Exceptional Unification of Families and Forces

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Abstract

This work considers the remarkable suggestion that the three families of quarks and leptons may be unified, together with the Higgs and gauge fields of the Standard Model (SM), into a single “particle”, namely the 248 vector superfield of a ten-dimensional $E_8$ super Yang Mills (SYM) theory. Towards a realistic model along these lines, a class of orbifoldings based on $T^6/(Z_N \times Z_M)$ are proposed and explored, that can in principle break $E_8$ SYM down to the minimal supersymmetric standard model (MSSM), embedded in a larger group such as $E_6$, $SO(10)$ or $SU(5)$, together with other gauge group factors which can be broken by Wilson lines. A realistic model based on $T^6/(Z_6 \times Z_2)$ is presented. The orbifold breaks $E_8$ SYM down to a Pati-Salam gauge group in 4d, together with other gauge groups, which are further broken to the SM by Wilson lines in the right-handed sneutrino directions, yielding proto-realistic fermion mass matrices, and experimental signals associated with a low Pati-Salam gauge group breaking scale.
1 Introduction

Grand Unified Theories (GUTs) unify the three independent gauge interactions of the Standard Model (SM) gauge group \(G_{321}\) into a larger gauge symmetry \([1–6]\). GUTs also unify the representations of fermions (and scalars) into a smaller number of simpler ones. For example, the SM gauge group may be unified into a simple \(SU(5)\) gauge group, with quarks and leptons in three copies of the \(\bar{5}\) and \(10\) representations. The gauge group may be further enlarged to \(SO(10)\) which unifies the SM fermions into three copies of the irreducible \(16\) representation (predicting right-handed neutrinos). The even larger group \(E_6\) contains the previous features and extends them by including the Higgs in the same representation - provided one has \(\mathcal{N} = 1\) supersymmetry (SUSY).

The sequence of unified groups may be further extended to include the exceptional groups \(E_6, E_7, E_8\), familiar from the Dynkin diagram analysis of Lie groups \([7, 8]\),

\[
G_{321} \equiv SU(3)_C \times SU(2)_L \times U(1)_Y \subset SU(5) \subset SO(10) \subset E_6 \subset E_7 \subset E_8,
\]

where \(E_8\) is the largest finite exceptional Lie group. \(E_7\) and \(E_8\) are not normally used in GUTs. The main reason for this is that they do not have complex representations, which implies on the one hand, the presence of currently unobserved (and tightly constrained) mirror fermions, and on the other, the necessity of separating them to obtain the chiral fermion structure of the SM, where such separation is highly non-trivial. In spite of this general situation, interest in these groups as potential GUTs exists due in no small measure by the fact that they contain the following subgroup structure:

\[
SU(5) \times SU(3)_F \subset E_7, \quad \text{and} \quad SO(10) \times SU(3)_F \subset E_8,
\]

where \(SU(3)_F\) may be identified as a “family” or “flavour” symmetry. This flavour symmetry in the context of GUTs has been widely explored in the literature, within different unification settings. Such scenarios are sometimes referred to as “Flavoured GUTs” (possibly including SUSY \([9–16]\) and/or extra-dimensions (ED) \([17–27]\)). Flavoured GUTs have also been proposed based on other groups such as \(SU(7)\) \([28–30]\), \(SU(8)\) \([31, 32]\), \(SU(11)\) \([33, 34]\), \(SO(16)\) \([35, 36]\), \(SO(18)\) \([37–40]\). While there have been several models proposed based on \(E_6\) \([41–53]\), none of them includes a complete unification of families, Higgs and gauge bosons into a single multiplet.

The gauge group \(E_8\) is particularly attractive from the point of view of unification since its adjoint \(248\) representation is also the fundamental representation, and so all three families of fermions, together with Higgs scalars and gauge vector bosons may all lie in the same representation. This would be the ultimate unification: all matter, Higgs and gauge forces arising from one \(E_8\) “particle”, namely the \(248\) representation, together with one \(E_8\) gauge force. Such a supermultiplet suggests an \(\mathcal{N} = 4\) super Yang Mills (SYM) theory in four dimensions (4d), commonly regarded as the simplest quantum field theory, which has the property of being completely finite. Unfortunately, \(\mathcal{N} = 4\) in 4d is very far from the SM, since both the gauge forces and the extra supersymmetries need to be broken somehow, and it is not clear how to do this if one starts from 4d.

One interesting suggestion, is to start from \(\mathcal{N} = 1\) \(E_8\) SYM in 10d, compactified using a coset space reduction which breaks both the \(E_8\) and the would-be extended \(\mathcal{N} = 4\) SUSY in 4d, to the SM group with \(\mathcal{N} = 1\) SUSY in 4d, also removing the mirror fermions,
as desired \[54\]. The coset space reduction is achievable by orbifold compactification, and
the resulting 4d effective theory would be the minimal supersymmetric standard model (MSSM).
These ideas appeared just before the first superstring revolution \[55\], leading
to the heterotic superstring based on \(E_8 \times E_8\), followed by the second superstring rev-
olution \[56\] leading to \(M\)-theory and \(F\)-theory, all of which promised new insights into gravity. 
Although many of these approaches also involve \(E_8\) in 10d, the fundamental
starting point is very different, namely superstrings and branes as the basic objects \[57\],
and the goals and objectives of these theories are very different, namely to relate (quan-
tum) gravity to gauge theories in a unified structure. Consequently, the proposal of family 
unification based on \(\mathcal{N} = 1\ E_8\) SYM in 10d was partially eclipsed by the superstring
revolutions, and the original idea \[54\] has been largely neglected. In particular a realistic
model in which the three families of quarks and leptons are unified, together with the
Higgs and gauge fields of the Standard Model (SM) into a single \(248\) vector superfield
of an \(\mathcal{N} = 1\ E_8\) SYM theory in 10d, was never developed. This extraordinarily elegant
hypothesis is worthy of further exploration, and the development of a realistic model
along these lines is long overdue.

Towards a realistic model, this paper proposes and explores a class of orbifoldings
based on \(T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)\), which can in principle break \(E_8\) SYM down to the minimal
supersymmetric standard model (MSSM), possibly embedded in a larger group such as
\(E_6, SO(10)\) or \(SU(5)\), together with other gauge group factors which can in principle be
broken using Wilson lines. A promising example with \(T^6/(\mathbb{Z}_6 \times \mathbb{Z}_2)\) is found that breaks \(E_8\)
SYM down to a Pati-Salam gauge group in 4d \[5\], together with other gauge groups, which
are further broken to the SM by Wilson lines in the right-handed neutrino directions,
allowing proto-realistic fermion mass matrices. Thus the whole SM field content, together
with the forces, are unified in a single \(E_8\) SYM field. It is important to emphasise
that this work deals with a field theory based on point particles, and unlike apparently
related string theory \[58\] or \(F\)-theory models \[59\], gravity is not included in the present
framework. On the other hand, gravity might not be a fundamental force of nature and
could arise as an emergent phenomenon \[60–62\].

The layout of the remainder of the paper is as follows. In Section 2 the \(E_8\) SYM
theory is introduced, the \(\mathcal{N} = 4\) SUSY Lagrangian in \(R^4 \times T^6\) is presented, and it is
shown how the extended SUSY may be broken by orbifolding. Section 3 discusses
\(E_8\) gauge breaking, first by considering the orbifolding \(T^6/(\mathbb{Z}_N)\) which breaks the gauge
group \(E_8\) into different subgroups for different choices of \(N\), then by considering a general
\(T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)\) orbifolding, which preserves \(\mathcal{N} = 1\) SUSY, and finally by adding Wilson
lines to break the rank of the gauge group. In Section 4 some examples of \(E_8\) breaking
for various values of \(N,M\) are discussed, including: \(E_6 \times SU(3)_F\) from \(T^6/\mathbb{Z}_3\); \(E_8\) from
\(T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)\); \(SO(10)\) and \(SU(5)\) cases from \(T^6/\mathbb{Z}_6\); and finally a report on a general
search for the MSSM from \(T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)\) is given. Section 5 considers a Pati-Salam model
from \(T^6/(\mathbb{Z}_6 \times \mathbb{Z}_2)\). It shows how the remaining symmetry can be broken by Wilson lines,
in a right-handed sneutrino direction, such that only the SM gauge group remains, with
proto-realistic fermion mass matrices and distinctive signatures. Final comments, lines
of interest for future explorations, and the conclusions are presented in section 6.
2 $E_8$ SYM in 10d

The theory of interest is $\mathcal{N} = 1$ SYM theory in 10d for the gauge group $E_8$. It is assumed that all SM matter and gauge fields are unified into one 10d vector gauge superfield $\mathcal{V}(x, z_1, z_2, z_3)$ (where $x$ denotes the uncompactified 4d coordinates in $R^4$ and the $z_i$ denote three complex coordinates of the remaining compact 6d space) that decomposes into a 10d vector field and a 10d Majorana fermion (which in 10d is also a Weyl fermion). The basic hypothesis is that all SM matter, Higgs and gauge fields are unified into a single “particle”, namely the 10d vector superfield: the $248 \sim (248)$ representation of $E_8$. In $E_8$ the $248$ is both the adjoint and the fundamental representation and it is real. The 10d vector superfield $\mathcal{V}$ decomposes into a 4d vector superfield $V$ and three 4d chiral superfield multiplets $\phi_{1,2,3}$. In general, $\mathcal{N} = 1$ SUSY in $n = 7, 8, 9, 10$ dimensions, implies $\mathcal{N} = 4$ SUSY in 4d after compactification [63,64]. In particular this implies that $E_8$ SYM theory with $\mathcal{N} = 1$ SUSY in 10d in principle has $\mathcal{N} = 4$ SUSY in 4d after compactification to a compact 6d torus $T^6$. The $\mathcal{N} = 4$ SYM Lagrangian in $R^4 \times T^6$ is displayed first, followed by a description of how the extra supersymmetries may be broken by orbifolding.

2.1 The $\mathcal{N} = 4$ SYM Lagrangian in $R^4 \times T^6$

The extra dimensional space $T^6$ (assumed here to be a six dimensional torus) is parametrized with the three complex coordinates $z_{1,2,3}$ as $(T^2)^3$. These are called symmetric toroidal orbifolds. After compactification to the torus $T^6$, the 10d real vector and 10d Majorana fermion components of the supermultiplet decompose into one 4d real vector $A_\mu$, six 4d real scalars $X$ and four 4d Weyl fermions $\lambda$. The compactified $\mathcal{N} = 4$ SYM (4d) Lagrangian is

$$\mathcal{L} = \frac{1}{2g^2} F_{\mu
u} F^{\mu\nu} + \frac{\theta_I}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} - i \bar{\lambda} \sigma^{\mu\nu} D_\mu \lambda - D_\mu X^i D^\mu X^i + g C_{iab} \lambda_a [X^i, \lambda_b] + g C_{iab} \bar{\lambda}_b [X^i, \bar{\lambda}_a] + \frac{g^2}{2} [X^i, X^j]^2,$$

(3)

where $i, j = 1...6; a, b = 1...4$, and $C_{iab}$ represent the $SU(4)_R$ structure constants. The $SU(4)_R \simeq SO(6)_R$ symmetry comes as a remnant of the 6d rotation group of the extra dimensions $O(6)$. There are only two gauge coupling constants $g$ and $\theta_I$, and all the vertices are completely defined by them.

As the compactification will actually be in $(T^2)^3$, the theory can be more conveniently seen in terms of simple $\mathcal{N} = 1$ SUSY by having one gauge vector supermultiplet $V$ and three chiral supermultiplets $\phi^i$:

$$\mathcal{L} = \frac{1}{32} \int d^2 \theta \ W^a W_a + \int d^2 \theta d^2 \tilde{\theta} e^{2gV} \phi^i - \int d^2 \theta \sqrt{2} g \phi_1 [\phi_2, \phi_3] + h.c.,$$

(4)

where $i = 1, 2, 3$. Note the explicit $SU(3)_R \times U(1)_R$ symmetry remaining from the rotation between the three complex coordinates and the complex rotation in all of them. This is particularly helpful since one can relate each chiral supermultiplet to the degrees of freedom of the vector superfield that come from each complex extra dimension.
\section{2.2 SUSY breaking by orbifolding}

The extra dimensions are assumed to be orbifolded by a discrete group \( F \) so that the actual extra dimensional space is \( T^6/F \). In general, there are six extra dimensions with Poincaré symmetry \( O(6) \times T^6/\Gamma \), where \( O(6) \) are rotations, and \( T^6 \) are the translations. The translation group is modded by the lattice vectors \( \Gamma = \mathbb{Z}^6 \) which makes it compact \( \mathbb{R}^6 \to T^6 \cong \mathbb{R}^6/\Gamma \). Orbifolding means modding the rotation group by a discrete subgroup \( F \in O(6) \). The group \( F \) must be a symmetry of the lattice \( F\Gamma = \Gamma \) to consistently define an orbifold. The rotation group is \( O(6) \cong SO(6) \times Z_2 \cong SU(4) \times Z_2 \). If one desires to keep some SUSY after orbifolding, then it can be done only by a discrete subgroup of \( SU(3) \) [65][66]. A simple \( Z_2 \), for example, would break \( \mathcal{N} = 4 \to \mathcal{N} = 2 \) SUSY.

The simplest way to break \( \mathcal{N} = 4 \to \mathcal{N} = 1 \) SUSY is with the \( Z_3 \) orbifolding,

\[
(x, z_1, z_2, z_3) \sim (x, \omega z_1, \omega z_2, \omega z_3),
\]

where \( \omega = e^{2\pi i/3} \) and \( R(\omega) \) is the representation of the \( Z_3 \) rotation acting on the vector superfield. In terms of the components of this superfield, the orbifolding acts as

\[
V(x, z_1, z_2, z_3) = V(x, \omega z_1, \omega z_2, \omega z_3), \quad \phi^i(x, z_1, z_2, z_3) = \omega \phi^i(x, \omega z_1, \omega z_2, \omega z_3).
\]

Note that, as desired, this orbifolding leaves invariant the Lagrangian in eq. (4). Each chiral multiplet is associated with the extra dimensional degrees of freedom of the vector supermultiplet. Multiplying a complex coordinate must also multiply the corresponding components of a vector, which in this case are the chiral supermultiplets. This orbifolding would only leave as a zero mode the \( \mathcal{N} = 1 \) vector multiplet \( V \), therefore providing a pure \( \mathcal{N} = 1 \) SYM at low energies. In order to get the SM (or MSSM) one needs to consider more general orbifoldings than \( Z_3 \).

A more general orbifolding that preserves “simple” \( (\mathcal{N} = 1) \) SUSY is

\[
F \cong Z_N \subset SU(3),
\]

with a positive integer \( N \). A general \( Z_N \) orbifolding can be defined as

\[
(x, z_1, z_2, z_3) \sim (x, e^{2\pi i n_1/N} z_1, e^{2\pi i n_2/N} z_2, e^{2\pi i n_3/N} z_3)
\]

\[
V(x, z_1, z_2, z_3) = R(e^{2\pi i n_1/N}, e^{2\pi i n_2/N}, e^{2\pi i n_3/N}) V(x, e^{2\pi i n_1/N} z_1, e^{2\pi i n_2/N} z_2, e^{2\pi i n_3/N} z_3),
\]

where, as before, \( R \) is the representation of the \( Z_N \) rotation acting on the 10d vector superfield. For the transformation to belong to \( SU(3) \) it must satisfy

\[
n_1 + n_2 + n_3 = 0 \mod N,
\]

so that it has determinant 1. If we want SUSY to be preserved, the actual constraint would be

\[
n_1 + n_2 + n_3 = 0 \mod 2N,
\]

since fermions and scalars are rotated together, while fermions rotate twice as slow [67]. Also note that can be \( n_i > N \). This orbifolding decomposes the fields as

\[
V(x, z_1, z_2, z_3) = V(x, e^{2\pi i n_1/N} z_1, e^{2\pi i n_2/N} z_2, e^{2\pi i n_3/N} z_3),
\]

\[
\phi^i(x, z_1, z_2, z_3) = e^{2\pi i n_i/N} \phi^i(x, e^{2\pi i n_1/N} z_1, e^{2\pi i n_2/N} z_2, e^{2\pi i n_3/N} z_3).
\]
By choosing the $n_i$, one chooses the chiral supermultiplets that do have zero modes and those that do not.

Finally note that the most general Abelian orbifolding that preserves $\mathcal{N} = 1$ SUSY is $F \simeq \mathbb{Z}_N \times \mathbb{Z}_M \subset SU(3)$, with positive integers $N, M$ (for discussion on the consistent Abelian and non Abelian orbifolds see [68–70]). If preservation of $\mathcal{N} = 1$ SUSY is not required, then the most general Abelian group available for orbifolding would be $F \simeq \mathbb{Z}_N \times \mathbb{Z}_M \times \mathbb{Z}_L \times \mathbb{Z}_2$. The additional $\mathbb{Z}_L$ belongs to $U(3)$, with general positive integer $L$, and it need not satisfy eq. (10). This orbifolding breaks SUSY completely by adding a phase (the non identity determinant of the corresponding operation) to the $\theta$ coordinate and thus killing all fermionic zero modes. The extra $\mathbb{Z}_2$ is related to reflection in the extra dimensions, and in this notation would involve complex conjugation of the $z_i$. Thus $\mathcal{N} = 1$ SUSY is broken while preserving an R parity identified as $\mathbb{Z}_2$. This could be used to stabilize a dark matter candidate, for example.

3 \ E_8 breaking

This section contains the proposal and discussion of the general orbifold $T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$ and how it can break the gauge group $E_8$ in various directions, while preserving $\mathcal{N} = 1$ SUSY. To recap, the starting point is $\mathcal{N} = 1$ SYM with a single 10d vector superfield in the adjoint representation: $V_{248} \sim (248)$. In $E_8$ this is also the fundamental representation and it is real. As previously seen, assuming that $\mathcal{N} = 1$ SUSY is preserved in the 4d theory, the 10d vector superfield $\mathcal{V}$ decomposes into a 4d vector superfield $V$ and three 4d chiral superfield multiplets $\phi_{1,2,3}$,

$$V_{248} \rightarrow \{V, \phi_{1,2,3}\}_{248} \quad (12)$$

where the chiral scalar superfields $\phi_{1,2,3}$ arise from the extra dimensional components of the original 10d gauge field $\mathcal{V}$. The task now is to also break the $E_8$ gauge theory into different subgroups, such that the 248 splinters into representations of the smaller subgroup, with some components surviving as zero (massless) modes and other components only having (very) heavy massive modes, making them practically unobservable, allowing suitable for applications to particle physics with the massless modes identified as the starting point for various 4d models.

The model requires the extra dimensions to form the orbifold $T^6/F$, where $F$ is a discrete subgroup of $F \subset SU(4) \times \mathbb{Z}_2$. Assuming the orbifold to be Abelian and to preserve $\mathcal{N} = 1$ SUSY, one is led to consider the general orbifold,

$$F \simeq \mathbb{Z}_N \times \mathbb{Z}_M \subset SU(3), \quad (13)$$

for positive integers $N$ and $M$, as defined in the next subsection. However, the analysis is performed by first considering a single $T^6/\mathbb{Z}_N$ orbifolding in the next subsection.

3.1 \ E_8 breaking by $T^6/\mathbb{Z}_N$ orbifolding

The standard mechanism to break a gauge symmetry through a $\mathbb{Z}_N$ orbifolding is by adding a gauge transformation to the orbifold transformation. Since $\mathbb{Z}_N \subset U(1)$, one can assume that $\mathbb{Z}_N$ is accompanied by a specific $U(1)_a \subset E_8$ transformation. This would
break the original gauge symmetry into the subgroup that commutes with $U(1)_a$. Let $q_a^f$ be the charge of a field $f$ under the chosen $U(1)_a$. Applying the boundary condition

$$f \sim e^{2i\pi q_a^f/N}f,$$

(14)

breaks the symmetry consistently into a subgroup preserving the $U(1)_a$ (e.g. the multiplicative phase would correspond to a simple parity of ±1 for the simplest examples based on an orbifold parity of $\mathbb{Z}_2$).

A general $\mathbb{Z}_N$ orbifolding is defined by identifying

$$(x, z_1, z_2, z_3) \sim (x, e^{2i\pi n_1/N}z_1, e^{2i\pi n_2/N}z_2, e^{2i\pi n_3/N}z_3),$$

(15)

with arbitrary integers $n_i$ satisfying $n_1 + n_2 + n_3 = 0 \mod 2N$. The decomposed 10d superfield is transformed as

$$V(x, z_1, z_2, z_3) = e^{2i\pi q_i^V/N}V(x, e^{2i\pi n_1/N}z_1, e^{2i\pi n_2/N}z_2, e^{2i\pi n_3/N}z_3),$$

$$\phi^i(x, z_1, z_2, z_3) = e^{2i\pi q_i^\phi/N}e^{2i\pi q_i^V/N}\phi^i(x, e^{2i\pi n_1/N}z_1, e^{2i\pi n_2/N}z_2, e^{2i\pi n_3/N}z_3),$$

(16)

so that each multiplet is multiplied by a phase associated to the multiplet itself and to its charge. The representation of $V$ is that of the adjoint of the unbroken gauge group, i.e. the fields with identity boundary conditions. The representation of the light chiral superfield $\phi_i$ is that of the fields with charge $q_i^\phi = -n_i \mod N$. One then chooses the $n_i$ to leave the desired light fields. This defines the orbifolding.

$E_8$ has rank 8 and the orbifolding must preserve the SM gauge symmetry, $SU(3)_C \times SU(2)_L \times U(1)_Y$, which has rank 4. This means that there are four different $U(1)$ groups (in addition to $U(1)_Y$, of course) that commute with the SM. One may define them by following the exceptional sequence [71]

$$E_8 \supset E_7 \times U(1)_F$$

$$\supset E_6 \times U(1)_{F'} \times U(1)_F$$

$$\supset SO(10) \times U(1)_{X'} \times U(1)_{F'} \times U(1)_F$$

$$\supset SU(5) \times U(1)_{X} \times U(1)_{X'} \times U(1)_{F'} \times U(1)_F$$

$$\supset SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \times U(1)_{X'} \times U(1)_{F'} \times U(1)_F,$$

(17)

where any of the (or a linear combination) can be chosen to be the $\mathbb{Z}_N$ orbifold operation

$$\mathbb{Z}_N \subset U(1)_Y \times U(1)_X \times U(1)_{X'} \times U(1)_{F'} \times U(1)_F.$$

(18)

Depending on which one is chosen, and the order $N$, one obtains a certain preserved group. This is shown in detail in Table [I] where the breakings are presented for all the different choices from $\mathbb{Z}_2$ to $\mathbb{Z}_7$ (any orbifolding with a group larger than $\mathbb{Z}_7$ would not break further the symmetry).

By considering the orbifold $T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$, with combinations of $N, M$ selected from Table [I] various patterns of symmetry breaking can be achieved consistently while preserving an $\mathcal{N} = 1$ SUSY. The idea is that the preserved group is the intersection of the preserved groups shown for individual values of $N, M$ chosen from Table [I]. However achieving the desired symmetry breaking pattern is not enough: it is also necessary to obtain the required chiral matter multiplets, and this can only be determined case by case. The strategy consists of searching specific numbers that define the orbifold that breaks the $\mathcal{N} = 4$ SUSY $E_8$ into a smaller group. Particular examples are discussed later.
SU field content is made of 19 multiplets \( (g_3.2, E_8) \). It is clear that the SM field content is contained in the single 10d vector superfield compactification.

Since the initial symmetry is strongly restrictive, it is up to the structure of the orbifold to weed out the extra fields and generate the effective couplings after compactification.

### 3.2 \( E_8 \) breaking by a more general \( T^6/(\mathbb{Z}_N \times \mathbb{Z}_M) \) orbifolding

It is clear that the SM field content is contained in the single 10d vector superfield \( \mathcal{V}_{248} \), however there are many more fields present. The SM has an \( SO(3, 1) \) Lorentz symmetry and \( SU(3)_C \times SU(2)_L \times U(1)_Y \) gauge symmetry. Under these symmetries, the SM field content is made of 19 multiplets \( (g_\mu, W_\mu, B_\mu, Q_i, L_i, e_i, d_i^c, u_i^c, H) \).

A viable model should resemble the SM, with its field content and their couplings, at low energies. Since the initial symmetry is strongly restrictive, it is up to the structure of the orbifold to weed out the extra fields and generate the effective couplings after compactification.

Consider a more general orbifold \( T^6/(\mathbb{Z}_N \times \mathbb{Z}_M) \), where\(^4\)

\[
\mathbb{Z}_N \times \mathbb{Z}_M \subset U(1)_Y \times U(1)_X \times U(1)_{X'} \times U(1)_{F'} \times U(1)_{F}. \tag{19}
\]

This orbifold is defined by linear combinations of charges

\[
\mathbb{Z}_N : \phi \rightarrow e^{2\pi i (a\phi_Y + b\phi_X + c\phi_{X'} + d\phi_{F'} + e\phi_{F})/N} \phi, \quad \mathbb{Z}_M : \phi \rightarrow e^{2\pi i (g\phi_Y + h\phi_X + i\phi_{X'} + j\phi_{F'} + k\phi_{F})/M} \phi, \tag{20}
\]

which are applied as

\[
(x, z_1, z_2, z_3) \sim (x, e^{2im_1\pi/N} z_1, e^{2im_2\pi/N} z_2, e^{2im_3\pi/N} z_3),
\]

\[
(x, z_1, z_2, z_3) \sim (x, e^{2im_1\pi/M} z_1, e^{2im_2\pi/M} z_2, e^{2im_3\pi/M} z_3), \tag{21}
\]

with arbitrary positive integers \( N \) and \( M \), arbitrary integers \( (a, b, c, d, e, g, h, j, k, l) \), and \( n_1 + n_2 + n_3 = m_1 + m_2 + m_3 = 0 \mod 2N \), preserving SUSY.

In the next subsection, it is shown that, although the orbifolding by itself can never break to the SM gauge group directly, nevertheless this can in principle be achieved by Wilson lines.

### 3.3 Wilson Lines and Scalar Vacuum Expectation Values

It is apparent that no orbifolding can ever break \( E_8 \rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y \) directly. The reason is clear: orbifolding by itself cannot reduce the rank of the group \([72, 74]\). However, it is possible to add non trivial gauge transformations to whole loop translation in each circle. Adding these phases creates Wilson lines that generate effective vacuum expectation values (VEVs) for the scalars coming from the extra dimensional part of gauge fields, dynamically breaking the symmetry \([75]\).

The setup has six different circles and six independent phases \( U_i \) with \( i = 1, \ldots, 6 \) defined by

\[
U_i = e^{i\alpha_i \omega_i T_a}, \tag{22}
\]

\(^4\)Note such orbifolds are also constrained by lattice consistency \([69]\), which we check case by case.
where $\omega_i$ are the basis vectors for each tori. All phases have to commute

$$[U_i, U_j] = 0. \quad (23)$$

The Wilson lines generate VEVs in the extra dimensional components of the gauge vectors. Since preserving SUSY is desired in this case, the six extra-dimensional components of the gauge vector arrange themselves into the three complex scalar components of the resulting three chiral supermultiplets. SUSY preservation forbids the six VEVs to be independent and only three independent complex ones are obtained:

$$\langle \phi^a_i \rangle = \frac{\alpha^a_i}{2\pi R_i}, \quad (24)$$

with $i = 1, 2, 3$ and where there are only three independent radii coming from the three independent tori. Note that for consistency with the orbifold, the VEVs must lie in chiral supermultiplets with a zero mode: the VEVs must be aligned in the gauge representations that have a zero mode. By integrating out the other fields, one can obtain an effective potential for the fields that get a VEV $[76–81]$. The $D = 0$ and $F = 0$ flatness conditions must also be satisfied by the VEVs if SUSY is to be preserved.

In conclusion, using the Wilson line mechanism one can give VEVs for the chiral supermultiplets with zero modes, and perhaps even the massive modes, through three independent commuting phases $\alpha^a_i$. Such continuous Wilson lines, not in the direction of the adjoint representation of the remaining gauge group, reduce the rank. Since orbifolding does not reduce rank, the symmetry breaking induced by the continuous Wilson lines is a crucial ingredient of the model.\footnote{Note that discrete Wilson lines (not considered here) would also not reduce the rank, even though they could break the gauge symmetry.}

### 4 Examples of $E_8$ breaking

The biggest challenge when building a unification model based on $E_8$ is the fact that it only has real representations. This is a problem because fermions in the SM are chiral, and so far there is no evidence for the existence of mirror fermions. Orbifolding gives a way to overcome this problem. The simplest model found, that obtains a chiral representation from $E_8$, ends up with $E_6$ as the remaining gauge symmetry after compactification. To see how this happens, it is convenient to analyze how the representations decompose into the different subgroup representation due to a particular orbifold. This gives the information needed to determine what fields survive in the low energy theory. This section contains a series of examples to illustrate the procedure one must follow to obtain a realistic model from the preceding formalism in the previous section, starting with the simplest $E_6$ example, before moving on to other examples.

#### 4.1 $E_6 \times SU(3)_F$ from $T^6/Z_3$

An interesting possibility is the orbifold $T^6/Z_3 [82–84]$, where the $Z_3 \subset U(1)_X$: orbifolding from Table 1 breaks $E_8 \rightarrow E_6 \times SU(3)_F$, where

$$248 \rightarrow (78, 1) + (1, 8) + (27, 3) + (27, \bar{3}). \quad (25)$$
The $U(1)_{X'}$ charges are identified by considering the hypothetical (but unachievable as shown below) further decomposition into $SO(10) \times U(1)_{X'} \times SU(3)_F$ representations as

\[
(78, 1) \rightarrow (45, 0, 1) + (1, 0, 1) + (16, -3, 1) + (16, 3, 1),
\]
\[
(1, 8) \rightarrow (1, 0, 8),
\]
\[
(27, 3) \rightarrow (16, 1, 3) + (10, -2, 3) + (1, 4, 3),
\]
\[
(27, 3) \rightarrow (16, -1, 3) + (10, 2, 3) + (1, -4, 3),
\]

where one can see that each $E_6 \times SU(3)_F$ representation has the same $U(1)_{X'}$ charge mod 3, corresponding to the $\mathbb{Z}_3$ symmetry. For example, $(78, 1)$ and $(1, 8)$ have $U(1)_{X'}$ charge zero mod 3, corresponding to a $\mathbb{Z}_3$ singlet (denoted as 1). Therefore this orbifold indeed breaks $E_8 \rightarrow E_6 \times SU(3)_F$ and not $E_8 \rightarrow SO(10) \times U(1)_{X'} \times SU(3)_F$.

Under $E_8 \rightarrow E_6 \times SU(3)_F$, the single 10d vector superfield in the adjoint representation $\mathcal{V}_{248} \sim 248$ is separated into $E_6 \times SU(3)_F$ multiplets with three different $\mathbb{Z}_3$ charges, listed in each line

\[
\mathcal{V}_{248} \rightarrow \mathcal{V}_{(78, 1)} + \mathcal{V}_{(1, 8)} \\
+ \mathcal{V}_{(27, 3)} \\
+ \mathcal{V}_{(27, 3)},
\]

where $\mathcal{V}_{(78, 1)}$ and $\mathcal{V}_{(1, 8)}$ are $\mathbb{Z}_3$ charge 1 (singlets), while $\mathcal{V}_{(27, 3)}$ has one unit of $\mathbb{Z}_3$ charge $\omega = e^{2\pi i/3}$, and $\mathcal{V}_{(27, 3)}$ has $\mathbb{Z}_3$ charge $\omega^2$.

The action of the orbifold is defined as

\[
(x, z_1, z_2, z_3) \sim (x, \omega^2 z_1, \omega^2 z_2, \omega^2 z_3),
\]

where $\omega = e^{2\pi i/3}$ as well as

\[
\mathbb{Z}_3 : \mathcal{V} \rightarrow e^{2i\pi X'/3} \mathcal{V}.
\]

The orbifold decomposes the 10d vector superfield to $N = 1$ vector and chiral superfields,

\[
\mathcal{V} \rightarrow \{ V, \phi_{1,2,3} \},
\]

where the chiral superfields $\phi_{1,2,3}$ are associated with the extra dimensions $z_1, z_2, z_3$, respectively. The resultant charges, under the combined action of the orbifold and $\mathbb{Z}_3$, for each $N = 1$ multiplet are listed in table 2. This is a simple application of the general formula in eq. (15). Therefore after compactification, at low energies, one has $N = 1$ SUSY with $E_6 \times SU(3)_F$, where the final - low energy - field content is composed of the fields in table 2 with charge 1. Note that there are fermions with zero modes contained in $\phi_{(27, 3)}$. Note also that, importantly, there are no mirror fermions with zero modes and one then obtains chiral matter from a real representation $S[5]$.  

Thus, orbifolding has produced a chiral representation from a real one. This is good news, however it is not quite what one needs yet. As table 2 shows, the orbifolding leaves three copies of triplets, $\phi_{i(27, 3)}$ with $i = 1, 2, 3$, therefore providing nine light families, which are too many.  

\[\text{Since there are no antitriplets this theory also suffers under an SU(3)}^3 \text{ gauge anomaly. In string theory, this gauge anomaly is cancelled by 81 anti-triplets of SU(3) that are localized at the fixed points of the Z}_3 \text{ orbifold. However here we only consider matter in the bulk arising from the } \mathcal{V}_{248} \sim 248.\]
Table 2: Charges of each $\mathcal{N} = 1$ superfield $E_6 \times SU(3)_F$ multiplet under the $\mathbb{Z}_3$ orbifolding. Only the charge singlets (denoted by charges $1$) have massless zero modes and survive in the low energy theory.

| $\mathcal{V}$         | $V$ | $\phi_1$ | $\phi_2$ | $\phi_3$ |
|-----------------------|-----|----------|----------|----------|
| $\mathcal{V}_{(78,1)}$| 1   | $\omega^2$ | $\omega^2$ | $\omega^2$ |
| $\mathcal{V}_{(1,8)}$ | 1   | $\omega^2$ | $\omega^2$ | $\omega^2$ |
| $\mathcal{V}_{(27,3)}$| $\omega$ | 1 | 1 | 1 |
| $\mathcal{V}_{(27,3)}$| $\omega^2$ | $\omega$ | $\omega$ | $\omega$ |

Table 3: Charges of each $\mathcal{N} = 1$ superfield $E_6 \times U(1)_{F'} \times U(1)_{F}$ multiplet under a $\mathbb{Z}_3 \times \mathbb{Z}_3'$ orbifolding. Only the fields with both charges equal to unity have zero modes.

| $\mathcal{V}$         | $V$ | $\phi_1$ | $\phi_2$ | $\phi_3$ |
|-----------------------|-----|----------|----------|----------|
| $\mathcal{V}_{(27,1,1)}$| $\omega, \omega^2$ | 1 | $\omega^2$ | 1, 1 | 1, $\omega$ |
| $\mathcal{V}_{(27,1,-1)}$| $\omega, 1$ | 1 | 1 | $\omega$, $\omega^2$ |
| $\mathcal{V}_{(27,-2,0)}$| $\omega, \omega$ | 1 | $\omega$, $\omega^2$ | 1 | 1 |
| $\mathcal{V}_{(27,-1,-1)}$| $\omega^2, \omega$ | $\omega$, $\omega^2$ | $\omega$, $\omega^2$ | $\omega$, $\omega^2$ |
| $\mathcal{V}_{(27,-1,1)}$| $\omega^2, 1$ | $\omega$, $\omega$ | $\omega$, $\omega$ | $\omega$, $\omega^2$ |
| $\mathcal{V}_{(27,2,0)}$| $\omega^2, \omega^2$ | $\omega^2, \omega^2$ | $\omega$, $\omega$ | $\omega$, $\omega^2$ |

4.2 $E_6$ based model from $T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3')$

A further problem of the previous example, is that the family symmetry $SU(3)_F$ is unbroken. Starting from the setup in the previous section, another orbifolding $\mathbb{Z}_3$ is desired that breaks $SU(3)_F \to U(1)_{F'} \times U(1)_{F}$ with the representation decomposition

\[
\begin{align*}
3 &\to (1, 1) + (1, -1) + (-2, 0), \\
8 &\to (0, 0) + (0, 0) + (3, 1) + (3, -1) + (0, -2) + (-3, -1) + (-3, 1) + (0, 2).
\end{align*}
\] (31)

Therefore we are led to consider $T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3')$, where the first $\mathbb{Z}_3$ orbifolding is as in the previous subsection and the extra $\mathbb{Z}_3'$ orbifold operation is defined by

\[
(x, z_1, z_2, z_3) \sim (x, \omega^3 z_1, \omega z_2, \omega^2 z_3),
\] (32)

with

\[
\mathbb{Z}_3' : \phi \to e^{i\pi(q_\phi+q_{\phi'})/3} \phi.
\] (33)

Note that the coordinate $z_1$ is rotated completely by $\omega^3 = 1$ to comply with eq.\ 10.

Following the same procedure as discussed in subsection 4.1.1, this leaves only three light chiral superfields as shown in table 3: $\phi_1(27,1,-1), \phi_2(27,1,1), \phi_3(27,-2,0)$.

After compactification, each $27$, contains a full $16$ component field of fermions plus a $10$ component multiplet which contains one pair of Higgs doublets per family, plus exotics, plus a singlet $1$. The resulting field theory after compactification, (although it does not preserve SUSY [69], resembles the $E_6$SSM [66] in its minimal version [77] (for a recent review see [88]). Thus with $T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3')$ the full SM (plus additional states of an $E_6$ GUT) can result from the single 10d $248$ vector superfield.
\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
& $V$ & $\phi_1$ & $\phi_2$ & $\phi_3$ & $V$ & $\phi_1$ & $\phi_2$ & $\phi_3$
\hline
$\phi_{(45,1,0)}$ & 1 & $\alpha^5$ & $\alpha^2$ & $\alpha^5$ & $\phi_{(16,1,-3)}$ & $\alpha^4$ & $\alpha^2$ & $\alpha^5$ & $\alpha^4$
$\phi_{(1,1,0)}$ & 1 & $\alpha^5$ & $\alpha^2$ & $\alpha^5$ & $\phi_{(16,1,3)}$ & $\alpha^4$ & $\alpha^2$ & $\alpha^5$ & $\alpha^2$
$\phi_{(1,8,0)}$ & 1 & $\alpha^5$ & $\alpha^2$ & $\alpha^5$ & $\phi_{(1,3,4)}$ & $\alpha^4$ & $\alpha^3$ & 1 & $\alpha^3$
$\phi_{(16,3,1)}$ & $\alpha$ & 1 & $\alpha^3$ & 1 & $\phi_{(10,3,-2)}$ & $\alpha^4$ & $\alpha^3$ & 1 & $\alpha^3$
$\phi_{(1,3,-4)}$ & $\alpha^2$ & $\alpha$ & $\alpha^4$ & $\alpha$ & $\phi_{(16,3,-1)}$ & $\alpha^5$ & $\alpha^2$ & $\alpha$ & $\alpha^4$
$\phi_{(10,3,2)}$ & $\alpha^2$ & $\alpha$ & $\alpha^4$ & $\alpha$ & & & & & \\
\hline
\end{tabular}
\end{center}

Table 4: Charges of each $N=1$ superfield $SO(10) \times SU(3)_F \times U(1)_{X'}$ multiplet under a $Z_6$ orbifolding. Only the fields with unit charges have zero modes, leading to an $SO(10)$ model after compactification.

4.3 $SO(10)$ based model from $T^6/Z_6$

We now consider using $Z_6 \in U(1)_{X'}$ for the orbifold. The 248 representation decomposes under $SO(10) \times SU(3)_F \times U(1)_{X'}$ as

\begin{equation}
(248) \rightarrow (45,1,0) + (1,1,0) + (1,8,0) + (16,3,1) + (1,3,-4) + (10,3,2) + (16,1,-3) + (16,1,3) + (1,3,4) + (10,3,-2) + (16,3,-1).
\end{equation}

The fields in each of the six lines of eq. 34 have a common $Z_6$ charge: 1, $\alpha$, $\alpha^2$, $\alpha^3$, $\alpha^4$, $\alpha^5$, respectively, where $\alpha = e^{2\pi i/6}$. The orbifolding can be applied as

\begin{equation}
(x, z_1, z_2, z_3) \sim (x, \alpha^5 z_1, \alpha^2 z_2, \alpha^5 z_3),
\end{equation}

where $\alpha = e^{2\pi i/6}$ and

\begin{equation}
Z_6 : \phi \rightarrow e^{2\pi q_{X'}/6} \phi.
\end{equation}

Following the same procedure as discussed in subsection 4.1, this gives the charge of each $N=1$ superfield as shown in Table 4.

Although $\phi_{2(10,3,-2)}$ contains the MSSM Higgses, both $\phi_{1(16,3,1)}$ and $\phi_{3(16,3,1)}$ have a zero mode and each contain three SM fermion families. The model needs another orbifolding to further break the symmetry, and halve the fermion content. Although the other orbifolding cannot break the flavour symmetry as in sec. 4.2, the model does contain a flavon $\phi_{2(1,3,4)}$ which can do the job. The ingredients of the above $T^6/Z_6$ orbifold, when supplemented by a further $Z_4$ orbifolding, are the basis of the realistic model, described in sec. 5.
4.4 \( SU(5) \) based from \( T^6/Z_6 \)

One may also consider using \( Z_6 \in U(1)_X \) for the orbifold. The 248 representation decomposes under \( E_8 \rightarrow SU(5) \times SU(4) \times U(1)_X \) as

\[
(248) \rightarrow (24, 1, 0) + (1, 1, 0) + (1, 15, 0) \\
+ (1, 4, -5) + (\bar{10}, \bar{4}, 1) \\
+ (10, 1, -4) + (5, 6, 2) \\
+ (5, 4, 3) + (5, 4, -3) \\
+ (10, 1, 4) + (5, 6, -2) \\
+ (10, 4, -1) + (1, 4, 5),
\]

where the fields in the same line share the same \( Z_6 \) charge: 1, \( \alpha \), \( \alpha^2 \), \( \alpha^3 \), \( \alpha^4 \), \( \alpha^5 \), respectively, where \( \alpha = e^{2i\pi/6} \). One may now consider various choices of geometrical orbifoldings, leading to various massless modes. We have considered many such possibilities, but in the interests of brevity we do not display the results here. Instead we highlight a common challenge to all such models.

Once a particular \( SU(5) \) model is considered, one must consider the usual GUT problem of Doublet-Triplet splitting. Since there is only a single 248 representation in the beginning, the field content and its symmetries are fixed. There are no representations to allow the Missing Partner mechanism [89], nor the shaping symmetry to make the Dimopolous-Wilczeck mechanism [90]. The only way to achieve it in this setup is through orbifolding. The next available \( Z_M \) orbifolding should break \( SU(5) \) and achieve the Doublet-Splitting. However this leaves an unbroken \( SU(4) \) flavour symmetry, and therefore any such setup generically contains 4 families at low energies.

4.5 \( SM \) based model from \( T^6/(Z_N \times Z_M) \)

Finally, we discuss the results of a general scan of \( T^6/(Z_N \times Z_M) \) orbifolds which can give rise to the SM gauge group factor, together with other gauge group factors. We find that imposing the requirement that the field content with zero modes must have three families of fermions does not allow the breaking of the \( SU(3)_F \) family symmetry from orbifolding. Therefore, without loss of generality, one can choose linear combinations of charges restricted such that

\[
Z_N \times Z_M \subset U(1)_Y \times U(1)_X \times U(1)_{X'}. \tag{38}
\]
The decomposition under $E_8 \rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \times U(1)_{X'} \times SU(3)_F$ of the 248 representation is (using a colour coding explained below):

$$(248) \rightarrow (8, 1, 0, 0, 0, 1) + (1, 3, 0, 0, 0, 1) + (1, 1, 0, 0, 0, 8)$$
$$+ (1, 1, 0, 0, 0, 1) + (1, 1, 0, 0, 0, 1) + (1, 1, 0, 0, 0, 1)$$
$$+ (1, 1, 6, 4, 0, 1) + (3, 1, -4, 4, 0, 1) + (3, 2, 1, 4, 0, 1)$$
$$+ (1, 1, -6, -4, 0, 1) + (3, 1, 4, -4, 0, 1) + (3, 2, -1, -4, 0, 1)$$
$$+ (1, 1, 6, -1, -3, 1) + (3, 1, -4, -1, -3, 1) + (3, 2, 1, -1, -3, 1)$$
$$+ (1, 1, -6, 1, 3, 1) + (3, 1, 4, 1, 3, 1) + (3, 2, -1, 1, 3, 1)$$
$$+ (1, 2, -3, 3, -3, 1) + (3, 1, 2, 3, -3, 1) + (3, 2, -5, 0, 0, 1)$$
$$+ (1, 2, 3, -3, 3, 1) + (3, 1, -2, -3, 3, 1) + (3, 2, 5, 0, 0, 1)$$
$$+ (1, 1, 6, -1, 1, 3) + (3, 1, -4, -1, 1, 3) + (3, 2, 1, -1, 1, 3)$$
$$+ (1, 2, -3, 3, 1, 3) + (3, 1, 2, 3, 1, 3) + (1, 1, 0, -5, 1, 3)$$
$$+ (1, 1, -6, 1, -1, 3) + (3, 1, 4, 1, -1, 3) + (3, 2, -1, 1, -1, 3)$$
$$+ (1, 2, 3, -3, -1, 3) + (3, 1, -2, -3, -1, 3) + (1, 1, 0, 5, -1, 3)$$
$$+ (1, 2, 3, 2, -2, 3) + (3, 1, -2, 2, -2, 3) + (1, 2, -3, -2, -2, 3) + (3, 1, 2, -2, -2, 3)$$
$$+ (1, 2, -3, -2, 2, 3) + (3, 1, 2, -2, 2, 3) + (1, 2, 3, 2, 2, 3) + (3, 1, -2, 2, 2, 3)$$
$$+ (1, 1, 0, -5, -3, 1) + (1, 1, 0, 5, 3, 1) + (1, 1, 0, 0, -4, 3) + (1, 1, 0, 0, 4, 3).$$

where the electric charge generator is given by $Q = T_3L + Y/6$ in our normalisation.

Thus one finds a unique (colour coded) embedding of the SM where:

- **Blue** corresponds to the SM fermions and one family triplet must have a zero mode.
- **Green** corresponds to the two MSSM Higgses which are flavour triplets and must have zero modes.
- **Violet** correspond to right handed neutrinos and may or may not have a zero mode.
- **Magenta** correspond to adjoint scalars that could obtain a VEV through the Wilson line. They may or may not have zero modes.
- **Yellow** corresponds to a scalar that could get a VEV that would generate Majorana masses for RHNs. It may or may not have a zero mode.
- **Red** corresponds to mirror families of the SM fermions. In general the number of massless blue modes minus the number of massless red modes must equal one.
- **Orange** correspond to Higgses that would force a GUT-scale $\mu$ term. Without them there is no $\mu$ term and it must be generated dynamically.
- **Brown** correspond to multiplets that could obtain a VEV through a Wilson line and break the remaining symmetry. They may or may not have zero modes.

---

7Multiplying every flavour triplet by 3 and flavour adjoint by 8, one finds 99 SM multiplets and hence 1089 SM states ($99 \times (6$ scalars $+ 4$ fermions $+ 1$ vector)).
• Black correspond to fields that must not exist at low energies. They are 4th and 5th families of fermions, leptoquarks and the triplets usually the Higgses in GUTs. Either they do not have zero modes or the zero mode is a vector-like pair.

A systematic scan of all the integers that define the orbifold was performed to find one that fulfills all the previous requirements. The whole parameter space for the orbifold was scanned for \( N, M \leq 7 \) (larger numbers did not yield a different result). While there are many setups for the integers, they are all physically equivalent. After finding a candidate solution we apply the extra constraint that there has to be a compatible lattice with the orbifolding and the remaining SUSY, as discussed in \[59\]. We find that this leads to a unique choice of the orbifold \( T^6/(\mathbb{Z}_6 \times \mathbb{Z}_2) \). In these cases solutions were found which satisfy the criteria that the field content of zero modes has three families of fermions. However such solutions always have a larger Pati-Salam symmetry, as discussed in Appendix A. This motivates a dedicated analysis of an Exceptional Pati-Salam orbifold model in the next section.

5 Exceptional Pati-Salam Model

5.1 The orbifold \( T^6/(\mathbb{Z}_6 \times \mathbb{Z}_2) \)

In this section, a particular orbifold \( T^6/(\mathbb{Z}_6 \times \mathbb{Z}_2) \) is considered where \( \mathbb{Z}_6 \in U(1)_{X'} \) as in subsection 4.3 and \( \mathbb{Z}_2 \in U(1)_Y \).

Under \( E_8 \to SU(4)_{PS} \times SU(2)_L \times SU(2)_R \times U(1)_{X'} \times SU(3)_F \) the decomposition of the adjoint representation is (using a similar colour coding as before):

\[
(248) \to (15, 1, 1, 0, 1) + (1, 3, 1, 0, 1) + (1, 1, 3, 0, 1) + (1, 1, 1, 0, 1) + (1, 1, 1, 0, 8) \\
+ (6, 2, 2, 0, 1) + (4, 2, 1, -3, 1) + (4, 1, 2, -3, 1) + (4, 2, 1, 3, 1) + (4, 1, 2, 3, 1) \\
+ (4, 2, 1, 1, 3) + (4, 1, 2, 1, 3) + (1, 2, 2, -2, 3) + (6, 1, 1, -2, 3) + (1, 1, 1, 4, 3) \\
+ (4, 2, 1, -1, 3) + (4, 1, 2, -1, 3) + (1, 2, 2, 2, 3) + (6, 1, 1, 2, 3) + (1, 1, 1, -4, 3).
\]

The same color coding as in eq. \[39\] has been used, with the difference that some fields that were black are now part of the adjoints in magenta. Also the right handed neutrinos that were in violet \( \nu^c \sim (1, 1, 0, -5, 1, 3) \) are now part of the two PS fermion multiplets in blue (and their conjugate representations in yellow \( \bar{\nu}^c \sim (1, 1, 0, 5, -1, 3) \) now being part of the red ones).

This decomposition corresponds to that of eq. \[31\] with the further breaking

\[
SO(10) \to SU(4)_{PS} \times SU(2)_L \times SU(2)_R \tag{41}
\]

\[
10 \to (6, 1, 1) + (1, 2, 2), \quad 16 \to (4, 2, 1) + (4, 1, 2), \quad 45 \to (15, 1, 1) + (1, 3, 1) + (1, 1, 3) + (6, 2, 2). \tag{42}
\]

The SM group would be obtained from PS by the subsequent breakings

\[
SU(4)_{PS} \to SU(3)_C \times U(1)_{B-L} \tag{43}
\]

\[\text{For a non-SUSY Pati-Salam model see Appendix B.}\]
\[ 4 \rightarrow (3, 1/3) + (1, -1), \quad 6 \rightarrow (3, -2/3) + (\overline{3}, 2/3), \]
\[ 15 \rightarrow (8, 0) + (1, 0) + (3, 4/3) + (\overline{3}, -4/3) \]  \hspace{1cm} (44)\]

and
\[ SU(2)_R \rightarrow U(1)_{T_R^8} \]  \hspace{1cm} (45)\]
\[ 2 \rightarrow (1/2) + (-1/2), \quad 3 \rightarrow (1) + (0) + (-1), \]  \hspace{1cm} (46)\]

which together would yield the QCD and hypercharge gauge groups \[^9\]
\[ SU(4)_{PS} \times SU(2)_R \rightarrow SU(3)_C \times U(1)_Y \times U(1)_X \]  \hspace{1cm} (47)\]
\[ (4, 1) \rightarrow (3, 1, -1) + (1, -3, 3), \]
\[ (6, 1) \rightarrow (3, -2, 2) + (\overline{3}, 2, -2), \]
\[ (15, 1) \rightarrow (8, 0, 0) + (1, 0, 0) + (3, 4, -4) + (\overline{3}, -4, 4), \]
\[ (1, 2) \rightarrow (1, 3, 3) + (1, -3, -3), \]
\[ (1, 3) \rightarrow (1, 6, 6) + (1, 0, 0) + (1, -6, -6), \]
\[ (4, 2) \rightarrow (3, 4, 2) + (1, 0, 6) + (3, -2, -4) + (1, -6, 0), \]
\[ (6, 2) \rightarrow (3, 1, 5) + (3, 5, 1) + (3, -5, -1) + (\overline{3}, -1, -5), \]  \hspace{1cm} (48)\]

where in our normalisation \( Y/6 = T_R^R + (B - L)/2 \) and \( X/6 = T_R^R - (B - L)/2 \).

The \( T^6/(\mathbb{Z}_6 \times \mathbb{Z}_2) \) orbifold which achieves the breaking in eq. (40) is defined as: \[^4\]
\[ Z_6 : \phi \rightarrow e^{2\pi i q x'/6} \phi, \quad Z_2 : \phi \rightarrow e^{2\pi i q y/2} \phi, \]  \hspace{1cm} (49)\]
\[ (x, z_1, z_2, z_3) \sim (x, \alpha^2 z_1, \alpha^5 z_2, \alpha^5 z_3), \]
\[ (x, z_1, z_2, z_3) \sim (x, -z_1, -z_2, (-1)^2 z_3), \]  \hspace{1cm} (50)\]

where \( \alpha = e^{2\pi i/6} \) and \( -1 = e^{2\pi i/2} \). The \((-1)^2\) implies a full rotation on the \( z_3 \) to comply with eq. (10).

The \( Z_6 \) orbifolding is based on \( U(1)_{Y'} \) which commutes with \( SO(10) \) and \( SU(3)_F \).
This orbifold operation breaks \( E_8 \rightarrow SO(10) \times U(1)_{Y'} \times SU(3)_F \), as can be seen in table \[^4\] and subsection \[^4\]. The \( Z_2 \) orbifolding is based on \( U(1)_Y \), the hypercharge. This orbifolding breaks \( E_8 \rightarrow E_7 \times SU(2) \) as seen in table \[^4\]. This \( E_7 \) does not contain the previous \( SO(10) \) and its intersection is the Pati-Salam group. The breaking of \( SO(10) \) in eq. (41) occurs since all the gauge bosons in PS have even hypercharge parity \( \mathbb{Z}_2 \), unlike the \( 45 \) of \( SO(10) \) which contains \( 6, 2, 2 \) which has odd hypercharge parity \( \mathbb{Z}_2 \). The orbifolds charges of all the resulting \( \mathcal{N} = 1 \) superfields can be seen in table \[^5\].

The massless zero modes in this Pati-Salam setup are only the \( \mathcal{N} = 1 \) superfields with both charges equal to unity (the singlets \( 1, 1 \)) from table \[^5\].

\[ V_\mu : (15, 1, 1, 0, 1) + (1, 3, 1, 0, 1) + (1, 1, 3, 0, 1) + (1, 1, 1, 0, 1) + (1, 1, 1, 0, 8), \]
\[ \phi_1 : (1, 2, 2, -2, 3), \]
\[ \phi_2 : (4, 2, 1, 1, 3), \]
\[ \phi_3 : (4, 1, 2, 1, 3). \]  \hspace{1cm} (51)\]

\[^9\] This is equivalent to one of the cases tabulated in [69].
The massless superfields can be named as

\[ V_\mu : G_\mu + W_\mu^L + W_\mu^R + Z_\mu + F_\mu, \]
\[ \phi_1 : H, \]
\[ \phi_2 : F, \]
\[ \phi_3 : F^c. \] (52)

The chiral superfields decompose into the SM fields as

\[ H \rightarrow h_u + h_d, \]
\[ F \rightarrow Q + L, \]
\[ F^c \rightarrow u^c + d^c + \nu^c + e^c. \] (53)

The SM gauge group is subsequently achieved by appealing to Wilson line breaking of the PS and other gauge group factors in addition to the orbifold breaking. The resulting model is dubbed the Exceptional Pati-Salam (EPS) model.

The only renormalizable term in the superpotential from eq. (1) becomes

\[ \mathcal{W} \sim \phi_1 \phi_2 \phi_3 = HFF^c, \] (54)

which are the Yukawa couplings.

In summary, \( E_8 \) has been broken down to Pati-Salam with an \( SU(3)_F \) flavour group. The field content is that of the SM with the only addition of the SM singlet right handed neutrinos. The Higgs doublets are now flavour triplets. There are no other fields with zero modes, i.e. no mirror fermions, exotics and effective doublet-triplet splitting. In order to achieve three (and only three) chiral superfields with the correct SM field content required at least 10d, where this conclusion is unrelated to String Theory.

This is the unique orbifold that eliminates mirror fermions, solves the DT splitting and preserves simple SUSY \[^{[57]}^{[69]}.\]

\[^{10}\]Although the considered theory with an \( E_8 \) symmetry and only 10d bulk matter arising from the \( V_{248} \sim 248 \) ensures that the high energy theory is anomaly free, the zero mode effective field content at low energies need not be. While the Pati-Salam group is anomaly free, the \( U(1)_X \times SU(3)_F \) are not. In a string theory such anomalies would be cancelled by extra matter located at the fixed points, however here we do not consider such states. Instead we ignore such anomalies since these groups are broken by a Wilson line which is naturally close to the compactification scale, as discussed in the next subsection.
\section{Wilson line breaking via the right-handed sneutrino VEV}

The orbifolding has broken $E_8 \rightarrow SU(4)_{PS} \times SU(2)_L \times SU(2)_R \times U(1)_{X'f} \times SU(3)_F$, which is still rank 8. Orbifolding by itself cannot reduce the rank and Wilson lines are needed to further break the PS symmetry as in eq. (17) and also to break $U(1)_X \times U(1)_{X'} \times SU(3)_F$.

We first recall that the right handed neutrinos $\nu^c \sim (1, 1, 0, -5, 1, 3)$ in eq. (39) are part of the PS fermion multiplet $F^c \sim (4, 1, 2, 1, 3)$. The superfield $\nu^c$, being a singlet under the SM symmetry, makes it the only candidate to obtain a VEV through a Wilson line. It is a flavour triplet. In general one can give an arbitrary complex VEV to the scalar components $\bar{\nu}^c$, although it would naturally be at the compactification scale.

Thus we give a VEV to the scalar component of the RH neutrino $\nu^c \sim (1, 1, 0, -5, 1, 3)$ in eq. (39) through a Wilson line in $\phi_2$. When imposing a Wilson line to give a VEV, one can obtain an effective potential that drives that VEV \cite{29,31}. Since $N = 1$ SUSY is preserved after compactification, the effective superpotential would be $W^{eff}_{\nu^c} (\bar{\nu}^c \nu^c)$, it is safe to assume that the VEV in the scalar component of the complex conjugated representation in eq. (39) $\bar{\nu}^c \sim (1, 1, 0, 5, -1, 3)$, would be induced due to D flatness, even though they are massive chiral superfield KK modes in that representation. These VEVs break the $SU(4)_{PS} \times SU(2)_L \times SU(2)_R \times U(1)_{X'} \times SU(3)_F \rightarrow SU(3)_C \times SU(2)_L \times U(1)_Y$ as desired.

The effective Yukawa terms become

\begin{equation}
W_Y \sim FH F^c + FH F^c (\bar{\nu}^c \nu^c) + (F \bar{\nu}^c) (\nu^c HF^c) + (FH \nu^c) (\bar{\nu}^c F^c) + \ldots \tag{55}
\end{equation}

where the higher order terms are mediated by KK modes with corresponding mass scales not shown. Assuming that both the RH sneutrinos and the Higgses get a VEV $\langle h_u^{u,d} \rangle = v_i^{u,d}$, these terms generate the mass matrices

\begin{align*}
M_{u,v,e,d} & \sim v_1^{u,d} \left[ \begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & -1 - \langle \bar{\nu}_i^l \nu_i^l \rangle \\
0 & 1 + \langle \bar{\nu}_i^l \nu_i^l \rangle & 0
\end{array} \right] + v_2^{u,d} \left[ \begin{array}{ccc}
0 & 0 & 1 + \langle \bar{\nu}_i^l \nu_i^l \rangle \\
-1 - \langle \bar{\nu}_i^l \nu_i^l \rangle & 0 & 0 \\
0 & 0 & 0
\end{array} \right] + v_3^{u,d} \left[ \begin{array}{ccc}
0 & -1 - \langle \bar{\nu}_i^l \nu_i^l \rangle & 0 \\
1 + \langle \bar{\nu}_i^l \nu_i^l \rangle & 0 & 0 \\
0 & 0 & 0
\end{array} \right],
\end{align*}

Assuming that the VEVs are of order the KK mass scale leads to a democratic structure. Even though there are no available free Yukawa couplings from the original theory, they can be effectively generated by the VEVs $v_i^{u,d}, \langle \bar{\nu}^c \rangle, \langle \bar{\nu}^c \rangle$ which introduce eleven complex free parameters. To reproduce the hierarchical pattern of SM fermion masses and mixing angles will require some tuning due to the democratic structure.

A VEV in the right handed sneutrino generates the effective bilinear R-parity violating term (since the chiral superfields $\phi_{2,3}$ have odd R-parity)

\begin{equation}
L h_u \langle \bar{\nu}^c \rangle. \tag{57}
\end{equation}
Assuming an even larger Higgs mass superpotential term $\mu h_u h_d$, where $\mu > \langle \tilde{\nu}^c \rangle$ (the various scales are discussed in the next subsection) the effective bilinear R-parity violating term in Eq.\ref{eq:57} can be rotated into the Yukawa matrices by a unitary transformation \cite{92},

$$
(L, h_d) \rightarrow V_{ij}^a (L, h_d),
$$

where $a = 1, 2$ for $L$ and $h_d$ respectively, and $i, j = 1, 2, 3$. The rotation would only leave R-parity violating terms in the Yukawa sector. It is very important to note that this rotation does not affect the up quark mass matrix nor the charged lepton mass matrix, since it is unitary and it involves both $L$ and $h_d$.

This rotation $V$ however, changes the down quark mass matrix by changing

$$
v^d_{ij} \rightarrow v^d_{ij} V^2_{ij},
$$

therefore generating a difference between the down quark mass matrix and unchanged the charged lepton mass matrix. Furthermore it also changes the neutrino mass matrix

$$
M_{\nu} \rightarrow V^1 M_{\nu},
$$

and thus the VEV $\langle \tilde{\nu}^c \rangle$ breaks the universal mass matrix structure, giving a pathway for a realistic set of SM fermion mass matrices (including neutrinos).

This VEVs would generate a right handed neutrino Majorana mass term

$$
\nu^c \nu^c \langle \tilde{\nu}^c \rangle \langle \tilde{\nu}^c \rangle,
$$

and the Majorana mass matrix

$$
M_{NN} \sim \begin{pmatrix}
2 \langle \tilde{\nu}^c_1 \tilde{\nu}^c_1 \rangle & \langle \tilde{\nu}^c_2 \tilde{\nu}^c_3 \rangle & \langle \tilde{\nu}^c_2 \tilde{\nu}^c_2 \rangle + \langle \tilde{\nu}^c_3 \tilde{\nu}^c_3 \rangle \\
\langle \tilde{\nu}^c_2 \tilde{\nu}^c_3 \rangle & 2 \langle \tilde{\nu}^c_3 \tilde{\nu}^c_3 \rangle & \langle \tilde{\nu}^c_3 \tilde{\nu}^c_3 \rangle \\
\langle \tilde{\nu}^c_1 \tilde{\nu}^c_3 \rangle & \langle \tilde{\nu}^c_2 \tilde{\nu}^c_3 \rangle & 2 \langle \tilde{\nu}^c_1 \tilde{\nu}^c_3 \rangle
\end{pmatrix},
$$

that can generate small left handed neutrino masses through the type-1 seesaw mechanism.

Finally one could be worried about the fact that the rotation generates the R-parity violating terms

$$
(V^1 \nu^c)QLd^c.
$$

Fortunately the bounds associated with them are easily satisfied for a large SUSY breaking scale, and also typically they are of the order of the down quark Yukawa couplings \cite{92}. It is important to note also that these terms by themselves do not induce proton decay.

In summary, the VEV $\langle \tilde{\nu}^c \rangle$ induced through a Wilson line can break the remaining symmetry into the SM, break the universality of masses and give Majorana masses to right handed neutrinos. This is at the cost of introducing R-parity violating terms which, however, should be naturally well below the experimental bounds.

Lastly, since SUSY has been respected by the orbifold, the R-symmetry breaking due to $\langle \tilde{\nu}^c \rangle$ would induce SUSY breaking \cite{93}.

It seems that the existence of right-handed neutrinos, within the context of this fully unified setup, turns out to be crucial since it plays a fundamental role in the whole process. Usually, right-handed neutrinos are included to fill in representations and/or to help generate light masses for the observed neutrinos but have limited theoretical/phenomenological relevance beyond that. In this setup, they become essential, leading to smoking gun experimental signatures, as discussed in the next subsection.
5.3 Experimental Signatures

As discussed above, this potentially troublesome R-parity violating term in Eq. (57) can be rotated into the Yukawa sector by the unitary transformation in (58) so that it leaves R-parity violating terms only in the d-type quark and Higgs fields. However it was already noted that such a rotation presumes a larger Higgs $\mu$ term. In this subsection, the various symmetry breaking scales and experimental signatures in the EPS model are discussed.

First note that when the right-handed sneutrino gets a VEV, the R-parity violating term in (57) leads to a Dirac mass term coupling the $L$ to the $\tilde{h}_u$. In the absence of other mass terms, these states will therefore be very heavy (close to the Pati-Salam breaking scale $\langle \tilde{\nu}_c \rangle$) and will thus decouple from the low energy theory. To make sure this does not happen in the model, it is necessary to consider a larger Higgs mass $\mu h_u, h_d$, where $\mu > \langle \tilde{\nu}_c \rangle$ term. In order to achieve correct electroweak symmetry breaking this further implies a similarly large SUSY breaking scale $M_{SUSY} \sim O(\mu)$. Since the current bound on the Pati-Salam breaking scale $\langle \tilde{\nu}_c \rangle$ from the non-observation of $K_L \to \mu e$, $n - \bar{n}$ oscillations and $B_{d,s} \to \mu e$ is around $10^6$ GeV [94], it must be assumed that $\mu$ is larger than that value (i.e. SUSY breaking must happen around the Pati-Salam breaking scale). Note that an effective $\mu$ term must be obtained radiatively. Putting all this together one is led to the suggestive pattern of scales in the EPS model,

$$M_{SUSY} \sim O(\mu) \gtrsim \langle \tilde{\nu}_c \rangle \gtrsim 10^6 \text{ GeV}. \quad (64)$$

While Wilson line breaking would suggest that $\langle \tilde{\nu}_c \rangle$ be very large, of order the compactification scale, the imposed requirement that SUSY is preserved in the low energy theory, suggests that some compromise should be achieved with the PS breaking scale near its experimental limit $\langle \tilde{\nu}_c \rangle \sim 10^6$ GeV, leading to smoking gun signals of PS breaking expected in the not too distant future.

It also turns out that this pattern of scales is also desirable from the point of view of the cosmological implications of the model. R-parity violating terms are very strongly constrained by matter-antimatter asymmetry, since they can in principle wash out any asymmetries from earlier cosmological epochs. This is in fact the case when the asymmetry arises above the SUSY breaking scale. In the present model, since it is assumed that the SUSY breaking scale is above the Pati-Salam breaking scale, the stringent constraints on such parameters are automatically avoided [95].

Along similar lines, the model does not lead to proton decay. This is due to the fact that baryon parity is automatic (where all quark superfields change sign while others remain the same) and the R-parity violating term in (57) does not involve baryon number. As discussed in [95], this is reminiscent of models with spontaneous R symmetry breaking.

Although in the present work no attempt has been made to include supergravity, given that R-parity is broken, a possible candidate for dark matter is the gravitino, provided its lifetime is large enough. Since the gravitino decay is associated with the gravitational constant, it can be very weak, leading to a lifetime much longer than the age of the universe [95].

One can see that the EPS model leads to a very distinctive scenario with possible phenomenological smoking gun signatures of PS breaking at $10^6$ GeV, namely $K_L \to \mu e$, $n - \bar{n}$ oscillations and $B_{d,s} \to \mu e$ [94], and other low energy consequences of the right-handed sneutrino VEVs arising from R-parity violation [95, 97].
6 Conclusion

This paper investigated the extraordinarily elegant hypothesis that the three families of quarks and leptons may be unified, together with the Higgs and gauge fields of the Standard Model (SM), into a single “particle”, namely the $248$ vector superfield of a ten-dimensional $E_8$ super Yang Mills (SYM) theory. There are no free coupling constants beyond the unique gauge coupling. Although this idea was proposed some time ago [54], it was never developed into realistic model, and the goal of the present paper is to make progress in that direction. Although the theory is necessarily formulated in 10d, it is based on field theory and point particles rather than string theory, and therefore gravity is ignored in this approach.

Towards a realistic model along these lines, a class of orbifoldings based on $T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$ have been proposed and explored, that can in principle break $E_8$ SYM down to the SM gauge group, embedded in a larger group such as $E_6$, $SO(10)$ or $SU(5)$, together with other gauge group factors which can be broken by Wilson lines. A discussion has been presented for some examples of $E_8$ breaking for various values of $N, M$, including: $E_6 \times SU(3)_f$ from $T^6/\mathbb{Z}_3$; $E_6$SSM from $T^6/(\mathbb{Z}_3 \times \mathbb{Z}_3)$; $SO(10)$ from $T^6/\mathbb{Z}_6$ and $SU(5)$ from a different $T^6/\mathbb{Z}_6$. Also, a general search for the SM from $T^6/(\mathbb{Z}_N \times \mathbb{Z}_M)$ was performed, that led to the result that no solutions of this kind exist, apart from those which include a Pati-Salam (PS) gauge symmetry.

The most promising example seems to be an Exceptional Pati-Salam (EPS) model based on $T^6/(\mathbb{Z}_6 \times \mathbb{Z}_2)$ which breaks the 10d $E_8$ SYM down to a SUSY PS gauge group in 4d, together with other gauge groups, namely $E_8 \rightarrow SU(4)_{PS} \times SU(2)_L \times SU(2)_R \times U(1)_X \times SU(3)_F$. In the EPS model, all the SM gauge fields, fermions, Higgs fields plus right handed neutrinos come from the same original $248$ vector superfield of $E_8$ SYM with no free parameters and no light exotics. The RH sneutrinos are assumed to obtain large VEVs through a Wilson line, generating the flavour structure in the fermion mass matrices and a mixing between the lepton doublet and the down type Higgs. This distinguishes between charged lepton and down quark masses, giving Majorana masses to the right handed neutrino and breaking the PS gauge group down to that of the SM. It also implies R-parity violation, and experimental signatures associated with a low PS gauge group breaking scale.

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A  Pati-Salam symmetry from a general scan

The general scan for the search for the SM gauge group included an orbifold \( T^6/(\mathbb{Z}_6 \times \mathbb{Z}_2) \) defined by

\[
\begin{align*}
\mathbb{Z}_6 : & \phi \rightarrow e^{2i\pi x_1/6} \phi, \\
\mathbb{Z}_2 : & \phi \rightarrow e^{2i\pi y/2} \phi,
\end{align*}
\]

which are applied respectively as

\[
\begin{align*}
(x, z_1, z_2, z_3) & \sim (x, \alpha^2 z_1, \alpha^2 z_2, \alpha^2 z_3), \\
(x, z_1, z_2, z_3) & \sim (x, -z_1, -z_2, (1)^2 z_3),
\end{align*}
\]

where \( \alpha = e^{2i\pi/6} \) and \( -1 = e^{2i\pi/2} \).

This orbifold leaves the zero modes, in the colour-coded notation of Eq (39) which transform under \( SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X \times SU(3)_F \) as:

\[
\begin{align*}
V_\mu : & (8, 1, 0, 0, 0, 1) + (1, 3, 0, 0, 0, 1) + (1, 1, 0, 0, 0, 8) \\
& + (1, 1, 0, 0, 0, 1) + (1, 1, 0, 0, 0, 1) + (1, 1, 0, 0, 0, 1) \\
& + (1, 1, 6, 4, 0, 1) + (3, 1, -4, 4, 0, 1) + (1, 1, -6, -4, 0, 1) + (3, 1, 4, -4, 0, 1) \\
\phi_1 : & (1, 2, 3, 2, -2, 3) + (1, 2, -3, -2, -2, 3), \\
\phi_2 : & (3, 2, 1, -1, 1, 3) + (1, 2, -3, 3, 1, 3), \\
\phi_3 : & (1, 1, 6, -1, 1, 3) + (3, 1, -4, -1, 1, 3) + (3, 1, 2, 3, 1, 3) + (1, 1, 0, -5, 1, 3),
\end{align*}
\]

where the electric charge generator is given by \( Q = T_{3L} + Y/6 \) in our normalisation.

The chiral multiplets can be written as

\[
\begin{align*}
\phi_1 : & h_u + h_d, \\
\phi_2 : & Q + L, \\
\phi_3 : & e^c + u^c + d^c + \nu^c.
\end{align*}
\]

Note that the \( V_\mu \) contains some fields seemingly not in the adjoint representation. This means that symmetry breaking is really not to the SM, but instead corresponds to an enhanced PS gauge group \( SU(4)_{PS} \times SU(2)_L \times SU(2)_R \), together with the other gauge group factors. The group embedding is \( SU(4)_{PS} \rightarrow SU(3)_C \times U(1)_{B-L} \) where \( 4 \rightarrow (3, 1/3) + (1, -1) \) and the hypercharge generator is given by \( Y/6 = T_{3R} + (B - L)/2 \) in our normalisation.

B  Non-SUSY Pati-Salam Model

Consider a Pati-Salam model broken SUSY in an alternative orbifold \( T^6/(\mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \). The orbifolding \( \mathbb{Z}_6 \times \mathbb{Z}_2 \in SU(3)_R \) breaks the extended SUSY. It preserves the remaining \( U(1)_R \). There is an available extra orbifolding \( \mathbb{Z}_L \in U(1)_R \) that would break the usual remaining \( U(1)_R \) and therefore breaking simple SUSY.

In the previous \( E_8 \rightarrow SU(4)_{PS} \times SU(2)_L \times SU(2)_R \times SU(3)_F \times U(1)_{X'} \) orbifold breakings one can see that \( \phi_1 \) decomposes into the Higgses and flavons, that must be scalars, the \( \phi_2 \) into right handed fermions and \( \phi_3 \) left handed fermions. They are already separated so one can easily impose the extra \( \mathbb{Z}_2 \) orbifolding

\[
(x, z_1, z_2, z_3) \sim (x, z_1^*, -z_2^*, -z_3^*),
\]

21
that breaks the remaining SUSY and \( U(1)_R \rightarrow \mathbb{Z}_2^R \) respecting the usual R parity. This orbifolding leaves the zero modes as
\[
\begin{align*}
V_\mu &: \text{real vector : } G_\mu + W_\mu^L + W^R_\mu + Z'_\mu + F_\mu, \\
\phi_1 &: \text{complex scalar : } h_u + h_d, \\
\phi_2 &: \text{Weyl fermion : } Q + L, \\
\phi_3 &: \text{Weyl fermion : } e^c + u^c + d^c + \nu^c,
\end{align*}
\]
which is exactly the representations needed.

Therefore the orbifold \( T^6/(\mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \) can break into Pati-Salam without SUSY. This SUSY breaking happens at the compactification scale which is identified with the GUT scale. Although this is an interesting possibility, it was not pursued in the main text.

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