RANDOM WALK WITH EQUIDISTANT MULTIPLE FUNCTION BARRIERS

THEO VAN UEM

ABSTRACT. We obtain expected number of arrivals, absorption probabilities and expected time before absorption for a discrete random walk on the integers with an infinite set of equidistant multiple function barriers.

1. Introduction

Random walk can be used in various disciplines: in medicine and biology where absorbing barriers give a natural model for a wide variety of phenomena, in physics as a simplified model of Brownian motion, in ecology to describe individual animal movements and population dynamics. Random walks have been studied for decades on regular structures such as lattices. Percus [1] considers an asymmetric random walk, with one or two boundaries, on a one-dimensional lattice. At the boundaries, the walker is either absorbed or reflected back to the system. Using generating functions the probability distribution of being at position m after n steps is obtained, as well as the mean number of steps before absorption. El-Shehawey [2] [3] obtains absorption probabilities at the boundaries for a random walk between one or two partially absorbing boundaries as well as the conditional mean for the number of steps before stopping given the absorption at a specified barrier, using conditional probabilities. In this paper we obtain expected number of arrivals, absorption probabilities and expected time before absorption for a discrete random walk on the integers with an infinite set of equidistant multiple function barriers. A multiple function barrier (MFB) is a state that can absorb, reflect, let through or hold for a moment. In each MFB we have probabilities $p_0, q_0, r_0, s_0$ for moving forward and backward, staying for a moment in the MFB and absorption in the MFB, where $p_0 + q_0 + r_0 + s_0 = 1$, $p_0 q_0 s_0 > 0$. MFB’s of type $p_0 q_0 r_0 s_0$ are defined in each barrier $kN$ ($k \in \mathbb{Z}$, $N > 1$). The random walk between the MFB’s is of $pqr$ type, where $p$ is the one-step forward probability, $q$ one-step backward probability ($pq > 0$) and $r = 1 - p - q$ the probability to stay for a moment in the same position. We start in $i_0$ ($0 \leq i_0 < N$).
2. Random walk on the MFB’s

We define the expected number of arrivals in state $j$ when starting in state $i$:

$$x_j = x_{i,j} = \sum_{k=0}^{\infty} p_{ij}^{(k)}$$

Let $\rho = \frac{p}{q}$ and $\lambda_1$ and $\lambda_2$ ($\lambda_1 \geq \lambda_2$) are the solutions of

$$q\lambda^2 - (1-r)\lambda + p = 0.$$ 

If $p > q$ then $\lambda_1 = \rho, \lambda_2 = 1$. If $p < q$ then $\lambda_1 = 1, \lambda_2 = \rho$. If $p = q$ then $\lambda_1 = \lambda_2 = 1$. We start with a $pqr$ random walk on the integers:

**Lemma 1.**

(1) $x_n = \delta(n, i_0) + px_{n-1} + qx_{n+1} + rx_n \quad (n \in \mathbb{Z}) \quad \rho \neq 1$

has solution:

(2) $x_n = \begin{cases} \frac{\lambda^{n-i_0}}{\sqrt{(1-r)^2-4pq}} & (n \leq i_0) \\ \frac{\lambda^{n-i_0}}{\sqrt{(1-r)^2-4pq}} & (n \geq i_0) \end{cases}$

**Proof.** Let

$$G(s) = \sum_{k=-\infty}^{\infty} x_k s^k \quad (|s| < 1)$$

Using $\square$ we obtain:

$$G(s) = s^{i_0} + psG(s) + qs^{-1}G(s) + rG(s)$$

$$G(s) = \frac{s^{i_0}}{1 - ps - qs^{-1} - r}$$

We use the inverse z-transform: $x_n = \frac{1}{2\pi i} \oint H(z)z^{n-1}dz$, where the integration is along the circle $|z| = 1$ and anticlockwise. we have:

$$H(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n} = G(z^{-1}) = \frac{z^{-i_0}}{1 - pz^{-1} - qz^{-r}}.$$ 

So,

$$x_n = \frac{1}{2\pi i} \oint \frac{z^{n-i_0}}{1 - pz^{-1} - qz^{-r}} = \frac{1}{2\pi i} \oint \frac{-z^{n-i_0}}{q(z - \lambda_1)(z - \lambda_2)}dz$$

Apply the residue theorem. $\square$

**Theorem 2.** The random walk on the subset of equidistant MFB’s is described by the difference equations: CASE $\rho \neq 1$.

$$(\lambda_1 - \lambda_2)q_0x_{(k+1)N} + \omega_0x_{kN} + (\lambda_1 - \lambda_2)p_0\rho^{N-1}x_{(k-1)N} =$$

(3) $$(\lambda_2^{N-i_0} - \lambda_1^{N-i_0})\delta(k,0) + (\lambda_1^{i_0} - \lambda_2^{i_0})\rho^N\delta(k,1) \quad (k \in \mathbb{Z})$$

where

(4) $$\omega_0 = (\lambda_2^N - \lambda_1^N)(1-r_0) + (\lambda_1^{N-1} - \lambda_2^{N-1})(p_0 + q_0\rho)$$
CASE $\rho = 1$.

(5) $q_0 x_{(k+1)N} = (p_0 + q_0 + N s_0) x_{kN} + p_0 x_{(k-1)N} = (i_0 - N) \delta (k, 0) - i_0 \delta (k, 1)$

Proof. We start with $0 < i_0 < N$. Random walk on interval $[kN + 1, (k + 1)N - 1]$:

(6) $(1 - r) x_{kN + n} = \delta (k, 0) \delta (n, i_0) + p x_{kN + n - 1} + q x_{kN + n + 1}$ \hspace{1cm} (n = 2, 3, \ldots, N - 2)

Characteristic equation:

$$q \lambda^2 -(1-r) \lambda + p = 0$$

A general solution of (6) is (use Lemma 1):

(7) $x_{kN + n} = \begin{cases} \frac{\lambda_1^{n-i_0} \delta (k, 0)}{(1-r)^2 - 4pq} + a_k \lambda_1^n + b_k \lambda_2^n & (n = 1, \ldots, i_0) \\ \frac{\lambda_2^{n-i_0} \delta (k, 0)}{(1-r)^2 - 4pq} + a_k \lambda_1^n + b_k \lambda_2^n & (n = i_0, \ldots, N - 1) \end{cases}$

Let $\zeta = [(1-r)^2 - 4pq]^{-\frac{1}{2}}$. By focusing on states $kN + 1$ and $(k + 1)N - 1$ we get:

$x_{kN+1} = p_0 x_{kN} + q x_{kN+2} + r x_{kN+1}$

$x_{(k+1)N-1} = p x_{(k+1)N-2} + q_0 x_{(k+1)N} + r x_{(k+1)N-1}$

$$p_0 x_{kN} = p [\zeta \lambda_1^{i_0} \delta (k, 0) + a_k + b_k]$$

$$q_0 x_{(k+1)N} = q [\zeta \lambda_2^{N-i_0} \delta (k, 0) + a_k \lambda_1^N + b_k \lambda_2^N]$$

$$(\lambda_2^N - \lambda_1^N) a_k = \lambda_2^N \frac{p_0}{p} x_0 - \frac{q_0}{q} x_N + \zeta \lambda_2^N (\lambda_2^{-i_0} - \lambda_1^{-i_0}) \delta (k, 0)$$

$$(\lambda_2^N - \lambda_1^N) b_k = -\lambda_1^N \frac{p_0}{p} x_0 + \frac{q_0}{q} x_N + \zeta \lambda_1^N \lambda_2^{N-i_0} - \lambda_2^{N-i_0}) \delta (k, 0)$$

Focusing on state $kN$:

$$x_{kN} = px_{kN-1} + qx_{kN+1} + r_0 x_{kN}$$

After some calculations, we get (3)

CASE $\rho = 1$ We use the same method, where (verified by substitution):

(8) $x_{kN + n} = \begin{cases} a_k n + b_k + \frac{n-i_0}{p} & (n = 1, \ldots, i_0) \\ a_k n + b_k n & (n = i_0, \ldots, N - 1) \end{cases}$

The special case where we start in $i_0 = 0$ can be handled in the same way, resulting in (4) and (3) with $i_0 = 0$ when $\rho \neq 1$ respectively $\rho = 1$. $\Box$

Theorem 3. The RW on the MFB’s is symmetric if and only if $(i_0 = 0) \land (q_0 = p_0 \rho^{N-1})$

Proof. See (3) and (6) $\Box$

Notice that $p_0 \rho^{N-1} = q_0 \rho^{N-1}$ can be interpreted as: direct probability from a MFB to it’s right neighbor equals direct probability in the reverse direction.
3. Value of the MFB game

We define a moment generating function on the MFB’s:

\[ F(s) = \sum_{k=-\infty}^{\infty} xN^k \quad (|s| < 1) \]

**Theorem 4.** \( \text{CASE } \rho \neq 1 \):

\[ F(s) = \frac{\lambda_2^{N-i_0} - \lambda_1^{N-i_0} + (\lambda_1^{i_0} - \lambda_2^{i_0})\rho^N s}{(\lambda_1 - \lambda_2)q_0 s^{-1} + \omega_0 + (\lambda_1 - \lambda_2)p_0\rho^{N-1}s} \]

**CASE \( \rho = 1 \):**

\[ F(s) = \frac{i_0 - N - i_0 s}{q_0 s^{-1} - (p_0 + q_0 + Ns_0) + p_0 s} \]

**Proof.** Use [3] and [6].

**Theorem 5.** Probability of absorption in a MFB is 1:

\[ \sum_{k=-\infty}^{\infty} s_0 xN^k = 1 \]

**Proof.** In both cases we have: \( F(1) = \sum_{k=-\infty}^{\infty} xN^k = \frac{1}{s_0} \).

We define the value \( v \) of the MFB game as: \( v = \sum_{k=-\infty}^{\infty} kxN^k \).

**Theorem 6.** \( \text{CASE } \rho \neq 1 \):

\[ v = \frac{(\lambda_1^{N-i_0} - \lambda_2^{N-i_0})\rho^N}{(\lambda_2^N - \lambda_1^N)s_0} +\]

\[ \frac{\lambda_1 - \lambda_2)(q_0 - p_0\rho^{N-1})[\lambda_1^{i_0} - \lambda_2^{i_0})\rho^N + \lambda_2^{N-i_0} - \lambda_1^{N-i_0}]}{(\lambda_2^N - \lambda_1^N)^2 s_0^2} - \]

**CASE \( \rho = 1 \):**

\[ v = \frac{p_0 - q_0 + i_0 s_0}{N s_0^2} \]

**Proof.** \( v = \frac{dF}{ds}|_{s=1} \).

Notice that the symmetric random walk on the MFB’s has value 0.

4. Expected number of arrivals

**Theorem 7.** The expected number of arrivals to the MFB’s is: \( \text{CASE } \rho \neq 1 \):

\[ xN^k = \begin{cases} \{(\lambda_1^{N-i_0} - \lambda_2^{N-i_0})\xi_1 + \rho^N(\lambda_2^{i_0} - \lambda_1^{i_0})\}\Omega^{k-1}_{1} & (k \leq 0) \\ \{(\lambda_1^{N-i_0} - \lambda_2^{N-i_0})\xi_2 + \rho^N(\lambda_2^{i_0} - \lambda_1^{i_0})\}\Omega^{k-1}_{2} & (k \geq 1) \end{cases} \]

where

\[ (\lambda_1 - \lambda_2)q_0 \xi_i^2 + \omega_0 \xi_i + (\lambda_1 - \lambda_2)p_0\rho^{N-1} = 0 \quad (i = 1, 2) \quad \xi_1 > 1 > \xi_2 > 0 \]
Apply the residue theorem. CASE \( \rho \):

Along the same lines as in Theorem 7, using 7 and 8.

Proof. CASE \( x \)

Using 3 we get:

\[(17)\]

Theorem 8. CASE \( \rho \)= 0

Proof. CASE \( \rho \)= 0 We use the inverse z-transform:

\( x_{kN} = \frac{1}{2\pi i} \oint H(z)z^{k-1}dz \), where the integration is along the circle \(|z| = 1\) and anticlockwise. Using 8 we get:

\[
H(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n} = F(z^{-1}) = \frac{(\lambda^N_2 - \lambda^N_1)z + (\lambda^{-i}_1 - \lambda^{-i}_2)\rho^n}{(\lambda_1 - \lambda_2)q_0z^2 + \omega_0z + (\lambda_1 - \lambda_2)p_0\rho^N}
\]

So,

\[
x_n = \frac{1}{2\pi i} \oint \frac{(\lambda^N_2 - \lambda^N_1)z^n + (\lambda^{-i}_1 - \lambda^{-i}_2)\rho^nz^{n-1}}{(\lambda_1 - \lambda_2)q_0(z - \xi_1)(z - \xi_2)}dz
\]

Apply the residue theorem. CASE \( \rho = 1 \) Using 8 we get:

\[(16)\]

Use \( x_{kN} = \frac{1}{2\pi i} \oint F(z^{-1})z^{k-1}dz \) and the residue theorem.

\[
(1 - \rho^N)x_{kN+n} = \]

\[
\begin{cases}
\frac{p_0}{p}[\rho^{n-kN} - \rho^N]x_{kN} + \frac{q_0}{q}[1 - \rho^{-kN}]x_{(k+1)N} + \frac{(1-\rho^n)(\rho^{-i_0}-1)}{p-q}\delta(k,0) \\
(n = 1, \ldots, i_0) \\
\frac{p_0}{p}[\rho^{n-kN} - \rho^N]x_{kN} + \frac{q_0}{q}[1 - \rho^{-kN}]x_{(k+1)N} + \frac{(\rho^n-\rho^N)(1-\rho^{-i_0})}{p-q}\delta(k,0) \\
(n = i_0, \ldots, N - 1)
\end{cases}
\]

CASE \( \rho = 1 \):

\[(18)\]

Proof. Along the same lines as in Theorem 7 using 7 and 8.
Mean absorption time

Let \( m_i \) be the mean absorption time (in any MFB) when starting in state \( i \) where \( i \in \mathbb{Z} \).

**Theorem 9.**

\[
m_i = m_i \mod N \quad (i \in \mathbb{Z})
\]

**CASE** \( \rho \neq 1 \). If \( 0 \leq i \leq N \):

\[
m_i = \frac{N\rho^{-i}}{(q-p)(1-\rho^{-N})} + \frac{i}{q-p} + \frac{1}{s_0} + \frac{p_0 + q_0(N-1)}{(q-p)s_0} + \frac{N[p_0\rho^{-1} + q_0\rho^{1-N} + r_0 - 1]}{(q-p)(1-\rho^{-N})s_0}
\]

**CASE** \( \rho = 1 \). If \( 0 \leq i \leq N \):

\[
m_i = \frac{i(N-i)}{2p} + \frac{1}{s_0} + \frac{p_0 + q_0(N-1)}{2ps_0}
\]

**Proof.**

\[
m_i = p(m_{i+1} + 1) + q(m_{i-1} + 1) + r(m_i + 1) \quad (1 \leq i \leq N - 1)
\]

\[
m_0 = p_0(m_1 + 1) + q_0(m_{-1} + 1) + r_0(m_0 + 1) + s_0.1
\]

Because of

\[
m_i = m_i \mod N \quad (i \in \mathbb{Z})
\]

we have:

\[
(1 - r)m_i = pm_{i+1} + qm_{i-1} + 1 \quad (1 \leq i \leq N - 1)
\]

\[
m_0 = m_N
\]

\[
(1 - r_0)m_0 = p_0m_1 + q_0(m_{N-1} + 1) + 1
\]

Use \( m_i = ap^{-i} + b + \frac{i}{q-p} \) (case \( \rho \neq 1 \)) or \( m_i = ai + b - \frac{i^2}{2p} \) (case \( \rho = 1 \)) where \( 0 \leq i \leq N \) because \( m_0 \) and \( m_N \) are part of the difference pattern [19].

Notice that the results for \( \rho = 1 \) can also be obtained by applying l'Hospital's rule in the result for \( \rho \neq 1 \) (except Theorem 9 where we need l'Hospital's rule twice).

**References**

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[2] El-Shehawey M A 1992 On absorption probabilities for a random walk between two different barriers. Annales de la faculté des sciences de Toulouse 6e série, tome 1, no 1 , 95-103

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Amsterdam University of Applied Sciences, Amsterdam, The Netherlands.

Email address: tjvanuem@gmail.com