Determination of Steady-State Amplitudes in the Region of Dynamic Instability of a Viscoelastic Plate

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Abstract. The problem of determining the amplitude of oscillations in the region of dynamic instability of a rectangular plate, the material of which obeys the hereditary law of deformation, in a geometrically nonlinear formulation is considered. The differential equation of plate oscillations under the action of constant and variable loads in its plane was obtained. It has been found also the amplitude of steady-state oscillations and was drawn amplitude-frequency curve.

1. Introduction
In the article [1], the areas of dynamic instability of a rectangular plate loaded in its plane by forces uniformly distributed around the edges were investigated,

\[ N_x = N_{x0} + N_{x\theta} \cos \theta t, \quad N_y = N_{y0} + N_{y\theta} \cos \theta t \] (1)

the material of which obeys the hereditary Boltzmann-Volterra law with a weakly singular creep core [2]:

\[ R(t - s) = A e^{-\beta(t-s)} (t - s)^{\alpha-1} (0 < \alpha < 1). \] (2)

Three core parameters allow you to describe the behaviour of many polymeric materials.

The paper considers the solution of the problem of determining the amplitudes of oscillations in the region of dynamic instability, which requires taking into account additional factors leading to a nonlinear differential equation.

2. Problem setting
Consider a rectangular plate loaded with forces uniformly distributed over the ends. It is possible to determine the amplitude of oscillations of the plate within the instability regions, if we take into account the stretching of the middle surface. We assume that the plate deflections are finite and small enough so that the angles of rotation of the plate elements can be neglected compared to unity. This assumption corresponds to the equations of T. Karman:
\[ \nabla^2 \nabla^2 w = \frac{h}{D} \left( \frac{g^2}{h^2} + \frac{\partial^2 \Phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 \Phi}{\partial x \partial y} \frac{\partial w}{\partial x} + \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right) \]

\[ \nabla^2 \nabla^2 \Phi = E \left[ \frac{\partial^2 w}{\partial x^2} \right]^2 \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \]  

where \( \Phi \) is the stress function, \( \sigma_x = \frac{\partial^2 \Phi}{\partial y^2}, \sigma_y = \frac{\partial^2 \Phi}{\partial x^2}, \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y}, \)

\( E \) - modulus of elasticity, \( D \) - cylindrical stiffness, \( h \) – plate thickness.

We transform equations (3). Taking \( V \) constant, the relationship between strain and stress can be written in the following form:

\[ \varepsilon_x = \frac{1}{E} \left( \sigma_x - \nu \sigma_y \right) + \frac{1}{E} \int_0^t K(t - \theta) \left( \sigma_x - \nu \sigma_y \right) d\theta, \]

\[ \varepsilon_y = \frac{1}{E} \left( \sigma_y - \nu \sigma_x \right) + \frac{1}{E} \int_0^t K(t - \theta) \left( \sigma_y - \nu \sigma_x \right) d\theta, \]

\[ \gamma_{xy} = \frac{2(1 + \nu)}{E} \tau_{xy} + \frac{2(1 + \nu)}{E} \int_0^t K(t - \theta) \tau_{xy} d\theta, \]

\( E \) - instantaneous modulus of elasticity, \( K(t - s) \) - kernel resolution (2). We write the equations of continuity of deformations of the plane problem:

\[ \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{\theta \theta}}{\partial x \partial y} = \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2}, \]

\[ \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, \quad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \]

Substituting the deformations (5) with regard to relations (4) into equation (6), then instead of the second equation (3) we will have:

\[ \nabla^2 \nabla^2 \Phi + \int_0^t K(t - s) \nabla^2 \nabla^2 \Phi ds = E \left[ \frac{\partial^2 w}{\partial x^2} \right]^2 \]  

\[ \nabla^2 \nabla^2 w = \frac{h}{D} \left( \frac{\partial^2 \Phi}{\partial y^2} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - 2 \frac{\partial^2 \Phi}{\partial x \partial y} \frac{\partial w}{\partial x} + \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \frac{m \partial^2 w}{h} \right) \]

Equations (8) and (9) are used to solve the problem.

3. Solution technique
Consider a hinged plate at the edges. Suppose that the sides of the size remain fixed, one of the sides \( b \) \((x = a)\) freely moves relative to the second (\( x = 0 \)), fixed in the \( xOy \) plane and both sides \( b \) continue to be straight when the plate is moving. Load applied by side \( b \) \( x = a \):

\[
p_x = p_{x0} + p_{x1} \cos \theta t, \quad p_{y0} = N_{y0}/h, \quad p_{y0} = N_{y1}/h
\]

For the main region of instability, we look for an approximate solution in the form:

\[
w = f(t) \sin \frac{\pi x}{a} \cos \frac{\pi y}{b} = f(t) \bar{w}
\]

We substitute this solution into equation (8):

\[
\nabla^2 \nabla^2 \Phi + \int_0^t K(t-s) \nabla^2 \nabla^2 \Phi ds = E \frac{\pi^4}{2a^2 b^2} f^2(t) \left( \cos \frac{2\pi x}{a} + \sin \frac{2\pi y}{b} \right)
\]

We look for the solution of this equation in the following form:

\[
\Phi = -0.5(p_{x0} + p_{x1} \cos \theta t)y^2 - 0.5v(p_{y0} + p_{y1} \cos \theta t)x^2 + A(t) \cos \frac{2\pi x}{a} + B(t) \cos \frac{2\pi y}{b}
\]

Substituting (13) into (12), we obtain the equation:

\[
A(t) \left( \frac{2\pi}{a} \right)^4 \cos \frac{2\pi x}{a} + B(t) \left( \frac{2\pi}{b} \right)^4 \cos \frac{2\pi y}{b} + \int_0^t K(t-s)(A(s) \left( \frac{2\pi}{a} \right)^4 \cos \frac{2\pi x}{a} +
\]

\[
B(s) \left( \frac{2\pi}{b} \right)^4 \cos \frac{2\pi y}{b} ds = \frac{E \pi^4}{2a^2 b^2} f^2(t) \left( \cos \frac{2\pi x}{a} + \cos \frac{2\pi y}{b} \right)
\]

Equate the coefficients facing trigonometric functions in the left and right sides of the equation:

\[
A(t) + \int_0^t K(t-s)A(s)ds = \frac{E \ a^2}{32 \ b^2} f^2(t), \quad B(t) + \int_0^t K(t-s)B(s)ds = \frac{E \ b^2}{32 \ a^2} f^2(t)
\]

Solving integral equations (15), we find:

\[
A(t) = \frac{E \ a^2}{32 \ b^2} F(t), \quad B(t) = \frac{E \ b^2}{32 \ a^2} F(t), \quad \text{где} \quad F(t) = f^2(t) - \int_0^t f^2(s) R(t-s)ds
\]

Thus, the stress function takes the form:

\[
\Phi = -(p_{x0} + p_{x1} \cos \theta t)y^2 - (p_{y0} + p_{y1} \cos \theta t)x^2 +
\]

\[
\frac{E}{32} F(t) \left( \frac{a^2}{b^2} \cos \frac{2\pi x}{a} + \frac{b^2}{a^2} \cos \frac{2\pi y}{b} \right)
\]

Substituting the obtained stress function (17) into equation (9), we obtain the integro-differential equation for plate oscillations:
Using the Bubnov-Galerkin method, we obtain an equation for the definition of a function \( f(t) \) that can be converted to a dimensionless form:

\[
z'' - \epsilon \Phi \int_0^\tau (\tau - s) z(s) ds + \Phi (1 - 2 \mu \cos \tau) z + \eta \epsilon \Phi \left( z^2 - \int_0^\tau (\tau - s) z^2(s) ds \right) = 0,
\]

where the following notation is entered: \( z = f/h, \tau = \theta t \) - dimensionless time, the prime denotes a no-dimensional time derivative \( \tau \),

\[
I_1 = \int_0^b \int_0^a \frac{\pi^2}{8\alpha^2} \cos 2\pi \frac{\partial^2 \bar{w}}{\partial \alpha^2} \partial^2 \bar{w} \partial x^2 dxdy, \quad I_2 = \int_0^b \int_0^a \bar{w}^2 dxdy, \quad I_3 = \int_0^b \int_0^a \frac{\partial^2 \bar{w}}{\partial \alpha^2} \partial x^2 dxdy, \quad I_4 = \int_0^b \int_0^a \frac{\partial^2 \bar{w}}{\partial \alpha^2} \partial x^2 dxdy,
\]

\[
\omega^2 = DI_1/mI_2 \quad \text{- natural frequency of an unloaded plate}, \quad N_{1s} = I_1/DI_3 \quad \text{and} \quad N_{2s} = I_1/DI_4 \quad \text{- critical values of statically applied forces} \ N_{s0}, N_{s0} \ \text{with their independent action}, \quad N_* = \frac{N_{1s}N_{2s}}{N_{s0} + \nu N_{1s}} \quad \text{- critical parameter of longitudinal static load type} \ N_{s0} = -N_0, N_{y0} = -\nu N_0, N_{s0} = 0 [3]. \ \text{The edges} \ y = 0 \ \text{and} \ y = b \ \text{are fixed, therefore there are efforts} \ N_{y0}, \ \Omega^2 = \omega^2 \left( 1 - \frac{N_{s0}}{N_*} \right) \quad \text{- proper vibration frequency of the plate exposed to the constant components of longitudinal forces}, \mu = \frac{1}{2} \frac{N_{s0}}{N_* - N_{s0}} \quad \text{- excitation coefficient}, \eta = 12 \left( 1 - v^2 \right) \frac{I_5}{I_1}, \quad \epsilon = \omega^2/\Omega^2, \quad \phi = \Omega^2/\theta^2. \ \text{The amplitude of oscillations is found from the equation (19).}

4. Results

Consider the oscillations occurring at the main resonance \( \theta \approx 2\Omega \). In the first approximation, the steady-state solution is sought in the form:

\[
z(\tau) = a_i \sin \frac{\tau}{2} + b_i \cos \frac{\tau}{2}
\]

\[
(21)
\]
Substitute the solution (21) in equation (19):

\[
- \frac{a_i}{4} \sin \frac{\tau}{2} - \frac{b_i}{4} \cos \frac{\tau}{2} - \epsilon \phi \int_0^\tau R(\tau - s) \left( a_i \sin \frac{s}{2} + b_i \cos \frac{s}{2} \right) ds + \\
\phi(1 - 2\mu \cos \tau) \left( a_i \sin \frac{\tau}{2} + b_i \cos \frac{\tau}{2} \right) + \eta \phi \left( a_i \sin \frac{\tau}{2} + b_i \cos \frac{\tau}{2} \right)^3 - \\
\eta \phi \left( a_i \sin \frac{\tau}{2} + b_i \cos \frac{\tau}{2} \right) \int_0^\tau R(\tau - s) \left( a_i \sin \frac{s}{2} + b_i \cos \frac{s}{2} \right)^2 ds = 0.
\]

Equation (22)

Perform the necessary transformations of equation (22), we equal the coefficients at \( \sin \frac{\tau}{2} \) and at \( \cos \frac{\tau}{2} \) zero, we obtain the system of equations for \( a_i, b_i \):

\[
- p^2 a_i - \epsilon (a_iB_i + b_iA_i) + a_i(1 + \mu) + 0.25 \eta \epsilon C_i^2 (ga_i - b_iA_i) = 0, \\
- p^2 b_i - \epsilon (b_iB_i - a_iA_i) + b_i(1 - \mu) + 0.25 \eta \epsilon C_i^2 (gb_i + a_iA_i) = 0.
\]

In (23) the notation is used:

\[
R_0 = \int_0^\tau R(x) dx, \quad A_k = \int_0^\tau R(x) \sin \frac{kx}{2} dx, \quad B_k = \int_0^\tau R(x) \cos \frac{kx}{2} dx, \quad k = 1, 2
\]

\( C_i = \sqrt{a_i^2 + b_i^2} \) - steady state amplitude, \( g = 3 - 2R_0 - B_2. \)

To find nonzero solutions, we compose the determinant of system (23) with respect to \( a_i, b_i \):

\[
\begin{vmatrix}
- p^2 + \epsilon B_i - (1 + \mu) - 0.25 \eta \epsilon C_i^2 g & \epsilon (A_i + 0.25 \eta C_i^2 A_i) \\
\epsilon (A_i + 0.25 \eta C_i^2 g) & - p^2 + \epsilon B_i - (1 - \mu) - 0.25 \eta \epsilon C_i^2 g
\end{vmatrix} = 0.
\]

Opening the determinant and solving the resulting equation with respect to the steady-state amplitude, we find:

\[
C_i = \frac{2}{\sqrt{\eta}} \sqrt{g \left( p^2 - 1 + \epsilon B_i \right) - \epsilon A_i A_2 \pm \sqrt{\mu^2 \left( g^2 + A_i^2 \right) - \left( \left( p^2 - 1 + \epsilon B_i \right) A_2 + \epsilon A_i g \right)^2}}
\]

(26)

Calculate using formula (26) for the values of the kernel parameters \( \alpha = 0.25, \quad \beta = 3.37, \quad A = 0.25, \quad \epsilon = 1.25 \) \([1]\). The result of the calculations is shown in figure 1, where amplitude-frequency curves are constructed for two values of excitation coefficients \( \mu = 0.05, \quad \mu = 0.1; \quad C_i = C_i \sqrt{\eta}/2 \). It should be noted that in the considered example, the oscillation dragging occurs in the direction of increasing frequencies.
5. Conclusions
The paper presents the solution of the problem, as a result of which the steady-state oscillation amplitude was found in a geometrically non-linear formulation in the main region of the dynamic instability of a viscoelastic plate. For specific values of the parameters of the core, amplitude-frequency curves are constructed, by the example of which it is shown that the oscillation is delayed in the direction of increasing frequencies. Recently, articles have been published in which new results have been reported on the investigation of viscoelastic properties of materials and the development of methods for calculating the corresponding structures [4-5]

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