RICH DYNAMICS IN SOME GENERALIZED DIFFERENCE EQUATIONS

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Abstract. There has been an increasing interest in the study of fractional discrete difference since Miller and Ross introduced the v-th fractional sum and the fractional integral was given as a fractional sum in 1989. It is known that fractional discrete difference equations hold discrete memory effects and can describe the long interaction of all the last states during evolution. Therefore the QR factorization algorithm described by Eckmann et al. in 1986 can not be directly applied to determine chaotic or nonchaotic behaviour in such a system, which becomes an interesting problem. Motivated by this, in this study, we propose a direct way to calculate the finite-time local largest Lyapunov exponent. Compared with those in the literature, we find that the test for determining the presence of chaos is reliable. Moreover, bifurcation diagrams which depends on the given fractional order parameter are given in Captuto like discrete Hénon maps and Logistic maps, which was not discussed in the literature. A transient behaviour in chaotic fractional Logistic maps is also discovered.

1. Introduction. Since 1970s [15, 22, 24], discrete dynamic systems and its applications have been paid a great deal of attention [11, 26, 27]. In 1989, Miller and Ross [25] introduced the v-th fractional sum and the fractional integral was given as a fractional sum. Since then, especially in recent ten years, discrete fractional models, as generalized discrete dynamic systems, has become a hot area of research. The theory of the fractional difference equations has been greatly developed, including basic theory [10], the initial value problems [4, 16], the discrete calculus of variations [5, 6], the Laplace transform [17], the properties of the Caputo and the Riemann-Liouville difference [2, 4, 16], and existence and uniqueness results [9, 14] etc.

At the same time, the chaotic aspects of fractional-order systems was also reported in Refs. [8, 7]. As to discrete-time fractional-order systems, a route to chaos by periodic doubling can be observed in [23, 30] and the Jacobian matrix algorithm

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[8] was applied to plot the distribution of the largest Lyapunov exponent in [30]. But it is known that fractional discrete difference equations hold discrete memory effects and can describe the long interaction of all the last states during evolution (actually, such a system is usually time-varying with a long-time memory), the QR factorization (Jacobian matrix) algorithm described by [13] can not be directly applied. Motivated by these, we write this paper.

The Wolf’s algorithm [29] can be used to estimate the largest Lyapunov exponent from time series data after performing a phase space reconstruction [1, 12, 28, 29]. However, it is not easy to choose the embedding dimension and the delay parameter inherent to the phase space reconstruction. Recently, D. Cafagna et al. [7] proposed a reliable binary test for detecting the presence of chaos in fractional-order nonlinear differential systems. The method does not require the phase space reconstruction in the underlying fractional system. It consists of obtaining the data series, constructing a random walk-type process and studying how the variance of the random walk scales with time. A long time computation ($10^4 - 10^7$ time evolution) is to be performed.

The method proposed in this paper can be applied to calculate the largest Lyapunov exponent based on the ratio of the distance on the different trajectories of the attractor at the $i$-th time step starting at two nearest neighboring points. This is to give the estimation of the temporal separation evolution of two nearby points on the attractor. This procedure is repeated by selecting the new neighboring points on the attractor and sufficiently large iteration number $N$ until the value of the exponent converges to a constant. It does not require phase space reconstruction of fractional systems. We find that a short time evolution is enough. As its application, we give a detailed study of rich dynamics in two fractional difference equations as follows:

(i) The logistic map of fractional order [30, 31] as

$$x(n) = x(0) + \frac{\mu}{\Gamma(v)} \sum_{j=1}^{n} \frac{\Gamma(n-j+v)}{\Gamma(n-j+1)} x(j-1)(1-x(j-1)), \quad (1)$$

which implies that there exist discrete memory effects and a long-time interaction of all the last states during its evolution when $v$ is not a positive integer. For $0 < v \leq 1$, Eq. (1) can be viewed as a solution of the following $v$-th Caputo like fractional [3] difference equation

$$C^a \Delta^v_x u(t) = \mu u(t + v - 1)(1 - u(t + v - 1)), \quad t \in \mathbb{N}_{a+1-v}, \quad u(a) = c, \quad 0 < v \leq 1,$$

where $C^a \Delta^v_x u(t)$ is the left Caputo-like delta difference, $\mathbb{N}_a$ denotes the isolated time scale and $\mathbb{N}_a = \{a, a + 1, a + 2, \ldots\}$ ($a \in \mathbb{R}$ fixed, here we choose $a = 0$). For the function $f(n)$, the delta difference operator $\Delta$ is defined as $\Delta f(n) = f(n+1) - f(n)$.

(ii) The Hénon map of fractional order [23] as

$$\begin{cases} x(n) & = x(0) + \frac{1}{\Gamma(v)} \sum_{j=1}^{n} \frac{\Gamma(n-j+v)}{\Gamma(n-j+1)} (1 - x(j-1)(1 + \mu_1 x(j-1)) + y(j-1)) \\
y(n) & = \mu_2 x(n-1), \quad x(0) = c_1, y(0) = c_2, \end{cases} \quad (2)$$
where \( \mu_i (i = 1, 2) \) are constants. For \( 0 < v \leq 1 \), Eq. (2) is a solution of the following \( v \)-th Caputo like fractional difference equation

\[
\begin{aligned}
C_{\Delta_v}^a x(t) &= 1 - x(t + v - 1)(1 + \mu_1 x(t + v - 1)) + y(t + v - 1),
\quad y(t) = \mu_2 x(t - 1),
\quad t \in \mathbb{N}_{a+1-v}, \quad 0 < v \leq 1,
\quad x(a) = c_1, \quad y(a) = c_2,
\end{aligned}
\]

which is a generalized form of classic Hénon map.

In this paper, we propose a direct way based on the computation of the largest Lyapunov exponent to determine chaotic or nonchaotic behaviour in some generalized (actually, fractional) difference equations. Compared with those discussed in [7], we find that the test for determining the presence of chaos is reliable. Moreover, bifurcation diagrams which depends on the given parameter (called by the fractional order) are given in Fractional Hénon maps and Logistic maps, which was not discussed in the literature. This is why we write this paper.

This paper is organized in what follows: In section 2, a direct way of computing the largest Lyapunov exponent is proposed. Numerical results about bifurcation diagrams versus the fractional order \( v \) and the largest Lyapunov exponent in system (1) or (2) are shown in Sec. 3, a transient behaviour is to be discussed for chaotic fractional Logistic maps. The conclusions are drawn in Sec. 4.

2. Chaos or not and its determination. There exist rich dynamics in chaotic systems and their neighbour trajectories which started even at the same initial conditions can diverge with an exponential rate [13]. This property is called sensitive dependence to the initial conditions. Another characteristic of chaotic systems is that their attractor has a fractal geometry and dimension [13]. Chaotic process is characterized by calculating the chaotic measures such as Lyapunov exponents and the so-called Kaplan-Yorke dimension (or Lyapunov dimension) [18].

Lyapunov exponents describe how a small change at the point \( x(0) \) propagates to the final point \( x(n) \), quantify the sensitivity to initial conditions of a dynamic system and predict that the considered system is chaotic or not. A system in the \( m \)-dimensional phase space has \( m \) Lyapunov exponents. It is well-known that positive Lyapunov exponents do not necessarily imply chaos, e.g. the differential equation \( x' = x \) has one positive LE but is not chaotic. Related to the so-called Perron effect of the LE sign reversals, we refer the reader to [21]. However a chaotic system has at least one positive Lyapunov exponent. It is very important to calculate the largest Lyapunov exponent (LLE) in order to detect the possibility of chaos [1, 28, 29]. Therefore, in this study, the LLE is estimated as an indicator of the chaotic/nonchaotic behaviour of fractional difference equations as follows:

\[
\lambda_{\text{max}} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \ln \frac{ED(i)}{ED(i-1)},
\]

i.e.,

\[
\lambda_{\text{max}} = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} (\ln(ED(i)) - \ln(ED(i-1))),
\]

where \( N \) is the iteration number and \( ED(i) \) is the Euclidean distance calculated between the two nearest neighboring points on the different trajectories of the attractor at the \( i \)-th time step. The ratio of these distances gives the estimation of the temporal separation evolution of two nearby points on the attractor. This procedure is repeated by selecting the new neighboring points on the attractor and
sufficiently large iteration number \( N \) until the value of the exponent converges to a constant. It is well-known that the value of the exponent can converge to a constant if the iteration number is sufficiently large, which is also its average value and this value is known as the true LLE value. By using the technique of considering two trajectories with nearby initial conditions, the trajectories diverge, on average, at an exponential rate characterized by the largest Lyapunov exponent.

3. Rich dynamics: Bifurcations and chaos. In this section we are to apply the method proposed in Sec. 2 to estimate the largest Lyapunov exponent (LLE). In the LLE analysis, the exponential evolution of the phase space trajectories can be found by calculating the differences approximately for sufficiently large \( N \) (\( \geq 8000 \)). By computing the average value of these differences, one can obtain the LLE. In the implementation of this algorithm, if the exponent estimate converges, it is generally not necessary to evolve for a long time. In this study, we find that the LLE can converge to a constant in fewer iteration than that provided by the method in Ref. [7]. Although the estimates of the LLE can reach almost a constant value when the iteration number \( N \) was about 4000, this calculation is performed for 8000 iterations to provide the convergence of the variation of estimation to the constant value which is actually the average value of the estimates in the long run.

In order to illustrate the effectiveness of the method proposed in this study, another approach is based on the reliable and efficient binary test for chaos, called 0−1 test, which has been recently proposed and applied to fractional-order systems in [7]. The idea underlying the test is to construct a random walk-type process from the discrete data (sampled at times \( n = 1, 2, \ldots, N \)) and then to examine how the variance of the random walk scales with time. Specifically, the method is based on the computation of the asymptotic growth rate \( K \): When \( K \) converges to 0, the motion is classified as regular (i.e. periodic or quasiperiodic) whereas if \( K \) converges to 1, the motion is classified as chaotic.

For example, in Fig. 1(a), the values of \( K \) obtained from (1) stay the neighborhood of 1 (in common sense, \( K > 0.5 \)) for \( v < 0.5 \), indicating that system (1) exhibits chaotic dynamics which agrees with the test consisting in computing the maximum Lyapunov exponent by Wolf algorithm proposed in Sec. 2. The computed value \( \lambda_{\text{max}} \) is positive for \( v < 0.5 \) in Fig. 1(b), confirming the chaotic behaviour of the considered fractional discrete system (1). As the fractional parameter \( v \) increases further (\( v > 0.6 \)) shown in Fig. 1(b), the values of \( K \) stay the neighborhood of 0 (in common sense, \( K < 0.5 \)), indicating that system (1) exhibits nonchaotic behaviour, which also agrees with the test consisting in computing the maximum Lyapunov exponent by Wolf algorithm in Sec. 2. The computed value \( \lambda_{\text{max}} \) becomes negative for \( v > 0.6 \) shown in Fig. 1(b), confirming the regular periodic motion of the considered fractional discrete system (1). Bifurcation diagrams which depends on the fractional order \( v \) are also given in Fig. 1(c), which implies that there is a route to chaos by period doubling bifurcations for fractional difference equations. Similar phenomenon can also be discovered in System (2) drawn in Fig. 2. Please note that a larger range of the parameter \( 0 < v < 2 \) than those in [23, 30] is discussed in this paper. It can be observed that there exists a stable fixed point in system (1) or (2) for some \( v > 1 \) in Figs. 1 and 2(c), which may be an interesting problem to be confirmed in an analytic way.

Moreover, one can find that when the motion of (1) or (2) is regular, the computed value \( K \) stays far away from zero because of less sufficiently long iterations. But
\( \lambda_{\text{max}} \) is negative, which illustrates the effectiveness of our method proposed in this paper.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Fractional Logistic maps (\( \mu = 2.18 \)): (a) \( K \) (Chaotic (=1) or nonchaotic behaviour (= 0)); (b) the largest Lyapunov exponent (LLE) (Chaotic (\( > 0 \)) or nonchaotic behaviour (\( \leq 0 \))); (c) Bifurcation diagrams.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Fractional Hénon maps (\( \mu_1 = 0.8, \mu_2 = 0.3 \)): (a) \( K \) (Chaotic (=1) or nonchaotic behaviour (= 0)); (b) the largest Lyapunov exponent (Chaotic (\( > 0 \)) or nonchaotic behaviour (\( \leq 0 \))); (c) Bifurcation diagrams.}
\end{figure}

Furthermore, we find that there can occur transient behaviour in fractional Logistic maps under different initial conditions for (a) \( x(0) = 0.6 \) and (b) \( x(0) = 0.1 \) \( (\nu = 0.01, \mu = 2.18) \) as is shown in Fig. 3(a) and (b) respectively. There exists a chaotic or regular (transient behaviour) motion which depends on its initial condition. This is called the transient behaviour in a chaotic system. A bifurcation diagram is drawn in Fig. 3(c) with last 50 iterations after hundreds of iterations under the initial condition \( x(0) = 0.1 \). There occurs a transient periodic-doubling reversal phenomenon [27].

In the end, we state some properties about systems (1) and (2):

**Proposition 1.** There exist two fixed points \( x^* = 0 \) and \( x^* = 1 \) in system (1).
Figure 3. Transient behaviour in fractional Logistic map with different initial conditions for (a) \( x(0) = 0.6 \), (b) \( x(0) = 0.1 \), and (c) \( x(0) = 0.1 \).

Proposition 2. If \((x^*, y^*)\) is a solution of the following equation

\[
\begin{cases}
1 - x(1 + \mu_1 x) + y = 0, \\
y = \mu_2 x,
\end{cases}
\]

Then \((x^*, y^*)\) is a fixed point of system (2).

The proofs can be finished by direct computation and they are omitted.

4. Conclusions. In this paper, we propose a simple and direct way to calculate the finite-time local largest Lyapunov exponent [19, 20] to determine chaotic or nonchaotic behaviour in some generalized difference equations. Compared with those discussed in [7], we find that the test for determining the presence of chaos is reliable. Moreover, bifurcation diagrams which depends on the given parameter (called by the fractional order [3, 4, 16, 25]) are given in Captuto like discrete Hénon maps [23] and Logistic maps [30, 31], which was not discussed in the literature.

One can find that in fractional map (1) or (2), \( x(n) \) depends on the past information \( x(0), x(1), x(2), \ldots, x(n-1) \) and this is called discrete memory effect in fractional discrete systems. The memory effects in discrete fractional maps imply that their present evolution state depends on all their past time states. Therefore the QR factorization algorithm developed by [13] can not be directly applied. In fact, related to fractional-order systems, there is no analytical algorithm to calculate Lyapunov exponents and no effective way to explore their rich dynamics but numerical simulation.

Fractional discrete-time dynamical systems has provided a great number of interesting topics and a further study including an analytic way is to be given in the future.

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