Observability estimate for the wave equation with variable coefficients

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Abstract

This paper is devoted to a study of observability estimate for the wave equation with variable coefficients \((h^{jk}(x))_{n \times n} (n \in \mathbb{N})\). We consider both the observation point lies outside the domain and the observation point lies inside the domain. Based on a Carleman estimate for the ultra-hyperbolic operator and a delicate treatment of observation region, we obtain two observability estimates with explicit observability constants. The key improvements are: (1) we improve the requirement of waiting time \(T\); (2) we improve the size of the observation region (see Figure 1 and Figure 2 for the case of \((h^{jk}(x))_{n \times n} = I_n\)).

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1 Introduction

Given \(T > 0\) and a bounded domain \(\Omega\) of \(\mathbb{R}^n (n \in \mathbb{N})\) with \(C^2\) boundary \(\Gamma\), put \(Q = (0, T) \times \Omega, \Sigma = (0, T) \times \Gamma\).

Let \(h^{jk}(\cdot) \in C^2(\overline{\Omega})\) be fixed satisfying \(h^{jk}(x) = h^{kj}(x), \quad \forall x \in \overline{\Omega}, \ j, k = 1, \ldots, n,\) (1.1)

and for some constant \(h_0 > 0,\)

\[
\sum_{j,k=1}^{n} h^{jk}(x) \xi_j \xi_k \geq h_0 |\xi|^2, \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n, \quad \xi = (\xi^1, \ldots, \xi^n). \tag{1.2}
\]

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Let us consider the following wave equation with variable coefficients:

\[
\begin{align*}
    w_{tt} - \sum_{j,k=1}^{n} (h^{jk} w_{x_j})_{x_k} &= qw + \sum_{k=1}^{n} q_1^k w_{x_k} + q_2 w_t, & \text{in } Q, \\
    w &= 0, & \text{on } \Sigma, \\
    w(0) &= w_0, \quad w_t(0) = w_1, & \text{in } \Omega,
\end{align*}
\]

(1.3)

where \((w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega), q \in L^\infty(\Omega)\) and \(q_1^k \in W^{1,\infty}(\Omega)\) \((k = 1, \cdots, n), q_2 \in W^{1,\infty}(\Omega)\). By the method of ([13, Ch. I, Th 4.2]) it follows that the existence and the uniqueness of the solution \(w\) of (1.3) lies in the class \(w \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))\).

The main purpose of this paper is to study the internal observability problem of (1.3), by which we mean the following: given \(T > 0\) and \(K\) be a sub-domain of \(Q\), find (if possible) a constant \(C = C(q, \{q_1^k\}_{k=1}^{n}, q_2) > 0\) such that the corresponding solution \(w\) of (1.3) satisfies

\[
|w_0|_{L_2(\Omega)}^2 + |w_1|_{H^{-1}(\Omega)}^2 \leq C(q, \{q_1^k\}_{k=1}^{n}, q_2) \int_{K} |w|^2 dx dt, \quad \forall (w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega). \tag{1.4}
\]

The inequality (1.4) is called an observability estimate for (1.3). This inequality means that the initial energy of a solution in the time \(t = 0\) can be bounded by its partial energy in the local observation region \(K\). Such kind of inequalities are closely related to control and state observability problems of wave equations with constant or variable coefficients. For example, they can be applied to a study of the controllability (e.g. [1, 2, 6, 11, 13, 15, 20, 21]), the stabilization of some locally damped semilinear wave equation with variable coefficients (e.g. [17]) and also inverse problems (e.g. [8, 10]). In this respect, there exist numerous works devoted to observability estimates of wave equations with constant or variable coefficients, we refer to [3, 5, 9, 12, 16, 18, 19] and rich references therein.

Note that the constant \(C\) in (1.4) depends on the lower-order term coefficients \(q(\cdot), q^k_1(\cdot)(k = 1, 2, \cdots, n), q_2(\cdot)\), the observation domain \(K\) and the waiting time \(T\) in (1.3). In this paper, the explicit estimate of \(C\), the critical value of waiting time \(T\), the size of the observation region \(K\) are parts of the problems we concerned.

For the case \((h^{jk}(\cdot))_{n \times n} = I_n, q(\cdot) \in L^\infty(\Omega), q^k_1(\cdot) = 0 (k = 1, 2, \cdots, n), q_2(\cdot) = 0\), it was proved in Zhang [18] that explicit observability estimate (1.4) holds for

\[
T > 2 \max_{x \in \Omega} |x - x_0|, \quad K = K_1 \triangleq (0, T) \times \omega, \quad C(q) = C \exp(\exp(\exp(C|q(\cdot)|_{L^\infty(\Omega)}))), \tag{1.5}
\]

where \(x_0 \in R^n \setminus \Omega, \omega\) is some given neighborhood of \(\Gamma_0\), and \(\Gamma_0\) is part of the boundary of \(\Gamma\) satisfying certain conditions, which will be specified later. For the case \((h^{jk}(\cdot))_{n \times n} = I_n, q(\cdot) \in C^\infty(\Omega), \{q^k_1(\cdot)\}_{k=1}^{n} \in C^\infty(\Omega), q_2(\cdot) \in C^\infty(\Omega)\), it was proved in Jena [7] that observability estimate (1.4) holds for

\[
x_0 \in R^n \setminus \Omega, \quad T > 2 \max_{x \in \Omega} |x - x_0|, \quad K = K_2 \triangleq K_1 \cap \{ (t, x) \in R^{1+n} \mid |x - x_0|^2 > t^2 \},
\]

and

\[
x_0 \in \Sigma, \quad T > 2 \max_{x \in \Omega} |x - x_0|, \quad K = K_3,
\]
where $\mathcal{K}_3$ is some neighborhood of $\mathcal{K}_2$. We also refer to the related references [16] for boundary observability estimate of linear wave equation (i.e. $(h^{jk} \cdot \cdot \cdot)^n = I_n$) on time-dependent domains with a smaller boundary observation region. In the case of wave equation with variable coefficients, $q(\cdot) \in L^\infty(0, T; L^n(\Omega))$, $q_k^x(\cdot) = 0$ ($k = 1, 2, \ldots, n$), $q_2(\cdot) = 0$, it was proved in [6] that explicit observability estimate (1.4) holds for larger waiting time and observation region $\mathcal{K} = \mathcal{K}_1$ satisfying (1.5). In this paper, we consider the explicit observability inequality of (1.3) with variable coefficients. Based on a Carleman estimate for the ultra-hyperbolic operator and a delicate treatment on the region of observation, we improve not only the requirement of waiting time $T$ but also the size of the observation region (see Theorem 2.1 and Theorem 2.2 in detail). Particularly, for the case $(h^{jk}(\cdot)_{n \times n} = I_n$, the waiting time $T$ can be improved to $T > 2 \max_{x \in \Omega \setminus \omega} |x - x_0|$, the changes of the observation regions can be seen in Remark 2.1, i.e. Figure 1 and Figure 2.

The rest of this paper is organized as follows. In Section 2, we give the statement of our main results. In Section 3, we collect some preliminaries we needed. Finally, we give the proofs of our main results in Section 4 and Section 5, respectively.

2 Statement of the main results

To begin with, we introduce the following condition:

**Condition 2.1** There exists a function $d(\cdot) \in C^2(\bar{\Omega})$ satisfying the following:

(i) For some constant $\mu_0 > 0$, it holds

$$\sum_{j,k=1}^n \left\{ \sum_{j',k'=1}^n \left[ 2h^{jk'}(h^{j'k}d_{x,j'})x_{j'} - h^{jk}h^{j'k'}d_{x,j'} \right] \right\} \xi_j \xi_{k'} \geq \mu_0 \sum_{j,k=1}^n h^{jk} \xi_j \xi_k,$$

\[ \forall (x, \xi_1, \cdots, \xi_n) \in \bar{\Omega} \times \mathbb{R}^n. \tag{2.1} \]

(ii) There is no critical point of function $d(\cdot)$ in $\bar{\Omega}$, i.e.,

$$\min_{x \in \bar{\Omega}} |\nabla d(x)| > 0. \tag{2.2}$$

Note that for the case $(h^{jk})_{n \times n} = I_n$, and any given $x_0 \in \mathbb{R}^n \setminus \bar{\Omega}$, by choosing $d(x) = |x - x_0|^2$, we have (2.1) with $\mu_0 = 4$. In this case,

$$\Gamma_0 = \left\{ x \in \Gamma \mid (x - x_0) \cdot \nu(x) > 0 \right\}. \tag{2.3}$$

We refer to [6, 14] for more examples and explanations on Condition 2.1.

For the function $d(\cdot)$ satisfying Condition 2.1, we introduce the following set:

$$\Gamma_0 \triangleq \left\{ x \in \Gamma \mid \sum_{j,k=1}^n h^{jk}d_{x,j} \nu_k > 0 \right\}. \tag{2.4}$$
It is easy to check that, if $d_0(\cdot) \in C^2(\Omega)$ satisfies (2.1), then for any given constants $a \geq 1$ and $b \in \mathbb{R}$, the function
\[
\hat{d} = \hat{d}(x) \triangleq ad_0(x) + b
\] (2.5)
still satisfies Condition 2.1 with $\mu_0$ replaced by $a\mu_0$, meanwhile, the scaling and translating $d_0(x)$ do not change the set $\Gamma_0$. Hence, by scaling and translating $d(x)$, if necessary, we may assume without loss of generality that
\[
\begin{cases}
(2.1) \text{ holds with } \mu_0 \geq 4, \\
\frac{1}{4} \sum_{j,k=1}^{n} h^{jk}(x)d_{x_j}d_{x_k} \geq d(x) > 0, \quad \forall x \in \Omega.
\end{cases}
\] (2.6)

For any set $M \subset \mathbb{R}^n$ and $\delta > 0$, we define $\mathcal{O}_\delta(M) = \{ x \in \mathbb{R}^n \mid |x - x'| < \delta \text{ for } x' \in M \}$. Let $\omega, \omega_0$ be proper open non-empty subsets of $\Omega$ satisfying $\omega_0 \subset \subset \omega$. We assume that there exist constants $0 < \delta_0 < \delta$ such that
\[
\omega \triangleq \mathcal{O}_{\delta}(\Gamma_0) \bigcap \Omega, \quad \omega_0 \triangleq \mathcal{O}_{\delta_0}(\Gamma_0) \bigcap \Omega, \quad Q_{\omega \setminus \omega_0} \triangleq (0, T) \times \{ \omega \setminus \omega_0 \}.
\] (2.7)

In what follows, we set
\[
R_0 \triangleq \min_{x \in \Omega} \sqrt{d(x)}, \quad R_1 \triangleq \max_{x \in \Omega} \sqrt{d(x)}, \quad T_* \triangleq 2\inf \left\{ R_1 \mid d(\cdot) \text{ satisfies (2.6)} \right\}.
\] (2.8)

For the function $d(\cdot)$ satisfying (2.8), for any constant $\delta_1 \in (0, 1/2)$, we define
\[
\begin{align*}
\mathcal{D} & \triangleq \left\{ (t, x) \in Q \mid d(x) - (t - T/2)^2 > 0 \right\}, \\
\mathcal{K} & \triangleq \left\{ \left( \frac{T}{2} - \delta_1 T, \frac{T}{2} + \delta_1 T \right) \times \omega_0 \right\} \cup \{ Q_{\omega \setminus \omega_0} \cap \mathcal{D} \}.
\end{align*}
\] (2.9)

Throughout of this paper, we shall denote by $C = C(\Omega, n, (h^{jk})_{n \times n}, T)$ a generic positive constant, which may change from line to line (unless otherwise stated). We have the following observability result.

**Theorem 2.1** Let $h^{jk}(\cdot) \in C^2(\Omega)$ satisfy (1.1) and (1.2), $q(\cdot) \in L^{\infty}(Q)$ and $q^k(\cdot) \in W^{1, \infty}(Q)$ ($k = 1, \ldots, n$), $q_2(\cdot) \in W^{1, \infty}(Q)$. Let the function $d(\cdot)$ satisfies Condition 2.1 and (2.6). Then for any $T > T_*$, the weak solution $w(\cdot) \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ of equation (1.3) satisfies that
\[
|w_0|_{L^2(\Omega)}^2 + |w_1|_{H^{-1}(\Omega)}^2 \leq C(r) \int_{\mathcal{K}} |w|^2 \, dx dt, \quad \forall (w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega),
\] (2.10)

where
\[
\begin{align*}
C(r) &= C \exp(\exp(Cr)), \\
r & \triangleq \max \left\{ |q(\cdot)|_{L^{\infty}(Q)}, \left\{ |q^k(\cdot)|_{W^{1, \infty}(Q)} \right\}_{k=0}^{n}, |q_2(\cdot)|_{W^{1, \infty}(Q)} \right\}.
\end{align*}
\] (2.11)
**Remark 2.1** In the case of \((h^{jk})_{n \times n} = I_n\), we refer to Figure 1 for the changes of observation regions in one dimensional case, and Figure 2 for the changes of observation regions in multidimensional case.

![Figure 1](image1.png)

Figure 1: For the wave equation in one dimensional case, the red regions in (a) and (b) are the observation regions developed in [18, Theorem 2.1] and [7, Theorem 1.2], respectively. The red region in (c) is our result stated in Theorem 2.1.

![Figure 2](image2.png)

Figure 2: The changes of the observation regions for the wave equations in multidimensional case.

Note that Theorem 2.1 holds provided the function \(d(\cdot)\) no critical point in \(\overline{\Omega}\), i.e., \(\min_{x \in \overline{\Omega}} |\nabla d(x)| > 0\). In the case that there is a critical point of function \(d(\cdot)\) in \(\overline{\Omega}\), we can also establish suitable observability estimate. To this aim, we introduce the following Condition.

**Condition 2.2** There exists a function \(b(\cdot) \in C^2(\overline{\Omega})\) satisfying the following:

(i) For some constant \(\mu_0 > 0\), \((2.1)\) holds for \(b(\cdot)\).

(ii) There is only one critical point \(x_0 \in \overline{\Omega}\), i.e.,

\[
\min_{x \in \overline{\Omega} \setminus \{x_0\}} |\nabla b(x)| > 0, \quad |\nabla b(x_0)| = 0.
\]

(iii) There exists constant \(s > 0\) such that

\[
\min_{x \in \overline{\Omega}} b(x) = b(x_0) = 0, \quad \lim_{x \to x_0} \frac{\sum_{j,k=1}^{n} h^{jk}(x) b_{x_j}(x) b_{x_k}(x)}{b(x)} = s.
\]

5
Let $\zeta (|\zeta| \neq 0)$ be sufficient small vector in $\mathbb{R}^n$ satisfying $x_0 - \zeta \in \overline{\Omega}$ and there exist constant $0 < \delta_2 < \delta_0$, such that $\omega_0 \supset \mathcal{O}_{\delta_2} \left( \Gamma_{0, \zeta} \right) \cap \Omega$ where

$$\Gamma_{0, \zeta} \triangleq \left\{ x \in \Gamma \left| \sum_{j, k=1}^{n} h_{jk} d_{x_j}(x) (x + \zeta) \nu(x) > 0 \right. \right\}. \quad (2.14)$$

We have that if function $d(\cdot), d(\cdot + \zeta) \in C^2(\overline{\Omega})$ satisfy Condition 2.2, we can also establish suitable observability estimate. Similar to the analysis of (2.6), without loss generality, we assume the following hold:

$$\begin{cases} (2.1) \text{ holds with } \mu_0 > 4, \\ \frac{1}{4} \sum_{j, k=1}^{n} h_{jk}(x) d_{x_j}(x) d_{x_k}(x) > d(x) > 0, \quad \forall x \in \Omega \setminus \{ x_0 \}, \\ \frac{1}{4} \sum_{j, k=1}^{n} h_{jk}(x) d_{x_j}(x + \zeta) d_{x_k}(x + \zeta) > d(x + \zeta) > 0, \quad \forall x \in \Omega \setminus \{ x_0 - \zeta \}. \end{cases} \quad (2.15)$$

In what follow, we set

$$R_1 \triangleq \max_{x \in \overline{\Omega} \setminus \omega} \{ \sqrt{d(x)}, \sqrt{d(x + \zeta)} \}, \quad T^* \triangleq 2 \inf \left\{ R_1 \left| d(\cdot) \text{ satisfies (2.15)} \right. \right\}, \quad (2.16)$$

and

$$D_\zeta \triangleq \left\{ (t, x) \in Q \left| d(x + \zeta) - (t - T/2)^2 > 0 \right. \right\}, \quad (2.17)$$

$$\mathcal{K}_\zeta \triangleq \left\{ \left[ \frac{T}{2} - \epsilon T, \frac{T}{2} + \epsilon T \right] \times \omega_0 \right\} \cup \{ Q_{\omega \setminus \omega_0} \cap D_\zeta \}. \quad (2.18)$$

We have the following observability result.

**Theorem 2.2** Let $h_{jk}(\cdot) \in C^2(\overline{\Omega})$ satisfy (1.1) and (1.2), $q(\cdot) \in L^\infty(Q)$ and $q^k(\cdot) \in W^{1, \infty}(Q)\,(k = 1, \cdots, n)$, $q_2(\cdot) \in W^{1, \infty}(Q)$. Let $d(\cdot), d(\cdot + \zeta)$ hold Condition 2.2 and (2.15), $\mathcal{K}$ and $\mathcal{K}_\zeta$ be given by (2.9) and (2.17). For any open domain $W$ satisfies $\mathcal{K} \cup \mathcal{K}_\zeta \subset W \subset (-T, T) \times \Omega, \, T > T^*$, the weak solution $w(\cdot) \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ of equation (1.3) satisfies

$$|w_0|_{L^2(\Omega)}^2 + |w_1|_{H^{-1}(\Omega)}^2 \leq C(r) \int_{W} |w|^2 \, dx \, dt, \quad \forall (w_0, w_1) \in L^2(\Omega) \times H^{-1}(\Omega), \quad (2.18)$$

where $C(r)$ is the same as (2.11).

**Remark 2.2** For the case $(h_{jk})_{n \times n} = I_n$, by choosing $d(x) = |x - x_0|^2$ ($x_0$ inside $\overline{\Omega}$), for any open domain $(0, T) \times \Omega \supset W \supset \mathcal{K}$ and $T > T^*$, there exist sufficient small $\zeta$ with $x_0 - \zeta \in \overline{\Omega}$ such that $W \supset \mathcal{K} \cup \mathcal{K}_\zeta$ and $T > \tilde{T}^*$ where

$$\tilde{T}^* \triangleq 2 \max_{x \in \overline{\Omega} \setminus \omega} |x - x_0|. \quad (2.19)$$
Notice that $|x - x_0|^2$ and $|x + \zeta - x_0|^2$ satisfy Condition 2.2 and (2.1), so that (2.18) hold with $W$.

3 Some preliminaries

In this section, we collect some preliminaries we needed. First, based on a weighted identity established in [5], we have the following pointwise inequality for the ultra-hyperbolic operator $\partial^2_t + \partial^2_s - \sum_{j,k=1}^n \partial_x(h^{jk}x_k)$:

**Corollary 3.1** Let $u \in C^2(\mathbb{R}^{2+n}; \mathbb{R}), \ell \in C^3(\mathbb{R}^{2+n}; \mathbb{R})$ and $\Psi \in C^1(\mathbb{R}^n; \mathbb{R})$. Set $\theta = e^\ell$ and $v = \theta u$. Then,

$$\theta^2 \left|u_{tt} + u_{ss} - \sum_{j,k=1}^n (h^{jk}u_{x_j})_{x_k}\right|^2 + 2 \text{div} V + 2M_t + 2N_s$$

$$\geq 2 \left[\ell_{tt} - \ell_{ss} + \sum_{j,k=1}^n (h^{jk}\ell_{x_j})_{x_k} + \Psi\right]v_t^2 - 8 \sum_{j,k=1}^n h^{jk}\ell_{tx_j}v_{x_k}v_t + 8\ell_{st}v_sv_t$$

$$- 8 \sum_{j,k=1}^n h^{jk}\ell_{sx_j}v_{x_k}v_s + 2 \left[\ell_{ss} - \ell_{tt} + \sum_{j,k=1}^n (h^{jk}\ell_{x_j})_{x_k} + \Psi\right]v_s^2$$

$$+ 2 \sum_{j,k=1}^n c^{jk}v_{x_j}v_{x_k} - 2 \sum_{j,k=1}^n h^{jk}\Psi v_{x_j}v_{x_k} + Bv^2,$$

where

$$A = \sum_{j,k=1}^n (h^{jk}\ell_{x_j}\ell_{x_k} - h^{jk}\ell_{x_k}\ell_{x_j} - h^{jk}\ell_{x_j,x_k}) - \ell_t^2 - \ell_s^2 + \ell_{tt} + \ell_{ss} - \Psi,$$

$$c^{jk} = \sum_{j',k'=1}^n \left[2h^{jk'}(h^{j'k}\ell_{x_{j'}})_{x_{k'}} - (h^{jk}h^{j'k'}\ell_{x_{j'}})_{x_{k'}}\right] + h^{jk}(\ell_{tt} + \ell_{ss} - \Psi),$$

$$B = 2 \left[A\Psi - (A\ell_t)_t - (A\ell_s)_s + \sum_{j,k=1}^n (Ah^{jk}\ell_{x_j})_{x_k}\right].$$
and

$$
\begin{align*}
V &= [V^1, \cdots, V^k, \cdots, V^n], \\
V^k &= 2 \sum_{j,j',k=1}^n h^{jk} h^{j'k'} \ell_{x,j} v_{x,j} x_{x,k} + \sum_{j=1}^n h^{jk} A(\ell_{x,j} v^2 - \Psi v) \sum_{j=1}^n h^{jk} v_{x,j} \\
&\quad - \sum_{j,j',k=1}^n h^{jk} h^{j'k'} \ell_{x,j} v_{x,j} v_{x,k} - 2(\ell_{x,t} v_t + \ell_s v_s) \sum_{j=1}^n h^{jk} v_{x,j} \\
&\quad + \sum_{j=1}^n h^{jk} \ell_{x,j} (v^2_t + v^2_s),
\end{align*}
$$

(3.3)

$$
\begin{align*}
M_t &= \ell_t \left(v^2_t - v^2_s + \sum_{j,k=1}^n h^{jk} v_{x,j} v_{x,k}\right) - 2 \sum_{j,k=1}^n h^{jk} \ell_{x,j} v_{x,k} v_t + 2 \ell_s v_s v_t \\
&\quad + \Psi v v_t - A(\ell_t v^2), \\
N_s &= \ell_s \left(v^2_s - v^2_t + \sum_{j,k=1}^n h^{jk} v_{x,j} v_{x,k}\right) - 2 \sum_{j,k=1}^n h^{jk} \ell_{x,j} v_{x,k} v_s + 2 \ell_t v_s v_t \\
&\quad + \Psi v v_s - A(\ell_s v^2).
\end{align*}
$$

Proof. In [5, Theorem 1.1], we choose

$$(a^{jk})_{(n+2)\times(n+2)} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & (h^{jk})_{n\times n}
\end{pmatrix}.
$$

(3.4)

By elementary computation, one can obtain Corollary 3.1 immediately. For the reader’s convenience, we also give a detailed proof of Corollary 3.1 in Appendix A.

Next, we recall the following known result.

Lemma 3.1 ( [4] ) Denote $L \triangleq \sum_{j,k=1}^n \partial_{x,j} (h^{jk} \partial_{x,k})$. There exist a constant $\Lambda$, such that for all $\Re \lambda \geq \Lambda$, equation $(-L + \lambda)u = f$ have unique solution $u \in H^1_0(\Omega)$ for all $f \in H^{-1}(\Omega)$ and there exist positive constants $C_1$ and $C_2$ such that

$$
C_1 |u|_{H^2_0(\Omega)} \leq |f|_{H^{-1}(\Omega)} \leq C_2 |u|_{H^2_0(\Omega)}.
$$

(3.5)

Put

$$
E(t) \triangleq \frac{1}{2} \left[ |w(t, \cdot)|^2_{H^{-1}(\Omega)} + |w(t, \cdot)|^2_{L^2(\Omega)} \right].
$$

(3.6)

We have the following energy estimate.

Lemma 3.2 Let $T > 0$, $q(\cdot) \in L^\infty(Q)$ and $q^k(\cdot) \in W^{1,\infty}(Q)$ ( $k = 1, \cdots, n$), $q_2(\cdot) \in W^{1,\infty}(Q)$, $w_0(\cdot) \in L^2(\Omega)$ and $w_1(\cdot) \in H^{-1}(\Omega)$. Then the weak solution $w(\cdot) \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ of (1.3) satisfies

$$
E(t) \leq CE(s)e^{Cr}, \quad \forall t, s \in [0, T],
$$

(3.7)
where \( r \) is given by (2.11).

**Proof of Lemma 3.2.** By Lemma 3.1, we know that there exists a constant \( \lambda_0 \), such that operator \(-L + \lambda_0\) is a reversible operator. Put \( f = (-L + \lambda_0)^{-1} w_t \), we have

\[
    w_t(-L + \lambda_0)^{-1}(w_{tt} + (-L + \lambda_0)w) = (-L + \lambda_0)f f_t + w_tw_t
\]

\[
    = -\left( \sum_{j,k=1}^{n} h^{jk} f_{x_j} f_t \right)_{x_k} + \left( \frac{1}{2} \sum_{j,k}^{n} h^{jk} f_{x_j} f_{x_k} \right)_t + \frac{1}{2} \left( \lambda_0 f^2 + w^2 \right)_t
\]

\[
    = w_t(-L + \lambda_0)^{-1} \left( qw + \sum_{j,k}^{n} q^{jk}_1 w_{x_k} + q_2 w_t \right).
\]

Integrating it on \( \Omega \), using integration by parts, by (3.5) and the Hölder inequality, the Sobolev embedding theorem, we obtain

\[
    \frac{dE(t)}{dt} \leq 2C \int_{\Omega} \left[ w_t(-L + \lambda_0)^{-1} \left( qw + \sum_{k=1}^{n} q^{jk}_1 w_{x_k} + q_2 w_t + \lambda_0 w \right) \right] dx
\]

\[
    \leq CrE(t), \quad \forall \ t \in [0, T],
\]

which can yield (3.7). \( \square \)

Finally, we recall the following energy estimate.

**Lemma 3.3** Let \( 0 \leq S_1 < S_2 < T_2 < T_1 \leq T \) and \( q(\cdot) \in L^\infty(\Omega) \) and \( q^k(\cdot) \in L^\infty(\Omega) \) \((k = 1, \cdots, n)\), \( q_2(\cdot) \in L^\infty(\Omega) \). Then the weak solution \( w(\cdot) \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega)) \) of (1.3) satisfies

\[
    \int_{S_2}^{T_2} E(t)dt \leq C(1 + r^2) \int_{S_1}^{T_1} |w(t, \cdot)|^2_{L^2(\Omega)}dt.
\]

**Proof of Lemma 3.3.** Denote

\[
    \phi(t) = \begin{cases} 
    1, & t \in [S_2, T_2], \\
    0, & t \in \left(-\infty, \frac{S_1 + S_2}{2}\right] \cup \left[\frac{T_1 + T_2}{2}, +\infty\right). 
\end{cases}
\]
Put \( g = (−L + \lambda_0)^{-1}w \) and by the following Pointwise identity,

\[
\phi w(-L + \lambda_0)^{-1}\left(w_{tt} - \sum_{j,k=1}^{n} (h^{jk}w_{x_j})_{x_k}\right)
= (\phi w g_t)_{t} - \phi w_t g_t - \phi_t w g_t + \phi w^2 - \lambda_0 \phi w g
\]

\[
= (\phi w g_t)_{t} - \phi(-L + \lambda_0)g_t g_t - \phi_t(-L + \lambda_0)g g_t + \phi w^2 - \lambda_0 \phi w g
\]

\[
= (\phi w g_t)_{t} + \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \phi h^{jk} g_{tx_j} g_t \right)_{x_k} - \left( \sum_{j,k=1}^{n} \phi h^{jk} g_{tx_j} g_{tx_k} \right) - \lambda_0 \phi g_t^2
\]

\[
+ \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \phi h^{jk} g_{x_j} g_t \right)_{x_k} - \left( \frac{1}{2} \sum_{j,k=1}^{n} \phi_t h^{jk} g_{x_j} g_{x_k} \right)_{t} + \frac{1}{2} \phi_{tt} \left( \sum_{j,k=1}^{n} h^{jk} g_{x_j} g_{x_k} \right)
\]

\[
- \left( \frac{1}{2} \lambda_0 \phi_t g_t^2 \right)_{t} + \frac{1}{2} \lambda_0 \phi_{tt} g_t^2 + \phi w^2 - \lambda_0 \phi w g,
\]

integrating by part, we have

\[
\int_{S_1}^{T_1} \int_{\Omega} \phi w(-L + \lambda_0)^{-1}\left(w_{tt} - \sum_{j,k=1}^{n} (h^{jk}w_{x_j})_{x_k}\right) dxdt 
= -\int_{S_1}^{T_1} \int_{\Omega} \sum_{j,k=1}^{n} \phi h^{jk} g_{tx_j} g_{tx_k} + \lambda_0 \phi g_t^2 dxdt
\]

\[
+ \int_{S_1}^{T_1} \int_{\Omega} \frac{1}{2} \phi_{tt} \left( \sum_{j,k=1}^{n} h^{jk} g_{x_j} g_{x_k} + \lambda_0 g_t^2 \right) dxdt
\]

\[
- \int_{S_1}^{T_1} \int_{\Omega} \phi(w^2 - \lambda_0 w g) dxdt.
\]
Thus, by Lemma 3.1, Poincaré inequality and equation (1.3), we get

\[ C_1 h_0 \int_{S_2}^{T_2} |w(t, \cdot)|^2_{H^{-1}(\Omega)} dt \]

\[ \leq \int_{S_1}^{T_1} \int_{\Omega} n \sum_{j,k=1}^{n} \phi h^{jk} g_{tx_j} g_{tx_k} + \lambda_0 \phi g_t^2 dx dt \]

\[ = - \int_{S_1}^{T_1} \int_{\Omega} \phi w(-L + \lambda_0)^{-1}(qw + \sum_{k=1}^{n} q_1^k w_{x_k} + q_2 w_t + \lambda_0 w) dx dt \]

\[ + \frac{1}{2} \int_{S_1}^{T_1} \int_{\Omega} \phi_{tt} \left( \sum_{j,k=1}^{n} h^{jk} g_{x_j} g_{x_k} + \lambda_0 g_t^2 \right) dx dt \]

\[ + \int_{S_1}^{T_1} \int_{\Omega} \phi (w^2 - \lambda_0 w g) dx dt \]

\[ \leq C(1 + r^2) \int_{S_1}^{T_1} |w(t, \cdot)|^2_{L^2(\Omega)} dt + \frac{C_1 h_0}{2} \int_{S_1}^{T_1} \phi |w_t(t, \cdot)|^2_{H^{-1}(\Omega)} dt. \]

which implies the desired result immediately. \( \square \)

4 Proof of Theorem 2.1

In this Section, we will give the proof of Theorem 2.1. We divide the proof into several steps.

**Step 1.** Put

\[ z(t, s, x) \triangleq \int_{s}^{t} w(\tau, x) d\tau, \quad \forall (t, s, x) \in (0, T) \times Q. \]

(4.1)

Then, it is easy to see that \( z \) satisfies the following ultra-hyperbolic equation:

\[ \begin{aligned}
& z_{tt} + z_{ss} - \sum_{j,k=1}^{n} (h^{jk} z_{x_j} z_{x_k}) = F, \quad \text{in} \ (0, T) \times Q, \\
& z = 0, \quad \text{on} \ (0, T) \times \Sigma,
\end{aligned} \]

where

\[ F \triangleq \int_{s}^{t} q(\tau, x) z_t(\tau, s, x) + \sum_{k=1}^{n} q_1^k(\tau, x) z_{x_k,t}(\tau, s, x) + q_2(\tau, x) z_{tt}(\tau, s, x) d\tau. \]

(4.3)

For \( \lambda > 0 \), we introduce the following weight functions:

\[ \begin{aligned}
& \theta(t, s, x) = e^{\ell(t, s, x)}, \quad \ell(t, s, x) = \lambda \phi(t, s, x), \\
& \phi(t, s, x) = d(x) - \alpha(t - T/2)^2 - \alpha(s - T/2)^2,
\end{aligned} \]

(4.4)
where \( \alpha \in (0, 1) \) and \( d(x) \) satisfies Condition 2.1 and (2.6).

**Step 2.** Denote

\[
\begin{align*}
Q & \triangleq (0, T) \times Q, & S & \triangleq (0, T) \times \Gamma, \\
T_i & \triangleq T/2 - \varepsilon_i T, & T_i' & \triangleq T/2 + \varepsilon_i T, \\
Q_i & \triangleq (T_i, T_i') \times (T_i, T_i') \times \Omega, & Q_i' & \triangleq (T_i, T_i') \times (T_i, T_i') \times \omega_1, \\
S_i & \triangleq (T_i, T_i') \times (T_i, T_i') \times \Gamma, & S_{0i} & \triangleq (T_i, T_i') \times (T_i, T_i') \times \Gamma_0.
\end{align*}
\] (4.5)

where \( \omega_1 \) is some neighborhood of \( \Gamma_0 \) satisfying \( \omega_0 \subset \omega_1 \subset \omega \), and \( 0 < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \frac{1}{2} \).

Recalling (2.8) the definitions of \( R_0 \) and \( R_1 \), since \( \min_{x \in \Omega} d(x) > 0 \), we see that \( R_0 > 0 \). We can choose a sufficiently small \( c \in (0, R_0) \) and an \( \alpha \in (0, 1) \) close to 1 such that

\[
\min_{x \in \Omega} d(x) > c^2, \quad (4.6)
\]

and

\[
1 - \frac{2c^2}{T^2} < \alpha < 1. \quad (4.7)
\]

Hence, if \( t = s = \frac{T}{2} \), for any \( x \in \overline{\Omega} \), we have

\[
\phi\left(\frac{T}{2}, \frac{T}{2}, x\right) = d(x) > c^2. \quad (4.8)
\]

For any \( b > 0 \), define

\[
Q(b) \triangleq \left\{ (t, s, x) \in (-\infty, +\infty) \times (-\infty, +\infty) \times \{\Omega \setminus \omega_1\} \mid \phi(t, s, x) > b^2 \right\}. \quad (4.9)
\]

Notice that for any \( x \in Q(c) \), by (4.7), we have

\[
d(x) - (t - T/2)^2 - (s - T/2)^2 > \phi(t, s, x) + (\alpha - 1)(t - T/2)^2 + (\alpha - 1)(s - T/2)^2
\]

\[
> \phi(t, s, x) + (\alpha - 1)T^2/2 > 0. \quad (4.10)
\]

Noting that \( T > 2R_1 \). Choose \( \omega_1 \) and \( \omega \) closely enough to make \( d(x) < T^2/4 \) for any \( x \in \Omega \setminus \omega_1 \). Now take \( \varepsilon_0 \) sufficiently small and \( \varepsilon_1 \in (0, \frac{1}{2}) \) sufficiently close to \( \frac{1}{2} \) such that

\[
Q_0 \setminus Q_0' \subset Q(c) \subset D' \subset Q_1 \setminus Q_1', \quad (4.11)
\]

where

\[
D' \triangleq \left\{ (t, s, x) \in \{Q \setminus (0, T) \times (0, T) \times \omega_1\} \mid d(x) - (t - T/2)^2 - (s - T/2)^2 > 0 \right\}. \quad (4.12)
\]
and

\[(T_0, T'_0) \times (T_0, T'_0) \times \Omega \subset D'' \triangleq \{(t, s, x) \in \Omega \mid d(x) - (t - T/2)^2 - (s - T/2)^2 > 0\}. \quad (4.13)\]

We refer to Figure 3 for the relations of (4.11) in one dimensional case.

![Figure 3](image)

Figure 3: (a) represent \(Q_0 \setminus Q'_0\), (b) represent \(Q(c)\), (c) represent \(Q_1 \setminus Q'_1\).

Choosing \(\varepsilon\) small enough, we can still hold

\[Q_0 \setminus Q'_0 \subset Q(c + 2\varepsilon) \subset Q(c + \varepsilon) \subset Q(c) \subset D' \subset Q_1 \setminus Q'_1. \quad (4.14)\]

Next, we introduce the following cut-off function:

\[\eta(t, s, x) = \begin{cases} 
1, & (t, s, x) \in Q(c + 2\varepsilon) \setminus \{(0, T) \times (0, T) \times w_2\}, \\
0, & (t, s, x) \in Q_1 \setminus Q(c + \varepsilon),
\end{cases} \quad (4.15)\]

where \(w_2\) is also some neighborhood of \(\Gamma_0\) satisfying \(\omega_1 \subset \omega_2 \subset \omega\). Set

\[u(t, s, x) \triangleq \eta(t, s, x)z(t, s, x).\]

It is easy to check that \(u\) satisfies

\[\begin{align*}
    u_{tt} + u_{ss} - \sum_{j, k=1}^{n} (h^{jk} u_{x_j})_{x_k} \\
    = \eta F + \left[ \eta_{tt} + \eta_{ss} - \sum_{j, k=1}^{n} (h^{jk} \eta_{x_j})_{x_k} \right] z + 2\eta_z \eta_t + 2\eta_s \eta_s - 2 \sum_{j, k=1}^{n} h^{jk} \eta_{x_j} \eta_{x_k}z_{x_k},
\end{align*}\]

where \(F\) is given by (4.3).
Step 3. By (4.4), it is easy to see that
\[\begin{align*}
\ell_t &= -2\lambda\alpha(t - T/2), \quad \ell_s = -2\lambda\alpha(s - T/2), \\
\ell_{tt} &= \ell_{ss} = -2\lambda\alpha, \\
\ell_{xj} &= \lambda d_{xj}, \quad \ell_{xjxk} = \lambda d_{xjxk}, \quad j, k = 1, \ldots, n.
\end{align*}\]

(4.17)

In the sequel, we denote \(O(\lambda^k)\) by a function of order \(\lambda^k\) for large \(\lambda\). Recall Corollary 3.1, put
\[\Psi = -\lambda \sum_{j,k=1}^{n} (h^{jk}d_{xj})_{xk} + 2\lambda(1 - \alpha).\]

By (2.1), (2.6) and (4.17), we have
\[2\left[\ell_{tt} - \ell_{ss} + \sum_{j,k=1}^{n} (h^{jk}\ell_{xj})_{xk} + \Psi\right] = 4\lambda(1 - \alpha), \tag{4.18}\]

and
\[2 \sum_{j,k=1}^{n} c^{jk}v_{xj}v_{xk}\]
\[= 2 \sum_{j,k=1}^{n} \left\{ \sum_{j',k'=1}^{n} \left[ 2h^{jk'}(h^{j'k}\ell_{xj'})_{xk'} - (h^{jk}h^{j'k'}\ell_{xj'})_{xk'} \right] + h^{jk}(\ell_{tt} + \ell_{ss} - \Psi) \right\} v_{xj}v_{xk}\]
\[\geq 2\mu_0 \sum_{j,k=1}^{n} h^{jk}v_{xj}v_{xk} - 8\lambda\alpha \sum_{j,k=1}^{n} h^{jk}v_{xj}v_{xk} - 4\lambda(1 - \alpha) \sum_{j,k=1}^{n} h^{jk}v_{xj}v_{xk}\]
\[\geq 4\lambda(1 - \alpha)h_0|\nabla v|^2,\]

Next, recall (3.2) for the definition of \(A\) and \(B\), we have
\[A = \sum_{j,k=1}^{n} \left( h^{jk}\ell_{xj}\ell_{xk} - h^{jk}\ell_{xj} - h^{jk}\ell_{xjxk} \right) - \ell^2_t - \ell^2_s + \ell_{tt} + \ell_{ss} - \Psi\]
\[= \lambda^2 \left[ \sum_{j,k=1}^{n} h^{jk}d_{xj}d_{xk} - 4\alpha^2(t - T/2)^2 - 4\alpha^2(s - T/2)^2 \right] + O(\lambda). \tag{4.20}\]
\begin{align}
B &= 2A \left[ \Psi - \ell_{tt} - \ell_{ss} + \sum_{j,k=1}^{n} (h_{jk} \ell_{x_j})_{x_k} \right] + 2 \left( \sum_{j,k=1}^{n} h_{jk} \ell_{x_j} A_{x_k} - A_{t} \ell_{t} - A_{s} \ell_{s} \right) \\
&= 4\lambda (1 + \alpha) A + 2\lambda^3 \sum_{j,k=1}^{n} h_{jk} d_{x_j} \left( \sum_{j',k'=1}^{n} h_{j'k'} d_{x_{j'}} d_{x_{k'}} \right)_{x_k} \\
&\quad - 32\lambda^3 \alpha^3 (t - T^2) - 32\lambda^3 \alpha^3 (s - T^2)^2 + O(\lambda^2) \\
&\geq 4(3 + \alpha)^3 \sum_{j,k=1}^{n} h_{jk} d_{x_j} d_{x_k} - 4\alpha^2 (t - T^2) - 4\alpha^2 (s - T^2)^2 + O(\lambda^2),
\end{align}

where we used
\begin{align}
\sum_{j,k=1}^{n} h_{jk} d_{x_j} \left( \sum_{j',k'=1}^{n} h_{j'k'} d_{x_{j'}} d_{x_{k'}} \right)_{x_k} \\
= \sum_{j,k=1}^{n} \sum_{j',k'=1}^{n} \left[ 2h_{j'k'} (h_{j'k'} d_{x_{j'}} x_{k'} - h_{j'k} h_{j'k'} d_{x_{j'}}) \right] d_{x_j} d_{x_k} \\
\geq \mu_0 \sum_{j,k=1}^{n} h_{jk} d_{x_j} d_{x_k} \geq 4 \sum_{j,k=1}^{n} h_{jk} d_{x_j} d_{x_k}.
\end{align}

Further, notice that
\begin{align}
\left| -2 \sum_{j,k=1}^{n} h_{jk} \Psi_{x_j} v v_{x_k} \right| &\leq C \sum_{j,k=1}^{n} h_{jk} v v_{x_k} + O(\lambda^2) v^2.
\end{align}

By (4.19), (4.21) and (4.23), we conclude that there is a \( \lambda_0 > 0 \), for any \( \lambda > \lambda_0 \), we have
\begin{align}
\theta^2 \left| u_{tt} + u_{ss} - \sum_{j,k=1}^{n} (h_{jk} u_{x_j})_{x_k} \right|^2 + 2 \text{div} V + 2M_t + 2N_s \\
\geq 2\lambda (1 - \alpha) \left( u_t^2 + v_s^2 + h_0 |\nabla v|^2 \right) + 8(3 + \alpha)^3 \left[ d(x) - \alpha^2 (t - T^2)^2 - \alpha^2 (s - T^2)^2 \right] v^2,
\end{align}

Now, integrate (4.24) on \( Q_1 \), we get
\begin{align}
\lambda \int_{Q(\varepsilon + 2\varepsilon) \setminus \{(0,T) \times (0,T) \times u_2\}} \theta^2 \left[ z_t^2 + z_s^2 + |\nabla z|^2 + \lambda^2 z^2 \right] dx dt ds \\
\leq C \int_{Q_1} \theta^2 \left| u_{tt} + u_{ss} - \sum_{j,k=1}^{n} (h_{jk} u_{x_j})_{x_k} \right|^2 dx dt ds.
\end{align}
Step 4. Let us estimate “$$\int_{Q_1} \theta^2 \left| u_{tt} + u_{ss} - \sum_{j,k=1}^n (h_{jk} u_{x_j})_{x_k} \right|^2 dx dt ds$$”. Recall the definition of \( \eta \), noting that \( \eta \equiv 1 \) on \( Q(\bar{c} + 2\varepsilon) \setminus \{(0, T) \times (0, T) \times \omega_2 \} \), by (4.3) and (4.16), we have

\[
\int_{Q_1} \theta^2 \left| u_{tt} + u_{ss} - \sum_{j,k=1}^n (h_{jk} u_{x_j})_{x_k} \right|^2 dx dt ds \\
\leq C \int_{Q_1} \theta^2 \left| \int_t^T q(\tau, x) z_t(\tau, s, x) d\tau \right|^2 dx dt ds \\
+ C \int_{Q_1} \theta^2 \left| \int_t^T \sum_{k=1}^n q_k^1(\tau, x) z_{x_k, x}(\tau, s, x) + q_2(\tau, x) z_{tt}(\tau, s, x) d\tau \right|^2 dx dt ds \\
+ Ce^{(c+\varepsilon)^2} \int_{Q_1} \left( z_t^2 + z_s^2 + |\nabla z|^2 + z^2 \right) dx dt ds \\
+ C \int_{(T_1, T_1') \times (T_1, T_1') \times \omega_2 \cap Q(c+2\varepsilon)} \theta^2 \left( z_t^2 + z_s^2 + |\nabla z|^2 + z^2 \right) dx dt ds.
\] (4.26)

Notice that

\[
\int_{Q_1} \theta^2 \left| \int_{T/2}^t z_t^2(\tau, s, x) d\tau \right| dt ds dx \\
= \int_{\Omega} dx \int_{T_1}^{T_1'} ds \left\{ \int_{T_1}^{T_1'} e^{\lambda \left( d(x) - \alpha(t-T/2)^2 - \alpha(s-T/2)^2 \right)} \left| \int_t^T z_t^2(\tau, s, x) d\tau \right| dt \right\} \\
= \int_{\Omega} dx \int_{T_1}^{T_1'} ds \left\{ \int_{T_1}^{T_1'} dt \left( e^{\lambda \left( d(x) - \alpha(t-T/2)^2 - \alpha(s-T/2)^2 \right)} \int_t^T z_t^2(\tau, s, x) d\tau \right) \\
+ \int_{T/2}^{T_1'} dt \left( e^{\lambda \left( d(x) - \alpha(t-T/2)^2 - \alpha(s-T/2)^2 \right)} \int_{T/2}^t z_t^2(\tau, s, x) d\tau \right) \right\} \\
\leq C \int_{\Omega} dx \int_{T_1}^{T_1'} ds \left\{ \int_{T_1}^{T_1'} \int_{T/2}^T e^{\lambda \left( d(x) - \alpha(t-T/2)^2 - \alpha(s-T/2)^2 \right)} z_t^2(\tau, s, x) d\tau d\tau \right\} \\
\leq C \int_{Q_1} \theta^2 z_t^2 dx dt ds.
\] (4.27)

Similarly,

\[
\int_{Q_1} \theta^2 \left| \int_{T/2}^T z_t^2(\tau, x) d\tau \right| dx dt ds \leq C \int_{Q_1} \theta^2 z_s^2 dx dt ds.
\] (4.28)

Combining (4.27) and (4.28), noting that \( z_t(\tau, s, x) = -z_s(t, \tau, x) \), we have

\[
\int_{Q_1} \theta^2 \left| \int_s^t q(\tau, x) z_t(\tau, s, x) d\tau \right|^2 dx dt ds \leq C r^2 \int_{Q_1} \theta^2 \left( z_t^2 + z_s^2 \right) dx dt ds.
\] (4.29)
Now, we consider the next term in (4.26). Using integration by parts, we have

$$\int_Q \theta^2 \left| \frac{d}{d\tau} \left( \sum_{k=1}^n q_k^t(\tau, x)z_{xk}(\tau, s, x) + q_2(\tau, x)z_{t}(\tau, s, x) \right) \right|^2 dx dt ds$$

$$\leq \int_Q \theta^2 \left| \sum_{k=1}^n q_k^k(t, x)z_{xk}(t, s, x) \right|^2 + \theta^2 \left| q_2(t, x)z_{t}(t, s, x) \right|^2 dx dt ds$$

$$+ \int_Q \theta^2 \left| \int_t^s \left( \sum_{k=1}^n q_{k,t}^k(\tau, x)z_{xk}(\tau, s, x) + q_{2,t}(\tau, x)z_{t}(\tau, s, x) \right) d\tau \right|^2 dx dt ds. \quad (4.30)$$

Similar to the argument in (4.27) and (4.28), we have

$$\int_Q \theta^2 \left| \int_t^s \left( \sum_{k=1}^n q_{k,t}^k(\tau, x)z_{xk}(\tau, s, x) + q_{2,t}(\tau, x)z_{t}(\tau, s, x) \right) d\tau \right|^2 dx dt ds \leq Cr^2 \int_Q \theta^2 \left| \nabla z \right|^2 + z^2 dx dt ds. \quad (4.31)$$

Combining (4.26), (4.29) and (4.31), we have

$$\int_Q \theta^2 \left| u_{tt} + u_{ss} - \sum_{j,k=1}^n (h^{jk}u_{xj})_{xk} \right|^2 dx dt ds \leq C(1 + r^2)e^{(c+2\varepsilon)^2\lambda} \int_Q \left( z_t^2 + z_s^2 + \left| \nabla z \right|^2 + z^2 \right) dx dt ds$$

$$+ C(1 + r^2) \int_{(T_1, T_1') \times (T_1, T_1') \times \omega_2 \cap \Omega(c+2\varepsilon)} \theta^2 \left( z_t^2 + z_s^2 + \left| \nabla z \right|^2 + z^2 \right) dx dt ds$$

$$+ C r^2 \int_{\Omega(c+2\varepsilon) \setminus \{(T_1, T_1') \times (T_1, T_1') \times \omega_2\}} \theta^2 \left( z_t^2 + z_s^2 + \left| \nabla z \right|^2 \right) dx dt ds. \quad (4.32)$$

Combining (4.25) and (4.32), when $\frac{\lambda}{2} \geq Cr^2$, we can yield

$$\lambda \int_{\Omega(c+2\varepsilon) \setminus \{(0,T) \times (0,T) \times \omega_2\}} \theta^2 \left[ \left( z_t^2 + z_s^2 + \left| \nabla z \right|^2 \right) + \lambda^2 z^2 \right] dx dt ds \leq C(1 + r^2)e^{\lambda(c+2\varepsilon)^2} \int_Q \left( z_t^2 + z_s^2 + \left| \nabla z \right|^2 + z^2 \right) dx dt ds$$

$$+ C(1 + r^2)e^{C\lambda} \int_{(T_1, T_1') \times (T_1, T_1') \times \omega_2 \cap \Omega(c+2\varepsilon)} \left( z_t^2 + z_s^2 + \left| \nabla z \right|^2 + z^2 \right) dx dt ds. \quad (4.33)$$
Adding $\lambda \int_{(T_0, T_0') \times (T_0, T_0')} \theta^2 (z_t^2 + z_s^2) dx dt ds$ on the each side of (4.33), we have

$$\lambda e^{\lambda(c+2\epsilon)}^2 \int_{Q_0} \left( z_t^2 + z_s^2 \right) dx dt ds \leq \lambda \int_{Q_0} \theta^2 (z_t^2 + z_s^2) dx dt ds \leq C(1 + r^2) e^{\lambda(c+2\epsilon)}^2 \int_{Q_1} \left( z_t^2 + z_s^2 + |\nabla z|^2 + z^2 \right) dx dt ds$$

(4.34)

$$+ C(1 + r^2) \lambda \int_{(T_1, T_1') \times (T_1, T_1')} (z_t^2 + z_s^2 + |\nabla z|^2 + z^2) dx dt ds$$

$$+ C \lambda \int_{(T_0, T_0') \times (T_0, T_0')} \theta^2 (z_t^2 + z_s^2) dx dt ds.$$

**Step 5.** Now, we estimate $\int_{Q_1} |\nabla z|^2 dx dt ds$ and

$$\int_{\{(T_1, T_1') \times (T_1, T_1')\} \cap Q(c+2\epsilon)} |\nabla z|^2 dx dt ds$$

from the right-hand side of (4.34) by standard method which can be found, for example, in [5, 18]. Set

$$\chi(t, s) = (t - T_2)(T_2' - t)(s - T_2)(T_2' - s),$$

(4.35)

so that

$$\int_{Q_2} \chi z F dx dt ds$$

$$= \int_{Q_2} \chi z \left( z_t + z_s - \sum_{j,k=1}^n (h_{jk} z_{x_j x_k}) \right) dx dt ds \geq -\int_{Q_2} \left[ z_t (\chi z_t + \chi z_s) + z_s (\chi z_t + \chi z_s) \right] dx dt ds + h_0 \int_{Q_2} \chi |\nabla z|^2 dx dt ds.$$

(4.36)

and

$$\int_{Q_2} \chi z F dx dt ds$$

$$= \int_{Q_2} \chi z \left( \int_s^t q(\tau, x) z_t(\tau, s, x) d\tau \right) dx dt ds$$

$$+ \int_{Q_2} \chi z \left( \int_s^t -\partial_\tau q_2(\tau, x) z_t(\tau, s, x) d\tau + q_2(t, x) z_t(t, s, x) \right) dx dt ds$$

$$+ \int_{Q_2} \chi z \left( \int_s^t \sum_{k=1}^n q_{1k}^1(t, x) z_{tx_k}(\tau, s, x) d\tau \right) dx dt ds,$$

(4.37)
Combining with

\[
\int_{Q_2} \chi z \left( \int_s^t \sum_{k=1}^n q_k^*(\tau, x) z_{tsk}^{*}(\tau, s, x) d\tau \right) dxdtds
\]

\[
= \int_{Q_2} \int_s^t \sum_{k=1}^n \left( \chi(t, s) z(t, s, x) q_k^*(\tau, x) z_t(\tau, s, x) \right) d\tau dxdtds
\]

\[
- \int_{Q_2} \int_s^t \left[ \chi(t, s) z_t(\tau, s, x) \left( \sum_{k=1}^n z_{sk}(t, s, x) q_k^*(\tau, x) \right) \right] d\tau dxdtds
\]

\[
- \int_{Q_2} \int_s^t \left[ \chi(t, s) z(t, s, x) z_t(\tau, s, x) \left( \sum_{k=1}^n \partial_{sk} q_k^*(\tau, x) \right) \right] d\tau dxdtds
\]

\[
\leq C r^2 \int_{Q_2} \left( z_t^2 + z_s^2 \right) dxdtds + c_0 \int_{Q_2} \chi(t, s) |\nabla z|^2 dxdtds
\]

by choosing \( c_0 \) sufficiently small, we get

\[
\int_{Q_1} |\nabla z|^2 dxdtds \leq C (1 + r^2) \int_{Q_2} \left( z_t^2 + z_s^2 + z^2 \right) dxdtds. \tag{4.39}
\]

Similarly, recall \( D'' = \left\{ (t, s, x) \in Q \mid d(x) - (t - T/2)^2 - (s - T/2)^2 > 0 \right\} \) and notice that

\[
\left\{ (T_1, T'_1) \times (T_1, T'_1) \times \omega_2 \right\} \cap Q(c + 2\varepsilon) \subset D'' \cap \left\{ (0, T) \times (0, T) \times \{\omega \setminus \omega_0\} \right\}. \tag{4.40}
\]

Putting

\[
\varrho(t, s, x) = \begin{cases} 
1, & (t, s, x) \in (T_1, T'_1) \times (T_1, T'_1) \times \omega_2 \cap Q(c + 2\varepsilon), \\
0, & (t, s, x) \in Q \setminus \{D'' \cap (0, T) \times (0, T) \times \{\omega \setminus \omega_0\}\},
\end{cases} \tag{4.41}
\]

we have

\[
\int_Q \varrho z F dxdtds
\]

\[
= \int_Q \varrho \left( z_{tt} + z_{ss} - \sum_{j,k=1}^n (h^{jk} x_j x_k) \right) dxdtds
\]

\[
\geq - \int_Q \left[ z(z_{tt} + z_{ss} + z_s(z_t + z_{s} + z_{t})) \right] dxdtds + h_0 \int_Q \varrho |\nabla z|^2 dxdtds
\]

\[
+ \int_Q \frac{1}{2} z^2 \sum_{j,k=1}^n (h^{jk} x_j x_k) dxdtds,
\]

\[19\]
\[
\int_Q \varrho z F dx dt ds \\
= \int_Q \varrho z \left( \int_s^t q(\tau, x) z_l(\tau, s, x) d\tau \right) dx dt ds \\
+ \int_Q \varrho z \left[ \left( \int_s^t \partial_\tau q_2(\tau, x) z_l(\tau, s, x) d\tau \right) + q_2(t, x) z_l(t, s, x) \right] dx dt ds \\
+ \int_Q \varrho z \left( \int_s^t \sum_{k=1}^n q_1^k(\tau, x) z_{tx_k}(\tau, s, x) d\tau \right) dx dt ds. 
\] (4.43)

Combining (4.41) and (4.43) with

\[
\int_Q \varrho z \left( \sum_{k=1}^n q_1^k(\tau, x) z_{tx_k}(\tau, s, x) d\tau \right) dx dt ds \\
= \int_Q \int_s^t \sum_{k=1}^n \left( q(t, s, x) z(t, s, x) q_1^k(\tau, x) z_l(\tau, s, x) \right) d\tau dx dt ds \\
- \int_Q \int_s^t \left[ \left( \sum_{k=1}^n z_{tx_k}(\tau, s, x) q_1^k(\tau, x) \right) \varrho(t, s, x) z_l(t, s, x) \right] d\tau dx dt ds \\
- \int_Q \int_s^t \left[ q(t, s, x) z(t, s, x) z_l(\tau, s, x) \left( \sum_{k=1}^n \partial_x q_1^k(\tau, x) \right) \right] d\tau dx dt ds \\
- \int_Q \int_s^t \left[ \left( \sum_{k=1}^n q_1^k(\tau, x) \right) z(t, s, x) z_l(\tau, s, x) \right] d\tau dx dt ds \\
\leq C r^2 \int_Q \left( z^2 + z_l^2 \right) dx dt ds + c_1 \int_Q \varrho |\nabla z|^2 dx dt ds, 
\] (4.44)

by choosing \( c_1 \) sufficiently small, we yield

\[
\int_{\left\{ (T_1, T_1') \times (T_1, T_1') \times \omega_2 \right\} \cap Q(c+2\varepsilon)} |\nabla z|^2 dx dt ds \\
\leq C(1 + r^2) \int_{D'' \cap \left\{ (0,T) \times (0,T) \times \left\{ \omega \setminus \omega_0 \right\} \right\}} z_s^2 + z_l^2 + z^2 dx dt ds. 
\] (4.45)
Combining (4.34), (4.39) and (4.45), we get
\[
\lambda \int_{Q_0} (z_t^2 + z_s^2) dx dt ds \\
\leq C(1 + r^4) \int_{Q_2} (z_t^2 + z_s^2 + z^2) dx dt ds \\
+ C(1 + r^4) \lambda e^{C\lambda} \int_{\mathcal{D} \cap \{(0,T) \times (0,T) \times \{\omega \setminus \omega_0\}\}} (z_s^2 + z_t^2 + z^2) dx dt ds \\
+ \lambda e^{C\lambda} \int_{(T_0, T_0') \times (0,T) \times \omega_2} (z_t^2 + z_s^2) dx dt ds
\]

(4.46)

**Step 6.** Let us return to the function \(w\). Recall that \(z(t, s, x) = \int_s^t w(\tau, x) d\tau\). By (4.46), we have
\[
\lambda \int_{T_0}^{T_0'} \int_{\Omega} w^2 dx dt \\
\leq C(1 + r^4) \lambda \int_{Q_2} \left[ w^2(t, x) + \left( \int_s^t w(\tau) d\tau \right)^2 \right] dx dt ds \\
+ C\lambda e^{C\lambda} (1 + r^4) \int_{\mathcal{D} \cap \{(0,T) \times (0,T) \times \{\omega \setminus \omega_0\}\}} \left( w^2(t, x) + \left( \int_s^t w^2(\tau, x) d\tau \right) \right) dx dt ds \\
+ \lambda e^{C\lambda} \int_{(T_0, T_0') \times \omega_2} w^2(t, x) dx dt \\
\leq C(1 + r^4) \int_{Q} w^2(t, x) dx dt + C(1 + r^4) \lambda e^{C\lambda} \int_{\mathcal{D} \cap \{(0,T) \times \{\omega \setminus \omega_0\}\}} w^2(t, x) dx dt \\
+ \lambda e^{C\lambda} \int_{(T_0, T_0') \times \omega_2} w^2(t, x) dx dt,
\]

(4.47)

where we used
\[
\int_{\mathcal{D} \cap \{(0,T) \times (0,T) \times \{\omega \setminus \omega_0\}\}} \left( \int_s^t w(\tau, x) d\tau \right)^2 dx dt ds \\
\leq \int_{\mathcal{D} \cap \{(0,T) \times (0,T) \times \{\omega \setminus \omega_0\}\}} \int_{\frac{t}{2} + \sqrt{d(x)}}^{\frac{t}{2} - \sqrt{d(x)}} w^2(\tau, x) d\tau dx dt ds
\]

(4.48)

where \(\mathcal{D} \triangleq \{ (t, x) \in Q \mid d(x) - (t - T/2)^2 > 0 \}\) defined in (2.9). Recall that \(\omega = \mathcal{O}_\delta(\Gamma_0)\) and
\( \omega_0 = O_\delta(\Gamma_0) \), let \( \varepsilon_0 < \delta_1 \) so that
\[
D \cap \left\{ (0, T) \times \{ \omega \setminus \omega_0 \} \right\} \cup \left\{ (T_0, T'_0) \times \omega_2 \right\} \subset K. \tag{4.49}
\]

On the other hand, set \( S_0 \in (T_0, \frac{1}{2} T) \) and \( S'_0 \in (\frac{1}{2} T, T'_0) \). Applying lemma 3.2 and lemma 3.3, we immediately get
\[
\lambda \int_{S'_0}^S E(t) dt \leq C \lambda e^{C \lambda} (1 + r^6) \int_K w^2(t, x) dx dt + C (1 + r^6) \int_0^T E(t) dt. \tag{4.50}
\]
Combining with energy estimate (3.7), we get
\[
\left( C \lambda e^{C r} - C (1 + r^6) e^{C r} \right) E(0) \leq C \lambda e^{C \lambda} (1 + r^6) \int_K w^2(t, x) dx dt. \tag{4.51}
\]
So that, we have
\[
E(0) \leq \lambda e^{C \lambda} (1 + r^6) \int_K w^2(t, x) dx dt, \tag{4.52}
\]
when \( \lambda \geq C (1 + r^6) e^{C r} \), which yields (2.10).

\section{5 Proof of Theorem 2.2}

In this section, we will give the proof of Theorem 2.2.

\textbf{Proof.} Denote
\[
Q_\zeta(b) \triangleq \left\{ (t, s, x) \in (-\infty, +\infty) \times (-\infty, +\infty) \times \Omega \setminus \omega_1 \mid \phi(t, s, x + \zeta) > b^2 \right\}. \tag{5.1}
\]
Notice that when \( t = \frac{T}{2} \), we have
\[
d(x) + d(x + \zeta) > 0, \quad \forall x \in \Omega. \tag{5.2}
\]
Set \( R_0 \triangleq \min_{x \in \Omega} \left\{ \frac{1}{2} \left( d(x) + d(x + \zeta) \right) \right\} \). Recall the note in (4.5) and (4.9), similar to the argument in (4.6), (4.7) and (4.8), we can still choose a sufficiently small \( c \in (0, R_0) \) and \( \alpha \in (0, 1) \) close to 1 and take \( \varepsilon_0 \) small enough, \( \varepsilon_1 \in (0, \frac{1}{2}) \) sufficiently close to \( \frac{1}{2} \) such that
\[
Q_0 \setminus Q'_0 \subset Q(c) \cup Q_\zeta(c) \subset \mathcal{D}' \cup \tilde{\mathcal{D}}' \subset Q_1 \setminus Q'_1, \tag{5.3}
\]
where \( \mathcal{D}' \) is defined in (4.12) and
\[
\tilde{\mathcal{D}}' \triangleq \left\{ (t, s, x) \in Q \setminus \{(0, T) \times (0, \frac{T}{2}) \times \omega_1 \} \mid d(x + \zeta) - \left( t - T/2 \right)^2 - \left( s - T/2 \right)^2 > 0 \right\}. \tag{5.4}
\]
Choosing \( \varepsilon \) small enough, we have the following relations similar to (4.14),
\[
Q_0 \setminus Q'_0 \subset Q(c + 2\varepsilon) \cup Q_\zeta(c + 2\varepsilon) \subset Q(c + \varepsilon) \cup Q_\zeta(c + \varepsilon) \subset Q(c) \cup Q_\zeta(c), \tag{5.5}
\]
and put \( \omega_1 \subset \omega_2 \subset \omega \), by the same way we yield (4.33), we have

\[
\begin{align*}
\int_{Q_\zeta(c+2\varepsilon) \setminus \{(0,T) \times (0,T) \times \omega_2\}} \theta^2(t, s, x + \zeta) \left( \lambda (z_t^2 + z_s^2 + |\nabla z|^2) + \lambda^3 z^2 \right) dx dt ds \\
\leq C(1 + r^2) e^{\lambda(c+2\varepsilon)^2} \int_{Q_1} \left( z_t^2 + z_s^2 + |\nabla z|^2 + z^2 \right) dx dt ds \\
+ C(1 + r^2) e^{\lambda(c+2\varepsilon)} \int_{\{(T_1, T_1') \times (T_1, T_1') \times \omega_2\} \cap Q_\zeta(c+2\varepsilon)} \left( z_t^2 + z_s^2 + |\nabla z|^2 + z^2 \right) dx dt ds.
\end{align*}
\]

By the definition of \( \theta \) and \( Q_\zeta(c + 2\varepsilon) \), we have

\[
\begin{align*}
e^{\lambda(c+2\varepsilon)^2} \int_{Q_\zeta(c+2\varepsilon) \setminus \{(0,T) \times (0,T) \times \omega_2\}} \left( \lambda (z_t^2 + z_s^2 + |\nabla z|^2) + \lambda^3 z^2 \right) dx dt ds \\
\leq C(1 + r^2) e^{\lambda(c+2\varepsilon)^2} \int_{Q_1} \left( z_t^2 + z_s^2 + |\nabla z|^2 + z^2 \right) dx dt ds \\
+ C(1 + r^2) e^{\lambda(c+2\varepsilon)} \int_{\{(T_1, T_1') \times (T_1, T_1') \times \omega_2\} \cap Q_\zeta(c+2\varepsilon)} \left( z_t^2 + z_s^2 + |\nabla z|^2 + z^2 \right) dx dt ds.
\end{align*}
\]

Combining with (4.33) and add \( \lambda \int_{(T_0, T_0') \times (T_0, T_0') \times \omega_2} \left( z_t^2 + z_s^2 \right) dx dt ds \), we have

\[
\begin{align*}
\int_{\{Q(c+2\varepsilon) \cup Q_\zeta(c+2\varepsilon)\} \setminus \{(0,T) \times (0,T) \times \omega_2\}} \left( \lambda (z_t^2 + z_s^2 + |\nabla z|^2) + \lambda^3 z^2 \right) dx dt ds \\
+ \lambda \int_{(T_0, T_0') \times (T_0, T_0') \times \omega_2} \left( z_t^2 + z_s^2 \right) dx dt ds \\
\leq C(1 + r^2) \int_{Q_1} \left( z_t^2 + z_s^2 + |\nabla z|^2 + z^2 \right) dx dt \\
+ C(1 + r^2) \lambda e^{\lambda(c+2\varepsilon)} \int_{\{(T_1, T_1') \times (T_1, T_1') \times \omega_2\} \cap (Q(c+2\varepsilon) \cup Q_\zeta(c+2\varepsilon))} \left( z_t^2 + z_s^2 + |\nabla z|^2 + z^2 \right) dx dt ds \\
+ \lambda \int_{(T_0, T_0') \times (T_0, T_0') \times \omega_2} \left( z_t^2 + z_s^2 \right) dx dt ds.
\end{align*}
\]
Recall that $Q_0 \setminus Q_0' \subset Q(c + 2\varepsilon) \cup Q_\zeta(c + 2\varepsilon)$, so that

$$
\int_{Q_0} \lambda(z_t^2 + z_s^2) dx dt ds \\
\leq C(1 + r^2) \int_{Q_1} (z_t^2 + z_s^2 + |\nabla z|^2 + z^2) dx dt \\
+ C(1 + r^2) \lambda e^{C\lambda} \int \{ (T_i, T_i') \times (T_j, T_j') \times \omega_2 \} \setminus \{ Q(c + 2\varepsilon) \cup Q_\zeta(c + 2\varepsilon) \} (z_t^2 + z_s^2 + |\nabla z|^2 + z^2) dx dt ds \\
+ \lambda \int_{(T_i, T_i') \times (T_j, T_j') \times \omega_2} (z_t^2 + z_s^2) dx dt ds.
$$

(5.9)

Notice that

$$
(T_1, T_1') \times (T_1, T_1') \times \omega_2 \cap \{ Q(c + 2\varepsilon) \cup Q_\zeta(c + 2\varepsilon) \} \\
\subset \{ \mathcal{D}' \cup \tilde{\mathcal{D}}' \} \cap \{ (0, T) \times (0, T) \times \omega \setminus \omega_0 \}.
$$

(5.10)

By the same process of Step 5 in proof of Theorem 2.1, we get

$$
\int_{Q_0} (\lambda(z_t^2 + z_s^2) dx dt ds \\
\leq C(1 + r^4) \int_{Q_2} (z_t^2 + z_s^2 + z^2) dx dt \\
+ C(1 + r^4) \lambda e^{C\lambda} \int \{ \mathcal{D}' \cup \tilde{\mathcal{D}}' \} \cap \{ (0, T) \times (0, T) \times \omega \setminus \omega_0 \} (z_t^2 + z_s^2 + z^2) dx ds dt \\
+ \lambda \int_{(T_0, T_0') \times (T_0, T_0') \times \omega_2} (z_t^2 + z_s^2) dx dt ds.
$$

(5.11)

Notice that

$$
(T_0, T_0') \times \omega_2 \cup \{ \mathcal{D} \cup \mathcal{D}_\zeta \} \cap \{ (0, T) \times \omega \setminus \omega_0 \} \subset \mathcal{K} \cup \mathcal{K}_\zeta \subset W.
$$

(5.12)

using the same argument from Step 6 of the proof of Theorem 2.1, we can yield (2.18). □
6 Appendix A

Proof of Corollary 3.1. Recall that $\theta = e^\ell$ and $v = \theta z$. Some elementary calculations yield that

$$\theta \left( z_{tt} + z_{ss} - \sum_{j,k=1}^{n} (h^{jk} z_{x_j})_{x_k} \right)$$

$$= v_{tt} + v_{ss} - \sum_{j,k=1}^{n} (h^{jk} v_{x_j})_{x_k} - 2\ell_t v_t - 2\ell_s v_s + 2 \sum_{j,k=1}^{n} h^{jk} \ell_{x_j} v_{x_k}$$

$$+ \ell_t^2 v + \ell_s^2 v - \sum_{j,k=1}^{n} h^{jk} \ell_{x_j} \ell_{x_k} v - \ell_{tt} v - \ell_{ss} v + \sum_{j,k=1}^{n} (h^{jk} \ell_{x_j})_{x_k} v$$

$$= I_1 + I_2,$$

where

$$I_1 \triangleq v_{tt} + v_{ss} - \sum_{j,k=1}^{n} (h^{jk} v_{x_j})_{x_k} - Av,$$

$$I_2 \triangleq -2\ell_t v_t - 2\ell_s v_s + 2 \sum_{j,k=1}^{n} h^{jk} \ell_{x_j} v_{x_k} - \Psi v,$$

$$A \triangleq -\ell_t^2 - \ell_s^2 + \sum_{j,k=1}^{n} h^{jk} \ell_{x_j} \ell_{x_k} + \ell_{tt} v + \ell_{ss} v - \sum_{j,k=1}^{n} (h^{jk} \ell_{x_j})_{x_k} - \Psi.$$

Then

$$\theta^2 \left| z_{tt} + z_{ss} - \sum_{j,k=1}^{n} (h^{jk} z_{x_j})_{x_k} \right|^2 = |I_1|^2 + |I_2|^2 + 2I_1 I_2 \geq 2I_1 I_2.$$ (6.3)

Let us compute $2I_1 I_2$. Denote the terms in the right-hand side of $I_1$ and $I_2$ by $I_1^j (j = 1, 2, 3, 4)$ and $I_2^k (k = 1, 2, 3, 4)$, respectively. Then

$$I_1^1 I_2^2 = \left( -\ell_t v_t^2 - 2\ell_s v_s v_t + 2 \sum_{j,k=1}^{n} h^{jk} \ell_{x_j} v_{x_k} v_t - \Psi v v_t \right)$$

$$+ \ell_{tt} v_t^2 + 2\ell_{tt} v_s v_t + (\ell_s v_t^2)_s - \ell_{ss} v_t^2 - 2 \sum_{j,k=1}^{n} h^{jk} \ell_{tx_j} v_{x_k} v_t$$

$$- \text{div} \left( \sum_{j=1}^{n} h^{jk} \ell_{x_j} v_t^2 \right) + \sum_{j,k=1}^{n} (h^{jk} \ell_{x_j})_{x_k} v_t^2 + \Psi v_t^2.$$ (6.4)
For $I_1^2 I_2$, we have
\[
I_1^2 I_2 = \left( -\ell_s v_s^2 - 2\ell_t v_s v_t + 2 \sum_{j,k=1}^{n} h^{jk} \ell_{x_j} v_{x_k} v_s - \Psi v v_s \right)_s \\
+ \ell_s v_s^2 + 2\ell_{st} v_s v_t + (\ell_t v_s^2)_t - \ell_{tt} v_s^2 - 2 \sum_{j,k=1}^{n} h^{jk} \ell_{sx_j} v_{x_k} v_s \\
- \text{div} \left( \sum_{j=1}^{n} h^{jk} \ell_{x_j} v_s^2 \right) + \sum_{j,k=1}^{n} (h^{jk} \ell_{x_j})_{x_k} v_s^2 + \Psi v_s^2.
\]

For $I_1^3 I_2$, we have
\[
I_1^3 I_2 = - \sum_{j,k=1}^{n} (h^{jk} v_{x_j})_{x_k} \left( -2\ell_t v_t - 2\ell_s v_s + 2 \sum_{j,k=1}^{n} h^{jk} \ell_{x_j} v_{x_k} - \Psi v \right) \\
= \text{div} \left( \sum_{j,k=1}^{n} 2h^{jk} v_{x_j} \ell_{tx_k} v_t \right) - \sum_{j,k=1}^{n} 2h^{jk} v_{x_j} \ell_{txt} v_t - \left( \sum_{j,k=1}^{n} h^{jk} \ell_{tx_j} v_{x_k} \right)_t \\
+ \sum_{j,k=1}^{n} h^{jk} \ell_{tt} v_{x_j} v_{x_k} + \text{div} \left( \sum_{j,k=1}^{n} 2h^{jk} v_{x_j} \ell_s v_s \right) - \sum_{j,k=1}^{n} 2h^{jk} v_{x_j} \ell_{sx_k} v_s \\
- \left( \sum_{j,k=1}^{n} h^{jk} \ell_s v_{x_j} v_{x_k} \right)_s + \sum_{j,k=1}^{n} h^{jk} \ell_{ss} v_{x_j} v_{x_k} \\
- \text{div} \left( 2 \sum_{j,j',k=1}^{n} h^{jk} h^{j'k'} \ell_{x_j} v_{x_{j'}} v_{x_{k'}} \right) + 2 \sum_{j,k,j',k'=1}^{n} h^{jk} (h^{j'k'} \ell_{x_{j'}})_{x_{k'}} v_{x_j} v_{x_k} \\
+ \text{div} \left( \sum_{j,j',k=1}^{n} h^{jk} h^{j'k'} \ell_{x_j} v_{x_{j'}} v_{x_{k'}} \right) - \sum_{j,k,j',k'=1}^{n} (h^{jk} h^{j'k'} \ell_{x_{j'}})_{x_{k'}} v_{x_j} v_{x_k} \\
+ \text{div} \left( \sum_{j,k=1}^{n} h^{jk} v_{x_j} \Psi v \right) - \sum_{j,k=1}^{n} h^{jk} \Psi v_{x_j} v_{x_k} - \sum_{j,k=1}^{n} h^{jk} \Psi_{x_j} v_{x_{k'}} v_{x_k}.
\]

For $I_1^4 I_2$, we have
\[
I_1^4 I_2 = -Av\left( -2\ell_t v_t - 2\ell_s v_s + 2 \sum_{j,k=1}^{n} h^{jk} \ell_{x_j} v_{x_k} - \Psi v \right) \\
= (A\ell_t v^2)_t - (A\ell_t) v^2 + (A\ell_s v^2)_s - (A\ell_s)_s v^2 \\
- \text{div} \left( \sum_{j,k=1}^{n} h^{jk} A\ell_{x_j} v^2 \right) + \sum_{j,k=1}^{n} (A h^{jk} \ell_{x_j})_{x_k} v^2 + A\Psi v^2.
\]

Finally, combining (6.3)–(6.7), we complete the proof of Corollary 3.1. \qed
References

[1] C. Bardos, G. Lebeau, J. Rauch, *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*. SIAM J. Control Optim., 30 (1992), 1024–1065.

[2] J. M. Coron, *Control and Nonlinearity*. Mathematical Surveys and Monographs. American Mathematical Society, Providence, 136 (2007).

[3] T. Duyckaerts, X. Zhang, E. Zuazua, *On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials*. Ann. Ann. Inst. H. Poincaré Anal. Non Linéaire., 25 (2008), 1–41.

[4] C. Evans, *Partial differential equations*. Graduate studies in Mathematics, 19, American Mathematical Society, 1998.

[5] X. Fu, Q. Lü, X. Zhang, *Carleman Estimates for Second Order Partial Differential Operators and Applications*. Springer, 2019.

[6] X. Fu, J. Yong, X. Zhang, *Exact controllability for the multidimensional semilinear hyperbolic equations*. SIAM J. Control Optim., 22 (2016), 1578–1614.

[7] V. K. Jena, *Carleman estimate for ultrahyperbolic operators and improved interior control for wave equations*. J. Differential Equations., 302 (2021), 273–333.

[8] M. V. Klibanov, *Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems*. J. Inverse Ill-Posed Probl., 21 (2013), 477–560.

[9] I. Lasiecka, R. Triggiani, P. F. Yao, *Inverse/observability estimates for second-order hyperbolic equations with variable coefficients*. J. Math. Anal. Appl., 235 (1999), 13–57.

[10] M. M. Lavent’ev, V. G. Romanov, S. P. Shishat · skii , *Ill-Posed Problems of Mathematical Physics and Analysis*. Translations of Mathematical Monographs, American Mathematical Society Providence, 64 (1986).

[11] A. López, X. Zhang, E. Zuazua, *Null controllability of the heat equation as singular limit of the exact controllability of dissipative wave equations*. J. Math. Pure. Appl., 79 (2000), 741–808.

[12] J. Li, Q. Lü, *State observation problem for general time reversible system and applications*. Appl. Math. Comput., 217 (2010), 2843–2856.

[13] J. L. Lions, *Contrôlabilité exacte, perturbations et systèmes distribués*. vol. 1, RMA no. 8. Paris: Masson, 1988.

[14] Y. Liu, *Some sufficient conditions for the controllability of wave equations with variable coefficients*. Acta. Appl. Math., 128 (2013), 181–191.

[15] D. L. Russell, *Controllability and stabilizability theory for linear partial differential equation: Recent progress and open questions*. SIAM Rev., 20 (1978), 639–739.
[16] A. Shao, *On Carleman and observability estimates for wave equations on time dependent domains*. Proc. Lond. Math. Soc., 67 (2019), 998–1064.

[17] L. Tebou, *A Carleman estimates based approach for the stabilization of some locally damped semilinear hyperbolic equations*. ESAIM Control Optim. Calc. Var., 14 (2008), 561–574.

[18] X. Zhang, *Explicit observability estimate for the wave equation with potential and its application*. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 456 (2000), 1101–1115.

[19] X. Zhang, *Explicit observability inequalities for the wave equation with lower order terms by means of Carleman inequalities*. SIAM J. Control. Optim., 39 (2000), 812–834.

[20] E. Zuazua, *Exact controllability for semilinear wave equations in one space dimension*. Ann. Inst. H. Poincaré Anal. Non Linéaire., 10 (1993), 109–129.

[21] E. Zuazua, *Exact controllability of the semilinear wave equation*. J. Math. Pure. Appl., 69 (1990), 1–31.