Combinatorial and Structural Results for $\gamma$-$\Psi$-dimensions

Yann Guermeur
LORIA-CNRS
Campus Scientifique, BP 239
54506 Vandœuvre-lès-Nancy Cedex, France
(e-mail: Yann.Guermeur@loria.fr)

September 20, 2018

Running Title: Combinatorial and Structural Results for $\gamma$-$\Psi$-dimensions

Keywords: margin multi-category classifiers, guaranteed risks, scale-sensitive combinatorial dimensions, $\gamma$-$\Psi$-dimensions

Mathematics Subject Classification: 68Q32, 62H30
Abstract

One of the main open problems of the theory of margin multi-category pattern classification is the characterization of the way the confidence interval of a guaranteed risk should vary as a function of the three basic parameters which are the sample size $m$, the number $C$ of categories and the scale parameter $\gamma$. This is especially the case when working under minimal learnability hypotheses. In that context, the derivation of a bound is based on the handling of capacity measures belonging to three main families: Rademacher/Gaussian complexities, metric entropies and scale-sensitive combinatorial dimensions. The scale-sensitive combinatorial dimensions dedicated to the classifiers of interest are the $\gamma$-$\Psi$-dimensions. This article introduces the combinatorial and structural results needed to involve them in the derivation of guaranteed risks. Such a bound is then established, under minimal hypotheses regarding the classifier. Its dependence on $m$, $C$ and $\gamma$ is characterized. The special case of multi-class support vector machines is used to illustrate the capacity of the $\gamma$-$\Psi$-dimensions to take into account the specificities of a classifier.

1 Introduction

In the framework of agnostic learning, one of the main open problems of the theory of margin multi-category pattern classification is the characterization of the way the confidence interval of an upper bound on the probability of error should vary as a function of the three basic parameters which are the sample size $m$, the number $C$ of categories and the scale parameter $\gamma$ (see [21] for a survey). This is especially the case when working under minimal learnability hypotheses. In that context, the derivation of such a bound, also called guaranteed risk, is based on the handling of capacity measures belonging to three main families: Rademacher/Gaussian complexities [5], metric entropies [20] and scale-sensitive combinatorial dimensions [18, 1, 14]. The scale-sensitive combinatorial dimensions dedicated to the classifiers of interest are the $\gamma$-$\Psi$-dimensions [14]. Their usefulness to derive guaranteed risks rests on the availability of two types of results. Combinatorial results [1, 28, 14, 16], also known as Sauer type lemmas even though they are variants of Theorem 1 in [35], connect them to metric entropies. Structural results [14, 22, 23, 25, 17] perform the transition with the $\gamma$-dimension [18] of function classes including the classes of component functions (roughly speaking from the multi-class case to the bi-class one). This article introduces such results and incorporate them in the derivation of a guaranteed risk holding under minimal hypotheses regarding the classifier. The dependence of its confidence interval on $m$, $C$ and $\gamma$ is characterized. The special case of multi-class support vector machines (M-
SVMs) [15, 10] is used to illustrate the benefits springing from dedicating the structural results to the classifier of interest.

The organization of the paper is as follows. Section 2 deals with the theoretical framework and the margin multi-category classifiers, focusing on their capacity measures. Section 3 highlights the connections between these measures. Section 4 complements this set of connections by introducing the new combinatorial and structural results. This contribution is assessed in Section 5, where these tools are included in the derivation of the new guaranteed risk. At last, we draw conclusions and outline our ongoing research in Section 6.

To make reading easier, all technical lemmas and proofs have been gathered in appendix.

2 Margin multi-category classifiers

We work under minimal assumptions on the data, corresponding to a standard setting named agnostic learning [19]. The classifiers considered exhibit one single important feature: for each description, they return one score per category. This basic framework, used for instance in [38, 14], is summarized below.

2.1 Theoretical framework

We consider the case of $C$-category pattern classification problems [9] with $C \in \mathbb{N} \setminus [0; 2]$. Each object is represented by its description $x \in \mathcal{X}$ and the set $\mathcal{Y}$ of the categories $y$ can be identified with the set of indices of the categories: $[1; C]$. We assume that $(\mathcal{X}, \mathcal{A}_X)$ and $(\mathcal{Y}, \mathcal{A}_Y)$ are measurable spaces and denote by $\mathcal{A}_X \otimes \mathcal{A}_Y$ the tensor-product sigma-algebra on the Cartesian product $\mathcal{X} \times \mathcal{Y}$. We make the hypothesis that the link between descriptions and categories can be characterized by an unknown probability measure $P$ on the measurable space $(\mathcal{X} \times \mathcal{Y}, \mathcal{A}_X \otimes \mathcal{A}_Y)$. Let $Z = (X, Y)$ be a random pair with values in $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$, distributed according to $P$. The only access to $P$ is via an $m$-sample $Z_m = (Z_i)_{1 \leq i \leq m} = ((X_i, Y_i))_{1 \leq i \leq m}$ made up of independent copies of $Z$ (in short $Z_m \sim P^m$). The classifiers considered are based on classes of vector-valued functions with one component function per category.

As in [16], we add an hypothesis to that framework: the classes of component functions are uniform Glivenko-Cantelli. The definition of this property calls for the introduction of an intermediate definition.

**Definition 1 (Empirical probability measure)** Let $(\mathcal{T}, \mathcal{A}_T)$ be a measurable space and let $T$ be a random variable with values in $\mathcal{T}$, distributed according to a probability measure
For $n \in \mathbb{N}^*$, let $T_n = (T_i)_{1 \leq i \leq n}$ be an $n$-sample made up of independent copies of $T$. The empirical measure supported on this sample, $P_{T_n}$, is given by

$$P_{T_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{T_i},$$

where $\delta_{T_i}$ denotes the Dirac measure centered on $T_i$.

**Definition 2 (Uniform Glivenko-Cantelli class [13])** Let the probability measures $P_T$ and $P_{T_n}$ be defined as in Definition 1. Let $\mathcal{F}$ be a class of measurable functions on $T$. Then $\mathcal{F}$ is a uniform Glivenko-Cantelli class if for every $\epsilon \in \mathbb{R}_+^*$,

$$\lim_{n \to +\infty} \sup \{ \mathbb{P} \left( \sup_{n' \geq n} \sup_{f \in \mathcal{F}} \left| \mathbb{E}_{T' \sim P_{T_n'}} [f(T')] - \mathbb{E}_{T' \sim P_T} [f(T)] \right| > \epsilon \right) \} = 0,$$

where $\mathbb{P}$ denotes the infinite product measure $P_T^\infty$.

Henceforth, we shall refer to uniform Glivenko-Cantelli classes by the abbreviation $uGC$ classes. Those classes must be uniformly bounded up to additive constants (see for instance Proposition 4 in [13]). We replace this property by a slightly stronger one that does not affect the generality of the study (does not assume any coupling between the outputs...): the vector-valued functions take their values in a hypercube of $\mathbb{R}^C$. The definition of a margin multi-category classifier is thus the following one.

**Definition 3 (Margin multi-category classifiers)** Let $\mathcal{G} = \prod_{k=1}^{C} \mathcal{G}_k$ be a class of functions from $\mathcal{X}$ into $[-M_G, M_G]^C$ with $M_G \in [1, +\infty)$. The classes $\mathcal{G}_k$ of component functions are supposed to be $uGC$ classes. For each function $g = (g_k)_{1 \leq k \leq C} \in \mathcal{G}$, a margin multi-category classifier on $\mathcal{X}$ is obtained by application of the decision rule $dr$, mapping $g$ to $dr_g \in (Y \cup \{\ast\})^\mathcal{X}$, and defined as follows:

$$\forall x \in \mathcal{X}, \begin{cases} \left| \text{argmax}_{1 \leq k \leq C} \ g_k(x) \right| = 1 \implies dr_g(x) = \text{argmax}_{1 \leq k \leq C} \ g_k(x) \\ \left| \text{argmax}_{1 \leq k \leq C} \ g_k(x) \right| > 1 \implies dr_g(x) = \ast \end{cases}$$

where $|\cdot|$ returns the cardinality of its argument and $\ast$ stands for a dummy category.

In words, $dr_g$ returns either the index of the component function whose value is the highest, or the dummy category $\ast$ in case of ex æquo. The qualifier *margin* refers to the fact that the generalization capabilities of such classifiers can be characterized by means of the values taken by the differences of the corresponding component functions. With this definition at hand, the aim of the learning process is to minimize over $\mathcal{G}$ the probability of error $P(dr_g(X) \neq Y)$. This probability can be reformulated in a handy way thanks to the introduction of the class of margin functions.
Definition 4 (Class $\rho_G$ of margin functions) Let $G$ be a function class satisfying Definition 3. For every $g \in G$, the margin function $\rho_g$ from $\mathcal{Z}$ into $[-M_G, M_G]$ is defined by:

$$\forall (x, k) \in \mathcal{Z}, \quad \rho_g(x, k) = \frac{1}{2} \left( g_k(x) - \max_{l \neq k} g_l(x) \right).$$

Then, the class $\rho_G$ is defined as follows:

$$\rho_G = \{ \rho_g : g \in G \}.$$

The probability of error is an instance of risk.

Definition 5 (Risks) Let $G$ be a function class satisfying Definition 3 and let $\phi$ be the standard indicator loss function given by:

$$\forall t \in \mathbb{R}, \quad \phi(t) = \mathbf{1}_{\{t \leq 0\}}.$$ 

The expected risk of any function $g \in G$, $L(g)$, is given by:

$$L(g) = \mathbb{E}_{(X,Y) \sim P} [\phi \circ \rho_g(X,Y)] = P(d_{rg}(X) \neq Y).$$

Its empirical risk measured on the $m$-sample $\mathcal{Z}_m$ is:

$$L_m(g) = \mathbb{E}_{Z' \sim P_m} [\phi \circ \rho_g(Z')] = \frac{1}{m} \sum_{i=1}^{m} \phi \circ \rho_g(Z_i)$$

(where $P_m$ is the empirical measure supported on $\mathcal{Z}_m$).

In order to take benefit from the fact that the classifiers of interest are margin ones, the sample-based estimate of performance which is actually used (involved in the different guaranteed risks) is obtained by substituting to $\phi$ a (dominating) margin loss function $\phi_{\gamma}$ (parameterized by $\gamma \in (0, 1]$). A risk computed by substituting to $\phi$ a function $\phi_{\gamma}$ is named a margin risk.

Definition 6 (Margin risks) Let $G$ be a function class satisfying Definition 3. Given a class of margin loss functions $\phi_{\gamma}$ parameterized by $\gamma \in (0, 1]$, for every (ordered) pair $(g, \gamma) \in G \times (0, 1]$, the risk with margin $\gamma$ of $g$, $L_{\gamma}(g)$, is defined as:

$$L_{\gamma}(g) = \mathbb{E}_{Z \sim P} [\phi_{\gamma} \circ \rho_g(Z)].$$

$L_{\gamma,m}(g)$ designates the corresponding empirical risk, measured on the $m$-sample $\mathcal{Z}_m$. 

4
The form taken by the guaranteed risk, more precisely its dependence on $m$, $C$ and $\gamma$, is governed by the choice of $\phi_\gamma$. This study makes use of the classical $\frac{1}{\gamma}$-regular loss \[24\]: the parameterized truncated hinge loss.

**Definition 7 (Parameterized truncated hinge loss $\phi_{2,\gamma}$)** For $\gamma \in (0, 1]$, the parameterized truncated hinge loss $\phi_{2,\gamma}$ is defined by
\[
\forall t \in \mathbb{R}, \quad \phi_{2,\gamma}(t) = \mathbb{I}_{\{t \leq 0\}} + \left(1 - \frac{t}{\gamma}\right) \mathbb{I}_{\{t \in (0, \gamma]\}}.
\]

When using a margin loss function, the behavior of the margin functions outside the interval $[0, \gamma]$ becomes irrelevant to characterize the generalization performance. The idea to exploit this property by means of a combination with a piecewise-linear squashing function can be traced back to \[4\]. The piecewise-linear squashing function that fits best with $\phi_{2,\gamma}$ is the function $\pi_\gamma$.

**Definition 8 (Piecewise-linear squashing function $\pi_\gamma$)** For $\gamma \in (0, 1]$, the piecewise-linear squashing function $\pi_\gamma$ is defined by:
\[
\forall t \in \mathbb{R}, \quad \pi_\gamma(t) = t \mathbb{I}_{\{t \in (0, \gamma]\}} + \gamma \mathbb{I}_{\{t > \gamma\}}.
\]

Thus, when possible, we replace $\rho_G$ with the following function class.

**Definition 9 (Function class $\rho_{G,\gamma}$)** Let $\mathcal{G}$ be a function class satisfying Definition\[3\] and $\rho_G$ the function class deduced from $\mathcal{G}$ according to Definition\[4\]. For every pair $(g, \gamma) \in \mathcal{G} \times (0, 1]$, the function $\rho_{g,\gamma}$ from $\mathcal{Z}$ into $[0, \gamma]$ is defined by:
\[
\rho_{g,\gamma} = \pi_\gamma \circ \rho_g.
\]

Then, the class $\rho_{G,\gamma}$ is defined as follows:
\[
\rho_{G,\gamma} = \{\rho_{g,\gamma} : g \in \mathcal{G}\}.
\]

The rationale for the introduction of $\rho_{G,\gamma}$ is elementary. On the one hand, it does not affect the data-fit term of the guaranteed risk ($\forall \gamma \in (0, 1], \quad \phi_\gamma \circ \pi_\gamma = \phi_\gamma$). On the other hand, it can improve its confidence interval, if one can derive an upper bound on the capacity of $\rho_{G,\gamma}$ which is lower than the upper bound on the capacity of $\rho_G$. Thus, deriving sharp upper bounds on the capacity of $\rho_{G,\gamma}$, i.e., making the best of the use of $\pi_\gamma$, is a major goal of the present study.
In the sequel, we make use of the floor function $\lfloor \cdot \rfloor$ and the ceiling function $\lceil \cdot \rceil$ defined by:
\[
\forall u \in \mathbb{R}, \quad \begin{cases} 
\lfloor u \rfloor = \max \{ j \in \mathbb{Z} : j \leq u \} \\
\lceil u \rceil = \min \{ j \in \mathbb{Z} : j \geq u \}
\end{cases}.
\]

2.2 Scale-sensitive capacity measures

Whatever the choice of the margin loss function and the pathway followed to derive the corresponding guaranteed risk, the capacity measures involved belong to the three following families: Rademacher/Gaussian complexities [5], metric entropies [20] and scale-sensitive combinatorial dimensions [13, 1, 14]. We start by giving the definition of the Rademacher complexity since our choice of margin loss function implies that it appears first among the capacity measures involved in the derivation of our new guaranteed risk. For $n \in \mathbb{N}^*$, a Rademacher sequence $\sigma_n$ is a sequence $(\sigma_i)_{1 \leq i \leq n}$ of independent and identically distributed random variables taking the values $-1$ and $1$ with probability $\frac{1}{2}$ (Rademacher random variables).

**Definition 10 (Rademacher complexity)** Let $(\mathcal{T}, \mathcal{A}_T)$ be a measurable space and let $T$ be a random variable with values in $\mathcal{T}$, distributed according to a probability measure $P_T$ on $(\mathcal{T}, \mathcal{A}_T)$. For $n \in \mathbb{N}^*$, let $T_n = (T_i)_{1 \leq i \leq n}$ be an $n$-sample made up of independent copies of $T$ and let $\sigma_n = (\sigma_i)_{1 \leq i \leq n}$ be a Rademacher sequence. Let $\mathcal{F}$ be a class of real-valued functions with domain $\mathcal{T}$. The empirical Rademacher complexity of $\mathcal{F}$ given $T_n$ is:
\[
\hat{R}_n(\mathcal{F}) = \mathbb{E}_{\sigma_n} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \sigma_i f(T_i) \mid T_n \right].
\]
The Rademacher complexity of $\mathcal{F}$ is:
\[
R_n(\mathcal{F}) = \mathbb{E}_{T_n} \left[ \hat{R}_n(\mathcal{F}) \right] = \mathbb{E}_{T_n, \sigma_n} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \sigma_i f(T_i) \right].
\]

**Remark 1** The fact that the function classes $\mathcal{F}$ of interest can be uncountable calls for a specification. We make use of the standard convention (see for instance Formula (2.2) in [33]). Let $(T_s)_{s \in \mathcal{S}}$ be a stochastic process. Then,
\[
\mathbb{E} \left[ \sup_{s \in \mathcal{S}} T_s \right] = \sup_{\{S \subseteq \mathcal{S} : |S| < +\infty\}} \mathbb{E} \left[ \max_{s \in S} T_s \right].
\]
The concept of covering number (metric entropy), as well as the underlying concepts of $\epsilon$-cover and $\epsilon$-net, can be traced back to [20].
Definition 11 (covering numbers and metric entropy) Let \((\mathcal{E}, \rho)\) be a pseudo-metric space, \(\mathcal{E}' \subset \mathcal{E}\) and \(\epsilon \in \mathbb{R}_+^*\). An \(\epsilon\)-cover of \(\mathcal{E}'\) is a coverage of \(\mathcal{E}'\) with open balls of radius \(\epsilon\) the centers of which belong to \(\mathcal{E}\). These centers form an \(\epsilon\)-net of \(\mathcal{E}'\). An internal/proper \(\epsilon\)-net of \(\mathcal{E}'\) is an \(\epsilon\)-net of \(\mathcal{E}'\) included in \(\mathcal{E}'\). If \(\mathcal{E}'\) has an \(\epsilon\)-net of finite cardinality, then its covering number \(\mathcal{N}(\epsilon, \mathcal{E}', \rho)\) is the smallest cardinality of its \(\epsilon\)-nets. If there is no such finite net, then the covering number is defined to be infinite. The corresponding binary logarithm, \(\log_2(\mathcal{N}(\epsilon, \mathcal{E}', \rho))\), is called the metric entropy of \(\mathcal{E}'\). \(\mathcal{N}^\text{int}(\epsilon, \mathcal{E}', \rho)\) will designate a covering number of \(\mathcal{E}'\) obtained by considering internal \(\epsilon\)-nets only.

There is a close connection between covering and packing properties of bounded subsets in pseudo-metric spaces.

Definition 12 (\(\epsilon\)-separation and packing numbers [20]) Let \((\mathcal{E}, \rho)\) be a pseudo-metric space and \(\epsilon \in \mathbb{R}_+^*\). A set \(\mathcal{E}' \subset \mathcal{E}\) is \(\epsilon\)-separated if, for any subset \(\{e, e'\}\) of \(\mathcal{E}'\), \(\rho(e, e') \geq \epsilon\). Its \(\epsilon\)-packing number, \(\mathcal{M}(\epsilon, \mathcal{E}', \rho)\), is the maximal cardinality of its \(\epsilon\)-separated subsets, if such maximum exists. Otherwise, the \(\epsilon\)-packing number of \(\mathcal{E}'\) is defined to be infinite.

In this study, the function classes met are endowed with empirical (pseudo-)metrics induced by the \(L_p\)-norms.

Definition 13 (Pseudo-distance \(d_{p,t_n}\)) Let \(\mathcal{F}\) be a class of real-valued functions on \(\mathcal{T}\). For \(n \in \mathbb{N}^*\), let \(t_n = (t_i)_{1 \leq i \leq n} \in \mathcal{T}^n\). Then,

\[
\forall p \in [1, +\infty), \forall (f, f') \in \mathcal{F}^2, \quad d_{p,t_n}(f, f') = \left\| f - f' \right\|_{L_p(\mu_{t_n})} = \left( \frac{1}{n} \sum_{i=1}^{n} |f(t_i) - f'(t_i)|^p \right)^{\frac{1}{p}}
\]

and

\[
\forall (f, f') \in \mathcal{F}^2, \quad d_{\infty,t_n}(f, f') = \left\| f - f' \right\|_{L_\infty(\mu_{t_n})} = \max_{1 \leq i \leq n} |f(t_i) - f'(t_i)|,
\]

where \(\mu_{t_n}\) denotes the uniform (counting) probability measure on \(\{t_i : 1 \leq i \leq n\}\).

Definition 14 (Uniform covering numbers [37] and uniform packing numbers [41]) Let \(\mathcal{F}\) be a class of real-valued functions on \(\mathcal{T}\) and \(\mathcal{F} \subset \mathcal{F}\). For \(p \in [1, +\infty]\), \(\epsilon \in \mathbb{R}_+^*\), and \(n \in \mathbb{N}^*\), the uniform covering number \(\mathcal{N}_p(\epsilon, \mathcal{F}, n)\) and the uniform packing number \(\mathcal{M}_p(\epsilon, \mathcal{F}, n)\) are defined as follows:

\[
\begin{align*}
\mathcal{N}_p(\epsilon, \mathcal{F}, n) &= \sup_{t_n \in \mathcal{T}^n} \mathcal{N}(\epsilon, \mathcal{F}, d_{p,t_n}) \\
\mathcal{M}_p(\epsilon, \mathcal{F}, n) &= \sup_{t_n \in \mathcal{T}^n} \mathcal{M}(\epsilon, \mathcal{F}, d_{p,t_n})
\end{align*}
\]
We define accordingly $\mathcal{N}^\text{int}_p(\epsilon, \mathcal{F}, n)$ as:

$$\mathcal{N}^\text{int}_p(\epsilon, \mathcal{F}, n) = \sup_{t_n \in \mathcal{T}} \mathcal{N}^\text{int}(\epsilon, \mathcal{F}, d_p t_n).$$

Our combinatorial result relates packing numbers of $\rho_G, \gamma$ to two $\gamma$-$\Psi$-dimensions of $\rho_G$. It is to be compared with the state-of-the-art result of this kind, which involves another scale-sensitive combinatorial dimension: the $\gamma$-dimension. All these dimensions are now defined.

**Definition 15 ($\gamma$-$\Psi$-dimensions [13])** Let $\mathcal{F}$ be a class of real-valued functions on $\mathcal{Z}$. Let $\Psi$ be a family of mappings from $\mathcal{Y}$ into $\{-1, 1, \ast\}$, where $\ast$ is a null element. For $\gamma \in \mathbb{R}^+_s$, a subset $s_{\mathcal{Z}^n} = \{z_i = (x_i, y_i) : 1 \leq i \leq n\}$ of $\mathcal{Z}$ is said to be $\gamma$-$\Psi$-shattered by $\mathcal{F}$ if there is a vector $\psi_n = (\psi^{(i)})_{1 \leq i \leq n} \in \Psi^n$ satisfying for every $i \in \{1; n\}$, $\psi^{(i)}(y_i) = 1$, and a vector $b_n = (b_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ such that, for every vector $s_n = (s_i)_{1 \leq i \leq n} \in \{-1, 1\}^n$, there is a function $f_{s_n} \in \mathcal{F}$ satisfying

$$\forall i \in \{1; n\}, \begin{cases} \text{if } s_i = 1, \exists k : \psi^{(i)}(k) = 1 \text{ and } f_{s_n}(x_i, k) - b_i \geq \gamma. \\ \text{if } s_i = -1, \exists l : \psi^{(i)}(l) = -1 \text{ and } f_{s_n}(x_i, l) + b_i \geq \gamma. \end{cases}$$

$\mathcal{F}$ is also said to $\gamma$-$\Psi$-shatter the triplet $(s_{\mathcal{Z}^n}, \psi_n, b_n)$ and the pair $(\psi_n, b_n)$ is called a witness to the shattering. The $\gamma$-$\Psi$-dimension of $\mathcal{F}$, denoted by $\gamma$-$\Psi$-$\dim(\mathcal{F})$, is the maximal cardinality of a subset of $\mathcal{Z}$ $\gamma$-$\Psi$-shattered by $\mathcal{F}$, if such maximum exists. Otherwise, $\mathcal{F}$ is said to have infinite $\gamma$-$\Psi$-dimension.

The $\gamma$-$\Psi$-dimensions are scale-sensitive extensions of the “multi-class” extensions of the Vapnik-Chervonenkis (VC) dimension [33]: the $\Psi$-dimensions [7]. Furthermore, setting $C = 2$ in their definition provides us with the definition of the standard scale-sensitive extension of the VC dimension: the fat-shattering or $\gamma$-dimension.

**Definition 16 ($\gamma$-dimension [18])** Let $\mathcal{F}$ be a class of real-valued functions on $\mathcal{T}$. For $\gamma \in \mathbb{R}^+_s$, a subset $s_{\mathcal{T}^n} = \{t_i : 1 \leq i \leq n\}$ of $\mathcal{T}$ is said to be $\gamma$-shattered by $\mathcal{F}$ if there is a vector $b_n = (b_i)_{1 \leq i \leq n} \in \mathbb{R}^n$ such that, for every vector $s_n = (s_i)_{1 \leq i \leq n} \in \{-1, 1\}^n$, there is a function $f_{s_n} \in \mathcal{F}$ satisfying

$$\forall i \in \{1; n\}, s_i (f_{s_n}(t_i) - b_i) \geq \gamma.$$ 

The $\gamma$-dimension of the class $\mathcal{F}$, $\gamma$-$\dim(\mathcal{F})$, is the maximal cardinality of a subset of $\mathcal{T}$ $\gamma$-shattered by $\mathcal{F}$, if such maximum exists. Otherwise, $\mathcal{F}$ is said to have infinite $\gamma$-dimension.
We have precisely:

\[ C = 2 \implies \gamma \Psi \dim (\rho_G) = \gamma \dim (\rho_G). \]

In this study, we focus on the extensions of the two main \(\Psi\)-dimensions: the Graph dimension and the Natarajan dimension [31]. They are associated with two decomposition methods, respectively named one-against-all and one-against-one.

Definition 17 (Graph dimension with margin \(\gamma\)) Let \(\mathcal{F}\) be a class of real-valued functions on \(\mathcal{Z}\). For \(\gamma \in \mathbb{R}^*_+,\) a subset \(s_{\mathcal{Z}^n} = \{z_i = (x_i, y_i) : 1 \leq i \leq n\}\) of \(\mathcal{Z}\) is said to be \(\gamma\)-G-shattered by \(\mathcal{F}\) if there is a vector \(b_n = (b_i)_{1 \leq i \leq n} \in \mathbb{R}^n\) such that, for every vector \(s_n = (s_i)_{1 \leq i \leq n} \in \{-1, 1\}^n\), there is a function \(f_{s_n} \in \mathcal{F}\) satisfying

\[
\forall i \in [1; n], \quad \begin{cases} 
  f_{s_n}(x_i, y_i) - b_i \geq \gamma \\
  f_{s_n}(x_i, c_i) + b_i \geq \gamma 
\end{cases}
\]

The Graph dimension with margin \(\gamma\) of \(\mathcal{F}\), denoted by \(\gamma\)-G-dim(\(\mathcal{F}\)), is the maximal cardinality of a subset of \(\mathcal{Z}\) \(\gamma\)-G-shattered by \(\mathcal{F}\), if such maximum exists. Otherwise, \(\mathcal{F}\) is said to have infinite Graph dimension with margin \(\gamma\).

Definition 18 (Natarajan dimension with margin \(\gamma\)) Let \(\mathcal{F}\) be a class of real-valued functions on \(\mathcal{Z}\). For \(\gamma \in \mathbb{R}^*_+,\) a subset \(s_{\mathcal{Z}^n} = \{z_i = (x_i, y_i) : 1 \leq i \leq n\}\) of \(\mathcal{Z}\) is said to be \(\gamma\)-N-shattered by \(\mathcal{F}\) if there is a vector \(c_n = (c_i)_{1 \leq i \leq n} \in \mathcal{Y}^n\) satisfying for every \(i \in [1; n]\), \(c_i \neq y_i\), and a vector \(b_n = (b_i)_{1 \leq i \leq n} \in \mathbb{R}^n\) such that, for every vector \(s_n = (s_i)_{1 \leq i \leq n} \in \{-1, 1\}^n\), there is a function \(f_{s_n} \in \mathcal{F}\) satisfying

\[
\forall i \in [1; n], \quad \begin{cases} 
  f_{s_n}(x_i, y_i) - b_i \geq \gamma \\
  f_{s_n}(x_i, c_i) + b_i \geq \gamma 
\end{cases}
\]

The Natarajan dimension with margin \(\gamma\) of \(\mathcal{F}\), denoted by \(\gamma\)-N-dim(\(\mathcal{F}\)), is the maximal cardinality of a subset of \(\mathcal{Z}\) \(\gamma\)-N-shattered by \(\mathcal{F}\), if such maximum exists. Otherwise, \(\mathcal{F}\) is said to have infinite Natarajan dimension with margin \(\gamma\).

It springs from Definition[3] that all the scale-sensitive capacity measures considered in the sequel (whatever the value of the scale parameter) are finite.

3 Connections between the capacity measures

In the theoretical framework of interest, the main building blocks of the derivation of a guaranteed risk are a basic supremum inequality and connections between the capacity
measures defined in the preceding section. These connections are of two kinds. A first group corresponds to a change of capacity measure. It includes the combinatorial results. The second group is that of the structural results. The basic supremum inequality appears (for instance) as a partial result in the proof of Theorem 8.1 in [29] (with $\rho_G$ replaced with $\rho_{G,\gamma}$).

**Theorem 1 (Theorem 5 in [16])** Let $G$ be a function class satisfying Definition 3. For $\gamma \in (0, 1]$, let $\rho_{G,\gamma}$ be the function class deduced from $G$ according to Definition 9. For a fixed $\gamma \in (0, 1]$ and a fixed $\delta \in (0, 1)$, with $P^m$-probability at least $1 - \delta$,

$$
\sup_{g \in G} (L(g) - L_{\gamma,m}(g)) \leq \frac{2}{\gamma} R_m(\rho_{G,\gamma}) + \sqrt{\frac{\ln(\frac{1}{\delta})}{2m}},
$$

(1)

where the margin loss function defining the empirical margin risk is the parameterized truncated hinge loss (Definition 7).

### 3.1 Rademacher complexity

The sharpest structural result for classes of vector-valued functions is due to Maurer [25]. It is an improvement of the one introduced in [24].

**Lemma 1 (After Corollary 4 in [25])** Let $G$ be a function class satisfying Definition 3. For $n \in \mathbb{N}^*$, let $F = \{f_i: 1 \leq i \leq n\}$ be a class of real-valued functions on $[-M_G, M_G]^C$ which are $L_F$-Lipschitz continuous with respect to the $\ell_2$-norm. Then

$$
\mathbb{E}_{\sigma_n} \left[ \sup_{g \in G} \sum_{i=1}^{n} \sigma_i f_i \circ g(x_i) \right] \leq \sqrt{2} L_F \mathbb{E}_{\sigma_{n,C}} \left[ \sup_{g \in G} \sum_{i=1}^{n} \sum_{k=1}^{C} \sigma_{i,k} g_k(x_i) \right],
$$

where $\sigma_{n,C} = (\sigma_{i,k})_{1 \leq i \leq n, 1 \leq k \leq C}$ is a Rademacher random matrix.

For $\gamma \in (0, 1]$, let us apply Lemma 1 by defining the functions $f_i$ in such a way that

$$
\forall i \in \llbracket 1; n \rrbracket, \ f_i \circ g(x_i) = \rho_{g,\gamma}(z_i),
$$

and making a double assumption:

- all the classes of component functions are identical ($\forall k \in \llbracket 1; C \rrbracket, \ G_k = \bigcup_{l=1}^{C} G_l$);
- there is no coupling among the component functions of the functions $g$.

Then, since $L_F$ can be set equal to 1, we obtain, up to the multiplicative factor $\sqrt{2}$, an older structural result which is a direct consequence of the proof of Theorem 3 in [22].
Lemma 2  Let $\mathcal{G}$ be a function class satisfying Definition 3. For $\gamma \in (0, 1]$, let $\rho_{\mathcal{G}, \gamma}$ be the function class deduced from $\mathcal{G}$ according to Definition 4. Then

$$\forall n \in \mathbb{N}^*, \ R_n (\rho_{\mathcal{G}, \gamma}) \leq CR_n \left( \bigcup_{k=1}^{C} \mathcal{G}_k \right).$$

Several results are available to bound from above the expected suprema of empirical processes. A standard approach, especially efficient in the case of Rademacher processes, is the application of Dudley’s chaining method [12, 33].

Theorem 2 (Dudley’s metric entropy bound, Theorem 9 in [16])  Let $\mathcal{F}$ be a class of bounded real-valued functions on $\mathcal{T}$. For $n \in \mathbb{N}^*$, let $t_n \in \mathcal{T}^n$ and let $\text{diam} (\mathcal{F}) = \sup_{(f, f') \in \mathcal{F}^2} \| f - f' \|_{L_2 (\mu_{t_n})}$ be the diameter of $\mathcal{F}$ in the $L_2 (\mu_{t_n})$ seminorm. Let $h$ be a positive and decreasing function on $\mathbb{N}$ such that $h (0) \geq \text{diam} (\mathcal{F})$. Then for $N \in \mathbb{N}^*$,

$$\hat{R}_n (\mathcal{F}) \leq h (N) + 2 \sum_{j=1}^{N} (h (j) + h (j - 1)) \sqrt{\frac{\ln (N^{\text{int}} (h (j), \mathcal{F}, d_{2.t_n}))}{n}}$$

(2)

and

$$\hat{R}_n (\mathcal{F}) \leq 12 \int_{0}^{\frac{1}{2} \cdot \text{diam} (\mathcal{F})} \sqrt{\frac{\ln (N^{\text{int}} (\epsilon, \mathcal{F}, d_{2.t_n}))}{n}} d\epsilon.$$  (3)

In the framework of this study, the chaining method can be applied either to $\rho_{\mathcal{G}, \gamma}$ or to the classes of component functions, according to the level at which the decomposition is performed.

3.2 Covering and packing numbers

Several authors have established instances of the following structural result, for different values of $p$.

Lemma 3 (Lemma 1 in [16])  Let $\mathcal{G}$ be a function class satisfying Definition 3 and $\rho_{\mathcal{G}}$ the function class deduced from $\mathcal{G}$ according to Definition 4. For $\gamma \in (0, 1]$, let $\rho_{\mathcal{G}, \gamma}$ be the function class deduced from $\mathcal{G}$ according to Definition 9. Then, for $\epsilon \in \mathbb{R}_+^*$, $n \in \mathbb{N}^*$, and $z_n = ((x_i, y_i))_{1 \leq i \leq n} \in \mathcal{Z}^n$,

$$\forall p \in [1, +\infty], \ N^{\text{int}} (\epsilon, \rho_{\mathcal{G}, \gamma}, d_{p, z_n}) \leq N^{\text{int}} (\epsilon, \rho_{\mathcal{G}}, d_{p, x_n}) \leq \prod_{k=1}^{C} N^{\text{int}} \left( \epsilon C^p, \mathcal{G}_k, d_{p, x_n} \right),$$

(4)

where $x_n = (x_i)_{1 \leq i \leq n}$.
We have indicated in Section 2.2 that there is a close connection between covering and packing numbers and our new combinatorial result involves packing numbers. The transition is provided by a well-known lemma.

**Lemma 4 (After Theorem IV in [20])** Let \((\mathcal{E}, \rho)\) be a pseudo-metric space. For every totally bounded set \(\mathcal{E}' \subset \mathcal{E}\) and \(\epsilon \in \mathbb{R}_+^*\),

\[
\mathcal{M}(2\epsilon, \mathcal{E}', \rho) \leq N^{\text{int}}(\epsilon, \mathcal{E}', \rho) \leq \mathcal{M}(\epsilon, \mathcal{E}', \rho).
\]

The main combinatorial results for margin classifiers connect the packing numbers of interest to the corresponding \(\gamma\)-dimension (the function class is the same on both sides). They differ according to the choice of the \(L_p\)-norm at the basis of the pseudo-metric. The literature provides us with two lemmas based on the uniform convergence norm and involving a \(\gamma\)-\(\Psi\)-dimension: Lemma 39 in [14] (margin Natarajan dimension) and Lemma 8 in [17] (margin Graph dimension). Both are extensions of Lemma 3.5 in [1]. However, in view of our choice of margin loss function and the corresponding sequence of capacity measures, this study calls for the use of a combinatorial result based on the \(L_2\)-norm. All the bounds of this kind currently available involve the \(\gamma\)-dimension, and the sharpest of them is Theorem 1 in [28].

**Lemma 5 (After Theorem 1 in [28])** Let \(\mathcal{F}\) be a class of functions from \(\mathcal{T}\) into \([-M_\mathcal{F}, M_\mathcal{F}]\) with \(M_\mathcal{F} \in \mathbb{R}_+^*\). \(\mathcal{F}\) is supposed to be a uGC class. For \(\epsilon \in (0, M_\mathcal{F}]\), let \(d(\epsilon) = \epsilon\text{-dim}(\mathcal{F})\).

Then for \(\epsilon \in (0, 2M_\mathcal{F}]\) and \(n \in \mathbb{N}^*\),

\[
\mathcal{M}_2(\epsilon, \mathcal{F}, n) \leq \left(\frac{12M_\mathcal{F}}{\epsilon}\right)^{20d\left(\frac{\epsilon}{M_\mathcal{F}}\right)}.
\]

(5)

### 3.3 Scale-sensitive combinatorial dimensions

To the best of our knowledge, only one decomposition result is available for a scale-sensitive combinatorial dimension, namely the \(\gamma\)-dimension: Lemma 6.2 in [11] (see also Theorem 3 in [3]). Lemma 6 is its dedication to the function class \(\rho_\mathcal{G}\), where the values of the constants have been explicited.

**Lemma 6 (After Lemma 6.2 in [11])** Let \(\mathcal{G}\) be a function class satisfying Definition 3 and \(\rho_\mathcal{G}\) the function class deduced from \(\mathcal{G}\) according to Definition 4. Then

\[
\forall \gamma \in (0, M_\mathcal{G}], \quad \gamma\text{-dim}(\rho_\mathcal{G}) \leq \frac{320}{\ln(2)} \ln \left(\frac{24M_\mathcal{G}\sqrt{C}}{\gamma}\right) \sum_{k=1}^{C} \left(\frac{\gamma}{96\sqrt{C}}\right)^{-\text{dim}(\mathcal{G}_k)}.
\]

(6)
3.4 Discussion

None of the decompositions exposed above appears utterly satisfactory. Indeed, under the assumption that there is no coupling among the component functions of the functions in \( \mathcal{G} \), then the decomposition involving Rademacher complexities (Lemmas 1 and 2) produces a confidence interval (upper bound on the right-hand side of Formula (11)) that scales linearly with the number of categories, whereas the decomposition involving covering numbers (Lemma 3) can lead to a sublinear dependence, as was proved in [16] (Theorem 7). Furthermore, the decomposition involving Rademacher complexities makes almost no use of the function \( \pi_\gamma \). This function “vanishes” when using Lemma 1 because its Lipschitz constant is 1, whereas Lemma 2 remains true if the double clipping performed by the function \( \pi_\gamma \) is replaced with a single one: that of the values above \( \gamma \). By the way, this is precisely what was done in the proof of Theorem 3 in [22]. Turning to the decomposition involving covering numbers, the transition through \( \mathcal{N}^{\text{int}}(\epsilon, \rho_\mathcal{G}, d_p, \mathbf{z}_m) \) in Inequality (4) shows that Lemma 3 makes no use of \( \pi_\gamma \). Thus, when delaying the decomposition a this level, the introduction of \( \pi_\gamma \) is only exploited upstream, by the chaining formulas, through the definition of the function \( h \). At last, the decomposition involving \( \gamma \)-dimensions (Lemma 6) is clearly unsatisfactory, since a substitution of (6) into (5) (for \( \mathcal{F} = \rho_\mathcal{G} \)) produces an upper bound on the metric entropy of interest, \( \ln \left( \mathcal{N}^{\text{int}}_2(\epsilon, \rho_\mathcal{G}, \gamma, m) \right) \), which is worse than the one obtained by combining Lemma 3 with the same combinatorial result:

\[
\forall \epsilon \in (0, \gamma], \quad \ln \left( \mathcal{N}^{\text{int}}_2(\epsilon, \rho_\mathcal{G}, \gamma, m) \right) \leq 20 \ln \left( \frac{12 M_\mathcal{G} \sqrt{C}}{\epsilon} \right) \sum_{k=1}^{C} \left( \frac{\epsilon}{48 \sqrt{C}} \right)^{-\dim(\mathcal{G}_k)} . \tag{7}
\]

These observations raise a double question: can a change of combinatorial dimension (replacing \( \gamma \)-dim \( (\rho_\mathcal{G}) \) with a \( \gamma \)-\( \Psi \)-dimension of \( \rho_\mathcal{G} \)) better account for the introduction of the double clipping? If so, how are the dependences on \( m \) and \( C \) affected? The second part of the question calls for a double answer, depending on whether there is a coupling among the component functions of the functions in \( \mathcal{G} \).

4 New results for \( \gamma \)-\( \Psi \)-dimensions

The main scale-sensitive combinatorial dimensions are connected through a simple ordering whose knowledge is crucial to evaluate the results involved in the derivation of our new guaranteed risk.
4.1 Basic ordering on the scale-sensitive combinatorial dimensions

**Proposition 1** Let \( G \) be a function class satisfying Definition 3 and \( \rho \) the function class deduced from \( G \) according to Definition 4. For \( \gamma \in (0, 1] \), let \( \rho_{G, \gamma} \) be the function class deduced from \( G \) according to Definition 9. Then,

\[
\forall \gamma \in (0, M_G], \; \gamma-N\text{-dim}(\rho) \leq \gamma-G\text{-dim}(\rho) \leq \gamma\text{-dim}(\rho) \tag{8}
\]

and

\[
\forall \epsilon \in (0, \gamma 2], \; \epsilon\text{-dim}(\rho_{G, \gamma}) \leq \epsilon\text{-dim}(\rho) \tag{9}
\]

The first utility of Proposition 1 regards the assessment of our combinatorial result.

4.2 Combinatorial result

Our combinatorial result is an extension of Lemma 5 meeting the requirements exposed at the end of Section 3.2, i.e., involving both the \( L_2 \)-norm and \( \gamma\text{-}\Psi \)-dimensions.

**Lemma 7** Let \( G \) be a function class satisfying Definition 3 and \( \rho \) the function class deduced from \( G \) according to Definition 4. For \( \gamma \in (0, 1] \), let \( \rho_{G, \gamma} \) be the function class deduced from \( G \) according to Definition 9. For \( \epsilon \in (0, M_G] \), let \( d_G(\epsilon) = \epsilon\text{-G-dim}(\rho) \) and \( d_N(\epsilon) = \epsilon\text{-N-dim}(\rho) \). Then for \( \epsilon \in (0, \gamma] \) and \( n \in \mathbb{N}^* \),

\[
\mathcal{M}_2(\epsilon, \rho_{G, \gamma}, n) \leq \left( \frac{6 \gamma}{\epsilon} \right)^{20d_G(\frac{\epsilon}{\gamma})} \tag{10}
\]

and

\[
\mathcal{M}_2(\epsilon, \rho_{G, \gamma}, n) \leq \left( \frac{6 \gamma}{\epsilon} \right)^{240 \log^2(C)(F(C))^{\beta(C)} (\frac{\epsilon}{\gamma})^{\alpha(C)}(\frac{\epsilon}{\gamma})^{\beta(C)} N(\epsilon^{48})} \tag{11}
\]

where \( F(C) = 4(C - 1) \), \( \alpha(C) = 2 + \frac{2}{2 \ln(F(C)) - 1} \) and \( \beta(C) = 1 + \frac{1}{4 \ln(F(C)) - 2} \).

Lemma 7 shares with Lemma 5 the property to be dimension-free (both upper bounds are independent of \( n \)). Formula (10) directly compares with the application of Lemma 5 to \( \rho \) and \( \rho_{G, \gamma} \). The first of these applications gives:

\[
\mathcal{M}_2(\epsilon, \rho, n) \leq \left( \frac{12 M_G}{\epsilon} \right)^{20(\frac{\epsilon}{\gamma})},
\]

which is obviously worse than Formula (10) since \( \gamma < 2 M_G \) and Proposition 4 states that \( \epsilon\text{-G-dim}(\rho) \leq \epsilon\text{-dim}(\rho) \). It is less simple to characterize the difference with the application of Lemma 5 to \( \rho_{G, \gamma} \). Indeed the only known relationship between \( \epsilon\text{-G-dim}(\rho) \)
and $\epsilon$-dim$(\rho_{G,\gamma})$, provided once more by Proposition 1, is that they are both smaller than $\epsilon$-dim$(\rho_G)$. However, to the best of our knowledge, (9) is the only upper bound on $\epsilon$-dim$(\rho_{G,\gamma})$ available, and we have seen in Section 3.4 that the sole bound on $\epsilon$-dim$(\rho_G)$ provided by the literature, Lemma 6, does not lead to a satisfactory upper bound on $\ln(N_{2\text{int}}(\epsilon, \rho_G, \gamma, m))$. Thus, the use of the margin Graph dimension could have practical motives, if a structural result improving Lemma 6 is derived for this capacity measure. In that respect, the usefulness of Inequality (11) should result from the fact that the margin Natarajan dimension is easier to upper bound than the margin Graph dimension (see the following subsection). Regarding the coefficients appearing in this inequality, $\alpha$ and $\beta$ are monotonous decreasing functions, that go respectively to 2 and 1 as $C$ goes to infinity. Looking at the proof of Lemma 15, precisely the use of (37), it is noteworthy that these functions can be replaced with functions that they dominate and are arbitrarily close to 2 and 1. The consequence on the guaranteed risk is an improved convergence rate and a worsened dependence on $C$.

4.3 Structural results

In this section, we introduce results that can be seen as counterparts of Lemmas 2, 3 and 6 dealing with $\gamma$-$\Psi$-dimensions. The basic structural result for the margin Graph dimension is an improvement of Lemma 6.

Lemma 8 Let $\mathcal{G}$ be a function class satisfying Definition 3 and $\rho_G$ the function class deduced from $\mathcal{G}$ according to Definition 4. Then

$$\forall \gamma \in (0, M_G], \gamma$-G-dim$(\rho_G) \leq \frac{10K_C \log_2(2C)}{\ln(2)} \ln\left(\frac{48M_G \log_2^2(2C)}{\gamma}\right) \sum_{k=1}^{C} \left(\frac{\gamma}{144 \log_2(2C)}\right)^{-\dim(G_k)} ,$$

where $K_C = \min \left\{ 4 \left(\frac{C}{C-2}\right)^2, 16 \right\}$.

The basic structural result for the margin Natarajan dimension is a straightforward generalization of Theorem 48 in [14].

Lemma 9 Let $\mathcal{G}$ be a function class satisfying Definition 3 and $\rho_G$ the function class deduced from $\mathcal{G}$ according to Definition 4. Then

$$\forall \gamma \in (0, M_G], \gamma$-N-dim$(\rho_G) \leq \sum_{k=1}^{C-1} \sum_{l=k+1}^{C} \gamma$-dim$\left(\text{absconv}(G_k \cup G_l)\right) ,$$

where absconv returns the symmetric convex hull of its argument.
In the succession of lemmas constituting the derivation of a guaranteed risk involving a \( \gamma - \Psi \)-dimension, the use of one of those two structural results could constitute a bottleneck. Indeed, Lemma 8 is a straightforward improvement of Lemma 6 and, as such, it still holds true for the \( \gamma \)-dimension of \( \rho_\mathcal{G} \) (the proof uses the right-hand side inequality of Formula (8)), so that it does not make the best of the nature of the combinatorial dimension considered. As for Lemma 9, it prevents the dependence of the confidence interval on \( C \) from being sublinear. The obvious way to cope with these limitations is to dedicate the formulas (the proofs) to the classifier of interest, making it possible, for instance, to take benefit from a coupling among classes. We illustrate the expected gain with a standard example, that of \( C \)-category SVMs.

4.4 Structural result for M-SVMs

We base the definition of the \( C \)-category SVMs on that of reproducing kernel Hilbert space (RKHS) \( \mathbb{S} \) of \( \mathbb{R}^C \)-valued functions.

Definition 19 (RKHS of \( \mathbb{R}^C \)-valued functions \( H_{\kappa, C} \), after Section 6 of [36]) Let \( \kappa \) be a real-valued positive type function on \( X^2 \) and let \( \left( H_{\kappa}, \langle \cdot, \cdot \rangle_{H_{\kappa}} \right) \) be the corresponding RKHS. Let \( \tilde{\kappa} \) be the real-valued positive type function on \( Z^2 \) deduced from \( \kappa \) as follows:

\[
\forall \left( (x, k), (x', l) \right) \in Z^2, \quad \tilde{\kappa} \left( (x, k), (x', l) \right) = \delta_{k,l} \kappa \left( (x, x') \right),
\]

where \( \delta \) is the Kronecker symbol. For every \( (x, k) \in Z \), let us define the \( \mathbb{R}^C \)-valued function \( \tilde{\kappa}^{(C)}_{x,k}(\cdot) \) on \( X \) by the formula

\[
\tilde{\kappa}^{(C)}_{x,k}(\cdot) = \left( \tilde{\kappa} \left( (x, k), (\cdot, l) \right) \right)_{1 \leq l \leq C}.
\] (14)

The RKHS of \( \mathbb{R}^C \)-valued functions at the basis of a \( C \)-category SVM whose kernel is \( \kappa \), \( \left( H_{\kappa, C}, \langle \cdot, \cdot \rangle_{H_{\kappa, C}} \right) \), consists of the linear manifold of all finite linear combinations of functions of the form (14) as \( (x, k) \) varies in \( Z \), and its closure with respect to the inner product

\[
\forall \left( (x, k), (x', l) \right) \in Z^2, \quad \left\langle \tilde{\kappa}^{(C)}_{x,k}, \tilde{\kappa}^{(C)}_{x',l} \right\rangle_{H_{\kappa, C}} = \tilde{\kappa} \left( (x, k), (x', l) \right).
\]

With the definition of this RKHS a hand, the function class at the basis of a \( C \)-category SVM is specified through the introduction of a condition controlling the capacity. We consider the standard one, used for instance in [24].

Definition 20 (Function class \( H_\Lambda \)) Let \( \kappa \) be a real-valued positive type function on \( X^2 \) and let \( \Lambda \in \mathbb{R}^+ \). Let \( \left( H_{\kappa, C}, \langle \cdot, \cdot \rangle_{H_{\kappa, C}} \right) \) be the RKHS of \( \mathbb{R}^C \)-valued functions spanned by
κ according to Definition 19. Then the function class $\mathcal{H}_\Lambda$ associated with the $C$-category SVM parameterized by $(\kappa, \Lambda)$ is:

$$\mathcal{H}_\Lambda = \left\{ h = (h_k)_{1 \leq k \leq C} \in \mathcal{H}_{\kappa,C} : \sum_{k=1}^{C} h_k = 0_{\mathcal{H}_\kappa} \text{ and } \|h\|_{\mathcal{H}_{\kappa,C}} \leq \Lambda \right\}.$$ 

Then, Lemma 10 provides a sharper bound on the margin Natarajan dimension of $\rho_{\mathcal{H}_\Lambda}$ than a direct application of Lemma 9.

**Lemma 10** For $\Lambda \in \mathbb{R}^*_+$, let $\mathcal{H}_\Lambda$ be a function class satisfying Definition 20. Suppose that for every $x \in \mathcal{X}$, $\kappa_x$ belongs to the closed ball of radius $\Lambda_{\mathcal{X}}$ about the origin in $\mathcal{H}_\kappa$. Then,

$$\forall \gamma \in (0, \Lambda_{\mathcal{X}}], \gamma\text{-N-dim} (\rho_{\mathcal{H}_\Lambda}) \leq C \left( \frac{\Lambda_{\mathcal{X}}}{2\gamma} \right)^2. \quad (15)$$

A comparison of Formulas (13) and (15) establishes that the $\gamma$-$\Psi$-dimensions can prove malleable enough to take into account standard features of the classifiers so as to improve significantly the dependence of the confidence interval on one of the basic parameters (here the number of categories).

### 4.5 Bounds on the metric entropy

The best way to evaluate the upper bounds on the metric entropy of $\rho_{G,\gamma}$ consists in comparing the corresponding confidence intervals, obtained by application of the chaining method. This is due to the fact that this application can be optimized case by case, by an appropriate choice of function $h$. This calls for an additional hypothesis on the behavior of the $\gamma$-dimensions of the classes of component functions. We use the standard hypothesis in learning theory [27, 16], and beyond in the theory of empirical processes [34]: that of polynomial $\gamma$-dimensions. Regarding the bound obtained through the use of the margin Natarajan dimension, the significant difference between the two structural results involving this measure, Lemmas 9 and 10, suggests the use of a generic expression for the decomposition formula. These two choices are gathered in the following hypothesis.

**Hypothesis 1** We consider classes of functions $\mathcal{G}$ satisfying Definition 3 plus the fact that there exists a quadruplet $(d_{\mathcal{G},C}^{d_{\mathcal{G},\gamma}}, \tilde{K}_{\rho_{G}}, K_{\mathcal{G}}) \in (0, 2] \times (\mathbb{R}^*_+)^3$ such that

$$\forall \epsilon \in (0, M_{\mathcal{G}}], \begin{cases} \epsilon\text{-N-dim} (\rho_{G}) \leq \tilde{K}_{\rho_{G}} C^{d_{\mathcal{G},C}} \max_{1 \leq k \leq C} \epsilon\text{-dim} (\mathcal{G}_k) \quad (16a) \\ \max_{1 \leq k \leq C} \epsilon\text{-dim} (\mathcal{G}_k) \leq K_{\mathcal{G}} \epsilon^{-d_{\mathcal{G},\gamma}}. \quad (16b) \end{cases}$$
In the sequel, we set $K_{\rho G} = K_{\rho G}^*K_{\rho G}$, so that a substitution of (16b) into (16a) leads to
\[
\forall \epsilon \in (0, M_G], \quad \epsilon\text{-N-dim} (\rho G) \leq K_{\rho G} C_{\rho G}^{\rho G} e^{-\rho G \gamma}.
\] (17)

Under Hypothesis [1] by substitution of (16b) into (7), applying the decomposition with covering numbers gives:
\[
\forall \epsilon \in (0, \gamma], \quad \ln (N_{\rho G}^{\rho G} (\epsilon, \rho G, \gamma, m)) \leq 20K_G C\left(\frac{48\sqrt{C}}{\epsilon}\right)^{\rho G \gamma} \ln \left(\frac{12M_G \sqrt{C}}{\epsilon}\right).
\] (18)

The decomposition with the margin Graph dimension (substitution of (12) into (10)) leads to
\[
\forall \epsilon \in (0, \gamma], \quad \ln (N_{\rho G}^{\rho G} (\epsilon, \rho G, \gamma, m)) \leq \frac{3200K_G \log_2(2)}{C \ln(2)} \left(\frac{2304M_G \log_2(2C)}{\epsilon}\right)\left(\frac{6912 \log_2(2C)}{\epsilon}\right)^{d_{\rho G} \gamma} \ln \left(\frac{6\gamma}{\epsilon}\right).
\] (19)

At last, to obtain the formula corresponding to the decomposition with the margin Natarajan dimension, we substitute (17) into (11), which gives after some algebra:
\[
\forall \epsilon \in (0, \gamma], \quad \ln (N_{\rho G}^{\rho G} (\epsilon, \rho G, \gamma, m)) \leq 240K_{\rho G}^{\rho G} C^{\rho G} (F(C)) \left(\frac{48}{\epsilon}\right)^{\rho G \gamma} \ln \left(\frac{6\gamma}{\epsilon}\right) \leq 240 e^{d_{\rho G} \gamma} K_{\rho G}^{\rho G} C^{\rho G} (F(C)) \left(\frac{48}{\epsilon}\right)^{\rho G \gamma} \ln \left(\frac{6\gamma}{\epsilon}\right) \leq 396K_{\rho G}^{\rho G} C^{\rho G} (F(C)) \left(\frac{48}{\epsilon}\right)^{\rho G \gamma} \ln \left(\frac{6\gamma}{\epsilon}\right) \leq 396K_{\rho G}^{\rho G} C^{\rho G} (F(C)) \left(\frac{48}{\epsilon}\right)^{\rho G \gamma} \ln \left(\frac{6\gamma}{\epsilon}\right).
\] (20)

Since the dependence of (19) on $C$ is better than that of (18) and we know that the decomposition with covering numbers produces a confidence interval that scales sublinearly with $C$ (see Section 3.4), then this property remains true with the margin Graph dimension. Thus, an alternative way to get a sublinear dependence on $C$ without making any assumption on the coupling of the component functions has been highlighted. However, the extra logarithmic factor ends up worsening the corresponding convergence rate. Thus, the inequality associated with the margin Natarajan dimension appears as a better candidate for the derivation of the guaranteed risk.

5 New guaranteed risk

With Formula (1) at hand, the derivation of the guaranteed risk boils down to the derivation of the upper bound on $R_m (\rho G, \gamma)$. 
5.1 Characterization of the confidence interval

To upper bound the Rademacher complexity of interest, we proceed in two steps. The first one is the chaining method (Theorem 2), that relates it to the metric entropy. The second one is the upper bound on the metric entropy obtained through the use of the margin Natarajan dimension: Inequality (20). This gives:

\[
R_m (\rho_{G, \gamma}) \leq h (N) + 2 \sum_{j=1}^{N} (h (j) + h (j - 1)) \sqrt{\frac{\ln (N^{\text{int}} (h (j) + h (j - 1)))}{m}}
\]

\[
\leq h (N) + 40 \sqrt{\frac{F_1 (C)}{m}} \sum_{j \in J} \frac{h (j) + h (j - 1)}{\rho_{G, \gamma}^{(j)}} \sqrt{\ln \left(\frac{6 \gamma}{h (j)}\right)}
\]

where

\[
F_1 (C) = 48 \beta (C) d_{G, \gamma} K_{\rho_{G, \gamma}}^{\beta (C)} C_{d_{G, \gamma}}^{\alpha (C)} \log_2 (F (C)),
\]

(21)

with the functions \( F, \alpha \) and \( \beta \) being defined in Lemma 7, and \( J = \{ j \in [1; N] : h (j) \leq \gamma \} \).

With the last formula at hand, the derivation of the guaranteed risk amounts to studying the phase transition first highlighted by Mendelson in [26].

**Theorem 3** Let \( G \) be a function class satisfying Hypothesis 7. For \( \gamma \in (0, 1] \), let \( \rho_{G, \gamma} \) be the function class deduced from \( G \) according to Definition 7.

If \( \beta (C) d_{G, \gamma} \in (0, 2) \), then

\[
R_m (\rho_{G, \gamma}) \leq 80 \frac{1 + 2 \frac{2^{\beta (C) d_{G, \gamma}}}{2 - \beta (C) d_{G, \gamma}}} {\sqrt{2 (1 - \frac{\beta (C) d_{G, \gamma}}{2})}} \sqrt{F_1 (C)} \left[ \sqrt{\ln (F_2 (C))} + \sqrt{\frac{1}{4 \ln (F_2 (C))}} \right],
\]

(22)

where \( F_1 (C) \) is given by Equation (21) and \( F_2 (C) = 2 \cdot 6^{\frac{2 - \beta (C) d_{G, \gamma}}{2}} \).

If \( \beta (C) d_{G, \gamma} = 2 \), then

\[
R_m (\rho_{G, \gamma}) \leq \frac{\gamma}{\sqrt{m}} + 120 \sqrt{\frac{F_1 (C)}{m}} \left[ \frac{1}{2} \log_2 (m) \right] \sqrt{\ln (6 \sqrt{m})}.
\]

At last, if \( \beta (C) d_{G, \gamma} > 2 \), then

\[
R_m (\rho_{G, \gamma}) \leq \gamma \left( \frac{\log_2 (m)}{m} \right)^{\frac{1}{\beta (C) d_{G, \gamma}}} \left[ 1 + 80 \left( 1 + 2 \frac{2^{\beta (C) d_{G, \gamma}}}{2 - \beta (C) d_{G, \gamma}} \right) \frac{\beta (C) d_{G, \gamma}}{2} \sqrt{\frac{F_1 (C)}{\log_2 (m)}} \ln \left( \frac{m}{\log_2 (m)} \left( \frac{1}{\log_2 (m)} \right)^{\frac{1}{\beta (C) d_{G, \gamma}}} \right) \right].
\]
5.2 Discussion

The bounds of Theorem 3 compare with those resulting from using (18), corresponding to Theorem 7 in [16]. The comparison can rest on the dependences on the three basic parameters: \( m \), \( C \) and \( \gamma \). The function \( \beta \) affects the convergence rate in two ways. First, it introduces a small shift to the phase transition, which occurs now earlier, for \( d_{G,\gamma} = 2\beta^{-1}(C) \) (instead of \( d_{G,\gamma} = 2 \)). The convergence rates remain those of Theorem 18 in [27] for each phase except the last one (where \( \left( \frac{\log_2(m)}{m} \right)^\frac{1}{\pi m d_{G,\gamma}} \) replaces \( \left( \frac{\log_2(m)}{m} \right)^\frac{1}{d_{G,\gamma}} \)), and we have seen in the discussion on Lemma 1 that this loss could be made arbitrarily small.

Since the occurrence of the phase transition depends on both \( d_{G,\gamma} \) and \( C \), it is slightly more difficult to characterize the dependence on the number of categories. Keeping in mind that for most of the main margin classifiers, such as the multi-layer perceptrons [4, 2] or the SVMs [6], \( d_{G,\gamma} \geq 2 \), we restrict the discussion to the case \( d_{G,\gamma} > 2\beta^{-1}(C) \). Then, the dependence on \( C \) is a \( O(\left( C^{-\frac{d_{G,\gamma}}{2}} \ln (C) \right) \) (it does not depend on \( d_{G,\gamma} \) anymore), which implies that the confidence interval scales sublinearly with \( C \) as soon as \( d_{G,C} < 2 \). At last, the bounds of Theorem 7 in [16] all provide an upper bound on \( \gamma^{-1} R_{m}(\rho_{G,\gamma}) \) growing with \( \gamma^{-1} \) as a \( O(\left( \frac{1}{\gamma} \right)^{\frac{d_{G,\gamma}}{2}} \sqrt{\ln (\frac{1}{\gamma})}) \), whereas those of Theorem 3 provide an upper bound on the same quantity growing with \( \gamma^{-1} \) as a \( O(\left( \frac{1}{\gamma} \right)^{\frac{\beta(C)d_{G,\gamma}}{2}}) \). Thus, our new bounds can represent an improvement compared to those of [16] with respect to the dependence on \( \gamma \), for large values of \( C \) (and an appropriate choice of the degree of freedom \( \beta \)).

6 Conclusions and ongoing research

This article has introduced the tools needed to derive guaranteed risks for margin multi-category classifiers using two \( \gamma\)-\( \Psi \)-dimensions: the margin Graph dimension and the margin Natarajan dimension. The main contributions are a new combinatorial result dedicated to the dimensions of interest, and structural results connecting them to the \( \gamma \)-dimensions of functions classes including the classes of component functions. The structural result dedicated to the margin Graph dimension improves that of [11]. The dependence of the confidence interval of the guaranteed risk on the number of categories is always sublinear when the margin Graph dimension is used. With the other combinatorial dimension, the sublinearity is obtained as soon as a coupling among the component functions of the classifier can be taken into account. Whether this is the case or not, the dependence is no
longer a function of the degree $d_{G, \gamma}$ of the polynomial upper bounding the $\gamma$-dimensions of the classes of component functions.

There is obviously room for improvements of the combinatorial result dedicated to the margin Natarajan dimension and the structural result dedicated to the margin Graph dimension. We conjecture that such improvements should underline the usefulness of the $\gamma$-$\Psi$-dimensions to control the dependence of the confidence interval on the scale parameter. Our current work consists in upper bounding the margin Natarajan dimension of the main margin classifiers in the literature.

Acknowledgements This work was partly funded by a CNRS research grant. It was presented at the Dagstuhl Seminar 18291 “Extreme Classification”. The author would like to thank the organizers for their invitation.

A Proof of Proposition 1

Proof For $\gamma \in (0, M_G]$, let $s_{\mathbb{Z}^n} = \{z_i = (x_i, y_i) : 1 \leq i \leq n\}$ be a subset of $\mathbb{Z}$ $\gamma$-N-shattered by $\{\rho_{g^n} : s_n \in \{-1, 1\}^n\} \subset \rho_G$ and let $(b_n, c_n)$ be a witness to this shattering. To prove the left-hand side inequality of Formula (8), it suffices to notice that for a given vector $s_n$, the function $\rho_{g^n} \in \rho_G$ satisfying

$$
\forall i \in \llbracket 1; n \rrbracket, \begin{cases} 
  \text{if } s_i = 1, & \rho_{g^n}(x_i, y_i) - b_i \geq \gamma \\
  \text{if } s_i = -1, & \rho_{g^n}(x_i, c_i) + b_i \geq \gamma 
\end{cases}
$$

also satisfies

$$
\forall i \in \llbracket 1; n \rrbracket, \begin{cases} 
  \text{if } s_i = 1, & \rho_{g^n}(x_i, y_i) - b_i \geq \gamma \\
  \text{if } s_i = -1, & \max_{k \neq y_i} \rho_{g^n}(x_i, k) + b_i \geq \gamma 
\end{cases}.
$$

Keeping the notations above, proving the right-hand side inequality boils down to establishing that $\max_{k \neq y_i} \rho_{g^n}(x_i, k) \leq -\rho_{g^n}(x_i, y_i)$. Indeed,

$$
\max_{k \neq y_i} \rho_{g^n}(x_i, k) = \frac{1}{2} \max_{k \neq y_i} \left( g_{k}^{s_n}(x_i) - \max_{l \neq k} g_{l}^{s_n}(x_i) \right) \\
\leq \frac{1}{2} \left( \max_{k \neq y_i} g_{k}^{s_n}(x_i) - g_{y_i}^{s_n}(x_i) \right) \\
= -\rho_{g^n}(x_i, y_i).
$$

The proof of Formula (9) can be found in [23].
B Proof of the combinatorial result

The sketch of the proof of Lemma 7 is that of Lemma 5. It involves several concepts and modules which are gathered in this appendix. Each of the combinatorial results in the literature is built upon a basic lemma that involves two (possibly identical) function classes whose domain and codomain are finite sets (so that their cardinalities are also finite). It upper bounds the cardinality of one of them in terms of a combinatorial dimension of the other. In the case of margin classifiers, the combinatorial dimension of the basic lemma is a variant of the scale-sensitive dimension of the combinatorial result, variant designed to take benefit from the aforementioned restrictions. The first capacity measure of this kind is a variant of the $\gamma$-dimension: the strong dimension (Definition 3.1 in [1]). The strong $\Psi$-dimensions extend the $\gamma$-$\Psi$-dimensions according to the same principle.

Definition 21 (Strong $\Psi$-dimensions) Let $\mathcal{F}$ be a class of functions from $\mathcal{Z}$ into $[\!-M_F; M_F\!]$ with $M_F \in \mathbb{N}^\ast$. Let $\Psi$ be a family of mappings from $\mathcal{Y}$ into \{-1, 1, *\}, where * is a null element. A subset $s_{\mathcal{Z}^n} = \{z_i = (x_i, y_i) : 1 \leq i \leq n\}$ of $\mathcal{Z}$ is said to be strongly $\Psi$-shattered by $\mathcal{F}$ if there is a vector $\psi_n = (\psi^{(i)})_{1 \leq i \leq n} \in \Psi^n$ satisfying for every $i \in [1; n]$, $\psi^{(i)}(y_i) = 1$, and a vector $b_n = (b_i)_{1 \leq i \leq n} \in [\!-M_F + 1; M_F - 1\!]$ such that, for every vector $s_n = (s_i)_{1 \leq i \leq n} \in \{-1, 1\}^n$, there is a function $f_{s_n} \in \mathcal{F}$ satisfying

$$\forall i \in [1; n], \begin{cases} 
\text{if } s_i = 1, & \exists k : \psi^{(i)}(k) = 1 \text{ and } f_{s_n}(x_i, k) - b_i \geq 1 \\
\text{if } s_i = -1, & \exists l : \psi^{(i)}(l) = -1 \text{ and } f_{s_n}(x_i, l) + b_i \geq 1
\end{cases}.$$  

The strong $\Psi$-dimension of $\mathcal{F}$, denoted by $S$-$\Psi$-$\dim(\mathcal{F})$, is the maximal cardinality of a subset of $\mathcal{Z}$ strongly $\Psi$-shattered by $\mathcal{F}$, if such maximum exists. Otherwise, $\mathcal{F}$ is said to have infinite strong $\Psi$-dimension.

The finiteness of the domain is simply obtained by application of a restriction to the data at hand. As for the finiteness of the codomain, if needed, it is obtained by application of a discretization operator. The present study makes use of the one introduced in [14].

Definition 22 ($\eta$-discretization operator, Definition 33 in [14]) Let $\mathcal{F}$ be a class of functions from $\mathcal{T}$ into the interval $[M_{F-}, M_{F+}]$. For $\eta \in \mathbb{R}^+_\ast$, define the $\eta$-discretization as an operator on $\mathcal{F}$ such that:

$$(\cdot)^{(\eta)} : \mathcal{F} \rightarrow \mathcal{F}^{(\eta)}$$

$$f \mapsto f^{(\eta)}$$
\( \forall t \in \mathcal{T}, \; f^{(n)}(t) = \text{sign}(f(t)) \cdot \left\lfloor \frac{|f(t)|}{\eta} \right\rfloor. \)

The following lemma extends the first proposition of Lemma 3.2 in [1].

**Lemma 11** Let \( \mathcal{F} \) be a class of functions from \( \mathcal{Z} \) into \( [-M_F, M_F] \) with \( M_F \in \mathbb{R}_+^* \). For every \( \eta \in (0, M_F) \) and every \( \epsilon \in (0, \frac{\eta}{2}) \),

\[
\begin{align*}
S \cdot G \cdot \dim \left( \mathcal{F}(\eta) \right) &\leq \epsilon \cdot G \cdot \dim (\mathcal{F}) \tag{23a} \\
S \cdot N \cdot \dim \left( \mathcal{F}(\eta) \right) &\leq \epsilon \cdot N \cdot \dim (\mathcal{F}). \tag{23b}
\end{align*}
\]

**Proof** To prove (23a), it is enough to establish that any set strongly \( G \)-shattered by \( \mathcal{F}(\eta) \) is also \( G \)-shattered with margin \( \frac{\eta}{2} \) by \( \mathcal{F} \). Suppose that the subset \( \mathcal{Z} = \{ z_i = (x_i, y_i) : 1 \leq i \leq n \} \) of \( \mathcal{Z} \) is strongly \( G \)-shattered by \( \mathcal{F}(\eta) \). Then, according to Definitions \( \text{21 and 22} \) there exists a vector \( \mathbf{b}_n = (b_i)_{1 \leq i \leq n} \in \left[ -\frac{M_F}{\eta} \right] + 1 \text{; } \left[ \frac{M_F}{\eta} \right] - 1 \right]^n \) and a set \( \{ f_{s_n} : s_n \in \{-1, 1\}^n \} \subset \mathcal{F} \) such that

\[
\forall s_n = (s_i)_{1 \leq i \leq n} \in \{-1, 1\}^n, \; \forall i \in [1; n], \quad \begin{cases} 
\text{if } s_i = 1, & f_{s_n}(x_i, y_i) - b_i \geq 1 \\
\text{if } s_i = -1, & \max_{k \neq y} f_{s_n}(x_i, k) + b_i \geq 1
\end{cases}
\]

As a consequence, a proof is obtained by exhibiting a vector \( \mathbf{b}_n' = (b'_i)_{1 \leq i \leq n} \in \mathbb{R}^n \) such that

\[
\forall s_n = (s_i)_{1 \leq i \leq n} \in \{-1, 1\}^n, \; \forall i \in [1; n], \quad \begin{cases} 
\text{if } s_i = 1, & f_{s_n}(x_i, y_i) - b'_i \geq \frac{\eta}{2} \\
\text{if } s_i = -1, & \max_{k \neq y} f_{s_n}(x_i, k) + b'_i \geq \frac{\eta}{2}
\end{cases}
\]

A feasible solution consists in setting \( b'_i = \eta \left( b_i + \frac{1}{2} \right) \) for every indice \( i \in [1; n] \) such that \( b_i \geq 0 \) and \( b'_i = \eta \left( b_i - \frac{1}{2} \right) \) otherwise. The same line of reasoning can be used to prove (23b).

In short, if \( (\mathbf{b}_n, \mathbf{c}_n) \) is a witness to the strong \( N \)-shattering, then \( (\mathbf{b}_n', \mathbf{c}'_n) \) is a witness to the \( \frac{\eta}{2} \)-\( N \)-shattering, where the vector \( \mathbf{b}_n' \) is deduced from \( \mathbf{b}_n \) as above and \( \mathbf{c}'_n = \mathbf{c}_n \). \hfill \Box

The following lemma extends the second proposition of Lemma 3.2 in [1].

**Lemma 12** Let \( \mathcal{F} \) be a class of functions from \( \mathcal{T} \) into the interval \([0, M_F]\) with \( M_F \in \mathbb{R}_+^* \). For \( n \in \mathbb{N}^* \), let \( \mathbf{t}_n = (t_i)_{1 \leq i \leq n} \in \mathcal{T}^n \). Let \( N \) be a positive integer. For every \( \epsilon \in (0, M_F] \) and every \( \eta \in (0, \frac{\epsilon}{2N}) \),

\[
\forall (f, f') \in \mathcal{F}^2, \; d_{2 \cdot \mathbf{t}_n} (f, f') \geq \epsilon \Longrightarrow d_{2 \cdot \mathbf{t}_n} \left( f^{(n)}, f'^{(n)} \right) \geq N, \tag{24}
\]

with the consequence that if the subset \( \mathcal{F} \) of \( \mathcal{F} \) is \( \epsilon \)-separated with respect to the pseudo-metric \( d_{2 \cdot \mathbf{t}_n} \), then it is in bijection with the subset \( \mathcal{F}^{(n)} \) of \( \mathcal{F}^{(n)} \), which is \( N \)-separated with respect to the same pseudo-metric.
Proof For \( f \in \mathcal{F} \) and \( i \in \llbracket 1; n \rrbracket \), let us denote the Euclidean division of \( f(t_i) \) by \( \eta \) as follows:

\[ \forall i \in \llbracket 1; n \rrbracket, \quad f(t_i) = \eta f^{(\eta)}(t_i) + r_i. \]

With the notation introduced above,

\[
d_{2,t_n}(f, f')^2 = \frac{1}{n} \sum_{i=1}^{n} (f(t_i) - f'(t_i))^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \eta \left(f^{(\eta)}(t_i) - f'^{(\eta)}(t_i)\right) + r_i - r'_i\right)^2.
\]

For \( i \in \llbracket 1; n \rrbracket \), let \( \delta_i = \left| f^{(\eta)}(t_i) - f'^{(\eta)}(t_i) \right| \).

\[
\left(d_{2,t_n}(f, f')^2 \geq \epsilon^2 \right) \quad \text{and} \quad \left( \eta \in \left(0, \frac{\epsilon}{2N}\right) \right) \quad \Rightarrow \quad \frac{1}{n} \sum_{i=1}^{n} (\eta \delta_i + |r_i - r'_i|)^2 \geq \epsilon^2
\]
\[
\Rightarrow \quad \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\delta_i + 1}{2N} \epsilon \right)^2 \geq \epsilon^2
\]
\[
\Rightarrow \quad \frac{1}{n} \sum_{i=1}^{n} (\delta_i + 1)^2 \geq 4N^2
\]
\[
\Rightarrow \quad \frac{1}{n} \left( \sum_{i=1}^{n} \delta_i^2 + 2 \sum_{i=1}^{n} \delta_i \right) \geq 4N^2 - 1
\]
\[
\Rightarrow \quad \frac{3}{n} \sum_{i=1}^{n} \delta_i^2 \geq 3N^2
\]
\[
\Rightarrow \quad d_{2,t_n}\left( f^{(\eta)}, f'^{(\eta)} \right) \geq N.
\]

To sum up, we have established (24), i.e., the lemma.

The following results are based on the concept of \( \epsilon \)-separating tree.

**Definition 23 (\( \epsilon \)-separating tree, Definition 7 in [28])** Let \( \mathcal{F} \) be a class of real-valued functions with domain \( \mathcal{T} \). A tree of nonempty subsets \( \text{Tr}(\mathcal{F}) \) of \( \mathcal{F} \) is a finite collection of subsets of \( \mathcal{F} \) such that every two elements in \( \text{Tr}(\mathcal{F}) \) are either disjoint or one contains the other. A son of an element \( \bar{\mathcal{F}} \in \text{Tr}(\mathcal{F}) \) is a maximal (with respect to inclusion) proper subset of \( \bar{\mathcal{F}} \) which belongs to \( \text{Tr}(\mathcal{F}) \). For \( \epsilon \in \mathbb{R}_+ \), an \( \epsilon \)-separating tree \( \text{Tr}(\mathcal{F}, \epsilon) \) of \( \mathcal{F} \) is a tree of nonempty subsets of \( \mathcal{F} \) whose only root is \( \mathcal{F} \) and such that every element \( \bar{\mathcal{F}} \in \text{Tr}(\mathcal{F}, \epsilon) \) which is not a leaf has exactly two sons \( \bar{\mathcal{F}}_+ \) and \( \bar{\mathcal{F}}_- \) and, for some \( t \in \mathcal{T} \),

\[
\forall (f_+, f_-) \in \bar{\mathcal{F}}_+ \times \bar{\mathcal{F}}_-, \quad f_+(t) > f_-(t) + \epsilon.
\]
Proposition 2 (After Proposition 8 in [28]) Let $T = \{t_i : 1 \leq i \leq n\}$ be a finite set and $t_n = (t_i)_{1 \leq i \leq n}$. Let $\mathcal{F}$ be a finite class of real-valued functions on $T$. Suppose that for some $\epsilon \in \mathbb{R}_+^*$, $\mathcal{F}$ is $\epsilon$-separated in the pseudo-metric $d_{\mathcal{F},t_n}$. Then there exists an $\frac{\epsilon}{5}$-separating tree of $\mathcal{F}$ with at least $|\mathcal{F}|^\frac{1}{2}$ leaves.

Lemma 13 Let $G$ be a function class satisfying Definition 3 and $\rho_G$ the function class deduced from $G$ according to Definition 4. For $\gamma \in (0, 1]$, let $\rho_{G, \gamma}$ be the function class deduced from $G$ according to Definition 9. Let $\eta \in (0, \frac{1}{2}]$. Suppose that there exist $(g', g) \in G^2$ and $z = (x, y) \in \mathcal{Z}$ such that

$$\rho_{g', \gamma}^{(\eta)}(z) - \rho_{g, \gamma}^{(\eta)}(z) \geq 2. \quad (25)$$

Then the set $\left\{ \rho_{g'}^{(\eta)}, \rho_g^{(\eta)} \right\}$ strongly $G$-shatters the pair $\{z\}, b$ and strongly $\mathcal{N}$-shatters the triplets $\{z\}, b, c$ with $b = \rho_{g', \gamma}^{(\eta)}(z) - 1$ and $c = \arg\max_{k \neq y} \rho_{g'}^{(\eta)}(x, k)$.

Proof It springs from (25) that $\rho_{g', \gamma}^{(\eta)}(z) > 0$ and thus $\rho_g^{(\eta)}(z) \geq \rho_{g', \gamma}^{(\eta)}(z)$. Consequently,

$$\rho_g^{(\eta)}(z) - b \geq \rho_{g', \gamma}^{(\eta)}(z) - b = 1.$$

Furthermore, $\rho_{g', \gamma}^{(\eta)}(z) < \left\lfloor \frac{2}{\eta} \right\rfloor$ implies that $\rho_g^{(\eta)}(z) \leq \rho_{g', \gamma}^{(\eta)}(z)$ and thus $\max_{k \neq y} \rho_{g'}^{(\eta)}(x, k) \geq -\rho_{g', \gamma}^{(\eta)}(z)$, leading to

$$\max_{k \neq y} \rho_{g'}^{(\eta)}(x, k) + b \geq -\rho_{g', \gamma}^{(\eta)}(z) + \rho_{g, \gamma}^{(\eta)}(z) - 1 \geq 1. \quad (26)$$

The strong $G$-shattering of $\{z\}, b$ by $\left\{ \rho_{g'}^{(\eta)}, \rho_g^{(\eta)} \right\}$ has been established. The strong $\mathcal{N}$-shattering of $\{z\}, b, c$ springs from (26) and the definition of $c$. $\blacksquare$

The following proposition extends Proposition 10 in [28].

Proposition 3 Let $G$ be a function class satisfying Definition 3 and $\rho_G$ the function class deduced from $G$ according to Definition 4. For $\gamma \in (0, 1]$, let $\rho_{G, \gamma}$ be the function class deduced from $G$ according to Definition 9. For $\tilde{G} \subset G$, $\tilde{s}_{\gamma} = \{z_i : 1 \leq i \leq n\} \subset \mathcal{Z}$, $\gamma \in (0, 1]$ and $\eta \in (0, \frac{1}{2}]$, let $\mathcal{F} = \left( \rho_{\tilde{G}}|_{\tilde{s}_{\gamma}} \right)^{(\eta)}$ and let $\mathcal{F}_\gamma = \left( \rho_{\tilde{G}, \gamma}|_{\tilde{s}_{\gamma}} \right)^{(\eta)}$. Then the number of pairs $(s'_{2^z}, b'_u)$ with $s'_{2^z} \subset s_{2^n}$ and $b'_u \in \left[ -\frac{M_G}{\eta} ; \frac{M_G}{\eta} - 1 \right]^u$ strongly $G$-shattered by $\mathcal{F}$ is at least the number of leaves in any 1-separating tree of $\mathcal{F}_\gamma$ minus one.
Proof. For every subset $\mathcal{G}$ of $\mathcal{G}$, denote by $s(\mathcal{G})$ the number of pairs $(s'_Z, b'_u)$ strongly $G$-shattered by $\mathcal{F} = (\rho_{s_Z} |_{s_Z})^{(n)}$. To prove Proposition 3, we establish that if $\mathcal{F}_\gamma = (\rho_{G_\gamma} |_{s_Z})^{(n)}$ is an inner node of a 1-separating tree of $\mathcal{F}_\gamma$, $\mathcal{F}_{\gamma,+}$ and $\mathcal{F}_{\gamma,-}$ are its two sons and $\mathcal{G}_+$ and $\mathcal{G}_-$ are two subsets of $\mathcal{G}$ of respective cardinalities $|\mathcal{F}_{\gamma,+}|$ and $|\mathcal{F}_{\gamma,-}|$ such that $\mathcal{F}_{\gamma,+} = (\rho_{G_{\gamma,+}} |_{s_Z})^{(n)}$ and $\mathcal{F}_{\gamma,-} = (\rho_{G_{\gamma,-}} |_{s_Z})^{(n)}$, then

$$s(\mathcal{G}) \geq s(\mathcal{G}_+) + s(\mathcal{G}_-) + 1. \quad (27)$$

Indeed, a simple induction on the number of leaves proves that (27) implies the proposition. Obviously, any pair strongly $G$-shattered by either $\mathcal{F}_+ = (\rho_{G_+} |_{s_Z})^{(n)}$ or $\mathcal{F}_- = (\rho_{G_-} |_{s_Z})^{(n)}$ is also strongly $G$-shattered by $\mathcal{F}$. Let $i_0 \in [1; n]$ be an index such that

$$\forall (f_{\gamma,+}, f_{\gamma,-}) \in \mathcal{F}_{\gamma,+} \times \mathcal{F}_{\gamma,-}, \ f_{\gamma,+}(z_{i_0}) > f_{\gamma,-}(z_{i_0}) + 1,$$

i.e.,

$$\forall (f_{\gamma,+}, f_{\gamma,-}) \in \mathcal{F}_{\gamma,+} \times \mathcal{F}_{\gamma,-}, \ f_{\gamma,+}(z_{i_0}) - f_{\gamma,-}(z_{i_0}) \geq 2.$$

Let us set $b = \min_{f_{\gamma,+} \in \mathcal{F}_{\gamma,+}} f_{\gamma,+}(z_{i_0}) - 1$. Note that neither the set $\mathcal{F}_+$ nor the set $\mathcal{F}_-$ strongly $G$-shatters the pair $\{z_{i_0}\}, b$. Indeed, if the functions $f_+ \in \mathcal{F}_+$ and $f_{\gamma,+} \in \mathcal{F}_{\gamma,+}$ are associated with the same function $g_+ \in \mathcal{G}_+$ (the bijection between the sets $\mathcal{F}_{\gamma,+}$ and $\mathcal{G}_+$ has been introduced precisely to avoid any ambiguity at this level), then

$$f_{\gamma,+}(z_{i_0}) \geq 2 \implies f_+(z_{i_0}) \geq f_{\gamma,+}(z_{i_0}) \implies \max_{k \neq y_{i_0}} f_+(x_{i_0}, k) \leq -f_{\gamma,+}(z_{i_0}).$$

Symmetrically, if the functions $f_- \in \mathcal{F}_-$ and $f_{\gamma,-} \in \mathcal{F}_{\gamma,-}$ are associated with the same function $g_- \in \mathcal{G}_-$, then

$$f_{\gamma,-}(z_{i_0}) \leq \min_{f_{\gamma,+} \in \mathcal{F}_{\gamma,+}} f_{\gamma,+}(z_{i_0}) - 2 \implies f_{\gamma,-}(z_{i_0}) < \left\lfloor \frac{\gamma}{\eta} \right\rfloor \implies f_- (z_{i_0}) \leq f_{\gamma,-}(z_{i_0}).$$

Consequently,

$$\forall f_+ \in \mathcal{F}_+, \ f_{\gamma,+}(z_{i_0}) \geq b + 1 \implies \max_{k \neq y_{i_0}} f_+(x_{i_0}, k) \leq -b - 1$$

$$\implies \max_{k \neq y_{i_0}} f_+(x_{i_0}, k) + b < 1$$

and

$$\forall f_- \in \mathcal{F}_-, \ b - f_{\gamma,-}(z_{i_0}) \geq 1 \implies f_- (z_{i_0}) - b < 1.$$
Furthermore, the pair \( \{ \{ z_i \} \} , b \) has been built in such a way that its strong G-shattering by \( F \) is a direct consequence of Lemma 13. Next, assume that a pair \( (s'_{Zu}, b'_u) \) is strongly G-shattered by both \( F_+ \) and \( F_- \). Let us set \( s'_{Zu} = \{ z_i' : 1 \leq i \leq u \} \), with

\[
\forall (i,j) : 1 \leq i < j \leq u, \left( z_i', z_j' \right) = (z_v, z_w) \implies 1 \leq v < w \leq n.
\]

Let us consider the pair \( (s''_{Zu+1}, b''_{u+1}) \) such that \( s''_{Zu+1} = s'_{Zu} \cup \{ z_i \} \) and the vector \( b''_{u+1} \) is deduced from \( b'_u \) by inserting the component \( b \) at the right place. We denote by \( i_1 \) the index such that \( z_{i_1}' = z_i \) (so that \( b_{i_1}' = b \)). Observe that \( (s''_{Zu+1}, b''_{u+1}) \) is strongly G-shattered by \( F \). Indeed, since both \( F_+ \) and \( F_- \) strongly G-shatter \( (s'_{Zu}, b'_u) \), for every vector \( s_{u+1} = (s_i)_{1 \leq i \leq u+1} \in \{-1,1\}^{u+1} \) such that \( s_{i_1} = 1 \), there is a function \( f^{s_{u+1}}_+ \in F_+ \) satisfying

\[
\forall i \in [1; u+1] \setminus \{ i_1 \}, \begin{cases} 
\text{if } s_i = 1, \quad f^{s_{u+1}}_+ (x_i'', y_i'') - b_{i_1}'' \geq 1 \\
\text{if } s_i = -1, \quad \max_{k \neq y_i''} f^{s_{u+1}}_+ (x_i'', k) + b_{i_1}'' \geq 1 
\end{cases}
\]

and for every vector \( s_{u+1} = (s_i)_{1 \leq i \leq u+1} \in \{-1,1\}^{u+1} \) such that \( s_{i_1} = -1 \), there is a function \( f^{s_{u+1}}_- \in F_- \) satisfying

\[
\forall i \in [1; u+1] \setminus \{ i_1 \}, \begin{cases} 
\text{if } s_i = 1, \quad f^{s_{u+1}}_- (x_i'', y_i'') - b_{i_1}'' \geq 1 \\
\text{if } s_i = -1, \quad \max_{k \neq y_i''} f^{s_{u+1}}_- (x_i'', k) + b_{i_1}'' \geq 1 
\end{cases}
\]

Clearly, neither \( F_+ \) nor \( F_- \) strongly G-shatters \( (s''_{Zu+1}, b''_{u+1}) \), simply because they do not strongly G-shatter the pair \( \{ z_i \}, b \). Summarizing, for each pair \( (s'_{Zu}, b'_u) \) strongly G-shattered by both \( F_+ \) and \( F_- \), we can exhibit by means of an injective mapping a pair \( (s''_{Zu+1}, b''_{u+1}) \) strongly G-shattered by \( F \) but not by \( F_+ \) or \( F_- \). Besides, \( \{ z_i \}, b \) is strongly G-shattered by \( F \). This concludes the proof of (27) and thus the proof of the proposition.

**Corollary 1** Let \( \mathcal{G} \) be a function class satisfying Definition 9 and \( \rho_{\mathcal{G}} \) the function class deduced from \( \mathcal{G} \) according to Definition 4. For \( \gamma \in (0,1] \), let \( \rho_{\mathcal{G},\gamma} \) be the function class deduced from \( \mathcal{G} \) according to Definition 9. For \( \tilde{\mathcal{G}} \subseteq \mathcal{G} \), \( s_{2^n} = \{ z_i : 1 \leq i \leq n \} \subseteq \mathcal{Z} \), \( \gamma \in (0,1] \) and \( \eta \in (0, \frac{1}{2}] \), let \( F = \left( \rho_{\tilde{\mathcal{G}}} \right)_{s_{2^n}}^{(\eta)} \) and let \( F_\gamma = \left( \rho_{\tilde{\mathcal{G}},\gamma} \right)_{s_{2^n}}^{(\eta)} \). If \( F_\gamma \) is \( 6 \)-separated in the pseudo-metric \( d_{2^n,n} \), then \( F \) strongly G-shatters at least \( |F_\gamma|^{\frac{1}{2}} - 1 \) pairs.
Proof Corollary 1 directly results from combining Propositions 2 and 3.

Proposition 4 Let $\mathcal{G}$ be a function class satisfying Definition 3 and $\rho_\mathcal{G}$ the function class deduced from $\mathcal{G}$ according to Definition 4. For $\gamma \in (0, 1]$, let $\mathcal{F}_\gamma = \left( \rho_{\mathcal{G}, \gamma} \right)^{(\eta)}$ and let $\mathcal{F}_\gamma = \left( \rho_{\mathcal{G}, \gamma} \right)^{(\eta)}$. If $\mathcal{F}_\gamma$ is 6-separated in the pseudo-metric $d_{2n}$, then

$$|\mathcal{F}_\gamma| \leq \left( \frac{(M_\gamma - 1)en}{d_G} \right)^{2d_G}$$

where $M_\gamma = \left\lfloor \frac{\gamma}{\eta} \right\rfloor$ and $d_G = S-G\text{-dim}(\mathcal{F})$.

Proof We first notice that according to Lemma 13, $d_G \geq 1$. By definition, the maximal cardinality of a subset of $s_{2n}$ strongly $G$-shattered by $\mathcal{F}$ is bounded from above by $d_G$. As a consequence, the number of pairs $(s_{2n}, b_u)$ strongly $G$-shattered by $\mathcal{F}$ is bounded from above by

$$\Sigma = \sum_{u=1}^{d_G} \binom{n}{u} (M_\gamma - 1)^u.$$

Making use of Corollary 1 and a well-known computation (see for instance the proof of Corollary 3.3 in [29]) thus provides us with:

$$|\mathcal{F}_\gamma| \leq \Sigma + 1 \leq (M_\gamma - 1)^{d_G} \sum_{u=0}^{d_G} \binom{n}{u} \leq \left( \frac{(M_\gamma - 1)en}{d_G} \right)^{d_G}.$$

The following lemma, a slight improvement of Lemma 13 in [28], implements a probabilistic extraction principle. Its proof uses an abuse of notation that will be repeated in the sequel: the symbol $\mathbb{P}$ designates different probability measures, some of which implicitly defined.

Lemma 14 Let $\mathcal{T} = \{t_i : 1 \leq i \leq n\}$ be a finite set and $t_n = (t_i)_{1 \leq i \leq n}$. Let $\mathcal{F}$ be a finite class of functions from $\mathcal{T}$ into $[0, M_\mathcal{F}]$ with $M_\mathcal{F} \in \mathbb{R}^*_+$. Assume that for some $\epsilon \in (0, M_\mathcal{F}]$, $\mathcal{F}$ is $\epsilon$-separated with respect to the pseudo-metric $d_{2, t_n}$, and let

$$r = \frac{\ln(|\mathcal{F}|)}{K_\epsilon e^\epsilon}.$$

28
with

\[ K_\varepsilon = \frac{3}{112 M_\varepsilon^2}. \]

Then, there exists a subvector \( t_q \) of \( t_n \) of size \( q \leq r \) such that \( F \) is \( \varepsilon \)-separated with respect to the pseudo-metric \( d_{2,t_q} \).

**Proof** We first note that the statement is trivially true for \( r \geq n \) (it suffices to set \( t_q = t_n \)). Thus, we proceed under the hypothesis \( r \in [1, n) \). Let us set \( F = \{ f_j : 1 \leq j \leq |F| \} \) and \( D_F = \{ f_j - f_{j'} : 1 \leq j < j' \leq |F| \} \). The set \( D_F \) has cardinality \( |D_F| < \frac{1}{2} |F|^2 \). Let \( (\epsilon_i)_{1 \leq i \leq n} \) be a sequence of \( n \) independent Bernoulli random variables with common expectation \( \mu = \frac{r}{2n} \). Then, by application of the \( \epsilon \)-separation property, for every \( \delta_f \) in \( D_F \),

\[ P \left( \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \delta_f (t_i)^2 < \frac{\epsilon^2 \mu}{2} \right) \leq P \left( \frac{1}{n} \sum_{i=1}^{n} (\mu - \epsilon_i) \delta_f (t_i)^2 > \frac{\epsilon^2 \mu}{2} \right). \]  

(28)

Since by construction, for every \( i \in [1;n] \), \( E \left[ (\mu - \epsilon_i) \delta_f (t_i)^2 \right] = 0 \) and \( |\mu - \epsilon_i| \delta_f (t_i)^2 \leq M_\varepsilon^2 (1 - \mu) < M_\varepsilon^2 \) with probability one, the right-hand side of (28) can be bounded from above thanks to Bernstein’s inequality. Given that

\[ \frac{1}{n} \sum_{i=1}^{n} E \left[ (\mu - \epsilon_i)^2 \delta_f (t_i)^4 \right] \leq M_\varepsilon^4 \mu (1 - \mu) < M_\varepsilon^4 \mu, \]

we obtain

\[ P \left( \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \delta_f (t_i)^2 < \frac{\epsilon^2 \mu}{2} \right) \leq P \left( \frac{1}{n} \sum_{i=1}^{n} (\mu - \epsilon_i) \delta_f (t_i)^2 > \frac{\epsilon^2 \mu}{2} \right) \leq \exp \left( -\frac{3 \mu n \epsilon^4}{4 (6 M_\varepsilon^4 + M_\varepsilon^2 \epsilon^2)} \right) \leq \exp \left( -\frac{3 r \epsilon^4}{56 M_\varepsilon^4} \right) = |F|^{-2}. \]

Therefore, given the assumption on \( r \), applying the union bound provides us with:
\[
\mathbb{P}\left( \exists \delta_f \in D_F : \left( \frac{1}{r} \sum_{i=1}^{n} \epsilon_i \delta_f (t_i) \right)^2 < \frac{\epsilon}{2} \right) = \mathbb{P}\left( \exists \delta_f \in D_F : \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \delta_f (t_i)^2 < \frac{\epsilon^2 \mu}{2} \right) \\
\leq \sum_{\delta_f \in D_F} \mathbb{P}\left( \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \delta_f (t_i)^2 < \frac{\epsilon^2 \mu}{2} \right) \\
\leq |D_F| \cdot |F|^{-2} \\
< \frac{1}{2}.
\] (29)

Moreover, if \( S_1 \) is the random set \( \{ i \in [1; n] : \epsilon_i = 1 \} \), then by Markov’s inequality,
\[
\mathbb{P}\left( |S_1| > r \right) = \mathbb{P}\left( \sum_{i=1}^{n} \epsilon_i > r \right) \leq \frac{1}{2}.
\] (30)

Combining (29) and (30) by means of the union bound provides us with
\[
\mathbb{P}\left\{ \left( \exists \delta_f \in D_F : \left( \frac{1}{r} \sum_{i=1}^{n} \epsilon_i \delta_f (t_i) \right)^2 < \frac{\epsilon}{2} \right) \text{ or } (|S_1| > r) \right\} < 1
\]
or equivalently
\[
\mathbb{P}\left\{ \left( \forall \delta_f \in D_F : \left( \frac{1}{r} \sum_{i=1}^{n} \epsilon_i \delta_f (t_i) \right)^2 \geq \frac{\epsilon}{2} \right) \text{ and } (|S_1| \leq r) \right\} > 0
\]
which implies that
\[
\mathbb{P}\left\{ \left( \forall \delta_f \in D_F : \| \delta_f \|_{L_2\left( \mu_{(t_i)_{i \in S_1}} \right)} \geq \frac{\epsilon}{2} \right) \text{ and } (|S_1| \leq r) \right\} > 0.
\]

This translates into the fact that there exists a subvector \( t_q \) of \( t_n \) of size \( q \leq r \) such that the class \( F \) is \( \frac{\epsilon}{2} \)-separated with respect to the pseudo-metric \( d_{2,t_q} \), i.e., our claim. \[ \square \]

The following lemma can be seen as a scale-sensitive extension of Theorem 10 in [7].

**Lemma 15** Let \( G \) be a function class satisfying Definition 3 and \( \rho_G \) the function class deduced from \( G \) according to Definition 4. For \( \bar{G} \subset G \), \( s_{2^n} = \{ z_i : 1 \leq i \leq n \} \subset \mathcal{Z} \) and \( \eta \in (0, M_G] \), let us set \( d_G = S-G-dim\left( \rho_G^\eta_{s_{2^n}} \right) \) and \( d_N = S-N-dim\left( \rho_G^\eta_{s_{2^n}} \right) \).

Then,
\[
d_G \leq 12 \log_2^{\alpha(C)} (F(C)) d_N^{\beta(C)},
\] (31)

where \( F(C) = 4(C - 1) \), \( \alpha(C) = 2 + \frac{2}{2 \ln(F(C)) - 1} \) and \( \beta(C) = 1 + \frac{1}{4 \ln(F(C)) - 2} \).
By the pigeonhole principle, the same example is picked for at least $K$ points in $s_{2G}$ if needed, find $K$ such that:

$$s_{2G}^1 \in S(F) \iff \forall \{f, f'\} \subset F, \exists z_i \in s_{2G}^1 : \begin{cases} f(z_i) - b_i \geq 1 \\ \max_{k \neq y_i} f'(x_i, k) + b_i \geq 1 \end{cases} \text{ or vice versa.}$$

The meaning of the formula $z_i \in s_{2G}^1$ is the obvious one, i.e., $\exists z_j^1 \in s_{2G}^1 : z_i = z_j^1$.

Notice first that $s_{2G}^1$ belongs to all the sets $S(F)$. For $F \subset F_0$ satisfying $|F| \geq 2$ and $s_{2G}^1 \in S(F)$, let $h(F, s_{2G}^1)$ be the number of triplets $(s_{2G}^1, b_u^{(2)}, c_u^{(2)})$ satisfying:

$$\begin{cases} s_{2G}^1 = \left\{ z_i^1 = \left( x_i^1, y_i^1 \right) : 1 \leq i \leq u \right\} \subset s_{2G}^1 \\ \forall (i, j) : 1 \leq i < j \leq u, \left( z_i^1, z_j^1 \right) = (z_v, z_w) \implies 1 \leq v < w \leq d_G \\ \forall (i, j) \in [1; u] \times [1; d_G], \ z_i^1 = z_j \implies b_i^{(2)} = b_j \\ \forall i \in [1; u], \ c_i^{(2)} \in \mathcal{Y} \setminus \left\{ y_i^{(2)} \right\} \end{cases}$$

which are strongly $N$-shattered by $F$. The ordering on the cardinalities considered is thus characterized by $u \leq \min \{d_N, r\}$ and $\max \{d_N, r\} \leq d_G$. Furthermore, the bound

$$h(F, s_{2G}^1) \geq 1 \tag{32}$$

is a direct consequence of (the proof of) Lemma 13. Let $F$ and $s_{2G}^1$ be defined as above. Suppose that there exists $K \in \mathbb{N}^*$ such that $|F| = 4K(C-1)r$. Split $F$ arbitrarily into $2K(C-1)r$ pairs $(f_+, f_-)$. For each pair, transposing the names of the functions if needed, find $z_i \in s_{2G}^1$ such that

$$\begin{cases} f_+ (z_i) - b_i \geq 1 \\ \max_{k \neq y_i} f_- (x_i, k) + b_i \geq 1 \end{cases}$$

By the pigeonhole principle, the same example is picked for at least $2K(C-1)$ pairs. Let $z_{i_0}$ be such an example. A new application of the pigeonhole principle implies that there exists a value $c_{i_0} \in \mathcal{Y} \setminus \{y_{i_0}\}$ such that among the aforementioned pairs, $2K$ of them satisfy:

$$\begin{cases} f_+ (z_{i_0}) - b_{i_0} \geq 1 \\ f_- (x_{i_0}, c_{i_0}) + b_{i_0} \geq 1 \end{cases}$$

31
Let $\mathcal{F}_+$ and $\mathcal{F}_-$ be the subsets of $\mathcal{F}$ respectively gathering those functions $f_+$ and $f_-$ ($|\mathcal{F}_+| = |\mathcal{F}_-| = 2K$) and let $s_{2^{r-1}}^{(2)} = s_{2^{r-1}}^{(1)} \setminus \{z_i\}$. By construction, $s_{2^{r-1}}^{(2)} \in S(\mathcal{F}_+) \cap S(\mathcal{F}_-)$. Clearly, $\mathcal{F}$ strongly N-shatters all the triplets $(s_{2^r}^{(3)}, b_u^{(3)}, c_u^{(3)})$ with $s_{2^r}^{(3)} \subset s_{2^{r-1}}^{(2)}$ strongly N-shattered by either $\mathcal{F}_+$ or $\mathcal{F}_-$ plus $(\{z_i\}, b_i, c_i)$. Moreover, if the triplet $(s_{2^r}^{(3)}, b_u^{(3)}, c_u^{(3)})$ is strongly N-shattered by both $\mathcal{F}_+$ and $\mathcal{F}_-$, then $\mathcal{F}$ strongly N-shatters the triplet $(s_{2^{r+1}}^{(4)}, b_{u+1}^{(4)}, c_{u+1}^{(4)})$ deduced from $(s_{2^r}^{(3)}, b_u^{(3)}, c_u^{(3)})$ by inserting the components of $(\{z_i\}, b_i, c_i)$ at the right place. Since by construction, $s_{2^{r+1}}^{(4)} \subset s_{2^r}^{(2)}$ and $s_{2^{r+1}}^{(4)} \not\subset s_{2^{r-1}}^{(2)}$, it follows that:

$$h(\mathcal{F}, s_{2^r}^{(1)}) \geq h(\mathcal{F}_+, s_{2^r}^{(2)}) + h(\mathcal{F}_-, s_{2^r}^{(2)}) + 1.$$ (33)

Let $K \in \mathbb{N}$ be given by

$$2 \left(2 (C - 1)\right)^K \prod_{j=0}^{K-1} (d_G - j) \leq 2^{d_G} < 2 \left(2 (C - 1)\right)^K \prod_{j=0}^{K} (d_G - j).$$

Then,

$$d_G - 1 < (K + 1) \log_2 (2 (C - 1)) d_G.$$ (34)

The upper bound on $K + 1$ results from a combination of (32) and (33):

$$2^{K+1} - 1 \leq h(\mathcal{F}_0, s_{2^d_G}) \leq \sum_{u=1}^{d_N} \binom{d_G}{u} (C - 1)^u \leq \left(\frac{(C - 1) e d_G}{d_N}\right)^{d_N} - 1,$$

so that

$$K + 1 \leq d_N \log_2 \left(\frac{(C - 1) e d_G}{d_N}\right).$$ (35)

By substitution of (35) into (34),

$$d_G - 1 < d_N \log_2 \left(\frac{(C - 1) e d_G}{d_N}\right) \log_2 (2 (C - 1)) d_G,$$

and consequently

$$d_G \ln^2 (2) \leq d_N \ln \left((C - 1) e \frac{d_G}{d_N}\right) \ln (4 (C - 1) d_G) \leq d_N \ln \left(F(C) \frac{d_G}{d_N}\right) \ln (F(C) d_G).$$ (36)

To bound from above the right-hand side of Inequality (36), we resort to the following statement:

$$\forall (u, u_0) \in [1, +\infty)^2, \ \ln (u) \leq 2 u_0 u^\frac{1}{u_0},$$ (37)

32
with \( u_0 = \ln(F(C)) \). We then obtain
\[
\begin{align*}
\ln \left( F(C) \frac{d_G}{d_N} \right) & \leq 2e^{\frac{1}{2}} \ln(F(C)) \left( \frac{d_G}{d_N} \right)^{\frac{1}{2}\ln(F(C))} \\
\ln(F(C)) d_G & \leq 2e^{\frac{1}{2}} \ln(F(C)) \left( \frac{d_G}{d_N} \right)^{\frac{1}{2}\ln(F(C))}
\end{align*}
\]
By substitution into (36), it gives:
\[
d_G \leq 4\sqrt{e} \log_2^2(F(C)) \left( \frac{d_G}{d_N} \right)^{\frac{1}{2}\ln(F(C))} d_N^{1 - \frac{1}{2}\ln(F(C))} \\
= (4\sqrt{e})^{\frac{2\ln(F(C))}{\ln(F(C)) - 1}} \log_2(F(C))^{\frac{4\ln(F(C))}{\ln(F(C)) - 1}} d_N^{\frac{4\ln(F(C)) - 1}{\ln(F(C)) - 2}} \\
\leq 12 \log_2(F(C))^{\frac{4\ln(F(C))}{\ln(F(C)) - 1}} d_N^{\frac{4\ln(F(C)) - 1}{\ln(F(C)) - 2}}.
\]

The proof of Lemma 7 is the following one.

**Proof** Since (10) and (11) trivially hold true for \( M_2(\epsilon, \rho G_\gamma, n) < 2 \), we establish the proof under the opposite hypothesis. Let us consider any vector \( z_n = (z_i)_{1 \leq i \leq n} \in \mathcal{Z}^n \) satisfying \( M(\epsilon, \rho G_\gamma, d_{2z_n}) \geq 2 \) and whose components are all different, so that \( s_{2z_n} = \{z_i: 1 \leq i \leq n\} \) is a subset of \( \mathcal{Z} \) (of cardinality \( n \)). By definition, \( M(\epsilon, \rho G_\gamma, d_{2z_n}) = M(\epsilon, \rho G_\gamma|_{s_{2z_n}}, d_{2z_n}) \), where \( \rho G_\gamma|_{s_{2z_n}} \) is the set of the restrictions to \( s_{2z_n} \) of the functions in \( \rho G_\gamma \). Let \( \tilde{G} \) be a subset of \( G \) of cardinality \( M(\epsilon, \rho G_\gamma, d_{2z_n}) \) such that \( \rho G_\gamma|_{s_{2z_n}} \) is \( \epsilon \)-separated with respect to \( d_{2z_n} \) and in bijection with \( \tilde{G} \). By definition,
\[
|\rho G_\gamma|_{s_{2z_n}} = |\tilde{G}| = M(\epsilon, \rho G_\gamma, d_{2z_n}). \tag{38}
\]
By application of Lemma 14 with \( F = \rho G_\gamma|_{s_{2z_n}} \), corresponding to \( K_\epsilon = \frac{3}{12\gamma} \), there exists a subvector \( z_q \) of \( z_n \) of size
\[
q \leq \frac{\ln(|\tilde{G}|)}{K_\epsilon \epsilon^4} \tag{39}
\]
such that \( \rho G_\gamma|_{s_{2z_q}} \) is \( \frac{1}{2} \)-separated with respect to the pseudo-metric \( d_{2z_q} \) and its cardinality is once more that of \( \tilde{G} \). Applying Lemma 12 with \( F = \rho G_\gamma|_{s_{2z_q}} \), \( N = 6 \) and the corresponding largest possible value for \( \eta_i \), \( \frac{1}{24} \), it appears that the set \( (\rho G_\gamma|_{s_{2z_q}}) \) is 6-separated with respect to the pseudo-metric \( d_{2z_q} \). Consequently, Proposition 4 applies. Taking into account (38), it gives:
\[
|\tilde{G}| \leq \left( \frac{24\gamma \epsilon q}{\epsilon \cdot S-G-dim(F)} \right)^{2S-G-dim(F)}, \tag{40}
\]
33
\[ \tilde{F} = \left( \rho \tilde{g} \big|_{sZq} \right)^{(\frac{\gamma}{\epsilon})} \]. A substitution of the upper bound on \( q \) provided by (39) into (40) gives:

\[
\left| \tilde{G} \right| \leq \left( K_1 \left( \frac{\gamma}{\epsilon} \right)^5 \frac{\ln \left( \left| \tilde{G} \right| \right)}{\text{S-G-dim} \left( \tilde{F} \right)} \right)^{2 \cdot \text{S-G-dim} \left( \tilde{F} \right)}
\]

with \( K_1 = 896e \). In order to upper bound \( \ln \left( \left| \tilde{G} \right| \right) \), we resort once more to (37), this time with \( u_0 = 1 \). Thus,

\[
\left| \tilde{G} \right| \leq \ln^2 \left( \left| \tilde{G} \right| \right) \left( K_1 \left( \frac{\gamma}{\epsilon} \right)^5 \right)^2
\]

and \( \left| \tilde{G} \right| = \mathcal{M} (\epsilon, \rho \tilde{G}, d_{2,z_n}) \) imply that

\[
\mathcal{M} (\epsilon, \rho \tilde{G}, d_{2,z_n}) \leq \left( K_2 \left( \frac{\gamma}{\epsilon} \right)^5 \right)^{4 \cdot \text{S-G-dim} \left( \tilde{F} \right)}
\]

with \( K_2 = 2K_1 \). Due to the construction of \( \tilde{F} \), which makes it possible to apply Formula (23a),

\[
\text{S-G-dim} \left( \tilde{F} \right) = \text{S-G-dim} \left( \left( \rho \tilde{g} \big|_{sZq} \right)^{(\frac{\gamma}{\epsilon})} \right)
\]

\[
\leq \left( \frac{\epsilon}{48} \right) \cdot \text{G-dim} \left( \rho \tilde{g} \big|_{sZq} \right)
\]

\[
\leq \left( \frac{\epsilon}{48} \right) \cdot \text{G-dim} \left( \rho \tilde{g} \right)
\]

\[
\leq \left( \frac{\epsilon}{48} \right) \cdot \text{G-dim} \left( \rho \tilde{g} \right).
\]

By substitution of (42) into (41), we obtain that for every vector \( z_n \in Z^n \) whose components are all different,

\[
\mathcal{M} (\epsilon, \rho \tilde{G}, d_{2,z_n}) \leq \left( K_2 \left( \frac{\gamma}{\epsilon} \right)^5 \right)^{4d_G \left( \frac{\gamma}{\epsilon} \right)}.
\]

Obviously, Inequality (43) still holds true if the cardinality \( n' \) of the smallest subset \( s_{Z''} \) of \( Z \) containing all the components of \( z_n \) is (strictly) inferior to \( n \). Then, the proof is basically the same, with \( s_{Z''} \) replaced with \( s_{Z'n'} \), and \( z_n \) replaced with the corresponding vector \( z_{n'} \).

At last, (43) implies (10) since its right-hand side does not depend on \( z_n \). The proof of (11) follows that of (10) up to Formula (41). Then, applying in sequence Lemma 15 and
Formula (23b) provides us with

\[
\text{S-G-dim} \left( \tilde{\mathcal{F}} \right) = \text{S-G-dim} \left( \left( \rho_{\mathcal{G}|sz} \right)^{\beta(C)} \right)
\]
\[
\leq 12 \log_2^2 (F(C)) \text{S-N-dim} \left( \left( \rho_{\mathcal{G}|sz} \right)^{\beta(C)} \right)
\]
\[
\leq 12 \log_2^2 (F(C)) \left( \frac{\epsilon}{48} \right)^{-N-dim} \left( \rho_{\mathcal{G}|sz} \right)^{\beta(C)}. \tag{44}
\]

After a substitution of (44) into (41), the end of the proof of (11) is the same as that of (10).

\[ \blacksquare \]

C Proofs of the structural results

This appendix gathers the proofs of the upper bounds on margin Graph dimensions and margin Natarajan dimensions.

C.1 Margin Graph dimension

The proof of Lemma 6 makes use of Proposition 1.4 in [32], Lemma 3 and Lemma 5. Lemma 8 is an improvement of Lemma 6 obtained by optimizing the use of Lemma 3, i.e., choosing the value of \( p \) as a function of that of \( C \). This calls for extensions of Proposition 1.4 in [32] and Lemma 5 holding for the \( L_p \)-norms with \( p \in \mathbb{N} \setminus \{0, 1\} \) (instead of simply \( p = 2 \)). Those extensions are respectively Proposition 5 and Lemma 16.

**Proposition 5** Let \( \mathcal{F} \) be a class of real-valued functions on \( \mathcal{T} \). For every \( \gamma \in \mathbb{R}^+ \) satisfying \( \gamma-dim(\mathcal{F}) > 0 \), \( n \in [1; \gamma-dim(\mathcal{F})] \) and \( p \in \mathbb{N} \setminus \{0, 1\} \),

\[ n \leq K_p \log_2 (\mathcal{M}_p (\gamma; \mathcal{F}, n)) \]

with \( K_p = \left( \frac{2^p}{2^p - 1} \right)^2 \).

**Proof** Suppose that for \( \gamma \in \mathbb{R}^+ \), the subset \( s_{\mathcal{T}^n} = \{ t_i : 1 \leq i \leq n \} \) of \( \mathcal{T} \) is \( \gamma \)-shattered by \( \mathcal{F} \) and \( \mathbf{b}_n = (b_i)_{1 \leq i \leq n} \in \mathbb{R}^n \) is a witness to this shattering. By definition, there exists a subset \( \tilde{\mathcal{F}} = \{ f_{s_n} : s_n \in \{-1, 1\}^n \} \) of \( \mathcal{F} \) satisfying

\[
\forall i \in [1; n], \quad s_i (f_{s_n} (t_i) - b_i) \geq \gamma. \tag{45}
\]
Let $t_n = (t_i)_{1 \leq i \leq n}$. To prove the proposition, it suffices to establish that
\[ n \leq K_p \log_2 \left( M(\gamma, \vec{F}, d_{p,t_n}) \right). \tag{46} \]

For $(s_n, s'_n) \in (\{-1, 1\}^n)^2$, let $S(s_n, s'_n)$ be the subset of $[1; n]$ defined by:
\[ S(s_n, s'_n) = \{ i \in [1; n] : s_i \neq s'_i \}. \]

Then, making use of (45), we obtain that
\[
d_{p,t_n}(f_{s_n}, f_{s'_n}) \geq \left( \frac{1}{n} \sum_{i \in S(s_n, s'_n)} (2\gamma)^p \right)^{\frac{1}{p}} = 2\gamma \left( \frac{d_H(s_n, s'_n)}{n} \right)^{\frac{1}{p}},
\]
where $d_H$ stands for the Hamming distance. Thus, a sufficient condition for $d_{p,t_n}(f_{s_n}, f_{s'_n}) \geq \gamma$ is $d_H(s_n, s'_n) \geq \left( \frac{1}{2} \right)^p n$. As a consequence, to prove (46), it suffices to establish that there is a subset of the set of vertices of the hypercube $Q_n$ of cardinality $\left\lceil \frac{n}{2^p} \right\rceil$ which is $\left( \frac{1}{2} \right)^p n$-separated with respect to the Hamming distance (the separation is well-defined since $\left\lceil \frac{n}{2^p} \right\rceil \geq 2$). To do so, a probabilistic approach similar to that of the proof of Lemma 14 is implemented. Let $\epsilon_{q,n} = (\epsilon_{j,i})_{1 \leq j \leq q, 1 \leq i \leq n}$ be a Bernoulli random matrix (its entries $\epsilon_{j,i}$ are independent Bernoulli random variables with common expectation $\frac{1}{2}$).

Then, by application of the union bound,
\[
P \left( \exists (j, j') \in [1; q]^2 : 1 \leq j < j' \leq q \text{ and } \sum_{i=1}^{n} \mathbb{1}_{\{\epsilon_{j,i} \neq \epsilon_{j',i}\}} < n \left( 1 - \left( \frac{1}{2} \right)^p \right) \right) \leq \frac{q}{2} P \left( \sum_{i=1}^{n} \epsilon_i > n \left( 1 - \left( \frac{1}{2} \right)^p \right) \right),
\]
where $(\epsilon_i)_{1 \leq i \leq n}$ is a Bernoulli random vector. To upper bound the tail probability on the right-hand side, we resort to Hoeffding’s inequality, which gives
\[
P \left( \sum_{i=1}^{n} \epsilon_i - \frac{n}{2} > \frac{n}{2} \left( 1 - \left( \frac{1}{2} \right)^{p-1} \right) \right) \leq \exp \left( -\frac{n}{2} \left( 1 - \left( \frac{1}{2} \right)^{p-1} \right)^2 \right).
\]

By transitivity, this implies that a sufficient condition for
\[
P \left( \exists (j, j') \in [1; q]^2 : 1 \leq j < j' \leq q \text{ and } \sum_{i=1}^{n} \mathbb{1}_{\{\epsilon_{j,i} \neq \epsilon_{j',i}\}} < n \left( 1 - \left( \frac{1}{2} \right)^p \right) \right) < 1
\]
is
\[
\frac{q}{2} \exp \left( -\frac{n}{2} \left( 1 - \left( \frac{1}{2} \right)^{p-1} \right)^2 \right) < 1
\]

36
and thus
\[ q \leq \left\lceil \frac{2}{2nKp} \right\rceil, \]
which is precisely the value announced and thus concludes the proof.

Theorem 10 in [27] is the extension of Lemma 5 to the \( L_p \)-norms of interest. The following lemma specifies the value of its constants (absolute or depending on \( p \)).

**Lemma 16 (After Theorem 2 in [30])** Let \( F \) be a class of functions from \( T \) into \([-M_F, M_F]\) with \( M_F \in \mathbb{R}^*_+ \). \( F \) is supposed to be a uGC class. For \( \epsilon \in (0, M_F] \), let \( d(\epsilon) = \epsilon \cdot \dim(F) \).

Then for \( \epsilon \in (0, 2M_F] \), \( n \in \mathbb{N}^* \) and \( p \in \mathbb{N} \setminus \{0, 1\} \),
\[ M_p(\epsilon, F, n) \leq \left( \frac{12M_Fp}{\epsilon} \right)^{10p(\frac{o}{n^{0p}})} . \]

With Proposition 5 and Lemma 16 at hand, Lemma 8 can be established as follows.

**Proof** Applying in sequence Proposition 1, Proposition 5, Lemma 4 (left-hand side inequality), Lemma 3, Lemma 4 (right-hand side inequality) and Lemma 16 gives:

\[ \forall \gamma \in (0, M_G], \; \gamma \cdot \text{G-dim} (\rho_G) \leq \left\lceil \log_2 (\frac{C}{\gamma}) \right\rceil \leq 10Kp \ln(2) \ln \left( \frac{24M_Gp^2 C_{\gamma}^2}{\gamma} \right) \sum_{k=1}^{C} \left( \frac{\gamma}{72pC_{\gamma}} \right)^{-\dim(G_k)} . \]

Let us set \( p = \lceil \log_2 (C) \rceil \) (which is possible since \( C \geq 3 \) implies \( p \geq 2 \)). Then, \( C^{\frac{1}{p}} \leq 2 \), so that for every \( \gamma \in (0, M_G] \),
\[ \gamma \cdot \text{G-dim} (\rho_G) \leq \frac{10K_{\lceil \log_2 (C) \rceil} \log_2 (2C)}{\ln(2)} \ln \left( \frac{48M_G \log_2^2 (2C)}{\gamma} \right) \sum_{k=1}^{C} \left( \frac{\gamma}{72pC_{\gamma}} \right)^{-\dim(G_k)} . \]

To finish the proof, it suffices to notice that
\[ \forall C \in \mathbb{N} \setminus \{0; 2\}, \; K_{\lceil \log_2 (C) \rceil} \leq \min \left\{ 4 \left( \frac{C}{C - 2} \right)^2 , 16 \right\} . \]
C.2 Margin Natarajan dimension of $\rho_G$

The proof of Lemma 9 is based on a restricted definition of the margin Natarajan dimension whose use is justified by the following lemma.

Lemma 17 The definition of the Natarajan dimension with margin $\gamma$ is not affected by the introduction of the restriction: $\forall i \in [1; n], y_i < c_i$.

Proof For a given $\gamma \in \mathbb{R}_+^*$, let $s_{\mathcal{Z}} = \{(x_i, y_i) : 1 \leq i \leq n\}$ be a subset of $\mathcal{Z}$ $\gamma$-$\mathcal{N}$-shattered by $\mathcal{F}$ and let $(b_n, c_n)$ be a witness to this shattering. By definition, for every vector $s_n = (s_i)_{1 \leq i \leq n} \in \{-1, 1\}^n$, there is a function $f_{s_n} \in \mathcal{F}$ satisfying

$$\forall i \in [1; n], \begin{cases} 
    \text{if } s_i = 1, \ f_{s_n} (x_i, y_i) - b_i \geq \gamma \\
    \text{if } s_i = -1, \ f_{s_n} (x_i, c_i) + b_i \geq \gamma 
\end{cases}.$$ 

Let us define the triplet $(s'_{\mathcal{Z}}n, b'_n, c'_n)$ in the following way:

$$\forall i \in [1; n], \begin{cases} 
    \text{if } y_i < c_i, \ (z'_i, b'_i, c'_i) = (z_i, b_i, c_i) \\
    \text{else } (z'_i, b'_i, c'_i) = ((x_i, c_i), -b_i, y_i) 
\end{cases}.$$ 

Then for the points satisfying $y_i < c_i$,

$$\begin{cases} 
    \text{if } s'_i = 1, \ f_{s_n} (x'_i, y'_i) - b'_i \geq \gamma \\
    \text{if } s'_i = -1, \ f_{s_n} (x'_i, c'_i) + b'_i \geq \gamma 
\end{cases}.$$ 

whereas for the points satisfying $c_i < y_i$,

$$\begin{cases} 
    \text{if } s'_i = 1, \ f_{s_n} (x'_i, c'_i) + b'_i \geq \gamma \\
    \text{if } s'_i = -1, \ f_{s_n} (x'_i, y'_i) - b'_i \geq \gamma 
\end{cases}.$$ 

To sum up, for the vector $s'_n = (s'_i)_{1 \leq i \leq n} \in \{-1, 1\}^n$ such that for every $i \in [1; n], s'_i = s_i$ if $y_i < c_i$, and $s'_i = -s_i$ otherwise, we obtain

$$\forall i \in [1; n], \begin{cases} 
    \text{if } s'_i = 1, \ f_{s_n} (x'_i, y'_i) - b'_i \geq \gamma \\
    \text{if } s'_i = -1, \ f_{s_n} (x'_i, c'_i) + b'_i \geq \gamma 
\end{cases}.$$
Consequently, the function \( f_{s_n} \) contributes to the \( \gamma \)-N-shattering of \( s_{Z^n} \) with \( (b'_n, c'_n) \) as witness, for the vector \( s'_n \). Thus, for symmetry reasons, the set \( \{ f_{s_n} : s_n \in \{-1, 1\}^n \} \) \( \gamma \)-N-shatters the set \( s'_{Z^n} \) and \( (b'_n, c'_n) \) is a witness to this shattering.

With Lemma \([17]\) at hand, the proof of Lemma \([9]\) is straightforward.

**Proof** Suppose that for \( \gamma \in (0, M_G] \), the subset \( s_{Z^n} = \{ z_i : 1 \leq i \leq n \} \) of \( Z \) is \( \gamma \)-N-shattered by \( \rho_G \). By definition, there exists a subset \( G = \{ g^n : s_n \in \{-1, 1\}^n \} \) of \( G \) and a pair \( (b_n, c_n) \in \mathbb{R}^n \times \gamma^m \) such that \( s_{Z^n} \) is \( \gamma \)-N-shattered by \( \rho_G = \{ \rho_{g^n} : s_n \in \{-1, 1\}^n \} \) and \( (b_n, c_n) \) is a witness to this shattering. Furthermore, according to Lemma \([17]\) without loss of generality, we can make the assumption that for every \( i \in [1; n] \), \( y_i < c_i \). For every pair \( (k, l) \in [1; C]^2 \) satisfying \( k < l \), let \( S_{k,l} \) be the subset of \( [1; n] \) defined as follows:

\[
S_{k,l} = \{ i \in [1; n] : y_i = k \text{ and } c_i = l \}
\]

and let \( n_{k,l} \in [0; n] \) be its cardinality. By construction, \( P = \{ S_{k,l} : n_{k,l} > 0 \} \) is a partition of \( [1; n] \). For every vector \( s_n = (s_i)_{1 \leq i \leq n} \in \{-1, 1\}^n \), the function \( g^n \) satisfies:

\[
\forall i \in [1; n], \quad \begin{cases} 
\text{ if } s_i = 1, & \rho_{g^n} (x_i, y_i) - b_i \geq \gamma \\
\text{ if } s_i = -1, & \rho_{g^n} (x_i, c_i) + b_i \geq \gamma
\end{cases}
\]

For a fixed \( S_{k,l} \in P \), this implies that

\[
\forall i \in S_{k,l}, \quad \begin{cases} 
\text{ if } s_i = 1, & \frac{1}{2} (g^n_k (x_i) - g^n_l (x_i)) - b_i \geq \gamma \\
\text{ if } s_i = -1, & \frac{1}{2} (g^n_l (x_i) - g^n_k (x_i)) + b_i \geq \gamma
\end{cases}
\]

To sum up,

\[
\forall i \in S_{k,l}, \quad s_i \left( \frac{1}{2} (g^n_k (x_i) - g^n_l (x_i)) - b_i \right) \geq \gamma.
\]

Thus, the class \( \left\{ \frac{1}{2} (g^n_k - g^n_l) : s_n \in \{-1, 1\}^n \right\} \subset \text{absconv} (G_k \cup G_l) \) \( \gamma \)-shatters a set of cardinality \( n_{k,l} \), with the consequence that

\[
n_{k,l} \leq \gamma \cdot \text{dim} \left( \text{absconv} (G_k \cup G_l) \right).
\]

Then, the conclusion springs from summing over all the elements of the partition \( P \).

**C.3 Margin Natarajan dimension of \( \rho_{H_p} \)**

The proof of Lemma \([10]\) makes use of that of Theorem 4.6 in \([6]\).
Proof This proof reuses the notations of the proof of Lemma\textsuperscript{9} with \( \mathcal{G} \) being instantiated by \( \mathcal{H}_\Lambda \). By application of Lemma 4.3 in \textsuperscript{[6]}, there exists a vector \( s'_n = (s'_i)_{1 \leq i \leq n} \in \{-1, 1\}^n \) satisfying

\[
\forall S_{k,l} \in \mathcal{P}, \quad \left\| \sum_{i \in S_{k,l}^+} \kappa_{x_i} - \sum_{i \in S_{k,l}^-} \kappa_{x_i} \right\|_{H_n} \leq \sqrt{n_{k,l} \Lambda_x}, \tag{47}
\]

where the sets \( S_{k,l}^+ \) and \( S_{k,l}^- \) are defined as follows:

\[
\forall S_{k,l} \in \mathcal{P}, \quad \begin{cases} 
S_{k,l}^+ = \{ i \in S_{k,l} : s'_i = 1 \} \\
S_{k,l}^- = S_{k,l} \setminus S_{k,l}^+
\end{cases}
\]

For every vector \( s_n = (s_i)_{1 \leq i \leq n} \in \{-1, 1\}^n \), the function \( h^{s_n} \) satisfies:

\[
\forall i \in \llbracket 1; n \rrbracket, \quad \begin{cases} 
\text{if } s_i = 1, \quad \rho_{h^{s_n}}(x_i, y_i) - b_i \geq \gamma \\
\text{if } s_i = -1, \quad \rho_{h^{s_n}}(x_i, c_i) + b_i \geq \gamma
\end{cases}
\]

For a fixed \( S_{k,l} \in \mathcal{P} \),

\[
\forall i \in S_{k,l}, \quad \begin{cases} 
\text{if } s_i = 1, \quad \frac{1}{2} \langle h_{k}^{s_n} - h_{l}^{s_n}, \kappa_{x_i} \rangle_{H_n} - b_i \geq \gamma \\
\text{if } s_i = -1, \quad \frac{1}{2} \langle h_{l}^{s_n} - h_{k}^{s_n}, \kappa_{x_i} \rangle_{H_n} + b_i \geq \gamma
\end{cases}
\]

implying by application of the reproducing property that

\[
\forall i \in S_{k,l}, \quad \begin{cases} 
\text{if } s_i = 1, \quad \frac{1}{2} \langle h_{k}^{s_n} - h_{l}^{s_n}, \kappa_{x_i} \rangle_{H_n} - b_i \geq \gamma \\
\text{if } s_i = -1, \quad \frac{1}{2} \langle h_{l}^{s_n} - h_{k}^{s_n}, \kappa_{x_i} \rangle_{H_n} + b_i \geq \gamma
\end{cases}
\]

Let us specify the vector \( s_n \) in the following way: \( \forall i \in S_{k,l}, s_i = s'_i \). By summation over \( i \in S_{k,l} \), it results from (48) that:

\[
\frac{1}{2} \left\langle h_{k}^{s_n} - h_{l}^{s_n}, \sum_{i \in S_{k,l}^+} \kappa_{x_i} \right\rangle_{H_n} - \sum_{i \in S_{k,l}^+} b_i + \frac{1}{2} \left\langle h_{l}^{s_n} - h_{k}^{s_n}, \sum_{i \in S_{k,l}^-} \kappa_{x_i} \right\rangle_{H_n} + \sum_{i \in S_{k,l}^-} b_i \geq n_{k,l} \gamma,
\]

which simplifies into

\[
\frac{1}{2} \left\langle h_{k}^{s_n} - h_{l}^{s_n}, \sum_{i \in S_{k,l}^+} \kappa_{x_i} - \sum_{i \in S_{k,l}^-} \kappa_{x_i} \right\rangle_{H_n} - \sum_{i \in S_{k,l}^+} b_i + \sum_{i \in S_{k,l}^-} b_i \geq n_{k,l} \gamma.
\]

Conversely, consider any vector \( s_n \) such that: \( \forall i \in S_{k,l}, s_i = -s'_i \). Then,

\[
\frac{1}{2} \left\langle h_{l}^{s_n} - h_{k}^{s_n}, \sum_{i \in S_{k,l}^+} \kappa_{x_i} - \sum_{i \in S_{k,l}^-} \kappa_{x_i} \right\rangle_{H_n} + \sum_{i \in S_{k,l}^+} b_i - \sum_{i \in S_{k,l}^-} b_i \geq n_{k,l} \gamma.
\]
As a consequence, if \( \sum_{i \in S_{k,l}^+} b_i - \sum_{i \in S_{k,l}^-} b_i \geq 0 \), there are functions \( h^{s_n} \) such that

\[
\frac{1}{2} \left( h^{s_n}_k - h^{s_n}_l, \sum_{i \in S_{k,l}^+} \kappa x_i - \sum_{i \in S_{k,l}^-} \kappa x_i \right) \geq n_{k,l} \gamma,
\]

whereas if \( \sum_{i \in S_{k,l}^+} b_i - \sum_{i \in S_{k,l}^-} b_i < 0 \), there are (different) functions \( h^{s_n} \) such that

\[
\frac{1}{2} \left( h^{s_n}_l - h^{s_n}_k, \sum_{i \in S_{k,l}^+} \kappa x_i - \sum_{i \in S_{k,l}^-} \kappa x_i \right) \geq n_{k,l} \gamma.
\]

Applying the Cauchy-Schwarz inequality to both (49) and (50) yields

\[
\frac{1}{2} \left\| h^{s_n}_k - h^{s_n}_l \right\|_{H_n} \left\| \sum_{i \in S_{k,l}^+} \kappa x_i - \sum_{i \in S_{k,l}^-} \kappa x_i \right\|_{H_n} \geq n_{k,l} \gamma.
\]

Consequently, whatever the sign of \( \sum_{i \in S_{k,l}^+} b_i - \sum_{i \in S_{k,l}^-} b_i \), there are functions \( h^{s_n} \) specified only by the components of \( s_n \) whose indices belong to \( S_{k,l} \) such that (51) holds true. To sum up, we have exhibited an algorithm taking in input \( s_{x^n}, (b_n, c_n) \) and \( s'_n \), and returning a vector \( s_n \in \{-1, 1\}^n \) such that the function \( h^{s_n} \in \bar{H}_\Lambda \) satisfies:

\[
\forall S_{k,l} \in \mathcal{P}, \quad \frac{1}{2} \left\| h^{s_n}_k - h^{s_n}_l \right\|_{H_n} \left\| \sum_{i \in S_{k,l}^+} \kappa x_i - \sum_{i \in S_{k,l}^-} \kappa x_i \right\|_{H_n} \geq n_{k,l} \gamma.
\]

By substitution of (47) into (52), this function also satisfies:

\[
\forall S_{k,l} \in \mathcal{P}, \quad n_{k,l} \leq \left( \frac{1}{2} \left\| h^{s_n}_k - h^{s_n}_l \right\|_{H_n} \Lambda x \right)^2 \frac{\Lambda x}{\gamma}.
\]

By summation over all the elements of the partition \( \mathcal{P} \),

\[
n \leq \left( \frac{\Lambda x}{2 \gamma} \right)^2 \sum_{k<l} \left\| h^{s_n}_k - h^{s_n}_l \right\|_{H_n}^2 \geq \sum_{k<l} \left\| h^{s_n}_k - h^{s_n}_l \right\|_{H_n}^2.
\]

Now, since by hypothesis, \( \sum_{k=1}^C h_k = 0 \)

\[
\sum_{k<l} \left\| h^{s_n}_k - h^{s_n}_l \right\|_{H_n}^2 = C \sum_{k=1}^C \left\| h^{s_n}_k \right\|_{H_n}^2 = C \left\| h^{s_n} \right\|_{H_{n,C}}^2 \leq C \Lambda^2.
\]

A substitution of (54) into (53) then concludes the proof. \( \blacksquare \)
D Proof of the guaranteed risk

The proof of Theorem 3 is the following one.

Proof

First case: \( \beta(C) \, d_{\mathcal{G}, \gamma} \in (0, 2) \)

This case is the only one for which the entropy integral exists. It is derived from (2) rather than from (3), to keep the choice of function \( h \) as a degree of freedom. Setting for every \( j \in \mathbb{N}, h(j) = \gamma 2^{-\beta(C) \, d_{\mathcal{G}, \gamma} \cdot j} \), we obtain

\[
R_m(\rho_{\mathcal{G}, \gamma}) \leq 80 \left( 1 + 2 \cdot 2^{-\beta(C) \, d_{\mathcal{G}, \gamma}} \right) \gamma \left( 1 - \frac{\beta(C) \, d_{\mathcal{G}, \gamma}}{2} \right) \sqrt{\ln \left( \frac{6}{\epsilon^2 - \beta(C) \, d_{\mathcal{G}, \gamma}} \right) / \epsilon} \int_0^1 \sqrt{\ln \left( \frac{6}{\epsilon^2 - \beta(C) \, d_{\mathcal{G}, \gamma}} \right) / \epsilon} \, d\epsilon.
\]  
(55)

Let us define the integral \( I(C) \) as follows:

\[
I(C) = \int_0^1 \sqrt{\ln \left( \frac{F_2(C)}{2 \epsilon} \right) / \epsilon} \, d\epsilon.
\]

The computation of the integral gives

\[
I(C) = \frac{1}{2} \left[ \sqrt{\ln(F_2(C))} + F_2(C) \frac{\sqrt{\pi}}{2} \text{erfc} \left( \frac{\sqrt{\ln(F_2(C))}}{2} \right) \right],
\]  
(56)

where \( \text{erfc} \) is the complementary error function. If \( T \) is a random variable following a standard normal distribution, then

\[
\mathbb{P}(T \geq t) \leq \frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2 / 2}.
\]

A substitution of this classical tail bound in (56) provides us with:

\[
I(C) \leq \frac{1}{2} \left[ \sqrt{\ln(F_2(C))} + \sqrt{\frac{1}{4 \ln(F_2(C))}} \right].
\]  
(57)

A substitution of (57) into (55) concludes the proof of (22).

Second case: \( \beta(C) \, d_{\mathcal{G}, \gamma} = 2 \)

\[
R_m(\rho_{\mathcal{G}, \gamma}) \leq h(N) + 40 \sqrt{\frac{F_1(C)}{m}} \sum_{j \in \mathcal{J}} \frac{h(j) + h(j - 1)}{h(j)} \sqrt{\ln \left( \frac{6 \gamma}{h(j)} \right)}.
\]
For $N = \lceil \frac{1}{2} \log_2 (m) \rceil$, we set $h(j) = \gamma m^{-\frac{1}{2}} 2^{-j-N}$. Then,

$$R_m(\rho_G, \gamma) \leq \frac{\gamma}{\sqrt{m}} + 120 \sqrt{F_1(C)} \sum_{j=1}^{N} \sqrt{\ln (6\sqrt{m} \cdot 2^{j-N})}$$

$$\leq \frac{\gamma}{\sqrt{m}} + 120 \sqrt{\frac{F_1(C)}{m}} \left[ \frac{1}{2} \log_2 (m) \right] \sqrt{\ln (6\sqrt{m})}.$$

**Third case: $\beta(C) d_{\rho_G, \gamma} > 2$**

For $N = \lceil \frac{\beta(C) d_{\rho_G, \gamma} - 2}{2 \pi(C) d_{\rho_G, \gamma} \log_2 (m)} \rceil$, let us set $h(j) = \gamma \left( \frac{\log_2 (m)}{m} \right) \frac{1}{\pi(C) d_{\rho_G, \gamma} 2^{\frac{\beta(C) d_{\rho_G, \gamma}}{2}}} (-j+N)$. We then get

$$R_m(\rho_G, \gamma) \leq \gamma \left( \frac{\log_2 (m)}{m} \right) \frac{1}{\pi(C) d_{\rho_G, \gamma}} \left[ 1 + 40 \left( 1 + 2 \frac{\pi(C) d_{\rho_G, \gamma}}{2} \right) \left( \frac{1}{\gamma} \right)^{\frac{\beta(C) d_{\rho_G, \gamma}}{2}} \sqrt{\frac{F_1(C)}{\log_2 (m) S_N}} \right]$$

with

$$S_N = \sum_{j=1}^{N} 2^{j-N} \sqrt{\ln \left( \frac{6\gamma}{h(j)} \right)}$$

$$\leq \sqrt{\ln \left( \frac{6\gamma}{h(N)} \right)} \sum_{j=1}^{N} 2^{j-N}$$

$$< 2 \sqrt{\ln \left( 6 \left( \frac{m}{\log_2 (m)} \right)^{\frac{1}{\pi(C) d_{\rho_G, \gamma}}} \right)}.$$
[5] P.L. Bartlett and S. Mendelson. Rademacher and Gaussian complexities: Risk bounds and structural results. *Journal of Machine Learning Research*, 3:463–482, 2002.

[6] P.L. Bartlett and J. Shawe-Taylor. Generalization performance of support vector machines and other pattern classifiers. In B. Schölkopf, C.J.C. Burges, and A. Smola, editors, *Advances in Kernel Methods - Support Vector Learning*, chapter 4, pages 43–54. The MIT Press, Cambridge, MA, 1999.

[7] S. Ben-David, N. Cesa-Bianchi, D. Haussler, and P.M. Long. Characterizations of learnability for classes of \( \{0, \ldots, n\} \)-valued functions. *Journal of Computer and System Sciences*, 50(1):74–86, 1995.

[8] A. Berlinet and C. Thomas-Agnan. *Reproducing Kernel Hilbert Spaces in Probability and Statistics*. Kluwer Academic Publishers, Boston, 2004.

[9] L. Devroye, L. Györfi, and G. Lugosi. *A Probabilistic Theory of Pattern Recognition*. Springer-Verlag, New York, 1996.

[10] Ü. Doğan, T. Glasmachers, and C. Igel. A unified view on multi-class support vector classification. *Journal of Machine Learning Research*, 17(45):1–32, 2016.

[11] H.H. Duan. Bounding the fat shattering dimension of a composition function class built using a continuous logic connective. *The Waterloo Mathematics Review*, 2(1):1–21, 2012.

[12] R.M. Dudley. The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *Journal of Functional Analysis*, 1(3):290–330, 1967.

[13] R.M. Dudley, E. Giné, and J. Zinn. Uniform and universal Glivenko-Cantelli classes. *Journal of Theoretical Probability*, 4(3):485–510, 1991.

[14] Y. Guermeur. VC theory of large margin multi-category classifiers. *Journal of Machine Learning Research*, 8:2551–2594, 2007.

[15] Y. Guermeur. A generic model of multi-class support vector machine. *International Journal of Intelligent Information and Database Systems*, 6(6):555–577, 2012.

[16] Y. Guermeur. \( L_p \)-norm Sauer-Shelah lemma for margin multi-category classifiers. *Journal of Computer and System Sciences*, 89:450–473, 2017.
[17] Y. Guermeur. Rademacher complexity of margin multi-category classifiers. *Neural Computing and Applications*, 2018. (accepted).

[18] M.J. Kearns and R.E. Schapire. Efficient distribution-free learning of probabilistic concepts. *Journal of Computer and System Sciences*, 48(3):464–497, 1994.

[19] M.J. Kearns, R.E. Schapire, and L.M. Sellie. Toward efficient agnostic learning. *Machine Learning*, 17(2-3):115–141, 1994.

[20] A.N. Kolmogorov and V.M. Tihomirov. $\epsilon$-entropy and $\epsilon$-capacity of sets in functional spaces. *American Mathematical Society Translations, series 2*, 17:277–364, 1961.

[21] A. Kontorovich and R. Weiss. Maximum margin muliclass nearest neighbors. In ICML’14, 2014.

[22] V. Kuznetsov, M. Mohri, and U. Syed. Multi-class deep boosting. In *NIPS 27*, pages 2501–2509, 2014.

[23] F. Lauer. *Optimisation et apprentissage statistique pour la régression lisse par morceaux et à commutations*. Habilitation à diriger des recherches, Lorraine University, 2019. (to appear).

[24] Y. Lei, Ü. Doğan, A. Binder, and M. Kloft. Multi-class SVMs: From tighter data-dependent generalization bounds to novel algorithms. In *NIPS 28*, pages 2026–2034, 2015.

[25] A. Maurer. A vector-contraction inequality for Rademacher complexities. In ALT’16, pages 3–17, 2016.

[26] S. Mendelson. Rademacher averages and phase transitions in Glivenko-Cantelli classes. *IEEE Transactions on Information Theory*, 48(1):251–263, 2002.

[27] S. Mendelson. A few notes on statistical learning theory. In S. Mendelson and A.J. Smola, editors, *Advanced Lectures on Machine Learning*, chapter 1, pages 1–40. Springer-Verlag, Berlin, Heidelberg, New York, 2003.

[28] S. Mendelson and R. Vershynin. Entropy and the combinatorial dimension. *Inventiones mathematicae*, 152:37–55, 2003.

[29] M. Mohri, A. Rostamizadeh, and A. Talwalkar. *Foundations of Machine Learning*. The MIT Press, Cambridge, MA, 2012.
[30] K. Musayeva, F. Lauer, and Y. Guermeur. Rademacher complexity and generalization performance of multi-category margin classifiers. *Neurocomputing*. (in press).

[31] B.K. Natarajan. On learning sets and functions. *Machine Learning*, 4(1):67–97, 1989.

[32] M. Talagrand. Vapnik-Chervonenkis type conditions and uniform Donsker classes of functions. *The Annals of Probability*, 31(3):1565–1582, 2003.

[33] M. Talagrand. *Upper and Lower Bounds for Stochastic Processes: Modern Methods and Classical Problems*. Springer-Verlag, Berlin Heidelberg, 2014.

[34] A.W. van der Vaart and J.A. Wellner. *Weak Convergence and Empirical Processes, With Applications to Statistics*. Springer Series in Statistics. Springer-Verlag, New York, 1996.

[35] V.N. Vapnik and A.Ya. Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities. *Theory of Probability and its Applications*, XVI(2):264–280, 1971.

[36] G. Wahba. Multivariate function and operator estimation, based on smoothing splines and reproducing kernels. In M. Casdagli and S. Eubank, editors, *Nonlinear Modeling and Forecasting, SFI Studies in the Sciences of Complexity*, volume XII, pages 95–112. Addison-Wesley, 1992.

[37] R.C. Williamson, A.J. Smola, and B. Schölkopf. Generalization performance of regularization networks and support vector machines via entropy numbers of compact operators. *IEEE Transactions on Information Theory*, 47(6):2516–2532, 2001.

[38] T. Zhang. Statistical analysis of some multi-category large margin classification methods. *Journal of Machine Learning Research*, 5:1225–1251, 2004.