ON A MODEL OF FORCED AXISYMMETRIC FLOWS.

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Abstract. In this work, we consider a model of forced axisymmetric flows which is derived from the inviscid Boussinesq equations. What makes these equations unusual is the boundary conditions they are expected to satisfy and the fact that the boundary is part of the unknown. We show that these flows give rise to an unusual Monge-Ampere equations for which we prove the existence and the uniqueness of a variational solution. We take advantage of these Monge-Ampere equations and construct a solution to the model.

Key words. Mass transport, Duality, Wasserstein metric, Boussinesq.

AMS subject classifications. 35Q35, 35R35, 49K20.

1. Introduction. In this paper, we consider a model of forced axisymmetric flows in the absence of viscosity. This model was introduced by [6] to study the structure of tropical cyclones. The solution can be regarded as an axisymmetric vortex which is stable to axisymmetric perturbations and which evolves slowly in time under the action of forcing, [7] [8]. Shutts et al. [14] developed a discrete procedure for solving this problem within a rigid axisymmetric boundary. We extend this procedure to the continuous case by using mass transportation methods, as reviewed by [5]. We also propose and solve a novel free boundary version which is more physically appropriate, as it allows the vortex to evolve within an ambient fluid at rest. Mass transportation methods have been applied successfully to a free boundary problem by [3], but our problem differs in important respects from theirs. The time dependent domain where the fluid evolves, in the cylindrical coordinates (λ, r, z), is of the form

\[ \Gamma_\varsigma = \{ (\lambda, r, z) : 0 \leq \lambda \leq 2\pi, \ 0 \leq z \leq H, \ r_0 \leq r \leq \varsigma(t, \lambda, z) \} \]

where the boundary \( r = \varsigma \) is a material surface and \( r_0, H \) are positive real numbers. We have used the notation \( S_t = S(t, \cdot, \cdot) \). The temperature \( \theta' \) within the domain of the vortex (where the PDEs are considered) is higher than the temperature in the ambient fluid which is maintained at a constant temperature \( \theta_0 \) in a rotating framework where the Coriolis coefficient is \( \Omega > 0 \). We denote by \( \mathbf{u} = (u, v, w) \) the velocity of the fluid in cylindrical coordinates. The material derivative associated to this velocity in cylindrical coordinates takes the form

\[ \frac{D}{Dt} := \frac{\partial}{\partial t} + u \frac{\partial}{\partial \lambda} + v \frac{\partial}{\partial r} + w \frac{\partial}{\partial z} \].

The pressure inside the vortex is denoted by \( \varphi \).

We follow procedures proposed by Craig [4] to solve the time evolution of the vortex under axisymmetric forcing. The unknown of the problem are \( \mathbf{u} = (u, v, w), \theta', \varphi, \varsigma \). We start by writing the equations for forced almost axisymmetric flows, as derived by Craig [4]:

\[
\begin{align*}
\frac{Du}{Dt} + uv + 2\Omega v + \frac{1}{r} \frac{\partial \varphi}{\partial \lambda} &= \frac{1}{r} F(t, \lambda, r, z), \\
\frac{D\theta'}{Dt} &= S(t, \lambda, r, z), \\
u^2 + 2\Omega u &= \frac{\partial \varphi}{\partial r}, \\
\frac{\partial \varphi}{\partial z} - g \frac{\theta'}{\theta_0} &= 0, \\
\frac{1}{r} \frac{\partial r}{\partial t} (u) + \frac{1}{r} \frac{\partial v}{\partial t} (rv) + \frac{\partial w}{\partial z} &= 0.
\end{align*}
\]

We consider these equations with Neumann conditions imposed on the rigid boundary while a kinematic boundary condition is imposed on the free boundary and the pressure is required at each time \( t \) to vanish at \( \{r = \varsigma(t)\} \cap \{z = H\} \). In Craig [4], the free boundary condition was replaced by a decay condition as \( r \to \infty \). \( F(t, \lambda, r, z) \) and \( S(t, \lambda, r, z) \) are prescribed forcing terms of the system.

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Though being an approximation to the inviscid Boussinesq equations, the equations for almost axisymmetric flow (when $F = S = 0$ in (1.2) ) have retained quite the same level of complexity and formally are known to have kept the infinite dimensional Hamiltonian structure already present in the inviscid Boussinesq equations. From a physical perspective, we are interested in solutions that are stable in the sense that they correspond to a minimum energy state with respect to parcel displacements that preserve the angular momentum and the potential temperature (see [7]). As suggested in [13], one of the main obstructions we run into while implementing the solution procedure we propose in solving the almost axisymmetric flow equations comes from our inability to find adequate regularity properties for the pressure $\phi$ with respect to $\lambda$ as the system evolves in time. In this paper, we set aside this difficulty by considering the equations for forced axisymmetric flow. The forcing terms can be considered as representing the effects of the non-axisymmetric parts of the real flow on the axisymmetric part. The solution, as we will show, shares the same stability property as the full system of equations and thus sheds some light into the structure of almost axisymmetric flows.

1.1. The Axisymmetric Model. We assume that the quantities and operators involved in (1.2) do not depend on $\lambda$ (in particular, here $\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{v} \frac{\partial}{\partial r} + \bar{w} \frac{\partial}{\partial z}$) and thus obtain the 2-dimensional system of equations:

\[
\begin{align*}
\text{(1.3a)} & \quad \frac{D\bar{u}}{Dt} + \frac{\bar{u}\bar{v}}{r} + 2\Omega\bar{v} = \frac{1}{r}\bar{F}(t,r,z), \\
\text{(1.3b)} & \quad \frac{D\bar{\theta}'}{Dt} = \bar{S}(t,r,z), \\
\text{(1.3c)} & \quad \frac{\bar{u}^2}{r} + 2\Omega\bar{u} = \frac{\partial \bar{z}}{\partial r}, \\
\text{(1.3d)} & \quad \frac{\partial \bar{\phi}}{\partial z} - g\bar{\theta}' = 0 \\
\text{(1.3e)} & \quad \frac{1}{r} \left( \frac{\partial}{\partial r}(r\bar{v}) + \frac{\partial \bar{w}}{\partial z} \right) = 0,
\end{align*}
\]

The above equations are to be solved in the moving domain

\[
\Gamma_\zeta = \{(r,z) : 0 \leq z \leq H, \quad r_0 \leq r \leq \zeta(t,z)\}
\]

where $\zeta$ is a free boundary and $r_0, H$ are positive real numbers. The conditions on the boundary are given by

\[
\begin{align*}
\text{(1.5)} & \quad \langle (\bar{v}_t, \bar{w}_t); n_t \rangle = 0 \quad \text{on} \quad \{r = r_0\} \cup \{z = 0\} \cup \{z = H\} \\
& \quad \frac{\partial \bar{\phi}}{\partial r} + \bar{w} \frac{\partial \bar{\phi}}{\partial z} = \bar{v} \quad \text{on} \quad \{r = \zeta(t,z)\}
\end{align*}
\]

along with

\[
\text{(1.6)} \quad \bar{\phi}(t,\zeta(t,H),H) = 0.
\]

Here $n_t$ is the unit outward normal vector field at time $t$. $\bar{F}(t,r,z)$ and $\bar{S}(t,r,z)$ are prescribed functions. In order to obtain stable solutions discussed above, we require the pressure to satisfy the following stability condition:

\[
\text{(1.7)} \quad \nabla_{r,z} \left( \bar{\phi} + \Omega^2 \frac{\bar{r}^2}{2} \right) \text{ is invertible}
\]
These equations are supplemented by the initial conditions

\[(u, v, w)|_{t=0} = (u_0, v_0, w_0); \quad \bar{\theta}'|_{t=0} = \bar{\theta}'_0 \quad \bar{\varphi}|_{t=0} = \varphi_0; \quad \bar{\zeta}|_{t=0} = \zeta_0\]

that are required to satisfy \((\text{1.3}), (\text{1.5})\); \((\text{1.7})\), the first equation in \((\text{1.5}), (\text{1.6})\) and \((\text{1.7})\).

### 1.2. Continuity equation corresponding to the 2D Axisymmetric Flows with Forcing Terms.

In this section, we provide a brief discussion of how the equations for forced axisymmetric flows can be reformulated as a continuity equation in a set of transformed variables. A more thorough discussion can be found in \([\text{13}]\). Let \(\bar{\varphi}\) be smooth and define

\[(1.8) \quad P_t(s, z) = \bar{\varphi}_t(r, z) + \frac{\Omega^2_r}{2} \quad \text{with} \quad 2s = r_0^2 - r^2.\]

We denote by \(\Psi_t := \Psi_t(T, Z)\) the Legendre transform of \(P_t\) for each \(t\) fixed. We assume in addition that \(\bar{\varphi}_t\) is such that \(P_t\) is convex and \(\bar{\varphi}\) satisfies \((\text{1.7})\) so that \(\nabla_{s, z} P_t\) is invertible for each \(t\) fixed. Let \(\bar{\zeta}\) be smooth and \(r \chi_{T=\bar{\zeta}(r, z)}\) a probability density function for each \(t\) fixed. We define \(h_t : [0, H] \mapsto [0; 1/(2r_0^2)]\) to be the function induced by the change of variable \(2s = r_0^2 - r^2\) such that

\[(1.9) \quad r \chi_{T=\bar{\zeta}}, drdz = e(s)\chi_{D_{\bar{h}_t}} dsdz\]

with

\[D_{\bar{h}_t} = \{(s, z) : 0 \leq s \leq h_t(z), z \in [0, H]\} \quad \text{and} \quad e(s) = r_0^4/(1 - 2sr_0)^2 \quad \text{for} \quad 0 \leq 2r_0^2s < 1.\]

We consider the family of Borel measures \(\{\sigma_t\}_{t \in [0, T]}\) on \(\mathbb{R}^2\) defined as the push forward of \(e(s)\chi_{D_{\bar{h}_t}}\) by \(\nabla_{s, z} P_t\). Assume in the sequel that \(\{\sigma_t\}_{t \in [0, T]}\) are absolutely continuous with respect to Lebesgue. As \(\nabla_{s, z} P_t\) is invertible, \(\sigma_t\) is equivalently defined by

\[e(\partial_T \Psi) \det(\nabla_{T, z}^2 \Psi) = \sigma \quad \nabla_{T, z} \Psi(spt(\sigma_t)) = D_{\bar{h}_t}\]

If equations \((\text{1.3}), (\text{1.5})\) admit a solution \(\bar{\zeta}, \bar{\varphi}\), then \(t \mapsto \sigma_t\) is an absolutely continuous curve in the space of Borel probability measures and

\[(1.10) \quad \begin{cases} \frac{\partial \sigma}{\partial t} + \text{div}(\sigma V_t) = 0 & (0, T) \times \mathbb{R}^2 \\ \sigma|_{t=0} = \sigma_0. \end{cases}\]

in the sense of distribution with

\[(1.11) \quad V_t = \begin{pmatrix} 2 \sqrt{T} \bar{F}_t \left( \frac{r_0}{\sqrt{1 - 2r_0^2s}} ; \frac{\partial \psi}{\partial z} \right) , \frac{\partial \psi}{\partial r} \bar{S}_t \left( \frac{r_0}{\sqrt{1 - 2r_0^2s}}, \frac{\partial \psi}{\partial z} \right) \end{pmatrix}\]

Conversely, let \(\{\sigma_t\}_{t \in [0, T]}\) be a family of \(\mathcal{P}(\mathbb{R}^2)\), the set of Borel probability measures, with supports in a fixed ball of \(\mathbb{R}^2\) and absolutely continuous with respect to Lebesgue. Assume \((P_t^\varphi, \Psi_t^\varphi, h_t^\varphi)\) are smooth such that \((P_t^\varphi, \Psi_t^\varphi)\) are Legendre transforms of each other for each \(t\) fixed and satisfy for each \(t\) fixed

\[(1.12) \quad \begin{cases} e(\partial_T \Psi) \det(\nabla_{T, z}^2 \Psi) = \sigma \\ \nabla_{T, z} \Psi(spt(\sigma)) = D_{\bar{h}_t} \\ P(h_t(z), z) = \frac{\Omega^2_r}{2(1 - 2r_0^2h_t(z))} \quad \text{on} \quad \{h_t > 0\} \end{cases}\]
Define $\varphi^\sigma$ and $\epsilon^\sigma$ respectively through (1.8) and (1.9). If $\sigma$ solve (1.10) with $V[\sigma]$ as in (1.11) when $\Psi$ is replaced by $\Psi^\sigma$ then $\varphi^\sigma$ and $\epsilon^\sigma$ solve (1.13)-(1.6). If in addition $\nabla_{s,z}P^\sigma$ is invertible then $\varphi^\sigma$ satisfies (1.7).

Equations (1.13)-(1.6) come along with a Hamiltonian from which we derive the following functional

\begin{equation}
(1.13) \quad \sigma \in \mathcal{P}(\mathbb{R}^2) \rightarrow H_*(\sigma) := \inf_{\gamma \in \Gamma(\chi_{D_{h_*}} L^2, \sigma)} I(h, \gamma) = \inf_h \tilde{I}[\sigma](h)
\end{equation}

where $I$ and $\tilde{I}$ will be defined respectively in (3.7) and (3.5).

The minimization problem in (1.13) has a dual formulation

\begin{equation}
(1.14) \quad H_*(\sigma) = \sup \left( \int_{\mathbb{R}^2} \left( \frac{Y}{2r_0^2} - \Omega \sqrt{\nabla - \Psi} \right) \sigma(dq) + \inf_{h \in H_0} \int_0^H \int_0^{h(z)} \left( \frac{\Omega^2 r_0^2}{2(1 - 2sr_0^2)} - P(s, z) \right) e(s)ds \right)
\end{equation}

Here, $H_0$ consists of all Borel functions $h : [0, H] \rightarrow [0, 1/(2r_0^2)]$. The supremum in (1.14) is taken over the set

\begin{equation}
(1.15) \quad \mathcal{U} := \left\{ (\Psi, P) \in C(\mathbb{R}^2_+) \times C(\bar{\Delta}_{r_0}) : P(p) + \Psi(q) \geq (p, q) \text{ for all } (p, q) \in \Delta_{r_0} \times \mathbb{R}^2_+ \right\}
\end{equation}

It turns out that if $\tilde{h}^\sigma$ is a minimizer in (1.13) and $(\tilde{P}^\sigma, \tilde{\Psi}^\sigma)$, Legendre transforms of each other, is a maximizer of (1.14) then $(\tilde{h}^\sigma, \tilde{P}^\sigma, \tilde{\Psi}^\sigma)$ solves (1.12) and $\nabla_{s,z} \tilde{P}^\sigma$ is invertible $e(s)\chi_{D_{h_*}} L^2$ a.e.

1.3. Challenges and Plan of the paper. We show that (1.12) admits a unique variational solution $(P^\sigma, \Psi^\sigma, h^\sigma)$ in the sense of (1.13) and (1.12) which we exploit to get that the operator $\sigma \mapsto V_t[\sigma]$ is continuous. The difficulty in obtaining the existence of a minimizer in (1.13) lies in the fact that the set of functions $\{\chi_{D_h(s, z)}\}_{h \in H_0}$ is not closed in the $L^\infty$ weak* topology. This is an obstacle we bypass by observing that

$\tilde{I}[\sigma](h^\#) \leq \tilde{I}[\sigma](h)$

If $h^\#$ is a monotone rearrangement of $h$ (see 13). The existence follows easily from the fact that the monotone functions are precompact with respect to pointwise topology. But the uniqueness proved extremely challenging in the sense that we don’t know any strict convexity property for the functional with respect to any interpolation we could think of. In section 5 we resort to a duality argument and discover a twist condition for a certain functional which ensures uniqueness in (1.12) and furthermore, we show that the boundary of the domain $D_{h^\sigma}$ is a finite union of graphs of Lipschitz functions. Before that, we fix the notations and give some definitions in section 2. We establish some stability results in section 4 which are used to construct a global solution in time to the continuity equation (1.10)-(1.11) in section 6 following a scheme pioneered by Ambrosio-Gangbo [2]. It is to emphasize that we have defined the velocity field via a Riesz representation when $\sigma$ is not absolutely continuous with respect to Lebesgue.

2. Notations and Definitions. In this section we introduce some notations and recall some standard definitions. Let $d \in \mathbb{N}$.

- For any real number $x$, $[x]$ denotes the integer part of $x$.
- For $A \subset \mathbb{R}^d$, $A$ is the closure of $A$.
- If $f : A \subset \mathbb{R}^d \rightarrow \mathbb{R}$ is a Lipschitz continuous then $\text{Lip}(f) \equiv \inf_{x, y \in A} \frac{|f(x) - f(y)|}{|x - y|}$ denotes the lipschitz constant of $f$.
- $\mathcal{P}(\mathbb{R}^2)$ is the set of all Borel probability measures on $\mathbb{R}^d$. If $R > 0$ then $\mathcal{P}[R]$ will denote the subset of $\mathcal{P}(\mathbb{R}^2)$ consisting of borel probability measures supported in $[0, R]^2$. For $\sigma \in \mathcal{P}(\mathbb{R}^2)$, $\text{spt}(\sigma)$ will denote the support of $\sigma$. 

AC_1 ((a,b) ; \mathcal{P}(\mathbb{R}^d)) is the set of 1–absolutely continuous curves on (a, b) in \mathcal{P}(\mathbb{R}^d).

* Given \( \mu_0 \) and \( \mu_1 \) \in \mathcal{P}(\mathbb{R}^d), we denote by \( \Gamma(\mu_0, \mu_1) \) the set of all Borel measures on \( \mathbb{R}^d \times \mathbb{R}^d \) whose first and second marginal are respectively \( \mu_0 \) and \( \mu_1 \). We say that a Borel map \( T \) pushes forward \( \mu_0 \) onto \( \mu_1 \) and write \( T_{\#} \mu_0 = \mu_1 \) if \( \mu_1(A) = \mu_0(T^{-1}(A)) \) for any \( A \in \mathbb{R}^d \) borel.

If in addition \( \mu_0, \mu_1 \) are of \( p– \) finite moments then the \( (p-th) \) Wasserstein distance between the Borel measures \( \mu_0 \) and \( \mu_1 \) is defined by

\[
W_p^p(\mu_0, \mu_1) = \min \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, d\gamma : \gamma \in \Gamma(\mu_0, \mu_1) \right\}
\]

The set of minimizers in (2.1) is denoted \( \Gamma_0(\mu_0, \mu_1) \).

* Throughout this manuscript, \( R_0, r_0, H \) are positive constants. Let \( \Delta \) be a closed subset of \([0, R_0]^2\) and set

\[
\Delta_{r_0} := [0, 1/2r_0^2] \times [0, H]
\]

### 3. Duality Methods and Monge-Ampere Problem

In this section, we show the existence and uniqueness for the minimization problem (1.13) by coming up with a dual problem. This provides a unique solution to the Monge Ampere equation (1.12) in some sense. Furthermore, this dual formulation helps establish some regularity result for the domain \( D_h \) in (1.12).

Let \( \sigma \in \mathcal{P}_{[R_0]} \), we consider a system of PDEs, where the unknown are

\[
(3.1) \quad \Psi : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}, \quad P : [0, 1/2r_0^2] \times [0, H] \rightarrow \mathbb{R}, \quad h : [0, H] \rightarrow [0, 1/2r_0^2].
\]

We impose that \( \Psi \) and \( P \) are Legendre transforms of each other and these functions solve the system of equations

\[
(3.2) \quad \begin{cases}
\det(\nabla^2 \Psi) = \sigma \\
\nabla \Psi(\text{spt}(\sigma)) = D_h \\
P(h(z), z) = \frac{\Omega^2 z^2}{2(1 - 2r_0^2 h(z))} \quad \text{on} \quad \{ h > 0 \}
\end{cases}
\]

**Definition 3.1.** (i) Assume that \( \sigma = \rho \mathcal{L}^2 \). Let \( P, \Psi, h \) be as in (3.1) such that \( P, \Psi \) are Legendre transforms of each other. We say that \( P, \Psi \) and \( h \) solve equation (3.2) in a weak sense if

\[
(3.3) \quad \begin{cases}
\nabla \Psi(\#) = e(s) \chi_{D_h}(s, z) \mathcal{L}^2 \\
P(h(z), z) = \frac{\Omega^2 z^2}{2(1 - 2r_0^2 h(z))} \quad \text{on} \quad \{ h > 0 \}
\end{cases}
\]

(ii) We say that \( P, \Psi \) and \( h \) solve equation (3.2) in the dual weak sense if

\[
(3.4) \quad \begin{cases}
\nabla P(\#) = e(s) \chi_{D_h}(s, z) \mathcal{L}^2 = \sigma \\
P(h(z), z) = \frac{\Omega^2 z^2}{2(1 - 2r_0^2 h(z))} \quad \text{on} \quad \{ h > 0 \}
\end{cases}
\]

Our main result in this section is the following:

**Theorem 3.2.** Let \( \sigma \in \mathcal{P}_{[R_0]} \) such that \( \text{spt}(\sigma) = \Delta \). Then (3.2) admits a unique variational solution \((\bar{\Psi}, \bar{P}, \bar{h})\) in the sense that \((\Psi, P)\) is obtained as the unique maximizer in (3.14) and \( \bar{h} \) is monotone and obtained as the unique minimiser in (3.8). Moreover, if the support of \( \sigma \) is contained in \([1/2r_0, R_0] \times [0, R_0]\) then \( \partial D_{\bar{h}} \) is a finite union of the graphs of Lipschitz continuous functions.
3.1. Primal and Dual formulation of the problem. Let $\sigma \in \mathcal{P}_{[r_0]}$. We denote by $\mathcal{H}_{dom}$ the set of all $h$ for which $e(s)\chi_{D_h}$ is a probability density function. For $h \in \mathcal{H}_{dom}$, we set

$$
\bar{I}[\sigma](h) := \frac{1}{2}W^2_2(\sigma, e(s)\chi_{D_h}) + \int_{D_h} \left( \frac{\Omega^2 r_0^2}{2(1 - 2sr_0^2)} - \frac{s^2 + z^2}{2} \right) e(s)dsdz
$$

(3.5)

$$
+ \int_{\mathbb{R}^2} \left( \frac{\gamma}{2r_0^2} - \Omega \sqrt{\gamma} - \frac{\gamma^2 + Z^2}{2} \right) \sigma(\gamma, dz)
$$

with

$$
D_h = \{(s, z) : 0 \leq s \leq h(z), z \in [0, H]\} \quad \text{and} \quad e(s) = r_0^2/(1 - 2sr_0^2)^2 \quad \text{for} \quad 0 \leq 2r_0^2s < 1.
$$

As $\sigma$ is compactly supported, we easily show that there exists $c_0 = c_0(H, R_0, r_o)$ such that

$$
c_0 \leq \bar{I}[\sigma](h) \quad \text{for any} \quad h \in \mathcal{H}_{dom}
$$

We rewrite the functional in (3.5) as

$$
\bar{I}[\sigma](h) = \inf_{\gamma \in \Gamma(\sigma, e(s)\chi_{D_h}(s, z))} I(h, \gamma)
$$

(3.6)

here,

$$
I(h, \gamma) := \int_{D_h \times \Delta} \left( -\langle p, q \rangle + \frac{\gamma}{2r_0^2} - \Omega \sqrt{\gamma} + \frac{r_0^2 \Omega^2}{2(1 - 2sr_0^2)} \right) \gamma(dp, dq)
$$

(3.7)

where

$$
p = (s, z) \quad \text{and} \quad q = (\gamma, Z)
$$

so that

$$
\inf_{h \in \mathcal{H}_{dom}} \bar{I}[\sigma](h) = \inf_{(h, \gamma) \in \mathcal{L}_\sigma} I(h, \gamma)
$$

(3.8)

where

$$
\mathcal{L}_\sigma = \{(h, \gamma) : h \in \mathcal{H}_0, \int_{D_h} e(s)dsdz = 1, \gamma \in \Gamma(e(s)\chi_{D_h}, \mathcal{L}^2, \sigma)\}
$$

(3.9)

$\mathcal{H}_0$ is the set of all Borel measurable functions $h : [0, H] \rightarrow [0, \frac{1}{2r_0^2}]$. To study the minimization problem in (3.8), we will introduce what will turn out to be its dual formulation by setting:

$$
J[\sigma](\Psi, P) = \int_{\mathbb{R}^2} \left( \frac{\gamma}{2r_0^2} - \Omega \sqrt{\gamma} - \Psi \right) \sigma(dp, dq) + j(P); \quad j(P) = \inf_{h \in \mathcal{H}_0} \int_0^H \Pi_P(h(z), z)dz.
$$

(3.10)

$J[\sigma]$ is defined on

$$
U := \{(\Psi, P) \in C(\mathbb{R}_+^2) \times C(\Delta_{r_0}) : P(p) + \Psi(q) \geq \langle p, q \rangle \text{ for all } (p, q) \in \Delta_{r_0} \times \mathbb{R}_+^2\}
$$

(3.11)

To $P : \Delta_{r_0} \rightarrow \mathbb{R}$ we have associated

$$
\Pi_P(\rho, z) = \int_0^\rho \left( \frac{\Omega^2 r_0^2}{2(1 - 2sr_0^2)} - P(s, z) \right) e(s)ds \quad \text{for} \quad 0 \leq 2r_0^2\rho < 1.
$$

(3.12)
We observe that if $P_1 \leq P_2$ then $\Pi_{P_1} \geq \Pi_{P_2}$ and also that if $P$ is a constant function that is equal to $C$ in (3.12) then

$$\Pi_C(\rho, z) := \Pi_P(\rho, z) = \frac{\Omega^2 r_0^6 (1 - \rho r_0^2) \rho}{2(1 - 2\rho r_0^2)^2} - \frac{Cr_0^4 \rho}{1 - 2\rho r_0^2}$$

The dual problem we will be looking at is the following:

$$\sup_{(\Psi, P) \in \mathcal{U}} J[\sigma](\Psi, P)$$

### 3.2. Existence of a minimizer for $\Pi_P(\cdot, z)$ and Twist condition.

Let’s denote by $\mathcal{U}_0$ the subset of $\mathcal{U}$ consisting of pairs $(\Psi, P)$ such that

$$P(p) = \sup_{q \in \Delta} (\langle p, q \rangle - \Psi(q)), \quad p \in \Delta_{r_0} \quad \text{and} \quad \Psi(q) = \sup_{p \in \Delta_{r_0}} (\langle p, q \rangle - P(p)) \quad q \in \mathbb{R}^2_+.$$

We note that if $P$ and $\Psi$ satisfy (3.15) then $P$ and $\Psi$ are convex as supremum of convex functions and

$$\partial P(p) \subset \Delta \quad \text{for all} \quad p \in \Delta_{r_0} \quad \text{and} \quad \partial \Psi(q) \subset \bar{\Delta}_{r_0} \quad \text{for all} \quad q \in \mathbb{R}^2_+$$

We denote the expression at the right hand side of the first equation in (3.15) by $\Psi^*$ and the one at the right hand side of the second equation in (3.15) by $P^*$.

We consider functions $P : \Delta_{r_0} \to \mathbb{R}$ Lipschitz such that

$$0 \leq \frac{\partial P}{\partial z}(s, z) \leq R_0 \quad \text{and} \quad 0 \leq \frac{\partial P}{\partial s}(s, z) \leq R_0$$

**Lemma 3.3.** Let $A \in \mathbb{R}_+$. Suppose $\hat{P} : \Delta_{r_0} \to \mathbb{R}$ such that $\hat{P} \leq \hat{P}(0, 0) + A$. Then there exists a constant $M_{\hat{P}}$ depending on $\hat{P}(0, 0)$ such that $2r_0^4 M_{\hat{P}} < 1$ and

$$\sup_{0 \leq z \leq H} \sup_{0 \leq r \rho < 1} \{ \rho \mid \Pi_{\hat{P}}(\rho, z) \leq 0 \} \leq M_{\hat{P}}.$$

**Proof:** We use $\hat{P} \leq \hat{P}(0, 0) + A$ to establish that for any $z \in [0, H]$ fixed,

$$0 \geq \Pi_{\hat{P}}(\rho, z) \geq \frac{\Omega^2 r_0^6 (1 - \rho r_0^2) \rho}{2(1 - 2\rho r_0^2)^2} - \frac{(\hat{P}(0, 0) + A)r_0^4 \rho}{1 - 2\rho r_0^2}$$

It is straightforward to check that

$$M_{\hat{P}} := \sup_{0 \leq r \rho < 1} \left\{ \frac{\Omega^2 r_0^6 (1 - \rho r_0^2) \rho}{2(1 - 2\rho r_0^2)} \leq \left[ \hat{P}(0, 0) + A \right] r_0^4 \rho \right\}$$

satisfies the above requirements.
Lemma 3.4. Assume \( P_n, P : \bar{\Delta}_{r_0} \to \mathbb{R} \) satisfy the hypotheses in Lemma 3.3 and are continuous.

(i) Given \( z \in [0, H] \), the set \( \text{Argmin} P(\cdot, z) \) consisting of \( h \) minimizing \( P(\cdot, z) \) over \( [0, 1/(2r_0^2)] \) is non empty. Moreover,

\[
\bigcup_{0 \leq z \leq H} \text{Argmin} P(\cdot, z) \subset [0, M_P]
\]

where \( M_P \) is as in lemma 3.5.

(ii) Suppose \( \{P_n\}_{n=1}^\infty \) converges uniformly to \( P \) on \( \bar{\Delta}_{r_0} \). Then

\[
2r_0^2 \sup_n M_{P_n} < 1.
\]

If in addition, \( \{z_n\}_{n=1}^\infty \subset [0, H] \) converges to \( z \) and we assume that \( h_n \in \text{Argmin} P_{n_i}(\cdot, z_n) \) and that \( \{h_n\}_{n=1}^\infty \) converges to \( h \) then

\[
\lim_{n \to \infty} P_{n_i}(h_n, z_n) = P(h, z) \quad \text{and} \quad h \in \text{Argmin} P(\cdot, z).
\]

In particular, for each \( z \in [0, H] \) the set \( \text{Argmin} P(\cdot, z) \) is a compact subset of \( \mathbb{R} \).

(iii) Assume in addition that \( P(\rho, \cdot) \) is Lipschitz and the first equation in (3.17) holds a.e on \( (0, H) \) for each \( \rho \) fixed. Let \( z_1, z_2 \in [0, H] \) be such that \( z_1 < z_2 \). If \( h_i \in \text{Argmin} P_{n_i}(\cdot, z_i) \) then \( h_1 \leq h_2 \).

Proof: (i) Let \( z \in [0, H] \). As \( P(0, z) = 0 \), in light of Lemma 3.3 minimizing \( P(\cdot, z) \) over \( [0, 1/(2r_0^2)] \) is equivalent to minimizing \( P(\cdot, z) \) over \( [0, M_P] \). We observe that \( P(\cdot, z) \) is continuous on \( [0, M_P] \). Hence, it admits a minimum there and \( \text{Argmin} P(\cdot, z) \subset [0, M_P] \). This establishes (3.19).

(ii) The convergence propriety of \( \{P_n\}_{n=1}^\infty \) ensures that \( \{P_n(0, 0)\}_{n=1}^\infty \) is bounded above by one of its terms say \( P_{n_0}(0, 0) \) or \( P(0, 0) \). The monotonicity result in Lemma 3.3 ensures that \( M_{P_n} \leq M_{P_{n_0}} < \frac{1}{2r_0^2} \) or \( M_{P_n} \leq M_P < \frac{1}{2r_0^2} \) for all \( n \geq 1 \). Thus, (3.20) holds.

Let \( \{z_n\}_{n=1}^\infty \subset [0, H] \) be a sequence converging to \( z \) and assume \( h_n \in \text{Argmin} P_{n_i}(\cdot, z_n) \) and is such that \( \{h_n\}_{n=1}^\infty \) converges to \( h \) and let \( \rho \in [0, 1/(2r_0^2)] \). We choose \( M \) such that \( M_{P_n}, \sup_n M_{P_n}, \rho \leq M < \frac{1}{2r_0^2} \) so that \( K := [0, M] \times [0, H] \) is a compact subset of \( \Delta_{r_0} \). We observe that \( \{P_{n_i}\}_{n=1}^\infty \) converges uniformly to \( P \) on \( K \). This combined with the fact that \( h_n \) minimizes \( P_{n_i}(\cdot, z_n) \), and \( P \) is continuous, yields

\[
P(h, z) = \lim_{n \to \infty} P_{n_i}(h_n, z_n) \leq \lim_{n \to \infty} P_{n_i}(\rho, z_n) = P(\rho, z).
\]

Since this holds for any \( \rho \in [0, 1/(2r_0^2)] \), we have that \( h \in \text{Argmin} P(\cdot, z) \).

(iii) For each \( z \in [0, H] \), \( P(\cdot, z) \) is differentiable on \( (0, 1/(2r_0^2)) \) and its derivative is the integrand of \( P \). As \( P(\rho, \cdot) \) is Lipschitz, \( \partial P/\partial \rho(\cdot, \cdot) \) is differentiable almost everywhere on \( (0, H) \) and

\[
\frac{\partial^2 P}{\partial \rho \partial z}(\rho, z) = -c(\rho)\frac{\partial P}{\partial z}(\rho, z) \leq 0.
\]

We have used the first equation in (3.17). This means that \( P \) satisfies the so-called twist condition. Let \( z_1 \in [0, H] \) and \( h_i \in \text{Argmin} P_i(\cdot, z_i) \) \( i = 1, 2 \). We use the minimality condition on \( h_1, h_2 \) and the fact that \( P(\rho, \cdot) \) is Lipschitz to obtain

\[
0 \leq \left( P_i(h_2, z_1) - P_i(h_1, z_1) \right) + \left( P_i(h_1, z_2) - P_i(h_2, z_2) \right) = -\int_{h_1}^{h_2} d\rho \int_{z_1}^{z_2} \frac{\partial^2 P}{\partial \rho \partial z}(\rho, z) dz.
\]
If $z_1 < z_2$, then we use (3.22) and (3.23) to get $h_1 \leq h_2$. \hfill \Box

**Remark 3.1.** Let $z \in [0, H]$ and $h \in \text{Argmin} \Pi_P(\cdot, z)$. If $h > 0$ then $\partial \Pi/\partial h(h, z) = 0$ that is,

(3.24) \quad P(h, z) = \frac{\Omega^2 r^2_0}{2(1 - 2hr^2_0)}.

Let $P : \Delta_{r_0} \to \mathbb{R}$ be such that $\text{Argmin} \Pi_P(\cdot, z)$ is compact for each $z \in [0, H]$. We define

$h^+(z) = \max_{h \in \text{Argmin} \Pi_P(\cdot, z)} h, \quad h^-(z) = \min_{h \in \text{Argmin} \Pi_P(\cdot, z)} h$

**Lemma 3.5.** Assume $P$ satisfies the hypotheses in Lemma 3.3 and the first equation in (3.17). Then, the following hold:

(i) $h^-$ is lower semi-continuous and $h^+$ is upper semi-continuous.

(ii) $h^-, h^+$ is monotone nondecreasing.

(iii) Let $z_1, z_2 \in [0, H]$ be such that $z_1 < z_2$. Then $h^+(z_1) \leq h^-(z_2)$.

(iv) $h^-$ is left continuous and $h^+$ is right continuous.

(v) Let $z \in [0, H)$. If $h^-$ is continuous at $z$ then $h^-(z) = h^+(z)$.

**Proof:** (i) is a consequence of the continuity property in Lemma 3.3(ii). (ii) and (iii) come from Lemma 3.3(iii). We use the fact that $h^-$ is monotone nondecreasing and lower semi-continuous to obtain that $h^-$ is left continuous. A similar argument gives that $h^+$ is right continuous. Let $z_0 \in [0, H)$. If $h^-$ is continuous at $z$ then $h^-(z) = h^+(z)$. Let $z \in [0, H)$ be such that $h^-(z_0) < h^+(z_0)$. We note that, as $h^-$ is monotone nondecreasing, it has a right limit. For $\delta > 0$ small enough, we use Part (iii) to obtain that $h^+(z_0) \leq h^-(z_0 + \delta)$ and so

$$h^-(z_0) < h^+(z_0) \leq \lim_{\delta \to 0^+} h^-(z_0 + \delta).$$

This implies that $h^-$ is discontinuous at $z_0$ which proved (v). \hfill \Box

**Corollary 3.6.** There exists a countable set $\mathcal{N} \subset [0, H]$ such that for every $z \notin \mathcal{N}$, $\text{Argmin} \Pi_P(\cdot, z)$ has a unique element.

**Remark 3.2.** If $P$ is Lipschitz and satisfies (3.17) then $P$ satisfies the hypotheses in Lemma 3.3. The compactness result in Lemma 3.3(ii) combined with the definition of $h^-$ ensure that $h^-$ is a minimizer in the second equation of (3.10). Note that by (3.19),

(3.25) \quad 0 \leq h^-(z) \leq M_P < \frac{1}{2r_0^2}.

for $z \in [0, H]$. Let $\{P_n\}_{n=1}^\infty \subset C(\Delta_{r_0})$ be a sequence of Lipschitz functions uniformly convergent on $\Delta_{r_0}$ and satisfying (3.17). By (3.26) and (3.27),

(3.26) \quad 0 \leq h^+_n(z) \leq \sup_n M_{P_n} < \frac{1}{2r_0^2}

for $z \in [0, H]$ and all $n \geq 1$. \hfill \Box
3.3. Existence of a minimizer for the functional $I$.

**Remark 3.3.** Let $(\Psi, P) \in \mathcal{U}_0$. We recall that $P$ and $\Psi$ are Lipschitz and $P$ satisfies (3.14). If in addition $P(0, 0) = 0$ then, in view of (3.10)

$$
(3.27) \quad |P(p)| \leq R_0 \left( \frac{1}{2r_0^2} + H \right) =: R_0H_0.
$$

We note that $0 \leq \langle p, q \rangle \leq R_0H_0$ for $q \in \Delta$ and $p \in \Delta_{r_0}$. This combined with the second equation in (3.15) and (3.27) yields that $\Psi$ is bounded on $\Delta$. More precisely

$$
-2R_0H_0 < -R_0H_0 \leq \Psi(q) \leq 2H_0R_0
$$

for $q \in \Delta$. As a consequence

$$
\left| \int_{\Delta} \left( \frac{\nabla \Psi}{2r_0^2} - \Omega \sqrt{\nabla Y} - \Psi \right) \sigma(dq) \right| \leq \int_{\Delta} \left( \frac{\nabla \Psi}{2r_0^2} + \Omega \sqrt{\nabla Y} \right) \sigma(dq) + 2H_0R_0 =: C(R_0) + 2H_0R_0 < \infty.
$$

**Lemma 3.7.** Let $C_0 \in \mathbb{R}$. There exists $C_1 \in \mathbb{R}$ satisfying the following: whenever $(P, \Psi) \in \mathcal{U}_0$ with $P(0, 0) = 0$, $\lambda \in \mathbb{R}$ and $\sigma \in \mathcal{P}(R_0)$ are such that $-\bar{C}_0 \leq J[\sigma](\Psi + \lambda, P - \lambda)$ then $|\lambda| \leq C_1$.

**Proof:** By (3.27) $-R_0H_0 < P(p)$ for $p \in \Delta_{r_0}$ so that

$$
\Pi\lambda - \lambda \leq \Pi - H_0, R_0 - \lambda
$$

Therefore, if $J[\sigma](\Psi + \lambda, P - \lambda) \geq -C_0$ then

$$
-C_0 \leq -\lambda + \int_{\Delta} \left( \frac{\nabla \Psi}{2r_0^2} - \Omega \sqrt{\nabla Y} - \Psi \right) \sigma(dq) + \int_0^H \Pi - H_0, R_0 - \lambda(h(z), z)dz
$$

for all $h \in \mathcal{H}_0$. Hence, using $C(R_0)$ as given in Remark 3.3 and setting $h$ to be a constant function $\bar{h}_0$ we obtain

$$
-C_0 \leq -\lambda + C(R_0) + 2H_0R_0 + \int_0^H \Pi - H_0, R_0 - \lambda(\bar{h}_0, z)dz
$$

We use (3.13) to get

$$
-\bar{C}_0 := -C_0 - C(R_0) - 2H_0R_0 \leq -\lambda + \frac{\Omega^2 r_0^4 (1 - \bar{h}_0^2) \bar{h}_0 H}{2(1 - 2\bar{h}_0^2)^2} - \frac{(-H_0R_0 - \lambda)H r_0^4}{2(1 - 2\bar{h}_0^2)^2}
$$

we rewrite this as

$$
\lambda \left( 1 - H \frac{r_0^4 \bar{h}_0}{1 - 2\bar{h}_0^2} \right) \leq \bar{C}_0 + H \frac{r_0^4 H_0 \bar{h}_0}{1 - 2\bar{h}_0^2} + H \frac{\Omega^2 r_0^4 (1 - \bar{h}_0^2) \bar{h}_0}{2(1 - 2\bar{h}_0^2)^2}
$$

set $\bar{h}_0 = 0$ in (3.28) then

$$
\lambda \leq \bar{C}_0
$$

when the constant value of $\bar{h}_0$ is chosen in $[0, \frac{1}{2r_0^2})$ (for instance close enough to $\frac{1}{2r_0^2}$) so that the factor of $\lambda$ in (3.28) is negative then there exists a constant $\bar{C}_1$ such that

$$
\lambda \geq \bar{C}_1
$$

We combine (3.29) and (3.30) to get the result. \qed
Lemma 3.8.
(i) Let $P \subset C(\bar{\Delta}_{r_0})$ be a Lipschitz function satisfying (3.14). Then, $h^-$ is the unique minimizer in (3.10) (up to a set of zero lebesgue measure).

(ii) Assume that $\{P_n\}_{n=1}^\infty \subset C(\bar{\Delta}_{r_0})$ is a sequence of Lipschitz functions satisfying (3.14) such that $\{P_n\}_{n=1}^\infty$ converges uniformly to $P$. Then

$$j(P_n) \text{ converges to } j(P).$$

Proof: The function $h^-$ is a minimizer in (3.10) as stated in remark 3.2. Corollary 3.8 ensures the uniqueness which proves (i). We note that as $\{P_n\}_{n=1}^\infty$ is uniformly Lipschitz and converges uniformly to $P$, we have that $P$ is Lipschitz. Let $h^*_n$ be the minimizer in the second equation of (3.10) when $P$ is replaced by $P_n$. By Helly's theorem there exists a subsequence of $\{h^*_n\}_{n=1}^\infty$ that we denote again by $\{h^*_n\}_{n=1}^\infty$ and $h$ monotone nondecreasing such that $\{h^*_n\}_{n=1}^\infty$ converges to $h$ pointwise. It is straightforward that $\{P_n\}_{n=1}^\infty$ converges uniformly to $P$ on compact subsets of $\Delta$. This, in view of (3.26), (3.21) and Part (i), yields $h^-=h$ a.e. In light of (3.20) again, we next use the fact that $\{P_n\}_{n=1}^\infty$ is uniformly bounded to obtain $\sup_{z \in [0,H]} \sup_n |P_n(h^*_n(z),z)| < \infty$. A simple Lebesgue dominated convergence theorem yields (ii). \hfill $\square$

3.4. A step towards the Proof of the main Theorem.

Proposition 3.4. Let $\sigma \in \mathcal{P}(\bar{\Omega})$.

(i) The set of maximizers $\mathcal{M}$ of $J[\sigma]$ over $\mathcal{U}$ is such that $\mathcal{M} \cap \mathcal{U}_0$ is non empty. $\mathcal{U}_0$ is defined by (3.20).

(ii) $I(\gamma, h) \geq J[\sigma](\Psi, P)$ for all $(\Psi, P) \in \mathcal{U}_0$ and all $(\gamma, h) \in \mathcal{L}_\sigma$. The equality holds if and only if $\text{id} \times \nabla P$ pushes forward $e(s)\chi_{D_{\bar{\Omega}}(s,z)} \mathcal{L}^2$ onto $\gamma$ and $h(z)$ minimizes $\Pi_P(z)$ for almost every $z \in [0,H]$. If in addition $\sigma$ is absolutely continuous with respect to Lebesgue $\mathcal{L}^2$, then the first condition for equality could be replaced by $\nabla \Psi \times \text{id}$ pushes forward $\sigma$ onto $\gamma$.

(iii) $I$ has a unique minimizer $(\bar{\gamma}_0, \bar{h}_0)$ over $\mathcal{L}_\sigma$. Moreover, if $(\Psi_0, P_0) \in \mathcal{U}_0$ maximizes $J[\sigma]$ on $\mathcal{U}$ then $J[\sigma](\Psi_0, P_0) = I(\bar{\gamma}_0, \bar{h}_0)$ and $\bar{h}_0$ is monotone non decreasing on $[0,H]$ satisfying (3.26) and

$$2(1 - 2r_0^2 \bar{h}_0(z))P(\bar{h}_0(z), z) = r_0^2 \Omega^2 \text{ on } \{\bar{h}_0 > 0\}.$$ 

If in addition $\sigma$ is absolutely continuous with respect to Lebesgue then $\nabla \Psi_0 \times \text{id}$ pushes $\sigma$ onto $\gamma_0$ and

$$\nabla \Psi_0 \circ \nabla P_0 = \text{id} e(s)\chi_{D_{\bar{\Omega}}(s,z)} \mathcal{L}^2 \text{ a.e } \nabla P_0 \circ \nabla \Psi_0 = \text{id} a.e \sigma$$

(iv) $J[\sigma]$ has a unique maximizer $(\bar{\Psi}_0, P_0)$ on $\mathcal{U}_0$ in the sense that if $J[\sigma](\bar{\Psi}_0, P_0) = J[\sigma](\bar{\Psi}_1, P_1)$ and $(\bar{\Psi}_1, P_1) \in \mathcal{U}_0$ then $P_1 = P_0$ on $D_{\bar{\Omega}}$ and $\Psi_1 = \Psi_0$ on $\Delta$.

Proof: 1. Set

$$\bar{c}_0 = \sup_{(p,q) \in \bar{\Delta} \times \bar{\Delta}_{r_0}} \langle p, q \rangle, \quad \bar{P}_0(p) = \bar{c}_0, \quad \bar{\Psi}_0(q) = 0$$

so that

$$(\bar{\Psi}_0, \bar{P}_0) \in \mathcal{U} \quad \text{and} \quad -C_0 := J[\sigma](\bar{\Psi}_0, \bar{P}_0) - 1 \text{ is finite}.$$ 

Let $\{(\bar{\Psi}_n, \bar{P}_n)\}_{n=1}^\infty \subset \mathcal{U}$ be a maximizing sequence for $J[\sigma]$ over $\mathcal{U}$. We note that whenever $(\bar{\Psi}, \bar{P}) \in \mathcal{U}$, by the double convexification trick (cfr. [15] Page 51), we have

$$(P^*, \bar{P}_{**}) \in \mathcal{U}_0 \text{ and } J[\sigma](\bar{\Psi}, \bar{P}) \leq J[\sigma](P^*, \bar{P}_{**}).$$
This shows, on the one hand, that if the set of maximizers $\mathcal{M}$ of $J[\sigma]$ over $\mathcal{U}$ is nonempty then so is $\mathcal{M} \cap \mathcal{U}_0$ and, on the other hand, that we may assume without loss of generality that $\{(\Psi_n, P_n)\}_{n=1}^{\infty}$ is contained in $\mathcal{U}_0$. We assume so and set

$$\Psi_n = \bar{\Psi}_n + \bar{P}_n(0,0), \quad \lambda_n = -\bar{P}_n(0,0), \quad P_n = \bar{P}_n - \bar{P}_n(0,0)$$

we easily check that $\{(\Psi_n, P_n)\}_{n=1}^{\infty} \subset \mathcal{U}_0$ and

$$\lim_{n \to \infty} J[\sigma](\Psi_n + \lambda_n, P_n - \lambda_n) = \lim_{n \to \infty} J[\sigma](\bar{\Psi}_n, P_n) = \sup_{\mathcal{U}} J[\sigma]$$

and so, for $n$ large enough

$$(3.33) \quad - C_0 \leq J[\sigma](\Psi_n + \lambda_n, P_n - \lambda_n).$$

Therefore, as $P_n(0,0) = 0$ by Lemma 3.7, we obtain that $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{R}$ is bounded. Hence, up to a subsequence, $\{\lambda_n\}_{n=1}^{\infty}$ converges to a real number $\lambda_*$. 

In view of (3.19) and (3.24), we have that the sequences $\{P_n\}_{n=1}^{\infty} \subset C(\tilde{\Delta})$ and $\{\Psi_n\}_{n=1}^{\infty} \subset C(\tilde{\Delta})$ are uniformly bounded and uniformly Lipschitz. We then use Ascoli- Arzelà to conclude that there exists a subsequence of $\{(\Psi_n, P_n)\}_{n=1}^{\infty}$ converging uniformly to some $(\Psi_*, P_*) \in C(\Delta) \times C(\Delta)$. In the sequel, we assume without loss of generality that

$$\{\lambda_n\}_{n=1}^{\infty} \text{ converges to } \lambda_* \quad \text{and } \quad \{(\Psi_n, P_n)\}_{n=1}^{\infty} \text{ converges uniformly to } (\Psi_*, P_*)$$

We set

$$P_0 := P_* - \lambda_*, \quad \Psi_0 := \Psi_* + \lambda_*$$

Therefore

$$\{(\bar{\Psi}_n, \bar{P}_n)\}_{n=1}^{\infty} \text{ converges uniformly to } (\Psi_0, P_0).$$

Note that $\{\bar{P}_n\}_{n=1}^{\infty}$ are Lipschitz and satisfies (3.17). We use the fact that $\{(\bar{\Psi}_n)\}_{n=1}^{\infty}$ converges uniformly to $\Psi_0$, $\sigma$ is a probability measure and Lemma 3.8 (ii) to obtain that

$$\{J[\sigma](\bar{\Psi}_n, \bar{P}_n)\}_{n=1}^{\infty} \text{ converges to } J[\sigma](\Psi_0, P_0).$$

This established that $(\Psi_0, P_0)$ is a maximizer of $J[\sigma]$ over $\mathcal{U}$.

2. Let $(\Psi, P) \in \mathcal{U}_0$ and $(\gamma, h) \in \mathcal{L}_\sigma$. Then $\Psi, P$ are Lipschitz and $P(p) + \Psi(q) \geq \langle p, q \rangle$. We note that

$$(3.34) \quad j(P) \leq \int_0^H \Pi_P(h(z), z)dz = \int_{D_h} \left( \frac{\Omega^2 r_0^2}{2(1 - 2r_0^2)} - P(s, z) \right) e(s)d\sigma dz$$

As $\gamma \in \Gamma \left( (e(s)\chi_{D_h}) \mathcal{L}^2, \sigma \right)$ and $P(p) + \Psi(q) \geq \langle p, q \rangle$ we use (3.34) to get

$$(3.35) \quad J[\sigma](\Psi, P) \leq \int_{D_h \times \mathbb{R}^2} \left( -\langle p, q \rangle + \frac{\gamma}{2r_0^2} \Omega \sqrt{\gamma} + \frac{r_0^2 \Omega^2}{2(1 - 2r_0^2)} \right) \gamma(dp, dq) = I(h, \gamma)$$

Note that equality holds in (3.35) if and only if equality holds in (3.34) and $P(p) + \Psi(q) = \langle p, q \rangle$ for $\gamma$ almost every $\langle p, q \rangle$. The first condition means that $h(z) \in \text{Argmin}_{\Pi_P}(\cdot, z)$ for almost every $z \in [0, H]$ by using Lemma 3.8 (i). As the first projection of $\gamma$ is absolutely continuous with respect to $\mathcal{L}^2$, the second condition means that $q = \nabla P(p)$ for $\gamma$ almost every $\langle p, q \rangle$ and so, $\gamma$ is concentrated on the graph of $\nabla P$. This implies that $\gamma$ is the push forward of $e(s)\chi_{D_h}(s, z)\mathcal{L}^2$ by $\text{id} \times \nabla P$. 


3. Assume that \((P_0, \Psi_0) \in \mathcal{U}_0\) is a maximizer of \(J[\sigma]\) over \(\mathcal{U}\). Let \(g \in C_c(\mathbb{R}^2)\). For any \(\delta \in (-1, 1)\), we set
\[
\Psi_\delta = \Psi_0 + \delta g \quad \text{and} \quad P_\delta = \Psi_\delta.
\]
We note that \(\{P_\delta\}_{-1 < \delta < 1} \subset C(\bar{\Delta})\). It can be shown that (cfr. \cite{9} \cite{10})
\[
\|P_\delta - P_0\|_\infty \leq |\delta|\|g\|_\infty \quad \text{and} \quad \lim_{\delta \to 0} \frac{P_\delta(p) - P_0(p)}{\delta} = -g(\nabla P_0(p))
\]
for all \(p \in \text{dom}(\nabla P_0)\). As \(P_0\) is Lipschitz, the second equation in (3.36) holds almost everywhere with respect to \(L^2\).

Fix \(z \in [0, H]\). Let \(\{\delta_n\}_{n=1}^\infty \subset (-1, 1)\) converging to 0. We note that the first equation in (3.36) ensures that \(\{P_{\delta_n}\}_{n=1}^\infty\) uniformly converges to \(P_0\). For each \(n \geq 1\), as \(\text{Argmin} \Pi_{P_{\delta_n}}(\cdot, z)\) is compact (cfr Lemma 3.4), let \(h_{\delta_n}(z)\) denote its smallest element. For the same reasons, let \(h_0(z)\) denote the smallest element of \(\text{Argmin} \Pi_{P_0}(\cdot, z)\). Then \(\{h_{\delta_n}(z)\}_{n=1}^\infty\) is bounded in light of (3.20) and so, without loss of generality we assume that \(\{h_{\delta_n}(z)\}_{n=1}^\infty\) converges. If \(z\) is a continuity point for \(h_0\) then Lemma 3.5 (v) ensures that \(h_0(z)\) is the unique element of \(\text{Argmin} \Pi_{P_0}(\cdot, z)\) and so by using Lemma 3.4 (ii) we obtain
\[
\lim_{n \to \infty} h_{\delta_n}(z) = h_0(z).
\]
As \(\{\delta_n\}_{n=1}^\infty\) is arbitrary, we obtain denoting by \(h_\delta(z)\) the smallest element of \(\text{Argmin} \Pi_{P_\delta}(\cdot, z)\)
\[
\lim_{\delta \to 0} h_\delta(z) = h_0(z).
\]
In light of Corollary 3.6 the equation (3.37) holds for almost every \(z \in [0, H]\).

Fix \(\delta \in (-1, 1)\). By definition of \(h_0(z)\) we have \(\Pi_{P_\delta}(h_0(z), z) \leq \Pi_{P_\delta}(h_\delta(z), z)\) and so
\[
\Pi_{P_\delta}(h_0(z), z) - \Pi_{P_\delta}(h_\delta(z), z) \leq \Pi_{P_\delta}(h_\delta(z), z) - \Pi_{P_\delta}(h_\delta(z), z) = \int_0^{h_\delta(z)} (P_\delta(s, z) - P_0(s, z))e(s)ds.
\]
Similarly, we establish that
\[
\Pi_{P_\delta}(h_\delta(z), z) - \Pi_{P_\delta}(h_0(z), z) \leq \Pi_{P_\delta}(h_0(z), z) - \Pi_{P_\delta}(h_0(z), z) = -\int_0^{h_0(z)} (P_\delta(s, z) - P_0(s, z))e(s)ds.
\]
Let again \(\{\delta_n\}_{n=1}^\infty \subset (-1, 1)\) converging to 0. We use the definition of \(j\) in (3.11), with (3.38), (3.39) to obtain that
\[
\int_0^H \int_0^{h_0(z)} (P_{\delta_n}(s, z) - P_0(s, z))e(s)dsdz \leq j(P_0) - j(P_{\delta_n}) \leq \int_0^H \int_0^{h_{\delta_n}(z)} (P_{\delta_n}(s, z) - P_0(s, z))e(s)dsdz.
\]
In view of (3.36), \(\{P_{\delta_n}\}_{n=1}^\infty\) converges uniformly to \(P_0\), so (3.20) holds for \(\{P_{\delta_n}\}_{n=1}^\infty\). We then choose \(M\) such that
\[
2r_0^2 \sup_n M_{P_{\delta_n}} \leq 2r_0^2 M < 1
\]
We note that \(\partial_u P_{\delta_n} \subset \bar{\Delta}\) and so \(\{P_{\delta_n}\}_{n=1}^\infty\) is uniformly Lipschitz and satisfies (3.17). By (3.20),
\[
0 \leq h_{\delta_n}(z) \leq M
\]
for $z \in [0, H]$ a.e and $n \geq 1$. This ensures that the integrals in (3.40) are finite for $n \geq 1$. We rewrite (3.40) as

$$0 \leq j(P_0) - j(P_{\delta_n}) - \int_0^H dz \int_0^{h_{\delta_n}(z)} (P_{\delta_n}(s, z) - P_0(s, z))e(s)ds$$

(3.42)

$$\leq \int_0^H \int_0^{h_{\delta_n}(z)} (P_{\delta_n}(s, z) - P_0(s, z))e(s)dsdz$$

We use the fact $e$ is bounded on $[0, M]$, the first equation in (3.36) and apply the Lebesgue dominated convergence, using (3.37) and (3.41) to obtain that

$$\limsup_{n \to \infty} \int_0^H dz \int_0^{h_{\delta_n}(z)} \frac{(P_{\delta_n}(s, z) - P_0(s, z))e(s)}{\delta_n}ds \leq \max \|g\|_{\infty} \limsup_{n \to \infty} \int_0^H |h_{\delta_n}(z) - h_0(z)|dz = 0$$

By the Lebesgue dominated theorem, (3.36) implies that

$$\lim_{n \to \infty} \int_0^H dz \int_0^{h_{\delta_n}(z)} \frac{(P_{\delta_n}(s, z) - P_0(s, z))e(s)}{\delta_n}ds = - \int_0^H g(\nabla P_0(p))e(s)dsdz$$

We note that

$$\frac{J[\sigma](P_{\delta_n}, \Psi_{\delta_n}) - J[\sigma](P_0, \Psi_0)}{\delta_n} = - \int_\Delta gd\sigma + \frac{j(P_{\delta_n}) - j(P_0)}{\delta_n}$$

(3.45)

We use the fact that $\{\delta_n\}_{n=1}^\infty$ is an arbitrary sequence that converges to 0 and combine (3.42)-(3.45) to get

$$\lim_{\delta \to 0} \frac{J[\sigma](P_0, \Psi_0)}{\delta} = - \int_\Delta gd\sigma + \int_0^H dz \int_0^{h_0(z)} g(\nabla P_0(p))e(s)dsdz$$

Since $(P_0, \Psi_0)$ maximizes $J[\sigma]$ over $U$ and $(P_0, \Psi_0) \in U$, (3.46) implies that

$$\int_\Delta gd\sigma = \int_0^H dz \int_0^{h_0(z)} g(\nabla P_0(p))e(s)dp$$

(3.47) holds for any $g \in C_c(\mathbb{R}^2)$ which means that $\nabla P_0$ pushes $e(s)\chi_{D_{h_0}}(s, z)\mathcal{L}^2$ forward to $\sigma$.

4. We recall that $(P_0, \Psi_0) \in U_0$ is a maximizer of $J[\sigma]$ over $U$, that for $z \in [0, H]$, $h_0(z)$ is the smallest element of $\text{Argmin}P_{\delta_n}(\cdot, z)$ and we set

$$\gamma_0 := (\text{id} \times \nabla P_0) e(s)\chi_{D_{h_0}}(s, z)\mathcal{L}^2$$

Then, by part (ii) of the theorem we have $I(h_0, \gamma_0) = J[\sigma](P_0, \Psi_0)$ which ensures that $(h_0, \gamma_0)$ is a minimizer in (3.8).

Let $(\bar{h}, \bar{\gamma})$ be another minimizer in (3.8). Then $I(\bar{h}, \bar{\gamma}) = I(h_0, \gamma_0)$ and so $I(\bar{h}, \bar{\gamma}) = J[\sigma](P_0, \Psi_0)$. Again by part (ii) $\bar{\gamma}$ is the push forward of $e(s)\chi_{D_{h_0}}\mathcal{L}^2$ by $(\text{id} \times \nabla P_0)$ and $\bar{h}(z) \in \text{Argmin}P_{\delta_n}(\cdot, z)$ for a.e $z \in [0, H]$.

We use Corollary 3.6 to obtain that $\bar{h}(z) = h_0(z)$ a.e. These prove that the minimizer in (3.8) is unique. By Remark 3.11 equation (3.31) holds. In light of lemma 3.15(i), $h_0$ is monotone and satisfies (3.25). The equation (3.32) is well known (see [12]). 5. Assume $(P_1, \Psi_1)$ is another maximizer of $J[\sigma]$ in $U_0$. Then

$$\gamma_0 = (\text{id} \times \nabla P_1) e(s)\chi_{D_{h_0}}(s, z)\mathcal{L}^2 = (\text{id} \times \nabla P_1) e(s)\chi_{D_{h_0}}(s, z)\mathcal{L}^2$$

This implies that $\nabla P_1 = \nabla P_0 e(s)\chi_{D_{h_0}}(s, z)\mathcal{L}^2$ a.e and so the equality holds $\mathcal{L}^2$-a.e on $D_{h_0}$ as $e > 0$. As, $D_{h_0}$ is connected and $P_0$ and $P_1$ are Lipschitz continuous satisfying (3.31), we conclude that $P_1 = P_0$ on $D_{h_0}$ and without loss of generality we take $P_1 = P_0$ on $\Delta_{r_0}$. Consequently, $\Psi_1 = \Psi_2$ on $\Delta$. 

\[\square\]
3.5. Regularity property of the domain $D_h$. In this section we consider the functions $P$ Lipschitz that satisfy

$$\frac{\partial P}{\partial s}(s, z) \leq R_0 \quad \text{and} \quad \frac{1}{R_0} \leq \frac{\partial P}{\partial z}(s, z) \leq R_0$$

As a consequence, $\text{Argmin}_{\Pi_P} (\cdot, z)$ is compact. We recall that for such $P$,

$$h^+(z) = \max_{h \in \text{Argmin}_{\Pi_P} (\cdot, z)} h, \quad h^-(z) = \min_{h \in \text{Argmin}_{\Pi_P} (\cdot, z)} h$$

**Lemma 3.9.** Assume $P$ is Lipschitz and satisfies (3.48). The set $Z = \{z \in [0, H] : 0 \in \text{Argmin}_{\Pi_P} (\cdot, z)\}$ when non empty, is a closed interval of the form $[0, z^*]$. In the case $Z$ is empty we set $z^* = 0$.

**Proof:** Assume $Z$ is non empty and set $z^*$ to be its supremum. By definition of $z^*$, $Z \subset [0, z^*]$. Conversely, let $\{z_n\}_{n=1}^\infty$ be a sequence in $Z$ such that $\{z_n\}_{n=1}^\infty$ converges to $z^*$. Then we use Lemma 3.4 (ii) to obtain that $0 \in \text{Argmin}_{\Pi_P} (\cdot, z^*)$, that is $z^* \in Z$. Lemma 3.4 (iii) ensures that $[0, z^*) \subset Z$. □

**Lemma 3.10.** Assume $P$ is Lipschitz and satisfies (3.48) and let $z^*$ be as in Lemma 3.9.

(i) There exists $c_0 > 0$ such that if $z^* \leq z_1 \leq z_2 \leq H$ and $h_i \in \text{Argmin}_{\Pi_P} (\cdot, z_i)$ $i = 1, 2$, then

$$z_2 - z_1 \leq c_0 (h_2 - h_1)$$

(ii) For any $z_1, z_2 \in [z^*, H]$, $\text{Argmin}_{\Pi_P} (\cdot, z_1) \cap \text{Argmin}_{\Pi_P} (\cdot, z_2) = \emptyset$ if $z_1 \neq z_2$

(iii) $h^-$, $h^+$ are strictly increasing on $[z^*, H]$.

**Proof:** Let $m(s) = \frac{\Omega^2 s^2}{2(1 - 2\pi c_0 s)}$. Note that $m$ is Lipschitz continuous on $[0, M_P]$. Here, $M_P$ is defined as in Lemma 3.3.

Set

$$\alpha(s, z) = m(s) - P(s, z)$$

As $P$ satisfies the first equation in (3.48), we have

$$-\text{Lip}(m) - R_0 \leq \partial_s \alpha(s, z) \leq \text{Lip}(m)$$

Let $z_1, z_2 \in (z^*, H)$ such that $z_1 < z_2$ and $h_i \in \text{Argmin}_{\Pi_P} (\cdot, z_i)$ $i = 1, 2$. Remark 3.1 ensures that $\alpha(h_2, z_2) = \alpha(h_1, z_1) = 0$ and so

$$\alpha(h_2, z_1) - \alpha(h_2, z_2) = \alpha(h_2, z_1) - \alpha(h_1, z_1).$$

We exploit the second equation in (3.48) to obtain that

$$\alpha(h_2, z_1) - \alpha(h_2, z_2) = \int_{z_1}^{z_2} \partial_z \alpha(h_2, z)dz = \int_{z_1}^{z_2} \partial_z P(h_2, z)dz \geq \frac{1}{R_0} (z_2 - z_1).$$

The second inequality in (3.51) leads to

$$\alpha(h_2, z_1) - \alpha(h_1, z_1) = \int_{h_1}^{h_2} \partial_s \alpha(s, z_1)ds \leq \text{Lip}(m)(h_2 - h_1).$$

We combine (3.52), (3.51) to conclude that

$$(z_2 - z_1) \leq R_0 \text{Lip}(m)(h_2 - h_1) = c_0(h_2 - h_1).$$
for all \( z^* < z_1 \leq z_2 \leq H \). Note that if \( Z = \emptyset \), the argument above is still valid when \( z_1 = z^* \). In the sequel, we assume that \( Z \neq \emptyset \). To obtain the inequality (3.50) when \( z_1 = z^* \), we consider a sequence \( \{\tilde{z}^n\}_{n=1}^\infty \) in \((z^*, H]\) such that \( \tilde{z}^n > z^* \) and \( \{\tilde{z}^n\}_{n=1}^\infty \) converges to \( z^* \). Let \( h^n \in \text{Argmin}\Pi_P(\cdot, \tilde{z}^n) \subset [0, M_P] \) so that (3.50) holds for \((z_1, h_1)\) and \((\tilde{z}^n, h^n)\). Since \( \text{Argmin}\Pi_P(\cdot, \tilde{z}^n) \subset [0, M_P] \), we assume without generality that sequence \( \{h^n\}_{n=1}^\infty \) converges to \( z^* \), Lemma 3.4 (ii) ensures that \( \{h^n\}_{n=1}^\infty \) converges to an element of \( \text{Argmin}\Pi_P(\cdot, z^*) \). We let \( n \) go to \( \infty \) in \( z_2 - \tilde{z}^n \leq c_0(h_2 - h^n) \) to obtain the desired result. (ii) and (iii) follow directly from (3.50).

Let \( P \) lipschitz and satisfying (3.48) and \( h^- \) be as in (3.49), we define

\[
(3.55) \quad a(s) := \inf A(s) \quad \text{with} \quad A(s) := \{ z \in [z^*, H] : h^-(z) \geq s \}
\]

for \( s \in [0, h^-(H)] \).

**Remark 3.5.**

1. Let \( 0 \leq s_1 \leq s_2 \leq h^-(H) \). If \( h^-(z) \geq s_2 \) then \( h^-(z) \geq s_1 \) and so \( A(s_2) \subset A(s_1) \).

2. Let \( \varepsilon > 0 \) small enough. By Lemma 3.4 (iii), \( h^-(z^* + \varepsilon) \geq h^+(z^*) \) and so \( z^* + \varepsilon \in A(h^+(z^*)) \). As \( z^* \)

is a lower bound for \( A(h^+(z^*)) \), we conclude that \( z^* \leq \inf A(h^+(z^*)) \leq z^* + \varepsilon \). By the arbitrariness of \( \varepsilon \) we obtain \( z^* = a(h^+(z^*)) \).

3. Let \( h^-(z^*) \leq s \leq h^+(z^*) \). By part (1) of this remark, \( A(h^-(z^*)) \subset A(s) \subset A(h^+(z^*)) \) and so \( a(h^-(z^*)) \leq a(s) \leq a(h^+(z^*)) \). We easily checked that \( a(h^-(z^*)) = z^* \). In view of part (2) of this remark, we obtain that \( a(h^-(z^*)) = a(s) = a(h^+(z^*)) = z^* \).

**Lemma 3.11.** Assume \( P \) satisfies (3.48). Let \( z^* \) be as in Lemma 3.4 (iv) such that \( z^* < H \).

(i) \( a \) is non decreasing on \( e \subset [0, h^-(H)] \).

(ii) If \( s \in (h^+(z^*), h^-(H)) \) then \( a(s) \) is an interior point of \([z^*, H]) \).

(iii) If \( s \in [h^-(z^*), h^-(H)) \) then \( s \in [h^-(A(s)), h^+(A(s))] \). Moreover, if \( s \in [h^-(z), h^+(z)] \) for some \( z \in [z^*, H] \) then \( a(s) = z \).

**Proof:** (i) is immediate from remark 3.5 (1).

Let \( s \in (h^+(z^*), h^-(H)) \). As \( s < h^-(H) \), we have that \( H \in A(s) \). We next choose \( z \in (a(s), H) \). The characterization of the infimum in (3.55) ensures that there exists \( \hat{z} \in A(s) \) such that \( a(s) \leq \hat{z} < z \). \( \hat{z} \in A(s) \) implies that \( h^-(\hat{z}) \geq s \) and as \( h^- \) increasing, \( h^-(\hat{z}) \leq h^-(z) \). We conclude that \( h^-(z) \geq s \) and so \( z \in A(s) \). Hence \( (a(s), H) \subset A(s) \).

Let \( \{a_n\}_{n=1}^\infty \) be a sequence in \([z^*, H]) \) such that \( \{a_n\}_{n=1}^\infty \) converges to \( z^* \). We use the right continuity of \( h^+ \) (cf Lemma 3.3 (iv)) to obtain that \( \{h^+(a_n)\}_{n=1}^\infty \) converges to \( h^+(z^*) \). As \( s > h^+(z^*) \) we obtain

\[
(3.56) \quad s > h^+(a_n) > h^+(z^*)
\]

for \( n \) big enough. We next choose \( a_n \) in (3.55) to be points of continuity of \( h^- \) so that \( h^+(a_n) = h^-(a_n)(\text{cfr Lemma 3.3 (v)}) \). Therefore (3.56) becomes

\[
(3.57) \quad s > h^-(a_n) > h^+(z^*)
\]

for \( n \geq n_0 \) for some \( n_0 \in \mathbb{N} \). In light of the definition of \( A(s) \), the first inequality in (3.57) implies that \( a_n \in (z^*, H) \setminus A(s) \) and so in view of (3.55), \( a_n \leq a(s) \) for all \( n \geq n_0 \). Since \( \{a_n\}_{n=1}^\infty \) converges to \( z^* \), there exists \( p_0 > n_0 \) such that \( a_{p_0} < a(s) \). The second inequality in (3.57) implies that \( h^-(a_{p_0}) > h^+(z^*) \). This, combined with \( h^+(z^*) > h^-(z^*) \), gives \( h^-(a_{p_0}) > h^-(z^*) \). By Lemma 3.3 \( h^- \) is strictly increasing on \([z^*, H]) \) and so \( a_{p_0} > z^* \). We conclude that

\[
(3.58) \quad z^* < a(s)
\]
Set $b_n = H - \frac{1}{n}$. By the left continuity of $h^-(cfr\ Lemma\ 3.5\ iv), \{h^-(b_n)\}^\infty_{n=1}$ converges to $h^-(H)$. This, with the fact that $s < h^-(H)$ yields

$$s < h^-(b_n) \leq h^-(H)$$

for $n$ big enough. For such $n$, $b_n \in A(s)$ so that $a(s) \leq b_n$. This, combined with $b_n < H$, yields

$$a(s) < H$$

From (3.58) and (3.60) we conclude that $a(s) \in (z^*, H)$ which proves (ii). Thus, there exists a sequence $\{z_n\}_n$ in $(z^*, H)$ such that $a(s) < z_n$ and $\{z_n\}_n$ converges to $a(s)$. As $(a(s), H) \subset A(s)$, we have that $z_n \in A(s)$ and so $h^-(z_n) \geq s$. Without loss of generality take $\{z_n\}_n$ to be points of continuity of $h^-$ so that $h^-(z_n) = h^+(z_n)$. Therefore, as $h^+$ is right continuous, $h^+(a(s)) = \lim_{n \to \infty} h^+(z_n) = \lim_{n \to \infty} h^-(z_n) \geq s$. On the other hand, let $\{\tilde{z}_n\}_n$ be a sequence in $(z^*, H)$ such that $\{\tilde{z}_n\}_n$ converges to $a(s)$ and $\tilde{z}_n < a(s)$ so that $\tilde{z}_n \notin A(s)$. Then, necessarily $h^-(\tilde{z}_n) < s$. Hence $h^-(a(s)) = \lim_{n \to \infty} h^-(\tilde{z}_n) \leq s$ by using the left continuity of $h^-$. We conclude that

$$s \in [h^-(a(s)), h^+(a(s))]$$

Note that $[h^-(a(s)), h^+(a(s))]$ is an element of the family $\{[h^-(z), h^+(z)]\}_{z^* < z < H}$ in which elements are disjoint from each other thanks to Lemma 3.10(ii). If $s \in [h^-(z_0), h^+(z_0)]$ for some $z_0 \in (z^*, H)$ then necessarily $h^-(z_0) = h^-(a(s))$ and so $z_0 = a(s)$ in light of the fact that $h^-$ is strictly increasing on $[z^*, H]$. Note that $a(s) = z^*$ for any $s \in [h^-(z^*), h^+(z^*)]$ by Remark 3.5(3). This concludes the proof of (iii). □

**Corollary 3.12.** Assume the hypotheses in Lemma 3.11 hold. The function $a$ is Lipschitz continuous. **Proof:** We first note that as $a$ is non decreasing, we only need to show that

$$a(s_2) - a(s_1) \leq c_0(s_2 - s_1)$$

for all $s_2 \geq s_1$ in $[0, h^-(H)]$ and some constant $c_0$.

(a) Assume $h^+(z^*) < s_1 < h^-(H)$. Lemma 3.11(iii) ensures that $s_1 \leq h^+(a(s_1))$ so that $h^+(z^*) < h^+(a(s_1))$. As $h^+$ is strictly increasing on $[z^*, H]$ (see Lemma 3.10), we obtain that $z^* < a(s_1)$. Let $s_2 \in [0, h^-(H)]$ such that $s_1 < s_2$. As $a$ is non decreasing, $a(s_1) \leq a(s_2)$. Thus, $z^* < a(s_1) \leq a(s_2)$. If $a(s_1) = a(s_2)$ then (3.62) holds. In the sequel, we assume $z^* < a(s_1) < a(s_2)$. Choose $z^n > a(s_1)$ such that $\{z^n\}_n$ converges to $a(s_1)$ and $z^n$ are points of continuity of $h^-$, that is, $h^-(z^n) = h^-(\tilde{z}_n)$. We use the fact that $h^+$ is non decreasing to obtain

$$h^+(a(s_1)) \leq h^+(z^n) = h^-(z^n).$$

This, with the fact that $s_1 \leq h^+(a(s_1))$ implies that $s_1 \leq h^-(z^n)$ which we use along with (3.59) and the fact that $h^-(a(s_2)) \leq s_2$ (see Lemma 3.11(iii)) to get

$$a(s_2) - z^n \leq c_0(h^-(a(s_2)) - h^-(z^n)) \leq c_0(s_2 - s_1)$$

By letting $n \to \infty$ we obtain (3.62) for $h^+(z^*) < s_1 < s_2 \leq h^-(H)$.

By Lemma 3.11(iii), $a(s) = z^*$ for all $s \in [h^-(z^*), h^+(z^*)]$. To show that $a$ is Lipschitz continuous on $[h^-(z^*), h^+(z^*)]$, it suffices to show that $a$ is continuous at $h^+(z^*)$ and more precisely right continuous at $h^+(z^*)$.

Let $h^+(z^*) \leq s_n \leq h^-(H)$ such that $\{s_n\}_n$ converges to $h^+(z^*)$. By Lemma 3.11(iii), $s_n \in [h^-(a(s_n)), h^+(a(s_n))]$ so that $h^-(a(s_n))$ converges to $h^+(z^*)$. We use (3.59) to obtain that

$$0 \leq a(s_n) - z^n \leq c_0(h^-(a(s_n)) - h^+(z^n))$$

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As $h^{-}(a(s_{n}))$ converges to $h^{+}(z^{*})$, (3.62) implies that $a(s_{n})$ converges to $z^{*} = a(h^{+}(z^{*}))$. We conclude that $a$ is continuous at $h^{+}(z^{*})$ and so Lipschitz on $[h^{-}(z^{*}), h^{-}(H)]$. In the case where $h^{-}(z^{*}) > 0$, we have $z^{*} = 0$ by definition of $z^{*}$. But since $0 = z^{*} \leq a(s) \leq a(h^{-}(z^{*})) = z^{*} = 0$ for any $s \in [0, h^{-}(z^{*})]$, we conclude that (3.62) holds on $[0, h^{-}(H)]$. □

Set

$$D_{h^{-}} = \{(s, z) : z^{*} \leq z \leq H, 0 \leq s \leq h^{-}(z)\}$$

and

$$Q = \{(s, z) : z^{*} < z < H, 0 < s < h^{-}(H), \ z^{*} < a(s) < z\}$$

**Lemma 3.13.** Assume the hypotheses in Lemma 3.11 hold. Then $Q \subset D_{h^{-}} \subset Q$ and $Q$ is open. **Proof:** As $a$ is continuous, $Q$ is open. $Q \subset D_{h^{-}} \subset Q$ is straightforward. □

We observe that the boundary of $Q$ is the union of the following sets:

$$Q_{1} = \{(s, z) : 0 \leq s \leq h^{-}(H), \ z = a(s)\} \quad Q_{2} = \{(s, z) : z = H, \ 0 \leq s \leq h^{-}(H)\}$$

$$Q_{3} = \{(s, z) : s = 0 \quad z^{*} \leq z \leq H\} \quad Q_{4} = \{(s, z) : z = z^{*}, \ 0 \leq s \leq h^{-}(z^{*})\}$$

**Proposition 3.6.** Assume the hypotheses in Lemma 3.11 hold. The boundary of the domain $D_{h^{-}}$ is the union of the graphs of Lipschitz continuous functions. **Proof:** Lemma 3.13 ensures that $Q$ and $D_{h^{-}}$ have the same boundary and clearly,

$$D_{h^{-}} = D_{h^{-}} \cup \{(s, z) : s = 0, \ 0 \leq z \leq z^{*}\}$$

The result follows immediately. □

**3.6. Existence and uniqueness of a solution in the Monge-Ampere equation.** Here we prove the main theorem of the section. **Proof of Theorem 3.2.** Proposition 3.4 (iii) and (iv) show that (3.8) has a unique minimizer $\tilde{h}$ and (3.11) has a unique maximizer $(\Psi, P) \in \mathbb{U}_{0}$ so that (3.22) has a solution. This variational solution is weak if $\sigma \ll \mathcal{L}^{2}$ and weak dual if $\sigma \ll \mathcal{L}^{2}$. Proposition 3.4 (iii) and then (ii) guarantee that $I(\tilde{\gamma}, \tilde{h}) = J[\sigma](\Psi, P)$ where $\tilde{\gamma}$ is the push forward of $e(s)\chi_{D_{h}}$ by $\text{id} \times \nabla P$. In view of Proposition 3.4 (ii), we can assume without loss of generality that $\tilde{h}(z)$ is the smallest value of $\text{Argmin}\Pi P(\cdot, z)$ for all $z \in [0, H]$. As $e(s)\chi_{D_{h}}$ is a probability measure and $\tilde{h}$ monotone non decreasing, $\{\tilde{h} > 0\}$ is of positive Lebesgue measure so that $\tilde{z} < H$ ($z^{*}$ is as defined in Lemma 3.10). Note that if $\text{spt}(\sigma) \subset [\frac{1}{R_{0}}, R_{0}] \times [0, R_{0}]$ then by (3.10), $P$ satisfies (3.8). In this case, we use Lemma 3.6 to conclude that $\partial D_{h}$ is the finite union of graphs of Lipschitz continuous functions. □

**4. Some stability results.** Theorem 3.2 generates two operators $\mathcal{H}$, $\hat{\mathcal{H}}$ defined in the following way: To any $\sigma \in \mathcal{P}[R_{0}]$ the operator $\mathcal{H}$ associates $h$, the minimizer in (3.8) and $\hat{\mathcal{H}}$ associates the convex functions $(P, \Psi) \in \mathcal{U}_{0}$, the maximizer in (3.11).

**Remark 4.1.** Let $\sigma \in \mathcal{P}[R_{0}]$. If $h = \mathcal{H}(\sigma)$, $(P, \Psi) = \hat{\mathcal{H}}(\sigma)$ then Proposition 3.4 (iii) ensures that (3.20) holds for $h$ a.e. If in addition we set $\gamma = (\nabla P \times \text{id})\#e(s)\chi_{D_{h}}$ then Proposition 3.4 and (3.8) yield that

$$I[\sigma](h) = I(h, \gamma) = J[\sigma](\Psi, P)$$
Proof: Let (4.1), we obtain a constant $c$ and let $\sigma_n$ be elements in $\mathcal{P}(R^d)$ such that $\{\sigma_n\}_{n=1}^\infty$ converges to $\sigma$ narrowly. If $(P_n, \Psi_n) = \bar{H}(\sigma_n)$ then $\{P_n\}_{n=1}^\infty$ and $\{\Psi_n\}_{n=1}^\infty$ are precompact respectively in $C(\bar{\Delta}_{\tau_0})$ and $C([0, R_0]^2)$. Moreover, if in addition (4.2), we obtain a constant $c_0 \in \mathbb{R}$ such that

$$c_0 \leq \bar{I}[\sigma_n](h_n) = J[\sigma_n](P_n, \Psi_n) = J[\sigma_n](P_n + \lambda_n, \Psi_n - \lambda_n)$$

In view of Lemma 3.7, $|\lambda_n| \leq C_0$ for some constant $C_0$. The result follows by Arzela-Ascoli.

The following lemma uses the Helly theorem, standard compactness results for optimal plans and uniqueness results in theorem 2.2. For more details, we refer the reader to [13].

Lemma 4.2. Let $\{\sigma_n\}_{n=1}^\infty$, $\sigma$ be elements in $\mathcal{P}(R^d)$ and let $h_n = \bar{H}(\sigma_n)$, $h = \bar{H}(\sigma)$, $\bar{H}(\sigma) = (P, \Psi)$ and $\bar{H}(\sigma_n) = (P_n, \Psi_n)$ for $n \geq 1$. Assume that $\{\sigma_n\}_{n=1}^\infty$ converges narrowly to $\sigma$. Then

(i) $\{h_n\}_{n=1}^\infty$ converges pointwise to $h$ and so $e(s) \chi_{D_{\tau_0}}$ converges narrowly to $e(s) \chi_{D_n}$. Moreover, if $\{P_n\}_{n=1}^\infty$ is uniformly convergent in $C(\Delta_{\tau_0})$ then there exists $M > 0$ such that

$$2r_0^2 M < 1 \quad \text{and} \quad 0 \leq h_n, h < M \quad \text{for } n \geq 1.$$  \hspace{1cm} (4.2)

(ii)

$$\nabla P_n \rightarrow \nabla P \quad L^2 - \text{a.e. in } \Delta_{\tau_0}.$$  \hspace{1cm} (4.3)

Moreover, if in addition $\{\sigma_n\}_{n=1}^\infty$, $\sigma$ are absolutely continuous with respect to Lebesgue then

$$\nabla \Psi_n \rightarrow \nabla \Psi \quad L^2 - \text{a.e. in } \mathbb{R}^2.$$  \hspace{1cm} (4.4)

5. Continuity equation for the forced Axisymmetric Model. Our goal in this section is to solve the continuity equation (1.10) corresponding to the forced axisymmetric flow discussed in the introduction, under two different sets of assumptions on the initial data. Throughout this section, we assume that $g, \theta_0$ are positive constants and $R_0 > 1$.

5.1. Existence of a solution for initial data that are absolutely continuous with respect to Lebesgue. In this section, $\Sigma$ denotes the set of all Borel probability measures $\sigma$ on $\mathbb{R}^2$ that are absolutely continuous with respect to Lebesgue and whose support is contained in $[0, R_0]^2$.

We consider the functions $F = F_1(r, z), S = S_1(r, z)$ such that $S, F \in C^1((0, \infty) \times \mathbb{R}^2)$ and satisfy the following conditions:

• (A1) $0 \leq F, \frac{\partial S}{\partial z} \leq M$ for some positive constant $M$.
• (A2) $\frac{\partial F}{\partial z} = \frac{\partial S}{\partial z}$ and $\frac{\partial F}{\partial z} = 0$.
• (A3) $\frac{\partial F}{\partial z}, \frac{\partial S}{\partial z} > 0$.

The next lemma is a well known result and can be found in [13].

Lemma 5.1. We consider a family $\sigma = \sigma L^2$, $\sigma^n = \sigma^n L^2 \in \mathcal{P}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ $n \geq 1$ that is equi-integrable and let $\{\psi^n\}_{n=1}^\infty : \mathbb{R}^2 \rightarrow \mathbb{R}$ be Borel measurable such that $|\psi^n| \leq M_0$ a.e where $M_0$ is a positive constant. Assume $\{\sigma^n\}_{n=1}^\infty$ converges narrowly to $\sigma$ and $\psi^n$ converges to $\psi$ a.e. Then

$$\psi^n \sigma^n \rightarrow \psi \sigma \quad \text{in the sense of distribution.}$$

Lemma 5.2. Let $a, \tau > 0$ and $L_a > 1$. Let $\sigma_a = \sigma_a L^2 \in \mathcal{P}(L_a)$ be a Borel probability measure that is absolutely continuous with respect to Lebesgue. Assume that $\Psi(a, \cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}$ is convex and that whenever
We assume that $\Psi(a, \cdot)$ exists, it has values in $[0, \frac{1}{2r_0^2}] \times [0, H]$. Set
\begin{equation}
(5.1) \quad \mathbf{v}_t(q) = \left( 2\sqrt{\Upsilon} F_t \left( \frac{r_0}{\sqrt{1-2r_0^2\tau^2(a,q)}}, \frac{\partial \psi}{\partial Z}(a,q) \right), \frac{2}{\sqrt{\Upsilon}} \tilde{S}_t \left( \frac{r_0}{\sqrt{1-2r_0^2\tau^2(a,q)}}, \frac{\partial \psi}{\partial Z}(a,q) \right) \right)
\end{equation}
with $q = (\Upsilon, Z)$. Assume that $(A_1), (A_2)$ and $(A_3)$ hold. Then, there exists a family of measures $\sigma_t = \delta_t \mathcal{L}^2 \in \mathcal{P}(\mathbb{R}^2)$ absolutely continuous with respect to Lebesgue such that
\[ \text{spt} \sigma_t \subset [0, L_{a+\tau}]^2 \quad \text{for} \ t \in [a, a+\tau] \text{ with } 1 < L_{a+\tau} := L_a(1 + M\tau)^2 \]
satisfying the following:
(a) $\int_{\mathbb{R}^2} \psi_t^0 dq \leq \int_{\mathbb{R}^2} \psi_0^0 dq$ for any $r \geq 1$ and $t \in [a, a+\tau]$.
(b) $t \mapsto \sigma_t \in AC_1 \left( \left( [a, a+\tau]; \mathcal{P}(\mathbb{R}^2) \right) \right)$ and
\begin{equation}
(5.2) \quad \begin{cases}
\frac{\partial \sigma}{\partial t} + \text{div}(\sigma \mathbf{v}_t) = 0, & (t, q) \in (a, a+\tau) \times \mathbb{R}^2 \\
\sigma|_{t=a} = \sigma_a
\end{cases}
\end{equation}
holds in the sense of distribution.
(c) $t \mapsto \sigma_t$ is Lipschitz continuous with respect to the $1-\text{Wasserstein distance}$ with Lipschitz constant less than $c_0 := M\sqrt{4L_a + 1}$ in $[a, a+\tau]$.

Remark 5.1. Since $\Psi(a, \cdot)$ is convex, $\nabla \Psi(a, \cdot)$ exists $\mathcal{L}^2$ a.e so that $\mathbf{v}_t$ is defined $\mathcal{L}^2$ a.e. As $\sigma_a$ is absolutely continuous with respect to $\mathcal{L}^2$, $\mathbf{v}_t$ is defined a.e $\sigma_a$.

Proof: We subdivide the proof into several steps.

Step 1 We assume that $\Psi(a, \cdot)$ is $C^2(\mathbb{R}^*_+)$.

We observe that the vector field $\mathbf{v}$ is smooth in $(0, \infty) \times (0, \infty)^2$ and define the associated flow by
\begin{equation}
(5.3) \quad \dot{\phi}_t = \mathbf{v}_t \circ \phi_t \quad \text{and} \quad \phi_a = \text{id} \quad \text{for} \ t \in (a, a+\tau).
\end{equation}

We note that $\sigma_t = \phi_t \# \sigma_a$ solves the continuity equation $\text{(5.2)}$. In view of $(A_2)$, A simple computation gives
\[ \text{div} \ [\mathbf{v}_t] = \frac{1}{\sqrt{\Upsilon}} F + \frac{r_0^3 \sqrt{\Upsilon}}{(1-2r_0^2\tau^2)^{\frac{3}{2}}} \frac{\partial^2 \psi}{\partial r^2} \frac{\partial F}{\partial r} + \frac{\partial^2 \psi}{\partial Z^2} \frac{\partial S}{\partial z} \]
Since $\Psi_a$ is convex, its second partial derivatives with respect to $\Upsilon$ and $Z$ are all non negative. This, combined with $(A_3)$ leads to
\[ \text{div} \ [\mathbf{v}_t] \geq 0 \]
which ensures that $t \mapsto \det(\nabla \phi_t)$ is non decreasing and so
\begin{equation}
(5.4) \quad \det(\nabla \phi_t) \geq \det(\nabla \phi_a) = 1
\end{equation}

2. We use $(A_1)$, the fact that $L_a > 1$ and the definition of the flow in $\text{(5.3)}$ to establish a bound on the range of $\phi_t$ for $t$ fixed:
\[ \phi_t([0, L_a]^2) \subset [0, L_a(1 + M(t-a))^2]^2 \]
Therefore, as \( \sigma_t = \phi_t \# \sigma_a \) and \( \phi_t \) is continuous,

\[
spt(\sigma_t) = \phi_t(spt(\sigma_a)) \subset \phi_t([0, L_a]^2) \subset [0, L_a(1 + M(t - a))^2]^2
\]

3. In view of \([5,4]\), \( \sigma_t = \phi_t \# \sigma_a \) is absolutely continuous with respect to the Lebesgue measure \( L^2 \) and its density function \( \varrho_t \) satisfies

\[
\varrho_t \circ \phi_t = \frac{\varrho_a}{\det(\nabla \phi_t)} \leq \varrho_a
\]

Using (5.5) and the fact that \( \det(\nabla \phi_t) \geq 1 \), we obtain

\[
\int_{\mathbb{R}^2} \varrho_t^r dq \leq \int_{\mathbb{R}^2} \varrho_a^r \circ \varphi^{-1} \det[\nabla \varphi]^{-1} \circ \varphi^{-1} dq = \int_{\mathbb{R}^2} \varrho_a^r dq \quad r \geq 1.
\]

This establishes (a). We easily check \( |v| \leq M \sqrt{4L_a + 1} = c_0 \) and so, by \([8.3.1]\) in \([I]\)

\[
W_1(\sigma_t, \sigma_s) \leq \int_s^t \|v_r\|_{L^1(\sigma_r)} dr \leq c_0(t - s) \quad \text{for all } a \leq s \leq t \leq a + \tau
\]

Therefore \( t \to \sigma_t \) is \( c_0 \)-Lipschitz continuous on \([a, a + \tau] \). Thus,

\[
W_1(\sigma_t, \sigma_a) \leq c_0(t - a) \leq c_0 \tau
\]

for all \( t \in [a, a + \tau] \). As a consequence \( \{\sigma_t\}_{t \in [a, a + \tau]} \) is bounded in the \( 1 \)-Wasserstein space.

**Step 2** We consider now the general case where \( \Psi(a, \cdot) \) is not necessary smooth. We note that, as \( \Psi(a, \cdot) \) is convex, \( \Psi(a, \cdot) \) is locally Lipschitz and so \( \Psi(a, \cdot) \in W^{1,1}_{loc}(\mathbb{R}^2) \). We set

\[
\Psi^n(a, \cdot) := \Psi(a, \cdot) * j_n
\]

Here, \( \{j_n\}_{n=1}^\infty \) are the standard mollifiers. We obtain that \( \Psi^n(a, \cdot) \) converges to \( \Psi(a, \cdot) \) in \( W^{1,1}_{loc}(\mathbb{R}^2) \). This convergence guarantees that up to a subsequence \( \nabla \Psi^n(a, \cdot) \) converges \( \nabla \Psi(a, \cdot), a.e \in \mathbb{R}^2 \).

Let’s denote by \( v^n \) the velocity field when \( \Psi \) is replaced by \( \Psi^n \) in (5.1). Without loss of generality, we have that

\[
v^n \to v_1 \quad a.e
\]

Let \( \sigma^n = \varrho^n L^2 \) denotes the solution of (5.2) when \( v \) is replaced by \( v^n \). Then \( \sigma^n \) satisfies (5.6) and the conditions (a), (b) and (c). We obtain that the family \( t \to \sigma^n_t \) is equi-Lipschitz on \([a, a + \tau]\) with respect to \( W_1 \) and (5.7) ensures that it is equi-bounded in \( P(\mathbb{R}^2) \) with respect to \( W_1 \). Therefore, there exists a subsequence that we still denote by \( t \to \sigma^n_t \) ( \( n \) is independent of \( t \)) such that \( \{\sigma^n_t\}_{n=1}^\infty \) converge narrowly to \( \sigma_t \) for each \( t \in [0, \tau] \).

Since the Wasserstein distance is lower semi-continuous with respect to narrow convergence and \( \sigma^n_t \) satisfy (5.6), \( \sigma_t \) also satisfies (5.6), that is, \( \sigma_t = c_0 \)-Lipschitz continuous on \([a, a + \tau]\), which proves (c). By condition (a), \( \{\varrho^n_t\}_{n=1}^\infty \) is equibounded in \( L^r, r \geq 1 \) and so, as \( \{\varrho^n_t\}_{n=1}^\infty \) converges weakly* to \( \varrho_t \), the Dunford-Pettis theorem guarantees that \( \sigma_t \) is absolutely continuous with respect to Lebesgue , that is \( \sigma_t = \varrho_t L^2 \). Also, as \( \{\varrho^n_t\}_{n=1}^\infty \) satisfy the condition (a), the weakly lower semi-continuity of the \( L^r \) norms ensures that \( \varrho_t \) satisfy the condition (a) as well.

To obtain the continuity equation in (b), we only need to show that \( \{\varrho^n_t \sigma^n_t\}_{n=1}^\infty \) converges to \( \varrho_t \sigma_t \) in the sense of distribution for each \( t \) fixed, as the fact that \( \{\varrho^n_t \sigma^n_t dt\}_{n=1}^\infty \) converges to \( \varrho_t \sigma_t dt \) in the sense of distribution will be obtained by a simple application of Lebesgue dominated convergence. We note that the
inequality in (a) ensures that \( \{ \varrho_i^n \}_{n=1}^\infty \) is equi-integrable. As \( \mathbf{v}_t^n \) converges to \( \mathbf{v}_t \) a.e and \( \sigma^n = \varrho_t L^2 \) narrowly to \( \sigma = \varrho L^2 \) we use Lemma 5.1 to obtain the desired result.

If \( \sigma \in \Sigma \) and \( (P, \Psi) = \tilde{\mathcal{H}}(\sigma) \) then we define

\[
(5.8) \quad V_t[\sigma] := \left( 2\sqrt{\mathcal{T}} \left[ \mathcal{F}_t \left( \frac{\mathcal{G}_t}{\sqrt{1-2\mathcal{G}_t^2 \mathcal{S}_t}}, \frac{\partial \Psi}{\partial \mathcal{S}_t} \right) \right), \frac{\varrho \nabla \tilde{\mathcal{S}}_t}{\sqrt{1-2\mathcal{G}_t^2 \mathcal{S}_t}} \right)
\]

**Theorem 5.2.** Assume that (A1), (A2) and (A3) hold. Assume \( 1 < L_0 < R_0 \) and let \( \tilde{\varrho}_0 = \tilde{\varrho}_0 L^2 \in \Sigma \) such that

\[
\text{spt}(\tilde{\varrho}_0) \subset [0, L_0]^2
\]

Let \( T > 0 \) such that \( L_0 e^{6MT} < R_0 \). Then, there exists \( \sigma_t = \varrho_t L^2 \in \Sigma \) satisfying:

(a) \( \int_{\mathbb{R}^2} \varrho_t dq \leq \int_{\mathbb{R}^2} \varrho_0 dq \) for any \( r \geq 1 \)

(b) \( t \mapsto \sigma_t \in AC_1([0,T); \mathcal{P}(\mathbb{R}^2)) \) and

\[
(5.9) \quad \begin{cases} \quad \frac{\partial \sigma_t}{\partial t} + \text{div}(\sigma_t V_t[\sigma]) = 0, & (0, T) \times \mathbb{R}^2 \\ \sigma_{t=0} = \tilde{\varrho}_0 & \end{cases}
\]

holds in the sense of distribution.

(c) \( t \mapsto \sigma_t \) is Lipschitz continuous with Lipschitz constant less than \( c_0 = M \sqrt{4L_0 + 1} \).

**Proof:** We fix a non-negative integer \( N \) and divide the interval \([0,T]\) into \( N \) intervals with equal length \( \tau = \frac{T}{N} \).

We first show that we can construct a discrete function \( \sigma_t^N = \varrho_t L^2 \) satisfying the following properties:

(a1) \( \int_{\mathbb{R}^2} \varrho_t^N dq \leq \int_{\mathbb{R}^2} \varrho_0 dq \) for any \( r \geq 1 \)

(b1) The “delayed” equation

\[
(5.10) \quad \begin{cases} \quad \frac{\partial \sigma_t^N}{\partial t} + \text{div}(\sigma_t^N \mathbf{v}_t^N) = 0, & (0, T) \times \mathbb{R}^2 \\ \sigma_{t=0} = \tilde{\varrho}_0 & \end{cases}
\]

holds in the sense of distribution with \( \mathbf{v}_t^N = V_t[\sigma_t^N] \) for all \( t \in [0,T] \).

(c1) \( t \mapsto \sigma_t^N \) is Lipschitz continuous with respect to \( W_1 \) with Lipschitz constant less than \( c_0 \).

The construction of \( \sigma_t^N \) goes as follows: we start off by setting \( \sigma_0^N = \tilde{\varrho}_0 \) and \( \mathbf{v}_t^N = V_t[\tilde{\varrho}_0] \) for \( t \in [0, \tau] \). we use Lemma 5.2 to obtain a solution \( \sigma_t^N \) on \([0, \tau]\). We repeat inductively the same process \((N-1)\) times by setting \( \sigma_t^{N-1} := \sigma_t^N \) and \( \mathbf{v}_t^{N} = \mathbf{v}_t[\sigma_t^{N}] \) for \( t \in [i\tau, (i-1)\tau] \) and using Lemma 5.2 to obtain \( \sigma_t^N \) on \([i\tau, (i+1)\tau] \). In view of Lemma 5.2 we note that the process described above works as long as \( \{\sigma_{i\tau}\}_{1 \leq i \leq N} \) stays absolutely continuous with respect to Lebesgue and compactly supported in \( \mathbb{R}^2 \). We next show that \( \{\sigma_{i\tau}\}_{1 \leq i \leq N} \subset \Sigma \).

We first observe that by construction, Lemma 5.2 guarantees that \( \{\sigma_{i\tau}\}_{1 \leq i \leq N} \) are absolutely continuous with respect to Lebesgue in \( \mathbb{R}^2 \). Define

\[
L_i := \max \left( \sup \{ \mathbf{Y} : (\mathbf{Y}, Z) \in \text{spt}(\sigma_{i\tau}) \} ; \sup \{ Z : (\mathbf{Y}, Z) \in \text{spt}(\sigma_{i\tau}) \} \right)
\]

for \( 1 \leq i \leq N \). By Lemma 5.2

\[
L_{i+1} \leq L_i(1 + M\tau)^2 \leq L_0(1 + M\tau)^2(i+2) < L_0(1 + M\tau)^6 = L_0(1 + M\tau \frac{T}{N})^6 \leq L_0 e^{6MT}.
\]
With the constraint $L_0e^{6MT} < R_0$, on $T$, we obtain that for any $0 \leq i \leq N$, $\text{spt}(\sigma_{i\tau})$ is contained in $[0, R_0]^2$. Therefore the above construction of $\sigma_t^N$ is thoroughly justified. We easily check that the conditions (a1) and (c1) follow from the condition (a) and (c) of Lemma 5.2.

**Step 2** By (c1), $t \mapsto \sigma_t^N$ are equi-Lipschitz continuous on $[0, T]$ and since $\sigma_0^N = \sigma_0$ for all $N$ they are equibounded in the 1-Wasserstein space. Thus, there exists a subsequence of $t \mapsto \sigma_t^N$ (N independent of $t$) such that $\{\sigma_t^N\}_{N=1}^\infty$ converges narrowly to $\sigma_t$ for any $t \in [0, T]$.

In view of (a1), the theorem of Dunford-Pettis ensures that $\sigma_t = \rho_t \mathcal{L}^2$. The weakly lower semi -continuity of the $L'$- norms leads to (a). We next show that $\sigma_t$ satisfies (5.9). As $\{\sigma_t^N\}_{N=1}^\infty$ converges narrowly to $\sigma_t$ we only need to show that $\{\mathbf{v}_t^N\}_{N=1}^\infty$ converges to $V_t[\sigma_t] \sigma_t$ in the sense of distribution for each $t$ fixed, as the fact that $\{\mathbf{v}_t^N\}_{N=1}^\infty$ converges to $\mathbf{v}_t[\sigma_t] \sigma_t dt$ in the sense of distribution will be obtained by a simple application of Lebesgue dominated convergence. By (c1)

$$W_1(\sigma_t^N, \sigma_{[\tau]}^N) \leq |t - \frac{t}{\tau}| \leq \frac{T}{N}$$

In light of this, $\{\sigma_t^N\}_{N=1}^\infty$ converges narrowly to $\sigma_t$ implies that $\{\sigma_{[\tau]}^N\}_{N=1}^\infty$ converges narrowly to $\sigma_{[\tau]}$. Thus, for each $t$ fixed, $\{\mathbf{v}_t^N\}_{N=1}^\infty$ converges $V_t[\sigma_t] \mathcal{L}^2 - a.e$ by Lemma 5.1(ii). We use Lemma 5.1 to obtain that $\{\mathbf{v}_t^N\}_{N=1}^\infty$ converges to $V_t[\sigma_t] \sigma_t$ in the sense of distribution for each $t$ fixed. This concludes the proof $\square$

### 5.2. Existence of a solution for general initial data

In this section, we impose the following conditions on the forcing terms $\mathcal{F}$ and $\mathcal{S} : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$.

- $(B_1)$ $\mathcal{F}$ and $\mathcal{S}$ are continuous and bounded.
- $(B_2)$ $\mathcal{F} \geq 0$ and $\mathcal{S} \geq 0$

Set

$$\mathcal{F}_t = \left( \mathcal{F}_t \circ \mathbf{d}, \frac{\partial}{\partial_0} \mathcal{S}_t \circ \mathbf{d} \right) \quad \text{with} \quad \mathbf{d}(s, z) = \left( \frac{r_0}{\sqrt{1 - 2r_0^2 s}}, z \right)$$

As $\mathcal{F}$ and $\mathcal{S}$ are bounded, there exists a constant $C_0$ (independent of $t$) such that

$$\sqrt{R_0}||\mathcal{F}_t||_{\infty} \leq C_0$$

for all $t \geq 0$. To any function $G = (G_1, G_2)$ we associate $\mathcal{A}[G]$ defined by

$$\mathcal{A}[G](\Upsilon, Z) = \left( \sqrt{1} G_1(\Upsilon, Z), G_2(\Upsilon, Z) \right).$$

Note that if $G \in C([0, R_0]^2; \mathbb{R}^2)$ then

$$\mathcal{A}[G] \in C([0, R_0]^2; \mathbb{R}^2) \quad \text{with} \quad ||\mathcal{A}[G]||_{\infty} \leq \sqrt{R_0}||G||_{\infty}$$

For $\sigma \in \mathcal{P}_{\{R_0\}}$, if $h = \mathcal{H}(\sigma)$ and $(P, \Psi) = \mathcal{H}(\sigma)$ then we define

$$L_t[\sigma](G) := \int_{\mathbb{R}^2} \langle \mathcal{A}[G] \circ \nabla P, \mathcal{F}_t \rangle e(s) \chi_D h(s, z) ds dz$$

for all $G$ Borel measurable. Note that if $G \in L^1(\sigma, \mathbb{R}^2)$ then
Remark 5.3. For $G \in L^1(\sigma; \mathbb{R}^2)$ such that $G_1 \geq 0, G_2 \geq 0 \quad \sigma \text{ a.e}$ we have that $L_t[\sigma](G) \geq 0$.

Lemma 5.3. Fix $t > 0$. There exists $V_t: \mathcal{P}_{[R_0]} \rightarrow \bigcup_{\sigma \in \mathcal{P}_{[R_0]}} L^\infty(\sigma; \mathbb{R}^2)$ such that for any $\sigma \in \mathcal{P}_{[R_0]}$,

\[
V_t[\sigma] := (V_t^1[\sigma], V_t^2[\sigma]) \in L^\infty(\sigma; \mathbb{R}^2) \quad \text{and}
\]

\[\sigma \in \mathcal{P}_{[R_0]} \]

\[
\begin{align*}
\left| L_t[\sigma](G) \right| &\leq \|A[G] \|_{L^1(\sigma; \mathbb{R}^2)} \|G\|_{L^1(\sigma; \mathbb{R}^2)} \\
&\leq \|R_0\| \|G\|_{L^1(\sigma; \mathbb{R}^2)} \\
\end{align*}
\]

The proof is similar to the proof of Lemma 5.3, and hence is omitted.

Remark 5.4. If $\sigma \in \mathcal{P}_{[R_0]}$, $h = \mathcal{H}(\sigma)$ and $(P, \Psi) = \mathcal{H}(\sigma)$ then for $G \in L^1(\sigma; \mathbb{R}^2)$ and any $t, r \geq 0$

\[
L_t[\sigma](G) - L_r[\sigma](G) = \int_{\mathbb{R}^2} \langle A[G] \circ \nabla P, F_t - F_r \rangle e(s) \chi_{D_h}(s, z) dsdz
\]

and so in view of Lemma 5.3,

\[
\left| \int_{\mathbb{R}^2} \langle V_t[\sigma] - V_r[\sigma], G \rangle d\sigma \right| = |L_t[\sigma](G) - L_r[\sigma](G)|
\]

\[
\leq \|A[G]\| \sup_{p \in \Delta_{R_0}} |F_t(p) - F_r(p)| \int_{\mathbb{R}^2} e(s) \chi_{D_h}(s, z) dsdz
\]

\[
\leq \sqrt{R_0} \|G\| \sup_{p \in \Delta_{R_0}} |F_t(p) - F_r(p)|
\]
Lemma 5.4. Let \( t \geq 0 \) and \( V_t \) as provided by Lemma 5.3. Let \( \{\sigma_n\}_{n=1}^{\infty} \) and \( \sigma \) be elements of \( \mathcal{P}_{(R_0)}(\mathbb{R}^2) \) such that \( \{\sigma_n\}_{n=1}^{\infty} \) converges narrowly to \( \sigma \). Then \( V_t[\sigma_n]\sigma_n \) converges to \( V_t[\sigma]\sigma \) in the sense of distributions.

**Proof:** Let \( (P_n, \Psi_n) = \mathcal{H}(\sigma_n) \), \( (P, \Psi) = \mathcal{H}(\sigma) \), \( h_n = \mathcal{H}(\sigma_n) \) and \( h = \mathcal{H}(\sigma) \). As \( \{\sigma_n\}_{n=1}^{\infty} \) converges narrowly to \( \sigma \), Lemma 4.19 ensures that there exists a subsequence \( \{n_k\}_{k=1}^{\infty} \) of integers such that \( \{P_{n_k}\}_{k=1}^{\infty} \) converges uniformly. Hence, by Lemma 4.2 (i), \( 0 \leq h_{n_k} \leq M_0 < \frac{1}{2\varepsilon} \) for some constant \( M_0 \) and so \( \{e(s)\chi_{D_{n_k}}\}_{k=1}^{\infty} \) is equi-integrable. Lemma 4.2(ii) ensures that \( \{\nabla P_{n_k}\}_{k=1}^{\infty} \) converges a.e to \( \nabla P \). Let \( G \in C([0,R_0]^2) \). Then \( A[G] \) is continuous on \([0,R_0]^2 \) and \( (\nabla P_{n_k};F_t) \) converges a.e to \( (A[G] \circ \nabla P;F_t) \). Moreover, as \( G \) is bounded function, \( A[G] \) is bounded. In addition, since \( F \) is bounded, there exists \( M > 0 \) such that \( |(A[G] \circ \nabla P_{n_k};F_t)| \leq M \) for all \( k \geq 1 \) and \( t > 0 \). Using Lemma 5.19, we obtain that

\[
\lim_{k \to \infty} \int \langle A[G] \circ \nabla P_{n_k};F \rangle e(s)\chi_{D_{n_k}}(s,z)dsdz = \int \langle A[G] \circ \nabla P;F \rangle e(s)\chi_{D_n}(s,z)dsdz
\]

This, in the light of (5.12), becomes

(5.17)

\[
\lim_{k \to \infty} \int_{\mathbb{R}^2} \langle G, V_t[\sigma_{n_k}] \rangle d\sigma_{n_k} = \int_{\mathbb{R}^2} \langle G, V_t[\sigma] \rangle d\sigma
\]

As \( \{\sigma_n\}_{n=1}^{\infty} \), \( \sigma \in \mathcal{P}_{(R_0)}(\mathbb{R}^2) \), (5.17) still holds for \( G \in C_c(\mathbb{R}^2) \). Thus, we obtain that \( \{V_t[\sigma_{n_k}]\sigma_{n_k}\}_k \) converges to \( V_t[\sigma]\sigma \) in the sense of distributions. Since the limit \( V_t[\sigma]\sigma \) is independent of the extracted subsequence of \( \{V_t[\sigma_n]\sigma_n\}_n \), we conclude that the whole sequence \( \{V_t[\sigma_n]\sigma_n\}_n \) converges narrowly to \( V_t[\sigma]\sigma \).

**Definition 5.5.** Let \( T > 0 \). \( t \to \sigma_t \) is an absolutely continuous path in \( \mathcal{P}_{(R_0)} \).

Let \((P(t,\cdot),\Psi(t,\cdot)) = \mathcal{H}(\sigma_t) \) and \( h_t = \mathcal{H}(\sigma_t) \) We say that

\[
\sigma_t = \chi_{\mathcal{H}}(\sigma_t) \quad t \in (0,T)
\]

in the weak dual sense if

(5.18)

\[
\int_0^T dt \int_{\mathbb{R}^2} \frac{\partial \varphi}{\partial t} \nabla P(t,\cdot)e(s)\chi_{D_n}dsdz + \int_0^T L_t[\sigma](\nabla \varphi)dt = 0
\]

for all \( \varphi \in C^1((0,T) \times \mathbb{R}^2) \).

**Remark 5.5.** In view of Lemma 5.19, (5.18) becomes

\[
\int_0^T dt \int_{\mathbb{R}^2} \frac{\partial \varphi}{\partial t} + \langle \nabla \varphi, V_t[\sigma] \rangle d\sigma_t dt = 0
\]

for all \( \varphi \in C^1((0,T) \times \mathbb{R}^2) \). That is,

(5.19)

\[
\frac{\partial \sigma}{\partial t} + \text{div}(\sigma V_t[\sigma]) = 0, \quad (0,T) \times \mathbb{R}^2
\]

holds in the distribution sense.

**Theorem 5.6.** Assume \( \tilde{F} \) and \( \tilde{S} \) satisfy (B1) and (B2). Assume \( 0 < L_0 < R_0 \). Let \( \tilde{\sigma}_0 \in \mathcal{P}_{[L_0]} \). Let \( T > 0 \) such that \( L_0 + C_0T < R_0 \). Then there exists \( \sigma_t : [0,T] \to \mathcal{P}(\mathbb{R}^2) \) \( C_0 \)-Lipschitz continuous such that \( \text{spt}(\sigma_t) \subset [0,R_0]^2 \) and

(5.20)

\[
\begin{align*}
\dot{\sigma}_t &= \chi_{\mathcal{H}}(\sigma_t) \\
\sigma_{t=0} &= \tilde{\sigma}_0
\end{align*}
\]
We combine (5.22) and (5.23) to obtain (5.25)

\[ \sigma \in P \]

Converges narrowly to some \( \sigma \)

By using the facts that equivalently (5.19) in view of Remark 5.5. For this purpose, we only have to show that up to some subsequence (b) \( W_1(\sigma_t^N, \sigma_0) \leq C_0 T \) and (c) \( spt(\sigma_t^N) \subset [0, R_0]^2 \) for \( t \in [0, T] \)

d) \( t \mapsto \sigma_t^N \) satisfies

\[
\begin{cases}
\frac{\partial \sigma^N_t}{\partial t} + \text{div}(\sigma_t^N V[\sigma_t^N]) = 0, & (0, T) \times \mathbb{R}^2 \\
\sigma^N_{t=0} = \sigma_0
\end{cases}
\]

holds in the distributional sense with

(5.21)

\[
V_t[\sigma_t^N] \sigma_t^N = \left( \text{id} + (t - \left[ \frac{t}{\tau} \right] \tau) V[\sigma_t^N] \right) \# \left( V[\sigma_t^N] \sigma_t^N \right)
\]

In view of (a) and (b), there exists a subsequence of \( \{ \sigma_t^N \}_N \) still denoted by \( \{ \sigma_t^N \}_N \) such that \( \{ \sigma_t^N \}_N \)

converges narrowly to some \( \sigma_t \) for each \( t \) fixed independently of \( N \). We next show that \( \sigma_t \) solves (5.20) or equivalently (5.19) in view of Remark 5.3. For this purpose, we only have to show that up to some subsequence, \( \{ V_t[\sigma_t^N] \sigma_t^N dt \}_N \) converges in the sense of distribution to \( V_t[\sigma_t] \sigma_t dt \).

Let \( \phi \in C_c^\infty((0, T) \times \mathbb{R}^2, \mathbb{R}^2) \). We use (5.21) to obtain

(5.22)

\[
\left| \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t, x); V_t[\sigma_t^N] \rangle d\sigma_t^N - \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t, x); V_{\tau[\sigma_t^N]} \rangle d\sigma_t^N \right| 
\]

\[
\leq \sum_{k=1}^{N} \int_{(k-1)\tau}^{k\tau} dt \int_{\mathbb{R}^2} \left| \phi \left( t, x \right) + (t - \left[ \frac{t}{\tau} \right] \tau) V_{\tau[\sigma_t^N]} \right| d\sigma_t^N
\]

By using the facts that \( \sigma_t^N \) has its support in \([0, R_0]^2\), \( \phi \) is Lipschitz on \([0, T] \times [0, R_0]^2\) and \( V \) is bounded, the right-hand side of (5.22) can be estimated by

(5.23)

\[
\sum_{k=1}^{N} \int_{(k-1)\tau}^{k\tau} dt \int_{[0, R_0]^2} \text{Lip}(\phi) \left| t - \left[ \frac{t}{\tau} \right] \tau \right| \left| V_{\tau[\sigma_t^N]} \right| d\sigma_t^N \leq C_0^2 \text{Lip}(\phi) \frac{T^2}{2N}
\]

We combine (5.22) and (5.23) to obtain

(5.24)

\[
\limsup_{N \to \infty} \left| \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t, x); V_t[\sigma_t^N] \rangle d\sigma_t^N - \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t, x); V_{\tau[\sigma_t^N]} \rangle d\sigma_t^N \right| = 0
\]

By (5.19),

(5.25)

\[
\left| \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t, x); V_{\tau[\sigma_t^N]} - V_{\tau[\sigma_t^N]} \rangle d\sigma_t^N \right| \leq \sqrt{R_0} \sup_{p \in \partial \Omega} |F_t(p) - F_{\tau[p]}(p)| dt
\]
As \( |t - \tau \left( \frac{1}{N} \right) | \leq \tau = \frac{T}{N} \) and \( F \) is continuous and bounded on \([0, T] \times \Delta_{r_0}\), we use the Lebesgue dominated convergence theorem to obtain that

\[
\limsup_{N \to \infty} \left| \int_0^T \int_{\mathbb{R}^2} \langle \phi(t, x); V_{r_1[\frac{1}{N}]}[\sigma_{r_1[\frac{1}{N}]}^N] - V_t[\sigma_{r_1[\frac{1}{N}]}^N] \rangle \, d\sigma_{r_1[\frac{1}{N}]} \right| \\
\leq \sqrt{R_0} \| \phi \|_\infty \limsup_{N \to \infty} \int_0^T \sup_{p \in \Delta_{r_0}} \left| F_{r_1[\frac{1}{N}]}(p) - F_t(p) \right| \, dt = 0
\]

We note that

\[
\left| \int_{\mathbb{R}^2} \langle \phi(t, x); V_t[\sigma_{r_1[\frac{1}{N}]}^N] \rangle \, d\sigma_{r_1[\frac{1}{N}]} \right| \leq C_0 \| \phi \|_\infty
\]

Using (a), we have

\[
W_1 \left( \sigma_{r_1[\frac{1}{N}]}^N, \sigma_t^N \right) \leq C_0 \left| t - \tau \left( \frac{t}{N} \right) \right| \leq \frac{C_0 T}{N}
\]

And so, as \( N \) goes to \( \infty \), \( \left\{ \sigma_{r_1[\frac{1}{N}]}^N \right\}_N \) converges narrowly to \( \sigma_t \) and lemma 5.4 ensures that

\[
\int_{\mathbb{R}^2} \langle \phi(t, x); V_t[\sigma_{r_1[\frac{1}{N}]}^N] \rangle \, d\sigma_{r_1[\frac{1}{N}]} \text{ converges a.e to } \int_{\mathbb{R}^2} \langle \phi(t, x); V_t[\sigma_t] \rangle \, d\sigma_t
\]

We combine (5.27) and (5.28) and use the Lebesgue dominated convergence theorem to obtain that

\[
\int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t, x); V_t[\sigma_{r_1[\frac{1}{N}]}^N] \rangle \, d\sigma_{r_1[\frac{1}{N}]} \text{ converges to } \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t, x); V_t[\sigma_t] \rangle \, d\sigma_t
\]

In view of (5.21), (5.20) and (5.29), we have

\[
\limsup_{N \to \infty} \left| \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t, x); V_t[\sigma_{r_1[\frac{1}{N}]}^N] \rangle \, d\sigma_{r_1[\frac{1}{N}]} - \int_0^T dt \int_{\mathbb{R}^2} \langle \phi(t, x); V_t[\sigma_t] \rangle \, d\sigma_t \right| = 0
\]

As \( \phi \) is arbitrary, we obtain that \( \left\{ V_t[\sigma_t^N] \sigma_t^N dt \right\}_N \) converges in the sense of distribution to \( V_t[\sigma_t] \sigma_t \, dt \) which concludes the proof. \( \square \)

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