MODULI SPACE OF SHEAVES AND CATEGORIZED COMMUTATOR OF FUNCTORS

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Abstract. Neguț constructed an action of the quantum toroidal algebra action on the K-theory of the smooth moduli space of stable sheaves over an algebraic surface, which generalized the action studied by Nakajima, Grojnowski and Baranovsky in cohomology. In this paper, we construct a weak categorification of Neguț's action. The new ingredient is two intersection-theoretic descriptions of the quadruple moduli space of stable sheaves.

The Earl of the West was imprisoned but made the Zhou Changes.

Sima Qian, The Letter to Ren An

1. Introduction

1.1. Description of our results. Let $S$ be a smooth projective algebraic surface over $\mathbb{C}$, with an ample line bundle $H$. Let $q := K_S$ be the canonical line of $S$. Fixing $(r,c_1) \in \mathbb{N}^> \times H^2(S, \mathbb{Z})$, let $\mathcal{M}$ be the moduli space of Gieseker $H$-stable sheaves $F$ such that $\text{rank}(F) = r, c_1(F) = c_1$.

Assuming the assumptions in Section 1.2, Neguț [12, 14] constructed a quantum toroidal algebra $U_{q_1,q_2}(\mathfrak{gl}_1)$ action on the Grothendieck group of $\mathcal{M}$. It generalized the construction of Schiffmann-Vasserot [19, 18] when $S = \mathbb{P}^2$ and $\mathcal{M}$ is the moduli space of framed sheaves and Feigin-Tsymbaliuk [5] when $S = \mathbb{A}^2$ and $\mathcal{M}$ is the Hilbert scheme of points.

The purpose of this paper is to construct a weak categorification of Neguț's action. Given any integer $k$, let $\mathcal{M}_k$ be the subspace of $\mathcal{M}$ which consists of Gieseker $H$-stable sheaves $F$ such that $c_2(F) = k$, with a universal sheaf $U_k$ over $\mathcal{M}_k \times S$. Let $\mathfrak{Z}_{k,k+1}$ be the closed subscheme of $\mathcal{M}_k \times \mathcal{M}_{k+1} \times S$ which consists of

\[
\{(F_{-1} \subset x) F_0 | F_0 \in \mathcal{M}_k, x \in S\},
\]

where $F_{-1} \subset x F_0$ means that $F_{-1} \subset F_0$ and $F_{-1} / F_0 \cong \mathbb{C}_x$. There is a tautological line bundle $\mathcal{L}$ on $\mathfrak{Z}_{k,k+1}$, such that for each closed point $(F_0 \subset x F_1)$, the fiber of $\mathcal{L}$ at this closed point is $F_1 / F_0$.

We define the Fourier-Mukai transform associated to a surface in Definition 2.5. Given an integer $i$, we consider the Fourier-Mukai kernels $e_i, f_i \in D^b(\mathcal{M} \times \mathcal{M} \times S)$ as the disjoint union of $\mathcal{L}^i \mathcal{O}_{3_{k,k+1}} \in D^b(\mathcal{M}_k \times \mathcal{M}_{k+1} \times S)$ and $\mathcal{L}^{i-k} \mathcal{O}_{3_{k,k+1}} \in D^b(\mathcal{M}_{k+1} \times \mathcal{M}_k \times S)$ for all $k$ respectively. Let $\Delta_S : \mathcal{M} \times S \to \mathcal{M} \times \mathcal{M} \times S \times S$ be the diagonal embedding and $\iota : \mathcal{M} \times \mathcal{M} \times S \times S \to \mathcal{M} \times \mathcal{M} \times S \times S$ be the involution morphism which maps $(x, z, s_1, s_2)$ to $(x, z, s_2, s_1)$.
Theorem 1.1 (See Theorem 4.1 for the details). We have
\[ f_{-i}e_i \cong R\pi_* e_{-i}f_i \bigoplus_{a=-r+1}^0 R\Delta S_*(q^{-a}\det(U_k)^{-1}\mathcal{O}_{M_k\times S})(1-2a-r). \]

When \( i + j \neq 0 \), there exists explicit morphisms \( e_i f_j \to R\pi_*(f_j e_i) \) when \( i + j < 0 \) and morphisms \( R\pi_*(f_j e_i) \to e_i f_j \) when \( i + j > 0 \) such that the cones are filtered by combinations of symmetric and wedge product of the universal sheaf \( U_k \) and its derived dual.

The relations of \( e_i \) actions were lifted to derived categories by Theorem 1.1 of [14]. Our paper lifts the commutators of \( e_i \) and \( f_j \) actions to the derived categories and thus could be regarded as a weak categorification of Neguț’s action.

The quantum toroidal algebra could also be presented by the elliptic Hall algebra by Schiffmann [17] and Burban-Schiffmann [1]. It contains more commutator relations, which we do not know how to categorify right now. See [14] for more discussions about it.

1.2. Technical Assumptions. We assume the following assumptions:

(1) For any short exact sequence which does not split:
\[ 0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{C}_x \to 0, \]
the sheaf \( \mathcal{F}' \in \mathcal{M}_k \) if and only if \( \mathcal{F} \in \mathcal{M}_{k+1} \).

(2) There exists a constant \( \text{const} \) which only depends on \( S, H, r, c_1 \), such that \( \mathcal{M}_k \) is a smooth projective scheme of dimension \( \text{const} + 2rk \).

In [12, 14], Neguț proved that the above assumptions are satisfied if \( \gcd(r, c_1 \cdot H) = 1 \), \( q \cong \mathcal{O}_S \) or \( c_1(q) \cdot H < 0 \).

1.3. Strategy of the proof. The geometry of nested moduli space of sheaves was studied by Neguț [12, 13, 15, 14]. We adapt this framework and consider the triple and quadruple moduli spaces \( Z^-, Z^+ \) and \( Z \) in Section 3.2. Different from the case \( r = 1 \) which had been constructed by the author [29], when \( r > 1 \) the moduli space \( Z^+ \) is neither equi-dimensional nor Cohen-Macaulay, and thus the dg/derived enhancement has to be considered. In this paper, we observe new descriptions of \( Z \) in Theorem 3.5 through blow-ups, which induce the vanishing property we need.

1.4. Related Work.

1.4.1. The categorified Hall algebra. The incarnation with derived algebraic geometry is inspired by the work of Porta-Sala [16], Diaconescu-Porta-Sala [3] and Toda [22, 23, 24, 25] on the categorified Hall algebra. Toda [25] proved a recent conjecture of Jiang [9], which obtained a semiquotient decomposition of the derived category of Grassmanians over a cohomological dimension 1 coherent sheaf. As an application, Koseki [11] obtained a categorical blow-up formula for Hilbert schemes of points on a smooth algebraic surface.

1.4.2. The categorification of the \( A_{q,t} \) algebra. The quantum toroidal algebra is a subalgebra of the \( A_{q,t} \)-algebra, which was invented by Carlsson-Mellit [2] to solve the shuffle conjecture. Gonzalez-Hogancamp [6] obtained a skein theoretic formulation of the polynomial representation of the \( A_{q,t} \) algebra at \( q = t^{-1} \) and obtained a categorification thereof using the derived trace of the Hecke category.
1.4.3. **The Loop Categorification of quantum loop \( sl_2 \).** Shan-Varagnolo-Vasserot [21] constructed an equivalence of graded Abelian categories from a category of representations of the quiver-Hecke algebra of type \( A_1^{(1)} \) to the category of equivariant perverse coherent sheaves on the nilpotent cone of type \( A \). It gives a representation-theoretic categorification of the preprojective \( K \)-theoretic Hall algebra considered by Schiffmann-Vasserot [20] when the quiver is type \( A_1 \). For affine type quivers, Varagnolo-Vasserot [28] proved that the \( K \)-theoretic Hall algebra of a preprojective algebra is isomorphic to the positive half of a quantum toroidal quantum group.

1.4.4. **The Derived Projectivizations of Complexes.** After we finished the first version of our paper, Qingyuan Jiang sent us his paper [10]. This paper constructed the projectivizations of complexes in the setting of the derived algebraic geometry, and proved the generalized Serre theorem. The construction of Hecke correspondences was also obtained in this paper in the setting of the derived algebraic geometry.

1.5. **The Organization of The Paper.** In Section 2, we introduce the background and notations in this paper, and review the content of derived algebraic geometry. In Section 3, we recall the nested, triple and quadruple moduli space of stable sheaves, and prove new descriptions of the quadruple moduli space of stable sheaves through blow-ups. In Section 4, we prove the main theorem.

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2. **Notations And Backgrounds**

2.1. **Projectivization of Two Term Complexes.** Given a coherent sheaf \( U \) on a scheme \( X \), let \( \mathbb{P}_X(U) \) be the Grothendieck’s projectivization of \( U \) on \( X \). It has the functorial property that for any \( X \)-scheme \( f : T \to X \), the set of \( X \)-scheme morphisms

\[
Hom_X(T, \mathbb{P}_X(U)) = \{ f^*U \to E | E \in Pic(T) \text{ and } f^*U \to E \text{ is surjective} \}.
\]

Let \( pr_U : \mathbb{P}_X(U) \to X \) be the projection morphism. The identity morphism of \( P_X(U) \) is characterized by a universal line bundle, which we denote as \( O_{\mathbb{P}_X(U)}(1) \), and a surjective morphism, which we denote as

\[
taut_U : pr_U^*U \to O_{\mathbb{P}_X(U)}(1).
\]

For a two term complex \( U := \{ W \xrightarrow{\phi} V \} \) of locally free sheaves over a scheme \( X \), we denote \( \mathbb{P}_X(U) := \mathbb{P}_X(\text{coker}(s)) \) and \( O_{\mathbb{P}_X(U)}(1) := O_{\mathbb{P}_X(\text{coker}(s))}(1) \). We denote the morphism of coherent sheaves on \( \mathbb{P}_X(V) \)

\[
taut_s : pr_V^*W \otimes O_{\mathbb{P}_X(V)}(-1) \to O_{\mathbb{P}_X(V)}
\]

as the composition of following morphisms

\[
pr_V^*W \otimes O_{\mathbb{P}_X(V)}(-1) \xrightarrow{\text{pr}_V^*s \otimes O_{\mathbb{P}_X(V)}(-1)} pr_V^*V \otimes O_{\mathbb{P}_X(V)}(-1) \xrightarrow{taut_V} O_{\mathbb{P}_X(V)}.
\]
A morphism of locally free sheaves \( f : V \to \mathcal{O}_X \) on a scheme \( X \) induces a global section \( f^\vee \in \Gamma(X, V^\vee) \) and we denote its zero locus as \( Z(f^\vee) \). We denote the Koszul complex \( \wedge^* f \) as the complex

\[
0 \to \wedge^{\text{rank}(V)} V \to \cdots \to V \overset{f}{\to} \mathcal{O}_X \to 0
\]

We say that \( f \) is regular, if \( \mathcal{O}_{Z(f^\vee)} \) is resolved by the Koszul complex \( \wedge^* f \). If \( X \) is smooth and \( \dim(Z(f^\vee)) = \dim(X) - \dim(V) \), then \( Z(f^\vee) \) is a complete intersection variety and thus \( f \) is regular. We say that \( \mathbb{P}_X(U) \) is regular for a two term complex \( U := \{ W \overset{f}{\to} V \} \), if \( \text{taut}_s \) is regular on \( \mathbb{P}_X(V) \).

**Lemma 2.1** (Proposition 9.7.9 of [4]). Given a two term complex \( U := \{ W \overset{f}{\to} V \} \) of locally free sheaves over a scheme \( X \), then \( \mathbb{P}_X(U) \) is the zero locus of \( \text{taut}_s \) on \( \mathbb{P}_X(V) \) and \( \mathcal{O}_{\mathbb{P}_X(U)}(1) = \mathcal{O}_{\mathbb{P}_X(V)}(1)|_{\mathbb{P}_X(U)} \).

We adapt the cohomological grading for complexes and associate \( C \) to the closed embedding and \( R \) to the projection morphism. We could also obtain \( \wedge^* \) by Proposition 2.2. By induction we have

\[
\begin{align*}
& S^mU & m \geq 0 \\
& 0 & -\text{rank}(U) < m < 0 \\
& \det(U)^{-1} \wedge^{-m-\text{rank}(U)} (U^\vee)[1-\text{rank}(U)] & m \leq -\text{rank}(U)
\end{align*}
\]

**Remark 2.3.** The case that \( V = 0 \) in Proposition 2.2 is given in [7], Exercise III.8.4.

2.2. **Blow-ups Along an Ambient Variety.** Given a closed embedding \( f : Y \to X \) of smooth varieties, let \( B_0 Y \) be the blow up of \( Y \) in \( X \) and \( \text{pr}_f : B_0 Y \to X \) be the projection morphism. It is well-known (see [8], Page 144-145) that \( R_{\text{pr}_f, \mathcal{O}_{B_0 Y}} \cong \mathcal{O}_X \). Moreover, let \( R := \mathbb{P}_Y(N^\vee_{Y/X}) \) be the exceptional divisor of \( B_0 Y \). The conormal bundle of \( Y \) in \( X \) then

\[
R_{\text{pr}_f, \mathcal{O}_{B_0 Y}}(mR)/\mathcal{O}((m-1)R) \cong R_{\text{pr}_f, \mathcal{O}_{\mathbb{P}_Y(N^\vee_{Y/X})}}(m) \cong 0
\]

if \( 0 < m < \dim(X) - \dim(Y) \) by Proposition 2.2. By induction we have

\[
R_{\text{pr}_f, \mathcal{O}(mR)} \cong \mathcal{O}_X, \quad \text{if } 0 \leq m < \dim(X) - \dim(Y).
\]

Let \( f : V \to \mathcal{O}_X \) be a morphism of locally free sheaves over \( X \) and ideal sheaf \( \mathcal{I} \) be the image of \( f \). Let \( Y \) be the zero locus of \( f^\vee \). Then \( \mathcal{I} \) is the ideal sheaf of \( Y \). Let \( J \subset \mathcal{I} \) be the ideal sheaf of another smooth closed subscheme \( Z \). Since \( \mathcal{I}^{-1} \cap \mathcal{J} \), we denote \( B_0 Z Y := \text{Proj}_X(\bigoplus_{k=0}^\infty \mathcal{I}^k/\mathcal{I}^{k-1} \mathcal{J}) \), and \( B_0 Z Y \) is a closed subscheme of \( B_0 Z Y \).

We could also obtain \( B_0 Z Y \) through the intersection theory. Let \( g : Z \subset X \) be the closed embedding and \( \bar{R} = \mathbb{P}_Z(N^\vee_{Z/X}) \) be the exceptional divisor of \( B_0 Z X \). The
Lemma 2.4. If $P$ is smooth of dimension $\dim P$. In this situation we say that $P$ is smooth of dimension $\dim P$. Then $Bl^Y_Z Y$ is the zero locus of $(f_R)^\nu$. Let

$$T^* f|_Z : V|_Z \to J/\mathcal{J}^2 \cong N^P_{Z/X}$$

be the restriction of $f$ to $Z$. By restricting $f_R$ to $R$, we have

$$f_R|_R = \tau_{T^* f|_Z}$$

and thus $Bl^Y_Z Y \cap R = \mathbb{P}_Z(T^* f|_Z)$.

**Lemma 2.4.** If $Y - Z$ is smooth of dimension $\dim(X) - \rank(V)$ and $\mathbb{P}_Z(T^* f|_Z)$ is smooth of dimension $\dim(X) - \rank(V) - 1$, then $Bl^Y_Z Y$ is smooth and $f_R$ is regular. In this situation we say that $\mathbb{P}_Z(T^* f|_Z)$ is the exceptional divisor of $Bl^Y_Z Y$. Moreover, if $\dim(Z) < \dim(Y - Z)$, then $f$ is also regular and $Rpr_g\mathcal{O}_{Bl^Y_Z Y} = \mathcal{O}_Y$.

**Proof.** The morphism $f_R$ is regular because $Bl^Y_Z Y$ is the disjoint union of $Y - Z$ and $\mathbb{P}_Z(T^* f|_Z)$ and its dimension is $\dim(X) - \dim(V)$. The smoothness of $Bl^Y_Z Y$ following from the fact that given a Noetherian local ring $A$ with maximal ideal $m$ and $f \in m$, if $A/(f)$ is regular and $\dim(A/(f)) = \dim(A) - 1$, then $A$ is also regular. The above fact holds since $\dim(m/m^2) \leq \dim(m/m^2 + (f)) + 1 = \dim(A)$.

If $\dim(Y - Z) > \dim(Z)$, then $\dim(Y) = \dim(X) - \rank(V)$ and $f$ is regular by the dimension counting. The structure sheaves $\mathcal{O}_{Bl^Y_Z Y}$ and $\mathcal{O}_Y$ are resolved by the Koszul complexes of $f$ and $f_R$ respectively. Thus $Rpr_g\mathcal{O}_{Bl^Y_Z Y} = \mathcal{O}_Y$ as $Rpr_g\mathcal{O}(\mathbb{P}_R) \cong \mathcal{O}_X$ when $0 \leq k \leq \rank(V) < \dim(X) - \dim(Z)$ by (2.3). $\square$

2.3. **Fourier-Mukai transforms associated to an algebraic surface.** Given a scheme $X$, we denote $D^b(X)$ as the bounded derived category of coherent sheaves on $X$.

**Definition 2.5** (Definition 6.1 of [29]). Let $S_1, S_2$ be two copies of $S$, in order to emphasize the factors of $S \times S$. Given smooth varieties $X, Y, Z$ be and Fourier-Mukai kernels $P \in D^b(X \times Y \times S_1)$ and $Q \in D^b(Y \times Z \times S_2)$, we define the composition $QP \in D^b(X \times Z \times S_1 \times S_2)$ by

$$QP := R\pi_{13*}(L\pi_{12*}P \otimes L\pi_{23*}Q),$$

where $\pi_{12}, \pi_{23}$ and $\pi_{13}$ are the projections from $X \times Y \times Z \times S_1 \times S_2$ to $X \times Y \times S_1$, $Y \times Z \times S_2$ and $X \times Z \times S_1 \times S_2$ respectively.

2.4. **Derived Algebraic Geometry.** We collect some facts about dg-schemes and derived schemes, which will only be used in Section 4.

**Definition 2.6** (dg-scheme and derived schemes). A dg-scheme consists of a scheme $X_0$ and quasi-coherent sheaves $\mathcal{O}_X := \{\mathcal{O}_{X,i}\}_{i \geq 0}$ on $X_0$ such that $\mathcal{O}_{X,0} = \mathcal{O}_{X_0}$, equipped with a cdga structure, i.e $\delta : \mathcal{O}_{X,i} \to \mathcal{O}_{X,i+1}$ and $\bullet : \mathcal{O}_{X,i} \otimes \mathcal{O}_{X,j} \to \mathcal{O}_{X,i+j}$ which respects the differentials and the multiplication. We denote $\mathcal{O}_{\pi_0 X}$ as the closed subscheme of $X_0$ such that $\mathcal{O}_{\pi_0 X} = H^0(\mathcal{O}_X)$.

A derived scheme $X$ consists of a pair $(X, \mathcal{O}_X)$, where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf of commutative simplicial rings on $X$ such that the ringed space $(X, \mathcal{O}_X)$ is a scheme and the homotopy sheaf $\mathcal{O}_i(\mathcal{O}_X)$ is a quasi-coherent sheaf over $(X, \mathcal{O}_X)$ for all $i > 0$.

Given a dg-scheme $(X_0, \mathcal{O}_X)$, there exists a canonical closed embedding $i : \pi_0 X \to X_0$ and $(\pi_0 X, i^{-1} \mathcal{O}_X)$ is a derived scheme.
Given any scheme $X$, $(X, \mathcal{O}_X)$ is also a derived scheme, which we call as a classical scheme in our paper. Let $dSch$ be the $\infty$-category of derived schemes. For any two morphisms $f : X \to Z, g : Y \to Z$ in $dSch$, by [27] the Cartesian product in $dSch$ exists, which we denote as $X \times^L_Z Y$.

**Lemma 2.7.** Given $f : X \to Z, g : Y \to Z$ regular embeddings of smooth classical schemes, if $\dim(X \times_Z Y) = \dim(X) + \dim(Y) - \dim(Z)$, then $X \times_Z Y \cong X \times^L_Z Y$.

**Proof.** It follows by Proposition 3.2 of [14] that $\text{Tor}^Z_X(\mathcal{O}_X, \mathcal{O}_Y) = 0$ when $i > 0$. □

**Example 2.8** (Derived zero locus of a section of a vector bundle). Let $X$ be a scheme and $s \in \Gamma(X, V)$, where $V$ is a locally free sheaf on $X$. The Koszul complex

$$\wedge^{\text{rank}V} V^\vee \to \cdots \to V^\vee \xrightarrow{s^\vee} \mathcal{O},$$

forms a cdga structure on $X$ and thus induces a dg-scheme which we denote as $\mathbb{R}Z(s)$. Regarding as a derived scheme, $\mathbb{R}Z(s)$ is the derived Cartesian product of $s : X \to V$ and the zero section $0 : X \to V$.

**Definition 2.9** (The bounded derived category). Given a derived scheme $\mathfrak{X}$, we define the dg-category

$$(2.6) \quad L_{\text{qcoh}}(\mathfrak{X}) := \lim_{\mathfrak{U} \to \mathfrak{X}} L_{\text{qcoh}}(\mathfrak{U}).$$

Here $\mathfrak{U} = \text{Spec} A$ is an affine derived scheme for a cdga $A$ and the category $L_{\text{qcoh}}(\mathfrak{U})$ is the dg-category of dg-modules over $A$ localized by quasi-isomorphisms. The homotopy category of $L_{\text{qcoh}}(\mathfrak{X})$ contains a triangulated subcategory $D^b_{\mathcal{O}_X}(\mathfrak{X})$ which consists of objects with bounded coherent cohomologies.

**Example 2.10.** Given a dg-scheme $(X_0, \mathcal{O}_X)$ and let $\mathfrak{X}$ be the derived scheme $(\pi_0 X, i^{-1} \mathcal{O}_X)$, where $i : \pi_0 X \to X$ is the canonical embedding. Given a dg-$\mathcal{O}_X$-module $\mathcal{F}$ such that cohomology sheaves are bounded and coherent $\mathcal{O}_{\pi_0 X}$-modules, then $i^{-1} \mathcal{F} \in D^b_{\text{coh}}(\mathfrak{X})$. Given dg-$\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}, \mathcal{H}$ such that the cohomology sheaves are bounded and coherent $\mathcal{O}_{\pi_0 X}$-modules, with $\mathcal{O}_X$-morphisms

$$(2.7) 0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0,$$

which is exact at any degree, then (2.7) could be lifted to a triangle in $D^b_{\text{coh}}(\mathfrak{X})$.

**Definition 2.11** (Quasi-smooth morphisms). A morphism of derived scheme $f : \mathfrak{F} \to \mathfrak{M}$ is called quasi-smooth if the $f$-relative cotangent complex $\mathbb{L}_f$ is perfect such that for any point $x \in \pi^0 \mathfrak{F}$ the restriction $\mathbb{L}_f|_x$ is of cohomological amplitude $[-1, 1]$.

Morphisms between classical smooth varieties are always l.c.i and thus quasi-smooth. Finally we introduce the pull-back and the push-forward functor:

**Lemma 2.12** (Section 4.2 of [16] and Proposition 1.4 of [26]). Given a morphism of derived schemes $f : X \to Y$, there exists a natural pull-back functor $Lf^* : D^b_{\text{coh}}(Y) \to D^b_{\text{coh}}(X)$ if $f$ is quasi-smooth and a natural push forward functor $Rf_* : D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(Y)$ if $f$ is proper. Moreover, for any derived square

$$\begin{array}{ccc}
X' & \xrightarrow{q} & X \\
\downarrow q & & \downarrow p \\
Y' & \xrightarrow{f} & Y
\end{array}$$
such that all derived schemes are quasi-compact and quasi-separated, and \( f \) is proper and \( p \) is quasi-smooth, it canonically induces an isomorphism of functors:

\[
f^*p_* \Rightarrow q_*g^* : D_{coh}^b(X) \rightarrow D_{coh}^b(Y').
\]

3. Nested, Triple and Quadruple Moduli Space of Sheaves

In this section, we first recall the geometry of the nested, triple and quadruple moduli space of sheaves from [12, 14]. Then we introduce a new description of the quadruple moduli spaces in Theorem 3.5. In this paper, we will always denote \( \text{const} \) as the constant in Section 1.2.

3.1. Nested Moduli Space of Sheaves.

Given an integer \( k \), we recall the definition of \( Z_{k,k+1} \) in (1.1) and define \( Z_{k-1,k,k+1} \):

\[
Z_{k-1,k,k+1} = \{(F_1 \subset F_0) | \text{\( F_1, F_0 \) are stable, } c_2(F_0) = k \}.
\]

The projections to \( F_0, F_1, x \) respectively induce projection morphisms

\[
p_k : Z_{k,k+1} \rightarrow M_k, \quad q_k : Z_{k,k+1} \rightarrow M_{k+1}, \quad \pi_k : Z_{k,k+1} \rightarrow S.
\]

The functors by forgetting \( F_1 \) and \( F_0 \) respectively induce morphisms:

\[
p_{k-1,k} : Z_{k-1,k,k+1} \rightarrow Z_{k-1,k}, \quad q_{k,k+1} : Z_{k-1,k,k+1} \rightarrow Z_{k,k+1}.
\]

There are tautological line bundles \( L \) on \( Z_{k,k+1} \) and \( L_1, L_2 \) on \( Z_{k-1,k,k+1} \) such that the fibers of each closed point are \( F_0/F_1, F_0/F_0, F_0/F_1 \) respectively.

**Proposition 3.1** (Proposition 2.14, 2.19 and 2.26 of [14]). There exists a resolution of the universal sheaf \( U_k \) on \( M_k \times S \) by locally free sheaves

\[
0 \rightarrow W_k \xrightarrow{\psi_k} V_k \rightarrow U_k \rightarrow 0.
\]

We abuse the notation to denote \( U_k := \{W_k \xrightarrow{\psi_k} V_k\} \). Then the scheme \( Z_{k,k+1} \) is smooth of dimension \( \text{const} + (2k+1)r + 1 \) and

\[
Z_{k,k+1} = \mathbb{P}M_k \times S(U_k), \quad Z_{k-1,k} \cong \mathbb{P}M_k \times S(q!U'_k[-1])
\]

as regular projectivizations, with \( L \cong \mathcal{O}_{\mathbb{P}M_k \times S(U_k)}(1) = \mathcal{O}_{\mathbb{P}M_k \times S(q!U'_k[1])}(-1) \). The morphisms \( p_k \times \pi_k \) and \( q_k \times \pi_k \) are the respective projection morphisms.

**Proposition 3.2** (Section 2.20 of [14]). Let \( \text{id} \times \pi_k : Z_{k,k+1} \rightarrow Z_{k,k+1} \times S \) be the graph of \( \pi_k \), and we regard \( Z_{k,k+1} \) as a closed subscheme of \( Z_{k,k+1} \times S \). There exists
the following diagram:

\[
\begin{array}{cccccccc}
0 & \rightarrow & W_{k+1} & \xrightarrow{w_{k,k+1}} & W_k & \rightarrow & B_{k,k+1} \otimes \mathcal{I}_{3,k,k+1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{V}_{k+1} & \xrightarrow{v_{k,k+1}} & \mathcal{V}_k & \rightarrow & B_{k,k+1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & U_{k+1} & \xrightarrow{u_{k,k+1}} & U_k & \rightarrow & B_{k,k+1} \otimes \mathcal{O}_{3,k,k+1} & \rightarrow & 0 \\
\end{array}
\]

(3.2)

such that all the rows and columns are short exact sequences and

- all $\mathcal{B}_{k,k+1}, \mathcal{W}_k, \mathcal{V}_k, \mathcal{W}_{k+1}, \mathcal{V}_{k+1}$ are locally free sheaves on $3_{k,k+1} \times S$;
- $\mathcal{B}_{k,k+1}|_{3_{k,k+1}} \cong \mathcal{L}$;
- $\psi_k$ and $\psi_{k+1}$ are the pull back of resolutions of $\mathcal{U}_k$ on $\mathcal{M}_k \times S$ and $\mathcal{U}_{k+1}$ on $\mathcal{M}_{k+1} \times S$ respectively.

Restricting (3.2) to $3_{k,k+1}$, there exists the following commutative diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & q\mathcal{L} & \rightarrow & W_{k+1} & \xrightarrow{w_{k,k+1}} & W_k & \rightarrow & \mathcal{L}T_S^* & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{V}_{k+1} & \xrightarrow{v_{k,k+1}} & \mathcal{V}_k & \rightarrow & 0 & \rightarrow & \mathcal{L} & \rightarrow & 0.
\end{array}
\]

(3.3)

where all rows are long exact sequences. There exists a unique morphism $\psi_{k,k+1} : W_k \rightarrow \mathcal{V}_{k+1}$ which makes (3.3) commute.

**Proposition 3.3** (Proposition 2.19 and 2.27 of [14]). The scheme $3_{k-1,k,k+1}^*$ is smooth of dimension $\text{const} + 2rk + 1$ and

\[
3_{k-1,k,k+1}^* \cong \mathbb{P}_{3_{k-1,k}}(\psi_{k-1,k} \oplus w_{k-1}) \cong \mathbb{P}_{3_{k,k+1}}(q\mathcal{L}\psi_{k,k+1}^\vee),
\]

where

\[
\psi_{k-1,k} \oplus w_{k-1} : W_{k-1} \rightarrow V_k \oplus \mathcal{L}T_S^*, \quad q\mathcal{L}\psi_{k,k+1}^\vee : q\mathcal{L}\mathcal{V}_{k+1}^\vee \rightarrow q\mathcal{L}W_k^\vee
\]

are morphisms of locally free sheaves on $3_{k-1,k}$ and $3_{k,k+1}$ respectively, defined in (3.3). Both two projectivizations are regular, and $p_{k-1,k}$ and $q_{k,k+1}$ are the respective projection morphisms. Moreover, we have

\[
\mathcal{O}_{3_{k-1,k}}(\psi_{k-1,k} \oplus w_{k-1})^*(1) \cong \mathcal{L}_1, \quad \mathcal{O}_{3_{k,k+1}}(q\mathcal{L}\psi_{k,k+1}^\vee)^*(1) \cong \mathcal{L}_1 \mathcal{L}_2^{-1}.
\]

Proposition 2.2 and Proposition 3.1 induce the following formula:

\[
R(p_k \times \pi_k)_*(\mathcal{L}^m) = \begin{cases}
S^m(U_k) & m \geq 0 \\
0 & -r < m < 0 \\
\text{det}(U_k)^{-1} \wedge^{-m-r} (U_k^\vee)[1-r] & m \leq -r
\end{cases}
\]

(3.4)

By Proposition 2.2, Proposition 3.3 and the fact that $(p_k \times \pi_k) \circ q_{k,k+1} = (q_{k-1} \times \pi_{k-1}) \circ p_{k-1,k}$, we have

\[
R((p_k \times \pi_k) \circ q_{k,k+1})_* \mathcal{L}_i^j \mathcal{L}_2^j = 0, \quad \text{if} \ -i \leq r < 0.
\]

(3.5)
3.2. Triple and Quadruple Moduli Spaces. Given an integer $k$, we denote

$$3_{-} = \{(F_0 \subset y, F_1, F'_1 \subset x) | F_0, F'_1, F_1 \text{ are stable}, c_2(F_0) = k\}$$

$$3_{+} = \{(F_0 \supset x, F'_0 \supset y, F_1, F'_1 \supset x) | F_0, F'_0, F_1, F'_1 \text{ are stable}, c_2(F_0) = k\}$$

$$\mathcal{Y} = \{(F_{-} \subset x, F_0 \subset y, F_1, F_{-} \subset y, F'_0 \subset x, F'_1) | F_0, F'_0, F_1, F'_1 \text{ are stable}, c_2(F_0) = k\}.$$

The schemes $3_{-1,k}, 3_{k+1}$ and $3_{-1,k,k+1}^*$ are closed subschemes of $3_{-}, 3_{+}$ and $\mathcal{Y}$ respectively which contain triples and quadruples that $F_0 \cong F'_0$. We denote $\Delta_{\mathcal{Y}}$ as $3_{-1,k,k+1}^*$ regarding as a closed subscheme of $\mathcal{Y}$. There are tautological line bundles $L_1, L_2, L_1', L_2'$ over the above moduli spaces such that the fibers of each closed point are $F_0/F_{-}, F_1/F_0, F'_0/F_{-}, F'_1/F'_0$ respectively. By forgetting $F_{-}$ and $F_1$ respectively, we induce the morphisms $\alpha_+ : \mathcal{Y} \to 3_{-}$ and $\alpha_- : 3_{+} \to \mathcal{Y}$.

**Proposition 3.4** (Proposition 2.39 and 2.41 of [14], Claim 3.8 of [12]). The scheme $\mathcal{Y}$ is smooth of dimension $dim(3_{-1,k,k+1}) + 1$. For any integers $a, b, c, d$, we have

$$L_1^a L_2^b L_1'^c L_2'^d = L_1^{a+d} L_1'^{b+c} O((c + d) \Delta_{\mathcal{Y}}).$$

Moreover, the morphisms $\alpha_+ and \alpha_- are isomorphisms when restricting to $\mathcal{Y} - \Delta_{\mathcal{Y}}$.

As the preimage of $3_{-1,k}$ in $\alpha_-$ and the preimage of $3_{k+1}$ in $\alpha_+$ are both $\Delta_{\mathcal{Y}}$, the morphism $\alpha_-$ and $\alpha_+$ factors through morphisms

$$\bar{\alpha}_- : \mathcal{Y} \to Bl_{3_{-1,k}} 3_{-}, \quad \bar{\alpha}_+ : 3_{+} \to Bl_{3_{k+1}} 3_{+}.$$ 

In Section 3.3, we will prove that both $\bar{\alpha}_-$ and $\bar{\alpha}_+$ are isomorphisms.

3.3. Description of the Triple Moduli Space. Given a resolution of $U_k$ and $U_{k+1}$ on $M_{k,k+1} \times S$ and a resolution of $U_k$ and $U_{k-1}$ on $3_{k-1,k} \times S$ as (3.2), by Proposition 3.1 we have

$$3_{+} = \mathbb{P}_{3_{k,k+1} \times S}(qU_k'), 3_{-} = \mathbb{P}_{3_{k-1,k} \times S}(U_{k-1})$$

and thus $3_+$ and $3_-$ are the zero locus of $\text{taut}_{qV_{k+1}}$ on $\mathbb{P}_{3_{k,k+1} \times S}(qV_{k+1})$ and $\text{taut}_{qV_{k-1}}$ on $\mathbb{P}_{3_{k-1,k} \times S}(V_{k-1})$ respectively. As $Bl_{3_{k,k+1}} 3_{+}$ and $Bl_{3_{k-1,k}} 3_{-}$ are closed subschemes of $Bl_{3_{k,k+1}} \mathbb{P}_{3_{k,k+1} \times S}(qV_{k+1}) 3_{+}$ and $Bl_{3_{k-1,k}} \mathbb{P}_{3_{k-1,k} \times S}(V_{k-1}) 3_{-}$ respectively, we abuse the notation to denote

$$\bar{\alpha}_- : \mathcal{Y} \to Bl_{3_{k,k+1}} \mathbb{P}_{3_{k,k+1} \times S}(qV_{k+1}) 3_{+}, \quad \bar{\alpha}_+ : 3_{+} \to Bl_{3_{k-1,k}} \mathbb{P}_{3_{k-1,k} \times S}(V_{k-1}) 3_{+}.$$ 

**Theorem 3.5.** The morphisms $\alpha_+ and \alpha_-$ are isomorphisms.

**Proof.** Taking the dual of $w_{k,k+1}$ in (3.2), we have

$$\text{coker}(qW_{k+1}) = \mathcal{E}xt^1(B_{k,k+1} \otimes I_{k,k+1}, O_{3_{k,k+1} \times S}) = qB_{k,k+1}^{-1} \otimes (q^{-1} O_{3_{k,k+1}}) = \mathcal{E}^{-1} O_{3_{k,k+1}}.$$ 

Thus $3_{k,k+1}$ is the zero locus of $\text{taut}_{qW_{k+1}}$ on $\mathbb{P}_{3_{k,k+1} \times S}(qW_{k+1})$:

$$3_{k,k+1} = \mathbb{P}_{3_{k,k+1} \times S}(\mathcal{E}^{-1} O_{3_{k,k+1}}) = \mathbb{P}_{3_{k,k+1} \times S}(qW_{k+1})$$

and the conormal bundle of $3_{k,k+1}$ in $\mathbb{P}_{3_{k,k+1} \times S}(qW_{k+1}) is q\mathcal{L}W_{k+1}$. Recalling that $3_{+}$ is the locus of $\text{taut}_{qW_{k+1}}$, we have

$$T^* \text{taut}_{qW_{k+1}}|3_{k,k+1} = q\mathcal{L}W_{k+1}.$$ 

On the other hand, the short exact sequence $0 \to \mathcal{V}_k \to \mathcal{V}_{k-1} \to \mathcal{B}_{k-1,k} \to 0$ in (3.2) induces a global section of $\mathcal{F}_k \times S$ in $\mathbb{P}_{3k-1,k} \times S(\mathcal{V}_{k-1})$ such that $\mathcal{F}_k \times S(\mathcal{V}_{k-1}) \cap \mathbb{P}_{3k+1}(\mathcal{V}_{k-1}) = \mathcal{F}_k$. Hence the conormal bundle of $\mathcal{F}_k$ in $\mathbb{P}_{3k-1,k} \times S(\mathcal{V}_{k-1})$ is $\mathcal{L}^{-1} \mathcal{V}_k \oplus \mathcal{T} \mathcal{S}$ and

$$T^* \mathcal{V}_k |_{\mathcal{F}_k} = \mathcal{L}^{-1}(\psi_{k-1,k} \oplus \mathcal{W}_{k-1}),$$

where $\mathcal{F}_k$ is the zero locus of $\mathcal{V}_k$. By (3.7), (3.8), Lemma 2.4 and Proposition 3.3, both two schemes $Bl_{\mathcal{F}_k} \mathbb{P}_{3k+1}(\mathcal{V}_{k-1})$ and $Bl_{\mathcal{F}_k} \mathbb{P}_{3k-1,k}$ are smooth and their exceptional divisors are

$$\mathcal{L}^{-1}(\psi_{k-1,k} \oplus \mathcal{W}_{k-1}).$$

Thus $\mathcal{L}^{-1}(\psi_{k-1,k} \oplus \mathcal{W}_{k-1})$. By (3.7), Lemma 2.4 and Proposition 3.3, both two schemes $Bl_{\mathcal{F}_k} \mathbb{P}_{3k+1}(\mathcal{V}_{k-1})$ and $Bl_{\mathcal{F}_k} \mathbb{P}_{3k-1,k}$ are smooth and their exceptional divisors are

$$\mathcal{L}^{-1}(\psi_{k-1,k} \oplus \mathcal{W}_{k-1}).$$

Thus $\alpha_- \text{ and } \alpha_+$ are isomorphism when restricting to $\Delta_{\mathbb{R}}$. As $\alpha_- \text{ and } \alpha_+$ are isomorphisms outside of $\Delta_{\mathbb{R}}$ by Proposition 3.4, they are birational proper etale morphisms and thus isomorphisms.

One considers the morphism $\theta_- : \mathbb{P}_{3k-1,k} \to \mathcal{M}_k \times \mathcal{M}_k \times \mathcal{S} \times \mathcal{S}$

$$(F_0 \subset \mathcal{F}_1, F_0' \subset \mathcal{F}_1) \to (F_0, F_0', y, x),$$

and the morphism $\theta_+ = \iota \circ \theta_0 \circ \alpha_- : \mathbb{P}_{3k-1,k} \to \mathcal{M}_k \times \mathcal{M}_k \times \mathcal{S} \times \mathcal{S}$:

$$(F_0 \subset \mathcal{F}_1, F_0' \subset \mathcal{F}_1) \to (F_0, F_0', x, y).$$

Noticing that $\dim(\mathcal{F}_k) = \text{const}+2rk+1-r < \text{const}+2rk+2 = \dim(\mathcal{F}_k) - \Delta_{\mathbb{R}})$, by Lemma 2.4 and Theorem 3.5 the projectivization $\mathcal{F}_k = \mathbb{P}_{3k-1,k} \times S(\mathcal{U}_{k-1})$ is regular and $\mathcal{R} \mathcal{D}_{\mathbb{R}} \mathcal{O}_{\mathbb{R}} \cong \mathcal{O}_{\mathbb{R}}$. Thus we induce a formula of $\mathcal{R} \mathcal{D}_{\mathbb{R}} \mathcal{O}_{\mathbb{R}}(i+j-r)\Delta_{\mathbb{R}}$

$$\mathcal{R} \mathcal{D}_{\mathbb{R}} \mathcal{O}_{\mathbb{R}}(i+j-r)\Delta_{\mathbb{R}} \cong \mathcal{R} \mathcal{D}_{\mathbb{R}} \mathcal{O}_{\mathbb{R}}(i+j-r)\Delta_{\mathbb{R}}.$$
The complexes $h_{i}^{a\pm}$ and $e_{i}^{a,b}$. Recalling a resolution of universal sheaves in (3.2) and the fact that $\mathcal{Z}_{+}$ is the zero locus of $\text{taut}^{\vee}_{\mathbb{P}(\mathcal{W}_{k+1}^{\vee})}$ on $\mathbb{P}(\mathcal{W}_{k+1}^{\vee})$. By Theorem 3.5, $\mathcal{Z}$ is a closed subscheme of $Bl_{t_{k}} := Bl_{3,k} \times_{\mathbb{R}^{3,k+1}} \mathbb{P}(\mathcal{W}_{k+1}^{\vee})$.

Other than $\mathcal{Z}_{+}$ and $\mathcal{Z}$, we also define line bundles $L_{1}$ and $L'_{1}$ on $\mathbb{P}(\mathcal{W}_{k+1}^{\vee})$ and $Bl_{t_{k}}$, so that $L_{1}$ and $L'_{1}$ are compatible with pull backs of morphisms: on $\mathbb{P}(\mathcal{W}_{k+1}^{\vee})$ we define $L_{1}$ as the pull back of $L$ from $\mathcal{O}_{3,k+1} \times \mathbb{P}(\mathcal{W}_{k+1}^{\vee})(-1)$, and on $Bl_{t_{k}}$ we define $L'_{1}$ and $L_{1}$ as the pull back of $L'_{1}$ and $L$ from $\mathbb{P}(\mathcal{W}_{k+1}^{\vee})$.

Let $D := \mathbb{P}(\mathcal{W}_{k+1}^{\vee})$ be the exceptional divisor of $Bl_{t_{k}}$. We denote

$$\text{taut}_{t_{k}} : qL_{1}^{\vee} \to \mathcal{O}_{Bl_{t_{k}}}$$

as the pull back of $\text{taut}_{t_{k}}^{\vee}$, and consider the derived scheme $\mathbb{R}^{3}_{+} := \mathbb{R}Z(\text{taut}_{t_{k}}^{\vee})$. The image of $\text{taut}_{t_{k}}$ is in $\mathcal{O}(-D)$ and induces

$$\text{taut}_{D}^{\vee} : qL_{1}^{\vee} \otimes \mathcal{O}(D) \to \mathcal{O}_{Bl_{t_{k}}}.$$ 

By Theorem 3.5, $\mathcal{Z}$ is the zero locus of $\text{taut}_{D}^{\vee}$. By (3.9), $\text{taut}_{D}^{\vee} | D = \text{taut}_{t_{k}}^{\vee}$. We denote $\mathcal{O}_{D}(1)$ as $\mathcal{O}_{\mathbb{P}(\mathcal{W}_{k+1}^{\vee})}(1)$.

**Definition 4.2.** Given an integer $a$, we define the complex of coherent sheaves $\mathcal{F}_{a}^{+}$ and $\mathcal{F}_{a}^{-}$ on $D = \mathbb{P}(\mathcal{W}_{k+1}^{\vee})$, such that the cohomological degree $m$ element is

$$(\mathcal{F}_{a}^{+})_{m} = \begin{cases} 0 & m \geq a, \\ \wedge^{m}(qL_{1}^{\vee})\mathcal{O}_{D}(m-a) & m < a; \end{cases}$$

$$(\mathcal{F}_{a}^{-})_{m} = \begin{cases} 0 & m \geq a, \\ \wedge^{m}(qL_{1}^{\vee})\mathcal{O}_{D}(m-a) & m < a; \end{cases}$$

and the differential is induced by $\wedge^{m}\text{taut}_{q\psi_{k+1}}^{\vee}$.

We define $\mathcal{Z}_{t_{k}}^{a\pm} := L_{1}^{\vee}\mathcal{F}_{a}^{\pm}$ and $h_{i}^{a\pm} := R((p_{k} \times \pi_{k}) \circ pr_{q\psi_{k+1}})(\mathcal{Z}_{t_{k}}^{a\pm})$.

Let $\Delta_{\mathcal{M}_{k+1}} : \mathcal{M}_{k+1} \to \mathcal{M}_{k+1} \times \mathcal{M}_{k+1}$ be the diagonal embedding. Let $pr_{t_{k}} : Bl_{t_{k}} \to \mathbb{P}(\mathcal{W}_{k+1}^{\vee})$ be the projection morphism. Let $\tau_{k} : \mathbb{P}(\mathcal{W}_{k+1}^{\vee}) \to \mathbb{P}(\mathcal{W}_{k+1}^{\vee})$ be the closed embedding induced by Proposition 3.1. And we define the morphism

$$\zeta : 3_{k,k+1} \times \mathbb{P}(\mathcal{W}_{k+1}^{\vee}) \to \mathcal{M}_{k+1} \times \mathcal{M}_{k+1}$$

$$(\mathcal{F}_{-1} \subset_{x} \mathcal{F}_{0} \subset (\mathcal{F}_{-1}', \mathbb{R}), e) \to (\mathcal{F}_{-1}, \mathbb{R})$$

where $e \in \mathcal{M}_{k+1} \times \mathbb{P}(\mathcal{W}_{k+1}^{\vee})$. By Lemma 2.7, we have the following diagram where all squares are derived Cartesian diagrams:

$$\begin{array}{ccc}
\mathbb{R}^{3}_{+} & \xrightarrow{\dagger} & \mathbb{P}(\mathcal{W}_{k+1}^{\vee}) \\
\downarrow & & \downarrow \text{id} \times \tau_{k} \\
Bl_{t_{k}} & \xrightarrow{pr_{t_{k}}} & \mathbb{P}(\mathcal{W}_{k+1}^{\vee}) \\
\downarrow & & \downarrow \\
\mathcal{M}_{k+1} & \xrightarrow{\Delta_{\mathcal{M}_{k+1}}} & \mathcal{M}_{k+1} \times \mathcal{M}_{k+1}
\end{array}$$

(4.7)
where $\gamma$ is the pull back of $\Delta_{\mathcal{M}_{k+1}}$ and $\sharp$ is the pull back of $\gamma \circ \pi_{1k}$. Consider the following morphism:

$$
\pi := p_k^1 \times p_k^2 \times \pi_k^1 \times \pi_k^2 : \mathcal{M}_{k,k+1} \times \mathcal{M}_{k,k+1} \to \mathcal{M}_k \times \mathcal{M}_k \times S \times S
$$

$$(F_1 \subset_x F_0, F_{-1} \subset_y F_0) \to (F_0, F_0, x, y)$$

**Definition 4.3.** Given integers $i, j, a \leq b$, we define the complex of coherent sheaves $\mathfrak{B}^{a,b}_{i,j}$ on $Bl_{i,j}$ such that the cohomological degree $m$ element

$$
(\mathfrak{B}^{a,b})_m = L^1_i L^j_l \otimes \left\{ \begin{array}{ll}
\wedge^{-m}(qL^1_i V^\vee_{k+1}) \otimes \mathcal{O}((a-m)D) & m < a \\
\wedge^{-m}(qL^1_i V^\vee_{l+1}) & a \leq m \leq b \\
\wedge^{-m}(qL^1_l V^\vee_{k+1}) \otimes \mathcal{O}((b-m)D) & m > b
\end{array} \right.
$$

The differential $d_m : (\mathfrak{B}^{a,b})_m \to (\mathfrak{B}^{a,b})_{m+1}$ is induced by

$$
L^1_i L^j_l \otimes \left\{ \begin{array}{ll}
\wedge^{-m+t}\text{aut}_{Bl} & a \leq m < b, \\
\wedge^{-m+t}\text{aut}_{Bl}^D & m < a \text{ or } m \geq b.
\end{array} \right.
$$

where all the notations are defined in (4.3) and (4.4). For any integer $a$, we have $\mathfrak{B}^{a,0}_{i,j} \cong \mathfrak{F}^{a,1}_{i,j} \cong \mathfrak{F}^{a,2}_{i,j} = \cdots$. For any integer $b$, we have $\mathfrak{B}^{a,b}_{i,j} = \mathfrak{B}^{a-1,b}_{i,j}$ when $a$ is sufficient small and denote it as $\mathfrak{B}^{a-\infty,b}_{i,j}$. Given any pair of integers $(a_1 \leq b_1), (a_2 \leq b_2)$, let $b_{i,j}^{(a_1,b_1),(a_2,b_2)} : \mathfrak{B}^{a_1,b_1}_{i,j} \to \mathfrak{B}^{a_2,b_2}_{i,j}$ by the inclusion morphism. By Example 2.10, $\mathfrak{B}^{a,b}_{i,j}$ and $b_{i,j}^{(a_1,b_1),(a_2,b_2)}$ are elements and morphisms in $D^b_{\text{coh}}(\mathbb{R}S^+)$ respectively.

We define $\mathfrak{c}^{a,b}_{i,j} := R(\pi \circ \sharp)^*(\mathfrak{B}^{a,b}_{i,j})$ and $\mathfrak{c}^{a_1,b_1}_{i,j}, (a_2,b_2) := R(\pi \circ \sharp)^*b_{i,j}^{(a_1,b_1),(a_2,b_2)}$. We define $\mathfrak{c}^{a,b+}_{i,j} := \mathfrak{c}^{(a-1),b}_{i,j}, (a,b)$ and $\mathfrak{c}^{a,b-}_{i,j} := \mathfrak{c}^{(a,b-1),a}_{i,j}$.

Let $\tau_D$ be the closed embedding from $D$ to $Bl_{i,j}$. Recalling $\mathfrak{H}_{i}^{a\pm}$ in Definition 4.2, we have $\pi \circ \sharp \circ \tau_D = \Delta_S \circ (p_k \times \pi_k) \circ \text{pr}_{q\mathcal{L}W^+_{k}}$ and thus $R(\pi \circ \sharp \circ \tau_D)_{\mathfrak{H}_{i}^{a\pm}} = R\Delta_S h_{i+}^{a\pm}$. Moreover, we have $\mathfrak{B}^{a,b}_{i,j} / \mathfrak{B}^{a-1,b}_{i,j} = \tau_D_{\ast} \mathfrak{H}_{i+1,j}^{a\pm}$ and $\mathfrak{B}^{a,b}_{i,j} / \mathfrak{B}^{a,b-1}_{i,j} = \tau_D_{\ast} \mathfrak{H}_{i+1,j}^{a\pm}$ by definition 4.2. Thus for any two integers $a, b$ we have the triangles:

\[ \cdots \to \mathfrak{c}^{a,b+}_{i,j} \to \mathfrak{c}^{a,b}_{i,j} \to R\Delta_S h_{i+}^{a\pm} \to \cdots \]

\[ \cdots \to \mathfrak{c}^{a,b-}_{i,j} \to \mathfrak{c}^{a,b}_{i,j} \to R\Delta_S h_{i+}^{b\pm} \to \cdots \]

**4.2. The Explicit Formula of $h_{i}^{a\pm}$ and Proof of Theorem 4.1.** By Proposition 3.3, Definition 4.2 induces a canonical morphism $\mathbb{H}_i^a : \mathfrak{H}_{i}^{a\pm}[-1] \to \mathfrak{H}_{i}^{a\pm}$ such that the cone resolves the sheaf $L^1_i L^j_l [-a] \otimes \Delta_0$. By (3.5), the morphism

$$
(R((p_k \times \pi_k) \circ \text{pr}_{q\mathcal{L}W^+_{k}}), \mathbb{H}_i^a) : h_{i}^{a\pm}[-1] \to h_{i}^{a\pm}
$$

is an isomorphism when $-r \leq a < 0$. Now we compute $h_{i}^{a\pm}$:

**Proposition 4.4.** Equations (4.1) and (4.2) hold. Moreover, we have $h_{i}^{a\pm} \cong 0$ if $a \leq -r$ and $h_{i}^{-a} \cong 0$ if $i < a \leq i + r, i + r \neq 0$. When $i = -r$ and $-r < a \leq 0$:

$$
h_{i}^{-a} \cong q^{-a} \text{det}((4c)^{-1}) \mathcal{O}_{\mathcal{M}_k \times S}[1 - 2a - r].
$$

Moreover, (4.11) is the image of the truncation morphism $h_{i-r}^{a\pm} \to \wedge^{-a}(q\mathcal{L}W_{k+1})^\vee[-a]$ under the functor $R((p_k \times \pi_k) \circ \text{pr}_{q\mathcal{L}W^+_{k}})$:

$$
h_{i}^{-a} \cong R((p_k \times \pi_k) \circ \text{pr}_{q\mathcal{L}W^+_{k}})(\mathcal{L}^{-r} \wedge^{-a}(q\mathcal{L}W_{k+1})^\vee[-a]).
$$
Proof. The short exact sequence
\[ 0 \to \mathcal{L}^{-1}V_{k+1} \to \mathcal{L}^{-1}V_k \to \mathcal{O}_D \to 0 \]
on D induces long exact sequences on D for any m > 0:
\begin{align}
(4.13) & \quad 0 \to \mathcal{O} \to \mathcal{L}V_k^\vee \cdots \to \wedge^m(\mathcal{L}V_k^\vee) \to \mathcal{L}^m(\mathcal{L}V_k^\vee) \to 0, \\
(4.14) & \quad 0 \to \wedge^m(\mathcal{L}^{-1}V_{k+1}) \to \wedge^m(\mathcal{L}^{-1}V_k) \to \cdots \to \mathcal{L}^{-1}V_k \to \mathcal{O} \to 0.
\end{align}
Recalling the morphism \( \mathcal{L}^{-1}\psi_{k,k+1} : \mathcal{W}_k \to \mathcal{W}_k+1 \) in (3.3), by (4.13) and (4.14):
\begin{align}
(4.15) & \quad S^m(\mathcal{L} \otimes \psi_{k,k+1}^\vee) \cong \{ \mathcal{O} \to \cdots \to S^{m-1}(\mathcal{L}U_k^\vee) \to S^m(\mathcal{L}U_k^\vee)[m] \\
(4.16) & \quad \wedge^m(\mathcal{L}^{-1} \otimes \psi_{k,k+1}) \cong \{ \wedge^m(\mathcal{L}^{-1}U_k) \to \wedge^{m-1}(\mathcal{L}^{-1}U_k) \to \cdots \to \mathcal{O} \}[-m].
\end{align}
By Proposition 2.2, if \( a \leq -r \), \( \text{Rpr}_{w_k^+} \delta^{a+} \cong 0 \) and hence \( h^{a+}_i \equiv 0 \). When \( a > -r \), \( \text{Rpr}_{w_k^+} \delta^{a+} \) is
\begin{align}
& \quad \text{det}(q\mathcal{L}V_{k+1}^\vee)\text{det}(q\mathcal{L}V_k^\vee)^{-1} \wedge^{\text{rank}(\mathcal{W}_{k+1})-\text{rank}(\mathcal{W}_k)+a} (q^{-1}\mathcal{L}^{-1}\psi_{k,k+1})[1-a] \\
\cong & \quad q^{-1}\mathcal{L}^{-1}\psi_{k,k+1} \wedge^{\text{rank}(\mathcal{W}_k)+a} (q^{-1}\mathcal{L}^{-1}\psi_{k,k+1})[1-a] \\
\cong & \quad q^{-a}\mathcal{L}^{-1}\psi_{k,k+1}^\vee \wedge^{r+1-a} \wedge^{r+1+a} (q^{-1}\mathcal{L}^{-1}\psi_{k,k+1})[1-a] \\
\cong & \quad q^{-a}\mathcal{L}^{-1}\psi_{k,k+1}^\vee \wedge^{r+1+a} (q^{-1}\mathcal{L}^{-1}\psi_{k,k+1})[1-a] \\
\cong & \quad q^{-a} \{ \mathcal{O} \to \cdots \to \mathcal{O} \}[-2a].
\end{align}
Thus (4.1) is induced by (3.4). When \( a \leq 0 \), by Proposition 2.2 and (4.15), we have
\begin{align}
(4.17) & \quad \text{Rpr}_{w_k^+} \delta^{-a} \cong q^{-a}S^{-a}(\mathcal{L} \psi_{k,k+1}^\vee) \cong q^{-a}\{ \mathcal{O} \to \cdots \to S^{-a}(\mathcal{L}U_k^\vee) \}[-a] \\
\text{and we get (4.2) and (4.11). When } & \quad i < a \leq i + r \text{ and } i + r \neq 0, \text{ by (4.17)} \\
& \quad h^{-a}_i \equiv q^{-a}\wedge^{i+r}(U_k^\vee) \to S^{i+r}(U_k^\vee) \equiv 0.
\end{align}
Finally we prove (4.12). Noticing that by (4.13),
\begin{align}
\text{Rpr}_{\mathcal{L} \mathcal{W}_k} (\mathcal{L}^{-1} \wedge^{i+r} (q\mathcal{L}V_{k+1}^\vee)[1-a]) & \equiv \mathcal{L}^{-1} \wedge^{-a} (q\mathcal{L}V_{k+1}^\vee)[1-a] \\
\equiv & \quad q^{-a} \{ \mathcal{O} \to \cdots \to \wedge^{-a} \mathcal{L}^m(\mathcal{L}V_k^\vee) \}[-a].
\end{align}
The truncation morphism \( S^m(\mathcal{L}U_k^\vee) \to \wedge^m(\mathcal{L}V_k^\vee) \) induces the morphism
\[ R(\text{pr}_{\mathcal{L} \mathcal{W}_k}^\vee), \delta^{i+r}_- \to R(\text{pr}_{\mathcal{L} \mathcal{W}_k}^\vee), (\mathcal{L}^{-1} \wedge^{i+r} (q\mathcal{L}V_{k+1}^\vee)[1-a]). \]
Hence by (3.4) and Proposition 4.4, we get the isomorphism (4.12).

\[ \square \]

Proof of Theorem 4.1. By Definition 4.3 and Theorem 3.5, as elements in \( D^{b}_{\text{coh}}(\mathbb{R}\mathcal{F}^\vee) \), we have \( \mathcal{L}_1^i \mathcal{L}_1^j \mathcal{O}(\mathcal{D}) \otimes \mathcal{O} \cong \mathcal{B}_{i,j}^0 \). Thus by (3.10), we have
\begin{align}
(4.18) & \quad R_{*} e_j f_i \cong R(\pi \circ \frac{1}{2})_{*}(\mathcal{B}_{i-r,j}^{-i+j-r,i+j-r}) \cong \mathcal{C}_{i-r,j}^{-i+j-r,i+j-r}. \\
\text{On the other hand, we have} & \quad (4.19) \quad f_i e_j \cong \text{R}_{\pi} \circ L(id \times \tau_k)^* \circ R_{\gamma} \circ (\mathcal{L}_1^{i-r} \mathcal{L}_1^j) \\
\cong & \quad \text{R}_{\pi} \circ L(id \times \tau_k)^* \circ R(\gamma \circ \text{pr}_{\mathcal{L}})_{*} \circ (\mathcal{L}_1^{i-r} \mathcal{L}_1^j) \\
\cong & \quad R(\pi \circ \frac{1}{2})_{*}(\mathcal{B}_{i-r,j}^{-i+j-0}) \cong \mathcal{C}_{i-r,j}^{-i+j-0}.
\end{align}
by Lemma 2.12

\[ \square \]
By Proposition 4.4, \( h_{t_{i-r_j}}^{a} = 0 \) when \( a \leq -r \). Thus for any \( l \leq -r \), \( C_{i-r_j}^{l_1} \cong C_{i-r_j}^{0} \). Specifically, \( C_{i-r_j}^{-\infty} \cong C_{i-r_j}^{r_0} \) if \( i + j > 0 \) and \( C_{i-r_j}^{-\infty} \cong C_{i-r_j}^{i+j-r_0} \) if \( i + j < 0 \). On the other hand, if \( i + j \neq 0 \), still by Proposition 4.4, \( h_{a}^{i+j-r_0} = 0 \) when \( i + j - r < a \leq i + j \). Thus \( C_{i-r_j}^{i+j-r_0} \cong C_{i-r_j}^{i+j-r_0} \) for all \( i + j - r \leq l \leq i + j \) if \( i + j \neq 0 \).

Specifically, \( C_{i-r_j}^{i+j-r_0} \cong C_{i-r_j}^{i+j-r_0} \) if \( i + j > 0 \) and \( C_{i-r_j}^{i+j-r_0} \cong C_{i-r_j}^{i+j-r_0} \) if \( i + j < 0 \). Hence we prove the case that \( i + j \neq 0 \).

When \( i + j = 0 \), we have the exact triangle:

\[
\begin{align*}
\cdots & \to C_{i-r_j}^{r_0} \to C_{i-r_j}^{r_0} \to R(\pi \circ \delta)_{*}(\mathbb{B}_{i-r_j}^{r_0}/\mathbb{B}_{i-r_j}^{r_0}) \to \cdots.
\end{align*}
\]

By (4.10), we have \( R\pi_{*}R\pi_{*}(\mathbb{B}_{i-r_j}^{r_0}/\mathbb{B}_{i-r_j}^{r_0}) = 0 \) and thus

\[
\begin{align*}
\chi_{i-r_j}^{r_0}(0,0) & \cong \chi_{i-r_j}^{r_0}(0,0) \circ \chi_{i-r_j}^{r_0}(0,0) : C_{i-r_j}^{r_0} \to C_{i-r_j}^{r_0},
\end{align*}
\]

is an isomorphism. Hence \( \chi_{i-r_j}^{r_0}(0,0) \) has a left inverse and (4.20) splits. Moreover, the restriction morphism \( L_{1}^{r} \otimes L_{1}^{r} \) \( L_{1}^{r} \otimes L_{1}^{r} \) \( qL_{1}^{r} \otimes L_{1}^{r} \otimes \mathcal{O}_{D} \) induces a morphism of dg-modules \( \mathbb{B}_{i-r_j}^{r_0} \to \oplus_{a=0}^{\infty} L_{1}^{r} \otimes L_{1}^{r} \otimes \mathcal{O}_{D} \) such that the image of \( \mathbb{B}_{i-r_j}^{r_0} \) is 0. Hence it induce a morphism:

\[
\begin{align*}
\mathbb{B}_{i-r_j}^{r_0}/\mathbb{B}_{i-r_j}^{r_0} \to \bigoplus_{a=0}^{\infty} L_{1}^{r} \otimes L_{1}^{r} \otimes \mathcal{O}_{D}.
\end{align*}
\]

By Proposition 4.4, we have

\[
\begin{align*}
R(\pi \circ \delta)_{*}(\mathbb{B}_{i-r_j}^{r_0}/\mathbb{B}_{i-r_j}^{r_0}) & \cong \bigoplus_{a=0}^{\infty} R(\pi \circ \delta)_{*} L_{1}^{r} \otimes L_{1}^{r} \otimes \mathcal{O}_{D}.
\end{align*}
\]

\[
\begin{align*}
& \cong \bigoplus_{a=0}^{\infty} R\Delta_{S}^{*}(q^{-a}\text{det}(\mathcal{U}_{k})^{-1}\mathcal{O}_{M_{k} \times S})[1 - 2a - r].
\end{align*}
\]

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