Model-driven reconstruction for highly-oversampled MRI

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Abstract—The Nyquist-Shannon theorem states that the information accessible by discrete Fourier protocols saturates when the sampling rate reaches twice the bandwidth of the detected continuous time signal. This maximum rate (the NS-limit) plays a prominent role in Magnetic Resonance Imaging (MRI). Nevertheless, reconstruction methods other than Fourier analysis can extract useful information from data oversampled with respect to the NS-limit, given that relevant prior knowledge is available. Here we present OverSampled MRI (OS-MRI), a method that exploits explicit prior knowledge of the highly controlled physical interactions between electromagnetic fields and the sample spins in MRI systems. Our simulations indicate that OS-MRI can be used for scan acceleration and suppression of noise effects in relevant scenarios by oversampling along frequency-encoded directions, which is innocuous in MRI systems under reasonable conditions. We find situations in which the reconstruction quality can be higher than with NS-limited acquisitions and traditional Fourier reconstruction. Besides, we compare the performance of a variety of encoding pulse sequences as well as image reconstruction protocols, and find that accelerated spiral trajectories in k-space combined with algebraic reconstruction techniques are particularly advantageous.

I. INTRODUCTION

The Nyquist-Shannon (NS) theorem specifies that a sampling rate twice the emission bandwidth of a continuous time-dependent signal suffices to recover all the information accessible by Fourier protocols [1]. This is relevant to Magnetic Resonance Imaging (MRI) because Fourier Transforms (FT) provide an efficient mapping between spatial frequency space (or k-space), which is a natural domain of representation for the detected signals, and the sought image space [2]. The NS-limit in k-space can be formulated as \( \delta k_i = \frac{2\pi}{\Delta x_i} \), where \( \Delta x_i \) is the spatial extent of the Region of Interest (RoI) along the i-axis \( (i \in \{x,y,z\}) \). \( \delta k_i \) is the separation between k-space points along \( k_i \), and k-space values are given in units of rad/m. From a simple FT perspective, undersampling \( (\delta k_i > \frac{2\pi}{\Delta x_i}) \) produces unwanted aliasing effects in the reconstruction, whereas oversampling \( (\delta k_i < \frac{2\pi}{\Delta x_i}) \) is pointless, since there is no useful information to recover at frequencies beyond the NS-limit. However, reconstruction techniques more elaborate than FTs exploit undersampling for scan acceleration [3], and oversampling for avoiding aliasing from active spins outside the RoI as well as increasing the signal-to-noise ratio (SNR) and dynamic range of the analog-to-digital converters (ADC) outputs [4]. We are not aware of the use of oversampling for scan acceleration in the existing literature. This is the topic of this work.

Prior knowledge about the MRI scanner, the sample or the interactions between them can be used to bypass some of the limitations of traditional Fourier methods and NS-limited acquisitions. Examples of prominent techniques exploiting prior knowledge are parallel imaging (PI), compressed sensing (CS) and non-linear gradient (NLG) encoding. In PI, where a phased array of multiple detectors is employed, k-space is typically undersampled along phase-encoded directions [5]. Although an FT reconstruction on the signal of any individual detection coil would be aliased, incorporating additional information about the unique spatial sensitivity of every coil in the array allows to recover unaliased images. PI therefore exploits prior knowledge about the scanner. CS, on the other hand, utilizes information about the scanned object. With the advent of CS, it became apparent that the number of required data samples can be related to information content rather than signal bandwidth [6]. If the former is sparse in some basis, then fewer samples are necessary, reducing scan times without necessarily sacrificing image quality. Such bases exist, exploiting, for instance, non-local features in the properties of real sampled objects [7], [8] or their induced signals in k-space [9]. Finally, in NLG scanners, where the encoding gradient fields are inhomogeneous and hence spatial frequency and position are not conjugate variables, reconstruction tools other than FTs are a must [10], [11]. In this case, it is useful to construct an Encoding Matrix (EM) with a priori information about how spin phases are expected to evolve in time depending on their position. The physical model for the interaction between the electromagnetic fields and the sample spins in an MRI system is simple, analytical and highly reliable, so image reconstruction can be performed by inverting the EM and having it act on the discretized signal vector. However, the size of the EM in typical acquisitions is too large for direct inversion and iterative methods are required, so this model-driven approach to reconstruction is used mostly when FTs are not a viable option [12].

In this paper we consider an MRI method where the acquisition is sampled at rates significantly higher than the Nyquist-Shannon limit and the reconstruction is based on prior knowledge about the physical interactions that take place during an MRI scan. We call this method OS-MRI, which stands for OverSampled MRI. In OS-MRI, abundant data (along frequency-encoded directions) combined with prior knowledge of the interaction model (embodied in an Encoding
Matrix), provides useful information for the reconstruction. We find OS-MRI a generally valid approach and of added value in some relevant scenarios. In Sec. II we present the theory behind OS-MRI. We then benchmark the performance of OS-MRI with simulations for a variety of encoding pulse sequences and reconstruction methods (Sec. III). We first show that sampling at high rates in a Cartesian acquisition can compensate for a reduction in the acquired $k$-space lines along a phase-encoded direction, thereby providing a means of scan acceleration based on prior knowledge distinct from existing approaches such as in parallel imaging or compressed sensing. Besides, we find that OS-MRI can also perform better than traditional encoding and reconstruction methods for fully sampled $k$-space trajectories in terms of image quality and resilience to noise.

II. Theory

A. Prior knowledge for spectral reconstruction - an example

It has been long known that fitting measured data to a presumed image shape may outperform FT-based reconstruction in MRI. For instance, fitting a collection of boxcar functions eliminates Gibbs ringing and enables superresolution [13]. This line of reasoning developed into the field of linear prediction [14], with specific methods such as Low-Rank Modeling of Local $k$-space Neighborhoods (LORAKS, [15]). The main insight behind this performance enhancement is that the Nyquist-Shannon sampling limit underlying Discrete Fourier Transform (DFT) methods is agnostic of any prior knowledge of the physical acquisition model: it simply takes equispaced $k$-space data as an input, and performs an orthogonal transformation. However, the time evolution of spins during an MRI scan is known a priori, and fitting the acquired signal using an interaction model can result in higher fidelity reconstructions.

For illustration purposes, let us consider a single spin (located at $x_0$) and a constant 1D magnetic gradient of strength $g$ (in T/m). After excitation, the detected signal is expected to evolve as $e^{-i\gamma g x_0 t}$, with $g$ the gyromagnetic factor ($\approx 2\pi \cdot 42 \text{ MHz/T}$ for protons). FT reconstruction requires $\delta k = 2\pi/\Delta x = \gamma g \delta t$ (with $\delta t$ the dwell time of the ADC) for a time $t_{\text{acq}} = k_{\text{max}}/(\gamma g) = 2\pi/(\gamma g \delta x)$ to determine the position of the spin with a resolution $\delta x$. Increasing the sampling rate above the NS-limit will not improve the reconstruction, whereas a simple fit to the known model can yield better results. This is illustrated in Fig. 1 where we follow three different protocols: (1) FT reconstruction of an NS-limited acquisition, (2) fit with an EM ($M$) that relates the signal $\tilde{s} = \{s(t=\iota \delta t)\}, \iota \in [t_{\text{acq}}/\delta t]$ and the image $\tilde{\rho} = \{\rho(x = -\Delta x/2 + j \cdot \delta x), j \in [0, \Delta x/\delta x]\}$, such that $\tilde{s} = M \tilde{\rho}$, and invert $M$ to obtain $\tilde{\rho} = M^{-1} \tilde{s}$, (3) idem but oversampling with respect to the NS-limit. The latter clearly outperforms the reconstruction capabilities of FT methods (Fig. 1).

This simplified example does not fully capture the complexity of real-life MRI signals and procedures, but it does show how prior knowledge can be exploited. Figures 2 and 3 in Sec. II-A show the performance of OS-MRI in more realistic (even if simulated) scenarios.

B. Interaction model and reconstruction in OS-MRI

In model-driven reconstruction, it is common to express the acquired discrete signal $\{s(t)\}$ as a vector $\tilde{s}$ of length equal to the number of time steps $N_t$, and the sought spin density as a vector $\tilde{\rho}$ of length equal to the number of voxels $N_v$. The sensor (\tilde{s}) and image (\tilde{\rho}) domains are related by $M(\tilde{\gamma}, t) = e^{-i\gamma k(t) \cdot \tilde{s}}$ as $\tilde{s} = M \tilde{\rho}$, where $\tilde{k}(t)$ is the time integral of the applied gradient up to time $t$. Thus, $M$ is the Encoding Matrix that stems from the physical interaction model and acts as a prior in OS-MRI and other model-driven methods (such as NLG encoding). For instance, for a 2D NS-limited acquisition with a sought resolution $\delta x, \delta y$ and size of the ROI $(\Delta x, \Delta y)$, where $x$ (y) is a frequency (phase) encoded direction, the EM size is equal to the square of the number of pixels, where $N_r = N_x N_y = (\Delta x/\delta x)(\Delta y/\delta y)$, the first EM row corresponds to $t = 0$ and the last one to $t_{\text{acq}} = 2\pi N_r/(\gamma G_x \delta x)$, in increments of $\delta t = 2\pi/(\gamma G_x \delta x)$.

One can solve for $\tilde{\rho}$ by direct inversion of the Encoding Matrix as $\tilde{\rho} = M^{-1} \tilde{s}$, or by any other means of solving the system of linear equations, e.g. by iterative algorithms such as algebraic reconstruction techniques (ART, [16]) or conjugate gradient methods [17]. In order to avoid the linear problem from being under-determined, $N_r \geq N_v$ is required (for FT protocols $N_t \geq N_v$ must be equal). Thus, $M$ has dimension at least $N_t \times N_v \geq N_v^2$, and scales as dim$(M) \geq N_v^n$ for an $n$-dimensional acquisition with $N_v$ pixels per dimension. A $170 \times 170$ image can be reconstructed in around two hours in a table computer with $\approx 40$ GB Random Access Memory. However, this approach is not scalable, and matrix inversion in the presence of noise can lead to strong noise amplification if the EM’s condition number is high. Instead, most of the below reconstructions result from running the Kaczmarz method [18, 19] in a Graphics Processing Unit with Cuda [20]. This method is a modality of ART which updates the density at each time step as

$$\tilde{\rho}^{(n+1)} = \tilde{\rho}^{(n)} + \lambda S_t - \tilde{M}_t \cdot \tilde{\rho}^{(n)} \frac{\tilde{M}_t}{\tilde{M}_t^2},$$  \hspace{1cm} (1)
where $\lambda$ is the update parameter, * denotes complex conjugation, and $M_r$ is the vector formed by the $t$-th row of the Encoding Matrix $M$, with components $(M_r)_t := M_{t,r}$. The method, hence, runs over all time steps, and we iterate $N_{\text{ths}}$ times, so that the total updates to $\rho$ are $N_t \times N_{\text{ths}}$. Since ART is merely an $l_2$-norm minimization method, one can include also terms penalizing the update step $\| \Delta \rho \|^2$. We discuss this possibility in Sec. III-E. Besides, we find it generally useful to update $\rho$ to its absolute value at each step in Eq. (1).

### III. Simulation results

In the remainder of this paper we use the following minimal settings unless otherwise explicitly stated: single-coil reception, no regularization (total variation, or others), and a direct comparison between fully-sampled FT with an oversampled-in-time algebraic reconstruction technique (see below). All simulated signals are generated from a phantom of higher resolution than the reconstructions, chosen so that the number of pixels in the former is non-divisible by the pixels in the latter. Besides, pixels are considered dense spin distributions rather than a single “heavy” spin. These distributions are integrated over to find the contribution to the overall signal from every individual pixel. We take these precautions to avoid “inverse crime” situations [22].

#### A. Imaging with reduced $k$-space coverage

We first check the performance of OS-MRI with standard Cartesian sampling at the NS rate, but removing every second phase-encoded $k$-space line (Fig. 2). In the language of Parallel Imaging this corresponds to a two-fold undersampling or acceleration, and requires the use of two signal detectors with complementary sensitivity regions. The images in Fig. 2 show OS-MRI reconstructions assuming a single detector. These show that sampling at high rates in a Cartesian acquisition can compensate for a reduction in the acquired $k$-space lines along a phase-encoded direction, thereby accelerating the acquisition with a model-driven approach. The reconstruction quality increases with oversampling in time, i.e. along the frequency-encoded direction. These acquisitions are simulated with small random displacements of the scanned $k$-space lines which suppress aliasing artifacts. In a different set of simulations, we have also successfully reconstructed images from acquisitions with variable density sampling, where the central region of $k$-space is NS-limited along the phase-encoded directions and the outer portions of $k$-space are undersampled. This is typical in scans with reconstructions based on compressed sensing [8].

OS-MRI can also be used to reconstruct images from spiral sequences with radial undersampling (accelerated by a factor $\alpha$), i.e. where instead of using $k_x(t) = \theta(t) \cos \theta(t)$ (and sine for $k_y$) with a slew-rate $S$, we use $k_x(t) = a \theta(t) \cos \theta(t)$ and $S/\alpha$, to keep the effective slew-rate at the same value [23]. In Fig. 3a) we compare OS-MRI reconstructions of a knee with NS-limited Echo Planar Imaging (EPI, [24]) and spiral acquisitions. Images d)-e) correspond to spiral trajectories with $\alpha = 2$ and oversampling in time by a factor of $\times 120$ (i.e. a sampling frequency of 10 MHz, well within reach of standard ADC electronics). Images b)-c) result from two NS-limited acquisitions reconstructed with Fourier protocols: EPI-DFT (b) and Spiral $\times 1$ DFT (c). The latter was regridded into a Cartesian $k$-space before applying the DFT. The quality of the OS-MRI reconstructions is visually better even for half the acquisition time (Fig. 3b). A quantitative study of the structural similarity index (SSIM, [25]) between the EPI-DFT and OS-MRI reconstructions (as a function of $\tau_{\text{acq}}$) and the original phantom, showed that the latter approach for $\tau_{\text{acq}} \approx 23$ ms yields the same as the former for 100 ms. This suggests accelerations factors around $\times 4$ are realistic for single-shot OS-MRI acquisitions. The point spread function (PSF), i.e. the resulting image for a single spin, for each sequence and reconstruction method is also shown. These highlight the enhanced resolution and reduction of side-lobes for OS-MRI. A similar analysis on fully-sampled Cartesian acquisitions comes next, in Sec. III-B.

Doubling the $k$-space step with Fourier protocols is equivalent to halving the RoI, so aliasing artifacts are expected, as seen in Fig. 4. For the left image, we recast the data into a Cartesian grid to use DFT protocols, but this does not prevent aliasing due to the radial undersampling. In contrast, the ART reconstruction appears unaliased when the signal is sufficiently oversampled in time.

#### B. OS-MRI with fully-sampled $k$-space trajectories

We find OS-MRI can perform better than standard methods also for fully sampled (non-accelerated) Cartesian trajectories. In Fig. 5, the dashed black line shows the SSIM of a Shepp-Logan phantom imaged with a single-shot EPI sequence and reconstructed with DFT, as a function of the readout duration $\tau_{\text{acq}}$. An ART reconstruction from the same data shows a slightly worse behavior with $\lambda = 0.1$ and $N_{\text{ths}} = 10$ (solid blue). When we oversample in time by a factor of 12 ($\delta t = 1 \mu$s), ART reconstructions result in a higher SSIM (solid purple). For reference, the DFT line crosses an SSIM of 0.9 at $\tau_{\text{acq}} \approx 35$ ms, whereas the ART line for $\delta t = 1 \mu$s reaches the same value at $\approx 14$ ms. Further oversampling does not significantly improve reconstruction quality with these parameters (solid red line).

Images b)-d) in Fig. 5 correspond to the pixel-by-pixel difference between the phantom and reconstructed images. The SSIM for the left (DFT) and middle (ART 0.1$\mu$s) images are both $\approx 0.9$, and their total absolute errors (normalized squared sum of the deviations in pixel brightness) are both at the 4 %
level, although the middle plot was acquired in less than half the time (14 vs 35 ms). A 35 ms oversampled acquisition can be reconstructed with ART with an SSIM ≈ 0.95 and a total absolute error ≈ 2.5 %. We observe also that the nature of the reconstruction deviations differ for both methods. When we apply a DFT to the NS-limited signal, there appears a pronounced Gibbs ringing artifact mostly external to the phantom, whereas the ringing is concentrated inside the phantom for oversampled ART reconstructions. The PSFs indicate that oversampling at 14 ms results in reduced sidelobes and a similar spatial resolution to DFT at 35 ms.

C. Noise sensitivity

One consequence of using orthogonal transformations (such as FT-based protocols) for reconstruction is Parseval’s theorem, which states that total power remains unchanged after transformation. The aftermath for MRI is that the integrated noise contribution in the acquired k-space data is conserved in the resulting image [20]. In contrast, ART estimates an image ρ(ϑ) from a set of measurement data. This image is updated at each time step based on the physical model of the signal and is consistency-checked with the measured data at that time (see Eq. (1)). Thus, deviations from the ideal noiseless signal

![Image](https://example.com/image1.png)

**Fig. 3.** a) Knee phantom. b) NS-limited DFT reconstruction of EPI acquisition on 120x120 pixels, at $t_{acq} = 100$ ms and $δt = 12 \mu s$. c) NS-limited DFT reconstruction of regridded spiral acquisition with $t_{acq} = 100$ ms and $δt = 12 \mu s$. d) Image reconstructed by x2 accelerated Spiral-ART with 120x120 pixels, at $t_{acq} = 50$ ms, $δt = 0.1 \mu s$. e) Image reconstructed by x2 accelerated spiral-ART with 320x320 pixels, at $t_{acq} = 100$ ms, $δt = 0.1 \mu s$. For all plots $g = 10$ mT/m, $Δx = Δy = 20$ cm. ART executed with $λ = 0.1$ and 10 iterations. The spiral has a slew rate $S = 100$ mT/m/s. The PSFs for each sequence and reconstruction method are shown at the bottom. f) Black: EPI-DFT k-space for a 100 ms acquisition (case b), reaching $k_{max} = 1450$ rad/m; blue: spiral x2 accelerated for an acquisition of 23 ms.

![Image](https://example.com/image2.png)

**Fig. 4.** Shepp-Logan phantom reconstruction with spiral pulse sequence accelerated by ×2 with $t_{acq} = 50$ ms. Left: DFT reconstruction with $δt = 12 \mu s$. Right: ART reconstruction with $δt = 0.12 \mu s$, $λ = 0.1$, $N_{fs} = 10$.

![Image](https://example.com/image3.png)

**Fig. 5.** a) Reconstructed image quality (SSIM) as a function of acquisition time. Black-dashed: NS-limited EPI with DFT ($δt = 12 \mu s$, zero-filled to reach 120x120 pixels). Continuous lines: EPI with ART at $δt = 12 \mu s$ (blue), ×12 oversampling (1µs, purple), and ×120 oversampling (0.1µs, red). b) Reconstruction difference with the original phantom, for NS-limited DFT at $t_{acq} = 35$ ms. Total absolute error = 4 %, c) ×120 oversampling and ART at $t_{acq} = 14$ ms, $δt = 0.1 \mu s$. Total absolute error ≈ 3.8 %. d) ×120 oversampling and ART at $t_{acq} = 35$ ms, $δt = 0.1 \mu s$. Total absolute error ≈ 2.5 %. For all plots $g = 100$ mT/m, $R_0 = (2 \text{ cm})^2$ and reconstruction on 120x120 pixels. The insets show the NS-limited DFT and oversampled ART reconstructions. The PSFs for each sequence and reconstruction method are shown at the bottom.
pull the image in different directions at different time steps. Consequently, noisy time steps compete against each other and the final image is one which maximizes compatibility with all of them.

Figure 6 provides a quantitative example for this effect. Here we study the noise resilience of NS-limited EPI-DFT and three OS-MRI acquisitions with ×120 oversampling: EPI-ART, fully sampled spiral and ×2 accelerated spiral. The signal-to-noise ratio (SNR) is defined as the ratio between the maximum signal strength (at \( t = 0 \)) and the amplitude of the added noise (of white spectral distribution in these studies). The images show that noise has a significantly stronger impact on the EPI-DFT than on OS-MRI reconstructions for a Shepp-Logan phantom. We observe a similar behavior for knee and brain phantoms.

Increasing the readout bandwidth generally leads to an increased noise, since less averaging is allowed to take place for every time bin. In principle, oversampling increases the acquired noise proportionally to \( 1/\sqrt{\delta t} \) [25]. However, the resonant character of the detector spectral response in MRI systems means that noise at frequencies far from resonance will be strongly suppressed at the ADC input. In realistic settings, the spectral noise distribution can be approximated to have the same shape as the resonant radio-frequency detector. We use 100 mT/m and an RoI of (2 cm)\(^2\) in the simulations in this work, corresponding to an emission bandwidth of \( \approx 83 \) kHz. We assume our detector resonance to have a width comparable to the emission bandwidth, and we sample at significantly higher frequencies (up to 100 MHz), so we generally neglect the extra noise sampled by short dwell-time acquisitions.

### D. ART parameters

The ART reconstructions in Figs. 2-6 all use \( \lambda = 0.1 \) and 10 iterations. We explore here the influence of these parameters on the quality of the reconstruction by comparing the SSIM of ART outputs to an original Shepp-Logan phantom (see Fig. 7). As a rule of thumb, we find satisfactory results for OS-MRI with ART when \( \lambda \times N_{\text{its}} \approx 1 \), in agreement with the bottom plot. For NS-limited acquisitions (\( \delta t = 0.1 \mu s \)) with \( t_{\text{acq}} = 20 \) ms, the inset shows the SSIM along the \( \lambda \times N_{\text{its}} = 1 \) line.

In some cases, such as the ×2 undersampled EPI reconstructions in Fig. 2, there is a trade-off between time-oversampling and ART iterations: increasing the latter can sometimes reduce the amount of oversampling required for a given image quality. It is possible that these two aspects both arise from the prior information provided to the system by the model-driven approach, although further research is due.

### E. Penalties

The ART algorithm (Eq. 1) is a gradient descent method where the data consistency condition \( \bar{s} = M\bar{\rho} \) is enforced. Consider the \( l_2 \)-norm squared cost function \( ||\bar{s} - M\bar{\rho}||^2 \); every ART step implements its gradient at a given time, and advances along \( M^* \) the reconstructed image \( \bar{\rho}^{(n)} \) by an
amount given by the data consistency error at that time step $s_i - M_i \cdot \vec{p}^{(n)}$. Since $M$ is a Vandermonde matrix, its rows $M_i$ are linearly independent vectors, and thus span a whole set of directions [28]. Similarly, other $l_2$-norm penalties, e.g. Tikhonov regularization with $||\vec{p}||^2$, may be included by incorporating its derivative into the ART steps (i.e. adding a term $\beta \vec{p}$, where $\beta$ is the new update parameter, see Ref. [12]).

Adding $l_1$-norm penalties by the gradient method is ill-defined because they imply expressions of the form $||x||_1 = \sum |x_i|$, whose derivative has singularities. One possibility is to replace the gradient operator by a proximal operator as in compressed sensing. Another is to add a small $\epsilon$ term ($|x_i| \approx \sqrt{x_i^2 + \epsilon}$) and build it into the ART step. An example of such procedure is the total variation (TV), which quantifies the total spatial derivative of $\vec{p}$. The $l_1$-norm of total variation can be expressed as

$$||\vec{p}||_{TV} = \sum_{i,j} \sqrt{(\rho_{i+1,j} - \rho_{i,j})^2 + (\rho_{i,j+1} - \rho_{i,j})^2 + \epsilon}$$

and its gradient can be incorporated into Eq. (1) as $\beta \nabla ||\vec{p}||_{TV}$ [29]. Adding TV regularization penalizes stark brightness differences among neighboring pixels, which smoothens (low-pass filters) the reconstructed image. Figure 8 shows an example where Gibbs ringing effects are alleviated as we increase $\beta$, but this also blurs the edges.

![Fig. 8. Shepp-Logan phantom reconstruction with an oversampled EPI pulse sequence and a total variation penalty added to the ART algorithm. Here, $t_{acq} = 50$ ms, $\delta t = 3$ ms, $\lambda = 0.1$, $N_{ix} = 10$ and $\beta = 0, 0.1, 0.5$ from left to right (see text).](image)

An open research direction is to explore the combination of ART with oversampling and rigorous $l_1$-norm regularization. This could be relevant in the context of CS, where an $l_1$-norm penalty is imposed on e.g. the wavelet-basis coefficients of the reconstructed image, which are typically few because natural images have a sparse representation in that basis. In order to incorporate $l_1$-norm penalties without the $\epsilon$-regularization of the $l_1$-norm, one may use a generalized concept of gradient (the proximal operator), leading to more elaborate methods such as Alternating Direction Method of Multipliers (ADMM [30]) or Split-Bregman [31] protocols.

**IV. Conclusion**

In this paper we have presented OS-MRI, an encoding and reconstruction method combining data sampling at rates well above the Nyquist-Shannon limit, with algebraic reconstruction techniques. We have demonstrated unaliased reconstructions from accelerated acquisitions (with reduced $k$-space coverage), higher quality images from fully sampled $k$-space trajectories than are possible with Nyquist-Shannon-limited acquisitions and Fourier-based reconstruction, and a substantial resilience of OS-MRI against noise in the acquired data. These results are possible because OS-MRI profits from prior knowledge of the physical model that connects the sensor and image spaces, and which concretizes in the form of an Encoding Matrix.

![Fig. 9. Number of significant pixels that can be obtained from the encoding matrix, calculated as the square root of the number of singular values $\geq 10\%$ of the maximum singular value, plotted as a function of $\delta t$. Continuous lines: EPI with $t_{acq} = 30$ ms. Dashed lines: spiral with $t_{acq} = 30$ ms. Acceleration factors are $\times 1$ (black) and $\times 2$ (blue). Here, $g = 100$ mT/m, $\Delta x = \Delta y = 2$ cm.](image)
nature, suggesting that a combination should be possible for enhanced performance with respect to the results we have presented. Finally, all these ideas need to be experimentally validated. A critical requirement in this sense is the direct access to raw, unfILTERED data from the readout electronics. This is not necessarily a given in many MRI laboratories.

Contributions

Simulations performed by FG; data analysis performed by FG, JA and JMA; paper written by FG and JA, with input from all authors; work conceived by FG, JA and JMB.

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