EQUITABLE PARTITION OF GRAPHS
INTO INDUCED FORESTS

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Abstract. An equitable partition of a graph $G$ is a partition of the vertex-set of $G$ such that the sizes of any two parts differ by at most one. We show that every graph with an acyclic coloring with at most $k$ colors can be equitably partitioned into $k-1$ induced forests. We also prove that for any integers $d \geq 1$ and $k \geq 3^{d-1}$, any $d$-degenerate graph can be equitably partitioned into $k$ induced forests.

Each of these results implies the existence of a constant $c$ such that for any $k \geq c$, any planar graph has an equitable partition into $k$ induced forests. This was conjectured by Wu, Zhang, and Li in 2013.

An equitable partition of a graph $G$ is a partition of the vertex-set of $G$ such that the sizes of any two parts differ by at most one. Hajnal and Szemerédi [4] proved the following result, which was conjectured by Erdős (see also [5] for a shorter proof).

Theorem 1. For any integers $\Delta$ and $k \geq \Delta + 1$, any graph with maximum degree $\Delta$ has an equitable partition into $k$ stable sets.

Note that there is no constant $c$, such that for any $k \geq c$, any star can be equitably partitioned into $k$ stable sets. Wu, Zhang, and Li made the following two conjectures [9].

Conjecture 2. There is a constant $c$ such that for any $k \geq c$, any planar graph can be equitably partitioned into $k$ induced forests.

Conjecture 3. For any integers $\Delta$ and $k \geq \lceil \frac{\Delta+1}{2} \rceil$, any graph of maximum degree $\Delta$ can be equitably partitioned into $k$ induced forests.

A proper coloring of a graph $G$ is acyclic if any cycle of $G$ contains at least 3 colors. We first prove the following result.

Theorem 4. Let $k \geq 2$. If a graph $G$ has an acyclic coloring with at most $k$ colors, then $G$ can be equitably partitioned into $k-1$ induced forests.

Proof. The proof proceeds by induction on $k \geq 2$. If $k = 2$ then $G$ itself is a forest and the result trivially holds, so we can assume that $k \geq 3$. Let $V_1, \ldots, V_k$ be the color classes in some acyclic $k$-coloring of $G$ (note that some sets $V_i$ might be empty). Let $n$ be the number of vertices of $G$. Without loss of generality, we can assume that $V_1$ contains at most $\frac{n}{k} \leq \frac{n}{k-1}$ vertices. Observe that the sum of the number of vertices in $V_1 \cup V_i$, $2 \leq i \leq k$, is $n + (k-2)|V_1| \geq n$. It follows that there is exists a color class, say $V_2$, such that $V_1 \cup V_2$ contains at least $\frac{n}{k}$ vertices. Let $S$ be a set of vertices of $G$.
consisting of $V_1$ together with $\left\lfloor \frac{n}{k-1} \right\rfloor - |V_1|$ vertices of $V_2$, and let $H$ be the graph obtained from $G$ by removing the vertices of $S$. Note that $S$ induces a forest in $G$, and $H$ has an acyclic coloring with at most $k-1$ colors. By the induction hypothesis, $H$ has an equitable partition into $k-2$ induced forests, and therefore $G$ has an equitable partition into $k-1$ induced forests. □

It was proved by Borodin [2] that any planar graph has an acyclic coloring with at most 5 colors. Therefore, Theorem 9 implies Corollary 5, which is a positive answer to Conjecture 2.

**Corollary 5.** For any $k \geq 4$, any planar graph can be equitably partitioned into $k$ induced forests.

We now prove stronger results in two different ways. We first show the induced forests can be chosen to be very specific. We then show that graphs from a class that is much wider than the class of planar graphs can also be equitably partitioned into constantly many induced forests.

A **star coloring** of a graph $G$ is a proper coloring of the vertices of $G$ such that any two color classes induce a star forest. Using the same proof as that of Theorem 4, it is easy to show the following result.

**Theorem 6.** Let $k \geq 2$. If a graph $G$ has a star coloring with at most $k$ colors, then $G$ can be equitably partitioned into $k-1$ induced star forests.

It was proved by Albertson et al. [1] that every planar graph has a star coloring with at most 20 colors. The next corollary follows as an immediate consequence.

**Corollary 7.** For any $k \geq 19$, any planar graph can be equitably partitioned into $k$ induced star forests.

Indeed, if one is not too regarding on the constant, a stronger result holds. An **orientation** of a graph $G$ is a directed graph obtained from $G$ by orienting each edge in either of two possible directions. An **out-star** (resp. **in-star**) is the orientation of a star such that every edge is oriented from the center of the star to the leaf (resp. from the leaf to the center of the star).

**Theorem 8.** For any $k \geq 319$, any orientation of a planar graph can be equitably partitioned into $k$ induced forests of in- and out-stars.

**Proof.** It was proved by Raspaud and Sopena [8] that every orientation of a planar graph has an acyclic coloring with at most 80 colors such that for any two colors classes $V_i$ and $V_j$, if there is an arc $(u,v)$ with $u \in V_i$ and $v \in V_j$, then there is no arc $(x,y)$ with $y \in V_i$ and $x \in V_j$. Using [1, Theorem 4.3] and the Four Color Theorem, this acyclic coloring can be refined into a star coloring, with the same additional property, using no more than $80 \cdot 4 = 320$ colors. In particular, every two color classes induce a forest of in- and out-stars. The remainder of the proof follows the same lines as the proofs of Theorems 4 and 6. □

It is known that graphs with bounded acyclic chromatic number also have bounded star chromatic number [1] and bounded oriented chromatic number [8], so it follows that the results of Theorems 6 and 8 hold for any
class of graphs with bounded acyclic chromatic number (with possibly larger constants).

A graph $G$ is $d$-degenerate if every subgraph of $G$ contains a vertex of degree at most $d$. In the remainder of this article, we prove the following result.

**Theorem 9.** For any integers $d \geq 1$ and $k \geq 3^{d-1}$, any $d$-degenerate graph can be equitably partitioned into $k$ induced forests.

It follows from Euler’s formula that every planar graph is 5-degenerate. Therefore, Theorem 9 also implies Conjecture 2 (with $c = 81$ instead of $c = 4$ in Corollary 5). For a graph $G$, let $\chi_a(G)$ denote the least integer $k$ such that $G$ has an acyclic coloring with $k$ colors. It is known that there is a function $f$ such that every graph $G$ is $f(\chi_a(G))$-degenerate [3]. However, there exist families of 2-degenerate graphs with unbounded acyclic chromatic number. It follows that Theorem 9 can be applied to wider classes of graphs than Theorems 4, 6, and 8.

A class of graphs is hereditary if it is closed under taking induced subgraphs.

**Lemma 10.** Let $\ell$ be an integer and $C$ be a hereditary class of graphs such that every graph in $C$ can be equitably partitioned into $\ell$ induced forests. Then for any $k \geq \ell$, any graph in $C$ can be equitably partitioned into $k$ induced forests.

**Proof.** Let $G$ be a graph of $C$, and let $n$ be the number of vertices of $G$. Let $n = kq + s$, with $0 \leq s < k$. Note that an equitable partition of $G$ into $k$ sets consists of $s$ sets of size $\lceil n/k \rceil$ and $k - s$ sets of size $\lfloor n/k \rfloor$.

Let $G_0 = G$. For any $1 \leq i \leq k - \ell$, we inductively define $G_i$ as a graph obtained from $G_{i-1}$ by removing a set $S_{i-1}$ of $\lceil n/k \rceil$ vertices (if $i \leq s$) or $\lfloor n/k \rfloor$ vertices (otherwise) inducing a forest in $G_{i-1}$. The existence of such an induced forest follows from the fact that for any $n' \geq \frac{n}{\ell}$, any induced subgraph of $G$ on $n'$ vertices contains an induced forest on at least $\lceil n'/\ell \rceil \geq \lceil n/k \rceil$ vertices. By assumption, the graph $G_{k-\ell}$ can be equitably partitioned into $\ell$ induced forests (each on $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ vertices). Combining these induced forests with $S_0, S_1, \ldots, S_{k-1}$, we obtain an equitable partition of $G$ into $k$ induced forests. \qed

The following result was proved in [6].

**Theorem 11.** Let $k \geq 3$ and $d \geq 2$. Then every $d$-degenerate graph can be equitably partitioned into $k$ $(d-1)$-degenerate graphs.

We now give a short proof of Theorem 9 using Lemma 10 and Theorem 11.

**Proof of Theorem 9.** By Lemma 10 it is enough to show that any $d$-degenerate graph has an equitable partition into $3^{d-1}$ induced forests.

We prove this result by induction on $d \geq 1$. If $d = 1$, the result follows from the fact that a 1-degenerate graph is a forest. Assume that $d \geq 2$. By Theorem 11 $G$ has an equitable partition into three $(d-1)$-degenerate graphs. By the induction, each of these graphs has an equitable partition.
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into $3^{d-2}$ induced forests, therefore $G$ has an equitable partition into $3 \cdot 3^{d-2} = 3^d - 1$ induced forests. \hfill \qed

**Open problems.** It remains to determine whether every planar graph has an equitable partition into three induced forests. By Theorems 4 and 9, a possible counterexample must have acyclic chromatic number equal to 5 and cannot be 2-degenerate.

It was proved by Poh [7] that every planar graph has a partition into three induced *linear forests* (i.e. graphs in which each connected component is a path). A natural question is the following.

**Question 12.** *Is there a constant $c$ such that for any $k \geq c$, any planar graph has an equitable partition into $k$ induced linear forests?*

It was pointed out to us by Yair Caro that the (outer)planar graph obtained from a large path by adding a universal vertex shows that Question 12 has a negative answer.

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