ÉTALE COVERS AND LOCAL ALGEBRAIC FUNDAMENTAL GROUPS

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Abstract. Let $X$ be a normal noetherian scheme and $Z \subseteq X$ a closed subset of codimension $\geq 2$. We consider here the local obstructions to the map $\hat{\pi}_1(X \setminus Z) \to \hat{\pi}_1(X)$ being an isomorphism. Assuming $X$ has a regular alteration, we prove the equivalence of the obstructions being finite and the existence of a Galois quasi-étale cover of $X$, where the corresponding map on fundamental groups is an isomorphism.

1. Introduction

Suppose that $X$ is a normal variety over $\mathbb{C}$ and $Z \subseteq X$ is a closed subset of codimension 2 or more. Then a natural question to pose is whether the surjective map of fundamental groups $\pi_1(X \setminus Z) \to \pi_1(X)$ is an isomorphism. For general normal schemes we can ask the same question for étale fundamental groups. For a regular scheme the Zariski-Nagata theorem on purity of the branch locus implies the above map on étale fundamental groups is an isomorphism (see [10], [8]). For a general normal scheme however this map need not be an isomorphism so that étale covers of $X \setminus Z$ need not extend to all of $X$.

The next question to ask is what are the obstructions to the above map being an isomorphism. Restricting any cover to a neighborhood of a point, we see that in order for it to be étale, it must restrict to an étale cover locally. Hence each point of $Z$ gives rise to a possible obstruction determined by the image of the local étale fundamental group into the étale fundamental group of $X \setminus Z$. Assuming these all vanish, the above map will be an isomorphism.

Even if they do not vanish, we can still hope for something in the case where all the obstructions are finite. A first guess might be that this would imply that the kernel of the above map is finite, yet Example 1 of the singular Kummer surface shows that this can be far from true in general. What is true however is that after a finite cover that is étale in codimension 1 the corresponding map is an isomorphism. The main theorem here is that this is in fact equivalent to finiteness of the obstructions and a couple other similar conditions:

Theorem 1. Suppose that $X$ is a normal noetherian scheme of finite type over an excellent base $B$ of dimension $\leq 2$. Let $Z = \text{Sing}(X) \subseteq X$. Then the following are equivalent.

(i) For every geometric point $x \in Z$ the image $G_x := \text{im}[\pi_1^\text{ét}(X_x \setminus Z_x) \to \pi_1^\text{ét}(X \setminus Z)]$ is finite (see §2 for definitions of $X_x$ and $Z_x$).

(ii) There exists a finite index closed normal subgroup $H \subseteq \pi_1^\text{ét}(X \setminus Z)$ such that $G_x \cap H$ is trivial for every geometric point $x \in X$.

(iii) For every tower of quasi-étale Galois covers of $X$

$$X \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$$
$X_{i+1} \rightarrow X_i$ are étale for $i$ sufficiently large.

(iv) There exists a finite, Galois, quasi-étale cover $Y \rightarrow X$ by a normal scheme $Y$ such that any étale cover of $Y_{\text{reg}}$ extends to an étale cover of $Y$.

As hinted to above, we can rephrase the question of the $\pi_1^{\text{ét}}(X\setminus Z) \rightarrow \pi_1^{\text{ét}}(X)$ being an isomorphism as a purity of the branch locus statement of $X$. Both are equivalent to the fact that any étale cover of $X\setminus Z$ extends to an étale cover of $X$. By purity of the branch locus for $X_{\text{reg}}$, any étale cover of $X\setminus Z$ will at least extend to $X_{\text{reg}}$. Hence it is enough to consider the case where $Z = X_{\text{reg}}$, as we have done in the theorem. From this point of view, (iv) says that we can obtain purity after a finite, Galois, quasi-étale cover.

The following example of the singular Kummer surface then elucidates what is going on in the above theorem.

**Example 1.** Consider the quotient $\pi : A \rightarrow A/\pm = X$, where $A$ is an abelian surface and $X$ is a singular Kummer surface over $\mathbb{C}$. Away from the 16 2-torsion points this map is étale, but at the 2-torsion points it ramifies. Each of these 2-torsion points gives rise to a nontrivial $\mathbb{Z}/2\mathbb{Z}$ obstruction. In particular any étale cover of $A$ will give a cover of $X$ not satisfying purity of the branch locus, and in particular there are infinitely many such covers. On the other hand the fundamental group of $X$ is trivial, which can be seen as $X$ will be diffeomorphic to a standard Kummer surface given as a singular nodal quartic in $\mathbb{P}^3$ with the maximum number of nodes. In particular its étale fundamental group is trivial, and there are no étale covers of $X$. Note that on the other hand as $A$ is smooth we obtain purity on a finite cover of $X$ that is étale away from a set of codimension 2 as in (iv) of the above theorem. In this sense, although the kernel of the map $\hat{\pi}_1(X_{\text{reg}}) \rightarrow \hat{\pi}_1(X)$ is large it is not far away from satisfying purity.

The recent history of studying these two problems started with a paper of Xu [9] showing that the local obstructions are finite for all klt singularities over $\mathbb{C}$. From this Greb, Kebekus, and Peternell [5] were able to show the global statements of (iii) and (iv) in the above theorem for klt singularities. Their proof used essentially the existence of a Whitney stratification, which allowed them to check only finitely many strata to prove a cover is étale. Then in positive characteristic, Caravajal-Rojas, Schwede, and Tucker [3] proved again that the local obstructions are finite for strongly $F$-regular singularities (which are considered a close analogue of klt singularities in positive characteristic). Using a bound on the size of the local fundamental groups from this paper, Bhatt, Carvalaj-Rojas, Graf, Tucker and Schwede [1] were able to construct a stratification that enabled them to run a similar local to global argument to deduce statements of the form (ii), (iii), and (iv) for strongly $F$-regular varieties. It is also worth noting in the recent preprint of Bhatt, Gabber, and Olsson [2] they are able to reprove the results in characteristic 0 by spreading out to characteristic $p$.

The proof of Theorem 1 can be broken down essentially into two parts. One is the construction of a stratification that allows us to deal with only finitely many obstructions. The second is a completely group theoretic fact about profinite groups, namely if we have a finite collection of finite subgroups of a profinite group, then there exists a closed finite index subgroup which intersects all of these groups trivially. Note that the assumption that the covers are Galois and the finite index subgroup is normal is essential. In fact due
to the choice of basepoint that we have suppressed above (note that $x$ is not the basepoint of these fundamental groups), $G_x$ is actually only a conjugacy class of a finite subgroup inside of the group $\pi_1^{et}(X\setminus Z)$.

Finally it is worth noting that in Xu’s paper a different local fundamental group, $\pi_1^{et}(X_x\setminus \{x\})$ was shown to be finite. If we instead defined our obstruction groups $G_x$ as Xu did then in fact the theorem is false, as will be shown in the following example. There are two ways around this for klt singularities: either showing the larger fundamental groups $\pi_1^{et}(X_x\setminus Z_x)$ are finite in the klt case, or show that a similar implication (i) $\Rightarrow$ (ii),(iii) will hold as long as the smaller local fundamental groups of all covers étale in codimension 1 remain finite, which is the case for klt singularities. We will discuss this issue further in the last section.

Example 2. Consider $X = CS$ the cone over a Kummer surface $S = A/\pm$ where $A$ is an abelian surface. Then there are three types of singular points:

First consider the case where $x$ is the generic point of the cone over one of the nodes. Then $X_x$ has a regular double quasi-étale cover ramifying at $x$ (note that since we have localized there is no difference between the two possible fundamental groups). This shows $\hat{\pi}_{1}^{loc}(X_x\setminus \{x\}) = \hat{\pi}_{1}^{loc}(X_x\setminus Z_x) \cong \mathbb{Z}/2\mathbb{Z}$, and there is no ambiguity in which definition we choose. In general this will work for the generic point of any irreducible component of the singular locus.

The next type of point where we start to see a difference is when $x$ is a closed point in the cone over a node of $S$. Then in this case $\hat{\pi}_{1}^{loc}(X_x\setminus \{x\})$ is trivial while $\hat{\pi}_{1}^{loc}(X_x\setminus \text{Sing}(X)_x) \cong \mathbb{Z}/2\mathbb{Z}$. Although they are different they are at least both finite. On the other hand if we desire for these groups to behave well under specialization it is clear that $\hat{\pi}_{1}^{loc}(X_x\setminus \text{Sing}(X)_x)$ is the better choice.

The last type of point, where the real problem occurs, is the cone point $x \in CS$. First consider the fundamental group $\hat{\pi}_{1}^{loc}(X_x\setminus \{x\})$. Then this will be isomorphic to $\hat{\pi}_{1}(S) \cong 0$ by the Lefschetz hyperplane theorem. In particular all the local fundamental groups defined in this sense are finite. So if this version of the theorem were true then any tower as above would stabilize. On the other hand $\hat{\pi}_{1}(S_{\text{reg}})$ is infinite, giving an infinite tower of cones $X = CS \leftarrow CS_1 \leftarrow CS_2 \leftarrow \cdots$ Galois over $X$ and quasi-étale. In particular finiteness of all the local fundamental groups $\hat{\pi}_{1}^{loc}(X_x\setminus \{x\})$ does not imply finiteness of the local fundamental groups $\hat{\pi}_{1}^{loc}(X_x\setminus \text{Sing}(X)_x)$. Note that in this case the singularity at the origin is not klt.

Notation. Given a finite morphism $f : Y \to X$ the branch locus, written $\text{Branch}(f)$, is the locus over which $f$ fails to be étale. A finite morphism $f : Y \to X$ is quasi-étale if it is étale in codimension 1 or in other words the branch locus has codimension $\geq 2$. By purity of the branch locus for normal schemes quasi-étale is equivalent to being étale over the regular locus.

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2. LOCAL FUNDAMENTAL GROUPS

In this section we review some basic facts and definitions about the fundamental groups we will be considering.

**Definition.** Suppose that \((R, m)\) is a strictly Henselian local normal domain and that \(Z \subseteq \text{Spec}(R)\) is a closed subset of codimension \(\geq 2\). Then we define the algebraic local fundamental with respect to \(Z\) to be \(\pi^\text{ét}_{1}(\text{Spec}(R)/\text{slash.right}Z)\). If \(x\) is a normal geometric point of an irreducible scheme \(X\) and \(Z \subseteq X\) has codimension \(\geq 2\), we define the local space \(X_{x} = \text{Spec}(\mathcal{O}_{X,x})\), the spectrum of the strict Henselization of the local ring. This comes with a map \(\iota: X_{x} \to X\) and we define \(Z_{x} = \iota^{-1}(Z)\). We then define the local fundamental group at \(x\) with respect to \(Z\) to be the algebraic local fundamental group of the strict Henselization of \(\mathcal{O}_{X,x}\) with respect to the closed set \(Z\) and use the notation \(\hat{\pi}_{1}^\text{loc}(X_{x}/\text{slash.right}Z_{x})\). For any geometric point \(x \in Z\) we define \(G_{x} := \text{im}[\hat{\pi}_{1}^\text{loc}(X_{x}/Z_{x}) \to \hat{\pi}_{1}(X/Z)]\), the obstructions occurring in theorem 1.

**Note.** The definition above depends on a choice of both strict Henselization (requiring a choice of separable closure of the residue field of \(x\)) and a choice of base point. In particular the choice of base point implies that the groups \(G_{x}\) are only defined up to conjugacy. It is for this reason that we need to take Galois morphisms in the main theorem. For counterexamples when the morphisms are not Galois see \([5]\). Also note the difference between the definition here and that used in \([9]\).

The following basic lemma shows the purpose in using Henselizations when defining the local fundamental group.

**Lemma 1** ([1] Claim 3.5). Suppose that \(f : Y \to X\) is a quasi-étale morphism of normal schemes, that is étale away from some subset \(Z\) of codimension \(\geq 2\). Then \(f\) is étale over a geometric point \(x \in X\) if and only if the pull back of the map to \(U_{x} := X_{x}/\text{slash.right}Z_{x}\) is trivial.

**Proof.** It is enough to prove that the map is étale once we pull back to the strict Henselization of the local ring. Now in this case if the map \(f\) is étale then it induces a trivial cover of \(\text{Spec}(\mathcal{O}_{X,x})\) and hence of the open set \(V\). On the other hand if the cover of \(U_{x}\) is trivial, then so is the cover of \(\text{Spec}(\mathcal{O}_{X,x})\) since the varieties are normal. Hence the morphism is étale. \(\square\)

3. THE BRANCH LOCUS OF A QUASI-ÉTALE MORPHISM

In this section we consider the question of where a quasi-étale cover of a normal variety \(X\) branches. We will see that there are finitely many locally closed subsets of \(X\) such that any branch locus is the union of some subcollection of these subsets. Moreover we will show that such a stratification is possible to compute in terms of any alteration. We start with the criterion for telling if a morphism is étale from an alteration.

**Alterations.** ([4] A map \(\pi : \tilde{X} \to X\) is an alteration if it is proper, dominant and generically finite. A regular alteration will be an alteration where \(\tilde{X}\) is regular. In his work, de Jong showed that regular alterations exist for noetherian schemes of finite type over an excellent base of dimension \(\leq 2\). We will say that a divisor \(E \subset \tilde{X}\) is exceptional if \(\pi(E)\)
Lemma 2. Suppose that \( X \) is normal Noetherian scheme, with a regular alteration \( \pi : \hat{X} \to X \). Let \( f : Y \to X \) be a finite morphism of Noetherian schemes, and denote by \( \tilde{f} : \tilde{Y} \to \hat{X} \) the normalized fiber product of the maps. Then \( f \) is \'etale if and only if \( \tilde{f} \) is \'etale and for any geometric point \( x \in X \), \( \tilde{f} \) induces a trivial cover of \( \pi^{-1}(x) \).

Proof. First suppose that \( f : Y \to X \) is \'etale. Then the base change \( Y \times_X \hat{X} \to \hat{X} \) is \'etale, so that the fiber product was already normal. Hence it follows \( \tilde{f} \) is \'etale. Then since \( \tilde{Y} \) is just the fiber product and \( f \) is \'etale, for any of the \( d \) points \( q \in Y \) mapping to \( p \in X \) we see that \( \sigma^{-1}(q) \cong \pi^{-1}(p) \times_{k(p)} k(q) \). Therefore \( \tilde{f} \) is just the trivial degree \( d \) cover on every fiber \( \pi^{-1}(p) \).

Now suppose that \( \tilde{f} \) is \'etale and induces a trivial cover on every fiber of \( \pi \). Then in particular for any \( x \in X \), \( f^{-1}(x) \) will have \( \deg(f) \) geometric connected components. In particular it must be \'etale at \( x \) ([7], V.7).

Example 3. Each of the two conditions in the lemma above are easily seen to be necessary. For example we can let \( X \) be the cone over a smooth conic. This has a quasi-\'etale double cover \( f : \mathbb{A}^2 \to X \). Blowing up the origins gives a map of the normalized fiber-products \( \text{BL}_0 \mathbb{A}^2 \to \text{BL}_0 X \), that will ramify along the exceptional divisors.

On the other hand we can take the \( X \) to be the cone over an elliptic curve \( E \). Take an \'etale cover \( E' \to E \), which will induce a quasi-\'etale cover \( X' \to X \) of their cones. After blowing up the origins we obtain the map of normalized fiber products which is \'etale, but induces a nontrivial cover of the exceptional divisors.

Using the above lemma we can check whether \( f \) is \'etale based on a single regular alteration. Our next goal will be to show that based on this alteration we really only need to check that \( f \) is \'etale at finitely many points. To identify what are the points we need to check we require the following condition, which roughly says that the reduced fibers of a morphism fit together in a flat family.

Condition *. Suppose that \( g : Z \to S \) is a proper morphism of Noetherian schemes with \( S \) integral. Then \( g \) satisfies this condition if there exists a purely inseparable morphism \( i : S' \to S \) such that if \( Z' = Z \times_S S' \to S' \) is the base change, then \( Z'_{\text{red}} \to S' \) is flat with geometrically reduced fibers.

We now show that there exists a stratification of \( X \) such that \( \pi \) will satisfy the above condition * over each of the strata.

Lemma 3. Let \( \pi : \hat{X} \to X \) be a morphism of Noetherian schemes. Then there exists a stratification \( X = \bigcup S_i \), where the \( S_i \) are irreducible locally closed subsets, such that \( \pi^{-1}(S_i) \to S_i \) satisfies condition (*) for all \( i \).

Proof. We will proceed by Noetherian induction on \( X \). Take an irreducible component \( S \) of \( X \). Consider the map \( \pi^{-1}(S)_{\text{red}} \to S \). Taking an irreducible component \( W \) of \( \pi^{-1}(S)_{\text{red}} \) if \( W \to S \) is not separable, we can take some high enough power of the Frobenius so that the pullback by the map is separable. Doing this for every irreducible component of
\( \pi^{-1}(S)_{\text{red}} \) we may assume that the general fiber is reduced. Then taking an open subset \( U \) of \( S \) we may assume that every fiber of \( \pi^{-1}(U)_{\text{red}} \to U \) is reduced and that this morphism is flat. Continuing on will give the desired stratification.

**Lemma 4** (e.g. [3] 7.8.6). Suppose that \( g : Z \to S \) is a morphism of Noetherian schemes satisfying condition \((*)\) with \( S \) integral. Then the number of connected components of geometric fibers are constant.

**Proof.** We have a purely inseparable morphism \( S' \to S \) such that \( Z' \to S' \) is flat with geometrically reduced fibers. Since \( S' \to S \) is a universal homeomorphism, it follows that \( Z' \to Z \) is a homeomorphism. Hence the number of connected components remains the same, so we can assume from the beginning that \( Z \to S \) is flat with geometrically reduced fibers.

Now in this case we will show that the Stein factorization of \( g : Z \to S \) factors as \( Z \to \hat{S} \to S \) where \( \hat{S} \to S \) is étale. Taking the strict Henselization of the local ring at any point we can reduce to the case where \( S \) is the spectrum of a strictly Henselian local ring. In this case \( \hat{S} \) is a product of finitely many local rings. Our goal is to show that these are isomorphic to \( S \). Now consider a connected component \( W \) of \( Z \), so that the map \( g : W \to S \) is flat and proper, with geometrically reduced fibers. Now since \( W \) is connected and \( S \) is the spectrum of strictly Henselian ring, the special fiber \( W_0 \) is also connected. But then since \( W_0 \) is reduced \( H^0(W_0, O_{W_0}) = k(0) \). Hence we see by the theorem of Grauert that \( O_S \to g_* O_W \) is an isomorphism. This implies that \( \hat{S} \to S \) is thus étale, so in particular the number of connected components of the geometric fibers are constant.

**Theorem 2.** Suppose that \( X \) is a normal Noetherian scheme and \( \pi : \tilde{X} \to X \) a regular alteration. Then there exists a stratification \( X = \bigcup_{i \in I} Z_i \) into locally closed subsets such that for any \( f : Y \to X \) quasi-étale, with \( Y \) a normal Noetherian scheme, \( \text{Branch}(f) = \bigcup_{i \in J \subseteq I} Z_i \).

**Proof.** The above lemma gives a stratification \( X = \bigcup_i S_i \) such that \( \pi^{-1}(S_i) \to S_i \) satisfies condition \((*)\). Moreover we a finite number of exceptional divisors \( E_i \) giving closed subsets \( \pi(E_i) \) on \( X \). Putting these together gives our desired stratification of \( X \). Our goal is then to show that any branch locus of a quasi-étale morphism is a union of these strata.

Consider \( \tilde{Y} = (X \times_X Y)^n \) the normalized fiber product which comes with a morphism \( \tilde{f} : \tilde{Y} \to \tilde{X} \) that is étale away from the exceptional locus. Now by purity of the branch locus \( \text{Branch}(\tilde{f}) = \bigcup_i E_i \) where the \( E_i \) are some subset of the exceptional divisors. In particular the branch locus of \( f \) will include \( B = \bigcup_i \pi(E_i) \), which will be a union of some strata. Now looking on the complement of \( B \), and replacing \( X \) by \( X \setminus B \) we can assume that \( \tilde{f} \) is in fact étale. In particular \( \tilde{f}^{-1}(\pi^{-1}(S_i)) \to \pi^{-1}(S_i) \to S_i \) satisfies condition \((*)\). Hence the number of connected components of the fibers are constant. This implies that for any point \( s \in S_i \) that if the cover of \( \pi^{-1}(s) \) is geometrically trivial, then the corresponding cover for other point in \( S_i \) is also trivial. Hence we see that the branch locus must be a union of the strata.

**Remark.** In the proof of (i) implying (iii) of the main theorem it would be nice to apply this theorem directly on \( X \). However when we take the normalized pullback of
an alteration we may not get another alteration. To remedy this we will need to take alterations of varieties that are further along in the tower.

4. Proof of the Main Theorem

In this section we prove the different implications in the main theorem.

(i) ⇒ (ii).

Proof. Consider a regular alteration $\pi : \hat{X} \to X$. This will give us a stratification $X = \bigcup_i Z_i$. Now for each of the finitely many generic points $\eta_i$ of the different strata consider the finitely many finite groups $G_i = G_{\eta_i}$. Then as $\pi_1^{\text{et}}(U)$ is profinite there exists some finite index closed normal subgroup $H$ intersecting all of these $G_i$ trivially. This corresponds to a quasi-étale cover $\gamma : Y \to X$ that is étale over $U$. Moreover by our choice of stratification for any geometric point $x$ we will also have that $G_x \cap H$ is trivial. Hence such a finite index normal subgroup $H$ can be taken uniformly for all $x \in X$. □

(ii) ⇒ (i).

Proof. Our assumption (ii) gives a closed finite index normal subgroup $H \subseteq \pi_1^{\text{et}}(U)$ such that $G_x \cap H = \{1\}$. Then in particular $G_x \cong G_x/H \subseteq \pi_1^{\text{et}}(U)/H$ which is finite. Hence $G_x$ is finite as well. □

(i) ⇒ (iii).

Proof. We proceed by Noetherian induction. Consider our tower of finite morphisms denoted by $\gamma_k : X_{k+1} \to X_k$, and consider the collection $\mathcal{U}$ of open sets $U \subseteq X$ such that when we restrict the tower over $U$ the morphisms are eventually étale. The assumption that all the morphisms are quasi-étale implies that $X_{\text{reg}} \in \mathcal{U}$. Since $X$ is assumed to be Noetherian this collection has a maximal element and our goal is to show that this must be all of $X$.

Therefore we need to show that if $U \in \mathcal{U}$ and $U \neq X$ then we can find a larger $U' \in \mathcal{U}$. To do this take any $x$ a generic point of an irreducible component of $X \setminus U$. Consider $X_x = \text{Spec}(\mathcal{O}_{X,\eta})$ and restrict the tower of $X_i$ over $X_x$ to get a tower

$$\text{Spec}(\mathcal{O}_{X,\eta}) = X_{x,0} \leftarrow X_{x,1} \leftarrow X_{x,2} \leftarrow X_{x,3} \leftarrow \cdots$$

Now using the assumption (i) applied to the point $x$, it follows that eventually the covers will be trivial when restricted over the regular locus and hence will be étale. This then shows that there exists some $N \gg 0$ such that $\gamma_n$ is étale over $\eta$ for $n \geq N$ and they are étale over the open set $U$ coming from Noetherian induction.

Now take a regular alteration $\pi : \hat{X}_N \to X_N$. Then using $\pi$ we construct a stratification $X_n = \bigcup_i Z_i$ as before. Then any of the maps $X_{N+k} \to X_N$ must be étale over $U$ and $\eta$. But because the branch locus must be a union of strata it follows that these are all étale over some open set $U' \ni U$ with $U' \ni \eta$. Hence such a larger $U' \in \mathcal{U}$ exists and by Noetherian induction we see that $X \in \mathcal{U}$. This proves property (iii). □

(iii) ⇒ (iv).
Proof. Assuming that no such cover exists, we inductively construct a tower $X \leftarrow X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$ as in (iii) of the main theorem using Galois closures, such that none of the $X_{i+1} \rightarrow X_i$ are étale. This will contradict our assumption, so eventually every étale cover of one of the $X_{i,\text{reg}}$ will extend to an étale cover of $X_i$. This gives the desired cover satisfying purity.

(iv) $\Rightarrow$ (i).

Proof. Consider a geometric point $x$ of $X$. Take a cover $f : Y \rightarrow X$ as in (iv), and a geometric point $y$ of $Y$ mapping to $x$. Denote by $U$ the regular locus of $X$ and $Z = X \setminus U$ the singular locus. This gives rise to the following commutative diagram of fundamental groups.

$$
\begin{array}{ccc}
\hat{\pi}^\text{loc}_1(Y \setminus f^{-1}(Z)) & \longrightarrow & \hat{\pi}_1(f^{-1}(U)) \\
\downarrow & & \downarrow \\
\hat{\pi}^\text{loc}_1(X \setminus Z) & \longrightarrow & \hat{\pi}_1(U)
\end{array}
$$

Now the assumption on $Y$ implies that the top map is zero. On the other hand, the image of the map on the left is a finite index normal subgroup. Hence looking at the images in $\hat{\pi}_1(U)$, we see that $G_x$ has a trivial finite index subgroup and hence must be finite.

5. Applications

Using our main theorem we can recover the results of [5] and [11].

**Corollary 1.** Suppose that $X$ is a normal klt variety over $\mathbb{C}$. Then $X$ satisfies the condition (ii).

Proof. We want to show that $X$ satisfies condition (i). There are two issues to deal with if we wish to apply Xu’s result [9]. First is the problem that in this paper the local fundamental groups are defined in terms of links instead of the local spaces $X_x$ given by Henselization. The second is that Xu proves the finiteness of $\hat{\pi}^\text{loc}_1(X \setminus \{x\})$ and we saw that this is not enough to guarantee (ii) in general.

There are two ways to get around this. The first is to strengthen the result of Xu to prove the finiteness of algebraic local fundamental groups as considered in this paper. In the proof of his main theorem, Xu cuts down to a surface. It is then possible to consider only quasi-étale covers instead of étale covers of $X_x \setminus \{x\}$, as these will agree after cutting down. Also you would need an equivalence of the algebraic local fundamental group defined in terms of links and Henselizations. Once this is done though (ii) will follow immediately from the main theorem. Note that also the recent result of [2] is strong enough to apply directly.

The second way to prove this is to note that we can get around the issue of which fundamental group we consider when we work in a class of normal varieties $\mathcal{R}$ satisfying the following. We want for every $X \in \mathcal{R}$, and every quasi-étale cover $Y \rightarrow X$ that $Y \in \mathcal{R}$, and also for every $x$ a geometric point of $X \in \mathcal{R}$ that $\hat{\pi}^\text{loc}_1(X_x \setminus \{x\})$ is finite. In particular klt singularities satisfy both these conditions by [9]. Then under these assumptions, the same argument for (i) implies (ii) works with the fundamental groups $\hat{\pi}^\text{loc}_1(X_x \setminus \{x\})$. This approach is used in [5].

□
Corollary 2. Suppose that $X$ is a normal $F$-finite strongly $F$-regular variety over a field of characteristic $p$. Then $X$ satisfies the condition (ii).

Proof. In this case the result of Carvajal-Rojas, Schwede, and Tucker [3] applies directly to the main theorem without any changes. □

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