SINGULAR RIEMANNIAN FOLIATIONS WITH SECTIONS, TRANSNORMAL MAPS AND BASIC FORMS

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Abstract. A singular riemannian foliation $\mathcal{F}$ on a complete riemannian manifold $M$ is said to admit sections if each regular point of $M$ is contained in a complete totally geodesic immersed submanifold $\Sigma$ that meets every leaf of $\mathcal{F}$ orthogonally and whose dimension is the codimension of the regular leaves of $\mathcal{F}$.

We prove that the algebra of basic forms of $M$ relative to $\mathcal{F}$ is isomorphic to the algebra of those differential forms on $\Sigma$ that are invariant under the generalized Weyl pseudogroup of $\Sigma$. This extends a result of Michor for polar actions. It follows from this result that the algebra of basic function is finitely generated if the sections are compact.

We also prove that the leaves of $\mathcal{F}$ coincide with the level sets of a transnormal map (generalization of isoparametric map) if $M$ is simply connected, the sections are flat and the leaves of $\mathcal{F}$ are compact. This result extends previous results due to Carter and West, Terng, and Heintze, Liu and Olmos.

1. Introduction

The main results of this paper are the following two theorems. The first one generalizes previous results of Carter and West [9], Terng [20] and Heintze, Liu and Olmos [12] for isoparametric submanifolds. It can also be viewed as a converse to the main result in [1], and as a global version of one of the results in [2].

**Theorem 1.1.** Let $\mathcal{F}$ be a singular riemannian foliation with sections on a complete simply connected riemannian manifold $M$. Assume that the leaves of $\mathcal{F}$ are compact and $\mathcal{F}$ admits a flat section of dimension $n$. Then the leaves of $\mathcal{F}$ are given by the level sets of a transnormal map $F : M \to \mathbb{R}^n$.

The second theorem generalizes a result of Michor for basic forms relative to polar actions [14, 15], and will be used to prove Theorem 1.1.

**Theorem 1.2.** Let $\mathcal{F}$ be a singular riemannian foliation with sections on a complete riemannian manifold $M$, and let $\Sigma$ be a section of $\mathcal{F}$. Then the immersion of $\Sigma$ into $M$ induces an isomorphism between the algebra of basic...
differential forms on $M$ relative to $F$ and the algebra of differential forms on $\Sigma$ which are invariant under the generalized Weyl pseudogroup of $\Sigma$.

Theorem 1.2 of course includes the important case of basic functions on $M$. In [19] G. Schwarz proved that the algebra of basic functions relative to the orbits of a smooth action of a compact group on a compact manifold $M$ is finitely generated. Using this result, Theorem 1.2 and a result of Töben [23], we get the following consequence:

**Corollary 1.3.** Let $F$ be a singular riemannian foliation with sections on a complete riemannian manifold $M$. Assume that the sections of $F$ are compact submanifolds of $M$. Then the algebra of basic functions on $M$ relative to $F$ is finitely generated.

**Singular riemannian foliations with sections** (s.r.f.s., for short) are singular riemannian foliations in the sense of Molino [16] which admit transversal complete immersed manifolds that meet all the leaves and meet them always orthogonally, see section 2 for the definition. These were introduced by Boualem [3], and then by the first author [1, 2] as a simultaneous generalization of orbital foliations of polar actions of Lie groups (see e.g. Palais and Terng [15]), isoparametric foliations in simply-connected space forms (see e.g. Terng [20]), and foliations by parallel submanifolds of an equifocal submanifold with flat sections in a simply connected compact symmetric space (see e.g. Terng and Thorbergsson [22]). S.r.f.s. were further studied in [3, 4, 5], by Töben [23], and also by Lytchak and Thorbergsson [13]. By using suspensions of homomorphisms, one can construct examples of s.r.f.s. with nonembedded or exceptional leaves, and also inhomogeneous examples [2]. Other techniques that are used to construct examples of s.r.f.s. on nonsymmetric spaces are surgery and suitable changes of metric [5].

An isoparametric submanifold in a simply-connected space form can always be described as a regular level set of an isoparametric polynomial map [20]. More generally, as proved by Heintze, Liu and Olmos [12], an equifocal submanifold with flat sections in a simply connected compact symmetric space can always be described as a regular level set of an analytic transnormal map (a smooth map is called transnormal if it is an integrable riemannian submersion in a neighborhood of any regular level set, see Definition 2.5). In the case of s.r.f.s., recently it has been proved a local version of this result, in that the plaques of a s.r.f.s. can always be described as level sets of a locally defined transnormal map [2]; Theorem 1.1 thus appears as the corresponding global statement. It is also worth noting that there is a global converse to the quoted result from [12], namely, the regular leaves of an analytic transnormal map on a complete analytic riemannian manifold are equifocal manifolds and leaves of a s.r.f.s. [1].

Let $F$ be a s.r.f.s. on a complete riemannian manifold $M$. A smooth function on $M$ is called a basic function if it is constant along the leaves of $F$. The definition of basic function can be extended to differential forms,
in that a differential form \( \omega \) on \( M \) is called a \textit{basic form} if both \( \omega \) and \( d\omega \) vanish whenever at least one of the arguments of \( \omega \) (resp. \( d\omega \)) is a vector tangent to a leaf of \( F \). Palais and Terng [17] considered the case of a polar action of a Lie group \( G \) on a complete riemannian manifold \( M \) and proved that the restriction from \( M \) to a section \( \Sigma \) induces an isomorphism between the algebra of basic functions on \( M \) relative to the orbital foliation and the algebra of functions on \( \Sigma \) which are invariant under the generalized Weyl group of \( \Sigma \). Michor [14, 15] extended Palais and Terng’s result to basic forms relative to a polar action. In this context, Theorem 1.2 is a generalization of these results to s.r.f.s. Here it is important to remark that a s.r.f.s. admits a generalized Weyl pseudogroup which acts on a section, but, in general, this is not a Weyl group, see section 2.

We finish this introduction with some remarks about Wolak’s claim to have proven Theorem 1.2 in [24] under the additional hypothesis that the leaves be compact. In our opinion, there are two problems with his arguments. The first one is that he has used a Weyl pseudogroup but has not defined it properly. It would appear that he has used the pseudogroup constructed by Boualem in [6]. Even if this is the case, Boualem’s pseudogroup is often smaller than the needed pseudogroup, which is correctly defined in [2 Definition 2.6] (see also remarks in [2, 5]). In fact, in order to properly define the Weyl pseudogroup, one needs the equifocal property or something equivalent to it. The second problem that we found with Wolak’s arguments is related to a property of s.r.f.s.. He incorrectly claimed at the end of Proposition 1 in [24] that the restriction of the foliation to a slice must be homogeneous. This claim is false since there exist many examples of isoparametric foliations with inhomogenous leaves in euclidean space [11].

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2. Facts about s.r.f.s.

In this section, we recall some results about s.r.f.s. that will be used in this text. Details can be found in [2, 5]. Throughout this section, we assume that \( F \) is a singular riemannian foliation with sections on a complete riemannian manifold \( M \); we start by recalling its definition.

\textbf{Definition 2.1.} A partition \( F \) of a complete riemannian manifold \( M \) by connected immersed submanifolds (the \textit{leaves}) is called a \textit{singular riemannian foliation with sections} of \( M \) (s.r.f.s., for short) if it satisfies the following conditions:

\begin{itemize}
  \item[(a)] \( F \) is \textit{singular}, i.e. the module \( \mathcal{X}_F \) of smooth vector fields on \( M \) that are tangent at each point to the corresponding leaf acts transitively on each leaf. In other words, for each leaf \( L \) and each \( v \in TL \) with footpoint \( p \), there exists \( X \in \mathcal{X}_F \) with \( X(p) = v \).
  \item[(b)] The partition is \textit{transnormal}, i.e. every geodesic that is perpendicular to a leaf at one point remains perpendicular to every leaf it meets.
\end{itemize}
(c) For each regular point \( p \), the set \( \Sigma := \text{exp}_p(\nu_p L_p) \) is a complete immersed submanifold that meets all the leaves and meets them always orthogonally. The set \( \Sigma \) is called a section.

**Remark 2.2.** In [6] Boualem dealt with a singular riemannian foliation \( F \) on a complete manifold \( M \) such that the distribution of normal spaces of the regular leaves is integrable. It was proved in [4] that such an \( F \) must be a s.r.f.s. and, in addition, the set of regular points is open and dense in each section.

A typical example of a s.r.f.s is the partition formed by parallel submanifolds of an isoparametric submanifold \( N \) of an euclidean space. A submanifold \( N \) of an euclidean space is called *isoparametric* if its normal bundle is flat and the principal curvatures along any parallel normal vector field are constant. Theorem 2.3 below shows how s.r.f.s. and isoparametric foliations are related to each other. In order to state this theorem, we need the concepts of slice and local section. Let \( q \in M \), and let \( \text{Tub}(P_q) \) be a tubular neighborhood of a plaque \( P_q \) that contains \( q \). Then the connected component of \( \text{exp}_q(\nu P_q) \cap \text{Tub}(P_q) \) that contains \( q \) is called a *slice* at \( q \) and is usually denoted by \( S_q \). A local section \( \sigma \) (centered at \( q \)) of a section \( \Sigma \) is a connected component \( \text{Tub}(P_q) \cap \Sigma \).

**Theorem 2.3 ([2]).** Let \( F \) be a s.r.f.s. on a complete riemannian manifold \( M \). Let \( q \) be a singular point of \( M \) and let \( S_q \) a slice at \( q \).

(a) Let \( \epsilon \) be the radius of the slice \( S_q \). Denote \( \Lambda(q) \) the set of local sections \( \sigma \) containing \( q \) such that \( \text{dist}(p,q) < \epsilon \) for each \( p \in \sigma \). Then \( S_q = \bigcup_{\sigma \in \Lambda(q)} \sigma \).

(b) \( S_x \subset S_q \) for all \( x \in S_q \).

(c) \( F|S_q \) is a s.r.f.s. on \( S_q \) with the induced metric from \( M \).

(d) \( F|S_q \) is diffeomorphic to an isoparametric foliation on an open subset of \( \mathbb{R}^n \), where \( n \) is the dimension of \( S_q \).

From (d), it is not difficult to derive the following corollary.

**Corollary 2.4.** Let \( \sigma \) be a local section. Then the set of singular points of \( F \) that are contained in \( \sigma \) is a finite union of totally geodesic hypersurfaces. These hypersurfaces are mapped by a diffeomorphism to the focal hyperplanes contained in a section of an isoparametric foliation on an open subset of an euclidean space.

We will call the set of singular points of \( F \) contained in \( \sigma \) the *singular stratification of the local section \( \sigma \).* Let \( M_r \) denote the set of regular points in \( M \). A *Weyl Chamber* of a local section \( \sigma \) is the closure in \( \sigma \) of a connected component of \( M_r \cap \sigma \). One can prove that a Weyl Chamber of a local section is a convex set.

It also follows from Theorem 2.3 that the plaques of a s.r.f.s. are always level sets of a transnormal map, whose definition we recall now.
Definition 2.5 (Transnormal Map). Let $M^{n+q}$ be a complete riemannian manifold. A smooth map $F = (f_1, \ldots, f_q) : M^{n+q} \to \mathbb{R}^q$ is called a transnormal map if the following assertions hold:

0. $F$ has a regular value.
1. For every regular value $c$, there exists a neighborhood $V$ of $F^{-1}(c)$ in $M$ and smooth functions $b_{ij}$ on $F(V)$ such that, for every $x \in V$, $\langle \text{grad } f_i(x), \text{grad } f_j(x) \rangle = b_{ij} \circ F(x)$.
2. There is a sufficiently small neighborhood of each regular level set on which, for every $i$ and $j$, the bracket $[\text{grad } f_i, \text{grad } f_j]$ is a linear combination of $\text{grad } f_1, \ldots, \text{grad } f_q$, where the coefficients are functions of $F$.

This definition is equivalent to saying that the map $F$ has a regular value and for each regular value $c$ there exists a neighborhood $V$ of $F^{-1}(c)$ in $M$ such that $F|_V : F(V)$ is an integrable riemannian submersion, where the metric $(g_{ij})$ of $F(V)$ is the inverse matrix of $(b_{ij})$. In particular, a transnormal map $F$ is said to be an isoparametric map if $V$ can be chosen to be $M$ and $\Delta f_i = a_i \circ F$, where $a_i$ are smooth functions. As we have remarked in the introduction, each isoparametric submanifold in an euclidian space can always be described as a regular level set of an isoparametric polynomial map (see [20] or [18]).

In [22], Terng and Thorbergsson introduced the concept of equifocal submanifolds with flat sections in symmetric spaces in order to generalize the definition of isoparametric submanifolds in euclidean space. Next we review the slightly more general definition of equifocal submanifolds in riemannian manifolds.

Definition 2.6. A connected immersed submanifold $L$ of a complete riemannian manifold $M$ is called equifocal if it satisfies the following conditions:

(a) The normal bundle $\nu(L)$ is flat.
(b) $L$ has sections, i.e. for each $p \in L$, the set $\Sigma := \exp_p(\nu_p L_p)$ is a complete immersed totally geodesic submanifold.
(c) For each parallel normal field $\xi$ on a neighborhood $U \subset L$, the derivative of the map $\eta_\xi : U \to M$ defined by $\eta_\xi(x) := \exp_x(\xi)$ has constant rank.

The next theorem relates s.r.f.s. and equifocal submanifolds.

Theorem 2.7 ([2]). Let $L$ be a regular leaf of a s.r.f.s. $\mathcal{F}$ of a complete riemannian manifold $M$.

(a) Then $L$ is equifocal. In particular, the union of the regular leaves that have trivial normal holonomy is an open and dense set in $M$ provided that all the leaves are compact.
(b) Let $\beta$ be a smooth curve of $L$ and $\xi$ a parallel normal field to $L$ along $\beta$. Then the curve $\eta_\xi \circ \beta$ belongs to a leaf of $\mathcal{F}$.
(c) Suppose that $L$ has trivial holonomy and let $\Xi$ denote the set of all parallel normal fields on $L$. Then $\mathcal{F} = \{\eta_\xi(L)\}_{\xi \in \Xi}$. 

The above theorem allows us to define the singular holonomy map, which will be very useful to study $\mathcal{F}$.

**Proposition 2.8** (Singular holonomy). Let $L_p$ be a regular leaf, let $\beta$ be a smooth curve in $L_p$ and let $[\beta]$ denote its homotopy class. Let $U$ be a local section centered at $p = \beta(0)$. Then there exists a local section $V$ centered at $\beta(1)$ and an isometry $\varphi_{[\beta]} : U \rightarrow V$ with the following properties:

1) $\varphi_{[\beta]}(x) \in L_x$ for each $x \in U$.
2) $d\varphi_{[\beta]}\xi(0) = \xi(1)$, where $\xi$ is a parallel normal field along $\beta$.

An isometry as in the above proposition is called the singular holonomy map along $\beta$.

We remark that, in the definition of the singular holonomy map, singular points can be contained in the domain $U$. If the domain $U$ and the range $V$ are sufficiently small, then the singular holonomy map coincides with the usual holonomy map along $\beta$.

Theorem 2.3 establishes a relation between s.r.f.s. and isoparametric foliations. Similarly as in the usual theory of isoparametric submanifolds, it is natural to ask if we can define a (generalized) Weyl group action on $\sigma$. The following definitions and results deal with this question.

**Definition 2.9** (Weyl pseudogroup $W$). The pseudosubgroup generated by all singular holonomy maps $\varphi_{[\beta]}$ such that $\beta(0)$ and $\beta(1)$ belong to the same local section $\sigma$ is called the generalized Weyl pseudogroup of $\sigma$. Let $W_\sigma$ denote this pseudogroup. In a similar way, we define $W_\Sigma$ for a section $\Sigma$. Given a slice $S$, we define $W_S$ as the set of all singular holonomy maps $\varphi_{[\beta]}$ such that $\beta$ is contained in the slice $S$.

**Remark 2.10.** Regarding the definition of pseudogroups and orbifolds, see Salem [16, Appendix D].

**Proposition 2.11.** Let $\sigma$ be a local section. Then the reflections in the hypersurfaces of the singular stratification of the local section $\sigma$ leave $\mathcal{F}|_\sigma$ invariant. Moreover these reflections are elements of $W_\sigma$.

By using the technique of suspension, one can construct an example of a s.r.f.s. such that $W_\sigma$ is larger than the pseudogroup generated by the reflections in the hypersurfaces of the singular stratification of $\sigma$. On the other hand, a sufficient condition to ensure that both pseudogroups coincide is that the leaves of $\mathcal{F}$ have trivial normal holonomy and be compact. So it is natural to ask under which conditions we can guarantee that the normal holonomy of regular leaves are trivial. The next result is concerned with this question.

**Theorem 2.12** ([5]). Let $\mathcal{F}$ be a s.r.f.s. on a simply connected riemannian manifold $M$. Suppose also that the leaves of $\mathcal{F}$ are compact. Then

(a) Each regular leaf has trivial holonomy.

(b) $M/\mathcal{F}$ is a simply connected Coxeter orbifold.
Let $\Sigma$ be a section of $\mathcal{F}$ and let $\Pi : M \to \tilde{M}$ be the canonical projection. Denote by $\Omega$ a connected component of the set of regular points in $\Sigma$. Then $\Pi : \Omega \to \tilde{M}$ and $\Pi : \Omega \to \tilde{M}$ are homeomorphisms, where $M_r$ denotes the set of regular points in $M$. In addition, $\Omega$ is convex, i.e. for any two points $p$ and $q$ in $\Omega$, every minimal geodesic segment between $p$ and $q$ lies entirely in $\Omega$.

3. Proof of Theorem 1.2

Throughout this section we assume that $\mathcal{F}$ is a s.r.f.s. on a complete riemannian manifold $M$ and prove Theorem 1.2. We start by recalling the definition of basic forms.

**Definition 3.1** (Basic forms). A differential $k$-form $\omega$ is said to be **basic** if, for all $X \in \mathcal{X}_\mathcal{F}$, we have:

- (a) $i_X \omega = 0$,
- (b) $i_X d\omega = 0$.

In the course of the proof of the theorem, we will also need the concept of differential form invariant by holonomy. As soon as Theorem 1.2 is proved, it will be clear that these two concepts are in fact equivalent.

**Definition 3.2.** A differential $k$-form $\omega$ is said to be **invariant by holonomy** if:

- (a) $i_X \omega = 0$ for all $X \in \mathcal{X}_\mathcal{F}$.
- (b) Let $\sigma$ and $\tilde{\sigma}$ be local sections and $\varphi : \sigma \to \tilde{\sigma}$ a singular holonomy map. Let $I : \sigma \to M$ and $\tilde{I} : \tilde{\sigma} \to M$ be the inclusions of $\sigma$ and $\tilde{\sigma}$ in $M$. Then $\varphi^*(\tilde{I}^* \omega) = I^* \omega$.

A $k$-form $\omega$ is said to be **invariant by regular holonomy** if it satisfies the definition above with the condition that the map $\varphi : \sigma \to \tilde{\sigma}$ can be only a regular holonomy.

**Lemma 3.3.**

(a) Forms invariant by holonomy are basic forms.

(b) Basic forms are invariant by regular holonomy.

**Proof.** (a) Let $\omega$ be a form invariant by holonomy and let $X \in \mathcal{X}_\mathcal{F}$. We want to prove that

\begin{equation}
    i_X d\omega = 0.
\end{equation}

First we prove this equation for regular points. Let $p$ be a regular point and let $P_p$ be a plaque of $\mathcal{F}$ that contains $p$. Using Theorem 2.4 and the normal exponential map $\exp^\nu : \nu(P_p) \to M$, we can construct a vector field $\tilde{X}$ on a neighborhood of $p$ such that

- (a) $\tilde{X}(y) = X(y)$ for $y \in P_p$.
- (b) $\varphi_t := \psi_t|_{\sigma_0}$ is a regular holonomy map, where $\psi_t$ is the flow of $\tilde{X}$ and $\sigma_0$ is a local section that contains $p$. 


Define \( \sigma_t := \varphi_t(\sigma_0) \), and let \( I_t : \sigma_t \to M \) denote the inclusion of \( \sigma_t \) in \( M \). Since \( \omega \) is invariant by holonomy, we have

\[
I_0^* \psi_t^* \omega = \varphi_t^* I_t^* \omega = I_0^* \omega.
\]

On the other hand, it follows from \( \varphi_t(x) \in L_x \) that

\[
i_Y(p) \psi_t^* \omega = 0
\]

for each \( Y \in \mathcal{X}_F \). Putting together Equations (3.2) and (3.3), we get that

\[
\psi_t^* \omega = \omega
\]

for small \( t \).

Equation (3.4) and the definition of \( \tilde{X} \) now yield that

\[
0 = L_{\tilde{X}}(p) = i_{\tilde{X}(p)} d\omega + d(i_{\tilde{X}})(p) = i_{X}(p) d\omega.
\]

We have shown that Equation (3.1) holds on the set of regular points of \( F \). Since this set is dense in \( M \), this finishes the proof.

(b) Let \( \omega \) be a basic form and let \( \varphi_{[\beta]} : \sigma_0 \to \sigma_1 \) be a regular holonomy, where \( \sigma_0 \) and \( \sigma_1 \) are local sections that contains only regular points and \( \beta \) is a curve contained in a regular leaf such that \( \beta(0) \in \sigma_0 \) and \( \beta(1) \in \sigma_1 \). Let \( 0 = t_0 < \cdots < t_n = 1 \) be a partition such that \( \beta_i := \beta|_{[t_{i-1}, t_i]} \) is a curve contained in a distinguished neighborhood \( U_i \), i.e., the plaques of \( F \) in \( U_i \) are fibers of a submersion. Since \( \varphi_{\beta} = \varphi_{\beta_n} \circ \cdots \circ \varphi_{\beta_1} \), in order to see that \( \omega \) is invariant by regular holonomy, it suffices to prove that

\[
\varphi_{[\beta]}^*(I_t^* \omega) = I_t^* \omega,
\]

where \( I_t : \sigma_t \to M \) is the inclusion in \( M \) of a local section \( \sigma_t \) that contains \( \beta(t_i) \).

Since \( \beta_i \) is contained in a distinguished neighborhood, we can construct a field \( X \) such that \( \varphi_{[\beta_i]} = \psi_{t_i}|_{\sigma_{t_i-1}} \), where \( \psi \) is the flow of \( X \). The fact the \( \omega \) is basic gives that \( L_X \omega = 0 \). Therefore \( \psi_t^* \omega = \omega \), and this implies Equation (3.5). \( \square \)

Let \( I : \Sigma \to M \) be the immersion of the section \( \Sigma \) in \( M \). We divide the proof of Theorem 1.2 into the following two lemmas.

**Lemma 3.4.** The map \( I^* : \Omega^k(B)(M) \to \Omega^k(W) \) is well defined and injective, where \( \Omega^k(B)(M) \) denotes the algebra of basic \( k \)-forms on \( M \) and \( \Omega^k(W) \) denotes the algebra of \( W \)-invariant \( k \)-forms on \( \Sigma \).

**Proof.** Let \( \omega \in \Omega^k(B)(M) \). We first verify that

\[
I^* \omega \in \Omega^k(W).
\]

In fact, Lemma 3.3(b) implies that

\[
\varphi^*(I^* \omega)|_{\Sigma \cap M_r} = I^* \omega|_{\Sigma \cap M_r},
\]
for a singular holonomy $\varphi \in W_\Sigma$. Equation (3.10) then follows from (3.7), as the set $M_r \cap \Sigma$ is dense in $\Sigma$. To show that $I^*\omega = 0$ implies $\omega = 0$. This follows again from Lemma (3.3(b)) and the denseness of $M_r$ in $M$.

**Lemma 3.5.** The map $I^* : \Omega^k_b (M) \to \Omega^k (\Sigma)^{W_s}$ is surjective.

**Proof.** According to Lemma (3.3(a)), it suffices to check that

$$I^* : \Omega^k_b (M)^{\text{Holsing}} \to \Omega^k (\Sigma)^{W_s}$$

is surjective, where $\Omega^k_b (M)^{\text{Holsing}}$ is the algebra of differential $k$-forms on $M$ that are invariant by holonomy.

Let $\omega \in \Omega^k (\Sigma)^{W_s}$. We will construct $\tilde{\omega} \in \Omega^k (M)^{\text{Holsing}}$ such that $I^*\tilde{\omega} = \omega$. Let $\tilde{q}$ be a point of $M$. To begin with, we set $i_Y \tilde{\omega} = 0$ for all $Y \in T_{\tilde{q}}P_{\tilde{q}}$, where $P_{\tilde{q}}$ is the plaque that contains $\tilde{q}$. Next, we must define $\tilde{\omega}|_{T_{\tilde{q}}S_{\tilde{q}}}$, where $S_{\tilde{q}}$ is a slice at $\tilde{q}$.

Suppose first that $\tilde{q}$ is a regular point. In this case, the slice $S_{\tilde{q}}$ is a local section. Let $\sigma \subset \Sigma$ be a local section that contains a point $q \in L_{\tilde{q}} \cap \Sigma$ and let $\varphi : \sigma \to S_{\tilde{q}}$ be a singular holonomy. We define $\tilde{\omega}|_{S_{\tilde{q}}} = (\varphi^{-1})^* \omega$. This definition does not depend on $\sigma$ and $\varphi$ since $\omega \in \Omega^k (\Sigma)^{W_s}$.

Next, suppose that $\tilde{q}$ is a singular point. In this case, the slice $S_{\tilde{q}}$ is no longer a local section, but, on the contrary, it is the union of the local sections that contain $\tilde{q}$ (see Theorem (2.3(a))). This leads us to consider the intersection of all those local sections that contain $\tilde{q}$. Denote by $T$ the connected component of the minimal stratum of the foliation $\mathcal{F} \cap S_{\tilde{q}}$ that contains $\tilde{q}$. It follows from Theorem (2.3) and from the theory of isoparametric submanifolds that $T$ is a union of singular points of the foliation $S_{\tilde{q}} \cap \mathcal{F}$, and $T$ is the intersection of the local sections that contain $\tilde{q}$.

In order to motivate the next step in the construction of $\tilde{\omega}$, we remark that for a point $\tilde{q} \in \Sigma$ and $Y(\tilde{q}) \in \nu_\tilde{q} (T)$, we have that $i_Y \omega = 0$. For the purpose of checking this remark, consider a local section $\sigma$ such that $Y \in T_{\tilde{q}}\sigma$. Then the theory of isoparametric submanifolds and Theorem (2.3) allow us to choose a basis $\{Y_i\}$ of $\nu_\tilde{q} T \cap T_{\tilde{q}}\sigma$ such that each $Y_i$ is orthogonal to a wall $H_i$. Due to the invariance of $\omega$ under the reflection on $H_i$, we have that $i_{Y_i} \omega|_{H_i} = i_{-Y_i} \omega|_{H_i}$. Hence $i_{Y_i} \omega = 0$. Since $\{Y_i\}$ is a basis, we conclude that $i_Y \omega = 0$, as wished.

Based on the previous remark, we set $i_Y \tilde{\omega} = 0$ for $Y(\tilde{q}) \in \nu(T)$ and arbitrary $\tilde{q} \in M$.

It remains to define $\tilde{\omega}|_T$. To do that, choose a local section $\tilde{\sigma}$ that contains $\tilde{q}$, a point $q \in L_{\tilde{q}} \cap \Sigma$, a local section $\sigma \subset \Sigma$ that contains $q$, and a holonomy $\varphi : \sigma \to \tilde{\sigma}$. Then we set $\tilde{\omega}|_T = (\varphi^{-1})^* \omega|_T$. Let us show that the definition does not depend on $\sigma$, $\tilde{\sigma}$ and $\varphi$ by using that $\omega \in \Omega^k (\Sigma)^{W_s}$. Indeed, for $i = 1, 2$, let $\sigma_i$ be a local section that contains a point $q_i \in L_{\tilde{q}} \cap \Sigma$, let $\tilde{\sigma}_i$ be a local section of $\tilde{q}_i$, and let $\varphi_i : \sigma_i \to \tilde{\sigma}_i$ be a singular holonomy. Denote by $\varphi_{21} : \tilde{\sigma}_1 \to \tilde{\sigma}_2$ a singular holonomy in $W_{S_{\tilde{q}}}$ (see Definition (2.3)). Then Theorem (2.3) and the theory of isoparametric submanifolds force $\varphi_{21}|_T$ to
be the identity. Owing to our assumption on \( \omega \), \((\varphi_2^{-1} \circ \varphi_1) \circ \varphi_1)^* \omega = \omega \).  

Hence \((\varphi_2^{-1})^* \omega|_{\mathcal{T}} = (\varphi_2^{-1})^* \omega|_{\mathcal{T}}\)  

We have already constructed the \( k \)-form \( \tilde{\omega} \). It also follows from the construction that \( \tilde{\omega} \) is invariant by holonomy in the sense that it satisfies the conditions (a) and (b) in Definition 3.2. Now it only remains to prove that \( \tilde{\omega} \) is smooth. It suffices to prove that in a neighborhood of an arbitrary point \( \tilde{q} \).  

If \( \tilde{q} \) is a regular point, then there exists only one (germ of) local section \( \sigma \) that contains \( \tilde{q} \). By construction, \( \tilde{\omega}|_{\sigma} \) is smooth. Since \( \tilde{\omega} \) is invariant by holonomy, we deduce that \( \tilde{\omega} \) is smooth in a distinguished neighborhood of \( \tilde{q} \).  

Next, we suppose that \( \tilde{q} \) is a singular point. Let \( \psi : S_{\tilde{q}} \to U \subset \mathbb{R}^n \) be the diffeomorphism that sends the s.r.f.s. \( \mathcal{F} \cap S_{\tilde{q}} \) into an isoparametric foliation \( \tilde{\mathcal{F}} \) in the open set \( U \subset \mathbb{R}^n \) (where \( n \) is the dimension of \( S_{\tilde{q}} \)). Note that \( \tilde{\omega}|_{S_{\tilde{q}}} \) is invariant by each holonomy \( \varphi[\beta] \), where \( \beta \) is a curve contained in the slice \( S_{\tilde{q}} \). Since the diffeomorphism \( \psi \) sends local sections into local sections, we conclude that \( (\psi^{-1})^*(\tilde{\omega}|_{S_{\tilde{q}}}) \) is invariant by the holonomy of the foliation \( \tilde{\mathcal{F}} \).  

Fix a section \( V \) of the isoparametric foliation \( \tilde{\mathcal{F}} \), set \( \dim V = l \), and select a minimal set of homogeneous generators \( \kappa_1, \ldots, \kappa_l \) of the algebra \( \mathbb{R}[V]^W \) of \( W \)-invariant functions of \( V \), where \( W \) is the Coxeter group of the isoparametric foliation \( \tilde{\mathcal{F}} \) (see e.g. [17]).  

By construction, we have that \( \tilde{\omega} \) restricted to a local section that contains \( \tilde{q} \) is smooth. Since \( \psi^{-1}(V) \) is a local section, \( \tilde{\omega}|_{\psi^{-1}(V)} \) is smooth. Therefore \( (\psi^{-1})^*(\tilde{\omega}|_{S_{\tilde{q}}})|_V \) is smooth and invariant by \( W \). Then it follows as in Michor [14] Lemma 3.3 and proof of Theorem 3.7 that  

\[
(\psi^{-1})^*(\tilde{\omega}|_{S_{\tilde{q}}})|_V = \sum \eta_{i_1 \ldots i_j} d\kappa_{i_1} \wedge \cdots \wedge d\kappa_{i_j},
\]

where \( \eta_{i_1 \ldots i_j} \) is a smooth \( W \)-invariant function. In view of Schwarz [19], we can write \( \eta_{i_1 \ldots i_j} = \lambda_{i_1 \ldots i_j}(\kappa_1, \ldots, \kappa_l) \) for a smooth function \( \lambda_{i_1 \ldots i_j} \) on \( \mathbb{R}^l \). By [18], we can extend each \( \kappa_i \) to a \( \tilde{\mathcal{F}} \)-invariant function \( \tilde{\kappa}_i \) on \( U \). Since \( (\psi^{-1})^*(\tilde{\omega}|_{S_{\tilde{q}}}) \) is invariant by the holonomy of the foliation \( \tilde{\mathcal{F}} \), we have  

\[
(\psi^{-1})^*(\tilde{\omega}|_{S_{\tilde{q}}}) = \sum \lambda_{i_1 \ldots i_j}(\tilde{\kappa}_1, \ldots, \tilde{\kappa}_l) d\tilde{\kappa}_{i_1} \wedge \cdots \wedge d\tilde{\kappa}_{i_j}.
\]

Therefore  

\[
\tilde{\omega}|_{S_{\tilde{q}}} = \sum \lambda_{i_1 \ldots i_j}(f_1, \ldots, f_l) df_{i_1} \wedge \cdots \wedge df_{i_j},
\]

where we have set \( f_i = \tilde{\kappa}_i \circ \psi \). This equation already shows that \( \tilde{\omega}|_{S_{\tilde{q}}} \) is smooth on \( S_{\tilde{q}} \). Finally, extend each \( f_i \) to a function defined on a tubular neighborhood of \( \tilde{q} \) and denoted by the same letter by setting it to be constant along each plaque in that neighborhood. Then the preceding equation and the invariance by holonomy imply that  

\[
\tilde{\omega} = \sum \lambda_{i_1 \ldots i_j}(f_1, \ldots, f_l) df_{i_1} \wedge \cdots \wedge df_{i_j},
\]
on a tubular neighborhood of $\tilde{q}$, which finally shows that $\tilde{\omega}$ is smooth in a neighborhood of $\tilde{\omega}$. Hence $\tilde{\omega}$ is smooth.

4. Proof of Corollary 1.3

We rewrite the statement of Corollary 1.3 as follows.

**Corollary 4.1.** Let $F$ be a s.r.f.s. on a complete riemannian manifold $M$, and let $\mathcal{E}(M)^F$ denote the space of basic functions. Suppose that the sections of $F$ are compact submanifolds of $M$. Then there exist functions $f_1, \ldots, f_n \in \mathcal{E}(M)^F$ such that

$$F^* \mathcal{E}(\mathbb{R}^n) = \mathcal{E}(M)^F,$$

where $F = (f_1, \ldots, f_n)$.

**Proof.** According to Töben [23], there exists a section $\Sigma$ such that the Weyl pseudogroup $W\Sigma$ is in fact a group. By Ascoli’s theorem, $W\Sigma$ is a compact Lie group. Let $\mathcal{E}(\Sigma)^W\Sigma$ be the space of smooth $W\Sigma$-invariant functions on $\Sigma$. It follows from Schwarz’s theorem [19] that there exist $f_1, \ldots, f_n \in \mathcal{E}(\Sigma)^W\Sigma$ such that

$$F^* \mathcal{E}(\mathbb{R}^n) = \mathcal{E}(\Sigma)^W\Sigma,$$

where $F = (f_1, \ldots, f_n)$. Finally, we use Theorem 1.2 to extend each function $f_i$ to a basic function on $M$ denoted by the same letter and this finishes the proof.

5. Proof of Theorem 1.1

Throughout this section we assume that $F$ is a s.r.f.s. on a complete simply connected riemannian manifold $M$ with compact leaves and a flat section $\Sigma$, and we prove Theorem 1.1.

**Lemma 5.1.** Let $\Omega_0$ and $\Omega_1$ be connected components of $M_r \cap \Sigma$ such that $\partial \Omega_0 \cap \partial \Omega_1$ contains a wall $N$. Let $\sigma_0$ be a local section that intersects $\Omega_0$. Then there exists a local section $\sigma_1$ which intersects $\Omega_1$ and an isometry $\phi: \sigma_0 \cup \Omega_0 \to \sigma_1 \cup \Omega_1$ such that $\phi(x) \in L_x$ for all $x \in \sigma_0 \cup \sigma_0$. In particular, $\phi$ coincides with each holonomy with source in $\sigma_0 \cup \Omega_0$ and target in $\sigma_1 \cup \Omega_1$.

**Proof.** Let $\beta$ be a curve contained in a regular leaf such that $\beta(0) \in \Omega_0$ and $\beta(1) \in \Omega_1$. To begin with, we want to extend the singular holonomy $\varphi[\beta]$ to an isometry $\varphi: \Omega_0 \to \Omega_1$. Since $\Omega_0$ is convex, flat and simply connected (see Theorem 2.12), there exists a unique vector $\xi$ such that $\exp_{\beta(0)}(\xi) = x$. Let $\xi(\cdot)$ be the normal parallel transport of the vector $\xi$ along the curve $\beta$. We define $\varphi(x) = \exp_{\beta(1)}(\xi(1))$. It follows from Theorem 2.7 that $\exp_{\beta(t)}(\xi(t)) \in L_x$ and hence $\varphi(x) \in L_x$. Since $\varphi(x) \in L_x$ and $\Omega_1$ is the interior of a fundamental domain, the map $\varphi$ restricted to a neighborhood of each regular point of $\Omega_0$ coincides with a regular holonomy. This implies that $\varphi$ is an isometry, and hence $\varphi$ is smooth.

Next, we want to extend $\varphi$ so that it is also defined on $\sigma_0$. Since the restriction of $\varphi$ to a neighborhood of each regular point of $\Omega_0$ coincides with
a regular holonomy, it suffices to prove that \( \sigma_0 \cap \Omega_0 \) has only one connected component. If \( \sigma_0 \) is centered at a regular point \( q \), then the fact that it is contained in the slice of \( q \) implies that it is a ball contained in \( \Omega_0 \). Suppose now that \( \sigma_0 \) is centered at a singular point \( q \). Since it is contained in the slice of \( q \), it follows from Theorem 2.3 that the intersection of \( \sigma_0 \) with the singular stratification of \( \Sigma \) has only one connected component and also that \( q \in \partial \Omega_0 \). These facts together with the fact that \( \Omega_0 \) is flat and simply connected imply that also in this case \( \sigma_0 \cap \Omega_0 \) has only one connected component. \( \square \)

**Proposition 5.2.** Let \( \Pi : \mathbb{R}^n \to \Sigma \) be the riemannian universal covering map of the section \( \Sigma \). Then:

(a) There exists a locally finite family \( \mathcal{H} \) of hyperplanes in \( \mathbb{R}^n \) which is invariant under the action of the group of isometries \( \tilde{W} \) generated by the reflections in the hyperplanes of \( \mathcal{H} \). In addition, the projection \( \Pi(\mathcal{H}) \) is the singular stratification on \( \Sigma \).

(b) The group of covering transformations of \( \Pi \) is a subgroup of \( \tilde{W} \).

(c) Each singular holonomy \( \varphi_{[\beta]} \in W_\Sigma \) is a restriction of a local isometry given by the composition of a finite number of reflections in the walls of the singular stratification of \( \Sigma \).

**Proof.** (a) The existence of the locally finite family \( \mathcal{H} \) of hyperplanes in \( \mathbb{R}^n \) follows from the facts that \( \Pi \) is a covering map and the singular stratification in the section \( \Sigma \) is locally finite. We still need to prove that \( \mathcal{H} \) is invariant under the action of the group of isometries \( \tilde{W} \). Let \( \tilde{H}_0 \) be an hyperplane in \( \mathcal{H} \), and let \( \tilde{w} \) be the reflection in \( \tilde{H}_0 \). Given an hyperplane \( \tilde{H}_1 \), we want to prove that \( \tilde{w}(\tilde{H}_1) \) is an hyperplane in \( \mathcal{H} \).

Let \( \tilde{\gamma} \) be the segment of line that joins a point \( \tilde{\gamma}(0) \in \tilde{H}_0 \) to a point \( \tilde{\gamma}(1) \in \tilde{H}_1 \) such that \( \tilde{\gamma} \) is orthogonal to \( \tilde{H}_1 \) at \( \tilde{\gamma}(1) \). Let us define \( \gamma \) as the geodesic segment \( \Pi(\tilde{\gamma}) \). Then we cover \( \tilde{\gamma} \) (respectively, \( \gamma \)) by neighborhoods \( \tilde{U}_0, \ldots, \tilde{U}_n \) (respectively, by neighborhoods \( U_0, \ldots, U_n \)) so that \( \Pi : \tilde{U}_i \to U_i \) is an isometry, \( \tilde{\gamma}(0) \in \tilde{U}_0 \) and \( \tilde{\gamma}(1) \in \tilde{U}_n \).

Define the singular holonomy \( \varphi_0 : U_0 \to U_0 \) such that \( \varphi_0 \Pi|_{\tilde{U}_0} = \Pi \tilde{w}|_{\tilde{U}_0} \). By induction, we define a singular holonomy \( \varphi_n : U_n \to U_n \) such that \( \varphi_{n-1}|_{U_{n-1}\cap U_n} = \varphi_n|_{U_{n-1}\cap U_n} \). Owing to \( \varphi_{n-1}\Pi|_{\tilde{U}_{n-1}} = \Pi \tilde{w}|_{\tilde{U}_{n-1}} \), we conclude that

\[
\varphi_n\Pi|_{\tilde{U}_n} = \Pi \tilde{w}|_{\tilde{U}_n}.
\]

Let \( H_1 \) be a wall whose closure contains \( \gamma(1) \) and such that \( H_1 \subset \Pi(\tilde{H}_1) \). Since \( \varphi_n \) is a singular holonomy, \( \varphi_n(H_1) \) is contained in the singular stratification of \( \Sigma \). This fact together with Equation (5.1) yield that \( \tilde{w}(\tilde{H}_1) \in \mathcal{H} \).

(b) Let \( \gamma \) be a loop so that \( \gamma(0) = x_0 = \gamma(1) \), and \( \tilde{\gamma} \) be the lift of \( \gamma \) such that \( \tilde{\gamma}(0) = \tilde{x}_0 \). Without loss of generality, we may assume that \( \gamma \) meets the singular stratification only in the walls and always transversally to them.

Let \( 0 = t_0 < \cdots < t_{n+1} = 1 \) be a partition such that
(i) $\gamma|_{(t_{i-1},t_i)}$ has only regular points,
(ii) $\gamma(t_i)$ belongs to a wall for $0 < i < n + 1$.

By induction define $x_i$ (respectively $\tilde{x}_i$) as the reflection of $x_{i-1}$ (respectively $\tilde{x}_{i-1}$) in the wall that contains $\gamma(t_i)$ (respectively $\tilde{\gamma}(t_i)$). Lemma 5.1 implies that $x_i \in L_{x_0}$ and $\Pi(\tilde{x}_i) = x_i$. By construction, $\tilde{x}_n$ and $\tilde{\gamma}(1)$ both belong to the same Weyl chamber $\tilde{\Omega}$. Note that $\Pi : \tilde{\Omega} \to \Omega$ is a diffeomorphism, where $\Omega$ is the connected component of $M_r \cap \Sigma$ that contains $x_0$. Since $\Omega$ is a fundamental domain, $L_{x_0}$ meets $\Omega$ only at $x_0$. We conclude that $\Pi(\tilde{x}_n) = x_0 = \Pi(\tilde{\gamma}(1))$. Since $\Pi : \tilde{\Omega} \to \Omega$ is a diffeomorphism, we have that $\tilde{x}_n = \tilde{\gamma}(1)$. Therefore $\tilde{\gamma}(1) = \tilde{x}_n = g_n \cdots g_1 \cdot \tilde{x}_0 = g_n \cdots g_1 \cdot \tilde{\gamma}(0)$, where $g_i$ is a reflection in the wall that contains $\tilde{\gamma}(t_i)$. In other words, we conclude that the covering transformation that sends $\tilde{\gamma}(0)$ to $\tilde{\gamma}(1)$ is $g_n \cdots g_1$.

(c) Let $\gamma$ be a curve in $\Sigma$ such that $\gamma(0) = \beta(0)$ and $\gamma(1) = \beta(1)$. As in item (b), we choose a partition $0 = t_0 < \cdots < t_{n+1} = 1$ such that $\gamma|_{(t_{i-1},t_i)}$ has only regular points, and $\gamma(t_i)$ belongs to a wall for $0 < i < n + 1$.

Let $w_i$ be the singular holonomy that is the reflection in the wall that contains $\gamma(t_i)$. According to Lemma 5.1 we can define the source of each $w_i$ so that $w_n \cdots w_1 : U \to \Sigma$ is well defined, where $U$ is the source of $\varphi[\beta]$. We want to prove that

\begin{equation}
\varphi[\beta] = w_n \cdots w_1|_U.
\end{equation}

For $i = 0$ and $i = 1$, define $\Omega_i$ to be the connected component of $M_r \cap \Sigma$ that contains $\beta(i)$. Since $\Omega_1$ is the interior of a fundamental domain, $L_{\beta(0)} \cap \Omega_1 = \{\beta(1)\}$. This fact together with the properties of singular holonomies imply that $w_n \cdots w_1 \beta(0) = \beta(1)$. We conclude that $(w_n \cdots w_1)^{-1} \varphi[\beta]$ is an holonomy that fixes $\beta(0)$. Since $\Omega_0$ is the interior of a fundamental domain, we get that $(w_n \cdots w_1)^{-1} \varphi[\beta] = I$, where $I$ is the identity with germ at $\beta(0)$, and this implies Equation (5.2). \hfill \Box

It follows from Terng [21, App.] that $\tilde{W}$ is a Coxeter group, i.e. the subgroup of isometries $\tilde{W}$ is generated by reflections, the topology induced in $\tilde{W}$ from the group of isometries of $\mathbf{R}^n$ is discrete and the action on $\mathbf{R}^n$ is proper. Since $\mathcal{H}$ is invariant by the action of $\tilde{W}$, we have the following results (see Bourbaki [7], Ch. V §3 Propositions 6, 7, 8, and 10, and Remarque 1 on p.86; Ch. VI §2 Proposition 8 and Remarque 1 on p.180).

**Proposition 5.3.** Let $\tilde{W}$ be the Coxeter group defined in Proposition 5.2. Then:

(a) We have that $\tilde{W}$ is a direct product of irreducible Coxeter groups $\tilde{W}_i$ $(1 \leq i \leq s)$ and, after a possible adjustment of the origin, there exists a decomposition of $\mathbf{R}^n$ into an orthogonal direct sum of subspaces $E_i$ $(0 \leq i \leq s)$ such that $\tilde{w}(x_0,x_1,\ldots,x_s) = (\tilde{w}_0(\tilde{x}_0),\ldots,\tilde{w}_s(\tilde{x}_s))$ for $\tilde{w} = \tilde{w}_1 \cdots \tilde{w}_s \in \tilde{W}$ with $\tilde{w}_i \in \tilde{W}_i$. 


(b) Let $\mathcal{H}_i$ be the set of hyperplanes of $E_i$ whose reflections generate $\tilde{W}_i$. Then the set $\mathcal{H}$ consists of hyperplanes of the form

$$H = E_0 \times E_1 \times \cdots \times E_{i-1} \times H_i \times E_{i+1} \times \cdots \times E_s$$

with $H_i \in \mathcal{H}_i$ and $i = 1, \ldots, s$.

(c) Every chamber $C$ is of the form $E_0 \times C_1 \times \cdots \times C_s$, where, for each $i$, the set $C_i$ is a chamber defined in $E_i$ by the set of hyperplanes $H_i$.

(d) Each $C_i$ is an open simplicial cone or an open simplex if $W_i$ is finite or infinite respectively.

(e) If $\tilde{W}_i$ is infinite, then it is an affine Weyl group, i.e. there exists a unique root system $\Delta_i$ in $E_i$ such that $\tilde{W}_i$ can be written as the semidirect product $\tilde{W}_i \rtimes \Gamma_i$, where $\tilde{W}_i$ is the Weyl group associated to $\Delta_i$ and $\Gamma_i$ is a group of translations of $E_i$ of rank equal to $\dim E_i$.

**Proposition 5.4.** Let $i$ be an index such that $\tilde{W}_i$ is an affine Weyl group acting on $E_i$ as in item (e) of Proposition 5.3. Then $\mathcal{E}(E_i)\tilde{W}_i$ contains a free polynomial subalgebra $A$ on $\dim E_i$ generators which is dense in the sup-norm.

**Proof.** The following argument is extracted from [12], Theorem 7.6 and Corollary 7.7. Since the index $i$ is fixed, throughout the proof we will drop it from the notation; we also identify $E_i$ with $\mathbb{R}^m$. Now the affine Weyl group $\tilde{W}$ is the semi-direct product $\tilde{W} \times \Gamma$, where $\tilde{W}$ is the isotropy subgroup at zero and $\Gamma$ is a lattice of translations of $\mathbb{R}^m$. Here zero is assumed to belong to one hyperplane from each family of parallel singular hyperplanes. Since $\Gamma$ is a normal subgroup of $\tilde{W}$, the algebra of $\tilde{W}$-invariant smooth functions on $\mathbb{R}^m$ can be written

$$\mathcal{E}((\mathbb{R}^m)\tilde{W}) \cong (\mathcal{E}(\mathbb{R}^m)\Gamma)\tilde{W} \cong \mathcal{E}(\mathbb{R}^m/\Gamma)\tilde{W}.$$

Note that $\mathbb{R}^m/\Gamma$ is a compact torus. Since $\tilde{W}$ is a finite group, $\mathcal{E}(\mathbb{R}^m/\Gamma)\tilde{W}$ separates the $\tilde{W}$-orbits in $\mathbb{R}^m/\Gamma$, and this implies that $\mathcal{E}(\mathbb{R}^m)\tilde{W}$ separates the $\tilde{W}$-orbits in $\mathbb{R}^m$.

Denote by $\Gamma^*$ the dual lattice of $\Gamma$ in $\mathbb{R}^{m*}$. For each $\gamma \in \Gamma^*$, $x \in \mathbb{R}^m \mapsto e^{2\pi \sqrt{-1}\gamma(x)} \in \mathbb{C}$ induces a complex-valued smooth function on $\mathbb{R}^m/\Gamma$ which is denoted by $e^{2\pi \sqrt{-1}\gamma}$. Let $\mathbb{C}[\Gamma]$ denote the complex algebra consisting of finite linear combinations with complex coefficients of elements of $\{e^{2\pi \sqrt{-1}\gamma}\}_{\gamma \in \Gamma^*}$, and let $\mathbb{R}[\Gamma]$ denote the subalgebra of $\mathbb{C}[\Gamma]$ consisting of real-valued functions. Since $\{e^{2\pi \sqrt{-1}\gamma}\}_{\gamma \in \Gamma^*}$ is an orthogonal basis of $L^2(\mathbb{R}^m/\Gamma)$ consisting of eigenfunctions of the Laplacian of $\mathbb{R}^m/\Gamma$, it is known that $\mathbb{C}[\Gamma]$ is a dense subalgebra of $\mathcal{E}(\mathbb{R}^m/\Gamma) \otimes \mathbb{C}$ with respect to the sup-norm. By taking real parts and averaging, we get that $\mathcal{A} := \mathbb{R}[\Gamma]\tilde{W}$ is a dense subalgebra of $\mathcal{E}(\mathbb{R}^m/\Gamma)\tilde{W}$. It remains to prove that $\mathcal{A}$ is a free polynomial algebra on $m$ generators.
The $W$-action on $\mathcal{E}(\mathbb{R}^m/\Gamma)$ extends $\mathbb{C}$-linearly to $\mathcal{E}(\mathbb{R}^m/\Gamma) \otimes \mathbb{C}$, and, it follows from [7], Ch. VI §3 Théorème 1, that $\mathbb{C}[\Gamma]^{\mathbb{W}}$ is a free polynomial algebra on $m$ generators. Moreover, as explained in that book, the generators can be chosen in a special way, as follows.

Let $\Delta$ be the root system in $\mathbb{R}^{m*}$ associated to $W$. For each root $\alpha \in \Delta$, the corresponding inverse root $\overline{\alpha} \in \mathbb{R}^m$ is defined as being $\overline{\alpha} = 2h_\alpha/||h_\alpha||^2$, where $h_\alpha$ is the element of $\mathbb{R}^m$ satisfying $\langle h_\alpha, x \rangle = \alpha(x)$ for all $x \in \mathbb{R}^m$. Identifying the translations of $\Gamma$ with elements of $\mathbb{R}^m$, we have that the lattice of inverse roots coincides with $\Gamma$ [7, Ch. VI §2 Proposition 1]. It follows that $\Gamma^*$ coincides with the dual lattice of the lattice of inverse roots, which is by definition the lattice of weights. Chosen a Weyl chamber, there is a distinguished basis $\gamma_1, \ldots, \gamma_n$ of $\Gamma^*$ whose elements are called the fundamental weights. The $W$-action on $\mathbb{R}^{m*}$ permutes the elements of $\Gamma^*$, and, plainly, $w \cdot e^{2\pi i \sqrt{-1} \gamma} = e^{2\pi i \sqrt{-1}(w \cdot \gamma)}$ for $w \in W$. The averaging operator is the $\mathbb{C}$-linear map $S : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]^{\mathbb{W}}$ given by $S(e^{2\pi i \sqrt{-1} \gamma}) = \frac{1}{|W|} \sum_{w \in W} e^{2\pi i \sqrt{-1}(w \cdot \gamma)}$. Then free generators $x_1, \ldots, x_n$ of $\mathbb{C}[\Gamma]^{\mathbb{W}}$ can be taken to be $x_i = S(e^{2\pi i \sqrt{-1} \gamma_i})$.

Finally, we construct free generators for $A$. Note that $-\gamma_i$ and $\gamma_i$ belong to the same $W$-orbit if and only if $-\gamma$ and $\gamma$ belong to the same $W$-orbit for all $\gamma$ in the $W$-orbit of $\gamma_i$, and this holds if and only if $x_i$ is real-valued. In general, according to [8], there is an involution $\varrho$ of the set $\{1, \ldots, n\}$ such that $-\gamma_i \in \overline{W}(\gamma_{\varrho(i)})$. By rearranging the indices, we may thus assume that $\varrho(i) = i$ for $i = 1, \ldots, p$ and $\varrho(p + i) = p + q + i$ for $i = 1, \ldots, q$, where $p + 2q = n$. It follows that $x_1, \ldots, x_p$ are real-valued, and $x_{p+i}, x_{p+q+i}$ are complex-conjugate for $i = 1, \ldots, q$. Therefore real-valued free generators $y_1, \ldots, y_n$ of $\mathbb{C}[\Gamma]^{\mathbb{W}}$ can be chosen so that $y_1 = x_i$ for $i = 1, \ldots, p$ and $y_{p+i} = \Re x_{p+i}, y_{p+q+i} = \Im x_{p+q+i}$ for $i = 1, \ldots, q$. It is clear that $y_1, \ldots, y_n$ generate $A$ as an algebra over $\mathbb{R}$.

**Proposition 5.5.** Let $\Xi$ be the Coxeter group defined in Proposition 5.2. Then there exists a map $\Xi : \mathbb{R}^n \to \mathbb{R}^n$ which is $W$-invariant and separates the $W$-orbits.

**Proof.** As remarked in Proposition 5.3, the Coxeter group $\Xi$ is the direct product of irreducible Coxeter groups $\Xi_i$ with $1 \leq i \leq s$. If $\Xi_i$ is finite, follows from Chevalley [10] that there exists a map $\Xi_i : E_i \to E_i$ which is $\Xi_i$ invariant and separates the $\Xi_i$-orbits. On the other hand, if $\Xi_i$ is infinite, Proposition 5.3 implies the existence of a map $\Xi_i$ with the same properties as above. Finally we define $\Xi : E_0 \oplus E_1 \oplus \cdots \oplus E_s \to E_0 \oplus E_1 \oplus \cdots \oplus E_s$ to be $\Xi(x_0, x_1, \ldots, x_s) = (x_0, \Xi_1(x_1), \ldots, \Xi_s(x_s))$. \hfill \square

**Proof of Theorem 1.4.** It follows from Proposition 5.2 b) and Proposition 5.5 that there exists a map $\Xi : \Sigma \to \mathbb{R}^n$ such that $\Xi \circ \Pi = \Xi$. Now Proposition 5.3 and Proposition 5.2 c) imply that the map $\Xi$ is $W_\Sigma$-invariant, where
$W_\Sigma$ is the Weyl pseudogroup of $\Sigma$. We can apply Theorem 1.2 to extend the map $\hat{F}$ to a map $F: M \to \mathbb{R}^n$ such that the leaves of $\mathcal{F}$ coincide with the level sets of $F$. Finally, the transnormality of $F$ follows from the fact that the regular leaves of $\mathcal{F}$ form a riemannian foliation with sections (see Molino [16, p. 77]), and this finishes the proof of the theorem. □

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