Calderón’s inverse problem with an imperfectly known boundary in two and three dimensions

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Abstract. We show how to eliminate the error caused by an incorrectly modeled boundary in electrical impedance tomography (EIT). In practical EIT measurements one usually lacks the exact knowledge of the boundary. Because of this the numerical reconstruction from the measured EIT data has to be computed using a model domain that represents the best guess for the true domain. However, it has been noticed in simulations and practical experiments that the errors in the model of the boundary cause severe errors to the reconstructions. We consider the two dimensional and higher dimensional cases separately. In the two dimensional case we review recent algorithms for finding a deformed image of the original isotropic conductivity based on the theory of Töichmüller spaces. For the higher dimensional case, we compare the higher dimensional and the two dimensional results and observe that the properties of the problem change in a radical way when the dimension changes.

1. Introduction.
Let us consider the electrical impedance tomography (EIT) problem, that is, the determination of an unknown conductivity distribution inside a domain from voltage and current measurements made on the boundary. EIT has several medical and non-medical applications. The medical applications include, for example, detection of tumors from breast tissue and monitoring of pulmonary function. Mathematically the EIT problem is formulated as follows: Let \( \Omega \subset \mathbb{R}^n \), \( n \geq 2 \) be open sufficiently smooth domain, and denote by \( \gamma = (\gamma^{ij})_{i,j=1}^{n} \in \mathbb{R}^{n \times n} \) the symmetric matrix describing the conductivity in \( \Omega \). We assume that the matrix has components in \( L^\infty(\Omega) \) and that it is strictly positive definite, that is, for some \( C_0 > 0 \) we have

\[
\frac{1}{C_0} \| \xi \|^2 \leq \langle \xi, \gamma(x) \xi \rangle \leq C_0 \| \xi \|^2, \quad \text{for a.e. } x \in \Omega.
\]

We denote by \( C_0(\gamma) \) the smallest \( C_0 \) for which (1) is valid. The electrical potential \( u \) satisfies in \( \Omega \) the equation

\[
\nabla \cdot \gamma \nabla u = 0.
\]

We consider the inverse problem of finding \( \gamma \) from boundary measurements. The usual idealization of boundary measurements is to formulate them using the Dirichlet-to-Neumann
Figure 1. An example of incorrect modeling of the boundary. True domain $\Omega$ is shown on left and the model domain $\Omega_m$ on right. The map $F_m: \Omega \to \Omega_m$ correspond to the inaccurate boundary modeling. The map $f_m = F_m|_{\partial \Omega}$ maps the electrodes $e_k$ on the true boundary $\partial \Omega$ to the boundary $\partial \Omega_m$ of the model domain.

or the Robin-to-Neumann map operating to the boundary values of the function $u$. Here, we will use the Robin-to-Neumann map (defined below) since it corresponds more accurately to the measurements done in practice. To consider the practical measurements we start with the model for electrode measurements. The physically realistic measurements are usually modeled by the following complete electrode model (see [6, 29]): Let $e_k \subset \partial \Omega$, $k = 1, \ldots, K$ be disjoint open sets of the boundary with smooth boundaries modeling the electrodes that are used for the measurements. The electrodes are denoted by thick surface patches in Figure 1. Let $v$ solve the equation

\[
\nabla \cdot \gamma \nabla v = 0 \quad \text{in } \Omega,
\]

(3)

\[
z_k \nu \cdot \gamma \nabla v + v |_{e_k} = V_k,
\]

(4)

\[
\nu \cdot \gamma \nabla v|_{\partial \Omega \cup \bigcup_{k=1}^{K} e_k} = 0,
\]

(5)

where $V_k$ are constants representing the electric potentials on the electrodes $e_k$. The values of $z_k$ are the electrode contact impedances modeling the voltage losses caused by electro-chemical phenomena at the interface of the skin and the measurement electrodes [6]. $\nu$ is the unit Euclidean exterior normal vector of $\partial \Omega$. The observed electric currents on the electrodes are given by

\[
J_k = \int_{e_k} \nu \cdot \gamma \nabla v(x) \, ds(x), \quad k = 1, \ldots, K.
\]

Thus the electrode measurements are given by map $M_E: \mathbb{R}^K \to \mathbb{R}^K$, $M_E(V_1, \ldots, V_K) = (J_1, \ldots, J_K)$. We say that $M_E$ is the electrode measurement matrix for $(\partial \Omega, \gamma, e_1, \ldots, e_K, z_1, \ldots, z_K)$. We associate to the electrode measurement matrix the corresponding quadratic forms $M_E: \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$

\[
M_E[V, V'] = \sum_{k=1}^{K} (M_E V_k) V'_k, \quad V = (V_k), \quad V' = (V'_k) \in \mathbb{R}^K.
\]

(6)

This form evaluated on diagonal $V = V'$, that is, the quantity $M_E(V, V)$ corresponds to the power needed to maintain the voltages $V$ in electrodes. Let us next formulate an analogous continuous model for the above electrode measurement model. Assume that the electrical potential $u$ satisfies

\[
\nabla \cdot \gamma \nabla u = 0, \quad x \in \Omega,
\]

(7)

\[
(z \nu \cdot \gamma \nabla u + u)|_{\partial \Omega} = h,
\]
where \( h \) is the Robin-boundary value of the potential and \( z \) is a function describing the contact impedance on the boundary. The perfect boundary measurements are modeled by the Robin-to-Neumann map \( R = R_{z,\gamma} \) given by

\[
R : h \mapsto \nu \cdot \gamma \nabla u|_{\partial \Omega}.
\]

This defines a bounded operator \( R : H^s(\partial \Omega) \to H^s(\partial \Omega) \), with \( s \in [s_0, s_1] \), where \( s_0, s_1 \) depends on smoothness of \( \partial \Omega \). Here we consider the case when the boundary is at least \( C^2 \) smooth and choose \( s = 1/2 \). The corresponding quadratic form is

\[
R[h_1, h_2] = \int_{\partial \Omega} (Rh_1) h_2 \, dS_E, \quad h_1, h_2 \in H^{1/2}(\partial \Omega)
\]

where \( dS_E \) is the Euclidean volume (or area or length) element on the boundary. Applying Green’s formula we observe that

\[
R[h, h] = \int_{\partial \Omega} (u + z \nu \cdot \gamma \nabla u) \nu \cdot \gamma \nabla u \, dS_E = \int_{\Omega} \gamma \nabla u \cdot \nabla u \, dx + \int_{\partial \Omega} z |\nu \cdot \gamma \nabla u|^2 \, dS_E
\]

where \( h \in H^{1/2}(\partial \Omega) \), \( u \) solves (7). The value \( R[h, h] \) corresponds to the power needed to maintain the potential \( h \) on the boundary. We note that in the case when \( z = 0 \) the boundary measurements \( R_{\gamma, z} \) correspond to the Dirichlet–Neumann map \( \Lambda_{\gamma} \) taking the Dirichlet boundary values to the corresponding Neumann boundary values of the solution to (7).

\[
\Lambda_{\gamma} : u|_{\partial \Omega} \mapsto \nu \cdot \nabla u|_{\partial \Omega}.
\]

In the case when \( z \) and \( \partial \Omega \) are known, the knowledge of \( R_{z,\gamma} \) and \( \Lambda_{\gamma} \) are equivalent. When \( \gamma \) is a scalar valued function times identity matrix we say that the conductivity is isotropic. Otherwise, the conductivities that are matrix-valued are referred to as anisotropic conductivities. The EIT problem, that is, reconstruction of \( \gamma \) from the knowledge of Dirichlet-to-Neumann map was originally proposed by Calderón [5] in 1980 and then solved in dimensions three and higher for isotropic conductivities which are \( C^\infty \)-smooth in [34] and [24]. The smoothness requirements have been since relaxed, and currently the best known result is [26] with unique determination of conductivities in \( W^{3/2, \infty} \), see also [8] for conductivities of lower regularity with conormal type of singularities. In two dimensions the first global result is due to Nachman ([25]), and later Astala and Päivärinta showed in [3] that uniqueness holds also for general isotropic \( L^\infty \)-conductivities. For the corresponding anisotropic case, see [2, 19, 20, 21] and numerical implementations of the methods with simulated and real data, see [12, 30, 31]. In this paper, we will concentrate to the important case where the domain \( \Omega \) is not known accurately, see Figure 1. As we will see, the wrong modeling of the domain can be viewed as a deformation of the original domain. Thus, let us next consider what happens to the conductivity equation when the domain \( \Omega \) is deformed to a domain \( \tilde{\Omega} \). Assume that \( F : \Omega \to \tilde{\Omega} \) is a sufficiently smooth orientation preserving map with sufficiently smooth inverse \( F^{-1} : \tilde{\Omega} \to \Omega \). Let \( f : \partial \Omega \to \partial \tilde{\Omega} \) be the restriction of \( F \) on the boundary. When \( u \) is a solution of \( \nabla \cdot \gamma \nabla u = 0 \) in \( \Omega \), then \( \tilde{u}(\tilde{x}) = u(F^{-1}(\tilde{x})) \) and \( \tilde{h}(x) = h(f^{-1}(x)) \) satisfy the conductivity equation

\[
\nabla \cdot \tilde{\gamma} \nabla \tilde{u} = 0, \quad \text{in} \, \tilde{\Omega},
\]

\[
\tilde{z} \tilde{\nu} \cdot \tilde{\gamma} \nabla \tilde{u} + \tilde{u}|_{\partial \tilde{\Omega}} = \tilde{h}.
\]

Here \( \tilde{h}(x) = h(f^{-1}(x)) \), \( \tilde{\nu} \) is the unit normal vector of \( \partial \tilde{\Omega} \), \( \tilde{\gamma} \) is the conductivity

\[
\tilde{\gamma}(x) = \frac{DF(y) \gamma(y) (DF(y))^T}{|\det DF(y)|} \bigg|_{y=F^{-1}(x)},
\]

where \( y \) is the Robin-boundary value of the potential and \( z \) is a function describing the contact impedance on the boundary. The perfect boundary measurements are modeled by the Robin-to-Neumann map \( R = R_{z,\gamma} \) given by

\[
R : h \mapsto \nu \cdot \gamma \nabla u|_{\partial \Omega}.
\]
where $DF$ is the Jacobian of the map $F$, and $\tilde{z}$ is the contact impedance

$$\tilde{z}(\tilde{x}) = (\det Df(x))z(x), \quad \tilde{x} = f(x)$$

(12)

where $f = F|_{\partial \Omega}$ and $Df$ is the Jacobian matrix of the map $f : \partial \Omega \to \partial \tilde{\Omega}$ mapping the tangent space of $\partial \Omega$ to the tangent space of $\partial \tilde{\Omega}$. Note that matrix $Df$ in (12) is represented in the Euclidean coordinates. We denote $\tilde{\gamma} = F_* \gamma$ and $\tilde{z} = f_* z$ and say that $F_* \gamma$ and $f_* z$ are push-forwards of $\gamma$ and $z$ in $F$ and $f$, correspondingly. For the transformation rule (11), see [33] and for the transformation rules (12), see [14]. Also, note that if $\gamma$ is isotropic, the deformed conductivity $\tilde{\gamma}$ can be anisotropic. On $\partial \tilde{\Omega}$ we define the quadratic form

$$\tilde{R}[h, h] = R[h \circ f, h \circ f], \quad h \in H^{1/2}(\partial \tilde{\Omega})$$

and denote it by $\tilde{R} = f_* R_{\gamma, z}$ and say that $\tilde{R}$ is the push-forward of $R_{\gamma, z}$ in $f$. The important property of the push-forward $\tilde{R}$ of the Robin-to-Neumann form $R$ (see [14]) is that it can be represented in the form

$$f_* R_{\gamma, z} = R_{\tilde{\gamma}, \tilde{z}}$$

where $\tilde{\gamma} = F_* \gamma$ and $\tilde{z} = f_* z$.

2. Inaccurately modeled boundary

In practice, one of the key difficulties in solving the EIT problem is that the domain $\Omega$ may not be known accurately. All the traditional approaches to the EIT problem assume that the boundary $\partial \Omega$ is known a priori, and the only unknown in the problem is the conductivity. Since there are no practically reliable measurement methods available for the determination of the boundary, the EIT image reconstruction problem has to be solved using an approximate model domain $\Omega_m$, which represents our best guess for the shape of the true body $\Omega$. However, it has been noticed that the use of slightly incorrect model for $\Omega$, i.e., a slightly incorrect model of the boundary causes serious errors in reconstructions, see e.g. [1, 9, 16]. As an example case, consider the EIT measurements from the human thorax. The measurement electrodes are attached on the skin of the patient around the thorax. In principle, an exact parameterization for the shape of the thorax could be obtained from other medical imaging modalities such as magnetic resonance imaging (MRI) or computerized tomography (CT). However, in practical situations such information may not be available, and one has to use some approximate thorax model in the computations. Further, the shape of the thorax varies between breathing states, and it is also dependent on the orientation of the patient. Therefore, the thorax boundary would be known inaccurately even in the best possible situations, and this would lead to errors in the reconstructed images. Next we consider a method to overcome the problem that the boundary and its parameterization are not exactly known. The set-up of the problem we consider is the following (see also Figure 1): Consider the task of recovering the conductivity $\gamma$ in $\Omega$ from the measurements of Robin-to-Neumann map, where we assume a priori that $\gamma$ is isotropic. We assume $\partial \Omega$ and $R_{\gamma, z}$ are not known. Instead, let $\Omega_m$, called the model domain, be a guess for the domain and let $f_m : \partial \Omega \to \partial \Omega_m$ be a diffeomorphism modeling the approximate knowledge of the boundary. As we do not know $\partial \Omega$ but only $\partial \Omega_m$, we have to interpret the boundary measurements as modified measurements on the model boundary $\partial \Omega_m$. To define them, we define that the boundary measurement corresponding to a function $\tilde{h} \in H^{1/2}(\partial \Omega_m)$ means applying the potential $\tilde{h} = \tilde{h} \circ f_m^{-1}$ on infinitesimally small electrodes located at the boundary $\partial \Omega$. For this potential we measure the power $R(\tilde{h}, \tilde{h}) = R(h, h)$.

$$((f_m)_* R)(\tilde{h}, \tilde{h}) = R(h, h).$$
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In the compete electrode model (3–5) the above measurement corresponds to the assumptions that we know the number $K$ of electrodes but not their locations and measure the power needed to maintain the voltages $V = (V_k^j)_{k=1}^{K}$ in the electrodes with all possible voltage configurations. The relation of continuous measurement and the discretized electrode measurements on inaccurately known boundary is analysed in more detail in [13, 14]. Let us summarize the definition of boundary measurements with an inaccurately known boundary in following definition.

**Definition 2.1** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a $C^2$ smooth domain, $\gamma$ be a scalar valued conductivity in $L^\infty(\Omega)$, and $\mathcal{z}: \partial \Omega \to \mathbb{R}$ be $C^1$-smooth contact impedance on $\partial \Omega$. Let $f_m : \partial \Omega \to \partial \Omega_m$ be a $C^2$-diffeomorphism corresponding to the modeling of the boundary. The measurements corresponding to conductivity $\gamma$ and the contact impedance $\mathcal{z}$ and boundary modeling map $f_m$ are

$$\partial \Omega_m \quad \text{and the map} \quad \tilde{h} \mapsto \tilde{R}[\tilde{h}, \tilde{h}], \quad \tilde{h} \in H^{1/2}(\partial \Omega_m),$$

where $\tilde{R}$ is the quadratic form $\tilde{R} = (f_m)_*R$.

3. **Inverse problem with an imperfectly known boundary in two dimensions**

Assume we are given boundary measurements with an inaccurately known boundary as in definition 2.1. Our next goal is to find a conductivity in $\Omega_m$ that is as close as possible to being isotropic and has the correct boundary boundary measurements. To formulate this result, we define the concept of maximal anisotropy.

**Definition 3.1** Let $\gamma^{jk}(x)$ be an $L^\infty(\Omega)$-smooth matrix valued conductivity in $\Omega$ and let $\lambda_1(x)$ and $\lambda_2(x)$, $\lambda_1(x) \geq \lambda_2(x)$ be the eigenvalues of matrix $\gamma^{jk}(x)$. We define the maximal anisotropy of a conductivity to be $K(\gamma)$ given by

$$K(\gamma) = \sup_{x \in \Omega} K(\gamma, x), \quad \text{where} \quad K(\gamma, x) = \frac{\sqrt{L(x)} - 1}{\sqrt{L(x)} + 1}, \quad L(x) = \frac{\lambda_1(x)}{\lambda_2(x)}.$$

We call the function $K(\gamma, x)$ the anisotropy of $\gamma$ at $x$. Here $\sup$ denotes the essential supremum.

Sometimes, to indicate the domain $\Omega$, we denote $K(\gamma) = K_\Omega(\gamma)$. As a particularly important example needed later, we consider the conductivity matrices of the form

$$\hat{\gamma}(x) = \eta(x)R_{\theta(x)} \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix} R_{\theta(x)}^{-1},$$

where $\lambda \geq 1$ is a constant, $\eta(x) \in \mathbb{R}_+$ is a real valued function, $R_{\theta(x)}$ is a rotation matrix corresponding to angle $\theta(x)$, where

$$R_{\theta} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$  

We denote such conductivities by $\hat{\gamma} = \hat{\gamma}_{\lambda, \theta, \eta}$. These conductivities have the anisotropy $K(\hat{\gamma}, x) = c_\lambda = (\lambda^{1/2} - 1)/(\lambda^{1/2} + 1)$ at every point and thus their maximal anisotropy is $K = c_\lambda$. We call such conductivities $\hat{\gamma}$ *uniformly anisotropic conductivities*. The next result says that there is a unique minimally anisotropic conductivity coinciding with the boundary measurements. Even though we are mainly interested about isotropic $\gamma$, we formulate the result for a general anisotropic conductivity.
Theorem 3.2 [13, 14] Let \( \Omega \) be a bounded, simply connected \( C^2 \)-domain. Assume that 
\( \gamma \in C^1(\overline{\Omega}) \) is a (possibly) anisotropic conductivity, \( z \in C^1(\partial \Omega) \) is the contact impedance and let 
\( R_{\gamma,z} \) be the corresponding Robin-to-Neumann map. Let \( \Omega_m \) be a model of the domain satisfying 
the same regularity assumptions as \( \Omega \), and \( f_m : \partial \Omega \to \partial \Omega_m \) be a \( C^2 \)-smooth diffeomorphism. 
Assume that we are given \( \partial \Omega_m \) and \( R = (f_m)_*, R_{\gamma,z} \). Then 

(i) We can determine the function \( \tilde{z} = (f_m)_*z \) on \( \partial \Omega_m \) uniquely.

(ii) There is a unique anisotropic conductivity \( \tilde{\gamma} \in L^\infty(\Omega_m, \mathbb{R}^{2 \times 2}) \) such that if \( \gamma_1 \) is an 
anisotropic conductivity in \( \Omega_m \) for which \( R_{\gamma_1,z} = \tilde{R} \) then \( K(\gamma_1) \geq K(\tilde{\gamma}) \).

(iii) Let \( \lambda \geq 1 \) be such that \( K(\tilde{\gamma}) = (\lambda^{1/2} - 1)/(\lambda^{1/2} + 1) \). Then for the conductivity \( \tilde{\gamma} \) defined 
in 2. there are unique \( \theta \in L^\infty(\Omega_m, S^1) \) and \( \eta \in L^\infty(\Omega_m, \mathbb{R}^+) \) such that \( \tilde{\gamma} = \tilde{\gamma}_{\lambda, \theta, \eta} \).

Theorem 3.2 means that we can find a unique conductivity in \( \Omega_m \) that is as close as possible 
to being isotropic and has the special form \( \tilde{\gamma} = \tilde{\gamma}_{\lambda, \theta, \eta} \). The theorem has a following corollary 
claiming that we can find the function \( \det(\tilde{\gamma}(x))^{1/2} \) in \( \Omega_m \) that represents a deformed image 
of original conductivity \( \gamma \). We emphasize that the deformation depends only on the error made in 
modeling the boundary, not on the conductivity in \( \Omega \). This is important in medical imaging if 
changes inside the body are to be imaged.

Corollary 3.3 Let \( \Omega, \Omega_m \) and \( f_m \) be as in Theorem 3.2. There is a unique map 
\[
F_e : \Omega \to \Omega_m, \quad F_e|_{\partial \Omega} = f_m
\]
depending only on \( f_m : \partial \Omega \to \partial \Omega_m \) such that for any isotropic conductivity \( \gamma \) in \( \Omega \) satisfying 
asumptions of Theorem 3.2 the corresponding minimally anisotropic conductivity \( \tilde{\gamma} \) satisfies 
\[
\det(\tilde{\gamma}(x))^{1/2} = \gamma(F_e^{-1}(x)), \quad x \in \Omega_m.
\]

The proof of Theorem 3.2 and corollary 3.3 presented in [13, 14] are based on the theory of 
quasiconformal maps. The quasiconformal maps have also a geometric definition. They are 
generalizations of conformal maps that take infinitesimal disks at \( z \) to infinitesimal disks at 
\( f(z) \), and the radii get dilated by \( |f'(z)| \). Analogously, a homeomorphic map is quasiconformal 
on a domain \( \Omega \) if infinitesimal disks at any \( z \in \Omega \) get mapped to infinitesimal ellipsoids at \( f(z) \). 
The ratio of the larger semiaxis to smaller semiaxis is called the dilation of \( f \) at \( z \), and the 
supremum of dilatations over \( \Omega \) is the maximal dilation. This dilatation of infinitesimal discs is 
in fact the reason why isotropic conductivities change to anisotropic ones in push-forwards with 
quasiconformal maps. The dilatation of \( F \) at \( x \) is the absolute value of the Beltrami coefficient 
of \( F \) defined by 
\[
\mu(x) = \frac{\overline{\partial}F(x)}{\partial F(x)}, \quad (15)
\]
where the complex derivatives are \( \partial = \frac{1}{2}(\overline{\partial} - i \overline{\partial} \eta) \), \( \overline{\partial} = \frac{1}{2}(\overline{\partial} + i \overline{\partial} \eta) \). An important ingredient 
of the proof of Theorem 3.2 is the result of Strebel [32], that roughly speaking says that among 
all quasiconformal self–maps of the unit disk to itself with a given sufficiently smooth boundary 
value there is a unique one with the minimal maximal dilation. Also, one observers that if \( \sigma \) is an 
isotropic conductivity and \( F : \Omega \to \Omega_m \) is a quasiconformal map, then the maximal anisotropy 
of \( F_\sigma \sigma \) satisfy 
\[
K_{\Omega_m}(F_\sigma \sigma) = \|\mu_F\|_{L^\infty(\Omega)}, \quad (16)
\]
where \( \mu_F \) is the Beltrami coefficient of \( F \). Thus the problem of finding the least anisotropic 
conductivity is closely related to the problem of finding a quasiconformal map with minimal
dilatation. One should note that even if the original conductivity \( \gamma \) is \( C^\infty \) smooth, it is possible that the minimally anisotropic conductivity \( \hat{\gamma} \) is only \( L^\infty \) smooth. The reason for this is that the Beltrami coefficient \( \mu_{F_e} \) of the quasiconformal map \( F_e \) with minimal dilatation is of the form

\[
\mu_{F_e}(z) = \|\mu_{F_e}\|_{L^\infty(\Omega)} \frac{\phi(z)}{|\phi(z)|},
\]

(17)

where \( \phi : \Omega \to \mathbb{C} \) is holomorphic when \( \Omega \) is considered as a subset of \( \mathbb{C} \), and has thus discrete set of zeros. Near these zeros \( \mu_{F_e} \) is not continuous implying by (15) that \( F \) is not \( C^1 \) but only Lipschitz. Thus \( (F_e)\circ\gamma \) is not even continuous but possibly only \( L^\infty \)-smooth. Because of this non-smoothness of conductivities, the last important ingredient is the uniqueness of anisotropic inverse problem \cite{2}. To formulate this result, we use the notation

\[
\Sigma(\Omega) = \{ \sigma \in L^\infty(\Omega; \mathbb{R}^{2\times2}) : \sigma \text{ is symmetric and } C_0(\sigma) < \infty \},
\]

where \( C_0(\sigma) \) is defined in (1).

**Theorem 3.4 (Astala-Lassas-Päivärinta)** Let \( \Omega \subset \mathbb{R}^2 \) be a simply connected bounded domain and \( \sigma \) be a symmetric matrix, \( \sigma \in L^\infty(\Omega; \mathbb{R}^{2\times2}) \). Suppose that the assumptions (1) are valid. Then the Dirichlet-to-Neumann map \( \Lambda_\sigma \) determines the equivalence class

\[
E_\sigma = \{ \sigma_1 \in \Sigma(\Omega) \mid \sigma_1 = F_\ast \sigma, \ F : \Omega \to \Omega \text{ is } W^{1,2}-\text{diffeomorphism and } F|_{\partial\Omega} = I \}.
\]

We note that Theorem 3.2 makes it possible to build an algorithm to find the conductivity \( \hat{\gamma} \). Indeed, the conductivity \( \sigma = \tilde{\gamma}_{\lambda, \eta, \theta} \) can be obtained by solving the minimization problem

\[
\min_{(\lambda, \theta, \eta) \in S} \lambda, \quad \text{where } S = \{ (\lambda, \theta, \eta, z) : R\tilde{\gamma}(\lambda, \theta, \eta, z) = \tilde{R} \}.
\]

In implementation of the algorithm (see \cite{13}) we can approximate this problem by

\[
\min_{\lambda, \theta, \eta, z} \|R\tilde{\gamma}(\lambda, \theta, \eta, z) - \tilde{R}\|^2 + \varepsilon_1|\lambda - 1|^2 + \varepsilon_2(\|\theta\|^2_{W^{s,p}(\Omega_m)} + \|\eta\|^2_{W^{s,p}(\Omega_m)}),
\]

where \( \varepsilon_1, \varepsilon_2 > 0 \) are small numbers and \( W^{s,p}(\Omega_m) \) are Sobolev spaces. The numerical results of such an algorithm can be found in \cite{13}. For an example on the numerical results, see Figure 2.

**4. Higher dimensional case**

In higher dimensions the results are very different. This has several reasons. First of all, the non-uniqueness due to anisotropy is not understood, except in the case when both the domain and the conductivity function are real analytic (\cite{18}, \cite{21}, \cite{20}). Because of this the previous results for the case when the boundary is not known need that both the conductivity and the domain are real analytic. For such techniques, see e.g. \cite{23}. Also, as we already mentioned, in the plane case one could use the theory of quasiconformal maps to break the non-uniqueness. The higher dimensional analogue of this is unknown. Finally, there is no analogue of the Riemann mapping theorem that we could use. Due to the above difficulties, in higher dimensional case we have to assume that the domain is convex and that in addition to the previously used data we know the contact impedances on boundary. In discrete model, this corresponds measuring separately the contact impedances of the electrodes. With this data, we can also determine the domain up to translation and rotation.
Figure 2. Numerical test case with EIT data from an arbitrary domain $\Omega$. Top left: True conductivity distribution $\gamma$. Top right: Reconstruction of the conductivity $\gamma$ with isotropic model in the correct geometry $\Omega$. Bottom left: Reconstruction of $\gamma$ with the isotropic model in incorrectly modeled geometry. The reconstruction domain $\Omega_m$ was the unit disk. Bottom right: Reconstruction of the parameter $\eta$ with the uniformly anisotropic model in the same unit disk geometry.

Theorem 4.1 [14] Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded, strictly convex, $C^\infty$-domain. Assume that $\gamma \in C^\infty(\Omega)$ is an isotropic conductivity, $z \in C^\infty(\partial \Omega)$, $z > 0$ is the contact impedance, and $R_{\gamma,z}$ the corresponding Robin-to-Neumann map. Let $\Omega_m$ be a model of the domain satisfying the same regularity assumptions as $\Omega$, and $f_m : \partial \Omega \to \partial \Omega_m$ be a $C^\infty$-smooth orientation preserving diffeomorphism. Assume that we are given $\partial \Omega_m$, the values of the contact impedance $z(f_m^{-1}(\tilde{x}))$, $\tilde{x} \in \partial \Omega_m$ and the map $\tilde{R} = (f_m)_* R_{\gamma,z}$. Then we can determine $\Omega$ up to a rigid motion $T$ and the conductivity $\gamma \circ T^{-1}$ on the reconstructed domain $T(\Omega)$.

We recall also that rigid motion is an affine isometry $T : \mathbb{R}^n \to \mathbb{R}^n$. To describe the idea of the proof, let $\gamma$ be the isotropic conductivity on $\Omega$, and consider $C^\infty$ diffeomorphism $F_m : \Omega \to \Omega_m$ satisfying $F_m|_{\partial \Omega} = f_m$, and $\tilde{\gamma} = (F_m)_* \gamma$. Let $\tilde{g}_{jk}$ be the metric in $\Omega_m$ corresponding to the conductivity $\tilde{\gamma}$, that is, $\tilde{\gamma}^{jk} = (\det(\tilde{g}_{jk}))^{1/2} \tilde{g}^{jk}$ where $\tilde{g}^{jk}$ is the inverse matrix of $\tilde{g}_{jk}$. First, the map $\tilde{R}$ has representation $\tilde{R} = R_{\tilde{\gamma},\tilde{z}}$ and one can show that it determines the contact impedance $\tilde{z}$ and the metric $\tilde{g}$ on boundary $\partial \Omega_m$. Using knowledge of $\tilde{z}(x)$ and $z(f_m^{-1}(x))$ we can determine $\gamma \circ f_m^{-1}$ on boundary $\partial \Omega_m$. Then, we can find on $\partial \Omega_m$ the metric corresponding to the Euclidean metric of $\partial \Omega$. By the Cohn-Vossen rigidity theorem, intrinsically isometric $C^2$-smooth surfaces that are boundaries of a strictly convex body are congruent in a rigid motion. For uniqueness,
see e.g. [28, Thm. V and VI] and also [10, 11]. Thus we can find the strictly convex set \( \Omega \) up to a rigid motion \( T \). After these steps, we can reduce the problem to that an isotropic inverse problem in a known domain \( T(\Omega) \) that can be solved.

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