A classification theorem and a spectral sequence for a locally free sheaf cohomology of a supermanifold

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Abstract. This paper is based on the paper [3], where two classification theorems for locally free sheaves on supermanifolds were proved and a spectral sequence for a locally free sheaf of modules $\mathcal{E}$ was obtained. We consider another filtration of the locally free sheaf $\mathcal{E}$, the corresponding classification theorem and the spectral sequence, which is more convenient in some cases. The methods, which we are using here, are similar to [2, 3].

The first spectral sequence of this kind was constructed by A.L. Onishchik in [2] for the tangent sheaf of a supermanifold. However, the spectral sequence considered in this paper is not a generalization of Onishchik’s spectral sequence from [2].

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1. Main definitions and classification theorems

1.1. Main definitions

Let $(M, \mathcal{O})$ be a supermanifold of dimension $n|m$, i.e. a $\mathbb{Z}_2$-graded ringed space that is locally isomorphic to a superdomain in $\mathbb{C}^n|m$. The underlying complex manifold $(M, \mathcal{F})$ is called the reduction of $(M, \mathcal{O})$. The simplest class of supermanifolds constitute the so-called split supermanifolds. We recall that a supermanifold $(M, \mathcal{O})$ is called split if $\mathcal{O} \simeq \bigwedge \mathcal{G}$, where $\mathcal{G}$ is a locally free sheaf of $\mathcal{F}$-modules on $M$. With any supermanifold $(M, \mathcal{O})$ one can associate a split supermanifold $(M, \hat{\mathcal{O}})$ of the same dimension which is called

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the retract of \((M, \mathcal{O})\). To construct it, let us consider the \(\mathbb{Z}_2\)-graded sheaf of ideals \(J = J_0 \oplus J_1 \subset \mathcal{O}\) generated by odd elements of \(\mathcal{O}\). The structure sheaf of the retract is defined by
\[
\tilde{\mathcal{O}} = \bigoplus_{p \geq 0} \tilde{\mathcal{O}}_p, \text{ where } \tilde{\mathcal{O}}_p = \mathcal{J}^p / \mathcal{J}^{p+1}, \mathcal{J}^0 := \mathcal{O}.
\]
Here \(\tilde{\mathcal{O}}_1\) is a locally free sheaf of \(\mathcal{F}\)-modules on \(M\) and \(\tilde{\mathcal{O}}_p = \Lambda^p \tilde{\mathcal{O}}_1\). By definition, the following sequences
\[
0 \to \mathcal{J} \cap \mathcal{O}_0 \to \mathcal{O}_0 \xrightarrow{\pi} \tilde{\mathcal{O}}_0 \to 0,
0 \to \mathcal{J}^2 \cap \mathcal{O}_1 \to \mathcal{O}_1 \xrightarrow{\tau} \tilde{\mathcal{O}}_1 \to 0.
\]
are exact. Moreover, they are locally split. The supermanifold \((M, \mathcal{O})\) is split if both sequences are globally split.

Denote by \(S_0\) and \(S_1\) the even and the odd parts of a \(\mathbb{Z}_2\)-graded sheaf of \(\mathcal{O}\)-modules \(S\) on \(M\), respectively; by \(\Pi(S)\) we denote the same sheaf of \(\mathcal{O}\)-modules \(S\) equipped with the following \(\mathbb{Z}_2\)-grading: \(\Pi(S)_0 = S_1, \Pi(S)_1 = S_0\). A \(\mathbb{Z}_2\)-graded sheaf of \(\mathcal{O}\)-modules on \(M\) is called free (locally free) of rank \(p q, p, q \geq 0\) if it is isomorphic (respectively, locally isomorphic) to the \(\mathbb{Z}_2\)-graded sheaf of \(\mathcal{O}\)-modules \(\mathcal{O}^p \oplus \Pi(\mathcal{O})^q\). For example, the tangent sheaf \(\mathcal{T}\) of a supermanifold \((M, \mathcal{O})\) is a locally free sheaf of \(\mathcal{O}\)-modules.

Let now \(\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1\) be a locally free sheaf of \(\mathcal{O}\)-modules of rang \(p q\) on an arbitrary supermanifold \((M, \mathcal{O})\). We are going to construct a locally free sheaf of the same rank on \((M, \mathcal{O})\). First, we note that \(\mathcal{E}_{\text{red}} := \mathcal{E} / \mathcal{J} \mathcal{E}\) is a locally free sheaf of \(\mathcal{F}\)-modules on \(M\). Moreover, \(\mathcal{E}_{\text{red}}\) admits the \(\mathbb{Z}_2\)-grading \(\mathcal{E}_{\text{red}} = (\mathcal{E}_{\text{red}})_0 \oplus (\mathcal{E}_{\text{red}})_1\), by two locally free sheaves of \(\mathcal{F}\)-modules
\[
(\mathcal{E}_{\text{red}})_0 := \mathcal{E}_0 / \mathcal{J} \mathcal{E} \cap \mathcal{E}_0 \text{ and } (\mathcal{E}_{\text{red}})_1 := \mathcal{E}_1 / \mathcal{J} \mathcal{E} \cap \mathcal{E}_1
\]
of ranks \(p\) and \(q\), respectively. Further, the sheaf \(\mathcal{E}\) possesses the filtration
\[
\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \mathcal{E}_2 \supset \ldots, \text{ where } \mathcal{E}_p = \mathcal{J}^p \mathcal{E}_0 + \mathcal{J}^{p-1} \mathcal{E}_1, p \geq 1. \tag{1.2}
\]
Using this filtration, we can construct the following locally free sheaf of \(\tilde{\mathcal{O}}\)-modules on \(M\):
\[
\tilde{\mathcal{E}} = \bigoplus_p \tilde{\mathcal{E}}_p, \text{ where } \tilde{\mathcal{E}}_p = \mathcal{E}_p / \mathcal{E}_{(p+1)}.
\]
The sheaf \(\tilde{\mathcal{E}}\) is also a locally free sheaf of \(\mathcal{F}\)-modules. In other words, \(\tilde{\mathcal{E}}\) is a sheaf of sections of a certain vector bundle. The following exact sequence gives a description of \(\tilde{\mathcal{E}}\).
\[
0 \to \tilde{\mathcal{O}}_p \otimes (\mathcal{E}_{\text{red}})_0 \to \tilde{\mathcal{E}}_p \to \tilde{\mathcal{O}}_{p-1} \otimes (\mathcal{E}_{\text{red}})_1 \to 0.
\]
We also have the following two exact sequences, which are locally split:
\[
0 \to \mathcal{E}_1^0 \to \mathcal{E}_0^0 \xrightarrow{\alpha} \tilde{\mathcal{E}}_0 \to 0; 
0 \to \mathcal{E}_1^1 \to \mathcal{E}_1^1 \xrightarrow{\beta} \tilde{\mathcal{E}}_1 \to 0. \tag{1.3}
\]
The total space of the bundle corresponding to a locally free sheaf $E$ is the orbit of the unit element $\epsilon$ denoted $H$. The retract $\tilde{H}$ has the following property: The retract $\tilde{H}$ leaves invariant the subsheaves $A$. Let $\Phi : O \to O'$ be a locally free sheaf of $O$-modules on $M$, respectively. Suppose that $\Psi : O \to O'$ is a superalgebra sheaf morphism. A vector space sheaf morphism $\Phi : E \to E'$ is called a quasi-morphism if

$$\Phi(fv) = \Psi(f)\Phi(v), \quad f \in O, \quad v \in E.$$ 

As usual, we assume that $\Phi(\epsilon_i) \subset \epsilon_i^i$, $i \in \{0, 1\}$. An invertible quasi-morphism is called a quasi-isomorphism. A quasi-isomorphism $\Phi : E \to E'$ is also called a quasi-automorphism of $E$. Denote by $\text{Aut}E$ the sheaf of quasi-automorphisms of $E$. It has a double filtration by the subsheaves

$$\text{Aut}_{(p)(q)}E := \{\Phi \in \text{Aut}E \mid \Phi(v) \equiv v \mod E(p), \quad \Psi(f) = f \mod J^q \quad \text{for } v \in E, \quad f \in O\}, \quad p, q \geq 0.$$ 

We also define the following subsheaf of $\text{Aut}\tilde{E}$:

$$\tilde{\text{Aut}}\tilde{E} := \{\Phi \mid \Phi \in \text{Aut}(\tilde{E}), \quad \Phi \text{ preserves the } \mathbb{Z}\text{-grading of } \tilde{E}\}. \quad (1.4)$$

If $\Phi \in \tilde{\text{Aut}}\tilde{E}$, then $\Psi : \tilde{O} \to \tilde{O}$ also preserves the $\mathbb{Z}$-grading. The 0-th cohomology group $H^0(M, \tilde{\text{Aut}}\tilde{E})$ acts on the sheaf $\text{Aut}\tilde{E}$ by the automorphisms $\delta \mapsto a \circ \delta \circ a^{-1}$, where $a \in H^0(M, \tilde{\text{Aut}}\tilde{E})$ and $\delta \in \text{Aut}\tilde{E}$. It is easy to see that this action leaves invariant the subsheaves $\text{Aut}_{(p)(q)}\tilde{E}$ and hence induces an action of $H^0(M, \tilde{\text{Aut}}\tilde{E})$ on the cohomology set $H^1(M, \text{Aut}_{(p)(q)}\tilde{E})$. The unit element $\epsilon \in H^1(M, \text{Aut}_{(p)(q)}\tilde{E})$ is a fixed point with respect to the action of $H^0(M, \text{Aut}\tilde{E}')$.

Let $E$ be a locally free sheaf of $O$-modules on $M$. Denote

$$[E] = \{E' \mid E' \text{ is quasi-isomorphic to } E\}.$$ 

The total space of the bundle corresponding to a locally free sheaf $E$ will be denoted $\tilde{E}$. It is a supermanifold. The locally free sheaf $\tilde{E}$ corresponding to $E$ has the following property: The retract $\tilde{E}$ of $E$ is the total space of the bundle corresponding to $\tilde{E}$.

**Theorem 1.1.** Let $(M, O')$ be a split supermanifold and $E'$ be a locally free sheaf of $O'$-modules on $M$ such that $E' \simeq \tilde{E}'$. Then

$$\{[E] \mid \tilde{O} = O', \quad \tilde{E} = E'\} \xleftarrow{\text{1-1}} H^1(M, \text{Aut}_{(2)(2)}E')/H^0(M, \tilde{\text{Aut}}E').$$

The orbit of the unit element $\epsilon$, which is $\epsilon$ itself, corresponds to $E'$. 

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Proof. Let \( \mathcal{E} \) be a locally free sheaf of \( \mathcal{O} \)-modules on \((M, \mathcal{O})\) and \( U = \{ U_i \} \) be an open covering of \( M \) such that \([1.1]\) and \([1.3]\) are split (hence exact) over \( U_i \) and \( \mathcal{E}|U_i \) are free. In this case, \( \mathcal{E}|U_i \) are free sheaves of \( \mathcal{O} \)-modules. We fix homogeneous bases (even and odd, respectively) \( (\tilde{e}^i_j) \) and \( (\tilde{f}^i_j) \) of the free sheaves of \( \mathcal{O} \)-modules \( \tilde{\mathcal{E}}|U_i, U_i \in \mathcal{U} \). Without loss of generality, we may assume that \( \tilde{e}^i_j \in \tilde{E}_0 \) and \( \tilde{f}^i_j \in \tilde{E}_1 \). We are going to define an isomorphism \( \delta_i : \mathcal{E}|U_i \rightarrow \tilde{\mathcal{E}}|U_i \).

Let \( e^i_j \in \mathcal{E}_{(0)\overline{0}} \) be such that \( \alpha(e^i_j) = \tilde{e}^i_j \) and \( f^i_j \in \mathcal{E}_{(0)\overline{1}} \) be such that \( \beta(f^i_j) = \tilde{f}^i_j \), see \([1.3]\). Then \( (e^i_j, f^i_j) \) is a local basis of \( \mathcal{E}|U_i \). A splitting of \([1.1]\) determines a local isomorphism \( \sigma_i : \mathcal{O}|U_i \rightarrow \mathcal{O}|U_i, \) see \([\|]\). We put

\[
\delta_i(\sum h_j e^i_j + \sum g_j f^i_j) = \sum \sigma_i(h_j) \tilde{e}^i_j + \sum \sigma_i(g_j) \tilde{f}^i_j, h_j, g_j \in \mathcal{O}.
\]

Obviously, \( \delta_i \) is an isomorphism. We put \( \gamma_{ij} := \sigma_i \circ \sigma_j^{-1} \) and \( (g_{ij}) \gamma_{ij} := \delta_i \circ \delta_j^{-1} \). Moreover, \((\gamma_{ij}) \in \mathcal{Z}^1(\mathcal{U}, \text{Aut}_{(2)} \mathcal{O}), \) see \([\|]\) for more details. We want to show that

\[
((g_{ij}) \gamma_{ij}) \in \mathcal{Z}^1(\mathcal{U}, \text{Aut}_{(2)(2)} \tilde{\mathcal{E}}).
\]

Let us take \( v \in \tilde{\mathcal{E}}|U_j, v = \sum h_k \tilde{e}^i_k + \sum g_k \tilde{f}^i_k, h_k, g_k \in \mathcal{O}. \) Then by definition we have

\[
\delta_j^{-1}(v) = \sum \sigma_j^{-1}(h_k) \tilde{e}^i_k + \sum \sigma_j^{-1}(g_k) \tilde{f}^i_k.
\]

The transition functions of \( \tilde{\mathcal{E}} \) may be expressed in \( U_i \cap U_j \) as follows:

\[
e^i_k = \sum a^k_s e^i_s \quad f^i_k = \sum c^k_s e^i_s + \sum d^k_s f^i_s, \quad a^k_s, d^k_s \in \mathcal{O}_0, \quad b^k_s, c^k_s \in \mathcal{O}_1.
\]

Further,

\[
\alpha(e^i_k) = \tilde{e}^i_k, \quad \beta(f^i_k) = \tilde{f}^i_k, \quad \gamma_{ij}(a^k_s) = \pi(a^k_s), \quad \gamma_{ij}(c^k_s) = \tau(c^k_s).
\]

We have

\[
\delta_j \circ \delta_j^{-1}(v) = \sum_k \gamma_{ij}(h_k)(\sum_s \sigma_i(a^k_s) \tilde{e}^i_s + \sum_s \sigma_i(b^k_s) \tilde{f}^i_s) + \sum_k \gamma_{ij}(g_k)(\sum_s \sigma_i(c^k_s) \tilde{e}^i_s + \sum_s \sigma_i(d^k_s) \tilde{f}^i_s) = \sum_k \delta_{ij}(g_k)(\sum_s \sigma_i(c^k_s) \tilde{e}^i_s + \sum_s \sigma_i(d^k_s) \tilde{f}^i_s) \mod \tilde{\mathcal{E}}(2) = v \mod \tilde{\mathcal{E}}(2) = \tilde{v}.
\]

The rest of the proof is a direct repetition of the proof of Theorem 2 from \([3]\). \( \square \)

2. The spectral sequence

2.1. Quasi-derivations

Quasi-derivations were defined in \([3]\). Let us briefly recall that construction. Consider a locally free sheaf \( \mathcal{E} \) on a supermanifold \((M, \mathcal{O})\). An even vector space sheaf morphism \( A_r : \mathcal{E} \rightarrow \mathcal{E} \) is called a quasi-derivation if \( A_r(fv) = \Gamma(f)v + f A_1(v) \), where \( f \in \mathcal{O}, v \in \mathcal{E} \) and \( \Gamma \) is a certain even super vector field. Denote by \( \text{Der} \mathcal{E} \) the sheaf of quasi-derivations. It is a sheaf of Lie algebras
with respect to the commutator \([A_{\Gamma}, B_{\Gamma}] := A_{\Gamma} \circ B_{\Gamma} - B_{\Gamma} \circ A_{\Gamma}\). The sheaf \(\text{Der} \mathcal{E}\) possesses a double filtration

\[
\begin{align*}
\text{Der}_{(0)(0)} \mathcal{E} & \supset \text{Der}_{(2)(0)} \mathcal{E} \supset \cdots \\
\text{Der}_{(0)(2)} \mathcal{E} & \supset \text{Der}_{(2)(2)} \mathcal{E} \supset \cdots ,
\end{align*}
\]

where

\[
\text{Der}_{(p)(q)} \mathcal{E} := \{ A_{\Gamma} \in \text{Der} \mathcal{E} \mid A_{\Gamma}(\mathcal{E}_{(r)}) \subset \mathcal{E}_{(r+p)}, \; \Gamma(\mathcal{J}^s) \subset \mathcal{J}^{s+q}, \; r, s \in \mathbb{Z}\},
\]

where \(p, q \geq 0\). The map defined by the usual exponential series

\[
\exp : \text{Der}_{(p)(q)} \mathcal{E} \to \text{Aut}_{(p)(q)} \mathcal{E}, \; p, q \geq 2,
\]

is an isomorphism of sheaves of sets, because operators from \(\text{Der}_{(p)(q)} \mathcal{E}, \; p, q \geq 2\), are nilpotent. The inverse map is given by the logarithmic series. Define the vector space subsheaf \(\text{Der}_{k,k} \tilde{\mathcal{E}}\) of \(\text{Der}_{(k)(k)} \tilde{\mathcal{E}}\) for \(k \geq 0\) by

\[
\text{Der}_{k,k} \tilde{\mathcal{E}} := \{ A_{\Gamma} \in \text{Der}_{(k)(k)} \tilde{\mathcal{E}} \mid A_{\Gamma}(\tilde{\mathcal{E}}_r) \subset \tilde{\mathcal{E}}_{r+k}, \; \Gamma(\tilde{\mathcal{O}}_s) \subset \tilde{\mathcal{O}}_{s+k}, \; r, s \in \mathbb{Z}\}.
\]

For an even \(k \geq 2\), define a map

\[
\mu_k : \text{Aut}_{(k)(2)} \tilde{\mathcal{E}} \to \text{Der}_{k,k} \tilde{\mathcal{E}}, \quad \mu_k(a_\gamma) = \bigoplus_q \text{pr}_{q+k} \circ A_{\Gamma} \circ \text{pr}_q,
\]

where \(a_\gamma = \exp(A_{\Gamma})\) and \(\text{pr}_k : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}_k\) is the natural projection. The kernel of this map is \(\text{Aut}_{(k+2)(2)} \tilde{\mathcal{E}}\). Moreover, the sequence

\[
0 \to \text{Aut}_{(k+2)(2)} \tilde{\mathcal{E}} \longrightarrow \text{Aut}_{(k)(2)} \tilde{\mathcal{E}} \xrightarrow{\mu_k} \text{Der}_{k,k} \tilde{\mathcal{E}} \to 0,
\]

where \(k \geq 2\) is even, is exact. Denoting by \(H_{(k)}(\tilde{\mathcal{E}})\) the image of the natural mapping \(H^1(M, \text{Aut}_{(k)(2)} \tilde{\mathcal{E}}) \to H^1(M, \text{Aut}_{(2)(2)} \tilde{\mathcal{E}})\), we get the filtration

\[
H^1(M, \text{Aut}_{(2)(2)} \tilde{\mathcal{E}}) = H_{(2)}(\tilde{\mathcal{E}}) \supset H_{(4)}(\tilde{\mathcal{E}}) \supset \ldots .
\]

Take \(a_\gamma \in H_{(2)}(\tilde{\mathcal{E}})\). We define the order of \(a_\gamma\) to be the maximal number \(k\) such that \(a_\gamma \in H_{(k)}(\tilde{\mathcal{E}})\). The order of a locally free sheaf \(\mathcal{E}\) of \(\mathcal{O}\)-modules on a supermanifold \((M, \mathcal{O}_M)\) is by definition the order of the corresponding cohomology class.

### 2.2. The spectral sequence.

A spectral sequence connecting the cohomology with values in the tangent sheaf \(\mathcal{T}\) of a supermanifold \((M, \mathcal{O})\) with the cohomology with values in the tangent sheaf \(\mathcal{T}_{gr}\) of the retract \((M, \mathcal{O})\) was constructed in \([2]\). Here we use similar ideas to construct a new spectral sequence connecting the cohomology with values in a locally free sheaf \(\mathcal{E}\) on a supermanifold \((M, \mathcal{O})\) with the cohomology with values in the locally free sheaf \(\tilde{\mathcal{E}}\) on \((M, \mathcal{O})\). Note that our spectral sequence is not a generalization of the spectral sequence obtained in \([2]\) because \(\mathcal{T}_{gr}\) is not in general isomorphic to \(\mathcal{T}\).

Let \(\mathcal{E}\) be a locally free sheaf on a supermanifold \((M, \mathcal{O})\) of dimension \(n|m\). We fix an open Stein covering \(\mathcal{U} = (U_i)_{i \in I}\) of \(M\) and consider the
corresponding Čech cochain complex \( C^*(\mathcal{U}, \mathcal{E}) = \bigoplus_{p \geq 0} C^p(\mathcal{U}, \mathcal{E}) \). The \( \mathbb{Z}_2 \)-grading of \( \mathcal{E} \) gives rise to the \( \mathbb{Z}_2 \)-gradings in \( C^*(\mathcal{U}, \mathcal{E}) \) and \( H^*(M, \mathcal{E}) \) given by

\[
C_0(\mathcal{U}, \mathcal{E}) = \bigoplus_{q \geq 0} C^{2q}(\mathcal{U}, \mathcal{E}_0) \oplus \bigoplus_{q \geq 0} C^{2q+1}(\mathcal{U}, \mathcal{E}_1),
\]

\[
C_1(\mathcal{U}, \mathcal{E}) = \bigoplus_{q \geq 0} C^{2q}(\mathcal{U}, \mathcal{E}_1) \oplus \bigoplus_{q \geq 0} C^{2q+1}(\mathcal{U}, \mathcal{E}_0).
\]

\[
H_0(M, \mathcal{E}) = \bigoplus_{q \geq 0} H^{2q}(M, \mathcal{E}_0) \oplus \bigoplus_{q \geq 0} H^{2q+1}(M, \mathcal{E}_1),
\]

\[
H_1(M, \mathcal{E}) = \bigoplus_{q \geq 0} H^{2q}(M, \mathcal{E}_1) \oplus \bigoplus_{q \geq 0} H^{2q+1}(M, \mathcal{E}_0).
\]

(2.1)

The filtration \((\mathcal{E})\) for \( \mathcal{E} \) gives rise to the filtration

\[
C^*(\mathcal{U}, \mathcal{E}) = C_{(0)} \supset \ldots \supset C_{(p)} \supset \ldots \supset C_{(m+2)} = 0
\]

(2.2)
of this complex by the subcomplexes

\[
C_{(p)} = C^*(\mathcal{U}, \mathcal{E}_{(p)}).
\]

Denoting by \( H(M, \mathcal{E})_{(p)} \) the image of the natural mapping \( H^*(M, \mathcal{E}_{(p)}) \rightarrow H^*(M, \mathcal{E}) \), we get the filtration

\[
H^*(M, \mathcal{E}) = H(M, \mathcal{E})_{(0)} \supset \ldots \supset H(M, \mathcal{E})_{(p)} \supset \ldots
\]

(2.3)

Denote by \( \text{gr} \; H^*(M, \mathcal{E}) \) the bigraded group associated with the filtration \((2.3)\); its bigrading is given by

\[
\text{gr} \; H^*(M, \mathcal{E}) = \bigoplus_{p,q \geq 0} \text{gr}_p H^q(M, \mathcal{E}).
\]

By the (more general) Leray procedure, we get a spectral sequence of bigraded groups \( E_r \) converging to \( E_\infty \simeq \text{gr} \; H^*(M, \mathcal{E}) \). For convenience of the reader, we recall the main definitions here.

For any \( p, r \geq 0 \), define the vector spaces

\[
C_r^p = \{ c \in C_{(p)} \mid dc \in C_{(p+r)} \}.
\]

Then, for a fixed \( p \), we have

\[
C_{(p)} = C_0^p \supset \ldots \supset C_r^p \supset C_{r+1}^p \supset \ldots
\]

The \( r \)-th term of the spectral sequence is defined by

\[
E_r = \bigoplus_{p=0}^{m} E_r^p, \ r \geq 0, \text{ where } E_r^p = C_r^p/C_r^{p+1} + dC_{r-1}^{p-r+1}.
\]

Since \( d(C_r^p) \subset C_r^{p+r} \), \( d \) induces a derivation \( d_r \) of \( E_r \) of degree \( r \) such that \( d_r^2 = 0 \). Then \( E_{r+1} \) is naturally isomorphic to the homology algebra \( H(E_r, d_r) \). The \( \mathbb{Z}_2 \)-grading \((2.1)\) in \( C^*(\mathcal{U}, \mathcal{E}) \) gives rise to certain \( \mathbb{Z}_2 \)-gradings in \( C_r^p \) and \( E_r^p \), turning \( E_r \) into a superspace. Clearly, the coboundary operator \( d \) on \( C^*(\mathcal{U}, \mathcal{E}) \) is odd. It follows that the coboundary \( d_r \) is odd for any \( r \geq 0 \).
The superspaces $E_r$ are also endowed with a second $\mathbb{Z}$-grading. Namely, for any $q \in \mathbb{Z}$, set
\[ C^{p,q}_r = C^r \cap C^{p+q}(\mathcal{U}, \mathcal{E}), \quad E^{p,q}_r = C^{p,q}_r / C^{p+1,q-1}_r + dC^{p-r+1,q+r-2}_r. \]
Then
\[ E_r = \bigoplus_{p,q} E^{p,q}_r \quad \text{and} \quad d_r(E^{p,q}_r) \subset E^{p+r,q+r-1}_r \quad \text{for any} \ r, p, q. \] (2.4)

Further, for a fixed $q$, we have $d(C^{p,q}_r) = 0$ for all $p \geq 0$ and all $r \geq m+2$. This implies that the natural homomorphism $E^{p,q}_r \to E^{p,q}_{r+1}$ is an isomorphism for all $p$ and $r \geq r_0 = m+2$. Setting $E^{p,q}_\infty = E^{p,q}_{r_0}$, we get the bigraded superspace
\[ E_\infty = \bigoplus_{p,q} E^{p,q}_\infty. \]

**Lemma 2.1.** The first two terms of the spectral sequence $(E_r)$ can be identified with the following bigraded spaces:
\[ E_0 = C^*(\mathcal{U}, \tilde{\mathcal{E}}), \quad E_1 = E_2 = H^*(M, \tilde{\mathcal{E}}). \]

More precisely,
\[ E_0^{p,q} = C^{p+q}(\mathcal{U}, \tilde{\mathcal{E}})_p, \quad E_1^{p,q} = E_2^{p,q} = H^{p+q}(M, \tilde{\mathcal{E}})_p. \]

We have $d_{2k+1} = 0$ and, hence, $E_{2k+1} = E_{2k+2}$ for all $k \geq 0$.

**Proof.** The proof is similar to the proof of Proposition 3 in [2]. \hfill \Box

**Lemma 2.2.** There is the following identification of bigraded algebras:
\[ E_\infty = \operatorname{gr} H^*(M, \mathcal{E}), \quad \text{where} \ E^{p,q}_\infty = \operatorname{gr}_p H^{p+q}(M, \mathcal{E}). \]

If $M$ is compact, then $\dim H^k(M, \mathcal{E}) = \sum_{p+q=k} \dim E^{p,q}_r$.

**Proof.** The proof is a direct repetition of the proof of Proposition 4 in [2]. \hfill \Box

Now we prove our main result concerning the first non-zero coboundary operators among $d_2$, $d_4$, ... Assume that the isomorphisms of sheaves $\delta_i : \mathcal{E}|U_i \to \tilde{\mathcal{E}}|U_i$ from Theorem [1.1] are defined for each $i \in I$. By Theorem [1.1] a locally free sheaf of $\mathcal{O}$-modules $\mathcal{E}$ on $M$ corresponds to the cohomology class $a_\gamma$ of the 1-cocycle $((a_\gamma)_{ij}) \in Z^1(\mathcal{U}, \operatorname{Aut}_{(2)}(\tilde{\mathcal{E}}))$, where $(a_\gamma)_{ij} = \delta_i \circ \delta_i^{-1}$. If the order of $(a_\gamma)_{ij}$ is equal to $k$, then we may choose $\delta_i$, $i \in I$, in such a way that $(a_\gamma)_{ij} \in Z^1(\mathcal{U}, \operatorname{Aut}_{(k)}(\tilde{\mathcal{E}}))$. We can write $a_\gamma = \exp A_\Gamma$, where $A_\Gamma \in C^1(\mathcal{U}, \operatorname{Der}_{(k)}(\tilde{\mathcal{E}}))$.

We will identify the superspaces $(E_0, d_0)$ and $(C^*(\mathcal{U}, \tilde{\mathcal{E}}), d)$ via the isomorphism of Lemma [2.1]. Clearly, $\delta_i : \mathcal{E}|_{U(p)}|U_i \to \tilde{\mathcal{E}}|_{U(p)}|U_i = \sum_r \mathcal{E}_r|U_i$ is an isomorphism of sheaves for all $i \in I$, $p \geq 0$. These local sheaf isomorphisms permit us to define an isomorphism of graded cochain groups
\[ \psi : C^*(\mathcal{U}, \mathcal{E}) \to C^*(\mathcal{U}, \tilde{\mathcal{E}}) \]
such that
\[ \psi : C^*(\mathcal{U}, \mathcal{E}_{(p)}) \to C^*(\mathcal{U}, (\tilde{\mathcal{E}})_{(p)}), \quad p \geq 0. \]
We put
\[ \psi(c)_{i_0 \ldots i_q} = \delta_{i_0} (c_{i_0 \ldots i_q}) \]
for any \((i_0, \ldots, i_q)\) such that \(U_{i_0} \cap \ldots \cap U_{i_q} \neq \emptyset\). Note that \(\psi\) is not an isomorphism of complexes. Nevertheless, we can explicitly express the coboundary \(d\) of the complex \(C^*(\Omega, E)\) by means of \(d_0\) and \(a_\gamma\).

The following theorem permits to calculate the spectral sequence \((E_r)\) whenever \(d_0\) and the cochain \(a_\gamma\) are known. It also describes certain coboundary operators \(d_r, r \geq 1\).

**Theorem 2.3.** For any \(c \in C^*(\Omega, \tilde{E}_q) = E^q_0\), we have
\[
(\psi(d\psi^{-1}(c)))_{i_0 \ldots i_{q+1}} = (d_0c)_{i_0 \ldots i_{q+1}} + ((a_\gamma)_{i_0i_1} - \text{id})(c_{i_1 \ldots i_{q+1}}).
\]

Suppose that the locally free sheaf of \(\mathcal{O}\)-modules \(E\) on \(M\) has order \(k\) and denote by \(a_\gamma\) the cohomology class corresponding to \(E\) by Theorem 1.1. Then \(d_r = 0\) for \(r = 1, \ldots, k - 1\), and \(d_k = \mu_k(a_\gamma)\).

**Proof.** The proof is similar to the proof of Theorem 7 in [3]. \(\square\)

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