The arithmetic geometry of $\text{AdS}_2$
and its continuum limit

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Abstract

We present and study in detail the construction of a discrete and finite arithmetic geometry $\text{AdS}_2[N]$. We show that an appropriate scaling limit exists, as $N \to \infty$, that can be identified with the universal $\text{AdS}_2$ radial and time near horizon geometry of extremal black holes.

The $\text{AdS}_2[N]$ geometry has been proposed as a toy model for describing the nonlocal and chaotic dynamics of the horizon microscopic degrees of freedom, that carry the finite black hole entropy. In particular, it supports exact quantum mechanical bulk-boundary holography for single particle wave packet probes, that possess an $N$–dimensional Hilbert space of states.

This construction amounts, in fact, to a compression of the information about the continuous $\text{AdS}_2$ geometry and it provides an example of a framework for the study of quantum complexity of spacetime geometries.

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1 Introduction

At Planck scale energies, quantum mechanics and gravity, imply that the notion of spacetime itself becomes ill-defined, through the appearance of black holes, with horizons (and singularities) of Planck length size.

Indeed, the above remarks imply that probably one has to abandon continuity of spacetime, locality of interactions and regularity of dynamics. Indeed there are arguments that quantization of gravity implies discretization of space time [1,2].

On the other hand, at this point, we would like to ponder about an old question, that is the “continuum” vs. “discretum” nature of spacetime at Planck scale distances. It is argued that, for resolving power of our experiments, much smaller than that of the Planck scale, the continuum description of all physical phenomena on spacetime is perfectly adequate.

As the resolving power increases about the Planck scale, scattering experiments of any sort of particle–like objects will produce black holes and the strength of the gravitational interaction will be of O(1), which leads to a breakdown of the usual continuum spacetime description [3,4].

In string theory the underlying hypothesis is that spacetime geometry and, thus, gravity, is an emergent phenomenon. This has been shown successfully only for distances much larger than the Planck scale distances. However perturbative string theory breaks down at Planck distances [5,8].

Various efforts, using duality symmetries, between string compactifications at weak and strong coupling, have produced very interesting results in the direction of the unification of all superstring
theories to M-theory \cite{9} with the M(atrix) model approach \cite{10,11}, for the moment, being the only candidate to represent M-theory.

In the framework of the AdS/CFT correspondence, where unitarity is manifest on both sides, strong coupling–closed–string theory, in the bulk, is relevant for Planck scale curvature of the AdS spacetime(s). In the same limit, the holographically dual conformal field theory (CFT) on the boundary, becomes a free field theory and the problem then boils down to determining infinitely "complicated" operators, in order to represent "local events" in the bulk, as well as the local diffeomorphisms within the bulk, in the language of the boundary theory (the so-called "problem of locality" in the AdS/CFT correspondence—for instance in \cite{12}).

Although the Planck scale energy frontier is very far from any imaginable experimentally accessible scale, there do exist instances in Nature, where the understanding of Planck scale physics becomes imperative. The most important one is the Planck scale spacetime regions at the creation of the Universe (the vicinity of the Big Bang singularity). Although there are interesting theoretical efforts in string theory cosmology \cite{13,14}, they avoid, in essence, addressing the question of the structure of spacetime at sub-Planck scales, by assuming that spacetime doesn’t exist as a fundamental notion at all.

The second instance, where Planck scale physics plays a vital role, is in understanding the near horizon geometry of black holes, as well as the question of quantum mechanical resolution of the (classical) black hole singularity.

In this paper we shall focus on the study of the near horizon geometry of black holes, assuming some general principles, which have emerged during the last forty years.

These principles are:

- Black holes behave like thermodynamic systems, in particular they can be assigned a finite entropy, proportional to the horizon area, which is a universal law for all black holes (the Bekenstein–Christodoulou–Hawking entropy) \cite{15–19}.

- All the information about the “interior” and “exterior” phenomena, with respect to the black hole horizon, is carried by degrees of freedom, that live at a Planck scale “distance” from the horizon (this region is known as the “stretched horizon”). This statement has become known as the “holographic principle” \cite{20–23}.

- The holographic principle implies that there is a finite number of bits of information that can be assigned to any Planck scale area \cite{24,25}.

- The interactions of the horizon degrees of freedom cannot be described by any local field theory, they are strongly chaotic and exhibit maximal mixing \cite{26,28}.

These general principles have motivated very active research, in order to construct explicit models, that incorporate them, while remaining consistent with the general principles of quantum mechanics, in particular unitarity, thus resolving the old “information paradox”, as it was stated by Hawking \cite{29,30}.
The most intense activity, in attempting to resolve the paradox, has taken place in the framework of the AdS/CFT correspondence. It consists of constructing a dictionary in order to describe the calculation of physical observables, either in the language of conformal field theory, which lives on the flat boundary of the AdS spacetime, or in the language of gravity, which lives in the bulk \[31,32\].

The success of this program has been immense and the hope is that it will be, indeed, the appropriate framework for understanding black hole physics, in full generality \[33,34\].

A less intense, but more adventurous, activity is to construct, directly, in the bulk and, in particular, in the vicinity of the black hole horizon, non-local models, that do capture the chaotic and mixing dynamics of the degrees of freedom that reside within Planck scale distance of the horizon as well as deriving general, rigorous, constraints from quantum information theory and entanglement.

In our framework space-time at Planck scales is intrinsically discrete, random (following \[1\]) and is not emergent from anything else; on the contrary it is from these properties that, at large distances, continuous spacetime geometry can be described as an infrared limit thereof.

This assumption amounts to using appropriate mathematical tools, that can describe the properties and dynamics of discrete geometries as well as the emergence of continuous geometries as their infrared limits.

Of paramount importance for the definition of a geometry is its isometry group. This is true, locally, for any known spacetime geometry and, globally, for the simplest ones (prominent examples are provided by the spacetimes of constant curvature such as de Sitter and anti-de Sitter). For discrete geometries, the relevant mathematical framework is arithmetic geometry, that is, geometries over the rationals (\(\mathbb{Q}\)), the \(p\)-adic (\(\mathbb{Q}_p\)) and over finite Galois fields (\(\text{Gal}(\mathbb{F}_{p^n})\)) \[35–39\]. The relation between these arithmetic geometries and the continuous ones (over the real, \(\mathbb{R}\), or the complex, \(\mathbb{C}\) numbers) is studied by the Langlands program \[40\].

In our previous work we have proposed a discrete and finite model which underlies the continuous AdS\(_2\) geometry at Planck scale distances from the horizon of extremal black holes. We have called it AdS\(_2[\mathcal{N}]\), for any, positive, integer \(\mathcal{N}\). This geometry is defined as the set of points of integer entries, \((k,l,m)\), that satisfy the relation \(k^2 + l^2 - m^2 \equiv 1 \text{ mod } \mathcal{N}\). Thus, we have replaced, in the definition of the continuous AdS\(_2\) geometry, the real numbers with the integers mod \(\mathcal{N}\).

AdS\(_2[\mathcal{N}]\), defined this way, has a random structure, due to the modular arithmetic. As explained in \[41–43\], this particular discretization is chosen because it provides a way of constructing a holographic correspondence between the bulk, AdS\(_2[\mathcal{N}]\), and its boundary, \(\mathbb{RP}^1[\mathcal{N}]\), the discrete projective line.

The reason this discrete holography exists at all is that it is possible to realize in two ways the action of the discrete and finite symmetry group of AdS\(_2[\mathcal{N}]\), which is PSL\(_2[\mathcal{N}]\): Firstly it acts as an isometry group of the bulk and secondly it acts as the (Möbius) conformal group on the boundary.

This geometry provides a natural framework for describing the single particle dynamics via observers with time evolution operators that are elements of the isometry group. This is a discrete analog of the superconformal quantum mechanics of probes near the horizon of large extremal black holes \[44\].

We have studied the quantum mechanics of single particle probes in the AdS\(_2[\mathcal{N}]\) geometry and we have found that, for observers whose time evolution operator is the Arnol’d cat map the scrambling
of initially Gaussian wavepackets occurs in a way consistent with the Eigenstate Thermalization Hypothesis. Interestingly when the dimension of the Hilbert space takes values in the set of Fibonacci integers, the “fast scrambling bound” which is obtained for the case of single particle probes \cite{26,27,45,49} is saturated.

In our approach a long–standing question has been the existence of a continuum limit of the finite and random modular geometry AdS$_2[N]$, for $N \to \infty$. In the present paper we shall demonstrate that this limit exists and in fact the continuous geometry of AdS$_2$ emerges from the discrete AdS$_2[N]$ geometry as an infrared limit in two steps. The first step involves the discretization of AdS$_2$, using an appropriate spacetime lattice in the embedding 2+1–dimensional, Minkowski, spacetime. This requires introducing an ultraviolet cutoff, by taking the limit $\lim_{M \to \infty} M$. This set defines the integral lattice of AdS$_2$, and then imposing periodic boundary conditions. In the above continuous limit we arrive at the integral lattice to its arithmetic discrete subgroup $SO(2,1,Z)$ \cite{50,51}. The Minkowski spacetime lattice induces moreover, on the continuum AdS$_2$, an infinite set of integral points with isometry group $SO(2,1,Z)$. This set defines the integral lattice of AdS$_2$, which we shall call henceforth AdS$_2[Z]$.

On the other hand, the introduction of the periodic box of size $L = Na$ identifies all the points of the integral lattice, whose coordinates differ by integer multiples of $N$.

This equivalence relation implies that all the points of AdS$_2[Z]$ which lie outside the box, can be identified with points inside the box, that, however, need not lie on the part of AdS$_2[Z]$ that’s enclosed by this box. We observe that the IR cutoff, $N$, deforms the $SO(2,1,Z)$ symmetry of the integral lattice to its mod $N$ reduction, $SO(2,1,Z_N)$.

The images of all points $(k,l,m)$ of AdS$_2[Z]$, inside this box, satisfy the equation $k^2 + l^2 - m^2 \equiv M^2 \mod N$.

In order to identify the solutions of this equation with the elements of AdS$_2[N]$, it is necessary to impose that $M^2 \equiv 1 \mod N$. This condition provides a relation between the points of AdS$_2[Z]$ and those of AdS$_2[Z_N]$, as well as an important constraint between the UV and IR cutoffs, $M$ and $N$.

Finally, we shall demonstrate that the continuum limit entails finding infinite sequences of UV/IR cutoff pairs $\{(M_n,N_n)\}$, where $M_n < N_n$ for all $n$. When $n \to \infty$, $M_n$ and $N_n$ will tend to infinity in such a way that $0 < \lim_{n \to \infty} M_n/N_n = R_{AdS_2}/L \equiv \gamma < 1$.

Another result is the explicit construction of such sequences, using properties of generalized $k$–Fibonacci numbers and the ratio $\gamma$ turns out to be the golden ($k = 1$) or, more generally, the “$k$–silver” ratios (for $k = 2,3,...$). Introducing the $k$–Fibonacci sequence will allow us to remove the IR cutoff, by taking the limit $k \to \infty$, while keeping $R_{AdS_2}$ fixed.

The plan of our paper is as follows:
In section 2, we first recall the salient, standard, features of the geometry of AdS$_2$ (in 2.1), as a ruling surface and as a coset space; then we introduce the UV cutoff and the integral lattice of AdS$_2$ as well as its discrete isometry group $SO(2,1,\mathbb{Z})$. The structure of this discrete Lorentz group is important for the study of hyperbolic algebras [52], since it plays the role of the Weyl group of its root space.

Since the ruling property of the continuous geometry is lost for AdS$_2[\mathbb{Z}]$, we provide a detailed construction of the light cone lines, that contain the integral points. We show that these lines emanate from the rational points of the circle, on the throat of AdS$_2$. This construction is relevant for understanding the causality structure of this lattice (known as causal diamonds) and for counting the points of this lattice.

Due to the randomness, that is inherent in the factorization of integers, there is a corresponding randomness in the distribution of the lattice points of AdS$_2[\mathbb{Z}]$, which is a well known subject in arithmetic geometry [53].

In subsection 2.3 we introduce, also, the IR cutoff, by enclosing (a part of) AdS$_2$, in a symmetric way, within a cube of size $L$ and imposing periodic boundary conditions.

In section 3 we construct the discrete geometry AdS$_2[N]$, by recovering the ruling property, thanks to the mod $N$ operation. Moreover we show that the for all of its points there are “precursors” in AdS$_2[\mathbb{Z}]$. This implies that AdS$_2[N]$ is the set of equivalence classes of points of AdS$_2[\mathbb{Z}]$ under the kernel of the mod $N$ reduction of $SO(2,1,\mathbb{Z})$ to $SO(2,1,\mathbb{Z}_N)$.

Furthermore, we actually count the number of points of the finite geometry AdS$_2[N]$, for every $N$ and we show that, as $N \to \infty$, this number grows like $N^2$. This is important since it demonstrates that most of its points are at close to the boundary, allowing the holographic description of the bulk.

In section 4 we study the key equation, $M^2 \equiv 1 \mod N$, which we solve, for all $N$, using the Chinese Remainder Theorem. Furthermore, we use the properties of the Fibonacci sequence, as well as their generalization, the $k$–Fibonacci sequences ($k=1,2,3,...$) in order to find the infinite sequence of pairs $(M,N)$, that define the correct continuum limit for the infinite folding of the continuous AdS$_2$ geometry, inside the periodic box.

Finally, we show that by using the $k$–Fibonacci sequence the IR cutoff can be removed and we obtain the full, infinite, single copy of the continuum AdS$_2$ geometry, by taking the limit $k \to \infty$.

In section 5 we draw our conclusions and present our ideas for further inquiry.

2 Continuum AdS$_2$ near horizon BH geometry and the AdS$_2$ integral lattice

2.1 AdS$_2$ geometry as a ruling surface and a coset space

In this section we review the salient features of the geometry of AdS$_2$, as the single–sheeted hyperboloid, considered as a ruled surface and a coset space [54,55].

In the near horizon region of spherically symmetric 4d extremal black holes the geometry is known to be of the form $\text{AdS}_2 \times S^2$, where the $\text{AdS}_2 = SL(2,\mathbb{R})/SO(1,1,\mathbb{R})$, factor describes the geometry
of the radial and time coordinates and $S^2$ is the horizon surface.

The AdS$_2$ spacetime, is a one-sheeted hyperboloid defined through its global embedding in Minkowski spacetime with one space– and two time–like dimensions by the equation \[56, 57\].

\[ x_0^2 + x_1^2 - x_2^2 = R_{\text{AdS}_2}^2 \]  

(2.1)

We shall work in units where $R_{\text{AdS}_2} = 1$.

The boundaries of AdS$_2$ consist of two time–like disconnected circles, where AdS$_2$ approaches, asymptotically, the light cone of $\mathcal{M}^{2,1}$,

\[ x_0^2 + x_1^2 - x_2^2 = 0 \]  

(2.2)

AdS$_2$ can be, also, described as the homogeneous space, $SO(2, 1)/SO(1, 1)$. This case is special, in that $SO(2, 1)$ has a double cover, $SL(2, \mathbb{R})$, so we have $\text{AdS}_2 = SL(2, \mathbb{R})/SO(1, 1)$.

In order to establish our notation and conventions, we proceed with the Weyl construction of the double covering group, $SL(2, \mathbb{R})$.

To every point, $x_\mu \in \text{AdS}_2$, $\mu = 0, 1, 2$, we assign the traceless, real, $2 \times 2$ matrix

\[ M(x) \equiv \begin{pmatrix} x_0 & x_1 + x_2 \\ x_1 - x_2 & -x_0 \end{pmatrix} \]  

(2.3)

Its determinant is, $\det M(x) = -x_0^2 - x_1^2 + x_2^2 = -1$.

The action of any element $A$ of the isometry group $SL(2, \mathbb{R})$ on $\text{AdS}_2$ is defined through the mapping

\[ M(x') = AM(x)A^{-1} \]  

(2.4)

This induces an $SO(2, 1)$ transformation on $(x_0, x_+, x_-)$, where $x_\pm = x_1 \pm x_2$.

\[ x' \equiv \Lambda(A)x \]  

(2.5)

More concretely, when

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]  

(2.6)

then the induced Lorentz transform, $\Lambda(A)$, in the light cone basis $(x_0, x_+, x_-)$, is given by the expression.

\[ \Lambda(A) = \begin{pmatrix} ad + bc & -ac & bd \\ -2ab & a^2 & -b^2 \\ 2cd & -c^2 & d^2 \end{pmatrix} \]  

(2.7)

Choosing as the origin of coordinates, the base point $p \equiv (1, 0, 0)$, its stability group $SO(1, 1)$, is the group of Lorentz transformations in the $x_0 = 0$ plane of $\mathcal{M}^{2,1}$ or equivalently, the “scaling” subgroup, $D$, of $SL(2, \mathbb{R})$

\[ D \ni S(\lambda) \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \]  

(2.8)
for $\lambda \in \mathbb{R}^*$.

For this choice of the stability point, we define the coset, $h_A$, by decomposing $A$ as

$$A = h_A S(\lambda_A) \quad (2.9)$$

Thus, we associate uniquely to every point $x \in \text{AdS}_2$ the corresponding coset representative $h_A(x)$.

We introduce now, the global coordinate system, defined by the straight lines that generate $\text{AdS}_2$ and for which it can be checked easily that they form its complete set of light cones.

Consider the two lines, $l_\pm(p)$, passing through the point $p \in M^{2,1}$, orthogonal to the $x_0$ axis and at angles $\pm \pi/4$ to the $x_1 = 0$ plane. They are defined by the intersection of $\text{AdS}_2$ and the plane $x_0 = 1$ cf. fig. [1].

The coordinates of any point, $q_+ \in l_+(p)$, $q_- \in l_-(p)$ are given as $(1, \mu_\pm, \pm \mu_\pm)$, $\mu_\pm \in \mathbb{R}$ correspondingly.

We can parametrize any point $x_\mu$, of $\text{AdS}_2$, by the intersection of the local light cone lines, $l_\pm(x)$, with coordinates $\mu_\pm$ and $\phi_\pm$ through the relations

$$x_0 = \cos \phi_\pm - \mu_\pm \sin \phi_\pm$$
$$x_1 = \sin \phi_\pm + \mu_\pm \cos \phi_\pm$$
$$x_2 = \pm \mu_\pm \quad (2.10)$$

These can be inverted as follows:

$$e^{i\phi_\pm} = \frac{x_0 + ix_1}{1 \pm ix_2} \quad \mu_\pm = \pm x_2 \quad (2.11)$$

The geometric meaning of the coordinates $\phi$ and $\mu$ is that $\mu$ parametrizes the $x_2$, space–like, coordinate and, thus, $\mu_\pm \sqrt{2}$ parametrizes the light cone lines $l_\pm(x)$. The angle $\phi_\pm$ is the azimuthal angle of the intersection of $l_\pm(x)$ with the plane $(x_0, x_1)$. From eq. (2.11), by re-expressing numerator and denominator in polar coordinates, we find

$$\phi = \tau - \sigma \quad (2.12)$$

where $\tau$ and $\sigma$ are the arguments of the complex numbers $x_0 + ix_1$ and $1 + ix_2$.

The corresponding coset parametrization (group coset motion which brings the origin to the point $x$) is:

$$h(\mu_\pm, \phi_\pm) = R(\phi_\pm) T_\pm(\mu_\pm) \quad (2.13)$$

where

$$R(\phi) = \begin{pmatrix} \cos \phi/2 & -\sin \phi/2 \\ \sin \phi/2 & \cos \phi/2 \end{pmatrix} \quad (2.14)$$

and

$$T_+(\mu) = [T_-(\mu)]^T = \begin{pmatrix} 1 & -\mu \\ 0 & 1 \end{pmatrix} \quad (2.15)$$

It is easy to see also, that $T_\pm(\mu_\pm)$, acting on the base point $X(p)$, generate the light cone $l_\pm(p)$, so we identify these one parameter subgroups with the light cones at $p$. 

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Figure 1: The light cone of $\text{AdS}_2$ at $p = (1, 0, 0)$. 
In the literature the study of fields on AdS$_2$ requires an extension, to the universal covering of this spacetime, $\tilde{\text{AdS}}_2$, together with appropriate boundary conditions, in order to avoid closed time-like geodesics and reflection of waves from the boundary. This extension can be parametrized using as time coordinate the azimuthal angle $\tau$, by extending its range, $(-\pi, \pi)$ to $(-\pi + 2\pi k, \pi + 2\pi(k + 1))$, $k = \pm 1, \pm 2, \ldots$ and by the space coordinate $\sigma \in (-\pi/2, \pi/2)$, defined in eqs. (2.12). The extension of the range of $\tau$ parametrizes the infinitely-sheeted Riemann surface of the function $\log(\cdot)$, used in deriving eq. (2.12).

These parametrizations induce specific metrics on AdS$_2$ and $\tilde{\text{AdS}}_2$, namely:

$$ds^2 = (1 + \mu^2) d\phi^2 + 2d\phi d\mu$$

and

$$ds^2 = \frac{1}{\cos^2 \sigma} [-d\tau^2 + d\sigma^2]$$

The latter metric, with $\tau \equiv \sigma + \phi \in \mathbb{R}$ describes the Einstein strip, that has two disconnected boundaries at $\sigma \equiv \pm \pi/2$.

The Einstein strip is the union of an infinite number of copies of AdS$_2$. It is interesting to note that the coset structure of AdS$_2$ can be elevated to $\tilde{\text{AdS}}_2$ by using the universal covering group of $SL(2, \mathbb{R})$, $\tilde{SL}(2, \mathbb{R})$, which has been explicitly constructed in ref. [58].

### 2.2 AdS$_2$ integral lattice and its discrete Lorentz group $SO(2,1,\mathbb{Z})$

isometry. We shall now present and study in detail the the lattice of integral points of AdS$_2$, which we denote henceforth as AdS$_2[\mathbb{Z}]$, along with its isometries. This is the first step towards the modular discretization of AdS$_2$.

As we shall show, this lattice is sufficient for the reconstruction of the continuous AdS$_2$ spacetime, in an appropriate scaling limit.

In previous work [41–43] we have embarked on a program to address the question of “continuum” vs. “discretum” for specific toy model geometries, e.g. AdS$_2$, based on the idea that spacetime is not emergent, but that the continuum description of spacetime, as a whole, emerges as an infrared limit of a discrete description of spacetime itself, beyond the Planck scale. This can be expressed as the “fractionalization” of spacetime, beyond the Planck scale, in the spirit of the parton model discussed in ref. [10].

The starting point of our construction is that, for extremal black holes, the entropy is, not, only, finite, but fixed, since they don’t radiate. Therefore the dimension of the Hilbert space of single-particle probes of the black hole microstates is finite—we shall call it $N$ in what follows. We interpret this finite number as describing a discrete number of points in the spatial direction of AdS$_2$.

Now AdS$_2$ is a continuous spacetime and, before we arrive at such a finite discretization, we study, in some detail, the problem of describing and counting the integral points on it, i.e. with coordinates $k, l, m \in \mathbb{Z}$, satisfying the constraint

$$k^2 + l^2 - m^2 = 1$$ (2.18)
The solution to this problem isn’t known in closed form; in the literature there has been considerable effort in counting the number of solutions, in particular the asymptotics of the density of such points \[53,59,61\]. This problem can be mapped to a problem whose solution is known, namely the Gauss circle problem, that of finding the number \(r_2(m)\), of solutions to the equation \(k^2 + l^2 = 1 + m^2\). This number is determined by factorizing \(1 + m^2\) into its prime factors \[59\] and counting the number of primes, \(p_i\), of the form \(p_i \equiv 1 \text{ mod } 4\) (this is described in detail in \[62\]) (p. 285 and following).

This factorization procedure generates a sequence of primes that contains an element of inherent randomness—and it is this property that captures the random distribution of the integral points on \(\text{AdS}_2\)—this is illustrated in figs. 2.

![Figure 2: Integral points on AdS\(_2\).](image)

Therefore, from these facts, the number of integral points of the hyperboloid, up to height \(m\), is given by the expression

\[
\text{Sol}(m) = 4 + 2 \sum_{j=1}^{m} r_2(j)
\]  

(2.19)

We plot this function—in fig. 3 when \(m\) runs from \(-200\) to \(200\) (due to the symmetry, \(m \leftrightarrow -m\), we plot only the positive values of \(m\).)

It is, indeed, striking that the result is a straight line (up to corrections).

We shall now discuss how to actually construct these points, using the property that they belong to light–cone lines, which emerge from the rational points of the circle on the throat of \(\text{AdS}_2\).

For, using the ruling property of \(\text{AdS}_2\), \(\text{viz.}\)

\[
\begin{align*}
  k &= \cos \phi - \mu \sin \phi \\
  l &= \sin \phi + \mu \cos \phi \\
  m &= \mu
\end{align*}
\]  

(2.20)
Figure 3: The number of integral points, on AdS$_2$, as a function of the height, $m$. Due to symmetry, $m \leftrightarrow -m$, we plot only the positive values of $m$.

we may repack these as follows

$$x_0 + ix_1 = k + il = e^{i\phi}(1 + i\mu) = e^{i\phi}(1 + im) \Leftrightarrow e^{i\phi} = \frac{k + il}{1 + im}$$

hence

$$\cos \phi = \frac{k + lm}{1 + m^2} \quad \text{and} \quad \sin \phi = \frac{l - mk}{1 + m^2}$$

We remark that these are rational numbers—therefore they label rational points on the circle $[63]$.

The light cone lines at $(k, l, m)$ are, therefore, parametrized as

$$x_0 = \frac{k + lm}{1 + m^2} - \mu \frac{l - mk}{1 + m^2}$$
$$x_1 = \frac{l - mk}{1 + m^2} + \mu \frac{k + lm}{1 + m^2}$$
$$x_2 = \mu$$

(When $\mu = x_2 = m$, $x_0 = k$ and $x_1 = l$.)

On these specific light-cone lines we shall show that there exist infinitely many integral points, when $\mu$, that labels the space–like direction $x_2$, takes integer values.

Proof. We write

$$x_0(\mu) + ix_1(\mu) = e^{i\phi}(1 + i\mu)$$

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where $\phi$ is defined by eq. (2.22).

If $\mu = n \in \mathbb{Z}$, we would like to show that $x_0(n)$ and $x_1(n)$ are, also, integers.

This implies that

$$x_0(n) + i x_1(n) = \frac{k + il}{1 + im}(1 + in)$$

(2.25)

should be a Gaussian integer, i.e. $(1 + im)/(1 + in) = a + ib$ with $a, b \in \mathbb{Z}$.

Therefore

$$1 + in = (a - mb) + i(am + b) \iff \begin{cases} 1 = a - mb \\ n = am + b \end{cases}$$

(2.26)

These expressions imply, in turn, that

$$
\begin{align*}
x_0 &= k + b(km - l) \\
x_1 &= l + b(k + lm) \\
x_2 &= n = m + b(1 + m^2)
\end{align*}
$$

(2.27)

This completes the dictionary between the rational points on the circle and the integral points on AdS$_2$. In these expressions $b$ can take any integer value.

Conversely, any rational point on the unit circle,

$$e^{i\phi} \equiv \frac{a + ib}{a - ib} \iff x_0 + ix_1 = \frac{a + ib}{a - ib}(1 + in)$$

(2.28)

with $a, b \in \mathbb{Z}$, to obtain an integral point, for $\mu = n$, we must have

$$\frac{1 + in}{a - ib} = d + ic$$

(2.29)

with $c, d \in \mathbb{Z}$

We immediately deduce that

$$\begin{align*}
1 &= ad - bc \\
n &= ac + bd
\end{align*}$$

(2.30)

These expressions imply that, given the integers $a$ and $b$, it’s possible to find the integers $c$ and $d$ and to express the coordinates $x_0, x_1$ and $x_2$ as

$$
\begin{align*}
x_0 &= ad + bc \\
x_1 &= ac - bd \\
x_2 &= ac + bd
\end{align*}
$$

(2.31)

The Diophantine equation $1 = ad - bc$ is solved for $c$ and $d$, given two coprime integers $a$ and $b$, by the Euclidian algorithm–which seems to lead to a unique solution, implying that the point $(x_0, x_1, x_2)$ is unique.

However there’s a subtlety! There are infinitely many solutions $(c, d)$, to the equation $ad - bc = 1$! The reason is that, given any one solution $(c, d)$, the pair $(c + \kappa a, d + \kappa b)$, with $\kappa \in \mathbb{Z}$, is, also, a solution, as can be checked by substitution.
Therefore there is a one–parameter family of points, labeled by the integer $\kappa$:

\[
\begin{align*}
    x_0 &= ad + bc + 2\kappa ab \\
    x_1 &= ac - bd + \kappa(a^2 - b^2) \\
    x_2 &= ac + bd + \kappa(a^2 + b^2)
\end{align*}
\] (2.32)

We remark, however, that the vector $(2ab, a^2 - b^2, a^2 + b^2)$ is light–like, with respect to the $(+ + -)$ metric: $(2ab)^2 + (a^2 - b^2)^2 - (a^2 + b^2)^2 = 0$. So eq. (2.32) describes a shift of the, original, point $(ad + bc, ac - bd, ac + bd)$, along a light–like direction; and since the shift is linear in the “affine parameter”, $\kappa$, this generates a light–like line, passing through the original point.

In this way we have established the dictionary between the rational points of the circle and the integral points of the hyperboloid.

Now we proceed with the study of the discrete symmetries of the integral, $\mathcal{M}^{2,1}$, Lorentzian lattice, where the lattice of integral points on AdS$_2$ is embedded. $\mathcal{M}^{2,1}$, with one space-like and two time-like dimensions, carries as isometry group the group of integral Lorentz boosts $\text{SO}(2, 1, \mathbb{Z})$, as well as integral Poincaré translations. The double cover of this infinite and discrete group is $\text{SL}(2, \mathbb{Z})$, the modular group. This has been shown by Schild [50,51] in the 1940s. The group $\text{SO}(2, 1, \mathbb{Z})$ can be generated by reflections, as has been shown by Coxeter [64], followed by Vinberg [65] and, finally, by Kac in his famous book [52], where he introduced the notion of hyperbolic, infinite dimensional, Lie algebras. The characteristic property of such algebras is that the discrete Weyl group of their root space is an integral Lorentz group. Generalization from $\text{SL}(2, \mathbb{Z})$ to other normed algebras has been studied in [66].

The fundamental domain of $\text{SO}(2, 1, \mathbb{Z})$ is the minimum set of points of the integral lattice of $\mathcal{M}^{2,1}$, which are not related by any element of the group and from which, all the other points of the lattice can be generated by repeated action of the elements of the group. It turns out that the fundamental region is an infinite set of points and can be generated by repeated action of reflections in the following way:

Using the metric $h \equiv \text{diag}(1, 1, -1)$ on $\mathcal{M}^{2,1}$ the generating reflections, elements of $\text{SO}(2, 1, \mathbb{Z})$, are given by the matrices

\[
R_1 = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_4 = \begin{pmatrix} 1 & -2 & -2 \\ 2 & -1 & -2 \\ -2 & 2 & 3 \end{pmatrix}
\] (2.33)

If $(k, l, m)$ are the coordinates of the integral lattice, the fundamental domain of $\text{SO}(2, 1, \mathbb{Z})$ can be defined by the conditions $m \geq k + l \geq 0$ and $k \geq l \geq 0$. This fundamental domain, restricted on AdS$_2[\mathbb{Z}]$, defines the corresponding fundamental domain of $\text{SO}(2, 1, \mathbb{Z})$, acting on AdS$_2[\mathbb{Z}]$. This region of AdS$_2[\mathbb{Z}]$ lies in the positive octant of $\mathcal{M}^{2,1}$ and between the two planes, that define the conditions—cf.fig. 4. It is of infinite extent.
Figure 4: The fundamental domain of SO(2,1,\mathbb{Z}) on AdS₂[\mathbb{Z}] is the dark green part of the hyperboloid, in the positive octant, that lies between the two planes, \( m \geq k + l \geq 0 \) and \( k \geq l \geq 0 \).

### 2.3 UV/IR cutoff of AdS₂

Having introduced the lattice of integral points on AdS₂, which we consider as defining a UV cutoff, we proceed, now, to impose an infrared (IR) cutoff. The crucial reason for such a cutoff is the interpretation of AdS₂ as a phase space of single particles, due to the symplectic nature of the isometry \( SL(2, \mathbb{R}) = Sp(2, \mathbb{R}) \) and the requirement to realize chaotic and mixing dynamics for geodesic infalling observers (fast scrambling) [41].

Finiteness of the volume in phase space is a necessary condition for mixing (for a detailed discussion of this point cf. [67]).

Having embedded the AdS₂ hyperboloid,

\[
x_0^2 + x_1^2 - x_2^2 = R_{\text{AdS}_2}^2
\]

in \( \mathcal{M}^{2,1} \), the IR cutoff, \( L \) is defined by periodically identifying all the spacetime points of \( \mathcal{M}^{2,1} \), if the difference of their coordinates is an integral vector \( \times L \):

\[
x \sim y \Leftrightarrow x - y = (k, l, m) L
\]

where \( k, l, m \in \mathbb{Z} \). In this way we have compactified \( \mathcal{M}^{2,1} \) to the three-dimensional torus, of size \( L \), \( \mathbb{T}^3(L) \).

More concretely, \( \mathbb{T}^3(L) \) is the fundamental domain of the group of integral translations, \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \), acting on \( \mathcal{M}^{2,1} \). To describe this geometric property by the algebraic operation, mod \( L \), that acts on the coordinates of \( \mathcal{M}^{2,1} \), we are led to identify the fundamental domain with the positive octant of \( \mathcal{M}^{2,1} \), i.e. \( x_0, x_1, x_2 \geq 0 \).

After this compactification, the spacetime geometry of AdS₂ becomes a foliation of the 3-torus, with leaves the images of AdS₂ under the operation mod \( L \). So the equation, whose solutions define
the points of the compactified AdS$_2$, is
\[x_0^2 + x_1^2 - x_2^2 \equiv R_{AdS_2}^2 \mod L\] (2.36)

where \((x_0, x_1, x_2) \in \mathbb{T}^3(L)\).

It is obvious, that inside the 3-torus, there is a part of the AdS$_2$ surface, which corresponds to solutions of eq. (2.36), without the mod $L$ operation. On the other hand, the infinite part of AdS$_2$, that lies outside the torus, is partitioned in infinitely many pieces, which belong to images of $\mathbb{T}^3(L)$ in $\mathcal{M}^{2,1}$, which, by the mod $L$ operation are brought inside the torus.

It is illuminating, for visualizing this complex foliation of the 3–torus into AdS$_2$ pieces, to consider the effect of the periodic compactification on the construction of AdS$_2$ by the rotation of light cone lines around the circle, which is the throat of AdS$_2$, in the \((x_0, x_1)\) plane. This is the ruling mechanism, which was described in the previous subsection. This is displayed in fig. 5. It is easy to observe, for

Figure 5: Rotating lines that rule AdS$_2$.

the initial position of the light cone line $l_+ (p)$ the mod $L$ operation generates copies of the parts of the (infinite) line outside the 3-torus, which are parallel and coplanar with the part of the line, which lies inside the torus, cf. fig. 6. This happens for every position of the light cone line, after rotation around the throat of AdS$_2$.

Let us provide an explicit construction.

From the equations
\[\begin{align*}
x_0 &= \cos \phi - \mu \sin \phi \\
x_1 &= \sin \phi + \mu \cos \phi \\
x_2 &= \mu
\end{align*}\] (2.37)
where $\phi \in [0, 2\pi)$ labels the point on the throat and $\mu$ the height along the space-like direction, $x_2$, we observe that, for fixed value of $\phi$ the light cone line, passing through any such point, is given, in parametric form by these equations.

For any fixed value of $\phi$, such that $\cos \phi = p/r$ and $\sin \phi = q/r$, where $p, q, r$ are integers, with $(p, r) = 1 = (q, r)$ and $p^2 + q^2 = r^2$, then, as $\mu$ takes values from $-\infty$ to $+\infty$,

$$
\begin{align*}
    x_0 &= \frac{p}{r} - \mu \frac{q}{r} \\
    x_1 &= \frac{q}{r} + \mu \frac{p}{r} \\
    x_2 &= \mu
\end{align*}
$$

we split the range of $\mu$ into segments of length $L$, which is, not only, the size of the box, but, in particular, its height, along the space-like direction $x_2$. Therefore, $\mu = kL + \nu$, where $k$ integer, and $0 \leq \nu < L$.

We observe that, when $k = nr$, with $n$ integer, then eqs. (2.38) take the form

$$
\begin{align*}
    x_0 &= \left(-nqL + \frac{p}{r}\right) - \nu \frac{q}{r} \\
    x_1 &= \left(npL + \frac{q}{r}\right) + \nu \frac{p}{r} \\
    x_2 &= nrL + \nu
\end{align*}
\text{mod } L =
\begin{align*}
    x_0 &= \frac{p}{r} - \nu \frac{q}{r} \\
    x_1 &= \frac{q}{r} + \nu \frac{p}{r} \\
    x_2 &= \nu
\end{align*}
$$

But the segment, described by the second set of equations, is identical to that when $k = 0$ and the
number of parallel segments is \( r \), that sets the period of the splitting of the line, as it extends beyond the cube.

Upon rotating all the line segments inside the cube, around the throat, we generate all the leaves of the image of \( \text{AdS}_2 \) under the mod \( L \) operation. However, as \( \phi \), then takes values, for which \( \cos \phi \) or \( \sin \phi \) aren’t both rational, this implies that the number of segments, inside the cube, is infinite and, indeed, densely fills the cube. This illustrates a generalization of the well–known torus case, where a line with rational slope is folded in a periodic way inside the torus; otherwise, it covers the torus densely [67].

The next step for defining and elucidating the modular discretization, \( \text{AdS}_2[N] \), of \( \text{AdS}_2 \), entails introducing a UV cutoff, which will reproduce the lattice of integral points of \( \text{AdS}_2 \) constructed in the previous subsection.

The physical lengthscale in our problem is the radius of the \( \text{AdS}_2 \) spacetime, \( R_{\text{AdS}_2} \). We set \( R_{\text{AdS}_2} = 1 \) and we divide it into \( M \) segments, of length \( a = R_{\text{AdS}_2}/M \). This defines \( a \) as the UV cutoff (lattice spacing) and \( M \in \mathbb{N} \) and, hence, a lattice in \( \mathcal{M}^{2,1} \).

The continuum limit is defined by taking \( M \to \infty \) and \( a \to 0 \) with \( R_{\text{AdS}_2} = 1 \) fixed.

The global, embedding, coordinates \((x_0, x_1, x_2)\) of this lattice are \((ka, la, ma) = a(k, l, m)\), where \( k, l, m \in \mathbb{Z} \), so are measured in units of the spacing \( a \). Therefore the lattice points, that lie on \( \text{AdS}_2 \) satisfy the equation

\[
k^2 + l^2 - m^2 = M^2
\]

whose solutions define \( \text{AdS}_2[\mathbb{Z}] \), the set of all integral points of \( \text{AdS}_2[\mathbb{Z}] \), with integer radius \( M \).

Now we choose the IR cutoff \( L \) in units of \( a \), so that \( L = aN \), where \( N \) is an integer, independent of \( M \). It is constrained by \( N > M \), since the cube should contain, at least, the throat of \( \text{AdS}_2 \).

So the scaling limit will be to take \( M \to \infty \), \( N \to \infty \), but keeping \( L \), also, fixed.

The periodic nature of the IR cutoff implies that, finally, we must take the images of all integral points of \( \text{AdS}_2[\mathbb{Z}] \) under the mod \( N \) operation, inside the cubic lattice of \( N^3 \) points.

The set of these images satisfy the equations

\[
k^2 + l^2 - m^2 \equiv M^2 \mod{N}
\]

Our definition for \( \text{AdS}_2[N] \) in our previous work was similar to the one given here, the only difference being that the RHS of eq. (2.41) was \( 1 \mod{N} \), which was chosen for convenience, not for any intrinsic reason. We remark that the two definitions are consistent iff \( M^2 \equiv 1 \mod{N} \).

The solutions of eq. (2.41), when \( M^2 \equiv 1 \mod{N} \), define \( \text{AdS}_2[N] \). This is a random set of points, thereby defining a random geometry, through the mod \( N \) operation.

For fixed lattice spacing \( a \) and large \( N \), this random set defines a limiting distribution on \( \mathcal{M}^{2,1} \) and, as we shall see later, the number of points of \( \text{AdS}_2[N] \) grows as \( N^2 \). This implies that “most” of its points are close to the boundary.
3 Reduction mod $N$ of the AdS$_2$ integral lattice

3.1 The discrete modular geometry AdS$_2[N]$ and its precursors

In this section we describe, in detail, the properties of the random set of points, that constitute AdS$_2[N]$, defined previously.

All the points of AdS$_2[N]$ live inside the cube, of side $L = Na$. Some of them are integral points of AdS$_2$. The rest are the mod $N$ reduction of integral points of AdS$_2$, outside the cube cf. fig[7]. Although, in the continuum, AdS$_2$ is generated by the rotation of a light cone line around the circle of the throat, for the integral points this isn’t true. Rather, as shown in the previous section, we can identify a set of light cone lines, emerging from the rational points of this circle, on which all the integral points lie.

This means that a globally defined ruling parametrization of the integral points doesn’t exist.

It is interesting to notice, though, that, for the mod $N$ reduction of AdS$_2[Z]$, i.e. AdS$_2[N]$, it is possible to define such a global ruling parametrization for $N = p^r$, where $p$ is a prime of the form (a) $p \equiv 3 \mod 4$, while when (b) $p \equiv 1 \mod 4$, we need two charts to obtain all such points.

Below we shall make this construction explicit.

We, therefore, start, by parametrizing the points of AdS$_2[N]$ by the ruling of the discrete line

\[ \frac{k^2 + l^2 - m^2 \equiv 1 \mod 47}{18} \]
$l = (1, \mu, \mu)$ around the discrete circle of the throat of AdS$_2[N]$:

\[
x_0 = a - \mu b \\
x_1 = b + \mu a \\
x_2 = \mu
\]

where $a, b, \mu \in \mathbb{Z}_N$ and $a^2 + b^2 \equiv 1 \mod N$. cf. fig.8

Figure 8: The points of the discrete circle, $a^2 + b^2 \equiv 1 \text{ mod } 1001$.

This parametrization suffices to generate all the points, for case (a), as an explicit comparison with direct counting confirms; for case (b), we must add a second parametrization, by exchanging $x_0$ and $x_1$. The reason this is necessary is that, in case (b), given $x_0$ and $x_1$ it’s not possible to obtain $a$ and $b$, since there exists a $\mu = \mu_0$, such that $\mu_0^2 \equiv -1 \mod N$ in this case.

When $N = p_1^{r_1} \times \cdots \times p_k^{r_k}$, we can use the factorization properties of AdS$_2[N]$ and choose the appropriate system of charts to generate the points for each factor separately.

Having described AdS$_2[N]$, we shall now show how it’s related with AdS$_2[\mathbb{Z}]$.

First of all, we remark that, given any point

\[
Y = \begin{pmatrix} k & l + m \\ l - m & -k \end{pmatrix}
\]

(3.2)
of AdS$_2[\mathbb{Z}]$, its mod $N$ reduction, $X \equiv Y \mod N$ is a point of AdS$_2[N]$. This means that AdS$_2[\mathbb{Z}] \mod N \subset$ AdS$_2[N]$. 

We shall, now, show that, in fact, these two sets are equal. To this end, we shall show that, given any point $X \in$ AdS$_2[N]$, there exists, at least one, point $Y \in$ AdS$_2[\mathbb{Z}]$, such that $X \equiv Y \mod N$, i.e. that AdS$_2[N] \subset$ AdS$_2[\mathbb{Z}] \mod N$.

It should be noted that, if there exists one such point, this implies that there exist infinitely many points, related through the action of elements $\gamma \in \Gamma[N]$, as $Y \rightarrow \gamma Y \gamma^{-1}$. $\Gamma[N]$ is the *principal congruent subgroup* of SL$_2[\mathbb{Z}]$, defined as

$$\Gamma(N) = \{ A \in \text{SL}_2[\mathbb{Z}] | A \equiv I \mod N \}$$ (3.3)

From this definition it’s straightforward to deduce that $\Gamma(N_1) \subset \Gamma(N_2)$ if $N_1 | N_2$.

We shall call the points of AdS$_2[\mathbb{Z}]$ the “precursors” of AdS$_2[N]$. To find them, it is crucial to use the ruling parametrization (3.1).

The way to determine explicitly the precursors of AdS$_2[N]$ is simplified if we consider the precursors of the light cone lines $l_{\pm} = (1, \mu, \pm \mu)$, and, independently, the precursors of the points of the throat of AdS$_2[N]$, i.e. $x_2 = 0$.

The precursors of the lines $l_{\pm}$ are the integral points of the generating light cones of AdS$_2$. In group theoretical terms we have to find the precursor of group element

$$T(\mu) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$ (3.4)

in $SL(2, \mathbb{Z}_N)$.

This precursor in SL$_2(\mathbb{Z})$ is defined by

$$p(T(\nu)) = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$$ (3.5)

where $\nu \in \mathbb{Z}$ and where $\mu \equiv \nu \mod N$.

The precursors of the discrete circle $C_N = \{(a, b) | a^2 + b^2 \equiv 1 \mod N\}$ are determined as follows: The points of this discrete circle are in 1-to-1 correspondence with the elements of the abelian group

$$SO_2(N) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \bigg| a^2 + b^2 \equiv 1 \mod N \right\}$$ (3.6)

It is possible to show that there exist elements of SL$_2(\mathbb{Z})$, whose mod $N$ reduction generate SO$_2(N)$. These elements define an, infinite, abelian, subgroup of SL$_2(\mathbb{Z})$.

We work out in detail below the construction of these precursors in the case where $N$ is a prime, $p$ (and this can be straightforwardly generalized to powers of odd primes). In this case the group SO$_2(p)$ is cyclic and its generator has been found, for the first 2000 primes, of the form $p = 4k - 1$, by random search [68]. It suffices, therefore, to find the precursors of the generator.

For primes $p = 4k + 1$, the generator is the diagonal element of SL$_2(\mathbb{Z}_N)$ corresponding to the primitive element of $\mathbb{Z}_N$. 

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For any prime of the form $4k - 1$, we choose the corresponding generator $R_0$, corresponding to $(a_0, b_0)$, and we find a precursor in $SL_2(\mathbb{Z})$. The precursors of all other elements of $SO_2(p)$ are powers of this element.

If $(a_0, b_0)$ are relative primes, then, by the Euclidian algorithm, there exist two integers $c_0$ and $d_0$ such that $a_0d_0 - b_0c_0 = 1$.

Thus, a putative precursor of $R_0$ in $SL_2(\mathbb{Z})$, is the element

$$p(R_0) \equiv \begin{pmatrix} a_0 & c_0 \\ b_0 & d_0 \end{pmatrix}$$

(3.7)

For this element to be a true precursor, the following constraints must, also, be satisfied:

- $c_0 \equiv -b_0 \mod N$ (3.8)
- $d_0 \equiv a_0 \mod N$ (3.9)

To satisfy these constraints, we start with any “seed” solution, $(c_0, d_0)$ and add to it $\kappa(a_0, b_0)$. Any such pair, $(c_0 + \kappa a_0, d_0 + \kappa b_0)$ is a solution of $a_0d_0 - b_0c_0 = 1$ and it is possible to find $\kappa$ such that the other constraints are, also, satisfied.

If $(a_0, b_0)$ aren’t relative primes, then we replace the previous element by

$$p(R_0) \equiv \begin{pmatrix} a_0 + kp & -b_0 + lp \\ b_0 + mp & a_0 + np \end{pmatrix}$$

(3.10)

where $k, l, m, n$, are integers to be found, are such that the determinant is 1.

Therefore, the precursors of points of the discrete circle $\mathcal{C}_N$, which is the neck of $AdS_2[N]$ and which belong to $AdS_2$ are defined by

$$p(\mathcal{C}_N) = \{p(R_0)^l \sigma_3 p(R_0)^{-l} | l \in \{0, 1, \ldots, p + 1\}\}$$

(3.11)

Explicitly, the element $Y = (y_0, y_1, y_2)$, corresponding to $p(R_0)$, where $c_0 = c_\kappa$ and $d_0 = d_\kappa$ is given by the expression

$$y_0 = a_0d_\kappa + b_0c_\kappa \quad y_1 = b_0d_\kappa - a_0c_\kappa \quad y_2 = -a_0c_\kappa - b_0d_\kappa$$

(3.12)

Upon reducing mod $N$, we find the expressions for $X = (x_0, x_1, x_2) \equiv Y \mod N$, that take the suggestive form

$$x_0 \equiv (a_0^2 - b_0^2) \mod N \quad x_1 \equiv 2a_0b_0 \mod N \quad x_2 \equiv 0 \mod N$$

(3.13)

These expressions for $X$ imply that the generator $R_0$, through the Weyl action on the basis point $x_0 = 1, x_1 = 0, x_2 = 0$ produces points, that are labeled by twice the “angle” defined by $R_0$. That is it
produces half the number of required points. We can’t, therefore, upon acting on \( \sigma_3 \), produce points in \( \text{AdS}_2[\mathbb{Z}] \), with odd multiples of the angle defined by \( R_0 \). It is easy to check, however, that the other half of the points can be found by using the symmetry \( x_0 \to x_1, x_1 \to x_0 \), that is to say from the Weyl action of all the powers of \( R_0 \), that act on the new origin \((0, 1, 0)\). Indeed the powers of \( R_0 \) on \((0, 1, 0)\) cannot produce points produced from its action on \((1, 0, 0)\), since there doesn’t exist a rotation mod \( N \), which connects the two points—therefore, they produce new points, which contribute to the other “half” and it is possible to show that the union of these two sets does have the correct cardinality.

Therefore, we produce by the above construction the precursors of all the points, defined by (3.1), by the ruling parametrization of \( \text{AdS}_2[N] \).

### 3.2 Counting points of \( \text{AdS}_2[N] \)

The finite geometry, \( \text{AdS}_2[p] \), has as isometry group the finite projective modular group, \( \text{PSL}_2[p] \). This group is obtained as the reduction mod \( p \), of all elements of \( \text{PSL}(2, \mathbb{Z}) \). The kernel of this homomorphism is the “principal congruent subgroup”, \( \Gamma_p \). The order of \( \text{PSL}_2[p] \) is \( p(p^2 - 1)/2 \) and the order of its dilatation subgroup is \( (p - 1)/2 \), thus, the number of points of \( \text{AdS}_2[p] \) is \( p(p + 1) \).

It is easy to find the number of points of \( \text{AdS}_2[N] \), for any integer \( N \).

Numerical experiments suggest the following recursion relation for the number of points of \( \text{AdS}_2[p^k], \text{Sol}(p^k) \),

\[
\text{Sol}(p^k) = p^{2(k-1)}\text{Sol}(p) \Rightarrow \text{Sol}(p^k) = p^{2k-1}(p + 1)
\]

where \( \text{Sol}(p) = p(p + 1) \) and \( k = 1, 2, \ldots \) for any prime integer \( p \).

The validity of the above counting can be proved directly by using the coset property of \( \text{AdS}_2[p^n] \) and then using factorization of integers for any \( N \) [69]. Indeed the rank of the group \( \text{PSL}_2[p^n] \) is known to be \( p^{3n-2}(p^2 - 1)/2 \) and its dilatation subgroup \( \text{PSO}(1,1,p^n) \), \( p^{n-1}(p - 1)/2 \), since it is equal to the number of invertible numbers modulo \( p^n \) divided by \( 2 \) (due to its projective structure). Thus since \( \text{AdS}_2[p^n] \) is identified with the coset geometry \( \text{PSL}_2[p^n]/\text{PSO}(1,1,p^n) \), we get the promised result, \( p^{2n-1}(p + 1) \).

For \( N = 2^n \) we find \( \text{Sol}(2) = 4, \text{Sol}(4) = 24, \text{Sol}(2^k) = 4\text{Sol}(2^{k-1}) \), for \( k \geq 3 \). We remark that \( N = 4 \) is an exception. The solution is \( \text{Sol}(2^k) = 2^{2k+1} \), for \( k \geq 3 \). We display the results of exact enumeration in fig. 9 for \( 3 \leq N \leq 29 \). We notice that there are peaks for composite values of \( N \). Therefore we have many more points inside the box, for \( \text{AdS}_2[N] \), than on \( \text{AdS}_2[\mathbb{Z}] \). The additional points count the equivalence classes of points of \( \text{AdS}_2[\mathbb{Z}] \mod N \).

From these results we deduce that, for large \( N \), the number of solutions, mod \( N \), scales like the area, i.e. \( N^2 \). So most of the points of \( \text{AdS}_2[N] \) are close to its boundary and holography is possible in this case too [70].
Figure 9: The number of solutions to $k^2 + l^2 - m^2 = 1$ (blue curve) and $k^2 + l^2 - m^2 \equiv 1 \mod N$ (yellow curve), for $3 \leq N \leq 29$ obtained by exact enumeration.

4 Continuum limit for large $N$

4.1 All solutions of $M^2 \equiv 1 \mod N$

In section 2.3 we constructed the discrete geometry $\text{AdS}_2[N]$ by introducing, first, a UV cutoff ($a = R_{\text{AdS}}/M$, with $M$ integer) and, also, an IR cutoff $L = N a$, with $N$ another integer, bigger than $M$.

The continuum limit is defined by any sequence of pairs of integers, $(M_n, N_n)$, $n = 1, 2, 3, \ldots$, such that, for any $n$, (a) $N_n > M_n$, (b) $M_n^2 \equiv 1 \mod N_n$, and (c) the limit of the ratio $N_n/M_n$ takes a finite value, $> 1$ (as $n \to \infty$), which we can identify with $L/R_{\text{AdS}}$.

Below we shall present the general solution to the equation $M^2 \equiv 1 \mod N$. Subsequently, we shall select those solutions that satisfy the other requirements.

The first step is to factor $N$ into (powers of) primes, $N = N_1 \times N_2 \times \cdots \times N_l = q_1^{k_1} q_2^{k_2} \cdots q_l^{k_l}$. Then the equation $M^2 \equiv 1 \mod N$, is equivalent to the system

$$M_I^2 \equiv 1 \mod q_I^{k_I} \tag{4.1}$$

where $I = 1, 2, \ldots, l$. The Chinese Remainder Theorem [62] then implies that all the solutions of eq. (4.1) can be used to construct $M$, with $M = M_1 m_1 n_1 + \cdots M_l m_l n_l$, where $M_I \equiv M \mod N_I$, $m_I = N/N_I$, $n_I \equiv m_I^{-1} \mod N_I$. 

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When $q_I \neq 2$, the solutions are $M_I = 1$ and $q^n_I - 1$. When $q_I = 2$, there exist four solutions, $M_I = 1, 2n_I - 1, 2n_I - 1 \pm 1$.

Now we must choose sequences, $N_n$ and determine the corresponding $M_n$, satisfying the constraints listed above.

In the next subsection(s) we shall present interesting sequences of pairs, $(M_n, N_n)$ of such solutions, whose limiting ratio, $\lim_{n \to \infty} N_n/M_n$, is the “golden” or “silver” ratios.

### 4.2 Fibonacci sequences for the UV/IR cutoffs

Although it is easy to demonstrate the existence of such sequences—for example, $N_n = 2^n$ and $M_n = 2^{n-1} \pm 1$, where $M_n^2 \equiv 1 \mod N_n$ and $N_n/M_n \to 2$, in this section we focus on a particular class of sequences, based on the Fibonacci numbers, $f_n$ [62]. This case is of particular interest, since, in our previous paper [41], where we studied fast scrambling, we found that, for geodesic observers, moving in $\text{AdS}_2[N]$, with evolution operator the Arnol’d cat map, the fast scrambling bound is saturated, when $N$ is a Fibonacci integer.

The Fibonacci sequence, defined by
\[
\begin{align*}
  f_0 &= 0; f_1 = 1 \\
  f_{n+1} &= f_n + f_{n-1}
\end{align*}
\]  

(4.2)

can be written in matrix form
\[
\begin{pmatrix}
  f_n \\
  f_{n+1}
\end{pmatrix}
= \begin{pmatrix}
  0 & 1 \\
  1 & 1
\end{pmatrix}
\begin{pmatrix}
  f_{n-1} \\
  f_n
\end{pmatrix}
\]

(4.3)

We remark that the famous Arnol’d cat map can be written as
\[
\begin{pmatrix}
  1 & 1 \\
  1 & 2
\end{pmatrix}
= A^2
\]

(4.4)

Since the matrix $A$ doesn’t depend on $n$, we can solve the recursion relation in closed form, by setting $f_n \equiv C\rho^n$ and find the equation, satisfied by $\rho$

\[
\rho^{n+1} = \rho^n + \rho^{n-1} \iff \rho^2 - \rho - 1 = 0 \iff \rho \equiv \rho_\pm = \frac{1 \pm \sqrt{5}}{2}
\]

Therefore, we may express $f_n$ as a linear combination of $\rho_+^n$ and $\rho_-^n = (-)^n\rho_+^{-n}$.

\[
f_n = A_+\rho_+^n + A_-\rho_-^n \iff \begin{cases} 
  f_0 = A_+ + A_- = 0 \\
  f_1 = A_+\rho_+ + A_-\rho_- = 1
\end{cases}
\]

(4.5)

whence we find that
\[
A_+ = -A_- = \frac{1}{\rho_+ - \rho_-} = \frac{1}{\sqrt{5}}
\]

whence we find that
therefore,

\[ f_n = \frac{\rho^n_+ - (-)^n \rho^n_-}{\sqrt{5}} \]  \hspace{1cm} (4.6)

It’s quite fascinating that the LHS of this expression is an integer!

The eigenvalue \( \rho_+ > 1 \) is known as the “golden ratio” (often denoted by \( \phi \)) and it’s straightforward to show that \( f_{n+1}/f_n \rightarrow \rho_+ \), as \( n \rightarrow \infty \).

Furthermore, it can be shown, by induction, that the elements of \( A^n \) are, in fact, the Fibonacci numbers themselves, arranged as follows:

\[ A^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix} \]  \hspace{1cm} (4.7)

One reason this expression is useful is that it implies that \( \det A^n = (-)^n = f_{n-1}f_{n+1} - f_n^2 \).

For \( n = 2l + 1 \), we remark that this relation takes the form \( f_{2l+1}^2 = 1 + f_{2l}f_{2l+2} \).

Now, since \( f_{2l+1} \) and \( f_{2l+2} \) are successive iterates, they’re coprime, which implies, that \( f_{2l+1}^2 \equiv 1 \mod f_{2l+2} \).

Therefore, the sequence of pairs, \( (M_l = f_{2l+1}, N_l = f_{2l+2}) \), where \( l = 1, 2, 3, \ldots \), satisfy all of the requirements and the corresponding limiting ratio, \( L/R_{AdS_2} \), can be found analytically. It is, indeed, equal to \( \rho_+ = (1 + \sqrt{5})/2 \), the golden ratio.

We recall here that the periods of the Fibonacci sequence mod \( N \), for any integer \( N \), has been analyzed in the literature (cf. \[71\]) and, in the case when \( N \), the IR cutoff, is, itself, a Fibonacci integer, then the period of the corresponding Arnol’d cat map grows logarithmically with \( N \) and this is the reason for the saturation of the fast scrambling bound \[41\]. In the next subsection we shall consider the so-called \( k \)-Fibonacci sequences, which will be important for removing the IR cutoff.

### 4.3 Generalized k-Fibonacci sequences and UV/IR cutoffs

It’s possible to generalize the Fibonacci sequence in the following way:

\[ g_{n+1} = kg_n + g_{n-1} \]  \hspace{1cm} (4.8)

with \( g_0 = 0 \) and \( g_1 = 1 \) and \( k \) an integer. This is known as the “\( k \)-Fibonacci” sequence \[72\].

We may solve for \( g_n \equiv C\rho^n \); the characteristic equation for \( \rho \), now, reads

\[ \rho^2 - k\rho - 1 = 0 \Leftrightarrow \rho_{\pm}(k) = \frac{k \pm \sqrt{k^2 + 4}}{2} \]  \hspace{1cm} (4.9)

and express \( g_n \) as a linear combination of the \( \rho_{\pm} \):

\[ g_n = A_+\rho_+(k)^n + A_-\rho_-(k)^n = \frac{\rho_+(k)^n - (-)^n\rho_+(k)^{-n}}{\sqrt{k^2 + 4}} \]  \hspace{1cm} (4.10)

that generalizes eq. \[4.6\].
In matrix form

\[
\begin{pmatrix}
g_n \\ g_{n+1}
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} \begin{pmatrix} g_n \\ g_{n-1}
\end{pmatrix}
\]  \hspace{1cm} (4.11)

Similarly as for the usual Fibonacci sequence, we may show, by induction, that

\[
A(k)^n = \begin{pmatrix} g_{n-1} & g_n \\ g_n & g_{n+1}
\end{pmatrix}
\]  \hspace{1cm} (4.12)

We find that \( \det A(k)^n = (-1)^n \), therefore that \( g_{2l+1}^2 \equiv 1 \mod g_{2l+2} \); thus, \( g_{2l+2}/g_{2l+1} \to L/R_{\text{AdS}_2} = \rho_+(k) \), where the eigenvalue of \( A(k) \), \( \rho_+(k) \), that’s greater than 1, of course, depends on \( k \). At this point we have determined \( L \), the IR cutoff, in terms of \( R_{\text{AdS}_2} \). This limiting procedure has removed the UV cutoff, since \( a \to 0 \), however the IR cutoff, \( L \) is, still, present.

What is remarkable is that, using the additional parameter, \( k \), of the \( k \)–Fibonacci sequence, it is possible to remove the IR cutoff, as well, since it is possible to send \( L \to \infty \), as \( k \to \infty \), keeping \( R_{\text{AdS}_2} \) fixed.

If \( k \) remains finite, the periodic box cannot be removed and, in the continuum limit, \( a \to 0 \), we obtain infinitely many foldings of the \( \text{AdS}_2 \) surface inside the box due to the mod \( L \) operation.

The Fibonacci sequence, taken mod \( N \), is periodic, with period \( T(N) \); this turns out to be a “random” function of \( N \). The “shortest” periods, as has been shown by Falk and Dyson \[71\], occur when \( N = F_l \), for any \( l \). In that case, \( T(F_l) = 2l \).

We may, thus, ask the same question for the \( k \)–Fibonacci sequence, where the ratio of its successive elements, \( g_{n+1}/g_n \) tend to the so-called “\( k \)–silver ratio”,

\[
\rho_+(k) = \frac{k + \sqrt{k^2 + 4}}{2}
\]  \hspace{1cm} (4.13)

(the “silver ratio” is \( \rho_+(k = 2) \))

From eq. (4.12), taking mod \( g_l \) on both sides, we find that, when \( n = l \), the matrix becomes \( \pm \) (the identity matrix), so \( T(g_l) = l \) or \( 2l \), respectively; thereby generalizing the Falk–Dyson result for the \( k \)–Fibonacci sequences. Since, for large \( l \), \( g_l \sim e^{l \log \rho_+(k)} \), \( \log \rho_+(k) \) can be identified with the Lyapunov exponent of the dynamics of the \( A(k) \) map. What is interesting in this generalization is that \( \rho_+(k) \sim k \), so, for, large \( k \), can become significantly larger than the Lyapunov exponent of the Arnol’d cat map–so the scrambling time is significantly shorter.

5 Discussion and open issues

In this work we have proposed a construction of the continuum \( \text{AdS}_2 \) radial and time near horizon geometry of extremal black holes from a finite and arithmetic geometry, \( \text{AdS}_2[N] \), for every integer \( N \). This entails the introduction of UV and IR cutoffs, respectively \( a = R_{\text{AdS}_2}/M \) and \( L = aN \), where \( L > R_{\text{AdS}_2} \) is the size of the periodic box, that encloses the one–sheeted hyperboloid.
The periodic box and the UV cutoff deform the SL(2, \mathbb{R}) isometry of AdS$_2$ to the finite group, $SL_2(\mathbb{Z}_N)$, which is the mod $N$ reduction of SL$_2[\mathbb{Z}]$.

The elements of this finite group are discrete maps and describe the evolution operators of the avatars of infalling observers, with proper time the iteration time of the corresponding maps.

For particular maps, such as the Arnol’d cat map and its $k$–Fibonacci generalizations, we find that they are chaotic, mixing and their mixing time time saturates the scrambling time bound [26,27].

It seems today that theoretical high energy physics is passing through a paradigm shift from the elementary harmonic oscillator and relativistic quantum mechanics to the qubit of information and the measurement of complexity for discrete or continuous quantum systems. This was necessary in order to understand the gravitational force as an information measure of complex quantum systems. The prominent example is the AdS/CFT correspondence and the relation of quantum entanglement of the boundary conformal field theories with purely geometric objects of classical theories of gravitation. This holographic duality is indeed in analogy with the redundancy and the associated compression of information necessary for fast processing in complex systems.

The notion of locality in gravity is expressed in terms of the diffeomorphism invariance of the gravitational action. This implies the absence of local observables and only in the case of well defined asymptotic behavior of the metric, conformal or not, do there exist globally defined observables, that can characterize the gravitational background. In the case of the AdS/CFT correspondence, the holographic dualities are restricted by the UV/IR correspondence and locality is lost both in the boundary as well as in the bulk.

On the other hand, the present efforts to understand the near horizon region, as well as the interior and the exterior of black holes, which are asymptotically anti-de Sitter, rely exclusively on the boundary CFT point of view. This approach, however, reaches its limit when attempting to resolve features, beyond the Planck scale, where no formalism for performing reliable calculations is, to date, available.

For these reasons our program for using the arithmetic of finite geometries has an intrinsic interest as an alternative way for reconstructing bulk spacetimes, as emerging in an appropriate scaling limit thereof. Among the main advantages are:

- As shown in this paper this scaling limit is the correct one, in that the usual, continuum, AdS$_2$ geometry is recovered–this is a very important sanity check.

- The relation of finite geometries to quantum information theory and their representation as quantum circuits with measurable complexity [73,76]. It, also, provides a framework for quantitatively studying the Eigenstate Thermalization Hypothesis [77] and the fast scrambling bound [41].

- Due to the modular arithmetic, an intrinsic number theoretic randomness appears in the geometry itself, as well as in the dynamics of wave packets with finite dimensional Hilbert space [68].

In the present work we established the modular geometry AdS$_2[N]$ as a useful toy model that realizes many of the basic properties, for the near horizon geometries of extremal/near extremal black
holes, such as holography, non–locality of the geometry and the fast mixing dynamics of particle probes, that saturate the scrambling time bound.

Along the way, we discussed interesting methods to localize and count the integral points of the AdS$_2$ continuous geometry and to characterize the points of AdS$_2[N]$ as equivalence classes of the AdS$_2$ integral points modulo the congruent modular group $\Gamma[N]$. The continuous limit of the modular geometry AdS$_2[N]$ was constructed explicitly, using infinite sequences of UV/IR cutoffs $(M_n, N_n), n = 1, 2, \ldots$, taken from the integer sequences of the $k$–Fibonacci numbers.

The sequence of UV cutoffs, $N_n$ describes the dimension of the Hilbert space of states of single–particle probes and, in the case of $k$–Fibonacci sequence, $k = 1, 2, \ldots$, the dynamics of the corresponding cat maps saturates the scrambling time bound with a Lyapunov exponent that grows logarithmically with $k$.

Among the open issues of our approach we may mention:

- The extension to modular discretizations of higher dimensional AdS/CFT duals, using the corresponding arithmetic isometry groups.
- The extension to the BTZ black hole.
- Describing de Sitter spacetimes [78] using arithmetic geometry.
- Many–body probe systems and the ensuing questions related to their entanglement and the time behaviour of their OTOC bulk quantum correlators.

These issues are technically feasible and physically interesting with available tools.

We should stress that we are still far from proposing a complete framework suitable to study the fluctuations, due to quantum gravity itself, on such discrete backgrounds. It will be necessary to express the metric in information-theoretic terms, or more directly in the case of AdS$_2[N]$ through the relation of geodesic distance and the quantum mechanical coherent states [43], as well as to formulate a discrete (gravitational) action principle at Planckian scales. In our discrete time setting we should rather construct a unitary evolution operator in the space of metrics. Another possible direction to this end could be the relation of the modular with the $p$–adic AdS$_2$ geometry, [79,80] and references therein.

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