Restriction estimates for the flat disks over finite fields

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Abstract. In this paper we study the restriction estimate for the flat disk over finite fields. Mockenhaupt and Tao initially studied this problem but their results were addressed only for dimensions \(n = 4, 6\). We improve and extend their results to all dimensions \(n \geq 6\). More precisely, we obtain the sharp \(L^2 \rightarrow L^{r'}\) estimates, which cannot be proven by applying the usual Stein-Tomas argument over a finite field even with the optimal Fourier decay estimate on the flat disk. One of main ingredients is to discover and analyze an explicit form of the Fourier transform of the surface measure on the flat disk. In addition, based on the recent results on the restriction estimates for the paraboloids, we address improved restriction estimates for the flat disk beyond the \(L^2\) restriction estimates.

1. Introduction

Let \(\mathbb{F}^n_q\) be an \(n\)-dimensional vector space over a finite field \(\mathbb{F}_q\) with \(q\) elements, where \(q\) is odd. Let \(n = 2d\) for an integer \(d \geq 2\). In this paper we investigate the restriction estimate for the following variety

\[(1.1) \quad \mathcal{F} := \{ (\alpha, \alpha \cdot \alpha, \beta, \alpha \cdot \beta) : \alpha, \beta \in \mathbb{F}^{d-1}_q \} \subset \mathbb{F}^{2d}_q ,\]

where \(\alpha \cdot \beta\) is the usual inner product of \(\alpha\) and \(\beta\). This variety \(\mathcal{F}\) is referred to as the flat disk over a finite field.

In 2004, Mockenhaupt and Tao \cite{24} initially studied the finite field analogue of the restriction problem for various algebraic varieties including the flat disk over finite fields. Since their work was introduced, follow-up studies have been extensively conducted, but most of them have focused on hyper-surfaces such as the paraboloid, the sphere, and the cone (see, for instance, \cite{9, 22, 19, 12, 26, 25, 10, 16, 15}). On the other hand, there are only few known concrete results on the restriction problem for the flat disk over the finite field. For example, Mockenhaupt and Tao \cite{24} addressed some partial results in the specific case when \(n = 6\), and settled it when \(n = 4\).

The purpose of this paper is to present a systematic study on restriction theory related to the flat disk, and improved results in all dimensions \(n \geq 6\). To this end, we begin by recalling notation regarding the restriction problem for the flat disk. We endow \(\mathbb{F}^n_q\) with counting measure “\(dm\)”. Let
$d\sigma$ be normalized “surface measure” on the flat disk $\mathcal{F}$ living in the dual space of $\mathbb{F}_q^n$:

$$\int_{\mathcal{F}} f(x)d\sigma(x) := \frac{1}{|\mathcal{F}|}\sum_{x \in \mathcal{F}} f(x).$$

Here, we note that $|\mathcal{F}| = q^{n-2}$, which denotes the cardinality of $\mathcal{F}$. The normalized surface measure $d\sigma$ can be interpreted as the following:

$$d\sigma(x) = \frac{q^n}{|\mathcal{F}|}1_{\mathcal{F}}(x)dx = q^21_{\mathcal{F}}(x)dx,$$

where $1_{\mathcal{F}}$ denotes the indicator function of $\mathcal{F}$ and we endow the dual space of $\mathbb{F}_q^d$ with normalized counting measure “$dx$”. Hence, we can identify $d\sigma$ as a function $q^21_{\mathcal{F}}$ on the dual space of $\mathbb{F}_q^n$.

For $1 \leq p, r \leq \infty$, we define $R^*_\mathcal{F}(p \to r)$ to be the smallest constant such that the extension estimate

$$\| (fd\sigma)^\vee \|_{L^r(\mathbb{F}_q^d, dm)} \leq R^*_\mathcal{F}(p \to r) \| f \|_{L^p(\mathcal{F}, d\sigma)}$$

holds true for all complex valued functions $f$ on $\mathcal{F}$. Here, the inverse Fourier transform of the measure $f d\sigma$ is defined by

$$(fd\sigma)^\vee (m) = \int \chi(x \cdot m) f(x)d\sigma(x) = \frac{1}{|\mathcal{F}|}\sum_{x \in \mathcal{F}} \chi(x \cdot m)f(x),$$

where $\chi$ denotes the canonical additive character of $\mathbb{F}_q$ (see Definition 2.3). By duality, $R^*_\mathcal{F}(p \to r)$ is the same as the smallest constant such that the restriction estimate

$$(1.2) \quad \| \hat{g} \|_{L^{p'}(\mathcal{F}, d\sigma)} \leq R^*_\mathcal{F}(p \to r) \| g \|_{L^p(\mathbb{F}_q^d, dm)}$$

holds for all functions $g$ on $\mathcal{F}$, where $p'$ denotes the Hölder conjugate of $p$, namely $1/p + 1/p' = 1$. The proof of the duality over a finite field can be found in Theorem 4.1 of Appendix in [11]. Recall that the Fourier transform of $g$, denoted by $\hat{g}$, is defined by

$$\hat{g}(x) = \sum_{m \in \mathbb{F}_q^d} \chi(-x \cdot m)g(m).$$

The restriction problem for $\mathcal{F}$ is to determine all exponents $1 \leq p, r \leq \infty$ such that

$$R^*_\mathcal{F}(p \to r) \lesssim 1.$$

Here, and throughout this paper, we use $A \lesssim B$ if there is a constant $C$ independent of $q$ such that $A \leq CB$. We also use the notation $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

Similar to the definition of $R^*_\mathcal{F}(p \to r)$, one can define $R^*_{V}(p \to r)$ for any algebraic variety $V$ in $\mathbb{F}_q^a, \alpha \geq 2$. We say that the $L^p \to L^r$ estimate for $V$ holds if $R^*_{V}(p \to r) \lesssim 1$.

REMARK 1.1. Using Hölder’s inequality and the nesting properties of $L^p$-norms over finite fields, one can observe that $R^*_{V}(p_1 \to r) \leq R^*_{V}(p_2 \to r)$ for $p_1 \geq p_2$, and $R^*_{V}(p \to r_1) \leq R^*_{V}(p \to r_2)$ for $r_1 \geq r_2$, which will allow us to reduce the proofs of our results to certain endpoint estimates.

Over the last few decades, various methods have been developed in the study of the restriction problem in the Euclidean setting, but only a few of them have been applied to that in the finite field setting. Among such applicable methods, the most well-known method is the Stein-Tomas argument, which enables us to deduce the “$r$” index such that $R^*_{V}(2 \to r) \lesssim 1$. To be precise, Mockenhauput and Tao [24] addressed the following formula (see the paragraph given after the proof of Lemma 1.6 in [24]).
Lemma 1.2 ([24]). Let $d\sigma_v$ denote normalized surface measure on the algebraic variety $V$ in $\mathbb{F}_q^n$, $\alpha \geq 2$. Suppose that for some integers $0 < k, s < \alpha$, it satisfies that

$$|V| \sim q^s \quad \text{and} \quad \max_{m \in \mathbb{F}_q^n \setminus \{0\}} |(d\sigma_v)^∨(m)| \lesssim q^{−\frac{k}{2}}.$$  

Then $R^*_V(2 \to r) \lesssim 1$ whenever $r \geq 2 + \frac{4(\alpha-s)}{k}$.

One curious question that naturally arises from the above lemma is whether the value of $k$, which gives the optimal Fourier decay estimate on the surface measure, yields the optimal $r$ index for the $L^2 \to L^r$ estimate for $V$. This question is the same as follows. If $|V| \sim q^s$ and $\max_{m \in \mathbb{F}_q^n \setminus \{0\}} |(d\sigma_v)^∨(m)| \sim q^{-\frac{k}{2}}$, then for all $1 \leq r < 2 + \frac{4(\alpha-s)}{k}$, is it impossible that $R^*_V(2 \to r) \lesssim 1$? In the finite field setting, it turns out that the answer is, in general, “No”. For instance, let $d\sigma_P$ be the normalized surface measure on the paraboloid $P$ in $\mathbb{F}_q^d, d \geq 2$,

$$P := \{x \in \mathbb{F}_q^d : x_1^2 + x_2^2 + \cdots + x_{d-1}^2 = x_d\}.$$  

Mockenhaupt and Tao [24] observed that for all dimensions $d \geq 2$,

$$|P| = q^{d-1} \quad \text{and} \quad \max_{m \in \mathbb{F}_q^d \setminus \{0\}} |(d\sigma_P)^∨(m)| \sim q^{-\frac{d-1}{2}}.$$  

Hence, if we invoke Lemma [1.2] by taking $\alpha = d, s = k = d - 1$, then we obtain

$$R^*_P \left(2 \to \frac{2d+2}{d-1} \right) \lesssim 1.$$  

This result is referred to as the Stein-Tomas result, which provides the sharp $L^2 \to L^r$ exponent for the piece of the paraboloid in the Euclidean case in the sense that the exponent $r$ for the estimate $R^*_V(2 \to r) \lesssim 1$ cannot be lower than $\frac{2d+2}{d-1}$. However, in the finite field setting, the Stein-Tomas exponent is not sharp except for the following specific cases:

- $d \geq 3$ is odd and $-1$ is a square number in $\mathbb{F}_q$.
- $d = 4\ell + 1$ for $\ell \in \mathbb{N}$, and $-1$ is not a square number in $\mathbb{F}_q$.

For any other cases including the even dimensions $d \geq 2$, the Stein-Tomas result can be significantly improved to much smaller exponents (for example, see Theorems [5.2] and [5.4] in Section [5]).

In the study of the restriction problem for algebraic varieties over finite fields, there are currently two important research trends.

Regarding the above question and examples, one of the main concerns is to verify the optimal $L^2 \to L^r$ estimate for a variety $V$, where the Stein-Tomas argument, Lemma 1, fails to yield the optimal $L^2 \to L^r$ estimate for $V$. Our first result below is closely related to this issue for the flat variety $F$ in $\mathbb{F}_q^n$. We establish the sharp $L^2 \to L^r$ restriction estimate for the flat disk.

Theorem 1.3. Let $F$ be the flat disk in $\mathbb{F}_q^n, n = 2d \geq 4$, defined as in (1.1). Then we have

$$R^*_F \left(2 \to \frac{2n+4}{n-2} \right) = R^*_F \left(2 \to \frac{2d+2}{d-1} \right) \lesssim 1.$$  

- By the nesting property of norms over finite fields (see Remark [1.1]), Theorem 1.3 implies that $R^*_F(2 \to r) \lesssim 1$ for all $r$ with $\frac{2n+4}{n-2} = \frac{2d+2}{d-1} \leq r \leq \infty$. Moreover, it provides the sharp $L^2 \to L^r$ restriction estimate for the flat disk. This can be shown from Lemma [2.2] in the following section. More precisely, the proof of the sharpness will be given in Remark [2.4].
Theorem 1.3 cannot be obtained from a direct application of the Stein-Tomas argument, namely Lemma 1.2. Indeed, it follows from Proposition 3.1 that \( \max_{m \in \mathbb{F}_q \setminus \{0\}} |(d \sigma)^n(m)| = q^{-\frac{d}{2}} \) with \( k = \frac{n-2}{2} \). In addition, we see that \( |\mathcal{F}| = q^{n-2} \). Hence, applying Lemma 1.2 with \( s = n-2, k = \frac{n-2}{2} \), we get \( R^*_F(2 \to \frac{2n+12}{n-2}) \lesssim 1 \), which, however, is much weaker than Theorem 1.3.

As a main idea to derive Theorem 1.3, we decompose the surface measure on \( \mathcal{F} \) as the 6 surface measures such that they have pairwise disjoint supports and each of them has a constant Fourier decay on the support, which makes our analysis much more efficient and simpler.

**Remark 1.4.** One of main ingredients to deduce Theorem 1.3 is based on the explicit Fourier transform on the surface measure \( d \sigma \) of the flat disk \( \mathcal{F} \), which will be given as Proposition 5.1. The key idea to compute it is to view the flat disk \( \mathcal{F} \) in \((1.1)\) as the set of common solutions of the following two equations:

\[
(1.5) \quad x_d = x_1^2 + x_2^2 + \cdots + x_{d-1}^2, \quad x_{2d} = x_1x_{d+1} + x_2x_{d+2} + \cdots + x_{d-2}x_{2d-2} + x_{d-1}x_{2d-1}.
\]

Then, adapting the argument by the discrete Fourier analysis due to Iosevich and Rudnev [13], we are able to relate the equations defining the flat disk to certain exponential sums, which essentially reduce to the well-understood Gauss sums.

The other interesting research trend is to deduce a new \( L^p \to L^r \) restriction estimate for \( V \) such that the exponent \( p \) is not based on \( \sim 2^n \). Here, and throughout, a new \( L^p \to L^r \) restriction result means any restriction result which cannot be obtained as a direct corollary of the optimal \( L^2 \to L^r \) estimate for \( V \).

In the Euclidean setting, various methods have been developed to induce new \( L^p \to L^r \) estimates (see, for example, [28, 30, 27, 7, 8, 29]). However, there are many limitations to the application of such techniques in the finite field. It has been considered as an extremely hard problem to deduce a new \( L^p \to L^r \) result for an algebraic variety. However, Mockenhaupt and Tao [24] proposed a new method to deduce a new \( L^p \to L^r \) estimate for the flat disk \( \mathcal{F} \) in \( \mathbb{F}_q^n \) with \( n \geq 4 \) even. More precisely, they related the problem to the Kakeya maximal estimate and the restriction estimates for the paraboloids in \( \mathbb{F}_q^{n/2} \). As a result, they addressed a new \( L^{36/13} \to L^{18/5+\varepsilon} \) estimate for any \( \varepsilon > 0 \) in the case when \( n = 6 \) and \(-1\) is not a square number of \( \mathbb{F}_q \).

Since the Mockenhauput and Tao’s work [24], much improvement on the restriction estimate for paraboloids has been made. Furthermore, the maximal Kakeya conjecture over finite fields was settled by Ellenberg-Oberlin-Tao [6]. Hence, improved new \( L^p \to L^r \) restriction estimates should be obtained. As such possible improvements have not been addressed in the literature, in this paper we will clearly indicate the improved new \( L^p \to L^r \) restriction results for the flat disk. To this end, we will formulate a proposition, which makes it possible to derive new \( L^p \to L^r \) restriction estimates for the flat disk \( \mathbb{F}_q^{2d} \) directly from restriction results for paraboloids in \( \mathbb{F}_q^d \) (see Proposition 5.1).

Now we state our new \( L^p \to L^r \) restriction estimates for the flat disk \( \mathcal{F} \) in \( \mathbb{F}_q^n = \mathbb{F}_q^d, d \geq 2 \). For even integers \( d \geq 2 \), we have the following consequences.

**Theorem 1.5.** Let \( \mathcal{F} \) be the flat disk in \( \mathbb{F}_q^{2d} \).

1. If \( d = 2 \), then \( R^*_F(4 \to 4) \lesssim 1 \).
2. If \( d = 4 \), then \( R^*_F \left( \frac{28}{11} \to \frac{28}{9} \right) \lesssim 1 \).
3. If \( d = 4 \) and \( q \) is prime, then \( R^*_F \left( \frac{8}{3} \to 3 \right) \lesssim 1 \).
4. If \( d = 6 \), then \( R^*_F \left( \frac{80+30\varepsilon}{34+15\varepsilon} \to \frac{8}{3} + \varepsilon \right) \lesssim 1 \) for all \( \varepsilon > 0 \).
(5) If \( d \geq 8 \) is even, then \( R^*_{d} \left( \frac{2d^2+2d-4}{d^2-2} \rightarrow \frac{2d+4}{d} \right) \leq 1. \)

For odd integers \( d \geq 3 \), we obtain the following restriction estimates.

**Theorem 1.6.** Let \( F \) be the flat disk in \( \mathbb{F}_q^d \).

1. If \( d = 3 \) and \( q \equiv 3 \pmod{4} \), then \( R^*_{d} \left( \frac{36-10\varepsilon}{13+5\varepsilon} \rightarrow \frac{18}{5} - \varepsilon \right) \leq 1 \) for some \( \varepsilon > 0 \).
2. If \( d = 3 \) and \( q \equiv 3 \pmod{4} \) is prime, then \( R^*_{d} \left( \frac{376+106\varepsilon}{135+53\varepsilon} \rightarrow \frac{188}{53} + \varepsilon \right) \leq 1 \) for all \( \varepsilon > 0 \).
3. If \( d \geq 3 \) is odd and \( q \equiv 1 \pmod{4} \), then \( R^*_{d} \left( \frac{2d+2}{d-1} \rightarrow \frac{2d+2}{d-1} \right) \leq 1 \).
4. If \( d = 4\ell + 1 \) with \( \ell \in \mathbb{N} \), and \( q \equiv 3 \pmod{4} \), then \( R^*_{d} \left( \frac{2d+2}{d-1} \rightarrow \frac{2d+2}{d-1} \right) \leq 1 \).
5. If \( d = 4\ell + 3 \), with \( \ell \in \mathbb{N} \), and \( q \equiv 3 \pmod{4} \), then \( R^*_{d} \left( \frac{2d^2+2d-4}{d^2-2} \rightarrow \frac{2d+4}{d} \right) \leq 1 \).

- Notice from Remark 1.1 that the smaller exponent implies the better restriction result for \( F \) in \( \mathbb{F}_q^n \), with \( n = 2d \geq 4 \). In order to deduce further results from a known restriction estimate, one can invoke the interpolation theorem (Theorem 1.2) with the trivial estimate \( R^*_{d}(1 \rightarrow \infty) \lesssim 1 \). Hence, Conjecture 2.3 in the following section shows that to settle the restriction problem for the flat disk \( F \) in \( \mathbb{F}_q^d \), with \( n = 2d \geq 4 \), it suffices to establish the critical endpoint estimate:

\[
R^*_{d} \left( \frac{2n}{n-2} \rightarrow \frac{2n}{n-2} \right) = R^*_{d} \left( \frac{2d}{d-1} \rightarrow \frac{2d}{d-1} \right) \lesssim 1.
\]

- Observe that the first part of Theorem 1.3 is the solution of the restriction problem for the flat disk \( F \subset \mathbb{F}_q^d \). This was first pointed out by Mockenhaupt and Tao \[24\] but the detail proof was not given.

- The third and fourth parts of Theorem 1.6 are not new \( L^p \rightarrow L^r \) restriction estimates for \( F \) as the sharp \( L^2 \rightarrow L^r \) result, Theorem 1.3 already implies those results. However, all other results including Theorem 1.5 are new \( L^p \rightarrow L^r \) estimates.

- As we will see from Conjecture 2.3, the conjectured exponents for \( R^*_{d}(p \rightarrow r) \) to be bounded are irrelevant of the ground field \( \mathbb{F}_q \). Hence, it is natural to expect that one can remove the conditions on \( q \) appearing in the statement of Theorem 1.6.

- One can obtain further results by interpolating the sharp \( L^2 \rightarrow L^r \) estimate of Theorem 1.3 and the results of Theorems 1.5 and 1.6. For example, the previously known estimate \( R^*_{d} \left( \frac{36}{13} \rightarrow \frac{18}{5} + \varepsilon \right) \leq 1 \) for \( n = 6 \) (or \( d = 3 \)), due to Mockenhaupt and Tao, can be improved to \( R^*_{d} \left( \frac{36}{13} \rightarrow \frac{12}{20+5\varepsilon} \right) \leq 1 \), which follows by interpolating the first part of Theorem 1.6 and the result \( R^*_{d}(2 \rightarrow 4) \lesssim 1 \), which is Theorem 1.3 for \( d = 3 \).

**Remark 1.7.** Theorems 1.5 and 1.6 are much weaker than the conjectured results (Conjecture 2.3) except for the first result of Theorem 1.5. We notice that one cannot settle this question by using our method in this paper. As we shall see, our results follow by applying the \( L^2 \rightarrow L^r \) restriction estimate for the paraboloid in \( \mathbb{F}_q^{n/2} \) (see Proposition 5.1). Even using the optimal \( L^2 \rightarrow L^r \) restriction estimate for the paraboloid, it fails to produce the conjectured result (see, for example, Remark 5.3). For this reason, it leaves the question of finding a new methodology to completely solve this problem. In addition, it would be interesting to extend our work to general quadratic surfaces of co-dimension bigger than one. In the Euclidean case, such problems have been extensively studied. We refer the reader to \[3, 1, 2, 18\]. However, it seems that the Euclidean arguments do not work in the finite field case.

The remaining part of this paper will be essentially designed to give proofs of our main results.

**Notation.** Throughout this paper, we will use the following notation:
• For any integer \( \alpha \geq 1 \), we use \( \mathbf{0} \in \mathbb{F}_q^\alpha \) to denote the zero vector in \( \mathbb{F}_q^\alpha \). In particular, we write 0 for \( \mathbf{0} \) when \( \alpha = 1 \). We write \( \delta_0 \) for the indicator function of \( \{0\} \), namely, \( \delta_0(\alpha) = 1 \) for \( \alpha = 0 \), and 0 otherwise.

• For a vector \( m \in \mathbb{F}_q^\alpha \), we write \( m_j \) to denote the \( j \)-th coordinate of \( m \). For example, we have \( m = (m_1, \ldots, m_\alpha) \in \mathbb{F}_q^\alpha \). We also define

\[
||m|| := \sum_{j=1}^\alpha m_j^2.
\]

• For a simple notation, we identify a set \( E \) with the indicator function \( 1_E \) of the set \( E \), where \( 1_E(x) = 1 \) for \( x \in E \), and 0 otherwise. For example, we write \( \hat{E} \) for \( 1_E \), the Fourier transform of the indicator function \( 1_E \).

• For \( 1 \leq r \leq \infty \), the Hölder conjugate of \( r \) is denoted by \( r' \), namely, \( 1/r + 1/r' = 1 \).

\section{Preliminaries}

Some necessary conditions for \( R^*_p(p \to r) \) to be bounded can be determined by the size of the underlying variety \( V \) and the size of any maximal affine subspace lying in \( V \).

\textbf{Lemma 2.1} (Mockenhaupt-Tao, \cite{24}). Let \( V \) be an algebraic variety in \( \mathbb{F}_q^\alpha, \alpha \geq 2 \), with the size \( |V| = q^\alpha, 0 < s < \alpha \). In addition, assume that the variety \( V \) contains an affine subspace \( H \) with the size \( |H| = q^k, 0 < k \leq s \). If \( R^*_p(p \to r) \lesssim 1 \) for some \( 1 \leq p, r \leq \infty \), then we have

\[
r \geq \frac{2\alpha}{s} \quad \text{and} \quad r \geq \frac{p(\alpha - k)}{(p-1)(s-k)}.
\]

\textbf{Proof.} The proof of the above lemma can be found on pages 41–42 in \cite{24}.

\section{1. Necessary conditions for the bound of \( R^*_p(p \to r) \).}

From now on we always assume that the flat disk \( \mathcal{F} \) is the variety lying in \( \mathbb{F}_q^n \) with \( n = 2d \geq 4 \) even integer.

\textbf{Lemma 2.2.} Suppose that \( R^*_p(p \to r) \lesssim 1 \) for some \( 1 \leq p, r \leq \infty \). Then we have

\[
r \geq \frac{2n}{n-2} \quad \text{and} \quad r \geq \frac{p(n+2)}{(p-1)(n-2)}.
\]

\textbf{Proof.} Since \( \mathcal{F} \) is contained in \( \mathbb{F}_q^n \), it is not hard to see that the cardinality of \( \mathcal{F} \) is \( q^{n-2} \). We note that when \( n = 2d, d \geq 2 \), the flat disk \( \mathcal{F} \) is the set of common solutions of the following equations:

\[
x_d = x_1^2 + x_2^2 + \cdots + x_{d-1}^2, \quad x_{2d} = x_1x_{d+1} + x_2x_{d+2} + \cdots + x_{d-2}x_{2d-2} + x_{d-1}x_{2d-1}.
\]

Setting \( H = \{0\} \times \mathbb{F}_q^{d-1} \times \{0\} \subset \mathbb{F}_q^d \times \mathbb{F}_q^{d-1} \times \mathbb{F}_q \), it is easily seen that \( H \) satisfies that \( |H| = q^{d-1} = q^{n-2} \), and is a subspace lying on the flat disk \( \mathcal{F} \). Hence, invoking Lemma \ref{2.1} with taking \( \alpha = n, s = n-2 \), and \( k = (n-2)/2 \), we obtain the required necessary conditions for the estimate \( R^*_p(p \to r) \lesssim 1 \).

It can be conjectured that the above necessary conditions are in fact sufficient conditions for the bound of \( R^*_p(p \to r) \). In other words, we conjecture the following statement.

\textbf{Conjecture 2.3.} If \((1/p, 1/r)\) is contained in the convex hull of points \((0,0), (n-2)/2n, (n-2)/2n, (n-2)/2n, \) and \((1,0)\), then

\[
R^*_p(p \to r) \lesssim 1.
\]
Remark 2.4. Theorem [13] shows that the above conjecture is true when the $p$ index is 2. Indeed, letting $x = 1/p, y = 1/r$, the line segment passing through $(1, 0)$ and $(\frac{n-2}{2n}, \frac{n-2}{2n})$ is given by the equation

$$y = -\frac{(n-2)x}{n+2} + \frac{n-2}{n+2}, \quad \left(\frac{n-2}{2n} \leq x \leq 1\right).$$

Hence, when $x = 1/p = 1/2$, it follows by a direct computation that $y = 1/r = \frac{n-2}{2n+4}$, which is exactly corresponding to Theorem 1.3.

2.2. The standard Gauss sum. One of the ingredients to prove Theorem 1.3 will be to deduce an explicit Fourier transform of the surface measure $d\sigma$ on the flat disk $\mathcal{F}$ in $\mathbb{F}_q^n$. To do this, by means of the Fourier analysis over finite fields, we shall reduce the matter to the estimate of an explicit Gauss sum. Here we review basics for the Gauss sum.

We begin by reviewing the definition of the canonical additive character of $\mathbb{F}_q$, which is given in [23]. Let $p$ be the characteristic of $\mathbb{F}_q$ with $q = p^\ell$ for some positive integer $\ell$. The absolute trace function $Tr$ is a function from $\mathbb{F}_q$ to $\mathbb{F}_p$, defined by

$$Tr(t) = t + t^p + t^{p^2} + \cdots + t^{p^{\ell-1}}.$$  

It is shown in Section 3 of [23] that the absolute trace function is well-defined.

Definition 2.5 (Canonical additive character and the quadratic character, [23]). The function $\chi$ defined by

$$\chi(c) = e^{2\pi i Tr(c)/p} \quad \text{for } c \in \mathbb{F}_q$$

is called the canonical additive character of $\mathbb{F}_q$. On the other hand, the multiplicative character $\eta$ is a function from $\mathbb{F}_q^* \rightarrow \mathbb{R}$, defined by

$$\eta(a) = \begin{cases} 1 & \text{if } a \text{ is a square number of } \mathbb{F}_q^*, \\ -1 & \text{if } a \text{ is not a square number of } \mathbb{F}_q^*. \end{cases}$$

It is known that $\eta(-1) = -1$ if and only if $q \equiv 3 \pmod{4}$, and $\eta(-1) = 1$ if and only if $q \equiv 1 \pmod{4}$ (for example, see Remark 5.13, [23]).

The orthogonality of characters $\chi$ and $\eta$ states that

$$\sum_{t \in \mathbb{F}_q} \chi(at) = \begin{cases} q & \text{if } a = 0, \\ 0 & \text{if } a \neq 0, \end{cases} \quad \text{and} \quad \sum_{t \in \mathbb{F}_q^*} \eta(at) = 0 \quad \text{if } a \neq 0.$$

The standard Gauss sum determined by $\chi$ and $\eta$ is defined by

$$G = G(\eta, \chi) := \sum_{t \in \mathbb{F}_q^*} \eta(t)\chi(t).$$

We will invoke the following well-known property of the standard Gauss sum. Here we provide an elementary proof.

Lemma 2.6. We have

$$G^2 = G(\eta, \chi)^2 = \eta(-1)q.$$  

Proof. Since $\eta = \overline{\eta}$ and $\overline{\chi(t)} = \chi(-t)$, it is seen by a change of variables that

$$G(\eta, \chi) = \eta(-1)\overline{G(\eta, \chi)}.$$
Hence, $\mathcal{G}(\eta, \chi)^2 = \eta(-1)|\mathcal{G}(\eta, \chi)|^2$, so the problem is reduced to showing that $|\mathcal{G}(\eta, \chi)|^2 = q$. Indeed, it follows that

$$|\mathcal{G}(\eta, \chi)|^2 = \left(\sum_{a \neq 0} \eta(a) \chi(a)\right) \left(\sum_{t \neq 0} \eta(t) \chi(t)\right) = \sum_{a, t \neq 0} \eta(at^{-1}) \chi(a - t).$$

By a change of variables, letting $b = at^{-1}$ for any fixed $t \neq 0$,

$$= \sum_{t \neq 0} \sum_{b \neq 0} \eta(b) \chi((b - 1)t) = \sum_{b \neq 0} \eta(b) \left(-1 + \sum_{t \in \mathbb{F}_q} \chi((b - 1)t)\right).$$

By the orthogonality of $\chi$ and $\eta$, we obtain the required estimate $|\mathcal{G}(\eta, \chi)|^2 = q$, where we also used the simple fact that $\eta(1) = 1$. This completes the proof of the lemma.

It is not hard to note that for any non-zero $a \in \mathbb{F}_q^*$,

$$\sum_{t \in \mathbb{F}_q} \chi(at^2) = \eta(a) \mathcal{G}.$$

Completing the square and using a simple change of variables, the above formula can be generalized to the formula below: For any non-zero $a \in \mathbb{F}_q^*$ and any $b \in \mathbb{F}_q$, we have

$$(2.1) \quad \sum_{t \in \mathbb{F}_q} \chi(at^2 + bt) = \eta(a) \mathcal{G} \left(\frac{b^2}{-4a}\right).$$

2.3. Ellenberg-Oberlin-Tao Kakeya maximal theorem over finite fields. We review the connection between the restriction estimate for the flat disk $\mathcal{F}$ and the Kakeya maximal estimate over finite fields. To deduce Theorem 1.5 and Theorem 1.6, we will heavily use the connection as well as recently established restriction estimates for paraboloids in $\mathbb{F}_q^d$.

We begin with some notation related to the Kakeya maximal problem over finite fields. Consider a direction $v \in \mathbb{F}_q^{d-1}$, $d \geq 2$, and a vector $z_0 \in \mathbb{F}_q^{d-1}$. Then the line $l(z_0, v)$ in $\mathbb{F}_q^d$ is defined by

$$l(z_0, v) := \{(z_0 + tv, t) : t \in \mathbb{F}_q\} \subset \mathbb{F}_q^d.$$  

For a function $h$ on $\mathbb{F}_q^d$, the Kakeya maximal function $h^*$ is defined on $\mathbb{F}_q^{d-1}$, the space of directions:

$$h^*(v) := \max_{z_0 \in \mathbb{F}_q^{d-1}} \sum_{\eta \in l(z_0, v)} |h(\eta)|.$$

Let $dv$ denote the normalized surface measure on the space of directions, which assigns $q^{-(d-1)}$ to each point in $\mathbb{F}_q^{d-1}$.

For $1 \leq p, r \leq \infty$, we define $K(p \to r)$ to be the smallest constant such that the estimate

$$(2.2) \quad \|h^*\|_{L^r(\mathbb{F}_q^{d-1}, dv)} \leq K(p \to r) \|h\|_{L^p(\mathbb{F}_q^d, dm)}$$

holds for all functions $h$ on the space $\mathbb{F}_q^d$ with the counting measure $dm$. The Kakeya maximal problem over finite fields is to determine all exponents $1 \leq p, r \leq \infty$ such that

$$K(p \to r) \lesssim 1.$$  

This problem was initially posed by Mockenhaupt and Tao [24], and was settled by Ellenberg, Oberlin, and Tao [6], who ingeniously applied the polynomial method of Dvir [5]. More precisely, they proved the following critical estimate.
Theorem 2.7 (Ellenberg-Oberlin-Tao, [6]). Let $K(p \to r)$ be defined as in [22]. Then we have

$$K(d \to d) \lesssim 1.$$ 

Here, we refer the readers to Lewko’s paper [20], which indicates how the sharp maximal kakeya estimate can be used to deduce the restriction results beyond the Stein-Tomas result for the paraboloid, where $-1$ is a square in $\mathbb{F}_q$.

It turned out that there is a strong connection among the restriction estimate for $F$ in $\mathbb{F}_q^{2d}$, the restriction estimate for the paraboloid $P$ in $\mathbb{F}_q^d$, and the Kakeya maximal estimate in $\mathbb{F}_q^d$.

Theorem 2.8 (Theorem 9.1, [23]). Let $P$, $F$, and $K$ denote the paraboloid in $\mathbb{F}_q^d$, $d \geq 2$, the flat disk in $\mathbb{F}_q^{2d}$, and the Kakeya operator in $\mathbb{F}_q^d$, respectively. Then, for $2 \leq p, r \leq \infty$, we have

$$R_F^*(p \to r) \leq R_P^*(2 \to r)K \left( \left( \frac{r}{2} \right)' \to \left( \frac{p}{2} \right)' \right)^{1/2}.$$ 

3. The Fourier transform of the surface measure on $F$

In this section, we deduce an explicit inverse Fourier transform of the normalized surface measure $d\sigma$ on the flat disk $F$. We begin by setting up some notation.

Notation. Let $m$ denote a vector in $\mathbb{F}_q^{2d}$ with $d \geq 2$ an integer. For $i = 0, 1, 2, 3, 4, 5$, we define $\Omega_i \subset \mathbb{F}_q^d$ as follows:

- $\Omega_0 = \{ 0 \}$
- $\Omega_1 = \{ m : m_d = 0 = m_{2d}, m \neq 0 \}$
- $\Omega_2 = \{ m : m_d = 0, m_{2d} \neq 0 \}$
- $\Omega_3 = \{ m : m_d \neq 0, m_{2d} = 0, \delta_0(m_{d+1}, \ldots, m_{2d-1}) = 0 \}$
- $\Omega_4 = \{ m : m_d \neq 0, \delta_0(m_{d+1}, \ldots, m_{2d-1}, m_{2d}) = 1 \}$
- $\Omega_5 = \{ m : m_d \neq 0, m_{2d} \neq 0 \}$.

Using the above notation, the inverse Fourier transform $(d\sigma)^\vee$ of the surface measure on $F$ takes the following form.

Proposition 3.1. Let $d\sigma$ be normalized surface measure on the flat disk $F$ in $\mathbb{F}_q^n$, $n = 2d \geq 4$. Then, for any $m = (m_1, \ldots, m_d, m_{d+1}, \ldots, m_{2d}) \in \mathbb{F}_q^n$, we have

$$(d\sigma)^\vee(m) = \begin{cases} 1 & \text{if } m \in \Omega_0, \\ 0 & \text{if } m \in \Omega_1, \\ q^{1-d} \chi \left( \frac{m_1m_{d+1}+\cdots+m_{d-1}m_{2d-1}}{-m_{2d}} \right) & \text{if } m \in \Omega_2, \\ 0 & \text{if } m \in \Omega_3, \\ q^{1-d} \eta(m_d)^{d-1}(d-1)^{d-1/2} \chi \left( \frac{m_1^2+\cdots+m_{2d-1}^2+4m_{2d}m_1}{-4m_{2d}} \right) & \text{if } m \in \Omega_4, \\ q^{1-d} \chi \left( \frac{m_1m_{d+1}+\cdots+m_{d-1}m_{2d-1}}{m_{2d}} \right) \chi \left( \frac{m_1m_{d+1}+\cdots+m_{d-1}m_{2d-1}}{-m_{2d}} \right) & \text{if } m \in \Omega_5. \\ \end{cases}$$

Proof. Let us fix $m \in \mathbb{F}_q^{2d}$. Since $|F| = q^{2d-2}$, we can write by the definition of $(d\sigma)^\vee$ that

$$(d\sigma)^\vee(m) = \frac{1}{q^{2d-2}} \sum_{x \in F} \chi(m \cdot x).$$

Notice that each $x \in F$ satisfies the both equations in (1.5), so whenever we fix $x_1, \ldots, x_{d-1}, x_{d+1}, \ldots, x_{2d-1} \in \mathbb{F}_q,$
the variables $x_d$ and $x_{2d}$ are determined as in (1.5). Thus, $(d\sigma)^\vee(m)$ is
\[
\frac{1}{q^{2d-2}} \sum_{x_d+1, \ldots, x_{2d-1} \in \mathbb{F}_q} \chi(m_1 x_{d+1} + \cdots + m_{2d-1} x_{2d-1}) \prod_{i=1}^{d-1} \sum_{x_i \in \mathbb{F}_q} \chi(m_i x_i^2 + (m_i + m_d x_{d+i}) x_i).
\]
Rearranging the general term and using Fubini's theorem, we are also able to write $(d\sigma)^\vee(m)$ as
\[
\frac{1}{q^{2d-2}} \sum_{x_{d+1}, \ldots, x_{2d-1} \in \mathbb{F}_q} \chi(m_d x_{d+1} + \cdots + m_{2d-1} x_{2d-1}) \prod_{i=1}^{d-1} \sum_{x_i \in \mathbb{F}_q} \chi((m_i + m_d x_{d+i}) x_i).
\]

**Case 1:** Suppose that $m_d = 0$ and $m_{2d} = 0$ (in this case, $m \in \Omega_0$ or $m \in \Omega_1$). Then, we see by the orthogonality of $\chi$ that
\[
(d\sigma)^\vee(m) = \delta_0(m_1, \ldots, m_{d-1}, m_{d+1}, \ldots, m_{2d-1}).
\]
Hence, $(d\sigma)^\vee(m) = 1$ for $m \in \Omega_0$, and $(d\sigma)^\vee(m) = 0$ for $m \in \Omega_1$, as required.

**Case 2:** Assume that $m_d = 0$ and $m_{2d} \neq 0$, which is the case when $m \in \Omega_2$. Then it follows by the orthogonality of $\chi$ that
\[
(d\sigma)^\vee(m) = \frac{1}{q^{2d-2}} \sum_{x_{d+1}, \ldots, x_{2d-1} \in \mathbb{F}_q} \chi(m_d x_{d+1} + \cdots + m_{2d-1} x_{2d-1}) \prod_{i=1}^{d-1} \sum_{x_i \in \mathbb{F}_q} \chi((m_i + m_d x_{d+i}) x_i)
\]
\[
= \frac{q^{d-1}}{q^{2d-2}} \sum_{x_{d+1}, \ldots, x_{2d-1} \in \mathbb{F}_q: x_{d+i} = -\frac{m}{m_d}} \chi(m_d x_{d+1} + \cdots + m_{2d-1} x_{2d-1}).
\]
Hence, we obtain the desired estimate
\[
(d\sigma)^\vee(m) = q^{1-d} \chi \left( \frac{m_1 m_{d+1} + \cdots + m_{d-1} m_{2d-1}}{-m_{2d}} \right).
\]

**Case 3:** Suppose that $m_d \neq 0$ and $m_{2d} = 0$, which corresponds to either $m \in \Omega_3$ or $m \in \Omega_4$. It follows that
\[
(d\sigma)^\vee(m) = \frac{1}{q^{2d-2}} \sum_{x_{d+1}, \ldots, x_{2d-1} \in \mathbb{F}_q} \chi(m_d x_{d+1} + \cdots + m_{2d-1} x_{2d-1}) \prod_{i=1}^{d-1} \sum_{x_i \in \mathbb{F}_q} \chi(m_i x_i^2 + m_i x_i).
\]
Since $m_d \neq 0$, we can invoke the formula (2.1) to compute the product term above:
\[
\prod_{i=1}^{d-1} \sum_{x_i \in \mathbb{F}_q} \chi(m_i x_i^2 + m_i x_i) = \eta^{d-1}(m_d) g^{d-1} \chi \left( \frac{m_1^2 + \cdots + m_{d-1}^2}{-4m_d} \right).
\]
Also note by orthogonality of $\chi$ that the sum over $x_{d+1}, \ldots, x_{2d-1} \in \mathbb{F}_q$ is $q^{d-1} \delta_0(m_{d+1}, \ldots, m_{2d-1})$. Then it is seen that
\[
(d\sigma)^\vee(m) = q^{-d+1} \eta^{d-1}(m_d) g^{d-1} \chi \left( \frac{m_1^2 + \cdots + m_{d-1}^2}{-4m_d} \right) \delta_0(m_{d+1}, \ldots, m_{2d-1}).
\]
Thus, by the definitions of $\Omega_3$ and $\Omega_4$, it is clear that $(d\sigma)^\vee(m) = 0$ for $m \in \Omega_3$, and we have
\[
(d\sigma)^\vee(m) = q^{\frac{1-d}{2}} \eta^{d-1}(m_d) \eta^{(d-1)/2} (-1)^d \chi \left( \frac{m_1^2 + \cdots + m_{d-1}^2}{-4m_d} \right) \delta(m_{d+1}, \ldots, m_{2d-1})
\]
for $m \in \Omega_4$, where we used the observation that $g^{d-1} = (g^2)^{(d-1)/2} = (\eta(-1)q)^{(d-1)/2}$ by Lemma 2.6.
Case 4: Assume that \( m_d \neq 0 \) and \( m_{2d} \neq 0 \), which corresponds to the case when \( m \in \Omega_5 \). By the formula (2.1), we are able to observe that
\[
\prod_{i=1}^{d-1} \sum_{x_i, x_{d+i} \in \mathbb{F}_q} \chi(m_d x_i^2 + (m_i + m_{2d} x_{d+i}) x_i) = \eta^{d-1}(m_d) G^{d-1} \prod_{i=1}^{d-1} \chi \left( \frac{(m_i + m_{2d} x_{d+i})^2}{-4m_d} \right).
\]
Hence, \((d\sigma)^\vee(m)\) becomes
\[
\frac{1}{q^{2d-2}} \eta^{d-1}(m_d) G^{d-1} \prod_{x_{d+i} \in \mathbb{F}_q} \chi(m_{d+1} x_{d+1} + \cdots + m_{d-1} x_{2d-1}) \prod_{i=1}^{d-1} \chi \left( \frac{(m_i + m_{2d} x_{d+i})^2}{-4m_d} \right).
\]
Applying a change of variables by letting \( y_{d+i} = m_i + m_{2d} x_{d+i} \) for each \( i = 1, \ldots, d-1 \), this becomes
\[
\frac{1}{q^{2d-2}} \eta^{d-1}(m_d) G^{d-1} \prod_{y_{d+i} \in \mathbb{F}_q} \chi \left( \frac{y_{d+i}^2}{-4m_d} + m_{d+i} y_{d+i} + m_{d+i} m_i \right).
\]
Once again we apply the formula (2.1), and obtain by a direct algebra that
\[
(d\sigma)^\vee(m) = q^{-2d+2} \eta^{d-1}(-1) G^{2d-2} \chi \left( \frac{m_d^2 m_{d+1}^2 + \cdots + m_{2d-1}^2}{m_{2d}^2} \right) \chi \left( \frac{m_1 m_{d+1} + \cdots + m_{d-1} m_{2d-1}}{-m_{2d}} \right).
\]
By Lemma 2.6 notice that \( q^{-2d+2} \eta^{d-1}(-1) G^{2d-2} = q^{1-d} \). Hence, we obtain the desired estimate of \((d\sigma)^\vee(m)\) for \( m \in \Omega_5 \). We have finished the proof of the lemma.

4. Proof of the \( L^2 \to L^r \) estimate (Theorem 1.3)

The proof will proceed by modifying ideas from the Stein-Tomas argument. It should be pointed out, however, that there exist additional difficulties that arise from a lack of the Fourier decay on the flat disk. Fortunately, the obstacles will be removed by observing certain cancellation property of the Fourier transform on some part of the domain, where the Fourier decay is not good.

Notice that the Hölder conjugates of 2 and \( \frac{2n+4}{n+6} \) are 2 and \( \frac{2n+4}{n+6} \), respectively. Hence, the proof of Theorem 1.3 by duality in (1.2), will be complete once we prove the following theorem.

**Theorem 4.1.** Let \( \mathcal{F} \) be the flat disk in \( \mathbb{F}_q^n, n = 2d \geq 4 \). Then the restriction estimate
\[
\|\hat{g}\|_{L^2(\mathcal{F}, d\sigma)} \lesssim \|g\|_{L^{\frac{2n+4}{n+6}}(\mathbb{F}_q^n, dm)}
\]
holds for all functions \( g \) on \( \mathcal{F} \).

4.1. **Standard tools from harmonic analysis.** We begin by reviewing some skills of harmonic analysis over finite fields, whose proofs can be found in been Green’s lecture note [4]. Plancherel’s theorem below can be easily proven by the orthogonality of \( \chi \):
\[
\|\hat{g}\|_{L^2(\mathbb{F}_q^n, dx)} = \|g\|_{L^2(\mathbb{F}_q^n, dx)}.
\]
Here, one should note that \( g \) is a function on \( \mathbb{F}_q^n \) with the counting measure \( dm \), but its Fourier transform \( \hat{g} \) is defined on the dual space of \( \mathbb{F}_q^n \) with the normalized counting measure \( dx \). Since \( \mathbb{F}_q^n \) is isomorphic to its dual space as an abstract group, we will use the same notation \( \mathbb{F}_q^n \) for both the space \( \mathbb{F}_q^n \) and its dual space. However, there will be no confusion since each of them has been given
For functions $g_1, g_2$ on $(\mathbb{F}^d_q, dm)$, the convolution function of $g_1$ and $g_2$ is defined on $(\mathbb{F}^d_q, dm)$:

$$g_1 * g_2(m) := \sum_{m' \in \mathbb{F}^d_q} g_1(m - m')g_2(m').$$

It can be seen that $\widehat{g_1 * g_2} = \widehat{g_1}\widehat{g_2}$. Young’s inequality for convolutions states that if $1 \leq p_1, p_2, r \leq \infty$ satisfy $1/r = 1/p_1 + 1/p_2 - 1$, then

$$\|g_1 * g_2\|_{L^r(\mathbb{F}^d_q, dm)} \leq \|g_1\|_{L^{p_1}(\mathbb{F}^d_q, dm)}\|g_2\|_{L^{p_2}(\mathbb{F}^d_q, dm)}.$$  

On the other hand, if functions $f_1, f_2$ are functions on $(\mathbb{F}_q^n, dx)$ with the normalized counting measure $dx$, then the convolution of $f_1$ and $f_2$ are defined on the space $(\mathbb{F}_q^n, dx)$:

$$f_1 * f_2(x) := \frac{1}{q^n} \sum_{y \in \mathbb{F}_q^n} f_1(x - y)f_2(y).$$

A powerful tool in harmonic analysis is the interpolation theorem.

**Theorem 4.2 (Riesz-Thorin).** Let $1 \leq p_0, p_1, r_0, r_1 \leq \infty$, $p_0 \leq p_1$, and $r_0 \leq r_1$. Suppose that $T$ is a linear operator, and satisfies the following two estimates:

$$\|Tg\|_{L^{r_0}} \leq M_0\|g\|_{L^{p_0}} \quad \text{and} \quad \|Tg\|_{L^{r_1}} \leq M_1\|g\|_{L^{p_1}}.$$  

Then we have

$$\|Tg\|_{L^r} \leq M_0^{1-\theta}M_1^\theta\|g\|_{L^p}$$

for any $0 \leq \theta \leq 1$, where

$$\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \quad \text{and} \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Now we introduce the $RR^*$ method, which is a key tool for the Stein-Tomas argument. Let $R, R^*$ denote the restriction operator and the extension operator for a variety $V$ in $(\mathbb{F}_q^n, dx)$, namely, $Rg = \widehat{g}|_V$ and $R^*f = (fd\sigma_v)^\vee$. The inner product of functions is defined as follows:

$$< Rg, Rg >_{L^2(V, d\sigma_v)} := \|Rg\|^2_{L^2(V, d\sigma_v)} \quad \text{and} \quad < g, R^*Rg >_{L^2(\mathbb{F}_q^n, dm)} := \sum_{m \in \mathbb{F}_q^n} g(m)\overline{R^*Rg(m)}.$$

The $RR^*$ method states that

$$< Rg, Rg >_{L^2(V, d\sigma_v)} < g, R^*Rg >_{L^2(\mathbb{F}_q^n, dm)}.$$

Since $R^*Rg = (\widehat{gd\sigma_v})^\vee$, the $RR^*$ method implies that

$$\|Rg\|^2_{L^2(V, d\sigma_v)} = \sum_{m \in \mathbb{F}_q^n} g(m)(\widehat{gd\sigma_v})^\vee(m) = < g, (\widehat{gd\sigma_v})^\vee >_{L^2(\mathbb{F}_q^n, dm)}.$$  

**4.2. Reduction for the proof of Theorem 4.1.** We start proving Theorem 4.1. We aim to show that

$$\|\widehat{g}\|^2_{L^2(\mathcal{F}, d\sigma)} \lesssim \|g\|^2_{L^{\frac{2n+q}{n+1}}(\mathbb{F}_q^n, dm)}.$$  

It can be seen, by using the inequality (4.2) with $V = \mathcal{F}$, that

$$\|\widehat{g}\|^2_{L^2(\mathcal{F}, d\sigma)} = < g, \ast (d\sigma)^\vee >_{L^2(\mathbb{F}_q^n, dm)}.$$  

For $j = 0, 1, 2, 3, 4, 5$, define

$$K_j = (d\sigma)^\vee\Omega_j.$$
By Proposition 3.1, it is clear that

\[ (d\sigma)^\vee = \delta_0 + \sum_{j=1}^{5} K_j. \]

So the equality in (4.4) is, by the linearity of the inner product and the convolution,

\[ ||g||^2_{L^2(F, d\sigma)} = \langle g, g \ast \delta_0 \rangle + \sum_{j=1}^{5} \langle g, g \ast K_j \rangle. \]

Here, and throughout, we write a simple notation \( \langle \cdot, \cdot \rangle \) for \( L^2(\mathbb{F}_q, dm) \). Since \( g \ast \delta_0 = g \), we observe that

\[ \langle g, g \ast \delta_0 \rangle = ||g||^2_{L^2(\mathbb{F}_q, dm)} \leq ||g||^2_{L^{\frac{2n+4}{n+6}}(\mathbb{F}_q^5, dm)}, \]

where the last inequality is valid by the nesting property of the norm with the fact that \( \frac{2n+4}{n+6} \leq 2 \).

Also observe that \( K_j \equiv 0 \) for \( j = 1, 3 \), which is an easy consequence of Proposition 3.1 and the definition of \( K_j \). It follows from these observations that, to achieve our goal in (4.3), it will be enough to show that

\[ \sum_{j=2, 4, 5} | \langle g, g \ast K_j \rangle | \leq ||g||^2_{L^{\frac{2n+4}{n+6}}(\mathbb{F}_q^5, dm)}. \]

Applying Hölder’s inequality to the general term of the left hand side, the matter is reduced to establishing the following: For all \( j = 2, 4, 5 \)

\[ ||g \ast K_j||_{L^{\frac{2n+4}{n+6}}(\mathbb{F}_q^5, dm)} \leq ||g||_{L^{\frac{2n+4}{n+6}}(\mathbb{F}_q^5, dm)}. \]

(4.5)

In the following subsection, we will prove this inequality. Hence, the proof of Theorem 4.1 is complete, which in turn finishes the proof of our main theorem, Theorem 1.3.

### 4.3. Proof of the inequality (4.5).

By interpolating estimates of \( L^2 \) and \( L^\infty \) norms of \( g \ast K_j \), we will deduce the inequality (4.5). To do this, we begin by proving the following claim that for each \( j = 2, 4, 5 \), we have

\[ ||g \ast K_j||_{L^\infty(\mathbb{F}_q^5, dm)} \leq \max_{m \in \mathbb{F}_q^5} |K_j(m)| ||g||_{L^1(\mathbb{F}_q^5, dm)}, \]

and

\[ ||g \ast K_j||_{L^2(\mathbb{F}_q^5, dm)} \leq \max_{x \in \mathbb{F}_q^5} |\widehat{K}_j(x)| ||g||_{L^2(\mathbb{F}_q^5, dm)}. \]

(4.6) and (4.7)

Since \( ||g||_{L^\infty(\mathbb{F}_q^5, dm)} := \max_{m \in \mathbb{F}_q^5} |g(m)| \), the first inequality (4.6) follows immediately from Young’s inequality for convolutions. The second inequality (4.7) follows by Plancherel’s theorem, the property of the Fourier transform of convolution functions, and Hölder’s inequality:

\[ ||g \ast K_j||_{L^2(\mathbb{F}_q^5, dm)} = ||\widehat{g} \widehat{K}_j||_{L^2(\mathbb{F}_q^5, dx)} \leq \max_{x \in \mathbb{F}_q^5} |\widehat{K}_j(x)| ||g||_{L^2(\mathbb{F}_q^5, dm)}. \]

4.3.1. Proof of the inequality (4.5) for \( j = 2 \). First, let us estimate the value \( \max_{m \in \mathbb{F}_q^5} |K_2(m)| \). By Proposition 3.1, it is clear that \( ||(d\sigma)^\vee(m)|| = q^{1-d} = q^{\frac{2-n}{2}} \) for all \( m \in \Omega_2 \). Hence,

\[ \max_{m \in \mathbb{F}_q^5} |K_2(m)| = \max_{m \in \mathbb{F}_q^5} |(d\sigma)^\vee(m)||\Omega_2(m)| = q^{\frac{2-n}{2}}. \]

So, we obtain from (4.6) with \( j = 2 \) that

\[ ||g \ast K_2||_{L^\infty(\mathbb{F}_q^5, dm)} \leq q^{\frac{2-n}{2}} ||g||_{L^1(\mathbb{F}_q^5, dm)}. \]
Next, let us estimate an upper bound of $\max_{x \in \mathbb{F}_q^n} |\widehat{K}_2(x)|$. Fix $x \in \mathbb{F}_q^n$. It follows that

$$\widehat{K}_2(x) = d\sigma \ast \Omega_2(x).$$

As mentioned in Introduction, we can identify $d\sigma$ as a function $q^2 1_{\mathcal{F}}$ on $(\mathbb{F}_q^n, dx)$. Hence, it can be seen from the definition of the convolution function in (4.1) that

$$\widehat{K}_2(x) = q^2 \mathcal{F} \ast \Omega_2(x) = \frac{1}{q^{n-2}} \sum_{y \in \mathbb{F}_q^n} \mathcal{F}(x - y) \widehat{\Omega}_2(y).$$  \tag{4.9}

Now, by the definition of $\Omega_2$ and the orthogonality of $\chi$, it follows that for each $y \in \mathbb{F}_q^n = \mathbb{F}_q^{2d}$,

$$\widehat{\Omega}_2(y) = \sum_{m \in \Omega_2} \chi(-y \cdot m) = q^{2d-2} \delta_0(y_1, \ldots, y_{d-1}, y_{d+1}, \ldots, y_{2d-1}) \sum_{m_{2d} \neq 0} \chi(-y_{2d}m_{2d}) = q^{n-2} \delta_0(y_1, \ldots, y_{d-1}, y_{d+1}, \ldots, y_{2d-1})(q\delta_0(y_{2d}) - 1).$$

Combining this estimate with the above sum for $\widehat{K}_2(x)$, we get

$$\widehat{K}_2(x) = \sum_{y_{d}, y_{2d} \in \mathbb{F}_q} \mathcal{F}(x_1, \ldots, x_{d-1}, x_d - y_d, x_{d+1}, \ldots, x_{2d-1}, x_{2d} - y_{2d})(q\delta_0(y_{2d}) - 1)$$

$$= q \sum_{y_d \in \mathbb{F}_q} \mathcal{F}(x_1, \ldots, x_{d-1}, x_d - y_d, x_{d+1}, \ldots, x_{2d})$$

$$- \sum_{y_{d}, y_{2d} \in \mathbb{F}_q} \mathcal{F}(x_1, \ldots, x_{d-1}, x_d - y_d, x_{d+1}, \ldots, x_{2d-1}, x_{2d} - y_{2d}).$$

Since $\mathcal{F}(\cdot)$ denotes the indicator function of the flat disk $\mathcal{F}$, we can use the trivial estimate, $|\mathcal{F}(\cdot)| \leq 1$. Hence, it is obvious that

$$\max_{x \in \mathbb{F}_q^n} |\widehat{K}_2(x)| \leq q^2. \tag{4.10}$$

Combining this with the inequality in (4.7), we get

$$\|g \ast K_2\|_{L^2(\mathbb{F}_q^n, dm)} \lesssim q^2 \|g\|_{L^2(\mathbb{F}_q^n, dm)}.$$  \tag{4.11}

Finally, interpolating this and the estimate in (4.8), we obtain the required inequality (4.5) for $j = 2$.

4.3.2. Proof of the inequality (4.5) for $j = 5$. We will follow the same steps and obtain the same estimates as in the case of $j = 2$. To be precise, we first claim that the following estimates are valid:

$$\|g \ast K_5\|_{L^\infty(\mathbb{F}_q^n, dm)} \lesssim q^{2 - \frac{n}{2}} \|g\|_{L^1(\mathbb{F}_q^n, dm)}, \tag{4.11}$$

and

$$\|g \ast K_5\|_{L^2(\mathbb{F}_q^n, dm)} \lesssim q^2 \|g\|_{L^2(\mathbb{F}_q^n, dm)} \tag{4.12}.$$  

Let us assume that, for a moment, the above claim is true. Then interpolating the above two estimates yields the desired inequality (4.5) for $j = 5$. Hence, it remains to prove the claimed inequalities (4.11) and (4.12). However, from the inequalities (4.6) and (4.7), it will be enough to establish the following two estimates:

$$\max_{m \in \mathbb{F}_q^n} |K_5(m)| \lesssim q^{\frac{2 - n}{2}} \tag{4.13}$$

and

$$\max_{x \in \mathbb{F}_q^n} |\widehat{K}_5(x)| \lesssim q^2. \tag{4.14}$$
The first inequality (4.13) can be easily proved by the same method of deriving the inequality (4.8). The second inequality (4.14) can be also obtained in a similar way to the argument used to deduce the inequality (4.10). Indeed, as in (4.9), it follows that
\[ \hat{K}_5(x) = \frac{1}{q^{n-2}} \sum_{y \in \mathbb{F}_q^n} \mathcal{F}(x - y)\hat{\Omega}_5(y), \]
and one can observe that for each \( y \in \mathbb{F}_q^n \),
\[ \hat{\Omega}_5(y) = q^{n-2}\delta_0(y_1, \ldots, y_{d-1}, y_{d+1}, \ldots, y_{2d-1})(q\delta_0(y_d) - 1)(q\delta_0(y_{2d}) - 1). \]
Putting the above two equations together, it follows by a direct computation that
\[ \hat{K}_5(x) = q^2\mathcal{F}(x) - q \sum_{y_d \in \mathbb{F}_q} \mathcal{F}(x_1, \ldots, x_{d-1}, x_d - y_d, x_{d+1}, \ldots, x_{2d-1}, x_{2d}) \]
\[ - q \sum_{y_d \in \mathbb{F}_q} \mathcal{F}(x_1, \ldots, x_{d-1}, x_d, x_{d+1}, \ldots, x_{2d-1}, x_{2d} - y_{2d}) \]
\[ + \sum_{y_d, y_{2d} \in \mathbb{F}_q} \mathcal{F}(x_1, \ldots, x_{d-1}, x_d - y_d, x_{d+1}, \ldots, x_{2d-1}, x_{2d} - y_{2d}). \]
Since \( \mathcal{F}(\cdot) \) denotes the indicator function of the flat disk \( \mathcal{F} \), the inequality (4.14) follows immediately from this estimate.

4.3.3. Proof of the inequality (4.5) for \( j = 4 \). We will complete the proof for \( j = 4 \) using the similar argument as in the proof of the cases for \( j = 2, 5 \). However, different kinds of estimates will be required. Indeed, we begin by assuming that the following estimates hold:
\[ \|g * K_4\|_{L^\infty(\mathbb{F}_q^d, dm)} \lesssim q^{2 - \frac{n}{d}} \|g\|_{L^1(\mathbb{F}_q^d, dm)}, \]
and
\[ \|g * K_4\|_{L^2(\mathbb{F}_q^d, dm)} \lesssim q \|g\|_{L^2(\mathbb{F}_q^d, dm)}. \]
Notice that interpolating the above two inequalities, we are able to obtain the desired inequality (4.5) for \( j = 4 \). Thus, it suffices to prove the estimates (4.15) and (4.16). However, these estimates will follow immediately from the inequalities (4.17) and (4.18) with \( j = 4 \), if we are able to prove the following:
\[ \max_{m \in \mathbb{F}_q^n} |K_4(m)| \lesssim q^{2 - \frac{n}{d}} \]
and
\[ \max_{x \in \mathbb{F}_q^n} |\hat{K}_4(x)| \lesssim q. \]
Hence, our problem is reduced to establishing these estimates. Since \( n = 2d \), the first inequality (4.17) follows by applying Proposition 3.1 with the definition that \( K_4 = (d\sigma)^{\gamma} \Omega_4 \). Now, we prove (4.18). As in (4.9), we note that for each \( x \in \mathbb{F}_q^n \),
\[ \hat{K}_4(x) = \frac{1}{q^{n-2}} \sum_{y \in \mathbb{F}_q^n} \mathcal{F}(x - y)\hat{\Omega}_4(y). \]
Now, by the definition of \( \Omega_4 \) and the orthogonality of \( \chi \), we see that for each \( y \in \mathbb{F}_q^n \),
\[ \hat{\Omega}_4(y) = q^{d-1}\delta_0(y_1, \ldots, y_{d-1})(q\delta_0(y_d) - 1). \]
Inserting this into the above equation and using the definition of $\delta_0$, we have

$$\hat{K}_4(x) = \frac{1}{q^{n/2}} \sum_{y_d, \ldots, y_{2d} \in \mathbb{F}_q} \mathcal{F}(x_1, \ldots, x_{d-1}, x_d - y_d, \ldots, x_{2d} - y_{2d})(q\delta_0(y_d) - 1).$$

Using a change of variables, $z_i = x_i - y_i$ for $i = d, d+1, \ldots, 2d$, it follows that

$$\hat{K}_4(x) = \frac{1}{q^{n/2}} \sum_{z_d, \ldots, z_{2d} \in \mathbb{F}_q} \mathcal{F}(x_1, \ldots, x_{d-1}, z_d, \ldots, z_{2d})(q\delta_0(x_d - z_d) - 1)$$

$$= \frac{q}{q^{n/2}} \sum_{z_{d+1}, \ldots, z_{2d} \in \mathbb{F}_q} \mathcal{F}(x_1, \ldots, x_{d-1}, x_d, z_{d+1} \ldots, z_{2d})$$

$$- \frac{1}{q^{n/2}} \sum_{z_d, \ldots, z_{2d} \in \mathbb{F}_q} \mathcal{F}(x_1, \ldots, x_{d-1}, z_d, \ldots, z_{2d}).$$

Given $x_1, \ldots, x_{d-1}, x_d \in \mathbb{F}_q$, it is not hard to observe that the following two statements hold true by the definition of the flat disk $\mathcal{F}$:

- In the first sum above, to satisfy that $\mathcal{F}(x_1, \ldots, x_{d-1}, x_d, z_{d+1} \ldots, z_{2d}) \neq 0$, (namely, it is 1), the vector $(z_{d+1}, \ldots, z_{2d}) \in \mathbb{F}_q^d$ must be contained in the plane $z_{2d} = x_1z_{d+1} + \cdots + x_{d-1}z_{2d-1}$ with $q^{d-1}$ elements.
- In the second sum above, to satisfy that $\mathcal{F}(x_1, \ldots, x_{d-1}, z_d, \ldots, z_{2d}) = 1$, we must take $z_d = x_1^2 + \cdots + x_{d-1}^2$, and the vector $(z_{d+1}, \ldots, z_{2d}) \in \mathbb{F}_q^d$ must lie on the plane $z_{2d} = x_1z_{d+1} + \cdots + x_{d-1}z_{2d-1}$ with $q^{d-1}$ elements.

Hence, we obtain that

$$\max_{x \in \mathbb{F}_q^n} |\hat{K}_4(x)| \leq \frac{q}{q^{n/2}} q^{d-1} + \frac{1}{q^{n/2}} q^{d-1} \leq 2 \frac{q}{q^{n/2}} q^{d-1} = 2q,$$

since $n = 2d$. Thus, we finish the proof of the inequality (4.18), as required.

5. Proofs of Theorems 1.5 and 1.6

We begin by deducing the relationship of the restriction phenomena between the paraboloid and the flat disk.

**Proposition 5.1.** Let $P, \mathcal{F}$ denote the paraboloid in $\mathbb{F}_q^d, d \geq 2$, and the flat disk in $\mathbb{F}_q^{2d}$, respectively. Suppose that $R^*_P(2 \to r) \lesssim 1$ for some $\frac{2d}{d-1} \leq r \leq \infty$. Then we have

$$R^*_\mathcal{F}\left(\frac{2r(d-1)}{rd - r - 2} \to r\right) \lesssim 1.$$  (5.1)

**Proof.** We shall invoke the following facts:

$$R^*_\mathcal{F}(p \to r) \leq R^*_P(2 \to r)K\left(\frac{r}{2}\right) \to \left(\frac{P}{2}\right)^{1/2}$$

for $2 \leq p, r \leq \infty$,  (5.2)

$$K(d \to d) \lesssim 1$$

for all $d \geq 2$,  (5.3)

and

$$K(1 \to \infty) \lesssim 1$$

for all $d \geq 2$.  (5.4)
The equality (5.2) is Theorem 2.8 due to Mockenhaupt and Tao, and the estimate (5.3) is Theorem 2.7 proven by Ellenberg, Oberlin, and Tao [6]. The equality (5.4) is obvious as we have

$$|f^*(v)| = \sup_{m_0 \in \mathbb{F}_q^{d-1}} \sum_{m \in \ell(m_0,v)} |f(m)| \leq \sum_{m \in \mathbb{F}_q^d} |f(m)| = ||f||_{L^1(\mathbb{F}_q^d, dm)}.$$

Taking $p = \frac{2r(d-1)}{rd-r-2}$ in (5.2) gives us that

$$R_F^*(\frac{2r(d-1)}{rd-r-2} \to r) \leq R_F^*(2 \to r) K \left( \frac{r}{r-2} \to \frac{r(d-1)}{2} \right)^{1/2}.$$

Since $R_F^*(2 \to r) \leq 1$ with $r \geq \frac{2d}{d-1}$ by hypothesis, to complete the proof, it is enough to show that

$$K \left( \frac{r}{r-2} \to \frac{r(d-1)}{2} \right) \leq 1.$$

But this Kakeya maximal estimate follows by interpolating the estimates (5.3) and (5.2). Thus the proof is complete. □

Notice from Proposition 5.1 that a better restriction result for the flat disk $F$ in $\mathbb{F}_q^{2d}$ can be obtained by finding a $r$ index as small as possible that satisfies the $L^2 \to L^r$ restriction estimate for the paraboloid $P$ in $\mathbb{F}_q^d$. However, in recent years, considerably advanced results have been revealed for the $L^2 \to L^r$ restriction problem for the paraboloid. Here, we collect such results, which shall be combined with Proposition 5.1 to finish the proofs of Theorems 1.5 and 1.6.

In even dimensions, the following results are known.

**Theorem 5.2.** Let $P$ be the paraboloid in $\mathbb{F}_q^d$, $d \geq 2$, defined as in (1.3). Then the following estimates hold.

1. $R_F^*(2 \to 4) \leq 1$ for $d = 2$.
2. $R_F^*(2 \to \frac{28}{3}) \leq 1$ for $d = 4$.
3. $R_F^*(2 \to 3) \leq 1$ if $d = 4$ and $q$ is prime.
4. $R_F^*(2 \to \frac{8}{3} + \varepsilon) \leq 1$ for $d = 6$ and for all $\varepsilon > 0$.
5. $R_F^*(2 \to \frac{2d+4}{d}) \leq 1$ for $d \geq 8$ even.

The first part of the theorem was introduced by Mockenhaupt and Tao [24]. More general version of the first part can be found in Theorem 1.1 in [17]. The second, the fourth, and the fifth parts of the above theorem were proven by Iosevich, Koh, and Lewko (see the proof of Theorem 1.1 in [12]). The third part of the theorem due Rudnev and Shkredov was given as Theorem 1 in [25].

**5.1. Proof of Theorem 1.5.** We are able to complete the proof by combining Theorem 5.2 with Proposition 5.1. Indeed, taking $d = 2, r = 4$ in (5.1) of Proposition 5.1 yields the first part of Theorem 1.5, namely, $R_F^*(4 \to 4) \leq 1$ for $d = 2$. Now, we can take $d = 4, r = \frac{28}{3}$ in (5.1) of Proposition 5.1 so that we obtain the second part of Theorem 1.5, namely, $R_F^*(\frac{28}{11} \to \frac{28}{9}) \leq 1$ for $d = 4$. On the other hand, by putting $d = 4, r = 3$ in (5.1) of Proposition 5.1 we obtain the third part of the theorem, which states $R_F^*(\frac{18}{7} \to 3) \leq 1$ for $d = 4$ with $q$ prime. Next, taking $d = 6, r = \frac{8}{3} + \varepsilon$ in (5.1) of Proposition 5.1 we obtain the fourth part of Theorem 1.5, which states $R_F^*(\frac{80+36\varepsilon}{34+12\varepsilon} \to \frac{8}{3} + \varepsilon) \leq 1$ for all $\varepsilon > 0$. Finally, when $d \geq 8$ is even, taking $r = \frac{2d+4}{d}$ in (5.1) of Proposition 5.1 gives the fifth part of Theorem 1.5 which is $R_F^*(\frac{2d+4-4}{d^2-2} \to \frac{2d+4}{d}) \leq 1$ for $d \geq 8$ even.
Remark 5.3. The first, the third, and the fifth parts of Theorem 5.2 are sharp in the sense that each of them provides the optimal $r$ index such that $R_P^r(2 \to r) \lesssim 1$ (see, for example, Conjecture 1.2 in [14]). Hence, the first, the third, and the fifth parts of Theorem 1.3 are the best possible results that can be obtained by applying Proposition 5.1 and new ideas are required to further improve the results.

In odd dimensions, the following consequences are the best known results for the restriction estimate for the paraboloid $P$ in $\mathbb{F}_q^d$.

**Theorem 5.4.** Let $P$ be the paraboloid in $\mathbb{F}_q^d$. Then the following statements are valid.

1. If $d = 3$ and $q \equiv 3 \pmod{4}$, then $R_P^r(2 \to \frac{18}{q} - \varepsilon) \lesssim 1$ for some $\varepsilon > 0$.
2. If $d = 3$ and $q \equiv 3 \pmod{4}$ is prime, then $R_P^r(2 \to \frac{18}{q} + \varepsilon) \lesssim 1$ for all $\varepsilon > 0$.
3. If $d \geq 3$ is odd and $q \equiv 1 \pmod{4}$, then $R_P^r(2 \to \frac{2d+2}{d-1}) \lesssim 1$.
4. If $d = 4\ell + 1$ with $\ell \in \mathbb{N}$, and $q \equiv 3 \pmod{4}$, then $R_P^r(2 \to \frac{2d+2}{d-1}) \lesssim 1$.
5. If $d = 4\ell + 3$, with $\ell \in \mathbb{N}$, and $q \equiv 3 \pmod{4}$, then $R_P^r(2 \to \frac{2d+4}{d-1}) \lesssim 1$.

The first and the second parts of the above theorem were given as Theorem 1 in [19], and Theorem 5 in [21], respectively. The third and the fourth parts are consequences of the Stein-Tomas argument (see (1.4)). The fifth part of the theorem was given as Theorem 1.4 in [16].

**5.2. Proof of Theorem 1.6** As in the proof of Theorem 1.5 after combining Theorem 5.4 with Proposition 5.1, a direct computation gives the desirable results. We leave the detail to readers.

Remark 5.5. It can be seen from Conjecture 1.2 in [14] that the third and the fourth parts of Theorem 5.4 are the sharp $L^2 \to L'$ restriction estimates for the paraboloid $P$ in $\mathbb{F}_q^d$. Hence, their corresponding parts of Theorem 1.6 are the best possible results that can be achieved by applying Proposition 5.1. On the other hand, when $d \geq 3$ is odd and $-1 \in \mathbb{F}_q$ is a square number, it has been conjectured in [14] that the results of Theorem 5.3 can be improved to the estimate $R_P^r(2 \to \frac{2d+2}{d-1}) \lesssim 1$. In those cases, it may be possible to get further improvement of Theorem 1.6 by applying Proposition 5.1.

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