CONSTRUCTION OF SCHEMOIDS FROM POSETS

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Abstract. A schemoid is a generalization of association schemes from the point of view of small categories. In this article, we discuss schemoid structures for two kinds of small categories; the canonical small category defined by a poset, and another small category which arises a poset. We also discuss the schemoid algebra, that is an analogue of the Bose–Mesner algebra for an association scheme, for them.

1. Introduction

We call a pair of a finite set \( X \) and a partition \( S = \{ R_0, R_1, \ldots, R_n \} \) of \( X \times X \) an \( n \)-class association scheme if \( S \) satisfies the following:

1. \( R_0 = \{ (x, x) \mid x \in X \} \).
2. If \( R \in S \), then \( \{ (y, x) \mid (x, y) \in R \} \in S \).
3. For \( i, j, k \in \{ 0, 1, \ldots, n \} \), there exists \( p_{i,j}^k \) such that

\[
p_{i,j}^k = \# \{ ((x, y), (y, z)) \in R_i \times R_j \}
\]

for every \( (x, z) \in R_k \).

Association schemes are introduced by Bose and Shimamoto [3] in their study of design of experiments. Since we can regard association schemes as a generalization of combinatorial designs, groups, and so on (see also [1]), many authors study association schemes from the view point of algebraic combinatorics. Bose and Mesner introduced an algebra which arises an association scheme [2]. The algebra is called the Bose–Mesner algebra, and plays an important role for algebraic study for association schemes. In [5], Kuribayashi and Matsuo introduced an association schemoid and a quasi-schemoid, which are generalizations of an association scheme from the viewpoint of small categories. A quasi-schemoid, we call it a schemoid for short in this paper, is defined to be a pair of small category \( C \) and a partition \( S \) of the morphisms of \( C \) satisfying the following condition:

- \( \{ (f, g) \in \sigma \times \tau \mid f \circ g = h \} \) and \( \{ (f, g) \in \sigma \times \tau \mid f \circ g = k \} \) have the same cardinality for \( \sigma, \tau, \mu \in S \) and \( h, k \in \mu \).

Let \( (X, \{ R_0, R_1, \ldots, R_n \}) \) be an \( n \)-class association scheme. Consider the codiscrete groupoid \( C_X \) on \( X \), i.e., the small category such that Obj\( (C_X) = X \) and Hom\( (C_X) (x, y) = \{ (y, x) \} \). For \( R_i \in S \), we define \( \sigma_i \) to be the set of morphisms \( (x, y) \) in \( C_X \) such that \( (x, y) \in R_i \). Since the compositions of morphisms in \( C_X \) are defined by \( (x, y) \circ (y, z) = (x, z) \), the pair \( \langle C_X, \{ \sigma_0, \sigma_1, \ldots, \sigma_n \} \rangle \) is a schemoid. Hence we can identify an association scheme with a schemoid. Moreover we can also construct a schemoid from a coherent configuration in the same manner. A schemoid requires the condition which is analogue of Condition 3 in the definition of an association scheme. An association schemoid is a schemoid satisfying the other conditions in the definition of an association scheme. Kuribayashi and Matsuo also introduce a subalgebra of the category algebra which arises a schemoid. The algebra

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is an analogue of the Bose–Mesner algebra for an association scheme. Kuribayashi [4] and Kuribayashi–Momose [6] develop homotopy theory for schemoids.

The purpose of this article is to give examples of schemoids. In this paper, we discuss schemoid structures for two kinds of small categories: the canonical small category obtained from a poset, and another acyclic small category obtained from a poset. The organization of this article is the following: We define schemoids and schemoid algebras in Section 2. In Section 3, we discuss schemoid structures on two kinds of small categories which arises a poset.

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2. Definition

Here we recall the definition of small categories, schemoids and schemoid algebras.

First we recall small categories and functors. A small category \( C \) is a quintuple of the set \( \text{Obj}(C) \) of objects, the set \( \text{Mor}(C) \) of morphisms, maps \( s \): \( \text{Mor}(C) \to \text{Obj}(C) \) and \( t \): \( \text{Mor}(C) \to \text{Obj}(C) \), and the operation \( \circ \) of composition, satisfying the following properties: For each morphism \( f \in \text{Mor}(C) \), \( s(f) \) is called the source of \( f \), and \( t(f) \) is called the target of \( f \). A morphism \( f \) is called an endomorphism if \( s(f) = t(f) \). For \( x, y \in \text{Obj}(C) \), define \( \text{Hom}_{\text{C}}(x, y) = \{ f \in \text{Mor}(C) \mid s(f) = x, \ t(f) = y \} \). A sequence \( (f_i, f_{i-1}, \ldots, f_1) \) of morphisms is called a nerve if \( t(f_i) = s(f_{i+1}) \) for \( i = 1, \ldots, n - 1 \). The composition \( g \circ f \) is defined for each nerve \((g, f)\) of length 2. The composition \( g \circ f \) is in \( \text{Hom}_{\text{C}}(x, z) \) for \( g \in \text{Hom}_{\text{C}}(y, z) \) and \( f \in \text{Hom}_{\text{C}}(x, y) \).

Moreover the operation satisfies \( (h \circ g) \circ f = h \circ (g \circ f) \) for every nerve \((h, g, f)\) of length 3. For \( x \in \text{Obj}(C) \), a morphism from \( x \) to \( x \) is called an endomorphism on \( x \). For each \( x \in \text{Obj}(C) \), there uniquely exists an endomorphism \( \text{id}_x \) on \( x \) such that \( \text{id}_x \circ f = f \) for every \( f \) with \( t(f) = x \) and \( g \circ \text{id}_x = g \) for every \( g \) with \( s(g) = x \). The morphism \( \text{id}_x \) is called the identity on \( x \).

Let \( C \) and \( C' \) be small categories. We call a pair \( \varphi = (\varphi_{\text{Obj}}, \varphi_{\text{Mor}}) \) a functor from \( C \) to \( C' \) if the map \( \varphi_{\text{Obj}} \) from \( \text{Obj}(C) \) to \( \text{Obj}(C') \) and the map \( \varphi_{\text{Mor}} \) from \( \text{Mor}(C) \) to \( \text{Mor}(C') \) satisfy the following:

1. \( \varphi_{\text{Mor}}(\text{id}_x) = \text{id}_{\varphi_{\text{Obj}}(x)} \) for each \( x \in \text{Obj}(C) \).
2. \( \varphi_{\text{Mor}}(f \circ g) = \varphi_{\text{Mor}}(f) \circ \varphi_{\text{Mor}}(g) \) for all nerve \((f, g)\) of length 2 in \( C \).

Next we define schemoids. In this paper, we define a schemoid as the pair of a small category and a map \( \pi \) from the set \( \text{Mor}(C) \) of morphisms to a set \( I \). For a map \( \pi \), we obtain a partition \( \{ \pi^{-1}(\{ i \}) \mid i \in I \} \) of \( \text{Mor}(C) \). On the other hand, for a partition \( S \) of \( \text{Mor}(C) \), we obtain the canonical surjection from \( \text{Mor}(C) \) to \( S \). Via this translation, the following definition is equivalent to the original definition of a schemoid.

**Definition 2.1.** Let \( C \) be a small category, \( I \) a set, and \( \pi \) a map from the set \( \text{Mor}(C) \) of morphisms in \( C \) to the set \( I \). For \( i, j \in I \) and \( h, k \in \text{Mor}(C) \), we define \( N^{i,j}_h \) to be

\[
\left\{ (f, g) \in \text{Mor}(C) \times \text{Mor}(C) \mid \begin{array}{l}
\pi(f) = i, \\
\pi(g) = j,
\end{array} f \circ g = h. \right\}
\]

We call the triple \((C, I, \pi)\) a schemoid if

\[\pi(h) = \pi(k) \iff N^{i,j}_h \text{ and } N^{i,j}_k \text{ have the same cardinality}\]

for each \( i, j \in I \) and \( h, k \in \text{Mor}(C) \).
For a morphism \( f \) of a small category \( C \), we write \( C_f \) to denote the minimum subcategory of \( C \) such that \( \text{Mor}(C_f) \) contains
\[
\{ g \in \text{Mor}(C) \mid f_1 \circ g \circ f_2 = f \text{ for some } f_1, f_2 \in \text{Mor}(C) \}.
\]
Then we can show the following lemma.

Lemma 2.2. Let \( C \) be a small category which does not contain any endomorphism except identities. Let \( \pi \) be a map from the set \( \text{Mor}(C) \) of morphisms to a set \( I \).

If the following condition holds, then \( (C, I, \pi) \) is a schemoid: For all morphisms \( f \) and \( g \) such that \( \pi(f) = \pi(g) \), there exists a functor \( \varphi_{f,g} \) from \( C_f \) to \( C_g \) such that
\[
\begin{align*}
(1) \quad & \varphi_{f,g}^{\text{Mor}} \text{ is a bijection;} \\
(2) \quad & \pi(f') = \pi(\varphi_{f,g}^{\text{Mor}}(f')) \text{ for each morphism } f' \text{ in } C_f; \text{ and} \\
(3) \quad & \varphi_{f,g}^{\text{Mor}}(f) = g.
\end{align*}
\]

Proof. Let \( h, k \in \text{Mor}(C) \) satisfy \( \pi(h) = \pi(k) \). The map \( \varphi_{h,k}^{\text{Mor}} \) induces a bijection from \( N^{\text{h,j}} \) to \( N^{\text{k,j}} \) for each \( i, j \in I \). Hence the triple \( (C, I, \pi) \) is a schemoid. \( \square \)

For a small category \( C \) and a field \( \mathbb{K} \), define \( \mathbb{K}[C] \) to be the \( \mathbb{K} \)-vector space whose basis is \( \text{Mor}(C) \). We define the product by
\[
g \cdot f = \begin{cases} 
  g \circ f & \text{if } s(g) = t(f) \\
  0 & \text{if } s(g) \neq t(f)
\end{cases}
\]
for \( g, f \in \text{Mor}(C) \). Moreover, for \( \sum_{f \in \text{Mor}(C)} \alpha_f f \) and \( \sum_{g \in \text{Mor}(C)} \beta_g g \in \mathbb{K}[C] \), we define the product of them by
\[
(\sum_{f \in \text{Mor}(C)} \alpha_f f) \cdot (\sum_{g \in \text{Mor}(C)} \beta_g g) = \sum_{f \in \text{Mor}(C)} \sum_{g \in \text{Mor}(C)} (\alpha_f \beta_g) f \cdot g.
\]
If \( \text{Obj}(C) \) is a finite set, then \( \mathbb{K}[C] \) is a \( \mathbb{K} \)-algebra with the unit \( \sum_{x \in \text{Obj}(C)} \text{id}_x \). Let \( \lambda \) be a map from \( \text{Mor}(C) \) to a set \( I \). Assume that \( \pi^{-1}(\{ i \}) \) is finite for every \( i \in I \). For \( i \in I \), we define \( \mathbb{T} \) to be \( \sum_{f \in \text{Mor}(C)} \text{id}_x \) for \( \mathbb{K}[C] \). We define \( \mathbb{K}(C, I, \pi) \) to be the vector subspace of \( \mathbb{K}[C] \) spanned by \( \{ \mathbb{T} \mid i \in I \} \). For a schemoid \( (C, I, \pi) \) such that \( \pi^{-1}(\{ i \}) \) is finite for every \( i \in I \), \( \mathbb{K}(C, I, \pi) \) is a subalgebra of \( \mathbb{K}[C] \). (The subalgebra \( \mathbb{K}(C, I, \pi) \) may not have the unit.) We call \( \mathbb{K}(C, I, \pi) \) a schemoid algebra.

3. Schemoids constructed from posets

Here we consider two kinds of small categories defined from a poset. The prototypical example of them is a schemoid structure for the \( n \)-th Boolean lattice \( 2^n \), i.e., the poset consisting of all subsets of \( \{ 1, \ldots, n \} \) ordered by inclusion. For \( X, Y \in 2^n \), we can consider the set difference \( X \setminus Y \). In 3.1, we discuss a poset with the operation which is analogue of the operation of set difference. The operation induces a schemoid structure for the canonical small category obtained from the poset. On the other hand, for \( X, Y \in 2^n \) with \( X \cap Y = \emptyset \), a greater element \( X \cup Y \) than \( X \) is obtained from \( X \) by adding \( Y \). By an analogue of the operation, we introduce an acyclic small category obtained from a poset with some conditions in 3.2. The category has also a schemoid structure.

3.1. Posets as a small category. Let \( P \) be a poset with respect to \( \leq \). For \( x, y \in P \), we define the interval \([x, y]\) from \( x \) to \( y \) by \([x, y] = \{ z \mid x \leq z \leq y \}\). We can naturally regard the poset \( P \) as the following small category \( C_P \): the set \( \text{Obj}(C_P) \) of objects is \( P \) and the set \( \text{Mor}(C_P) \) of morphisms is the relation \( \geq \), i.e., \( \{ (y, x) \mid x \leq y \} \subset P \times P \). For \( x \leq y \in P \), \( \text{Hom}_{C_P}(x, y) \) consists of \( (y, x) \). For \( (y, x) \in \text{Hom}_{C_P}(x, y) \) and \( (z, y) \in \text{Hom}_{C_P}(y, z) \), it follows by definition that \( x \leq z \).
We define the composition \((z, y) \circ (y, x)\) by \((z, y) \circ (y, x) = (z, x)\). For \(x \in P\), \(\text{id}_x\) is \((x, x)\).

Here we consider a poset \(P\) with a difference operation \(\delta\) defined as follows:

**Definition 3.1.** Let \(o\) be an element in the poset \(P\), and \(\delta\) a map from the set \(\{ (y, x) \in P^2 \mid x \leq y \}\) to \(P\). We say that \(\delta\) is a difference operation with the base point \(o\) if there exists a family

\[
\{ \varphi_{x,y} : [x,y] \to [o, \delta(y,x)] \mid x \leq y \}
\]

of maps satisfying the following:

1. Each \(\varphi_{x,y}\) is a bijection from the interval \([x,y]\) to the interval \([o, \delta(y,x)]\).
2. \(\varphi_{o,\delta(y,z)} = \delta(x,z)\) for \(x \leq z \leq y\).

Let \(\delta\) be a difference operation of poset \(P\). In this case, we have bijections \(\varphi_{x,y}\). If we fix an interval \([x, y]\), then we can translate each element in the interval \([x, y]\) into some interval from the base point \(o\) via the bijection \(\varphi_{x,y}\). Fix \(x \in P\) and consider two intervals \([x, y]\) and \([x, y']\). For \(z \in [x, y] \cap [x, y']\), it follows by Condition 2 that \(\delta(o, \varphi_{x,y}(z)) = \delta(o, \varphi_{x,y'}(z))\). In this sense, Condition 2 implies that the translation depends not on the interval but only on the minimum of the interval.

Since the difference operation induces functors \(\varphi_{f,g}\) from \((C_P)_{f}\) to \((C_P)_{g}\), Theorem 3.2 follows from Lemma 2.2.

**Theorem 3.2.** For a poset \(P\) with the difference operation \(\delta\), the triple \((C_P, P, \delta)\) is a schemoid.

**Example 3.3.** Let \(P\) be the \(n\)-th Boolean lattice, i.e., \(2^{[n]}\) ordered by inclusion. For \(x \leq y \in P\), we define \(\delta(y,x)\) to be \(y \setminus x\). The map \(\delta\) is a difference operation with the base point \(\emptyset\). Hence \((C_P, P, \delta)\) is a schemoid. In this case, the schemoid algebra \(\mathbb{K}(C_P, P, \delta)\) is isomorphic to \(\mathbb{K}[x_i \mid i \in P]/(x_i^2 \mid i \in P)\).

**Example 3.4.** Let \(P\) be a Coxeter groups ordered by the Bruhat order. For \(x \leq y \in P\), we define \(\delta(y,x)\) to be \(gx^{-1}\). The map \(\delta\) is a difference operation with the base point \(\emptyset\). Hence \((C_P, P, \delta)\) is a schemoid. In this case, the schemoid algebra \(\mathbb{K}(C_P, P, \delta)\) is isomorphic to the NilCoxeter algebra.

**Example 3.5.** Let \(\Delta\) be a simplicial complex on the vertex set \(V\). Consider the lattice \(P\) of faces of the simplicial complex \(\Delta\). (We regard \(\emptyset\) as a face of \(\Delta\).) For \(x, y \in P\), we define \(\delta(y,x)\) to be \(y \setminus x\). The map \(\delta\) is a difference operation with the base point \(\emptyset\). Hence \((C_P, P, \delta)\) is a schemoid. Let \(I_{\Delta}\) be an ideal of \(\mathbb{K}[x_i \mid i \in V]\) generated by \(\{ x_{v_1} \cdots x_{v_l} \mid \{ v_1, \ldots, v_l \} \notin \Delta \}\). The quotient ring \(\mathbb{K}[x_i \mid i \in V]/I_{\Delta}\) is called the Stanley–Reisner ring. Let \(I_{\Delta} = I_{\Delta} + (x_i^2 \mid i \in V)\). The schemoid algebra \(\mathbb{K}(C_P, P, \delta)\) is isomorphic to \(\mathbb{K}[x_i \mid i \in V]/I_{\Delta}\).

**Remark 3.6.** In Appendix of [6], Kuribayashi and Momose discuss schemoids in Example 3.5 from the point of view of the category theory.

### 3.2. Yet another small category obtained from posets

Here we introduce another kind of small categories obtained from a poset. We also introduce a schemoid structure for it.

Let \(P\) be a poset with respect to \(\leq\). Assume that the number of minimal elements in \(\{ z \in P \mid x \leq z, \, y \leq z \}\) is 1 or 0 for each pair \(x, y \in P\). We write \(x \lor y\) to denote the minimum element in \(\{ z \in P \mid x \leq z, \, y \leq z \}\) if \(\{ z \in P \mid x \leq z, \, y \leq z \}\) is 0.

Assume the following conditions:

1. \(P\) has the minimum element \(\emptyset\).
2. \(P\) is a ranked poset with the rank function \(\rho\).
3. \(\rho(x \lor y) \leq \rho(x) + \rho(y)\) for \(x, y \in P\).
We define a small category $\tilde{P}$ whose set of objects is $P$. For $x, y, d \in P$ such that $\rho(y) = \rho(x) + \rho(d)$ and $y = x \lor d$, we define a morphism $f_{x,y}^d$ from $x$ to $y$. Or equivalently,

$$\text{Hom}_{\tilde{P}}(x, y) = \left\{ f_{x,y}^d \mid \begin{array}{l} \text{ } \mid d \in P; \text{ } \mid y = x \lor d. \text{ } \mid \rho(y) = \rho(x) + \rho(d). \end{array} \right\}$$

for $x, y \in P$. If $f_{x,y}^d$ and $f_{y,z}^c$ are in $\text{Mor}(\tilde{P})$, then $\rho(y) = \rho(x) + \rho(c) = \rho(z) = \rho(x) + \rho(d)$. Hence $\rho(z) = \rho(x) + \rho(c) + \rho(d)$. Since $z = x \lor (c \lor d)$, we have $\rho(x) + \rho(c) + \rho(d) = \rho(x) \leq \rho(x) + \rho(c \lor d)$. On the other hand, $\rho(x) + \rho(c \lor d) \leq \rho(x) + \rho(c) + \rho(d)$ since $\rho(c \lor d) \leq \rho(c) + \rho(d)$. Hence $\rho(x) = \rho(x) + \rho(c \lor d) = \rho(x) + \rho(c) + \rho(d)$. Since $f_{x,z}^d$ is in $\text{Mor}(\tilde{P})$, we define the composition $f_{x,z}^d \circ f_{z,y}^c$ to be $f_{x,y}^{d+c}$.

**Example 3.7.** Let $P = \{ 0, 1, 2 \} \times \{ 0, 1 \}$. For $(x, y), (x', y') \in P$, $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. In this case, the set of morphisms of $\tilde{P}$ consists of the following:

$$f_{(0,0),(1,0)}, f_{(0,1),(1,0)}, f_{(0,1),(2,0)}, f_{(1,0),(2,0)}, f_{(1,0),(1,1)}, f_{(2,0),(1,1)}, f_{(2),(0,1)}, f_{(2),(1,1)}, f_{(1),(1)}, f_{(0,0),(1,0)}, f_{(0,0),(2,1)}, f_{(1),(2,1)}, f_{(2),(2,1)}$$

and identities. See also Figure 1.

**Theorem 3.8.** For the map $\alpha$ from $\text{Mor}(\tilde{P})$ to $P$ defined by $\alpha(f_{x,y}^d) = d$, the triple $(\tilde{P}, P, \alpha)$ is a schemoid.

**Proof.** Let $f_{x,y}^d$ and $f_{x',y'}^d$ be in $\text{Mor}(\tilde{P})$. In this case, it follows from the definition of morphisms in $\tilde{P}$ that

$$\rho(x \lor d) = \rho(x) + \rho(d),$$
$$\rho(x' \lor d) = \rho(x') + \rho(d).$$

Let $f_{x',y'}^d = f_{x,y}^d \circ f_{x',y}^d \circ f_{x,d}^d$. In this case, we have $d = d_1 \lor d_2$ and $\rho(d) = \rho(d_1) + \rho(d_2)$. If $\rho(x' \lor d_1) < \rho(x') + \rho(d_1)$ or $\rho(x' \lor d_1 \lor d_2) < \rho(x' \lor d_1) + \rho(d_2)$, then we have

$$\rho(x' \lor d_1 \lor d_2) < \rho(x') + \rho(d_1) + \rho(d_2) = \rho(x') + \rho(d_1 \lor d_2),$$

Figure 1. The small category in Example 3.7.
which contradicts \( \rho(x' \vee d) = \rho(x') + \rho(d) \). Hence morphisms \( f^{d_2}_{x',x'\vee d_1} \) and \( f^{d_1}_{x',x'} \) satisfies \( f^{d_1}_{x',x'\vee d_1} (f^{d_1}_{x',x'}(x')) = f^{d_1}_{x',x'} \circ f^{d_1}_{x',x'\vee d_1} \). Therefore there exists a bijection between \( N^{d_1,d_2}_{f_{x',x'}} \) and \( N^{d_1,d_2}_{f_{x',x'}} \). Hence the triple \((\hat{P},P,\pi)\) is a schemoid. □

Now we discuss the schemoid algebra. Consider the polynomial ring \( \mathbb{K}[X_x \mid x \in P] \) in variables corresponding to elements in \( P \). Define \( G_i \) by

\[
G_0 = \{ X_0 = 1 \} \\
G_1 = \{ X_x X_y \mid \rho(x \vee y) < \rho(x) + \rho(y) \} \\
G'_1 = \{ X_x X_y \mid \{ z \mid z \geq x, z \geq y \} = \emptyset \} \\
G_2 = \{ X_x X_y - X_{x \vee y} \mid \rho(x \vee y) = \rho(x) + \rho(y) \}. 
\]

Let \( I \) be the ideal generated by \( G_0 \cup G_1 \cup G'_1 \cup G_2 \), and \( R_P \) the quotient ring \( \mathbb{K}[X_x \mid x \in P]/I \). The ring \( R_P \) is the same as the ring defined in the following manner: \( R_P \) is the \( \mathbb{K} \)-vector space whose basis is \( \{ X_x \mid x \in P \} \)

\[
\begin{cases} 
X_x X_y - X_{x \vee y} & \text{(if there exists } x \vee y \text{ and } \rho(x \vee y) = \rho(x) + \rho(y)), \\
X_x X_y = 0 & \text{(otherwise)}.
\end{cases}
\]

**Theorem 3.9.** For the map \( \pi \) from \( \text{Mor}(\hat{P}) \) to \( P \) defined by \( \pi(f^{d}_x) = d \), the schemoid algebra for the schemoid \((\hat{P},P,\pi)\) is isomorphic to \( R_P \).

**Example 3.10.** Let \( P \) be the \( n \)-th Boolean lattice \( 2^{[n]} \). In this case, the set of morphisms is

\[
\{ f^{d}_{x',x'\vee d} \mid x, d \subset [n], x \cap d = \emptyset \}.
\]

Hence \((\hat{P},P,\pi)\) is \((G_P,P,\pi)\) in Example 3.4.

**Example 3.11.** Let \( K \) be a finite field. Consider the poset \( P \) of all subspaces in \( K^n \) ordered by the inclusion. In this case, the set of morphisms is

\[
\{ f^W_{V,W \cap V} \mid V, W \in P, V \cap W = 0 \}.
\]

**Example 3.12.** Let \( P \) be the poset of flats of a matroid \( M \) ordered by inclusion. Assume that \( P \) satisfies the conditions in this section. In this case, the schemoid algebra for \((\hat{P},P,\pi)\) is isomorphic to the algebra defined in Maeno–Numata [7], which is Möbius algebra with the relations \( x_i^2 = 0 \) for all variables \( x_i \).

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