DOMINANT DIMENSION AND TILTING MODULES

VAN C. NGUYEN, IDUN REITEN, GORDANA TODOROV, AND SHIJIE ZHU

Abstract. We study which algebras have tilting modules that are both generated and cogenerated by projective-injective modules. Crawley-Boevey and Sauter have shown that Auslander algebras have such tilting modules; and for algebras of global dimension 2, Auslander algebras are classified by the existence of such tilting modules.

In this paper, we show that the existence of such a tilting module is equivalent to the algebra having dominant dimension at least 2, independent of its global dimension. In general such a tilting module is not necessarily cotilting. Here, we show that the algebras which have a tilting-cotilting module generated-cogenerated by projective-injective modules are precisely 1-Auslander-Gorenstein algebras.

When considering such a tilting module, without the assumption that it is cotilting, we study the global dimension of its endomorphism algebra, and discuss a connection with the Finitistic Dimension Conjecture. Furthermore, as special cases, we show that triangular matrix algebras obtained from Auslander algebras and certain injective modules, have such a tilting module. We also give a description of which Nakayama algebras have such a tilting module.

Contents

Introduction 2

1. Projective-injectives and the subcategory $C_{\Lambda}$
   1.1. General properties of the subcategory $C_{\Lambda}$
   1.2. Tilting modules in $C_{\Lambda}$

2. Dominant dimension and tilting modules (or cotilting modules)
   2.1. Numerical condition
   2.2. Maps from $X$ to projective non-injective modules
   2.3. Existence of tilting modules in $C_{\Lambda}$ in terms of dominant dimension
   2.4. Existence of cotilting modules in $C_{\Lambda}$

3. Homological applications of the tilting module in $C_{\Lambda}$
   3.1. More on dominant dimension
   3.2. The endomorphism algebra of the tilting module
   3.3. A relation to the Finitistic Dimension Conjecture

4. Special class: Extensions of Auslander algebras by injective modules
   4.1. General triangular matrix construction
   4.2. Triangular matrix construction from Auslander algebras

5. Special class: Tilting modules in $C_{\Lambda}$ for Nakayama algebras
   5.1. Nakayama algebras
   5.2. Cyclic-Nakayama algebras with dominant dimension at least 2
   5.3. Linear-Nakayama algebras with dominant dimension at least 2
   5.4. The tilting module $T_{C}$ for Nakayama algebras

References 24

Date: October 4, 2017.

2010 Mathematics Subject Classification. 16G10; 16G20; 16G70; 16S50; 16S70.

Key words and phrases. tilting modules, dominant dimension, Auslander algebras, Nakayama algebras.
Introduction

Let $\Lambda$ be an artin algebra and let $\text{mod} \Lambda$ be the category of finitely generated left $\Lambda$-modules. Throughout the paper, $\text{gldim} \Lambda$ denotes the global dimension of $\Lambda$, $\text{domdim} \Lambda$ denotes its dominant dimension (c.f. Definition 2.3.1), and $\text{Gdim} \Lambda$ denotes its Gorenstein dimension (c.f. Remark 2.4.6). For any $\Lambda$-module $M$, $\text{projdim} M$ denotes its projective dimension and $\text{injdim} M$ denotes its injective dimension.

Let $\tilde{Q}$ be the direct sum of representatives of the isomorphism classes of all indecomposable projective-injective $\Lambda$-modules. Let $C_{\Lambda} := (\text{Gen} \tilde{Q}) \cap (\text{Cogen} \tilde{Q})$ be the full subcategory of $\text{mod} \Lambda$ consisting of all modules generated and cogenerated by $\tilde{Q}$. When $\text{gldim} \Lambda = 2$, Crawley-Boevey and Sauter showed in [10, Lemma 1.1] that the algebra $\Lambda$ is an Auslander algebra if and only if there exists a tilting $\Lambda$-module $T_C$ in $C_{\Lambda}$. In fact, $T_C$ is the direct sum of representatives of the isomorphism classes of indecomposable modules in $C_{\Lambda}$. Furthermore $T_C$ is the unique tilting module in $C_{\Lambda}$ and it is also a cotilting module.

There is another characterization of Auslander algebras as algebras $\Lambda$ such that $\text{gldim} \Lambda \leq 2$ and $\text{domdim} \Lambda \geq 2$. From the above result in [10], it follows that in global dimension 2, the existence of such a tilting module is equivalent to $\text{domdim} \Lambda \geq 2$. In this paper, we show that the existence of such a tilting module is equivalent to $\text{domdim} \Lambda \geq 2$ without any condition on the global dimension of $\Lambda$, and we give a precise description of such a tilting module (see Corollary 2.4.2, Corollary 2.2.9 and Remark 2.4.3):

**Theorem 1.** Let $\Lambda$ be an artin algebra. Let $\tilde{Q}$ be the projective-injective $\Lambda$-module as above.

1. The following statements are equivalent:
   (a) $\text{domdim} \Lambda \geq 2$,
   (b) $C_{\Lambda}$ contains a tilting $\Lambda$-module $T_C$,
   (c) $C_{\Lambda}$ contains a cotilting $\Lambda$-module $C_C$.

2. If a tilting module $T_C$ exists, then $T_C \cong \tilde{Q} \oplus \bigoplus_i (\Omega^{-1}P_i)$, where $\Omega^{-1}P_i$ is the cosyzygy of $P_i$ and the direct sum is taken over representatives of the isomorphism classes of all indecomposable projective non-injective $\Lambda$-modules $P_i$.

3. If a cotilting module $C_C$ exists, then $C_C \cong \tilde{Q} \oplus \bigoplus_i \Omega I_i$, where $\Omega I_i$ is the syzygy of $I_i$ and the direct sum is taken over representatives of the isomorphism classes of all indecomposable injective non-projective $\Lambda$-modules $I_i$.

Dominant dimensions of algebras under derived equivalences induced by tilting modules were studied by Chen and Xi; in particular they looked at a special class of the so-called canonical tilting modules [9, p.385] (or canonical $k$-tilting modules to specify the projective dimension being $k$, c.f. Remark 1.2.3). Recently, the same tilting modules, also called $k$-shifted modules, are studied by Pressland and Sauter in [23]. They show that the existence of a $k$-shifted module is equivalent to the dominant dimension of the algebra being at least $k$. We remark that our tilting module $T_C$ in $C_{\Lambda}$ is a canonical 1-tilting module. However when $\text{domdim} \Lambda \leq 1$, a canonical 1-tilting module never belongs to $C_{\Lambda}$.

In this paper, we concentrate on the existence and properties of the classical tilting module $T_C$ in the subcategory $C_{\Lambda}$; in addition to its description, we also consider classes of algebras $\Lambda$ which have such tilting modules in $C_{\Lambda}$.

Theorem 1 is proved and discussed in detail in Sections 2.3 and 2.4. As a generalization of [10, Lemma 1.1], we describe Auslander algebras as algebras $\Lambda$ with finite global dimension such that there exists a tilting-cotilting module in $C_{\Lambda}$ (see Corollary 2.4.13). More generally, we characterize a larger class, of 1-Auslander-Gorenstein algebras (c.f. Definition 2.4.10) as:

**Theorem 2.** Let $\Lambda$ be an artin algebra. Then the subcategory $C_{\Lambda}$ contains a tilting-cotilting module if and only if $\Lambda$ is a 1-Auslander-Gorenstein algebra.
The (sub)structures of classes of such algebras with their homological properties are described in the following diagram (see Definition 2.4.10 and Remark 2.4.11 for some definitions):

In Section 3.1, we gather further properties of algebras with dominant dimension at least 2. From the results in [20, 21, 25], it follows that such algebras are isomorphic to $\text{End}_\Lambda(X)^{\text{op}}$ for some algebra $\Lambda$ and a $\Lambda$-module $X$ which is a generator and a cogenerator; we recall what these algebras $\Lambda$ and modules $X$ should look like and also give a precise description of the tilting module $T_C$ in terms of $\Lambda$ and $X$ in Proposition 3.1.6. In Section 3.2, given an artin algebra $\Lambda$ with $\text{gldim} \Lambda = d$ and a tilting module $T_C \in \mathcal{C}_\Lambda$ (if it exists), we study the endomorphism algebra $B_C := \text{End}_\Lambda(T_C)^{\text{op}}$. We show that $d - 1 \leq \text{gldim} B_C \leq d$, and $\text{gldim} B_C = d - 1$ if and only if $\text{projdim}(\tau T_C) < d$, (see Corollary 3.2.5 and Theorem 3.2.9). Applying this together with the description of algebras of dominant dimension at least 2 in Section 3.1, we obtain a result about the Finitistic Dimension Conjecture for a certain class of artin algebras of representation dimension at most 4 in Corollary 3.3.9.

In Section 4, we construct classes of algebras closely related to Auslander algebras which have tilting modules in the subcategory $\mathcal{C}_A$ of mod $\Lambda$. More precisely we have:

**Theorem 3.** Let $A$ be an Auslander algebra. Let $E$ be an injective $A$-module such that $\text{End}_A(E)$ is a semisimple algebra and $\text{Hom}_A(E, Q) = 0$, for all projective-injective $A$-modules $Q$. Then $A[E]$, the triangular matrix algebra of $A$ and the $A$-$\text{End}_A(E)^{\text{op}}$-bimodule $E$, has a tilting module in the subcategory $\mathcal{C}_A$.

In Section 5, we use a numerical condition to give a characterization of Nakayama algebras $\Lambda$ which have a tilting module in $\mathcal{C}_A$. This class of algebras has been classified by Fuller in [11, Lemma 4.3]; we give a combinatorial approach using Auslander-Reiten theory:

**Theorem 4.** Let $\Lambda$ be a Nakayama algebra with $n$ simple modules. Let $c$ be an admissible sequence of a given Kupisch series. Let the set $P_c$ label all indecomposable projective non-injective $\Lambda$-modules, the set $Q_c$ label all indecomposable projective-injective $\Lambda$-modules, and $c_j$ be the length of the indecomposable module $P_j$. Then there exists a tilting module in $\mathcal{C}_A$ if and only if $P_c \subseteq \{j - c_j \in \mathbb{Z}_n \mid j \in Q_c\}$.

The description of such a tilting module is given in Section 5.4.

**Acknowledgement:** We would like to thank Rene Marczinzik, Matthew Pressland, and Julia Sauter for helpful conversations and remarks, especially Matthew for pointing out a
Corollary 1.1.2. Let $X$ and injdim $X$ be in $\Lambda$. Then the projective cover $P(X)$ of $X$ is injective, and the injective envelope $I(X)$ of $X$ is projective. Hence, $P(X)$ and $I(X)$ are in $\mathcal{C}_A$.

1. Projective-injectives and the subcategory $\mathcal{C}_A$

Let $\Lambda$ be an artin algebra and let mod $\Lambda$ be the category of finitely generated left $\Lambda$-modules.

Definition 1.0.1. A $\Lambda$-module $X$ is called a generator if for any $\Lambda$-module $M$, there is an epimorphism $X^m \to M$, for some $m$. A $\Lambda$-module $X$ is called a cogenerator if for any $\Lambda$-module $M$, there is a monomorphism $M \to X^m$, for some $m$.

We denote by $\text{Gen}(X)$, respectively $\text{Cogen}(X)$, the full subcategories of $\text{mod}\Lambda$ consisting of modules generated by $X$, respectively cogenerated by $X$. Notice that $X$ is a generator-cogenerator if and only if each indecomposable projective $\Lambda$-module and indecomposable injective $\Lambda$-module is isomorphic to a direct summand of $X$.

Definition 1.0.2. Let $\tilde{Q} := \bigoplus_{i=1}^{t} Q_i$, where the $Q_i$ are representatives of the isomorphism classes of all indecomposable projective-injective $\Lambda$-modules. Let $\mathcal{C}_\Lambda := (\text{Gen}\tilde{Q}) \cap (\text{Cogen}\tilde{Q})$ be the full subcategory of $\text{mod}\Lambda$ consisting of all modules generated and cogenerated by $\tilde{Q}$.

We are going to investigate when there exists a tilting module in $\mathcal{C}_\Lambda$.

1.1. General properties of the subcategory $\mathcal{C}_\Lambda$. We now describe some basic homological properties of the modules in the subcategory $\mathcal{C}_\Lambda$ for artin algebras $\Lambda$.

Proposition 1.1.1. Let $\Lambda$ be an artin algebra with $\text{gldim}\Lambda = d$. Let $\mathcal{C}_\Lambda = (\text{Gen}\tilde{Q}) \cap (\text{Cogen}\tilde{Q})$, where $\tilde{Q}$ is the above projective-injective $\Lambda$-module. Let $X$ be any module in $\mathcal{C}_\Lambda$. Then $\text{projdim} X \leq d - 1$ and $\text{injdim} X \leq d - 1$.

Proof. Since $X$ is in $\mathcal{C}_\Lambda$, there exist short exact sequences:

$$0 \to N \to Q_0 \to X \to 0$$

and

$$0 \to X \to Q_0' \to L \to 0,$$

with $Q_0$ and $Q_0'$ projective-injective $\Lambda$-modules. Then there are induced long exact sequences:

$$\cdots \to \text{Ext}^d_\Lambda( , N) \to \text{Ext}^d_\Lambda( , Q_0) \to \text{Ext}^d_\Lambda( , X) \to \text{Ext}^{d+1}_\Lambda( , N) \to \cdots,$$

and

$$\cdots \to \text{Ext}^d_\Lambda(L, ) \to \text{Ext}^d_\Lambda(Q_0', ) \to \text{Ext}^d_\Lambda(X, ) \to \text{Ext}^{d+1}_\Lambda(L, ) \to \cdots,$$

which show that $\text{Ext}^j_\Lambda( , X) = 0$ and $\text{Ext}^j_\Lambda(X, ) = 0$, for all $j \geq d$. Hence, $\text{projdim} X \leq d - 1$ and $\text{injdim} X \leq d - 1$. $\square$

As an Auslander algebra $A$ has $\text{gldim} A \leq 2$, we obtain the following consequence:

Corollary 1.1.2. Let $A$ be an Auslander algebra. Let $X$ be in $\mathcal{C}_\Lambda$. Then $\text{projdim} X \leq 1$ and $\text{injdim} X \leq 1$.

Proposition 1.1.3. Let $\Lambda$ be an artin algebra. Then:

1. If $P$ is projective and $P$ is in $\mathcal{C}_\Lambda$, then $P$ is projective-injective.

   If $I$ is injective and $I$ is in $\mathcal{C}_\Lambda$, then $I$ is projective-injective.

2. Let $X$ be in $\mathcal{C}_\Lambda$. Then the projective cover $P(X)$ of $X$ is injective, and the injective envelope $I(X)$ of $X$ is projective. Hence, $P(X)$ and $I(X)$ are in $\mathcal{C}_\Lambda$. 

Proof. (1) Let \( P \) be projective in \( \mathcal{C}_\Lambda \). Then \( P \) is a quotient of a projective-injective module \( Q \). Since \( P \) is projective, it is a summand of \( Q \), and therefore it is injective as well. Dually the injective module \( I \in \mathcal{C}_\Lambda \) is also projective.

(2) Since \( X \) is in \( \mathcal{C}_\Lambda \), there is a projective-injective module \( Q_0 \) which maps onto \( X \). Thus, the projective cover \( P(X) \) is a direct summand of \( Q_0 \) and so it is injective. Similarly, \( I(X) \) is projective by a dual argument. \( \square \)

**Lemma 1.1.4.** Let \( \Lambda \) be an artin algebra. Let \( X \) be in \( \mathcal{C}_\Lambda \). Let \( Y \) be a \( \Lambda \)-module with projdim\( Y = 1 \). Then \( \text{Ext}^1_\Lambda(Y, X) = 0 \).

**Proof.** Let \( 0 \to K \to P_0 \to X \to 0 \) be an exact sequence with \( P_0 \) the projective cover of \( X \). Consider the induced exact sequence

\[
\cdots \to \text{Ext}^1_\Lambda(Y, P_0) \to \text{Ext}^1_\Lambda(Y, X) \to \text{Ext}^2_\Lambda(Y, K) \to \cdots.
\]

Here \( \text{Ext}^1_\Lambda(Y, P_0) = 0 \) since \( P_0 \) is injective, and \( \text{Ext}^2_\Lambda(Y, K) = 0 \) since projdim\( Y = 1 \). Therefore, we have \( \text{Ext}^1_\Lambda(Y, X) = 0 \) as claimed. \( \square \)

### 1.2. Tilting modules in \( \mathcal{C}_\Lambda \).

Usually, there will be only partial tilting modules in \( \mathcal{C}_\Lambda \), and in general there could be no tilting module in \( \mathcal{C}_\Lambda \). In this subsection, we show some of the properties of a tilting module in \( \mathcal{C}_\Lambda \), if it exists. We recall here the definition of tilting and cotilting modules, since both will be studied extensively in this paper:

**Definition 1.2.1.** Let \( \Lambda \) be an artin algebra. A basic \( \Lambda \)-module \( T \) is called **partial tilting** if it satisfies conditions (1) and (2). It is called **tilting module** if it satisfies (1), (2), and (3).

1. \( \text{projdim}_\Lambda T \leq 1 \)
2. \( \text{Ext}^1_\Lambda(T, T) = 0 \)
3. There is an exact sequence \( 0 \to \Lambda \to T_0 \to T_1 \to 0 \), where \( T_0, T_1 \in \text{add} \ T \).

A \( \Lambda \)-module \( C \) is called **cotilting** if it satisfies conditions (1)', (2)', and (3)'.

1. \( \text{injdim}_\Lambda C \leq 1 \)
2. \( \text{Ext}^1_\Lambda(C, C) = 0 \)
3. There is an exact sequence \( 0 \to C_0 \to C_1 \to D \Lambda \to 0 \), where \( C_0, C_1 \in \text{add} \ C \).

**Remark 1.2.2 (\cite{H}, Corollary VI. 4.4).** Let \( n \) be the number of non-isomorphic simple \( \Lambda \)-modules. Let \( T \) be a partial tilting module. Then the condition (3) is equivalent to:

(3)’ The number of non-isomorphic indecomposable summands of \( T \) is \( n \).

**Remark 1.2.3.** To avoid confusion, we clarify the use of terminology “tilting module” here.

- In our definition, “tilting” means the classical tilting module as in Definition 1.2.1 with projdim\( \Lambda T \leq 1 \). In particular, we denote a classical tilting module by \( T_\mathcal{C} \) if it lies in \( \mathcal{C}_\Lambda \). We will prove later that \( T_\mathcal{C} \) is unique, if it exists.

- In the literature, some authors use the terminology “tilting modules” for **generalized tilting modules** (e.g. Happel \cite{H}): (1) \( \text{projdim} T < \infty \), (2) \( \text{Ext}^i(T, T) = 0 \), for all \( i > 0 \), (3) There is an exact sequence \( 0 \to \Lambda \to T_0 \to T_1 \to \cdots \to T_m \to 0 \) for some \( m > 0 \) and \( T_i \in \text{add} \ T \) for all \( 0 \leq i \leq m \). A generalized tilting module \( T \) with projdim\( T = k \) is also called **\( k \)-tilting module** in \cite{M} Definition 2.3.

- For an algebra \( \Lambda \) with dominant dimension at least \( k \), Chen and Xi defined in \cite{CX} the **canonical \( k \)-tilting module** as follows: Let \( \bar{Q} \) be the direct sum of representatives of the isomorphism classes of all indecomposable projective-injective \( \Lambda \)-modules, and

\[
0 \to \Lambda \xrightarrow{d_0} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \to \cdots
\]
be a minimal injective resolution of Λ. Then the module $T_{(k)} := \tilde{Q} \oplus \text{Im} d_k$ is a basic $k$-tilting module and it is called the canonical $k$-tilting module. A canonical $k$-cotilting module is defined dually.

**Lemma 1.2.4.** Let Λ be an artin algebra. Let $n$ be the number of non-isomorphic simple Λ-modules. Let $\{X_i\}_{i \in I}$ be a set of indecomposable modules such that $X_i \not\cong X_j$ for all $i \neq j$ and $\text{Ext}^1_{\Lambda}(X_i, X_j) = 0$ for all $i, j$. Assume that $\text{projdim} X_i = 1$ for all $i \in I$. Then the set $I$ is finite and has at most $n$ elements.

*Proof.* Let $X_1, \ldots, X_s$ be any $s$ modules in this set. Then $\bigoplus_{i=1}^s X_i$ is a partial tilting module. Every partial tilting module can be completed to a tilting module (see [3]). A tilting module has $n$ non-isomorphic indecomposable summands. Therefore, $s \leq n$. So there are at most $n$ modules in the set $\{X_i\}_{i \in I}$.

**Proposition 1.2.5.** Let Λ be an artin algebra. Let $\tilde{Q}$ be the projective-injective module defined above and let $\mathcal{C}_\Lambda = (\text{Gen} \tilde{Q}) \cap (\text{Cogen} \tilde{Q})$. Let $\{X_i\}_{i \in I}$ be the set of representatives of the indecomposable modules in $\mathcal{C}_\Lambda$ such that $\text{projdim} X_i = 1$. Then:

1. The set $\{X_i\}_{i \in I}$ is finite, that is, $I = \{1, 2, \ldots, s\}$ for some $s < \infty$.
2. Let $X = \bigoplus_{i=1}^s X_i$. Then $\tilde{Q} \oplus X$ is a partial tilting module.
3. If there is a tilting module $T_C$ in $\mathcal{C}_\Lambda$, then $T_C = \tilde{Q} \oplus X$.
4. If there is a tilting module $T_C$ in $\mathcal{C}_\Lambda$, then $T_C$ is unique.

*Proof.* (1) It follows from Lemma 1.1.4 that $\text{Ext}_{\Lambda}^1(X_i, X_j) = 0$, for all $i \neq j$. Since $\text{projdim} X_i = 1$, it follows from Lemma 1.2.4 that there are at most $n$ modules $X_i$, where $n$ is the number of non-isomorphic simple Λ-modules.

(2) Follows from the definition of partial tilting module.

(3) This follows since all other modules in $\mathcal{C}_\Lambda$ have projective dimension $\geq 2$.

(4) It follows from (3) that $T_C = \tilde{Q} \oplus X$, hence it is unique.

The following proposition is about the add $T_C$-approximations of projective modules.

**Proposition 1.2.6.** Let Λ be an artin algebra and $P$ be a projective Λ-module. Suppose there exists a tilting module $T_C$ in $\mathcal{C}_\Lambda$. Let $f_P : P \to T_P$ be a minimal left add $T_C$-approximation of $P$. Then $T_P$ is projective-injective.

*Proof.* Let $f_P : P \to T_P$ be a minimal add $T_C$-approximation of $P$. Then $T_P = Q_P \oplus M_P$, where $Q_P$ is projective-injective and $\text{projdim} M_P = 1$ and $f_P = (s, \rho) : P \to Q_P \oplus M_P$. Let $\sigma : Q'_P \to M'_P$ be the projective cover of $M_P$; here $Q'_P$ is projective-injective by Proposition 1.1.3(2). Then $\rho$ factors through $\sigma$, i.e. $\rho = \sigma a$ for some $a : P \to Q'_P$. It is easy to check that $(s, a) : P \to Q_P \oplus Q'_P$ is an add $T_C$-approximation. Hence $T_P$ is a direct summand of $Q_P \oplus Q'_P$ and therefore it is projective-injective.

2. Dominant dimension and tilting modules (or cotilting modules)

In this section we show that the existence of a tilting module (or a cotilting module) in the subcategory $\mathcal{C}_\Lambda$ is equivalent to the dominant dimension of Λ being at least 2.

2.1. Numerical condition. We now state a numerical condition which will be necessary and sufficient for the existence of a tilting module in $\mathcal{C}_\Lambda$.

Let $\mathcal{Q} := \text{add} \tilde{Q}$ be the subcategory of $\mathcal{C}_\Lambda$ consisting of projective-injective modules where $\tilde{Q} = \bigoplus_{i=1}^s Q_i$, as in Definition 1.0.2. Let $\mathcal{X} := \text{add} X$ be the subcategory of $\mathcal{C}_\Lambda$ consisting of modules with projective dimension 1, where $X = \bigoplus_{i=1}^s X_i$ as in Proposition 1.2.5. We denote
by $n_Q$ the number of non-isomorphic indecomposable modules in $Q$ and by $n_\mathcal{X}$ the number of non-isomorphic indecomposable modules in $\mathcal{X}$. Hence, $n_Q = t$ and $n_\mathcal{X} = s$.

**Remark 2.1.1.** Let $n$ be the number of non-isomorphic simple $\Lambda$-modules. Since by Proposition 1.2.3(2), $Q \oplus X$ is a partial tilting module, it follows that $n_Q + n_\mathcal{X} \leq n$.

Combining this remark and Proposition 1.2.3 we obtain the following important numerical condition for the existence of a tilting module in $\mathcal{C}_\Lambda$.

**Corollary 2.1.2.** Let $\Lambda$ be an artin algebra with $n$ non-isomorphic simple modules. Let $Q$ and $\mathcal{X}$ be the above subcategories. Then there is a tilting module in $\mathcal{C}_\Lambda$ if and only if $n_Q + n_\mathcal{X} = n$.

**Proof.** If there is a tilting module $T_C$ in $\mathcal{C}_\Lambda$ then it has $n_Q$ summands from $Q$ and $n_\mathcal{X}$ summands from $\mathcal{X}$ by Proposition 1.2.3(3).

### 2.2. Maps from $\mathcal{X}$ to projective non-injective modules

In this part we define a mapping $\Omega : \text{ind}\mathcal{X} \to \text{ind}\mathcal{P}$, where $\mathcal{P}$ is the subcategory of projective non-injective $\Lambda$-modules. This mapping will be a bijection exactly when there is a tilting module in $\mathcal{C}_\Lambda$, which will be shown in Corollary 2.2.3. This will be used in a very essential way in the proof of the main Theorem 2.3.4. We need some preparation:

**Lemma 2.2.1.** [3, II, Lemma 4.3] Let $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ be a non-split exact sequence in an additive category $\mathcal{C}$. Then:

1. If $\text{End}_\mathcal{C}(A)$ is local, then $f : B \to C$ is right minimal in $\mathcal{C}$.
2. If $\text{End}_\mathcal{C}(C)$ is local, then $g : A \to B$ is left minimal in $\mathcal{C}$.

**Lemma 2.2.2.** Let $0 \to Y \xrightarrow{g} Q \xrightarrow{f} X \to 0$ be a non-split exact sequence.

1. Suppose $Y$ is indecomposable, $g$ is left minimal and $Q$ is projective. Then $X$ is indecomposable and $f$ is right minimal.
2. Suppose $X$ is indecomposable, $f$ is right minimal and $Q$ is injective. Then $Y$ is indecomposable and $g$ left minimal.

**Proof.** (1) By Lemma 2.2.1 $Y$ being indecomposable implies that $f$ is right minimal. Hence $f$ is a projective cover of $X$. To show that $X$ is indecomposable, suppose $X = X_1 \oplus X_2$ where $X_1$ and $X_2$ are both non-zero. Consider the projective covers $Q_1$ and $Q_2$ of $X_1$ and $X_2$ respectively and the associated exact sequences:

$$0 \to Y_1 \to Q_1 \to X_1 \to 0,$$

$$0 \to Y_2 \to Q_2 \to X_2 \to 0.$$

Then $Q_1 \oplus Q_2$ is the projective cover of $X_1 \oplus X_2 \cong X$. Because the projective cover of $X$ is unique up to isomorphism it follows that $Q \cong Q_1 \oplus Q_2$. Therefore we have $Y \cong Y_1 \oplus Y_2$. Since $Y$ is indecomposable, either $Y_1 = 0$ or $Y_2 = 0$. If $Y_1 = 0$, then $X_1 \cong Q_1 \not\cong 0$, which contradicts the fact that $g$ is left minimal. A similar contradiction is drawn if we assume $Y_2 = 0$. Therefore, $X$ is indecomposable.

(2) This is the dual statement of (1).

**Corollary 2.2.3.** Let $0 \to Y \xrightarrow{g} Q \xrightarrow{f} X \to 0$ be a non-split exact sequence, where $Q$ is a projective-injective module. Then the following statements are equivalent:

1. $X$ is indecomposable and $f$ is a projective cover,
2. $Y$ is indecomposable and $g$ is an injective envelope.

Applying Corollary 2.2.3 recursively, we have the following result:
Corollary 2.2.4. Suppose \( 0 \to X \to I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \cdots \) is a minimal injective resolution of an indecomposable module \( X \). If \( I_0, I_1, \ldots, I_k \) are also projective, then \( \text{Im} \, d_i \) are indecomposable for all \( 0 \leq i \leq k \).

Lemma 2.2.5. Let \( X \) be an indecomposable module in \( \mathcal{X} \) and let

\[
0 \to P \xrightarrow{i} Q \xrightarrow{p} X \to 0
\]

be the minimal projective resolution of \( X \). Then the following statements hold:

1. The module \( Q \) is projective-injective.
2. The syzygy \( \Omega X = P \) is indecomposable, projective and non-injective.
3. The map \( i : P \to Q \) is an injective envelope of \( P \).

Proof. (1) By Proposition 1.1.3, \( Q \) is also injective.
(2) It is clear that \( P \) is projective non-injective. By Corollary 2.2.3, \( P \) is indecomposable.
(3) The fact that the map \( i \) is an injective envelope also follows from Corollary 2.2.3. \( \square \)

Definition 2.2.6. Let \( \mathcal{P} \) be the subcategory of projective non-injective modules in \( \text{mod} \, \Lambda \). Denote by \([M] \) the isomorphism class of an \( \Lambda \)-module \( M \). Then by Lemma 2.2.5, we know that \( \Omega([X]) := [\Omega X] = [P] \) defines a set-theoretic map: \( \Omega : \text{ind} \, \mathcal{X} \to \text{ind} \, \mathcal{P} \).

Now we show the main Lemma:

Lemma 2.2.7. Let \( \Lambda \) be an artin algebra. Let \( n \) be the number of non-isomorphic simple \( \Lambda \)-modules. Then

1. \( \Omega : \text{ind} \, \mathcal{X} \to \text{ind} \, \mathcal{P} \) is an injection of sets,
2. \( n_Q = n_X \leq n \),
3. \( n_Q + n_{\mathcal{X}} = n \) if and only if \( \Omega \) is a bijection.

Proof. (1) Injectivity of \( \Omega \): Suppose \( X_1 \not\cong X_2 \) in \( \text{ind} \, \mathcal{X} \). We will show that \( \Omega([X_1]) \neq \Omega([X_2]) \).
In fact, taking the minimal projective resolution of \( X_1 \) and \( X_2 \), we get

\[
0 \to P_1 \to Q_1 \to X_1 \to 0,
\]

\[
0 \to P_2 \to Q_2 \to X_2 \to 0.
\]
Assume \( P_1 \cong P_2 \). By Corollary 2.2.3, \( Q_1 \) and \( Q_2 \) are injective envelopes of \( P_1 \) and \( P_2 \) respectively. Hence \( Q_1 \cong Q_2 \), and then \( X_1 \cong X_2 \) which is a contradiction.
(2) It follows from (1) that \( n_X \leq n_\mathcal{P} \). Therefore \( n_Q + n_X \leq n_Q + n_\mathcal{P} = n \). Then (3) is clear. \( \square \)

Corollary 2.2.8. Let \( \Lambda \) be an artin algebra with \( n \) simple modules. Then there is a tilting module \( T_C \) in \( \mathcal{C}_\Lambda \) if and only if \( \Omega \) is a bijection.

Corollary 2.2.9. If a tilting module \( T_C \) exists, then \( T_C \cong \tilde{Q} \oplus (\bigoplus \Omega^{-1} P_i) \), where \( \Omega^{-1} P_i \) is the cosyzygy of \( P_i \) and the direct sum is taken over representatives of the isomorphism classes of all indecomposable projective non-injective \( \Lambda \)-modules \( P_i \).

2.3. Existence of tilting modules in \( \mathcal{C}_\Lambda \) in terms of dominant dimension. We now prove the main theorem. Recall that the dominant dimension of a (left) \( \Lambda \)-module \( M \) is defined as follows.

Definition 2.3.1. Let \( 0 \to \Lambda M \to I_0 \to I_1 \to \cdots \to I_m \to \cdots \) be a minimal injective resolution of \( M \). Then \( \text{domdim} \, \Lambda M = \sup \{ k \mid I_i \text{ is projective, for all } 0 \leq i < k \} \). The left dominant dimension of the algebra \( \Lambda \) is defined to be \( \text{domdim} \, \Lambda \).
Remark 2.3.2. The dominant dimension of a right module and the right dominant dimension of the algebra are defined similarly. It is well known that \( \text{dom dim } \Lambda = \text{dom dim } \Lambda \) for any algebra \( \Lambda \) (see e.g. [21, Theorem 4]). So for the rest of this paper, we will denote both left and right dominant dimension of \( \Lambda \) by \( \text{dom dim } \Lambda \) and call it the dominant dimension of \( \Lambda \).

Remark 2.3.3. Here are some basic properties of dominant dimension:

1. Let \( Q \) be a projective-injective module. Then \( \text{dom dim } Q = \infty \).
2. \( \text{dom dim } \Lambda = \min \{ \text{dom dim } \Lambda P \mid \Lambda P \text{ is indecomposable projective} \} \).
3. If \( \Lambda \) is selfinjective, then \( \text{dom dim } \Lambda = \infty \).

Theorem 2.3.4. Let \( \Lambda \) be an artin algebra. Then the following statements are equivalent:

1. The subcategory \( \mathcal{C}_\Lambda \) contains a tilting \( \Lambda \)-module \( T \),
2. \( \text{dom dim } \Lambda \geq 2 \).

Proof. (1) \( \Rightarrow \) (2). Let \( P \) be an indecomposable projective \( \Lambda \)-module. If \( P \) is projective-injective then \( \text{dom dim } P = \infty \) by Remark 2.3.3. If \( P \) is projective non-injective we will show that in the minimal injective copresentation of \( P \):

\[
0 \to P \to I_0 \to I_1 \to I_2 \to I_3 \to \ldots,
\]

both \( I_0 \) and \( I_1 \) are projective-injective. To show this we use the assumption that there is a tilting module \( T \) in \( \mathcal{C}_\Lambda \). By property (3) in Definition 1.2.1 of tilting modules, for each projective \( P \) there is an exact sequence

\[
0 \to P \xrightarrow{a} T_0 \xrightarrow{f} T_1 \to 0,
\]

where \( T_0 \) and \( T_1 \) are in \( \text{add } T \) and the map \( g \) is a minimal left \( \text{add } T \)-approximation of \( P \). It follows by Proposition 1.2.6 that \( T_0 \) is projective-injective. Call it \( Q_0 \). Let \( T_1 \xrightarrow{g} Q_1 \) be the injective envelope of \( T_1 \). Since \( T_1 \) is in \( \text{add } T \subseteq \mathcal{C}_\Lambda \), the injective envelope \( Q_1 \) is also projective-injective by Proposition 1.1.3(2). Combining these two statements, we get the minimal injective copresentation of \( P \):

\[
0 \to P \xrightarrow{a} Q_0 \xrightarrow{f} Q_1,
\]

where \( Q_0 \) and \( Q_1 \) are projective-injective modules. Hence, \( \text{dom dim } P \geq 2 \). By Remark 2.3.3 we have \( \text{dom dim } \Lambda \geq 2 \).

(2) \( \Rightarrow \) (1). We use the fact that \( \text{dom dim } \Lambda \geq 2 \) in order to show that the map

\[
\Omega : \text{ind } \mathcal{X} \to \text{ind } \mathcal{P}
\]

is a bijection. Then apply Corollary 2.2.8 to conclude (1). It follows from Lemma 2.2.7 that \( \Omega \) is an injective map. To show that it is surjective, we consider \( P \in \text{ind } \mathcal{P} \) and find \( X \in \text{ind } \mathcal{X} \) so that \( \Omega X \cong P \). Let \( P \xrightarrow{a} Q \) be the injective envelope of \( P \). The module \( Q \) is projective-injective since \( \text{dom dim } P \geq 2 \). Consider the induced short exact sequence

\[
0 \to P \xrightarrow{a} Q \xrightarrow{b} X \to 0.
\]

Let \( X \xrightarrow{c} I \) be the injective envelope of \( X \). We have a minimal injective copresentation of \( P \):

\[
0 \to P \xrightarrow{a} Q \xrightarrow{b} X \xrightarrow{c} I.
\]

The assumption that \( \text{dom dim } P \geq 2 \) implies that \( I \) is projective-injective. Therefore \( X \) is a submodule of a projective-injective module. Since \( X \) is also a quotient of \( Q \) and \( \text{proj dim } X = 1 \), it follows that \( X \) is in \( \mathcal{X} \). Furthermore, \( X \) is indecomposable since \( P \) is indecomposable, by Corollary 2.2.3. Therefore \( X \) in \( \text{ind } \mathcal{X} \) and \( \Omega X \cong P \). Thus \( \Omega \) is a surjection and therefore a bijection. By Corollary 2.2.8, it follows that there is a tilting module \( T \) in \( \mathcal{C}_\Lambda \). \( \square \)
Using this theorem, we can deduce the following result of Crawley-Boevey and Sauter [10].

**Corollary 2.3.5.** If \( \text{gldim} \Lambda = 2 \), then \( C_\Lambda \) contains a tilting \( \Lambda \)-module if and only if \( \Lambda \) is an Auslander algebra.

**Proof.** An Auslander algebra is an algebra \( A \) with \( \text{gldim} A = 2 \) and \( \text{domdim} A = 2 \). □

More directly by Theorem 2.3.3 for \( m \)-Auslander algebras \( \Lambda \), we can always guarantee the existence of such a tilting module in \( C_\Lambda \). Recall that Iyama introduced the notion of higher Auslander algebras (see [17, 2.2]): an artin algebra \( \Lambda \) is called \( m \)-Auslander if \( \text{gldim} \Lambda \leq m + 1 \leq \text{domdim} \Lambda \). It is easy to see that \( m \)-Auslander algebras are either semisimple or satisfy \( \text{gldim} \Lambda = \text{domdim} \Lambda \). Notice that Auslander algebras are precisely 1-Auslander algebras.

**Corollary 2.3.6.** For any integer \( m \geq 1 \) and any \( m \)-Auslander algebra \( \Lambda \), its subcategory \( C_\Lambda \) always contains a tilting \( \Lambda \)-module.

**Example 2.3.7.** In this example, we illustrate Corollary 2.3.6 for a 2-Auslander algebra. Let \( \Lambda \) be the Nakayama algebra given by the following quiver and relations \( \alpha \gamma = \gamma \beta = 0 \).

\[
\begin{array}{c}
\alpha \\
\downarrow \\
2 \\
\beta \\
\downarrow \\
3 \\
\gamma
\end{array}
\quad \text{with the AR-quiver}
\]

\[
\begin{array}{cccc}
1 & 3 & 2 & 1 \\
\end{array}
\]

The subcategory \( C_\Lambda \) is add\{ \frac{3}{1}, \frac{1}{3}, 1, 3 \}, where there is a tilting \( \Lambda \)-module \( T_C = \frac{3}{1} \oplus \frac{1}{3} \oplus 1 \).

### 2.4. Existence of cotilting modules in \( C_\Lambda \)

Notice that a left \( \Lambda \)-module \( \Lambda T \) is tilting if and only if \( D(T)_\Lambda \) as a right \( \Lambda \)-module is cotilting. As a dual statement, we provide a result on the existence of a cotilting module here.

**Theorem 2.4.1.** Let \( \Lambda \) be an artin algebra. Then the following statements are equivalent:

1. \( C_\Lambda \) contains a cotilting \( \Lambda \)-module,
2. \( C_{\Lambda^{\text{op}}} \) contains a tilting \( \Lambda^{\text{op}} \)-module,
3. \( \text{domdim} \Lambda^{\text{op}} \geq 2 \).

On the other hand, by definition, we have \( \text{domdim} \Lambda^{\text{op}} = \text{domdim} \Lambda \) which is the same as \( \text{domdim} \Lambda \) as we mentioned before (see Remark 2.3.2). So combining our results, we have:

**Corollary 2.4.2.** Let \( \Lambda \) be an artin algebra. Then the following statements are equivalent:

1. \( \text{domdim} \Lambda \geq 2 \),
2. \( C_\Lambda \) contains a tilting \( \Lambda \)-module \( T_C \),
3. \( C_{\Lambda} \) contains a cotilting \( \Lambda \)-module \( C_C \).

**Remark 2.4.3.** If a cotilting module \( C_C \) exists, then \( C_C \simeq \tilde{Q} \oplus (\bigoplus I_i) \), where \( \Omega I_i \) is the syzygy of \( I_i \) and the direct sum is taken over the representatives of the isomorphism classes of all indecomposable injective non-projective \( \Lambda \)-modules \( I_i \).

**Remark 2.4.4.** By [23, Proposition 2.6], for \( k \geq 1 \), the existence of the canonical \( k \)-tilting (or \( k \)-cotilting) modules is equivalent to the dominant dimension of the algebra being at least \( k \). One can see that the implications (1) \( \implies \) (2) and (1) \( \implies \) (3) in Corollary 2.4.2 follow immediately from the existence of the canonical 1-tilting (or 1-cotilting) modules. However, our proof of the equivalence of statements (1),(2),(3) is done using a different approach.
In general, the tilting module and the cotilting module in $C_{\Lambda}$ from Corollary 2.4.2 are not necessarily the same module. Now we discuss when $C_{\Lambda}$ contains a module which is both tilting and cotilting. We call this module the tilting-cotilting module in $C_{\Lambda}$.

**Definition 2.4.5.** An artin algebra $\Lambda$ is called **Gorenstein** if both $\text{injdim}_{\Lambda} \Lambda < \infty$ and $\text{injdim}_{\Lambda} \Lambda < \infty$.

**Remark 2.4.6.** It is conjectured that for an artin algebra $\Lambda$, $\text{injdim}_{\Lambda} \Lambda < \infty$ is equivalent to $\text{injdim}_{\Lambda} \Lambda < \infty$ (Gorenstein Symmetry Conjecture). But we know that if $\text{injdim}_{\Lambda} \Lambda$ and $\text{injdim}_{\Lambda} \Lambda$ are both finite, then $\text{injdim}_{\Lambda} \Lambda = \text{injdim}_{\Lambda} \Lambda$ (e.g. see [27]); in this case, we call this number **Gorenstein dimension**, denoted as $\text{Gdim}_{\Lambda} := \text{injdim}_{\Lambda} \Lambda = \text{injdim}_{\Lambda} \Lambda$. An artin Gorenstein algebra $\Lambda$ is called **Iwanaga-Gorenstein of Gorenstein dimension** $m$, if $\text{Gdim}_{\Lambda} = m$. To avoid confusion, we point out that in the literature, there is an original notion of $m$-Gorenstein algebra (e.g. see [5]) which is different from the notion of Iwanaga-Gorenstein algebra of Gorenstein dimension $m$.

It is well known that selfinjective algebras and algebras of finite global dimensions are Gorenstein. For the convenience of the readers, we show:

**Proposition 2.4.7.** Let $\Lambda$ be an artin algebra with $\text{gldim}_{\Lambda} \Lambda = d < \infty$. Then there exists an indecomposable projective $\Lambda$-module $P$ with $\text{injdim} P = d$.

That is, if $\text{gldim}_{\Lambda} \Lambda = d < \infty$ then $\Lambda$ is Iwanaga-Gorenstein with $\text{Gdim}_{\Lambda} = d$.

**Proof.** Since $\text{gldim}_{\Lambda} \Lambda = d$, $\text{injdim} P \leq d$ for all projective $\Lambda$-module $P$. Here, we claim that at least one $P$ satisfies $\text{injdim} P = d$. Otherwise, since any $\Lambda$-module $M$ has a finite projective resolution $0 \to P_d \to \cdots \to P_0 \to M \to \cdots \to 0$, with each $\text{injdim} P_i < d$, then $\text{injdim} M < d$ and hence $\text{gldim}_{\Lambda} \Lambda < d$, a contradiction. So $\text{gldim}_{\Lambda} \Lambda = d$ implies $\text{injdim}_{\Lambda} \Lambda = d$. Similarly, we have $\text{projdim} D(\Lambda) = d = \text{injdim} \Lambda$. Therefore, $\Lambda$ is Iwanaga-Gorenstein with $\text{Gdim}_{\Lambda} = d$. \(\Box\)

Recently Iyama and Solberg defined $m$-Auslander-Gorenstein algebra in [18]. They also showed that the notion of $m$-Auslander-Gorenstein algebra is left and right symmetric.

**Definition 2.4.8.** [18] An artin algebra $\Lambda$ is called **$m$-Auslander-Gorenstein** if $\text{injdim}_{\Lambda} \Lambda \leq m + 1 \leq \text{domdim} \Lambda$.

**Proposition 2.4.9.** [18, Proposition 4.1] Let $\Lambda$ be an artin algebra.

1. If $\Lambda$ is an $m$-Auslander-Gorenstein algebra, then either $\text{injdim}_{\Lambda} \Lambda = m + 1 = \text{domdim} \Lambda$ holds or $\Lambda$ is selfinjective.

2. An algebra $\Lambda$ is an $m$-Auslander-Gorenstein if and only if $\Lambda^{op}$ is $m$-Auslander-Gorenstein.

**Remark 2.4.10.** Let $\Lambda$ be an artin algebra. We have the following equivalent characterizations of $m$-Auslander-Gorenstein algebras:

1. $\Lambda$ is $m$-Auslander-Gorenstein,
2. $\Lambda$ is Iwanaga-Gorenstein with $\text{Gdim}_{\Lambda} \Lambda \leq m + 1 \leq \text{domdim} \Lambda$,
3. $\Lambda$ is selfinjective or $\text{injdim}_{\Lambda} \Lambda = \text{injdim} \Lambda = m + 1 = \text{domdim} \Lambda$,
4. $\Lambda$ is selfinjective or $\text{injdim}_{\Lambda} \Lambda = m + 1 = \text{domdim} \Lambda$,
5. $\Lambda$ is selfinjective or $\text{injdim}_{\Lambda} \Lambda = m + 1 = \text{domdim} \Lambda$,
6. $\text{injdim}_{\Lambda} \Lambda \leq m + 1 \leq \text{domdim} \Lambda$,
7. $\text{injdim}_{\Lambda} \Lambda \leq m + 1 \leq \text{domdim} \Lambda$.

In particular, a 1-Auslander-Gorenstein algebra is either a selfinjective algebra or a Gorenstein algebra satisfying $\text{injdim}_{\Lambda} \Lambda = 2 = \text{domdim} \Lambda$.

---

The authors called it “minimal $m$-Auslander-Gorenstein” in the introduction of [18]. Later in the paper, they called it “$m$-Auslander-Gorenstein” for simplicity.
Remark 2.4.11. The algebras satisfying the condition $\text{injdim}_\Lambda \Lambda = \text{domdim} \Lambda = 2$ are called $DTr$-selfinjective algebras and they were classified by Auslander and Solberg in [7].

We have a characterization of 1-Auslander-Gorenstein algebras in terms of the existence of the tilting-cotilting module in $C_\Lambda$:

**Theorem 2.4.12.** Let $\Lambda$ be an artin algebra. Then the following statements are equivalent:

1. $\Lambda$ is 1-Auslander-Gorenstein,
2. $C_\Lambda$ contains a tilting-cotilting module.

**Proof.** (1) $\implies$ (2). Assume that (1) holds, then $\text{domdim} \Lambda \geq 2$. By Corollary 2.4.2, the subcategory $C_\Lambda$ contains a tilting module $T_C$. It suffices to show that $\text{injdim} T_C \leq 1$.

In fact, let $T_0$ be any non-injective indecomposable summand of $T_C$ (if it exists). Then by Proposition 1.1.3, $T_0$ is neither projective nor injective with $\text{projdim} T_0 = 1$. Moreover, $T_0$ has a minimal projective resolution: $0 \to P_1 \to P_0 \to T_0 \to 0$, where $P_0$ is projective-injective. Because $\text{domdim} \Lambda \geq 2$, $T_0$ is a submodule of a projective-injective module $I_0$. But $I_0/T_0$ must be injective since $\text{injdim}_\Lambda \Lambda \leq 2$. Hence $\text{injdim} T_0 = 1$. Therefore $\text{injdim} T_C \leq 1$ and $T_C$ is a cotilting module.

(2) $\implies$ (1). Assume that (2) holds, then Corollary 2.4.2 implies that $\text{domdim} \Lambda \geq 2$. Let $P$ be an indecomposable projective non-injective module (if it exists). Let $f : P \to I(P)$ be an injective envelope of $P$. Then we know that $X \cong \text{Coker} f$ is a non-injective summand of the tilting module $T_C$. Since $T_C$ is also cotilting, we have that $\text{injdim} X = 1$, which implies $\text{injdim} P = 2$. Hence, we show $\text{injdim}_\Lambda \Lambda \leq 2$.

Therefore $\text{injdim}_\Lambda \Lambda \leq 2 \leq \text{domdim} \Lambda$ which means that $\Lambda$ is 1-Auslander-Gorenstein. □

We have the following statement which generalizes Crawley-Boevey and Sauter’s result [10, Lemma 1.1] from an algebra $\Lambda$ with $\text{gldim} \Lambda = 2$ to an algebra $\Lambda$ with any finite $\text{gldim} \Lambda$.

**Corollary 2.4.13.** Let $\Lambda$ be an artin algebra with $\text{gldim} \Lambda < \infty$. Then the subcategory $C_\Lambda$ contains a tilting-cotilting module if and only if $\Lambda$ is an Auslander algebra.

**Proof.** Assume $C_\Lambda$ contains a tilting-cotilting module, then $\Lambda$ is 1-Auslander-Gorenstein by Theorem 2.4.12. It follows from Remark 2.4.10 that $\Lambda$ is Iwanga-Gorenstein with Gdim $\Lambda = 2$. By Proposition 2.4.7, this forces $\text{gldim} \Lambda = 2$. Also $\text{domdim} \Lambda \geq 2$ follows from Theorem 2.3.4. Thus, $\Lambda$ is an Auslander algebra. The converse holds by using Theorem 2.4.12 and the fact that an Auslander algebra is 1-Auslander-Gorenstein. □

**Remark 2.4.14.**

- Theorem 2.4.12 and Corollary 2.4.13 suggest that 1-Auslander-Gorenstein algebras are generalizations of Auslander algebras in the sense of the existence of a tilting-cotilting module in $C_\Lambda$.

- A more general situation is considered by Pressland and Sauter (c.f. [23, Proposition 3.7, Theorem 3.9]). They show that $\Lambda$ is an $m$-Auslander-Gorenstein algebra if and only if canonical $k$-tilting modules coincide with canonical $(m+1-k)$-cotilting modules, for all $0 \leq k \leq m + 1$.

**Example 2.4.15.** In this example, we present a 1-Auslander-Gorenstein algebra which contains a tilting-cotilting module in $C_\Lambda$ but is not an Auslander algebra, since its $\text{gldim} \Lambda = \infty$. Let $\Lambda$ be the Nakayama algebra given by the following quiver and relations $\gamma \beta \alpha = \delta \gamma \beta = \alpha \epsilon = \epsilon \delta = 0$. We omit the modules when drawing the AR-quiver.
There exists a tilting-cotilting module $T_C = P_1 \oplus P_2 \oplus P_4 \oplus P_5 \oplus S_4$ in $\mathcal{C}_A$.

3. Homological applications of the tilting module in $\mathcal{C}_A$

3.1. More on dominant dimension. Algebras with dominant dimension at least 2 have been studied since the 1960’s by Morita [20], Tachikawa [25], Mueller [21], Ringel [24] and many others. Morita and Tachikawa showed that any artin algebra of dominant dimension at least 2 is an endomorphism algebra of a generator-cogenerator of another artin algebra. This gives us a full machinery for producing algebras whose dominant dimension is at least 2, and hence, algebras which have a tilting module in $\mathcal{C}_A$.

Theorem 3.1.1. [20, 25, 21, Theorem 2] For an artin algebra $\Gamma$, the following statements are equivalent:

1. $\text{domdim } \Gamma \geq 2$,
2. $\Gamma \simeq \text{End}_\Lambda(X)^{\text{op}}$, where $X$ is a generator-cogenerator of an artin algebra $\Lambda$.

Furthermore, there is a more precise result on the dominant dimension:

Lemma 3.1.2. [21, Lemma 3] Let $\Gamma = \text{End}_\Lambda(X)^{\text{op}}$, where $X$ is a generator-cogenerator of an artin algebra $\Lambda$. Then $\text{domdim } \Gamma \geq m + 2$ if and only if $\text{Ext}^i_\Lambda(X,X) = 0$, for all $1 \leq i \leq m$, $m \in \{0, 1, 2, \ldots\}$.

From this lemma, the following well-known results can be deduced.

Corollary 3.1.3. Let $X$ be a generator-cogenerator of an artin algebra $\Lambda$ and $\Gamma = \text{End}_\Lambda(X)^{\text{op}}$.

1. If $X$ is a summand of a module $Y$, then $\text{domdim } \text{End}_\Lambda(X)^{\text{op}} \geq \text{domdim } \text{End}_\Lambda(Y)^{\text{op}}$.
2. Suppose $\Lambda$ is non-selfinjective. If $\text{injdim}_\Lambda \Lambda \leq m$ or $\text{injdim} \Lambda \leq m$, then $\text{domdim } \Gamma \geq m + 1$.
3. If $\text{gldim } \Lambda \leq m$, then $\text{domdim } \Gamma \leq m + 1$.
4. If $\Lambda$ is non-semisimple hereditary, then $\text{domdim } \Gamma = 2$.

Next, we recall how to construct the algebra $\Lambda$ and $\Lambda$-module $X$ such that $\Gamma = \text{End}_\Lambda(X)^{\text{op}}$ is of dominant dimension at least 2, under the assumption that both $\Gamma$ and $\Lambda$ are basic. In general, we emphasize that such an algebra $\Lambda$ is only unique up to Morita equivalence, see also [2, IV], [24] for details.

Proposition 3.1.4. Let $X$ be a module over an artin algebra $\Lambda$ and $\Gamma = \text{End}_\Lambda(X)^{\text{op}}$. Then:

1. $\text{Hom}_\Lambda(X, X_i)$ are all the indecomposable projective $\Gamma$-modules, where $X_i$ runs through the non-isomorphic indecomposable direct summands of $X$.
2. If $X$ is a cogenerator of $\Lambda$, then the $\text{Hom}_\Lambda(X, I_i)$ are all the indecomposable projective-injective $\Gamma$-modules, where $I_i$ runs through the non-isomorphic indecomposable injective $\Lambda$-modules.

Proposition 3.1.5. Let $\Gamma$ be a basic algebra of dominant dimension at least 2. Let $\tilde{Q}$ be the direct sum of representatives of the isomorphism classes of all indecomposable projective-injective $\Gamma$-modules. Then the algebra $\Lambda$ and $\Lambda$-module $X$ can be chosen to be $\Lambda = \text{End}_\Gamma(\tilde{Q})^{\text{op}}$ and $X = \text{Hom}_\Gamma(\tilde{Q}, D\Gamma)$ so that $\Gamma \simeq \text{End}_\Lambda(X)^{\text{op}}$. 
We now describe the tilting module $T_C$ (whose existence is given by Theorem\ref{thm:tilting}) in terms of the algebra $\Lambda$ and a generator-cogenerator $X$.

**Proposition 3.1.6.** Let $\Gamma$ be a basic algebra with $\text{domdim} \Gamma \geq 2$. Let $T_C$ be the tilting module in $\mathcal{C}_T$. Let $\Gamma \cong \text{End}_A(X)^{op}$, for an artin algebra $A$ and a $\Lambda$-generator-cogenerator $X$. Then:

1. $T_C \cong (\bigoplus_i \text{Hom}_A(X, I_i)) \oplus (\bigoplus_j \text{Hom}_A(X, I_0(X_j)/X_j))$, where $\{I_i\}$ are the indecomposable injective $\Lambda$-modules, $\{X_j\}$ are the non-injective indecomposable direct summands of $X$, and $\{I_0(X_j)\}$ are the corresponding injective envelopes of $\{X_j\}$.

2. In part (1), the first summand is a projective-injective $\Gamma$-module isomorphic to $\tilde{Q}$, and the second summand is a non-projective-injective $\Gamma$-module isomorphic to $\left( \bigoplus_i \Omega^{-1}P_i \right)$, as described in Corollary\ref{cor:projective-injective}.

### 3.2. The endomorphism algebra of the tilting module

Using the Morita-Tachikawa correspondence as in Theorem\ref{thm:morita} for any artin algebra $\Lambda$, taking a generator-cogenerator $X \in \text{mod} \Lambda$, the algebra $\Gamma = \text{End}_A(X)^{op}$ is an artin algebra of dominant dimension at least 2. So by Theorem\ref{thm:tilting} there exists a unique tilting module $T_C \in \mathcal{C}_T$. We are going to study the global dimension of the endomorphism algebra $B_C := \text{End}_F(T_C)^{op}$ and in the next section, we obtain its relationship with the Finitistic Dimension Conjecture.

We recall some facts about tilting modules and torsion classes due to Brenner and Butler (see\cite[VI, §3, §4]{BrennerButler} for more details and proofs). Let $\Lambda$ be an artin algebra and $T$ be any tilting $\Lambda$-module. Then there is a torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\text{mod} \Lambda$:

- $\mathcal{T}(T) := \text{Gen}_\Lambda(T) = \{ M \in \text{mod} \Lambda \mid \text{Ext}_\Lambda^1(T, M) = 0 \}$ and \((*)\)
- $\mathcal{F}(T) := \text{Sub}_\Lambda(\tau T) = \{ M \in \text{mod} \Lambda \mid \text{Hom}_\Lambda(T, M) = 0 \}$. \((**)\)

Let $B := \text{End}_A(T)^{op}$, then $T$ is also a tilting $B^{op}$-module. We have a torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$:

- $\mathcal{X}(T) := D \text{Gen}_B(T) = \{ M \in \text{mod} B \mid \text{Hom}_B(M, DT) = 0 \}$ and
- $\mathcal{Y}(T) := D \text{Sub}_B(\tau T) = \{ M \in \text{mod} B \mid \text{Ext}_B^1(M, DT) = 0 \}$.

In all four descriptions above, the first equalities may be considered as definitions and the second equalities are consequences of the results stated in\cite[VI, §3, §4]{BrennerButler}.

**Theorem 3.2.1** (Brenner-Butler Tilting Theorem).\cite[VI, §3]{BrennerButler}

1. The functors $\text{Hom}_\Lambda(T, -)$ and $- \otimes_B T$ induce quasi-inverse equivalences:

$$
\begin{array}{ccc}
\mathcal{T}(T) & \overset{\text{Hom}_\Lambda(T, -)}{\leftarrow} & \mathcal{F}(T) \\
\otimes_B T & \overset{(-)}{\rightarrow} & \mathcal{Y}(T).
\end{array}
$$

2. The functors $\text{Ext}_\Lambda^1(T, -)$ and $\text{Tor}_B^1(-, T)$ induce quasi-inverse equivalences:

$$
\begin{array}{ccc}
\mathcal{F}(T) & \overset{\text{Ext}_\Lambda^1(T, -)}{\leftarrow} & \mathcal{X}(T) \\
\text{Tor}_B^1(-, T) & \overset{(-)}{\rightarrow} & \mathcal{Y}(T).
\end{array}
$$

We will first state and prove several general lemmas about tilting modules.

**Lemma 3.2.2.** Let $\Lambda$ be an artin algebra, $T$ be any tilting $\Lambda$-module, and $B := \text{End}_A(T)^{op}$.

1. Let $U$ be a $B$-module. Then the first syzygy $\Omega U$ of $U$ is in $\mathcal{Y}(T)$.
2. There exists a $\Lambda$-module $M \in \mathcal{T}(T)$ such that $\Omega U \cong \text{Hom}_\Lambda(T, M)$.

**Proof.** Let $U$ be a $B$-module and $\text{Hom}_\Lambda(T, T_0) \rightarrow U$ be its projective cover. In the exact sequence $0 \rightarrow \Omega U \rightarrow \text{Hom}_\Lambda(T, T_0) \rightarrow U \rightarrow 0$, the middle term $\text{Hom}_\Lambda(T, T_0)$ is in $\mathcal{Y}(T)$ by Theorem\ref{thm:tilting} (a), and $\mathcal{Y}(T)$ is a torsion-free class so it is closed under submodules. Therefore, the first syzygy $\Omega U$ of $U$ is in $\mathcal{Y}(T)$. So $\Omega U \cong \text{Hom}_\Lambda(T, M)$, for some $M \in \mathcal{T}(T) = \text{Gen}(T)$. \qed
**Example 3.2.7.** This is an example of $\Gamma$ such that $\text{gldim} \mathcal{N} \text{akayama algebra}$ given by the following quiver and relations $\gamma \beta \alpha = \delta \gamma \beta = \alpha \epsilon = 0$. We

**Lemma 3.2.3.** Let $\Lambda$ be an artin algebra, $T$ be any tilting $\Lambda$-module, and $B := \text{End}_\Lambda(T)^{op}$.

1. Let $M$ be a $\Lambda$-module where $M \in \mathcal{T}(T)$. Then $\text{projdim}_B(\text{Hom}_\Lambda(T, M)) \leq \text{projdim}_\Lambda M$.
2. Let $U$ be a non-projective $B$-module and $M$ be a $\Lambda$-module such that $\Omega U \cong \text{Hom}_\Lambda(T, M)$.

Then $\text{projdim}_B U \leq \text{projdim}_\Lambda M + 1$.

3. $\text{gldim} B \leq \text{gldim} \Lambda + 1$.
4. $\text{gldim} \Lambda \leq \text{gldim} B + 1$.
5. $| \text{gldim} B - \text{gldim} \Lambda | \leq 1$.

**Proof.** (1) follows from [11, VI, Lemma 4.1].
(2) $\text{projdim}_B U \leq \text{projdim}_B \Omega U + 1 = \text{projdim}_B(\text{Hom}_\Lambda(T, M)) + 1 \leq \text{projdim}_\Lambda M + 1$.
(3) This is a consequence of (2).
(4) Since $B^T$ is a tilting $B$-module and $\Lambda \cong \text{End}_B(T)^{op}$, it follows from (3) that $\text{gldim} \Lambda \leq \text{gldim} B + 1$. Finally, (5) is just a combination of (3) and (4).

Now for our unique tilting module $T_C$ in $\mathcal{C}_\Gamma$, we have more precise results: 3.2.4 and 3.2.5.

**Lemma 3.2.4.** Let $\Gamma$ be an artin algebra with $\text{domdim} \Gamma \geq 2$, $T_C$ be the unique tilting module in $\mathcal{C}_\Gamma$, and $B_C = \text{End}_\Gamma(T_C)^{op}$. If $M \in \text{Gen}(T_C)$ and $\text{projdim}_\Gamma M \geq 1$, then

$$\text{projdim}_{B_C}(\text{Hom}_\Gamma(T_C, M)) = (\text{projdim}_\Gamma M) - 1.$$ 

**Proof.** We use induction on $\text{projdim}_\Gamma M$. First, assume $\text{projdim}_\Gamma M = 1$. Since $M \in \text{Gen}(T_C) = \text{Gen}(\tilde{Q})$, we know there is some $Q_0 \in \text{add} \tilde{Q}$ such that $Q_0 \rightarrow M$ is an epimorphism. Since $\text{projdim}_\Gamma M = 1$, $M$ has a projective resolution:

$$0 \rightarrow P \rightarrow Q_0 \rightarrow M \rightarrow 0.$$ 

Since $P \in \text{add} \Gamma$ and $\text{domdim} \Gamma \geq 2$, the module $M$ is a submodule of a projective-injective $\Gamma$-module $Q_1$. Therefore $M$ is in $\mathcal{C}_\Gamma$. Additionally, as $\text{projdim}_\Gamma M = 1$ by assumption, it follows from Proposition 1.2.5 that $M \in \text{add} T_C$. Hence $\text{projdim}_{B_C}(\text{Hom}_\Gamma(T_C, M)) = 0$.

Now assume $\text{projdim}_\Gamma M = d > 1$. In particular, $M \notin \text{add} T_C$. There is an exact sequence:

$$0 \rightarrow L \rightarrow T_0 \xrightarrow{f} M \rightarrow 0,$$

where $f$ is a right $\text{add} T_C$-approximation and $L \neq 0$. This induces an exact sequence of $B_C$-modules:

$$0 \rightarrow \text{Hom}_\Gamma(T_C, L) \rightarrow \text{Hom}_\Gamma(T_C, T_0) \rightarrow \text{Hom}_\Gamma(T_C, M) \rightarrow 0.$$ 

It follows that, $\text{Ext}^1_\Gamma(T_C, L) = 0$, which implies $L \in \text{Gen}(T_C)$ by (*)).

Since $\text{projdim}_\Gamma T_0 \leq 1$ and $\text{projdim}_\Gamma M = d > 1$, it follows that $\text{projdim}_\Gamma L \leq d - 1$ by the standard arguments. By induction hypothesis, $\text{projdim}_{B_C}(\text{Hom}_\Gamma(T_C, L)) = d - 2$. Therefore, due to the above exact sequence, $\text{projdim}_{B_C}(\text{Hom}_\Gamma(T_C, M)) = d - 1$. \[\Box\]

**Corollary 3.2.5.** Let $\Gamma$ be an artin algebra with $\text{domdim} \Gamma \geq 2$, $T_C$ be the unique tilting module in $\mathcal{C}_\Gamma$, and $B_C = \text{End}_\Gamma(T_C)^{op}$. Then $\text{gldim} B_C \leq \text{gldim} \Gamma$.

**Proof.** Let $U$ be a $B_C$-module. Then by Lemma 3.2.2(2) and Lemma 3.2.4 it follows that $\text{projdim}_{B_C} U \leq \text{projdim}_{B_C}(\text{Hom}_\Gamma(T_C, M)) + 1 = (\text{projdim}_\Gamma M - 1) + 1 \leq \text{gldim} \Gamma$. \[\Box\]

**Remark 3.2.6.** Let $\Gamma$, $T_C$, and $B_C$ be as in Corollary 3.2.5.

1. The sharp inequality $\text{gldim} B_C < \text{gldim} \Gamma$ does not always hold, see the Example 3.2.7.
2. By Lemma 3.2.3(5), either $\text{gldim} B_C = \text{gldim} \Gamma$ or $\text{gldim} B_C = \text{gldim} \Gamma - 1$.

**Example 3.2.7.** This is an example of $\Gamma$ such that $\text{gldim} B_C = \text{gldim} \Gamma$. Let $\Gamma$ be the Nakayama algebra given by the following quiver and relations $\gamma \beta \alpha = \delta \gamma \beta = \alpha \epsilon = 0$. We
omit the modules when drawing the AR-quiver.

\[
\begin{array}{ccc}
2 & & 1 \\
\delta & \alpha & \\
3 & & 4
\end{array}
\]

with the AR-quiver

The subcategory \( C \) contains a tilting module \( T_C = P_1 \oplus P_4 \oplus P_5 \oplus S_4 \oplus S_3 \). One can check that \( \text{domdim} \Gamma = 2 \), \( \text{gldim} \Gamma = 4 \), and \( \text{gldim} \text{End}_\Gamma(T_C) = 4 \).

In the next discussion, we are going to show exactly when it holds that \( \text{gldim} B_C < \text{gldim} \Gamma \).

The proof relies on the following easy fact about homological dimensions.

**Lemma 3.2.8.** Let \( \Lambda \) be an artin algebra with \( \text{gldim} \Lambda = d \). If \( N \) is a \( \Lambda \)-submodule of \( M \) with \( \text{projdim} N = d \), then \( \text{projdim} M = d \).

**Theorem 3.2.9.** Let \( \Gamma \) be an artin algebra with \( \text{domdim} \Gamma \geq 2 \), \( T_C \) be the unique tilting module in \( C \), and \( B_C = \text{End}_\Gamma(T_C)^{op} \). Then

\[
\text{gldim} B_C < \text{gldim} \Gamma \text{ if and only if } \text{projdim}_\Gamma(\tau T_C) < \text{gldim} \Gamma.
\]

**Proof.** Let \( \text{gldim} \Gamma = d \). Then:

\( \Leftrightarrow \): Suppose \( \text{projdim}_\Gamma(\tau T_C) < d \). We prove by contradiction: assume \( \text{gldim} B_C = d \). So, there is a \( B_C \)-module \( U \) with \( \text{projdim}_{B_C} U = d \). We have an exact sequence of \( B_C \)-modules:

\[
0 \to \Omega U \to \text{Hom}_\Gamma(T_C, T_0) \to U \to 0.
\]

Then the first syzygy \( \Omega U \cong \text{Hom}_\Gamma(T_C, M) \), for some \( M \in \text{Gen}(T_C) \) by Lemma 3.2.2. Thus, \( \text{projdim}_{B_C}(\text{Hom}_\Gamma(T_C, M)) = d - 1 \) and \( \text{projdim}_\Gamma M = d \) due to Lemma 3.2.4.

As \( \text{Hom}_{B_C}(\text{Hom}_\Gamma(T_C, M), \text{Hom}_\Gamma(T_C, T_0)) \cong \text{Hom}_\Gamma(M, T_0) \), the embedding \( \text{Hom}_\Gamma(T_C, M) \to \text{Hom}_\Gamma(T_C, T_0) \) is induced by a morphism \( f : M \to T_0 \). Since \( T_0 \) is in \( \text{Sub}(\tilde{Q}) \) and \( \text{projdim}_\Gamma M = d \), the map \( f \) is not a monomorphism. But \( \text{Hom}_\Gamma(T_C, \text{Ker}(f)) = 0 \) since \( \text{Hom}_\Gamma(T_C, M) \to \text{Hom}_\Gamma(T_C, T_0) \) is a monomorphism. Hence, \( \text{Ker}(f) \in \mathcal{F}(T_C) = \text{Sub}(\tau T_C) \). From Lemma 3.2.8 and our assumption \( \text{projdim}_\Gamma(\tau T_C) < d \), we know \( \text{projdim}_\Gamma \text{Ker}(f) < d \).

Consider the exact sequences:

\[
0 \to \text{Ker}(f) \to M \to \text{Im}(f) \to 0, \text{ and} \\
0 \to \text{Im}(f) \to T_0 \to \text{Coker}(f) \to 0.
\]

Since \( \text{projdim}_\Gamma \text{Ker}(f) < d \) and \( \text{projdim}_\Gamma M = d \), we have \( \text{projdim}_\Gamma \text{Im}(f) = d \). However, by Lemma 3.2.8 and the above second exact sequence, it implies that \( \text{projdim}_\Gamma T_0 = d \geq 2 \), which is a contradiction. Therefore, we must have \( \text{gldim} B_C < d \).

\( \Leftrightarrow \): By Proposition 2.4.7, it follows that \( \text{Gdim} \Gamma = d \). Since \( T_C \) is the direct sum of the first cosyzygy of the injective resolution of \( \Gamma \) and \( \tilde{Q} \), it follows that \( \text{injdim} T_C = d - 1 \). According to [13, Theorem 3.2], it follows that \( \text{projdim}_\Gamma \tau T_C < d \).

**Example 3.2.10.** This example illustrates Theorem 3.2.9. Let \( Q \) be the quiver

\[
1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \leftarrow 5
\]

and \( \Gamma = kQ / \text{rad}^2(kQ) \). Then \( \text{gldim} \Gamma = 4 = \text{domdim} \Gamma \), \( \text{projdim}_\Gamma(\tau T_C) = 0 \) and \( \text{gldim} B_C = 3 \).

Finally, we give a digression on the condition \( \text{projdim}_\Gamma(\tau T_C) < \text{gldim} \Gamma \).
Lemma 3.2.11. Suppose $\Gamma$ is an artin algebra with $\text{gldim} \Gamma = d$. Assume the tilting module $T_C$ exists in $C_\Gamma$, then the following statements are equivalent:

1. $\text{projdim} (\tau T_C) < d$,
2. $\text{Ext}^d_C(\tau T_C, M) = 0$ for all $\Gamma$-modules $M$,
3. $\text{Ext}^d_C(\tau T_C, S) = 0$ for all simple $\Gamma$-modules $S$ such that $\text{injdim} S = d$,
4. $\tau^{-1}(\Sigma^{d-1}S) \in \text{Gen}(\tilde{Q})$, for all simple $\Gamma$-modules $S$ satisfying $\text{injdim} S = d$, where $\tilde{Q}$ is an additive generator of projective-injective $\Gamma$-modules and $\Sigma^{d-1}S$ is the $(d-1)$-th syzygy of $S$ in the minimal injective resolution.

Proof. (1) $\iff$ (2), (2) $\implies$ (3) Trivial.

(3) $\implies$ (4) Notice that $\text{Ext}^d_C(\tau T_C, S) \simeq \text{Ext}^1_C(\tau T_C, \Sigma^{d-1}S)$. Because $\text{projdim} T_C = 1$ and $\Sigma^{d-1}S = 1$, it follows by [I, IV,2.14] that $\text{Ext}^1_C(\tau T_C, \Sigma^{d-1}S) \simeq \text{Ext}^1_C(T_C, \tau^{-1}(\Sigma^{d-1}S))$. Hence $\text{Ext}^d_C(\tau T_C, S) = 0$ if and only if $\text{Ext}^1_C(T_C, \tau^{-1}(\Sigma^{d-1}S)) = 0$ if and only if $\tau^{-1}(\Sigma^{d-1}S) \in \text{Gen}(T_C) = \text{Gen}(\tilde{Q})$.

Remark 3.2.12. Notice that if a simple $\Gamma$-module $S$ satisfies $\text{injdim} S = \text{gldim} \Gamma < \infty$, then its projective cover $P(S)$ is not injective.

Let $0 \to S \to I_0(S) \to I_1(S) \to \cdots$ be the minimal injective resolution of a simple module $S$, $\nu = D \text{Hom}_\Gamma(-, \Gamma)$ be the Nakayama functor and $\nu^{-1} = \text{Hom}_\Gamma(D\Gamma, -)$ be its quasi-inverse.

Proposition 3.2.13. Suppose $\Gamma$ is an artin algebra with $\text{gldim} \Gamma = d$. Assume the tilting module $T_C$ exists in $C_\Gamma$, then $\text{projdim}_C(\tau T_C) < d$ if and only if $\nu^{-1}I_d(S)$ is injective, for any simple $\Gamma$-module $S$ with $\text{injdim} S = d$.

Proof. By Lemma 3.2.11 $\text{projdim}_C(\tau T_C) < d$ if and only if $\tau^{-1}(\Sigma^{d-1}S) \in \text{Gen}(\tilde{Q})$, for all simple modules $S$ such that $\text{injdim} S = d$.

Applying $\nu^{-1}$ to the following minimal injective resolution

$$0 \to \Sigma^{d-1}S \to I_{d-1}(S) \to I_d(S) \to 0,$$

we have an exact sequence:

$$0 \to \nu^{-1}\Sigma^{d-1}S \to \nu^{-1}I_{d-1}(S) \to \nu^{-1}I_d(S) \to \tau^{-1}(\Sigma^{d-1}S) \to 0.$$

Hence the assertion follows from the fact that $\tau^{-1}(\Sigma^{d-1}S) \in \text{Gen}(\tilde{Q})$ if and only if its projective cover $\nu^{-1}I_d(S)$ is injective.

Proposition 3.2.14. Let $\Lambda$ be an artin algebra and $M \simeq \bigoplus_i M_i$ be a generator-cogenerator, where the $M_i$ are non-isomorphic indecomposable $\Lambda$-modules. Denote $\Gamma = \text{End}_\Lambda(M)^{op}$. Then

1. The complete set of representatives of non-isomorphic indecomposable projective $\Gamma$-modules is given by $\{ D \text{Hom}_\Lambda(M_i, M) \}$.
2. There are $\Gamma$-module isomorphisms $\nu^{-1}D \text{Hom}_\Lambda(M_i, M) \simeq \text{Hom}_\Lambda(M_i, M_i)$.
3. The module $\nu^{-1}D \text{Hom}_\Lambda(M_i, M)$ is injective if and only if $M_i$ is an injective $\Lambda$-module.
3.3. A relation to the Finitistic Dimension Conjecture. In this section, we obtain an application of the results in Section 3.2 to the Finitistic Dimension Conjecture for a certain class of artin algebras of representation dimension at most 4. First, let us recall:

**Definition 3.3.1.** Let $\Lambda$ be an artin algebra. Then the **finitistic dimension** of $\Lambda$ is:

$$\text{findim} \Lambda := \sup \{\text{projdim} M \mid M \in \text{mod} \Lambda \text{ and } \text{projdim} M < \infty\}.$$  

The **representation dimension** of $\Lambda$ is:

$$\text{repdim} \Lambda := \inf \{\text{gldim} \text{End}_\Lambda(X) \mid X \text{ is a generator-cogenerator in mod } \Lambda\}.$$  

Iyama [16] proved that for any artin algebra $\Lambda$, its $\text{repdim} \Lambda < \infty$ always. On the other hand, the long-standing Finitistic Dimension Conjecture says that for any artin algebra $\Lambda$, its $\text{findim} \Lambda$ is finite. In [15], Igusa and Todorov proved a partial result of this conjecture. In particular, they proved that $\text{findim} \Lambda < \infty$ provided $\text{repdim} \Lambda \leq 3$. Their proof relied on the following result using the IT function $\psi$ defined as:

**Definition 3.3.2.** Let $\Lambda$ be an artin algebra. Let $K_0$ be the abelian group generated by $[X]$, for all finitely generated $\Lambda$-modules $X$, modulo the relations:

(a) $[C] = [A] + [B]$ if $C = A \oplus B$ and (b) $[P] = 0$ for projective $\Lambda$-modules $[P]$.

Let $L : K_0 \to K_0$ be the group homomorphism defined by $L[X] = [\Omega X]$.

For any $\Lambda$-module $M$, denote by $(\text{add } M)$ the subgroup of $K_0$ generated by $[M_i]$ where $M_i$’s are indecomposable summands of $M$. Then the **IT-functions** are:

$$\phi(M) := \min \{m \mid L^m(\text{add } M) \cong L^{m+1}(\text{add } M)\}$$  

$$\psi(M) := \phi(M) + \sup \{\text{projdim} X \mid \text{projdim} X < \infty, X \text{ is a summand of } \Omega^{\phi(M)} M\}.$$  

**Remark 3.3.3.** For any $\Lambda$-module $M$, the IT-functions $\phi(M)$ and $\psi(M)$ are always finite. When $\text{projdim}_\Lambda M < \infty$, it is easy to see that $\phi(M) = \psi(M) = \text{projdim}_\Lambda M$. So IT-functions are generalizations of projective dimension.

The next lemma is also a generalization of the well-known result about projective dimensions: $\text{projdim} C \leq \text{projdim}(A \oplus B) + 1$ when $\text{projdim} A$ and $\text{projdim} B$ are finite.

**Lemma 3.3.4.** Suppose that $0 \to A \to B \to C \to 0$ is a short exact sequence of finitely generated $\Lambda$-modules and $C$ has finite projective dimension. Then $\text{projdim}_\Lambda C \leq \psi(A \oplus B) + 1$.

**Corollary 3.3.5.** [15] Let $\Gamma$ be an artin algebra with $\text{gldim} \Gamma \leq 3$. Let $\Lambda = \text{End}_\Gamma(P)^{\text{op}}$, where $P$ is a projective $\Gamma$-module. Then $\text{findim} \Lambda \leq \psi(\text{Hom}_\Gamma(P, \Gamma)) + 3$, where $\text{Hom}_\Gamma(P, \Gamma)$ is considered as a $\Lambda$-module.

Motivated by Igusa-Todorov’s result, Wei introduced in [26] the notion of $m$-IT algebra, for any non-negative integer $m$:

**Definition 3.3.6.** [26] Let $\Lambda$ be an artin algebra. Then $\Lambda$ is said to be $m$-IT if there exists a module $V$ such that for any $\Lambda$-module $M$ there is an exact sequence

$$0 \to V_1 \to V_0 \to \Omega^n M \oplus P \to 0,$$

where $V_0, V_1 \in \text{add } V$ and $P$ is a projective $\Lambda$-module.

Applying Lemma 3.3.4 it is easy to see that the finitistic dimension of an $m$-IT algebra $\Lambda$ is bounded as $\text{findim} \Lambda \leq m + 1 + \psi(V)$. Consequently, the Finitistic Dimension Conjecture holds for $m$-IT algebras, [26, Theorem 1.1].

It was shown that the class of 2-IT algebras is closed under taking endomorphism algebras of projective modules, [26, Theorem 1.2]. Artin algebras with global dimension $d$ are $(d-1)$-IT.
If repdim $\Lambda \leq 3$, then $\Lambda$ is 2-IT. However, there exist algebras which are not IT algebras, for example, the exterior algebra of a 3-dimensional vector space.

In the following, we are going to see how the endomorphism algebra $B_C := \text{End}_\Gamma(T_C)^{\text{op}}$ studied in Section 3.2 relates to 2-IT algebras.

To fix the notation, let $\Lambda$ be a basic artin algebra. Since repdim $\Lambda < \infty$, there exists a (multiplicity-free) generator-cogenerator $X \in \text{mod} \Lambda$ such that $\Gamma = \text{End}_\Lambda(X)^{\text{op}}$ has finite global dimension, say gldim $\Gamma = d$. Additionally, $\Gamma$ has dominant dimension at least 2. Let $T_C$ be the unique tilting module in $\mathcal{C}_\Gamma$. Denote by $\tilde{Q}$ the sum of the representatives of isomorphism classes of indecomposable projective-injective $\Gamma$-modules. We have $\Lambda \cong \text{End}_\Gamma(\tilde{Q})^{\text{op}}$ by Proposition 3.1.5.

From the endomorphism algebra $B_C := \text{End}_\Gamma(T_C)^{\text{op}}$, we can also recover the algebra $\Lambda$ as follows: Let $R := \text{Hom}_\Gamma(T_C, \tilde{Q})$ which is a projective $B_C$-module. Then

**Lemma 3.3.7.** $\text{End}_{B_C}(R)^{\text{op}} \cong \Lambda$.

**Proof.** $\text{End}_{B_C}(R) = \text{Hom}_{B_C}(\text{Hom}_\Gamma(T_C, \tilde{Q}), \text{Hom}_\Gamma(T_C, \tilde{Q})) \cong \text{Hom}_\Gamma(\tilde{Q}, \tilde{Q}) = \text{End}_\Gamma(\tilde{Q})$; see [11 VI, §3.2]. Thus, $\Lambda \cong \text{End}_\Gamma(\tilde{Q})^{\text{op}} \cong \text{End}_{B_C}(R)^{\text{op}}$. □

Combining this fact with Corollary 3.3.3 and [26, Theorem 1.2], we have:

**Corollary 3.3.8.** Suppose gldim $B_C \leq 3$. Then $B_C$ is 2-IT and hence $\Lambda \cong \text{End}_{B_C}(R)^{\text{op}}$ is also 2-IT, and findim $\Lambda \leq \psi(\text{Hom}_{B_C}(R, B_C)) + 3$.

As a consequence of this result and Theorem 3.2.9, we conclude with the following special case of the Finitistic Dimension Conjecture: for an artin algebra $\Lambda$, let $X$ be its Auslander generator-cogenerator. Let $\Gamma := \text{End}_\Lambda(X)^{\text{op}}$, then $\Gamma$ is of dominant dimension at least 2 and satisfies gldim $\Gamma = \text{repdim} \Lambda$. Let $T_C$ be the unique tilting module in $\mathcal{C}_\Gamma$. Then:

**Corollary 3.3.9.** If $\text{repdim} \Lambda \leq 4$ and $\text{projdim}_{\Gamma}(\tau T_C) \leq 3$, then findim $\Lambda < \infty$.

**Proof.** With our setting and assumptions, gldim $\Gamma = \text{repdim} \Lambda \leq 4$ and $\text{projdim}_{\Gamma}(\tau T_C) \leq 3$. Let $B_C = \text{End}_\Gamma(T_C)^{\text{op}}$. By Theorem 3.2.9 we must have gldim $B_C \leq 3$. It follows from Corollary 3.3.8 that findim $\Lambda \leq \psi(\text{Hom}_{B_C}(R, B_C)) + 3$, which is finite. □

**Example 3.3.10.** We give an example of an algebra of representation dimension 4 which satisfies the hypothesis in Corollary 3.3.9 and hence the set of such algebras is not empty. Let $\Lambda$ be the Beilinson algebra with 3 vertices, that is, it is defined by the following quiver

\[
\begin{array}{c}
\circ & \circ & \circ \\
3 & 2 & 1 \\
\downarrow x_1 & \downarrow x_2 & \downarrow x_3 \\
\end{array}
\]

with relations $I = \langle x_i x_j - x_j x_i \rangle$ for all $1 \leq i, j \leq 3$. It was studied and shown by Krause-Kussin, Iyama, and Oppermann that repdim $\Lambda = 4$ (see for example [22 Examples 7.3 and A.8]). To check that $\Lambda$ satisfies the hypothesis in Corollary 3.3.9, one needs to apply and check Propositions 3.2.13 and 3.2.14(3).

4. **Special case: Extensions of Auslander algebras by injective modules**

We now describe a procedure to create algebras $\Lambda$ which will have tilting modules that are generated and cogenerated by projective-injective modules, that is, those tilting modules are in $\mathcal{C}_\Lambda$. In particular, we describe a class of algebras, constructed from Auslander algebras by “extending” the Auslander algebras by certain injective modules.
4.1. General triangular matrix construction. We now recall the construction of an algebra \( \Lambda \) from algebras \( R \) and \( S \) and a bimodule \( sM_R \) as investigated in \cite{12}. We also recall some basic properties of these algebras, together with the description of modules over such algebras.

**Definition 4.1.1.** Let \( R \) and \( S \) be finite dimensional algebras over the field \( k \). Let \( sM_R \) be an \( S \)-\( R \)-bimodule. Define an algebra \( \Lambda \), which we will often denote by \( \Lambda = T(R, S, sM_R) := \begin{bmatrix} R & 0 \\ sM_R & S \end{bmatrix} \),

where the multiplication is defined using the bimodule structure of \( sM_R \).

A convenient way of viewing \( \Lambda \)-modules is using the category of triples \( T \) of \( \Lambda \)-modules respectively, and \( R \)-\( S \)-modules. Define an algebra \( \Lambda \), which we will often denote by \( \Lambda = T(R, S, sM_R) \):

\[
\Lambda = T(R, S, sM_R) := \begin{bmatrix} R & 0 \\ sM_R & S \end{bmatrix},
\]

Remark 4.1.3. Let \( \Lambda = T(R, S, sM_R) \) and let \( T \) be the associated category of triples. Then the categories \( \text{mod} \Lambda \) and \( T \) are equivalent. Using this fact we refer to triples as \( \Lambda \)-modules.

**Proposition 4.1.4.** \cite{6} Proposition III.2.5, \cite{12} Let \( \Lambda = T(R, S, sM_R) \) and let \( T \) be the associated category of triples.

1. Indecomposable projective objects in \( T \) are \((0, sP, 0)\) and \((RQ, sM_R \otimes_R Q, \text{Id}_{sM_R \otimes_R Q})\), where \( sP \) and \( RQ \) are indecomposable projective \( S \)-modules and \( R \)-modules respectively.

2. Indecomposable injective objects in \( T \) are \((RJ, 0, 0)\) and \((\text{Hom}_S(M, I), sI, \eta: sM_R \otimes \text{Hom}_S(M, I) \xrightarrow{\cong} sI)\), where \( RJ \) and \( sI \) are indecomposable injective \( R \)-modules and \( S \)-modules respectively and \( \eta(m \otimes f) := f(m) \) is an \( S \)-isomorphism.

The above proposition has a description of all indecomposable projective and injective objects in \( T \) and hence, using equivalence, projective and injective \( \Lambda \)-modules. In addition to this, we will also be using the following functor which relates categories of \( S \) and \( \Lambda \)-modules.

**Proposition 4.1.5.** Let \( \Lambda = T(R, S, sM_R) \). Then \( \Psi(sX) := (0, sX, 0) \) defines a functor \( \Psi: \text{mod} S \to \text{mod} \Lambda \), which has the following properties:

1. \( \Psi \) is fully-faithful.

2. \( \Psi \) preserves kernels.

3. \( \Psi \) preserves projective resolutions.

4.2. Triangular matrix construction from Auslander algebras. In this section, we will look at the triangular matrix where \( S = A \) is an Auslander algebra, \( A \) is a special injective \( A \)-module and \( R = \text{End}_A(E)^{op} \), then \( AE_R \) is an \( A \)-\( R \)-bimodule. For the simplicity of notation we will denote the algebra \( T(R, A, AE_R) \) by \( A[E] \).

**Definition 4.2.1.** Let \( A \) be an Auslander algebra and let \( Q = \bigoplus_{i=1}^l Q_i \), where \( \{Q_1, \ldots, Q_l\} \) is a set of representatives of isomorphism classes of all indecomposable projective-injective \( A \)-modules. We choose an \( A \)-module \( E \) which satisfies the following conditions:

1. \( E = I_1 \oplus \cdots \oplus I_r \), where the \( I_i \) are indecomposable injective \( A \)-modules for all \( i \),

2. \( \text{End}_A(I_i) = K_i \), where \( K_i \) is a field,
Let $A$ be the algebra described in Definition 4.1.1. Then there is a tilting module in $\mathcal{C}_A$ that is a partial tilting module $T_0$ in $\mathcal{C}_A$.

**Lemma 4.2.4.** Let $T_C$ be a tilting module in $\mathcal{C}_A$. Then $(0, T_C, 0)$ is a partial tilting module in $\mathcal{C}_{A[E]}$, with $n_A$ summands, where $n_A$ is the number of non-isomorphic simple $A$-modules.

**Theorem 4.2.6.** Let $A$ be an Auslander algebra and $A[E]$ be the algebra described in Definition 4.2.1. Then there is a tilting module in $\mathcal{C}_{A[E]}$.

**Proof.** Let $T_C$ be a tilting module in $\mathcal{C}_A$. Then $T_{C_A[E]} := (0, T_C, 0) \oplus (\bigoplus_{i=1}^r Y_i)$ is a tilting module in $\mathcal{C}_{A[E]}$, where $Y_i = (K_i, A I_i, \eta_i : A \otimes K_i A I_i)$ is a tilting module in $\mathcal{C}_{A[E]}$. The number of indecomposable summands of $T_{C_A[E]}$ equals $n_A + r$ which is the number of non-isomorphic simple $A[E]$-modules.
5. Special class: Tilting modules in $C_\Lambda$ for Nakayama algebras

5.1. Nakayama algebras. In this section, let $\Lambda$ be any Nakayama algebra. We will show criteria for the subcategory $C_\Lambda$ to contain a tilting module $T_C$. Due to Theorem 2.3.1, it is equivalent to finding Nakayama algebras with dominant dimension at least 2. Notice that such a class of algebras has been classified by Fuller in [11, Lemma 4.3] in a module theoretic way. However, using Auslander-Reiten theory, our descriptions in Corollary 5.2.5 and Theorem 5.3.1 can be regarded as a combinatorial approach.

First, we recall some well known facts about Nakayama algebras. A module $M$ over an artin algebra is called a uniserial module if the set of its submodules is totally ordered by inclusion, or equivalently, there is a unique composition series of $M$. An artin algebra $\Lambda$ is said to be Nakayama algebra if both the indecomposable projective and indecomposable injective modules are uniserial. One can show that all indecomposable modules over a Nakayama algebra are uniserial [6, VI, Theorem 2.1].

Moreover, we have the following classification of Nakayama algebras [11, Theorem V.3.2]: A basic connected artin algebra $\Lambda$ is a Nakayama algebra if and only if its ordinary quiver $Q_\Lambda$ is either a quiver of type $A_n$ with straight orientation or a complete oriented cycle. According to [19], Nakayama algebras whose ordinary quiver is $A_n$ with straight orientation are called Linear-Nakayama algebras and Nakayama algebras whose ordinary quiver is a complete oriented cycle are called the Cyclic-Nakayama algebras. In this section, we always assume $\Lambda$ to be basic and connected.

For any $\Lambda$-module $M$, denote by $l(M)$ the length of $M$. For a Nakayama algebra $\Lambda$, there exists an ordering $\{P_1, P_2, \ldots, P_n\}$ of non-isomorphic indecomposable projective $\Lambda$-modules such that:

(a) $P_{i+1}/\text{rad } P_{i+1} \cong \tau^{-1}(P_i/\text{rad } P_i)$, for $1 \leq i \leq n - 1$; and if $l(P_1) \neq 1$, then $P_1/\text{rad } P_1 \cong (P_n/\text{rad } P_n)$,

(b) $l(P_i) \geq 2$, for $2 \leq i \leq n$,

(c) $l(P_{i+1}) \leq l(P_i) + 1$, for $1 \leq i \leq n - 1$ and $l(P_1) \leq l(P_n) + 1$.

Such an ordering is called a Kupisch series for $\Lambda$ and $(l(P_1), l(P_2), \ldots, l(P_n))$ is called the corresponding admissible sequence for $\Lambda$.

Remark 5.1.1.

(1) The Kupisch series (and hence the admissible sequence) for a Nakayama algebra is always unique up to a cyclic permutation (or simply unique if $l(P_1) = 1$).

(2) Let $(c_1, c_2, \ldots, c_n)$ be a sequence of integers such that $c_j \geq 2$ for all $j \geq 2$, and $c_{j+1} \leq 1 + c_j$ for $j \leq n - 1$, and $c_1 \leq c_n + 1$. There is a Nakayama algebra $\Lambda$ such that $(c_1, c_2, \ldots, c_n)$ is the admissible sequence for $\Lambda$.

5.2. Cyclic-Nakayama algebras with dominant dimension at least 2. Suppose $\Lambda$ is a Cyclic-Nakayama algebra with $n$ simple modules. We always label the vertices of its ordinary quiver in such a way that arrows are $(i + 1 \to i)$ for $1 \leq i \leq n - 1$ and $(1 \to n)$. It is easy to check that $\{P_1, P_2, \ldots, P_n\}$ is a Kupisch series, where $P_i := P(S_i)$ is the projective cover of the simple module $S_i$.

Let $\Lambda$ be a Cyclic-Nakayama algebra with Kupisch series $(P_1, P_2, \ldots, P_n)$. Then we can view the corresponding admissible sequence $(c_1, c_2, \ldots, c_n)$ as a function $c : \mathbb{Z}_n \to \mathbb{Z}$ sending $i \mapsto c_i$ and satisfying $c_{i+1} \leq c_i + 1$ and $c_i \geq 2$. On the other hand, each such function gives rise to a Cyclic-Nakayama algebra.

According to our labeling, it is easy to see:

**Lemma 5.2.1.** Let $P_i := P(S_i)$ be the projective cover of the simple module $S_i$. Then...
(1) \( S_i \cong \tau S_{i+1} \).
(2) \( \soc P_i \cong S_{i-c_i+1} \), where the index \( i-c_i+1 \) is regarded as an element in \( \mathbb{Z}_n \).

**Lemma 5.2.2.** Suppose \( P_i := P(S_i) \) is projective non-injective. The injective envelope \( I(P_i) \) is a projective-injective module. Then \( \soc I(P_i)/P_i \cong S_{i+1} \), where the index \( i+1 \) is regarded as an element in \( \mathbb{Z}_n \).

**Proof.** The exact sequence: \( 0 \to P_i \to I(P_i) \to I(P_i)/P_i \to 0 \) suggests that \( \soc I(P_i)/P_i \cong \tau^{-1} \top P_i = \tau^{-1} S_i \). Then the assertion follows from Lemma 5.2.1. \( \square \)

**Definition 5.2.3.** Define \( Q_c := \{ i \in \mathbb{Z}_n \mid c_i+1 \leq c_i \} \) and \( P_c := \{ i \in \mathbb{Z}_n \mid c_{i+1} = c_i + 1 \} \).

By definition, \( Q_c \cup P_c = \mathbb{Z}_n \). An indecomposable projective module \( P_i \) is also injective if and only if \( i \in Q_c \).

**Theorem 5.2.4.** Let \( P_i \) be an indecomposable projective module with the index \( i \in P_c \). Then \( \domdim P_i \geq 2 \) if and only if \( i \in \{ j-c_j \in \mathbb{Z}_n \mid j \in Q_c \} \).

**Proof.** The dominant dimension \( \domdim P_i \geq 2 \) if and only if \( I(P_i)/P_i \) is a submodule of \( P_j \) for some \( j \in Q_c \), which is equivalent to \( \soc I(P_i)/P_i \cong \soc P_j \).

By Lemmas 5.2.1 and 5.2.2, it is equivalent to say \( i+1 = j-c_j+1 \). Therefore, \( \domdim P_i \geq 2 \) if and only if \( i \in \{ j-c_j \in \mathbb{Z}_n \mid j \in Q_c \} \). \( \square \)

**Corollary 5.2.5.** Suppose \( \Lambda \) is a Cyclic-Nakayama algebra with \( n \) simple modules. Then \( \domdim \Lambda \geq 2 \) if and only if \( P_c \subseteq \{ j-c_j \in \mathbb{Z}_n \mid j \in Q_c \} \).

**Corollary 5.2.6.** If \( \domdim \Lambda \geq 2 \) then \( \vert Q_c \vert \geq \frac{n}{2} \).

At last, we point out that this provides us with a method to find all Cyclic-Nakayama algebras with dominant dimension at least 2.

Suppose \( c \) and \( c' \) are admissible sequences of Cyclic-Nakayama algebras \( \Lambda \) and \( \Lambda' \). We say that \( c \) and \( c' \) are in the same difference class (see [19]) if \( c'_i = c_i + n \) for all \( i \). From Corollary 5.2.5, it is easy to see that if \( c \) and \( c' \) are in the same difference class, then \( \domdim \Lambda \geq 2 \) if and only if \( \domdim \Lambda' \geq 2 \). In fact, the dominant dimension of \( \Lambda \) only depends on the difference class of the admissible sequence [19, Theorem 1.1.4].

Therefore, to find all the Cyclic-Nakayama algebras with dominant dimension at least 2, it is enough to find those with “minimal” admissible sequences.

**Definition 5.2.7.** Suppose \( c \) is an admissible sequence of Cyclic-Nakayama algebras \( \Lambda \). We say that \( c \) is **elementary** if \( \min_{1 \leq i \leq n} \{ c_i \} \leq n + 1 \), and \( c \) is **absolutely elementary** if \( \min_{1 \leq i \leq n} \{ c_i \} = 2 \).

**Example 5.2.8.** When \( n = 3 \), the following are all the (absolutely) elementary admissible sequences for Cyclic-Nakayama algebras with dominant dimension at least 2 (up to cyclic permutations):

Absolutely elementary: \( (2, 2, 2), (2, 2, 3) \).
Elementary: \( (2, 2, 2), (3, 3, 3), (4, 4, 4), (2, 2, 3), (3, 3, 4) \).

5.3. **Linear-Nakayama algebras with dominant dimension at least 2.** Most of the results for Cyclic-Nakayama algebras also works for Linear-Nakayama algebras. For completeness, we will state the criteria for Linear-Nakayama algebras \( \Lambda \) having \( \domdim \Lambda \geq 2 \).

Suppose \( \Lambda \) is a Nakayama algebra whose underlying quiver is of type \( A_n \) with \( n \) simple modules. We always label the vertices of its quiver in such a way that arrows are \((i+1 \to i)\)
for \(1 \leq i \leq n-1\). It is easy to check that \(\{P_1, P_2, \ldots, P_n\}\) is a Kupisch series, where \(P_i := P(S_i)\) is the projective cover of simple module \(S_i\).

The corresponding admissible sequence \((c_1, c_2, \ldots, c_n)\) satisfies \(c_1 = 1\), \(c_{i+1} \leq c_i + 1\), and \(c_i \geq 2\) for \(2 \leq i \leq n\). On the other hand, each such sequence gives rise to a Linear-Nakayama algebra. Define \(Q_c := \{i \mid c_{i+1} \leq c_i\}\) and \(P_c := \{i \mid c_{i+1} = c_i + 1\}\) as in Definition 5.2.3. Notice that for Linear-Nakayama algebras \(c_2 = 2\), \(c_i \leq i\) and \(1 \in P_c\).

**Theorem 5.3.1.** Suppose \(\Lambda\) is a Linear-Nakayama algebra with \(n\) simple modules. Then \(\text{domdim} \Lambda \geq 2\) if and only if \(P_c \subseteq \{j - c_j \mid j \in Q_c\}\).

5.4. **The tilting module \(T_C\) for Nakayama algebras.** Lastly, for a Nakayama algebra \(\Lambda\), we give a description of the tilting module \(T_C\) in the subcategory \(C_\Lambda\) if it exists.

Let \(\Lambda\) be a Nakayama algebra with Kupisch series \((P_1, P_2, \ldots, P_n)\) and admissible sequence \((c_1, c_2, \ldots, c_n)\). Let \(Q_c\) and \(P_c\) be the sets as defined in the previous sections. Then for each \(i \in P_c\), define \(\delta(i) := \min\{k \in \mathbb{N} \mid i + k \in Q_c\}\).

According to Corollary 2.2.4 we have the following description of the tilting module \(T_C\):

**Theorem 5.4.1.** Let \(\Lambda\) be a Nakayama algebra with Kupisch series \((P_1, P_2, \ldots, P_n)\) and admissible sequence \((c_1, c_2, \ldots, c_n)\). If the tilting module \(T_C\) exists in \(C_\Lambda\), then

\[
T_C \simeq \left( \bigoplus_{j \in Q_c} P_j \right) \oplus \left( \bigoplus_{i \in P_c} T_i \right),
\]

where each \(T_i\) is a uniserial module with socle \(\text{soc} T_i = i + 1\) and length \(|T_i| = \delta(i)|).
[15] K. Igusa, G. Todorov, *On finitistic global dimension conjecture for artin algebras*, Rep. of algebras and related topics, Fields Inst. Commun., 45, Amer. Math. Soc., Providence, RI (2005), 201–204.

[16] O. Iyama, *Finiteness of representation dimension*, Proc. Amer. Math. Soc., 131 (2003), 1011–1014.

[17] O. Iyama, *Auslander-Reiten theory revisited*, Trends in representation theory of algebras and related topics, 349–397, EMS Ser. Congr. Rep., Eur. Math. Soc., Zurich, 2008.

[18] O. Iyama and Õ. Solberg, *Auslander-Gorenstein algebras and precluster tilting*, arXiv:1608.04179.

[19] R. Marczinzik, *Upper bounds for the dominant dimension of Nakayama and related algebras*, arXiv:1605.09653.

[20] K. Morita, *Duality for modules and its applications in the theory of rings with minimum condition*, Sci. Rep. Tokyo Daigaku A, 6 (1958), 83–142.

[21] B.J. Mueller, *The classification of algebras by dominant dimension*, Canadian Journal of Mathematics, 20 (1968), 398–409.

[22] S. Oppermann, *Lower bounds for Auslander’s representation dimension*, Duke Math J., 148 (2009), 211–249.

[23] M. Pressland and J. Sauter, *Special tilting modules for algebras with positive dominant dimension*, arXiv:1705.03367.

[24] C.M. Ringel, *Artin algebras of dominant dimension at least 2*, preprint manuscript Selected Topics CMR.

[25] H. Tachikawa, *On dominant dimension of QF-3 algebras*, Trans. Amer. Math. Soc., 112 (1964), 249–266.

[26] J. Wei, *Finitistic dimension and Igusa-Todorov algebras*, Adv. Math., 222 (2009), 2215–2226.

[27] A. Zaks, *Injective dimension of semiprimary rings*, J. Algebra, 13 (1969), 73–86.

**Department of Mathematics, Hood College, Frederick, MD 21701, USA**

*E-mail address*: nguyen@hood.edu

**Institutt for matematiske fag, Norges Teknisk-Naturvitenskapelige Universitet, N-7491 Trondheim, Norway**

*E-mail address*: idun.reiten@ntnu.no

**Department of Mathematics, Northeastern University, Boston, MA 02115, USA**

*E-mail address*: g.todorov@northeastern.edu

**Department of Mathematics, Northeastern University, Boston, MA 02115, USA**

*E-mail address*: zhu.shi@husky.neu.edu