EXISTENCE OF KIRILLOV–RESHETIKHIN CRYSTALS FOR MULTIPLICITY FREE NODES

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ABSTRACT. We show that the Kirillov–Reshetikhin crystal $B^{r,s}$ exists when $r$ is a node such that the Kirillov–Reshetikhin module $W^{r,s}$ has a multiplicity free classical decomposition.

1. INTRODUCTION

Kirillov–Reshetikhin (KR) modules are an class of finite-dimensional representation of an affine quantum group $U_q^+(g)$ without the degree operator that is classified by their Drinfel’d polynomials that have received significant attention. We denote a KR module by $W_{r,s}$, where $r$ is a node of the classical (i.e. underlying finite type) Dynkin diagram and $s \in \mathbb{Z}_{>0}$. One construction of a KR module $W_{r,s}$ is by computing the minimal affinization of the highest weight $U_q(\mathfrak{g}_0)$-module $V(s\Lambda_r)$ [Cha95, CP95a, CP96a, CP96b], where $\mathfrak{g}_0$ is the classical Lie algebra. Another method is by using the fusion construction of [KKM+92] from the image under an $R$-matrix of an $s$-fold tensor product of the fundamental module $W_{r,1}$ (see, e.g., [Kas02]). KR modules are also known to have special properties. The classical decomposition, the branchingle rule of $W_{r,s}$ to a $U_q(\mathfrak{g}_0)$-module, is given by a fermionic formula [DFK08, Her10], which leads to the (virtual) Kleber algorithm [Kle98, OS03]. The characters (resp. $q$-characters) of KR modules also satisfy the $Q$-system (resp. $T$-system) relations [Her10, Nak03]. Furthermore, the graded characters of (Demazure submodules of) a tensor product of fundamental modules are (nonsymmetric) Macdonald polynomials at $t = 0$ [LNS+15, LNS+16b] (LNS+17).

One important (conjectural) property [HKO+99, HKO+02] is that the KR module $W_{r,s}$ admits a crystal base [Kas90, Kas91], which is known as a Kirillov–Reshetikhin (KR) crystal and denoted by $B_{r,s}$. Kashiwara showed that all fundamental modules $W_{r,1}$ have crystal bases [Kas02]. It was shown that $B_{r,s}$ exists in all nonexceptional types in [Oka07, OS08] and in types $G_2^{(1)}$ and $D_4^{(3)}$ in [KMOY07, Nao18, Yam98]. For all affine types, the existence of $B_{r,s}$ has been proven when $r$ is adjacent to 0 or in the orbit of 0 under a Dynkin automorphism (equivalently, $W_{r,s}$ is irreducible as $U_q(\mathfrak{g})$-module) [KKM+92].

Our main result is that the KR module $W_{r,s}$ has a crystal base whenever its classical decomposition is multiplicity free in all affine types. We do this by showing the existence of $B_{r,s}$ in the cases not covered by [KKM+92, Oka07, OS08]. More explicitly, we show this for $r = 3,5$ in type $E_6^{(1)}$, for $r = 2,6$ in type $E_7^{(1)}$, for $r = 7$ in type $E_8^{(1)}$, and for $r = 4$ in types $F_4^{(1)}$ and $E_6^{(2)}$, where we label the Dynkin diagrams following [Bou02] (see also Figure 1 for the labeling). Using the techniques developed in [KKM+92], our proof shows the existence of a crystal pseudobase $(L,B)$ by using the fusion construction of $W_{r,s}$ and is similar to [Oka07, OS08] by calculating the prepolarization for certain vectors. From there, we can construct the associated crystal by $B/\{\pm 1\}$.

Let us describe some possible applications of our results. The $X = M$ conjecture [HKO+99, HKO+02] arises from mathematical physics relating vertex models and the Bethe ansatz of Heisenberg spin chains, and the $X$ side requires the existence of KR crystals. A uniform model for $B^{r,1}$ was given using quantum and projected level-zero LS paths [LNS+15, LNS+16a, LNS+16b, NS06, NS08a, NS08b]. Since the KR crystal $B^{r,s}$ exists, we have a partial (conjectural) combinatorial description from [LS18] using $(B^{r,1})^{\otimes s}$, partially mimicking the fusion construction.

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After completion of this paper, we learned that Naoi independently proved all cases in type $E_{6,7}^{(1)}$ using similar techniques [Nao19].

This paper is organized as follows. In Section 2, we give the necessary background. In Section 3, we show our main result: that the KR modules $W^{r,s}$ has a crystal pseudobase whenever $W^{r,s}$ has a multiplicity free classical decomposition.

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2. Background

In this section, we provide the necessary background. Let $\mathfrak{g}$ be an affine Kac–Moody Lie algebra with index set $I$, Cartan matrix $A = (A_{ij})_{i,j \in I}$, simple roots $(\alpha_i)_{i \in I}$, simple coroots $(h_i)_{i \in I}$, fundamental weights $(\Lambda_i)_{i \in I}$, weight lattice $P$, dominant weights $P^+$, coweight lattice $P^\vee$, and canonical pairing $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Z}$ given by $\langle h_i, \alpha_j \rangle = A_{ij}$. We note that we follow the labeling given in [Bou02] (see Figure 1 for the exceptional types and their labellings). Let $\mathfrak{g}_0$ denote the canonical simple Lie algebra given by the index set $I_0 = I \setminus \{0\}$. Let $\nabla$ denote the natural projection of $\lambda \in P$ onto the weight lattice $P_0$ of $\mathfrak{g}_0$, so $\{\nabla \cdot \}_{r \in I_0}$ are the fundamental weights of $\mathfrak{g}_0$. Let $\omega_r = \Lambda_r - \langle c, \Lambda_r \rangle \Lambda_0$, where $c$ is the canonical central element of $\mathfrak{g}$, denote the level-zero fundamental weights. Let $q$ be an indeterminate, and we denote

$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [k]_q! = [k]_q[k-1]_q \cdots [1]_q, \quad \begin{bmatrix} m \end{bmatrix}_q = \frac{[m]_q[m-1]_q \cdots [m-k+1]_q}{[k]_q!}$

for $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$. Let $q_i = q^{s_i}$ and $K_i = q^{s_i h_i}$, where $(s_1, \ldots, s_n)$ is the diagonal symmetrizing matrix of $A$.

2.1. Quantum groups. Let $U'_q(\mathfrak{g}) = U_q(\mathfrak{g}, g)$ denote the quantum group of the derived subalgebra of $\mathfrak{g}$. More specifically, the quantum group $U'_q(\mathfrak{g})$ is the associative $\mathbb{Q}(q)$-algebra generated by $e_i, f_i, q^s$, where

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dynkin_diagrams.png}
\caption{Dynkin diagrams for affine type $E_{6,7,8}^{(1)}$, $F_4^{(1)}$, and $E_6^{(2)}$.}
\end{figure}
Let $U$ is a combination of $\lambda_i$ with $e$, we will abuse notation and denote the $U$-dependence relation on the simple roots in $U$.

For a $U$-module, we denote the highest weight $\lambda$, then we say $\text{wt}(v) = \lambda$. For $\lambda \in P_0^+$, we denote the highest weight $U_{\lambda}$-module by $V(\lambda)$.

### 2.2. Crystal (pseudo)bases and polarizations

Let $A$ denote the subring of $\mathbb{Q}(q)$ of rational functions without poles at 0. A crystal base of an integrable $U_q(g)$-module $M$ is a pair $(L, B)$, where $L$ is a free $A$-module and $B$ is a basis of the $\mathbb{Q}$-vector space $L/qL$, such that

(1) $M \cong \mathbb{Q}(q) \otimes_A L$,
(2) $L \cong \bigoplus_{\lambda \in P} L_\lambda$ with $L_\lambda = L \cap M_\lambda$,
(3) $e_iL \subseteq L$ and $f_iL \subseteq L$ for all $i \in I$,
(4) $B = \bigcup_{\lambda \in P} B_\lambda$ with $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,
(5) $e_iB \subseteq B \cup \{0\}$ and $f_iB \subseteq B \cup \{0\}$,
(6) $f_i b = b'$ if and only if $e_i b' = b$ for all $b, b' \in B$ and $i \in I$.

We say $(L, B)$ is a crystal pseudobase of $M$ if it satisfies the conditions above for $B = B' \cup (-B')$, where $B'$ is a basis of $L/qL$. For a $U_q(g)$-module $M$, a prepolarization is a symmetric bilinear form $(\ , \ ) : M \times M \rightarrow \mathbb{Q}(q)$ that satisfies

\begin{equation}
(q^h v, w) = (v, q^h w), \quad (e_i v, w) = (v, q^{-1} K_i^{-1} f_i w), \quad (f_i v, w) = (v, q^{-1} K_i e_i w),
\end{equation}

for all $i \in I$, and $v, w \in M$.\footnote{For $U_q(g)$-modules $M, N$, a pairing $(\ , \ ) : M \times N \rightarrow \mathbb{Q}(q)$ that satisfies (2.1) is often called admissible.} Denote $\|v\|^2 = (v, v)$. If a prepolarization is positive definite with respect to the total order on $\mathbb{Q}(q)$

$$f > g \text{ if and only if } f - g \in \bigcup_{n \in \mathbb{Z}} \{q^n (d + qA) \mid d \in \mathbb{Q}_{>0}\}$$

(with $f \geq g$ defined as $f = g$ or $f > g$) then it is called a polarization.
Proposition 2.1. Let $M$ be a finite-dimensional integrable $U_q'(\mathfrak{g})$-module. Suppose $M$ has a prepolarization $(\ ,\ )$ and a $U_q'(\mathfrak{g})_{K_2}$-submodule $M_{K_2}$ such that $(M_{K_2}, M_{K_2}) \subseteq K_2$. Assume $M \cong \bigoplus_{k=1}^m V(\lambda_k)$ as $U_q(\mathfrak{g}_0)$-modules, with $\lambda_k \in P_0^+$ for all $k$, such that there exists $u_k \in (M_{K_2})_{\lambda_k}$ such that $(u_k, u_k) \in \delta_{kl} + qA$ and $\|e_i u_k\|^2 \in q^{-2}(h_i, \lambda_k)^{-2} qA$. Then $(\ ,\ )$ is a polarization and for

$$L = \{ v \in M \mid \|v\|^2 \in A \} , \quad B = \{ b \in (M_{K_2} \cap L)/(M_{K_2} \cap qL) \mid (b, b)_0 = 1 \},$$

where $(\ ,\ )_0: L/qL \rightarrow Q$ is the bilinear form induced by $(\ ,\ )$, the pair $(L, B)$ is a crystal pseudobase of $M$.

For an indeterminate $z$, let $M_z$ denote the $U_q'(\mathfrak{g})$-module $Q(q)[z, z^{-1}] \otimes M$, where $e_i$ and $f_i$ act by $z^{\delta_{0i}} \otimes e_i$ and $z^{-\delta_{0i}} \otimes f_i$ called the **affinization module** of $M$. For $a \in Q(q)$, define the **evaluation module** $M_a = M_z/(z - a)M_z$. For $v \in M$, let $u_a$ denote the corresponding element in $M_a$ (i.e., the projection of $1 \otimes v$). Let $W(\varpi_r)$ denote the **fundamental module** from [Kas02].

Proposition 2.2 ([Kas02, Prop. 9.3]). Consider nonzero $a, b \in Q(q)$ such that $a/b \in A$. Then for any $r \in I_0$, there exists a unique nonzero $U_q'(\mathfrak{g})$-module homomorphism

$$R_{a,b}: W(\varpi_r)_a \otimes W(\varpi_r)_b \rightarrow W(\varpi_r)_a \otimes W(\varpi_r)_b,$$

that satisfies $R_{a,b}(u_a \otimes u_b) = u_0 \otimes u_a$ for some nonzero $u \in W(\varpi_r)_{\varpi_r}$. The map $R_{a,b}$ is called the (normalized) $R$-matrix and satisfies the Yang–Baxter equation.

Denote

$$W(\varpi_r; a_1, a_2, \ldots, a_m) = W(\varpi_r)_{a_1} \otimes W(\varpi_r)_{a_2} \otimes \cdots \otimes W(\varpi_r)_{a_m}.$$

Let $\kappa = s_i$ if $\mathfrak{g}$ is of untwisted affine type and $\kappa = 1$ if $\mathfrak{g}$ is of twisted affine type. Since the $R$-matrix satisfies the Yang–Baxter equation, we can define the map

$$R_\kappa: W(\varpi_r; q^{\kappa(s-1)}, q^{\kappa(s-3)}, \ldots, q^{\kappa(1-s)}) \rightarrow W(\varpi_r; q^{\kappa(s-1)}, q^{\kappa(3-s)}, q^{\kappa(1-s)})$$

by applying the $R$-matrix on every pair of factors according to the long element of the symmetric group on $s$ letters $(q^{\kappa(s-1)}, q^{\kappa(s-3)}, \ldots, q^{\kappa(1-s)})$. Let $W_{r,s}$ denote the image of $R_\kappa$, which is a simple $U_q'(\mathfrak{g})$-module [Kas02], and we call $W_{r,s}$ a **Kirillov–Reshetikhin (KR) module**. From [CP95b, CP98], the module $W_{r,s}$ satisfies the Drinfeld’s polynomial characterization of the usual definition of a KR module.

Lemma 2.3 ([KKM+92, Lemma 3.4.1]). Let $M_j$ and $N_j$, for $j = 1, 2$, be $U_q'(\mathfrak{g})$-modules such that there exists a pairing $(\ ,\ )_j: M_j \times N_j \rightarrow Q(q)$ satisfying (2.1). Then there exists a pairing $(\ ,\ ): (M_1 \otimes M_2) \times (N_1 \otimes N_2) \rightarrow Q(q)$ defined by

$$(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)(u_2, v_2),$$

for all $u_j \in M_j$ and $v_j \in N_j$ with $j = 1, 2$, that satisfies (2.1).

Proposition 2.4 ([KKM+92, Prop. 3.4.3]). Let $u \in W(\varpi_r)_{\varpi_r}$ be a vector such that $\|u\|^2 = 1$.

1. The pairing $(\ ,\ ) : W_{r,s} \times W_{r,s} \rightarrow Q(q)$ constructed using Lemma 2.3 and the prepolarization on $W_{r,1}$ (see [Kas02]) is a nondegenerate prepolarization on $W_{r,s}$.

2. $\|R_u(u^{\otimes s})\|^2 = 1$.

3. $(W_{r,s})_{K_2}, (W_{r,s})_{K_2} \subseteq K_2$, where

$$(W_{r,s})_{K_2} = R_u\left((U_q'(\mathfrak{g})_{K_2}u)^{\otimes s}\right) \cap \left((U_q'(\mathfrak{g})_{K_2}u)^{\otimes s}\right)$$

is a $U_q'(\mathfrak{g})_{K_2}$-submodule of $W_{r,s}$.

3. Existence of KR crystals

This section is devoted to proving our main result.

Theorem 3.1. Let $r$ be such that $W_{r,s}$ is multiplicity free as a $U_q(\mathfrak{g}_0)$-module for all $s \in \mathbb{Z}_{>0}$. Then $W_{r,s}$ admits a crystal pseudobase. Moreover, the KR crystal $B_{r,s}$ exists.
We prove Theorem 3.1 case-by-case. When \( r \) is adjacent to 0 or in the orbit of 0 under a Dynkin diagram automorphism, Theorem 3.1 was shown in [KKM+92]. Theorem 3.1 was shown in nonexceptional affine types [Oka07, OS08]. Thus, it remains to show Theorem 3.1 for the values given in Table 1.

From Proposition 2.4 and Proposition 2.1, it is sufficient to show for the \( U_q(\mathfrak{g}_0) \)-module decomposition \( W^{r,s} \cong \bigoplus_{k=1}^M V(\lambda_k) \) (where \( \lambda_k \in P_0^+ \)), there exists \( u_k \in (M_{K_k})_{\lambda_k} \) such that

(i) \((u_k, u_\ell) \in \delta_{k\ell} + qA\) and

(ii) \(\|e_i u_k\|^2 \in q^{-2(h_i, \lambda_k)-2}qA\).

Thus, we have

\[
\begin{array}{c|c|c|c|c|c}
g & E_6^{(1)} & E_7^{(1)} & E_8^{(1)} & F_4^{(1)} & E_6^{(2)} \\
\hline 3 & 5 & 2 & 6 & 7 & 4 & 4
\end{array}
\]

Table 1. The nodes \( r \) such that we show \( B^{r,s} \) exists.

The \( U_q(\mathfrak{g}_0) \)-module decomposition of \( W^{r,s} \) is given in [Cha01].

We require the following facts. Since the decomposition is multiplicity free, we have \((u_k, u_\ell) = 0 \) for all \( k \neq \ell \) since \( \text{wt}(u_k) \neq \text{wt}(u_\ell) \). Note that

\[
[m] \in q^{1-m}A, \quad \left[\begin{array}{c} m \\ k \end{array}\right]_q \in q^{-k(m-k)}A.
\]

Let \( M \) be a \( U'_q(\mathfrak{g}) \)-module. We will use this variant of Equation (2.1):

\[
(e_i^{(k)} v, w) = q_i^{k(h_i, \mu)} (v, f_i^{(k)} w), \quad (3.1a)
\]

\[
(f_i^{(k)} v, w) = q_i^{k(h_i, \mu)} (v, e_i^{(k)} w), \quad (3.1b)
\]

for all \( v \in M_{\mu} \). We also require

\[
f_i^{(a)} e_i^{(b)} v = \sum_{k=0}^{\min(a,b)} \left[ a-b - \langle h_i, \mu \rangle k \right]_q e_i^{(b-k)} f_i^{(a-k)} v, \quad (3.2)
\]

for any \( v \in M_{\mu} \), which follows from applying the defining relation on \([e_i, f_i]\). By applying Equation (3.1), Equation (3.2), and the bilinearity of \((\ , \ )\), we have for any \( v \in M_{\mu} \):

\[
\|e_i v\|^2 = q_i^{-1(h_i, \mu)} (v, f_i e_i v) \\
= q_i^{-1(h_i, \mu)} (v, e_i f_i v + [\langle h_i, \mu \rangle] q_i v) \\
= q_i^{-1(h_i, \mu)} ((v, e_i f_i v) + [\langle h_i, \mu \rangle] q_i (v, v)) \\
= q_i^{-1(h_i, \mu)} \left(q_i^{-(1+h_i, \mu)} \|f_i v\|^2 + [\langle h_i, \mu \rangle] q_i \|v\|^2\right)
\]

Thus, we have

\[
\|e_i v\|^2 = q_i^{-2(h_i, \mu)} \|f_i v\|^2 + q_i^{-1(h_i, \mu)} [\langle h_i, \mu \rangle] q_i \|v\|^2. \quad (3.3)
\]

For the remainder of the proof, we let \( u \in W^{r,s}_{\text{inv}} \) be such that \( \|u\|^2 = 1 \). We have

\[
\|f_i u\|^2 = q_i^{1+\delta_{ir,s}} (u, e_i f_i u) = q_i^{1+\delta_{ir,s}} (u, [\delta_{ir,s}] q_i u) = q_i^{1+\delta_{ir,s}} [\delta_{ir,s}] q_i, \quad (3.4)
\]

for all \( i \in I_0 \) by Equation (3.1a), the defining relation on \([e_i, f_i]\) (or Equation (3.2)), and \( e_i u = 0 \). So we have \( \|f_i u\|^2 \in q_i^2 A \) (note \( f_i u = 0 \) for all \( i \neq r \)).
3.1. **Type $E_6^{(1)}$, $r = 3$.** We claim the elements

$$ u_k := e_6^{(k)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u $$

are the desired elements, where $0 \leq k \leq s$. We have

$$ \text{wt}(u_k) = \lambda_k := (s - k) \Lambda_3 + k \Lambda_6 - (2s - k) \Lambda_6, $$

and from [Cha01], the classical decomposition is $W^{3,s} \cong \bigoplus_{k=0}^{s} V ((s - k) \Lambda_3 + k \Lambda_6)$. Thus, we need to show $u_k$ satisfies (i) and (ii).

We first show (i). We have

$$ \|u_k\|^2 = q_0^{k(k-k)} (e_6^{(k)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u, f_6^{(k)} u_k) $$

from Equation (3.1a). Next, we have

$$ f_6^{(k)} u_k = f_6^{(k)} (e_6^{(k)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u) $$

$$ = \sum_{m=0}^{k} \left[ \begin{array}{c} k \\ m \end{array} \right] e_6^{(k-m)} f_6^{(k-m)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u $$

$$ = e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u, $$

where the second equality comes from Equation (3.2) and the third equality follows from the fact $e_i f_j = f_j e_i$ for all $i \neq j$ and $f_6 w = 0$ (so only the $m = k$ term is nonzero). By computations similar to Equation (3.5), we have

$$ \|u_k\|^2 = (e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u, e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u) = \|e_0^{(k)} u\|^2. $$

Moreover, similar to Equation (3.5), we have

$$ \|e_0^{(k)} u\|^2 = (e_0^{(k)} u, e_0^{(k)} u) = q_0^{k(k+2s-2k)} (u, f_0^{(k)} e_0^{(k)} u) $$

$$ = q_0^{k(2s-k)} \sum_{m=0}^{k} \left[ \begin{array}{c} 2s \\ m \end{array} \right] (u, e_0^{(k-m)} f_0^{(k-m)} u) = q_0^{k(2s-k)} \left[ \begin{array}{c} 2s \\ k \end{array} \right] (u, u) $$

since $f_0 u = 0$. Hence, we have

$$ \|u_k\|^2 = q_0^{k(2s-k)} \left[ \begin{array}{c} 2s \\ k \end{array} \right] \in 1 + qA. \quad (3.6) $$

Next, we show (ii). Fix some $i \in I_0$. From Equation (3.3), it remains to compute $\|f_i u_k\|^2$. We compute $\|f_i u_k\|^2$ depending on the value of $i$. We note that the case of $k = 0$ is done by Equation (3.4). Therefore, we assume $k \geq 1$. For $i = 6$, we have

$$ f_6^{u_k} = \left[ \begin{array}{c} 1 - k + k \\ 1 \end{array} \right] q_6 e_6^{(k-1)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u + e_6^{(k-1)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u = e_6^{(k-1)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} u \quad (3.7) $$

by Equation (3.2) and the fact $f_6 u = 0$. Hence, similar to the computation for $\|u_k\|^2$, we have

$$ \|f_6 u_k\|^2 = q_6^{k-1} \left[ \begin{array}{c} k \\ k-1 \end{array} \right] \left[ \begin{array}{c} k-1 \\ 2s \end{array} \right] $$

$$ = q_6^{k-1} \left[ \begin{array}{c} k \\ k-1 \end{array} \right] q_6^{k(2s-k)} \left[ \begin{array}{c} 2s \\ k \end{array} \right] q_0. $$

For $i = 1$, we have $f_1 u_k = e_6^{(k)} e_5^{(k)} e_4^{(k)} e_2^{(k)} e_0^{(k)} f_1 u = 0$, and so $\|f_1 u_k\|^2 = 0$. For $i = 5, 4, 2$, we have $f_i u_k = 0$ by applying Equation (3.2) and the Serre relations (e.g., a straightforward calculation shows $e_4^{(k)} e_2^{(k-1)} e_0^{(k)} u = 0$.
by repeatedly applying the Serre relations). Finally, we have $f_{3u_k} = e_{6}^{(k)} e_{5}^{(k)} e_{4}^{(k)} e_{2}^{(k)} e_{0}^{(k)} f_{3u}$. Therefore, we have $\|f_{3u_k}\|^2 = \left\| e_{4}^{(k)} e_{2}^{(k)} e_{0}^{(k)} f_{3u} \right\|^2$ similar to Equation (3.5). However, for removing $e_{4}^{(k)}$, we obtain

\[
(e_{4}^{(k)} e_{2}^{(k)} e_{0}^{(k)} f_{3u}, e_{4}^{(k)} e_{2}^{(k)} e_{0}^{(k)} f_{3u}) = q_{4}^{(k-(k+1))} (e_{2}^{(k)} e_{0}^{(k)} f_{3u}, f_{4}^{(k)} e_{2}^{(k)} e_{0}^{(k)} f_{3u})
\]

by Equation (3.1a). Furthermore, we have

\[
f_{4}^{(k)} e_{2}^{(k)} e_{0}^{(k)} f_{3u} = \sum_{m=0}^{k} \binom{k-1}{m} q_{4}^{(k-m)} e_{2}^{(k)} e_{0}^{(k)} f_{3u} = q_{4}^{(k-1)} e_{2}^{(k)} e_{0}^{(k)} f_{3u} + \binom{k-1}{k} q_{4}^{(k)} e_{0}^{(k)} f_{3u} = e_{4}^{(k)} e_{0}^{(k)} f_{4f_{3u}},
\]

where we note that $\binom{k-1}{k} q_{4} = 0$ (recall that we assumed $k \geq 1$). Thus, by applying Equation (3.1a), we obtain

\[
\left\| e_{4}^{(k)} e_{2}^{(k)} e_{0}^{(k)} f_{3u} \right\|^2 = q_{4}^{-(k-1)} \left( e_{2}^{(k)} e_{0}^{(k)} f_{3u}, e_{4}^{(k)} e_{2}^{(k)} e_{0}^{(k)} f_{4f_{3u}} \right) = q_{4}^{-(k-1)} \left\| e_{2}^{(k)} e_{0}^{(k)} f_{4f_{3u}} \right\|^2.
\]

Next, we have that

\[
\left\| e_{2}^{(k)} e_{0}^{(k)} f_{4f_{3u}} \right\|^2 = \left\| e_{0}^{(k)} f_{2f_{4f_{3u}}} \right\|^2 = q_{0}^{(2s-1-k)} (f_{2f_{4f_{3u}}}, f_{0}^{(k)} e_{0}^{(k)} f_{2f_{4f_{3u}}}).
\]

We note that $f_{0} f_{2f_{4f_{3u}}} = 0$ for any $w \in W^{3,1}$ from weight considerations and the classical decomposition. So $f_{0} f_{2f_{4f_{3u}}} (w_{1} \otimes \cdots \otimes w_{s}) = 0$ for any $w_{1}, \ldots, w_{s} \in W^{3,1}$ from applying the coproduct $\Delta(f_{i}) = f_{i} \otimes 1 + K_{i} \otimes f_{i}$. Thus, we have $f_{0} f_{2f_{4f_{3u}}} = 0$ from the construction of $u$ and $W^{3,s}$. Therefore, we compute

\[
f_{0}^{(k)} e_{0}^{(k)} f_{2f_{4f_{3u}}} = \sum_{m=0}^{k} \binom{2s-1}{m} q_{0}^{(2s-1-k)} e_{0}^{(k)} f_{2f_{4f_{3u}}} = \binom{2s-1}{k} q_{0}^{(2s-1-k)} f_{2f_{4f_{3u}}}
\]

similar to Equation (3.5) and using the Serre relations. Thus, we have

\[
\left\| e_{0}^{(k)} f_{2f_{4f_{3u}}} \right\|^2 = q_{0}^{(2s-1-k)} \binom{2s-1}{k} q_{0}^{(2s-1-k)} \left\| f_{2f_{4f_{3u}}} \right\|^2.
\]

Next, we see

\[
\left\| f_{2f_{4f_{3u}}} \right\|^2 = q_{2}^{-1} (f_{4f_{3u}}, e_{2f_{4f_{3u}}}) = (f_{4f_{3u}}, [1] q_{2} f_{4f_{3u}}) = q_{4}^{-1} (f_{3u}, e_{4f_{3u}}) = (f_{3u}, [1] q_{4} f_{3u}) = \|f_{3u}\|^2
\]

by a similar computation to Equation (3.4). Hence, we have

\[
\|f_{3u_k}\|^2 = q_{0}^{(2s-1-k)} \binom{2s-1}{k} q_{0}^{(2s-1-k)} \binom{2s-1}{k} q_{0}^{1+s[s] q_{3}^{2} A}
\]

where the last equality is by Equation (3.4). Hence, we have

\[
\|f_{3u_k}\|^2 = q_{0}^{(2s-1-k)} \binom{2s-1}{k} q_{0}^{(2s-1-k)} \binom{2s-1}{k} q_{0}^{1+s[s] q_{3}^{2} A},
\]

where $\langle h_{i}, \lambda_{k} \rangle \geq 0$.

3.2. Type $E_{6}^{(1)}$, $r = 5$. The following are the desired elements in $W^{5,s}$:

\[
u_{k} := e_{1}^{(k)} e_{3}^{(k)} e_{2}^{(k)} e_{0}^{(k)} u_{0} \in W^{5,s}_{(s-k) \otimes w_{s} + k \otimes w_{k}},
\]

where $0 \leq k \leq s$. The proof is the same as $r = 3$ after applying the order 2 diagram automorphism that fixes 0.
3.3. **Type $E_r^{(1)}$, $r = 2$.** The following are the desired elements in $W^{2,s}$:

$$u_k := e_7^{(k)} e_0 e_5 e_4 e_3 e_1 e_0 u_0 \in W^{2,s}_{(s-k)\pi_2 + k\pi_7},$$

where $0 \leq k \leq s$. The proof is similar to the $W^{3,s}$ in type $E_6^{(1)}$, where we compute

$$\|u_k\|^2 = q_0^{2(2s-k)} \left\| \frac{2s}{k} \right\|_{q_1},$$

$$\|f_7 u_k\|^2 = q_{11}^{k-1} \left[ \frac{k}{k-1} q_{s} \|u_k\|^2, \right.$$  

$$\|f_i u_k\|^2 = 0 \quad (i = 6, 5, 4, 3, 1),$$  

$$\|f_2 u_k\|^2 = q_0^{2(2s-k)} \left\| \frac{2s-1}{k} \right\|_{q_2} \|f_2 u\|^2.$$  

3.4. **Type $E_r^{(2)}$, $r = 4$.** We claim

$$u_{k',k} := e_0 e_1 e_2 e_3 e_0 u$$

are the desired elements, where $0 \leq k' \leq k \leq s$. We note that

$$\text{wt}(u_{k',k}) = \lambda_{k',k} := (s-k)\Lambda_4 + (k-k')\Lambda_1 - (2s-2k')\Lambda_0.$$  

To obtain the parameterization of the classical decomposition

$$W^{4,s} \cong \bigoplus_{t_1, t_2 \geq 0, t_1 + t_2 \leq s} V(t_1\overline{\Lambda}_4 + t_2\overline{\Lambda}_1)$$

given in [Scr17, Prop. 9.31], we set $t_1 = s - k$ and $t_2 = k - k'$ (which is forced by weight considerations). Note that $t_1 \geq 0$ if and only if $k \leq s$; $t_2 \geq 0$ if and only if $k' \leq k$; and $t_1 + t_2 \leq s$ if and only if $0 \leq k'$ (as $t_1 + t_2 = s - k'$). Hence, we have the same classical decomposition.

To show (i), we have

$$\|u_{0,k}\|^2 = q_1^{k(k-k)} \left( e_2 e_3 e_2 e_1 e_0 u, f_1(k) u_{0,k} \right).$$

Next, we compute

$$f_1^{(k)} u_{0,k} = f_1^{(k)} e_1 e_2 e_3 e_2 e_1 e_0 u$$  

$$= \sum_{m=0}^{k} \left[ \begin{array}{c} k \\ m \end{array} \right] e_1^{(k-m)} f_1^{(k-m)} e_2^{(k)} e_3^{(k)} e_2^{(k)} e_1^{(k)} e_0^{(k)} u$$  

$$= \sum_{m=0}^{k} \left[ \begin{array}{c} k \\ m \end{array} \right] e_1^{(k-m)} e_3^{(k)} e_2^{(k)} e_2^{(k)} \sum_{p=0}^{k-m} \left[ \begin{array}{c} k-m \\ p \end{array} \right] e_1^{(k-p)} f_1^{(k-m-p)} e_0^{(k)} u$$  

$$= \sum_{m=0}^{k} \left[ \begin{array}{c} k \\ m \end{array} \right] \left[ \begin{array}{c} k-m \\ m \end{array} \right] e_1^{(k-m)} e_3^{(k)} e_2^{(k)} e_2^{(k)} e_1^{(m)} e_0^{(k)} u$$  

$$= e_2 e_3 e_2 e_1 e_0 u,$$

where the last equality follows from the fact $e_2^{(k)} e_1^{(m)} e_0^{(k)} u = 0$ for all $k > m$ by the Serre relations and $e_2 u = 0$. Hence, we have

$$\|u_{0,k}\|^2 = \left\| e_2 e_3 e_2 e_1 e_0 u \right\|^2 = q_1^{k(k-k)} \left( e_2 e_3 e_2 e_1 e_0 u, f_2 u_{0,k} \right).$$
Now, similar to the previous computation, we obtain
\[ f_{2}^{(k)} e_{2}^{(k)} e_{3}^{(k)} e_{1}^{(k)} e_{0}^{(k)} u = \sum_{m=0}^{k} \left[ \frac{k}{m} e_{2}^{(k-m)} f_{2}^{(k-m)} e_{3}^{(k)} e_{1}^{(k)} e_{0}^{(k)} u \right. \]
\[ = \sum_{m=0}^{k} \left[ \frac{k}{m} e_{2}^{(k-m)} e_{3}^{(k)} \sum_{p=0}^{k-m} \left[ \frac{k-m}{p} e_{2}^{(k-p)} f_{2}^{(k-p-m)} e_{1}^{(k)} e_{0}^{(k)} u \right. \right. \]
\[ = \sum_{m=0}^{k} \left[ \frac{k}{m} \left( - \frac{k-m}{m} \right) e_{2}^{(k-m)} e_{3}^{(k)} e_{2}^{(m)} e_{1}^{(k)} e_{0}^{(k)} u \right. \]
\[ = e_{3}^{(k)} e_{2}^{(m)} e_{1}^{(k)} e_{0}^{(k)} u \]

since \( e_{3}^{(k)} e_{2}^{(m)} e_{1}^{(k)} e_{0}^{(k)} u = 0 \) for all \( k > m \) by the Serre relations (recall that \( A_{2} = -1 \) and \( e_{3}u = 0 \). Hence, we have
\[ \|u_{0,k}\|^2 = \|e_{3}^{(k)} e_{2}^{(m)} e_{1}^{(k)} e_{0}^{(k)} u\|^2 = q_0(2k-k') \left[ \frac{2s}{k} \right]_{q_0} \in 1 + qA, \]

where the last equality is shown similar to Equation (3.6).

Next, we consider
\[ \|u_{k',k}\|^2 = q_0^{k' (2s-2k')} (u_{0,k}, f_0^{(k')} u_{k',k}). \]

We compute
\[ f_0^{(k')} u_{k',k} = f_0^{(k')} e_{0}^{(k')} u_{0,k} = \sum_{m=0}^{k',-m} \left[ \frac{2s}{m} e_{0}^{(k'-m)} f_0^{(k'-m)} u_{0,k} \right. \]
\[ = \sum_{m=0}^{k'-m} \left[ \frac{k'-m-k+2s}{p} e_{0}^{(k-p)} f_0^{(k'-m-p)} u_{0,k} \right] \left[ \frac{k'-m-k+2s}{k'-m} e_{0}^{(k'-m+1)} u_{0,k} \right. \]

as \( k' - m \leq k \) (since \( k' \leq k \) and \( m \geq 0 \)) and \( f_0 u = 0 \). Next, we have \( e_{1}^{(k)} e_{0}^{(m)} u = 0 \) for all \( k > m \) by the Serre relations and \( e_{1} u = 0 \), and so the only term that is nonzero in Equation (3.10) is when \( m = k' \). Therefore, we have
\[ \|u_{k',k}\|^2 = q_0^{k' (2s-k')} \left[ \frac{2s}{k'} \right]_{q_0} \|u_{0,k}\|^2 = q_0^{k' (2s-k')} \left[ \frac{2s}{k'} \right]_{q_0} q_0^{k (2s-k)} \left[ \frac{2s}{k} \right]_{q_0} \in 1 + qA. \]

To show (ii), it remains to compute \( |f_i u_{k',k}|^2 \) by Equation (3.3), and by Equation (3.4), we can assume \( k \geq 1 \). For \( i \in I_0 \), we have \( f_i u_{k',k} = e_{0}^{(k')} f_i u_{0,k} \), and by the above, we have
\[ \|f_i u_{k',k}\|^2 = q_0^{k' (2s-\delta_1-k')} \left[ \frac{2s - \delta_1}{k'} \right]_{q_0} \|f_i u_{0,k}\|^2. \]

Next, similar to the computation in Equation (3.7), we have
\[ f_{1} u_{0,k} = e_{1}^{(k)} e_{2}^{(k)} e_{3}^{(k)} e_{1}^{(k)} e_{0}^{(k)} u + e_{1}^{(k)} e_{2}^{(k)} e_{3}^{(k)} e_{2}^{(k)} e_{1}^{(k)} e_{0}^{(k)} u \]
\[ = e_{1}^{(k)} e_{2}^{(k)} e_{3}^{(k)} e_{1}^{(k)} e_{0}^{(k)} u + e_{1}^{(k)} e_{2}^{(k)} e_{3}^{(k)} e_{2}^{(k)} e_{1}^{(k)} e_{0}^{(k)} u \]
\[ = e_{1}^{(k)} e_{2}^{(k)} e_{3}^{(k)} e_{2}^{(k)} e_{1}^{(k)} e_{0}^{(k)} u, \]

where the last equality is using \( e_{2}^{(k)} e_{1}^{(m)} e_{0}^{(k)} u = 0 \) for all \( k > m \). Therefore, we have
\[ \|f_{1} u_{0,k}\|^2 = q_1^{k-1} \left[ \frac{k}{k-1} \right] q_0^{k (2s-k)} \left[ \frac{2s}{k} \right]_{q_0} \]

by a computation similar to Equation (3.6). Similar to Equation (3.9), we have
\[ \|f_{4} u_{0,k}\|^2 = q_0^{k (2s-1-k)} \left[ \frac{2s-1}{k} \right] \|f_{4} u\|^2. \]
We also have $f_2u_{0,k} = f_3u_{0,k} = 0$ by applying the Serre relations. Thus, we see that (ii) holds.

3.5. **Type $E^{(1)}_7$, $r = 6$.** The following are the desired elements in $W^{6,s}$:

$$u_{k',k} := e_0^{(k')} c_1^{(k')} c_3^{(k')} c_4^{(k')} c_5^{(k')} c_6^{(k')} c_8^{(k')} c_0^{(k')} u \in W^{6,s}_{(s-t_1-t_2)\varpi_6+t_2\varpi_1},$$

where $0 \leq k' \leq k \leq s$. Then $\text{wt}(u_{k',k}) = (s-k)\Lambda_6 + (k-k')\Lambda_1 - (2s-2k')\Lambda_0$. Showing the classical decomposition is the same as in [Cha01] is similar to the $r = 4$ case for type $E^{(2)}_6$. Moreover, it is similar to show that

$$\|u_{k',k}\|^2 = q_0^{k'(2s-k')} \left[\frac{2s}{k'}\right]^{k(2s-k)} \left[\frac{2s}{k}\right]_{q_0},$$

$$\|f_i u_{k',k}\|^2 = q_0^{k'(2s-\delta_i-k')} \left[\frac{2s-\delta_i}{k'}\right]_{q_0} \|f_i u_{0,k}\|^2 \quad (i \in I_0),$$

$$\|f_1 u_{0,k}\|^2 = q_8^{k-1} \left[\frac{k}{k-1}\right]_{q_8} \|u_{0,k}\|^2,$$

$$\|f_2 u_{0,k}\|^2 = 0 \quad (i = 2, 3, 4, 5, 7),$$

$$\|f_6 u_{0,k}\|^2 = q_0^{k(2s-1-k)} \left[\frac{2s-1}{k}\right]_{q_0} \|f_6 u\|^2.$$

3.6. **Type $E^{(1)}_8$, $r = 1$.** The following are the desired elements in $W^{1,s}$:

$$u_{k',k} := e_0^{(k')} e_8^{(k')} e_7^{(k')} e_6^{(k')} e_5^{(k')} e_4^{(k')} e_3^{(k')} e_2^{(k')} e_1^{(k')} e_0^{(k')} u \in W^{1,s}_{(s-t_1-t_2)\varpi_8+t_2\varpi_1},$$

where $0 \leq k' \leq k \leq s$. Then $\text{wt}(u_{k',k}) = (s-k)\Lambda_1 + (k-k')\Lambda_8 - (2s-2k')\Lambda_0$. Showing the classical decomposition is the same as in [Cha01] is similar to the $r = 4$ case for type $E^{(2)}_6$. Moreover, it is similar to show that

$$\|u_{k',k}\|^2 = q_0^{k'(2s-k')} \left[\frac{2s}{k'}\right]^{k(2s-k)} \left[\frac{2s}{k}\right]_{q_0},$$

$$\|f_i u_{k',k}\|^2 = q_0^{k'(2s-\delta_i-k')} \left[\frac{2s-\delta_i}{k'}\right]_{q_0} \|f_i u_{0,k}\|^2 \quad (i \in I_0),$$

$$\|f_8 u_{0,k}\|^2 = q_8^{k-1} \left[\frac{k}{k-1}\right]_{q_8} \|u_{0,k}\|^2,$$

$$\|f_2 u_{0,k}\|^2 = 0 \quad (i = 2, 3, 4, 5, 6, 7),$$

$$\|f_1 u_{0,k}\|^2 = q_0^{k(2s-1-k)} \left[\frac{2s-1}{k}\right]_{q_0} \|f_1 u\|^2.$$

3.7. **Type $F^{(1)}_4$, $r = 4$.** The following are the desired elements in $W^{4,s}$:

$$u_{k',k} := e_0^{(k')} e_1^{(k')} e_2^{(2k')} e_3^{(k')} e_4^{(k')} e_5^{(k')} e_6^{(k')} e_8^{(k')} e_0^{(k')} u \in W^{4,s}_{(s-2k)\varpi_4+(k-k')\varpi_1},$$

where $0 \leq k' \leq k \leq s/2$. Then $\text{wt}(u_{k',k}) = (s-2k)\Lambda_4 + (k-k')\Lambda_1 - (s-2k')\Lambda_0$. To obtain the parameterization of the classical decomposition

$$W^{4,s} \cong \bigoplus_{t_2=0}^{s/2} \bigoplus_{t_1=0}^{t_2} V((s-2t_2)\Lambda_4 + t_1 \Lambda_1)$$

given in [Cha01], we take $t_1 = k - k'$ and $t_2 = k$. Indeed, we have $t_2 \leq s/2$ if and only if $k \leq s/2$; $t_1 \geq 0$ if and only if $k \leq k'$; and $t_1 \leq t_2$ if and only if $0 \leq k'$. 
Moreover, it is similar to the $r = 4$ case for type $E_6^{(2)}$ to show that
\[
\|u_0^{k',-k}\|^2 = q_0^{k'(2s-k')} \left[\begin{array}{c} 2s \\ k' \end{array}\right] q_0^{(2s-k)} \left[\begin{array}{c} 2s \\ k \end{array}\right], \\
\|f_i u_0^{k',-k}\|^2 = q_0^{k'(2s-\delta_i-1-k')} \left[\begin{array}{c} 2s - \delta_i \\ k' \end{array}\right] q_0 \|f_i u_0\|^2, \\
\|f_1 u_0\|^2 = q_1^{k} \left[\begin{array}{c} k \\ k-1 \end{array}\right] q_1 \|u_0\|^2, \\
\|f_2 u_0\|^2 = \|f_3 u_0\|^2 = 0, \\
\|f_4 u_0\|^2 = q_0^{k(2s-1-k)} \left[\begin{array}{c} 2s - 1 \\ k \end{array}\right] q_0 \|f_4 u\|^2.
\]

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