Morse index theorem for heteroclinic, homoclinic and halfclinic orbits of Lagrangian systems

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Abstract

The aim of this paper is to prove a new and general version of the Morse index theorem for heteroclinic, homoclinic and halfclinic solutions of general Lagrangian systems. In the last section we compute the Morse index of some heteroclinic and halfclinic solutions of some classical mechanical systems like the mathematical pendulum, the Nagumo equation and a competition-diffusion system in dimension 4.

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1 Introduction, description of the problem and main results

Morse Index theory for Lagrangian systems describes the relation intertwining the Morse index of the index form at a critical point of a variational problem and the symplectic oscillation properties of the associated variational differential equation of the critical point. The origin of this topic could be traced back to M. Morse who firstly found an explicit formula describing the relation intertwining the index of a geodesic (seen as critical point of the geodesic action functional) and the total number of conjugate points counted with their multiplicities. This result has been generalized in the last decades by Edwards, Simons and Smale to systems of higher order, minimal surfaces, and partial differential systems, respectively.

Despite of its popularity almost all of the existing results regard orbits of Hamiltonian systems parametrized on compact intervals. For heteroclinics, homoclinics and halfclincs (h-clinic in shorthand notation) of a Lagrangian system, to the author’s knowledge, very few results are known. Recently in [HP17], authors constructed an index theory for h-clinic motions of a (general) Hamiltonian system and in [BHPT19] the authors provided an ad-hoc generalization for an important class of asymptotic motions in weakly singular Lagrangian systems (including the gravitational n-body problem).

Starting from the spectral flow formula proved in [HP17] we were able to construct an index theory for h-clinic motions. Such a theory consists in establishing an equality between the Morse index for an h-clinic solution and a geometric index defined in terms of a Maslov-type index, plus a correction term.

Finally, we apply the main results proved in the paper for computing the Morse index of some heteroclinic and halfclinic solutions in the classical models of

- The mathematical pendulum
- The Nagumo reaction equation modeling the impulse propagation along a nerve fiber and finally
- A reaction-diffusion system in \( \mathbb{R}^4 \).

1.1 Description of the problem and main results

Let \( T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \) be denote the tangent bundle of \( \mathbb{R}^n \) whose elements will be denoted by \((q,v)\), with \( q \in \mathbb{R}^n \) and \( v \in T_q\mathbb{R}^n \cong \mathbb{R}^n \).

Let \( L : \mathbb{R} \times T\mathbb{R}^n \to \mathbb{R} \) be a smooth non-autonomous (Lagrangian) function satisfying the following Legendre convexity condition

\[
\text{(L1)} \quad L \text{ is } C^2\text{-convex on the fibers of } T\mathbb{R}^n, \quad \text{meaning that the quadratic form} \quad \|D^2_{q,v}L(t,q,v)\| \geq \ell_0 I > 0 \quad \forall (t,q,v) \in \mathbb{R} \times T\mathbb{R}^n.
\]

In what follows we will denote by \( u^-, u^+ \in \mathbb{R}^n \), two restpoints for the Lagrangian vector field \( \nabla L \); thus, we have

\[
\nabla L(t,u^+,0) = 0 \quad \text{for every} \quad t \in \mathbb{R}.
\]

**Definition 1.1.** An *heteroclinic orbit* \( u \) asymptotic to \( u^- \) or a connecting orbit between \( u^\pm \) is a \( C^2 \)-solution of the following boundary value problem

\[
\begin{cases}
\frac{d}{dt} \partial_{(t,u)} L(t,u(t),\dot{u}(t)) = \partial_{q} L(t,u(t),\dot{u}(t)) & t \in \mathbb{R} \\
\lim_{t \to -\infty} u(t) = u^- \quad \text{and} \quad \lim_{t \to +\infty} u(t) = u^+
\end{cases}
\] (1.1)

If \( u^- = u^+ \) we will refer to the connecting orbit \( u \) as *homoclinic orbit*. 
Definition 1.2. A future (resp. past) halfclinic solution $u$, starting at the Lagrangian subspace $L_0 \in L(n)$, is a solution of the following boundary value problem

$$\begin{align*}
\frac{d}{dt} \partial_u L(t, u(t), \dot{u}(t)) &= \partial_q L(t, u(t), \dot{u}(t)) && t \in \mathbb{R}^+ \text{ (resp. } t \in \mathbb{R}^-) \\
(\partial_u L(0, u(0), \dot{u}(0)), u(0))^T &= L_0 \\
\lim_{t \to +\infty} u(t) &= u^+ \quad \text{(resp. } \lim_{t \to -\infty} u(t) = u^-) \end{align*}$$

(1.2)

Notation 1.3. In shorthand notation, we refer to a solution of Equation (1.1) or Equation (1.2), as h-clinic orbit and we set $I := \mathbb{R}^- \text{ or } \mathbb{R}^+ \text{ or } \mathbb{R}$.

By linearizing the Euler-Lagrangian equation

$$\frac{d}{dt} \partial_u L(t, u(t), \dot{u}(t)) = \partial_q L(t, u(t), \dot{u}(t)) \quad t \in I$$

along a h-clinic solution $u$ and by setting

$$P(t) := \partial_{uv} L(t, u(t), \dot{u}(t)), \quad Q(t) := \partial_{uu} L(t, u(t), \dot{u}(t)) \quad \text{and } R(t) := \partial_{uu} L(t, u, \dot{u}(t))$$

we get the following variational equation defining the Sturm-Liouville differential operator:

$$\mathcal{A} w := -\frac{d}{dt}(P(t) \dot{w}(t) + Q(t) w(t)) + (Q(t))^T \dot{w}(t) + R(t) w(t) \quad t \in I$$

(1.3)

and in the halfclinic case the linearizing boundary condition at $t = 0$ reads as follows:

$$(P(0) \dot{w}(0) + Q(0) w(0), w(0))^T \in L_0.$$ 

We observe that $P$ and $R$ are path of symmetric matrices. Let us now introduce the following condition:

(L2) The matrices $P(t), Q(t)$ and $R(t)$ converge to $P_{\pm}, Q_{\pm}$ and $R_{\pm}$ at $\pm \infty$, respectively. Moreover we assume that there exist positive constants $C_1, C_2$ and $C_3$ such that

$$\|P(t)\| \geq C_1, \quad \|Q(t)\| \leq C_2 \quad \text{and} \quad \|R(t)\| \leq C_3 \quad \forall t \in I.$$

By a symplectic change of coordinates the variational equation defined in Equation (1.3), fits into the following Hamiltonian system

$$\dot{z} = JB(t)z \quad \text{where} \quad B(t) := \begin{pmatrix} P^{-1}(t) & -P^{-1}(t)Q(t) \\ -Q^T(t)P^{-1}(t) & Q^T(t)P^{-1}(t)Q(t) - R(t) \end{pmatrix}.$$ 

Let us now introduce the following conditions.

(H1) The limit matrices $JB(-\infty)$ and $JB(\infty)$ are hyperbolic meaning that their spectrum is out from the imaginary axis

(H2) The matrices $\begin{pmatrix} P_- & Q_- \\ Q_T^- & R_- \end{pmatrix}$ and $\begin{pmatrix} P_+ & Q_+ \\ Q_T^+ & R_+ \end{pmatrix}$ are both positive definite.

We let

$$E := W^{2,2}(\mathbb{R}, \mathbb{R}^n) \quad \text{and} \quad E_{L_0}^\pm := \left\{ w \in W^{2,2}(\mathbb{R}^\pm, \mathbb{R}^n) \mid (P(0) \dot{w}(0) + Q(0) w(0))^T \in L_0 \right\}$$

and we set

$$\mathcal{A} := \mathcal{A}|_E \quad \text{and} \quad \mathcal{A}_{L_0}^\pm := \mathcal{A}|_{E_{L_0}^\pm}.$$ 

Notation 1.4. In what follows, we will denote by $\mathcal{A}_{L_0}^\pm$ (resp. $\mathcal{A}_{M_0}^\pm$) the minimal (resp. maximal) operator in $L^2$ defined by $\mathcal{A}$; namely the restriction of $\mathcal{A}$ onto $W^{2,2}(\mathbb{R}^\pm, \mathbb{R}^n)$ (resp. $W^{2,2}(\mathbb{R}^\pm, \mathbb{R}^n)$). We denote by $m^-(\mathcal{A})$ and $m^-(\mathcal{A}_{L_0}^\pm)$ the Morse index of $\mathcal{A}$ and $\mathcal{A}_{L_0}^\pm$, respectively.
Analogously, we define the Hamiltonian operator $\mathcal{F}$ as follows

$$\mathcal{F} := -J \frac{d}{dt} B(t) \quad t \in I$$

and by setting $W = W^{1,2}(\mathbb{R}, \mathbb{R}^{2n})$ and $W^\pm_{L_0} := \{ z \in W^{1,2}(\mathbb{R}^\pm, \mathbb{R}^{2n}) \mid z(0) \in L_0 \}$, we define the first order differential operators

$$\mathcal{F} := \mathcal{F}|_W \quad \text{and} \quad \mathcal{F}^\pm_{L_0} := \mathcal{F}|_{W^\pm_{L_0}}.$$

**Notation 1.5.** As before, we will denote by $\mathcal{F}^+_n$ (resp. $\mathcal{F}^-_n$) the minimal (resp. maximal) operator defined by $\mathcal{F}$; namely the restriction of $\mathcal{F}$ onto $W^{1,2}_n(\mathbb{R}^\pm, \mathbb{R}^{2n})$ (resp. $W^{1,2}(\mathbb{R}^\pm, \mathbb{R}^{2n})$).

**Remark 1.6.** It is well-known that the operator $\mathcal{F}$ is Fredholm if and only if the Hamiltonian matrices $JB(\pm 0)$ are hyperbolic. (Cfr. [RS95] and references therein, for further details). The same characterization holds also for the operators $\mathcal{F}^\pm$. (Cfr. [RS05a, RS05b] for further details).

Borrowing the notation from [HP17], we denote by $\gamma_\tau$ be the (matrix) solution of the following (linear) Hamiltonian system

$$\begin{cases}
\dot{\gamma_\tau}(t) = JB(t) \gamma_\tau(t), \quad t \in \mathbb{R} \\
\gamma_\tau(\tau) = I
\end{cases}$$

and we define respectively the stable and unstable subspaces as follows

$$E^s(\tau) := \left\{ v \in \mathbb{R}^{2n} \mid \lim_{t \to \pm \infty} \gamma_\tau(t)v = 0 \right\} \quad \text{and} \quad E^u(\tau) := \left\{ v \in \mathbb{R}^{2n} \mid \lim_{t \to \pm \infty} \gamma_\tau(t)v = 0 \right\}$$

and we observe that, for every $\tau \in \mathbb{R}$, $E^s(\tau), E^u(\tau) \in L(n)$. (For further details, we refer the interested reader to [CH07, HP17] and references therein). Denoting by $E^s(\pm \infty)$ the stable spaces at $\pm \infty$ defined by

$$E^s(+\infty) := \left\{ v \in \mathbb{R}^{2n} \mid \lim_{t \to +\infty} \exp(tB(+\infty)) v = 0 \right\} \quad \text{and} \quad E^s(-\infty) := \left\{ v \in \mathbb{R}^{2n} \mid \lim_{t \to -\infty} \exp(tB(-\infty)) v = 0 \right\}$$

and assuming that condition (H1) holds, then we get that

$$\lim_{\tau \to +\infty} E^s(\tau) = E^s(+\infty) \quad \text{and} \quad \lim_{\tau \to -\infty} E^u(\tau) = E^u(-\infty)$$

where the convergence is meant in the gap (norm) topology of the Lagrangian Grassmannian. (Cfr. [AM03] for further details). It is well-known (Cfr. [HP17] and references therein) that the path $\tau \mapsto E^s(\tau)$ and $\tau \mapsto E^u(\tau)$ are both Lagrangian and to each ordered pair of Lagrangian paths we can assign an integer known in literature as Maslov index of the pair. Following authors in [HP17] we now associate to each h-clinic orbit an integer named geometrical index.

**Definition 1.7.** [HP17] We define the geometrical index of

- The (heteroclinic) solution $u$ given at Equation (1.1) as the integer given by
  $$\iota(u) = -\iota^{CLM}_G(E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+)$$

- The (future halfclinic) solution $u$ given at Equation (1.2) as the integer given by
  $$\iota^+_0(u) = -\iota^{CLM}_G(E^s(\tau), L_0; \tau \in \mathbb{R}^+)$$

- The (past halfclinic) solution $u$ given at Equation (1.2) as the integer given by
  $$\iota^-_0(u) = -\iota^{CLM}_G(L_0, E^u(-\tau); \tau \in \mathbb{R}^+)$$

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where \( \iota_{\text{CLM}} \) denotes the Cappell-Lee-Miller index. (Cfr. Appendix A and references therein).

Given three Lagrangian subspaces \( L_1, L_2, L_3 \), we denote by \( \iota(L_1, L_2, L_3) \) the \textbf{triple index} (Cfr. Appendix A.2 for the definition and the basic properties). Bearing this notation in mind we are in position to state the main result of this paper in the case of heteroclinic solutions.

**Theorem 1.** Let \( u \) be a (heteroclinic) solution of the Lagrangian system given at Equation (1.1) and we assume conditions (L1)-(L2) and (H1) are fulfilled. Then we get

\[
\iota^- (u) = \iota (u) + \iota (E^u(-\infty), E^u(+\infty); L_D)
\]

where \( \iota^- (u) = \iota (A) \) denotes the Morse index of \( u \) and where \( L_D := \mathbb{R}^n \times (0) \) denotes the Dirichlet Lagrangian.

**Corollary 1.** Under the assumptions of Theorem 1 and if (H2) holds, then we get

\[
\iota^- (A) = \iota (u).
\]

The next result of this paper provides a Morse index theorems in the cases of past and future halfclinic solutions of a Legendre convex Lagrangian system with general Lagrangian boundary conditions. The following result holds.

**Theorem 2.** Let \( u \) be a solution either of Equation (1.2). If conditions (L1)-(L2) and (H1) are fulfilled, then we get

\[
\begin{align*}
\text{(future halfclinic case) } \\
\iota^- (u, L_0, +) &= \iota^+ (u, L_0) + \iota (L_D, L_0; E^u(+\infty)) \quad (1.4) \\
\text{(past halfclinic case) } \\
\iota^- (u, L_0, -) &= \iota^- (u, L_0) + \iota (E^u(-\infty), L_0; L_D) \quad (1.5)
\end{align*}
\]

where \( \iota^- (u, L_0, +) = \iota^+ (A_{L_0}) \) (resp. \( \iota^- (u, L_0, -) = \iota^- (A_{L_0}) \)) denote the Morse index of the future (resp. past) halfclinic solution \( u \).

As direct consequence of Theorem 2, we provide a characterization of the difference of the Morse indices of a halfclinic orbit for a Legendre convex Lagrangian system once the Lagrangian boundary condition has been replaced by the Dirichlet one, in terms of the triple index and of the relative position of the Lagrangian subspaces \( L_0, L_D \) and finally \( E^u(0) \) or \( E^u(0) \).

**Corollary 2.** Let \( u \) be a solution of the boundary value problem given in Equation (1.2). If (L1)-(L2)-(H1)-(H2) conditions hold, then we get

\[
\begin{align*}
\text{(future halfclinic orbit) } \\
\iota^- (u, L_0, +) - \iota^- (u, L_D, +) &= \iota (L_D, L_0; E^u(0)) \quad (1.6) \\
\text{(past halfclinic orbit) } \\
\iota^- (u, L_0, -) - \iota^- (u, L_D, -) &= \iota (E^u(0), L_0; L_D) \quad (1.7)
\end{align*}
\]

**Example 1.8. (The scalar case)** In this example, we consider the Sturm-Liouville operator \( \mathcal{A}_\lambda \) defined in Equation (1.3) obtained by replacing \( R \) with \( R_\lambda = R + \lambda \) and we assume that the dimension \( n = 1 \). As direct consequence of the results proved in Section D, we infer that \( \mathcal{A}_\lambda \) is Fredholm if and only the matrices \( JB_\lambda(\pm \infty) \) are both hyperbolic or which is the same that \( R_\pm > 0 \). By observing that \( P(t), Q(t) \) and \( R(t) \) are scalar functions and by performing this computation at \( \infty \), we get

\[
\det (\mu - JB_\lambda(+\infty)) = \det \begin{pmatrix} \mu - Q_+ P_+^{-1} & -Q^2_+ P_+^{-1} - (R_+ + \lambda) \\ -P_+^{-1} & \mu + P_+^{-1} Q_+ \end{pmatrix} = \mu^2 - P_+^{-1} (R_+ + \lambda).
\]
Thus, we get that $\pm \sqrt{P_+^{-1}(R_+ + \lambda)}$ are the eigenvalues of $JB_\lambda(+\infty)$, and it is easy to check that
\[
JB_\lambda(+\infty) \left( Q_+ \pm \sqrt{P_+^{-1}(R_+ + \lambda)} \right) = \pm \sqrt{P_+^{-1}(R_+ + \lambda)} \left( Q_+ \pm \sqrt{P_+^{-1}(R_+ + \lambda)} \right).
\]
So, we get that
\[
E^+_\lambda(+\infty) = V^-(JB_\lambda(+\infty)) = \text{span} \left\{ \left( Q_+ - \sqrt{P_+^{-1}(R_+ + \lambda)} \right) \right\}.
\]
Similarly, we have that
\[
E^-_\lambda(-\infty) = \text{span} \left\{ \left( Q_- + \sqrt{P_-^{-1}(R_- + \lambda)} \right) \right\}.
\]

(a) In this case $\lambda \mapsto L^+_\lambda$ is approaching the $x$-axis in the counter-clockwise direction whilst $\lambda \mapsto L^-_\lambda$ is approaching the $x$-axis clockwise direction so, no coincidence times on $[0, \bar{\lambda}]$.

(b) In this case $\lambda \mapsto L^+_\lambda$ is approaching the $x$-axis in the clockwise direction whilst $\lambda \mapsto L^-_\lambda$ is approaching the $x$-axis counter-clockwise direction and so, only one coincidence time on $[0, \bar{\lambda}]$.

Then $E^+_\lambda(+\infty)$ represents the straight line $L^+_\lambda$ through the origin with a slope (with respect to the $y$ axis) of $Q_+ - \sqrt{P_+^{-1}(R_+ + \lambda)}$ and $E^-_\lambda(-\infty)$ represents the straight line $L^-_\lambda$ through the origin with slope (with respect to the $y$ axis) of $Q_- + \sqrt{P_-^{-1}(R_- + \lambda)}$. It is immediate to observe that $L^+_\lambda$ (resp. $L^-_\lambda$) approaches the $x$-axis in the counter-clockwise (resp. clockwise) direction as $\lambda$ increases to $\infty$ respectively. So $\ell^{\text{CLM}} \left( E^+_\lambda(+\infty), E^-_\lambda(-\infty), \lambda \in [0, \bar{\lambda}] \right)$ equal to the total number of counting the coincidence times (overlapping times) between $L^+_\lambda$ and $L^-_\lambda$ as $\lambda$ increases from $0$ up to $\bar{\lambda}$. We consider the following two cases.

- **Case 1** (as shown in Figure 1a). If condition (H2) holds, then we get that $\sqrt{P_\pm R_\pm + Q_\pm} > 0$ and so the line $L^+_0$ (resp. $L^-_0$) lies on the left (resp. right) half-plane bounded by the $y$-axis. By the above discussion we get that the coincidence times between $L^+_\lambda$ and $L^-_\lambda$ is 0, and then $\ell^{\text{CLM}} \left( E^+_\lambda(+\infty), E^-_\lambda(-\infty), \lambda \in [0, \bar{\lambda}] \right) = 0$. So, by the Morse index formula, we get $m^-(u) = \iota(u)$.

- **Case 2** (as shown in Figure 1b). If $\sqrt{P_\pm R_\pm + Q_\pm} < 0$ , $Q_+ > 0$ and $Q_- < 0$, it is easy to see that the line $L^+_0$ (resp. $L^-_0$) lies on the right (resp. left) half-plane bounded by the $y$-axis. By the above discussion we get that the coincidence times between $L^+_\lambda$ and $L^-_\lambda$ is 1 since $L^+_\lambda$ and $L^-_\lambda$ will overlap just once as $\lambda \to +\infty$. Thus, we get $\ell^{\text{CLM}} \left( E^+_\lambda(+\infty), E^-_\lambda(-\infty), \lambda \in [0, \bar{\lambda}] \right) = -1$ and $m^-(u) = \iota(u) + 1$.

Notation
For the sake of the reader, let us introduce some common notations that we shall use henceforth throughout the paper.

- $\mathbb{R} := \mathbb{R} \cup \{\infty, +\infty\}$, $\mathbb{R}^+ := [0, +\infty)$, $\mathbb{R}^- := (-\infty, 0)$. The pair $(\mathbb{R}^n, (\cdot, \cdot))$ denotes the $n$-dimensional Euclidean space
• # stands for denoting the derivative of # with respect to the time variable t

• I_{X}$ or just $I$ will denote the identity operator on a space $X$ and we set for simplicity $I_{k} := I_{k,k}$ for $k \in \mathbb{N}$

• $TR^{n} \simeq R^{n} \times R^{n}$ denotes the tangent of $R^{n}$ and $T^{*}R^{n} \simeq R^{n} \times R^{n}$ the cotangent of $R^{n}$. $\omega$ stands for the standard symplectic form and the pair $(T^{*}R^{n},\omega)$ denotes the standard symplectic space. $J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ denotes the standard symplectic matrix and $\omega(u,v) = (Ju,v)$.

• $L(n)$ denotes the Lagrangian Grassmannian manifold. $L_{D} := R^{n} \times \{0\}$ and $L_{N} = \{0\} \times R^{n}$ and we refer to as Dirichlet and Neumann Lagrangian subspace.

• $\text{Mat}(n, \mathbb{R})$ the set of all $n \times n$ matrices; $\text{Sym}(n)$ the set of all $n \times n$ symmetric matrices, $\text{Sym}^{-}(n)$ the set of all $n \times n$ positive definite and symmetric matrices. $V^{+}$ and $V^{-}$ denotes the positive and negative spectral spaces, respectively. $E^{+}, E^{-}$ the stable and unstable space respectively.

• Given the linear subspaces $L_{0}, L_{1}$ we write $L_{0} \oplus L_{1}$ meaning that $L_{0} \cap L_{1} = \{0\}$.

• $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ denotes a real separable Hilbert space. $\mathcal{F}(\mathcal{H})$ denotes the Banach space of all bounded and linear operators. $C^{\infty}(\mathcal{H})$ be the set of all (closed) densely defined and selfadjoint operators. We denote by $\mathcal{F}_{sa}(\mathcal{H})$ the space of all closed selfadjoint and Fredholm operators equipped with the gap topology. $\sigma(#)$ denotes the spectrum of the linear operator #. $\text{rge}(#)$ stands for the range of selfadjoint Fredholm operators #. $\text{rge}(#)$ denotes the spectral flow of the path of selfadjoint Fredholm operators #. $\text{rge}(#)$ of the path of selfadjoint Fredholm operators.

• $\omega(J, \mathbb{R})$ denotes the standard symplectic space.

• $\iota^{\text{CLM}}$ denotes the Maslov index of a pair of Lagrangian paths. $\iota(#) \ i(#)$ denotes the triple index. $\iota(u)$ (resp. $\iota^{-}$) are the geometrical indices for heteroclinic (resp. future or past halfclinic orbit) $u$.

• $\mathcal{F}_{m}^{\pm}(\#)$ are the geometrical indices for heteroclinic (resp. future or past halfclinic orbit) $u$.

• $\mathcal{F}_{m}^{\pm}$ the minimal operators associated to $\mathcal{F}$ and $\mathcal{F}^{\pm}$, respectively.

• $\mathcal{F}_{m}^{\pm}$ the minimal operators associated to $\mathcal{F}$ and $\mathcal{F}^{\pm}$, respectively.

2 Fredholmness, hyperbolicity and spectral flows

This section is devoted to introduce the main ingredients behind the spectral flow formulas and the Morse index theorem. We refer to Appendix D for some technical details.

We start with the following classical results which relates the Fredholmness properties of the operator $\mathcal{A}$ and $\mathcal{A}_{L_{0}}^{\pm}$ to the hyperbolicity of the limit matrices.

**Lemma 2.1.** The operator $\mathcal{A}_{L_{0}}^{\pm}$ (resp. $\mathcal{A}_{L_{0}}^{-}$) is Fredholm if and only if the matrix $JB(+\infty)$ (resp. $JB(-\infty)$) is hyperbolic.

**Proof.** From [Kat80, Chapter IV, Theorem 5.35], Lemma D.7 and Corollary D.9, we get that $\mathcal{F}_{m}^{+}$ is Fredholm if and only if the matrix $JB(+\infty)$ is hyperbolic. The conclusion now follows by invoking Lemma D.6 and Lemma D.2.

We now consider the operator $\tilde{\mathcal{A}}_{L_{0}} := \mathcal{A}_{L_{0}}^{-} \oplus \mathcal{A}_{L_{0}}^{+}$ on dom $\tilde{\mathcal{A}}_{L_{0}} := \text{dom}\mathcal{A}_{L_{0}}^{-} \oplus \text{dom}\mathcal{A}_{L_{0}}^{+} \subset L^{2}(\mathbb{R}^{-}, \mathbb{R}^{n}) \oplus L^{2}(\mathbb{R}^{+}, \mathbb{R}^{n})$ and we set

$$\tilde{E} = \left\{ (u,v) \in W^{2,2}(\mathbb{R}^{-}, \mathbb{R}^{n}) \oplus W^{2,2}(\mathbb{R}^{+}, \mathbb{R}^{n}) \mid \begin{pmatrix} \tilde{u}(0) \\ \tilde{v}(0) \end{pmatrix} = \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} \right\}.$$  

We observe that $\mathcal{A}$ is the restriction of $\tilde{\mathcal{A}}_{L_{0}}$ on $\tilde{E}$.

**Lemma 2.2.** The operator $\mathcal{A}$ is Fredholm if and only if $JB(\pm)$ are both hyperbolic.

**Proof.** First, we assume that the operator $\mathcal{A}$ is Fredholm. Since $\mathcal{A} = \tilde{\mathcal{A}}_{L_{0}}|_{E}$, then we get

$$\text{codim}\mathcal{A}_{L_{0}}^{+} + \text{codim}\mathcal{A}_{L_{0}}^{-} = \text{codim}\tilde{\mathcal{A}}_{L_{0}} \leq \text{codim}\mathcal{A} < +\infty.$$  

Then $\text{codim}\mathcal{A}_{L_{0}}^{+}$ and $\text{codim}\mathcal{A}_{L_{0}}^{-}$ are both finite. By taking into account Lemma D.1, $\text{rge}\mathcal{A}_{L_{0}}^{\pm}$ are closed, so $\mathcal{A}_{L_{0}}^{\pm}$ are Fredholm operators. By taking into account Lemma 2.1, we get that $JB(\pm)$ are hyperbolic. Conversely, if $JB(\pm)$ are hyperbolic, then from [RS95], we get that $\mathcal{A}$ is a Fredholm operator.
The following result gives a characterization of the Fredholmness of $A$ (resp. $A^\pm$) in terms of condition (H1).

**Proposition 2.3.** Under the above notation we get that

$A$ is Fredholm $\iff$ $F$ is Fredholm $\iff$ $JB(\pm\infty)$ are both hyperbolic

$A^\pm_{L_0}$ is Fredholm $\iff$ $F^\pm_{L_0}$ is Fredholm $\iff$ $JB(\pm\infty)$ are hyperbolic.

**Proof.** The proof of the equivalence between the hyperbolicity and the Fredholmness of the operators $A$ and $A^\pm_{L_0}$ is a direct consequence of Lemma 2.1 and Lemma 2.2. The equivalence between the Fredholmness of $A$ and $F$ has been proved in [RS95] whilst the equivalence between the Fredholmness of $A^\pm_{L_0}$ and $F^\pm_{L_0}$ has been done in [RS05a, RS05b].

In the next results we will construct suitable deformations of the Hamiltonian boundary value problem that together with the stratum homotopy property of the spectral flow and the Maslov index greatly simplify the proof of the main results.

So, let us start to consider the continuous path $[0,1] \ni \lambda \mapsto R_{\lambda}(t) \in \text{Sym}(n)$ and we define the one parameter family of Hamiltonian systems, defined as follows

$$\dot{z} = JB_{\lambda}(t)z \quad \text{for} \quad B_{\lambda}(t) := \begin{pmatrix} P^{-1}(t) & -P^{-1}(t)Q(t) \\ -Q(t)P^{-1}(t) & Q(t)P^{-1}(t)Q(t) - R_{\lambda}(t) \end{pmatrix}$$

and we set $H_{\lambda}(t) := JB_{\lambda}(t)$.

**Notation 2.1.** We denote by $A_{\lambda}, A^\pm_{L_0,\lambda}, F_{\lambda}, F^\pm_{L_0,\lambda}$ the operators defined above once replacing the matrix $R$ by $R_{\lambda}$.

We introduce the following condition.

**H3** There exist two continuous paths of hyperbolic Hamiltonian matrices, namely $\lambda \mapsto H_{\lambda}(+\infty)$ and $\lambda \mapsto H_{\lambda}(-\infty)$ such that

$$H_{\lambda}(+\infty) = \lim_{t \to +\infty} JB_{\lambda}(t) \quad \text{and} \quad H_{\lambda}(-\infty) = \lim_{t \to -\infty} JB_{\lambda}(t)$$

uniformly with respect to $\lambda$.

By condition (H3) and by Proposition 2.3 we get that $A_{\lambda}, A^\pm_{L_0,\lambda}, F_{\lambda}, F^\pm_{L_0,\lambda}$ are Fredholm operators. Moreover, it is also well-known that they are also closed and selfadjoint with dense domain in $L^2$ and in particular, it is possible to associate to each of these paths the spectral flow. The following result holds.

**Proposition 2.5.** If condition (H3) holds, then we get the following equalities

$$\text{Sf} \ (A_{\lambda}; \lambda \in [0,1]) = \text{Sf} \ (F_{\lambda}; \lambda \in [0,1])$$
$$\text{Sf} \ (A^+_L, A^-_{L_0,\lambda}; \lambda \in [0,1]) = \text{Sf} \ (F^+_L, F^-_{L_0,\lambda}; \lambda \in [0,1])$$
$$\text{Sf} \ (A^+_{L_0,\lambda}; \lambda \in [0,1]) = \text{Sf} \ (F^+_{L_0,\lambda}; \lambda \in [0,1]). \quad (2.1)$$

**Proof.** We only prove the first equality in Equation (2.1) being the proof of the others completely similar. We start by introducing the continuous map

$$f : \mathcal{C} \mathcal{F}^{sa} (L^2(\mathbb{R}, \mathbb{R}^n)) \to \mathcal{C} \mathcal{F}^{sa} (L^2(\mathbb{R}, \mathbb{R}^{2n})) \text{ defined by } f(A_{\lambda}) := F_{\lambda}.$$ 

We let $h(\lambda, s) = f(A_{\lambda} + sI)$ for $(\lambda, s) \in [0,1] \times [0,\delta]$ and we observe that for every $\lambda \in [0,1]$, $s \mapsto h(\lambda, s)$ is a positive curve (Cfr. Definition 2.2 for further details). Let $\lambda_0 \in [0,1]$ be a crossing instant for the path $\lambda \mapsto A_{\lambda}$ and let us consider the positive path $s \mapsto A_{\lambda_0} + sI$. Since 0 is an isolated eigenvalue in the spectrum, then there exists $\delta > 0$ such that $\ker (A_{\lambda_0} + \delta I) = \{0\}$ or which is equivalent to $\ker h(\lambda_0, \delta) = \{0\}$. Now, since the set of selfadjoint invertible Fredholm operators is open, then there exists $\delta_1 > 0$ such that $\ker (A_{\lambda_0} + \delta I) = \{0\}$ for every $\lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]$. By this argument, we get that $\ker h(\lambda, \delta) = \{0\}$ for every $\lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]$. Then, we have

$$\text{Sf} \ (A_{\lambda_0} + \delta I, \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) = 0 \quad \Rightarrow \quad \text{Sf} \ (h(\lambda, \delta), \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) = 0.$$
By the homotopy invariance of the spectral flow, we get that
\[
\text{Sf} (A_\lambda, \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) = \text{Sf} (A_{\lambda_0 - \delta_1} + sI, s \in [0, \delta]) - \text{Sf} (A_{\lambda_0 + \delta_1} + sI, s \in [0, \delta]) \quad (2.2)
\]
and so
\[
\text{Sf} (h(\lambda, 0), \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) = \text{Sf} (h(\lambda_0 - \delta_1, s), s \in [0, \delta]) - \text{Sf} (h(\lambda_0 - \delta_1, s), s \in [0, \delta]) \quad (2.3)
\]
We now observe that \( s \rightarrow A_{\lambda_0 \pm \delta_1} + sI \) and \( s \rightarrow h(\lambda_0 \pm \delta_1, s) \) are both positive curves. Thus, we get
\[
\text{Sf} (A_{\lambda_0 \pm \delta_1} + sI, s \in [0, \delta]) = \sum_{0 < s \leq \delta} \text{dim ker} (A_{\lambda_0 \pm \delta_1} + sI) = \sum_{0 < s \leq \delta} \text{dim ker} h(\lambda_0 \pm \delta_1, s) = \text{Sf} (h(\lambda_0 \pm \delta_1, s), s \in [0, \delta]) \quad (2.4)
\]
By taking into account Equation (2.2), Equation (2.3) and finally Equation (2.4), we conclude that
\[
\text{Sf} (A_\lambda, \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) = \text{Sf} (h(\lambda, 0), \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) = \text{Sf} (\mathcal{F}_\lambda, \lambda \in [\lambda_0 - \delta_1, \lambda_0 + \delta_1]) \quad (2.5)
\]
Equation (2.5) together with the concatenation property of the spectral flow concludes the proof. \( \square \)

### 2.1 Spectral flow formulas

Borrowing the notation of [HP17] for \( \lambda \in [0, 1] \) we denote by \( \gamma_{(\tau, \lambda)} \) be the (primary) fundamental solution of the following linear Hamiltonian system
\[
\begin{cases}
\dot{\gamma}(t) = J B_\lambda(t) \gamma(t), & t \in \mathbb{R} \\
\gamma(\tau) = I
\end{cases}
\]
and we define, respectively, the stable and unstable subspaces as follows
\[
E^s_\lambda(\tau) := \left\{ v \in \mathbb{R}^{2n} \mid \lim_{t \to +\infty} \gamma_{(\tau, \lambda)}(t) v = 0 \right\} \quad \text{and} \quad E^u_\lambda(\tau) := \left\{ v \in \mathbb{R}^{2n} \mid \lim_{t \to -\infty} \gamma_{(\tau, \lambda)}(t) v = 0 \right\}.
\]

We observe that, for every \((\lambda, \tau) \in [0, 1] \times \mathbb{R}, E^s_\lambda(\tau), E^u_\lambda(\tau) \in L(n)\). (For further details, we refer the interested reader to [CH07, HP17] and references therein). Setting
\[
E^s_\lambda(+\infty) := \left\{ v \in \mathbb{R}^{2n} \mid \lim_{t \to +\infty} \exp(t B_\lambda(+\infty)) v = 0 \right\},
\]
\[
E^u_\lambda(-\infty) := \left\{ v \in \mathbb{R}^{2n} \mid \lim_{t \to -\infty} \exp(t B_\lambda(-\infty)) v = 0 \right\}
\]
and assuming that condition (H3) holds, then we get that
\[
\lim_{\tau \to +\infty} E^s_\lambda(\tau) = E^s_\lambda(+\infty) \quad \text{and} \quad \lim_{\tau \to -\infty} E^u_\lambda(\tau) = E^u_\lambda(-\infty)
\]
where the convergence is meant in the gap (norm) topology of the Lagrangian Grassmannian. (Cfr. [AM03] for further details).

**Remark 2.6.** Let \( \gamma \) be the fundamental matrix of equation \( \dot{z} = J B(t)z \). Assume that condition (H1) holds. Then each solution \( x \) of \( \mathcal{F}_M^+x = 0 \) decay exponentially fast as \( t \to +\infty \). So, if \( v \in \mathbb{R}^{2n} \), then \( t \to \gamma(t)v \) is a solution of the system \( \mathcal{F}_M^+x = 0 \) if and only if \( v \in E^s(0) \).

A similar argument holds for each solution \( x \) of the system \( \mathcal{F}_M^-x = 0 \), \( x(0) \in E^u(0) \) and each \( v \in E^u(0) \) determines the solution through \( x(0) \). In particular, we get also that \( \ker \mathcal{F}_M^+ \) and \( \ker \mathcal{F}_M^- \) are determined by \( E^s(0) \cap L_0 \) and \( E^u(0) \cap L_0 \) respectively. Starting by this observation, authors in [HP17] proved some general spectral flow formulas as summarized in the following result.
Proposition 2.7. Under the previous notation and if condition (H3) holds, then we get:

$$\text{Sf}(A_{\lambda} ; \lambda \in [0, 1]) = \epsilon^{\text{CLM}} (E_{1}^{*}(\tau), E_{1}^{\dagger}(\tau) ; \tau \in \mathbb{R}^{+}) - \epsilon^{\text{CLM}} (E_{0}^{*}(\tau), E_{0}^{\dagger}(\tau) ; \tau \in \mathbb{R}^{+})$$

and

$$\text{Sf}(A_{\lambda}^{\dagger} ; \lambda \in [0, 1]) = \epsilon^{\text{CLM}} (L_{0}, E_{0}^{*}(-\tau) ; \tau \in \mathbb{R}^{+}) - \epsilon^{\text{CLM}} (L_{0}, E_{0}^{\dagger}(-\tau) ; \tau \in \mathbb{R}^{+})$$

(2.6)

$$\text{Sf}(A_{\lambda}^{\dagger} ; \lambda \in [0, 1]) = \epsilon^{\text{CLM}} (L_{0}, E_{0}^{*}(\tau) ; \tau \in \mathbb{R}^{+}) - \epsilon^{\text{CLM}} (L_{0}, E_{0}^{\dagger}(\tau) ; \tau \in \mathbb{R}^{+})$$

(2.7)

Proof. The proof directly follows by [HP17, Theorem 1] and by Proposition 2.5.

Remark 2.8. We now assume that conditions (L1), (H1) and (H2) hold and let us consider the path \( \lambda \mapsto A_{\lambda} := A + \lambda I \). As proved in Section D, the path \( \lambda \mapsto A_{\lambda} \) of selfadjoint Fredholm operators is positive and there exists \( \lambda \in \mathbb{R}^{+} \) such that ker \( A_{\lambda} \) = 0 for \( \lambda \geq \lambda \). It is well-known that for positive paths of closed selfadjoint Fredholm operators, the spectral flow is equal to the difference of the Morse index of the operator at the starting instant \( \lambda = 0 \) minus the Morse index of the operator at the final instant \( \lambda = \lambda \). Thus, we get

\[
\text{m}^{-}(A) = \text{Sf}(A_{\lambda}, \lambda \in [0, \lambda]), \quad \text{and} \quad \text{m}^{+}(A_{\lambda}^{\dagger}) = \text{Sf}(A_{\lambda}^{\dagger}, \lambda \in [0, \lambda]).
\]

3 Transversality between invariant subspaces

The aim of this section is to give some sufficient conditions on the coefficients of the Sturm-Liouville operators in order the get the corresponding differential operators being nondegenerate. This conditions are crucial in proving the well-posedness of the indices.

3.1 Transversality for heteroclinics

Lemma 3.1. We assume condition (L1) & (L2) are fulfilled. Then \( K_{\lambda}(t) := \begin{pmatrix} P(t) & Q(t) \\ Q(t)^{T} & R(t) + \lambda I \end{pmatrix} \) is positive definite for all \( (t, \lambda) \in \mathbb{R} \times \left[ \frac{2c_{2}}{c_{1}} + C_{3}, +\infty \right) \) where \( C_{1}, C_{2}, C_{3} \) are the same constants defined at condition (L2).

Proof. Let \( \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{2n} \). As direct consequence of condition (L2) and the Cauchy-Schwarz inequality, we get that

\[
\begin{pmatrix} u \\ v \end{pmatrix}^{T} \begin{pmatrix} P(t) & Q(t) \\ Q(t)^{T} & R(t) + \lambda I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\
= \langle P(t)u, u \rangle + 2 \langle Q(t)v, u \rangle + \langle R(t)v, v \rangle + \lambda \| v \|^{2} \\
\geq C_{1} \| u \|^{2} - 2 C_{2} \left( \varepsilon \| u \|^{2} + \frac{1}{\varepsilon} \| v \|^{2} \right) - C_{3} \| v \|^{2} + \lambda \| v \|^{2} \\
= (C_{1} - 2 \varepsilon C_{2}) \| u \|^{2} + \left( \lambda - C_{3} - \frac{2}{\varepsilon} C_{2} \right) \| v \|^{2} \quad (3.1)
\]

We let \( \varepsilon := \frac{C_{1}}{4C_{2}} \) and we observe that if \( \lambda \geq \frac{8C_{2}^{2}}{C_{1}} + C_{3} \), by Equation (3.1), we get that

\[
\begin{pmatrix} u \\ v \end{pmatrix}^{T} \begin{pmatrix} P(t) & Q(t) \\ Q(t)^{T} & R(t) + \lambda I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} > 0 \quad \text{for all} \quad 0 \neq \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^{2n}
\]

and this concludes the proof.  \( \Box \)
Lemma 3.2. We assume conditions (L1)-(L2) hold and let $D$ be a positive constant. Then we get that the matrix
\[
\begin{pmatrix}
P(t) & Q(t) + DI \\
Q(t)^T + DI & R(t) + \lambda I
\end{pmatrix}
\]
is positive definite for every $(t, \lambda) \in \mathbb{R} \times \left[ 2(C_2 + C_3)^2 + C_3, +\infty \right)$.

Proof. The proof of this result readily follows by arguing precisely as in Lemma 3.1. \qed

For any fixed $s \in \mathbb{R}^+$, let us introduce the following closed operators in $L^2(\mathbb{R}^+, \mathbb{R}^n)$ having dense domain $W^{2,2}(\mathbb{R}^+, \mathbb{R}^n)$, as follows:
\[
A_{s,M} = -\frac{d}{dt} \left[ P(t + s) \frac{d}{dt} Q(t + s) \right] + Q^T(t + s) \frac{d}{dt} + R_\lambda(t + s)
\]
where $R_\lambda(t) = R(t) + \lambda I$. The next result provides a lower bound on $\lambda$ for the non-invertibility of the operators $A_{s,M}$.

Lemma 3.3. We assume conditions (L1)-(L2) hold. If $\lambda \geq \frac{2C_2^2}{C_1} + C_3$, then the system
\[
\begin{cases}
A_{s,M}^+ x_1(t) = 0 \\
A_{s,M}^- x_2(t) = 0
\end{cases}
\]
\[
x_1(0) = x_2(0), \quad P(s)x_1(0) + Q(s)x_1(0) = P(-s)x_2(0) + Q(-s)x_2(0)
\]

admits only the trivial solution.

Proof. Arguing by contradiction we assume that $t \mapsto (x_1(t), x_2(t))$ is a non-trivial solution of the system given at Equation (3.2). Then, we have
\[
\langle A_{s,M}^+ x_1(t), x_1(t) \rangle_{L^2} + \langle A_{s,M}^- x_2(t), x_2(t) \rangle_{L^2} = 0.
\]

Integrating by part, we get
\[
\langle A_{s,M}^+ x_1(t), x_1(t) \rangle_{L^2} = I_1 + \langle P(s)\ddot{x}_1(0) + Q(s)x_1(0), x_1(0) \rangle
\]
\[
\langle A_{s,M}^- x_2(t), x_2(t) \rangle_{L^2} = I_2 - \langle P(-s)\dot{x}_2(0) + Q(-s)x_2(0), x_2(0) \rangle
\]

where
\[
I_1 = \langle P(t + s)\ddot{x}_1(t), \dot{x}_1(t) \rangle_{L^2} + \langle Q(t + s)x_1(t), \dot{x}_1(t) \rangle_{L^2}
\]
\[
+ \langle Q^T(t + s)\dot{x}_1(t), x_1(t) \rangle_{L^2} + \langle R_\lambda(t + s)x_1(t), x_1(t) \rangle_{L^2}
\]
and
\[
I_2 = \langle P(t - s)\dot{x}_2(t), \ddot{x}_2(t) \rangle_{L^2} + \langle Q(t - s)x_2(t), \dot{x}_2(t) \rangle_{L^2}
\]
\[
+ \langle Q^T(t - s)\ddot{x}_2(t), x_2(t) \rangle_{L^2} + \langle R_\lambda(t - s)x_2(t), x_2(t) \rangle_{L^2}.
\]

By using the second boundary condition given at Equation 3.2, we get
\[
I_1 + I_2 = 0.
\]

Setting $K_\lambda(t + s) = \begin{pmatrix} P(t + s) & Q(t + s) \\ Q(t + s)^T & R(t + s) + \lambda I \end{pmatrix}$ and by using Lemma 3.1, we get that for $\lambda \geq \frac{2C_2^2}{C_1} + C_3$ the matrices are both positive definite and the following holds
\[
I_1 = \int_0^\infty \langle \ddot{x}_1(t), x_1(t) \rangle_{L^2} \left( K_\lambda(t + s) \right) dt
\]
and
\[
I_2 = \int_0^\infty \langle \ddot{x}_2(t), x_2(t) \rangle_{L^2} \left( K_\lambda(t - s) \right) dt.
\]

By this computation, we infer that $I_1 + I_2 = 0$ if and only if $\dot{x}_i(t) = x_i(t) = 0$ for $i = 1, 2$ and for every $t \in [0, +\infty)$ concluding the proof. \qed
Corollary 3.4. Assuming condition (L2), the operator $A_{L_0,\lambda}$ is non-degenerate for every $\lambda \geq \frac{2C_2^2}{C_1} + C_3$.

Let us now consider the first order differential operators $F_{s,M}^{\pm,\lambda}$ associated to $A_{s,M}^{\pm,\lambda}$. By arguing precisely as in Lemma 3.3 the following result holds.

Lemma 3.5. We assume that condition (L2) is fulfilled and $\lambda \geq \frac{2C_2^2}{C_1} + C_3$. Then, the initial value problem
\[
\begin{cases}
F_{s,M}^{+,\lambda} z_1(t) = 0 \\
F_{s,M}^{-,\lambda} z_2(t) = 0 \\
z_1(0) = z_2(0)
\end{cases}
\]

admits only the trivial solution.

Proposition 3.6. Under the assumption (L2), the following transversality condition holds:
\[
E_\lambda^\pm(-\tau) \cap E_\lambda^\pm(\tau) = \{0\} \quad \text{for all} \quad (\tau, \lambda) \in \mathbb{R}^+ \times \left[\frac{2C_2^2}{C_1} + C_3, +\infty\right).
\]

Proof. The stable subspace of the equation $F_{s,M}^{+,\lambda}$ at 0 is $E_\lambda^+(\tau)$ and the unstable subspace of the equation $F_{s,M}^{-,\lambda}$ at 0 is $E_\lambda^-(\tau)$. As already observed in Remark 2.6, there exists a linear bijection from the set of solutions of the system
\[
\begin{cases}
F_{s,M}^{+,\lambda} z_1(t) = 0 = F_{s,M}^{-,\lambda} z_2(t) \\
z_1(0) = z_2(0)
\end{cases}
\]

with the subspace $E_\lambda^+(\tau) \cap E_\lambda^-(\tau)$. By invoking Lemma 3.5, we conclude that the initial value problem only admits the trivial solution for every $\lambda \geq \frac{2C_2^2}{C_1} + C_3$. So $E_\lambda^+(\tau) \cap E_\lambda^-(\tau) = \{0\}$ concluding the proof.

So, the conclusion of Proposition 3.6 is that as soon as $\lambda$ is sufficiently large then $E_\lambda^+(\tau)$ and $E_\lambda^-(\tau)$ meet transversally.

3.2 Transversality for Halfclinic case

As we will show, a similar result holds also in the halfclinic case. In fact, let now consider boundary value problems on the half-line with selfadjoint boundary condition. Let $(y, x)^T \in L_0$ and denoting by $L_N := \{0\} \times \mathbb{R}^n$ the Neumann Lagrangian, we define the following subspace
\[
V(L_0) := (L_0 + L_D) \cap L_N \subset \mathbb{R}^n \times \mathbb{R}^n
\]
and we observe that the elements of $V(L_0)$ are of the form $(0, x)$ for some $x \in \mathbb{R}^n$. By using the decomposition $\mathbb{R}^n = V(L_0) \oplus V^\perp(L_0)$, then we have $y = y_1 + y_2$, where $y_1 \in V(L_0)$ and $y_2 \in V^\perp(L_0)$. By choosing a basis in $V(L_0)$, we define the matrix $A$ acting on vectors of $V(L_0)$ such that $y_1 = Ax$, since $L_0$ is a Lagrangian subspace, it is easy to see that $A$ is symmetric. Then we have
\[
\langle y, x \rangle = \langle y_1, x \rangle = \langle Ax, x \rangle \quad \forall (y, x) \in L_0.
\]
In particular, there exists a positive constant $C_0$ such that
\[
|\langle Ax, x \rangle| \leq C_0 |x|^2 \quad \forall (0, x) \in V(L_0) \quad x \in \mathbb{R}^n.
\]
Then
\[
|\langle y, x \rangle| = |\langle Ax, x \rangle| \leq C_0 |x|^2.
\]
The following result gives a sufficient condition on the matrix $K_\lambda(t)$ to be positive definite.

The next result is analogous of Lemma 3.3 and provides a lower bound on $\lambda$ in order the operator $A_{L_0,\lambda}^{\pm}$ is non-degenerate.
Lemma 3.7. If condition (L2) holds, then the matrix $A_{L_{0}, \lambda}^{\pm} \pm$ is non-degenerate for every $\lambda \geq \frac{2(C_{2} + C_{0})^{2}}{C_{1}} + C_{0}$ where $C_{0} > 0$ is the constant defined in Equation (3.3).

Proof. We only prove the claim for the operator $A_{L_{0}, \lambda}^{+}$ being the proof of the claim for $A_{L_{0}, \lambda}^{-}$ completely similar. We start by observing that $A_{L_{0}, \lambda}^{+} x = 0$ if and only if $x$ is solution of the boundary value problem

$$\begin{cases}
A_{M, \lambda}^{+} x = 0 \\
(P(0) \dot{x}(0) + Q(0) x(0), x(0))^{T} \in L_{0}
\end{cases}$$

where

$$A_{M, \lambda}^{+} := -\frac{d}{dt} \left[ P(t) \frac{d}{dt} + Q(t) \right] + Q^{T}(t) \frac{d}{dt} + R(t) + \lambda I : W^{2,2}(\mathbb{R}^{+}, \mathbb{R}^{n}) \subset L^{2}(\mathbb{R}, \mathbb{R}^{n}) \rightarrow L^{2}(\mathbb{R}, \mathbb{R}^{n})$$

By a direct integration by parts and by using condition (L2) as well as the inequality given at Equation (3.4), we get

$$\langle A_{M, \lambda}^{+}(t), x(t) \rangle_{L^{2}} = (P(t) \dot{x}(t), \dot{x}(t))_{L^{2}} + \langle Q(t) x(t), \dot{x}(t) \rangle_{L^{2}} + \langle Q(t) \dot{x}(t), x(t) \rangle_{L^{2}} + \langle Q(t) x(t), \dot{x}(t) \rangle_{L^{2}} + \langle Q(t) \dot{x}(t), x(t) \rangle_{L^{2}} + \langle (R(t) + \lambda I) x(t), x(t) \rangle_{L^{2}} - C_{0} |x(0)|^{2}.$$  

Since $x \in \text{dom}(A_{M, \lambda}^{+})$, we infer also that

$$|x(0)|^{2} = -\int_{0}^{+\infty} \frac{d}{dt} |x|^{2} dt = -2 \langle \dot{x}, x \rangle_{L^{2}}. \quad (3.5)$$

Moreover, by using Equation (3.5), we get

$$0 = \langle A_{M, \lambda}^{+}(t), x(t) \rangle_{L^{2}} \geq (P(t) \dot{x}(t), \dot{x}(t))_{L^{2}} + \langle Q(t) x(t), \dot{x}(t) \rangle_{L^{2}} + \langle Q(t) \dot{x}(t), x(t) \rangle_{L^{2}} + \langle Q(t) x(t), \dot{x}(t) \rangle_{L^{2}} + \langle (R(t) + \lambda I) x(t), x(t) \rangle_{L^{2}} - C_{0} |x(0)|^{2}$$

$$= \int_{0}^{+\infty} \left[ \langle \dot{x}, \dot{x} \rangle + \left( P(t) \frac{d}{dt} + Q(t) \right) + \left( Q(t) \frac{d}{dt} + C_{0} \right) + \langle R(t) + \lambda I, \dot{x} \rangle \right] dt > 0$$

where the last inequality directly follows by Lemma 3.2 concluding the proof.

By arguing precisely as in Proposition 3.6 and by using Lemma 3.7, the following result holds.

Lemma 3.8. We assume conditions (L1) and (L2) hold. Then we have

$$E_{\lambda}^{s}(\tau) \cap L_{0} = \{0\} \quad \text{and} \quad E_{\lambda}^{s}(\tau) \cap L_{0} = \{0\} \quad \forall (\tau, \lambda) \in \mathbb{R}^{+} \times \left[ \frac{2(C_{2} + C_{0})^{2}}{C_{1}} + C_{3}, +\infty \right].$$

3.3 Computation of the $t^{\text{CLM}}$-index of the (un)stable paths at infinity

The aim of this section is to explicitly compute the index

$$t^{\text{CLM}}(E_{\lambda}^{s}(+\infty), E_{\lambda}^{u}(-\infty); \lambda \in [0, \hat{\lambda}]).$$

As direct consequence of Lemma A.6 this computation can be performed by computing the following triple indices

$$t(E_{\lambda}^{s}(-\infty), E_{\lambda}^{s}(+\infty); L_{D}) \quad \text{and} \quad t(E_{\lambda}^{u}(-\infty), E_{\lambda}^{u}(+\infty); L_{D}).$$

Since $E_{\lambda}^{s}(+\infty)$ is transversal to $L_{D}$, then $E_{\lambda}^{s}(-\infty) \cap (E_{\lambda}^{s}(+\infty) \oplus L_{D}) = E_{\lambda}^{s}(-\infty)$ and we let

$$\begin{pmatrix} N_{\lambda}^{u} u \\ u \end{pmatrix} \in E_{\lambda}^{u}(\infty)$$

where $N_{\lambda}$ is a symmetric matrix.
Setting
\[
\begin{pmatrix}
N \lambda u \\
u
\end{pmatrix} = \begin{pmatrix} M \lambda u \\
N \lambda u - M \lambda u \end{pmatrix}
\]
by a direct calculation and by taking into account of Equation (A.3), we conclude that
\[
Q (E^\alpha_{\lambda}(-\infty), E^\alpha_{\lambda}(+\infty), L_B) \left( \begin{pmatrix} N \lambda u \\
u \end{pmatrix}, \begin{pmatrix} (N \lambda - M \lambda) u \\
0 \end{pmatrix} \right)
= \left( \begin{pmatrix} 0 \\
-I \end{pmatrix} \begin{pmatrix} M \lambda u \\
u \end{pmatrix}, \begin{pmatrix} (N \lambda - M \lambda) u \\
0 \end{pmatrix} \right)
= \left( \begin{pmatrix} -u \\
M \lambda u \end{pmatrix}, \begin{pmatrix} (N \lambda - M \lambda) u \\
0 \end{pmatrix} \right) = \langle (M \lambda - N \lambda) u, u \rangle. \tag{3.6}
\]

We now consider the operator \( J^+_{\lambda} := -J \frac{d}{dt} - B_{\lambda}(+\infty) \) and the associated second order differential operator \( A^+_{\lambda} \) and let \( x \in \ker A^+_{\lambda,M} \) where \( A^+_{\lambda,M} \) denotes the corresponding maximal operator. Then the map \( x \mapsto (P_+ \tilde{x}(0) + Q_+ x(0), x(0)) \) provides a linear bijection between \( \ker A^+_{\lambda,M} \) and \( E^\alpha_{\lambda}(0) = V^-(J B_{\lambda}(+\infty)) \). By a direct calculation, we get
\[
0 = \langle A^+_{\lambda,M} x(t), x(t) \rangle_{L^2} = \langle P_+ \dot{x}(t), \dot{x}(t) \rangle_{L^2} + \langle Q_+ x(t), \dot{x}(t) \rangle_{L^2}
+ \langle Q^+_\lambda \dot{x}(t), x(t) \rangle_{L^2} + ((R_+ + \lambda I)x(t), x(t))_{L^2} + (P_+ \dot{x}(0) + Q_+ x(0), x(0))
= \langle P_+ \dot{x}(t), \dot{x}(t) \rangle_{L^2} + \langle Q_+ x(t), \dot{x}(t) \rangle_{L^2} + \langle Q^+_\lambda \dot{x}(t), x(t) \rangle_{L^2}
+ ((R_+ + \lambda I)x(t), x(t))_{L^2} + \langle M_\lambda x(0), x(0) \rangle.
\]

Let \( x \in \ker A^+_{\lambda,M} \) be the solution with \( x(0) = v \). Then, we get
\[
\langle M_\lambda v, v \rangle = -\int_0^{\infty} \left< K_\lambda \left( \frac{\dot{x}(t)}{x(t)} \right), \left( \frac{\dot{x}(t)}{x(t)} \right) \right> dt \quad \text{where} \quad K_\lambda = \begin{pmatrix} P_+ & Q_+ \\ Q^+_\lambda & R_+ + \lambda I \end{pmatrix}. \tag{3.7}
\]
If \( K_\lambda \) is positive definite, then \( \langle M_\lambda v, v \rangle < 0 \) for each nonzero \( v \in \mathbb{R}^n \). Similarly \( \langle N_\lambda v, v \rangle > 0 \) for each nonzero \( v \in \mathbb{R}^n \).

**Lemma 3.9.** If the condition (H2) holds, then \( M_0 \) is negative and \( N_0 \) is positive definite. Moreover, if also condition (L1) holds, then \( M_\lambda \) and \( N_\lambda \) are respectively negative and positive definite for all \( \lambda \geq \frac{2C_2^2}{C_1} + C_3 \).

**Proof.** The proof is a direct consequence of Lemma 3.1. \( \square \)

Let now fix \( \lambda \in \mathbb{R}^+ \), \( L_0 \) be a Lagrangian subspace and we assume that \( (u, v)^T \in L_0, v \neq 0 \) and let \( x \in \ker A^+_{\lambda,M} \) be the solution such that \( x(0) = v \). From Equation (3.3) and Equation (3.7), we get
\[
\langle M_\lambda v, v \rangle - \langle u, v \rangle \leq -\int_0^{\infty} \left< K_\lambda \left( \frac{\dot{x}(t)}{x(t)} \right), \left( \frac{\dot{x}(t)}{x(t)} \right) \right> dt + C_0 |v|^2
= -\int_0^{\infty} \left< K_\lambda + C_0 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \left( \frac{\dot{x}(t)}{x(t)} \right), \left( \frac{\dot{x}(t)}{x(t)} \right) \right> dt - C_0 \int_0^{\infty} \frac{d}{dt}(x(t), x(t)) dt
= -\int_0^{\infty} \left< \left< K_\lambda + C_0 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \left( \frac{\dot{x}(t)}{x(t)} \right), \left( \frac{\dot{x}(t)}{x(t)} \right) \right> \right> dt.
\]

**Lemma 3.10.** If the condition (L1) \theta (L2) hold and \( (u, v) \in L_0 \) then we have
\[
\langle M_\lambda v, v \rangle - \langle u, v \rangle \leq 0 \quad \text{for all} \quad \lambda \geq \frac{2(C_2^2 + C_0)^2}{C_1} + C_3.
\]

**Proof.** Arguing as in Lemma 3.2, we get that \( K_\lambda + C_0 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \) is positive for all \( \lambda \geq \frac{2(C_2^2 + C_0)^2}{C_1} + C_3. \) \( \square \)
Remark 3.11. We assume that $\lambda \geq \frac{2(C_2 + C_0)^2}{C_4} + C_3$ and let $(u_0) \in L_D \cap (L_0 + E^s_\lambda(+\infty))$. We set
\[
\begin{pmatrix} u' \\ 0 \end{pmatrix} = \begin{pmatrix} -M_\lambda u + u \\ -v \end{pmatrix} + \begin{pmatrix} M_\lambda v \\ v \end{pmatrix} \quad \text{where} \quad \begin{pmatrix} -M_\lambda u + u \\ -v \end{pmatrix} \in L_0.
\]
Then we have
\[
Q(L_D, L_0, E^s_\lambda(+\infty)) \begin{pmatrix} u' \\ 0 \end{pmatrix} = \omega \left( \begin{pmatrix} -M_\lambda u + u \\ -v \end{pmatrix}, \begin{pmatrix} M_\lambda v \\ v \end{pmatrix} \right) = (u, v) = \langle M_\lambda v, v \rangle - \langle M_\lambda v - u, v \rangle.
\]
Since $\langle M_\lambda v, v \rangle < 0$ and since by Lemma 3.10 $\langle M_\lambda v - u, v \rangle < 0$, then we get that
\[
\langle M_\lambda v, v \rangle - \langle M_\lambda v - u, v \rangle \leq 0
\]
and so $Q(L_D, L_0, E^s_\lambda(+\infty))$ is non-positive definite. In particular
\[
m^+(Q(L_D, L_0, E^s_\lambda(+\infty))) = 0.
\]
Similarly we have that
\[
m^+ (Q(E^u_0(-\infty), L_0, L_D)) = 0.
\]
**Corollary 3.12.** If condition (H2) holds, then we get
\[
\iota(E^u_0(-\infty), E^s_0(+\infty); L_D) = 0.
\]
**Proof.** Since condition (H2) holds, by using Equation (3.6) evaluated at $\lambda = 0$ and by Lemma 3.9, we get that
\[
m^+ (Q(E^u_0(-\infty), E^s_0(+\infty); L_D)) = m^+ (M_0 - N_0) = 0.
\]
By this computation, and since of the transversality condition $E^u_0(-\infty) \cap L_D = \{0\}$, by using Equation (A.3) we get
\[
\iota(E^u_0(-\infty), E^s_0(+\infty); L_D) = 0
\]
concluding the proof. \(\square\)

### 4 Proof of the main results

In this section we provide the complete proofs of the results stated in Section 1.

#### 4.1 Proof of Theorem 1

We start by letting $R_\lambda := R + \lambda I$ and by choosing $\hat{\lambda} := \frac{2C_2^2}{C_4} + C_3$. By taking into account Remark 2.8, we get that $m^-(u) = \text{St} (A\lambda; \lambda \in [0, \hat{\lambda}])$ and by Equation (2.6), we infer that
\[
m^-(u) = \iota^{\text{CLM}}(E^u_\lambda(\tau), E^u_\lambda(-\tau); \tau \in \mathbb{R}^+) - \iota^{\text{CLM}}(E^u_\lambda(\tau), E^s_\lambda(-\tau); \tau \in \mathbb{R}^+)
\]
\[
- \iota^{\text{CLM}}(E^s_\lambda(+\infty), E^u_\lambda(-\infty); \lambda \in [0, \hat{\lambda}]) \quad (4.1)
\]
By invoking Lemma A.6, the third term in the (RHS) of Equation (4.1) can be written as follows
\[
\iota^{\text{CLM}}(E^s_\lambda(+\infty), E^u_\lambda(-\infty); \lambda \in [0, \hat{\lambda}]) = \iota \left( E^u_\lambda(-\infty), E^s_\lambda(+\infty); L_D \right) - \iota \left( E^u_\lambda(-\infty), E^s_0(+\infty); L_D \right).
\]
By taking into account Lemma 3.9, Equation (3.6), Equation (A.3) and Lemma 3.8, we finally get that
\[
\iota \left( E^u_\lambda(-\infty), E^s_\lambda(+\infty); L_D \right) = 0
\]
and so, we infer that
\[ \iota^{\text{CLM}} \left( E^{u}_{\lambda}(+\infty), E^{w}_{\lambda}(-\infty); \lambda \in [0, \hat{\lambda}] \right) = -\iota \left( E^{u}_{0}(-\infty), E^{w}_{0}(+\infty); L_{D} \right). \]

Moreover, by using Proposition 3.6, we conclude that the first term in the (RHS) of Equation (4.1) vanishes
\[ \iota^{\text{CLM}} \left( E^{u}_{\lambda}(\tau), E^{w}_{\lambda}(-\tau); \tau \in \mathbb{R}^{+} \right) = 0 \]
Thus, Equation (4.1) reduces to
\[ m^{+}(u) = \iota^{\text{CLM}} \left( E^{u}_{\lambda}(\tau), E^{w}_{\lambda}(-\tau); \tau \in \mathbb{R}^{+} \right) + \iota \left( E^{u}_{0}(-\infty), E^{w}_{0}(+\infty); L_{D} \right) \]
where the last equality follows by Definition 1.7. This concludes the proof.

\[ \square \]

4.2 Proof of Theorem 2

We start by proving Equation (1.4) corresponding to the future halfclinic orbit. We start observing that under the assumptions, we get that $\text{Sf} (A^{+}_{\lambda}, \lambda \in [0, \hat{\lambda}]) = m^{+}(u, L_{0}, +)$. Moreover, by using Equation (2.7), we get the following relation
\[ m^{+}(u, L_{0}, +) = \iota \left( u, L_{0}, + \right) \]
By choosing $\hat{\lambda} := \frac{2(C_{2} + C_{0})^{2}}{C_{1}} + C_{3}$, then we get that the operator $\mathscr{A}^{+}_{\hat{\lambda}}$ satisfies (H2) and
\[ \left( \begin{array}{cc}
P_{+} & Q_{+} + C_{0}I \\
Q_{+}^{T} + C_{0}I & R_{+} + \hat{\lambda}I \end{array} \right) \]
is positive definite. By Lemma 3.8, we get
\[ \iota^{\text{CLM}} \left( E^{u}_{\lambda}(\tau), L_{0}; \tau \in \mathbb{R}^{+} \right) = 0 \]
and by Lemma C.6 we get $E^{u}_{\lambda}(+\infty) \cap L_{D} = \{0\}$ for all $\lambda \geq 0$. This implies that
\[ \iota^{\text{CLM}} \left( E^{u}_{\lambda}(+\infty), L_{D}; \lambda \in [0, \hat{\lambda}] \right) = 0. \]
By taking into account Remark 3.11 we get that
\[ m^{+} \left( Q \left( L_{D}, L_{0}, E^{w}_{\lambda}(+\infty) \right) \right) = 0 \] for all $\lambda \geq \frac{2(C_{2} + C_{0})^{2}}{C_{1}} + C_{3}$, and so by Equation (4.2), Definition A.3, Equation (A.4) and finally Equation (A.3), we get
\[ m^{+}(u, L_{0}, +) = t^{+}_{L_{0}}(u) - \iota^{\text{CLM}} \left( E^{u}_{\lambda}(+\infty), L_{0}; \lambda \in [0, \hat{\lambda}] \right) \]
\[ = t^{+}_{L_{0}}(u) - \left( \iota^{\text{CLM}} \left( E^{u}_{\lambda}(+\infty), L_{0}; \lambda \in [0, \hat{\lambda}] \right) - \iota^{\text{CLM}} \left( E^{u}_{\lambda}(+\infty), L_{D}; \lambda \in [0, \hat{\lambda}] \right) \right) \]
\[ = t^{+}_{L_{0}}(u) - \iota(L_{D}, L_{0}, E^{u}_{\lambda}(+\infty), E^{w}_{\lambda}(+\infty)) \]
\[ = t^{+}_{L_{0}}(u) - \iota(L_{D}, L_{0}, E^{u}_{\lambda}(+\infty)) + \iota(L_{D}, L_{0}, E^{w}_{\lambda}(+\infty)) \]
\[ = t^{+}_{L_{0}}(u) - m^{+} \left( Q \left( L_{D}, L_{0}, E^{w}_{\lambda}(+\infty) \right) \right) - \dim \left( L_{D} \cap E^{w}_{\lambda}(+\infty) \right) \]
\[ + \dim \left( L_{D} \cap L_{0} \cap E^{w}_{\lambda}(+\infty) \right) + \iota \left( L_{D}, L_{0}, E^{w}_{\lambda}(+\infty) \right) \]
\[ = t^{+}_{L_{0}}(u) + \iota(L_{D}, L_{0}, E^{w}_{\lambda}(+\infty)) \]
Arguing as before and by using once again Equation (2.6) and Remark 3.11, we have
\[
m'(u, L_0, -) = \iota_{L_0}^-(u) - \iota_{L_0}^{\text{CLM}} (L_0, E_0^u(-\infty); \lambda \in [0, \hat{\lambda}])
\]
\[
= \iota_{L_0}^-(u) - \iota (E_0^u(-\infty), L_0; L_D) + \iota (E_0^u(-\infty), 0; L_D)
\]
\[
= \iota_{L_0}(u) - m^+ \left( Q \left( E_0^u(-\infty), L_0, L_D \right) \right) + \iota (E_0^u(-\infty), 0; L_D)
\]
\[
= \iota_{L_0}(u) + \iota (E_0^u(-\infty), 0; L_D).
\]
This concludes the proof. \(\square\)

4.3 Proof of Corollary 2

We start by proving Equation (1.6). By invoking Equation (1.4), Definition A.3 and Equation (A.4), we get that
\[
m'(u, L_0, +) - m'(u, L_D, +) = \iota_{L_0}^+(u) + \iota (L_D, L_0, E_0^u(+\infty)) - \iota_{L_D}^+(u)
\]
\[
= - \iota (E_0^u(t), L_0; t \in \mathbb{R}^+) - \iota (E_0^u(t), L_D; t \in \mathbb{R}^+) + \iota (L_D, L_0, E_0^u(+\infty))
\]
\[
= - s (L_D, L_0, E_0^u(0), E_0^u(+\infty)) + \iota (L_D, L_0, E_0^u(0)) + \iota (L_D, L_0, E_0^u(+\infty))
\]
\[
= \iota (L_D, L_0, E_0^u(0)).
\]
We finally prove Equation (1.7). Once again, by invoking Equation (1.5), Definition A.3 and Equation (A.4), we get
\[
m'(u, L_0, -) - m'(u, L_D, -) = \iota_{L_0}^-(u) + \iota (E_0^u(-\infty), L_0; L_D) - \iota_{L_D}^-(u)
\]
\[
= \iota (L_D, E_0^u(-t); t \in \mathbb{R}^+) - \iota (L_D, E_0^u(-t); t \in \mathbb{R}^+) + \iota (E_0^u(-\infty), 0; L_D)
\]
\[
= s (E_0^u(0), E_0^u(-\infty); L_0, L_D) + \iota (E_0^u(0), L_0; L_D)
\]
\[
= \iota (E_0^u(0), L_0, L_D) - \iota (E_0^u(-\infty), L_0; L_D) + \iota (E_0^u(-\infty), 0; L_D)
\]
\[
= \iota (E_0^u(0), L_0, L_D).
\]
This concludes the proof. \(\square\)

5 Some classical examples

The aim of this section is to compute the Morse index of heteroclinic and halfclinic solutions in the case of pendulum equation, Nagumo equation and reaction-diffusion systems. In particular in Subsection 5.1 we compute the indices for the heteroclinic solutions of the pendulum equation, Nagumo equation and reaction-diffusion systems. In particular in Subsection 5.2 we investigate the heteroclinic solutions for the Nagumo equation whilst in Subsection 5.3 we investigate some classical reaction-diffusion system in \(\mathbb{R}^3\).

5.1 Mathematical pendulum

Consider the mathematical pendulum consisting of a light rod of length \(l\) to which is attached a ball of mass \(m\). The other end of the rod is attached to a wall at a point so that the ball of the pendulum moves on a circle centered at this point. The position of the ball at time \(t\) is completely described by the angle \(\theta(t)\) of the ball from the straight down position and measured in the counterclockwise direction. The system can be expressed by a second Hamiltonian system
\[
\dot{\theta}(t) = V'(\theta), \tag{5.1}
\]
where \(V(\theta) = \frac{g}{l} \cos \theta\) and prime denotes the gradient of \(V\). The Lagrangian function associated with (5.1) is the following
\[
L(\theta, \dot{\theta}) = \frac{1}{2}(\dot{\theta})^2 + V(\theta).
\]
Heteroclinic solutions

From paper [BY11, Section 5], the Equation (5.1) has an exact heteroclinic solution, given by

\[ \hat{\theta}(t) = 4 \arctan \left( \tanh \left( \frac{1}{2} \sqrt{\frac{g}{l}}t \right) \right) \]

from \(-\pi\) to \(\pi\).

Linearizing the Equation (5.1) about \(\hat{\theta}\), we get the following Morse-Sturm system

\[
\begin{cases}
-\frac{d^2}{dt^2}\phi + \frac{g}{l} \left( 1 - 2 \sech^2 \left( \sqrt{\frac{g}{l}}t \right) \right) \phi = 0, \\
\lim_{t \to -\infty} \phi(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \phi(t) = 0
\end{cases}
\]  
(5.2)

and we observe that conditions (L1), (L2), (H1), (H2) hold.

| \(P(t)\) | \(Q(t)\) | \(R(t)\) | \(R_\pm\) |
|-----|-----|-----|-----|
| \(I\) | 0 | \(\frac{g}{l} \left( 1 - 2 \sech^2 \left( \sqrt{\frac{g}{l}}t \right) \right)\) | \(\frac{g}{l}\) |

Table 1: In the table are displayed the coefficients of the linearized associated Sturm-Liouville operator for the heteroclinic solution of the mathematical pendulum.

Let \(\Phi = \begin{pmatrix} \dot{\phi} \\ \phi \end{pmatrix}\) and we observe that the Sturm-Liouville boundary value problem given in (5.2) fits into the Hamiltonian boundary value problem given by

\[
\begin{cases}
\dot{\Phi}(t) = J B(t) \Phi(t), \\
\lim_{t \to -\infty} \Phi(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} \Phi(t) = 0,
\end{cases}
\]  
(5.3)

where \(B(t) = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{g}{l} \left( 1 - 2 \sech^2 \left( \sqrt{\frac{g}{l}}t \right) \right) \end{pmatrix}\).

By Corollary 1, we get that

\[ m^-(\hat{\theta}) = -e^{CLM} \left( E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+ \right), \]

where \(m^-(\hat{\theta})\) denotes for the Morse index of operator

\[ -\frac{d^2}{dt^2} + \frac{g}{l} \left( 1 - 2 \sech^2 \left( \sqrt{\frac{g}{l}}t \right) \right) \]

while \(E^u(\tau)\) denotes the stable subspace and unstable subspace of (5.3). Moreover, it is easy to check that \(\begin{pmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \end{pmatrix}\) satisfies the equation (5.3). By a straightforward calculation we get that

\[ E^s(\tau) = \left\{ -\sqrt{\frac{g}{l}} \tanh \left( \sqrt{\frac{g}{l}} \tau \right) v \right\} \quad v \in \mathbb{R} \}
\]

\[ E^u(\tau) = \left\{ -\sqrt{\frac{g}{l}} \tanh \left( \sqrt{\frac{g}{l}} \tau \right) v \right\} \quad v \in \mathbb{R} \}
\]

and

\[ E^s(+\infty) = \left\{ -\sqrt{\frac{g}{l}} v \right\} \quad v \in \mathbb{R} \}
\]

\[ E^u(-\infty) = \left\{ \sqrt{\frac{g}{l}} v \right\} \quad v \in \mathbb{R} \}
\]
By these we get that
\[ E^s(\tau) \cap L_D = \{0\} \quad \text{and} \quad E^u(-\tau) \cap L_D = \{0\} \quad \tau \in \mathbb{R}^+. \]
By taking into account Lemma A.6, we get that
\[ m^-(\hat{\theta}) = -\iota_{\text{CLM}}(E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+) = \iota(E^u(0), E^s(0); L_D) - \iota(E^u(-\infty), E^s(+\infty); L_D) \]
From the definition of quadratic form \( Q \) defined in (A.1), by a simple calculation, we have that
\[ Q(E^u(0), E^s(0); L_D)(v, v) = 0 \quad Q(E^u(-\infty), E^s(+\infty); L_D)(v, v) = -2 \sqrt{\frac{g}{l}} v^2 \quad \forall v \in \mathbb{R}, \]
and so
\[ m^-(\hat{\theta}) = -\iota_{\text{CLM}}(E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+) = \iota(E^u(0), E^s(0); L_D) - \iota(E^u(-\infty), E^s(+\infty); L_D) = 0. \]
This implies the following result
- All the eigenvalues of \( \frac{d^2}{dt^2} + \frac{g}{l} \left( 1 - 2 \text{sech}^2 \left( \sqrt{\frac{g}{l}} \right) \right) \) are non-negative.

**Halfclinic solutions**

Let us start by considering
\[ \hat{\theta}(t) = 4 \arctan \left( \text{tanh} \left( \frac{1}{2} \sqrt{\frac{g}{l}} t \right) \right) \]
which is a future halfclinic solution of the Equation (5.1) satisfying the following boundary condition:
\[ \left( \hat{\theta}(0), \theta(0) \right)^T \in L_D \quad \lim_{t \to +\infty} \theta(t) = \pi. \]
Linearizing Equation (5.1) about \( \hat{\theta} \), we get the following Morse-Sturm system
\[
\begin{align*}
\left\{ 
- \frac{d^2}{dt^2} \phi + \frac{g}{l} \left( 1 - 2 \text{sech}^2 \left( \sqrt{\frac{g}{l}} t \right) \right) \phi &= 0, \\
\left( \dot{\phi}(0), \phi(0) \right)^T &\in L_D \quad \text{and} \quad \lim_{t \to +\infty} \phi(t) = 0.
\end{align*}
\] (5.4)
As before conditions (L1), (L2), (H1) and finally (H2) hold. Setting \( \Phi = \left( \begin{array}{c} \dot{\phi} \\ \phi \end{array} \right) \), we get that the Sturm-Liouville boundary value problem given at Equation (5.4) corresponds to the Hamiltonian boundary value problem given by
\[
\begin{align*}
\Phi(t) &= JB(t)\Phi(t), \\
\Phi(0) &\in L_D \quad \text{and} \quad \lim_{t \to +\infty} \Phi(t) = 0,
\end{align*}
\]
where \( B(t) = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{g}{l} \left( 1 - 2 \text{sech}^2 \left( \sqrt{\frac{g}{l}} t \right) \right) \end{pmatrix} \). By using Theorem 2, we have that
\[ m^-(\hat{\theta}, L_D, +) = -\iota_{\text{CLM}}(E^s(\tau), L_D; \tau \in \mathbb{R}^+) + \iota(L_D, L_D; E^s(+\infty)) = -\iota_{\text{CLM}}(E^s(\tau), L_D; \tau \in \mathbb{R}^+). \]
Moreover, by using the previous computations provided in the heteroclinic case, we get
\[ E^s(\tau) = \left\{ -\sqrt{\frac{g}{l}} \tanh \left( \sqrt{\frac{g}{l}} \tau \right) \nu \bigg| \nu \in \mathbb{R} \right\} \]
and 
\[ E^s(+\infty) = \left\{ -\frac{\sqrt{g}}{v} \right\} \bigg| v \in \mathbb{R} \right\}. \]

In particular, we get that 
\[ E^s(\tau) \cap L_D = \{0\} \quad \text{for all} \quad \tau \in \mathbb{R}^+, \]
and this implies that 
\[ m^-(\hat{\theta},L_D,+) = -\iota^{\text{CLM}}(E^s(\tau),L_D;\tau \in \mathbb{R}^+). \]

Let \( L_0 = \left( \begin{array}{c} av \\ v \end{array} \right) \) be a Lagrangian subspace of the standard symplectic space \((\mathbb{R}^2, \omega)\), where \( a, v \in \mathbb{R} \).

By Corollary 2, we have that 
\[ m^-(\hat{\theta},L_0,+) = \iota(L_D,L_0;E^s(0)). \]

By the definition of quadratic form \( Q \) defined in (A.1), by a simple calculation, we have that 
\[ Q(L_D,L_0;E^s(0))(V,V) = \left( \sqrt{\frac{g}{l}} - a \right) v^2, \quad \forall V = \left( \begin{array}{c} v \\ 0 \end{array} \right) \in L_D \cap (L_0 + E^s(0)), \]
so, we have that 
\[ m^+(Q(L_D,L_0;E^s(0))) = \begin{cases} 1 & \text{if } a < \sqrt{\frac{g}{l}} \\ 0 & \text{if } a \geq \sqrt{\frac{g}{l}} \end{cases} \]
and so 
\[ m^-(\hat{\theta},L_0,+) = \iota(L_D,L_0;E^s(0)) = \begin{cases} 1 & \text{if } a < \sqrt{\frac{g}{l}} \\ 0 & \text{if } a \geq \sqrt{\frac{g}{l}} \end{cases} \]

Summarizing the previous computations, the following holds.

**Proposition 5.1.** Let \( \hat{\theta}(t) = 4 \arctan \left( \tanh \left( \frac{1}{2} \sqrt{\frac{g}{l}} t \right) \right) \) be an heteroclinic orbit of mathematical pendulum given at Equation (5.1) and connecting \(-\pi\) and \(\pi\). Then we have:
\[ m^-(\hat{\theta}) = 0. \]

Let \( \hat{\theta}(t) = 4 \arctan \left( \tanh \left( \frac{1}{2} \sqrt{\frac{g}{l}} t \right) \right) \) be a future halfclinic solution of mathematical pendulum given at Equation (5.1) and satisfying satisfying the following boundary condition:
\[ \left( \dot{\theta}(0), \theta(0) \right)^T \in L_D \quad \lim_{t \to +\infty} \theta(t) = \pi. \]

Then we have:
\[ m^-(\hat{\theta},L_0,+) = \begin{cases} 1 & \text{if } a < \sqrt{\frac{g}{l}} \\ 0 & \text{if } a \geq \sqrt{\frac{g}{l}} \end{cases} \]
where \( L_0 = \left( \begin{array}{c} av \\ v \end{array} \right) \) is a Lagrangian subspace of the standard symplectic space \((\mathbb{R}^2, \omega)\), and \( a, v \in \mathbb{R} \).

### 5.2 Nagumo equation

An important scalar reaction-diffusion equation is known as Nagumo equation
\[ u_t = u_{xx} + u(u-a)(1-u), \quad -1 \leq a \leq 1. \]

This equation arises, for example, in phase separation of binary alloys and from a simplification of the FitzHugh-Nagumo system modelling impulse propagation along a nerve fiber\[ M78, IF06 \].
Heteroclinic solutions

From [KT83], for \( a = \frac{1}{2} \), there exists a steady solitary wave solution

\[
\hat{u}(x) = \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{\sqrt{2}}{4} x \right), \quad x \in \mathbb{R}
\]

which satisfies the following steady equation

\[
u_{xx} + u \left( u - \frac{1}{2} \right) (1 - u) = 0.
\] (5.5)

By linearizing the Equation (5.5) about \( \hat{u} \), we get the following Sturm-Liouville system

\[
\begin{aligned}
\frac{d^2}{dx^2} w(x) + \left( \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} \right) w(x) &= 0 \quad x \in \mathbb{R} \\
\lim_{x \to -\infty} w(x) &= 0 \quad \text{and} \quad \lim_{x \to +\infty} w(x) = 0.
\end{aligned}
\] (5.6)

Let \( W = \begin{pmatrix} \dot{w} \\ w \end{pmatrix} \), then Equation (5.6) can be written as the following Hamiltonian system

\[
\begin{aligned}
\dot{W}(x) &= JB(x)W(x), \\
\lim_{x \to -\infty} W(x) &= 0 \quad \text{and} \quad \lim_{x \to +\infty} W(x) = 0,
\end{aligned}
\] (5.7)

where \( B(x) = \begin{pmatrix} 1 & 0 \\ 0 & 3 \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} \end{pmatrix} \) where the coefficients of the matrix \( B \) are given in the Table 2.

| \( P(x) \) | \( Q(x) \) | \( R(x) \) | \( R_\pm \) |
|---|---|---|---|
| 1 | 0 | \( \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} \) | \( 1 \) | \( \frac{1}{2} \) |

Table 2: In the table are displayed the coefficients of the linearized associated Sturm-Liouville operator of the steady solitary wave solution of the Nagumo equation.

By taking into account Corollary 1, we get that

\[
m \hat{m} \left( \hat{u} \right) = -\epsilon_{\text{CLM}} \left( E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+ \right)
\]

where \( m \hat{m} \left( \hat{u} \right) \) denotes the Morse index of operator

\[
A := -\frac{d^2}{dx^2} + \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4}
\]

on \( W^{2,2}(\mathbb{R}, \mathbb{R}) \) and \( E^s(\tau) \) and \( E^u(\tau) \) respectively denotes the stable subspace and unstable subspace of the Hamiltonian system

\[
\dot{W}(x) = JB(x)W(x).
\] (5.8)

Moreover, it is easy to check that the matrix

\[
W(x) := \begin{pmatrix}
- \tanh \left( \frac{\sqrt{2}}{4} x \right) \sech^2 \left( \frac{\sqrt{2}}{4} x \right) \\
\sqrt{2} \sech^2 \left( \frac{\sqrt{2}}{4} x \right)
\end{pmatrix}
\]

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satisfies the equation (5.8). So, we get that
\[
E_s(\tau) = \left\{ \left( -\tanh \left( \frac{\sqrt{2} v}{4} \right) v \right) \, | \, v \in \mathbb{R} \right\}, \quad E_u(\tau) = \left\{ \left( -\tanh \left( \frac{\sqrt{2} v}{4} \right) v \right) \, | \, v \in \mathbb{R} \right\}
\]
and
\[
E_s(+\infty) = \left\{ \left( -v \sqrt{2} v \right) \, | \, v \in \mathbb{R} \right\}, \quad E_u(-\infty) = \left\{ \left( v \sqrt{2} v \right) \, | \, v \in \mathbb{R} \right\}.
\]
Moreover, we have that
\[
E_s(\tau) \cap L_D = \{0\} \quad \text{and} \quad E_u(-\tau) \not\subset L_D, \ \tau \in \mathbb{R}^+.
\]
By invoking Lemma A.6, we get that
\[
m_-(\hat{u}) = -i^{\text{CLM}}(E_s(\tau), E_u(-\tau); \tau \in \mathbb{R}^+) = i(E_u(0), E_s(0); L_D) - i(E_u(-\infty), E_s(+\infty); L_D)
\]
and by a direct computation, we infer that
\[
Q(E_u(0), E_s(0); L_D) = 0, \quad Q(E_u(-\infty), E_s(+\infty); L_D) = -2\sqrt{2}v^2, \quad \forall v \in \mathbb{R},
\]
which imply that
\[
m_+(Q(E_u(0), E_s(0); L_D)) = m_+(Q(E_u(-\infty), E_s(+\infty); L_D)) = 0.
\]
So, from Equation (A.3), we have that
\[
m_-(\hat{u}) = i(E_u(0), E_s(0); L_N) - i(E_u(-\infty), E_s(+\infty); L_N)
\]
\[
= m_+(Q(E_u(0), E_s(0); L_N)) - m_+(Q(E_u(-\infty), E_s(+\infty); L_N)) = 0.
\]
This implies that all the eigenvalues of the operator
\[
-\frac{d^2}{dx^2} + \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4}
\]
are non-negative.

\section*{Halfclinic solutions of the Nagumo equation}

We consider the following boundary value
\[
\begin{cases}
    u_{xx} + u(u - \frac{1}{2})(1 - u) = 0 \\
    (\dot{u}(0), u(0))^T \in L_0 \quad \lim_{x \to +\infty} u(x) = 0,
\end{cases}
\quad (5.9)
\]
where \( L_0 = \left\{ \left( \frac{v}{2\sqrt{2}v} \right) \, | \, v \in \mathbb{R} \right\} \). It is immediate to check that
\[
\hat{u}(x) = \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{\sqrt{2}}{4} x \right) \quad x \in [0, +\infty)
\]
is a solution of the Equation (5.9). Linearizing Equation (5.9) about \( \hat{u} \), we get the following Sturm-Liouville system
\[
\begin{cases}
    \frac{d^2}{dx^2} w(x) + \left( \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} \right) w(x) = 0 \\
    (\dot{w}(0), w(0))^T \in L_0 \quad \lim_{x \to +\infty} w(x) = 0
\end{cases}
\quad (5.10)
\]
It is worth observing that the conditions (L1), (L2), (H1) and (H2) hold.

Let $W = \begin{pmatrix} \dot{w} \\ w \end{pmatrix}$. Then Equation (5.10) reduces to the following Hamiltonian system

$$
\begin{align*}
\dot{W}(x) &= JB(x)W(x) e \in (-\infty, 0] \\
\dot{W}(0), W(0) &\in L_0 \quad \text{and} \quad \lim_{x \to -\infty} W(x) = 0,
\end{align*}
$$

where $B(x) =
\begin{pmatrix}
1 & 0 \\
0 & \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4}
\end{pmatrix}$. By Corollary 1, we get that

$$
m^-(\hat{u}, L_0, +) = -\iota_{CLM}(E^s(\tau), L_0; \tau \in \mathbb{R}^+)
$$

where $m^-(\hat{u}, L, +)$ denotes for the Morse index of operator

$$
-\frac{d^2}{dx^2} + \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4}
$$
on the domain $W_{L_0} = \{ u \in W^{2,2}(\mathbb{R}^+, \mathbb{R}) \mid u(0) \in L_0 \}$ and $E^s(\tau)$ denotes the stable subspace of the Equation (5.7). Thus we get that

$$
E^s(\tau) = \begin{cases} 
-v \sqrt{2} \tau \bigg| v \in \mathbb{R} 
\end{cases}
$$

and $E^s(+\infty) = \begin{cases} 
-v \sqrt{2} \tau \bigg| v \in \mathbb{R} 
\end{cases}$.

So, we get that

$$
E^s(\tau) \cap L_D = \{ 0 \} \quad \forall \tau \in \mathbb{R}^+
$$

which implies that $\iota_{CLM}(E^s(\tau), L_D; \tau \in \mathbb{R}^+) = 0$.

Let $L_1 = \begin{pmatrix} av \\ v \end{pmatrix}$ be a Lagrangian subspace of the standard symplectic space $(\mathbb{R}^2, \omega)$, where $a, v \in \mathbb{R}$. From Theorem 2, we have that

$$
m^-(\hat{u}, L_1, +) = -\iota_{CLM}(E^s(\tau), L_1; \tau \in \mathbb{R}^+) + \iota(L_D, L_1; E^s(+\infty)),
$$

where $m^-(\hat{u}, L_1, +)$ stands for the Morse index of $-\frac{d^2}{dx^2} + \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4}$ on the domain $W_{L_1}^+ := \{ w \mid w \in W^{2,2}(\mathbb{R}^+, \mathbb{R}^2), w(0) \in L_1 \}$.

From Corollary 2, we get that

$$
m^-(\hat{u}, L_1, +) = \iota(L_D, L_1; E^s(0)).
$$

From the definition of quadratic form $Q$ defined in (A.1), by a simple calculation, we have that

$$
m^+(Q(L_D, L_1, E^s(0))) = \begin{cases} 
1 & \text{if } a < 0 \\
0 & \text{if } a \geq 0.
\end{cases}
$$

this implies that

$$
m^-(\hat{u}, L_1, +) = \iota(L_D, L_1; E^s(0)) = \begin{cases} 
1 & \text{if } a < 0; \\
0 & \text{if } a \geq 0.
\end{cases}
$$

Summing up the previous computations, the following result holds.
Proposition 5.2. Let \( \hat{u}(x) = \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{\sqrt{2}}{4} x \right) \) be a heteroclinic solution of the Equation (5.5).

Then we get

\[
m^-(\hat{u}) = 0.
\]

Let \( \hat{u}(x) = \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{\sqrt{2}}{4} x \right) \) be a future halfclinic solution of Equation (5.5) satisfying the following boundary conditions \((\hat{u}(0), u(0))^T \in L_0 \) and \( \lim_{x \to +\infty} u(x) = 0 \). Then, we get:

\[
m^-(\hat{u}, L_1) = \begin{cases} 
1 & \text{if } a < 0 \\
0 & \text{if } a \geq 0
\end{cases}
\]

where

\[
L_0 = \left\{ \left( \frac{v}{2 \sqrt{2} v} \right) \bigg| v \in \mathbb{R} \right\} \quad \text{and} \quad L_1 = \left( \begin{array}{c} a v \\ v \end{array} \right) \quad a, v \in \mathbb{R}
\]

are Lagrangian subspaces of the standard symplectic space \((\mathbb{R}^2, \omega)\).

5.3 The coupled reaction-diffusion system in \( \mathbb{R}^4 \)

Let us start to consider the reaction-diffusion system

\[
\begin{cases}
    u_t = u_{xx} + u(u - \frac{1}{2})(1 - u) + c(u - v) \\
v_t = v_{xx} + v(v - \frac{1}{2})(1 - v) + c(v - u)
\end{cases}
\]  

(5.11)

where \( c \), the coupling constant, is a non-zero real parameter, restricted to the values \( c < \frac{1}{4} \). The steady state equation is given by the second order Lagrangian system

\[
w''(x) = \nabla V(w(x)) \quad x \in \mathbb{R}
\]  

(5.12)

where \( w = \begin{pmatrix} u \\ v \end{pmatrix} \) and the potential function is given by

\[
V(w) = \frac{1}{4}(u^4 + v^4) - \frac{1}{2}(u^3 + v^3) + \frac{1}{4}(u^2 + v^2) - \frac{c}{2}(u - v)^2.
\]

The corresponding Lagrangian function is given by

\[
L(w, w') = \frac{1}{2}|w'|^2 + V(w)
\]

The Newton equation given at (5.12) admits the exact heteroclinic orbit

\[
\hat{w}(x) = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + \frac{1}{2} \tanh \left( \frac{\sqrt{2}}{4} x \right) \\
\frac{1}{2} + \frac{1}{2} \tanh \left( \frac{\sqrt{2}}{4} x \right) \end{pmatrix}
\]

which connects between \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). For \( c < 1/4 \), it is straightforward to check that the restpoints \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) are both stable. By linearizing the Equation (5.11) about \( \hat{w} \), we get the following Sturm-Liouville system

\[
\begin{cases}
    -\frac{d^2}{dx^2} w(x) + D^2 V(\hat{w}) w(x) = 0 \\
    \lim_{x \to -\infty} w(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} w(x) = 0
\end{cases}
\]

(5.13)
where the Hessian $D^2V(\tilde{w})$ is given by

$$D^2V(\tilde{w}) = \begin{pmatrix} \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} - c & c \\ c & \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} - c \end{pmatrix}.$$ 

| $P(x)$ | $Q(x)$ | $R(x)$ | $R_\pm$ |
|--------|--------|--------|--------|
| $I$   | 0      | $\begin{pmatrix} \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} - c & c \\ c & \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} - c \end{pmatrix}$ | $\begin{pmatrix} 1/2 - c & c \\ c & 1/2 - c \end{pmatrix}$ |

Table 3: In the table are displayed the matrix coefficients of the linearized associated Sturm-Liouville operator of the heteroclinic solution of the reaction-diffusion equation.

In Table 3 are displayed the coefficients of the Sturm-Liouville operator associated to the linearized Sturm-Liouville operator of about the heteroclinic solution $\tilde{w}$. It is worth observing that also in this case the conditions (L1), (L2), (H1) and (H2) are fulfilled.

Let $W = \begin{pmatrix} \dot{w} \\ \ddot{w} \end{pmatrix}$, then Equation (5.13) fits into the following Hamiltonian system

$$\begin{cases} \dot{W}(x) = JB(x)W(x) \\ \lim_{x \to -\infty} W(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} W(x) = 0, \end{cases} \quad x \in \mathbb{R}$$

where $B(x) = \begin{pmatrix} I & 0 \\ 0 & -D^2V(\tilde{w}) \end{pmatrix}$. By invoking Corollary 1, we have that

$$m^-(\tilde{w}) = -CLM(E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+)$$

where $m^-(\tilde{w})$ denotes the Morse index of operator

$$\mathcal{A} = -\frac{d^2}{dx^2} + D^2V(\tilde{w}).$$

Set $T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, then $T^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$, we have

$$\widehat{\mathcal{A}} := T^{-1} \mathcal{A} T = -\frac{d^2}{dx^2} + \begin{pmatrix} \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} - 2c \end{pmatrix}.$$ 

By this computation, we readily get that $\mathcal{A}$ and $\widehat{\mathcal{A}}$ have the same spectrum. Let us now consider the following eigenvalue equation

$$-\phi_{xx} + \left( \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4} \right) \phi = \lambda \phi. \quad (5.14)$$

It is easy to check that Equation (5.14) has two explicit linearly independent solutions given by

$$\phi(x) = \begin{cases} 1 - \tanh^2 \left( \frac{\sqrt{2}}{4} x \right), & 6x + \sqrt{2} \sinh \left( \sqrt{2} x \right) + 8 \sqrt{2} \sinh \left( \frac{x}{\sqrt{2}} \right) \sech^2 \left( \frac{x}{2 \sqrt{2}} \right) \text{ if } \lambda = 0 \\ e^{\pm \sqrt{2} \lambda x/2} \left( \pm 2 \sqrt{1 - 2 \lambda \tanh \left( \frac{x}{2 \sqrt{2}} \right)} + \frac{1}{3} (3 - 8 \lambda) + \tan^2 \left( \frac{x}{2 \sqrt{2}} \right) \right) \text{ if } \lambda < \frac{1}{2} \text{ and } \lambda \neq 0. \end{cases}$$
Then we get that 0 is the only eigenvalue of the operator

\[
\frac{d^2}{dx^2} + \frac{3}{4} \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) - \frac{1}{4}
\]

whose corresponding eigenfunction is given by \(1 - \tanh^2 \left( \frac{\sqrt{2}}{4} x \right)\).

By this argument we conclude that 0 and \(-2c\) are only the eigenvalues of the operator \(\hat{A}\) with eigenfunctions

\[
\begin{pmatrix}
1 - \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) \\
0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
0 \\
1 - \tanh^2 \left( \frac{\sqrt{2}}{4} x \right)
\end{pmatrix}
\]

In particular

- If \(c \leq 0\), the operator \(\hat{A}\) is non-negative
- If \(0 < c < \frac{1}{4}\), then \(\lambda = -2c\) is the only negative eigenvalue of \(\hat{A}\) with an eigenfunction being
- \(\begin{pmatrix} 0 \\ 1 - \tanh^2 \left( \frac{\sqrt{2}}{4} x \right) \end{pmatrix}\).

So, we have that

\[
m^-(A) = m^-(\hat{A}) = \begin{cases}
0 & \text{if } c \leq 0 \\
1 & \text{if } 0 < c < \frac{1}{4}
\end{cases}
\]

and by this we infer that

- If \(c \leq 0\), then \(\iota_{CLM} (E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+) = 0\)
- If \(0 < c < \frac{1}{4}\), then \(\iota_{CLM} (E^s(\tau), E^u(-\tau); \tau \in \mathbb{R}^+) = -1\).

Summing up this computation, the following result holds.

**Proposition 5.3.** Let \(\hat{w}(x) = \begin{pmatrix}
\frac{1}{2} + \frac{1}{2} \tanh \left( \frac{\sqrt{2}}{4} x \right) \\
\frac{1}{2} + \frac{1}{2} \tanh \left( \frac{\sqrt{2}}{4} x \right)
\end{pmatrix}\) be a solitary wave solution of the Equation (5.11). Then we get

\[
\iota(\hat{w}) = \begin{cases}
0 & \text{if } C \leq 0 \\
1 & \text{if } 0 < c < \frac{1}{4}
\end{cases}
\]

### A Maslov, Hörmander and triple index

This last section is devoted to recall some basic definitions, well-known results and main properties of the Maslov index and interrelated invariants that we used throughout the paper. Our basic reference is [HP17] and references therein.
A.1 The Cappell-Lee-Miller index

In the standard symplectic space $\mathbb{R}^{2n}$, we denote by $L(n)$ the set of all Lagrangian subspaces of $\mathbb{R}^{2n}$ and we refer to as the Lagrangian Grassmannian of $\mathbb{R}^{2n}$. For $a, b \in \mathbb{R}$ with $a < b$, we denote by $\mathcal{P}([a, b]; \mathbb{R}^{2n})$ the space of all ordered pairs of continuous maps of Lagrangian subspaces $L : [a, b] \ni t \mapsto L(t) : L_1(t), L_2(t) \in L(n) \times L(n)$ equipped with the compact-open topology. Following authors in [CLM94] we are in position to briefly recall the definition of the Maslov index for pairs of Lagrangian subspaces, that will be denoted throughout the paper by the symbol $\iota_{\text{CLM}}$. Loosely speaking, given the pair $L = (L_1, L_2) \in \mathcal{P}([a, b]; \mathbb{R}^{2n})$, this index counts with signs and multiplicities the number of instants $t \in [a, b]$ such that $L_1(t) \cap L_2(t) \neq \emptyset$.

**Definition A.1.** The CLM-index is the unique integer valued function

$$\iota_{\text{CLM}} : \mathcal{P}([a, b]; \mathbb{R}^{2n}) \ni L \mapsto \iota_{\text{CLM}}(L(t); t \in [a, b]) \in \mathbb{Z}$$

satisfying the Properties I-VI given in [CLM94, Section 1].

A.2 The triple index and the Hörmander index

In this section we quickly recap some basic definitions and fact about the triple and the Hörmander index. Our basic reference for this section is the paper [ZWZ18] and references therein. Given three isotropic subspaces $\alpha, \beta$ and $\delta$ in $\mathbb{R}^{2n}$, we define the quadratic form $Q$ as follows

$$Q := Q(\alpha, \beta; \delta) : \alpha \cap (\beta + \delta) \to \mathbb{R} \text{ given by } Q(x_1, x_2) = \omega(y_1, z_2),$$

(A.1)

where for $j = 1, 2$, $x_j = y_j + z_j \in \alpha \cap (\beta + \delta)$ and $y_j \in \beta$, $z_j \in \delta$. By invoking [ZWZ18, Lemma 3.3], in the particular case in which $\alpha, \beta, \delta$ are Lagrangian subspaces, we get

$$\ker Q(\alpha, \beta; \delta) = \alpha \cap \beta + \alpha \cap \delta.$$

We are in position to define the **triple index** in terms of the quadratic form $Q$ defined above.

**Definition A.2.** Let $\alpha, \beta$ and $\kappa$ be three Lagrangian subspaces of symplectic vector space $\mathbb{R}^{2n}$. Then the triple index of the triple $(\alpha, \beta, \kappa)$ is defined by

$$\iota(\alpha, \beta, \kappa) = m^-(Q(\alpha; \beta)) + m^-(Q(\beta; \kappa)) - m^-(Q(\alpha; \kappa)),$$

(A.2)

where $\delta$ is a Lagrangian subspace such that $\delta \cap \alpha = \delta \cap \beta = \delta \cap \kappa = \emptyset$.

By [ZWZ18, Lemma 3.13], the triple index given in Equation (A.2) can be characterized as follows

$$\iota(\alpha, \beta, \kappa) = m^+(Q(\alpha; \beta; \kappa)) + \dim(\alpha \cap \kappa) - \dim(\alpha \cap \beta \cap \kappa).$$

(A.3)

Another closely related symplectic invariant is the so-called Hörmander index whose vocation is to measure the difference of the (relative) Maslov index computed with respect to two different Lagrangian subspaces.

Let $V_0, V_1, L_0, L_1$ be four Lagrangian subspaces and $L \in \mathcal{C}^0([0, 1], L(n))$ be such that $L(0) = L_0$ and $L(1) = L_1$.

**Definition A.3.** Let $L, V \in \mathcal{C}^0([0, 1], L(n))$ be such that $L(0) = L_0$, $L(1) = L_1$, $V(0) = V_0$ and $V(1) = V_1$, the Hörmander index is the integer defined by

$$s(L_0, L_1; V_0, V_1) = \iota_{\text{CLM}}(V_1, L(t); t \in [0, 1]) - \iota_{\text{CLM}}(V_0, L(t)); t \in [0, 1]$$

$$= \iota_{\text{CLM}}(V(t), L_1; t \in [0, 1]) - \iota_{\text{CLM}}(V(t), L_0; t \in [0, 1]).$$

Remark A.4. As direct consequence of the fixed endpoints homotopy invariance of the $\iota_{\text{CLM}}$-index, is actually possible to prove that Definition A.3 is well-posed, meaning that it is independent on the path $L$ joining the two Lagrangian subspaces $L_0, L_1$. (Cfr. [RS93] for further details).
Let now be given four Lagrangian subspaces, namely \(\lambda_1, \lambda_2, \kappa_1, \kappa_2\) of symplectic vector space \((\mathbb{R}^{2n}, \omega)\). By [ZWZ18, Theorem 1.1], the Hörmander index \(s(\lambda_1, \lambda_2; \kappa_1, \kappa_2)\) can be expressed in terms of the triple index as follows

\[
s(\lambda_1, \lambda_2; \kappa_1, \kappa_2) = \iota(\lambda_1, \lambda_2, \kappa_2) - \iota(\lambda_1, \lambda_2, \kappa_1) = \iota(\lambda_1, \kappa_1, \kappa_2) - \iota(\lambda_2, \kappa_1, \kappa_2).
\]

(A.4)

In particular, by using Equation (A.4) the following result holds.

**Lemma A.5.** [HWY18] Let \(L_0, L \in \mathcal{L}(n)\) and \(l \in \mathcal{C}^0([0, 1], \mathcal{L}(n))\). If \(t \mapsto l(t)\) is transversal to \(L\) for every \(t \in [0, 1]\) (meaning that \(l(t) \cap L = \{0\}\)), then we get

\[
\iota_{\text{CLM}}(L_0, l(t); t \in [0, 1]) = \iota(l(1), L_0; L) - \iota(l(0), L_0; L).
\]

Being the triple index a symplectic invariant, meaning that if \(\alpha, \beta, \kappa \in \mathcal{L}(n)\) and \(\phi \in \text{Sp}(2n, \mathbb{R})\), then we get that \(\iota(\phi \alpha, \phi \beta, \phi \kappa) = \iota(\alpha, \beta, \kappa)\). By using the symplectic invariance, it is not difficult to generalize Lemma A.5 to the case of a pair of Lagrangian paths. More precisely the following result holds.

**Lemma A.6.** Let \(l_1\) and \(l_2\) be two continuous paths in \(\mathcal{L}(n)\) with \(t \in [0, 1]\) and we assume that \(l_1(t)\) and \(l_2(t)\) are both transversal to the (fixed) Lagrangian subspace \(L\). Then we get

\[
\iota_{\text{CLM}}(l_1(t), l_2(t); t \in [0, 1]) = \iota(l_1(1), l_2(1); L) - \iota(l_2(0), l_1(0); L).
\]

**Proof.** Let \(\mathbb{R}^{2n} = L \oplus JL\). Since \(l_1(t)\) and \(l_2(t)\) both transversal to \(L\), then we get that \(l_1(t)\) and \(l_2(t)\) both have the Lagrangian frames given by \(\begin{bmatrix} M(t) & I \\ I & I \end{bmatrix}\) and \(\begin{bmatrix} N(t) & I \\ I & I \end{bmatrix}\) respectively, where \(M(t)\) and \(N(t)\) are both symmetric matrices for all \(t \in [0, 1]\). We define another path of symplectic matrices as follows \(T(t) := \begin{bmatrix} I & M(0) - M(t) \\ 0 & I \end{bmatrix}\). By a straightforward calculations we get

\[
T(t)L_1(t) = \begin{bmatrix} M(0)v \\ M_1v \end{bmatrix}\quad \text{and} \quad T(t)L_2(t) = \begin{bmatrix} N(t)v - M(t)v + M(0)v \\ N_1v \end{bmatrix}.
\]

Thus we get

\[
\iota_{\text{CLM}}(l_1(t), l_2(t); t \in [0, 1]) = \iota_{\text{CLM}}(T(t)l_1(t), T(t)l_2(t); t \in [0, 1])
\]

\[
= \iota(T(1)l_2(1), T(1)l_1(1), L) - \iota(T(0)l_2(0), T(0)l_1(0), L)
\]

\[
= \iota(l_2(1), l_1(1), L) - \iota(l_2(0), l_1(0), L).
\]

\( \square \)

### B  Spectral flow for paths of selfadjoint Fredholm operators

Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a real separable Hilbert space and we denote by \(\mathcal{CF}^a(\mathcal{H})\) the space of all closed selfadjoint and Fredholm operators equipped with the *gap topology*. Given \(T \in \mathcal{CF}^a(\mathcal{H})\) and \(a, b \notin \sigma(T)\), we set

\[
P_{[a, b]}(T) := \Re \left( \frac{1}{2\pi i} \int_{\gamma} (\lambda I - T^*)^{-1} d\lambda \right)
\]

where \(\gamma\) stands for the circle of radius \((b - a)/2\) around the point \((a + b)/2\). We recall that if \(\sigma(T)\) consists of isolated eigenvalues of finite multiplicity, then

\[
\text{rge } P_{[a, b]}(T) = E_{[a, b]}(T) := \bigoplus_{\lambda \in [a, b]} \ker(\lambda I - T).
\]

Let \(A : [a, b] \to \mathcal{CF}^a(\mathcal{H})\) be a continuous path. As a direct consequence of [BLP05, Proposition 2.10], for every \(t \in [a, b]\), there exists \(\epsilon > 0\) and an open connected neighborhood \(\mathcal{N}_{t,a} \subset \mathcal{CF}^a(\mathcal{H})\) of \(A(t)\) such that \(\pm \epsilon \notin \sigma(T)\) for all \(T \in \mathcal{N}_{t,a}\). The map \(\mathcal{N}_{t,a} \in T \mapsto P_{[-\epsilon, \epsilon]}(T) \in \mathcal{L}(\mathcal{H})\) is continuous and hence the rank of \(P_{[-\epsilon, \epsilon]}(T)\) does not depends on \(T \in \mathcal{N}_{t,a}\). Now, let us consider
the open covering of the interval $I$ given by the pre-images of the neighborhoods $N_{t, a}$ through $A$ and by choosing a sufficiently fine partition of the interval $[a, b]$ having diameter less than the Lebesgue number of the covering, we can find $a := t_0 < t_1 < \cdots < t_n := b$, operators $T_i \in C \mathcal{F}^{sa}(\mathcal{H})$ and positive real numbers $\alpha_i$, $i = 1, \ldots, n$ in such a way the restriction of the path $A$ on the interval $[t_{i-1}, t_i]$ belongs to the the neighborhood $N_{t, a_i}$ and hence the dim $E_{[-a, a_i]}(A_i)$ is constant for $t \in [t_{i-1}, t_i]$, $i = 1, \ldots, n$.

**Definition B.1.** The spectral flow of $A$ on the interval $[a, b]$ is defined by

$$\text{Sf} (A; \lambda \in [a, b]) := \sum_{i=1}^{N} \dim E_{[0, a_i]}(A_i) - \dim E_{[0, a_i]}(A_{i-1}) \in \mathbb{Z}.$$ 

Before closing this section, closely following [HW18], we recall the definition of positive curve.

**Definition B.2.** [HW18] Let $A : [a, b] \rightarrow C \mathcal{F}^{sa}(\mathcal{H})$ be a continuous curve. The curve $A$ is named positive curve if $\{ \lambda \mid \ker A_\lambda \neq 0 \}$ is finite and

$$\text{Sf} (A; \lambda \in [a, b]) = \sum_{a<\lambda\leq b} \dim \ker A_\lambda.$$ 

### C Hyperbolicity

In this section, we provide some sufficient conditions about the hyperbolicity of the Hamiltonian matrix.

Our first result is a characterization of the hyperbolicity of the matrix $JB$ for

$$B = \begin{pmatrix} P^{-1} & -P^{-1}Q \\ -Q^TP^{-1} & Q^TP^{-1}Q - R \end{pmatrix}$$

(C.1)

in terms of a non-vanishing determinant of a suitable matrix.

**Lemma C.1.** We assume that $P$ invertible. Then $JB$ is hyperbolic if and only if

$$\det (a^2P + ia(Q^T - Q) + R) \neq 0 \quad \text{for all } a \in \mathbb{R}.$$ 

**Proof.** $JB$ is hyperbolic if and only if $\det(JB + iaI) \neq 0$ for all $a \in \mathbb{R}$. By a straightforward calculation, we get

$$\det(JB + iaI) = \det(B - iaJ) = \det \begin{pmatrix} P^{-1} & -P^{-1}Q + iaI \\ -Q^TP^{-1} & Q^TP^{-1}Q - R \end{pmatrix} = \det \left[ \begin{pmatrix} I & 0 \\ (Q^TP^{-1} + iaI) & P \end{pmatrix} \begin{pmatrix} P^{-1} & -P^{-1}Q + iaI \\ -Q^TP^{-1} & Q^TP^{-1}Q - R \end{pmatrix} \right] = \det \begin{pmatrix} P^{-1} & -P^{-1}Q + iaI \\ 0 & -iaQ + i a^2P - R \end{pmatrix} = \det P^{-1} \det(-a^2P - R - ia(Q - Q^T)).$$

Thus, we get that $\det(JB + iaI) \neq 0$ if and only if $\det(-a^2P - R - ia(Q - Q^T)) \neq 0$ and this concludes the proof.

The following result will be useful later and gives a sufficient condition on the hyperbolicity of $JB$ in terms of a symmetric matrix constructed through the coefficient matrices of the Sturm-Liouville operators.

**Corollary C.2.** If $\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix}$ is positive definite then $JB$ is hyperbolic.
Proof. We start by observing that the following equality holds:

\[ a^2P + ia(Q^T - Q) + R = (iaI, I) \begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} \begin{pmatrix} -iaI \\ I \end{pmatrix}. \]

By assumption

\[ \begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix} \]

is positive definite, so the matrix \( a^2P + ia(Q^T - Q) + R \) is positive definite as well; in particular, its determinant doesn’t vanish. Then the result readily follows direct by Lemma C.1.

The following result gives a sufficient condition about the hyperbolicity of a special path of Hamiltonian matrices if the starting point matrix is hyperbolic.

Lemma C.3. Let \([0, 1] \ni \lambda \mapsto R_\lambda := R + \lambda I \in \text{Sym}(n)\) and let \(JB_\lambda\) where \(B_\lambda\) is obtained from \(B\) by replacing \(R\) with \(R_\lambda\) in Equation (C.1). We assume \(P \in \text{Sym}^+(n)\) and that \(JB_\lambda\) is hyperbolic for \(\lambda = 0\). Then we get that \(JB_\lambda\) is hyperbolic for all \(\lambda \in \mathbb{R}^+\).

Proof. We start by introducing the following one parameter family of matrix functions defined as following

\[ f_\lambda(a) := a^2P + ia(Q^T - Q) + R + \lambda I. \]

By assumption and by taking into account Lemma C.1 we infer that \(f_0(a) \neq 0\) for all \(a \in \mathbb{R}\) and since for a sufficiently large, then \(f_0(a)\) is positive definite, then \(f_0(a)\) is positive definite for all \(a \in \mathbb{R}\).

Now, for every \(\lambda > 0\), we have that \(f_\lambda(a) = f_0(a) + \lambda I > f_0(a)\) and so \(f_\lambda(a)\) is positive definite for all \((\lambda, a) \in \mathbb{R}^+ \times \mathbb{R}\) as well. The thesis directly follows.

Corollary C.4. Let \(R_\lambda = R + \lambda I\) and we assume condition (L2) holds. Then, for \(\lambda \geq \frac{C_2}{C_1} + C_3\), the associated matrix \(JB_\lambda\) is hyperbolic.

Proof. By invoking Corollary C.2, we need to prove that the matrix \( \begin{pmatrix} P & Q \\ Q^T & R_\lambda \end{pmatrix} \) is positive definite and this is an immediate consequence of Lemma 3.1, concluding the proof.

A similar result holds by replacing the path \(\lambda \mapsto R + \lambda I\) with the path \(\lambda \mapsto R_\lambda := \lambda P\) which is very useful in very many situations.

Corollary C.5. Assume that \(P\) invertible and we let \(\lambda \mapsto R_\lambda := \lambda P\). Then, there exists \(\hat{\lambda} > 0\) such that, for every \(\lambda > \hat{\lambda}\) the matrix \(JB_\lambda\) is hyperbolic.

Proof. We start by observing that

\[ a^2P + ia(Q^T - Q) + R_\lambda = (a^2 + \lambda)P + ia(Q^T - Q) = P(a^2 + \lambda) \left( I + \frac{ia}{a^2 + \lambda}P^{-1} [Q^T - Q] \right). \]

For positive \(\lambda\) we get \(\left| \frac{a}{a^2 + \lambda} \right| \leq \frac{1}{2\sqrt{\lambda}}\). So, there exists \(\hat{\lambda} > 0\) such that for each \(\lambda > \hat{\lambda}\)

\[ \left\| \frac{ia}{a^2 + \lambda}P^{-1} [Q^T - Q] \right\| < 1. \]

This inequality directly implies that the matrix \(a^2P + ia(Q^T - Q) + R_\lambda\) is invertible for every \(\lambda > \hat{\lambda}\). By invoking Lemma C.1, we get the thesis.

The next transversality result establish the mutual position of the spectral eigenspaces of an hyperbolic matrix with respect to the Dirichlet Lagrangian.
Lemma C.6. We assume that $JB_0$ is hyperbolic. Then the positive and negative spectral subspaces of $JB_\lambda$ are both transversal to the horizontal (Dirichlet) Lagrangian; in symbols

$$V^\pm(JB_\lambda) \cap L_D = \{0\} \text{ for all } \lambda \in \mathbb{R}^+.$$ 

Proof. We provide the proof of $V^+(JB_\lambda) \cap L_D = \{0\}$ being the other completely similar. First of all, we start by observing that as a direct consequence of Lemma C.3, the matrix $JB_\lambda$ is hyperbolic for all $\lambda \in \mathbb{R}^+$. In particular, $V^+(JB_\lambda)$ are Lagrangian subspaces. (Cfr. [HP17] and references therein). We let $(u, 0)^T \in V^+(JB_\lambda) \cap L_D$ and we observe that, since $V^+(JB_\lambda)$ is invariant under $JB_\lambda$, then $JB_\lambda \left(\begin{array}{c} u \\ 0 \end{array}\right) \in V^+(JB_\lambda)$. A direct computation yields

$$0 = \omega \left( JB_\lambda \left(\begin{array}{c} u \\ 0 \end{array}\right), \left(\begin{array}{c} u \\ 0 \end{array}\right) \right) = - \langle Pu, u \rangle$$

and since $P$ is positive definite, we conclude that $u = 0$ or which is the same that $V^+(JB_\lambda) \cap L_D = \{0\}$ concluding the proof. \hfill \Box

D First order differential operators and Fredholmness

This section is devoted to collect the results about the Fredholmsness of the Sturm-Liouville operators both on the line and on the half-line as well as for the associated (first order) Hamiltonian differential operators that we need in the construction of the index theory.

We start by recalling a classical abstract result (consequence of the closed graph theorem) useful for comparing operators Sturm-Liouville and Hamiltonian operators on different domains.

Lemma D.1. [Kre82, Theorem 2.4] Let $X, Y$ be two Banach spaces and let $L : \text{dom} \ L \subset X \to Y$ be a closed linear operator with a dense domain $\text{dom} \ L$. We assume that there exists a close subspace $V$ of $Y$ such that $\text{rge} \ L \oplus V = Y$. Then we have that $\text{rge} \ L$ is closed in $Y$. Moreover, if $\text{codim} \ rge \ L < +\infty$, then the $\text{rge} \ L$ is closed in $Y$.

Lemma D.2. Under the previous notation, the operator $A^+_L$ is Fredholm if and only if the operators $A^+_m$ and $A^+_T$ are Fredholm, where $L_0$ denotes the selfadjoint boundary condition at the initial instant.

Proof. We start to consider the (maximal) Sturm-Liouville operator $A^+$ on $W^{2,2}$. Since $A^+$ and $A^+_m$ are conjugated with respect to the $L^2$ scalar product, we get that $A^+_m$ is a Fredholm operator if and only if $A^+$ is. Moreover the following inclusion holds

$$\ker \left( A^+_m \right) \subset \ker \left( A^+_L \right) \subset \ker \left( A^+_T \right) \text{ and } \text{rge} \left( A^+_m \right) \subset \text{rge} \left( A^+_L \right) \subset \text{rge} \left( A^+_T \right).$$

($\Leftarrow$) We assume that $A^+_m$ is Fredholm and we want to prove that $A^+_L$ is Fredholms. For, we start by observing that $\text{codim} \left( A^+_L \right) \leq \text{codim} \left( A^+_m \right) < +\infty$. Being $A^+_T$ a closed operator and by using Lemma D.1 and since $\text{codim} \left( A^+_T \right) < +\infty$, we get that $\text{rge} \left( A^+_L \right)$ is closed. So, we get that $A^+_L$ is a Fredholm operator.

($\Rightarrow$) The converse implication goes through the same arguments and is left to the reader. \hfill \Box

By arguing precisely as before, the following result holds.

Lemma D.3. The operator $\mathcal{F}^+_s$ is Fredholm if and only if the operator $\mathcal{F}^+_m$ is Fredholm.

Now, for each $s \in \mathbb{R}$, we let $\mathcal{F}^+_s : L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \to L^2(\mathbb{R}^+, \mathbb{R}^{2n})$ be the first order differential operator defined by

$$\mathcal{F}^+_s := -J \frac{d}{dt} - B_s(t)$$

on $W^{1,2}_0(\mathbb{R}^+, \mathbb{R}^{2n})$ and we let

$$B_s(t) = \begin{pmatrix} P^{-1}(t) & -P^{-1}(t)Q(t) \\ -Q^T(t)P^{-1}(t) & Q^T(t)P^{-1}(t)Q(t) - sP(t) \end{pmatrix}. $$

We define $B^+_s$ as the uniform limit of $B_s(t)$ when $t \to +\infty$. 

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Lemma D.4. The operator $\mathcal{F}_{0,s}^+$ is non-degenerate for every $s \in \mathbb{R}$.

Proof. We consider the following Hamiltonian system:

$$\begin{cases}
\dot{z}(t) = JB_z(t)z(t) & t \in \mathbb{R}^+
\end{cases}$$

and we observe that that $\ker \mathcal{F}_{0,s}^+$ consists of all solutions of Equation (D.1). By the existence and uniqueness theorem for odes, the Hamiltonian system given in quation (D.1) has only the trivial solution for every $s \in \mathbb{R}$. This implies that $\mathcal{F}_{0,s}^+$ is non-degenerate for all $s \in \mathbb{R}$. □

Lemma D.5. There exists $\hat{s}$ such that $JB_z^+$ is hyperbolic for every $s \geq \hat{s}$.

Proof. This result is a direct consequence of Corollary C.5. □

The following result gives a characterization of the Fredholmness of the operator $\mathcal{A}_m$ (resp. $\mathcal{A}_m^+$) in terms of $\mathcal{F}_m$ (resp. $\mathcal{F}_m^+$).

Proposition D.6. The operator $\mathcal{A}_m^+: W^{2,2}_0(\mathbb{R}^+; \mathbb{R}^n) \subset L^2(\mathbb{R}^+; \mathbb{R}^n) \to L^2(\mathbb{R}^+; \mathbb{R}^n)$ is Fredholm if and only if the operator $\mathcal{F}_m^+$ is Fredholm. Moreover $\text{ind } \mathcal{A}_m^+ = \text{ind } \mathcal{F}_m^+$.

Proof. We start by observing that $\mathcal{F}_m^+$ and $\mathcal{A}_m^+$ are both symmetric operators whose adjoint are respectively $\mathcal{F}_m^+$ and $\mathcal{A}_m^+$. Thus, we get

$$\ker(\mathcal{F}^+) = \text{rge}(\mathcal{F}_m^+)^\perp \quad \text{and} \quad \ker(\mathcal{A}^+) = \text{rge}(\mathcal{A}_m^+)^\perp.$$ 

Moreover, it is well-known that

$$\dim \ker(\mathcal{A}^+) = \dim \ker(\mathcal{F}^+) \leq 2n \quad \text{and} \quad \dim \ker(\mathcal{A}_m^+) = \dim \ker(\mathcal{F}_m^+) = 0.$$ 

So, in order to conclude the proof of the first claim, we only need to prove that

• $\text{rge}(\mathcal{A}_m^+)$ is closed if and only if $\text{rge}(\mathcal{F}_m^+)$ is closed.

Let us consider the closed subspaces

$$H_1 := \left\{ (v,0)^T \mid v \in L^2(\mathbb{R}^+, \mathbb{R}^n) \right\} \quad \text{and} \quad H_2 := \left\{ (0,u)^T \mid u \in L^2(\mathbb{R}^+, \mathbb{R}^n) \right\}$$

and we observe that $L^2 = H_1 \oplus H_2$.

First claim. The following implication holds

$$\text{rge}(\mathcal{F}_m^+) \text{ is closed } \Rightarrow \text{rge}(\mathcal{A}_m^+) \text{ is closed}$$

In fact, a straightforward computation shows that

$$\mathcal{F}_m^+ \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ h \end{pmatrix} \iff y = Px + Qx \text{ and } A_m^+x = h.$$  

Equation (D.2) implies that

$$h \in \text{rge}(A_m^+) \iff \begin{pmatrix} 0 \\ h \end{pmatrix} \in \text{rge}(\mathcal{F}_m^+).$$

It follows that $H_2 \cap \text{rge}(\mathcal{F}_m^+)$ is isomorphic to $\text{rge}(A_m^+)$. So, if $\text{rge}(\mathcal{F}_m^+)$ is closed, then $\text{rge}(A_m^+)$ is closed as well.
Second claim. We now show that

\[
\text{rge} (A_m^+) \text{ is closed } \implies \text{rge} (\mathcal{F}_m^+) \text{ is closed}
\]

We assume that \( \text{rge} (A_m^+) \) is closed. To conclude, it is enough to show that \( H_2 \cap \text{rge} \mathcal{F}_m^+ \) is closed in \( L^2(\mathbb{R}, \mathbb{R}^{2n}) \). Let now \( s > \tilde{s} \) (where \( \tilde{s} \) is given in Lemma D.5). By Lemma D.4 and Lemma D.5, we get that \( \mathcal{F}_{0,s}^+ \) is a Fredholm operator (being invertible on its domain) and \( \text{ker} (\mathcal{F}_{0,s}^+) = \{0\} \). By the closed graph theorem we conclude that \( \mathcal{F}_{0,s}^+ \) has a bounded inverse on \( \text{rge} \mathcal{F}_{0,s}^+ \). Let \( f \in \text{rge} \mathcal{F}_{0,s}^+ \). Then, we have

\[
f - \mathcal{F}_m^+ (\mathcal{F}_{0,s}^+)^{-1} f = (\mathcal{F}_{0,s}^+ - \mathcal{F}_m^+) (\mathcal{F}_{0,s}^+)^{-1} f = \begin{pmatrix} 0 & 0 \\ 0 & R_+ - sP_+ \end{pmatrix} (\mathcal{F}_{0,s}^+)^{-1} f \in H_2.
\]

Let \( T = I - \mathcal{F}_m^+ (\mathcal{F}_{0,s}^+)^{-1} \). Then \( T \) is a continuous operator from \( \text{rge} \mathcal{F}_{0,s}^+ \) to \( H_2 \) and \( Tf \in \text{rge} \mathcal{F}_m^+ \) if and only if \( f \in \text{rge} \mathcal{F}_{m}^+ \). It follows that

\[
\text{rge} (\mathcal{F}_{0,s}^+) \cap \text{rge} (\mathcal{F}_m^+) = T^{-1}(H_2 \cap \text{rge} \mathcal{F}_m^+).
\]

So \( \text{rge} \mathcal{F}_{0,s}^+ \cap \text{rge} \mathcal{F}_m^+ \) is closed. Let \( X = L^2(\mathbb{R}, \mathbb{R}^{2n}) \). We have

\[
\text{rge} \mathcal{F}_m^+ / (\text{rge} \mathcal{F}_{0,s}^+ \cap \text{rge} \mathcal{F}_m^+) \cong (\text{rge} \mathcal{F}_{0,s}^+ + \text{rge} \mathcal{F}_m^+) / \text{rge} \mathcal{F}_{0,s}^+
\]

Note that \( \dim ((\text{rge} \mathcal{F}_{0,s}^+ + \text{rge} \mathcal{F}_m^+) / \text{rge} \mathcal{F}_{0,s}^+) \leq \text{codim} \mathcal{F}_{0,s}^+ \). So \( \dim ((\text{rge} \mathcal{F}_m^+ / (\text{rge} \mathcal{F}_{0,s}^+ \cap \text{rge} \mathcal{F}_m^+) ) < \infty \). Then \( \text{rge} \mathcal{F}_{m}^+ \) is a direct sum of \( \text{rge} \mathcal{F}_{0,s}^+ \cap \text{rge} \mathcal{F}_m^+ \) with a finite dimensional space. So \( \text{rge} \mathcal{F}_m^+ \) is closed since \( \text{rge} \mathcal{F}_{0,s}^+ \cap \text{rge} \mathcal{F}_m^+ \) is closed.

The second claim is a straightforward consequence of the previous equalities. By these the result follows. \( \square \)

We let \( B(+) := \begin{pmatrix} P_+^{-1} & -P_+^{-1}Q_+ \\ -Q_+^TP_+^{-1} & Q_+^TP_+^{-1}Q_+ - R_+ \end{pmatrix} \) where \( P_+, Q_+, R_+ \) are the matrices appearing in condition (H2). We define the following operators

\[
\begin{align*}
\mathcal{F}_{L_0}^+ := -J \frac{d}{dt} - B(+) : W^1_{L_0}(\mathbb{R}^+, \mathbb{R}^{2n}) &\subset L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \to L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \\
\mathcal{F}_m^+ := -J \frac{d}{dt} - B(+) : W^1_{L_m}(\mathbb{R}^+, \mathbb{R}^{2n}) &\subset L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \to L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \\
\mathcal{F}_m^+ := -J \frac{d}{dt} - B(+) : W^1_{L_0}(\mathbb{R}^+, \mathbb{R}^{2n}) &\subset L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \to L^2(\mathbb{R}^+, \mathbb{R}^{2n})
\end{align*}
\]

Lemma D.7. The operator

\[
\mathcal{F}^+ = -J \frac{d}{dt} - B(t) : W^1_{L}(\mathbb{R}^+, \mathbb{R}^{2n}) \subset L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \to L^2(\mathbb{R}^+, \mathbb{R}^{2n})
\]

is a relatively compact perturbation of the operator \( \mathcal{F}_{L_0}^+ \) given in Equation (D.3).

Proof. We fix \( \lambda \) in the resolvent set of \( \mathcal{F}_{L_0}^+ \), and we need to prove that the operator

\[
L_\lambda = (\mathcal{F}_{L_0}^+ - \mathcal{F}_{L_0}^+) \circ (\mathcal{F}_{L_0}^+ - \lambda I)^{-1} : L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \to L^2(\mathbb{R}^+, \mathbb{R}^{2n})
\]

is compact.

Define a sequence \( \{\chi_k\}_{k \in \mathbb{N}} \) of \( \mathcal{C}^\infty \) functions on \( \mathbb{R}^+ \) such that \( \sup_{t \in \mathbb{R}^+} |\chi(t)| \leq 1 \), \( \sup_{t \in \mathbb{R}^+} |\chi'(t)| \leq 1 \) and

\[
\chi_k(t) = \begin{cases} 
1 & \text{if } t \in [0, k - 1] \\
0 & \text{if } t \in [k, +\infty).
\end{cases}
\]

We define the bounded multiplication operator through the action of \( \chi_k \) by

\[
\alpha_k : L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \ni x \mapsto \chi_k x \in L^2(\mathbb{R}^+, \mathbb{R}^{2n})
\]

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and let us consider the operators defined by
\[ L_{k,\lambda} = (\mathcal{F}_{L_0}^+ - \mathcal{F}_{L_0}^-) \circ \alpha_k \circ (\mathcal{F}_{L_0}^+ - \lambda I)^{-1} : L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \rightarrow L^2(\mathbb{R}^+, \mathbb{R}^{2n}). \]

Then, we have
\[ \left((\mathcal{F}_{L_0}^+ - \mathcal{F}_{L_0}^-) (\alpha_k - I) x\right)(t) = (B(\infty) - B(t)) (\chi_k(t) - 1) x(t) \]
and \( \lim_{k \to \infty} (B(\infty) - B(t)) (\chi_k(t) - 1) = 0 \) uniformly with respect to \( t \).

So, we get that \( (\mathcal{F}_{L_0}^+ - \mathcal{F}_{L_0}^-) (\alpha_k - I) \) converge to \( 0 \) in the operator norm topology and this, in particular, implies that the operator \( L_{k,\lambda} \) converges to \( L_{\lambda} \) in the operator norm for \( k \to +\infty \).

Thus, in order to conclude, we need to prove that the operator \( L_{k,\lambda} \) is compact for all \( k \) (being, in this case, the set of compact operators, a closed ideal of the linear bounded operators onto \( L^2 \)).

We observe that the operator
\[ (\mathcal{F}_{L_0}^+ - \lambda I)^{-1} : L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \rightarrow W^{1,2}(\mathbb{R}^+, \mathbb{R}^{2n}) \]
is bounded. Since \( \chi_k(t) = 0 \) for all \( t \geq k \), \( \sup_{t \in \mathbb{R}^+} |\chi(t)| \leq 1 \) and \( \alpha_k : W^{1,2}(\mathbb{R}^+, \mathbb{R}^{2n}) \rightarrow W^{1,2}([0, k], \mathbb{R}^{2n}) \)
is a bounded operator, then, by the Sobolev embedding theorem, we get that the embedding of \( W^{1,2}([0, k], \mathbb{R}^{2n}) \) into \( L^2([0, k], \mathbb{R}^{2n}) \) is compact. So, \( \alpha_k \circ (\mathcal{F}_{L_0}^+ - \lambda I)^{-1} \) is a compact operator from \( L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \) to \( L^2(\mathbb{R}^+, \mathbb{R}^{2n}) \). Since \( \mathcal{F}_{L_0}^+ - \mathcal{F}_{L_0}^- \) is bounded, then the operator \( L_{\lambda,k} \) is compact. This concludes the proof. \( \square \)

**Lemma D.8.** The operator \( \mathcal{F}^{+\infty} \) defined by Equation (D.3) is Fredholm if and only if the matrix \( JB(+\infty) \) is hyperbolic. In this case, its Fredholm index is equal to the dimension of the negative spectral space of the matrix \( JB(+\infty) \), namely \( \text{ind} \mathcal{F}^{+\infty} = \dim V^-(JB(+\infty)) \).

**Proof.** By [RS05b, Theorem 2.3], we get that the operator given by
\[ G^{+\infty} := \frac{d}{dt} - JB(+\infty), \]
on \( W^{1,2}(\mathbb{R}^+, \mathbb{R}^{2n}) \) is Fredholm if and only if the matrix \( JB_+ \) is hyperbolic. Moreover, if the operator \( G^{+\infty} \) is Fredholm, then the following equality holds, \( \text{ind} G^{+\infty} = \dim V^-(JB(+\infty)) \).

It is straightforward to check that \( \text{rge} \mathcal{F}^{+\infty} = - \text{rge} G^{+\infty} \), which implies that \( \text{codim} \mathcal{F}^{+\infty} = \text{codim} G^{+\infty} \) and \( \text{ker} \mathcal{F}^{+\infty} = \text{ker} G^{+\infty} \). The thesis readily follows by Lemma D.1 and [RS05b, Lemma 2.1]. \( \square \)

**Corollary D.9.** The operator \( \mathcal{F}_{m}^{+\infty} \) defined in Equation (D.3) is Fredholm if and only if the matrix \( JB_+ \) is hyperbolic.

**Proof.** We recall that \( \mathcal{F}_{m}^{+\infty} \) and \( \mathcal{F}^{+\infty} \) are conjugated and so, the thesis readily follows by Lemma D.8. \( \square \)

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