Regularization of $f(T)$ gravity theories and local Lorentz transformation

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We regularized the field equations of $f(T)$ gravity theories such that the effect of Local Lorentz Transformation (LLT), in the case of spherical symmetry, is removed. A “general tetrad field”, with an arbitrary function of radial coordinate preserving spherical symmetry is provided. We split that tetrad field into two matrices; the first represents a LLT, which contains an arbitrary function, the second matrix represents a proper tetrad field which is a solution to the field equations of $f(T)$ gravitational theory, (which are not invariant under LLT). This “general tetrad field” is then applied to the regularized field equations of $f(T)$. We show that the effect of the arbitrary function which is involved in the LLT invariably disappears.

1. Introduction

Amended gravitational theories have become very interesting due to their ability to provide an alternative framework for understanding the nature of dark energy. This is done through the modifications of the gravitational Lagrangian so as it render an arbitrary function of its original argument, for instance $f(R)$ instead of Ricci scalar $R$ in the Einstein-Hilbert action [1, 2, 3, 4].

Indeed there exists an equivalent construction of General Relativity (GR) dependent on the concept of parallelism. The idea is initially done by Einstein who had tried to make a unification between electromagnetism and gravity fields using absolute parallelism spacetime [5, 6]. This goal was frustrated by the lack of a Schwarzschild solution. Much later, the theory of absolute parallelism gained much attention as a modification theory of gravity, refereed to “teleparallel equivalent of general relativity” (TEGR) (cf. [7, 8, 9, 10, 11]). The basic block in TEGR is the tetrad field. The tetrad field consists of fields of orthonormal bases which belong to the tangent space of the manifold. Note that the contravariant tetrad field, $h_i^\mu$, has sixteen components while the metric tensor has only ten. However, the tetrads are invariant under local Lorentz rotations.

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The aim of the modification is to treat a more general manifold which comprises in addition to curvature a quantity called “torsion”. The curvature tensor, consists of a part without torsion plus part with torsion, is vanishing identically. One can generally use either the torsion-free part or the torsion part to represent the gravitational field. The most suitable way is to deal with the covariant tetrad field, $h^i_\mu$, and the so-called Weitzenböck spacetime \[12\]. The tetrad field describes fields of orthonormal bases, which are related to the tangent spacetime of the manifold with spacetime coordinates $x^\mu$. This tangent spacetime is Minkowski spacetime with the metric $\eta_{ij}$ that can be defined at any given point on the manifold.

Recently, modifications of TEGR have been studied in the domain of cosmology \[13\ [14\ [15\]. This is known as $f(T)$ gravity and is built from a generalized Lagrangian \[13\ [14\ [15\]. In such a theory, the gravitational field is not characterized by curved spacetime but with torsion. Moreover, the field equations are only second order unlike the fourth order equations of the $f(R)$ theory.

Many of $f(T)$ gravity theories had been analyzed in \[16\--\[28\]. It is found that $f(T)$ gravity theory is not dynamically equivalent to TEGR Lagrangian through conformal transformation \[29\]. Many observational constraints had been studied \[30\--\[33\]. Large-scale structure in $f(T)$ gravity theory had been analyzed \[34\ [35\]; perturbations in the area of cosmology in $f(T)$ gravity had been demonstrated \[36\--\[40\]; Birkhoff’s theorem, in $f(T)$ gravity had been studied \[41\]. Stationary solutions having spherical symmetry have been derived for $f(T)$ theories \[42\ [43\ [44\ [45\]. Relativistic stars and the cosmic expansion derived in \[46\ [47\].

Nevertheless, a major problem of $f(T)$ gravitational theories is that they are not locally Lorentz invariant and appear to harbour extra degrees of freedom.

The goal of this study is to regularize the field equations of $f(T)$ gravitational theory so that we remove the effect of LLT. We then apply a “general tetrad” field, which consists of two matrices the first is a solution to the non-invariant field equation of $f(T)$ and the second matrix is a local Lorentz transformation, to the amended field equations and show that the effect of LLT disappears.

In §2, a brief survey of the $f(T)$ gravitational theory is presented.

In §3, a “general tetrad” field, having spherically symmetric with an arbitrary function of the radial coordinate $r$ is applied to the field equations of the $f(T)$ which are not invariant under LLT. It is shown that the arbitrary function has affect in this application.

In §4, we derive the field equations of $f(T)$ which are invariant under LLT. We then apply these amended field equations to a “general tetrad” field. We show the effect of the arbitrary function invariably disappears.

The final section is devoted to discussion.
2. Brief review of $f(T)$

In the Weitzenböck spacetime, the fundamental field variables describing gravity are a quadruplet of parallel vector fields $h_i^\mu$, which we call the tetrad field. This is characterized by:

$$ D_\nu h_i^\mu = \partial_\nu h_i^\mu + \Gamma^\mu_{\lambda\nu} h_i^\lambda = 0, $$

where $\Gamma^\mu_{\lambda\nu}$ defines the nonsymmetric affine connection:

$$ \Gamma^\lambda_{\mu\nu} \overset{\text{def.}}{=} h_i^\lambda h^j_{\mu\nu}, $$

with $h_{i\mu\nu} = \partial_\nu h_{i\mu}^{\mu\nu}$.

Equation (1) leads to the metricity condition and the identical vanishment of the curvature tensor defined by $\Gamma^\lambda_{\mu\nu}$, given by equation (2). The metric tensor $g_{\mu\nu}$ is defined by

$$ g_{\mu\nu} \overset{\text{def.}}{=} \eta_{ij} h_i^\mu h_j^\nu, $$

with $\eta_{ij} = (-1, +1, +1, +1)$ that is metric of Minkowski spacetime. We note that, associated with any tetrad field $h_i^\mu$ there is a metric field defined uniquely by (3), while a given metric $g^{\mu\nu}$ does not determine the tetrad field completely, any LLT of the tetrad $h_i^\mu$ leads to a new set of tetrad which also satisfies (3).

The torsion components and the contortion are defined as:

$$ T^\alpha_{\mu\nu} \overset{\text{def.}}{=} \Gamma^\alpha_{\nu\mu} - \Gamma^\alpha_{\mu\nu} = h_a^\alpha (\partial_\mu h_a^\alpha - \partial_\nu h_a^\alpha), $$

$$ K^\mu_{\nu\rho} \overset{\text{def.}}{=} -\frac{1}{2} (T^\mu_{\nu\rho} - T^\nu_{\mu\rho} - T^\rho_{\nu\mu}), $$

where the contortion equals the difference between Weitzenböck and Levi-Civita connection, i.e., $K^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} - \{\mu_{\nu\rho}\}$.

One can defined the skew-symmetric tensor $S^\mu_{\alpha\nu}$ as

$$ S^\mu_{\alpha\nu} \overset{\text{def.}}{=} \frac{1}{2} (K^\mu_{\alpha\nu} + \delta^\mu_{\alpha} T^\beta_{\beta\nu} - \delta^\nu_{\beta} T^\beta_{\beta\mu}), $$

which is skew symmetric in the last two indices. The torsion scalar is defined as

$$ T \overset{\text{def.}}{=} T^\alpha_{\mu\nu} S^\mu_{\alpha\nu}. $$

Similar to the $f(R)$ theory, one can define the action of $f(T)$ theory as

$$ \mathcal{L}(h^\mu_\nu) = \int d^4x h \left[ \frac{1}{16\pi} f(T) \right], \quad \text{where} \quad h = \sqrt{-g} = \text{det} (h^i_\mu), $$

\*spacetime indices $\mu$, $\nu$, $\cdots$ and SO(3,1) indices $a$, $b$ $\cdots$ run from 0 to 3. Time and space indices are indicated by $\mu = 0, i$, and $a = (0), (i)$.\*
(assuming units in which $G = c = 1$). Considering the action in equation (7) as a function of the fields $h^i_\mu$ and putting the variation of the function with respect to the field $h^i_\mu$ to be vanishing, one can obtain the following equations of motion:

$$S^\mu_\nu T^\nu_\rho f(T)TT + \left[h^{-1}h^i_\mu \partial_\rho (hh_i^\alpha S^\alpha_\nu) - T^\alpha_\lambda T^\lambda_\mu \right] f(T)T - \frac{1}{4} S^\nu_\mu f(T) = -4\pi T^\nu_\mu,$$

where $T^\rho_\nu = \frac{\partial f(T)}{\partial f(T)}$, $f(T)T = \frac{\partial^2 f(T)}{\partial T^2}$ and $T^\nu_\mu$ is the energy momentum tensor.

In this study we are interested in studying the vacuum case of $f(T)$ gravity theory, i.e., $T^\nu_\mu = 0$.

### 3. Spherically symmetric solution in f(T) gravity theory

Assuming that the manifold is a stationary and spherically symmetric $(h^i_\mu)$ has the form:

$$\begin{pmatrix}
LA + HA_2 & LA_1 + HA_3 & 0 & 0 \\
-(LA_2 + HA) \sin \theta \cos \phi & -(LA_3 + HA_1) \sin \theta \cos \phi & -r \cos \theta \cos \phi & r \sin \theta \sin \phi \\
-(LA_2 + HA) \sin \theta \sin \phi & -(LA_3 + HA_1) \sin \theta \sin \phi & -r \cos \theta \sin \phi & -r \sin \theta \cos \phi \\
-(LA_2 + HA) \cos \theta & -(LA_3 + HA_1) \cos \theta & r \sin \theta & 0
\end{pmatrix},$$

where $A(r), A_1(r), A_2(r)$ and $A_3(r)$ are four unknown functions of the radial coordinate $r$, $L = L(r) = \sqrt{H(r)^2 + 1}$ and $H = H(r)$ is an arbitrary function. Tetrad fields (9) transform as

$$(h^i_\mu) = (A^i_j) (h^j_\mu)_1,$$

where $(h^j_\mu)_1$ is given by

$$\begin{pmatrix}
A(r) & A_1(r) & 0 & 0 \\
A_2(r) \sin \theta \cos \phi & A_3(r) \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
A_2(r) \sin \theta \sin \phi & A_3(r) \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
A_2(r) \cos \theta & A_3(r) \cos \theta & -r \sin \theta & 0
\end{pmatrix}.$$

The tetrad field (10) has been studied and it has been shown that the solution to the $f(T)$ gravitational theory has the form:

$$A = 1 - \frac{M}{r}, \quad A_1 = \frac{M}{r(1 - \frac{M}{r})}, \quad A_2 = \frac{M}{r}, \quad A_3 = \frac{1 - \frac{M}{r}}{1 - \frac{2M}{r}},$$

(11)
where $M$ is the gravitational mass. Equation (11) is an exact vacuum solution to field equations of $f(T)$ gravitational theory provided that
\[
  f(0) = 0, \quad f_T(0) \neq 0, \quad f_{TT} \neq 0. \tag{12}
\]

The LLT ($\Lambda^i_j$) has the form:
\[
  (\Lambda^i_j) = \begin{pmatrix}
    L & H \sin \theta \cos \phi & H \sin \theta \sin \phi & H \cos \theta \\
    -H \sin \theta \cos \phi & 1 + H_1 \sin^2 \theta \cos^2 \phi & H_1 \sin \theta \sin \phi \cos \phi & H_1 \sin \theta \cos \theta \sin \phi \\
    -H \sin \theta \sin \phi & H_1 \sin \theta \sin \phi \cos \phi & 1 + H_1 \sin^2 \theta \sin^2 \phi & H_1 \sin \theta \cos \theta \sin \phi \\
    -H \cos \theta & H_1 \sin \theta \sin \phi \cos \phi & H_1 \sin \theta \cos \theta \sin \phi & 1 + H_1 \cos^2 \theta
  \end{pmatrix}.
\tag{13}
\]

From the general spherically symmetric local Lorentz transformation, Eq. (13), one can generate the previous spherically symmetric solution [36].

Using Eq. (11) in Eq. (9), one can obtain $h = \text{det}(h^\mu_{\ a}) = r^2 \sin \theta$ and, with the use of Eqs. (4) and (5), we obtain the torsion scalar and its derivatives in terms of $r$
\[
  T(r) = \frac{4[(1 - MH')L + HH'[M - r] - L^2]}{r^2 L}, \quad \text{where} \quad H' = \frac{\partial H(r)}{\partial r},
\]
\[
  T'(r) = \frac{\partial T(r)}{\partial r} = -\frac{4\{r^2 H''[(r - M)H + ML] - r(M - r)H'^2 - 2MH'L^2[L - H] + 2L^3(1 - L)\}}{r^3 L^3}.
\tag{14}
\]

The field equations (7) have the form
\[
  4\pi T_0^0 = -\frac{f_{TT} T'[(M - r)H - ML]}{r(r - 2M)} + \frac{f_T}{r^2 L} \left[\frac{L(1 - MH') + HH'(M - r) - L^2}{r^2 L}\right] + \frac{f}{4},
\tag{15}
\]
\[
  4\pi T_0^1 = \frac{4f_{TT} T'[(M - r)H - ML]}{r},
\tag{16}
\]
\[
  4\pi T_1^1 = \frac{f_T\{(1 - MH')L + HH'(M - r) - L^2\}}{r^2 L} + \frac{f}{4},
\tag{17}
\]
\[
  4\pi T_2^2 = 4\pi T_3^3 = -\frac{f_{TT} T'\{M(1 + H) - r + L(r - M)\}}{2r^2} + \frac{f_T\{(1 - MH')L + (M - r)H H' - L^2\}}{r^2 L} + \frac{f}{4}.
\tag{18}
\]

Equations (14)-(18), show that the field equations of $f(T)$ are effected by the inertia which is located in the LLT given by Eq. (13). This effect is related to the non-invariance of the field equations of $f(T)$ gravitational theory under LLT.
4. Regularization of \( f(T) \) gravitational theory under LLT

The tetrad field \( (h^i_\mu) \) transforms under LLT as:

\[
(h^i_\mu) = (\Lambda^i_j(x))(h^j_\mu).
\] (19)

The derivatives of \( (h^i_\mu) \) has the form:

\[
\frac{\partial (h^i_\mu)}{\partial x^\nu} = (h^i_{\mu, \nu}) = (\Lambda^i_j(x))_{, \nu}(h^j_\mu) + (\Lambda^i_j(x))(h^j_\mu, \nu).
\] (20)

The non-symmetric affine connection constructed from the tetrad field \( (h^i_\mu) \) has the form

\[
\bar{\Gamma}^\mu_{\nu\rho} = \eta^{ij}(h^i_\mu)(h_{j\nu, \rho}).
\] (21)

Using equations (19) and (20) in (21) one gets

\[
\bar{\Gamma}^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + (\Lambda^i_j(x))(h^i_\mu)(\Lambda^j_k(x)),_{\rho}(h^k_\nu),
\] (22)

where \( \Gamma^\mu_{\nu\rho} \) is the non-symmetric affine connection constructed from the tetrad field \( (h^i_\mu) \) which is assumed to satisfy the field equation of \( f(T) \). Therefore, for \( \bar{\Gamma}^\mu_{\nu\rho} \) (which is effected by LLT) to be identical with \( \Gamma^\mu_{\nu\rho} \) (which is assumed to satisfy the field equation of \( f(T) \)) we must have

\[
(\bar{\Gamma}^\mu_{\nu\rho})_{\text{Regularized}} = \eta^{ij}(h^i_\mu)(h_{j\nu, \rho}) - (\Lambda^i_j(x))(h^i_\mu)(\Lambda^j_k(x)),_{\rho}(h^k_\nu).
\] (23)

Eq. (23) means that the affine connection is invariant under LLT in the linear case, i.e., \( f(T) = T \), which means that the extra degrees of freedom, six ones, are controlled. Also Eq. (23) breaks the restriction of teleparallelism. From Eq. (23) we have

\[
(\bar{\Gamma}^\mu_{\nu\rho})_{\text{Regularized}} \equiv \Gamma^\mu_{\nu\rho}.
\]

Therefore, if \( \Gamma^\mu_{\nu\rho} \) satisfies the field equations of \( f(T) \) then \( (\bar{\Gamma}^\mu_{\nu\rho})_{\text{Regularized}} \) need not to be a solution to the field equations of \( f(T) \) given by Eq. (8). The main reason for this is the second term in Eq. (8), i.e., \( h^{-1}h^i_\mu \partial_\rho(hh_\alpha^\alpha S_\alpha^{\mu\nu}) \). This term depends on the choice of the tetrad field. Using Eq. (23), the torsion, the contortion and \( \bar{S}_\alpha^{\mu\nu} \) tensors have the form

\[
\begin{align*}
(\bar{T}_\nu^\mu)_{\text{Regularized}} &= \bar{T}_\nu^\mu + (h^i_\mu)(\Lambda^i_j(x))\left\{ (\Lambda^j_k(x)),_{\nu}(h^k_\rho) - (\Lambda^j_k(x)),_{\rho}(h^k_\nu) \right\}, \\
(\bar{K}_\alpha^{\mu\nu})_{\text{Regularized}} &= -\frac{1}{2}\left[ (\bar{T}_\alpha^{\mu\nu})_{\text{Regularized}} - (\bar{T}_\alpha^{\nu\mu})_{\text{Regularized}} - (\bar{T}_\alpha^{\mu\nu})_{\text{Regularized}} \right], \\
(\bar{S}_\alpha^{\mu\nu})_{\text{Regularized}} &= \frac{1}{2}\left[ (\bar{K}_\alpha^{\mu\nu})_{\text{Regularized}} + \delta_\alpha^{\mu\nu} (\bar{T}_\beta^{\nu\beta})_{\text{Regularized}} - \delta_\alpha^{\nu\beta} (\bar{T}_\beta^{\mu\beta})_{\text{Regularized}} \right].
\end{align*}
\] (24)

Equation (24) shows that the torsion tensor (and all tensors constructed from it) is invariant under LLT. Using equation (24) in the field equations of \( f(T) \), one can easily see that the first,
third and fourth terms of the field equations (8) will be invariant under LLT, but the second term, \( \partial_\rho (h \bar{h}^\alpha_\alpha \tilde{S}^{\rho \nu}_\alpha) \), which depends on the derivative must take the following form

\[
(\partial_\rho \left[ h \bar{h}^\alpha_\alpha \tilde{S}^{\rho \nu}_\alpha \right])_{\text{Regularized}} = \partial_\rho \left( h \bar{h}^\alpha_\alpha \tilde{S}^{\rho \nu}_\alpha - h \left( \Lambda_b^a \right)_\rho \left( \bar{h}^\alpha_\alpha \tilde{S}^{\rho \nu}_\alpha \right) \right).
\]

(25)

Using equations (24) and (25) the invariance field equations of \( f(T) \) gravitational theory under LLT take the form:

\[
\left( \tilde{S}^{\rho \nu}_\mu \right)_{\text{Regularized}} \left( \bar{T}_\mu \rho \right)_{\text{Regularized}} f(\bar{T})_{\bar{T}\bar{T}} + \left[ h^{-1} \bar{h}^\nu_\mu \left( \partial_\rho \left[ h \bar{h}^\alpha_\alpha \tilde{S}^{\rho \nu}_\alpha \right] \right)_{\text{Regularized}} - \left( \bar{T}_\alpha^\alpha \lambda_\mu \right)_{\text{Regularized}} \left( \tilde{S}^{\nu \lambda}_\alpha \right)_{\text{Regularized}} f(\bar{T})_{\bar{T}} + \frac{1}{4} \delta^\nu_\mu f(\bar{T}) = 4\pi T^{\nu \mu},
\]

(26)

where \( (\bar{T})_{\text{Regularized}} = (\bar{T}_\mu^\nu \bar{S}^{\mu \nu}_\alpha \bar{S}^{\alpha \nu}_\mu)_{\text{Regularized}} \).

Let us check if Eq. (26) when applied to the tetrad field (9) will indeed remove the effect of the inertia which appears in the LLT (13). Calculating the necessary components of the modified field equations (26) we get a vanishing quantity of the left hand side. This means that the tetrad field (9) is a solution to the \( f(T) \) field equations (26) which is invariant under LLT.

5. Discussion and conclusion

In this paper we have addressed the problem of the invariance of the field equations of \( f(T) \) gravitational theory under LLT. We first used a “general tetrad field” which contained five unknown functions in \( r \). This tetrad field has been studied \[45\] and a special solution has been obtained. This solution is characterized by its scalar torsion is vanishes.

We rewrite this tetrad field, “general tetrad field”, into two matrices. The first matrix represent a tetrad fields contains four unknown functions in \( r \). This tetrad field has been studied before in \[43\] and has been shown that it represent an exact solution within the framework of \( f(T) \) gravitational theories. The second matrix represent a LLT that satisfies

\[
(\Lambda^j_i) \eta_{jk} (\Lambda^k_m) = \eta_{im},
\]

(27)

and contains an arbitrary \( H(r) \).

We have applied the field equations of \( f(T) \) which are not invariant under LLT to the general tetrad field. We have obtained a set of non-linear differential equations which depend on the \( H(r) \). Therefore, We have regularized the field equations of \( f(T) \) gravitational theory such that it has been became invariant under LLT. Then, we have applied these invariant field equations to the generalized tetrad field. We have shown that this generalized tetrad field is an exact solution to the regularized field equations of \( f(T) \) gravitational theory.

\[ \text{§The details calculations of the non-vanishing components of the necessary quantities of the modified field equations (26) are given in Appendix A.} \]
The problem of the non-invariance of the field equations of \( f(T) \) under LLT is not a trivial task to tackle. The main reason for this is the following: We have the following known relation between the Ricci scalar tensor and the scalar torsion [35]

\[
R = -T - 2\nabla^\mu T^\rho_{\mu\rho} = T - \frac{2}{R} \partial^\mu (hT^\rho_{\mu\rho}). \tag{28}
\]

Last term in the R.H.S. of Eq. (28) is a total divergence term which has no effect on the field equations of TEGR, i.e., \( L(h^\rho_\mu) = \int d^4x h \left[ \frac{1}{16\pi} T \right] \), from this fact comes the well known name teleparallel equivalent of general relativity. However, this term, divergence term is the main reason that makes the field equations of \( f(T) \) non invariance under LLT. Let us explain this for some specific form of \( f(T) \). If

\[
f(R) = R + R^2 \equiv [-T - 2\nabla^\mu T^\rho_{\mu\rho}] + [-T - 2\nabla^\mu T^\rho_{\mu\rho}]^2
\]

\[
= -T - 2\nabla^\mu T^\rho_{\mu\rho} + T^2 + 4[\nabla^\mu T^\rho_{\mu\rho}]^2 + 4T\nabla^\mu T^\rho_{\mu\rho}, \tag{29}
\]

last term in the R.H.S. of Eq. (29) is not a total derivative term. Therefore, this term is responsible to make \( f(R) = R + R^2 \) when written in terms of \( T \) and \( T^2 \) is not invariant under LLT in contrast to the linear case, i.e., the form of Eq. (28). Same discussion can be applied to the general form of \( f(R) \) and \( f(T) \) which shows in general a different between the \( f(R) \) and \( f(T) \) gravitational theories that makes the field equation of \( f(R) \) to be of fourth order and invariant under LLT while \( f(T) \) is of second order and not invariant under LLT. Here in this study we tackle the problem of the invariance of the field equations of \( f(T) \) under LLT for specific symmetry, spherical symmetry. Although the method achieved in this study can be done for any symmetry however, we do not have the general local Lorentz transformation that has axial symmetry or homogenous and isotropic, \( \cdots \). This will be study elsewhere.

Appendix A

Calculations of the non-vanishing components of the necessary quantities of the modified field equations (26)

The non-vanishing components of \((\Lambda^b_a(x))_\phi:\)

\[
(\Lambda^0_{0,\phi}) = \frac{H(\Lambda^0_{1,\phi})}{L\sin \theta \cos \phi} = \frac{H'(\Lambda^0_{1,\phi})}{L\cos \theta \sin \phi} = \frac{-H'(\Lambda^0_{2,\phi})}{L\sin \theta \sin \phi} = \frac{H(\Lambda^0_{2,\phi})}{L\cos \theta \sin \phi} = \frac{-H'(\Lambda^0_{2,\phi})}{L\sin \theta \sin \phi} \]

\[
= \frac{-H'(\Lambda^0_{2,\phi})}{L\sin \theta \cos \phi} = \frac{H(\Lambda^0_{3,\phi})}{L\cos \theta} = \frac{-H'(\Lambda^0_{3,\phi})}{L\sin \theta} = \frac{H(\Lambda^0_{1,\phi})}{L\cos \theta \sin \phi} = \frac{-H'(\Lambda^0_{1,\phi})}{L\sin \theta \cos \phi} = \frac{-H'(\Lambda^0_{1,\phi})}{L\sin \theta \sin \phi} \]

\[
= -\frac{(\Lambda^1_{1,\phi})}{\sin^2 \theta \cos^2 \phi} = -\frac{(\Lambda^1_{2,\phi})}{\sin^2 \theta \cos \phi \sin \phi} = -\frac{H(\Lambda^2_{0,\phi})}{L\sin \theta \sin \phi} = \frac{-H'(\Lambda^2_{0,\phi})}{L\cos \theta \sin \phi} = -\frac{H'(\Lambda^2_{0,\phi})}{L\sin \theta \cos \phi} \]

\[
= -\frac{(\Lambda^3_{1,\phi})}{\sin \theta \cos \theta \sin \phi} = -\frac{(\Lambda^3_{1,\phi})}{\sin^2 \theta \cos \phi \sin \phi} = -\frac{(\Lambda^3_{1,\phi})}{\sin \theta \cos \theta \cos \phi} = -\frac{(\Lambda^3_{1,\phi})}{\sin \theta \cos \theta \cos \phi} \]
\[
\frac{(\Lambda^3_{2,r})}{\sin \theta \cos \theta \sin \phi} = \frac{(\Lambda^3_{3,r})}{\cos^2 \theta} = -\frac{H}{L} \frac{(\Lambda^3_{0,r})}{\cos \theta} = -\frac{H}{L} \frac{(\Lambda^3_{0,0})}{L \sin \theta} = \frac{HH'}{L},
\]

\[
(\Lambda^1_{1,0}) = -\cot \theta \cot \phi (\Lambda^1_{1,0}) = -\cot \theta \cot \phi (\Lambda^2_{2,0}) = \cot \phi (\Lambda^1_{1,0}) = \cot \phi (\Lambda^2_{1,0})
\]

\[
= 2 \sec 2 \phi \cos^2 \phi \cot \theta (\Lambda^1_{2,0}) = 2 \sec 2 \phi \cos^2 \phi \cot \theta (\Lambda^2_{2,0}) = \tan 2 \theta \cos \phi (\Lambda^1_{3,0})
\]

\[
= -2 \cos^2 \phi \csc \phi (\Lambda^1_{3,0}) = 2 \cot^2 \phi \cot \theta (\Lambda^2_{2,0}) = \cot^2 \phi (\Lambda^2_{2,0}) = \tan 2 \theta \cos^2 \phi \csc \phi (\Lambda^2_{3,0})
\]

\[
= 2 \cos \phi (\Lambda^2_{3,0}) = \tan 2 \theta \cos \phi (\Lambda^3_{1,0}) = -2 \cos^2 \phi \csc \phi (\Lambda^3_{1,0}) = 2 \tan 2 \theta \cos^2 \phi \csc \phi (\Lambda^3_{2,0})
\]

\[
= 2 \cos \phi (\Lambda^3_{2,0}) = -\cos^2 \phi (\Lambda^3_{3,0}) = -\sin 2 \theta \cos^2 \phi (L - 1).
\]

The non-vanishing components of the non-symmetric affine connection \((\tilde{\Gamma}^\mu_{\nu \rho})_{\text{Regularized}}\):

\[
(r_{01})_{\text{Regularized}} = -\frac{(r - 2M)}{r} \tilde{\Gamma}^{0}_{11} = \frac{1}{r^2} \tilde{\Gamma}^{0}_{22} \text{Regularized} = \frac{1}{r^2 \sin^2 \theta} \tilde{\Gamma}^{0}_{33} \text{Regularized} = \frac{M}{r(r - 2M)}.
\]

\[
(\tilde{\Gamma}^{1}_{01})_{\text{Regularized}} = -\frac{(r - 2M)}{r} \tilde{\Gamma}^{1}_{22} \text{Regularized} = -\frac{(r - 2M)}{r} \tilde{\Gamma}^{1}_{33} \text{Regularized} = -\frac{(r - 2M)}{r} \tilde{\Gamma}^{1}_{33} \text{Regularized} = -\frac{(r - 2M)}{r} \tilde{\Gamma}^{1}_{33} \text{Regularized}
\]

\[
= \frac{(r - 2M)}{r} \tilde{\Gamma}^{1}_{11} \text{Regularized} = \frac{M}{r^2 (r - M)} \tilde{\Gamma}^{1}_{22} \text{Regularized} = \frac{M}{r^2 \sin^2 \theta (r - M)} = \frac{M}{r^2},
\]

\[
(\tilde{\Gamma}^{2}_{12})_{\text{Regularized}} = \frac{(r - M)}{r} \tilde{\Gamma}^{2}_{13} \text{Regularized} = \frac{r - M}{r^2 (r - 2M)} \tilde{\Gamma}^{2}_{21} \text{Regularized} = \frac{r - M}{r^2 \sin^2 \theta (r - M)} = \frac{r - M}{r^2 \sin^2 \theta (r - M)}
\]

\[
= -\sin \theta \cos \theta, \quad (\tilde{\Gamma}^{3}_{23})_{\text{Regularized}} = -\frac{(r - 2M)}{r} \tilde{\Gamma}^{3}_{32} \text{Regularized} = \frac{1}{r},
\]

\[
(\tilde{\Gamma}^{3}_{33})_{\text{Regularized}} = \cot \theta.
\]

The non-vanishing components of the torsion \((\tilde{T}^\mu_{\nu \rho})_{\text{Regularized}}\):

\[
(\tilde{T}^{1}_{10})_{\text{Regularized}} = -(\tilde{T}^{1}_{01})_{\text{Regularized}} = -(\tilde{T}^{2}_{20})_{\text{Regularized}} = (\tilde{T}^{2}_{02})_{\text{Regularized}} = -(\tilde{T}^{3}_{30})_{\text{Regularized}}
\]

\[
= (\tilde{T}^{3}_{03})_{\text{Regularized}} = -\frac{M}{r^2},
\]

\[
(\tilde{T}^{0}_{01})_{\text{Regularized}} = -(\tilde{T}^{0}_{10})_{\text{Regularized}} = (\tilde{T}^{2}_{12})_{\text{Regularized}} = -(\tilde{T}^{2}_{21})_{\text{Regularized}} = (\tilde{T}^{3}_{13})_{\text{Regularized}} = \frac{M}{r^2}
\]

\[
= -(\tilde{T}^{3}_{31})_{\text{Regularized}} = -\frac{M}{r^2 (r - 2M)}.
\]

The non-vanishing components of \(\tilde{S}^{\mu \nu}_{\alpha}\):

\[
(\tilde{S}^{0}_{10})_{\text{Regularized}} = -(\tilde{S}^{0}_{01})_{\text{Regularized}} = -\frac{M}{r^2},
\]

\[
(\tilde{S}^{0}_{01})_{\text{Regularized}} = -(\tilde{S}^{0}_{10})_{\text{Regularized}} = -\frac{M}{r^2}.
\]
\[ (\tilde{S}_1^{10})_{Regularized} = - (\tilde{S}_1^{10})_{Regularized} = - \frac{M}{r(r - 2M)}. \]  

(33)

The non-vanishing components of \( \partial_\lambda [h_{\alpha}^a \tilde{S}_\alpha^{\rho\nu}] \) are:

\[ N_0^{01},_\theta = -N_0^{10},_\theta = M(L - H) \cos \theta, \]

\[ N_1^{10},_\theta = -N_1^{01},_\theta = \cot \phi N_2^{10},_\theta = - \cot \phi N_2^{01},_\theta = M(L + 1 - H) \sin \theta \cos \theta \cos \phi, \]

\[ N_3^{10},_\theta = -N_3^{01},_\theta = M(L - \tan^2 \theta - H) \cos^2 \theta, \]

\[ N_1^{10},_\phi = -N_1^{01},_\phi = - \tan \phi N_2^{10},_\phi = \tan \phi N_2^{01},_\phi = -M \sin^2 \theta \sin \phi. \]  

(34)

Using Eq (32) in Eq. (25) we get a vanishing components of \( (\partial_\rho [h_{\alpha}^a \tilde{S}_\alpha^{\rho\nu}])_{Regularized} \).

References

[1] S. Nojiri and S.D. Odintsov, arXiv:0807.0685.
[2] T.P. Sotiriou and V. Faraoni, Rev. Mod. Phys. 82 (2010), 451.
[3] A. De Felice and S. Tsujikawa, Living Rev. Rel. 13 (2010), 3.
[4] R. Durrer, R. Maartens, Published in Dark energy: Observational and theoretical approaches, ed. P Ruiz-Lapuente (Cambridge UP), (2010). arXiv:0811.4132 [astroph].
[5] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl., (1928) 217.
[6] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl., (1930) 401.
[7] C. Møller, Mat. Fys. Medd. Dan. Vid. Selsk. 39 (1978), 13.
[8] K. Hayashi and T. Shirafuji Phys. Rev. D19 (1979), 3524.
[9] K. Hayashi and T. Shirafuji Phys. Rev. D24, 3312 (1981).
[10] M.I. Mikhail and M.I. Wanas, Proc. Roy. Soc. London A 356 (1977), 471.
[11] C. Pellegrini and J. Plebański, Mat. Fys. Scr. Dan. Vid. Selsk. 2 (1963), no.4.
[12] R. Weitzenbock, Invariance Theorie, Nordhoff, Groningen, 1923.
[13] R. Ferraro and F. Fiorini, Phys. Rev. D 75 (2007), 084031.
[14] R. Ferraro and F. Fiorini, D 78 (2008), 124019.
[15] R. Ferraro and F. Fiorini, \textbf{D84} (2011), 083518.

[16] G. R. Bengochea and R. Ferraro, \textit{Phys. Rev. D} \textbf{79} (2009), 124019.

[17] Rong-Xin Miao, M. Li and Yan-Gang Miao, \textit{JCAP} \textbf{11} (2011), 033.

[18] H. Wei, Xiao-Jiao Guo and Long-Fei Wang, \textit{Phys. Lett. B707} (2012), 298.

[19] P. A. Gonzalez, E. N. Saridakis and Y. Vasquez, \textit{JHEP 1207} (2012), 053.

[20] R. Ferraro and F. Fiorini, \textit{IJMP (Conference Series)} \textbf{3} (2011), 227.

[21] L. Iorio and E. N. Saridakis, \textit{Mon. Not. Roy. Astron. Soc.} \textbf{427} (2012), 1555.

[22] N. Tamanini and C. G. Boehmer, \textit{Phys. Rev.} \textbf{D86} (2012), 044009.

[23] F.W. Hehl, P. Von Der Heyde, G. D. Kerlick and J. M. Nester, \textit{Rev. Mod. Phys.} \textbf{48} (1976), 393.

[24] F.W. Hehl, J. D. McCrea, E. W. Mielke and Y. Neeman, \textit{Phy. Rept.} \textbf{258} (1995), 1.

[25] E. V. Linder, \textit{Phys. Rev. D} \textbf{81} (2010), 127301.

[26] R. Myrzakulov, \textit{Eur. Phys. J. C} \textbf{71} (2011), 1752.

[27] K. Bamba, C.-Q. Geng, C.-C. Lee, and L.-W. Luo, \textit{J. Cosmol. Astropart. Phys.} \textbf{01} (2011), 021.

[28] P. Wu and H. Yu, \textit{Eur. Phys. J. C 71} (2011), 1552.

[29] R.-J. Yang, \textit{Europhys. Lett.} \textbf{93} (2011), 60001.

[30] G. R. Bengochea, \textit{Phys. Lett. B} \textbf{695} (2011), 405.

[31] P. Wu and H. Yu, \textit{Phys. Lett. B} \textbf{693} (2010), 415.

[32] Y. Zhang, H. Li, Y. Gong, and Z.-H. Zhu, \textit{J. Cosmol. Astropart. Phys.} \textbf{07} (2011), 015.

[33] H. Wei, X.-P. Ma, and H.-Y. Qi, \textit{Phys. Lett. B} \textbf{703} (2011), 74.

[34] B. Li, T. P. Sotirious and J. D. Barrow, \textit{Phys. Rev. D} \textbf{83} (2011), 064035.

[35] T. P. Sotirious, B. Li, J. D. Barrow, \textit{Phys. Rev. D} \textbf{83} (2011), 104030.

[36] T. Shirafuji, G.G. L. Nashed, and K. Hayashi, \textit{Prog. Theor. Phys.} \textbf{95} (1996), 665.

[37] J. B. Dent, S. Dutta, and E. N. Saridakis, \textit{J. Cosmol. Astropart. Phys.} \textbf{01} (2011), 009.

[38] S.-H. Chen, J. B. Dent, S. Dutta, and E. N. Saridakis, \textit{Phys. Rev. D} \textbf{83} (2011), 023508.
[39] R. Zheng and Q.-G. Huang, *J. Cosmol. Astropart. Phys.* **03** (2011), 002 (2011).

[40] Y.-F. Cai, S.-H. Chen, J. B. Dent, S. Dutta, and E. N. Saridakis, *Class. Quantum Grav.* **28** (2011), 215011.

[41] X.-h. Meng and Y.-b. Wang, [arXiv:1107.0629](http://arxiv.org/abs/1107.0629).

[42] T. Wang, *Phys. Rev. D* **84** (2011), 024042.

[43] G. G. L. Nashed, *Gen. Relat. Grav.* **45** (2013a), 1887.

[44] G. G. L. Nashed, *Phys. Rev. D* **88** (2013b), 104034.

[45] G. G. L. Nashed, *Astrophysics and space science* **348** (2013c), 591.

[46] C. Deliduman and B. Yapiskan, [arXiv:1103.2225](http://arxiv.org/abs/1103.2225).

[47] S. Capozziello, V. F. Cardone, H. Farajollahi, and A. Ravanpak, *Phys. Rev. D* **84** (2011), 043527.

[48] S. Capozziello, P. A. González E. N. Saridakis and Y. Vásquez, *JHEP* **1302** (2013), 039.