ON THE QUANTIZATION OF CONJUGACY CLASSES

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Abstract. Let \( G \) be a compact, simple, simply connected Lie group. A theorem of Freed-Hopkins-Teleman identifies the level \( k \geq 0 \) fusion ring \( R_k(G) \) of \( G \) with the twisted equivariant \( K \)-homology at level \( k + h^\vee \), where \( h^\vee \) is the dual Coxeter number of \( G \). In this paper, we will review this result using the language of Dixmier-Douady bundles. We show that the additive generators of the group \( R_k(G) \) are obtained as \( K \)-homology push-forwards of the fundamental classes of pre-quantized conjugacy classes in \( G \).

1. Introduction

A classical result of Dixmier-Douady [10] states that the integral degree three cohomology group \( H^3(X) \) of a space \( X \) classifies bundles of \( C^* \)-algebras \( A \to X \), with typical fiber the compact operators on a Hilbert space. For any such Dixmier-Douady bundle \( A \to X \), one defines the twisted \( K \)-homology and \( K \)-cohomology groups of \( X \) as the \( K \)-groups of the \( C^* \)-algebra of sections of \( A \), vanishing at infinity:

\[
K_q(X, A) := K^q(\Gamma_0(X, A)), \quad K^q(X, A) := K_q(\Gamma_0(X, A)).
\]

If a group \( G \) acts by automorphisms of \( A \), one has definitions of \( G \)-equivariant \( K \)-groups.

The twisted \( K \)-groups have attracted a lot of interest in recent years, mainly due to their applications in string theory. For the case of torsion twistings, they were pioneered by Donovan-Karoubi [11] in 1963, while the general case was developed by Rosenberg [34] in 1989. Rosenberg also gave an alternative characterization of \( K^0(X, A) \) as homotopy classes of sections of a bundle of Fredholm operators; this viewpoint was further explored by Atiyah-Segal [4] (see [6, 41] for alternative approaches).

One of the most natural examples of an integral degree three cohomology class comes from Lie theory. Let \( G \) be a compact, simple, simply connected Lie group, acting on itself by conjugation. The generator of \( H^3_G(G) = \mathbb{Z} \) is realized by a \( G \)-Dixmier-Douady bundle \( A \to G \). Let \( h^\vee \) be the dual Coxeter number of \( G \), and \( k \geq 0 \) a non-negative integer (the level). A beautiful result of Freed-Hopkins-Teleman [16, 15] (see also [14, 13, 17]) asserts that the twisted equivariant \( K \)-homology at the shifted level \( k + h^\vee \) coincides with the level \( k \) fusion ring (Verlinde algebra) of \( G \):

\[
K^0_k(G, A^{k + h^\vee}) = R_k(G).
\]

Here \( R_k(G) \) may be defined as the ring of positive energy level \( k \) representations of the loop group \( LG \), or equivalently as the quotient \( R_k(G) = R(G)/I_k(G) \) of the usual representation ring by the level \( k \) fusion ideal. The quotient map \( R(G) \to R_k(G) \) is realized on the \( K \)-homology side as push-forward under inclusion \( \{e\} \to G \), while the product on \( R_k(G) \) is given by push-forward under group multiplication.

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As a \(\mathbb{Z}\)-module, the fusion ring \(R_k(G)\) is freely generated by the set \(\Lambda^*_k\) of level \(k\) weights of \(G\). In this paper the isomorphism \(R_k(G) = \mathbb{Z}[\Lambda^*_k]\) is realized as follows. Given \(\mu \in \Lambda^*_k \subset \mathfrak{t}^*\), (where \(\mathfrak{t}\) is the Lie algebra of a maximal torus), let \(\mathcal{C}\) be the conjugacy class of the element \(\exp(\mu/k) \in G\), where the basic inner product is used to identify \(\mathfrak{t}^* \cong \mathfrak{t}\). We will show that there is a canonical stable isomorphism between of the restriction \(\mathcal{A}^{k+h^\vee}|_C\) with the Clifford algebra bundle \(\text{Cl}(\mathcal{C})\). This then defines a push-forward map in twisted \(K\)-homology, and the image of the \(K\)-homology fundamental class \([C] \in K^0_G(\mathcal{C}, \text{Cl}(\mathcal{C}))\) under the push-forward is exactly the generator of \(R_k(G)\) labeled by \(\mu\). This is parallel to the fact that the generators of \(R(G) = \mathbb{Z}[\Lambda^*_+]\) are obtained by geometric quantization of the coadjoint orbits through dominant weights. In fact, as shown in [15] the generators of \(R_k(G)\) can also be obtained by geometric quantization of coadjoint orbits of the loop group of \(G\). Hence, our modest observation is that this can also be carried out in finite-dimensional terms. In a forthcoming paper with A. Alekseev [2], we will discuss more generally the quantization of group-valued moment maps [1] along similar lines.

A second theme in this paper is the construction of a canonical resolution of \(R_k(G)\) in the category of \(R(G)\)-modules,

\[
0 \longrightarrow C_l \xrightarrow{\partial} C_{l-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \xrightarrow{\epsilon} R_k(G) \longrightarrow 0
\]

where \(l = \text{rank}(G)\). In more detail, let \(\{0, \ldots, l\}\) label the vertices of the extended Dynkin diagram of \(G\). For each non-empty subset \(I \subset \{0, \ldots, l\}\), let \(G_I \subset G\) be the maximal rank subgroup whose Dynkin diagram is obtained by deleting the vertices labeled by \(I\). These groups have canonical central extensions \(1 \rightarrow U(1) \rightarrow \hat{G}_I \rightarrow G_I \rightarrow 1\) (described below). Let \(R(\hat{G}_I)_k\) denote the Grothendieck group of all \(\hat{G}_I\)-representations for which the central circle acts with weight \(k\). Define

\[
C_p = \bigoplus_{|I| = p+1} R(\hat{G}_I)_k.
\]

The differentials \(\partial\) in (2) are given by holomorphic induction maps relative to the inclusions \(\hat{G}_I \hookrightarrow \hat{G}_J\) for \(J \subset I\). As we will explain, the chain complex \((C_\bullet, \partial)\) arises as the \(E^1\)-term of a spectral sequence computing \(K^G_*(G, \mathcal{A}^{k+h^\vee})\), and the exactness of (2) implies that the spectral sequence collapses at the \(E^2\)-term. Since \(R_k(G)\) is free Abelian, there are no extension problems, and one recovers the equality \(K^G_0(G, \mathcal{A}^{k+h^\vee}) = R_k(G)\) as \(R(G)\)-modules, and hence also as rings.

This article does not make great claims of originality. In particular, I learned that a very similar computation of the twisted equivariant \(K\)-groups of a Lie group had appeared in the article *Thom Prospects for loop group representations* by Kitchloo-Morava [25]. The argument itself may be viewed as a natural generalization of the Mayer-Vietoris calculation for \(G = \text{SU}(2)\), as explained by Dan Freed in [13]. Independently, the chain complex had been obtained by Christopher Douglas (unpublished), who used it to obtain information about the algebraic structure of the fusion ring \(R_k(G)\).
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CONTENTS

1. Introduction
2. Review of twisted equivariant $K$-homology
   2.1. Dixmier-Douady bundles
   2.2. Dixmier-Douady bundles related to central extensions
   2.3. Twisted $K$-homology
3. The Dixmier-Douady bundle over $G$
   3.1. Pull-back to the maximal torus
   3.2. The centralizers $G^i$
   3.3. Construction of the Dixmier-Douady bundle $\mathcal{A} \to G$
4. Conjugacy classes
   4.1. Pull-back to conjugacy classes
   4.2. Pre-quantization of conjugacy classes
   4.3. The $h^\vee$-th power of the Dixmier-Douady bundle
   4.4. Quantization of conjugacy classes
   4.5. Twisted $K$-homology of the conjugacy classes
5. Computation of $K^G_\bullet(G, \mathcal{A}^{k+h^\vee})$
   5.1. The spectral sequence for $K^G_\bullet(G, \mathcal{A}^{k+h^\vee})$
   5.2. The induction maps in terms of weights
   5.3. Fusion ring
   5.4. A resolution of the $R(G)$-module $R_k(G)$
   5.5. Proof of Theorem 5.6
Appendix A. Morita isomorphisms and stable isomorphisms
Appendix B. Relative Dixmier-Douady bundles
Appendix C. Review of Kasparov $K$-homology
References

2. Review of twisted equivariant $K$-homology

Throughout this paper, all Hilbert spaces $\mathcal{H}$ will be taken to be separable, but not necessarily infinite-dimensional. All (topological) spaces $X$ will be assumed to allow the structure of a countable CW-complex (respectively $G$-CW complex, in the equivariant case).

2.1. Dixmier-Douady bundles. [11] [33] [34] For any Hilbert space $\mathcal{H}$, we denote by $U(\mathcal{H})$ the unitary group, with the strong operator topology. Let $\mathbb{K}(\mathcal{H})$ be the $C^*$-algebra of compact operators, that is, the norm closure of the finite rank operators. The conjugation action of the unitary group on $\mathbb{K}(\mathcal{H})$ descends to the projective unitary group, and provides an isomorphism, $\text{Aut}(\mathbb{K}(\mathcal{H})) = \text{PU}(\mathcal{H})$. A Dixmier-Douady bundle $\mathcal{A} \to X$ is a locally trivial bundle of $C^*$-algebras, with typical fiber $\mathbb{K}(\mathcal{H})$ and structure group $\text{PU}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. That is,

\begin{equation}
\mathcal{A} = \mathcal{P} \times_{\text{PU}(\mathcal{H})} \mathbb{K}(\mathcal{H})
\end{equation}
for a principal $PU(H)$-bundle $P \to X$. Dixmier-Douady bundles of finite rank are also known as Azumaya bundles [26, 27]. A gauge transformation of $A$ is a bundle automorphism inducing the identity on $X$, and whose restriction to the fibers are $C^*$-algebra automorphisms. Equivalently, the group of gauge transformations consists of sections of the associated group bundle, $\text{Aut}(A) = P \times_{PU(H)} \text{Aut}(K(H))$. This group bundle has a central extension

\begin{equation}
1 \to X \times U(1) \to \tilde{\text{Aut}}(A) \to \text{Aut}(A) \to 1,
\end{equation}

where $\tilde{\text{Aut}}(A) = P \times_{PU(H)} U(H)$.

If $A_1, A_2$ are Dixmier-Douady bundles modeled on $K(H_1), K(H_2)$, then their (fiberwise) $C^*$-tensor product $A_1 \otimes A_2$ is a Dixmier-Douady bundle modeled on $K(H_1 \otimes H_2)$. Also, the (fiberwise) opposite $A^{opp}$ of a Dixmier-Douady bundle modeled on $K(H)$ is a Dixmier-Douady bundle modeled on $K(H^{opp})$. Here the Hilbert space $H^{opp}$ is equal to $H$ as an additive group, but with the new scalar multiplication by $z \in \mathbb{C}$ equal to the old scalar multiplication by $\overline{z}$.

A Morita trivialization of $A \to X$ is a pair $(E, \psi)$, consisting of a Hilbert space bundle $E \to X$ and an isomorphism $\psi : K(E) \to A$. A Morita isomorphism from $A \to X$ to $B \to X$ is a Morita trivialization $(E, \psi)$ of $B \otimes A^{opp}$. We will write $A \simeq B$ to indicate a Morita isomorphism.

Morita isomorphism is an equivalence relation: For instance, given $A$ as in [4], the bundle of ‘Hilbert-Schmidt operators’

\begin{equation}
\mathcal{A}_{HS} := P \times_{PU(H)} (H \otimes H^{opp})
\end{equation}

defines a Morita isomorphism $\mathcal{A} \simeq A$. A Morita trivialization of $A$ amounts to a lift of the structure group from $PU(H)$ to $U(H)$. The obstruction to the existence of such a lift is given by the Dixmier-Douady class

\[ DD(A) \in H^3(X). \]

The Dixmier-Douady class satisfies $DD(A_1 \otimes A_2) = DD(A_1) + DD(A_2)$ and $DD(A^{opp}) = -DD(A)$. It hence gives an isomorphism of $H^3(X)$ with the Morita isomorphism classes of Dixmier-Douady bundles.

**Example 2.1.** Let $V \to X$ be an oriented Euclidean vector bundle of rank $k$, and let $\text{Cl}(V) \to X$ be the complex Clifford algebra bundle. If $k$ is even, the bundle $\text{Cl}(V)$ is a Dixmier-Douady bundle. A Morita trivialization of $\text{Cl}(V)$ is equivalent to the choice of a spinor module $S \to X$, which in turn is equivalent to the choice of a Spin$_c$ structure on $V$. For details, see Plymen [32]. The Dixmier-Douady class $DD(\text{Cl}(V))$ is the image of the Stiefel-Whitney class $w_2(V) \in H^2(X, \mathbb{Z}_2)$ under the Bockstein homomorphism. The fact that this class is 2-torsion may be seen directly, since the canonical anti-involution identifies $\text{Cl}(V)^{opp} \cong \text{Cl}(V)$. In the case of $k$ odd, the even part $\text{Cl}^+(V)$ is a Dixmier-Douady bundle, and a similar discussion applies.

Two Morita trivializations $(E, \psi)$ and $(E', \psi')$ of $A$ will be called equivalent if there exists an isomorphism of Hilbert bundles $E \to E'$ intertwining $\psi, \psi'$. Thus, $(E, \psi)$ and $(E', \psi')$ are equivalent if and only if the Hermitian line bundle

\[ L = \text{Hom}_A(E, E') \to X \]

(fiberwise $A$-module homomorphisms) is isomorphic to the trivial line bundle. Conversely, given a Hermitian line bundle $L \to X$, one may ‘twist’ a Morita trivialization $(E, \psi)$ by setting

\footnote{We take all cohomology groups with integer coefficients, unless indicated otherwise.}
Given a central extension of \( G \) to \( \hat{L} \) associated line bundles (defined in terms of tensor products of line bundles). Thus, the central extensions (7) form an Abelian group, with unit the trivial extension. The group of gauge transformations of a \( G \) or any central extension \( \hat{E} \) on \( G \) was proved by Atiyah-Segal [4].

The stabilization of a Dixmier-Douady bundle \( A \to X \) is defined as \( \mathcal{A}^{st} = A \otimes \mathbb{K} \), where \( \mathbb{K} \) denotes the compact operators on a fixed infinite-dimensional Hilbert space \( \mathbb{H} \). A stable isomorphism between \( A, B \to X \) is an isomorphism \( \mathcal{A}^{st} \xrightarrow{\simeq} \mathcal{B}^{st} \). Two stable isomorphisms are called equivalent if they are homotopic. It is known (cf. Appendix A) that equivalence classes of stable isomorphisms are in 1-1 correspondence with equivalence classes of Morita isomorphisms.

Given a compact Lie group \( G \) acting on \( X \), one may similarly define \( G \)-equivariant Dixmier-Douady bundles. All of the above extends to this equivariant setting: In particular, there is a \( G \)-equivariant Dixmier-Douady class \( \text{DD}_G(A) \in H^2_G(X) \), which classifies \( G \)-equivariant Dixmier-Douady bundles up to \( G \)-Morita isomorphism \( (\mathcal{E}, \psi) \). (That is, \( \mathcal{E} \) is a \( G \)-Hilbert space bundle \( \mathcal{E} \to X \) and \( \psi : B \otimes \mathcal{A}^{opp} \to K(\mathcal{E}) \) is a \( G \)-equivariant isomorphism.) The stabilization of a \( G \)-equivariant Dixmier-Douady bundle is defined as \( \mathcal{A} \otimes K_G \), where \( K_G \) is the \( G \)-\( C^* \)-algebra of compact operators on a fixed \( G \)-Hilbert space \( \mathbb{H}_G \), containing all irreducible finite-dimensional unitary \( G \)-representations with infinite multiplicity. The extension of the Dixmier-Douady theorem to the \( G \)-equivariant case was proved by Atiyah-Segal [4].

Still more generally, one can also consider \( \mathbb{Z}_2 \)-graded \( G \)-Dixmier-Douady bundles \( A \to X \). Here, isomorphisms and tensor products are understood in the \( \mathbb{Z}_2 \)-graded sense, and the Hilbert space bundles in the definition of stable isomorphism are \( \mathbb{Z}_2 \)-graded. If \( A \to X \) is such a bundle, and \( \text{DD}_G(A) = 0 \) (so that there exists a \( G \)-equivariant isomorphism \( K(\mathcal{E}) \to A \), ignoring the \( \mathbb{Z}_2 \)-grading), there is an obstruction in \( H^1(X, \mathbb{Z}_2) \) for the existence of a compatible \( \mathbb{Z}_2 \)-grading on \( \mathcal{E} \). That is, the map from Morita equivalence classes of \( \mathbb{Z}_2 \)-graded \( G \)-Dixmier-Douady bundles to those of ungraded \( G \)-Dixmier-Douady bundles (forgetting the \( \mathbb{Z}_2 \)-grading) is onto, with kernel \( H^1(X, \mathbb{Z}_2) \). See [4] for details.

2.2. Dixmier-Douady bundles related to central extensions. We assume that \( G \) is compact and connected. Then \( H^2_G(pt) = 0 \), while \( H^2_G(pt) = \text{Hom}(G, U(1)) \). Consider central extensions of \( G \) by \( U(1) \),

\[
1 \to U(1) \to \hat{G} \to G \to 1.
\]

For any central extension \( \hat{G} \), there is an associated line bundle \( L = \hat{G} \times_{U(1)} \mathbb{C} \to G \). Powers \( \hat{G}^{(l)} \), \( l \in \mathbb{Z} \) are obtained as unit circle bundles inside the corresponding powers \( L^l \) of the associated line bundles (defined as \( (L^*)^l \) for \( l < 0 \)), similarly products of central extensions are defined in terms of tensor products of line bundles. Thus, the central extensions (7) form an Abelian group, with unit the trivial extension. The group of gauge transformations of a central extension \( \hat{G} \) (i.e. group automorphims covering the identity on \( G \)) is \( \text{Hom}(G, U(1)) \).

It is well-known that the group of isomorphism classes of central extensions is isomorphic to \( H^2_G(pt) \). From the interpretation via Dixmier-Douady bundles, this may be seen as follows: Given a \( G \)-equivariant Dixmier-Douady algebra \( \mathcal{A} \), (viewed as a bundle over pt), one obtains a central extension of \( G \) as a pull-back of \( U(\mathcal{H}) \to \text{Aut}(K(\mathcal{H})) \) by the homomorphism \( G \to \mathbb{U} \).
operator, acting on the sections of a

\[ \tau \]

2.3. Twisted K-homology. The input for the twisted equivariant K-homology of a G-space X is a Z_2-graded G-Dixmier-Douady bundle A \to X. From now on, we usually omit explicit mention of the Z_2-grading (which may be trivial), with the understanding that all tensor products are in the Z_2-graded sense, isomorphisms should preserve the Z_2-grading, and so on.

Given A \to X, the space A = \Gamma_0(X, A) of continuous sections of A vanishing at infinity is a (Z_2-graded) G - C*-algebra, with norm ||s|| = \sup_{x \in X} ||s_x||_{A_0}. Following J. Rosenberg [34], we define the twisted equivariant K-homology and K-cohomology groups as the equivariant C*-algebra K-homology and K-cohomology groups of A:

\[ K^G_q(X, A) := K^G_q(\Gamma_0(X, A)), \quad K^G_q(X, A) := K^G_q(\Gamma_0(X, A)) \]

In this paper, we will mostly work with the K-homology groups. See the appendix for a quick review of the K-homology of C*-algebras, and some examples. For instance, if X = M is a compact manifold, and A has finite rank, then every G-invariant first order elliptic differential operator, acting on the sections of a G-equivariant bundle of A-modules, defines a twisted
equivariant $K$-homology class. Some of the basic properties of the $K$-homology groups are as follows.

(a) **Stability.** The twisted $K$-groups are unchanged under stabilization:

$$K^G_q(X,A) = K^G_q(X, A \otimes \mathbb{K}_G).$$

In more detail, recall that $\mathbb{K}_G$ denotes the compact operators on a fixed stable $G$-Hilbert space. Let $p \in \mathbb{K}_G$ denote the projection operator onto a 1-dimensional invariant subspace. Then the map $A \to A \otimes \mathbb{K}_G$, $a \mapsto a \otimes p$ induces an isomorphism in twisted $K$-homology. Since $p$ is unique up to homotopy, the induced map in $K$-homology does not depend on the choice of $p$.

(b) **Morphisms.** The morphisms in the category of $G$-Dixmier-Douady bundles $(X,A)$ are the equivariant $C^*$-algebra bundle map $A_1 \to A_2$ for which the induced map on the base $f: X_1 \to X_2$ is proper. Any such morphism induces a morphism of $G-C^*$-algebras $f^*: \Gamma_0(X_2,A_2) \to \Gamma_0(X_1,A_1)$, hence a push-forward in $K$-homology

$$K^G_q(f): K^G_q(X_1,A_1) \to K^G_q(X_2,A_2).$$

Then $K^G_\bullet$ becomes a covariant functor, invariant under proper $G$-homotopies. More generally, using (a) it suffices to have a stable morphism between $A_1,A_2$, i.e. a morphism of their stabilizations $A_i \otimes \mathbb{K}_G$.

(c) **Excision.** For any closed, invariant subset $Y \subset X$, with complement $U = X \setminus Y$, there is a long exact sequence

$$\cdots \to K^G_q(Y,A|_Y) \to K^G_q(X,A) \to K^G_q(U,A|_U) \to K^G_{q-1}(Y,A|_Y) \to \cdots$$

Here the restriction map $K^G_q(X,A) \to K^G_q(U,A|_U)$ is induced by the $C^*$-algebra morphism $\Gamma_0(U,A|_U) \to \Gamma_0(X,A)$, given as extension by 0. More generally, one obtains a spectral sequence for any filtration of $X$ by closed, invariant subspaces.

(d) **Products.** Suppose $A \to X$ and $B \to Y$ are two $G$-Dixmier-Douady bundles. Then the exterior tensor product $A \boxtimes B \to X \times Y$ is again a $G$-Dixmier-Douady bundle. Its space of sections is the $C^*$-tensor product of the spaces of sections of $A,B$. As a special case of the Kasparov product in $K$-homology, one has a natural associative cross product,

$$K^G_q(X,A) \otimes K^G_\bullet(Y,B) \to K^G_\bullet(X \times Y, A \boxtimes B).$$

(e) **Module structure.** The group $K^G_\bullet(pt)$ is canonically identified with the representation ring $R(G)$. The ring structure on $K^G_0(pt)$ is defined by the cross product for $\mathbb{C} \boxtimes \mathbb{C} \to \text{pt} \times \text{pt}$. Similarly, if $A \to X$ is a $G$-Dixmier-Douady bundle, the cross product for $\mathbb{C} \boxtimes A \to \text{pt} \times X$ makes $K^G_q(X,A)$ into a module over $R(G)$. The maps $K^G_q(f)$ are $R(G)$-module homomorphisms.

If $M$ is a manifold one has the **Poincaré duality isomorphism** relating twisted $K$-homology and $K$-cohomology,

$$K^G_q(M,A) \cong K^G_q(M, A^{opp} \otimes \text{Cl}(TM)).$$

Here $\text{Cl}(TM)$ is the Clifford algebra bundle for some choice of invariant metric. For $A = \mathbb{C}$ the Poincaré duality was proved by Kasparov in [21, Section 8]; the result in the twisted case was

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Note that $K$-homology is analogous to Borel-Moore homology (homology with non-compact supports), rather than ordinary homology.
obtained by J.-L. Tu [39, Theorem 3.1]. (See also [9, Section 2]). The image of $1 \in K^0_G(M)$ under this isomorphism is Kasparov’s $K$-homology fundamental class $[M] \in K^0_G(M, \text{Cl}(TM))$.

**Remark 2.3.** Note that $\text{Cl}(TM)$ is a Dixmier-Douady bundle only if $\dim M$ is even. However, the definition of the twisted $K$-groups works for arbitrary bundles of $C^*$-algebras, and the isomorphism [8] holds in this sense (but with $\mathcal{A}$ a Dixmier-Douady bundle). Alternatively, one may state the result in terms of Dixmier-Douady bundles, using $\text{Cl}(TM) = \text{Cl}^+(TM) \otimes \text{Cl}(\mathbb{R})$ and the isomorphism the isomorphism $K^G_{q+1}(M, \mathcal{B}) = K^G_q(M, \mathcal{B} \otimes \text{Cl}(\mathbb{R}))$.

The following basic computations in twisted equivariant $K$-homology may be deduced from their $K$-theory counterparts, using Poincaré duality.

(a) If $M = \text{pt}$, the twisted $K$-homology is

$$K^G_0(\text{pt}, \mathcal{A}) = R(\widehat{G})_{-1},$$

while $K^G_1(\text{pt}, \mathcal{A}) = 0$. Here $\widehat{G}$ is the central extension defined by the action $G \to \text{Aut}(\mathcal{A})$, and $R(\widehat{G})_{-1}$ is the Grothendieck group of $\widehat{G}$-representations where the central $U(1)$ acts with weight $-1$.

(b) Suppose $H$ is a closed subgroup of $G$. For any $H$-Dixmier-Douady bundle $\mathcal{B} \to Y$, there is a natural *induction isomorphism*,

$$\text{Ind}^G_H : K^H_q(Y, \mathcal{B} \otimes \text{Cl}(\mathfrak{g}/\mathfrak{h})) \cong - \rightarrow K^G_q(G \times_H Y, G \times_H \mathcal{B}),$$

which is Poincaré dual to the isomorphism $K^G_q(G \times_H Y, G \times_H \mathcal{B}^\text{opp}) \cong K^H_{q+1}(Y, \mathcal{B}^\text{opp})$. If $Y = \text{pt}$, the left hand side may be evaluated as in (a). If $H \subset H' \subset G$ are closed subgroups, we have

$$\text{Ind}^G_{H'} = \text{Ind}^G_H \circ \text{Ind}^{H'}_H.$$

Here we are identifying $\text{Cl}(\mathfrak{g}/\mathfrak{h}) \cong \text{Cl}(\mathfrak{g}/\mathfrak{h}') \otimes \text{Cl}(\mathfrak{h}'/\mathfrak{h})$, and we are using the canonical isomorphism $H' \times_H \text{Cl}(\mathfrak{g}/\mathfrak{h}') \cong H'/H \times \text{Cl}(\mathfrak{g}/\mathfrak{h}')$.

(c) Let $\mathcal{A} \to \text{pt}$ be a $G$-Dixmier-Douady algebra as in [39], and let $H$ be a closed subgroup of $G$. Then $G \times_H \mathcal{A}$ is canonically isomorphic to $\pi^* \mathcal{A}$, the pull-back under the map $\pi : G/H \to \text{pt}$. By composing the map $\text{Ind}^G_H$ with the push-forward $K^G_0(\pi)$, we obtain an *induction map*

$$\text{ind}^G_H : K^H_q(\text{pt}, \mathcal{A} \otimes \text{Cl}(\mathfrak{g}/\mathfrak{h})) \to K^G_0(\text{pt}, \mathcal{A}).$$

An $H$-invariant complex structure on $\mathfrak{g}/\mathfrak{h}$ defines a stable trivialization of $\text{Cl}(\mathfrak{g}/\mathfrak{h})$; the resulting map

$$\text{ind}^G_H : K^H_0(\text{pt}, \mathcal{A}) = R(\widehat{H})_{-1} \to K^G_0(\text{pt}, \mathcal{A}) = R(\widehat{G})_{-1}$$

is *holomorphic induction* for the complex structure on $G/H = \widehat{G}/\widehat{H}$.

For other examples of calculations of twisted $K$-groups, see [39, Section 8].
3. The Dixmier-Douady bundle over G

For the rest of this paper, G will denote a compact, simply, simply connected Lie group, acting on itself by conjugation. Then $H^2_G(G)$ is canonically isomorphic to $\mathbb{Z}$. Hence there exists a $G$-Dixmier-Douady bundle $\mathcal{A} \to G$, unique up to Morita isomorphism, such that $DD_G(G, \mathcal{A})$ corresponds to the generator $1 \in \mathbb{Z}$. Any two bundles $\mathcal{A}, \mathcal{A}' \to G$ representing the generator are related by a $G$-equivariant Morita isomorphism, unique up to equivalence (since $H^2_G(G) = 0$).

The quickest construction of $\mathcal{A}$ is as an associated bundle

$$\mathcal{A} = P_G \times_{L_G} \mathbb{K}(\mathcal{H}),$$

where $P_G$ is the space of based paths in $G$, $L_G = LG \cap P_G$ the based loop group, and $\mathcal{H}$ a representation of the standard central extension $\hat{LG}$ of $LG$ where the central circle acts with weight $-1$. The construction given in this Section is essentially just a slow-paced version of this model for $\mathcal{A}$, avoiding technicalities such as the choice of topology on the loop group. Our strategy is to first give a direct construction of the family of central extensions of the centralizers $G_g \subset G$, corresponding to their action on $\mathcal{A}$.

3.1. Pull-back to the maximal torus. Let $T \subset G$ be a maximal torus of $G$, with Lie algebra $\mathfrak{t}$. Consider the map

$$H^3_G(G) \to H^3_T(T)$$

obtained by first restricting the action to $T$ and then pulling back to $T$. Since the map $H^3(G) \to H^3(T)$ defined by the inclusion is just the zero map, the image of $[\theta]$ lies in the kernel of the map $H^3_T(T) \to H^3(T)$, i.e. it is contained in the subgroup

$$H^3_T(pt) \otimes H^1(T) \subset H^3_T(T).$$

We will compute the image of the generator of $H^3_G(G)$ under this map. Denote by $\Lambda \subset \mathfrak{t}$ the integral lattice (i.e. the kernel of $\exp : \mathfrak{t} \to T$). Recall that the basic inner product $B$ on the Lie algebra $\mathfrak{g}$ is the unique invariant inner product, with the property that the smallest length of a non-zero element $\lambda \in \Lambda$ equals $\sqrt{2}$. One of the key properties of $B$ is that it restricts to an integer-valued bilinear form on $\Lambda$. That is, $B|_\Lambda \in \Lambda^* \otimes \Lambda^*$ where $\Lambda^* = \text{Hom}(\Lambda, \mathbb{Z}) \subset \mathfrak{t}^*$ is the (real) weight lattice.

Lemma 3.1. The map $[\theta]$ is injective, and takes the generator of $H^3_G(G)$ to the element

$$B|_\Lambda \in \Lambda^* \otimes \Lambda^* \cong H^2_T(pt) \otimes H^1(T)$$

given by the basic inner product.

Proof. Since $H^3_G(G)$ and $H^3_T(T)$ have no torsion, we may pass to real coefficients, and hence work with Cartan’s equivariant de Rham model $\Omega^\bullet_G(M) = \bigoplus_{2i+j=p} \Omega^i(M)^G$ for the equivariant cohomology $H^*_G(M, \mathbb{R})$ of a $G$-manifold, with differential $(d_G\alpha)(\xi) = d\alpha(\xi) - \iota(\xi_M)\alpha(\xi)$ where $\xi_M$ is the vector field defined by $\xi \in \mathfrak{g}$. Let $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ be the left-, right- invariant Maurer Cartan forms. The generator of $H^3_G(G)$ is represented by an equivariant de Rham form (see e.g. [29]),

$$\eta_G(\xi) = \frac{1}{12}B(\theta^L, [\theta^L, \theta^L]) + \frac{1}{2}B(\theta^L + \theta^R, \xi).$$
Its image under the map $\Omega_G(G) \to \Omega_T(T)$ is

$$i^*\eta(\xi) = B(\theta_T, \xi) \in \Omega^2_T(pt) \otimes \Omega^1(T) = t^* \otimes \Omega^1(T)$$

with $\theta_T \in \Omega^1(T, t)$ the Maurer-Cartan form for $T$. The inclusion $t^* \to \Omega^1(T)$, $\mu \mapsto \langle \mu, \theta_T \rangle$ induces an isomorphism in cohomology, $t^* \cong H^1(T, \mathbb{R})$. Since $B(\theta_T, \cdot)$ represents the image of $B|_t \in t^* \otimes t^* \subset t^* \otimes \Omega^1(T)$, the proof is complete. \hfill $\Box$

As discussed in Section 2.2 elements of $H^2_G(pt) \otimes H^1(T)$ are realized as the holonomy of a family of central extensions. For any $\mu \in \Lambda^*$ let $T \to U(1)$, $t \mapsto t^\mu$ be the corresponding homomorphism. Let the lattice $\Lambda$ act on $\hat{T} = T \times U(1)$ as

$$\Lambda \times \hat{T} \to \hat{T}, \quad (\lambda; h, z) \mapsto (h, h^{-B^\mu(\lambda)} z).$$

Then the holonomy of the family

$$t \times_\Lambda \hat{T} \to t/\Lambda = T.$$

is the element $-B|_t$. Lemma 3.1 implies:

**Lemma 3.2.** Suppose $A \to G$ is a $G$-Dixmier-Douady bundle, where the action on the base $G$ is given by conjugation. Then $DD_G(A)$ represents the generator of $H^3_G(G)$, if and only if the family of central extensions of $T$, defined by the $T$-action on $A|_T$, is isomorphic to the opposite of the family (12) (i.e., using the action $(h, z) \mapsto (h, h^{B^\mu(\lambda)} z)$).

Let $\hat{T}(t)$ be the fiber of (12) over $t \in T$. The choice of $\xi$ with $\exp \xi = t$ defines a trivialization

$$T \to \hat{T}(t) \subset t \times_\Lambda \hat{T}, \quad h \mapsto [(\xi; h, 1)]$$

where the brackets indicate equivalence classes. Shifting $\xi$ by $\lambda \in \Lambda$ changes the trivialization by the homomorphism $T \to U(1)$, $h \mapsto h^{-B^\mu(\lambda)}$. There is the following equivalent description of $\hat{T}(t)$.

**Lemma 3.3.** For any $t \in T$ there is a canonical isomorphism $\hat{T}(t) = t \times_\Lambda U(1)$, where $\Lambda$ acts on $U(1)$ by the homomorphism

$$g_t : \Lambda \to U(1), \quad \lambda \mapsto t^{B^\mu(\lambda)}.$$

In terms of this identification, the trivialization of $\hat{T}(t) = t \times_\Lambda U(1)$ defined by the choice of $\xi \in t$, $\exp \xi = t$ reads,

$$T = t/\Lambda \to t \times_\Lambda U(1), \quad [(\zeta)] \mapsto [(\zeta, e^{-2\pi \sqrt{-1} B(\xi, \zeta)})].$$

**Proof.** The choice of $\xi$ with $\exp \xi = t$ defines an isomorphism

$$t \times_\Lambda U(1) \to \hat{T}(t), \quad [(\zeta, z)] \mapsto [(\zeta; \exp \zeta, e^{2\pi \sqrt{-1} B(\xi, \zeta)} z)].$$

It is straightforward to check that this map is well-defined, and independent of the choice of $\xi$. Its composition with (15) is $[\zeta] \mapsto [(\zeta; \exp \zeta, 1)]$; the trivialization of $\hat{T}(t)$ defined by the choice of $\xi$. \hfill $\Box$
The action of the Weyl group $W = N(T)/T$ on $T$ lifts to an action on the bundle $[12]$, given as
$$w.[(\xi; h, z)] = [(w\xi; wh, z)].$$
If $w$ fixes $\exp \xi$, so that $w\xi = \xi - \lambda$ for some $\lambda \in \Lambda$, the formula may be written
$$w.[(\xi; h, z)] = [(\xi - \lambda; wh, z)] = [(\xi; wh, h^{-B^0(\lambda)})].$$
This leads us to another description of the bundle $[12]$. Let $t_+ \subset t$ be the choice of a closed Weyl chamber, and let $\Delta \subset t_+$ be the corresponding closed Weyl alcove. Recall that $\Delta$ labels the $W$-orbits in $T$, in the sense that every orbit contains a unique point in $\exp(\Delta)$. Label the vertices of $\Delta$ by 0 and for every non-empty subset $I \subset \{0, \ldots, l\}$ let $\Delta_I$ denote the closed simplex spanned by the vertices in $I$, and let $W_I \subset W$ denote the subgroup fixing $\exp(\Delta_I) \subset T$. Then the maps $W/W_I \times \Delta_I \to T$, $(wW_I, \xi) \mapsto w\exp \xi$ define an isomorphism
$$T \cong \coprod_I W/W_I \times \Delta_I/\sim$$
using the identifications,
$$\phi^I_J: \Delta_J \to \Delta_I \quad \text{for } J \subset I.$$
Here $\phi^I_J: \Delta_J \to \Delta_I$ is the natural inclusion, giving rise to an inclusion $W_I \hookrightarrow W_J$ of Lie groups and hence to projection $\phi^I_J: W/W_I \to W/W_J$. With similar identifications we have,
$$t \times \lambda \to \hat{T} = \coprod_I (W \times W_I \hat{T}) \times \Delta_I/\sim.$$
Here $W_I$ acts on $\hat{T}$ by
$$w.(h, z) = (w.h, h^{-B^0(\lambda)}z)$$
where $\lambda \in \Lambda$ is the unique lattice vector with $\lambda + w\Delta_I = \Delta_I$.

3.2. The centralizers $G_I$. Let $G_I \subset G$ be the subgroup of $G$ fixing $\exp(\Delta_I) \subset G$. Equivalently, $G_I$ is the centralizer of any element $t = \exp \xi$ with $\xi \in t$ in the interior of the face $\Delta_I$. Each $G_I$ is a connected subgroup containing $T$, and we have $W_I = N_{G_I}(T)/T$. For $J \subset I$ we have $G_I \subset G_J$. The description (16) of the maximal torus extends to the group $G$:
$$G \cong \coprod_I G/G_I \times \Delta_I/\sim$$
using the equivalence relations (17) for the natural maps $\phi^I_J: G/G_I \to G/G_J$ for $J \subset I$.

Lemma 3.4. There are distinguished central extensions
$$1 \to U(1) \to \widehat{G}_I \to G_I \to 1,$$
with lifts $\widehat{i}^I_J: \widehat{G}_I \to \widehat{G}_J$ of the inclusions $i^I_J: G_I \hookrightarrow G_J$ for $J \subset I$, such that
(a) $\widehat{G}_{\{0, \ldots, l\}} = \hat{T}$,
(b) the lifted inclusions satisfy the coherence condition $\widehat{i}^K_J = \widehat{i}^K_J \circ \widehat{i}^I_J$ for $K \subset J \subset I$,
(c) the $W_I$-action on $\hat{T} \subset \widehat{G}_I$ (cf. (18)) is induced by the conjugation action of $N_{G_I}(T)$. 

Proof. Recall $\pi_1(G_I) = \Lambda/\Lambda_I$, where $\Lambda_I$ is the co-root lattice of $G_I$ [8, Theorem (7.1)]. But
$$\lambda \in \Lambda_I, \quad t \in \exp(\Delta_I) \Rightarrow t^{D^\prime(\lambda)} = 1$$
(see [28, Proposition 5.4]). Hence, for $t \in \exp(\Delta_I)$, the homomorphism $\varrho_t$ defined in (14) descends to a homomorphism
$$\varrho_{t,I} : \pi_1(G_I) \to U(1).$$
We therefore obtain a family of central extensions $\hat{G}_{I,(t)} = G_I \times_{\pi_1(G_I)} U(1)$ parametrized by the points of $\exp(\Delta_I)$. Since $\exp(\Delta_I)$ is contractible, we may use the flat connection on the family of central extensions (cf. Section 2.2) to identify all $\hat{G}_{I,(t)}$ to $\hat{G}_{I,\Delta_I}$ to identify all $\hat{G}_{I,(t)}$. The resulting $\hat{G}_{I}$ has the desired properties. In particular, if $J \subset I$ and $t \in \exp(\Delta_J) \subset \exp(\Delta_I)$, the homomorphism $\varrho_{t,I}$ is given by the inclusion $\pi_1(G_I) \to \pi_1(G_J)$ followed by $\varrho_{t,J}$. This defines an inclusion $\hat{G}_{I,(t)} \hookrightarrow \hat{G}_{J,(t)}$, compatible with the flat connection and (hence) satisfying the coherence condition. \hfill $\square$

Remarks 3.5. (a) The inclusion of $\hat{T} = T \times U(1)$ into $\hat{G}_I \cong \hat{G}_{I,(t)}$ is explicitly given as (see Equation (15))
$$\exp_T(\zeta, z) \mapsto [\exp_{\hat{G}_I}(\zeta, e^{-2\pi \sqrt{-1}B(\zeta, \zeta)})].$$
(b) The lifts $\hat{i}_I^J$ of $i_I^J : G_I \to G_J$ intertwine the inclusions $\hat{T} \hookrightarrow \hat{G}_I$, and are the unique lifts having this property.
(c) The central extensions $\hat{G}_I$ are sometimes non-trivial, even though of course their Lie algebra $\hat{g}_I$ is a trivial central extension of $g_I$. The choice of any $t \in \exp(\Delta_I)$ identifies
$$\hat{g}_I \cong \hat{g}_{I,(t)} = g_I \times \mathbb{R},$$
by the definition of $\hat{G}_{I,(t)}$ as a quotient of $\hat{G}_I \times U(1)$.

3.3. Construction of the Dixmier-Douady bundle $A \to G$. Our construction of the Dixmier-Douady bundle $A \to G$ involves a suitable Hilbert space $\mathcal{H}$.

Lemma 3.6. There exists a Hilbert space $\mathcal{H}$, equipped with unitary representations of the central extensions $\hat{G}_I$ such that (i) the central $U(1)$ acts with weight $-1$, and (ii) for $J \subset I$ the action of $\hat{G}_J$ restricts to the action of $\hat{G}_I$.

One may construct such an $\mathcal{H}$ using the theory of affine Lie algebras. Let $\mathcal{L}(g) = \mathfrak{g}^C \otimes \mathbb{C}[z, z^{-1}]$ be the loop algebra associated to $g$. For all roots $\alpha$ of $G$, let $e_\alpha \in \mathfrak{g}^C$ be the corresponding root vector. Then $\mathfrak{g}^C$ is spanned by $t^C$ together with the root vectors $e_\alpha$ such that $\langle \alpha, \xi \rangle \in \mathbb{Z}$ for $\xi \in \Delta_I$. The map $j_I : \hat{g}_I^C = \hat{L}(g)$ given by $\zeta \mapsto \zeta \otimes 1$ for $\zeta \in t^C$ and
$$e_\alpha \mapsto e_\alpha \otimes z^{\langle \alpha, \xi \rangle},$$
for $\langle \alpha, \xi \rangle \in \mathbb{Z}$ is an injective Lie algebra homomorphism (independent of $\xi$). Consider the standard central extension $\hat{\mathcal{L}}(g) = \mathcal{L}(g) \oplus \mathbb{C}c$, with bracket
$$[c_1 \otimes f_1 + s_1 c, \quad c_2 \otimes f_2 + s_2 c] = ([c_1, c_2] \otimes f_1 f_2] + B(\zeta_1, \zeta_2) \text{Res}(f_1 d f_2) c.$$
Its restriction to constant loops is canonically trivial, thus $\hat{\mathcal{L}}^C$ is embedded in $\mathcal{L}(g^C)$ by the map $(\zeta, s) \mapsto \zeta + sc$. The inclusions $j_I$ lift to inclusions $\hat{j}_I : \hat{\mathfrak{g}}_I \hookrightarrow \hat{\mathcal{L}}(g)$ extending the given inclusion of $\mathfrak{g}_I$. To see this, take $\xi \in \Delta_I$ (defining a trivialization $\hat{g}_I \cong \hat{g}_{I,(\exp \xi)} = g_I \times \mathbb{R}$). Then the desired lift reads,
$$\hat{j}_I, \xi : \hat{g}_{I,(\exp \xi)} \to \hat{\mathcal{L}}(g), \quad \hat{j}_I, \xi (\zeta, s) = j_I(\zeta) + (s + B(\xi, \zeta)) c.$$
By the theory of affine Lie algebras \([20]\), there exists a unitarizable \(\hat{L}_G\)-module where the central element \(\xi\) acts as \(-1\). Unitarizability means in particular that the \(L\)-action exponentiates to a unitary \(\hat{T}\)-action, and hence all \(\hat{g}_T\)-actions exponentiate to unitary \(\hat{G}_T\)-actions.

**Remark 3.7.** If \(G = SU(2)\), there is a much simpler construction. We have unique trivializations \(\hat{G}_0 \cong G \times U(1) \cong \hat{G}_1\). The inclusion of \(\hat{G}_0 = \hat{T}\) into \(\hat{G}_1\) is given by \((t,z) \mapsto (t,t^r z)\) where \(\rho \in \Lambda^*\) generates the weight lattice. Let \(\mathcal{H} = L^2(G)\), and denote by \(\mathcal{H}(r)\) the subspace on which \(T\) acts with weight \(r\). Since all \(\mathcal{H}(r)\) are infinite-dimensional, there exists a unitary transformation \(\Psi\) with \(\Psi: \mathcal{H}(r) \to \mathcal{H}(r+1)\). Extend the given \(G\) action to \(\hat{G}_0 = G \times U(1)\) by letting \(U(1)\) acts with weight \(-1\), and let \(\hat{G}_1 = G \times U(1)\) act by the conjugate of this action by the automorphism \(\Psi\). Let \(\hat{G}_0\) act as a subgroup of \(G_0\). Then these actions satisfy (ii).

With \(\mathcal{H}\) as in the Lemma, put \(A_I = G \times_{G_I} \mathbb{K}(H)\). For \(J \subset I\), the map \(\phi^J_I: G/G_J \to G/G_J\) is covered by a homomorphism of Dixmier-Douady bundles, \(A_I \to A_J\). Hence we may define a \(G\)-Dixmier-Douady bundle,

\[
A = \coprod_I (A_I \times \Delta_I)/ \sim
\]

with identifications similar to those in (19). By construction, the central extension of \(G_I\) defined by the restriction \(A_I|_{\exp(\Delta_I)}\) coincides with \(\hat{G}_I\). Hence, the Dixmier-Douady class \(DD_G(G,A)\) is a generator of \(H^3_G(G) \cong \mathbb{Z}\).

4. *Conjugacy classes*

As is well-known, coadjoint orbits \(O \subset g^*\) carry a distinguished invariant complex structure, hence a \(\text{Spin}_c\)-structure. If \(O\) admits a pre-quantum line bundle \(L \to O\) (i.e. a line bundle with curvature equal to the symplectic form), one may twist the original \(\text{Spin}_c\)-structure by the conjugate of this line bundle. The resulting equivariant index is the irreducible representation parametrized by \(O\). In this Section, we will describe a similar picture for conjugacy classes \(C \subset G\).

4.1. **Pull-back to conjugacy classes.** Given \(\xi \in \Delta\), define a \(G\)-equivariant map \(\Psi: G/T \to G\), \(gT \mapsto \text{Ad}_g(\exp \xi)\). The pull-back \(\Psi^*A\) admits a canonical Morita trivialization, defined by the Hilbert space bundle \(G \times_T \mathcal{H}\). More generally, for any \(l \in \mathbb{Z}\) and any weight \(\mu \in \Lambda^*\) there is a Morita trivialization,

\[
\mathbb{K}(E) \cong \Psi^* A^l, \quad E = G \times_T (\mathcal{H}^l \otimes \mathbb{C}_\mu)
\]

where \(\mathbb{C}_\mu\) is the 1-dimensional 1-dimensional \(T\)-representation of weight \(\mu\). Dixmier-Douady bundles over \(G\), together with Morita trivializations of their pull-backs by \(\Psi\) are classified by the relative cohomology group \(H^3_G(\Psi)\). (See Appendix [12]) The map \(\Psi =: \Psi_1\) is equivariantly homotopic to the constant map \(\Psi_0: gT \mapsto e\), by the homotopy \(\Psi_t(gT) = \exp(t \text{Ad}_g(\xi))\). Hence \(H^3_G(\Psi) = H^3_G(\Psi_0) = H^3_G(G/T) \oplus H^3_G(G)\). Identifying \(H^3_G(G/T) = H^3_T(\text{pt}) = \Lambda^*\) and \(H^3_G(G) = \mathbb{Z}\), we obtain an isomorphism

\[H^3_G(\Psi) = \Lambda^* \oplus \mathbb{Z},\]

The element \((\mu,l) \in H^3_G(\Psi)\) is realized by the Morita trivialization (22).
Proposition 4.3. The conjugacy class $C$ at level $k$ quantization is isomorphic to the class $\Psi_k = \Phi \circ \pi$. We obtain a map of long exact sequences in relative cohomology,

$$
\cdots \longrightarrow 0 \longrightarrow H^2_G(C) \longrightarrow H^2_G(\Phi) \longrightarrow H^2_G(G) \longrightarrow H^2_G(C) \longrightarrow \cdots
$$

Since the map $H^2_G(C) \to H^2_G(G/T)$ is injective, the 5-Lemma implies that the map $H^3_G(\Phi) \to H^3_G(\Psi)$ is injective. The exact sequence in the bottom row splits (see Section 4.1), and hence we obtain an injective map, $H^2_G(\Phi) \to H^2_G(\Psi) = \Lambda^* \oplus \mathbb{Z}$.

By a parallel discussion with real coefficients, there is an isomorphism $H^3_G(\Psi, \mathbb{R}) = t^* \oplus \mathbb{R}$ and an inclusion of $H^3_G(\Phi, \mathbb{R})$.

4.2. Pre-quantization of conjugacy classes. We return to Cartan’s de Rham model for $H^*_G(M, \mathbb{R})$ (cf. the proof of Lemma 3.1) with $\eta_G \in \Omega^1_G(G)$ representing the generator of $H^1_G(G)$. The conjugacy class $C$ carries a unique invariant 2-form $\omega \in \Omega^2(C)^G \subset \Omega^2_G(C)$ with the property \[11, 13,\]

$$
d_{G}\omega = \Phi^* \eta_G.
$$

The triple $(C, \omega, \Phi)$ is an example of a quasi-Hamiltonian $G$-space in the terminology of [1]. Equation (23) together with $d_G \eta G = 0$ say that $(\omega, \eta_G) \in \Omega^2_G(\Phi)$ is a relative equivariant cocycle. Let $[(\omega, \eta_G)]$ be its class in $H^3_G(\Phi, \mathbb{R})$.

Lemma 4.1. The inclusion $H^3_G(\Phi, \mathbb{R}) \to t^* \oplus \mathbb{R}$ takes the class $[(\omega, \eta_G)]$ to the element $(B^3(\xi), 1)$.

Proof. Let $h_t: \Omega^*_G(G) \to \Omega^{*-1}_G(G/T)$ be the homotopy operator defined by the family of maps $\Psi_t$. Thus $d \circ h_t + h_t \circ d = \Psi_t^* - \Psi_0^*$. Then

$$
\Omega^*_G(\Psi_t) \to \Omega^*_G(\Psi_0), \quad (\alpha, \beta) \mapsto (\alpha - h_t(\beta), \beta)
$$

is an isomorphism of chain complexes, inducing the isomorphism $H^*_G(\Psi_t, \mathbb{R}) \to H^*_G(\Psi_0, \mathbb{R})$. Hence, the isomorphism $H^3_G(\Psi_1, \mathbb{R}) \to H^*_G(\Psi_0, \mathbb{R})$ takes $[(\omega, \eta_G)]$ to $[(\omega - h^*_1 \eta_G, \eta_G)]$.

The family of maps $\Psi_t$ is a composition of the map $f: G/T \to g$, $gT \mapsto \text{Ad}_g(\xi)$ with the family of maps $g \to G$, $\zeta \mapsto \exp(t \zeta)$. Let $j_t: \Omega^*_G(G) \to \Omega^{*-1}_G(g)$ be the homotopy operator for the second family of maps. Then $h_t = f^* \circ j_t$. By [23], we have $j_t \eta_G = \varpi_G$, where $\varpi_G \in \Omega^2_G(g)$ is of the form $\varpi_G(\zeta) = \varpi_0(\zeta) - B(\xi, \cdot)$. It follows that the image of $[(\omega, \eta_G)]$ under the map to $t^* \oplus \mathbb{R}$ is $(B^3(\xi), 1)$.

As a special case of pre-quantization of group-valued moment maps [2], we define:

**Definition 4.2.** A level $k \in \mathbb{Z}$ pre-quantization of a conjugacy class $C$ is a lift of the class $k [(\omega, \eta_G)] \in H^3_G(\Phi, \mathbb{R})$ to an integral class.

By the long exact sequence in relative cohomology, if $C$ admits a level $k$ pre-quantization, then the latter is unique (since $H^2_G(C)$ has no torsion).

**Proposition 4.3.** The conjugacy class $C$ of the element $\exp \xi$ with $\xi \in \Delta$ admits a pre-quantization at level $k$ if and only if $(B^3(k \xi), k) \in \Lambda^* \times \mathbb{Z}$. 
Proof. According to the Lemma, \( k[(\omega, \eta_G)] \) maps to \((B^{3}(k\xi), k) \) \( \in t^* \times \mathbb{R} \). Since all maps in the commutative diagram

\[
\begin{array}{ccc}
H^3_G(\Phi) & \longrightarrow & \Lambda^* \oplus \mathbb{Z} \\
\downarrow & & \downarrow \\
H^3_G(\Phi, \mathbb{R}) & \longrightarrow & t^* \oplus \mathbb{R}
\end{array}
\]

are injective, it follows that \( k[(\omega, \eta_G)] \) is integral if and only if \((B^{3}(k\xi), k) \) \( \in \Lambda^* \times \mathbb{Z} \). \( \square \)

Geometrically, a level \( k \) pre-quantization is given by a \( G \)-equivariant Morita trivialization of \( \Phi^*A^k \). This can be seen explicitly, as follows.

Lemma 4.4. Let \( \xi \in \Delta_I \), and suppose that \( B^{3}(k\xi) \in \Lambda^* \). Then the \( k \)-th power of the central extension of \( G_I \) admits a unique trivialization \( G_I \rightarrow \tilde{G}^{(k)}_I \) extending the map

\[
(24) \quad T \rightarrow \tilde{T}^{(k)} = T \times U(1), \ h \mapsto (h, hB^{3}(k\xi)).
\]

Proof. Uniqueness is clear, since a trivialization \( G_I \rightarrow \tilde{G}^{(k)}_I \) is uniquely determined by its restriction to \( T \). For existence, recall that \( \xi \in \Delta_I \) determines an identification \( \tilde{G}_I \cong \tilde{G}_{I,(t)} = \tilde{G}_I \times_{\pi_1(G_I)} U(1) \), where \( t = \exp \xi \), and using the homomorphism \( \varrho_{t,I} : \pi_1(G_I) = \Lambda/\Lambda_I \rightarrow U(1) \), \( \lambda \mapsto t^{B^3(\lambda)} \). The powers \( \tilde{G}^{(l)}_I \) are obtained similarly, using the \( l \)-th powers of the homomorphism \( \varrho_{t,I} \). Since \( B^3(k\xi) \) is a weight, we have \( (\varrho_{t,I})^k = 1 \). This defines a trivialization,

\[
\tilde{G}^{(k)}_I \cong \tilde{G}_{I,(t)} = G_I \times U(1).
\]

According to (21), this isomorphism intertwines the standard inclusion of \( \tilde{T}^{(k)} \rightarrow \tilde{G}^{(k)}_I \) with the map

\[
\tilde{T} = T \times U(1) \rightarrow G_I \times U(1), \ (h, z) \mapsto (h, h^{-B^3(k\xi)}z).
\]

The composition of this map with (24) is \( h \mapsto (h, 1) \), as required. \( \square \)

Let \( \Phi : C \hookrightarrow G \) be the conjugacy class of \( t = \exp \xi \), and let \( I \) be the unique index set such that \( \xi \) lies in the relative interior of \( \Delta_I \). If \( C \) is pre-quantizable at level \( k \), so that \( B^3(k\xi) \in \Lambda^* \), the Lemma defines a trivialization of \( G^{(k)}_I \). Hence, its action on \( \mathcal{H}^k \) descends to an action of \( G_I \), and the Hilbert bundle \( \mathcal{E} = G \times_G \mathcal{H}^k \) defines a Morita trivialization of \( \Phi^*A^k \).

Proposition 4.5. The relative Dixmier-Douady class \( DD_G(A, \mathcal{E}, \psi) \in H^3_G(\Phi) \) is an integral lift of the class \( k[(\omega, \eta_G)] \in H^3_G(G, C, \mathbb{R}) \).

Proof. We have to show that the image of \( DD_G(A, \mathcal{E}, \psi) \) in \( H^3_G(\Phi) = \Lambda^* \oplus \mathbb{Z} \) is \((B^3(k\xi), k) \). But this follows from the discussion in the last Section, since the pull-back of \( \mathcal{E} \) to \( G/T \) is

\[
\pi^* \mathcal{E} = G \times_T (\mathcal{H}^k \otimes C_{B^3(k\xi)}).
\]

\( \square \)
4.3. The $h^\vee$-th power of the Dixmier-Douady bundle. For any coadjoint orbit $O \subset g^*$, the compatible complex structure defines a $G$-invariant Spin$_c$-structure, i.e. Morita trivialization of $\text{Cl}(TO)$. We show that similarly, for all conjugacy classes $C \subset G$, there is a distinguished Morita isomorphism between $\text{Cl}(TC)$ and $A^{h^\vee}|_C$, where $h^\vee$ is the dual Coxeter number. That is, conjugacy classes carry a canonical ‘twisted Spin$_c$-structure’. There are examples of conjugacy classes that do not admit invariant Spin$_c$-structures, let alone invariant complex structures.

We will need some additional notation. Let $\mathcal{S}_0 = \{\alpha_1, \ldots, \alpha_l\}$, $l = \text{rank}(G)$, be a set of simple roots for $g$, relative to our choice of fundamental Weyl chamber. We denote by $\alpha_0 = -\alpha_{\text{max}}$ minus the highest root, and let

$$\mathcal{S} = \mathcal{S}_0 \cup \{\alpha_0\} = \{\alpha_0, \ldots, \alpha_l\}.$$ 

Thus $\Delta \subset t_+$ is the $l$-simplex cut out by the inequalities $\langle \alpha_i, \cdot \rangle + \delta_{i,0} \geq 0$ for $i = 0, \ldots, l$, and $t_+$ is cut out by the hyperplanes with $i > 0$. The roots of $G_I$ are those roots $\alpha$ of $G$ for which $\langle \alpha, \xi \rangle \in \mathbb{Z}$, and a set of simple roots is

$$\mathcal{S}_I = \{\alpha_i \in \mathcal{S} \mid i \notin I\}.$$ 

That is, the Dynkin diagram of $G_I$ is obtained from the extended Dynkin diagram of $G$ by removing the vertices labeled by $i \in I$. Let $\rho$ be the half-sum of positive roots of $G$, $\rho^\sharp = B^\sharp(\rho)$, and let

$$h^\vee = 1 + \langle \alpha_{\text{max}}, \rho^\sharp \rangle$$

be the dual Coxeter number.

**Theorem 4.6.** For any conjugacy class $\Phi : C \hookrightarrow G$, there is a distinguished $G$-equivariant Morita isomorphism $\Phi^* A^{h^\vee} \simeq \text{Cl}(TC)$.

**Proof.** We have to construct a Morita trivialization $(E, \psi)$ of $\Phi^* A^{h^\vee} \otimes \text{Cl}(TC)$ (recall that Clifford algebras satisfy $\text{Cl}(V) \cong \text{Cl}(V)^{opp}$). Let $\xi \in \Delta \cong G \backslash \text{Ad}(G)$ be the point of the alcove corresponding to $C$, and $I$ the index set such that $\xi$ lies in the interior of $\Delta_I$. By construction, $\Phi^* A^{h^\vee} = G \times_{G_I} \mathbb{K}(\mathcal{H}^{h^\vee})$, while $\text{Cl}(TC) = G \times_{G_I} \text{Cl}(g_I^\perp)$ where $g_I^\perp$ is the orthogonal complement of $g_I$ in $g$. Hence

$$\Phi^* A^{h^\vee} \otimes \text{Cl}(TC) = G \times_{G_I} (\mathbb{K}(\mathcal{H}^{h^\vee}) \otimes \text{Cl}(g_I^\perp)),$$

and the Theorem is reduced to the following Lemma. \hfill \Box

**Lemma 4.7.** For each $I$ there is a canonical $G_I$-equivariant Morita trivialization

$$C \simeq \mathbb{K}(\mathcal{H}^{h^\vee}) \otimes \text{Cl}(g_I^\perp).$$

**Proof.** The central extension of $\text{SO}(g_I^\perp)$ defined by its homomorphism to $\text{Aut}(\text{Cl}(g_I^\perp))$ is by definition the group Spin$_c(g_I^\perp)$. Hence, the central extension of $G_I$ defined by its action on $\text{Cl}(g_I^\perp)$ equals the pull-back of Spin$_c(g_I^\perp)$ under the homomorphism $G_I \to \text{SO}(g_I^\perp)$. That is, it is of the form

$$\tilde{G}_I \times_{\tau_1(G_I)} U(1).$$
where $\tilde{G}_I$ is the universal covering group, and the homomorphism $\pi_1(G_I) \to U(1)$ is defined by the commutative diagram,
\[
\begin{array}{cccccc}
1 & \longrightarrow & \pi_1(G_I) & \longrightarrow & \tilde{G}_I & \longrightarrow & G_I & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow & \\
1 & \longrightarrow & U(1) & \longrightarrow & \text{Spin}_c(g_I^\perp) & \longrightarrow & \text{SO}(g_I^\perp) & \longrightarrow & 1 \\
\end{array}
\]

Let $\Lambda_I$ be the co-root lattice of $G_I$, so that $\pi_1(G_I) = \Lambda/\Lambda_I$. By a direct calculation (cf. Sternberg [38, Section 9.2]), the homomorphism $\pi_1(G_I) \to U(1)$ is
\[
(25) \quad \pi_1(G_I) = \Lambda/\Lambda_I \to U(1), \quad \lambda \mapsto e^{2\pi\sqrt{-1}(\rho - \rho_I)}
\]
where $\rho$ is the half-sum of positive roots of $G$, and $\rho_I$ is the half-sum of positive roots of $G_I$, relative to the given system $\mathfrak{S}_I$ of simple roots. Let
\[
(26) \quad \nu_I = \frac{1}{h^\vee}(\rho - \rho_I), \quad \nu_I^\perp = B^2(\nu_I).
\]
The element $\nu_I^\perp$ is contained in the the interior of the face $\Delta_I$ (see e.g. [30]). Hence, the homomorphism (25) is just the $h^\vee$-th power of the homomorphism $\varphi_{t,I}$, $t = \exp\nu_I^\perp$ in the definition of $\tilde{G}_{I,(t)} \cong \tilde{G}_I$. That is, we have a pull-back diagram
\[
\begin{array}{cccccc}
\tilde{G}_I^{(h^\vee)} & \longrightarrow & \text{Spin}_c(g_I^\perp) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
G_I & \longrightarrow & \text{SO}(g_I^\perp)
\end{array}
\]
There is an explicit spinor module $S_I$ for $\text{Cl}(g_I^\perp)$, constructed as follows. Let $n_+ \subset g^C$ and $n_{I,+} \subset g_I^C$ be the sum of root spaces for positive roots of $G$ and $G_I$, respectively. (Here positivity is defined by the respective sets $\mathfrak{S}_0, \mathfrak{S}_I$ of simple roots.) Since projection to the real part identifies $n_+ \cong t^\perp$ (as real vector spaces), $S = \wedge n_+$ is a spinor module for $\text{Cl}(t^\perp)$. Similarly $S_I = \wedge n_{I,+}$ is a spinor module for $\text{Cl}(g_I \cap t^\perp)$. The spinor module $S_I$ over $\text{Cl}(g_I^\perp)$ is
\[
(27) \quad S_I = \text{Hom}_{\text{Cl}(g_I \cap t^\perp)}(S_I, S).
\]
The Clifford action on $S_I$ restricts to a unitary representation of $\text{Spin}_c(g_I^\perp) \subset \text{Cl}(g_I^\perp)$ where the central circle acts with weight 1. The $\mathbb{K}(H^{h^\vee}) \otimes \text{Cl}(g_I^\perp)$-module $H^{h^\vee} \otimes S_I$ with the resulting action of $G_I$ gives the desired Morita trivialization.

The spinor module (27) is $T$-equivariant. This has the following consequence:

**Proposition 4.8.** Let $C$ be the conjugacy class of $\exp \xi$, $\xi \in \Delta$. The pull-back of $\text{Cl}(TC)$ under the projection map
\[
\pi: G/T \to C, \quad gT \mapsto \text{Ad}_g(\exp(\xi)).
\]
admits a canonical $G$-equivariant Morita trivialization
\[
(28) \quad C \simeq \pi^\ast \text{Cl}(TC).
\]

**Proof.** Let $I$ be the index set such that $G_I$ is the stabilizer of $\exp \xi$. We have $\pi^\ast \text{Cl}(TC) = \text{Cl}(\pi^\ast TC) = G \times_T \text{Cl}(g_I^\perp)$. Hence we need a $T$-equivariant Morita trivialization of $\text{Cl}(g_I^\perp)$, and this is provided by $S_I$. \qed
If the conjugacy class $C$ is pre-quantized at level $k$, the Morita equivalences $Cl(TC) \simeq \Phi^* A^h$ and $C \simeq \Phi^* A^k$, combine to a Morita equivalence

\[(29)\quad Cl(TC) \simeq \Phi^* A^{k+h}\]

Recall $\Psi = \Phi \circ \pi : G/T \to G$. The composition of the Morita equivalences (28) and $Cl(TC) \simeq \Phi^* A^h$ is the Morita trivialization $C \simeq \Psi^* A^h$ defined by the bundle $G \times_T H^h$. It is thus labeled by $(0, h^\vee) \in \Lambda^* \oplus \mathbb{Z}$. Hence, in the pre-quantized case, the composition of (28) and (29) is the Morita trivialization of $\Psi^* A^{k+h}$ parametrized by $(B^\phi(k\xi), k+h^\vee) \in \Lambda^* \oplus \mathbb{Z}$. 

### 4.4. Quantization of conjugacy classes

The twisted equivariant $K$-homology group

$$K^G_\bullet (G, A^{k+h^\vee})$$

carries a ring structure, with product given by the cross-product for $G \times G$, followed by push-forward under group multiplication $\text{Mult}: G \times G \to G$. Indeed, since $\text{Mult}^* x = \text{pr}^* x + \text{pr}_2^* x$ for all $x \in H^3_G(G, \mathbb{Z})$, the pull-back bundle $\text{Mult}^* A^{k+h^\vee}$ over $G \times G$ is (stably) isomorphic to $\text{pr}_1^* A^{k+h^\vee} \otimes \text{pr}_2^* A^{k+h^\vee}$. The choice of a stable isomorphism defines a push-forward map

$$K^G_\bullet (\text{Mult}): K^G_\bullet (G, A^{k+h^\vee}) \otimes K^G_\bullet (G, A^{k+h^\vee}) \to K^G_\bullet (G, A^{k+h^\vee}),$$

where we identify $K^G_\bullet (G, A^{k+h^\vee}) \otimes K^G_\bullet (G, A^{k+h^\vee}) \cong K^G_\bullet (G \times G, \text{pr}_1^* A^{k+h^\vee} \otimes \text{pr}_2^* A^{k+h^\vee})$. Since $H^3_G(G \times G) = 0$ this map in $K$-homology is independent of the choice of stable isomorphism, and by a similar reasoning the resulting product is commutative and associative. (For non-simply connected groups $G$, the existence of ring structures on the twisted $K$-homology is a much more subtle matter [40].)

The inclusion $i: \{e\} \to G$ of the group unit induces a ring homomorphism

\[(30)\quad K^G_\bullet (i): R(G) = K^G_\bullet (\text{pt}) \to K^G_\bullet (G, A^{k+h^\vee}).\]

**Theorem 4.9** (Freed-Hopkins-Teleman). For all non-negative integers $k \geq 0$ the ring homomorphism (30) is onto, with kernel the level $k$ fusion ideal $I_k(G) \subset R(G)$. That is, $K^G_0 (G, A^{k+h^\vee}) = 0$, while $K^G_0 (G, A^{k+h^\vee})$ is canonically isomorphic to the level $k$ fusion ring, $R_k(G) = R(G)/I_k(G)$.

We will explain a proof of this Theorem in Section 5 below.

**Remark 4.10.** It is also very interesting to consider the non-equivariant twisted $K$-homology rings $K_\bullet (G, A^{k+h^\vee})$. These are studied are in the work of V. Braun [7] and C. Douglas [12].

The ring $R_k(G)$ may be defined as the ring of level $k$ projective representations of the loop group $LG$, or in finite-dimensional terms (cf. [8]): Let

$$\Lambda^*_k = \Lambda^* \cap B^2(k\Delta)$$

be the set of *level $k$ weights*. Identify $R(G)$ with ring of characters of $G$. Then $R_k(G) = R(G)/I_k(G)$, where $I_k(G)$ is the vanishing ideal of the set of elements $\{t_\nu \in T, \nu \in \Lambda^*_k\}$ where

$$t_\nu = \frac{1}{k+h^\vee} B^2(\nu + \rho).$$

It turns out that as an additive group, $R_k(G)$ is freely generated by the images of irreducible characters $\chi_\mu$ for $\mu \in \Lambda^*_k$. Thus $R_k(G) = \mathbb{Z}[\Lambda^*_k]$ additively.
Remark 4.11. If $G$ has type $ADE$ (so that all roots have equal length), the lattice $B^2(\Lambda^*) \subset \mathfrak{t}$ is identified with the set of elements $\xi \in \mathfrak{t}$ with $\exp \xi \in Z(G)$, the center of $G$. Hence the ideal $I_k(G)$ may be characterized, in this case, as the vanishing ideal of the set of all $g \in G^{reg}$ such that $g^{k+h^\vee} \in Z(G)$.

Remark 4.12. Freed-Hopkins-Teleman compute twisted $K$-homology groups of $G$ for arbitrary compact groups, not necessarily simply connected. The case of simple, simply connected groups considered here is considerably easier than the general case.

Suppose $\Phi: \mathcal{C} \hookrightarrow G$ is the conjugacy class of $\exp \xi$, $\xi \in \Delta$, pre-quantized at level $k \geq 0$. Thus $\mu := B^h(k\xi)$ is a weight. The Morita equivalence (29) defines a push-forward map in K-homology,

$$K^G_0(\Phi): K^G_0(\mathcal{C}, \text{Cl}(T\mathcal{C})) \to K^G_0(G, A^{k+h^\vee})$$

where $\Phi: \mathcal{C} \hookrightarrow G$ is the inclusion.

Theorem 4.13. The push-forward map (31) takes the fundamental class $[\mathcal{C}] \in K^G_0(\mathcal{C}, \text{Cl}(T\mathcal{C}))$ to the equivalence class of the character $\chi_\mu$ in $R_k(G) = R(G)/I_k(G)$.

Proof. Let $\pi: G/T \to \mathcal{C}$ and $\Psi = \Phi \circ \pi: G/T \to G$ be as in Section 4.11 and recall that $\Psi = \Psi_1$ is equivariantly homotopic to the constant map $\Psi_0$ onto $e \in G$. That is, the diagram

$$
\begin{array}{ccc}
G/T & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
\text{pt} & \longrightarrow & G,
\end{array}
$$

commutes up to a $G$-equivariant homotopy. As discussed at the end of Section 4.3, the Morita trivialization $\mathcal{C} \simeq \pi^* \text{Cl}(T\mathcal{C})$ from (28) combines with (29) to a Morita trivialization of

$$
\Psi^* A^{k+h^\vee} \cong \mathbb{K}(G \times_T (C_\mu \otimes \mathcal{H}^{k+h^\vee})).
$$

It is related to the obvious trivialization $\Psi_0^* A^{k+h^\vee} \cong \mathbb{K}(G \times_T \mathcal{H}^{k+h^\vee})$ by a twist by the line bundle $G \times_T C_\mu$. We obtain a commutative diagram

$$
\begin{array}{ccc}
K^G_0(G/T) = R(T) & \longrightarrow & K^G_0(\mathcal{C}, \text{Cl}(T\mathcal{C})) \\
\downarrow & & \downarrow \\
K^G_0(\text{pt}) = R(G) & \longrightarrow & K^G_0(G, A^{k+h^\vee}) = R_k(G),
\end{array}
$$

where the lower horizontal map is defined by the ‘obvious’ trivialization $\iota_e^* A^{k+h^\vee} \cong \mathbb{K}(\mathcal{H}^{k+h^\vee})$, and the left vertical map is twist by the line bundle $G \times_T C_\mu$ (acting as an automorphism of $K^G_0(G/T)$, followed by the push-forward.

Using the Morita trivialization $\text{Cl}(T(G/T)) \cong \mathbb{K}(G \times_T S)$ (where $S = \wedge n_+$) to identify $K^G_0(G/T) = K^G_0(G/T, \text{Cl}(T(G/T)))$, the left vertical map takes the fundamental class $[G/T]$ to the irreducible character $\chi_\mu \in R(G) = K^G_0(\text{pt})$. On the other hand, the upper horizontal map takes $[G/T]$ to $[\mathcal{C}]$, which then goes to $\Phi_\mu [\mathcal{C}]$. We conclude that $\Phi_\mu [\mathcal{C}] \in R_k(G)$ is the image of $\chi_\mu$ under the quotient map $R(G) \to R_k(G)$. □
4.5. **Twisted $K$-homology of the conjugacy classes.** Suppose $\Phi : C \to G$ is an arbitrary conjugacy class (not necessarily pre-quantized) corresponding to $\xi \in \Delta$. Let $I$ be the index set such that $\xi$ is in the interior of $\Delta_I$, thus $C = G/G_I$. The twisted $K$-homology group $K^G_q(C, \Phi^* A^{k+h'}) = K^G_q(G/G_I, A_I^{k+h'})$ is computed as in Section 2.3 using Lemma 4.7

$$K^G_q(G/G_I, A_I^{k+h'}) = K^G_q(pt, \mathbb{K}(H)^{k+h'} \otimes \operatorname{Cl}(g_I^\bot))$$

$$= K^G_q(pt, \mathbb{K}(H)^k).$$

The central extension of $G_I$ defined by its action on $\mathbb{K}(H)^k$ is $\hat{G}_I^{(-k)}$. Hence, the last group is isomorphic to the Grothendieck group of $\hat{G}_I^{(-k)}$-representations where the central circle acts with weight $-1$, or equivalently the Grothendieck group $R(\hat{G}_I)_k$ of $\hat{G}_I$-representations where the central circle acts with weight $k$. That is, $K^G_q(G/G_I, A_I^{k+h'}) = 0$, while

$$K^G_0(G/G_I, A_I^{k+h'}) \cong R(\hat{G}_I)_k$$

as $R(G)$-modules. (The module structure is given by the restriction homomorphism $R(G) \to R(G_I)_0$, which acts on $R(\hat{G}_I)$ by multiplication.) If $J \subset I$, we have a natural map $\phi_I^J : G/G_I \to G/G_J$ covered by a map of Dixmier-Douady bundles $A_I \to A_J$. Hence we obtain a push-forward map,

$$(33) \quad K^G_0(\phi_I^J) : K^G_0(G/G_I, A_I^{k+h'}) \to K^G_0(G/G_J, A_J^{k+h'})$$

On the other hand, the collections of simple roots $S_J \subset S_I$ determine complex structures on the homogeneous spaces $G_J/G_I = \hat{G}_J/\hat{G}_I$. This defines holomorphic induction maps $\operatorname{ind}_I^J : R(\hat{G}_I) \to R(\hat{G}_J)$, restricting to $R(G)$-module homomorphisms

$$\operatorname{ind}_I^J : R(\hat{G}_I)_k \to R(\hat{G}_J)_k.$$  

From the discussion in Section 2.3 we see:

**Proposition 4.14.** The identifications $K^G_0(G/G_I, A_I^{k+h'}) \cong R(\hat{G}_I)_k$ intertwine the push-forward maps (33) with the holomorphic induction maps (34).

5. **Computation of $K^G_*(G, A^{k+h'})$**

The Dixmier-Douady bundle $A \to G$, as described in (21), may be viewed as the geometric realization of a co-simplicial Dixmier-Douady bundle, with non-degenerate $p$-simplices the bundle $\coprod_{|I|=p+1} A_I$ over $\coprod_{|I|=p+1} G/G_I$. This defines a spectral sequence computing the $K$-homology group $K^G_*(G, A^{k+h'})$, in terms of the known $K$-homology groups $K^G_*(G/G_I, A_I^{k+h'}) = R(\hat{G}_I)_k$ and the holomorphic induction maps between these groups. As it turns out, the spectral sequence collapses at the $E_2$-stage, and computes the level $k$ fusion ring.

5.1. **The spectral sequence for $K^G_*(G, A^{k+h'})$.** The construction (21) of $A \to G$ as a quotient of $\coprod_I A_I \times \Delta_I \to \coprod_I G/G_I \times \Delta_I$ may be thought of as the geometric realization of a ‘co-simplicial Dixmier-Douady bundle’. See [35] and [31] for background on co-simplicial (semi-simplicial) techniques. Here the $G$-Dixmier-Douady bundles

$$\coprod_{|I|=p+1} A_I \to \coprod_{|I|=p+1} G/G_I$$
are the non-degenerate $p$-simplices; the full set of $p$-simplices is a union $\bigsqcup_f A_f([p]) \to \bigsqcup_f G/G_f([p])$ over all non-decreasing maps $f : [p] = \{0, \ldots, p\} \to \{0, \ldots, l\}$. By the theory of co-simplicial spaces (see [35 Section 5]), one obtains a spectral sequence $E^1_{p,q} \Rightarrow K^G_{p+q}(G, \mathcal{A}^{k+h^\vee})$ where

$$E^1_{p,q} = \bigoplus_{|I|=p+1} K^G_q(G/G_I, \mathcal{A}^{k+h^\vee}_I).$$

The differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ is given on $K^G_q(G/G_I, \mathcal{A}^{k+h^\vee}_I)$ as an alternating sum,

$$d^1 = \sum_{r=0}^p (-1)^r K^G_{q-1}(\phi_r^I).$$

Here $\delta_r I$ is obtained from $I$ by omitting the $r$-th entry: $\delta_r I = \{i_0, \ldots, i_r, \ldots, i_p\}$ for $I = \{i_0, \ldots, i_p\}$ with $i_0 < \cdots < i_p$. Recall that $\phi^I_r : G/G_I \to G/G_{I_r}$ are the natural maps for $J \subset I$.

By mod 2 periodicity of the $K$-homology, we have $E^1_{p,q} = E^1_{p,q+2}$. Since the groups $G_I$ are connected, and since $\dim G/G_I$ is even, one has $K^G_q(G/G_I, \mathcal{A}^{k+h^\vee}_I) = 0$, thus $E^1_{1,q} = 0$. Hence, the $E^1$-term is described by a single chain complex $(C\bullet, \partial)$, where

$$C_p = E^1_{p,0}, \quad \partial = d^1.$$  

The map $R(G) \to K^G_q(G, \mathcal{A}^{k+h^\vee})$ defined by the inclusion $\iota : e \hookrightarrow G$ may also be described by the spectral sequence. Think of $\iota$ as the geometric realization of a map of co-simplicial manifolds, given as the inclusion of $\{e\} = G/G_{(0)}$ into $\coprod_i G/G_{(i)}$. The co-simplicial map gives to a morphism of spectral sequences, $E^\bullet \to E^\bullet$, where

$$\tilde{E}^1_{p,q} = \begin{cases} K^G_q(\text{pt}, \mathbb{C}) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

At the $E^1$-stage, this boils down to a chain map

$$R(G) \to C\bullet$$

where $R(G) = E^1_{0,0}$ carries the zero differential. Our goal is to show that the homology of $(C\bullet, \partial)$ vanishes in positive degrees, while the induced map in homology $R(G) \to H_0(C, \partial)$ is onto, with kernel $I_k(G)$.

5.2. The induction maps in terms of weights. To get started, we express the chain complex in terms of weights of representations. Recall that $R(T)$ is isomorphic to the group ring $\mathbb{Z}[\Lambda^*]$. The restriction map $R(G) \to R(T)$ is injective, and identifies

$$R(G) \cong \mathbb{Z}[\Lambda^*]^W.$$  

Let us next describe $R(\hat{G}_T)_k$ in terms of weights. Each $\hat{G}_T$ has maximal torus $\hat{T} = T \times U(1)$, hence the weight lattice is

$$\hat{\Lambda}^* = \Lambda^* \times \mathbb{Z} \subset \mathfrak{t}^* = \mathfrak{t}^* \times \mathbb{R}.$$  

The simple roots for $\hat{G}_T$ are $(\alpha_i, 0)$ with $\alpha_i \in \mathfrak{S}_T$, the corresponding co-roots are

$$\alpha_i^\vee, \delta_i, 0) \in \mathfrak{t} = \mathfrak{t} \times \mathbb{R}, \quad \alpha_i \in \mathfrak{S}_T.$$
These define a fundamental Weyl chamber

\[ \tilde{U}_{I,+} = \{ (\nu, s) \mid \langle \nu, \alpha_i^\vee \rangle + s\delta_i,0 \geq 0, \ \alpha_i \in \mathcal{S}_I \} \]

The elements \( \nu_I \) satisfy \( \langle \nu_I, \alpha_i^\vee \rangle + \delta_i,0 = 0 \). Hence, \((\nu, s) \in \tilde{U}_{I,+}\) is and only if \( \nu - s\nu_I \in \tilde{U}_{I,+}\).

Let \( \Lambda_{I,k}^* \) be the intersection of \((38)\) with \( \Lambda^* \times \{ k \} \cong \Lambda^* \). Thus

\[ \Lambda_{I,k}^* = \{ \nu \in \Lambda^* \mid \langle \nu, \alpha_i^\vee \rangle + k\delta_i,0 \geq 0, \ i \notin I \} \]

labels the irreducible \( \hat{G}_I \)-representations for which the central circle acts with weight \( k \). The Weyl group \( W_I \) of \( G_I \) is also the Weyl group of \( \hat{G}_I \). Its action on \( \hat{\Lambda}^* \) preserves the levels \( \Lambda^* \times \{ k \} \), hence it takes the form \( w.(\nu, k) = (w \bullet_k \nu, k) \) for a level \( k \)-action \( \nu \mapsto w \bullet_k \nu \) on \( \Lambda^* \).

Explicitly,

\[ w \bullet_k \nu = w(\nu - k\nu_I) + k\nu_I. \]

Fix \( k \), and denote by \( Z[\Lambda^*]^{W_I-\text{as}} \) the anti-invariant part for the \( W_I \)-action \( \nu \mapsto w \bullet_{k+h^\vee} \nu \) at the shifted level \( k + h^\vee \). Observe that this space is invariant under the action of \( Z[\Lambda^*]^W \). Let

\[ \text{Sk}^I : Z[\Lambda^*] \rightarrow Z[\Lambda^*]^{W_I-\text{as}}, \nu \mapsto \sum_{w \in W_I} (-1)^{\text{length}(w)} w \bullet_{k+h^\vee} \nu \]

denote skew-symmetrization relative to the action at level \( k + h^\vee \). For \( \mu \in \Lambda^*_k \), let \( \chi^I_{\mu} \in R(\hat{G}_I)_k \) be the character of the irreducible \( \hat{G}_I \)-representation of weight \((\mu, k)\).

**Lemma 5.1.** The map \( \chi^I_{\mu} \mapsto \text{Sk}^I(\mu + \rho) \) extends to an isomorphism

\[ R(\hat{G}_I)_k \rightarrow Z[\Lambda^*]^{W_I-\text{as}}. \]

Under this isomorphism, the \( R(G) \cong Z[\Lambda^*]^W \)-module structure is given by multiplication in the group ring. Furthermore, the identification \((40)\) intertwines the holomorphic induction maps \( \text{ind}^J_I : R(\hat{G}_I)_k \rightarrow R(\hat{G}_J)_k \) for \( J \subset I \) with skew-symmetrizations

\[ \text{Sk}^J_I = \frac{1}{|W_I|} \text{Sk}_J : Z[\Lambda^*]^{W_I-\text{as}} \rightarrow Z[\Lambda^*]^{W_J-\text{as}}. \]

Note that the statement involves a shift by \( \rho \), rather than \( \rho_I \). Thus, even in the case \( I = \{0, \ldots, l\} \) where \( G_I = T \) and \( W_I = \{1\} \), \( \rho_I = 0 \), the identification \( R(\hat{T})_k \rightarrow Z[\Lambda^*] \) involves a \( \rho \)-shift.

**Proof.** Let \( \Lambda_{I,k+h^\vee}^{\text{reg}} \) be the intersection of \( \Lambda^* \times \{ k + h^\vee \} \) with \( \text{int}(\tilde{U}_{I,+}) \). Since obviously \( R(\hat{G}_I)_k = Z[\Lambda_{I,k}^*] \), the first part of the Lemma amounts to the assertion that

\[ \mu \in \Lambda_{I,k}^* \iff \mu + \rho \in \Lambda_{I,k+h^\vee}^{\text{reg}}. \]

We have \( \mu \in \Lambda_{I,k}^* \) if and only if \( \langle \mu, \alpha_i^\vee \rangle + k\delta_i,0 \geq 0 \) for \( i \notin I \). Since \( \langle \rho, \alpha_i^\vee \rangle + h^\vee \delta_i,0 = 1 \) this is equivalent to \( \langle \mu + \rho, \alpha_i^\vee \rangle + (k+h^\vee)\delta_i,0 \geq 1, \ i \notin I \), i.e. \( \mu + \rho \in \Lambda_{I,k+h^\vee}^{\text{reg}} \) as claimed. The assertion about the \( R(G) \)-module structure is obvious. Finally, for \( J \subset I \) the holomorphic induction map \( \text{ind}^J_I \) is given by

\[ \text{ind}^J_I(\chi^I_{\mu}) = (-1)^{\text{length}(w)} \chi^{J}_{w\bullet_k(\mu+\rho)-\rho_J}. \]
if there exists \( w \in W_J \) with \( w \cdot_k (\mu + \rho_J) - \rho_J \in A_{J,k}^* \), while \( \text{ind}_J^I(\nu_J) = 0 \) if there is no such \( w \). Using \( (39) \) together with \( \rho_J - k \nu_I = \rho - (k + h^\vee) \nu_I \) (by the definition of \( \nu_I \)), this may be re-written in terms of the action at level \( k + h^\vee \):

\[
w \cdot_k (\mu + \rho_J) - \rho_J = w \cdot_{k+h^\vee} (\mu + \rho) - \rho.
\]

\( \square \)

By combining this discussion with Proposition 4.14, we have established a commutative diagram

\[
\begin{array}{ccc}
K_0^G(G/G_J, A_{J,k}^{k+h^\vee}) & \xrightarrow{\approx} & R(\wG_J)_k \xrightarrow{\approx} \mathbb{Z}[\Lambda^*]^{W_J-\text{as}} \\
\uparrow K_0(\phi_J^I) & & \uparrow \text{ind}_J^I \\
K_0^G(G/G_I, A_{I,k}^{k+h^\vee}) & \xrightarrow{\approx} & R(\wG_I)_k \xrightarrow{\approx} \mathbb{Z}[\Lambda^*]^{W_I-\text{as}}
\end{array}
\]

(41)

We can thus re-express the chain complex \((C_\bullet, \partial)\) in terms of weights:

\[
C_p = \bigoplus_{|I| = p+1} \mathbb{Z}[\Lambda^*]^{W_I-\text{as}}, \quad \partial \phi^I = \sum_{r=0}^p (-1)^r \text{Sk}_I^\delta \omega(\phi^I),
\]

for \( \phi^I \in \mathbb{Z}[\Lambda^*]^{W_I-\text{as}} \). The map \( R(G) \to C_0 \subset C_\bullet \) given by \( (36) \) is expressed as the inclusion of \( \mathbb{Z}[\Lambda^*]^{W-\text{as}} \), as the summand corresponding to \( I = \emptyset \). By construction, \( C_\bullet \) is a complex of \( R(G) \)-modules, and the map \( (36) \) is an \( R(G) \)-module homomorphism.

5.3. **Fusion ring.** Let us also describe the fusion ring in terms of weights. The subset \( B^\mu(k\Delta) \subset \mathfrak{t}^* \) defining the set \( \Lambda_k^* = \Lambda^* \cap B^\mu(k\Delta) \) of level \( k \) weights is cut out by the inequalities

\[
\langle \nu, \alpha_i^\vee \rangle + k \delta_i,0 \geq 0.
\]

It is a fundamental domain for the level \( k \) action \( \nu \mapsto w \cdot_k \nu \) of the affine Weyl group, generated by the simple affine reflections

\[
\nu \mapsto \nu - \langle \nu, \alpha_i^\vee \rangle + \alpha_i, \quad i = 0, \ldots, l.
\]

This is consistent with our earlier notation: The level \( k \) action of \( W_{aff} \) restricts to the level \( k \) action of the subgroup \( W_I \), generated by the affine reflections with \( i \notin I \).

Let \( \mathbb{Z}[\Lambda^*] \) be the \( \mathbb{Z}[\Lambda^*] \)-module consisting of all functions \( \Lambda^* \to \mathbb{Z} \), not necessarily of finite support. Let

\[
\text{Sk}_{aff} : \mathbb{Z}[\Lambda^*] \to \mathbb{Z}[\Lambda^*]^{W_{aff}-\text{as}}, \quad \nu \mapsto \sum_{w \in W_{aff}} (-1)^{\text{length}(w)} \nu \cdot_{k+h^\vee} w
\]

be skew-symmetrization, using the action at the shifted level \( k + h^\vee \). The map \( \mu \mapsto \text{Sk}_{aff}(\mu + \rho) \) extends to an isomorphism, \( \mathbb{Z}[\Lambda_k^*] \to \mathbb{Z}[\Lambda^*]^{W_{aff}-\text{as}} \). This identifies

\[
R_k(G) \cong \mathbb{Z}[\Lambda^*]^{W_{aff}-\text{as}}
\]

as an abelian group. For any \( I \) we have \( R(G) = \mathbb{Z}[\Lambda^*]^{W-\text{module}} \) homomorphisms \( R(\wG_I)_k \to R_k(G) \),

\[
\mathbb{Z}[\Lambda^*]^{W_I-\text{as}} \to \mathbb{Z}[\Lambda^*]^{W_{aff}-\text{as}}, \quad \phi_I \mapsto \frac{1}{|W_I|} \text{Sk}_{aff} \phi_I.
\]
For \( I = \{0\} \) we may use the obvious trivialization \( \hat{G} = G \times U(1) \) to identify \( R(G) = R(\hat{G}_0)_k \). The following is clear from the description of the quotient map \( R(G) \to R_k(G) \) (see e.g. [25]):

**Lemma 5.2.** The identifications \( R(G) = \mathbb{Z}[[\Lambda^*]]^{W-as} \) and \([13]\) intertwine the quotient map \( R(G) \to R_k(G) \) with the skew-symmetrization map,

\[
\frac{1}{|W|} \mathrm{Sk}_{aff}: \mathbb{Z}[[\Lambda]]^{W-as} \to \mathbb{Z}[[\Lambda^*]]^{W_{aff-as}}.
\]

In particular, \([13]\) is an isomorphism of \( R(G) \cong \mathbb{Z}[[\Lambda^*]]^W \)-modules.

In fact, we could define the ideal \( I_k(G) \subset R(G) \) as the kernel of the map \([15]\). Let \( \epsilon: C_0 \to R_k(G) \) be the direct sum of the morphisms \([44]\) for \( |I| = 1 \).

### 5.4. A resolution of the \( R(G) \)-module \( R_k(G) \).

**Theorem 5.3.** For all \( k \geq 0 \) the chain complex \((C_\bullet, \partial)\) defines a resolution

\[
0 \to C_l \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \xrightarrow{\epsilon} R_k(G) \to 0
\]

of \( R_k(G) \) as an \( R(G) \)-module.

The proof will be given below. As mentioned in the introduction, Theorem 5.3 is implicit in the work of Kitchloo-Morava [25].

**Remark 5.4.** If \( G \) is of type \( A_n \) or \( C_n \), the central extensions \( \hat{G}_I \) are all trivial, thus \( R(\hat{G}_I)_k \cong R(G_I) \) as an \( R(G) \)-module. By the Pittie-Steinberg theorem [37], if \( K \) is a maximal rank subgroup of a compact simply connected Lie group \( G \), then \( R(K) \) is free as a module over \( R(G) \). Thus \( C_\bullet \) is a free resolution of the \( R(G) \)-module \( R_k(G) \) in those cases.

**Remark 5.5.** Theorem 5.3 implies the Freed-Hopkins-Teleman theorem [1]: By acyclicity of the chain complex \( C_\bullet \) the spectral sequence \( E^p \) collapses at the \( E^2 \)-term, with

\[
E^2_{p,q} = E^\infty_{p,q} = \begin{cases} R_k(G) & \text{if } p = 0 \text{ and } q \text{ even} \\
0 & \text{otherwise} \end{cases}
\]

Since \( R_k(G) \) is free Abelian as a \( \mathbb{Z} \)-module, there are no extension problems and we conclude \( K_1^G(G, \mathcal{A}^{k+h^V}) = 0 \), while

\[
K_0^G(G, \mathcal{A}^{k+h^V}) = R_k(G)
\]

as modules over \( R(G) \). This isomorphism takes the ring homomorphism \( R(G) \to K_0^G(G, \mathcal{A}^{k+h^V}) \) to the quotient map \( R(G) \to R_k(G) \), hence \([46]\) is an isomorphism of rings.

The statement of Theorem 5.3 can be simplified. Indeed, the chain complex \( C_\bullet \) breaks up as a direct sum of sub-complexes \( C_\bullet(\mu), \mu \in \Lambda^*_k \), given as

\[
C_p(\mu) = \bigoplus_{|I|=p+1} \mathbb{Z}[W_{aff} \bullet k+h^V \mu]^{W_{I-as}}.
\]

Similarly the map \( \epsilon: C_0 \to R_k(G) \) splits into a direct sum of maps

\[
\epsilon: C_0(\mu) \to \mathbb{Z}[W_{aff} \bullet k+h^V \mu]^{W_{aff-as}} = \begin{cases} \mathbb{Z} & \text{for } \mu \in \Lambda^*_{k+h^V} \\
0 & \text{otherwise} \end{cases}
\]
Finally the chain map $R(G) \hookrightarrow C_\bullet$ splits into inclusions of $\mathbb{Z}[W_{\text{aff}} \bullet_{k+h^\vee} \mu]^W$ as the term corresponding to $I = \{0\}$. Clearly, $(C_\bullet(\mu), \partial)$ depends only on the open face $B^g((k+h^\vee) \Delta_J)$ of $B^g((k+h^\vee) \Delta)$ containing $\mu$. Indeed, since $\mathbb{Z}[W_{\text{aff}} \bullet_{k+h^\vee} \mu] = \mathbb{Z}[W_{\text{aff}}/W_J]$ we have

$$C_p(J) = \bigoplus_{|I|=p+1} \mathbb{Z}[W_{\text{aff}}/W_J]^{W_I}_{-\text{as}}.$$

The differential $\partial$ is again given by anti-symmetrization as in [42], but with $\phi^J$ now an element of $\mathbb{Z}[W_{\text{aff}}/W_J]^{W_J}_{-\text{as}}$. The map $\epsilon: C_0 \to R_k(G)$ translates into the zero map $C_0(J) \to \mathbb{Z}$ unless $J = \{0, \ldots, l\}$, in which case it becomes a map $\epsilon: C_0(J) \to \mathbb{Z}$, given as the direct sum for $i = 0, \ldots, l$ of the maps,

$$Z[W_{\text{aff}}]^{W_i}_{-\text{as}} \to \mathbb{Z}, \quad \sum_w n_w w \mapsto \sum_{W} n_w (-1)^{\text{length}(w)}.$$

The map $R(G) \to C_\bullet$ is again the inclusion of the summand of $C_0(J)$ corresponding to $I = \{0\}$.

Theorem 5.3 is now reduced to the following simpler statement:

**Theorem 5.6.** The homology $H_\bullet(J)$ of the chain complex $C_\bullet(J)$ vanishes in degree $p > 0$, while

$$H_0(J) = \begin{cases} 0 & \text{if } J \neq \{0, \ldots, l\} \\ \mathbb{Z} & \text{if } J = \{0, \ldots, l\} \end{cases}$$

In the second case, the isomorphism is induced by the augmentation map $\epsilon: C_0(J) \to \mathbb{Z}$.

5.5. **Proof of Theorem 5.6.** Throughout this Section, we consider a given face $\Delta_J$ of the alcove. We may think of $W_{\text{aff}}/W_J$ as the $W_{\text{aff}}$-orbit of a point in the interior of the face $\Delta_J$, under the standard action of $W_{\text{aff}}$ on $t$. To be concrete, let us take the point $\nu_J^\sharp$. Denote its orbit by

$$V = W_{\text{aff}}.\nu_J^\sharp \subset t.$$

We introduce a length function $\text{length}: V \to \mathbb{Z}$, defined in terms of the function on $W_{\text{aff}}$ as

$$\text{length}(x) = \min\{\text{length}(w) | w \in W_{\text{aff}}, x = w.\nu_J^\sharp\}, \quad x \in V.$$

Geometrically, $\text{length}(x)$ is the number of affine root hyperplanes in the Stiefel diagram, crossed by a line segment from the a point in the interior of $\Delta$ to the point $x$.

For any $I$ let $t_{I,+}$ be defined by the inequalities $\langle \alpha_i, \cdot \rangle + \delta_{i,0} \geq 0$ for $\alpha_i \in \mathfrak{S}_I$. (Equivalently, it is the affine cone over $\Delta$ at $\nu_J^\sharp$.) Then $t_{I,+}$ is a fundamental domain for the $W_I$-action. Let $V_I \subset \nabla_I \subset V$ be the subsets,

$$V_I = V \cap \text{int}(t_{I,+}), \quad \nabla_I = V \cap t_{I,+}.$$

Every $W_I \subset W_{\text{aff}}$-orbit contains a unique point in $\nabla_I$. Thus, if $x \in V$, we may choose $u \in W_I$ with $u.x \in \nabla_I$. Then

$$\text{length}(u.x) \leq \text{length}(x),$$

with equality if and only if $x \in \nabla_I$ and hence $u.x = x$.

The elements

$$\beta_I(x) = \text{Sk}^I(x), \quad x \in V_I$$

are
form a basis of the \( \mathbb{Z}\)-module \( \mathbb{Z}[V]^{W_j - \text{as}} \). (Note that if \( x \in V^I \setminus V^I \) then \( \text{Sk}^I(x) = 0 \).) Let us describe the differential in terms of this basis. For \(|I| = p + 1\) and \( x \in V^I \) we have,

\[
\partial \beta_I(x) = \sum_{r=0}^{p} (-1)^r \text{Sk}^{\delta_r I}(x).
\]

In general, the terms \( \text{Sk}^{\delta_r I}(x) \) are not standard basis elements, since \( x \) need not lie in \( V^{\delta_r I} \). Letting \( u_r \in W_{\delta_r I} \) be the unique element such that \( u_r x \in V^{\delta_r I} \), we have

\[
\partial \beta_I(x) = \sum_{r=0}^{n} (-1)^{r + \text{length}(u_r)} \beta_{\delta_r I}(u_r x).
\]

(48)

5.5.1. Computation of \( H_0(J) \). Consider \( C_0(J) = \bigoplus_{i=0}^{p} \mathbb{Z}[V]^{W_i - \text{as}} \). For all \( i, j \) and all \( x \), the elements \( \text{Sk}^i(x), \text{Sk}^j(x) \) are homologous since they differ by the boundary of \( \text{Sk}^j(x) \in C_1(J) \). Together with \( \text{Sk}^i(x) = (-1)^{\text{length}(w)} \text{Sk}^j(wx) \) for \( w \in W_j \), this implies

\[
\text{Sk}^i(x) \sim (-1)^{\text{length}(w)} \text{Sk}^j(wx)
\]

for \( w \in W_j \). Since the subgroups \( W_j \) generate \( W_{\text{aff}} \), this holds in fact for all \( w \in W_{\text{aff}} \). Thus

\[
\text{Sk}^i(w, u_j^\pm) \sim \text{Sk}^i(w, v_j^\pm) \sim (-1)^{\text{length}(w)} \text{Sk}^i(u_j^\pm)
\]

for all \( i, j \), and all \( w \in W_{\text{aff}} \). If \( J \neq \{0, \ldots, l\} \), the choice of any \( i \not\in J \) gives \( \text{Sk}^i(v_j^\pm) = 0 \). This proves \( H_0(J) = 0 \). Suppose now \( J = \{0, \ldots, l\} \). The augmentation map \( C_0(J) \to \mathbb{Z} \) is described in terms of the basis by \( \beta_i(x) \mapsto (-1)^{\text{length}(x)} \). It has a right inverse \( \mathbb{Z} \to C_0(J), \ 1 \mapsto \beta_0(v_0^\pm) \). Hence the induced map in homology \( \mathbb{Z} \to H_0(J) \) is injective, but also surjective since \( \text{Sk}^i(x) \sim (-1)^{\text{length}(x)} \beta_0(v_0^\pm) \). Thus \( H_0(J) = \mathbb{Z} \) in this case.

5.5.2. Computation of \( H_l(J) \). Suppose \( \phi \in C_l(J) = \mathbb{Z}[V] \). Then \( \partial \phi = 0 \) if and only if \( \text{Sk}^{0 \ldots i \ldots l} \phi = 0 \) for all \( i \). That is, \( \phi \) is invariant under every reflection \( \sigma_i \in W_{\text{aff}} \), hence under the full affine Weyl group \( W_{\text{aff}} \). But since \( \phi \) has finite length this is impossible unless \( \phi = 0 \). This shows \( H_l(J) = 0 \).

5.5.3. Computation of \( H_p(J) \), \( 0 < p < l \). To simplify notation, we will write \( C_\bullet \) instead of \( C_{\bullet}(J) \). (This should of course not be confused with the chain complex \( C_\bullet \) considered in previous sections.) Introduce a \( \mathbb{Z} \)-filtration

\[
0 = F_{-1} C_\bullet \subset F_0 C_\bullet \subset F_1 C_\bullet \subset \cdots
\]

where \( F_N C_p \) is spanned by basis elements \( \{17\} \) with \(|I| = p + 1\) and \( \text{length}(x) \leq N \). Formula \( \{18\} \) shows that for any basis element \( \beta_I(x) \in F_N C_p \),

\[
\partial \beta_I(x) = \sum_r (-1)^r \beta_{\delta_r I}(x) \mod F_{N-1} C_{p-1}
\]

(49)

where the sum is only over those \( r \) for which \( x \in V^{\delta_r I} \subset V^I \), i.e. \( u_r = 1 \) (other terms lower the filtration degree since \( \text{length}(u_r x) < \text{length}(x) \) unless \( x = u_r x \)). In particular, \( \partial \) preserves the filtration. Define operators \( h_i : C_p \to C_{p+1} \) on basis elements, as follows:

\[
h_i \beta_I(x) = \begin{cases} 
(-1)^r \beta_{I \cup \{i\}}(x) & \text{if } i_{r-1} < i < i_r, \\
0 & \text{if } i = i_r, \text{ some } r.
\end{cases}
\]
Note that $h_i$ preserves the filtration: $h_i(F_N C_p) \subset F_N C_{p+1}$. Let
$$A_i = \text{id} - h_i \partial - \partial h_i.$$  
Then $A_i$ is a chain map, which is homotopic to the identity map.

**Lemma 5.7.** Let $p > 0$. For any basis element $\beta_I(x) \in F_N C_p$ we have $A_i \beta_I(x) \in F_{N-1} C_p$ unless $i \in I$ and $x \notin V_I^{-\{i\}}$. In the latter case,
$$A_i \beta_I(x) = \beta_I(x) \mod F_{N-1} C_p.$$  

**Proof.** Write $I = \{i_0, \ldots, i_p\}$ where $i_0 < \cdots < i_p$. Using (49) we obtain
$$h_i \partial h_I (x) = \sum_r' (-1)^r h_i \beta_{I-r}(x) \mod F_{N-1} C_p,$$
summing over indices with $x \in V_\delta^I \subset V^I$. The calculation of $A_i \beta_I (x)$ divides into two cases:

**Case 1:** $i \in I$. Thus $i = i_s$ for some index $s$, and $(-1)^r h_i \beta_{I-r}(x) = 0$ unless $r = s$, in which case one obtains $\beta_I(x)$. Hence all terms in the sum (50) vanish, except possibly for the term $r = s$ which appears if and only if $x \in V_\delta^I = V^I^{-\{i\}}$. That is,
$$h_i \partial h_I (x) = \begin{cases} \beta_I(x) \mod F_{N-1} C_p & \text{if } x \in V^I^{-\{i\}} \\ 0 & \text{if } x \notin V^I^{-\{i\}} \end{cases}$$

(using the assumption $p > 0$). Since $h_i \beta_I(x) = 0$ this shows $A_i \beta_I(x) \in F_{N-1} C_p$ unless $x \notin V^I^{-\{i\}}$, in which case $A_i \beta_I(x) = \beta_I(x) \mod F_{N-1} C_p$.

**Case 2:** $i \notin I$. Exactly one of the terms in $\partial h_i \beta_I(x)$ reproduces $\beta_I(x)$. The remaining terms are organized in a sum similar to (48):
$$\partial h_i \beta_I(x) = \beta_I(x) - \sum r'' (-1)^r h_i \beta_{I-r}(x) \mod F_{N-1} C_p,$$
where the sum is over all $r$ such that $x \in V^{I \cup \{i\} - \{i_r\}}$. But $x \in V_\delta^I \iff x \in V^{I \cup \{i\} - \{i_r\}}$, since $V_\delta^I = V^{I \cup \{i\} - \{i_r\}} \cap V^I$.

Hence the sum $\sum r'$ and $\sum r''$ are just the same. This proves $A_i \beta_I(x) \in F_{N-1} C_p$. □

Consider now the product $A := A_0 \cdots A_l$. By iterated application of the Lemma, we find that if $0 < p < l$, then $A \beta_I(x) \in F_{N-1} C_p$ (because at least one index $i$ is not in $I$). Thus
$$A: F_N C_p \to F_{N-1} C_p$$
for $0 < p < l$. The chain map $A$ is chain homotopic to the identity, since each of its factors are. Thus, if $\phi \in F_N C_p$ is a cycle,
$$\phi \sim A \phi \sim \cdots A^{N+1} \phi = 0.$$  
This proves $H_p(J) = 0$ for $0 < p < l$, and concludes the proof of Theorem 5.6.

**Remark 5.8.** N. Kitchloo pointed out a more elegant proof of Theorem 5.6 along the lines of Kitchloo-Morava [25]. His argument produces an inclusion of $C_\bullet(J)$ as a direct summand of $S_\bullet \otimes_{\mathbb{Z}[W_J]} \mathbb{Z}$, where $S_\bullet$ is the simplicial complex with respect to the Stiefel diagram, and $\mathbb{Z}[W_J]$ acts on $\mathbb{Z}$ by the sign representation. The acyclicity of $C_\bullet(J)$ then follows from the $W_J$-equivariant acyclicity of $S_\bullet$. 

APPENDIX A. MORITA ISOMORPHISMS AND STABLE ISOMORPHISMS

Let $\mathbb{H}_G$ be the stable $G$-Hilbert space, i.e. the unique (up to isomorphism) $G$-Hilbert space containing all finite-dimensional unitary $G$-representations with infinite multiplicity. (As a model, one may take for instance $\mathbb{H}_G = L^2(G) \otimes L^2(\mathbb{R})$.) Given a closed subgroup $H \subset G$, the space $\mathbb{H}_G$ with the restricted action is a model for $\mathbb{H}_H$. If $\mathcal{H}$ is any $G$-Hilbert space, its stabilization $\mathcal{H}^{st} = \mathcal{H} \otimes \mathbb{H}_G$ is isomorphic to $\mathbb{H}_G$. This generalizes to bundles:

**Lemma A.1.** For any $G$-Hilbert bundle $\mathcal{E} \to X$, the stabilization $\mathcal{E}^{st} = \mathcal{E} \otimes \mathbb{H}_G$ is equivariantly isomorphic to the trivial bundle $X \times \mathbb{H}_G$, with diagonal $G$-action. Moreover, the isomorphism is unique up to a $G$-equivariant homotopy.

*Proof.* Using induction over the cells of a $G$-CW decomposition of $X$, it is enough to show that for any $G$-Dixmier-Douady bundle $\mathcal{E} \to G/H \times D^r$, a given $G$-equivariant trivialization over $G/H \times \partial D^r$ extends to all of $G/H \times D^r$, and that the extension is unique up to homotopy. Since $G \times H \mathbb{H}_G \cong G/H \times \mathbb{H}_G$, it suffices to extend the trivialization over $eH \times \partial D^r$ to an $H$-equivariant isomorphism between $\mathcal{E}$ and $\mathbb{H}_G$ over $eH \times D^r$. By the contractibility of the unitary group in the strong operator topology, and since the $H$-action on the base is trivial, such an extension exists and is unique up to homotopy. □

**Lemma A.2.** For any $G$-Dixmier-Douady bundle $\mathcal{A}$, there is a canonical equivariant Morita isomorphism $(\mathcal{E}, \psi) : \mathcal{A} \simeq \mathcal{A}^{st}$.

*Proof.* Letting $\mathcal{A}_{HS}$ be the Hilbert bundle implementing the Morita isomorphism $\mathcal{A} \simeq \mathcal{A}$. Then $\mathcal{A}_{HS}^{st} = \mathcal{A}_{HS} \otimes \mathbb{H}_G$ defines a Morita isomorphism $\mathcal{A} \simeq \mathcal{A}^{st} = \mathcal{A} \otimes \mathbb{K}_G$. □

The following Lemma is a very simple special case of the Brown-Green-Rieffel Theorem.

**Lemma A.3.** For any two $G$-Dixmier-Douady bundles $\mathcal{A}, \mathcal{B}$, there is a 1-1 correspondence between equivalence classes of equivariant stable isomorphisms $\mathcal{A}^{st} \to \mathcal{B}^{st}$ and equivalence classes of equivariant Morita isomorphisms $\mathcal{A} \simeq \mathcal{B}$.

*Proof.* Suppose $(\mathcal{E}, \psi) : \mathcal{A} \simeq \mathcal{B}$ is a $G$-Morita isomorphism. That is, $\psi : \mathbb{K}(\mathcal{E}) \to \mathcal{B} \otimes \mathcal{A}^{opp}$ is a $G$-equivariant isomorphism. Tensoring with $\mathcal{A}$, and using the isomorphism $\mathcal{A} \otimes \mathcal{A}^{opp} \cong \mathbb{K}(\mathcal{A}_{HS})$, we obtain an isomorphism $\mathcal{A} \otimes \mathbb{K}(\mathcal{E}) \to \mathcal{B} \otimes \mathbb{K}(\mathcal{A}_{HS})$. Stabilize by tensoring with $\mathbb{K}_G$. The choice of isomorphisms $\mathcal{E} \otimes \mathbb{H}_G = \mathbb{H}_G$ and $\mathcal{A}_{HS} \otimes \mathbb{H}_G = \mathbb{H}_G$ produces an isomorphism $\mathcal{A} \otimes \mathbb{K}_G \cong \mathcal{B} \otimes \mathbb{K}_G$, hence a $G$-equivariant isomorphism $\mathcal{A}^{st} \cong \mathcal{B}^{st}$.

Conversely, given a stable equivariant isomorphism, we obtain a Morita isomorphism by composition,

$\mathcal{A} \simeq \mathcal{A}^{st} \simeq \mathcal{B}^{st} \simeq \mathcal{B}$.

It is easily checked that this construction gives a bijection on equivalence classes. □

APPENDIX B. RELATIVE DIXMIER-DOUady BUNDLES

For any map $f : Y \to X$, and $\text{cone}(f)$ its mapping cone, obtained by gluing $\text{cone}(Y) = Y \times I / Y \times \{0\}$ with $X$ by the identification $(y, 1) \sim f(y)$. Let $H^\bullet(f) = H^\bullet(\text{cone}(f))$ denote the relative cohomology of $f$. Equivalently $H^\bullet(f)$ is the cohomology of the algebraic mapping cone $C^\bullet(f)$ of the cochain map $C^\bullet(Y) \to C^\bullet(X)$, i.e. $C^p(f) = C^{p-1}(Y) \oplus C^p(X)$ with differential
The group $H^3(f)$ has a geometric interpretation as isomorphism classes of relative line bundles, i.e. pairs $(L, \psi_Y)$, where $L$ is a Hermitian line bundle over $X$, and $\psi_Y : Y \times \mathbb{C} \to f^* L$ is a unitary trivialization of its pull-back to $Y$. (The class of a relative line bundle is the Chern class of the line bundle $L \to \text{cone}(f)$, obtained by gluing $\text{cone}(Y) \times \mathbb{C}$ with $L$ via $\psi_Y$.)

Similarly, $H^3(f)$ is interpreted in terms of relative Dixmier-Douady bundles, i.e. triples $(\mathcal{A}, \mathcal{E}_Y, \psi_Y)$, where $\mathcal{A} \to X$ is a Dixmier-Douady bundle, $\mathcal{E}_Y \to Y$ is a Hilbert space bundle, and $\psi_Y : \mathcal{E}_Y \to \psi_Y^* \mathcal{A}$ a Morita trivialization of its pull-back to $Y$. Given such a triple, one may construct a Dixmier-Douady bundle $\tilde{\mathcal{A}} \to \text{cone}(f)$: First, use the Morita trivialization $\psi_Y$ to define a stable trivialization $Y \times \mathbb{K} \cong \mathbb{K} (\mathcal{E}_Y^{st}) \cong f^* \mathcal{A}^{st}$, then define $\tilde{\mathcal{A}}$ by gluing the trivial bundle $\text{cone}(Y) \times \mathbb{K}$ with $\mathcal{A}^{st}$ using this identification. We define the relative Dixmier-Douady class $DD(\mathcal{A}, \mathcal{E}_Y, \psi_Y) := DD(\tilde{\mathcal{A}})$.

Tensor products and opposites of relative Dixmier-Douady bundles are defined in the obvious way. A Morita trivialization of $(\mathcal{A}, \mathcal{E}_Y, \psi_Y)$ is a triple $(\mathcal{E}_X, \psi_X, h_Y)$, consisting of a Morita trivialization $\psi_X : \mathcal{E}_X \to \mathcal{A}$ and an isomorphism $h_Y : \mathcal{E}_Y \to f^* \mathcal{E}_X$ intertwining $\psi_Y$ and $f^* \psi_X$. A Morita isomorphism between two triples is a Morita trivialization of their quotient. From the usual Dixmier-Douady theorem, one deduces that $DD(\mathcal{A}, \mathcal{E}_Y, \psi_Y)$ classifies the Morita isomorphism classes.

More generally, one may define relative equivariant Dixmier-Douady bundles; these are classified by an equivariant class $DD_G(\mathcal{A}, \mathcal{E}_Y, \psi_Y) \in H^3_G(f) := H^3(f_G)$. Here $f_G : Y_G \to X_G$ is the induced map of Borel constructions.

**Appendix C. Review of Kasparov K-homology**

In this Section we review Kasparov’s definition of K-homology [23, 24] for $C^*$-algebras. Excellent references for this material are the books by Higson-Roe [19] and Blackadar [5]. Suppose $\mathcal{A}$ is a $\mathbb{Z}_2$-graded $C^*$-algebra, equipped with an action of a compact Lie group $G$ by automorphisms. An equivariant Fredholm module over $\mathcal{A}$ is a triple $x = (\mathcal{H}, \varrho, F)$, where $\mathcal{H}$ is a $G$-equivariant $\mathbb{Z}_2$-graded Hilbert space, $\varrho : \mathcal{A} \to L(\mathcal{H})$ is a morphism of $\mathbb{Z}_2$-graded $G$-$C^*$-algebras, and $F \in L(\mathcal{H})$ is a $G$-invariant odd operator such that for all $a \in \mathcal{A}$,

$$(F^2 - I)\varrho(a) \sim 0, \quad (F^* - F)\varrho(a) \sim 0, \quad [F, \varrho(a)] \sim 0.$$ 

Here $\sim$ denotes equality modulo compact operators. There is an obvious notion of direct sum of Fredholm modules over $\mathcal{A}$. One defines a semi-group $K^G_0(\mathcal{A})$, with generators $[x]$ for each Fredholm module over $\mathcal{A}$, and equivalence relations

$$[x] + [x'] = [x \oplus x'],$$

and

$$[x_0] = [x_1]$$

provided $x_0, x_1$ are related by an ‘operator homotopy’ $x_t = (\mathcal{H}, \varrho, F_t)$ (cf. [5, 19]). One then proves that every element in this semi-group has an additive inverse, so that $K^G_0(\mathcal{A})$ is actually a group. More generally, for $q \leq 0$ one defines $K^G_q( \mathcal{A} ) = K^G_0(\mathcal{A} \otimes \text{Cl}(\mathbb{R}^q))$. This has the mod 2 periodicity property $K^G_{q+2}(\mathcal{A}) = K^G_q(\mathcal{A})$, which is then used to extend the definition to all
$q \in \mathbb{Z}$. The assignment $A \rightarrow K^q_G(A)$ is homotopy invariant, contravariant functor, depending only on the Morita equivalence class of $A$. It has the stability property, $K^q_G(A \otimes \mathbb{K}_G) = K^q_G(A)$.

With this definition, let us now review some basic examples of twisted $K$-homology groups $K^G_q(X, A) = K^q_G(\Gamma_0(X, A))$ for Dixmier-Douady bundles $A \rightarrow X$.

**Example C.1.** Let $A \rightarrow pt$ be a $G$-equivariant Dixmier-Douady bundle over a point. Thus $A \cong \mathbb{K}(\mathcal{E})$ for some Hilbert space $\mathcal{E}$. As in Section [22] the action $G \rightarrow \operatorname{Aut}(A)$ defines a central extension $\hat{G}$ of $G$ by $U(1)$. The group $\hat{G}$ acts on $\mathcal{E}$, in such a way that the central circle acts with weight $1$. Let $V$ be a $\hat{G}$-module where the central circle acts with weight $-1$. Then the Hilbert space $\mathcal{H} = V \otimes \mathcal{E}$ is a $G$-module. Letting $\rho: \mathbb{C} \rightarrow L(\mathcal{H})$ be the action by scalar multiplication, the triple $(\mathcal{H}, \varrho, 0)$ is a $G$-equivariant Fredholm module over $C(pt) = \mathbb{C}$. This construction realizes the isomorphism $R(\hat{G})_{-1} \rightarrow K^G_0(pt, A)$.

**Example C.2.** Let $M$ be a compact Riemannian $G$-manifold, and $D$ an invariant first order elliptic operator acting on a $G$-equivariant $\mathbb{Z}_2$-graded Hermitian vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. Suppose also that a finite rank $\mathbb{Z}_2$-graded $G$-Dixmier-Douady bundle $A \rightarrow M$ acts on $\mathcal{E}$, where the action is equivariant and compatible with the grading. Let $\mathcal{H}$ be the space of $L^2$-sections of $\mathcal{E}$, with the natural representation $\varrho$ of $\Gamma(M, A)$, and $F = D(1 + D^2)^{-1/2} \in L(\mathcal{H})$. The commutator of $F$ with elements $\varrho(a)$ for $a \in \Gamma(M, A)$ are pseudo-differential operators of degree $-1$, hence are compact. Thus $(\mathcal{H}, \varrho, F)$ is an equivariant Fredholm module over $\Gamma(M, A)$, defining a class in $K^G_0(M, A)$.

**Example C.3.** [24 page 114] Let $M$ be a compact Riemannian $G$-manifold, and $A = \operatorname{Cl}(TM)$ its Clifford bundle. Take $\mathcal{E} = \wedge^*TM$, $\mathcal{H}$ its space of $L^2$-sections, and $\varrho$ the usual action of sections of $\Gamma(M, \operatorname{Cl}(TM))$. Let $D = d + d^*$ be the de-Rham Dirac operator. By [22] above, we obtain a Fredholm module $(\mathcal{H}, \varrho, F)$ over $\Gamma(M, \operatorname{Cl}(TM))$, defining a class $[M] \in K^G_0(M, \operatorname{Cl}(TM))$. This is the Kasparov fundamental class of $M$. (Actually, $\operatorname{Cl}(TM)$ is a Dixmier-Douady bundle only if dim $M$ is even. If dim $M$ is odd, one can use the isomorphism $K^G_0(M, \operatorname{Cl}(TM)) = K^G_1(M, \operatorname{Cl}^+(TM))$ if needed.)

**Example C.4.** Let $H$ be a closed subgroup of $G$, and that $B \rightarrow pt$ be an $H$-Dixmier-Douady bundle of finite rank. As explained in [21] any class in $K^H_0(pt, \operatorname{Cl}(g/h) \otimes B)$ is realized by a Fredholm module of the form $(\mathcal{E}, \varrho, 0)$ where $\mathcal{E}$ is a Hilbert space of finite dimension. Let $\hat{\mathcal{E}} = G \times_H \mathcal{E}$. The action of $\operatorname{Cl}(T(G/H))$ defines a Dirac operator, which together with the action of $\operatorname{Ind}^G_H(B)$ yields a Fredholm module and hence an element of $K^G_0(G/H, \operatorname{Ind}^G_H(B))$. This construction realizes the isomorphism $K^H_0(pt, B \otimes \operatorname{Cl}(g/h)) \rightarrow K^G_0(G/H, \operatorname{Ind}^G_H(B))$ if $B$ has finite rank. Note that since $H^3_B(pt)$ is torsion, all $H$-Dixmier-Douady bundles over pt are Morita isomorphic to finite rank ones.

**References**

[1] A. Alekseev, A. Malkin, and E. Meinrenken, *Lie group valued moment maps*, J. Differential Geom. 48 (1998), no. 3, 445–495.

[2] A. Alekseev and E. Meinrenken, *On the quantization of group-valued moment maps*, in preparation.

[3] A. Alekseev, E. Meinrenken, and C. Woodward, *The Verlinde formulas as fixed point formulas*, J. Symplectic Geom. 1 (2001), no. 1, 1–46.

[4] M. Atiyah and G. Segal, *Twisted $K$-theory*, Ukr. Mat. Visn. 1 (2004), no. 3, 287–330.

[5] B. Blackadar, *$K$-theory for operator algebras*, second ed., Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998.
[6] P. Bouwknegt, A. Carey, V. Mathai, M. Murray, and D. Stevenson, Twisted K-theory and K-theory of bundle gerbes, Comm. Math. Phys. 228 (2002), no. 1, 17–45.

[7] V. Braun, Twisted K-theory of Lie groups, J. High Energy Phys. (2004), no. 3, 029, 15 pp. (electronic).

[8] T. Bröcker and T. tom Dieck, Representations of compact Lie groups, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, Berlin-Heidelberg-New York, 1985.

[9] J. Brodzki, V. Mathai, J. Rosenberg, and R. Szabo, D-Branes, RR-Fields and Duality on Noncommutative Manifolds, ESI-1813, HWM-06-30, EMPG-06-06.

[10] J. Dixmier and A. Douady, Champs continus d’espaces hilbertiens et de C∗-algèbres, Bull. Soc. Math. France 91 (1963), 227–284.

[11] P. Donovan and M. Karoubi, Graded Brauer groups and K-theory with local coefficients, Inst. Hautes Études Sci. Publ. Math. (1970), no. 38, 5–25.

[12] J. Dixmier and A. Douady, Champs continus d’espaces hilbertiens et de C∗-algèbres, Bull. Soc. Math. France 91 (1963), 227–284.

[13] J. Dixmier and A. Douady, Champs continus d’espaces hilbertiens et de C∗-algèbres, Bull. Soc. Math. France 91 (1963), 227–284.

[14] J. Dixmier and A. Douady, Champs continus d’espaces hilbertiens et de C∗-algèbres, Bull. Soc. Math. France 91 (1963), 227–284.

[15] J. Dixmier and A. Douady, Champs continus d’espaces hilbertiens et de C∗-algèbres, Bull. Soc. Math. France 91 (1963), 227–284.

[16] J. Dixmier and A. Douady, Champs continus d’espaces hilbertiens et de C∗-algèbres, Bull. Soc. Math. France 91 (1963), 227–284.

[17] J. Dixmier and A. Douady, Champs continus d’espaces hilbertiens et de C∗-algèbres, Bull. Soc. Math. France 91 (1963), 227–284.

[18] K. Guruprasad, J. Huebschmann, L. Jeffrey, and A. Weinstein, Group systems, groupoids, and moduli spaces of parabolic bundles, Duke Math. J. 89 (1997), no. 2, 377–412.

[19] N. Higson and J. Roe, Analytic K-homology, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000, Oxford Science Publications.

[20] V. Kac, Infinite-dimensional Lie algebras, second ed., Cambridge University Press, Cambridge, 1985.

[21] J. Kalkman, Cohomology rings of symplectic quotients, J. Reine Angew. Math. 485 (1995), 37–52.

[22] G. G. Kasparov, Topological invariants of elliptic operators. I. K-homology, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 4, 796–838.

[23] G. G. Kasparov, Topological invariants of elliptic operators. I. K-homology, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 4, 796–838.

[24] G. G. Kasparov, Topological invariants of elliptic operators. I. K-homology, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 4, 796–838.

[25] N. Kitchloo and J. Morava, Thom prospectra for loopgroup representations, Elliptic Cohomology (D. Ravenel H. Miller, ed.), London Mathematical Society Lecture Note Series, vol. 342, Cambridge University Press, Cambridge, 1995, pp. 101–146.

[26] V. Mathai, R. Melrose, and I. Singer, Fractional analytic index, arXiv:math.DG/0402329.

[27] V. Mathai, R. Melrose, and I.M. Singer, The index of projective families of elliptic operators. I. K-homology, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 4, 796–838.

[28] E. Meinrenken, The basic gerbe over a compact simple Lie group, Enseign. Math. (2) 49 (2003), no. 3-4, 307–333.

[29] E. Meinrenken, The basic gerbe over a compact simple Lie group, Enseign. Math. (2) 49 (2003), no. 3-4, 307–333.

[30] E. Meinrenken, The basic gerbe over a compact simple Lie group, Enseign. Math. (2) 49 (2003), no. 3-4, 307–333.

[31] E. Meinrenken, The basic gerbe over a compact simple Lie group, Enseign. Math. (2) 49 (2003), no. 3-4, 307–333.

[32] E. Meinrenken, The basic gerbe over a compact simple Lie group, Enseign. Math. (2) 49 (2003), no. 3-4, 307–333.
[36] E. H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1981, Corrected reprint.

[37] R. Steinberg, *On a theorem of Pittie*, Topology 14 (1975), 173–177.

[38] S. Sternberg, *Lie algebras*, Lecture notes, available at http://www.math.harvard.edu/~shlomo/.

[39] J.-L. Tu, *Twisted K-theory and Poincare duality*, [arXiv:math.KT/0609556](http://arxiv.org/abs/math.KT/0609556).

[40] J.-L. Tu and P. Xu, *The ring structure for equivariant twisted K-theory*, [arXiv:math.KT/0604160](http://arxiv.org/abs/math.KT/0604160).

[41] J.-L. Tu, P. Xu, and C. Laurent-Gengoux, *Twisted K-theory of differentiable stacks*, Ann. Sci. École Norm. Sup. (4) 37 (2004), no. 6, 841–910.