THE WORK OF MIKE HOCHMAN ON MULTIDIMENSIONAL SYMBOLIC DYNAMICS AND BOREL DYNAMICS

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ABSTRACT. We review the impact of Mike Hochman’s work on multidimensional symbolic dynamics and Borel dynamics.

1. INTRODUCTION

In an ancient story, several blind men encounter an elephant. One feels the ears, one the trunk, one the tusks, and so on. All features are dramatic and important. The blind men have quite different opinions about what makes an elephant.

Mike Hochman’s research is like the elephant – with several dramatic features, quite varied; including important work on fractals, multidimensional symbolic dynamics, Borel dynamics, ergodic theory and more. In this short note, I only look quickly at parts of two features of Hochman’s elephant – his most important work (so far) in multidimensional symbolic dynamics and Borel dynamics.

2. MULTIDIMENSIONAL SYMBOLIC DYNAMICS

I’ll focus on the content and impact of the papers [23] (joint with Tom Meyerovitch) and [19].

2.1. Definitions. “Multidimensional symbolic dynamics” refers here to $\mathbb{Z}^d$ subshifts, $(X,\sigma)$, $d = 2, 3, \ldots$. Here $X$ is a closed, shift invariant subset of $A^{\mathbb{Z}^d}$, for some finite alphabet $A$, with the shift action by $\mathbb{Z}^d$: for $v \in \mathbb{Z}^d$, $(\sigma^v x)(n) = x(n+v)$.

$A^{\mathbb{Z}^d}$ has the product topology of the discrete topology on $A$, and $X$ has the relative topology. An element $x$ of $X$ is a function from $\mathbb{Z}^d$ into some finite alphabet $A$; $x$ is visualized as a way of filling the $\mathbb{Z}^d$ lattice with symbols from $A$.

Let $C_n(x)$ be the restriction of $x$ to the cube $\{0, 1, \ldots, n-1\}^d$. The $\mathbb{Z}^d$ topological entropy of $(X,\sigma)$ is

$$\lim_{n} (1/n^d) \log |\{C_n(x) : x \in X\}| .$$

$X$ is a $\mathbb{Z}^d$ shift of finite type (SFT) if there is some positive integer $n$ and some finite set $F$ such that $X$ is the subset of $A^{\mathbb{Z}^d}$ consisting of the $x$ such that for all $v$ in $\mathbb{Z}^d$, $C_n(\sigma^v x) \notin F$. When $X = A^{\mathbb{Z}^d}$, the SFT is the full $\mathbb{Z}^d$ shift on $|A|$ symbols.

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There is other excellent work on multidimensional symbolic dynamics.
A $d$ dimensional cellular automaton is a continuous shift-commuting map from a full $\mathbb{Z}^d$ shift into itself.

2.2. Before Hochman and Meyerovitch. Let’s sketch the main features of the state of knowledge of the possible entropies of $\mathbb{Z}^d$ SFTs before the Hochman-Meyerovitch paper [23]:

1. ($d = 1$) The set of possible topological entropies of a $\mathbb{Z}$ SFT was known to be the set of logarithms of an easily understood class of algebraic integers [32].
2. For $d > 1$, a limited collection of $\mathbb{Z}^d$ SFT entropies had been computed exactly, by (highly nontrivial) work of mathematical physicists (e.g. [4, 29, 31]).
3. For $d > 1$, it was well known that there can be no algorithm which takes a finite presentation of a general $\mathbb{Z}^d$ SFT or $d$-dimensional cellular automata and computes its entropy (or, roughly speaking, any nontrivial property–see e.g. [27, 28]).

2.3. The main theorems.

**Theorem 2.1.** [23] Suppose $h$ is a nonnegative real number and $d$ is an integer, $d \geq 2$. Then the following are equivalent.

1. $h$ is the topological entropy of a $\mathbb{Z}^d$ shift of finite type.
2. There is a Turing machine which produces a sequence of nonnegative numbers $h_n$ such that $h = \inf h_n$.

The countable class of numbers in (2) properly contains the nonnegative real numbers $h$ which can be algorithmically approximated to arbitrary precision. For example: $\pi, e, \gamma,$ algebraic numbers ... But a general procedure for estimating numbers in this class can only only produce converging upper bounds – no general procedure will give an error bound for the estimate.

Hochman’s paper [19] showed Theorem 2.1 to be part of a systematic approach. Here is an example.

**Theorem 2.2.** [19] Suppose $h$ is a nonnegative real number and $d \geq 3$ is an integer. The following are equivalent.

1. $h$ is the topological entropy of a $\mathbb{Z}^d$ cellular automaton.
2. There is a Turing machine which produces a sequence of nonnegative numbers $h_n$ such that $h = \lim \inf h_n$.

For $h = \inf h_n$, from the sequence $(h_n)$ you at least get at any finite input an upper bound to $h$. For $h = \lim \inf h_n$, from any finite set of terms from the sequence $(h_n)$ you obtain no bound at all on $h$.

For Theorems 2.1 and 2.2 the proof that (1) implies (2) is not hard. So, there were two difficulties. Initially, to have the audacity to imagine the theorems. Then, to produce actual constructions showing (2) $\Rightarrow$ (1). A starting point for these constructions was the 1989 paper [36] of Shahar Mozes, which used ideas from Raphael Robinson’s 1971 paper [39] to construct $\mathbb{Z}^2$ SFT presentations for many planar substitution tilings.

Theorem 2.2 is dramatic and easily stated, but the paper [19] fundamentally is about simulating effective subshifts with $\mathbb{Z}^d$ shifts of finite type ($d \geq 3$). For

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*This theorem was later extended to $d = 2$ in the papers [3, 16], and then to $d = 1$ in [17].*
example, in [19] this gives (for \( d \geq 3 \)) a recursion theoretic characterization of the possible dynamics of a \( d \)-dimensional cellular automaton on its limit set (up to a modest equivalence relation). Simulation results for \( d \geq 3 \) were generalized to \( d = 2 \) (optimal) in the papers [3, 16].

2.4. Impact of the papers. Before [19, 23], workers were very aware of the “swamp of undecidability” as an obstacle to proving something (perhaps anything) general and interesting about multidimensional cellular automata and SFTs. The papers [19, 23] did “mathematical judo” on recursion theory: making it a friend instead of an enemy. It seems fair to call this a paradigm shift. The papers offered a blueprint for characterizing the range of invariants of \( \mathbb{Z}^d \) SFTs (or sofic shifts, or effective shifts):

1. Find the “obvious” recursion theoretic obstruction.
2. Make constructions to prove there is no other obstruction.

Work by various people has since been carried out in this vein, to characterize for example the possible entropy dimensions of \( \mathbb{Z}^d \) SFTs, for \( d \geq 3 \) [35]; periods of \( \mathbb{Z}^d \) SFTs, for \( d \geq 2 \) [24]; sets of expansive directions for \( \mathbb{Z}^2 \) SFTs [14]; sets of limit measures of cellular automata [13, 18]; and so on. It now seems that to a large extent the landscape of possibilities for general multidimensional SFTs (or sofic shifts, or effective shifts) has a recursion theoretic description. “Effective subshifts” have also been considered for more general groups [2]. For more, see the surveys [22, 25] of Hochman and Jeandel.

The papers [19, 23] also furthered the rich interdisciplinarity of the multidimensional symbolic dynamics area, as it spurred contributions from logicians and experts in recursive tiling constructions (e.g. [15, 16, 41]).

3. Borel dynamics

3.1. Definitions. A standard Borel space is a pair \((X, \Sigma)\) such that \(X\) is a set, and \(\Sigma\) is the \(\sigma\)-algebra (the Borel \(\sigma\)-algebra) generated by the open sets defined by a given complete, separable metric on \(X\). A Borel set is an element of \(\Sigma\). We usually suppress the \(\sigma\)-algebra from the notation. A morphism in the Borel category is a map for which the inverse image of every Borel set is a Borel set. Standard Borel spaces of equal cardinality are isomorphic.

In this article, a Borel system \((X, T)\) is a Borel automorphism \(T: X \to X\) of a standard Borel space. (So, we restrict to \(\mathbb{Z}\) actions.) We say a Borel system \((X, T)\) is free if the \(\mathbb{Z}\) action generated by \(T\) is free, i.e., \(T\) has no periodic point.

A null set for a Borel system \((X, T)\) is a set which has measure zero for every \(T\)-invariant Borel probability. A full set is the complement of a null set. \(M(T)\) is the set of \(T\)-invariant Borel probabilities.

For a Borel system \((X, T)\), pick a countable collection of sets generating the Borel \(\sigma\)-algebra; if \((X, T)\) comes with a topology, the countable collection is chosen to be a basis for the topology. We say a point in \(X\) is generic for a measure \(\mu\) in \(M(T)\) if under \(T\) and \(T^{-1}\) it it visits all sets in the countable collection with...
ergodic-theorem frequencies. Let \( X_g \) be the union over \( \mu \) in \( M(T) \) of the \( \mu \)-generic points. Let \( X_p \) be the set of \( T \)-periodic points, a subset of \( X_g \). Set \( X_d = X \setminus X_g \).

A Borel system \((X, T)\) is the disjoint union of subsystems given by restriction to \( X_p, X_g \setminus X_p \) and \( X_d \). Then \( X_g \) is a \( T \)-invariant full set, and \( X_g \setminus X_p \) supports all nonatomic measures in \( M(T) \). A complete Borel invariant for \((X_p, T)\) is simply the function giving the cardinality of the set of orbits of size \( n \), for \( n \in \mathbb{N} \). So, understanding of the Borel system amounts to understanding the free systems \((X_g \setminus X_p, T)\) and \((X_d, T)\). By ergodic decomposition, the system \((X_d, T)\) supports no \( T \)-invariant Borel probability.

3.2. Borel dynamics off null sets, before Hochman. It is fundamental in dynamics to consider a homeomorphism \( T : X \to X \) with respect to some \( T \)-invariant Borel probability \( \mu \), neglecting the sets of \( \mu \) measure zero. More ambitiously, one could try to understand \( T \) simultaneously with respect to a large subset of \( M(T) \). In this spirit, motivated by his study of low dimensional piecewise smooth dynamics, Buzzi proposed the following definition.

**Definition 3.1.** [10] Suppose \( S, T \) are Borel systems with equal and finite topological entropy \( h \). Then \( S \) and \( T \) are *entropy conjugate* if for some \( \epsilon > 0 \) they are isomorphic modulo sets of measure zero for all ergodic invariant Borel probabilities of entropy \( \geq h - \epsilon \).

Buzzi showed certain piecewise smooth systems of equal entropy are entropy conjugate to (countable state) Markov shifts, and asked whether recurrent Markov shifts of equal entropy are entropy conjugate. Mixing shifts of finite type of equal entropy were shown to be entropy conjugate [7]. Entropy conjugacy was not understood even for general positive recurrent Markov shifts.

3.3. Borel dynamics on full sets, after Hochman. To understand the impact of Hochman’s paper [20] on this topic, we begin with some definitions. For \( t > 0 \), let \( B_t \) be the collection of Borel systems which are free and have no invariant Borel probability of entropy \( \geq t \). A Borel system \((X, T)\) is strictly \( t \)-universal if it is in \( B_t \) and every member of \( B_t \) embeds modulo null sets to a subsystem of \((X, T)\). The entropy of a Borel system \((X, T)\) is defined to be the supremum of \( h_\mu(T) \), over \( T \)-invariant Borel probabilities \( \mu \).

For a Borel system \((X, T)\), form \( X' \) by removing periodic points and points generic for any ergodic measure of maximal entropy.

**Theorem 3.2.** [20] Suppose a Borel system \((X, T)\) of finite entropy \( t > 0 \) contains mixing shifts of finite type with entropies arbitrarily close to \( t \).

1. Then \((X', T)\) is strictly \( t \)-universal.

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5In a similar spirit, Rufus Bowen in his 1973 classic [5] introduced a closely related notion, which he also called “entropy conjugacy”. However, a map implementing Bowen’s entropy conjugacy is required to be continuous, not only Borel, on a suitably large set.

6The SPR shifts are the natural subclass of positive recurrent Markov shifts whose properties most closely resemble those of finite state irreducible Markov shifts. They are the positive recurrent Markov shifts for which the measure of maximal entropy is exponentially recurrent. For characterizations of this class in terms of subsystems and local zeta functions, see [7, Proposition 2.3 and the definition (4) above it].
(2) Strictly $t$-universal systems are isomorphic mod null sets.

Proof. We give a brief indication of how the proof goes.
(2) Not hard. (Cantor-Bernstein argument of set theory works in this category.)
(1) As follows:
(I) Broad strategy. B. Weiss showed a free Borel system has a countable generator, hence embeds to a subsystem of $((1, 2, ...)^2, \sigma)$, the full shift over a countable alphabet. Then, it’s not hard to show, given $t > t - \epsilon > 0$ and a mixing SFT of entropy $t$, that it suffices to find $B$ a Borel subsystem of $((1, 2, ...)^2, \sigma)$ supporting all the ergodic measures of entropy $\leq t - \epsilon$, and Borel embed $B$ into that SFT.

(II) Finer strategy. “Observe”: the argument of the Krieger generator theorem gives a Borel map which on the generic points of every ergodic Borel measure, individually, is injective with finitary inverse.

(III) Hard construction step. Augment the coding of that map to make it injective on the union of all those supports.

□

Corollary 3.3. [20] For every $(X, T)$ of equal positive entropy $t$ from the following collection, $(X', T)$ is the same strictly $t$-universal system: mixing SFTs, mixing countable state Markov shifts, mixing finitely presented systems, Anosov diffeomorphisms ...

Corollary 3.3 leaves open the nature of measures of maximal entropy for the listed systems. For example, the mixing Markov shifts which are not positive recurrent have no measure of maximal entropy. But, the other systems listed have a unique measure of maximal entropy, which is Bernoulli. This gives the following.

Corollary 3.4. [20] Entropy is a complete invariant of Borel isomorphism modulo periodic points and null sets among systems in the following collection: mixing SFTs, mixing positive recurrent Markov shifts, mixing finitely presented systems, Anosov diffeomorphisms ...

We see that Hochman not only resolved Buzzi’s entropy conjugacy question, but with a more insightful viewpoint proved something much more general and fundamental.

There have been further developments in this vein. Recall that Hochman’s argument begins with the Krieger generator theorem [30], using mixing SFTs. Quas and Soo [55] proved a generator theorem for a much more general class, the homeomorphisms of compact metric spaces satisfying (i) almost weak specification, (ii) the small boundaries property and (iii) asymptotic h-expansiveness (or more generally upper semicontinuity of the entropy map on invariant Borel probabilities). The Quas-Soo theorem was striking both in its generality and in its liberation from shifts of finite type – it applies to many systems, such as quasihyperbolic toral automorphisms, which contain no infinite shift of finite type.

B. Weiss (unpublished) showed the requirement of (iii) in the Quas-Soo Theorem could be dropped. Recently, Chandgotia and Meyerovitch (to be posted) removed the requirement (ii) and, dramatically, prove the universality result of Hochman, but with mixing systems with almost weak specification in place of mixing SFTs.

Note added in proof: This result is also contained in the work [8] of David Burguet, appearing in the 2019 arxiv post [9]. The work [11] of Chandgotia and Meyerovitch was presented at an August 2018 PIMS workshop; the Burguet paper [8] was submitted to ETDS in August 2017. Neither of the independent works [8, 11] contains the other.
Chandgotia and Meyerovitch also generalize the result to $\mathbb{Z}^d$ actions. Altogether, these works comprise a remarkable advance in our understanding of how positive entropy in a class of systems can force dynamical complexity.

Away from mixing systems, Hochman’s universality approach to isomorphism on full sets can become more complicated, but remain useful (see e.g. [6]).

3.4. Borel dynamics beyond probabilities. For a Borel system $(X, T)$, we have seen the success of Hochman’s universality approach to the action of $T$ on $X_g$. What about $X_d$? Although $(X_d, T)$ supports no invariant Borel probability, it may still admit complicated dynamics (for example, $\sigma$-finite or nonsingular invariant measures). Commonly, $X_d$ will be a dense $G_\delta$ in a topological space $X$. One might think of $X_d$ as the “dark matter”: $X_d$ is often large, and the dynamics of $(X_d, T)$ can be hard to see.

Let $D$ be the class of Borel systems admitting no invariant Borel probability. The study of the class $D$ began with Shelah and Weiss [40, 43, 44], and then Kechris and others (see [14, 24]). Earlier, Krengel had shown that for any infinite measure preserving ergodic conservative system, there is a two-set generator. (This is a.e., neglecting measure zero sets.) Weiss’s 1989 paper [44] included the following question (this question was later asked for a general countable group in [24, Problem 5.7]).

**Question 3.5.** [44] Suppose a Borel system admits no invariant Borel probability. Must it have a finite generator? A 2-set generator?

This was a rather fundamental, longstanding question. The title of Hochman’s paper [21] gives the answer: “Every Borel automorphism without finite invariant measure admits a two-set generator.”

We note here the related theorem of Tserunyan:

**Theorem 3.6.** [42] Suppose $G$ is an arbitrary countable group, acting by homeomorphisms on a $\sigma$-compact Polish space, which does not admit an invariant Borel probability. Then $G$ has a finite generator. (In fact, a 32-generator.)

Tserunyan’s theorem is remarkable for the generality of the acting group. However, not every $\mathbb{Z}$-action on a Borel space is Borel conjugate to a continuous action on a $\sigma$-compact space [12]. The two theorems seem quite different. Certainly, the proofs are very different.

Hochman’s proof of the 2-generator theorem is much harder than the proof of Theorem 3.2 [20]. A big problem is to even find a strategy. We refer to [21] for a clear explanation, and a list of compelling open problems. The paper also contains the following striking result.

**Theorem 3.7.** [21] If a Borel system $(X, T)$ contains an infinite mixing SFT, then $(X_d, T)$ is the unique (up to Borel conjugacy) Borel system in $D$ into which every system in $D$ embeds.

So, for many familiar systems, (e.g., those on the list of Corollary 3.3, $(X_d, T)$ is the same universal “dark matter” Borel dynamics. (Whatever that is . . . )

**Corollary 3.8.** [21] Suppose two homeomorphisms of equal finite positive entropy are mixing and lie in any of following classes: SFT, sofic shift, positive recurrent countable state Markov shift, finitely presented system (e.g. Anosov homeomorphism, or Axiom A on a basic set).
Then their restrictions to the complement of the periodic points are Borel isomorphic.

There is now an obvious question.

**Question 3.9.** Suppose \((X, T)\) is a mixing homeomorphism of an infinite compact metric space satisfying almost weak specification. Must \((X_d, T)\) be the universal “dark matter” dynamics?

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