An effective lowest Landau level treatment of demagnetization in superconducting mesoscopic disks

J. J. Palacios
Departamento de Física Aplicada, Universidad de Alicante, San Vicente del Raspeig, Alicante 03080, Spain.

F. M. Peeters and B. J. Baelus
Departement Natuurkunde, Universiteit Antwerpen (UIA), B-2610 Antwerpen, Belgium.
(March 21, 2022)

Demagnetization, which is inherently present in the magnetic response of small finite-size superconductors, can be accounted for by an effective $\kappa$ within a two-dimensional lowest Landau level approximation of the Ginzburg-Landau functional. We show this by comparing the equilibrium magnetization of superconducting mesoscopic disks obtained from the numerical solution of the three-dimensional Ginzburg-Landau equations with that obtained in the “effective” LLL approximation.

I. INTRODUCTION

The magnetic response of a three-dimensional superconductor depends on intrinsic parameters such as the coherence length $\xi$ and the penetration length $\lambda$ through the ratio $\kappa = \lambda / \xi$, but also depends critically on the shape and orientation with respect to the applied magnetic field $H$. The latter dependence makes it very difficult to extract information about the true nature of the superconductor from magnetic measurements and it can completely mask the intrinsic magnetic response. This phenomenon is known as demagnetization. The best known example of how demagnetization affects the magnetic properties of a finite-size superconductor is the appearance of the intermediate state $\text{III}$ in type-I superconductors. In very simple terms the intermediate state develops due to the tendency for the expelled and “over-stretched” magnetic flux lines to penetrate back into the otherwise perfectly diamagnetic material. For instance, macroscopic type-I superconducting disks in perpendicular to the field or thickness of the disk, which are still the focus of experimental and theoretical studies $\text{III}$, exhibit an apparent type-II magnetic response due to the formation of the intermediate state $\text{III}$. Thin mesoscopic Aluminum disks (with radii ranging typically between 0.1 and 2$\mu$m) have also received a lot of attention lately, in part due to recent breakthroughs in magnetic $\text{III}$ and resistive $\text{III}$ measurement techniques. Demagnetization in these mesoscopic disks is the focus of this paper.

From the theoretical point of view the demagnetization presented by macroscopic ellipsoidal samples can be simply accounted for by a demagnetizing factor, $N$, which is a scalar if $H$ lies parallel to one of the principal axis of the ellipsoid $\text{III}$. $N$ runs from $N = 1$ (maximum demagnetization) for the case of an infinite ellipsoid perpendicular to $H$ to $N = 0$ (zero demagnetization) for the same ellipsoid parallel to $H$, taking any value in between for intermediate situations like the spherical geometry ($N = 1/3$). When the samples are not ellipsoidal the situation is more complicated. Still, various effective models have been proposed to account for demagnetizing effects in macroscopic samples with simple forms like strips, disks or finite cylinders with various orientations with respect to the field $\text{III}$, and very accurate analytical results have been obtained in certain limits $\text{III}$. Compared to macroscopic disks, the situation for mesoscopic disks $\text{III}$ gets complicated by the fact that $\lambda$ can be comparable to the dimensions of the sample; in particular, to the dimension parallel to the field or thickness of the disk, $d$. When this is the case the penetration length is naively expected to get renormalized to $\lambda = \lambda^2 / d$ $\text{III}$ and the new $\kappa$ can be larger than $1/\sqrt{2}$, value that separates type-I from type-II behavior. Whether or not thin mesoscopic Aluminum disks exhibit truly type-II behavior and vortices form in the condensate can only be unambiguously answered with imaging techniques $\text{III}$ since demagnetization is always present and complicates the interpretation of the experimental results $\text{III}$.

In principle, this non-trivial interplay between dimensions, geometry, and intrinsic parameters can only be addressed theoretically within a three-dimensional Ginzburg-Landau framework:

$$\frac{1}{2m^*} \left( -i\hbar \vec{\nabla} - \frac{e}{c} \vec{A}(\vec{r}) \right)^2 \Psi(\vec{r}) = -\alpha \Psi(\vec{r}) - \beta \Psi(\vec{r}) |\Psi(\vec{r})|^2, \quad (1)$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A}(\vec{r}) = \frac{4\pi}{c} \vec{j}, \quad (2)$$

where $\vec{j}$ is the superconducting current $\text{III}$. These equations are obtained after extremizing the phenomenological Ginzburg-Landau energy functional:
where \( G_n \) is the Gibbs free energy of the normal state and 
\[-i\hbar \nabla - e^* \vec{A}(\vec{r})/c \] 
\( \Psi(\vec{r}) \) is the Cooper pair wave function or complex order parameter and the coefficients \( \alpha \) and \( \beta \) have the usual values \([3]\) which scale phenomenologically with temperature \([3]\).

The difficult task of solving the coupled Ginzburg-Landau differential equations \([11]\) and \([12]\) for three-dimensional mesoscopic disks has been undertaken numerically by Peeters and co-workers \([11]\) (see also Ref. \([10]\) for the equivalent problem of cylinders). While, typically, the order parameter within the disk can be considered uniform in the direction of the field, i.e., effectively two-dimensional, the three-dimensional magnetic flux structure needs to be taken fully into account. Considering the expected limitations of the effective Ginzburg-Landau theory, the results \([9]\) reproduce to a large extent the experimental findings \([8]\). Akkermans and co-workers have undertaken similar studies from a different point of view. They have been able to minimize analytically the Ginzburg-Landau functional close to the dual point \( \kappa = 1/\sqrt{2} \) \([13]\) and in the London limit \( \kappa \to \infty \) \([12]\), although demagnetizing effects have been considered only at a phenomenological level.

The aim of this work is to show that for small disks there is an effective way to incorporate demagnetization in the solution of the Ginzburg-Landau equations without considering in detail the three-dimensional structure of the magnetic field lines. Essentially, we do this by introducing an effective \( \kappa \), which we will denote by \( \tilde{\kappa} \), that is geometry dependent. Adequately chosen, the use of \( \tilde{\kappa} \) instead of the intrinsic \( \kappa \) can account, in part, for demagnetizing effects. This is conveniently done within the framework of the lowest Landau level (LLL) approximation which has been broadly used in the case of bulk superconductors \([13,14]\) and mesoscopic systems \([15,16]\). It will turn out that this procedure works rather well if we define two different effective \( \kappa \), one for the low magnetic field region, i.e., the Meissner state, and one for magnetic fields near \( H_{c2} \).

In Sec. \([1]\) we briefly review how the standard LLL approximation must be modified to obtain the magnetic response of finite-size superconductors where surface effects are dominant. We also compare our LLL approximation with traditional \([17]\) results for magnetization in the case of bulk systems. In Sec. \([1]\) we present results for the equilibrium magnetization of mesoscopic disks obtained from the full numerical solution of the Ginzburg-Landau differential equations \([1]\). We finally introduce the effective \( \kappa \) in our LLL approximation and make a comparison between the exact results and those obtained in our “effective” LLL approach.

II. MAGNETIZATION IN THE LOWEST LANDAU LEVEL APPROXIMATION

A. The basics

A fully two-dimensional alternative to the approaches mentioned in the introduction for solving the Ginzburg-Landau equations has been proposed by one of the authors \([15]\). One expands the order parameter in a set of functions that are the lowest level solutions of the linearized version of Eq. \([1]\):

\[
\Psi(\vec{r}) = \sum_{L=0}^{\infty} C_L \mathcal{Y}_L(\vec{r}) = \sum_{L=0}^{\infty} C_L \frac{1}{\sqrt{2\pi}} e^{-iL\theta} \Phi_L(r).
\]

It is essential for the radial part of the expansion functions, \( \Phi_L(r) \), to obey the standard boundary conditions of zero current through the surface \([13]\). The presence of the boundary implies that, for large enough angular momentum \( L \), an infinite number of bulk Landau levels participate (numerical work is usually required at this point). Nevertheless, we will still refer to this expansion as a LLL expansion. The other central idea behind this approximation is to consider the magnetic induction to be spatially constant, \( B \), but not necessarily equal to the external field \( H \), as it has been usually the case in the context of the LLL approximation for bulk systems \([3]\). The magnetic induction \( B \) sets the scale of the magnetic length \( \ell = \sqrt{e^*\hbar/cB} \) and, thus, the scale of the expansion functions. Obviously, to consider a constant value of \( B \) across the superconductor is strictly valid only in the limit \( B \to H \) and it does not capture correctly the magnetic induction profile in the Meissner phase. On the other hand, the LLL approximation is not expected to be valid below \( B \approx 0.3H_{c2} \), although this naive expectation has been shown to be higher than the real limit \([13]\). Still, as we will show in the next section, this approach gives qualitative and quantitatively good results for small and thin Aluminum disks in the whole range of fields as long as \( \tilde{\kappa} \) is adequately chosen.

When one substitutes this expansion into the Ginzburg-Landau functional \([3]\) one obtains an algebraic expression in terms of the complex coefficients \( C_L \):

\[
G_s = G_n + \sum_{L=0}^{\infty} (\alpha + \epsilon_L)|C_L|^2 + \frac{\beta}{2} \sum_{L_1, L_2, L_3, L_4=0}^{\infty} C^*_{L_1} C_{L_2} C_{L_3} C_{L_4} \int d\vec{r} \mathcal{Y}^*_{L_1} \mathcal{Y}^*_{L_2} \mathcal{Y}_{L_3} \mathcal{Y}_{L_4} + V(B-H)^2/8\pi.
\]

The effect of the surface on the condensate is cast in the Cooper pair energy \( \epsilon_L \), which, unlike bulk systems,
presents a dip for states $L$ close to the surface $\xi$, and in the “interaction” integrals. In this form the Ginzburg-Landau functional can be easily minimized or, in general, extremized with respect to $C_L$ and $B$. This can even be done analytically when not too many terms in the expansion (4) participate [15,16].

B. Bulk magnetization

It is illustrative at this point to consider a bulk system within our LLL approximation. Let us assume an ellipsoidal form or a long cylinder for which no demagnetization exists ($N = 0$). In the spirit of Abrikosov’s seminal work, it is straightforward to obtain an expression for the magnetic induction

$$B = H - \frac{H_{c2} - H}{2\beta_A\kappa^2 - 1},$$

and the free energy density

$$G_s - G_n/V = \frac{(H_{c2} - H)^2}{8\pi(2\beta_A\kappa^2 - 1)}.$$  

In the above expressions $V$ is the volume and we have made use of the Abrikosov parameter $\beta_A = \langle \Psi(\vec{r})^2 \rangle / \langle \Psi(\vec{r})^4 \rangle$ [17] which measures the uniformity of the superconducting density. Notice that the expression that we obtain for the magnetic induction is different from that obtained by Abrikosov [17]:

$$\langle h(\vec{r}) \rangle = H - \frac{H_{c2} - H}{2\beta_A\kappa^2 - \beta_A}.$$  

The difference lies essentially in two facts. (i) We consider a uniform magnetic induction $B$ while, in Abrikosov’s work, $h(\vec{r})$ is not uniform; instead, the spatial average $\langle h(\vec{r}) \rangle$ is calculated. (ii) Our LLL expansion functions are evaluated for a value of $B$ that needs to be determined after minimization; on the other hand, Abrikosov’s expansion functions are calculated at the upper critical field $H_{c2}$. These two significant differences account for the difference in the magnetization expressions although, our approach is considerably simpler. Considering that when one minimizes with respect to the structure of the vortex lattice one obtains $\beta_A \approx 1.16$, which corresponds to a triangular arrangement of vortices [1], both approaches give a remarkably similar result for the bulk magnetization.

III. DEMAGNETIZATION IN THE LOWEST LANDAU LEVEL APPROXIMATION

We now compare our LLL approximation with the exact magnetization results for mesoscopic disks of typical sizes $R \sim \xi$. Here, only one state $\Upsilon_L$ participates in the expansion (4) for any value of the external field or, in other words, the order parameter has a well-defined value of the angular momentum and forms a giant vortex [20]. Minimizing Eq. 5 with respect to $C_L$ one obtains

$$G_s - G_n = -\frac{[1 - B\epsilon_L]^2}{\kappa^2 BR^2} + (B - H)^2,$$  

where the minimal value of $B$ is obtained numerically. The free energy is given in units of $H_{c2}^2 V/8\pi$, $\epsilon_L(B)$ is given in units of the LLL energy $\hbar\omega_c/2$ (with $\omega_c = eB/m^*c$), $R$ is in units of $\xi$, and $B$ and $H$ are given in units of $H_{c2}$. Notice that in the denominator of the above expression we have written $\kappa$ instead of $\kappa$. As mentioned in the introduction, finite thickness affects the real value of $\kappa$ through the expression $\lambda^2/d$, but also does demagnetization in a more complicated way. We propose to consider $\kappa$ as an effective parameter that depends on the geometry, i.e., on $R$ and $d$ in such a way that the exact magnetic response is approximately reproduced (within the limits of the LLL approximation). Figure 1 shows the equilibrium [21] magnetization of a disk characterized by $R = 3\xi$ and $\kappa = 1$ for four different values of the disk thickness: $d/\xi = 0.01(a), 0.1(b), 0.5(c), 1.0(d)$. Dashed lines correspond to the LLL approximation and solid lines to the three-dimensional numerical solution of Eqs. (1) and (3).

![FIG. 1. Equilibrium magnetization of a disk of radius $R = 3\xi$ and bulk $\kappa = 1$ for four different thicknesses: $d/\xi = 0.01(a), 0.1(b), 0.5(c), 1.0(d)$. Solid lines correspond to the full numerical solution and dashed lines correspond to the LLL approximation using different values of $\kappa$: (a) $\kappa = 32$, (b) $\kappa = 6.1$, (c) $\kappa = 2.9$, and (d) $\kappa = 2.15$.](image)
it is a simple and unambiguous criterion for choosing \( \tilde{\kappa} \) and we will use it in what follows. In this figure one can appreciate that the magnetization obtained in the LLL approximation follows closely the “exact” magnetic response, both in magnitude and position of the jumps. As the thickness increases and the disk resembles more and more a long cylinder, the value of \( \tilde{\kappa} \) approaches the intrinsic one and the effective LLL approximation works better. Notice, however, how demagnetization manifests itself in that the value of \( \tilde{\kappa} \) \((a) \tilde{\kappa} = 32, \ (b) \tilde{\kappa} = 6.1, \ (c) \tilde{\kappa} = 2.9, \) and \( (d) \tilde{\kappa} = 2.15 \)\) is different from the one expected from the simple scaling \( \lambda^2 / d \) (see Fig. 4). This manifestation is more clear for larger disks as will be shown below.

Confident in the validity of the effective LLL approximation to describe the actual magnetic response of small type-II disks (we have repeated the analysis just described for different disk radii with similar encouraging results), we now try to push it further to include the description of type-I mesoscopic disks like those of Geim’s experiments \[16\]. Figure 2 shows the equilibrium magnetic response of an Aluminum disk with \( d = 0.1\xi \) for four different values of \( R / \xi = 2.0(a), 3.0(b), 4.0(c), 5.0(d) \). The same convention as before for fixing \( \tilde{\kappa} \) has been followed (dashed lines). Although for \( d = 0.1\xi \) a renormalized \( \kappa \) is already expected, it is clear from the results that, as the radius increases, one needs to increase \( \tilde{\kappa} \) \((a) \tilde{\kappa} = 1.45, \ (b) \tilde{\kappa} = 2.2, \ (c) \tilde{\kappa} = 2.7, \) and \( (d) \tilde{\kappa} = 2.7 \) in order to match the position of the magnetization jumps and, to some extent, the magnitude of them. The fact that we have to use higher values of \( \tilde{\kappa} \) as the disk radius increases (whereas the thickness remains constant) constitutes again a manifestation of demagnetizing effects which decrease the magnetic response of the disk as the radius grows bigger (i.e., the disk resembles more and more a thin film in perpendicular geometry). Even for the larger disk considered the agreement of the LLL results with the exact solution is fairly satisfactory. However, now it becomes more difficult to match the overall magnitude of the magnetization in the whole range of fields with a single value of \( \tilde{\kappa} \). At large fields, where the LLL approximation is expected to work better, the agreement can be improved considering a different set of values for \( \tilde{\kappa} \) (dotted lines). Unlike the previous case, the fitting criterion is a little ambiguous at high fields, but one can clearly appreciate the improvement. As Fig. 4 shows, in all cases considered \( \tilde{\kappa} \) behaves almost linearly with \( R \). One last caveat: Multiple-vortex states \[16\] could be stable in the condensate of the larger disk, but we are not considering here such a possibility since it is not relevant for our discussion.

Finally, Fig. 3 shows the equilibrium magnetization of an Aluminum disk with \( d = 0.5\xi \) for different radii: \( R / \xi = 2.0(a), 3.0(b), 4.0(c), 5.0(d) \). Now it becomes more difficult to find an effective \( \kappa \) that allows the LLL approximation to reproduce, even if only qualitatively, the exact results. This is not surprising since, for such thickness, the type-I behavior of Aluminum manifests itself more pronouncedly. Still, the number of magnetization jumps is approximately reproduced in all cases shown, but the the magnitude of the magnetization is appreciably underestimated with the set of values for \( \tilde{\kappa} \) that fit the Meissner phase (dashed lines). Again, a different set can be chosen so that the high-field response is approximately

\[ \text{FIG. 2. Equilibrium magnetization of a disk of thickness } d = 0.1\xi \text{ and bulk } \kappa = 0.28 \text{ for different radii: } R / \xi = 2.0(a), 3.0(b), 4.0(c), 5.0(d). \] 

\[ \text{FIG. 3. Equilibrium magnetization of a disk of thickness } d = 0.5\xi \text{ and bulk } \kappa = 0.28 \text{ for different radii: } R / \xi = 2.0(a), 3.0(b), 4.0(c), 5.0(d). \]
reproduced (dotted lines), although the agreement is not completely satisfactory for large disks.

As conclusion, and for illustration purposes, we plot in Fig. 4 \( \tilde{\kappa} \) as a function of thickness and radius, including the cases considered above. With the help of the calculations presented in this paper, we leave it to the reader to judge by himself when the effective LLL can be expected from the scaling \( \lambda^2 / d \). (b) \( \tilde{\kappa} \) as a function of \( R \) for \( \kappa = 0.28 \) and \( d = 0.1 \xi \) (upper curves) and \( d = 0.5 \xi \) (lower curves). Black symbols denote fitting to the Meissner phase and white symbols to the high-field magnetic response.

FIG. 4. (a) Black dots: \( \tilde{\kappa} \) as a function of the thickness for \( \kappa = 1 \) and \( R = 3 \xi \). White dots: \( \tilde{\kappa} \) expected from the scaling \( \lambda^2 / d \). (b) \( \tilde{\kappa} \) as a function of \( R \) for \( \kappa = 0.28 \) and \( d = 0.1 \xi \) (upper curves) and \( d = 0.5 \xi \) (lower curves). Black symbols denote fitting to the Meissner phase and white symbols to the high-field magnetic response.

ACKNOWLEDGMENTS

Part of this work was supported by the Flemish Science Foundation (FWO-Vl), the "Onderzoeksraad van de Universiteit Antwerpen", IUAP-IV, the European ESF-project on Vortex Matter, the Spanish CICYT under Grant No. 1FD97-1358, and the Generalitat Valenciana under Grant No. GV00-151-01. Discussions with E. Akkermans, A. Geim, and K. Mallick are gratefully acknowledged. JJP also acknowledges L. van Look for discussions and hospitality.

[1] See for instance M. Tinkham, *Introduction to Superconductivity*, 2nd Ed. (McGraw-Hill, New York, 1996).
[2] A. V. Kuznetsov, D. V. Eremenko, and V. N. Trofimov, Phys. Rev B 57, 5412 (1998).
[3] F. M. Araujo-Moreira, C. Navau, and A. Sanchez, cond-mat/9910063.
[4] E. H. Brandt, cond-mat/9904191, ibid., Phys. Rev B 58, 6523 (1998); ibid., Phys. Rev B 58, 6506 (1998); ibid., Phys. Rev B 54, 4246 (1996); E. H. Brandt, M. Indenbom, and A. Forkl, Europhys. Lett. 22, 735 (1993); J. R. Clem, and A. Sanchez, Phys. Rev B 50, 9355 (1994).
[5] A. K. Geim, I.V. Grigorieva, S.V. Dubonos, J.G.S. Lok, J.C. Maan, A.E. Filippov, and F.M. Peeters, Nature 390, 259 (1997); A. K. Geim, S.V. Dubonos, J.G.S. Lok, M. Henini, and J.C. Maan, Nature 396, 144 (1998); A. K. Geim, S. V. Dubonos, J. J. Palacios, I.V. Grigorieva, M. Henini, and J.J. Schermer, Phys. Rev. Lett. 85, 1528 (2000); A.K. Geim, S.V. Dubonos, I.V. Grigorieva, K.S. Novoselov, F.M. Peeters and V.A. Schweigert, Nature (London) 407, 55 (2000).
[6] V.V. Moshchalkov, L. Gelen, C. Strunk, R. Jonckheere, X. Qiu, C. Van Haesendonck, and Y. Bruynseraede, Nature (London) 373, 319 (1995).
[7] J. Pearl, Appl. Phys. Lett. 5, 65 (1964).
[8] N. Moussey, H. Courtois, and B. Panneteri, cond-mat/0007128.
[9] V. A. Schweigert, F. M. Peeters, and P. S. Deo, Phys. Rev. Lett. 81, 2783 (1998); P. S. Deo, V. A. Schweigert, and F. M. Peeters, Phys. Rev. B 59, 6039 (1999); P. S. Deo, V. A. Schweigert, F. M. Peeters, and A. K. Geim, Phys. Rev. Lett. 79, 4653 (1997); V. A. Schweigert and F. M. Peeters, Phys. Rev. B 57, 13 817 (1998).
[10] V. V. Moshchalkov, X. G. Qiu, and V. Bruyndoncx, Phys. Rev. B 55, 11 793 (1997); G. F. Zharkov, V. G. Zharkov, and A. Y. Zvetkov, cond-mat/0008217.
[11] E. Akkermans and K. Mallick, J. Phys. A 32, 7133 (1999); E. Akkermans, D. M. Gangardt, and K. Mallick, cond-mat/0005542.
[12] E. Akkermans, D. M. Gangardt, and K. Mallick, cond-mat/0008289.
[13] I. F. Herbut and Z. Tesanovic, Phys. Rev. Lett. 76, 4588 (1996).
[14] J. Hu and A. H. MacDonald, Phys. Rev. B 56, 2788 (1997).
[15] J. J. Palacios, Phys. Rev. Lett. 84, 1796 (2000); ibid., Phys. Rev. B 58, 5948 (1998); ibid., Physica B 256-258, 610 (1998); ibid., Phys. Rev. B 57, 10873 (1998).
[16] V. A. Schweigert and F. M. Peeters, Phys. Rev. Lett. 83, 2409 (1999); S.V. Yampolskii and F.M. Peeters, Phys. Rev. B 62, 9663 (2000).
[17] A.A. Abricosov, *Fundamentals of the Theory of Metals* (North-Holland, Amsterdam, 1988).
[18] P.G. de Gennes, *Superconductivity of Metals and Alloys* (Addison-Wesley, New York, 1994).
[19] D. Li and B. Rosenstein, Phys. Rev. B 60, 9704 (1999).
[20] For earlier work on giant vortices see H. J. Fink and A. G. Presson, Phys. Rev. 168, 399 (1968).
[21] Non-equilibrium magnetization curves (see Ref. [13]) could also be used for comparison purposes.

Electronic address: francois.peeters@ua.ac.be.