POISSON BRACKETS AFTER JACOBI AND PLÜCKER

PANTELIS A. DAMIANOU

Abstract. We construct a symplectic realization and a bi-hamiltonian formulation of a 3-dimensional system whose solution are the Jacobi elliptic functions. We generalize this system and the related Poisson brackets to higher dimensions. These more general systems are parametrized by lines in 3-dimensional projective space. For these systems the Jacobi identity is satisfied only when the Plücker relations hold. Two of these Poisson brackets are compatible only if the corresponding lines in projective space intersect. We present several examples of such systems.

1. Introduction

Jacobi’s elliptic functions have several applications in Geometry, Mechanics and Physics. They are called elliptic since they appear in some integrals needed to calculate the perimeter of the ellipse. The formula for the length of an ellipse has the form

\[ \int_{0}^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \]

and the integral is usually called an elliptic integral of the second type. Elliptic integrals of the first kind have the form

\[ F(t, k) = \int_{0}^{t} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}} = \int \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}. \]

Ever since 1790 Gauss observed that the inverse function of

\[ \sin^{-1}(t) = \int_{0}^{t} \frac{dx}{\sqrt{1 - x^2}} \]

gives simply periodic function while the inverse function of \( F(t, k) \) is a doubly periodic function.

Thus, in the same manner that we define the sine as the inverse function of this integral, we define the function \( sn(k, t) \) as the inverse of \( F(t, k) \). The elliptic functions which were discovered again later by Abel and Jacobi are denoted by \( sn(t, k) \), \( cn(t, k) \) and \( dn(t, k) \). The real number \( k \) takes values in the interval between 0 and 1. These three functions satisfy the identities

\[ sn^2(t, k) + cn^2(t, k) = 1 \]

\[ k^2 sn^2(t, k) + dn^2(t, k) = 1. \]
These functions are generalizations of the trigonometric functions, e.g. \( sn(t, 0) = \sin t \) and \( cn(t, 0) = \cos t \).

2. A DYNAMICAL APPROACH TO JACOBI ELLIPTIC FUNCTIONS

There are many ways to define trigonometric functions. Let us examine, for example the definition of sine. Usually the first definition that we learn in high school is geometric. It is the length of the opposite side over the hypotenuse. We can also use Taylor series or define it as the real part of \( e^{i\theta} \). Our approach is to define \( \sin t \), \( \cos t \) as the solutions of the system

\[
\frac{dx}{dt} = y \\
\frac{dy}{dt} = -x ,
\]

with initial conditions \( x(0) = 0, x'(0) = 1, y(0) = 1, y'(0) = 0 \).

The function \( f(x, y) = x^2 + y^2 \) is a constant of motion since

\[
\dot{f} = 2x\dot{x} + 2y\dot{y} = 2xy + 2y(-x) = 0 .
\]

Therefore we have \( x^2(t) + y^2(t) = c \). Putting \( t = 0 \) we conclude that \( c = 1 \) and

\[
\sin^2 t + \cos^2 t = 1 .
\]

**Definition 1.** Inspired by the previous remarks, and following [7], [10], we define the functions \( sn(t, k) \), \( cn(t, k) \), \( dn(t, k) \) to be the solutions of the equations

\[
\begin{align*}
\dot{x} &= yz \\
\dot{y} &= -xz \\
\dot{z} &= -k^2 xy .
\end{align*}
\]

with initial conditions \( sn(0, k) = x(0) = 0, cn(0, k) = y(0, k) = 1, dn(0, k) = z(0) = 1 \).

**Remark 1.** Right away we obtain the formulas for the derivatives of these functions, e.g.

\[
\frac{d}{dt} sn(t, k) = cn(t, k) \ dn(t, k) .
\]

**Proposition 1.**

\[
\begin{align*}
sn(t, 0) &= \sin t \\
cn(t, 0) &= \cos t \\
ds(t, 0) &= 1 .
\end{align*}
\]
Proof. When $k = 0$ then $z$ is constant and using the initial conditions $z = 1$. We end-up with the system $\dot{x} = y$, $\dot{y} = -x$ which, as we know has the solution $x = \sin t$, $y = \cos t$. □

Proposition 2. Equations (2) have two constants of motion:

\[ F = x^2 + y^2 \quad G = k^2 x^2 + z^2 \]

Corollary 1. For $0 < k < 1$, and for every $t$

\[ sn^2 + cn^2 = 1, \quad k^2 sn^2 + dn^2 = 1 \]

Observation

\[ \frac{dx}{dt} = yz = cn(t, k) \ dn(t, k) = \sqrt{1 - x^2} \sqrt{1 - k^2 x^2} \]

Therefore

\[ \frac{dt}{dx} = \frac{1}{\sqrt{(1 - x^2)(1 - k^2 x^2)}}. \]

Thus we end-up with the classical definition using elliptic integrals.

3. Hamiltonian Approach

We define

\[ H = \frac{1}{2} (1 + k^2) p_1^2 + \frac{1}{2} p_2^2 + \frac{1}{2} (p_1 q_2 - p_2 q_1)^2 \]

with $k \in \mathbb{R}$.

Hamilton’s equations become

\[
\begin{aligned}
\dot{q}_1 &= \frac{\partial H}{\partial p_1} = (1 + k^2) p_1 + (p_1 q_2 - p_2 q_1) q_2 \\
\dot{q}_2 &= \frac{\partial H}{\partial p_2} = p_2 - (p_1 q_2 - p_2 q_1) q_1 \\
\dot{p}_1 &= -\frac{\partial H}{\partial q_1} = (p_1 q_2 - p_2 q_1) p_2 \\
\dot{p}_2 &= -\frac{\partial H}{\partial q_2} = -(p_1 q_2 - p_2 q_1) p_1
\end{aligned}
\]

We see right away that the function $\frac{1}{2} (p_1^2 + p_2^2)$ is a constant of motion and therefore the system is integrable.

We define the following change of variables from $\mathbb{R}^4$ to $\mathbb{R}^3$. 
\[ x = p_1 \]
\[ y = p_2 \]
\[ z = p_1 q_2 - p_2 q_1 . \]

Then \( \dot{x} = \dot{p}_1 = p_2 (p_1 q_2 - p_2 q_1) = yz \), similarly \( \dot{y} = \dot{p}_2 = -xz \) and \( \dot{z} = -k^2 xy \). The Hamiltonian in the new coordinates is transformed to

\[ \dot{H} = \frac{1}{2} (1 + k^2) x^2 + \frac{1}{2} y^2 + \frac{1}{2} z^2 . \]

The second constant of motion becomes \( \frac{1}{2} (x^2 + y^2) \). The difference of this function from the Hamiltonian is \( \frac{1}{2} (k^2 x^2 + z^2) \). Therefore, ignoring the factor \( \frac{1}{2} \) we obtain the two functions \( F, G \) of last section. The new Hamiltonian has the form \( \frac{1}{2} (F + G) \).

The standard symplectic bracket in \( \mathbb{R}^4 \) is mapped onto the Poisson bracket

\[ \{x, y\} = 0 \quad \{x, z\} = y \quad \{y, z\} = -x . \]

In this Poisson bracket \( F = x^2 + y^2 \) is a Casimir, i.e. the bracket of \( F \) with every other function is 0 and therefore, we can take \( G = \frac{1}{2} (k^2 x^2 + z^2) \) as the Hamiltonian. Hamilton’s equations have the form

\[ \dot{x}_i = \{x_i, G\} . \]

For example,

\[ \dot{x} = \{x, G\} = \{x, \frac{1}{2} (k^2 x^2 + z^2)\} = yz . \]

In other words we end-up with equations (2).

Define the vector \( X \) by

\[ X = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \]

and let the vector \( \nabla H_1 \) be the gradient vector of \( H_1 \), i.e.

\[ \nabla H_1 = \begin{bmatrix} k^2 x \\ 0 \\ z \end{bmatrix} . \]

Then

\[ X = \pi_1 \nabla H_1 \]

is equivalent to equations (2).

The system is bi-hamiltonian. We can take as Hamiltonian the function \( H_2 = \frac{1}{2} F = \frac{1}{4} (x^2 + y^2) \) and define the Poisson bracket \( \pi_2 \) by

\[ \{x, y\} = z \quad \{x, z\} = 0 \quad \{y, z\} = k^2 x . \]
Then we easily verify that
\begin{equation}
X = \pi_1 \nabla H_1 = \pi_2 \nabla H_2.
\end{equation}

Another way to obtain a bi-hamiltonian formulation is to choose $\frac{1}{2}(x^2 + y^2)$ and the following Poisson bracket
\begin{equation}
\{x, y\} = z \quad \{x, z\} = \frac{1}{2}k^2 y \quad \{y, z\} = \frac{1}{2}k^2 x.
\end{equation}

4. A GENERALIZATION

We generalize system (2) as follows: Define the Hamiltonian function in $\mathbb{R}^6$ with coordinates $(q, p)$
\begin{equation}
H = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2}p_3^2 + \frac{1}{2}(p_1q_2 - p_2q_1)^2 + \frac{1}{2}(p_1q_3 - p_3q_1)^2 + \frac{1}{2}(p_2q_3 - p_3q_2)^2.
\end{equation}

This system is integrable. Extending our earlier construction we define the following Poisson map from $\mathbb{R}^6 \to \mathbb{R}^6$:
\begin{align*}
x_1 &= p_1 \\
x_2 &= p_2 \\
x_3 &= p_3 \\
y_1 &= p_3q_2 - p_2q_3 \\
y_2 &= p_1q_3 - p_3q_1 \\
y_3 &= p_2q_1 - p_1q_2.
\end{align*}

The symplectic bracket of rank 6 maps onto the Poisson bracket of rank 4 with Poisson matrix
\begin{equation}
\pi = \begin{pmatrix} 0 & X \\ -X & Y \end{pmatrix},
\end{equation}

where
\begin{equation}
X = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix},
\end{equation}

and
\begin{equation}
Y = \begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}.
\end{equation}

The bracket $\pi$ has two Casimirs $f_1 = x_1^2 + x_2^2 + x_3^2$ and $f_2 = x_1y_1 + x_2y_2 + x_3y_3$. The Poisson bracket is easily seen to be the Lie-Poisson bracket on $\mathfrak{e}(3)$ which is a semi-direct product. The system is a very
special case of the Clebsch system where all parameters are set equal to 1, see [1].

More generally, by introducing coefficients in the Hamiltonian $H$ and using the transformation (8) we get a new system with the Clebsch Hamiltonian

$$h = \frac{1}{2} \left( \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \kappa_1 y_1^2 + \kappa_2 y_2^2 + \kappa_3 y_3^2 \right).$$

The mapping (8) is a symplectic realization of the Clebsch system. The Poisson bracket $\pi$ is the Lie-Poisson structure on $e(3)$ with Casimirs $f_1, f_2$ as before. When the constants satisfy the condition

$$\frac{\lambda_2 - \lambda_3}{\kappa_1} + \frac{\lambda_3 - \lambda_1}{\kappa_2} + \frac{\lambda_1 - \lambda_2}{\kappa_3} = 0$$

we obtain one of the known integrable cases of geodesic flow on $e(3) = so(3) \times \mathbb{R}^3$. The equations of motion $\dot{y} = \{y, h\}, \dot{x} = \{x, h\}$ can be written in the vector form

$$\dot{y} = x \wedge \frac{\partial h}{\partial x} - y \wedge \frac{\partial h}{\partial y}$$

$$\dot{x} = x \wedge \frac{\partial h}{\partial y}$$

where $\wedge$ denotes the cross product on $\mathbb{R}^3$. The extra integral found by Clebsch is

$$f_3 = c(y_1^2 + y_2^2 + y_3^2 + \kappa_1 x_1^2 + \kappa_2 x_2^2 + \kappa_3 x_3^2),$$

where

$$c = \frac{\kappa_1 (\kappa_2 - \kappa_3)}{\lambda_2 - \lambda_3} = \frac{\kappa_2 (\kappa_3 - \kappa_1)}{\lambda_3 - \lambda_1} = \frac{\kappa_3 (\kappa_1 - \kappa_2)}{\lambda_1 - \lambda_2}.$$ 

For more details on this system see [1], [2], [3], [11], [12], [19].

5. Plücker coordinates

Let $U$ be a $k$-dimensional subspace of the $n$-dimensional Euclidean space $V = k^n$ where $k$ is a field. The set of all such $U$ is the Grassmanian $G(k,n)$ or $G(k,V)$. Let $N = \binom{n}{k} - 1$. Suppose that $U$ is spanned by the columns of a $n \times k$ matrix $A$. For every subset $I = \{1, \ldots, n\}$ of cardinality $k$, let $p_I$ be the $k \times k$ subdeterminant of $A$ defined by the rows of $I$. The vector $p = (p_I), |I| = k$ is called the vector of Plücker coordinates of $U$. This defines a point in $\mathbb{P}^N$. In this paper, the most relevant case is when $k = 2$. In the following running example we take $n = 4$.

Example 1. Consider the matrix $A$ given by
Then the Plücker vector $p = (p_{ij})$ has six components. The coordinate $p_{ij}$ corresponds to the sub-determinant defined by rows $i$ and $j$.

(8)\[ p_{ij} = x_i y_j - x_j y_i, \quad 1 \leq i < j \leq 4. \]

We think of this vector as a point in $\mathbb{P}^5$. Not every vector in $\mathbb{P}^5$ is the Plücker vector of a 2-dimensional subspace of $k^4$.

Another way to view the Plücker imbedding is to consider the map 

$$\gamma : G(k, n) \to \mathbb{P} \left( \bigwedge^k V \right)$$

given by 

$$\text{Span}(v_1, \ldots, v_k) \to [v_1 \wedge \cdots \wedge v_k].$$

This map $\gamma$ is injective and the image of $G(k, n)$ is closed. An element in the image of $\gamma$ is called totally decomposable and it can be written in the form $v_1 \wedge \cdots \wedge v_k$. A simple argument shows that if $n = 3$ then every non-zero element of $\bigwedge^2 V$ is totally decomposable. In dimension $n = 4$ this is not the case.

**Example 2.** Suppose $\dim V = 4$ and take a basis $e_1, e_2, e_3, e_4$. Then every element $v \in \bigwedge^2 V$ can be written as

$$v = p_{12}(e_1 \wedge e_2) + p_{13}(e_1 \wedge e_3) + p_{14}(e_1 \wedge e_4) + p_{23}(e_2 \wedge e_3) + p_{24}(e_2 \wedge e_4) + p_{34}(e_3 \wedge e_4).$$

In order for $v$ to be totally decomposable, we should have $v \wedge v = 0$, i.e.,

$$v \wedge v = 2(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) = 0.$$

Therefore the image of the Plücker embedding of $G(2, 4)$ into $\mathbb{P}^5$ satisfies the homogeneous quadric equation

(9)\[ p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0. \]

Let $U$ be a $k$-dimensional subspace of $V$ spanned by $v_1, \ldots, v_k$. Then the vector of coefficients $p_{i_1 \ldots i_k}$, where $i_1 < i_2 < \cdots < i_k$, in the basis representation

$$v_1 \wedge \cdots \wedge v_k = \sum p_{i_1 \ldots i_k} e_{i_1} \wedge \cdots \wedge e_{i_k},$$

equals the Plücker coordinates of $U$ in homogeneous coordinates, i.e. both vectors denote the same point in $\mathbb{P}^N$.

Given a fixed $\omega \in \bigwedge^k V$ we examine the linear map

$$f_\omega : V \to \bigwedge^{k+1} V,$$
given by

\[ v \rightarrow v \wedge \omega. \]

By choosing canonical bases for \( V \) and \( \bigwedge^{k+1} V \) in lexicographic order, we obtain the corresponding matrix of \( f_\omega \) which we denote by \( A_\omega \). Then one can prove:

**Proposition 3.** The following are equivalent

(i) \( \omega \) is totally decomposable

(ii) \( \text{Ker} f_\omega = k \)

(iii) \( \text{rank} A_\omega = n - k \).

**Example 3.** We look again at the case \( G(2, 4) \) for an illustration. A vector \( \omega \in \bigwedge^2 V \) can be written in the form

\[ v = p_{12}(e_1 \wedge e_2) + p_{13}(e_1 \wedge e_3) + p_{14}(e_1 \wedge e_4) + p_{23}(e_2 \wedge e_3) + p_{24}(e_2 \wedge e_4) + p_{34}(e_3 \wedge e_4). \]

Using the basis \( e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4 \) of \( \bigwedge^3 V \) the representation matrix \( A_\omega \) is

\[
A_\omega = \begin{pmatrix}
p_{23} & -p_{13} & p_{12} \\
p_{24} & -p_{14} & 0 \\
p_{34} & 0 & -p_{14} \\
0 & p_{34} & -p_{24} & p_{23}
\end{pmatrix}.
\]

The vector \( \omega \) defines a Plücker vector of a line in \( \mathbb{P}^3 \) if and only if this matrix \( A_\omega \) has rank 2. That means that all \( 3 \times 3 \) minors of this matrix should vanish. The image of \( G(2, 4) \) under the Plücker embedding is the common zero locus of the sixteen \( 3 \times 3 \) minors of \( A_\omega \). For example

\[
\begin{vmatrix}
p_{23} & -p_{13} & p_{12} \\
p_{24} & -p_{14} & 0 \\
p_{34} & 0 & -p_{14}
\end{vmatrix} = p_{14}(p_{14}p_{23} - p_{13}p_{24} + p_{12}p_{34}) = 0.
\]

Indeed, four of the minors are zero, and the rest are all multiples of the quadratic polynomial \( p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} \) which we found earlier.

More generally the Plücker relations for \( G(2, n) \) are given by the vanishing of quadratic polynomials of the form

\[
p_{ij}p_{kl} - p_{ik}p_{jl} + p_{jk}p_{il} = 0
\]

where \( 1 \leq i < j < k < l \leq n \). For later use we note that a line \( l \) intersects a line \( l' \) in \( \mathbb{P}^3 \) if their Plücker coordinates \( p \) and \( p' \) satisfy

\[
p_{12}p'_{34} - p_{13}p'_{24} + p_{14}p'_{23} + p_{23}p'_{14} - p_{24}p'_{13} + p_{34}p'_{12} = 0.
\]

See [15] for details.
6. Generalization to Dimension 4

We generalize the Poisson bracket (6) from three to four dimensions as follows:

\[
\begin{align*}
\{x_1, x_2\} &= \pi_{12} x_3 x_4 \\
\{x_1, x_3\} &= \pi_{13} x_2 x_4 \\
\{x_1, x_4\} &= \pi_{14} x_2 x_3 \\
\{x_2, x_3\} &= \pi_{23} x_1 x_4 \\
\{x_2, x_4\} &= \pi_{24} x_1 x_3 \\
\{x_3, x_4\} &= \pi_{34} x_1 x_2 .
\end{align*}
\]

(12)

A simple calculation shows that the Jacobi identity is satisfied if and only the following condition holds

\[
\pi_{12}\pi_{34} - \pi_{13}\pi_{24} + \pi_{14}\pi_{23} = 0 .
\]

(13)

We recognize this equation as the Plücker relation (9).

Remark 2. This type of bracket appears in [23, p. 23] as a generalization of the Sklyanin bracket, and where it is noted that it gives a 5-dimensional family of quadratic Poisson structures of rank two.

Example 4. Define

\[
\pi_{12} = 1, \quad \pi_{13} = 2, \quad \pi_{14} = 0, \quad \pi_{23} = -2, \quad \pi_{24} = -k^2, \quad \pi_{34} = 2k^2 .
\]

Take \( H = \frac{1}{2} (x_1^2 + x_2^2 + x_3^2) \). Then the equations of motion become

\[
\begin{align*}
\dot{x}_1 &= x_2 x_3 x_4 \\
\dot{x}_2 &= -x_1 x_3 x_4 \\
\dot{x}_3 &= 0 \\
\dot{x}_4 &= -k^2 x_1 x_2 x_3 .
\end{align*}
\]

(14)

The functions \( f = k^2 x_1^2 + x_4^2 \) and \( g = x_1^2 + x_2^2 + \frac{1}{2} x_3^2 \) are Casimirs of the Poisson bracket. Note that the choice \( x_3 = 1, x_1 = x, x_2 = y \) and \( x_4 = z \) gives equations (3).

Example 5. The Sklyanin bracket [20]

The choice

\[
\pi_{12} = j_2 - j_3, \quad \pi_{13} = j_1 - j_3, \quad \pi_{14} = j_1 - j_2, \quad \pi_{23} = 1, \quad \pi_{34} = 1 \quad \pi_{24} = 1 ,
\]

leads to the Sklyanin bracket. The Casimirs are \( x_1^2 + j_1 x_2^2 + j_2 x_3^2 + j_3 x_4^2 \) and \( x_2^2 + x_3^2 + x_4^2 \).

The Poisson bracket in the two examples can be obtained in a different way using a Jacobian type Poisson formula. This type of Poisson structure first appeared in [5], who attributed the formula to H.
Flaschka and T. Ratiu. For explicit proofs see [14], [22]. This formula may be obtained also from Nambu brackets by fixing some variables to constant. Nambu defined this generalization of Poisson bracket in three dimensional space $\mathbb{R}^3$. The Nambu brackets were generalized and given a geometrical interpretation by Takhtajan in [22]. See also, [9], [10].

Let us point out how such a Jacobian Poisson structure is defined. Let $f_1, \ldots, f_{n-2}$ be $n-2$ independent functions in $n$ variables $x_1, \ldots, x_n$. Then a Poisson structure is defined on $\mathbb{R}^n$ by

$$\{F, G\} := \det(\nabla F, \nabla G, \nabla f_1, \ldots, \nabla f_{n-2}),$$

where $F$ and $G$ are arbitrary smooth functions. It is clear that each of the $f_i$ is a Casimir of $\{\cdot, \cdot\}$, so that in particular the generic rank of $\{\cdot, \cdot\}$ is two.

In the case $n = 4$ the formula takes the form

$$\{F, G\} = \det(\nabla F, \nabla G, \nabla f, \nabla g),$$

where $f$ and $g$ are two arbitrary functions.

Let us take $f = \frac{1}{\beta}(\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 + \alpha_4 x_4^2)$ and $g = \frac{1}{\beta}(\beta_1 x_1^2 + \beta_2 x_2^2 + \beta_3 x_3^2 + \beta_4 x_4^2)$. Consider the Jacobian bracket generated by $f$ and $g$. For example,

$$\{x_1, x_2\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 x_1 \\ \alpha_2 x_2 \\ \alpha_3 x_3 \\ \alpha_4 x_4 \end{pmatrix} = (\alpha_3 \beta_4 - \alpha_4 \beta_3)x_3 x_4.$$

**Proposition 4.** The Jacobian Poisson bracket generated by the two Casimirs $f$, $g$ is of the form (12). Conversely, the Poisson bracket (12) is a Jacobian type bracket generated by two Casimirs.

**Proof.** Using (15) we obtain

$$\pi_{12} = (\alpha_3 \beta_4 - \alpha_4 \beta_3) = \pi'_{34},$$

$$\pi_{13} = (\alpha_4 \beta_2 - \alpha_2 \beta_4) = -\pi'_{24},$$

$$\pi_{14} = (\alpha_2 \beta_3 - \alpha_3 \beta_2) = \pi'_{23},$$

$$\pi_{23} = (\alpha_1 \beta_4 - \alpha_4 \beta_1) = \pi'_{14},$$

$$\pi_{24} = (\alpha_3 \beta_1 - \alpha_1 \beta_3) = -\pi'_{13},$$

$$\pi_{34} = (\alpha_1 \beta_2 - \alpha_2 \beta_1) = \pi'_{12}. $$

Then we have

$$\pi_{12} \pi_{34} - \pi_{13} \pi_{24} + \pi_{14} \pi_{23} = \pi'_{34} \pi'_{12} - \pi'_{24} \pi'_{13} + \pi'_{23} \pi'_{14} = 0.$$

Therefore the Jacobi identity is satisfied. We obtain a Jacobian Poisson bracket of rank 2 and of the from (12).

Conversely, if
\[ \pi = \begin{pmatrix} 0 & \pi_{12} x_3 x_4 & \pi_{13} x_2 x_4 & \pi_{14} x_2 x_3 \\ -\pi_{12} x_3 x_4 & 0 & \pi_{23} x_1 x_4 & \pi_{24} x_1 x_3 \\ -\pi_{13} x_2 x_4 & -\pi_{23} x_1 x_4 & 0 & \pi_{34} x_1 x_2 \\ -\pi_{14} x_2 x_3 & -\pi_{24} x_1 x_3 & -\pi_{34} x_1 x_2 & 0 \end{pmatrix}, \]

then \( \det \pi \) is equal to \( x_1^2 x_2^2 x_3^2 x_4^2 (\pi_{12} \pi_{34} - \pi_{13} \pi_{24} + \pi_{14} \pi_{23})^2 \). The bracket has rank 2 if and only if \( \pi_{12} \pi_{34} - \pi_{13} \pi_{24} + \pi_{14} \pi_{23} = 0 \). Therefore it is Poisson if and only if it has rank 2. It follows from the fact that the Plücker map is injective that we can define \( \pi \) Poisson if and only if it has rank 2. Using this Poisson bracket and taking as Hamiltonian the function \( f \) and \( g \). We prove this explicitly in the case \( \pi_{23} \neq 0 \). We take \( \alpha_1 = 0 \) and \( \beta_4 = 0 \). Then we have

\[
\begin{align*}
\pi_{12} &= -\alpha_4 \beta_3 \\
\pi_{13} &= \alpha_4 \beta_2 \\
\pi_{14} &= \alpha_4 \beta_3 - \alpha_3 \beta_2 \\
\pi_{23} &= -\alpha_4 \beta_1 \\
\pi_{24} &= \alpha_3 \beta_1 \\
\pi_{34} &= -\alpha_4 \beta_1.
\end{align*}
\]

The fourth and fifth equation imply that \( \alpha_3 \pi_{23} = -\alpha_4 \pi_{24} \). We choose \( \alpha_3 = -\pi_{24} \) and \( \alpha_4 = \pi_{23} \). The first equation gives \( \beta_3 = -\frac{\alpha_3}{\pi_{23}} \) and the second gives \( \beta_2 = \frac{\pi_{13}}{\pi_{23}} \). The fourth equation implies that \( \beta_1 = -1 \) and the last equation \( \alpha_2 = \pi_{34} \). The consistency of the third equation follows from the Plücker relation. Therefore we may choose \( f = \pi_{34} x_2^2 - \pi_{24} x_3^2 + \pi_{23} x_4^2 \) and \( g = \pi_{23} x_1^2 - \pi_{13} x_2^2 + \pi_{12} x_3^2 \). This completes the proof.

Given a system of the form

\begin{equation}
\begin{align*}
\dot{x}_1 &= c_1 x_2 x_3 x_4 \\
\dot{x}_2 &= c_2 x_1 x_3 x_4 \\
\dot{x}_3 &= c_3 x_1 x_2 x_4 \\
\dot{x}_4 &= c_4 x_1 x_2 x_3
\end{align*}
\end{equation}

where the \( c_i \) are different than zero, it has the obvious integrals \( f_1 = c_2 x_1^2 - c_1 x_2^2 \) and \( f_2 = c_4 x_3^2 - c_3 x_4^2 \). Using the Jacobian formula (generated by \( f_1 \) and \( f_2 \) we obtain a Poisson bracket of the form (12) where

\[
\pi_{12} = 0, \quad \pi_{13} = -4c_1 c_3, \quad \pi_{14} = -4c_1 c_4, \quad \pi_{23} = -4c_2 c_3, \quad \pi_{24} = -4c_2 c_4, \quad \pi_{34} = 0.
\]

Using this Poisson bracket and taking as Hamiltonian the function

\[ H = \frac{1}{16} \left( \frac{x_1^2}{c_1} + \frac{x_2^2}{c_2} - \frac{x_3^2}{c_3} - \frac{x_4^2}{c_4} \right) \]
we obtain equations (16). Therefore the system (16) is always a completely integrable Hamiltonian system. This procedure, of course, generalizes easily to higher dimensions.

Recall the definition of compatible Poisson brackets. Let \((M, \pi_1), (M, \pi_2)\) be two Poisson structures on \(M\). If \(\pi_1 + \pi_2\) is also Poisson then the two tensors form a Poisson pair on \(M\). The corresponding Poisson brackets are called compatible. We would like to find a condition so that two brackets of the form (12) are compatible. Consider two brackets in \(R^4\) of the form (12) defined by functions \(\pi_{ij}\) and \(\pi'_{ij}\).

When are they compatible? A simple computation of the Jacobi identity shows that the following identity should be satisfied:

\[
\pi_{12}\pi'_{34} - \pi_{13}\pi'_{24} + \pi_{14}\pi'_{23} - \pi_{24}\pi'_{13} + \pi_{34}\pi'_{12} = 0 .
\]

Comparing this equation with equation (11) we immediately have the following:

**Proposition 5.** Consider two Poisson brackets \(\pi_{ij}\) and \(\pi'_{ij}\) associated to two lines \(l\) and \(l'\) in \(P^3\). Then \(\pi_{ij}\) and \(\pi'_{ij}\) are compatible if and only if the line \(l\) intersects the line \(l'\) in \(P^3\).

The Poisson brackets of examples (4) and (5) are not compatible.

### 7. Generalization to \(R^n\)

More generally we can consider on \(R^n\) a bracket of the form

\[
\{x_i, x_j\} = \pi_{ij}x_1x_2\cdots\hat{x}_i\cdots\hat{x}_j\cdots x_n .
\]

As usual the notation \(\hat{x}_i\) means that the variable \(x_i\) is omitted. As in dimension 4 the Jacobi identity is satisfied if and only if the Plücker relations are satisfied.

**Theorem 1.** The bracket (17) is Poisson if and only if the \(\pi_{ij}\) satisfy the Plücker relations.

**Proof.** We may assume \(i < j < k\). We check the Jacobi identity for coordinate functions \(x_j\).
\( \{ x_i, \{ x_j, x_k \} \} + \{ x_j, \{ x_k, x_i \} \} + \{ x_k, \{ x_i, x_j \} \} = \)
\( \{ x_i, \pi_{jk}x_1x_2 \cdots \hat{x}_l \cdots x_k \cdots x_n \} - \{ x_j, \pi_{ik}x_1x_2 \cdots \hat{x}_l \cdots x_i \cdots x_k \cdots x_n \} + \{ x_k, \pi_{ij}x_1x_2 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x \}
\)
\( = x_i \pi_{jk} \left( \sum_{l \neq j, k} \{ x_i, x_l \} x_1x_2 \cdots \hat{x}_l \cdots \hat{x}_j \cdots x_k \cdots x_n \right) - \)
\( - x_j \pi_{ik} \left( \sum_{l \neq i, k} \{ x_j, x_l \} x_1x_2 \cdots \hat{x}_l \cdots \hat{x}_i \cdots \hat{x}_k \cdots x_n \right) + \)
\( x_k \pi_{ij} \left( \sum_{l \neq i, j} \{ x_k, x_l \} x_1x_2 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_k \cdots x_n \right) \)

We examine this expression for fixed \( l \). It is equal to
\( x_i \pi_{jk} \left( \pi_{il} x_1x_2 \cdots \hat{x}_i \cdots \hat{x}_l \cdots x_n \right)_{l \neq j, k} x_1x_2 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_k \cdots x_n - \)
\( - x_j \pi_{ik} \left( \pi_{jl} x_1x_2 \cdots \hat{x}_j \cdots \hat{x}_l \cdots x_n \right)_{l \neq i, k} x_1x_2 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_k \cdots x_n + \)
\( + x_k \pi_{ij} \left( \pi_{kl} x_1x_2 \cdots \hat{x}_i \cdots \hat{x}_l \cdots x_n \right)_{l \neq i, j} x_1x_2 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_k \cdots x_n \)
\( = \left( \pi_{jk} \pi_{il} - \pi_{ik} \pi_{jl} + \pi_{ij} \pi_{kl} \right) x_1^2 x_2^2 \cdots \hat{x}_i^2 \cdots \hat{x}_j^2 \cdots x_n x_i x_j x_k . \)

Since the expression involves homogeneous polynomials, the Jacobi identity is satisfied only if the coefficients all vanish. Therefore we get
\( \pi_{jk} \pi_{il} - \pi_{ik} \pi_{jl} + \pi_{ij} \pi_{kl} = 0 . \)

Example 6. Double-elliptic system. (see \cite{1}, \cite{18}.)

The Hamiltonian of the two-particle double-elliptic system in the form of \cite{5} is
\[
H(p, q) = \alpha(q|k) \text{cn} \left( p \beta(q|k) \frac{\tilde{k} \alpha(q|k)}{\beta(q|k)} \right)
\]
where \( \alpha(q|k) = \sqrt{1 + \frac{g^2}{sn^2(q|k)}} \) and \( \beta(q|k) = \sqrt{1 + \frac{g^2 k}{sn^2(q|k)}} \) coincides with \( x_5 \).

We choose the following system of four quadrics in \( \mathbb{R}^6 \)
\[
\begin{align*}
x_1^2 - x_2^2 &= 1 \\
x_1^2 - x_3^2 &= k^2 \\
-g^2 x_1^2 + x_4^2 - x_5^2 &= 1 \\
-g^2 x_1^2 + x_4^2 + k^{-2} x_6^2 &= \tilde{k}^{-2} .
\end{align*}
\]
Using the bracket (15) we obtain a Poisson bracket of the form (17). Up to a factor of $\frac{16}{k^2}$ some non-trivial brackets are
\[
\{x_1, x_5\} = -x_2 x_3 x_4 x_6 \\
\{x_2, x_5\} = -x_1 x_3 x_4 x_6 \\
\{x_3, x_5\} = -x_1 x_2 x_4 x_6 \\
\{x_4, x_5\} = -g^2 x_1 x_2 x_3 x_6.
\]

The fact that $\{x_5, x_6\} = 0$ shows that $x_6$ is a constant of motion. Setting the integrals $x_5$ and $x_6$ to constants we obtain the following system of equations in $\mathbb{R}^4$:
\[
\dot{x}_1 = x_2 x_3 x_4 \\
\dot{x}_2 = x_1 x_3 x_4 \\
\dot{x}_3 = x_1 x_2 x_4 \\
\dot{x}_4 = g^2 x_1 x_2 x_3.
\]

This system has the form of Fairlie elegant integrable system (13). Of course the integrals survive after reduction, i.e. $x_1^2 - x_2^2$, $x_1^2 - x_3^2$, $x_4^2 - g^2 x_1^2$ are first integrals of the reduced system.

References

[1] Adler M., Van Moerbeke P., Vanhaecke P., Algebraic integrability, Painlevé geometry and Lie algebra Ergebnisse der Mathematik und ihrer grenzgebiete 3.folge 47 (2004) Springer-Verlag, Berlin Heidelberg.

[2] Borisov, A. V.; Mamaev, I. S. The dynamics of a rigid body, Regul. Chaotic Dyn. Izhevsk, 2001. 379 pp. ISBN: 5-93972-055-7 [in Russian].

[3] Borisov, A. V.; Mamaev, I. S. Isomorphisms of geodesic flows on quadrics. Regul. Chaotic Dyn. 14 (2009), no. 4-5, 455–465.

[4] Braden, H. W.; Gorsky, A.; Odesskii, A.; Rubtsov, V. Double-elliptic dynamical systems from generalized Mukai-Sklyanin algebras. Nuclear Phys. B 633 (2002), no. 3, 414–442.

[5] Braden, H. W.; Marshakov, A.; Mironov, A.; Morozov, A. The Ruijsenaars-Schneider model in the context of Seiberg-Witten theory. Nuclear Phys. B 558 (1999), no. 1-2, 371–390.

[6] Borisov Tsiganov 2009

[7] R. H. Cushman, L. M. Bates, Global aspects of classical integrable systems, Birkhauser, Berlin, 1997.

[8] P. A. Damianou, Nonlinear Poisson brackets, Ph.D. thesis, 1989.

[9] Damianou P.A. Transverse Poisson structures of coadjoint orbits, Bull.Sci.Math. 120 (1996), 525-534.

[10] Damianou, Pantelis A.; Sabourin, Herv Vanhaecke. Pol. Transverse Poisson structures to adjoint orbits in semisimple Lie algebras. Pacific J. Math. 232 (2007), no. 1, 111–138.

[11] Dragovic, Vladimir; Gajic, Borislav. On the cases of Kirchhoff and Chaplygin of the Kirchhoff equations. Regul. Chaotic Dyn. 17 (2012), no. 5, 431–438.

[12] Dubrovin, B. A. Theta-functions and nonlinear equations. (Russian) With an appendix by I. M. Krichever. Uspekhi Mat. Nauk 36 (1981), no. 2(218), 11–80.

[13] Fairlie, D. B. An elegant integrable system. Phys. Lett. A 119 (1987), no. 9, 438–440.

[14] J. Grabowski, G. Marmo, and A. M. Perelomov, Poisson structures: towards a classification, Mod. Phys. Lett. A 8:18 (1993), 17191713.

[15] Joewing, Michael; Theobald, Thorsten. Polyhedral and algebraic methods in computational geometry. Universitext. Springer, London, 2013.

[16] K. R. Meyer, Jacob Elliptic Funions from a Dynamical Point of View, Amer. Math. Monthly, v. 108 729–737. (2001).

[17] Nambu, Yoichiro. Generalized Hamiltonian dynamics. Phys. Rev. D (3) 7 (1973), 2405–2412.
[18] Odesskii, A. V.; Rubtsov, V. N. Polynomial Poisson algebras with a regular structure of
symplectic leaves. (Russian) ; translated from Teoret. Mat. Fiz. 133 (2002), no. 1, 3–23
Theoret. and Math. Phys. 133 (2002), no. 1, 1321–1337
[19] Perelomov, A. M. Some remarks on the integrability of the equations of motion of a rigid
body in an ideal fluid. (Russian) Funktsional. Anal. i Prilozhen. 15 (1981), no. 2, 83–85.
[20] Sklyanin, E. K., Some algebraic structures connected with the Yang-Baxter equation. (Rus-
sian) Funktsional. Anal. i Prilozhen. 16 (1982), no. 4, 27–34, 96.
[21] 2018
[22] L. Takhtajan, On foundation of the generalized Nambu mechanics, Comm. Math. Phys. 160:2
(1994), 295315.
[23] Vanhaecke, Pol. Integrable systems in the realm of algebraic geometry. Second edition. Lecture
Notes in Mathematics, 1638. Springer-Verlag, Berlin, 2001.

Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537,
1678 Nicosia, Cyprus
E-mail address: Damianou@ucy.ac.cy