The Dual Horospherical Radon Transform
for Polynomials

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The dual horospherical Radon transform for polynomials

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1 Introduction

A Radon transform is generally associated to a double fibration

\[ p \quad \text{Z} \quad q \]
\[ \quad X \quad \quad \quad \quad Y, \]
where one may assume without loss of generality that the maps \( p \) and \( q \) are surjective and \( Z \) is embedded into \( X \times Y \) via \( z \mapsto (p(z),q(z)) \). Let some measures be chosen on \( X,Y,Z \) and on the fibers of \( p \) and \( q \) so that

\[
\int_X \left( \int_{p^{-1}(x)} f(u) \, du \right) \, dx = \int_Z f(z) \, dz = \int_Y \left( \int_{q^{-1}(y)} f(v) \, dv \right) \, dy. \tag{1}
\]

Then the Radon transform \( R \) is the linear map assigning to a function \( \varphi \) on \( X \) the function on \( Y \) defined by

\[
(R\varphi)(y) = \int_{q^{-1}(y)} (p^*\varphi)(v) \, dv,
\]
where we have set \( p^*\varphi := \varphi \circ p \). In a dual fashion, one defines a linear transform \( R^* \) from functions on \( Y \) to functions on \( X \) via

\[
(R^*\psi)(x) = \int_{p^{-1}(x)} (q^*\psi)(u) \, du.
\]

It is dual to \( R \). Indeed, formally,

\[
(R\varphi, \psi) = \int_Y \left( \int_{q^{-1}(y)} (p^*\varphi)(v) \, dv \right) \psi(y) \, dy
= \int_Y \left( \int_{q^{-1}(y)} (p^*\varphi)(q^*\psi)(v) \, dv \right) \, dy
= \int_Z (p^*\varphi)(z) (q^*\psi)(z) \, dz
\]

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\[ = \int_X \varphi(x) \left( \int_{p^{-1}(x)} (q^* \psi)(u) \, du \right) \, dx \\
= : (\varphi, R^* \psi). \]

In particular, if \( X = G/K \) and \( Y = G/H \) are homogeneous spaces of a Lie group \( G \), one can take \( Z = G/(K \cap H) \), with \( p \) and \( q \) being \( G \)-equivariant maps sending \( e(K \cap H) \) to \( eK \) and \( eH \), respectively. Assume that there exist \( G \)-invariant measures on \( X \), \( Y \) and \( Z \). If such measures are fixed, one can uniquely define measures on the fibers of \( p \) and \( q \) so that the condition (1) holds. (Here, speaking about a measure on a smooth manifold, we mean a measure defined by a differential form of top degree.) In this setting we can consider the transforms \( R \) and \( R^* \), and if \( \dim X = \dim Y \), one can hope that they are invertible. The basic example is provided by the classical Radon transform, which acts on functions on the Euclidean space \( \mathbb{E}^n = (\mathbb{R}^n \times SO(n))/SO(n) \) and maps them to functions on the space \( H\mathbb{E}^n = (\mathbb{R}^n \times SO(n))/((\mathbb{R}^{n-1} \times O(n-1)) \) of hyperplanes in \( \mathbb{E}^n \).

For a semisimple Riemannian symmetric space \( X = G/K \) of noncompact type, one can consider the horospherical Radon transform as proposed by I.M. Gelfand and M.I. Graev in [GG59]. Namely, generalizing the classical notion of a horosphere in Lobachevsky space, one can define a horosphere in \( X \) as an orbit of a maximal unipotent subgroup of \( G \). The group \( G \) naturally acts on the set \( \text{Hor} \, X \) of all horospheres. This action is transitive, so we can identify \( \text{Hor} \, X \) with some quotient space \( G/S \) (see §5 for details). It turns out that \( \dim X = \dim \text{Hor} \, X \). Moreover, the groups \( G, K, S \) and \( K \cap S = M \) are unimodular, so there exist \( G \)-invariant measures on \( X \), \( \text{Hor} \, X \) and \( Z = G/M \). The Radon transform \( R \) associated to the double fibration

\[
\begin{array}{ccc}
p & & q \\
\downarrow & & \downarrow \\
G/M & \rightarrow & G/S = \text{Hor} \, X \\
\end{array}
\]

is called the horospherical Radon transform.

The space \( Z = G/M \) can be interpreted as the set of pairs \( (x, \mathcal{H}) \in X \times \text{Hor} \, X \) with \( x \in \mathcal{H} \), so that \( p \) and \( q \) are just the natural projections. The fiber \( q^{-1}(\mathcal{H}) \) with \( \mathcal{H} \in \text{Hor} \, X \) is then identified with the horosphere \( \mathcal{H} \), and the fiber \( p^{-1}(x) \) with \( x \in X \) is identified with the submanifold \( \text{Hor}_x \, X \subseteq \text{Hor} \, X \) of all horospheres passing through \( x \). Note that, in contrast to the horospheres, all submanifolds \( \text{Hor}_x \, X \) are compact, since \( \text{Hor}_x \, X \) is the orbit of the stabilizer of \( x \) in \( G \), which is conjugate to \( K \).

In this paper, we describe the dual horospherical Radon transform \( \mathcal{R}^* \) in terms of its action on polynomial functions. Here a differentiable function \( \varphi \) on a homogeneous space \( Y = G/H \) of a Lie group \( G \) is called polynomial, if the linear span of the functions \( g \varphi \) with \( g \in G \) is finite dimensional. The polynomial functions constitute an algebra denoted by \( \mathbb{R}[Y] \).

For \( X = G/K \) as above, the algebra \( \mathbb{R}[X] \) is finitely generated and \( X \) is naturally identified with a connected component of the corresponding affine real algebraic variety (the real spectrum of \( \mathbb{R}[X] \)). The natural linear representation of \( G \) in \( \mathbb{R}[X] \) decomposes into a sum of mutually non-isomorphic absolutely irreducible finite dimensional representations whose highest weights \( \lambda \) form a semigroup \( \Lambda \). Let \( \mathbb{R}[X]_{\lambda} \) be the irreducible component of \( \mathbb{R}[X] \) with highest weight \( \lambda \), so

\[
\mathbb{R}[X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[X]_{\lambda}. \tag{2}
\]

Denote by \( \varphi_{\lambda} \) the highest weight function in \( \mathbb{R}[X]_{\lambda} \) normalized by the condition

\[
\varphi_{\lambda}(o) = 1,
\]
where $o = eK$ is the base point of $X$. Then the subgroup $S$ is the intersection of the stabilizers of all $\varphi_\lambda$’s. Its unipotent radical $U$ is a maximal unipotent subgroup of $G$.

The algebra $\mathbb{R}[\text{Hor } X]$ is also finitely generated. The manifold $\text{Hor } X$ is naturally identified with a connected component of a quasi-affine algebraic variety, which is a Zariski open subset in the real spectrum of $\mathbb{R}[\text{Hor } X]$. The natural linear representation of $G$ in $\mathbb{R}[\text{Hor } X]$ is isomorphic to the representation of $G$ in $\mathbb{R}[X]$. Let $\mathbb{R}[\text{Hor } X]_\lambda$ be the irreducible component of $\mathbb{R}[\text{Hor } X]$ with highest weight $\lambda$, so that

$$\mathbb{R}[\text{Hor } X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[\text{Hor } X]_\lambda.$$  \hfill (3)

Denote by $\psi_\lambda$ the highest weight function in $\mathbb{R}[\text{Hor } X]_\lambda$ normalized by the condition

$$\psi_\lambda(suo) = 1,$$

where $s$ is the symmetry with respect to $o$ and $suo = (sU_s^{-1})o$ is considered as a point of $\text{Hor } X$.

The decomposition (2) defines a filtration of the algebra $\mathbb{R}[X]$ (see §4 for the precise definition). Let $\text{gr } \mathbb{R}[X]$ be the associated graded algebra. There is a canonical $G$-equivariant algebra isomorphism

$$\Gamma : \text{gr } \mathbb{R}[X] \to \mathbb{R}[\text{Hor } X]$$

mapping each $\varphi_\lambda$ to $\psi_\lambda$. As a $G$-module, $\text{gr } \mathbb{R}[X]$ is canonically identified with $\mathbb{R}[X]$, so we can view $\Gamma$ as a $G$-module isomorphism from $\mathbb{R}[X]$ to $\mathbb{R}[\text{Hor } X]$.

The horospherical Radon transform $\mathcal{R}$ is not defined for polynomial functions on $X$ but its dual transform $\mathcal{R}^*$ is defined for polynomial functions on $\text{Hor } X$, since it reduces to integrating along compact submanifolds. Moreover, as follows from the definition of polynomial functions, it maps polynomial functions on $\text{Hor } X$ to polynomial functions on $X$. Obviously, it is $G$-equivariant.

Thus, we have $G$-equivariant linear maps

$$\mathbb{R}[X] \overset{\Gamma}{\to} \mathbb{R}[\text{Hor } X] \overset{\mathcal{R}^*}{\to} \mathbb{R}[X],$$

Their composition $\mathcal{R}^* \circ \Gamma$ is a $G$-equivariant linear operator on $\mathbb{R}[X]$, so

$$(\mathcal{R}^* \circ \Gamma)(\varphi) = c_\lambda \varphi \quad \forall \varphi \in \mathbb{R}[X]_\lambda,$$

where the $c_\lambda$ are constants. To give a complete description of $\mathcal{R}^*$, it is therefore sufficient to find these constants. Our main result is the following theorem.

**Theorem 1** $c_\lambda = \mathbf{c}(\lambda + \rho)$, where $\mathbf{c}$ is the Harish-Chandra $c$-function and $\rho$ is the half-sum of the positive roots of $X$ (counted with multiplicities).

The Harish-Chandra $c$-function governs the asymptotic behavior of the zonal spherical functions on $X$. A product formula for the $c$-function was found by S.G. Gindikin and F.I. Karpelevich [GK62]: for a rank-one Riemannian symmetric space of the noncompact type, the $c$-function is a ratio of gamma functions involving only the root multiplicities; in the general case, it is the product of the $c$-functions for the rank-one symmetric spaces defined by the indivisible roots of the space. Thus, known the root structure of the symmetric space, the product formula makes the $c$-function, and hence our description of the dual horospherical Radon transform, explicitly computable.

For convenience of the reader, we collect some crucial facts about the $c$-function in an appendix to this paper.

The following basic notation will be used in the paper without further comments.
• Lie groups are denoted by capital Latin letters, and their Lie algebras by the corresponding small Gothic letters.

• The dual space of a vector space $V$ is denoted by $V^*$.

• The complexification of a real vector space $V$ is denoted by $V(\mathbb{C})$.

• The centralizer (resp. the normalizer) of a subgroup $H$ in a group $G$ is denoted by $Z_G(H)$ (resp. $N_G(H)$).

• The centralizer (resp. the normalizer) of a subalgebra $\mathfrak{h}$ in a Lie algebra $\mathfrak{g}$ is denoted by $Z_{\mathfrak{g}}(\mathfrak{h})$ (resp. by $N_{\mathfrak{g}}(\mathfrak{h})$).

• If a group $G$ acts on a set $X$, we denote by $X^G$ the subset of fixed points of $G$ in $X$.

2 Groups, spaces, and functions

For any connected semisimple Lie group $G$ admitting a faithful (finite-dimensional) linear representation, there is a connected complex algebraic group defined over $\mathbb{R}$ such that $G$ is the connected component of the group of its real points. Among all such algebraic groups, there is a unique one such that all the others are its quotients. It is called the complex hull of $G$ and is denoted by $G(\mathbb{C})$. The group of real points of $G(\mathbb{C})$ is denoted by $G(\mathbb{R})$. If $G(\mathbb{C})$ is simply connected, then $G = G(\mathbb{R})$.

The restrictions of polynomial functions on the algebraic group $G(\mathbb{R})$ to $G$ are called polynomial functions on $G$. They are precisely those differentiable functions $\varphi$ for which the linear span of the functions $ge \cdot \varphi$, $g \in G$, is finite dimensional (see e.g. [CSM95], §II.8). (Here $G$ is supposed to act on itself by left multiplications.) The polynomial functions on $G$ constitute an algebra which we denote by $\mathbb{R}[G]$ and which is naturally isomorphic to $\mathbb{R}[G(\mathbb{R})]$.

For any subgroup $H \subseteq G$, we denote by $H(\mathbb{C})$ (resp. $H(\mathbb{R})$) its Zariski closure in $G(\mathbb{C})$ (resp. $G(\mathbb{R})$). If $H$ is Zariski closed in $G$, i.e. $H = H(\mathbb{R}) \cap G$, then $H$ is a subgroup of finite index in $H(\mathbb{R})$; if $H$ is a semidirect product of a connected unipotent group and a compact group, then $H = H(\mathbb{R})$.

For a homogeneous space $Y = G/H$ with $H$ Zariski closed in $G$, set $Y(\mathbb{C}) = G(\mathbb{C})/H(\mathbb{C})$. This is an algebraic variety defined over $\mathbb{R}$, and $Y$ is naturally identified with a connected component of the variety $Y(\mathbb{R})$ of real points of $Y(\mathbb{C})$. We call $Y(\mathbb{C})$ the complex hull of $Y$.

If $H$ is reductive, then also $H(\mathbb{C})$ is reductive and the variety $Y(\mathbb{C})$ is affine, the algebra $\mathbb{C}[Y(\mathbb{C})]$ being naturally isomorphic to the algebra $\mathbb{C}[G(\mathbb{C})]/\mathbb{C}[H(\mathbb{C})]$ of $H(\mathbb{C})$-right-invariant polynomial functions on $G(\mathbb{C})$ (see e.g. [VP89], Section 4.7 and Theorem 4.10). Correspondingly, the algebra $\mathbb{R}[Y(\mathbb{R})]$ is naturally isomorphic to $\mathbb{R}[G(\mathbb{R})]/\mathbb{R}^H(\mathbb{R})$.

In general, the functions on $Y(\mathbb{R})$ arising from $H(\mathbb{R})$-right-invariant polynomial functions on $G(\mathbb{R})$ are called polynomial functions on $Y(\mathbb{R})$, and their restrictions to $Y$ are called polynomial functions on $Y$. They are precisely those differentiable functions $\varphi$ for which the linear span of the functions $g\varphi$, $g \in G$, is finite dimensional. They form an algebra which we denote by $\mathbb{R}[Y]$.

In the following we consider a semisimple Riemannian symmetric space $X = G/K$ of non-compact type. This means that $G$ is a connected semisimple Lie group without compact factors and $K$ is a maximal compact subgroup of $G$. We do not assume that the center of $G$ is trivial, so the action of $G$ on $X$ may be non-effective. We do, however, require that $G$ has a faithful linear representation. According to the above, the space $X$ is then a connected component of the affine algebraic variety $X(\mathbb{R})$. 
3 Subgroups and subalgebras

We recall some facts about the structure of Riemannian symmetric spaces of noncompact type (see [Hel78] for details). Let $X = G/K$ be as above and $\theta$ be the Cartan involution of $G$ with respect to $K$, so $K = G^\theta$. Let $\mathfrak{a}$ be a Cartan subalgebra for $X$, i.e. a maximal abelian subalgebra in the $(-1)$-eigenspace of $d\theta$. Its dimension $r$ is called the rank of $X$. Under any representation of $G$, the elements of $\mathfrak{a}$ are simultaneously diagonalizable. The group $A = \exp \mathfrak{a}$ is a maximal connected abelian subgroup of $G$ such that $\theta(a) = a^{-1}$ for all $a \in A$. It is isomorphic to $(\mathbb{R}^*_+)^r$. Its Zariski closure $A(\mathbb{R})$ in $G(\mathbb{R})$ is a split algebraic torus which is isomorphic to $(\mathbb{R}^*_+)^r$. Let $X(A)$ denote the (additively written) group of real characters of the torus $A(\mathbb{R})$. It is a free abelian group of rank $r$. We identify each character $\chi$ with its differential $d\chi \in \mathfrak{a}^*$. The root decomposition of $\mathfrak{g}$ with respect to $A$ (or with respect to $A(\mathbb{R})$, which is the same) is of the form

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0 = \mathfrak{z}_G(\mathfrak{a})$. If $\mathfrak{m} := \mathfrak{z}_G(\mathfrak{a}_0)$, then $\mathfrak{g}_0 = \mathfrak{m} + \mathfrak{a}$.

The set $\Delta \subset X(A)$ is the root system of $X$ (or the restricted root system of $G$) with respect to $A$ and $\mathfrak{g}_\alpha$ is the root subspace corresponding to $\alpha$. The dimension of $\mathfrak{g}_\alpha$ is called the multiplicity of the root $\alpha$ and is denoted by $m_\alpha$. By the identification of a character with its differential, we will consider $\Delta$ as a subset of $\mathfrak{a}^*$. Choose a system $\Delta^+$ of positive roots in $\Delta$. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset \Delta^+$ be the corresponding system of simple roots. Then

$$C = \{x \in \mathfrak{a} : \alpha_i(x) \geq 0 \text{ for } i = 1, \ldots, r\}$$

is called the Weyl chamber with respect to $\Delta^+$. The subspace

$$\mathfrak{u} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

is a maximal unipotent subalgebra of $\mathfrak{g}$.

Set

$$G_0 = Z_G(A), \quad M = Z_K(A).$$

Then $G_0 = M \times A$ and the Lie algebras of $G_0$ and $M$ are $\mathfrak{g}_0$ and $\mathfrak{m}$, respectively. Clearly, $G_0$ is Zariski closed in $G$.

The group $U = \exp \mathfrak{u}$ is a maximal unipotent subgroup of $G$. It is normalized by $A$ and the map

$$U \times A \times K \to G, \quad (u, a, k) \mapsto uak,$$

is a diffeomorphism. The decomposition $G = UAK$ (or $G = KAU$) is called the Iwasawa decomposition of $G$. Since every root subspace is $G_0$-invariant, $G_0$ normalizes $U$, so

$$P := UG_0 = U \times G_0$$

is a subgroup of $G$. Moreover, $P = N_G(U)$ (see e.g. [War72], Proposition 1.2.3.4), so $P$ is Zariski closed in $G$.

We say that a Zariski closed subgroup of $G$ is parabolic, if its Zariski closure in $G(\mathbb{C})$ is a parabolic subgroup of $G(\mathbb{C})$. Then $P$ is a minimal parabolic subgroup of $G$. The subgroup

$$S := UM = U \times M$$

of $P$ is normal in $P$ and $P/S$ is isomorphic to $A$. It follows from the Iwasawa decomposition that $K \cap S = M$. 

5
4 Representations

For later use we collect some well-known facts about finite-dimensional representations of $G$.

The natural linear representation of $G$ in $\mathbb{R}[X]$ decomposes into a sum of mutually non-isomorphic absolutely irreducible finite-dimensional representations called (finite-dimensional) spherical representations (see e.g. [GW98], chap. 12).

**Theorem 2** (see [Hel84], § V.4) An irreducible finite dimensional representation of $G$ on a real vector space $V$ is spherical if and only if the following equivalent conditions hold:

1. $V^K \neq \{0\}$.
2. $V^S \neq \{0\}$.

If these conditions hold, then $\dim V^K = \dim V^S = 1$ and the subspace $V^S$ is invariant under $P$.

For a spherical representation, the group $P$ acts on $V^S$ via multiplication by some character of $P$ vanishing on $S$. The restriction of this character to $A$ is called the highest weight of the representation. A spherical representation is uniquely determined by its highest weight. The highest weights of all irreducible spherical representations constitute a subsemigroup $\mathbb{X}$. The semigroup $\mathbb{X}$ is freely generated by $\omega_1, \ldots, \omega_r$.

**Proposition 3** (see [Hel94], Proposition 4.23) The semigroup $\Lambda$ is freely generated by $\omega_1, \ldots, \omega_r$. 

For $\lambda \in \Lambda$, let $\mathbb{R}[X]_\lambda$ denote the irreducible component of $\mathbb{R}[X]$ with highest weight $\lambda$. Then we have

$$\mathbb{R}[X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[X]_\lambda.$$ 

We denote by $\varphi^S_\lambda$ the highest weight function of $\mathbb{R}[X]_\lambda$ normalized by the condition

$$\varphi^S_\lambda(o) = 1.$$ 

(Since $Po = PKo = Go = X$, we have $\varphi^S_\lambda(o) \neq 0$.) Obviously,

$$\varphi^S_{\lambda + \mu} \varphi^S_\mu = \varphi^S_{\lambda + \mu}.$$ 

In general, the multiplication in $\mathbb{R}[X]$ has the property

$$\mathbb{R}[X]_\lambda \mathbb{R}[X]_\mu \subset \bigoplus_{\nu \leq \lambda + \mu} \mathbb{R}[X]_\nu.$$
where “≤” is the ordering in the group $X(A)$ defined by the subsemigroup generated by the simple roots. In other words, the subspaces
\[ \mathbb{R}[X]_{≤λ} = \bigoplus_{μ≤λ} \mathbb{R}[X]_μ \]
constitute a $X(A)$-filtration of the algebra $\mathbb{R}[X]$ with respect to the ordering “≤”.

The functions $φ ∈ \mathbb{R}[X]_λ$ vanishing at $o$ constitute a $K$-invariant subspace of codimension 1. The $K$-invariant complement of it is a 1-dimensional subspace, on which $K$ acts trivially. Let $φ^K_λ$ denote the function of this subspace normalized by the condition $φ^K_λ(o) = 1$. It is called the zonal spherical function of weight $λ$.

**Lemma 4** (see e.g. [Hel84], p. 537) *For any finite dimensional irreducible representation of $G$ on a real vector space $V$ there is a positive definite scalar product $(\cdot | \cdot)$ on $V$ such that
\[ (gx | θ(g)y) = (x | y) \quad \text{for all} \ g ∈ G \ \text{and} \ x, y ∈ V. \]
This scalar product is unique up to a scalar multiple.*

The scalar product given by Lemma 4 is called $G$-skew-invariant. Note that it is $K$-invariant.

With respect to the $G$-skew-invariant scalar product on $\mathbb{R}[X]_λ$, the zonal spherical function $φ^K_λ$ is orthogonal to the subspace of functions vanishing at $o ∈ X$. Let $α_λ$ denote the angle between $φ^K_λ$ and $φ_S^λ$. Then the projection of $φ^K_λ$ to $\mathbb{R}[X]_λ^S = \mathbb{R}[X]_λ^S$ is equal to $(\cos^2 α_λ)φ_S^λ$ (see Figure 1). In particular, we see that $φ^K_λ$ and $φ_S^λ$ are not orthogonal.

![Figure 1](image-url)

The weight decomposition of $φ^K_λ$ is of the form
\[ φ^K_λ = (\cos^2 α_λ)φ_S^λ + \sum_{μ<λ} φ_{λ,μ}, \]
for some weight vector $φ_{λ,μ}$ of weight $μ$ in $\mathbb{R}[X]_λ$. This gives the asymptotic behavior of $φ^K_λ$ on $(\exp(-C^o))o$, where $C^o$ is the interior of the Weyl chamber $C$ in $a$. More precisely, for $ξ ∈ C^o$ we have
\[ φ^K_λ((\exp(-tξ))o) = ((\exp tξ)φ^K_λ)(o) \sim_{t→+∞} (\cos^2 α_λ)e^{tλ(ξ)}. \]
But it is known (see [Hel84], § IV.6) that the same asymptotics is described in terms of the Harish-Chandra c-function:
\[ φ^K_λ((\exp(-tξ))o) \sim_{t→+∞} c(λ + ρ)e^{tλ(ξ)}, \]
where $ρ = \frac{1}{2} \sum_{α∈Δ^+} m_α α$ is the half-sum of positive roots. This shows that
\[ \cos^2 α_λ = c(λ + ρ). \]
5 Horospheres

Definition 5 A horosphere in \( X \) is an orbit of a maximal unipotent subgroup of \( G \).

Since all maximal unipotent subgroups are conjugate to \( U \), any horosphere is of the form \( gUx \) \((g \in G, x \in X)\). Moreover, since \( X = P_0 \) and \( P \) normalizes \( U \), any horosphere can be represented in the form \( gUo \) \((g \in G)\). In other words, the set \( \text{Hor}_X \) of all horospheres is a homogeneous space of \( G \). The following lemma shows that the \( G \)-set \( \text{Hor}_X \) is identified with \( G/S \) if we take the horosphere \( Uo \) as the base point for \( \text{Hor}_X \).

Lemma 6 (see also [Hel94], Theorem 1.1, p. 77) The stabilizer of the horosphere \( Uo \) is the algebraic subgroup
\[
S = UM = U \Delta M.
\]

Proof. Obviously, \( S \) stabilizes \( Uo \). Hence the stabilizer of \( Uo \) can be written as \( \tilde{S} = U\tilde{M} \), where
\[
M \subset \tilde{M} \subset K.
\]
Since \( U \) is a maximal unipotent subgroup in \( G \) (and hence in \( \tilde{S} \)), it contains the unipotent radical \( \tilde{U} \) of \( \tilde{S} \). The reductive group \( \tilde{S}/\tilde{U} \) can be decomposed as
\[
\tilde{S}/\tilde{U} = (U/\tilde{U})\tilde{M},
\]
so the manifold \((\tilde{S}/\tilde{U})/(U/\tilde{U})\) is compact. But then the Iwasawa decomposition for \( \tilde{S}/\tilde{U} \) shows that the real rank of \( \tilde{S}/\tilde{U} \) equals 0, that is \( \tilde{S}/\tilde{U} \) is compact. This is possible only if \( U = \tilde{U} \). Hence,
\[
\tilde{S} \subset N(U) = P = S \Delta A.
\]
It follows from the Iwasawa decomposition \( G = AUK \) that \( \tilde{S} \cap A = \{e\} \). Thus \( \tilde{S} = S \).

As follows from [VP72], the \( G \)-module structure of \( \mathbb{R}[\text{Hor}_X] \) is exactly the same as that of \( \mathbb{R}[X] \), but in contrast to the case of \( \mathbb{R}[X] \), the decomposition of \( \mathbb{R}[\text{Hor}_X] \) into the sum of irreducible components is a \( \mathbf{X}(A) \)-grading.

Let \( \mathbb{R}[\text{Hor}_X]\lambda \) be the irreducible component of \( \mathbb{R}[\text{Hor}_X] \) with highest weight \( \lambda \), so
\[
\mathbb{R}[\text{Hor}_X] = \bigoplus_{\lambda \in \Lambda} \mathbb{R}[\text{Hor}_X]\lambda.
\]

Denote by \( \psi^S_\lambda \) and \( \psi^K_\lambda \) the highest weight function and the \( K \)-invariant function in \( \mathbb{R}[\text{Hor}_X]\lambda \) normalized by
\[
\psi^S_\lambda(sUo) = \psi^K_\lambda(sUo) = 1.
\]
To see that this is possible, note that the horosphere \( sUo \) is stabilized by \( sSs^{-1} = \theta(S) \). Hence the subspace \( V_0 \) of functions in \( \mathbb{R}[\text{Hor}_X]\lambda \) vanishing at \( sUo \) is \( \theta(S) \)-invariant. Its orthogonal complement is therefore \( S \)-invariant and must coincide with \( \mathbb{R}[\text{Hor}_X]^{K}\lambda \). This implies \( \psi^K_\lambda(sUo) \neq 0 \), so we can normalize \( \psi^K_\lambda \) as asserted. Since \( \mathbb{R}[\text{Hor}_X]^{K}\lambda \) and \( \mathbb{R}[\text{Hor}_X]^{K}\mu \) are not orthogonal in \( \mathbb{R}[\text{Hor}_X]^{K} \), we have \( \mathbb{R}[\text{Hor}_X]^{K}\lambda \cap V_0 = \{0\} \) and we can normalize also \( \psi^K_\lambda \) as asserted. Notice that the horospheres passing through \( o \) form a single \( K \)-orbit and therefore \( \psi^K_\lambda \) takes the value 1 on each of them.

Consider the \( \mathbf{X}(A) \)-graded algebra \( \text{gr} \mathbb{R}[X] \) associated with the \( \mathbf{X}(A) \)-filtration of \( \mathbb{R}[X] \) defined in §4. As a \( G \)-module, \( \text{gr} \mathbb{R}[X] \) can be identified with \( \mathbb{R}[X] \), but when we multiply elements \( \varphi_\lambda \in \mathbb{R}[X]\lambda \) and \( \varphi_\mu \in \mathbb{R}[X]_\mu \) in \( \text{gr} \mathbb{R}[X] \), only the highest term in their product in \( \mathbb{R}[X] \) survives. Moreover, there is a unique \( G \)-equivariant linear isomorphism
\[
\Gamma : \mathbb{R}[X] = \text{gr} \mathbb{R}[X] \rightarrow \mathbb{R}[\text{Hor}_X]
\]
such that \( \Gamma(\varphi_\lambda) = \psi^S_\lambda \).
Proposition 7 The map $\Gamma$ is an isomorphism of the algebra $gr \mathbb{R}[X]$ onto the algebra $\mathbb{R}[Hor X]$. 

Proof. For any semisimple complex algebraic group, the tensor product of the irreducible representations with highest weights $\lambda$ and $\mu$ contains a unique irreducible component with highest weight $\lambda + \mu$. It follows that, if we identify the irreducible components of $\mathbb{R}[X]$ with the corresponding irreducible components of $\mathbb{R}[Hor X]$ via $\Gamma$, the product of functions $\varphi_\lambda \in R[X]_\lambda$ and $\varphi_\mu \in R[X]_\mu$ in $gr \mathbb{R}[X]$ differs from their product in $\mathbb{R}[Hor X]$ only by some factor $a_{\lambda\mu}$ depending only on $\lambda$ and $\mu$. Taking $\varphi_\lambda = \varphi^S_\lambda$ and $\varphi_\mu = \varphi^S_\mu$, we conclude that $a_{\lambda\mu} = 1$. \hfill $\square$

Remark 8 The definition of $\Gamma$ makes use of the choice of a base point $o$ in $X$ and a maximal unipotent subgroup $U$ of $G$, but it is easy to see that all such pairs $(o,U)$ are $G$-equivalent. It follows that $\Gamma$ is in fact canonically defined.

6 Proof of the Main Theorem

Consider the double fibration

\[
\begin{array}{ccc}
\mathbb{R}/M & \xrightarrow{p} & G/M \\
\downarrow & & \downarrow \\
X = G/K & \xrightarrow{q} & G/S = Hor X
\end{array}
\]

Since all the involved groups are unimodular, there are invariant measures on the homogeneous spaces $X$, $Hor X$, $G/M$ and on the fibers of $p$ and $q$, which are the images under the action of $G$ of $K/M$ and $S/M$, respectively. Let us normalize these measures so that:

1. the volume of $K/M$ is 1;
2. the measure on $G/M$ is the product of the measures on $K/M$ and $X$;
3. the measure on $G/M$ is the product of the measures on $S/M$ and $Hor X$.

(This leaves two free parameters).

Consider the dual horospherical Radon transform

\[ R^* : \mathbb{R}[Hor X] \to \mathbb{R}[X]. \]

Combining it with the map $\Gamma$ defined in §5, we obtain a $G$-equivariant linear isomorphism

\[ R^* \circ \Gamma : \mathbb{R}[X] \to \mathbb{R}[X]. \]

In view of absolute irreducibility, Schur’s lemma shows that $R^* \circ \Gamma$ acts on each $\mathbb{R}[X]_\lambda$ by scalar multiplication. The scalars are given by the following theorem:

Theorem 9 For $\varphi \in \mathbb{R}[X]_\lambda$,

\[ (R^* \circ \Gamma)(\varphi) = c(\lambda + \rho)\varphi, \]

where $c$ is the Harish-Chandra c-function.

Proof. We test the map at the zonal spherical function $\varphi^K_\lambda \in \mathbb{R}[X]_\lambda$. The map $\Gamma$ takes it to $c_\lambda \psi^K_\lambda$ for some $c_\lambda \in \mathbb{R}$. Since the function $\psi^K_\lambda$ has value 1 on the horospheres passing through $o$, the map $R^*$ takes it to $\varphi^K_\lambda$. Thus we have

\[ (R^* \circ \Gamma)(\varphi^K_\lambda) = c_\lambda \varphi^K_\lambda. \]

Identifying $\mathbb{R}[X]_\lambda$ and $\mathbb{R}[Hor X]_\lambda$ via $\Gamma$, we now find $c_\lambda = \cos^2 \alpha_\lambda$ (see Figure 2), and this proves the claim.
A Appendix: The c-function

Because of the Iwasawa decomposition $G = KAU$, every $g \in G$ can be written as $g = k \exp H(g)u$ for a uniquely determined $H(g) \in a$. Let $U := \theta(U)$, and let $d\pi$ denote the invariant measure on $U$ normalized by the condition

$$\int_U e^{-2\rho(H(\pi))} d\pi = 1.$$ 

The c-function has been defined by Harish-Chandra as the integral

$$c(\lambda) := \int_U e^{-(\lambda + \rho)(H(\pi))} d\pi,$$

which absolutely converges for all $\lambda \in a(C)^*$ satisfying $\text{Re}(\lambda, \alpha) > 0$ for all $\alpha \in \Delta^+$. The computation of the integral gives the so-called Gindikin–Karpelevich product formula [GK62] (see also [Hel84], Section IV.6.4, or [GV88], p. 179):

$$c(\lambda) = \kappa \prod_{\alpha \in \Delta^{++}} \frac{2^{-\lambda_\alpha}}{\Gamma \left( \frac{\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{1}{2} \right) \Gamma \left( \frac{\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{m_\kappa}{2} + \frac{1}{2} \right)},$$

(5)

where $\Delta^{++}$ denotes the set of indivisible roots in $\Delta^+$, $\lambda_\alpha := (\lambda, \alpha) / (\alpha, \alpha)$, and the constant $\kappa$ is chosen so that $c(\rho) = 1$. This formula provides the explicit meromorphic continuation of $c$ to the entire $a(C)^*$.

Formula (5) simplifies in the case of a reduced root system (i.e. when $\Delta^{++} = \Delta^+$), since the duplication formula

$$\Gamma(2z) = 2^{2z-1} \sqrt{\pi} \Gamma(z)\Gamma(z+1/2)$$

(6)

for the gamma function yields

$$c(\lambda) = \kappa \prod_{\alpha \in \Delta^+} \frac{\Gamma(\lambda_\alpha)}{\Gamma(\lambda_\alpha + m_\alpha/2)},$$

(7)

with

$$\kappa = \prod_{\alpha \in \Delta^+} \frac{\Gamma(\rho_\alpha + m_\alpha/2)}{\Gamma(\rho_\alpha)}.$$
If, moreover, all the multiplicities \( m_\alpha \) are even (which is equivalent to the property that all Cartan subalgebras of \( g \) are conjugate), then the functional equation \( z \Gamma(z) = \Gamma(z + 1) \) implies

\[
c(\lambda) = \prod_{\alpha \in \Delta^+} \frac{\rho_\alpha (\rho_\alpha + 1) \cdots (\rho_\alpha + m_\alpha/2 - 1)}{\lambda_\alpha (\lambda_\alpha + 1) \cdots (\lambda_\alpha + m_\alpha/2 - 1)}.
\]

Finally, suppose that the group \( G \) admits a complex structure. In this case the root system is reduced and \( m_\alpha = 2 \) for every root \( \alpha \), and (7) reduces to

\[
c(\lambda) = \prod_{\alpha \in \Delta^+} \frac{\rho_\alpha}{\lambda_\alpha}.
\]

**Example 10** For \( n \)-dimensional Lobachevsky space, there is only one positive root \( \alpha \), with \( m_\alpha = n - 1 \), so \( \rho = (n - 1)\alpha/2 \). Let \( \lambda = la \). Then formula (7) gives

\[
c(\lambda) = \frac{\Gamma(n - 1)\Gamma(l)}{\Gamma(l + n - 1)}.
\]

The semigroup \( \Lambda \) is generated by \( \alpha \). For \( \lambda = la \ (l \in \mathbb{N}) \), we obtain

\[
c_\lambda = c(\lambda + \rho) = \frac{\Gamma(n - 1)\Gamma(l + \frac{n-1}{2})}{\Gamma(l + \frac{n-1}{2})\Gamma(n + l - 1)} = \begin{cases} 
\frac{(n + l - 1)(n + l) \cdots (n + 2l - 3)}{2^{2l-1} (\frac{l}{2} + 1) \cdots (\frac{l}{2} + l - 2)}, & n \text{ even,} \\
\frac{(\frac{n+1}{2} + 1)(\frac{n+1}{2} + 2) \cdots (\frac{n+1}{2} + l - 1)}{2n(n+1) \cdots (n+l-2)}, & n \text{ odd.}
\end{cases}
\]

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