 \sigma_i^1 \sigma_j^1 \) and the string operator is

\[ W_{ir} = t_{ij} t_{jk} \ldots t_{qr} \propto (\sigma_i^1 \sigma_j^1)(\sigma_j^1 \sigma_k^1)\ldots(\sigma_q^1 \sigma_r^1) = \sigma_i^1 \sigma_r^1 \]

The operator \( W_{ir} \) clearly creates a particle at position \( i \) and destroys one at position \( r \). However, as we see above, it is trivial and has no string-like structure. The particle creation and annihilation operators are completely local.

Our claim is that this can never happen in the case of emerging fermions. That is, the string operator \( W_{ir} \) can never be written as a product

\[ W_{ir} = A_i B_r \]

where \( A_i, B_r \) are operators which act only within the local Hilbert spaces near sites \( i, r \). Physically, this means that the fermion creation and annihilation operators are never local in the underlying bosonic degrees of freedom. Instead, they naturally appear in pairs, with a string-like structure connecting them. [31]

**Proof:** We wish to use the algebraic formula (4) from the previous section. That formula relied on the locality of the hopping operators (3). Here we would like to use a weaker locality condition:

\[ [t_{ij}, t_{kl}] = 0 \]

(5)
if $i$, $j$, are far from $k$, $l$. One can make a similar argument using this weaker assumption. One arrives at a slightly weaker algebraic relation:

$$W_{kl}W_{ki}W_{ij} = -W_{ij}W_{kl}W_{il}$$

if $i$, $l$, $k$ are sufficiently far from each other. Since $W_{kl}W_{ij} = W_{kj}$, we can rewrite this as

$$W_{kl}W_{kj} = -W_{ij}W_{kl}$$

(6)

Condition (6) is a direct consequence of the fermionic nature of the quasi-particles. We will now show that it implies that the strings are nontrivial. Indeed, if the $W_{ir}$ could be written as $W_{ir} = A_{i}B_{r}$, then

$$W_{il}W_{kj} = A_{i}B_{l}A_{k}B_{j}$$

$$W_{ij}W_{kl} = A_{i}B_{j}A_{k}B_{l}$$

But the $A$'s and $B$'s are local operators, so they commute with each other when they are well separated. We can therefore rearrange the operators in these two equations to obtain

$$W_{il}W_{kj} = W_{ij}W_{kl}$$

This directly contradicts the fermion condition (6). We conclude that the string operator is always nontrivial. □

The presence of this nontrivial string indicates that fermions always appear together with some kind of gauge field. One way to see this is to consider a closed loop of hopping operators $t_{ij}t_{jk}...t_{pq}t_{qi}$. This string can be interpreted as a Wilson loop operator, since its phase is precisely the accumulated phase of the particle when it traverses a loop. The fact that it is nontrivial (that is, not equal to the identity operator) means that the particle is coupled to a nontrivial gauge field.

V. A 2D EXAMPLE

In this section, we present an exactly soluble lattice spin model with fermionic excitations. The exactly soluble model provides a concrete realization of the string picture discussed above. The emerging fermions turn out to be coupled to a $Z_{2}$ gauge field.

In this spin-1/2 system, proposed by Kitaev[15], the spins live on the links of a square lattice (Fig. 2). The Hamiltonian is

$$H = -U \sum_{i} (\prod_{c_{i}'} \sigma_{j}^{c_{i}'}) - g \sum_{p} (\prod_{c_{p}'} \sigma_{j}^{3})$$

(7)

Here $i$ labels the links, $I$ labels the sites, and $p$ labels the plaquettes of the square lattice. Also, $C_{i}'$ denotes the loop connecting the four spins adjacent to site $I$, while $C_{p}'$ denotes the loop connecting the four spins adjacent to plaquette $p$ (see Fig. 2).

This model is exactly soluble since all the terms in the Hamiltonian commute with each other. The ground state satisfies

$$\prod_{c_{i}'} \sigma_{j}^{c_{i}'} = 1$$

for all sites $I$, and

$$\prod_{c_{p}'} \sigma_{j}^{3} = 1$$

for all plaquettes $p$. There are two types of (localized) excited states. We can have a site where

$$\prod_{c_{i}'} \sigma_{j}^{c_{i}'} = -1$$

or we can have a plaquette where

$$\prod_{c_{p}'} \sigma_{j}^{3} = -1$$

We call the first type of excitation a “charge” and the second type of excitation a “flux.” Static charge and flux configurations are exact eigenstates of the above Hamiltonian. Thus, the charge and flux quasi-particles have no dynamics.

This lack of dynamics is a special feature of the above model. However, we are interested in the properties of a generic Hamiltonian in the same quantum phase. Thus, we need to perturb the system, and analyze the resulting dynamics. The simplest nontrivial perturbation is

$$H' = H + J_{1} \sum_{i} \sigma_{i}^{1} + J_{3} \sum_{i} \sigma_{i}^{3}$$

(8)

It’s not hard to see that the first term allows the fluxes to hop from plaquettes to adjacent plaquettes, while the second term allows the charges to hop from sites to neighboring sites.

We now calculate the statistics of the fluxes. To do this, we restrict our Hamiltonian to the low energy subspace with $n$ fluxes and zero charges. Within this subspace, our Hamiltonian reduces to

$$H'_{eff} = J_{1} \sum_{i} \sigma_{i}^{1}$$

To make contact with our previous formalism, we write this as

$$H'_{eff} = \sum_{<pq>} (t_{pq} + t_{qp})$$
where the sum is taken over adjacent plaquettes, \( p, q \). \( t_{pq} \) is defined by \( t_{pq} = \frac{i}{2} \sigma^1_p \) and \( i \) the link joining \( p \) and \( q \).

To calculate the statistics, we need to compare \( t_{pq} t_{rp} / ps \) with \( t_{ps} t_{rp} / pq \). Well, it’s obvious from the definition that all the hopping operators \( t_{pq} \) commute with one another. Therefore,

\[
t_{pq} t_{rp} / ps = t_{ps} t_{rp} / pq
\]

We conclude that the fluxes are bosons. In the same way, one can show that the charges are also bosons.

Next, we consider the bound state of a flux and a charge. That is, we consider excitations with a flux through some plaquette \( p \) and also a charge at one of the sites \( I \) adjacent to \( p \).

These bound states are not actually stable for the above Hamiltonian (8) - the charge and flux will separate from one another over time. However, one can imagine modifying the Hamiltonian so that charges and fluxes prefer to be adjacent to each other. In this case, the bound state is a true quasi-particle.

Suppose we’ve made such a modification. We can then consider the statistics of the bound state. If we restrict our Hamiltonian (8) to the low energy subspace with \( n \) bound states, then our Hamiltonian reduces to

\[
H_{\text{eff}}' = J_1 \sum_i \sigma^1_i + J_3 \sum_i \sigma^3_i
\]

To understand the effect of these terms, imagine we have a bound state with a flux at \( p \) and a charge at \( I \). It’s not hard to see that the first term allows the flux to hop to the two neighboring plaquettes which are also adjacent to \( I \). Similarly, the second term allows the charge to hop to the two neighboring sites, adjacent to \( p \). All other hopping destroys the bound state and is therefore forbidden for energetic reasons. Thus, the perturbation gives rise to 4 types of hopping operators - 2 corresponding to fluxes and 2 corresponding to charges. (See Fig. 3).

Formally, we can write our low-energy Hamiltonian as

\[
H_{\text{eff}}' = \sum_{< (p, I) (q, J)>} (t_{(p, I)(q, J)} + t_{(q, J)(p, I)})
\]

where the hopping operators are defined by

\[
t_{(p, I)(q, J)} = \frac{J_1}{2} \sigma^1_i \quad \text{or} \quad \frac{J_3}{2} \sigma^3_i
\]

depending on whether \((p, I), (q, J)\) differ by a flux hop or a charge hop. In the first case, \( i \) is defined to be the link joining \( p \) and \( q \), while in the second case, \( i \) is the link joining \( I \) and \( J \).

To calculate the statistics we need to compare a product of the form \( t_{1} t_{2} t_{3} \) with the product \( t_{3} t_{2} t_{1} \), where the \( t_k \) are hopping operators involving a single bound state at position \((p, I)\). (We could write out these expressions precisely, but it’s messy and not very enlightening).

Now, as we discussed above, there are four different ways that a bound state at \((p, I)\) can hop - two charge hops, and two flux hops. Each of the flux hopping operators can be paired with a corresponding charge operator which involves the same link \( i \). It’s easy to see that each of the flux operators anti-commutes with the corresponding charge operator (since one involves a \( \sigma^1 \), while the other involves a \( \sigma^3 \)). However, everything else commutes. (See Fig. 3).

With these facts in mind, we can compare \( t_{1} t_{2} t_{3} \) with \( t_{3} t_{2} t_{1} \). By our discussion above, any set of 3 hopping operators involving \((p, I)\) must contain exactly 2 which anti-commute. Thus, exactly 2 of \( t_{1}, t_{2}, t_{3} \) anti-commute. This implies that

\[
t_{1} t_{2} t_{3} = -t_{3} t_{2} t_{1}
\]

We conclude that the bound states are fermions. Of course, this result is not that surprising once we notice that the charges and fluxes have relative statistics \( e^{i \theta_{\text{rel}}} = -1 \).

We can easily check that the charges and fluxes have relative statistics \(-1\). Normally, we would just use the formula (A1) derived in the appendix. However, this relation assumes that the two types of particles both hop on the same lattice. In our case, the charges hop on the square lattice, while the fluxes hop on the dual lattice.

It’s not hard to see that in our case the formula (A1) needs to be modified to

\[
t_{IJ} t_{pq} = e^{i \theta_{\text{rel}}} t_{pq} t_{IJ}
\]

where \( I, J \) are neighboring sites, \( p, q \) are neighboring plaquettes, and the links connecting \( I, J \), and \( p, q \) are the same. From our previous calculations we know that the \( t_{pq} = \frac{i}{2} \sigma^1_p \) and \( t_{IJ} = \frac{i}{2} \sigma^3_I \). Therefore, \( t_{pq}, t_{IJ} \) anti-commute, and the relative statistics are \( e^{i \theta_{\text{rel}}} = -1 \).

We now see why the two particles were called “charges” and “fluxes” - they have the same statistics and relative statistics as \( Z_2 \) charges and fluxes. It turns out that this connection with \( Z_2 \) gauge theory extends beyond the low energy regime - in fact, all the way to the lattice scale. One can show that the Kitaev model is exactly equivalent to standard \( Z_2 \) gauge theory coupled to a \( Z_2 \) Higgs field.

We argued earlier that whenever fermions or anyons occur in a bosonic system, they are always created in

\[\text{FIG. 3: A bound state of a charge, denoted by a dark circle, and a flux, denoted by a shaded square. The arrows illustrate the four ways the bound state can hop. The two charge hops are denoted by solid arrows, and have hopping operator } \sigma^1. \text{ The flux hops are denoted by dotted arrows, and have hopping operator } \sigma^3. \text{ Notice that each charge hopping operator anti-commutes with a corresponding flux hopping operator, but everything else commutes.}\]
pairs, and the pair creation operator has a string-like structure. The above exactly soluble model (7) provides a good example of this phenomenon.

We begin with the charges. We can construct the string operators associated with these particles by taking products of their hopping operators along some path $P = I_1, ..., I_n$ on the lattice. We find

$$W(P) = \prod_{i=1}^{n} \sigma^3_{i_1} \sigma^3_{i_2} ... \sigma^3_{i_{n-2}} \sigma^3_{i_{n-1}}$$

where $i_1, ..., i_{n-1}$ are the links along the path (see Fig. 4). Notice that $W(P)$ is non-trivial - it acts non-trivially on the spin degrees of freedom along the string $P$.

It is also a pair creation operator: If we apply $W(P)$ to the ground state, the resulting state is an exact eigenstate of the Hamiltonian except for a charge hop, respectively (see Fig. 4). Once again, one can show that the string $W(P)$ is a creation operator for a pair of bound states at the ends of $P$.

In each of these examples, the creation operator for fermions or anyons has an extended string-like structure. It is important to note that the position of this string is completely observable. That is, in each case, the excited state $W(P)|0\rangle$ is independent of the position of $P$; it only depends on the position of the endpoints of $P$.

VI. EXACTLY SOLUBLE 3D MODEL

In two dimensional systems, one can create a fermion by binding a $Z_2$ vortex to a $Z_2$ charge. This is how we obtained the fermion in the above spin-1/2 model on the square lattice. Since both the $Z_2$ vortex and $Z_2$ charge appear as the ends of open strings, the fermions also appear as the ends of strings. However, in three dimensions, we cannot change a boson into a fermion by attaching a vortex to a vortex. Thus we have a string-like structure. This example demonstrates that the string picture for fermions is more general than the flux-charge picture.

Our model has four states for each site of a cubic lattice. Thus we call it a spin-3/2 model. Let $\gamma^{ab}$, $a, b \in \{x, \bar{x}, y, \bar{y}, z, \bar{z}\}$, a $\neq$ b, be $4 \times 4$ hermitian matrices that satisfy

$$\gamma^{ab} = -\gamma^{ba} = (\gamma^{ab})^\dagger$$

$$[\gamma^{ab}, \gamma^{cd}] = 0, \text{ if } a, b, c, d \text{ are all different}$$

$$\gamma^{ab}\gamma^{bc} = i\gamma^{ac}, \text{ if } a \neq c$$

$$\gamma^{ab}\gamma^{ab} = 1$$

A solution of the above algebra can be constructed by taking pairwise products of Majorana fermion operators $\lambda^a$, $a \in \{x, \bar{x}, y, \bar{y}, z, \bar{z}\}$:

$$\gamma^{ab} = i\lambda^a\lambda^b$$

$$\{\lambda^a, \lambda^b\} = 2\delta_{ab}$$

The six Majorana fermion operators naturally require a space of dimension $2^6/2 = 8$, but if we restrict them to the space $\bigwedge^k \lambda^a = 1$, we obtain the desired $4 \times 4$ hermitian matrices. Alternatively, a more concrete description of the $\gamma^{ab}$ is given the appendix, where we express the $\gamma^{ab}$ in terms of Dirac matrices.

FIG. 4: Three examples of string-like creation operators. The string operators are drawn together with the particles they create at their endpoints. The thick solid line (drawn on the lattice) is an example of a charge string operator. The dotted line (drawn on the dual lattice) is an example of a flux string operator. At the bottom, we give an example of a string operator for the bound state of a charge and flux. It is essentially a combination of a charge string and a flux string.
In terms of $\gamma_{i}^{ab}$, the exactly soluble spin-3/2 Hamiltonian can be written as

$$H = -g \sum_{p} F_{p}$$  \hspace{1cm} (14)$$

where $p$ labels all the square plaquettes in the cubic lattice, and the $F_{p}$ are “flux” operators defined by

$$F_{p} = \gamma_{i}^{xy} \gamma_{i+\hat{x}}^{yz} \bar{\gamma}_{i+\hat{y}}^{zx} \gamma_{i+\hat{z}}^{\hat{x}y}$$

or $\gamma_{i}^{yz} \gamma_{i+\hat{y}}^{\hat{z}x} \bar{\gamma}_{i+\hat{z}}^{\hat{x}y} \gamma_{i+\hat{x}}^{zx}$

or $\gamma_{i}^{zx} \gamma_{i+\hat{z}}^{\hat{x}y} \bar{\gamma}_{i+\hat{y}}^{\hat{z}x} \gamma_{i+\hat{y}}^{\hat{x}y}$ \hspace{1cm} (15)$$

depending on the orientation of the plaquette $p$. Just as in the Kitaev model,\[15\] all the $F_{p}$ commute with each other and all the $F_{p}$ have only two eigenvalues: $\pm 1$. Thus, we can solve the Hamiltonian by simultaneously diagonalizing all the $F_{p}$. If $\{\{f_{p}\}\}$ is a common eigenstate of $F_{p}$ with $F_{p}\{\{f_{p}\}\} = f_{p}\{\{f_{p}\}\}$, $f_{p} = \pm 1$, then it is also an energy eigenstate with energy

$$E(\{f_{p}\}) = -g \sum_{p} f_{p}$$

The one subtlety is that the $f_{p}$ are not all independent. If $C$ is the surface of a unit cube, then we have the operator identity

$$\prod_{p \in C} F_{p} = 1$$

It follows that

$$\prod_{p \in C} f_{p} = 1$$

for all cubes $C$. This constraint means that the spectrum of our model is identical to a $Z_{2}$ gauge theory on a cubic lattice. The ground state can be thought of as a state with no flux: $f_{p} = 1$ for all $p$. Similarly, the elementary excitations are small flux loops where

$$f_{p} = -1$$

for the four plaquettes $p$ adjacent to some link $(ij)$. We can think of these excitations as quasi-particles which live on the links of the cubic lattice.

We would like to compute the statistics of these excitations. It is tempting to assume that they are bosons since the model is almost the same as a $Z_{2}$ gauge theory. However, as we will show, this superficial similarity is misleading: the flux loops are actually fermions.

As in the Kitaev model, the quasi-particles are exact eigenstates of our Hamiltonian. Thus, they have no dynamics and it is difficult to compute their statistics. However, this is a special feature of our model. We need to perturb the theory to understand generic states in the same quantum phase. The simplest perturbation is

$$H' = H + t \sum_{i,a,b} \gamma_{i}^{ab}$$

for the four plaquettes $p$ adjacent to some link $(ij)$. We can think of these excitations as quasi-particles which live on the links of the cubic lattice.

FIG. 5: A fermion on link 1 can hop onto 10 different links: 2 - 11. The hopping 1 \rightarrow 2 is generated by $\gamma_{1}^{xz}$, the hopping 1 \rightarrow 10 by $\gamma_{1}^{yz}$, and the hopping 1 \rightarrow 4 by $\gamma_{1}^{zx}$.

To compute the statistics, we need to restrict ourselves to the low energy subspace with $n$ quasi-particles. In this subspace, $H'$ reduces to

$$H'_{\text{eff}} = t \sum_{i,a,b} \gamma_{i}^{ab}$$

The effect of each term $\gamma_{i}^{ab}$ is to allow the quasi-particles to hop between the two links $(i(i+\hat{a}))$ and $(i(i+\hat{b}))$ adjacent to site $i$. Here $\hat{a} = \hat{x}$ if $a = x$, $\hat{a} = -\hat{x}$ if $a = x$, etc. (See Fig. 5.) Thus, our Hamiltonian can be written in the standard hopping form:

$$H'_{\text{eff}} = \sum_{i,a,b} (t_{(i(i+\hat{a}))}(i(i+\hat{b})) + t_{(i(i+\hat{b}))}(i(i+\hat{a})))$$

where $t_{(i(i+\hat{a}))}(i(i+\hat{b})) = \frac{1}{2} \gamma_{i}^{ab}$.

To calculate the statistics we need to compare a product of the form $t_{12}t_{3}t_{1}t_{2}$ with the product $t_{3}t_{2}t_{1}$, where the $t_{k}$ are hopping operators involving a single link $(ij)$. (As in the Kitaev model, we could write out these expressions explicitly, but it’s not very enlightening).

Note that a quasi-particle on the link $(ij)$ can hop to any of the 10 neighboring links, 5 of which are adjacent to site $i$, and 5 of which are adjacent to site $j$ (see Fig. 5). Using the algebra (12) or the Majorana fermion representation, we find that the 5 hopping operators associated with site $i$ all anti-commute with each other and similarly for site $j$. On the other hand, each of the operators associated with $i$ commute with each of the operators associated with $j$.

With these facts in mind, we can now compare $t_{12}t_{3}t_{1}$ with $t_{3}t_{2}t_{1}$. There are essentially two cases: either all three of the $t_{k}$’s are associated with a single site $i$ or $j$, or two involve one site and one involves the other. In the first case, all the $t_{k}$’s anti-commute; in the second case, a pair of the $t_{k}$’s anti-commute and everything else commutes. In either case, we have

$$t_{12}t_{3}t_{1} = -t_{3}t_{2}t_{1}$$

We conclude that the quasi-particles are indeed fermions.
This string corresponds to the operator $W(P) \propto \gamma^{\hat{z}}_{i_1} \gamma^{\hat{z}}_{i_2} ... \gamma^{\hat{z}}_{i_n}$, where $b_m, a_{m+1}$ is the pair of indices associated with the link $(i_m, i_{m+1})$. (See Fig.6.) Notice that the string operator commutes with the Hamiltonian, except at the ends. Thus, when we apply $W(P)$ to the ground state, the only effect is to flip the signs of $f_p$ for the plaquettes $p$ adjacent to the links $(i_1, i_2), (i_{n-1}, i_n)$. This means that the open string operator creates a pair of fermions at its two ends.

VII. CONCLUSION

In this paper we have derived a general relation (4) between the statistics of a lattice particle and the algebra of its hopping operators. This relation allows us to analyze emerging fermions in three dimensions. We have also shown that there is a close connection between strings and fermions. The statistical algebra (4) fundamental to this connection since it allows us to determine the statistics of the ends of strings from the structure of the string operator.

It is interesting to put this string picture of emerging fermions in context. Indeed, it is well-known that gauge theories and strings are closely related - the strings correspond to electric flux lines in the associated gauge theory. [23–29] Thus, it appears that the fundamental concepts of Fermi statistics, gauge theory, and strings are all connected. They are all just different aspects of a kind of order - topological order - in bosonic lattice systems.

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APPENDIX A: RELATIVE STATISTICS

In this section, we derive an algebraic formula, analogous to (4), for the relative statistics of distinguishable particles. We begin with a Hilbert space which describes two types of hard-core particles hopping on a 2D lattice. For concreteness, say that there are $m$ particles of type 1 and $n$ particles of type 2. The states can be labeled by listing the positions of the two types of particles: $|i_1, ..., i_m, j_1, ..., j_n\rangle$.

A typical Hamiltonian for this system is of the form

$$H = \sum_{<ij>} (t_{ij}^1 + t_{ji}^1 + t_{ij}^2 + t_{ji}^2)$$

where $t_{ij}^1, t_{ij}^2$ are hopping operators for the two types of particles.

We wish to calculate the relative statistics of the two types of particles. As before, the statistics are related to the simple algebraic properties of the hopping operators. Specifically, we will show that the particles have relative statistics $e^{i\phi}$ if

$$(t_{ip}^1 t_{pj}^2) (t_{ip}^1 t_{pj}^1) = e^{i\phi} (t_{kp}^1 t_{pl}^1) (t_{ip}^2 t_{pj}^2)$$

(A1)

Here, $i, k, j, l$ are (distinct) neighbors of $p$ oriented in the clockwise direction.

We begin with the state $|i; j\ldots\rangle$ which contains a type 1 particle at $i$, a type 2 particle at $j$, and other particles which are far away.

Imagine that we wind particle 1 around particle 2 using an appropriate product of hopping operators. One expression for the final wound state is:

$$|i; j, \text{wound}\rangle = (t_{j\alpha}^2)(t_{\alpha j}^1)(t_{i l}^1...t_{m o}^1)$$

$$t_{a n}^1...t_{p q}^1 t_{a q}^1)(t_{j o}^1)(t_{j o}^1)(i; j\ldots)$$

Here, we’ve grouped the hopping operators into 6 terms. The cumulative effect of these terms is to wind one particle around the other (see Fig. 7a).
Similarly, we can construct a state where the two particles don’t wind around each other (see Fig. 7b):

\[ |i; j, \text{unwound}\rangle = (t_{jl}^1 \ldots t_{ma}^1)(t_{an}^1 \ldots t_{pj}^1 t_{ql}^2)(t_{ja}^2)
\]

As before, the two states involve the same total path in configuration space. Thus, the phase difference between the two states is precisely the relative statistics of the particles. This can be made rigorous using an argument similar to the exchange statistics case.

We now calculate this phase difference. We use the assumed algebraic relation

\[ (t_{ja}^2 t_{ja'}^2)(t_{ma}^1 t_{an}^1) = e^{i\phi}(t_{ma}^1 t_{an}^1)(t_{ja}^2 t_{ja'}^2) \]

Applying this relation to the wound state, and reordering the hopping operators using locality, we find

\[ |i; j, \text{wound}\rangle = e^{i\phi} |i; j, \text{unwound}\rangle \]

This establishes the desired result: the particles have relative statistics \( e^{i\phi} \).

A good consistency check for this formula can be obtained by considering the relative statistics of two particles of the same type. In this case, we expect the angle for the relative statistics to be exactly twice the angle for the exchange statistics:

\[ e^{i\phi_{\text{rel.}}} = e^{2i\theta_{\text{stat.}}} \]

One way to see this is that exchanging the particles twice in the same direction is topologically equivalent to winding one particle around the other.

This result, which has a topological character to it, can be derived algebraically from our two formulas. We start with the expression \((t_{ip}t_{pj})(t_{kp}t_{pl})\). Applying the exchange statistics formula Eq. (4), we find:

\[
\begin{align*}
(t_{ip}t_{pj})(t_{kp}t_{pl}) &= t_{ip}(t_{pj}t_{kp}t_{pl}) \\
&= t_{ip}(e^{i\theta_{\text{stat.}}} t_{pl}t_{kp}t_{pj})
\end{align*}
\]

Applying the formula again gives

\[
\begin{align*}
t_{ip}(t_{pl}t_{kp}t_{pj}) &= (t_{ip}t_{pl}t_{kp})t_{pj} \\
&= (e^{i\theta_{\text{stat.}}} t_{kp}t_{pl}t_{ip})t_{pj}
\end{align*}
\]

Combining these two equations gives

\[
(t_{ip}t_{pj})(t_{kp}t_{pl}) = e^{2i\theta_{\text{stat.}}} (t_{kp}t_{pl})(t_{ip}t_{pj})
\]

Comparing this with the relative statistics formula Eq. (A1), we see that \( e^{i\phi_{\text{rel.}}} = e^{2i\theta_{\text{stat.}}} \), as claimed.

**APPENDIX B: DIRAC MATRIX REPRESENTATION OF \( \gamma^{ab} \)**

In the following, we represent \( \gamma^{ab} \)'s in terms of the Dirac matrices. Note that Eq. (12) implies that

\[
\{\gamma^{ac}, \gamma^{bc}\} = 2\delta_{ab}
\]

Thus \( \gamma^{xz}, \gamma^{zx}, \gamma^{yz}, \gamma^{zy} \) satisfy the algebra of Dirac matrices. Introducing four Dirac matrices

\[
\begin{align*}
\gamma^x &= \sigma^1 \otimes \sigma^1, & \gamma^\bar{x} &= \sigma^2 \otimes \sigma^1 \\
\gamma^y &= \sigma^3 \otimes \sigma^1, & \gamma^\bar{y} &= \sigma^3 \otimes \sigma^2,
\end{align*}
\]

we can write

\[
\gamma^{az} = \gamma^a, \quad a = x, \bar{x}, y, \bar{y}. \quad (B1)
\]

Similarly, \( \gamma^{\bar{x}z}, \gamma^{\bar{x}z}, \gamma^{\bar{y}z}, \gamma^{\bar{y}z} \) satisfy the algebra of Dirac matrices. We can express those operators as

\[
\gamma^{\bar{x}z} = i\gamma^a \gamma^5, \quad a = x, \bar{x}, y, \bar{y}. \quad (B2)
\]

where \( \gamma^5 = \gamma^x \gamma^y \gamma^z \). Finally, for \( a, b = x, \bar{x}, y, \bar{y} \), we have

\[
\gamma^{ab} = i\gamma^a \gamma^b \quad (B3)
\]

In this way, we can express all the \( \gamma^{ab} \), \( a, b \in \{x, \bar{x}, y, \bar{y}, z, \bar{z}\} \), \( a \neq b \) in terms of Dirac matrices.

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[30] More precisely, the $t_{ij}$ are local at low energies: the $t_{ij}$ can be written as $t_{ij} = P\tilde{t}_{ij}P$ where $\tilde{t}_{ij}$ is local and commutes with the projection operator $P$.
[31] A more precise statement of our claim is that $W_{ir}$ cannot be written as a product $P_sA_iB_rP_s$ where $P_s$ is the projection operator onto the $n$ fermion subspace with no particles at any of the intermediate sites $j, k, ..., q$, and $A_i, B_r$ are local operators commuting with $P$. This statement is slightly stronger then the one made in the text, but the same proof applies.