Sine-Gordon system with hysteretic links

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Abstract. In this work we consider a nonlinear dynamical system which is a set of nonlinear oscillators coupled by springs with hysteretic blocks (modified sine-Gordon system or hysteretic sine-Gordon model where the hysteretic nonlinearity is simulated by the Bouc-Wen model). We investigate the wave processes (namely, the solitonic solutions) in such a system taking into account the hysteretic nonlinearity in the coupling.

1. Introduction

Oscillatory processes are widely used in various fields of both fundamental and applied science. The theory of oscillations, which studies oscillations occurring in various systems, is an intensively developing field of modern mathematics and physics [1, 2, 3, 4]. The main models of the theory of oscillations are the linear and nonlinear oscillators, rotators, RLC circuit, etc. These are used in modeling of physical processes in various real-life systems. New features of oscillatory processes appear in the cases when there is a large number of interacting subsystems. The standard model of wave processes is a finite and infinite chain of coupled (interacting) oscillators. Such chains are often used in radio engineering as filters that allocate or suppress signals with frequencies lying in a certain band. From the fundamental point of view, chains of oscillators are used as models of solid media with oscillations and waves with various properties [5, 6, 7]. The oscillatory processes of a large number of such elements are called waves. Wave phenomena are widespread in nature: waves on the surface of a fluid, sound waves in a gas, compression-expansion waves in a solid, vibrations of a string and membrane, electromagnetic waves, etc.

Note that, in addition to nonlinear oscillations, there are also nonlinear waves the behaviour of which is described by nonlinear partial differential equations. Within the framework of the theory of nonlinear waves there exist the standard models, similar to the reference models in the theory of oscillations, namely, simple waves, shock waves, as well as the solitary waves (solitons), that play significant roles in the theory of nonlinear processes. One of the basic models for studying the nonlinear processes is the sine-Gordon model (a physical realization of which is a chain of nonlinear oscillators connected by coil springs) [8].
Another example of a strong nonlinear system playing a significant role in modern research is hysteresis. Hysteretic behavior is typical both for the characteristics of substances (ferroelectrics, ferromagnetics, piezoelectrics etc.), and for the dynamics of many mechanical systems (backlash, stop, etc.). In the mechanical systems hysteretic nonlinearities arise due to an aging of the material and must be taken into account at the modeling level for the corresponding mechanical systems. The hysteresis in such systems leads to the problem of investigation of nonlinear operator-differential equations, which is an extremely complex problem.

The purpose of this paper is to investigate the dynamics of oscillatory system taking into account the hysteretic coupling conditions between individual links of such a system. We consider the sine-Gordon model in the case when the links between pendulums contain hysteresis nonlinearities (modified sine-Gordon model). On the basis of numerical modeling, the dynamics of soliton-like solutions in such a system is studied.

2. Bouk-Wen model

Dependencies of hysteretic type are determined by input-output correspondences, when the output depends not only on the instantaneous value of the input, but also on its behavior in the preceding moments of time (memory effect). Mathematical models of mechanical properties of many building materials, such as reinforced concrete, steel, wood, as well as the damping materials, usually include a nonlinear hysteresis mechanism that takes into account the restoring properties of these structures.

Mathematical models of hysteresis-type phenomena are rather diverse and include both design models (backlash, stop, non-ideal relay, as well as their continuous analogues, namely the Ishlinskii and Preisach models [9]), and phenomenological models (S-converter, Duhem model, Bouc-Wen model etc. [10, 11, 12, 13]). In this paper, we focus on the phenomenological approach based on the Bouc-Wen model for describing the hysteresis nonlinearity [13, 14, 15].

Let us consider the equation of motion of a single-degree-of-freedom (figure 1) system:

\[ \mu \ddot{u}(t) + F(u, z) = f(t), \]  

where \( \mu \) is the mass, \( u(t) \) is the displacement, \( F(u, z) \) is the restoring force and \( f(t) \) is the excitation force (hereafter the overdot shows the derivative with respect to time). Following the
Bouc-Wen approach the restoring force is presented as (the corresponding function depends on
the input and output states)

\[ F(u, z) = \alpha ku(t) + (1 - \alpha)kz(t). \]  

(2)

From (2) it follows that the restoring force \( F(u, z) \) can be divided into elastic and hysteretic parts,
where \( k \) is the yielding stiffness, \( \alpha \) is the ratio of post-yield to pre-yield (elastic) stiffnesses and
\( z(t) \) is the non-dimensional hysteretic parameter that obeys the following nonlinear differential
equation with zero initial condition (\( z(0) = 0 \)):

\[ \dot{z}(t) = [A - |z(t)|^n(\beta + \text{sign}(z(t)\dot{u}(t))\gamma)]\dot{u}(t), \]

(3)

where, \( A, \beta, \gamma \) and \( n \) are non-dimensional parameters controlling the behavior of the model and
\( \text{sign}(\cdot) \) is the standard signum-function. For small values of the positive exponential parameter
\( n \) the transition from elastic to post-elastic branch is smooth, whereas for large values of this
parameter the transition becomes abrupt, approaching that of a bilinear model. Parameters \( \beta \) and \( \gamma \) control the size and shape of the hysteretic loop. Thus, such a multi-parameter model
describes wide class of hysteretic systems [13, 14, 15].

3. Sine-Gordon model with hysteretic nonlinearity

The most well-known and well-studied equations in mathematical physics are equations
describing the propagation of waves in a linear medium. For a nonlinear medium with hysteresis
properties, there are no ready-made methods for solution of such equations.

One of the interesting results of the analysis of wave propagation processes in nonlinear media
is the existence of soliton solutions – solitary waves behaving like particles. One of the models
that has a soliton solution is the sine-Gordon system. This system can be presented as a chain
of nonlinear pendulums with elastic torsion-tied links. This model is widely used both in biology
and in physics. This system has many applications, including the propagation of crystal defects
and domains in ferromagnetic and ferroelectric materials, the propagation of splay waves on
biological (lipid) membranes, one-dimensional model of elementary particles and propagation of
magnetic flux quanta in the long Josephson junction [8].

![Figure 2. Sine-Gordon system model.](image)

In what follows we consider a mechanical system with hysteretic links. The physical model
of such a system is shown in figure 2. It is a chain of identical pendulums strung on a string
and connected by springs [16]. Pendulums oscillate transversely to the direction of the chain. The principal feature of the mechanical system under consideration is that the backlash-type hysteretic nonlinearity [17] is included in the connection between two neighboring pendulums. This system is a modification of the classical mechanical sine-Gordon system and can be called hysteretic sine-Gordon system.

Let \( \mu \) be the mass of the pendulum, \( \mu l^2 \) is the moment of inertia, \( l \) is the length, and \( \kappa \) is the torsion constant of the spring. When the deviation of the pendulum with number \( m \) from the equilibrium point by an angle \( \theta_m \) takes place, the gravitational force moment \( -\mu gl \sin \theta_m \) acts on the pendulum alongside the torsional moment acting on the side of adjacent springs \( -\kappa(\theta_m - \theta_{m-1}) + \kappa(\theta_{m+1} - \theta_m) \). Since the hysteretic nonlinearity is included in the system, the equation of motion can be presented as:

\[
\begin{align*}
\mu l^2 \ddot{\theta}_m &= -\mu gl \sin \theta_m + \omega_m^{left} + \omega_m^{right}, \\
\omega_m^{left} &= L[\omega_m^{left}(t_0); y_m^{left}(t_0)]y_m^{left}(t), \\
y_m^{left} &= -\kappa(\theta_m - \theta_{m-1}), \\
\omega_m^{right} &= L[\omega_m^{right}(t_0); y_m^{right}(t_0)]y_m^{right}(t), \\
y_m^{right} &= \kappa(\theta_{m+1} - \theta_m),
\end{align*}
\]

(4)

where the time-dependent outputs \( \omega_m^{left}, \omega_m^{right} \) and inputs \( y_m^{left}, y_m^{right} \) (these inputs are the corresponding moments affecting single pendulum from the left and right sides relative to neighbor pendula respectively) are the corresponding outputs and inputs for the physically realizable converter \( L[\cdot] \) in the frame of Krasnosel’skii and Pokrovskii approach [9], and \( \omega_m(t_0), y_m(0) \) are the corresponding initial states (output and input, respectively) of the converter.

4. Numerical simulation

It is known that the operator interpretation of the hysteretic nonlinearity implies the non-smoothness of the corresponding operator. Therefore, in our numerical simulation we use the approach to hysteresis based on the Bouc-Wen phenomenological model. In this case, the sine-Gordon system with the hysteretic nonlinearity in the links takes the following form:

\[
\begin{align*}
\mu l^2 \ddot{\theta}_m &= -\mu gl \sin \theta_m - \alpha \kappa(\theta_m - \theta_{m-1}) - (1 - \alpha) \kappa z_m^{left} + \alpha \kappa(\theta_{m+1} - \theta_m) + (1 - \alpha) \kappa z_m^{right}, \\
z_m^{left} &= [A - |z_m^{left}|^\kappa(\beta + \text{sign}(z_m^{left}(\dot{\theta}_m - \dot{\theta}_{m-1})))\gamma](\dot{\theta}_m - \dot{\theta}_{m-1}), \\
z_m^{right} &= [A - |z_m^{right}|^\kappa(\beta + \text{sign}(z_m^{right}(\dot{\theta}_{m+1} - \dot{\theta}_m)))\gamma](\dot{\theta}_{m+1} - \dot{\theta}_m).
\end{align*}
\]

(5)

We performed the numerical simulation of the dynamics of the mechanical system described by Equations (5). Namely, we obtained the numerical solutions to the Cauchy’s problem for (5) using the 4-th order Runge-Kutta method (our numerical results obtained using MATLAB® system). For example, for results presented in figure 3 (right panel) the model time is \( t = 200 \) and the corresponding time-step is \( h = 0.1 \). In such a system appearance of soliton-like solutions is expected (in the same manner as for the classical sine-Gordon system). The solitary wave, which is the solution to (5), is treated as a dynamical object that retains energy for a long time.
The chain has a finite length \( m = 100 \) (we recall that we consider a discrete system), and its ends are fixed. The initial conditions for pendulums

\[ \theta_1(t_0), \dot{\theta}_1(t_0), \theta_2(t_0), \dot{\theta}_2(t_0), \ldots, \theta_{m-1}(t_0), \dot{\theta}_{m-1}(t_0), \theta_m(t_0), \dot{\theta}_m(t_0) \]

generate a family of solitonic solutions moving with different velocities along the \((m, t)\)-plane with reflection at the ends of the chain \((m = 1, m = 100)\). For the parameters of the hysteretic blocks formalized by means of the Bouc-Wen model, \( z_{m, \text{left}}^{\text{left}}(t_0), z_{m, \text{right}}(t_0) \), the initial conditions are zero by default.

**Figure 3.** Simulation of the collision of two soliton-like solutions without hysteresis (left panel) and with hysteresis (right panel) in the links.

**Figure 4.** Simulation of the dynamics of localized oscillations of pendulums in a chain without hysteresis (left panel) and with hysteresis (right panel) in the links.

Let us have a look on the dynamics of two solitonic solutions launched from opposite ends of the chain, as shown in figure 3. The corresponding initial conditions

\[ \theta_1(t_0) = 2\pi, \dot{\theta}_1(t_0) = 1, \theta_2(t_0) = 0, \ldots, \dot{\theta}_{m-1}(t_0) = 0, \theta_m(t_0) = 2\pi, \dot{\theta}_m(t_0) = 1 \]
generate two pulses, moving towards each other. During the simulation, two solitary waves collide in situations without \((\alpha = 1\), left panel\) and with \((\alpha = 0.75, \beta = 0.1, \gamma = 0.9\), right panel\) hysteresis in the links. The interaction of two pulses can demonstrate the nature of the colliding formations, since solitons interacting with each other, show special properties (similar to particle behavior). As follows from the numerical results presented in figure 3 (left panel), the dynamics of solutions demonstrates all the properties of soliton-like objects (they do not change their shape and speed). In the case when there are hysteretic connections between the pendulums (right panel in figure 3), soliton-like solution changes the speed (as can be seen by breaking the symmetry of the reflection process at the ends of the chain), retaining its shape, as well as the nature of interaction.

In order to study the influence of hysteresis bonds in the system, we consider the case in which the vibrations of 25th \((\theta_{25}(t_0) = 2\pi, \dot{\theta}_{25}(t_0) = 0)\) and 75th \((\theta_{75}(t_0) = \pi, \dot{\theta}_{75}(t_0) = 0)\) pendulums are excited with the corresponding initial conditions. Under these initial conditions, the oscillations of the corresponding components are excited in the chain (figure 4 (left panel)). In the case when the hysteresis in the links is taken into account (figure 4 (right panel)), spatial localization of oscillations is observed.

Let us consider in more detail the evolution of the states of the components of the chain \((\theta(t), \dot{\theta}(t))\) in the neighborhood of 25th and 75th pendulums. Figures 5 and 6 show the phase portraits for 24th, 25th, 26th, 74th, 75th, 76th pendulums, respectively together with corresponding hysteretical loops (such loops are obtained as a numerical solution to Equation 3 of the Bouc-Wen model). As follows from these figures, in the absence of hysteretic bonds \((\alpha = 1)\) the dynamics of pendulums demonstrates a complex oscillatory structure. However, in the presence of hysteresis in the bonds \((\alpha = 0.5, \beta = 0.1, \gamma = 0.9)\), the dynamics in the neighborhood of the 25th pendulum is regularized and the stable limit cycle can be seen. Note a similar behavior for the 75th pendulum (figure 6).

![Figure 5](image)

**Figure 5.** Phase portraits of 24th, 25th, 26th pendulums without hysteresis (left panel) and with hysteresis (right panel) in the links. The input (right panel) shows the corresponding hysteretic loop obtained as a solution to Equation 3.

Also, we investigated the influence of the hysteretic blocks in the connections between pendulums by using the methods of spectral analysis. We performed the Fourier transform for the 25th pendulum in the presence of hysteretic block \((\alpha = 0.5, \beta = 0.1, \gamma = 0.9)\) and without \((\alpha = 1)\) it. The corresponding results are shown in figure 7. As it follows from the results presented in this figure, the oscillation spectrum changes after inclusion of hysteretic
Figure 6. Phase portraits of 74th, 75th, 76s pendulums without hysteresis (left panel) and with hysteresis (right panel) in the links. The input (right panel) shows the corresponding hysteretic loop obtained as a solution to Equation 3.

Figure 7. The oscillation spectrum of the 25th pendulum without hysteresis (left panel) and with hysteresis (right panel) in the link.

bonds. Thus we can conclude that the hysteresis in such a system plays a role of a "filter" that quenches frequencies corresponding to small-amplitude oscillations and releases the main frequency.

5. Conclusions
In this paper we studied the dynamics of an oscillatory system with many degrees of freedom under conditions of hysteretic blocks in the coupling between the individual parts of the system. The system under consideration can be classified as a modified mechanical model of the sine-Gordon system in the case when the connections between the pendulums contain a hysteretic nonlinearity. The hysteretic nonlinearity was formalized by means of the Bouc-Wen model which allows a fairly simple numerical realization of the solution to the system of equations corresponding to the mechanical system under consideration. On the basis of
numerical simulations, the dynamics of the solitonic solution for this system was studied taking into account the hysteretic nature of the coupling. It was demonstrated that the presence of hysteretic coupling leads to a change in the speed of propagation of the solitary solution while maintaining the character of interaction between various solitary solutions. Also, the results of numerical simulation demonstrate the regularizing role of hysteresis bonds in the character of oscillatory motions of the system under consideration. The filtering properties of hysteretic bonds are inferred from the spectral analysis of the oscillatory motions of individual components of the system.

6. Reference
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