Non-surjective pullbacks of graph $C^*$-algebras from non-injective pushouts of graphs

Alexandru Chirvasitu, Piotr M. Hajac and Mariusz Tobolski

Abstract

We find a substantial class of pairs of $*$-homomorphisms between graph $C^*$-algebras of the form $C^*(E) \hookrightarrow C^*(G) \twoheadleftarrow C^*(F)$ whose pullback $C^*$-algebra is an AF graph $C^*$-algebra. Our result can be interpreted as a recipe for determining the quantum space obtained by shrinking a quantum subspace. There are numerous examples from noncommutative topology, such as quantum complex projective spaces (including the standard Podleś quantum sphere) and quantum teardrops, that instantiate the result. Furthermore, to go beyond AF graph $C^*$-algebras, we consider extensions of graphs over sinks and prove an analogous theorem for the thus obtained graph $C^*$-algebras.

1. Introduction

The classical two-sphere $S^2$ can be obtained by shrinking the boundary of the disc $B^2$ to a point. In other words, there is a pushout diagram in the category of topological spaces:

$$
\begin{array}{ccc}
S^2 & \xrightarrow{\ast} & B^2 \\
\downarrow & & \downarrow \\
S^1 & \xleftarrow{\ast} & S^1
\end{array}
$$

(1.1)

Due to the contravariant duality of algebras and spaces, the diagram (1.1) amounts to an isomorphism $C(S^2) \cong C(B^2 \sqcup S^1 \{\ast\}) = C(B^2) \oplus C(S^1)$ of $C^*$-algebras of complex-valued continuous functions on the two-sphere and on the pushout $B^2 \sqcup S^1 \{\ast\}$, respectively.

At the same time, the Toeplitz algebra $T$ can be viewed as a noncommutative deformation of $C(B^2)$ (e.g., see [13, Theorem IV.7]). Therefore, the $C^*$-algebra $C(S^2_{q0})$ of the standard Podleś quantum sphere [16, (3a)] provides a noncommutative deformation of the diagram (1.1), namely we have the following pullback diagram in the category of $C^*$-algebras:

$$
\begin{array}{ccc}
C(S^2_{q0}) & \xleftarrow{C} & T \\
\downarrow & & \downarrow \\
C(S^1) & \xrightarrow{C} & C(S^1)
\end{array}
$$

(1.2)
The aim of this paper is to generalize the above pullback construction using the concept of the C*-algebra $C^*(E)$ of a directed graph $E$ (e.g., see [3]). Graph C*-algebras provide powerful tools in noncommutative topology, and many C*-algebras representing noncommutative deformations of topological spaces are isomorphic with C*-algebras of graphs [5, 11, 12]. For a non-zero deformation parameter, these isomorphisms are usually quite complicated. However, when such an isomorphism is established, it is easier to obtain solutions to many problems, especially concerning K-theory.

Our starting point is that all the C*-algebras in the diagram (1.2) can be viewed as C*-algebras of graphs. We present this pictorially as follows:

\[ C^* \quad (\infty) \quad C^* \quad (\bullet) \quad C^* \quad (\bigcirc) \]

The graph-algebraic decomposition (1.3) manifests a certain general phenomenon that can be explained in terms of non-injective pushouts of graphs. The goal of this paper is to explore this phenomenon to arrive at a general setting. To this end, we search for a new concept of morphisms of graphs, so as to ensure that, in the thus defined category of graphs, the assignment of graph algebras to graphs becomes a contravariant functor translating pushouts of graphs into pullbacks of graph algebras. For injective pushouts of finite graphs, this task was initiated by Hong and Szymański in [12], where they obtained a pullback presentation of even-dimensional equatorial quantum spheres viewed as graph C*-algebras. This work was continued by Robertson and Szymański in [17], where the authors give conditions on pairs of C*-correspondences to yield a pullback of the associated Cuntz–Pimsner algebras. Independently, the task of writing down the conditions for injective pushouts of graphs without breaking vertices to yield the pullbacks of associated graph C*-algebras was completed in [10] (cf. [14, Corollary 3.4] for higher rank graphs).

Herein we handle the non-injective case of the type as in (1.3). To accommodate this naturally occurring non-injectivity, we replace the standard idea of mapping vertices to vertices and edges to edges by the more flexible idea of mapping finite paths to finite paths. We arrive at a general result for a class of unital AF graph C*-algebras including the standard Podleś quantum sphere, the Vaksman–Soibelman quantum complex projective spaces [19, Definition, p. 109], and quantum teardrops [4]. Finally, we go beyond AF graph C*-algebras by extending their acyclic graphs over sinks.

2. Graph-algebraic preliminaries

A directed graph $E$ is a quadruple $(E^0, E^1, s, r)$, where $E^0$ is the set of vertices, $E^1$ is the set of edges (arrows), and $s, r : E^1 \to E^0$ are the source map and the range (target) map, respectively. Throughout the paper, we consider only directed graphs with countable sets of vertices and edges, and we will often simply refer to them as graphs.
DEFINITION 2.1 (Graph $C^*$-algebra). The graph $C^*$-algebra $C^*(E)$ of a directed graph $E$ is the universal $C^*$-algebra generated by mutually orthogonal projections $\{P_v \mid v \in E^0\}$ and partial isometries $\{S_e \mid e \in E^1\}$ satisfying the following conditions:

$$S^*_e S_f = \delta_{e,f} P_{r(e)}$$

for all $e, f \in E^1$,  
(\text{GA1})

$$\sum_{e \in s^{-1}(v)} S_e S^*_e = P_v$$

for all $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$,  
(\text{GA2})

$$S_e S^*_e \leq P_{s(e)}$$

for all $e \in E^1$.  
(\text{GA3})

Now we can explain a graph-algebraic presentation of the main motivating example of the paper, that is, the diagram (1.3). Let $q \in [0, 1]$. The $C^*$-algebra $C(S^2_q)$ of the standard Podleś quantum sphere coincides with the $C^*$-algebra of the Vaksman–Soibelman quantum complex projective line $C(CP^1_q)$ [19, p.109], which has a graph-algebraic presentation as the graph $C^*$-algebra of the graph given below (see [11, Section 2.3]):

$$v_1 \rightarrow (\infty) \rightarrow v_2.$$  
(2.1)

Here the arrow decorated by $(\infty)$ denotes countably infinitely many arrows with the source $v_1$ and the range $v_2$. The one-vertex and one-loop graphs obviously yield the $C^*$-algebras $\mathbb{C}$ and $C(S^1)$, respectively. The remaining right-hand side graph is well known to give the Toeplitz algebra: the sum of the partial isometries corresponding to edges can be identified with the $C^*$-algebra generated by mutually orthogonal projections $\{S_e \mid e \in E^1\}$ of the standard Podleś $\mathbb{C}$-algebra of the graph given below (see [3, Equation (1)]):

$$I_H = \overline{\text{span}}\{S_{\alpha} S^*_\beta \mid \alpha, \beta \in \text{Path}(E), r(\alpha) = r(\beta) \in H\}.  
(2.2)$$

Here $\overline{\text{span}}$ denotes the closed linear span.

Furthermore, we will need the following known lemmas.

**Lemma 2.2.** Let $\alpha$ and $\beta$ be paths in an arbitrary graph $E$. Then

$$S^*_\alpha S_\beta \neq 0 \iff (\alpha \preceq \beta \text{ or } \beta \preceq \alpha).$$

**Lemma 2.3.** Let $\alpha$ be a path in an arbitrary graph $E$. Then

$$S_\alpha S^*_\alpha \leq P_{s(\alpha)} \in C^*(E).$$

We end this section by recalling some standard results that we will use throughout the paper. Let $E$ be any graph. A subset $H \subseteq E^0$ is called hereditary iff, for any $v \in H$ such that there is an edge (or a path) starting at $v$ and ending at $w$, we have $w \in H$. If $H$ is hereditary, then the ideal $I_H$ generated by the projections associated with the elements of $H$ is of the form (see [3, Equation (1)]):

$$I_H = \overline{\text{span}}\{S_{\alpha} S^*_\beta \mid \alpha, \beta \in \text{Path}(E), r(\alpha) = r(\beta) \in H\}.  
(2.2)$$

Here $\overline{\text{span}}$ denotes the closed linear span.

Assume additionally that there are no vertices that emit infinitely many arrows into $H$ and finitely many (but not zero) arrows outside of $H$. (These are called breaking vertices.)
Assume also that $H$ is saturated, that is, there does not exist a regular vertex $v \not\in H$ such that $r(s^{-1}(v)) \subseteq H$. Then, the quotient algebra $C^*(E)/I_H$ is again a graph $C^*$-algebra (see the discussion below Equation (1) in [3]):

$$C^*(E)/I_H \cong C^*(E/H), \quad \text{where } E/H := (E^0 \setminus H, r^{-1}(E^0 \setminus H), s_H, r_H)$$

(2.3)

and $s_H$ and $r_H$ are the restrictions of $s$ and $r$, respectively.

3. Non-surjective pullbacks of graph $C^*$-algebras

In this section, we prove a non-surjective pullback theorem generalizing the statement that the diagram (1.3) is a pullback diagram of $C^*$-algebras. First, we need some preliminaries on graphs and their morphisms.

Let $D = (D^0, D^1, s_D, r_D)$ and $E = (E^0, E^1, s_E, r_E)$ be directed graphs. A morphism of graphs $f : D \to E$ is a pair of mappings $f^0 : D^0 \to E^0$ and $f^1 : D^1 \to E^1$ satisfying

$$f^0 \circ s_D = s_E \circ f^1, \quad f^0 \circ r_D = r_E \circ f^1.$$  

(3.1)

If there is an injective morphism of graphs $D \to E$ (i.e. both the vertex map and the edge map are injective), we say that $D$ is a subgraph of $E$ and write $D \subseteq E$.

**Definition 3.1.** An injective graph morphism $\iota : D \to E$ is called an admissible inclusion if and only if the following conditions are satisfied:

(A1) $E^0 \setminus \iota^0(D^0)$ is hereditary and saturated;

(A2) $\iota^1(D^1) = r_E^{-1}(\iota^0(D^0))$;

(A3) no vertex in $E^0$ emits infinitely many edges into $E^0 \setminus \iota^0(D^0)$ while emitting finitely many (but not zero) edges into $\iota^0(D^0)$.

Next, let us state the following fundamental fact corresponding to (2.3) and the discussion preceding it.

**Proposition 3.2.** Let $D \subseteq E$ be an admissible inclusion. Then we have the canonical isomorphism of graph $C^*$-algebras

$$C^*(D) \cong C^*(E/(E^0 \setminus D^0)).$$

(3.2)

To phrase our main result, it is convenient to view graphs as small categories whose objects are vertices and morphisms are finite paths. Then functors between such categories are what we view as morphisms between graphs. Using the thus understood functors as morphisms, we generalize the directed-graph category [8, Section 2.1] (cf. [1, Definition 1.6.2]) by allowing edges to be mapped to finite paths instead of only edges.

**Lemma 3.3.** Let $f : F \to E$ be a functor between graphs such that

(1) $f$ is compatible with the prolongation relation as follows:

$$f(\alpha) \leq f(\beta) \implies \alpha \leq \beta;$$

(2) for any regular vertex $v$, $f$ restricts to a bijection

$$s_F^{-1}(v) \to s_E^{-1}(f(v)).$$

Then $f$ induces a $*$-homomorphism $f_* : C^*(F) \to C^*(E)$ given by

$$\forall v \in F^0 : f_*(P_v) := P_{f(v)} \quad \text{and} \quad \forall x \in F^1 : f_*(S_x) := S_{f(x)}.$$
Proof. Since graph $C^*$-algebras are universal, it suffices to show that all defining relations are preserved. For starters, since the condition (1) implies the injectivity of $f$, we infer that the set of mutually orthogonal projections is sent to the set of mutually orthogonal projections:

$$P_{f(v)}P_{f(w)} = \delta_{f(v),f(w)}P_{f(v)} = \delta_{v,w}P_{f(v)}. \quad (3.3)$$

Next, to show that (GA1) is preserved, it suffices to prove the implication

$$S_{f(e_1)}^*S_{f(e_2)} \neq 0 \Rightarrow e_1 = e_2, \quad (3.4)$$

which follows from combining Lemma 2.2 with the condition (1). Finally, showing that (GA2) and (GA3) are preserved is also straightforward: the former follows directly from the condition (2) and the latter from Lemma 2.3. □

We are now ready to prove our first main result:

**Theorem 3.4.** Let $F_i \subseteq E_i$, $i = 1, 2$, be admissible inclusions of graphs such that

1. $E_1$ has no loops, $E_2$ has no short loops at vertices in $E_2^0 \setminus F_2^0$, and $E_1^0 = E_2^0$, $F_1^0 = F_2^0$;
2. there is a functor $f : E_1 \to E_2$ such that $f$ satisfies the condition (1) in Lemma 3.3, $f$ is id on the set of objects, and the image of $f$ is the set of all pointed paths.

Then the induced $*$-homomorphisms exist and render the diagram

$$
\begin{array}{ccc}
C^*(E_1) & \xrightarrow{f_*} & C^*(E_2) \\
\pi_1 & & \pi_2 \\
C^*(F_1) & \xleftarrow{f|_*} & C^*(F_2)
\end{array}
$$

a pullback diagram of $C^*$-algebras. (If $E_1^0$ is finite, then this is a pullback diagram of unital $C^*$-algebras.) Here $\pi_1$ and $\pi_2$ are the canonical surjections defining (3.2), $f : F_1 \to F_2$ is a restriction of $f$, and $f_*$ and $f|_*$ are induced $*$-homomorphisms of Lemma 3.3.

Proof. We begin by proving that $f_*$ and $f|_*$ are well-defined injective $*$-homomorphisms. To see that $f_*$ is well defined, by Lemma 3.3 and the assumption (2), it suffices to check the condition (2) of Lemma 3.3. To this end, take any regular vertex $v \in E_1^0$ and any edge $e \in s^{-1}(v)$. Then, as the image of $f$ is the set of pointed paths, $f(e)$ is a pointed path from $v$ to $r(e)$.

Suppose now that $f(e)$ factorizes through a third vertex $w$. Then we can write $f(e) = \alpha \beta$, where $\alpha$ is a pointed path from $v$ to $w$ and $\beta$ is a pointed path from $w$ to $r(e)$. Indeed, deleting any initial subpath from a pointed path always yields a pointed path, and making all loops based at $w$ part of $\beta$ makes $\alpha$ a pointed path. Furthermore, as $f$ is surjective on the set of pointed paths, we can write $f(e) = \alpha \beta = f(\alpha')f(\beta') = f(\alpha'\beta')$. Combining this with the injectivity of $f$, which follows from the condition (1) in Lemma 3.3, we get a contradiction $e = \alpha'\beta'$ (the edge $e$ is not a path factorizing through the vertex $w$). Hence $f(e)$ is a pointed path from $v$ to $r(e)$ that does not factorize through any third vertex.

If there is a loop in $E_2$ based at $v$, then there are infinitely many non-factorizing pointed paths from $v$ to $r(e)$, and (because $f$ is a functor) none of them can be the image of a path that factorizes through a third vertex. Consequently, as $f$ is bijective when we restrict its codomain to the set of pointed paths, and there are no loops in $E_1$, there must be infinitely many edges...
in $E_1$ from $v$ to $r(e)$, which contradicts the assumption that $v$ is a regular vertex in $E_1$. Hence, there is no loop in $E_2$ based at $v$, so $f(e)$ is an edge.

Next, if $f(\alpha) \in E_2^1$, then $\alpha \in E_1^1$ because $f$ is an injective functor. Indeed, suppose that $\alpha = e_1 \cdots e_n$, where each $e_i$ is an edge. Then $f(\alpha) = f(e_1 \cdots e_n) = f(e_1) \cdots f(e_n)$ is of length at least $n$, as $f(e_i)$ cannot be a vertex. Hence, $n = 1$, that is $\alpha$ is an edge, so any edge emitted from $v$ in $E_2$ comes from an edge emitted from $v$ in $E_1$. Combining this with the injectivity of $f$ and the above established fact that $f(e)$ is an edge, we conclude that the condition (2) in Lemma 3.3 is satisfied. Thus, we obtain a well-defined $*$-homomorphism $f_*$ that is injective by [18, Corollary 1.3] because $E_1$ has no loops and $f_*(P_v) = P_{f(v)} \neq 0$ for all $v$.

Furthermore, as $f_*(\ker \pi_1) \subseteq \ker \pi_2$ by (2.2) and (2.3), $f_*$ corestricts to the $*$-homomorphism $\overline{f}_* : C^*(F_1) \to C^*(F_2)$ given on generators by $\overline{f}_*(P_v) = P_{f(v)} = P_v$ and $\overline{f}_*(S_e) = \pi_2(S_{f(e)})$. Since, by the admissibility condition Definition 3.1(A2), it is clear that $f$ restricted to the subgraph $F_1$ takes values in $F_2$ yielding a morphism of graphs $f : F_1 \to F_2$, we infer that $\pi_2(S_{f(e)}) = S_{f(e)}$. The restricted morphism $f|_E$ induces a $*$-homomorphism $f_*|_{E}$ as in Lemma 3.3 because $f_*|_{E}$ coincides on generators with $\overline{f}_*$. Consequently, the diagram (3.5) is commutative because $f_*|_{E} = \overline{f}_*$. To complete the picture, note that the injectivity of $f_*|_{E}$ is proven by the same argument as the argument proving the injectivity of $f_*$.\)

Furthermore, as $\pi_1$ and $\pi_2$ are surjective and $f_*$ and $f_*|_{E}$ are injective, due to [15, 3.1. Proposition], to show that (3.5) is a pullback diagram, it suffices to prove that

$$\ker \pi_2 \subseteq f_*(\ker \pi_1). \quad (3.6)$$

To obtain the above inclusion, we use the characterization of ideals associated to hereditary subsets (2.2):

$$\ker \pi_1 = \operatorname{span}\{S_\alpha S_\beta^* \mid \alpha, \beta \in \operatorname{Path}(E_1), \ r(\alpha) = r(\beta) \in E_1^0 \setminus F_1^0\}, \quad (3.7)$$

$$\ker \pi_2 = \operatorname{span}\{S_\gamma S_\delta^* \mid \gamma, \delta \in \operatorname{Path}(E_2), \ r(\gamma) = r(\delta) \in E_2^0 \setminus F_2^0\}. \quad (3.8)$$

By the assumption (1), all paths in $E_2$ terminating in $E_2^0 \setminus F_2^0$ are pointed, so they are in the image of $f$. Therefore, as $E_2^0 \setminus F_2^0 = E_1 \setminus F_1$ by the assumption (1) and $f$ is id on the set of vertices by the assumption (2), we conclude that the inclusion (3.5) holds at the algebraic level. However, as the image of any $*$-homomorphism between $C^*$-algebras is closed, the inclusion at the algebraic level implies the desired inclusion at the $C^*$-level, so (3.5) is a pullback diagram.

Finally, the unitality claim is immediate remembering that the unit in a unital graph $C^*$-algebra is the sum of all its vertex projections.

\]

4. Extending graphs over sinks

To generalize the diagram (1.2) even further (e.g., to allow loops in $E_1$ in the pullback theorem of the previous section), we first need to determine suitable conditions under which one can extend graphs over sinks preserving the pullback property.

The general-setup (GS) assumptions are as follows:

- $E$ and $H$ are graphs;
- $X$ is a set regarded as a graph with no edges;
- $\iota_E : X \to E^0$ and $\iota_H : X \to H^0$ are injective maps defining the pushout

\[
E^0 \uplus X \xrightarrow{\iota_E} E^0 \xleftarrow{\iota_H} X \xrightarrow{\iota_H} H^0
\]

\[
(4.1)
\]
• \( E \sqcup_X H := (E^0 \sqcup_X H^0, E^1 \sqcup H^1, \pi \circ (s_E \sqcup s_H), \pi \circ (r_E \sqcup r_H)) \), where \( \pi \) is the canonical quotient map.

Next, let \( \iota_H^* : C^*(X) \to C^*(E) \) and \( \iota_E^* : C^*(X) \to C^*(H) \) be the induced \( \ast \)-homomorphisms (see Lemma 3.3). Define

\[
C^*(E) \quad \ast \quad C^*(H) := (C^*(E) \ast C^*(H))/\langle \lambda \iota_E^* \rangle
\]

Here we divide the amalgamated free product by the ideal generated by the product of non-identified projections.

**Lemma 4.1.** Assume that at least one of the maps \( \iota_H \) and \( \iota_E \) takes its values in the sinks of the respective graph. Then the natural assignment of elements defines an isomorphism of \( C^* \)-algebras:

\[
C^*(E) \quad \ast \quad C^*(H) \longrightarrow C^*(E \sqcup_X H).
\]

**Proof.** Since \( \iota_E : X \to E^0 \) or \( \iota_H : X \to H^0 \) takes values in the sinks of \( E \) or \( H \), respectively, all edge relations in \( C^*(E \sqcup_X H) \) involving vertices in the image of \( X \) are of one of two types: either they refer to edges only in \( E^1 \), or to edges only in \( H^1 \). Hence, there are \( \ast \)-homomorphisms

\[
\iota_E^* : C^*(E) \longrightarrow C^*(E \sqcup_X H) \quad \text{and} \quad \iota_H^* : C^*(H) \longrightarrow C^*(E \sqcup_X H)
\]

given by the natural assignment of elements. Furthermore, as \( (E \sqcup_X H)^0 = E^0 \sqcup_X H^0 \) and \( (E \sqcup_X H)^1 = E^1 \sqcup H^1 \), they induce a surjective \( \ast \)-homomorphism

\[
\pi_{\sqcup} : C^*(E) \ast C^*(H) \longrightarrow C^*(E \sqcup_X H).
\]

Finally, with the help of [1, Corollary 1.5.12], one can verify that the kernel of \( \pi_{\sqcup} \) coincides with the kernel of the defining surjection

\[
\pi_{\ast} : C^*(E) \ast C^*(H) \longrightarrow C^*(E) \ast C^*(H),
\]

so the claim follows. \( \square \)

Next, consider three graphs \( E_1, E_2 \) and \( H \) with injective maps \( \iota_{E_1} : X \to E_1^0 \), \( \iota_{E_2} : X \to E_2^0 \), and \( \iota_H : X \to H^0 \). Assume also that \( \iota_{E_1}(X) \) and \( \iota_{E_2}(X) \) consist of sinks of the graphs \( E_1 \) and \( E_2 \), respectively. Now, consider a \( C^* \)-algebra homomorphism

\[
\delta : C^*(E_1) \longrightarrow A
\]

annihilating the vertex projections assigned to the vertices in \( \iota_{E_1}(X) \subseteq E_1^0 \). Then \( \delta \) and the zero map \( C^*(H) \to A \) induce a \( \ast \)-homomorphism on the amalgamated product that annihilates the kernel of \( \pi_{\ast} \). Hence, by Lemma 4.1, \( \delta \) extends to

\[
\delta' : C^*(E_1 \sqcup_X H) \longrightarrow A.
\]

**Lemma 4.2.** Let \( j_H^1 : C^*(H) \to C^*(E_1 \sqcup_X H) \) be the map defined in (4.3). Then

\[
\ker \delta' = j_{E_1}^1(\ker \delta) + \langle j_H^1(C^*(H)) \rangle.
\]

**Proof.** Since \( \delta' \) annihilates \( C^*(H) \), it factors as

\[
C^*(E_1 \sqcup_X H) \to C^*(E_1 \sqcup_X H)/\langle j_H^1(C^*(H)) \rangle \cong C^*(E_1)/\langle \iota_{E_1}^*(C^*(X)) \rangle \to A,
\]

where the last map is induced by \( \delta : C^*(E_1) \to A \). This proves the lemma because the kernel of the induced map is \( \ker \delta/\langle \iota_{E_1}^*(C^*(X)) \rangle \cong j_{E_1}^1(\ker \delta)/\langle j_H^1(C^*(H)) \rangle \). \( \square \)
Observe also that
\[ \langle j_{E_1}(\ker \delta) \rangle = j_{E_1}(\ker \delta) + \langle j_{H}(C^*(H)) \rangle \] (4.9)
implies
\[ \langle j_{E_1}(\ker \delta) \rangle + \langle j_{H}(C^*(H)) \rangle = j_{E_1}(\ker \delta) + \langle j_{H}(C^*(H)) \rangle. \] (4.10)
Assume now that, under the GS assumptions and the assumptions preceding (4.6), we have a pullback diagram
\[
\begin{array}{ccc}
C^*(E_1) & \xrightarrow{\phi} & C^*(E_2) \\
\downarrow{\delta} & & \downarrow{\theta} \\
A & \xrightarrow{p} & B \end{array}
\] (4.11)
of *-homomorphisms of $C^*$-algebras. Here $\phi$ is an injective *-homomorphism sending all vertex projections to vertex projections and all partial isometries associated with edges to partial isometries associated with paths. We also assume that $\phi$ intertwines $\iota_{E_1*}$ with $\iota_{E_2*}$:
\[ \phi \circ \iota_{E_1*} = \iota_{E_2*}. \] (4.12)
Furthermore, here $\delta$ is a *-homomorphism annihilating the vertex projections labeled by $\iota_{E_1}(X)$, and $A$ and $B$ are arbitrary $C^*$-algebras fitting into the pullback diagram (4.11) for some *-homomorphisms $\theta$ and $\rho$.
It follows from the intertwining (4.12) and the injectivity of $\phi$ that $\phi$ sends the projections labeled by $E_1^0 \setminus \iota_{E_1}(X)$ to projections labeled by $E_2^0 \setminus \iota_{E_2}(X)$. It also follows that all partial isometries in $C^*(E_2)$ associated with paths ending in $\iota_{E_2}(X)$ are in the image of $\phi$. Indeed, for starters, $\delta \circ \iota_{E_1*} = 0$ combined with (4.12) implies that $\theta \circ \iota_{E_2*} = 0$, so all partial isometries associated to paths ending in $\iota_{E_2}(X)$ are in $\ker \theta$. Finally, since (4.11) is a pullback diagram, we have $\phi(\ker \delta) = \ker \theta$, so they are in the image of $\phi$.
Note that the assumptions made on $\phi$ allow us to define its extension
\[ \psi : C^*(E_1 \sqcup \underline{X}) \to C^*(E_2 \sqcup \underline{X}). \] (4.13)
Indeed, we can use the isomorphism (4.2) and observe that the conditions on $\phi$ allow us to extend it by $\id : C^*(H) \to C^*(H)$ to
\[ C^*(E_1)_{C^*(X)} \to C^*(E_2)_{C^*(X)} \] (4.14)
This brings us to the second main result of the paper:

**Theorem 4.3.** Under the GS assumptions and the additional assumptions preceding (4.6), the pullback diagram (4.11) of *-homomorphisms of $C^*$-algebras induces the following pullback diagram of *-homomorphisms of $C^*$-algebras:
\[
\begin{array}{ccc}
C^*(E_1 \sqcup \underline{X}) & \xrightarrow{\psi} & C^*(E_2 \sqcup \underline{X}) \\
\downarrow{\delta'} & & \downarrow{\theta'} \\
A & \xrightarrow{\rho} & B \end{array}
\] (4.15)
Here $\delta'$ and $\theta'$ are defined by (4.7), and $\psi$ is defined by (4.13).
Proof. The commutativity of the diagram (4.15) is immediate by construction. To prove that it is a pullback diagram, first we establish the injectivity of \( \psi \).

For starters, as \( \text{id} \) is an injection, we infer from [18, Theorem 1.2] that, if \( \alpha \) is a vertex-simple loop without an exit (i.e. a loop \((e_1, \ldots, e_n)\) without a self-crossing and without a vertex \( v \) such that \( e_k, f \in s^{-1}(v) \) and \( e_k \neq f \) for \( 1 \leq k \leq n \)), then the spectrum of \( S_\alpha \) contains the entire unit circle. Consequently, again by [18, Theorem 1.2], we conclude that the injectivity of \( \psi \) follows from the fact that it maps all vertex projections to vertex projections and all partial isometries of vertex-simple loops without exits to partial isometries of vertex-simple loops without exits.

The former property is clear as both \( \phi \) and \( \text{id} \) map all vertex projections to vertex projections.

To verify the latter property, observe first that, since \( E_i \) over sinks of \( E_i \), \( i = 1, 2 \), any loop with or without an exit in \( E_i \cup X H \) is either a loop with or without an exit in \( E_i \) or a loop with or without an exit in \( H \). Finally, as both \( \phi \) and \( \text{id} \) are injective, employing [18, Theorem 1.2] one more time, we infer that \( \psi \) maps all partial isometries of vertex-simple loops without exits to partial isometries of vertex-simple loops without exits, as desired.

Next, using the injectivity of \( \psi \) and appealing to [15, 3.1. Proposition], we note that to conclude the proof of the theorem, it suffices to check the following two conditions:

\[
\rho^{-1}\left( \theta\left(C^*(E_2 \cup H)\right) \right) = \delta'\left(C^*(E_1 \cup_X H)\right),
\]

\[
\ker \theta' \subseteq \psi(\ker \delta').
\]

The first condition is immediate from our assumption that (4.11) is a pullback. Indeed, the analogous equation holds for the unprimed maps \( \theta \) and \( \delta \), and the images of these maps coincide with those of \( \theta' \) and \( \delta' \), respectively, because the primed maps are obtained from the unprimed maps by extending them by the zero map on \( C^*(H) \).

To show the second condition, we apply Lemma 4.2 and (4.10) to obtain

\[
\ker \delta' = \langle j_{E_1}(\ker \delta) \rangle + \langle j_H^1(C^*(H)) \rangle,
\]

\[
\ker \theta' = j_{E_2}(\ker \theta) + \langle j_H^2(C^*(H)) \rangle.
\]

Here both \( j_{E_1} \) and \( j_{E_2} \) are defined as in (4.3). Now, since (4.11) is a pullback diagram, we have \( \ker \theta \subseteq \phi(\ker \delta) \). Furthermore, it follows from the construction of \( \psi \) and \( \delta' \) that

\[
\psi(\ker \delta) = \psi(j_{E_1}(\ker \delta)) \subseteq \psi(\ker \delta').
\]

Hence, \( j_{E_2}(\ker \theta) \subseteq \psi(\ker \delta') \), so, by (4.19), we are left having to argue that

\[
\langle j_H^2(C^*(H)) \rangle \subseteq \psi(\ker \delta').
\]

To this end, note first that

\[
j_H^2(C^*(H)) \subseteq \psi(\ker \delta')
\]

because \( j_H^1(C^*(H)) \subseteq \ker \delta' \) and \( \psi \circ j_H^1 = j_H^2 \). Furthermore, observe that

\[
\langle j_H^2(C^*(H)) \rangle = \langle j_H^2(C^*(H)) \rangle + \langle j_H^2(C^*(H)) \rangle \cap \langle j_{E_2}(C^*(E_2)) \rangle
\]

and

\[
\langle j_H^2(C^*(H)) \rangle \cap \langle j_{E_2}(C^*(E_2)) \rangle
\]

\[
= \langle j_H^2(C^*(H)) \rangle \langle j_{E_2}(C^*(E_2)) \rangle \subseteq \langle \{j_{E_2}(P_v) \mid v \in \iota_{E_2}(X)\} \rangle.
\]

Next, since all partial isometries in \( C^*(E_2) \) associated with paths ending in \( \iota_{E_2}(X) \) are in the image of \( \phi \) (see the paragraph above (4.13)), we can prove that

\[
\langle \{j_{E_2}(P_v) \mid v \in \iota_{E_2}(X)\} \rangle = \psi(\langle \{j_{E_1}(P_v) \mid v \in \iota_{E_1}(X)\} \rangle).
\]
Indeed, this boils down to showing that

\[
\langle \psi(\{j_{E_1}(P_v) \mid v \in \iota_{E_1}(X)\}) \rangle \subseteq \psi(\langle \{j_{E_1}(P_v) \mid v \in \iota_{E_1}(X)\} \rangle).
\]  (4.26)

To this end, note first that, since any graph \(C^*\)-algebra is the closed linear span of elements of the form \(S_\alpha S^*_\beta\) with \(r(x) = r(y)\) (see [1, Corollary 1.5.12]), we are looking at \(\alpha, \beta, \mu, \nu \in \text{Path}(E_2 \cup_X H)\) and \(P_v\) with \(v \in \iota_{E_2}(X)\) such that \(S_\alpha S^*_\beta j_{E_2}(P_v)S_\mu S^*_\nu\) can be non-zero. Hence, we must have \(s(\beta) = v = s(\mu)\). However, as \(v\) is a sink in \(E_2\), we infer that \(\beta, \mu \in \text{Path}(H)\) and \(r(\alpha), r(\nu) \in H^0\). Furthermore, any maximal \(E_2\)-subpath \(\gamma\) of any path ending in \(H^\beta\) has to end in \(\iota_{E_2}(X)\). Consequently, by the paragraph above (4.13), \(S_\gamma \in \phi(C^*(E_1))\), so \(j_{E_2}(S_\gamma) \in \psi(C^*(E_1 \cup_X H))\). Combining this with the fact that \(j_H^2(S_{\gamma'}) \in \psi(C^*(E_1))\) for any \(\gamma' \in \text{Path}(H)\), we conclude that all elements \(S_\alpha S^*_\beta j_{E_2}(P_v)S_\mu S^*_\nu\) are in the image of \(\psi\). Hence, the inclusion (4.26) holds because its right-hand side is closed.

Finally, taking advantage of the assumption that \(\delta(P_v) = 0\) for any \(v \in \iota_{E_1}(X)\), we infer that

\[
\langle \{j_{E_1}(P_v) \mid v \in \iota_{E_1}(X)\} \rangle \subseteq \langle j_{E_1}(\ker \delta) \rangle.
\]  (4.27)

Hence, by (4.18),

\[
\psi(\langle \{j_{E_1}(P_v) \mid v \in \iota_{E_1}(X)\} \rangle) \subseteq \psi(\langle j_{E_1}(\ker \delta) \rangle) \subseteq \psi(\ker \delta').
\]  (4.28)

Now, combining (4.24), (4.25) and (4.28), we arrive at

\[
\langle j_H^2(C^*(H)) \rangle \cap \langle j_{E_2}(C^*(E_2)) \rangle \subseteq \psi(\ker \delta'),
\]  (4.29)

which, together with (4.22) and (4.23) proves (4.21). \(\square\)

5. Examples and applications

This section is devoted to the study of special cases of Theorem 3.4 and Theorem 4.3 leading to interesting examples in noncommutative topology.

5.1. The standard Podleś quantum sphere

Observe that the assumptions of Theorem 3.4 are true for the standard Podleś quantum sphere. Here \(C^*(E_1) = C(S^2_{q0}), C^*(F_1) = \mathbb{C}, C^*(E_2) = T\) and \(C^*(F_2) = C(S^1)\) (see the diagram (1.3)).

Our next example generalizes a simple gluing construction in topology. Recall that the real projective plane \(\mathbb{R}P^2\) may be represented as a closed hemisphere with the antipodal points on the equator identified. If we further identify all those antipodal points, we obtain the sphere \(S^2\). Here we present a \(q\)-deformed analog of this procedure.

The \(C^*\)-algebra \(C(\mathbb{R}P^2_{q})\) of the quantum real projective plane \(\mathbb{R}P^2_{q}\) [9, Section 4] admits a graph-algebraic presentation (see [11, Section 3.2]) as the \(C^*\)-algebra of the graph

\[
(5.1)
\]
Due to Theorem 3.4 and $C(S^1) \cong C(\mathbb{R}P^1)$, we have the following pullback diagram:

\[
\begin{array}{ccc}
C(S^2_{q_0}) & \xrightarrow{\sim} & C(\mathbb{R}P^2_q) \\
\downarrow & & \downarrow \\
C^* & \xrightarrow{\sim} & C(\mathbb{R}P^1)
\end{array}
\] (5.2)

Observe that the diagram (5.2) reflects the aforementioned procedure of shrinking the copy of $\mathbb{R}P^1$ inside $\mathbb{R}P^2$ to a point.

5.2. The quantum teardrop $\mathbb{W}P^1_q(1, 2)$

The classical teardrop $\mathbb{W}P^1(1, 2)$ may be represented as the wedge of two spheres, namely we have the following pushout diagram:

\[
\begin{array}{ccc}
\mathbb{W}P^1(1, 2) & \xrightarrow{\sim} & S^2 \\
\downarrow & & \downarrow \\
\{\ast\} & \xrightarrow{\sim} & S^1
\end{array}
\] (5.3)

To obtain a noncommutative counterpart of the diagram (5.3), we need to introduce a different kind of a noncommutative sphere. The C*-algebra $C(S^2_{q_{\infty}})$ of the equatorial Podleś quantum sphere $S^2_{q_{\infty}}$ \([16, (3b)]\) admits a graph-algebraic presentation (see [11, Section 3.1]) as the C*-algebra of the graph

\[
\begin{array}{ccc}
\ast & \xrightarrow{\sim} & S^2 \\
\downarrow & & \downarrow \\
\ast
\end{array}
\] (5.4)

The C*-algebra $C(\mathbb{W}P^1_q(1, 2))$ is the graph C*-algebra of the graph (5.11) with $n = 2$. Theorem 3.4 applies, and we obtain the pullback diagram.

\[
\begin{array}{ccc}
C(\mathbb{W}P^1_q(1, 2)) & \xrightarrow{\sim} & C(S^2_{q_{\infty}}) \\
\downarrow & & \downarrow \\
C^* & \xrightarrow{\sim} & C(S^1)
\end{array}
\] (5.5)

which can be regarded as a noncommutative deformation of the diagram (5.3).
5.3. The Vaskman–Soibelman quantum complex projective spaces $\mathbb{C}P^n_q$

The CW-complex decomposition of complex projective spaces may be described in terms of pushout diagrams

\[
\begin{array}{ccc}
\mathbb{C}P^n & \xleftarrow{\phi_1} & \mathbb{C}P^{n-1} \\
\downarrow{\phi_2} & & \downarrow{\phi_3} \\
B^{2n} & \xrightarrow{\psi_1} & S^{2n-1}
\end{array}
\]

(5.6)

Let us recall the graph-algebraic presentation of $q$-deformations of the spaces in the diagram (5.6).

- The $C^*$-algebra $C(\mathbb{C}P^n_q)$ of the quantum complex projective space $\mathbb{C}P^n_q$ [19] is the graph $C^*$-algebra of a graph that, for $n = 3$, is given below (see [11, Section 4.3]).

\[
\begin{array}{ccc}
(\infty) & \xleftarrow{\phi_1} & (\infty) \\
\downarrow{\phi_2} & & \downarrow{\phi_3} \\
(\infty) & \xrightarrow{\psi_1} & (\infty)
\end{array}
\]

(5.7)

- The $C^*$-algebra $C(B^{2n}_q)$ of the Hong–Szymański quantum even-dimensional ball $B^{2n}_q$ [12] is the graph $C^*$-algebra of a graph that, for $n = 3$, is given below (see [12, Section 3.1]).

\[
\begin{array}{ccc}
\infty & \xleftarrow{\phi_1} & \infty \\
\downarrow{\phi_2} & & \downarrow{\phi_3} \\
\infty & \xrightarrow{\psi_1} & \infty
\end{array}
\]

(5.8)

- The $C^*$-algebra $C(S^{2n-1}_q)$ of the Vaskman–Soibelman quantum odd-dimensional sphere $S^{2n-1}_q$ [19, Definition, p. 106] is the graph $C^*$-algebra of a graph that, for $n = 4$, is given below (see [11, Section 4.1]).

\[
\begin{array}{ccc}
\infty & \xleftarrow{\phi_1} & \infty \\
\downarrow{\phi_2} & & \downarrow{\phi_3} \\
\infty & \xrightarrow{\psi_1} & \infty
\end{array}
\]

(5.9)

Applying Theorem 3.4, we obtain the pullback diagram

\[
\begin{array}{ccc}
C(\mathbb{C}P^n_q) & \xleftarrow{\phi_1} & C(\mathbb{C}P^{n-1}_q) \\
\downarrow{\phi_2} & & \downarrow{\phi_3} \\
C(B^{2n}_q) & \xrightarrow{\psi_1} & C(S^{2n-1}_q)
\end{array}
\]

(5.10)

Note that the diagram (5.10) was obtained in [2, Proposition 4.1] using equivariant pullback structures.
5.4. The quantum teardrops $\mathcal{W}P_q^1(1, n)$

Let $n \in \mathbb{N} \setminus \{0\}$. Consider the following graph $W_n$:

\begin{equation}
(5.11)
\end{equation}

Observe that $C^*(W_1) \cong C(S^2_q)$. Moreover, one can show (see [5, Proposition 3.1]) that, in general, the graph $C^*$-algebra $C^*(W_n)$ is isomorphic with the $C^*$-algebra $C(\mathcal{W}P_q^1(1, n))$ [4, Section 3]. We will also need the following $n$-sink extension $R_n^m$ of the Hawaiian-earring graph with $m$ short loops:

\begin{equation}
(5.12)
\end{equation}

Here the notation $(i_j)$ means that there are $i_j \in \mathbb{N} \setminus \{0\}$ many edges from $r_0$ to $r_j$. Now, due to Theorem 3.4, we obtain the pullback diagram

\begin{equation}
(5.13)
\end{equation}

where $O_m$ is the Cuntz algebra with $m$ generators [7].

Let us now consider the graph $G_n$ defined as a pushout of $W_n$ (see (5.11)) and another graph $H$ over the sinks of $W_n$. The only restriction on the graph $H$ is that there exists an inclusion $\{r_1, \ldots, r_n\} \subseteq H^0$. The graph $G_n$ is represented pictorially as follows:

\begin{equation}
(5.14)
\end{equation}

Next, we consider an analogous construction for the graph $R_n^m$ (see (5.12)) using the same graph $H$, and we denote the resulting graph by $E_n^m$. The graph $E_n^m$ is represented pictorially as follows:
Theorem 4.3 applies, and, for any \( n, m \in \mathbb{N} \setminus \{0\} \), we obtain the following pullback diagram:

\[
C^*(G^n) \quad \xymatrix{ \mathbb{C} \ar[r] \ar[d] & C^*(P^n_m) \ar[d] \quad (5.16) \\ \mathcal{O}_m & } \]

**Acknowledgements.** It is a pleasure to thank Sarah Reznikoff for a helpful discussion and a referee for pointing to us the paper [17]. P.M. Hajac is also grateful to SUNY Buffalo for its hospitality and financial support.

**References**

1. G. Abrams, P. Ara and M. Siles Molina, *Leavitt path algebras*, Lecture Notes in Mathematics 2191 (Springer, London, 2017).
2. F. Arici, F. D’Andrea, P. M. Hajac and M. Tobolski, ‘An equivariant pullback structure of trimmable graph \( C^* \)-algebras’, J. Noncommut. Geom., to appear.
3. T. Bates, J. H. Hong, I. Raeburn and W. Szymański, ‘The ideal structure of the \( C^* \)-algebras of infinite graphs’, *Illinois J. Math.* 46 (2002) 1159–1176.
4. T. Brzeziński and S. A. Fairfax, ‘Quantum teardrops’, *Comm. Math. Phys.* 316 (2012) 151–170.
5. T. Brzeziński and W. Szymański, ‘The \( C^* \)-algebras of quantum lens and weighted projective spaces’, *J. Noncommut. Geom.* 12 (2018) 195–215.
6. L. A. Coburn, ‘The \( C^* \)-algebra generated by an isometry’, *Bull. Amer. Math. Soc.* 73 (1967) 722–726.
7. J. Cuntz, ‘Simple \( C^* \)-algebras generated by isometries’, *Comm. Math. Phys.* 57 (1979) 173–185.
8. K. R. Goodearl, ‘Leavitt path algebras and direct limits’, *Rings, modules and representations*, Contemporary Mathematics 480 (American Mathematical Society, Providence, RI, 2009) 165–187.
9. P. M. Hajac, R. Matthes and W. Szymański, ‘Quantum real projective space, disc and spheres’, *Algebr. Represent. Theory* 6 (2003) 169–192.
10. P. M. Hajac, S. Reznikoff and M. Tobolski, ‘Pullbacks of graph \( C^* \)-algebras from admissible pushouts of graphs’, *Banach Center Publ.* 120 (2020) 169–178.
11. J. H. Hong and W. Szymański, ‘Quantum spheres and projective spaces as graph algebras’, *Comm. Math. Phys.* 232 (2002) 157–188.
12. J. H. Hong and W. Szymański, ‘Noncommutative balls and mirror quantum spheres’, *J. Lond. Math. Soc.* (2) 77 (2008) 607–626.
13. S. Klimek and A. Leśniewski, ‘Quantum Riemann surfaces. I. The unit disc’, *Comm. Math. Phys.* 146 (1992) 103–122.
14. A. A. Kumjian, D. A. Pask, A. D. Sims and M. F. Whittaker, ‘Topological spaces associated to higher-rank graphs’, *J. Combin. Theory Ser. A* 143 (2016) 19–41.
15. G. K. Pedersen, ‘Pullback and pushout constructions in \( C^* \)-algebra theory’, *J. Funct. Anal.* 167 (1999) 243–344.
16. P. Podleś, ‘Quantum spheres’, *Lett. Math. Phys.* 14 (1987) 193–202.
17. D. Robertson and W. Szymański, ‘\( C^* \)-algebras associated to \( C^* \)-correspondences and applications to mirror quantum spheres’, *Illinois J. Math.* 55 (2013) 845–870.
18. W. Szymański, ‘General Cuntz-Krieger uniqueness theorem’, *Internat. J. Math.* 13 (2002) 549–555.
19. L. L. Vaksman and Y. S. Soibelman, ‘Algebra of functions on the quantum group SU(\( n + 1 \)), and odd-dimensional quantum spheres’, *Algebra i Analiz* 2 (1990) 101–120 (Russian).
