Confinement of $\mathcal{N} = 1$ Super Yang-Mills from Supergravity

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We calculate circular Wilson loop of pure $\mathcal{N} = 1$ super Yang-Mills from Klebanov-Strassker-Tseytlin solution of supergravity and proposed gauge/gravity duality. The calculation is performed numerically via searching minimal surface of string worldsheet. It is shown that Wilson loop exhibits area law for large radius such that $\mathcal{N} = 1$ super Yang-Mills is confined at large distance. Meanwhile, Wilson loop exhibits logarithmic behavior for small radius and indicates asymptotical freedom of $\mathcal{N} = 1$ super Yang-Mills.

I. INTRODUCTION

Physicists now believe non-abelian gauge theory exhibits some dramatic behaviors at long distance, such as confinement and mass gap, etc.. It is difficult to study these properties using traditional quantum field theory (QFT) method due to its nonpertubative characters. However, the remarkable success of AdS/CFT correspondence reveals a practical approach to study strong coupling gauge theory\[1, 2, 3\]. The similar correspondence is believed to exist in broad class of theories even with less supersymmetry and non-conformal theories (so-called gauge/gravity duality). According to continuously breaking supersymmetry, we can expect to study strong-coupling properties of non-abelian gauge theory via dual weak-coupling gravity. So far, several nonsingular supergravity (SUGRA) solution have been constructed. They preserve 1/2 or 1/4 of the maximal supersymmetry and are conjectured dual to $d = 4$, $\mathcal{N} = 2$\[4\] or $\mathcal{N} = 1$\[5, 6, 7\] super Yang-Mills (SYM) theories. In particular, the case $\mathcal{N} = 1$ super SYM is very interesting, because it possesses the common features of non-abelian gauge theory at long distance but is hard to be study using usual QFT method. For this SYM, two dual SUGRA solution were found:

- The Maldacena-Nuñez (MN) solution\[5\]: It corresponds to a large ($N$) number of D5-branes wrapped on a supersymmetric two-cycle inside a Calabi-Yau threefold. Meanwhile, 6d SYM theory onto D5-branes is reduced to pure $d = 4$, $\mathcal{N} = 1$ $SU(N)$ SYM in the IR.

- The Klebanov-Strassler-Tseytlin (KST) solution\[6, 7\]: It describes the geometry of the warped deformed conifold when one places M D3-branes and N fractional D3-branes at the apex of the conifold. It is dual to a certain $\mathcal{N} = 1$ supersymmetric $SU(N + M) \times SU(M)$ gauge theory. If $M$ is a multiple of $N$, then this theory flows to $SU(N)$ in the IR, via a chain of duality cascade which reduces the rank of the gauge group by $N$ units at each cascade jump. Thus at the end of the duality cascade the gauge theory is effectively pure $\mathcal{N} = 1$ $SU(N)$ SYM.

The purpose of this paper is to study confinement of $\mathcal{N} = 1$ SYM from KST solution. We will give a brief comments on MN solution at the end of the paper.

The natural criterion for confinement of pure Yang-Mills theory is expectation value of large Wilson loop satisfying area law\[8\]. The evaluation on Wilson loop of $\mathcal{N} = 4$ SYM from supergravity in AdS space has been studied in various aspects. It was proposed in\[9, 10\] that Wilson loops of the CFT can be described in AdS by

$$< W(C) > \sim \lim_{\Phi \to \infty} e^{-\Phi},$$

(1)

where $S$ is the proper area of a fundamental string world-sheet which lies on the loop $C$ on the boundary of AdS, $l$ is the total length of the Wilson loop and $\Phi$ is the mass of the $W$ boson. The presence of the term $l\Phi$ is to subtract
divergence from infinity mass of the W boson. It corresponds to UV divergence in QFT. The evaluation for the rectangular Wilson loop was performed in refs. 1, 10 and for simpler circular loop was performed in refs. 11, 12. There are no any surprise results due to conformal invariance. The more non-trivial results is from calculation of operator product expansion for the Wilson loop when probed from a distance much larger than the size of the loop 11, and from calculation on Wilson loop correlator 13. The extension of Eq. (1) to KST background is in principle direct. However, since KST background is much more complicated than AdS space, we can not expect to obtain any analytic results. So that although there is a simple scaling argument that large Wilson loops of $N = 1$ SYM resulted by KST solution satisfies the area law 14, the direct verification is still lacked so far. In this paper we use numerical tools to search minimal surface of string world-sheet in KST background which lies on the Wilson loop on the boundary of KST space. In order to simplify the calculation, we consider circular Wilson loop of $N = 1$ SYM in Euclidean space. The proper area of string world-sheet can be expressed as a function of radius of circular loop. We will evaluate both of cases for small and large radius. It can be expected that behavior of Wilson loop exhibits the following conclusions:

- The confinement of $N = 1$ SYM at large distance.
- The asymptotical freedom of $N = 1$ SYM at short distance.
- Phase transition from short distance to large distance.

This study can also be treated as an examination for duality between SUGRA KST solution and $N = 1$ SYM.

The paper is organized as follows. In Sec.II we extend to calculation on Wilson loop in AdS space to KST background and derive Euler-Lagrangian equation which governs minimal surface of string world-sheet. In Sec.III we show some numerical results such as curve of Wilson loop vs. radius of the loop. The physical results are obtained via those numerical results. A brief summary will be devoted in Sec.IV and several details on numerical evaluation are presented in the Appendix.

II. WILSON LOOP OF $N = 1$ SYM FROM KST SOLUTION

The Wilson loop operator in $N = 1$ SYM is

$$ W(\mathcal{C}) = \frac{1}{N} Tr P e^{i \oint \mathcal{C} A} , \tag{2} $$

where $\mathcal{C}$ denotes a closed loop in spacetime, and the trace is taken over the fundamental representation of the gauge group. The expectation of the Wilson loop can be calculated by supergravity in terms of Eq. (1) directly, with action $S$ is given by

$$ S = \frac{1}{2\pi\alpha'} \int d^2 \sigma \sqrt{-g} , \tag{3} $$

where $\sigma^1$ and $\sigma^2$ parameterize fundamental string (Euclidean) world-sheet,

$$ g_{\alpha\beta} = G_{MN} \partial_\alpha X^M \partial_\beta X^N , \tag{4} $$

is induced metric on the world-sheet. In this paper, we focus our attention on nonsingular KST background such that $G_{MN}$ is given by

$$ ds_{10}^2 = h^{-1/2}(\tau) dx_n dx_n + h^{1/2}(\tau) ds_6^2 , $$

$$ h(\tau) = \alpha \int_\tau^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh (2x) - 2 x)^{1/3} = \alpha I(\tau) , \tag{5} $$

where $\tau$ is the radial parameter of the transverse space, $\alpha = 2^{2/3} (g_s N \alpha')^2 \epsilon^{-8/3}$. $ds_6^2$ is the metric of the deformed conifold 12:

$$ ds_6^2 = \frac{1}{2} \epsilon^{4/3} K(\tau) \left[ \frac{1}{3K'(\tau)} (ds^2 + (g^5)^2) + \cosh^2 (\frac{\tau}{2}) \sum_{i=3}^4 (g_i)^2 + \sinh^2 (\frac{\tau}{2}) \sum_{i=1}^2 (g_i)^2 \right] \tag{6} $$

where $\epsilon$ parameterizes length of the deformed conifold and possesses mass dimension $-3/2$, $K(\tau)$ is defined by

$$ K(\tau) = \frac{(\sinh (2\tau) - 2 \tau)^{1/3}}{2^{1/3} \sinh \tau} , \tag{7} $$
and the 1-form basis $g^i$ parameterize transverse five-dimension space. To simplify complicated calculation we do the following notions:

- Set angle parameters of transverse space as constants such that $g^i \equiv 0$. The Wilson loop in $\mathcal{N} = 4$ SYM with non-constant angle parameters has been considered in ref.\[16\]. When we extend this consideration to $\mathcal{N} = 1$ SYM calculation becomes very difficult.
- To define a new dimensionless radial parameter $r = e^{-\tau/3}$, $0 \leq r \leq 1$. The boundary of KST space is $r = 0$.
- We consider that $\mathcal{N} = 1$ SYM is defined in Euclidean space, and circular Wilson loop is located at $x^1 - x^2$ plane. Since we are aiming at finding a minimal surface, the string world-sheet can be considered as a rotational surface. Then we define

$$
x^1 = \frac{3}{2} \alpha'^{1/3} f(r) \cos \theta, 
$$

$$
x^2 = \frac{3}{2} \alpha'^{1/3} f(r) \sin \theta, 
$$

with a dimensionless function $f(r)$. Consequently the string world-sheet is parameterized by $r$ and $\theta$,

$$
ds^2 = \frac{3}{2} \sqrt{\alpha'}/3 I^{-1/2}(r) \left\{ \left[f'^2(r) + \frac{1}{r^2} I(r) K^{-2}(r) \right] dr^2 + f'^2(r) d\theta^2 \right\},
$$

where prime denotes derivative on $r$.

According to the above notions, the action defined in Eq. (3) is rewritten to

$$
S = \frac{3}{2^{2/3}\sqrt{\alpha'} g_s N} \int dr \cdot \sqrt{f'^2 + G(r)},
$$

with

$$
G(r) = \frac{1}{r^2} I(r) K^{-2}(r).
$$

From action (10) we obtain the Euler-Lagrangian equation which govern minimal surface of string world-sheet,

$$
ff'' - \frac{1}{2}(\ln IG)' f' - \frac{I'}{2IG} f'^3 - \frac{1}{f} (f'^2 + G) = 0.
$$

The minimal surface implies the initial conditions of Eq. (12) should be

$$
f(0) = R, \quad f'(0) = 0.
$$

In order to show that this initial condition is taken care of properly, let us find asymptotical solution of Eq. (12) for $r \to 0$. Noticing the expressions,

$$
I(r) \xrightarrow{r \to 0} -\frac{3}{16 \cdot 2 \cdot 7} r^4 (1 + 12 \ln r) + O(r^10), 
$$

$$
K(r) \xrightarrow{r \to 0} 2^{1/3} r + O(r^7)
$$

at limit $r \to 0$, the Euler-Lagrangian equation (12) approaches to

$$
ff'' - \left(\frac{2}{r} + \frac{1}{r \ln r}\right) f f' + \frac{9}{8} \ln r + \frac{3}{32} = 0.
$$

Considering the initial condition (13), we obtain asymptotical solution of the above equation as follows

$$
f(r \to 0) = R + \frac{9}{16R} r^2 \ln r - \frac{15}{64R} r^2 + O(r^3),
$$

$$
f'(r \to 0) = \frac{9}{8R} r \ln r + \frac{3}{32R} r + O(r^2).
$$
In addition, the minimal surface also implies that \( f(r) \) must vanish at a point \( r = r_0 \) with \( 0 < r \leq r_0 < 1 \), but \( f'(r) \) is divergent at this point. So that the numerical evaluation will not be valid near this point, and we have to find asymptotical solution analytically for \( r \to r_0 \),

\[
f(r \to r_0) = c(r_0 - r)^{1/2} + O((r_0 - r)^{3/2}),
\]

with constant \( c^2 = 8I(r_0)G(r_0)/I'(r_0) \).

Another aspect which need to be dealt with manually is renormalization of divergence. It is caused by integration in Eq. (10) when \( r \to 0 \). This divergence should be subtracted from numerical result on proper area of string world-sheet. Explicitly, using Eqs. (14) (16) and taking a cut-off \( \lambda \to 0 \), we obtain the divergence as follows

\[
\int dr \cdot r^{-1/2} f \sqrt{f'^2 + G(r)} = \frac{R}{2^{1/3} \lambda} + \frac{9}{2^{1/3} \cdot 32 R} \lambda + O(\lambda^2) + \text{other terms independent of } \lambda.
\]

Here we include sub-sub leading term which is proportional to \( \lambda/R \) such that it will strictly vanish at limit \( \lambda \to 0 \). In numerical evaluation, however, the cut-off \( \lambda \) is no longer very small. Then this term will play a role for small \( R \).

According to the above discussions, the Euler-Lagrangian equation (12) can be solved numerically. The consistence of numerical calculation will be checked via match initial conditions (13) and (17). Consequently, the proper area (10) of string world can be obtained after subtracting \( \lambda \)-dependent terms in Eq. (18). Since difference equation associating equation (12) is ill-defined for \( s \to s_0 \), some tricks on numerical evaluation are needed. We put all details on numerical evaluation in appendix of this paper.

### III. NUMERICAL RESULTS AND DISCUSSIONS

We first show some numerical results:

- Fig. 1 includes some curves of \( f(r) \) vs. \( r \), which represents some solutions of Euler-Lagrangian equation (12) with different initial values \( R \).

![Fig. 1: Some solutions of Euler-Lagrangian equation (12) with different initial values. The dash line denotes \( R = 0.32 \), the dot line denotes \( R = 2.09 \) and solid line denotes \( R = 5.06 \).](image)

- Fig. 2 denotes curve of \( r_0 \) vs. \( R \), where \( r_0 \) is zero-point of \( f(r) \) and \( R \) is initial value.

![Fig. 2: Curve of \( r_0 \) vs. \( R \), where \( r_0 \) is zero-point of \( f(r) \) and \( R \) is initial value.](image)

- Fig. 3 and fig. 4 show that the resulted numerical data and related fit curves on proper area \( A \) of string world-sheet vs. radius \( R \) of Wilson loop at large distance and short distance respectively. The unit of area \( A \) is taken over \( 3g_s N/2^{2/3} \).

The main conclusions of this paper are included in fig. 3 and fig. 4. In order to precisely describe behavior of \( A \sim \ln <W(C)> \) with variation of radius \( R \) of Wilson loop, we fit numerical data with several different functions and pick up the functions with the smallest \( \chi^2 \) to approach the behaviors of Wilson loop.
The resulted functions for long distance (larger $R$) are listed in Eq. (19-a)-(19-c), which fit the numerical data in fig. 3.

\[
\begin{align*}
A &= -1.5126 + 0.2941 R + 1.04 R^2 - 2.162 \ln R \quad \chi^2 = 0.0013 \quad (19-a) \\
A &= -0.7835 - 1.0274 R + 1.1367 R^2 \quad \chi^2 = 0.0046 \quad (19-b) \\
A &= -0.0464 - 1.717 R + 1.342 R^2 - 0.014 R^3 \quad \chi^2 = 0.0017 \quad (19-c)
\end{align*}
\]

We can see that Eq. (19-a) is the best approach on Wilson loop at large distance. Therefore, it is obvious that Wilson loop exhibits area law for large $R$. In Eq. (19-a) we also meet linear term and logarithmic term of $R$ which are unexpected. These extra terms may be induced by the following reasons:

1. Due to error bar of numerical evaluation, the subtraction on divergence in Eq. (18) is not precise.
2. Other error bar of numerical evaluation.
3. Due to lack of powerful computers, in our numerical evaluation $R$ is not taken very large. Then resulted Wilson loop does not lie in pure confinement phase. These extra may implies some “mixing-phase”.

However, it is unambiguous that Wilson loop will be dominant by $R^2$-term for very large $R$. Consequently $\mathcal{N} = 1$ SYM is confined at long distance.

Now let us pay attention to short distance (smaller $R$). The numerical data and fit curve are shown in fig. 4, and fit functions are listed as follows,

\[
\begin{align*}
A &= -1.511 + 0.829 R + 0.0753 \ln R \quad \chi^2 = 6.4 \times 10^{-4} \quad (20-a) \\
A &= -1.682 + 1.128 R - 0.131 R^2 \quad \chi^2 = 0.001 \quad (20-b)
\end{align*}
\]

It is unambiguous that Eq. (20-a) is the best fit. Therefore, Wilson loop of $\mathcal{N} = 1$ SYM exhibits logarithmic behavior at short distance. It indicates that $\mathcal{N} = 1$ SYM is asymptotical freedom or approach to Coulomb phase at short distance. In addition, we see that the constants in Eq. (19-a) and in Eq. (20-a) are almost same. Then this constant is harmless. It should be associated to renormalization or normalization of Wilson loop. This also indicates fitting in Eq. (19-a) and in Eq. (20-a) are consistent.

Another interesting issue is that a phase transition occurs when $R$ varies from small ones to large ones. This phase transition occurs between the phase of asymptotical freedom and the phase of confinement of non-abelian gauge theory. The transition point lies in region $0.5 < R < 2$ in unit $3g_sN/2^{2/3}$. Consequently from Eq. (5) we can define $\Lambda_{\text{SYM}} = e^{2/3}/\alpha'$ as a scale associating to confinement. Using the results listed in ref. [7, 14, 18], we have:
The masses of glueball and Kaluza-Klein (KK) states scale as

\[ m_{\text{glueball}} \sim m_{\text{KK}} \sim \frac{2^{2/3}}{g_s N \alpha'} \sim \Lambda_{\text{SYM}}/g_s N. \]  

(21)

We usually expect \( m_{\text{glueball}} \sim m_{\text{KK}} > \Lambda_{\text{SYM}} \) such that it requires smaller \( g_s \sim g_{\text{YM}}^2 \). In other words, the glueball is formed at near perturbative region. Using NSVZ \( \beta \) function[17] for \( N = 1 \) SYM, we have

\[ \frac{1}{g_{\text{YM}}^2 N} = \frac{3}{16 \pi^2} \ln \frac{\mu^2}{2 \Lambda_{\text{SYM}}^2} + c_0 + 1 - 2 \ln \left( \frac{\ln \frac{\mu^2}{2 \Lambda_{\text{SYM}}^2} + c_0}{\ln x} \right) \ln \frac{\mu^2}{2 \Lambda_{\text{SYM}}^2} + c_0, \]  

(22)

where \( c_0 \) is a small constant regularizing divergence at \( \mu = \Lambda_{\text{SYM}} \) and \( \mu \) is the scale that glueball or KK states are produced (roughly we can set \( \mu \simeq m_{\text{glueball}} \)). Since the singularity in KST solution is removed through the blowing-up of the \( S^3 \) of \( T^{1,1} \), more precisely, we can include relevant coefficient to Eq. (21),

\[ m_{\text{KK}} \simeq 3 I^{-1/2}(s_0) \Lambda_{\text{SYM}}/g_s N, \]  

(23)

where \( I(s_0) \) was defined Eq. (5) and \( s_0 \), which denotes the lowest energy detected by certain Wilson loop, is zero-point of \( f(s) \). Considering radius/energy-scale relation for KST background[19], in our nations \( s \sim \mu/\mu_0 \), where the typical scale \( \mu_0 \sim \Lambda_{\text{SYM}} \) consequently it corresponds to \( R_0 \sim 1 \) and \( s_0 \sim 0.8 \). Then defining \( x = m_{\text{KK}}/\Lambda_{\text{SYM}} \simeq \mu/\Lambda_{\text{SYM}} \), we achieve the following equation

\[ x \simeq 3 I^{-1/2}(s_0/x) \frac{3}{16 \pi^2} \ln x^2 - 1 - 2 \ln \left( \frac{\ln x^2}{\ln x} \right). \]  

(24)

The solution of the above equation is \( x \simeq 2.6 \) such that we obtain

\[ m_{\text{glueball}} \simeq m_{\text{KK}} \simeq 2.6 \Lambda_{\text{SYM}}. \]  

(25)

• The mass of the baryon scales as

\[ M_b \sim N \frac{e^{2/3}}{\alpha'} \sim N \Lambda_{\text{SYM}}. \]  

(26)

• The gluino condensate of \( N = 1 \) SYM scales as

\[ <\lambda \lambda> \sim N \frac{e^2}{\alpha'^3} \sim N \Lambda_{\text{SYM}}^3. \]  

(27)

In QCD we knew that there is a so-called “confinement scale”, \( \Lambda_{\text{QCD}} \simeq 0.3 \text{GeV} \). Extend to the above results to QCD, we have

\[ m_{\text{glueball}} \simeq 2.6 \Lambda_{\text{QCD}} \simeq 0.8 \text{GeV}, \]

\[ M_{\text{baryon}} \simeq N_c \Lambda_{\text{QCD}} \simeq 0.9 \text{GeV}, \]

\[ <\lambda \lambda> \simeq \Lambda_{\text{QCD}}^3 \simeq (0.3 \text{GeV})^3, \]  

(28)

where we use gluino condensate to mimic quark condensate in QCD, but we ignore \( N_c \) in Eq. (27) since those chiral quarks are fundamental representation of gauge group. It is surprised that the above results agree with phenomenology results of QCD well. It apparently means that pure \( N = 1 \) SYM mimics some low energy behaviors of QCD even without chiral multiplets.

### IV. SUMMARY AND COMMENTS

Circular Wilson loop of \( N = 1 \) SYM from supergravity KST solution. The Wilson loop exhibits area law at long distance and logarithmic law at short distance. Therefore, \( N = 1 \) SYM lies in confinement phase at low energy, and is asymptotical freedom or approach to Coulomb phase at high energy. The phase transition occurs when energy varies.
We also discuss glueball mass, baryon mass and gluino condensate. It is shown that pure $\mathcal{N} = 1$ SYM mimics some low energy behaviors of QCD even without chiral multiplets.

We would like to make a comment on MN background. In principle, the extension the calculation of this paper to MN background is directly. The Euler-Lagragian equation in MN solution, however, does not possesses initial condition like Eq. (13). Instead, it will be \( f \sim f' \rightarrow \infty \) at boundary. Since another initial condition at zero-point \( r = r_0 \) of \( f(r) \) is \( f' \rightarrow 0 \) fixed, \( f' \rightarrow -\infty \), the numerical evaluation on solution of Euler-Lagragian is very difficult according to criterion in Appendix. The study on this issue, however, is valuable because it can help us to understand which phase of $\mathcal{N} = 1$ SYM in UV is dual to MN background. Some more discussions on MN background and its noncommutative extension are in ref. [20].

APPENDIX: NUMERICAL EVALUATIONS

The main difficulty on numerical evaluations are from near-zero point behavior of \( f(r) \), i.e., \( f'(r \rightarrow r_0) \rightarrow -\infty \). It makes Euler-Lagragian equation \( f(r) \) be stiff for \( r \rightarrow r_0 \). If we use Eq. (18), or more precisely, Eq. (16) with very small \( r \) as initial conditions of numerical evaluation. Due to inevitable error on initial condition in numerical evaluation, this error will be enlarged rapidly for \( r \rightarrow r_0 \). Consequently we can not obtain a convergent solution. Fortunately, since we can analytically obtain solution of Eq. (12) for \( r \rightarrow r_0 \), we can use Eq. (17) as initial condition of numerical evaluation. In this case, the error bar induced by numerical approximation on initial condition is controllable, and it will decrease when \( r \) is evaluated from \( r_0 \) to zero. So that we obtain the convergent result.

In our evaluation, we adopt the fourth order Runge-Kutta method. We take initial condition at \( r_0 - r = 10^{-6} \) and interval \( 2 \times 10^{-9} \). According to Eq. (17), the error bar on initial condition is about \( 10^{-6} \). When we let initial value vary 0.1%, the resulted \( f(0) = R \) and area only vary 0.0001%. It indicates that the numerical evaluation is indeed convergent, and subleading order on initial condition (17) can be ignored consistently.

Another detail is to fix the value of cut-off \( \lambda \) defined in Eq. (18). Theoretically, function \( f'(r) \) is single-valued in region \( 0 \leq r \leq r_0 \), and increases from \(-\infty \) to zero when \( r \) varies from \( r_0 \) to zero. When string world-sheet lies across branes, i.e., in region \( r < 0 \), \( f'(r) \) is again single-valued and increases from \(-\infty \) to zero when \( r \) varies from \(-r_0 \) to zero. It means that \( f'(r = 0) \) is maximal value of \( f'(r) \) if we extend space to \(-r_0 \leq r \leq r_0 \). Numerically, however, we can not expect to obtain the maximal value of \( f'(r) \) at \( r = 0 \) exactly and maximal value of \( f'(r) \) is not precise zero. Then we can take maximal-value point \( f'(r) \) as a natural cut-off. In our evaluation, the value of the cut-off \( \lambda \) varies from 0.0015 to 0.0033 when \( R = f(0) \) varies from 0.1 to 5.5. It implies that we indeed consider the second term in Eq. (18) for smaller \( R \) in order to achieve high precision fit.

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