TROPES, TORELLI AND THETA CHARACTERISTICS

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Abstract

This paper concerns the geometry associated to the 2-torsion subgroup $A[2]$ of a principally polarized abelian variety (ppav) $(A, \lambda)$. The main results are these.

1. (Theorem 1.2.) If the characteristic is 2 then $(A, \lambda)$ is ordinary if and only if, for every symmetric theta divisor $\Phi$ on $A$, there is a 2-torsion point that does not lie on $\Phi$.

2. (Theorem 2.8 and Proposition 2.9.) We give a geometrical description, if either the characteristic of the ground field $k$ is zero or if the genus of the curve is 3, of the quadratic twist observed by Serre [LS] that arises when $(A, \lambda)$ is geometrically isomorphic to a Jacobian.

3. (Theorem 3.2.) Suppose that $C$ is a non-hyperelliptic curve of genus 3 in characteristic $\neq 2$ and that $Z$ is the set of its odd theta characteristics. Then $Z$ is naturally embedded in $\mathbb{P}^6$ and the intersection of quadrics though $Z$ is a normal del Pezzo surface $S$ of degree 2. The curve $C$ can then be recovered as the normalization of the unique anti-sexcanonical curve on $S$ that is singular at each point of $Z$.

(1) is due to Laszlo and Pauly [LP] when $(A, \lambda)$ is a Jacobian.

The geometrical description referred to in (2) is in terms of a theorem of Welters [W] concerning the geometry of the linear system $|2\Theta|$. The hypothesis that the ground field be of characteristic zero underlies much of the literature in this area and we have not undertaken the task of trying to remove it. In genus 3 Beauville and Ritzenthaler [BR] have found a different geometrical description, under certain additional hypotheses.

The index $r(S)$ of the del Pezzo surface in (3) is 1 if $\text{char } k \neq 3$ and 2 if $\text{char } k = 3$. A del Pezzo surface of degree 2 and index 1 is anticanonically a double cover of $\mathbb{P}^2$ branched along a quartic; however, the quartic that appears here is not $C$ but rather the contravariant $K_1$ given in symbolical terms [GY] by $K_1 = (abu)^4$ when the quartic $f$ defining $C$ is given symbolically by $f = a_2^4 = b_2^4$.

Lehavi [L] has given another way of recovering $C$ from its bitangents; our result appears to be complementary to his.

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1 Theta characteristics in characteristic two

We start by recalling some basic results and notation.

Suppose first that \((Y, \Theta)\) is a principal symmetric abelian torsor (psat) over a base \(S\), with associated ppav \((A, \lambda)\); recall (see [SB], for example) that this means that \(Y\) is projective over \(S\), that \(\Theta\) is an effective ample Cartier divisor on \(Y\), \(Y\) is a torsor under \(A := \text{Aut}^0\), that \(\lambda : A \to \text{Pic}^0_y \cong \text{Pic}^0_A\) defined by \(\lambda(a) = r_a^* \Theta - \Theta\) is an isomorphism, there is given an extension of the action of \(A\) on \(Y\) to an action of the split extension \(A \times [-1_A]\) and that \(\Theta\) is preserved by \(\iota = [-1_A]\). The standard example is \(Y = \text{Pic}^g_C\), \(\Theta\) is the locus of effective classes in \(Y\) and \(A = \text{Pic}^0_C\), where \(C\) is a curve of genus \(g \geq 2\).

Of course, a point \(y\) on \(Y\) defines an isomorphism \(y' : Y \to A\) given by subtraction. In particular, if \(y \in T\), then \(y'\) takes \(\Theta\) to a symmetric theta divisor \(\Phi_y\) on \(A\), and all symmetric theta divisors on \(A\) arise in this way.

Although, when \(g \geq 2\), a psat is not naturally a ppav and a ppav is not naturally a psat, the natural morphism from the stack \(\mathcal{Y}_g\) of psat’s \((Y, \Theta)\) to the stack \(\mathcal{A}_g\) of ppav’s \((A, \lambda)\) is an isomorphism. Then the scheme of theta characteristics, defined as the fixed locus \(T := \text{Fix}_Y = \text{Fix}_{[-1_A], Y}\) of \([-1_A]\) in \(Y\), is a torsor under the 2-torsion \(P := A[2]\). We also identify \(T\) with the scheme that parametrizes symmetric theta divisors \(\Phi\) on \(A\): given a point \(t\) of \(T\), the corresponding divisor \(\Phi_t\) is \(\Phi_t = \Theta - t\).

Let \(T \to \mathcal{Y}_g \cong \mathcal{A}_g\) denote the universal scheme of theta characteristics.

The main result of this section is that, in characteristic 2, a ppav is ordinary if and only if on the corresponding psat, there is a theta characteristic that does not lie on the theta divisor.

The following result is merely a restatement in a way that emphasizes psat’s of the discussion on pp. 132-135 of [FC]. Their stack \(N_g\) of ppav’s with a symmetric theta divisor is isomorphic to the stack \(T\) of psat’s \(Y\) with a point of \(\text{Fix}_Y\). (As usual, \(e_n\) is the Weil pairing defined by \((Y, \Theta)\) on the \(n\)-torsion \(A[n]\).)

So fix a \(g\)-dimensional psat \((Y, \Theta) \to S\), corresponding to the ppav \((A, \lambda) \to S\). Put \(P = A[2]\). Let \(T'\) denote the scheme of morphisms \(t : P \to \mu_{2, S}\) such that

\[ t(p + q)t(p) = e_2(p, q) \]

That is, the points of \(T'\) are the \(\mu_{2, S}\)-valued quadratic forms on \(P\) whose polarization is \(e_2\); cf. [I], p. 214.

**Proposition 1.1** (1) \(T'\) is naturally identified with \(T\).

(2) There is a morphism \(\text{Arf} : T \to \mu_2\) such that \(\text{Arf}(t)\text{Arf}(t+p) = t(p)\).

(3) Over any geometric point \(\sigma\) of \(S\) of characteristic \(\neq 2\), \(\text{Arf}(t)\) is the usual Arf invariant of \(t\), regarded as a quadratic form on the symplectic \(\mathbb{F}_2\)-vector space \(P(\sigma)\).

**PROOF:** We first construct the morphism \(\text{Arf}\).
The fact that \( \iota(\Theta) = \Theta \) means that we can identify \( \mathcal{O}_Y(-\Theta) \) with a subsheaf, preserved by \( \iota \), of the structure sheaf \( \mathcal{O}_Y \), so with an \( \langle \iota \rangle \)-linearized invertible sheaf on \( Y \). So the sheaf \( \mathcal{L} = \mathcal{O}_Y(\Theta) \) is \( \langle \iota \rangle \)-linearized.

Restrict to the subscheme \( T \) of \( Y \) and, locally on \( T \), pick a generator \( s \) of \( \mathcal{L}|_T \). Then \( \iota^*(s) = u.s \) for some \( u \in \mathcal{O}_T \) and \( s = \iota^*(u).\iota(s) = u^2.s \) since \( \iota \) acts trivially on \( T \). So \( u^2 = 1 \). If \( \tilde{s} \) is another local generator of \( \mathcal{L}|_T \), then \( \tilde{s} = w.s \) for some local section \( w \) of \( \mathcal{O}_T^* \), and then

\[
\iota^* \tilde{s} = \iota^*(w)\iota^*(s) = w.u.s = u.\tilde{s}
\]

and therefore \( u \) is defined as a morphism \( u : T \to \mu_{2,S} \); define \( \text{Arf} = u \). That is, for any local generator \( s \) of \( \mathcal{L}|_T \) and for \( t \in T \), \( (\iota^*(s))(t) = \text{Arf}(t)s(t) \).

Note that this construction commutes with base change.

(For any field \( k \) of characteristic \( \neq 2 \) and any \( k \)-point \( t \) of \( T \), \( \text{Arf}(t) = (-1)^{\text{mult}_t(\Theta)} \). This is proved in [MEq], bottom of p. 307.)

Now define a morphism \( T \to \text{Mor}(P, \mu_2) \) by \( t(p) = \text{Arf}(t)Arf(t + p) \) for all points \( t \) (not necessarily geometric) of \( T \). We need to show that \( t(p)t(q)t(p + q) = e_2(p, q) \), i.e., that \( t \) is a \( \mu_2 \)-valued quadratic form that returns the alternating form \( e_2 \). To prove a formula such as this, nothing is lost by making a faithfully flat base change \( S' \to S \), so we can assume that \( T \) has an \( S \)-point \( t_0 \) and that therefore \( Y = A \), where \( t_0 \) is identified with \( 0_A \), and that \( \Theta \) is symmetric on \( A \). This is the context of [MEq].

The \( \langle \iota \rangle \)-linearization of \( \mathcal{L} \) is an isomorphism \( \phi : \mathcal{L} \to \iota^*[\mathcal{L}] \) covering \([-1_A] \). Normalize \( \phi \) by demanding that \( \phi(0) = 1 \). Then Mumford’s morphism \( e^\xi_p : P \to \mu_2 \), defined by \( e^\xi_p(p) = \phi(p) \), is exactly \( e^\xi_p(p) = \text{Arf}(t_0)Arf(t_0 + p) \). Mumford proves, in all characteristics except 2, that

\[
e^\xi_p(p + q) = e^\xi_p(p)e^\xi_p(q)e_2(p, q).
\]

So we can assume that \( S \) has a closed point \( \sigma \) of characteristic 2, and then that \( (Y, \Theta) \to S \), or \( (A, \lambda) \to S \), is versal at \( \sigma \) and that \( S \) is integral. Then the formula to be proved holds over the generic point of \( S \), and so over all of \( S \).

So the morphism \( T \to \text{Mor}(P, \mu_2) \) is a morphism \( T \to T' \). There is an action of \( P \) on \( T' \) given by \( (p(t'))(q) = t'(q)e_2(p, q) \) for \( t' \in T' \); this makes \( T' \) into a torsor under \( P \), and \( T \to T' \) is then \( P \)-equivariant, so an isomorphism.

The identification of \( \text{Arf} \) with the usual \( \text{Arf} \) invariant (in characteristic not 2) is proved, for Jacobians, in [MTh]. From this, an argument involving lifting to characteristic zero and the irreducibility of \( A_g \) (only required here in characteristic zero) completes the proof.  

\( \square \)

In all characteristics except 2, it follows from the properties of the \( \text{Arf} \) invariant, as is well known, that there are just \( 2g-1(2g-1) \) points of \( T \) with odd multiplicity on \( \Theta \) and \( 2g-1(2g+1) \) with even multiplicity. That is, provided that \( \text{char} \, k \neq 2 \), \( T = T^+ \coprod T^- \) where \( T^\pm = \text{Arf}^{-1}(\pm 1) \), \( T^+ \) has order \( 2g-1(2g+1) \) and \( T^- \) has order \( 2g-1(2g-1) \).
The next result is due to Lazslo and Pauly [LP] in the case where \((A, \lambda)\) is a Jacobian.

**Theorem 1.2** Assume that the base is a field \(k\) of characteristic 2.

1. There is at most one geometric point of \(T\) that does not lie on \(\Theta\).
2. \((T \otimes \overline{k})_{\text{red}}\) is contained in \(\Theta \otimes \overline{k}\) if and only if \(A\) is not ordinary.
3. \(A\) is ordinary if and only if, for every symmetric theta divisor \(\Phi_t\) on \(A\), there is a geometric point of \(A[2]\) that does not lie on \(\Phi_t\).

**PROOF:** We may assume that \(k = \overline{k}\).

Since \(H^0(X, \mathcal{O}(\Theta))\) is 1-dimensional, any section \(\theta\) satisfies \(\iota^*(\theta) = \pm \theta\). So in characteristic 2, \(\iota^*(\theta) = \theta\). Recall that \(A\) is ordinary if and only if the identity connected component \(P^0\) of \(P\) has order at most \(2^g\) (so exactly \(2^g\)).

Assume that \(T_{\text{red}}\) does not lie in \(\Theta\), so that there is a non-empty union \(T^0 = \sqcup T^0_i\) of connected components \(T^0_i\) of \(T\) that are disjoint from \(\Theta\). Then \(P^0\) preserves each \(T^0_i\) and \(P^0\) has order \(\geq 2^g\). Also \(\theta|_{T^0}\) generates \(\mathcal{O}(\Theta)|_{T^0}\), and then, from the definition, \(\text{Arf} = 1\) on \(T^0\).

If \(T^0\) is not connected, then there is a subgroup \(P_1\) of \(A[2]\) that contains \(P^0\) as a subgroup of index 2 and which preserves the union \(T^0_i \sqcup T^0_j\) of two distinct components \(T^0_i\) and \(T^0_j\).

Suppose that \(R\) is a \(k\)-algebra, that \(t \in T^0(R)\) and that \(p, q \in P^0(R)\). Then

\[
e_2(p, q) = \text{Arf}(t)\text{Arf}(t + p)\text{Arf}(t + q)\text{Arf}(t + p + q);
\]

each factor on the right equals 1, and so \(P^0\) is totally isotropic. Then \(P^0\) has order \(\leq 2^g\) and \(A\) is ordinary.

Moreover, if \(T^0\) is not connected, then take \(t \in (T^0_i \sqcup T^0_j)(R)\). The same argument shows that \(P_1\) is totally isotropic; this is impossible, since its order is \(2^{g+1}\). So \(T^0\) is connected, which proves (1).

Conversely, suppose that \(A\) is ordinary. It remains to prove that \(T_{\text{red}}\) does not lie in \(\Theta\).

Laszlo and Pauly [LP] showed that on any ordinary ppav \(A\) over an algebraically closed field of characteristic \(p\), with symmetric theta divisor \(\Phi = (\theta)_0\), the translated powers \(t^x\theta^p\) form a basis of \(H^0(A, \mathcal{O}(p\Phi))\), where \(x\) runs over the \(k\)-points of \(A[p]_{\text{red}}\). So if \(\Phi\) contains \(A[p]_{\text{red}}\), then \(A[p]_{\text{red}}\) is contained in the base locus of the linear system \(|p\Phi|\). But (Lefschetz) this base locus is empty, and (2) is proved.

(3) follows at once from (2) and the correspondence between the symmetric theta divisors \(\Phi_t\) on \(A\) and the geometric points \(t\) of \(T\). \(\square\)

Let \(A_g^{\text{ord}}\) denote the ordinary locus in \(A_g \otimes \mathbb{F}_2\) (regarded as the stack of psat’s in characteristic 2) and let \(\mathcal{T} \to A_g^{\text{ord}}\) denote the universal scheme of theta characteristics. (As mentioned above, this is isomorphic to the stack \(N_g^{\text{ord}}\) of ordinary ppav’s with a symmetric theta divisor.) Then \(\mathcal{T} = T^1 \coprod T^2\), where \(T^1\) parametrizes theta characteristics that do not lie in the universal \(\Theta\) divisor and
\( \mathcal{T}^2 \) parametrizes those that do. So the restriction of \( Arf \) to \( \mathcal{T}_1 \) is identically 1 and \( \mathcal{T}^1 \rightarrow A^{ord}_2 \) is a torsor under the connected part of the 2-torsion subgroup of the universal abelian variety. On the other hand, the morphism \( Arf : \mathcal{T}^2 \rightarrow \mu_2 \) is smooth ([FC], p. 134).

The next lemma extends Lemma 23 of [1], p. 214, to characteristic 2.

**Lemma 1.3** A decomposition \( P = L \oplus M \) into Lagrangian subgroups determines a bilinear form \( f : P \times P \rightarrow \mu_2 \) such that \( e_2(p, q) = f(p, q)f(q, p) \) and an even theta characteristic \( \delta \) in \( T(k) \) such that \( \delta(p) = f(p, p) \).

For ordinary abelian varieties in characteristic two, \( \delta \) is the unique theta characteristic that does not lie on \( \Theta \).

**PROOF:** We can identify \( M = L^\vee \), the Cartier dual of \( L \). Define a bilinear morphism \( f : P \times P \rightarrow \mu_2 \) by \( f((a, \alpha), (b, \beta)) = \beta(a) \), so that \( e_2(p, q) = f(p, q)f(q, p) \), and then define \( \delta(p) = f(p, p) \). So \( \delta \in T(k) \). Igusa’s argument shows that \( arf(\delta) = 1 \) if \( p \neq 2 \), while if \( p = 2 \) then \( arf(t) = 1 \) for any \( k \)-point \( t \) of \( T \).

If \( A \) is ordinary in characteristic two, then the local-étale decomposition of \( A[2] \) is \( A[2] = \mu_2^2 \times (\mathbb{Z}/2)^g \), and so determines \( \delta \). The image of the monodromy group on \( A[2] \) is \( GL_g(\mathbb{Z}/2) \), so \( \delta \) is the unique globally defined theta characteristic.

\( \square \)

As showed to us by Bjorn Poonen, the machinery of theta characteristics also illuminates the finite Heisenberg subgroupschemes of a level 2 theta group, as follows.

Recall that if \( (X, \Theta) \) is a psat over some base \( S \) and \( (A, \lambda) \) is the corresponding ppav then the theta group \( G_n = G_{(X, \Theta), n} \) of level \( n \) attached to \( (X, \Theta) \) is an \( S \)-groupscheme that is a central extension

\[
1 \rightarrow G_m \rightarrow G_n \xrightarrow{\pi} A[n] \rightarrow 0
\]

from which the Weil pairing \( e_n : A[n] \times A[n] \rightarrow \mu_n \hookrightarrow G_m \) is then constructed as the commutator pairing. The points of \( G_n \) are pairs \( (\phi, x) \) where \( x \in A[n] \) and \( \phi : t_2^*O_X(n\Theta) \rightarrow O_X(n\Theta) \) is an isomorphism of line bundles. This description is given in [MAV] when \( X \) is identified with \( A \).

Part of the data of \( (X, \Theta) \) is an involution \( \iota \) of \( X \) that preserves \( \Theta \) and is compatible with \([-1_A] \) acting on \( A = Aut_X^0 \). This involution defines an involution \( \bar{\iota} \) of \( G_n \) given by \( \bar{\iota}(\phi, x) = (\phi \circ \iota, -x) \). The extended theta group \( G_n^e \) is the split extension \( G_n \rtimes \langle \bar{\iota} \rangle \).

A finite Heisenberg group of level \( n \) and type \( (A[n], e_n) \) is a central extension

\[
1 \rightarrow \mu_n \rightarrow K \rightarrow A[n] \rightarrow 0
\]

whose commutator pairing is \( e_n \).

**Proposition 1.4** (Poonen) The scheme \( \text{Heis}_2 = \text{Heis}_{(X, \Theta), 2} \) that classifies finite Heisenberg subgroups of \( G_{(X, \Theta), 2} \) that are of level 2 and type \( (A[2], e_2) \) is a torsor...
over $S$ under $A[2]$. It is isomorphic, as a torsor under $A[2]$, to the scheme $T$ of theta characteristics.

PROOF: Set $Z_2 = \mu_2 \subset \mu_4 \subset \mathbb{G}_m$ and $\mathcal{H}_4 = \{g \in G_2 | g^4 = 1\}$. It is easy to check that $\mathcal{H}_4$ is a subgroup scheme of $G_2$ and that $g^2 \in \mu_2$ for all $g \in \mathcal{H}_4$. Moreover, there is a central extension

$$1 \to \mu_4 \to \mathcal{H}_4 \to A[2] \to 0$$

and the finite Heisenberg subgroupschemes of level 2 and type $(A[2], e_2)$ of $G_2$ are exactly the subgroupschemes $G$ of $\mathcal{H}_4$ such that $G \cap \mu_4 = Z_2$ and $G \to A[2]$ is surjective. So $Heis_2$ is a locally closed subfunctor of the Hilbert functor of $\mathcal{H}_4$ and is therefore representable. The extension

$$1 \to \mu_4/Z_2 \to \mathcal{H}_4/Z_2 \to A[2] \to 0 \quad (*)$$

is an extension of commutative groupschemes of type $(2, 2, 2, ...)$ and, via replacing $G$ by $\overline{G} = G/Z_2$ and $\mathcal{H}_4$ by $\overline{\mathcal{H}_4} = \mathcal{H}_4/Z_2$, we see that $Heis_2$ is the scheme that parametrizes the splittings of this last extension $(*)$. This exhibits the structure of $Heis_2$ as a pseudo-torsor under $\text{Hom}(A[2], \mu_4/Z_2)$, so, via the natural isomorphism $\mu_4/Z_2 \to \mu_2 : s \mapsto s^2$ and the pairing $e_2$, as a torsor under $A[2]$. To show that this pseudo-torsor is a torsor we can assume that $S$ is the spectrum of an algebraically closed field. We must find a splitting $\overline{G}$ of the surjection $\overline{\mathcal{H}_4} \to A[2]$. This is equivalent to showing that the exact sequence

$$0 \to A[2] \to \overline{\mathcal{H}_4} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

that arises as the Cartier dual of the sequence $(*)$ is split. For this, just lift the element 1 of $\mathbb{Z}/2\mathbb{Z}$ to any $S$-point of $\overline{\mathcal{H}_4}$; that $S$-point is killed by 2 since the whole group is. The existence of this $S$-point shows that the pseudo-torsor is a torsor.

We shall next construct a morphism $\alpha : Heis_2 \to T$. Take a subgroup $G$ of $\mathcal{H}_4$ that is a point of $Heis_2$. Given $x \in A[2]$, choose $\tilde{x} \in G$ with $\pi(\tilde{x}) = x$; then $\tilde{x}^2 \in \mu_2$. Note that if $\tilde{x}' \in G$ and $\pi(\tilde{x}') = x$, then $\tilde{x}' = u\tilde{x}$ for some $u \in \mu_2$, so that $\tilde{x}'^2 = \tilde{x}^2$. So there is a morphism $t_G : A[2] \to \mu_2$ defined by $t_G(x) = \tilde{x}^2$.

Verifying that $t_G(x + y)t_G(x)t_G(y) = e_2(x, y)$ for $x, y \in A[2]$ is immediate, and so there is a morphism $\alpha : Heis_2 \to T$ defined by $G \mapsto t_G$. We need to verify that $\alpha$ is $A[2]$-equivariant.

Let $\sigma_G : G \to \overline{G}$ and $\rho : \mu_4 \to \mu_4/Z_2$ be the quotients by $Z_2$, and $\overline{\sigma_G} : \overline{G} \to A[2]$ the induced isomorphism. Given a character $\chi : A[2] \to \mu_4/Z_2$, put

$$G_{\chi} = \{gs | g \in G, \ s \in \mu_4, \ \rho(s) = \chi \circ \overline{\sigma_G} \circ \sigma_G(g)\}.$$ 

Then $G_{\chi} \cap \mu_4 = Z_2$, so that $G_{\chi}$ is also a point of $Heis_2$. Since $G_{\chi}^\psi = (G_{\chi})^\psi$, we have an action of $A[2]$ on $Heis_2$. This is the action referred to above.
Let $p \in A[2]$ and pick $g \in G$ such that $\pi \sigma_G(g) = p$. Then $t_G(p) = g^2$ and $\pi \sigma_G(s) = p$ for some $s \in \mu_4$ with $\rho(s) = \chi \pi \sigma_G(g)$, so that $t_G, (p) = s^2g^2$. Since the composite homomorphism $\mu_4 \xrightarrow{\rho} \mu_4/Z_2 \cong \mu_2$ equals the homomorphism $\mu_4 \to \mu_2: s \mapsto s^2$, it follows that $\chi \pi \sigma_G(g) = s^2$. So $\rho(s) = \chi(p)$ and $t_G, (p) = \chi t_G$, as required.

The morphism $Arf : Heis_2 \to \mu_2$ induced by this isomorphism $\alpha : Heis_2 \to T$ is, over an algebraically closed field of characteristic $\neq 2$, the morphism that distinguishes between the two classes of extraspecial 2-groups of order $2^{1+2g}$.

2 Torelli’s theorem and Serre’s quadratic twist

A crude version of the Torelli theorem for curves states that, over an algebraically closed field, a curve can be recovered from its Jacobian. A more precise version is given by Oort and Steenbrink [OS]. We need some notation to state it. They used the language of level structures but we shall use that of stacks.

Recall that if $\mathcal{X}$ is an algebraic stack with finite stabilizers (for example, a Deligne–Mumford stack) then $[\mathcal{X}]$ denotes its coarse, or geometric, quotient.

On any pp abelian scheme $A \to S$, there is an involution $[-1]$. So $\mathbb{Z}/2$ acts on the moduli stack $A_g$. Stacks can be defined as equivalence classes of groupoids; from this point of view we define the quotient stack $\tilde{A}_g = A_g/(\mathbb{Z}/2)$ as follow. Recall that $A_g$ is the quotient $X/R$ where $X$ is the disjoint union of finitely many schemes of finite type over $\text{Spec} \mathbb{Z}$, $X$ is the base of an everywhere versal family of pp abelian schemes $(A, \lambda) \to X$, as in [FC], and $R \to X \times_{\text{Spec} \mathbb{Z}} X$ is the Isom scheme $R = \text{Isom}_{X \times X}(pr^*_1(A, \lambda), pr^*_2(A, \lambda))$. Then define $\tilde{R} = R/(-1)$ and $\tilde{A}_g = X/\tilde{R}$. The quotient morphism $\rho : A_g \to \tilde{A}_g$ is a commutative gerbe banded by $\mathbb{Z}/2$, so that, locally in the étale topology on $\tilde{A}_g$, $A_g \cong \tilde{A}_g \times B(\mathbb{Z}/2)$.

Equivalently, define the prestack $\text{pre}_{\tilde{A}_g}$ by $\text{Ob}(\text{pre}_{\tilde{A}_g}) = \text{Ob}(A_g)$ and

$$\text{Mor}_{\text{pre}_{\tilde{A}_g}}((A, \lambda), (B, \mu)) = \text{Mor}_{A_g}((A, \lambda), (B, \mu))/(-1).$$

Then $\tilde{A}_g$ is the stack associated to $\text{pre}_{\tilde{A}_g}$.

Assume that $g \geq 2$, and consider the jacobian morphism $j_g : \mathcal{M}_g \to A_g$. This is given by sending a curve $C$ of genus $g$ to either the psat $(\text{Pic}_C^{2g-1}, \Theta)$ or the ppav $(\text{Pic}_C^0, \lambda)$.

In a neighbourhood of a non-hyperelliptic curve, $j_g$ is isomorphic to a morphism $X/G \to Y/(G \times (\mathbb{Z}/2))$, of quotient stacks, where $X$ and $Y$ are smooth, of dimensions $3g-3$ and $g(g+1)/2$ respectively, while in a neighbourhood of a hyperelliptic curve $j_g$ is isomorphic to $X/H \to Y/H$ where $\mathbb{Z}/2$ is a normal subgroup of $H$. So there is a quotient stack $\pi : \mathcal{M}_g \to \mathcal{M}_g$, given locally by $X/G \xrightarrow{id} X/G$ or $X/H \to [X/(\mathbb{Z}/2)]/(H/(\mathbb{Z}/2))$ such that $\mathcal{M}_g$ is normal, and is relatively normal.
over $\text{Spec} \mathbb{Z}[1/2]$, and $\pi$ is an isomorphism on the non-hyperelliptic locus over $\text{Spec} \mathbb{Z}$. Moreover, there is a 2-commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}^{nh}_g & \xrightarrow{j_g} & \mathcal{A}^{irred}_g \\
\cong \downarrow & \pi \downarrow & \rho \downarrow \\
\tilde{\mathcal{M}}^{nh}_g & \xrightarrow{\tilde{j}_g} & \tilde{\mathcal{A}}^{irred}_g \\
\downarrow & \downarrow & \downarrow \\
M^{nh}_g & \xrightarrow{[j_g]} & \mathcal{A}^{irred}_g
\end{array}
$$

where, as usual, $M_g = [\mathcal{M}_g]$, $A_g = [\mathcal{A}_g]$, the superscripts $nh$ and $irred$ refer to non-hyperelliptic curves and geometrically irreducible ppav’s, respectively, and $\xrightarrow{\sim}$ denotes an open embedding. (A psat $(Y, \Theta) \to S$ is irreducible if and only if $\Theta$ is geometrically irreducible, in the sense that every geometric fiber $\Theta_s$ is irreducible; this is equivalent to the corresponding ppav $(A, \lambda) \to S$ being geometrically irreducible as a ppav.)

Suppose that $\mathcal{X}$ is a Deligne–Mumford stack over $\text{Spec} \mathbb{Z}$, that $\mathcal{Y}$ is a closed substack of $\mathcal{X}$ and that $\mathcal{Y}$ is smooth over $\text{Spec} \mathbb{Z}$. Then we say that $\mathcal{X}$ has Veronese singularities along $\mathcal{Y}$ if locally in the étale topology there is

1. a short exact sequence

$$1 \to \mathbb{Z}/2 \to G \to H \to 1$$

of finite étale groups;
2. a smooth $\mathbb{Z}$-scheme $Y$;
3. an equivariant action of $G$ on $Y \times \mathbb{A}^N \to Y$ that preserves $Y \times \{0\}$ such that $\mathbb{Z}/2$ acts trivially on $Y \cong Y \times \{0\}$ and freely on $Y \times (\mathbb{A}^N \setminus \{0\})$;
4. and an isomorphism $\mathcal{X} \to [(Y \times \mathbb{A}^N)/(\mathbb{Z}/2)]/H$ such that the composite morphism

$$\mathcal{Y} \to \mathcal{X} \to [(Y \times \mathbb{A}^N)/(\mathbb{Z}/2)]/H \to Y/H$$

is an isomorphism.

Let $\tilde{\mathcal{J}}_g$ denote the image of $\tilde{j}_g$. Then Oort and Steenbrink’s version of the Torelli theorem [OS] is this.

**Theorem 2.1 (Torelli)** (1) The morphisms $j_g : M_g \to A^{irred}_g$ and $\tilde{j}_g : M_g \to \tilde{A}^{irred}_g$ are finite and separate geometric points.
(2) $\tilde{j}_g$ induces an isomorphism of automorphism group schemes.
(3) $\tilde{j}_g$ is a closed embedding over $\text{Spec} \mathbb{Z}[1/2]$. 


(4) \( \tilde{j}_g \) is an embedding of the non-hyperelliptic locus \( \tilde{M}^{nh}_g \).

(5) \( j_g \) and \( \tilde{j}_g \) induce closed embeddings of \( M^h_g \) and \( \tilde{M}^h_g \), respectively. So \( \pi \) induces an isomorphism of the hyperelliptic loci.

(6) \( \tilde{M}_g \) has Veronese singularities along the hyperelliptic locus.

(7) At every closed point of \( \tilde{J}^h_g \), the Zariski tangent spaces of \( \tilde{J}_g \) and \( \tilde{A}_g \) coincide.

PROOF: Except for the statements concerning \( \tilde{A}_g \), this is nothing more than a translation of [OS]. The rest is then a simple observation.

This has concrete corollaries, as follows.

**Corollary 2.2** Over \( \mathbb{C} \), we can write \( A_g = \mathfrak{H}_g/Sp_{2g}(\mathbb{Z}) \) and \( \tilde{A}_g = \mathfrak{H}_g/PSp_{2g}(\mathbb{Z}) \), where \( \mathfrak{H}_g \) is the Siegel upper half-space of degree \( g \). The locus of period matrices in \( \mathfrak{H}_g \) (that is, the inverse image of \( \tilde{J}_g \) in \( \mathfrak{H}_g \)) has Veronese singularities along the hyperelliptic locus.

**Corollary 2.3** (Serre) Suppose that \( (A, \lambda) \) is a ppav over a field \( k \) and that \( K/k \) is a field extension for which there is a curve \( C \) over \( K \) such that the Jacobian \( JC \) is isomorphic to \( (A, \lambda) \otimes K \). Then there is a curve \( C_0 \) over \( k \) such that \( C \) is \( K \)-isomorphic to \( C_0 \otimes K \) and \( (A, \lambda) \) is \( k \)-isomorphic to the étale quadratic twist of \( JC_0 \) given by a unique quadratic character \( \epsilon \). If also \( C \) is hyperelliptic, then no twist is necessary.

Moreover, \( C_0 \) is unique up to a unique \( k \)-isomorphism.

Conversely, if \( C \) is a non-hyperelliptic curve and \( (A, \lambda) \) is a non-trivial quadratic twist of \( JC \), then \( (A, \lambda) \) is not a Jacobian.

In fact this can be stated slightly more generally, as follows.

**Corollary 2.4** Suppose that \( S \) is a normal scheme and that a family \( (A, \lambda) \to S \) of irreducible ppav’s over \( S \) is given, defining \( f : S \to A_g^{irred} \). Suppose also that there is a dense open subscheme \( i : S_0 \hookrightarrow S \) whose image in \( A_g^{irred} \) lies in the image of \( M^{nh}_g \).

Then there is a curve \( C \) over \( S \) such that \( JC \) is \( S \)-isomorphic to an étale quadratic twist of \( (A, \lambda) \to S \). This twist is trivial along the inverse image of the hyperelliptic locus in \( S \).

PROOF: We show first that there is a morphism \( h : S \to M_g \) such that \( \rho \circ f \) is isomorphic to \( \rho \circ j_g \circ h \).

Since \( M_g \) is the normalization of its image in \( A_g^{irred} \), \( S \) maps to \( M_g \) in \( A_g^{irred} \).

**Lemma 2.5** \( \tilde{M}_g \) is identified with the normalization of \( (M_g \times A_g \tilde{A}_g^{irred})_{red} \).

PROOF: Since \( \tilde{j}_g \) is finite the morphism \( \alpha : \tilde{M}_g \to M_g \times A_g^{irred} \tilde{A}_g^{irred} = M_g \times A_g \tilde{A}_g \) is finite. Since \( \alpha \) induces an isomorphism on geometric points and \( \tilde{M}_g \) is normal and \( \tilde{j}_g \) induces an isomorphism of stabilizers, the lemma follows.
Since $S$ is normal $S \to M_g$ factors through $\widetilde{M}_g$, say via $r : S \to \widetilde{M}_g$. Put
$$\mathcal{Y} = \widetilde{M}_g \times \widetilde{A}_g = \widetilde{M}_g \times \widetilde{A}_g^{irred}.$$ Then there is a 2-commutative diagram

Now $S_0 \hookrightarrow S \twoheadrightarrow \widetilde{M}_g$ factors through some morphism $h_0 : S_0 \to M_g^{nh}$, since $M_g^{nh} \to \widetilde{M}_g^{nh}$ is an isomorphism, so there is a second 2-commutative diagram

Since the composite $j_g = pr_2 \circ (\pi, j_g) : M_g \to \widetilde{A}_g^{irred}$ is finite, $(\pi, j_g)$ is also finite. Therefore $\widetilde{pr}_2$ is finite, and so, since $S$ is normal, $(h_0, i)$ extends to a morphism $\delta : S \to M_g \times \mathcal{Y}$. That is, there is a morphism $h : S \to M_g$ that extends $h_0$, and $\delta = (h, 1_S)$. So the previous diagram can be extended to a third 2-commutative diagram

Comparison of the first and third diagrams shows that
$$\rho \circ f \simeq \tilde{j}_g \circ pr_1 \circ (r, f) \simeq \tilde{j}_g \circ pr_1 \circ (r, f) \circ 1_S \simeq \tilde{j}_g \circ pr_1 \circ (\pi, j_g) \circ h \simeq \rho \circ pr_2 \circ (\pi, j_g) \circ h \simeq \rho \circ j_g \circ h,$$ as stated above.
Say \( h(S) = (C \to S) \) and \((B, \mu) = JC\). Put \( I = Isom_S((A, \lambda), (B, \mu)) \); then \( I \to S \) is finite and \((-1)\) acts freely on it. An isomorphism \( \rho \circ f \to \rho \circ j_g \circ h \) gives an \( S \)-point of \( I/(−1) \). Let \((X, \Theta)\) and \((Y, \Phi)\) be the psat’s corresponding to \((A, \lambda)\) and \((B, \mu)\), respectively; then \( I \cong Isom_S((X, \Theta), (Y, \Phi)) \). There are actions of \((-1) = \langle \iota \rangle \) on \((X, \Theta)\) and \((Y, \Phi)\) and \( I \); for \( \gamma \in I \), \( \iota(g) = [−1_B] \circ \gamma \).

Write \( \widetilde{I} = I/(\iota) \to S \). We know that \( \widetilde{I} \) has an \( S \)-point; fix one such, say \( S \to \widetilde{I} \), and put \( T = I \times_{\widetilde{I}} S \). This is a closed subscheme of \( I \) and gives

\[
T \times_S T = \gamma_1 \cup \gamma_2 \hookrightarrow I_T = Isom_T((X, \Theta)_T, (Y, \Phi)_T),
\]

where \( \gamma_{3−i} = [−1_B] \circ \gamma_i \).

Say \( Gal(T/S) = \langle \sigma \rangle \). Then \( \sigma \) acts on \((X, \Theta) \times_S T \) and \((Y, \Phi) \times_S T \) by \( \sigma(x, t) = (x, \sigma(t)), \sigma(y, t) = (y, \sigma(t)) \). So \( \sigma \circ \gamma_i = \gamma_{3−i} \circ \sigma \).

Let \((Y', \Phi')\) denote the quadratic twist of \((Y, \Phi)\) by \( T \to S \). Then \((Y', \Phi') \times_S T \cong (Y, \Phi) \times_S T \) and the action \( \ast \) of \( \sigma \) on \((Y, \Phi) \times_S T \) is given by \( \ast = [−1_B] \circ \sigma \). So

\[
\sigma \ast \circ \gamma_i = [−1_B] \circ \sigma \circ \gamma_i = [−1_B] \circ \gamma_{3−i} \circ \sigma = \gamma_i \circ \sigma.
\]

Therefore each \( \gamma_i \) descends to an \( S \)-isomorphism \( \delta_i : (X, \Theta) \to (Y', \Phi') \).

This completes the proof of Corollary 2.4. \( \square \)

A result of Welters [W] gives a geometrical description of Serre’s quadratic character \( \epsilon \), at least in characteristic zero or in genus three, as follows. Welters’ paper and many of those to which it refers assume that the base field is \( \mathbb{C} \); we have not checked whether this hypothesis is necessary.

Consider the subtraction map \( s : C \times C \to \text{Jac}^0 : (P, Q) \mapsto [P − Q] \). If \( C \) is non-hyperelliptic, then \( s \) is birational to its image \( \Sigma \), which has a unique singularity, the image of the diagonal \( \Delta \).

**Lemma 2.6** If \( C \) is non-hyperelliptic, then \( \Sigma \) is normal and \( C \times C \) is the blow-up of \( \Sigma \) at the origin.

**Proof:** Put \( \mathfrak{m} = \mathcal{I}_{J,0} \). It is clear that \( \mathfrak{m}.\mathcal{O}_{C \times C} \subset \mathcal{I}_\Delta \), where \( \mathcal{I}_\Delta \) is the ideal sheaf of \( \Delta \); it is enough to show that this is an equality.

Since the natural map

\[
s^*\Omega^1_J \to \Omega^1_{C \times C} = pr^*_1\Omega^1_C \oplus pr^*_2\Omega^1_C,
\]

induces \( \omega \mapsto (\omega, −\omega) \) at the level of global sections, where \( H^0(J, \Omega^1_J) \) is identified with \( H^0(C, \Omega^1_C) \), it follows that the homomorphism \( \mathfrak{m}/\mathfrak{m}^2 \to H^0(\Delta, \mathcal{I}_\Delta/\mathcal{I}_\Delta^2) = H^0(C, \Omega^1_C) \) is the identity. So \( \mathfrak{m}.\mathcal{O}_{C \times C} + \mathcal{I}_\Delta^2 = \mathcal{I}_\Delta \), and we are done, by Nakayama’s lemma. \( \square \)

In the natural \( 2\Theta \) linear system on \( JC \), let \( \Gamma_{00} \) be the linear subsystem consisting of those members that vanish to order at least 4 at the origin 0. Consider the intersection of the members of \( \Gamma_{00} \), a subscheme of \( JC \). Welters shows that, up to embedded points and some 0-dimensional material, this subscheme is
the surface $\Sigma$, and $C$ is recovered as the exceptional divisor over 0 in the minimal desingularization $S \to \Sigma$. Moreover, the normality of $\Sigma$ means that $S$ is just the blow up of $\Sigma$ at the origin, so this desingularization exists in families of non-hyperelliptic curves.

**Lemma 2.7** Suppose that $C$ is a curve of genus $g \geq 2$ and that $S = C \times C$. Then the only morphisms from $S$ to a curve whose generic fibre has arithmetic genus $\gamma \leq g$ are the two projections $pr_i : S \to C$.

**PROOF:** Suppose that the generic fibre $\phi$ of $q : S \to B$ has arithmetic genus $\gamma \leq g$ and that $f_i$ is a fibre of $pr_i$. Then, by the adjunction formula,

$$2g - 2 = K_S.\phi + \phi^2 = (2g - 2)(f_1 + f_2).\phi,$$

so that, without loss of generality, $f_1.\phi = 0$ and $\phi$ is a fibre of $pr_1$, so that $q = pr_1$. \hfill \Box

**Theorem 2.8** In characteristic zero Serre’s quadratic character $\epsilon$ equals that given by the Galois action on the pair of projections $pr_i : S \to C$, $i = 1, 2$.

**PROOF:** It is clear that the involution $[-1_A]$ exchanges the projections. \hfill \Box

We now verify this for curves of genus 3 in all characteristics.

**Proposition 2.9** Suppose that $C$ is a non-hyperelliptic curve of genus 3 over a field of any characteristic. Then $\Sigma$ is the unique member of $|2\Theta|$ on $A = \text{Jac}_C^0$ that vanishes to order at least 4 at 0. The curve $C$ is recovered by resolving the singularity $(\Sigma, 0)$ and $\epsilon$ has the same description as in Theorem 2.8.

**PROOF:** First, it is well known that in characteristic zero, $\Sigma \in |2\Theta|$. Then the same result holds in characteristic $p$, by specialization. Since $\text{mult}_0 \Sigma = 4$, because the singularity arises by contracting the plane quartic $\Delta$, $\Sigma$ lies in $\Gamma_{00}$, the subspace of 2nd order theta functions vanishing to order at least 4 at 0. It is enough to prove that $\dim \Gamma_{00} = 1$.

Suppose first that $p \neq 2$. Let $\Gamma_0$ be the space of 2nd order theta functions vanishing at 0; then $\dim \Gamma_0 = 7$ and every member of $\Gamma_0$ is even, so singular at 0. Since, in characteristic not 2, the second order thetas provide an embedding of the Kummer variety, the natural homomorphism $\Gamma_0 \to H^0(\mathbb{P}^2, \mathcal{O}(2))$ is surjective.

The kernel is $\Gamma_{00}$, so that $\dim \Gamma_{00} = 1$.

Now suppose that $p = 2$. Since $\mathcal{O}_A(2\Theta).\mathcal{O}_{C \times C} = \mathcal{O}_{C \times C}(pr_1^*K_C + pr_2^*K_C + 2\Delta)$ and $I_{A,0}.\mathcal{O}_{C \times C} = \mathcal{O}_{C \times C}(-\Delta)$, it follows that

$$I_{A,0}^4.\mathcal{O}_A(2\Theta).\mathcal{O}_{C \times C} = \mathcal{O}_{C \times C}(pr_1^*K_C + pr_2^*K_C - 2\Delta),$$

so it’s enough to show that $H^0(C \times C, \mathcal{O}_{C \times C}(pr_1^*K_C + pr_2^*K_C - 2\Delta)) = 0$.

There are exact sequences

$$0 \to \mathcal{O}(pr_1^*K_C + pr_2^*K_C - \Delta) \to \mathcal{O}(pr_1^*K_C + pr_2^*K_C) \to \mathcal{O}_\Delta(2K_\Delta) \to 0,$$
0 \to \mathcal{O}(pr_1^*K_C + pr_2^*K_C - 2\Delta) \to \mathcal{O}(pr_1^*K_C + pr_2^*K_C - \Delta) \to \mathcal{O}_\Delta(3K_\Delta) \to 0.

Taking $H^0$ of the first sequence identifies $H^0(C \times C, \mathcal{O}(pr_1^*K_C + pr_2^*K_C - \Delta))$ with the kernel of the natural multiplication map $H^0(C, \mathcal{O}(K_C)) \otimes H^0(C, \mathcal{O}(K_C)) \to H^0(C, \mathcal{O}(2K_C))$, so with $\bigwedge^2 H^0(C, \mathcal{O}(K_C))$. (Recall that, in any characteristic, the kernel of the natural projection $\otimes^2 V \to \text{Sym}^2 V$ is identified with $\bigwedge^2 V$ for any finite-dimensional vector space $V$.)

Taking $H^0$ of the second sequence then identifies the vector space

$$H^0(C \times C, \mathcal{O}_{C \times C}(pr_1^*K_C + pr_2^*K_C - 2\Delta))$$

with the kernel of the Wahl-Gauss map

$$\phi : \bigwedge^2 H^0(C, \mathcal{O}(K_C)) \to H^0(C, \mathcal{O}(3K_C)) : s \wedge t \mapsto sdt - tds.$$

Note that, because $h^0(C, \mathcal{O}(K_C)) = 3$, every element of $\bigwedge^2 H^0(C, \mathcal{O}(K_C))$ is of the form $s \wedge t$. Fix $0 \neq \omega \in H^0(C, K_C)$, so that if $s, t \in H^0(C, K_C)$, then $s = f\omega$ and $t = g\omega$ for rational functions $f, g$ on $C$. Then $s \wedge t \in \ker \phi$ if and only if $\frac{df}{f} = \frac{dg}{g}$.

Now $(f) = D - F$ and $(g) = E - F$, with $D, E, F \in |K_C|$ and $F = (\omega)_0$. Since $C$ is a double cover of its Frobenius twist $C^{(1)}$, there are rational functions $p, q, r, s, t$ on $C$ such that $f = p^2 + q^t$ and $g = r^2 + s^t$; then $df/f = dg/g$ if and only if $r/s = p/q$. This implies that $f/g$ is a square; say $f/g = h^2$, with $h \in k(C)$. Then there is an equality $2(h) = D - E$ of divisors on $C$; since $C$ is non-hyperelliptic, this leads at once to a contradiction. \hfill \Box

**Corollary 2.10** When $g = 3$ the commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_{3}^{\text{nh}} & \xrightarrow{\cong} & \mathcal{A}_3^{\text{irred}} \\
\downarrow & & \downarrow \rho \\
\tilde{\mathcal{J}}_{3}^{\text{nh}} & \xrightarrow{\rho} & \tilde{\mathcal{A}}_{3}^{\text{irred}}
\end{array}
$$

shows that the gerbe $\rho : \mathcal{A}_3 \to \tilde{\mathcal{A}}_3$ is neutral over the open substack $\tilde{\mathcal{J}}_{3}^{\text{nh}}$, so that the stack $\mathcal{A}_3^{\text{smooth}}$ of 3-dimensional psat’s whose theta divisor is smooth is isomorphic to $\mathcal{M}_3^{\text{nh}} \times B(\mathbb{Z}/2)$. However, $\rho$ is not neutral over $\tilde{\mathcal{A}}_{3}^{\text{irred}}$ since $\mathcal{M}_3 \to \tilde{\mathcal{J}}_3$ is not an isomorphism in any neighbourhood of the hyperelliptic locus.

### 3 Tropes in genus 3

*In this section the characteristic is not 2.*

Suppose that $(\mathcal{Y}, \Theta)$ is a psat of dimension $g$ and that $(A, \lambda)$ is the corresponding ppav.
Assume also that \((Y, \Theta)\) is irreducible as a psat (equivalently, that \((A, \lambda)\) is irreducible as a ppav). Then the complete linear system \(|2\Theta|\) embeds the Kummer variety \(Km(Y) := [Y/\sim] \) into \(\mathbb{P}^{2g-1}\).

There is a unique hyperplane \(H\) in \(\mathbb{P}^{2g-1}\) such that \(H. Km(Y) = 2X\), where \(X = [\Theta/\sim] \). Both \(H\) and \(X\) are known as the trope either of \(Y\) or of \(Km(Y)\). Also, the 2-torsion subgroupscheme \(A[2] \) of \(A\) acts projectively on \(Km(Y) \subset \mathbb{P}^{2g-1}\), and every translate of \(H\) or \(X\) by a geometric point in \(A[2]\) is also called a trope. However, we focus on psat’s rather than ppav’s, and then there is a well defined choice of trope, as above. The singular locus of \(Km(Y)\) is exactly the image of \(Fix_Y\) and the singular locus of \(X\) is, if \(g \geq 3\), exactly the image of \(Sing \Theta \cup (\Theta \cap Fix_Y)\).

If \(C\) is a curve of genus \(g\) and \(Y = Jac^{g-1}_C\), so that \(A = Jac^0_C\), then we write \(Km(C)\) instead of \(Km(A)\).

When \(g = 2\), that is, when \(A\) is the Jacobian of a genus 2 curve \(C\), then the trope is a conic \(\Gamma\) passing through 6 of the 16 nodes (ordinary double points, of type \(A_1\)) on \(Km(A)\), and \(C\) can be recovered, up to a quadratic twist, as the double cover of \(\Gamma\) ramified in the 6 points \([Hu]\). Note that the ambiguity concerning \(C\) in this quadratic twist is the same ambiguity in recovering \((Y, \Theta)\) from the Kummer surface and its trope. In fact, it is enough just to know the position of the 6 nodes in \(\mathbb{P}^3\), since then \(\Gamma\) is the unique conic through them; it is the intersection of the quadrics in \(\mathbb{P}^3\) through them.

We now show that in genus 3 things are similar.

Recall that, by definition, an index 1 del Pezzo surface is a reduced and irreducible Gorenstein surface \(S\) whose anti-canonical class \(-K_S\) is ample. Its degree \(deg(S)\) is \(K^2_S\). Normal del Pezzo surfaces \(S\) of index 1 can be divided into two classes, as follows, where \(\widetilde{S} \to S\) is the minimal resolution.

1. Those with only rational double points (RDPs). In this case either \(deg(S) = 8\) and \(S\) is a quadric or \(1 \leq deg(S) \leq 9\) and \(\widetilde{S}\) is a blow-up of \(\mathbb{P}^2\) in \(9 - deg(S)\) points.

2. Those with a simply elliptic singularity. The degree may be any positive integer \(d\) and \(\widetilde{S}\) is a \(\mathbb{P}^1\)-bundle over an elliptic curve \(E\). The exceptional locus of \(\widetilde{S} \to S\) is a section \(E_0\) of the bundle and \(d = -E_0^2\); \(S\) can be regarded as the cone over an elliptic curve \(E\) of degree \(d\).

If \(S\) is a del Pezzo surface of index 1 and degree 2, then \(|-K_S|\) has no base points and defines \(S\) as a double cover of \(\mathbb{P}^2\) branched in a quartic.

We shall also need to consider normal del Pezzo surfaces of degree 2 and of index 2: for our purposes, these are defined as normal surfaces \(S\) that are quadric sections of the cone \(\tilde{V}\) in \(\mathbb{P}^6\) over a Veronese surface \(V\) and that contain the vertex \(v\) of \(\tilde{V}\). For any such surface \(-2K_S\) is a very ample Cartier divisor, \(K^2_S = 2\), and \(S\) has a single non-Gorenstein singularity, at \(v\).

Recall the following result about tropes in genus 3; see, for example, Remark 6 on p. 189 of [DO]
Proposition 3.1 Suppose that $C$ is a non-hyperelliptic curve of genus 3 and that $X$ is a trope of $C$.

(1) $X$ is a surface whose singular locus $Z$ consists of 28 nodes. Its canonical class $K_X$ is ample, $K_X^2 = 3$, $p_g(X) = 3$, $q(X) = 0$ and the complete linear system $|K_X|$ has no base points. The canonical ring $R(X)$ is a hypersurface $k[x_1, x_2, x_3, y]/(f)$, where $\deg x_i = 1$, $\deg y = 2$ and $\deg f = 6$.

(2) The embedding $X \to H \cong \mathbb{P}^5$ is defined by $|2K_X|$.

(3) The intersection of the quadrics containing $X$ in $H$ is a copy $\tilde{V}$ of the cone over a Veronese surface $V$ in $\mathbb{P}^5$.

(4) The vertex $v$ of $\tilde{V}$ does not lie on $X$ and the morphism $X \to V \overset{\alpha}{\to} \mathbb{P}^2$ given by composing the projection from $v$ with an isomorphism $\alpha$ is defined by $|K_X|$.

Remark: Horikawa showed [Ho] that, in characteristic zero if $X$ is any canonical surface with $K_X^2 = 3$, $p_g(X) = 3$, $H^1(X, \mathcal{O}_X) = 0$ and whose canonical system $|K_X|$ has no base points, then $R(X)$ is as described. Ekedahl’s results [Ek] are enough to show that Horikawa’s argument applies to prove this in all characteristics.

Theorem 3.2 (1) The intersection in $H$ of the quadrics through $Z$ is a normal del Pezzo surface $S$ of degree 2, of index 1 or 2 and embedded in $H$ by $|−2K_S|$.

(2) If $\text{char} \ k \neq 3$, then $v \notin S$ and $S$ is of index 1.

(3) If $\text{char} \ k = 3$, then $v \in S$ and $S$ is of index 2.

(4) The curve $C$ is the normalization of the curve $R = X \cap S$. Moreover, $R$ is the unique anti-sexcanonical curve lying in $|−6K_S − 2Z|$. That is, $R$ is the unique anti-sexcanonical curve and singular at $Z$.

Proof: Let $\pi : Y = \text{Pic}_C^2 \to \text{Km}(C)$ be the quotient morphism and consider the canonical morphism $\alpha = \phi_{|K_X|} : X \to \mathbb{P}^2$, which we identify with projection from $v$ after fixing an isomorphism of the Veronese surface $V$ with $\mathbb{P}^2$. According to Andreotti, $\alpha$ is separable and $\deg \alpha = \frac{1}{2}(2g−2) = 3$.

Suppose that $R \subset X$ is the ramification divisor and $B \subset \mathbb{P}^2$ the branch curve, the image of $R$, each given its reduced structure. Put $\tilde{R} = \pi^{-1}(R)$. Then, by Andreotti’s argument, $\tilde{R} = R_1 + R_2$ where $R_1$ is the set of divisor classes $\mathcal{O}(p + q)$ such that the line $(p, q)$ is tangent to $C$ at a third point $r$, so that $K_C − p − q \sim 2r$, and $R_2 = \{\mathcal{O}(2r) | r \in C\}$, the image of the diagonal. The involution $\iota = [−1_A]$ exchanges $R_1$ and $R_2$, so that each morphism $R_i \to R$ is an isomorphism outside the fixed locus of $\iota$.

The doubling morphism $d : C \to R_2 : r \mapsto \mathcal{O}(2r)$ fits into a commutative diagram

$$
\begin{array}{ccc}
C & \xrightarrow{d} & R_2 \\
\downarrow & & \downarrow \\
\text{Pic}_C & \xrightarrow{[2]} & \text{Pic}_C^2
\end{array}
$$
where the vertical arrows are injective and [2] is étale, and so separates tangent vectors. Since $d$ separates geometric points ($C$ is not hyperelliptic) $d : C \to R_2$ is an isomorphism.

Therefore the composite $\nu : C \to R_2 \to R$ is an isomorphism except that $\nu(P) = \nu(Q)$ if $P + Q$ is a bitangent and in this case $R$ has a node at $\nu(P)$, and $R$ has a unibranched singularity at $\nu(P)$ if $P$ is a hyperflex of $C$. So $R$ is smooth outside $Z$ and has points of multiplicity 2 everywhere on $Z$. Moreover, $\nu$ is birational, so that we shall be able to recover $C$ as the normalization of $R$.

Note that, if $C$ is general, then it has no hyperflexes, so that $R$ has only nodes and is smooth outside $Z$. To see this, regard $C$ as the branch locus of a general del Pezzo surface of degree 2.

As Andreotti proved, it follows that $B$ is the projective dual of the plane quartic $C$.

Recall that if $B$ is the dual curve of $C$ and $\widetilde{B}$ is the normalization of $B$, then $C \to \widetilde{B}$ is some iterate of the Frobenius, so that $pg(B) = pg(C) = 3$ and

$$12 = p^r \deg B$$

for some power $p^r$ of char $k$. So either $\deg B = 12$, in which case the morphisms $C \to R$ and $R \to B$ are both birational, or $\deg B = 4$, $p = 3$, $C \to R$ is birational and $R \to B$ is birationally equivalent to the Frobenius, so that $\deg(R \to B) = 3$. ($If p = 2 then p^r \geq 2$, so that $\deg B = 6.$)

**Lemma 3.3** If $C \to B$ is not birational, then $p = 3$ and $C$ is isomorphic to the Fermat quartic.

**Proof:** Assume that $C \to B$ is not birational. If $C$ is defined by $f = 0$, then all the first partial derivatives of $f$ are $p$th powers, so $p = 3$ and the derivatives are cubes of linear forms. Let $V_d$ denote the space of ternary $d$-ics, regarded as a representation of $GL_3$. Then $f$ lies in the sub-representation $W = V_1 \otimes V_1^{(1)}$ of $V_4$. Since $C$ is smooth, the non-vanishing of the discriminant and the finiteness of the automorphism group show that $f$ is a semi-stable point of $W$ whose stabilizer, modulo the centre of $GL_3$, is finite. So $f$ is a stable point. Since dim $W = \dim GL_3$, there is only one stable orbit.\[\Box\]

Moreover, the Plücker formulae give the following result.

**Lemma 3.4** Suppose that $C$ has $\delta_1$ ordinary bitangents, $\delta_2$ hyperflexes and $\kappa$ flexes.

(1) $\delta_1 + \delta_2 = 28$.

(2) The ordinary bitangents give nodes on $B$.

(3) If char $k \neq 3$ then $2\delta_2 + \kappa = 24$. The hyperflexes give unibranched singularities whose value semigroup is generated by $\{3, 4\}$ and the flexes give unibranched singularities whose value semigroup is generated by $\{2, 3\}$.

(4) If char $k = 3$ and $C$ is not the Fermat quartic then $\delta_2 + \kappa = 8$. The hyperflexes give unibranched singularities whose value semigroup is generated
by \{3, 5\} and the flexes give unibranch singularities whose value semigroup is generated by \{3, 4\}. In particular, the non-nodal singularities of \(B\) are of multiplicity 3.

**PROOF:** (1) is the basic result about odd theta characteristics and the rest is based on the fact that, if \(v(t) = (X : Y : Z)\) is a local parametrization of \(C\), then \(v(t) \wedge v'(t)\) is a parametrization of \(B\).

Let \(\ell\) be the class of a line in \(\mathbb{P}^2\). So

\[
\alpha^* \ell \sim K_X \sim \alpha^* K_{\mathbb{P}^2} + e'R
\]

for some \(e' \geq 1\), so that \(4\alpha^* \ell \sim e'R\) and \(e'(R.\alpha^* \ell) = 12\). Moreover, \(R.\alpha^* \ell = \deg(R \to B).\deg B\), so that \(e' = 1\) or 3.

Suppose first that \(e' = 3\). Then \(\deg B = 4\) and \(B\) is smooth, and \(R \to B\) is birational, so that \(R\) is also smooth; this is a contradiction.

Therefore \(e' = 1\). Then \(R\) is a quadric section of \(X\) and \(R.\alpha^* \ell = 12\). Since \(R\) is singular at every point of \(Z\) and \(\deg Z = 28\), a manipulation of intersection numbers on the minimal resolution \(\tilde{X}\) of \(X\) shows that \(R\) is the unique member of \(|4K_X - Z|\) (the point is that \((4K_X)^2 < 28.2\)). That is, \(R\) is the unique quadric section of \(X\) passing through \(Z\).

Regard \(X\) as a Cartier divisor on the 3-fold \(\hat{V}\) and let \(\mathcal{O}(1)\) denote the line bundle defining the embeddings of \(X\) and \(\hat{V}\) into \(H\). So \(\mathcal{O}_X(1) = \alpha^* \mathcal{O}_{\mathbb{P}^2}(2)\). Then \(H^0(\hat{V}, \mathcal{O}_{\hat{V}}(2)) \to H^0(X, \mathcal{O}_X(2))\) is an isomorphism, so that \(R\) is cut out on \(X\) by a unique member \(S\) of \(|\mathcal{O}_{\hat{V}}(2)|\). Then \(S\) is the unique member of \(|\mathcal{O}_{\hat{V}}(2)|\) that passes through \(Z\), and so is the intersection in \(H\) of the quadrics through \(Z\).

The reducedness and irreducibility of \(S\) follow from the fact that \(R = S.X\) is a reduced, irreducible and ample Cartier divisor on \(S\).

**Lemma 3.5** \(S\) is smooth along \(R\).

**PROOF:** \(R\) is smooth outside \(Z\) and is Cartier on \(S\), so \(S\) is smooth along \(R - Z\). At points of \(Z\) the multiplicity of \(R\) is 2 and the multiplicity of \(X\) is 2. So \(S\) has multiplicity 1 along \(Z\).

**Corollary 3.6** \(S\) is normal.

**PROOF:** \(S\) is Cartier on \(\hat{V}\), so Cohen–Macaulay, so satisfies Serre’s condition \((S_2)\) and is smooth along an ample Cartier divisor, so satisfies \((R_1)\).

**Lemma 3.7** (1) \(S\) contains \(v\) if and only if \(\text{char } k = 3\).

(2) \(S\) is a degree 2 del Pezzo surface embedded by \(|-2K_S|\). Its index is 2 if \(\text{char } k = 3\) and 1 if \(\text{char } k \neq 3\).

**PROOF:** Write \(\hat{V} = \text{Proj } k[x_1, y_1, z_1, w_2]\), where the suffixes indicate the degrees of the variables. Then \(X\) is defined in \(\hat{V}\) by the vanishing of a homogeneous sextic polynomial

\[
F = w^3 + w^2 f_2(x, y, z) + wg_4(x, y, z) + h_6(x, y, z).
\]
It follows that $S$ is defined in $\hat{V}$ by the vanishing of the homogeneous quartic
\[ \frac{\partial F}{\partial w} = 3w^2 + 2wf + g, \]
so that $v \in S$ if and only if $\mathrm{char} \ k = 3$.

Finally, consider $R$ as a curve on $S$. Since $R = S.X$ and is singular at $Z$, it is a member of $|O_S(3) - 2Z|$; i.e., it is a member of $|-6K_S - 2Z|$. Since $(6K_S)^2 < 4.28$, $R$ is the only member of this linear system. We have already remarked that $C$ is the normalization of $R$.

This concludes the proof of Theorem 3.2.

**Proposition 3.8** Assume that $\mathrm{char} \ k \neq 3$.

(1) $C_1$ is the unique quartic in the dual plane passing through the 24 cusps of the dual curve $B$ of $C$.

(2) In symbolic terms [GY], if $C = a_x^4 = b_x^4 = c_x^4$, then $C_1$ is given by $C_1 = (abu)^4$.

**Proof:** We have already observed that $C_1$ is the unique quartic passing through the 24 cusps of $B$.

Define contravariants $K_1$ and $K_2$ by
\[ K_1 = (abu)^4, \quad K_2 = (abu)^2(acu)^2(bcu)^2; \]
then it is well known that $B = K_1^3 - 6K_2^2$. Here is a proof of this formula.

Consider a binary quartic $\alpha_y^4 = \beta_y^4 = \gamma_y^4$. Its ring of invariants is generated by $S$ and $T$, where $S = (\alpha\beta)^4$ and $T = (\alpha\beta)^2(\alpha\gamma)^2(\beta\gamma)^2$, and its discriminant is $S^3 - 6T^2$. Therefore (this is the Clebsch transfer principle) $K_1$ is the locus of lines $L$ such that $S(L \cap C) = 0$, $K_2$ is the locus such that $T(L \cap C) = 0$ and, by definition, $B$ is the locus such that $L \cap C$ is singular; it follows that $B = K_1^3 - 6K_2^2$.

In particular, it is clear from this formula that $B$ has cusps at every point of $K_1 \cap K_2$ and that the intersection $K_1 \cap K_2$ is transverse (otherwise $B$ would have a singularity worse than a cusp, which is not the case). So $K_1 \cap C_1$ contains at least 24 points, while $K_1.C_1 = 16$. So $K_1 = C_1$, as required.

**Remark:** For example, if $\mathrm{char} \ k \neq 2, 3$ and $C$ is the curve
\[ X_1^3X_2 + X_2^4 + X_3^4 = 0, \]
then $C_1$ consists of 4 lines through the point $(0, 1, 0)$ in the dual plane, so that the del Pezzo surface $S$ has a simply elliptic singularity. This shows that the del Pezzo surface $S$ is not enough to determine $C$, as follows.

Suppose that $C$ is a contravariant of $C_1$. Every invariant of $C$ is then an invariant of $C_1$, so vanishes, since $C_1$ is unstable. So $C$ is also unstable, contradiction.

**Remark:** The expanded form of the symbolical expression for $C_1$ is as follows. For non-negative integers $d, j_1, ..., j_r$, write $j = (j_1, ..., j_r)$ and let $\binom{d}{j}$ denote
the multinomial coefficient \( \binom{d}{j} = \frac{d!}{j! \cdots j!} \) if \( \sum j_i = d \) and \( \binom{d}{j} = 0 \) otherwise. If \( C = \sum (i) A_i x^i \), where \( i = (i_1, i_2, i_3) \) and \( x = (x_1, x_2, x_3) \), then

\[
C_1 = \sum_{\ell} \left( \binom{4}{\ell} \right) (-1)^{\ell_2+\ell_4+\ell_6} A_{\ell_1+\ell_2+\ell_3+\ell_4+\ell_5+\ell_6} A_{\ell_4+\ell_5+\ell_1+\ell_2+\ell_3} u_{\ell_3+\ell_6+\ell_2+\ell_5+\ell_1+\ell_4}
\]

where \( \ell = (\ell_1, \ldots, \ell_6) \) and \( u = (u_1, u_2, u_3) \).

**Remark:** As mentioned above, Lehavi [L] has already given an explicit way of recovering a smooth plane quartic \( C \) from its bitangents. The comparison between his result and the one given above is this: suppose given the locus \( Z \) of 28 points in the trope \( H = \mathbb{P}^6 \). Lehavi’s result says that, if there is given also the vertex \( v \) of the Veronese cone \( \tilde{V} \) that contains \( X \), then there is an explicit procedure for recovering \( C \) from the image of \( Z \) in the Veronese surface \( V \) that arises under projection from \( v \). On the other hand, our result does not demand the knowledge of \( v \), and it is not clear how to determine \( v \) from knowledge solely of the 28 points in \( \mathbb{P}^6 \).

Now suppose that \( C \) is hyperelliptic. Most of the details are similar to those in the non-hyperelliptic case. However, they are slightly more intricate because the surfaces \( X \) and \( S \) now pass through the vertex \( v \) of the Veronese cone \( \tilde{V} \) and have non-Gorenstein singularities there. Recall that a singularity \( X \) of type \([-4]\) is a normal surface singularity the exceptional locus of whose minimal resolution \( \pi : \tilde{X} \to X \) consists of one smooth rational curve \( E \) with \( E^2 = -4 \). Such a singularity is a quotient \( X = \mathbb{A}^2 / \mu_4 \), so rational, \( 2K_X \) is Cartier and \( 2K_X \sim \pi^*(2K_X) - E \).

**Theorem 3.9** Suppose that \( C \) is a hyperelliptic curve of genus 3 and \( X \) its trope.

1. \( X \) is a surface whose singular locus is a set \( Z \) of 28 nodes and one further point \( v \) at which it has a singularity of type \([-4]\).

2. Let \( \pi : \tilde{X} \to X \) be the minimal resolution of the singular point \( v \). Then \( K_{\tilde{X}} \) is ample, \( \tilde{X} \) has just 28 nodes, \( K_{\tilde{X}}^2 = 2 \), \( p_g(\tilde{X}) = 3 \), \( q(\tilde{X}) = 0 \) and \( |K_{\tilde{X}}| \) has no base points. The canonical morphism \( \phi = \phi|_{K_{\tilde{X}}} : \tilde{X} \to \mathbb{P}^2 \) is of degree 2 and the branch locus \( B \) in \( \mathbb{P}^2 \) is the union of 8 lines \( B_i \) that are tangent to a common conic \( D \). The bicanonical map of \( \tilde{X} \) is identified with the projection of \( X \) from \( v \) to \( V \).

3. The composite morphism \( \tilde{X} \to X \to H \cong \mathbb{P}^6 \) is defined by the complete linear system \( |2K_{\tilde{X}} + E| \).

4. \( X \) lies in a copy \( \hat{V} \) of a cone, with vertex \( v \), over a Veronese surface \( V \).

5. The intersection of the quadrics containing \( X \) is \( \hat{V} \) and \( X \) is a cubic section of \( \hat{V} \). In particular, \( X \) is a Cartier divisor on \( \hat{V} \).

6. The intersection of the quadrics through \( Z \) is a normal surface \( S \) that passes through \( v \) and has a singularity of type \([-4]\) there. It is a del Pezzo surface of index 2 and degree 2. It is obtained by first taking the blow-up \( \tilde{S} \to \mathbb{P}^2 \) of \( \mathbb{P}^2 \). In
8 distinct points $P_1, \ldots, P_8$ that lie on a conic $\overline{D}$ and then contracting the strict transform $D$ of $\overline{D}$. The rational map $\mathbb{P}^2 \to \mathbb{P}^6$ is defined by the linear system $|4L - \sum P_i|$ of quartics through the $P_i$.

(7) The embedding $S \in H \cong \mathbb{P}^6$ is defined by $| - 2K_S|$.

(8) $X \cap S$ is the union of 8 twisted cubics $\overline{R}_i$, each passing through $v$. As curves on $S$, each $\overline{R}_i$ is the strict transform of the tangent line to $\overline{D}$ at $P_i$ under the birational map referred to in (6).

(9) The divisor $\sum \overline{R}_i$ is Cartier on $S$ and is the unique member of the linear system $|3H - 2Z|$.

(10) The curve $C$ is recovered by blowing up $S$ at $v$ to get an exceptional curve $E$ and then taking the double cover of $E$ ramified at the points where $E$ meets the strict transforms of the curves $\overline{R}_i$.

PROOF: By the Riemann–Kempf singularity theorem, $\Theta$ has a node $P$ at the point in $A[2]$ corresponding to the half–canonical $g^1_2$ on $C$. There are three possible quotients of a node $(\Theta, P)$ by $[-1]$, namely, a smooth point, an RDP of type $A_3$ and a $[-4]$ singularity, and they are distinguished by whether $[-1]$ acts on the Zariski tangent space as a diagonal matrix $(-1,1,1), (-1,-1,1)$ or $(-1,-1,-1)$. Since $[-1]$ acts on the tangent space $T_P A$ as $(-1)$, it follows that the quotient $(X, v) = (\Theta, P)/[-1]$ is a $[-4]$ singularity. This is not Gorenstein, but since $[-1]$ is an involution $2K_X$ is Cartier. The remaining singularities of $X$ are a set $Z$ of 28 nodes, as before.

Let $\pi : \Theta \to X$ be the quotient. Since $[-1]$ acts freely in codimension one, $\pi^*(2K_X) = 2K_\Theta$. Moreover, exactly as in the non-hyperelliptic case, the morphism $\Theta \to H = \mathbb{P}^6$ induced by $|2\Theta|$ factors through $X$ and $X \to H$ is defined by $|2K_X|$ and is an embedding. Also, the Gauss map on $\Theta$ is again, tautologically, defined by $|K_\Theta|$ and factors through $X$. Since $X$ has just rational singularities, it follows that $p_g(\widetilde{X}) = 3$.

From the description of the singularities on $X$, we get $2K_{\widetilde{X}} \sim \pi^*(2K_X) - E$. Suppose that $X^\dagger \to \widetilde{X}$ is the minimal resolution and that $F$ is a $(-1)$-curve in $X^\dagger$. Then $K_{X^\dagger}.F = -1$ and $\pi^*(2K_X).F > 0$, so that $F.E \geq 3$. However, contracting $F$ gives a curve, the image of $E$, that cannot live on a surface with $p_g > 0$. Hence $\widetilde{X}$ is minimal, with $p_g = 3$ and $K^2 = 2$. The classification of such surfaces combined with Andreotti’s proof of Torelli in the hyperelliptic genus 3 case gives (1) - (3).

We next verify that $X$ is a cubic section of $\widehat{V}$; this will complete the proof of (4) to (5).

Let $\mathcal{O}(1)$ denote the hyperplane bundle on $\mathbb{P}^6$. Then $\mathcal{O}_{\widetilde{X}}(3) = \mathcal{O}_{\widetilde{X}}(6K_{\widetilde{X}} + 3E)$, so that, by Riemann–Roch, $\chi(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(1)) = 49$. Considering exact sequences of the form

$$0 \to \mathcal{O}_{\widetilde{X}}(6K_{\widetilde{X}} + (n - 1)E) \to \mathcal{O}_{\widetilde{X}}(6K_{\widetilde{X}} + nE) \to \mathcal{O}_E(6K_{\widetilde{X}} + nE) \to 0$$

shows that $H^1(\widetilde{X}, \mathcal{O}(6K_{\widetilde{X}} + 3E)) = 0$, so that $h^0(X, \mathcal{O}_X(3)) = 49$. Since $\widehat{V} \cong \mathbb{P}^6$.
\(\mathbb{P}(1, 1, 1, 2)\), it follows that \(h^0(\hat{V}, \mathcal{O}(3)) = 50\). So \(X\) lies in a non-trivial cubic section of \(\hat{V}\). Since \(\deg(X) = 12 = 3\deg(\hat{V})\), \(X\) is a cubic section of \(\hat{V}\). In particular, it is a Cartier divisor there.

Suppose that \(R \subset \hat{X}\) is the ramification curve of \(\alpha : \hat{X} \to \mathbb{P}^2\). Since \(2R = \alpha^*B\), it follows that \(R = \sum R_i\) with \(2R_i = \alpha^*B_i\). Since \(B\) is tangent to \(D\), we have \(\alpha^{-1}(D) = E + E'\), where \(E\) and \(E'\) are exchanged by \([-1]\). So \(E^2 = E'^2 = -4\) and \(E.E' = 8\). Also, if \(L\) is a line in \(\mathbb{P}^2\), then \(K_{\hat{X}} \sim \alpha^*L\) and

\[
2 = \alpha^*B_i.\alpha^*L = 2R_i.K_{\hat{X}},
\]

so that \(R_i.K_{\hat{X}} = 1\).

Since \(2R_i.(E + E') = \alpha^*B_i.\alpha^*D = 4\) and \(R_i.E = R_i.E'\), by symmetry under \([-1]\), we get \(R_i.E = 1\). Hence \(R_i\) maps to a twisted cubic in \(\mathbb{P}^6\). Also, \(2R \sim \alpha^*(8L)\), so that \(R + 2E \sim 2\pi^*H|_\mathbb{P}^6\). Define \(\overline{R}\) to be the image of \(R\) in \(\mathbb{P}\); then \(\overline{R} \sim 2H|_X\). That is, \(\overline{R}\) is a quadric section, passing through \(Z\) and \(v\).

Suppose that \(T\) is another quadric section of \(X\) through \(Z\), not necessarily through \(v\). Then in \(\hat{X}\) there are two members of \(|2K_{\hat{X}} + E - Z|\), namely \(R + 2E\) and \(\pi^*T\). Since \((2K_{\hat{X}} + E)^2 = 12\) and smooth curves meeting transversely at a node have local intersection number \(1/2\) there, we have \(\overline{R} \sim T\). So there is a unique quadric section \(S\) of \(\hat{V}\) such that \(X.S = R\) in \(\hat{V}\).

Hence \(v \in S\) and \(S\) is the intersection in \(\mathbb{P}^6\) of the quadrics through \(Z\).

We show next that \(S\) is irreducible. For this, let \(\gamma : \tilde{V} \to \hat{V}\) be the blow-up at \(v\), with exceptional divisor \(V_0 \cong V\). Let \(\tilde{X}, \tilde{S}\) be the strict transforms of \(X, S\). Then \(\gamma^*X \sim \tilde{X} + E\) and \(\tilde{X}.V_0 = E\). The strict transform \(\tilde{R}_i\) of \(R_i\) lies on \(X\) and on \(S\), so that \(\tilde{R}_i \subset \tilde{X} \cap \tilde{S}\) and \(\tilde{R}_i\) meets \(E\). Hence \(E\) and \(S\) have at least \(8\) points in common.

We have \(\gamma^*S = \tilde{S} + aV_0\) for some \(a \geq 1\). Now \(E.V_0 = -4\), since \(E \leftrightarrow V_0 \cong \mathbb{P}^2\) realizes \(E\) as a conic, so \(0 = \tilde{S}.E = 4a\).

Suppose that \(E\) is not contained in \(\tilde{S}\); then \(E.\tilde{S} \geq 8\), so that \(a \geq 2\). We know, by looking at \(X\), that projection \(\beta\) from \(v\) maps \(R_i\) to a line in \(V \cong \mathbb{P}^2\), so \(\beta(S)\) contains at least \(8\) distinct curves. Let \(\phi\) denote a fibre of the projection \(\hat{V} \to V\). Then \(\phi.V_0 = 1\) and \(\phi\) maps to a line in \(\mathbb{P}^6\), so that \(\phi.\gamma^*S = 2\). So, if \(a \geq 2\), then \(\phi.\tilde{S} = 0\). That is, \(\beta(S) \neq V\). Then \(S\) is of degree \(8\) and has at least \(8\) components. However, \(\hat{V}\) contains no planes, and so \(a = 1\) and \(\phi.\tilde{S} = 1\). Moreover, \(E\) is contained in \(\tilde{S}\).

So if \(S\) is reducible, then \(S = S_1 + S_2\) with \(S_1\) irreducible, \(v \in S_2\), \(S_1\) is a linear section of \(\tilde{V}\) and \(S_2\) is a cone. So \(S_1 \cong V\) and \(S_2\) is a quartic cone. Since \(E\) lies in \(\tilde{S}\), \(S_2\) is irreducible. Then \(R_1, \ldots, R_8\) are twisted cubics lying in \(S_2\), which is impossible on the quartic cone \(S_2\).

Having proved irreducibility, we move on to normality. Since \(S\) is an irreducible quadric section of \(\tilde{V}\) through \(v\), the projection \(\beta : S \dashrightarrow \tilde{V}\) is birational. Thus \(S\) can only fail to be normal along generators of the cone \(\hat{V}\).
We have $\gamma^*S \sim \tilde{S} + V_0$, $\gamma^*X \sim \tilde{X} + V_0$, $\tilde{X}.V_0 = E$ and $\tilde{S} \cap V_0 \supset E$. So $\tilde{S}.V_0 = bE + F$ for some $b \geq 1$ and some curve $F \subset V_0$. So $\tilde{X}.(bE + F) = \tilde{X}.\tilde{S}.V_0 = (\gamma^*X - V_0).(\gamma^*S - V_0).V_0 = V_0^3 = 4$.

Now $\tilde{X}.E = -V_0.E = 4$, so that $b = 1$ and $F = 0$. Hence $\tilde{S}$ is normal along $E$, and so $S$ is normal in a neighbourhood of $v$, and so everywhere.

Now the identification $\tilde{V} = \mathbb{P}(1,1,1,2)$ and the adjunction formula show that $-2K_S \sim H|_S$. Projection from $v$ completes the proof of (6) and (7).

The only thing left is to show that $\overline{R}$ is the unique member of $|3H - 2Z|$ on $S$. This is a straightforward calculation of intersection numbers on $\tilde{S}$, as in the non-hyperelliptic case.

Remark: Suppose that the non-hyperelliptic curve $C_\eta$ specializes to a smooth hyperelliptic curve $C_0$ and consider the corresponding specializations of tropes $X_\eta$ to $X_0$ and degree 2 del Pezzo surfaces $S_\eta$ to $S_0$. Then $X_0$ and $S_0$ have singularities of type $[-4]$, which are not Gorenstein, but from the viewpoint of the moduli of algebraic surfaces of general type this is a well known picture: $X_0$ has semi-log canonical singularities and the family $\{X_t\}$ is a relative canonical model, so that the surface $X_0$ appears on the boundary of the separated and compact moduli space for stable surfaces with $K^2 = 3$ and $\chi(O) = 4$.

Remark: Tacit in this discussion so far has been the assumption that we began with the Jacobian of some curve, and then sought to recover the curve. However, there is another point of view.

Suppose that we begin with a curve $\Gamma$ of genus 3 over a field $k$, construct its Jacobian $J\Gamma = A$ and then take the quotient $A/L$ of $A$ by a Lagrangian subgroup $L$ of $A[n]$ for some $n \geq 2$. Given an equation for $\Gamma$ and sufficient knowledge of torsion points on $A$, it is possible to write down equations for $A$ such that the Kummer variety of the principally polarized quotient $A/L$ can also be written down as the image of $A/L$ under its $2\Theta$ linear system.

Then, by using the results of this paper, we can construct a genus 3 curve $C$ such that $A/L$ is a quadratic twist of $JC$ (provided that $A/L$ is geometrically irreducible as a ppav). An analysis of the singular surface $\Sigma$ in the linear system $[2\Theta]$ will determine the corresponding quadratic character. However, over a finite field $k$ this can often be done more easily by point-counting: the trace of Frobenius on $H^1(C \otimes \overline{k}, \mathbb{Q}_\ell)$ is $\pm$ the trace of Frobenius on $H^1(\Gamma \otimes \overline{k}, \mathbb{Q}_\ell)$, so if this trace is non-zero the triviality or otherwise of the quadratic character is determined by a comparison of the two traces.
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