COUNTEREXAMPLE TO THE HODGE CONJECTURE

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Abstract. We construct a K3 surface whose transcendental lattice has a self-isomorphism which is not a linear combination of self-isomorphisms over \(\mathbb{Q}\) which preserve cup products up to nonzero multiples. Products of it with itself give candidates for counterexamples to the Hodge conjecture which may be of interest.

1. Introduction

The Hodge conjecture is that every cohomology class over the rational numbers of a nonsingular projective algebraic variety \(V^n\) and has Hodge filtration \(p, p\) is represented by a rational linear combination of classes of subvarieties of complex dimension \(p\). Finding such subvarieties is important in many settings such as the study of fibrations whose fibres are abelian varieties, and of sections of those fibrations. There are more general categorical implications, such as the question of a semisimple category of motives, and the role that ordinary homology plays in stable \(A^1\) homotopy \([32],[33],[28]\) and related theories \([1]\). The definition of the Hodge structure of the complex cohomology of algebraic varieties will not be given here, see for instance \([8],[34]\). This conjecture is a theorem of Lefschetz in complex dimensions \(1, n - 1\). Griffiths \([5],[6],[7]\) proved that there are examples of algebraic cycles which are equivalent in homology but are not algebraically equivalent; a more recent extension of this result is \([20]\), and this shows that a kind of counterpart to the Hodge conjecture is false.

In general the Hodge conjecture is unknown even for abelian varieties, that is, algebraic varieties which are tori differentiably, admitting the group operation: a discussion is given in the Appendix by Brent Gordon to \([15]\). In particular it is unknown in general for what are called abelian varieties of Weil type, though Chad Schoen \([26]\) proved the Hodge conjecture for a few of them. The Hodge structures of abelian varieties are closely related \([31]\) to the Hodge structures of a famous class of algebraic surfaces: K3 surfaces are complex structures differentiably equivalent to a generic quartic hypersurface in \(CP^3\). Yuri Zarhin \([36]\) proved a number of basic results pertaining to the Hodge structures of K3 surfaces. Jordan Rizov \([25]\) gives a description of topics related to complex multiplication for the Hodge structures of K3 surfaces, and proves a fundamental theorem. A general exposition on complex surfaces including the basics on K3 surfaces is \([30]\).

Shigeru Mukai \([19]\) proved that any Hodge isometry over \(\mathbb{Q}\) for K3 surfaces \(X, Y\) is induced by an algebraic cycle \(Z \subset X \times Y\). This was used by José Mari \([16]\) to prove the Hodge conjecture for certain products of K3 surfaces. For our example surfaces related to K3 surfaces are constructed such that the Hodge structure has a linear equivalence to itself over \(\mathbb{Q}\), which is not a linear combination of self-maps that

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preserve the cup product quadratic forms. If there were a suitable partial converse to Mukai’s result in this special situation, then that would give a counterexample to the Hodge conjecture. In the general case, a K3 Hodge structure will determine the cup products up to rational multiples, but there are exceptions.

This present method is not related to our earlier remarks [13].

2. Background on Hodge structures, K3 surfaces, and cohomology transfer; polarizations

Much of this material will follow the references by Gordon [15] and Rizov [25].

Definition. A Riemann form on a complex torus $V/L$ is a nondegenerate, skew-symmetric $\mathbb{R}$-bilinear form $E: L \times L, V \times V \to \mathbb{R}$ such that $E(x, iy)$ is symmetric and positive definite.

Complex tori with a Riemann form are abelian varieties.

Definition. A rational Hodge structure of weight $n$ on a $\mathbb{Q}$-vector space $V$ consists of a decreasing filtration $F^r(V_\mathbb{C})$ of the complexification $V_\mathbb{C}$ such that $F^r(V_\mathbb{C}) \oplus F^{n-r+1}(V_\mathbb{C}) = V_\mathbb{C}$, $F^0(V_\mathbb{C}) = V_\mathbb{C}, F^{n+1}(V_\mathbb{C}) = 0$.

This is equivalent to having a Hodge decomposition $V_\mathbb{C} = \bigoplus V_{p,q} = F^p(V_\mathbb{C}) \cap F_q(V_\mathbb{C})$ and is equivalent also to a homomorphism $h_1: U(1) \to GL(V_\mathbb{R})$ of real algebraic groups, given by multiplication by $u^{-p}$ on $V_{p,q}$, or a homomorphism $h: \mathbb{C}^* \to GL(V_\mathbb{R})$ of real algebraic groups given by multiplication by $z^{-p}z^{-q}$.

Definition. A morphism of rational Hodge structures is a rational linear map preserving filtration (equivalently, any of the other structures in the last paragraph).

Definition. A polarization of a Hodge structure $(V, h)$ is a morphism $b: V \otimes V \to \mathbb{Q}(-n)$ which as a bilinear form satisfies $b(x, iy), x, y \in V_\mathbb{R}$ is symmetric and positive definite; $\mathbb{Q}(-n)$ is the weight 2 $n$ Hodge structure on $\mathbb{Q}$ whose only non-zero summand is $\mathbb{Q}^{n,n}$.

Likewise for an abelian variety, a polarization is an isogeny to the dual variety; it is said to be principal if it is an isomorphism of abelian varieties. Polarizations of abelian varieties are not unique but the first homology of curves do have canonical polarizations, and those polarizations determine the curves uniquely. There exist polarizations of the cohomology of all nonsingular projective algebraic varieties, but again, they may not be unique. The standard polarizations on primitive cohomology have the form $b(x, y)$ is the evaluation of $xyw$ on the fundamental class of the manifold, for some $w$.

Polarizations exist naturally on tensor products of polarized Hodge structures, and on substructures of polarized Hodge structures.

We will be considering situations in which an $H^2$ summand is spanned by cup products from $H^1$ and the automorphisms of the $H^1$ Hodge structure form a Clifford algebra which preserves the $H^2$ polarization.

Unless stated otherwise, coefficients in topological homology and cohomology will be $\mathbb{Q}$.

We will relate these K3 surfaces to the Kuga-Satake construction as treated by van Geemen [31], [14]. This construction shows that every polarized weight 2 Hodge structure with a single holomorphic 2 form which has suitable signature arises as a Hodge structure from a 1 dimensional polarized Hodge structure. The 1-dimensional Hodge structure gives a certain abelian variety and the 2-dimensional Hodge structure will then be a summand on the 2-dimensional cohomology of the abelian variety.
This construction goes as follows. Start with a vector space $V$ with a bilinear form $b$ giving the polarization. Take the Clifford algebra $C$ of this form, and its even degree part $C^+$. Let $f_1 + if_2$ be a holomorphic 2 form. Then multiplication in the Clifford algebra by $f_1f_2$ suitably normalized can be taken as multiplication by $i$ in a complex structure; this extends linearly to an action of $a + bi$ and hence a Hodge structure of weight 1.

The Clifford algebra has an anti-involution $i_a$ which reverses the order of multiplication in monomials from $V$ and a linear map $Tr : C \to \mathbb{Q}$, the trace of right multiplication. Let $e_1,e_2$ be two basis elements for $V$, in a basis such that the quadratic form is diagonalized, so that $e_1e_2 = -e_2e_1$, and such that the signs of $e_1^2, e_2^2$ are negative. Then a polarization of the Hodge structure on $C^+$ is given by $Tr(\pm e_1e_2i_a(v)w)$, for some choice of sign.

This form $E$ allows $C$ to be identified with its dual, and therefore the natural isomorphism $C^+ \otimes C^{++} \to \text{End}(C^+, C^+)$ gives an identification of $\text{End}(C^+, C^+)$ and $C^+ \otimes C^+$. Embedded in $C^+ \otimes C^+$ are copies of the original Hodge structure on $V$: for a fixed invertible element $e \in E$ there is a mapping $V \to \text{End}(C^+, C^+)$ which for $v \in V$ sends $x \in C^+$ to $vxe$. This map $V \to C^+ \otimes C^+$ commutes with multiplication by the complex numbers and thus gives a sub-Hodge structure.

Note that for weight 2 Hodge structures, the action of $i$ corresponds to the square of the action for weight 1 Hodge structures.

Definition. The Hodge group of $V$ is the smallest algebraic subgroup of $GL(V)$ defined over $\mathbb{Q}$, containing $h(U(1))$. The smallest algebraic subgroup containing $h(C^+)$ is the Mumford-Tate group.

The Mumford-Tate group consists of scalar multiples of elements in the Hodge group. Its representations determine those aspects of Hodge structures which are important for the Hodge conjecture; on any tensor product of $V$ and its dual, the rational sub-Hodge structures are those subspaces which are subrepresentations of the representation of the Mumford-Tate group. For a polarizable Hodge structure, both groups will be reductive. For Hodge structures occurring as the homology of a nonsingular algebraic variety in some dimension, the Hodge group will preserve a bilinear form obtained from cup products up to scalar multiples, which makes it something like an orthogonal or symplectic group times scalars.

Algebraic representations of a reductive group are semisimple (even though reductive algebraic groups are not called semisimple as algebraic groups) for instance, $U$. K3 surfaces are 4 dimensional topologically and simply connected, their middle cohomology groups are a sum of 22 copies of $\mathbb{Z}$, the ranks of $H^{0,2}, H^{1,1}, H^{2,0}$ are 1,20,1.

A quadratic lattice is a lattice provided with a bilinear form. $U$ is the hyperbolic plane, i.e. it has basis $e,f$ with $\langle e,e \rangle = \langle f,f \rangle = 0, \langle e,f \rangle = 1$. $E_8$ is the positive quadratic lattice corresponding to the Dynkin diagram of type $E_8$. Then the 2-dimensional cohomology of a K3 surface has type a sum of 3 copies of $U$ and 2 of $-E_8$. This lattice and its form are denoted $L_0, \psi$. $L_{2d}, \psi_{2d}$ are the sublattices spanned by $e_1 + df_1$ in the first $U$ summand, together with all other summands.

$A_X$ is the subgroup of $L_0$ for a K3 surface, generated by Chern classes of line bundles, equivalently by Hodge (1,1)-cycles or algebraic cycles, and its orthogonal complement is called $T_X$, the transcendental lattice.
Definition. A polarized Hodge structure of degree $d$ of K3 type is a homomorphism $h : \mathbb{C}^* \rightarrow SO(V, \psi_{\mathbb{R}})$ such that $V, \psi$ as an orthogonal space is equivalent to $(V_{2d}, \psi_{2d})$, $\psi$ is a polarization of this Hodge structure, $h$ defines a rational Hodge structure of type $(-1,1),(0,0),(1,-1)$, and the Hodge numbers (ranks) are 1, 19, 1.

In the case of $H^2$ polarizations of surfaces and transcendental lattices of K3 surfaces, polarizations of Hodge structures are given by the quadratic forms given by cup products; the different signs for the cup product form vs. the polarization correspond to the different $C^*$ action on $H^{2,0}, H^{0,2}$.

Zarhin proved that for Hodge structures of K3 surfaces, the complex multiplication, that is, the algebra $E$ of endomorphisms of the Hodge structure on the transcendental lattice, must be either a totally real field or a quadratic imaginary extension of such a field. He also proved that the Hodge group is the special orthogonal or unitary group of a form arising from cup products. These also imply irreducibility of the Hodge structure of the transcendental lattice. Thus when any Hodge structure of a nonsingular projective algebraic variety is decomposed as a sum of indecomposable Hodge structures, each component will either be isomorphic to that transcendental lattice or have no nonzero homomorphisms to or from it.

The global Torelli theorem for K3 surfaces was first proved by I. Piatetski-Shapiro and I. Shafarevich [23]. Friedman [4] states the global Torelli theorem, which completely determines the set of possible polarized Hodge structures including cup products over $\mathbb{Z}$, for all K3 surfaces, in slightly different notation. In effect pairs consisting of a K3 surface and a polarization, a choice of primitive, numerically effective line bundle, are in one-to-one correspondence with lines
\[ Cv \subset L_{2k} \otimes \mathbb{C} \ni v^2 = 0, v \cdot \overline{v} > 0 \]
modulo automorphisms of $L_0$ over $\mathbb{Z}$. The lines can be identified the space of holomorphic 2 forms. Complex multiplication by a subring of a field $E$ will correspond to an embedding of $E$ in $\mathbb{C}$ called the reflex field, acting multiplicatively on the vector $v$, such there is a compatible ring of additive endomorphisms of $\Lambda$. A version of global Torelli due to D. Burns and M. Rapaport [2] also describes mappings among K3 surfaces.

We also require the fact that any birational map of surfaces can be realized as a regular map by blowups of points each of which introduces only new algebraic cycles in the middle dimensional cohomology [9], [11]. Such blowups map to the original space and will not affect transcendental summands of the Hodge structure, nor products from them into $H^4$ except up to integer nonzero multiples, nor $H^1$ nor products from $H^1$ into $H^2$ modulo algebraic summands of the Hodge structure.

3. Quadratic forms

Some background on quadratic forms over the algebraic number fields is also needed: By the Hasse-Minkowski theorem, two quadratic forms over an algebraic number field $K$ are isomorphic if and only if they are isomorphic over every completion of $K$. Every form can be diagonalized over the $K$, that is, it can be written in the form
\[ \sum_{i=1}^{n} c_i x_i^2. \]
A complete set of invariants for isomorphism of quadratic forms over $K$ is given by the dimension, the signature, the discriminant (determinant of the matrix of the
form up to squares) and the Hasse invariant. The Hasse invariant of a diagonalized form \( \sum c_i x_i^2 \) is the Brauer group class of a tensor product of quaternion algebras \( \otimes_{i \leq j} (c_i, c_j) \) over the rational numbers, where \((c_i, c_j)\) is a quaternion algebra whose norm form in terms of a standard basis is \( x_1^2 + c_i x_2^2 + c_j x_3^2 + c_k x_4^2 \). It is simplest to compute the Hasse invariant locally at each prime, where it can be considered as a Hilbert symbol, is \( \pm 1 \), and then multiply the Hilbert symbols for all pairs \((c_i, c_j), i \leq j\). There is a reciprocity formula for Hilbert symbols, hence Hasse invariants, so that knowing the Hasse invariant at all but one prime for a quadratic form determines its value at the remaining prime. The Hasse invariant of a sum of two quadratic forms is the product of their Hasse invariants times a quantity which depends only on their discriminants.

It will suffice to consider and embed a transcendental lattice over \( \mathbb{Q} \), to study the \( \mathbb{Q} \) Hodge structure. That is we take the quadratic form over \( \mathbb{Z} \), over \( \mathbb{Q} \) find a summand of this form, take a mapping, and carry the period vector with the mapping to construct a K3 surface with a suitable transcendental lattice. Within the transcendental lattice, the coordinates of \( z \) must be linearly independent over \( \mathbb{Q} \), since any linear combination which is 0 corresponds to a class in \( H^1 \) which is a Hodge cycle, and the transcendental lattice will not contain any such classes. For background in algebraic number theory, see [35].

Proposition 3.1. For any quadratic forms \( q_1, q_2 \) over \( \mathbb{Q} \) such that \( q_1 \) embeds in \( q_2 \) over \( \mathbb{R} \) with codimension at least 4, \( q_1 \) will embed in \( q_2 \) over \( \mathbb{Q} \).

Proof. It will suffice to construct a form of degree 4 whose sum gives arbitrary invariants at all but one prime. This can be done prime by prime, including the real prime. By consideration of the invariants of a sum of two quadratic forms, it is then enough to construct a 4 dimensional form with arbitrary discriminant and Hasse invariants, except at one non-archimedean prime. By Theorem 73.1 of [21] it is possible to do this globally if it is possible to do so at each prime, given the finiteness of the number of primes involved in the Hasse invariant and discriminant. There are 4 possibilities for the local discriminant (up to multiplication by local units) and Hasse invariant and they can be realized by forms as follows

\[
(1, -1, 1, -1), (1, -a, \pi, -a\pi), (1, -1, 1, \pi), (1, -1, a, \pi)
\]

where \( \pi \) is the (odd) prime and \( a \) is a nonquadratic residue modulo it. \( \Box \)

Proposition 3.2. Let \( E \) be a totally real algebraic number field with basis \( 1, b_2, \ldots, b_n \). Consider a Hodge structure with a single holomorphic 2 form having periods

\[
v = b_1, b_2, \ldots, b_n, b_1 x_2, x_2 b_2, \ldots, x_2 b_n, b_1 x_3, x_3 b_2, \ldots, x_3 b_n \ldots, x_d b_n.
\]

where \( x_i \) are linearly independent over \( E \). Consider a bilinear form given by a block diagonal matrix \( D \) whose blocks are matrices of elements of \( E \) under the regular representation \( F_{11}, \ldots, F_{d_0} \). For the regular representation the basis \( b_i \) is used, and it is also assumed that the regular representation consists of symmetric matrices. (1) Consider the effect of complex multiplication by a block diagonal matrix whose blocks are some \( 3 \times 3 \) matrix \( A \) of an element of \( E \). This multiplication preserves the period vector and its orthogonal complement up to complex multiples, and is therefore an isomorphism of Hodge structures. (2) A quadratic form associated to a group element has signature given by the number of positive and negative Galois conjugates of that element (its image under all real embeddings of
The only complex multiplications which preserve the cup product form up to rational multiples are by those elements whose squares are rational numbers.

Proof. The properties of the regular representation directly implies that multiplication by the matrix $A$ on the period vector has the effect of multiplying by the corresponding field element $a$ embedded in the complex numbers:

$$a(b_1, b_2, b_3) = (ab_1, ab_2, ab_3)$$

For the orthogonal complement of $z$, we will be looking at $wDz$; the effect of multiplying by $A$ is to change this to $wDAz = wD(az)$ and the orthogonality condition for $w$ is not changed.

The quadratic form given by a symmetric matrix has signature given by its eigenvalues, and these will be the Galois conjugates of the field element in a regular representation.

If a complex multiplication by $A$ corresponding to field element $a \in E$ sends all cup products to $k$ times themselves, $k \in \mathbb{Q}$, then that is also true for complex cohomology, and in particular for the inner product $z\bar{z}$. But that inner product under the complex multiplication goes to $az\bar{az}$, so that $a^2 = k$. □

Proposition 3.3. There exists a totally real cubic field and elements $f_1, f_2, f_3$ with the following properties. The field has a regular representation by symmetric matrices. The algebraic numbers $f_1, f_2$ have exactly one positive Galois conjugate each and $f_3$ have no positive Galois conjugate. The positive Galois conjugates of $f_1, f_2$ are much larger than the other numbers and approximately equal, to any desired degree, e.g. the positive conjugates exceed 1 and for any $\epsilon > 0$, they differ by at most $\epsilon$, and the other conjugates are at most $\epsilon$ in absolute value. The only elements of the field whose squares are rational numbers are those which are already rational.

Proof. The statement about Galois conjugates follows by looking at the Minkowski embedding of the cubic field in $\mathbb{R}^3$, and its having dense image. Strong approximation makes this condition compatible with the other order conditions. Regarding the symmetric representation, it suffices to take a field generated by an eigenvalue of some symmetric $3 \times 3$ matrix whose characteristic polynomial is irreducible over the rational numbers. The last statement is true in general since irrational square roots generate quadratic fields which do not embed in cubic fields. □

4. Construction of K3 surfaces

Proposition 4.1. There exists a K3 surface $X_b$ having transcendental lattice of rank 9 over $K$, such that there exists an isomorphism of Hodge structures from the lattice to itself, which is not a linear combination of isomorphisms each of which sends the cup product quadratic form over $\mathbb{Q}$ to a rational multiple of itself.

Proof. The Hodge structure will have complex multiplication by a totally real cubic field $E$ of the last proposition. The transcendental lattice $T_X$ over $\mathbb{Q}$ will be a sum of 3 copies of $E$, as regards complex multiplication. As in the previous propositions we can choose a family of vectors $v$, where $x_i$ remain unspecified, which admit this complex multiplication and also admit cup product forms in which $v$ has the same orthogonal complement, and which have the correct signature to embed in $L_{2d}$ for any $d$ as quadratic forms over $\mathbb{Q}$. Given any such embeddings the natural mappings between coordinates of $v$ will give a map which sends $v$ to $\bar{v}$ and its
orthogonal complement under that bilinear form to its orthogonal complement. This is then a homomorphism of Hodge structures of two K3 surfaces.

The vectors $v$ are obtained by taking solving for $x_2$ to be close to 1, pure imaginary and transcendental giving a solution for $x_3$ which is to be a very small real number. Then the conditions of Proposition 3.3 will ensure positivity of $v\tau$. There will be no larger field of complex multiplication.

By Proposition 3.2 complex multiplications preserving the cup product form up to rational multiples must be square roots of rational numbers, and by Proposition 3.3 they are already rational, so complex multiplications by irrational elements of the cubic field will not be linear combinations of isomorphisms of the Hodge structure, which must be some complex multiplications, which preserve cup products.

□

5. Kuga-Satake correspondence

At this point we return to the Kuga-Satake correspondence [14], [15] which associates polarized 1-dimensional Hodge structures $G_1$ with polarized 2-dimensional Hodge structures $G_2$ in which the holomorphic 2-forms are all multiples of a single form $f$, and satisfying a signature condition.

It will be shown that if the Kuga-Satake construction is applied to the transcendental lattices of the K3 surfaces of Proposition 4.1 then the endomorphisms of the 1-dimensional Hodge structure are given by right multiplications in the Clifford subalgebras $C^2$ have unique polarization up to right multiplication, which will not affect the $H^2$ summands.

Our reason for doing this is the hope that algebraic cycles in the Kuga-Satake abelian variety $A$ can be studied in terms of algebraic cycles giving an equivalence between their 2-dimensional summands and the 2-dimensional summands of K3 surfaces.

To do that in turn it will suffice to show that the Mumford-Tate groups acting on these summands admit no nontrivial (linear) endomorphisms other than those given by right multiplication (which will not affect $H^2$). To do that it will suffice to prove the same for the algebra of endomorphisms spanned by the Mumford-Tate group. This will be the algebra of endomorphisms spanned by right-multiplication by $f_1f_2$ taken in terms of a rational basis. It will suffice to show these generate the even part of the Clifford algebra considered as left multiplications.

Proposition 5.1. For specific transcendental lattices of the K3 surfaces of the last section, there are no nontrivial endomorphisms except for right multiplication within the Clifford algebras.

Proof. This is a computer calculation using van Geemen’s constructions and a specific example. The matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

has irreducible characteristic polynomial $x^3 - 2x^2 - x + 1$. The elements $b_i$ are chosen so that the regular representation of the field generated by $A$ with basis $b_i$ will give the symmetric matrix $A$, and hence the entire regular representation is by symmetric matrices. Choose $b_3 = 1$; the 3rd row of the matrix of $A$ means we
should choose \( b_1 \) as \( A \). Then the first row means \( AA \) should be the sum of the 3 basis elements, \( A + I \) plus the 2nd basis element, so the latter should be \( A^2 - A - I \).

The period vector \( f \) will be chosen in as generic a way as possible and \( f_1, f_2 \) will be its real and complex parts. We will assume some real multiple is chosen to give van Geemen’s normalization in terms of \( f \); our results hold for multiples of \( f \) by any real number.

So \( f \) is

\[
b_1, b_2, 1, x_1 b_1, x_1 b_2, x_1, x_2 b_1, x_2 b_2, x_2
\]
or in terms of a basis \( e_i \) for the vector space, later to be generators of the Clifford algebra,

\[
b_1 e_1 + b_2 e_2 + e_3 + x_1 b_1 e_4 + x_1 b_2 e_5 + x_1 e_6 + x_2 b_1 e_7 + x_2 b_2 e_8 + x_2 e_9
\]

We take \( x_i \) to be approximations to some solution of \( f^2 = 0, f \mathcal{J} > 0 \) which are as general as possible. One choice of elements \( \phi_i \) of the cubic field to give the diagonal matrices \( F_i \) of the quadratic form is \( 10 I - A - A^2, 2 I - 7 A + 2 A^2, -A - I \). When the \( x_i \) are chosen in a generic way, then their real and imaginary parts and products between them will be linearly independent over the cubic field.

The algebraic numbers \( \phi_i \) which determine the quadratic form satisfy only order conditions and will range over some open neighborhood within \( n \)-tuples of rational numbers. By expanding in terms of a basis, one can see that if there is a choice of \( \phi_i \) within the field such that the algebras generated by \( x_i \) have rank 256, then the rank will be 256 on the complement of some lower dimensional algebraic variety for each form, and this complement has elements within any nonempty open set. So it is enough to find any field elements such that the algebra has rank 256. The particular choice of \( \phi_i \) in our computer programs was \( A, A^2 - A - I, I \).

To compute the van Geemen Clifford algebra, now take the product \( f_1 f_2 \)

\[
(b_1 e_1 + b_2 e_2 + e_3 + x_1 b_1 e_4 + x_1 b_2 e_5 + x_1 e_6 + x_2 b_1 e_7 + x_2 b_2 e_8 + x_2 e_9)
\]

\[
(x_1 b_1 e_4 + x_1 b_2 e_5 + x_1 e_6 + x_2 b_1 e_7 + x_2 b_2 e_8 + x_2 e_9)
\]

A basis for it over the rational numbers will give elements of the algebra generated by the Hodge group, since it represents multiplication by \( i \) in the Hodge structure. For such a basis we can expand out in terms of real and imaginary parts \( x_{1r}, x_{2r}, x_{1i} x_{2i} \) and then expand the products of the \( b_i \) in terms of the basis \( I, A, A^2 \) for the cubic field. This gives nine generators for the algebra generated by the Hodge group. They are symmetric to the first case in terms of the \( x \) expansion, that is, the coefficients of \( x_{1r} \) in \( f_1 f_2 \), which are

\[
(b_1 e_1 + b_2 e_2 + e_3) (b_1 e_4 + b_2 e_5 + b_3 e_6)
= (A e_1 + (A^2 - A - I) e_2 + e_3) (A e_4 + (A^2 - A - I) e_5 + e_6)
\]

\[
= A^2 e_1 e_4 + A (A^2 - A - I) (e_1 e_5 + e_2 e_4) + (A^2 - A - I^2) e_2 e_5 + A (e_1 e_6 + e_3 e_4)
\]

\[
+ (A^2 - A - I) (e_2 e_6 + e_3 e_5) + e_3 e_6
= A^2 e_1 e_4 + (A^2 - I) (e_1 e_5 + e_2 e_4) + (A + I) e_2 e_5 + A (e_1 e_6 + e_3 e_4)
\]

\[
+ (A^2 - A - I) (e_2 e_6 + e_3 e_5) + e_3 e_6
\]

This yields 3 generators

\[
e_1 e_4 + e_2 e_6 + e_3 e_5 + e_1 e_5 + e_2 e_4
\]

\[
e_2 e_5 + e_1 e_6 + e_3 e_4 - e_2 e_6 - e_3 e_5
\]

\[
- e_1 e_5 - e_2 e_4 + e_2 e_5 - e_2 e_6 - e_3 e_5 + e_3 e_6.
\]
In the Clifford algebra, their products will be given by the quadratic form using $\phi_i$. This proposition will be established if it can be shown that these elements (left multiplications) generate the Clifford algebra, since then no maps but right multiplications will commute with them, and right multiplications will have trivial effects on $H^2$.

The remaining generators differ from the 3 given ones by replacing $e_1, e_2, e_3$ by $e_4, e_5, e_6$ and $e_4, e_5, e_6$ by $e_7, e_8, e_9$.

A first program computes the regular representation of $e_i$ by $512 \times 512$ matrices for a general quadratic form; then making use of the fact that these matrices are very sparse, it computes products of $e_i e_j, i < j$. Then it adds together the sums listed together to get the regular representation of the 9 generators of the subalgebra.

A second program computes the top row of all 4-fold products of the 9 generators in which the last generator is one of the first four, and stores this as a $512 \times 729$ array. A final program puts these matrices modulo 101 into row echelon form. The computed rank of these products is 247. By Prop.4.7 the Clifford algebra $C^+$ is either a full matrix algebra over a quaternion algebra. By Prop.2.5 in the Appendix to [15], the Hodge and Mumford-Tate groups are reductive and therefore the subalgebra is semisimple and will be a sum of matrix algebras over division rings over fields. In comparing the algebras it is enough to consider their dimensions over the complex numbers and a simple algebra of $16 \times 16$ matrices will have no proper subalgebras of rank 247 which are sums of full matrix algebras. The rank being 256 means the algebra generated by these elements includes all left multiplications. Therefore there are no symmetries but right multiplications. □

Proposition 5.2. In the Kuga-Satake construction, begin with a polarized space $V$ of weight two dimension, constructing a Clifford algebra $C$, a polarized weight weight Hodge structure identified with $C^+$, and then an embedding of $V \subset C^+ \otimes C^+$. The polarization with $V$ inherits from this embedding is a positive rational multiple of the original Hodge structure. Thus it is independent of all choices made in the definition of the Kuga-Satake polarization, and is invariant under right multiplication by the Clifford algebra.

Proof. A dense subset of polarized Hodge structures on $V$ will have no complex multiplication except by $\mathbb{Q}$, and hence a unique polarization, in the following sense. The Hodge structures are represented in a way similar to those for K3 surfaces: take diagonalized quadratic forms and general period vectors with $zz = 0, z^2 > 0$. We can take neighborhoods bounded away from zero, in which the Kuga-Satake construction is continuous, and for a general solution $z$ there will be no complex multiplication. The unique polarization must be both the original polarization, and with some normalization to be within bounds, the new. One normalization on a given summand might choose a fixed nonzero integer cohomology class and assume the polarizations have a certain nonzero integer value on it paired with itself. Limits of these will have the same property. □

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