Dirac Operators on Coset Spaces

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Abstract

The Dirac operator for a manifold $Q$, and its chirality operator when $Q$ is even dimensional, have a central role in noncommutative geometry. We systematically develop the theory of this operator when $Q = G/H$, where $G$ and $H$ are compact connected Lie groups and $G$ is simple. An elementary discussion of the differential geometric and bundle theoretic aspects of $G/H$, including its projective modules and complex, Kähler and Riemannian structures, is presented for this purpose. An attractive feature of our approach is that it transparently shows obstructions to spin- and spin$^c$-structures. When a manifold is spin$^c$ and not spin, $U(1)$ gauge fields have to be introduced in a particular way to define spinors [1][2]. Likewise, for manifolds like $SU(3)/SO(3)$, which are not even spin$^c$, we show that $SU(2)$ and higher rank gauge fields have to be introduced to define spinors. This result has potential consequences for string theories if such manifolds occur as $D$-branes. The spectra and eigenstates of the Dirac operator on spheres $S^n = SO(n+1)/SO(n)$, invariant under $SO(n+1)$, are explicitly found. Aspects of our work overlap with the earlier research of Cahen et al. [2].
1 Introduction

When a group $G$ acts transitively on a manifold $Q$ with stability group $H$ at a point $p$, we can identify $Q$ with the coset space $G/H$. Such spaces are important in the description of Goldstone modes created by the spontaneous breakdown of $G$ to $H$. Models of spacetime such as the Minkowski spacetime $M^{3,1}$ or its compact Euclidean version $S^4$ are also of this sort. The group $G$ in these cases is the Poincaré group and $SO(5)$ respectively, while $H$ is the Lorentz group and $SO(4)$ respectively. In addition, coset spaces like $CP^N$ and $S^N$ have begun to proliferate as $D-$branes in string and boundary conformal field theories.

The Dirac operator for a manifold $Q$, and its chirality operator when $Q$ is even–dimensional, have a central role in noncommutative geometry. That is a good motivation for their study. This work focuses on this enterprise when $Q$ is a coset space. In addition, in a subsequent paper, we shall develop fuzzy versions of certain coset spaces and their Dirac and chirality operators, primarily as a device to regularize quantum field theories thereon, and what we do here is also a preparation for it.

We assume throughout that $G$ is a simple compact connected Lie group and $H$ is a compact connected group. Without loss of generality we assume also that $G$ is simply connected. These restrictions on $G$ and $H$ can be relaxed somewhat, $G$ can be semi–simple for instance, and certain noncompact Lie groups $G$ too seem approachable by our methods.

Not all $G/H$ admit a spin−, or even a spin$\_c$−structure [3]. One attractive aspect of our approach is that obstructions to spin− and spin$\_c$−structure show up transparently and we can also easily see when and how we can overcome them using suitable generalized spin−structures, spin$\_K$. The latter in general involve groups $K$ of any dimension, whereas spin$\_c$ (= spin$\_U(1)$ in our notation) uses $U(1)$ of dimension 1. The role of $K$ is roughly that of a gauge group, so insisting on the existence of spinors introduces nontrivial gauge symmetry and internal degrees of freedom. In addition, typically, spin$\_K$− theories are chiral, left- and right–chiral spinors transforming differently under $K$. This suggests that there may be a clever way to use fuzzy spaces to get the chiral fermions of the standard model.

There is a simple global approach to differential geometry on $G/H$. We introduce this formalism after setting up the preliminaries in Section 2. We follow this up in Section 3 introducing spin− and spin$\_K$−structures. Their Dirac and chirality operators are formulated in Section 4. We call this version of the Dirac operator 'Kähler–Dirac operator', as it is similar to the operator with the same name on a complex manifold. There is another equivalent version using projective modules equally useful for fuzzy physics,
which we have decided to call the projective Dirac operator. The ‘Dirac’ operator then refers to either of these two versions. Section 5 takes this up and also establishes its equivalence to the Kähler–Dirac operator. Along the way, the differential geometry of Section 2 is also translated to the language of projective modules. The cut–off versions of these expressions have an important role in fuzzy physics. In Section 5 we also explicitly consider the spheres $S^n$ and $\mathbb{C}P^n$. In particular, for spheres, we compute the curvature and Dirac spectrum for the maximally symmetric metric. Section 6 extends the preceding considerations to gravity on $G/H$. Finally Section 7 discusses the complex and Kähler structures of coset manifolds.

2 Differential Geometry on $G/H$

2.1 Preliminaries

$G$ is a simple, simply connected, connected, compact Lie group with Lie algebra $G$. $H$ is a subalgebra of $G$ which by exponentiation generates a compact connected Lie group $H$.

We think of $G$ concretely as $N \times N$ unitary matrices. The Lie algebra $H$ then has a basis $\{T(\alpha)\}$ of hermitean matrices (we follow physics conventions, more correctly $\{iT(\alpha)\}$ span $H$), which are trace orthogonal:

$$\text{Tr} T(\alpha)T(\beta) = c \delta_{\alpha\beta} \quad , \quad c = \text{constant} > 0 . \quad (2.1)$$

Using trace to define scalar product, $G$ can be decomposed as the orthogonal direct sum

$$G = H \oplus G/H . \quad (2.2)$$

Let $\{S(i)\}$ be a basis for $G/H$ with

$$\text{Tr} S(i)S(j) = c \delta_{ij} . \quad (2.3)$$

We also of course have

$$\text{Tr} S(i)T(\alpha) = 0 . \quad (2.4)$$

We denote the elements of the basis $\{T(\alpha), S(i)\}$ collectively as $\Sigma_A, \ A \in \{\alpha, i\}$.

Let $Ad$ denote the adjoint representation of $G$. Then $H \subset G$ leaves $G/H$ invariant in this representation:

$$h S(i) h^{-1} = S(j)(Ad \ h)_{ji} \quad , \quad h \in H . \quad (2.5)$$
We call this representation of $H$ on $G/H$ as $\text{Ad}_{G/H}$, and the corresponding representation of $H$ as $\text{ad}_{G/H}$. $\text{Ad}_{G/H}(h)$ are real matrices as the hermitean conjugation of (2.5) shows. They are also orthogonal as conjugation leaves the relation (2.3) invariant. Thus if $|G|, |H|$ and $|G/H| = |G| - |H|$ denote the dimensions of $G$, $H$ and $G/H$, $\{\text{Ad}_{G/H}(h)\}$ is a subgroup of $SO(|G/H|)$:

$$\{\text{Ad}_{G/H}(h)\} \subseteq SO(|G/H|).$$  \hspace{1cm} (2.6)

The above discussion implies the following commutation relations:

$$[T(\alpha), T(\beta)] = ic_{\alpha\beta\gamma}T(\gamma),$$
$$[T(\alpha), S(i)] = ic_{\alpha ij}S(j),$$
$$[S(i), S(j)] = ic_{ij\alpha}T(\alpha) + ic_{ijk}S(k).$$  \hspace{1cm} (2.7)

The structure constants $c_{ABC}$ are real and totally antisymmetric.

We will call $c_{ijk}$ the torsion of the space $G/H$. Below we will see that it plays exactly the role of the usual torsion for the canonical covariant derivative on $G/H$ [5]. If $c_{ijk} = 0$, the homogeneous space $G/H$ is said to be ‘symmetric’ [6]. In that case, $G$ admits the involutive automorphism:

$$\sigma : T(\alpha) \rightarrow T(\alpha), \quad S(i) \rightarrow -S(i)$$  \hspace{1cm} (2.9)

leaving $H$ fixed. $\sigma$ lifts to an involutive automorphism $\Sigma$ of $G$ leaving $H$ fixed, $\Sigma$ being defined from

$$\Sigma : e^{i\theta_{\alpha}}T(\alpha) \rightarrow e^{i\theta_{\alpha}}T(\alpha), \quad e^{i\theta_{i}}S(i) \rightarrow e^{-i\theta_{i}}S(i).$$  \hspace{1cm} (2.10)

\section*{2.2 Tensor Fields on $G/H$}

Let $W$ be a fixed vector space with an orthonormal basis $\{e_i\}$ which carries the representation $\text{Ad}_{G/H}$ of $H$, $h : e_i \rightarrow e_j\text{Ad}_{G/H}(h)_{ji}$. The vector space $W^{\otimes n} = W \otimes W \otimes \ldots \otimes W$ ($n$ factors) carries the tensor product representation $\text{Ad}_{G/H}^{\otimes n} = \text{Ad}_{G/H} \otimes \text{Ad}_{G/H} \otimes \ldots \otimes \text{Ad}_{G/H}$ ($n$ factors). Let $\mathbb{C} \equiv W^{\otimes 0}$ also denote the one–dimensional complex vector space carrying the trivial representation $\text{Ad}_{G/H}^{\otimes 0} : h \rightarrow 1$.

Tensor fields of rank $n$ on $G/H$ can be defined to be equivariant functions on $G$ with values in $W^{\otimes n}$. That means the following: for $n = 0$ we have scalar fields $f^{(0)}$, complex (or $W^{\otimes 0}$) valued functions on $G$ invariant under the right–action of $H$ on $G$ (equivariance):

$$f^{(0)} = \text{scalar fields} : f^{(0)}(gh) = f^{(0)}(g), \quad \forall h \in H.$$  \hspace{1cm} (2.11)
A tensor field \( f^{(1)} \) of rank 1 has values in \( W \); we can write it as:

\[
f^{(1)} = f^{(1)}_i e_i, \quad f^{(1)}_i : g \rightarrow f^{(1)}_i(g) \in \mathbb{C}.
\]  
(2.12)

Equivariance for \( n = 1 \) means the following transformation property under the right action of \( H \) on \( G \):

\[
f^{(1)}_i(gh)e_i = f^{(1)}_i(g) \text{Ad}_{G/H}(h)_{ij}e_j.
\]  
(2.13)

Therefore

\[
f^{(1)}_i(gh) = f^{(1)}_j(g) \text{Ad}_{G/H}(h)_{ji}.
\]  
(2.14)

Let \( J \) label the inequivalent irreducible representations of \( G \) by unitary matrices \( \{D^J(g)\} \); their matrix elements in a convenient orthonormal basis are \( D^J_{mn}(g) \). We have that

\[
D^J_{mn}(gh) = D^J_{mn}(g) D^J_{n' n}(h).
\]  
(2.15)

If the representation \( h \rightarrow D^J(h) \) contains the identity representation of \( H \), we can choose the basis in the representation space so that the index \( n \) in (2.15) transforms trivially when \( n \in \) an appropriate index set \( I_0 \):

\[
D^J_{mio}(gh) = D^J_{mio}(g), \quad \forall \ i_0 \in I_0.
\]  
(2.16)

From this, (2.11) and Peter–Weyl theorem it follows that we can expand \( f^{(0)} \) in the form

\[
f^{(0)}(g) = \sum \xi^J_{mio} D^J_{mio}(g), \quad \xi^J_{mio} \in \mathbb{C}.
\]  
(2.17)

\( \xi^J_{mio} \) is zero if \( h \rightarrow D^J(h) \) fails to contain the trivial representation of \( H \).

Henceforth we assume for notational simplicity that the identity representation occurs only once in the restriction of the irreducible representations \( J \) of \( G \) to \( H \), and so drop the index \( i_0 \) from \( \xi^J_{mio} \). Otherwise a degeneracy index has to be included here and elsewhere.

In the same way, if the representation \( h \rightarrow D^J(h) \) contains \( \text{Ad}_{G/H} \), we can choose the basis in the representation space so that the index \( n \) in (2.15) transform by \( \text{Ad}_{G/H} \) if \( i,j \) belong to an appropriate index set \( I \):

\[
D^J_{mi}(gh) = D^J_{mj}(g)[\text{Ad}_{G/H}(h)]_{ji}, \quad i, j \in I.
\]  
(2.18)

(For notational simplicity we are assuming that \( \text{Ad}_{G/H} \) occurs only once in the representation \( J \), otherwise a degeneracy index has to be added here and elsewhere.) Then we can expand \( f^{(1)}_i \) in the form

\[
f^{(1)}_i(g) = \sum \xi^J_m D^J_{mi}(g), \quad \xi^J_m \in \mathbb{C}.
\]  
(2.19)
\( \xi_m^I \) now is zero if \( h \to D^I(h) \) fails to contain \( \text{Ad}_{G/H} \).

Continuing in this vein we see that tensor fields of rank \( n \) in component form look like \( f^{(n)}_{i_1...i_n} \) and have the expansion

\[
 f^{(n)}_{i_1...i_n}(g) = \sum \xi_m^{i} D_{m,(i_1...i_n)}(g) , \quad i_k \in I ,
 \]

(2.20)

\[
 D_{m,(i_1...i_n)}(gh) = D_{m,(j_1...j_n)}(g)[\text{Ad}_{G/H}(h)]_{j_1i_1}...[\text{Ad}_{G/H}(h)]_{j_ni_n} .
 \]

(2.21)

We have used a convenient multi–index notation for the second index of \( D^{i} \). The rest should be clear. Tensor fields of diverse permutation symmetries are readily constructed along similar lines.

### 2.3 Covariant Derivative.

Let \( \mathcal{T}^{(n)} \) denote the space of tensor fields of rank \( n \), with a typical member \( f^{(n)} = \{ f^{(n)}_{i_1...i_n} \} \). \( \mathcal{T}^{(0)} \) consists of functions, and it is also an algebra under pointwise multiplication. All \( \mathcal{T}^{(n)} \) are \( \mathcal{T}^{(0)} \)-modules. The covariant derivative \( \nabla \) is a map

\[
 \nabla : \mathcal{T}^{(n)} \to \mathcal{T}^{(n+1)} , \quad f^{(n)} \to \nabla f^{(n)} ,
 \]

(2.22)

where \( \nabla f^{(n)} \) has components \( (\nabla f^{(n)})_{i_1...i_n} \). It has in addition to fulfill the following important derivation property. Note that we can take tensor products of \( \mathcal{T}^{(n)} \)-s (over \( \mathcal{T}^{(0)} \), \( \mathcal{T}^{(n)} \) being \( \mathcal{T}^{(0)} \) modules):

\[
 \mathcal{T}^{(n)} \otimes \mathcal{T}^{(m)} = \mathcal{T}^{(n+m)} , \quad f^{(n)} \otimes f^{(m)} = f^{(n+m)} ,
 \]

(2.23)

where

\[
 f^{(n+m)}_{i_1...i_nj_1...j_m} = f^{(n)}_{i_1...i_n} f^{(m)}_{j_1...j_m} .
 \]

(2.24)

Then we require that

\[
 \nabla (f^{(n)} \otimes f^{(m)}) = \nabla f^{(n)} \otimes f^{(m)} + f^{(n)} \otimes \nabla f^{(m)} .
 \]

(2.25)

There is a natural choice for the covariant derivative in our case. We call it hereafter as \( X \). The action of \( X \) on functions is:

\[
 [X f^{(0)}]_i (g) = \frac{d}{dt} f^{(0)}(ge^{itS(i)}) \bigg|_{t=0} .
 \]

(2.26)

\( \nabla f^{(0)} \) transforms correctly in view of (2.5). In the same way the action on \( f^{(n)} \) is:

\[
 [X f^{(n)}]_{i_1...i_n} (g) = \frac{d}{dt} f^{(n)}_{i_1...i_n}(ge^{itS(i)}) \bigg|_{t=0} .
 \]

(2.27)
The right–hand side defines a vector field \( X_i \). Using \( X_i \) the covariant derivative in components is \( f_{i_1...i_n}^{(n)} \rightarrow X_i f_{i_1...i_n}^{(n)} = \text{R.H.S. of (2.27)} \).

The torsion of the covariant derivative vanishes only if \( [X_i, X_j] f^{(0)} = 0 \). From the definition and (2.7), we have

\[
[X_i, X_j] f^{(0)} = -c_{ijk} X_k f^{(0)}.
\]  

(2.28)

So there is torsion if \( c_{ijk} \neq 0 \). But there is an easy way to construct the torsion–free covariant derivative \( \overline{X}_i \). Set

\[
\overline{X}_i f^{(0)} = X_i f^{(0)},
\]

\[
\overline{X}_i f_j^{(1)} = X_i f_j^{(1)} + \frac{1}{2} c_{ijk} f_k^{(1)},
\]

\[
\ldots
\]

\[
\overline{X}_i f_{j_1...j_n}^{(n)} = X_i f_{j_1...j_n}^{(n)} + \frac{1}{2} c_{ij_1j_1'} f_{j_2...j_n}^{(n)} + \frac{1}{2} c_{ij_2j_2'} f_{j_1j_3...j_n}^{(n)} + \ldots + \frac{1}{2} c_{ij_nj_n'} f_{j_1...j_n}^{(n)}.
\]

Then

\[
[\overline{X}_i, \overline{X}_j] f^{(0)} = [X_i, X_j] f^{(0)} + c_{ijk} X_k f^{(0)} = 0,
\]  

(2.30)

just as we want.

Gauge fields will certainly have a central role in further developments. So we briefly indicate what they are here. Let us first consider \( U(1) \) gauge fields. The general gauge potential is \( A_i = \sum \xi_M^I D_M^i \), \( \xi_M^I \in \mathbb{C} \). It is subject to the reality condition \( \overline{A}_i = -A_i \). Then if \( f^{(n)} \) has charge \( e \), its covariant derivative is \( (\overline{X}_i + e A_i) f_{i_1...i_n}^{(n)} \), where \( A_i \) acts by pointwise multiplication. This definition is compatible with equivariance. We can substitute \( X_i \) for \( \overline{X}_i \) at the cost of possible torsion.

The gauge covariant derivative for a general gauge group as usual only involves regarding \( eA_i(g) \), that is \( e\xi_M^I \), to be Lie algebra valued, its action on \( f^{(n)} \) in (2.27) is then dictated by the representation content of the latter.

### 3 Spin– and Spin\(_K\)–structures

Spinorial fields are essential for physics. We can go about constructing them as follows. The orthogonal group \( SO(|G/H|) \) has a double cover \( Spin(|G/H|) \). Associated with \( SO(|G/H|) \), there is also a Clifford algebra \( \mathbb{C}l(|G/H|) \) with generators \( \gamma_1, \gamma_2, \ldots, \gamma_{|G/H|} \):

\[
\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} \mathbb{I}.
\]  

(3.1)

Here \( \mathbb{I} \) denotes unit matrix. (Its dimension should be clear from the context). \( \mathbb{C}l(|G/H|) \) has one or two inequivalent IRR’s of dimension \( 2^n \) if \( |G/H| = 2n \) or \( |G/H| = 2n + 1 \).
In the latter case, the two IRR’s are related by a change of sign of all $\gamma_i$’s. In either case, they generate a unique faithful representation of $Spin(|G/H|)$ with generators

$$\Sigma_{ij} = \frac{1}{4}(\gamma_i \gamma_j - \gamma_j \gamma_i)$$

which we call $Spin_{cl}(|G/H|)$.

A recursive scheme for constructing anticommuting sets of hermitian $\gamma$—matrices goes as follows. We start with a set of $2^{n-1} \times 2^{n-1}$ matrices $\gamma_i$, $i = 1, \ldots, 2n - 1$, satisfying eq.(3.1), and such that $(-i)^{n-1}\gamma_1 \ldots \gamma_{2n-1} = 1$, e.g. for $n = 2$, the three Pauli matrices. Then a set of $2^n \times 2^n$ matrices $\Gamma_\lambda$, $\lambda = 1, \ldots, 2n + 1$, satisfying eq.(3.1), and such that $(-i)^n\Gamma_1 \ldots \Gamma_{2n+1} = I$ is given by

$$\Gamma_i = \begin{pmatrix} 0 & \gamma_i \\ \gamma_i & 0 \end{pmatrix}, \quad i = 1, \ldots, 2n - 1, \quad \Gamma_{2n} = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \quad \Gamma_{2n+1} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (3.2)$$

The matrices $\gamma_1, \ldots, \gamma_{2n-1}$ span $\mathbb{C}l(2n - 1)$, and the matrices $\Gamma_1, \ldots, \Gamma_{2n}$ span $\mathbb{C}l(2n)$, whereas $\Gamma_1, \ldots, \Gamma_{2n+1}$ span $\mathbb{C}l(2n + 1)$.

### 3.1 Spin Manifolds

We say that $G/H$ is a spin manifold if the commutative diagram of Fig.1 exists, arrows being homomorphisms (which need not be onto):

$$
\begin{array}{ccc}
G & \supset & H \\
\downarrow & & \downarrow \\
Ad_{G/H} & \subset & SO(|G/H|)
\end{array}
$$

Fig. 1

The vertical homomorphisms are there by construction, so what is to be verified is the existence of the horizontal arrow. If it exists, a general spinor can be constructed as follows. We can reduce $Spin_{cl}(|G/H|)$ restricted to $H$ into a direct sum $\oplus \rho$ of unitary irreducible representations of $H$. Let $g \to D_J^I(g)$ be the unitary matrix of $g$ in a representation of $G$ which on restriction to $H$ contains $\oplus \rho$. Then we can restrict its second index $a$ to an index set $I$ so that it transforms by $\oplus \rho$ under $g \to gh$:

$$D_J^I_{oa}(gh) = D_J^I_{ab}(g)D_J^I_{ba}(h) \quad , \quad a, b \in I. \quad (3.3)$$

By construction we know how the Clifford algebra acts on the index $a \in I$. A general spinor $\psi$ then is a function on $G$ with components

$$\psi_a = \sum \xi_M^J D_J^I_{Ma}. \quad (3.4)$$

Let us look at examples.
Example 1: \( \mathbb{CP}^1 = SO(3)/SO(2) = [Spin(3) = SU(2)]/[Spin(2) = U(1)] \). So \( G = SU(2), \ H = U(1) = \{e^{i\sigma_i \theta/2}\}, \sigma_A \) the Pauli matrices. Then \( S(i) = \sigma_i, \ i = 1, 2, \) and

\[
e^{i\sigma_i \theta/2} \sigma_j e^{-i\sigma_i \theta/2} = \sigma_j R_{ji}(\theta), \quad R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in SO(2). \tag{3.5}
\]

\( Spin^Ct([G/H]) \) is just \( H = \{e^{i\sigma_i \theta/2}\} \), the homomorphism \( Spin^Ct(2) \to SO(2) \) being \( e^{i\sigma_i \theta/2} \to R(\theta) \). Thus Fig.1 exists and \( SU(2)/U(1) = S^2 \simeq \mathbb{CP}^1 \) is spin.

For a thorough treatment of noncommutative geometry and Dirac operator on \( S^2 \), see [4].

Example 2: Similar arguments show that all the spheres \( S^N = SO(N + 1)/SO(N) = Spin(N + 1)/Spin(N) \) are spin. \( G \) for \( S^N \) is \( Spin(N + 1) \) while \( H = Spin(N) \). \( Ad_{G/H} \) is \( SO(N) \), the \( \mathbb{Z}_2 \)-quotient of \( Spin(N) \). Since \( Spin^Ct([G/H]) \) is isomorphic to \( Spin(N) \), \( S^N \) is spin.

Example 3: \( \mathbb{CP}^2 = SU(3)/U(2) \). So \( G = SU(3), \ H = U(2) \). A basis for the 3–dimensional \( SU(3) \)-Lie algebra consists of the Gell-Mann matrices \( \lambda_A \). The \( U(2) \) Lie algebra has basis \( \lambda_1, \lambda_2, \lambda_3, \lambda_8 \), the hypercharge \( Y \) being \( \frac{1}{\sqrt{3}} \lambda_8 \). The \( S(i) \) are \( \lambda_4, \lambda_5, \lambda_6, \lambda_7 \).

Under \( U(2) \), they transform as \( (K^+, K^0) \) or \( (-\overline{K}^0, K^-) \) in particle physics notation. That means that \( Ad_{G/H} = U(2) \). Regarding \( U(2) \) as \( 2 \times 2 \) unitary matrices \( U \), we can embed \( U(2) \) in \( SO(4) \) by the map

\[
U \to \frac{1}{2} \begin{pmatrix} U + U^* & i(U - U^*) \\ -i(U - U^*) & U + U^* \end{pmatrix} \tag{3.6}
\]

\( Spin^Ct(4) \) is the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) representation of \( SU(2) \otimes SU(2) \). It is the double cover of \( SO(4) \). Now \( H \) is \( U(2) \) and \( Spin^Ct(4) \) has no \( U(2) \) subgroup. So \( \mathbb{CP}^2 \) is not spin [1].

Example 4: \( G = SU(3), \ H = SO(3) \). With \( G \) as \( 3 \times 3 \) unitary matrices, \( H \) consists of all real orthogonal matrices and corresponds to the spin 1 representation of \( SO(3) \). \( G/H \) is of dimension 5. It carries the spin 2 representation of \( SO(3) \), isomorphic (but not equivalent!) to the spin 1 representation. There is no homorphism \( SO(3) \to Spin^Ct(5) \) compatible with Fig.1, so that \( SU(3)/SO(3) \) is not spin [3].

Let us show this result in more detail. We can show it by establishing that the \( 2\pi \)-rotation in \( SO(3) \) becomes a noncontractible loop in \( SO(5) \) under the embedding in Fig.2. Then the inverse image of \( SO(3) \) under the homomorphism \( Spin^Ct(5) \to SO(5) \) is \( SU(2) \) giving us the result.

Now \( SO(3) \) acts on real symmetric traceless \( 3 \times 3 \) matrices \( T = (T_{ij}) \) according to \( T \to RTR^T \). This is its spin 2 representation. We can eliminate say \( T_{33} \) using \( \text{Tr} T = 0 \), thereby representing it as real transformations on \( (T_{11}, T_{12}, T_{13}, T_{22}, T_{23}) \). \( SO(5) \) consists
of real transformations on this five-dimensional vector, so we now have the needed explicit embedding of $SO(3)$ in $SO(5)$. Let

$$R(\theta) : \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

(3.7)

It generates the $2\pi$-rotation loop in $SO(3)$ as $\theta$ increases from 0 to $2\pi$. Consider $T' = R(\theta)TR(\theta)^T$. Then as $\theta$ increases from 0 to $2\pi$ we have a $2\pi$-rotation in the $T_{13} - T_{23}$ plane. But by the time $\theta = \pi$, $T_{ab}$ ($a, b \leq 2$) return to $T_{ab}$ so that the rotations in their planes are by $4\pi$ or/and 0 (in fact $\delta_{ab}(T_{11} + T_{22})$ undergoes no change and $T_{ab} - \frac{1}{2}\delta_{ab}(T_{11} + T_{22})$ undergoes $4\pi$-rotation). The corresponding loop of matrices in $SO(5)$ is the product of an odd number of $2\pi$-rotations and hence cannot be deformed to a point in $SO(5)$. That concludes the proof.

We will now explain the Dirac operators for Spin– and Spin$_K$–manifolds after discussing Spin$_K$–structures.

### 3.2 Spin$_K$–Manifolds

$K$ and $\mathcal{H}$ are compact connected Lie groups in what follows. We say that $G/H$ is a Spin$_K$–manifold if the commutative diagram of Fig. 2 exists.

$$
\begin{array}{rcl}
G & \supset & H \\
\downarrow & & \downarrow^K \\
Ad_{G/H} & \subset & SO(|G/H|) \\
\end{array}$$

Fig. 2

$K$ and $\mathbb{Z}_2$ on the arrows are to show that they are the kernels of those homomorphisms.

Spin$_{U(1)}$ in our language is what mathematicians call Spin$_c$.

The intersection Spin$_{ct}(|G/H|) \cap K$ clearly contains $\mathbb{Z}_2$. It cannot be larger, for that would mean that the kernel for the slanting arrow exceeds $\mathbb{Z}_2$. Thus $\mathcal{H} \supset [\text{Spin}_{ct}(|G/H|) \times K] / \mathbb{Z}_2$. Its quotient by $K$ being exactly $SO(|G/H|)$, we conclude that

$$\mathcal{H} = [\text{Spin}_{ct}(|G/H|) \times K] / \mathbb{Z}_2,$$  

(3.8)
giving Fig. 3, also a commutative diagram:

\[
\begin{array}{ccc}
G \supset H & \longrightarrow & \mathcal{H} = [\text{Spin}^\text{Cl}(|G/H|) \times K]/\mathbb{Z}_2 \supset \text{Spin}^\text{Cl}(|G/H|) \\
\downarrow & & \downarrow K \\
\text{Ad}_{G/H} & \subset & \text{SO}(|G/H|)
\end{array}
\]

Fig. 3

Let us denote the generators of $\mathbb{Z}_2$ in $\text{Spin}^\text{Cl}(|G/H|)$ and $K$ by $z_{\text{spin}}$ and $z_K$, they square to the respective identities. The inclusion of $\text{Spin}^\text{Cl}(|G/H|)$ in Fig. 3 is to be understood as follows. The elements of $\mathcal{H}$ are the equivalence classes

\[
<s, k> = <z_{\text{spin}}s, z_Kk> , \quad s \in \text{Spin}^\text{Cl}(|G/H|) , \quad k \in K . \quad (3.9)
\]

Then the top inclusion is via the isomorphism

\[
s \rightarrow <s, e_K> , \quad e_K = \text{identity of } K . \quad (3.10)
\]

As we think of $\mathcal{H}$ as the concrete matrix group obtained by tensoring $\text{Spin}^\text{Cl}(|G/H|)$ with a faithful unitary representation of $K$ where $z_K$ is represented by $-I$

\[
\mathcal{H} = \text{Spin}^\text{Cl}(|G/H|) \otimes K \quad (3.11)
\]

we can write $-1$ for $z_{\text{spin}}$ and $z_K$. The inclusion of $\text{Spin}^\text{Cl}(|G/H|)$ is then just $s \rightarrow s \otimes I$.

Let us motivate the new requirements in Figures 2 and 3. For a physicist, a spinor changes sign under ‘2π–rotation’. $\mathcal{H}$ is the group acting on $\text{Spin}_K$–spinors. We have required it to contain $\text{Spin}^\text{Cl}(|G/H|)$, so we can check this requirement by looking at the action of $2\pi$–rotation $\in \text{Spin}^\text{Cl}(|G/H|) \subset \mathcal{H}$. As for asking that $H \rightarrow \mathcal{H}$, we can reduce the representation of $\mathcal{H}$ into a direct sum $\oplus \rho$ of irreducible representations $\rho$ of $H$ just as in the discussion of spin structures. The action of the Clifford algebra on $\oplus \rho$ by construction is known. The wave functions of $\text{Spin}_K$–spinors are then given by linear spans of representations of $G$ induced by $\oplus \rho$, see (3.4). Later we shall see how the Dirac operator can be defined on these wave functions.

Example 5. $G = SU(3)$, $H = U(2)$, $G/H = CP^2$. Here we choose $K = U(1)$. Elements of $U(2)$ can be written as the equivalence classes

\[
<s, u> = <-s, -u> , \quad s \in SU(2) , \quad u \in U(1) , \quad (3.12)
\]
where we identify $SU(2)$ with $2 \times 2$ unitary matrices of unit determinant and $u$ with a phase. $Spin^{Cl}(4)$ is $SU(2) \otimes SU(2)$ and $\mathcal{H}$ consists of the equivalence classes

\[ < s_1, s_2, u > = < -s_1, -s_2, -u > . \] (3.13)

With elements of $SO(4)$ represented as $< s_1, s_2 > = < -s_1, -s_2 >$, the homomorphism $\mathcal{H} \to SO(4)$ is $< s_1, s_2, u > \to (s_1, s_2)$. The homomorphism $H \to \mathcal{H}$ is also simple:

\[ [s, u] \to < s, s, u > . \] (3.14)

Thus $\mathbb{C}P^2$ is $Spin_{U(1)}$ or $Spin_c$.

**Example 6.** $G = SU(3), \ H = SO(3)$.

We return to the choices $G = SU(3), \ H = SO(3)$. Choosing $K = U(1)$ is not helpful now, as we lack a suitable homomorphism $H = SO(3) \to \mathcal{H} = Spin^{Cl}(5) \otimes \mathbb{Z}_2 U(1)$. So $SU(3)/SO(3)$ is not even $Spin_{U(1)}$, a result originally due to Landweber and Stong [3]. A better choice is $K = SU(2)$. Then we can find the homomorphism $H \to \mathcal{H}$ as follows. The image of $Ad_{G/H}$ in $SO(5)$ is an $SO(3)$ subgroup $SO(3)'$. Its inverse image in $Spin^{Cl}(5)$ is an $SU(2)$ subgroup $SU(2)'$. Let $\vec{\Sigma}$ and $\vec{T}$ be the angular momentum generators of $SU(2)'$ and $K$. If $\vec{L}$ are the angular momentum generators of $H$, the map at the level of Lie algebras is just $\vec{L} \to \vec{\Sigma} + \vec{T}$. Hence $SU(3)/SO(3)$ is $Spin_{SU(2)}$. More such examples can be found.

### 3.3 What is $\mathbf{X}(i)$ now?

We need the extension of the torsion–free connection with components $X_i$ to spinors on general spin$K$–manifolds. The first step in this direction is the extension of $X_i$.

A spinor field $\psi = (\psi_a)$ on a $Spin_K$–manifold has the expansion

\[ \psi_a = \sum \xi^L_M D^L_{Ma} , \] (3.15)

where $a$ carries the action of the Clifford algebra. The definition of $X_i$ on $\psi$ is immediate from (2.26):

\[ (X_i \psi)(g) = \frac{d}{dt} \sum \xi^L_M D^L_{Ma}(ge^{itS(i)})\bigg|_{t=0} . \] (3.16)

The definition of $\mathbf{X}_i$ involves the extension of $c_{ijk}$ to spinors so that it can act on the index $a$. Now, the generators of the $SO(|G/H|)$-Lie algebra are $M_{ij}$ where:

\[ (M_{ij})_{kl} = -i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) . \] (3.17)

Its image in the spinor representation is $\frac{1}{4\iota}[\gamma_i, \gamma_j]$. 

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Now
\[ c_{ij'k'}(M_{j'k'})_{kj}f_k^{(1)} = -2ic_{ikj}f_k^{(1)}. \]  

(3.18)

Hence
\[ \overline{X}_i f_j^{(1)} = X_i f_j^{(1)} + \frac{1}{4i}c_{ij'k'}(M_{j'k'})_{kj}f_k^{(1)}. \]  

(3.19)

From this follows the definition of \( \overline{X}_i \) on spinors:
\[ \overline{X}_i \psi_a = X_i \psi_a - \frac{1}{16}c_{ijk} ([\gamma_j, \gamma_k])_{ba} \psi_b. \]  

(3.20)

A tensor formed of spinors will then transform correctly.

The introduction of gauge fields follows the earlier discussion.

4 The Dirac and Chirality Operators

The massless Dirac operator for the torsion free connection \( \overline{X}_i \) is just:
\[ D_W = -i\gamma^R \overline{X}_i, \]  

(4.1)

where the superscript \( R \) indicates that the \( \gamma \)'s act on spinors on the right. It is self-adjoint. This expression can be gauged, and \( \overline{X}_i \) can be substituted by \( X_i \) if we can tolerate torsion.

If \(|G/H|\) is even, e.g. if \( G/H \) is a [co]adjoint orbit, there is also a chirality operator \( \gamma \) anticommuting with \( D_W \):
\[ \gamma = (-i)^{\frac{1}{2}|G/H|} \gamma_1 \ldots \gamma_{|G/H|} = \gamma^\dagger, \quad \gamma^2 = \mathbb{I}. \]  

(4.2)

The subscript ‘\( W \)’ is to indicate that it is the form of the Dirac operator used by the Watamuras \[8\]. For even \(|G/H|\) there is also the unitarily equivalent Dirac operator \[9\]
\[ D = e^{i\gamma^R \pi/4} D_W e^{-i\gamma^R \pi/4} = i\gamma^R D_W', \]  

(4.3)

which is central to fuzzy physics.

5 Projective Modules and their Dirac Operator

5.1 Projective Modules.

In the algebraic approach to vector bundles, their sections are substituted by elements of projective modules (‘of finite type’) \[10\]. A projective module is constructed as follows.
Let $A$ be an algebra. It can be the commutative algebra $A$ of $C^\infty$–functions on a manifold $M$ if our interest is in the algebraic description of its vector bundles. But it can also be a noncommutative algebra, in which case there is no evident correspondence with sections of differential geometric vector bundles. Consider $A^N \equiv A \otimes_C \mathbb{C}^N$ with elements $a = (a_1, \ldots, a_N)$, $a_i \in A$. Let $P$ be an $N \times N$ projector with coefficient in $A$:

$$P_{ij} \in A, \quad P^\dagger = P = P^2. \quad (5.1)$$

Then $A^N P$ (whose elements are vectors $\alpha$ with components $a_j P_{ji}$) is a projective module. The Serre–Swan theorem ([10]) establishes that sections of any vector bundle can be got from some $N$ and $P$.

It is very helpful for subsequent developments to have a projective module description of vector bundles. We can find the appropriate projectors by a known method described nicely by Landi [11]. It goes as follows.

Consider for example a rank 1 tensor field and any particular $D^J$ matrix occurring in its expansion, with elements $D^J_{\rho i}$. We have

$$(D^J)_{i\rho}^\dagger(g) D^J_{\rho j}(g) = \delta_{ij}. \quad (5.2)$$

Let

$$P^J(g)_{\rho \sigma} = D^J_{\rho i}(g)(D^J)_{i \sigma}^\dagger(g). \quad (5.3)$$

Since $P^J(g h) = P^J(g)$ if $h \in H$, $P^J_{\rho \sigma}$ are functions on $G/H$. In view of (5.2), they are projectors too. If $P^J(g)$ are $|J| \times |J|$ matrices, a projective module describing rank 1 tensor fields is

$$A^{\lvert J \rvert} P^J = \langle \alpha^J = (\alpha^J_1, \ldots, \alpha^J_{\lvert J \rvert}), \alpha_i^J = a_k^J P^J_{ki} \rangle. \quad (5.4)$$

There is no unique correspondence between projective modules and vector bundles. Thus for each $J$, we can find a projector and its module. But all such modules are equivalent, since there are elements $\alpha^J = a^J P^J$ and $\alpha^K = a^K P^K$ which naturally correspond for different $J$ and $K$:

$$\alpha^J_\rho = \sum \xi^L_M D^L_{M_i}(D^J)_{i \rho}^\dagger, \quad \alpha^K_\sigma = \sum \xi^L_M D^L_{M_i}(D^K)_{i \sigma}^\dagger. \quad (5.5)$$

### 5.2 Differential Geometry

There is much to be said on the differential geometry on projective modules, but for reasons of brevity we limit ourselves to indicating how to extend the definitions of $X(i)$ and $\overline{X}(i)$. 

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Let us first focus on the tensorial case. Let
\[ P = (P_{\lambda\rho}) , \quad P_{\lambda\rho} = (D_J^I)_{i\rho}^i \]
be a projector appropriate for rank 1 tensors. Then what substitutes for the torsion–free \( X_i \) acting on \( \alpha^J \) is \( \nabla_\lambda \), which is defined by
\[ \nabla_\lambda \alpha^J_\rho(g) = \sum \xi_M^L \frac{d}{dt} D_M^I (ge^{it\Sigma(A)}) \bigg|_{t=0} (D_J^I)_{i\rho}^i (D_J^J)_{j\lambda}^j (g) . \]  
It belongs to the projective module for rank 2 tensors with the natural choice \( P \otimes P \) of their projectors. A more compact expression for the covariant derivative can be given in terms of the right-invariant vector fields of \( G \), defined by
\[ (L_A f)(g) = -i \frac{d}{dt} f(e^{-it\Sigma(A)}) \bigg|_{t=0} , \]
so that
\[ (L_A)(D_J^I)_{\lambda\rho}^i = (D_J^I)_{\lambda\rho}^i (\Sigma_A)_{j\rho}^J \] .
If functions on \( G/H \) are regarded as functions on \( G \) invariant by right action of \( H \) on \( G \), \( L_A \) on these correspond to ‘orbital’ operators of angular momentum.

The vector fields \( L_A \) are related to the left–invariant vector fields \( X_A \) by:
\[ (X_A f)(g) = \frac{d}{dt} f(ge^{it\Sigma(A)}) \bigg|_{t=0} = -i(L_B f)(g) (Ad g)_{BA} . \]
From this we may derive the following expression for the covariant derivative:
\[ \nabla_\lambda = -i Ad g_{Ai} (D_J^I)_{iA}^i J_A , \quad J_A = \text{‘total angular momentum’} = L_A - \Sigma_A^R , \]
where \( \Sigma_A^R \) are the generators appropriate for representation \( J \) acting on the right. In fact, applying \( (5.10) \) and then \( (5.9) \) to the definition \( (5.7) \) we find
\[ \nabla_\lambda \alpha^J_\rho = -i (L_A \sum \xi_M^L D_M^I_{Mi}) (Ad g)_{Aj} D_J^J_{j\lambda} = -i Ad g_{Aj} D_J^J_{j\lambda} (\Sigma_A^R)_{j\lambda} \] .
\( \nabla_\lambda \) maps tensors of rank \( k \) to \( k + 1 \). It also has the correct derivation property so that it is a covariant differentiation. Also \( (5.7) \) shows that it corresponds to the operator \( X_i \).

We can define the covariant derivative \( \nabla_\rho \) on spinors corresponding to \( X_i \) in the same way, just changing the index \( i \) to \( a \) in \( (5.13) \), and accordingly changing the choice of \( J \) as well.

The canonical torsion \( c_{ijk} \) generalizes for tensors to
\[ C_{\lambda\rho\sigma} = (D_J^I)_{i\lambda}^i (D_J^J)_{j\rho}^j c_{ijk} D_J^J_{\sigma\lambda} . \]
A torsion–free covariant derivative on tensors when $c_{ijk} \neq 0$ is then defined from
\[ \nabla_{\lambda} \alpha^{J}_\rho = \nabla_{\lambda} \alpha^{J}_\rho + \frac{1}{2} C_{\lambda\rho\sigma} \alpha^{J}_\sigma. \] (5.14)

As for spinors, following (3.20), we define a spinorial torsion which is twice the expression
\[ -\frac{i}{4} c_{ijk} \frac{1}{2i} (\gamma_j \gamma_k)_{ba} = -\frac{1}{8} c_{ijk} (\gamma_j \gamma_k)_{ba}. \] (5.15)

Let $J_s$ be the representation of choice for the projective module of spinors, and $J_T$ for rank 1 tensors. The transform of (5.15) onto spinorial modules is:
\[ -\frac{1}{8} c_{ijk} [D^{J_s} (\gamma_j \gamma_k) (D^{J_s})^\dagger]_{\sigma' \sigma} (D^{J_T})^\dagger_{i\rho}, \] (5.16)

while the torsion–free covariant derivative $\nabla_{\rho}$ acts on a spinor $\alpha^{J_s}$ represented as an element of a projective module as follows:
\[ \nabla_{\rho} \alpha^{J_s}_\sigma = \nabla_{\rho} \alpha^{J_s}_\sigma - \frac{1}{8} \alpha^{J_s}_\sigma' c_{ij'k'} (D^{J_s} (\gamma_j \gamma_k') (D^{J_s})^\dagger)_{\sigma' \sigma} (D^{J_T})^\dagger_{i\rho}. \] (5.17)

### 5.3 The Projective Dirac Operator for Spheres

The equations (5.15) tell us the invertible transformation of a spinor field of § 3.3 to an element of a projective module. So we can transform the Dirac operator $D$ to one acting on this $\mathcal{A}$–module. The result is not illuminating except in special cases like spheres and $\mathbb{C}P^N$, so we take them up first.

a) **Even Spheres**

For $G/H = S^{2n}$, we can choose $G = Spin^{Cl}(2n + 1) = \{g\}$, $H = Spin^{Cl}(2n) = \{h\}$, identifying them with the representations given by $\gamma$–matrices, $Spin^{Cl}(2n + 1)$ and $Spin^{Cl}(2n)$. We denote the $\gamma$-matrices of $H$ by $\gamma_i$, $i = 1, .., 2n$, and by $\gamma = (-i)^n \gamma_1 \ldots \gamma_{2n}$ the additional gamma matrix of $G$, and call them collectively as $\Gamma_{\lambda} = (\gamma_i, \gamma)$, $\lambda = 1, .., 2n + 1$. The generators of $H$ are $\Sigma_{ij} = \frac{1}{2i} [\gamma_i, \gamma_j]$, which together with $\Sigma_{2n+1,i} = \frac{1}{2i} \gamma \gamma_i$ make up the full set of generators $\Sigma_{\mu\nu}$ of $G$.

The $\Gamma_{\lambda}$ transform as vectors under conjugation by $G$. That lets us introduce coordinate functions $x = (x_\lambda)$ for $S^{2n}$, starting from an ‘origine’ $x^0 = (0, ..., 0, 1)$, as follows:
\[ \Gamma_{\lambda} x_{\lambda} = g \Gamma_{2n+1} g^{-1}, \quad g \in Spin(2n + 1), \quad x_\lambda x_{\lambda} = 1. \] (5.18)

We let subscript $A = (\mu \nu)$, $\mu > \nu$ stand for either of the multi-indices $(ij)$, $(\alpha$ of Sec.2), or $(2n + 1, i)$, $(i$ of Sec.2). For $A = (2n + 1, i)$, $X_A$ gives back $X_{2n+1,i} \equiv X_i$ of Sec.2, which is now torsionless, $G/H$ being symmetric.
Since $\Gamma_{2n+1}$ commutes with $\Sigma_{ij}$, $D_W$ can be written as:

$$D_W = -i\gamma^R_i x_i = [\Gamma_{2n+1}, \Sigma_A]^R X_A = [\Gamma_\lambda x_\lambda, \Sigma_A]^R X_A , \text{ at } x = x^0 ,$$  

(5.19)

while

$$D = i\Gamma^R_{2n+1} D_W = i\Gamma^R_\lambda x_\lambda D_W , \text{ at } x = x^0 .$$  

(5.20)

We choose $J$ to correspond to the preceding Clifford representation to fix the spinorial projective module. We now show that on this module the above Dirac operators have the beautiful forms

$$D_W = i[\Gamma^R_\lambda x_\lambda, \Sigma_A]^R J_A , \quad D = i\Gamma^R_\lambda x_\lambda D_W ,$$  

(5.21)

$J_A$ being again the total 'orbital' plus 'spin' generators $L_A$ and $-\Sigma_A^R$ of $G$. The matrices $\Gamma^R_\lambda$, $\Sigma_A^R$ act on the index $a$ of the spinor $\chi_a = \sum \xi^K_M D^K_M b^\dagger_{ba}$

on the right as in $(\Gamma^R_\lambda \chi)_a = \chi_b (\Gamma_\lambda)_{ba}$.

In fact, if we apply (5.21), since by (5.9) $J_A (D_C^\ell)_{a'}^\dagger = 0$, we can see that

$$(D_W \chi)_a = i \sum (\mathcal{L}_A (\xi^K_M D^K_M b)) (D_C^\ell)_{ba}^\dagger [\Sigma_A, \Gamma_\lambda x_\lambda]_{ca} .$$  

(5.23)

Inserting

$$[\Sigma_A, \Gamma_\lambda x_\lambda] = (Ad g)_{AB} D_C^\ell (g) [\Sigma_B, \Gamma_{2n+1}] (D_C^\ell (g))^\dagger =$$  

$$= i (Ad g)_{A,(2n+1,i)} D_C^\ell (g) \gamma_i (D_C^\ell (g))^\dagger ;$$  

(5.24)

we get

$$(D_W \chi)_a = -\sum (\mathcal{L}_A \xi^K_M D^K_M b)(Ad g)_{A,(2n+1,i)} (\gamma_i)_{ba'} (D_C^\ell (g))_{a'a} .$$  

(5.25)

But the right-invariant vector fields are related to the left-invariant ones by eq.(5.10), so

$$(D_W \chi)_a = -i(X_i \sum \xi^K_M D^K_M b)(\gamma_i)_{bc}(D_C^\ell (g))_{ca} .$$  

(5.26)

Writing $\psi_a = \chi_a D_C^\ell_{a'a}$, (5.24) shows that under $D_W : \psi_a \rightarrow \psi_a (D_W \psi)_a$, which is the action (5.19). So $D_W$ is equivalent to $D_W$. In a similar manner $D$ is seen to be equivalent to $D$.

When acting on functions on $S^{2n}$, we can use our coordinates to express the right-invariant vector fields in the form

$$\mathcal{L}_{\mu \nu} = -i (x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}) ,$$  

(5.27)
and therefore the Dirac operators as
\[ D_W = -x^\mu \Gamma^\nu (\mathcal{L}_{\mu\nu} - \Sigma^{\mu\nu}) \quad , \quad D = -\Sigma^{\mu\nu} \mathcal{L}_{\mu\nu} + n \, . \] (5.28)

To determine the spectrum and eigenspinors of the Dirac operator we need to be more explicit about the group $Spin(2n + 1)$. It has rank $n$, and IRR’s that can be labeled by the components of the highest weight $(m_1, ..., m_n)$, with the $m_i$’s all integers or all half integers, and $m_1 \geq m_2 \geq ... \geq m_n \geq 0$. The Clifford representation $Spin^{\mathfrak{c}l}$ has highest weight $(\frac{1}{2}, ..., \frac{1}{2})$, dimension $2^n$ and quadratic Casimir operator $C_2(\mathfrak{c}l) \equiv C_2(Spin^{\mathfrak{c}l}) = \frac{1}{2} \Sigma_{\mu\nu} \Sigma_{\mu\nu} = \frac{1}{4} n(2n + 1)$. We indicate by $L$ an IRR associated with the set $I_0$ of §2.2; it has highest weight $(l, 0, ..., 0)$, where $l$ is an integer, and dimension and quadratic Casimir operator
\[ d(L) = 2l + 2n - 1 \frac{(l + 2n - 1)!}{l + 2n - 1 \cdot l!(2n - 1)!} \quad , \quad C_2(L) = l(l + 2n - 1) \, . \] (5.29)

The final piece of required information is
\[ L \otimes \mathfrak{c}l = (l + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) \otimes (l - \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) \] (5.30)
with
\[ d(j, \frac{1}{2}, ..., \frac{1}{2}) = 2^n \frac{(j + 2n - \frac{3}{2})!}{(j - \frac{1}{2})!(2n - 1)!} \quad , \quad C_2(j, \frac{1}{2}, ..., \frac{1}{2}) = j(j + 2n - 1) + \frac{1}{2}(n - 1)(n - \frac{1}{2}) \, . \] (5.31)

With this background it is easy to show that the eigenspinors of $D$ are of the form
\[ \chi_a^{JL} = \sum \xi^{JL}_M < JM|LN, \mathfrak{c}l \ a > D_{N\iota_0}^L \quad \text{with} \quad J = (l \pm \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) \, , \] (5.32)
where $M$, $N$ and $\iota_0$ , and $a$ label vectors in the IRR’s $J, L$ and $\mathfrak{c}l$. In fact
\[ (D\chi)_a = \sum \xi^{JL}_M < JM|LN', \mathfrak{c}l \ a'> (\Sigma_{\mu\nu})_{a'a}(\Sigma^{L}_{\mu\nu})_{N'N} D_{N\iota_0}^L + n\chi_a \, , \] (5.33)
where $\Sigma^{L}_{\mu\nu}$ is the representative of $\mathcal{L}_{\mu\nu}$ in the IRR $L$.'Completing the square’ in this equation, one finds
\[ < JM|LN', \mathfrak{c}l \ a'> (\Sigma_{\mu\nu})_{a'a}(\Sigma^{L}_{\mu\nu})_{N'N} = (C_2(J) - C_2(L) - C_2(\mathfrak{c}l) < JM|LN, \mathfrak{c}l \ a > . \] (5.34)

Using the expressions for the various quadratic Casimir operators, the eigenvalues corresponding to the eigenspinors (5.32) are found to be
\[ \rho = \pm(j + n - \frac{1}{2}) \quad , \quad \text{for} \quad j = l \pm \frac{1}{2} \, . \] (5.35)
b) Odd Spheres

An odd sphere $S^{2n-1} = SO(2n)/SO(2n - 1)$ differs from an even sphere $S^{2n}$ in important details. The Clifford algebra $\mathbb{C}l(2n-1)$ has two inequivalent $2^{n-1}$-dimensional representations, with $(-i)^{n-1}\gamma_1...\gamma_{2n-1} = \mathbb{I}$ and $(-i)^{n-1}\tilde{\gamma}_1...\tilde{\gamma}_{2n-1} = -\mathbb{I}$; we may take $\tilde{\gamma}_i = -\gamma_i$, which makes clear that they give a single IRR’s of $Spin(2n-1)$, with generators $(\frac{1}{2i}[\gamma_i, \gamma_j])$. They do give however two inequivalent IRR’s of $Spin(2n)$, with generators $(\frac{1}{4i}[\gamma_i, \gamma_j], -\frac{1}{2}\gamma_i)$ and $(\frac{1}{4i}[\gamma_i, \gamma_j], \frac{1}{2}\gamma_i)$, let us label them $\mathbb{C}l^+$ and $\mathbb{C}l^-$. For covariance it is better to put these two representations together and work with the $2^n$-dimensional $\Gamma_\mu, \mu = 1, ..., 2n$, built from the $\gamma_i$-s as indicated in (3.2); that particular construction gives

$$\Gamma_{2n+1} = (-i)^n\Gamma_1...\Gamma_{2n} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \quad \tilde{\Gamma}_{2n-1} = (-i)^{n-1}\Gamma_1...\Gamma_{2n-1} = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}.$$ 

For $\mathbb{C}l(2n-1)$, $\Gamma_\mu$ splits into the two inequivalent IRR’s of $Spin(2n-1)$, with generators $(\frac{1}{2}(\mathbb{I} \pm \tilde{\Gamma}_{2n-1})\Gamma_j \equiv \Gamma_j^{(1,2)}, 1 \leq j \leq 2n - 1$. The corresponding generators of $Spin^{\mathbb{C}l}(2n)$ are:

$$\Sigma_{\mu\nu} = \frac{1}{4i}[\Gamma_\mu, \Gamma_\nu] = \begin{cases} \Sigma_{ij} = \begin{pmatrix} \frac{1}{4i}[\gamma_i, \gamma_j] & 0 \\ 0 & \frac{1}{4i}[\gamma_i, \gamma_j] \end{pmatrix}, \quad \Sigma_{2n, i} = \begin{pmatrix} -\frac{1}{2}\gamma_i & 0 \\ 0 & \frac{1}{2}\gamma_i \end{pmatrix} \end{cases}, \quad (5.36)$$

and give, as expected, the direct sum $\mathbb{C}l^+ \oplus \mathbb{C}l^-$; for the group elements of $Spin(2n)$ we have $g = \begin{pmatrix} D^{\mathbb{C}l^+} & 0 \\ 0 & D^{\mathbb{C}l^-} \end{pmatrix}$, split by the projectors $\frac{1}{2}(\mathbb{I} \pm \Gamma_{2n+1})$.

Spinors carry the direct sum of these two IRR’s on their index, and we can use either of the Dirac operators

$$D_W^{(1,2)} = \frac{1}{2}(\mathbb{I} \pm \tilde{\Gamma}_{2n-1})^R(-i \Gamma_i^R)X_i, \quad (5.37)$$

or else we can accept fermion doubling and work with $D_W^{(1)} + D_W^{(2)}$.

There is no chirality in odd dimensions, but $\Gamma_{2n}$ plays a role in space(time)-reflection, and can be used to give Dirac operators equivalent to $D_W^{(1,2)}$ (5.9):

$$D_R^{(1,2)} = e^{i\Gamma_2 \pi/4}D_W^{(1,2)}e^{-i\Gamma_2 \pi/4} = \frac{1}{2}(\mathbb{I} \pm \Gamma_{2n+1}^R) \Gamma_{2n}^R \Gamma_i^R X_i. \quad (5.38)$$

We can introduce coordinates for $S^{2n-1}$, starting from $x^0 = (0, .., 1)$, by

$$\Gamma_\lambda x_\lambda = g\Gamma_\lambda x_\lambda^0g^{-1} = g\Gamma_{2n}g^{-1}, \quad x_\lambda x_\lambda = 1. \quad (5.39)$$

Hence at $x = x^0$,

$$D^{(1,2)}_W = p(x^0_R)^{(1,2)}[\Gamma_\lambda x_\lambda^0, \Sigma_A]^R X_A, \quad D^{(1,2)}_R = \frac{1}{2}(\mathbb{I} \pm \tilde{\Gamma}_{2n-1})^R i\Gamma_\lambda^R x_\lambda^0[\Gamma_\lambda x_\lambda^0, \Sigma_A]^R X_A,$n

$$p(x^0)^{(1,2)} = \frac{1}{2}(\mathbb{I} \pm \tilde{\Gamma}_{2n-1}) = \frac{1}{2} \left( \mathbb{I} \pm \frac{(-i)^{n-1}}{(2n - 1)!} \epsilon_{\mu_1...\mu_{2n}} \Gamma_{\mu_1}...\Gamma_{\mu_{2n-1}} x_{\mu_{2n}} \right). \quad (5.40)$$
Their covariant forms follow:

\[ \mathcal{D}_{(1,2)}^{(1,2)} = p(x)_{(1,2)}^R \left[ \Gamma_{\lambda}^{R} x_{\lambda}, \Sigma_{A}^{R} \right] \mathcal{J}_A, \quad \mathcal{D}^{(1,2)} = -\frac{1}{2} (\mathbb{I} \mp \Gamma_{2n+1}^{R} \Gamma_{x_{1}}^{R} x_{1}, \Sigma_{A}^{R} ) \mathcal{J}_A , \]

\[ p(x)_{(1,2)} = \frac{1}{2} \left( \mathbb{I} \pm \frac{(-i)^{n-1}}{(2n-1)!} \varepsilon_{\mu_1 \ldots \mu_2} \Gamma_{\mu_1} \ldots \Gamma_{\mu_{2n-1}, x_{\mu_{2n}}} \right) = g p(x^{0}) g^{-1}. \quad (5.41) \]

\( \mathcal{J}_A \) is defined as before.

Proceeding as we did for even spheres, with \( \mathcal{L}_{\mu \nu} \) as in eq. (5.27) the Dirac operators can be rewritten in the form

\[ \mathcal{D}_{W}^{(1,2)} = -p(x)_{(1,2)}^R \Gamma_{\mu}^{R} (\mathcal{L}_{\mu \nu} - \Sigma_{\mu \nu}^{R} ) , \quad \mathcal{D}^{(1,2)} = \frac{1}{2} (\mathbb{I} \mp \Gamma_{2n+1}^{R} (\Sigma_{\mu \nu}^{R} \mathcal{L}_{\mu \nu} + n - \frac{1}{2}) ) . \quad (5.42) \]

Given their form, it is easy to find one set of eigenvalues and eigenspinors for the Dirac operators \( \mathcal{D}^{(1,2)} \), by the same argument that led us to eq. (5.32). The IRR’s of \( \text{Spin}(2n) \) are labeled by highest weights \( (m_1, \ldots, m_n) \), \( m_1 \geq m_2 \ldots \geq |m_n| \geq 0 \) with the \( m_i \) all integers or all half integers. The two \( 2^{n-1} \)-d spinor representations \( \mathbb{C}^{\ell \pm} \) have \( (\frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2}) \), with quadratic Casimir \( C_{2}(\mathbb{C}^{\ell \pm}) = \frac{1}{2} \Sigma_{\mu \nu} \Sigma_{\mu \nu} = \frac{1}{4} n(2n-1) \). The IRR’s associated with the set \( I_0 \) of section 2.2 are \( L = (l, 0, \ldots) \), with dimension \( d(L) = \frac{(l+n-1)(l+2n-3)!}{(n-1)(2n-3)!} \), and quadratic Casimir \( C_{2}(L) = l(l+2n-2) \). Finally

\[ L \otimes \mathbb{C}^{\ell \pm} = (l + \frac{1}{2}, \ldots, \pm \frac{1}{2}) \oplus (l - \frac{1}{2}, \frac{1}{2}, \ldots, \pm \frac{1}{2}) . \quad (5.43) \]

For these last representations we have \( C_{2}(j, \frac{1}{2}, \ldots, \pm \frac{1}{2}) = (j - \frac{1}{2}) (j + 2n - \frac{3}{2}) + \frac{1}{4} n(2n-1) \).

With all this information, the analogues of (5.33), (5.34) give for the eigenvalues

\[ \rho_{\pm} = \pm (j_{\pm} + n - 1) , \quad \text{with } j_{\pm} = l \pm \frac{1}{2} . \quad (5.44) \]

There are eigenstates for each of the two inequivalent representations of the Clifford algebra. They are given by

\[ \chi_{1,a,\pm}^{JL} = (0, \sum \xi_{J}^{M} < (j_{\pm}, \frac{1}{2}, \ldots, -\frac{1}{2}) , M|L, N', \mathbb{C}^{\ell^{-}} a > D_{N'ia}^{J} ) , \]

\[ \chi_{2,a,\pm}^{JL} = (\sum \xi_{J}^{M} < (j_{\pm}, \frac{1}{2}, \ldots, \frac{1}{2}) , M|L, N', \mathbb{C}^{\ell^{+}} a > D_{N'ia}^{J} , 0 ) . \quad (5.45) \]

However, from eq. (5.38), we have

\[ \Gamma_{\lambda}^{R} x_{\lambda} \mathcal{D}^{(1,2)} = -\mathcal{D}^{(2,1)} \Gamma_{\lambda}^{R} x_{\lambda} , \quad (5.46) \]

and this implies that there is another set of eigenvectors, with the same eigenvalues, given by

\[ \tilde{\chi}_{1,a,\pm}^{JL} = \Gamma_{\lambda} x_{\lambda} \chi_{2,a,\pm}^{JL} = (0, \sum \xi_{J}^{M} < (j_{\pm}, \frac{1}{2}, \ldots, \frac{1}{2}) , M|D^{J}|L, i_{0}', \mathbb{C}^{\ell^{+}} a > D_{\alpha a}^{\ell^{+}} , 0 ) , \]

\[ \tilde{\chi}_{2,a,\pm}^{JL} = \Gamma_{\lambda} x_{\lambda} \chi_{1,a,\pm}^{JL} = (\sum \xi_{J}^{M} < (j_{\pm}, \frac{1}{2}, \ldots, -\frac{1}{2}) , M|D^{J}|L, i_{0}', \mathbb{C}^{\ell^{-}} a > D_{\alpha a}^{\ell^{-}} , 0 ) . \quad (5.47) \]
5.4 The Projective Dirac Operators on \( \mathbb{CP}^N \)

For reasons of brevity, we focus on \( \mathbb{CP}^2 \), a case we have already treated in [12]. \( \mathbb{CP}^2 \) is \( SU(3)/U(2) \). If \( \lambda_\alpha \) are the Gell-Mann matrices, it is the orbit of \( \lambda_8 \) under \( SU(3) \):

\[
\mathbb{CP}^2 : \{ g\lambda_8 g^{-1} , \, g \in SU(3) \} .
\] (5.48)

Writing \( g\lambda_8 g^{-1} = \lambda_A \xi_A \) analogously to (5.18), we can regard those \( \xi \in \mathbb{R}^8 \) given by (5.48), as points of \( \mathbb{CP}^2 \). The stability group at \( \lambda_8 \), or equally well at \( \xi^0 = (0, \ldots, 0, 1) \) is \( U(2) \). Its generators are \( \lambda_1, \lambda_2, \lambda_3, \lambda_8 \).

If we can achieve a covariant–looking form for \( D \) and \( D_W \) looking like (5.19), (5.20), we can find covariant \( D \) and \( D_W \). Towards this end we introduce the Clifford algebra with eight generators \( \gamma_A \). They can be transformed by the adjoint representation of \( SU(3) \) without disturbing their anticommutators:

\[
\gamma'_A = Ad g \gamma_B \rightarrow \{ \gamma'_A, \gamma'_B \} = 2 \delta_{AB} .
\] (5.49)

The generators \( \Sigma_A \) in this representation can actually be written using \( \gamma_A \):

\[
\Sigma_A = \frac{1}{4i} f_{ABC} \gamma_B \gamma_C .
\] (5.50)

Consider the action \( \gamma_A \rightarrow [\Sigma_8, \gamma_A] \) of \( \Sigma_8 \) on \( \gamma_A \). For this action, the eigenvalues of \( \Sigma_8 \) are \( \pm \frac{\sqrt{2}}{2} \) and 0. The 0 eigenvalues are for \( \gamma_A \) with \( A = 1, 2, 3, 8 \), thus:

\[
[\Sigma_8, [\Sigma_8, \gamma_A]] = 0 \quad \text{if} \quad A = 1, 2, 3, 8 ,
\]

\[
= \frac{3}{4} \gamma_A \quad \text{if} \quad A = 4, 5, 6, 7 .
\] (5.51)

This lets us write the Dirac operator in 'covariant' form

\[
D_W = -i \frac{4}{3} [\Sigma \cdot \xi^0, [\Sigma \cdot \xi^0, \gamma_A]]^R X_A , \quad (X_A f)(g) = \frac{d}{dt} f(ge^{it\Sigma_A})|_{t=0} .
\] (5.52)

The role of \( \Sigma_A \) and \( \gamma_A \) are reversed here for covariantization as compared to spheres.

For the projective module, for the representation \( D' \), we have the one given by \( \Sigma_A \). It is \( 2^8/2 = 16 \)-dimensional. The transform \( D_W \) of \( D_W \) onto this module is immediate:

\[
D_W = -i \frac{4}{3} [\Sigma \cdot \xi, [\Sigma \cdot \xi, \gamma_A]]^R J_A .
\] (5.53)

In addition to \( D_W \) we can also write the Dirac operators

\[
D' = -i \frac{2}{\sqrt{3}} [\Sigma \cdot \xi^0, \gamma_A] X_A , \quad D = i \Gamma(\xi^0) D_W , \quad \Gamma(\xi^0) = -\gamma_4 \gamma_5 \gamma_6 \gamma_7 .
\] (5.54)
$D'$ becomes $D' = -i \frac{2}{\sqrt{3}} [\Sigma \cdot \xi, \gamma A] J_A$ on the projective module. To find $D$ we need to find the chirality operator $\Gamma(\xi)$ for all $\xi$. This is in [12] and is just

$$\Gamma(\xi) = -\frac{1}{4!} \epsilon_{ABCD}(\xi) \gamma A \gamma B \gamma C \gamma D \quad , \quad \epsilon_{ABCD}(\xi) = 4(\text{ad} \cdot \xi)_{[AB}(\text{ad} \cdot \xi)_{CD]}$$  \hspace{1cm} (5.55)

([ ] =antisymmetrization). We cannot have an $a$ in $\chi_a$ take values from 1 to 16: that would give 4 spinors. We must have it taking just 4 values and carrying the representation of just $\gamma A$, $A = 4, 5, 6, 7$. The explanation of how this is done takes up some space in [12].

### 6 On Riemannian Structure and Gravity.

An inverse metric $(\eta^{ij})$ is a symmetric nondegenerate field, which defines a map $\mathcal{T}^{(1)} \otimes \mathcal{T}^{(1)} \rightarrow \mathcal{T}^{(0)}$ via $f \otimes f' \rightarrow \eta^{ij} f_i f'_j$. As the $f$’s transform by $Ad_{G/H}$ under $g \rightarrow gh$, $(\eta^{ij})$ transform by the product $Ad_{G/H}^{-1}_g \otimes Ad_{G/H}^{-1}_g$ of its contragradient representation. Or, the metric $(\eta_{ij})$ itself transforms by $Ad_{G/H} \otimes Ad_{G/H}$.

A particular metric is $(\hat{\delta}_{ij})$, where $\hat{\delta}_{ij}(g)$ is $\delta_{ij}$ ($\delta = $ Kronecker $\delta$). The torsion–free covariant derivative compatible with $\hat{\delta}$ is $\overline{X}$:

$$\overline{X} \hat{\delta} = 0 \ .$$  \hspace{1cm} (6.1)

The corresponding curvature tensor of $G/H$ can be calculated in terms of the structure constants of $G$. From their form we have that for any vector field $f_i$ tangent to $G/H$,

$$[\overline{X}_i, \overline{X}_j] = R_{ijkl} f_l = (c_{ij a} c_{akl} + \frac{1}{4} (2 c_{ijk} c_{k'kl} - c_{ik k'i} c_{k'jl} - c_{jk k'i} c_{k'jl})) f_l \ .$$  \hspace{1cm} (6.2)

The scalar curvature is then $R = R_{ijij} = c_{ij a} c_{a ij} + \frac{1}{4} c_{ijk} c_{k'ij}$. For $S^n = Spin(n + 1)/Spin(n)$, we found in section 5.3 that in a Clifford representation, $[\Sigma_{n+1,i}, \Sigma_{n+1,j}] = i \Sigma_{ij}$. So, with the correspondences $i \leftrightarrow (n + 1, i)$, $\alpha \leftrightarrow (i, j)$ we have that $c_{ijk} = 0$, and the curvature is $R = n(n - 1)$.

A more general $H$-invariant metric $\eta$ can be defined as follows. Let us decompose $G/H$ into irreducible subspaces under $Ad_{G/H}$ and let $\{S_n^{(\sigma)}\}$ be a basis for the unitary representation $\sigma$ such that

$$\text{Tr} S_m^{(\sigma')} S_n^{(\sigma')} = c \delta_{\sigma \sigma'} \delta_{mn} \ .$$  \hspace{1cm} (6.3)

(Here $\sigma$ and $\sigma'$ can be equivalent representations). Let $X_{m}^{(\sigma)}$ be the corresponding (in general complex) vector field, it is a linear combination of $X_i$. Then a general $H$-invariant metric $\eta$ on vector fields $X_m^{(\sigma)\dagger}$ and $X_n^{(\sigma')}$ is the constant function defined by

$$\eta(X_m^{(\sigma)\dagger}, X_n^{(\sigma')})(g) = \lambda_{\sigma} \delta_{\sigma \sigma'} \delta_{mn} \ , \ \lambda_{\sigma} \text{ a positive constant} \ ,$$  \hspace{1cm} (6.4)
independent of \( g \). Such metrics are essential for certain Kähler structures as we shall see in Section 7.

The general covariant differential \( \nabla \) can be defined in the usual way:

\[
\nabla_i \eta_{jk} = \overline{X}_i \eta_{jk} + \Gamma^l_{ij} \eta_{lk} + \Gamma^l_{ik} \eta_{lj}.
\]

(6.5)

The formula shows that \( \Gamma \) transforms by \( Ad_G/H \otimes Ad_G/H \otimes Ad_{G/H}^{-1} \) under the structure group \( H \). As \( \overline{X} \) is torsion–free, so is \( \nabla \) if as usual \( \Gamma^k_{ij} = \Gamma^k_{ji} \). A standard calculation gives the metric–compatible torsion–free \( \nabla \), its \( \Gamma \) being given by

\[
\Gamma^k_{ij} = -\frac{1}{2} \eta^{kk'} (\overline{X}_i \eta_{jk'} + \overline{X}_j \eta_{ik'} - \overline{X}_{k'} \eta_{ij}).
\]

(6.6)

These \( \Gamma^k_{ij} \) are not Christoffel symbols, for example they vanish if \( c_{ijk} = 0 \). Christoffel symbols are defined with respect to some local coordinates \( x^a \) on \( G/H \).

Next, introduce \(|G/H|\)-beins or soldering forms \( e_i^a \) such that

\[
\eta_{ij} = \eta(X_i, X_j) = e_i^a e_j^b \eta_{ab}.
\]

(6.7)

The Christoffel symbols are defined from \( \eta_{ab} \) in the usual way.

The spin connection is defined by:

\[
\nabla_i e_j^a = \overline{X}_i e_j^a + \Gamma^b_{ij} e_k^a + e_j^b (\omega_i)_{ba} = 0,
\]

(6.8)

where \( (\omega_i)_{ba} = -(\omega_i)_{ab} \) and transforms as a tensor field in \( i \) under \( H \). The solution for \( \omega_i \) is standard:

\[
(\omega_i)_{ca} = -E_c^i [\overline{X}_i e_j^a + \Gamma^b_{ij} e_k^a] , \quad E_c^i e_j^a = \delta_c^a \quad \text{or} \quad E_c^i = \eta^{ik} e_k^a \eta_{ac}.
\]

(6.9)

The covariant derivative on spinors \( \psi \) is given by

\[
(\nabla_i \psi)_a = (\overline{X}_i \psi)_a - \frac{1}{4} (\omega_i)_{cd} (\gamma_c \gamma_d \psi)_{a}.
\]

(6.10)

The Dirac operator in the presence of a gravity field \((\eta_{ij})\) is thus:

\[
D = \eta^{ij} e_j^a \gamma_a \nabla_j.
\]

(6.11)

All this stuff is very natural. It remains to transport it to projective modules. In the module picture \( \eta_{ij} \) gets transformed to

\[
G_{\lambda \rho} = \eta_{ij} (D^j)^i_{j \lambda} (D^j)^i_{j \rho}.
\]

(6.12)
while \( \eta^{ij} \) becomes

\[ G^\lambda \rho = \eta^{ij} D_{\lambda i}^J D_{\rho j}^J. \quad (6.13) \]

The projector for the module is

\[ P^\lambda_\sigma G^\rho \sigma = D_{\lambda i}^J (D^J)_{i \rho}^\dagger. \quad (6.14) \]

The projective module analogue of \( \overline{\nabla}_i \) is the \( \nabla_\rho \) defined in section 5.2. Adding the action of \( \Gamma^\nu_{\lambda \mu} = \Gamma^k_{ij} (D^J)_{j \lambda}^\dagger (D^J)_{i \nu}^\dagger (D^J)_{k \nu}^\dagger \) to \( \nabla_\rho \) defines the action of \( \nabla_\rho \), the metric-compatible torsion-free covariant derivative on tensors \( (\nabla_\rho G^\mu \nu = 0) \).

The action of \( \nabla_\rho \) on spinorial modules follows from \( (6.10) \). We let \( J_\rho \) be the total angular momentum for the representation \( J_S \) chosen for spinors, and

\[ C^{(S)}_{\rho \lambda \sigma} = D_{\lambda \sigma}^J (c_{ijk} \gamma_i \gamma_j \gamma_k)_{ba} (D^J)_{a \sigma}^\dagger (D^J)_{i \rho}^\dagger , \]
\[ \Omega_{\rho \lambda \sigma} = D_{\lambda \sigma}^J ((\omega_{i})_{jk} \gamma_i \gamma_j \gamma_k)_{ba} (D^J)_{a \sigma}^\dagger (D^J)_{i \rho}^\dagger , \]
\[ \chi_\sigma = \psi_a (D^J)_{a \sigma}^\dagger. \quad (6.16) \]

Then, as can easily be shown from \( (5.17) \) above,

\[ (\nabla_\rho \chi)_\sigma = -i (J_\rho \chi)_\sigma - \chi_\lambda \left( \frac{1}{8} C^{(S)} + \frac{1}{4} \Omega \right)_{\rho \lambda \sigma}. \quad (6.17) \]

7 Complex Structures and Kähler Manifolds

In favourable circumstances, we can push this program ahead and define more refined ideas like complex and Kähler structures on tensors \( T^{(n)} \) and on their projective modules. We indicate how to treat them briefly.

We consider adjoint orbits only for \( G/H \). Thus let \( \mathbf{k} \) be a fixed element of \( G \) from the Cartan subalgebra \( C(G) \), and \( H \) its stability group:

\[ H = \{ h \in G : h \mathbf{k} h^{-1} = \mathbf{k} \} , \quad [\mathbf{k}, T(\alpha)] = 0 \forall \alpha , \quad [\mathbf{k}, S(i)] \neq 0 \forall i. \quad (7.1) \]

The Cartan subalgebra of \( H \), \( C(H) = C(G) \), since any element of \( G \) which commutes with \( \mathbf{k} \) is in \( H \). The manifold \( G/H \), being an adjoint orbit of the simple Lie group \( G \), has even dimension. These observations have the following implications.

Consider the eigenvalue equation

\[ [\mathbf{k}, E_a] = \lambda_a E_a. \quad (7.2) \]
Then $\lambda_a \neq 0$. The $E_a$ will be of the form $\sum_i \xi_{ai} S(i)$, $\xi_{ai} \in \mathbb{C}$, and span the complexification $(G/H)_c$ of $(G/H)$.

By (2.7), $(G/H)_c$ is invariant under the adjoint action of $k$. Also, as $Ad_{G/H}$ is a real, orthogonal representation, the eigenvalues $\lambda_a$ are real, while of course the $S(i)$ are hermitean. So the adjoint of (7.2) shows that $E_a^\dagger$ corresponds to the eigenvalue $-\lambda_a$, and that each positive eigenvalue is paired with a negative one. The eigenvalues $\lambda_a$ may be degenerate.

We choose $E_a$, $a = 1, \ldots, \tfrac{1}{2}(|G|-|H|)$, to be solutions of (7.2) with $\lambda_a > 0$, $E_{-a} = E_a^\dagger$, and the normalization

$$\text{Tr } E_a E_b = \delta_{a+b,0} . \quad (7.3)$$

So, if $E_a = \xi_{ai} S(i)$ (for both signs of $a$), $\xi_{-ai} = \xi_{ai}^*$. We choose $c = 1$ in (2.1) and (2.3). Then (2.3) and (7.3) show that the matrix $\{\xi_{ai}\}$ is unitary as well.

Let $(G/H)_{c}^{\pm}$ denote the span of the eigenvectors $E_{\pm|a|}$ (where note that $|a| > 0$). The subspaces $(G/H)_{c}^{\pm}$ are of precisely the same dimension and

$$(G/H)_c = (G/H)_c^{+} \oplus (G/H)_c^{-} \quad (7.4)$$

The elements $E^+ = \sum_i \xi^i S(i) \in (G/H)_c^+$ generate vector fields $X^+ = \xi^i X_i$ which we define to be holomorphic. Let $\mathcal{H}^+$ denote the space of holomorphic vector fields. Likewise $(G/H)_c^-$ gives rise to the space $\mathcal{H}^-$ of antiholomorphic vector fields. This splitting of the space of fields $\mathcal{H}$ as the direct sum $\mathcal{H}^+ \oplus \mathcal{H}^-$ gives us the complex structure. The $(1,1)$ tensor $J$ of complex analysis is $\pm i$ on $\mathcal{H}^\pm$: for a vector field $X = \xi^i X_i$, $JX = \xi^j \sum_{a>0} i (\xi^*_a \xi_{ai} - \xi_{aj} \xi^*_{ai}) X_i$.

This complex structure is Kähler. To show it, let us introduce the Maurer-Cartan forms $\theta^A$, defined by $g^{-1} dg = i \Sigma_A \theta^A$, or, setting $c = 1$ in (2.1), by

$$\theta^A = -i \text{Tr } \Sigma_A g^{-1} dg . \quad (7.5)$$

They are dual to the vector fields $X_A = (X_i, X_a)$, so that for example $\theta^i(X_j) = \delta_{ij}$, and fulfill

$$d\theta^A = -\frac{i}{2} \text{Tr} (\Sigma_A [\Sigma_B, \Sigma_C]) \theta^B \wedge \theta^C = \frac{1}{2} c_{BCA} \theta^B \wedge \theta^C . \quad (7.6)$$

Consider the particular Maurer-Cartan form

$$\Theta = -i \text{Tr } k g^{-1} dg = \text{Tr } (k \Sigma_A) \theta^A = \text{Tr } (k T(\alpha)) \theta^a . \quad (7.7)$$

In the last step we used (2.4). For $d\Theta$ we have

$$d\Theta = -\frac{i}{2} \text{Tr } (k [\Sigma_B, \Sigma_C]) \theta^B \wedge \theta^C . \quad (7.8)$$
But using (7.1) we have $\text{Tr} \, k[T(\alpha), T(\beta)] = \text{Tr} [k, T(\alpha)] T(\beta) = 0$ and $\text{Tr} \, k[T(\alpha), S(i)] = \text{Tr} [k, T(\alpha)] S(i) = 0$. Therefore

$$d\Theta = -\frac{i}{2} \text{Tr} \left( k [S(i), S(j)] \right) \theta^i \wedge \theta^j .$$  \hspace{1cm} (7.9)

Remembering that the matrix $\{\xi_{ai}\}$ (with $a$ of both signs) is unitary, we may set

$$S(i) \theta^i = E_a \xi_{ai} \theta^i = E_a \theta^a$$  \hspace{1cm} (7.10)

(with implied sum over $a$ of both signs), and rewrite (7.9) as

$$d\Theta = -\frac{i}{2} \lambda_a \text{Tr} \left( E_a E_b \right) \theta^a \wedge \theta^b = -i \sum_{a>0} \lambda_a \theta^a \wedge \theta^{-a} ,$$  \hspace{1cm} (7.11)

where we have used (7.2) and (7.3). The vector fields $X_a = \xi_{ai} X_i$ are dual to $\theta^a$: $\theta^a (X_b) = \delta_{ab}$. Consequently, the two-form $\Omega = d\Theta$ can be specified by

$$\Omega(X_a, X_b) = d\Theta(X_a, X_b) = -i \lambda_a \delta_{a+b,0} .$$ \hspace{1cm} (7.12)

Since all $\lambda_a \neq 0$, it follows from (7.11) that $\Omega$ is a symplectic (i.e. closed and non-degenerate) form on $G/H$. It has been extensively discussed in [13] where its physical implications are also explained. It fulfills the Kählerian condition

$$\Omega(JX_a, JX_b) = \Omega(X_a, X_b) .$$  \hspace{1cm} (7.13)

For vector fields $X = \xi^i X_i$, $Y = \eta^j X_j$, we have $\Omega(X, Y) = \sum_{a>0} (i \lambda_a)(\xi_{ai} \xi_{aj}^* - \xi_{aj} \xi_{ai}^*) \xi^i \eta^j$.

The Kähler metric $\eta$ on vector fields $(X_a, X_b)$ is given by

$$\eta(X_a, X_b) = \Omega(JX_a, X_b) = |\lambda_a| \delta_{a+b,0} .$$ \hspace{1cm} (7.14)

The Levi-Civita connection corresponding to this metric is the torsionless connection compatible with $\eta$. Its coefficients are given by the formula (6.6):

$$\frac{1}{2} c_{abc} + \Gamma^c_{ab} = \frac{1}{2} c_{abc} - \frac{|\lambda_a| - |\lambda_b|}{2|\lambda_c|} c_{abc} .$$ \hspace{1cm} (7.15)

Note from (7.3) and (2.8) that $c_{abc} = \text{Tr}[E_a, E_b] E_{-c}$. So we have the symmetries

$$c_{abc} = c_{b,-c,-a} = c_{-c,a,-b} .$$ \hspace{1cm} (7.16)

Also from $\text{Tr} [k, [E_a, E_b] E_{-c}] = 0$ and (7.2) we have that

$$c_{abc} \text{ and } (7.15) = 0 \text{ if } \lambda_a + \lambda_b - \lambda_c \neq 0 .$$ \hspace{1cm} (7.17)
Finally, we shall show that the Kähler metric on \( G/H \) can be derived from a Kähler potential \( \Phi_\zeta \). It is a function on \( G/H \) and depends on a parameter \( \zeta \). It has the property

\[
X_a X_{-b} \Phi_\zeta = \eta(X_a, X_{-b})
\]

for \( \lambda_a \) and \( \lambda_b \) of the same sign and \( |\lambda_a| \leq |\lambda_b| \). The ordering is needed because of the torsion term in (7.8). It can be discarded when the torsion is zero, that is for symmetric spaces. Note that \( \Phi_\zeta \) can in general be only locally defined on \( G/H \).

The construction of \( \Phi_\zeta \) involves the member of a specific class of unitary representations \( \Sigma_K : g \rightarrow \Sigma_K(g) \) of \( G \). Let \( \sigma_K \) be the associated representation of \( G \). Any such representation contains a normalized highest weight vector \( |K> \) with eigenvalue \( K \) for \( \sigma_K(k) \), which is annihilated by the orthogonal complement of \( k \) in \( H \) and the positive roots \( E_a \):

a) \( \sigma_K(k)|K> = K|K> \), \( K > 0 \),
b) \( \sigma_K(T(\alpha))|K> = 0 \) if \( \text{Tr}(\alpha)k = 0 \),
c) \( \sigma_K(E_a)|K> = 0 \) for \( \forall \, a > 0 \).

(7.19)

A representation of \( G \) fulfilling a) and b) always exists: it is induced from the unitary one-dimensional representation of \( H \) given by a) and b):

\[
\Sigma_K(e^{i\xi T(\alpha)})|K> = e^{i\frac{K}{\text{Tr}k^2}\text{Tr}(k\xi T(\alpha))}|K> .
\]

(7.20)

Here we have used

\[
\sigma_K(\xi T(\alpha))|K> = \frac{K}{\text{Tr}k^2}\text{Tr}(k\xi T(\alpha))|K> .
\]

(7.21)

As for a), b) and c) together, it gives the representation of the group \( G_c \) generated by \( H \) and \( E_a, \, a \geq 0 \), induced from the representation \( \sigma_K(H), \sigma_K(E_a), \, a > 0 \).

Let us fix an orthonormal basis \( \{e_1, e_2, \ldots, e_M\} \) in the representation space of dimension \( M \) (say) of \( \Sigma_K(G) \). Choose a vector \( |\zeta> = \sum_{i=1}^M \zeta_i e_i, \, \zeta_i \in \mathbb{C} \), \( \zeta = (\zeta_1, \zeta_2, \ldots, \zeta_M) \), so that \( (|\zeta> |\Sigma_K(g)>|K> \neq 0 \) when \( g \) belongs to some open set \( \mathcal{O} \). Such a \( |\zeta> \) exists since \( (|\zeta> |\Sigma_K(g)>|K> = 1 \) for \( |\zeta> = \Sigma_K(g)|K> \). Further choose \( \mathcal{O} \) so that it is invariant under \( H \)-action. That is always possible since for \( (|\zeta> |\Sigma_K(g)>|K> changes only by a phase under this action by (7.20). \)

Now the function \( \omega_\zeta \) defined by

\[
\omega_\zeta(g) = <\zeta|\Sigma_K(g)|K>, \quad g \in \mathcal{O}
\]

(7.22)
has the properties
\[
(X_{a\omega}(g) = (X_{-a\bar{\omega}})(g) = 0 \quad \text{for } \forall a > 0 ,
\]
\[
\frac{1}{\omega(g)}(X_{a\omega})\zeta(g) = -\frac{1}{\omega(g)}(X_{a\bar{\omega}})(g) = i\frac{K}{2\kappa^2} \Tr(kT(\alpha)) .
\] (7.23)

Here the bar denotes complex conjugation. The first line is a direct consequence of the fact that \(|K|\) is the highest weight vector, the second line follows from the relation \(\omega(g) = \omega(g)\Sigma_K(h)\) valid for any \(g \in G, h \in H\), with the phase factor \(\Sigma_K(h)\) given by (7.20).

If \(g \in O\), the Kähler potential is given by the formula
\[
\Phi = -\frac{\Tr k^2}{2K} \log \omega(g)\bar{\omega}(g) .
\] (7.24)

\(\Phi\) is a function on \(O/H \subseteq G/H\), since in the product
\[
\omega(g)\bar{\omega}(gh) = \omega(g)\Sigma_K(h)\bar{\omega}(gh)
\]
the phase factors \(\Sigma_K(h)\) and \(\bar{\Sigma}_K(h)\) cancel. The Kähler potential is closely related to the one-form \(\Theta\) introduced in (7.7). Thus the exterior derivative \(d\) on \(G\) can be written as
\[
d = d_+ + d_- + d_0, \quad d_{+} f(g) = (X_{|a|}f)(g)\theta^{\pm|a|} , \quad d_{-} f(g) = (X_{\alpha}f)(g)\theta^{\alpha} .
\] (7.25)

Now, using (7.23) one obtains
\[
i(d_+ - d_-)\Phi = i(X_{|a|}\Phi \theta^{|a|} - i(X_{-|a|}\Phi \theta^{-|a|})
\]
\[
= -i\frac{\Tr k^2}{2K}\omega(g)\theta^{|a|} + i\frac{\Tr k^2}{2K}\omega(g)\theta^{-|a|}
\]
\[
= i\frac{\Tr k^2}{2K} d \log \omega(g)\bar{\omega}(g) + \Tr(kT(\alpha))\theta^\alpha = i\frac{\Tr k^2}{2K} d \log \omega(g)\bar{\omega}(g) + \Theta .
\] (7.26)

It follows that
\[
di(d_+ - d_-)\Phi = d\Theta = \Omega .
\] (7.27)

The left hand side of (7.26) can be evaluated using the first line of (7.26) and (7.6). Calculating its values on \(iX_{a} \otimes X_{-b}\) for \(0 < \lambda_a \leq \lambda_b\) and \(0 < -\lambda_a \leq -\lambda_b\), we get (7.18). For this calculation, it is also important that \(c_{abc} = 0\) if \(\lambda_a + \lambda_b \neq \lambda_c\).

In another open set \(O' \subset G\), we may have to work with the Kähler potential \(\Phi_{\eta}\). Then if \(O \cap O' \neq \emptyset\), the two potentials on \(O \cap O'\) are related by
\[
\Phi_{\eta} = \Phi + \frac{\Tr k^2}{2K} \log \frac{\omega_{\eta}}{\omega_{\zeta}} + \frac{\Tr k^2}{2K} \log \frac{\bar{\omega}_{\eta}}{\bar{\omega}_{\zeta}} .
\] (7.28)
The mapping $\Phi_\zeta$ to $\Phi_\eta$ is often called a gauge transformation.

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