Quantization of Massive Non-Abelian Gauge Fields
in The Hamiltonian Path Integral Formalism

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It is argued that the massive non-Abelian gauge field theory without involving Higgs bosons may be well established on the basis of gauge-invariance principle because the dynamics of the field is gauge-invariant in the physical space defined by the Lorentz constraint condition. The quantization of the field can readily be performed in the Hamiltonian path-integral formalism and leads to a quantum theory which shows good renormalizability and unitarity.

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It is the prevailing viewpoint that in the non-Abelian case, a renormalizable and unitary massive gauge field theory could not be built up if without introducing the Higgs mechanism[1,2]. This letter will show that this viewpoint is not always true. Originally, the Proca-type Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{a} F_{\mu\nu}^{a} + \frac{1}{2} m^2 A_{\mu}^{a} A_{\mu}^{a}$$

(1)

was used, as a starting point, to establish the massive non-Abelian gauge field theory. However, as was widely discussed in the literature[3-9], this Lagrangian is not gauge-invariant and gives rise to an unrenormalizable quantum theory. Although the Lagrangian may be recast in a gauge-invariant Stueckelberg version [8-16], the scalar Stueckelberg function introduced in the mass term has no physical meaning and unavoidably much complicates the theory. In the previous studies, the Lagrangian mentioned above was considered to form a complete description for the massive gauge field dynamics. This concept, we note, actually is not reasonable because the Lagrangian contains redundant unphysical degrees of freedom.

As is well-known, a massive vector field has only three polarization states. They can completely be described by the Lorentz-covariant transverse vector potential $A_{\mu}^{a}$ such that

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{a} F_{\mu\nu}^{a} + \frac{1}{2} m^2 A_{\mu}^{a} A_{\mu}^{a}$$

(2)

In order to express the dynamics through the full vector potential $A_{\mu}^{a}$ as shown in Eq.(1), it is necessary to impose an appropriate constraint condition on the Lagrangian (1). A suitable constraint is the covariant Lorentz condition

$$\varphi^{a} \equiv \partial^{\mu} A_{\mu}^{a} = 0$$

(3)
which implies vanishing of the redundant longitudinal vector potential $A_\mu^0(x)$. It is obvious that only the Lagrangian (2), or instead, the Lagrangian (1) plus the constraint (3) can be viewed as complete for formulating the dynamics of the massive gauge field.

An important fact is that in the physical space, the dynamics of the massive gauge field is gauge-invariant. In fact, under the gauge-transformation[6]

$$
\delta A_\mu^a = D_\mu^{ab} A_\mu^b
$$

where

$$
D_\mu^{ab} = \delta^{ab} \partial_\mu - gf^{abc} A_\mu^c
$$

and the condition (3), the action given by the Lagrangian (1) is invariant

$$
\delta S = -m^2 \int d^4x \theta^a \partial_\mu A_\mu^a = 0
$$

Saying equivalently, the action made of the Lagrangian (2) is invariant with respect to the gauge-transformation as shown in Eqs.(4) and (5) with the full vector potential being replaced by the transverse one.

The constraint (3) may be incorporated in the Lagrangian (1) by the Lagrange undetermined multiplier method to give a generalized Lagrangian. In the first order formalism, this Lagrangian is written as [6.18]

$$
L = \frac{1}{4} F^{\mu\nu}_a F^a_{\mu\nu} - \frac{1}{2} F^{\mu\nu}_a (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c) + \frac{1}{2} m^2 A^a_\mu A^a_\mu - \lambda^a \partial^a F^a_\mu
$$

where $\lambda^a$ are the Lagrange multipliers. Using the conjugate variables which are defined by

$$
\pi^a_\mu(x) = \frac{\partial L}{\partial \dot{A}^a_\mu} = F^a_\mu - \lambda^a \delta_\mu^0 = \begin{cases} E^a_k, & \text{if } \mu = k = 1, 2, 3; \\ -\lambda^a = -E^a_0, & \text{if } \mu = 0 \\ \end{cases}
$$

the Lagrangian (7) may be rewritten in the canonical form

$$
\mathcal{L} = E^a_\mu \dot{\pi}^a_\mu + A^a_0 \pi^a_0 - E^a_0 \phi^a - \mathcal{H}
$$

where

$$
C^a = \partial_\mu E^a_\mu + gf^{abc} A^b_\mu E^c_k + m^2 A^a_0
$$

$$
\mathcal{H} = \frac{1}{2} (E^a_k)^2 + \frac{1}{4} (F^a_\mu)^2 + \frac{1}{2} m^2 (A^a_0)^2 + \mathcal{H}^a
$$

here $E^a_\mu = (E^a_0, E^a_k)$ is a Lorentz vector, $\mathcal{H}$ is the Hamiltonian density. In the above, the Lorentz and the three-dimensional spatial indices are respectively denoted by the Greek and Latin letters. From the stationary condition of the action constructed by the Lagrangian (7) or (9), one may derive the equations of motion as follows

$$
\dot{A}^a_k = \partial_k A^a_0 + gf^{abc} A^b_k A^c_0 - E^a_k
$$

$$
\dot{E}^a_k = \partial^i F^a_{ik} + gf^{abc} (E^b_k A^c_0 + F^b_{ki} A^c_i) + m^2 A^a_k + \partial_k E^a_0
$$
\[ C^a(x) \equiv \partial^\mu E^a_\mu + g f^{abc} A^b_k E^{ck} + m^2 A^a_0 = 0 \quad (14) \]

and the equation (3). Eqs.(12) and (13) are the equations of motion satisfied by the independent canonical variables \( A^a_0 \) and \( E^a_0 \), while, Eqs.(3) and (14) can only be regarded as the constraint equations obeyed by the constrained variables \( A^a_0 \) and \( E^a_0 \). Eq.(9) clearly shows that these constraints have already been incorporated in the Lagrangian by the Lagrange undetermined multiplier method. Especially, the Lagrange multipliers are just the constrained variables themselves in this case.

Along the general line by Dirac[17], we shall examine the evolution of the constraints \( \varphi^a \) and \( C^a \) with time. Taking the derivative of the both equations (3) and (14) with respect to time and making use of the equations of motion:

\[
\dot{A}^a_\mu (x) = \frac{\delta H}{\delta E^{a\mu}(x)} - \int d^4 y [A^b_\mu (y) \frac{\delta C^b(y)}{\delta E^{a\mu}(x)} - E^b_\mu (y) \frac{\delta \varphi^b(y)}{\delta E^{a\mu}(x)}] 
\]

\[
\dot{E}^a_\mu (x) = - \frac{\delta H}{\delta A^{a\mu}(x)} + \int d^4 y [A^b_\mu (y) \frac{\delta C^b(y)}{\delta A^{a\mu}(x)} - E^b_\mu (y) \frac{\delta \varphi^b(y)}{\delta A^{a\mu}(x)}] 
\]

which are obtained from the stationary condition of the action given by the Lagrangian (9), one may derive the following consistence equations[17,18]

\[
\{ H, \varphi^a (x) \} + \int d^4 y [\{ \varphi^a (x), C^b (y) \} A^b_\mu (y) - \{ \varphi^a (x), \varphi^b (y) \} E^b_\mu (y)] = 0 \quad (17)
\]

\[
\{ H, C^a (x) \} + \int d^4 y [\{ C^a (x), C^b (y) \} A^b_\mu (y) - \{ C^a (x), \varphi^b (y) \} E^b_\mu (y)] = 0 \quad (18)
\]

where Eqs.(3) and (14) have been used. In the above, H is the Hamiltonian defined by an integral of the Hamiltonian density shown in Eq.(11) over the coordinate \( x \) and \( \{ F, G \} \) represents the Poisson bracket which is defined as

\[
\{ F, G \} = \int d^4 x \left( \frac{\delta F}{\delta A^a_\mu (x)} \frac{\delta G}{\delta E^{a\mu}(x)} - \frac{\delta F}{\delta E^a_\mu (x)} \frac{\delta G}{\delta A^{a\mu}(x)} \right) 
\]

The Poisson brackets in Eqs.(17) and (18) are easily calculated. The results are

\[
\{ C^a (x), C^b (y) \} = D^{ab}_\mu (x) \partial^\mu \delta^4 (x - y) 
\]

\[
\{ \varphi^a (x), \varphi^b (y) \} = 0 
\]

\[
\{ C^a (x), \varphi^b (y) \} = m^2 [g f^{abc} A^b_\mu (x) - 2 \delta^{ab} \partial^\mu] \delta^4 (x - y) 
\]

\[
\{ H, \varphi^a (x) \} = \partial^k E^a_k (x) 
\]

\[
\{ H, C^a (x) \} = m^2 [\partial^\mu A^a_\mu (x) + \partial^k A^a_k (x)] 
\]

It is pointed out that by the requirement of Lorentz-covariance, in the computation of the above brackets, the second term in Eq.(10) has been written in a Lorentz-covariant form.
$$gf^{abc}A^{b\mu}_T E^c_{\mu}$$

We are allowed to do it because the added term $gf^{abc}A^{b\mu}_T E^c_{\mu}$ only gives a vanishing contribution to the term $A^b E^a_0$ in Eq. (9) due to the identity $f^{abc}A^b_0 E^c_0 = 0$. Particularly, the determinant of the matrix which is constructed by the Poisson bracket denoted in Eq. (20) is not singular. This indicates that Eqs. (17) and (18) can directly be derived from Eqs. (3) and (12)-(14). In addition, we mention that if the Hamiltonian is defined by $\hat{H} = H - A^b_0 E^a_0$, the equations (25) and (26) can also be obtained from the equations $\{\dot{H}, \varphi^a(x)\} = 0$ and $\{\dot{H}, C^a(x)\} = 0$ respectively.

Now let us turn to formulate the quantization performed in the Hamiltonian path-integral formalism for the massive non-Abelian gauge field theory. In accordance with the general procedure of the quantization\[18-20\], we firstly write the generating functional of Green’s functions via the independent canonical variables $A^a_T$ and $E^{a\mu}_T$.

$$Z[J] = \frac{1}{N} \int D(A^a_T, E^{a\mu}_T) \exp\{i \int d^4x [E^{a\mu}_T \dot{A}^a_T - \mathcal{H}^*(A^{a\mu}_T, E^{a\mu}_T) + J^{a\mu}_T A^a_T]\}$$ \hspace{1cm} (27)

where $\mathcal{H}^*(A^{a\mu}_T, E^{a\mu}_T)$ is the Hamiltonian which is obtained from the Hamiltonian (11) by replacing the constrained variables $A^{a\mu}_L$ and $E^{a\mu}_L$ with the solutions of Eqs. (3) and (14). As mentioned before, Eq. (3) leads to $A^a_L = 0$. If setting $E^{a\mu}_L(x) = \partial^\mu Q^a(x)$ where $Q^a(x)$ is a scalar function, one may get from Eq. (14) an equation obeyed by the scalar function

$$K^{ab}(x)Q^b(x) = w^a(x)$$ \hspace{1cm} (28)

where

$$K^{ab}(x) = \delta^{ab} \Box_x - gf^{abc}A^c_T(x)\partial^\mu_{\mu}$$ \hspace{1cm} (29)

and

$$w^a(x) = gf^{abc}E^{c\mu}_T(x)A^b_T(x) - m^2 A^{ab}_T(x)$$ \hspace{1cm} (30)

With the aid of the Green’s function $G^{ab}(x-y)$ (the ghost particle propagator) which satisfies the following equation,

$$K^{ac}(x)G^{cb}(x-y) = \delta^{ab}\delta^4(x-y)$$ \hspace{1cm} (31)

one may find the solution to the equation (28) as follows

$$Q^a(x) = \int d^4y G^{ab}(x-y)w^b(y)$$ \hspace{1cm} (32)

In order to express the generating functional in terms of the variables $A^a_T$ and $E^{a\mu}_T$, it is necessary to insert the following $\delta$-functional into the generating functional in Eq. (27)[18-20]
where \( M \) is the matrix whose elements are

\[
M^{ab}(x, y) = \{C^a(x), \varphi^b(y)\}
\]

which was given in Eq.(20). The relation in Eq.(33) is easily derived from Eqs.(3) and (14) by applying the property of \( \delta \)-functional. Upon inserting Eq.(33) into Eq.(27) and utilizing the representation

\[
\delta[C^a] = \int D(\eta^a / 2\pi)e^{i\int d^4x \eta^a C^a}
\]

we have

\[
Z[J] = \frac{1}{N} \int D(A^a_\mu, E^a_\mu, \eta^a)\det M\delta[\partial^\mu A^a_\mu] \times \exp\{i \int d^4x [E^a_\mu \dot{A}^a_\mu
\]

\[
+ \gamma^a C^a - \mathcal{H}(A^a_\mu, E^a_\mu) + J^{a\mu}(\bar{A}^a_\mu)]\}
\]

(36)

In the above exponential, there is a \( E^a_0 \)-related term \( E^a_0(\partial_0 A^a_0 - \partial_0 \eta^a) \) which permits us to perform the integration over \( E^a_0 \), giving a \( \delta \)-functional \( \delta[\partial_0 A^a_0 - \partial_0 \eta^a] = \det[\partial_0]^{-1}\delta[A^a_0 - \eta^a] \). The determinant \( \det[\partial_0]^{-1} \), as a constant, may be put in the normalization constant \( N \) and the \( \delta \)-functional \( \delta[A^a_0 - \eta^a] \) will disappear when the integration over \( \eta^a \) is carried out. The integral over \( E^a_0 \) is of Gaussian-type and hence easily calculated. After these manipulations, we arrive at

\[
Z[J] = \frac{1}{N} \int D(A^a_\mu, E^a_\mu, \eta^a)\det M\delta[\partial^\mu A^a_\mu]\exp\{i \int d^4x [\frac{1}{4} F^{a\mu\nu} F^{a}_{\mu\nu}
\]

\[
+ \frac{1}{2} m^2 A^{a\mu} A^a_\mu + J^{a\mu}(\bar{A}^a_\mu)]\}
\]

(37)

When employing the familiar expression\(\text{[18,19]}\)

\[
det M = \int D(C^a, \bar{C}^a)\gamma^a(x)M^{ab}(x,y)C^b(y)
\]

(38)

where \( C^a(x) \) and \( \bar{C}^a(x) \) are the ghost field variables and the following limit for the Fresnel functional

\[
\delta[\partial^\mu A^a_\mu] = \lim_{\alpha \to 0} C[\alpha]e^{-\frac{\alpha}{2\alpha} \int d^4x (\partial^\mu A^a_\mu)^2}
\]

(39)

where \( C[\alpha] \sim \prod_x (\frac{1}{2\alpha})^{1/2} \) and supplementing the external source terms for the ghost fields, the generating functional in Eq.(37) is finally given in the form

\[
Z[J, \bar{K}, K] = \frac{1}{N} \int D(A^a_\mu, C^a, \bar{C}^a)\exp\{i \int d^4x [\mathcal{L}_{eff} + J^{a\mu} A^a_\mu + K^a C^a + \bar{C}^a K^a]\}
\]

(40)

where

\[
\mathcal{L}_{eff} = -\frac{1}{4} F^{a\mu\nu} F^{a}_{\mu\nu} + \frac{1}{2} m^2 A^{a\mu} A^a_\mu - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 - \partial^\mu \bar{C}^a D^{a\mu} C^b
\]

(41)

which is the effective Lagrangian for the quantized massive gauge field. In Eq.(40), the limit \( \alpha \to 0 \) is always implied. Certainly, the theory may be given in general gauges(\(\alpha \neq 0\)).
In this case, the ghost particle will acquire a mass \( \mu = \sqrt{\alpha m} \) (we leave the details of this subject in other papers). However, we note that the Landau gauge is truly physical gauge for the massive gauge field which we need to work in only for practical calculations. In passing, we note that in the zero-mass limit, Eqs.(40) and (41) immediately go over to the corresponding results for the massless gauge field theory established in the Lorentz gauge condition.

From the generating functional (40), one may derive the massive gauge boson propagator like this [6,18]

\[
iD^{ab}_{\mu \nu}(k) = -i\delta^{ab}g_{\mu \nu} - \frac{k_\mu k_\nu}{k^2} - \frac{m^2}{k^2} + i\varepsilon
\]

which, as we see, can not make trouble with the renormalizability of the theory. The ghost particle propagator and vertices derived from the above generating functional are the same as those given in the massless theory. Furthermore, the BRST transformation [21] under which the effective action appearing in Eq.(40) is invariant and the Ward-Takahashi identity [6,22] obeyed by the generating functional are formally identical to those for the massless theory. Therefore, by the reasoning much similar to that given in the massless theory, the renormalizability and unitarity of the theory can be exactly proved to be no problems. All these issues will be discussed in detail in the separate papers. We end this letter with an emphasis that the massive gauge field theory can, indeed, be set up on the gauge-invariance principle without relying on the Higgs mechanism. The key point to achieve this conclusion is that the massive gauge field must be viewed as a constrained system and the Lorentz condition must be introduced from the beginning and imposed on the Proca Lagrangian.

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