Topological defects in lattice models
and affine Temperley–Lieb algebra

J. Belletête\textsuperscript{1,7}, A.M. Gainutdinov\textsuperscript{2,6}, J.L. Jacobsen\textsuperscript{1,3,4}, H. Saleur\textsuperscript{1,5}, T.S. Tavares\textsuperscript{1}

\textsuperscript{1} Institut de Physique Théorique, Université Paris Saclay, CEA, CNRS, 91191 Gif-sur-Yvette, France
\textsuperscript{2} Institut Denis Poisson, CNRS, Université de Tours, Université d’Orléans, Parc de Grandmont, 37200 Tours, France
\textsuperscript{3} Laboratoire de Physique Théorique, Département de Physique de l’ENS, École Normale Supérieure, Sorbonne Université, CNRS, PSL Research University, 75005 Paris, France
\textsuperscript{4} Sorbonne Université, École Normale Supérieure, CNRS, Laboratoire de Physique Théorique (LPT ENS), 75005 Paris, France
\textsuperscript{5} Department of Physics, University of Southern California, Los Angeles, CA 90089-0484, USA
\textsuperscript{6} National Research University Higher School of Economics, Myasnitskaya Ulitsa, 20, Moscow, Russia
\textsuperscript{7} Laboratoire de Physique Théorique et Modélisation, CY Cergy Paris Université, CNRS, F-95302 Cergy-Pontoise, France

Abstract

This paper is the first in a series where we attempt to define defects in critical lattice models that give rise to conformal field theory topological defects in the continuum limit. We focus mostly on models based on the Temperley-Lieb algebra, with future applications to restricted solid-on-solid (also called anyonic chains) models, as well as non-unitary models like percolation or self-avoiding walks. Our approach is essentially algebraic and focusses on the defects from two points of view: the “crossed channel” where the defect is seen as an operator acting on the Hilbert space of the models, and the “direct channel” where it corresponds to a modification of the basic Hamiltonian with some sort of impurity. Algebraic characterizations and constructions are proposed in both points of view. In the crossed channel, this leads us to new results about the center of the affine Temperley-Lieb algebra; in particular we find there a special basis with non-negative integer structure constants that are interpreted as fusion rules of defects. In the direct channel, meanwhile, this leads to the introduction of fusion products and fusion quotients, with interesting algebraic properties that allow to describe representations content of the lattice model with a defect, and to describe its spectrum.
# Contents

1 Introduction .............................................. 3

2 The affine TL algebra and its center .................. 6
   2.1 Two definitions .................................. 6
   2.2 Diagrammatic presentation ....................... 8
   2.3 Standard modules ................................ 10
   2.4 The center of $\mathfrak{a}T\mathfrak{l}_n(q)$ ............. 12
   2.5 Tower structure .................................. 17
   2.6 Tile formalism and the transfer matrix .......... 18

3 Lattice topological defects: crossed channel ........ 20
   3.1 The algebra of defects $Y$ and $\bar{Y}$ ........... 20
   3.2 Higher-spin operators $Y_j$ and $\bar{Y}_j$ .......... 21
   3.3 Relation to symmetric polynomials. ............... 23

4 Lattice topological defects: direct channel ........ 24
   4.1 The fusion product ................................ 25
   4.2 The fusion quotient ................................ 28
   4.3 Dualities between the two fusions ................. 29
   4.4 Fusion and the Hamiltonian ....................... 30
   4.5 Example of quotient: the twisted XXZ spin chain ... 31

5 Conclusion: connection to CFT ......................... 33

6 Declarations ............................................. 34

A Proofs and rigors ......................................... 35
   A.1 Topological defects with a higher spin. .......... 35
   A.2 The Jucys-Murphy elements ....................... 36
   A.3 Fusion with standard modules ...................... 37
      A.3.1 $W_{k,z}(n+2) \underset{u/o}{\div} S_0(2)$ .......... 38
      A.3.2 $W_{k,z}(n) \underset{u/o}{\times} S_0(2)$ .......... 38
      A.3.3 $W_{k,z}(n+1) \underset{f}{\div} S_{1/2}(1), (k \neq 0)$ 39
      A.3.4 $W_{k,z}(n+1) \underset{f}{\times} S_{1/2}(1), (k \neq 0)$ 40
      A.3.5 $W_{0,z}(n) \underset{f}{\times} S_{1/2}(1)$ .......... 42
      A.3.6 $W_{0,z}(n+1) \underset{f}{\div} S_{1/2}(1)$ .......... 43

B Comparison with the other fusion types ............ 43
   B.1 The GS fusion .................................. 43
   B.2 The BSA fusion ................................ 44

C Alternative proof of the relations (114)-(115) ....... 45

D Finding the basis in (105) to (107) .................. 47
1 Introduction

A defect—or interface—in conformal field theory is generally defined as a non-contractible line separating two a priori different conformal field theories (CFTs), with matching conditions between the two sides of the line. Various situations can be encountered in this general context. We will restrict here to the case of so-called topological defects, where the two CFTs are identical, and the stress-energy tensor is continuous across the defect line. In this case, correlation functions for fields inserted away from the defect line are unchanged when the line is continuously deformed, as long as the line is not taken across the field insertions: hence the name “topological”.

Defects in CFT appear in a variety of physical problems, both in two-dimensional statistical mechanics, e.g. in the context of Kramers–Wannier duality [1, 2], and in imaginary-time one-dimensional quantum mechanics, e.g. in the context of quantum impurity problems such as the Kondo problem [3, 4]. The problem of classifying topological defects has received considerable attention, in particular in the case of rational CFTs [5, 6]. For such theories with diagonal modular invariants, for instance, it has been shown that the set of defects is in bijection with the set of representations of the chiral algebra. Many results for non-diagonal invariants, or for non-rational unitary theories such as Liouville [7] are also known.

Meanwhile, the general question of relating structures within the CFTs with properties of underlying lattice models has also attracted much attention. Work in this direction has included attempts to define lattice versions of the Virasoro algebra [8–10], to define fusion of primary fields in terms or representation theory of lattice algebras [11–14, 16, 17], to calculate modular transformations from lattice partition functions [18], and to build topological defect lines directly on the lattice [19, 20, 30]. Many of these attempts drew from the pioneering works [21, 31].

The present work is motivated by our interest in non-unitary (in particular, logarithmic) conformal field theory (LCFT) [22]. Decisive progress has been realized in this difficult subject by turning to lattice models—in particular, to understand better the indecomposable properties of the Virasoro-algebra representations involved. In view of the close relationship between defects, primary fields and fusion in the unitary case, it is natural to continue the program set out in [23] by trying to define topological defects on the lattice using an algebraic approach. While such endeavor has been partially completed in the case of restricted solid-on-solid models—whose associated CFTs are rational, and which are closely related to “anyonic” spin chains [19, 27, 28]—we will be interested here in the profoundly different case of loop models, which provide regularizations of the simplest known LCFTs. This paper will discuss the first part of our study, where we will focus on the definition and mathematical properties of a certain kind of lattice topological defects.

The correspondence between CFTs and lattice models is often best handled by thinking of the CFT in radial quantization, where, after the usual logarithmic mapping, (imaginary) time propagation occurs along the axis of a cylinder, and space is periodic. In this point of view, the non-contractible line for the topological defect can either run along the infinite cylinder, or be a non-contractible loop winding around it. We will refer to these two situations as a defect in the “direct” or in the “crossed” channel, respectively, see Fig. 1.

In the crossed channel, the defect can be associated with an operator $X$ acting on the Hilbert space of the bulk CFT. The defect is topological if $X$ commutes with the chiral $\text{Vir}$ and the anti-chiral $\overline{\text{Vir}}$ Virasoro generators [5, 6]:

\[ [L_n, X] = 0 = [\overline{L}_n, X] . \]  (1)
Figure 1: The two possible geometries for a defect line after mapping the plane to the cylinder.

Our strategy to identify the possible choices of operators $X$ is based on the identification of (representations of) the Virasoro algebra via the continuum limit of the Temperley-Lieb (TL) algebra—an idea that has been used in several works on related topics [8, 24, 25]. This “identification” must be qualified. First, since we are dealing with bulk CFTs, we must think of the product of the chiral and anti-chiral Virasoro algebras, $Vir \otimes \overline{Vir}$. Similarly, since the lattice models are defined on a cylinder, the proper lattice algebra is a “periodicized” version—the affine Temperley-Lieb algebra $aTL$: strictly speaking, the continuum limit of this algebra is known to be larger than $Vir \otimes \overline{Vir}$, and has been identified as the “interchiral algebra” in [24].

In the typical physical interpretation of the (affine) Temperley-Lieb algebras on $n$ sites, the nodes on the top and bottom of the TL diagrams should be interpreted as a chain of $n$ subsystems whose interactions are determined by the TL generators, but whose internal sub-structure is not—it is determined by the specific model chosen, which also fixes the $aTL$ representation corresponding to the chain. The simplest examples of these are the various kinds of spin-chains, like the twisted XXZ model. We will therefore start our search for lattice analogues of topological defects by demanding the closest lattice equivalent of (1), that is by looking for operators $X$ on the lattice that commute with the interactions in the chain, or, in a model-independent setting, that are central in $aTL$. We will follow this model-independent point of view on lattice defects as central elements satisfying certain nice properties, e.g. having a well-defined fusion. This is discussed in Section 3 after the algebraic preliminaries of Section 2 where we recall the usual definition of $aTL$ together with a less standard formulation using a blobbed set of generators. In this last formulation, the lattice meaning of $X$ turns out to be very simple: it just consists in passing a line “above” or “below” the non-contractible loops by using solutions of the spectral-parameter independent Yang-Baxter equation exchanging spin-$1/2$ (the value relevant for bulk loops) and spin-$j$ (the value relevant for the defect lines) representations. The topological nature of this defect is obvious, as the Yang-Baxter equation allows one to move and deform the defect line at will without changing neither the partition function, nor the correlation functions if operators are inserted. The simplest example of such a defect operator $X$ is given by a diagram corresponding to a single non-contractible loop passing over the bulk, see Fig. 2 where we denote this operator by $Y$. This operator and its powers are manifestly in the center of the affine TL algebra. We define similarly operators $\bar{Y}$ where the non-contractible loop is passing under the bulk. The two operators generate an interesting algebra of defects.

Let us describe this type of defect operators in more precise mathematical terms. First of all, the $aTL$ algebras depend on $n$ (the number of sites) and a loop parameter $q + q^{-1}$. One of our
main mathematical results is Theorem 2.1 stating that for any non-zero value of $q$, that is not a root of unity, the two central elements $Y$ and $\bar{Y}$ generate the center $Z(\mathfrak{aTL})$ of the affine TL algebra $\mathfrak{aTL}$. As a by-product we obtain an expression of symmetric Laurent polynomials in the Jucys-Murphy elements in terms of $Y$ and $\bar{Y}$: the power-sum polynomials have a simple relation to the Chebychev polynomials in $Y$ and $\bar{Y}$ of 1st kind, this is discussed in Section 3.3. Moreover, we show that for $q$ a generic complex number$^1$ products of Chebyshev polynomials in $Y$ and $\bar{Y}$ provide a “canonical” basis in $Z(\mathfrak{aTL})$ with non-negative integer structure constants. The spin-$j$ defect operators mentioned above are defined as the $2j$-th Chebyshev polynomials and they thus form a special basis in the center, i.e. a product of two defect operators is decomposed onto defect operators again, and with non-negative multiplicities. These multiplicities are interpreted as fusion rules of the defects.

Of course, the line passing above or below the loops can as well be taken to run along the axis of the cylinder, i.e. along the time direction. This corresponds to having the defect in the direct channel. In this setting, the presence of the defect line leads to a modified Hilbert space where an extra representation of spin $j$ is introduced, together with a Hamiltonian suitably modified by corresponding “defect” terms. This is discussed in Section 4 with the main result in Theorem 4.1, where we relate spectral properties of such a modified Hamiltonian (which is hard to study directly) to a clear and precise algebraic construction within the representation theory of $\mathfrak{aTL}$ algebras—namely the fusion product and fusion quotient. The first is based on a certain induction, while the second is dual to it and practically very convenient for actual calculations. In simple terms, the spectrum of the spin-$j$ defect Hamiltonian is given by the spectrum of the standard affine TL Hamiltonian with no defects however acting on the fusion quotient of an $\mathfrak{aTL}$ representation by the spin-$j$ standard TL representation. The advantage of this construction is that it allows us to perform precise calculations, as we demonstrate in several examples, including the case of the twisted XXZ model.

In the last section 5, we provide conclusions and discuss a CFT interpretation together with further steps that will be discussed in the next papers, like the analysis of modular $S$-transformation in infinite lattices and the continuum limit from a more physical point of view. In Section 5, we also make an attempt to give a precise mathematical definition of lattice defects studied in this work. Finally, several appendices contain proofs of our mathematical results and auxiliary calculations, such as examples of fusion products and fusion quotients.

$^1$i.e. for $q$ not a root of unity, we leave the root of unity case discussion for a forthcoming paper.
2 The affine TL algebra and its center

In this section, we fix our notations and conventions. We first give a definition of the affine Temperley-Lieb algebra $\mathfrak{aTL}$ in terms of generators, and in terms of diagrams. We give the definition both in terms of the translation generator, which is very standard, and a new one in terms of the so-called blob and hoop generators; the blob formulation is significantly more convenient when discussing topological defects. We then discuss the standard modules and give the eigenvalues of the central elements on them. The key result of this section is Theorem 2.1 where we describe the center of $\mathfrak{aTL}$.

2.1 Two definitions

The affine Temperley-Lieb algebras $\{\mathfrak{aTL}_n(q)\}$ form a family of infinite dimensional associative $\mathbb{C}$-algebras, indexed by a positive integer $n$ – number of sites – and a non-zero complex number $q$. They can be defined in many ways but we chose three particular presentations for their relevance in physics. Each of these are described in terms of generators with relations and were chosen because they lighten the notation in particular sub-sections of this work.

The first set of generators, which we shall refer to as the periodic set of generators, is the one appearing in the original literature on these algebras: two shift generators $u, u^{-1}$, and $n$ arc generators $e_1, \ldots, e_n$, with the defining relations ($n > 2$) [35]

\[
e_i e_i = (q + q^{-1}) e_i,
\]

\[
e_i e_{i \pm 1} e_i = e_i,
\]

\[
e_i e_j = e_j e_i \quad \text{if } |i - j| \geq 2,
\]

\[
ue_i = e_{i+1} u,
\]

\[
u^2 e_{n-1} = e_1 \ldots e_{n-1},
\]

which stands for all $i$, and we defined $e_0 \equiv e_n, e_{n+1} \equiv e_1$. If $n = 2$, one must remove the relations $e_i e_{i \pm 1} e_i = e_i$, but the other relations are unchanged. If $n = 1$, one must remove all the arc generators, keeping only the shift generators with the defining relations $uu^{-1} = u^{-1}u = 1$. One notices immediately that this set of generators is not minimal, since for instance $e_i = u^{i-1}e_1u^{1-i}$ for all $i \geq 1$. Furthermore, the elements $u^{\pm n}$ are both obviously central. The sub-algebra generated by $\{e_1, \ldots, e_n\}$ is often called the periodic Temperley-Lieb algebra, while the one generated by $\{e_1, \ldots, e_{n-1}\}$ is called the regular Temperley-Lieb algebra.

The second set of generators, which we shall refer to as the blobbed set of generators, is significantly less known: there are two blob generators $b, b^{-1}$, and $n-1$ arc generators $e_i, 1 \leq i \leq n-1$ (so if $n = 1$ there are no arc generators), with defining relations

\[
e_i e_i = (q + q^{-1}) e_i,
\]

\[
e_i e_{i \pm 1} e_i = e_i,
\]

\[
e_i e_j = e_j e_i \quad \text{if } |i - j| \geq 2,
\]

\[
e_i b = be_i \quad \text{if } i \geq 2,
\]

\[
e_1 b e_1 = (q b + q^{-1} b^{-1}) e_1 = e_1(q b + q^{-1} b^{-1}),
\]

which stands for all $i$, and we defined $e_0 \equiv e_n, e_{n+1} \equiv e_1$. If $n = 2$, one must remove the relations $e_i e_{i \pm 1} e_i = e_i$, but the other relations are unchanged. If $n = 1$, one must remove all the arc generators, keeping only the shift generators with the defining relations $uu^{-1} = u^{-1}u = 1$. One notices immediately that this set of generators is not minimal, since for instance $e_i = u^{i-1}e_1u^{1-i}$ for all $i \geq 1$. Furthermore, the elements $u^{\pm n}$ are both obviously central. The sub-algebra generated by $\{e_1, \ldots, e_n\}$ is often called the periodic Temperley-Lieb algebra, while the one generated by $\{e_1, \ldots, e_{n-1}\}$ is called the regular Temperley-Lieb algebra.


which stands for all $i$ such that these expressions make sense; note that in this case we have no
generator $e_n$. We also note that the element $Y \equiv -q b - q^{-1} b^{-1}$ introduced in the above relations
is central, it will be called the hoop operator\(^2\). The epithet blob for the generators $b^{\pm 1}$ is due to
the relation with the so-called blob algebra [34], which is a finite-dimensional algebra. This latter
algebra is obtained by taking the quotient of $\mathfrak{TL}_n(q)$ by the two-sided ideal $\mathfrak{TL}_n(q) \cdot (Y - y I)$
for some $y \in \mathbb{C}$ such that $y^2 \neq 4$, or in simple words the blob algebra is obtained via fixing
the eigenvalue of $Y$. We must however point out that our invertible blob operator is not
the pre-image of the blob operator appearing in the blob algebra, which is an idempotent, see also
a similar discussion of relations between the affine Hecke and type $B$ Hecke algebras in [29]. In more
details, for the quotient with $Y = (\delta + \delta^{-1}) I$ for $\delta \neq \pm 1$ the idempotent blob is $\bar{b} = \frac{qb + q^{-1}}{\delta - \delta^{-1}}$, and the
blob weight is
\[
\bar{y} = \frac{\delta q^{-1} - q \delta^{-1}}{\delta - \delta^{-1}},
\]
i.e. after the quotient we have $\bar{b}^2 = \bar{b}$ and $e_1 \bar{b} e_1 = \bar{y} e_1$. The special case with $\delta = \pm 1$ (or
$y = \pm 2$) corresponds to the “generalized” blob algebra [15] with defining relations $(b')^2 = 0$ and $e_1 b' e_1 = \mp(q - q^{-1}) e_1$ where we set $b' = q b \pm 1$. It is also related to the boundary seam algebra [39].

We note that the connection of the blob type generators with the first description, i.e. in terms
of “periodic” type generators is (here, we place periodic type generators in RHS)
\[
e_i = e_i, \quad 1 \leq i \leq n - 1,
\]
\[
b = (-q)^{-3/2} g_1^{-1} \ldots g_{n-1}^{-1} u^{-1},
\]
\[
b^{-1} = (-q)^{3/2} u g_{n-1} \ldots g_1,
\]
where we introduced the braid generators
\[
g_i^{\pm 1} = (-q)^{\pm 1/2} 1 + (-q)^{\mp 1/2} e_i.
\]
It is straightforward to check the braid relations
\[
g_i g_{i \pm 1} g_i = g_{i \pm 1} g_i g_{i \pm 1}.
\]
The normalization\(^3\) in (7) was chosen such that
\[
g_i^{\pm 1} g_{i+1}^{\pm 1} e_i = e_{i+1} g_i^{\pm 1} g_{i+1}^{\pm 1} = e_{i+1} e_i,
\]
\[
g_{i+1}^{\pm 1} g_i^{\pm 1} e_{i+1} = e_i g_{i+1}^{\pm 1} g_i^{\pm 1} = e_i e_{i+1}.
\]
These relations are used to prove the equivalence of the relations in equations (3) and (2), specifically
when verifying those involving $b$, $u$, or $e_n$.

We note that an expression of periodic generators in terms of the blobbed ones is obtained as
follows: the shift generators $u^{\pm 1}$ are obtained multiplying both sides of (5)-(6) with appropriate
$g_i^{\pm 1}$'s, then the generator $e_n$ is formally defined as $u^{-1} e_1 u$. It is then rather straightforward,
however tedious, to show that the defining relations (2) are equivalent to those in (3). We give

---

\(^2\)The name will be justified via its diagrammatic presentation that we discuss below.

\(^3\)The normalization used here for $g_i$ will also become useful when doing graphical calculations.
one example of such computations as the others are all quite similar; recall the proposed form for $b$ in (5), we then verify that

$$(qb)^2 e_1 = (-q)^{-1} u g_{n-1}^{-1} \ldots g_1^{-1} u g_{n-1}^{-1} \ldots g_1^{-1} e_1,$$

$$= (-q)^{3/2} e_1$$

$$= -u g_{n-1}^{-1} \ldots g_2^{-1} (1 - q e_1) u g_{n-1}^{-1} \ldots g_2^{-1} e_1,$$

$$= -u^2 g_{n-2}^{-1} \ldots g_1^{-1} g_{n-2}^{-1} \ldots g_2^{-1} e_1 + q b e_1 b e_1,$$

$$= -u^2 e_{n-1} e_{n-2} \ldots e_1 + q e_1 b e_1,$$

$$= -e_1 + q b e_1 b e_1,$$

and then multiplying both sides by $b^{-1}$ from the left yields the identity $e_1 b e_1 = (qb + q^{-1} b^{-1}) e_1$ from the list in (3). We also note that in the context of blob algebras (recalled above as the quotients), the relation between periodic and blobbed generators reflects what was called “braid translation” in [26, 34].

We will also use another relation between the periodic and blobbed set of generators: Because the algebra is invariant under the substitution $b \rightarrow b^{-1}$, $q \rightarrow q^{-1}$, i.e. it provides an algebra automorphism, there is a second way to write the blob generators in terms of the generators of “periodic” type:

$$\bar{b} = (-q)^{-3/2} u g_{n-1}^{-1} \ldots g_1^{-1},$$

$$\bar{b}^{-1} = (-q)^{3/2} g_1 \ldots g_{n-1} u^{-1}.\quad (10)$$

We turn now to introduction of diagrammatical presentations for both types of generators, and it is much easier to check such an equivalence (or isomorphism of the two algebras) by doing standard diagram calculations.

### 2.2 Diagrammatic presentation

We now introduce the graphical presentation of the algebra [41], which can be used to write words in the algebra in a very compact and intuitive form. Each of the classical generators gets associated to a diagram with $2n$ nodes connected by $n$ strands, or lines:

$$e_i = \begin{array}{l}
\vdots \quad \vdots \\
\frac{i-1}{n-1} \quad \frac{n-1}{n-1}
\end{array}, \quad e_n = \begin{array}{l}
\vdots \quad \vdots \\
\frac{n-1}{n-1} \quad \frac{1}{1}
\end{array}, \quad i = 1, \ldots, n-1, \quad (11)$$

$$u = \begin{array}{l}
\vdots \quad \vdots \\
\frac{n}{n}
\end{array}, \quad u^{-1} = \begin{array}{l}
\vdots \quad \vdots \\
\frac{n}{n}
\end{array}, \quad \mathbf{1}_{\text{TL}_n} = \begin{array}{l}
\vdots \quad \vdots \\
\frac{i-1}{n-1} \quad \frac{n-i-1}{n-i-1}
\end{array}, \quad (12)$$

where the opposing vertical sides are identified, so these drawings should be imagined as being drawn on a cylinder, with the top and bottom black lines resting on its top and bottom edge, respectively. Strands that connects both edges of the cylinder are called through lines. One can show that every diagram which can be drawn on this cylinder with $n$ non-intersecting strings represents a non-zero element of the algebra, and every such element is represented by a unique
\[
\left( \begin{array}{c}
\vdots
\end{array} \right) \equiv (-q)^{1/2} \left( \begin{array}{c}
\vdots
\end{array} \right) + (-q)^{-1/2} \left( \begin{array}{c}
\vdots
\end{array} \right)
\]

\[g_i = (-q)^{1/2} 1 + (-q)^{-1/2} e_i = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}\]

\[g_i^{-1} = (-q)^{-1/2} 1 + (-q)^{1/2} e_i = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}\]

**Figure 3:** Braid notations

diagram, up to isotopy of the strands which is ambient on the boundary. Sums of elements of the algebra can be understood as formal sums of diagrams, and products in the algebra are computed using **diagram composition**\(^4\): the diagram \(ab\) is defined by putting the diagram for \(a\) on top of the diagram for \(b\) and joining the strands that meet. A closed arc that is homotopic to a point is simply removed and replaced by a factor \(q + q^{-1}\). For instance, here are some of the defining relations of the algebra in the diagrammatic presentation (for \(n = 3\)):

\[e_1e_1 = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} = (q + q^{-1})e_1, \quad e_1e_2e_1 = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} = e_1.\]

(13)

For graphical presentation of the blobbed generators, we introduce first the **braid notation** for the overlapping strands in Fig. 3, as well as the diagram presentation of \(g_i^{\pm 1}\) introduced in (7). Then using (12) we get by stacking the diagrams:

\[g_1^{-1} \ldots g_{n-1} u^{-1} = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}\]

(14)

and a similar calculation for \(ug_{n-1} \ldots g_1\). Therefore, the blob generators \(b\) and \(b^{-1}\) from (5)-(6) can be represented as

\[b = (-q)^{-3/2} \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}, \quad b^{-1} = (-q)^{3/2} \begin{array}{c}
\begin{array}{c}
\vdots
\end{array}
\end{array}.\]

(15)

It is then straightforward to check the relations (3) using the standard graphical manipulations together with the relations (9).

We recall the central element \(Y = -(qb + q^{-1}b^{-1})\). In the diagram basis, it can be written as

\(^4\)In this work, product of operators are read from right to left, and diagrams are read from top to bottom. In some of the authors previous work, for instance in [16], the opposite convention is used so operators were multiplied left to right and diagrams read from bottom to top.
\[ Y = (-q)^{-\frac{1}{2}} \begin{array}{c}
\cdots \\
n-1
\end{array} + (-q)^{\frac{1}{2}} \begin{array}{c}
\cdots \\
n-1
\end{array} = \begin{array}{c}
\cdots \\
n
\end{array}, \]

where for the last equality we also used the braid conventions in Fig. 3. That \( Y \) is central is easy to check using the diagrammatic calculation as in Fig. 2: generators \( e_j \) obviously commute with the insertion of a line going “above” or “under” the system, the same applies for the commutativity with the shift operators where one just uses the braid relations.

Recall now the algebra automorphism \( b \rightarrow b^{-1}, q \rightarrow q^{-1} \) discussed above (10). The diagram presentation for the second set of blobbed generators is

\[ \bar{b} = (-q)^{-3/2} \begin{array}{c}
\cdots \\
n-1
\end{array}, \quad \bar{b}^{-1} = (-q)^{3/2} \begin{array}{c}
\cdots \\
n-1
\end{array}. \quad (16) \]

The second representative of the blob generators \( \bar{b} \) and \( \bar{b}^{-1} \) allows us to identify the second distinct central element \( \bar{Y} \):

\[ \bar{Y} \equiv -(q\bar{b} + q^{-1}\bar{b}^{-1}) = \begin{array}{c}
\cdots \\
n
\end{array}. \quad (17) \]

We will show in section 2.4 that for any generic value of \( q \) the two central elements \( Y \) and \( \bar{Y} \) generate the center of \( aTL_n(q) \).

Finally, note that the definition of the hoop operator is strikingly similar to that of a central element in the regular Temperley-Lieb algebra [42], \( F_n \), represented by the diagram:

\[ F_n = \begin{array}{c}
\cdots \\
n
\end{array}. \quad (18) \]

In terms of generators, it can also be written \( F_n = -q\bar{b} - q^{-1}\bar{b}^{-1} \), with

\[ \bar{b}^{-1} = q^3g_1g_2\cdots g_{n-1}g_{n-2}g_{n-3}\cdots g_1, \quad (19) \]

and it satisfies the relations

\[ e_1\bar{b}e_1 = F_ne_1, \quad (20) \]
\[ e_i\bar{b} = \bar{b}e_i \quad i = 2, \ldots n - 1. \quad (21) \]

### 2.3 Standard modules

We present a brief overview of the most common class of \( aTL_n \equiv aTL_n(q) \) modules (see [32] for details): the standard modules \( W_{k,z}(n) \); these are indexed by a non-negative integer \( 2k \leq n \) (so \( k \) is a half-integer), of the same parity as \( n \), and a non-zero complex number \( z \). The simplest way of describing their basis is in terms of diagrams having \( n \) (2\( k \)) nodes on their top (bottom) side, and having exactly 2\( k \) through lines. One simply takes the formal sums of every such diagrams, and use diagram composition to describe the action of the algebra (by stacking an algebra diagram
on the top), understanding that if the composition produces a diagram with less than \(2k\) through lines, it is identified with the zero element. For instance, in the case of \(k = 1\) we have

\[
\begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} = \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array}, \quad \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} = 0.
\]

(22)

This is the way standard modules \(S_k(n)\) are defined for the regular Temperley-Lieb algebra \(\mathcal{TL}_n(q)\), by simply excluding the diagrams with strings crossing the imaginary boundary on each side of the diagrams; while for \(\mathcal{TL}_n(q)\) such diagrams form a finite dimensional module, it is not true for the affine version \(\mathfrak{a}\mathcal{TL}_n(q)\), as e.g. the translation generators \(u^{\pm 1}\) produce states with arbitrary winding of through lines. To get a finite dimensional module for \(\mathfrak{a}\mathcal{TL}_n(q)\), one must also fix the eigenvalues of the two central elements identified in the previous subsection:

\[
\begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} = q \cdot \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} + q^{-1} \cdot \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array}, \quad \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} = q^{-1} \cdot \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} + \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array}.
\]

(23)

where the left side of the first equality is the right action of \(u\) while the left side of the second equality is the right action of \(u^{-1}\). The eigenvalue of the central element \(u^n\) is thus \(z^n\). It was shown in [32] that the endomorphism ring of standard modules is one dimensional, so any central element must act like a multiple of the identity on a standard module; finding the eigenvalue is then simply a matter of choosing a convenient element \(x\) such that computing \(Yx\) is easy. For example, using \(x\) which is filled by non-nested arcs from the right and the rest are the \(2k\) through lines, we find that the choice (23) for the action of \(u\) also fixes the eigenvalues of the central elements \(Y\) and \(\bar{Y}\) as follows:

\[
\begin{align*}
Y &= -(qb + q^{-1}b^{-1}) = z(-q)^{k} + z^{-1}(-q)^{-k}, \\
\bar{Y} &= -(q\bar{b} + q^{-1}\bar{b}^{-1}) = z(-q)^{-k} + z^{-1}(-q)^{k}.
\end{align*}
\]

(24)

To see this, we first recall the diagram presentation for \(b\) in (15). Applying then \(-qb\) to the chosen \(x\) and expanding the braid-crossings according to the rules in Figure 3, only one configuration has a non-zero contribution that corresponds to the factor \(z^{-1}(-q)^{-k}\). As an example of such a calculation for \(k = 1, n = 4\), we have

\[
\begin{align*}
(-q)b \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} &= (-q)^{-\frac{1}{2}} \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} = (-q)^{-\frac{1}{2}} \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array}, \\
&= (-q)^{-1} \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array} + \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array}, \\
&= z^{-1}(-q)^{-1} \begin{array}{c}
\quad \\
\quad \\
\quad \\
\end{array}.
\end{align*}
\]

(25)

(26)

(27)

A similar calculation can be done for \(\bar{b}^{\pm 1}\) confirming the result in (24).

It shall be convenient in what follows to use the notation

\[
\begin{align*}
W_{\pm |k|,\delta}^{\mu}(n) &\equiv W_{|k|,\delta \pm 1(-q)^{-k}}(n), & W_{\pm |k|,\mu}^{\nu}(n) &\equiv W_{|k|,\mu \pm 1(-q)^{k}}(n).
\end{align*}
\]

(28)
which fixes the eigenvalue of \( Y = \delta + \delta^{-1} \), or \( \bar{Y} = \mu + \mu^{-1} \), respectively, and the superscript \( o/u \) refers here to the central element being fixed: the one with a horizontal line going over (\( Y \)) or under (\( \bar{Y} \)) all others.

We conclude this section with a description of the structure of these modules.

**Proposition 2.1.** For \( q \in \mathbb{C}^* \), there exists a non-zero homomorphism \( f: W_{s,w}(n) \rightarrow W_{r,z}(n) \) if and only if \( s \geq r \) and \( Y, \bar{Y} \) have the same eigenvalues on both modules; furthermore any such morphism is proportional to the identity (for \( s = r \)) or to a unique injective map (for \( s > r \)).

The proof is straightforward and follows from the results in [32]. Indeed we observe that the condition on equality of the eigenvalues of \( Y \) and \( \bar{Y} \) is equivalent to the Graham-Lehrer condition [32]:

\[
    z = \begin{cases} 
        w(-q)^{r-s} & \text{if } (-q)^{2(r-s)} = 1 \text{ or } w^2 = (-q)^{-2r}, \\
        w^{-1}(-q)^{r+s} & \text{if } (-q)^{2(r+s)} = 1 \text{ or } w^2 = (-q)^{2r}.
    \end{cases}
\]

(29)

Let us also note that each standard module has a unique simple quotient denoted by \( \bar{W}_{r,z}(n) \), and these form a complete set of irreducible modules [32].

**2.4 The center of \( \mathfrak{aTL}_n(q) \)**

In this section we prove one of our main results about the algebra \( \mathfrak{aTL}_n(q) \).

**Theorem 2.1.** For all \( q \in \mathbb{C}^* \) not an integer root of 1, the center of \( \mathfrak{aTL}_n(q) \) is generated by the two hoop operators, \( Y \) and \( \bar{Y} \).

Before proving this theorem, we establish a few technical results.

**Definition 2.1.** Let \( z \in \mathbb{C} \), define \( \pi_z \) as the canonical projection from \( \mathfrak{aTL}_n \) onto the quotient algebra \( \mathfrak{aTL}_n/\mathfrak{l}_z \), where \( \mathfrak{l}_z \equiv \mathfrak{aTL}_n(Y - z\mathbb{I}_{\mathfrak{aTL}_n}) \).

**Lemma 2.1.** When seen as a \( \mathbb{C}[Y] \)-module, \( \mathfrak{aTL}_n \) is finite dimensional, and admits a basis \( \{ x_\lambda \}_{\lambda \in \Lambda} \) such that \( \{ \pi_z(x_\lambda) \}_{\lambda \in \Lambda} \) is a basis of the \( \mathbb{C} \)-module \( \mathfrak{aTL}_n/\mathfrak{l}_z \) for all \( z \in \mathbb{C} \).

**Proof.** As a \( \mathbb{C} \)-algebra, \( \mathfrak{aTL}_n \) contains all finite sums of finite products of the generators \( b, b^{-1}, e_1, \ldots, e_{n-1} \) subject to the defining relations (3). However, recall that by definition \( Y = -qb - q^{-1}b^{-1} \), so that \( b^{-1} = -q^2b - qY \). It thus follows that one can remove the generator \( b^{-1} \) and replace it, as a generator, with \( Y \). In particular, this means that, as a \( \mathbb{C}[Y] \)-algebra, \( \mathfrak{aTL}_n \) is generated solely by the generators \( b, e_1, \ldots, e_{n-1} \). In other words, as a \( \mathbb{C}[Y] \)-module, \( \mathfrak{aTL}_n \) is spanned by a set \( \{ x_\lambda \}_{\lambda \in \Lambda} \), for some yet unknown set \( \Lambda \), composed of all elements corresponding to distinct diagrams\(^5\) which can be constructed without using the generator \( b^{-1} \), relying solely on the generators \( b, e_1, e_2, \ldots \).

A generic element \( a \in \mathfrak{aTL}_n \) can thus be written

\[
    a = \sum_{\lambda \in \Lambda} p_\lambda(Y)x_\lambda, \quad p_\lambda(Y) \in \mathbb{C}[Y] \forall \lambda \in \Lambda.
\]

(30)

\(^5\)Note that a generic element of \( \mathfrak{aTL}_n \) can be understood as a sum of diagrams (see section 2.2); we are here specifically looking at elements which can be written as a single diagram.
We must now find \(|\Lambda|\); note that a generic element \(x_\lambda\) which contains the generator \(b\) can be written in the form
\[
x_\lambda = a_1 ba_2 ba_3 \ldots, \quad \text{or} \quad x_\lambda = a_1,
\]
where \(a_1, a_2, a_3, \ldots\) can be written using only \(e_1, e_2, \ldots, e_{n-1}\). However, each \(a_k\) can be put in (reverse) Jones normal form:
\[
a_k = e_{i_1} e_{i_1+1} \ldots e_{i_1+r_1} e_{i_2} e_{i_2+1} \ldots e_{i_2+r_2} \ldots,
\]
where \(i_1 > i_2 > i_3 > \ldots\) and \(i_1 + r_1 > i_2 + r_2 > \ldots\). In particular, \(e_1\) appears at most once. Since \(b\) commutes with all the \(e_i\)s except \(e_1\), it thus follows that one can write
\[
x_\lambda = (\prod_{j=1}^{r}(e_{i_j} \ldots e_{k_j}))(be_1 \ldots e_{k_{r+1}})(be_1 \ldots e_{k_{r+2}}) \ldots,
\]
where \(r \geq 0, i_1 > i_2 > \ldots > i_r, k_1 > k_2 > \ldots\), and we understand that if \(r = 0\), the product corresponds to the identity element. In order to count how many elements of the form (33) there are, we encode them as path on a \((n+1) \times (n+1)\) square lattice and a subset of \(\mathbb{Z}_k\) in the following way; the corresponding path is
\[
(n, n) \rightarrow (i_1, n) \rightarrow (i_1, k_1) \rightarrow (i_2, k_1) \rightarrow (i_2, k_2) \rightarrow \ldots \rightarrow (i_r, k_r) \rightarrow (0, k_r),
\]
while the corresponding set of integers is simply
\[
(k_{r+1}, k_{r+2}, \ldots).
\]
The corresponding path always starts at \((n, n)\), moving along the lattice either left or down, and ends at \((0, k_r)\) arriving from the right, never crossing the diagonal (since \(i_j < k_j\)). Furthermore, one can see that every such path corresponds to a unique product of the form \(\prod_{j=1}^{r}(e_{i_j} \ldots e_{k_j})\) where \((i_j, k_j)\) are the corners of the path. Let thus \(d_{i,k}[n]\) be the number of paths starting at \((n, n)\) and ending at \((i, k)\), moving along the lattice either left or down without crossing the diagonal; since every such path must pass by \((i + 1, k)\) or \((i, k + 1)\),
\[
d_{i,k}[n] = d_{i+1,k}[n] + d_{i,k+1}[n], \quad d_{n,n} = 1,
\]
with boundary conditions \(d_{i,j} = 0\) whenever \(i, j < 0, i, j > n, \text{ or } i > j\). Solving this recurrence system, we find that
\[
d_{1,k}[n] = \left(\frac{2n - k - 1}{n - 1}\right) \frac{k}{n},
\]
Since there are \(2^k\) distinct subset of \(\mathbb{Z}_k\), including the empty set, one concludes that
\[
\dim_{\mathbb{C}[\gamma]}(aTL_n) = |\Lambda| = \sum_{k=1}^{n} \left(\frac{2n - k - 1}{n - 1}\right) \frac{k}{n} 2^k.
\]
Note now that for any \(z \in \mathbb{C}, \{\pi_z(x_\lambda)\}_{\lambda \in \Lambda}\) must generate \(aTL_n/L_z\); to show that this set is a basis however, one must show that the dimension of this algebra is equal to \(|\Lambda|\). Since these
algebras all have the same dimension for all values of $z$ (as $\mathbb{C}$-module), it is sufficient to prove the result for generic values of $z$. For $q, z$ generic, one has

$$\dim_\mathbb{C}(aTL_n/I_z) \geq \sum_{k=-n}^{n} \binom{n}{n-k}^2$$

(38)

$$= \sum_{p=0}^{n} \binom{n}{p} \binom{n}{n-p},$$

(39)

$$= \binom{2n}{n}.$$  

(40)

To see this, note that the standard modules $W_{\omega, \delta}^{\omega}[n]$ is a non-trivial $aTL_n/I_z$ module if and only if $\delta + \delta^{-1} = z$, and it’s dimension is $\binom{n}{n-\delta}$. Since the standard modules are simple for generic values of $q, z$, any complete set of simple modules for the quotient algebra must include them. A simple combinatorial argument\(^6\) then shows that

$$\binom{2n}{n} = \sum_{k=1}^{n} \binom{2n-k-1}{n-1} \frac{k}{2} 2^k.$$  

(41)

and thus $\{\pi_z(x_\lambda)\}_{\lambda \in \Lambda}$ must be a basis of $aTL_n/I_z$. □

**Lemma 2.2.** Let $\{z_\delta\}_{\delta \in \Delta}$ be an infinite set of distinct complex numbers, then

$$\bigcap_{\delta \in \Delta} I_{z_\delta} = 0.$$  

(42)

**Proof.** Let $a \in \bigcap_{\delta \in \Delta} I_{z_\delta}$, according to lemma 2.1, there must exist $\{p_\lambda(Y)\}_{\lambda \in \Lambda} \subset \mathbb{C}[Y]$ such that

$$a = \sum_{\lambda \in \Lambda} p_\lambda(Y)x_\lambda.$$  

(43)

Furthermore, since $\{\pi_{z_\delta}(x_\lambda)\}_{\lambda \in \Lambda}$ is a basis of $aTL_n/I_{z_\delta}$, it follows that

$$\pi_{z_\delta}(p_\lambda(Y)) = p_\lambda(z_\delta) = 0, \quad \forall \delta \in \Delta,$$  

(44)

so that the polynomial $p_\lambda(\ )$ must have infinitely many roots. The only such polynomial is identically zero, and thus $a = 0$. □

**Lemma 2.3.** Let $p(-)$ be some non-zero polynomial in one variable, and $a \in aTL_n$ be such that $p(Y)a = 0$, then $a = 0$. In particular, $p(Y)a$ is central if and only if $a$ is.

**Proof.** Let $\{z_i\}_{i \in \mathbb{N}} \subset \mathbb{C}$ be a set of non-zeroes of $p$, i.e. numbers such that $p(z_i) \neq 0$. However, since the canonical projection $\pi_{z_i}$ is a homomorphism of algebras,

$$0 = \pi_{z_i}(p(Y)a) = \pi_{z_i}(p(Y))\pi_{z_i}(a) = p(z_i)\pi_{z_i}(a),$$  

(45)

\(^6\)The expression on the left counts the number of sequences of $2n + 1$ consecutive integers, starting and ending with 0; the expression on the right counts the same sequences by first grouping them in subsets where 0 appears $k + 1$ times.
and thus \( a \in I_{z_i} \) for all \( i \in \mathbb{N} \). It thus follows that
\[
a \in \bigcap_{i \in \mathbb{N}} I_{z_i}.
\] (46)

Applying lemma 2.2 then yields \( a = 0 \). Furthermore, since \( Y \) is central, then
\[
p(Y)ab - bp(Y)a = p(Y)(ab - ba), \quad \forall b \in a\mathbb{TL}_n.
\] (47)
The left side of this equation is zero for all \( b \) if and only if \( p(Y)a \) is central, while the right one is if and only if \( a \) is central, by the first part of this lemma.

**Lemma 2.4.** For all \( n \in \mathbb{N} \),
\[
\prod_{k=n \mod 2}^n Q_k = 0,
\] (48)
where
\[
Q_k \equiv \begin{cases} 
Y - \bar{Y} & k = 0 \\
n^{2} + \bar{n}^{2} - ((-q)^{k} + (-q)^{-k})Y\bar{Y} + (q^{k} - q^{-k})^{2} & k \neq 0
\end{cases} \in a\mathbb{TL}_n.
\] (49)

**Proof.** Using (24), one verifies that \( Q_k \) takes the eigenvalue 0 on \( \mathbb{W}_{k,w}(n) \) for all \( n \in \mathbb{N} \) and all \( w \in \mathbb{C}^* \). In particular, it follows that \( Q = \prod_{k=n \mod 2}^n Q_k \) takes the eigenvalue zero on all simple \( a\mathbb{TL}_n \)-modules. Since there are infinitely many values of \( z \) for which \( a\mathbb{TL}_n/I_z \) is semisimple, it follows that \( Q \) is in the intersection of infinitely many \( I_z \), and must therefore be zero by lemma 2.2.

**Proof of theorem 2.1.** Let \( \{c_i\}_{i=1}^N \) be a basis of the center of \( a\mathbb{TL}_n \), seen as a \( \mathbb{C}[Y] \)-module; such a basis exists by lemma 2.1. Let then \( z \in \mathbb{C} \) be such that \( \{\pi_z(c_i)\}_{i=1}^N \) is not linearly independent over \( \mathbb{C} \), i.e. there exists \( \{y_i^{(z)}\}_{i=1}^N \subset \mathbb{C} \) such that
\[
\sum_{i=1}^N y_i^{(z)} \pi_z(c_i) = 0.
\] (50)
By definition of the canonical projection, it thus follows that there exist some \( a \in a\mathbb{TL}_n \) such that
\[
\sum_{i=1}^N y_i^{(z)} c_i = a(Y - z).
\] (51)
However, by lemma 2.3, \( a \) must be central, so there exist a set \( \{p_i^{(z)}(Y)\}_{i=1}^N \subset \mathbb{C}[Y] \) such that
\[
(Y - z) \sum_{i=1}^N p_i^{(z)}(Y)c_i = \sum_{i=1}^N y_i^{(z)} c_i,
\] (52)
and since by assumption \( \{c_i\}_{i=1}^N \) is a basis, one must have
\[
(Y - z)p_i^{(z)}(Y) = y_i^{(z)}.
\] (53)
Since \((Y - z)\) does not have an inverse in \(\mathbb{C}[Y]\), it thus follows that \(y^{(z)}_i = 0\) for all \(i = 1, \ldots, N\). We have thus proven that the dimension of the center of \(aTL_n/I_z\), seen as a \(\mathbb{C}\)-module, must be greater than the dimension of the center of \(aTL_n\) seen as a \(\mathbb{C}[Y]\)-module.

However, if \(q\) is not a root of 1, for generic values of \(z\), i.e. not of the form \(z_t = -(q^t - q^{-t})\) for some \(t \in \mathbb{Z}\), the quotient algebra \(aTL_n/I_z\) is isomorphic to the finite dimensional blob algebra, which is then semisimple (see the discussion below (3) as well as [36] for the structure of this algebra). In particular, it has \(n + 1\) distinct simple modules, so its center has dimension \(n + 1\). It then follows that the center of \(aTL_n\), seen as a \(\mathbb{C}[Y]\)-algebra, is at most of dimension \(n + 1\).

Finally, note that for generic values of \(z\), \(\pi_z(\hat{Y})\) has different eigenvalues on all simple \(\mathbb{C}\)-algebra, and then finding it’s preimage in \(aTL_n\), however, one could extend this result to the root of unity case by finding the center of the Blob algebra, and then finding it’s preimage in \(aTL_n\).

Also of note is that, by lemma 2.4, \(\hat{Y}^{n+1}\) is a polynomial of order \(n + 1\) in \(Y\) and of order \(n\) in \(\hat{Y}\); since the center has to have dimension \(n + 1\) as a \(\mathbb{C}[Y]\)-module, it follows that there cannot be any identity of lower order linking powers of \(Y\) and \(\hat{Y}\).

Note also that while the theorem states that central elements can be expressed as a polynomial in \(Y\) and \(\hat{Y}\), finding such an expression for a given central element can be very challenging. For instance, two well known central elements are \(u[n]^n\) and its inverse \(u[n]^{-n}\), which can be expressed in terms of Jucys-Murphy elements\(^7\) as

\[
    u[n]^n = (-q)^{3n/2} J_n J_{n-1} \cdots J_1, \quad u[n]^{-n} = (-q)^{3n/2} M_1 M_2 \cdots M_n, \quad (54)
\]

which is readily verified by simply drawing the corresponding diagrams; for instance if \(n = 3\),

\[
    (-q)^{3/2} J_3 J_2 J_1 = \begin{array}{c}
    \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots
    \end{array}
    \end{array}, \quad \text{and} \quad (-q)^{3/2} M_1 M_2 M_3 = \begin{array}{c}
    \begin{array}{c}
    \vdots \\
    \vdots \\
    \vdots
    \end{array}
    \end{array}.
\]

Using the formulas in section 3.3 to express the Jucys-Murphy elements in terms of the hoop operators, we find

\[
    u[2]^2 = I_{aTL_2} + \frac{Z_2Z_0}{(q^2 - q^{-2})(q - q^{-1})}, \quad (56)
\]

\[
    u[2]^{-2} = I_{aTL_2} + \frac{Z_2Z_0}{(q^2 - q^{-2})(q - q^{-1})}, \quad (57)
\]

\[
    u[3]^3 = \frac{(Z_3 Z_1 Z_{-1} - (q^2 - q^{-2})^2(q + q^{-1})Z_1 - (q - q^{-1})^2 Z_{-1})}{(q^3 - q^{-3})(q^2 - q^{-2})(q - q^{-1})}, \quad (58)
\]

\[
    u[3]^{-3} = -\frac{(Z_3 Z_1 Z_{-1} - (q^2 - q^{-2})^2(q + q^{-1})Z_{-1} - (q - q^{-1})^2 Z_1)}{(q^3 - q^{-3})(q^2 - q^{-2})(q - q^{-1})}, \quad (59)
\]

---

\(^7\)See section 3.3
where
\[ Z_k = (-q)^{k/2}Y - (-q)^{-k/2}ar{Y}, \quad k \in \mathbb{Z}. \] (60)

Note that these expressions may not be well-defined if \( q \) is a root of 1, so the condition that \( q \) be generic is necessary. As an example, one can verify directly that in the twisted XXZ spin chain (see section 4.5) at \( q = 1 \), \( u[n]^n \) is algebraically independent of the two hoop operators.

Let us also mention here that the center of \( aTL_n(q) \) has two interesting properties, which will be proven in Sections 3.3, and 3.2, respectively:

1. The center is the image of the center of the affine Hecke algebra under the standard covering map: \( \hat{H}_n(q) \to aTL_n(q) \), \( T_i \mapsto g_i \), \( J_i \mapsto J_i \), see Section 3.3 for definitions. Recall that the center of \( \hat{H}_n(q) \) is spanned by symmetric polynomials in the Jucys-Murphy elements \( J_i \), for \( 1 \leq i \leq n \) (see more about the affine Hecke algebra in [33, Sec. 3] where the Jucys-Murphy elements are denoted by \( X_i \)).

2. There is a special “canonical” basis (made of Chebyshev polynomials) such that the structure constants in the center are non-negative integers, i.e. the center of \( aTL_n(q) \) endowed with this basis is a Verlinde algebra.

The second point is very important for our defect operators construction, and we will see below in Section 3 that the central elements in the canonical basis provide (on certain \( aTL \) representations) operators that represent topological defects in the crossed channel.

Finally one should point out that for a decomposable module, the centralizer is often very different from the center of the algebra. For instance, for any positive integer \( N \), the centralizer of \( M_N = \bigoplus_{i=1}^{N} W_{2z}(2) \) is isomorphic to the algebra of \( N \times N \) matrices with complex coefficients, but any central element will have the same eigenvalue on each copy of \( W_{2z}(2) \).

### 2.5 Tower structure

The family of affine Temperley-Lieb algebras admits inclusions of the form (we will often abbreviate \( aTL_n \equiv aTL_n(q) \))
\[ aTL_n \subset aTL_{n+1} \subset aTL_{n+2} \subset \ldots , \quad n \geq 1 , \]
giving the structure of a tower of algebras [13, Sec. 3.3]. Some of these inclusions will play a role in our construction of topological defects so we describe them here. We now assume that \( k \) is a positive integer, and define a morphism\(^8\) of algebras
\[ \phi_{n,k}^u : aTL_n \to aTL_{n+k} , \] (61)

by its action on the various sets of generators of the algebra. For clarity, we add a superscript to the generators to indicate which algebra they belong to; for instance \( u^{(n)} \) is the shift generator in \( aTL_n \), while \( u^{(n+2)} \) is the shift generator in \( aTL_{n+2} \), etc. With this notation, the map \( \phi_{n,k}^u \) on the blobbed set of generators is
\[ \phi_{n,k}^u : \begin{align*}
(b^{(n)})^{\pm 1} & \mapsto (b^{(n+k)})^{\pm 1}, \\
e_i^{(n)} & \mapsto e_i^{(n+k)}.
\end{align*} \] (62)

\[ \phi_{n,k}^{\epsilon_i} : e_i^{(n)} \mapsto e_i^{(n+k)} . \] (63)

\(^8\)For brevity, we will use the term “morphism” instead of the more standard “homomorphism”.

17
It is straightforward to verify that $\phi_{n,k}^u$ defines an inclusion of algebras. We note that this definition is parallel to what was done in affine Hecke algebra terms in [13, Sec. 4.4.2]. While the map is very simple with the blobbed generators, it is more complicated when expressed on the periodic set of generators, for instance

$$\phi_{n,k}^u: \quad u^{(n)} \mapsto u^{(n+k)} g_{n+k-1} g_{n+k-2} \cdots g_n = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \quad (64)$$

which agrees with [13, Eq. (3.9)]. One therefore sees that, in terms of diagrams, the morphism $\phi_{n,k}^u$ consists in adding $k$ through lines on the right side of each diagrams, going under every lines that wraps around the cylinder (hence the superscript $u$ on the morphism).

Similarly, one defines another morphism of algebras

$$\phi_{n,k}^o: \quad \mathfrak{aTL}_n \to \mathfrak{aTL}_{n+k} \quad (65)$$

by adding the $k$ lines over the lines that wrap around the cylinder, i.e. on the periodic set of generators the map is

$$\phi_{n,k}^o: \quad u^{(n)} \mapsto u^{(n+k)} (g_{n+k-1})^{-1} (g_{n+k-2})^{-1} \cdots (g_n)^{-1} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array} \quad (66)$$

On the blobbed set of blob type generators, the map is simply

$$\phi_{n,k}^o: \quad (\bar{b}^{(n)})^\pm 1 \mapsto (\bar{b}^{(n+k)})^\pm 1, \quad e_i^{(n)} \mapsto e_i^{(n+k)}. \quad (67)$$

$$\psi_{n,k}^{u/o}: \quad \mathfrak{aTL}_n \to \mathfrak{aTL}_{n+k}, \quad (69)$$

for the corresponding under and over versions. We then notice that the two subalgebras $\phi_{n,k}^u(\mathfrak{aTL}_n)$ and $\psi_{k,n}^o(\mathfrak{aTL}_k)$ commute with each others; this can be seen by a direct calculation as in [13] or showing that

$$\phi_{n,k}^u(\bar{b}^{(n)}) \propto J_1^{(n+k)}, \quad \psi_{k,n}^o(\bar{b}^{(k)}) \propto J_{n+1}^{(n+k)}, \quad (68)$$

where $J_i$ is the Jucys-Murphy element\(^9\) of the affine Temperley-Lieb algebra (see Section 3.3). This fact can be exploited to define a monoidal structure on the affine Temperley-Lieb category [13], see also [14] for the corresponding fusion calculation. We note that $\phi_{n,k}^o(\mathfrak{aTL}_n)$ and $\psi_{k,n}^u(\mathfrak{aTL}_k)$ also commute.

### 2.6 Tile formalism and the transfer matrix

While we formulate most of our results in terms of diagrams with strings and arcs on a cylinder, a very significant body of work on this subject is written in terms of planar tiles (see for instance [37, \footnote{The Jucys-Murphy elements form a commutative subalgebra.}].
we present here a brief translation between the two formalisms and use it to introduce the usual transfer matrix.

The planar tile with spectral parameter \( x \) is defined by (where the corner mark allows to keep track of the orientation of the tile)\(^{10}\)

\[
\begin{align*}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\end{align*}
\]

These satisfy three particular relations:

\[
\begin{align*}
\begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\end{align*}
\]

These are respectively called the inversion and the Yang-Baxter identities, and the crossing symmetry.

The transfer matrix \( T_n(\vec{x}) \) can then be defined as

\[
T_n(\vec{x}) = \prod_{i=1}^{n} x_i = (\prod_{i=1}^{n} x_i)^{-1}, \quad \vec{x} = \{x_1, x_2, \ldots, x_n\},
\]

where there are \( n \) tiles and the opposing vertical sides are identified so that this defines an element of \( aTL_n(q) \) for each \( n \)-dimensional vector \( \vec{x} \). If \( x_1 = x_2 = \ldots = x_n \) the transfer matrix is said to be homogeneous and is simply written \( T_n(x_1) \). Using the three previous identities, one readily shows that homogeneous transfer matrices commute with each others, i.e. \([T_n(x), T_n(y)] = 0\), and thus define families of integrable lattice models.

We note four specific cases of the homogeneous transfer matrix. Setting the spectral parameter \( x \) to 1 or \( q \) gives the translation operators \( u^\pm 1 \):

\[
\begin{align*}
T_n(1) &= (q - q^{-1})^n = (q - q^{-1})^n u^{-1}, \\
T_n(q) &= (q - q^{-1})^n = (q - q^{-1})^n u,
\end{align*}
\]

while taking the limits in the spectral parameter to zero or infinity produces the two hoop operators \( Y \) and \( \bar{Y} \):

\[
\lim_{x \to 0} ((-q^{-\frac{1}{2}} x)^n T_n(x)) = \prod_{i=1}^{n} x_i = Y,
\]

\[\text{This tile is often divided by } (q - q^{-1}) \text{ to normalize it, but then the natural defect operator would be } (q - q^{-1})^{-n}Y \text{ instead of } Y.\]
\[
\lim_{x \to \infty} \left( \left( -q^{-1} \right)^{-n} T_n(x) \right) = \begin{array}{ccccccccccc}
\circ & \cdots & \circ
\end{array} = Y. \tag{78}
\]
Finally, the standard periodic Temperley-Lieb Hamiltonian \( H_n = \sum_{j=1}^n e_j \) can be obtained as the logarithmic derivative evaluated at \( x = 1 \) of the homogeneous transfer matrix \( T_n(x) \), see e.g. [40].

3 Lattice topological defects: crossed channel

In this section, we formulate our lattice topological defects in terms of the affine TL algebra using the hoop operators – the central elements \( Y \) and \( \bar{Y} \) introduced in the previous section – and describe their fusion rules. In more mathematical terms, we prove that the center \( Z(\text{aTL}) \) of \( \text{aTL}_n \) admits a certain basis with non-negative integer structure constants. Interestingly, at least for generic values of \( q \), the structure constants do not depend on \( n \) or \( q \). Then we also show that \( Z(\text{aTL}) \) agrees with the algebra of symmetric Laurent polynomials in the famous Jucys-Murphy elements, and give a precise relations in terms of Chebyshev polynomials of \( Y \) and \( \bar{Y} \).

3.1 The algebra of defects \( Y \) and \( \bar{Y} \)

Recall that the hoop operators defined in Section 2.2 can be represented by diagrams with a single closed string wrapping over or under all the other strings:

\[
Y = - (qb + q^{-1}b^{-1}) = \begin{array}{ccccccccccccccc}
\circ & \cdots & \circ \\
\end{array}, \quad \bar{Y} = - (qb + q^{-1}b^{-1}) = \begin{array}{ccccccccccccccc}
\circ & \cdots & \circ \\
\end{array}, \tag{79}
\]

and these are central elements in \( \text{aTL}_n(q) \). This wrapping string can be isotopically deformed at will without changing the spectrum of the transfer matrix from Section 2.6, and it thus can be thought of as a defect line (in the crossed channel). We are interested in the algebra generated by these hoop operators, and first study their powers.

Taking powers of the hoop operators will increase the width of the defects by increasing the number of lines going across the system; one can then imagine Temperley-Lieb operators acting horizontally on the defect. For instance, we can introduce the operators

\[
Y^2(e_1) = \begin{array}{ccccccccccccccc}
\circ & \cdots & \circ \\
\end{array}, \quad \bar{Y}^2 = (q + q^{-1})1_{\text{aTL}_n}, \tag{80}
\]

\[
Y^3(e_1e_2) = \begin{array}{ccccccccccccccc}
\circ & \cdots & \circ \\
\end{array} = Y. \tag{81}
\]

One recognize that this corresponds to taking a Markov trace in the horizontal direction; in particular, the operator \( Y^m \) can be seen as a map from \( \text{TL}_m \) to the ring of endomorphisms of \( \text{aTL}_n \):

\[
Y^m : \text{TL}_m \to \text{End}_{\text{aTL}_n}(\text{aTL}_n), \tag{82}
\]

where a given element in \( \text{TL}_m \) considered as a diagram is just placed on the \( m \) horizontal strands, as in Fig. 4. It is easy to see that the image of this map lives in the center of \( \text{aTL}_n \), and the central elements provide an endomorphism via the multiplication. We similarly define the mapping

\[
\bar{Y}^m : \text{TL}_m \to \text{End}_{\text{aTL}_n}(\text{aTL}_n) \tag{83}
\]
whose image is also in the center of $a\mathbf{TL}_n$, and that can be represented graphically similarly to Fig. 4, however with horizontal lines going under the vertical ones.

### 3.2 Higher-spin operators $Y_j$ and $\bar{Y}_j$

Instead of applying the defect operators $Y^m$ on individual elements of the Temperley-Lieb algebra, we can have them act on an entire ideal, sending each to a sub-ring of the ring of endomorphisms of $a\mathbf{TL}_n$. If $q$ is generic, every indecomposable left-ideal of $\mathbf{TL}_m$ is isomorphic to one of the form $S_j(m) = \mathbf{TL}_m P_j$, where $P_j$ is an idempotent of spin $j$ (a primitive idempotent in $\mathbf{TL}_m$ such that $\mathbf{TL}_m P_j$ is isomorphic to the standard representation with $2j$ through lines). When $j = m/2$ one can use the Jones-Wenzl projectors

$$P_{m/2} = W^m_{1/1},$$

defined recursively through the following formula:

$$W^1_i(n) \equiv W^2_i(n) \equiv 1_{\mathbf{TL}_n},$$

$$W^m_i(n) \equiv W^{m-1}_{i+1}(n) \left( 1_{\mathbf{TL}_n} - \frac{q^{m-2} - q^{2-m}}{q^{m-1} - q^{1-m}} e_i \right) W^{m-1}_{i+1}(n),$$

where the index $m$ is related to the spin as above, and $i$ is just the lattice position.

Recall that $P_j$ is an idempotent, i.e. $P_j P_j = P_j$, and the map $Y^m$ has the property of a trace, we then have

$$Y^m(x P_j) = Y^m(P_j x P_j)$$

for all $x \in \mathbf{TL}_m$. By construction, $P_j x P_j$ is an endomorphism of the ideal $S_j(m)$ (by multiplication on the right), which is simple whenever $q$ is generic; it follows that $P_j x P_j = \lambda^e P_j$ for some $\lambda^e \in \mathbb{C}$, and thus that

$$Y^m(S_j(m)) = \lambda^e Y_j,$$

where we introduced a special central element

$$Y_j := Y^{2j}(W^{2j+1}_1).$$

Here, we used the fact that the trace of $P_j$ is independent both of $m$, and of the particular choice of $P_j$ we made (see Appendix A.1 for details of the proof). In particular, the identity (86) makes sense and is true for any valid value of $m$ when the ideal $S_j(m)$ is non-zero.
Using the recurrence relation (85) for the Jones-Wenzl projectors, we find in Appendix A.1 that
\[ Y_j = U_{2j}(Y/2), \] 
(88)
where \( U_k(x) \) is the Chebyshev polynomial of the second kind, of order \( k \). For instance, we have
\[
Y_{1/2} = Y,
Y_1 = (Y_{1/2})^2 - 1,
Y_{3/2} = (Y_{1/2})^3 - 2Y_{1/2},
Y_2 = (Y_{1/2})^4 - 3(Y_{1/2})^2 + 1.
\]
Recall that \( Y \) acts on \( W_{k,\delta}^\circ \) as \((\delta + \delta^{-1})\); writing \( \delta = e^{i\theta} \), the higher-spin operator eigenvalues are thus
\[ Y_j = \frac{\sin((2j + 1)\theta)}{\sin \theta}. \]
(88)

The important observation is that the properties of the Chebyshev polynomials allow us to decompose products of \( Y_j \)'s:
\[ Y_j \cdot Y_k = \sum_{r=|j-k|}^{j+k} Y_r. \] 
(89)

We finally note that the whole construction of this section would work equally well if the defect had been going under the strings instead of over them, by simply replacing \( Y \) with \( \bar{Y} \) everywhere it appears. We begin with the map \( \bar{Y}_m \) defined in (83). Its properties are identical to those of the map \( Y^m \) in every way; applying it to the ideals \( S_j(m) \) yields higher-spin defect operators \( \bar{Y}_j \) whose eigenvalues on \( W_{k,\delta}^\circ \) are
\[ \bar{Y}_j = \frac{\sin((2j + 1)(\phi))}{\sin \phi}, \]
where \( \delta \equiv e^{i\phi} \). And they have similarly the fusion
\[ \bar{Y}_j \cdot \bar{Y}_k = \sum_{r=|j-k|}^{j+k} \bar{Y}_r. \] 
(90)

Using Theorem 2.1, we see that the algebra generated by \( Y_j \) and \( \bar{Y}_j \), for all non-negative half-integer \( j \), is identified with the whole center \( Z(a\mathcal{TL}) \).

Finally, we note that by theorem 2.1, the center of the algebra is spanned by polynomials in \( Y \) and \( \bar{Y} \); since the Chebyshev polynomials form a basis of the space of polynomials in one variable, one can then conclude that any element in the center can be written as a linear combination of \( Y_j, \bar{Y}_k, \) and \( Y_j \cdot \bar{Y}_k \) using (89) and (90). We finally note that for the “mixed” fusion \( Y_j \cdot \bar{Y}_k \) there is, to our knowledge, no interesting decomposition: the relation in lemma 2.4 between \( Y \) and \( \bar{Y} \) is polynomial of order \( n+1 \), so any decomposition of \( Y_j \cdot \bar{Y}_k \) would heavily depend on the size of the lattice. In particular, in the continuum limit of the model, one would then expect the two hoop operators to become algebraically independent.
3.3 Relation to symmetric polynomials.

While identifying the topological defect operators with the hoop operator is an intuitive choice, there are many other known central elements, which could also lead to topological defects. These are built from the so-called Jucys-Murphy elements; let

\[ J_1 \equiv b, \quad J_i \equiv g_{i-1}J_{i-1}g_{i-1} = (-q)^{-3/2} \quad \text{for} \quad i = 2, \ldots, n, \quad (91) \]

\[ M_1 \equiv b, \quad M_i \equiv g_{i-1}M_{i-1}g_{i-1} = (-q)^{-3/2} \quad \text{for} \quad i = 2, \ldots, n. \quad (92) \]

It is straightforward, though tedious, to prove that all \( J_i \), for \( 1 \leq i \leq n \), commute with each others and so do the \( M_i \), see [29, 33]. Furthermore, \( J_i \) and \( M_i \) are invertible and if \( P(x_1, \ldots, x_n) \) is a symmetric Laurent polynomial, then \( P((-q)J_1, \ldots, (-q)^nJ_n) \) and \( P((-q)^1M_1, \ldots, (-q)^nM_n) \) are central in \( aTL_n \). All of these can be generated from the power-sum symmetric polynomials

\[ C_k(n) = \sum_{i=1}^{n} ((-q)^{i+1}M_i)^k, \quad C_k(n) = \sum_{i=1}^{n} ((-q)^{i+1}J_i)^k, \quad k \in \mathbb{Z}. \quad (93) \]

However, it turns out that these are related to the hoop operators through the following relations:

\[ C_k(n) + C_{-k}(n) = (-q)^{-nk} \tilde{C}_k(n) + (-q)^{nk} \tilde{C}_{-k}(n) = 2[n]_k \mathbb{T}_k(Y/2), \quad (94) \]

\[ \tilde{C}_k(n) + \tilde{C}_{-k}(n) = (-q)^{-nk}C_k(n) + (-q)^{nk}C_{-k}(n) = 2[n]_k \mathbb{T}_k(Y/2), \quad (95) \]

where we defined

\[ [n]_k \equiv \frac{(-q)^{kn} - (-q)^{-kn}}{(-q)^{k} - (-q)^{-k}}. \]

and here it is understood that \([n]_0 \equiv n\), and \( \mathbb{T}_k(x) \) is the \( k \)th Chebyshev polynomial of the first kind. The proof of these identities can be found in Appendix A.2 and it uses the important relations that hold in \( aTL_n \):\(^{11}\)

\[ (-q)^2J_i + (-q)^{-2}J_i^{-1} = (-q)J_{i+1} + (-q)^{-1}J_{i+1}^{-1}. \]

If \((-q)^{nk} \neq 1\), the relations (94)-(95) can be combined to find

\[ ((-q)^k - (-q)^{-k})C_k(n) = 2 ((-q)^{kn}\mathbb{T}_{[k]}(Y/2) - \mathbb{T}_{[k]}(Y/2)), \quad (96) \]

\[ ((-q)^k - (-q)^{-k})C_k(n) = 2 ((-q)^{kn}\mathbb{T}_{[k]}(Y/2) - \mathbb{T}_{[k]}(Y/2)). \quad (97) \]

Finally, using the properties of the Chebyshev polynomials it follows that

\[ Y_{k/2} = \sum_{j=-k}^{k} \frac{1}{[n]_j} \tilde{C}_j(n), \quad \tilde{Y}_{k/2} = \sum_{j=-k}^{k} \frac{1}{[n]_j} C_j(n). \quad (98) \]
4 Lattice topological defects: direct channel

In this section, we are interested in interpretation of previously introduced defects $Y_j$ and $\bar{Y}_j$ in the direct channel, or in their Hamiltonian realization. We will consider only the case of generic $q$. The action of the defect $Y_{1/2}$ in the direct channel can be inferred by a simple modular transformation – that is, a rotation by $90^\circ$ as in Fig. 5. What this means microscopically is that we should have a system where, on top of the usual TL interaction terms, we have an extra line that simply goes over/under the others, and this contributes to defect terms in the Hamiltonian that we also call “impurities”. Below, we calculate explicitly the Hamiltonians with impurities that correspond to the defect operators $Y_j$ and $\bar{Y}_j$, and study their spectral problem.

The Hamiltonian with $m$ defect lines can be obtained as a logarithmic derivative evaluated at $x = 1$ of the transfer matrix $T_n(x; m)$ given in Fig. 6. In this case, we obtain the Hamiltonian on $n + m$ sites, as an element in $\mathfrak{aTL}_{n+m}$:

$$H_{n,m}^u = \sum_{j=1}^{n-1} e_j^{(n+m)} + \mu_{n,m}^{-1} e_n^{(n+m)} \mu_{n,m},$$

where $\mu_{n,m} = g_n g_{n+1} \cdots g_{n+m}$, with $g_i$ defined in Fig. 3. This derivation is similar to the standard calculation with the homogeneous transfer matrix $T_n(x)$, recall the remark below (78). We are interested in the spectral problem of $H_{n,m}^u$ acting on some module $M$ over $\mathfrak{aTL}_{n+m}$. It is important to note that this Hamiltonian can be written as:\footnote{However note that there relations do not hold in the corresponding affine Hecke algebra that covers $\mathfrak{aTL}_n$.}

$$H_{n,m}^u = \phi_{n,m}^u \left( \sum_{j=1}^{n} e_j^{(n)} \right), \quad (99)$$

i.e. as the image of the standard periodic TL Hamiltonian

$$H_n = \sum_{j=1}^{n} e_j \quad (100)$$

\footnote{This is even clearer on the level of transfer matrices: $T_n(x; m) = \phi_{n,m}^u (T_n(x))$, by definition of $\phi_{n,m}^u$.}
on \( n \) sites under the embedding map \( \phi^u_{n,m} \) defined in Section 2.5. In order to simplify the solution of this spectral problem as much as possible, it is customary to start by searching for elements of \( \text{End}_C(M) \) which commute with \( H^u_{n,m} \). With the property of the embeddings \( \phi^u \) and \( \psi^o \) discussed after (69) and using (99), it is clear that for any \( a \in \mathbb{TL}_m \), \( \psi^o(a) \) commutes with \( H^u_{n,m} \). It follows in particular that, as matrices\(^{13}\),

\[
H^u_{n,m} = \sum_{\lambda \in \Lambda_m} H^u_{n,m}[\lambda],
\]

(101)

where \( \Lambda_m \) is a complete set of primitive orthogonal idempotents of \( \mathbb{TL}_m \), and

\[
H^u_{n,m}[\lambda] \equiv \psi^o_{m,n}(\lambda)H^u_{n,m} = H^u_{n,m}\psi^o_{m,n}(\lambda).
\]

(102)

Note that because the \( \lambda \)'s are orthogonal, these matrices are themselves orthogonal to each others. The spectral problem of \( H^u_{n,m} \) is thus reduced to that of \( H^u_{n,m}[\lambda] \). To solve this reduced spectral problem, we present an algebraic construction linking the spectrum of \( H^u_{n,m}[\lambda] \) with the spectrum of the standard Hamiltonian \( H_n \) acting on a certain “fusion quotient” module of \( a\mathbb{TL}_n(q) \). This requires certain preparation and an algebraic discussion below. We then come back to the spectral problem in Section 4.4 with the final result formulated in Theorem 4.1, and then provide an explicit example based on the twisted XXZ chains in Section 4.5.

Let us begin with the idea that stays behind the two algebraic constructions formulated below. Adding the extra lines/defects in the direct channel can be realised as a functor that combine a module of \( a\mathbb{TL} \) (the bulk model) with a module of \( \mathbb{TL} \) (the defect) into a new module of \( a\mathbb{TL} \) (the bulk model with a defect); it turns out that there are (at least) two natural ways of doing this: one can add new strands carrying the defect to the module, a process we call the fusion product, or one can impose the defect on an existing part of the module, a process we call the fusion quotient.

### 4.1 The fusion product

*This section uses the notation introduced in Section 2.5.*

Let \( m, k \) both be positive integers, we give \( a\mathbb{TL}_{m+k} \) the structure of a \((a\mathbb{TL}_{m+k}, a\mathbb{TL}_m \otimes_C \mathbb{TL}_k)\) bimodule by letting \( a\mathbb{TL}_{m+k} \) act on the left through the natural representation, and \( a\mathbb{TL}_m \otimes_C \mathbb{TL}_k \) acts on the right by the morphism \( \phi^{u/o}_{m,k} \otimes_C \psi^{o/u}_{k,m} \), where we identified \( \mathbb{TL}_k \) with its image in \( a\mathbb{TL}_k \). For \( M \) an \( a\mathbb{TL}_m \) module, and \( V \) a \( \mathbb{TL}_k \)-module, our definition of the fusion product can then be written

\[
M \times_{f}^{u/o} V \equiv a\mathbb{TL}_{m+k} \otimes_{a\mathbb{TL}_m \otimes_C \mathbb{TL}_k} (M \otimes_C V),
\]

(103)

\(^{13}\)We can introduce similarly the versions \( H^o_{n,m} \) and \( H^o_{n,m}[\lambda] \) using the over lines map \( \phi^o \).

---
where the superscript \( u/o \) denotes which one of \( \phi_{m,k}^{u/o} \) we used to define the bimodule structure of \( a\mathsf{TL}_{m+k} \). From a more physical point of view, this corresponds to having a bulk model described by \( M \) which contains an isolated sub-system \( V \), such that they are both entirely blind to each others so that the Hilbert space of the system is simply the tensor product of the Hilbert spaces of \( M \otimes \mathbb{C} V \); at some point one then remove the barrier between the two sub-systems and thus letting \( V \) propagate freely inside \( M \). Note also that this fusion is related, though different, to ones introduced previously in the literature \([13,16]\). We discuss this more in Appendix B where also important properties of the hoop operators are studied in relation to the tower homomorphisms \( \phi^{u/o} \) and \( \psi^{o/u} \).

Before giving the general result for the fusion we give a small example and compute the fusion product of two standard modules \( W_{1/2,z}(3) \times^o_f S_{1/2}(1) \). Since the standard modules are cyclic, their fusion is also, and thus \( W_{1/2,z}(3) \times^o_f S_{1/2}(1) = a\mathsf{TL}_4y \), with

\[
y = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}.
\]

(104)

where we also introduced our diagram notation for the fusion product: the diagram at the top is the element of \( a\mathsf{TL}_4 \), the one on the bottom left corner is the element of \( W_{1/2,z}(3) \), and the one on the bottom right corner is the element of \( S_{1/2}(1) \). Since this module is cyclic, we can choose a basis of the form \( \{a_i y | i = 1, \ldots \} \) for some subset \( \{a_i\} \subset a\mathsf{TL}_4 \); in the case at hand the simplest choice is\(^{14}\)

\[
a_1 = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}, \quad
a_2 = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}, \quad
a_3 = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}, \quad
a_4 = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array},
\]

\[
a_5 = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}, \quad
a_6 = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}, \quad
a_7 = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}, \quad
a_8 = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}, \quad
a_9 = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}, \quad
a_{10} = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}.
\]

(105-107)

It is not trivial at all to show that this set is sufficient, for instance, can \( e_4 a_2 y \) really be expressed as a linear combination of \( a_i y \)? Indeed it can:

\[
e_4 a_2 y = \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array} = z^{-1} \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array} = z^{-1} \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array} = -(-q)^{3/2} z^{-1} a_6 y,
\]

(108)

where the last equality was obtained by using the closed braid identity

\[
\begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array} = -(-q)^{3/2} \begin{array}{c}
\begin{array}{|c|c|}
\hline
& \\
\hline
\end{array}
\end{array}.
\]

(109)

Using similar tricks, every elements can be brought to a linear combination of the \( a_i y \).

\(^{14}\)See appendix D to see how this particular choice was obtained.
Note that it is clear that any element $a \in a T L_4$ acting on a linear combination of $a_5x$, $a_6x$, \ldots, $a_{10}x$ will result in another linear combination of those same basis elements; in other words, \{\(a_i x \mid i = 5, 6, \ldots, 10\}\) generate a submodule of $W_{1/2,z}(3) \times_f S_{1/2}(1)$, which we immediately recognize as $W_{0,z-}(4)$ for some $z- \in \mathbb{C}^*$. By definition, $z_- + z_-^{-1}$ is the weight of the non-contractible loops in the standard module, which must be equal to the eigenvalue of $\bar{Y}$; however $\phi^o_{3,1}(Y^{(3)}) = \bar{Y}^{(4)}$, so this eigenvalue must be $z(-q)^{-1/2} + z^{-1}(-q)^{1/2}$ (the eigenvalue of $\bar{Y}$ on $W_{1/2,z}(3)$). We thus conclude\(^{15}\) that $z_- = z(-q)^{-1/2}$.

Similarly, the quotient of $W_{1/2,z}(3) \times_f S_{1/2}(1)$ by this submodule yields the standard module $W_{1,z+}(4)$. By definition, in $W_{1,z+}(4)$

\[
\begin{array}{ccc}
\Xi & \equiv & z_+ \\
\end{array}
\]

By contrast, in $W_{1/2,z}(3) \times_f S_{1/2}(1)$

\[
\begin{array}{ccc}
\Xi & = & q \Xi + z(-q)^{1/2} \Xi \\
\end{array}
\]

where we used the same trick as in equation (108). We thus conclude that $z_+ = (-q)^{1/2}z$.

Finally, the eigenvalue of $Y^{(4)}$ on $W_{0,z-}(4)$ is $z_- + z_-^{-1}$, while on $W_{1,z+}$ it is $(-q)z_+ + (-q)^{-1}z_+^{-1}$, so this fusion product cannot be indecomposable unless

\[
(-q)z_+ + (-q)^{-1}z_+^{-1} = z_- + z_-^{-1} \iff z^2 = (-q)^2 \text{ or } (-q)^2 = 1.
\]

It follows that, if $q$ and $z$ are generic

\[
W_{1/2,z}(3) \times_f S_{1/2} \cong W_{0,z-}(4) \oplus W_{1,z+}(4).
\]

What if the parameters are not generic? If $z^2 = (-q)^2$, a direct calculation shows that the defect operator $Y$ has a Jordan block linking the two standard modules, and this fusion product is indecomposable. However, these new indecomposable modules are, for the moment, largely unclassified, and we plan to come back to this question in the close future.

More generally, we find that:

**Proposition 4.1.** For all $k \in \mathbb{Z}$, $T \in \mathbb{Z}_{\geq 0}$, $\delta, q \in \mathbb{C}^*$ such that $\delta^2$ is not an integer power of $-q$,

\[
W^u_{k/2,\delta} \times_f S_{T/2} \cong \bigoplus_{i=k-T \text{ step } 2}^{k+T} W^u_{i/2,\delta} \cong \bigoplus_{i=k-T \text{ step } 2}^{k+T} W_{i/2,\delta}(-q)^{(i-k)},
\]

\[
W^o_{k/2,\delta} \times_f S_{T/2} \cong \bigoplus_{i=k-T \text{ step } 2}^{k+T} W^o_{i/2,\delta} \cong \bigoplus_{i=k-T \text{ step } 2}^{k+T} W_{i/2,\delta}(-q)^{(k-i)}.
\]

\(^{15}\)By definition $W_{0,z} = W_{0,z-1}$ so there is no ambiguity here.
These formulas were obtained using the technique and the results outlined in Appendix A.3. However, as shown in Appendix C, these results can also be derived by following the approach in [14], that is, by first establishing the branching rules from $\text{aTL}_{n_1+n_2}$ to $\text{aTL}_{n_1} \otimes \text{TL}_{n_2}$ and then inferring the corresponding fusion product from Frobenius reciprocity. Finally, note that the dimension of $W_{k/2,z}[n]$ is $\binom{n}{k}$, so the dimensions of the various fusion products are

$$\dim(W_{k/2,z}[n_1] \times_f^{o/a} S_{T/2}[n_2]) = \sum_{i=k-T}^{k+T} \left( \binom{n_1 + n_2 - 2}{2} \right).$$

(166)

### 4.2 The fusion quotient

The fusion product defined in the previous section implicitly assumed that the affine module $M$ was only a left $\text{aTL}_m$-module. If $M$ is an $\text{aTL}_m$-bimodule, then the fusion product $M \times_f^{u/o} V$ will also be a $(\text{aTL}_{m+k}, \text{aTL}_m)$ bimodule; given $W$ a left $\text{aTL}_{m+k}$ module, and $V$ a left $\text{TL}_k$ module, we define the fusion quotient by

$$W \overset{u/o}{\times} f V \equiv \text{Hom}_{\text{aTL}_{m+k}} \left( \text{aTL}_m \times_f^{u/o} V, W \right).$$

(177)

Because $\text{aTL}_m \times_f^{u/o} V$ is a right $\text{aTL}_m$ module, this Hom space is naturally a left $\text{aTL}_m$-module. Here’s an example to show that the construction is actually quite natural despite its abstract definition. Let $V = \text{TL}_k$ seen as the regular left $\text{TL}_k$-module; one finds

$$\text{aTL}_m \times_f^{u/o} V = \{ax_0 | a \in \text{aTL}_{k+m}\}, \quad x_0 = \mathbb{I}_{\text{aTL}_{k+m}} \otimes_{\text{aTL}_m} \mathbb{C} \text{TL}_k \left( \mathbb{I}_{\text{aTL}_m} \otimes \mathbb{C} \mathbb{I}_{\text{TL}_k} \right),$$

and then by definition we have an isomorphism of vector spaces

$$W \overset{u/o}{\times} f \text{TL}_k \equiv \{g_w : ax_0 \mapsto aw | w \in W\} \cong W,$$

where the isomorphism is simply $g_w \mapsto w$. We recall then that by definition of the tensor product $x_0 b = \phi_{m,k}^{u/o}(b)x_0$, and thus the left $\text{aTL}_m$ action is then simply

$$[b \cdot g_w](x_0) \equiv g_w(ax_0b) = g_w(a\phi_{m,k}^{u/o}(b)x_0) = a\phi_{m,k}^{u/o}(b)x_0 \equiv g_{\phi_{m,k}^{u/o}(b)w}(ax_0).$$

(188)

It follows that for any $b$ in $\text{aTL}_m$, $b \cdot g_w = g_{\phi_{m,k}^{u/o}(b)w}$, and thus that $W \overset{u/o}{\times} f \text{TL}_k$ is simply the restriction of $W$ to the subalgebra $\phi_{m,k}^{u/o}(\text{aTL}_m) \simeq \text{aTL}_m$.

More generally, if there exists an idempotent $a_0 \in \text{TL}_k$ such that $V = \text{TL}_ka_0$ then for the fusion quotient $W \overset{u/o}{\times} f \text{TL}_ka_0$ the previous arguments can be essentially repeated replacing $x_0$ by the cyclic vector $\psi_{k,m}^{u/o}(a_0)$ and we obtain the following result (note that the action of $\text{aTL}_m$ on $W$ commutes with $\psi_{k,m}^{u/o}(a_0)$):

**Proposition 4.2.** For an $\text{aTL}_{m+k}$-module $W$ and a $\text{TL}_k$-module $V$ such that $V = \text{TL}_ka_0$ for some idempotent $a_0$, the fusion quotient $W \overset{u/o}{\times} f V$ is isomorphic to the $\text{aTL}_m$-submodule $\psi_{k,m}^{u/o}(a_0)W$ of the restriction of $W$ to $\text{aTL}_m$.  

28
We thus recognize that quotients by using this duality, and vice versa. It follows in particular that if one knows every fusion product, one can get back all the fusion types of fusion are of independent interest for the representation theory of

Aside from their possible interpretation as algebraic realisations of topological defects, the two

The element \( x_2 \) was chosen so that

\[
\phi_{2,1}(e_1^{(2)}) x_2 = ((-q)^{-1/2} z + (-q)^{1/2} z^{-1}) x_1, \quad \phi_{2,1}(e_2^{(2)}) x_2 = (q + q^{-1}) x_2, \quad \phi_{2,1}(u^{(2)}) x_2 = x_1. \tag{120}
\]

We thus recognize that \( \{ x_1, x_2 \} \) span a submodule isomorphic to \( W_{0,z(-q)^{-1/2}}(2) \). Furthermore,

\[
\phi_{2,1}(e_1^{(2)}) x_3 = x_1, \quad \phi_{2,1}(e_2^{(2)}) x_3 = (-q)^{-3/2} z^{-1} x_2, \quad \phi_{2,1}(u^{(2)}) x_3 = (-q)^{1/2} z x_3 + q x_2, \tag{121}
\]

so the quotient \( (W_{1/2,z}(3) \div f S_{1/2}(1)) \div (W_{0,z(-q)^{-1/2}}(2)) \) is isomorphic to \( W_{1, (-q)^{1/2}}(2) \). Finally, comparing the eigenvalues of \( Y \) on the two standard modules yields the conclusion: if \( q \) and \( z \) are generic

\[
W_{1/2,z}(3) \div f S_{1/2}(1) \simeq W_{0,z(-q)^{-1/2}}(2) \oplus W_{1,z(-q)^{1/2}}(2). \tag{122}
\]

More generally, we find that:

**Proposition 4.3.** For all \( k \in \mathbb{Z}, \ T, n, m \in \mathbb{Z}_{\geq 0}, \ \delta, q \in \mathbb{C}^* \) such that \( \delta^2 \) is not an integer power of \( -q \), and \( |k| \leq n + m, \ T \leq m \), we have

\[
W_{k/2,\delta}(n + m) \div f S_{T/2}(m) \simeq \bigoplus_{i=k-T \atop \text{step}=2}^{k+T} \mathbb{W}_{i/2,\delta}(n) \simeq \bigoplus_{i=k-T \atop \text{step}=2}^{k+T} \mathbb{W}_{i/2,\delta(-q)(i-k)}(n), \tag{123}
\]

\[
W_{k/2,\delta}(n + m) \div o S_{T/2}(m) \simeq \bigoplus_{i=k-T \atop \text{step}=2}^{k+T} \mathbb{W}_{i/2,\delta}(n) \simeq \bigoplus_{i=k-T \atop \text{step}=2}^{k+T} \mathbb{W}_{i/2,\delta(-q)(k-i)}(n). \tag{124}
\]

In these expressions it should be understood that all modules of the form \( W_{k/2,z}(n) \) with \( n < k \) should be identified with the zero module. These formulas were obtained using the technique and the results outlined in Appendix A.3.

### 4.3 Dualities between the two fusions

Aside from their possible interpretation as algebraic realisations of topological defects, the two types of fusion are of independent interest for the representation theory of \( aTL_n \). As such, we mention here certain properties which they have, and which can be used to compute them. The first such property is that the fusion product and the fusion quotients are duals as functors, i.e. for any \( aTL_{n+m} \) module \( W \), \( aTL_n \) module \( V \) and \( TL_m \) module \( U \), there is a natural isomorphism\(^{16}\)

\[
\text{Hom}_{aTL_{n+m}} \left( V \times f U^o, W \right) \simeq \text{Hom}_{aTL_n} \left( V, W \div f U^o \right). \tag{125}
\]

It follows in particular that if one knows every fusion product, one can get back all the fusion quotients by using this duality, and vice versa.

\(^{16}\)Note that this follows directly from the fact that the tensor product and the Hom functors form an adjoint pair.
The second property we mention is the associativity: for all $aTL_n$ module $W$, $TL_k$ module $V$ and $TL_m$ module $U$,
\[
(W \times_f^{u/O} V) \times_f^{u/O} U \simeq W \times_f^{u/O} (V \times_f^r U) \simeq (W \times_f^{u/O} U) \times_f^{u/O} V,
\]
(126)
where $\times_f^r$ is the fusion product in the regular Temperley-Lieb algebra, which was studied in detail in [12,17]. Similarly, for all $aTL_{n+k+m}$ module $W$, $TL_k$ module $V$ and $TL_m$ module $U$,
\[
(W \div_f^{u/O} V) \div_f^{u/O} U \simeq W \div_f^{u/O} (V \times_f^r U) \simeq (W \div_f^{u/O} U) \div_f^{u/O} V.
\]
(127)

As an example, if we assume that $q$ is generic then for all $k \geq 0$
\[
S_k(n) \times_f^r S_0(2) \simeq S_k(n + 2), \quad S_k(n) \times_f^r S_{1/2}(1) \simeq S_{k-1/2}(n + 1) \oplus S_{k+1/2}(n + 1).
\]
(128)
It follows that for a given $aTL_n$ module $W$, knowing its fusion product (or quotient) with $S_0(2m)$ and $S_{1/2}(1)$ is enough to compute the fusion with all other standard modules by recurrence. Equations (114)-(115) and (123)-(124) were obtained in this manner (see appendix A.3).

### 4.4 Fusion and the Hamiltonian

We now go back to the problem of studying the defects in the direct channel, or the spectrum problem of the Hamiltonians $H^{u,m}_{n,m}$ introduced in the beginning of this section 4. Recall that we reduced this problem to studying $H^{u,m}_{n,m}[^\rho]$ from (102) for an idempotent $\rho$.

Let $W$ be an $aTL_{n+m}$-module, and $\rho$ be a non-zero idempotent of $TL_m$, then the Hamiltonians with impurities (where we also introduced the over lines version)
\[
H^{u,m}_{n,m}[\rho] = \phi^{u,m}_{n,m}(H_n) \psi^{o,u}_{m,n}(\rho) = (\sum_{j=1}^{n-1} e_j^{(n+m)} + \mu_{n,m}^{-1} e_n^{(n+m)} \mu_{n,m}) \psi^{o,u}_{m,n}(\rho),
\]
(129)
\[
H^{o,u}_{n,m}[\rho] = \phi^{o,u}_{n,m}(H_n) \psi^{u}_{m,n}(\rho) = (\sum_{j=1}^{n-1} e_j^{(n+m)} + \nu_{n,m} e_n^{(n+m)} \nu_{n,m}^{-1}) \psi^{u}_{m,n}(\rho),
\]
(130)
where $\mu_{n,m} = g_n g_{n+1} \ldots g_{n+m}$ and $\nu_{n,m} = g_{n+m} \ldots g_n$, acting on $W$ are the standard periodic TL Hamiltonian (i.e. without defects) $H_n = \sum_{j=1}^n e_j^{(n)}$ acting on the $aTL_n$-submodule $\psi^{o,u}_{m,n}(\rho)W$ in the restriction of $W$ to $aTL_n$ under $\phi^{u,o}_{n,m}$. And this submodule is just the fusion quotient $W \div_f^{u/O}(TL_m \rho)$, due to Proposition 4.2.

We note that choosing the idempotent $\rho$ to correspond to a representation of spin $m/2$, or the standard module on $m$ strands with $m$ through lines, e.g. the Jones-Wenzl idempotents $\rho = P_{m/2}$ defined in (84), the expressions (129) and (130) are precisely the expressions for the Hamiltonians with impurities that correspond in the direct channel to the defect operators $Y_{m/2}$ and $\bar{Y}_{m/2}$, respectively.

As a corollary of the preceding discussion we formulate our main result on the spectral problem of the defect Hamiltonians:

**Theorem 4.1.** Let $\rho \in TL_m$ be an idempotent such that $TL_m \rho \simeq V$, then for any $aTL_{n+m}$-module $M$, the Hamiltonian $H^{u,m}_{n,m}[\rho]$ acting on $M$ is similar (as a matrix) to the direct sum of the standard Hamiltonian $H_n$ acting on $M \div_f^{u/O} V$ and a zero matrix of dimension dim($\mathbb{I}_{aTL_{n+m}} - \psi^{o,u}_{m,n}(\rho)M$).
Proof. By definition,

\[ H_{n,m}^{u/o}[\rho] = \phi_{n,m}(H_n)\psi_{m,n}^{o/u}(\rho), \]

and since \( \rho \) is an idempotent,

\[ M \simeq (\psi_{m,n}^{o/u}(\rho)M) \oplus (I_{aTL_{n+m}} - \psi_{m,n}^{o/u}(\rho))M. \]

The first summand is isomorphic to \( M \div_{f}^{u/o} V \), by Proposition 4.2, while one quickly sees that \( H_{n,m}^{u/o}[\rho] \) is identically zero on the second. Furthermore, by definition \( H_n \) acts on \( \psi_{m,n}^{o/u}(\rho)M \) by left multiplication by \( \phi_{n,m}(H_n) \).

As an application of this theorem, we can calculate the spectrum of the Hamiltonian with impurities corresponding to the defect operator \( Y_j \) in the direct channel as follows: the set of eigenvalues on \( M \) is the set of eigenvalues of the standard periodic TL Hamiltonian \( H_n \) on the \( aTL_n \)-module \( M \div_f^{u/o} S_j \). The case of the Hamiltonian for \( \tilde{Y}_j \) is analogous. Taking into account the general results on the fusion from Proposition 4.3 and that the spectrum of \( H_n \) is known on all the standard \( aTL_n \)-modules \( W_{k,z} \), we thus solved the spectrum problem for the Hamiltonians with the impurities.

Similarly, the transfer matrix acting on this fused module \( M \div_{f}^{u/o} S_j \) is precisely the one obtained by adding a cluster of lines going under (or over) the other lines in the lattice. This strongly suggests that the fusion quotient is indeed the right algebraic construction for these defects. However it should be mentioned that for generic values of the parameters the fusion product and quotients are equivalent for large values of \( n \), in the sense that

\[ W_k[n+2m] \div_{i}^{u/o} S_{i}[m] \simeq W_k[n] \times_{i}^{u/o} S_{i}[m], \]

provided that \( W_k[n] \neq 0 \). It follows that while the Hamiltonian acting on the fusion product does not have such a simple interpretation it will produce the same spectrum (unless \( n \) is too small).

### 4.5 Example of quotient: the twisted XXZ spin chain

The twisted XXZ spin chain on \( n \) sites can be realized by the Hamiltonian \( H_n(Q) \) expressed in terms of the usual Pauli matrices acting on \((\mathbb{C}^2)^{\otimes n}:

\[ H_n(Q) = \sum_{j=1}^{n} \left( \sigma_j^+ \sigma_{j+1}^- + \sigma_{j+1}^+ \sigma_j^- + \frac{q+q^{-1}}{4} \left( \sigma_j^z \sigma_{j+1}^z - 1 \right) \right) = -\sum_{j=1}^{n} e_j, \]

where \( \sigma_j^{\pm} = 1/2(\sigma_j^x \pm i\sigma_j^y) \) are the usual ladder operators, \( Q \) is a non-zero complex number, and the boundary conditions are

\[ \sigma_{n+1}^{\pm} \equiv \sigma_1^{\pm}, \quad \sigma_{n+1}^{\pm} \equiv Q^{\mp 2} \sigma_1^{\pm}. \]

The model is unitary if \( Q \) is on the unit circle in \( \mathbb{C} \). The Temperley-Lieb generators are

\[ -e_j \equiv \sigma_j^+ \sigma_{j+1}^- + \sigma_{j+1}^+ \sigma_j^- + \frac{q+q^{-1}}{4} \left( \sigma_j^z \sigma_{j+1}^z - 1 \right) + \frac{q-q^{-1}}{4}(\sigma_j^z - \sigma_{j+1}^z), \]

with the twist

\[ u = (-1)^{n/2}Q^{-\sigma_1} s_1 \ldots s_{n-1}, \quad s_j = \sigma_j^- \sigma_{j+1}^+ + \sigma_j^+ \sigma_{j+1}^- + \frac{1}{2}(\sigma_j^z \sigma_{j+1}^z + 1). \]
A quick calculation shows that the hoop operators are\(^{17}\)

\[
Y[n] = (-1)^n \left( q^{S_z} Q^{-1} + q^{-S_z} Q \right), \quad \bar{Y}[n] = q^{S_z} Q + q^{-S_z} Q^{-1}, \quad (138)
\]

with \( S_z = \frac{1}{2} \sum_{j=1}^n \sigma_j^z \) the total spin. Our goal is now to impose a defect of spin 1/2 on this chain, which according to our formalism (see Theorem 4.1) consist in computing the fusion quotient of its Hilbert space by \( S_1(1) = \mathcal{TL}_1 \). This specific defect corresponds to a simple restriction from \( a\mathcal{TL}_n \) to \( a\mathcal{TL}_{n-1} \), so the new Hamiltonian with a defect is either

\[
H_{n-1,1}^u(Q) = -\sum_{j=1}^{n-1} \delta_{n-1,1}^u(e_j^{(n-1)}) = -\sum_{j=1}^{n-2} e_j^{(n)} - g_n^{(n)} e_{n-1}^{(n)} (g_n^{(n)})^{-1}, \quad (139)
\]

or

\[
H_{n-1,1}^o(Q) = -\sum_{j=1}^{n-1} \delta_{n-1,1}^o(e_j^{(n-1)}) = -\sum_{j=1}^{n-2} e_j^{(n)} - (g_n^{(n)})^{-1} e_{n-1} g_n^{(n)}, \quad (140)
\]

for a defect that goes under or over the other lines, respectively. Using the explicit construction of \( e_{n-1} \) and \( g_n \) one finds

\[
H_{n-1,1}^u(Q) = \sum_{j} (a_j^- a_{j+1}^+ + a_{j+1}^- a_j^+ + \frac{q + q^{-1}}{4} (a_j^z a_{j+1}^z - 1)) + (1 - q^{2\alpha j}) a_{n-1}^- + Q^2 (1 - q^{-2\alpha n-1}) a_1^+ \sigma_n^z,
\]

where we defined new operators \( a_j^k = \sigma_j^k, k = z, \pm, j = 1, 2, \ldots n - 1 \), with boundary conditions

\[
a_n^z \equiv a_1^z, \quad a_n^\pm \equiv (Q^2 q^{-\alpha n} \mp 1) a_1^\pm.
\]

It follows that

\[
H_{n-1,1}^u(Q) \sim \begin{pmatrix} (\ldots) \otimes | \uparrow \rangle & (\ldots) \otimes | \downarrow \rangle \\ \Delta & H_{n-1}(-Qq^{1/2}) \end{pmatrix}, \quad \Delta = (1 - q^{2\alpha j}) a_{n-1}^- + Q^2 (1 - q^{-2\alpha n-1}) a_1^-.
\]

A straightforward calculation then shows that the defect operators are now:

\[
Y[n-1] = (-1)^n (q^{S_z} Q^{-1} + q^{-S_z} Q) = (-1)^{n-1} \left( q^{S_z - \frac{1}{2} \sigma_n^z} \left( -Q q^{-\frac{1}{2} \sigma_n^z} \right)^{-1} + q^{-S_z + \frac{1}{2} \sigma_n^z} \left( -Q q^{-\frac{1}{2} \sigma_n^z} \right) \right),
\]

\[
\bar{Y}[n-1] \sim \begin{pmatrix} (\ldots) \otimes | \uparrow \rangle & (\ldots) \otimes | \downarrow \rangle \\ Q_+ q^{\tilde{S}_z} + Q^{-1} q^{-\tilde{S}_z} & Q(q - q^{-1})^2 \tilde{S}_- \\ 0 & Q_+ q^{\tilde{S}_z} + Q^{-1} q^{-\tilde{S}_z} \end{pmatrix},
\]

where \( Q_\pm = -Q q^{\pm 1/2} \), and \( \tilde{S}_- \), \( q^{\pm \tilde{S}_z} \) are the standard \( U_q(\mathfrak{sl}_2) \) generators on \( n - 1 \) spins

\[
\tilde{S}_- = \sum_{j=1}^{n-1} (q) \sigma_j^- = \sum_{j=i+1}^{n-1} \sigma_j^z, \quad q^{\pm \tilde{S}_z} = q \sum_{j=1}^{n-1} \sigma_j^z/2.
\]

\(^{17}\)In everything that follows, one should understand that for all matrix \( A \), \( q^A = (-q)^A (-1)^{-A} \). We simplify these expressions to lighten the notation but one should be careful when verifying these results numerically.
Note that $\bar{Y}[n-1]$ can be diagonalized if and only if $(Q - q^{-S_z})(Q + q^{-S_z})$ is an invertible matrix, which can be verified by comparing its eigenvalues in the $\sigma^z_n = \pm 1$ sectors. It follows in particular that the Hamiltonian (142) cannot have a Jordan block linking the $\sigma^z_n = \pm 1$ sectors if $Q$ is generic.

Similarly, one finds

$$H_{n-1,1}(Q) \sim \begin{pmatrix} (\ldots) \otimes | \uparrow \rangle & (\ldots) \otimes | \downarrow \rangle \\ \Delta & H_{n-1}(Qq^{1/2}) \end{pmatrix},$$

$$\Delta = (1 - q^{2a^+_i})a^+_{n-1} + Q^{-2}(1 - q^{-2a^+_{n-1}})a^+_1.$$

Note that in each of these expressions, the off-diagonal term $\Delta$ can only link sectors of $H_{n-1}(Qq^{\pm 1/2})$ corresponding to different total spin $(\sum_{j=1}^{n-1} a_j)$ because of the ladder operators appearing in it.

5 Conclusion: connection to CFT

In order to provide a lattice analogue of CFT topological defects $X$ satisfying (1), we have defined and studied in a model-independent way operators on the lattice that commute with the local interactions given by the TL elements—the central elements $Y$ and $\bar{Y}$ in $aTL_n$—and have demonstrated their interesting properties. From the crossed-channel point of view, these defect operators generate an algebra spanned by $Y_j$, $\bar{Y}_j$, and their products, that has structure constants or fusion rules (89) and (90) resembling the chiral and anti-chiral fusion rules of Virasoro Kac modules of type $(1,s)$ where $s = 2j + 1$. We recall that the Kac modules are obtained as quotients of Verma modules of the conformal weight $h_{1,s}$ by the submodule generated by the singular vector at the level $h_{1,s} + s$.

The analogy with CFT goes further: Recall that at least in rational CFT a topological defect can be seen as a map from the set of chiral primary fields to the ring of endomorphisms of the Hilbert space of the full non-chiral CFT. In much the same way our maps $Y^m$ and $\bar{Y}^m$ from Fig. 4 defining the defect operators send ideals in the open or regular TL algebra (which are known to correspond to chiral primary fields of conformal weight $h_{1,s}$) to central elements in affine TL algebra which are realized as endomorphisms of the bulk lattice model, e.g. of periodic spin-chains.

We saw that the higher-spin defects $Y_j$ and $\bar{Y}_j$ (87)-(88) carry some sort of internal structure “living” on the horizontal non-contractible loops. From the direct-channel point of view, or after a modular transformation, this internal structure was realized in Section 4 as some sort of impurities in the spatial direction. Therefore, we have just rewritten the defects $Y_j$ and $\bar{Y}_j$ in the Hamiltonian formulation. Interestingly, the problem of spectrum with impurities was reformulated in algebraic terms as a rather simple fusion product of affine and regular TL representations which is a combination of the constructions in [13,14] and [16] that we review in Appendix B.

So far we have defined and studied lattice defects that do not depend on a spectral parameter. Let us call these defects of first type. However, there is some evidence that there should be a second type of (lattice) defects that do depend on a spectral parameter. Though they are not central in $aTL_n$, but possibly become topological defects $X$, i.e. they satisfy (1), in the continuum limit only. We will address studying these defects of the second type in our next paper where an identification with Virasoro Kac modules of the type $(r,1)$ is expected.

It is important for several reasons to try to define what we call lattice defects in a precise mathematical way and in higher generality, for a possible application to more general lattice
models not necessarily based on TL interactions. For the first kind of defects, from the results obtained in this work, we are approaching a mathematical definition of (an algebra of) defects for general lattice algebras (e.g. $aTL_n(q)$, Birman-Wenzl-Murakami, Brauer algebras, etc):

**Definition:** In a lattice algebra $A$, the space of defects $D$ of 1st type is the center of $A$ with the structure of a Verlinde algebra.

Note that not every central element in a lattice algebra corresponds to a defect operator: it should also have nice properties that reflect known properties from the CFT side. That is why we demand that the space of defects forms a Verlinde algebra. First of all this implies the presence of a special basis in this algebra with structure constants being non-negative integers. Secondly, the idea is that these integer numbers should correspond to fusion rules of corresponding representations of an (anti-)chiral algebra, e.g. Virasoro.

We have indeed recovered these two aspects in our case of $A = aTL_n(q)$, where we identified $D$ as the center $Z(aTL)$ of $aTL_n(q)$, and the latter as a Verlinde algebra generated by $Y_j$ and $\bar{Y}_j$ where the structure constants do not depend on $n$, for non-mixed products, and correspond to fusion rules of chiral and anti-chiral Virasoro representations of type $\{1, s\}$.\(^\text{18}\) This is here shown to be true for the generic $q$ case where the fusion rules might look rather trivial, since they are $sl(2)$ type fusion after all. The situation is not so trivial in degenerate cases (where $q$ is a root of unity) that we will describe in one of our forthcoming papers on the subject, with applications to minimal models as well as LCFTs. There, a connection to Virasoro fusion rules also holds, although it is much less evident due to more involved representation theory.

However, this is not the end of the story. Any Verlinde algebra has the third aspect: it admits a modular $S$-transformation that “diagonalizes” the fusion rules. For the moment we have concentrated on the first two aspects only. It is, of course, an important problem to properly define and analyze such $S$-transformations in a precise algebraic way, and hopefully it will reflect the modular transformation on the lattice. We hope to come back to this problem soon.

### 6 Declarations

**Funding.** This work was supported by the Institut Universitaire de France and the European Research Council (advanced grant NuQFT). We are also grateful to the CRM in Montreal and to the organizers of the conference *Algebraic methods in mathematical physics* in Montreal in 2018 where a part of this work was done. The work of AMG was supported by CNRS, and partially by ANR grant JCJC ANR-18-CE40-0001 and the RSF Grant No. 20-61-46005. AMG is also grateful to IPHT Saclay for kind hospitality in 2017 and 2018.

The authors have no competing interests to declare that are relevant to the content of this article.

**Acknowledgments.** We acknowledge interesting discussions with Yvan St Aubin, and we thank D. Bulgakova for discussions and early collaboration on this project. We would also like to thank an anonymous referee for their helpful comments, and for pointing out a problem in the original proof of theorem 2.1.

**Data Availability.** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

---

\(^{18}\)Strictly speaking our algebra of defects is the product of two Verlinde algebras, for chiral and anti-chiral Virasoro representations, modulo non-linear algebraic relations between $Y$ and $\bar{Y}$.
## A Proofs and rigors

We collect in this appendix the proofs of certain technical results used in this work.

### A.1 Topological defects with a higher spin.

We show here how to obtain the expressions for topological defects with higher-spins given in section 3.2, i.e.

\[ Y_j = U_{2j}(Y/2). \]  

(147)

First we show that the result is independent of the choice of idempotent we make.

Let \( A \) be some finite dimensional \( \mathbb{C} \)-algebra, \( \rho_1, \rho_2 \) be two idempotents such that \( A\rho_1 \cong A\rho_2 \) as left \( A \) modules, and let \( F \) be any function defined on \( A \) such that for all \( a, b \in A \), \( F(ab) = F(ba) \) (in other words \( F \) is cyclic). We know that

\[ \text{Hom}_A(A\rho_1, A\rho_2) \cong \rho_1 A \rho_2, \quad \text{Hom}_A(A\rho_2, A\rho_1) \cong \rho_2 A \rho_1, \]  

(148)

where the isomorphism is obtained by right-multiplication. For instance,

\[ (f : A\rho_1 \to A\rho_2) \mapsto \rho_1 f(\rho_1) \rho_2, \quad \rho_1 a \rho_2 \mapsto (b \rho_1 \mapsto b \rho_1 a \rho_2). \]  

(149)

Because \( A\rho_1 \cong A\rho_2 \), it follows that there exists \( a, b \in A \) such that \( \rho_1 a \rho_2 \) gives the \( A \)-linear isomorphism \( A\rho_1 \to A\rho_2 \) and \( \rho_2 b \rho_1 \) gives the isomorphism \( A\rho_2 \to A\rho_1 \). In particular, if \( A\rho_1 \) is irreducible this means that the compositions of the two isomorphisms applied to \( \rho_1 \) and \( \rho_2 \) are respectively \( \rho_1 a \rho_2 b \rho_1 = \alpha \rho_1 \) and \( \rho_2 b \rho_1 a \rho_2 = \gamma \rho_2 \) for some non-zero complex numbers \( \alpha, \gamma \). However one quickly verifies that

\[ \gamma^2 \rho_2 = (\rho_2 b \rho_1 a \rho_2)^2 = \rho_2 b (\rho_1 a \rho_2 b \rho_1) a \rho_2 = \alpha \gamma \rho_2, \]

so \( \alpha = \gamma \) and we can thus choose \( a, b \) such that \( \alpha = \gamma = 1 \). Now by hypothesis the function \( F \) is cyclic and \( \rho_1 \) and \( \rho_2 \) are idempotents so

\[ F(\rho_1) = F(\rho_1 a \rho_2 b \rho_1) = F(\rho_2 b \rho_1 a \rho_2) = F(\rho_2). \]  

(150)

Next, we remark that for any elements \( a \in TL_n, b \in TL_m, Y^{n+m}(a \otimes TL b) = Y^n(a) Y^m(b) \) where \( \otimes TL \) is the tensor product in the Temperley-Lieb category, obtained by joining diagrams side by side. For instance,

\[ e_1 \otimes TL e_1 \equiv \begin{array}{c|c} & \end{array} \begin{array}{c|c} 1 & \end{array} \equiv \begin{array}{c} \begin{array}{c|c} & \end{array} \begin{array}{c} 1 \end{array} \end{array} \]

(151)

It follows in particular that for any idempotent \( a \in TL_n \),

\[ Y^{n+1}(a \otimes TL 1_{TL_1}) = Y_1/2 Y^n(a). \]  

(152)

Now if the idempotent \( a \) is such that \( TL_n a \simeq S_{k/2}(n) \), one can show that there exists a decomposition of the idempotent \( (a \otimes TL 1_{TL_1}) = a_- + a_+ \) in \( TL_{n+1} \), where \( a_\pm \) are orthogonal idempotents such that \( TL_{n+1} a_\pm \simeq S_{(k\pm 1)/2}(n+1) \). This follows from the decomposition of the induced \( TL_{n+1} \)-module (see e.g. [12, 17]). We therefore have for generic values of \( q \) the following relations:

\[ Y_{(k+1)/2} = Y_{1/2} Y_{k/2} - Y_{(k-1)/2}, \quad Y_0 \equiv 1_{a TL_n}, \]  

(153)

which is the Chebyshev recurrence relation, giving (147).
A.2 The Jucys-Murphy elements

Here, we prove the identities (94) and (95) involving $C_k(n)$ and $\mathcal{C}_k(n)$ defined in (93). As the proofs for the identities involving $Ms$ are identical to those involving the $Js$, we only prove the two identities involving $\mathcal{C}_k(n)$. The proof of this result relies on two key observations; the first is the identity

$$[i]_k(-q)^{\pm k} - [i - 1]_k = (-q)^{\pm ki}, \quad i = 0, 1, \ldots.$$  

Furthermore, defining for $i = 1, \ldots, n - 1$,

$$\bar{X}_{i+1} \equiv (-q)^2 J_i + (-q)^{-2} J_i^{-1},$$  

$$X_{i+1} \equiv (-q)^2 M_i + (-q)^{-2} M_i^{-1},$$

the second observation is that for all $i = 1, \ldots, n - 1$ we have the relations in $\mathcal{aTL}_n(q)$:

$$\bar{X}_{i+1} = (-q) J_{i+1} + (-q)^{-1} J_{i+1}^{-1},$$  

$$X_{i+1} = (-q) M_{i+1} + (-q)^{-1} M_{i+1}^{-1},$$  

$$Y = (-q)^2 J_n + (-q)^{-2} J_n^{-1},$$  

$$\bar{Y} = (-q)^2 M_n + (-q)^{-2} M_n^{-1}.$$  

These can all be proven in the same way, by showing that both sides of these equality correspond to the same diagram. For instance

$$(-q)^2 J_n + (-q)^{-2} J_n^{-1} = (-q)^{1/2} \underbrace{\ldots \ldots}_{n-1} + (-q)^{-1/2} \underbrace{\ldots \ldots}_{n-1} = \underbrace{\ldots \ldots}_{n-1} \equiv Y,$$

where we used the definition of the braids. Putting the two observations together gives the following relations, for all $i = 1, \ldots, n - 1$,

$$((-q)^{i+1} J_i)^k + ((-q)^{i+1} J_i)^{-k} = [i]_k \big(((-q)^2 J_i)^k + ((-q)^2 J_i)^{-k}\big) - [i - 1]_k \big(((-q) J_i)^k + ((-q) J_i)^{-k}\big),$$

$$= 2[i]_k T_k \left(\frac{(-q)^2 J_i + (-q)^{-2} J_i^{-1}}{2}\right) - 2[i - 1]_k T_k \left(\frac{(-q) J_i + (-q)^{-1} J_i^{-1}}{2}\right),$$

$$= 2[i]_k T_k \left(\frac{X_{i+1}}{2}\right) - 2[i - 1]_k T_k \left(\frac{X_i}{2}\right),$$

$$((-q)^{n+1} J_n)^k + ((-q)^{n+1} J_n)^{-k} = 2[n]_k T_k \left(\frac{Y}{2}\right) - 2[n - 1]_k T_k \left(\frac{X_n}{2}\right),$$

where we used the fact that for all non-zero $x$

$$2T_k((x + x^{-1})/2) = x^k + x^{-k}.$$
Then, it follows that

\[ \tilde{C}_k(n) + \tilde{C}_{-k}(n) = \sum_{i=1}^{n} \left( ((-q)^{i+1}J_i)^k + ((-q)^{i+1}J_i)^{-k} \right) \]

\[ = 2[1]_k T_k(\tilde{X}_2/2) + \sum_{i=2}^{n} \left( ((-q)^{i+1}J_i)^k + ((-q)^{i+1}J_i)^{-k} \right) \]

\[ = 2[2]_k T_k(\tilde{X}_3/2) + \sum_{i=3}^{n} \left( ((-q)^{i+1}J_i)^k + ((-q)^{i+1}J_i)^{-k} \right) \]

\[ = 2[n]_k T_k(Y/2). \]

Using very similar arguments, we have

\[ ((-q)^{i+1-n}J_i)^k + ((-q)^{i+1-n}J_i)^{-k} = -[n - i]_k \left( ((-q)^2J_i)^k + ((-q)^2J_i)^{-k} \right) \]

\[ + [n + 1 - i]_k \left( ((-q)J_i)^k + ((-q)J_i)^{-k} \right) \]

\[ = 2[n + 1 - i]_k T_k(\tilde{X}_{i}/2) - 2[n - i]_k T_k(\tilde{X}_{i+1}/2) \]

\[ ((-q)^1J_n)^k + ((-q)^1J_n)^{-k} = 2T_k(\tilde{X}_n/2), \]

which give

\[ (-q)^{-nk}C_k(n) + (-q)^{nk}C_{-k}(n) = \sum_{i=1}^{n} \left( ((-q)^{i+1-n}J_i)^k + ((-q)^{i+1-n}J_i)^{-k} \right) \]

\[ = 2[n + 1 - n]_k T_k(\tilde{X}_n/2) + \sum_{i=1}^{n-1} \left( ((-q)^{i+1-n}J_i)^k + ((-q)^{i+1-n}J_i)^{-k} \right) \]

\[ = 2[n + 1 - 1]_k T_k(\tilde{X}_1/2) = 2[n]_k T_k(Y/2), \]

where we used the fact that \( X_1 \equiv Y \) by definition.

### A.3 Fusion with standard modules

We explain here how to compute the fusion product/quotient of standard modules. The final results from Propositions 4.1 and 4.3 rely on two key facts:

1. Fusions of the same type \((u/o)\) are associative (see (126) and (127)), i.e. for any aTL-module \(M\), and TL-modules \(V_1, V_2\), we have isomorphisms

   \[ (M \times_{u/o} V_1) \times_{u/o} V_2 \cong M \times_{u/o} (V_1 \times_f V_2), \]  

   \[ (M \div_{u/o} V_1) \div_{u/o} V_2 \cong M \div_{u/o} (V_1 \times_f V_2). \]  

2. For any \( t \in \mathbb{N} \) and for generic \( q \):

   \[ S_{t/2}(n) \times_f S_{0}(2) \cong S_{t/2}(n + 2), \]  

   \[ S_{t/2}(n) \times_f S_{1/2}(2) \cong S_{(t-1)/2}(n + 1) \oplus S_{(t+1)/2}(n + 1), \]

   where it is understood that \( S_{(-1)/2}(n + 1) \equiv 0. \)

Putting these two facts together, it follows that if we can compute the fusions of aTL-modules with \( S_{0}(2) \) and \( S_{1}(1) \), fusions with \( S_{t/2}(m), t \geq 0, m \geq 1 \) can be computed easily by recurrence. In this appendix, we thus present how these four necessary fusions are calculated.
A.3.1 \( \mathcal{W}_{k,z}(n + 2) \div_f^{u/o} S_0(2) \)

Assuming that \( q^2 \neq -1 \), the primitive idempotent corresponding to the projective module \( S_0(2) \) is \( \rho_0 \equiv (q + q^{-1})^{-1} e_1 \). According to our definition of the fusion quotient, we must consider the subspace \( W_0 \equiv \rho_0 \mathcal{W}_{k,z}(n + 2) \) with the action of \( aTL_n \) obtained from the morphism of algebras \( \phi_{n,2}^{u/o} \); however, in this case there is a map \( \psi: \mathcal{W}_{k,z}(n) \to W_0 \) which consists in adding two positions linked with an arc on the right of every diagram in \( \mathcal{W}_{k,z}(n) \), for instance

\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example_diagram1.png}
\end{array}
\rightarrow
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example_diagram2.png}
\end{array}.
\]

One sees directly that this map defines a morphism of modules, and that the resulting sub-module of \( W_0 \) is the same for both types of fusion. Furthermore, any diagram in \( \mathcal{W}_{k,z}(n + 2) \) is sent to one of the form \( \psi(x) \) by the action of the idempotent \( \rho_0 \), so this morphism is surjective. Because the map \( \psi \) is obviously injective as well, it must be an isomorphism and we thus get

\[
\mathcal{W}_{k,z}(n + 2) \div_f^{u/o} S_0(2) \simeq \mathcal{W}_{k,z}(n),
\]

where it is understood that \( k \leq n/2 \) because otherwise the \( e_{n+1} \) would acts as zero on \( \mathcal{W}_{k,z}(n + 2) \).

A.3.2 \( \mathcal{W}_{k,z}(n) \times_f^{u/o} S_0(2) \)

Because both \( \mathcal{W}_{k,z}(n) \) and \( S_0(2) \) are cyclic, so is their fusion product; in particular one can write

\[
\mathcal{W}_{k,z}(n) \times_f^{u/o} S_0(2) = aTL_{n+2} x
\]

with

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{example_diagram3.png}
\end{array}.
\]

We also recall our diagram notation for the fusion product: the diagram at the top is the element of \( aTL_{n+2} \), the one on the bottom left corner is the element of \( \mathcal{W}_{k,z}(n) \), and the one on the bottom right corner is the element of \( S_0(2) \). Note that closing together any of the through lines in the top diagram of \( x \) will yield the zero element, because it will be able to pass through the tensor product. It follows that this fusion product is isomorphic to a standard module of the form \( \mathcal{W}_{k,z'}(n + 2) \) for some \( z' \). Note also that because of the duality between the two types of fusion

\[
\text{Hom}_{aTL_{n+2}} \left( \mathcal{W}_{k,z}(n) \times_f^{u/o} S_0(2), \mathcal{W}_{k,z}(n + 2) \right) \simeq \text{Hom}_{aTL_n} \left( \mathcal{W}_{k,z}(n), \left[ \mathcal{W}_{k,z}(n + 2) \div_f^{u/o} S_0(2) \right] \right) \simeq \mathbb{C},
\]

so we conclude that \( z' = z \), and thus

\[
\mathcal{W}_{k,z}(n) \times_f^{u/o} S_0(2) \simeq \mathcal{W}_{k,z}(n + 2).
\]
A.3.3 $W_{k,z}(n + 1) \divides_{y}$ S$_{1/2}(1)$, \( k \neq 0 \)

Both types of fusion are very similar so we focus on the \( u \)-type. The primitive idempotent corresponding to S$_{1/2}(1)$ is simply the identity so the fusion quotient is the full restriction of $W_{k,z}(n+1)$. There is then a map $\psi: W_{k-1/2,z'}(n) \rightarrow W_{k,z}(n + 1)$ which consists in adding a single position on the right of every diagram in $W_{k-1/2,z'}(n)$ and adding a through line to it which passes under every other line. For instance,

\[
\begin{array}{c|c}
\hline
\text{Diagram 1} & \text{Diagram 2} \\
\hline
\begin{array}{c}
\text{Diagram 1}
\end{array} & \rightarrow & \begin{array}{c}
\text{Diagram 2}
\end{array} \\
\hline
\end{array}
\]

One can see that this map (extended linearly) indeed defines an injective morphism of $\text{aTL}_n$ module if $z' = (-q)^{1/2}z$; this condition on $z$ can be seen by observing that

\[
\begin{array}{c|c}
\hline
\text{Diagram 1} & \text{Diagram 2} \\
\hline
\begin{array}{c}
\text{Diagram 1}
\end{array} = z' & \begin{array}{c}
\text{Diagram 2}
\end{array} = (-q)^{1/2}z \\
\hline
\end{array}
\]

Next, consider the map $\phi: W_{k,z}(n + 1) \rightarrow W_{k+1/2,z(-q)^{-1/2}}(n + 2)$, defined by $\phi = (q + q^{-1})^{-1}e_{n+1}$. In other words we add an extra through line, going under all others, at the right of each diagram in $W_{k,z}(n + 1)$ then multiply the result by the idempotent $(q + q^{-1})^{-1}e_{n+1}$. For instance, we get

\[
\begin{array}{c|c}
\hline
\text{Diagram 1} & \text{Diagram 2} \\
\hline
\begin{array}{c}
\text{Diagram 1}
\end{array} \rightarrow & \begin{array}{c}
\text{Diagram 2}
\end{array} \\
\hline
\end{array}
\]

Based on the previous results of this section, we recognize the image of this map is $W_{k+1/2,z(-q)^{-1/2}}(n + 2) \divides_{y} S_0(2) \simeq W_{k+1/2,z(-q)^{-1/2}}(n)$. It can then be shown that this map (extended linearly) defines a surjective morphism of $\text{aTL}_n$ modules. Furthermore, one can see that the image of the first map $\psi$ is contained in the kernel of the second map $\phi$:

\[
\begin{array}{c|c}
\hline
\text{Diagram 1} & \text{Diagram 2} \\
\hline
\begin{array}{c}
\text{Diagram 1}
\end{array} \rightarrow & \begin{array}{c}
\text{Diagram 2}
\end{array} \\
\hline
\end{array}
\]

where we used the fact that diagrams with less than $k/2$ through lines are equivalent to the zero element in $W_{k,z}(n)$.

Finally, note that

\[
\text{Dim}(W_{k,z}(n + 1) \divides_{y} S_1(1)) = \text{Dim}(W_{k,z}(n + 1)) = \text{Dim}(W_{k-1/2,z'}(n)) + \text{Dim}(W_{k+1/2,z''}(n)),
\]

so the image of $\psi$ is exactly the kernel of $\phi$. Since the eigenvalues of $\bar{Y}$ are different on $W_{k\pm 1/2,z(-q)^{\pm 1/2}}(n)$, if $z$ is generic, it follows that

\[
W_{k,z}(n + 1) \divides_{y} S_{1/2}(1) \simeq W_{k-1/2,z(-q)^{1/2}}(n) \oplus W_{k+1/2,z(-q)^{-1/2}}(n).
\]

What about when $z$ is not generic? The eigenvalues of $\bar{Y}$ will be the same on the modules appearing in the previous direct sum if and only if $(-q)^2 = 1$ or $z^2 = (-q)^{1/2}$; one can verify directly that $\bar{Y}$ has a Jordan block when acting on the fusion quotient only in the later case. This can be seen by taking any $x \in W_{k,z}(n + 1) \divides_{y} S_{1/2}(1)$ that is not in the kernel of $\phi$ and verifying that $(\bar{Y} - \lambda)x \neq 0$, where $\lambda$ is the eigenvalue of $\bar{Y}$ on $W_{k+1/2,z(-q)^{-1/2}}(n)$. The modules appearing in the fusion quotient is then the indecomposable extension of two standard modules; while these are not yet classified, we shall show that they are indeed unique in our forthcoming work.
A.3.4 $W_{k,z}(n+1) \times_{j}^{\mathfrak{gl}} S_{1/2}(1)$, $(k \neq 0)$

Both types of fusion are very similar so we again focus on the $u$-type. Since standard modules are always cyclic, so is their fusion and $W_{k,z}(n) \times_{j}^{\mathfrak{gl}} S_{1/2}(1) = aTL_{n+1}x$ with

\begin{align*}
x \equiv \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\otimes aTL_n \\
\vdots \\
\vdots \\
\end{array}.
\end{align*}

The module can be decomposed by defining diagrams

\begin{align*}
v_+ = e_{2k+2}e_{2k+4} \cdots e_n, \quad v_- = e_{2k}e_{2k+2} \cdots e_n.
\end{align*}

These were chosen such that

\begin{align*}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\otimes aTL_n \\
\vdots \\
\vdots \\
\end{array}, \quad
\begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\otimes aTL_n \\
\vdots \\
\vdots \\
\end{array}.
\end{align*}

One verifies easily that $aTL_{n+1}v_-x$ is a sub-module: any diagram acting on $v_-x$ will either contract two (or more) through lines together or shuffle around the arcs on the bottom boundary. However, the former will be trivial because, for instance,

\begin{align*}
e_1v_-x = \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\otimes aTL_n \\
\vdots \\
\vdots \\
\end{array} = 0,
\end{align*}

where we used the definition of the tensor product, and the definition of the standard modules. We thus conclude that $aTL_{n+1}v_-x$ is a sub module isomorphic to $W_{k-1/2,z'}(n+1)$ for some $z'$.
where the isomorphism is obtained by simply cutting off the top of the diagrams, i.e.

\[
\begin{array}{cccc}
2k & & & n-2k+2 \\
\vdots & \ddots & \ddots & \ddots \\
2k & & & 1 \\
\otimes_{a\text{TL}_n} & & & \to \end{array}
\begin{array}{c}
2k & & & n-2k+2 \\
\vdots & \ddots & \ddots & \ddots \\
2k & & & 1 \\
\end{array}
\] (180)

To find \( z' \) we use the duality between the two fusion (125): 

\[
\text{Hom}_{a\text{TL}_{n+1}}\left( W_{k,z}(n) \times \mathbb{S}_2^{1}(1), W_{k-\frac{1}{2},z'}(n+1) \right) \cong \text{Hom}_{a\text{TL}_n}\left( W_{k,z}(n), W_{k-\frac{1}{2},z'}(n+1) \right)
\]

\[
\cong W_{k,z'}^{(q\frac{3}{2})} \oplus W_{k-1,z'}^{(-q)\frac{1}{2}} \cong \delta_{z,z'}^{(-q)^{\frac{3}{2}}}. \]

so it follows that \( z' = z(-q)^{1/2} \).

Note now that acting on \( v_x \) with some diagram \( a \) can do two things: it can shuffle the arcs on the bottom boundary, and it can close pairs of through lines. One can verify that closing the two rightmost through lines together will produce an element of the form \( av_x \), and the same thing happens when closing the leftmost line with the rightmost one. To see this last assertion, observe that in \( W_{k,z} \)

\[
\begin{array}{cccc}
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
2k & & & 1 \\
\otimes_{a\text{TL}_n} & & & \to \end{array}
\begin{array}{c}
\vdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
2k & & & 1 \\
\end{array}
\] = \( z^{-1} \) (181)

it thus follows that

\[
\begin{array}{cccc}
2k+1 & & & n-2k \\
\vdots & \ddots & \ddots & \ddots \\
2k & & & 1 \\
\otimes_{a\text{TL}_n} & & & \to \end{array}
\begin{array}{c}
2k+1 & & & n-2k \\
\vdots & \ddots & \ddots & \ddots \\
2k & & & 1 \\
\end{array}
\] = \( z^{-1} \) (182)

\[
\begin{array}{cccc}
2k+1 & & & n-2k \\
\vdots & \ddots & \ddots & \ddots \\
2k & & & 1 \\
\otimes_{a\text{TL}_n} & & & \to \end{array}
\begin{array}{c}
2k+1 & & & n-2k \\
\vdots & \ddots & \ddots & \ddots \\
2k & & & 1 \\
\end{array}
\] = \( -(-q)^{3/2} z^{-1} \) (183)
where we used the closed braid identity

\[ \bigotimes = (-q)^{3/2} \].

(184)

Repeating the arguments leading to the identification of \( aTL_{n+1}v_+x \), we finally obtain

\[
(W_{k,z}(n) \times_f S_{1/2}(1))/(aTL_{n+1}v_- x) = aTL_{n+1}(v_+x + aTL_{n+1}v_- x) \simeq W_{k+1/2,z(-q)^{-1/2}}(n + 1). \quad (185)
\]

To complete the decomposition, we need to figure out if this quotient splits, i.e. if the fusion product is the direct sum of two standard modules, or if it’s their indecomposable extension. However, if \( z \) is generic then \( \bar{Y} \) has distinct eigenvalues on \( W_{k \pm 1/2,z(-q)^{\mp 1/2}}(n + 1) \) so the quotient must split, and thus

\[
W_{k,z}(n) \times_f S_{1/2}(1) \simeq W_{k-1/2,z(-q)^{1/2}}(n + 1) \oplus W_{k+1/2,z(-q)^{-1/2}}(n + 1). \quad (186)
\]

If \( z^2 = (-q)^k \) then the quotient does not split and the fusion product is then the indecomposable extension of the two standard modules appearing in the previous direct sum.

**A.3.5 \( W_{0,z}(n) \times_f S_{1/2}(1) \)**

By the results in the previous sections and associativity of the fusion product (126), we know that

\[
W_{0,z}(2m) \times_f S_{1/2}(1) \simeq (W_{0,z}(2) \times_f S_{0}(2(m - 1))) \times_f S_{1/2}(1) \\
\simeq (W_{0,z}(2) \times_f S_{1/2}(1)) \times_f S_{0}(2(m - 1)),
\]

so we focus on the case \( n = 2 \) and the other cases will follow directly from it. Using the same reasoning as in the \( k \neq 0 \) cases, we find that \( W_{0,z}(2) \times_f S_{1/2}(1) = aTL_3x \), with

\[ x \equiv \bigotimes_{aTL_3}. \]

(187)

Now acting on \( x \) with any diagram can only move around the arc at the bottom of \( x \), so the fusion product is generated by elements of the form \( u^i x \) for \( i \in \mathbb{Z} \). However,

\[
Ye_1 = \bigotimes_{aTL_3} = (-q)^{1/2} \bigotimes_{aTL_3} + (-q)^{-1/2} \bigotimes_{aTL_3} = ((-q)^{-1/2}u^3 + (-q)^{1/2}u^{-3})e_1. \quad (188)
\]

Since by construction \( Yx = (z + z^{-1})x \), it follows that \((u^3 - (-q)^{-1/2}z)(u^3 - (-q)^{-1/2}z^{-1})x = 0\), and thus that the fusion product has dimension six. Furthermore, if \( z^2 \neq 1 \) then \( u^3 \) must have the two eigenvalues \((-q)^{-1/2}z^{\pm 1}\) so

\[
W_{0,z}(2) \times_f S_{1/2}(1) \simeq W_{1/2,(-q)^{-1/2}z}(3) \oplus W_{1/2,(-q)^{-1/2}z^{-1}}(3). \quad (189)
\]

Note that if \( z^2 = 1 \) then the two standard modules \( W_{1/2,(-q)^{-1/2}z^{\pm 1}}(3) \) are isomorphic, and one can show that the fusion product is then the self-extension of \( W_{1/2,(-q)^{-1/2}z^{\pm 1}}(3) \).
A.3.6 \( W_{0,z}(n + 1) \div_u f S_{1/2}(1) \)

Here again the two fusion types are similar so we focus only on the \( u \)-type. Here we must restrict to \( z \) generic right from the start; it is possible to present a unified proof for the \( z \) generic or not cases, but doing so requires more sophisticated tools which we haven’t introduced here. We thus start by noticing that

\[
\text{Hom}_{aTL_n}(W_{k,z'}(n), W_{0,z}(n + 1) \div_u f S_{1/2}(1)) \simeq \text{Hom}_{aTL_{n+1}}(W_{k,z'}(n) \times_u f S_{1/2}(1), W_{0,z}(n + 1))
\]

\[
\simeq \delta_{k,1/2}(\delta_{z',(-q)^{1/2}z^1} + \delta_{z',(-q)^{1/2}z^{-1}}) C,
\]

where we used the duality between the fusion product and the fusion quotient together with the formulas for the fusion product of standard modules obtained in the previous section. Furthermore, for generic values of \( z \) the standard modules \( W_{1/2,(-q)^{1/2}z^1}(n) \) are simple and non-isomorphic, so these morphisms must be injective. Finally, we have

\[
\text{Dim}(W_{0,z}(n + 1)) = 2 \text{Dim}(W_{1/2,z'}(n)),
\]

and we thus conclude that

\[
W_{0,z}(n + 1) \div_u f S_{1/2}(1) \simeq W_{1/2,z(-q)^{1/2}z^1}(n) \oplus W_{1/2,z^{-1}(-q)^{1/2}z}(n).
\]

B Comparison with the other fusion types

We discuss briefly how the defect operators – the hoop elements \( Y \) and \( \bar{Y} \) – introduced in this work behave with respect to the various previously defined fusion products, in particular the one introduced by Gainutdinov and Saleur in [13] and the one introduced by Belletête and Saint-Aubin [16].

B.1 The GS fusion

For this fusion, we endow \( aTL_{n+m} \) \((n, m \) positive integers) with the structure of a left \((aTL_n, aTL_m)\) module through the injection \( \Phi \equiv \phi_{n,m}^u \otimes \psi_{m,n}^o : aTL_n \otimes aTL_m \to aTL_{n+m} \). In terms of diagrams, this corresponds to gluing the two cylinders on which \( aTL_n \) and \( aTL_m \) lives into a pair of \textit{pants}; Fig. 7 illustrates how the hoop operators behave under this gluing.

Given an \( aTL_n \)-module \( U \) and an \( aTL_m \)-module \( V \), their fusion is defined as

\[
U \times_{GS} V \equiv aTL_{n+m} \otimes aTL_n \otimes aTL_m U \otimes_C V.
\]

Note that by construction (see Figs. 7a and 7b)

\[
Y^{(n+m)} = \phi_{n,m}^u(Y^{(n)}), \quad \bar{Y}^{(n+m)} = \psi_{m,n}^o(\bar{Y}^{(m)}),
\]

so that eigenvalues of the central elements \( Y, \bar{Y} \) on \( U \times_{GS} V \) are fully determined from their value on \( U \) and \( V \), respectively. It follows in particular that this fusion product is not commutative in general. Furthermore, the construction also gives (see Fig. 7c)

\[
\phi_{n,m}^u(\bar{Y}^{(n)}) = \psi_{m,n}^o(Y^{(m)}),
\]

43
which imposes severe constraints on the combinations of modules leading to non-trivial fusion products.

To illustrate just how restrictive equations (193) and (194) are, we compute the fusion of two standard modules $U \simeq W_{r,\delta}^o$ and $V \simeq W_{s,\mu}^o$. First, equation (194) imposes $\mu = \delta \pm 1$. Second, if the fusion product is non-zero then it must have at least one non-trivial simple quotient and, for generic values of $q$ and $\delta$, all of the simple modules of $\mathfrak{a}T\mathbb{L}_n$ are isomorphic to standard modules $W_{k,\nu}^o$ for some integer $k$ and $\nu \in \mathbb{C}^*$. Combining this observation with equation (193) gives the conditions
\[ \nu = (\delta (q^{-2r}) \pm 1, \quad \text{and} \quad \nu = (\mu (q^{-2s}) \pm 1 (-q)^{-2k}, \]
where all the $\pm$ are independent. In the generic cases where $\delta^2$ is not an integer power of $(-q)$, and $q$ is not a root of unity, this leaves the possibilities:
\[ k = (r + \epsilon s), \quad \delta = \mu \epsilon, \]
or, using a more usual notation
\[ W_{r,z}(n) \times_{GS} W_{s,w}(m) \simeq \begin{cases} a_+ W_{r+s,z(-q)^{-s}}(n) & \text{if } z = w (-q)^{r+s} \\ a_- W_{r-s,z(-q)^s}(n) & \text{if } z = w^{-1}(-q)^{r-s} \\ 0 & \text{otherwise} \end{cases}, \]
where $a_\pm$ are (unknown) non-negative integers (because each module could appear multiple times) and it is understood that if $s > r$, $W_{r-s,z(-q)^s}(n) \equiv W_{s-r,z^{-1}(-q)^{-s}}(n)$. It therefore simply remains to find the value of $a_\pm$; more extensive calculations \[13\] yields $a_\pm = 1$.

### B.2 The BSA fusion

For this fusion, we endow $\mathfrak{a}T\mathbb{L}_{n+m}$ ($n, m$ positive integers) with the structure of a left $(T\mathbb{L}_n, T\mathbb{L}_m)$ module through the injection $\Phi \equiv \phi^\nu_{n,m} \otimes \psi^\mu_{m,n} : T\mathbb{L}_n \otimes T\mathbb{L}_m \to \mathfrak{a}T\mathbb{L}_{n+m}$, where we identified the regular algebras with their images inside the affine algebra. In terms of diagrams, this corresponds

---

19This follows because the fusion of cyclic modules is also cyclic.
to taking the two strips on which the regular algebras live and stitching them into a cylinder, introducing no particular relations for the hoop operators. Given \( U, V \) a \( TL_n \) and a \( TL_m \) module, respectively, their fusion is defined as

\[
U \times_{BSA} V \equiv aTL_{n+m} \otimes_{TL_n \otimes TL_m} U \otimes_{C} V.
\] (198)

Note that because no relations were introduced for the hoop operators, the fusion of any non-zero module is always infinite dimensional. For the particular case of standard modules (and \( q \) generic), one finds [16]

\[
S_r \times_{BSA} S_s \simeq \bigoplus_{k=|r-s|}^{r+s} P_k,
\] (199)

where the \( P_k \) are the projective indecomposable modules of the affine Temperley-Lieb algebra. The algebraic structure of these modules is quite complicated and described in some details in [16].

We simply mention that there is a family of inclusions \( P_k \subset P_{k+2} \subset P_{k+4} \subset \ldots \) that is such that \( P_k/P_{k-2} \) is an indecomposable submodule of the direct product of the standard modules \( W_{k,z} \) for all non-zero \( z \). In other (more heuristic) words, \( P_k \) is an indecomposable collage of all \( W_{r,z} \) with \( r \leq k, z \in \mathbb{C}^* \).

\section{Alternative proof of the relations (114)-(115)}

We first establish the branching rules from \( aTL_{n_1+n_2} \) to \( aTL_{n_1} \otimes TL_{n_2} \), by an adaptation of the working of [14] that dealt with the case of \( aTL_{n_1} \otimes aTL_{n_2} \). The algebras \( aTL_{n_1} \) and \( TL_{n_2} \) can be embedded into \( aTL_n \) on \( n = n_1 + n_2 \) sites, by defining the periodic generator \( e_0^{(1)} \) and the shift operator \( u^{(1)} \) for the first subalgebra by braid translation in \( aTL_n \) (see [14]), while all other generators simply carry over.

Using the techniques of [14], and in particular the operator \( \tau_j^{(1)} \) defined there, we then find the branching rules (actually, one has just to restrict the second tensor factor in branching rules of [14] to the subalgebra \( TL_{n_2} \subset aTL_{n_2} )

\[
W_{j,z}(n)|_{aTL_{n_1} \otimes TL_{n_2}} \cong \bigoplus_{j_1,j_2} W_{j_1,z_1}(n_1) \otimes \left( \bigoplus_{k \geq j_2} S_k(n_2) \right),
\] (200)

with the following values of the momenta:

- For \( j = j_1 + j_2 \) and any values of \( j_1, j_2 \): \( z_1 = (i\sqrt{q})^{-2j_2 z+1} \).
- For \( j = j_1 - j_2 \) and either \( j = 0 \) or \( j_2 > 0 \): \( z_1 = (i\sqrt{q})^{+2j_2 z+1} \).
- For \( j = j_2 - j_1 \) and either \( j = 0 \) or \( j_1 > 0 \): \( z_1 = (i\sqrt{q})^{+2j_2 z-1} \).

Notice that this closely parallels the main result of [14], the only difference being that the right tensorands \( W_{j_2,z_2} \) have been replaced by \( \bigoplus_{k \geq j_2} S_k \), which is exactly the restriction of \( W_{j_2,z_2} \) to the subalgebra \( TL_{n_2} \subset aTL_{n_2} \). In particular, these modules have the same dimensions.

We can now read off the corresponding fusion rules by Frobenius reciprocity:

\[
\text{Hom}_{aTL_{n+m}}(W_{k,w}(n) \times_f S_j(m), W_{j,z}(n+m)) \simeq \text{Hom}_{aTL_{n} \otimes TL_{m}}(W_{k,w}(n) \otimes S_j(m), W_{j,z}(n+m)),
\]

45
where on the right side of the equation $W_{i,z}(n + m)$ is seen as a $aTL_n \otimes TL_m$-module, i.e. is identified with the branching rules (200). If $q$ and $z$ are generic, the standard modules $W_{j,z}(n + m)$ are simple, so the dimension of this homomorphism group is the number of copies of the standard module appearing as direct summands of the fusion product. There is however a slight subtlety in computing these dimensions, which is conveniently illustrated in the example $(n_1, n_2, n) = (2, 4, 6).$ The branching rules read in this case

$$W_{0,z}(6) = W_{0,z}(2) \otimes (S_2(4) \oplus S_1(4) \oplus S_0(4)) \oplus W_{1,\frac{\sqrt{2}}{2}z}(2) \otimes (S_2(4) \oplus S_1(4)) \oplus W_{1,\frac{\sqrt{2}}{2}z}(2) \otimes (S_2(4) \oplus S_1(4)),$$

$$W_{1,z}(6) = W_{0,\frac{\sqrt{2}}{2}z}(2) \otimes (S_2(4) \oplus S_1(4)) \oplus W_{1,\frac{\sqrt{2}}{2}z}(2) \otimes (S_2(4) \oplus S_1(4)) \oplus W_{1,\frac{\sqrt{2}}{2}z}(2) \otimes S_2(4),$$

$$W_{2,z}(6) = W_{0,\frac{\sqrt{2}}{2}z}(2) \otimes S_2(4) \oplus W_{1,\frac{\sqrt{2}}{2}z}(2) \otimes (S_2(4) \oplus S_1(4))$$

$$W_{3,z}(6) = W_{1,\frac{\sqrt{2}}{2}z}(2) \otimes S_2(4).$$ (201)

At first sight it appears that one would have fusion rules like

$$W_{0,z} \times_f S_1 = W_{0,z} \oplus W_{1,(-q)z},$$

$$W_{0,z} \times_f S_2 = W_{0,z} \oplus W_{1,(-q)z} \oplus W_{2,(-q)^2z}.$$ (202)

This is however not quite correct. Indeed, we should be careful when the left tensorand in the fusion product is $W_{0,z},$ since we have to take into account the isomorphism $W_{0,z} \cong W_{0,z}.$ Therefore the corresponding terms in the branching rules (201) can also be written

$$W_{1,z}(6) = W_{0,\frac{\sqrt{2}}{2}z}(2) \otimes (S_2(4) \oplus S_1(4)) \oplus \ldots,$$

$$W_{2,z}(6) = W_{0,\frac{\sqrt{2}}{2}z}(2) \otimes S_2(4) \oplus \ldots. (203)$$

This implies that we have a few extra terms, and (202) should be corrected into

$$W_{0,z} \times_f S_1 = W_{0,z} \oplus W_{1,(-q)z} \oplus W_{1,(-q)z}^{-1} \oplus W_{2,(-q)^2z} \oplus W_{1,(-q)z}^{-1} \oplus W_{2,(-q)^2z}^{-1}.$$

Taking into account this subtlety, the general result comes out as

$$W_{j_1,z} \times_f S_{j_2} = \bigoplus_{j = \text{max}(j_1 - j_2, j_2)}^{j_1 + j_2} W_{j,(-q)^{j - j_2}} \oplus \bigoplus_{j = j_2}^{j_2 - j_1} W_{j,(-q)^{j + j_2}}. (204)$$

where we have defined $j_2 = (j_1 - j_2) \mod 1,$ and $j_2' = (j_2 - j_1) \mod 1.$ After some amount of rewriting, this can be shown to lead to (114)-(115) in the main text, as claimed.

\footnote{Note that this implicitly assumes that the fusion product of standard modules is semisimple, this can be shown by using the fact that the fusion product preserves the spectrum of at least one of the hoop operators. It follows that if a module factors through the blob algebra, then so will its fusion with any other module; for generic values of $q, w,$ the blob algebra through which the standard module $W_{k,w}(n)$ factors through is semisimple, therefore so is its fusion.}
D Finding the basis in (105) to (107)

We explain here how we arrived at the basis of $W_{1/2,z}(3) \times_S S_{1/2}(1) = aTL_4x$ shown in section 4.1. The general procedure can be used to find bases of the fusion product of arbitrary standard modules, but the calculations quickly become very difficult to track when the size of the modules increases.

Recall that since $W_{1/2,z}(3)$ and $S_{1/2}(1)$ are both cyclic, so is their fusion, and we can thus write $W_{1/2,z}(3) \times_S S_{1/2}(1) = aTL_4y$ with

$$y = y.$$  \hspace{1cm} (205)

To start, observe that

$$\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}.  \hspace{1cm} (206)
$$

It thus follows that $e_2e_3y = y$, and thus, when looking for a basis of $W_{1/2,z}(3) \times_S S_{1/2}(1)$, we can restrict our search to elements of the form $ay$ where $a \in aTL_4$ is a sum of diagrams with at most 2 through lines (lines connecting the top and bottom of the diagram), and such that $ae_1 = \beta a$ (since $ay = ae_2e_1y$).

Furthermore, one also has

$$\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array},  \hspace{1cm} (207)
$$

and since

$$\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array},  \hspace{1cm} (208)
$$

it follows that

$$zy = \left( (-q)^{-1/2}ue_1 + (-q)^{1/2}ue_3 \right) y.  \hspace{1cm} (209)$$

Let’s consider then $M_0$, the submodule of $W_{1/2,z}(3) \times_S S_{1/2}(1)$ generated by elements of the form $ay$, where $a$ is a diagram with no through lines, no closed loop wrapping around the cylinder\footnote{Note that for any diagram $a$ with no through lines, $Ya = aY = \tilde{Y}a = a\tilde{Y}$, therefore the weight of non-contractible loops is always fixed by the parameter $z$ in $W_{1/2,z}(3)$.}, and such that $ae_1 = \beta a$. We find 12 such diagrams:

\begin{align}
&\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array} 
\end{align}

\begin{align}
&\begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}, \begin{array}{c}
\end{array}.
\end{align}
However, we then have

\[ \frac{\text{diagram}}{\text{diagram}} = -(-q)^{3/2} \frac{\text{diagram}}{\text{diagram}}, \]  \tag{212} \]

where we used the closed braid identity

\[ \frac{\text{diagram}}{\text{diagram}} = -(-q)^{3/2} \frac{\text{diagram}}{\text{diagram}}. \]  \tag{213} \]

Using equation (209), it follows that

\[ \frac{\text{diagram}}{\text{diagram}} y = -z(-q)^{3/2} \frac{\text{diagram}}{\text{diagram}} y. \]  \tag{214} \]

Applying the same reasoning to the other diagrams in (211), one concludes that \( M_0 \) is generated entirely by elements of the form \( a(y) \) with \( a \) one of the diagrams in (210). Furthermore, these elements are clearly linearly independent since the bottom side of the diagrams in (210) are all identical, and any relation of the form \( a(y) = aby \) for some \( b \in a\text{TL}_4 \) cannot affect the top side of the diagram \( a \), since it acts on its bottom (provided that \( a \) has no through line, which is the case here). We thus conclude that this set of elements spans \( M_0 \).

We now consider \( M_2 \), the subspace of \( W_{1/2, z}(3) \times_f S_{1/2}(1) \) generated by elements of the form \( a(y) \), where \( a \) is a diagram with exactly two through lines, and such that \( ae_1 = \beta a \). Such diagrams are all those which can be written in the form \( u^k e_{j_m} e_{j_{m-1}} \ldots e_{j_0} \), where \( k \in \mathbb{Z} \) and \( \{j_i\} \) is a set of \( m \in \mathbb{N} \) consecutive integers such that \( j_0 = 1 \).

However, from (209),

\[ e_1 y = z^{-1} e_1 \left( (-q)^{-1/2} u e_1 + (-q)^{1/2} u e_3 \right) y, \]  \tag{215} \]
\[ = z^{-1}(-q)^{-1/2} u e_4 e_1 y + a_0 y, \]  \tag{216} \]

where \( a_0 \) is a diagram with zero through lines (so \( a_0 y \in M_0 \)). It thus follows that

\[ u^{-1} e_1 y = w^{-1} e_4 e_1 y + b_1 y, \]  \tag{217} \]

where \( w = z(-q)^{1/2} \) and \( b_1 \) is a diagram with no through lines. Repeating this argument, we find

\[ u^{-2} e_1 y = w^{-2} u e_3 e_4 e_1 y + b_2 y, \]  \tag{218} \]
\[ u^{-3} e_1 y = w^{-3} u^2 e_3 e_2 e_3 e_4 e_1 y + b_3 y, \]  \tag{219} \]
\[ u^{-4} e_1 y = w^{-4} u^3 e_1 e_2 e_3 e_4 e_1 y + b_4 y, \]  \tag{220} \]

where \( b_j, j = 1, \ldots, 4 \) are diagrams with no through lines. Note that from the algebra relations (2) one has \( e_1 e_2 e_3 = u^2 e_3 \), so that

\[ u^{-4} e_1 y = w^{-4} u^2 e_3 e_4 e_1 y + b_4 y, \]  \tag{221} \]
\[ = w^{-2} e_1 y + b_4 y - w^{-2} b_2 y. \]  \tag{222} \]

It follows that \( M_2 \), the quotient of \( W_{1/2, z}(3) \times_f S_{1/2}(1) \) by the submodule \( M_0 \), is spanned by the set of equivalence classes \( \{ u^{-k} e_1 (y + M_0) | k = 0, 1, 2, 3 \} \).

We thus conclude that a basis of \( W_{1/2, z}(3) \times_f S_{1/2}(1) \) can be chosen as \( \{a_i| i = 5, \ldots, 10\} \cup \{ u^{-k} e_1 y | k = 0, \ldots, 3\} \), where the diagrams \( a_i \) are given in equation (210). Note that it is often more convenient to work with the set \( e_1 y, e_2 e_1 y, \ldots \) instead of \( e_1 y, u^{-1} e_1 y, \ldots \) as the resulting diagrams are easier to draw.
References

[1] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Kramers-Wannier duality from conformal defects*, Phys. Rev. Lett. 93, 070601 (2004).

[2] J. Fröhlich, J. Fuchs, I. Runkel and C. Schweigert, *Duality and defects in rational conformal field theory*, Nucl. Phys. B763 354-430 (2007).

[3] C. Bachas and M. Gaberdiel, *Loop operators and the Kondo problem*, JHEP 0411 (2004) 065.

[4] M. Oshikawa and I. Affleck, *Boundary conformal field theory approach to the two-dimensional critical Ising model with a defect line*, Nucl. Phys. B 495, 533–582 (1997).

[5] V.B. Petkova and J.-B. Zuber, *Generalized twisted partition functions*, Phys. Lett. B 504, 157 (2001).

[6] V. Petkova, *Topological defects in CFT*, in the Proceedings of the conference SYMPHYS XV, Yerevan, 25-29 July 2011, Physics of Atomic Nuclei 76, 10 (2013) 1329-1333.

[7] G. Sarkissian, *Defects and permutation branes in the Liouville field theory*, Nucl. Phys. B 821, 607 (2009).

[8] W.M. Koo and H. Saleur, *Representations of the Virasoro algebra from lattice models*, Nucl. Phys. B 426, 459 (1994).

[9] A. Milsted and G. Vidal, *Extraction of conformal data in critical quantum spin chains using the Koo-Saleur formula*, Phys. Rev. B 96, 245105 (2017).

[10] M. Zini and Z. Wang, *Conformal field theories as scaling limit of anyonic chains*, arXiv:1706.08497.

[11] N. Read and H. Saleur, *Associative-algebraic approach to logarithmic conformal field theories*, Nucl. Phys. B 777, 316 (2007).

[12] A.M. Gainutdinov and R. Vasseur, *Lattice fusion rules and logarithmic operator product expansions*, Nucl. Phys. B 868, 223–270 (2013).

[13] A.M. Gainutdinov and H. Saleur, *Fusion and braiding in finite and affine Temperley-Lieb categories*, arXiv:1606.04530, preprint Hamburger Beiträge zur Mathematik 596.

[14] A.M. Gainutdinov, J.L. Jacobsen and H. Saleur, *A fusion for the periodic Temperley-Lieb algebra and its continuum limit*, J. High Energ. Phys. 2018:117 (2018).

[15] A. M. GAINUTDINOV, J. L. JACOBSEN, H. SALEUR, AND R. VASSEUR, *A physical approach to the classification of indecomposable Virasoro representations from the blob algebra*, Nucl. Phys. B, 873 (2013), pp. 614–681.

[16] J. Belletête and Y. Saint-Aubain, *On the computation of fusion over the affine Temperley-Lieb algebra*, Nucl. Phys. B 937, 333–370 (2018).
[17] J. Belletête, The fusion rules for the Temperley-Lieb algebra and its dilute generalization, J. Phys. A: Math. Theor. 48, 395205 (2015).

[18] V. Pasquier and H. Saleur, Common structures between finite systems and conformal field theories through quantum groups, Nucl. Phys. B 330, 52 (1990).

[19] D. Aasen, R.S.K. Mong and P. Fendley, Topological defects on the lattice. I: The Ising model, J. Phys. A: Math. Theor. 49, 354001 (2016).

[20] M. Hauru, G. Evenbly, W. Wei Ho, D. Gaiotto, G. Vidal, Topological conformal defects with tensor networks, Phys. Rev. B 94, 115125 (2016).

[21] L.P. Kadanoff and H. Ceva, Determination of an operator algebra for the two-dimensional Ising model, Phys. Rev. B 3, 3918 (1971).

[22] A. Gainutdinov, D. Ridout and I. Runkel (guest editors), Special issue on logarithmic conformal field theory, J. Phys. A 46 (2013).

[23] A.M. Gainutdinov, J.L. Jacobsen, N. Read and R. Vasseur, Logarithmic conformal field theory: a lattice approach, J. Phys. A: Math. Theor. 46, 494012 (2013).

[24] A.M. Gainutdinov, N. Read and H. Saleur, Associative algebraic approach to logarithmic CFT in the bulk: the continuum limit of the gl(1|1) periodic spin chain, Howe duality and the interchiral algebra, Comm. Math. Phys. 341, 35–103 (2016).

[25] A.M. Gainutdinov, N. Read, H. Saleur and R. Vasseur, The periodic sl(2|1) alternating spin chain and its continuum limit as a bulk LCFT at c = 0, JHEP (2015) 114.

[26] J. Belletête, A.M. Gainutdinov, J.L. Jacobsen, H. Saleur and R. Vasseur, On the correspondence between boundary and bulk lattice models and (logarithmic) conformal field theories, J. Phys. A: Math. Theor. 50, 484002 (2017).

[27] A. Feiguin, S. Trebst, A.W.W. Ludwig, M. Troyer, A. Kitaev, Z. Wang and M.H. Freedman, Interacting Anyons in Topological Quantum Liquids: The Golden Chain, Phys. Rev. Lett. 98, 160409 (2007); 2010.1007/s00220-008-0419-4.

[28] S. Trebst, E. Ardonne, A. Feiguin, D. A. Huse, A. W. W. Ludwig, and M. Troyer, Collective States of Interacting Fibonacci Anyons, Phys. Rev. Lett. 101, 050401 (2008).

[29] T. Halverson, M. Mazzocco and A. Ram, Commuting families in Hecke and Temperley-Lieb algebras, Nagoya Math. J. 195, 125–152 (2009).

[30] C. Chui, C. Merkat, P. Orrick and P.A. Pearce, Integrable lattice realizations of conformal twisted boundary conditions, Phys. Lett. B 517, 429–435 (2001).

[31] F.C. Alcaraz, U. Grimm and V. Rittenberg, The XXZ Heisenberg chain, conformal invariance and the operator content of c < 1 systems, Nucl. Phys. B 316, 735 (1989).

[32] J.J. Graham and G.I. Lehrer, The representation theory of affine Temperley-Lieb algebras, L’Ens. Math. 44, 173 (1998).
[33] J. J. Graham and G. I. Lehrer, *Diagram algebras, Hecke algebras, and decomposition numbers at roots of unit*, Ann. Sci. E.N.S., 36 (2003), 479-524.

[34] P. Martin and H. Saleur, *The blob algebra and the periodic Temperley-Lieb algebra*, Lett. Math. Phys. 30, 189 (1994).

[35] P. Martin and H. Saleur, *An algebraic approach to higher dimensional statistical mechanics*, Comm. Math. Phys., 158, 1555–190 (1993).

[36] P.P. Martin and D. Woodcock, *On the structure of the blob algebra*, J. of Algebra 225, 957-988 (2000).

[37] P.A. Pearce, J. Rasmussen and S.P. Villani, *Solvable critical dense polymers on the cylinder*, J. Stat. Mech. 1002: P02010 (2010).

[38] A. Morin-Duchesne, P.A. Pearce and J. Rasmussen, *Fusion hierarchies, T-systems and Y-systems of logarithmic minimal models*, J. Stat. Mech. P05012 (2014).

[39] A. Morin-Duchesne, J. Rasmussen, D. Ridout, *Boundary algebras and Kac modules for logarithmic minimal models*, Nucl. Phys. B 899, 677–769 (2015).

[40] R.J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London, 1982.

[41] C.K. Fan, R.M. Green, *On the Affine Temperley-Lieb Algebras*, J. London Math. Soc. (2), 60, 366-380 (1999)

[42] D. Ridout and Y. Saint-Aubin, *Standard modules, induction and the structure of the Temperley-Lieb algebra*, Adv. Theor. Math. Phys. 18, no.5, 957-1041 (2014)