A General Scheme for Solving Systems of Linear First-Order Differential Equations Based on the Differential Transform Method

Ahmed Hussein Msmali, A. M. Alotaibi, M. A. El-Moneam, Badr S. Badr, and Abdullah Ali H. Ahmadini

1Mathematics Department, Faculty of Science, Jazan University, Jazan, Saudi Arabia
2Department of Mathematics, Faculty of Science, University of Tabuk, P.O. Box 741, Tabuk 71491, Saudi Arabia

Correspondence should be addressed to M. A. El-Moneam; mabdelmeneam2014@yahoo.com

Received 25 August 2020; Accepted 17 August 2021; Published 27 August 2021

This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this study, we develop the differential transform method in a new scheme to solve systems of first-order differential equations. The differential transform method is a procedure to obtain the coefficients of the Taylor series of the solution of differential and integral equations. So, one can obtain the Taylor series of the solution of an arbitrary order, and hence, the solution of the given equation can be obtained with required accuracy. Here, we first give some basic definitions and properties of the differential transform method, and then, we prove some theorems for solving the linear systems of first order. Then, these theorems of our system are converted to a system of linear algebraic equations whose unknowns are the coefficients of the Taylor series of the solution. Finally, we give some examples to show the accuracy and efficiency of the presented method.

1. Introduction

The differential transform was first introduced by Zhou [1], and up to now, the DT method has been developed for solving various kinds of differential and integral equations in many literatures. For example, Ali [2] has developed the DT method for solving partial differential equations and Ayaz [3, 4] has applied this method to differential algebraic equations. Arikoglu and Ozkol [5] have solved the integrodifferential equations with boundary value conditions by the DT method. Odibat [6] has used the DT method for solving Volterra integral equations with separable kernels. Tari and Ziyyae [7] have solved the system of two-dimensional nonlinear Volterra integrodifferential equations by the DT method. The systems of integral and integrodifferential equations, the multiorder fractional differential equations, the system of fractional differential equations, the singularly perturbed Volterra integral equations, and the time-fractional diffusion equation have been solved by the DT method in [2, 6, 8–10]. Also, the DT method has been applied to nonlinear parabolic-hyperbolic partial differential equations, and a modified approach of DT has been developed to nonlinear partial differential equations in studies by Biazarand Abdul Halim-Haasan [8]. Patil and Kembayat [11] have solved the two-dimensional Fredholm integral equations. Abdewahid [12] introduced a new basic formula for the one-dimensional differential transform. The main aim of this work is to introduce new useful algorithms depending on the DT method to solve systems of linear differential equations.

2. Analysis of Differential Transform

The basic definition and the fundamental theorems of the DTM and its applicability for various kinds of differential equations are given in [13–20]. For the convenience of the reader, we will present a review of the DTM. To do this, we assume that \( f(x) \in C^{\infty}(I) \); then, for any point...
Let $f(x)$ be an analytic function about $x_0$; then, the $k^{th}$ order differential transform of $f(x)$ is defined as

$$ DT[f(x)] = F(k) = \frac{f^{(k)}(x_0)}{k!}. $$

From equations (1) and (2), we get

$$ f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k, $$

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, the relative derivatives are calculated by an iterative way which are described by the transformed equations of the original function. In real applications, the function $f(x)$ is expressed by a finite series, and equation (3) can be written as

$$ f(x) = \sum_{k=0}^{n} \frac{d^k f(x_0)}{k!} (x-x_0)^k, $$

where $n$ is decided by the convergence of natural frequency. The fundamental operations performed by differential transform can be readily obtained and are given.

**Theorem 1.** Let $U(k)$ and $V(k)$ be the differential transformations of the functions $u(x)$ and $v(x)$, respectively; then, we have the following properties:

1. If $f(x) = au(x) \pm bv(x)$, then $F(k) = aU(k) \pm bV(k)$
2. If $f(x) = u(x) \cdot v(x)$, then $F(k) = \sum_{l=0}^{\infty} V(l)U(k-l)$
3. If $f(x) = f'(x)$, then $F(k) = (k+1)U(k+1)$
4. If $f(x) = (d^m u(x)/dx^m)$, then $F(k) = (k+1)(k+2)\ldots(k+m)U(k+m)$
5. If $f(x) = \int_{x_0}^{x} u(t)dt$, then $F(k) = \left(U(k-1)k\right)$
6. If $f(x) = x^m$, then $F(k) = \delta(k-m)$
7. If $f(x) = e^{ax}$, then $F(k) = (a^k/k!)$
8. If $f(x) = \sin(\alpha x + \alpha)$, then $F(k) = (\alpha k/k!)\sin((kr/2) + \alpha)$
9. If $f(x) = \cos(\alpha x + \alpha)$, then $F(k) = (\alpha k/k!)\cos((kr/2) + \alpha)$
10. If $f(x) = x^m f^{(n)}(x)$, then $F(k) = \sum_{i=0}^{k} \delta_{im} ((k+n-i)!(k-i)!F(k+n-i)$
11. If $f(x) = x^n f^{(n)}(x)$, then $F(k) = \prod_{i=0}^{n-1} (k-i) F(k)$
12. If $f(x) = e^{ax} f^{(n)}(x)$, then $F(k) = \sum_{i=0}^{k} (a^i(k+n-i)!(k-i)!F(k+n-i)$
13. If $f(x) = \left(\{x f^{(n)}(x)\}\right)$, then $F(k) = (k+1)!F(k+n)$
14. If $f(x) = \left(\{x^m f^{(n)}(x)\}\right)$, then $F(k) = (k+1)(k+n-m+i)!(k-m+i)!F(k+n-m+i)$

The proof of this theorem can be found in [2–5, 8, 10, 12–15, 20].

**3. Description of the Method**

In the first part of this section, we introduce a general algorithm depending on the differential transform method to solve systems of $n$ linear differential equations with constant coefficients. The systems are assumed to be autonomous, which means that the independent variable $t$ is not present explicitly. Such a system has the appearance

$$ X'(t) = AX(t), $$

$$ X(0) = X_0, $$

where $A$ is an $n \times n$ matrix given by

$$ A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, $$

$$ X'(t) = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}, $$

$$ X(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}. $$

**Theorem 2.** Let $X(k)$ be the $k^{th}$ differential transform of $X(t)$ and $A^k$ be the $k^{th}$ power of the matrix $A$. Then, the solution of the system given by (5) can be expressed as

$$ X(t) = \sum_{k=0}^{\infty} X(k)t^k = \sum_{k=0}^{\infty} \frac{A^k}{k!} X(0)t^k. $$

**Proof.** By using the differential transform property (3) of Theorem 1, we get

$$ (k+1)X(k+1) = AX(k), \quad k = 0, 1, 2, \ldots. $$

From this equation, we have the following recurrence relation:

$$ X(k+1) = \frac{A}{(k+1)} X(k). $$

Then, for $k = 0, 1, 2, \ldots$ we get

$$ X(1) = \frac{A}{1!} X(0), $$

$$ X(2) = \frac{A}{2!} X(1) = \frac{A^2}{2!} X(0), $$

$$ X(3) = \frac{A}{3!} X(2) = \frac{A^3}{3!} X(0), $$

$$ X(4) = \frac{A}{4!} X(3) = \frac{A^4}{4!} X(0), $$

$$ \vdots $$

$$ X(n) = \frac{A^n}{n!} X(0), $$

$$ \vdots $$

$$ X(t) = \sum_{k=0}^{\infty} X(k)t^k = \sum_{k=0}^{\infty} \frac{A^k}{k!} X(0)t^k. $$
Theorem 3. Let \( X(k) \) and \( W(k) \) be the \( k \)th differential transforms of \( X(t) \) and \( W(t) \), respectively, and \( A^k \) be the \( k \)th power of the matrix \( A \). Then, the solution of the system given by (13) can be expressed as

\[
X(t) = \sum_{k=0}^{\infty} \left\{ \frac{1}{k!} A^k X(0) \right\} t^k + \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k-1} \frac{(j)!}{(k)!} A^{k-1-j} W(j) \right\} t^k.
\]

Proof. By taking the \( k \)th differential transform of both sides of (13), we get

\[
DT\{X'\} = DT\{AX\} + DT\{W\}.
\]

According to Theorem 1 and the operations of differential transform, we have the following recurrence relation:

\[
X(k + 1) = \frac{1}{k + 1} \{AX(k) + W(k)\}, \quad k = 0, 1, 2, \ldots
\]

Consequently,

\[
X(1) = \frac{1}{1!} \{AX(0) + W(0)\}, \quad W(0) = W(t)_{t=0}
\]

\[
X(2) = \frac{1}{2!} \{AX(1) + W(1)\} = \frac{1}{2!} A^2 X(0) + \frac{1}{2!} AW(0) + \frac{1}{2!} A^3 W(1), \quad W(1) = \frac{1}{1!} \left\{ \frac{dW(t)}{dt} \right\}_{t=0},
\]

\[
X(3) = \frac{1}{3!} \{AX(2) + W(2)\} = \frac{1}{3!} A^3 X(0) + \frac{1}{3!} A^2 W(0) + \frac{1}{3!} AW(1) + \frac{1}{3!} A^4 W(2), \quad W(2) = \frac{1}{2!} \left\{ \frac{d^2W(t)}{dt^2} \right\}_{t=0},
\]

\[
X(4) = \frac{1}{4!} \{AX(3) + W(3)\} = \frac{1}{4!} A^4 X(0) + \frac{1}{4!} A^3 W(0) + \frac{1}{4!} A^2 W(1) + \frac{2}{4!} AW(2) + \frac{3!}{4!} A^5 W(3),
\]

\[
X(5) = \frac{1}{5!} \{AX(4) + W(4)\} = \frac{1}{5!} A^5 X(0) + \frac{1}{5!} A^4 W(0) + \frac{1}{5!} A^3 W(1) + \frac{2}{5!} A^2 W(2) + \frac{3!}{5!} AW(3) + \frac{4!}{5!} A^6 W(4),
\]

\[
W(4) = \frac{1}{4!} \left\{ \frac{d^4W(t)}{dt^4} \right\}_{t=0}.
\]

Therefore, this leads to the general form of \( X(k) \), where

\[
X(k) = \frac{1}{(k)!} A^k X(0) + \sum_{j=0}^{k-1} \frac{(j)!}{(k)!} A^{k-1-j} W(j).
\]

Then, the general solution of the system is given by

\[
X(t) = \sum_{k=0}^{\infty} \left\{ \frac{1}{k!} A^k X(0) \right\} t^k + \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k-1} \frac{(j)!}{(k)!} A^{k-1-j} W(j) \right\} t^k.
\]

4. Applications and Numerical Results

In order to illustrate the advantages and the accuracy of the results given by Theorems 2 and 3 for solving homogeneous and nonhomogeneous problems, we have applied the method to the following examples.

Example 1. Consider the following system of homogeneous differential equations.
\[ x_1' = x_1 + x_2, \quad x_1(0) = 5, \]
\[ x_2' = 0, \quad x_2(0) = 7, \quad (20) \]
\[ x_3' = -x_3, \quad x_3(0) = 6. \]

This system can be written in the matrix form as
\[
\begin{bmatrix}
x_1' \\
x_2' \\
x_3'
\end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad (21)
\]
or
\[
X' = AX,
\]
where
\[
A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (22)
\]
\[
X(0) = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix}.
\]

Applying the result of Theorem 2, we get
\[
X(t) = \sum_{k=0}^{\infty} \frac{A^k X(0)}{k!} t^k = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix} t
\]
\[
+ \frac{1}{2!} \begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix} t^2 + \frac{1}{3!} \begin{bmatrix} 11 \\ -6 \\ 0 \end{bmatrix} t^3 + \frac{1}{4!} \begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix} t^4
\]
\[
+ \frac{1}{5!} \begin{bmatrix} 11 \\ -6 \\ 0 \end{bmatrix} t^5 + \frac{1}{6!} \begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix} t^6 + \frac{1}{7!} \begin{bmatrix} 11 \\ -6 \\ 0 \end{bmatrix} t^7 + \ldots.
\]

Then, the solutions are given by
\[
x_1(t) = 5 + 11t + \frac{5}{2} t^2 + \frac{11}{3!} t^3 + \frac{5}{4!} t^4 + \frac{11}{5!} t^5 + \frac{5}{6!} t^6 + \frac{11}{7!} t^7 + \ldots = 8e^t - 3e^{-t},
\]
\[
x_2(t) = 7,
\]
\[
x_3(t) = 6(1 - t) + \frac{1}{2!} t^2 - \frac{1}{3!} t^3 + \frac{1}{4!} t^4 - \frac{1}{5!} t^5 + \frac{1}{6!} t^6 - \frac{1}{7!} t^7 + \ldots = 6e^{-t},
\]
which is the exact solution of the given problem.

**Example 2.** Consider the system
\[
X' = AX, \quad \text{where } A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad X(0) = \begin{bmatrix} 5 \\ 3 \\ 9 \end{bmatrix}.
\]

Applying the result of Theorem 2, consequently, the solutions of the given system are given as
\[
\begin{bmatrix}
459 \\
8 \\
0
\end{bmatrix} + \begin{bmatrix} 891 \\ 20 \\ 40 \end{bmatrix} t + \begin{bmatrix} 1161 \\ 40 \\ 80 \end{bmatrix} t^2 + \begin{bmatrix} 81 \\ 5 \\ 560 \end{bmatrix} t^3 + \begin{bmatrix} 81 \\ 5 \\ 560 \end{bmatrix} t^4 + \begin{bmatrix} 81 \\ 5 \\ 560 \end{bmatrix} t^5 + \begin{bmatrix} 81 \\ 5 \\ 560 \end{bmatrix} t^6 + \begin{bmatrix} 81 \\ 5 \\ 560 \end{bmatrix} t^7 + \ldots.
\]
\[ x_1(t) = 7 + 26t + 48t^2 + 60t^3 + \frac{459}{8}t^4 + \frac{891}{20}t^5 + \frac{1161}{40}t^6 + \frac{81}{5}t^7 + \cdots, \]
\[ x_2(t) = 5 + 18t + 36t^2 + \frac{99}{2}t^3 + \frac{405}{8}t^4 + \frac{81}{2}t^5 + \frac{1053}{40}t^6 + \frac{8019}{560}t^7 + \cdots, \]
\[ x_3(t) = 3 + 18t + \frac{81}{2}t^2 + 54t^3 + \frac{405}{8}t^4 + \frac{729}{20}t^5 + \frac{1701}{80}t^6 + \frac{729}{70}t^7 + \cdots, \]
\[ x_4(t) = 9 + 27t + \frac{81}{2}t^2 + \frac{81}{2}t^3 + 243t^4 + \frac{729}{40}t^5 + \frac{729}{80}t^6 + \frac{2187}{560}t^7 + \cdots. \]

The solutions can be written in matrix form as
\[
X(t) = e^{3t}
\begin{bmatrix}
5 & 0 & 3 & 7 \\
3 & 9 & 0 & 5 \\
0 & 0 & 9 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
t^0 \\
t^1 \\
t^2 \\
t^3
\end{bmatrix}
\]
\[ X(t) = e^{3t}
\begin{bmatrix}
7 + 5t + 1.5t^2 + 1.5t^3 \\
5 + 3t + 4.5t^2 \\
3 + 9t \\
9
\end{bmatrix}, \]

which is the exact solution of the given problem.

Now, we will give some examples of nonhomogeneous initial value problems.

**Example 3.** Consider the following system:
\[
x'_1 = t^2, \quad x_1(0) = 5, \\
x'_2 = x_2 + t, \quad x_2(0) = 7, \\
x'_3 = 2x_3 + \sin t, \quad x_3(0) = 9.
\]

The exact solutions are obtained by [17] as follows
\[
x_1(t) = 5 + \frac{1}{3}t^3, \\
x_2(t) = \frac{1}{8}e^t - t - 1, \\
x_3(t) = \frac{48}{5}e^{t^2} - \frac{2}{5}\sin t - \frac{1}{5}\cos t.
\]

To find the solutions of the given system applying the results in Theorem 3, the approximate solutions up to \( k = 6 \) are given by
\[
X(t) = X(0) + [AX(0) + W(0)]t + \frac{1}{2!}[A^2X(0) + AW(0) + W(1)]t^2 + \frac{1}{3!}[A^3X(0) + A^2W(0) + AW(1) + 2W(2)]t^3 + \frac{1}{4!}[A^4X(0) + A^3W(0) + A^2W(1) + 2AW(2) + 6A^0W(3)]t^4 + \frac{1}{5!}[A^5X(0) + A^4W(0) + A^3W(1) + 2A^2W(2) + 6AW(3) + 24A^0W(4)]t^5 + \frac{1}{6!}[A^6X(0) + A^5W(0) + A^4W(1) + 2A^3W(2) + 6A^2W(3) + 24AW(4) + 120A^0W(5)]t^6.
\]
Therefore,

\[
X(t) = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 18 & 7 & 0 & 0 & 0 \\ 2 & 72 & 7 & 0 & 0 \\ 0 & 4 & 144 & 7 & 0 \\ -1 & 0 & 0 & 288 & 7 \\ 0 & 0 & 0 & 16 & 7 \\ 0 & 0 & 0 & 0 & 576 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} t^2 \\
+ \frac{1}{3!} \begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} t^3 \\
+ \frac{1}{4!} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} t^4 \\
+ \frac{1}{5!} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} t^5 \\
+ \frac{1}{6!} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} t^6 \\
+ \frac{1}{7!} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} t^7 + \cdots.
\]

X(t) is the solution of the given problem.

\[x_1(t) = 5 + \frac{1}{3}t^3, \]
\[x_2(t) = 7 + 7t + 8 \left( \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \frac{1}{6!}t^6 + \cdots \right) = 8e^t - t - 1, \]
\[x_3(t) = 9 + 18t + \frac{37}{2!}t^2 + \frac{74}{3!}t^3 + \frac{147}{4!}t^4 + \frac{294}{5!}t^5 + \frac{589}{6!}t^6 = \frac{46}{5}e^{2t} - \frac{2}{3} \sin t - \frac{1}{5} \cos t, \]

which is the exact solution of the given problem.

**Example 4.** Consider the following system:

\[x_1' = 5x_1 + x_2 + t^2, \quad x_1(0) = 5, \]
\[x_2' = 5x_2 + x_3 + t, \quad x_2(0) = 7, \]
\[x_3 = 5x_3 + \sin t, \quad x_3(0) = 9. \]

The exact solution of this system is given by [17] as follows:

\[x_1 = \left(5 + \frac{74}{26}\right)e^{st} + \left(7 + \frac{10}{26}\right)t e^{st} \]
\[+ \frac{1}{2}\left(9 + \frac{1}{26}\right)t^2 e^{st} - \frac{2}{26}(55 \sin t + 37 \cos t) - \frac{1}{5}t - \frac{1}{25}t, \]
\[x_2 = \left(7 + \frac{1}{25} - \frac{10}{26}\right)e^{st} + \frac{1}{2}\left(9 + \frac{1}{26}\right)t e^{st} \]
\[+ \frac{2}{26}(12 \sin t + t \cos t) - \frac{1}{5}t - \frac{1}{25}t, \]
\[x_3 = \left(9 + \frac{1}{26}\right)e^{st} - \frac{5}{26} \sin t - \frac{1}{26} \cos t. \]

To find the approximate solutions of the given system, applying the results of Theorem 3, we have

(32)

(33)

(34)
\[
X(t) = X(0) + [AX(0) + W(0)]t + \frac{1}{2!} \left[ A^2 X(0) + AW(0) + W(1) \right] t^2
+ \frac{1}{3!} \left[ A^3 X(0) + A^3 W(0) + AW(1) + 2W(2) \right] t^3 + \\
+ \frac{1}{4!} \left[ A^4 X(0) + A^4 W(0) + A^4 W(1) + 2AW(1) + 6A^2 W(3) \right] t^4
+ \frac{1}{5!} \left[ A^5 X(0) + A^5 W(0) + A^5 W(1) + 2A^2 W(2) + 6AW(3) + 24A^2 W(4) \right] t^5
+ \frac{1}{6!} \left[ A^6 X(0) + A^6 W(0) + A^6 W(1) + 2A^3 W(2) + 6A^3 W(3) + 24AW(4) + 120A^2 W(5) \right] t^6.
\]

Therefore,

\[
X(t) = \left[ \begin{array}{c} 5 \\ 7 \\ 9 \end{array} \right] + \left[ \begin{array}{c} 32 \\ 44 \\ 45 \end{array} \right] t + \frac{1}{2!} \left[ \begin{array}{c} 204 \\ 265 \\ 225 \end{array} \right] t^2
+ \frac{1}{3!} \left[ \begin{array}{c} 1285 \\ 1550 \\ 1125 \end{array} \right] t^3 + \left[ \begin{array}{c} 7975 \\ 8875 \\ 5625 \end{array} \right] t^4
+ \frac{1}{4!} \left[ \begin{array}{c} 48750 \\ 50000 \\ 28125 \end{array} \right] t^5 + \left[ \begin{array}{c} 293750 \\ 278125 \\ 140625 \end{array} \right] t^6,
\]

\[
X(t) = \left[ \begin{array}{c} 5 \\ 7 \\ 9 \end{array} \right] + \left[ \begin{array}{c} 32 \\ 44 \\ 45 \end{array} \right] t + \frac{1}{2!} \left[ \begin{array}{c} 204 \\ 266 \end{array} \right] t^2.
\]
\[
\begin{align*}
&\frac{1}{3!} \begin{bmatrix} 1288 \\ 1556 \\ 1130 \end{bmatrix} + \frac{1}{4!} \begin{bmatrix} 7996 \\ 8910 \\ 5649 \end{bmatrix} t^4 + \frac{1}{5!} \begin{bmatrix} 48890 \\ 50199 \\ 28245 \end{bmatrix} t^5 \\
&\frac{1}{6!} \begin{bmatrix} 293750 \\ 278125 \\ 140625 \end{bmatrix} t^6 + \cdots.
\end{align*}
\]

Then, the solution of the given system is

\[
\begin{align*}
x_1(t) &= 5 + 32t + \frac{204}{2!} t^2 + \frac{1288}{3!} t^3 + \frac{7996}{4!} t^4 + \frac{48890}{5!} t^5 + \frac{293750}{6!} t^6 + \cdots, \\
x_2(t) &= 7 + 44t + \frac{266}{2!} t^2 + \frac{1556}{3!} t^3 + \frac{8910}{4!} t^4 + \frac{50199}{5!} t^5 + \frac{278125}{6!} t^6 + \cdots, \\
x_3(t) &= 9 + 45t + \frac{226}{2!} t^2 + \frac{1127}{3!} t^3 + \frac{5649}{4!} t^4 + \frac{28245}{5!} t^5 + \frac{140625}{6!} t^6 + \cdots,
\end{align*}
\]

which are the exact solutions of the given system.

5. Conclusions

In this work, we successfully apply the differential transform method to find useful formulas for solving homogeneous or nonhomogeneous systems of \( n \) first-order differential equations. The present methods reduce the computational difficulties of the traditional methods, and all the calculations can be made simple manipulations. Several examples were tested by applying the results of Theorems 2 and 3, and the results have shown a remarkable performance.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] J. K. Zhou, *Differential Transform and its Applications for Electrical Circuits*, Huazhong University Press, Wuhan, China, 1986.

[2] J. Ali, “One dimensional differential transform method for some higher order boundary value problems in finite domain,” *International Journal of Contemporary Mathematical Sciences*, pp. 263–275, 2012.

[3] F. Ayaz, “Solutions of the system of differential equations by differential transform method,” *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 547–567, 2004.

[4] F. Ayaz, “Applications of differential transform method to differential-algebraic equations,” *Applied Mathematics and Computation*, vol. 152, no. 3, pp. 649–657, 2004.

[5] A. Arikoglu and I. Ozkol, “Solution of difference equations by using differential transform method,” *Applied Mathematics and Computation*, vol. 173, no. 1, pp. 126–136, 2006.

[6] Z. M. Odibat, C. Bertelle, M. A. Aziz-Alaoui, and G. H. E. Duchamp, “A multi-step differential transform method and application to non-chaotic or chaotic systems,” *Computers & Mathematics with Applications*, vol. 59, no. 4, pp. 1462–1472, 2010.

[7] F. Ziyaee and A. Tari, “Differential transform method for solving the two dimensional Fredholm Integral Equation,” *Application and Applied Mathematics*, vol. 10, no. 2, pp. 852–863, 2015.

[8] I. H. Abdel-Halim Hassan, “Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems,” *Chaos, Solitons & Fractals*, vol. 36, no. 1, pp. 53–65, 2008.

[9] K. Raslan and Z. F. Abu Sheer, “Differential transform method for solving nonlinear systems of partial differential equations,” *International Journal of the Physical Sciences*, vol. 8, no. 38, pp. 1880–1884, 2013.

[10] I. Hassan and V. Erturk, “Solutions of different types of the linear and nonlinear higher order boundary value problems by differential transform method,” *European Journal of Pure and Applied Mathematics*, vol. 2, no. 3, pp. 426–447, 2009.

[11] N. Patil and A. Khembayat, “Differential transform method for solving system of linear differential equations,” *Journal of Mathematics and Statistics*, vol. 2, no. 3, pp. 1–10, 2014.

[12] E. Abdewahid, “Useful formulas for one dimensional differential transform,” *British Journal of Applied Science & Technology*, vol. 18, no. 3, pp. 1–8, 2016.

[13] A. Arikoglu and I. Ozkol, “Solution of boundary value problems for integro-differential equations by using differential transform method,” *Applied Mathematics and Computation*, vol. 168, pp. 1145–1158, 2006.

[14] K. Batıha and B. Batıha, “A new algorithm for solving linear ordinary differential equations,” *World Applied Sciences Journal*, vol. 15, no. 12, pp. 1774–1779, 2011.
[15] N. Bildik, A. Konuralp, F. Küçükarslan, and S. Kucukarslan, "Solution of different type of the partial differential equation by differential transform method and Adomian’s decomposition method," Applied Mathematics and Computation, vol. 172, no. 1, pp. 551–567, 2006.

[16] D. Kincaid and C. Ward, Numerical Analysis Mathematics of Scientific Computing, The University of Texas at Austin, Austin, TX, USA, 1990.

[17] R. Rizkalla, T. Seham, and M. Taha, "Application on differential transform method for some nonlinear functions and for solving Volterra integral equation involving fresnel integrals," Journal of Fractional Calculus and Application, vol. 5, no. 35, pp. 1–14, 2014.

[18] A. Yazdani, J. Vahidi, and S. Ghasempour, "Comparison between differential transform method and Taylor series method for solving linear and nonlinear ordinary differential equations," International Journal of Mechatronics, Electrical and computer Technology, vol. 6, no. 20, pp. 2872–2877, 2016.

[19] E. M. E. Zayed, S. R. Grace, H. El-Metwally, and M. A. El-Moneam, "The oscillatory behavior of second order nonlinear functional differential equations," Arabian Journal for Science and Engineering, vol. 31, no. 1A, pp. 23–30, 2006.

[20] E. M. E. Zayed and M. A. El-Moneam, "Some oscillation criteria for second order nonlinear functional ordinary differential equations," Acta Mathematica Scientia, vol. 27, no. 3, pp. 602–610, 2007.