Moduli stacks of vector bundles on curves and the King–Schofield rationality proof

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Introduction

Let $C$ be a connected smooth projective curve of genus $g \geq 2$ over an algebraically closed field $k$. Consider the coarse moduli scheme $\mathcal{B}un_{r,d}$ (resp. $\mathcal{B}un_{r,L}$) of stable vector bundles on $C$ with rank $r$ and degree $d \in \mathbb{Z}$ (resp. determinant isomorphic to the line bundle $L$ on $C$).

Motivated by work of A. Tyurin [10, 11] and P. Newstead [7, 8], it has been believed for a long time that $\mathcal{B}un_{r,L}$ is rational if $r$ and the degree of $L$ are coprime. Finally, this conjecture was proved in 1999 by A. King and A. Schofield [4]; they deduce it from their following main result:

**Theorem 0.1 (King–Schofield).** $\mathcal{B}un_{r,d}$ is birational to the product of an affine space $\mathbb{A}^n$ and $\mathcal{B}un_{h,0}$ where $h$ be the highest common factor of $r$ and $d$.

The present text contains the complete proof of King and Schofield translated into the language of algebraic stacks. Following their strategy, the moduli stack $\mathcal{B}un_{r,d}$ of rank $r$, degree $d$ vector bundles is shown to be birational to a Grassmannian bundle over $\mathcal{B}un_{r_1,d_1}$ for some $r_1 < r$; then induction is used. However, this Grassmannian bundle is in some sense twisted. Mainly for that reason, King and Schofield need a stronger induction hypothesis than 0.1. They add the condition that their birational map preserves a certain Brauer class $\psi_{r,d}$ on $\mathcal{B}un_{r,d}$. One main advantage of the stack language here is that this extra condition is not needed: The stack analogue of theorem 0.1 is proved by a direct induction.

(In more abstract terms, this can be understood roughly as follows: A Brauer class corresponds to a gerbe with band $\mathbb{G}_m$. But the gerbe on $\mathcal{B}un_{r,d}$ corresponding to $\psi_{r,d}$ is just the moduli stack $\mathcal{B}un_{r,d}$. Thus a rational map of coarse moduli schemes preserving this Brauer class corresponds to a rational map of the moduli stacks.)

This paper consists of four parts. Section 1 contains the precise formulation of the stack analogue 1.2 to theorem 0.1; then the original results of King and Schofield are deduced. Section 2 deals with Grassmannian bundles over stacks

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because they are the main tool for the proof of theorem 1.2 in section 3. Finally, appendix A summarizes the basic properties of the moduli stack $B_{un,r,d}$ that we use. In particular, a proof of Hirschowitz' theorem about the tensor product of general vector bundles on $C$ is given here, following Russo and Teixidor [9].

The present article has grown out of a talk in the joint seminar of U. Stuhler and Y. Tschinkel in Göttingen. I would like to thank them for the opportunity to speak and for encouraging me to write this text. I would also like to thank J. Heinloth for some valuable suggestions and for many useful discussions about these stacks.

1 The King-Schofield theorem in stack form

We denote by $B_{un,r,d}$ the moduli stack of vector bundles of rank $r$ and degree $d$ on our smooth projective curve $C$ of genus $g \geq 2$ over $k = \bar{k}$. This stack is algebraic in the sense of Artin, smooth of dimension $(g-1)r^2$ over $k$ and irreducible; these properties are discussed in more detail in the appendix.

Our main subject here is the birational type of $B_{un,r,d}$. We will frequently use the notion of a rational map between algebraic stacks; it is defined in the usual way as an equivalence class of morphisms defined on dense open substacks. A birational map is a rational map that admits a two-sided inverse.

Definition 1.1. A rational map of algebraic stacks $\mathcal{M} \rightarrow \mathcal{M}'$ is birationally linear if it admits a factorization

$$\mathcal{M} \rightleftharpoons \mathcal{M}' \times A^n \xrightarrow{pr_1} \mathcal{M}'$$

into a birational map followed by the projection onto the first factor.

Now we can formulate the stack analogue of the King-Schofield theorem 0.1; its proof will be given in section 3.

Theorem 1.2. Let $h$ be the highest common factor of the rank $r \geq 1$ and the degree $d \in \mathbb{Z}$. There is a birationally linear map of stacks

$$\mu : B_{un,r,d} \rightarrow B_{un,h,0}$$

and an isomorphism between the Picard schemes $\text{Pic}^d(C)$ and $\text{Pic}^0(C)$ such that the following diagram commutes:

$$\begin{array}{ccc}
B_{un,r,d} & \xrightarrow{\mu} & B_{un,h,0} \\
\downarrow \text{det} & & \downarrow \text{det} \\
\text{Pic}^d(C) & \xrightarrow{\sim} & \text{Pic}^0(C)
\end{array}$$

(1)

Remark 1.3. One cannot expect an isomorphism of Picard stacks here: If (1) were a commutative diagram of stacks, then choosing a general vector bundle...
$E$ of rank $r$ and degree $d$ would yield a commutative diagram of automorphism groups

\[
\begin{array}{ccc}
G_m & \xrightarrow{\sim} & G_m \\
\downarrow \scriptstyle{\omega^r} & & \downarrow \scriptstyle{\omega^h} \\
G_m & \xrightarrow{\sim} & G_m
\end{array}
\]

which is impossible for $r \neq h$.

**Remark 1.4.** In the theorem, we can furthermore achieve that $\mu$ preserves scalar automorphisms in the following sense:

Let $E$ and $E' = \mu(E)$ be vector bundles over $C$ that correspond to a general point in $\mathcal{B}un_{r,d}$ and its image in $\mathcal{B}un_{h,0}$. Then $E$ and $E'$ are stable (because we have assumed $g \geq 2$) and hence simple. The rational map $\mu$ induces a morphism of algebraic groups

\[
\mu^E : G_m = \text{Aut}(E) \longrightarrow \text{Aut}(E') = G_m
\]

which is an isomorphism because $\mu$ is birationally linear. Thus $\mu^E$ is either the identity or $\lambda \mapsto \lambda^{-1}$; it is independent of $E$ because $\mathcal{B}un_{r,d}$ is irreducible. Modifying $\mu$ by the automorphism $E' \mapsto E'^{\text{dual}}$ of $\mathcal{B}un_{h,0}$ if necessary, we can achieve that $\mu^E$ is the identity for every general $E$.

Clearly, the map $\mu$ in the theorem restricts to a birationally linear map between the dense open substacks of stable vector bundles. But any rational (resp. birational, resp. birationally linear) map between these induces a rational (resp. birational, resp. birationally linear) map between the corresponding coarse moduli schemes; cf. proposition A.6 in the appendix for details. Hence the original theorem of King and Schofield follows:

**Corollary 1.5 (King–Schofield).** Let $\mathcal{B}un_{r,d}$ be the coarse moduli scheme of stable vector bundles of rank $r$ and degree $d$ on $C$. Then there is a birationally linear map of schemes

\[
\mu : \mathcal{B}un_{r,d} \longrightarrow \mathcal{B}un_{h,0}.
\]

Of course, this is just a reformulation of the theorem 0.1 mentioned in the introduction.

**Remark 1.6.** As mentioned before, King and Schofield also prove that the rational map $\mu : \mathcal{B}un_{r,d} \longrightarrow \mathcal{B}un_{h,0}$ preserves their Brauer class $\psi_{r,d}$. This is equivalent to the condition that $\mu$ induces a rational map between the corresponding $G_m$-gerbes, i.e. a rational map $\mathcal{B}un_{r,d} \longrightarrow \mathcal{B}un_{h,0}$ that preserves scalar automorphisms in the sense of remark 1.4.

We recall the consequences concerning the rationality of $\mathcal{B}un_{r,L}$. Because the diagram (1) commutes, $\mu$ restricts to a rational map between fixed determinant moduli schemes; thus one obtains:
Corollary 1.7 (King–Schofield). Let $L$ be a line bundle on $C$, and let $\text{Bun}_{r,L}$ be the coarse moduli scheme of stable vector bundles of rank $r$ and determinant $L$ on $C$. Then there is a birationally linear map of schemes

$$\mu : \text{Bun}_{r,L} \rightarrow \text{Bun}_{h,O}$$

where $h$ is the highest common factor of $r$ and $\deg(L)$.

In particular, $\text{Bun}_{r,L}$ is rational if the rank $r$ and the degree $\deg(L)$ are coprime; this proves the conjecture mentioned in the introduction. More generally, it follows that $\text{Bun}_{r,L}$ is rational if $\text{Bun}_{h,O}$ is. For $h \geq 2$, it seems to be still an open question whether $\text{Bun}_{h,O}$ is rational or not.

## 2 Grassmannian bundles

Let $\mathcal{V}$ be a vector bundle over a dense open substack $\mathcal{U} \subseteq \text{Bun}_{r,d}$. Recall that a part of this datum is a functor from the groupoid $\mathcal{U}(k)$ to the groupoid of vector spaces over $k$. So for each appropriate vector bundle $E$ over $C$, we do not only get a vector space $\mathcal{V}_E$ over $k$, but also a group homomorphism $\text{Aut}_{\mathcal{O}_C}(E) \rightarrow \text{Aut}_k(\mathcal{V}_E)$. Note that both groups contain the scalars $k^*$.

**Definition 2.1.** A vector bundle $\mathcal{V}$ over a dense open substack $\mathcal{U} \subseteq \text{Bun}_{r,d}$ has weight $w \in \mathbb{Z}$ if the diagram

$$
\begin{array}{ccc}
\text{k}^* & \rightarrow & \text{Aut}_{\mathcal{O}_C}(E) \\
\downarrow & & \downarrow \\
\text{k}^* & \rightarrow & \text{Aut}_k(\mathcal{V}_E)
\end{array}
$$

commutes for all vector bundles $E$ over $C$ that are objects of the groupoid $\mathcal{U}(k)$.

**Example 2.2.** The trivial vector bundle $\mathcal{O}^n$ over $\text{Bun}_{r,d}$ has weight 0.

We denote by $\mathcal{E}_{\text{univ}}$ the universal vector bundle over $C \times \text{Bun}_{r,d}$, and by $\mathcal{E}^\text{univ}_p$ its restriction to $\{p\} \times \text{Bun}_{r,d}$ for some point $p \in C(k)$.

**Example 2.3.** $\mathcal{E}^\text{univ}_p$ is a vector bundle of weight 1 on $\text{Bun}_{r,d}$, and its dual $(\mathcal{E}^\text{univ}_p)^\text{dual}$ is a vector bundle of weight $-1$.

For another example, we fix a vector bundle $F$ over $C$. By semicontinuity, there is an open substack $\mathcal{U} \subseteq \text{Bun}_{r,d}$ that parameterizes vector bundles $E$ of rank $r$ and degree $d$ over $C$ with $\text{Ext}^1(F,E) = 0$; we assume $\mathcal{U} \neq \emptyset$. The vector spaces $\text{Hom}(F,E)$ are the fibres of a vector bundle $\text{Hom}(F,\mathcal{E}^\text{univ})$ over $\mathcal{U}$ according to Grothendieck’s theory of cohomology and base change in EGA III.

Similarly, there is a vector bundle $\text{Hom}(\mathcal{E}^\text{univ},F)$ defined over an open substack of $\text{Bun}_{r,d}$ whose fibre over any point $[E]$ with $\text{Ext}^1(E,F) = 0$ is the vector space $\text{Hom}(E,F)$.
Example 2.4. \( \text{Hom}(F, \mathcal{E}^{\text{univ}}) \) is a vector bundle of weight 1, and \( \text{Hom}(\mathcal{E}^{\text{univ}}, F) \) is a vector bundle of weight \(-1\).

Note that any vector bundle of weight 0 over an open substack \( \mathcal{U} \subseteq \text{Bun}_{r,d} \) contained in the stable locus descends to a vector bundle over the corresponding open subscheme \( \mathcal{U} \subseteq \text{Bun}_{r,d} \) of the coarse moduli scheme, cf. proposition A.6. Vector bundles of nonzero weight do not descend to the coarse moduli scheme.

Proposition 2.5. Consider all vector bundles \( \mathcal{V} \) of fixed weight \( w \) over dense open substacks of a fixed stack \( \text{Bun}_{r,d} \). Assume that \( \mathcal{V}_0 \) has minimal rank among them. Then every such \( \mathcal{V} \) is generically isomorphic to \( \mathcal{V}_0^n \) for some \( n \).

Proof. The homomorphism bundles \( \text{End}(\mathcal{V}_0) \) and \( \text{Hom}(\mathcal{V}_0, \mathcal{V}) \) are vector bundles of weight 0 over dense open substacks of \( \text{Bun}_{r,d} \). Hence they descend to vector bundles \( A \) and \( M \) over dense open subschemes of \( \text{Bun}_{r,d} \), cf. proposition A.6. The algebra structure on \( \text{End}(\mathcal{V}_0) \) and its right(!) action on \( \text{Hom}(\mathcal{V}_0, \mathcal{V}) \) also descend; they turn \( A \) into an Azumaya algebra and \( M \) into a right \( A \)-module.

In particular, the generic fibre \( M_K \) is a right module under the central simple algebra \( A_K \) over the function field \( K := k(\text{Bun}_{r,d}) \). By our choice of \( \mathcal{V}_0 \), there are no nontrivial idempotent elements in \( A_K \); hence \( A_K \) is a skew field.

We have just constructed a functor \( \mathcal{V} \mapsto M_K \) from the category in question to the category of finite-dimensional right vector spaces over \( A_K \). This functor is a Morita equivalence; its inverse is defined as follows:

Given such a right vector space \( M_K \) over \( A_K \), we can extend it to a right \( A \)-module \( M \) over a dense open subscheme of \( \text{Bun}_{r,d} \), i.e. to a right \( \text{End}(\mathcal{V}_0) \)-module of weight 0 over a dense open substack of \( \text{Bun}_{r,d} \); we send \( M_K \) to the vector bundle of weight \( w \) \( V := M \otimes_{\text{End}(\mathcal{V}_0)} \mathcal{V}_0. \)

Using this Morita equivalence, the proposition follows from the corresponding statement for right vector spaces over \( A_K \). \( \square \)

Corollary 2.6. There is a vector bundle of weight \( w = 1 \) (resp. \( w = -1 \)) and rank \( h = \text{hcf}(r,d) \) over a dense open substack of \( \text{Bun}_{r,d} \).

Proof. Because the case of weight \( w = -1 \) follows by dualizing the vector bundles, we only consider vector bundles of weight \( w = 1 \). Here \( \mathcal{E}^{\text{univ}}_r \) is a vector bundle of rank \( r \) over \( \text{Bun}_{r,d} \), and \( \text{Hom}(L^\text{dual}, \mathcal{E}^{\text{univ}}) \) is a vector bundle of rank \( r(1-g+\deg(L)) + d \) over a dense open substack if \( L \) is a sufficiently ample line bundle on \( C \). Consequently, the rank of \( \mathcal{V}_0 \) divides \( r \) and \( r(1-g+\deg(L)) + d \); hence it also divides their highest common factor \( h \). \( \square \)

To each vector bundle \( \mathcal{V} \) over a dense open substack \( \mathcal{U} \subseteq \text{Bun}_{r,d} \), we can associate a Grassmannian bundle

\[ \text{Gr}_j(\mathcal{V}) \rightarrow \mathcal{U} \subseteq \text{Bun}_{r,d}. \]
By definition, $\text{Gr}_j(V)$ is the moduli stack of those vector bundles $E$ over $C$ which are parameterized by $U$, endowed with a $j$-dimensional vector subspace of $V_E$. $\text{Gr}_j(V)$ is again a smooth Artin stack locally of finite type over $k$, and its canonical morphism to $U$ is representable by Grassmannian bundles of schemes.

If $V$ is a vector bundle of some weight, then all scalar automorphisms of $E$ preserve all vector subspaces of $V$. This means that the automorphism groups of the groupoid $\text{Gr}_j(V)(k)$ also contain the scalars $k^*$. In particular, it makes sense to say that a vector bundle over $\text{Gr}_j(V)$ has weight $w \in \mathbb{Z}$: There is an obvious way to generalize definition 2.1 to this situation.

To give some examples, we fix a point $p \in C(k)$. Let $\text{Par}_{r,d}^m$ be the moduli stack of rank $r$, degree $d$ vector bundles $E$ over $C$ endowed with a quasiparabolic structure of multiplicity $m$ over $p$. Recall that such a quasiparabolic structure is just a coherent subsheaf $E' \subseteq E$ with the property that $E/E'$ is isomorphic to the skyscraper sheaf $\mathcal{O}_C(mp)$.

**Example 2.7.** $\text{Par}_{r,d}^m$ is canonically isomorphic to the Grassmannian bundle $\text{Gr}_m((\mathcal{E}_p^{\text{univ}})_{\text{dual}})$ over $\text{Bun}_{r,d}$.

Here we have regarded a quasiparabolic vector bundle $E^* = (E' \subseteq E)$ as the vector bundle $E$ together with a dimension $m$ quotient of the fibre $E_p$. But we can also regard it as the vector bundle $E'$ together with a dimension $m$ vector subspace in the fibre at $p$ of the twisted vector bundle $E'(p)$. Choosing a trivialization of the line bundle $\mathcal{O}_C(p)$ over $p$, we can identify the fibres of $E'(p)$ and $E'$ at $p$; hence we also obtain:

**Example 2.8.** $\text{Par}_{r,d}^m$ is isomorphic to the Grassmannian bundle $\text{Gr}_m(\mathcal{E}_{p_{\text{univ}}})$ over $\text{Bun}_{r,d-m}$ where $\mathcal{E}_{\text{univ}}$ is the universal vector bundle over $C \times \text{Bun}_{r,d-m}$.

These two Grassmannian bundles

$\text{Bun}_{r,d} \xleftarrow{\theta_1} \text{Par}_{r,d}^m \xrightarrow{\theta_2} \text{Bun}_{r,d-m}$

form a correspondence between $\text{Bun}_{r,d}$ and $\text{Bun}_{r,d-m}$, the Hecke correspondence. Its effect on the determinant line bundles is given by

$$\det \theta_1(E^*) = \det(E) \cong \mathcal{O}_C(mp) \otimes \det(E') = \mathcal{O}_C(mp) \otimes \det \theta_2(E^*)$$

(2)

for each parabolic vector bundle $E^* = (E' \subseteq E)$ with multiplicity $m$ at $p$.

**Proposition 2.9.** Let $V$ and $W$ be two vector bundles of the same weight $w$ over dense open substacks of $\text{Bun}_{r,d}$. If $j \leq \text{rk}(W) \leq \text{rk}(V)$, then there is a birationally linear map

$$\rho: \text{Gr}_j(V) \dashrightarrow \text{Gr}_j(W)$$

over $\text{Bun}_{r,d}$. 
Proof. According to proposition 2.5, there is a vector bundle \( W' \) of weight \( w \) such that \( V \cong W \oplus W' \) over some dense open substack \( \mathcal{U} \subseteq \text{Bun}_{r,d} \). We may assume without loss of generality that \( \mathcal{U} \) is contained in the stable locus and denote by \( \mathcal{U} \subseteq \text{Bun}_{r,d} \) the corresponding open subscheme, cf. proposition A.6.

We use the following simple fact from linear algebra: If \( W \) and \( W' \) are vector spaces over \( k \) with \( \dim(W) \geq j \), then every \( j \)-dimensional vector subspace of \( W \oplus W' \) whose image \( S \) in \( W \) also has dimension \( j \) is the graph of a unique linear map \( S \rightarrow W' \).

This means that \( \text{Gr}_j(W \oplus W') \) contains as a dense open subscheme the total space of the vector bundle \( \text{Hom}(S_{\text{univ}}, W') \) over \( \text{Gr}_j(W) \) where \( S_{\text{univ}} \) is the universal subbundle of the constant vector bundle \( W \) over \( \text{Gr}_j(W) \).

In our stack situation, these considerations imply that \( \text{Gr}_j(V) \) is birational to the total space of the vector bundle \( \text{Hom}(S_{\text{univ}}, W') \) over \( \text{Gr}_j(W) \) where \( S_{\text{univ}} \) is the universal subbundle of the pullback of \( W \) over \( \text{Gr}_j(W) \). This defines the rational map \( \rho \).

The vector bundle \( \text{Hom}(S_{\text{univ}}, W') \) has weight 0 because \( S_{\text{univ}} \) and \( W' \) both have weight \( w \). Since the scalars act trivially, we can descend \( \text{Gr}_j(W) \) and this vector bundle over it to a Grassmannian bundle over \( \mathcal{U} \) and a vector bundle over it, cf. proposition A.6. In particular, our homomorphism bundle is trivial over a dense open substack of \( \text{Gr}_j(W) \). This proves that \( \rho \) is birationally linear.

Corollary 2.10. Let \( V \) be a vector bundle of weight \( w = \pm 1 \) over a dense open substack of \( \text{Bun}_{r,d} \). If \( j \) is divisible by \( \text{hcf}(r,d) \), then the Grassmannian bundle

\[
\text{Gr}_j(V) \longrightarrow \text{Bun}_{r,d}
\]

is birationally linear.

Proof. By corollary 2.6, there is a vector bundle \( W \) of weight \( w \) and rank \( j \). Due to the proposition, \( \text{Gr}_j(V) \) is birationally linear over \( \text{Gr}_j(W) \cong \text{Bun}_{r,d} \).

\[3\] Proof of theorem \[1.2\]

The aim of this section is to prove theorem \[1.2 \] i.e. to construct the birationally linear map \( \mu : \text{Bun}_{r,d} \rightarrow \text{Bun}_{h,0} \) where \( h \) is the highest common factor of the rank \( r \) and the degree \( d \). We proceed by induction on \( r/h \).

For \( r = h \), the theorem is trivial: Tensoring with an appropriate line bundle defines even an isomorphism of stacks \( \text{Bun}_{r,d} \cong \text{Bun}_{h,0} \) with the required properties. Thus we may assume \( r > h \).

Lemma 3.1. There are unique integers \( r_F \) and \( d_F \) that satisfy

\[
(1-g)r_F r + r_F d - rd_F = h \tag{3}
\]

and

\[
r < hr_F < 2r. \tag{4}
\]
Proof. \[3\] has an integer solution \(r_F, d_F\) because \(h\) is also the highest common factor of \(r\) and \((1 - g)r + d\); here \(r_F\) is unique modulo \(r/h\). Furthermore, \(r_F\) is nonzero modulo \(r/h\) since \(-rd_F = h\) has no solution. Hence precisely one of the solutions \(r_F, d_F\) of \[3\] also satisfies \[4\].

We fix \(r_F, d_F\) and define

\[
\begin{align*}
    r_1 &:= hr_F - r, \\
    d_1 &:= hd_F - d, \\
    h_1 &:= \text{hcf}(r_1, d_1).
\end{align*}
\]

Then \(r_1 < r\), and \(h_1\) is a multiple of \(h\). In particular, \(r_1/h_1 < r/h\).

Lemma 3.2. There is an exact sequence

\[
0 \rightarrow E_1 \rightarrow F \otimes_k V \rightarrow E \rightarrow 0
\]  
(5)

where \(E_1, F, E\) are vector bundles over \(C\) and \(V\) is a vector space over \(k\) with

\[
\begin{align*}
    \text{rk}(E_1) &= r_1, \\
    \text{rk}(F) &= r_F, \\
    \text{rk}(E) &= r, \\
    \dim(V) &= h \\
    \deg(E_1) &= d_1, \\
    \deg(F) &= d_F, \\
    \deg(E) &= d
\end{align*}
\]

such that the following two conditions are satisfied:

i) \(\text{Ext}^1(F, E) = 0\), and the induced map \(V \rightarrow \text{Hom}(F, E)\) is bijective.

ii) \(\text{Ext}^1(E_1, F) = 0\), and the induced map \(V^\text{dual} \rightarrow \text{Hom}(E_1, F)\) is injective.

Proof. We may assume \(h = 1\) without loss of generality: If there is such a sequence for \(r/h\) and \(d/h\) instead of \(r\) and \(d\), then the direct sum of \(h\) copies is the required sequence for \(r\) and \(d\).

By our choice of \(r_F\) and \(d_F\) and Riemann-Roch, all vector bundles \(F\) and \(E\) of these ranks and degrees satisfy

\[
\chi(F, E) := \dim_k \text{Hom}(F, E) - \dim_k \text{Ext}^1(F, E) = h = 1.
\]

If \(F\) and \(E\) are general, then

\[
\text{Hom}(F, E) \cong k \quad \text{and} \quad \text{Ext}^1(F, E) = 0
\]

according to a theorem of Hirschowitz [2, section 4.6], and there is a surjective map \(F \rightarrow E\) by an argument of Russo and Teixidor [9]. Thus we obtain an exact sequence

\[
0 \rightarrow E_1 \rightarrow F \rightarrow E \rightarrow 0
\]  
(6)

that satisfies condition i (with \(V = k\)).

(For the convenience of the reader, a proof of the cited results is given in the appendix, cf. theorem [A.7].)

Furthermore, all vector bundles of the given ranks and degrees satisfy

\[
\chi(E_1, F) = \chi(F, E) - \chi(E, E) + \chi(E_1, E_1) > \chi(F, E) = h = 1
\]

because \(r_1 < r\). Now we can argue as above: For general \(E_1\) and \(F\), we have \(\text{Ext}^1(E_1, F) = 0\) by Hirschowitz, and there is an injective map \(E_1 \rightarrow F\) with
torsionfree cokernel by Russo-Teixidor; cf. also theorem A.7 in the appendix. Thus we obtain an exact sequence (6) that satisfies condition ii (with $V = k$).

Finally, we consider the moduli stack of all exact sequences (6) of vector bundles with the given ranks and degrees. As explained in the appendix (cf. corollary A.5), it is an irreducible algebraic stack locally of finite type over $k$. But i and ii are open conditions, so there is a sequence that satisfies both. □

From now on, let $F$ be a fixed vector bundle of rank $r_F$ and degree $d_F$ that occurs in such an exact sequence (5).

**Definition 3.3.** The rational map of stacks

$$\lambda_F : \text{Bun}_{r,d} \dashrightarrow \text{Bun}_{r_1,d_1}$$

is defined by sending a general rank $r$, degree $d$ vector bundle $E$ over $C$ to the kernel of the natural evaluation map

$$\epsilon_E : \text{Hom}(F, E) \otimes_k F \rightarrow E.$$

We check that this does define a rational map. Let $U_F \subseteq \text{Bun}_{r,d}$ be the open substack that parameterizes all $E$ for which $\text{Ext}^1(F, E) = 0$ and $\epsilon_E$ is surjective. Then the $\epsilon_E$ are the restrictions of a surjective morphism $\epsilon_{\text{univ}}$ of vector bundles over $C \times U_F$. So the kernel of $\epsilon_{\text{univ}}$ is also a vector bundle; it defines a morphism $\lambda_F : U_F \rightarrow \text{Bun}_{r_1,d_1}$. This gives the required rational map because $U_F$ is nonempty by our choice of $F$.

For later use, we record the effect of $\lambda_F$ on determinant line bundles:

$$\det \lambda_F(E) \cong \det(F)^{\otimes h} \otimes \det(E)^{\text{dual}}. \quad (7)$$

Following [4], the next step is to understand the fibres of $\lambda_F$. We denote by $\text{Hom}(E_{\text{univ}}, F)$ the vector bundle over an open substack of $\text{Bun}_{r_1,d_1}$ whose fibre over any point $[E_1]$ with $\text{Ext}^1(E_1, F) = 0$ is the vector space $\text{Hom}(E_1, F)$.

**Proposition 3.4.** $\text{Bun}_{r,d}$ is over $\text{Bun}_{r_1,d_1}$ naturally birational to the Grassmannian bundle $\text{Gr}_h(\text{Hom}(E_{\text{univ}}^1, F))$.

**Proof.** If $E$ is a rank $r$, degree $d$ vector bundle over $C$ for which $\text{Ext}^1(F, E) = 0$ and the above map $\epsilon := \epsilon_E$ is surjective, then the exact sequence

$$0 \rightarrow \ker(\epsilon) \rightarrow \text{Hom}(F, E) \otimes_k F \overset{\epsilon}{\rightarrow} E \rightarrow 0$$

satisfies the condition i of the previous lemma. This identifies the above open substack $U_F \subseteq \text{Bun}_{r,d}$ with the moduli stack of all exact sequences (6) that satisfy i.

Similarly, let $U'_F \subseteq \text{Gr}_h(\text{Hom}(E_{\text{univ}}^1, F))$ be the open substack that parameterizes all pairs $(E_1, S \subseteq \text{Hom}(E_1, F))$ for which $\text{Ext}^1(E_1, F) = 0$ and the natural map $\alpha : E_1 \rightarrow S^{\text{dual}} \otimes_k F$ is injective with torsionfree cokernel. For such a pair $(E_1, S)$, the exact sequence

$$0 \rightarrow E_1 \overset{\alpha}{\rightarrow} S^{\text{dual}} \otimes_k F \rightarrow \text{coker}(\alpha) \rightarrow 0$$
satisfies the condition ii of the previous lemma. This identifies \( U'_F \) with the moduli stack of all exact sequences \( \mathcal{E}^\text{univ}_1 \) that satisfy ii.

Hence both \( \text{Bun}_{r,d} \) and \( \text{Gr}_h(\text{Hom}(\mathcal{E}^\text{univ}_1, F)) \) contain as an open substack the moduli stack \( U'_F \) of all exact sequences \( \mathcal{E}^\text{univ}_1 \) that satisfy both conditions i and ii. But \( U'_F \) is nonempty by our choice of \( F \), so it is dense in both stacks; thus they are birational over \( \text{Bun}_{r_1,d_1} \).

Still following [4], the proof of theorem 1.2 can now be summarized in the following diagram; it is explained below.

\[
\begin{array}{c}
\text{Bun}_{r,d} \xrightarrow{\rho} \text{Gr}_h(\mathcal{W}) \xrightarrow{\theta_1} \text{Bun}_{h,0} \\
\downarrow \phi \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{Bun}_{r_1,d_1} \xrightarrow{\mu_1} \text{Bun}_{h_1,0}
\end{array}
\]

Here \( \mu_1 \) and \( \mu_2 \) are the birationally linear maps given by the induction hypothesis. \( (\theta_1, \theta_2) \) is the Hecke correspondence explained in the previous section; note that \( \theta_2 \) is birationally linear by corollary 2.10.

The square in this diagram is cartesian, so \( \tilde{\mu}_1 \) is the pullback of \( \mu_1 \), and \( \mathcal{W} := \mu_1^!(\mathcal{E}^\text{uni}_{p})^\text{dual} \) is the pullback of the vector bundle \( (\mathcal{E}^\text{uni}_{p})^\text{dual} \) over \( \text{Bun}_{h_1,0} \) to which \( \theta_1 \) is the associated Grassmannian bundle. Using remark 1.4 we may assume that \( \mu_1 \) preserves scalar automorphisms, i.e. that \( \mathcal{W} \) has the same weight \(-1\) as \( (\mathcal{E}^\text{uni}_{p})^\text{dual} \). Then we can apply proposition 2.9 to obtain the birationally linear map \( \rho \). Now we have the required birationally linear map \( \mu := \mu_2 \circ \theta_2 \circ \tilde{\mu}_1 \circ \rho : \text{Bun}_{r,d} \dashrightarrow \text{Bun}_{h,0} \); it satisfies the determinant condition in theorem 1.2 due to equations (7), (2) and the corresponding induction hypothesis on \( \mu_1, \mu_2 \).

**A Moduli stacks of sheaves on curves**

This section summarizes some well-known basic properties of moduli stacks of vector bundles and more generally coherent sheaves on curves. For the general theory of algebraic stacks, we refer the reader to [5] or the appendix of [12]. We prove that the moduli stacks in question are algebraic, smooth and irreducible. Then we discuss descent to the coarse moduli scheme. Finally, we deduce Hirschowitz’ theorem [2] and a refinement by Russo and Teixidor [9] about morphisms between general vector bundles.

Recall that we have fixed an algebraically closed field \( k \) and a connected smooth projective curve \( C/k \) of genus \( g \). We say that a coherent sheaf \( F \) on \( C \) has type \( t = (r, d) \) if its rank \( \text{rk}(F) \) (at the generic point of \( C \)) equals \( r \) and its degree \( \text{deg}(F) \) equals \( d \).

If \( F' \) and \( F \) are coherent sheaves of types \( t = (r, d) \) and \( t' = (r', d') \) on \( C \), then the Euler characteristic

\[
\chi(F', F) := \dim_k \text{Hom}(F', F) - \dim_k \text{Ext}^1(F', F)
\]
satisfies the Riemann-Roch theorem $\chi(F', F) = \chi(t', t)$ with
$$\chi(t', t) := (1 - g)r'r + r'd - rd'.$$

Note that $Ext^n(F', F)$ vanishes for all $n \geq 2$ since $\dim(C) = 1$.

We denote by $\mathcal{C}oh_t$ the moduli stack of coherent sheaves $F$ of type $t$ on $C$. More precisely, $\mathcal{C}oh_t(S)$ is for each $k$-scheme $S$ the groupoid of all coherent sheaves on $C \times S$ which are flat over $S$ and whose fibre over every point of $S$ has type $t$.

Now assume $t = t_1 + t_2$. We denote by $\mathcal{E}xt(t_2, t_1)$ the moduli stack of exact sequences of coherent sheaves on $C$

$$0 \to F_1 \to F \to F_2 \to 0$$

where $F_i$ has type $t_i = (r_i, d_i)$ for $i = 1, 2$. This means that $\mathcal{E}xt(t_2, t_1)(S)$ is for each $k$-scheme $S$ the groupoid of short exact sequences of coherent sheaves on $C \times S$ which are flat over $S$ and fibrewise of the given types.

**Proposition A.1.** The stacks $\mathcal{C}oh_t$ and $\mathcal{E}xt(t_2, t_1)$ are algebraic in the sense of Artin and locally of finite type over $k$.

**Proof.** Let $\mathcal{O}(1)$ be an ample line bundle on $C$. For $n \in \mathbb{Z}$, we denote by

$$\mathcal{C}oh_t^n \subseteq \mathcal{C}oh_t \quad (\text{resp. } \mathcal{E}xt(t_2, t_1)^n \subseteq \mathcal{E}xt(t_2, t_1))$$

the open substack that parameterizes coherent sheaves $F$ on $C$ (resp. exact sequences $0 \to F_1 \to F \to F_2 \to 0$ of coherent sheaves on $C$) such that the twist $F(n) = F \otimes \mathcal{O}(1)^{\otimes n}$ is generated by global sections and $H^1(F(n)) = 0$.

By Grothendieck’s theory of Quot-schemes, there is a scheme $\text{Quot}_t^n$ of finite type over $k$ that parameterizes such coherent sheaves $F$ together with a basis of the $k$-vector space $H^0(F(n))$. Moreover, there is a relative Quot-scheme $\text{Flag}(t_2, t_1)^n$ of finite type over $\text{Quot}_t^n$ that parameterizes such exact sequences $0 \to F_1 \to F \to F_2 \to 0$ together with a basis of $H^0(F(n))$.

Let $N$ denote the dimension of $H^0(F(n))$. According to Riemann-Roch, $N$ depends only on $t$, $n$ and the ample line bundle $\mathcal{O}(1)$, but not on $F$.

Changing the chosen basis defines an action of $\text{GL}(N)$ on $\text{Quot}_t^n$, and $\mathcal{C}oh_t^n$ is precisely the stack quotient $\text{Quot}_t^n/\text{GL}(N)$. Similarly, $\mathcal{E}xt(t_2, t_1)^n$ is precisely the stack quotient $\text{Flag}(t_2, t_1)^n/\text{GL}(N)$. Hence these two stacks are algebraic and of finite type over $k$.

By the ampleness of $\mathcal{O}(1)$, the $\mathcal{C}oh_t^n$ (resp. $\mathcal{E}xt(t_2, t_1)^n$) form an open covering of $\mathcal{C}oh_t$ (resp. $\mathcal{E}xt(t_2, t_1)$).

**Remark A.2.** In general, $\mathcal{C}oh_t$ is not quasi-compact because the family of all coherent sheaves on $C$ of type $t$ is not bounded.

**Proposition A.3.** i) $\mathcal{C}oh_t$ is smooth of dimension $-\chi(t, t)$ over $k$. 

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ii) If we assign to each exact sequence \(0 \to F_1 \to F \to F_2 \to 0\) the two sheaves \(F_1, F_2\), then the resulting morphism of algebraic stacks
\[
\mathcal{E}xt(t_2, t_1) \to \text{Coh}(t_1) \times \text{Coh}(t_2)
\]
is smooth of relative dimension \(-\chi(t_2, t_1)\), and all its fibres are connected.

iii) \(\mathcal{E}xt(t_2, t_1)\) is smooth of dimension \(-\chi(t_2, t_2) - \chi(t_2, t_1) - \chi(t_1, t_1)\) over \(k\).

Proof. i) By standard deformation theory, \(\text{Hom}(F, F)\) is the automorphism group of any infinitesimal deformation of the coherent sheaf \(F\), \(\text{Ext}^1(F, F)\) classifies such deformations, and \(\text{Ext}^2(F, F)\) contains the obstructions against extending deformations infinitesimally, cf. [3, 2.A.6]. Because \(\text{Ext}^2\) vanishes, deformations of \(F\) are unobstructed and hence \(\text{Coh}_t\) is smooth; its dimension at \(F\) is then \(\dim \text{Ext}^1(F, F) - \dim \text{Hom}(F, F) = -\chi(t, t)\).

ii) The fibre of this morphism over \([F_1, F_2]\) is the moduli stack of all extensions of \(F_2\) by \(F_1\); it is the stack quotient of the affine space \(\text{Ext}^1(F_2, F_1)\) modulo the trivial action of the algebraic group \(\text{Hom}(F_2, F_1)\). Hence this fibre is smooth of dimension \(-\chi(t_2, t_1)\) and connected.

More generally, consider a scheme \(S\) of finite type over \(k\) and a morphism \(S \to \text{Coh}(t_1) \times \text{Coh}(t_2)\); let \(F_1\) and \(F_2\) be the corresponding coherent sheaves over \(C \times S\). By EGA III, the object \(\text{RHom}(F_2, F_1)\) in the derived category of coherent sheaves on \(S\) can locally be represented by a complex of length one \(V^0 \xrightarrow{\delta} V^1\) where \(V^0, V^1\) are vector bundles. This means that the pullback of \(\mathcal{E}xt(t_2, t_1)\) to \(S\) is locally the stack quotient of the total space of \(V^1\) modulo the action of the algebraic group \(\text{Hom}(F_2, F_1)\). Hence this pullback is smooth over \(S\); this proves ii.

iii) follows from i and ii. \(\square\)

Proposition A.4. The stacks \(\text{Coh}_t\) and \(\mathcal{E}xt(t_2, t_1)\) are connected.

Proof. Proposition A.3 implies that \(\mathcal{E}xt(t_2, t_1)\) is connected if \(\text{Coh}_{t_1}\) and \(\text{Coh}_{t_2}\) are. We prove the connectedness of the latter by induction on the rank (and on the degree for rank zero).

\(\text{Coh}_t\) is connected for \(t = (0, 1)\) because there is a canonical surjection \(C \to \text{Coh}_t\); it sends a point \(P\) to the skyscraper sheaf \(\mathcal{O}_P\). Now consider \(t = (0, d)\) with \(d \geq 2\) and write \(t = t_1 + t_2\). By induction hypothesis and A.3 \(\mathcal{E}xt(t_1, t_2)\) is connected. But there is a canonical surjection \(\mathcal{E}xt(t_1, t_2) \to \text{Coh}_t\); it sends an extension \(0 \to F_1 \to F \to F_2 \to 0\) to the sheaf \(F\). This shows that \(\text{Coh}_t\) is also connected; now we have proved all connectedness assertions in rank zero.

If \(F\) and \(F'\) are two coherent sheaves on \(C\) of type \(t = (r, d)\) with \(r \geq 1\), then there is a line bundle \(L\) on \(C\) such that both \(L^{dual} \otimes F\) and \(L^{dual} \otimes F'\) have a generically nonzero section. In other words, there are injective morphisms \(L \to F\) and \(L \to F'\). Let \(t_L\) be the type of \(L\); then \(F\) and \(F'\) are both in the image of the canonical morphism \(\mathcal{E}xt(t - t_L, t_L) \to \text{Coh}_t\). But \(\mathcal{E}xt(t - t_L, t_L)\) is connected by the induction hypothesis and A.3. This shows that any two points \(F\) and \(F'\) lie in the same connected component of \(\text{Coh}_t\), i.e. \(\text{Coh}_t\) is connected. \(\square\)
Corollary A.5. The stacks $\text{Coh}_t$ and $\text{Ext}(t_2,t_1)$ are reduced and irreducible.

The moduli stack $\text{Bun}_t$ of vector bundles, the moduli stack $\text{Stab}_t$ of semi-stable vector bundles and the moduli stack $\text{Stab}_t$ of (geometrically) stable vector bundles on $C$ of type $t = (r,d)$ are open substacks

$$\text{Stab}_t \subseteq \text{Stab}_t \subseteq \text{Bun}_t \subseteq \text{Coh}_t.$$ 

Hence these stacks are all irreducible and smooth of the same dimension $-\chi(t,t)$ if they are nonempty. $\text{Stab}_t$ is known to be nonempty for $g \geq 2$ and $r \geq 1$. Moreover, $\text{Stab}_t$ and $\text{Stab}_t$ are quasi-compact (and thus of finite type) because the family of (semi-)stable vector bundles of given type $t$ is bounded.

Proposition A.6. Assume $g \geq 2$. Let $\text{Stab}_t \rightarrow \text{Bun}_t$ be the coarse moduli scheme of stable vector bundles of type $t$, and let $V$ be a vector bundle of some weight $w \in \mathbb{Z}$ over an open substack $U \subseteq \text{Stab}_t$.

i) $U$ descends to an open subscheme $U \subseteq \text{Bun}_t$.

ii) $\text{Gr}_j(V)$ descends to a (twisted) Grassmannian scheme $\text{Gr}_j(V)$ over $U$.

iii) If $V$ has weight $w = 0$, then it descends to a vector bundle over $U$.

iv) More generally, any vector bundle of weight 0 over $\text{Gr}_j(V)$ descends to a vector bundle over $\text{Gr}_j(V)$.

v) Any birationally linear map of stacks $\text{Stab}_t \rightarrow \text{Bun}_t$ induces a birationally linear map of schemes $\text{Bun}_t \rightarrow \text{Bun}_t$.

Proof. We continue to use the notation introduced in the proof of proposition A.1. By boundedness, there is an integer $n$ such that $\text{Stab}_t$ is contained in $\text{Coh}^n_t$; hence $\text{Stab}_t = \text{Quot}^{\text{stab}}_t/\text{GL}(N)$ where $\text{Quot}^{\text{stab}}_t \subseteq \text{Quot}^n_t$ is the stable locus. Here the center of $\text{GL}(N)$ acts trivially: by Geometric Invariant Theory [6], $\text{Quot}^{\text{stab}}_t$ is a principal $\text{PGL}(N)$-bundle over $\text{Bun}_t$.

i) Let $U \subseteq \text{Quot}^{\text{stab}}_t$ be the inverse image of $U$. Then $U$ is a $\text{PGL}(N)$-stable open subscheme in the total space of this principal bundle and hence the inverse image of a unique open subscheme $U \subseteq \text{Bun}_t$.

ii) Let $V$ be the pullback of $V$ to $U$; it is a vector bundle with $\text{GL}(N)$-action. Hence its Grassmannian scheme $\text{Gr}_j(V) \rightarrow U$ also carries an action of $\text{GL}(N)$. But here the center acts trivially: $\lambda \cdot \text{id} \in \text{GL}(N)$ acts as the scalar $\lambda^w$ on the fibres of $V$ and hence acts trivially on $\text{Gr}_j(V)$. Thus we obtain an action of $\text{PGL}(N)$ on our Grassmannian scheme $\text{Gr}_j(V)$ over $U$. Since this action is free, $\text{Gr}_j(V)$ descends to a Grassmannian bundle $\text{Gr}_j(V)$ over $U$ (which may be twisted, i.e. not Zariski-locally trivial).

iii) The assumption $w = 0$ means that the scalars in $\text{GL}(N)$ act trivially on the fibres of $V$. Hence $\text{PGL}(N)$ acts on $V$ over $U$ here. Again since this action is free, $V$ descends to a vector bundle over $U$.

iv) Here weight 0 means that the scalars in $\text{GL}(N)$ act trivially on the pullback of the given vector bundle to $\text{Gr}_j(V)$. Hence $\text{PGL}(N)$ acts on this
Conversely, two extensions $0 \to \phi : \mathcal{U} \to \mathcal{V}$ between dense open substacks $\mathcal{U} \subseteq \text{Stab}_\nu \times \mathbb{A}^2$ and $\mathcal{U} \subseteq \text{Stab}_\nu$. By i, $\mathcal{U}$ corresponds to an open scheme $\mathcal{U} \subseteq \text{Bun}_\nu$; the proof of i shows that $\mathcal{U}$ is a coarse moduli scheme for the stack $\mathcal{U}$. A straightforward generalization of this argument shows that $\mathcal{U}'$ corresponds to an open subscheme $\mathcal{U}' \subseteq \text{Bun}_\nu \times \mathbb{A}^2$ and that $\mathcal{U}'$ is again a coarse moduli scheme for $\mathcal{U}'$. By the universal property of coarse moduli schemes, $\phi$ induces an isomorphism $\mathcal{U}' \to \mathcal{U}$ and thus the required birationally linear map of schemes.

**Theorem A.7** (Hirschowitz, Russo-Teixidor). Assume $g \geq 2$. Let $F_1$ and $F_2$ be a general pair of vector bundles on $C$ with given types $t_1 = (r_1,d_1)$ and $t_2 = (r_2,d_2)$.

1. If $\chi(t_1,t_2) \geq 0$, then $\dim \text{Hom}(F_1,F_2) = \chi(t_1,t_2)$ and $\text{Ext}^1(F_1,F_2) = 0$.
2. If $\chi(t_1,t_2) \geq 1$ and $r_1 > r_2$ (resp. $r_1 = r_2$, resp. $r_1 < r_2$), then a general morphism $F_1 \to F_2$ is surjective (resp. injective, resp. injective with torsionfree cokernel).

**Proof.** The cases $r_1 = 0$ and $r_2 = 0$ are trivial, so we may assume $r_1, r_2 \geq 1$; then $\text{Stab}_{t_1} \neq \emptyset \neq \text{Stab}_{t_2}$. By semicontinuity, there is a dense open substack $\mathcal{U} \subseteq \text{Stab}_{t_1} \times \text{Stab}_{t_2}$ of stable vector bundles $F_1, F_2$ with $\dim \text{Hom}(F_1,F_2)$ minimal, say equal to $m$. According to Riemann-Roch, $m \geq \chi(t_1,t_2)$; part i of the theorem precisely claims that we have equality here.

Let $\text{Hom}(\mathcal{F}_1^{\text{univ}},\mathcal{F}_2^{\text{univ}})$ be the vector bundle of rank $m$ over $\mathcal{U}$ whose fibre over $F_1, F_2$ is $\text{Hom}(F_1,F_2)$. By generic flatness (cf. EGA IV, §6.9), there is a dense open substack $\mathcal{V}$ in the total space of $\text{Hom}(\mathcal{F}_1^{\text{univ}},\mathcal{F}_2^{\text{univ}})$ such that the cokernel of the universal family of morphisms $F_1 \to F_2$ is flat over $\mathcal{V}$. Then its image and kernel are also flat over $\mathcal{V}$; we denote the types of cokernel, image and kernel by $t_Q = (r_Q,d_Q)$, $t = (r,d)$ and $t_K = (r_K,d_K)$.

If $r = 0$, then the theorem clearly holds: In this case, the general morphism $\phi : F_1 \to F_2$ has generic rank zero, so $\phi = 0$: this means $m = 0$. Together with $m \geq \chi(t_1,t_2)$, this proves i and shows that the hypothesis of ii cannot hold here. Henceforth, we may thus assume $r \neq 0$.

Note that $t_1 = t_K + t$ and $t_2 = t + t_Q$; moreover, we have a canonical morphism of moduli stacks

$$\Phi : \mathcal{V} \to \mathcal{E}xt(t,t_K) \times \text{Coh}, \mathcal{E}xt(t_Q,t)$$

that sends a morphism $\phi : F_1 \to F_2$ to the extensions

$$0 \to \ker(\phi) \to F_1 \to \text{im}(\phi) \to 0 \quad \text{and} \quad 0 \to \text{im}(\phi) \to F_2 \to \text{coker}(\phi) \to 0.$$ 

Conversely, two extensions $0 \to K \to F_1 \to I \to 0$ and $0 \to J \to F_2 \to Q \to 0$ together with an isomorphism $I \to J$ determine a morphism $\phi : F_1 \to F_2$. Thus $\Phi$ is an isomorphism onto the open locus in $\mathcal{E}xt \times \text{Coh} \mathcal{E}xt$ where both extension
sheaves $F_1, F_2$ are stable vector bundles and $\dim \text{Hom}(F_1, F_2) = m$. Hence the stack dimensions coincide, i.e.

$$m - \chi(t_1, t_1) - \chi(t_2, t_2) = -\chi(t_1, t_K) - \chi(t, t) - \chi(t_Q, t_2).$$

Since $\chi$ is biadditive, this is equivalent to

$$m - \chi(t_1, t_2) = -\chi(t_K, t_Q). \quad (8)$$

In particular, $\chi(t_K, t_Q) \leq 0$ follows.

Now suppose that $t_K$ and $t_Q$ were both nonzero. Since the general vector bundles $F_1$ and $F_2$ are stable, we then have

$$\frac{d_K}{r_K} < \frac{d_1}{r_1} < \frac{d}{r} < \frac{d_2}{r_2} < \frac{d_Q}{r_Q}.$$  

Using the assumption $\chi(t_1, t_2) \geq 0$, we get

$$\frac{\chi(t_K, t_Q)}{r_Kr_Q} = 1 - g - \frac{d_K}{r_K} + \frac{d_Q}{r_Q} > 1 - g - \frac{d_1}{r_1} + \frac{d_2}{r_2} = \frac{\chi(t_1, t_2)}{r_1r_2} \geq 0$$

and hence $\chi(t_K, t_Q) > 0$. This contradiction proves $t_K = 0$ or $t_Q = 0$.

(In some sense, this argument also covers the cases $r_K = 0$ and $r_Q = 0$. More precisely, $r_K = 0$ implies $t_K = 0$ because every rank zero coherent subsheaf of a vector bundle $F_1$ is trivial. On the other hand, $r_Q \neq 0$ and $t_Q = (0, d_Q) \neq 0$ would imply $\chi(t_K, t_Q) = r_Kd_Q > 0$ which is again a contradiction.)

In particular, we get $\chi(t_K, t_Q) = 0$; together with equation (8), this proves part i of the theorem.

If $r_1 > r_2$ (resp. $r_1 \leq r_2$), then $r_K > r_Q$ (resp. $r_K \leq r_Q$) and hence $r_K \neq 0 = r_Q$ (resp. $r_K = 0$); we have just seen that this implies $t_Q = 0$ (resp. $t_K = 0$), i.e. the general morphism $\varphi : F_1 \to F_2$ is surjective (resp. injective).

Furthermore, the morphism of stacks $V \to \text{Coh}_{t_Q}$ that sends a morphism $\varphi : F_1 \to F_2$ to its cokernel is smooth (due to the open embedding $\Phi$ and proposition A.3). If $r_1 < r_2$, then $r_Q \geq 1$, so $\text{Bun}_{t_Q}$ is open and dense in $\text{Coh}_{t_Q}$; this implies that the inverse image of $\text{Bun}_{t_Q}$ is open and dense in $V$, i.e. the general morphism $\varphi : F_1 \to F_2$ has torsionfree cokernel.

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