STABLE AND REAL RANK FOR CROSSED PRODUCTS BY AUTOMORPHISMS WITH THE TRACIAL ROKHLIN PROPERTY

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Abstract. We introduce the tracial Rokhlin property for automorphisms of stably finite simple unital C*-algebras containing enough projections. This property is formally weaker than the various Rokhlin properties considered by Herman and Ocneanu, Kishimoto, and Izumi. Our main results are as follows. Let $A$ be a stably finite simple unital C*-algebra, and let $\alpha$ be an automorphism of $A$ which has the tracial Rokhlin property. Suppose $A$ has real rank zero and stable rank one, and suppose that the order on projections over $A$ is determined by traces. Then the crossed product algebra $C^*(\mathbb{Z}, A, \alpha)$ also has these three properties.

We also present examples of C*-algebras $A$ with automorphisms $\alpha$ which satisfy the above assumptions, but such that $C^*(\mathbb{Z}, A, \alpha)$ does not have tracial rank zero.

0. Introduction

We introduce the tracial Rokhlin property for automorphisms of stably finite simple unital C*-algebras containing enough projections. This property is formally weaker than the various Rokhlin properties which have appeared in the literature, such as in [12], [18], and [13], at least for C*-algebras which are tracially AF in the sense of [20], in roughly the same way that being tracially AF is weaker than the local characterization of AF algebras (Theorem 2.2 of [4]).

Our main results are as follows. Let $A$ be a stably finite simple unital C*-algebra, and let $\alpha$ be an automorphism of $A$ which has the tracial Rokhlin property. Suppose $A$ has real rank zero and stable rank one, and suppose that the order on projections over $A$ is determined by traces (Blackadar’s Second Fundamental Comparability Question, 1.3.1 of [2], for $M_\infty(A)$). Then $C^*(\mathbb{Z}, A, \alpha)$ also has these three properties. In fact, we will see that not all the hypotheses on $A$ are needed for all the conclusions.

The proofs are adapted from [27]. The arguments here are more difficult for several reasons. First, in [27] there is a single “large” AF subalgebra, and the properties of the reduced groupoid C*-algebra are obtained by comparison with this subalgebra. In this paper, we are not able to choose nested approximating subalgebras; moreover, even if we were, the direct limit would not be AF. Second, we assume that the order on projections over $A$ is determined by all traces, but only the invariant traces extend to the crossed product. Third, in [27] we relied...
on previous work to get from the Rokhlin property to the existence of appropriate subalgebras, but in the present paper we must do the analogous construction from scratch.

Kishimoto has proved (Theorem 6.4 of [19]) that if $A$ is a simple unital AT algebra with real rank zero which has a unique tracial state, and if $\alpha \in \text{Aut}(A)$ has the Rokhlin property and satisfies a kind of approximate innerness assumption, then $C^*(\mathbb{Z}, A, \alpha)$ is again a simple unital AT algebra with real rank zero. As this paper was in progress, H. Lin and the first author have generalized this [22]. Let $A$ be a simple separable unital C*-algebra which satisfies the Universal Coefficient Theorem, which has tracial rank zero, and which has a unique tracial state. If $\alpha \in \text{Aut}(A)$ has the Rokhlin property and if $\alpha^n$ is an approximately inner for some $n > 0$, then $C^*(\mathbb{Z}, A, \alpha)$ is a simple AH algebra with no dimensional growth and real rank zero. It seems reasonable to hope that whenever $A$ is tracially AF and $\alpha$ has the tracial Rokhlin property, then $C^*(\mathbb{Z}, A, \alpha)$ is again tracially AF. However, the theorems in this paper also apply to automorphisms of C*-algebras which are not tracially AF, and for which the crossed products are also not tracially AF. We give some examples in Section 6.

Our motivating examples are the noncommutative Furstenberg transformations, which are automorphisms of the irrational rotation algebras analogous to Furstenberg transformations on the torus, and the automorphisms in the crossed product descriptions of the simple quotients of the C*-algebras of discrete subgroups of nilpotent Lie groups studied in [24] and [25]. These automorphisms do not satisfy the hypotheses in [22], although in these cases we believe that the crossed products are in fact tracially AF. We treat these examples in a separate paper [26]. In that paper we also show that an automorphism of a simple unital tracially AF C*-algebra $A$ with unique trace $\tau$ has the tracial Rokhlin property if and only if all nontrivial powers of the corresponding automorphism of the factor $\pi_\tau(A)'$, obtained from the Gelfand-Naimark-Segal representation associated with $\tau$, are outer.

This paper is organized as follows. In Section 1 we introduce the tracial Rokhlin property, and prove, under reasonable conditions, that it is implied by various forms of the Rokhlin property in the literature. We also obtain several elementary consequences. In Section 2 we use the tracial Rokhlin property to construct “large” subalgebras of $C^*(\mathbb{Z}, A, \alpha)$ which are stably isomorphic to $A$. The next three sections treat, in order, the order on projections, real rank zero, and stable rank one. These are the analogs of Sections 3, 4, and 5 of [27]. It is in Section 4 that the weaker conditions satisfied by the subalgebras cause the greatest additional difficulty. We also obtain several other results: the restriction map from tracial states on $C^*(\mathbb{Z}, A, \alpha)$ to invariant tracial states on $A$ is bijective, and, under suitable hypotheses, $C^*(\mathbb{Z}, A, \alpha)$ satisfies the local approximation property of Popa [30]. The last section gives some examples, but the ones we are most interested in are in [26].

1. The tracial Rokhlin property

We begin by defining the tracial Rokhlin property for single automorphisms (actions of $\mathbb{Z}$). It is closely related to, but slightly weaker than, the approximate Rokhlin property of Definition 4.2 of [10]. To our knowledge, the idea was first
It is closely related to the tracial Rokhlin property for actions of finite cyclic groups, as in [28].

**Definition 1.1.** Let $A$ be a stably finite simple unital C*-algebra and let $\alpha \in \text{Aut}(A)$. We say that $\alpha$ has the **tracial Rokhlin property** if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that:

1. $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n - 1$.
2. $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.
3. With $e = \sum_{j=0}^{n} e_j$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of $A$ generated by $x$.

We do not say anything about $\alpha(e_n)$.

It is not completely clear that Condition (3) is the right condition in the general case. We return to this point, and to the comparison of our definition with others, after some preliminaries.

**Notation 1.2.** Let $A$ be a unital C*-algebra. We denote by $T(A)$ the set of all tracial states on $A$, equipped with the weak* topology. For any element of $T(A)$, we use the same letter for its standard extension to $M_n(A)$ for arbitrary $n$, and to $M_\infty(A) = \bigcup_{n=1}^\infty M_n(A)$ (no closure).

**Definition 1.3.** Let $A$ be a unital C*-algebra. We say that the **order on projections over $A$ is determined by traces** if whenever $n \in \mathbb{N}$ and $p, q \in M_\infty(A)$ are projections such that $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \preceq q$.

This is Blackadar’s Second Fundamental Comparability Question for $M_\infty(A)$. See 1.3.1 in [2].

In all applications so far, in addition to the conditions in Definition 1.1, the algebra $A$ has real rank zero (Section 1 of [7]), and the order on projections over $A$ is determined by traces. In this case, we can replace the third condition by one involving traces:

**Lemma 1.4.** Let $A$ be a stably finite simple unital C*-algebra such that $\text{RR}(A) = 0$ and the order on projections over $A$ is determined by traces. Let $\alpha \in \text{Aut}(A)$. Then $\alpha$ has the tracial Rokhlin property if and only if for every finite set $F \subset A$, every $\varepsilon > 0$, and every $n \in \mathbb{N}$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that:

1. $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n - 1$.
2. $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.
3. With $e = \sum_{j=0}^{n} e_j$, we have $\tau(1 - e) < \varepsilon$ for all $\tau \in T(A)$.

**Proof.** If the condition of Definition 1.1 holds and $\varepsilon$, $n$, and $F$ are given, then we can use Theorem 1.1(a) of [37] to find a projection $x \in A$ such that $\tau(x) < \varepsilon$ for all $\tau \in T(A)$. Then apply Definition 1.1 with this $x$ and with $\varepsilon$, $n$, and $F$ as given. Conversely, assume the condition of the lemma, and let $\varepsilon$, $n$, $F$, and $x$ be given. Choose a nonzero projection $q \in x_{\mathbb{R}} A x$, and apply the condition of the lemma with $\varepsilon$ replaced by $\min(\varepsilon, \inf_{\tau \in T(A)} \tau(q))$. The assumption that the order on projections over $A$ is determined by traces implies that $1 - e \preceq q$, giving (3) of Definition 1.1.

We now want to relate the tracial Rokhlin property to forms of the Rokhlin property which have appeared in the literature. The most important of these is
as follows. (See, for example, Definition 2.5 of [13], and Condition (3) in Proposition 1.1 of [18].)

**Definition 1.5.** Let $A$ be a simple unital C*-algebra and let $\alpha \in \text{Aut}(A)$. We say that $\alpha$ has the Rokhlin property if for every finite set $F \subset A$, every $\varepsilon > 0$, every $n \in \mathbb{N}$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_{n-1}, f_0, f_1, \ldots, f_n \in A$

such that:

1. $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-2$ and $\|\alpha(f_j) - f_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$.
2. $\|e_j a - ae_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $a \in F$, and $\|f_j a - af_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.
3. $\sum_{j=0}^{n-1} e_j + \sum_{j=0}^{n} f_j = 1$.

We will also consider analogs of the original version for C*-algebras, in for example the first definition of [12], in which Condition (3) of Definition 1.1 is replaced by $\sum_{j=0}^{n-1} e_j = 1$, but in which, as in Lemma 1.6 below, the towers are only required to exist for all $n$ in an unbounded subset $S \subset \mathbb{N}$ which does not depend on $\varepsilon$ and $F$.

We do not know whether these properties imply the tracial Rokhlin property in the generality we have considered so far. We prove that they do under the following sets of hypotheses, in all of which we assume that $A$ is stably finite, simple, and unital:

- $\text{RR}(A) = 0$, the order on projections over $A$ is determined by traces, and the homeomorphism $\tau \mapsto \tau \circ \alpha$ of $T(A)$ has finite order.
- $A$ is approximately divisible in the sense of [3], all quasitraces on $A$ are traces, and projections in $A$ distinguish the traces on $A$.
- $A$ has tracial rank zero.

Together, these cover most of the interesting cases in which $A$ has real rank zero. Note that $\tau \mapsto \tau \circ \alpha$ has finite order whenever all tracial states are $\alpha$-invariant (in particular, whenever $\alpha$ is approximately inner or $\text{RR}(A) = 0$ and $\alpha$ is trivial on K-theory), and also whenever there are only finitely many extreme tracial states.

We can obtain a version of the tracial Rokhlin property which is implied by the Rokhlin property in full generality by allowing two Rokhlin towers in Definition 1.5 as is done in Definition 1.6, but still allowing a remainder projection. The proofs of our main theorems should all still work under this condition. Another possibility, motivated by Proposition 2.4, is to merely require that there be $q \in \pi_0(A)$ such that $1 - e \sim q$, with equivalence in $C^*(\mathbb{Z}, A, \alpha)$ rather than $A$. We have not checked whether our proofs still work with this assumption. Our motivation for using the definition as stated is Theorem 2.14 of [25], which under certain conditions relates the tracial Rokhlin property to a property having the form of the Rokhlin property for automorphisms of factors of type II$_1$.

There are immediate K-theoretic obstructions to any version of the Rokhlin property involving only one tower and requiring $\sum_{j=0}^{n} e_j = 1$, as in 12. However, we know of no K-theoretic obstructions for an automorphism of a simple unital C*-algebra with real rank zero to have the Rokhlin property as in Definition 1.5. It is in fact implicit in several proofs in the literature, in particular in the proof of Theorem 4.1 of [16], that, under suitably restrictive hypotheses, the tracial Rokhlin property implies the Rokhlin property. The hypotheses include the assumption
that there are no infinitesimals in the $K_0$-group. We know of no examples of automorphisms with the tracial Rokhlin property which do not have the Rokhlin property of Definition 1.6.

Kishimoto’s definition of the approximate Rokhlin property, Definition 4.2 of [16], specifies that $A$ is an AF algebra, and, instead of having a finite set $F$, it assumes a finite dimensional subalgebra $B$ is given; in place of approximate commutativity with a finite set, it requires that every $e_j$ commute exactly with every element of $B$. More significantly, that definition also requires that $\|\alpha(e_n) - e_0\|$ be small, while we make no assumption on $\alpha(e_n)$. It furthermore has a slightly different version of Condition (3). When the order on projections is determined by traces, the analog of the approximate Rokhlin property in our case is formally stronger than the tracial Rokhlin property as we defined it. In Lemma 4.4 of [16], Kishimoto explicitly proves that on a simple unital AF algebra whose $K_0$-group is finitely generated and contains no infinitesimal elements, the approximate Rokhlin property implies the Rokhlin property.

We now prove that the Rokhlin property implies the tracial Rokhlin property under the first of the sets of hypotheses discussed above. We need a lemma.

**Lemma 1.6.** Let $A$ be a stably finite simple unital C*-algebra such that $\text{RR}(A) = 0$ and the order on projections over $A$ is determined by traces. Let $\alpha \in \text{Aut}(A)$, and suppose that the homeomorphism of $T(A)$ given by $\tau \mapsto \tau \circ \alpha$ has finite order. Then $\alpha$ has the tracial Rokhlin property if and only if there is an unbounded set $S \subset \mathbb{N}$ and a constant $C > 0$ such that for every finite set $F \subset A$, every $\varepsilon > 0$, and every $n \in S$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ such that:

1. $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n - 1$.
2. $\|e_0 a - a e_0\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.
3. With $e = \sum_{j=0}^{n} e_j$, we have $\tau(1 - e) \leq C(n + 1)^{-1}$ for all $\tau \in T(A)$.

**Proof.** That the tracial Rokhlin property implies the condition of the lemma is clear from Lemma 1.4. Conversely, assume the conditions of the lemma. We prove the condition in Lemma 1.4 for $\varepsilon$, $n$, and $F$ as there. Without loss of generality $\varepsilon < 1$. Let $k$ be a positive integer such that $\tau \circ \alpha^k = \tau$ for all $\tau \in T(A)$. Thus, if $p_0, p_1, \ldots, p_N$ are projections such that $\|\alpha(p_j) - p_{j+1}\| < 1$ for $l \leq j \leq l + k - 1$, then $\tau(p_{l+k}) = \tau(p_l)$ for all $\tau \in T(A)$. Choose $N \in S$ with

$$N \geq \max \left( \frac{2k(n+1)}{\varepsilon}, \frac{2C}{\varepsilon} \right).$$

Apply the condition of this lemma with $N$ in place of $n$ and with $\frac{1}{2}\varepsilon N^{-1}$ in place of $\varepsilon$, to find mutually orthogonal projections $p_0, p_1, \ldots, p_N \in A$, and set $p = \sum_{m=0}^{N} p_m$. Thus $\tau(1 - p) \leq C(N + 1)^{-1} < \frac{1}{2}\varepsilon$. Write $N + 1 = rk(n + 1) + s$ with $r \in \mathbb{N}$ and $0 \leq s < k(n + 1)$, so that $r > 2\varepsilon^{-1}$. For $0 \leq j \leq n$ set

$$e_j = p_j + p_{j+n+1} + \cdots + p_{j+(rk-1)(n+1)},$$

and set $e = \sum_{j=0}^{n} e_j$. We easily get

$$\|\alpha(e_j) - e_{j+1}\| < rk \cdot \frac{1}{2}\varepsilon N^{-1} \leq \varepsilon \quad \text{and} \quad \|e_0 a - a e_0\| < rk \cdot \frac{1}{2}\varepsilon N^{-1} \leq \varepsilon$$

for $0 \leq j \leq n$ and $a \in F$. For the trace estimate, for $0 \leq l \leq r - 1$ set

$$q_l = p_{k(n+1)l} + p_{k(n+1)(l+1)} + \cdots + p_{k(n+1)(l+1)-1}.$$
Then
\[ \epsilon = q_0 + q_1 + \cdots + q_{r-1} \quad \text{and} \quad p - \epsilon = p_{k(n+1)r} + p_{k(n+1)r+1} + \cdots + p_N. \]
Using periodicity of \( \alpha \) on \( T(A) \) of order \( k(n+1) \), and \( N + 1 - k(n+1)r < k(n+1) \), we get for every \( \tau \in T(A) \)
\[ \tau(q_0) = \tau(q_1) = \cdots = \tau(q_{r-1}) > \tau(p - \epsilon), \]
so
\[ \tau(p - \epsilon) < \frac{1}{r+1} < \frac{\epsilon}{2}. \]
Thus \( \tau(1 - \epsilon) < \epsilon \). \( \blacksquare \)

**Proposition 1.7.** Let \( A \) be a stably finite simple unital C*-algebra such that \( RR(A) = 0 \) and the order on projections over \( A \) is determined by traces. Let \( \alpha \in \text{Aut}(A) \), and suppose that the homeomorphism of \( T(A) \) given by \( \tau \mapsto \tau \circ \alpha \) has finite order. If \( \alpha \) has the Rokhlin property of Definition 1.5, then \( \alpha \) has the tracial Rokhlin property.

**Proof.** Let \( k \) be a positive integer such that \( \tau \circ \alpha^k = \tau \) for all \( \tau \in T(A) \). We verify the condition of Lemma 1.6 with
\[ S = \{ k - 1, 2k - 1, \ldots \} \quad \text{and} \quad C = k. \]
Let \( r \in \mathbb{N} \). Apply the Rokhlin property with \( \min \left( 1, \frac{1}{r} \epsilon \right) \) in place of \( \epsilon \), with \( F \) as given, and with \( n = rk \). Let
\[ p_0, p_1, \ldots, p_{rk-1}, q_0, q_1, \ldots, q_{rk} \in A \]
be the resulting projections, and take \( e_j = p_j + q_j \) for \( 0 \leq j \leq rk - 1 \). Then
\[ 1 - \sum_{j=0}^{rk-1} e_j = q_{rk}. \]
By periodicity of \( \alpha \) on \( T(A) \), we have, for all \( \tau \in T(A) \),
\[ \tau(q_{rk}) = \tau(q_0) \leq \tau(q_0 + q_1 + \cdots + q_{k-1}) < \frac{1}{r} \leq \frac{C}{rk}. \]
This completes the proof. \( \blacksquare \)

Now we prove that the Rokhlin property implies the tracial Rokhlin property under the second and third of the sets of hypotheses discussed above. In the proofs above, the “leftover projection” in the tracial Rokhlin property was the sum of the projections in a small part of the tower obtained from the Rokhlin property. Without something like \( \tau \mapsto \tau \circ \alpha \) having finite order, we don’t see how to make such a proof work. Instead, we must divide a tower in parallel towers of the same height, and arrange to omit different projections in each, so that altogether the “leftover” consists of a small part of each of the projections in the original towers. This is a bit messy to write down.

We need a preparatory lemma for each set of hypotheses.

**Lemma 1.8.** Let \( A \) be a simple unital infinite dimensional C*-algebra with tracial rank zero. Let \( p \in A \) be a nonzero projection, let \( F \subset pAp \) be a finite set, let \( m \in \mathbb{N} \) be a power of two, and let \( \epsilon > 0 \). Then there exist projections \( p_0, p_1, \ldots, p_m \in A \) such that
\[ \sum_{r=0}^{m} p_r = p, \quad p_1 \sim p_2 \sim \cdots \sim p_m, \quad \text{and} \quad p_0 \lesssim p_1, \]
and such that \( \| [p_r, a] \| < \epsilon \) for \( 0 \leq r \leq m \) and all \( a \in F \).
Proof. We have $\text{RR}(A) = 0$ by Theorem 3.4 of [20]. It now follows easily from Theorem 1.1(a) of [37] that there is a nonzero projection $q \in pAp$ such that $pAp$ contains $2m+3$ mutually orthogonal projections, each Murray-von Neumann equivalent to $q$. It follows from Theorem 3.12 of [20] that $pAp$ also has tracial rank zero. Therefore there is a finite dimensional subalgebra $E \subset pAp$, with identity $e \leq p$, such that $\|[e,a]\| < \frac{1}{6}\varepsilon$ for all $a \in F$, such that for every $a \in F$ there is $b \in E$ with $\|b - eae\| < \frac{1}{6}\varepsilon$, and such that $p - e \lessgtr q$.

Let
\[
B = E' \cap eAe = \{ x \in eAe : xc = cx \text{ for every } c \in E \}.
\]
Write $e = \sum_{k=1}^{n} e_k$ as a sum of minimal central projections of $E$. Let $f_k \in E$ be a minimal projection with $f_k \leq e_k$. Then $B = \bigoplus_{k=1}^{n} e_k Be_k$, and $e_k Be_k \cong f_k A f_k$ is simple and has real rank zero. Since $2m$ is also a power of two, Theorem 1.1(a) of [37] therefore provides projections $q_{k,0}, q_{k,1}, \ldots, q_{k,2m} \in e_k Be_k$ such that
\[
\sum_{r=0}^{2m} q_{k,r} = e_k, \quad q_{k,1} \sim q_{k,2} \sim \cdots \sim q_{k,2m}, \quad \text{and} \quad q_{k,0} \lessgtr q_{k,1}.
\]

Then define
\[
p_0 = p - e + \sum_{k=1}^{n} q_{k,0},
\]
and, for $0 \leq r \leq m$,
\[
p_r = \sum_{k=1}^{n} (q_{k,2r-1} + q_{k,2r}).
\]

We prove that these projections satisfy the conclusion of the lemma.

It is clear that $\sum_{r=0}^{n} p_r = p$ and $p_1 \sim p_2 \sim \cdots \sim p_m$. To prove that $p_0 \lessgtr p_1$, we use the fact that, by Theorems 5.8 and 6.8 of [21], the order on projections over $A$ is determined by traces. We certainly have $\sum_{k=1}^{n} q_{k,0} \lessgtr \sum_{k=1}^{n} q_{k,1}$. Let $\tau \in T(A)$. Set $\beta = \sum_{k=1}^{n} \tau(q_{k,2})$. Then $\sum_{k=1}^{n} \tau(q_{k,0}) \leq \beta$ and $\tau(p_r) = 2\beta$ for $1 \leq r \leq m$. So
\[
\tau(p) = \tau(p - e) + \sum_{k=1}^{n} \tau(q_{k,0}) + 2m\beta \leq \tau(p - e) + (2m + 1)\beta.
\]

Using the choice of $q$, we get
\[
\tau(p - e) \leq \tau(q) < \frac{\tau(p)}{2m + 2}.
\]
It follows that $\beta > \tau(p)/(2m + 2)$. Therefore $\tau(p - e) < \beta$. Since this is true for all $\tau \in T(A)$, we conclude that $p - e \lessgtr \sum_{k=1}^{n} q_{k,2}$. Combining that with our first observation gives $p_0 \lessgtr p_1$, as desired.

It remains to estimate $\|[p_r, a]\|$ for $0 \leq r \leq m$ and $a \in F$. Choose $b \in E$ such that $\|b - eae\| < \frac{1}{6}\varepsilon$. Then
\[
\|[b + (p - e)a(p - e)] - a\| \leq \|eae - b\| + \|(p - e)ae\| + \|ea(p - e)\| < 3 \left(\frac{1}{6}\varepsilon\right) = \frac{1}{2}\varepsilon,
\]
and $b + (p - e)a(p - e)$ commutes with $p_r$, so $\|[p_r, a]\| < \varepsilon$.

**Lemma 1.9.** Let $A$ be a simple separable unital approximately divisible C*-algebra. Let $p \in A$ be a nonzero projection, let $F \subset pAp$ be a finite set, let
$m \in \mathbb{N}$, and let $\varepsilon > 0$. Then there exist projections $p_0, p_1, \ldots, p_m \in A$ such that
\[
\sum_{r=0}^{m} p_r = p, \quad p_1 \sim p_2 \sim \cdots \sim p_m, \quad \text{and} \quad p_0 \gtrsim p_1,
\]
and such that $\|[p_r, a]\| < \varepsilon$ for $0 \leq r \leq m$ and all $a \in F$.

**Proof.** It follows from Corollary 2.10 of [3] that there is a finite dimensional unital subalgebra $E \subset A$ such that $\|[a, x]\| \leq \frac{1}{2}\|x\|$ for all $x \in E$ and $a \in F$, and such that $E \cong \bigoplus_{l=1}^{t} M_{n(l)}$ with $n(l) \geq m^2$ for $1 \leq l \leq t$. Let $(e_{j,k}^{(l)})_{1 \leq j,k \leq n(l)}$ be a system of matrix units for the $l$-th summand of $E$. Write $n(l) = d(l)m + z(l)$ with $0 \leq z(l) \leq m - 1$. Note that $d(l) \geq m$ for all $l$. Then define
\[
p_r = \sum_{l=1}^{t} \sum_{j=(r-1)d(l)+1}^{rd(l)} e_{j,j}^{(l)},
\]
for $1 \leq r \leq m$, and
\[
p_0 = \sum_{l=1}^{t} \sum_{j=md(l)+1}^{zd(l)} e_{j,j}^{(l)}.
\]
The commutator estimates follow because $p_r \in E$ for $0 \leq r \leq m$, and all the remaining statements are clear. \]

**Lemma 1.10.** Let $A$ be a stably finite simple unital C*-algebra and let $\alpha \in \text{Aut}(A)$. Assume either that $A$ has tracial rank zero, or that $A$ is approximately divisible, every quasitrace on $A$ is a trace, and projections in $A$ distinguish the tracial states of $A$. Suppose that there is an unbounded subset $S \subset \mathbb{N}$ such that for every finite set $F \subset A$, every $\varepsilon > 0$, and every $n \in S$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_{n-1}, f_0, f_1, \ldots, f_n \in A$ satisfying:

1. $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-2$ and $\|\alpha(f_j) - f_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$.
2. $\|e_ja - ae_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $a \in F$, and $\|f_ja - af_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.
3. $\tau\left(1 - \sum_{j=0}^{n-1} e_j + \sum_{j=0}^{n} f_j\right) < \varepsilon$ for every $\tau \in T(A)$.

Then $\alpha$ has the tracial Rokhlin property.

**Proof.** If $A$ has tracial rank zero, then RR($A$) = 0 by Theorem 3.4 of [20], and the order on projections over $A$ is determined by traces by Theorems 5.8 and 6.8 of [21]. Under the other hypotheses, these conclusions follow from Corollary 3.9(b) and Theorem 1.4(c) of [3]. Accordingly, we verify the conditions of Lemma 1.1.

Let $F \subset A$ be a finite set, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Without loss of generality $\|a\| \leq 1$ for every $a \in F$. Choose $m \in \mathbb{N}$, of the form $m = 2^{m_0}$, and so large that $\frac{1}{m} < \frac{\varepsilon}{3}$. Choose $N \in S$ with $N > n[m(n+1) + 1]$. Set
\[
\varepsilon_0 = \frac{\varepsilon}{4(N + 2)m}.
\]
Choose $\varepsilon_1 > 0$ with
\[
\varepsilon_1 \leq \min\left(\frac{\varepsilon_0}{2}, \frac{\varepsilon}{3}, \frac{\varepsilon}{2(N + 2)^2}\right),
\]
and also so small that whenever $B$ is a unital $C^*$-algebra, whenever

$$e_1, e_2, \ldots, e_{2N-1} \quad \text{and} \quad f_1, f_2, \ldots, f_{2N-1}$$

are sets of orthogonal projections in $B$ with $\|e_k - f_k\| < \varepsilon_1$ for $1 \leq k \leq 2N - 1$, then there is a unitary $u \in B$ such that $\|u - 1\| < \varepsilon_0$ and $ue_ku^* = f_k$ for $1 \leq k \leq 2N - 1$.

Apply the hypotheses with $N$ in place of $n$, with $F$ as given, and with $\varepsilon_1$ in place of $\varepsilon$. Let $p_0, p_1, \ldots, p_{N-1}, q_0, q_1, \ldots, q_N \in A$ be the resulting projections. Set

$$r = 1 - \sum_{k=0}^{N-1} p_k - \sum_{k=0}^{N} q_k.$$ 

By the choice of $\varepsilon_1$, there is a unitary $u \in A$ such that $\|u - 1\| < \varepsilon_0$, such that $u\alpha(p_k)u^* = p_{k+1}$ for $1 \leq k \leq N - 2$, and such that $u\alpha(q_k)u^* = q_{k+1}$ for $1 \leq k \leq N - 1$. Set $\beta = \operatorname{Ad}(u) \circ \alpha$, giving $\beta(p_k) = p_{k+1}$ and $\beta(q_k) = q_{k+1}$ for appropriate $k$.

For $a \in A$ define

$$E(a) = rar + \sum_{k=0}^{N-1} p_k ap_k + \sum_{k=0}^{N} q_k aq_k.$$ 

If $a \in F$, then we can write $a$ as a sum of $(2N + 2)^2$ terms of the form $r a p_k$, $p_j a p_k$, etc., of which $2N + 2$ appear in the formula for $E(a)$ and all the rest have norm dominated by $\max_k \| [p_k, a] \|$ or $\max_k \| [q_k, a] \|$. Accordingly, $\| E(a) - a \| < (2N + 2)^2 \varepsilon_1 \leq \frac{1}{2} \varepsilon$.

We now carry out a construction involving the projections $p_0, p_1, \ldots, p_{N-1}$. We do the same thing with $q_0, q_1, \ldots, q_N$, but only describe the outcome afterwards.

Set

$$F_0 = \bigcup_{k=0}^{N-1} \{ \beta^{-k}(p_k ap_k): a \in F \}.$$ 

Use Lemma 1.8 or Lemma 1.9 depending on what we are assuming about $A$, to write $p_0$ as a sum of orthogonal projections,

$$p_0 = p_{0,0} + p_{0,1} + \cdots + p_{0,m}$$

with

$$p_{0,0} \sim p_{0,1} \sim p_{0,2} \sim \cdots \sim p_{0,m},$$

and such that $\| [p_{0,j}, b] \| < \varepsilon_0$ for $0 \leq j \leq m$ and $b \in F_0$. For $1 \leq k \leq N - 1$ and $0 \leq j \leq m$, set $p_{k,j} = \beta^k(p_{0,j}) \leq p_k$.

We require estimates involving the $p_{k,j}$. First,

$$\| \alpha(p_{k,j}) - p_{k+1,j} \| \leq 2 \| u - 1 \| < 2 \varepsilon_0.$$ 

Second, we claim that if $a \in F$ then $\| [p_{k,j}, a] \| < 2 \varepsilon_0$. To see this, write

$$\| [p_{k,j}, a] \| \leq \| p_{k,j} \| \cdot \| p_k a - p_k ap_k \| + \| p_k a p_k - p_k ap_k \| + \| p_k ap_k - ap_k \| \cdot \| p_{k,j} \|$$

$$\leq \| p_{k,j} \| + \| [\beta^{-k}(p_k ap_k), p_{0,j}] \| + \| [p_k, a] \|$$

$$< \varepsilon_1 + \varepsilon_0 + \varepsilon_1 \leq 2 \varepsilon_0.$$ 

This proves the claim.

Set $N_0 = m(n + 1) + 1$. We define subsets

$$Y, I_0, I_1, \ldots, I_n \subset \{0, 1, \ldots, N_0 - 1\} \times \{0, 1, \ldots, m\},$$

which form a partition of this set, as follows. Set

$$Y = \{(0, 0), (n + 1, 1), \ldots, (m(n + 1), m)\}.$$
For $0 \leq j \leq m$ define
\[ I_0^{(j)} = \{(0, j), (n + 1, j), \ldots, ((j - 1)(n + 1), j), (j(n + 1) + 1, j), \ldots, ((m - 1)(n + 1) + 1, j)\}. \]
Thus, \[ I_0^{(0)} = \{(1, 0), (n + 2, 0), \ldots, ((m - 1)(n + 1) + 1, 0)\} \]
and \[ I_0^{(m)} = \{(0, m), (n + 1, m), \ldots, ((m - 1)(n + 1), m)\}. \]
Then set \[ I_0 = I_0^{(0)} \cup I_0^{(1)} \cup \cdots \cup I_0^{(m)} \]
and for $1 \leq l \leq n$ set \[ I_l = \{(k + l, j): (k, j) \in I_0\}. \]
There is one more important property: for $0 \leq k \leq N_0 - 1$, there is at most one $j$ such that $(k, j) \in Y$.

Now write $N = d(n + 1) + s$ with $0 \leq s \leq n$. Set $d_0 = d - sm$. The condition on $N$ guarantees that $d_0 \geq 0$.

We define subsets \[ Z, L_0, L_1, \ldots, L_n \subset \{0, 1, \ldots, N - 1\} \times \{0, 1, \ldots, m\}, \]
which form a partition of this set, as follows. Set
\[ Z = \{(k + tN_0, j): (k, j) \in Y \text{ and } 0 \leq t \leq s - 1\}. \]

Set \[ J_l = \{(k + tN_0, j): (k, j) \in I_l \text{ and } 0 \leq t \leq s - 1\}. \]
These sets form a partition of \[ \{0, 1, \ldots, sN_0 - 1\} \times \{0, 1, \ldots, m\}. \]

Further set \[ K_0 = \{(sN_0 + t(n + 1), j): 0 \leq t \leq d_0 - 1 \text{ and } 0 \leq j \leq m\} \]
and \[ K_l = \{(k + l, j): (k, j) \in K_0\} \]
for $1 \leq l \leq n$. Then set $L_l = J_l \cup K_l$. Note that $L_l = \{(k + l, j): (k, j) \in L_0\}$.

We now introduce the notation $p_T = \sum_{(k, j) \in T} p_{k, j}$ for any subset \[ T \subset \{0, 1, \ldots, N - 1\} \times \{0, 1, \ldots, m\}. \]

Define \[ f_0 = p_{L_0}, \quad f_1 = p_{L_1}, \quad \ldots, \quad f_n = p_{L_n}, \quad \text{and} \quad f = p_Z. \]
These are orthogonal projections which add up to $\sum_{k=0}^{n-1} p_k$. For $0 \leq l \leq n - 1$, we have
\[ \|\alpha(f_l) - f_{l+1}\| \leq \sum_{(k, j) \in L_0} \|\alpha(p_{k+1, j}) - p_{k+1, j}\| < 2\text{card}(L_0)\varepsilon_0 \leq 2Nm\varepsilon_0, \]
and for $0 \leq l \leq n$ and $a \in F$ we have
\[ \|\{a, f_l\}\| \leq \sum_{(k, j) \in L_0} \|p_{k+1, j}, a\| < 2\text{card}(L_0)\varepsilon_0 \leq 2Nm\varepsilon_0. \]
Furthermore, for $\tau \in T(A)$ we can estimate $\tau(f)$ as follows. For $0 \leq k \leq N - 1$, there is at most one $j$ with $(k, j) \in \mathbb{Z}$. We have

$$p_k, \delta \preceq p_k, 1 \sim p_k, 2 \sim \cdots \sim p_k, m,$$

so that $\tau(p_{k, j}) \leq \frac{1}{m} \tau(p_k)$. Therefore

$$\tau(f) \leq \frac{1}{m} \sum_{k=0}^{N-1} \tau(p_k) \leq \frac{1}{m} < \frac{\varepsilon}{3}.$$

Applying the same construction to $q_0, q_1, \ldots, q_N$, we obtain orthogonal projections $g, g_0, g_1, \ldots, g_n$ which add up to $\sum_{k=0}^{N} g_k$, and such that $\|\alpha(g) - g_{l+1}\| < 2(N + 1)m\varepsilon_0$ for $0 \leq l \leq n - 1$, such that $\|\|g_1, a\|| < 2(N + 1)m\varepsilon_0$ for $0 \leq l \leq n$ and $a \in F$, and such that $\tau(g) < \frac{\varepsilon}{3}$ for $\tau \in T(A)$.

Now set $e_l = f_l + g_l$ for $0 \leq l \leq n$, and set $e = f + g + r = 1 - \sum_{l=0}^{n} e_l$. This gives

$$\|\alpha(e_l) - e_{l+1}\| < 2N m \varepsilon_0 + 2(N + 1)m\varepsilon_0 \leq \varepsilon$$

for $0 \leq l \leq n - 1$,

$$\|\|e_l, a\|| < 2N m \varepsilon_0 + 2(N + 1)m\varepsilon_0 \leq \varepsilon$$

for $0 \leq l \leq n$ and $a \in F$, and $\tau(e) = \tau(f) + \tau(g) + \tau(r) < \varepsilon$ for $\tau \in T(A)$.

As corollaries, we obtain the next two results. The main difference between the first and Lemma 1.6 is that we do not assume that $\tau \mapsto \tau \circ \alpha$ has finite order, but we require more of the algebra.

**Proposition 1.11.** Let $A$ be a stably finite simple unital C*-algebra and let $\alpha \in \text{Aut}(A)$. Assume either that $A$ has tracial rank zero, or that $A$ is approximately divisible, every quasitrace on $A$ is a trace, and that projections in $A$ distinguish the tracial states of $A$. Suppose that there is an unbounded subset $S \subset \mathbb{N}$ such that for every finite set $F \subset A$, every $\varepsilon > 0$, and every $n \in S$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_n \in A$ satisfying:

1. $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n - 1$.
2. $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n$ and all $a \in F$.
3. $\tau \left(1 - \sum_{j=0}^{n} e_j\right) < \varepsilon$ for every $\tau \in T(A)$.

Then $\alpha$ has the tracial Rohlin property.

**Proof.** This is the special case of Lemma 1.10 in which always $e_j = 0$ for all $j$.

**Theorem 1.12.** Let $A$ be a stably finite simple unital C*-algebra and let $\alpha \in \text{Aut}(A)$. Assume either that $A$ has tracial rank zero, or that $A$ is approximately divisible, every quasitrace on $A$ is a trace, and that projections in $A$ distinguish the tracial states of $A$. Suppose that $\alpha$ has the Rokhlin property in the sense of Definition 1.5. Then $\alpha$ has the tracial Rohlin property.

**Proof.** This is the special case of Lemma 1.11 in which always $\sum_{j=0}^{n-1} e_j + \sum_{j=0}^{n} f_j = 1$.

We finish this section by giving several elementary consequences of the tracial Rohlin property.

**Lemma 1.13.** Let $A$ be a stably finite simple unital C*-algebra and let $\alpha \in \text{Aut}(A)$ have the tracial Rohlin property. Then $\alpha^n$ is outer for all $n \neq 0$. 
Proof. It suffices to consider $n > 0$. Let $n > 0$ and let $u \in A$ be unitary; we show $\alpha^n \neq \text{Ad}(u)$. We may clearly assume $A \neq C$. Apply Definition 1.1 with this value of $n$, with $\epsilon = \frac{1}{n+2}$, with $F = \{u\}$, and with some noninvertible $x$. Let $e_0, e_1, \ldots, e_n$ be the resulting projections.

We claim that $e_j \neq 0$ for all $j$. If $e_j = 0$ for some $j$, relation (1) in Definition 1.1 implies that $e_0 = e_1 = \cdots = e_n = 0$. Then relation (3) in Definition 1.1 shows that $1 = 1 - \sum_{j=0}^{n} e_j$ is Murray-von Neumann equivalent to a projection in $xAx$. Since $xAx$ is a proper hereditary subalgebra, this contradicts stable finiteness, and the claim follows.

Orthogonality now implies $\|e_n - e_0\| = 1$. Furthermore, we get

$$\|\alpha^n(e_0) - e_n\| < n\epsilon = \frac{n}{n+2},$$

so

$$\|\alpha^n(e_0) - e_0\| > \frac{2}{n+2}.$$ However, by construction we have $\|e_0u - ue_0\| < \epsilon$, so

$$\|ue_0u^* - e_0\| < \epsilon = \frac{1}{n+2}.$$ Therefore $\alpha^n(e_0) \neq ue_0u^*$. 

Corollary 1.14. Let $A$ be a stably finite simple unital C*-algebra and let $\alpha \in \text{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(\mathbb{Z}, A, \alpha)$ is simple.

Proof. Using Lemma 1.13, this is immediate from Theorem 3.1 of [15].

2. Rokhlin towers and subalgebras

In this section, we prove the basic approximation lemma for actions with the tracial Rokhlin property. Our first step is Proposition 2.4: if $A$ has real rank zero and if the order on projections over $A$ is determined by traces, then for any crossed product $C^*(\mathbb{Z}, A, \alpha)$, the order with respect to $C^*(\mathbb{Z}, A, \alpha)$ on projections over $A$ is determined by traces on $C^*(\mathbb{Z}, A, \alpha)$, equivalently, by $\alpha$-invariant traces on $A$.

Since the proof works just as easily for arbitrary countable amenable groups, and since we intend to study actions of more general groups in future work, we give it in that generality.

Notation 2.1. For any compact convex set $\Delta$ in a topological vector space, we let $\text{Aff}(\Delta)$ be the set of all real valued continuous affine functions on $\Delta$.

We are, of course, particularly interested in $\text{Aff}(T(A))$.

The proof of Proposition 2.4 requires two lemmas.

Lemma 2.2. Let $A$ be a unital C*-algebra, and let $\alpha: \Gamma \to \text{Aut}(A)$ be an action of a countable amenable group. Let $f_1, \ldots, f_l \in \text{Aff}(T(A))$ have the property that $f_j(\tau) > 0$ for all $\Gamma$-invariant $\tau \in T(A)$. Then there exist $n$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that for all $\tau \in T(A)$ we have

$$\frac{1}{n} \sum_{k=1}^{n} f_j(\tau \circ \alpha_{\gamma_k}^{-1}) > 0$$

for $1 \leq j \leq l$. 
Proof: The action of $\Gamma$ on $T(A)$ will be denoted by $(\gamma \tau)(a) = \tau(\alpha^{-1}_\gamma(a))$.

Since $\Gamma$ is amenable, there exists a Følner sequence in $\Gamma$, that is, a sequence of nonempty finite subsets $F_n \subset \Gamma$ such that

\[
\lim_{n \to \infty} \frac{\text{card}(F_n \triangle \gamma F_n)}{\text{card}(F_n)} = 0
\]

for all $\gamma \in \Gamma$. Define $S_n : T(A) \to T(A)$ by

\[
S_n(\tau) = \frac{1}{\text{card}(F_n)} \sum_{\gamma \in F_n} \gamma \tau.
\]

Define

\[
Z_n = \bigcup_{k=n}^{\infty} S_n(T(A)).
\]

Then each $Z_n$ is a compact subset of $T(A)$, and $Z_1 \supset Z_2 \supset \cdots$.

We claim that if $\tau \in \bigcap_{n=1}^{\infty} Z_n$, then $\gamma \tau = \tau$ for all $\gamma \in \Gamma$. So let $\tau \in \bigcap_{n=1}^{\infty} Z_n$, let $\gamma \in \Gamma$, let $a \in A$, and let $\varepsilon > 0$. Choose $N$ so large that if $n \geq N$ then

\[
\frac{\text{card}(F_n \triangle \gamma F_n)}{\text{card}(F_n)} < \frac{\varepsilon}{3\|a\|}
\]

By the definition of the weak* topology, there is $\sigma \in \bigcup_{n=N}^{\infty} S_n(T(A))$ such that

\[
|\sigma(a) - \tau(a)| < \frac{\varepsilon}{3} \quad \text{and} \quad |\sigma(\alpha^{-1}_\gamma(a)) - \tau(\alpha^{-1}_\gamma(a))| < \frac{\varepsilon}{3}.
\]

Write $\sigma = S_n(\rho)$ for some $n \geq N$ and $\rho \in T(A)$. Then

\[
|\tau(\alpha^{-1}_\gamma(a)) - \tau(a)| < \frac{2}{3} \varepsilon + |\sigma(\alpha^{-1}_\gamma(a)) - \sigma(a)|
\]

\[
= \frac{2}{3} \varepsilon + \frac{1}{\text{card}(F_n)} \left| \sum_{\eta \in F_n} \rho(\alpha^{-1}_\gamma \circ \alpha^{-1}_\eta(a)) - \sum_{\eta \in F_n} \rho(\alpha^{-1}_\eta(a)) \right|
\]

\[
= \frac{2}{3} \varepsilon + \frac{\text{card}(F_n \triangle \gamma F_n)}{\text{card}(F_n)} \|a\| < \varepsilon.
\]

Since $\varepsilon > 0$ is arbitrary, it follows that $\gamma \tau = \tau$.

Now set

\[
Y_n = Z_n \cap \{ \tau \in T(A) : f_j(\tau) \leq 0 \text{ for some } j \text{ with } 1 \leq j \leq l \}.
\]

Then each $Y_n$ is compact, and $Y_1 \supset Y_2 \supset \cdots$. Moreover, $\bigcap_{n=1}^{\infty} Y_n = \emptyset$, because any element $\tau$ of this set is an invariant tracial state such that $f_j(\tau) \leq 0$ for some $j$. Therefore there is $n$ such that $Y_n = \emptyset$. Now $f_j(S_n(\tau)) > 0$ for $1 \leq j \leq l$ and all $\tau \in T(A)$. Since each $f_j$ is affine, we have

\[
\frac{1}{\text{card}(F_n)} \sum_{\gamma \in F_n} f_j(\gamma \tau) = f_j(S_n(\tau)) > 0,
\]

as required. |
Lemma 2.3. Let $A$ be a simple unital infinite dimensional C*-algebra with real rank zero. Let $p \in A$ be a projection, and let $n \in \mathbb{N}$. Then there exist projections $p_0, p_1, \ldots, p_n \in A$ such that

$$\sum_{k=0}^{n} p_k = p, \quad p_1 \sim p_2 \sim \cdots \sim p_n, \quad \text{and} \quad p_0 \not\sim p_1.$$ 

Proof. Choose $m \in \mathbb{N}$ such that $2^m > n^2$. Set $N = 2^m$, and write $N = ln + r$ for integers $r$ and $l$ such that $0 \leq r < n$. Note that $l \geq n$. Apply Theorem 1.1(a) of [37], obtaining projections $e_0, e_1, \ldots, e_N \in A$ such that

$$\sum_{k=0}^{N} e_k = p, \quad e_1 \sim e_2 \sim \cdots \sim e_N, \quad \text{and} \quad e_0 \not\sim e_1.$$ 

Define $p_0 = e_0 + e_{n+1} + \cdots + e_N$ and, for $1 \leq k \leq n$, define $p_k = e_{(k-1)l+1} + e_{(k-1)l+2} + \cdots + e_{kl}$. The conditions $\sum_{k=0}^{n} p_k = p$ and $p_1 \sim p_2 \sim \cdots \sim p_n$ in the conclusion are obvious, and $p_0 \not\sim p_n$ follows from $e_0 \not\sim e_{(k-1)l+1}$ and the fact that there are $r + 1 \leq n \leq l$ terms in the sum defining $p_0$. □

Proposition 2.4. Let $A$ be a simple unital infinite dimensional C*-algebra with real rank zero, and assume that the order on projections over $A$ is determined by traces. Let $\alpha : \Gamma \to \text{Aut}(A)$ be an action of a countable amenable group. Let $p, q \in M_\infty(A)$ be projections such that $\tau(p) < \tau(q)$ for every $\Gamma$-invariant tracial state $\tau$ on $A$. (We extend $\tau$ to $M_\infty(A)$ in the obvious way.) Then there is $s \in M_\infty(C^*(\Gamma, A, \alpha))$ such that

$$ss^* = p, \quad ss^* \leq q, \quad \text{and} \quad ss^* \in M_\infty(A).$$

In particular, $p \not\sim q$ in $M_\infty(C^*(\Gamma, A, \alpha))$.

Proof. Throughout the proof, we regard elements of $T(A)$ as being defined on all of $M_\infty(A)$ in the obvious way.

Define $f \in \text{Aff}(T(A))$ by $f(\tau) = \tau(q) - \tau(p)$. Use Lemma 2.2 on this function $f$ to find $n \in \mathbb{N}$ and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that for all $\tau \in T(A)$ we have

$$g(\tau) = \frac{1}{n} \sum_{k=1}^{n} [\tau(\alpha_\gamma^{-1}(q)) - \tau(\alpha_\gamma^{-1}(p))] > 0.$$ 

Then set $\varepsilon = \inf_{\tau \in T(A)} g(\tau)$, which is strictly positive because $T(A)$ is compact. Also set $M = \sup_{\tau \in T(A)} \tau(q)$, which is finite for the same reason. Choose $N \in \mathbb{N}$ such that $M/N \varepsilon < \frac{\varepsilon}{3}$.

Use Lemma 2.3 on $p$ with $Nn - 1$ in place of $n$, calling the resulting projections $p_0, p_1, \ldots, p_{Nn-1}$, and on $q$ with $Nn$ in place of $n$, calling the resulting projections $q_0, q_1, \ldots, q_{Nn}$.

We now claim that

$$\sum_{k=1}^{n} \tau(\alpha_\gamma^{-1}(p_1)) < \sum_{k=1}^{n} \tau(\alpha_\gamma^{-1}(q_1)).$$
for all $\tau \in T(A)$. To see this, use

$$p_1 \sim p_2 \sim \cdots \sim p_{n-1} \quad \text{and} \quad q_0 \preceq q_1 \sim q_2 \sim \cdots \sim q_n$$

to get

$$\frac{1}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(p)) \geq \frac{Nn - 1}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(p_1))$$

and

$$\frac{1}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(q)) \leq \frac{Nn + 1}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(q_1)).$$

We also have

$$\frac{2}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(q_1)) \leq \frac{2}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(q)) \leq \frac{2M}{Nn} < \frac{2\varepsilon}{3}.$$ 

Using this result at the last step, we get

$$\frac{Nn - 1}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(p_1)) \leq \frac{1}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(p)) \leq \frac{1}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(q)) - \varepsilon$$

$$
\leq \frac{Nn + 1}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(q_1)) - \varepsilon \leq \frac{Nn - 1}{n} \sum_{k=1}^{n} \tau(\alpha_{\gamma_k}^{-1}(q_1)) - \frac{\varepsilon}{3}.
$$

The claim follows by dividing by $(Nn - 1)/n$.

Now for $1 \leq k \leq n$ define

$$e_k = p(k-1)N + p(k-1)N + \cdots + p(k-1)N$$

and

$$f_k = q(k-1)N + q(k-1)N + \cdots + q(k-1)N.$$ (We do not use $q_0$.) Regarding $e$ and $f$ as elements of $M_{\tau}(A)$ for suitable $r$, further define

$$e = \text{diag}(e_1, \ldots, e_n) \quad \text{and} \quad \overline{e} = \text{diag}(\alpha_{\gamma_1}(e_1), \ldots, \alpha_{\gamma_n}(e_n)),$$

and

$$f = \text{diag}(f_1, \ldots, f_n) \quad \text{and} \quad \overline{f} = \text{diag}(\alpha_{\gamma_1}(f_1), \ldots, \alpha_{\gamma_n}(f_n)),$$

which are all projections in $M_{\tau}(A)$. By construction, in $M_{\tau}(A)$ we have

$$p \sim e \quad \text{and} \quad f \sim \sum_{m=1}^{Nn} q_m = q - q_0.$$

For $\gamma \in \Gamma$ let $u_\gamma$ be the standard unitary in $C^*(\Gamma, A, \alpha)$ which implements $\alpha_{\gamma}$. Set

$$v = \text{diag}(1_r \otimes u_{\gamma_1}, \ldots, 1_r \otimes u_{\gamma_n}) \in M_{\tau}(A),$$

so that $vev^* = \overline{e}$ and $vf^*v^* = \overline{f}$. The claim proved above implies that $\tau(\overline{e}) \prec \tau(\overline{f})$ for all $\tau \in T(A)$, whence $\overline{e} \not\preceq \overline{f}$ in $M_{\tau}(A)$.

We now have enough to get $p \preceq q$ in $M_{\tau}(C^*(\Gamma, A, \alpha))$, but we need more to get the stronger statement in the conclusion. Since $M_{\tau}(A)$ has real rank zero, Theorem 1.1 of [20] implies that projections in $M_{\tau}(A)$ satisfy Riesz decomposition, so there are projections $g_1, \ldots, g_n \in M_{\tau}(A)$ such that $g_k \preceq \alpha_{\gamma_k}(f_k)$ for all $k$ and, with $g = \text{diag}(g_1, \ldots, g_n)$, we have $\overline{g} \sim g$ in $M_{\tau}(A)$. Then

$$\overline{g} = v^*gv = \text{diag}(\alpha_{\gamma_1}^{-1}(g_1), \ldots, \alpha_{\gamma_n}^{-1}(g_n)) \in M_{\tau}(A)$$
satisfies \( \overline{g} \sim g \) in \( C^*(\Gamma, A, \alpha) \) and \( \overline{g} \leq f \). Since \( f \sim q - q_0 \) in \( M_\infty(A) \), there is a projection \( h \in M_r(A) \) such that \( h \leq q - q_0 \) and \( \overline{g} \sim h \) in \( M_\infty(A) \). Thus, in \( M_\infty(C^*(\Gamma, A, \alpha)) \) we have
\[
p \sim e \sim \overline{e} \sim g \sim \overline{g} \sim h \leq q - q_0 \leq q,
\]
with \( h \in M_r(A) \).

**Lemma 2.5.** Let \( A \) be a stably finite simple unital \( C^* \)-algebra with real rank zero such that the order on projections over \( A \) is determined by traces. Let \( \alpha \in \text{Aut}(A) \) have the tracial Rokhlin property. Let \( \iota: A \to C^*(\mathbb{Z}, A, \alpha) \) being the inclusion map. Then for every finite set \( F \subset C^*(\mathbb{Z}, A, \alpha) \), every \( \varepsilon > 0 \), every \( N \in \mathbb{N} \), every nonzero positive element \( z \in C^*(\mathbb{Z}, A, \alpha) \), and every sufficiently large \( n \in \mathbb{N} \) (depending on \( F, \varepsilon, N, \) and \( z \)), there exist a projection \( e \in A \subset C^*(\mathbb{Z}, A, \alpha) \), a unital subalgebra \( D \subset eC^*(\mathbb{Z}, A, \alpha)e \), a projection \( p \in D \), a projection \( f \in A \), and an isomorphism \( \varphi: M_n \otimes fAf \to D \), such that:

1. With \((e_{j,k})\) being the standard system of matrix units for \( M_n \), we have \( \varphi(e_{1,1} \otimes a) = \iota(a) \) for all \( a \in fAf \) and \( \varphi(e_{k,k} \otimes 1) = \iota(1) \) for \( 1 \leq k \leq n \).
2. With \((e_{j,k})\) as in (1), we have \( \| \varphi(e_{j,j} \otimes a) - a^{j-1}(\iota(a)) \| \leq \varepsilon \| a \| \) for all \( a \in fAf \).
3. For every \( a \in F \) there exist \( b_1, b_2 \in D \) such that \( \| pa - b_1 \| < \varepsilon, \| ap - b_2 \| < \varepsilon \), and \( \| b_1 \|, \| b_2 \| \leq \| a \| \).
4. There is \( m \in \mathbb{N} \) such that \( 2m/n < \varepsilon \) and \( p = \sum_{j=m+1}^{n-m} \varphi(e_{j,j} \otimes 1) \).
5. The projection \( 1 - p \) is Murray-von Neumann equivalent in \( C^*(\mathbb{Z}, A, \alpha) \) to a projection in the hereditary subalgebra of \( C^*(\mathbb{Z}, A, \alpha) \) generated by \( z \).
6. There are \( N \) mutually orthogonal projections \( f_1, f_2, \ldots, f_N \in pDp \), each of which is Murray-von Neumann equivalent in \( C^*(\mathbb{Z}, A, \alpha) \) to \( 1 - p \).

**Proof.** We first make a simplification: We need not check the estimates \( \| b_1 \|, \| b_2 \| \leq \| a \| \) in Condition (3) of the conclusion. To prove this, without loss of generality \( \| a \| \leq 1 \) for all \( a \in F \). Apply the weaker statement with \( \frac{1}{2} \varepsilon \) in place of \( \varepsilon \), and with all other parameters the same. Let \( c_1 \) and \( c_2 \) be the resulting elements in Condition (3) of the conclusion. Then \( \| c_1 \|, \| c_2 \| \leq 1 + \frac{\varepsilon}{2} \). Set
\[
b_1 = \left( \frac{1}{1 + \frac{\varepsilon}{2}} \right) c_1 \quad \text{and} \quad b_2 = \left( \frac{1}{1 + \frac{\varepsilon}{2}} \right) c_2.
\]
One checks that \( \| b_1 - c_1 \| \leq \frac{\varepsilon}{2} \), so \( \| b_1 - pa \| < \varepsilon \). Similarly \( \| b_2 - ap \| < \varepsilon \). This proves the reduction.

Now we do the main part of the proof. Let \( \varepsilon > 0 \), and let \( F \subset C^*(\mathbb{Z}, A, \alpha) \) be a finite set. Let \( N \in \mathbb{N} \), and let \( z \in C^*(\mathbb{Z}, A, \alpha) \) be a nonzero positive element.

Let \( u \) be the standard unitary in the crossed product \( C^*(\mathbb{Z}, A, \alpha) \). We regard \( A \) as a subalgebra of \( C^*(\mathbb{Z}, A, \alpha) \) in the usual way. Choose \( m \in \mathbb{N} \) such that for every \( x \in F \) there are \( a_l \in A \) for \(-m \leq l \leq m\) such that
\[
\left\| x - \sum_{l=-m}^{m} a_l u^l \right\| < \frac{\varepsilon}{2}.
\]
For each \( x \in F \) choose one such expression, and let \( S \subset A \) be a finite set which contains all the coefficients used for all elements of \( F \). Let \( M = 1 + \sup_{a \in S} \| a \| \).

Since \( A \) has Property (SP), and since (by Lemma 1.1) all nontrivial powers of \( \alpha \) are outer, we can apply Theorem 4.2 of [13], with \( N = \{ 1 \} \), to find a nonzero projection \( q \in A \) which is Murray-von Neumann equivalent in \( C^*(\mathbb{Z}, A, \alpha) \) to a
projection in $\mathcal{Z}(Z, A, \alpha)$. Moreover, Lemma 2.6 provides nonzero orthogonal Murray-von Neumann equivalent projections $g_0, g_1, \ldots, g_{2m} \in qAq$.

Since $A$ is simple, $g_0$ is a nonzero projection, and the tracial state space $\tau(A)$ of $A$ is weak* compact, we have $\delta = \inf_{T \in \tau(A)} \tau(g_0) > 0$. Now let $n \in \mathbb{N}$ be any integer such that

$$n > \max \left( \frac{1}{\delta}, (N + 2)(2m + 1), \frac{2m}{\varepsilon} \right).$$

Set $\varepsilon_0 = \frac{\varepsilon}{10(2m + 1)n^2M}$.

Choose $\varepsilon_1 > 0$ so small that whenever $e_1, e_2, \ldots, e_n$ are mutually orthogonal projections in a unital $\mathcal{C}^*$-algebra $B$ and $u \in B$ is a unitary such that $\|ue_ju^* - e_{j+1}\| < \varepsilon_1$ for $1 \leq j \leq n$, then there is a unitary $v \in B$ such that $\|v - u\| < \varepsilon_0$ and $ve_jv^* = e_{j+1}$ for $1 \leq j \leq n$. Further use Lemma 2.6 to find nonzero orthogonal Murray-von Neumann equivalent projections $h_1, h_2, \ldots, h_{n+2} \leq g_0$.

Apply the tracial Rokhlin property (Definition 1.1) with $n-1$ in place of $n$, with $\min(1, \varepsilon_1, \varepsilon_0)$ in place of $\varepsilon$, with $S$ in place of $F$, and with $h_1$ in place of $x$. Call the resulting projections $e_1, e_2, \ldots, e_n$, and let $e = \sum_{j=0}^n e_j$. Apply the choice of $\varepsilon_1$ to these projections and the standard unitary $u$, obtaining a unitary $v \in \mathcal{C}^*(Z, A, \alpha)$ as in the previous paragraph.

Set $f = e_1$. The elements $e_jv^{j-k}e_k$, for $1 \leq j, k \leq n$, can be seen to satisfy the relations for matrix units $e_{j,k}$. So there is a unique injective homomorphism $\varphi: M_n \otimes fAf \to \mathcal{C}^*(Z, A, \alpha)$ such that $\varphi(e_{1,1} \otimes a) = a$ for $a \in fAf$ and $\varphi(e_{j,k} \otimes f) = e_jv^{j-k}e_k$ for $1 \leq j, k \leq n$. Let $D$ be the range of $\varphi$, so that $\varphi: M_n \otimes fAf \to D$ is an isomorphism, as required. Condition (1) of the conclusion is immediate. For $1 \leq j, k \leq n$ and $a, b \in fAf$ we have

$$\varphi(e_{j,k} \otimes a) = \varphi(e_{1,1} \otimes 1)\varphi(e_{1,k} \otimes 1) = e_jv^{j-1}av^{1-k}e_k = v^{j-1}av^{1-k}.$$

In particular, if $j = k$ then

$$\|\varphi(e_{j,j} \otimes a) - a^{j-1}(a)\| \leq 2\|a\| \cdot \|v^{j-1} - w^{j-1}\| \leq 2\|a\| \cdot (j - 1)\|v - w\| \leq 2n\varepsilon_0\|a\| \leq \varepsilon\|a\|.$$

This is Condition (2) of the conclusion.

Let $p = \sum_{j=m}^{m+n-1} e_j$, and note that $\sum_{j=m+1}^{m+n} \varphi(e_{j,j} \otimes 1) = p$. Condition (4) of the conclusion now follows from the choice of $n$.

We now claim that if $y = \sum_{l=-m}^{m} a_lv^l$ with $a_l \in A$ for $-m \leq l \leq m$, and if $[e_j, a_l] = 0$ for $-m \leq l \leq m$ and $1 \leq j \leq n$, then there are $d_1, d_2 \in D$ such that

$$\|py - d_1\|, \|yp - d_2\| < 2Mn(n-2m)(2m + 1)\varepsilon_0.$$

We produce $d_1$: the proof for $d_2$ is essentially the same. We write

$$py = \sum_{j=-m}^{m} \sum_{l=-m}^{m} e_ja_lv^l = \sum_{j=m+1}^{m+n} \sum_{l=-m}^{m} (e_ja_l)e_j(e_jv^le_{j-l}).$$

Since $v^{j-1}e_1v^{j+1} = e_j$, we have

$$\|\varphi(e_{j,j} \otimes f\alpha^{j+1}(a) f) - e_ja_ie_j\| = \|e_jv^{j-1}e_1v^{j+1}a_lv^{j-1}e_1v^{j+1}e_j - e_ja_ie_j\| \leq 2\|a_i\| \cdot \|u^{j-1} - v^{j-1}\| < 2M(j - 1)\varepsilon_0 \leq 2Mn\varepsilon_0,$$
so
\[ \| \varphi(e_{j,-1} \otimes f \alpha^{-j+1}(a_i)) - (e_j a_t e_j)(e_j v^i e_{j-1}) \| < 2Mn \varepsilon_0. \]

Therefore
\[ \sum_{j=m+1}^{n-m} \sum_{l=-m}^{m} (e_j a_t e_j)(e_j v^i e_{j-1}) \]
differs from an element of \( D \) by less than \( 2Mn(n-2m)(2m+1) \varepsilon_0 \). The claim follows.

We next prove Condition (3) of the conclusion. Let \( x \in F \). Choose
\[ b_{-m}, b_{-m+1}, \ldots, b_m \in S \]
such that
\[ \| x - \sum_{l=-m}^{m} b_l v^l \| < \frac{\varepsilon}{2}. \]

For \(-m \leq l \leq m\) define
\[ a_l = (1 - e)b_l(1 - e) + \sum_{j=1}^{n} e_j b_l e_j. \]

We write
\[ b_l - a_l = \sum_{j=1}^{n} [e_j a_l(1 - e) + (1 - e)a_l e_j] + \sum_{i=1}^{n} e_i a_l e_j, \]
so that the estimate \( \| [a_l, e_j] \| < \varepsilon_1 \leq \varepsilon_0 \) implies
\[ \| b_l - a_l \| < [2n + n(n - 1)] \varepsilon_0 < 2n^2 \varepsilon_0. \]

Moreover, from \( \| v - u \| < \varepsilon_0 \) we get \( \| v^l - u^l \| < m \varepsilon_0 \) for \(-m \leq l \leq m\). Therefore, with \( y = \sum_{l=-m}^{m} a_l v^l \), we get
\[ \| x - y \| \leq \left\| x - \sum_{l=-m}^{m} b_l v^l \right\| + \sum_{l=-m}^{m} \| b_l \| \cdot \| v^l - u^l \| + \sum_{l=-m}^{m} \| b_l - a_l \| \]
\[ < \frac{1}{2} \varepsilon + (2m + 1) Mm \varepsilon_0 + (2m + 1) \cdot 2n \varepsilon_0. \]

According to our claim, there is \( d \in D \) such that \( \| py - d \| < 2Mn(n-2m)(2m+1) \varepsilon_0 \). Then
\[ \| px - d \| < \frac{1}{2} \varepsilon + (2m + 1)[Mm + 2n^2 + 2Mn(n - 2m)] \varepsilon_0 \]
\[ \leq \frac{1}{2} \varepsilon + (2m + 1) \cdot 5Mn^2 \varepsilon_0 \leq \varepsilon. \]

This is one half of Condition (3) of the conclusion. The other half is proved similarly.

It remains to verify Conditions (5) and (6) of the conclusion. We have
\[ 1 - p = 1 - e + \sum_{j=1}^{m} e_j + \sum_{j=n-m+1}^{n} e_j. \]

By construction we have \( 1 - e \lesssim h_1 \leq g_0 \). Now let \( \tau \) be any \( \alpha \)-invariant tracial state on \( A \). Then \( \tau(e_j) = \tau(e_1) \) for all \( j \), whence \( \tau(e_j) \leq \frac{1}{n} \). The inequality
\[ n > \frac{1}{2} \geq \frac{1}{\tau(g_0)} \]
therefore implies \( \tau(e_j) < \tau(g_0) \). Since all \( g_j \) are Murray-von Neumann equivalent, it follows that for any \( \alpha \)-invariant tracial state \( \tau \) we have
\[ \tau(e_j) < \tau(g_j) \quad \text{and} \quad \tau(e_{n-j}) < \tau(g_{m+j}) \]
for $1 \leq j \leq m$. So Proposition 2.4 implies that

\[ e_j \not\asymp g_j \quad \text{and} \quad e_{n-j} \not\asymp g_{m+j} \]

in $C^* (\mathbb{Z}, A, \alpha)$ for $1 \leq j \leq m$. Thus

\[ 1 - p \not\asymp \sum_{j=0}^{2m} g_j \leq q, \]

which is Murray-von Neumann equivalent in $C^* (\mathbb{Z}, A, \alpha)$ to a projection in the hereditary subalgebra $zC^* (\mathbb{Z}, A, \alpha)z$. This is Condition (5) of the conclusion.

Finally, we prove (6). Let $\tau \in T (A)$ be $\alpha$-invariant. By construction, we have $1 - e \not\asymp h_1$. Since $h_1, h_2, \ldots, h_{n+2} \leq g_0 \leq 1$ are orthogonal Murray-von Neumann equivalent projections, we get $\tau(h_1) \leq (n + 2)^{-1}$ for all $\tau \in T (A)$, and in particular $\tau(1 - e) \leq (n + 2)^{-1}$. Since all $\tau(e_j)$ are equal, we have

\[ \tau(e_j) \geq \frac{1}{n} \left( 1 - \frac{1}{n+2} \right) > \frac{1}{n+2}. \]

So Proposition 2.4 provides a projection in $e_j A e_j \subset e_j D e_j$ which is Murray-von Neumann equivalent in $C^* (\mathbb{Z}, A, \alpha) / 1 - e$. Therefore, for every $k \geq 0$ with $(2m + 1)(k + 2) \leq n$,

\[ 1 - p = 1 - e + \sum_{j=1}^{m} e_j + \sum_{j=n-m+1}^{n} e_j \not\asymp \sum_{j=(2m+1)k+1}^{(2m+1)(k+1)} e_j \leq p, \]

and the projection Murray-von Neumann equivalent to $1 - p$ can be chosen to be in $p D p$. Since $n \geq (N + 2)(2m + 1)$, there are at least $N$ such projections. They are orthogonal, so Condition (6) of the conclusion is verified.

Given objects satisfying part (1) of the conclusion of Lemma 2.5, we can make a useful homomorphism into $C^* (\mathbb{Z}, A, \alpha)$ which should be thought of as a kind of twisted inclusion of $A$.

**Lemma 2.6.** Let $A$ be any simple unital C*-algebra, let $\alpha \in \text{Aut} (A)$, and let $\iota : A \rightarrow C^* (\mathbb{Z}, A, \alpha)$ be the inclusion. Let $c, f \in A$ be a projections, and let $n \in \mathbb{N}$. Assume that there is an injective unital homomorphism $\varphi : M_n \otimes f Af \rightarrow \iota(e)C^* (\mathbb{Z}, A, \alpha)\iota(e)$ such that, with $(e_{i,k})$ being the standard system of matrix units for $M_n$, we have $\varphi (e_{1,1} \otimes a) = \iota(a)$ for all $a \in f Af$. Then there is a corner $A_0 \subset M_{n+1} \otimes A$ which contains

\[ \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in (1 - e)A(1 - e) \text{ and } b \in M_n \otimes f Af \right\} \]

as a unital subalgebra, and an injective unital homomorphism $\psi : A_0 \rightarrow C^* (\mathbb{Z}, A, \alpha)$ such that

\[ \psi \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \iota(a) + \varphi(b) \]

for $a \in (1 - e)A(1 - e)$ and $b \in M_n \otimes f Af$.

Moreover, for every $\alpha$-invariant tracial state $\tau$ on $A$ there is a tracial state $\sigma$ on $C^* (\mathbb{Z}, A, \alpha)$ such that the extension $\overline{\tau}$ of $\tau$ to $M_{n+1} \otimes A$ satisfies $\overline{\tau}|_{A_0} = \sigma \circ \psi$. 
Proof. Set
\[ q = \text{diag}(1 - e, f, \ldots, f) \in M_{n+1} \otimes A, \]
and set
\[ A_0 = q(M_{n+1} \otimes A)q \quad \text{and} \quad e_0 = \text{diag}(0, f, \ldots, f) \in A_0. \]
In \( M_{n+1} \), call the matrix units \( e_{j,k} \) for \( 0 \leq j, k \leq n \). Then \( q - e_0 = e_{0,0} \otimes (1 - e) \).
Define \( \psi : A_0 \to C^*(\mathbb{Z}, A, \alpha) \) as follows.

(1) For \( a \in (q - e_0)A_0(q - e_0) \), write \( a = e_{0,0} \otimes x \) with \( x \in (1 - e)A(1 - e) \), and set \( \psi(a) = \iota(x) \).

(2) For \( a \in e_0A_0e_0 \), write \( a = \sum_{j,k=1}^n e_{j,k} \otimes x_{j,k} \) with \( x_{j,k} \in fAf \) for all \( j \) and \( k \). Regard this sum as an element of \( M_n \otimes fAf \) in the obvious way, and set \( \psi(a) = \varphi(a) \).

(3) For \( a \in (e_{j,j} \otimes f)A_0(q - e_0) \) for some \( j \) with \( 1 \leq j \leq n \), write \( a = e_{j,0} \otimes x \) with \( x \in fA(1 - e) \), and set \( \psi(a) = \varphi(e_{j,1} \otimes f)\iota(x) \).

(4) For \( a \in (q - e_0)A_0(e_{j,j} \otimes f) \) for some \( j \) with \( 1 \leq j \leq n \), set \( \psi(a) = \psi(a^*)^* \) using (3).

Then extend by linearity.

To prove the first part of the lemma, it suffices to prove that \( \psi \) defined this way is in fact a homomorphism. It is clear that \( \psi \) is linear and that \( \psi(a^*) = \psi(a)^* \) for all \( a \in A_0 \), so we prove multiplicativity. We show that \( \psi(ab) = \psi(a)\psi(b) \) in four cases:

(5) \( a \in (e_{j,j} \otimes f)A_0(q - e_0) \) and \( b \in (q - e_0)A_0(q - e_0) \).

(6) \( a \in (e_{j,j} \otimes f)A_0(q - e_0) \) and \( b \in (q - e_0)A_0(e_{j,j} \otimes f) \).

(7) \( a \in (q - e_0)A_0(e_{j,j} \otimes f) \) and \( b \in e_0A_0e_0 \).

(8) \( a \in (q - e_0)A_0(e_{j,j} \otimes f) \) and \( b \in (e_{k,k} \otimes f)A_0(q - e_0) \).

The other 12 cases are all of three kinds: both \( \psi(ab) \) and \( \psi(a)\psi(b) \) are easily seen to be zero; the formula \( \psi(ab) = \psi(a)\psi(b) \) follows from the fact that \( \iota \) is a homomorphism or \( \varphi \) is a homomorphism; or the case follows from one of the four cases above by taking adjoints.

For (5), write \( a = e_{j,0} \otimes x \) as in (3) and write \( b = e_{0,0} \otimes y \) analogously to (1). Then \( ab = e_{j,0} \otimes xy \) analogously to (3), so
\[ \psi(a)\psi(b) = \varphi(e_{j,1} \otimes f)\iota(x)\varphi(y) = \varphi(e_{j,1} \otimes f)\iota(xy) = \psi(ab). \]

For (6), the analogous computation is: \( a = e_{j,0} \otimes x, b = e_{0,j} \otimes y \), and, using \( xy \in fAf \) so that \( \iota(xy) = \varphi(e_{1,1} \otimes xy) \),
\[ \psi(a)\psi(b) = \varphi(e_{j,1} \otimes f)\iota(x)\varphi(e_{1,1} \otimes f) = \varphi(e_{j,1} \otimes f)\iota(xy)\varphi(e_{1,1} \otimes f) = \varphi(e_{j,1} \otimes f)\varphi(e_{1,1} \otimes xy)\varphi(e_{1,1} \otimes f) = \psi(ab). \]

Similarly, in (7) write \( a = e_{0,j} \otimes x \) with \( x \in (1 - e)Af \) and \( b = \sum_{j,k=1}^n e_{j,k} \otimes y_{j,k} \) with all \( y_{j,k} \in fAf \); then
\[ ab = \sum_{k=1}^n e_{0,k} \otimes xy_{j,k} \]
with $x y_{j,k} \in (1 - e)A f$, and

$$\psi(a)\psi(b) = \sum_{k=1}^{n} \iota(x) \varphi(e_{1,j} \otimes f) \varphi(e_{j,k} \otimes y_{j,k}) = \sum_{k=1}^{n} \iota(x) \varphi(e_{1,1} \otimes y_{j,k}) \varphi(e_{1,k} \otimes f)$$

$$= \sum_{k=1}^{n} \iota(y_{j,k}) \varphi(e_{1,1} \otimes f) = \psi(ab).$$

Finally, in (8) if $j \neq k$ one easily gets $\psi(a)\psi(b) = 0 = \psi(ab)$, and otherwise one writes $a = e_{0,j} \otimes x$, $b = e_{j,0} \otimes y$, and

$$\psi(a)\psi(b) = \iota(x) \varphi(e_{1,j} \otimes f) \varphi(e_{j,1} \otimes f) \iota(y) = \iota(x) \varphi(e_{1,1} \otimes f) \iota(y)$$

$$= \iota(x) \iota(f) \iota(y) = \iota(xy) = \psi(ab).$$

It remains to prove the statement about the tracial states. So let $\tau$ be an $\alpha$-invariant tracial state on $A$. Let $E : C^*({\mathbb Z}, A, \alpha) \to A$ be the standard conditional expectation, and let $\sigma = \tau \circ E$ be the induced tracial state on $C^*({\mathbb Z}, A, \alpha)$. If $f = 0$ then $A_0 = A$ and $\psi = \iota$, so the statement is immediate. Otherwise, for $a \in f A f$, we have

$$\sigma \circ \psi(e_{1,1} \otimes a) = \sigma \circ \varphi(e_{1,1} \otimes a) = \sigma \circ \iota(a) = \tau(a).$$

Therefore $\sigma \circ \psi$ and $\tau$ agree on the full corner $(e_{1,1} \otimes f)(M_{n+1} \otimes A)(e_{1,1} \otimes f)$ of $A_0$. So $\sigma \circ \psi = \tau$. □

3. TRACES AND ORDER ON PROJECTIONS IN CROSSED PRODUCTS

In this section, we prove that if $A$ is a simple unital C*-algebra with real rank zero such that the order on projections over $A$ is determined by traces, and if $\alpha \in \text{Aut}(A)$ has the tracial Rokhlin property, then the order on projections over $C^*({\mathbb Z}, A, \alpha)$ is determined by traces. The methods are adapted from Section 3 of [27], and originally came from [31]. We make one small improvement. In previous versions of this argument, the conclusion was only that the order on $K_0(C^*({\mathbb Z}, A, \alpha))$ is determined by traces, and the result on the order on projections was then obtained using stable rank one. Here, we obtain the full result even if $C^*({\mathbb Z}, A, \alpha)$ does not have stable rank one.

We begin with a comparison lemma for projections in crossed products by actions with the tracial Rokhlin property.

**Lemma 3.1.** Assume the hypotheses of Lemma 2.6 and assume in addition that $A$ has real rank zero and that the order on projections over $A$ is determined by traces. Let $\psi : A_0 \to C^*({\mathbb Z}, A, \alpha)$ be as in the conclusion of Lemma 2.6. Suppose that $p, q \in \psi(A_0)$ are projections such that $\tau(p) < \tau(q)$ for all tracial states $\tau$ on $C^*({\mathbb Z}, A, \alpha)$. Then there exists a projection $r \in \psi(A_0)$ such that $r \leq q$ and $r$ is Murray-von Neumann equivalent to $p$ in $C^*({\mathbb Z}, A, \alpha)$.

**Proof.** If the projection $f$ as in Lemma 2.6 is zero, then $A_0 = A$ and $\psi = \iota$. So the statement follows from Proposition 2.4.

Otherwise, following the proof of Lemma 2.6 let $e_{i,k}$, for $0 \leq j, k \leq n$, be the matrix units in $M_{n+1}$. Also let $\iota : A \to C^*({\mathbb Z}, A, \alpha)$ be the inclusion, and let $D = \iota(A)$ and $D_0 = \psi(A_0)$. Since $a \in f A f$ implies $\iota(a) = \psi(e_{1,1} \otimes a)$, the algebra $E = \iota(f A f)$ is a hereditary subalgebra of both $D$ and $D_0$. 

Now let \( p, q \in D_0 \) be projections as in the hypotheses. Since \( D_0 = \psi(A_0) \) is simple, there is \( m \) such that
\[
(1, 0, \ldots, 0) \preceq (\iota(f), \iota(f), \ldots, \iota(f))
\]
in \( M_m(D_0) \). We identify \( D \) and \( D_0 \) with corners in \( M_m(D) \) and \( M_m(D_0) \) in the usual way. Then, in particular, there exist projections
\[
p_0, q_0 \leq (\iota(f), \iota(f), \ldots, \iota(f))
\]
in \( M_m(D_0) \) such that \( p \sim p_0 \) and \( q \sim q_0 \) in \( M_m(D_0) \). Clearly \( p_0, q_0 \in M_m(E) \subset M_m(D) \), and satisfy \( \tau(p_0) < \tau(q_0) \) for \( \tau \in T(C^*(\mathbb{Z}, \tau, \alpha)) \). Because \( D = \iota(A) \), Proposition 24 provides \( r_0 \in M_m(D) \) such that \( p_0 \sim r_0 \) in \( M_m(C^*(\mathbb{Z}, A, \alpha)) \) and \( r_0 \not\sim q_0 \). Then \( r_0 \in M_m(E) \subset M_m(D_0) \). Choose \( s \in M_m(D_0) \) such that \( s^*s = q_0 \) and \( ss^* = q \). Then \( r = sr_0s^* \in M_m(D_0) \) satisfies \( p \sim r \) in \( M_m(C^*(\mathbb{Z}, A, \alpha)) \) and \( r \not\sim q \). Since \( p, q \in C^*(\mathbb{Z}, A, \alpha) \), we in fact get \( p \sim r \) in \( C^*(\mathbb{Z}, A, \alpha) \).

**Lemma 3.2.** Let \( A \) be a C*-algebra, let \( p, q \in A \) be projections, let \( \tau \) be a tracial state on \( A \), and let \( g : [0, 1] \to \mathbb{R} \) be a continuous function. Then \( \tau(g(pqp)) = \tau(g(qpp)) \).

**Proof.** The conclusion is true when \( g(t) = t^n \), since
\[
\tau((pqp)^n) = \tau((pqp)^{n-1}(pq)(qp)) = \tau((qp)(pqp)^{n-1}(pq)) = \tau((qpq)^n).
\]
So it also holds for any polynomial and therefore, by approximation, for any continuous function \( g \).

**Lemma 3.3.** Let \( g : [0, 1] \to [0, 1] \) be a continuous function such that \( g(1) = 1 \). Then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that whenever \( A \) is a unital C*-algebra, \( \tau \) is a tracial state on \( A \), and \( p, q \in A \) are projections such that \( \tau(p) > 1 - \delta \), then \( \tau(g(pqp)) > \tau(q) - \varepsilon \) and \( \tau(g(ppp)) > \tau(q) - \varepsilon \).

**Proof.** We prove the result for the inequality \( \tau(g(pqp)) > \tau(q) - \varepsilon \). Choose \( \delta_0 \in (0, 1) \) such that \( g(t) > 1 - \frac{1}{2\varepsilon} \) for all \( t \in [1 - \delta_0, 1] \). Then set \( \delta = \frac{1}{2\varepsilon} \delta_0 \). Let \( A, \tau, p, q \) be as in the hypotheses.

We first estimate \( \tau(qpq) \), as follows. We have \( \tau(qpq) + \tau(q(1-p)q) = \tau(q) \) and
\[
\tau(q(1-p)q) = \tau((1-p)q(1-p)) \leq \tau(1-p) < \delta,
\]
so that \( \tau(qpq) > \tau(q) - \delta \).

Now let \( \mu \) be the measure on \( X = \text{sp}(qqp) \) corresponding to the functional on \( C(X) \) defined by \( h \to \tau(h(qpq)) \), with the functional calculus evaluated in \( qAg \). This measure has total mass \( \tau(q) \). With \( E = [1 - \delta_0, 1] \), we have
\[
\tau(q) - \delta < \tau(qpq) = \int_0^1 t \, d\mu(t) = (1 - \delta_0)\mu([0, 1] \setminus E) + \mu(E)
\]
\[
= (1 - \delta_0)\tau(q - \mu(E)) + \mu(E) = (1 - \delta_0)\tau(q) + \delta_0\mu(E).
\]
Rearranging this gives
\[
\mu(E) > \tau(q) - \frac{\delta}{\delta_0} = \tau(q) - \frac{1}{2\varepsilon} \varepsilon.
\]
Since \( g(t) > 1 - \frac{1}{2} \varepsilon \) for \( t \in E \), we now get

\[
\tau(g(qpq)) = \int_0^1 g(t) \, d\mu(t) \geq \left( 1 - \frac{1}{2} \varepsilon \right) \mu(E) \\
> \left( 1 - \frac{1}{2} \varepsilon \right) (\tau(g) - \frac{1}{2} \varepsilon) \geq \tau(q) - \frac{1}{2} \varepsilon - \frac{1}{2} \varepsilon \tau(q).
\]

Since \( \tau(q) \leq 1 \), this gives \( \tau(g(qpq)) > \tau(q) - \varepsilon \), as desired.

The result with the inequality \( \tau(g(qpq)) > \tau(q) - \varepsilon \) now follows from Lemma 3.4.

**Lemma 3.4.** Let \( \delta > 0 \). Then there exists a continuous function \( g : [0, 1] \to [0, 1] \) such that \( g(0) = 0 \), \( g(1) = 1 \), and whenever \( A \) is a \( C^* \)-algebra with real rank zero and \( a \in A \) is a positive element with \( \|a\| \leq 1 \), then there is a projection \( e \in aAa \) such that \( g(a)e = e \) and \( \|ea - a\| < \delta \).

**Proof.** Choose \( t_0 \) such that \( 0 < t_0 < \frac{1}{4} \delta \). Let \( g_0 : [0, 1] \to [0, 1] \) be a continuous function which vanishes on \( [0, t_0] \) and satisfies \( |g_0(t) - t| < \frac{1}{4} \delta \) for all \( t \in [0, 1] \).

Let \( g : [0, 1] \to [0, 1] \) be any continuous function such that \( g(0) = 0 \), \( g(1) = 1 \), and \( gg_0 = g_0 \).

Let \( A \) be a \( C^* \)-algebra with real rank zero and let \( a \in A \) be a positive element with \( \|a\| \leq 1 \). Since \( A \) has real rank zero, there is a projection \( e \in g_0(a)Ag_0(a) \) such that \( \|eg_0(a) - g_0(a)\| < \frac{1}{4} \delta \). Since \( \|a - g_0(a)\| < \frac{1}{4} \delta \), we get \( \|ea - a\| < \delta \). From \( gg_0 = g_0 \) we get \( g(a)g_0(a) = g_0(a) \), whence \( g(a)e = e \). \[ \square \]

The proof of the following theorem is adapted from the proofs of Theorem 3.5 and Lemma 3.3 of [31], which in turn are based on Section 3 of [31]. However, the construction of the projection \( q_0 \) in the proof is new. It enables us to prove directly that the order on projections over \( C^*(Z, A, \alpha) \) is determined by traces, rather than merely that the order on \( K_0(C^*(Z, A, \alpha)) \) is determined by traces.

**Theorem 3.5.** Let \( A \) be a simple unital \( C^* \)-algebra with real rank zero, and suppose that the order on projections over \( A \) is determined by traces. Let \( \alpha \in \text{Aut}(A) \) have the tracial Rokhlin property. Then the order on projections over \( C^*(Z, A, \alpha) \) is determined by traces.

**Proof.** We claim that it suffices to show that if \( q, r \in C^*(Z, A, \alpha) \) are projections such that \( \tau(q) < \tau(r) \) for all tracial states \( \tau \) on \( C^*(Z, A, \alpha) \), then \( q \precsim r \). Indeed, it is easy to check that the action \( \text{id}_{M_n} \otimes \alpha \) on \( M_n \otimes A \) again has the tracial Rokhlin property, so the result applies to projections in \( M_n \otimes C^*(Z, A, \alpha) \) as well, and this version implies the statement of the theorem.

Accordingly, let \( q, r \in C^*(Z, A, \alpha) \) be projections such that \( \tau(q) < \tau(r) \) for all tracial states \( \tau \) on \( C^*(Z, A, \alpha) \). Since the tracial state space is weak* compact, there is \( \varepsilon > 0 \) such that \( \tau(r) - \tau(q) > \varepsilon \) for all tracial states \( \tau \).

Choose \( \eta > 0 \) so small that whenever \( B \) is a \( C^* \)-algebra and \( e, f \in B \) are projections such that \( \|ef - f\| < \eta \), then \( f \precsim e \). Choose continuous functions \( g_1, g_2 : [0, 1] \to [0, 1] \) such that

\[
g_1(0) = g_2(0) = 0, \quad g_1(1) = g_2(1) = 1, \quad g_1g_2 = g_2,
\]

and \( |g_1(t) - t| < \frac{1}{4} \eta \) for all \( t \in [0, 1] \). Choose a continuous function \( g : [0, 1] \to [0, 1] \) as in Lemma 3.4 with \( \frac{\eta}{2} \) in place of \( \delta \).
Choose \( \delta > 0 \) so small that whenever \( B \) is a \( C^* \)-algebra and \( a, b \in B \) are positive elements with
\[
\|a\|, \|b\| \leq 1 \quad \text{and} \quad \|a - b\| < \delta,
\]
then
\[
\|g_1(a) - g_1(b)\| < \frac{1}{4} \eta, \quad \|g_2(a) - g_2(b)\| < \frac{1}{21} \epsilon, \quad \text{and} \quad \|g(a) - g(b)\| < \frac{1}{6} \epsilon.
\]
We also require \( \delta < \frac{1}{2} \eta \).

Apply Lemma 2.8 with \( g_1 \) in place of \( g \) and with \( \frac{1}{16} \) in place of \( \epsilon \), obtaining a number \( \delta_0 > 0 \). Choose an integer \( N \geq \max(\delta_0^{-1}, 6\epsilon^{-1}) \).

Apply Lemma 2.8 with \( \{q, r\} \) in place of \( F \), with \( \frac{1}{2} \) in place of \( \epsilon \), with \( N \) as given, and with 1 in place of \( z \). We obtain a projection \( \epsilon \in A \subset C^* (Z, A, \alpha) \), a unital subalgebra \( D \subset eC^* (Z, A, \alpha) \), a projection \( p \in D \cap A \), a projection \( f \in A \), and an isomorphism \( \varphi: M_n \otimes fAf \to D \), satisfying the conditions (1) through (6) there.

In the next several paragraphs, we construct a projection \( r_0 \in D \) such that \( r_0 \leq r \) and \( \tau(r_0) > \tau(r) - \frac{1}{2} \epsilon \) for every tracial state \( \tau \) on \( C^* (Z, A, \alpha) \).

By the choice using Lemma 2.8 there exists \( x \in D \) such that \( \|xp - x\| < \frac{1}{4} \epsilon \) and \( \|x\| \leq 1 \), so that \( \|x^*xx^* - xx^*\| < \delta \).

We claim that \( r_0 \leq r \), and we prove this by showing that \( \|rr_0 - r_0\| < \eta \). The choice of \( \eta \) and the estimate \( \|ppp - xx^*\| < \delta \) imply that \( \|g_1(rpp) - g_1(xx^*)\| < \frac{1}{4} \eta \). So \( g_1(x^*x)r_0 = r_0 \) implies \( \|g_1(rpp)r_0 - r_0\| < \frac{1}{4} \eta \), and from \( |g_1(t) - t| < \frac{1}{4} \eta \) we then get \( \|rppr_0 - r_0\| < \frac{1}{4} \eta \).

Now
\[
\|rr_0 - r_0\| \leq \|\|r_0\| \cdot \|r_0 - rppr_0\| + \|r^2ppr_0 - r_0\| < \eta,
\]
as desired. This proves the claim.

Now let \( \tau \) be a tracial state on \( C^* (Z, A, \alpha) \). We obtain a lower bound on \( \tau(r_0) \).

The choice of \( \delta \) and the estimate \( \|ppp - xx^*\| < \delta \) imply that \( \|g_2(rpp) - g_2(xx^*)\| < \frac{1}{4} \epsilon \). So \( \|r_0g_2(x^*x) - g_2(x^*x)\| < \frac{1}{32} \epsilon \) implies \( \|r_0g_2(rpp) - g_2(rpp)\| < \frac{1}{21} \epsilon \), whence \( \|r_0g_2(rpp)r_0 - g_2(rpp)\| < \frac{1}{27} \epsilon \). Therefore
\[
\tau(r_0) \geq \tau(r_0g_2(rpp)r_0) > \tau(g_2(rpp)) - \frac{1}{27} \epsilon.
\]
Now the choice using Lemma 2.8 implies \( \tau(1 - p) \leq N^{-1} \tau(p) \leq N^{-1} < \delta_0 \), so \( \tau(p) > 1 - \delta_0 \), and the choice using Lemma 3.3 gives \( \tau(g_2(rpp)) > \tau(r) - \frac{1}{4} \epsilon \). Thus \( \tau(r_0) > \tau(r) - \frac{1}{4} \epsilon = \tau(r) - \frac{1}{4} \epsilon \). We have proved that \( r_0 \) is the required projection.

We now construct a projection \( q_0 \in (1 - p) + pDp \) such that \( q \leq q_0 \) and \( \tau(q_0) < \tau(q) + \frac{1}{4} \epsilon \) for every tracial state \( \tau \) on \( C^* (Z, A, \alpha) \). The method is similar to the construction of \( r_0 \) but is a bit more complicated.

By the choice using Lemma 2.8 there exists \( x \in D \) such that \( \|p - x\| < \frac{1}{4} \delta \) and \( \|x\| \leq 1 \). Replacing \( x \) by \( px \), we may assume in addition that \( px = x \). Then \( xx^* \in pDp \) and \( \|ppp - xx^*\| < \delta \). Since \( D \cong M_n \otimes fAf \), which has real rank zero, we may apply the choice of \( g \) to find a projection \( q_1 \in xx^*Dxx^* \subset pDp \) such that
\[
g(xx^*)q_1 = q_1 \quad \text{and} \quad \|q_1xx^* - xx^*\| < \frac{1}{8} \eta^2.
\]
Now set \( q_0 = 1 - p + q_1 \in (1 - p) + pDp \).
Lemma 4.1. Let $\eta$ be the choice of $q$ at the fourth step, Lemma 3.2 at the fifth step, and $\xi$ at the sixth step. So $\|q_0 - q\| < \frac{1}{2} \epsilon$. Now, using inequality in the $C^*$-algebra at the third step, this estimate at the fourth step, Lemma 3.2 at the fifth step, and $g(pq) \leq q$ at the sixth step, we get

$$\tau(q_0) = \tau(1 - p) + \tau(q_1) < \tau(q) + \frac{1}{2} \epsilon.$$  

We have proved that $q_0$ is the required projection.

Apply Lemma 2.6 with $\varphi: M_n \otimes fA f \to D$ and the projection $e$ as given. We obtain $A_0$ and a unital homomorphism $\psi: A_0 \to C^*(Z, A, \alpha)$. We note that $\psi(A_0)$ contains $D$, and hence $r_0$, also $1, p \in \psi(A_0)$ so $q_0 \in (1 - p) + pDp \subset \psi(A_0)$. For every tracial state $\tau$ on $C^*(Z, A, \alpha)$ we have

$$\tau(r_0) - \tau(q_0) > (\tau(r) - \frac{1}{2} \epsilon) - (\tau(q) + \frac{1}{2} \epsilon) > \frac{1}{2} \epsilon.$$  

Therefore Lemma 3.1 applies, and shows that $q_0 \lesssim r_0$ in $C^*(Z, A, \alpha)$. Thus,

$$q \lesssim q_0 \lesssim r_0 \lesssim r,$$

as was to be proved.

4. REAL RANK OF CROSSED PRODUCTS

In this section, we prove that if $A$ is a simple unital $C^*$-algebra with real rank zero such that the order on projections over $A$ is determined by traces, and if $\alpha \in \text{Aut}(A)$ has the tracial Rokhlin property, then $C^*(Z, A, \alpha)$ has real rank zero, and every tracial state on $C^*(Z, A, \alpha)$ is induced from an $\alpha$-invariant tracial state on $A$. The methods are adapted from Section 4 of [27]. The lemma used to find a suitable projection which approximately commutes with a selfadjoint element is considerably harder in our context. To prove it, we start with the following lemma. It will follow from arguments in [9], where it is proved that if $\epsilon \in A$ then $c^*Ac \cong cAc^*$. 

Lemma 4.1. Let $A$ be a $C^*$-algebra, let $p \in A$ be a projection, and let $e \in A$. Suppose that $c^*Ac \subset pAp$. Then for any projection $r \in cAc^*$, we have $r \lesssim p$.

Proof. For each $\epsilon > 0$ define a continuous function $f_\epsilon: [0, \infty) \to [0, 1]$ by

$$f_\epsilon(t) = \begin{cases} 
0 & t \leq \frac{\epsilon}{2} \\
\frac{\epsilon}{2} (t - \frac{\epsilon}{2}) & \frac{\epsilon}{2} \leq t \leq \epsilon \\
1 & \epsilon \leq t.
\end{cases}$$
Recall that $|c| = (c^*c)^{1/2}$, so that $|c^*| = (cc^*)^{1/2}$, and (analogously to 1.2 of [9])

$$cAc^* = \bigcup_{r > 0} f_r(|c^*|)Af_r(|c^*|).$$

Now let $r \in cAc^*$ be a projection; we prove $r \preceq p$. Without loss of generality there is $\varepsilon > 0$ such that

$$r \in f_{\varepsilon/2}(|c^*|)Af_{\varepsilon/2}(|c^*)|.$$

Then $f_{\varepsilon/2}(|c^*|)r = r$, whence

$$r \in f_{\varepsilon/2}(|c^*|)Af_{\varepsilon/2}(|c^*|).$$

By 1.4 of [9] there is $z \in A$ such that the map $a \mapsto zaz^*$ is an isomorphism

$$f_{\varepsilon/2}(|c^*|)Af_{\varepsilon/2}(|c^*|) \rightarrow f_{\varepsilon/2}(|c|)Af_{\varepsilon/2}(|c|).$$

In particular, $zrz^*$ is a projection and $rz^*$ is a partial isometry from $r$ to $zrz^*$. Since

$$f_{\varepsilon/2}(|c|)Af_{\varepsilon/2}(|c|) \subset cAc \subset pAp,$$

we have $r \sim zrz^* \leq p$. $\blacksquare$

The following lemma is surely true for actions of (not necessarily abelian) compact groups, and may even be known.

**Lemma 4.2.** Let $A$ be a unital C*-algebra, let $G$ be a finite abelian group, let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of $G$ on $A$, and let $A^\alpha$ be the fixed point algebra. Then every approximate identity for $A^\alpha$ is also an approximate identity for $A$.

**Proof.** Let $(e_\lambda)_{\lambda \in \Lambda}$ be an approximate identity for $A^\alpha$. For $\tau \in \hat{G}$, let

$$A_\tau = \{ a \in A : \alpha_\tau(a) = \tau(g)a \text{ for all } g \in G \}.$$ 

Since $A$ is the direct sum of the subspaces $A_\tau$, it suffices to prove that $\lim_\lambda ae_\lambda = a$ for all $a \in A_\tau$.

So let $a \in A_\tau$. Then

$$\|(a - ae_\lambda)(a - ae_\lambda)\| = \|(a^*a - a^*ae_\lambda) - e_\lambda(a^*a - a^*ae_\lambda)\| \leq 2\|a^*a - a^*ae_\lambda\|.$$ 

Since $a^*a \in A^\alpha$, we have $\lim_\lambda \|a^*a - a^*ae_\lambda\| = 0$. $\blacksquare$

**Lemma 4.3.** Let $A$ be a C*-algebra, let $G$ be a topological group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of $G$ on $A$ which is inner in the sense that there is a strictly continuous group homomorphism $g \mapsto u(g) \in U(M(A))$, the unitary group of the multiplier algebra of $A$, such that $\alpha_g(a) = u(g)au(g)^*$ for all $g \in G$. Then the restriction of the action to any invariant hereditary subalgebra $B \subset A$ is also inner in the same sense.

**Proof.** It suffices to show that if $g \in G$ then the pair $(L, R)$ of linear maps on $B$, given by $L(x) = u(g)x$ and $R(x) = xu(g)$, is a multiplier of $B$. The only nontrivial part is that $L(B) \subset B$ and $R(B) \subset B$. For the first, $x \in B$ implies

$$L(x)L(x)^* = u(g)xx^*u(g)^* \in \alpha_g(B) = B$$

and

$$L(x)^*L(x) = x^*u(g)^*u(g)x = x^*x \in B,$$

whence $L(x) \in B$. The proof of the second is similar. $\blacksquare$
Lemma 4.4. Let $A$ be a unital C*-algebra with real rank zero. Let $m, n, N \in \mathbb{N}$ satisfy $(2n + 1)m \leq N$. Let $a \in M_N(A)$ be a selfadjoint element with $\|a\| \leq 1$. Let $(e_{i,j})_{i,j=1}^N$ be the standard system of matrix units of $M_N(\mathbb{C})$. Then there is a projection $q \in M_N(A)$ such that
\[
\sum_{k=1}^m e_{k,k} \otimes 1 \leq q, \quad q \succeq \sum_{k=1}^{(2n+1)m} e_{k,k} \otimes 1, \quad \text{and} \quad \|qa - aq\| < \frac{1}{n}.
\]

Proof. Set
\[
p = \sum_{k=1}^m e_{k,k} \quad \text{and} \quad q_0 = \sum_{k=1}^{(2n+1)m} e_{k,k} \otimes 1.
\]
Then we want $q$ with
\[
p \leq q \preceq q_0 \quad \text{and} \quad \|qa - aq\| < \frac{1}{n}.
\]

Since $A$ has real rank zero, so does $M_N(A)$. Set
\[
\varepsilon = \frac{1}{n} \left( \frac{1}{2n} - \frac{1}{2n+1} \right).
\]
Because
\[
\|a\| \leq 1 \quad \text{and} \quad \frac{1}{2n+1} + \varepsilon > \frac{1}{2n+1},
\]
we can approximate $a$ by a selfadjoint element in $M_N(A)$ with finite spectrum, and then perturb its spectrum, to get a selfadjoint element $b \in M_N(A)$ such that
\[
\|a - b\| < \frac{1}{2n+1} + \varepsilon \quad \text{and} \quad \text{sp}(b) \subseteq \left\{-\frac{2n}{2n+1}, -\frac{2n - 2}{2n+1}, \ldots, \frac{2n - 2}{2n+1}, \frac{2n}{2n+1}\right\}.
\]
Define $u = \exp(\pi ib)$. Then $u$ is a unitary in $M_N(A)$ with $u^{2n+1} = 1$. Letting log be the standard branch defined on the complement of the negative real axis, we furthermore have $b = -\frac{i}{\pi} \log(u)$.

Let $v \in M_N(A)$ be the following unitary block matrix, whose entries are the identity matrices of the appropriate sizes:
\[
v = \begin{pmatrix}
0 & 1_m \\
1_m & \ddots \\
\vdots & \ddots \\
1_m & 0
\end{pmatrix} \oplus 1_{N-(2n+1)m}.
\]
Define
\[
c = p + upu^* + \cdots + u^{2n}p(v^*)^{2n},
\]
and let $C = ce^{c^*}$, which is a hereditary subalgebra in $M_N(A)$.

For $0 \leq k, l \leq 2n$ we have
\[
v^k p(v^*)^l = \sum_{j=1}^m e_{km+j, lm+j}.
\]
Since $p(v^*)^l q_0 = p(v^*)^l$ for $0 \leq l \leq 2n$, we get $c q_0 = c$, so that $c^*c \in q_0 M_N(A) q_0$. It also follows that in the computation of $cc^*$ most of the terms cancel, and one gets
\[
cc^* = p + upu^* + \cdots + u^{2n}p(u^*)^{2n}.
\]
Since $u^{2n+1} = 1$, it follows that $ucc^*u^* = cc^*$. 

We claim that $uCu^* = C$. Indeed, if $x \in A$ then
\[ uex^*u^* = (ue)x(ue) = \|x\| \cdot \|uecx^*u^*\| = \|x\| \cdot \|cc^* \| \leq C. \]
This shows that $uCu^* \subset C$, and the reverse inclusion now follows from $u^{2n+1} = 1$. This proves the claim.

Write $Z_{2n+1} = Z/(2n+1)Z$. The automorphism $Ad(u)$ generates an inner action of $Z_{2n+1}$ on $M_N(A)$, and by invariance and Lemma 4.3, also an inner action $\alpha : Z_{2n+1} \to \text{Aut}(C)$. We claim that the fixed point algebra $C^\alpha$ has real rank zero. To see this, note that the Proposition in [12] shows that $C^\alpha$ is isomorphic to a hereditary subalgebra of the crossed product $C^*(Z_{2n+1}, C, \alpha)$. Since $\alpha$ is inner, this crossed product is isomorphic to the direct sum of $2n+1$ copies of $C$, so has real rank zero. Therefore so does every hereditary subalgebra. This proves the claim.

Choose $\delta_1 > 0$ such that whenever $D$ is a unital C*-algebra and an element $x \in D$ and a unitary $v \in D$ satisfy
\[ \|x\| \leq 1, \quad \|vx - vx\| < 4\delta_1, \quad \text{and} \quad \text{sp}(v) \subset \{ \exp(i\theta) : -\frac{2n}{2n+1} \leq \theta \leq \frac{2n}{2n+1} \}, \]
then $\|x \log(v) - \log(v)x\| < \pi \varepsilon$. Choose $\delta_2 > 0$ such that whenever $D$ is a unital C*-algebra and projections $e, f \in D$ satisfy $\|e - f\| < \delta_2$, then there exists a unitary $w \in D$ such that $\|w - 1\| < \delta_1$ and $weuw^* = f$. Choose $\delta_3 > 0$ such that $\delta_3 < \frac{1}{4} \min(\delta_2, 1)$.

We now claim that there is a projection $e \in C$ such that $ueu^* = e$ and $\|ep - p\| < \delta_3$. To prove this, first observe that the formula for $cc^*$ above shows that $p \leq cc^*$, whence $p \in C$. Next, use the fact that $C^\alpha$ has real rank zero to find (Theorem 2.6 of [11]) an approximate identity in $C^\alpha$ consisting of projections. Lemma 4.2 shows that any approximate identity for $C^\alpha$ is also an approximate identity for $C$. Since $\alpha(b) = ubu^*$ for $b \in C$, the claim follows.

From $\|ep - p\| < \delta_3$ we get
\[ \|(ep)^2 - ep\| = \|ep(e - p)e\| \leq \|ep\| \cdot \|ep - p\| \cdot \|e\| < \delta_3. \]
Since $\delta_3 < \frac{1}{4}$, a standard argument provides a projection $f \in (ep)eA(ep)e$ such that $\|ep - f\| < 2\delta_3$. Then
\[ \|p - f\| \leq \|p - ep\| + \|f - ep\| \cdot \|e\| + \|ep - f\| < 4\delta_3 < \delta_2. \]
Note that $ef = f$, so $e \geq f$; also, $e \geq q_0$ by Lemma 4.4.

By the choice of $\delta_2$ there is a unitary $w \in A$ such that $\|w - 1\| < \delta_1$ and $wfw^* = p$. Set $q = weuw^*$. Then $q \geq wfw^* = p$. Since $q \sim e \geq q_0$, we also have $q \geq q$. It remains to show that $\|aq - qa\| < \frac{1}{n}$. Since
\[ \|aq - qa\| = \|weu^*u - uwu^*u\| \leq 4\|w - 1\| < 4\delta_1, \]
the choice of $\delta_1$ gives
\[ \|qb - bq\| = \frac{1}{\varepsilon} \|q \log(u) - \log(u)q\| < \varepsilon. \]
Therefore
\[ \|qa - aq\| \leq 2\|a - b\| + \|qb - bq\| \leq 2 \left( \frac{1}{2n+1} + \varepsilon \right) + \varepsilon < \frac{1}{n}. \]
This completes the proof. \[ \square \]
The proof of the following theorem is analogous to that of Theorem 4.6 of [27], but is somewhat more complicated. In [27] we worked with a single “large” AF subalgebra; here, we only have subalgebras of real rank zero, obtained from Lemma 2.5, which play the role of algebras in a direct limit decomposition for the AF subalgebra of [27].

**Theorem 4.5.** Let \( A \) be a simple unital C*-algebra with real rank zero. Suppose that the order on projections over \( A \) is determined by traces and \( \alpha \in \text{Aut}(A) \) has the tracial Rokhlin property. Then \( C^*(\mathbb{Z}, A, \alpha) \) has real rank zero.

**Proof.** Set \( B = C^*(\mathbb{Z}, A, \alpha) \).

Let \( a \in B \) be selfadjoint with \( \|a\| \leq 1 \). Let \( \varepsilon > 0 \). We approximate \( a \) to within \( \varepsilon \) by an invertible selfadjoint element. If \( a \) is already invertible, there is nothing to prove. Therefore we assume \( 0 \in \text{sp}(a) \).

Set \( \varepsilon_0 = 1/12 \varepsilon \), and choose a continuous function \( g: [-1, 1] \to [0, 1] \) such that \( g(0) = 1 \) and \( \text{supp}(g) \subset (-\varepsilon_0, \varepsilon_0) \).

Recalling the notation \( T(B) \) from Notation 1.2, define

\[
\eta = \inf_{\tau \in T(B)} \tau(g(a)).
\]

The algebra \( B \) is simple by Corollary 1.14 so that every tracial state is faithful. Also, \( g(a) \) is a nonzero positive element, and \( T(B) \) is weak* compact. Therefore \( \eta > 0 \).

Use a polynomial approximation to the function \( g \) to choose \( \delta_0 > 0 \) such that whenever \( C \) is a unital C*-algebra and \( x, y \in C_{sa} \) satisfy \( \|x\|, \|y\| \leq 2 \) and \( \|x - y\| < \delta_0 \), then \( \|g(x) - g(y)\| < \frac{1}{6} \eta \). Choose \( n_0, N \in \mathbb{N} \) such that

\[
\frac{1}{N} < \frac{\eta}{12} \quad \text{and} \quad \frac{1}{n_0} < \frac{\delta}{2}.
\]

Since \( \alpha \) has the tracial Rokhlin property, we can use Lemma 2.5 to find a projections \( e, p \in A \), a projection \( f \in A \), integers \( n, m > 0 \), a unital C*-subalgebra \( D \subseteq eBe \), and an isomorphism \( \varphi: D \to M_n \otimes fAf \), such that \( p \in D \), such that

\[
\frac{2m}{n} < \min \left( \frac{1}{2n_0 + 1}, \frac{\eta}{12(2n_0 + 1)} \right),
\]

such that

\[
\varphi(p) = \sum_{j=m+1}^{n-m} e_{j,j} \otimes 1_{fAf} \in M_n \otimes fAf,
\]

such that

\[
\text{dist}(pa, D) < \frac{1}{2} \delta \quad \text{and} \quad \text{dist}(ap, D) < \frac{1}{2} \delta,
\]

and such that there are \( N \) mutually orthogonal projections \( f_1, f_2, \ldots, f_N \in pDp \), each of which is Murray-von Neumann equivalent in \( B \) to 1 - \( p \).

From the last condition, it is evident that for every \( \tau \in T(B) \) we have

\[
\tau(1-e) \leq \tau(1-p) \leq \frac{\tau(p)}{N} \leq \frac{1}{N} < \frac{\eta}{12}.
\]

Moreover,

\[
\tau(e-p) = \left( \frac{2m}{n} \right) \tau(e) \leq \frac{2m}{n} < \frac{\eta}{12(2n_0 + 1)}.
\]
Set
\[ x = a - (1 - p)a(1 - p) = pa + (1 - p)ap. \]
Choose \( x_1, x_2 \in D \) such that
\[ \|pa - x_1\| < \frac{1}{4} \delta \quad \text{and} \quad \|ap - x_2\| < \frac{1}{4} \delta. \]
We arrange to replace \( x_1 \) and \( x_2 \) by a single selfadjoint element. Since \( p \in D \) and \( D \) is a unital subalgebra of \( eBe \), we have
\[ (1 - p)x_2 = (1 - p)e x_2 = (e - p)x_2 \in D. \]
So \( px_1, (1 - p)x_2p \in D \). Set \( d = px_1 + (1 - p)x_2p \in D \), and observe that
\[ \|d - x\| \leq \|p\| \cdot \|pa - x_1\| + \|1 - p\| \cdot \|ap - x_2\| \cdot \|p\| < \frac{1}{4} \delta + \frac{1}{4} \delta = \delta. \]
Then set \( a_0 = a - x + \frac{1}{2}(d + d^*) \), which satisfies
\[ a_0^* = a_0, \quad a_0 - (1 - p)a_0(1 - p) = \frac{1}{2}(d + d^*) \in D, \quad \text{and} \quad \|a - a_0\| < \delta. \]
Next, we replace \( p \) by a smaller projection (which will be called \( 1 - q \)) which approximately commutes with \( a_0 \). Let \( z \in M_n \subseteq M_n \otimes fAf \) be a permutation unitary such that
\[ z\varphi(e - p)z^* = \sum_{j=1}^{2m} e_{j,j} \otimes 1_{fAf}. \]
Apply Lemma 4.4 with \( fAf \) in place of \( A \), with \( z\varphi(d)z^* \) in place of \( a \), with \( n \) in place of \( N \), with \( 2m \) in place of \( m \), and with \( n_0 \) in place of \( n \). Note that
\[ (2n_0 + 1) \cdot 2m = \left( \frac{2m}{n} \right) (2n_0 + 1)n < \left( \frac{1}{2n_0 + 1} \right) (2n_0 + 1)n = n, \]
as required in the hypotheses of Lemma 4.4. We obtain a projection \( q_0 \in M_n \otimes fAf \) such that \( z\varphi(e - p)z^* \leq q_0 \), such that \( [q_0] \leq (2n_0 + 1)\|\varphi(e - p)\| \) in \( K_0(fAf) \), and such that
\[ \|[q_0, z\varphi(d)z^*]\| < \frac{1}{m_0} < \frac{1}{4} \delta. \]
Set \( q = 1 - e + \varphi^{-1}(z^*q_0z) \). We estimate \( \|[q, a_0]\| \). Since \( z\varphi(e - p)z^* \leq q \), we get \( q \geq 1 - p \). In particular, \( [q, (1 - p)a_0(1 - p)] = 0 \), whence \( [q, a_0] = [q, d] = [\varphi^{-1}(z^*q_0z), d] \). Thus
\[ \|[q, a_0]\| = \|[q_0, z\varphi(d)z^*]\| < \frac{1}{4} \delta. \]
Let \( \tau \in T(B) \); we estimate \( \tau(q) \). We start with \( \tau(\varphi^{-1}(z^*q_0z)) \). We know \( [\varphi^{-1}(q_0)] \leq (2n_0 + 1)\|\tau(e - p)\| \) in \( K_0(B) \), so, using the estimate above on \( \tau(e - p) \), we get
\[ \tau(\varphi^{-1}(z^*q_0z)) = \tau(\varphi^{-1}(q_0)) \leq (2n_0 + 1)\tau(e - p) < \frac{1}{12} \eta. \]
Using also the estimate above on \( \tau(1 - e) \), we then get
\[ \tau(q) = \tau(1 - e) + \tau(\varphi^{-1}(z^*q_0z)) < \frac{1}{12} \eta + \frac{1}{12} \eta = \frac{1}{6} \eta. \]
Set \( y = (1 - q)a_0(1 - q) \), which is a selfadjoint element of \( D \) with \( \|y\| \leq \|a_0\| < \|a\| + \delta \leq 2 \). Let \( g(y) \) be the result of evaluating functional calculus in \( (1 - q)D(1 - q) \). Since \( D \) has real rank zero, there is a projection \( r \in g(y)Dg(y) \) such that \( \|rg(y) - g(y)\| < \frac{1}{3} \eta \).
Let \( \tau \in T(B) \); we claim that \( \tau(r) > \tau(q) \). First, \( \|rg(y)r - g(y)\| < \frac{1}{3} \eta \). Since \( g \leq 1 \) we get \( rg(y)r \leq r \), so that
\[ \tau(r) \geq \tau(rg(y)r) > \tau(g(y)) - \frac{1}{3} \eta. \]
Next,
\[\|a_0 - (qa_0q + y)\| \leq \|qa_0(1 - q)\| + \|(1 - q)a_0q\| \leq 2\|q, a_0\| < 2 \cdot \frac{1}{2} \delta = \delta \leq \delta_0.\]

Let \(g(qa_0q)\) be the result of evaluating functional calculus in \(qBq\). Then orthogonality of \(qa_0q\) and \(y\), together with the choice of \(\delta_0\), gives
\[\|g(a_0) - [g(qa_0q) + g(y)]\| = \|g(a_0) - g(qa_0q + y)\| < \frac{1}{\delta} \eta.\]

Since \(g(qa_0q) \leq q\), the estimate \(\tau(q) < \frac{1}{\delta} \eta\) implies
\[\tau(q(y)) > \tau(q(a_0)) - \tau(q(qa_0q)) - \frac{1}{\delta} \eta \geq \tau(q(a_0)) - \tau(q) - \frac{1}{\delta} \eta > \tau(q(a_0)) - \frac{1}{\delta} \eta.\]

Moreover, \(\|a - a_0\| < \delta \leq \delta_0\) gives \(\|g(a) - g(a_0)\| < \frac{1}{\delta} \eta\), whence \(\tau(q(a_0)) > \tau(q(a)) - \frac{1}{\delta} \eta\). By the choice of \(\eta\) we have \(\tau(q(a)) \geq \eta\). Putting everything together, we get
\[\tau(r) > \tau(q(y)) - \frac{1}{\delta} \eta > \tau(q(a_0)) - \frac{2}{\delta} \eta > \tau(q(a)) - \frac{2}{\delta} \eta \geq \frac{1}{\delta} \eta > \tau(q).\]

This proves the claim.

Since \(r \in g(y)Bq(y)\), the condition on \(\text{supp}(g)\) and Lemma 4.5 of [24] give
\[\|ry - yr\| < 2\varepsilon_0\quad\text{and}\quad\|r\| \leq \varepsilon_0.\]

Since \(r \leq 1 - q\) and \(y = (1 - q)a_0(1 - q)\), we have \(ra_0r = ryr\), whence \(\|ra_0r\| < \varepsilon_0\). Also,
\[\|r, a_0\| = \|ry + r(1 - q)a_0q - [yr + qa_0(1 - q)r]\| \leq \|r, y\| + 2\|a_0, q\| < 2\varepsilon_0 + 2\left(\frac{1}{\delta} \delta\right) \leq 3\varepsilon_0.\]

Define \(a_1 = (1 - q - r)a_0(1 - q - r) + qa_0q\).

We estimate \(\|a_1 - a\|\). We have
\[a_0 - a_1 = (1 - q - r)a_0q + qa_0(1 - q - r) + (1 - q - r)a_0r + ra_0(1 - q - r) + ra_0r + ra_0q + qa_0r.\]

So, using \(\|q, a_0\| \leq \frac{1}{2} \delta \leq \frac{1}{2} \varepsilon_0\), we get
\[\|a_0 - a_1\| \leq 4\|q, a_0\| + 2\|r, a_0\| + \|ra_0r\| < 2\varepsilon_0 + 6\varepsilon_0 + \varepsilon_0 = 9\varepsilon_0.\]

Since \(\|a_0 - a\| < \delta \leq \varepsilon_0\), it follows that
\[\|a_1 - a\| < 10\varepsilon_0.\]

Let \(A_0\) and \(\psi: A_0 \to C^*(Z, A, \alpha)\) be as in Lemma 24 using \(\varphi^{-1}\) in place of \(\varphi\) and with \(\varepsilon\) as above. Then \(\varphi^{-1}(z^*q_0z) \in D \subset \psi(A_0)\) and \(1 - e \in \psi(A_0)\), so \(q \in \psi(A_0)\); also \(r \in D \subset \psi(A_0)\) by construction. We proved above that \(\tau(r) > \tau(q)\) for all \(r \in T(B)\). So Lemma 24 implies \(q \not\prec r\) in \(B\). Therefore Lemma 8 of [11] provides an invertible selfadjoint element \(b_1 \in (q + r)B(q + r)\) such that \(\|b_1 - qa_0q\| < \varepsilon_0\). Also, by construction, we have
\[1 - q, r, y = (1 - q)a_0(1 - q) \in D,\]
so \((1 - q - r)a_0(1 - q - r) \in D\). Since \(D\) has real rank zero, there is an invertible selfadjoint element \(b_2 \in D\) such that
\[\|b_2 - (1 - q - r)a_0(1 - q - r)\| < \varepsilon_0.\]
Then $b_1 + b_2$ is an invertible selfadjoint element of $B$, and satisfies
\[
\| (b_1 + b_2) - a \| \leq \| a - a_1 \| + \| b_1 - qa_0 \| + \| b_2 - (1 - q - r) a_0 (1 - q - r) \|
\]
\[
< 10\varepsilon + \varepsilon_0 + \varepsilon = 12\varepsilon_0 = \varepsilon.
\]
This completes the proof.}

**Corollary 4.6.** Let $A$ be a simple unital C*-algebra with real rank zero, and suppose that the order on projections over $A$ is determined by traces. Let $\alpha \in \text{Aut}(A)$ have the tracial Rokhlin property. Then the restriction map is a bijection from the tracial states of $C^*(Z, A, \alpha)$ to the $\alpha$-invariant tracial states of $A$.

**Proof.** Since $C^*(Z, A, \alpha)$ has real rank zero by Theorem 4.5, this follows from Proposition 2.2 of [18].

**Corollary 4.7.** Let $A$ be a simple separable nuclear unital C*-algebra with tracial rank zero and satisfying the Universal Coefficient Theorem. Let $\alpha \in \text{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(Z, A, \alpha)$ satisfies the local approximation property of Popa [30] (is a Popa algebra in the sense of Definition 1.2 of [8]).

**Proof.** By Corollary 5.7 and Theorem 6.8 of [21], the order on projections over $A$ is determined by traces, and by Theorem 3.4 of [20], the algebra $A$ has real rank zero. So $C^*(Z, A, \alpha)$ has real rank zero by Theorem 4.5. It embeds in an AF algebra by Corollary 1 at the end of Section 3 of [23], and is hence quasidiagonal. That $C^*(Z, A, \alpha)$ satisfies the local approximation property of Popa now follows from Theorem 1.2 of [30].

5. Stable rank of crossed products

In this section, we prove that if $A$ is a simple unital C*-algebra with real rank zero and stable rank one, such that the order on projections over $A$ is determined by traces, and if $\alpha \in \text{Aut}(A)$ has the tracial Rokhlin property, then $C^*(Z, A, \alpha)$ has stable rank one. The methods are adapted from Section 5 of [27].

**Lemma 5.1.** Let $\delta > 0$. Then there exists a continuous function $g: [0, 1] \to [0, 1]$ such that $g(0) = 0, g(1) = 1$, and whenever $A$ is a C*-algebra with real rank zero and $a \in A$ is a positive element with $\|a\| \leq 1$, then there is a projection $e \in aAa$ such that $\|eg(a) - g(a)\| < \delta$ and $\|ae - e\| < \delta$.

**Proof.** Choose $t_0, t_1$ such that $1 - \delta < t_0 < t_1 < 1$. Let $g: [0, 1] \to [0, 1]$ be any continuous function which vanishes on $[0, t_1]$ and satisfies $g(1) = 1$.

Let $A$ be a C*-algebra with real rank zero and let $a \in A$ be a positive element with $\|a\| \leq 1$. Choose a continuous function $h: [0, 1] \to [0, 1]$ which vanishes on $[0, t_0]$ and satisfies $h(t) = 1$ for $t \in [t_1, 1]$. Since $A$ has real rank zero, there is a projection $e \in g(a)A g(a)$ such that $\|eg(a) - g(a)\| < \delta$. Moreover, from $hg = g$ we get $h(a)g(a) = g(a)$, whence $h(a)e = e$. We also have $\|ah(a) - h(a)\| < \delta$ because $|t - 1| \leq 1 - t_0 < \delta$ whenever $h(t) \neq 0$. Accordingly,
\[
\|ae - e\| = \|ah(a)e - h(a)e\| \leq \|ah(a) - h(a)\| \cdot \|e\| < \delta,
\]
as was to be proved.

**Lemma 5.2.** Let $A$ be a simple C*-algebra with real rank zero and such that the order on projections over $A$ is determined by traces. Let $\alpha \in \text{Aut}(A)$ have the tracial Rokhlin property. Let $q_1, \ldots, q_n \in C^*(Z, A, \alpha)$ be nonzero projections, let
Let \( a_1, \ldots, a_m \in C^*(\mathbb{Z}, A, \alpha) \) be arbitrary, and let \( \varepsilon > 0 \). Then there exists a unital subalgebra \( A_0 \subset C^*(\mathbb{Z}, A, \alpha) \) which is stably isomorphic to \( A \), a projection \( p \in A_0 \), nonzero projections \( r_1, \ldots, r_n \in pA_0p \), and elements \( b_1, \ldots, b_m \in C^*(\mathbb{Z}, A, \alpha) \), such that:

1. \( \|gr_k - r_k\| < \varepsilon \) for \( 1 \leq k \leq n \).
2. For \( 1 \leq k \leq n \) there is a projection \( g_k \in r_kA_0r_k \) such that \( 1 - p \sim g_k \) in \( C^*(\mathbb{Z}, A, \alpha) \).
3. \( \|a_j - b_j\| < \varepsilon \) for \( 1 \leq j \leq m \).
4. \( pb_jp \in pA_0p \) for \( 1 \leq j \leq m \).

**Proof.** Set \( B = C^*(\mathbb{Z}, A, \alpha) \).

Let

\[
\eta = \min_{1 \leq k \leq n} \left( \inf_{\tau \in T(B)} \tau(q_k) \right) > 0 \quad \text{and} \quad \varepsilon_0 = \min \left( \frac{\eta}{5}, \frac{\varepsilon^2}{2} \right) .
\]

Apply Lemma 5.1 with \( \varepsilon_0 \) in place of \( \delta \), obtaining a continuous function \( g: [0,1] \to [0,1] \). Apply Lemma 5.3 with this function \( g \) and with \( \varepsilon_0 \) in place of \( \varepsilon \), obtaining a number \( \delta > 0 \) such that whenever \( \tau \) is a tracial state on \( B \) and \( p, q \in B \) are projections such that \( \tau(q) > 1 - \delta \), then \( \tau(g(pq)) \geq \tau(p) - \varepsilon_0 \). Further choose \( \varepsilon_1 > 0 \) with \( \varepsilon_1 \leq \min(\varepsilon_0, \varepsilon) \) and so small that whenever \( x, y \in B \) are positive elements with \( \|x\|, \|y\| \leq 1 \) and \( \|x - y\| < \varepsilon_1 \), then \( \|g(x) - g(y)\| < \varepsilon_0 \). Then choose \( \varepsilon_2 > 0 \) with \( \varepsilon_2 \leq \varepsilon_1 \) and so small that if \( x, y \in B \) are selfadjoint elements with \( \|x\|, \|y\| \leq 1 \) and \( \|x - y\| < \varepsilon_2 \), then the positive parts \( x_+ \) and \( y_+ \) satisfy \( \|x_+ - y_+\| < \varepsilon_1 \). Apply Lemma 2.6 with \( F = \{q_1, \ldots, q_n, a_1, \ldots, a_m\} \), with \( \varepsilon_2 \) in place of \( \varepsilon \), with an integer \( N \) so large that \( 1/N < \min(\delta, \varepsilon_0) \), and with \( \delta = 1 \).

We obtain projections \( e \in A \subset B \) and \( f \in A \), a unital subalgebra \( D \subset eB_0 \), an isomorphism \( \varphi: M_n \otimes fAf \to D \), and a projection \( p \in D \), such that, in particular, there exist

\[
x_1, \ldots, x_n, c_1, \ldots, c_m \in D
\]

with \( \|pa_j - c_j\| < \varepsilon_2 \) for \( 1 \leq j \leq m \), and

\[
\|x_k\| \leq 1 \quad \text{and} \quad \|pq_k - x_k\| < \varepsilon_2
\]

for \( 1 \leq k \leq n \). Moreover, \( \tau(1 - p) < \min(\delta, \varepsilon_0) \) for every \( \tau \in T(B) \).

Apply Lemma 2.4 with \( \varphi: M_n \otimes fAf \to D \) and the projection \( e \) as given. We obtain a \( \mathrm{C}^* \)-algebra \( A_0 \) which is stably isomorphic to \( A \) and a unital homomorphism \( \psi: A_0 \to C^*(\mathbb{Z}, A, \alpha) \). The subalgebra \( \psi(A_0) \) will be the algebra \( A_0 \) called for in the statement of the lemma. We note that \( \psi(A_0) \) contains \( D \), and hence \( p \).

For \( 1 \leq j \leq m \), set \( b_j = a_j + p(c_j - a_j)p \), which satisfies

\[
\|b_j - a_j\| < \varepsilon_2 \leq \varepsilon_1 \leq \varepsilon
\]

and \( pb_jp = pc_jp \in D \subset \psi(A_0) \).

These are Parts (3) and (4) of the conclusion.

Next, for \( 1 \leq k \leq n \), observe that \( \frac{1}{2}(px_kp + px_k^*p) \) is a selfadjoint element of \( pDp \) of norm at most one such that

\[
\|pq_kp - \frac{1}{2}(px_kp + px_k^*p)\| < \varepsilon_2.
\]

So

\[
y_k = \frac{1}{2}(px_kp + px_k^*p)
\]

is a positive element of \( pDp \) of norm at most one such that \( \|pq_kp - y_k\| < \varepsilon_1 \).
By the choice of $g$ using Lemma \ref{lemma:choice_of_g}, there exists a projection $r_k \in pDp \subset \psi(A_0)$ such that
\[ \|r_k y_k - r_k\| < \varepsilon_0 \quad \text{and} \quad \|r_k g(y_k) - g(y_k)\| < \varepsilon_0. \]
Using $r_k \leq p$ at the second step, we now have
\[ (r_k q_k - r_k)(q_k r_k - r_k) = r_k - r_k q_k r_k = r_k - r_k p q_k p r_k. \]
Therefore
\[ \|r_k q_k - r_k\|^2 = \|r_k - r_k p q_k p r_k\| \leq \|r_k - r_k y_k\| \cdot \|y_k - p q_k p\| < \varepsilon_1 + \varepsilon_0 \leq \varepsilon^2, \]
so $\|r_k q_k - r_k\| < \varepsilon$, which is Part (1) of the conclusion.

We now estimate the traces on $r_k$. For every $\tau \in T(B)$, we have $\tau(r_k) \geq \tau(r_k g(\eta) r_k)$. By construction we have $\|r_k g(\eta) r_k - g(\eta)\| < 2\varepsilon_0$. From $\|y_k - p q_k p\| < \varepsilon_1$ and the choice of $\varepsilon_1$, we get $\|g(\eta) - g(p q_k p)\| < \varepsilon_0$. Since $\tau(p) > 1 - \delta$, the choice of $\delta$ using Lemma \ref{lemma:delta_choice} implies that $\tau(g(p q_k p)) > \tau(q_k) - \varepsilon_0$. Combining all these, we get $\tau(r_k) > \tau(q_k) - 4\varepsilon_0$. On the other hand, $\tau(1 - p) \leq \varepsilon_0$. Therefore, $\tau(1 - p) < \varepsilon_0$. Since $\varepsilon_0 \leq \frac{1}{13}\varepsilon$, we get $\tau(r_k) > \tau(1 - p)$. Since $\tau \in T(B)$ is arbitrary, and since $1 - p$ and $r_k$ are in $\psi(A_0)$, Lemma \ref{lemma:trace_estimates} gives Part (2) of the conclusion. \hfill \qed

**Theorem 5.3.** Let $A$ be a simple $C^*$-algebra with real rank zero and stable rank one, and such that the order on projections over $A$ is determined by traces. Let $\alpha \in \text{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(\mathbb{Z}, A, \alpha)$ has stable rank one.

**Proof.** Let $B = C^*(\mathbb{Z}, A, \alpha)$.

We are going to show that every two sided zero divisor in $B$ is a limit of invertible elements. That is, if $a \in B$ and there are nonzero $x, y \in B$ such that $xa = ay = 0$, then we show that for every $\varepsilon > 0$ there is an invertible element $c \in B$ such that $\|a - c\| < \varepsilon$. Because $B$ has a faithful tracial state, every one sided invertible element is invertible. Therefore Theorem 3.3(a) of \cite{32} will imply that any element is a limit of invertible elements, that is, $B$ has stable rank one.

So let $a \in B$, let $x, y \in B$ be nonzero elements such that $xa = ay = 0$, and let $\varepsilon > 0$. Without loss of generality $\|a\| \leq \frac{1}{2}$ and $\varepsilon \leq 1$. Since $B$ has real rank zero by Theorem \ref{thm:real_rank_zero}, there are are nonzero projections
\[ e \in x^* B x^* \quad \text{and} \quad f \in y B y^*, \]
and we have $ea = af = 0$. Apply Lemma \ref{lemma:traces_and_polytopes} to the nonzero projections $e$ and $f$ and the element $a$, with $\frac{1}{13}\varepsilon$ in place of $\varepsilon$. Call the resulting subalgebra $A_0$, the resulting projection $p_0$, the resulting nonzero projections $e_0$ and $f_0$, and the resulting element $x_0$. Thus
\[ e_0, f_0, p_0 x_0 p_0 \in p_0 A_0 p_0, \quad 1 - p_0 \not\prec e_0, f_0, \]
and
\[ \|e e_0 - e_0\|, \|f f_0 - f_0\|, \|a - x_0\| < \frac{1}{13}\varepsilon. \]

Define $a_0 = (1 - e_0) x_0 (1 - f_0)$. We clearly have $e_0 a_0 = a_0 f_0 = 0$, and we claim that $\|a - a_0\| < \frac{5}{13}\varepsilon$. First, using
\[ \|a\| \leq 1 \quad \text{and} \quad \|e_0 e - e_0\| = \|e e_0 - e_0\| < \frac{1}{13}\varepsilon, \]
we have
\[ \|e_0a_0\| \leq \|e_0\| \cdot \|x_0 - a\| + \|e_0 - e_0\| \cdot \|a\| + \|e_0e_0\| < \frac{1}{13}\varepsilon + \frac{1}{13}\varepsilon + 0 = \frac{2}{13}\varepsilon. \]

Similarly, \( \|x_0f_0\| < \frac{2}{13}\varepsilon \). Therefore
\[
\|a - a_0\| \leq \|a - x_0\| + \|x_0 - (1 - e_0)x_0(1 - f_0)\| \\
\leq \|a - x_0\| + \|e_0x_0\| + \|1 - e_0\| \cdot \|x_0f_0\| < \frac{1}{13}\varepsilon + \frac{2}{13}\varepsilon + \frac{2}{13}\varepsilon = \frac{5}{13}\varepsilon.
\]

This proves the claim. From \( \|a\| < \frac{1}{13} \) and \( \varepsilon \leq 1 \) we now get \( \|a_0\| \leq 1 \).

Since \( A \) has real rank zero and \( A_0 \) is stably isomorphic to \( A \), the algebra \( A_0 \) also has real rank zero. So Proposition 1.8 of [9] and Lemma 4.1 show that there is a nonzero projection \( r \leq e_0 \) such that \( r \not\leq f_0 \). Similarly, \( A_0 \) has stable rank one, so in fact there is a unitary \( v \in A_0 \) such that \( v^*rv \leq f_0 \). Then \( r(a_0v^*) = (a_0v^*)r = 0 \).

Apply Lemma 5.2 to the nonzero projection \( r \) and the element \( a_0v^* \), with \( \frac{1}{13}\varepsilon \) in place of \( \varepsilon \). Call the resulting subalgebra \( A_1 \), the resulting projection \( p_1 \), the resulting nonzero projection \( e_1 \), and the resulting element \( x_1 \). Thus
\[ e_1, p_1x_1p_1 \in p_1A_1p_1, \quad \|re_1 - e_1\|, \|a_0v^* - x_1\| < \frac{1}{13}\varepsilon, \quad \text{and} \quad 1 - p_1 \not\leq e_1.
\]

Define \( a_1 = (1 - e_1)x_1(1 - e_1) \). We clearly have \( e_1a_1 = a_1e_1 = 0 \). Also, \( p_1a_1p_1 = (1 - e_1)p_1x_1p_1(1 - e_1) \in p_1A_1p_1 \). Furthermore, since still \( \|a_0v^*\| \leq 1 \), the argument used above to prove \( \|a - a_0\| < \frac{1}{13}\varepsilon \) now shows that \( \|a_0v^* - a_1\| < \frac{1}{13}\varepsilon \). So \( \|v^* - a_1\| < \frac{10}{13}\varepsilon \). The conclusion of Lemma 5.2 provides \( s \in B \) such that
\[
ss = 1 - p_1, \quad ss^* \leq e_1, \quad \text{and} \quad ss^* \in A_1.
\]

Set \( e_2 = ss^* \) and \( w = s + s^* + p_1 - e_2 \). Since \( e_2 \leq e_1 \leq p_1 \), it follows that \( w \) is a unitary satisfying
\[ uw = 1 - p_1, \quad w(1 - p_1)^* = e_2, \quad \text{and} \quad w(p_1 - e_2) = p_1 - e_2.
\]

We now have \( e_2a_1w = 0 \) and \( a_1w(1 - p_1) = a_1e_2w = 0 \). Therefore, with respect to the decomposition of the identity
\[ 1 = e_2 \oplus (p_1 - e_2) \oplus (1 - p_1), \]
and with \( c = (p_1 - e_2)a_1w(p_1 - e_2) \) and suitable \( x, y, z \in B \), the element \( a_1w \) has the block matrix form
\[
a_1w = \begin{pmatrix} 0 & 0 & 0 \\ x & c & 0 \\ y & z & 0 \end{pmatrix}.
\]

Now use \( w(p_1 - e_2) = p_1 - e_2 \) and \( e_2 \leq p_1 \) to rewrite
\[ c = (p_1 - e_2)p_1a_1p_1(1 - e_2) \in (p_1 - e_2)A_1(p_1 - e_2).
\]

Since \( (p_1 - e_2)A_1(p_1 - e_2) \) has stable rank one, there exists an invertible element \( d \in (p_1 - e_2)A_1(p_1 - e_2) \) such that \( \|c - d\| < \frac{1}{13}\varepsilon \). Then
\[
a_2 = \begin{pmatrix} \frac{1}{13}\varepsilon e_2 & 0 & 0 \\ x & d & 0 \\ y & z & \frac{1}{13}(1 - p_1) \end{pmatrix}
\]
is an invertible element in \( B \), which satisfies \( \|a_2 - a_1w\| < \frac{3}{13}\varepsilon \). So also \( a_2w^*v \) is an invertible element in \( B \), and satisfies
\[
\|a_2w^*v - a\| \leq \|a_2 - a_1w\| + \|a_1 - aw^*\| < \frac{3}{13}\varepsilon + \frac{10}{13}\varepsilon = \varepsilon.
\]

This is the required approximation by an invertible element.
Corollary 5.4. Let $A$ be a simple C*-algebra with real rank zero and stable rank one, and such that the order on projections over $A$ is determined by traces. Let $\alpha \in \text{Aut}(A)$ have the tracial Rokhlin property. Then the projections in $M_\infty(C^*(Z, A, \alpha))$ satisfy cancellation: if $e, f, p \in M_\infty(C^*(Z, A, \alpha))$ are projections such that $e \oplus p$ is Murray-von Neumann equivalent to $f \oplus p$, then $e$ is Murray-von Neumann equivalent to $f$.

Proof. This follows from the fact that $C^*(Z, A, \alpha)$ has stable rank one (Theorem 5.3), using Proposition 6.5.1 of [1].

6. Examples

In this section we give some examples of crossed products by automorphisms with the tracial Rokhlin property. The examples we are most interested in require a longer treatment, and will appear separately [20].

We believe that if an action $\alpha$ of $Z$ on a simple C*-algebra $A$ has the tracial Rokhlin property, and if $A$ has tracial rank zero, then $C^*(Z, A, \alpha)$ should again have tracial rank zero. This would in particular imply that the crossed products by the Furstenberg transformations on irrational rotation algebras that we consider in [20] have tracial rank zero, and also that the crossed product in Example 6.3 has tracial rank zero. However, we give here some examples to which such a theorem can’t apply, because neither the original algebra nor the crossed product has tracial rank zero.

For easy reference, we state the following two results.

Proposition 6.1. There exists an automorphism $\beta$ of the $2^\infty$ UHF algebra $B$ which generates an action of $Z$ with the Rokhlin property and which is the identity on K-theory.

Proof. This is implicitly proved in Sections 4 and 5 of [9], although the Rokhlin property is not explicitly mentioned there. (See [17] for an explicit proof for the $n^\infty$ UHF algebra for arbitrary $n$. Note that every automorphism of $B$ is the identity on K-theory.)

Proposition 6.2. Let $A$ be a unital C*-algebra, and let $\alpha \in \text{Aut}(A)$ be arbitrary. Let $B$ be a unital C*-algebra, and let $\beta \in \text{Aut}(B)$ generate an action of $Z$ with the Rokhlin property. Then $\alpha \otimes \beta$ generates an action of $Z$ on $A \otimes_{\text{min}} B$ with the Rokhlin property.

Proof. Using density of the algebraic tensor product, one sees that it suffices to simply tensor appropriate systems of Rokhlin projections for $\beta$ with $1_A$.

Of course, the proof works for any tensor product on which $\alpha \otimes \beta$ extends to an automorphism, in particular for $A \otimes_{\text{max}} B$. The situation for the tracial Rokhlin property is much less clear.

The following example shows that the implication (1) implies (5) of Theorem 6.4 of [19] is no longer valid when the action is not approximately inner.

Example 6.3. We sketch an example of an automorphism $\alpha$ of a simple unital AF algebra $A$ which has the Rokhlin property but such that $C^*(Z, A, \alpha)$ is not an AT algebra.
Let $A$ be the simple unital AF algebra such that $K_0(A) \cong \mathbb{Z} \left[ \frac{1}{2} \right] \oplus \mathbb{Z} \left[ \frac{1}{2} \right]$, with the strict order from the first coordinate, and with $[1] \mapsto (1,0)$. (One checks that this is in fact a Riesz group. See Section 7.6 of [1].) For any $d \in \mathbb{Z}$, the matrix

$$
\begin{pmatrix}
1 & 0 \\
d & 1
\end{pmatrix}
$$

defines an automorphism of $K_0(A)$ as a scaled ordered group. Let $\alpha_0 \in \text{Aut}(A)$ induce this automorphism on $K$-theory.

This automorphism need not have the Rokhlin property. Let $B$ be the $2^\infty$ UHF algebra and let $\beta \in \text{Aut}(B)$ be as in Proposition 6.1. K-theory computations show that $A \otimes B \cong A$, and $\alpha = \alpha_0 \otimes \beta$ does have the Rokhlin property (by Proposition 6.2), and induces the same map on K-theory. Since $K_0(A)$ has a unique state, $A$ has a unique tracial state, so Proposition 1.7 shows that $\alpha$ has the tracial Rokhlin property.

The Pinchuk-Voičulescu exact sequence [29] shows that $K_0(C^*(\mathbb{Z}, A, \alpha))$ is isomorphic to the cokernel of the map on $K_0(A)$ induced by

$$
id - \alpha_* = \begin{pmatrix} 0 & 0 \\ -d & 0 \end{pmatrix}.
$$

If, say, $d = 3$, then this cokernel has torsion. Therefore $C^*(\mathbb{Z}, A, \alpha))$ is not an AT algebra.

On the other hand, Theorem 1.12 implies that $\alpha$ generates an action with the tracial Rokhlin property. So Theorem 1.12, Theorem 5.3, and Theorem 5.5 show that $C^*(\mathbb{Z}, A, \alpha)$ has real rank zero and stable rank one, and that the order on projections over this algebra is determined by traces.

The remaining examples are all on C*-algebras which do not have tracial rank zero.

**Example 6.4.** Let $n \in \{2,3,\ldots,\infty\}$, let $F_n$ be the free group on $n$ generators, and let $\alpha$ be any automorphism of $C^*_r(F_n)$. (An example which is particularly interesting in this context is to take $n = \infty$ and to take $\alpha$ to be induced by an infinite order permutation of the free generators of $F_n$. Another possibility is to have $\alpha$ multiply the $k$-th generating unitary by an irrational number $\lambda_k$.) Let $B$ be the $2^\infty$ UHF algebra and let $\beta \in \text{Aut}(B)$ be as in Proposition 6.1. Then $\alpha \otimes \beta$ generates an action with the Rokhlin property by Proposition 6.2. Since $C^*_r(F_n)$ has a unique tracial state, it follows from Corollary 6.6 of [32] that $C^*_r(F_n) \otimes B$ has stable rank one. Moreover, $C^*_r(F_n) \otimes B$ is exact, so every quasitrace is a trace (11), whence Theorem 7.2 of [33] implies that $C^*_r(F_n) \otimes B$ has real rank zero and Theorem 5.2(b) of [33] implies that the order on projections over $C^*_r(F_n) \otimes B$ is determined by traces. (In fact, $K_0(C^*_r(F_n) \otimes B))$ is $\mathbb{Z} \left[ \frac{1}{2} \right]$ with its usual order.) We can now use Proposition 1.7 to conclude that $\alpha \otimes \beta$ generates an action with the tracial Rokhlin property. On the other hand, the corollary to Theorem A1 of [35] shows that $C^*_r(F_n)$ is not quasidiagonal, so $C^*_r(F_n) \otimes B$ is not quasidiagonal either. Theorem 3.4 of [20] therefore shows that $C^*_r(F_n) \otimes B$ does not have tracial rank zero. Theorem 3.3, Theorem 5.3, and Theorem 5.5 show that the crossed product $C^*(\mathbb{Z}, C^*_r(F_n) \otimes B, \alpha \otimes \beta)$ has real rank zero and stable rank one, and that the order on projections over this algebra is determined by traces. However, it does not have tracial rank zero because it contains the nonquasidiagonal C*-algebra $C^*_r(F_n)$. 
Example 6.5. Let $A$ be the simple separable C*-algebra of Theorem 7.20 of [8]. This algebra has real rank zero and stable rank one, and the order on projections over $A$ is determined by traces. It also has a number of other nice properties: it is exact, it satisfies the Universal Coefficient Theorem, it is approximately divisible in the sense of [8], it is a direct limit of residually finite dimensional C*-algebras, and it satisfies the local approximation property of Popa (is a Popa algebra in the sense of Definition 1.2 of [8]) and is hence quasidiagonal (by Theorem 1.2 of [30]).

According to Corollary 7.21 of [8], the algebra $A$ is a direct limit of residually finite dimensional C*-algebras, and it satisfies the Universal Coefficient Theorem, it is approximately divisible (as in Example 6.5), and has a unique tracial state, but is not exact. It also does not have tracial rank zero, stable rank one, and that the order on projections over this algebra is determined by traces.

We claim that the crossed product $C^*(Z, A, \gamma)$ does not have tracial rank zero. Using the notation before Definition 3.1 of [8], we note that the proof of Theorem 7.20 of [8] gives a tracial state $\tau_0 \in T(A) \setminus T(A)_n$. Since $\tau_0$ is assumed to be invariant under $\gamma$, it extends to a tracial state $\tau$ on $C^*(Z, A, \gamma)$. Using the equivalence of Conditions (1) and (4) in Theorem 3.1 of [8], it follows that $\tau \in T(C^*(Z, A, \gamma)) \setminus T(C^*(Z, A, \gamma))_n$. So $C^*(Z, A, \gamma)$ fails to have tracial rank zero for the same reason that $A$ does.

We have not determined whether $C^*(Z, A, \gamma)$ is quasidiagonal, but it seems reasonable to hope that one can use the tracial Rokhlin property to show that it is.

Example 6.6. Let $A$ be a simple separable C*-algebra constructed as in Theorem 7.23 of [8]. This algebra has real rank zero and stable rank one, satisfies the local approximation property of Popa and is hence quasidiagonal (as in Example 6.5), and has a unique tracial state, but is not exact. It also does not have tracial rank zero. We show below that $A$ may be chosen such that in addition the order on projections over $A$ is determined by traces.

Let $B$ be the $2^\infty$ UHF algebra. It is easy to see that all the properties given above for $A$ carry over to $A \otimes B$. (To see that $A \otimes B$ does not have tracial rank zero, observe that the last paragraph of the proof of Theorem 7.23 of [8] applies just as well to $A \otimes B$ as to $A$.) Let $\alpha$ be any automorphism of $A$. Let $\beta$ be an automorphism of $B$ which generates an action with the Rokhlin property (Proposition 6.1). Then $\gamma = \alpha \otimes \beta$ generates an action of $Z$ on $A \otimes B$ with the Rokhlin property, by Proposition 6.2. Since $A \otimes B$ has a unique tracial state, it follows from Proposition 1.7 that $\gamma$ generates an action with the tracial Rokhlin property. Theorem 4.5, Theorem 5.3, and Theorem 8.5 now show that the crossed product $C^*(Z, A \otimes B, \gamma)$ has real rank zero, stable rank one, and a unique tracial state, and that the order on projections over this algebra is determined by traces.

We now show how to arrange that the order on projections over $A$ is determined by traces. This is done by adding one more condition to Conditions (1) through (5)
at the beginning of the proof of Theorem 7.23 of [3]. In addition to the dense sequence \( \{s_k^{(m)} \}_{k \in \mathbb{N}} \) in \( A_n \), we noted that \( \{p_k^{(m)} \}_{k \in \mathbb{N}} \) be a countable set of projections in \( \bigcup_{i=1}^{\infty} M_i \otimes A_n \) such that every projection in \( \bigcup_{i=1}^{\infty} M_i \otimes A_n \) is Murray-von Neumann equivalent to some \( p_k^{(m)} \). Then we require, in addition to Conditions (1) through (5), the existence of a finite set \( P_n \subset A_n \) such that whenever \( p \) and \( q \) are projections in

\[
\sigma_n,0 \left( \{p_k^{(0)} \}_{1 \leq k \leq n} \right) \cup \sigma_n,1 \left( \{p_k^{(1)} \}_{1 \leq k \leq n} \right) \cup \cdots \cup \sigma_n,n-1 \left( \{p_k^{(n-1)} \}_{1 \leq k \leq n} \right)
\]

such that \( \tau(p) < \tau(q) \) for all tracial states \( \tau \) on \( A_n \), then there is \( s \in \bigcup_{i=1}^{\infty} M_i \otimes A_n \), all of whose matrix entries are in \( P_n \), such that \( ss^* = p \) and \( s^*s \leq q \). To see that this can be done, at the step in the proof where the sets \( S \) and \( I \) are chosen, we observe that the order on projections over \( \prod_{j \in \mathbb{N}} M_{k(n(j))} \otimes \mathbb{C} \) is determined by traces, choose \( P \) accordingly, and include \( P \) along with \( U \), \( S \), and \( I \) when generating the next \( C^* \)-algebra.

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