Fractal Superconductivity near Localization Threshold

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Superconductivity v/s Localization

• Granular systems with Coulomb interaction
  K. Efetov 1980 et al

• Coulomb-induced suppression of Tc in uniform films
  A. Finkelstein 1987 et al

• Competition of Cooper pairing and localization (no Coulomb)
  Imry-Strongin, Ma-Lee, Kotliar-Kapitulnik, Bulaevsky-Sadovsky (mid-80’s)
  Ghosal, Randeria, Trivedi 2001

There will be no grains and no Coulomb in this talk!
Plan of the talk

1. Motivation from experiments

2. Hard-gap insulator due to electron pairing on localized states

3. BCS-like theory for critical eigenstates

4. S-I transition region and pseudogap
Experimental puzzle: Localized Cooper pairs

D. Shahar & Z. Ovadyahu
amorphous InO 1992

V. Gantmakher et al
D. Shahar et al
T. Baturina et al
InO InO TiN
Strongly insulating InO and nearly-critical TiN

Kowal-Ovadyahu 1994

\( d = 20 \text{ nm} \)
\( T_0 = 15 \text{ K} \)
\( R_0 = 20 \text{ kΩ} \)

Baturina et al 2007

\( d = 5 \text{ nm} \)
\( T_0 = 0.38 \text{ K} \)
\( R_0 = 20 \text{ kΩ} \)
Phase Diagram

- Phase: Mott-law Insulator
- Phase: Hard Gap Insulator
- Phase: Metal
- Point: $B_{MI}$
- Point: $B_C$
- Point: SC
Theoretical model

Simplest BCS attraction model, but for localized electrons

\[ H = H_{\text{kin}} - g \Psi_{\uparrow}^{\dagger} \Psi_{\uparrow}^{\dagger} \Psi_{\uparrow} \Psi_{\downarrow} \]

\[ \Psi = \sum c_j \Psi_j (r) \]

Basis of localized eigenfunctions

M. Ma and P. Lee (1985)
Localization length $L_{\text{loc}}$ is finite but large

\[ H = \sum_{j\sigma} \xi_j c_{j\sigma}^\dagger c_{j\sigma} - \frac{\lambda}{\nu_0} \sum_{i,j,k,l} M_{ijkl} c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger c_{k\uparrow} c_{l\downarrow}, \]

where

\[ M_{ijkl} = \int d^d r \psi_i(r) \psi_j(r) \psi_k(r) \psi_l(r) \]

\[ \lambda = g \nu_0 \]

\[ M_{ij} = \int \psi_i^2(r) \psi_j^2(r) d^d r \]

\[ M_j = \int \psi_j^4(r) d^d r \propto L_{\text{loc}}^{-D_2} \]

\[ D_2 = 1.30 \pm 0.05 \]

All other (off-diagonal) terms: beyond BCS MFA
INSULATING STATE AT LARGE $\delta_L = (\nu_0 L_{loc}^3)^{-1}$

Typical value of superdiagonal matrix element:

$$\bar{M} = L_0^{-3} (L_{loc}/L_0)^{-D_2}$$

where $L_0$ is the short-scale cutoff length of the fractal behaviour.
INSULATING STATE AT LARGE $\delta_L = (\nu_0 L_{loc}^3)^{-1}$

Typical value of superdiagonal matrix element:

$$\tilde{M} = L_0^{-3}(L_{loc}/L_0)^{-D_2}$$

where $L_0$ is the short-scale cutoff length of the fractal behaviour.

$$\Delta_P = \frac{\lambda}{2} E_0 \left( \frac{L_0}{L_{loc}} \right)^{D_2} \propto (E_m - E_F)^{D_2}$$

where

$$E_0 = \frac{1}{\nu_0 L_0^3} \ll E_F$$
Average Density of States

\[ \frac{1}{\mathcal{V}} \sum\int_{0}^{M} \Theta (\epsilon - \frac{M \bar{\lambda}}{2v}) dM \]

no coherence peak!

[Ghosal et al. 2001]

effective gap \( \overline{\Delta p} \)
P(M) distribution

A. Mildenberger and A. Mirlin, private correspondence
(Critical Ensemble)
Activation energy $T_i$ from Shahar-Ovadyahu exp. and fit to theory
Superconductivity at the Localization Threshold: $T_c \gg \delta_L$

Now we will consider the case of Fermi energy very close to the mobility edge: single-electron states are extended but fractal and populate small fraction of the whole volume.

How BCS theory should be modified to account for eigenstates fractality?
Mean-Field Eq. for $T_c$

\[ \Delta(r) = \int K_T(r, r') \Delta(r') d^d r' \]  \hspace{1cm} (9)

where kernel $\hat{K}_T$ is equal to

\[ K_T(r, r') = \frac{\lambda}{2\nu_0} \sum_{ij} \frac{\tanh \frac{\xi_i}{2T} + \tanh \frac{\xi_i}{2T}}{\xi_i + \xi_j} \psi_i(r) \psi_j(r) \psi_i(r') \psi_j(r') \]  \hspace{1cm} (10)

Standard averaging over space $\Delta(r) \to \overline{\Delta}$ leads to ”Anderson theorem” result: totally incorrect in the present situation.

The reason: critical eigenstates $\psi_j(r)$ are strongly correlated in real 3D space, they fill some small submanifold of the whole space only.
In fact one should define $T_c$ as the divergence temperature of the Cooper ladder

$$\mathcal{C} = \left(1 - \hat{K}\right)^{-1}$$

Thus averaging procedure should be applied to $\mathcal{C}$ instead of $K$.

We expand $\mathcal{C}$ in powers of $K$ and average over disorder realizations. Keeping main sequence of resulting diagramms only, we come to the following equation for determination of $T_c$:

$$\Phi(\xi) = \frac{\lambda}{2} \int d\xi' \frac{\tanh(\xi'/2T)}{\xi'} M(\xi - \xi') \Phi(\xi')$$  \hspace{1cm} (11)
\[ M(\omega) = \sqrt{M_{ij}} = \int \frac{\psi_i^2(r)\psi_j^2(r)}{d^d r} \quad \text{for} \quad |\xi_i - \xi_j| = \omega \]

For critical eigenstates

\[ L_{\text{loc}} \to \infty \]

one finds

\[ M(\omega) = \left( \frac{E_0}{\omega} \right)^\gamma \]

where

\[ \gamma = 1 - \frac{D_2}{d} \]

is a measure of fractality

Usual "dirty superconductor":

\[ M(\omega) = 1 \quad \gamma = 0 \]

3D Anderson model: \( \gamma = 0.57 \)
The above equation for $T_c$ is equivalent to the neglect in the Hamiltonian "off-diagonal" terms. We employ eigenfunction expansion of the gap function $\Delta(r)$ and use the idea that pairing amplitude

$$F_j = \langle c_{j\uparrow} c_{j\downarrow} \rangle = F(\xi_j)$$

is a smooth function of the bare energy $\xi_j$:

$$F(\xi) = \frac{\Delta(\xi)}{\sqrt{\Delta^2(\xi) + \xi^2}} \tanh \frac{\sqrt{\Delta^2(\xi) + \xi^2}}{2T}$$

where

$$\Delta(\xi) = \lambda \int d\xi' M(\xi - \xi') F(\xi')$$

Then local pairing amplitude:

$$F(r) = \sum_j \psi_j^2 \langle c_{j\uparrow} c_{j\downarrow} \rangle \equiv \sum_j \psi_j^2 F_j$$

fluctuates strong in real space.

Volume fraction \((T_c/E_0)\gamma \ll 1\)
Self-consistent "gap equation" in terms of $\Delta(\xi)$:

$$\Delta(\xi) = \lambda \int d\xi' M(\xi - \xi') \frac{\Delta(\xi)}{\sqrt{\Delta^2(\xi) + \xi^2}} \tanh \frac{\sqrt{\Delta^2(\xi) + \xi^2}}{2T}$$

Dimensional analysis of the Mean Field equation:

$$T_c = C(\gamma) E_0 \lambda^{1/\gamma}$$

$$\Delta(\xi = 0, T = 0) = D(\gamma) E_0 \lambda^{1/\gamma}$$

Functions $C(\gamma)$ and $D(\gamma)$ were found numerically:

Now we can relate collective gap $\Delta(0)$ and local pairing gap $\Delta_P$:

$$\Delta_P = \frac{1}{2D^\gamma(\gamma)} \delta_L \left( \frac{\Delta(0)}{\delta_L} \right)^\gamma$$

where $\delta_L = \frac{1}{\nu_0 L_{loc}^3}$ - typical level spacing inside localization volume.
Comparison with virial expansion result: disagreement by 10%
1. VERTEX CORRECTIONS

Matrix elements of Cooper interaction are in general

\[ M_{ijkl} = \int d^3r \psi_i(r) \psi_j(r) \psi_k(r) \psi_l(r) \]

Quantities \( M_{ijkl} \) are random with zero mean, In the second order

\[ \delta [\lambda M(\epsilon_1 - \epsilon_2)] = \lambda^2 \nu_0 \int \int d\epsilon_1 d\epsilon_2 R(\epsilon_i, \epsilon_j, \epsilon_1, \epsilon_2) \cdot \frac{\tanh(\beta \epsilon_1/2) - \tanh(\beta \epsilon_2/2)}{2(\epsilon_1 - \epsilon_2)} \]

\[ R(\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l) = V^3 M^2_{ijkl} \sim L^3 \left( \frac{E_0}{\omega} \right)^t \]

\[ t = 3(1 - d_4/d) \]

and \( d_4 \) is the exponent entering expression \( P_4 = \int dr (|\psi|^2(r))^4 = V^{-3d_4/d} \).
\[ \delta[\lambda M(\omega)] = \lambda^2 \left( \frac{E_0}{\omega} \right)^{2-3d_4/d} \]

The condition \( \delta[\lambda M(\omega)] \ll \lambda M(\omega) \) is fulfilled at \( \omega \sim T_c \) if the exponent \( x = d_4 - \frac{2}{3}d_2 > 0 \), since their ratio scales as

\[
\frac{\delta[\lambda M(T_c)]}{\lambda M(T_c)} \sim \left(\frac{T_c}{E_0}\right)^x
\]

\( x > 0 \) always !

3D Anderson: \( x = 0.17 \)
Numerics for 3D Anderson model:

Conclusion: “diagonal” approximation makes sense, but corrections are non-negligible.
Go deeper into insulator

Level spacing $\delta_L$ becomes comparable to $T_c$.

\[ M(\omega) = \left( \frac{E_0}{|\omega| + \delta_L} \right)^\gamma \]

\[ T_c = T_c^{(0)} T_\gamma \left( \frac{T_c^0}{\delta_L} \right) \]

However, this $M(\omega)$ is specific to 1D models only.
3D Anderson insulator: \( M(\omega) \)

No saturation at \( \omega < \delta_L \)

Superconductivity with \( T_c < \delta_L \) is possible

Then "local gap"

\[
\Delta_P = \frac{1}{2D^\gamma(\gamma)} \delta_L \left( \frac{\Delta(0)}{\delta_L} \right)^\gamma
\]

exceeds \( T_c \) !
Transition temperature \( v/s \) Pseudogap

Virial expansion results:

\[
T_c (B)/T_c^0 = \frac{\Delta_p}{T_c^0} = \frac{\omega_0}{b} + g
\]

With \( \omega_0 = 0.2, b = 0.38, g = 0.3 \)

\( \Delta_p = (g/2)P_2 \)
The condition $\Delta_p > T_c$ leads to “pseudogap phenomenology”: a spectral gap opens above superconducting transition. With increase of disorder $\Delta_p$ grows but $T_c$ decreases.

Baturina et al., PRL 2007
Major unsolved problems

• 1. Role of Coulomb enhancement near mobility edge? (this effect was treated by Finkelstein for thin-film case)

• 2. How to include magnetic field into the “fractal” scheme?
Conclusions

• Pairing of electrons on localized states leads to hard gap and Arrhenius resistivity
• Pairing on nearly-critical states produces fractal superconductivity with relatively high $T_c$ but very small superconductive density
• Pseudogap behaviour is expected near S-I transition, with “insulating gap” exceeding $T_c$