One-loop $\lambda \phi^4$ field theory in Robertson-Walker spacetimes: adiabatic regularization and analytic approximations

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The renormalization of a scalar field theory with a quartic self-coupling (a $\lambda \phi^4$ theory) via adiabatic regularization in a general Robertson-Walker spacetime is discussed. The adiabatic counterterms are presented in a way that is most conducive to numerical computations. A variation of the adiabatic regularization method is presented which leads to analytic approximations for the energy-momentum tensor of the field and the quantum contribution to the effective mass of the mean field. Conservation of the energy-momentum tensor for the field is discussed and it is shown that the part of the energy-momentum tensor which depends only on the mean field is not conserved but the full renormalized energy-momentum tensor is conserved as expected and required by the semiclassical Einstein’s equation. It is also shown that if the analytic approximations are used then the resulting approximate energy-momentum tensor is conserved. This allows a self-consistent backreaction calculation to be performed using the analytic approximations. The usefulness of the approximations is discussed.

I. INTRODUCTION

The study of free quantized fields in curved space has been a remarkably fruitful endeavor, particularly in the applications that have been made to black hole and cosmological spacetimes. Much less has been done regarding interacting fields in curved space. However, interacting fields are very important since all real fields in nature appear to have interactions. Interactions also play an important role in cosmological models such as inflation, being required in some cases for the inflaton potential to have the right form and also for the thermalization that is necessary to reheat the universe after inflation. Interactions can also significantly enhance the particle production that often occurs for free fields in curved space.

The study of interacting quantum fields in Robertson-Walker (RW) spacetimes is of great importance as well in understanding quantum fields in Minkowski space. It is well known that following a relativistic heavy ion collision, the quark-gluon plasma produced eventually undergoes a chiral phase transition. A good approximation to describing the dynamics of this system is provided by the linear $\sigma$ model, and by assuming that the expansion is mostly radial. Written in terms of the spherical hydrodynamical fluid coordinates the system is equivalent to an interacting quantum scalar field (mean field plus fluctuations) in a RW spacetime.

Perhaps the simplest interacting quantum field theory in four dimensions is a scalar field with a quartic self-coupling, often called the $\lambda \phi^4$ theory. There is a long history of the study of this theory in curved space. The original investigations centered on renormalization. Drummond, Birrell and Ford, Bunch, Panangaden, and Parker, and Bunch and Panangaden investigated the renormalization of the theory in various cosmological spacetimes using techniques such as dimensional regularization. Bunch and Parker showed that the theory is renormalizable in an arbitrary spacetime to second order in the coupling constant $\lambda$. Birrell extended their work by using momentum space techniques and computing self-energy graphs to second order in $\lambda$.

Along with studies of the renormalization of the theory, various calculations have been undertaken. For example, Ford and Toms investigated phase transitions caused by one-loop radiative corrections.
in an expanding universe. The one-loop finite temperature effective potential for a $\lambda\phi^4$ theory in a RW universe was calculated by Hu, under the assumptions that the rate of change of quantum fluctuations is much greater than that of the mean field and the expansion rate of the universe. Ringwald investigated the evolution of the expectation value of the quantum fluctuation $\langle \psi^2 \rangle$ at one-loop order in a spatially flat RW universe.

The quantity $\langle \psi^2 \rangle$ can be used to determine the backreaction of the quantum fluctuations on the mean (or classical) scalar field as it appears as an effective mass for the mean field at one-loop order. However, to determine the backreaction of the scalar field on the spacetime geometry, one must compute the renormalized energy-momentum tensor for the field. The renormalization of the energy-momentum tensor to one-loop order in a spatially flat RW spacetime was discussed by Paz and Mazzitelli. They displayed the renormalization counterterms which were obtained using point splitting, adiabatic regularization, and dimensional regularization. The divergent counterterms were displayed in the format of dimensional regularization. Mazzitelli, Paz, and El Hasi used this formalism in a calculation relating to the evolution of the inflaton and the reheating after inflation in the new inflationary scenario.

There are two ways that so called “nonperturbative” effects are usually taken into account. One is the Hartree approximation which works for a single scalar field. The other is the large $N$ approximation where a single scalar field is replaced with $N$ scalar fields which are coupled via a quartic interaction, which is invariant under the group $O(N)$ and is thus often called the $O(N)$ model. Mazzitelli and Paz considered the renormalization in both the Hartree and large $N$ approximations in an arbitrary background gravitational field, using point splitting techniques, and adiabatic and dimensional regularization. More recent work on the backreaction of a scalar field on the background geometry in a RW spacetime has been done for inflationary models by Boyanovsky, Cormier, de Vega, Holman, Kumar, Lee, Singh, and Srednicki and by Ramsey and Hu.

In this paper we derive a set of renormalized equations that can be used to determine the evolution of a quartically coupled scalar field with arbitrary mass and curvature coupling to one-loop order in a RW spacetime. We also derive expressions for the unique components of the renormalized energy-momentum tensor that can be used to determine the backreaction of the field on the spacetime geometry. To renormalize we use the method of adiabatic regularization which is particularly useful for deriving a set of equations which are to be solved numerically. We display the counterterms for the energy-momentum tensor and the quantum contribution to the effective mass of the mean field (for arbitrary mass and curvature coupling). Previously the adiabatic counterterms for the energy-momentum tensor have been displayed by Paz and Mazzitelli but only in the context of dimensional regularization, which makes it difficult to use them for numerical computations, and by Ramsey and Hu for the minimally coupled case (if one takes the one-loop limit of their $1/N$ expansion).

We discuss in more detail than has previously been done the conservation of the energy-momentum tensor for the full system (mean field plus quantum fluctuations). We show that the natural division of this tensor into a “classical” and a “quantum” piece leads to neither piece being separately conserved. We also show explicitly that the full energy-momentum tensor is conserved.

We present a variation on the method of adiabatic regularization which has been used by one of us to develop an analytic approximation for the energy-momentum tensor for a free scalar field in a RW spacetime. We use this method to derive analytic approximations to both the energy-momentum tensor of the quantum fluctuation and to the effective mass of the mean field. If the analytic approximation is used in the equation for the mean field and if it is also used for the “quantum” energy-momentum tensor, then the resulting set of equations results in a conserved approximate energy-momentum tensor. Thus the analytic approximations can be used in lieu of the full renormalized expressions in the mean field and semiclassical backreaction equations. They are useful for the investigation of vacuum polarization effects, but not particle production since particle production is a nonlocal phenomenon. Nevertheless, the approximations give important information, particularly if one wishes to estimate the conditions under which the loop expansion breaks down. We discuss the validity of the approximation and argue that it is likely to be most useful for massless fields.

It is well known that the problem of solving Einstein’s equation in the presence of quantum matter is not an easy one. On the left-hand-side of Einstein’s equation one needs higher order geometric tensors $^{(1)}H_{\mu\nu}$, $^{(2)}H_{\mu\nu}$, and $H_{\mu\nu}$ that involve up to fourth order time derivatives of the metric $g_{\mu\nu}$. On the right-hand-side one encounters the expectation value of the energy-momentum tensor of the quantum field in a certain quantum state (for which there is no a priori rule to be determined or chosen), a quantity that is ultraviolet divergent. One has to regularize and renormalize $\langle T_{\mu\nu} \rangle$ in
such a way that it remains covariantly conserved, as the left-hand-side of Einstein’s equation is. This is why the problem of a full backreaction of the quantum matter on the spacetime geometry is so difficult. One has to make use of regularization methods that are suited for a numeric computation (as the dynamical equation for the quantum field is most likely not to have analytic solutions), and at the same time guarantee the covariant conservation of the renormalized value of $\langle T_{\mu\nu} \rangle$. In this paper we use adiabatic subtraction to fulfill both requirements, and to set up all the formalism and techniques required to perform a backreaction calculation for an interacting theory (mean field plus fluctuations) in a general RW spacetime.

Before involving ourselves in analysing the backreaction, it will be helpful to estimate how big or small the quantum effects on the geometry and the mean field are. In this paper we introduce and describe the analytic approximation as such a tool. It yields a renormalized and covariantly conserved energy-momentum tensor for the quantum fluctuations, (that carries no information, whatsoever, about particle production effects), that can be evaluated in the spacetime geometry that is a solution of the Einstein’s equation determined by the mean field. This paper presents all the technical details for such a calculation. Future work will consist of evaluating the analytically approximated energy-momentum tensor of the quantum fluctuations in various scenarios, such as the reheating period of the inflationary regime of the early universe, and the spherical expansion of the quark-gluon plasma.

In section II we introduce the conventions to be used, the background geometry, and derive the one-loop equations for the mean and quantum fields. We also compute the energy-momentum tensor of the system at one-loop, and show that it splits naturally in two terms: a “classical” and a “quantum” energy-momentum tensor. In section III we discuss the method of adiabatic regularization when the quantum fluctuations have a time dependent mass, and derive the adiabatic order two and four counterterms that need to be subtracted from $\langle \psi^2 \rangle_u$ and $\langle T_{\mu\nu} \rangle_u$, respectively. We explicitly separate those new terms that were not present in the free case [24,25]. In section IV we introduce the analytic approximation as a way of estimating the importance of vacuum polarization effects, and as a first approximation to doing a full backreaction calculation. In section V we discuss the covariant conservation of the energy-momentum tensor. We show that the part of the energy-momentum tensor that depends only on the mean field is usually not conserved by itself, but that both the full energy-momentum tensor and its analytic approximation are covariantly conserved.

II. BACKGROUND, CONVENTIONS, AND NOTATION

We consider a quantum scalar field with self-interactions in a general RW spacetime. The metric of an RW spacetime can be written in the form

$$ds^2 = a^2(\eta) \left[ d\eta^2 - \frac{dr^2}{1 - \kappa r^2} - r^2 d\Omega^2 \right],$$

where $\eta$ is the conformal time coordinate, and $\kappa = -1, 0, +1$ is the three-dimensional spatial curvature, corresponding to spatial Cauchy hypersurfaces that have negative, zero, and positive spatial curvature, respectively.

The action of a scalar field with a quartic self-interaction is given by

$$S_{\text{matter}}[\Phi, g_{\mu\nu}] = -\frac{1}{2} \int d^4x \left[ \Phi(\Box + m^2 + \xi R)\Phi + \frac{\lambda}{12} \Phi^4 \right].$$

where $g$ is the determinant of the metric, $\Box$ the D’Alembert wave operator given by $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$, and $R$ the scalar curvature of the RW spacetime.

The equation of motion for the classical field (obtained by the principle of least action) is given by

$$\left( \Box + m^2 + \xi R + \frac{\lambda}{3!} \Phi^2 \right) \Phi = 0.$$

The classical energy-momentum tensor is

1Throughout this paper we use units such that $\hbar = c = 1$. The metric signature is $(+ − − −)$ and the conventions for curvature tensors are $R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\gamma,\delta} − ...$ and $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$. 


\[ T_{\mu\nu} = (1 - 2\xi)\partial_{\mu} \Phi \partial_{\nu} \Phi + (2\xi - 1/2)g_{\mu\nu} \partial_{\alpha} \phi \partial^{\alpha} \phi - 2\xi \phi \nabla_{\mu} \nabla_{\nu} \phi \]
\[ + 2\xi g_{\mu\nu} \Phi \Box \Phi - \xi G_{\mu\nu} \phi^{2} + \frac{m^{2}}{2} g_{\mu\nu} \phi^{2} + \frac{3\lambda}{4!} g_{\mu\nu} \phi^{4} . \]

If we quantize the theory \( \Phi \) becomes an operator. We then define the mean (or background) field \( \phi \) by the equations

\[ \Phi \equiv \phi + \psi , \]
\[ \phi \equiv \langle \Phi \rangle , \]

where the expectation value is taken with respect to the initial state of the system (in the Heisenberg representation).

Taking the expectation value of Eq. \((2.3)\) and noting that

\[ \langle \Phi^{3} \rangle = \phi^{3} + 3\phi^{2} \langle \psi \rangle + 3\phi \langle \psi^{2} \rangle + \langle \psi^{3} \rangle = \phi^{3} + 3\phi \langle \psi^{2} \rangle + \langle \psi^{3} \rangle , \]

we find the following equation for the mean field \( \phi \)

\[ (\Box + m^{2} + \xi R)\phi + \frac{\lambda}{3!} \left( \phi^{3} + 3\phi \langle \psi^{2} \rangle + \langle \psi^{3} \rangle \right) = 0 . \]

In the same way, by subtracting equation \((2.7)\) from equation \((2.3)\), we obtain the equation of motion for the quantum fluctuation \( \psi \)

\[ (\Box + m^{2} + \xi R)\psi + \frac{\lambda}{3!} \left( 3\phi^{2} \psi - 3\phi \langle \psi^{2} \rangle + 3\phi \psi^{2} + \psi^{3} - \langle \psi^{3} \rangle \right) = 0 . \]

If we truncate at one-loop (free field theory for the quantum fluctuation \( \psi \)), the equations of motion \((2.7)\) and \((2.8)\) become

\[ (\Box + m^{2} + \xi R)\phi + \frac{\lambda}{3!} \phi^{3} + \frac{\lambda}{2!} \phi \langle \psi^{2} \rangle = 0 , \]
\[ (\Box + m^{2} + \xi R)\psi + \frac{\lambda}{2!} \phi^{2} \psi = 0 . \]

The expectation value of the energy-momentum tensor can be broken into a “classical” and a “quantum” part. The classical part is given by

\[ \langle T_{\mu\nu} \rangle^{C} \equiv (1 - 2\xi)\partial_{\mu} \phi \partial_{\nu} \phi + (2\xi - 1/2)g_{\mu\nu} \partial_{\alpha} \phi \partial^{\alpha} \phi - 2\xi \phi \nabla_{\mu} \nabla_{\nu} \phi \]
\[ + 2\xi g_{\mu\nu} \phi \Box \phi - \xi G_{\mu\nu} \phi^{2} + \frac{m^{2}}{2} g_{\mu\nu} \phi^{2} + \frac{\lambda}{4!} g_{\mu\nu} \phi^{4} , \]

while the quantum part is

\[ \langle T_{\mu\nu} \rangle^{Q} = \langle T_{\mu\nu} \rangle_{w} \equiv (1 - 2\xi)\partial_{\mu} \psi \partial_{\nu} \psi + (2\xi - 1/2)g_{\mu\nu} \partial_{\alpha} \psi \partial^{\alpha} \psi - 2\xi \langle \psi \nabla_{\mu} \nabla_{\nu} \psi \rangle \]
\[ + 2\xi g_{\mu\nu} \langle \psi \Box \psi \rangle - \xi G_{\mu\nu} \langle \psi^{2} \rangle + \frac{m^{2}}{2} g_{\mu\nu} \langle \psi^{2} \rangle + \frac{\lambda}{4} g_{\mu\nu} \phi^{2} \langle \psi^{2} \rangle . \]

Equations \((2.9a)\) and \((2.10b)\) describe our system at one-loop order (mean field \( \phi \) plus quantum fluctuations \( \psi \), which contribute to the effective mass of \( \phi \)). It is well known \([26,24]\) that to make sense of these equations one needs to regularize the theory, that is, on the one hand, define a way to obtain from the bare parameters \((m, \lambda, \xi)\) the renormalized ones, and on the other hand, regularize the divergent quantities \( \langle \psi^{2} \rangle_{w} \) and \( \langle T_{\mu\nu} \rangle_{w} \), to obtain the physically finite energy-momentum tensor of the system. In the next section we discuss these issues.

### III. ADIABATIC REGULARIZATION

Using dimensional regularization it has been shown that for a \( \lambda \phi^{4} \) theory in a general spacetime, the bare and the renormalized parameters are related in the following way \([23]\)

\[ m_{B}^{2} \equiv m_{R}^{2} - \frac{3\lambda_{R}}{8\pi^{2}(n - 4)} m_{R}^{2} , \]
\[ \xi_{B} = \frac{1}{6} \xi_{R} - \frac{1}{6} \left( \frac{3\lambda_{R}}{8\pi^{2}(n - 4)} \xi_{R} - \frac{1}{6} \right) , \]
\[ \lambda_{B} = \lambda_{R} - \frac{9\lambda_{R}^{2}}{8\pi^{2}(n - 4)} . \]
Here $n$ is the number of dimensions the spacetime has been analytically continued to. Thus even for a general spacetime the counterterms for the renormalization of $m^2$, $\xi$, and $\lambda$ are constant in space and time as expected.

In principle one would prefer to renormalize at the level of the effective action and then vary that action with respect to $\phi$ and the metric $g_{\mu\nu}$ to obtain the renormalized equations of motion and energy-momentum tensor, respectively [27]. However the computation of the one-loop effective action for arbitrary mean fields $\phi$ and arbitrary spacetime geometries is quite involved, and will not yield an intrinsically different answer from that obtained by looking at the one-loop field equations. For this reason, we consider a different method called adiabatic regularization [28–31] which works at the level of the field equations. Another advantage of adiabatic regularization is that it is well suited to perform numerical calculations [32,33]. In adiabatic regularization the divergences in quantities such as $\langle \psi^2 \rangle_u$ and $\langle T_{\mu\nu} \rangle_u$ are computed using a WKB expansion for the modes of the quantized field $\psi$. These terms are then subtracted from the unrenormalized (bare) expressions with the result that

$$\langle \psi^2 \rangle_R = \langle \psi^2 \rangle_u - \langle \psi^2 \rangle_{ad},$$

$$\langle T_{\mu\nu} \rangle_R = \langle T_{\mu\nu} \rangle_u - \langle T_{\mu\nu} \rangle_{ad},$$

where the subscripts $u$ and $ad$ stand for the unrenormalized (bare) and the adiabatic value, respectively of $\langle \psi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$. This procedure has been shown to be equivalent to point splitting for free scalar fields in a RW spacetime [34,25]. For the quartically coupled scalar field Eqs. (2.9a) and (2.9b) then become

$$(\Box + m^2_R + \xi R)\phi + \frac{\lambda R}{3!} \phi^3 + \frac{\lambda R}{2!} \phi (\psi^2)_R = 0,$$

$$(\Box + m^2_R + \xi R)\psi + \frac{\lambda R}{2!} \phi^2 \psi = 0.$$  

In what follows we only consider the renormalized values of the coupling constants $m$, $\xi$, and $\lambda$, so we drop the subscript $R$ for these quantities.

We assume that the mean field is homogeneous $\phi = \phi(\eta)$, as a RW spacetime is homogeneous and isotropic. Then Eq. (2.9a) becomes

$$\phi'' + \frac{2}{a} \phi' + a^2 \left( m^2_R + \xi \phi + \frac{\lambda R}{3!} \phi^2 + \frac{\lambda R}{2!} (\psi^2) \right) \phi = 0.$$  

Here primes denote derivatives with respect to $\eta$. The $\eta\eta$ component of the classical renormalized energy-momentum tensor (2.10a) is given by

$$\langle T_{\eta\eta} \rangle_R = \frac{1}{2} \phi' + 6\xi a' \phi' + 3\xi \left( \frac{a^2}{a'^2} + \kappa \right) \phi^2 + \frac{a^2 m^2}{2} \phi^2 + \frac{a^2 \lambda}{4!} \phi^4,$$  

and the trace is

$$\langle T \rangle_R = (6\xi - 1) \frac{1}{a^2} \phi' + 6\xi a' \phi' + \frac{12}{a^2} \xi a' \phi' + \xi R \phi^2 + 2m^2 \phi^2 + \frac{\lambda R}{3!} \phi^4.$$  

To determine the renormalization counterterms used in adiabatic regularization, $(\langle \psi^2 \rangle_{ad}, \langle T_{\mu\nu} \rangle_{ad})$, we first review canonical quantization in a RW spacetime. We then discuss the WKB expansion for the modes of the quantum fluctuation field $\psi$, and compute the adiabatic counterterms needed to renormalize $\langle \psi^2 \rangle_u$ and $\langle T_{\mu\nu} \rangle_u$.

Since at the one-loop level the quantum field $\psi$ is a free field with an effective mass of the form $m^2 + \frac{\lambda \phi^2}{2}$, it can be expanded in the following manner [4]

$$\psi(x) = \frac{1}{a(\eta)} \int d\mu(k) \left[ a_k Y_k(x) f_k(\eta) + a_k^* Y_k^*(x) f_k^*(\eta) \right],$$  

where the measure is given by [21]

$$\int d\mu(k) \equiv \int d^3k \quad \text{if} \quad \kappa = 0 ,$$

$$\equiv \int_0^{+\infty} dk \sum_{l,m} \quad \text{if} \quad \kappa = -1 ,$$

$$\equiv \sum_{k,l,m} \quad \text{if} \quad \kappa = +1 .$$
The spatial part of the mode function, \( Y_k(x) \), obeys the equation
\[
\Delta^{(3)} Y_k(x) = -(k^2 - \kappa) Y_k(x),
\]
and the time dependent part \( f_k(\eta) \) is a solution to the equation
\[
f_k'' + \omega_k^2 + \frac{\lambda}{2} a^2 \phi^2 + (\xi - 1/6)a^2 R = 0. \tag{3.6}
\]
Here \( \omega_k^2 \equiv k^2 + m^2 a^2 \), primes denote derivatives with respect to the conformal time \( \eta \), and for spacetimes with the metric (2.1) the scalar curvature is given by
\[
R = 6 \left( \frac{a''}{a^3} + \frac{\kappa}{a^2} \right). \tag{3.7}
\]

The unrenormalized (bare) expression for the quantum part of the energy-momentum tensor (2.10b) is [24,25]
\[
\langle T^\mu_\nu \rangle_u = \frac{1}{2\pi^2 a^2} \int d\mu(k) \left\{ \left| f_k' \right|^2 + \left( k^2 + m^2 a^2 + \frac{\lambda}{2} a^2 \phi^2 \right) \left| f_k \right|^2 
+ 6 \left( \xi - \frac{1}{6} \right) \left[ \frac{a''}{a} (f_k f_k'' + f_k' f_k') - \left( \frac{a^2}{a^2} - \kappa \right) \left| f_k \right|^2 \right] \right\}, \tag{3.8a}
\]
\[
\langle T \rangle_u = \frac{1}{2\pi^2 a^2} \int d\mu(k) \left\{ \left( m^2 a^2 + \frac{\lambda}{2} a^2 \phi^2 \right) \left| f_k \right|^2 + 6 \left( \xi - \frac{1}{6} \right) \left[ \left| f_k' \right|^2 - \frac{a^2}{a} (f_k f_k'' + f_k' f_k') 
- \left( k^2 + m^2 a^2 + \frac{\lambda}{2} a^2 \phi^2 + \frac{a''}{a} - \frac{a^2}{a^2} \left( \xi - \frac{1}{6} \right) a^2 R \right) \left| f_k \right|^2 \right] \right\}. \tag{3.8b}
\]
Note that the pressure is not an independent quantity, and can be obtained from knowledge of the energy density and the trace.

The equation for the mean field (2.9a) also contains the quantity \( \langle \psi^2 \rangle \). The unrenormalized expression for it can be written in terms of the mode functions as follows
\[
\langle \psi^2 \rangle_u = \frac{1}{2\pi^2 a^2} \int d\mu(k) \left| f_k \right|^2. \tag{3.8c}
\]
In these expressions the measure is
\[
\int d\mu(k) \equiv \int_0^{+\infty} dk k^2 \quad \text{if} \quad \kappa = 0, -1,
\]
\[
\equiv \sum_{k=1}^{+\infty} k^2 \quad \text{if} \quad \kappa = +1.
\]

To determine the adiabatic counterterms needed to renormalize these expectation values we solve the mode equation (3.4) using a WKB expansion. To obtain this expansion we first make the variable transformation
\[
f_k = (2W_k)^{-1/2} \exp \left[ \int_0^\eta d\eta' W_k(\eta') \right]. \tag{3.9}
\]
Substituting Eq. (3.4) into Eq. (3.6) yields
\[
W_k^2 = \omega_k^2 + \frac{\lambda}{2} a^2 \phi^2 + \left( \xi - \frac{1}{6} \right) a^2 R - \frac{1}{2} \left( \frac{W_k''}{W_k} - \frac{3 W_k'^2}{2 W_k^2} \right). \tag{3.10}
\]
This equation is then solved iteratively with \( \omega_k \) being of adiabatic order zero and the next two terms on the right hand side being of adiabatic order two\(^2\). Thus to second adiabatic order the solution for \( W_k \) is
\[
W_k = \omega_k + \frac{1}{2\omega_k^2} \left[ \frac{\lambda}{2} a^2 \phi^2 + \left( \xi - \frac{1}{6} \right) a^2 R - \frac{1}{2} \left( \frac{\omega_k''}{\omega_k} - \frac{3 \omega_k'^2}{2 \omega_k^2} \right) \right]. \tag{3.11}
\]
\(^2\)The \( \lambda a^2 \phi^2 \) term is considered to be of second adiabatic order because only terms with up to two time derivatives of \( \phi \) are needed to cancel divergences in \( \langle T_{\mu\nu} \rangle_u \).
To renormalize \( \langle \psi^2 \rangle \) we use a second order adiabatic expansion for the modes. A fourth order expansion is necessary to cancel the divergences in \( \langle T_{\mu \nu} \rangle \) \[24,25\]. The renormalization counterterms for these quantities are

\[
\langle \psi^2 \rangle_{ad} = \langle \psi^2 \rangle_{ad} + \frac{1}{4\pi^2 a^4} \int d\mu(k) \frac{\lambda \phi^2 a^2}{4\omega_k^2},
\]

\[
\langle T^\eta \rangle_{ad} = \langle T^\eta \rangle_{ad} + \frac{1}{4\pi^2 a^4} \int d\mu(k) \left\{ \frac{\lambda \phi^2 a^2}{4\omega_k} \left[ \frac{\lambda \phi^2 a^2}{32\omega_k^4} + \frac{m^2 \lambda}{8\omega_k} \right] - \frac{5m^4 \lambda}{32\omega_k^4} \left( 2a^2 a'' - 2a^4 a'' + 4a^2 a'' - a^4 \right) + 3m^2 \lambda \left( 2a^2 a'' - a^4 \right) \right\} + \frac{9m^6 \lambda}{8\omega_k^4} \left( 2a^2 a'' - 2a^4 a'' + 4a^2 a'' - a^4 \right) - \frac{15m^4 \lambda}{8\omega_k^4} \left( 2a^2 a'' - 2a^4 a'' + 4a^2 a'' - a^4 \right) \right\},
\]

\[
\langle T \rangle_{ad} = \langle T \rangle_{ad} + \frac{1}{4\pi^2 a^4} \int d\mu(k) \left\{ \frac{m^2 a^2}{8\omega_k} \left[ \frac{3m^2 a^2}{8\omega_k} + \frac{5m^4 a^2}{16\omega_k^3} \right] + \frac{15m^4}{8\omega_k^4} \left( 2a^2 a'' - 2a^4 a'' + 4a^2 a'' - a^4 \right) \right\} + \frac{27m^2}{8\omega_k^4} \left( 2a^2 a'' - 2a^4 a'' + 4a^2 a'' - a^4 \right) \right\}.
\]

The adiabatic counterterms for the free field are \[24,25\]

\[
\langle \psi^2 \rangle_{F} = \frac{1}{4\pi^2 a^4} \int d\mu(k) \left\{ \frac{m^2 a^2}{8\omega_k} \left[ \frac{3m^2 a^2}{8\omega_k} + \frac{5m^4 a^2}{16\omega_k^3} \right] + \frac{15m^4}{8\omega_k^4} \left( 2a^2 a'' - 2a^4 a'' + 4a^2 a'' - a^4 \right) \right\} + \frac{27m^2}{8\omega_k^4} \left( 2a^2 a'' - 2a^4 a'' + 4a^2 a'' - a^4 \right) \right\}.
\]

Equations \[3.12\], \[3.12\], and \[3.12\] together with \[3.12\], \[3.12\], and \[3.12\] give the adiabatic counterterms needed to obtain the renormalized expectation value of the quantum fluctuations and of the quantum energy-momentum tensor for an interacting quantum scalar field in a general RW
spacetime, for the case of a homogeneous mean field $\phi(\eta)^3$.

IV. ANALYTIC APPROXIMATIONS

As it stands the method of adiabatic regularization can be used to compute the quantities $\langle \psi^2 \rangle_R$ and $\langle T_{\mu\nu} \rangle_R$. This then allows one to obtain a self-consistent solution to the mean field and mode equations (2.9a) and (2.9b) in a background RW spacetime (in the test field approximation) or to these equations plus the semiclassical backreaction equations in a RW spacetime. But before getting involved in the backreaction problem (and the difficulties this presents), it would be very useful to have a way of estimating how big or small the quantum corrections are beyond test field approximation. In this section we discuss this issue and present such an approximation.

For a free quantum scalar field in a general RW spacetime, one of us has already shown that there is a way of defining a certain approximate energy-momentum tensor that is covariantly conserved [2]. In the present paper we extend this method to interacting quantum fields, and discuss its advantages and limitations.

The analytic approximations result from a revision of the method of adiabatic regularization that simplifies the calculations in a RW spacetime with compact spatial sections, and in the process yields approximations for the quantities $\langle \psi^2 \rangle_R$ and $\langle T_{\mu\nu} \rangle_R$. These analytic approximations give information about vacuum polarization effects but not particle production, since particle production is a nonlocal phenomenon. However they also make it possible to solve the mean field, mode, and semiclassical backreaction equations in an approximate manner, which goes beyond the test field approximation, and does not involve a full backreaction calculation. This can be useful, for example, if one wishes to determine under what conditions vacuum polarization effects will be important and what influence they may have on the mean field and the spacetime geometry. In particular, we believe that these analytic approximations may provide important information for reheating calculations, just before particle production from the inflaton field takes place. The analytic approximations provide a natural way to estimate the change in the vacuum polarization energy of the inflaton field. During the inflationary regime the mean field $\phi$ dominates the energy density of the universe. It is reasonable to expect that when the vacuum polarization energy is of the order of the mean field energy density, the inflaton field will switch from the slow-roll regime to the oscillatory behavior, that will eventually lead to particle production. In this paper we restrict ourselves to presenting the analytic approximations and our method for obtaining them. In future work we will present the applications to reheating.

In order to derive the analytic approximation, we first improve on the method of adiabatic regularization by expanding the renormalization counterterms in inverse powers of $k$, keeping only terms which are ultraviolet divergent. For the case of compact spatial sections ($\kappa = +1$) the integral is also changed into a sum. We call the resulting expressions $\langle \psi^2 \rangle_d$ and $\langle T_{\mu\nu} \rangle_d$, respectively. In a general RW spacetime they have the form

$$\langle \psi^2 \rangle_d = \frac{1}{4\pi^2 a^2} \int d\mu(k) \left( k + \frac{1}{k} \right) \left( k \pm 1 \right) \left[ \frac{m^2 a^2}{2} + \frac{\lambda \phi^2 a^2}{4} + \left( \xi - \frac{1}{6} \right) \frac{a^2 R}{2} \right],$$

$$\langle T^\eta_\eta \rangle_d = \frac{1}{4\pi^2 a^2} \int d\mu(k) \left( k + \frac{1}{k} \right) \left( k \pm 1 \right) \left[ \frac{m^2 a^2}{2} + \frac{\lambda \phi^2 a^2}{4} - 3 \left( \xi - \frac{1}{6} \right) \left( \frac{a''}{a^2} - \kappa \right) \right]$$

$$+ \frac{1}{4\pi^2 a^2} \int d\mu(k) \left( k + \frac{1}{k} \right) \left( k \pm 1 \right) \left[ \frac{m^4 a^4}{8} - \frac{m^2 \lambda \phi^2 a^4}{8} - \frac{\lambda^2 \phi^4 a^4}{32} - \frac{3m^2 a^2}{2} \left( \frac{a''}{a^2} + \kappa \right) \right]$$

$$- 6 \left( \xi - \frac{1}{6} \right) \frac{\lambda}{8} \phi^2 a^2 + 2 \phi \phi' a' + \kappa \phi^2 a^2 \right) + \left( \xi - \frac{1}{6} \right)^2 \left( 1 \right) H^2 \kappa^{-1} a^4 \right],$$

$$\langle T \rangle_d = \frac{1}{4\pi^2 a^2} \int d\mu(k) \left( k + \frac{1}{k} \right) \left( k \pm 1 \right) \left[ \frac{m^2 a^2}{2} + \frac{\lambda \phi^2 a^2}{4} - 6 \left( \xi - \frac{1}{6} \right) \left( \frac{a''}{a} - \frac{a''}{a^2} \right) \right].$$
\begin{align*}
\frac{1}{4\pi^2a^2} & \int d\bar{\mu}(k) \frac{1}{k^3} \left[ -\frac{m^4a^4}{2} - \frac{m^2\lambda\phi^2a^4}{2} - \frac{\lambda^2\phi^4a^4}{8} \right] \\
& - \left( \xi - \frac{1}{6} \right) \left[ 3m^2a^2 \left( \frac{a''}{a} + \kappa \right) + \frac{\lambda}{4} (6\phi''a^2 + 6\phi'a'^2 + 12\phi'aa' + 6\phi^2a'' + 6\kappa\phi^2a^2) \right] \\
& + \left( \xi - \frac{1}{6} \right)^2 \left( 1 \right) \left[ \frac{a^4}{3} \right],
\end{align*}
with
\[
\int d\bar{\mu}(k) \equiv \int_{-\infty}^{+\infty} dk k^2 \quad \text{if} \quad \kappa = 0, -1,
\]
\[
\equiv \sum_{k=1}^{+\infty} k^2 \quad \text{if} \quad \kappa = +1.
\]

Here \( \varepsilon \) is an arbitrary lower limit cutoff and
\[
(1) \ H_{\mu\nu} = 2R_{\mu\nu} = 2g_{\mu\nu} \Box R - \frac{1}{2} g_{\mu\nu} R^2 + 2RR_{\mu\nu}.
\]

In a RW spacetime it has the components
\[
(1) \ H^\eta_\eta = \frac{36\alpha' a''}{a^6} + \frac{72a^2}{a^5} + \frac{18a''}{a^4} + \frac{36\kappa a'^2}{a^5} + \frac{18\kappa^2}{a^4},
\]
\[
(1) \ H = \frac{36a'' a''}{a^3} + \frac{144a'' a''}{a^2} + \frac{216a'^2}{a^3} + \frac{108a''}{a^2} + \frac{72\kappa a'^2}{a^3} - \frac{72\kappa a'}{a^2}.
\]

The renormalized energy-momentum tensor is then computed by subtracting and adding the quantity \( \left< T_{\mu\nu}\right>_d \) to Eq. (3.2b), with the result that
\[
\left< T_{\mu\nu}\right>_e \equiv \left< T_{\mu\nu}\right>_n + \left< T_{\mu\nu}\right>_{an},
\]
\[
\left< T_{\mu\nu}\right>_n \equiv \left< T_{\mu\nu}\right>_a - \left< T_{\mu\nu}\right>_{an},
\]
\[
\left< T_{\mu\nu}\right>_{an} \equiv \left< T_{\mu\nu}\right>_d - \left< T_{\mu\nu}\right>_ad.
\]

In general \( \left< T_{\mu\nu}\right>_n \) must be computed numerically while \( \left< T_{\mu\nu}\right>_{an} \) can always be computed analytically.

The result is
\[
(\psi^2)_{an} = (\psi^2)^F_{an} - \frac{\lambda\phi^2}{16\pi^2} \left\{ 1 - \log \left( \frac{2\varepsilon}{a\mu} \right) - \frac{\kappa(k+1)}{2}(\log \varepsilon + \gamma) \right\},
\]
\[
(\psi^2)^n_{an} = (\psi^2)^F_{an} + \frac{1}{4\pi^2} \left\{ -\frac{\lambda\phi^2}{48} \frac{\kappa(k+1)}{2a^2} - \frac{\lambda\phi^2}{16} \frac{m^2 + \lambda\phi^2}{2} + \frac{\lambda\phi^2}{8} \frac{m^2 + \lambda\phi^2}{4} \right\} \log \left( \frac{2\varepsilon}{a\mu} \right) - \frac{\kappa(k+1)}{2}(\log \varepsilon + \gamma)
\]
\[
+ \frac{(6\xi - 1)\lambda}{16\pi^2a^4} \frac{a^2\phi''^2 + a^2\phi' \phi'' + a^2\phi'}{2} \left[ \log \left( \frac{2\varepsilon}{a\mu} \right) - \frac{\kappa(k+1)}{2}(\log \varepsilon + \gamma) \right]
\]
\[
- \frac{x}{96a^4} \frac{a^2\phi''^2 + a^2\phi' \phi'' + a^2\phi'}{2} \left( \frac{\phi''}{\phi'} \right)
\]
\[
= (\psi^2)^F_{an} + \frac{1}{4\pi^2} \left\{ -\frac{\lambda\phi^2}{48} \frac{\kappa(k+1)}{2a^2} - \frac{\lambda\phi^2}{4} \frac{m^2 + \lambda\phi^2}{2} + \frac{\lambda\phi^2}{2} \frac{m^2 + \lambda\phi^2}{2} \right\} \log \left( \frac{2\varepsilon}{a\mu} \right) - \frac{\kappa(k+1)}{2}(\log \varepsilon + \gamma)
\]
\[
+ \frac{(6\xi - 1)\lambda}{16\pi^2a^4} \frac{a^2\phi''^2 + a^2\phi' \phi'' + a^2\phi'}{2} \left[ \log \left( \frac{2\varepsilon}{a\mu} \right) - \frac{\kappa(k+1)}{2}(\log \varepsilon + \gamma) \right]
\]
\[
- \frac{x}{96a^4} \frac{a^2\phi''^2 + a^2\phi' \phi'' + a^2\phi'}{2} \left( \frac{\phi''}{\phi'} \right)
\]
\[
= (\psi^2)^F_{an} + \frac{1}{4\pi^2} \left\{ \frac{\lambda\phi^2}{48} \frac{\kappa(k+1)}{2a^2} + \frac{\lambda\phi^2}{4} \frac{m^2 + \lambda\phi^2}{2} + \frac{\lambda\phi^2}{2} \frac{m^2 + \lambda\phi^2}{2} \right\} \log \left( \frac{2\varepsilon}{a\mu} \right) - \frac{\kappa(k+1)}{2}(\log \varepsilon + \gamma)
\]
\[
- \frac{(6\xi - 1)\lambda}{16\pi^2a^4} \frac{a^2\phi''^2 + a^2\phi' \phi'' + a^2\phi'}{2} \left[ \log \left( \frac{2\varepsilon}{a\mu} \right) - \frac{\kappa(k+1)}{2}(\log \varepsilon + \gamma) \right]
\]
\[
- \frac{(6\xi - 1)\lambda}{16\pi^2a^4} \frac{a^2\phi''^2 + a^2\phi' \phi'' + a^2\phi'}{2} \left( \frac{\phi''}{\phi'} \right),
\]
with
\[
(\psi^2)^F_{an} = \frac{a^2}{4\pi^2a^3} + \frac{1}{4\pi^2} \left\{ \frac{m^4}{4} \frac{\kappa(k+1)}{24a^2} + \frac{m^2}{2} \log \left( \frac{2\varepsilon}{a\mu} \right) - \frac{\kappa(k+1)}{2}(\log \varepsilon + \gamma) \right\}
\]
\[
- \frac{(6\xi - 1)}{8\pi^2} \left\{ 1 - \log \left( \frac{2\varepsilon}{a\mu} \right) - \frac{\kappa(k+1)}{2}(\log \varepsilon + \gamma) \right\},
\]
\[
(4.5)
\]
\[ (T^\eta_\eta)^F \equiv \frac{1}{288\pi^2} \left[ -\frac{1}{6} H^\eta_\eta + (3) H^\eta_\eta - \frac{3a}{a^4} \right] + \frac{m^2}{288\pi^2} G^\eta_\eta - \frac{m^2 \kappa (\kappa - 1)}{192\pi^2 a^2} \]
\[ \frac{m^4}{64\pi^2} \left[ \frac{1}{2} + 2 \log \left( \frac{\mu a}{2\varepsilon} \right) + \kappa(\kappa + 1)(\gamma + \log \varepsilon) \right] + \left( \xi - \frac{1}{6} \right) \left[ \frac{(1) H^\eta_\eta}{288\pi^2} + \frac{\kappa(\kappa - 1)}{32\pi^2 a^4} \left( 1 + \frac{a'^2}{a^2} \right) \right] \]
\[ + \frac{m^2}{16\pi^2} G^\eta_\eta \left( 3 + 2 \log \left( \frac{\mu a}{2\varepsilon} \right) + \kappa(\kappa + 1)(\gamma + \log \varepsilon) \right) + \frac{3\kappa m^2}{8\pi^2 a^2} \]
\[ + \left( \xi - \frac{1}{6} \right)^2 \left[ \frac{(1) H^\eta_\eta}{32\pi^2} \left( 2 + 2 \log \left( \frac{\mu a}{2\varepsilon} \right) + \kappa(\kappa + 1)(\gamma + \log \varepsilon) \right) \right] - \frac{9}{4\pi^2} \left( \frac{a'^2}{a^2} + \frac{\kappa a^2}{a^6} \right), \quad (4.6) \]

\[ \langle T \rangle^F_{\alpha\beta} = \frac{1}{288\pi^2} \left[ -\frac{1}{6} (1) H + (3) H \right] - \frac{m^2}{288\pi^2} G \left( 1 + 2 \log \left( \frac{\mu a}{2\varepsilon} \right) + \kappa(\kappa + 1)(\gamma + \log \varepsilon) \right) \]
\[ + \left( \xi - \frac{1}{6} \right)^2 \left[ \frac{(1) H}{32\pi^2} \left( 2 + 2 \log \left( \frac{\mu a}{2\varepsilon} \right) + \kappa(\kappa + 1)(\gamma + \log \varepsilon) \right) \right] - \frac{9}{8\pi^2} \left( 4a'^2 a'' \frac{a^6}{a^5} + 3a'' \frac{a^6}{a^5} + \frac{4\kappa a''}{a^5} \frac{a^6}{a^5} + \frac{\kappa^2}{a^4} \right). \quad (4.7) \]

Here \( G_{\mu\nu} \) is the Einstein tensor with components

\[ G^\eta_\eta = -\frac{3a'^2}{a^4} - \frac{3a}{a^2}, \quad (4.8a) \]
\[ G = -\frac{6a''}{a^3} - \frac{6\kappa}{a^2} = -R, \quad (4.8b) \]

and \( (3) H_{\mu\nu} \) is the tensor

\[ (3) H_{\mu\nu} = R_{\mu\nu} - \frac{2}{3} RR_{\mu\nu} - \frac{1}{2} R_{\rho\sigma} R^{\rho\sigma} g_{\mu\nu} + \frac{1}{4} R^2 g_{\mu\nu}, \quad (4.9a) \]

with components

\[ (3) H^\eta_\eta = \frac{3a'^4}{a^6} + \frac{6\kappa a'^2}{a^4} + \frac{3\kappa^2}{a^4}, \quad (4.9b) \]
\[ (3) H = \frac{12\kappa a'^2 a''}{a^8} + \frac{12a'^4}{a^8} + \frac{12\kappa a''}{a^6} - \frac{12\kappa a'^2}{a^6}. \quad (4.9c) \]

For a massive field \( \mu = m \), while for a massless field \( \mu \) is an arbitrary mass scale. However, in the massless case each of the terms containing \( \log \mu \) has as a coefficient a multiple of the tensor \( (1) H_{\mu\nu} \), which comes from an \( R^2 \) term in the gravitational Lagrangian. Thus the terms containing \( \log \mu \) simply correspond to a finite renormalization of the coefficient of the \( R^2 \) term in the gravitational action.

Note that if \( \kappa = +1 \), \( \langle \psi^2 \rangle_d \) and \( \langle T_{\mu\nu} \rangle_d \) contain a sum over \( k \), while \( \langle \psi^2 \rangle_{ad} \) and \( \langle T_{\mu\nu} \rangle_{ad} \) contain an integral over \( k \). Thus, either the integral must be converted to a sum or the sum to an integral. We have converted the sum to an integral using the Plana sum formula \[2, 8\]. This formula is

\[ \sum_{n=m}^{+\infty} f(n) = \frac{1}{2} \left( f(m) + \int_{m}^{+\infty} df(x) + i \int_{0}^{+\infty} dt e^{2\pi it} \right), \quad (4.10) \]

Because of the way \( \langle T_{\mu\nu} \rangle_d \) is defined, the third term in the Plana sum formula can be computed exactly. In the traditional form of adiabatic regularization one would convert the integral in the adiabatic counterterms to a sum using the Plana sum formula and then substitute the result into Eq. \[3.21\]. However, if this is done then, for a massive field, it is not possible to compute the third term in the Plana sum formula analytically. Thus the computation of the renormalized energy-momentum tensor is simplified by our method in the \( \kappa = +1 \) case. Clearly the same simplification would occur if one was using compact spatial sections for \( \kappa = 0 \) or \( \kappa = -1 \) RW spacetimes.
V. CONSERVATION OF THE ENERGY-MOMENTUM TENSOR

In this section we show that the renormalized energy-momentum tensor for the $\lambda \phi^4$ theory (at one-loop) in a RW spacetime is covariantly conserved. In a RW spacetime there is only one nontrivial conservation equation which is

$$\langle T^\eta_\eta \rangle^C_R + \frac{4a'}{a} \langle T^\eta_\eta \rangle^C_R - \frac{a'}{a} \langle T \rangle^C_R = 0.$$  \hspace{1cm} (5.1)

We have shown that the energy-momentum tensor for the field can be divided into a classical and a quantum part. First consider the classical part which in a RW spacetime has the components (3.3a) and (3.3b). Substituting into Eq. (5.1) and using (3.3a) we find

$$\langle T^{\eta\eta} \rangle + \frac{4a'}{a} \langle T^{\eta\eta} \rangle - \frac{a'}{a} T = -\frac{\lambda}{2} \phi \phi' \langle \psi^2 \rangle_R.$$  \hspace{1cm} (5.2)

Thus the classical energy-momentum tensor is conserved only if there is no quantum correction to the effective mass of the mean field.

The energy-momentum tensor for the quantum part consists of the difference between the unrenormalized part and the adiabatic counterterms. The components of the unrenormalized part are given in Eqs. (3.8a) and (3.8b). If they are substituted into Eq. (5.1) and the mode equation (3.6) is used then one finds

$$\langle T^{\eta\eta} \rangle_{ad, \eta} + \frac{4a'}{a} \langle T^{\eta\eta} \rangle_{ad} - \frac{a'}{a} \langle T \rangle_{ad} = \frac{\lambda}{2} \phi \phi' \langle \psi^2 \rangle_{ad}.$$  \hspace{1cm} (5.3)

If one substitutes the adiabatic counterterms (3.12a) and (3.12b) into Eq. (5.4), and compares the result with Eq. (3.2a) one finds

$$\langle T^{\eta\eta} \rangle_{ad, \eta} + \frac{4a'}{a} \langle T^{\eta\eta} \rangle_{ad} - \frac{a'}{a} \langle T \rangle_{ad} = \frac{\lambda}{2} \phi \phi' \langle \psi^2 \rangle_{ad}.$$  \hspace{1cm} (5.4)

Combining these results and using (3.2a) and (3.2b), one finds that the total renormalized energy-momentum tensor, classical plus quantum, is conserved.

One further finds that if the analytic approximation is used in place of $\langle \psi^2 \rangle_R$ in the equation for the mean field, and if it is used for the quantum energy-momentum tensor, then the analytic approximate energy-momentum tensor is conserved. This means that one can use the analytic approximation to define a consistent set of equations for the mean and the quantum fields, and to be the source in the right hand-side of Einstein’s equation to solve a first approximation to the backreaction problem.

Thus $\langle \psi^2 \rangle_{an}$ and $\langle T_{\mu\nu} \rangle_{an}$ can be used in place of $\langle \psi^2 \rangle_R$ and $\langle T_{\mu\nu} \rangle_R$ in Eqs. (2.9a), (2.10a), and (2.10b) to obtain an analytic approximation for the system. For $\kappa = 0, -1$ RW spacetimes the terms being approximated contain the arbitrary constant $\epsilon$. This means the approximation is not unique unless the coefficients of the log $\epsilon$ terms vanish. It is important to note that this is only true when using $\langle \psi^2 \rangle_{an}$ and $\langle T_{\mu\nu} \rangle_{an}$ as an analytic approximation. The $\epsilon$ dependent terms do not appear in the fully renormalized expressions of these quantities.

From the DeWitt-Schwinger expansion it is known that for a free quantum field in the large mass limit $\langle \psi^2 \rangle_R$ and $\langle T_{\mu\nu} \rangle_R$ have leading order terms proportional to $1/m^2$. Thus, the analytic approximations for these quantities are not good approximations in this limit. Previous numerical work indicates that the relevant condition is likely to be $ma \ll 1$. The analytically approximated quantities are also local in the sense that they depend on the scale factor and its derivatives at a given time $\eta$. Therefore, they cannot accurately describe particle production effects which are inherently nonlocal. However for massless fields they should allow one to estimate how important vacuum polarization effects are, and how they qualitatively effect the evolution of the system.

VI. SUMMARY

We have used adiabatic regularization to renormalize a scalar field theory with a quartic self-coupling of the form $\lambda \phi^4$ in an arbitrary RW spacetime. We have found that the energy-momentum tensor can be naturally split into two parts, a “classical” contribution (which corresponds to the energy-momentum tensor of a classical scalar field with a quartic interaction in a RW spacetime) and a “quantum” piece (which corresponds to the energy-momentum tensor of a free quantum scalar
field with the time dependent mass $m^2 + \lambda \phi^2$). We have displayed the renormalization counterterms for both the energy-momentum tensor and the contribution of the quantum fluctuations to the effective mass of the mean field at one-loop order. We have directly checked to see if the energy momentum tensor is covariantly conserved and found that while the entire tensor is conserved, its classical and quantum contributions are not separately conserved.

By using a variant on the adiabatic regularization method we have derived analytic approximations for the energy-momentum tensor and the contribution of the quantum fluctuations to the effective mass of the mean field. We have shown that the approximate energy-momentum tensor is covariantly conserved. Thus the analytic approximations can be used in a self-consistent way to find approximate solutions to the mean field and backreaction equations. The approximations can provide a useful tool for learning about vacuum polarization effects for massless fields. However, they are not useful for massive fields in the large mass limit. They do not give any significant amount of information about particle production, which is a nonlocal phenomenon.

The approximations could be of use in reheating calculations, in particular the transition from the slow-roll dynamics of the inflaton field to the oscillatory behavior around the minimum of the potential, (which is believed to produce particles of lighter masses), and in the context of relativistic heavy ion collision as a way of estimating the physical energy density and pressure of the vacuum and thermal excitations. The advantage of this approximation is that one obtains analytic expressions for the quantum piece of the energy-momentum tensor, without need to solve exactly the mode equation (which is the most difficult part to implement in numeric computations). In this way, it is relatively easy to study the backreaction problem of the full system (mean field, quantum fluctuations, and gravitational field).

We believe that this program can be carried forward, and improved easily. We plan to extend the present approach and approximations to two-loop order, and to the $1/N$ expansion. We also plan to perform some specific calculations with applications to reheating and early density perturbations.

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