Uniqueness of the $[\varphi, \vec{e}_3]$-catenary cylinders by their asymptotic behaviour

A.L. Martínez-Triviño and J.P. dos Santos

August 5, 2021

Abstract

We establish a uniqueness result for the $[\varphi, \vec{e}_3]$-catenary cylinders by their asymptotic behaviour. Well known examples of such cylinders are the grim reaper translating solitons for the mean curvature flow. For such solitons, F. Martín, J. Pérez-García, A. Savas-Halilaj and K. Smoczyk proved that, if $\Sigma$ is a properly embedded translating soliton with locally bounded genus, and $C^\infty$-asymptotic to two vertical planes outside a cylinder, then $\Sigma$ must coincide with some grim reaper translating soliton. In this paper, applying the moving plane method of Alexandrov together with a strong maximum principle for elliptic operators, we increase the family of $[\varphi, \vec{e}_3]$-minimal graphs where these types of results hold under different assumption of asymptotic behaviour.

2020 Mathematics Subject Classification: 53C42, 35J60

Keywords: $\varphi$-minimal surface, $\varphi$-catenary cylinder, uniqueness, asymptotic behaviour.

1 Introduction

Let $\varphi : \Omega \subset \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and $\Omega$ an open subset of $\mathbb{R}^3$. An orientable immersion $\Sigma$ in $\Omega$ is called $\varphi$-minimal if and only if the mean curvature $H$ verifies the following equation

\begin{equation}
H = -\langle \nabla \varphi, N \rangle,
\end{equation}
where $N$ is the Gauss map, $\nabla$ denotes the usual gradient in $\mathbb{R}^3$ and $\langle \cdot , \cdot \rangle$ stands the usual Euclidean metric. From the work of T.Ilmanen [7], any $\varphi$-minimal surface can be viewed as a minimal surface in a conformal Riemannian 3-manifold $\Omega^\varphi := (\Omega , \langle \cdot , \cdot \rangle^\varphi)$ whose metric is defined for any $p \in \Omega^\varphi$ by

$$\langle \cdot , \cdot \rangle_p^\varphi := e^{\varphi(p)} \langle \cdot , \cdot \rangle_p.$$ 

Moreover, they can be viewed as critical points of the weighted volume functional,

$$\mathcal{A}^\varphi(\Sigma) = \int_{\Sigma} e^\varphi d\Sigma,$$

where $d\Sigma$ is the induced volume element of $\Sigma$.

In this paper we are interested in the case when the function $\varphi$ is invariant under a two-parameter group of translations. Up to rigid motions, if we write $p = (x_1, x_2, x_3)$ we can assume $\varphi$ depending only on $x_3$. Specifically, we will consider $\Omega$ as an open subset $\mathbb{R}^2 \times I$ and $\varphi : I \subset \mathbb{R} \to \mathbb{R}$ a smooth function. In this case, the equation (1.1) is written as

$$(1.2) \qquad H = -\dot{\varphi}(\vec{e}_3, N),$$

where $\{\vec{e}_i\}_{i=1,2,3}$ is the usual orthonormal frame of $\mathbb{R}^3$ and $(\cdot)$ stands for the derivative with respect to $x_3$. A surface $\Sigma \subset \Omega$ which satisfies the equation (1.2) will be called $[\varphi, \vec{e}_3]$-minimal surface.

Some particular cases of these surfaces have been the key in the development of some issues of the differential geometry theory. We highlight the works [4, 5, 9, 14, 15] for Translating soliton for the mean curvature flow, when $\varphi$ is the identity and the works [2, 3, 8] for singular $\alpha$-minimal surfaces, when $\varphi(x_3) = \alpha \log(x_3)$ for $\alpha \neq 0$. In fact, the special case $\alpha = 1$, from a physically point of view [13], represents a membrane with intrinsic force vanishes under a gravitational field.

Next, consider a smooth function $u : \mathcal{O} \subset \mathbb{R}^2 \to \mathbb{R}$. We say that Graph$[u]$ is a $[\varphi, \vec{e}_3]$-minimal graph if and only if it solves the following elliptic equation

$$(1.3) \qquad (1 + u_x^2)u_{yy} + (1 + u_y^2)u_{xx} - 2u_xu_yu_{xy} = \dot{\varphi}(u)(1 + u_x^2 + u_y^2),$$

where $u_x$ and $u_y$ denotes the partial derivative with respect to the first and second coordinate, respectively. As consequence of the ellipticity of (1.3), the Hopf’s maximum principle hold [6]. The main class of $[\varphi, \vec{e}_3]$-minimal graphs that we will use in this paper are the $[\varphi, \vec{e}_3]$-catenary cylinders described in [11]. In fact, from the Theorem 3.7 of [11], they are the only complete flat examples together with the vertical planes and the tilted-$[\varphi, \vec{e}_3]$-catenary cylinders.

Let $\varphi : ]a, +\infty[ \to ]b, c]$ with $a, b \in \mathbb{R} \cup \{-\infty\}$ and $c \in \mathbb{R} \cup \{+\infty\}$ be an strictly monotone diffeomorphism. A $[\varphi, \vec{e}_3]$-catenary cylinder $\mathcal{C}^\varphi$ is given by the cartesian product Graph$[u] \times \mathbb{R}$ where $u$ only depends of one variable, $u = u(x)$, and from (1.3), it is solution of the following Cauchy’s problem

$$(1.4) \begin{cases} u''(x) = \dot{\varphi}(u)(1 + u'(x)^2) \\ u(0) = h, \quad u'(0) = 0. \end{cases}$$

2
The solution \( u \) of (1.4) is even and it is defined in the interval \([-\Lambda_h, \Lambda_h]\) with \( \Lambda_h \in \mathbb{R} \cup \{+\infty\} \). The main properties and the asymptotic behaviour of these surfaces can be summarized in the following two results, proved in Section 3 of [11].

**Theorem 1.1.** Let \( \varphi : \mathbb{R} \cup \{a, +\infty\} \rightarrow [b, c] \), \( a, b \in \mathbb{R} \cup \{-\infty\}, c \in \mathbb{R} \cup \{+\infty\} \) be a strictly increasing diffeomorphism, then the solution \( u \) of (1.4) is defined in \([-\Lambda_h, \Lambda_h]\), \( \Lambda_h \in \mathbb{R} \cup \{+\infty\} \), it is convex, symmetric about the \( y \)-axis and has a minimum at \( x = 0 \). Moreover,

1. if \( c < \infty \), then \( \Lambda_h = \infty \) and,
   \[
   \lim_{x \to \pm \infty} u(x) = \infty, \quad \lim_{x \to \pm \infty} u'(x) = \pm \sqrt{e^{2(c-\varphi(h))} - 1}.
   \]

2. if \( c = \infty \), \( \lim_{x \to \pm \Lambda_h} u(x) = \infty \), \( \lim_{x \to \pm \Lambda_h} u'(x) = \pm \infty \).

   In particular, if \( \Lambda_h < \infty \), the graph of \( u \) is asymptotic to two vertical lines. Moreover,

3. \( \Lambda_h < \infty \) if and only if \( e^{-\varphi} \in L^1([h, +\infty]) \), \( i.e \int_h^\infty e^{-\varphi(\lambda)} d\lambda < \infty \).

4. If \( \Lambda_\lambda < \infty \) and \( \dot{\varphi} \) is increasing (respectively, decreasing), then \( \Lambda_\lambda \) is decreasing (respectively, increasing) in \( \lambda \).

**Theorem 1.2.** Let \( \varphi : [a, +\infty] \rightarrow [b, c] \), \( a, b \in \{\mathbb{R}, +\infty\}, c \in \{\mathbb{R}, +\infty\} \) be a strictly decreasing diffeomorphism, then the solution \( u \) of (1.4) is defined in \([-\Lambda_h, \Lambda_h]\), \( \Lambda_h \in \mathbb{R} \cup \{+\infty\} \), it is concave, symmetric about the \( y \)-axis and has a maximum at \( x = 0 \). Moreover,

1. if \( c < \infty \), then \( \Lambda_h < \infty \) and,
   \[
   \lim_{x \to \pm \Lambda_h} u(x) = a, \quad \lim_{x \to \pm \Lambda_h} u'(x) = \pm \sqrt{e^{-2(c-\varphi(h))} - 1}.
   \]

2. if \( c = \infty \), \( \int_a^h e^{-\varphi(\lambda)} d\lambda < \infty \), and,
   \[
   \lim_{x \to \pm \Lambda_h} u(x) = a, \quad \lim_{x \to \pm \Lambda_h} u'(x) = \pm \infty.
   \]

**Remark 1.3.** In the hypothesis of Theorem 1.2, the graph of \( u \) is complete when \( a = -\infty \). But in this case, by changing \( \varphi \) by \(-\varphi\), we can also apply Theorem 1.1. Hereinafter, we always assume that \( \varphi \) is a strictly increasing diffeomorphism.
The goal of this paper is a uniqueness result for the \( [\varphi, \vec{e}_3] \)-catenary cylinders by their asymptotic behaviour. Roughly, if we have a surface \( \Sigma \) that looks like a \( [\varphi, \vec{e}_3] \)-catenary cylinder \( \mathcal{G}^h \) at infinity, in the sense of the definition 2.2 in Section 2, then our surface coincides with some \( [\varphi, \vec{e}_3] \)-catenary cylinder with the same behaviour \( \mathcal{G}^h \). Motivated by the work [9] of F. Martín, J. Pérez-García, A. Savas-Halilaj and K. Smoczyk for the grim reaper cylinder translating soliton, we increase the family of \( [\varphi, \vec{e}_3] \)-minimal surfaces where these types of results hold under different assumptions of asymptotic behaviour when \( \varphi \) is a strictly increasing convex diffeomorphism such that \( e^{-\varphi} \) is integrable. From [11, 12], this family of functions \( \varphi \) is the natural candidate to consider for generalizing the result of [9]. The statement of the main result of this paper tells us the following

**Theorem 1.4.** Let \( \varphi : [a, +\infty] \to [b, c] \), \( a, b \in \mathbb{R} \cup \{-\infty\} \) and \( c \in \mathbb{R} \cup \{+\infty\} \) be a strictly increasing convex diffeomorphism such that \( e^{-\varphi} \in L^1([a, +\infty[) \) and bounded quotient \( \dot{\varphi}/\varphi \). If \( \Sigma \) is a complete connected \( [\varphi, \vec{e}_3] \)-minimal graph \( \mathcal{C}^\infty \)-asymptotic to \( [\varphi, \vec{e}_3] \)-catenary cylinder \( \mathcal{G}^h \), outside a cylinder, for some \( h \in [a, +\infty[ \), then \( \Sigma \) coincides with some \( [\varphi, \vec{e}_3] \)-catenary cylinder with the same behaviour that \( \mathcal{G}^h \).

We would like to point out that we generalize this uniqueness for \( [\varphi, \vec{e}_3] \)-minimal surface assuming that \( \Sigma \) is a vertical graph. The main advantage of this hypothesis is that, we only need to prove that the angle function \( \langle N, \vec{e}_2 \rangle \) vanishes everywhere because in such case, \( \Sigma \) is invariant by translations in the direction \( \vec{e}_2 \) and so, from [11], it must coincide with some \( [\varphi, \vec{e}_3] \)-catenary cylinder. Otherwise, this conclusion is not true in general. The uniqueness of the grim reaper translating soliton is guaranteed by [9], since it was showed that the Gauss curvature vanishes everywhere and applying the Theorem B of [10], \( \Sigma \) must be a grim reaper cylinder.

The paper is organized as follows. In Section 2, we introduce the notation together with some fundamental equations that we will use throughout the work. In Section 3, we prove a compactness result to studying the infinity of our surfaces under horizontal translations. Finally, the proof of Theorem 1.4...
appears in Section 4. We will show that our graphs are invariant in the direction $\vec{e}_2$ because the function $\langle N, \vec{e}_2 \rangle$ vanishes everywhere due to the asymptotic behaviour of $\Sigma$.

2 Preliminaries

In this section, we introduce the notations, definitions and the fundamental equations that we will use throughout the paper.

For an orientable immersion $\Sigma$ in $\mathbb{R}^3$, we denote by $\nabla$, $\Delta$ the gradient and the Laplacian on $\Sigma$, respectively. The second fundamental form on $\Sigma$ will be denoted by $S$, so that $H = \text{trace}(S)$, $K$, and $|S|^2$ stand for the mean curvature, the Gauss curvature and the squared norm of the second fundamental form on $\Sigma$, respectively. Besides that, let us define the coordinate functions $x_i : \Sigma \to \mathbb{R}$ by $x_i(p) = \langle p, \vec{e}_i \rangle$ and the angle functions $\eta_i : \Sigma \to \mathbb{R}$ by $\eta_i(p) = \langle \vec{e}_i, N(p) \rangle$, for any $p \in \Sigma$.

Bearing in mind our goals, we will need a good control of our surfaces in the sense motivating the following definitions.

Definition 2.1. Consider a family $t$-parameter of vertical planes $\Pi(t) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 = t\}$ and $\Sigma$ be a subset of $\mathbb{R}^3$. The right part of $\Sigma$ with respect to the plane $\Pi(t)$ is defined by the subset $\Sigma_+(t) = \{(x_1, x_2, x_3) \in \Sigma : x_1 > t\}$, and the left part with respect to $\Pi(t)$ is defined by $\Sigma_-(t) = \Sigma \setminus \Sigma_+(t)$. The reflection of $\Sigma_+(t)$ with respect to the plane $\Pi(t)$ is given by $\Sigma^*_+(t) = \{(2t - x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2, x_3) \in \Sigma_+(t)\}$.

Taking into account the shape of the $[\varphi, \vec{e}_3]$-catenary cylinders given by the Theorem 1.1. The following definition is the key to understand the smooth convergence of a surface to $[\varphi, \vec{e}_3]$-catenary cylinder.

Definition 2.2. Let $\varphi : ]a, +\infty[ \to ]b, c[ \ a, b \in \mathbb{R} \cup \{-\infty\}, \ c \in \mathbb{R} \cup \{+\infty\}$ be a strictly increasing diffeomorphism such that $e^{-\varphi} \in L^1([a, +\infty[)$ and $G^h$ be a $[\varphi, \vec{e}_3]$-catenary cylinder in $\mathbb{R}^3_+ = \{p \in \mathbb{R}^3 : \langle p, \vec{e}_3 \rangle \geq a\}$ for some $h \in ]a, +\infty[$.

We will say that a smooth surface $\Sigma$ is $C^k$-asymptotic to the right part $G^h_+(0)$ of $G^h$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\Sigma$ can parametrized as a graph over $G^h$ as follows

$$\tilde{F} : T_{\delta,h}^+ \subset \mathbb{R}^2 \to \mathbb{R}^3 \quad \tilde{F} = F + \bar{u} N_F,$$

where $T_{\delta,h}^+ := \Lambda_h - \delta, \Lambda_h \times \mathbb{R}$, $F(x_1, x_2) = (x_1, x_2, u(x_1))$ parametrizes $G^h$ on $T_{\delta,h}^+$, $u$ is a solution of (1.4) with $u(0) = h$, $\bar{u} : T_{\delta,h}^+ \to \mathbb{R}$ is a function in

5
\( C^k(T_{\delta,h}(+)) \) such that

\[
\sup_{T_{\delta,h}^+} |\pi| < \varepsilon, \quad \sup_{T_{\delta,h}^+} |D^j\pi| < \varepsilon, \text{ for any } j \in \{1, \cdots, k\}.
\]

and \( N_F \) is the downwards unit normal of \( G^h \). Analogously, we will say that a smooth surface \( \Sigma \) is \( C^k \)-asymptotic to left left part \( G^h_{-}(0) \) of \( G^h \) if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \Sigma \) can be parametrized as a graph over \( G^h \) as follows

\[
\tilde{F}: T_{\delta,h}^{-} \subset \mathbb{R}^2 \to \mathbb{R}^3, \quad \tilde{F} = F + \pi N_F,
\]

where \( T_{\delta,h}^{-} := [-\Lambda_h, -\Lambda_h + \delta|x|, F(x_1, x_2) = (x_1, x_2, u(x_1)) \) parametrizes \( G^h \) on \( T_{\delta,h}^{-} \), \( u \) is a solution of \( (1.4) \) with \( u(0) = h \), \( \pi: T_{\delta,h}^{-} \to \mathbb{R} \) is a function in \( C^k(T_{\delta,h}^{-}) \) such that

\[
\sup_{T_{\delta,h}^{-}} |\pi| < \varepsilon, \quad \sup_{T_{\delta,h}^{-}} |D^j\pi| < \varepsilon, \text{ for any } j \in \{1, \cdots, k\}.
\]

In particular, we say that \( \Sigma \) is \( C^k \)-asymptotic to \( G^h \) if and only if \( \Sigma \) is \( C^k \)-asymptotic to both branches \( G^h_{+}(0) \) and \( G^h_{-}(0) \). Moreover, a smooth surface \( \Sigma \) is called \( C^k \)-asymptotic to \( G^h \), outside a cylinder, if there exists a solid cylinder \( c \) whose axis is \( G^h \cap \Pi(0) \) and the set \( \Sigma - c \) consists of two connected components \( \Sigma_1 \) and \( \Sigma_2 \) which are \( C^k \)-asymptotic to \( G^h_{+}(0) \) and \( G^h_{-}(0) \), respectively.

Next, we compute the fundamental equations that we will use in Section 4. Following the proof of Lemma 2.1 in [11], we get the following result.

**Lemma 2.3.** The following equations hold

\[
\begin{align*}
\Delta x_i + \langle \nabla \varphi, \nabla x_i \rangle &= 0 \quad \text{for } i = 1, 2 \\
\Delta x_i + \langle \nabla \varphi, \nabla x_3 \rangle &= \varphi, \\
\Delta \eta_i + \langle \nabla \varphi, \nabla \eta_i \rangle + |S|^2 \eta_i &= \varphi \eta_i \eta_i^2 \quad \text{for } i = 1, 2 \\
\Delta \eta_3 + \langle \nabla \varphi, \nabla \eta_3 \rangle + |S|^2 \eta_3 &= -\varphi \eta_3 |\nabla x_3|^2.
\end{align*}
\]

**Proof.** Fix \( p \in \Sigma \) and consider \( \{v_j\}_{j=1,2} \) an orthonormal frame of \( T_p \Sigma \). Then,

\[
\Delta x_i = \sum_{j=1}^{2} \langle \nabla_{v_j} \nabla x_i, v_j \rangle,
\]

where \( \nabla_{v_j} \) is the Levi-Civita connection on \( \Sigma \). Moreover, it is clear that

\[
\nabla x_i = \varepsilon_i^T = \varepsilon_i - \langle \varepsilon_i, N \rangle N \quad \text{and} \quad \nabla \eta_i = \nabla \varepsilon_i^T N.
\]

From (2.5), (2.6) and the Laplace-Beltrami equation, we have that

\[
\Delta x_i = -\langle \varepsilon_i, N \rangle \sum_{j=1}^{2} \langle \nabla_{v_j} N, v_j \rangle = -\langle \varepsilon_i, N \rangle H.
\]
Hence, from the definition (1.2) and the equation (2.7), it is proved that
\[ \Delta x_i = \dot{\varphi}(\dot{e}_i, N) \langle \ddot{e}_i, N \rangle = \dot{\varphi}(\dot{e}_i^2, \dot{e}_i^T) = -\langle \nabla \varphi, \nabla x_i \rangle \quad i = 1, 2. \]
\[ \Delta x_3 = \dot{\varphi}(\dot{e}_3, N)^2 = \dot{\varphi}(1 - |\nabla x_3|^2) = \dot{\varphi} - \langle \nabla \varphi, \nabla x_3 \rangle. \]

Next, from (1.2) and the well known equation \( \Delta N = \nabla H - |S|^2N \)
\[ \Delta N + \dot{\varphi} \nabla \eta_3 + \dot{\varphi} \eta_3 \nabla x_3 + |S|^2N = 0. \]

Consequently,
\[ (2.8) \quad \Delta \eta_i + |S|^2 \eta_i = -\dot{\varphi} \langle \nabla \eta_3, \ddot{e}_i \rangle - \dot{\varphi} \eta_3 \langle \nabla x_3, \dot{e}_i \rangle \quad \text{for } i = 1, 2 \]
\[ (2.9) \quad \Delta \eta_3 + |S|^2 \eta_3 = -\dot{\varphi} \langle \nabla \eta_3, \ddot{e}_3 \rangle - \dot{\varphi} \eta_3 \langle \nabla x_3 \rangle. \]

On the other hand, by a simple computation
\[ (2.10) \quad \dot{\varphi} \langle \nabla \eta_3, \ddot{e}_i \rangle = \dot{\varphi} S(\nabla x_3, \dot{e}_i) = \langle \nabla \varphi, \nabla \eta_i \rangle \quad \text{for } i = 1, 2, 3. \]

Then, the lemma follows from (2.8), (2.9), and (2.10).

From the previous expressions (2.1), (2.2), (2.3) and (2.4), we can prove the following result using the same equations as in [12].

**Proposition 2.4.** Let \( \varphi : [a, +\infty[ \rightarrow [b, c], a, b \in \mathbb{R} \cup \{-\infty\} \) and \( c \in \mathbb{R} \cup \{+\infty\} \) be a smooth function and \( \Sigma \) be a strictly mean convex \([\varphi, \dot{e}_3]\)-minimal immersion in \( \mathbb{R}^3_a \). Then,
\[ (2.11) \quad \Delta^\varphi \left( \frac{\eta_2}{\eta_3} \right) + 2 \langle \nabla \left( \frac{\eta_2}{\eta_3} \right), \nabla \frac{\eta_3}{\eta_3} \rangle = \dot{\varphi} \frac{\eta_2}{\eta_3}. \]

where \( \Delta^\varphi = \Delta + \langle \nabla \varphi, \nabla \cdot \rangle \) is the drift Laplacian of \( \Sigma \).

**Proof.** By a straightforward computation, for any smooth functions \( f, g : \Sigma \rightarrow \mathbb{R} \) with \( g \neq 0 \) everywhere we have that,
\[ (2.12) \quad \Delta^\varphi \left( \frac{f}{g} \right) + 2 \langle \nabla \left( \frac{f}{g} \right), \frac{\nabla g}{g} \rangle = \frac{g \Delta^\varphi f - f \Delta^\varphi g}{g^2}. \]

Consequently, the proof follows applying (2.12) in the equation (2.11) together with (2.1), (2.2), (2.3) and (2.4).

From the ellipticity of the equation (2.11) and bearing in mind the goal of proving that the function \( \eta_2 \) vanishes everywhere, we finish this section writing the previous quotient \( \eta_2/\eta_3 \) as a function over the \([\varphi, \dot{e}_3]\)-catenary cylinder to getting the control of this function at infinity. In Section 4, we will see that the quotient \( \eta_2/\eta_3 \) tends to zero and, from a strong maximum principle, we will prove that \( \eta_2/\eta_3 \) vanishes everywhere. In particular, the angle \( \eta_2 \) will always be zero as we wanted.
Consider a $[\varphi, e_3]$-catenary cylinder $G^h$ contained in the slab $]-\Lambda_h, \Lambda_h[ \times \mathbb{R}^2$ (possibly all $\mathbb{R}^3$) parametrized by

$$F(x_1, x_2) = (x_1, x_2, u(x_1)) \quad (x_1, x_2) \in ]-\Lambda_h, \Lambda_h[ \times \mathbb{R},$$

where $u$ is a smooth function satisfying (1.4) with $u(0) = h$ for some $h \in ]a, +\infty[$. Now, suppose that our immersion $\Sigma$ is a vertical $[\varphi, e_3]$-minimal graph $C^\infty$-asymptotic to $G^h$, outside a cylinder. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that the right part $\Sigma_+ (\Lambda_h - \delta)$ can be parametrized by $\tilde{F} : T_{\delta,h}^+ \to \mathbb{R}^3$ where $\tilde{F} = F + \overline{u} N_F$ and $\overline{u} : T_{\delta,h}^+ \to \mathbb{R}$ is a smooth function such that

$$\sup_{T_{\delta,h}^+} |\overline{u}| < \varepsilon \quad \text{and} \quad \sup_{T_{\delta,h}^+} |D^j \overline{u}| < \varepsilon \quad \text{for any} \quad j \in \mathbb{N},$$

and $N_F$ is the unit normal vector of $G^h$ given by,

$$N_F = \left( \frac{u'}{\sqrt{1 + u'^2}}, 0, -\frac{1}{\sqrt{1 + u'^2}} \right).$$

Next, we compute the Gauss map $N_{\tilde{F}}$ of $\Sigma$ with respect to the following orthogonal frame $\{ \partial F / \partial x_1, \partial F / \partial x_2, N_F \}$. Notice that,

$$\frac{\partial F}{\partial x_1} = (1, 0, u') \quad \text{and} \quad \frac{\partial F}{\partial x_2} = (0, 1, 0).$$

If we denote by

$$E_1 = \left( \frac{1}{\sqrt{1 + u'^2}}, 0, \frac{u'}{\sqrt{1 + u'^2}} \right) \quad \text{and} \quad E_2 = \frac{\partial F}{\partial x_2},$$

then,

$$\frac{\partial N_F}{\partial x_1} = \varphi E_1 \quad \text{and} \quad \frac{\partial N_F}{\partial x_2} = 0.$$

Consequently, from (2.14), (2.15) and (2.16) we get that

$$\frac{\partial \tilde{F}}{\partial x_1} = (\sqrt{1 + u'^2} + \overline{u} \varphi) E_1 + \overline{u}_x N_F,$$

$$\frac{\partial \tilde{F}}{\partial x_2} = E_2 + \overline{u}_x N_F,$$

$$\frac{\partial \tilde{F}}{\partial x_1} \wedge \frac{\partial \tilde{F}}{\partial x_2} = -\overline{u}_x E_1 - (\sqrt{1 + u'^2} + \overline{u} \varphi) \overline{u}_x E_2 + (\sqrt{1 + u'^2} + \overline{u} \varphi) N_F.$$

Thus, from (2.17), (2.18) and (2.19), the normal $N_{\tilde{F}}$ can be written as

$$N_{\tilde{F}} = \left( -\frac{\overline{u}_x}{1+u'^2} \right) \frac{\partial F}{\partial x_1} - \overline{u}_x \left( 1 + \frac{\varphi}{\sqrt{1+u'^2}} \right) \frac{\partial F}{\partial x_2} + \left( 1 + \frac{\varphi}{\sqrt{1+u'^2}} \right) N_F \sqrt{\frac{\overline{u}_x^2}{1+u'^2} + (1 + \overline{u}_x^2) \left( 1 + \frac{\varphi}{\sqrt{1+u'^2}} \right)^2}.$$
Finally, from (2.20), we compute the previous quotient $\eta_2/\eta_3$ over $G^h$ as we wanted. From the equations (2.13) and (2.14), we get that
\[
(2.21) \quad \langle \frac{\partial F}{\partial x_1}, \vec{e}_2 \rangle = 0, \quad \langle \frac{\partial F}{\partial x_2}, \vec{e}_2 \rangle = 1, \quad \langle N_F, \vec{e}_2 \rangle = 0,
\]
\[
(2.22) \quad \langle \frac{\partial F}{\partial x_1}, \vec{e}_3 \rangle = u', \quad \langle \frac{\partial F}{\partial x_2}, \vec{e}_3 \rangle = 0, \quad \langle N_F, \vec{e}_3 \rangle = -\frac{1}{\sqrt{1 + u'^2}}.
\]
Consequently, from (2.20), (2.21) and (2.22), we prove the following result.

**Proposition 2.5.** Under the assumptions above, for any $\varepsilon > 0$, there exists $\delta > 0$ small enough such that the following formula holds
\[
(2.23) \quad \eta_2 \eta_3 = u_x^2 \sqrt{1 + u'^2} \left( \frac{1 + \overline{\pi} \left( \frac{\varphi}{\sqrt{1 + u'^2}} \right)}{1 + \pi x_1 u' \sqrt{1 + u'^2} + \overline{\pi} \left( \frac{\varphi}{\sqrt{1 + u'^2}} \right)} \right) \text{ on } T_{s,h}^\pm,
\]
respectively, where $\overline{\pi} : T_{s,h}^\pm \times \mathbb{R} \to \mathbb{R}$ is a smooth function defined in 2.2 such that
\[
\sup_{T_{s,h}^\pm} \{|\overline{\pi}|\} < \varepsilon \text{ and } \sup_{T_{s,h}^\pm} \{|D^j \overline{\pi}|\} < \varepsilon \text{ for any } j \in \mathbb{N}.
\]

**Remark 2.6.** From Theorem (1.1) and equations (2.13)-(2.16), if $\varphi$ is a smooth function such that $e^{-\varphi}$ is not integrable and the $j$-th derivative of $\varphi$ is bounded for any $j \in \{1, \cdots, k\}$, then $\Sigma$ is closed to $G^h$ in $C^k$-topology.

## 3 Compactness Theorem from a Barrier Maximum Principle

A fundamental step for the uniqueness of the $[\varphi, \vec{e}_3]$-catenary cylinder is his behaviour under translations in the direction $\vec{e}_2$. In section 4, we will prove that the translations in the direction $\vec{e}_2$ of any $[\varphi, \vec{e}_3]$-minimal graph $\Sigma$ converge to some $[\varphi, \vec{e}_3]$-catenary cylinder if $\Sigma$ is $C^\infty$-asymptotic to some $G^h$, outside a cylinder. Our first result of this section is a direct application of a compactness theorem of B. White for minimal surfaces in 3-manifolds (Theorem 1.1 of [17]).

Next, we state a Barrier Maximum principle for the blow-up points of the area proved in [16]. Finally, as a consequence of these results together with the existence of $[\varphi, \vec{e}_3]$-Bowls given in [11], we prove a compactness result for surfaces $C^\infty$-asymptotic to $G^h$, outside either a cylinder or a compact set.

**Theorem 3.1.** Let $\Omega$ be an open subset of $\mathbb{R}^3$. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence of smooth functions on $\Omega$ converging smoothly to a $\varphi_\infty$. Let $\{\Sigma_n\}_{n \in \mathbb{N}}$ be a sequence of properly embedded minimal surface in $\Omega^{\varphi_\infty}$. Suppose also that the area and the genus of the sequence $\{\Sigma_n\}$ are uniformly bounded on compacts subsets of $\Omega$. Then, after passing to a subsequence, $\{\Sigma_n\}$ converges to a smooth properly embedded minimal $\Sigma_\infty$ in $\Omega^{\varphi_\infty}$ and the convergence is smooth away from a discrete set $\Gamma$. For each connected component $S$ of $\Sigma_\infty$ either,
1. the convergence to $S$ is smooth everywhere with multiplicity 1, or

2. the convergence to $S$ is smooth with some multiplicity $> 1$ away from $S \cap \Gamma$.

In this case, if $S$ is two-sided, then it must be stable.

If the total curvatures of $\Sigma_n$ are bounded by $\beta$, the set $\Gamma$ has at most $\beta/4\pi$ points.

A crucial assumption in the Theorem of B. White is that the sequence $\{\Sigma_n\}$ has uniformly bounded area on compact subset of $\Omega$. Next, we discuss when this hypothesis hold and prove a result of compactness.

**Remark 3.2.** In [12], it is proved that if $\phi$ is strictly increasing convex smooth function with $\sup\{\ddot{\phi} - \dot{\phi}^2\} < +\infty$, then any sequence $\{\Sigma_n\}$ of mean convex $[\phi, \vec{e}_3]$-minimal immersion has uniformly bounded intrinsic area on compact subsets of $\Omega$.

Let us denote by,

$$Z := \{p \in \Omega : \limsup_{n \to +\infty} A^\phi(\Sigma_n \cap B_r(p)) = +\infty \text{ for any } r > 0\},$$

where $B_r(p)$ is an Euclidean ball centered in $p$ of radius $r$. If we show that $Z = \emptyset$, then $\{\Sigma_n\}$ has locally uniformly bounded area. From the Theorems 2.6 and 7.3 in [16], White proved that $Z$ satisfies the following maximum principle as properly embedded minimal surfaces without boundary.

**Theorem 3.3.** (Strong Barrier Principle) Let $\Omega$ be an open subset of $\mathbb{R}^3$. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence of smooth function converging smoothly to a smooth functions $\varphi_\infty$ and $\{\Sigma_n\}_{n \in \mathbb{N}}$ be a sequence of minimal surfaces in $\Omega^{\varphi_\infty}$ such that the length of $\partial \Sigma_n$ are uniformly bounded on compact subsets of $\Omega$ and $D$ be a closed region of $\Omega$ with smooth connected boundary such that $Z \subset D$ and $\langle H_{\partial D}, \nu \rangle \leq 0$ where $\nu$ is the unit normal of $\partial D$ points into $D$ and $H_{\partial D}$ is the mean curvature of the boundary of $D$. If $Z$ contains any point of $\partial D$, then $\partial D \subset Z$.

As a consequence of the result above, we give the following compactness theorem showing that the set $Z = \emptyset$.

**Theorem 3.4.** Let $\varphi : [a, +\infty[ \to [b, c]$, $a, b \in \mathbb{R} \cup \{-\infty\}$ and $c \in \mathbb{R} \cup \{+\infty\}$ be a strictly increasing convex diffeomorphism and $\Sigma$ be a connected properly embedded $[\varphi, \vec{e}_3]$-minimal minimal surface with locally bounded genus $C^\infty$-asymptotic to $G^h$, outside a cylinder, for some $h \in [a, +\infty[$. Suppose that $\{b_n\}_{n \in \mathbb{N}}$ is a divergent sequence of real numbers and consider the sequence of $[\varphi, \vec{e}_3]$-minimal surfaces $\{\Sigma_n = \Sigma - (0, b_n, 0)\}_{n \in \mathbb{N}}$. Then, after passing to a subsequence, $\Sigma_n$ converge smoothly with multiplicity one to a properly embedded connected $[\varphi, \vec{e}_3]$-minimal $\Sigma_\infty$ which has the same asymptotic behaviour that $\Sigma$.

**Proof.** By assumption our surface $\Sigma$ is $C^\infty$-asymptotic to $G^h$, outside a cylinder $c$. Up to rigid motions, we can consider $c$ such that

$$c = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq r_0^2\}.$$
From Definition 2.2 each $\Sigma_n \setminus \mathcal{C}$ consists in two connected components $\Sigma_{n,1}$, $\Sigma_{n,2}$ $C^\infty$-asymptotic to $Q_h^+(0)$ and $Q_l^+(0)$, respectively. For an arbitrary $z \in \mathbb{R}$ denote by
\[ R_z^+ = \{ p \in \mathbb{R}^3 : \langle p, \tilde{e}_3 \rangle \geq z \} \text{ and } R_z^- = \mathbb{R}^3 \setminus R_z^+. \]
Split each surface $\Sigma_n$ of the surface into the parts
\[ \Sigma_{n,k}^+(z) = \Sigma_{n,k} \cap R_z^+ \text{ with } k \in \{1, 2\} \text{ and } \Sigma_{n,k}^-(z) = \Sigma_n \setminus \left( \bigcup_{k=1}^{2} \Sigma_{n,k}^+(z) \right). \]

**Claim 1.** There exists $z_1$ large enough such that for any $z \geq z_1$ the sequence $\{\Sigma_{n,k}^+(z)\}_{n \in \mathbb{N}}$ have uniformly bounded area on compact subsets of $\Omega$ with respect to the Ilmanen’s metric.

**Proof of the Claim 1.** We only argue on the case $k = 1$ since the case $k = 2$ is analogous. Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^3$ and $B(0, r)$ the ball of radius $r$ centered at the origin containing $\mathcal{K}$. Denote by $\mathcal{U}_{n,z}$ the projection of $\Sigma_{n,1}^+(z) \cap \mathcal{K}$ to $]-\Lambda_h, \Lambda_h[ \times \mathbb{R}$. From the definition 2.2 and taking $z_1$ large enough, we know that for any $z \geq z_1$ and any $\varepsilon > 0$ there exists $\delta$ (depending only of $\varepsilon$) such that we can parametrize $\Sigma_{n,1}^+(z) \cap \mathcal{K}$ by $\tilde{F}_n$ over $\mathcal{U}_{n,z} \cap T_{\delta,h}^+$. Hence, the area on this compact subset is given by,
\[ A^\mathcal{F}(\Sigma_{n,1}^+(z) \cap \mathcal{K}) = \int_{\mathcal{U}_{n,z} \cap T_{\delta,h}^+} e^{\varphi(\tilde{F}_n(x_1, x_2), \tilde{e}_3))} \left| \frac{\partial \tilde{F}_n}{\partial x_1} \wedge \frac{\partial \tilde{F}_n}{\partial x_2} \right| dx_1 dx_2. \]

Consequently, from the monotonicity of $\varphi$, the Theorem 1.1 and the expression 2.19, there must be a constant $C(R, \varepsilon) > 0$ (depending only of $R$ and $\varepsilon$) such that
\[ A^\mathcal{F}(\Sigma_{n,1}^+(z) \cap \mathcal{K}) \leq e^{\varphi(R+\varepsilon)} C(R, \varepsilon) A(\mathcal{U}_{n,z}) \leq e^{\varphi(R+\varepsilon)} C(R, \varepsilon) A(\mathcal{K}). \]

**Claim 2.** There exists $z_2 \geq z_1$ such that for any $z \geq z_2$ the sequence of surfaces $\{\Sigma_{n}^-(z)\}_{n \in \mathbb{N}}$ have uniformly bounded area on compact sets of $\Omega$ with respect to the Ilmanen’s metric.

**Proof of the Claim 2.** Proceeding as before, the sequence $\{\partial \Sigma_n^-(z)\}$ has uniformly bounded length on compact sets. Notice that each $\partial \Sigma_n^-(z)$ has two connected components $\partial \Sigma_n^-(z)$ with $k \in \{1, 2\}$. Again, we will only argue on $k = 1$ since the same reasoning works for $k = 2$. Fix a compact subset $\mathcal{K}$ of $\mathbb{R}^3$ and $B(0, r)$ the ball of radius $r$ centered in the origin containing $\mathcal{K}$. Taking $z_2 \geq z_1$ large enough, we get that for any $z \geq z_2$ and any $\varepsilon > 0$ there exists $\delta$ (depending only of $\varepsilon$) such that $\partial \Sigma_{n,1}^-(z) \cap \mathcal{K}$ can be represented as a planar curve $\gamma_n : I_n \subset \mathbb{R} \to \Pi(r_n)$ given by $\gamma_n(t) = \tilde{F}_n(r_n, t)$ with $r_n \in]\Lambda_h - \delta, \Lambda_h[\}, I_n$.
stands the projection of $\partial \Sigma_{n,1}(z) \cap \mathcal{K}$ to \{\(r_n\)\} $\times \mathbb{R}$ and $\tilde{F}_n$ defined in 2.2. Hence, from the equation (2.18), we can estimate the length as follows

$$\int_{I_n} e^{\varphi(\langle \gamma_n(t), \vec{e}_3 \rangle)} \| \gamma_n'(t) \| \, dt \leq e^{\varphi(R+\varepsilon)} \int_{I_n} \| \gamma_n'(t) \| \, dt \leq C(\varepsilon, r) e^{\varphi(R+\varepsilon)},$$

where $C(\varepsilon, r)$ is a positive constant that only depends on $\varepsilon$ and $r$.

On the other hand, from the Claim 1, we get that $Z$ is contained within a horizontal cylinder $c'$ of radius greater or equal to $z_2$. Notice that, the third coordinate $x_3$ is bounded in $Z$. From [11], we can assure the existence of a $[\varphi, \vec{e}_3]$-Bowl $B$ such that $B \cap c' = \emptyset$. Translating $B$ in the direction $-\vec{e}_3$, there exists a first $l > 0$ such that $B - l\vec{e}_3$ has a tangency point $p$ of contact with $Z$. Thus, from the estimate above, we can apply the Theorem 3.3 to showing that $B - l\vec{e}_3$ is contained in $c'$ which contradicts the asymptotic behaviour of $B$ given in [11]. It would like pointing out that the translated Bowl is a $[\varphi_l, \vec{e}_3]$-minimal graph, in particular mean convex, with $\varphi_l(x_3) = \varphi(x_3 + l)$.

From the Claims 1 and 2, the sequence $\Sigma_n$ have uniformly bounded area on compact subsets of $\mathbb{R}^3$ with respect to the Ilmanen’s metric. Consequently, from the Theorem 3.1 $\{\Sigma_n\}_{n \in \mathbb{N}}$ converges to a smooth properly embedded $[\varphi, \vec{e}_3]$-minimal $\Sigma_\infty$. Notice that the limit $\Sigma_\infty \neq \emptyset$ because $\{\Sigma_n\}$ have accumulation points due to their asymptotic behaviour. Since each $\Sigma_{n,h}(z)$ is a graph over $\mathcal{G}^h$ and $\Sigma_n$ is connected, we deduce that the multiplicity is one everywhere and so, the convergence is smooth. Moreover, observe that each component of $\Sigma_\infty \cap \mathcal{R}_z^+$ can be represented as the graph of a smooth function $\pi_\infty$ which is the limit of a sequence of graphs $\pi_n = \pi(s - b_n, t)$. Thus, $\Sigma_\infty$ has the same asymptotic behaviour that $\Sigma$. Finally, $\Sigma_\infty$ must be connected. Otherwise, there should exists a properly embedded connected component $\mathcal{S}_\infty$ lying inside $\mathcal{C}$ getting to contradiction since the third coordinate of $\Sigma$ must be bounded.

Remark 3.5. From the paper [11], the hypothesis of convexity for $\varphi$ is crucial for the existence of the $[\varphi, \vec{e}_3]$-Bowls.
4 Proof of the main result

In this section, we prove the uniqueness result for the \([\varphi, \vec{e}_3]\)-catenary cylinders by their asymptotic behaviour. The main tools for proving these theorems are the moving plane method of Alexandrov [1], the Hopf’s maximum principle and a strong maximum principle for elliptic equations [6]. The goal will be to prove that the angle function \(\eta_2\) vanishes everywhere. Hence, our graph \(\Sigma\) will be invariant by translations in the direction \(\vec{e}_2\) and so, it must be a \([\varphi, \vec{e}_3]\)-catenary cylinder.

**Lemma 4.1.** Let \(\varphi : [a, +\infty[ \to ] b, c]\), \(a, b \in \mathbb{R} \cup \{-\infty\}\), \(c \in \mathbb{R} \cup \{+\infty\}\) be a strictly increasing smooth function such that \(e^{-\varphi} \in L^1([a, +\infty[)\). If \(\Sigma\) is a \([\varphi, \vec{e}_3]\)-minimal immersion \(C^\infty\)-asymptotic to \([\varphi, \vec{e}_3]\)-catenary cylinder \(G^h\), outside a cylinder, for some \(h \in [a, +\infty[,\) then \(\Sigma\) is strictly contained in the slab \(]-\Lambda_h, \Lambda_h[ \times \mathbb{R}^2\).

**Proof.** Argue by contradiction. Let \(x_1\) be the first coordinate of \(\Sigma\) and assume that \(\lambda = \sup_{\Sigma} \{x_1\} > \Lambda_h\). Consider the following subset of \(\Sigma\),

\[
S_{h, \lambda} = \Sigma + \left( \frac{\Lambda_h}{2} + \frac{\lambda}{2} \right) \quad \text{with} \quad \partial S_{h, \lambda} = \Sigma \cap \Pi \left( \frac{\Lambda_h}{2} + \frac{\lambda}{2} \right).
\]

The asymptotic behaviour of \(\Sigma\) with respect to \(G^h_\varphi(0)\) implies that there must exist a local maximum in the interior of \(S_{h, \lambda}\). From the equation 2.1 we get that

\[
\sup_{\Sigma, \lambda} \{x_1\} = \sup_{\partial S_{h, \lambda}} \{x_1\}.
\]

Hence, there must be a point \(p \in \partial S_{h, \lambda}\) such that

\[
\frac{\Lambda_h}{2} + \frac{\lambda}{2} = x_1(p) \geq \lambda,
\]
getting to a contradiction since \(\lambda > \Lambda_h\). On the other hand, if the equality holds, comparing the vertical plane \(\Pi(\Lambda_h)\) with \(\Sigma\) we have again a contradiction applying the Hopf’s maximum principle [4]. Therefore, \(\sup_{\Sigma} \{x_1\} < \Lambda_h\).

Similarly, we can prove that \(\inf_{\Sigma} \{x_1\} > -\Lambda_h\) thanks to the asymptotic behaviour of \(\Sigma\) with respect to \(G^h_\varphi(0)\).

**Lemma 4.2.** Let \(\varphi : [a, +\infty[ \to ] b, c]\), \(a, b \in \mathbb{R} \cup \{-\infty\}\), \(c \in \mathbb{R} \cup \{+\infty\}\) be a strictly increasing smooth function such that \(e^{-\varphi} \in L^1([a, +\infty[)\). If \(\Sigma\) is a \([\varphi, \vec{e}_3]\)-minimal immersion \(C^\infty\)-asymptotic to \([\varphi, \vec{e}_3]\)-catenary cylinder \(G^h\), outside a cylinder, for some \(h \in [a, +\infty[,\) then \(\Sigma\) is symmetric with respect to the plane \(\Pi(0)\).

**Proof.** The main tool for proving the Lemma is the Alexandrov’s method of moving planes of [1] together with the arguments of [9] and [10]. Fix \(t > 0\) and consider the reflection of \(\Sigma_+(t)\) with respect to \(\Pi(t)\)

\[
\Sigma_+(t) = \{(2t - x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2, x_3) \in \Sigma_+(t)\}.
\]
From the previous Lemma 4.1 let us define the following set

\[ A = \{ t \in [0, \Lambda_h] : \Sigma_+(t) \text{ graph over } \Pi(0) \text{ and } \Sigma_+(t) \geq \Sigma_-(t) \}, \]

where \( \Sigma_+(t) \geq \Sigma_-(t) \) means that

\[
\inf[x_1(\pi^{-1}_t(x_1, x_2, x_3) \cap \Sigma_+(t))] \geq \sup[x_1(\pi^{-1}_t(x_1, x_2, x_3) \cap \Sigma_-(t))],
\]

for any point \((x_1, x_2, x_3) \in \Pi(t)\) such that \( \pi^{-1}_t((x_1, x_2, x_3)) \neq \emptyset \) and \( \pi^{-1}_t((x_1, x_2, x_3)) \cap \Sigma_-(t)) \neq \emptyset \) where \( \pi_t \) is the projection onto \( \Pi(t) \) given by \( \pi_t((x_1, x_2, x_3)) = (t, x_2, x_3) \). We will prove that \( 0 \in A \). In this case, we have that \( \Sigma_+(0) \geq \Sigma_-(0) \) and by a symmetric argument, we can show that \( \Sigma_-(0) \leq \Sigma_+(0) \) and so, \( \Sigma_+(0) = \Sigma_-(0) \).

Firstly, we prove that \( A \neq \emptyset \). From the asymptotic behaviour with respect to \( G^h_+(0) \), for any \( \varepsilon > 0 \) there exists \( t_0 > 0 \) such that \( \Sigma_+(t) \) can be parametrized by

\[
\tilde{F} : T_{t,h} :=|t, \Lambda_h| \times \mathbb{R} \rightarrow \mathbb{R}^3 \quad \tilde{F} = F + \pi N_F \quad \text{for any } t \geq t_0,
\]

where \( F \) is the parametrization of \( G^h \) given by the graph of \( u \) over \( T_t \) and \( \pi : T_{t,h} \rightarrow \mathbb{R} \) is a smooth function such that

\[
\sup_{T_{t,h}} |\pi| < \varepsilon, \quad \sup_{T_{t,h}} |D\pi| < \varepsilon.
\]

Consider,

\[
\tilde{u}(x_1) = \langle \tilde{F}, e_3 \rangle = u(x_1) - \frac{\pi(x_1, x_2)}{\sqrt{1 + u'(x_1)^2}}
\]

and define \( \tilde{u}^* \) the reflection with respect to \( \Pi(t) \) as follows

\[
\tilde{u}^*(x_1) = u(2t - x_1) - \frac{\pi(2t - x_1, x_2)}{\sqrt{1 + u'(2t - x_1)^2}}.
\]
From Theorem [1.1] and equations [4.1],[4.2] and [4.3], the following inequality holds
\[ \tilde{u}^*(x_1) - \tilde{u}(x_1) > u(2t - x_1) - u(x_1) - 2\varepsilon \]
Choosing a positive constant \( a \) (independent of \( t_0 \)) and \( t_1 \geq t_0 \) large enough such that \( u'(x_1) \geq \varepsilon/a \) for \( a < t_1 - x_1 \), then the following inequalities hold
\[ \tilde{u}^*(x_1) - \tilde{u}(x_1) > 2(u'(x_1)a - \varepsilon) > 0, \]
Consequently, from the inequality above, we get that
\[ \Sigma^+_{t}(t + a) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \leq t \} \]
\[ \geq \Sigma_-(t + a) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \leq t \}, \]
for any \( t \geq t_1 \). Moreover, from the fact that \( \Sigma_+(t) \) is a graph over \( \Pi(t) \) we deduce that
\[ \Sigma^+_{t}(t + a) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : t \leq x_1 \leq t + a \} \]
\[ \geq \Sigma_-(t + a) \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 : t \leq x_1 \leq t + a \}. \]
Hence, \( t_1 + a, \Lambda_h \subset A \). Therefore, it is clear that if \( s \in A \) then \( s, \Lambda_h \subset A \). Now, suppose by contradiction that \( s_0 = \min(A) > 0 \). Arguing as in the proof of Claim 3 in [9], there exists \( \varepsilon_1 \in ]0, \varepsilon_0] \) such that the surface \( \Sigma_+(s_0 - \varepsilon_1) \) can be represented as a graph over the plane \( \Pi(0) \). Consequently, \( \Sigma_+(t) \) is a graph over \( \Pi(0) \) for any \( t \geq s_0 - \varepsilon_1 \). Moreover for such \( \varepsilon_1 \) and using an analogous reasoning that the Claim 3 of [10], we can prove that \( \Sigma^+_{s_0}(s_0 - \varepsilon_1) \geq \Sigma_-(s_0 - \varepsilon_1) \) and so, \( s_0 - \varepsilon_1 \in A \) getting to contradiction.

**Lemma 4.3.** Let \( \varphi : ]a, +\infty[ \rightarrow ]b, c[ \), \( a, b \in \mathbb{R} \cup \{-\infty\} \), \( c \in \mathbb{R} \cup \{+\infty\} \) be a convex strictly increasing smooth function such that \( e^{-\varphi} \in L^1(]a, +\infty[) \) and \( \Sigma \) be a connected \( [\varphi, \varepsilon_3] \)-minimal immersion \( C^{\infty} \)-asymptotic to \( [\varphi, \varepsilon_3] \)-catenary cylinder \( \mathcal{G}^h \) outside a cylinder, for some \( h \in ]a, +\infty[ \). Consider the profile curve \( \Gamma = \Sigma \cap \Pi(0) \). If the function \( x_3|_\Gamma \) attains its global extremum on \( \Gamma \), then \( \Sigma \) is a \( [\varphi, \varepsilon_3] \)-catenary cylinder.

**Proof.** Suppose first that there exists a point \( p \in \Gamma \) such that \( \tau = \max_{\Gamma}\{x_3\} = x_3(p) \). Observe that,
\[ \partial \Sigma_+(0) \subset \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \leq \tau \}. \]
For any \( t \in \mathbb{R} \) consider an horizontal translation of \( \mathcal{G}^t_+(0) \) given by
\[ \mathcal{G}^t_+(0) = \{(x_1, x_2, u(x_1 - t)) \in \mathbb{R}^3 : x_1 \in [t, \Lambda_t + t]\} \]
where, from the Theorem [1.1] \( \Lambda_t \leq \Lambda_h \). Define the following set
\[ \mathcal{Q} = \{t \in ]-\infty, 0[ : \mathcal{G}^t_+(0) \cap \Sigma_+(0) = \varnothing\}. \]
Obviously, $Q \neq \emptyset$. Moreover, if $t \in Q$ then $] - \infty, t[ \subset Q$. We claim that

$$t_0 = \sup \{Q\} = 0.$$ 

Assume the opposite. If $t_0 < 0$ is not in $Q$, then there would exist an interior point of contact since, from Theorem [1.1], the boundaries of both surfaces do not touch when $t < 0$. Applying the Hopf’s maximum principle [6], we get that $\Sigma = G^{\tau, t_0}$ getting to contradiction with the asymptotic behaviour of $\Sigma$. Hence, we can assume that $t_0 \in Q$. In this case, because the distance between the boundaries is positive and $G^{\tau, t_0}_+ (0), \Sigma_+ (0)$ have different asymptotic behaviour, there exists a sequence $\{p_n = (p_{1,n}, p_{2,n}, p_{3,n})\} \in N \subset \Sigma_+ (0)$ such that $\{p_{3,n}\}$ is bounded, $\{p_{2,n}\}$ is unbounded and

$$\lim_{n \to +\infty} d(p_n, G^{\tau, t_0}_+(0)) = 0.$$ 

From Corollary [3.4], we get that the sequence $\Sigma_n = \Sigma - (0, p_{2,n}, 0)$ converges smoothly, after passing to a subsequence, to a properly embedded connected $[\varphi, \vec{e}_3]$-minimal surface $\Sigma_\infty$ with the same asymptotic behaviour that $\Sigma$. However, $\Sigma_\infty$ and $G^{\tau, t_0}_+(0)$ have an interior point of contact getting again to contradiction. Consequently, we can assume that $t_0 = 0$. Thus, $G^{\tau, t}_+(0)$ and $\Sigma_+ (0)$ have a boundary contact at $p$. Observe that the tangent plane at $p$ of both surfaces is horizontal by Lemma [4.2] and so, from Hopf’s maximum principle in the boundary, these surfaces coincide as we wanted. On the other hand, if there exists $q \in \Gamma$ such that $\sigma = \min \{x_3\} = x_3(q)$, the same reasoning works comparing $\Sigma_+ (0)$ with $G^{\sigma, -t}_+(0)$ for $t \geq 0$.

**Remark 4.4.** We would like pointing out that this argument is not true when $e^{-\varphi} \notin L^1([a, +\infty])$ because we cannot assure the existence of a first contact point of tangency since the inclination of the asymptotic plane of $G^h$, given by the Theorem [1.7], with respect any horizontal plane is decreasing with respect to the initial condition, that is, $\lim_{x_1 \to +\infty} u'(x_1)$ is decreasing with respect to $h$. 

![Diagram](image-url)
Proposition 4.5. Let $\varphi : [a, +\infty) \to [b, c]$ be a convex strictly increasing diffeomorphism with $e^{-\varphi} \in L^1([a, +\infty])$ and $\Sigma$ be a connected $[\varphi, \vec{e}_3]$-minimal immersion $C^\infty$-asymptotic to $[\varphi, \vec{e}_3]$-catenary cylinder $G^h$, outside a cylinder, for some $h \in [a, +\infty]$. For any sequence of points $\{(p_{1,n}, p_{2,n}, p_{3,n})\}$ of $\Sigma$ such that $\{p_{2,n}\}$ diverges and $\{p_{3,n}\}$ is bounded, the sequence $\{\Sigma_n = \Sigma - (0, p_{2,n}, 0)\}_{n \in \mathbb{N}}$ converge smoothly, after subsequence, to some $[\varphi, \vec{e}_3]$-catenary cylinder with the same asymptotic behaviour that $G^h$.

Proof. Up to subsequence, we can assume that $\{p_{3,n}\}$ is strictly monotone converging to either the supremum or infimum. Now, from the Corollary 3.4, we know that $\Sigma_n$ converge smoothly to a connected properly embedded $[\varphi, \vec{e}_3]$-minimal surface $\Sigma_\infty$ with the same asymptotic behaviour that $\Sigma$. Taking into account the way in which we have constructed the limit, we get that either

\[
\max_{\Pi(0) \cap \Sigma_\infty} x_3 = x_3|_{\Sigma_\infty}((0, 0, p_{3,n})) \quad \text{or,}
\]

\[
\min_{\Pi(0) \cap \Sigma_\infty} x_3 = x_3|_{\Sigma_\infty}((0, 0, p_{3,n})).
\]

Consequently, from the Lemma 4.3 and (4.4) or (4.5), we prove that $\Sigma_\infty$ coincides with some $[\varphi, \vec{e}_3]$-catenary cylinder. \hfill \Box

PROOF OF THEOREM 1.4

Proof. Let $\{a_n\}_{n \in \mathbb{N}}, \{b_n\}_{n \in \mathbb{N}}$ be a strictly increasing sequences of positive real numbers. Consider a compact exhaustion $\{\Lambda_n\}_{n \in \mathbb{N}}$ whose boundary consists in the following four curves,

$\Lambda_{1,n} = \{(x_1, x_2, x_3) \in \Sigma : x_1 > 0, \ -a_n \leq x_2 \leq b_n, \ x_3 = n\}$,

$\Lambda_{2,n} = \{(x_1, x_2, x_3) \in \Sigma : x_1 < 0, \ -a_n \leq x_2 \leq b_n, \ x_3 = n\}$,

$\Lambda_{3,n} = \{(x_1, x_2, x_3) \in \Sigma : x_2 = a_n, \ x_3 \leq n\}$,

$\Lambda_{4,n} = \{(x_1, x_2, x_3) \in \Sigma : x_2 = b_n, \ x_3 \leq n\}$. 

17
From the asymptotic behaviour given by the definition 2.2, fix $n_1$ large enough and consider an arbitrary $n \geq n_1$, for any $\varepsilon > 0$ there exists $\delta > 0$ (depending only of $\varepsilon$) such that a neighborhood of $\Lambda_{1,n}$ can parametrized as a graph over $G_h(0)$ by

$$\tilde{F} : T^+_{\delta,h} \to \mathbb{R}^3, \quad \tilde{F} = F + \pi N_F,$$

where $F(x_1, x_2) = (x_1, x_2, u(x_1))$ parametrizes $G_h$ on $T^+_{\delta,h}$ with $u$ solution of $(1.4), \pi : T^+_{\delta,h} \to \mathbb{R}$ is a smooth function such that

$$\sup_{T^+_{\delta,h}} |\overline{u}| < \varepsilon, \quad \sup_{T^+_{\delta,h}} |D^j \overline{u}| < \varepsilon, \quad \text{for any } j \in \mathbb{N},$$

and $N_F$ is the unit normal of $G_h$. From the equation (2.23), we can write

$$(4.6) \quad \frac{\eta_2}{\eta_3} = \pi x_2 \sqrt{1 + u'^2} \left( \frac{1 + \pi \left( \frac{\dot{\varphi}}{\sqrt{1 + u'^2}} \right)}{1 + \pi x_1 \frac{u'}{\sqrt{1 + u'^2}} + \pi \left( \frac{\dot{\varphi}}{\sqrt{1 + u'^2}} \right)} \right) \quad \text{on } T^+_{\delta,h}.$$ 

Let us examine at first the behaviour of $\eta_2/\eta_3$ along $\Lambda_{1,n}$. Because for any fixed $x_2$, we have that $\lim_{x_1 \to \Lambda_h} |\overline{u}| = \lim_{x_1 \to \Lambda_h} |D\overline{u}| = 0$, then

$$(4.7) \quad |\pi x_2(x_1, x_2)| = \left| - \int_{x_1}^{\Lambda_h} \pi x_2(s, x_2) \, ds \right| \leq (\Lambda_h - x_1) \varepsilon.$$ 

Moreover, from the Theorem 1.1 the L’Hôpital rule and assumptions of Theorem A, the following limits exists

$$(4.8) \quad \lim_{x_1 \to \Lambda_h} \frac{\dot{\varphi}}{\sqrt{1 + u'^2}} = \lim_{x_1 \to \Lambda_h} (\Lambda_h - x_1) \sqrt{1 + u'^2(x_1)}.$$ 

Consequently, from $(4.7)$ and $(4.8)$, there exists $n_2 \geq n_1$ large enough such that for any $n \geq n_2$, the following estimates holds for the equation $(4.6)$,

$$(4.9) \quad \sup_{\Lambda_{1,n}} |\eta_2/\eta_3| < \varepsilon.$$ 

Moreover, from the symmetry of $(4.2)$ the previous inequality $(4.9)$ holds in $\Lambda_{2,n}$. On the other hand, from the Proposition 4.5, we can argue analogously with the curves $\Lambda_{3,n}$ and $\Lambda_{4,n}$ because both curves are $C^\infty$-asymptotic to some $G^{h'}$ such that $G^{h'}$ has the same asymptotic behaviour that $G^h$, that is, $\Lambda_{h'} = \Lambda_h$. Hence, there exists $n_3 \geq n_2$ large enough such that for any $n \geq n_3$ a neighborhood of $\Lambda_{3,n}$ can be parametrized as a graph over $G^{h'}$ by

$$\tilde{F}_n : T^+_{\delta,h,n} \to \mathbb{R}^3 \quad \tilde{F}_n = F + \pi_n N_F,$$

where $T^+_{\delta,h,n} = ] - \Lambda_{h'} + \delta, \Lambda_{h'} - \delta \times [m_{1,n}, m_{2,n} \times ] \to \mathbb{R}$ is a smooth function, $\delta > 0$ only depends of $n_3$, $\{m_{1,n}\}_{n \in \mathbb{N}}, \{m_{2,n}\}_{n \in \mathbb{N}}$ are strictly monotonous sequences.
with \( m_{1,n} < m_{2,n} \) and each \( \overline{v}_n : T_{\delta,h,n}^+ \rightarrow \mathbb{R} \) is a smooth function satisfying the following inequalities,

\[
\sup_{T_{\delta,h,n}^+} |\overline{v}_n| < \varepsilon, \quad \sup_{T_{\delta,h,n}^+} |D^j \overline{v}_n| < \varepsilon, \quad \text{for any } j \in \mathbb{N}.
\]

Notice that, the existence of these sequence of functions is guaranteed by the convergence given in the Theorem 3.1. In these case, \( x_1 \) is not tending to \( \pm \Lambda_h \) and from the Theorem 1.1 \( \varphi' \) is bounded. Hence, the same argument works as above because the previous limits (4.8) exist for any divergence sequence of points and so, the inequality (4.9) holds over \( \Lambda_{4,n} \). Analogously, taking \( n_4 \geq n_3 \) large enough and parametrizing a neighborhood of \( \Lambda_{4,n} \) as a graph over \( G_{h'} \) by \( \overline{v}_n : T_{\delta,h,n}^+ = [\Lambda_{h'} + \delta, \Lambda_{h'} - \delta] \times -m_{2,n}, -m_{1,n} \rightarrow \mathbb{R} \) with

\[
\sup_{T_{\delta,h,n}^+} |\overline{v}_n| < \varepsilon, \quad \sup_{T_{\delta,h,n}^+} |D^j \overline{v}_n| < \varepsilon \quad \text{for any } j \in \mathbb{N},
\]

and the inequality (4.9) also holds over \( \Lambda_{4,n} \). Consequently, we get that the function \( \eta_2/\eta_3 \) tends to zero as \( p \rightarrow \infty \). In particular, there exists an interior point where the function \( \eta_2/\eta_3 \) attains either a local minimum or a local maximum. From the equation (2.11) and the convexity of \( \varphi \), we deduce that \( \eta_2 \) vanishes everywhere and then, \( \Sigma \) is invariant under translations in the direction \( \vec{e}_2 \) as we wanted.

**Remark 4.6.** Notice that if \( \Lambda_\lambda \) is strictly decreasing with respect to initial data \( \lambda \), then we can assure that \( \Sigma \) coincides with \( G^h \).

## 5 Concluding remarks

- As a direct consequence of the proof of Theorem 1.4, we can change the hypothesis of to be graph in the Theorem 1.4 by the following conditions.

- Suppose that \( \varphi \) is a strictly increasing convex smooth function with at most a linear growth such that \( \dot{\varphi} \leq 0 \) and assume that \( \Sigma \) is a connected complete mean convex properly embedded \([\varphi, \vec{e}_3]\)-minimal immersion with locally bounded genus \( C^\infty \)-asymptotic to some \([\varphi, \vec{e}_3]\)-catenary cylinder, outside a cylinder. Notice that, \( \dot{\varphi} \) and \( \ddot{\varphi} \) are bounded. Thus, \( \Sigma \) is closed to the \([\varphi, \vec{e}_3]\)-catenary cylinder in \( C^2 \)-topology. In particular the Gauss curvature of \( \Sigma \) is bounded. From the section 4. and the Theorem B of [12], \( \Sigma \) has locally uniformly bounded intrinsic area and \( K \geq 0 \) everywhere. If we prove that \( \eta_2 = 0 \) everywhere, then there exists a point where the Gauss curvature vanishes. Consequently, from the Theorems 2.5 and 3.7 of [11], \( \Sigma \) is flat and it must be coincide with some \([\varphi, \vec{e}_3]\)-catenary cylinder.

- On the other hand, an interesting question is whether a complete properly embedded \([\varphi, \vec{e}_3]\)-minimal surface with locally bounded genus \( C^\infty \)-asymptotic, outside a cylinder, to some \([\varphi, \vec{e}_3]\)-catenary cylinder is a vertical graph with \( \varphi \) under the conditions of theorem 1.4. In such case, we can generalize the result of F. Martín, J. Pérez-García, A. Savas-Halilaj and K. Smoczyk [9].

---

19
Finally, from the remark 4.4, it would be interesting to prove the veracity of the Theorem 1.4 when $\phi$ is strictly increasing convex smooth function but $e^{-\phi}$ is not integrable.

References

[1] Alexandrov, A.D.: Uniqueness theorems for surfaces in the large. Vestnik. Leninger. Univ. Math. 11, 5-17 (1956).

[2] U. Dierkes: Singular Minimal Surfaces, S. Hildebrandt et. al (eds.), Geometric Analysis and Nonlinear Partial Differential Equations, (2003) 177-193.

[3] U. Dierkes, G. Huisken: The N-dimensional analogue of the catenary: Prescribed area. J. Jost (ed) Calculus of Variations and Geometric Analysis. International Press, (1996) 1-13.

[4] D. Hoffman, T. Ilmanen, F. Martín and B. White: Notes on translating solitons for mean curvature flow, prePrint, arXiv: 1901.09101v1, (2019).

[5] D. Hoffman, T. Ilmanen, F. Martín and B. White: Graphical translators for mean curvature flow. Calc. Var. Partial Differential Equations 58 (2019), no. 4, Paper No. 117, 29 pp.

[6] Gilbarg, D., Trudinger. N.S: Elliptic Partial Differential Equations of Second Order, Classics in Mathematics. Springer, Berlin (2001). (Reprint of the 1998 Edition).

[7] T. Ilmanen: Elliptic regularization and partial regularity for motion by mean curvature. Men. Amer. Math. Soc, 108 (1994).

[8] R. López : Constant Mean Curvature Surfaces with Boundary. Springer Monographs in Mathematics, (2013).

[9] F.Martín, J. Pérez-García, A.Savas-Halilaj and K.Smoczyk: A characterization of the grim reaper cylinder, Journal fur die reine und angewandte Mathematik (2016), 1-28.

[10] Martín,F, A.Savas-Halilaj and Smoczyk,K: On the topology of translating solitons of the mean curvature flow, Cal. Var. 54 (2015):2853-2882.

[11] A. Martínez. and A. L. Martínez Triviño: Equilibrium of Surfaces in a Vertical Force Field. Preprint: arXiv: 1910.07795, (2019).

[12] A. Martínez, A.L. Martínez Triviño and J.P. dos Santos: Mean convex properly embedded $[\varphi, e_3]$-minimal surfaces in $\mathbb{R}^3$. arXiv: 2011.15029v1 (2020).

[13] S.D.Poisson. Sur les surfaces elastique. Men. CL. Sci. Math. Phys. Inst. Frace, deux, 167-225 (1975).
[14] J. Spruck, and L. Xiao: Complete translating solitons to the mean curvature flow in $\mathbb{R}^3$ with nonnegative mean curvature, Amer. J. Math. 142 (2020), no. 3, 993–1015.

[15] X. Wang: Convex solutions to the mean curvature flow of mean-convex sets, J. Amer. Math. Soc, pages 123-138, 16 (2003).

[16] B. White: Controlling area blow-up in minimal or bounded mean curvature varities . J.Differential Geom. Volume 102, no.3 (2016), 501-535.

[17] B. White: On the compactness theorem for embedded minimal surfaces in 3-manifolds with locally bounded area and genus Comm. Anal. Geom. 26 (2018), no. 3, 659–678.