Centralized Hierarchical Coded Caching Scheme over Two-Layer Networks

Yun Kong, Youlong Wu, and Minquan Cheng

Abstract

This paper considers a hierarchical caching system where a server connects with multiple mirror sites, each connecting with a distinct set of users, and both the mirror sites and users are equipped with caching memories. Although there already exist works studying this setup and proposing coded caching scheme to reduce transmission loads, two main problems are remained to address: 1) the optimal communication load under the uncoded placement for the first hop, denoted by $R_1$, is still unknown. 2) the previous schemes are based on Maddah-Ali and Niesen’s data placement and delivery, which requires high subpacketization level. How to achieve the well tradeoff between transmission loads and subpacketization level for the hierarchical caching system is unclear. In this paper, we aim to address these two problems. We first propose a new combination structure named hierarchical placement delivery array (HPDA), which characterizes the data placement and delivery for any hierarchical caching system. Then we construct two classes of HPDAs, where the first class leads to a scheme achieving the optimal $R_1$ for some cases, and the second class requires a smaller subpacketization level at the cost of slightly increasing transmission loads.

Index Terms

hierarchical placement delivery array, hierarchical coded caching scheme, transmission load, subpacketization.

I. INTRODUCTION

With the growing data demand especially the streaming media, there exists an extreme transmission pressure during the peak traffic hours in the wireless network. It is well known that caching system is an efficient way to reduce transmission during the peak traffic hours by shifting traffic from peak to off peak hours. That is, the central server can firstly place some contents into users’ memories during the off peak traffic hours. During the peak traffic hours, the central server would only transmit the contents which have not been cached by the users. In addition, Maddah-Ali and Niesen in [1] showed that the contents cached by the users can be used to further reduce the transmission load during the peak traffic hours since these contents could generate more multicast opportunities among the users. They first introduced the centralized $(K,M,N)$ caching system where a single server having access to a library containing $N$ files is connected to $K$ cache-aided users whose cache size is $M$ files through an error-free shared-link. An $F$-division $(K,M,N)$ coded caching scheme contains two phases. During the placement phase, each file is divided into $F$ packets, where $F$ is referred as the subpacketization, and the server places at most $MF$ packets into each user’s cache without any information about users’ demands. Since the packets are directly cached by the users, this placement strategy is called uncoded placement. In the delivery phase, each user requests a file from the server randomly and the server broadcasts some coded messages to users such that all the users can decode their requesting files with the help of their cached packets. We focus on the worst case where each user requests a distinct file. The transmission amount normalized by the size of file is defined as the “transmission load”.

To further reduce the transmission load of a $(K,M,N)$ caching system, Maddah-Ali and Niesen proposed the first centralized coded caching scheme [1] (MN scheme) and the first decentralized coded caching scheme [2] (MN decentralized scheme) respectively. It is worth noting that when $N \geq K$, MN scheme has the minimum transmission load under uncoded placement in [3]. However, the subpacketization $F$ of the MN scheme increases exponentially with the growing on user number $K$. In order to design a scheme with low subpacketization, the author in [4] proposed a combination structure named placement delivery array (PDA) to simultaneously characterize the placement and delivery phase of a coded caching scheme. The MN scheme could also be depicted by PDA referred to as MN PDA. It is worth noting that MN scheme has the minimum subpacketization for the fixed minimum transmission load among all the schemes which can be realized by PDAs [5]. Besides PDA, there also exists some other combination structures to describe a coded caching scheme aiming at reducing the subpacketization level, such as [6–8], etc.
A. Two-layer hierarchical network model

In reality, the caching systems consist multiple layers of caches in most cases, which means between the central server and the end users, there are some in-between devices, and all these devices are arranged in a tree-like hierarchy with the central server at the root node while the end users act as the leaf nodes. Between each layer, the parent node communicates with its children nodes. As illustrated in Fig. 1, a \((K_1, K_2; M_1, M_2; N)\) hierarchical caching system was first studied in [9]. That is, a two-layer hierarchical network consists of a single origin server and \(K_1\) cache-aided mirror sites and \(K_2\) cache-aided users where the server hosts a collection of \(N \geq K_1K_2\) files with equal size, each mirror site and each user have memories of size \(M_1\) files and \(M_2\) files respectively where \(M_1, M_2 \leq N\). The server is connected through an error-free shared link to each mirror site. Each mirror site is connected through an error-free broadcast link to \(K_2\) users and each user is connected to only one mirror site.

Due to the complexity of the above hierarchical caching system, there are only a few studies in [9]–[11]. In fact, all the existing schemes consist of two-subsystem model controlled by two parameters \(\alpha, \beta \in [0, 1]\), where the first subsystem includes the whole cache memory size of each mirror sites, an \(\alpha\) fraction of each file and a \(\beta\) fraction of users’ cache memory size, and the second subsystem includes the rest \(1 - \alpha\) fraction of each file and a \(1 - \beta\) fraction of users’ cache memory size. In [9], the authors directly used the MN decentralized scheme in each layer. Then the requested files are built in each layer, so it results in a high transmission load of \(R_1\). This scheme is referred as KNMD scheme. Without building the whole file in each layer, the authors in [10] proposed a hybrid scheme (the ZWXWL scheme) based on two MN schemes for the two layers, so it has a lower transmission load of \(R_1\) and its \(R_2\) achieves the minimum transmission load under uncoded placement. However, the ZWXWL scheme ignores the usefulness of users’ cache in the second layer when decoding the messages sent from the server. The author in [11] proposed an improved scheme (the WWCY scheme) by utilize two MN schemes. The WWCY scheme also achieves the minimum load of \(R_2\) under uncoded placement, while the load of the first layer \(R_1\) is smaller than the ZWXWL scheme. There are some other works on hierarchical caching model, such as the coded placement for hierarchical cache-enabled network [12], [13], the intension between two layers transmission loads [14] and the topology where there are some users directly connected to the server [15] etc.

B. Contribution and paper organization

In this paper we focus on the hierarchical caching model in [9]. From the introduction in the above subsection, the existing schemes already have a good performance on \(R_2\), while designing a scheme aiming at decreasing \(R_1\) still remains open. Further, [9]–[11] all utilize the MN scheme or MN decentralized scheme, whose subpacketization increases exponentially with the growing on user number, so it is meaningful to design the scheme with minimizing the transmission load for the first layer \(R_1\) or reducing the subpacketization. According to the above points, in this paper we obtain the following main results.

- Inspired by the concept of PDA, we propose a new combination structure referred to as hierarchical placement delivery array (HPDA), which could be used to characterize the placement and delivery phase of a hierarchical coded caching scheme. So designing a hierarchical coded caching scheme is transformed to constructing an appropriate HPDA.

Fig. 1: The \((K_1, K_2; M_1, M_2; N)\) hierarchical caching system with \(N = 4, K_1 = K_2 = 2, M_1 = 2\) and \(M_2 = 1\).
• We propose a class of HPDA by dividing a MN PDA into several subarrays. Then we have a class of hierarchical coded caching schemes which achieve the lower bound of the first layer transmission load $R_1$.
• We provide a general hybrid construction of HPDA based on any two PDAs. So we can get a scheme with flexible subpacketization when we choose the base PDAs with low subpacketization. In addition if the base PDAs are MN PDAs, then the scheme realized by our HPDA is exactly the scheme in [11].

The rest of this paper is organized as follows. The hierarchical caching model and some preliminary results are introduced in Section II. We review PDA and introduce the structure of HPDA in Section III. The main results and performance analysis are listed in Section IV. The proofs of our main results are proposed in Sections V and VI. Finally we conclude this paper in Section VII.

C. Notations

The following notations are used in this paper.
• For any positive integers $a$ and $b$ with $a < b$, let $[a : b] \triangleq \{a, a + 1, \ldots, b\}$, $[a : b) \triangleq \{a, a + 1, \ldots, b - 1\}$ and $[a) \triangleq \{1, 2, \ldots, a\}$. Let $[k : t] \triangleq \{V | V \subseteq [k], |V| = t\}$, for any positive integer $t \leq b$.
• Given an array $\mathbf{P} = (p_{j,k})_{j \in [F], k \in [K]}$ with alphabet $[S] \cup \{\ast\}$, we define $\mathbf{P} + a = (p_{j,k} + a)_{j \in [F], k \in [K]}$ and $\mathbf{P} \times a = (p_{j,k} \times a)_{j \in [F], k \in [K]}$ for any integer $a$, where $a + \ast = \ast, a \times \ast = \ast$.

II. PROBLEM DEFINITIONS AND PRIOR WORKS

In this section, we first describe the hierarchical caching system, and then review some existing works that motivate our work in this paper.

A. Hierarchical Caching System

Consider a $(K_1, K_2; M_1, M_2; N)$ hierarchical caching system as shown Fig. 1, which consists of a single server, $K_1$ mirror sites and $K_1 K_2$ users. The server connects with $K_1$ mirror sites via a shared link and each mirror site connects with $K_2$ users via another shared link. The server contains a collection of $N$ files, denoted by $\mathcal{W} = \{W_1, W_2, \ldots, W_N\}$, each of which is uniformly distributed over $[0,1]^B$. Each mirror site and user has memory size of $M_1 B$ and $M_2 B$ bits, respectively, for some $M_1, M_2 \geq 0$.

Denote the $k_2$-th user attached to the $k_1$-th mirror site as $U_{k_1,k_2}$, for $k_1 \in [K_1], k_2 \in [K_2]$, and the set of users attached to the $k_1$-th mirror site as $U_{k_1}$. An $F$-division $(K_1, K_2; M_1, M_2; N)$ coded caching scheme contains two phases:

• Placement phase: During the off peak traffic time, each file is divided into $F$ packets with equal size, i.e., $W_n = \{W_{n,j} | n \in [N], j \in [F]\}$ where $B$ is divisible by $F$. Then the mirror sites and users cache some packets of each file. In other words, we consider the uncoded cache placement. Denote the contents cached by the mirror site $k_1$ and user $U_{k_1,k_2}$ as $Z_{k_1}$ and $Z_{k_1,k_2}$, respectively. During the placement phase, we assume the server is not aware of the users' requests.
• Delivery phase: During the peak traffic time, each user requests one file from the file library $\mathcal{W}$ randomly. The demand vector is denoted by $\mathbf{d} = (d_{1,1}, d_{1,2}, \ldots, d_{k_1,k_2})$, i.e., user $U_{k_1,k_2}$, requests the $d_{k_1,k_2}$-th file where $d_{k_1,k_2} \in [N]$. The messages sent in the hierarchical network contain two parts:
  - The messages sent by the server: Based on the cached contents and the demand vector $\mathbf{d}$, the server broadcasts a message including $S(\mathbf{d})$ packets to $K_1$ mirror sites.
  - The messages sent by mirror site: Based on the messages sent by the server, the locally cached contents and the demand vector $\mathbf{d}$, each mirror site $k_1$ broadcasts a coded messages of size $S_{k_1}(\mathbf{d})$ packets to its attached users (i.e., users in $U_{k_1}$), such that all the users can recover their requested files.

In this paper, we consider the worst case where each user requests a distinct file. Given a hierarchical caching system described above, the transmission loads in terms of files for the first and second layer are defined as

$$R_1 = \max \left\{ \frac{S(\mathbf{d})}{F} \mid \mathbf{d} \in [N]^{K_1,K_2} \right\},$$
$$R_2 = \max \left\{ \frac{S_{k_1}(\mathbf{d})}{F} \mid k_1 \in [K_1], \mathbf{d} \in [N]^{K_1,K_2} \right\},$$

respectively. Define the optimal transmission loads of the first and second layer, denoted by $R_1^\ast$ and $R_2^\ast$, as the minimum transmission load of $R_1$ and $R_2$ under uncoded placement respectively, such that all users can recover their requesting files.
B. Prior Works

The authors in [9] first studied this hierarchy caching system and proposed a decentralized hierarchical coded caching scheme, namely the KNMD scheme, based on the decentralized coded caching scheme for cache-aided broadcast network [2]. The main idea of KNMD scheme is to divide the system into two independent subsystems for some fixed parameters $\alpha, \beta \in [0 : 1]$. The first subsystem includes the entire cache memory of each mirror site and a $\beta$ fraction of each user’s cache memory, which is responsible for caching and delivering the $\alpha$ parts of each file. The second subsystem includes the remaining $(1 - \beta)$ fraction of each user’s cache memory, and is responsible for caching and delivering the left $(1 - \alpha)$ parts of each file. In the first subsystem, the server first sends coded signals to $K_1$ mirror sites using the single-layer MN decentralized coded caching scheme [2] where each mirror site requests $K_2$ distinct files, without considering users’ cache contents. Then each mirror site decodes its intended $K_2$ files and applies again the single-layer decentralized coded caching scheme to broadcast a message to its attached users to satisfy their demands. In the second subsystem, the server ignores the cache contents of mirror sites, and applies the single-layer decentralized coded caching scheme to directly serve $K_1K_2$ users each of caching size $(1 - \beta)M_2$. By extending this scheme to the case with centralized data placement [1], we obtain the transmission loads of the first and second layer, denoted by $R_{1}^{KNMD}$ and $R_{2}^{KNMD}$, as

$$R_{1}^{KNMD}(\alpha, \beta) \triangleq \alpha \cdot K_2 \cdot r_c \left( \frac{M_1}{\alpha N}, K_1 \right) + (1 - \alpha) \cdot r_c \left( \frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_1K_2 \right),$$

$$R_{2}^{KNMD}(\alpha, \beta) \triangleq \alpha \cdot r_c \left( \frac{\beta M_2}{\alpha N}, K_2 \right) + (1 - \alpha) \cdot r_c \left( \frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_2 \right),$$

for some $\alpha$ and $\beta$, where

$$r_c \left( \frac{M}{N}, K \right) \triangleq \frac{K(1 - M/N)}{1 + KM/N}$$

is the transmission load of the $(K, M, N)$ MN scheme for any memory ration $\frac{M}{N} \in \{0, \frac{1}{N}, \frac{2}{N}, \ldots, 1\}$.

In the KNMD scheme, the server sends messages to the mirror sites while ignoring the users’ cache contents. This means that the server may send some information which has already been stored by the users, leading to redundant communication cost in the first layer. To address this problem, [11, Section VI] improved the transmission load of the first layer of DHCC scheme by concatenating two MN schemes, whose transmission loads of the first and second layer, denoted by $R_{1}^{WWCY}$ and $R_{2}^{WWCY}$, are

$$R_{1}^{WWCY}(\alpha, \beta) \triangleq \alpha \cdot r_c \left( \frac{M_1}{\alpha N}, K_1 \right) r_c \left( \frac{\beta M_2}{\alpha N}, K_2 \right) + (1 - \alpha)r_c \left( \frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_1K_2 \right),$$

$$R_{2}^{WWCY}(\alpha, \beta) \triangleq \alpha \cdot r_c \left( \frac{\beta M_2}{\alpha N}, K_2 \right) + (1 - \alpha) \cdot r_c \left( \frac{(1 - \beta)M_2}{(1 - \alpha)N}, K_2 \right).$$

Note that under uncoded placement, schemes in [9] and [11, Section VI] both achieve the optimal transmission load of the second layer when $\alpha = \beta$, i.e.,

$$R_{2}^{*} = r_c \left( \frac{M_2}{N}, K_2 \right).$$

An interesting question is what is the optimal transmission load of the first layer. In this paper, we aim to find novel centralized coded caching schemes to reduce the transmission load of the first layer (i.e., $R_1$), and establish its optimal value for some regimes.

III. HIERARCHY PLACEMENT DELIVERY ARRAY

In this section, we first briefly describe the vanilla PDA for the single-layer cache-aided broadcast network [4], and then introduce a novel PDA structure, namely HPDA, that would help characterize the placement and delivery of coded caching schemes for the hierarchical caching system in Fig. [1].

A. Placement Delivery Array

**Definition 1:** (i) For any positive integers $K, F, Z$ and $S$, an $F \times K$ array $P = (p_{j,k})_{j \in [F], k \in [K]}$ over alphabet set $\{\ast\} \cup [0, S)$ is called a $(K, F, Z, S)$ PDA if it satisfies the following conditions,

C1. The symbol “*” appears $Z$ times in each column;

C2. Each integer occurs at least once in the array;
C3. For any two distinct entries \( p_{j_1,k_1} \) and \( p_{j_2,k_2} \), \( p_{j_1,k_1} = p_{j_2,k_2} = s \) is an integer only if

a. \( j_1 \neq j_2 \), \( k_1 \neq k_2 \), i.e., they lie in distinct rows and distinct columns; and

b. \( p_{j_1,k_1} = p_{j_2,k_2} = * \), i.e., the corresponding \( 2 \times 2 \) subarray formed by rows \( j_1, j_2 \) and columns \( k_1, k_2 \) must be of the following form

\[
\begin{pmatrix}
  s & * \\
  * & s \\
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
  * & s \\
  s & * \\
\end{pmatrix}.
\]

**Example 1:** When \( K = F = S = 3 \) and \( Z = 1 \), we can see that the following array is a \((3, 3, 1, 3)\) PDA.

\[
A = \begin{pmatrix}
  * & 1 & 2 \\
  1 & * & 3 \\
  2 & 3 & *
\end{pmatrix}.
\]

The authors in [4] showed that a \((K, F, Z, S)\) PDA can be used to realize an \( F \)-division \((K, M, N)\) coded caching scheme with \( \frac{N}{M} = \frac{Z}{F} \) and transmission load \( R = \frac{S}{F} \) for the single-layer cache-aided broadcast network. Furthermore, the seminal coded caching scheme proposed in [1] can be represented by a special PDA which is referred to as MN PDA. That is the following result.

**Lemma 1:** (MN PDA [1]) For any positive integers \( K \) and \( t \) with \( t \leq K \), there exists a \((K, (K)_t, (K-1)_t, (K-1)_t+1)\) PDA which realizes a \((K, M, N)\) MN scheme with \( \frac{N}{M} = \frac{Z}{F} \), subpacketization \( F = (K)_t \) and transmission load \( R = \frac{K-1}{K-1+t} \).

Here we briefly review the construction of MN PDA as follows.

**Construction 1:** (MN PDA [1]) For any integer \( t \in [K] \), let \( F = (K)_t \). Then we have a \((K)_t \times K\) array \( P = (P(T, k))_{T \subseteq (K)} \) by

\[
P(T, k) = \begin{cases} 
  \phi_{t+1}(T \cup \{k\}), & \text{if } k \notin T \\
  *, & \text{otherwise,}
\end{cases}
\]

(4)

where \( \phi_{t+1}(\cdot) \) is a bijection from \([K]_{t+1}\) to \([K]_t\) and the rows are labelled by all the subsets \( T \in (K)_t \) listed in the order from the small to large.

Finally, we should point out that PDA has been widely studied. There are some schemes with lower subpacketization level based on PDA proposed in [4, 16–24]. In addition, the authors in [25] pointed out that all the proposed schemes in [7, 26–29] could be represented by appropriate PDAs.

**B. Hierarchical Placement Delivery Array**

The definition of hierarchy placement delivery array (HPDA) is given as follows.

**Definition 2:** For any given positive integers \( K_1, K_2, F, Z_1, Z_2 \) with \( Z_1 < F, Z_2 < F \) and any integer sets \( S_m \) and \( S_{k_1}, k_1 \in [K_1] \), an \( F \times (K_1 + K_1 K_2) \) array \( P = (P^{(0)}, P^{(1)}, \ldots, P^{(K_2)}) \), where \( P^{(0)} = (p_{j,k_1})_{j \in [F], k_1 \in [K_1]} \) is an \( F \times K_1 \) array consisting of \( * \) and null, and \( P^{(k_2)} = (p_{j,k_1})_{j \in [F], k_1 \in [K_1]} \) is an \( F \times K_2 \) array over \( \{\ast\} \cup S_{k_1}, k_1 \in [K_1] \), is a \((K_1, K_2, F; Z_1, Z_2; S_m, S_{k_1}, \ldots, S_{k_1})\) hierarchy placement delivery array (HPDA) if it satisfies the following conditions:

B1. Each column of \( P_0 \) has \( Z_1 \) stars;

B2. \( P^{(k_1)} \) is a \((K_2, F, Z_2, [S_{k_1}])\) PDA for each \( k_1 \in [K_1] \).

B3. Each integer \( s \in S_m \) occurs in exactly one subarray \( P^{(k_1)} \) where \( k_1 \in [K_1] \). And for each \( p_{j,k_2}^{(k_1)} = s \in S_m, j \in [F], k_1 \in [K_1], k_2 \in [K_2] \), \( p_{j,k_2}^{(0)} = * \);

B4. For any two entries \( p_{j,k_2}^{(k_1)} \) and \( p_{j',k_2}^{(k_1)} \) where \( k_1 \neq k'_1 \in [K_1], j, j' \in [F] \) and \( k_2, k'_2 \in [K_2] \), if \( p_{j,k_2}^{(k_1)} = p_{j',k_2}^{(k_1)} \) is an integer then

- \( p_{j,k_2}^{(0)} \) is an integer only if \( p_{j,k_2}^{(0)} = * \);
- \( p_{j,k_2}^{(0)} \) is an integer only if \( p_{j,k_2}^{(0)} = * \).

For any given HPDA, when we use \( P^{(0)} \) to indicate the data placement at mirror sites, and use \( P^{(k_1)} \) to indicate data placement at the users attached to \( k_1 \)-th mirror site (i.e., users in \( U_{k_1} \)) and the delivery strategy at the server and mirror sites (see detailed explanation in Remark [1], a hierarchical coded caching scheme can be obtained by Algorithm [1]).

First we use the following example to demonstrate the placement and delivery strategy of the hierarchical caching scheme by Algorithm [1] based on a HPDA.
Algorithm 1 Caching scheme based on \((K_1,K_2;F;Z_1,Z_2;S_m,S_1,\ldots,S_{K_3})\) HPDA P

1: \textbf{procedure} Placement(P, W)
2: \hspace{1em} Split each file \(W_n \in W\) into \(F\) packets, i.e., \(W_n = \{W_{n,j} | j \in [F]\}\).
3: \hspace{1em} for \(k_1 \in [K_1]\) do
4: \hspace{2em} \(Z_{k_1} \leftarrow \{W_{n,j} | p_{j,k_1}^{(0)} = \ast, n \in [N], j \in [F]\}\)
5: \hspace{1em} end for
6: \hspace{1em} for \((k_1,k_2), k_2 \in [K_1], k_2 \in [K_2]\) do
7: \hspace{2em} \(Z_{(k_1,k_2)} \leftarrow \{W_{n,j} | p_{j,k_2}^{(k_1)} = \ast, n \in [N], j \in [F]\}\)
8: \hspace{1em} end for
9: \textbf{end procedure}

10: \textbf{procedure} Delivery_Server(P, W, d)
11: \hspace{1em} for \(s \in \left(\bigcup_{k_1=1}^{K_1} S_{k_1}\right) \setminus S_m\) do
12: \hspace{2em} Server sends the following coded signal to the mirror sites:
13: \hspace{2em} \(X_s = \bigoplus_{p_{j,k_2}=s,j \in [F], k_1 \in [K_1], k_2 \in [K_2]} W_{d_{k_1,k_2}j}\)
14: \hspace{1em} end for
15: \textbf{end procedure}

16: \textbf{procedure} Delivery_Mirrors(P, W, d, X_s)
17: \hspace{1em} for \(k_1 \in [K_1], s \in S_{k_1} \setminus S_m\) do
18: \hspace{2em} After receiving \(X_s\), mirror site \(k_1\) sends the following coded signal to users in \(U_{k_1}\):
19: \hspace{2em} \(X_{k_1,s} = X_s \bigoplus \bigoplus_{p_{j,k_2}^{(k_1)} = s', j \in [F], k_2 \in [K_2]} W_{d_{k_1,k_2}j}\)
20: \hspace{1em} end for
21: \hspace{1em} for \(k_1 \in [K_1], s' \in S_{k_1} \cap S_m\) do
22: \hspace{2em} Mirror site \(k_1\) sends the following coded signal to users in \(U_{k_2}\):
23: \hspace{2em} \(X_{k_1,s'} = \bigoplus_{p_{j,k_2}^{(k_1)} = s', j \in [F], k_2 \in [K_2]} W_{d_{k_1,k_2}j}\)
24: \hspace{1em} end for
25: \textbf{end procedure}

Example 2: When \(K_1 = 3, K_2 = 2, F = 15, Z_1 = 6, Z_2 = 4, S_m = [7 : 42], S_1 = [1 : 18], S_2 = [1 : 6] \cup [19 : 30], S_3 = [1 : 6] \cup [31 : 42]\), one can check that the following array is a \((3, 2; 15; 6, 4; S_m, S_1, S_2, S_3)\) HPDA P in (5).

\[
P = (P_0, P_1, P_2, P_3) = \begin{pmatrix}
* & * & 7 & 8 & 19 & 20 & 1 & 2 \\
* & 9 & 10 & * & 1 & * & 3 \\
* & 11 & 12 & * & 2 & 3 & * \\
* & 13 & 14 & 1 & * & 4 & * \\
* & 15 & 16 & 2 & * & 4 & * \\
* & * & 17 & 18 & 3 & 4 & 31 & 32 \\
* & * & 1 & 21 & 22 & * & 5 \\
* & * & 2 & 23 & 24 & 5 & * \\
* & * & 3 & * & 5 & 33 & 34 \\
* & * & 4 & 5 & * & 35 & 36 \\
* & 1 & * & 25 & 26 & * & 6 \\
* & 2 & * & 27 & 28 & 6 & * \\
* & 3 & * & 6 & 37 & 38 \\
* & 4 & * & 6 & * & 39 & 40 \\
* & * & 5 & 6 & 29 & 30 & 41 & 42
\end{pmatrix}.
\]  

(5)

Based on \(P\) and by Algorithm 1 we can get a 15-(3, 2; 2.4, 1.6; 6) coded caching scheme in the following way.

- **Placement Phase**: From Line 2 in Algorithm 1 each file is divided into \(F = 15\) packets with equal size, i.e., \(W_n = \{W_{n,1}, W_{n,2}, \ldots, W_{n,15}\}, n \in [6]\). From lines 3-5 in Algorithm 1 and \(P_0\) in (5), the contents cached by mirror sites are
as follows:

\[ Z_1 = \{ W_{n,1}, W_{n,2}, W_{n,3}, W_{n,4}, W_{n,5}, W_{n,6} \mid n \in [6] \}, \]
\[ Z_2 = \{ W_{n,1}, W_{n,7}, W_{n,8}, W_{n,11}, W_{n,12}, W_{n,15} \mid n \in [6] \}, \]
\[ Z_3 = \{ W_{n,6}, W_{n,9}, W_{n,10}, W_{n,13}, W_{n,14}, W_{n,15} \mid n \in [6] \}. \]

From lines 6-8 in Algorithm 1 and \( P_{k_1} \) in \([5]\), \( k_1 \in [3] \), the packets cached by the users are as follows:

\[ Z_{(1,1)} = \{ W_{n,7}, W_{n,8}, W_{n,9}, W_{n,10} \mid n \in [6] \}, \]
\[ Z_{(1,2)} = \{ W_{n,11}, W_{n,12}, W_{n,13}, W_{n,14} \mid n \in [6] \}, \]
\[ Z_{(2,1)} = \{ W_{n,2}, W_{n,3}, W_{n,9}, W_{n,13} \mid n \in [6] \}, \]
\[ Z_{(2,2)} = \{ W_{n,4}, W_{n,5}, W_{n,10}, W_{n,14} \mid n \in [6] \}, \]
\[ Z_{(3,1)} = \{ W_{n,2}, W_{n,4}, W_{n,7}, W_{n,11} \mid n \in [6] \}, \]
\[ Z_{(3,2)} = \{ W_{n,3}, W_{n,5}, W_{n,8}, W_{n,12} \mid n \in [6] \}. \]

- **Delivery Phase**: Assume that \( d = (1, 2, 3, 4, 5, 6) \). From Algorithm 1 the messages sent to all users consist of two parts.

  - The messages \( S(d) \) sent by the server: From \([5]\) and Line 11, we have \( s \in \left( \bigcup_{k=1}^{3} S_{k_1} \right) \setminus S_m = [1 : 6] \). By Lines 11-14 in Algorithm 1 the server transmits the coded messages \( X_s \):

    \[
    W_{1,11} \oplus W_{2,7} \oplus W_{3,4} \oplus W_{4,2} \oplus W_{5,1}, \]
    \[
    W_{1,12} \oplus W_{2,8} \oplus W_{3,5} \oplus W_{4,3} \oplus W_{6,1}, \]
    \[
    W_{1,13} \oplus W_{2,9} \oplus W_{3,6} \oplus W_{5,3} \oplus W_{6,2}, \]
    \[
    W_{1,14} \oplus W_{2,10} \oplus W_{4,6} \oplus W_{5,5} \oplus W_{6,4}, \]
    \[
    W_{1,15} \oplus W_{3,10} \oplus W_{4,9} \oplus W_{5,8} \oplus W_{6,7}, \]
    \[
    W_{1,16} \oplus W_{3,14} \oplus W_{4,13} \oplus W_{5,12} \oplus W_{6,11},
    \]

to all mirror sites. So the transmission load of the first layer is \( R_1 = \frac{6}{15} = 0.4 \).

  - The messages \( S_{k_1}(d) \) sent by mirror site \( k_1 \) consists of the coded packets \( X_{k_1,s} \) generated by \( X_s \) from the server and the packets cached by mirror site \( k_1 \) where \( s \in S_{k_1} \setminus S_m \), and the coded packets \( X_{k_1,s'} \) generated only by the packets cached by mirror site \( k_1 \) where \( s' \in S_{k_1} \bigcap S_m \). From Lines 17-20 and \([5]\), we have \( S_1 \setminus S_m = [1 : 6] \), so the mirror site 1 sends the coded packets \( X_{1,s} \):

    \[
    (W_{1,11} \oplus W_{2,7} \oplus W_{3,4} \oplus W_{4,2} \oplus W_{5,1}) \oplus (W_{3,4} \oplus W_{4,2} \oplus W_{5,1}) = W_{1,11} \oplus W_{2,7}, \]
    \[
    (W_{1,12} \oplus W_{2,8} \oplus W_{3,5} \oplus W_{4,3} \oplus W_{6,1}) \oplus (W_{3,5} \oplus W_{4,3} \oplus W_{6,1}) = W_{1,12} \oplus W_{2,8}, \]
    \[
    (W_{1,13} \oplus W_{2,9} \oplus W_{3,6} \oplus W_{5,3} \oplus W_{6,2}) \oplus (W_{3,6} \oplus W_{5,3} \oplus W_{6,2}) = W_{1,13} \oplus W_{2,9}, \]
    \[
    (W_{1,14} \oplus W_{2,10} \oplus W_{4,6} \oplus W_{5,5} \oplus W_{6,4}) \oplus (W_{4,6} \oplus W_{5,5} \oplus W_{6,4}) = W_{1,14} \oplus W_{2,10}, \]
    \[
    W_{1,15} \oplus W_{3,10} \oplus W_{4,9} \oplus W_{5,8} \oplus W_{6,7}, \]
    \[
    W_{1,16} \oplus W_{3,14} \oplus W_{4,13} \oplus W_{5,12} \oplus W_{6,11},
    \]

  since it can receive the coded packets \( X_s \) from server and it has cached packets \( \{ W_{n,1}, W_{n,2}, W_{n,3}, W_{n,4}, W_{n,5}, W_{n,6} \mid n \in [6] \} \). Then user \( U_{1,1} \) can decode \( W_{1,11}, W_{1,12}, W_{1,13}, W_{1,14} \) and \( W_{1,15} \) from \( X_{1,s} \) since it has cached \( \{ W_{n,7}, W_{n,8}, W_{n,9}, W_{n,10} \mid n \in [6] \} \). Similarly, \( U_{1,2} \) can also recover some of its required file packets from \( X_{1,s} \) and its own cache memory respectively. Similarly, \( |S_1 \setminus S_m| = 6 \) there are 6 coded packets sent by mirror site 1. Now we see the coded packets \( X_{k_1,s'} \) sent by mirror site \( k_1 \). From Lines 21-24 and \([5]\), we have \( S_1 \bigcap S_m = [7 : 18] \), so mirror site 1 sends \( X_{1,s'} \):

    \[
    W_{1,1}, W_{1,2}, W_{1,3}, W_{1,4}, W_{1,5}, W_{1,6}, W_{2,1}, W_{2,2}, \]
    \[
    W_{2,3}, W_{2,4}, W_{2,5}, W_{2,6},
    \]

to users \( U_{1,1} \) and \( U_{1,2} \) from its own cached packets. Clearly each user can directly get the above packets and there are 12 packets. Then the transmission amount by mirror site 1 is \( \frac{12}{15} = 0.8 \), which is the transmission load of the second layer \( R_2 = 1.2 \).

Actually, \( P \) in \([5]\) is obtained by Theorem 2 whose \( R_1 \) achieves the minimum load under the restriction of parameters specified by \( P \) in \([9], [11]\), by the exhaustive computer searches for the values of \( \alpha \) and \( \beta \) to find the minimum transmission load of the first layer under the same circumstance, we have \( R_1^{NSMD} = 0.73 \) from \([1]\) and \( R_1^{WWCY} = 0.55 \) from \([2]\). Clearly \( R_1 < R_1^{WWCY} < R_1^{NSMD} \).
Remark 1: From Algorithm 1 and Example 2, we have the following relationship between \((K_1, K_2; F; Z_1, Z_2; S_m, S_1, \ldots, S_{K_1})\) HPDA and its realized \(F\)-division coded caching scheme for the \((K_1, K_2; M_1, M_2; N)\) hierarchical coded caching problem where \(\frac{M_1}{N} = \frac{Z_1}{F}, \frac{M_2}{N} = \frac{Z_2}{F}\).

- An \(F \times K_1\) mirror sites-placement array \(P^{(0)}\) consists of * and null entries. The column labels represent the mirror site indices while the row labels represent the packet indices. If entry \(P_{j,k_1}^{(0)} = *\), \(j \in [F]\) and \(k_1 \in [K_1]\), then mirror site \(k_1\) has already cached the \(j\)-th packet of all the files in server. All mirror sites have the same memory ratio \(\frac{M_1}{N} = \frac{Z_1}{F}\) according to B1 of Definition 2.

- An \(F \times K_1 K_2\) users-placement and delivery array \((P^{(1)}), \ldots, P^{(K_1)}\) consists of \(*\) \(\cup\{1\} \{S_m\}\}. The column labels represent the user indices while the row labels represent the packet indices. If entry \(P_{j,k_1}^{(0)} = *\), \(j \in [F]\), \(k_1 \in [K_1]\), \(k_2 \in [K_2]\), user \(U_{k_1,k_2}\) has already cached the \(j\)-th packet of all the files in server. All the users have the same memory ratio \(\frac{M_1}{N} = \frac{Z_1}{F}\) according to B2 of Definition 2. The integers in \(\{U_{k_1}^{K_1} \{S_m\}\} \{S_m\}\) indicate the broadcast packets transmitted by the server, and the integers in \(S_m\) \(k_1 \in [K_1]\), represent the broadcast packets sent by the mirror site \(k_1\). In addition the integers in \(S_m\) represent the multicast messages sent only by the mirror sites.

- The property B2 and B4 of Definition 2 guarantee that each user \(U_{k_1,k_2}\) can recover its requested packet, since user \(U_{k_1,k_2}\) or mirror site \(k_1\) has cached all the other packets in the broadcast message except the one requested by \(U_{k_1,k_2}\). More precisely, if entry \(P_{j,k_1}^{(0)} = *\), \(j \in [F]\), \(k_1 \in [K_1]\), \(k_2 \in [K_2]\), user \(U_{k_1,k_2}\) has already cached the \(j\)-th packet of all the files in server. In this case the server broadcasts a coded packet (i.e., the XOR of all the requested packets indicated by \(s\)) to the mirror sites. Assume that the packet requested by user \(U_{k_1,k_2}\), say \(W_{d_{k_1},k_2,j}\), and any other packet, say \(W_{d_{k_1},k_2,j'}\), are included in the coded signal \(X_s\) listed in Line 13 of Algorithm 1. Then we have \(P_{j,k_1}^{(0)} = P_{j',k_1}^{(0)} = *\). This implies that mirror site \(k_1\) has cached \(W_{d_{k_1},k_2,j}\). From Line 19 of Algorithm 1 the coded signal \(X_{k_1,s}\) which is transmitted to user \(U_{k_1,k_2}\) by mirror site \(k_1\), is generated by cancelling the \(W_{d_{k_1},k_2,j'}\) by mirror site \(k_1\). So \(X_{k_1,s}\) only contains one packet required by user \(U_{k_1,k_2}\), and the packets which have been cached by user \(U_{k_1,k_2}\). Clearly user \(U_{k_1,k_2}\) can decode its requiring packet \(W_{d_{k_1},k_2,j}\). So the number of packets transmitted by the server is \(\frac{|U_{k_1}^{K_1} | S_m | - |S_m|}{F}\). Then the transmission load from server to mirror site is \(R_1 = \frac{|U_{k_1}^{K_1} | S_m | - |S_m|}{F}\). While if \(s' \in S_m\) \(S_m\) by the Condition B3 of Definition 2 the mirror site \(k_1\) has already cached all the required packets labeled by \(s'\). So the mirror site can broadcast a multicast message \(X_{k_1,s'}\) (i.e. the XOR of all the requested packets indicated by \(s'\)) to the user in \(U_{k_1}\). Then the number of packets transmitted simply by the mirror site \(k_1\) is \(|S_m|\). This implies that the transmission load from mirror site \(k_1\) to its attached users in \(U_{k_1}\) is \(R_2 = \max_{k_1 \in [K_1]} \{ \frac{|S_m|}{F} \}\).

From the above investigations in Remark 1 we can obtain the following result.

**Theorem 1:** Given a \((K_1, K_2; F; Z_1, Z_2; S_m, S_1, \ldots, S_{K_1})\) HPDA \(P = (P_0, P_1, \ldots, P_{K_1})\), we can obtain an \(F\)-division \((K_1, K_2; M_1, M_2; N)\) coded caching scheme with \(\frac{M_1}{N} = \frac{Z_1}{F}, \frac{M_2}{N} = \frac{Z_2}{F}\) and transmission load \(R_1 = \frac{|U_{k_1}^{K_1} | S_m | - |S_m|}{F}\). \(R_2 = \max_{k_1 \in [K_1]} \{ \frac{|S_m|}{F} \}\).

From Theorem 1 we can obtain a hierarchical coded caching scheme by constructing an appropriate HPDA. So in this paper we focus on constructing HPDA to get its realized scheme with better performance compared with the previously known results.

IV. MAIN RESULTS

In this section, we first present new upper bounds on the optimal transmission loads \((R_1, R_2)\) based on two classes of HPDAs, and then compare these bounds with previously known results.

**Theorem 2:** For any positive integers \(K_1, K_2, t, K_2 < t < K_1 K_2\), there exists a \((K_1, K_2; (K_1 K_2) - (K_1 K_2 - K_2); (K_1 K_2 - K_2) - (K_1 K_2 - K_2 - K_2), S_m, S_1, \ldots, S_{K_1})\) HPDA, which leads to an \(F\)-division \((K_1, K_2; M_1, M_2; N)\) coded caching
We propose the following upper bound based on a new class of HPDA.

\[
M_1 = \frac{(K_1, K_2 - K_2)}{t - K_2}, \quad M_2 = \frac{t}{K_1 K_2} - \frac{(K_1, K_2 - K_2)}{t - K_2},
\]

Subpacketization: \( F = \frac{(K_1, K_2)}{t} \).

Transmission loads:

\[
R_1 = \frac{K_1 K_2 - t}{t + 1}, \quad R_2 = \frac{K_1 K_2 - t}{t + 1} - \frac{(K_1, K_2 - K_2)}{t - K_2} + \frac{(K_1, K_2 - K_2)}{t - K_2} K_2.
\]

**Theorem 2:** For any \((K_1, F_1, Z_1, S_1)\) PDA \(A\) and \((K_2, F_2, Z_2, S_2)\) PDA \(B\), there exists a \((K_1, K_2; F_1 F_2; Z_1 F_2, Z_2 F_1; S_m, S_1, \ldots, S_{K_1})\) HPDA, which leads to a \((K_1, K_2; M_1, M_2; N)\) coded caching scheme with memory ratios \(M_1/N = Z_1/F_1, M_2/N = Z_2/F_2\) and transmission loads

\[
R_1 = \frac{S_1 S_2}{F_1 F_2}, \quad R_2 = \frac{S_2}{F_2}.
\]

**Proof.** See the proof in Section [V].

**Corollary 1:** For a two-level hybrid network with memory ratios satisfying (6a), the transmission load \(R_1\) in (6c) is optimal under the uncoded data placement, i.e.,

\[
R_1 = \frac{K_1 K_2 (1 - \frac{M_1 + M_2}{N})}{K_1 K_2 \frac{M_1 + M_2}{N} + 1}.
\]

**Proof.** The achievability proof holds directly from Theorem 2. For the converse proof, please refer to Appendix [A].

It can be checked that the HPDA in (5) in Example 2 is in fact a specific HPDA of Theorem 2. In Section [V-A], we show how to construct the HPDA in (5) based on a (6, 15, 10, 6) MN PDA.

The memory ratios in Theorem 2 are constrained by combination numbers as shown in (6a), which means the rate \(R_1 = K_1 K_2 - t/(t + 1)\) may not always be achievable for general memory ratios \((M_1/N, M_2/N)\). In order to allow flexible memory ratios, we propose the following upper bound based on a new class of HPDA.

**Theorem 3:** For any \((K_1, F_1, Z_1, S_1)\) PDA \(A\) and \((K_2, F_2, Z_2, S_2)\) PDA \(B\), there exists a \((K_1, K_2; F_1 F_2; Z_1 F_2, Z_2 F_1; S_m, S_1, \ldots, S_{K_1})\) HPDA, which leads to a \((K_1, K_2; M_1, M_2; N)\) coded caching scheme with memory ratios \(M_1/N = Z_1/F_1, M_2/N = Z_2/F_2\) and transmission loads

\[
R_1 = \frac{S_1 S_2}{F_1 F_2}, \quad R_2 = \frac{S_2}{F_2}.
\]

**Proof.** See the proof in Section [VI].

**Remark 2:** By choosing different PDAs \(A\) and \(B\) to construct the HPDA in Theorem 3, we can obtain different transmission loads and subpacketization levels. In particular,

- when \(A\) and \(B\) are MN PDA, the corresponding scheme is the same as the WWCY scheme [11], which achieves the optimal transmission load of \(R_2^*\);
- when \(A\) and \(B\) are the PDA proposed in [4] and MN PDA, respectively, we could reduce the subpacketization level at cost of increasing communication loads. We name the corresponding scheme as Scheme I for Theorem 3;
- when both \(A\) and \(B\) are PDAs proposed in [4], we would further reduce the subpacketization level. We name the corresponding scheme as Scheme II for Theorem 3.

In Fig. 2, we compare the following schemes: 1) the KNMD scheme [9]; 2) the WWCY scheme [11]; 3) the Scheme for Theorem 2; 4) Scheme I for Theorem 3; 5) Scheme II for Theorem 3. Note that we can also design our new hybrid schemes like the previous works [9, 11], which divide the system into two subsystems with splitting parameters \((\alpha, \beta)\), and run the proposed schemes in the first subsystem, and the MN scheme in the second subsystem. Since the optimal \(\alpha\) and \(\beta\) are hard to determine due to a tradeoff between \(R_1\) and \(R_2\), and the second subsystem totally ignores mirror sites’ caching abilities, we only focus on schemes working in the first subsystem, i.e., compare all schemes with \(\alpha = \beta = 1\). Besides, due to the limitation on memory ratios (6a), it is hard to compare all schemes with general \(M_1, M_2 \in [0 : N]\). We thus evaluate the performance of various scheme with fixed parameters \((K_1, K_2, N) = (40, 20, 800)\), and varying parameters \((M_1, M_2)\) such that the ratios in (6a) are satisfied. More precisely, \(M_1/N\) takes the value from 0.2 to 0.9 regularly with step size 0.1, and \(M_2/N\) takes the value from 0.72 to 0.1 (without fixed step size but on a downward trend), which satisfies memory ratios (6a) in Theorem 2.
As mentioned in Remark 1, we let its element \( Q^{(k_1)} \) be a star row, and be null otherwise. To construct the array \( P^{(k_1)} \), we simply replace all the star entries \( P_{f,k_1}^{(k_1)} \) by \( k_1 \in [K_1] \). Then we have the following expressions.

\[
P^{(0)} = (p_{f,k_1}^{(0)})_{f \in [F], k_1 \in [K_1]}, \quad p_{f,k_1}^{(0)} \in \{*, \text{null}\}
\]

(8)

\[
P^{(k_1)} = (p_{f,k_2}^{(k_1)})_{f \in [F], k_2 \in [K_2]}, \quad p_{f,k_2}^{(k_1)} \in \{*, \cup S_{k_1}\}
\]

(9)

\[
Q^{(k_1)} = (q_{f,k_2}^{(k_1)})_{f \in [F], k_2 \in [K_2]}, \quad q_{f,k_2}^{(k_1)} \in [S], \quad k_1 \in [K_1].
\]

(10)

As mentioned in Remark 1, \( P^{(0)} \) and \( P^{(k_1)} \) indicates the data placement at mirror sites and users in \( U_{k_1} \), respectively. Given any array, we call a row of it a star row if this row contains only star entries. The construction of HPDA \( P = (P^{(0)}, P^{(1)}, \ldots, P^{(K_1)}) \) in Theorem 2 based on the array \((K, F, Z, S)\) MN PDA \( Q\). We partition \( Q \) into \( K_1 \) parts by column, i.e., \( Q = (Q^{(1)}, \ldots, Q^{(K_1)}) \). Then we have the following expressions.

In the follows, we first use an illustrative example to show the construction of \( P \), and then present our general proof of coded caching scheme based on the HPDA in Theorem 2.
A. Example of the Construction of HPDA in Theorem 2

Given a \((K_1, K_2, F, Z, S) = (3 \times 2, 15, 10, 6)\) MN PDA \(Q = (Q^{(1)}, \ldots, Q^{(K_1)})\), we will use a grouping method to construct a \((3, 2; 15; 6; 4; S_m, S_1, S_2, S_3)\) HPDA \(P\) in \([5]\), where

\[
S_m = [7 : 42], \; S_1 = [1 : 18], \; S_2 = [1 : 6] \cup [19 : 30], \; S_3 = [1 : 6] \cup [31 : 42]
\]

The construction includes the following three steps, as illustrated in Fig. 3.

**Step 1.** Construction of \(P^{(0)} = (P_{f, k}^{(0)})_{f \in [15], k \in [3]}\). Because the star rows of \(Q^{(1)}\) are row 1, 2, 3, 4, 5, and 6, we fill \(p_{1,1}^{(0)} = p_{2,1}^{(0)} = p_{3,1}^{(0)} = p_{4,1}^{(0)} = p_{5,1}^{(0)} = p_{6,1}^{(0)} = *, \) and for the rest entries in column 1 of \(P^{(0)}\), we fill them with null. Similarly, the columns 2 and 3 of \(P^{(0)}\) can be obtained by applying the same operation on \(Q^{(2)}\) and \(Q^{(3)}\) respectively. Then the resulting array is our required \(P^{(0)}\).

**Step 2.** Construction of \((P^{(1)}, P^{(2)}, P^{(3)}) = (p_{f, k}^{(i)})\) where \(f \in [15], k_2 \in [2], k_1 \in [3]\). Taking \(P^{(1)}\) as an example, we fill the entries in star rows of \(Q^{(1)}\) with distinct integers to get \(P^{(1)}\), i.e., \(q_{1,1}^{(1)} = 7, q_{1,2}^{(1)} = 8, q_{2,1}^{(1)} = 9, q_{2,2}^{(1)} = 10, q_{3,1}^{(1)} = 11, q_{3,2}^{(1)} = 12, q_{4,1}^{(1)} = 13, q_{4,2}^{(1)} = 14, q_{5,1}^{(1)} = 15, q_{5,2}^{(1)} = 16, q_{6,1}^{(1)} = 17, q_{6,2}^{(1)} = 18\). Similarly we can obtain \(P^{(2)}\) and \(P^{(3)}\) in Fig. 3. The integer sets of \(P^{(1)}, P^{(2)}\) and \(P^{(3)}\) in \([7]\) can be obtained directly from Fig. 3.

**Step 3.** Construction of \(P\). We get a \(15 \times 9\) array by arranging \(P^{(0)}\) and \(P^{(1)}, P^{(2)}\) and \(P^{(3)}\) horizontally, i.e. \(P = (P^{(0)}, P^{(1)}, P^{(2)}, P^{(3)})\).

Now we verify that the construction above leads to an HPDA defined in Definition 2.

- Each column of \(P^{(0)}\) has \(Z_1 = 6\) stars, satisfying Condition B1.
- \(P^{(1)}, P^{(2)}\) and \(P^{(3)}\) are \((2, 15, 4, 18)\) PDAs, satisfying Condition B2 of Definition 2.
- From Fig. 3 we have \(S_m = [7 : 42]\), whose integers only appear in one \(P_{f, k_1}^{(k_1)}, k_1 \in [3]\), and we can check that, if \(p_{f, k_2}^{(k_1)} = s \in S_m\), then \(p_{f, k_2}^{(k_1)} = *\), thus Condition B3 of Definition 2 holds.
- It can be checked that Condition B4 also holds. Take \(p_{f, k_2}^{(k_1)} = p_{f', k_2}^{(k_1)} = 1\) as an example. From \([5]\), we can see that \(p_{11,1}^{(1)} = p_{1,2}^{(2)} = p_{2,2}^{(2)} = p_{1,1}^{(3)} = 1\). When choosing \(f = 11, k_2 = 2\) and \(f' \in \{4, 2, 1\}\), the corresponding \(p_{f, k_1}^{(0)} = p_{f', k_1}^{(0)} = p_{4,1}^{(0)} = p_{2,1}^{(0)} = p_{1,1}^{(0)} = *\), satisfying Condition B4 of Definition 2.

B. General Proof of Theorem 2

Given a \((K, F, Z, S) = (K_1, K_2, (K_1, K_2)_{t \in [1, t_1]}, (K_1, K_2)_{t \in [t_1, 1]}), (K_1, K_2)_{t \in [t_1, t_2]}, (K_1, K_2)_{t \in [t_2, 1]}\) MN PDA \(Q = (Q^{(1)}, \ldots, Q^{(K_1)})\), \(t \in [K_2, K_1, K_2]\), we show how to construct a \((K_1, K_2; F; Z_1, Z_2; S_m, S_1, \ldots, S_{K_1}) = (K_1, K_2; (K_1, K_2)_{t \in [1, t_1]}, (K_1, K_2)_{t \in [t_1, t_2]}, (K_1, K_2)_{t \in [t_2, 1]}; S_m, S_1, \ldots, S_{K_1})\) HPDA. 

---

Fig. 3: The transformation from MN PDA \(Q\) to a HPDA \(P\) in Theorem 2.
\[ \ldots, \mathcal{S}_{K_1} \] HPDA \( \mathbf{P} = (\mathbf{P}^{(0)}, \mathbf{P}^{(1)}, \ldots, \mathbf{P}^{(K_1)}, \mathcal{S}_{m}, \mathcal{S}_{2}, k_1 \in [K_1], \) are listed in [13] and [15] respectively. For the sake of convenience, we use a set \( \mathcal{T} = ([K_1, K_2]) \) to represent the row index of an MN PDA defined in Construction 1 and the constructed HPDA, i.e., \( \mathcal{Q}^{(k_1)} \) is represented by \( \mathcal{Q}^{(k_1)} = (q^{(k_1)}_{x}, t \in ([K_1, K_2]), k_2 \in [(k_1 - 1)K_2 + 1 : k_1 K_2], k_1 \in [K_1]. \) The constructions of \( \mathbf{P}^{(0)} \) and \( \mathbf{P}^{(1)}, \ldots, \mathbf{P}^{(K_1)} \) are described as follows:

- **Step 1.** Construction of \( \mathbf{P}^{(0)} \). We construct the \( F \times K_1 \) mirror site's placement array \( \mathbf{P}^{(0)} = (p^{(0)}_{x}, t \in ([K_1, K_2]), k_2 \in [K_1], \) by the following rule:

\[
p^{(0)}_{x,k_1} = \begin{cases} * & \text{if } q^{(k_1)}_{x,k_2} = *, \forall k_2 \in [(k_1 - 1)K_2 + 1 : k_1 K_2] \\ \text{null} & \text{otherwise.} \end{cases}
\]

That is, \( p^{(0)}_{x,k_1} \) be a star if the row \( k \) of \( \mathcal{Q}^{(k_1)} \) is a star row, and be null otherwise.

- **Step 2.** Construction of \( \mathbf{P}^{(1)}, \ldots, \mathbf{P}^{(K_1)} \). For each \( k_1 \in [K_1], \mathcal{Q}^{(k_1)} \) is used to construct \( \mathbf{P}^{(k_1)} = (p^{(k_1)}_{x}, t \in ([K_1, K_2]), k_2 \in [(k_1 - 1)K_2 + 1 : k_1 K_2]. \) Note that there are in total \( K_1 Z_1 = K_1 \left( K_1 K_2 - K_2 \right) \) star rows in \( \mathcal{Q}^{(1)}, \ldots, \mathcal{Q}^{(K_1)} \) and each star row has \( K_2 \) star entries. Then we replace all these star entries in each star row of \( \mathcal{Q}^{(1)}, \ldots, \mathcal{Q}^{(K_1)} \) with consecutive integers from \( S + 1 \) to \( S + K_2 K_1 \left( K_1 K_2 - K_2 \right) \) to construct \( \mathbf{P}^{(1)}, \ldots, \mathbf{P}^{(K_1)} \), and all these integers form the set \( \mathcal{S}_m \) as follows.

\[
\mathcal{S}_m = \left[ S + 1 : S + K_2 K_1 \left( \frac{K_2 - K_1}{K_2 - K_2} \right) \right]
\]

- **Step 3.** Construction of \( \mathbf{P} \). We get an \( F_1 F_2 \times (K_1 + K_2 K_2) \) array by arranging \( \mathbf{P}^{(0)} \) and \( \mathbf{P}^{(1)}, \ldots, \mathbf{P}^{(K_1)} \) horizontally, i.e., \( \mathbf{P} = (\mathbf{P}^{(0)}, \mathbf{P}^{(1)}, \ldots, \mathbf{P}^{(K_1)}). \)

1) **Parameter computations:** The integer set \( \mathcal{S}_m \) can be directly obtained from [13], which has no intersection with \( [S] \).

Then we focus on \( \mathcal{S}_{K_1}. \) Clearly each \( \mathcal{Q}^{(k_1)} \) satisfies Conditions C1 and C3 of Definition 1. Now we consider the integer set of \( \mathcal{Q}^{(k_1)} \). Recall that \( \phi_{x} \) is a bijection from \( ([K_1, K_2]) \) to \( ([K_1, K_2]) \) in [4]. Then for any sub-array \( \mathcal{Q}^{(k_1)} \), integer \( s \in [S] \) is in \( \mathcal{Q}^{(k_1)} \) if and only if its inverse mapping \( S = \phi_{x}^{-1}(s) \) contains at least one integer of \( [(k_1 - 1)K_2 + 1 : k_1 K_2] \), i.e., \( S \cap [(k_1 - 1)K_2 + 1 : k_1 K_2] \neq \emptyset \). So the integer set of \( \mathcal{Q}^{(k_1)} \) is

\[
\mathcal{S}_{k_1} = \left\{ \phi_{x}^{-1}(S) \mid S \cap [(k_1 - 1)K_2 + 1 : k_1 K_2] \neq \emptyset, S \in ([K_1 K_2]) \right\},
\]

Furthermore, after adding up the integers used for the substitution in Step 2 of \( \mathcal{Q}^{(k_1)} \), the integer set of \( \mathcal{Q}^{(k_1)} \) is

\[
\mathcal{S}_{k_1} = \left[ S + (k_1 - 1)K_2 \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right) + 1 : S + k_1 K_2 \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right) \right] \bigcup \mathcal{S}_{k_1},
\]

and \( |\mathcal{S}_{k_1}| = K_2 \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right) + S - \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right) \).

2) **The properties of HPDA verification:** From [4] the row \( \mathcal{T} \) of \( \mathcal{Q}^{(k_1)} \) is a star row if and only if all the integers of \( [(k_1 - 1)K_2 + 1 : k_1 K_2] \) (i.e., indices of users in \( \mathcal{U}_{k_1} \)) are contained by \( \mathcal{T}. \) So there are \( \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right) \) star rows, i.e., each column of \( \mathbf{P}^{(0)} \) has \( Z_1 = \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right) \) stars, satisfying Condition B1 of Definition 2.

From [14] we have \( |\mathcal{S}_{k_1}| = S - \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right). \) So \( \mathcal{Q}^{(k_1)} \) is a \( (K_2, F, Z, S - \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right)) \) PDA. Actually, \( \mathbf{P}^{(k_1)} \) is obtained by replacing the star entries in star row of \( (K_2, F, Z, S - \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right)) \) PDA \( \mathcal{Q}^{(k_1)} \) by some unique integers which have no intersection with \( [S] \). So \( \mathbf{P}^{(k_1)} \) is a \( (K_2, F, Z, S - \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right) + K_2 \left( \frac{K_1 K_2 - K_2}{K_2 - K_2} \right)) \) PDA, satisfying Condition B2 of Definition 2.

From Step 2, we know that each integer \( s \in \mathcal{S}_m \) occurs only once in \( \mathbf{P}^{(1)}, \ldots, \mathbf{P}^{(K_1)} \). Clearly the first part of Condition B3 holds. For any \( p^{(k_1)}_{x,k_2} = s \in \mathcal{S}_m, \) we know row \( \mathcal{T} \) is a star row of \( \mathcal{Q}^{(k_1)}, \) then from (12) we have \( p^{(0)}_{x,k_1} = \ast \). The second part of Condition B3 holds, then Condition B3 holds.

We can show that Condition B4 holds by the following reason. Assume that there are two entries \( p^{(k_1)}_{x,k_2} = p^{(k_1)}_{x,k_2} = s \), where \( k_1 \neq k_1 \). Then \( s \notin \mathcal{S}_m \) since each integer of \( \mathcal{S}_m \) occurs exactly once. Furthermore if \( p^{(k_1)}_{x,k_2} = s \), then \( s \) must be the element of \( \mathcal{S}_m \), otherwise it contradicts our hypothesis that \( \mathcal{Q} \) is a PDA. From the construction of \( \mathbf{P}^{(k_1)}, \) \( s \in \mathcal{S}_m \) only if the row indexed by \( \mathcal{T} \) of \( \mathcal{Q}^{(k_1)} \) is a star row. Then from (12) we have \( p^{(0)}_{x,k_1} = \ast \).
From the above introduction, our expected HPDA is obtained. And the integer set $\bigcup_{k_1=1}^{K_1} S_{k_1} \setminus S_M$ is actually the integer set $[S]$, whose cardinality is $\binom{K_1+K_2}{t+1}$. Then by Theorem 1, the transmission loads

$$R_1 = \frac{|\bigcup_{k_1=1}^{K_1} S_{k_1}| - |S_M|}{F} = \frac{S}{F} = \frac{K_1 K_2 - t}{t + 1}$$

$$R_2 = \max \left\{ \frac{|S_{k_1}|}{F} \mid k_1 \in [K_1] \right\} = \frac{K_2 \binom{K_1+K_2}{t-K_2} + S - \binom{K_1+K_2}{t+1}}{\binom{K_1+K_2}{t}}$$

$$= \frac{K_1 K_2 - t}{t+1} - \frac{\binom{K_1+K_2}{t}}{\binom{K_1+K_2}{t-K_2}} + \frac{\binom{K_1+K_2}{t}}{\binom{K_1+K_2}{t}}$$

can be directly obtained.

VI. PROOF OF THEOREM 3

In this section, we describe how to construct the $(K_1, K_2; F_1, F_2; Z_1, F_2, Z_2 F_1; S_m, S_1, \ldots, S_{K_1})$ HPDA $P = (P^{(0)}, P^{(1)}, \ldots, P^{(K_1)})$ in Theorem 3 based on any $(K_1, F_1, Z_1, S_1)$ PDA $A$ and $(K_2, F_2, Z_2, S_2)$ PDA $B$, where

$$P^{(0)} = (p^{(0)}_{(f_1, f_2), k_1}), \quad P^{(0)}_{(f_1, f_2), k_1} \in \{*, \text{null}\},$$

$$P^{(k_1)} = (p^{(k_1)}_{(f_1, f_2), k_2}), \quad P^{(k_1)}_{(f_1, f_2), k_2} \in \{*\} \cup S_{k_1},$$

$(f_1, f_2)$ is a couple indicating the $(f_1 - 1) F_1 + f_2$-th row and where $f_1 \in [F_1]$, $f_2 \in [F_2]$, $k_1 \in [K_1]$, $k_2 \in [K_2]$. The construction of HPDA $P = (P^{(0)}, P^{(1)}, \ldots, P^{(K_1)})$ for Theorem 3 can be briefly described as follows: Given two PDAs, denoted by $A$, $B$ respectively, $P^{(0)}$ is obtained by deleting all the integers of $A$ and then simply expanding each row of it, and $(P^{(1)}, \ldots, P^{(K_1)})$ is obtained by using a hybrid method, in which $A$ acts as an outer array and the inner arrays are simply constructed from $B$.

In the follows, we first give an illustrative example to show the construction of $P$ based on two MN PDAs, and then present our general proof of coded caching scheme based on the HPDA in Theorem 3.

A. Example of the Construction of HPDA in Theorem 3

Given a $(K_1, F_1, Z_1, S_1) = (2, 2, 1, 1)$ MN PDA $A = (a_{f_1, k_1})_{f_1 \in [2], k_1 \in [2]}$ and a $(K_2, F_2, Z_2, S_2) = (3, 3, 1, 3)$ MN PDA $B = (b_{f_2, k_2})_{f_2 \in [3], k_2 \in [3]}$ where

$$A = \begin{pmatrix} * & 1 \\ 1 & * \end{pmatrix}, \quad B = \begin{pmatrix} * & 1 & 2 \\ 1 & * & 3 \\ 2 & 3 & * \end{pmatrix}.$$

We will use a hybrid method to construct a $(2, 3; 6; 3; 2; S_m, S_1, S_2)$ HPDA where

$$S_m = [4 : 9], \quad S_1 = [1 : 6], \quad S_2 = [1 : 3] \cup [7 : 9],$$

through the following three steps, as illustrated in Fig. 1.

1. **Step 1.** Construction of $P^{(0)}$ for mirror sites. We can get a $6 \times 2$ array $P^{(0)}$ by deleting all the integer entries of $A = (a_{f_1, k_1})_{f_1 \in [2], k_1 \in [2]}$ and then expanding each row 3 times.

2. **Step 2.** Construction of $(P^{(1)}, P^{(2)})$ for users. We replace the integer entries $a_{2,1}$ and $a_{1,2}$ by $B$, and replace $a_{1,1} = a_{2,2} = *$ by $B + 3, \quad B + 6$ respectively to get

$$P^{(1)} = \begin{pmatrix} B + 3 \\ B \end{pmatrix}, \quad P^{(2)} = \begin{pmatrix} B \\ B + 6 \end{pmatrix}.$$

3. **Step 3.** Construction of $P$. We get a $6 \times 8$ array by arranging $P^{(0)}$ and $(P^{(1)}, P^{(2)})$ horizontally, i.e., $P = (P^{(0)}, P^{(1)}, P^{(2)})$.

Now we verify that the construction above leads to a HPDA defined in Definition 2.

- Each column of $P^{(0)}$ has $Z_1 = 3$ stars, satisfying Condition B1 of Definition 2.
- $P^{(1)}$ and $P^{(2)}$ are $(3, 6, 2, 6)$ PDAs, satisfying Condition B2 of Definition 2.

1 This is the row number of the inner structure array which will be introduced in Step 2.
From Fig. 4 we have \( S_m = [4, 9] \), whose integers only appear in one \( P^{(k_1)}, k_1 \in [2] \), and we can check that, if \( p_{(f_1, f_2), k_2} \) is \( s \) in \( S_m \), then \( p_{(f_1, f_2), k_1} = * \), thus Condition B3 of Definition 2 holds.

Finally we claim that the Condition B4 of Definition 2 holds. Here we take \( p_{(f_1, f_2), k_2} = p_{(f'_1, f'_2), k'_2} = 1 \) as an example. From Fig. 4 we can see that \( p_{(2,2),1} = p^{(1)}_{(2,2),1} = p^{(2)}_{(2,1),1} = 1 \). When choosing \( (f_1, f_2) = (2, 2), k_1 = k_2 = 1 \) and \( (f'_1, f'_2) \in \{(1, 2), (1, 1)\} \), the corresponding \( p_{(f_1, f_2), k_1} \) equals to *, i.e., \( p_{(1,2),1} = p^{(0)}_{(1,1),1} = * \), satisfying Condition B4 of Definition 2.

**B. General Proof of Theorem 3**

In this subsection, we will prove Theorem 3 by constructing a \( (K_1, K_2; F_1 F_2; Z_1 F_2, Z_2 F_1; S_m, S_1, \ldots, S_{K_1}) \) HPDA \( P = (P^{(0)}, P^{(1)}, \ldots, P^{(K_1)}) \) with any \( (K_1, F_1, Z_1, S_1) \) PDA \( A = (a_{f_1, k_1})_{f_1 \in [F_1], k_1 \in [K_1]} \) and \( (K_2, F_2, Z_2, S_2) \) PDA \( B = (b_{f_2, k_2})_{f_2 \in [F_2], k_2 \in [K_2]} \), where \( S_m \) and \( S_{K_1} \) will be proposed in (24) and (25) respectively. The constructions of \( P^{(0)} \) and \( (P^{(1)}, \ldots, P^{(K_1)}) \) are described as follows:

**Step 1.** Construction of \( P^{(0)} \). We can get an \( F_1 F_2 \times K_1 \) array \( P^{(0)} \) by deleting all the integers in \( A \) and expanding each row by \( F_2 \) times. Then each entry \( p^{(0)}_{(f_1, f_2), k_1} \) can be written as follows.

\[
p^{(0)}_{(f_1, f_2), k_1} = \begin{cases} * & \text{if } a_{f_1, k_1} = *, \\ \\ \text{null} & \text{otherwise.} \end{cases}
\]

(19)

**Step 2.** Construction of \( (P^{(1)}, \ldots, P^{(K_1)}) \). The main idea of constructing \( (P^{(1)}, \ldots, P^{(K_1)}) \) is replacing the entries of \( A \) by inner array \( B \) and adjusting its integers. As \( A \) consists of integer-type entries and star-type entries, our construction consists of the following two parts. First we replace each integer entry \( a_{f_1, k_1} = s \) by an \( F_2 \times K_2 \) array

\[
I_1(s) = B + (s - 1) \times S_2.
\]

(20)

Secondly we replace each star entry \( a_{f_1, k_1} = * \) by an \( F_2 \times K_2 \) array

\[
I_2(k_1, f_1) = B + [(k_1 - 1)Z_1 + \varphi_{k_1}(f_1) - 1 + S_1] \times S_2,
\]

(21)

where \( \varphi_{k_1}(f_1) \) represents the order of the row labels from up to down among all the star entries in \( k_1 \)-th column of \( A \). Here we take \((P^{(1)}, P^{(2)})\) in Fig. 4 as an example. As \( a_{1,2} = a_{2,1} = s = 1 \), we have \( I_1(1) = B + (1 - 1) \times 3 = B \) in (16). While \( a_{1,1} = a_{2,2} = *, \varphi_1(1) = \varphi_2(2) = 1 \), we have \( I_2(1, 1) = B + [(1 - 1) \times 1 + 1 - 1 + 1] \times 3 = B + 3 \) and \( I_2(2, 1) = B + [(2 - 1) \times 1 + 1 - 1 + 1] \times 3 = B + 6 \) in (22), where

\[
B + 3 = \begin{pmatrix} * & 4 & 5 \\ 4 & * & 6 \\ 5 & 6 & * \end{pmatrix}, B + 6 = \begin{pmatrix} * & 7 & 8 \\ 7 & * & 9 \\ 8 & 9 & * \end{pmatrix}.
\]

(22)
From the above example, obviously the integer sets in $I_1(1) = B, I_2(1, 1) = \mathbb{B} + 3$ and $I_2(2, 2) = \mathbb{B} + 6$ have no common integer, and we can generalize the investigation to the general cases, as illustrated in Lemma 2.

**Lemma 2:** For any integers $k_1, k_1' \in [K_1], f_1, f_1' \in [F_1], s$ and $s'$ we have the following statements on the integer sets in $I_1(s)$, $I_1(s'), I_2(k_1, f_1)$ and $I_2(k_1', f_1')$.

- The integer sets in $I_1(s)$ and $I_1(s')$ have no common integer if and only if $s \neq s'$;
- When $k_1 = k_1'$, the integer sets in $I_2(k_1, f_1)$ and $I_2(k_1', f_1')$ have no common integer if and only if $\varphi_{k_1}(f_1) \neq \varphi_{k_1'}(f_1')$;
- When $k_1 \neq k_1'$, the integer sets in $I_2(k_1, f_1)$ and $I_2(k_1', f_1')$ have no common integer;
- When $s \in [S_1], I_1(s)$ and $I_2(k_1, f_1)$ have no common integer.

By Lemma 2, each entry $p_{(k_1), (f_1, f_2), k_2}$ in $(P^{(1)} \ldots P^{(K_1)})$ is determined uniquely and can be written as follows,

$$p_{(k_1), (f_1, f_2), k_2} = \begin{cases} b_{f_2, k_2} + (s-1)S_2, & \text{if } a_{f_1, k_1} = s, \\ b_{f_2, k_2} + [(k_1-1)Z_1 + \varphi_{k_1}(f_1)-1 + S_1]S_2, & \text{if } a_{f_1, k_1} = s'. \end{cases} \quad (23)$$

**Step 3. Construction of $P.$** We get an $F_1 F_2 \times (K_1 + K_1 K_2)$ array by arranging $P^{(0)}$ and $(P^{(1)}, \ldots, P^{(K_1)})$ horizontally, i.e., $P = (P^{(0)}, P^{(1)}, \ldots, P^{(K_1)})$.

1) **Parameter computations:** For a $(K, F, Z, S)$ PDA, we define $C_i$ as the integer set containing all the integers in the $i$-th column where $C_i \subset [S]$ and $|C_i| = F - Z$. Firstly we consider the integer set $S_m$, which is the union set of all integer sets of each $I_1(k_1, f_1)$. There are in total $K_1 Z_1$ stars in $A$, then from (21) and Lemma 2 we have

$$S_m = (S_1 S_2 : (S_1 + Z_1 K_1) S_2). \quad (24)$$

Obviously the cardinality of $S_m$ is $|S_m| = Z_1 K_1 S_2$.

Secondly we focus on $S_{k_1}$, i.e., the integer set of $P^{(k_1)}$ for each $k_1 \in [K_1]$. From (20), $P^{(k_1)}$ is composed of $F_1 - Z_1$ inner arrays $I_1(s)$ and $Z_1$ inner arrays $I_2(k_1, f_1)$. So the integer set $S_{k_1}$ is actually the union set of all integer sets of the $F_1$ inner arrays. Then from (20), (21) and Lemma 2 the integer set of $P^{(k_1)}$ is

$$S_{k_1} = (((k_1 - 1)Z_1 + S_1) S_2 : (k_1 Z_1 + S_1) S_2) \cup \bigcup_{s \in C_{k_1}, k_1 \in [K_1]} \bigcup_{s \in C_{k_1}, k_1 \in [K_1]} (0 + (s-1)S_2 : s S_2), \quad (25)$$

$k_1 \in [K_1]$. Because $s \in [S_1], S_{k_1}$, according to Lemma 2 all the $F_1$ integer sets of inner arrays used for composing $P^{(k_1)}$ do not have any common integer, so we have $|S_{k_1}| = F_1 S_2$.

2) **The properties of HPDA verification:** Because there are $Z_1$ stars in each column of $A$, from (19) each column of $P^{(0)}$ has exactly $Z_1 F_2$ stars, satisfying Condition B1 of Definition 1.

Then we focus on Condition B2, i.e., $P^{(k_1)}$ is a $(K_2, F_1 F_2, F_1 Z_2, F_1 S_2)$ PDA. Because $P^{(k_1)}$ is composed of $F_1$ inner arrays, each column of $P^{(k_1)}$ has $F_1 Z_2$ stars. So Condition C1 of Definition 1 holds. In the above we have $|S_{k_1}| = F_1 S_2$, obviously C2 of Definition 1 holds. Because all the arrays defined in (20) and (21) satisfy Condition C3 of Definition 1 and by Lemma 2 the intersection of the integer sets of any two of the $F_1$ inner arrays is empty, then $P^{(k_1)}$ also satisfies the Condition C3. Thus, each $P^{(k_1)}$ is a $(K_2, F_1 F_2, F_1 Z_2, F_1 S_2)$ PDA.

Now consider Condition B3. Recall that all the integers in $S_m$ are generated from (21). If $k_1 \neq k_1'$, by the third statement of Lemma 2, each integer in $S_m$ only exists in one $P^{(k_1)}$. When the entry $p_{(k_1), (f_1, f_2), k_2} = s \in S_m$, from (23) and (19) we have $a_{f_1, k_1} = *$, $P_{(f_1, f_2), k_2} = *$. Thus, Condition B3 holds.

Finally we consider the Condition B4. For any integers $k_1, k_1' \in [K_1], k_2, k_2' \in [K_2]$ and any couples $(f_1, f_2), (f_1', f_2')$, assume that $p_{(k_1), (f_1, f_2), k_2} = p_{(k_1', f_1', f_2'), k_2'} = s$ is an integer. By Lemma 2, the case $k_1 \neq k_1', f_1 = f_1'$ is impossible since the intersection of the integer sets of related inner arrays is empty. So we only need to consider the case where $k_1 \neq k_1', f_1 \neq f_1'$, and there are three conditions:

- $p_{(k_1), (f_1, f_2), k_2}$ and $p_{(k_1'), (f_1', f_2'), k_2'}$ are all in the arrays generated by (21). By the third statement of Lemma 2 this case is impossible since the intersection of the integer sets in related inner arrays is empty.
- $p_{(k_1), (f_1, f_2), k_2}$ and $p_{(k_1'), (f_1', f_2'), k_2'}$ are in the arrays generated by (20) and (21) respectively. By the forth statement of Lemma 2 this case is also impossible since the intersection of the integer sets in related inner arrays is empty.
- $p_{(k_1), (f_1, f_2), k_2}$ and $p_{(k_1'), (f_1', f_2'), k_2'}$ are all in the arrays generated by (20). Then we have $s = b_{f_2, k_2} + (s'-1) \times S_2 = b_{f_2', k_2'} + (s''-1) \times S_2$ and it’s true if and only if $(b_{f_2, k_2} = b_{f_2', k_2'}), a_{f_1, k_1} = s' = a_{f_1', k_1} = s'',$ because $b_{f_2, k_2}, b_{f_2', k_2'} \leq S_2$. Without loss of generality we assume that $p_{(k_1), (f_1, f_2), k_2}$ is an integer entry. Because $a_{f_1, k_1} = a_{f_1', k_1}' = s'$, and from Condition C3 of definition 1 we have $a_{f_1', k_1} = a_{f_1, k_1}' = *$. According to (19) we have $p_{(f_2', f_2), k_2} = *.$
From the above discussion, the Condition B4 of Definition 2 holds. Thus, \( P \) is our expected HPDA.

The integer set \( \bigcup_{k=1}^{K_1} s_k \setminus s_M \) is the union set of integer sets in \( s_1 \) inner arrays \( I_1(s), s \in [s_1] \), then we have \( |\bigcup_{k=1}^{K_1} s_k \setminus s_M| = s_1 s_2 \). From Theorem 1 we have the load for the first layer

\[
R_1 = \frac{|\bigcup_{k=1}^{K_1} s_k \setminus s_M|}{F} = \frac{s_1 s_2}{F_1 F_2}
\]

and the load for the second layer

\[
R_2 = \max \left\{ \frac{|s_k|}{F} \left| k_1 \in [K_1] \right. \right\} = \frac{F_1 s_2}{F_1 F_2} = \frac{s_2}{F_2}.
\]

VII. CONCLUSION

In this paper, we studied the hierarchical network model and introduced a new combination structure, referred as HPDA, which can be used to characterize both the placement and delivery strategy of the coded caching scheme. So the problem of designing a scheme for hierarchical network is transformed into constructing an appropriate HPDA. Firstly we propose a class of HPDAs, which achieves the lower bound of the first layer transmission load \( R_1 \) for non-trivial cases, by dividing the MN PDAs into several equal size groups. Due to the limitation of the system parameters in this class of HPDAs, we then proposed another class of HPDAs via a hybrid construction of two PDAs. Consequently, using any two PDAs, a new HPDA can be obtained which allows flexible system parameters and has a smaller subpacketization level compared with our first class of HPDAs.

APPENDIX A

LOWER BOUND OF \( R_1^* \)

Recall that during the data placement placement, the cached contents at user \( u_{k_1,k_2} \) and mirror site \( k_1 \) are \( Z_{k_1} \) and \( Z_{(k_1,k_2)} \), respectively.

Now we introduce an enhanced system where each user already knows the cache contents of its connected mirror site. For this enhanced system, denote the cache contents of the user \( u_{k_1,k_2} \) as \( Z_{(k_1,k_2)} = Z_{k_1} \cup Z_{(k_1,k_2)} \). In uncoded placement scenarios, each file can be viewed as a collection of \( 2^{K_1} K_2 \) packets as \( W_i = \{ W_{i,T} | T \subseteq [K_1] \times [K_2] \} \), where user \( u_{k_1,k_2} \) stores \( W_{i,T} \) if \( (k_1,k_2) \in T \). Consider one permutation of \([K_1] \times [K_2] \) denoted by \( \{(1,1),(1,2),\ldots,(K_1,K_2)\}\), \((k_1,k_2) \in \{K_1\times[K_2]\}\), and one demand vector \( d = \{d_{1,1},d_{1,2},\ldots,d_{K_1,K_2}\} \) where \( d_{k_1,k_2} \neq d_{k_1',k_2'} \) if \( k_1 \neq k_1' \) or \( k_2 \neq k_2' \). We then construct a genie-aided super-user with cached content

\[
\tilde{Z} = (\tilde{Z}_{(1,1)}, \tilde{Z}_{(1,2)}); (\tilde{Z}_{(2,1)} \cup W_{d_{1,1}}), \ldots, \\
\tilde{Z}_{(K_1,K_2)}); (\tilde{Z}_{(1,1)} \cup W_{d_{1,1}} \cup \tilde{Z}_{(1,2)} \cup W_{d_{1,2}} \cup \ldots \cup \tilde{Z}_{(K_1,K_2-1)} \cup W_{d_{K_1,K_2-1}}))
\]

(26)

The genie-aided super-user is able to recover \( W_{d_{1,1}}, W_{d_{1,2}},\ldots, W_{d_{K_1,K_2}} \) from \( X, X_{K_1}, \ldots, X_{K_1,K_2} \), where \( X \) and \( X_{k_1} \) are the signals sent by the server and mirror site \( k_1 \), respectively. Thus, we have

\[
\begin{align*}
H(W_{d_{1,1}}, W_{d_{1,2}}, \ldots, W_{d_{K_1,K_2}} & | \tilde{Z}) = H\left(W_{d_{1,1}}, X_{K_1}, \ldots, X_{K_1,K_2} | \tilde{Z} \right) + I\left(W_{d_{1,1}}, W_{d_{1,2}}, \ldots, W_{d_{K_1,K_2}}; X, X_{K_1}, \ldots, X_{K_1,K_2} | \tilde{Z} \right) \\
&= I\left(W_{d_{1,1}}, W_{d_{1,2}}, \ldots, W_{d_{K_1,K_2}}; X, X_{K_1}, \ldots, X_{K_1,K_2} | \tilde{Z} \right) \\
&\leq H(X, X_{K_1}, \ldots, X_{K_1,K_2} | \tilde{Z}) \\
= H(X | \tilde{Z})
\end{align*}
\]

(27)

where the last equality holds because \( \tilde{Z} \) contains all mirrors’ contents \( Z_{1,\ldots,K_1} \), leading to \( H(X_{k_1} | X, \tilde{Z}) = H(X_{k_1} | X, Z_{k_1}) = 0 \) for all \( k_1 \in [K_1] \).

Next we introduce a more powerful enhanced system where each user \( u_{k_1,k_2} \) has a caching size of \((M_1 + M_2)B\) bits, and denote its cached content as \( \tilde{Z}_{(k_1,k_2)} \). Note that this enhanced system can only result in smaller communication loads in \( R_1 \) and \( R_2 \) than that of the first enhanced system. This is because in the new enhanced system each users \( u_{k_1,k_2} \) is able to cache
any set of sub-files of \((M_1 + M_2)B\) bits, including the caching strategy of the first enhanced system \(\tilde{Z}_{(k_1, k_2)} = \tilde{Z}_{k_1} \cup \tilde{Z}_{(k_1, k_2)}\). We then construct a new genie-aided super-user with cached content
\[
\tilde{Z} = (\tilde{Z}_{(1,1)}, \tilde{Z}_{(1,2)} \setminus (\tilde{Z}_{(1,1)} \cup W_{d_{i_1}}), \ldots, \\
\tilde{Z}_{(K_2, K_2)} \setminus (\tilde{Z}_{(1,1)} \cup W_{d_{i_1}} \cup \tilde{Z}_{(1,2)} \cup W_{d_{i_2}} \cup \cdots \cup \tilde{Z}_{(K_1, K_2-1)} \cup W_{d_{K_1, K_2-1}})).
\]
(28)

Due to the stronger caching ability of the new genie-aided super-user, we have
\[
H(W_{d_{i_1}}, W_{d_{i_2}}, \ldots, W_{d_{K_1, K_2}}|\tilde{Z}) \leq H(W_{d_{i_1}}, W_{d_{i_2}}, \ldots, W_{d_{K_1, K_2}}|\bar{Z}) \leq H(X|\bar{Z}) \leq H(X).
\]
(29)

where (a) holds by (27). From (29), we obtain that
\[
R_1 \geq \sum_{(k_1, k_2) \in [K_1] \times [K_2]} \sum_{T \subseteq [K_1] \times [K_2] \setminus \{(1,1), \ldots, (k_1, k_2)\}} \frac{W_{d_{k_1, k_2}, T}}{B}.
\]
(30)

This is equivalent to a single-layer coded caching system where the server connects \(K\)-user each equipped with cache memory of \((M_1 + M_2)B\) bits. Now we follow the method in [30] to prove the lower bound of \(R_1^*\).

Summing all the inequalities in the form of (30) over all permutations of users and all demand vectors in which users have distinct demands, we obtain that
\[
R_1 \geq \sum_{t \in [0:K_1 K_2]} \frac{K_1 K_2 - t}{t + 1} \frac{X_t}{x_t}
\]
(31a)

where
\[
x_t = \sum_{t \in [N]} \sum_{T \subseteq [K_1] \times [K_2] : |T| = t} \frac{W_{i, T}}{B}.
\]
(31b)

Also, we have the following conditions due to the constraints on the file size and memory size
\[
\sum_{t \in [0:K_1 K_2]} x_t = N, \quad \sum_{t \in [0:K_1 K_2]} tx_t = K_1 K_2 (M_1 + M_2).
\]
(32)

Combining (31) and (32) and by Fourier Motzkin elimination, we obtain the lower bound of
\[
R_1^* \geq \frac{K_1 K_2 - t}{t + 1}, \quad \text{for } t \in [0 : K_1 K_2].
\]

APPENDIX B

PROOF OF LEMMA 2

Without loss of generality we assume that \(s' > s\). From (20) the first statement holds since the minimum integer of the integer set of \(I_1(s')\) minus the maximum integer of the integer set of \(I_1(s)\) is
\[
(1 + (s' - 2)S_2) - (S_2 + (s - 1)S_2) = [s' - (s + 1)]S_2 + 1 \\
\geq 1.
\]

While if \(s' = s\), \(I_1(s)\) and \(I_1(s')\) are the same array.

Without loss of generality we assume that \(\varphi_{k_1'}(f_1') > \varphi_{k_1}(f_1)\). From (21) the second statement holds since the minimum integer of the integer set of \(I_2(k_1', f_1')\) minus the maximum integer of the integer set of \(I_2(k_1, f_1)\) is
\[
1 + [(k_1' - 1)Z_1 + \varphi_{k_1'}(f_1') - 1 + S_1]S_2 - S_2 - [(k_1 - 1)Z_1 + \varphi_{k_1}(f_1) - 1 + S_1]S_2 \\
= (\varphi_{k_1'}(f_1') - \varphi_{k_1}(f_1)) - 1)S_2 + 1 \\
\geq 1.
\]

While if \(\varphi_{k_1'}(f_1') = \varphi_{k_1}(f_1)\), \(I_2(k_1', f_1')\) and \(I_2(k_1, f_1)\) are the same array.
Without loss of generality we assume that $k'_1 > k_1$. From (21) the third statement holds since the minimum integer of the integer set of $I_2(k'_1, f'_1)$ minus the maximum integer of the integer set of $I_2(k_1, f_1)$ is

$$1 + [(k'_1 - 1)Z_1 + \varphi_{k'_1}(f'_1) - 1 + S_1]S_2 - [(k_1 - 1)Z_1 + \varphi_{k_1}(f_1) - 1 + S_1]S_2$$

$$= 1 + [(k'_1 - k_1)Z_1 + \varphi_{k'_1}(f'_1) - \varphi_{k_1}(f_1) - 1]S_2$$

$$\geq 1 + [(k'_1 - k_1 - 1)Z_1]S_2$$

$$\geq 1.$$

From (20) and (21) the last statement holds since the minimum integer of the integer set of $I_2(k_1, f_1)$ minus the maximum integer of the integer set of $I_1(s)$ is

$$1 + [(k_1 - 1)Z_1 + \varphi_{k_1}(f_1) - 1 + S_1]S_2 - (S_1 - 1)S_2$$

$$= 1 + [(k_1 - 1)Z_1 + \varphi_{k_1}(f_1) - 1]S_2$$

$$\geq 1.$$

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