Likelihood ratio test for the mean of asymptotic spatial regression with the Brownian sheet noise

W Somayasa\textsuperscript{1,2}, R Sahupala\textsuperscript{2} and D K Sutiari\textsuperscript{3}

\textsuperscript{1,2)Department of Mathematics, Hala Oleo University
\textsuperscript{3)Department of Medical Electrical Engineering, Mandala Waluya School of Health Sciences

E-mail: \textsuperscript{1)wayan.somayasa@uho.ac.id (corresponding author),
\textsuperscript{2)rahmalia.sahupala@uho.ac.id,
\textsuperscript{3)sutiaridesak@gmail.com

Abstract. Likelihood ratio test (LR-test) is frequently applied in regression analysis when the observations are normally distributed. However, when the normality assumption is violated, such a test can not be directly adopted in practice. In this work, we propose an approach by firstly transforming the observations to a set-indexed partial sums process. The limit model under this transformation is presented as a deterministic trend plus a random noise given by the set-indexed Brownian sheet. We show that the problem of testing the appropriateness of a model can be handled step-wise by testing the validity of the trend in the limit model by defining an LR-test based on the ratio between the likelihood function under $H_0$ and under $H_0 \cup H_1$. The rejection region as well as the power of the size $\alpha$ LR-test are obtained based on the Cameron-Martin-Girsanov formula of the Radon-Nikodym derivative of the set-indexed partial sums limit process of the observation with respect to the set-indexed Brownian sheet. Simulation study shows that the proposed test behaves as a consistent test in that it maximizes the power when $H_1$ is true. Application of the method to real data can help in detecting valid regression model describing the variability of the maximum height of corn plants over the experimental region.

1. Introduction
Regression analysis is one method in statistics which has been widely used in many disciplines, such as in agriculture, economics, engineering, physics, chemistry, biology, environmental sciences and many other life sciences, see [1, 2] for references. Regression is very important in particular for the area of studies involving empirical model building as the base in drawing assessment and conclusion of some random quantities. For example, in response surface methodology, linear regression is used for prediction of future response and determination of the optimal conditions, cf. [3, 4, 5, 6]. One important inference procedure involved in modeling using regression analysis is model check.

To understand the problem, let us consider a spatial random process of observations

\[ \{Y(t, s) : (t, s) \in \mathbf{D} := [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2 \} \]

following a nonparametric regression model

\[ Y(t, s) = g(t, s) + \varepsilon(t, s), \quad (t, s) \in \mathbf{D}, \]

(1)
where \( g(t,s) \) is unknown-true regression function and \( \varepsilon(t,s) \) is the independent and identically distributed random error with \( \mathbb{E}(\varepsilon(t,s)) = 0 \) and \( \text{Var}(\varepsilon(t,s)) = \sigma^2 < \infty \). Let \( \mathbf{W} := [f_1, \ldots, f_q] \) and \( \mathbf{V} := [f_1, \ldots, f_q, f_{q+1}, \ldots, f_p] \) be linear subspaces generated respectively by linearly independent functions \( \{f_1, \ldots, f_q\} \) and \( \{f_1, \ldots, f_q, f_{q+1}, \ldots, f_p\} \), with \( q < p \), where for \( i = 1, \ldots, p, \ f_i \in \mathbb{E}\{\mathbf{D}, P_0\} \) and \( P_0 \) is a probability measure on measurable space \( \mathbb{D}, \mathcal{B}(\mathbb{D}) \). In model check for (1) we concerns with the problem of testing whether or not the regression functions \( g \) can be sufficiently represented by the bases of \( \mathbf{W} \). In other word we test the hypotheses

\[
H_0 : g \in \mathbf{W} \text{ versus } H_1 : g \in \mathbf{V}.
\]

(2)

Suppose Model (1) is observed over an experimental design with \( n_1 \times n_2 \) points, defined by

\[
\mathcal{E}_{n_1 \times n_2} = \{(t_{n_1 \ell}, s_{n_2 k}) : 1 \leq \ell \leq n_1, 1 \leq k \leq n_2\} \subseteq \mathbb{D},
\]

where the design points in \( \mathcal{E}_{n_1 \times n_2} \) are constructed according to \( P_0 \), see [7, 8]. Hence the corresponding realization of (1) when it is applied to \( \mathcal{E}_{n_1 \times n_2} \) will be given by the following triangular array of the observations

\[
\{Y(t_{n_1 \ell}, s_{n_2 k}) : 1 \leq \ell \leq n_1, 1 \leq k \leq n_2\}
\]

satisfying the following decomposition

\[
Y(t_{n_1 \ell}, s_{n_2 k}) = g(t_{n_1 \ell}, s_{n_2 k}) + \varepsilon(t_{n_1 \ell}, s_{n_2 k}), (t_{n_1 \ell}, s_{n_2 k}) \in \mathcal{E}_{n_1 \times n_2}.
\]

There are many methods which have been proposed in the literatures for testing the significance of a linear regression model. In the classical methods, when the sample is normally distributed the appropriateness of a proposed model is tested by conducting \( F \)-test which has been shown coincides with the LR-test, see [2, 9, 10]. However, when the normal assumption is not satisfied, nonparametric approach based on the rank of the residuals has been proposed, see [12]. Asymptotic procedure based on the ratio of the length of the residuals under \( H_0 \) and under \( H_1 \) has been investigated in [13]. Recently, [14, 15] proposed asymptotic test based on set-indexed partial sums process of least squares residuals, defined by utilizing the partial sums operator \( T_{n_1 \times n_2} : \mathcal{R}^{n_1 \times n_2} \rightarrow \mathcal{C}(\mathcal{B}(\mathbb{D})) \), defined by

\[
T_{n_1 \times n_2}(A_{n_1 \times n_2})(A) = \frac{1}{\sqrt{n_1 n_2}} \sum_{k=1}^{n_2} \sum_{\ell=1}^{n_1} \mathbf{1}_A(t_{n_1 \ell}, s_{n_2 k}) a_{\ell k}
\]

(3)

for every matrix \( A = (a_{\ell k})_{\ell=1, k=1}^{n_1, n_2} \subseteq \mathcal{R}^{n_1 \times n_2} \) and \( A \in \mathcal{B}(\mathbb{D}) \). The set \( \mathcal{C}(\mathcal{B}(\mathbb{D})) \) is the space of continuous set-functions with respect to the symmetric-difference metric \( d_{P_0} \), defined by \( d_{P_0}(A_1, A_2) := P_0(A_1 \Delta A_2) \), cf. [16]. The transformation causes no lost of information in the residuals, since the partial sums operator is one to one.

Although almost all methods mentioned above have a common approach in that they are based on the behavior of the residuals, in this paper we propose a slightly different method in that we firstly attach the array of the observations obtained from Model (1) by using \( T_{n_1 \times n_2} \) to a stochastic process with sample path in \( \mathcal{C}(\mathcal{B}(\mathbb{D})) \) and then we define a test based on the statistic given by the sum of the square of the Riemann-Stieltjes sum of the regression function in \( \mathbf{V} \mid \mathbf{W} \) with respect to \( T_{n_1 \times n_2}(\mathbf{Y}_{n_2 \times n_1})(\cdot) \),

\[
\mathcal{S}_{n_1 n_2}(\mathbf{Y}_{n_2 \times n_1}) := \sum_{i=q+1}^{p} \left( \sum_{k=1}^{n_2} \sum_{\ell=1}^{n_1} f_i(t_{n_1 \ell}, s_{n_2 k}) \Delta tk T_{n_1 \times n_2}(\mathbf{Y}_{n_2 \times n_1})(t_{n_1 \ell}, s_{n_2 k}) \right)^2,
\]
where
\[ \Delta_{\ell k} T_{n_1 \times n_2}(Y_{n_2 \times n_1}) = T_{n_1 \times n_2}(Y_{n_2 \times n_1})(t_{n_1 \ell}, s_{n_2 k}) - T_{n_1 \times n_2}(Y_{n_2 \times n_1})(t_{n_1 \ell-1}, s_{n_2 k}) - T_{n_1 \times n_2}(Y_{n_2 \times n_1})(t_{n_1 \ell}, s_{n_2 k-1}) + T_{n_1 \times n_2}(Y_{n_2 \times n_1})(t_{n_1 \ell-1}, s_{n_2 k-1}). \]

The hypothesis \( H_0 \) is rejected for large value of \( S_{n_1 \times n_2}(Y_{n_2 \times n_1}) \). This is a reasonable test statistic by the reason the value of \( S_{n_1 \times n_2}(Y_{n_2 \times n_1}) \) will be large when the observations support \( H_1 \). Conversely, \( S_{n_1 \times n_2}(Y_{n_2 \times n_1}) \) will takes small value when the observations support \( H_0 \). Thus our proposed test is different to those defined based on the Kolmogorov-Smirnov and Cramér-von Mises functionals of the partial sums process of the residuals proposed in [14, 15].

We organize the rest of the present paper as follows. In Section 2 we give an investigation to the limiting distribution of \( S_{n_1 \times n_2}(Y_{n_2 \times n_1}) \) under \( H_0 \) as well as under \( H_1 \), so that the rejection region and the power of size \( \alpha \) test can be fixed. In Section 3 we derive the rejection region and the power function of the LR-test of size \( \alpha \). We show that the test based on \( S_{n_1 \times n_2}(Y_{n_2 \times n_1}) \) is asymptotically a likelihood ratio test. The performance of the test is investigated by simulation in Section 4. Next, we discus an application of the test procedure in real data by considering corn plant data, see Section 5. Some conclusion and remarks for future research are summarized at the end of the paper.

2. Preliminary results
In this section we present some auxiliary results regarding the limiting behavior of the partial sums process of the spatial observations. The results will be shown important for deriving our main results in the next section.

**Theorem 2.1** Let \( Y_{n_2 \times n_1} = (Y(t_{n_1 \ell}, s_{n_2 k}))_{\ell=1}^{n_1} \), \( k=1 \) be the triangular array of the random observations obtained by observing Model 1 over the experimental design \( \mathcal{E}_{n_1 \times n_2} \). Let \( \hat{\sigma}_{n_1}^2 \) be a consistent estimator for \( \sigma^2 \). Suppose the regression function \( g \) is continuous on the closed rectangle \( \mathcal{D} \) and \( \varepsilon \)'s are independent and identically distributed with \( E(\varepsilon) = 0 \) and \( Var(\varepsilon) = \sigma^2 \).

Then, for \( n_1 \to \infty \) and \( n_2 \to \infty \), it holds when \( H_0 \) is true
\[
\frac{1}{\hat{\sigma}_{n_1 n_2}} S_{n_1 n_2}(Y_{n_2 \times n_1}) \Rightarrow \sum_{i=q+1}^{p} \left( \int_{\mathcal{D}} f_i(t, s) dZ(t, s) \right)^2.
\]

Conversely, when \( H_1 \) is true, we have
\[
\frac{1}{\hat{\sigma}_{n_1 n_2}} S_{n_1 n_2}(Y_{n_2 \times n_1}) \Rightarrow \infty, \text{ for } n_1 \to \infty \text{ and } n_2 \to \infty,
\]

where \( Z \) is the set-indexed Gaussian white noise (standard set-indexed Brownian sheet), see [16]. Here and throughout the paper the notation ” \( \Rightarrow \) ” stands for the convergence in distribution (in law).

**Proof:** By the linearity of \( T_{n_1 \times n_2} \) on \( \mathcal{R}^{n_2 \times n_1} \), we have
\[
\frac{1}{\hat{\sigma}_{n_1 n_2}} S_{n_1 n_2}(Y_{n_2 \times n_1}) = \sum_{i=q+1}^{p} \sum_{k=1}^{n_2} \sum_{\ell=1}^{n_1} f_i(t_{n_1 \ell}, s_{n_2 k}) \Delta_{\ell k} \frac{1}{\hat{\sigma}_{n_1 n_2}} T_{n_1 \times n_2}(g_{n_2 \times n_1})(t_{n_1 \ell}, s_{n_2 k})
\]
\[
+ \sum_{k=1}^{n_2} \sum_{\ell=1}^{n_1} f_i(t_{n_1 \ell}, s_{n_2 k}) \Delta_{\ell k} \frac{1}{\hat{\sigma}_{n_1 n_2}} T_{n_1 \times n_2}(\varepsilon_{n_2 \times n_1})(t_{n_1 \ell}, s_{n_2 k}) \right)^2
\]
\[
= \sum_{i=q+1}^{p} \left( \int_{\mathcal{D}} f_i(t, s) d \frac{1}{\hat{\sigma}_{n_1 n_2}} T_{n_1 \times n_2}(g_{n_2 \times n_1})(t, s) + \int_{\mathcal{D}} f_i(t, s) d \frac{1}{\hat{\sigma}_{n_1 n_2}} T_{n_1 \times n_2}(\varepsilon_{n_2 \times n_1})(t, s) \right)^2.
\]
Here the involved integrals are in the sense of Riemann-Stieltjes. Since \( \frac{1}{\sigma_{n_1 n_2}} T_{n_1 \times n_2}(g_{n_1})^2 \) converges to \( \frac{\sqrt{n_{1 n_2}}}{\sigma} \varphi_g(\cdot) \), where \( \varphi_g \) is a function in \( \mathcal{C}(D) \), defined by \( \varphi_g(A) := \int_A g(t, s) P_0(dt, ds) \), and by the invariant principle of [16], \( \frac{1}{\sigma_{n_1 n_2}} T_{n_1 \times n_2}(\xi_{n_2 \times n_1})^2 \) converges to \( Z \). Then by applying the well-known continuous mapping theorem, we get

\[
\frac{1}{\sigma_{n_1 n_2}} S_{n_1 n_2}(Y_{n_2 \times n_1}) \Rightarrow \sum_{i=q+1}^{p} \left( \frac{\sqrt{n_{1 n_2}}}{\sigma} \int_D f_i(t, s) d\varphi_g(t, s) + \int_D f_i(t, s) dZ(t, s) \right)^2.
\]

When \( H_0 \) is true, \( f_i \) is orthogonal to \( g \), for \( i = q + 1, \ldots, p \), so that \( \int_D f_i(t, s) d\varphi_g(t, s) P_0(dt, ds) = 0 \), for \( i = q + 1, \ldots, p \). Conversely, if \( H_1 \) is true, then \( \int_D f_i(t, s) d\varphi_g(t, s) P_0(dt, ds) > 0 \), for \( i = q + 1, \ldots, p \), provided \( g > 0 \). Hence, \( \frac{\sqrt{n_{1 n_2}}}{\sigma} \int_D f_i(t, s) g(t, s) P_0(dt, ds) \) converges to \( \infty \), as \( n_1 \to \infty \) and \( n_2 \to \infty \), finishing the proof.

By the result in the preceding theorem, we can formulate the rejection region of the asymptotic test based on \( S_{n_1 n_2}(Y_{n_2 \times n_1}) \). Since \( \sum_{i=q+1}^{p} \left( \int_D f_i(t, s) dZ(t, s) \right)^2 \) follows a chi-square distribution with \( p-q \) degrees of freedom, denoted by \( \chi_{2p-q}^2 \), then \( H_0 \) will be rejected a level \( \alpha \), if and only if \( \frac{1}{\sigma_{n_1 n_2}} S_{n_1 n_2}(Y_{n_2 \times n_1}) \geq \chi_{1-\alpha}^2 \), where \( \chi_{1-\alpha}^2 \) is the \((1-\alpha)\) lower quantile of \( \chi_{2p-q}^2 \) distribution. Furthermore, when \( H_1 \) is true and \( g \) is positive function, then the probability of the rejection of \( H_0 \) can be approximated as follows:

\[
P \left\{ \frac{1}{\sigma_{n_1 n_2}} S_{n_1 n_2}(Y_{n_2 \times n_1}) \geq \chi_{2p-q}^2 | H_1 \right\} \approx P \left\{ \sum_{i=q+1}^{p} \left( \int_D f_i(t, s) dZ(t, s) \right)^2 \geq \chi_{1-\alpha}^2 \right\} = 1.
\]

By the preceding result, it is seen that to get the power of the test, we need to re-scale the regression function in amount of \( \frac{1}{\sqrt{n_{1 n_2}}} \), so that Model 1 becomes

\[
Y^{scl}(t, s) = \frac{1}{\sqrt{n_{1 n_2}}} g(t, s) + \varepsilon(t, s), \ (t, s) \in D, \ n_1 \geq 1 \ and \ n_2 \geq 1.
\]

Let \( Y_{n_2 \times n_1} = (Y^{scl}(t, s_k))_{t=1, k=1}^{n_1 n_2} \) be the triangular array of the random observations of the re-scaled model observed over the experimental design \( E_{n_1 \times n_2} \). Then, by applying the method in [14], for \( n_1 \to \infty \) and \( n_2 \to \infty \), it holds

\[
\frac{1}{\sigma_{n_1 n_2}} T_{n_1 \times n_2}(Y_{n_2 \times n_1}) \Rightarrow \frac{1}{\sigma} \varphi_g + Z.
\]

In the sequel the process \( W := \{ W(A) = \frac{1}{\sigma} \varphi_g(A) + Z(A) \mid A \in B(D) \} \) will be called the partial sums limit model of (1). By applying the similar argument as that used in the proof of Theorem 2.1, we get the following convergence results

\[
\frac{1}{\sigma_{n_1 n_2}} S_{n_1 n_2}(Y_{n_2 \times n_1}) \Rightarrow \sum_{i=q+1}^{p} \left( \int_D f_i(t, s) dZ(t, s) \right)^2.
\]

when \( H_0 \) is true. Conversely, when \( H_1 \) is true, it holds

\[
\frac{1}{\sigma_{n_1 n_2}} S_{n_1 n_2}(Y_{n_2 \times n_1}) \Rightarrow \sum_{i=q+1}^{p} \left( \frac{1}{\sigma} \int_D f_i(t, s) d\varphi_g(t, s) + \int_D f_i(t, s) dZ(t, s) \right)^2.
\]
It can be said that the test based on unscaled array of observations and that based on the array of re-scaled observations are equivalent in the sense both have the same rejection region of size $\alpha$. However, the second test gives a byproduct in that the behavior of the test can be analyzed by investigating the limiting power under $H_1$ which is given by

$$\Xi(g) := P \left\{ \sum_{i=q+1}^p \left( \frac{1}{\sigma} \int_D f_i(t, s) d\varphi(t, s) + \int_D f_i(t, s) dZ(t, s) \right)^2 \geq \chi^2_{2p-q}\{g\} \right\}$$

It is clear that for the asymptotic test of size $\alpha$, we have $\Xi(g) = \alpha$, when $g \in W$. In case $\Xi(g) > \alpha$, for $g \in V$, we say that the test is consistent, cf. Lehmann and Romano [18]. In the next section we show that $\Xi(g)$ coincides with a likelihood ratio test for testing the mean of the partial sums limit model of (1).

3. Likelihood ratio test

In this section we derive a likelihood ratio test for testing $\varphi_g$. Corresponding to $W$ and $V$, let us define finite dimensional subspaces $W_{\mathcal{H}_Z} := [\varphi_{f_1}, \ldots, \varphi_{f_p}]$ and $V_{\mathcal{H}_Z} := [\varphi_{f_1}, \ldots, \varphi_{f_r}, \varphi_{f_{r+1}}, \ldots, \varphi_{f_s}]$, where $\varphi_{f_j}(A) = \int_A f_j(x, y) P_0(dx, dy)$. Both subspaces are subspaces of an infinite dimensional subspace $\mathcal{H}_Z$ which is called the reproducing kernel Hilbert space (RKHS) of $Z$, defined by

$$\mathcal{H}_Z := \\{ h : \exists u \in L_2(D, P_0), \ h(A) = \int_A u(t, s) P_0(dt, ds) \}.$$  

The inner product and norm on $\mathcal{H}_Z$ are given by

$$\langle h_1, h_2 \rangle_{\mathcal{H}_Z} := \langle u_1, u_2 \rangle_{L_2(D, P_0)} \text{ and } \| h \|^2_{\mathcal{H}_Z} := \| u \|^2_{L_2(D, P_0)}.$$

Some basic important facts about the space $\mathcal{H}_Z$ are summarized here. Since $L_2(D, P_0)$ is a Hilbert space, cf. [17], and the mapping $L_2(D, P_0) \ni u \mapsto \varphi_u \in \mathcal{H}_Z$ is one to one, then under the inner product and norm defined above, it is clear that $\mathcal{H}_Z$ is immediately a Hilbert space. Since both $W_{\mathcal{H}_Z}$ and $V_{\mathcal{H}_Z}$ are finite dimensional subspaces of $\mathcal{H}_Z$, hence they are closed subsets of $\mathcal{H}_Z$. Furthermore, $W_{\mathcal{H}_Z}$ and $V_{\mathcal{H}_Z}$ are bounded by the straightforward consequence of the fact that $W_{\mathcal{H}_Z}$ is isomorphic to $\mathcal{R}^q$ and $V_{\mathcal{H}_Z}$ is isomorphic to $\mathcal{R}^p$. The last two arguments ensure that $W_{\mathcal{H}_Z}$ as well as $V_{\mathcal{H}_Z}$ are compact subspaces.

**Proposition 2.2** For the space $W_{\mathcal{H}_Z}$, it holds $g \in W$ if and only if $\varphi_g \in W_{\mathcal{H}_Z}$. Similarly, $g \in V$ if and only if $\varphi_g \in V_{\mathcal{H}_Z}$. Furthermore, $\{ f_1, \ldots, f_q, f_{q+1}, \ldots, f_p \} \subset L_2(D, P_0)$ is orthonormal if and only if $\{ \varphi_{f_1}, \ldots, \varphi_{f_q}, \varphi_{f_{q+1}}, \ldots, \varphi_{f_p} \} \subset \mathcal{H}_Z$ is orthonormal.

**Proof:** Let $g \in W$, then there exist constants $\beta_1, \beta_2, \ldots, \beta_q$, such that $g = \sum_{j=1}^q \beta_j f_j$. For every $A \in \mathcal{B}(G)$, we have

$$\varphi_g(A) = \int_A g(t, s) P_0(dt, ds) = \int_A \sum_{j=1}^q \beta_j f_j(t, s) P_0(dt, ds)$$

$$= \sum_{j=1}^q \beta_j \int_A f_j(t, s) P_0(dt, ds) = \sum_{j=1}^q \beta_j \varphi_{f_j}(A).$$

Since the last equation holds for every $A \in \mathcal{B}(D)$, we conclude that $\varphi_g \in W_{\mathcal{H}_Z}$. Conversely, suppose that $\varphi_g \in W_{\mathcal{H}_Z}$, then there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_q$, such that $\varphi_g = \sum_{j=1}^q \alpha_j \varphi_{f_j}$.
Hence, for arbitrary \( A \in \mathcal{B}(\mathbf{D}) \), we have

\[
\int_A g(t, s) P_0(dt, ds) = \sum_{j=1}^q \alpha_j \int_A f_j(t, s) P_0(dt, ds) = \int_A \sum_{j=1}^q \alpha_j f_j(t, s) P_0(dt, ds)
\]

\[
\iff \int_A (g(t, s) - \sum_{j=1}^q \alpha_j f_j(t, s)) P_0(dt, ds) = 0
\]

\[
\iff g = \sum_{j=1}^q \alpha_j f_j, \ P_0 - \text{almost surely}.
\]

Thus there exist \( \alpha_1, \alpha_2, \ldots, \alpha_q \), such that \( g = \sum_{j=1}^q \alpha_j f_j \). This means that \( g \in \mathcal{W} \). Similar assessment for \( \mathcal{V}_{\mathcal{H}_Z} \) can be proved analogously. The last statement is immediate consequent of the definition of the inner product and norm on \( \mathcal{H}_Z \), establishing the proof.

Theorem 2.1 and Proposition 2.2 guarantee that testing Hypotheses 2 can also be handled asymptotically by testing the following hypotheses

\[
H_0 : \varphi_g \in \mathcal{W}_{\mathcal{H}_Z} \text{ versus } H_1 : \varphi_g \in \mathcal{V}_{\mathcal{H}_Z}
\]

while observing the partial sums limit model \( \mathcal{W} = \{ \mathcal{W}(A) \mid A \in \mathcal{B}(\mathbf{D}) \} \). In this paper we propose LR-test which is defined by comparing the likelihood function of the observation \( \mathcal{W} \) under \( H_0 \) and under \( H_0 \cup H_1 \), see [18]. For that we need to derive the probability density functions of the process \( \mathcal{W} \) under \( H_0 \) as well as under \( H_0 \cup H_1 \).

The following theorem gives the density function of the probability measure of the process \( \mathcal{W} \) with respect to the probability measure of \( Z \). The proof is omitted, since it can be straightforwardly verified by studying either Theorem 5.1 in Lifshits (2011) or the Cameron-Martin-Girsanov formula documented in [20, 21], see also [8] for multivariate process.

**Theorem 2.3** ([20, 21]) Let \( \mathbf{P}^{\varphi_g + Z} \) and \( \mathbf{P}^Z \) be the probability measure of \( \mathcal{W} \) and \( Z \), respectively, where for every \( B \in \mathcal{B}(\mathcal{C}(\mathbf{D})) \), we define \( \mathbf{P}^{\varphi_g + Z}(B) := \mathbf{P}^Z(B - \varphi_g) \). The space \( \mathcal{B}(\mathcal{C}(\mathbf{D})) \) is the Borel \( \sigma \)-algebra of the subsets of \( \mathcal{C}(\mathbf{D}) \). Then, the Radon-Nykodym density of \( \mathcal{W} \) with respect to \( Z \) is given by

\[
\frac{d\mathbf{P}^{\varphi_g + Z}}{d\mathbf{P}^Z}(\mathcal{W}) = \exp \left\{ \frac{1}{\sigma} \int_{\mathbf{D}} g(t, s) d\mathcal{W}(t, s) - \frac{1}{2\sigma^2} \| \varphi_g \|^2_{\mathcal{H}_Z} \right\},
\]

for \( \mathbf{P}^Z \)-almost all \( \mathcal{W} \in \mathcal{B}(\mathcal{C}(\mathbf{D})) \). The integral involved in this formula is interpreted in the sense of Wiener integral.

In this section we derive the LR-test for Hypotheses 4 based on the Radon-Nykodym density of the shifted measure \( \mathbf{P}^{\varphi + Z} \) with respect to \( \mathbf{P}^Z \). By referring to [18], the LR statistic is defined as the ratio between the likelihood for the condition under \( H_0 \) and that under \( H_0 \cup H_1 \). More precisely, let \( \Lambda \) be the LR statistic, then

\[
\Lambda := \sup_{\varphi \in \mathcal{W}_{\mathcal{H}_Z}} \exp \left\{ \frac{1}{\sigma} \int_{\mathbf{D}} g(t, s) d\mathcal{W}(t, s) - \frac{1}{2\sigma^2} \| \varphi \|^2_{\mathcal{H}_Z} \right\} / \sup_{\varphi \in \mathcal{W}_{\mathcal{H}_Z} \cup \mathcal{V}_{\mathcal{H}_Z}} \exp \left\{ \frac{1}{\sigma} \int_{\mathbf{D}} g(t, s) d\mathcal{W}(t, s) - \frac{1}{2\sigma^2} \| \varphi \|^2_{\mathcal{H}_Z} \right\}.
\]

Since, \( \mathcal{W}_{\mathcal{H}_Z} \subseteq \mathcal{V}_{\mathcal{H}_Z} \), the optimization over \( \mathcal{W}_{\mathcal{H}_Z} \cup \mathcal{V}_{\mathcal{H}_Z} \) reduces to that over \( \mathcal{V}_{\mathcal{H}_Z} \) only.
Theorem 3.1 Suppose that \( \{ \varphi_{f_1}, \ldots, \varphi_{f_q} \} \) and \( \{ \varphi_{f_{q+1}}, \ldots, \varphi_{f_p} \} \) are orthonormal bases for \( W_{H_x} \) and \( V_{H_x} \), respectively. The LR-test for Hypotheses 4 will reject \( H_0 \) at a level of significance \( \alpha \), if and only if

\[
\sum_{i=q+1}^{p} \left( \int_{D} f_i(t, s) dW(t, s) \right)^2 \geq \chi_{1-\alpha}^{2p-q},
\]

where \( \chi_{1-\alpha}^{2p-q} \) is the \((1-\alpha)\)th quantile of the chi-square distribution with \( p-q \) degrees of freedom.

Proof: Without loss of generality, let us assume throughout that \( \sigma^2 = 1 \), otherwise \( \sigma^2 \) is estimated by a consistent estimator \( \hat{\sigma}_{11}^{2} \). We notice that since \( W_{H_x} \) and \( V_{H_x} \) are compact, the optimization problem on both spaces are well defined. Next, since the basis is fixed, we can re-parameterize the problem. By recalling Proposition 2.2 and Theorem 2.3, we have

\[
\arg \sup_{\varphi_{\beta} \in W_{H_x}} \exp \left\{ \int_{D} g(t, s) dW(t, s) - \frac{1}{2} \| \varphi_{\beta} \|_{H_x}^2 \right\} = \arg \sup_{(\beta_1, \ldots, \beta_q) \in \mathbb{R}^q} \exp \left\{ \sum_{i=1}^{q} \beta_i \int_{D} f_i(t, s) dW(t, s) - \frac{1}{2} \sum_{i=1}^{q} \beta_i^2 \right\} = \arg \sup_{(\beta_1, \ldots, \beta_q) \in \mathbb{R}^q} \exp \left\{ \sum_{i=1}^{q} \beta_i \int_{D} f_i(t, s) dW(t, s) - \frac{1}{2} \sum_{i=1}^{q} \beta_i^2 \right\}.
\]

The last optimization problem is well defined by the compactness of the Euclidean space \( \mathbb{R}^q \). Let \( L \) be the likelihood function, given by

\[
L(\beta_1, \ldots, \beta_q) := \sum_{i=1}^{q} \beta_i \int_{D} f_i(t, s) dW(t, s) - \frac{1}{2} \sum_{i=1}^{q} \beta_i^2.
\]

Then by using the differential method, we get for \( i = 1, \ldots, q \),

\[
\frac{\partial L(\beta_1, \ldots, \beta_q)}{\partial \beta_i} = 0 \iff \int_{D} f_i(t, s) dW(t, s) - \beta_i = 0 \iff \beta_i = \int_{D} f_i(t, s) dW(t, s).
\]

The Hessian matrix is clearly given by a negative identity matrix of order \( q \times q \) which is negative definite. Hence the solution for \( \beta_i \) presented above constitutes a local maximum in \( \mathbb{R}^q \). This leads us to get the following result

\[
\sup_{(\beta_1, \ldots, \beta_q) \in \mathbb{R}^q} \exp \left\{ \sum_{i=1}^{q} \beta_i \int_{D} f_i(t, s) dW(t, s) - \frac{1}{2} \sum_{i=1}^{q} \beta_i^2 \right\} = \exp \left\{ \sum_{i=1}^{q} \left( \int_{D} f_i(t, s) dW(t, s) \right)^2 - \frac{1}{2} \sum_{i=1}^{q} \left( \int_{D} f_i(t, s) dW(t, s) \right)^2 \right\} = \exp \left\{ \frac{1}{2} \sum_{i=1}^{q} \left( \int_{D} f_i(t, s) dW(t, s) \right)^2 \right\}. \tag{5}
\]

By applying the similar manner as for the optimization over \( W_{H_x} \), we have

\[
\sup_{\varphi_{\beta} \in V_{H_x}} \exp \left\{ \int_{D} g(t, s) dW(t, s) - \frac{1}{2} \| \varphi_{\beta} \|_{H_x}^2 \right\}
\]
The LR-test of (4) will reject $H_0$ if

$$\{ \sum_{i=1}^{p} \beta_i \int_{D} f_i(t, s)dW(t, s) - \frac{1}{2} \sum_{i=1}^{p} \beta_i^2 \}_{\beta_i, \ldots, \beta_p} \in \mathbb{R}^p$$

is true. We get

$$\exp\left\{ \sum_{i=1}^{p} \left( \int_{D} f_i(t, s)dW(t, s) \right)^2 - \frac{1}{2} \sum_{i=1}^{p} \left( \int_{D} f_i(t, s)dW(t, s) \right)^2 \right\}$$

Thus, by combining both (5) and (6), we further obtain the following expression for $\Lambda$:

$$\Lambda = \sup_{\varphi \in W_{\mathbb{N}_Z}} \exp \left\{ \frac{1}{\sigma} \int_{D} g(t, s)dW(t, s) - \frac{1}{2\sigma^2} \| \varphi \|_{\mathbb{N}_Z}^2 \right\}$$

The LR-test of (4) will reject $H_0$ at a level $\alpha \in (0, 1)$, if and only if $\Lambda \leq k$, where $k$ is a positive constant that satisfies $\mathbb{P}\{ \Lambda \leq k | H_0 \} = \alpha$. This means that the solution for $k$ can be obtained by determining the probability distribution of $\Lambda$ when $H_0$ is true. We get

$$\mathbb{P}\{ \Lambda \leq k | H_0 \} = \alpha$$

$$\iff \mathbb{P}\left\{ \exp \left\{ -\frac{1}{2} \sum_{i=q+1}^{p} \left( \int_{D} f_i(t, s)dW(t, s) \right)^2 \right\} \leq k | H_0 \right\} = \alpha$$

$$\iff \mathbb{P}\left\{ \sum_{i=q+1}^{p} \left( \int_{D} f_i(t, s)dW(t, s) \right)^2 \geq -2\ln k | H_0 \right\} = \alpha$$

$$\iff \mathbb{P}\left\{ \sum_{i=q+1}^{p} \left( \int_{D} f_i(t, s)d\varphi(t, s) + \int_{D} f_i(t, s)dZ(t, s) \right)^2 \geq -2\ln k | H_0 \right\} = \alpha$$

$$\iff \mathbb{P}\left\{ \sum_{i=q+1}^{p} \left( \frac{\beta_j}{\sum_{j=1}^{q}} \int_{D} f_i(t, s)f_j(t, s)dP_0 + \int_{D} f_i(t, s)dZ(t, s) \right)^2 \geq -2\ln k \right\} = \alpha$$

The last result follows from the fact that when $H_0$ is true, then $f_i \perp f_j$ in $L_2(D, P_0)$, for $i = q + 1, \ldots, p$ and $j = 1, \ldots, q$. Next, since $\int_{D} f_i(t, s)dZ(t, s), i = q + 1, \ldots, p$ are independent and identically distributed as $\mathcal{N}(0, 1)$, then we get

$$-2\ln k = \chi^2_{1-\alpha}(p-q) \iff k = \exp\left\{ -\frac{1}{2} \chi^2_{1-\alpha} \right\}$$

We note that the stochastic independence of $\int_{D} f_i(t, s)dZ(t, s)$, for $i = q + 1, \ldots, p$ follows as a consequence of the orthogonality of $\{f_{q+1}, \ldots, f_p\}$. We are done.
It is not like Neymann-Pearson (NP) test which is certainly a most powerful (MP) test for simple hypotheses, LR-test is not necessarily a uniformly most powerful (UMP) test, cf. [18].

The behavior of the LR-test with respect to other test for the same hypotheses can be evaluated by comparing their powers. For our LR-test, the power is defined on \( W_{H_2} \cup V_{H_2} \), given by

\[
\Xi(\varphi_g) := P \left\{ \sum_{i=q+1}^{p} \left( \int_{D} f_i(t,s) dW(t,s) \right)^2 \geq \chi^2_{1-\alpha} | \varphi_g \right\} = P \left\{ \sum_{i=q+1}^{p} \left( \int_{D} f_i(t,s) d\varphi_g(t,s) + \int_{D} f_i(t,s) dZ(t,s) \right)^2 \geq 2^{2p-q} \right\}
\]

(7)

If \( \varphi_g \) is in \( W_{H_2} \), then it can be shown that (7) reduces to

\[
\Xi(\varphi_g) = P \left\{ \sum_{i=q+1}^{p} \left( \int_{D} f_i(t,s) dZ(t,s) \right)^2 \geq \chi^2_{1-\alpha} \right\} = \alpha.
\]

Conversely, if \( \varphi_g \) is in \( V_{H_2} \), then (7) becomes

\[
\Xi(\varphi_g) = P \left\{ \sum_{i=q+1}^{p} \sum_{j=1}^{p} \beta_j \int_{D} f_i(t,s) f_j(t,s) P_0(dt,ds) + \int_{D} f_i(t,s) dZ(t,s) \right)^2 \geq 2^{2p-q} \}
\]

where \( \chi^{2p-q}(\lambda) \) is the noncentral chi-square distribution with the noncentral parameter \( \lambda \), defined by \( \lambda := \sum_{i=q+1}^{p} \beta_i^2 \). A test of size \( \alpha \) is a good test if the power under \( H_1 \) is larger than \( \alpha \). In the comparison with other test, the LR-test is better then other test, if \( \Xi(\varphi_g) \) is greater than that of other test, for every \( \varphi_g \in V_{H_2} \), cf. [18, 22].

By the last result it can bee seen that the limiting power function of the test based on the statistic \( \sum_{i=q+1}^{p} S_{n_1 n_2} (Y_{n_2 \times n_1}^{\text{d}}) \) for testing \( H_0 : g \in W \) against \( H_1 : g \in V \) coincides with the power function of the LR-test for testing \( H_0 : \varphi_g \in W_{H_2} \) against \( H_1 : \varphi_g \in V_{H_2} \). By this reason, the first test is said to be asymptotically an LR-test.

4. Simulation study

The purpose of the present section is to demonstrate the behavior of the LR-test defined in the preceding section for testing the appropriateness of the mean of the asymptotic model. We study two different hypotheses: \( H_0 \) : constant model against \( H_1 \) : first-order model and \( H_0 \) :
Figure 1. The graphs of the empirical power function of size \( \alpha \) LR-test for testing constant against first-order model.

first-order model against \( H_1 \) : second-order model. To be more specific, in both cases we sample the observations according to \( n \times n \) regular lattice on the unit rectangle \([0, 1] \times [0, 1]\).

In the first case we test \( H_0 : Y(t, s) = \beta_0 + \varepsilon(t, s) \) against \( H_1 : Y(t, s) = \beta_0 + \beta_1 t + \beta_2 s + \varepsilon(t, s) \), for \((t, s) \in [0, 1] \times [0, 1]\) which is equivalent with testing of \( H_0 : g \in W \) against \( H_1 : g \in V \), where \( W = [f_1] \) and \( V = [f_1, f_2, f_3] \), with \( f_1(t, s) = 1 \), \( f_2(t, s) = t \) and \( f_3(t, s) = s \). Hence, by applying Gramm-Schmidt procedure, the orthonormal version of these functions in \( L^2(D, \lambda^2) \) are given by \( \tilde{f}_1(t, s) = 1 \), \( \tilde{f}_2(t, s) = \sqrt{3}(2t - 1) \) and \( \tilde{f}_3(t, s) = \sqrt{3}(2s - 1) \), where \( \lambda^2 \) is the Lebesque measure on \([0, 1] \times [0, 1]\). Thus, by Proposition 2.2, \( W_{\mathcal{H}_Z} = [\varphi_{f_1}] \) dan \( V_{\mathcal{H}_Z} = [\varphi_{f_1}, \varphi_{f_2}, \varphi_{f_3}] \). In this case we generate the observations according to the following re-scaled model:

\[
Y_{scl}^{\ell k} = \frac{1}{n} \left( 1 + \rho \left( \frac{0.5 \ell}{n} + \frac{2k}{n} \right) \right) + \varepsilon_{\ell k}, \quad 1 \leq \ell, k \leq n,
\]

where \( \varepsilon_{\ell k} \) is generated independently from the standard normal distribution. The real constant \( \rho \) is chosen to vary in the closed interval \([-10, 10]\). If \( \rho = 0 \), then the observations are clearly from \( H_0 \), conversely, if \( \rho \neq 0 \), then the observations are from \( H_1 \), for every \( n \geq 1 \). The power function of size \( \alpha \) LR-test is approximated by the probability

\[
\Xi(\varphi_g) \approx P \left\{ \sum_{i=2}^{3} \left( \sum_{\ell=1}^{n} \sum_{k=1}^{n} \tilde{f}_i(\ell/n, k/n) \Delta_{\ell k} T_{n \times n}(Y_{scl}^{\ell k})_{(t_{n\ell}, s_{nk})} \right)^2 \geq \chi^2_{1-\alpha}(2) | \varphi_g \right\},
\]

where for \( \alpha = 0.01 \), and \( \alpha = 0.05 \), we have \( \chi^2_{0.99}(2) = 9.21034 \) and \( \chi^2_{0.95}(2) = 5.99147 \). In this simulation the function \( \varphi_g \) varies in the space \( W_{\mathcal{H}_Z} \cup V_{\mathcal{H}_Z} \) because of the variability of \( \rho \) in
the interval $[-10, 10]$. Consequently, $\Xi(\varphi_2)$ reduces to $\Xi(\rho)$. The simulation result of the first scenario is presented in Figure 1 which exhibits the graphs of the empirical power functions of the LR-test for $\alpha = 0.01$ (left panel) and $\alpha = 0.05$ (right panel) generated for the sample size of $50 \times 50$ and 1000 runs. The graphs show that the power attains the pre-specified levels of the test at $\rho = 0$, that is $\Xi(0) = \alpha$ as it should be. Furthermore, the test shows a tendency that the larger the magnitude of $\rho$, the greater the power. The larger the distance of the model from $H_0$, the greater the probability of the test to reject $H_0$. This means that the test has good ability in detecting whether the model from $H_0$ or $H_1$.

In the second simulation we consider the case of testing $H_0 : Y(t, s) = \beta_0 + \beta_1 t + \beta_2 s + \varepsilon(t, s)$ against $H_1 : Y(t, s) = f(t, s) + \varepsilon(t, s)$. That is we test $H_0 : g \in W$ against $H_1 : g \in V$, where $W = [f_1, f_2, f_3]$ and $V = [f_1, f_2, f_3, f_4, f_5, f_6]$, where $f_1(t, s) = 1$, $f_2(t, s) = t$, $f_3(t, s) = s$, $f_4(t, s) = t^2$, $f_5(t, s) = ts$ and $f_6(t, s) = s^2$. The corresponding Gram-Schmidt orthonormal bases functions in $L_2(D, \lambda^2)$ are given by $f_1(t, s) = 1$, $f_2(t, s) = \sqrt{3}(2t - 1)$, $f_3(t, s) = \sqrt{5}(2s - 1)$, $f_4(t, s) = \sqrt{3}(6t^2 - 6t + 1)$, $f_5(t, s) = \frac{1}{3}(4ts - 2t - 2s + 1)$ and $f_6(t, s) = \sqrt{5}(6s^2 - 6s + 1)$. Thus, $W_{H_2} = [\varphi_1, \varphi_2, \varphi_3]$ and $V_{H_2} = [\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6]$. For this scenario we generate the array of the observation by the following model

$$Y_{\ell k}^{\text{sel}} = \frac{1}{n} \left[ 1 + 2 \frac{0.5 \ell}{n} + 2 \frac{k}{n} \right] + \rho \left[ 2 \left( \frac{\ell}{n} \right)^2 + 0.5 \left( \frac{\ell}{n} \right) + 2.5 \left( \frac{k}{n} \right)^2 \right] \frac{1}{n} + \varepsilon_{\ell k},$$

for $1 \leq \ell, k \leq n$. The random error $\varepsilon_{\ell k}$ is generated independently from the standard normal distribution. As in the first case, the multiplication with a constant $\rho$ is intended to make $\varphi_2$ varies in the space $W_{H_2} \cup V_{H_2}$. It is clear by construction that the observations are viewed as those from $H_0$ when $\rho = 0$, otherwise they are from $H_1$. Hence the power function of the LR-test of size $\alpha$ now represented by the following quantity:

$$\Xi(\rho) \approx \mathbb{P} \left\{ \sum_{i=4}^{6} \sum_{\ell=1}^{n} \sum_{k=1}^{n} f_i(\ell/n, k/n) \Delta_{\ell k} T_{n \times n}(Y_{\ell k}^{\text{sel}}(t_{n \ell}, s_{nk}))^2 \geq \chi_{1-\alpha}^2(3) | \rho \right\}, n \geq 1,$$

where $\chi_{0.99}^2(3) = 11.34487$ and $\chi_{0.95}^2(3) = 7.814728$, for $\alpha = 0.01$ and $\alpha = 0.05$, respectively. Figure 2 presents the graphs of the empirical power functions of the size $\alpha$ LR-test, for $\alpha = 0.01$ (left panel) and $\alpha = 0.05$ (right panel), simulated under $50 \times 50$ sample size with 1000 runs. Similar results as in the first simulation can be declared in that the power functions achieve the pre-specified level of significance when it is evaluated at $\rho = 0$ which coincides with the condition under $H_0$. On the other hand, when $\rho$ is different from zero, the empirical power attains the values larger than $\alpha$. More precisely, for this case the larger the values of $\rho$, the greater the powers. This means that the larger the distance of the model from that under $H_0$, the greater the ability of the test in detecting the false model. All of these assessments are the properties that must be fulfilled by a test so that it can be claimed as a consistent test.

5. Application

In this section we study the application of the LR-test method in real data in which we consider a data consists of the maximum height attained by corn plants planted in an area of farm land of size 12m width running from west to east and 15.75m length running from south to north. The corns have been planted following a design in a form of a regular lattice with 16 x 21 equally spaced points where the distance between nearby points are 0.75m, see also [15]. Under a condition where every corn plant obtained the same treatments, the maximum height (in cm) of 336 corn plants have been measured after they have been reaching the age of eight weeks, where the measurements have been initiated in the point with the coordinate (0, 0) lies in the southwest corner. The three dimensional drop-line scatter plot of the observations with respect
to their corresponding coordinate of the point is presented in Figure 3, where $x$ axis lies in the west to the east direction, $y$ axis lies in the south to the north direction and $z$ axis represents the logarithm of the maximum height, denoted by Ln-Height. It can be seen that the Ln-Height is slightly increasing as the observations have been made to move away from the origin along either the $x$ or the $y$ axis. In particular the Ln-Height is rather strictly increasing for the measurements along the diagonal line as it moves away from the origin. The largest Ln-Height is owned by the corn plants lie in the northeast region. This characteristic of the corn plants can also be seen in the contour plot of the Ln-Height presented in Figure 4. The regions having large Ln-Height are exhibited with clear blue color, whereas those having small Ln-Height are presented with dark-blue color.

Our purpose is to build a regression model describing how the logarithm of the maximum height of the corn plants vary with respect to their coordinate of the positions on the farm land. To check the appropriateness of a proposed model we propose to apply stepwise the LR-test studied in the preceding sections. Our method is totaly different from the classical significance test using F-test documented in many standard references of regression analysis in that our test is conducted recursively until the $H_0$ is not rejected. We also do not need to test the normality of the observations.

By inspecting Figure 3, we start by proposing under $H_0$ that a constant model is fit to the data, whereas under the competing alternative $H_1$ we assume a first-order model. For the Ln-Height of the corn plants, we get after computation, the value of the test statistic is 178.1133 with the corresponding approximated $p$–value is given by 0.00006. So, it is reasonable to reject $H_0$, leading us to the conclusion that the constant model is not significant. We notice that the
The drop-line scatter plot of the logarithm of the maximum height (Log Height) of corn plants.

$p$-value is computed by the formula $p - value \approx P\{\chi^2 \geq 178.1133\}$. Since when assuming constant model $H_0$ is not yet rejected, we test in the next step the hypothesis that a first-order is fit against the alternative that a second-order model is fit. To this hypotheses, we get the value of the test statistic 1.0317 and the corresponding approximated $p$-value given by 0.79358 which lead us to the acceptance of $H_0$. Thus, we conclude that a polynomial of first-degree is appropriate to describe the variability of the variable Ln-Height over the region.

The least squares fitted model is given by $\hat{Y}(t, s) = 4.4972 + 0.29157t + 0.11326s$, $(t, s) \in D$, whose three dimensional perspective plot is exhibited in Figure 5. By this fitted model it can be further assessed that the logarithm of the maximum height of the corn plant increases in amount of 0.29157 unit for every 1m increment of the distance of the point from the origin along the west to east direction. Similarly, the Ln Height increases in amount of 0.11326 unit for every 1m increment of the distance in the south to north direction. In agricultural context, the height of the corn plant can reflect the fertility level of the farm land. So the fitted model can be further interpreted as the fertility model for the land farm where the corn pants have been planted. The model shows that the fertility level of the land farm gets larger as the points move away from the origin. The region with the highest fertility level lies in the north east subregion of the land farm.

6. Concluding remark
We have successfully developed a method for verifying the validity of an assumed model by observing the limit of set-indexed partial sums process of the original observation, which is given by a deterministic trend plus the set-indexed Brownian sheet as the noise. The appropriateness of the trend has been checked by mean of the LR-test which is constructed based on the ratio between the likelihood function under $H_0$ and under $H_0 \cup H_1$. The rejection region as well as the power of the size $\alpha$ LR-test has been obtained based on the Cameron-Martin-Girsanov formula of the Radon-Nikodym derivative of the set-indexed partial sums limit process of the observation with respect to the set-indexed Brownian sheet.
Figure 4. The contour plot of the logarithm of the maximum height (Log Height) of corn plants.

Figure 5. The three dimensional perspective plot of the fitted first-order polynomial model for the logarithm of the maximum height (Log Height) of corn plants.

The optimization problem in finite dimensional linear subspaces has been solved by differential method after a re-parameterization. However, this approach seems to be rather crucial. In the future research we need a rather direct method in finding the maximum likelihood estimator of the deterministic trend under $H_0$ as well as under $H_1$ without beforehand transforming the optimization problem to the parametric model. For this context we need to study the
optimization theory in normed space.

The simulation results have shown that the LR-test investigated in this paper seems to be a consistent test in the sense it maximizes the power under the alternative. However, the variability of the trend are represented by the variability of a real constant over a chosen closed interval. It is suggested for the future research that the power should be evaluated over a set of functions so that the simulation will represent the more general situation.

The condition encountered in the application can be more complicated than the present given example. The observation may be a vector of random variable with inherent covariance among the components. Therefore, in a forthcoming paper we expand the present univariate method to multivariate regression model. We show that our test becomes a serious competitor for the tests based on the multivariate residual partial sums processes studied in [23].

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