Thermal fluctuations of charged black hole solution in Rastall theory

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We study the thermodynamics of a charged black hole surrounded by a perfect fluid in Rastall theory and investigate three different cases of quintessence, dust and radiation fields. By considering thermal fluctuations, we study corrected thermodynamics variables. We investigate the effects of thermal fluctuations on the black hole stability, phase transition and critical points. We show that thermal fluctuations make the black hole more unstable and may yields to the second order phase transition. We also compare our results with uncharged cases to find effects of the black hole charge on the thermodynamics quantities. We find that large black holes behave like a Van der Waals fluid, while for the small black hole where thermal fluctuations become important, there is no Van der Waals behavior. Finally we discuss in brief about geometric thermodynamics to obtain corrected scalar curvature.

\textbf{Keywords:} Black holes; Thermodynamic; Quantum corrections.

\section{I. INTRODUCTION}

In order to explain an accelerating phase of the universe, a generalization to Einstein general relativity (GR) is proposed by introducing the coupling between matter and gravitational fields in a non-minimal way, which violets energy-momentum conservation in the curved space-time \cite{1, 2}. In this modified GR, the covariant divergence of energy-momentum tensor should depend on the space-time curvature through a coupling parameter so that energy-momentum conservation can be recovered in the weak gravitational field limit or Minkowski flat space-time. In this consideration,

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the Einstein’s equation get modified to

\[ G_{\mu\nu} + \kappa \lambda g_{\mu\nu} R = \kappa T_{\mu\nu}, \]  

where \( \lambda \) is called the Rastall parameter. This field equation has found many exact solutions for both, astrophysical and cosmological frameworks [3–9]. In recent years, Rastall theory of modified GR has found great attention to many researches [10–12]. On the other hand, black holes are important objects which considered by astrophysics, cosmology and particle physics. In that case rotating black hole in Rastall theory considered already by the Ref. [13]. One of the best way to obtain information about black holes is thermodynamics study. In that case thermodynamics of black holes in Rastall gravity studied by the Ref. [14] and found that there is a lower bound for the horizon radius which satisfied by black holes stability condition.

One of the important issue in the black hole thermodynamics is consideration of the thermal fluctuations [15, 16]. Thermal fluctuations are due to the statistical perturbations which may be interpreted as quantum effects. Because these are important when the black hole size decreases due to the Hawking radiation [17, 18]. It means that when the black hole will be sufficiently small, then its temperature will be so large which can not omit the thermal fluctuations effect [19].

Investigation of quantum effects in a strong gravitational system is so important and interesting. Hence, study of thermal fluctuations in a given black hole help us to understand quantum theory of gravity. Recently, this subject take more attention so there are several works where thermal fluctuations of a given black hole has been studied [20–29]. Now, we would like follow Ref. [12] to study thermal fluctuations on thermodynamics of the black hole surrounded by perfect fluids in Rastall theory of gravity. We present this paper in following manner. In section II, we discuss a black hole solution surrounded by a perfect fluid in Rastall theory. Also, we shed light on the thermodynamics of this black hole solution in three special cases of perfect fluid, namely, quintessence, dust and radiation field. In section III, we give details of corrected entropy due to small statistical disturbance around equilibrium. Section IV is devoted to study the effects of thermal fluctuation on the various thermodynamical equations of state. The critical points and stability of such black hole are emphasized in section V. In section VI we briefly discuss about geometrothermodynamics to obtain thermal fluctuation effects on the black hole stability. Finally, we summarize results of our investigation in the last section.
II. BLACK HOLE SOLUTION

The metric of a charged black hole with mass $M$ and charge $Q$ surrounded by perfect fluid in Rastall theory with Rastall gravitation coupling constant $k$ and $\lambda$ given by [30],

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_s}{r^{\Lambda_s}}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_s}{r^{\Lambda_s}}\right)} + r^2 d\Omega^2,$$

where $N_s$ denotes the surrounding field structure parameter with the equation of state parameter $\omega_s$, and

$$\Lambda_s = \frac{1 + 3\omega_s - 6k\lambda(1 + \omega_s)}{1 - 3k\lambda(1 + \omega_s)}.$$  \hspace{1cm} (3)

The special case of $\lambda = 0$ and $k = 8\pi G$ reduced to the Reissner-Nordström black hole surrounded by a surrounding field [31].

The black hole event horizon $r_+$ given by the largest root of the following relation,

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_s}{r^{\Lambda_s}} = 0.$$  \hspace{1cm} (4)

In that case the black hole entropy given by,

$$S = \pi r_+^2.$$  \hspace{1cm} (5)

Also, the black hole temperature obtained by the following relation,

$$T = \frac{1}{4\pi} \frac{d}{dr} \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_s}{r^{\Lambda_s}}\right)_{r=r_+},$$  \hspace{1cm} (6)

By using the equation (4) the black hole mass given by,

$$M = \frac{r_+}{2} + \frac{Q^2}{2r_+} - \frac{N_s}{2r_+^{\Lambda_s-1}}.$$  \hspace{1cm} (7)

Also chemical potential (electrostatic potential) conjugated with the black hole charge is given by,

$$\Phi = \frac{Q}{r_+}.$$  \hspace{1cm} (8)

In that case the first law of black hole thermodynamics given by,

$$dM = T dS + \Phi dQ + \Theta dN_s,$$  \hspace{1cm} (9)
where
\[ \Theta = -\frac{1}{\Lambda_s - 1}, \] (10)
is generalized force corresponds to the surrounding field structure parameter \( N_s \). We can find that the first law of thermodynamics satisfied if we have the following condition,
\[ Q = \left[ \int (r^2 - \Lambda_s \frac{dN_s}{dr^+}) dr + c \right]^{\frac{1}{2}}, \] (11)
where \( c \) is an integration constant. It is clear that for the constant \( Q \) and \( N_s \) (as it is the case we will consider) the first law of thermodynamics satisfied.

A. Quintessence field

The black hole surrounded by the quintessence field given by setting \( \omega_s \equiv \omega_q = -\frac{2}{3} \) or \( \omega_q = -\frac{1}{3} \). Both of them yields to the similar results in our calculations. Hence, we only consider the case of \( \omega_q = -\frac{2}{3} \). In the case of \( k\lambda = \frac{1}{2} \) we have \( \Lambda_s \equiv \Lambda_q = -2 \). In that case the equation (11) reduced to the following relation,
\[ 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - N_q r^2 = 0. \] (12)

In the plots of Fig. 1 we can see horizon structure of the black hole surrounded by the quintessence field in Rastall theory. We can see extremal case corresponding to \( N_q = 0 \) and \( M = Q \), while there is naked singularity for the \( Q > M \) and regular black hole with \( M > Q \). It is possible to have \( r_{+,q} \) and \( r_{-,q} \) which is illustrated by the Fig. 1. Hence, by using the relations (5) and (7) one can obtain the following relation for the black hole mass \[ 12],
\[ M = \frac{\pi^2 Q^2 + \pi S - N_q S^2}{2\pi^2 \sqrt{S}}. \] (13)

In that case the black hole temperature given by,
\[ T = \frac{\pi S - 3N_q S^2 - \pi^2 Q^2}{4\sqrt{\pi^3 S^3}}. \] (14)

Also, the heat capacity given by,
\[ C = \frac{2S(\pi S - 3N_q S^2 - \pi^2 Q^2)}{3\pi^2 Q^2 - \pi S - 3N_q S^2}. \] (15)

In the Ref. \[ 12 \] it has been argued that the stable-unstable phase transition happen.
FIG. 1: Horizon structure of the black hole surrounded by the quintessence field with $\omega_q = -\frac{2}{3}$ and $k\lambda = \frac{1}{4}$.

(a) The case of $N_q = 0$. Solid red line drawn for $M = Q = 1$ which is extremal case. (b) Variation of quintessence parameter $N_q$ for $M = Q = 1$.

B. Dust field

The black hole surrounded by the dust field given by setting $\omega_s \equiv \omega_q = 0$ [30]. Also, we assume $k\lambda = \frac{2}{9}$ and find that the equation (4) reduced to the following relation,

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} - N_dr = 0,$$

where $N_s \equiv N_d$ is the dust field parameter and $\Lambda_s \equiv \Lambda_d = -1$. In that case the black hole event horizon radius given by,

$$r_{+,d} = \frac{X^\frac{1}{3}}{6N_d} + \frac{2(1 - 6MN_s)}{3N_dX^\frac{1}{3}} + \frac{1}{3N_d},$$

where

$$X = 108Q^2N^2 + 12\sqrt{3}\sqrt{27N^2Q^4 + 32M^3N - 36MNQ^2 - 4M^2 + 4Q^2N} - 72MN + 8. \quad (18)$$

Therefore, by using the relations [15] and [17] one can obtain the following relation for the black hole mass [12],

$$M = \frac{\pi^2Q^2 + \pi S - N_d\sqrt{\pi S^3}}{2\pi^\frac{3}{2}\sqrt{S}}. \quad (19)$$
In that case the black hole temperature given by,

\[ T = \frac{\pi S - 2N_d\sqrt{\pi S^3} - \pi^2 Q^2}{4\sqrt{\pi^3 S^3}}. \]  
(20)

Also, the heat capacity given by,

\[ C = \frac{2S(\pi S - 2N_d\sqrt{\pi S^3} - \pi^2 Q^2)}{3\pi^2 Q^2 - \pi S}. \]  
(21)

This case also leads to the black hole phase transition.

C. Radiation field

The black hole surrounded by the radiation field given by setting \( \omega_s \equiv \omega_r = \frac{1}{3} \). Assuming \( k\lambda = \frac{2}{3} \), the equation (4) takes the following form:

\[ 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_r}{r^2} = 0, \]  
(22)

where \( N_s \equiv N_r \) is the radiation field parameter and \( \Lambda_s \equiv \Lambda_r = 2 \). Hence, we can obtain,

\[ r_{\pm,r} = M \pm \sqrt{M^2 - Q^2 + N_r}. \]  
(23)

It means that the extremal case given by \( M^2 = Q^2 - N_r \), while naked singularity happen for \( M^2 + N_r < Q^2 \). Therefore, by using the relations (5) and (7) we obtain the following expression for the black hole mass corresponding to radiation field:

\[ M = \frac{S + \pi Q^2 - \pi N_r}{2\sqrt{\pi S}}. \]  
(24)

In this case, the Hawking temperature is calculated by

\[ T = \frac{S - \pi Q^2 + 3\pi N_r}{4\sqrt{\pi S^3}}. \]  
(25)

The heat capacity in this case is found by,

\[ C = \frac{2S(S - \pi Q^2 + 3\pi N_r)}{\pi Q^2 - S - 9\pi N_r}. \]  
(26)

In that case, it is clear from the condition (11) that the first law of thermodynamics satisfied if \( Q^2 = N_r + c \).
III. THERMAL FLUCTUATIONS

In this section, we discuss how thermal fluctuations around the thermal equilibrium affect the entropy of the black holes surrounded by perfect fluid in Rastall gravity. The density of states in terms of the partition function $Z(\beta)$ describing a black hole is written by

$$\rho(E) = \frac{1}{2\pi i} \int_{\beta_0-i\infty}^{\beta_0+i\infty} d\beta Z(\beta) e^{\beta E} = \frac{1}{2\pi i} \int_{\beta_0-i\infty}^{\beta_0+i\infty} d\beta e^{S(\beta)}.$$  \hspace{1cm} (27)

where $\beta = \frac{1}{T}$ since Boltzmann constant is unit here and $S = \ln Z(\beta) + \beta E$ is exact entropy for the black hole. For small sized black hole, we expand entropy around equilibrium and apply method of steepest descent (where $\frac{dS}{d\beta} = 0$ and $\frac{d^2S}{d\beta^2} > 0$) to get

$$S(\beta) = S + \frac{1}{2} (\beta - \beta_0)^2 \frac{d^2S}{d\beta^2} \bigg|_{\beta=\beta_0} + \text{(higher order terms)},$$  \hspace{1cm} (28)

where $S$ represents equilibrium entropy. By utilizing relations (28) and (27), we have

$$\rho(E) = e^S \sqrt{\frac{e^S}{2\pi \frac{d^2S}{d\beta^2}}}.$$  \hspace{1cm} (29)

Eventually, the logarithm of this density of states leads to the microcanonical entropy

$$S = S - \frac{1}{2} \ln \left( \frac{d^2S}{d\beta^2} \right).$$  \hspace{1cm} (30)

It is usual to add a constant coefficient to the logarithmic terms to track correction terms. But here, because our main results are based on numerical analysis, we don’t need to such constant coefficient. Below, we exam above entropy for different fields.

A. Quintessence field

In this case, entropy (30) reduced to the following expression,

$$S = \pi (r_{+q})^2 - \frac{1}{2} \ln \left( \frac{(3N_q(r_{+q})^4 -(r_{+q})^2 + Q^2)^3}{54\pi(r_{+q})^6 \left( N_q(r_{+q})^4 + \frac{(r_{+q})^2}{3} - Q^2 \right)^3} \right) X_q,$$  \hspace{1cm} (31)

where we defined,

$$X_q = 9N_q^2(r_{+q})^8 + 6N_q(r_{+q})^6 - (r_{+q})^4(1 + 30N_qQ^2) + 6Q^2(r_{+q})^2 - 3Q^4.$$  \hspace{1cm} (32)
B. Dust field

In this case, entropy (30) reduced to the following relation,
\[
S = \pi (r_{+,d})^2 - \frac{1}{2} \ln \left( \frac{2N \sqrt{\pi} (r_{+,d})^3 - (r_{+,d})^2 + Q^2}{2\pi^{\frac{12}{7}} (r_{+,d})^{13} (3Q^2 - (r_{+,d})^2)^3} X_d \right),
\]
where we defined,
\[
X_d = 15N Q^2 \pi^7 (r_{+,d})^{12} - 3N \pi^7 (r_{+,d})^{14} + \sqrt{\pi^{13}} (3Q^4 - 6Q^2 (r_{+,d})^2 + (r_{+,d})^4) (r_{+,d})^9.
\]

C. Radiation field

In this case, entropy (30) reduced to the following equation,
\[
S = \pi (r_{+,r})^2 - \frac{1}{2} \ln \left( \frac{27 (3N + (r_{+,r})^2 - Q^2)^3}{2\pi^3 (r_{+,r})^4 (9N + (r_{+,r})^2 - 3Q^2)^3} X_r \right),
\]
where we defined,
\[
X_r = \frac{\pi^2}{9} \left( \frac{(r_{+,r})^4}{3} + (3N - Q^2)((2r_{+,r})^2 + 3N - Q^2) \right).
\]

In the plots of the Fig. 2 we draw entropy in terms of horizon radius for three cases of quintessence, dust and radiation fields. The first plot shows the entropy of black hole surrounded by quintessence field in Rastall theory. We can see that corrected entropy may increases or decreases. Also, corrected entropy may negative for small radius. It means that in quantum level the black hole may unstable. Later we discuss about the black hole stability by analysing specific heat. We can see approximately similar behavior for the black hole surrounded by dust and radiation fields in Rastall theory which illustrated by the second and third plots. In all cases we can see an asymptotic behavior (dash dotted blue lines which is corresponding to $Q = 1$) which may be sign of phase transition. It should be verified by analyzing specific heat.

In order to compare three different cases of quintessence, dust and radiation entropy with each other we put them in single plot for $Q = 0$ and other plot for $Q = 2$ (see Fig. 3). Right plot of the Fig. 3 shows that charged system has the same behavior at small radius. In the case of $Q = 0$ (left plot of the Fig. 3) the black hole surrounded by radiation field yields to negative entropy at small radii which may be sign of instability at quantum level. It means that effects of thermal fluctuations is instability of black hole.
FIG. 2: Typical behavior of the entropy in terms of horizon radius, for $N_s = 0.02$.

FIG. 3: Typical behavior of the corrected entropy in terms of horizon radius, for $N_s = 0.02$.

IV. THERMODYNAMICS

In order to calculate thermodynamics variables we fix black hole volume (horizon radius). Having entropy and temperature, one can calculate other thermodynamics quantities like Helmholtz free energy given by,

$$ F = - \int SdT. $$

Then, internal energy given by,

$$ U = F + TS $$

(37)
Using internal energy we can obtain specific heat in constant volume,

\[ C_V = \left( \frac{\partial U}{\partial T} \right)_V. \]  

(39)

Analyzing its sign help us to study black hole stability and its asymptotic behavior tell us about the black hole phase transition.

Another interesting issue to study is behavior of \( p - V \) diagram. By using the black hole volume,

\[ V = \frac{4}{3} \pi r_+^3, \]  

(40)

one can calculate black hole pressure via the following relation,

\[ p = -\left( \frac{\partial F}{\partial V} \right)_T. \]  

(41)

Hence by drawing plot of \( p \) in terms of \( V \) we can study critical points which investigated in the next section.

By using the internal energy, pressure and black hole volume, one can obtain the enthalpy,

\[ H = U + pV, \]  

(42)

and use it to obtain Gibbs free energy,

\[ G = H - TS. \]  

(43)

Following we discuss about above points for three different cases of black hole surrounded by quintessence, dust and radiation fields.

### A. Quintessence field

Neglecting the thermal fluctuations and use the equation (37), one can obtain,

\[ F = \frac{(r_+q)^4 N_q + (r_+q)^2 + 3Q^2}{4r_+q}. \]  

(44)

In presence of thermal fluctuation we find that Helmholtz free energy increased. In some cases of charged black hole surrounded by quintessence field in Rastall theory with thermal fluctuation, there is a minimum bound for the horizon radius (see left plots of the Fig. 4) which may be sign of some instabilities which mentioned already.
FIG. 4: Typical behavior of the Helmholtz free energy in terms of horizon radius, for $N_q = 0.02$ (left plots), $N_d = 0$ (middle plots) and $N_r = 0.02$ (right plots).

B. Dust field

Neglecting the thermal fluctuations and use the equation (37), one can obtain Helmholtz free energy of black hole surrounded by dust field in Rastall theory as follow,

$$F = \left(\frac{8}{9} + \frac{3Q^2}{4r_{+,d}}\right).$$  \hspace{1cm} (45)

Surprisingly, it is independent of $N_d$, hence we draw behavior of the Helmholtz free energy in presence of thermal fluctuations for the case of $N_d = 0$ to compare with uncorrected case (see middle plots of the Fig. 4). Also, we can see apposite behavior with the previous case, now effect of thermal fluctuations are decreasing the Helmholtz free energy.

Other thermodynamics potentials like enthalpy, internal and Gibbs free energy have similar behavior.
C. Radiation field

Neglecting the thermal fluctuations and use the equation (37), one can obtain Helmholtz free energy of black hole surrounded by radiation field in Rastall theory as follow,

\[ F = \frac{(r_{+r})^2 + 3Q^2 - 9N_r}{4r_{+r}}. \]  

(46)

In that case we find that Helmholtz free energy is zero if \( r_{+r} = \sqrt{9N_r - 3Q^2} \). Combining it with the equation (23), one can obtain a condition on the black hole mass,

\[ M^4 - [2(4N_r - Q^2) + 1]M^2 + Q^2 - N_r = 0. \]  

(47)

Root of this equation give us special \( M \) where Helmholtz free energy is zero.

Effect of thermal fluctuations of this case illustrated by right plots of the Fig. 4). In this case we consider \( Q = 0.3 \), because for the cases of \( Q < 0.3 \) the situation is very similar to the uncharged black hole which drawn in upper right plot. Such behavior of free energy (a minimum with negative value) may be sign of some stabilities which verified in the next section. In the case of \( Q = 0 \) we can see new behavior. Thermal fluctuations, depend on horizon radius, may reduce or increase value of Helmholtz free energy. Charged black hole with enough charge behaves like previous case.

![Figure 5: Typical behavior of the temperature in terms of horizon radius, for \( N_s = 0.02 \).](image)

In order to find valid regions of horizon radius we draw plots of temperature with the fact that must be positive. In the plots of the Fig. 5 we can see that temperature of black holes surrounded
by dust and radiation fields are positive for $r_+ < 10$ hence for the selected values of our parameters there is no invalid regions, while for the black hole surrounded by quintessence field (left plot of the Fig. 5) we can see that temperature is negative for $r_+ \geq 4$, hence we should focus to this region.

V. CRITICAL POINTS AND STABILITY

In order to study black hole stability we should look at the specific heat at constant volume \[39\]. If $C > 0$, black hole is stable, and if $C < 0$, then black hole is unstable, while asymptotic behavior of the specific heat show phase transition. We can also check following conditions \[34\],
\[
\frac{\partial p}{\partial V} = 0, \\
\frac{\partial^2 p}{\partial V^2} = 0. 
\]
These relations hold at critical points which are inflection points.

A. Quintessence field

Now, we can discuss about the specific heat at constant volume, which is given by the equation \[39\] and yields to the following expression,
\[
C_V = \frac{1}{(3(r_{+q})^4 N_q + (r_{+q})^2 - 3Q^2)^2 B_q}, 
\]
where we defined,
\[
A_q = 162N_q^4 \pi (r_{+q})^{18} + 54 (2\pi - 3N_q) (r_{+q})^{16} N_q^3 - 36 \left(6(1 + 3\pi Q^2)N_q + \pi\right) (r_{+q})^{14} N_q^2 \\
+ 108 (7N_q + \pi) N_q^2 Q^2 - 90N_q^2 - 12\pi N_q \right) (r_{+q})^{12} \\
+ 252\pi N_q^3 Q^4 + 72 \left(13N_q + \frac{5\pi}{3}\right) Q^2 + 24N_q + 2\pi \right) (r_{+q})^{10} \\
- \left(12 (180N_q + 29\pi) N_q^4 Q^4 + 24 \left(10N_q + \frac{5\pi}{6}\right) Q^2 \right) (r_{+q})^8 \\
+ 12 \left(18\pi N_q (Q^6 + (52N_q + 5\pi) Q^4) (r_{+q})^6 - 6 \left(10 (3N_q + \pi) Q^2 + 1\right) (r_{+q})^4 Q^4 \right. \\
+ 6Q^6 (3\pi Q^2 - 4) (r_{+q})^2 + 18Q^8, 
\]
and
\[
B_q = 9(r_{+q})^8 N_q^2 + 6N_q(r_{+q})^6 - (1 + 30N_q Q^2)(r_{+q})^4 + 6Q^2(r_{+q})^2 - 3Q^4. 
\]
FIG. 6: Typical behavior of the specific heat in terms of horizon radius, for $N_s = 0.02$.

In the plots of the Fig. 6, we draw specific heat in terms of the black hole horizon radius to see effects of thermal fluctuations. We perform graphical analysis for both charged and uncharged cases. In the case of uncharged black hole surrounded by quintessence field, we can see completely unstable black hole without any phase transition in valid regions (see left plots of the Fig. 6). In that case, effect of thermal fluctuation is presence of a phase transition. Similar results obtain for the case of charged black hole surrounded by quintessence field in Rastall theory, however, this time effect of thermal fluctuation is presence of the second phase transition. So there is the first order phase transition in both corrected and uncorrected thermodynamics.

Our numerical analysis about critical points presented by Fig. 7. We plot $p$ in terms of $V$ for both cases of corrected (lower plots) and uncorrected (upper plots) black hole entropy. In the case of black hole surrounded by quintessence field in Rastall theory, we find that points affected by thermal fluctuations. Neglecting thermal fluctuations (upper left plot of the Fig. 7), we have critical points which denoted by thick solid green line of the upper left plot of the Fig. 7. In that case we can see Van der Waals like behavior. However, in presence of the thermal fluctuations, there is no critical
point or Van der Waals behavior. Instead we can see a maximum for the pressure of highly charged black hole.

![Graphs showing pressure vs. volume for different charged states](image)

**FIG. 7:** $p - V$ diagram, for $N_s = 0.02$.

### B. Dust field

Now, we consider black hole surrounded by dust field in Rastall theory. In absence of thermal fluctuations we can obtain the following specific heat,

$$C_V = \frac{2(r_{+d})^2 \left( \pi (r_{+d})^2 - 2N_d \sqrt{\pi} (r_{+d})^3 - \pi Q^2 \right)}{3Q^2 - (r_{+d})^2}. \quad (52)$$

In presence of thermal fluctuations, we have a complicated expression like (49), hence, we only give graphical analysis. In this case as previous we can see that effect of thermal fluctuation is existence of the second order phase transition (see middle plots of the Fig. 6). Upper plots of the Fig. 6 are corresponding to the uncharged cases, while lower plots show charged black holes.

Now, by using the equation (41) one can calculate pressure in terms of event horizon radius. Then
by using the relation (40) one can represent pressure in terms of volume. In absence of thermal fluctuations one can obtain,

$$pV^3 + f(V) = 0.$$  \hspace{1cm} (53)

If \( f(V) = a_1V^2 + a_2V + a_3 \), then we have Van der Waals equation of state, where \( a_1, a_2 \) and \( a_3 \) are some constants. However, in our case we find that,

$$f(V) \approx 0.05V^{2/3} - 0.4Q^2V^{2/3},$$  \hspace{1cm} (54)

which behaves approximately like Van der Waals equation of state. It is interesting to see that is independent of \( N_d \). This is illustrated by top middle plot of the Fig. 7. In that case, both conditions (48) yields to the following equation,

$$Y_d \equiv V_c^{2/3} - \frac{4}{3}V_c^{2/3} + \left(\frac{10}{21} - \frac{5}{7}V_c\right)Q^2\pi^{2/3}(48)^{1/3} = 0,$$  \hspace{1cm} (55)

where \( V_c \) is critical volume which yields to the critical horizon radius \( r_{+,d,c} \).

FIG. 8: Some possible values of critical volume of black hole surrounded by dust field in Rastall theory in absence of thermal fluctuations.

In the Fig. 8 we can see some possible values of critical volume of black hole surrounded by dust field in Rastall theory in absence of thermal fluctuations. For example, by choosing \( Q = 0.6 \)
one can obtain $V_c = 4$ which means $r_{+,d,c} \approx 1$. It may be larger or smaller than the event horizon radius which is also depend on $M$ and $N_d$.

In presence of thermal fluctuations, we can see similar behavior with the previous case (quintessence field), where there is no critical point or Van der Waals like behavior.

C. Radiation field

Finally, we discuss about stability of the black hole surrounded by radiation field in Rastall theory. In absence of thermal fluctuations we can obtain the following specific heat,

$$C_V = \frac{2\pi (r_{+,r})^2 \left( (r_{+,r})^2 + 3N_r - Q^2 \right)}{Q^2 - (r_{+,r})^2 - 9N},$$

while in presence of the thermal fluctuation, specific heat modified non-trivially if black hole charge be large. It is clear that the specific heat is zero if $(r_{+,r})^2 + 3N_r - Q^2 = 0$. In the top right plot of the Fig. 6 we can see that corrected and uncorrected specific heat are similar for the uncharged black hole.

In is illustrated that uncharged black hole surrounded by radiation field in Rastall theory is completely unstable without any phase transition. Also, we find that for the small charge, the same results obtained. For the large enough charge, we can see that effects of thermal fluctuation is instability of black hole.

Finally, we can investigate critical points of this model. Neglecting thermal fluctuations, one can obtain,

$$pV^3 + g(V) = 0.$$  \hspace{1cm} (57)

where

$$g(V) \approx 0.075V^7 + 1.2(N_r - 0.3Q^2)V^\frac{7}{3}. $$

In that case we find that the critical point exist about $Q = 0.4$ (for $N_r = 0.02$). It is illustrated by green thick line of top middle plot of the Fig. 7. In presence of thermal fluctuations, however, critical points may exists only for the highly charged black hole.
VI. GEOMETROTHERMODYNAMICS

In this section, following the works of [35–38] we study geometric formalism of the thermal system, and investigate the thermodynamics of a charged black hole surrounded by perfect fluids. In that case, we could calculate scalar curvature to obtain singular points. Then, we compare it with zeros of specific heat to obtain some information.

A. Quintessence field

In the Ref. [12], the relevant scalar curvature of Ruppiner formalism is found as

\[ R = -\frac{17\pi^2 Q^2 - 9S^2 N_q - 7\pi S}{4S(\pi^2 Q^2 + 3S^2 N_q - \pi S)} \]

(59)

where expression of \( S \) is given in (31). This equation still valid because the black hole temperature is not affected by thermal fluctuations and the first law of thermodynamics is valid in presence of corrections. From the left plots of the Fig. 9, it is clear that there is at least a singular point which coincides with zero of specific heat represented by left plots of the Fig. 6. In the case of charged black hole surrounded by quintessence field we can obtain information about the first and the second order phase transitions from asymptotic behaviors of scalar curvature. But in the case of uncharged black hole we have no information about the first order phase transition.

B. Dust field

In the case of black hole surrounded by dust field, the scalar curvature is obtained

\[ R = -\frac{13\pi^2 Q^2 - 4\sqrt{\pi}S^{3/2} N_d - 3\pi S}{4S(\pi^2 Q^2 + 2\sqrt{\pi}S^{3/2} N_q - \pi S)} \]

(60)

Here, \( S \) refers to corrected entropy given in the relation (33). Its behavior is illustrated by middle plots of the Fig. 9 which incidentally shows that there is no singular point for \( Q = 0 \). However, charged black holes have singular point coincident with phase transition point of the black hole (see Fig. 6).
FIG. 9: Curvature scalar of Ruppeiner metric corresponding to a black hole surrounded by perfect fluid in terms of horizon radius for $N_s = 0.02$.

C. Radiation field

For the black hole surrounded by the dust field, the scalar curvature is calculated by

$$R = \frac{39\pi Q^2 - 49S + 33\pi N_r}{4S(\pi Q^2 - S - 3\pi N_r)}, \quad (61)$$

where entropy $S$ refers to [33]. The right plots of the Fig. 9 show that there is no singular point, otherwise there is a maximum which is near the phase transition point.

VII. CONCLUSION

We have considered a charged black hole surrounded by perfect fluid in Rastall theory. Thermodynamics of such system has already been studied in the Ref. [12], where perfect fluid assumed as quintessence or dust fields. Here, we have considered radiation field also as a perfect fluid and
studied the effects of thermal fluctuations in this system. Thermal fluctuations become important when the size of black hole decreases due to the Hawking radiation and one expect that this may affect black hole stability and critical points. Therefore, in each section, we have performed our analysis separately for quintessence, dust and radiation fields. First of all we have reviewed some important properties (specially thermodynamics properties) of the system and obtain appropriate condition to verify the first law of thermodynamics. Here, we found that in the case of constant $Q$ and $N_s$ the first law of thermodynamics satisfied in absence of thermal fluctuations. However, this condition modified in presence of thermal fluctuation. We have also discussed about the horizon structure of the model and found the outer horizon which denoted by $r_+$. Then, as the main work of this paper, we obtained the corrected entropy. Correction term, which is logarithmic in nature, come from thermal fluctuations. Interestingly, we found that there is no differences between entropy of the charged black hole surrounded by quintessence, dust or radiation fields when the horizon radius is small. On the other hand, for the case of uncharged black hole, entropy of black hole surrounded by radiation field has different behavior at the small radius. By using the corrected entropy, we studied modified thermodynamics of the system described above. We have calculated Helmholtz free energy and specific heat in separate sections. Because some expressions are too large in presence of logarithmic correction, we give graphical analysis to see thermal fluctuation effects. Although the entropy of three different cases are approximately similar, but their Helmholtz free energy show different behavior. In the case of uncharged black hole surrounded by quintessence, dust and radiation fields, the effect of thermal fluctuations may be increasing, decreasing or both (depending value of the event horizon radius) for the Helmholtz free energy. In the case of charged black hole surrounded by quintessence field, the effect of thermal fluctuations is increasing for free energy with the same general behavior (linear variation with radius). On the other hand, in the cases of dust and radiation field, the effect of logarithmic correction is decreasing for the Helmholtz free energy. In all cases, there exists a minimum of the radius. For outside of this radius, the Helmholtz free energy is not defined. It may interpreted as black hole instability at small radius which is an effect of thermal fluctuations. It means that by decreasing black hole size, we have transition to the unstable phase due to thermal fluctuations. As discussed by the Ref. [12], the charged black hole surrounded by perfect fluid has one type of phase transition which is obtained by analyzing the specific heat. We have shown that the due to the effect of logarithmic term the second phase transition occurs. Hence, region of the stability modified due to the thermal fluctuation. By
analyzing critical points, we have found that the charged black hole surrounded by perfect fluid may behave like Van der Waals fluid in absence of thermal fluctuations, i.e., when the black hole event horizon radius is large. However, for the small black hole (in presence of thermal fluctuations) there is no Van der Waals behavior. We also created the Ruppiner geometric structure \[35\] for a black hole surrounded by quintessence, dust and radiation fields. We found that only in the case of the charged black hole surrounded by quintessence field, the singular point of scalar curvature coincides with zero of specific heat.

There is a possibility to generalize this work by considering higher-order corrections to the entropy and study the stability condition \[39\]–[41]. This will be the subject of future investigation.

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