The Query Complexity of Correlated Equilibria∗†

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February 25, 2022

Abstract

We consider the complexity of finding a correlated equilibrium of an $n$-player game in a model that allows the algorithm to make queries on players’ payoffs at pure strategy profiles. Randomized regret-based dynamics are known to yield an approximate correlated equilibrium efficiently, namely, in time that is polynomial in the number of players $n$. Here we show that both randomization and approximation are necessary: no efficient deterministic algorithm can reach even an

∗Dedicated to the memory of Lloyd S. Shapley: a giant in the field, a pioneering and inspiring figure, a supportive teacher and mentor, and a friend. The combination of game theory with operations research, combinatorics, probability, and computer science—all present in this paper—has been a cornerstone of Lloyd Shapley’s work. Interestingly, the edge iso-perimetric inequality (Hart 1976) that we use here came about in order to solve a problem posed by Lloyd in 1974 in connection with the Banzhaf value.

†This version: December 2016. Previous versions: May 2013, September 2013 (Center for Rationality DP-647). Part of this research was carried out at Microsoft Research, Silicon Valley. We thank Parikshit Gopalan for helpful discussions leading to the proof of Theorem B, Yakov Babichenko, Kevin Leyton-Brown, Christos Papadimitriou, Tim Roughgarden, Eva Tardos, and Ricky Vohra for useful discussions, and the referees and editor for their careful reading and comments.

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approximate correlated equilibrium, and no efficient randomized algorithm can reach an exact correlated equilibrium. The results are obtained by bounding from below the number of payoff queries that are needed.

_JEL Classification codes: C8, C7_

1 Introduction

The computational complexity of various notions of equilibrium in games is of interest in many different models of computation. In the present paper we focus on the important concept of correlated equilibrium, introduced by Aumann (1974). Perhaps the most striking positive result in this vein is the surprising power of “regret-based” algorithms in finding (approximate) correlated equilibria. These algorithms are obtained from a large family of natural dynamics that converge to correlated equilibria in any game, by emulating the computations carried out by the players in the game; for these dynamics, see, e.g., Foster and Vohra (1997), Hart and Mas-Colell (2000, 2001), Blum and Mansour (2007), and also the books of Cesa-Bianchi and Lugosi (2006) and Hart and Mas-Colell (2013). Some of these algorithms have been shown to converge efficiently (i.e., “quickly”; see Cesa-Bianchi and Lugosi 2003, 2006 and the Remark at the end of the Introduction). This is in contrast to the fact that there is no natural dynamic converging to Nash equilibria (Hart and Mas-Colell 2003, 2006; 2013) and, in fact, no efficient algorithm is believed to exist either (see Nisan et al. 2007 [Chapter 2]).

Looking at these dynamics from a strictly computational point of view, they yield algorithms that take the payoff (or utility) functions \((u_1, ..., u_n)\) of \(n\) players as input, and produce an (approximate) correlated equilibrium as output. Assuming that each player has \(m\) pure strategies, the input size is \(n \cdot m^n\), and yet these algorithms run in time that is polynomial in \(n\) and \(m\)—which is sub-linear (even poly-logarithmic) in the input size. Even though the output size \(m^n\) (a probability for each \(n\)-tuple of strategies) is similar to the input size, these algorithms nevertheless produce an output whose support is

\[1\] As well as in \(1/\varepsilon\), the inverse of the approximation parameter.
also small: polynomial in $n$ and $m$. These regret-based algorithms need only “black box” access to the payoff functions, namely, the possibility of making a sequence of queries $u_i(s_1, \ldots, s_n)$ for pure strategy profiles $(s_1, \ldots, s_n)$.

However, the regret-based algorithms have two undesirable aspects: first, they are *randomized*, and second, they produce only an *approximate* equilibrium. The present paper asks whether these shortcomings can be fixed.

While for many problems with sub-linear algorithms it is clear that both randomization and approximation are required, this is not the case here. Usually, the necessity of randomized approximation is already implied by the verification of the result itself, also known as “certificate complexity” or “non-deterministic complexity.” However, correlated equilibria can be verified in time that is polynomial in the size of their support, which itself can be polynomial in $n$ and $m$, and so the certificate (non-deterministic) complexity of correlated equilibrium is small.

Another indication that exact or deterministic algorithms may be possible comes from the known linear programming (LP) based algorithms for correlated equilibria (Papadimitriou and Roughgarden 2008, Jiang and Leyton-Brown 2011) that produce, in time that is polynomial in $m$ and $n$, a correlated equilibrium *exactly and deterministically*. However, this is obtained in the stronger model where the algorithm may query the payoff black boxes also at profiles of *mixed* strategies. We note that an even stronger query model...
appears in the “communication complexity” setup, where any function of the payoff matrices may be queried, e.g., Hart and Mansour (2010). In recent work, Babichenko and Barman (2013) showed that every deterministic algorithm that finds an exact correlated equilibrium requires a number of pure queries that is exponential in the number of players \( n \). This raises the question of whether the success of the regret-based algorithms is due to the power of randomization (which is known to be critical for achieving low regret), or to the relaxation that allows approximate equilibria rather than exact ones.

In the present paper we show that it is actually both of these, even when we limit ourselves to bi-strategy games \(^6\) (i.e., \( m = 2 \)). The following lower bounds apply in the query model (or “decision tree model”), which makes no assumptions on the algorithm (in particular not restricting its computational power) beyond the fact that it has black-box access to the payoff functions, which provide the values \( u_i(s_1, \ldots, s_n) \) for an adaptively chosen sequence of pure strategy profiles \( (s_1, \ldots, s_n) \) and players \( i \). In particular, this model abstracts away from issues having to do with the way that the input is accessed. \(^8\)

Our results are:

**Theorem A.** Every deterministic algorithm that finds a 1/2-approximate correlated equilibrium in every \( n \)-person bi-strategy game with payoffs in \( \{0, 1\} \) requires \( 2^{\Omega(n)} \) queries in the worst case. (Of course, this holds a fortiori for an \( \epsilon \)-approximate correlated equilibrium

\(^6\)For bi-strategy games coarse correlated equilibria (also known as “Hannan equilibria”) are equivalent to correlated equilibria, and so the lower bounds obtained here apply also to the easier problem of finding coarse correlated equilibria.

\(^7\)Since we are proving lower bounds, the strong (perhaps unrealistic) model only strengthens our results. We should note that all the regret-based algorithms mentioned above are actually effective and not only do they make few queries but also the rest of the computation requires only polynomial time.

\(^8\)Input queries presume “random access memory”; without it, all algorithms become at least linear in the input size, and thus exponential in \( n \).

\(^9\)We use the standard notations: \( f(n) = O(g(n)) \) when there is \( c > 0 \) such that \( f(n) \leq cg(n) \) for all \( n \), and \( f(n) = \Omega(g(n)) \) when there is \( c > 0 \) such that \( f(n) \geq cg(n) \) for all \( n \). Also, \( f(n) = o(g(n)) \) if \( f(n)/g(n) \to 0 \) as \( n \to \infty \).
for any $0 < \varepsilon \leq 1/2$.)

**Theorem B.** Every algorithm (randomized or deterministic) that finds an *exact* correlated equilibrium in every $n$-person bi-strategy game with payoffs specified as $b$-bit integers with $b = \Omega(n)$ incurs an $2^{\Omega(n)}$ expected cost in the worst case.

In Theorem B the “cost” includes the number of queries together with the size of the support of the output produced; see Section 2 (d) for a discussion of this issue. We do not know whether the result continues to hold for smaller payoffs, such as payoffs with $b = o(n)$ bits, or payoffs in $\{0, 1\}$ (i.e., $b = 1$); see Section 2 (e).

The following table summarizes the results on the number of queries—we refer to these as “query complexity bounds”:

| Algorithm                         | Randomized | Deterministic |
|-----------------------------------|------------|---------------|
| **Approximate Corr Eq**           | $O(n \log n)$ (regret-based) | $2^{\Omega(n)}$ (Theorem A) |
| **Exact Corr Eq**                 | $2^{\Omega(n)}$ (Theorem B) | $2^{\Omega(n)}$ (B&B 2013) |

**Remark:** Regret-based randomized algorithms for computing approximate correlated equilibria. Assume that each player has at most $m$ pure strategies. In Cesa-Bianchi and Lugosi (2003; 2006, Remark 7.6 in Section 7.4) it is shown that by running a regret-based procedure one finds, with probability at least $1/2$, an $\varepsilon$-correlated equilibrium in at most

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"Approximate Corr Eq" stands for $\varepsilon$-approximate correlated equilibrium for fixed (small enough) $\varepsilon > 0$; the number of strategies $m$ is also fixed ($m = 2$); and the “regret-based” algorithms are discussed in the Remark immediately below.
\[ T = 16 \ln(2nm)/\varepsilon^2 \] steps. Because each player makes no more than \( m \) payoff queries at each step, the total number of queries is \( \leq nmT \). After these \( T \) steps one checks whether the regrets are all \( \leq \varepsilon \) (no further queries are needed here, as the regrets are computed all along) and, if they are not, one starts the procedure afresh. Because the probability of success is at least \( 1/2 \), the expected number of repetitions is 2, and so the expected total number of queries is a most \( 2nmT = 32nm \ln(2nm)/\varepsilon^2 \), which is \( O(n \log n) \) for fixed \( m \) and \( \varepsilon > 0 \).

2 Extensions, Variations, and Open Problems

In this section we discuss a number of relevant issues and open problems.

(a) Query complexity of linear programming

Since computing a correlated equilibrium (CE) in an \( n \)-person bi-strategy game is a linear programming (LP) problem with \( N = 2^n \) nonnegative unknowns (the probabilities of the \( 2^n \) strategy profiles) and \( 2n + 1 \) linear constraints (two inequalities per player; in addition, the probabilities sum up to 1), it is appropriate to ask what is the query complexity of general LP problems of this size. Here, one queries the coefficients appearing in the various constraints.

Consider a linear programming problem with \( N \) nonnegative unknowns and just 2 constraints:

\[
\sum_{j=1}^{N} a_j x_j \geq 0, \quad \sum_{j=1}^{N} x_j = 1, \quad x \geq 0.
\]

(1)

Claim 1 Finding a 1/2-approximate solution to problem (1) requires \( \Omega(N) \) queries on the coefficients \( (a_j)_{j=1,...,N} \) in the worst case.

Proof. For every \( k = 1, ..., N \) let \( \Pi_k \) be the instance of problem (1) with \( a_k = 1 \) and \( a_j = -3 \) for all \( j \neq k \). For every \( \varepsilon \geq 0 \), all the \( \varepsilon \)-approximate solutions of \( \Pi_k \) (where the inequality is relaxed to \( \sum_{j=1}^{N} a_j x_j \geq -\varepsilon \)) satisfy \( x_k \geq (3-\varepsilon)/4 \) and \( \sum_{j \neq k} x_j \leq (1+\varepsilon)/4 \), and so for \( \varepsilon \leq 1/2 \) we get \( x_k \geq 5/8 > \)
\[ 3/8 \geq x_j \text{ for all } j \neq k. \] Therefore the algorithm must find \( k \) in \( \{1, ..., N\} \), which requires \( \Omega(N) \) queries (whether deterministic or randomized). ■

The query complexity of an LP of size comparable to the correlated equilibrium LP is thus \( \Omega(N) = \Omega(2^n) \), i.e., exponential in \( n \). This immediately implies that regret-based algorithms cannot be efficiently translated to general LP problems.

**(b) Correlated equilibrium as special linear programming**

As seen in (a) above, the correlated equilibrium LP must have a special structure that distinguishes it from general LP problems of similar size. What is that structure, and how does it help to get the fast (i.e., polynomial in \( n \)) convergence of randomized algorithms to approximate correlated equilibria?

One feature is that the dual LP of the correlated equilibrium LP decomposes into \( n \) separate LP problems, each one of size \( m \times m \) (where \( m \) is the number of strategies of each player). This “dual separability” feature lies at the basis of the existence proof of Hart and Schmeidler (1989), is used in the algorithm of Papadimitriou and Roughgarden (2008) and Jiang and Leyton-Brown (2011), and translates to “uncoupledness” in the world of game dynamics (cf. Hart and Mas-Colell 2003, 2006; 2013). While this feature distinguishes the correlated equilibrium LP from other LP problems, it does not explain why it helps in only one of the four cases (see the table at the end of the Introduction). We thus have:

**Open Problem 1.** Why does the special structure of the correlated equilibrium LP help only for randomized algorithms yielding approximate solutions (where the query complexity is polynomial rather than exponential in \( n \)), and not in any of the other cases (where the query complexity is exponential in \( n \))?

**(c) Support size of approximate correlated equilibria**

What is the minimal support size that guarantees existence of an \( \varepsilon \)-approximate correlated equilibrium in every \( n \)-person bi-strategy game? An \( \varepsilon \)-correlated equilibrium is just an \( \varepsilon \)-optimal strategy in a two-person zero-
sum game where the opponent has 2\(n\) strategies (that correspond to the correlated equilibrium inequalities; this zero-sum game is the “auxilliary game” of Hart and Schmeidler 1989). It follows, by using the result of Lipton and Young (1994), that there always exist \(\varepsilon\)-correlated equilibria with uniform support of size\(^\text{11} k = \log n / (2\varepsilon^2)\).

As for dynamics, it has been shown (Cesa-Bianchi and Lugosi 2003, 2006; see the Remark at the end of the Introduction) that there are regret-based procedures that in \(T = 16 \ln n (4n) / \varepsilon^2\) steps reach an \(\varepsilon\)-correlated equilibrium with probability at least 1/2; moreover, the resulting \(\varepsilon\)-correlated equilibrium has uniform support of size \(T\) (it gives equal weight of \(1/T\) to each one of the \(n\)-tuples of strategies played in the first \(T\) periods). The fact that \(T\) is no more than a constant multiple of \(k\) is remarkable, as it implies that regret-based algorithms yield \(\varepsilon\)-correlated equilibria with support that is essentially minimal, and so they converge as fast as theoretically possible (up to a constant factor). In terms of queries, this translates to an upper bound of \(O(n \log n)\) queries (because each period every one of the \(n\) players makes 2 queries; cf. Goldberg and Roth 2014). For improved bounds on the support size, see Babichenko, Barman and Peretz (2014).

(d) Cost does not include the size of the support of the output

As we have seen in the Introduction, if the output of the algorithm is a distribution that has small support (i.e., polynomial in \(n\)), then computing payoffs and verifying the correlated equilibrium inequalities requires only polynomially many queries. By contrast, if the support is large (i.e., exponential in \(n\)) these computations require exponentially many queries (and that is so even if the representation is succinct, e.g., the product of uniform mixed strategies). Therefore, our model counts the size of the support as part of the cost. Interestingly, for Theorem A it turns out that this issue does not matter (we show this in Section 3.3). But it may well matter for Theorem B. When the output’s support size is not counted, our proofs show that the number of queries is \(2^{\Omega(n)}\) in the worst case for randomized algo-

\(^{11}\)Uniform support of size \(k^n\) means that the support consists of \(k\) strategy profiles, not necessarily distinct, each one with weight \(1/k\) (alternatively, the probability weights are all integer multiples of \(1/k\)).
rithms that yield correct answers with probability one (see footnote 17 in Section 5). However, we do not know whether this is so also for randomized algorithms that are required to yield correct answers with high probability, and possibly incorrect answers otherwise (we conjecture that it is).

**Open Problem 2.** Does the result of Theorem B hold also for randomized algorithms that yield correct answers with high probability (rather than with probability one) and for which the size of the support of the output is not counted?

(e) Exact correlated equilibria for games with small payoffs

Our proof of Theorem B uses, for the worst case, games whose payoffs range up to $2^{\Omega(n)}$; we do not know whether this requirement is needed, and so we have:

**Open Problem 3.** Does the result of Theorem B hold also for payoffs with $b = o(n)$ bits, and even for payoffs in \{0, 1\} (i.e., $b = 1$)?

(f) Query complexity of Nash equilibria

The lower bounds of Theorems A and B apply also to the harder problem of finding a Nash equilibrium (since every Nash equilibrium is also a correlated equilibrium), but it is not difficult to see that, in contrast to the correlated case, for Nash equilibria these bounds are “trivial” as they apply also to the verification complexity. However, if we allow both randomization and approximation then the verification complexity of Nash equilibrium becomes polynomial in $n$ and $m$ (since we can verify that each player $i$ is approximately best-responding by sampling from the distributions of the other players). In earlier versions of the present paper we raised the following question:

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12Consider $n/2$ pairs of players, where each pair of players is playing their own matching pennies game. Clearly the unique Nash equilibrium has every player uniformly randomizing between his two strategies. However, verifying this equilibrium—even allowing randomized verification—requires looking at essentially all $2^n$ strategy profiles since if an adversary had changed the utility of a player in any single profile, this would no longer be an equilibrium. For deterministic verification the adversary could change the utility at all non-queried profiles so that this would no longer be even an approximate equilibrium.
Problem 4. Fix $\varepsilon > 0$; does there exist a randomized algorithm, with only black-box access to the players’ payoff functions, that finds an $\varepsilon$-Nash equilibrium for every $n$-player $m$-strategy game whose running time is polynomial in $n$ and $m$?

This problem was recently solved by Babichenko (2014) and Chen, Cheng, and Tang (2015), who proved that the answer is negative: the query complexity of approximate Nash equilibria is exponential.

3 Model and Preliminaries

3.1 Correlated Equilibrium

The setup and notations are standard:

- **Game:** We will consider games between $n$ players, where each player’s pure strategy set is $\{0, 1\}$. Our players’ payoffs are normalized between 0 and 1, and so a game is given by $n$ payoff functions $u_1, \ldots, u_n$, where for each $i$ we have $u_i : \{0, 1\}^n \rightarrow [0, 1]$.

- **Notation:** For a (pure) strategy profile $v \in \{0, 1\}^n$, we use two notations: $v^{(i)} \in \{0, 1\}^n$ denotes the result of flipping the $i$’th bit of $v$ (i.e., where player $i$ plays $1 - v_i$ and all other $j$ play $v_j$); and $v^{i \rightarrow b} \in \{0, 1\}^n$ denotes $v$ with the $i$’th bit set to $b$ (i.e., if $v_i = b$ then $v^{i \rightarrow b} = v$, and otherwise $v^{i \rightarrow b} = v^{(i)}$).

- **Regret:** Let $x$ be a probability distribution over the set of (pure) strategy profiles, i.e., $x : \{0, 1\}^n \rightarrow [0, 1]$ with $\sum_{v \in \{0, 1\}^n} x(v) = 1$. Take a player $1 \leq i \leq n$ and a possible strategy $b \in \{0, 1\}$ for $i$. We say that the regret of $i$ for not playing $b$ is:

$$\text{Regret}_{i \rightarrow b}(x) = \sum_{v \in \{0, 1\}^n} x(v)u_i(v^{i \rightarrow b}) - \sum_{v \in \{0, 1\}^n} x(v)u_i(v).$$

- **Correlated Equilibrium:** For $\varepsilon \geq 0$, we say that $x$ is an $\varepsilon$-correlated
equilibrium \((\varepsilon\text{-CE})\) if for all \(i = 1, \ldots, n\) and for all \(b \in \{0, 1\}\) we have \(\text{Regret}_{i \rightarrow b}(x) \leq \varepsilon\). When \(\varepsilon = 0\) it is a \textit{correlated equilibrium (CE)}.

3.2 The Computational Problem

The computational problem is as follows.

The (Approximate) Correlated Equilibrium (CE) problem:

- **Input:** The payoff functions \(u_i : \{0, 1\}^n \rightarrow [0, 1]\) for \(i = 1, \ldots, n\) and the desired approximation parameter \(\varepsilon \geq 0\).

- **Queries:** We assume only black-box access to the payoffs: a query is of the form \(v \in \{0, 1\}^n\), and the reply to the query is the \(n\)-tuple of player payoffs: \(u_1(v), \ldots, u_n(v)\).

- **Output:** An \(\varepsilon\)-correlated equilibrium of the game. The equilibrium is given by listing the probability \(x(v)\) for every strategy profile \(v\) in its support.

- **Cost:** The cost on a given input \(u_1, \ldots, u_n\) and \(\varepsilon \geq 0\) is the total number of queries made plus the size of the support of the equilibrium produced. The cost of the algorithm is the worst-case cost over all \(n\)-tuples of payoff functions.

3.3 Concise Equilibrium Representations

Notice that in the definition of the CE problem we counted the size of the support of the produced equilibrium toward the cost of the algorithm. We could alternatively talk about the \textit{weak-CE problem} where the algorithm is allowed to produce an equilibrium with arbitrarily large support whose size is not counted as part of the cost. Practically, the algorithm could use some concise representation of the equilibrium, e.g., some mixture of product distributions. We show however that this would not make the problem significantly easier, as any approximate CE algorithm that makes a small
Lemma 2 Given an algorithm that solves the weak-CE problem with error \( \varepsilon \) in \( T \) queries, there exists an algorithm that solves the (strong) CE problem with error \( O(\sqrt{\varepsilon}) \) and cost (including support size) \( O(T) \).

We first bound the probability that a CE algorithm may assign to unqueried profiles.

Lemma 3 Consider any \( \varepsilon \)-CE algorithm that on some input \((u_1, ..., u_n)\) queries a set \( Q \) of profiles and outputs an \( \varepsilon \)-CE \( x \). Denote \( Q' = \{ v | v \in Q \text{ or } v^{(1)} \in Q \} \) and \( \alpha = \max_{v \in Q'} u_1(v) \); then \( x \) puts probability at most \( 2(\alpha + \varepsilon) \) outside \( Q' \), i.e., \( \sum_{v \notin Q'} x(v) \leq 2(\alpha + \varepsilon) \).

Proof. Assume by way of contradiction that this is not the case, and furthermore assume without loss of generality that at least half of this weight is on \( v \)'s with \( v_1 = 0 \), i.e., \( \sum_{v \notin Q' \text{ and } v_1 = 0} x(v) > \alpha + \varepsilon \). Now consider changing the input payoffs in two different ways without changing the queried profiles. The first way just assigns \( u_1(v) = 0 \) for all \( v \notin Q' \). In the second way we put \( u_1(v) = 0 \) for all \( v \notin Q' \) with \( v_1 = 0 \) and \( u_1(v) = 1 \) for \( v \notin Q' \) with \( v_1 = 1 \). Clearly the gap between \( \text{Regret}_{1 \rightarrow 1} \) for these two cases is exactly \( \sum_{v \notin Q' \text{ and } v_1 = 0} x(v) > \alpha + \varepsilon \). However, notice that since \( \alpha = \max_{v \in Q'} u_1(v) \), in the first way, \( u_1(v) \) is bounded by \( \alpha \) for all \( v \) and thus \( |\text{Regret}_{1 \rightarrow 1}| \leq \alpha \); in the second way we therefore have \( \text{Regret}_{1 \rightarrow 1} > \varepsilon \). This contradicts the correctness of the \( \varepsilon \)-CE algorithm on the input obtained by the second way of changing the original input. □

We now complete the proof of the computational equivalence between the two versions of the approximate CE problem.

Proof of Lemma 2. Let us first run the weak-CE algorithm on \( \alpha \)-scaled payoffs \( u'_i(v) = \alpha u_i(v) \) to obtain an \( \varepsilon \)-CE \( x'(v) \) for the \( u \)'s (with \( \alpha \) to be determined below). By scaling back we have that \( x' \) is an \( (\varepsilon/\alpha) \)-CE for the original \( u \)'s: \( \sum_{v \in \{0,1\}^n} x'(v)(u_i(v^{i \rightarrow b}) - u_i(v)) \leq \varepsilon/\alpha \). As in Lemma 3...
denote $Q' = \{v | v \in Q \text{ or } v^{(1)} \in Q\}$, where $Q$ is the set of queries made by the algorithm. Our output will be $x(v) = 0$ for $v \notin Q'$ and $x(v) = \beta x'(v)$ for $v \in Q'$, where $\beta = 1/\sum_{v \in Q} x'(v)$ is the scaling factor ensuring that $\sum_{v \in \{0,1\}^n} x(v) = 1$. By Lemma 3, $\sum_{v \notin Q} x'(v) \leq 2(\alpha + \varepsilon)$, and so $\beta \leq 1/(1 - 2(\alpha + \varepsilon))$ and

$$\sum_{v \in \{0,1\}^n} x(v)(u_i(v^{i\to b}) - u_i(v)) = \beta \sum_{v \in Q'} x'(v)(u_i(v^{i\to b}) - u_i(v))$$

$$\leq \beta \sum_{v \in \{0,1\}^n} x'(v)(u_i(v^{i\to b}) - u_i(v))$$

$$+ \beta \sum_{v \notin Q'} x'(v)(u_i(v^{i\to b}) - u_i(v))$$

$$\leq \frac{1}{1 - 2(\alpha + \varepsilon)} \left( \frac{\varepsilon}{\alpha} + 2(\alpha + \varepsilon) \right)$$

Choosing $\alpha = O(\sqrt{\varepsilon})$ completes the proof. ■

4 Deterministic Algorithms for Approximate Correlated Equilibria

This section proves that deterministic algorithms require exponentially many queries to compute even an approximate equilibrium. Notice that due to Lemma 2 the same is also implied for the weak-CE problem that allows arbitrary concise representations of the output. Our result here is:

**Theorem A.** Every deterministic algorithm that finds a $1/2$-approximate correlated equilibrium in every $n$-person bi-strategy game with payoffs in $\{0,1\}$ requires $2^{\Omega(n)}$ queries in the worst case.

4.1 The Approximate Sink Problem

Our lower bound here will be based on analyzing the following combinatorial problem on the (Boolean) hypercube $\{0,1\}^n$.

The “Approximate Sink” (AS) problem:
• **Input:** A labeling of the edges of the hypercube by directions; i.e., for every edge \((v, v^{(i)})\) we have a weight (or direction) \(R(v, v^{(i)}) \in \{-1, 1\}\) such that \(R(v, v^{(i)}) = -R(v^{(i)}, v)\). We interpret \(R(v, v^{(i)}) = 1\) as the edge going from \(v\) to \(v^{(i)}\), and \(R(v^{(i)}, v) = -1\) as the edge going from \(v^{(i)}\) to \(v\).

• **Queries:** The algorithm queries a vertex \(v \in \{0, 1\}^n\) and gets the directions \(R(v, v^{(i)})\) of all edges adjacent to \(v\).

• **Output:** The algorithm wins when it queries a vertex \(v\) with in-degree no less than \(n/4\), i.e., \(\sum_i R(v, v^{(i)}) \leq n/2\).

We show that exponentially many queries are needed in order to find such a vertex. This implies Theorem A due to the following simple reduction:

**Lemma 4** If there exists a deterministic algorithm that in every game with payoffs in \(\{0, 1\}\) finds a 1/2-CE with at most \(T\) queries, then the AS problem can be solved with at most \(T\) queries.

**Proof.** Given an AS instance we build a CE instance with \(R(v, v^{(i)}) = u_i(v^{(i)}) - u_i(v)\) and run the approximate CE algorithm on it, translating every query the CE algorithm makes to an AS query. For \(R(v, v^{(i)}) = 1\) we set \(u_i(v^{(i)}) = 1\) and \(u_i(v) = 0\), while for \(R(v, v^{(i)}) = -1\) we set \(u_i(v^{(i)}) = 0\) and \(u_i(v) = 1\). Under this mapping, when the CE algorithm makes a query \((u_1(v), ..., u_n(v))\) it is immediately translated to the same query \(v\) on the original AS instance, and \(R(v, v^{(i)}) = 1\) means \(u_i(v) = 0\) while \(R(v, v^{(i)}) = -1\) means \(u_i(v) = 1\). When the CE algorithm produces an approximate equilibrium \(x\), we continue by querying all the profiles in the support of \(x\), whose number was, by definition, already counted toward the cost of the CE algorithm.

\[\sum_i R(v, v^{(i)})\] is the net outflow through \(v\), i.e., the out-degree \(O(v)\) of \(v\) minus its in-degree \(I(v)\); because \(O(v) + I(v) = n\), we have \(I(v) \geq n/4\) iff \(\sum_i R(v, v^{(i)}) \leq n/2\).

Edges are thus pointed in the direction of increasing payoff (i.e., positive regret): from \(v\) to \(v^{(i)}\) when \(u_i(v^{(i)}) > u_i(v)\), and from \(v^{(i)}\) to \(v\) when \(u_i(v) > u_i(v^{(i)})\).
Now take a 1/2-equilibrium output by the CE algorithm. Summing up the inequalities of the CE we get: 
\[ \sum_{i,b} \text{Regret}_{i \rightarrow b} = \sum_v x(v) \sum_i (u_i(v^{(i)}) - u_i(v)) \leq n/2, \] which implies that for some \( v \) in the support of \( x \) we have 
\[ \sum_i R(v, v^{(i)}) = \sum_i (u_i(v^{(i)}) - u_i(v)) \leq n/2, \] as needed. ■

4.2 Polite Algorithms

We now prove the lower bound for the AS problem. The core of our argument is to show that every relevant algorithm for the AS problem (see Lemma 9) can be transformed into an algorithm of the following form, without significantly increasing the number of queries.

**Definition 5** We call an algorithm for the AS problem polite if whenever a vertex \( v \) is queried, at least \( 3n/4 \) of its neighbors in the hypercube have not yet been queried.

It is quite easy to show that polite algorithms cannot solve AS.

**Lemma 6** No deterministic polite algorithm can solve the AS problem.

**Proof.** We provide an adversary argument: whenever a vertex is queried, the adversary answers with all edges that were previously not committed to pointing out. Since there are at least \( 3n/4 \) such edges, the in-degree is at most \( n/4 \), and so this vertex cannot be an answer. ■

4.3 Closure

To convert an algorithm to a polite form, we need to make sure that vertices are queried before too many of their neighbors are. We use the following notion:

**Definition 7** For a set \( V \subseteq \{0, 1\}^n \) of vertices in the hypercube, we define its closure \( V^* \subseteq \{0, 1\}^n \) to be the smallest set containing \( V \) such that for every \( v \not\in V^* \) at most \( n/8 \) of its neighbors are in \( V^* \).
The closure is well defined and can be obtained by starting with $V^* = V$ and repeatedly adding to $V^*$ any vertex $v$ that has more than $n/8$ of its neighbors already in $V^*$. Clearly the order of additions does not matter since the number of neighbors a vertex has in $V^*$ only increases as other vertices are added to $V^*$. When the process stops every $v \notin V^*$ has at most $n/8$ of its neighbors in $V^*$.

The point is that we will not need to continue this process of adding vertices for a long time.

**Lemma 8** If $|V| < 2^{n/8-1}$ then $|V^*| \leq 2|V|$.

**Proof.** Assume by way of contradiction that $|V^*| > 2|V|$, and denote by $U$ the set obtained during the process of building $V^*$ after adding exactly $|V|$ vertices; thus $|U| = 2|V|$. Let us denote by $e(U)$ the number of directed edges within $U$, i.e., $e(U) = |\{(u,i)|u \in U$ and $u^{(i)} \in U\}|$. We provide conflicting lower and upper bounds for $e(U)$. For the lower bound, notice that every vertex that we added during the process adds at least $n/4$ edges to $e(U)$ (of its own edges as well as the $n/8$ opposite ones), and so $e(U) \geq |U-V|n/4 = |V|n/4$. For the upper bound we use the edge-isoperimetric inequality on the hypercube (Hart 1976), which implies that for every subset of the hypercube $e(U) \leq |U| \log_2 |U|$. Thus we have $|V|n/4 \leq |U| \log_2 |U| = 2|V|(|\log_2 |V| + 1)$, and so $(1 + \log_2 |V|) \geq n/8$, contradicting the bound on the size of $V$. □

**4.4 A Polite Simulation**

We can now provide our general simulation by polite algorithms, which completes the proof of the theorem.

**Lemma 9** Every algorithm that makes at most $T = 2^{n/8-1}$ queries can be simulated by a polite algorithm that makes at most $2T$ queries.

**Proof.** For $t = 1, ..., T$ denote by $q_t$ the $t$th query made by the original algorithm, and let $Q_t = \{q_1, ..., q_t\}$ be the set of all queries made until time $t$. Our polite algorithm will simulate query $q_t$ by querying all vertices in $Q^*_t$, i.e., completing the closure implied by adding $q_t$. Notice that $Q^*_t = \{q_1, ..., q_t\}$.
\((Q_{t-1} \cup \{q_t\})^* = (Q_{t-1}^* \cup \{q_t\})^*\). The difficulty is that we need to add the vertices in \(Q_t^* - Q_{t-1}^*\) in a way that maintains politeness, i.e., such that each vertex is added before \(n/4\) of its neighbors are.

To see that this is possible let us look at the vertices in \(N_t = Q_t^* - Q_{t-1}^*\). First, the previous lemma implies that \(|N_t| \leq |Q_t| = t\). By the edge-isoperimetric inequality applied to \(N_t\) we have \(e(N_t) \leq t \log_2 t\), and so some vertex \(v \in N_t\) has at most \(\log_2 t < n/8\) neighbors in \(N_t\); this will be the last vertex our polite algorithm will query in this stage. Similarly, from the remaining elements \(N' = N_t \setminus \{v\}\) there is also a vertex \(v'\) with at most \(\log_2(t-1) < n/8\) neighbors in \(N'\), and this vertex will be asked just before \(v\). We continue so until we exhaust \(N_t\). Now we claim that this order maintains politeness: since, by definition, every vertex in \(N_t\) has fewer than \(n/8\) neighbors in \(Q_{t-1}^*\), when we add the fewer than \(n/8\) neighbors from \(N_t\) that appeared before it in the ordering of \(N_t\), we still get fewer than \(n/4\) neighbors preceding it. Finally, notice that the simulating algorithm queries, by Lemma 8, at most \(2T\) vertices in \(Q_T^*\), and so its running time is as required.

\section{Randomized Algorithms for Exact Correlated Equilibria}

This section provides the lower bound for randomized algorithms. Recall that randomized algorithms can in fact compute an approximate CE with polynomially many queries (using regret-based procedures). This section proves that they cannot compute an exact CE.

First let us formally define a randomized algorithm. A randomized algorithm is just a probability distribution over deterministic algorithms.\footnote{Which is the same as a “behavioral” algorithm that makes randomizations all along (cf. Kuhn’s mixed vs. behavioral strategies in games of perfect recall—which our algorithms clearly have, as we impose no restrictions such as finite automata).} For every input, this random choice of the algorithm results in the output being a random variable. We say that a randomized algorithm solves a search problem (like our problem of finding an equilibrium) if for every input, the
probability that the output is a correct solution is at least \( \frac{1}{2} \). The cost of a randomized algorithm on a given input is the expected cost made over the random choice of the algorithm, and the cost of a randomized algorithm is its cost for the worst-case input. So our theorem for this section is:

**Theorem B.** Every algorithm (randomized or deterministic) that finds an **exact** correlated equilibrium in every \( n \)-person bi-strategy game with payoffs specified as \( b \)-bit integers with \( b = \Omega(n) \) requires a \( 2^{\Omega(n)} \) expected cost in the worst case.

This theorem applies even to randomized algorithms that produce a CE with any non-negligible probability. It also applies to the weak-CE version of the problem defined in Section 3.3, but only for zero-error algorithms.\(^{17}\) Finally, it can be seen from the proofs below that it also applies to \( \varepsilon \)-CE, for \( \varepsilon \) that is exponentially small in \( n \).

### 5.1 The Non-Positive Vertex Problem

Similarly to the deterministic case, we here reduce the correlated equilibrium problem to the following combinatorial problem on the hypercube. It is essentially a weighted version of the Approximate Sink problem, with a stricter bound on the output quality.

**The “Non-Positive Vertex”** (NPV) problem:

- **Input:** A labeling of the directed edges of the hypercube by integers where the convention is that \( R(u, v) = -R(v, u) \).

- **Queries:** A query is a vertex \( v \) in the hypercube. The answer to this query is the tuple of labels on all adjacent edges: \( R(v, v(i)) \) for \( i = 1, ..., n \).

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\(^{16}\)With the complementary probability the output may be incorrect (and so this is not a “zero-error” algorithm).

\(^{17}\)This is because Lemma \( \mathcal{E} \) holds also for zero-error randomized algorithms as it can be applied to each deterministic algorithm in the support. (Our proof does not imply the extension to the weak-CE case for general randomized algorithms, even though we believe that the theorem itself does extend.)
The algorithm must output a vertex \( v \) in the hypercube with total non-positive weight, i.e., \( \sum_i R(v, v^{(i)}) \leq 0 \).

As in Lemma \[\text{Lemma 10}\] proving a randomized lower bound for the NPV problem implies a similar bound for the CE problem due to the following reduction.

**Lemma 10** The number of queries required for a randomized NPV algorithm to solve the NPV problem is at most the number of queries required for a randomized algorithm to solve the CE problem.

**Proof.** We convert a CE algorithm to an NPV one. Let \( m = \max_{u,v} R(u,v) \). Given an NPV instance we build a CE instance ensuring that \( u_i(v^{(i)}) - u_i(v) = R(v, v^{(i)})/m \): for positive \( R(v, v^{(i)}) \) we set \( u_i(v) = 0 \) and \( u_i(v^{(i)}) = R(v, v^{(i)})/m \), while for negative \( R(v, v^{(i)}) \) we set \( u_i(v) = -R(v, v^{(i)})/m \) and \( u_i(v^{(i)}) = 0 \). Under this mapping, when the CE algorithm makes a query \( v \) it is immediately translated to the same query \( v \) of the NPV black box, and the answer from the NPV black box directly provides the answer to the CE query.

Now take an equilibrium \( x \) output by the CE algorithm. Summing up the inequalities of the CE we get: \( \sum_{i,b} Regret_{i ightarrow b} = \sum_v x(v) \sum_i (u_i(v^{(i)}) - u_i(v)) \leq 0 \), which implies that for some \( v \) in the support of \( x \) we have \( \sum_i R(v, v^{(i)}) = \sum_i (u_i(v^{(i)}) - u_i(v)) \leq 0 \), as needed to provide an answer to the NPV problem.

We continue proving the lower bound for randomized algorithms that solve the NPV problem by exhibiting a distribution over NPV instances such that every deterministic algorithm requires exponentially many queries in order to succeed on a non-negligible fraction of inputs drawn according to this distribution; we appeal here to the so-called “Yao (1977) Principle,” an instance of von Neumann’s Minimax Theorem. The lower bound for randomized CE algorithms follows, thus completing the proof of Theorem B.
5.2 A Path Construction

We build hard instances of the NPV problem from paths in the hypercube. Let \((v_0, v_1, ..., v_L)\) be a (not necessarily simple) path in the hypercube; i.e., for each \(0 \leq j < L\), the vertex \(v_{j+1}\) is obtained from \(v_j\) by flipping a single random bit. The NPV instance we build from this path essentially gives weight \(-j\) to the edge \((v_{j-1}, v_j)\), with weights added over the possible multiple times the path goes through a single edge. This way every time the path passes a vertex \(v\) at step \(j\), the incoming edge gets weight \(-j\) while the outgoing edge gets weight \(j + 1\), adding one to the total net weight going out of \(v\). Formally:

**Definition 11** Let \((v_0, v_1, ..., v_L)\) be a (not necessarily simple) path in the hypercube. The path induces the following labeling of the hypercube: \(R(u, v) = \sum_{\{j|u = v_{j-1}, v = v_j\}} j - \sum_{\{j|u = v_j, v = v_{j-1}\}} j\).

**Lemma 12** For each \(v \neq v_L\) we have \(\sum_i R(v, v^{(i)}) = |\{j|v_j = v\}|\). For \(v_L\) we have \(\sum_i R(v_L, v^{(i)}_L) = |\{j|v_j = v_L\}| - L\).

**Proof.** Except for the last vertex, \(v_L\), whenever the path goes into \(v\) at step \(j\) and exits it in step \(j + 1\), the total values of \(R\)'s going out of this vertex increases by 1 (\(-j\) incoming and \(j + 1\) outgoing). As for \(v_L\), we need to subtract \(L\) due to the fact that the last edge goes in (and there is no edge that goes out). ■

This means that if the path covers the whole hypercube then the only non-positive vertex in it is the end of the path.

We now define a random distribution over paths that cover the whole hypercube and whose final end point is random. We start with some (fixed) Hamiltonian path in the hypercube. From that point on we continue with a random walk of length \(L = n \cdot 2^{n/3}\). Why would it be hard for an algorithm to find the end of the path? We show that an algorithm must essentially follow the path query by query. Otherwise it is looking for a needle of length \(L = n \cdot 2^{n/3}\) that is randomly hidden in a haystack of size \(2^n\). But probing places that are already known to be in the random part of the path only allows
the algorithm to advance sequentially over it, thus requiring exponential time to reach the end.

**Lemma 13** Any deterministic algorithm that runs in time \( T < 2^{n/3}/n \) is able to solve the NPV problem on at most a fraction of \( O(2^{-n/3}) \) of inputs drawn according to this distribution.

Since all the information in our path-based instances of the NPV problem are determined by the path, we can imagine that the queries directly ask for this path information. Since the only non-positive vertex in these instances is the end of the path, our algorithm really needs to find it. Not only is this hard to do, but it is hard even to find any vertex that is near the tail of the path. This is so since, on the one hand, the tail is a tiny fraction of the hypercube and so it can’t be found “at random,” and, on the other hand, the only possible “deliberate” way to find it is to follow the path step by step, which takes exponential time. The reason that no “shortcuts” are possible when following the path is that the random walk in the hypercube mixes rapidly, and so one gets no information about how the path continues beyond the very near vicinity. To formalize this line of reasoning we introduce a variant of the problem that explicitly provides the algorithm with any information that we think it may get a handle on. Specifically, we tell the algorithm everything about the path except for its tail, and, furthermore, give it an additional \( n^2 \) vertices at the beginning of this tail every step. Once this is given to the algorithm, we are able to show that the algorithm can never learn anything new.

### 5.3 The “Hit The Path” (HTP) problem

We formally define the HTP problem.

**The “Hit The Path” (HTP) problem:**

- **Input:** A random path \((v_0, ..., v_L)\) in the hypercube of length \( L = n2^{n/3} \), starting from a *revealed* vertex \( v_0 \).
• Queries: At each step \( t = 1, \ldots, T \):

1. The algorithm may query a vertex \( q_t \) of the hypercube, depending on the revealed information so far (which we will soon see is exactly the sequence \((v_0, \ldots, v_{(t-1)n^2})\)).

2. The next \( n^2 \) vertices of the path, i.e., \( v_{(t-1)n^2+1}, \ldots, v_{tn^2} \), are revealed (independently of the query).

• Output: The algorithm wins at time \( t \) if \( q_t \) is on the non-previously revealed part of the path, i.e., \( q_t \in \{v_{tn^2+1}, \ldots, v_L\} \).

Lemma 14 Every algorithm for the HTP problem with at most \( T \) queries wins with probability at most \((1 + o(1))n^{2-2n/3}T\).

Proof. Fix a (deterministic) algorithm for the HTP problem that makes at most \( T \) queries. If it wins, then for some step \( 1 \leq t \leq T \) it won (for the first time) by finding an unrevealed vertex \( v_j \) for \( tn^2 < j \leq L \). We bound this probability (over the random choice of the path) for a fixed \( t \) and \( j \), and then use the union bound to obtain an upper bound on the probability that the algorithm wins. Now let us look at the \( t \)'th query \( q_t \) made by the algorithm. If none of the previous queries won, then the only information the algorithm had when making this query was the revealed part of the path, i.e., \((v_0, v_1, \ldots, v_{(t-1)n^2})\), and so the query is just a function of these: \( q_t = q_t(v_0, v_1, \ldots, v_{(t-1)n^2}) \). What is the probability that for some function on these inputs we have \( q_t(v_0, v_1, \ldots, v_{(t-1)n^2}) = v_j \)? Note that our construction of random paths means that \( v_j \) is obtained by taking a random walk of length \( j - (t-1)n^2 \geq n^2 \) from vertex \( v_{(t-1)n^2} \). Now comes the crucial observation: as the mixing time of the hypercube is known to be \( O(n \log n) < n^2 \) (cf. Diaconis et al. 1990), this means that a random walk of length \( l \geq n^2 \) ends at an almost uniformly random vertex of the hypercube. Thus for any fixed \((v_0, v_1, \ldots, v_{(t-1)n^2})\), we have that \( v_j \) is almost uniformly distributed over the hypercube and so \( Pr[q_t(v_0, v_1, \ldots, v_{(t-1)n^2}) = v_j] = (1 + o(1))2^{-n} \). Multiplying this quantity by \( T < 2^{n/3}/n \) (for all possible values of \( t \)) and then by \( 2^{n/3}/n \) (for
all possible values of \( j \), we get the required upper bound for the probability of winning. ■

**Proof of Lemma 13.** First, note that Lemma 14 implies in particular that an HTP-algorithm with \( T < 2^{n/3}/n \) queries can win with probability at most \( O(2^{-n/3}) \).

Second, recalling the discussion immediately following the statement of Lemma 13, suppose that we have an algorithm that succeeds in solving the NPV problem on a larger fraction of inputs drawn according to this distribution; we use it to win instances of this HTP problem with at least the same probability. Whenever the NPV algorithm makes a query to \( v \) we make the same query in the HTP case. If we win, then we are done. Otherwise, we know that the non-revealed part of the path does not pass through \( v \) and so the reply to the NPV query is completely determined by the revealed part of the path, which we already have and can use for the reply. If the NPV algorithm succeeds then it must have found the last vertex on the path (the only non-positive one), which is on the path and is revealed only after \( L/n^2 = 2^{n/3}/n > T \) queries, and so is still unrevealed and thus our HTP algorithm wins too. ■

Combining Lemmas 10 and 13 proves Theorem B.

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