Elastic Scattering of Point Particles With Nearly Equal Masses

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Abstract

We show that for $n$ billiard particles on the line there exist three open sets in the product of phase space and the space of their masses, such that these particles exhibit exactly $n - 1$, $(\frac{n}{2})$ respectively $(\frac{n+1}{3})$ collisions. These open sets intersect any neighborhood of the diagonal in mass space.

1 Introduction

We consider a system of the $n$ hard spheres, that move on a straight line. The elastic collision between two spheres is to be the only interaction, being defined by conservation of energy and momentum.

Estimates of the number of collisions in hard ball systems, and more generally, in semidispersing billiards, have been studied for a long time. But it was only in 1998 when Burago, Ferleger and Kononenko [BFK98] could show the following statement, using tools from metric geometry:

In a system of $n$ hard spheres, that move in $\mathbb{R}^d$, the total number of collisions is bounded above uniformly in the initial conditions.

All known upper bounds on the collision number of a general system of hard spheres increase superexponentially in the number of particles $n$, even for equal masses.

One of the few systems that can be solved exactly is that for equal masses and $d = 1$. For $d = 1$ without loss of generality one can assume the particles to...
be pointlike. For equal masses scattering then just exchanges the labels of the particles, thus generically leading to exactly \( \binom{n}{2} \) collisions.

In this paper we are going to study of hard sphere systems in \( d = 1 \), having approximately equal masses. Chen could show in the papers [Che07, Che09] that subject to certain requirements for the masses, the upper quadratic bound of \( \binom{n}{2} \) remains true. Now we show that additionally to these cases, there are open sets in the product of phase space and the space \((0, \infty)^n\) of masses, such that these particles exhibit exactly \( n - 1 \), \( \binom{n}{2} \) respectively \( \binom{n+1}{3} \) collisions. These three open sets intersect any neighborhood of the diagonal in mass space.

This shows that — concerning the number of collisions — the case of exactly equal masses is nongeneric.

2 Nearly Equal Masses

The position of the \( k \)-th (pointlike) particle at time \( t \) is denoted by \( q_k(t) \), and we assume initial conditions with \( q_k(0) < q_{k+1}(0) \) \( (k = 1, \ldots , n - 1) \).

We begin with a trivial observation, valid only for \( d = 1 \).

**Lemma 2.1:** If for \( d = 1 \), all \( n \in \mathbb{N} \) and any mass distribution \((m_1, \ldots , m_n)\) the number of collisions is strictly smaller than \( n - 1 \), then all velocities are equal, so that no collision occurs.

**Proof.** Under the above assumption there is a particle, with number \( k < n \), not being involved in any collision with particle number \( k + 1 \), so that \( t \mapsto q_k(t) \) is convex and \( t \mapsto q_{k+1}(t) \) is concave. But as \( q_k(t) < q_{k+1}(t) \), both functions must be affine, with velocities \( v_{k+1} = v_k \). So particle number \( k \) experiences no collision with particle number \( k - 1 \) and particle number \( k + 1 \) experiences no collision with particle number \( k + 2 \). An iteration of the argument shows the assertion. ■

For \( d = 1 \) and \( n \) equal masses the open set of initial conditions

\[
U = U(n) := \{(q, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid q_k < q_l \text{ and } v_k > v_l \text{ for } 1 \leq k < l \leq n \}
\]

leads to exactly \( \binom{n}{2} \) two-body collisions (if one continues multi-body collisions in a way that the set of velocities \( v_k \) does not change), and all these collisions occur at positive times.

By Chen’s result [Che07] for \( d = 1 \) and any \( \varepsilon > 0 \) there exists an open set of \( n \) masses \((m_1, \ldots , m_n) \in (1 - \varepsilon , 1 + \varepsilon)^n\), leading to at most \( \binom{n}{2} \) collisions.

This is complemented by the following result:
Theorem 2.2: For $d = 1$, all $n \in \mathbb{N}$ and for all $\varepsilon > 0$ there exist non-empty open subsets $V_i := W_i \times U_i \subset (1 - \varepsilon, 1 + \varepsilon)^n \times (\mathbb{R}^n \times \mathbb{R}^n)$ $(i = 1, 2, 3)$ of the extended phase space such that for any initial data in the set the number of collisions

1) on $V_1$ equals $n - 1 = \binom{n-1}{1}$,

2) on $V_2$ equals $\binom{n}{2}$, and

3) on $V_3$ equals $\binom{n+1}{3}$.

Proof. Preliminary note: As the velocities are unchanged between collisions, we denote the velocity of the $i$–th particle between the $(\ell - 1)$–th and $\ell$–th collision by $v_i^{(\ell-1)}$. If the $\ell$–th collision involves particles $i$ and $i + 1$, then

$$
v_i^{(\ell)} = \frac{(m_i - m_{i+1})v_i^{(\ell-1)} + 2m_{i+1}v_{i+1}^{(\ell-1)}}{m_i + m_{i+1}}, \quad v_{i+1}^{(\ell)} = \frac{(m_{i+1} - m_i)v_{i+1}^{(\ell-1)} + 2m_i v_i^{(\ell-1)}}{m_i + m_{i+1}}. \tag{2.2}
$$

We define for $i = 1, \ldots, n - 1$ the velocity differences by $\Delta v_i^{(\ell)} := v_{i+1}^{(\ell)} - v_i^{(\ell)}$. During a collision between particles $i$ and $i + 1$ we have

$$
\Delta v_i^{(\ell)} = -\Delta v_{i+1}^{(\ell-1)}, \quad \Delta v_k^{(\ell)} = \Delta v_k^{(\ell-1)} \text{ if } |i - k| > 1,
\Delta v_{i-1}^{(\ell)} = \Delta v_{i-1}^{(\ell-1)} + \frac{2m_{i+1}}{m_i + m_{i+1}} \Delta v_i^{(\ell-1)} \quad \text{and} \quad \Delta v_{i+1}^{(\ell)} = \Delta v_{i+1}^{(\ell-1)} + \frac{2m_i}{m_i + m_{i+1}} \Delta v_i^{(\ell-1)}.
\tag{2.3}
$$

Later we will apply the following functions, indexed by $k \in \mathbb{N}$, to quotients of adjacent masses:

$$
g_1(x) := g(x) := 2x - 1, \quad g_{k+1} := g \circ g_k, \text{ thus } g_k(x) = 2^k(x - 1) + 1.
$$

$$
f_1(x) := f(x) := \frac{1 + x}{3 - x}, \quad f_{k+1} := f \circ f_k, \text{ thus } f_k(x) = \frac{k(2x - 1) - 2x}{k(x - 1) - 2}. \tag{2.4}
$$

For all these functions $\lim_{x \searrow 1} f_k(x) = \lim_{x \searrow 1} g_k(x) = 1$ and

$$
f_k(x) = g_k(x) = \frac{k + 2^{k+1} - 2}{k} \quad \text{for } x = \frac{k + 2^{k+1} - 2}{k}.
$$

The function $f_k$ has a pole at $(k + 2)/k$, and because

$$
\frac{d}{dx} f_k(x) = \frac{4}{(k(x - 1) - 2)^2} > 0 \quad \text{for all } x, k
$$

$f_k$ is strictly increasing in the interval $(1, (k + 2)/k)$. In addition $f_k$ is convex in that interval, because

$$
\frac{d^2}{dx^2} f_k(x) = -\frac{8k}{(k(x - 1) - 2)^3} > 0 \quad \text{if and only if } x < \frac{k + 2}{k}.
$$
By the validity of the inequalities $\frac{k+1}{k} \leq \frac{k+2}{k} - \frac{1}{k} \leq \frac{k+2}{k}$ for $k \in \mathbb{N}$, we see that for all $x$ in the subinterval $(1, (k + 1) / k)$ the following inequalities are also true:

$$g_k(x) > f_k(x) = x + O((x - 1)^2) > x.$$  \hspace{1cm} (2.5)

**Case 1)** It is the idea of the proof to find conditions on the masses and the initial velocities, so that the sequence of collisions is $(1, 2), (2, 3), (3, 4), \ldots, (n - 1, n)$.

We consider for $\delta := (1+\varepsilon)^{1/(n-1)}$ the non-empty open set $W_1 = W_1(n, \delta) := \{ (m_1, \ldots, m_n) \in (1 - \varepsilon, 1 + \varepsilon)^n | \forall k = 2, \ldots, n : \frac{m_k}{m_{k-1}} \in (1, \delta) \}$.

We prove the assertion by induction on the number $n$ of particles.

The case $n = 2$ is trivial. Now we assume that we found a neighbourhood $U_1(n-1)$ for the first $n - 1$ particles. Then for all initial conditions $x \in U_1(n-1)$ there exists a time $T(x) \in (0, \infty)$, so that no collisions occur after $T(x)$. $T(x)$ is chosen to be continuous on $U_1(n-1)$. Further there is a continuous function $Q : U_1(n-1) \to \mathbb{R}$, such that $Q(x) > \max\{ g_{n-1}(t) | t \in [0, T(x)] \}$.

We now consider the open set $\tilde{U}_1(n) := \{ (x, q_n, v_n) \in U_1(n-1) \times \mathbb{R}^2 | v_n < v_{n-1}, q_n > Q(x), q_n + v_n T(x) > Q(x) \}$.

According to our induction assumption the last collision took place between the particle $n - 2$ and $n - 1$, and the next one should occur between the particle $n - 1$ and $n$. We can see from the equation (2.3) of the preliminary note

\[
\Delta v^{(n-1)}_{n-2} = \Delta v^{(n-2)}_{n-2} + \frac{2m_n}{m_{n-1} + m_n} \Delta v^{(n-2)}_{n-1}
\]

\[
= -\Delta v^{(n-3)}_{n-2} + \frac{2m_n}{m_{n-1} + m_n} \left( \Delta v^{(n-3)}_{n-1} + \frac{2m_{n-2}}{m_{n-2} + m_{n-1}} \Delta v^{(n-3)}_{n-2} \right)
\]

\[
= \frac{4m_{n-2} m_{n-3} (m_{n-2} + m_{n-1}) (m_{n-1} + m_n)}{(m_{n-2} + m_{n-1}) (m_{n-1} + m_n)} \Delta v^{(n-3)}_{n-2} + \frac{2m_n}{m_{n-2} + m_{n-1}} \Delta v^{(n-3)}_{n-1}.
\]

A necessary condition is that particle $n - 2$ and $n - 1$ have no collision once again so that $\Delta v^{(n-1)}_{n-2} \geq 0$. Thus, inserting the definition of the $\Delta v^{(n)}_i$, and noting $v^{(n-3)}_{n-1} = v^{(0)}_{n-1}$ and $v^{(n-3)}_{n-2} = v^{(0)}_{n-2}$, we see that

\[
0 \leq \frac{4m_{n-2} m_{n-3} (m_{n-2} + m_{n-1}) (m_{n-1} + m_n)}{(m_{n-2} + m_{n-1}) (m_{n-1} + m_n)} \left( v^{(0)}_{n-1} - v^{(n-3)}_{n-2} \right) + \frac{2m_n}{m_{n-2} + m_{n-1}} \left( v^{(0)}_{n-2} - v^{(n-3)}_{n-1} \right).
\]

With the above choice of $\tilde{U}_1(n)$ and by solving the last inequality, we have

\[
0 > \Delta v^{(0)}_{n-1} \geq \left( \frac{3m_{n-1} - m_{n-2}}{2m_{n-2} + m_{n-1}} + \frac{m_{n-1} - m_{n-2}}{2m_{n-2} + m_{n-1}} \right) \left( v^{(0)}_{n-1} - v^{(n-3)}_{n-2} \right) + \left( \frac{3m_{n-1} - m_{n-2}}{2m_{n-2} + m_{n-1}} - \frac{m_{n-1} - m_{n-2}}{2m_{n-2} + m_{n-1}} \right) \left( v^{(0)}_{n-2} - v^{(n-3)}_{n-1} \right).
\]

\hspace{1cm} (2.6)
As \( v_{n-2}^{(n-3)} > v_{n-1}^{(n-3)} = v_{n-1}^{(0)} \) (otherwise there would be no \((n-2)\)-th collision), the inequality is satisfied if and only if
\[
m_n \leq \frac{m_{n-1}(m_{n-1} + m_{n-2})}{3m_{n-2} - m_{n-1}} \quad \text{for} \quad 3m_{n-2} > m_{n-1}. \tag{2.7}
\]

With \( f \) from (2.4) the quotient \( \frac{m_n}{m_{n-1}} \) has a recursive inequality
\[
\frac{m_n}{m_{n-1}} \leq \frac{m_{n-1} + m_{n-2}}{3m_{n-2} - m_{n-1}} = 1 + \frac{m_{n-1} - m_{n-2}}{3m_{n-2} - m_{n-1}} = f \left( \frac{m_{n-1}}{m_{n-2}} \right).
\]

By inequality (2.5) of the preliminary note we can find two initial conditions \( m_1 \) and \( m_2 \) in the interval \((1, (n+1)/n)\), such that
\[
1 < \frac{m_n}{m_{n-1}} < f_{n-2} \left( \frac{m_2}{m_1} \right) = \frac{(n-2) \left( \frac{m_2}{m_1} - 1 \right) - 2m_2/m_1}{(n-2) \left( \frac{m_2}{m_1} - 1 \right) - 2} < \delta = (1 + \varepsilon)^{1/(n-1)}.
\]

We choose a velocity \( v_{n-1}^{(0)} \) in the interval
\[
\left( \left( \frac{3m_{n-1} - m_{n-2}}{2(m_{n-2} + m_{n-1})} \right) v_{n-1}^{(0)}, \left( \frac{3m_{n-2} - m_{n-1}}{2(m_{n-2} + m_{n-1})} \right) v_{n-1}^{(0)} + \left( \frac{3m_{n-2} - m_{n-1}}{2(m_{n-2} + m_{n-1})} - \frac{m_{n-1}}{2m_2} \right) v_{n-2}^{(n-3)}, v_{n-1}^{(0)} \right).
\]

This is possible, since the length of the interval is positive, according to the formula (2.6) and to the condition (2.7) on the masses.

By the induction hypothesis we found open neighbourhoods
\[
V_1^{(0)} = (\underline{v}_{1}^{(0)}, \overline{v}_{1}^{(0)}), \ldots, V_{n-1}^{(0)} = (\underline{v}_{n-1}^{(0)}, \overline{v}_{n-1}^{(0)}),
\]
and thus we found a neighbourhood \( V_{n-2}^{(n-3)} = (\underline{v}_{n-2}^{(n-3)}, \overline{v}_{n-2}^{(n-3)}) \) for \( v_{n-2}^{(n-3)} \), too.
We can find a neighbourhood for \( v_{n-1}^{(0)} \) if and only if the length of the interval for \( v_{n-1}^{(0)} \) is greater than zero. Therefore we must have
\[
\min \left\{ v_{n-2}^{(n-3)} - v_{n-1}^{(0)} \mid v_{n-2}^{(n-3)} \in V_{n-2}^{(n-3)}, v_{n-1}^{(0)} \in V_{n-1}^{(0)} \right\} = \underline{v}_{n-2}^{(n-3)} - v_{n-1}^{(0)} > 0.
\]

One can achieve this after possibly reducing the size of the interval \( V_{n-2}^{(n-3)} \).

Now we find an explicit configuration space neighbourhood of the \( n \)-th particle. We have to choose the lower limit of the interval by
\[
\underline{q}_{j-1}^{(0)} \geq \frac{\overline{q}_{n-1}^{(0)} - \overline{q}_{n-2}^{(0)}}{\overline{q}_{n-2}^{(0)} - \underline{q}_{n-1}^{(0)}} \left( \underline{v}_{n-1}^{(0)} - \overline{v}_{n}^{(0)} \right) + \underline{q}_{j-1}^{(0)}.
\]
So we have the required sequence of collisions. We can choose the upper limit \( q_{\beta/2}^{(0)} \) greater than \( q_1^{(0)} \).

Therefore we found a non-empty open set \( V_1 := W_1 \times U_1 \subset \mathbb{R}_+^n \times \mathbb{R}^n \times \mathbb{R}^n \) in extended phase space, leading to \( n-1 = \binom{n-1}{1} \) collisions.

**Case 2)** Here we use a transversality argument, perturbing the case of equal masses \( m_1 = \ldots = m_n = 1 \). For equal masses there is an open bounded set \( U_2 \subset U(n) \subset \mathbb{R}^n \times \mathbb{R}^n \) (with \( U(n) \) from (2.1)) of initial conditions in phase space with the following properties: Both the minimal time between the (binary) collisions and the minimal collision angle

\[ \beta := \min_{1 \leq i < j \leq n} \arccos \frac{1+v_i^{(0)}v_j^{(0)}}{\sqrt{1+v_i^{(0)^2}} \sqrt{1+v_j^{(0)^2}}} > 0 \]

in the extended configuration space \( \mathbb{R}^n \times \mathbb{R}^n \) are bounded from below by positive constants for all \((q^{(0)}, v^{(0)}) \in U_2\).

The final positions and velocities for binary collisions depend continuously on the initial data and the masses, see (2.2). Thus by uniform continuity on compacts, we find non-empty open neighborhoods \( U_2 \subset U_2 \) and \( W_2 = (1 - \varepsilon, 1 + \varepsilon)^n \) for which the same statement holds true for the initial data in the subset \( W_2 \times U_2 \) of extended phase space. In particular, the number of collisions equals \( \binom{n}{2} \).

**Case 3)** We consider for \( \delta := (1 + \varepsilon)^{1/(n-1)} \) and \( \delta' \in (1, \delta) \) the non-empty open set \( W_3 = W_3(n, \delta, \delta') := \)

\[ \left\{(m_1, \ldots, m_n) \in (1 - \varepsilon, 1 + \varepsilon)^n \mid \forall k = 2, \ldots, n : \frac{m_k}{m_{k-1}} \in (\delta', \delta) \right\} \]

We prove the assertion again by induction on the number of the particles \( n \). The case for \( n = 2 \) is simple, since then \( \binom{n+1}{3} = 1 \).

In analogy to Case 1) we assume that we have already found a neighbourhood \( U_3(n-1) \) for the first \( n-1 \) particles. Moreover for any initial condition \( x \in U_3(n-1) \) there is a time \( T(x) \in (0, \infty) \), such that there are no more collisions after that time \( T(x) \). Further exists a continuous function \( Q : U_3(n-1) \to \mathbb{R} \), such that \( Q(x) > \max\{q_{n-1}(t) \mid t \in [0, T(x)]\} \).

Now we consider an open set \( U_3(n) := \)

\[ \{(x, q_n, v_n) \in U_3(n-1) \times \mathbb{R}^2 \mid v_n < v_{n-1}, q_n > Q(x), q_n + v_n T(x) > Q(x)\} \]

By these initial conditions particle \( n-1 \) and \( n \) will hit after time \( T \). We will show, that for an appropriate subset \( U_3(n) \subset U_3(n) \) there are exactly \( \binom{n}{3} \) collisions after time \( T \). This then proves the inductive step, since

\[ \binom{n}{3} + \binom{n}{2} = \binom{n+1}{3} \].
Assumed, that there is a collision after time \( T(x) \) for the initial conditions \( x \in (q, v) \in \bar{U}_3(n) \subset \mathbb{R}^n \times \mathbb{R}^n \), then the first collision of this kind will occur between particle \( n-1 \) and \( n \).

We denote the velocity difference of the particles \( k + 1 \) and \( k \) between the \( k \)-th and \( (k + 1)\)-th collision after time \( T \) by \( \Delta v_k^{(k)} \). By definition of \( \bar{U}_3(n) \) we start with \( \Delta v_k^{(k)} \geq 0 \) for \( k = 1, \ldots, n-2 \) and we assume that \( \Delta v_k^{(n-1)} < 0 \). That assumption is justified, since we still can freely choose the initial velocity of the \( n \)-th particle. Then we see

\[
\Delta v_{n-1}^{(1)} = -\Delta v_{n-1}^{(0)} \gg 0, \quad \Delta v_k^{(1)} = \Delta v_k^{(0)} \quad (k = 1, \ldots, n-3)
\]

and

\[
\Delta v_{n-2}^{(1)} = \Delta v_{n-2}^{(0)} + \frac{2m_n}{m_{n-1} + m_n} \Delta v_{n-1}^{(0)} \approx \frac{2m_n}{m_{n-1} + m_n} \Delta v_{n-1}^{(0)} < 0. \quad (2.8)
\]

If the next collision occurs between particle \( n-2 \) and \( n-1 \), then there is

\[
\Delta v_{n-1}^{(2)} = \Delta v_{n-1}^{(1)} + \frac{2m_{n-2}}{m_{n-2} + m_{n-1}} \Delta v_{n-2}^{(1)}
\]

\[
= -\Delta v_{n-1}^{(0)} + \frac{2m_{n-2}}{m_{n-2} + m_{n-1}} \left( \Delta v_{n-2}^{(0)} + \frac{2m_n}{m_{n-1} + m_n} \Delta v_{n-1}^{(0)} \right)
\]

\[
= \frac{4m_{n-2}m_n - (m_{n-2} + m_{n-1})(m_{n-1} + m_n)}{(m_{n-2} + m_{n-1})(m_{n-1} + m_n)} \Delta v_{n-1}^{(0)} + \frac{2m_{n-2} - m_n}{m_{n-2} + m_{n-1}} \Delta v_{n-1}^{(0)}.
\]

Thus, the coefficient of \( \Delta v_{n-1}^{(0)} \) is positive, the numerator must be positive, i.e. it must apply

\[
0 < 4m_{n-2}m_n - (m_{n-2} + m_{n-1})(m_{n-1} + m_n)
\]

\[
= (3m_{n-2} - m_{n-1})m_n - (m_{n-2} + m_{n-1})m_{n-1}. \quad (2.9)
\]

Thus, we obtain the following condition for the \( n \)-th mass

\[
m_n > \frac{(m_{n-2} + m_{n-1})m_{n-1}}{3m_{n-2} - m_{n-1}} \quad \text{for} \quad 3m_{n-2} > m_{n-1}. \quad (2.10)
\]

Now we consider the quotient \( \frac{m_{n}}{m_{n-1}} \), then by (2.4) and (2.10) the result is

\[
\frac{m_{n}}{m_{n-1}} > \frac{m_{n-2} + m_{n-1}}{3m_{n-2} - m_{n-1}} = \frac{1}{3} \frac{m_{n-1}}{m_{n-2}} = f \left( \frac{m_{n-1}}{m_{n-2}} \right).
\]

According to the preliminary note, we can find two initial conditions \( m_1 \) and \( m_2 \) in the interval \((1, (n+1)/n)\) for \( \frac{m_n}{m_{n-1}} \) so that

\[
1 < \delta' = \left( \frac{(n-2)(m_{n-1}^2 - m_{n-2})}{m_{n-1}} \right)^{1/2} < \frac{m_n}{m_{n-1}} < 2^n \left( \frac{m_2}{m_1} - 1 \right) + 1 = \delta = (1 + \epsilon)^{1/(n-1)}.
\]
By iteration, using (2.8), we have for $n-1$ collisions negative values for $\Delta v_\ell^{(n-1)}$ ($\ell = 2, \ldots, n-1$) and we have $\Delta v_{n-1}^{(0)} \ll 0$. Moreover, $\Delta v_1^{(n-1)} \gg 0$.

This means, however, that for the particles 2, \ldots, $n$ by Case 2) a neighbourhood $U_2(n-1)$ can be found, so that we have $(n-1)$ collisions. Since $n-1 + \binom{n-1}{2} = \binom{n}{2}$, we proved the assertion.

3 Numerical Example

By the proof of Theorem 2.2, Part 3) we can also produce $\binom{n+1}{3}$ collisions with mass distribution $m_1 > \ldots > m_{k-1} > m_k < m_{k+1} < \ldots < m_n$.

If we consider (2.9), then we can rewrite this equation to

$$4m_{n-2}m_n > (m_{n-2} + m_{n-1})(m_{n-1} + m_n)$$

and we see that this equation is symmetric in $m_{n-2}$ and $m_n$.

We can add masses, alternating between left and right, so we get $\binom{n+1}{3}$ collisions (see Figure 1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Numerical example of the theorem for $n = 7$ particles and mass distribution $m_1 > \ldots > m_3, m_3 < \ldots < m_7$. The $(n-1)$ collisions from the induction step are black-colored.}
\end{figure}
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