Time-dependent three-dimensional Oldroyd-B nanofluid flow due to bidirectional movement of surface with zero mass flux

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Abstract
Unsteady three-dimensional flow of an incompressible Oldroyd-B nanomaterial is reported in this article. The origin of flow is time-dependent surface spreading in lateral directions transversely taking nanoparticles with zero mass flux. The formulated partial differential system is reframed by similarity variables into ordinary differential system. The obtained system is solved by the process of homotopy analysis for dimensional temperature and concentration of nanoparticles. Physical parameter behavior on temperature and concentrations of nanoparticles is examined using graph and tabular data. The surface temperature is also measured and evaluated, and it is found that the temperature is reduced for greater unsteadiness parameter values. We found that the higher $\beta_1$ enhances the curves of nanoparticle concentration and temperature while these curves retard for the incrementing values of $\beta_2$. The increasing nature of Brownian movement $N_b$ and Lewis number $Le$ corresponds to lower profiles of nanoparticles concentration.

Keywords
Unsteady flow, Oldroyd-B fluid, nanoparticles, heat transfer, series solutions

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Introduction
Because of many uses in the fields of engineering, manufacturing, and biology, the study of non-Newtonian fluids has become significant in recent years. Such applications include industries such as meat, product processing, and cosmetics\textsuperscript{1}. The equations governing non-Newtonian fluids are very nonlinear because of the complex geometry. The empirical equations that determine the non-Newtonian fluid’s mathematical description give the fluid’s rheology. Due to their nature, the non-Newtonian materials are usually categorized into three forms, that is, the type of rate, the type of differential form, and the type of the integral form. The Maxwell fluid is the simplest non-Newtonian rate type material which describes the nature of relaxation time phenomenon. But it cannot predict the time effects of retardation. The Oldroyd-B liquid is a sub-category of rate type materials that defines both the retardation and relaxation stress features. For two-dimensional flows configuration under distinct non-Newtonian fluid models, studies\textsuperscript{2–7} are reported and many therein. However, sometimes the flow is three-dimensional in practical applications. The three-dimensional flow was

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studied by scientists of various flow geometries in view of this inspiration.\textsuperscript{7–10}

For better cooling efficiency, an advanced category nanotechnology has been proposed as fluid cooling is the main issue in automated processes. Because of its huge implications in industrial, chemical, and technological processes, nanotechnology is an exciting field of research. The growth of processes for heat transfer is the primary focus of researchers working in this direction. Choi and Eastman\textsuperscript{11} and Buongiorno\textsuperscript{12} suggested the nanofluid concept for improving thermal performance by moving nanoparticles in the base fluid. Thermophoresis, nanoparticles length, volume fraction, and Brownian movement factors are the major factors for enhancing thermal conductivity. With Buongiorno’s\textsuperscript{12} introduction of a credible nanofluid model, nanofluids have become a highly interesting subject to investigators in recent years.\textsuperscript{13–18} Soret effect on mixed convection nanofluid flow with convective boundary conditions was explored by RamReddy et al.\textsuperscript{19} Rashad et al.\textsuperscript{20} considered mixed convection of non-Newtonian nanofluid flow in vertical porous surface. The non-similar solution of mixed convection along a wedge of a non-Newtonian nanofluid flow in a porous medium was studied by Chamkha et al.\textsuperscript{21} Ghalambaz et al.\textsuperscript{22} executed the analysis of nanofluid flow under different impacts of nanoparticles shapes and sizes. Some recent studies can be found in the literature.\textsuperscript{23–28}

Multiple efforts have been presented and demonstrated to identify the behaviors of distinct Newtonian and non-Newtonian materials under various assumptions and geometries. For example, Wang\textsuperscript{29} first considered the three-dimensional flow due to stretching sheet. Heat transfer analysis due to bidirectional stretching sheet with variable thermal conditions was carried out by Liu and Andersson.\textsuperscript{30} Xu et al.\textsuperscript{31} performed the first analysis of time-dependent three-dimensional flows due to movement of surface. Hayat et al.\textsuperscript{32} reported the time-dependent nature of viscoelastic material flow induced by the movement of sheet. Awais et al.\textsuperscript{33} addressed the steady-state three-dimensional Maxwell fluid flow behavior. Magnetized time-dependent viscous fluid flow through porosity medium is executed by Ahmad et al.\textsuperscript{34} Ahmad et al.\textsuperscript{35,36} used the same principle to describe the nature of Maxwell and Oldroyd-B non-Newtonian fluids model. Due to bidirectional moving boundaries, there are limited works reported for nanofluid in literature. The effects of different thermal conditions on bidirectional stretching boundaries to analyze heat transport with nanoparticles are studied by Ahmad et al.\textsuperscript{37} for both Newtonian and non-Newtonian fluid steady and unstable boundary layer flows.

Such structure for time-dependent Oldroyd-B nanofluid’s flow has not been previously been published to our knowledge in the literature. Although some studies are available in the literature dealing with the flow of different fluids over a stretching surface, the Oldroyd-B fluid’s unsteady three-dimensional flow phenomenon and other interesting features are not yet published. Therefore, present scientific calculations are conducted to fill this gap, and the reported results may be useful in improving thermal extrusion processes, solar energy system, and biofuels. We preferred to use the boundary conditions of nanoparticles with no mass flux to model the equations. The homotopy analysis method (HAM)\textsuperscript{38–42} is used to achieve nonlinear differential governance solutions. Convergence analysis has been performed through graphs and tabular data for developed series solutions. The impact of physical parameters appears numerically and graphically in the governing equations.

**Modeling**

Because of the unsteady lateral stretch of the surface where \(y\) and \(x\) axes are taken along surface and \(z\)–axis is adopted along vertical to the surface of incompressible Oldroyd-B nanofluid along thermophoresis and Brownian effects, the problem is based on a three-dimensional approach. The material velocities are \(V_w\) and \(U_w\) in the \(y\) and \(x\) directions, respectively, and \(z = 0\). Geometric description of the problem is shown in Figure 1. The equations developed for this case are using the boundary layer approach. Schlichting\textsuperscript{43}

\[
\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0
\]  

\text{Figure 1. Physical adjustment of the problem.}
\[ \frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} + \lambda_1 \left( \frac{\partial^2 u_1}{\partial t^2} + \frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial y} \right) = \nu \frac{\partial^2 u_1}{\partial z^2} + \lambda_2 \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial y} \right) \]

\[ U_w = \frac{ax}{1 - ct}, \quad V_w = \frac{by}{1 - ct}, \quad T_w = T_\infty + \frac{T_0x}{1 - ct} \]

where \( a, b, x, C_0, \) and \( T_0 \) represent the constants. Introducing the dimensionless quantities as

\[ \zeta = \frac{a}{\nu(1 - ct)} \zeta, \quad u_1 = \frac{ax}{1 - ct} f'(\zeta), \]

\[ u_2 = \frac{ay}{1 - ct} g'(\zeta), u_3 = -\frac{a\nu}{1 - ct} (g(\zeta) + f(\zeta)) \]

\[ \theta(\zeta) = \frac{T - T_\infty}{T_w - T_\infty}, f(\zeta) = \frac{C - C_\infty}{C_w - C_\infty} \]

Equation (1) is similarly satisfied, and after dimensional analysis, equations (2)-(7) take the form

\[ f''' - f'' + A \left( \frac{\xi}{2} f'' + f' \right) + (f + g) f'' \]

\[ - \beta_1 \left( \frac{\xi}{2} f'' + \frac{\xi}{2} g''' - 2(f + g) f'' + \frac{g'}{\xi} f''' \right) = 0 \]

\[ + \beta_2 \left( \frac{\xi}{2} f'' + \frac{\xi}{2} g''' - (f + g) f'' + (f'' + g') f'' \right) \]

\[ g''' - g'' - A \left( \frac{\xi}{2} g'' + g' \right) + (f + g) g'' \]

\[ - \beta_1 \left( \frac{\xi}{2} g'' + \frac{\xi}{2} g''' - 2(f + g) g'' + \frac{g'}{\xi} g''' \right) = 0 \]

\[ + \beta_2 \left( \frac{\xi}{2} f'' + \frac{\xi}{2} g''' - (f + g) g'' + A \left( \frac{\xi g'}{2} + \frac{g''}{2} \right) \right) \]

\[ \theta'' + Pr \left( \frac{N_r}{N_b} \theta'' \right) + (f + g) \theta'' + N_r \phi'' - f' \theta + N_b \phi'' \]

The non-dimensionalized boundary conditions have the form
Here, $A, \beta_1, \beta_2, N_h, Pr, N_i, Le$, and $\lambda$ are the unsteadiness parameter, relaxation parameter, retardation parameter, Brownian coefficient, Prandtl number, thermophoresis coefficient, Lewis number, and stretching variable, respectively. The prime symbolization indicated the derivative w.r.t. $\xi$ and the above parameters are defined in dimensionless form as

$$A = \frac{c}{a}, \beta_1 = \frac{\lambda_1 a}{1 - ct}, \beta_2 = \frac{\lambda_2 a}{(1 - ct)},$$

$$\lambda = \frac{b}{a}, Pr = \frac{\nu}{\alpha}, N_h = \frac{(p_C)_1 D_0 (C_w - C_v)}{(p_C)_1 \nu},$$

$$N_i = \frac{(p_C)_1 D_T (T_v - T_\infty)}{(p_C)_1 T_v \nu}, Le = \frac{\alpha}{D_B} \tag{14}$$

The heat transport rate $Nu_t$ at the wall is the physical quantity and can be described as

$$Nu_t = -\frac{x}{(T - T_\infty)} \left( \frac{\partial T}{\partial z} \right) \bigg|_{z=0} \tag{15}$$

The dimensionless pattern off the above relationship is

$$Re^{1/2} Nu_t = -\theta'(0) \tag{16}$$

where $Re = ux/\nu$ is the Reynolds number.

### Homotopy analysis approach

The HAM $^{38-42}$ is used to solve the differential nonlinear equations (9)–(12) based on the limits (13). Using the HAM method, we select the initial assumptions $f_0(\xi), g_0(\xi), \theta_0(\xi)$, and $\phi_0(\xi)$ for the functions $f(\xi), g(\xi), \theta(\xi)$ and $\phi(\xi)$ meeting the limiting conditions (13)

$$f_0(\xi) = 1 - e^\xi, g_0(\xi) = \alpha (1 - e^{-\xi}),$$

$$\theta_0(\xi) = e^{-\xi}, \phi_0(\xi) = -\left( \frac{N_h}{N_i} \right) e^{-\xi} \tag{17}$$

and the auxiliary linear operator are

$$L_1 = f''' - f', \quad L_2 = \theta'' - \theta \tag{18}$$

Satisfying

$$L_1 (C_1 + C_2 e^\xi + C_3 e^{-\xi}) = 0, \quad L_2 (C_4 e^\xi + C_5 e^{-\xi}) = 0 \tag{19}$$

where $C_i$ are arbitrary constants. From equations (9)–(12), the expressions for nonlinear operatives $N_f, N_g, N_\theta$, and $N_\phi$ are

$$f'''(\xi, \kappa) - \left( \frac{f'}{(\xi, \kappa)} \right)^2 + \left( f(\xi, \kappa) + g(\xi, \kappa) \right) f''(\xi, \kappa)$$

$$\beta_1 \left( f(\xi, \kappa) + g(\xi, \kappa) \right)^2 f'''(\xi, \kappa)$$

$$\beta_2 \left( f(\xi, \kappa) + g(\xi, \kappa) \right)^2 g'''(\xi, \kappa)$$
The auxiliary parameters \( h_f, h_g, h_\theta, \) and \( h_{\phi} \) from equations (33)-(36) give the information of convergence of series solutions. Assuming that \( h_f, h_g, h_\theta, \) and \( h_{\phi} \) are selected such that the series in equations (33)-(36) is convergent at \( \kappa = 1 \). Thus

\[
\begin{align*}
\hat{f}(\xi, \kappa) &= f_0(\xi) + \sum_{m=1}^{\infty} f_m(\xi) \kappa^m, \quad f_m(\xi) = \frac{1}{m!} \partial^m \hat{g}(\xi, \kappa) \bigg|_{\kappa=0} \\
\hat{g}(\xi, \kappa) &= g_0(\xi) + \sum_{m=1}^{\infty} g_m(\xi) \kappa^m, \quad g_m(\xi) = \frac{1}{m!} \partial^m \hat{g}(\xi, \kappa) \bigg|_{\kappa=0} \\
\hat{\theta}(\xi, \kappa) &= \theta_0(\xi) + \sum_{m=1}^{\infty} \theta_m(\xi) \kappa^m, \quad \theta_m(\xi) = \frac{1}{m!} \partial^m \hat{\theta}(\xi, \kappa) \bigg|_{\kappa=0} \\
\hat{\phi}(\xi, \kappa) &= \phi_0(\xi) + \sum_{m=1}^{\infty} \phi_m(\xi) \kappa^m, \quad \phi_m(\xi) = \frac{1}{m!} \partial^m \hat{\phi}(\xi, \kappa) \bigg|_{\kappa=0} 
\end{align*}
\]

Equations (37)-(40) have the overall solutions in the forms

\[
\begin{align*}
f_m(\xi) &= f_m^{*}(\xi) + C_1 + C_2 e^{\xi} + C_3 e^{-\xi} \\
g_m(\xi) &= g_m^{*}(\xi) + C_4 + C_5 e^{\xi} + C_6 e^{-\xi} \\
\theta_m(\xi) &= \theta_m^{*}(\xi) + C_7 e^{\xi} + C_8 e^{-\xi} \\
\phi_m(\xi) &= \phi_m^{*}(\xi) + C_9 e^{\xi} + C_{10} e^{-\xi}
\end{align*}
\]

where \( f_m^{*}(\xi), g_m^{*}(\xi), \theta_m^{*}(\xi), \) and \( \phi_m^{*}(\xi) \) signify the special solutions.

**Convergence**

HAM’s developed equations (solutions) include auxiliary parameters such as \( h_f, h_g, h_\theta, \) and \( h_{\phi} \). Such parameters play a significant role in convergence and approximation rate. The right values for convergent solutions are the \( h \)-curves performed at
The convergence approximation order, and it can be noted that can see in Figure 2. Table 1 shows the solution convergence order, and it can be noted that the approach is compatible at approximations of the 25th order. The acceptable values ranges are \(-1.20 \leq h_f \leq -0.1\), \(-1.22 \leq h_g \leq 0\), \(-1.22 \leq h_0 \leq -0.5\), and \(-1.25 \leq h_R \leq h_R - 0.5\) as you can see in Figure 2. Table 1 shows the solution convergence approximation order, and it can be noted that the approach is compatible at approximations of the 25th order for the distributions of concentration and temperature while it converges at the 17th order for the flow analysis.

**Discussion**

The consequences of parameters arising such as unsteady parameter \(A\), ratio parameter \(\alpha\), Deborah numbers \(\beta_1\) and \(\beta_2\), thermophoresis \(N_t\), Brownian motion \(N_b\), and Prandtl number \(Pr\) on nanoparticle concentration \(\phi(\zeta)\) and temperature \(\theta(\zeta)\) fields are elaborated in this part. The nature of these constraints on the temperature \(\theta(\zeta)\) profile is sketched in Figures 3–9. The impact of unsteadiness constraint \(A\) is executed in Figure 3. This figure portrays a decrease in the temperature \(\theta(\zeta)\) and thermal layer of thickness by the improvement in the unsteadiness constraint. It relies on the thermal diffusivity due to the unsteady parameter. As we raise the unsteady parameter, the thermal diffusivity decreases and therefore temperature decreases. The influence of the stretching parameter \(\alpha\) on the temperature profile is examined via Figure 4. Here, temperature \(\theta(\zeta)\) decay and thickness of thermal layer are considered for larger stretching parameter values. Clearly, the layer thickness reduces due to the cooler-to-ambient liquid for greater values of stretching parameter. Figure 5 demonstrates the Deborah number \(\beta_1\) impacts on the temperature \(\theta(\zeta)\) distribution. This figure elucidates that the larger Deborah number values correspond to temperature rises. The higher relaxation time factor is responsible for augmentation in temperature \(\theta(\zeta)\). The nature of Deborah number \(\beta_2\) on temperature \(\theta(\zeta)\) is reported in Figure 6. Temperature \(\theta(\zeta)\) is a decreasing function of higher Deborah number values. This is because if we augment the Deborah number \(\beta_2\) values, the factor of retardation time is larger, which

| Approximate order | \(-f''(0)\) | \(-g''(0)\) | \(-\theta''(0)\) | \(-\phi''(0)\) |
|-------------------|-------------|-------------|----------------|-------------|
| 1                 | -0.8248028 | -0.361215   | -1.189583      | 1.18958     |
| 5                 | -0.8443760 | -0.360637   | -1.314035      | 1.31404     |
| 10                | -0.8443823 | -0.360497   | -1.312236      | 1.31224     |
| 15                | -0.8443835 | -0.360493   | -1.312291      | 1.31229     |
| 17                | -0.8443836 | -0.360491   | -1.312292      | 1.31229     |
| 20                | -0.8443836 | -0.360491   | -1.312294      | 1.31229     |
| 25                | -0.8443836 | -0.360491   | -1.312273      | 1.31227     |
| 30                | -0.8443836 | -0.360491   | -1.312273      | 1.31227     |
| 35                | -0.8443836 | -0.360491   | -1.312273      | 1.31227     |
| 40                | -0.8443836 | -0.360491   | -1.312273      | 1.31227     |

![Figure 2](image2.png) **Figure 2.** The \(h\)-curve for \(f(\zeta)\), \(g(\zeta)\), \(\theta(\zeta)\), and \(\phi(\zeta)\).

![Figure 3](image3.png) **Figure 3.** Variations of \(A\) on \(\theta(\zeta)\).
produces a decrease in the temperature $\theta(\zeta)$. It is important to illustrate here that $\beta_1 = 0 = \beta_2$ corresponds to the case of viscous fluid and $\beta_2 = 0$ represents the state of Maxwell fluid flow.

The behavior of temperature $\theta(\zeta)$ for distinct Brownian movement $N_b$ and thermophoresis $N_t$ values is designated in Figures 7 and 8. For greater Brownian movement constraint values, an augmentation is reported in the temperature $\theta(\zeta)$. The viscous forces decrease for higher values of Brownian movement and the Brownian diffusion factor improves due to which the boundary layer thickness and temperature $\theta(\zeta)$ increase. Figure 8 discloses that the temperature $\theta(\zeta)$ profile and thickness layer increase for higher thermophoresis values. Thermophoresis factor plays an important role in the temperature distribution. When we augment the thermophoresis, the thermophoretic forces enhance and the nanoparticles move from warm areas to cold areas because of these forces. The temperature and thickness of thermal layer decay due to enhancing values of Prandtl number $Pr$ (see Figure 9). In addition, the Prandtl number increases or decreases as a result of increase or decrease in the fluid’s thermal diffusivity. For greater values of Prandtl number, the thermal diffusivity of the fluid increases, and it leads to decrease in the temperature $\theta(\zeta)$.

To observe the parametric behavior of unsteady constraint $A$, ratio parameter $a$, Deborah numbers $\beta_1$ and $\beta_2$, Brownian movement $N_b$, thermophoresis $N_t$, Prandtl number $Pr$, and Lewis number $Le$ on nanoparticle concentration $\phi(\zeta)$ fields, we present Figures 10–17. The role of unsteady constraint $A$ is seen in Figure 10. As a consequence of higher values of unsteady constraint, the profile of concentration $\phi(\zeta)$ and thickness of concentration layer are decreased. Figure 11 shows the nature of stretching constraint $\alpha$ on $\phi(\zeta)$. The nanoparticles concentration $\phi(\zeta)$ decreases due to improved stretching constraint. The activity of Deborah number $\beta_1$ is addressed in Figure 12. Improvement in the concentration profile and boundary thickness is observed...
for higher Deborah number $\beta_1$ values. Figure 13 shows the effect of Deborah number $\beta_2$ on the concentration profile. With higher values of $\beta_2$, the concentration of nanoparticles $\phi(\xi)$ retards. Figure 14 elucidates that the higher factor of Brownian movement leads to decaying trend of concentration profile $\phi(\xi)$. In the case of the thermophoresis constraint $N_t$, the reverse behavior of $\phi(\xi)$ is noted (see Figure 15). It is investigated that the parameter of thermophoresis influences the nanomaterial more compared to the parameter of Brownian motion. The influence of Prandtl number on the nanoparticles concentration $\phi(\xi)$ is sketched in Figure 16. It is worth mentioning that the concentration profile of nanoparticles is decreasing due to the higher Prandtl number values. It is due to the increased concentration of nanoparticles $\phi(\xi)$ near the surface for higher values of Prandtl number decreases the adjunct thickness of the boundary layer. Lewis number $Le$ impact on $\phi(\xi)$ is displayed in Figure 17. A decrease in $\phi(\xi)$ is achieved for higher Lewis values. This happens because the diffusion factor is inversely related to Lewis number. Hence, weaker diffusion factor is occurred due to Larger Lewis number due to which nanoparticles concentration $\phi(\xi)$ profile is decreased.
Table 2 shows that the heat transport rate \(-\theta'(0)\) (Nusselt number) for distinct \(A, \alpha, \beta_1, \beta_2, Le, Nb, Nt,\) and \(Pr\). From tabular data, it can be seen that

\(-\theta'(0)\) increases for greater values of \(A, \alpha, \beta_2\) and decreases for the enlargement of the values of \(\beta_1, Le, Nt\). Table 2 clearly shows that the values of \(-\theta'(0)\) in case of Oldroyd-B fluid are higher as compared to Maxwell fluid. In order to check the accuracy of our method, the values of \(-f''(0), g''(0), f(\infty), g(\infty)\) for different values of stretching parameter are compared with Wang\textsuperscript{29} and Liu and Andersson\textsuperscript{30} for Newtonian fluids (Table 3). We observed that the solutions have excellent agreement with the previously published data in a limiting approach.

**Conclusion**

This work introduces the time-dependent phenomenon in three-dimensional Oldroyd-B nanomaterial flow generated by the unsteady bilateral moving sheet. Series solutions are obtained for the developed transformed differential expressions. The important points of this investigation are summarized as follows:
The decay in temperature \( u(z) \) and concentration \( f(z) \) distributions is significant for improving values of time-dependent constraint \( A \).

The larger Deborah number \( \beta_1 \) values strengthened the profiles of nanoparticle concentration \( f(z) \) and temperature \( u(z) \), while these curves are reducing for improving Deborah number \( \beta_2 \).

### Table 2. Heat transport rate \( -\theta'(0) \) (Nusselt number) for multiple values \( A, \alpha, \beta_1, \beta_2, Le, N_t, N_b, \) and \( Pr \).

| \( A \) | \( \alpha \) | \( \beta_1 \) | \( \beta_2 \) | \( Le \) | \( N_t \) | \( N_b \) | \( Pr \) | \( -\theta'(0) \) |
|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5 | 0.2 | 0.3 | 1.0 | 0.1 | 0.1 | 1.0 | 0.825084 |
| 0.5 | 0.5 | 0.5 | 1.0 | 1.5 | 1.232875 |
| 1.0 | 0.0 | 0.5 | 1.0 | 1.5 | 1.21124 |
| 1.5 | 0.0 | 0.0 | 1.0 | 1.5 | 1.29571 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.69026 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.79571 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.36879 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.26071 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.17786 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.11739 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.26071 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.37215 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.49512 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.57675 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.38586 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 0.938207 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 0.87859 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 0.77637 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.47228 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.44559 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.38295 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.28195 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.28193 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.28184 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.28176 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.28171 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.28171 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 0.99462 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 0.92444 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 0.87295 |
| 0.5 | 0.5 | 0.0 | 1.0 | 1.5 | 1.86627 |

### Table 3. Tabular data for the comparison of \( -f''(0), g''(0), f(\infty), \) and \( g(\infty) \) with Wang\textsuperscript{29} and Liu and Andersson\textsuperscript{30} for different values of stretching parameter \( \alpha \) in a limiting case when \( A = 0, \beta_1 = 0, \) and \( \beta_2 = 0 \).

| \( \alpha \) | \( f''(0) \) | \( g''(0) \) | \( f(\infty) \) | \( g(\infty) \) |
|---|---|---|---|---|
| Wang\textsuperscript{29} | 0 | -1 | 0 | 1 |
| Liu and Andersson\textsuperscript{30} | -1 | 0 | 1 | 0 |
| Present | -1 | 0 | 1 | 0 |
| Wang\textsuperscript{29} | 0.25 | -1.048811 | -0.194564 | 0.907075 |
| Liu and Andersson\textsuperscript{30} | -1.048811 | -0.194564 | 0.907046 | 0.257993 |
| Present | -1.048811 | -0.194564 | 0.907069 | 0.257989 |
| Wang\textsuperscript{29} | 0.50 | 1.093097 | -0.465205 | 0.842360 | 0.451663 |
| Liu and Andersson\textsuperscript{30} | 1.093097 | -0.465205 | 0.842361 | 0.451669 |
| Present | 1.093097 | -0.465205 | 0.842363 | 0.451669 |
| Wang\textsuperscript{29} | 1.0 | -1.173270 | -0.194564 | 0.907075 |
| Liu and Andersson\textsuperscript{30} | -1.173270 | -0.194564 | 0.907046 | 0.257993 |
| Present | -1.173270 | -0.194564 | 0.907069 | 0.257989 |
• An enhancement in Brownian movement $N_b$ and thermophoresis $N_t$ boost up the temperature $\theta(\xi)$ and its thermal thickness layer.
• Higher Brownian movement $N_b$ and Lewis number $Le$ correspond to weaker nanoparticle concentration profile.
• The situation of steady flow is retrieved for $A = 0$.
• Rate of heat transportation at the wall is increased for greater values of $N_t$, but remain constant for $N_b$.

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**Appendix I**

**Notation**

| Symbol | Description |
|--------|-------------|
| (a, b) | constants |
| A | unsteadiness parameter |
| C | concentration profile |
| C∞ | atmospheric concentration |
| DB | Brownian motion constant |
| DT | thermophoretic diffusion coefficient |
| Le | Lewis number |
| Nsb | Brownian motion |
| Nr | thermophoresis parameter |
| Pr | Prandtl number |
| T | nanoparticle temperature |
| T∞ | atmospheric temperature |
| (u1, u2, u3) | velocity component |
| α | stretching parameter |
| α1 | thermal diffusivity |
| β1 | Deborah number in terms of relaxation time |
| β2 | Deborah number in terms of retardation time |
| η | similarity variable |
| λ1 | retardation time coefficient |
| λ2 | retardation time |
| ρf | nanoparticle density |