LQR-based coupling gain for synchronization of linear systems

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Abstract

Synchronization control of coupled continuous-time linear systems is studied. For identical systems that are stabilizable, a linear feedback law obtained via algebraic Riccati equation is shown to synchronize any fixed directed network of any number of coupled systems provided that the coupling is strong enough. The strength of coupling is determined by the smallest distance of a nonzero eigenvalue of the coupling matrix to the imaginary axis. A dual problem where detectable systems that are coupled via their outputs is also considered and solved.

1 Introduction

Synchronization is, at one hand, a desired behaviour in many dynamical systems related to numerous technological applications [9, 10, 8]; and, at the other, a frequently-encountered phenomenon in biology [21, 25, 1]. On top of that, it is an important system theoretical topic on its own right. For instance, there is nothing keeping us from seeing a simple Luenberger observer [12] with decaying error dynamics as a two-agent system where the agents globally synchronize. Since people lack anything but good reasons to investigate synchronization, a wealth of literature has been formed, mostly in recent times [6, 22, 26].

An essential problem from a control theory point of view is to find conditions that imply synchronization of a number of coupled individual systems. This problem, which is usually studied under the name synchronization stability, has been attacked by many and from various angles. Two cases are of particular interest: (i) where the dynamics of individual systems are primitive (such as that of an integrator) yet the coupling between them is considered time-varying; and (ii) where the individual systems are let be more sophisticated, however, coupled via a fixed interconnection. The studies concentrated on the first case have resulted in the emergence of the area now known as consensus in multi-agent systems [15] [14, 7, 11] [18] [2] [5] [23] where fairly weak conditions on the interconnection have been established under which the states of individual systems converge to a common point that is fixed in space. The second case,
into which the problem studied in this note falls, has also accommodated important theoretical developments especially by using tools from algebraic graph theory [27]. Using Lyapunov functions, it has been shown that spectrum of the coupling matrix plays a crucial role in determining the stability of synchronization [29, 16] notwithstanding it need not necessarily be explicitly known [4]. It has also been shown that passivity theory can be useful in studying stability provided that the interconnection is symmetric [3, 17, 20].

In this note we consider identical individual system dynamics that are linear time-invariant ˙\(x = Ax + Bu\). Under the weakest possible assumption that pair \((A, B)\) is stabilizable, we search for a feedback law \(\kappa(A, B)\) which would guarantee asymptotic synchronization for any fixed (directed) interconnection of arbitrary number of coupled systems provided that the coupling is strong enough. Following the tradition, we use the spectral information of the coupling matrix to determine the strength of connectedness. Namely, the farther the second eigenvalue (with largest real part) from the imaginary axis the more connected the network. We show that a linear \(\kappa\) solving a linear quadratic regulation (LQR) problem performs the task.

It is worth noting that most of the existing work on synchronization focuses on analysis rather than design. Also, the individual system dynamics are usually taken to be stable (i.e. the trajectories of the uncoupled system are required to be bounded.) In those respects the issue we deal with in this work is relatively different. Of particular relevance to this note are the works [24] and [28]. The former provides, for linear individual system dynamics, a linear feedback law that guarantees synchronization for all connected interconnections (regardless of the strength of coupling) under the extra assumption that matrix \(A\) is neutrally stable. The latter establishes sufficient conditions for nonlinear individual system dynamics so that synchronization is achieved for strong enough coupling.

In the remainder of the paper we first provide notation and definitions. Then, in Section 3 we formalize the problem and state our objectives. In Section 4 we show via our main theorem that optimal control theory yields us a feedback law which serves our purpose, i.e. synchronizes coupled systems for all network topologies with strong enough coupling. Finally, in Section 5 a dual problem is formulated and solved.

2 Notation and definitions

Let \(\mathbb{R}_{\geq 0}\) denote set of nonnegative real numbers and \(|\cdot|\) 2-norm. For \(\lambda \in \mathbb{C}\) let \(\text{Re}(\lambda)\) denote the real part of \(\lambda\). Identity matrix in \(\mathbb{R}^{n\times n}\) is denoted by \(I_n\) and zero matrix in \(\mathbb{R}^{m\times n}\) by \(0_{m\times n}\). Conjugate transpose of a matrix \(A\) is denoted by \(A^H\). Matrix \(A \in \mathbb{C}^{n\times n}\) is Hurwitz if all of its eigenvalues have strictly negative real parts.\(^1\) Given \(B \in \mathbb{R}^{n\times m}\), \(C \in \mathbb{R}^{m\times n}\), and \(A \in \mathbb{R}^{n\times n}\); pair \((A, B)\) is stabilizable if there exists \(K \in \mathbb{R}^{m\times n}\) such that \(A - BK\) is Hurwitz; pair \((C, A)\)

\(^1\)Note that \(A\) is Hurwitz if there exists a symmetric positive definite matrix \(P\) such that \(A^H P + PA < 0\).
is detectable if \((A^T, C^T)\) is stabilizable. Let \(1 \in \mathbb{R}^p\) denote the vector with all entries equal to one.

**Kronecker product** of \(A \in \mathbb{C}^{m \times n}\) and \(B \in \mathbb{C}^{p \times q}\) is

\[
A \otimes B := \begin{bmatrix}
    a_{11}B & \cdots & a_{1n}B \\
    \vdots & \ddots & \vdots \\
    a_{m1}B & \cdots & a_{mn}B
\end{bmatrix}
\]

Kronecker product comes with the property \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\) (provided that products \(AC\) and \(BD\) are allowed.)

A (directed) graph is a pair \((\mathcal{N}, \mathcal{A})\) where \(\mathcal{N}\) is a nonempty finite set (of nodes) and \(\mathcal{A}\) is a finite collection of pairs (arcs) \((n_i, n_j)\) with \(n_i, n_j \in \mathcal{N}\). A path from \(n_1\) to \(n_\ell\) is a finite collection of nodes \(\{n_1, n_2, \ldots, n_\ell\}\) such that \((n_i, n_{i+1})\) is an arc for \(i \in \{1, 2, \ldots, \ell - 1\}\). A graph is connected if it has a node to which there exists a path from every other node.\(^3\)

The graph of a matrix \(\Gamma := [\gamma_{ij}] \in \mathbb{R}^{p \times p}\) is the pair \((\mathcal{N}, \mathcal{A})\) where \(\mathcal{N} = \{n_1, n_2, \ldots, n_p\}\) and \((n_i, n_j) \in \mathcal{A}\) iff \(\gamma_{ij} > 0\). Matrix \(\Gamma\) is said to be connected if it satisfies:

(i) \(\gamma_{ij} \geq 0\) for \(i \neq j\);

(ii) each row sum equals 0;

(iii) its graph is connected.

A connected \(\Gamma\) has an eigenvalue at \(\lambda = 0\) with eigenvector \(1\), i.e. \(\Gamma 1 = 0\), and all its other eigenvalues have real parts strictly negative.\(^3\) When we write \(\text{Re}(\lambda_2(\Gamma))\) we mean the real part of a nonzero eigenvalue of \(\Gamma\) closest to the imaginary axis.

Given maps \(\xi_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^n\) for \(i \in \{1, 2, \ldots, p\}\) and a map \(\bar{\xi} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n\), the elements of the set \(\{\xi_i(\cdot) : i = 1, 2, \ldots, p\}\) are said to synchronize to \(\bar{\xi}(\cdot)\) if \(|\xi_i(t) - \bar{\xi}(t)| \to 0\) as \(t \to \infty\) for all \(i\). The elements of the set \(\{\xi_i(\cdot) : i = 1, 2, \ldots, p\}\) are said to synchronize if they synchronize to some \(\bar{\xi}(\cdot)\).

### 3 Problem statement

#### 3.1 Systems under study

We consider \(p\) identical linear systems

\[
\dot{x}_i = Ax_i + Bu_i, \quad i = 1, 2, \ldots, p
\]

where \(x_i \in \mathbb{R}^n\) is the state and \(u_i \in \mathbb{R}^m\) is the input of the \(i\)th system. Matrices \(A\) and \(B\) are of proper dimensions. The solution of \(i\)th system at time \(t \geq 0\) is

\[^2\]Note that this definition of connectedness for directed graphs is weaker than strong connectivity and stronger than weak connectivity. In [28] an equivalent condition (for connectedness) is given as that the graph contains a spanning tree.

\[^3\]For the sake of completeness we provide a proof of this fact in Appendix.
denoted by \( x_i(t) \). In this paper we consider the case where at each time instant (only) the following information

\[
z_i = \sum_{j=1}^{p} \gamma_{ij}(x_j - x_i) \quad (2)
\]

is available to \( i \)th system to determine an input value where \( \gamma_{ij} \) are the entries of the matrix \( \Gamma \in \mathbb{R}^{p \times p} \) describing the network topology. Nondiagonal entries of \( \Gamma \) are nonnegative and each row sums up to zero. That is, the coupling between systems is diffusive.

### 3.2 Assumptions made

We only make the following assumption on systems (1) which will henceforth hold.

(A1) Pair \((A, B)\) is stabilizable.

### 3.3 Objectives

We have two objectives in this paper.

(O1) Show that for each \( \delta > 0 \) there exists a linear feedback law \( K \in \mathbb{R}^{m \times n} \) such that, for all \( p \) and connected \( \Gamma \in \mathbb{R}^{p \times p} \) with \(-\text{Re}(\lambda_2(\Gamma)) \geq \delta\), solutions of systems (1) for \( u_i = Kz_i \), where \( z_i \) is as in (2), globally (i.e. for all initial conditions) synchronize.

(O2) Compute one such \( K \).

### 4 Main result

For later use in this section we first borrow a well-known result from optimal control theory [19]: Given a stabilizable pair \((A, B)\), where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \), the following algebraic Riccati equation

\[
A^T P + PA + I_n - PB B^T P = 0 \quad (3)
\]

has a (unique) solution \( P = P^T > 0 \). One can rewrite (3) as

\[
(A - B B^T P)^T P + P(A - B B^T P) + (I_n + P B B^T P) = 0
\]

whence we infer that \( A - B B^T P \) is Hurwitz.

**Lemma 1** Let \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) satisfy (3) for some symmetric positive definite \( P \). Then for all \( \sigma \geq 1 \) and \( \omega \in \mathbb{R} \) matrix \( A - (\sigma + j \omega) B B^T P \) is Hurwitz.
Proof. Let $\varepsilon := \sigma - 1 \geq 0$. Write
\[
(A - (\sigma + j\omega)BB^TP)^H P + P(A - (\sigma + j\omega)BB^TP) = (A - (\sigma - j\omega)BB^TP)^T P + P(A - (\sigma - j\omega)BB^TP) = (A - (1 + \varepsilon)BB^TP)^T P + P(A - (1 + \varepsilon)BB^TP) = (A - BB^TP)^T P + P(A - BB^TP) - 2\varepsilon PBB^T P = -\ln - (1 + 2\varepsilon)PBB^T P. (4)
\]
Finally, observe that (4) is nothing but (complex) Lyapunov equation.

Below is our main result.

**Theorem 1** Consider systems (1). Let $K := BB^TP$ where $P$ is the solution to (3). Given $\delta > 0$, for all $p$ and connected $\Gamma \in \mathbb{R}^{p \times p}$ with $-\text{Re}(\lambda_2(\Gamma)) \geq \delta$, solutions $x_i(\cdot)$ for $i = 1, 2, \ldots, p$ and $u_i = \max\{1, \delta^{-1}\} K z_i$, where $z_i$ is as in (2), globally synchronize to $\tilde{x}(t) := (r^T \otimes e^{At}) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}$.

Proof. Let $K_\delta := \max\{1, \delta^{-1}\} K$. Combine (1) and (2) to obtain
\[
\dot{x}_i = Ax_i + BK_\delta \sum_{j=1}^{p} \gamma_{ij} (x_j - x_i). \tag{5}
\]
Stack individual system states as $x := [x_1^T \ x_2^T \ \ldots \ x_p^T]^T$. Then we can express (5) as
\[
x = (I_p \otimes A + \Gamma \otimes BK_\delta)x. \tag{6}
\]
Now let $Y \in \mathbb{C}^{p \times (p-1)}$, $W \in \mathbb{C}^{(p-1) \times p}$, $V \in \mathbb{C}^{p \times p}$, and upper triangular $\Delta \in \mathbb{C}^{(p-1) \times (p-1)}$ be such that
\[
V = [1 \ Y], \quad V^{-1} = \begin{bmatrix} r^T \\ W \end{bmatrix}
\]
and
\[
V^{-1} \Gamma V = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & \ddots & \Delta \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & 0 \end{bmatrix}
\]
Note that the diagonal entries of $\Delta$ are nothing but the nonzero eigenvalues of $\Gamma$ which we know have real parts no greater than $-\delta$. Engage the change of variables $\mathbf{v} := (V^{-1} \otimes I_n)x$ and modify (6) first into
\[
\dot{\mathbf{v}} = (I_p \otimes A + V^{-1} \Gamma V \otimes BK_\delta)\mathbf{v}
\]
and then into
\[
\dot{v} = \begin{bmatrix} A & 0_{n \times (p-1)n} \\ 0_{(p-1)n \times n} & I_{p-1} \otimes A + \Delta \otimes BK_\delta \end{bmatrix} v
\]  
(7)

Observe that \(I_{p-1} \otimes A + \Delta \otimes BK_\delta\) is upper block triangular with (block) diagonal entries of the form \(A + \lambda_i BK_\delta\) for \(i = 2, 3, \ldots, p\) with \(\text{Re}(\lambda_i) \leq -\delta\). Lemma 1 implies therefore that \(I_{p-1} \otimes A + \Delta \otimes BK_\delta\) is Hurwitz. Thus (7) implies
\[
\left| v(t) - \begin{bmatrix} e^{At} & 0_{n \times (p-1)n} \\ 0_{(p-1)n \times n} & 0_{(p-1)n \times (p-1)n} \end{bmatrix} v(0) \right| \to 0
\]
as \(t \to \infty\) which yields
\[
\left| x(t) - \left( 1_r \otimes e^{At} \right) x(0) \right| \to 0.
\]

Hence the result.

By Theorem 1 we attain our objectives. In the next section we provide a dual result which may be more useful in certain applications.

5 Dual problem

Let \(p\) identical linear systems be
\[
\dot{x}_i = A^T x_i + u_i, \quad y_i = B^T x_i, \quad i = 1, 2, \ldots, p
\]  
(8)

where \(x_i \in \mathbb{R}^n\) is the state, \(u_i \in \mathbb{R}^n\) is the input, and \(y_i \in \mathbb{R}^m\) is the output of the \(i\)th system. Matrices \(A^T\) and \(B^T\) are of proper dimensions and they make a detectable pair \((B^T, A^T)\). Now consider the case where at each time instant the following information
\[
z_i = \sum_{j=1}^{p} \gamma_{ij}(y_j - y_i)
\]  
(9)
is available to \(i\)th system to determine an input value. Not surprisingly, the following result accrues.

**Theorem 2** Consider systems (8). Let \(L := PB\) where \(P\) is the solution to (3). Given \(\delta > 0\), for all \(p\) and connected \(\Gamma \in \mathbb{R}^{p \times p}\) with \(-\text{Re}(\lambda_2(\Gamma)) \geq \delta\), solutions \(x_i(\cdot)\) for \(i = 1, 2, \ldots, p\) and \(u_i = \max\{1, \delta^{-1}\} L z_i\), where \(z_i\) is as in (9), globally synchronize to
\[
\bar{x}(t) := \left( r^T \otimes e^{A^T t} \right) \begin{bmatrix} x_1(0) \\ \vdots \\ x_p(0) \end{bmatrix}
\]
where \(r \in \mathbb{R}^p\) satisfies \(r^T \Gamma = 0\) and \(r^T 1 = 1\).
6 Conclusion

For identical (unstable) linear systems, we have shown that no more than stabilizability (detectability) is necessary for an LQR-based linear feedback law to exist under which coupled systems synchronize for all network topologies with strong enough coupling. We have considered directed networks and measured the strength of coupling via the real part of a nonzero eigenvalue (of the coupling matrix) closest to the imaginary axis.

A Proof of a fact

Fact 1 A connected $\Gamma \in \mathbb{R}^{p \times p}$ has an eigenvalue at $\lambda = 0$ with eigenvector $1$ and all its other eigenvalues have strictly negative real parts.

Proof. First part of the statement directly follows from the definition. Since the entries of each row of $\Gamma$ sum up to zero, it trivially follows that $\Gamma 1 = 0$.

Consider the dynamical system

$$\dot{x} = \Gamma x$$

where $x \in \mathbb{R}^p$ with entries $x_i \in \mathbb{R}$ for $i = 1, 2, \ldots, p$. Since $\Gamma 1 = 0$ we can write

$$\dot{x}_i = \sum_{j=1}^{p} \gamma_{ij}(x_j - x_i).$$

By definition $\gamma_{ij} \geq 0$ for $i \neq j$. That brings us the following. For each $\tau \geq 0$, $x_i(t) \in [\min_i x_i(\tau), \max_i x_i(\tau)]$ for all $i$ and $t \geq \tau$. In addition, since the graph of $\Gamma$ is connected, interval $[\min_i x_i(t), \max_i x_i(t)]$ must shrink to a point as $t \to \infty$. In other words, there exists $\bar{x} \in [\min_i x_i(0), \max_i x_i(0)]$ such that

$$\lim_{t \to \infty} x_i(t) = \bar{x} \quad (10)$$

for all $i$ (see [13]). That readily implies by simple stability arguments that if $\lambda \in \mathbb{C}$ is an eigenvalue of $\Gamma$ then $\text{Re}(\lambda) \leq 0$. Also, for $\text{Re}(\lambda) = 0$, the size of the associated Jordan block cannot be greater than unity. Possibility of a purely imaginary eigenvalue means sustaining oscillations and is therefore ruled out by (10). The only case left unconsidered is a second eigenvalue at the origin. That would imply, since it cannot be related to a Jordan block of size two or greater, that there exists a nonzero (eigen)vector $v \in \mathbb{R}^p$ such that $\Gamma v = 0$ and $v \neq \alpha 1$ for any $\alpha \in \mathbb{R}$. That contradicts (10), too. ■

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