HEREDITARILY HUREWICZ SPACES AND
ARHANGEL’SKIĬ SHEAF AMALGAMATIONS

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Abstract. A classical theorem of Hurewicz characterizes spaces with
the Hurewicz covering property as those having bounded continuous
images in the Baire space. We give a similar characterization for spaces
X which have the Hurewicz property hereditarily.

We proceed to consider the class of Arhangel’skiĭ α₁ spaces, for which
every sheaf at a point can be amalgamated in a natural way. Let C₀(X)
denote the space of continuous real-valued functions on X with the
topology of pointwise convergence. Our main result is that C₀(X) is
an α₁ space if, and only if, each Borel image of X in the Baire space
is bounded. Using this characterization, we solve a variety of problems
posed in the literature concerning spaces of continuous functions.

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1. Introduction

We are mainly concerned with spaces $X$ which are (homeomorphic to) sets of irrational numbers, and we recommend adopting this restriction for clarity. Our results (and proofs) apply to all topological spaces $X$ in which each open set is a union of countably many clopen sets, and the spaces considered are assumed to have this property

Fix a topological space $X$. Let $\mathcal{A}, \mathcal{B}$ be families of covers of $X$. The space $X$ may or may not have the following property $\mathcal{U}_{\text{fin}}(\mathcal{A}, \mathcal{B})$: Whenever $U_1, U_2, \cdots \in \mathcal{A}$ and none contains a finite subcover, there exist finite sets $F_n \subseteq U_n, n \in \mathbb{N}$, such that $\{\bigcup F_n : n \in \mathbb{N}\} \in \mathcal{B}$.

Let $O$ denote the collection of all countable open covers of $X$. A cover $U$ of $X$ is point-cofinite if $U$ is infinite and each $x \in X$ is a member of all but finitely many members of $U$. Let $\Gamma$ denote the collection of all open point-cofinite covers of $X$. Motivated by studies of Menger [26], Hurewicz [19] introduced the Hurewicz property $\mathcal{U}_{\text{fin}}(O, \Gamma)$.

Hurewicz [19] essentially obtained the following combinatorial characterization of $\mathcal{U}_{\text{fin}}(O, \Gamma)$ (see Rec/suppress law [29]). For $f, g \in \mathbb{N}^\mathbb{N}$, $f \leq^* g$ means $f(n) \leq g(n)$ for all but finitely many $n$. A subset $Y$ of $\mathbb{N}^\mathbb{N}$ is bounded if there is $g \in \mathbb{N}^\mathbb{N}$ such that $f \leq^* g$ for all $f \in Y$.

Theorem 1 (Hurewicz). $X$ satisfies $\mathcal{U}_{\text{fin}}(O, \Gamma)$ if, and only if, every continuous image of $X$ in $\mathbb{N}^\mathbb{N}$ is bounded.

This characterization has found numerous applications—see [38, 24, 41] and references therein. We give a similar characterization for hereditarily Hurewicz spaces, that is, spaces $X$ such that each subspace of $X$ satisfies $\mathcal{U}_{\text{fin}}(O, \Gamma)$.

The property of being hereditarily Hurewicz was studied in, e.g., [15, 28, 27]. Rubin introduced a property of subsets of $\mathbb{R}$ such that the existence of a set with this property is equivalent to the possibility of a certain construction of boolean algebras [31]. Miller [27] proved that the Rubin spaces are exactly the hereditarily Hurewicz spaces.

The property of being hereditarily Hurewicz also manifests itself as follows: A set $X \subseteq \mathbb{R}$ is a $\sigma'$ space [32] if for each $F_\sigma$ set $E$, there is an $F_\sigma$ set $F$ such that $E \cap F = \emptyset$ and $X \subseteq E \cup F$. This property was effectively used in studies of generalized metric spaces [13]. Recently, Sakai proved that $X$ is a $\sigma'$ space if, and only if, $X$ is hereditarily Hurewicz (Theorem 6 below).

There exist additional classes of hereditarily Hurewicz spaces in the literature. We describe some of them.

\footnote{Every perfectly normal space (open sets are $F_\sigma$) with upper inductive dimension 0 (disjoint closed sets can be separated by a clopen set) has the required property. Thus, the spaces considered in the references also have the required property.}

\footnote{If $X$ is Lindelöf, we can consider arbitrary open covers of $X$.}

\footnote{Traditionally, point-cofinite covers were called $\gamma$-covers [17].}
A topological space is Fréchet if each point in the closure of a subset of the space is a limit of a convergent sequence of points from that subset. The following concepts, due to Arhangel’ski˘ı [1, 2], are important in determining when a product of Fréchet spaces is Fréchet. Let \( Y \) be a general topological space (not necessarily Lindelöf or zero-dimensional). A sheaf at a point \( y \in Y \) is a family of sequences, each converging to \( y \). To avoid trivialities, we consider only sequences of distinct elements. We say that a countable set \( A \) converges to \( y \) if some (equivalently, each) bijective enumeration of \( A \) converges to \( y \). The space \( Y \) is an \( \alpha_1 \) space if for each \( y \in Y \), each countable sheaf \( \{ A_n : n \in \mathbb{N} \} \) at \( y \) can be amalgamated as follows: There are cofinite subsets \( B_n \subseteq A_n \), \( n \in \mathbb{N} \), such that the set \( B = \bigcup_n B_n \) converges to \( y \). The references dealing with \( \alpha_1 \) spaces are too numerous to be listed here; see [40] and the references therein for a partial list.

Fix a space \( X \). The space \( C_p(X) \) is the family of all continuous real-valued functions on \( X \), viewed as a subspace of the Tychonoff product \( \mathbb{R}^X \). A sequence of results by Bukovský–Reclaw–Repický [10], Reclaw [30], Sakai [33], and Bukovský–Haleš [9], culminated in the result that if \( C_p(X) \) is an \( \alpha_1 \) space, then \( X \) is hereditarily Hurewicz. Our main result is that if \( C_p(X) \) is an \( \alpha_1 \) space, then each Borel image of \( X \) in \( \mathbb{N}^\mathbb{N} \) is bounded. It is easy to see that the converse implication also holds, and we obtain a powerful characterization of spaces \( X \) such that \( C_p(X) \) is an \( \alpha_1 \) space.

Historically, the realization that if \( C_p(X) \) is an \( \alpha_1 \) space then \( X \) is hereditarily Hurewicz goes through QN spaces [10]: Let \( Y \) be a metric space. A function \( f : X \to Y \) is a quasi-normal limit of functions \( f_n : X \to Y \) if there are positive reals \( \epsilon_n, n \in \mathbb{N} \), converging to 0 such that for each \( x \in X \), \( d(f_n(x), f(x)) < \epsilon_n \) for all but finitely many \( n \). A topological space \( X \) is a QN space if whenever 0 is a pointwise limit of a sequence of continuous real-valued functions on \( X \), we have that 0 is a quasi-normal limit of the same sequence. QN spaces are studied in, e.g., [10, 30, 36, 28, 11, 33, 9]. In [33, 9] it was shown that \( X \) is a QN space if, and only if, \( C_p(X) \) is an \( \alpha_1 \) space. Thus, QN spaces are also characterized by having bounded Borel images in \( \mathbb{N}^\mathbb{N} \).

We use our main theorem to show that quite a few additional properties studied in the literature are equivalent to having bounded Borel images in \( \mathbb{N}^\mathbb{N} \), and consequently solve a variety of problems posed in the literature. To make the paper self-contained and accessible to a wide audience, we supply proofs for all needed results. Often, our proofs of known results are slightly simpler than those available in the literature.

2. A CHARACTERIZATION OF HEREDITARILY HUREWICZ SPACES

Let \( \overline{\mathbb{N}} = \mathbb{N} \cup \{ \infty \} \) be the one-point compactification of \( \mathbb{N} \), and endow \( \overline{\mathbb{N}}^\mathbb{N} \) with the Tychonoff product topology. An element \( f \in \overline{\mathbb{N}}^\mathbb{N} \) is eventually finite if there is \( m \) such that \( f(n) < \infty \) for all \( n \geq m \). Let \( \text{EF} \) be the subspace of
$\mathbb{N}^n$ consisting of all eventually finite elements of $\mathbb{N}^N$. The partial order $\leq^*$ extends to $\mathcal{EF}$ in the natural way.

**Theorem 2.** $X$ is hereditarily $\mathcal{U}_{\text{fin}}(O,\Gamma)$ if, and only if, every continuous image of $X$ in $\mathcal{EF}$ is bounded.

**Proof.** $(\Rightarrow)$ Assume that $\Psi : X \to \mathcal{EF}$ is continuous. For each $n$, define the following $(G_\delta)$ subset of $\Psi[X]$:

$$G_n = \{ f \in \Psi[X] : (\forall m \geq n) f(m) < \infty \}.$$  

Let $T^n : G_n \to \mathbb{N}^N$ be the shift transformation defined by $T^n(f)(m) = f(m + n)$ for all $m$.

For each $n$, $X_n = \Psi^{-1}[G_n] \subseteq X$, and therefore satisfies $\mathcal{U}_{\text{fin}}(O,\Gamma)$. By Theorem 1, $T^n[\Psi[X_n]] = T^n[G_n]$ is a bounded subset of $\mathbb{N}^N$. Thus, $G_n$ is a bounded subset of $\mathcal{EF}$, and therefore so is $\Psi[X] = \bigcup_n G_n$.

$(\Leftarrow)$ First, note that Hurewicz’s Theorem 1 and our assumption on $X$ imply that $X$ satisfies $\mathcal{U}_{\text{fin}}(O,\Gamma)$.

**Lemma 3.** If each $G_\delta$ subset of $X$ satisfies $\mathcal{U}_{\text{fin}}(O,\Gamma)$, then $X$ is hereditarily $\mathcal{U}_{\text{fin}}(O,\Gamma)$.

**Proof.** Let $Y \subseteq X$. Assume that $U_n, n \in \mathbb{N}$, are covers of $Y$ by open subsets of $X$, which do not contain finite subcovers. Each $U_n$ is an open cover of $G = \bigcap_n \bigcup U_n \supseteq Y$, and has no finite subcover of $G$. As $G$ is a $G_\delta$ subset of $X$, it satisfies $\mathcal{U}_{\text{fin}}(O,\Gamma)$. Thus, there are finite $F_n \subseteq U_n, n \in \mathbb{N}$, such that $\{ \bigcup F_n : n \in \mathbb{N} \}$ is a point-cofinite cover of $G$, and therefore of $Y$.

Assume that $G$ is a $G_\delta$ subset of $X$.

**Lemma 4** (Sakai [33]). For each $G_\delta$ subset $G$ of $X$, there is an open point-cofinite cover $\{ U_n : n \in \mathbb{N} \}$ of $X$ such that $G = \bigcap_n U_n$.

**Proof.** $G^c = \bigcup_n C_n$ with each $C_n$ closed. If $A$ is closed and $B$ is open, then $B$ is a union of countably many disjoint clopen sets, and therefore $A \cap B$ is a union of countably many disjoint closed sets. Thus, each of the disjoint sets $C_n \setminus (C_1 \cup \ldots \cup C_{n-1})$, $n \in \mathbb{N}$, is a union of countably many disjoint closed sets. Hence, $G^c = \bigcup_n \tilde{C}_n$ where the sets $\tilde{C}_n$ are closed and disjoint, and therefore $G = \bigcap_n \tilde{C}_n$, where $\{ \tilde{C}_n : n \in \mathbb{N} \}$ is an open point-cofinite cover of $X$.

So, let $\{ U_n : n \in \mathbb{N} \}$ be an open point-cofinite cover of $X$ such that $G = \bigcap_n U_n$. For each $n$, let $U_n = \bigcup_m C_m^n$, a union of disjoint clopen sets. Define $\Psi : X \to \mathcal{EF}$ by

$$\Psi(x)(n) = \begin{cases} m & m \in \mathbb{N}, x \in C_m^n \\ \infty & x \notin U_n \end{cases}$$

As $\{ U_n : n \in \mathbb{N} \}$ is a point-cofinite cover of $X$, $\Psi(x)$ is eventually finite for each $x \in X$. 


The function $\Psi$ is continuous: A basic open set in $\mathbb{E} \mathbb{F}$ has the form $\prod_{n} V_n$ such that there are finite $I_0, I_1 \subseteq \mathbb{N}$ and elements $m_n, n \in I_0 \cup I_1$, for which: For each $n \in I_0$, $V_n = \{m_n\}$, for each $n \in I_1$, $V_n = \{m_n, m_n + 1, \ldots\} \cup \{\infty\}$, and for each $n \notin I_0 \cup I_1$, $V_n = \mathbb{N} \cup \{\infty\}$. Now,

$$
\Psi^{-1}\left[\prod_{n \in \mathbb{N}} V_n\right] = \bigcap_{n \in I_0} C_{m_n}^n \cap \bigcap_{n \in I_1} \left(X \setminus \bigcup_{k < m_n} C_k^n\right)
$$

is open.

Thus, $\Psi[X]$ is bounded by some $g \in \mathbb{N}^{\mathbb{N}}$. Now,

$$
G = \{x \in X : (\forall n) \Psi(x)(n) < \infty\} = \Psi^{-1}[\{f \in \mathbb{N}^{\mathbb{N}} : f \leq^* g\}].
$$

The set $\{f \in \mathbb{N}^{\mathbb{N}} : f \leq^* g\}$ is an $F_\sigma$ subset of $\mathbb{E} \mathbb{F}$. Indeed, let $\{g_n : n \in \mathbb{N}\}$ enumerate all elements of $\mathbb{N}^{\mathbb{N}}$ which are eventually equal to $g$. Then $\{f \in \mathbb{N}^{\mathbb{N}} : f \leq^* g\} = \bigcup_n \{f \in \mathbb{E} \mathbb{F} : f \leq g_n\}$. Thus, $G$ is an $F_\sigma$ subset of $X$. As $U_{\text{fin}}(O, \Gamma)$ is hereditary for closed subsets and preserved by countable unions, $G$ satisfies $U_{\text{fin}}(O, \Gamma)$. □

Recall that a topological space $X$ is a $\sigma$ space if each $G_\delta$ subset of $X$ is an $F_\sigma$ subset of $X$. The proof of Theorem 2 actually shows that (2 $\Rightarrow$ 1), (2 $\Rightarrow$ 4), and (4 $\Rightarrow$ 2) in the following theorem (and therefore establishes it).

**Theorem 5.** The following are equivalent:

1. $X$ is hereditarily $U_{\text{fin}}(O, \Gamma)$.
2. Each $G_\delta$ subset of $X$ satisfies $U_{\text{fin}}(O, \Gamma)$.
3. Every continuous image of $X$ in $\mathbb{E} \mathbb{F}$ is bounded.
4. $X$ satisfies $U_{\text{fin}}(O, \Gamma)$ and is a $\sigma$ space. □

The implication (1 $\Rightarrow$ 4) in Theorem 5 was first proved by Fremlin and Miller [15]. The implication (4 $\Rightarrow$ 1) can be alternatively deduced from Theorem 3.12 of [11] and Corollary 10 of [8]. An additional equivalent formulation was discovered by Sakai. Recall the definition of $\sigma'$ space from the introduction (page 2).

**Theorem 6** (Sakai). Let $X \subseteq \mathbb{R}$. $X$ is a $\sigma'$ space if, and only if, $X$ is hereditarily $U_{\text{fin}}(O, \Gamma)$.

**Proof.** This follows from Theorem 5.7 of [22]: $X$ satisfies $U_{\text{fin}}(O, \Gamma)$ if, and only if, for each $G_\delta$ set $G \subseteq \mathbb{R}$ containing $X$, there is an $F_\sigma$ set $F \subseteq \mathbb{R}$ such that $X \subseteq F \subseteq G$.

$(\Rightarrow)$ As being a $\sigma'$ space is hereditary, it suffices to show that $X$ satisfies $U_{\text{fin}}(O, \Gamma)$. Indeed, for each $G_\delta$ set $G \subseteq \mathbb{R}$ containing $X$, let $E = \mathbb{R} \setminus G$, and take an $F_\sigma$ set $F \subseteq \mathbb{R}$ disjoint from $E$ such that $X \subseteq E \cup F$. Then $X \subseteq F \subseteq G$.

$(\Leftarrow)$ Let $E \subseteq \mathbb{R}$ be $F_\sigma$. As $X \setminus E$ satisfies $U_{\text{fin}}(O, \Gamma)$ and is a subset of the $G_\delta$ set $\mathbb{R} \setminus E$, there is an $F_\sigma$ set $F \subseteq \mathbb{R}$ such that $X \setminus E \subseteq F \subseteq \mathbb{R} \setminus E$. Then $E \cap F = \emptyset$ and $X \subseteq E \cup F$. □
To indicate the potential usefulness of Theorem 2, we use it to give slightly more direct proofs of two known theorems. Recall the definition of QN spaces from the introduction (page 3).

**Theorem 7** (Reclaw [30]). If \( X \) is a QN space, then \( X \) is hereditarily \( U_{\text{fin}}(O,\Gamma) \).

**Proof.** Let \( Y \subseteq EF \) be a continuous image of \( X \). Then \( Y \) is a QN space. By Theorem 2, it suffices to show that \( Y \) is bounded. For each \( n \), and each \( y \in Y \), define \( f_n(y) = \frac{1}{\min y^{-1}(n)} \), using the natural conventions that \( \min \emptyset = \infty \) and \( 1/\infty = 0 \). \( \lim_n f_n(y) = 0 \) for all \( y \in Y \). As \( Y \) is a QN space, there are positive \( \epsilon_n, n \in \mathbb{N} \), dominating this convergence. For each \( k \), let \( Y_k = \{ y \in Y : (\forall n \geq k) f_n(y) < \epsilon_n \} \).

\( Y = \bigcup_k Y_k \). We will show that each \( Y_k \) is bounded.

Fix \( k \). Take an increasing \( g \in \mathbb{N}^\mathbb{N} \) such that \( g(1) = k \) and for each \( n \), \( \epsilon_m < 1/n \) for all \( m \geq g(n) \). Then \( Y_k \) is bounded by \( g \): Let \( y \in Y_k \). Fix \( n \) such that \( y(n) < \infty \). If \( y(n) \leq k \), then \( y(n) \leq g(1) \leq g(n) \). Otherwise, \( y(n) > k \), and since \( y \in Y_k \), \( f_{y(n)}(y) < \epsilon_{y(n)} \). Thus,

\[
\frac{1}{n} \leq \frac{1}{\min y^{-1}(y(n)))} = f_{y(n)}(y) < \epsilon_{y(n)},
\]

and therefore \( y(n) \) cannot be greater than \( g(n) \). \( \square \)

**Theorem 8** (Reclaw [30]). If \( X \) is QN space, then \( X \) is a \( \sigma \) space.

**Proof.** Theorems 5 and 7.

\( \square \)

The following sections give a deeper reason for the last two theorems.

3. **Bounded Borel images**

Our main goal in this section is to establish the equivalence in the following Theorem 9. The implication \( 2 \Rightarrow 1 \) in this theorem is Proposition 9 of Scheepers [35]. The implication \( 1 \Rightarrow 2 \) is the more difficult one, and will be proved in a sequence of related results.

**Theorem 9.** The following are equivalent:

1. \( C_p(X) \) is an \( \alpha_1 \) space.
2. Each Borel image of \( X \) in \( \mathbb{N}^\mathbb{N} \) is bounded.

**Proof.** \( 2 \Rightarrow 1 \) Consider a sheaf \( \{ A_n : n \in \mathbb{N} \} \) at \( f \in C_p(X) \). For each \( n \), enumerate \( A_n = \{ f^n_m : m \in \mathbb{N} \} \) bijectively. Define a Borel function \( \Psi : X \to \mathbb{N}^\mathbb{N} \) by

\[
\Psi(x)(n) = \min\{ k : (\forall m \geq k) |f^n_m(x) - f(x)| \leq 1/n \}.
\]
Let $g \in \mathbb{N}^\mathbb{N}$ bound $\Psi[X]$, and take the amalgamation $B = \bigcup_m \{f_m^n : m \geq g(n)\}$. Then $B$ converges to $f$.

$$(1 \Rightarrow 2)$$ Assume that $C_p(X)$ is an $\alpha_1$ space. Then the subspace $C_p(X, \{0, 1\})$ of $C_p(X)$, consisting of all continuous functions $f : X \to \{0, 1\}$, is an $\alpha_1$ space. Each element of $C_p(X, \{0, 1\})$ has the form $\chi_U$, the characteristic function of a clopen set $U \subseteq X$. Immediately from the definition, a sequence $\chi_{U_n}$ of elements of $C_p(X, \{0, 1\})$ converges pointwise to the constant function 1 if, and only if, $\{U_n : n \in \mathbb{N}\}$ is a clopen point-cofinite cover of $X$. This gives the following, which is due to Bukovský–Haleš (cf. [9, Theorem 17]), and independently Sakai (cf. [33, Theorem 3.7]).

**Lemma 10.** The following are equivalent:

1. $C_p(X, \{0, 1\})$ is an $\alpha_1$ space;
2. For each family $\{U_n : n \in \mathbb{N}\}$ of pairwise disjoint clopen point-cofinite covers of $X$, there are cofinite $V_n \subseteq U_n$, $n \in \mathbb{N}$, such that $\bigcup_n V_n$ is a point-cofinite cover of $X$. \hfill $\Box$

A function $f$ with domain $X$ is a **discrete limit** of functions $f_n$, $n \in \mathbb{N}$, if for each $x \in X$, $f_n(x) = f(x)$ for all but finitely many $n$.

Each bijectively enumerated family $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ of subsets of a set $X$ induces a **Marczewski map** $\mathcal{U} : X \to P(\mathbb{N})$ defined by

$$\mathcal{U}(x) = \{n \in \mathbb{N} : x \in U_n\}$$

for each $x \in X$. The main step in our proof is the following.

**Lemma 11.** Assume that $C_p(X, \{0, 1\})$ is an $\alpha_1$ space, and $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a bijectively enumerated family of open subsets of $X$. Then the Marczewski map $\mathcal{U} : X \to P(\mathbb{N})$ is a discrete limit of continuous functions.

**Proof.** First, consider the case where for each $n$, $U_n$ is not clopen.

For each $n$, write $U_n$ as a union $\bigcup_m C_m^n$ of nonempty disjoint clopen sets. We may assume that the partitions are disjoint: Inductively, for each $n = 2, 3, \ldots$, consider the elements $C_m^n$, $m \in \mathbb{N}$, of the $n$th partition. For each $m$, if $C_m^n$ appears in the partition of $U_k$ for some $k < n$, merge (in the $n$th partition) $C_m^n$ with some other element of the $n$th partition. Continue in this manner until the $n$th partition is disjoint from all previous partitions.

Thus, the families $\mathcal{U}_n = \{(C_m^n)^c : m \in \mathbb{N}\}$ are disjoint clopen point-cofinite covers of $X$. By Lemma [11] there are $k_n$, $n \in \mathbb{N}$, and subsets $\mathcal{V}_n = \{(C_m^n)^c : m \geq k_n\} \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_n \mathcal{V}_n$ is a point-cofinite cover of $X$. In other words,

$$\mathcal{V} = \left\{\bigcap_{m=k_n}^\infty (C_m^n)^c : n \in \mathbb{N}\right\}$$

is a point-cofinite cover of $X$.

\footnote{In fact, by the methods of Gerlits–Nagy [13], the converse implication also holds. This fact will not be used in our proof.}
For each $n, m$, let
\[ U^n_m = \bigcup_{i=1}^{\max\{m, k_n\}} C^n_i. \]

For each $m$, define $\Psi_m : X \to P(\mathbb{N})$ by
\[ \Psi_m(x) = \{ n : x \in U^n_m \}. \]

As each $U^n_m$ is clopen, $\Psi_m$ is continuous. It remains to prove that, viewed as a Marczewski map, $\mathcal{U}$ is a discrete limit of the maps $\Psi_m$, $m \in \mathbb{N}$.

Fix $x \in X$. Let $N$ be such that $x \in \bigcap_{n=1}^{\infty} (C^n_m)^c$ for all $n \geq N$. For each $n < N$ with $x \in U^n_n$, let $m_n$ be such that $x \in U^n_{m_n}$. Set $M = \max\{m_n : n < N\}$.

Fix $m \geq M$. We show that $x \in U^n_m$ if, and only if, $x \in U_n$. One direction follows from $U^n_m \subseteq U_n$. To prove the other direction, assume that $x \in U_n$, and consider the two possible cases: If $n < N$, then $x \in U^n_m$ because $m \geq M \geq m_n$, and we are done. Thus, assume that $n \geq N$. Then $x \in \bigcap_{i=k_n}^{\infty} (C^n_i)^c$. As $x \in U^n = \bigcup_{n} C^n_m$, it follows that $x \in \bigcup_{i=k_n-1}^{\infty} C^n_i \subseteq U^n_m$.

Thus, for each $x \in X$ there is $M$ such that $\Psi_m(x) = \mathcal{U}(x)$ for all $m \geq M$. This completes the proof in the case that no $U_n$ is clopen.

For the remaining case, let $I \subseteq \mathbb{N}$ be the set of all $n$ such that $U_n$ is not clopen. The previous case shows that $\mathcal{U}_I = \{ U_n : n \in I \}$, viewed as a Marczewski function from $X$ to $P(I)$, is a discrete limit of continuous functions $\Psi_m : X \to P(I)$.

For each $m$, define $\Phi_m : X \to P(\mathbb{N})$ by
\[ \Phi_m(x) = \{ n : n \in I \text{ and } n \in \Psi_m(x) \} \text{ or } (n \notin I \text{ and } x \in U_n) \} \].

Then $\mathcal{U}$ is a discrete limit of the continuous functions $\Phi_m$, $m \in \mathbb{N}$. \hfill \Box

As $X$ satisfies item (2) of Lemma 10, it satisfies $\mathcal{U}_{\text{fin}}(O, \Gamma)$: Refine each given cover to a clopen cover, turn it to a clopen point-cofinite cover by taking finite unions, and make the point-cofinite covers disjoint.

Assume that $\mathcal{U}$ is a countable family of open subset of $X$. By Lemma 11, the Marczewski map $\mathcal{U} : X \to P(\mathbb{N})$ is a discrete limit of continuous functions $\Psi_n$.

Clearly, every discrete limit is a quasi-normal limit. The proof of [10, Theorem 4.8] actually establishes the following.

Lemma 12. Assume that $P$ is a property of topological spaces, which is preserved by taking closed subsets, continuous images and countable unions.

If $X$ has the property $P$ and $\Psi : X \to Y$ is a quasi-normal limit of continuous functions into a metric space $Y$, then $\Psi[X]$ has the property $P$.

Proof. Let $\Psi_n$, $n \in \mathbb{N}$, be continuous functions as in the premise of the lemma, and let $\epsilon_n$, $n \in \mathbb{N}$, be as in the definition of quasi-normal convergence. For each $k$,
\[ X_k = \{ x \in X : (\forall n, m \geq k) \ d(\Psi_n(x), \Psi_m(x)) \leq \epsilon_n + \epsilon_m \}. \]
is a closed subset of $X$, and the functions $\Psi_n$ converge to $\Psi$ uniformly on $X_k$. Thus, $\Psi$ is continuous on $X_k$, and therefore $\Psi[X_k]$ has the property $P$.

Now, $X = \bigcup_k X_k$, and therefore $\Psi[X] = \bigcup_k \Psi[X_k]$ has the property $P$. □

It follows that for each countable family $\mathcal{U}$ of open subsets of $X$, $\mathcal{U}[X]$ satisfies $\mathcal{U}_{\text{fin}}(O, \Gamma)$.

Let $F, B$ denote the families of all countable closed and all countable Borel covers of $X$, respectively. Similarly, let $F_\Gamma, B_\Gamma$ denote the families of all countable closed point-cofinite covers of $X$ and all Borel point-cofinite covers of $X$. Following is a striking result of Bukovský, Reclaw, and Repický [10]. In their terminology, it tells that the family of closed subsets of $X$ is weakly distributive if, and only if, the same holds for the family of Borel subsets of $X$. In the language of selection principles, this result has the following compact form.

**Lemma 13** (Bukovský–Reclaw–Repický [10]). $\mathcal{U}_{\text{fin}}(F, F_\Gamma) = \mathcal{U}_{\text{fin}}(B, B_\Gamma)$.

*Proof.* Assume that $X$ satisfies $\mathcal{U}_{\text{fin}}(F, F_\Gamma)$. We first show that $X$ is a $\sigma$ space [10, Theorem 5.2].

Assume that $G = \bigcap_n U_n$ where for each $n$, $U_n \supseteq U_{n+1}$ are open subsets of $X$. Write, for each $n$,

$$U_n = \bigcup_{m \in \mathbb{N}} C^n_m,$$

where for each $m$, $C^n_m \subseteq C^n_{m+1}$ are closed subsets of $X$. We may assume that the closed cover $\{C^n_m \cup (X \setminus U_n) : m \in \mathbb{N}\}$ of $X$ has no finite subcover.

As $X$ satisfies $\mathcal{U}_{\text{fin}}(F, F_\Gamma)$, and each given cover is monotone, there are $m_n, n \in \mathbb{N}$, such that $\{C^n_{m_n} \cup (X \setminus U_n) : n \in \mathbb{N}\}$ is a closed point-cofinite cover of $X$. For each $k$ define

$$Z_k = \bigcap_{n=k}^{\infty} C^n_{m_n}.$$

Then each $Z_k$ is a closed subset of $X$, and $G = \bigcup_k Z_k$ is $F_\sigma$. This shows that $X$ is a $\sigma$ space.

Now, assume that $\mathcal{U}_n \in B$, $n \in \mathbb{N}$. Then for each $n$, each element of $\mathcal{U}_n$ is $F_\sigma$ and can therefore be replaced by countably many closed sets. Applying $\mathcal{U}_{\text{fin}}(F, F_\Gamma)$ to the thus modified covers, we obtain a cover in $F_\Gamma$. For each $n$, extend each of the finitely many chosen elements of the $n$th cover to an $F_\sigma$ set from the original cover $\mathcal{U}_n$, to obtain an element of $B_\Gamma$ chosen in accordance with the definition of $\mathcal{U}_{\text{fin}}(B, B_\Gamma)$ □

**Lemma 14.** The following are equivalent:

1. $X$ satisfies $\mathcal{U}_{\text{fin}}(B, B_\Gamma)$;

5If there are infinitely many $n$ for which there is some $m_n$ with $C^n_{m_n} = U_n$, then $G = \bigcap_n C^n_{m_n}$ is closed and we are done. Otherwise, we can ignore finitely many $n$ and assume that there are no $n, m$ such that $C^n_m = U_n$ contains $G$.

6This argument, in more general form, appears in [11, Theorem 2.1].
(2) For each countable family \( U \) of open subsets of \( X \), \( U[X] \) satisfies \( U_{\text{fin}}(O, \Gamma) \);

(3) For each countable family \( C \) of closed subsets of \( X \), \( C[X] \) satisfies \( U_{\text{fin}}(O, \Gamma) \).

**Proof.** (2 \( \iff \) 3) Use the auto-homeomorphism of \( P(\mathbb{N}) \) defined by mapping a set to its complement.

(1 \( \Rightarrow \) 2) The Marczewski map \( U : X \to P(\mathbb{N}) \) is Borel. It is easy to see that \( U_{\text{fin}}(B, B_{\Gamma}) \) is preserved by Borel images \([39]\). Thus, \( U[X] \) satisfies \( U_{\text{fin}}(B, B_{\Gamma}) \), and in particular \( U_{\text{fin}}(O, \Gamma) \).

(3 \( \Rightarrow \) 1) By Lemma [13] it suffices to show that \( X \) satisfies \( U_{\text{fin}}(\Gamma, \Gamma) \).

For each \( C = \{C_n : n \in \mathbb{N}\} \in \Gamma \) which does not contain a finite subcover, 
\( \{\bigcup_{m \leq n} C_m : n \in \mathbb{N}\} \in \Gamma \). Thus, \( U_{\text{fin}}(\Gamma, \Gamma) = U_{\text{fin}}(\Gamma, \Gamma) [39] \) and we prove the latter property.

Let \( C_n = \{C^m_n : m \in \mathbb{N}\}, n \in \mathbb{N} \), be bijectively enumerated closed point-cofinite covers of \( X \) which do not contain finite subcovers. We may assume that these covers are pairwise disjoint \([35]\).

Let \( C = \bigcup_n C_n \), and consider the Marczewski map \( C : X \to P(\mathbb{N} \times \mathbb{N}) \) defined by

\[ C(x) = \{(n, m) : x \in C^m_n\} \]

for all \( x \in X \). For each \( (n, m) \), \( O_{(n,m)} = \{A \subseteq \mathbb{N} \times \mathbb{N} : (n, m) \in A\} \) is an open subset of \( P(\mathbb{N} \times \mathbb{N}) \), and for each \( n \), \( U_n = \{O_{(n,m)} : m \in \mathbb{N}\} \) is an open cover of \( C[X] \) that does not contain a finite subcover. As \( C[X] \) satisfies \( U_{\text{fin}}(O, \Gamma) \), there are \( k_n, n \in \mathbb{N} \), such that \( \{\bigcup_{m < k_n} O_{(n,m)} : n \in \mathbb{N}\} \) is a point-cofinite cover of \( C[X] \). Then \( \{\bigcup_{m < k_n} C^m_n : n \in \mathbb{N}\} \) is a point-cofinite cover of \( X \) (it is infinite because \( X \) does not appear there as an element). \footnote{This statement holds in a more general form \([22]\).}

By Lemma [14] \( X \) satisfies \( U_{\text{fin}}(B, B_{\Gamma}) \). It remains to observe the following. For the reader’s convenience, we reproduce the proof of the implication needed in the present proof.

**Lemma 15** (Bartoszyński–Scheepers \([4]\)). \( X \) satisfies \( U_{\text{fin}}(B, B_{\Gamma}) \) if, and only if, each Borel image of \( X \) in \( \mathbb{N}^\mathbb{N} \) is bounded.

**Proof.** (\( \Rightarrow \)) Assume that \( Y \subseteq \mathbb{N}^\mathbb{N} \) is a Borel image of \( X \). Then \( Y \) satisfies \( U_{\text{fin}}(B, B_{\Gamma}) \). By taking the image of \( Y \) under the continuous mapping \( f(n) \mapsto f(1) + \cdots + f(n) \) defined on \( \mathbb{N}^\mathbb{N} \), we may assume that all elements in \( Y \) are nondecreasing.

We first consider the trivial case: There is an infinite \( I \subseteq \mathbb{N} \) such that for each \( n \in I \), \( F_n = \{f(n) : f \in Y\} \) is finite. For each \( n \), let \( m \in I \) be minimal such that \( n \leq m \), and define \( g(n) = \max F_m \). Then \( Y \) is bounded by \( g \).

Thus, assume that there is \( N \) such that for each \( n \geq N \), \( \{f(n) : f \in Y\} \) is infinite. For all \( n, m \), consider the open set \( U^m_n = \{f \in Y : f(n) \leq m\} \).

\footnote{The argument is standard: For each finite family of proper subsets of \( X \), there is a finite subset of \( X \) not contained in any member of this family.}
Then for each \( n \geq N \), \( U_n = \{ U_m^n : m \in \mathbb{N} \} \) is an open point-cofinite cover of \( Y \). Apply \( U_{\text{fin}}(B, B \Gamma) \) to obtain for each \( n \geq N \) a finite set \( F_n \subseteq \mathbb{N} \), such that \( \bigcup_{m \in F_n} U_m^n : n \in \mathbb{N} \) is a point-cofinite cover of \( Y \). Define \( g \in \mathbb{N}^\mathbb{N} \) by \( g(n) = \max F_n \) for each \( n \geq N \) (and arbitrary for \( n < N \)). Then \( Y \) is bounded by \( g \).

This completes the proof of Theorem 9.

Remark 16. Let \( A \subseteq B \) mean that \( A \setminus B \) is finite. A semifilter is a family \( F \) of infinite subsets of \( \mathbb{N} \) such that for each \( A \in F \) and each \( B \subseteq \mathbb{N} \) such that \( A \subseteq B \), we have that \( B \in F \). In [43] it is proved that if in item (2) of Theorem 14 we replace \( U[X] \) with the semifilter it generates, then we obtain a characterization of \( U_{\text{fin}}(O, \Gamma) \). Theorem 14 shows that moving to the generated semifilter is essential to obtain this result, since \( U_{\text{fin}}(B, B \Gamma) \) is strictly stronger than \( U_{\text{fin}}(O, \Gamma) \).

4. Applications

4.1. QN spaces. We begin with a straightforward proof of one implication in the following theorem (which answers in the affirmative Problem 2 of Scheepers [36]). Because of the importance of this result, we also supply a proof for the other implication.

**Theorem 17** (Sakai [33], Bukovský–Haleš [9]). \( X \) is a QN space if, and only if, \( C_p(X) \) is an \( \alpha_1 \) space.

**Proof.** \((\Leftarrow)\) This is Theorem 4 of [36]. Using Theorem 9 this becomes straightforward: Assume that \( C_p(X) \) is an \( \alpha_1 \) space. Given \( f_n, n \in \mathbb{N} \), converging pointwise to 0, define a Borel function \( \Psi : X \to \mathbb{N}^\mathbb{N} \) by

\[
\Psi(x)(n) = \min\{k : \forall m \geq k \ | f_m(x) | < 1/n\}.
\]

By Theorem 9 \( \Psi[X] \) is bounded by some \( g \in \mathbb{N}^\mathbb{N} \). For each \( x \in X \) and all but finitely many \( n \), \( | f_m(x) | < 1/n \) for each \( m \geq g(n) \). For each \( n \) and each \( m \) with \( g(n) \leq m < g(n+1) \), take \( \epsilon_m = 1/n \).

\((\Rightarrow)\) Assume that \( X \) is a QN space, and \( \{ A_n : n \in \mathbb{N} \} \) is a countable sheaf at \( f \in C_p(X) \). We may assume that \( f \) is the zero function, and that the image of each member of each \( A_n \) is contained in the unit interval \([0, 1]\).

For each \( n \), enumerate \( A_n = \{ f_m^n : m \in \mathbb{N} \} \) bijectively. For each \( m \), define \( g_m \in C_p(X) \) by

\[
g_m(x) = \sup\{ f_m^n(x)/n : n \in \mathbb{N} \}
\]

for all \( x \in X \). Then \( \{ g_m : m \in \mathbb{N} \} \) converges pointwise to the constant zero function. As \( X \) is a QN space, there are positive \( \epsilon_m, m \in \mathbb{N} \), converging to 0, such that \( X \) it is the increasing union of the sets

\[
X_n = \{ x \in X : (\forall m \geq n) \ g_m(x) \leq \epsilon_m \}.
\]

For each \( n \), choose \( m_n \) such that \( n \epsilon_m \leq 1/n \) for all \( m \geq m_n \). We claim that the amalgamation \( B = \bigcup_n \{ f_m^n : m \geq m_n \} \) converges pointwise to the zero function. Indeed, fix \( x \in X \) and a positive \( \epsilon \). Take \( N \) such that \( x \) belong
to \(X_N\) (and thus to all \(X_k\) with \(k \geq N\)) and such that \(1/N \leq \epsilon\). For each \(n \geq N\) and each \(m \geq m_n\),
\[f^m_n(x) \leq n \cdot g_m(x) \leq n \epsilon_m \leq 1/n \leq \epsilon.\]
And for each \(n < N\), there are only finitely many \(m\) such that \(f^m_n(x) > \epsilon\).

Thus, for all but finitely many \(f \in B\), \(f(x) \leq \epsilon\).

A beautiful direct (but tricky) proof for \((\Leftarrow)\) of Theorem 17 was recently discovered by Bukovský [7].

Theorems 9 and 17 solve in the affirmative Problem 22 from [9].

**Corollary 18.** \(X\) is a QN space if, and only if, each Borel image of \(X\) in \(\mathbb{N}^\mathbb{N}\) is bounded.

**Theorem 19** (Recawlaw [30]). The QN property is hereditary.

**Proof.** The property of having bounded Borel images in \(\mathbb{N}^\mathbb{N}\) is hereditary. \(\square\)

Answering Question 5.8 of Shakhmatov [40] (attributed to Scheepers), Sakai [33] and independently Bukovský–Hales [9] gave a characterization of the QN property in terms of covering properties of \(X\). Their characterization uses the new Kočinac \(\alpha_1\) selection principle [25]. Theorem 15 and Corollary 18 give a new characterization in terms of the classical Hurewicz selection principle: \(U_{\text{fin}}(B, B_T)\). This selection hypothesis can be stated in a more elegant manner. For families of covers \(\mathcal{A}, \mathcal{B}\) of \(X\), define
\[S_1(\mathcal{A}, \mathcal{B})\]: Whenever \(U_1, U_2, \ldots \in \mathcal{A}\), there exist elements \(U_n \in U_n, n \in \mathbb{N}\), such that \(\{U_n : n \in \mathbb{N}\} \in \mathcal{B}\).

Then \(U_{\text{fin}}(B, B_T) = S_1(B_T, B_T)\) [30]. By Lemma 13 also \(U_{\text{fin}}(F, F_T) = S_1(F_T, F_T)\). (This can also be proved directly.) We obtain the following new characterizations.

**Corollary 20.** The following are equivalent:

1. \(C_p(X)\) is an \(\alpha_1\) space;
2. \(X\) is a QN space;
3. \(X\) satisfies \(S_1(F_T, F_T)\);
4. \(X\) satisfies \(S_1(B_T, B_T)\). \(\square\)

4.2. **Convergent sequences of Borel functions.** Let \(B_p(X)\) be the space of all Borel real-valued functions on \(X\), with the topology of pointwise convergence. We obtain the surprising result, that if \(C_p(X)\) is an \(\alpha_1\) space, then so is \(B_p(X)\). This is not provably the case for Arhangel’skiǐ’s properties \(\alpha_2\), \(\alpha_3\), and \(\alpha_4\). A topological space \(Y\) is an \(\alpha_2\) space if it satisfies \(S_1(\Gamma_y, \Gamma_y)\) for each \(y \in Y\), where \(\Gamma_y\) is the family of all sequences converging to \(y\). For the definitions of \(\alpha_3\) and \(\alpha_4\), see e.g. [36].

**Corollary 21.** The following are equivalent:

1. Each Borel image of \(X\) in \(\mathbb{N}^\mathbb{N}\) is bounded;
2. \(C_p(X)\) is an \(\alpha_1\) space;
3. \(B_p(X)\) is an \(\alpha_1\) space;
(4) $B_p(X)$ is an $\alpha_2$ space;
(5) $B_p(X)$ is an $\alpha_3$ space;
(6) $B_p(X)$ is an $\alpha_4$ space.

**Proof.** (1 $\Rightarrow$ 3) This is proved verbatim as the proof of (2 $\Rightarrow$ 1) in Theorem 9.
(3 $\Rightarrow$ 2) is evident.
(2 $\Rightarrow$ 1) is due to the mentioned result of Scheepers, and the equivalence of being a QN space and (1).
(4 $\Leftrightarrow$ 5 $\Leftrightarrow$ 6) is proved as in Gerlits–Nagy [18] or Scheepers’ [36] (in fact, the Borel case is easier).
(3 $\Rightarrow$ 4) is evident.
(4 $\Rightarrow$ 1) It suffices to show that $X$ satisfies $S_1(B_\Gamma, B_\Gamma)$. Given $U_n \in B_\Gamma$, $n \in \mathbb{N}$, we have that for each $n$, $A_n = \{ \chi_U : U \in U_n \} \subseteq B_p(X)$ converges pointwise to 0. Applying $\alpha_2$, let $U_n \in U_n$, $n \in \mathbb{N}$, be such that $\chi_{U_n}$ converges pointwise to 0. Then $\{ U_n : n \in \mathbb{N} \}$ is a point-cofinite cover of $X$. □

4.3. *Almost continuous functions.* A function $f : X \to Y$ is *almost continuous* [3] if for each nonempty $A \subseteq X$, the restriction of $f$ to $A$ has a point of continuity. $AC_p(X)$ is the space of all almost continuous real valued functions on $X$, with the topology of pointwise convergence [5].

If $X$ and $Y$ are Tychonoff and $f : X \to Y$ is almost continuous, then for each $A \subseteq X$ the set of points of continuity of the restriction of $f$ to $A$ is open dense in $A$ [5]. Each function with the latter property is Borel [42]. Thus, $C_p(X) \subseteq AC_p(X) \subseteq B_p(X)$.

**Corollary 22.** $AC_p(X)$ is an $\alpha_1$ space if, and only if, $C_p(X)$ is an $\alpha_1$ space.

4.4. *wQN spaces and the Scheepers Conjecture.* $X$ is a *wQN space* [10] if each sequence of continuous real-valued functions on $X$ converging pointwise to zero has a subsequence converging to zero quasi-normally.

Two fundamental problems concerning wQN spaces appear in the literature: In [36, page 269], [9, Problem 23], and [6, Problems 10.3–10.4], we are asked whether, consistently, every wQN space is a QN space. The *Scheepers Conjecture* [37] asserts that $X$ is a wQN space if, and only if, $X$ satisfies $S_1(\Gamma, \Gamma)$. It is still open whether the Scheepers Conjecture is provable. A striking result of Dow gives a positive answer to the first problem, and a consistently positive answer to the second.

**Theorem 23 (Dow [14]).** In the Laver model, each $\alpha_2$ space is an $\alpha_1$ space.

Let $C_\Gamma$ denote the family of all clopen point-cofinite covers of $X$. Clearly, $S_1(\Gamma, \Gamma)$ implies $S_1(C_\Gamma, C_\Gamma)$.

**Corollary 24.** In the Laver model:
(1) $S_1(B_\Gamma, B_\Gamma) = S_1(C_\Gamma, C_\Gamma)$.
(2) $X$ is a wQN space if, and only if, $X$ is a QN space.
(3) The Scheepers Conjecture holds.
In particular, these assertions are (simultaneously) consistent.

**Proof.** (1) Using the correspondence described just before Lemma 10, we have that $C_p(X)$ is an $\alpha_2$ space if, and only if, $X$ satisfies $S_1(C_\Gamma, C_\Gamma)$. Thus, if $X$ satisfies $S_1(C_\Gamma, C_\Gamma)$, then by Dow’s Theorem 23, $C_p(X)$ is an $\alpha_1$ space. By Theorem 9, $X$ satisfies $S_1(B_\Gamma, B_\Gamma)$.

(2) Assume that $X$ is a wQN space. Then $C_p(X)$ is an $\alpha_2$ space. By Dow’s Theorem 23, $C_p(X)$ is an $\alpha_1$ space. By Theorem 17, $X$ is a QN space.

(3) $S_1(\Gamma, \Gamma)$ implies (in ZFC) being a wQN space. Now, back in the Laver model, assume that $X$ is a wQN space. By (2), $X$ is a QN space. By Corollary 20, $X$ satisfies $S_1(B_\Gamma, B_\Gamma)$, and in particular $S_1(\Gamma, \Gamma)$.

**Remark 25.** In [33, 9] it is shown that $X$ is a wQN space if, and only if, $X$ satisfies $S_1(C_\Gamma, C_\Gamma)$. Using this, (2) and (3) follow immediately from Corollary 24(1).

### 4.5. QN spaces and M spaces.

$X$ is a **QN** space [11] if each real-valued function (not necessarily continuous) on $X$ which is a pointwise limit of a sequence of continuous functions, is in fact a quasi-normal limit of those functions.

The following result is immediate from Theorem 9 and [11, Theorem 5.10(9)]. For completeness, we give a simple, direct proof.

**Theorem 26.** The following are equivalent:

1. $X$ is a QN space;
2. $X$ is a QN space;
3. Each sequence of Borel functions converging pointwise to 0, converges to 0 quasi-normally;
4. Each sequence of Borel functions converging pointwise to any function, converges quasi-normally to this function.

**Proof.** (1 ⇒ 2) is immediate.

(2 ⇒ 3) Assume (2). By Theorem 9, each Borel image of $X$ in $\mathbb{N}^\mathbb{N}$ is bounded. Thus, an argument verbatim as in our proof of ($\Leftarrow$) of Theorem 9 gives (3).

(3 ⇒ 4) the limit function $f$ is also Borel, and $f_n - f$ converges to 0.

(4 ⇒ 1) is immediate. □

This shows that the first assumption in [11, Theorem 5.10(9)] is not needed. It also answers [11, Problem 6.11] in the positive. Based on [11, Theorem 6.9] and improving it, we also obtain the following solution of [11, Problem 6.10].

**Corollary 27.** Every QN space is an M space. □

The definition of M space is available at [11].
4.6. **wQN* spaces.** A space $X$ is wQN* if each sequence of lower semi-continuous real-valued functions on $X$ converging pointwise to zero has a subsequence converging to zero quasi-normally. In his talk at the Third Workshop on Coverings, Selections, and Games in Topology (Serbia, April 2007), Bukovský defined wQN* spaces and described his recent investigations of this property and its upper semi-continuous variant. The main problem he posed was: Is every QN space a wQN* space?

**Theorem 28.** Every QN space is a wQN* space.

**Proof.** Every lower semi-continuous function is Borel. Use Theorem 26. □

Bukovský has later proved that the converse implication in Theorem 28 also holds [7], and therefore the notions coincide (with one another and with having bounded Borel images).

4.7. **Bounded-ideal convergence spaces.** The notion of ideal convergence originates in works of Steinhaus and Fast on statistical convergence, and was generalized by Bernstein, Katětov, and others (see [16] for an introduction). The following definitions are as in Jasinski–Recław [20]. Let $D$ be a countable set, and $I \subseteq P(D)$ be an ideal (i.e., $I$ contains all singletons and is closed under taking subsets and finite unions). $I^*$ denotes the filter $\{D \setminus A : A \in I\}$ dual to $I$. A sequence $\{r_d\}_{d \in D}$ of real numbers $I$-converges to 0 if for each positive $\epsilon$, $\{d \in D : |r_d| < \epsilon\} \in I^*$. A sequence $\{f_d\}_{d \in D}$ of continuous real-valued functions on $X$ $I$-converges to 0 if for each $x \in X$, the sequence of real numbers $\{f_d(x)\}_{d \in D}$ $I$-converges to 0. A space $X$ has the $I$-convergence property if for each sequence $\{f_d\}_{d \in D}$ of continuous real-valued functions on $X$ which $I$-converges to 0, there is $A \in I^*$ such that $\{f_d\}_{d \in A}$ converges pointwise to 0.

We will use the following.

**Lemma 29.** In the definition of the $I$-convergence property, it suffices to consider only sequences of distinct elements.

**Proof.** Let $\{f_d\}_{d \in D}$ be given. Enumerate $D = \{d_n : n \in \mathbb{N}\}$ bijectively. For each $n$, as the functions $f_{d_n} + 1/m$, $m \in \mathbb{N}$, are all distinct, there is $m(d_n) \in \mathbb{N}$ such that $m(d_n) \geq n$ and $f_{d_n} + 1/m(d_n) \notin \{f_{d_1} + 1/m(d_1), \ldots, f_{d_{n-1}} + 1/m(d_{n-1})\}$.

It is easy to see that $\{f_d\}_{d \in D}$ $I$-converges to 0 if, and only if, $\{f_d + 1/m(d)\}_{d \in D}$ $I$-converges to 0. □

We use these definitions for $D = \mathbb{N} \times \mathbb{N}$. For $h \in \mathbb{N}^\mathbb{N}$, define $A_h = \{(n, m) : m \leq h(n)\}$. The family $\{A_h : h \in \mathbb{N}^\mathbb{N}\}$ is closed under finite intersections, and generates the **bounded-ideal**

$$I_h = \{B \subseteq \mathbb{N} \times \mathbb{N} : (\exists h \in \mathbb{N}^\mathbb{N}) B \subseteq A_h\}.$$  

$X$ has the **bounded-ideal convergence property** if it has the $I_h$-convergence property.
The bounded-ideal, which is also the Fubini product $\emptyset \times \text{Fin}$ of the trivial ideal and the ideal of finite sets, plays a central role in studies of ideal convergence. For each analytic $P$-ideal $\mathcal{I}$, if any $X \subseteq \mathbb{R}$ not having Lebesgue measure zero has the $\mathcal{I}$ ideal convergence, then $\mathcal{I}$ is isomorphic to $\mathcal{I}_b$ [21]. For additional uses of this ideal and its associated convergence, see [16].

Jasinski and Recław [20] proved that every Sierpiński set has the bounded-ideal convergence property, and that if $X$ has the bounded-ideal convergence property, then $X$ is a $\sigma$ space. Both of these assertions follow at once from the following.

**Theorem 30.** The following are equivalent:

1. $X$ has the bounded-ideal convergence property;
2. $C_p(X)$ is an $\alpha_1$ space;
3. Each Borel image of $X$ in $\mathbb{N}^\mathbb{N}$ is bounded.

**Proof.** By Theorem [9] it suffices to show that $(1 \iff 2)$.

(2 $\Rightarrow$ 1) Assume that $\{f_{(n,m)}\}_{(n,m) \in \mathbb{N} \times \mathbb{N}}$ $\mathcal{I}_b$-converges to 0. By Lemma [29] we may assume that the elements $f_{(n,m)}$, $(n, m) \in \mathbb{N} \times \mathbb{N}$, are distinct.

For each $x \in X$ and each positive $\epsilon$, $\{(n, m) : |f_{(n,m)}(x)| < \epsilon\} \in \mathcal{I}_b^*$, that is, there is $h \in \mathbb{N}^\mathbb{N}$ such that $\{(n, m) : |f_{(n,m)}(x)| < \epsilon\} \subseteq (\mathbb{N} \times \mathbb{N}) \setminus A_h$. Thus, $|f_{(n,m)}(x)| < \epsilon$ for all $n, m \in \mathbb{N}$ such that $h(n) < m$. It follows that for each $n$, $\{f_{(n,m)}\}_{m \in \mathbb{N}}$ converges pointwise to 0.

As $C_p(X)$ is an $\alpha_1$ space, there is for each $n$ a number $h(n) \in \mathbb{N}$ such that $\{f_{(n,m)} : n, m \in \mathbb{N}, m > h(n)\}$ converges pointwise to 0, and since its enumeration is bijective, the sequence $\{f_{(n,m)}\}_{(n,m) \in (\mathbb{N} \times \mathbb{N}) \setminus A_h}$ also converges pointwise to 0. As $(\mathbb{N} \times \mathbb{N}) \setminus A_h \in \mathcal{I}_b^*$, this shows that $X$ has the bounded-ideal convergence property.

(1 $\Rightarrow$ 2) Assume that for each $n$ the sequence $\{f_{(n,m)}\}_{m \in \mathbb{N}}$ converges pointwise to 0. For each $x \in X$, each positive $\epsilon$, and each $n$, there is $h(n) \in \mathbb{N}$ such that $|f_{(n,m)}(x)| < \epsilon$ for all $m > h(n)$. Thus, $\{(n, m) : |f_{(n,m)}(x)| < \epsilon\} \supseteq (\mathbb{N} \times \mathbb{N}) \setminus A_h$, that is, $\{f_{(n,m)}\}_{(n,m) \in \mathbb{N} \times \mathbb{N}}$ $\mathcal{I}_b$-converges to 0.

By the bounded-ideal convergence property, there is $h \in \mathbb{N}^\mathbb{N}$ such that $\{f_{(n,m)} : n, m \in \mathbb{N}, m > h(n)\}$ converges pointwise to 0, and therefore so does the sheaf amalgamation $\{f_{(n,m)} : n, m \in \mathbb{N}, n > h(n)\}$ (which can be enumerated as a subsequence of $\{f_{(n,m)}\}_{(n,m) \in (\mathbb{N} \times \mathbb{N}) \setminus A_h}$).

**Corollary 31.** The following are equivalent:

1. $X$ has the bounded-ideal convergence property;
2. For each sequence $\{f_d\}_{d \in \mathbb{N} \times \mathbb{N}}$ of Borel real-valued functions on $X$ which $\mathcal{I}_b$-converges to 0, there is $A \in \mathcal{I}_b^*$ such that $\{f_d\}_{d \in A} \mathcal{I}_b$-converges pointwise to 0;
3. For each sequence $\{f_d\}_{d \in \mathbb{N} \times \mathbb{N}}$ of Borel real-valued functions on $X$ which $\mathcal{I}_b$-converges to a Borel function $f$, there is $A \in \mathcal{I}_b^*$ such that $\{f_d\}_{d \in A}$ $\mathcal{I}_b$-converges pointwise to $f$. 

□
For each sequence \( \{ f_d \}_{d \in \mathbb{N} \times \mathbb{N}} \) of Borel real-valued functions on \( X \) which \( I_b \)-converges to a function \( f \), there is \( A \in I_b^* \) such that \( \{ f_d \}_{d \in A} \) converges pointwise to \( f \).

**Proof.** (1 \( \Rightarrow \) 2) Replace “continuous” by “Borel” in the proof of Theorem 30 and use Theorem 21.

(2 \( \Rightarrow \) 3) \( B_p(X) \) is a topological group, and in particular homogeneous.

(3 \( \Rightarrow \) 4) The assumption in (4) implies, in particular, that \( f \) is a pointwise limit of \( \{ f_{(1,m)} \}_{m \in \mathbb{N}} \). Thus, \( f \) is Borel. \( \square \)

### 4.8. Bounded Baire-class \( \alpha \) images.

Continuous functions and Borel functions are the extremal notions in the Baire hierarchy of functions: A real-valued function \( f \) is of Baire-class 0 if it is continuous. For \( 0 < \alpha \leq \aleph_1 \), \( f \) is of Baire-class \( \alpha \) if \( f \) is the pointwise limit of a sequence of functions, each of Baire-class smaller than \( \alpha \). \( f \) is Borel if, and only if, \( f \) is of Baire class \( \aleph_1 \) (see [23]). A natural question in light of our study is: Which spaces \( X \) have the property that each Baire-class \( \alpha \) image of \( X \) in \( \mathbb{N}^\mathbb{N} \) is bounded?

**Theorem 32.** For each \( \alpha > 0 \), the following are equivalent:

1. Each Baire-class \( \alpha \) image of \( X \) in \( \mathbb{N}^\mathbb{N} \) is bounded;
2. Each Borel image of \( X \) in \( \mathbb{N}^\mathbb{N} \) is bounded.

**Proof.** Assume that each Baire-class 1 image of \( X \) in \( \mathbb{N}^\mathbb{N} \) is bounded. Baire-class 1 functions are exactly the \( F_\sigma \)-measurable functions.

One way to proceed is using Lemma 14, since for each bijectively enumerated family of closed sets \( C = \{ C_n : n \in \mathbb{N} \} \), the corresponding Marczewski function is \( F_\sigma \)-measurable (and by the proof of Lemma 14 we may assume that for each \( x \in X \), \( C(x) \) is infinite).

However, there is a more direct proof. By Lemma 13 it suffices to prove that \( X \) satisfies \( U_{\text{fin}}(F, F_\Gamma) \). Assume that \( \mathcal{U}_n = \{ C^n_m : m \in \mathbb{N} \}, n \in \mathbb{N} \), are closed covers of \( X \) not containing a finite subcover. Define \( \Psi : X \to \mathbb{N}^\mathbb{N} \) by

\[
\Psi(x)(n) = \min\{m : x \in C^n_m\}
\]

for all \( n \in \mathbb{N} \). Each basic open subset of \( \mathbb{N}^\mathbb{N} \) is an intersection of finitely many sets of the form \( O^n_m = \{ f \in \mathbb{N}^\mathbb{N} : f(n) = m \} \). As \( \Psi^{-1}[O^n_m] = C^n_m \setminus \bigcup_{k < m} C^n_k \) is an \( F_\sigma \) set for all \( n \) and \( m \), \( \Psi \) is \( F_\sigma \)-measurable. Thus, \( \Psi[X] \) is bounded by some \( g \in \mathbb{N}^\mathbb{N} \). Then \( \bigcup_{m \leq g(n)} C^n_m : n \in \mathbb{N} \) is a point-cofinite cover of \( X \). \( \square \)

### 5. Closing the circle: Continuous bounded images again

The proof of Theorem 9 gives us the following analogue of Theorem 2.

Say that a set \( Y \subseteq \mathbb{N}^\mathbb{N} \) is bounded if there is \( g \in \mathbb{N}^\mathbb{N} \) such that for each \( f \in Y \) and all but finitely many \( n \), \( f(n) < \infty \) implies \( f(n) \leq g(n) \). This generalizes the standard notions of boundedness in \( \mathbb{N}^\mathbb{N} \) or \( EF \).

**Theorem 33.** The following are equivalent:

1. Each Borel image of \( X \) in \( \mathbb{N}^\mathbb{N} \) is bounded;
(2) Each continuous image of $X$ in $\mathbb{N}^\mathbb{N}$ is bounded.

Proof. (1 $\Rightarrow$ 2) Assume that $\Psi : X \to \mathbb{N}^\mathbb{N}$ is continuous. Define $d : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $d(x)(n) = x(n)$ if $x(n) < \infty$, and $d(x)(n) = 1$ if $x(n) = \infty$. Then $d \circ \Psi : X \to \mathbb{N}^\mathbb{N}$ is Borel, and therefore $d[\Psi[X]]$ is a bounded subset of $\mathbb{N}^\mathbb{N}$. Thus, $\Psi[X]$ is a bounded subset of $\mathbb{N}^\mathbb{N}$.

(2 $\Rightarrow$ 1) Assume that each continuous image of $X$ in $\mathbb{N}^\mathbb{N}$ is bounded. We first prove that for each bijectively enumerated family $U = \{U_n : n \in \mathbb{N}\}$ of open sets, $U$ is a discrete limit of continuous functions. The proof is similar to the proof of Lemma 11. As shown at the end of the proof of Lemma 11, we may assume that no $U_n$ is clopen.

For each $n$, write $U_n = \bigcup C_{n,i}$ as a union of disjoint clopen sets. Define $\Psi : X \to \mathbb{N}^\mathbb{N}$ by

$$
\Psi(x)(n) = \begin{cases} 
  m & x \in U_n^m \\
  \infty & x \notin U_n
\end{cases}
$$

Then $\Psi$ is continuous. Let $g \in \mathbb{N}^\mathbb{N}$ bound $\Psi[X]$. For each $n,m$, let

$$
U_n^m = \max\{m,g(n)\} \bigcup_{i=1}^{m,g(n)} C_{n,i}.
$$

For each $m$, define a continuous function $\Psi_m : X \to P(\mathbb{N})$ by

$$
\Psi_m(x) = \{n : x \in U_n^m\}.
$$

We claim that $U$ is a discrete limit of the maps $\Psi_m$, $m \in \mathbb{N}$.

Fix $x \in X$. Let $N$ be such that for all $n \geq N$, $\Psi(x)(n) < \infty$ implies $\Psi(x)(n) \leq g(n)$. For each $n < N$ with $x \in U_n$, let $m_n$ be such that $x \in U_n^{m_n}$. Set $M = \max\{m_n : n < N\}$.

Fix $m \geq M$. We show that $n \in \Psi_m(x)$ if, and only if, $x \in U_n$. One direction follows from $U_n^m \subseteq U_n$. To prove the other direction, assume that $x \in U_n$, and consider the two possible cases: If $n < N$, then $x \in U_n^m$ because $m \geq M \geq m_n$, and we are done. Thus, assume that $n \geq N$. Then $\Psi(x)(n) \leq g(n)$, and therefore $x \in \bigcup_{i=1}^{g(n)} C_{n,i} \subseteq U_n^m$.

Thus, for each $x \in X$ there is $M$ such that for all $m \geq M$, $\Psi_m(x) = U(x)$. Now, each continuous image of $X$ in $\mathbb{N}^\mathbb{N}$ is bounded because $\mathbb{N}^\mathbb{N}$ is a subspace of $\mathbb{N}^\mathbb{N}$. By the Hurewicz Theorem $X$ satisfies $U_{\text{fin}}(O,\Gamma)$, and by Lemma 12, so does $U[X]$. By Lemma 14, each Borel image of $X$ in $\mathbb{N}^\mathbb{N}$ is bounded.

We therefore obtain the aesthetically pleasing result, that the chain of properties

$$
U_{\text{fin}}(B,B_{\Gamma}) \quad \Rightarrow \quad \text{hereditarily-$U_{\text{fin}}(O,\Gamma)$} \quad \Rightarrow \quad U_{\text{fin}}(O,\Gamma)
$$

is obtained by requiring bounded continuous images in the chain of subspaces

$$
\mathbb{N}^\mathbb{N} \supseteq \text{EF} \supseteq \mathbb{N}^\mathbb{N},
$$
respectively.

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Note. After the present paper was accepted for publication, Bukovský and Šupina devised an alternative, more analytic and less combinatorial, proof of our main Theorem [9] [12].

References

[1] A. Arhangel’ski˘ı, The frequency spectrum of a topological space and the classification of spaces, Soviet Mathematical Doklady 13 (1972), 1186–1189.
[2] A. Arhangel’ski˘ı, The frequency spectrum of a topological space and the product operation, Transactions of the Moscow Mathematical Society 40 (1979), 171–206.
[3] A. Arhangel’ski˘ı, B. M. Bokalo, The tangency of topologies and tangential properties of topological spaces, Transactions of the Moscow Mathematical Society 1993, 139–163.
[4] T. Bartoszyński and M. Scheepers, A-sets, Real Analysis Exchange 19 (1993/94), 521–528.
[5] B. Bokalo, O. Malanyuk, Some properties of topological spaces of almost continuous mappings, Matematychni Studii 14 (2000), 197–201.
[6] L. Bukovský, Convergence of real functions and covering properties, in: Selection Principles and Covering Properties in Topology (L. Kočinac ed.), Quaderni di Matematica 18, Seconda Universita di Napoli, Caserta, 2006, 107–132.
[7] L. Bukovský, On wQN∗ and wQN∗ spaces, Topology and its Applications 156 (2008), 24–27.
[8] L. Bukovský and J. Haleš, On Hurewicz properties, Topology and its Applications 132 (2003), 71–79.
[9] L. Bukovský and J. Haleš, QN-space, wQN-space and covering properties, Topology and its Applications 154 (2007), 848–858.
[10] L. Bukovský, I. Reclaw, and M. Repický, Spaces not distinguishing pointwise and quasinormal convergence of real functions, Topology and its Applications 41 (1991), 25–41.
[11] L. Bukovský, I. Reclaw, M. Repický, Spaces not distinguishing convergences of real-valued functions, Topology and its Applications 112 (2001), 13–40.
[12] L. Bukovský and J. Šupina, Sequence selection principles for quasi-normal convergence, Topology and its Applications 159 (2012), 283–289.
[13] H. Chen, Compact-covering maps and k-networks, Proceedings of the American Mathematical Society 131 (2003), 2623–2632.
[14] A. Dow, Two classes of Fréchet-Urysohn spaces, Proceedings of the American Mathematical Society 108 (1990), 241–247.
[15] D. Fremlin and A. Miller, On some properties of Hurewicz, Menger and Rothberger, Fundamenta Mathematica 129 (1988), 17–33.
[16] R. Filipów, N. Mrówka, I. Reclaw, and P. Szuca, Ideal convergence of bounded sequences, Journal of Symbolic Logic 72 (2007), 501–512.
[17] J. Gerlits and Zs. Nagy, Some properties of C(X), I, Topology and its Applications 14 (1982), 151–161.
[18] J. Gerlits and Zs. Nagy, *On Frechet spaces*, Rendiconti del Circolo Matematico di Palermo, serie ii, supplm. 18 (1988), 51–71.
[19] W. Hurewicz, *Über Folgen stetiger Funktionen*, Fundamenta Mathematicae 9 (1927), 193–204.
[20] J. Jasinski and I. Recław, *Ideal convergence of continuous functions*, Topology and its Applications 153 (2006), 3511–3518.
[21] J. Jasinski and I. Recław, *Spaces with the ideal convergence property*, Colloquium Mathematicum 111 (2008), 43–50.
[22] W. Just, A. Miller, M. Scheepers, and P. Szeptycki, *The combinatorics of open covers II*, Topology and its Applications 73 (1996), 241–266.
[23] A. Kechris, *Classical Descriptive Set Theory*, Graduate Texts in Mathematics 156, Springer-Verlag, 1994.
[24] Lj. Kočinac, *Selected results on selection principles*, in: *Proceedings of the 3rd Seminar on Geometry and Topology* (Sh. Rezapour, ed.), July 15–17, Tabriz, Iran, 2004, 71–104.
[25] Lj. Kočinac, *Selection principles related to α₁-properties*, Taiwanese Journal of Mathematics 12 (2008), 561–572.
[26] K. Menger, *Eindeckeungssätze der Punktmengelehre*, Sitzungsberichte der Wiener Akademie 133 (1924), 421–444.
[27] A. Miller, *A hodgepodge of sets of reals*, Note di Matematica 27 (2007), 25–39.
[28] A. Nowik, *Additive properties and Uniformly Completely Ramsey sets*, Colloquium Mathematicum 82 (1999), 191–199.
[29] I. Recław, *Every Luzin set is undetermined in the point-open game*, Fundamenta Mathematicae 144 (1994), 43–54.
[30] I. Recław, *Metric spaces not distinguishing pointwise and quasinormal convergence of real functions*, Bulletin of the Polish Academy of Sciences 45 (1997), 287–289.
[31] M. Rubin, *A Boolean algebra with few subalgebras, interval Boolean algebras and retractionness*, Transactions of the American Mathematical Society 278 (1983), 65–89.
[32] M. Sakai, *A special subset of the real line and regularity of weak topologies*, Topology Proceedings 23 (1998), 281–287.
[33] M. Sakai, *The sequence selection properties of C_p(X)*, Topology and its Applications 154 (2007), 552–560.
[34] M. Scheepers, *A sequential property of C_p(X) and a covering property of Hurewicz*, Proceedings of the American Mathematical Society 125 (1997), 2789–2795.
[35] M. Scheepers, *Combinatorics of open covers I: Ramsey theory*, Topology and its Applications 69 (1996), 31–62.
[36] M. Scheepers, *C_p(X) and Arhangel’skii’s α₁ spaces*, Topology and its Applications 89 (1998), 265–275.
[37] M. Scheepers, *Sequential convergence in C_p(X) and a covering property*, East-West Journal of Mathematics 1 (1999), 207–214.
[38] M. Scheepers, *Selection principles and covering properties in topology*, Note di Matematica 22 (2003), 3–41.
[39] M. Scheepers and B. Tsaban, *The combinatorics of Borel covers*, Topology and its Applications 121 (2002), 357–382.
[40] D. Shakhmatov, *Convergence in the presence of algebraic structure*, in: *Recent progress in general topology, II*, 463–484, North-Holland, Amsterdam, 2002.
[41] B. Tsaban, *Some new directions in infinite-combinatorial topology*, in: *Set Theory* (J. Bagaria and S. Todorcevic, eds.), Trends in Mathematics, Birkhäuser, 2006, 225–255.
[42] V. Vinokurov, *Strong regularizability of discontinuous functions*, Soviet Mathematical Doklady 281 (1985), 265–269.
[43] L. Zdomskyy, A semifilter approach to selection principles, Commentationes Mathematicae Universitatis Carolinae 46 (2005), 525–539.

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