Initial self-embeddings of models of set theory

Ali Enayat\(^1\) and Zachiri McKenzie\(^2\)

\(^1\)University of Gothenburg, Gothenburg, Sweden
ali.enayat@gu.se

\(^2\)Department of Philosophy, Zhejiang University, Hangzhou, P. R. China
zach.mckenzie@gmail.com

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Abstract

By a classical theorem of Harvey Friedman (1973), every countable nonstandard model \(M\) of a sufficiently strong fragment of ZF has a proper rank-initial self-embedding \(j\), i.e., \(j\) is a self-embedding of \(M\) such that \(j[M] \subset M\), and the ordinal rank of each member of \(j[M]\) is less than the ordinal rank of each element of \(M \setminus j[M]\). Here we investigate the larger family of proper initial-embeddings \(j\) of models \(M\) of fragments of set theory, where the image of \(j\) is a transitive submodel of \(M\). Our results include the following three theorems.

In what follows, ZF\(^-\) is ZF without the power set axiom; WO is the axiom stating that every set can be well-ordered; WF(\(M\)) is the well-founded part of \(M\); and \(\Pi^1_\alpha\)–DC\(\alpha\) is the full scheme of dependent choice of length \(\alpha\).

Theorem A. There is an \(\omega\)-standard countable nonstandard model \(M\) of ZF\(^-\) + WO that carries no initial self-embedding \(j : M \rightarrow M\) other than the identity embedding.

Theorem B. Every countable \(\omega\)-nonstandard model \(M\) of ZF is isomorphic to a transitive submodel of the hereditarily countable sets of its own constructible universe \(L^M\).

Theorem C. The following three conditions are equivalent for a countable nonstandard model \(M\) of ZF\(^-\) + WO + \(\forall \alpha \Pi^1_\alpha\)–DC\(\alpha\).

(I) There is a cardinal in \(M\) that is a strict upper bound for the cardinality of each member of WF(\(M\)).

(II) WF(\(M\)) satisfies the powerset axiom.

(III) For all \(n \in \omega\) and for all \(b \in M\), there exists a proper initial self-embedding \(j : M \rightarrow M\) such that \(b \in \text{rng}(j)\) and \(j[M] \prec_n M\).

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1 Introduction

By a classical theorem of Friedman [Fri], every countable nonstandard model $M$ of ZF admits a proper rank-initial self-embedding, i.e., an embedding $j : M \rightarrow M$ such that $j[M] \subsetneq M$ and the ordinal rank of each member of $M \setminus j[M]$ (as computed in $M$) exceeds the ordinal rank of each member of $j[M]$ (some authors refer to this situation by saying that $M$ is a top extension of $j[M]$). Friedman’s work on rank-initial self-embeddings was refined by Ressayre [Res], who constructed proper rank-initial self-embeddings of models of set theory that pointwise fix any prescribed rank-initial segment of the ambient model determined by an ordinal of the model; and more recently by Gorbow [Gor], who vastly extended Ressayre’s work by carrying out a systematic study of the structure of fixed point sets of rank initial self-embeddings of models of set theory.

In another direction, Hamkins [Ham13] investigated the family of embeddings $j : M \rightarrow N$, where $M$ and $N$ are models of set theory, for which $j[M]$ is merely required to be a submodel of $N$. The main result of Hamkins’ paper shows that, surprisingly, every countable model $M$ of a sufficiently strong fragment of ZF is embeddable as a submodel of their own constructible universe $L^M$.

Here we investigate a family of self-embeddings that is wider than the family of rank-initial embeddings, but narrower than the family considered by Hamkins. More specifically, we study initial self-embeddings, i.e., embeddings $j : M \rightarrow M$ such that no member of $j[M]$ gains a new member in the passage from $j[M]$ to $M$ (some authors refer to this situation by saying that $M$ is an end extension of $j[M]$, or that $j[M]$ is a transitive submodel of $M$). Theorems A, B, and C of the abstract represent the highlights of our results. Theorem A is presented as Theorem 4.6; it shows that in contrast with Friedman’s aforementioned self-embedding theorem, the theory $ZF^- \neg$ has countable nonstandard models with no proper initial self-embeddings. Theorem B is presented as Theorem 5.4; it demonstrates that for $\omega$-nonstandard models, Hamkins’ aforementioned theorem can be refined so as to yield a proper initial embedding. Finally, Theorem C, which is presented as Theorem 5.18, gives necessary and sufficient conditions for the existence of proper initial self-embeddings whose images are $\Sigma_n$-elementary in the ambient model; these necessary and sufficient conditions reveal the subtle relationship between the existence of initial self-embeddings of a model $M$ of set theory and the way in which the well-founded part of $M$ “sits” in $M$.
2 Preliminaries

Throughout this paper $\mathcal{L}$ will denote the usual language of set theory whose only nonlogical symbol is the membership relation. Structures will usually be denoted using upper-case calligraphic Roman letters ($\mathcal{M}, \mathcal{N}$, etc.) and the corresponding plain font letter ($M, N$, etc.) will be used to denote the underlying set of that structure. If $\mathcal{M}$ is an $\mathcal{L}$-structure where $\mathcal{L}' \supseteq \mathcal{L}$ and $a \in M$, then we will use $a^*$ to denote the set \{ $x \in M \mid M \models (x \in a)$\} where the background model, $\mathcal{M}$, used in definition of $a^*$ will be clear from the context.

In addition to the Lévy classes of $\mathcal{L}$-formulae $\Delta_0 = \Sigma_1 = \Pi_0$, $\Sigma_1$, $\Pi_1$, etc., we will also have cause to consider the Takahashi classes $\Delta_0^p$, $\Sigma_1^p$, $\Pi_1^p$, etc. $\Delta_0^p$ is the smallest class of $\mathcal{L}$-formulae that contains all atomic formulae, contains all compound formulae formed using the connectives of first-order logic, and is closed under quantification in the form $\forall x \in y$ and $\forall x \subseteq y$ where $x$ and $y$ are distinct variables, and $\forall$ is $\exists$ or $\forall$. The classes $\Sigma_1^p, \Pi_1^p$, etc. are defined inductively from the class $\Delta_0^p$ in the same way that the classes $\Sigma_1, \Pi_1$, etc. are defined from $\Delta_0$. If $\Gamma$ is a collection of formulae and $T$ is a theory, then we will write $\Gamma^T$ for the collection of formulae that are $T$-provably equivalent to a formula in $\Gamma$. If $T$ is an $\mathcal{L}$-theory, then $\Delta_0^T$ is the collection of all $\mathcal{L}$-formulae that are $T$-provably equivalent to both a $\Sigma_n$-formula and a $\Pi_n$-formula. Similarly, $(\Delta_0^p)^T$ is the collection of all $\mathcal{L}$-formulae that are $T$-provably equivalent to both a $\Sigma_n^p$-formula and a $\Pi_n^p$-formula.

Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. We write $\mathcal{M} \equiv \mathcal{N}$ to indicate that $\mathcal{M}$ and $\mathcal{N}$ satisfy the same $\mathcal{L}$-sentences; and write $\mathcal{M} \subseteq \mathcal{N}$ to indicate that $\mathcal{M}$ is a substructure (also referred to as a submodel) of $\mathcal{N}$. If $\Gamma$ is a class of $\mathcal{L}$-formulae, then we will write $\mathcal{M} \prec \mathcal{N}$ if $\mathcal{M} \subseteq \mathcal{N}$ and for every finite tuple $\vec{a} \in M$, $\vec{a}$ satisfies the same $\Gamma$-formulae in both $\mathcal{M}$ and $\mathcal{N}$. In the case that $\Gamma$ is $\Pi_\infty$ (i.e., all $\mathcal{L}$-formulae) or $\Gamma$ is $\Sigma_n$, we will abbreviate this notation by writing $\mathcal{M} \prec \mathcal{N}$ and $\mathcal{M} \prec_n \mathcal{N}$ respectively. If $\mathcal{M} \subseteq \mathcal{N}$ and for all $x \in M$ and $y \in N$,

$$\text{if } \mathcal{N} \models (y \in x) \text{ then } y \in M,$$

then we say that $\mathcal{N}$ is an end-extension of $\mathcal{M}$ (equivalently: $\mathcal{M}$ is an initial submodel of $\mathcal{N}$, or $\mathcal{M}$ is a transitive submodel of $\mathcal{N}$) and write $\mathcal{M} \subseteq_e \mathcal{N}$. It is well-known that if $\mathcal{M} \subseteq_e \mathcal{N}$, then $\mathcal{M} \prec_0 \mathcal{N}$. The following is a slight generalisation of the notion of a powerset preserving end-extension that was first studied by Forster and Kaye in [FK].

**Definition 2.1** Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. We say that $\mathcal{N}$ is a **powerset preserving end-extension** of $\mathcal{M}$, and write $\mathcal{M} \subseteq^p_e \mathcal{N}$ if

(i) $\mathcal{M} \subseteq_e \mathcal{N}$, and

(ii) for all $x \in N$ and for all $y \in M$, if $\mathcal{N} \models (x \subseteq y)$, then $x \in M$.

Just as end-extensions preserve $\Delta_0$-properties, powerset preserving end-extensions preserve $\Delta_0^p$-properties. The following is a slight modification of a result proved in [FK]:

**Lemma 2.2** Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures that satisfy Extensionality. If $\mathcal{M} \subseteq^p_e \mathcal{N}$, then $\mathcal{M} \prec_{\Delta_0^p} \mathcal{N}$. \(\square\)

**Definition 2.3** Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. We say that $\mathcal{N}$ is a **topless powerset preserving end-extension** of $\mathcal{M}$, and write $\mathcal{M} \subseteq^\topless_p \mathcal{N}$ if
(i) \( M \subseteq^p N \), and
(ii) if \( c \in N \) and \( c^* \subseteq M \), then \( c \in M \).

Let \( \Gamma \) be a class of \( \mathcal{L} \)-formulae. The following define the restriction of some commonly encountered axiom and theorem schemes of ZFC to formulae in the class \( \Gamma \):

\((\Gamma\text{-Separation})\) For all \( \phi(x, \vec{z}) \in \Gamma \),
\[
\forall \exists \forall w \exists y \forall x(x \in y \iff (x \in w) \land \phi(x, \vec{z})).
\]

\((\Gamma\text{-Collection})\) For all \( \phi(x, y, \vec{z}) \in \Gamma \),
\[
\forall \exists \forall w((\forall x \in w) \exists y \phi(x, y, \vec{z}) \Rightarrow \exists c(\forall x \in w)(\exists y \in c)\phi(x, y, \vec{z})).
\]

\((\text{Strong } \Gamma\text{-Collection})\) For all \( \phi(x, y, \vec{z}) \in \Gamma \),
\[
\forall \exists \forall w \exists C(\forall x \in w)(\exists y \phi(x, y, \vec{z}) \Rightarrow (\exists y \in C)\phi(x, y, \vec{z})).
\]

\((\Gamma\text{-Foundation})\) For all \( \phi(x, \vec{z}) \in \Gamma \),
\[
\forall \exists(\exists x \phi(x, \vec{z}) \Rightarrow \exists y(\phi(y, \vec{z}) \land (\forall x \in y)\neg \phi(x, \vec{z}))).
\]

If \( \Gamma = \{x \in z\} \) then we will refer to \( \Gamma\text{-Foundation} \) as \textit{Set Foundation}.

We will use \( \bigcup x \subseteq x \) to abbreviate the \( \Delta^0_0 \)-formula that says “\( x \) is transitive”, i.e., \((\forall y \in x)(\forall z \in y)(z \in x)\). We will also make reference to the axiom of transitive containment (TCo), Zermelo’s well-ordering principle (WO), Axiom H and for all \( n \in \omega \), the axiom scheme of \( \Delta^1_n\)-Separation:

\((\text{TCo})\)
\[
\forall x \exists y \left( \bigcup y \subseteq y \land x \subseteq y \right).
\]

\((\text{WO})\)
\[
\forall x \exists r (r \text{ is a well-ordering of } x).
\]

\((\text{Axiom H})\)
\[
\forall u \exists t \left( \bigcup t \subseteq t \land \forall z \left( \bigcup z \subseteq z \land |z| \leq |u| \Rightarrow z \subseteq t \right) \right).
\]

\((\Delta^1_n\text{-separation})\) For all \( \Sigma^1_n \)-formulae \( \phi(x, \vec{z}) \) and \( \psi(x, \vec{z}) \),
\[
\forall \exists \forall w(\forall x(\phi(x, \vec{z}) \iff \neg \psi(x, \vec{z})) \Rightarrow \exists y \forall x(x \in y \iff (x \in w) \land \phi(x, \vec{z}))).
\]

For \( \alpha \) an ordinal, the \( \alpha\text{-dependent choice scheme} \ (\Pi^1_\infty \text{-DC}_\alpha) \) is the natural class version of Lévy’s axiom \( \text{DC}_\alpha \) \( \text{[Lévy]} \) that generalises Tarski’s Dependent Choice Principle by facilitating \( \alpha \)-sequences of dependent choices.

\((\Pi^1_\infty \text{-DC}_\alpha)\) For all \( \mathcal{L} \)-formulae \( \phi(x, y, \vec{z}) \),
\[
\forall \exists \left( \forall g(\forall \gamma \in \alpha)((g \text{ is a function}) \land (\text{dom}(g) = \gamma) \Rightarrow \exists y \phi(g(y, \vec{z})) \Rightarrow \right.
\]
\[
\exists f((f \text{ is a function}) \land (\text{dom}(f) = \alpha) \land (\forall \beta \in \alpha)\phi(f \upharpoonright \beta, f(\beta), \vec{z})) \right).
\]
We will have cause to consider the following subsystems of ZF C:

- $M^-$ is the $\mathcal{L}$-theory with axioms: Extensionality, Emptyset, Pair, Union, Infinity, TCo, $\Delta_0$-Separation and Set Foundation.
- $M$ is obtained from $M^-$ by adding the powerset axiom.
- Mac is obtained from $M$ by adding the axiom of choice.
- KPI is obtained from $M^-$ by adding $\Delta_0$-Collection and $\Pi_1$-Foundation.
- KP is obtained from KPI by removing the axiom of infinity.
- $K^P$ is obtained from $M^-$ by adding $\Delta_0^P$-Collection and $\Pi_1^P$-Foundation.
- MOST is obtained from Mac by adding $\Sigma_1$-Separation and $\Delta_0$-Collection.
- $ZF^-$ is obtained by adding $\Pi_\infty$-Collection to KPI.

The theories $M$, KPI, KP and $K^P$ are studied in [Mat01]. In contrast with the version of Kripke-Platek Set Theory studied in [Fri, Bar75], which includes $\Pi_\infty$-Foundation, we follow [Mat01], by only including $\Pi_1$-Foundation in the theories KP and KPI, and only including $\Pi_1^P$-Foundation in the theory $K^P$. The theory KPI, as defined here, plays a key role in [FLW], where it is referred to as $K^- + \text{infinity} + \Pi_1$-Foundation.

The results of Zarach and, more recently, [GHT] highlight the importance of axiomatising $ZF^- + \text{WO}$ using the collection scheme ($\Pi_\infty$-Collection) instead of the replacement scheme. The strength of Zermelo’s well-order principle WO in the $ZF^-$ context is revealed in [Zar82], which shows that, in the absence of the powerset axiom, the statement that every set of nonempty sets has a choice function does not imply WO\[1\].

Let $ZF^- + \text{GWO}(R)$ be the extension of $ZF^- + \text{WO}$ obtained as follows: introduce a new binary relation symbol, $R$, to the language of set theory, and then add an axiom asserting that $R$ is a bijection between the universe and the class of ordinals (a global well-order), and also extend the schemes of separation and collection so as to ensure that formulae mentioning $R$ can be used. As a consequence of a result of Flanagan [Fla, Theorem 7.1], $ZF^- + \text{GWO}(R)$ is a conservative extension of the theory $ZF^- + \text{WO} + \forall \alpha \Pi_\infty^1 - \text{DC}_\alpha$. Recent work of S. Friedman, Gitman and Kanovei [FGK] shows that $\Pi_\infty^1 - \text{DC}_\omega$ is independent of $ZF^- + \text{WO}$.

Next we record the following useful relationships between fragments of Collection, Separation and Foundation over the base theory $M^-$.\footnote{Zarach credits Z. Szczechaniak with first finding a model of $ZF^-$ in which every set of nonempty sets has a choice function, but in which WO fails.}

**Lemma 2.4** Let $\Gamma$ be a class of $\mathcal{L}$-formulae, and $n \in \omega$.

1. In the presence of $M^-$, $\Pi_n$-Separation is equivalent to $\Sigma_n$-Separation.
2. $M^- + \Gamma$-Separation $\vdash \Gamma$-Foundation.
3. $M^- + \Pi_n$-Collection $\vdash \Sigma_{n+1}$-Collection.
4. [FLW, Lemma 4.13] $M^- + \Pi_n$-Collection $\vdash \Delta_{n+1}$-Separation.
5. [McK19, Lemma 2.5] In the presence of $M^-$, $\Pi_n$-Collection+$\Sigma_{n+1}$-Separation is equivalent to Strong $\Pi_n$-Collection.

As indicated by the following well-known result, over the theory $M^-$, $\Pi_n$-Collection implies that the classes $\Pi_{n+1}$ and $\Sigma_{n+1}$ are essentially closed under bounded quantification (part (3) of Lemma 2.4 is used in the proof).

**Lemma 2.5** Let $\phi(x, \vec{z})$ be a $\Sigma_{n+1}$-formula and let $\psi(x, \vec{z})$ be a $\Pi_{n+1}$-formula. The theory $M^- + \Pi_n$-Collection proves that $(\forall x \in y)\phi(x, \vec{z})$ is equivalent to a $\Sigma_{n+1}$-formula and $(\exists x \in y)\psi(x, \vec{z})$ is equivalent to a $\Pi_{n+1}$-formula.

**Definition 2.6** A transitive set $M$ is said to be **admissible** if $\langle M, \in \rangle \models KP$.

The theory KPI and its variants that include the scheme of full class foundation have been widely studied [Fri, Bar75, Mat01, FLW]. One appealing feature of this theory is the fact that it is strong enough to carry out many of the fundamental set-theoretic constructions such as defining set-theoretic rank, proving the existence of transitive closures, defining satisfaction and constructing G"odel's $L$ hierarchy.

- For all sets $X$, we use $TC(X)$ to denote the $\subseteq$-least transitive set with $X$ as a subset. The theory KPI proves that the function $X \mapsto TC(X)$ is total. Moreover, the proof of [Mat01] Proposition 1.29 shows that the formulae “$x = TC(y)$” and “$x \in TC(y)$” with free variables $x$ and $y$ are $\Delta^1_{KP}$, and “$x = TC(y)$” is also $\Delta^0_P$.

- The theory KP is capable of defining and proving the totality of the rank function $\rho$ satisfying
  \[ \rho(a) = \sup \{ \rho(b) + 1 : b \in a \} \, . \]
The formula “$\rho(x) = y$” with free variables $x$ and $y$ is $\Delta^1_{KP}$ [Fri, Theorem 1.5].

- As verified in [Bar75] Section III.1, satisfaction in set structures is definable in KPI. In particular, if $\mathcal{N}$ is a set structure in a model $M$ of KPI, $\mathcal{L}(\mathcal{N})$ is the language of $\mathcal{N}$, $\vec{a}$ is an $\mathcal{M}$-finite sequence of members of $\mathcal{N}$, and $\phi$ is an $\mathcal{L}(\mathcal{N})$-formula in the sense of $\mathcal{M}$ whose arity agrees with the length of $\vec{a}$, then “$M \models \phi[\vec{v}/\vec{a}]$” is definable in $\mathcal{M}$ by a formula that is $\Delta^1_{KPI}$.

- As shown in [Bar75] Chapter II] the theory KPI is capable of constructing the levels of G"odel’s $L$ hierarchy. The following operation can be defined using a formula for satisfaction for set structures in KPI: for all sets $X$,
  \[ \text{Def}(X) = \{ Y \subseteq X \mid Y \text{ is a definable subclass of } \langle X, \in \rangle \} . \]
The levels of the $L$ hierarchy are then recursively defined by:
  
  \[ L_0 = \emptyset \text{ and } L_\alpha = \bigcup_{\beta < \alpha} L_\beta \text{ if } \alpha \text{ is a limit ordinal}, \]
  
  \[ L_{\alpha+1} = L_\alpha \cup \text{Def}(L_\alpha), \]
  
  \[ L = \bigcup_{\alpha \in \text{Ord}} L_\alpha. \]

The function $\alpha \mapsto L_\alpha$ is total and $\Delta^1_{KP}$. As usual, we will use $V = L$ to abbreviate the axiom that says that every set is a member of some $L_\alpha$, i.e., $\forall x \exists \alpha (x \in L_\alpha)$. 

The fact that KPI can express satisfaction in set structures can be used, in this theory, to express satisfaction for \( \Delta_0 \)-formulae in the universe via the definition below.

**Definition 2.7** The formula \( \text{Sat}_{\Delta_0}(q, x) \) is defined as

\[
(q \in \omega) \land (q = \langle \phi(v_1, \ldots, v_m) \rangle \land \text{where } \phi \text{ is } \Delta_0) \land (x = \langle x_1, \ldots, x_m \rangle) \land \\
\exists N (\cup N \subseteq N \land (x_1, \ldots, x_m \in N) \land ((N, \in) \models \phi[x_1, \ldots, x_m]))
\]

The absoluteness of \( \Delta_0 \) properties between transitive structures and the universe, and the availability of TCo in KPI imply that the formula \( \text{Sat}_{\Delta_0} \) satisfies the universal generalization of the following formula:

\[
\text{Sat}_{\Delta_0}(q, x) \iff \forall N (\cup N \subseteq N \land (x = \langle x_1, \ldots, x_m \rangle) \Rightarrow ((N, \in) \models \phi[x_1, \ldots, x_m]))
\]

Therefore, the fact that \( \langle N, \in \rangle \models \phi[\cdots] \) is \( \Delta^1_{KPI} \) implies that \( \text{Sat}_{\Delta_0}(q, x) \) is also \( \Delta^1_{KPI} \), and \( \text{Sat}_{\Delta_0}(q, x) \) expresses satisfaction for \( \Delta_0 \)-formulae in the theory KPI. We can now inductively define formulae \( \text{Sat}_{\Sigma_n}(q, x) \) and \( \text{Sat}_{\Pi_n}(q, x) \) that express satisfaction for formulae in the classes \( \Sigma_n \) and \( \Pi_n \).

**Definition 2.8** The formulae \( \text{Sat}_{\Sigma_n}(q, x) \) and \( \text{Sat}_{\Pi_n}(q, x) \) are defined recursively for \( n > 0 \). \( \text{Sat}_{\Sigma_{n+1}}(q, x) \) is defined as the formula

\[
\exists \bar{y} \forall k \exists b \left( (q = \langle \exists \bar{u} \phi(\bar{u}, v_1, \ldots, v_l) \rangle \land (x = \langle x_1, \ldots, x_l \rangle) \land (b = \langle \bar{y}, x_1, \ldots, x_l \rangle) \land (k = \langle \phi(\bar{u}, v_1, \ldots, v_l) \rangle) \land \text{Sat}_{\Pi_n}(k, b) \right)
\]

and \( \text{Sat}_{\Pi_{n+1}}(q, x) \) is defined as the formula

\[
\forall \bar{y} \forall k \forall b \left( (q = \langle \forall \bar{u} \phi(\bar{u}, v_1, \ldots, v_l) \rangle \land (x = \langle x_1, \ldots, x_l \rangle) \land ((b = \langle \bar{y}, x_1, \ldots, x_l \rangle) \land (k = \langle \phi(\bar{u}, v_1, \ldots, v_l) \rangle) \Rightarrow \text{Sat}_{\Sigma_n}(k, b)) \right)
\]

**Theorem 2.9** Suppose \( n \in \omega \) and \( m = \max \{1, n\} \). The formula \( \text{Sat}_{\Sigma_n}(q, x) \) (respectively \( \text{Sat}_{\Pi_n}(q, x) \)) is \( \Sigma^1_n \) (respectively \( \Pi^1_n \)), respectively. Moreover, \( \text{Sat}_{\Sigma_n}(q, x) \) (respectively \( \text{Sat}_{\Pi_n}(q, x) \)) expresses satisfaction for \( \Sigma_n \)-formulae (\( \Pi_n \)-formulae, respectively) in the theory KPI, i.e., if \( M \models \text{KPI} \), \( \phi(v_1, \ldots, v_k) \) is a \( \Sigma_n \)-formula, and \( x_1, \ldots, x_k \) are in \( M \), then for \( q = \langle \phi(v_1, \ldots, v_k) \rangle \), \( M \) satisfies the universal generalization of the following formula:

\[
x = \langle x_1, \ldots, x_k \rangle \Rightarrow (\phi(x_1, \ldots, x_k) \iff \text{Sat}_{\Sigma_n}(q, x)).
\]

The following result appears in [FLW, Theorem 3.8].

**Lemma 2.10** (Friedman, Li, Wong) The theory KP proves the Schröder-Bernstein Theorem, i.e., KP proves that if \( A \) and \( B \) are sets such that \( |A| \leq |B| \) and \( |B| \leq |A| \), then \( |A| = |B| \).

The following theorem highlights the important fact that the \( \Sigma^P \)-Recursion Theorem is provable in the theory KP\(^P\) [Mat01, Theorem 6.26].

**Theorem 2.11** (KP\(^P\)) Let \( G \) be a \( \Sigma^P_1 \)-definable class. If \( G \) is a total function, then there exists a \( \Sigma^P_1 \)-definable total class function \( F \) such that for all \( x \), \( F(x) = G(F \upharpoonright x) \).
Definition 2.12 We write “$V_\alpha$ exists” as an abbreviation for the sentence expressing that $\alpha$ is an ordinal, and there is a function $f$ whose domain is $\alpha+1$ that satisfies the following conditions (1) through (3) below.

1. $f(0) = \emptyset$.

2. $\forall \beta < \alpha \left( (\beta \text{ is a limit ordinal}) \Rightarrow f(\beta) = \bigcup_{\xi < \beta} f(\xi) \right)$.

3. $(\forall \beta \in \text{dom}(f))(\forall y)(y \in f(\beta+1) \iff y \subseteq f(\beta))$.

Note that under Definition 2.12 if $V_\alpha$ exists, then $V_\beta$ exists for all $\beta < \alpha$. The following consequence of the $\Sigma^P_1$-Recursion Theorem is Proposition 6.28 of [Mat01].

Corollary 2.13 The theory $\text{KP}^P$ proves that for all ordinals $\alpha$, $V_\alpha$ exists. Note that in particular, this theory proves that for all ordinals $\alpha$, there is a function $f$ with domain $\alpha$ such that for all $\beta \in \alpha$, $f(\beta) = V_\beta$. $\square$

Section 3 of [Mat01] contains the verification of the following lemma.

Lemma 2.14 MOST is the theory $\text{Mac} + \text{Axiom H}$.

We also record the following consequence of MOST that are proved in [Mat01] Section 3:

Lemma 2.15 The theory MOST proves

(i) every well-ordering is isomorphic to an ordinal,

(ii) every well-founded extensional relation is isomorphic to a transitive set,

(iii) for all cardinals $\kappa$, $\kappa^+$ exists, and

(iv) for all cardinals $\kappa$, the set $H_\kappa = \{x \mid |\text{TC}(x)| < \kappa\}$ exists.

The following result is [EKM] Lemma 3.3] combined with the refinement of a theorem due to Takahashi proved in [Mat01] Proposition Scheme 6.12:

Lemma 2.16 If $\mathcal{M}, \mathcal{N} \models \text{MOST}$ and $\mathcal{N} \subseteq^P \text{topless} \mathcal{M}$, then $\mathcal{N} \models \Pi_1^1\text{-Collection}$.

We next recall a remarkable absoluteness phenomenon unveiled by Lévy [Lév], which shows that, provably in ZF, $H^L_{\aleph_1}$ (i.e., the collection of sets that are hereditarily countable, as computed in the constructible universe) is a $\Sigma_1$-elementary submodel of the universe of sets.$^2$

Theorem 2.17 (Lévy-Shoenfield Absoluteness) Let $\theta(x, y)$ be a $\Sigma^ZF_1$-formula with no free variables except $x$ and $y$, then the universal generalization of the following formula is provable in ZF

$$(y \in H^L_{\aleph_1} \land \exists x \theta(x, y)) \Rightarrow \exists x (x \in H^L_{\aleph_1} \land \theta(x, y)).$$

$^2$The proof of this result relies heavily on the venerable Shoenfield Absoluteness Theorem. The original proof by Lévy of Theorem 2.17 presented this result as a theorem of ZF + DC (where DC here is axiom of dependent choice of length $\omega$). As pointed out by Kunen, DC can be eliminated by a forcing-and-absoluteness stratagem (see page 55 of [Bar71]). Later Barwise and Fischer gave a direct forcing-free proof in ZF [BF].
The Lévy-Shoenfield Absoluteness Theorem readily implies the following corollary that shows that the $\Sigma_1$-theory of every model of ZF coincides with the $\Sigma_1$-theory of $H_{\aleph_1}$ of the constructible universe of the same model.

**Corollary 2.18** Let $\delta(\vec{x})$ be a $\Delta^2_0$-formula, and $\mathcal{M}$ be a model of ZF. Then

$$\mathcal{M} \models \exists \vec{x} \delta(\vec{x}) \iff H^1_{\aleph_1} \models \exists \vec{x} \delta(\vec{x}).$$

Any model of KPI comes equipped with its well-founded part that consists of all sets in this structure whose rank is a standard ordinal, as indicated by the following definition.

**Definition 2.19** Let $\mathcal{M} \models KPI$. The **well-founded part** or **standard part** of $\mathcal{M}$, denoted $WF(\mathcal{M})$, is the substructure of $\mathcal{M}$ with underlying set

$$WF(\mathcal{M}) = \{ x \in M \mid \neg \exists f : \omega \to M \land (\forall n \in \omega)(f(0) = x \land f(n + 1) \in M \land f(n))\}.$$  

If $WF(\mathcal{M}) \neq \mathcal{M}$, then we say that $\mathcal{M}$ is **nonstandard**. The **standard ordinals** of $\mathcal{M}$, denoted $o(\mathcal{M})$, is the substructure of $\mathcal{M}$ with underlying set $o(\mathcal{M}) = WF(\mathcal{M}) \cap Ord^M$. If $\omega^M \in o(\mathcal{M})$, then we say that $\mathcal{M}$ is $\omega$-**standard**; otherwise $\mathcal{M}$ is said to be $\omega$-**nonstandard**. Mostowski’s Collapsing Lemma ensures that both $o(\mathcal{M})$ and $WF(\mathcal{M})$ are isomorphic to transitive sets. In particular, $o(\mathcal{M})$ is isomorphic to an ordinal that is called the **standard ordinal** of $\mathcal{M}$.

The following definition generalises the notion of standard system that plays an important role in the study of models of arithmetic.

**Definition 2.20** Let $\mathcal{M} \models KPI$. The **standard system** of $\mathcal{M}$ is the set

$$SSy(\mathcal{M}) = \{ y^* \cap WF(\mathcal{M}) \mid y \in M \}.$$  

If $A \in SSy(\mathcal{M})$ and $y \in M$ is such that $A = y^* \cap WF(\mathcal{M})$, then we say that $y$ codes $A$.

**Definition 2.21** Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An **embedding** of $\mathcal{M}$ into $\mathcal{N}$ is an injection $j : M \to N$ such that for all $x, y \in M$,

$$\mathcal{M} \models x \in y \text{ if and only if } \mathcal{N} \models j(x) \in j(y).$$

Note that we will often write $j : \mathcal{M} \to \mathcal{N}$ to indicate that $j$ is an embedding of $\mathcal{M}$ into $\mathcal{N}$. If $j : \mathcal{M} \to \mathcal{N}$ is an embedding of $\mathcal{M}$ into $\mathcal{N}$, then we write $j[M]$ for the substructure of $\mathcal{N}$ whose underlying set is $\text{rng}(j)$.

**Definition 2.22** Let $\mathcal{M}$ be an $\mathcal{L}$-structure and let $j : \mathcal{M} \to \mathcal{M}$ be an embedding of $\mathcal{M}$ into $\mathcal{M}$. The **fixed point set** of $j$ is the set $\text{Fix}(j) = \{ x \in M \mid j(x) = x \}$.

**Definition 2.23** Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. Let $j : \mathcal{M} \to \mathcal{N}$ be an embedding of the structure $\mathcal{M}$ into $\mathcal{N}$. We say that $j$ is an **initial embedding** if $j[M] \subseteq \mathcal{N}$. We say that $j$ is a $\mathcal{P}$-**initial embedding** if $j[M] \subseteq^P \mathcal{N}$. If $j : \mathcal{M} \to \mathcal{M}$ is a ($\mathcal{P}$-) initial embedding with $j[M] \neq \mathcal{M}$, then we say that $j$ is a **proper ($\mathcal{P}$-) initial self-embedding** of $\mathcal{M}$.

Next, we take advantage of the rank function available in KP to define the notion of rank extension, and the notion of rank-initial embedding.
Definition 2.24 Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures with $\mathcal{M} \subseteq_e \mathcal{N}$ and $\mathcal{N} \models \text{KP}$. We say that $\mathcal{N}$ is a **rank extension** of $\mathcal{M}$ if for all $\alpha \in \text{Ord}^\mathcal{M}$, if $x \in \mathcal{N}$ and $\mathcal{N} \models \rho(x) = \alpha$, then $x \in \mathcal{M}$.

Definition 2.25 Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures that satisfy $\text{KP}$. Let $j : \mathcal{M} \rightarrow \mathcal{N}$ be an embedding of the structure $\mathcal{M}$ into $\mathcal{N}$. We say $j$ is a **rank-initial embedding** if $j$ is an initial embedding and $\mathcal{M}$ is a rank extension of $j[\mathcal{M}]$. If $j : \mathcal{M} \rightarrow \mathcal{M}$ is a rank-initial embedding with $j[\mathcal{M}] \neq \mathcal{M}$, then we say that $j$ is a **proper rank-initial self-embedding** of $\mathcal{M}$.

Note that a rank-initial embedding $j : \mathcal{M} \rightarrow \mathcal{N}$, where $\mathcal{M}, \mathcal{N} \models \text{KP}$, is also $\mathcal{P}$-initial. The following result of Gorbow [Gor, Corollary 4.6.12] shows that if the source and target model of a $\mathcal{P}$-initial embedding both satisfy $\text{KP}$, then this embedding is also rank-initial.

**Lemma 2.26** Let $\mathcal{M}$ and $\mathcal{N}$ be models of $\text{KP}$. If $j : \mathcal{M} \rightarrow \mathcal{N}$ is a $\mathcal{P}$-initial embedding, then $j$ is a rank-initial embedding.

Note that, in any model of ZFC, $L$ is a powerset preserving end-extension of $H^L_\omega$ and $H^L_\omega$ satisfies $\text{MOST} + \Pi_\infty$-$\text{Separation}$. This example shows that the assumption that $\mathcal{M}$ satisfies $\Delta^P_0$-$\text{Collection}$ in Lemma 2.26 can not relaxed to $\Delta^P_0$-$\text{Collection}$ even in the presence of the full scheme of separation.

H. Friedman’s seminal [Fri] pioneered the study of rank-initial self-embeddings of $\text{KP}^P + \Pi_\infty$-$\text{Foundation}$. His work was refined and extended by Ressayre [Res], and more recently by Gorbow [Gor]. The following theorem of Gorbow guarantees the existence of proper rank-initial self-embeddings of countable nonstandard models of an extension of $\text{KP}^P$. Gorbow’s theorem refines [Fri, Theorem 4.3], and is a consequence of results proved in [Gor, Section 5.2].

**Theorem 2.27** (Gorbow) Every countable nonstandard model $\mathcal{M}$ of $\text{KP}^P + \Sigma^P_1$-$\text{Separation}$ has a proper rank-initial self-embedding. Moreover, given any $\alpha \in \text{Ord}^\mathcal{M}$ there exists a proper rank-initial self-embedding $j$ of $\mathcal{M}$ that fixes every element of $(V^\mathcal{M}_\alpha)^*$. 

We also note the following self-embedding theorem that is readily obtained by putting [EKM, Theorem 5.6] together [Mat01, Proposition Scheme 6.12].

**Theorem 2.28** Every countable recursively saturated model of $\text{MOST} + \Pi_1$-$\text{Collection}$ has a proper $\mathcal{P}$-initial self-embedding.

### 3 The well-founded part

In this section we present results about well-founded parts of models of set theory that are relevant to the proofs the main result of this paper. H. Friedman’s [Fri] systematically studied the structure of the well-founded part of a nonstandard model of Kripke-Platek Set Theory and [Fri, Theorem 2.1] showed that such a well-founded part must be isomorphic to an admissible set. This result is also a consequence of [Bar75, Lemma 8.4]. As we mentioned earlier, the versions of Kripke-Platek Set Theory studied in [Bar75] and [Fri] include $\Pi_\infty$-$\text{Foundation}$. An examination of these proofs reveals that the well-founded part of a model of $\text{KPI} + \Sigma_1$-$\text{Foundation}$ is isomorphic to an admissible set. Before proving this, we will first verify in the lemma below that any nonstandard model of KPI is a topless powerset preserving end-extension of its well-founded part.
Lemma 3.1 Let $\mathcal{M}$ be a nonstandard model of KPI. Then
\[ \text{WF}(\mathcal{M}) \subseteq_{\text{topless}} \mathcal{M}. \]

Proof It follows immediately from Definition 2.19 that $\text{WF}(\mathcal{M}) \subseteq_{e} \mathcal{M}$. The fact that $\mathcal{M}$ is nonstandard immediately means that $M \neq \text{WF}(\mathcal{M})$. Let $c \in M$ with $c^* \subseteq \text{WF}(\mathcal{M})$. Suppose, for a contradiction, that $f : \omega \to M$ witness the fact that $c \notin \text{WF}(\mathcal{M})$. But, $f(0) = c$ and $\mathcal{M} \models (f(1) \in c)$, so $f(1) \in \text{WF}(\mathcal{M})$. Define $g : \omega \to M$ by: for all $n \in \omega$ with $n \geq 1$, $g(n-1) = f(n)$. Now, $g$ witness the fact that $f(1) \notin \text{WF}(\mathcal{M})$, which is a contradiction. This shows that $\text{WF}(\mathcal{M}) \subseteq_{\text{topless}} \mathcal{M}$. ◻

Lemma 3.2 Let $\mathcal{M}$ be a nonstandard model of KPI. Then $\text{WF}(\mathcal{M})$ satisfies Extensionality, Emptyset, Pair, Union, $\Delta_0$-Separation and $\Pi_\infty$-Foundation. Moreover, for all $x \in M$,
\[ x \in \text{WF}(\mathcal{M}) \text{ if and only if } \rho^\mathcal{M}(x) \in \text{WF}(\mathcal{M}). \]

Proof The fact $\text{WF}(\mathcal{M}) \subseteq_{e} \mathcal{M}$ implies that $\text{WF}(\mathcal{M})$ satisfies Extensionality, Emptyset, Pair, Union, and $\Delta_0$-Separation. The fact that $\text{WF}(\mathcal{M})$ is well-founded ensures that $\Pi_\infty$-Foundation holds in $\text{WF}(\mathcal{M})$. Now, to prove the last statement, consider $a = \{x \in \text{WF}(\mathcal{M}) \mid \rho^\mathcal{M}(x) \notin \text{WF}(\mathcal{M})\}$. Suppose that $a \neq \emptyset$ and let $z \in a$ be a $\in^\mathcal{M}$-least member of $a$. Working inside $\mathcal{M}$, consider
\[ b = \{\rho(y) + 1 \mid y \in z\}. \]
Next we observe that $b$ is a set and $b^* \subseteq \text{WF}(\mathcal{M})$, so $b \in \text{WF}(\mathcal{M})$. It now follows that $\rho^\mathcal{M}(z) = (\sup b)^\mathcal{M} \in \text{WF}(\mathcal{M})$, which is a contradiction. This shows that if $x \in \text{WF}(\mathcal{M})$, then $\rho^\mathcal{M}(x) \in \text{WF}(\mathcal{M})$. Conversely, consider
\[ c = \{\rho^\mathcal{M}(x) \mid x \notin \text{WF}(\mathcal{M})\}. \]
Suppose, for a contradiction, that $c \cap \text{WF}(\mathcal{M}) \neq \emptyset$. Let $\alpha \in c$ be least and let $z \in M$ with $z \notin \text{WF}(\mathcal{M})$ be such that $\mathcal{M} \models \rho(z) = \alpha$. Since $\alpha$ is least, $z^* \subseteq \text{WF}(\mathcal{M})$. Since $\text{WF}(\mathcal{M}) \subseteq_{\text{topless}} \mathcal{M}$, this implies that $z \in \text{WF}(\mathcal{M})$, which is a contradiction. ◻

We next verify that under the additional assumption that the nonstandard model of KPI satisfies $\Sigma_1$-Foundation, the well-founded part also satisfies $\Delta_0$-Collection.

Theorem 3.3 Let $\mathcal{M}$ be a nonstandard model of KPI + $\Sigma_1$-Foundation. Then $\text{WF}(\mathcal{M})$ is isomorphic to an admissible set.

Proof We need to show that $\text{WF}(\mathcal{M})$ satisfies all of the axioms of KP. By Lemma 3.2, we are left to verify that $\text{WF}(\mathcal{M})$ satisfies $\Delta_0$-Collection. Let $\phi(x, y, \bar{z})$ be a $\Delta_0$-formula and let $a, \bar{b} \in \text{WF}(\mathcal{M})$ be such that
\[ \text{WF}(\mathcal{M}) \models (\forall x \in a) \exists y \phi(x, y, \bar{b}). \]
Since $\text{WF}(\mathcal{M}) \prec_{\Delta_0} \mathcal{M}$,
\[ \mathcal{M} \models (\forall x \in a) \exists y \phi(x, y, \bar{b}). \]
Consider \( \theta(\gamma, \vec{z}, w) \) defined by
\[
(\forall x \in w)(\exists \alpha \in \gamma)\exists y(\phi(x, y, \vec{z}) \land \rho(y) = \alpha).
\]
Recall that \( \Sigma_1 \)-Collection in \( \mathcal{M} \) implies that \( \theta(\gamma, \vec{z}, w) \) is equivalent to a \( \Sigma_1 \)-formula. Therefore, using \( \Sigma_1 \)-Foundation, let \( \delta \in M \) be the least element of
\[
A = \{ \beta \in M \mid \mathcal{M} \models \theta(\beta, \vec{b}, a) \}.
\]
Now, every nonstandard \( \mathcal{M} \)-ordinal is an element of \( A \) and so \( \delta \in \text{WF}(M) \). Let \( \psi(x, z, \gamma, \vec{w}) \) be the \( \Sigma_1 \)-formula
\[
\exists y \exists \alpha(\alpha = \langle y, \alpha \rangle \land (\alpha \in \gamma) \land \rho(y) = \alpha \land \phi(x, y, \vec{w})).
\]
Therefore,
\[
\mathcal{M} \models (\forall x \in a)\exists z \psi(x, z, \delta, \vec{b}).
\]
Work inside \( \mathcal{M} \). By \( \Sigma_1 \)-Collection, there exists \( d \) such that \( (\forall x \in a)(\exists z \in d)\psi(x, z, \delta, \vec{b}) \). Let \( c = \text{dom}(d) \). Note that by Lemma \( 3.2 \) \( c^* \subseteq \text{WF}(M) \) and so \( c \in \text{WF}(M) \). Hence
\[
\text{WF}(M) \models (\forall x \in a)(\exists y \in c)\phi(x, y, \vec{b}),
\]
and therefore \( \Delta_0 \)-Collection holds in \( \text{WF}(M) \). So, the Mostowski collapse of \( \text{WF}(M) \) witnesses the fact that \( \text{WF}(M) \) is isomorphic to an admissible set. \( \Box \)

In Definitions \( 3.4 \) and \( 3.5 \) we introduce two relationships between models of set theory and their well-founded parts that will be shown to be linked to the existence of proper initial self-embeddings in Sections \( 4 \) and \( 5 \).

**Definition 3.4** Let \( \mathcal{M} \models \text{KPI} \).

(a) The well-founded part of \( \mathcal{M} \) is **c-bounded** in \( \mathcal{M} \), where “c” stands for “cardinalitywise”, if there is some \( x \in M \) such that for all \( w \in \text{WF}(M) \) \( \mathcal{M} \models |x| > |w| \).

(b) The well-founded part of \( \mathcal{M} \) is **c-unbounded** in \( \mathcal{M} \) if the well-founded part of \( \mathcal{M} \) is not c-bounded in \( \mathcal{M} \), i.e., if for all \( x \in M \), there exists \( w \in \text{WF}(M) \) such that \( \mathcal{M} \models |x| \leq |w| \).

In Section \( 4 \) we will see that the well-founded part being c-bounded prevents a model of KPI from admitting a proper initial self-embedding. In contrast, the following condition will be used in Section \( 5 \) to show that nonstandard models of certain extensions of KPI are guaranteed to admit proper initial self-embedding.

**Definition 3.5** Let \( \mathcal{M} \models \text{KPI} \). We say that the **well-founded part of \( \mathcal{M} \) is contained** if there exists \( c \in M \) such that \( \text{WF}(M) \subseteq c^* \).

The next result shows that if the well-founded part of an \( \omega \)-standard model is contained, then Theorem \( 3.3 \) can be extended to show that the well-founded part satisfies all of the axioms of KP\(^P\).

**Theorem 3.6** Let \( \mathcal{M} \models \text{KPI} \) be \( \omega \)-standard. If the well-founded part of \( \mathcal{M} \) is contained, then \( \text{WF}(M) \models \text{KP}^P \).
Proof Suppose that the well-founded part of $\mathcal{M}$ is contained. Let $c \in M$ be such that $WF(M) \subseteq c^*$. Since $\omega \in WF(M)$, $WF(M)$ satisfies the axiom of infinity. By Lemma 3.2, we are left to verify that Powerset and $\Delta^P_0$-Collection hold in $WF(M)$. To see that Powerset holds, let $x \in WF(M)$. It follows from Lemma 3.2 that if $y \in M$ with $WF(M) \subseteq y \subseteq x$, then $y \in WF(M)$. Note that $\Delta^P_0$-Separation in $M$ ensures that $A = \{y \in c \mid y \subseteq x\}$ is a set. Moreover, $\mathcal{M} \models (A = P(x))$. Now, $A^* \subseteq WF(M)$ and so, by Lemma 3.1, $A \in WF(M)$. Therefore, $WF(M) \models$ Powerset. We are left to verify $\Delta^P_0$-Collection. Let $\phi(x,y,\vec{z})$ be a $\Delta^P_0$-formula. Work inside $\mathcal{M}$. Let $B = \{y \in c \mid (\exists x \in a)(\phi^c(x,y,\vec{b}) \land (\forall w \in c)(\phi^c(x,w,\vec{b}) \Rightarrow \rho(y) \leq \rho(w)))\}$. Now, $\Delta^0_1$-Separation ensures that $B$ is a set. Lemma 3.2 implies that $B^* \subseteq WF(M)$. Therefore, since $WF(M) \subseteq c^*$ and $WF(M) \subseteq P\topless \mathcal{M}$, for all $x, y, \vec{z} \in WF(M)$,

$$\mathcal{M} \models \phi(x,y,\vec{z}) \text{ if and only if } WF(M) \models \phi^c(x,y,\vec{z}).$$

We can now see that if the well-founded part of a model of KPI is contained, then the well-founded part is $c$-bounded in that model.

**Theorem 3.8** Let $\mathcal{M} \models$ KPI. If the well-founded part of $\mathcal{M}$ is contained, then the well-founded part of $\mathcal{M}$ is $c$-bounded in $\mathcal{M}$.\[\square\]
Proof Assume that the well-founded part of $\mathcal{M}$ is contained. Let $C \in M$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$. Suppose, for a contradiction, that the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$. Let $X \in \text{WF}(M)$ be such that $\text{WF}(M) \subseteq C^*$.
and so, by $\Pi^1_\alpha - \text{DC}_\kappa$, there exists $f \in M$ such that

$$\mathcal{M} \models (f \text{ is a function}) \land (\text{dom}(f) = \kappa) \land (\forall \alpha \in \kappa) \phi(f \upharpoonright \alpha, f(\alpha), Y, \kappa).$$

Work inside $\mathcal{M}$. If $\alpha \in \kappa$ is least such that $f(\alpha) \not\subseteq Y$, then $\phi(f \upharpoonright \alpha, f(\alpha), Y, \kappa)$ does not hold. Therefore for all $y \in \text{rng}(f)$, $y \subseteq Y$. If $\alpha \in \kappa$ is least such that there exists $\beta \in \alpha$ with $f(\alpha) = f(\beta)$, then $\phi(f \upharpoonright \alpha, f(\alpha), Y, \kappa)$ does not hold. This shows that $f$ is injective. Therefore, we have $\text{rng}(f)^* \subseteq \text{WF}(M)$ and $\mathcal{M} \models \kappa \leq |\text{rng}(f)|$. And, since $\text{WF}(M) \subseteq_{\text{topless}} M$, $\text{rng}(f) \in \text{WF}(M)$, and the fact that $\mathcal{M} \models \kappa \leq |\text{rng}(f)|$ contradicts our choice of $\kappa$. This shows that $\text{WF}(M)$ satisfies the powerset axiom.

**Lemma 3.12** Let $\mathcal{M}$ be a model of $\text{ZF}^- + \text{WO}$. If $\mathcal{M}$ is nonstandard and

$$\text{WF}(\mathcal{M}) \models \text{Powerset},$$

then the well-founded part of $\mathcal{M}$ is contained.

**Proof** Suppose that $\mathcal{M}$ is nonstandard and

$$\text{WF}(\mathcal{M}) \models \text{Powerset}.$$ 

Consider the formula $\phi(f, \alpha)$ that expresses that $f$ is a function with domain $\alpha$ such that $f(\beta) = V_{\beta}$ for all $\beta < \alpha$. Suppose that the class $\{\alpha \in \text{Ord}^M \mid \mathcal{M} \models \neg \exists f \phi(f, \alpha)\}$ is nonempty and, using foundation, let $\xi \in \text{Ord}^M$ be the least element of this class. We claim that $\xi \notin o(M)$. Suppose, for a contradiction, that $\xi \in o(M)$. Since $\text{WF}(\mathcal{M}) \models \text{Powerset}$, $\xi$ is not a successor ordinal and therefore must be a limit ordinal. Work inside $\mathcal{M}$. Using Collection and Separation in $\mathcal{M}$, the class

$$A = \{f \mid (\exists \alpha \in \xi) \phi(f, \alpha)\}$$

is a set. Now, $\bigcup A$ is a function that satisfies $\phi(\bigcup A, \xi)$, which contradicts our choice of $\xi$. This shows that if $\{\alpha \in \text{Ord}^M \mid \mathcal{M} \models \neg \exists f \phi(f, \alpha)\}$ is nonempty, then its least element cannot be standard. Therefore, since $\mathcal{M}$ is nonstandard, there exists $f \in M$ and $\gamma \in \text{Ord}^M \setminus o(M)$ such that $\mathcal{M} \models \phi(f, \gamma)$. Moreover, since $\text{WF}(\mathcal{M}) \subseteq_{\text{topless}} M$, there exists $\nu \in \gamma^* \setminus o(M)$. A standard induction argument inside $\mathcal{M}$ shows that

$$\mathcal{M} \models \forall x(x \in f(\nu) \iff \rho(x) < \nu).$$

Therefore, by Lemma 3.12 and the fact that $\nu$ is nonstandard, $\text{WF}(M) \subseteq f(\nu)^*$. This shows that the well-founded part of $\mathcal{M}$ is contained. □

**Theorem 3.13** Let $\mathcal{M}$ be a model of $\text{ZF}^- + \text{WO} + \forall \alpha \ \Pi^1_\alpha^- \text{DC}_\alpha$. If the well-founded part of $\mathcal{M}$ is $c$-bounded in $\mathcal{M}$, then the well-founded part of $\mathcal{M}$ is contained.

**Proof** Suppose that the well-founded part of $\mathcal{M}$ is $c$-bounded in $\mathcal{M}$. Therefore $\mathcal{M}$ is nonstandard and, by Lemma 3.11,

$$\text{WF}(\mathcal{M}) \models \text{Powerset}.$$ 

The fact that the well-founded part of $\mathcal{M}$ is contained now follows from Lemma 3.12 □
4 Obstructing initial self-embeddings

In this section we establish the first main result of the paper (Theorems 4.6) on the existence of countable nonstandard models of ZF$^- + \text{WO}$ with no nontrivial initial self-embeddings. Furthermore, in Theorem 4.7 we exhibit nonstandard uncountable models of ZF with no proper initial self-embeddings.

We begin with verifying that an initial embedding of a model of KPI must fix the well-founded part of this model. This result will allow us to show that models of KPI in which the well-founded part is $c$-unbounded (in the sense of Definition 3.4) do not admit proper initial self-embedding.

Lemma 4.1 Let $\mathcal{M} \models \text{KPI}$. If $j : \mathcal{M} \to \mathcal{M}$ is an initial self-embedding, then $j$ is the identity on $WF(\mathcal{M})$.

Proof Let $j : \mathcal{M} \to \mathcal{M}$ be a proper initial self-embedding. Suppose that $j$ is not the identity on $WF(\mathcal{M})$ and let $x \in WF(\mathcal{M})$ be $\in^\mathcal{M}$-least such that $j(x) \neq x$. Now, if $z \in M$ with $\mathcal{M} \models z \in x$ and $\mathcal{M} \models z \notin j(x)$, then $j(z) = z$ and $\mathcal{M} \models (z \in x) \land (j(z) \notin j(x))$, which is a contradiction. Similarly, if $z \in M$ with $\mathcal{M} \models z \notin x$ and $\mathcal{M} \models z \in j(x)$, then $j^{-1}(z) \neq z$ and $\mathcal{M} \models j^{-1}(z) \in x$, which contradicts the fact that $x$ is the $\in^\mathcal{M}$-least thing moved by $j$. 

Corollary 4.2 Let $\mathcal{M} \models \text{KPI}$. If $j : \mathcal{M} \to \mathcal{M}$ is a proper initial self-embedding, then $WF(\mathcal{M}) \subseteq \text{topless} \ j[M]$.

The next lemma shows that in addition to containing the well-founded part, the fixed point set $\text{Fix}(j)$ of a proper initial self-embedding $j : \mathcal{M} \to \mathcal{M}$ also contains all points that are $\Sigma_1$-definable in $\mathcal{M}$ from points in $\text{Fix}(j)$.

Lemma 4.3 Let $\mathcal{M} \models \text{KPI}$. Let $j : \mathcal{M} \to \mathcal{M}$ be an initial self-embedding. If $x \in M$ is definable in $\mathcal{M}$ by a $\Sigma_1$-formula with parameters from $\text{Fix}(j)$, then $x \in \text{Fix}(j)$.

Proof Suppose that $\phi(z, \bar{y})$ is a $\Sigma_1$-formula, $\bar{a} \in \text{Fix}(j)$ and $x \in M$ is the unique element of $M$ such that $\mathcal{M} \models \phi(x, \bar{a})$.

Therefore, since $\bar{a} \in \text{Fix}(j)$,

$j[M] \models \phi(j(x), \bar{a})$.

And, since $j[M] \subseteq e \mathcal{M}$ and $\phi$ is a $\Sigma_1$-formula,

$\mathcal{M} \models \phi(j(x), \bar{a})$.

By the uniqueness of $x$, $j(x) = x$ and $x \in \text{Fix}(j)$.

In particular, if $j : \mathcal{M} \to \mathcal{M}$ is a proper initial self-embedding and $x \in M$ is a point that is $\Sigma_1$-definable in $\mathcal{M}$ from points in the well-founded part of $\mathcal{M}$, then $x$ must be fixed by $j$. This observation allows us to show if the well-founded part of a nonstandard model of KPI is $c$-unbounded, then that model admits no proper initial self-embedding.

Theorem 4.4 Let $\mathcal{M} \models \text{KPI}$. If the well-founded part of $\mathcal{M}$ is $c$-unbounded in $\mathcal{M}$, and $j : \mathcal{M} \to \mathcal{M}$ is an initial self-embedding, then $j$ is the identity embedding.
Proof Assume that the well-founded part of \( M \) is c-unbounded in \( M \) and suppose that \( j : M \to M \) is an initial self-embedding. We will show that \( j \) must be the identity function. Let \( x \in M \). Let \( y \in M \) be such that

\[
\mathcal{M} \models y = \text{TC}^M(\{x\}).
\]

Let \( X \in \text{WF}(M) \) be such that

\[
\mathcal{M} \models |y| \leq |X|.
\]

Work inside \( M \). Let \( f : y \to X \) be injective. Let \( X' = \text{rng}(f) \). Consider \( Y = \{\langle u, v \rangle \in X \times X \mid (u, v \in \text{rng}(f)) \land (f^{-1}(u) \in f^{-1}(v))\} \).

Now, \( Y, X' \in \text{WF}(M) \). Consider \( \phi(z, W, Z) \) defined by

\[
\exists w \exists f \left( (f : w \to W \text{ is a bijection}) \land (\forall u, v \in w)(u \in v \iff \langle f(u), f(v) \rangle \in Z) \land (Z \subseteq W \times W) \land \bigcup w \subseteq w \land (z \in w) \land (\forall u \in w)(z \notin u) \right).
\]

Note that \( \phi(z, W, Z) \) is a \( \Sigma_1 \)-formula and \( x \) is the unique point in \( M \) such that

\[
\mathcal{M} \models \phi(x, X', Y).
\]

Therefore, by Lemmas 4.3 and 4.3, \( j(x) = x \). Since \( x \in M \) was arbitrary, this shows that \( j \) is the identity embedding, as desired. \( \square \)

This allows us to show that there are nonstandard \( \omega \)-standard models of ZF\(^-\) + WO that are not isomorphic to a transitive subclass of themselves. To build such a model we will employ the following consequence of [Fri, Theorem 2.2].

Lemma 4.5 Let \( T \) be a recursive \( \mathcal{L} \)-theory. If \( A \) is a countable admissible set such that \( \langle A, \in \rangle \models T \), then there exists a nonstandard \( \mathcal{L} \)-structure \( M \) such that \( M \models \text{KP} + T \) and \( o(M) = A \cap \text{Ord} \).

Theorem 4.6 There exists a countable nonstandard \( \omega \)-standard model \( M \models \text{ZF}^- + \text{WO} \) such that there is no initial self-embedding \( j : M \to M \) other than the identity embedding.

Proof Note that \( \langle H_{\aleph_1}, \in \rangle \models \text{ZF}^- + \text{WO} + \forall x(|x| \leq \aleph_0) \). Therefore, by the Downwards Löwenheim-Skolem Theorem and the Mostowski Collapsing Lemma, there exists a countable admissible set \( A \) such that \( \omega \in A \) and \( \langle A, \in \rangle \equiv \langle H_{\aleph_1}, \in \rangle \). So, by Lemma 4.3 and the Downwards Löwenheim-Skolem Theorem, there exists a countable \( \mathcal{L} \)-structure \( M \) such that \( M \models \text{ZF}^- + \text{WO} + \forall x(|x| \leq \aleph_0) \), \( M \) is nonstandard and \( o(M) = A \cap \text{Ord} \). Since \( \omega \in A \), the well-founded part of \( M \) is c-unbounded in \( M \) and so, by Theorem 4.4 there is no proper initial self-embedding \( j : M \to M \).

We conclude this section by exhibiting uncountable nonstandard models of ZF that carry no proper initial self-embeddings. Before doing so, let us note that it is well-known that every consistent extension of ZF has a model of cardinality \( \aleph_1 \) that carries no proper rank-initial self-embedding. To see this, recall that by a classical result due to Keisler and Morley (first established in [KM], and exposited as Theorem 2.2.18 of [CK]) every countable model of ZF has a proper elementary end-extension. It is easy to see that an elementary end-extension of a model of ZF is a rank-extension. Now if \( T \) is a consistent extension of ZF, we can readily build
a countable nonstandard model of $T$ and use the Keisler-Morley theorem $\aleph_1$-times (while taking unions at limit ordinals) to build a so-called $\aleph_1$-like model of $T$, i.e., a model $\mathcal{M}$ of power $\aleph_1$ such that $a^*$ is finite or countable for each $a \in \mathcal{M}$. It is evident that $\mathcal{M}$ is nonstandard. Moreover, $\mathcal{M}$ carries no proper rank initial embedding $j$ since any such embedding $j$ would have to have the property that $j[\mathcal{M}]$ is a submodel of some structure of the form $V_\alpha^\mathcal{M}$ for some “ordinal” $\alpha$ of $\mathcal{M}$, which is impossible, since $(V_\alpha^\mathcal{M})^*$ is countable thanks to the fact that $\mathcal{M}$ is $\aleph_1$-like.

**Theorem 4.7** Every consistent extension of $ZF + V = L$ has a nonstandard model of power $\aleph_1$ that carries no proper initial self-embedding.

**Proof** Let $T$ be a consistent extension of $ZF + V = L$, and $\mathcal{M}$ be a nonstandard $\aleph_1$-like model of $T$. Recall that, provably in $ZF + V = L$, there is a $\Sigma_1$-formula $\sigma(x, y)$ that describes the graph of a bijection $f$ between the class $V$ of sets and the class $\text{Ord}$ of ordinals; see, e.g., the proof of Lemma 13.19 of [3]. Suppose $j$ is an initial embedding of $\mathcal{M}$. We will show that $j$ is not a proper embedding by verifying that every element $m$ of $\mathcal{M}$ is in the image of $j$. Being an ordinal is a $\Delta_0$-property and thus preserved by $j$, so $j[\text{Ord}^\mathcal{M}] \subseteq \text{Ord}^\mathcal{M}$. Since $\mathcal{M}$ is $\aleph_1$-like, $j[\text{Ord}^\mathcal{M}]$ must be cofinal in $\text{Ord}^\mathcal{M}$. But, $j$ is initial, so $j[\text{Ord}^\mathcal{M}] = \text{Ord}^\mathcal{M}$. To show that $j[\mathcal{M}] = \mathcal{M}$, suppose $m \in \mathcal{M}$. Then there is a unique $\alpha \in \text{Ord}^\mathcal{M}$ such that $\mathcal{M} \models \sigma(m, \alpha)$. Since every ordinal of $\mathcal{M}$ is in $j[\text{Ord}^\mathcal{M}]$, there is some $\beta \in \text{Ord}^\mathcal{M}$ such that $j(\beta) = \alpha$. Let $m_0$ be the unique element of $\mathcal{M}$ such that $\mathcal{M} \models \sigma(m_0, \beta)$. Then since $j$ is an embedding, $j[\mathcal{M}] \models \sigma(j(m_0), j(\beta))$, and by the choice of $\beta$, $j[\mathcal{M}] \models \sigma(j(m_0), \alpha)$, which coupled with the fact that $\sigma$ is a $\Sigma_1$-formula, yields $\mathcal{M} \models \sigma(j(m_0), \alpha)$. So in light of the fact that $\sigma$ within $\mathcal{M}$ defines the graph of a bijection $f$ between $V$ and $\text{Ord}$, and $f(m) = \alpha$ (by the choice of $\alpha$), we can can conclude that $j(m_0) = m$, thereby showing that $j[\mathcal{M}] = \mathcal{M}$. □

5 Constructing initial self-embeddings

In the previous section we saw that if the well-founded part of a model $\mathcal{M}$ of KPI is $\epsilon$-unbounded (in the sense of Definition [3.1] in $\mathcal{M}$, then there is no proper initial self-embedding of $\mathcal{M}$. In this section we prove an adaption of H. Friedman’s Self-embedding Theorem [Fr1, Theorem 4.1] that ensures the existence of proper initial self-embeddings of models of extensions of KPI with contained well-founded parts.\(^3\)

We now turn to the investigation of conditions under which models of KPI with contained well-founded parts admit proper initial self-embeddings. We begin with the verification that $\Delta_1$-Separation ensures that $\Sigma_0$-types with parameters from the well-founded part that are realised are coded in the standard system; and that for $n > 0$, $\Sigma_n$-Separation is sufficient to ensure the corresponding condition for $\Sigma_n$-types.

**Lemma 5.1**\(^4\) Suppose $n \in \omega$ and $\mathcal{M}$ is a model of KPI + $\Sigma_n$-Separation such that the well-founded part of $\mathcal{M}$ is contained. If $\bar{a} \in \mathcal{M}$, then

$$\{ \langle \tau \phi(x, \bar{y}), b \rangle \mid \phi \in \Sigma_n, b \in \text{WF}(M) \text{ and } \mathcal{M} \models \phi(b, \bar{a}) \} \in \text{SSy}(\mathcal{M}).$$

\(^3\)It is known that KP + $\neg I + \Sigma_1$-separation is bi-interpretable with the fragment $\Sigma_1$ of PA (Peano arithmetic), where KP + $\neg I$ is KP plus the negation the axiom of infinity; the proof is implicit in the proof of the main result of Kaye and Wong [KW]. Moreover, the bi-interpretation at work makes it clear that the study of initial self-embeddings of models of KP + $\neg I + \Sigma_1$-Separation boils down to the study of initial self-embeddings of models of $\Sigma_1$. The interested reader can consult [BE] for a systematic study of initial self-embeddings of $\Sigma_1$.

\(^4\)We are grateful to Kameryn Williams for spotting an unnecessary assumption in an earlier version of this Lemma [5.1].
Proof Let \( C \in M \) be such that \( \text{WF}(M) \subseteq C^* \). Let \( a_1, \ldots, a_l \in M \). Work inside \( M \). Consider
\[
D = \{ (q, b) \in C \mid (q \in \omega) \land (a = \langle b, a_1, \ldots, a_l \rangle) \land \text{Sat}_{\Sigma_n}(q, a) \}.
\]
Thanks to Theorem 2.9, \( \Sigma_n \)-Separation (\( \Delta_1 \)-Separation when \( n = 0 \)) ensures that \( D \) is a set in \( M \). It is clear that \( D \) codes
\[
\{ \langle \ulcorner \phi(x, \vec{y}) \urcorner, b \rangle \mid \phi \text{ is } \Sigma_n, b \in \text{WF}(M) \text{ and } M \models \phi(b, \vec{a}) \}.
\]
\( \Box \)

As verified in the next lemma, in the special case when the model is not \( \omega \)-standard, in Lemma 5.1 the assumption that the well-founded part is contained can be dropped and the assumption that \( \Sigma_n \)-Separation holds can be replaced by a fragment of the collection scheme coupled with a fragment foundation scheme.

Lemma 5.2 Suppose that \( n \in \omega, m = \max\{1, n\} \), and \( M \) is an \( \omega \)-nonstandard model of \( \text{KPI} + \Pi_{m-1} \text{-Collection} + \Pi_{n+1} \text{-Foundation} \). If \( \vec{a} \in M \), then
\[
\{ \langle \ulcorner \phi(x, \vec{y}) \urcorner, b \rangle \mid \phi \text{ is } \Sigma_n, b \in \text{WF}(M) \text{ and } M \models \phi(b, \vec{a}) \} \in \text{SSy}(M).
\]

Proof By Lemma 3.9 the well-founded part of \( M \) is contained. Let \( C \in M \) be such that \( \text{WF}(M) \subseteq C^* \). Let \( a_1, \ldots, a_m \in M \). Let \( \psi_1(\alpha, f, C) \) be the formula:
\[
\left( \alpha \in \omega \land \text{dom}(f) = \alpha + 1 \land f(0) = \emptyset \land (\forall \beta \in \text{dom}(f))(\forall y \in C)(y \in f(\beta + 1) \iff y \subseteq f(\beta)) \right).
\]
Consider the formula \( \theta(\alpha, \omega, \vec{a}_1, \ldots, \vec{a}_m) \) defined by:
\[
\exists v(\exists f \in C) \left( \psi_1(\alpha, f, C) \land \psi_2(\alpha, f, C, v, a_1, \ldots, a_m) \right),
\]
where \( \psi_2(\alpha, f, C, v, a_1, \ldots, a_m) \) is:
\[
\left( \forall (q, b) \in C \left( \langle q, b \rangle \in v \land f(\alpha) \iff (q, b) \in f(\alpha) \land \text{Sat}_{\Sigma_n}(q, \langle b, a_1, \ldots, a_m \rangle) \right) \right).
\]
By \( \Pi_{m-1} \)-Collection, the formula \( \theta(\alpha, \omega, \vec{a}_1, \ldots, \vec{a}_m) \) is equivalent to a \( \Sigma_{n+1} \)-formula. Now, since \( \text{WF}(M) \) is isomorphic to \( V_\omega \), for all \( \alpha \in \text{o}(M) \),
\[
M \models \theta(\alpha, \omega, a_1, \ldots, a_m).
\]
Therefore, by \( \Pi_{n+1} \)-Foundation, there exists \( \gamma \in (\omega^M)^* \setminus \text{WF}(M) \) such that
\[
M \models \theta(\gamma, \omega, a_1, \ldots, a_m).
\]
Let \( v \in M \) be such that
\[
M \models (\exists f \in C) \left( \psi_1(\gamma, f, C) \land \psi_2(\gamma, f, C, v, a_1, \ldots, a_m) \right).
\]
Since \( \gamma \in (\omega^M)^* \) is nonstandard, it follows that \( v \) codes
\[
\{ \langle \ulcorner \phi(x, \vec{y}) \urcorner, b \rangle \mid \phi \text{ is } \Sigma_n, b \in \text{WF}(M) \text{ and } M \models \phi(b, \vec{a}) \} \in \text{SSy}(M).
\]
\( \Box \)
Lemma 5.1 allows us to prove the following theorem that gives a sufficient condition for nonstandard models of extensions of KPI to admit proper initial self-embeddings.

**Theorem 5.3** Let \( p \in \omega \), \( M \) be a countable model of KPI + \( \Sigma_{p+1} \)-Separation + \( \Pi_p \)-Collection, and let \( b, B \in M \) and \( c \in B^* \) with the following properties:

(I) \( M \models \bigcup B \subseteq B \).

(II) \( \text{WF}(M) \subseteq B^* \).

(III) for all \( \Pi_p \)-formulae \( \phi(\vec{x}, y, z) \) and for all \( a \in \text{WF}(M) \),

\[
\text{if } M \models \exists \vec{x}\phi(\vec{x}, a, b), \text{ then } M \models (\exists \vec{x} \in B)\phi(\vec{x}, a, c).
\]

Then there exists a proper initial self-embedding \( j : M \rightarrow M \) such that \( j[M] \subseteq c B^* \), \( j(b) = c \) and \( j[M] \prec_p M \).

**Proof** It follows from (II) that \( B \in M \) witnesses the fact that the well-founded part of \( M \) is contained. Let \( \langle d_i \mid i \in \omega \rangle \) be an enumeration of \( M \) such that \( d_0 = b \). Let \( \langle e_i \mid i \in \omega \rangle \) be an enumeration of \( B^* \) in which every element of \( B^* \) appears infinitely often. We will construct an initial embedding \( j : M \rightarrow M \) by constructing sequences \( \langle u_i \mid i \in \omega \rangle \) of elements of \( M \) and \( \langle v_i \mid i \in \omega \rangle \) of elements of \( B^* \) and defining \( j(u_i) = v_i \) for all \( i \in \omega \). After stage \( n \in \omega \), we will have chosen \( u_0, \ldots, u_n \in M \) and \( v_0, \ldots, v_n \in B^* \) and maintained

\[
(\dagger_n) \text{ for all } \Pi_p\text{-formulae, } \phi(\vec{x}, z, y_0, \ldots, y_n), \text{ and for all } a \in \text{WF}(M),
\]

\[
\text{if } M \models \exists \vec{x}\phi(\vec{x}, a, u_0, \ldots, u_n), \text{ then } M \models (\exists \vec{x} \in B)\phi(\vec{x}, a, v_0, \ldots, v_n).
\]

At stage 0, let \( u_0 = b \) and let \( v_0 = c \). By (III), this choice of \( u_0 \) and \( v_0 \) satisfy \((\dagger_0)\). Let \( n \in \omega \) with \( n \geq 1 \). Assume that we have chosen \( u_0, \ldots, u_{n-1} \in M \) and \( v_0, \ldots, v_{n-1} \in B^* \) and that \((\dagger_{n-1})\) holds.

**Case** \( n = 2k + 1 \) for \( k \in \omega \): This step will ensure that the embedding \( j : M \rightarrow M \) is initial. If

\[
M \models e_k \notin \text{TC}(\{v_0, \ldots, v_{n-1}\}),
\]

then let \( u_n = u_0 \) and \( v_n = v_0 \). This choice of \( u_n \) and \( v_n \) ensure that \( u_0, \ldots, u_n \in M \) and \( v_0, \ldots, v_n \in B^* \) satisfy \((\dagger_n)\). If

\[
M \models e_k \in \text{TC}(\{v_0, \ldots, v_{n-1}\}),
\]

then let \( v_n = e_k \) and we need to choose \( u_n \) to satisfy \((\dagger_n)\). By Lemma 5.1 \( \Sigma_{p+1}\)-separation, and \( \Pi_p \)-collection, there exists \( D \in M \) that codes the class

\[
\{\langle \pi(\vec{x}, z, y_0, \ldots, y_n)\rangle, a \mid a \in \text{WF}, \phi \text{ is } \Sigma_p \text{ and } M \models (\forall \vec{x} \in B)\phi(\vec{x}, a, v_0, \ldots, v_n)\} \in \text{SSy}(M).
\]

By Corollary 3.7, the well-founded part of \( M \) believes that ranks exist. For all \( \alpha \in \text{o}(M) \), let \( D_\alpha \in M \) be such that

\[
M \models D_\alpha = D \cap V_\alpha.
\]

Note that for all \( \alpha \in \text{o}(M) \), \( D_\alpha \in \text{WF}(M) \). We have that for all \( \alpha \in \text{o}(M) \),

\[
M \models \exists v((v \in \text{TC}(\{v_0, \ldots, v_{n-1}\})) \land (\forall (m, a) \in D_\alpha)(\forall \vec{x} \in B)\text{Sat}_{\Sigma_p}(m, \langle \vec{x}, a, v_0, \ldots, v_{n-1}, v \rangle)).
\]

(1)
Claim: For all $\alpha \in o(M)$,

$$
\mathcal{M} \models (\exists v \in TC(\{u_0, \ldots, u_{n-1}\}))(\forall \langle m, a \rangle \in D_\alpha)\forall \vec{x} \text{ Sat}_{\Sigma_p}(m, \langle \vec{x}, a, u_0, \ldots, u_{n-1}, v \rangle). \tag{2}
$$

To prove this claim, suppose not, and let $\alpha \in o(M)$ be such that

$$
\mathcal{M} \models (\forall v \in TC(\{u_0, \ldots, u_{n-1}\}))(\exists \langle m, a \rangle \in D_\alpha)\exists \vec{x} \text{ Sat}_{\Sigma_p}(m, \langle \vec{x}, a, u_0, \ldots, u_{n-1}, v \rangle).
$$

By $\Pi_p$-collection,

$$
\mathcal{M} \models \exists C(\forall v \in TC(\{u_0, \ldots, u_{n-1}\}))(\exists \langle m, a \rangle \in D_\alpha)(\exists \vec{x} \in C) \neg \text{ Sat}_{\Sigma_p}(m, \langle \vec{x}, a, u_0, \ldots, u_{n-1}, v \rangle).
$$

Now, $\Pi_p$-Collection implies that (3) is equivalent to a $\Sigma_{p+1}$-formula. Therefore, by (4),

$$
\mathcal{M} \models (\exists C \in B)(\forall v \in TC(\{v_0, \ldots, v_{n-1}\}))(\exists \langle m, a \rangle \in D_\alpha)(\exists \vec{x} \in C) \neg \text{ Sat}_{\Sigma_p}(m, \langle \vec{x}, a, v_0, \ldots, v_{n-1}, v \rangle).
$$

But then

$$
\mathcal{M} \models (\forall v \in TC(\{v_0, \ldots, v_{n-1}\}))(\exists \langle m, a \rangle \in D_\alpha)(\exists \vec{x} \in B) \neg \text{ Sat}_{\Sigma_p}(m, \langle \vec{x}, a, v_0, \ldots, v_{n-1}, v \rangle),
$$

which contradicts (1). This proves the claim.

Consider the formula $\theta(\alpha, D, B, u_0, \ldots, u_{n-1})$ defined by:

$$(\exists f \in B)(\psi_1(\alpha, f, B) \land \psi_2(\alpha, f, D, u_0, \ldots, u_{n-1})),
$$

where $\psi_1(\alpha, f, B)$ is:

$$
(\alpha \text{ is an ordinal}) \land \text{ dom}(f) = \alpha \land f(0) = \emptyset \land \left( (\forall \beta \in \text{ dom}(f)) \left( (\beta \text{ is a limit ordinal}) \Rightarrow f(\beta) = \bigcup_{\xi < \beta} f(\xi) \right) \land (\forall \beta \in \text{ dom}(f))(\forall y \in B)(y \subseteq f(\beta + 1) \iff y \subseteq f(\beta)) \right),
$$

and $\psi_2(\alpha, f, D, u_0, \ldots, u_{n-1})$ is:

$$(\exists v \in TC(\{u_0, \ldots, u_{n-1}\}))(\forall \langle m, a \rangle \in D \cap f(\alpha))\forall \vec{x} \text{ Sat}_{\Sigma_p}(m, \langle \vec{x}, a, u_0, \ldots, u_{n-1}, v \rangle).
$$

Now, for all $\alpha \in o(M)$,

$$
\mathcal{M} \models \theta(\alpha, D, B, u_0, \ldots, u_{n-1}).
$$

And $\Pi_p$-Collection implies that $\theta(\alpha, D, B, u_0, \ldots, u_{n-1})$ is equivalent to a $\Pi_{p+1}$-formula. Therefore, by $\Sigma_{p+1}$-Foundation, there exists $\gamma \in \text{ Ord}^M \setminus o(M)$ and $f \in M$ such that

$$
\mathcal{M} \models \psi_1(\gamma, f, B).
$$

and

$$
\mathcal{M} \models \psi_2(\gamma, f, D, u_0, \ldots, u_{n-1}).
$$

Let $v \in M$ be such that

$$
\mathcal{M} \models v \in TC(\{u_0, \ldots, u_{n-1}\}),
$$

and

$$
\mathcal{M} \models (\forall \langle m, a \rangle \in D \cap f(\gamma))\forall \vec{x} \text{ Sat}_{\Sigma_p}(m, \langle \vec{x}, a, u_0, \ldots, u_{n-1}, v \rangle).
$$

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Let $u_n = v$. This choice of $u_n$ ensures that $u_0, \ldots, u_n \in M$ and $v_0, \ldots, v_n \in B^*$ satisfy $(\dagger_n)$.

**Case** $n = 2k$ for $k \in \omega$: Let $u_n = d_k$. This choice will ensure that the domain of $j$ is all of $M$. By Lemma 5.1 there exists $A \in M$ that codes the class

$$\{\langle \phi(\bar{x}, z, y_0, \ldots, y_n), a \rangle \mid a \in \text{WF}, \phi \text{ is } \Pi_p \text{ and } M \models \exists \bar{x}\phi(\bar{x}, a, u_0, \ldots, u_n)\} \in \text{SSy}(M).$$

Now, by Corollary 3.2 the well-founded part of $M$ believes that ranks exist. For all $\alpha \in o(M)$, let $A_\alpha \in M$ be such that

$$M \models A_\alpha = V_\alpha \cap A.$$

Note that for all $\alpha \in o(M)$, $A_\alpha \in \text{WF}(M)$. We have that for all $\alpha \in o(M)$,

$$M \models (\forall \langle m, a \rangle \in A_\alpha) \exists \bar{x} \text{Sat}_{\Pi_p}(m, \langle \bar{x}, a, u_0, \ldots, u_{n-1}, v \rangle).$$

So, for all $\alpha \in o(M)$,

$$M \models \exists v(\forall \langle m, a \rangle \in A_\alpha) \exists \bar{x} \text{Sat}_{\Pi_p}(m, \langle \bar{x}, a, u_0, \ldots, u_{n-1}, v \rangle),$$

and, using $\Pi_p$-Collection, this formula is equivalent to a $\Sigma_{p+1}$-formula with parameters $A_\alpha \in \text{WF}(M)$ and $u_0, \ldots, u_{n-1}$. Therefore, by $(\dagger_{n-1})$ and (I), for all $\alpha \in o(M)$,

$$M \models (\exists v \in B)(\forall \langle m, a \rangle \in A_\alpha)(\exists \bar{x} \in B)\text{Sat}_{\Pi_p}(m, \langle \bar{x}, a, v_0, \ldots, v_{n-1}, v \rangle).$$

Consider the formula $\theta(\alpha, A, B, v_0, \ldots, v_{n-1})$ defined by

$$(\exists v, f \in B)(\psi_1(\alpha, f, B) \land \psi_2(\alpha, A, B, v_0, \ldots, v_{n-1}, v)),$$

where $\psi_1(\alpha, f, B)$ is as in the proof of Case $(2n = k + 1)$, and $\psi_2(\alpha, A, B, v_0, \ldots, v_{n-1}, v)$ is:

$$(\forall \langle m, a \rangle \in A \cap f(\alpha))(\exists \bar{x} \in B)\text{Sat}_{\Pi_p}(m, \langle \bar{x}, a, v_0, \ldots, v_{n-1}, v \rangle).$$

Note that $\theta(\alpha, A, B, v_0, \ldots, v_{n-1})$ is equivalent to a $\Pi_p$-formula and for all $\alpha \in o(M)$,

$$M \models \theta(\alpha, A, B, v_0, \ldots, v_{n-1}).$$

Therefore, by $\Sigma_p^*$-Foundation, there exists $\gamma \in \text{Ord}^M \setminus o(M)$ such that:

$$M \models \theta(\gamma, A, B, v_0, \ldots, v_{n-1}).$$

Let $f, v \in B^*$ be such that

$$M \models (\psi_1(\gamma, f, B^*) \land \psi_2(\gamma, f, A, B, v_0, \ldots, v_{n-1}, v)),$$

and let $v_n = v$. Therefore

$$M \models (\forall \langle m, a \rangle \in A \cap f(\gamma))(\exists \bar{x} \in B)\text{Sat}_{\Pi_p}(m, \langle \bar{x}, a, v_0, \ldots, v_n \rangle),$$

and this choice of $v_n$ ensures that $u_0, \ldots, u_n \in M$ and $v_0, \ldots, v_n \in B^*$ satisfy $(\dagger_n)$. This completes the case where $n = 2k$ and shows that we can construct sequences $\langle u_i \mid i \in \omega \rangle$ and $\langle v_i \mid i \in \omega \rangle$ while maintaining the conditions $(\dagger_n)$ at each stage of the construction. Now, define $j : M \rightarrow M$ by: for all $i \in \omega$, $j(u_i) = v_i$. Our “back-and-forth” construction ensures that $j$ is a proper initial self-embedding with $j[M] \subseteq B^*$, $j(b) = c$ and $j[M] \prec_p M$. □
In the proof of Theorem 5.3 the only use of $\Sigma_{p+1}$-Separation is to prove $\Sigma_{p+1}$- and $\Pi_{p+1}$-Foundation, and to satisfy the assumptions of Lemma 5.1. Therefore, in the special case where the model involved is $\omega$-nonstandard, we can replace Lemma 5.1 with Lemma 5.2 to obtain the following simplified variant of Theorem 5.3.

**Theorem 5.4** Let $p \in \omega$, $\mathcal{M}$ be a countable $\omega$-nonstandard model of KPI + $\Pi_p$-Collection + $\Pi_{p+2}$-Foundation, and let $b, B \in \mathcal{M}$ and $c \in B^*$ with the following properties:

(I) $\mathcal{M} \models \bigcup B \subseteq B$,

(II) $\text{WF}(B) \subseteq B^*$, and

(III) for all $\Pi_p$-formulae $\phi(x, z)$,

\[
\text{if } \mathcal{M} \models \exists x \phi(x, b), \text{ then } \mathcal{M} \models (\exists x \in B) \phi(x, c).
\]

Then there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $j[\mathcal{M}] \subseteq e B^*$, $j(b) = c$ and $j[\mathcal{M}] \prec_p \mathcal{M}$.

Equipped with Theorems 5.3 and 5.4 we are now able to demonstrate in Theorem 5.5 and Corollary 5.6 that a variety of nonstandard models of KPI are isomorphic to $\Sigma_n$-elementary transitive substructures of themselves.

**Theorem 5.5** Let $p \in \omega$, $\mathcal{M}$ be a countable nonstandard model of KPI + $\Sigma_{p+1}$-Separation + $\Pi_p$-Collection such that the well-founded part of $\mathcal{M}$ is contained, and let $b \in \mathcal{M}$. Then there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $b \in \text{rng}(j)$ and $j[\mathcal{M}] \prec_p \mathcal{M}$.

**Proof** Let $C \in \mathcal{M}$ be such that $\text{WF}(M) \subseteq C^*$. Work inside $\mathcal{M}$. Consider the formula $\theta(x, y, b)$ defined by

\[
x = (m, a) \land (m \in \omega) \land y = (y_1, \ldots, y_k) \land \text{Sat}_{\Pi_p}(m, (y_1, \ldots, y_k, a, b)).
\]

Note that if $p \geq 1$, then $\theta(x, y, b)$ is equivalent to a $\Pi_p$-formula, and if $p = 0$, then $\theta(x, y, b)$ is equivalent to a $\Sigma_1$-formula. By Lemma 2.4 Strong $\Pi_p$-Collection holds in $\mathcal{M}$. Therefore, there exists a set $D$ such that

\[
(\forall x \in C)(\exists y \theta(x, y) \Rightarrow (\exists y \in D) \theta(x, y)).
\]

Let $B = \text{TC}(D)$. Now, $B \in \mathcal{M}$ is transitive set in $\mathcal{M}$ with $\text{WF}(M) \subseteq B^*$, and for all $\Pi_p$-formulae $\phi(x, y, z)$ and for all $a \in \text{WF}(M)$,

\[
\text{if } \mathcal{M} \models \exists x \phi(x, a, b), \text{ then } \mathcal{M} \models (\exists x \in B) \phi(x, a, b).
\]

Therefore, by Theorem 5.3 there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $j[\mathcal{M}] \subseteq e B^*$, $j(b) = b$ and $j[\mathcal{M}] \prec_p \mathcal{M}$. □

**Corollary 5.6** Let $\mathcal{M}$ be a countable nonstandard model of $\text{ZF}^{-} + \text{WO}$ such that the well-founded part of $\mathcal{M}$ is contained. Then for all $p \in \omega$ and for all $b \in \mathcal{M}$, there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $b \in \text{rng}(j)$ and $j[\mathcal{M}] \prec_p \mathcal{M}$.

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Theorem 5.5 also yields the following results that provide two different sufficient conditions for models of KPI + $\Sigma_1$-Separation to admit proper initial self-embeddings.

**Corollary 5.7** Let $\mathcal{M}$ be a countable nonstandard model of KPI + $\Sigma_1$-Separation such that the well-founded part of $\mathcal{M}$ is contained and let $b \in M$. Then there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $b \in \text{rng}(j)$.

**Corollary 5.8** Let $\mathcal{M}$ be a countable $\omega$-nonstandard model of KPI + $\Sigma_1$-Separation and let $b \in M$. Then there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ with $b \in \text{rng}(j)$.

This allows us to give an example of a countable $\omega$-nonstandard model of MOST+$\Pi_1$-Collection that admits a proper initial self-embedding, but no proper $\mathcal{P}$-initial self-embedding.

**Example 5.9** Let $\mathcal{M}$ be a countable model of ZF + $V = L$ that is not $\omega$-standard. Let $\mathcal{N}$ be the substructure of $\mathcal{M}$ with underlying set

$$\mathcal{N} = \bigcup_{n \in \omega} (H^M_{\aleph_n})^*.$$ 

The fact that $\mathcal{N} \models \text{Mac}$ follows immediately from the fact that $\mathcal{N} \models \text{Powerset}$ and $\mathcal{N} \subseteq^\mathcal{P} \mathcal{M}$. Since $\mathcal{M}$ satisfies the Generalised Continuum Hypothesis, $\mathcal{N} \models \text{Axiom H}$. Therefore, by Lemma 2.14, $\mathcal{N} \models \text{MOST}$. It follows from Lemma 2.16 that $\mathcal{N} \models \text{MOST + } \Pi_1$-Collection. By Corollary 5.8, $\mathcal{N}$ admits a proper initial self-embedding. But this is impossible, because, since $j[\mathcal{N}] \subseteq^\mathcal{P} \mathcal{N}$, cardinals are preserved between $j[\mathcal{N}]$ and $\mathcal{N}$.

We are also able to find an example of a countable $\omega$-nonstandard model of MOST + $\Pi_1$-Collection that admits a proper $\mathcal{P}$-initial self-embedding, but no proper rank-initial self-embedding.

**Example 5.10** Let $\mathcal{N} = \langle N, e^M \rangle$ be the countable model of MOST + $\Pi_1$-Collection described in Example 5.9. Note that $\mathcal{N}$ satisfies the Generalised Continuum Hypothesis and the infinite cardinals of $\mathcal{N}$ are exactly $\aleph_n$ for each standard natural number $n$. In particular, for all $n \in \omega$,

$$\mathcal{N} \models (|\mathcal{P}^n(V_\omega)| = \aleph_n),$$

where $\mathcal{P}^n(V_\omega)$ is the powerset operation applied to $V_\omega$ $n$-times.

It follows that $\mathcal{N}$ satisfies

$$(\dagger) \text{ For all cardinals } \kappa, \text{ there exists a set } X \text{ with cardinality } \kappa \text{ and countable rank.}$$

Therefore, $\mathcal{N}$ shows that the theory MOST + $\Pi_1$-Collection + $(\dagger)$ is consistent. Now, let $\mathcal{K} = \langle K, e^K \rangle$ be a recursively saturated model of MOST + $\Pi_1$-Collection + $(\dagger)$. By Theorem 2.28, $\mathcal{K}$ has a proper $\mathcal{P}$-initial self embedding. Now, suppose $j : \mathcal{K} \rightarrow \mathcal{K}$ is a proper $\mathcal{P}$-initial self embedding. Since a bijection between an ordinal $\kappa$ and $\alpha \in \kappa$ is a subset of $\mathcal{P}(\kappa \times \kappa)$, for all $\kappa \in \text{rng}(j)$, $\kappa$ is a cardinal according to $\mathcal{K}$ if and only if $\kappa$ is a cardinal according to $j[\mathcal{K}]$. Similarly, if $R \in K$ and $\kappa \in \text{rng}(j)$ is a cardinal of $\mathcal{K}$ such that

$$\mathcal{K} \models (R \subseteq \kappa \times \kappa) \land (R \text{ is a well-founded extensional relation with a maximal element}),$$

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then $R \in \text{rng}(j)$ and

$$j[K] \models (R \text{ is a well-founded extensional relation with a maximal element})$$

Therefore, by Lemma 2.15, if $\kappa \in \text{rng}(j)$ is a cardinal, then $(\kappa^+)^K \in \text{rng}(j)$ and $H^j_{\kappa^+} = H^K_{\kappa^+}$. Now, since $j$ is proper, let $x \in K \setminus \text{rng}(j)$. Let $\kappa \in K$ be such that $K \models (|\text{TC}(|x|)| = \kappa)$. By the observations that we have just made, $\kappa \notin \text{rng}(j)$ and for all $y \in K$, if $K \models (|\text{TC}(|y|)| \geq \kappa)$, then $y \notin \text{rng}(j)$. Therefore, since $K \models (\dagger)$, there exists a set $y \in K$ with countable rank in $K$ such that $y \notin \text{rng}(j)$. This shows that $j$ is not a proper rank-initial self-embedding.

Theorem 5.5 combined with Lemma 3.10 also yields the following result that shows that every nonstandard model of $\text{KP} + \Sigma_1$-Separation admits a proper initial self-embedding.

**Corollary 5.11** Let $M$ be a countable nonstandard model of $\text{KP} + \Sigma_1$-Separation and let $b \in M$. Then there exists a proper initial self-embedding $j : M \rightarrow M$ with $b \in \text{rng}(j)$.

Note that the theory $\text{MOST} + \Pi_1$-Collection + $\Pi^P_1$-Foundation is obtained from $\text{KP} + \Sigma_1$-Separation by adding the Axiom of Choice. The following example shows that the assumptions of Corollary 5.11 cannot be weakened to saying that $M$ is a nonstandard model of $\text{MOST} + \Pi_1$-Collection.

**Example 5.12** Let $M$ be a countable model of $\text{ZF} + V = L$ that is $\omega$-standard but has a nonstandard ordinal that is countable according to $M$. Note that such a model can by obtained from the assumption that there exists a transitive model of $\text{ZF} + V = L$ using [KM Theorem 2.4] or, from the same assumption, using the Barwise Compactness Theorem as in the proof of [McK15, Theorem 4.5]. Let $N$ be the substructure of $M$ with underlying set

$$N = \bigcup_{\alpha \in \text{o}(M)} (H^M_{\aleph_\alpha})^*. $$

Using the same reasoning that was used in Example 5.9

$$N \models \text{MOST} + \Pi_1\text{-Collection}. $$

Moreover, $N$ is nonstandard. Since $M$ satisfies the Generalised Continuum Hypothesis, a straightforward induction argument inside $M$ shows that

$$M \models \forall \alpha (\alpha \text{ is an ordinal } \Rightarrow |V_{\omega + \alpha}| = \aleph_\alpha).$$

Therefore,

$$\text{WF}(N) = \bigcup_{\alpha \in \text{o}(N)} (V_{\omega + \alpha}^M)^* $$

and the well-founded part of $N$ is c-unbounded in $N$. So, by Theorem 4.4, $N$ admits no proper initial self-embedding.

We can also use Theorem 5.4 to prove the following variant of Theorem 5.5 for models of extensions of KPI that are not $\omega$-standard.
Theorem 5.13 Let $p \in \omega$, $\mathcal{M}$ be a countable $\omega$-nonstandard model of KPI + $\Pi_p$-Collection + $\Pi_{p+2}$-Foundation, and let $b \in M$. Then there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $b \in \text{rng}(j)$ and $j[\mathcal{M}] \prec_p \mathcal{M}$.

Proof Consider $\theta(C, n, b, \omega)$ defined by

$$(n \in \omega) \land (\forall m \in n)(\exists \bar{x} \text{ Sat}_{\Pi_p}(m, \langle \bar{x}, b \rangle) \Rightarrow (\exists \bar{x} \in C)\text{Sat}_{\Pi_p}(m, \langle \bar{x}, b \rangle)).$$

Note that $\Pi_p$-Collection implies that $\theta(C, n, b, \omega)$ is equivalent to a $\Pi_p^{+1}$-formula. Moreover, if $n \in \omega$, then there exists a finite set $C$ such that $\theta(C, n, b, \omega)$ holds. Therefore, for all $n \in \omega$,

$\mathcal{M} \models \exists C \ \theta(C, n, b, \omega).$

So, by $\Pi_p^{+2}$-Foundation, there exists a nonstandard $k \in (\omega^\mathcal{M})^*$ such that

$\mathcal{M} \models \exists C \ \theta(C, k, b, \omega).$

Let $C \in \mathcal{M}$ be such that

$\mathcal{M} \models \theta(C, k, b, \omega).$

And, working inside $\mathcal{M}$, let $B = \text{TC}(C \cup V_\omega)$. Therefore $\mathcal{M} \models \bigcup B \subseteq B$, $\text{WF}(M) \subseteq B^*$ and for all $\Pi_p$-formulae $\phi(\bar{x}, z)$,

if $\mathcal{M} \models \exists \bar{x} \phi(\bar{x}, b)$, then $\mathcal{M} \models (\exists \bar{x} \in B) \phi(\bar{x}, b)$.

Therefore, by Theorem 5.4, there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $j[\mathcal{M}] \subseteq e(\mathcal{H}_L^\mathcal{M})^*$, where $\kappa = (\aleph_1^L)^\mathcal{M}$.

Corollary 5.14 Let $\mathcal{M}$ be a countable $\omega$-nonstandard model of KPI + $\Pi_2$-Foundation, and let $b \in M$. Then there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $b \in \text{rng}(j)$.

Note that Corollaries 5.8 and 5.14 give two distinct extensions of KPI such that every countable $\omega$-nonstandard model of these extensions is isomorphic to a transitive proper initial segment of itself.

We will next use Theorem 5.4 together with Corollary 2.18 to verify the surprising result that every model of ZFC that is $\omega$-nonstandard is isomorphic to a transitive substructure of the hereditarily countable sets of its own $L$.

Theorem 5.15 Let $\mathcal{M}$ be a countable $\omega$-nonstandard model of ZF. Then there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $j[\mathcal{M}] \subseteq_e (\mathcal{H}_k^L)^*$, where $\kappa = (\aleph_1^L)^\mathcal{M}$.

Proof Let $\kappa = (\aleph_1^L)^\mathcal{M}$. Now, let $B = \mathcal{H}_L^\mathcal{M}$. It is clear that $\mathcal{M} \models \bigcup B \subseteq B$ and $\text{WF}(M) \subseteq B^*$. Note that $\emptyset \in B^* \cap M$. By Corollary 2.18 for all $\Delta_0$-formula $\phi(\bar{x}, z)$,

if $\mathcal{M} \models \exists \bar{x} \phi(\bar{x}, \emptyset)$, then $\mathcal{M} \models (\exists \bar{x} \in B) \phi(\bar{x}, \emptyset)$.

So, by Theorem 5.4, there exists a proper initial self-embedding $j : \mathcal{M} \rightarrow \mathcal{M}$ such that $j[\mathcal{M}] \subseteq_e B^*$. □

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Hamkins [Ham13] showed that if $\mathcal{M}$ is a countable model of ZF, then there exists an embedding of $\mathcal{M}$ into its own $L$. However, the embeddings produced in [Ham13] are not required to be initial embeddings. Theorem 5.15 shows that under the condition that $\mathcal{M}$ is a countable $\omega$-nonstandard model of ZF, there exists an embedding of $\mathcal{M}$ into its own $L$ that is also initial. Question 35 of [Ham13] asks whether every countable model of set theory can be embedded into its own $L$ by an embedding that preserves ordinals. Since initial embeddings preserve ordinals, Theorem 5.15 provides a positive answer to this question when $\mathcal{M}$ is a countable $\omega$-nonstandard model of ZF.

Theorem 5.15 immediately implies the corollary below that shows that every countable model of ZF that is not $\omega$-standard can be end-extended to a model of ZFC + $V = L$.

**Corollary 5.16** Let $\mathcal{M}$ be a countable $\omega$-nonstandard model of ZF. Then there exists structures $\mathcal{N}_1$ and $\mathcal{N}_2$ such that

1. $\mathcal{M} \subseteq_e \mathcal{N}_1 \subseteq_e \mathcal{N}_2$,
2. $\mathcal{N}_2 \models ZFC + V = L$, and
3. $\mathcal{N}_1 = \langle (H^N_{\aleph_1})^*, \in^\mathcal{N}_2 \rangle$.

Corollary 5.16 is a special case of [Bar71, Theorem 3.1], which shows that Corollary 5.16 holds for all countable models of ZF. Barwise used methods from infinitary logic; Hamkins has recently formulated a purely set-theoretic proof of the same result [Ham18].

We now turn to applying Theorem 5.3 to finding transitive partially elementary substructures of nonstandard models of ZF$^- + WO$. Despite the failure of reflection in ZF$^- + WO$ [FGK], Quinsey [Qui, Corollary 6.9] employed indicators and methods from infinitary logic to show the following:

**Theorem 5.17** (Quinsey) Let $n \in \omega$. If $\mathcal{M} \models ZF^-$, then there exists $\mathcal{N} \subseteq_e \mathcal{M}$ such that $\mathcal{N} \neq \mathcal{M}$, $\mathcal{N} \prec_n \mathcal{M}$ and $\mathcal{N} \models ZF^-$. Extensions of H. Friedman’s self-embedding result [Fri] proved by Gorbou [Gor] show that if the nonstandard model $\mathcal{M}$ in Theorem 5.17 is countable and satisfies ZF, then the conclusions of Theorem 5.17 can be strengthened to require that $\mathcal{N} \subseteq_p \mathcal{M}$ and $\mathcal{N} \cong \mathcal{M}$. In light of this, it natural to ask under what circumstances the conclusion of Theorem 5.17 can be strengthened to require that the $\Sigma_n$-elementary submodel be isomorphic to the original nonstandard model. Theorem 4.6 shows that such a strengthening of Theorem 5.17 does not hold in general, even when $n = 0$ and the model is countable. However, using Theorem 5.3 we can show that the countable nonstandard models of ZF$^- + WO + \forall \alpha \Pi^1_{\infty}DC_\alpha$ for which this strengthening of Quinsey’s result holds are exactly the models in which the well-founded part is $c$-bounded.

Our final result below shows that the $c$-boundedness of the well-founded part of a countable nonstandard model of ZF$^- + WO + \forall \alpha \Pi^1_{\infty}DC_\alpha$ is necessary and sufficient for $\mathcal{M}$ to admit a proper initial self-embedding.

**Theorem 5.18** Let $\mathcal{M}$ be a countable nonstandard model of ZF$^- + WO + \forall \alpha \Pi^1_{\infty}DC_\alpha$. Then the following are equivalent:

1. The well-founded part of $\mathcal{M}$ is $c$-bounded in $\mathcal{M}$,
2. $WF(\mathcal{M}) \models$ Powerset,
(III) For all $n \in \omega$ and for all $b \in M$, there exists a proper initial self-embedding $j : M \rightarrow M$ such that $b \in \text{rng}(j)$ and $j[M] \prec_n M$.

**Proof** (I)⇒(II) is Lemma 3.11. To see that (II)⇒(III), assume that (II) holds and note that Lemma 3.12 implies that the standard part of $M$ is contained. Therefore, by Corollary 5.6 (III) holds. Finally, (III)⇒(I) is the contrapositive of Theorem 4.4. □

6 Questions

**Question 6.1** Can Theorem 5.5 be strengthened by adding the requirement to the conclusion of the theorem that the self-embedding $j$ fixes every member of $b^*$?

- The above question is motivated by the “moreover” clause of Theorem 2.27.

**Question 6.2** Does every countable model of KPI that is not $\omega$-standard admit a proper initial self-embedding?

- The above question is prompted by Corollaries 5.7 and 5.14, which provide sufficient conditions for a countable model of KPI that is not $\omega$-standard to admit a proper initial self-embedding.

**Question 6.3** Is there a countable model of $\mathcal{M}$ of ZF− + WO such that the well-founded part of $\mathcal{M}$ is $c$-bounded and $\mathcal{M}$ does not admit any proper initial self-embedding?

- The key role played by the scheme $\forall\alpha \Pi^1_1-\text{DC}_\alpha$ in the proof of Theorem 5.18 suggests a positive answer to the above question.

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