Strong disorder for a certain class of directed polymers in a random environment

Philippe CARMONA, Francesco GUERRA, Yueyun HU and Olivier MEJANE

February 10, 2022

Abstract

We study a model of directed polymers in a random environment with a positive recurrent Markov chain, taking values in a countable space $\Sigma$. The random environment is a family $(g(i,x), i \geq 1, x \in \Sigma)$ of independent and identically distributed real-valued variables. The asymptotic behaviour of the normalized partition function is characterized: when the common law of the $g(\ldots)$ is infinitely divisible and the Markov chain is exponentially recurrent we prove that the normalized partition function converges exponentially fast towards zero at all temperatures.

1 Introduction

In the model of directed polymers in random environment, we study a random Gibbs measure defined on the set of paths (of given length $n$) of a stochastic process. Usually one chooses for the underlying process a simple random walk on $\mathbb{Z}^d$ (see for instance [7], [11] or [3]) or $\mathbb{R}^d$ (see [9]). In this paper:

- The stochastic process is an irreducible Markov chain $(S_n)_{n \in \mathbb{N}}$ with countable state space $\Sigma$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_x, x \in \Sigma)$ with $\mathbb{P}_x(S_0 = x) = 1$. 

---

*P.Carmona: Laboratoire Jean Leray, UMR 6629, Université de Nantes, 92208, F-44322, Nantes cedex 03, e-mail: philippe.carmona@math.univ-nantes.fr
†F.Guerra: Dipartimento di Fisica, Università di Roma “La Sapienza”, Instituto Nazionale di Fisica Nucleare, Sezione di Roma 1, Piazzale Aldo Moro, 2 I-00185 Roma, Italy, e-mail: francesco.guerra@roma1.infn.it
‡Y.Hu: Laboratoire de Probabilités et Modèles Aléatoires (CNRS UMR-7599), Université Paris VI, 4 Place Jussieu, F-75252 Paris cedex 05, e-mail: hu@proba.jussieu.fr
§O.Mejane: Laboratoire de Statistique et Probabilités, Université Paul Sabatier, 118 route de Narbonne F-31062 Toulouse cedex 04, France, e-mail: Olivier.Mejane@lsp.ups-tlse.fr
• The environment is a family \((g(i,x), i \geq 1, x \in \Sigma)\) of non-degenerate i.i.d. random variables, distributed as a fixed random variable \(g\), defined on a probability space \((\Omega^{(g)}, \mathcal{F}^{(g)}, \mathbf{P})\), having some exponential moments

\[
\exists \beta_0 \in (0, +\infty], \forall |\beta| < \beta_0 : \mathbb{E}[e^{\beta g}] = e^{\lambda(\beta)} < +\infty. \tag{1}
\]

• The random energy is the Hamiltonian, defined on the space \(\Omega_n\) of paths of length \(n\) by

\[
H_n(g, \gamma) = \sum_{i=1}^{n} g(i, S_i).
\]

(If \(\Pi(\cdot, \cdot)\) is the transition matrix of the chain \(S\) then

\[
\Omega_n = \{\gamma = (\gamma(1), \ldots, \gamma(n)) : \Pi(\gamma(i-1), \gamma(i)) > 0, 2 \leq i \leq n\}.
\]

• For a given inverse temperature \(\beta > 0\), we introduce the Gibbs measure \(\langle \cdot \rangle^{(n)}\) on \(\Omega_n\) and the normalized partition function \(Z_n(\beta)\) according to the definitions:

\[
\langle f \rangle^{(n)} \overset{\text{def}}{=} \frac{\mathbb{E}_{\omega_0}(f(S)e^{\beta H_n(g, S) - n\lambda(\beta)})}{Z_n(\beta)},
\]

\[
Z_n(\beta) \overset{\text{def}}{=} \mathbb{E}_{\omega_0}(e^{\beta H_n(S) - n\lambda(\beta)}),
\]

for any bounded function \(f\) from \(\Omega_n\) to \(\mathbb{R}\). We will denote by \(\langle \cdot \rangle^{(n)}_2\) the product probability measure \(\langle \cdot \rangle^{(n)} \otimes \langle \cdot \rangle^{(n)}\) on \(\Omega_n^2\).

It is elementary to check that \((Z_n(\beta))_{n \geq 0}\) is a \(((G_n)_{n \geq 0}, \mathbf{P})\) positive martingale, if \((G_n)_{n \geq 0}\) denotes the natural filtration: \(G_n = \sigma(g(k,x), 1 \leq k \leq n, x \in \Sigma)\) for \(n \geq 1\) and \(G_0 = \{\emptyset, \Omega^g\}\). Hence \(Z_n(\beta) \to Z_\infty(\beta) \geq 0\) almost surely.

Using the terminology of Comets and Yoshida [4], we say there is weak disorder if \(Z_\infty(\beta) > 0\) a.s., and strong disorder if a.s. \(Z_\infty(\beta) = 0\).

When \((S_n)\) is the simple random walk on \(\mathbb{Z}^d\) and when the environment \(g\) is Gaussian, the picture is the following:

• if \(d \geq 3\) and \(\beta > \beta_1\) for some \(\beta_1 > 0\), there is strong disorder and almost surely \(Z_n(\beta)\) converges to zero exponentially fast.

• if \(d \geq 3\) and \(\beta < \beta_2\) for some \(\beta_2 > 0\), then there is weak disorder.

• if \(d = 1, 2\) then for any \(\beta > 0\) there is strong disorder (see [2, 3]) with exponential convergence of \(Z_n(\beta)\) to 0 if \(\beta\) is large enough, but the rate of convergence is still unknown for small \(\beta\).
It is not difficult to prove, by the method of second moment, that there is weak disorder for a “transient” Markov chain when $\beta$ is small, here by “transient” we mean that $\sum_{n, x} \mathbb{P}_{x_0}(S_n = x)^2 < +\infty$.

The aim of this paper is to prove that for a large class of positive recurrent Markov chain, and for fairly general random environments, almost surely $Z_n(\beta)$ converges to zero exponentially fast.

>From now on, we shall assume that the Markov chain $(S_n)_{n \in \mathbb{N}}$ is positive recurrent, and that the first return time to $x_0$, $\tau_{x_0} = \inf \{ n \geq 1 : S_n = x_0 \}$, has small exponential moments

$$\exists x_0 \in \Sigma, \exists \kappa > 0, \quad \mathbb{E}_{x_0}[e^{\kappa \tau_{x_0}}] < \infty \quad (EM)$$

Define

$$p_n(\beta) = \frac{1}{n} \mathbb{E}[\log(Z_n(\beta))], \quad 0 \leq \beta < \beta_0, \quad \beta_0 \in (0, +\infty].$$

Our main result is the following theorem:

**Theorem 1.** If the Markov chain $(S_n)_{n \geq 0}$ is irreducible, positive recurrent and satisfies (EM) and if the law of the random environment is infinitely divisible and satisfies (1), then

(a) for small $\beta > 0$, the free energy $p(\beta) = \lim_{n \to \infty} p_n(\beta)$ exists and

$$\frac{1}{n} \log Z_n(\beta) \to p(\beta), \quad a.s. \text{ and in } L^1.$$  

(b) the function $\beta \in [0, \beta_0) \to p_n(\beta)$ is non increasing.

(c) for all $0 < \beta < \beta_0$,

$$\limsup_{n \to \infty} p_n(\beta) < 0, \quad \forall \beta \in (0, \beta_0).$$

In particular, for any $0 < \beta < \beta_0$, almost surely $Z_n(\beta)$ converges to zero exponentially fast.

This paper is inspired by the works of Francesco Guerra and Fabio Toninelli (see [6]), who developed an interpolation technique to study the high temperature behaviour of the Sherrington-Kirkpatrick mean field spin glass model. The principal ingredient of the proof on the exponential decay is the interpolation between the random Hamiltonian $H_n(g, \gamma)$ and a deterministic Hamiltonian.

The paper is organized as follows:

- In Section 2, we evaluate the exponential moments of some additive functionals whose first consequence is the existence of the free energy $p(\beta)$ for small $\beta > 0$. Concentration of measure implies then the a.s. convergence $\frac{1}{n} \log Z_n(\beta) \to p(\beta)$.
• We devote Section 3 to an integration by parts formula, a feature of infinitely
divisible distributions, which entails the monotonicity of free energy (b).

• The last section contains the proof of Theorem 1.

Unless stated otherwise, we assume in the sequel that $\beta \in [0, \beta_0)$ and the random
environment $g$ is centered.

2 Exponential moments

Recall that $(S_n)$ is a Markov chain taking values in a countable set $\Sigma$ satisf-
ying (EM), and the environment variables $(g(i, x))$ are centered and have some ex-
ponential moments (see (1)).

Let us omit the dependence on $x_0$ of $\tau$ and denote the successive return times to $x_0$
by $\tau_0 = 0 < \tau_1 < \tau_2 < \ldots < \tau_n < \ldots$ For a bounded function $f : \Sigma \to \mathbb{R}$, we define
$\overline{f} = \sup_{x \in \Sigma} f(x), \underline{f} = \inf_{x \in \Sigma} f(x)$ and $\|f\|_{\infty} = \sup_{x \in \Sigma} |f(x)|$. The main result of this
section is the following theorem:

**Theorem 2.** Let $f : \Sigma \to \mathbb{R}$ be a bounded measurable function and
$|\beta| < \beta_0$ is
sufficiently small such that $\lambda(\beta) + 2\|f\|_{\infty} < \kappa$.

(i) There exists a unique real number $c(\beta, f) \in [\underline{f}, \overline{f} + \lambda(\beta)]$ such that

$$\frac{1}{n} \log \mathbb{E}_{x_0} \exp \left( \beta \sum_{i=1}^{\tau_n} g(i, S_i) + \sum_{i=1}^{\tau_n} f(S_i) - c(\beta, f) \tau_n \right) \to 0, \quad \text{a.s. and in } L^1.$$

(ii) We have

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{x_0} e^{\beta \sum_{i=0}^{\tau_n} g(i, S_i) + \sum_{i=0}^{\tau_n} f(S_i)} = c(\beta, f), \quad \text{a.s. and in } L^1.$$

The constant $c(\beta, f)$ does not depend on the starting point $x_0$, see the forthcoming
Remark 9. Taking $\beta = 0$ in Theorem 2 we can evaluate the following Varadhan’s
type integral

**Proposition 3.** For any bounded function $f : \Sigma \to \mathbb{R}$ such that $\|f\|_{\infty} < \kappa/2$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{x_0} e^{\sum_{i=0}^{\tau_{x_0} - 1} f(S_i)} = c(f),$$

where $c(f) \in [\underline{f}, \overline{f}]$ is the unique real number such that

$$\mathbb{E}_{x_0} \exp \left( \sum_{i=0}^{\tau(x_0) - 1} f(S_i) - c(f) \tau(x_0) \right) = 1.$$
According to the theory of large deviations, Proposition 3 is well-known at least for the case when \((S_n)\) is a Markov chain with finite states, for example by combining Dembo and Zeitouni ([5], pp. 75) and Ney and Nummelin ([8], Lemma 4.1). See also de Acosta and Ney ([1]) and the references therein for the large deviation principles for a Markov chain.

Taking \(f = 0\) in Theorem 2, we obtain the existence of the free energy at high temperature (recalling that \(g\) is centered):

**Proposition 4.** Let \(\beta < \beta_0\) be sufficiently small such that \(\lambda(\beta) < \kappa\).

(i) There exists a unique real number \(c(\beta) \in [0, \lambda(\beta)]\) such that

\[
\frac{1}{n} \log \mathbb{E}_x \exp \left( \beta \sum_{i=1}^{\tau_n} g(i, S_i) - c(\beta) \tau_n \right) \to 0, \quad \text{a.s. and in } L^1.
\]

(ii) We have

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_x e^\beta \sum_{i=1}^{\tau_n} g(i, S_i) = c(\beta), \quad \text{a.s. and in } L^1.
\]

Before entering into the proof of Theorem 2, we establish a preliminary result on the concentration of measure, which is essentially adapted from Comets, Shiga and Yoshida ([3], Proposition 2.9).

**Lemma 5.** (Concentration of measure) Let \(f : \Sigma \to \mathbb{R}\) be a bounded measurable function and \(|\beta| < \beta_0\). Denote by \(D_n(\beta, f) = \beta \sum_{i=1}^{\tau_n} g(i, S_i) + \sum_{i=1}^{\tau_n} f(S_i)\) and \(\bar{f} = \sup_{x \in \Sigma} f(x)\).

(i) Assume that \(\bar{f} + \lambda(\beta) < \kappa\). For any \(\varepsilon > 0\), there exists a \(n_1 = n_1(\beta, f, \varepsilon) < \infty\) such that for all \(n \geq n_1\),

\[
\mathbb{P} \left( \left| \frac{1}{n} \log \mathbb{E}_x e^{D_n} - \frac{1}{n} \mathbb{E} \log \mathbb{E}_x e^{D_n} \right| > \varepsilon \right) \leq e^{-\varepsilon^2 / 3n^{2/3}/6}.
\]

(ii) For any \(\varepsilon > 0\), there exists a \(n_2 = n_2(\beta, f, \varepsilon) < \infty\) such that for all \(n \geq n_2\),

\[
\mathbb{P} \left( \left| \frac{1}{n} \log \mathbb{E}_x e^{D_n} - \frac{1}{n} \mathbb{E} \log \mathbb{E}_x e^{D_n} \right| > \varepsilon \right) \leq e^{-\varepsilon^2 / 3n^{2/3}/4}.
\]

(iii) Assume that \(\bar{f} + \lambda(\beta) < \kappa\) and fix \(1 \leq a \leq b < \infty\). Then for any \(\varepsilon > 0\), there exists a \(n_3 = n_3(a, b, \beta, f, \varepsilon) < \infty\) such that for all \(n \geq n_3\),

\[
\mathbb{P} \left( \left| \frac{1}{n} \log \mathbb{E}_x (e^{D_n} | an \leq \tau_n \leq bn) - \frac{1}{n} \mathbb{E} \log \mathbb{E}_x (e^{D_n} | an \leq \tau_n \leq bn) \right| > \varepsilon \right) \leq e^{-\varepsilon^2 / 3n^{2/3}/5},
\]

with convention \(\mathbb{E}_x (\cdot | \emptyset) \equiv 1\).
Proof of Theorem 2: Using the same arguments (martingale decomposition, large deviations for martingale) as that of Comets, Shiga and Yoshida pp. 720–721, we obtain (ii) and the following inequality: For any $\epsilon > 0$ and $b > 1$, there exists a $n_4 = n_4(b, \beta, f, \epsilon) < \infty$ such that for all $n \geq n_4$ with $\mathbb{P}_{x_0}(\tau_n = k(n)) > 0$ and $k(n) \leq bn$,

$$\mathbb{P} \left( \left| \frac{1}{n} \log \mathbb{E}_{x_0} \left( e^{D_{\tau_n} 1(\tau_n = k(n))} \right) - \frac{1}{n} \mathbb{E} \log \mathbb{E}_{x_0} \left( e^{D_{\tau_n} 1(\tau_n = k(n))} \right) \right| > \epsilon \right) \leq e^{-\epsilon^2/2} n^{2/3}/4.$$  

(2)

Observe that

$$0 \leq \log \mathbb{E}_{x_0} \left( e^{D_{\tau_n} 1(\tau_n \leq mn \leq bn)} \right) - \max_{an \leq k \leq b} \log \mathbb{E}_{x_0} \left( e^{D_{\tau_n} 1(\tau_n = k)} \right) \leq \log b + \log n,$$

and for any $u > 0$,

$$\left\{ \max_{an \leq k \leq b} \log \mathbb{E}_{x_0} \left( e^{D_{\tau_n} 1(\tau_n = k)} \right) - \max_{an \leq k \leq b} \mathbb{E} \log \mathbb{E}_{x_0} \left( e^{D_{\tau_n} 1(\tau_n = k)} \right) \bigg| > u \right\} \subset \bigcup_{an \leq k \leq b} \left\{ \left| \log \mathbb{E}_{x_0} \left( e^{D_{\tau_n} 1(\tau_n = k)} \right) - \mathbb{E} \log \mathbb{E}_{x_0} \left( e^{D_{\tau_n} 1(\tau_n = k)} \right) \bigg| > u \right\}$$

The above two observations together with (2) imply (iii). To prove (i), we remark that

$$\lim_{b \to \infty} \limsup_{j \to \infty} \frac{1}{j} \mathbb{E} \left( \mathbb{E}_{x_0} \left( e^{D_{\tau_j} 1(\tau_j \geq bj)} \right) \right) = -\infty.$$  

(3)

In fact, we have from Fubini’s theorem and Chebychev’s inequality that

$$\mathbb{E} \left( \mathbb{E}_{x_0} \left( e^{D_{\tau_j} 1(\tau_j \geq bj)} \right) \right) \leq \mathbb{E}_{x_0} \left( e^{(\lambda(\beta) + T)\tau_j 1(\tau_j \geq bj)} \right) \leq e^{-\delta_0 bj} \mathbb{E}_{x_0} \left( e^{(\lambda(\beta) + T + \delta_0)\tau_j} \right)^j,$$

where $\delta_0 > 0$ denotes a small constant such that $\lambda(\beta) + T + \delta_0 < \kappa$. This yields (3). Finally, applying (iii) to $a = 1$ (since $\tau_n \geq n$) and a sufficiently large $b > 0$, we obtain (i). \hfill $\square$

Proof of Theorem 2: (i) Let $c_+ = \kappa - \lambda(\beta) - T > 0$. We shall show that the following function $\psi : (-\infty, c_+) \to \mathbb{R}$ is well-defined: for any $-\infty < c < c_+$,

$$\psi(c) \overset{\text{def}}{=} \lim_{n \to \infty} \left( \text{a.s. and in } L^1 \right) \frac{1}{n} \log \mathbb{E}_{x_0} \exp \left( \beta \sum_{i=1}^{\tau_n} g(i, S_i) + \sum_{i=1}^{\tau_n} f(S_i) + c \tau_n \right).$$

To this end, we shall apply the subadditivity theorem. For notational convenience, denote by

$$D_n = D_n(g, S) = \beta \sum_{i=1}^{n} g(i, S_i) + \sum_{i=1}^{n} f(S_i) + c n.$$
Using the strong Markov property at $\tau_n$, we have

$$
\mathbb{E}_{x_0} e^{D_{\tau_n+m}} = \sum_j \mathbb{E}_{x_0} \left( e^{D_{\tau_n} 1(\tau_n = j)} \right) \mathbb{E}_{x_0} e^{D_{\tau_m}(\theta_j g, S)} ,
$$

where $\theta_j$ denotes the shift operator on $g$: $\theta_j g(i, x) = g(i + j, x)$. By concavity,

$$
\log \mathbb{E}_{x_0} e^{D_{\tau_n+m}} = \log \mathbb{E}_{x_0} e^{D_{\tau_n}} + \log \sum_j \frac{\mathbb{E}_{x_0} \left( e^{D_{\tau_n} 1(\tau_n = j)} \right) \mathbb{E}_{x_0} e^{D_{\tau_m}(\theta_j g, S)}}{\mathbb{E}_{x_0} e^{D_{\tau_n}}} \geq \log \mathbb{E}_{x_0} e^{D_{\tau_n}} + \log \mathbb{E}_{x_0} e^{D_{\tau_m}(\theta_j g, S)} .
$$

Hence

$$
\mathbb{E} \log \mathbb{E}_{x_0} e^{D_{\tau_n+m}} \geq \mathbb{E} \log \mathbb{E}_{x_0} e^{D_{\tau_n}} + \mathbb{E} \log \mathbb{E}_{x_0} e^{D_{\tau_m}} ,
$$

and

$$
\psi(c) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log \mathbb{E}_{x_0} e^{D_{\tau_n}} = \sup_{n \geq 1} \frac{1}{n} \mathbb{E} \log \mathbb{E}_{x_0} e^{D_{\tau_n}} .
$$

The a.s. convergence follows from Lemma 5 (i) since $\tau_n \geq n$. As limit of convex and nondecreasing functions, $\psi(\cdot)$ is convex and nondecreasing. Moreover, $\psi : (-\infty, c_+) \to \mathbb{R}$ is strictly increasing since $\tau_n \geq n$. By Jensen’s inequality,

$$
\psi(c) \leq \log \mathbb{E}_{x_0} e^{(\lambda(\beta) + c + \overline{\beta}) \tau_1} ,
$$

which implies that $\psi(-(\overline{\beta} + \lambda(\beta))) \leq 0$. Again using Jensen’s inequality and the fact that $g$ is centered, we have

$$
\psi(c) \geq \mathbb{E} \log \mathbb{E}_{x_0} e^{D_{\tau_1}} \geq \mathbb{E} \log \mathbb{E}_{x_0} e^{\beta \sum_{i=1}^n g(i, S_i) + (c + \overline{f}) \tau_1} \geq (c + f) \mathbb{E}_{x_0} \tau_1 ,
$$

hence $\psi(-f) \geq 0$. It follows that there exists a unique real number $c = c(\beta, f) \in [\overline{f}, \overline{\beta} + \lambda(\beta)]$ such that $\psi(-c) = 0$, proving (i).

(ii) Define

$$
D_n = D_n(g, S) = \beta \sum_{i=1}^n g(i, S_i) + \sum_{i=1}^n f(S_i) - c(\beta, f) n .
$$

Then by (i),

$$
\frac{1}{j} \log \mathbb{E}_{x_0} e^{D_{\tau_j}} \to 0, \quad \text{a.s. and in } L^1 .
$$

(4)

We are going to prove that

$$
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{x_0} e^{D_n} = 0 , \quad \text{a.s.}
$$

(5)
It is not difficult to show that the family \((\frac{1}{n}\log E_{x_0}e^{D_n}, n \geq 1)\) is bounded in \(L^2\), in fact, by Jensen’s inequality, \(\frac{1}{n}\log E_{x_0}e^{D_n} \geq \frac{\beta}{n}E_{x_0}\sum_i g(i, S_i) + f - c(\beta, f)\). On the other hand, since the function \(x(\in \mathbb{R}_+) \mapsto \log^2(x + e)\) is concave,

\[
E\left(\max(0, \frac{1}{n}\log E_{x_0}e^{D_n})\right)^2 \leq \frac{1}{n^2}\log^2(e + E\log E_{x_0}e^{D_n}) = O(1).
\]

Therefore, the family \((\frac{1}{n}\log E_{x_0}e^{D_n}, n \geq 1)\) is uniformly integrable, which in view of (5) implies that \(\frac{1}{n}E\log E_{x_0}e^{D_n} \rightarrow 0\). This proves the \(L^1\) convergence part of (ii). It remains to show (5), whose proof is divided into two parts.

**Upper bound of (5):** Notice that \(\tau_j \geq j\); therefore, we have

\[
E_{x_0}e^{D_n} = \sum_{j=0}^{n-1} E_{x_0}\left(e^{D_{\tau_j}}1_{(\tau_j < n})\right)
\]

\[
= \sum_{j=0}^{n-1} E_{x_0}\left(e^{D_{\tau_j}}1_{(\tau_j < n)}\right)E_{x_0}\left[e^{D_k(\theta_{k+g}S)}1_{(k \leq \tau_j)}\right]_{k=n-\tau_j}
\]

\[
\leq M_n \sum_{j=0}^{n-1} E_{x_0}\left(e^{D_{\tau_j}}\right),
\]

where

\[
M_n = \max_{1 \leq k \leq n} E_{x_0}\left[e^{D_k(\theta_{k+g}S)}1_{(k \leq \tau_1)}\right] .
\]

Observe that

\[
EM_n \leq \sum_{k=1}^{n} E\left(E_{x_0}\left[e^{D_k(\theta_{k+g}S)}1_{(k \leq \tau_1)}\right]\right)
\]

\[
\leq \sum_{k=1}^{n} E_{x_0}\left[e^{(\lambda(\beta) + T - c(\beta, f))k}1_{(k \leq \tau_1)}\right]
\]

\[
\leq C n,
\]

with

\[
C = E_{x_0}\left[e^{(\lambda(\beta) + T - c(\beta, f))\tau_1}\right] \leq E_{x_0}\left[e^{(\lambda(\beta) + 2\|f\|_\infty)\tau_1}\right] < \infty.
\]

By Borel-Cantelli’s lemma, almost surely for all large \(n\),

\[
M_n \leq n^3.
\]

This together with the a.s. convergence in (4) imply the upper bound:

\[
\limsup_{n \to \infty} \frac{1}{n}\log E_{x_0}e^{D_n} \leq 0, \quad \text{a.s.} \quad (6)
\]

**Lower bound of (5):** By means of (3), for sufficiently large \(b > 0\),

\[
E\left(E_{x_0}\left(e^{D_{\tau_j}}1_{(\tau_j \geq b)}\right)\right) \leq e^{-2j},
\]

8
which in view of Borel-Cantelli’s lemma yields that \( P(d\omega) \) a.s. for all large \( n \geq n_0(\omega) \),

\[
\mathbb{E}_{x_0}\left(e^{D_j 1_{(\tau_j \geq b_j)}}\right) \leq e^{-j}. \tag{7}
\]

Then by (4) and (7), a.s. for all large \( j \geq j_0(\varepsilon, \omega) \),

\[
\liminf_{j \to \infty} \frac{1}{j} \log \mathbb{E}_{x_0}\left(e^{D_j 1_{(\tau_j < b_j)}}\right) \geq 0. \tag{8}
\]

Let \( \varepsilon > 0 \) be small. We divide the interval \([1, b]\) into \( K = K(\varepsilon) = [b/\varepsilon] \) intervals \([a_1, a_2], \ldots, [a_{K-1}, a_K]\) with \( a_1 = 1, a_K = b \) and \( a_k + 1 - a_k = \frac{b-1}{K} < \varepsilon \) for \( 1 \leq k \leq K-1 \). For any \( 0 \leq k \leq K-1 \), we may repeat the similar argument of subadditivity in (i) and apply the concentration of measure (Lemma 5, (iii)). This yields that

\[
\frac{1}{j} \log \mathbb{E}_{x_0}\left(e^{D_j 1_{(a_k \leq \tau_j \leq a_{k+1})}}\right) \to \gamma_k \quad \text{a.s. and in } L^1,
\]

for some deterministic constant \( \gamma_k \in [-\infty, 0] \) (\( \gamma_k \leq 0 \) because of (4)). Note that \( \gamma_k = -\infty \) if and only if for all \( j \geq 1 \), \( P_j(\tau_j \in [ja_k, ja_{k+1}]) = 0 \).

We claim that

\[
\max_{0 \leq k \leq K} \gamma_k = 0. \tag{10}
\]

Otherwise, since \( \gamma_k < 0 \) for each \( k < K \), \( \mathbb{E}_{x_0}\left(e^{D_j 1_{(a_k \leq \tau_j \leq a_{k+1})}}\right) \) converges to 0 exponentially fast; then \( \mathbb{E}_{x_0}\left(e^{D_j 1_{(\tau_j < b_j)}}\right) = \sum_{k=0}^{K-1} \mathbb{E}_{x_0}\left(e^{D_j 1_{(a_k \leq \tau_j \leq a_{k+1})}}\right) \) would also converge to 0 exponentially fast, which is in contradiction with (8). Then we proved (10).

Now, we proceed to show the lower bound. Choose a fixed \( k \in [0, K-1] \) such that \( \gamma_k = 0 \). Let \( j = \left\lceil \frac{n}{a_{k+1}} \right\rceil \). We have

\[
\mathbb{E}_{x_0} e^{D_n} \geq \mathbb{E}_{x_0}\left(e^{D_n 1_{(ja_k \leq \tau_j < ja_{k+1})}}\right) = \mathbb{E}_{x_0}\left(e^{D_j 1_{(ja_k \leq \tau_j < ja_{k+1})}}\mathbb{E}_{x_0}\left[e^{D_j(\theta_{a_k \ell} S)}\right]_{\ell = n - \tau_j}\right) \geq m_n \mathbb{E}_{x_0}\left(e^{D_j 1_{(ja_k \leq \tau_j < ja_{k+1})}}\right), \tag{11}
\]

where by our choice of \( j \) and \( a_k, \ell = n - \tau_j \leq n - ja_k \leq 2\varepsilon n \) and

\[
m_n = \min_{1 \leq \ell \leq 2\varepsilon n} \mathbb{E}_{x_0}\left[e^{D_j(\theta_{a_k \ell} S)}\right].
\]

By Jensen’s inequality,

\[
\mathbb{E}_{x_0}\left[e^{D_j(\theta_{a_k \ell} S)}\right] \geq e^{\mathbb{E}_{x_0}\left[D_j(\theta_{a_k \ell} S)\right]}
\]

9
Since \( f(x) - c(\beta, f) \geq f - c(\beta, f) \geq -(2\|f\|_\infty + \lambda(\beta)) > -\kappa \), we have

\[
\mathbb{E}_{\mathbf{x}_0} \left[ Df_\mathbf{0} - D_g f_\mathbf{0}, S \right] \geq \beta \sum_{i=1}^{\ell} \sum_{x} g(i + n - \ell, x) q_i(x) - \kappa \ell,
\]  

where we write \( q_i(x) = \mathbb{P}_{\mathbf{x}_0}(S_i = x) \) for notational convenience. Observe that \( \sum_{i=1}^{\ell} \sum_{x} g(i(x)) = \sum_{i=1}^{\ell} \mathbb{P}_{\mathbf{x}_0}(S_i = \tilde{S}_i) \leq \ell \leq 2en \), where \( \tilde{S} \) denotes an independent copy of \( S \). By Chebychev’s inequality, for any \( \nu > 0 \),

\[
\mathbb{P} \left( \sum_{i=1}^{\ell} \sum_{x} g(i + n - \ell, x) q_i(x) < -n^{2/3} \right) \leq e^{-\nu n^{2/3}} e^{\sum_{i=1}^{\ell} \lambda_{1/2}(\nu q_i(x))} \leq e^{-\frac{1}{2\nu}(0)},
\]

where in the last inequality, we choose \( \nu = \frac{n^{1/3}}{\kappa_0} \) and use the fact that \( \lambda(u) \sim \frac{\lambda'(0)}{2} u^2 \) for small \( u \). It turns out that

\[
\mathbb{P} \left( \min_{1 \leq \ell \leq 2en} \sum_{i=1}^{\ell} \sum_{x} g(i + n - \ell, x) q_i(x) < -n^{2/3} \right) \leq n e^{-\frac{1}{2\nu}(0)},
\]

whose sum on \( n \) converges. Hence \( \mathbb{P} \) a.s. for all large \( n \geq n_0(\omega) \), \( \min_{1 \leq \ell \leq 2en} \sum_{i=1}^{\ell} \sum_{x} g(i + n - \ell, x) q_i(x) \geq -n^{2/3} \) and therefore

\[
m_n \geq e^{-\beta n^{2/3} - 2\kappa \kappa n}.
\]

Plugging this into (11) and using (9) with \( \gamma_k = 0 \) by our choice of \( k \), we obtain that a.s.

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{\mathbf{x}_0} e^{D_n} \geq -2\varepsilon \kappa,
\]

for any \( \varepsilon > 0 \). The lower bound of (5) follows by letting \( \varepsilon \to 0 \). This together with the upper bound (6) complete the proof of Theorem 2. \( \square \)

**Remark 6.** When \( \beta = 0 \), the value of \( a_k \) in (11) can be easily determined by a change of probability measure.

We shall need the following corollary:

**Lemma 7.** Assume (EM). Let \( f \) be a bounded function from \( \Sigma \) to \( \mathbb{R} \). Then for all \( |t| < t_0 = \frac{-\kappa}{2\|f\|_\infty} \), the limit

\[
c(t) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{\mathbf{x}_0} \left( e^{\sum_0^{n-1} f(S_i)} \right)
\]

exists. Moreover, \( c \) is differentiable at 0 with \( c'(0) = \sum_{x \in \Sigma} f(x) \mu(x) \), where \( \{\mu(x), x \in \Sigma\} \) denotes the invariant probability measure of \( S \).
Proof.
Indeed, for $|t| < t_0$, the limit $c(t) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_{x_0} \left( e^{\sum_{i=0}^{n-1} f(S_i)} \right)$ exists. It is the unique real number $c$ such that $\phi(c, t) = 1$ where $\phi$ is the function

$$
\phi(c, t) = \mathbb{E}_{x_0} \exp \left( t \sum_{0}^{\tau(x_0) - 1} f(S_i) - c \tau(x_0) \right).
$$

Since $\phi$ is continuously differentiable in $(-t_0, t_0) \times J$ with $J$ and open interval, with derivatives

$$
\frac{\partial \phi}{\partial c} = -\mathbb{E}_{x_0} \left[ \tau(x_0) \exp \left( t \sum_{0}^{\tau(x_0) - 1} f(S_i) - c \tau(x_0) \right) \right],
$$

$$
\frac{\partial \phi}{\partial t} = \mathbb{E}_{x_0} \left[ \sum_{0}^{\tau(x_0) - 1} f(S_i) \exp \left( t \sum_{0}^{\tau(x_0) - 1} f(S_i) - c \tau(x_0) \right) \right]
$$

the implicit function theorem entails that $c(t)$ is differentiable in a neighborhood of $t = 0$ and

$$
c'(0) = \frac{\mathbb{E}_{x_0} \left[ \sum_{0}^{\tau(x_0) - 1} f(S_i) \right]}{\mathbb{E}_{x_0} \left[ \tau(x_0) \right]}.
$$

Since $f$ is bounded, hence $\mu$-integrable, the ergodic theorem implies

$$
c'(0) = \frac{\mathbb{E}_{x_0} \left[ \sum_{0}^{\tau(x_0) - 1} f(S_i) \right]}{\mathbb{E}_{x_0} \left[ \tau(x_0) \right]} = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_{x_0} \left[ \sum_{i=1}^{n} f(S_i) \right] = \langle \mu, f \rangle,
$$

with $\langle \mu, f \rangle = \sum_{x \in \Sigma} f(x) \mu(x)$.

We now prove that the constant $c(f)$ appearing in Proposition 3 does not really depend on the starting point $x_0$. Let $(S_n)$ is a Markov chain taking values in a countable set $\Sigma$. For any $x \in \Sigma$ define

$$
\kappa(x) = \sup \left\{ \alpha > 0 : \mathbb{E}_x \left[ e^{\alpha \tau(x)} \right] < +\infty \right\}, \quad \text{with } \tau(x) = \inf \{ n \geq 1 : S_n = x \}.
$$

Let $f : \Sigma \to \mathbb{R}$ be a bounded function. If $\|f\|_{\infty} < \kappa(x)$ then the following limit exists

$$
c(x, f) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_x \left[ e^{A_n} \right], \quad \text{with } A_n = \sum_{i=0}^{n-1} f(S_i).
$$

Different state points $x, y$ need to communicate to have the same coefficient.

Lemma 8. If $(S_n)$ is irreducible recurrent and $\|f\|_{\infty} < \frac{1}{2} \inf (\kappa(x), \kappa(y))$ then $c(x, f) = c(y, f)$. 

11
Proof. Since $(S_n)$ is irreducible recurrent, $\mathbb{P}_x(\tau(y) < +\infty) = 1$ and there exists $p \geq 1$ such that $\mathbb{P}_x(\tau(y) = p) > 0$. Thanks to the strong Markov property,

$$\mathbb{E}_x[e^{A_n}] \geq \mathbb{E}_x[e^{A_{n+1}} \mathbf{1}_{\{\tau(y) = p\}}] = \mathbb{E}_x[e^{A_p} \mathbf{1}_{\{\tau(y) = p\}}] \mathbb{E}_y[e^{A_{p-1}}] \cdot$$

Let $\varepsilon > 0$. There exists $n_0$ such that for all $n \geq n_0$, $\frac{1}{n} \mathbb{E}_x[e^{A_n}] \geq c(y, f) - \varepsilon$.

Therefore, if $n \geq n_0 + p$, then

$$\mathbb{E}_x[e^{A_n}] \geq e^{n(c(y, f) - \varepsilon)} \mathbb{E}_x[e^{A_{n+1}} \mathbf{1}_{\{\tau(y) = p\}}]$$

and this yields

$$c(x, f) = \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}_x[e^{A_n}] \geq c(y, f) - \varepsilon.$$

Letting $\varepsilon \to 0$ we get $c(x, f) \geq c(y, f)$. Substituting $x$ for $y$, we obtain $c(x, f) = c(y, f)$. \qed

Remark 9. With the same argument we can prove that $c(x, \beta, f)$ is the same for the starting points $x$ ans $y$, as soon as $t(\beta) + 2\|f\|_\infty < \inf(\kappa(x), \kappa(y))$.

3 Integration by parts formula for infinitely divisible laws

Recall that the random variable $g$ has small exponential moments. We assume now that it is infinitely divisible, and hence we have a Levy Khinchine formula

$$1(\beta) = \log \mathbb{E}[e^{\beta g}] = c\beta + \frac{\sigma^2}{2} \beta^2 + \int \pi(du) \left( e^{\beta u} - 1 - \mathbf{1}_{|u| \leq 1} \beta u \right) \quad (|\beta| < \beta_0),$$

where $c \in \mathbb{R}$, $\sigma \geq 0$ are constants and $\pi$ is a measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int \pi(du) (1 \wedge u^2) < +\infty$.

Lemma 10. If $g$ satisfies (12), then for any bounded differentiable $f$ with bounded derivative, one has the following integration by parts formula:

$$\mathbb{E}[gf(g)] = c\mathbb{E}[f(g)] + \sigma^2 \mathbb{E}[f'(g)] + \int_{-\infty}^{+\infty} \pi(du) u \left[ \mathbb{E}[f(g+u)] - \mathbf{1}_{|u| \leq 1} \mathbb{E}[f(g)] \right]$$

(13)

Proof. As pointed out by Nicolas Privault, this Lemma can be seen as an easy consequence of much more general integration by parts formulas on the Poisson space (see Picard [10]). Let us give a short proof here: it suffices to prove the formula (13) for $f(x) = e^{\lambda x}$, the extension to more general functions following standard arguments. In that case, $\mathbb{E}[gf(g)] = \mathbb{E}[ge^{\lambda x}] = e^{\lambda(x)} \lambda'(i\theta)$. Since $\lambda'(i\theta) = c + \sigma^2 \lambda' + \int_{-\infty}^{+\infty} \pi(du) u e^{\lambda u} - \mathbf{1}_{|u| \leq 1}$, we obtain:

$$e^{\lambda(x)} \lambda'(i\theta) = c e^{\lambda(x)} + \sigma^2 \lambda' e^{\lambda(x)} + \mathbb{E}[e^{\lambda u}] \int_{-\infty}^{+\infty} \pi(du) u \left( e^{\lambda u} - \mathbf{1}_{|u| \leq 1} \right)$$
\[ = cE[f(g)] + \sigma^2 E[f'(g)] + \int_{-\infty}^{+\infty} \pi(du)u[E[f(g+u)] - 1_{|u| \leq 1}E[f(g)]] \]

We shall now link the derivative of the free energy to \( \langle L_n(S^1,S^2) \rangle_2^{(n)} \), where here and in the sequel, \( L_n(S^1,S^2) \) denotes the global correlation between the two independent configurations \( S^1 \) and \( S^2 \) (under the same polymers measure \( \langle \cdot \rangle^{(n)} \)). Recall that \( I \) is an open interval, chosen as big as possible, such that \( 0 \in I \subset \{ \beta : 1(\beta) < +\infty \} \).

**Proposition 11.** If \( g \) satisfies (12), then there exists \( c_1 > 0 \), depending on the law of \( g \) and on \( \beta \), such that \( \forall \beta \in I \cap (0,\infty) \)

\[ p'_n(\beta) \leq -\frac{c_1}{n} E\left[ \langle L_n(S^1,S^2) \rangle_2^{(n)} \right]. \tag{14} \]

In particular, for all \( n \geq 1 \), \( \beta \mapsto p_n(\beta) \) is non increasing.

Moreover if \( 2\beta \in I \cap (0,\infty) \), there exists \( c_2(\beta) > 0 \) such that

\[ p'_n(\beta) \geq -\frac{c_2}{n} E\left[ \langle L_n(S^1,S^2) \rangle_2^{(n)} \right]. \tag{15} \]

**Proof.** In the sequel we write \( \langle \cdot \rangle \) instead of \( \langle \cdot \rangle^{(n)} \). The first step is the following identity:

\[ np'_n(\beta) = E\left[ \sum_{i=1}^{n} g(i,S_i) \right] - n\lambda'(\beta) = \sum_{i,x} E[\{g(i,x)F_{i,x}(g(i,x))\}] - n\lambda'(\beta), \tag{16} \]

where we have set, for each \((i,x)\):

\[ F_{i,x}(u) \equiv \frac{E[1_{(S_i=x)} \exp(\beta \sum_{(j,y) \neq (i,x)} g(j,y) 1(S_j=y) + \beta u 1(S_i=x)) \bigg| \exp(\beta \sum_{(j,y) \neq (i,x)} g(j,y) 1(S_j=y) + \beta u 1(S_i=x))] \bigg]}{E[\exp(\beta \sum_{(j,y) \neq (i,x)} g(j,y) 1(S_j=y) + \beta u 1(S_i=x))]}, \quad u \in \mathbb{R}. \]

Since \( F_{i,x} \) is a random function depending only on \((g(j,y),(j,y) \neq (i,x))\), it is independent of \(g(i,x)\), so by Lemma 10 one has for each fixed \((i,x)\):

\[ E[g(i,x)F_{i,x}(g(i,x))] = cE[F_{i,x}(g(i,x))] + \sigma^2 E[F'_{i,x}(g(i,x))] \]

\[ + \int_{-\infty}^{+\infty} \pi(du)u[E[F_{i,x}(g(i,x)+u)] - 1_{|u| \leq 1}E[F_{i,x}(g(i,x))]] \tag{17} \]

Here one easily obtains that \( F'_{i,x}(u) = \beta F_{i,x}(u)[1 - F_{i,x}(u)] \). In particular, one has

\[ F'_{i,x}(g(i,x)) = \beta \langle 1_{(S_i=x)} \rangle \left( 1 - \langle 1_{(S_i=x)} \rangle \right). \]
Moreover,

\[ F_{i,x}(g(i,x) + u) = \frac{\langle 1_{(S_i=x)} \rangle e^{\beta u}}{\langle 1_{(S_i\neq x)} \rangle + \langle 1_{(S_i=x)} \rangle e^{\beta u}}, \]

so that formula (17) leads to:

\[
\mathbb{E}[g(i,x)F_{i,x}(g(i,x))] = c\mathbb{E}[\langle 1_{(S_i=x)} \rangle] + \sigma^2 \beta \mathbb{E}[\langle 1_{(S_i=x)} \rangle (1 - \langle 1_{(S_i=x)} \rangle)] \\
+ \int_{-\infty}^{+\infty} \pi(du)u \left[ \mathbb{E}\left[ \frac{\langle 1_{(S_i=x)} \rangle e^{\beta u}}{\langle 1_{(S_i\neq x)} \rangle + \langle 1_{(S_i=x)} \rangle e^{\beta u}} - \delta_{1}\right] \right].
\]

Then, using that \( \lambda'(\beta) = c + \sigma^2 \beta + \int_{-\infty}^{+\infty} \pi(du)u(e^{\beta u} - 1) \) and remembering that \( \sum_{i,x} \langle 1_{(S_i=x)} \rangle = n \), equation (16) becomes

\[
np'_n(\beta) = -\sigma^2 \beta \mathbb{E}\left[ \sum_{i,x} \langle 1_{(S_i=x)} \rangle^2 \right] - \sum_{i,x} \int_{-\infty}^{+\infty} \pi(du)u \mathbb{E}\left[ \frac{\langle 1_{(S_i=x)} \rangle^2 e^{\beta u}(e^{\beta u} - 1)}{1 + \langle 1_{(S_i=x)} \rangle (e^{\beta u} - 1)} \right].
\]

Now we prove that

\[
\inf_{0 \leq a \leq 1} \int_{-\infty}^{+\infty} \pi(du)u \frac{e^{\beta u}(e^{\beta u} - 1)}{1 + a(e^{\beta u} - 1)} > 0
\]

as soon as \( \pi(.) \neq 0 \). On the one hand, if \( \text{supp}(\pi) \cap \mathbb{R}_+ \neq \emptyset \), then for all \( 0 \leq a \leq 1 \),

\[
\int_{0}^{+\infty} \pi(du)u \frac{e^{\beta u}(e^{\beta u} - 1)}{1 + a(e^{\beta u} - 1)} = \int_{0}^{+\infty} \pi(du)u(e^{\beta u} - 1) > 0.
\]

On the other hand, if \( \text{supp}(\pi) \cap \mathbb{R}_- \neq \emptyset \), then for all \( 0 \leq a \leq 1 \),

\[
\int_{-\infty}^{0} \pi(du)|u| \frac{e^{\beta u}(1 - e^{\beta u})}{1 - a(1 - e^{\beta u})} = \int_{-\infty}^{0} \pi(du)e^{\beta u} |u|(1 - e^{\beta u}) > 0.
\]

In all cases, (18) is true for all \( \pi(.) \neq 0 \), so there exists \( c_1 > 0 \) such that

\[
np'_n(\beta) \leq -c_1 \mathbb{E}\left[ \sum_{i,x} \langle 1_{(S_i=x)} \rangle^2 \right],
\]

\( c_1 \) being positive because \( \pi(.) \neq 0 \) or \( \sigma > 0 \), since the law of \( X \) is non-degenerate. This leads the upper bound (14) thanks to the following identity:

\[
\sum_{i,x} \langle 1_{(S_i=x)} \rangle^2 = \langle L_n(S^1, S^2) \rangle^{(n)}_2.
\]

The lower bound (15) can be deduced in the same way, using that

\[
\sup_{0 \leq a \leq 1} \int_{-\infty}^{+\infty} \pi(du)u \frac{e^{\beta u}(e^{\beta u} - 1)}{1 + a(e^{\beta u} - 1)} = \int_{0}^{+\infty} \pi(du)ue^{\beta u}(e^{\beta u} - 1) \leq \int_{0}^{+\infty} \pi(du)ue^{\beta u}(e^{\beta u} - 1) \leq \int_{-\infty}^{0} \pi(du)|u|(1 - e^{\beta u}) \defeq c',
\]

because \( c' < \infty \) provided that \( 2\beta \in I \).
4 Strong disorder: Proof of Theorem 1

The part (a) of Theorem 1 follows from Proposition 4 whereas the part (b) from Proposition 11. To show the part (c), we make use of the monotonicity of $\beta \to p_n(\beta)$, then it suffices to prove that $\limsup p_n(\beta) < 0$ for $\beta > 0$ small enough.

Recall that $E(g) = \lambda'(0) = 0$. Then for all $q = (q(i,x), i \geq 1, x \in \Sigma) \in \mathbb{R}^N \times \Sigma$, one has, using Jensen’s inequality:

$$p_n(\beta) = \frac{1}{n} E \left[ \log E_{x_0} \left( \left( e^{\beta \sum_{i} g(i,x) 1_{x_i-\epsilon} - n\lambda(\beta)} \right) - \beta \sum_{i,x} g(i,x) q(i,x) \right) \right]$$

$$= \frac{1}{n} E \left[ \log E_{x_0} \left( \left( e^{\beta \sum_{i} g(i,x) 1_{x_i-\epsilon} - q(i,x) - n\lambda(\beta)} \right) \right) \right]$$

$$\leq \frac{1}{n} \log E_{x_0} \left( e^{\sum_{i} \lambda'(\beta 1_{x_i-\epsilon} - q(i,x)) - n\lambda(\beta)} \right),$$

Let us choose $q(i,x) = \mu(x)$ (the invariant probability measure of $S$) for all $(i,x) \in \mathbb{Z}^2 \times \Sigma$ and let us fix $\epsilon > 0$. There exists $\beta_* > 0$ such that $0 < \beta < \beta_* \iff \frac{1}{2} \beta^2 \lambda''(0) \leq \lambda'(\beta) \leq \frac{1}{2} \beta^2 \lambda''(0)$. Thus, for $0 < \beta < \beta_*$,

$$p_n(\beta) \leq \frac{1}{n} \log E_{x_0} \left( e^{\frac{1}{2} \beta^2 \lambda''(0) \sum_{i} 1_{x_i-\epsilon} - \mu(x)} - n \frac{1}{2} \beta^2 \lambda''(0) \right)$$

$$\leq \frac{1}{n} \log E_{x_0} \left( e^{-(1+\epsilon) \beta^2 \lambda''(0) \sum_{i} 1_{x_i-\epsilon} - \mu(x)} + \epsilon \lambda''(0) \beta^2 + \frac{1+\epsilon}{2} \beta^2 \lambda''(0) \|\mu\|^2, \right.$$ \with \( \|\mu\|^2 = \sum_{x \in \Sigma} \mu^2(x). \)

Applying Lemma 4 to $f(x) = -\mu(x)$, one deduces the existence of $J = [-t_0,t_0]$ and of $c_\mu$ defined on $J$ such that $\forall t \in J$, $\frac{1}{n} \log E_0 (e^{\sum_{i=1}^{n} \mu(S_i)} \to c_\mu(t)$ and $\forall t \in J$, $c_\mu(t) \leq -(1 - \epsilon) t \|\mu\|^2$, since $c_\mu(0) = -\|\mu\|^2$. Hence for $\beta$ small enough one concludes that $\limsup_{n \to \infty} p_n(\beta) \leq -\frac{1}{2} \beta^2 \|\mu\|^2 \lambda''(0)$ and thus $\limsup_{n \to \infty} p_n(\beta) < 0$ since $\lambda''(0) = \text{Var}(g) > 0$. Finally, for any $0 < \beta < \beta_0$, we deduce from the property of concentration of measure (Lemma 5 (ii)) that almost surely, $Z_n(\beta)$ converges to 0 exponentially fast. \qed

References

[1] A. de Acosta and P. Ney. Large deviation lower bounds for arbitrary additive functionals of a Markov chain. Ann. Probab. 26 (1998), no. 4, 1660–1682.

[2] Ph. Carmona and Y. Hu. On the partition function of a directed polymer in a Gaussian random environment. Probab. Theory Related Fields, 124(3):431–457, 2002.

[3] F. Comets, T. Shiga, and N. Yoshida. Directed polymers in random environment: path localization and strong disorder. Bernoulli (2003) 9(4):705–723, 2003.
[4] F. Comets and N. Yoshida. Brownian directed polymers in random environment. *preprint*, 2003.

[5] A. Dembo and O. Zeitouni. *Large deviations techniques and applications*. Second edition. Springer-Verlag, New York, 1998.

[6] F. Guerra and F. L. Toninelli. Quadratic replica coupling in the Sherrington-Kirkpatrick mean field spin glass model. *J. Math. Phys.*, 43(7):3704–3716, 2002.

[7] J. Z. Imbrie and T. Spencer. Diffusion of directed polymers in a random environment. *J. Statist. Phys.*, 52(3-4):609–626, 1988.

[8] P. Ney and E. Nummelin. Markov additive processes. I. Eigenvalue properties and limit theorems. *Ann. Probab.* 15 (1987), no. 2, 561–592.

[9] M. Petermann. Superdiffusivity of directed polymers in a random environment. part of PHD thesis, 2000.

[10] J. Picard. Formules de dualité sur l’espace de Poisson. *Ann. Inst. H. Poincaré Probab. Statist.*, 32(4):509–548, 1996.

[11] Y. Sinai. A remark concerning random walks with random potentials. *Fundamenta Mathematicae*, 147(2):173–180, 1995.