Gröbner Geometry of Schubert Polynomials Through Ice

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Based on joint work with Oliver Pechenik and Zachary Hamaker and work in progress with Patricia Klein.
Schubert Polynomials
The complete flag variety $GL(n)/B$ has
a special family of subvarieties
$\{ X_\omega : \omega \in S_n \}$ called Schubert varieties.

Each Schubert variety defines a Schubert class
$\sigma_\omega \in H^*(GL(n)/B)$. The Schubert classes are a
linear basis for this ring.

By Borel's isomorphism:

$H^*(GL(n)/B) \cong \mathbb{Z}[x_1, x_2, \ldots, x_n]/I_{S_n}$. 
Lascoux and Schützenberger (1982) defined Schubert polynomials \( \{ G_w(x) : w \in S_n \} \) which are a choice of coset representatives for the images of the Schubert classes under Borel's isomorphism.

There's also double Schubert polynomials \( \{ G_{w(x,y)} : w \in S_n \} \) which are enriched versions of single Schuberts and satisfy \( G_w(x) = G_{w(x;0)} \).
Pipe Dreams

On main antidiagonal

Above main antidiagonal

Below main antidiagonal

Fill $n \times n$ grid with $n$ pipes that start at the top and end at the left and pairwise cross at most one time.

$$w_t(P) = (x_1 - y_1)(x_1 - y_2)(x_1 - y_3)(x_2 - y_1)(x_2 - y_3)(x_3 - y_1)(x_3 - y_2)$$
Theorem (Fomin-Kirillov 1996 / Bergeron-Billey 1993): The double Schubert polynomial is

\[ G_w(x; y) = \sum_{P \in \text{Pipes}(w)} \text{wt}(P). \]

\[ G_{2143}(x; y) = (x_1-y_1)(x_3-y_3) + (x_1-y_1)(x_2-y_2) + (x_1-y_1)(x_1-y_2) \]
Fill an $n \times n$ grid with $n$ pipes that start at the bottom, end at the right and pairwise cross at most one time.

\[ \text{wt}(P) = (x_1 - y_1)(x_1 - y_2)(x - y_3) \cdot (x_2 - y_1)(x_2 - y_2)(x_3 - y_1)(x_3 - y_5) \]

no bumps allowed!
Theorem (Lam-Lee-Shimozono 2013):

The double Schubert polynomial is

\[ G_r (x_i y_j) = \sum_{P \in BPD(\omega)} w(P). \]

\[ G_{2143} (x_i y_j) = (x_1 - y_1) (x_3 - y_3) + (x_i - y_i) (x_2 - y_2) + (x - y_1) (x_1 - y_2) \]
Secretly, BPDs were hiding in an unpublished preprint of Lascoux (2002).

Lascoux gave a formula for double Grothendieck polynomials as a sum over states of the 6-vertex (ice) model.

See arXiv:2003.07342.
\[ \mathcal{S}_{2143}(x; y) = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_2) + (x_1 - y_1)(x_1 - y_2) \]

\[ \mathcal{S}_{2143}(x) = x_1 x_3 + x_1 x_2 + x_1^2 \]

\[ \mathcal{S}_{2143}(x; y) = (x_1 - y_1)(x_3 - y_3) + (x_1 - y_1)(x_2 - y_1) + (x_1 - y_2)(x_1 - y_2) \]
Matrix
Schubert
Varieties
Determinantal Ideals

Let $\text{Mat}(m,n)$ be the space of $m \times n$ matrices and

$$Z_{m,n} = \begin{bmatrix}
Z_{11} & Z_{12} & \cdots & Z_{1n} \\
Z_{21} & Z_{22} & \cdots & Z_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{m1} & Z_{m2} & \cdots & Z_{mn}
\end{bmatrix}$$

a generic matrix.

$$\mathbb{C}[Z_{m,n}] = \mathbb{C}[Z_{11}, Z_{12}, \ldots, Z_{mn}]$$ is the coordinate ring of $\text{Mat}(m,n)$. 
Let \( I_k(\mathbb{Z}^m_n) \) be the ideal generated by the minors of size \( k \) in \( \mathbb{Z}^m_n \). \( I_k(\mathbb{Z}^m_n) \) is a determinantal ideal.

Example:

\[
I_2 \left( \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{bmatrix} \right) = \langle \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} \rangle
\]

\( V( R / I_{k+1}(\mathbb{Z}^m_n) ) \) is the set of matrices \( M \in \text{Mat}(m,n) \) such that \( \text{rank}(M) \leq k \).
Given \( w \in S_n \), the Schubert determinantal ideal \( I_w \subseteq R = \mathbb{C}[Z_{n,n}] \) is generated by minors of varying size, determined by \( w \).

\[
I_{426135} = \langle z_{11}, z_{12}, z_{13}, z_{21}, z_{31} \rangle \\
+ \langle 2 \times 2 \text{ minors in } \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \rangle \\
+ \langle 3 \times 3 \text{ minors in } \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \end{bmatrix} \rangle 
\]
Matrix Schubert Varieties

Write $\overline{X_\omega} := \overline{i \left( \pi^{-1}(X_\omega) \right)}$.

Theorem (Fulton 1992): $I_\omega$ is prime and $\overline{X_\omega} = V(R/I_\omega)$.
A Recipe

1. Fix an antidiagonal term order $<_a$ on $R$.
2. Compute the initial ideal $\text{init}_a(I_w)$.
3. Take the primary decomposition.

Example:

$I_{2143} = \langle z_{11} \rangle,
\begin{bmatrix}
    z_{11} & z_{12} & z_{13} \\
    z_{21} & z_{22} & z_{23} \\
    z_{31} & z_{32} & z_{33}
\end{bmatrix}
> \text{init}_a(I_{2143}) = \langle z_{11}, z_{31}, z_{22}, z_{13} \rangle
= \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle
Example:

\[ I_{2143} = \langle z_{11}, \begin{bmatrix} z_{11} & z_{21} & z_{31} \\ z_{12} & z_{22} & z_{32} \\ z_{13} & z_{23} & z_{33} \end{bmatrix} \rangle \]

\[
\text{init}_2 (I_{2143}) = \langle z_{11}, z_{31}, z_{22}, z_{13} \rangle
\]

\[
= \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle
\]

\[
S_{2143}(x,y) = (x_1-y_1)(x_3-y_1) + (x_1-y_1)(x_2-y_2) + (x_1-y_1)(x_1-y_3)
\]
Theorem (Knutson - Miller 2005):

1. Fulton's generators are a Gröbner basis for $I_w$ under any antidiagonal term order.

2. $\text{init}_{\text{la}}(I_w)$ is radical.

3. $\text{init}_{\text{la}}(I_w) = \bigcap_{p \in \text{Pipes}(w)} I_p$
But wait, there’s more!

Each $\overline{X}$ defines a class in the $T \times T$ equivariant cohomology of $\text{Mat}(n)$.

To compute $[X]_{T \times T}$ you can use three axioms

1. Degeneration
2. Additivity
3. Normalization

Theorem (Knutson-Miller 2005):

$$[\overline{X}]_{T \times T} = \mathcal{G}_w(x; y).$$
Diagonal Degenerations
Knutson, Miller, and Yong (2009) worked out an analogous story for diagonal degenerations of vexillary permutations.

**Theorem (KMY 2009):**

1. Fulton's generators are a diagonal Gröbner basis if and only if $w$ is vexillary.
2. In this case, primes are indexed by flagged tableaux.
$$I_{1423} = \left< \begin{array}{c|c|c}
1 & 2_{11} & Z_{12} \\
1 & Z_{21} & 2_{22} \\
1 & 2_{21} & Z_{23} \end{array} \right>$$

$$\text{initialized} \left( I_{1423} \right) = \left< 2_{11} Z_{22}, Z_{11} Z_{23}, 2_{12} Z_{23} \right>$$

$$= \left< 2_{11}, 2_{12} \right> \cap \left< 2_{11}, Z_{23} \right> \cap \left< Z_{22}, Z_{23} \right>$$
\[ init_d(\Sigma_{1423}) = \langle z_{11}, z_{12} \rangle \cap \langle z_{11}, z_{23} \rangle \cap \langle z_{22}, z_{23} \rangle \]
Example:

\[
I_{2143} = \left< \begin{array}{c}
\bar{Z}_{11} \\
\bar{Z}_{21} \\
\bar{Z}_{31}
\end{array}, \begin{array}{ccc}
\bar{Z}_{11} & \bar{Z}_{22} & \bar{Z}_{33} \\
\bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{33} \\
\bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33}
\end{array} \right>
\]

\[
= \left< \bar{Z}_{11}, \begin{array}{ccc}
0 & \bar{Z}_{22} & \bar{Z}_{33} \\
\bar{Z}_{21} & \bar{Z}_{22} & \bar{Z}_{33} \\
\bar{Z}_{31} & \bar{Z}_{32} & \bar{Z}_{33}
\end{array} \right>
\]

\[
\text{initialize}(I_{2143}) = \langle \bar{Z}_{11}, \bar{Z}_{33}, \bar{Z}_{21}, \bar{Z}_{12} \rangle
\]

\[
= \langle \bar{Z}_{11}, \bar{Z}_{33} \rangle \land \langle \bar{Z}_{11}, \bar{Z}_{21} \rangle \land \langle \bar{Z}_{11}, \bar{Z}_{12} \rangle
\]
Example:

\[ I_{2143} = \langle z_{11}, \begin{vmatrix} z_{11} & z_{2} & z_{3} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \rangle > \]

\[ \quad \quad = \langle z_{11}, \begin{vmatrix} 0 & z_{2} & z_{3} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \rangle > \]

\[ \text{initial } (I_{2143}) = \langle z_{11}, z_{33}, z_{21}, z_{12} \rangle \]

\[ = \langle z_{11}, z_{33} \rangle \cap \langle z_{11}, z_{21} \rangle \cap \langle z_{11}, z_{12} \rangle \]

\[ \sigma_{2143}(x;y) = (x_{1}-y_{1})(x_{3}-y_{3}) + (x_{1}-y_{1})(x_{2}-y_{1}) + (x_{1}-y_{1})(x_{1}-y_{2}) \]
Caution: initiated $(I_w)$ might not be radical!

$V(\mathbb{k}/\langle z_{11}, z_{12}, z_{13}, z_{21}, z_{22}, z_{317} \rangle)$ shows up in $\text{Spec}(\mathbb{k}/\text{init}_{\mathbb{d}}(I_{321054}))$ with multiplicity!
Conjecture (Hamaker-Pechenik-W. 2020):

BPD’s for w label set theoretic components of $\text{Spec} (R/\text{init}_<(I_w))$ with multiplicity, i.e. the multiplicity of $V(R/\mathbb{Z}_{<j} : (i,j) \in D)$ is

$$\# \{ P \in \text{BPD}(w) : D \text{ indexes the blank tiles in } P^3 \}. $$
CDG Generators

$I_{426135} = \langle z_{11}, z_{12}, z_{13}, z_{21}, z_{31} \rangle$

+ $\langle 2 \times 2 \text{ minors in } \begin{bmatrix}
0 & 0 & 0 \\
0 & z_{22} & z_{23} \\
0 & z_{32} & z_{33}
\end{bmatrix} \rangle$

+ $\langle 3 \times 3 \text{ minors in } \begin{bmatrix}
0 & 0 & 0 & z_{14} & z_{15} \\
0 & z_{22} & z_{23} & z_{24} & z_{25} \\
0 & z_{32} & z_{33} & z_{34} & z_{35}
\end{bmatrix} \rangle$

Question: When are CDG generators diagonal Gröbner bases?
Predominant Permutations

A permutation is predominant if its Lehmer code is of the form $20^k n$ where $\lambda$ is a partition and $k, n \in \mathbb{Z}_{\geq 0}$.

Example: $426135$ has code $313000$.

\[ \lambda = (3, 1), \quad k = 0, \quad n = 3. \]
Skew Sums of BPDS

Lemma: $\text{BPD}(u) \times \text{BPD}(w) \rightarrow \text{BPD}(u \uplus w)$ is a bijection
We say \( w \in S_n \) is \textit{banner} if we can write

\[ w = u_1 u_2 \cdots u_k \]

where the \( u_i \)'s are (partial) permutations coming from Vexillary, predominant, or inverse predominant permutations.
Theorem (Hamaker–Pechenik–W. 2020):
If \( w \) is banner, then

1. The CDG generators are a diagonal Gröbner basis and

2. \( \text{init}_d(I_w) = \bigcap_{P \in \text{BPD}(w)} I_p \).
Theorem (Klein 2020): The CDG generators for $I_w$ are a diagonal Gröbner basis if and only if $w$ avoids the patterns:
13254, 21543, 214635, 215364, 241635,
315264, 215634, and 4261735.

(Conjectured by Hamaker - Pechenik - W.)
Theorem (Klein-W. 2021+): If $\text{init}_{zd}(Iw)$ is radical, $\text{init}_{zd}(Iw) = \bigcap_{P \in \text{BPD}(w)} I_P$.

Corollary: The conjecture holds for CDC permutations.
Transition

426135

425136

524136

452136
Thanks!
A Paradigm of Patterns

No duplicate sets of blank tiles

CDG

2143-avoiding

Vexillary

132-avoiding

Monomial ideals

Multiplicity free

Beyond