Holographic superconductivity in M-Theory

Jerome P. Gauntlett, Julian Sonner and Toby Wiseman
1Theoretical Physics Group, Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.
2The Institute for Mathematical Sciences, Imperial College, London SW7 2PE, U.K.

Using seven-dimensional Sasaki-Einstein spaces we construct solutions of $D = 11$ supergravity that are holographically dual to superconductors in three spacetime dimensions. Our numerical results indicate a new zero temperature solution dual to a quantum critical point.

INTRODUCTION

The AdS/CFT correspondence provides a powerful framework for studying strongly coupled quantum field theories using gravitational techniques. It is an exciting possibility that these techniques can be used to study classes of superconductors which are not well described by more standard approaches [1][2][3].

The basic setup requires that the CFT has a global abelian symmetry corresponding to a massless gauge field propagating in the $AdS$ space. We also require an operator in the CFT that corresponds to a scalar field that is charged with respect to this gauge field. Adding a black hole to the $AdS$ space describes the CFT at finite temperature. One then looks for cases where there are high temperature black hole solutions with no charged scalar hair but below some critical temperature black hole solutions with charged scalar hair appear and moreover dominate the free energy. Since we are interested in describing superconductors in flat spacetime we consider black holes with planar symmetry. In order to obtain a critical temperature, conformal invariance then implies that another scale needs to be introduced. This is achieved by considering electrically charged black holes which corresponds to studying the dual CFT at finite chemical potential.

Precisely this set up has been studied using a phenomenological model of gravity in $D = 4$ coupled to a single charged scalar field and it has been shown that, for certain parameters, the system manifests superconductivity in three spacetime dimensions, in the above sense [2]. It is important to go beyond such models and construct solutions in the context of string/M-theory so that there is a consistent underlying quantum theory and CFT dual. Also, as we shall see, the behaviour of the string/M-theory solutions will differ substantially from that of the phenomenological model [2] at low temperature. It was shown in [2] that the $D = 4$ phenomenological models of [2] arise, at the linearised level, after Kaluza-Klein (KK) reduction of $D = 11$ supergravity on a seven-dimensional Sasaki-Einstein space $SE_7$. Here we go beyond this linearised approximation by working with a consistent truncation of the $D = 4$ KK reduced theory presented in [2]. The truncation is consistent in the sense that any solution of this $D = 4$ theory, combined with a given $SE_7$ metric, gives rise to an exact solution of $D = 11$ supergravity. Here we shall use this $D = 4$ theory to construct exact solutions of $D = 11$ supergravity that correspond to holographic superconductivity.

THE KK TRUNCATION

We begin by recalling that any $SE_7$ metric can, locally, be written as a fibration over a six-dimensional Kähler-Einstein space, $KE_6$:

$$ds^2(SE_7) = ds^2(KE_6) + \eta \otimes \eta$$

Here $\eta$ is the one-form dual to the Reeb Killing vector satisfying $d\eta = 2J$ where $J$ is the Kähler form of $KE_6$. We denote the $(3,0)$ form defined on $KE_6$ by $\Omega$. For a regular or quasi-regular $SE_7$ manifold, the orbits of the Reeb vector all close, corresponding to compact $U(1)$ isometry, and the $KE_6$ is a globally defined manifold or orbifold, respectively. For an irregular $SE_7$ manifold, the Reeb-vector generates a non-compact $\mathbb{R}$ isometry and the $KE_6$ is only locally defined.

In the KK ansatz of [2] the $D = 11$ metric is written

$$ds^2 = e^{-6U-V} ds_4^2 + e^{2U} ds^2(KE_6) + e^{2V} (\eta + A_1) \otimes (\eta + A_1)$$

while the four-form is written

$$G_4 = 6e^{-18U-3V} (\epsilon + h^2 + |\chi|^2) vol_4 + H_3 \wedge (\eta + A_1) + H_2 \wedge J + dh \wedge J \wedge (\eta + A_1) + 2hJ \wedge J + \sqrt{3} [ \chi (\eta + A_1) \wedge \Omega - \frac{1}{2} D\chi \wedge \Omega + c.c. ]$$

where $ds_4^2$ is a four-dimensional metric (in Einstein frame), $U, V, h$ are real scalars, and $\chi$ is a complex scalar defined on the four-dimensional space. Furthermore, also defined on this four-dimensional space are $A_1$ a one-form potential, with field strength $F_2 \equiv dA_1$, two-form and three-form field strengths $H_2$ and $H_3$, related to one-form and two-form potentials via $H_3 = dB_2$ and $H_2 = dB_1 + 2B_2 + hF_2$. Finally $D\chi \equiv d\chi - 4i A_1 \chi$.

This is a consistent KK truncation of $D = 11$ supergravity in the sense that if the equations of motion for the 4d-fields $ds_4^2, U, V, A_1, H_2, H_3, h, \chi$ as given in [2] are satisfied then so are the $D = 11$ equations. The $D = 4$ equations of motion admit a vacuum solution with vanishing matter fields which uplifts to the $D = 11$ solution:

$$ds^2 = \frac{1}{4} ds^2(AdS_4) + ds^2(SE_7), \quad G_4 = e^{-\frac{3}{8}} Vol(AdS_4)$$

where $ds^2(AdS_4)$ is the standard unit radius metric. When $\epsilon = +1$, this $AdS_4 \times SE_7$ solution is supersymmetric and describes $M2$-branes sitting at the apex of
the Calabi-Yau four-fold (CY$^4$) cone whose base space is given by the $SE_7$. When $\epsilon = -1$ the solution is a “skew-whiffed” $AdS_4 \times SE_7$ solution, which describes anti-M2-branes sitting at the apex of the CY$^4$ cone. These solutions break all of the supersymmetry except for the special case when the $SE_7$ is the round seven-sphere, $S^7$, in which case it is maximally supersymmetric. Note that the skew-whiffed solutions with $SE_7 \neq S^7$ are perturbatively stable ⁷, despite the absence of supersymmetry. Thus such backgrounds should be dual to three-dimensional CFTs at least in the strict $N = \infty$ limit. We are most interested in the skew-whiffed case because it is for that case that the operator dual to $\chi$ has scaling dimensions $\Delta = 1$ or 2 ⁸ and, based on the work of ⁹, is when we expect holographic superconductivity.

The $D = 4$ equations of motion may be derived from a four-dimensional action given in ⁹. It is convenient to work with an action that is obtained after dualising the one-form $B_1$ to another one form $\tilde{B}_1$ and the two-form $B_2$ to a scalar $a$ as explained in section 2.3 of ⁸. The dual fields are related to the original fields via

$$
H_3 = -e^{-12U} \left[ da + 6(\tilde{B}_1 - \epsilon A_1) - \frac{3}{2} i (\chi^* D \chi - \chi D \chi^*) \right]
$$

$$
H_2 = (4 h^2 + e^{4U + 2V} r^2 - e^{-2U - V} \pi G \tau) \left( \tilde{H}_2 + h^2 F_2 \right)
$$

where $\tilde{H}_2 \equiv d \tilde{B}_1$. We now restrict to the (skew-whiffed) case $\epsilon = -1$. For this case we can make the following additional truncation of the $D = 4$ theory:

$$
a = h = 0, \quad V = -2U, \quad A_1 = -\tilde{B}_1, \quad e^{6U} = 1 - \frac{4}{3} |\chi|^2
$$

One can show that provided that we restrict to configurations satisfying $F_2 \wedge F_2 = 0$ we obtain equations of motion that can be derived from the action

$$
S = \frac{1}{16\pi G} \int d^4 x \sqrt{-g} \left[ R - \frac{1}{2} F_{\mu \nu} F^{\mu \nu} + (1 - \frac{1}{2} |\chi|^2)^{-2} \left( -|D \chi|^2 + 24(1 - \frac{3}{2} |\chi|^2) \right) \right]
$$

where $D \chi \equiv d \chi - 2 i \tilde{A}_1 \chi$, and we have defined $\tilde{A}_1 \equiv 2 A_1$, $\tilde{\chi} \equiv (3/2)^{1/2} \chi$. Linearizing in the complex scalar $\tilde{\chi}$ this gives the action considered in ⁸ (with their $L = 1/2$ and their $q = 2$). This non-linear action is in the class considered in ⁸ and in addition to the $AdS_4$ vacuum with $A_1 = 0$ and $\tilde{\chi} = 0$, which uplifts to ⁹, it also admits $AdS_4$ vacua with $\tilde{A}_1 = 0$ and constant $|\tilde{\chi}| = 1$, which uplift to the $D = 11$ solutions ¹² found in ⁸.

**BLACK HOLE SOLUTIONS**

The key result of the last section is that any solution to the $D = 4$ equations of motion of the action (7) with $\tilde{F} \wedge \tilde{F} = 0$, gives an exact solution of $D = 11$ supergravity for any $SE_7$ metric. To find solutions relevant for studying superconductivity via holography we consider the following ansatz

$$
ds^2 = -ge^{-\beta} dt^2 + g^{-1} dr^2 + r^2 (dx^2 + dy^2)
$$

$$
\tilde{A}_1 = \hat{\phi} dt, \quad \tilde{\chi} \equiv r \equiv x \in \mathbb{R}
$$

where $g, \beta, \hat{\phi}$ and $\sigma$ are all functions of $r$ only. Being purely electrically charged this satisfies the $\tilde{F} \wedge \tilde{F} = 0$ condition. After substituting into the equations of motion arising from (7), we are led to ordinary differential equations which can also be obtained from the action obtained by substituting the ansatz directly into (7).

$$
S = e \int dr e^{-\beta/2} \left[ -g'' + g' \left( \frac{3}{2} \beta' - \frac{4}{r} \right) \right.
$$

$$
+ g(\beta'' - \frac{1}{2} (\beta')^2 + 2 \frac{\beta'}{r} - \frac{2}{r^2}) + \frac{1}{2} e^\beta (\dot{\phi})^2
$$

$$
\left. + \left( 1 - \frac{1}{2} \sigma^2 \right)^2 \left( -g'(\sigma'^2) + 4 g^{-1} e^\beta \sigma^2 + 24(1 - \frac{3}{2} \sigma^2) \right) \right]
$$

where $c = (16\pi G)^{-1} \int dt dx dy$.

We next observe that the system admits the following exact AdS Reissner-Nordström type solution $\sigma = \beta = 0$

$$
g = 4 r^2 - \frac{1}{r} (4 r^3 + \frac{\alpha^2}{r^2}) + \frac{\alpha^2}{r^2}, \quad \tilde{\phi} = \alpha (\frac{1}{r} - \frac{1}{r})
$$

for some constants $\alpha, r_+$. The horizon is located at $r = r_+$ and for large $r$ it asymptotically approaches $1/4$ of a unit radius $AdS_4$ (see ⁸). This solution should describe the high temperature phase of the superconductor.

We are interested in finding more general black hole solutions with charged scalar hair, $\sigma \neq 0$. Let us examine the equations at the horizon and at infinity. At the horizon $r = r_+$ we demand that $g(r_+) = \tilde{\phi}(r_+) = 0$. One then finds that the solution is specified by 4 parameters at the horizon $r_+, \beta(r_+), \phi(r_+), \sigma(r_+)$. At $r = \infty$ we have the asymptotic expansion

$$
\beta = \beta_a + \ldots, \quad \frac{\sigma}{\sqrt{8\pi G}} = \frac{\sigma_1}{r} + \frac{\sigma_2}{r^2} + \ldots,
$$

$$
\frac{\tilde{\phi}}{\sqrt{16\pi G}} = e^{-\beta_a/2} (\tilde{\mu} - \frac{\tilde{q}}{r}) + \ldots
$$

$$
e^{-\beta} g = e^{-\beta_a} (4 r^2 - \frac{8 \pi G (m + \frac{4}{3} \sigma_1 \sigma_2)}{r}) + \ldots
$$

determined by the data $\beta_a, \sigma_{1,2}, m, \tilde{\mu}, \tilde{q}$. The scaling

$$
r \rightarrow a r, (t, x, y) \rightarrow a^{-1} (t, x, y), g \rightarrow a^2 g, \tilde{\phi} \rightarrow a \tilde{\phi}
$$

leaves the metric, $A_1$, and all equations of motion invariant.

**Action and thermodynamics**

We analytically continue by defining $\tau \equiv i t$. The temperature of the black hole is $T = e^{\epsilon \beta_a/2}/\Delta \tau$ where $\Delta \tau$ is
fixed by demanding regularity of the Euclidean metric at $r = r_+$. We find:

$$T = \frac{r_+ e^{\beta_s - \beta/2}}{4\pi} \left[ \frac{12}{(1 - \frac{4}{3}\sigma^2)^2} - \frac{4}{3}e^\beta \phi'^2 \right]_{r=r_+} \quad (13)$$

Defining $I \equiv -iS$, we can calculate the on-shell Euclidean action $I_{OS}$

$$I_{OS} = \frac{\Delta \tau v_0}{16\pi G} \int_{r_+}^{\infty} dr \left[ r^2 e^{-\beta/2} (g' - g\beta' - e^{\phi^2})' \right]'$$

$$= \frac{\Delta \tau v_0}{16\pi G} \int_{r_+}^{\infty} dr \left[ 2r e^{-\beta/2} \right]' \quad (14)$$

where $v_0 \equiv \int dx dy$. The latter expression only gets contributions from the on-shell functions at $r = \infty$ since $g(r_+) = 0$, while the former expression gets contributions from $r = r_+$ and $r = \infty$. The on-shell action diverges and we need to regulate by adding appropriate counter terms. We define $I_{Tot} \equiv I + I_{ct}$ and, for simplicity, we will focus on the following counter-term action $I_{ct}$:

$$I_{ct} = \frac{1}{16\pi G} \int d\tau d^2 x \sqrt{g_{\infty}} \left[ -2K + 8 + 2\sigma^2 \right] \quad (15)$$

where $K = \frac{\mu}{\sqrt{\infty}} \nabla \nu \nu_{\nu}$ is the trace of the extrinsic curvature. For our class of solutions we find

$$I_{ct} = \frac{\Delta \tau v_0}{16\pi G} \lim_{r \to \infty} e^{-\beta/2} \left[ - r^2 g' + r^2 g\beta' - 4gr + r^2 g^{1/2} (8 + 2\sigma^2) \right] \quad (16)$$

Notice that under a variation of the action $I_{Tot}$ with respect to $\beta, g, \phi$ yields the equations of motion together with surface terms. For an on-shell variation the only terms remaining are these surface terms, and after substituting the asymptotic boundary expansion (11) (higher order terms are also required) we find

$$[\delta I_{Tot}]_{OS} = \frac{\Delta \tau v_0}{16\pi} e^{-\beta/2/2} \left[ \left( -\frac{1}{2}m + \frac{1}{2}\hat{\mu} \hat{q} \right) \delta \beta_a - \hat{q} \delta \hat{\mu} - 4\sigma_2 \delta \sigma_1 \right] \quad (17)$$

Note that we are keeping $\Delta \tau$ fixed in this variation. Hence we see that $I_{Tot}$ is stationary for fixed temperature and chemical potential (ie. $\delta \beta_a = \delta \hat{\mu} = 0$) and for either $\sigma_2 = 0$ or fixed $\sigma_1$.

We also find that the on-shell total action is given by

$$[I_{Tot}]_{OS} = \frac{v_0}{T} \left[ m - \hat{\mu} \hat{q} - Ts \right]$$

$$= \frac{v_0}{T} \left[ -\frac{1}{2}m - 2\sigma_1 \sigma_2 \right] \quad (18)$$

where $s = \frac{T^4}{3\pi^2}$ is the entropy density of the solution and $m$ is the energy density. The two forms of the on-shell action come from the two ways of writing the action as a total derivative given above. We note that the equality of these expressions imply a Smarr-like relation. Also note that after using $\delta \beta_a = 2\delta T/T$ (since $\Delta \tau$ is held fixed) the equality of (17) and the variation of the first line of (13) imply a first law,

$$\delta m = T \delta s + \hat{\mu} \delta \hat{q} - 4\sigma_2 \delta \sigma_1 \quad (19)$$

Both this Smarr relation and the first law were used to confirm the accuracy of our numerical solutions below.

For simplicity we restrict discussion to solutions with boundary condition $\sigma_1 = 0$ (13) and we interpret $T_{I_{Tot}} = (v_0/2)(-m/2)$ as a thermodynamic potential, $\Omega(T, \mu)$. Note also that $\sigma_2$ then determines the vev of the operator dual to $\chi$. Recall from (14) that writing $U = -u + v/3$, $V = 6u + v/3$, the fields $u, v$ are dual to operators $O_{u,v}$ with dimensions $\Delta_u = 4, \Delta_v = 6$. The truncation (14) implies that the vevs of these dual operators are fixed by $\sigma_2$. The asymptotic expansion of $u$ to $o(1/r^4)$ and $v$ to $o(1/r^6)$ gives $\langle O_u \rangle \propto \sigma_2^2$ and $\langle O_v \rangle \propto \mu^2 \sigma_2^2$.

Numerical Results

Following (3) we solved the differential equations numerically using a shooting method. We used (12) to fix the scale $\mu = 1$. At high temperatures the black hole solutions have no scalar hair ($\sigma_2 = 0$) and are just the solutions given in (10). At a critical temperature $T_c \sim 0.042$ a new branch of solutions with $\sigma_2 \neq 0$ appears and moreover dominates the free energy. We refer to these as the unbroken and broken phase solutions, corresponding to normal and superconducting phases, respectively. In the figures we have plotted some features of our solutions and compared them with the solutions of the phenomenological model considered in (3).

While the results are in agreement near the critical temperature, as expected, we see marked differences as the temperature goes to zero. We have calculated the Ricci scalar and curvature invariant $\sqrt{R_{\alpha \beta \mu \nu} R^{\alpha \beta \mu \nu}}$ at $r = r_+$ which indicate that the solutions of (3) are becoming singular but our solutions are approaching a regular zero temperature solution, without horizon, holographically dual to a quantum critical point. Indeed as $r \to r_+$ we find $\sigma \sim 1, \beta \sim \text{const}, \phi \sim 0$ and $g \sim \frac{1}{2} \left( r^2 - r_+^2 / r \right)$ and fixing $\hat{\mu} = 1$ gives $r_+ \to 0$ in the extremal limit. In particular, the geometry near $r = r_+$ is consistent with being the exact $AdS_4$ solution with $\sigma = 1$, mentioned earlier, which uplifts to the $D = 11$ solution found in (9). For such a solution $R = -64$ and $\sqrt{R_{\alpha \beta \mu \nu} R^{\alpha \beta \mu \nu}} = 32$, agreeing with the low temperature limit seen in the figures. The full zero temperature solution thus appears to be a charged domain wall, of the type considered in (11), connecting two $AdS_4$ vacua of (7), one with $\sigma = 0$ and the other with $\sigma = 1$. Interestingly this implies the entropy of the solutions vanishes in the low temperature limit, unlike for the Reissner-Nordström solution (10), (13). The asymptotic charge appears to be derived from the scalar hair, with the region near $r = r_+$ carrying no flux.
CONCLUDING REMARKS

For any seven-dimensional Sasaki-Einstein space we have constructed solutions of $D = 11$ supergravity corresponding to holographic superconductors in three spacetime dimensions. We have studied electric black holes using the action (7) whose solutions lift to $D = 11$ when $F \wedge \tilde{F} = 0$. One may consider adding magnetic charge using the full consistent truncation of [5]. Our results indicate the existence of a regular zero temperature solution which is a charged domain wall connecting two $AdS_4$ vacua of (7) and dual to a new quantum critical point. An important open issue is whether or not there are additional unstable charged modes for skew-whiffed $AdS_4 \times SE_7$ solutions, which condense at higher temperatures. If they exist, and dominate the free energy, then the corresponding supergravity solutions would be the appropriate ones to describe the superconductivity and not the ones that we have constructed. However, it is plausible that we have found the dominant modes for large classes of $SE_7$, if not all. For the specific class of deformations of the four-form that were considered in [4], it was proven that the modes that we consider are in fact the only condensing modes. It would be worthwhile extending this result to cover other bosonic and/or fermionic modes.

Note added: after this work was completed we received [7] which constructs solutions of string theory that are dual to superconductors in four spacetime dimensions.

We are supported by EPSRC (JG,JS), the Royal Society (JG) and STFC (TW). We thank R. Emparan, S. Hartnoll, G. Horowitz, V. Hubeny, M. Rangamani, O. Varela and D. Waldram for helpful discussions.

[1] S. S. Gubser, Phys. Rev. D 78 (2008) 065034 [arXiv:0801.2977 [hep-th]].
[2] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, Phys. Rev. Lett. 101 (2008) 031601 [arXiv:0803.3295 [hep-th]].
[3] S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, JHEP 0812, 015 (2008) [arXiv:0810.1563 [hep-th]].
[4] F. Denef and S. A. Hartnoll, arXiv:0901.1160 [hep-th].
[5] J. P. Gauntlett, S. Kim, O. Varela and D. Waldram, JHEP 0904 (2009) 102 [arXiv:0901.0676 [hep-th]].
[6] M. J. Duff, B. E. W. Nilsson and C. N. Pope, Phys. Lett. B 139 (1984) 154.
[7] S. S. Gubser, C. P. Herzog, S. S. Pufu and T. Tesileanu, arXiv:0907.3531 [hep-th].
[8] S. Franco, A. Garcia-Garcia and D. Rodriguez-Gomez, arXiv:0906.1214 [hep-th].
[9] C. N. Pope and N. P. Warner, Phys. Lett. B 150 (1985) 352; C. N. Pope and N. P. Warner, Class. Quant. Grav. 2 (1985) L1.
[10] L. J. Romans, Phys. Lett. B 153 (1985) 392.
[11] S. S. Gubser and F. D. Rocha, Phys. Rev. Lett. 102 (2009) 061601 [arXiv:0807.1337 [hep-th]].
[12] There are analogous AdS$_5$ solutions of the theory considered in [7] which uplift to IIB solutions found in [10].
One may also consider fixing $\sigma_2 = 0$ with similar results. Non-zero $\sigma_1$ is less interesting as we want the scalar to condense without being sourced.

We thank Gary Horowitz for a discussion on this point.