AN APPLICATION OF TOPOLOGICAL EQUIVALENCE TO
MORSE THEORY

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Abstract. In a previous paper, under the assumption that the Riemannian
metric is special, the author proved some results about the moduli spaces and
CW structures arising from Morse theory. By virtue of topological equivalence,
this paper extends those results by dropping the assumption on the metric.
In particular, we give a strong solution to the following classical question:

Does a Morse function on a compact Riemannian manifold give rise to a CW
decomposition that is homeomorphic to the manifold?

1. Introduction

In a previous paper [21], the author proved some results on moduli spaces and
CW structures arising from Morse theory in the CF case. By the CF case, we mean
the Morse function satisfies the Palais-Smale Condition (C) on a complete Hilbert-
Riemannian manifold and its critical points have finite indices (see [21, def. 2.6]).
Those results include the manifold structure of the compactified moduli spaces,
orientation formulas, and the CW structure on the underlying manifold. (See [21]
for a detailed description and a bibliography.)

Most results in [21] are based on the assumption of that the Riemannian metric
(or the negative gradient vector field) is locally trivial (see Definition 2.3). This
means the vector field has the simplest form near each critical point.

In this paper, by virtue of topological equivalence (see Definition 2.9), we shall
extend those results by dropping the above assumption provided that the Morse
function is proper. Here the underlying manifold has to be finite dimensional but
not necessarily compact.

In order to apply topological equivalence, based on the idea outlined in the paper
by Newhouse and Peixoto [10], we shall prove the following main theorem stated
in Franks’ paper [3 prop. 1.6].

Theorem A. Suppose $f$ is a Morse function on a compact manifold $M$. Sup-
pose $X$ is a negative gradient-like field for $f$ (see Definition 2.7), and $X$ satisfies
transversality (see Definition 2.6). Then there is a regular path between $X$ and $Y$
such that $Y$ is also a negative gradient-like field for $f$. More importantly, $Y$ is
locally trivial. In particular, there is a topological equivalence between $X$ and $Y$.

In Theorem A by a regular path, we mean a continuous path of negative
gradient-like vector fields in which each single vector field on the path satisfies
transversality. A precise version of Theorem A is Theorem 4.1.

Key words and phrases. Morse theory, negative gradient-like dynamical system, topological
equivalence, Moduli space, compactification, orientation formula, CW structure.
The importance of Theorem A is that it can be combined with the results of \cite{21} to give an extension of those results to more general metrics. In particular, we give a strong solution to the following classical question which had been considered by Thom (\cite{27}), Bott (\cite{2, p. 104}) and Smale (\cite{25, p. 197}): Does a Morse function on a compact Riemannian manifold give rise to a CW decomposition that is homeomorphic to the manifold such that its open cells are the unstable manifolds of the negative gradient vector field? A corollary of Theorem 9.1 gives the following answer which strengthens the work in \cite{11} and \cite{12} (see also Remark 9.1):

**Theorem B.** Suppose $f$, $M$ and $X$ are the same as those in Theorem A. Then the compactified unstable manifolds of $X$ give a CW decomposition that is homeomorphic to $M$. The open cells of this CW complex are the unstable manifolds. Furthermore, the characteristic maps have explicit formulas.

The following is the reason for making the extension of results in \cite{21}. There are at least two disadvantages of the locally trivial metric assumed in \cite{21}. Firstly, local triviality is not a generic property. Sometimes, especially in the infinite dimensional setting such as in Floer theory, it is not usually the case that one can find a metric satisfying both the local triviality and transversality conditions. Secondly, the assumption of local triviality of the metric contradicts symmetry. Take for example a homogeneous Riemannian manifold. If the metric is locally trivial, then the curvature tensor must vanish near each critical point. Since the metric is homogeneous, the curvature tensor must vanish globally. Thus only a tiny class of homogeneous Riemannian manifolds have this type of metric.

Actually, the local triviality assumption on the metric was made in \cite{21} exclusively because of the techniques employed there. The theorems in \cite{21} thm. 3.3, 3.4, and 3.5 show that, under the assumption of a locally trivial metric, the compactified moduli spaces have smooth structures compatible with that of the underlying manifold. However, the example in \cite{21} example 3.1 shows that, if the metric is not locally trivial, there is no such compatibility (see also Remark 7.2). Thus the case of a locally trivial metric has several distinct features from the general case. In fact, the proofs of \cite{21} thm. 3.7 and 3.8] rely heavily on the compatibility.

In this situation, it’s natural to pose the following strategy for obtaining results about Morse moduli spaces in the case of a general metric. As a first step, we implement the subtle and technical arguments in the special case. In the second and final step, we try to convert the general case to the special case. The paper \cite{21} completes the first step. This paper achieves the second one.

Franks’ paper \cite[prop. 1.6]{5} proposes an excellent idea to reduce the general case to the special case as follows. The proof of \cite[lem. 2]{16} claims that there exists a regular path (i.e. each single vector field on the path satisfies transversality) as the one stated in Theorem A. Since a negative gradient-like vector field satisfying transversality is structurally stable, we get the topological equivalence in Theorem A which converts the general vector field $X$ to the locally trivial $Y$. (The argument in \cite{5} also shows the power of Theorem A)

However, the proof in \cite{16} does not provide sufficient details. It’s well known that, for negative gradient-like vector fields, transversality is preserved under small $C^1$ perturbations. However, the vector fields certainly change largely in the $C^1$ topology along the above path. How can we guarantee the transversality? Franks’ paper \cite{5} refers the proof to \cite{16}, and the latter outlines the construction of the
path. Both [5] and [16] indicate that the λ-Lemma in [18] verifies the transversality. Unfortunately, none of them explain why the λ-Lemma works in this setting.

The current paper supports the above idea in [16]. Precisely, following this idea, we shall give a self-contained and detailed proof of Theorem 4.1. However, the statement of Theorem 4.1 is slightly different from that in [16] such that it becomes better in the setting of Morse theory. (Actually, the papers [16] and [5] emphasize the setting of dynamical systems. However, our argument also proves the result in [16]. See Remark 4.3.)

The main body of this paper consists of two parts. The first part, Sections 3-5, consists of preparations for the application of topological equivalence. The main theorems in it are Theorems 3.1 and 4.1, which may be of independent interest. The second part consists of the subsequent sections and gives the application of topological equivalence. Theorem 6.7 shows that the compactified moduli spaces are invariants of topological equivalence, which is the base for our application. The theorems in Sections 7-9 are extensions of those in [21].

2. Preliminaries

In this section, we give some definitions, notation and elementary results mostly used throughout the paper.

Suppose $M$ is a finite dimensional smooth manifold, and $f$ is a proper Morse function on $M$. Let $M^{a,b}$ denote $f^{-1}([a,b])$. Let $M^a$ denote $f^{-1}((−\infty,a])$.

**Definition 2.1.** A vector field $X$ is negative gradient-like for $f$ if $Xf(x) < 0$ when $x$ is not a critical point, and, near each critical point $p$, $X$ is the negative gradient of $f$ for some metric.

By Definition 2.1, every negative gradient vector field is obviously a negative gradient-like vector field. On the contrary, Smale [26, remark after thm. B] gives the following fact (see also [21, lem. 7.12]).

**Lemma 2.2.** Every negative gradient-like field of a Morse function $f$ is actually a negative gradient field of $f$ for some metric.

By the Morse Lemma, there exists a local coordinate chart near a critical point $p$ such that $p$ has the coordinate $(0,0)$, and the function has the form

\[ f(x_1, x_2) = f(p) - \frac{1}{2} \langle x_1, x_1 \rangle + \frac{1}{2} \langle x_2, x_2 \rangle \]

in this chart. We call this chart a Morse chart.

**Definition 2.3.** We say the metric of $M$ is trivial near $p$ if the metric of $M$ coincides with the standard metric of a Morse chart near $p$. In other words, in this Morse chart, $−\nabla f$ has the simplest form $−\nabla f(x_1, x_2) = (x_1, −x_2)$. Similarly, we say a negative gradient-like field $X$ is trivial near $p$ if $X(x_1, x_2) = (x_1, −x_2)$ in a Morse chart. If the metric (or $X$) is trivial near each critical point, we say this metric (or $X$) is locally trivial.

**Remark 2.1.** Some papers in the literature include the local triviality of $X$ into the definition of a gradient-like vector field. We follow the style of [26] and exclude it.

We omit the proof of the following easy lemma which can be proved by inverse function theorem.
Lemma 2.4. Denote by $\phi_t(x)$ the flow generated by $X$ with initial value $x$ and time $t$. Suppose $a$ is a regular value of $f$. Suppose $\phi_t(x_0) \in f^{-1}(a)$ for some $t$. Then there exists a neighborhood $U$ of $x_0$ such that, for all $y \in U$, $\phi_{t(y)}(y) \in f^{-1}(a)$ for a unique $t(y)$. Furthermore, $t(y)$ and $\phi_{t(y)}(y)$ are smooth with respect to $y$.

Definition 2.5. Let $\phi_t(x)$ be the flow generated by $X$ with initial value $x$ and time $t$. Suppose $p$ is a critical point. Define the descending manifold of $p$ as $D(p) = \{x \in M \mid \lim_{t \to -\infty} \phi_t(x) = p\}$. Define the ascending manifold of $p$ as $A(p) = \{x \in M \mid \lim_{t \to +\infty} \phi_t(x) = p\}$. We call $D(p)$ and $A(p)$ the invariant manifolds of $p$.

We also let $D(p; X)$ denote $D(p)$ and $A(p; X)$ denote $A(p)$ in order to indicate the vector field $X$.

Clearly, $D(p)$ is the unstable manifold of $p$ with respect to $X$, and $A(p)$ is the stable manifold. They are smoothly embedded open disks in $M$. Furthermore, $\dim(D(p)) = \ind(p)$, where $\ind(p)$ is the Morse index of $p$. (See e.g. [24, lem. 3.8])

Definition 2.6. We say that $X$ satisfies transversality if $D(p)$ and $A(q)$ are transversal for all critical points $p$ and $q$. For critical points $p$ and $q$, we say that $p$ and $q$ are transversal if the invariant manifolds of $p$ are transverse to those of $q$. Suppose $U$ is a subset of $M$, and these invariant manifolds meet transversally at each point in $U$ (this includes the case that they don’t meet at that point). We say that $p$ and $q$ are transversal in $U$.

The following lemma is obvious.

Lemma 2.7. If $p$ and $q$ are transversal in $f^{-1}((a, b))$ and $p \in f^{-1}((a, b))$, then $p$ and $q$ are transversal. If $p$ and $q$ are transversal in $f^{-1}(a)$ and $f(q) < a < f(p)$, then $p$ and $q$ are transversal.

Definition 2.8. Suppose $p$ and $q$ are critical points. Define $q \preceq p$ if there exists a flow from $p$ to $q$. Define $q \prec p$ if $q \preceq p$ and $q \neq p$.

If $X$ satisfies transversality, then “$\preceq$” is a partial order on the set consisting of all critical points (see [19, p. 85, cor. 1]).

Now we introduce the definitions of topological conjugacy and topological equivalence in dynamical systems. The reader is to be forewarned that the definitions appearing in the literature are not uniform. We follow the terminology of [19, p. 26]. In this paper, a topological conjugacy is a relation strictly stronger than a topological equivalence. This is different from the definition in [5]. The “topological conjugacy” in [5, p. 201] is actually the “topological equivalence” in this paper. Although a topological equivalence is good enough for our application to Morse theory, we still introduce the notion of topological conjugacy in order to make the statement of Theorem 2.1 stronger.

Definition 2.9. Suppose $X_i$ $(i = 1, 2)$ is a vector field on $M_i$ and $\phi^i_t$ is the flow generated by $X_i$. Suppose $h : M_1 \to M_2$ is a homeomorphism. If $h \circ \phi^1_t = \phi^2_t \circ h$, then we call $h$ a topological conjugacy between $X_1$ and $X_2$. If $h$ maps the orbits of $X_1$ to the orbits of $X_2$ and $h$ preserves the directions of orbits, then we call $h$ a topological equivalence between $X_1$ and $X_2$.

Remark 2.2. In dynamical systems, people usually consider the topological equivalence (or conjugacy) of vector fields on one manifold $M$, i.e. $M_1 = M_2$ in Definition 2.9. However, it seems beneficial for topology to allow that $M_1$ is not diffeomorphic
to $M_2$. For example, choose a standard sphere $S^n$ and a twist sphere $\Sigma^n$. Let $f_1$ and $f_2$ be the height functions on $S^n$ and $\Sigma^n$ respectively. We can define a topological conjugacy between $-\nabla f_1$ and $-\nabla f_2$ as follows. Choose a homeomorphism (or even a diffeomorphism) $h_0 : S^{n-1} \to \Sigma^{n-1}$, where $S^{n-1}$ and $\Sigma^{n-1}$ are the equators of $S^n$ and $\Sigma^n$ respectively. Define $h$ such that $h\phi^t_1(x) = \phi^t_2h_0(x)$ for all $x \in S^{n-1}$, and $h$ maps the maximum (minimum) point to the maximum (minimum) point. Clearly, this topological conjugacy $h$ recovers the Alexander trick.

### 3. A Strengthened Morse Lemma

In this section, we shall present a Strengthened Morse Lemma which is useful for the proof of Theorem 3.1 (See Remarks 4.2 and 4.3).

Suppose $H$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and $U$ is an open subset of $H$. Define a smooth Riemannian metric (or smooth metric for brevity) on $U$ in the usual sense. In other words, for each $x \in U$, assign a bounded symmetric positive definite linear operator $A(x)$ such that $A(x)$ is a smooth function of $x$. For any $v$ and $w$ in $T_xU = H$, define $\langle v, w \rangle_{G(x)} = \langle A(x)v, w \rangle$.

**Theorem 3.1** (Strengthened Morse Lemma). Suppose $H$ is a Hilbert space, $U$ is an open neighborhood of $0 \in H$. Suppose $f$ is a smooth Morse function on $U$ with a critical point 0, and $G$ is a smooth metric on $U$. Let $-\nabla f$ be the negative gradient of $f$ with respect to $G$, and $\phi_t$ be the flow generated by $-\nabla f$. Suppose $H = H_1 \oplus H_2$, where $H_1$ and $H_2$ are the negative and positive spectral spaces of $\nabla^2 f(0)$ respectively. Then there exist $\epsilon > 0$, an open neighborhood $V$ of $0$ such that $V \subseteq U$, $B_1 = \{x \in H_1 \mid \|x\| < \epsilon\}$, $B_2 = \{x \in H_2 \mid \|x\| < \epsilon\}$, and a diffeomorphism $h : B_1 \times B_2 \to V$ such that the following holds. We have

\begin{equation}
\tag{3.1}
h^*f(x_1, x_2) = f(0) - \frac{1}{2}\langle x_1, x_1 \rangle + \frac{1}{2}\langle x_2, x_2 \rangle,
\end{equation}

\[ h(B_1) = D^u_V(0; -\nabla f) = \{x \in V \mid \phi((-\infty, 0), x) \subseteq V\} = \left\{ x \in V \mid \phi((-\infty, 0), x) \subseteq V, \lim_{t \to -\infty} \phi(t, x) = 0 \right\}, \]

and

\[ h(B_2) = A^u_V(0; -\nabla f) = \{x \in V \mid \phi([0, +\infty), x) \subseteq V\} = \left\{ x \in V \mid \phi([0, +\infty), x) \subseteq V, \lim_{t \to +\infty} \phi(t, x) = 0 \right\}. \]

Before proving it, we explain the statement of Theorem 3.1. In this theorem, $D^u_V(0; -\nabla f)$ is the local unstable (descending) manifold of 0 in the neighborhood $V$, and $A^u_V(0; -\nabla f)$ is the local stable (ascending) manifold. They certainly depend on the metric. The classical Morse Lemma shows that, by a coordinate transformation $h$, we get a new chart (we call it a Morse Chart) such that the function has the form (3.1) in it. Theorem 3.1 tells us more: No matter what the metric is, there exists a Morse chart such that the local invariant manifolds are standard in it. (Figure 1 illustrates this strengthened Morse chart, where the arrows indicate the directions of the flows.) This makes three objects, i.e. the function, the local invariant manifolds, and the coordinate chart fit well. In short, Theorem 3.1 strengthens the classical Morse Lemma by taking the dynamical system into account.
Proof of Theorem 3.1. We know that \( \phi_1 \) is a smooth map defined on \( U_0 \) with a hyperbolic fixed point 0, where \( U_0 \) is a neighborhood of 0. By the Local Invariant Manifold Theorem (see \cite{7} and \cite{8, thm. 28}), shrinking \( U_0 \) suitably, there exists a diffeomorphism \( h_1 : \bar{B}_1 \times \bar{B}_2 \to U_0 \) such that

\[
h_1(\bar{B}_1) = D_{U_0}(0; \phi_1) = \{ x \in U_0 | \forall n \leq 0, (\phi_1)^n(x) \in U_0 \}
\]

\[
h_1(\bar{B}_2) = A_{U_0}(0; \phi_1), \quad h_1(0) = 0.
\]

Here the definition of \( A_{U_0}(0; \phi_1) \) is similar to that of \( D_{U_0}(0; \phi_1) \), \( \bar{B}_1 \times \bar{B}_2 \subseteq H_1 \times H_2 \), and \( \bar{B}_1 \times \bar{B}_2 \) is an open neighborhood of \((0,0)\).

Clearly, \( h_1^* f \big|_{\bar{B}_1} \) and \( h_1^* f \big|_{\bar{B}_2} \) are Morse functions on \( \bar{B}_1 \) and \( \bar{B}_2 \) respectively. By the Morse Lemma, composing \( h_1 \) with a diffeomorphism if necessary, we may assume that

\[
h_1^* f \big|_{\bar{B}_1} = f(0) - \frac{1}{2} \langle x_1, x_1 \rangle, \quad h_1^* f \big|_{\bar{B}_2} = f(0) + \frac{1}{2} \langle x_2, x_2 \rangle.
\]

Define

\[
R(x) = h_1^* f(x) - \left( f(0) - \frac{1}{2} \langle x_1, x_1 \rangle + \frac{1}{2} \langle x_2, x_2 \rangle \right).
\]

Here \( x = (x_1, x_2) \). Denote the differential of \( R \) with order \( n \) by \( D^n R \). Then \( R(x_1,0) \equiv 0 \) and \( R(0,x_2) \equiv 0 \). In addition, since \( Df(0) = 0 \), for any \( v_1 \in H_1 \) and \( v_2 \in H_2 \), we have

\[
D^2(h_1^* f)(0)(v_1, v_2) = D^2 f(Dh_1(0) \cdot v_1, Dh_1(0) \cdot v_2)
\]

\[
= \langle \nabla^2_G f(0) Dh_1(0) \cdot v_1, Dh_1(0) \cdot v_2 \rangle_{G(0)}.
\]

By the Local Invariant Manifold Theorem again, we have the tangent spaces

\[
T_0 D_{U_0}(0; \phi_1) = H_1 \quad \text{and} \quad T_0 A_{U_0}(0; \phi_1) = H_2.
\]

Therefore \( Dh_1(0) \cdot v_1 \in H_1 \) and \( Dh_1(0) \cdot v_2 \in H_2 \). By the assumption, \( \nabla^2_G f(0) \) is symmetric with respect to \( G(0) \), and \( H_1 \) and \( H_2 \) are negative and positive spectral spaces of \( \nabla^2_G f(0) \) respectively. Thus \( D^2(h_1^* f)(0)(v_1, v_2) = 0 \). We infer \( D^2 R(0)(v_1, v_2) = 0 \) and \( D^2_{1,2} R(0) = 0 \), where \( D^2_{1,2} R(0) \) is the restriction of \( D^2 R(0) \) on \( H_1 \times H_2 \).

Figure 1. Strengthened Morse Chart
Now we have
\[ R(x_1, x_2) \]
\[ = R(x_1, 0) + \int_0^1 \frac{d}{dt} R(x_1, tx_2) dt \]
\[ = \int_0^1 D_2 R(x_1, tx_2) dt \cdot x_2 \quad \text{(because } R(x_1, 0) = 0) \]
\[ = \int_0^1 \left[ D_2 R(0, tx_2) + \int_0^1 \frac{d}{ds} D_2 R(sx_1, tx_2) ds \right] dt \cdot x_2 \]
\[ = \int_0^1 \int_0^1 D^2_{1,2} R(sx_1, tx_2) ds dt(x_1, x_2) \quad \text{(because } D_2 R(0, tx_2) = 0) \]
\[ = \int_0^1 \int_0^1 \left[ D^2_{1,2} R(0, 0) + \int_0^1 \frac{d}{dt} D^2_{1,2} R(\tau sx_1, \tau tx_2) d\tau \right] ds dt(x_1, x_2) \]
\[ = \int_0^1 \int_0^1 s D^3_{1,1,2} R(\tau sx_1, \tau tx_2) d\tau ds dt(x_1, x_1, x_2) \quad \text{(because } D^2_{1,2} R(0) = 0) \]
\[ + \int_0^1 \int_0^1 t D^3_{1,2,2} R(\tau sx_1, \tau tx_2) d\tau ds dt(x_1, x_2, x_2). \]

Since \( D^3 R \) is a bounded symmetric multilinear form, there exist bounded symmetric operators \( R_1(x) \) and \( R_2(x) \) on \( H_1 \) and \( H_2 \) respectively such that, for any \( v_1 \) and \( w_1 \) in \( H_1 \),
\[ \int_0^1 \int_0^1 \int_0^1 s D^3_{1,1,2} R(\tau sx_1, \tau tx_2) d\tau ds dt(v_1, w_1, x_2) = \frac{1}{2} \langle R_1(x_1, x_2) v_1, w_1 \rangle; \]
and, for any \( v_2 \) and \( w_2 \) in \( H_2 \),
\[ \int_0^1 \int_0^1 \int_0^1 t D^3_{1,2,2} R(\tau sx_1, \tau tx_2) d\tau ds dt(v_2, w_2, x_2) = \frac{1}{2} \langle R_2(x_1, x_2) v_2, w_2 \rangle. \]

Here \( R_1(x) \) and \( R_2(x) \) are smooth with respect to \( x \).

Clearly, \( R_1(0) = 0 \), \( R_2(0) = 0 \), and
\[ h^*_x f(x) = f(0) - \frac{1}{2} \langle (I - R_1)(x) x_1, x_1 \rangle + \frac{1}{2} \langle (I - R_2)(x) x_2, x_2 \rangle. \]

Since \( I - R_1 \) and \( I + R_2 \) are symmetric, \( I - R_1(0) = I \) and \( I + R_2(0) = I \), shrinking \( B^*_1 \times B^*_2 \) if necessary, we have \( I - R_1(x) = C_1(x)^2 \) and \( I + R_2(x) = C_2(x)^2 \). Here \( C_1(x) \) and \( C_2(x) \) are bounded symmetric and positive definite operators on \( H_1 \) and \( H_2 \) respectively, and they are smooth functions of \( x \). Thus
\[ h^*_x f(x) = f(0) - \frac{1}{2} \langle C_1(x)x_1, C_1(x)x_1 \rangle + \frac{1}{2} \langle C_2(x)x_2, C_2(x)x_2 \rangle. \]

Define \( h_2 : B^*_1 \times B^*_2 \to H_1 \times H_2 \) by \( h_2(x) = (C_1(x)x_1, C_2(x)x_2) \). Then \( h_2(B^*_1) \subseteq H_1 \) and \( h_2(B^*_2) \subseteq H_2 \). Clearly, \( C_1(0) = I \) and \( C_2(0) = I \). Since
\[ Dh_2(x; v_1, v_2) = (C_1(x)v_1 + DC_1(x; v_1, v_2)x_1, C_2(x)v_2 + DC_2(x; v_1, v_2)x_2), \]
we have
\[ Dh_2(0; v_1, v_2) = (C_1(0)v_1 + DC_1(0; v_1, v_2) \cdot 0, C_2(0)v_2 + DC_2(0; v_1, v_2) \cdot 0) = (v_1, v_2) \]
and then $Dh_2(0) = I$. There exists $B_1 \times B_2 \subseteq H_1 \times H_2$ such that $h_2^{-1}$ exists and is smooth on $B_1 \times B_2$. Then we get

$$(h_1 \circ h_2^{-1})^*f(x) = f(0) - \frac{1}{2}\langle x_1, x_1 \rangle + \frac{1}{2}\langle x_2, x_2 \rangle.$$  

Define $h = h_1 \circ h_2^{-1}$ and $V = h(B_1 \times B_2)$.

We see that $h^{-1}(U_0(0; \phi_3)) = B_1$ and $h^{-1}(A_{U_0}(0; \phi_1)) = B_2$. It is straightforward to prove that $h(B_1) = D_V(0; -\nabla_G f)$ and $h(B_2) = A_V(0; -\nabla_G f)$. \hfill \Box

4. A Regular Path

As mentioned in the Introduction, the purpose of this section is to present a detailed proof of Theorem 4.1 in order to support an idea outlined in [10, lem. 2]. In this proof, Lemma 4.2 plays a key role.

Theorem 4.1 (Regular Path). Suppose $f$ is a Morse function on a compact manifold $M$. Suppose $X$ is a negative gradient-like field for $f$, and $X$ satisfies transversality. Then there is a continuous path $\gamma : [0, 1] \to \mathcal{X}^\infty(M)$ such that, for all $s \in [0, 1]$, $\gamma_s$ is a negative gradient-like field for $f$, $\gamma_s$ satisfies transversality, $\gamma_0 = X$ and $\gamma_1$ is locally trivial. In particular, there exists a topological conjugacy $h$ between $X$ and $\gamma_1$ such that $h(p) = p$ for each critical point $p$. Here $\mathcal{X}^\infty(M)$ is the set with the Whitney $C^\infty$ topology consisting of $C^\infty$ vector fields on $M$.

We call a continuous path of negative gradient-like vector fields $\gamma : [a, b] \to \mathcal{X}^\infty(M)$ a regular path if $\gamma_s$ satisfies transversality for all $s$.

We need the following classical Comparison Theorem for ODEs (see [28, p. 96]).

Theorem 4.2 (well-known). Suppose $F(t, x)$ is a Lipschitz continuous function defined on $[t_0, t_1] \times [a, b]$. Let $x(t)$ be the solution of the equation $\dot{x} = F(t, x)$ with $x(t_0) = x_0$. Then

1. if $\dot{y} \leq F(t, y)$, then $y(t) \leq x(t)$ on $[t_0, t_1]$;
2. if $\dot{y} \geq F(t, y)$, then $y(t) \geq x(t)$ on $[t_0, t_1]$.

Suppose $H = H_1 \oplus H_2$ is a Hilbert space, $v = (v_1, v_2) \in H$, $v_1 \neq 0$, and $\lambda = \|v_2\|_v / \|v_1\|$. We call $\lambda$ the inclination of $v$ with respect to $H_1$. Suppose $L$ is a closed subspace of $H$, and $P : H \to H_1$ is the projection. If $P : L \to P(L)$ is a topological linear isomorphism, then there exists a bounded linear operator $A : P(L) \to H_2$ such that $L$ is the graph of $A$, i.e., for any $v \in L$, we have $v = (v_1, Av_1)$, where $v_1 \in P(L)$. We call the supremum of the inclinations of all non-zero vectors in $L$ the inclination of $L$ with respect to $H_1$. Clearly, the inclination of $L$ equals $||A||$, the norm of $A$.

Suppose $H$, $H_1$ and $H_2$ are Hilbert spaces as above. Suppose $A_0$ and $A_1$ are bounded linear operators on $H_1$, and $B$ is a bounded linear operator on $H_2$. Suppose further there exist positive numbers $a_0 > 0$, $a_1 > 0$ and $\beta > 0$ such that

$$a_0 \langle w, w \rangle \leq \langle A_i w, w \rangle \leq a_1 \langle w, w \rangle \quad (i = 0, 1),$$

and

$$\beta \langle w, w \rangle \leq \langle B w, w \rangle.$$

Let $\rho$ be a smooth bump function on $(-\infty, +\infty)$ such that $0 \leq \rho \leq 1$, $\rho(s) \equiv 1$ when $s \leq \frac{1}{2}$, and $\rho(s) \equiv 0$ when $s \geq 1$. Define $\rho_r(s) = \rho(s^2)$ for $r > 0$. For convenience, we denote $\rho_r(||x||)$ by $\rho_r(x_i)$, where $x_i \in H_1$. 

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Similarly, \( A \) not enter \( \gamma \) (Figure 2 illustrates this. The trajectory \( X \) once and stays for no more than \( \ln 2 \)). Thus \( D\phi_t \) acts on the tangent vectors at each point \( (x_1, x_2) \), where \( D\phi_t \) is the differential of \( \phi_t \) with respect to \( x = (x_1, x_2) \).

**Lemma 4.3.** For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that the following holds. For any \( r > 0 \) and \( v \in H \), if the inclination of \( v \) with respect to \( H_1 \) is less than \( \delta \), then we have the inclination of \( D\phi_t \cdot v \) with respect to \( H_1 \) is less than \( \epsilon \) for all \( t \geq 0 \). Here \( \delta \) only depends on \( \alpha_0, \alpha_1, \beta \) and \( \epsilon \), and \( \delta \) is independent of \( r \).

**Proof.** The flow \( \phi_t = (\phi_t^1, \phi_t^2) \) satisfies the following ordinary differential equation

\[
\begin{align*}
\dot{\phi}_t^1 &= \rho_r(\phi_t^1)\rho_r(\phi_t^2)A_0 \phi_t^1 + [1 - \rho_r(\phi_t^1)\rho_r(\phi_t^2)]A_1 \phi_t^1, \\
\dot{\phi}_t^2 &= -B \phi_t^2.
\end{align*}
\]

Denote \( \rho_r(\phi_t^1)\rho_r(\phi_t^2)A_0 + [1 - \rho_r(\phi_t^1)\rho_r(\phi_t^2)]A_1 \) by \( A(\phi_t^1, \phi_t^2) \). We have

\[
\frac{d}{dt} \langle \phi_t^1, \phi_t^1 \rangle = 2\langle \dot{\phi}_t^1, \phi_t^1 \rangle = 2(A(\phi_t^1, \phi_t^2)\dot{\phi}_t^1, \phi_t^1).
\]

By (4.1), we have

\[
0 \leq 2\alpha_0 \langle \phi_t^1, \phi_t^1 \rangle \leq \frac{d}{dt} \langle \phi_t^1, \phi_t^1 \rangle \leq 2\alpha_1 \langle \phi_t^1, \phi_t^1 \rangle.
\]

Thus \( \|\phi_t^1\| \) is increasing, and by Theorem 4.2 we have

\[
e^{\alpha_0 t} \|\phi_0^1\| \leq \|\phi_t^1\| \leq e^{\alpha_1 t} \|\phi_0^1\|.
\]

Similarly, \( \|\phi_t^2\| \) is decreasing, \( \phi_t^2 = e^{-Bt} \phi_0^2 \), and \( \|\phi_t^2\| \leq e^{-\beta t} \|\phi_0^2\| \).

Let \( D_1(r) = \{x_1 \in H_1 \mid \|x_1\| < r\} \), and \( D_2(r) = \{x_2 \in H_2 \mid \|x_2\| < r\} \). Clearly, \( A(x_1, x_2)|_{H_1 \cap \overline{D_1(r) \times D_2(r)}} = A_1 \), and \( A(x_1, x_2)|_{\overline{D_1(r) \times D_2(r)}} = A_0 \). Denote \( D_1(r) \times \overline{D_2(r)} - (D_1(\frac{r}{2}) \times D_2(\frac{r}{2})) \) by \( E(r) \). (This is illustrated by Figure 2.) The shadowed part is \( E(r) \). The arrows indicate the directions of flows.) When \( \phi([0,t], x) \) is out of \( E(r) \), we have \( \phi(t, x) = (e^{A_1 t} x_1, e^{-Bt} x_2) \), and

\[
D\phi_t = \begin{pmatrix}
e^{A_1 t} & 0 \\
0 & e^{-Bt}
\end{pmatrix}.
\]

Since \( e^{A_1 t} w \geq \|w\| \) and \( e^{-Bt} w \leq \|w\| \) for \( t \geq 0 \), we have that the inclination of \( D\phi_t \cdot v \) is decreasing when \( t \) is increasing. Thus it suffices to control the variation of the inclination when \( \phi_t(x) \) passes through \( E(r) \).

Suppose \( t \geq 0 \) and \( \|\phi_t^1\| = 2\|\phi_0^1\| \), then by (4.4), we have \( t \leq \frac{\ln 2}{\alpha_0} \). Similarly, if \( \|\phi_t^2\| = \frac{1}{2} \|\phi_0^2\| \), then \( t \leq \frac{\ln 2}{\beta} \). Since \( \|\phi_t^1\| \) is increasing and \( \|\phi_t^2\| \) is decreasing, we infer that \( \phi_t \) enters \( E(r) \) at most twice, and the time for it to stay in \( E(r) \) is no more than

\[
T = \frac{\ln 2}{\alpha_0} + \frac{\ln 2}{\beta}.
\]

(Figure 2 illustrates this. The trajectory \( \gamma_1 \) and the constant trajectory at 0 do not enter \( E(r) \). The trajectory \( \gamma_2 \) is in \( H_1 \). Both \( \gamma_3 \) and \( \gamma_4 \) enter \( E(r) \) once, and the time for them to stay in \( E(r) \) is no more than \( \frac{\ln 2}{\alpha_0} \). The trajectory \( \gamma_3 \) is in \( H_2 \), it enters \( E(r) \) once and stays for no more than \( \frac{\ln 2}{\beta} \). The trajectory \( \gamma_5 \) enters \( E(r) \) once and stays for no more than \( \frac{\ln 2}{\alpha_0} + \frac{\ln 2}{\beta} \). The trajectory \( \gamma_6 \) enters \( E(r) \) twice,
the first time for it to stay is no more than \(\frac{\ln 2}{\beta}\), and the second time is no more than \(\frac{\ln 2}{\alpha_{0}}\).

Suppose \(\phi([0, t], x) \subset E(r)\), we have \(0 \leq t \leq T\). Since \(\phi_{2}^{T}(x) = e^{-Bt}x_{2}\), we have

\[
D_{1}\dot{\phi_{1}}^{1} = 0, \quad D_{2}\phi_{2}^{2} = e^{-Bt}, \quad \text{and} \quad \|D_{2}\phi_{2}^{2} \cdot w\| \leq \|w\|.
\]

Since

\[
\dot{\phi}_{1}^{1} = A(\phi_{1}^{1}, e^{-Bt}x_{2})\phi_{1}^{1},
\]

we have

\[
D_{1}\dot{\phi}_{1}^{1} \cdot w = A(\phi_{1}^{1}, \phi_{2}^{2})(D_{1}\phi_{1}^{1} \cdot w) + D_{\rho_{r}}(\phi_{1}^{1})(D_{1}\phi_{1}^{1} \cdot w)\rho_{r}(\phi_{2}^{2})(A_{0} - A_{1})\phi_{1}^{1}.
\]

Thus

\[
\frac{d}{dt}(D_{1}\phi_{1}^{1} \cdot w, D_{1}\phi_{1}^{1} \cdot w) = 2(D_{1}\dot{\phi}_{1}^{1} \cdot w, D_{1}\phi_{1}^{1} \cdot w) = 2(A(\phi_{1}^{1}, \phi_{2}^{2})(D_{1}\phi_{1}^{1} \cdot w), D_{1}\phi_{1}^{1} \cdot w) + 2(D_{\rho_{r}}(\phi_{1}^{1})(D_{1}\phi_{1}^{1} \cdot w)\rho_{r}(\phi_{2}^{2})(A_{0} - A_{1})\phi_{1}^{1}, D_{1}\phi_{1}^{1} \cdot w).
\]

Clearly, \(D_{\rho_{r}}(\phi_{1}^{1}) = O(r^{-1})\), and \(\|\phi_{1}^{1}\| \leq r\) when \(D_{\rho_{r}}(\phi_{1}^{1}) \neq 0\). So there exists a constant \(C_{1} > 0\) which is independent of \(r\) such that

\[
|\langle D_{\rho_{r}}(\phi_{1}^{1})(D_{1}\phi_{1}^{1} \cdot w)\rho_{r}(\phi_{2}^{2})(A_{0} - A_{1})\phi_{1}^{1}, D_{1}\phi_{1}^{1} \cdot w\rangle| \leq C_{1}\|D_{1}\phi_{1}^{1} \cdot w\|^{2}.
\]

Combining the above inequality with (4.1), we get

\[
-2C_{1}(D_{1}\phi_{1}^{1} \cdot w, D_{1}\phi_{1}^{1} \cdot w) \leq \frac{d}{dt}(D_{1}\phi_{1}^{1} \cdot w, D_{1}\phi_{1}^{1} \cdot w).
\]

Since \(D_{1}\phi_{0}^{1} = I\) and \(\|D_{1}\phi_{1}^{1} \cdot w\| = \|w\|\), by Theorem 4.2 we have

\[
(4.7) \quad \|D_{1}\phi_{1}^{1} \cdot w\| \geq e^{-C_{1}t}\|w\| \geq e^{-C_{1}T}\|w\|.
\]
Similarly, we have
\[
\frac{d}{dt} \langle D_2 \phi_t^1 \cdot w, D_2 \phi_t^1 \cdot w \rangle = 2 \langle A(\phi_t^1, \phi_t^2)(D_2 \phi_t^1 \cdot w), D_2 \phi_t^1 \cdot w \rangle 
+ 2 \langle D_\rho(\phi_t^1)(D_2 \phi_t^1 \cdot w) \rho_r(\phi_t^2)(A_0 - A_1)\phi_t^1, D_2 \phi_t^1 \cdot w \rangle
\]
\[
+ 2 \langle \rho_r(\phi_t^1)D_\rho(\phi_t^2)e^{-Br}w(A_0 - A_1)\phi_t^1, D_2 \phi_t^1 \cdot w \rangle,
\]
and
\[
|\langle D_\rho(\phi_t^1)(D_2 \phi_t^1 \cdot w) \rho_r(\phi_t^2)(A_0 - A_1)\phi_t^1, D_2 \phi_t^1 \cdot w \rangle| \leq C\|D_2 \phi_t^1 \cdot w\|^2.
\]
In addition, \( \rho_r(\phi_t^1)D_\rho(\phi_t^2) = O(r^{-1}) \), and \( \|\phi_t^2\| \leq r \) when \( \rho_r(\phi_t^1)D_\rho(\phi_t^2) \neq 0 \). So there exists \( C_2 > 0 \) which is independent of \( r \) such that
\[
2\langle \rho_r(\phi_t^1)D_\rho(\phi_t^2)e^{-Br}w(A_0 - A_1)\phi_t^1, D_2 \phi_t^1 \cdot w \rangle
\]
\[
\leq 2C_2\|D_2 \phi_t^1 \cdot w\|\|w\| \leq C_2\|D_2 \phi_t^1 \cdot w\|^2 + C_2\|w\|^2.
\]
Thus by \((4.1)\), we infer
\[
\frac{d}{dt} \langle D_2 \phi_t^1 \cdot w, D_2 \phi_t^1 \cdot w \rangle \leq (2\alpha_1 + 2C_1 + C_2)\|D_2 \phi_t^1 \cdot w, D_2 \phi_t^1 \cdot w \| + C_2\|w\|^2.
\]
Since \( \|D_2 \phi_t^1 \cdot w\| = 0 \), by Theorem \(4.2\) again, there exists \( C_3 > 0 \) which is independent of \( r \) such that
\[
\|D_2 \phi_t^1 \cdot w\| \leq \left[ \frac{C_2}{C_3(e^{C_3T} - 1)} \right]^\frac{1}{2} \|w\|.
\]
By \((4.5), (4.7)\) and \((4.8)\), there exist \( K_1 > 0 \) and \( K_2 > 0 \), which are independent of \( r \), such that
\[
\|D_1 \phi_t^1 \cdot w\| \geq K_1\|w\|, \quad \text{and} \quad \|D_2 \phi_t^1 \cdot w\| \leq K_2\|w\|.
\]
Suppose \( v = (v_1, v_2) \in H_1 \oplus H_2 \), and its inclination is \( \lambda_0 = \frac{\|v_2\|}{\|v_1\|} \). By \((4.6)\) and \((4.9)\), we have the inclination of \( D\phi_t \cdot v \) is
\[
\lambda_1 = \frac{\|D_2 \phi_t^2 \cdot v_2\|}{\|D_1 \phi_t^1 \cdot v_1 + D_2 \phi_t^1 \cdot v_2\|} \leq \frac{\|D_2 \phi_t^2 \cdot v_2\|}{\|v_2\|} \leq \frac{\lambda_0}{K_1\|v_1\| - K_2\|v_2\|} = \frac{\lambda_0}{K_1 - K_2\lambda_0}.
\]
Thus \( \lambda_1 \) tends to 0 when \( \lambda_0 \) tends to 0.

We know that \( \phi_t(x) \) enters \( E(r) \) at most twice, and \( K_1 \) and \( K_2 \) are independent of \( r \). Thus, for an \( \epsilon \) in the statement of this lemma, we can find the desired \( \delta \) which is independent of \( r \).

**Remark 4.1.** Our Lemma \(4.3\) is similar to the classical \( \lambda \)-Lemma (\(19\) chap. 2, lem. 7.1 & 7.2) as both are to control the inclinations of tangent vectors. However, there is one remarkable difference between them: Lemma \(4.3\) deals with a family of vector fields \((4.3)\) parameterized by \( r \) and the desired \( \delta \) is independent of \( r \).

The following definition of filtration is a special case of that in hyperbolic dynamical systems (see \(17\) p. 1029)).

**Definition 4.4.** A compact submanifold \( M_1 \) with boundary inside \( M \) is a filtration for \( X \) if \( \dim(M_1) = \dim(M) \), \( \phi_t(M_1) \subseteq \text{Int}M_1 \) for \( t > 0 \), and \( X \) is transverse to \( \partial M_1 \). Here \( \text{Int}M_1 \) is the interior of \( M_1 \), and \( \phi_t \) is the flow generated by \( X \).
Lemma 4.5. Suppose $X$ satisfies transversality. If $p$ and $q$ are critical points such that $p \nleq q$, then there exists a filtration $M_1$ for $X$ such that $p \in M - M_1$ and $q \in \text{Int}M_1$.

Lemma 4.5 can be proved as follows. The transversality implies “$\leq$” is a partial order. We have $p \nleq q_1$ if $q_1 \preceq q$. Using [13] thm. 4.1 repeatedly, we can modify $f$ to be a Morse function $g$ such that $X$ is a negative gradient-like field for $g$ and $g(q) < g(p)$. The proof is finished.

By Definition 2.1 we have the following obvious lemma.

Lemma 4.6. Suppose $X_1$ and $X_2$ are negative gradient-like fields of $f$. Suppose $\sigma_1(x)$ and $\sigma_2(x)$ are nonnegative smooth functions on $M$ such that $\sigma_1 + \sigma_2 > 0$. Then $\sigma_1X_1 + \sigma_2X_2$ is also a negative gradient-like field for $f$.

Let $p$ be a critical point. Suppose there exists a Morse chart near $p$ (see (2.1)), and $X(x_1,x_2) = (Ax_1, -Bx_2)$, where $A$ and $B$ are symmetric positive definite linear operators. Similarly to Lemma 4.3 define

$$\mathcal{Y}_r(x_1,x_2) = (\rho_r(x_1)x_1 + [1 - \rho_r(x_1)^2]Ax_1 - Bx_2)$$

in this Morse chart and $\mathcal{Y}_r = X$ out of this Morse chart. For $s \in [0,1]$, define

$$\mathcal{Y}_{r,s} = (1 - s)X + s\mathcal{Y}_r.$$

By Lemma 4.6 for all $s \in [0,1]$, $\mathcal{Y}_{r,s}$ is a negative gradient-like field for $f$.

Lemma 4.7. Suppose in addition $X$ satisfies transversality. Then when $r$ is small enough, we have the following conclusion.

Suppose $q_1$ and $q_2$ are two critical points which are not of the following two cases: (1) $q_2 \prec p \preceq q_1$; or (2) $q_1 \prec p \preceq q_2$. Then we have that $q_1$ and $q_2$ are transversal with respect to $\mathcal{Y}_{r,s}$ for all $s \in [0,1]$. Here “$\prec$” is defined with respect to $X$.

Proof. Clearly, $\mathcal{Y}_{r,s}$ differs from $X$ only in a neighborhood $U_r$ of $p$. When $r$ tends to 0, $U_r$ shrinks to $p$.

We may assume that $f(q) \neq f(p)$ for any critical point $q$ such that $q \neq p$. If this is not true, perturb $f$ to be a Morse function $\tilde{f}$ such that $X$ is a negative gradient-like field for $\tilde{f}$, and $\tilde{f}(x) = f(x) + C$ in a neighborhood $U$ of $p$. Let $r$ be small enough such that $U_r \subseteq U$. Then $\mathcal{Y}_{r,s}$ is also a negative gradient-like field for $\tilde{f}$. For the rest of the proof we make the above assumption.

Suppose $U_r \subseteq M^{a,b}$ and $p$ is the unique singularity in $M^{a,b}$. As in Definition 2.5 we use notation $D(\ast; \ast)$ and $A(\ast; \ast)$ to indicate the vector fields.

It’s easy to see that $D(p; \mathcal{Y}_{r,s}) \cap U_r = D(p; X) \cap U_r$. Since $\mathcal{Y}_{r,s}$ is identical to $X$ outside $U_r$, we have $D(p; \mathcal{Y}_{r,s}) = D(p; X)$. Suppose that $q \in M^a$. Since $\mathcal{Y}_{r,s}$ is identical to $X$ in $M - M^{a,b}$, we have $A(q; \mathcal{Y}_{r,s}) \cap M^a = A(q; X) \cap M^a$. Since $X$ satisfies transversality, we infer that $p$ and $q$ are transversal in $M^a$ with respect to $\mathcal{Y}_{r,s}$. By Lemma 2.7, $p$ and $q$ are transversal globally. Similarly, if $q \in M - M^a$, $p$ and $q$ are also transversal. As a result, $p$ and $q$ are transversal.

By the discussion above, it remains to check the case that $q_1 \neq p$ and $q_2 \neq p$.

If $p \neq q$, by Lemma 4.5 there exists a filtration $M_1$ such that $q \in \text{Int}M_1$ and $p \in M - M_1$. Let $r$ be small enough such that $U_r \subseteq M - M_1$, then $\mathcal{Y}_{r,s}$ is identical to $X$ on $M_1$. So $D(q; \mathcal{Y}_{r,s}) = D(q; X)$. Similarly, if $q \neq p$, we can get $A(q; \mathcal{Y}_{r,s}) = A(q; X)$ when $r$ is small enough. Thus there exists $r_0 > 0$ such that the following holds. When $r < r_0$, we have, for all $s \in [0,1]$, $D(q; \mathcal{Y}_{r,s}) = D(q; X)$ if $p \neq q$, and $A(q; \mathcal{Y}_{r,s}) = A(q; X)$ if $q \neq p$. 


In order to complete this proof, we only need to check the following three cases.

1. Case 1: $q_1$ and $q_2$ are in $M^a$.
   Since $\mathcal{Y}_{r,s}$ is identical to $X$ on $M^a$ and $X$ satisfies transversality, we have $q_1$ and $q_2$ are transversal in $M^a$. By Lemma 2.7, they are transversal globally with respect to $\mathcal{Y}_{r,s}$.

2. Case 2: $q_1$ and $q_2$ are in $M - M^a$.
   By our assumption that $p$ is the unique critical point in $M^{a,b}$, we infer that $q_1$ and $q_2$ are actually in $M - M^b$. Similarly to Case (1), this case is also true.

3. Case 3: one of $q_1$ and $q_2$ is in $M - M^a$ and the other one is in $M^a$.
   We may presume $q_1 \in M - M^a$ and $q_2 \in M^a$. Then actually $q_1 \in M - M^b$. By the assumption of this lemma, we have either $p \not\in q_1$ or $q_2 \not\in p$. Suppose $p \not\in q_1$. We have $\mathcal{D}(q_1; \mathcal{Y}_{r,s}) = \mathcal{D}(q_1; X)$. Since $\mathcal{Y}_{r,s} = X$ on $M^a$ and $X$ satisfies transversality, we have $q_1$ and $q_2$ are transversal in $M^a$ with respect to $\mathcal{Y}_{r,s}$. By Lemma 2.7, they are transversal globally. Similarly, if $q_2 \not\in p$, this is also true. Thus Case 3 is also verified.

Since there are only finitely many critical points, we can find $r_0 > 0$ such that all $r < r_0$ satisfy the conclusion of this lemma.

We shall strengthen Lemma 4.7 to get the transversality of $\mathcal{Y}_{r,s}$. Recall a classical result on transversality at first.

Suppose $U$ is a neighborhood of $p$ such that $U$ is identified with a neighborhood of 0 in $T_pM = H_1 \oplus H_2$, and $p$ is identified with 0, where $H_1 = T_pA(p; X)$ and $H_2 = T_pA(p; X)$. Furthermore, suppose $\mathcal{D}(p; X) \cap U \subseteq H_1$ and $\mathcal{A}(p; X) \cap U \subseteq H_2$.

Then we have the following crucial fact: When $U$ is small enough, there exists $\Lambda > 0$ such that for any $q_1 \preceq p$ and any $x \in \mathcal{D}(q_1; X) \cap U$, there exists a linear space $V_z^d \subseteq T_x\mathcal{D}(q_1; X)$ such that $\dim(V_z^d) = \dim(H_1)$ and the inclination of $V_z^d$ with respect to $H_1$ is less than $\Lambda$. Similarly, for any $q_2 \succeq p$ and any $x \in \mathcal{A}(q_2; X) \cap U$, there exists $V_x^a \subseteq T_x\mathcal{A}(q_2; X)$ such that $\dim(V_x^a) = \dim(H_2)$ and the inclination of $V_x^a$ with respect to $H_2$ is also less than $\Lambda$. In addition, $\Lambda$ tends to 0 when $U$ shrinks to $p$. This fact follows from the transversality of $X$ and the estimate of the $\lambda$-Lemma. (Note: the $\lambda$-Lemma is also named the Inclination Lemma.)

On the contrary, we assume this fact holds but do not assume the transversality of $X$. If $\Lambda < 1$, then, for any $x \in \mathcal{D}(q_1; X) \cap \mathcal{A}(q_2; X) \cap U$, we have

$$T_xM = H_1 \oplus H_2 = V_z^d \oplus V_x^a = T_x\mathcal{D}(q_1; X) + T_x\mathcal{A}(q_2; X).$$

So we infer that $\mathcal{D}(q_1; X)$ and $\mathcal{A}(q_2; X)$ are transversal in $U$. The above argument is the key part of the proof of that, for Morse-Smale dynamical systems, transversality is preserved under small $C^1$ perturbations. All of these are addressed in [13] lem. 1.11 and thm. 3.5]. In the proof of the following lemma, we shall apply a similar argument to large $C^1$ perturbations of $X$.

**Lemma 4.8.** Suppose $X$ in addition satisfies transversality. When $r$ is small enough, we have $\mathcal{Y}_{r,s}$ satisfies transversality for all $s \in [0, 1]$.

**Proof.** By Lemma 4.7, it suffices to prove that $\mathcal{D}(q_1; \mathcal{Y}_{r,s})$ is transverse to $\mathcal{A}(q_2; \mathcal{Y}_{r,s})$ if $q_2 \prec p \prec q_1$.

Similarly to the proof of Lemma 4.7, we assume that $p$ is the unique critical point in $M^{f(p)-\varepsilon, f(p)+\varepsilon}$. Let $U$ be the neighborhood of $p$ in the argument before this lemma. Let $D$ be an open subset of $f^{-1}(f(p) + \varepsilon) \cap U$ such that $D \supseteq f^{-1}(f(p) + \varepsilon) \cap \mathcal{A}(p; X)$. Let $U_0 = [\phi([0, +\infty), D) \cup \mathcal{D}(p; X)] \cap M^{f(p)-\varepsilon, f(p)+\varepsilon}$. Then $U_0$ is a
neighborhood of \( p \) and is relatively open in \( M^{f(p)−\epsilon,f(p)+\epsilon} \). When \( \epsilon \) tends to 0 and \( D \) shrinks, \( U_0 \) shrinks to \( p \). (In Figure 3, the shadowed part is \( U_0 \), the arrows indicate the directions of the flows.) Denote the flow generated by \( \mathcal{Y}_{r,s} \) by \( \phi_{t,r,s}^* \).

By Lemma 4.3, there exists \( \delta > 0 \) for some \( \phi \) orbit of \( \phi \). Let \( r \) be small enough such that \( \mathcal{Y}_{r,s} \) is identical to \( X \) out of \( U_1 \). We infer that, for any \( z \in M^{f(p)−\epsilon,f(p)+\epsilon} − U_1 \), the orbit of \( \phi_{t,r,s}^*(z) \) also has no intersection with \( U_1 \). Let \( r \) be small enough such that the inclination of \( V \) and \( A_X \) is less than 1. It’s necessary to point out that \( \delta \) is independent of \( r \) and \( s \).

By Lemma 4.3, there exists \( \delta > 0 \) such that the following holds. Suppose \( x \in D(q_1;X) ∩ f^{-1}(f(p)+\epsilon) ∩ U_0 \), and \( V_z^d = T_z\mathcal{D}(q_1;X) \) is the space described before this lemma. If the inclination of \( V_z^d \) with respect to \( H_1 \) is less than \( \delta \), then, in \( U_0 \), the inclination of \( D\phi_{t,r,s}^*(V_z^d) \) with respect to \( H_1 \) is less than 1. It’s necessary to point out that \( \delta \) is independent of \( r \) and \( s \).

Clearly, \( D(q_1;X) ∩ f^{-1}([f(p)+\epsilon, +\infty)) = D(q_1;\mathcal{Y}_{r,s}) ∩ f^{-1}([f(p)+\epsilon, +\infty)) \) and \( A(q_2;X) ∩ M^{f(p)−\epsilon} = A(q_2;\mathcal{Y}_{r,s}) ∩ M^{f(p)−\epsilon} \). Since \( X \) satisfies transversality, by the argument before this lemma, we can choose \( U_0 \) be small enough such that the following holds. For any \( x \in D(q_1;X) ∩ f^{-1}(f(p)+\epsilon) ∩ U_0 \), the inclination of \( V_z^d \) with respect to \( H_1 \) is less than \( \delta \), and, for any \( y \in A(q_2;X) ∩ f^{-1}(f(p)−\epsilon) ∩ U_0 \), the inclination of \( V_y^a \) with respect to \( H_2 \) is less than 1. Here \( V_z^d = T_z\mathcal{D}(q_1;X) = T_z\mathcal{D}(q_1;\mathcal{Y}_{r,s}) \) and \( V_y^a = T_yA(q_2;X) = T_yA(q_2;\mathcal{Y}_{r,s}) \). Thus, if \( \phi_{t,r,s}^*(x) = y \), then

![Figure 3. Neighborhood of \( U_0 \)](image-url)
the inclination of $V_y^d = D\phi_t^g \cdot V_y^d$ with respect to $H_1$ is less than 1. Here $V_y^d \subseteq T_yD(q_1; y_{r,s})$. By the argument before this lemma again, we have $T_yM = V_y^d \oplus V_y^g$. So $D(q_1; y_{r,s})$ and $A(q_2; y_{r,s})$ are transversal in $f^{-1}(f(p) - \epsilon) \cap U_0$.

Furthermore, $D(q_1; X) \cap (M^{f(p)-\epsilon,f(p)+\epsilon} - U_1) = D(q_1; y_{r,s}) \cap (M^{f(p)-\epsilon,f(p)+\epsilon} - U_1)$ and $A(q_2; X) \cap (M^{f(p)-\epsilon,f(p)+\epsilon} - U_1) = A(q_2; y_{r,s}) \cap (M^{f(p)-\epsilon,f(p)+\epsilon} - U_1)$. Thus $D(q_1; y_{r,s})$ and $A(q_2; y_{r,s})$ are transversal in $M^{f(p)-\epsilon,f(p)+\epsilon} - U_1$.

In summary, $D(q_1; y_{r,s})$ and $A(q_2; y_{r,s})$ are transversal in $f^{-1}(f(p) - \epsilon)$. By Lemma 2.7, they are transversal globally.

Since there are only finitely many critical points, we can find $r_0 > 0$ such that all $r < r_0$ satisfy the conclusion of this lemma.

Proof of Theorem 4.4 First, we construct the regular path. Since the number of critical points is finite, it suffices to prove that, for any critical point $p$, we can construct a regular path $Y$ such that $Y_0 = X$ and $Y_1$ is locally trivial at $p$.

By Theorem 3.1, there exists a coordinate chart $U$ near $p$ such that $p$ has coordinate $(0, 0)$,

$$f(x_1, x_2) = f(p) - \frac{1}{2}(x_1, x_1) + \frac{1}{2}(x_2, x_2),$$

$D(p, X) \cap U = \{(x_1, 0)\}$ and $A(p, X) \cap U = \{(0, x_2)\}$. We immediately see that $DX(p)$ is a diagonal

$$(4.11) \quad DX(p) = \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix},$$

where $A$ and $B$ are symmetric and positive definite. By (4.11), the vector field $(Ax_1, -Bx_2)$ is the linearization of $X$ at $p$. By (4.10), it is also negative gradient-like for $f$ near $p$.

Let $\rho_r$ be the bump function defined before. For convenience, for all $x = (x_1, x_2)$, let $\rho_r(x)$ denote $\rho_r(\|x\|)$. Let $R(x) = X(x) - (Ax_1, -Bx_2)$. Then we have $\|\rho_r(x)R(x)\|$ and $\|D(\rho_r(x)R(x))\|$ tend to 0 when $r$ tends to 0. Since the transversality of $X$ is preserved under small $C^1$ perturbations, we have $Z_1 = X - s\rho_s R$ is a regular path when $r$ is small enough and $s \in [0, 1]$. By Lemma 4.6, $Z_1$ and then each $Z_s = (1 - s)X + s Z_1$ are negative gradient-like for $f$. Since $Z_1(x) = (Ax_1, -Bx_2)$ near $p$, by Lemma 4.8, we can construct a regular path $Z_{[1,2]}$ such that $Z_2(x) = (x_1, -Bx_2)$ near $p$. Since $-Z_2$ is a negative gradient-like field for $-f$, using Lemma 4.8 again, we can construct a regular path $-Z_{[2,3]}$ for $-f$ such that $-Z_3(x) = (-x_1, x_2)$ near $p$. We get the desired path by defining $Y_s = Z_{3s}$.

Second, we prove the existence of the conjugacy $h$.

By the proof in [20] thm. 5.2, we know that, for each $Y_{s_0}$, there is a topological equivalence $h_{s_0}$ between $Y_{s_0}$ and $Y_s$ such that $h_{s_0}(p) = p$ for all critical points $p$ when $s$ is close to $s_0$ enough. In addition, since the flow generated by $Y_{s_0}$ has no closed orbits, by the comment in [20] p. 231, we know that $h_{s_0}$ is actually a conjugacy. Thus it’s easy to get the desired conjugacy $h$.

Remark 4.2 In order to guarantee that the path $Y$ in Theorem 4.1 is negative gradient-like for $f$, we need to find a Morse chart satisfying both (4.10) and (4.11). Theorem 3.1 trivially yields this chart. (Actually, Theorem 3.1 provides us more than what we actually need.) It’s necessary to point out that a general Morse chart does not necessarily satisfy (4.11) because $DX(p)$ depends on the metric.
Thus the usual Morse Lemma is not sufficient for us. This special Morse chart may be constructed without using Theorem 3.1. Nevertheless, we present this theorem because it may be of independent interest.

Remark 4.3. The regular path in [16 lem. 2] consists of the Morse-Smale vector fields without closed orbits. In this case, \( DX(p) = \text{diag}(A, -B) \) for singularities \( p \), where \( A \) and \( B \) are linear isomorphisms whose eigenvalues have positive real parts. The paper [16] claims that there exists a regular path connecting \( X \) with \( Y \) such that \( Y(x_1, x_2) = (2x_1, -2x_2) \) near each singularity. Thus, in the setting of dynamical systems, this result is more general than Theorem 4.1. However, Theorem 4.1 has the advantage that its vector fields are negative gradient-like for \( f \). Furthermore, the argument in this paper can also be used to verify the result in [16]. This is because we can choose a metric near each critical point, for example, by the real Jordan canonical form, such that the above operators \( A \) and \( B \) satisfy (4.1) and (4.2).

5. A Reduction Lemma

In this paper, we shall prove theorems for noncompact manifolds with proper Morse functions. However, the manifold in Theorem 4.1 is required to be compact. (As we have seen, the proof of Theorem 4.1 heavily relies on the compactness of the manifold.) The following lemma reduces the proper case to the compact case.

**Lemma 5.1.** Suppose \( M \) is a compact manifold with boundary \( \partial M = M_1 \sqcup M_2 \). Here \( M_i \) (\( i = 1, 2 \)) may be empty. Suppose \( f \) is a Morse function on \( M \) such that \( f|_{M_1} \equiv a, f|_{M_2} \equiv b \), \( a \) and \( b \) are regular values of \( f \), and \( a < b \). Suppose \( X \) is a negative gradient-like vector field for \( f \), and \( X \) satisfies transversality. Then there exist a compact manifold \( \bar{M} \) without boundary and a smooth embedding \( i : M \hookrightarrow \bar{M} \) such that the following holds. There exist a Morse function \( \bar{f} \) and its negative gradient-like vector field \( \bar{X} \) on \( \bar{M} \). They are extensions of \( f \) and \( X \) respectively, and \( \bar{X} \) satisfies transversality. For any critical points \( p \) and \( q \) in \( M \), we have \( D(p; \bar{X}) \cap A(q; \bar{X}) = D(p; X) \cap A(q; X) \). Furthermore, \( D(p; \bar{X}) = D(p; X) \) and \( \bar{f}|_{\bar{M} - M} > b \) if \( M_1 = \emptyset \); and \( A(p; \bar{X}) = A(p; X) \) and \( \bar{f}|_{\bar{M} - M} < a \) if \( M_2 = \emptyset \).

The proof of Lemma 5.1 is based on Milnor’s sliding invariant (descending or ascending) manifolds in [14 thm. 5.2]. Suppose \( g \) is a Morse function on a compact manifold and \( \xi \) is a negative gradient-like field for \( g \). Basically, there are two ways of modifying \( \xi \) to get transversality. Method 1 is sliding the descending manifolds one by one with the order from critical points with lower values to those with higher values. In this case, one repeats using [14 thm. 5.2] to slide each \( D(p) \) such that \( D(p) \) becomes transverse to all \( A(q) \) if \( g(q) < g(p) \). On the contrary, Method 2 is sliding the ascending manifolds one by one with the order from critical points with higher values to those with lower values. A key point is that one only needs to change the filed \( \xi \) in an arbitrarily small neighborhood of \( p \) when sliding the invariant manifolds of \( p \). Our method is a combination of the above two methods.

**Proof of Lemma 5.1.** If \( \partial M = \emptyset \), let \( \bar{M} = M \), the proof is finished. Now we assume \( \partial M \neq \emptyset \).

Let \( \tilde{M} \) be the double of \( M \). Extend \( f \) to be a Morse function \( \tilde{f} \) such that \( a \) and \( b \) are regular values of \( \tilde{f} \), and extend \( X \) to be \( \tilde{X} \) which is a negative gradient-like field for \( \tilde{f} \). (Figure 4 illustrates the manifold \( \tilde{M} \), where the Morse function is the height.
function and the shadowed part is $M$.) We shall modify $\tilde{X}$ such that it satisfies transversality.

In this proof, we say two critical points $\tilde{p}$ and $\tilde{q}$ of $\tilde{f}$ are transversal if they are transversal with respect to $\tilde{X}$.

Step 1: we show the transversality between $p \in M$ and $q \in M$. Since $D(p; \tilde{X}) \subseteq M \cup \text{Int} \tilde{M}^a$, we have $D(p; \tilde{X}) \cap \tilde{M}^{a,b} = D(p; \tilde{X}) \cap M = D(p; X)$. Similarly, $A(p; \tilde{X}) \cap \tilde{M}^{a,b} = A(p; X)$. Since $X$ satisfies transversality, $p$ and $q$ are transversal in $\tilde{M}^{a,b}$. By Lemma 2.7, they are transversal globally. This shows the transversality between $p$ and $q$ does not depend on the extension of $X$. So, no matter how $\tilde{X}$ is changed outside of $M$, $p$ and $q$ are always transversal if they are in $M$.

Step 2: we modify $\tilde{X}$ in $\tilde{M}^a$. We make modifications near each critical point $\tilde{p}$ in $\tilde{M}^a$ with the order from critical points with higher values to those with lower values. Slide $A(\tilde{p}; \tilde{X})$ for each $\tilde{p} \in \tilde{M}^a$ such that $\tilde{p}$ is transverse to each $\tilde{q} \in M \cup M^a$ with $\tilde{f}(\tilde{q}) < \tilde{f}(\tilde{p})$. (Here, for all $\tilde{q} \in M$, we have $\tilde{f}(\tilde{q}) > \tilde{f}(\tilde{p})$.) Thus, for all $\tilde{p}$ and $\tilde{q}$ in $M \cup M^a$, they are transversal globally after these modifications. By Lemma 2.7, and Step 1, no matter how $\tilde{X}$ is changed outside of $M \cup \tilde{M}^a$, $\tilde{p}$ and $\tilde{q}$ are still transversal globally because they are still transversal in $\tilde{M}^a$.

Step 3: we modify $\tilde{X}$ in $\tilde{M}^b - [M \cup \tilde{M}^a]$. To do this, we slide the descending manifolds with the order from critical points with lower values to those with higher values. More precisely, slide $D(\tilde{p}; \tilde{X})$ for each $\tilde{p} \in \tilde{M}^b - [M \cup \tilde{M}^a]$ such that $\tilde{p}$ is transverse to all $\tilde{q} \in \tilde{M}^b - M$ with $\tilde{f}(\tilde{q}) < \tilde{f}(\tilde{p})$. (Here, for all $\tilde{q} \in \tilde{M}^a$, we have $\tilde{f}(\tilde{q}) < \tilde{f}(\tilde{p})$.) We claim that, for all $\tilde{p}$ and $\tilde{q}$ in $\tilde{M}^b$, they are transversal. By what we have already proved, it remains to show that, for each $p \in M$ and $\tilde{q} \in \tilde{M}^{a,b} - M$, we have $p$ and $\tilde{q}$ are transversal. Clearly, $D(\tilde{q}; \tilde{X}) \subseteq \tilde{M}^{b} - M$, thus $D(\tilde{q}; \tilde{X}) \cap \tilde{M}^{a,b} \subseteq \tilde{M}^{a,b} - M$. Since $A(p; \tilde{X}) \cap \tilde{M}^{a,b} \subseteq M$, we get $D(\tilde{q}; \tilde{X}) \cap A(p; \tilde{X}) \cap \tilde{M}^{a,b} = \emptyset$. So $D(\tilde{q}; \tilde{X}) \cap A(p; \tilde{X}) = \emptyset$. Similarly, $A(\tilde{q}; \tilde{X}) \cap D(p; \tilde{X}) = \emptyset$. We infer that $p$ and $\tilde{q}$ are transversal. The above claim is proved. By Lemma 2.7, again, no matter how $\tilde{X}$ is changed outside of $\tilde{M}^b$, all critical points in $\tilde{M}^b$ are still mutually transverse.

Step 4: we modify $\tilde{X}$ on $\tilde{M} - \tilde{M}^b$. Slide the descending manifolds with the order from critical points with lower values to those with higher values. We eventually get that $\tilde{X}$ satisfies transversality.
By the above argument, for all \( p \) and \( q \) in \( M \), we have \( \mathcal{D}(p; \tilde{X}) \subseteq M \cup \overline{f^{-1}}((-\infty, a)) \), \( \mathcal{A}(q; \tilde{X}) \subseteq M \cup \overline{f^{-1}}((b, +\infty)) \), \( \mathcal{D}(p; \tilde{X}) \cap M = \mathcal{D}(p; X) \) and \( \mathcal{A}(q; \tilde{X}) \cap M = \mathcal{A}(q; X) \).

Thus
\[
\mathcal{D}(p; \tilde{X}) \cap \mathcal{A}(q; \tilde{X}) = (\mathcal{D}(p; \tilde{X}) \cap M) \cap (\mathcal{A}(q; \tilde{X}) \cap M) = \mathcal{D}(p; X) \cap \mathcal{A}(q; X).
\]

Suppose \( M_1 = \emptyset \). Clearly, we can construct \( \tilde{f} \) such that \( \overline{f}|_{\overline{M} - M} > b \). Thus, for any \( p \in M \), we have \( \mathcal{D}(p; \tilde{X}) \subseteq M \) and \( \mathcal{D}(p; \tilde{X}) = \mathcal{D}(p; X) \). Similarly, the conclusion is true in the case of \( M_2 = \emptyset \).

\[\Box\]

6. MODULI SPACES AND TOPOLOGICAL EQUIVALENCE

In this section, we shall review the definitions of moduli spaces and their compactifications. These definitions are standard in the literature (see e.g. \[13\], \[3\], \[22\] and \[21\]). There are several ways to define the topology of these spaces. The definitions in this paper follow those in \[21\] thms. 3.3, 3.4 and 3.5.

The paper \[21\] focuses on the negative gradient vector fields. This paper deals with the negative gradient-like vector fields. By Lemma \[22\] there is no difference.

After this review, we shall prove Theorem 6.7. This theorem shows that topologically equivalent negative gradient-like fields have homeomorphic compactified moduli spaces. In other words, the compactified moduli spaces are invariants of topological equivalence. In this paper, the application of topological equivalence to Morse theory is based on this theorem.

Let \( M \) be a finite dimensional manifold. Let \( f \) be a proper Morse function on \( M \) and \( X \) be a negative gradient-like vector field for \( f \). Assume \( X \) satisfies transversality. Denote by \( \phi_t(x) \) the flow generated by \( X \) with initial value \( x \) and time \( t \). Define an equivalence relation on \( M \) by

\[ x \sim y \iff y = \phi_t(x) \text{ for some } t \in (-\infty, +\infty). \]

Then \( x \sim y \) if and only if \( x \) and \( y \) lie on the same flow line. Suppose \( p \) and \( q \) are critical points of \( f \). Define \( \mathcal{W}(p, q) = \mathcal{D}(p) \cap \mathcal{A}(q) \). Then \( \mathcal{W}(p, q) \) is a smoothly embedded submanifold of \( M \). Define \( \mathcal{M}(p, q) = \mathcal{W}(p, q)/\sim \). We define the smooth structure of \( \mathcal{M}(p, q) \) as follows. Choose a regular value \( a \in (f(q), f(p)) \). Then each flow line in \( \mathcal{W}(p, q) \) intersects \( f^{-1}(a) \) exactly at one point. This identifies \( \mathcal{M}(p, q) \) with \( \mathcal{W}(p, q) \cap f^{-1}(a) \) naturally. We transfer the smooth structure of \( \mathcal{W}(p, q) \cap f^{-1}(a) \) to \( \mathcal{M}(p, q) \) by this identification. Clearly, this definition does not depend on the choice of \( a \). Furthermore, the natural projection from \( \mathcal{W}(p, q) \) to \( \mathcal{M}(p, q) \) is a smooth submersion.

It’s well known that \( \dim(\mathcal{W}(p, q)) = \text{ind}(p) - \text{ind}(q) \) and \( \dim(\mathcal{M}(p, q)) = \text{ind}(p) - \text{ind}(q) - 1 \) if \( p > q \).

We shall generalize the concept of flow lines. Suppose \( \gamma \) is a flow line. If it passes through a singularity, it is a constant flow line. Otherwise, it is nonconstant. The following definitions follow \[21\] sec. 2

**Definition 6.1.** An ordered sequence of flow lines \( \Gamma = (\gamma_1, \cdots, \gamma_n) \), \( n \geq 1 \), is a generalized flow line if \( \gamma_i(\pm\infty) = \gamma_{i+1}(\mp\infty) \) and \( \gamma_i \) are constant or nonconstant alternatively according to the order of their places in the sequence. We call \( \gamma_i \) a component of \( \Gamma \).

**Definition 6.2.** Suppose \( x \) and \( y \) are two points in \( M \). A generalized flow line \((\gamma_1, \cdots, \gamma_n)\) connects \( x \) with \( y \) if there exist \( t_1, t_2 \in (-\infty, +\infty) \) such that \( \gamma_1(t_1) = x \) and \( \gamma_n(t_2) = y \).


Critical sequence because 

\[ r \quad \text{such that} \quad r \quad \text{the unique topology such that the evaluation map} \quad E \quad \text{one point} \quad (\alpha) \quad \text{We can identify} \quad x \quad \text{and} \quad x \quad \text{venient. If} \quad \alpha \quad \text{Here, if} \quad I \quad \text{on the choice of} \quad a \quad \text{of} \quad M \quad \text{Suppose the critical values of} \quad I \quad \text{Suppose} \quad \text{For} \quad p \quad \text{It’s easy to see that the definition of the topology of} \quad M \quad \text{As mentioned before, “} \geq \quad \text{is a partial order because of transversality.}

**Definition 6.3.** An ordered set \( I = \{r_0, r_1, \ldots, r_{k+1}\} \) is a critical sequence if \( r_i \quad (i = 0, \ldots, k + 1) \) are critical points and \( r_0 \geq r_1 \geq \cdots \geq r_{k+1} \). We call \( r_0 \) the head of \( I \), and \( r_{k+1} \) the tail of \( I \). The length of \( I \) is \( |I| = k \). In particular, if \( I = \{r_0\} \), then \( |I| = -1 \).

Suppose \( I = \{r_0, r_1, \ldots, r_{k+1}\} \) is a critical sequence. If \( k + 1 > 0 \), define

\[ \mathcal{M}_I = \bigcap_{i=0}^{k} \mathcal{M}(r_i, r_{i+1}). \]

On the contrary, if \( I = \{r_0\} \), then define \( \mathcal{M}_I \) as the one point set \( \{\beta(r_0)\} \), where \( \beta(r_0) \) is the constant flow line passing through \( r_0 \).

For \( p \succ q \), define a space \( \mathcal{M}(p, q) \) as

\[ \mathcal{M}(p, q) = \bigcup_I \mathcal{M}_I, \]

where the disjoint union is over all critical sequence with head \( p \) and tail \( q \).

We can give \( \mathcal{M}(p, q) \) another equivalent definition which is sometimes more convenient. If \( \alpha \in \mathcal{M}_I \subseteq \mathcal{M}(p, q) \), then \( \alpha = (\gamma_0, \ldots, \gamma_k) \), where \( \gamma_i \in \mathcal{M}(r_i, r_{i+1}) \), \( r_i = p \) and \( r_{k+1} = q \). Let \( \beta(r_i) \) denote the constant flow line passing through \( r_i \). We can identify \( \alpha \) with the generalized flow line \( (\beta(r_0), \gamma_0, \beta(r_1), \ldots, \gamma_k, \beta(r_{k+1})) \) connecting \( p \) with \( q \). Thus we get

\[ \overline{\mathcal{M}(p, q)} = \{\Gamma \mid \Gamma \text{ is a generalized flow line connecting } p \text{ with } q\}. \]

Suppose the critical values of \( f \) divide \([f(q), f(p)]\) into \( l + 1 \) intervals \([c_{i+1}, c_i]\) \( (i = 0, \ldots, l) \), where \( c_0 = f(p) \) and \( c_{l+1} = f(q) \). Choose a regular value \( a_i \in (c_{i+1}, c_i) \). The generalized flow line \( \Gamma \in \overline{\mathcal{M}(p, q)} \) intersects with \( f^{-1}(a_i) \) at exactly one point \( x_i(\Gamma) \). There is an evaluation map \( E : \overline{\mathcal{M}(p, q)} \to \prod_{i=0}^{l} f^{-1}(a_i) \) which is injective and is defined as

\[ E(\Gamma) = (x_0(\Gamma), \cdots, x_l(\Gamma)). \]

**Definition 6.4.** For \( p \succ q \), define the set \( \overline{\mathcal{M}(p, q)} \) as (6.1). Equip \( \overline{\mathcal{M}(p, q)} \) with the unique topology such that the evaluation map \( E : \overline{\mathcal{M}(p, q)} \to \prod_{i=0}^{l} f^{-1}(a_i) \) in (6.2) is a topological embedding. We call \( \overline{\mathcal{M}(p, q)} \) the compactified moduli space of \( \mathcal{M}(p, q) \).

It’s easy to see that the definition of the topology of \( \overline{\mathcal{M}(p, q)} \) does not depend on the choice of \( a_i \).

For \( p \succ q \), we compactify \( W(p, q) \) to be \( \overline{W(p, q)} \) as follows.

Suppose \( I_1 = (p, r_1, \cdots, r_s) \) and \( I_2 = (r_{s+1}, \cdots, r_k, q) \) are critical sequences such that \( r_s \succeq r_{s+1} \). Let \( (I, s) = (p, r_1, \cdots, q) \). Note that \( (I, s) \) is not necessarily a critical sequence because \( r_s \) may equal \( r_{s+1} \). Define

\[ W_{I_1,s} = \mathcal{M}_{I_1} \times W(r_s, r_{s+1}) \times \mathcal{M}_{I_2} \]

Here, if \( r_s = r_{s+1} \), then by definition \( W(r_s, r_{s+1}) = \{r_s\} \). Furthermore, if \( I_1 = \{p\} \), then \( (I, s) = (I, 0) \) and

\[ W_{I_1,s} = \{\beta(p)\} \times W(r_s, r_{s+1}) \times \mathcal{M}_{I_2}, \]
we naturally identify $W_{I,s}$ with $W(r_s, r_{s+1}) \times \mathcal{M}_{I_2}$. Similarly, if $I_2 = \{q\}$, we identify $W_{I,s}$ with $\mathcal{M}_{I_1} \times W(r_s, r_{s+1})$. Most particularly, if $I_1 = \{p\}$ and $I_2 = \{q\}$, we identify $W_{I,s} = W_{I,0}$ with $W(r_s, r_{s+1}) = W(p,q)$.

Define a space $\overline{W}(p,q)$ as
\begin{equation}
\overline{W}(p,q) = \bigcup_{(I,s)} W_{I,s},
\end{equation}
where the disjoint union is over all $(I,s) = (p, r_1, \cdots, r_k, q)$ such that $p \succ r_1 \succ \cdots \succ r_s \succeq r_{s+1} \succ \cdots \succ r_k \succ q$.

Suppose $(\alpha_1, x, \alpha_2) \in \mathcal{M}_{I_1} \times W(r_s, r_{s+1}) \times \mathcal{M}_{I_2} = W_{I,s}$. Then $x$ is on the unique generalized flow line $\Gamma \in \overline{\mathcal{M}(p,q)}$ whose components include those of $\alpha_1$ and $\alpha_2$ and, in addition, the flow line through $x$. Thus, identify $(\alpha_1, x, \alpha_2)$ with $(\Gamma, x)$, we get
\[
\overline{W}(p,q) = \{ (\Gamma, x) \in \overline{\mathcal{M}(p,q)} \times M \mid x \text{ is on } \Gamma \}.
\]

**Definition 6.5.** For $p \succ q$, define the set $\overline{W}(p,q)$ as (6.3). Define the topology of $\overline{W}(p,q)$ as the restriction of that of $\overline{\mathcal{M}(p,q)} \times M$. We call $\overline{W}(p,q)$ the compactified space of $W(p,q)$.

Clearly, the map $\overline{E} : \overline{W}(p,q) \rightarrow \prod_{i=0} f^{-1}(a_i) \times M$ is a topological embedding, where
\begin{equation}
\overline{E}(\Gamma, x) = (E(\Gamma), x).
\end{equation}
Thus the topology of $\overline{W}(p,q)$ in this paper is equivalent to that of [21, thm. 3.5].

Finally, we define the compactified space $\overline{D}(p)$ of $D(p)$. Suppose $f$ is bounded below.

Suppose $I = \{p, r_1, \cdots, r_k\}$ is a critical sequence. Define
\[
\mathcal{D}_I = \mathcal{M}_I \times D(\{r_k\}).
\]
In particular, if $I = \{p\}$, we naturally identify $\mathcal{D}_I$ with $D(p)$. Define a space $\overline{D}(p)$ as
\begin{equation}
\overline{D}(p) = \bigcup_I \mathcal{D}_I,
\end{equation}
where the disjoint union is over all critical sequences with head $p$.

Suppose $(\alpha, x) \in \mathcal{M}_I \times D(\{r_k\}) = \mathcal{D}_I$. We can identify $\alpha$ with a generalized flow line connecting $p$ with $r_k$. Adding the flow line passing through $x$ to the above generalized flow line, we get a generalized flow line connecting $p$ with $x$. Thus we get
\[
\overline{D}(p) = \{ (\Gamma, x) \mid \Gamma \text{ is a generalized flow line connecting } p \text{ with } x \}.
\]

The definition of the topology of $\overline{D}(p)$ is slightly complicated.

Suppose the critical values in $(-\infty, f(p)]$ are exactly $c_l < \cdots < c_0 = f(p)$. Define $U(i) \subseteq \overline{D}(p)$ ($i = 0, \cdots, l$) as
\begin{equation}
U(i) = \{ (\Gamma, x) \mid c_{i+1} < f(x) < c_{i-1} \},
\end{equation}
where $c_{l+1} = -\infty$ and $c_{-1} = +\infty$. Clearly, $\overline{D}(p) = \bigcup_i U(i)$. We have the following injection $E_i : U(i) \rightarrow \prod_{j=1}^{i-1} f^{-1}(a_j) \times M$ such that
\[
E_i(\Gamma, x) = (x_0(\Gamma), \cdots, x_{i-1}(\Gamma), x),
\]
where \( x_i(\Gamma) \) is the unique intersection point between \( \Gamma \) and \( f^{-1}(a_i) \). Equip \( U(i) \) with the unique topology such that \( E_i \) is a topological embedding. The paper [21, thm. 3.4] shows that these \( U(i) \) have compatible smooth structures under the assumption of the local triviality of the vector field. Follow that argument, we can prove that the topologies of these \( U(i) \) are compatible even if we drop the local triviality. Here compatibility means that \( U(i) \) and \( U(j) \) share the same topology on \( U(i) \cap U(j) \).

**Definition 6.6.** Define the set \( \overline{D(p)} \) as \([6.5]\). Define the topology of \( \overline{D(p)} = \bigcup_i U(i) \) as the coherent topology such that each \( U(i) \) is an open subspace of \( \overline{D(p)} \) (see \([6.6]\)). We call \( \overline{D(p)} \) the compactified space of \( D(p) \).

For the convenience of the reader, we include here an example in [21].

**Example 6.1.** Figure 3 shows a standard example on a torus \( T^2 = S^1 \times S^1 \), where the arrows indicate the directions of flows. Consider \( S^1 \) as the unit circle on the complex plane. Define a Morse function on \( T^2 \) by \( f(z_1, z_2) = \text{Re}(z_1) + \text{Re}(z_2) \). \( f \) has 4 critical points \( p, r, s \) and \( q \). Their indices are 2, 1, 1 and 0 respectively. Equip \( T^2 \) with the standard metric. The left part of Figure 5 shows the flow on \( T^2 \), where the opposite sides of the square are identified with each other. The right part is \( \overline{D(p)} \) which is an octagon. Here \( \mathcal{M}(p, r) \times D(r) \) (or \( \mathcal{M}(p, s) \times D(s) \)) consists of open edges containing \( r_i \) (or \( s_i \)), where \( i = 1, 2 \); \( \mathcal{M}(p, q) \times D(q) \) consists of the other 4 open edges; and \( (\mathcal{M}(p, r) \times \mathcal{M}(r, q) \times D(q)) \cup (\mathcal{M}(p, s) \times \mathcal{M}(s, q) \times D(q)) \) consists of the 8 vertices.

![Figure 5. Compactification of the Descending Manifolds](image)

Suppose \( f_1 \) and \( f_2 \) are Morse functions on \( M_1 \) and \( M_2 \). Suppose \( X_i \) is a negative gradient-like field for \( f_i \), and \( X_i \) satisfies transversality. Suppose \( h : M_1 \rightarrow M_2 \) is a topological equivalence between \( X_1 \) and \( X_2 \). If \( p \) is a critical point of \( f_1 \), then \( h(p) \) is a critical point of \( f_2 \). Furthermore, \( h(\overline{D(p)}) = \overline{D(h(p))} \), \( h(\mathcal{A}(p)) = \mathcal{A}(h(p)) \), and \( h(W(p, q)) = W(h(p), h(q)) \). Thus \( h \) naturally induces maps \( h_* : \mathcal{M}(p, q) \rightarrow \mathcal{M}(h(p), h(q)) \), \( h_* : W(p, q) \rightarrow W(h(p), h(q)) \), and \( h_* : \overline{D(p)} \rightarrow \overline{D(h(p))} \). Here, if \( \Gamma \in \mathcal{M}(h(p), h(q)) \), then \( h_* (\Gamma) = h(\Gamma) \); if \( (\Gamma, x) \in W(p, q) \) (or \( \overline{D(p)} \)), then \( h_* (\Gamma, x) = (h(\Gamma), h(x)) \). Clearly, \( h_* \) is a bijection and \( (h_*)^{-1} = (h^{-1})_* \).
Theorem 6.7. The maps $h_* : \mathcal{M}(p, q) \to \mathcal{M}(h(p), h(q))$, $h_* : \mathcal{D}(p) \to \mathcal{D}(h(p))$, and $h_* : \mathcal{W}(p, q) \to \mathcal{W}(h(p), h(q))$ are homeomorphisms.

Proof. It suffices to prove that $h_*$ is continuous because this implies $h_*^{-1}$ is also continuous.

(1). We consider the case of $h_* : \mathcal{M}(p, q) \to \mathcal{M}(h(p), h(q))$.

By the definition, $\mathcal{M}(p, q)$ is identified with a topological subspace of $\prod_{i=0}^{l} a_i^{-1}(i)$ and $\mathcal{M}(h(p), h(q))$ is identified with a topological subspace of $\prod_{i=0}^{k} h_i^{-1}(b_i)$. By this identification, for any $\Gamma \in \mathcal{M}(p, q)$, we have $\Gamma = (x_0(\Gamma), \cdots, x_l(\Gamma))$ and $h_*(\Gamma) = (y_0(h(\Gamma)), \cdots, y_k(h(\Gamma)))$. Suppose $x_0(\Gamma_0)$ is on $\gamma \in \mathcal{M}(p, r)$ and $\gamma$ is a component of $\Gamma_0$, then $h(x_0(\Gamma_0))$ is on $h(\gamma) \in \mathcal{M}(h(p), h(r))$. Suppose the regular values in $[f_2(h(r)), f_2(h(p))]$ are $b_0, \cdots, b_k$. Then $h(\gamma)$ intersects with $f_2^{-1}(b_i)$ $(0 \leq i \leq s)$ at $y_i(h(\Gamma_0))$. When $\Gamma \in \mathcal{M}(p, q)$ converges to $\Gamma_0$, we have $x_0(\Gamma)$ converges to $x_0(\Gamma_0)$, then $h(x_0(\Gamma))$ converges to $h(x_0(\Gamma_0))$. Thus, by Lemma 2.4 when $\Gamma$ is close to $\Gamma_0$ enough, the flow line passing through $h(x_0(\Gamma))$ intersects with $f_2^{-1}(b_i)$ $(0 \leq i \leq s)$ at $y_i(h(\Gamma))$ and $y_i(h(\Gamma))$ is continuous with respect to $\Gamma$.

By an induction, we can prove that, for all $0 \leq i \leq k$, $y_i(h(\Gamma))$ is continuous with respect to $\Gamma$. Thus $h_*$ is continuous.

(2). Since $\mathcal{W}(p, q)$ is a topological subspace of $\mathcal{M}(p, q) \times M_1$, by (1), we infer that $h_*$ is continuous on $\mathcal{W}(p, q)$.

(3). We consider the case of $h_* : \mathcal{D}(p) \to \mathcal{D}(h(p))$.

It suffices to check the continuity of $h_*$ on each $U(i)$. Suppose $(\Gamma_0, z_0) \in U(i)$ and $\bar{c}_{s+1} = f_2(h(z_0)) < \bar{c}_{M-1}$, where $\bar{c}_j$ are critical values of $f_2$. Then $h_s(\Gamma_0, z_0) \in \bar{U}(s)$, where $\bar{U}(s) \subseteq \mathcal{D}(h(p))$ is defined similarly to $U(i)$. Thus, when $(\Gamma, z)$ is close to $(\Gamma_0, z_0)$ enough, we have $h_s(\Gamma, z) \in \bar{U}(s)$. Identify $\bar{U}(s)$ with a topological subspace of $\prod_{j=0}^{s-1} f_2^{-1}(b_j) \times M_2$, we have $h_s(\Gamma, z) = (y_0(h(\Gamma)), \cdots, y_{s-1}(h(\Gamma)), h(z))$. By an argument similar to that in (1), we can prove that $y_j(h(\Gamma))$ is continuous with respect to $\Gamma$. Since $h(z)$ is continuous with respect to $z$, we infer $h_*$ is continuous.

7. Properties of Moduli Spaces

In this section, we establish the relevant properties of the compactified moduli spaces. Particularly, the manifold structures of these spaces will be emphasized.

When the metric is locally trivial, similar results can be found in the literature (see e.g. [13], [3] and [21]). Our results are extensions of those results to the case of a general metric provided that the Morse function $f$ is proper. In this case, every negative gradient-like vector field $X$ for $f$ satisfies the CF condition in [21, def. 2.6]. This extension needs Theorem 4.1, Lemma 5.1 and Theorem 6.7.

We introduce the concepts of manifolds with corners or faces. Our terminology follows that in [4, p. 2], [9, sec. 1.1] and [21].

Definition 7.1. A smooth manifold with corners is a space defined in the same way as a smooth manifold except that its atlases are open subsets of $[0, +\infty)^n$.

If $L$ is a smooth manifold with corners, $x \in L$, a neighborhood of $x$ is diffeomorphic to $(0, \epsilon)^{n-k} \times [0, \epsilon)^k$, then define $c(x) = k$. Clearly, $c(x)$ does not depend on the choice of atlas.

Definition 7.2. Suppose $L$ is a smooth manifold. We call $\{x \in L \mid c(x) = k\}$ the $k$-stratum of $L$. Denote it by $\partial^k L$. 

Clearly, \( \partial^k L \) is a submanifold without corners inside \( L \), its codimension is \( k \).

**Definition 7.3.** A smooth manifold \( L \) with faces is a smooth manifold with corners such that each \( x \) belongs to the closures of \( c(x) \) different components of \( \partial^1 L \).

Consider firstly the special case when \( M \) is compact. By Theorem 4.1 we can construct a negative gradient-like field \( Y \) for \( f \) such that \( Y \) is locally trivial and satisfies transversality. In addition, there exists a topological equivalence between \( X \) and \( Y \) such that \( h(p) = p \) for each critical point \( p \). Thus, by Theorem 6.7, \( X \) and \( Y \) have isomorphic compactified moduli spaces. Since the properties of these spaces for \( Y \) are proved in [21], we deduce certain properties of these spaces for \( X \).

More generally, suppose that \( f \) is proper but \( M \) is not necessarily compact. For any pair of critical points \( (p, q) \), choose regular values \( a \) and \( b \) such that \( M^{a,b} \) is compact and contains \( p \) and \( q \). By Lemma 5.1, we can embed \( M^{a,b} \) into \( \hat{M} \), extend \( f|_{M^{a,b}} \) to be \( \hat{f} \) on \( \hat{M} \), and extend \( X|_{M^{a,b}} \) to be \( \hat{X} \) on \( \hat{M} \). Furthermore, \( W(p, q; X) = W(p, q; \hat{X}) \). Thus we get \( \overline{M}(p, q; X) = \overline{M}(p, q; \hat{X}) \) and \( \overline{W}(p, q; X) = \overline{W}(p, q; \hat{X}) \).

If \( f \) is bounded below, we choose \( M^a \) such that \( p \in M^a \). Do the above extension again to get \( \overline{D}(p, X) = \overline{D}(p, \hat{X}) \). Thus Lemma 5.1 reduces the proper case to the compact case.

Before formulating the property of \( \overline{M}(p, q) \), we introduce a map. Suppose \( \Gamma_1 \in \overline{M}(p, r) \) is a generalized flow line connecting \( p \) with \( r \) and \( \Gamma_2 \in \overline{M}(r, q) \) is a generalized flow line connecting \( r \) with \( q \). Thus the combination of \( \Gamma_1 \) and \( \Gamma_2 \) gives a generalized flow line \( \Gamma \) connecting \( p \) with \( q \). So we have the natural inclusion \( i_{(p, r, q)} : \overline{M}(p, r) \times \overline{M}(r, q) \to \overline{M}(p, q) \).

**Theorem 7.4.** Suppose \( f \) is proper and \( X \) satisfies transversality. Then, for each critical sequence \( \{p, q\} \), the topological space \( \overline{M}(p, q) = \bigsqcup \mathcal{M}_I \) is defined by Definition 6.4. It has the following properties.

1. It is a compact topological manifold with boundary. Its interior is \( \mathcal{M}(p, q) \).
2. Its topology is compatible with that of each \( \mathcal{M}_I \), and the map \( i_{(p, r, q)} : \overline{M}(p, r) \times \overline{M}(r, q) \to \overline{M}(p, q) \) is a topological embedding.
3. The evaluation map \( E : \overline{M}(p, q) \to \prod_{i=0}^1 f^{-1}(a_i) \) is a topological embedding, where \( E \) is defined in \([6.2]\).
4. There exists a topological embedding \( i : \overline{M}(p, q) \to \prod_{i=0}^1 f^{-1}(a_i) \) such that \( i(M(p, q)) \) is a smoothly embedded submanifold with faces inside \( \prod_{i=0}^1 f^{-1}(a_i) \) and the \( k \)-stratum of \( i(M(p, q)) \) is \( \bigsqcup_{l=k} i_*(\mathcal{M}_l) \).

In particular, if \( M \) is compact, then there exist homeomorphisms \( h_i : f^{-1}(a_i) \to f^{-1}(a_i) \) such that \( i = \left( \prod_{i=0}^1 h_i \right) \circ E \) in \( (4) \).

**Theorem 7.5.** Under the assumption of Theorem 7.4, each \( \overline{M}(p, q) \) carries a smooth structure compatible with its topology such that \( \overline{M}(p, q) \) is a smooth manifold with faces and \( \partial^k \overline{M}(p, q) = \bigsqcup_{l=k} \mathcal{M}_l \). In particular, suppose \( M \) is compact, then \( i_{(p, r, q)} : \overline{M}(p, r) \times \overline{M}(r, q) \to \overline{M}(p, q) \) is a smooth embedding.

**Remark 7.1.** The (1) of Theorem 7.4 shows that we can add a boundary to \( \mathcal{M}(p, q) \) such that it becomes a compact manifold with boundary. The following theorems show that this is also true for \( W(p, q) \) and \( D(p) \). Thus moduli spaces are special open manifolds (if they are open) because there exist obstructions of adding a boundary to a general open manifold (see [23]).
Remark 7.2. The paper [21 example 3.1] shows that, if the metric is not locally trivial, then \( E(\mathcal{M}(p, q)) \) usually is even not a \( C^1 \) embedded submanifold of \( \prod_{i=0}^{l} f^{-1}(a_i) \). Here \( E \) is the evaluation map in the (3) of Theorem 7.4. However, the (4) of Theorem 7.4 shows that a suitable embedding \( \iota \) makes the image good.

Proof of Theorem 7.4. Choose regular values \( a \) and \( b \) such that \( M^{a, b} \) is compact and contains \( p \) and \( q \). As described in the above, construct \( \hat{M} \), \( f \) and \( \hat{X} \). We have \( \mathcal{M}(p, q; \hat{X}) = \mathcal{M}(p, q; X) \) and \( \mathcal{M}_I(\hat{X}) = \mathcal{M}_I(X) \) for all critical sequences \( I \) with head \( p \) and tail \( q \). There exists a negative gradient-like vector field \( Y \) on \( \hat{M} \) and a topological equivalence \( h : M \to \hat{M} \) which maps the orbits of \( \hat{X} \to X \) to those of \( Y \), where \( Y \) is locally trivial.

(1) By [21 thm. 3.3], we know that \( M(p, q; \hat{X}) \) is a compact smooth manifold with faces whose \( k \)-stratum is \( \bigsqcup_{|I|=k} \mathcal{M}_I(Y) \). Thus \( \mathcal{M}(p, q; \hat{X}) \) is a compact topological manifold with boundary, and its interior is \( \mathcal{M}(p, q; X) \). By Theorem 6.7 we know that \( h \) induces a homeomorphism \( h_* : \mathcal{M}(p, q; X) \to \mathcal{M}(p, q; \hat{X}) \) such that \( h_*(\mathcal{M}_I(\hat{X})) = \mathcal{M}_I(Y) \). This completes the proof of (1).

(2) The proof is easy and even does not need the comparison among \( \mathcal{M}(p, q; X) \), \( \mathcal{M}(p, q; \hat{X}) \) and \( \mathcal{M}(p, q; \hat{X}) \). Similar details is also included in the proof of [21 thm. 3.3].

(3) This is the definition of the topology of \( \overline{\mathcal{M}}(p, q; \hat{X}) \).

(4) Let \( E_Y : \overline{\mathcal{M}}(p, q; \hat{X}) \to \prod_{i=0}^{l} f^{-1}(a_i) \) be the evaluation map. By [21 thm. 3.3], we know \( E_Y \) is a smooth embedding.

Suppose \( r_1 \) and \( r_2 \) are in \( M^{a, b} \). It’s easy to see that \( D(r_1; Y) \subseteq M^{a, b} \cup f^{-1}((\infty, a_i)) \) and \( A(r_2; Y) \subseteq M^{a, b} \cup f^{-1}((b, +\infty)) \). We infer that \( W(r_1, r_2; Y) \subseteq M^{a, b} \). Thus \( \text{Im}(E_Y) \subseteq \prod_{i=0}^{l} f^{-1}(a_i) \) and \( \iota = E_Y \circ h_* \) is the desired map.

Finally, we consider the special case when \( M \) is compact. We construct \( Y \) on \( M \). The topological equivalence \( h : M \to \hat{M} \) induces the homeomorphism \( h_* : \overline{\mathcal{M}}(p, q; X) \to \overline{\mathcal{M}}(p, q; \hat{X}) \). We consider the relation between \( h(f^{-1}(a_i)) \) and \( f^{-1}(a_i) \). Denote by \( \phi^X_r \) the flow generated by \( X \) and by \( \phi^Y_r \) the flow generated by \( Y \). For any \( x \in f^{-1}(a_i) \), we have \( \phi^X_r(-\infty, x) = r_1 \) for some \( r_1 \in M - M^{a_i} \) and \( \phi^X_r(+\infty, x) = r_2 \) for some \( r_2 \in M^{a_i} \). Since \( h \) is a topological equivalence fixing \( r_1 \) and \( r_2 \), we know that \( \phi^Y_r(-\infty, h(x)) = r_1 \) and \( \phi^Y_r(+\infty, h(x)) = r_2 \). Thus, \( \phi^Y_r(t(x), h(x)) \in f^{-1}(a_i) \) for some \( t(x) \in (-\infty, +\infty) \). By Lemma 2.4 an isotopy along the flows generated by \( Y \) gives a homeomorphism \( \psi : h(f^{-1}(a_i)) \to f^{-1}(a_i) \).

We complete the proof by defining \( h_i = \psi \circ h \).

Proof of Theorem 7.5. The first half part of Theorem 7.5 is a corollary of Theorem 7.4. It remains to prove that, when \( M \) is compact, there exists a suitable smooth structure of \( \overline{\mathcal{M}}(r_1, r_2; X) \) for each pair of critical points \( r_1 \) and \( r_2 \) such that \( i_{(p, r, q)} \) is a smooth embedding.

Construct a locally trivial field \( Y \) as we did in the proof of Theorem 7.3. We get a topological equivalence \( h : M \to \hat{M} \) which induces homeomorphisms \( h_*^{r_1, r_2} : \overline{\mathcal{M}}(r_1, r_2; X) \to \overline{\mathcal{M}}(r_1, r_2; Y) \). Here we use the notation \( h_*^{r_1, r_2} \) instead of \( h_* \) to indicate the critical points. By [21 thm. 3.3], each \( \overline{\mathcal{M}}(r_1, r_2; Y) \) has a natural smooth structure. Define the smooth structure of \( \overline{\mathcal{M}}(r_1, r_2; X) \) such that \( h_*^{r_1, r_2} \) is
a diffeomorphism. We have the following commutative diagram:

\[
\begin{array}{ccc}
M(p, r; X) \times M(r, q; X) & \xrightarrow{i_{(p, r, q)}} & M(p, q; X) \\
\downarrow \quad h^p \times h^r & & \downarrow h^p \\
M(p, r; Y) \times M(r, q; Y) & \xrightarrow{i'_{(p, r, q)}} & M(p, q; Y),
\end{array}
\]

where \( i'_{(p, r, q)} \) is the natural inclusion similar to \( i_{(p, r, q)} \). By [21, thm. 3.3], we know that \( i_{(p, r, q)} \) is a smooth embedding. Since the vertical maps of this diagram are diffeomorphisms, we infer that \( i_{(p, r, q)} \) is a smooth embedding.

Since we define \( W(p, q) \) as a subspace of \( M(p, q) \times M \), we have the inclusion \( i : W(p, q) \to M(p, q) \times M \). Suppose \( p \succ r \succ q \). If \( (\Gamma, x) \in W(p, r) \) and \( \Gamma_2 \in M(r, q) \), then the combination of \( \Gamma_1 \) and \( \Gamma_2 \) gives an element in \( M(p, q) \) and \( x \) is on it. This defines a natural inclusion \( i_{(p, r, q)} : W(p, r) \times M(r, q) \to W(p, q) \).

Similarly, we can define a natural inclusion \( i^2_{(p, r, q)} : M(p, r) \times W(r, q) \to W(p, q) \).

Suppose \( f \) is bounded below. We define the evaluation map \( e : D(p) \to M \) as

\[ e(\Gamma, x) = x. \]

Clearly, the restriction of \( e \) to \( D_I = M_I \times D(r_k) \) is the coordinate projection onto \( D(r_k) \subset M \).

**Example 7.1.** In Example 6.1, the space \( \overline{D}(p) \) is illustrated by the right part of Figure 3. The evaluation map \( e \) maps the interior of \( \overline{D}(p) \) to \( D(p) \). On the boundary of \( \overline{D}(p) \), it maps the horizontal open line segment through \( r_i \) to \( D(r) \), maps the vertical open line segment through \( s_i \) to \( D(s) \), and maps the remaining part to \( q \). Furthermore, it maps \( r_1 \) to \( r \) and maps \( s_i \) to \( s \).

Suppose \( p \succ r \). If \( \Gamma_1 \in M(p, r) \) and \( (\Gamma, x) \in \overline{D}(r) \), then the combination of \( \Gamma_1 \) and \( \Gamma_2 \) is a generalized flow line connecting \( p \) with \( x \). This defines a natural inclusion \( i_{(p, r, q)} : M(p, r) \times \overline{D}(r) \to \overline{D}(p) \).

By [21] thms. 3.4, 3.5 and 3.7, using an argument similar to the proof of Theorem 7.6, we can get the following results. The proof of Theorem 7.6 needs the fact that the map \( E \) defined in (6.4) is a smooth embedding when the vector field \( X \) is locally trivial. Although this fact is not stated in [21], its easy to see that it is true from the proof of [21, thm. 3.5].

**Theorem 7.6.** Suppose \( f \) is proper and \( X \) satisfies transversality. Then, for each critical sequence \( \{p, q\} \), the topological space \( \overline{W}(p, q) = \bigsqcup_{(I, s)} W_{I, s} \) is defined by Definition 6.3. It has the following properties.

1. It is a compact topological manifold with boundary. Its interior is \( W(p, q) \).
2. Its topology is compatible with that of each \( W_{I, s} \). The maps \( i_{(p, r, q)} : W(p, r) \times M(r, q) \to \overline{W}(p, q) \) and \( i^2_{(p, r, q)} : M(p, r) \times \overline{W}(r, q) \to \overline{W}(p, q) \) are topological embeddings.
3. The inclusion \( i : \overline{W}(p, q) \to M(p, q) \times M \) and the map \( E : \overline{W}(p, q) \to \prod_{i=0}^k f^{-1}(a_i) \times M \) are topological embeddings, where \( E \) is defined in (6.4).
4. There exists a topological embedding \( \iota : \overline{W}(p, q) \to \prod_{i=0}^k f^{-1}(a_i) \times M \) such that \( \iota(\overline{W}(p, q)) \) is a smoothly embedded submanifold with faces inside \( \prod_{i=0}^k f^{-1}(a_i) \times M \).
and the $k$-stratum of $i(\mathcal{W}(p,q))$ is $\bigcup_{(I,s)} i(\mathcal{W}_{I,s})$, where $(I,s)$ contains $k+2$ components.

In particular, if $M$ is compact, then there exist homeomorphisms $h_i : f^{-1}(a_i) \to f^{-1}(a_i)$ such that $i = [(\prod_{i=0}^r h_i) \times h] \circ \tilde{E}$ in (4).

**Corollary 7.7.** Under the assumption of Theorem 7.6, $\mathcal{W}(p,q)$ carries a smooth structure compatible with its topology such that $\mathcal{W}(p,q)$ is a compact smooth manifold with faces and $\partial^k \mathcal{W}(p,q) = \bigcup_{(I,s)} \mathcal{W}_{I,s}$, where $(I,s)$ contains $k+2$ components.

**Theorem 7.8.** Suppose $f$ is proper and bounded below. Suppose $X$ satisfies transversality. Then, for each critical point $p$, the topological space $\overline{D}(p) = \bigcup \mathcal{D}_I$ is defined by Definition 6.6. It has the following properties.

1. It is homeomorphic to a closed disc. Its interior is $\mathcal{D}(p)$.
2. Its topology is compatible with that of each $\mathcal{D}_I$. The map $i_{(p,r)} : M(p,r) \times \overline{D}(r) \to \overline{D}(p)$ is a topological embedding.
3. The evaluation map $e : \overline{D}(p) \to M$ is continuous. Here, for a critical sequence $I$ with tail $r_k$, the restriction of $e$ to $\mathcal{D}_I = \mathcal{M}_I \times \mathcal{D}(r_k)$ is given by
   \[ e|_{\mathcal{D}_I} : \mathcal{M}_I \times \mathcal{D}(r_k) \to \mathcal{D}(r_k) \subseteq M \]
   \[ e(\alpha, x) = x, \]
   i.e. $e|_{\mathcal{D}_I}$ is the coordinate projection onto $\mathcal{D}(r_k)$. In particular, $e|_{\mathcal{D}(p)}$ is the identity inclusion.
4. It carries a smooth structure compatible with its topology such that it is a compact smooth manifold with faces and $\partial^k \overline{D}(p) = \bigcup_{|I|=k-1} \mathcal{D}_I$.

**8. Orientation Formulas**

In this section, we shall prove the following orientation formulas.

**Theorem 8.1 (Orientation Formulas).** Suppose $f$ is proper and $X$ satisfies transversality. As oriented topological manifolds, we have

1. $\partial^1 M(p,q) = \bigcup_{p \succ r \succ q} (-1)^{\text{ind}(p) - \text{ind}(r)} M(p,r) \times M(r,q)$;
2. $\partial^1 \overline{M}(p,q) = \bigcup_{p \succ r} M(p,r) \times \mathcal{D}(r)$, where $f$ is bounded below;
3. $\partial^1 \overline{W}(p,q) = \bigcup_{p \prec r \succ q} (-1)^{\text{ind}(p) - \text{ind}(r)+1} W(p,r) \times M(r,q) \cup \bigcup_{p \prec r \preceq q} M(p,r) \times W(r,q)$.

In the above, $\partial^1 \square$ are equipped with boundary orientations, $\square \times \square$ are equipped with product orientations, and $\text{ind}(\ast)$ is the Morse index of $\ast$.

In order to explain the concepts in Theorem 8.1, we need to review the definition of orientation at first.

Suppose $M$ is an $n$ dimensional smooth manifold. In algebraic topology, the orientation of $M$ at $x$ is a generator $\alpha \in H^n(M, M - \{x\})$. In differential topology, the orientation is an ordered base $\{e_1, \ldots, e_n\} \subseteq T_x M$. These two definitions are related as follows. Choose a smooth embedding $\varphi : V \to M$ such that $\varphi(0) = x$ and $D \varphi(0) = \text{Id}$, where $V$ is a neighborhood of $0$ in $T_x M$. Then $\varphi^\ast \alpha \in H^n(V, V - \{0\}) = H^n(T_x M, T_x M - \{0\})$ is a generator. Here $\varphi^\ast \alpha$ does not depend on the choice of $\varphi$. Actually, if $\varphi$ is another such embedding, then there exists an isotopy between
\( \varphi \) and \( \tilde{\varphi} \) in a smaller neighborhood of 0. Denote by \( \alpha_0 \) the preferred generator in \( H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \) (see [13] p. 266). The ordered base \( \{e_1, \ldots, e_n\} \) determines a linear isomorphism \( A : T_x M \rightarrow \mathbb{R}^n \), then \( A^* \alpha_0 \in H^n(T_x M, T_x M - \{0\}) \) is also a generator. We say that these two definitions give the same orientation if and only if \( \varphi^* \alpha = A^* \alpha_0 \).

Suppose \( L \) is a \( k \) dimensional embedded submanifold of \( M \) such that its normal bundle is orientable. Choose a neighborhood \( U \) of \( L \) such that \( L \) is closed in \( U \). Choose a Thom class \( \beta \in H^{n-k}(U, U - L) \). The Thom class \( \beta \) defines the normal orientation in the sense of algebraic topology. On the other hand, for any \( x \in L \), choose an ordered base \( \{\varepsilon_{k+1}, \ldots, \varepsilon_n\} \) of the normal space \( N_x(L, M) = T_x N_x \). This defines the normal orientation of \( L \) at \( x \) in the sense of differential topology. These two definitions are related as follows. Let \( \varphi : V \rightarrow M \) be a smooth embedding such that \( \varphi(0) = x \) and \( P \cdot D\varphi(0) = \text{Id} \), where \( V \) is a neighborhood of 0 in \( N_x(L, M) \) and \( P : T_x M \rightarrow T_x N_x \) is the projection. Then \( \varphi^* \beta \in H^{n-k}(V, V - \{0\}) = H^{n-k}(N_x, N_x - \{0\}) \) is a generator. Here \( \varphi^* \beta \) does not depend on the choice of \( \varphi \). The ordered base determines an isomorphism \( A : N_x \rightarrow \mathbb{R}^{n-k} \). So \( A^* \alpha_0 \) is a generator of \( H^{n-k}(N_x, N_x - \{0\}) \), where \( \alpha_0 \) is the preferred generator of \( H^{n-k}(\mathbb{R}^{n-k}, \mathbb{R}^{n-k} - \{0\}) \). These two definitions coincide if and only if \( \varphi^* \beta = A^* \alpha_0 \).

Suppose \( \{e_1, \ldots, e_k\} \subseteq T_x L \) represents the orientation of \( L \) and \( \{e_{k+1}, \ldots, e_n\} \subseteq T_x M \) represents the normal orientation of \( L \). We say the orientation \( \{e_1, \ldots, e_n\} \) of \( M \) is defined by the orientation and the normal orientation of \( L \). We have the following lemma whose proof is in the Appendix.

**Lemma 8.2.** Suppose \( M_i (i = 1, 2) \) is a smooth orientable manifold, \( L_i \) is an orientable submanifold and a closed subset of \( M_i \). Suppose the orientation and the normal orientation of \( L_i \) define the orientation of \( M_i \). Let \( \beta_i \in H^{n-k}(M_i, M_i - L_i) \) be the Thom class representing the normal orientation of \( L_i \). Let \( h : (M_1, L_1) \rightarrow (M_2, L_2) \) be a homeomorphism such that \( h \) preserves the orientation of \( M_i \) and \( h^* \beta_2 = \beta_1 \). Then \( h \) preserves the orientation of \( L_i \).

Following [21], we define the orientations of \( \mathcal{D}(p) \), \( W(p, q) \) and \( M(p, q) \). We review the definition by means of differential topology in [21] p. 500 as follows (see [21] for more details).

Assign an arbitrary orientation to \( \mathcal{D}(p) \) for each critical point \( p \). Since \( \mathcal{D}(q) \) and \( \mathcal{A}(q) \) are transversal, the orientation of \( \mathcal{D}(q) \) gives the normal bundle \( N(\mathcal{A}(q), M) = T_{\mathcal{A}(q)} M / T\mathcal{A}(q) \) an orientation. Since \( \mathcal{D}(p) \) is transverse to \( \mathcal{A}(q) \) and \( W(p, q) = \mathcal{D}(p) \cap \mathcal{A}(q) \), the orientation of \( N(\mathcal{A}(q), M) \) gives the normal bundle \( N(\mathcal{W}(p, q), \mathcal{D}(p)) \) an orientation. We choose the orientation of \( \mathcal{W}(p, q) \) such that the orientation and the normal orientation of \( \mathcal{W}(p, q) \) define the orientation of \( \mathcal{D}(p) \). Identify \( \mathcal{M}(p, q) \) with \( \mathcal{W}(p, q) \cap f^{-1}(a) \) for some regular value \( a \in (f(q), f(p)) \). The orientation of \( \mathcal{W}(p, q) \cap f^{-1}(a) \) is defined by the direction of the flow and the orientation of \( \mathcal{W}(p, q) \). This defines the orientation of \( \mathcal{M}(p, q) \). This definition does not depend on the choice of \( a \).

By Theorems [7,4] we know \( \overline{\mathcal{M}(p, q)} \) is a topological manifold with boundary, whose interior is \( \mathcal{M}(p, q) \). Thus the orientation of \( \mathcal{M}(p, q) \) gives \( \partial \overline{\mathcal{M}(p, q)} \) the **boundary orientation** in the usual sense. In differential topology, the combination of the outward normal direction and the boundary orientation of the boundary gives the orientation of the manifold. In algebraic topology, the boundary orientation is defined by [6] (28.7), (28.16)). Also by Theorem 7.4 we know that
\( \partial^1 \mathcal{M}(p, q) = \bigsqcup_{|I|=1} \mathcal{M}_I = \bigsqcup_{p>r>q} \mathcal{M}(p, r) \times \mathcal{M}(r, q) \) is an open subset of \( \partial \mathcal{M}(p, q) \).

Thus \( \partial^1 \mathcal{M}(p, q) \) has the boundary orientation. On the other hand, both \( \mathcal{M}(p, r) \) and \( \mathcal{M}(r, q) \) have orientations. Thus \( \mathcal{M}(p, r) \times \mathcal{M}(r, q) \) has the product orientation. We shall consider the relation between these two orientations. Similarly, \( \overline{\mathcal{M}}(p) \) and \( \overline{\mathcal{M}}(p, q) \) also have such issues. Theorem 8.1 indicates these relations.

Similarly to the previous section, by Lemma 4.1, we may assume that \( M \) is compact. By Theorem 4.1, we can construct the locally trivial field \( Y \) and the topological equivalence \( h \) mapping the orbits of \( X \) to those of \( Y \).

However, since \( h \) is not assumed differentiable, we have to use the algebraic method to describe the orientation of \( W(p, q; X) \). Choose an open tubular neighborhood \( U_q \) of \( \mathcal{A}(q; X) \) such that \( \mathcal{A}(q; X) \) is closed in \( U_q \). Suppose the index \( \text{ind}(q) = s \). We have the inclusion isomorphism

\[
H^*(U_q, U_q - \mathcal{A}(q; X)) \rightarrow H^*(U_q \cap D(q; X), U_q \cap D(q; X) - \{q\}),
\]

where \( H^*(U_q \cap D(q; X), U_q \cap D(q; X) - \{q\}) = H^*(D(q; X), D(q; X) - \{q\}) \). Thus the orientation of \( D(q; X) \), \( \alpha_q \in H^*(D(q; X), D(q; X) - \{q\}) \), determines a Thom class \( \beta_q \in H^*(U_q, U_q - \mathcal{A}(q; X)) \). Let \( U_{p,q} = D(p; X) \cap U_q \). Then \( U_{p,q} \) is open in \( D(p; X) \) and \( W(p, q; X) \) is closed in \( U_{p,q} \). By the inclusion monomorphism (it is an isomorphism if and only if \( U_{p,q} \) is connected)

\[
H^*(U_q, U_q - \mathcal{A}(q; X)) \rightarrow H^*(U_{p,q}, U_{p,q} - W(p, q; X)),
\]

we have that \( \beta_q \) determines a Thom class \( \beta_{p,q} \in H^*(U_{p,q}, U_{p,q} - W(p, q; X)) \). Clearly, \( U_{p,q} \) inherits the orientation from \( D(p; X) \). Thus \( \beta_{p,q} \) and the orientation of \( U_{p,q} \) give \( W(p, q; X) \) the orientation.

Since \( h(D(p; X)) = D(p; Y) \), we can define the orientation of \( D(p; Y) \) as \( \alpha'_p = (h^{-1})^* \alpha_q \in H^*(D(p; Y), D(p; Y) - \{p\}) \) for each \( p \). Then the orientations of \( W(p, q; Y) \) are defined. We also have \( h(W(p, q; X)) = W(p, q; Y) \).

**Lemma 8.3.** The topological equivalence \( h \) preserves the orientation of \( W(p, q; X) \).

**Proof.** Choose the open tubular neighborhood \( U_q \) of \( \mathcal{A}(q; X) \) and define \( U_{p,q} = D(p; X) \cap U_q \) as the above. Define \( U'_q = h(U_q) \) and \( U'_{p,q} = h(U_{p,q}) \). We may assume \( U_{p,q} \) is connected.

Suppose the orientation of \( D(q; Y) \) defines the Thom class \( \beta'_q \in H^*(U'_q, U'_q - \mathcal{A}(q; Y)) \) and the Thom class \( \beta'_{p,q} \in H^*(U'_{p,q}, U'_{p,q} - W(p,q; Y)) \). We have the following commutative diagram.

\[
\begin{array}{ccc}
H^*(U'_{p,q}, U'_{p,q} - W(p,q; Y)) & \xrightarrow{h^*} & H^*(U_{p,q}, U_{p,q} - W(p,q; X)) \\
\downarrow & & \downarrow \\
H^*(U'_q, U'_q - \mathcal{A}(q; Y)) & \xrightarrow{h^*} & H^*(U_q, U_q - \mathcal{A}(q; X)) \\
\downarrow & & \downarrow \\
H^*(U'_q \cap D(q; Y), U'_q \cap D(q; Y) - \{q\}) & \xrightarrow{h^*} & H^*(U_q \cap D(q; X), U_q \cap D(q; X) - \{q\})
\end{array}
\]

All of these maps are isomorphisms. The vertical maps are induced by inclusions. Since \( h^* \alpha'_q = \alpha_q \), we have \( h^* \beta'_q = \beta_q \). Thus we get \( h^* \beta'_{p,q} = \beta_{p,q} \).
We also know that \( h \) preserves the orientation of \( U_{p,q} \). By Lemma 8.3 the proof is completed. \( \square \)

As in Theorem 6.7 let \( h_* : \mathcal{M}(p,q;X) \to \mathcal{M}(p,q;Y) \), \( h_* : \mathcal{W}(p,q;X) \to \mathcal{W}(p,q;Y) \) be the maps induced by \( h \). Since \( h \) preserves the direction of flow, by Lemma 8.3 we get the following immediately.

**Lemma 8.4.** The map \( h_* \) preserves the orientation of \( \mathcal{M}(p,q;X) \).

**Proof of Theorem 8.1.** Consider the map \( h_* \) mentioned in the above. Clearly, \( h_* \) is identical to \( h \) on \( \mathcal{D}(p;X) \) and \( \mathcal{W}(p,q;X) \).

By the definition of the orientation of \( \mathcal{D}(p;Y) \), we know \( h_* \) preserves the orientation of \( \mathcal{D}(p,X) \). Combining this fact with Lemmas 8.3 and 8.4 we infer that \( h_* \) preserves both the boundary orientations and the product orientations. Thus \( h_* \) preserves the orientation relations. Since these formulas are proved in the case of \( Y \) in \([21]\) thm. 3.6], we infer that the orientation formulas are valid for \( X \). \( \square \)

### 9. CW Structures

In this section, We shall address the problem of the CW structures arising from a negative gradient-like dynamical system.

**Theorem 9.1.** Suppose \( f \) is proper and bounded below. Suppose \( X \) satisfies transversality. Suppose \( a \) is a regular value of \( f \). Define \( K^a = \bigcup_{f(p) \leq a} \mathcal{D}(p) \) with the topology induced from \( M \). Then \( K^a \) is a finite CW complex with characteristic maps \( e : \mathcal{D}(p) \to K^a \), and each \( e \) has the explicit formula (7.2). The inclusion \( K^a \hookrightarrow M^a \) is a simple homotopy equivalence. In fact, there is a CW decomposition of \( M^a \) such that \( K^a \) expands to \( M^a \) by elementary expansions.

**Theorem 9.2.** Under the assumption of Theorem 9.1 define \( K = \bigcup_{p \in M} \mathcal{D}(p) \).

Define the topology of \( K \) as the direct limit of that of \( K^a \) when \( a \) tends to \( +\infty \). Then \( K \) is a countable CW complex with characteristic maps \( e : \mathcal{D}(p) \to K \), and each \( e \) has the explicit formula (7.2). Furthermore, the inclusion \( i : K \hookrightarrow M \) is a homotopy equivalence.

As mentioned before, \( \text{dim}(\mathcal{M}(p,q)) = \text{ind}(p) - \text{ind}(q) - 1 \) when \( p > q \). If \( \text{ind}(q) = \text{ind}(p) - 1 \), then \( \mathcal{M}(p,q) \) is a 0 dimensional manifold. Actually, \( \mathcal{M}(p,q) \) consists of finitely many points because it is compact in this case.

**Theorem 9.3.** Let \( K^a \) (or \( K^f \)) be the CW complex in Theorem 9.1 (or 9.2). Let \( C_*(K^a) \) (or \( C_*(K^f) \)) be the associated cellular chain complex and \( [\mathcal{D}(p)] \) be the base element represented by the oriented \( \mathcal{D}(p) \) in \( C_*(K^a) \) (or \( C_*(K^f) \)). Then

\[
\partial [\mathcal{D}(p)] = \sum_{\text{ind}(q) = \text{ind}(p) - 1} \# \mathcal{M}(p,q)[\mathcal{D}(q)],
\]

where \( \# \mathcal{M}(p,q) \) is the sum of the orientations \( \pm 1 \) of all points in \( \mathcal{M}(p,q) \) defined in Theorem 8.1 and \( \text{ind}(\ast) \) is the Morse index of \( \ast \).

**Remark 9.1.** Consider the special case when \( M \) is compact. Theorem 9.1 shows that the compactified descending manifolds give a CW decomposition of \( M \). Before the invention of the theory of moduli spaces, this problem was addressed in \([11] \) thm. 1] and \([12] \) rem. 3], which show the existence of the characteristic maps under the assumption that the vector field is locally trivial. The paper \([10] \) sec. 4] (with
a correction in \cite[sec. 4.5]{11}) shows that \(i^a : K^a \hookrightarrow M^a\) is a deformation retract. Theorem 9.1 strengthens their solutions in three ways. Firstly, the characteristic maps here \(e : \mathcal{D}(p) \to M\) have the explicit formula (7.2). Secondly, we drop the assumption of the local triviality of the vector field. Thirdly, it shows that \(K^a\) expands to \(M^a\) by elementary expansions. In the case when \(f\) has only one critical point of index 0, the paper \cite[lem. 2.15]{11} also gives an answer similar to Theorem 9.1.

\begin{remark}
The above theorems show that \(C_*(K)\) computes the homology of \(M\), and its boundary operator \(\partial\) coincides with that of Morse homology. This shows Morse homology arises from a cellular chain complex. For Morse homology, see \cite[cor. 7.3]{14} and \cite{22}.
\end{remark}

\begin{proof}[Proof of Theorem 9.1]
By Theorem 7.8 \(\mathcal{D}(p)\) is a closed disc and \(e\) is continuous. Thus \(K^a\) is a finite CW complex with characteristic maps \(e\).

We shall construct the desired CW decomposition of \(M^a\).

Suppose \(M\) is not compact. By Lemma 5.1 we can embed \(M^a\) into \(\tilde{M}\) and extend \(f|_{M^a}\) to be \(\tilde{f}\) on \(\tilde{M}\) such that \(\tilde{f}|_{\tilde{M} - M^a} > a\). We get \(\tilde{M}^a = M^a\). As a result, we may assume \(M\) is compact.

By Theorem 4.1 we can construct a locally trivial field \(Y\) on \(M\) and a topological equivalence \(h\) which maps the orbits of \(Y\) to those of \(X\). Consequently, \(h(\mathcal{D}(p; Y)) = \mathcal{D}(p; X)\) and \(h(K^a(Y)) = K^a\) where \(K^a(Y) = \bigsqcup_{f(p) \leq a} \mathcal{D}(p; Y)\). By \cite[thm. 3.8]{21}, there exists a CW decomposition of \(M^a\) such that \(K^a(Y)\) expands to \(M^a\) by elementary expansions. Thus it suffices to prove that there exists a homeomorphism \(\tilde{h} : M^a \to M^a\) such that \(\tilde{h}\) and \(h\) coincide on \(K^a(Y)\).

Denote by \(\phi^X_t\) the flow generated by \(X\) and by \(\phi^Y_t\) the flow generated by \(Y\). For any \(x \in f^{-1}(a)\), we have \(\phi^Y_t(-\infty, x) = r_1\) for some \(r_1 \in M - M^a\) and \(\phi^Y_t(\infty, x) = r_2\) for some \(r_2 \in M^a\). Since \(h\) is a topological equivalence fixing \(r_1\) and \(r_2\), we have \(\phi^X_t(h(x))\) is a flow line between \(r_1\) and \(r_2\). Thus, for any \(x \in h(f^{-1}(a))\), we have \(\phi^X_t(t(x), x) \in f^{-1}(a)\) for some \(t(x)\) and, by Lemma 2.4, \(t(x)\) is continuous on \(h(f^{-1}(a))\). Since \(h(f^{-1}(a))\) is compact, there exists \(T > 0\) such that \(T > -t(x)\) for all \(x \in h(f^{-1}(a))\). As a result, \(\phi^X_T(M^a) \subseteq \text{Int}[h(M^a)]\). (This is illustrated by Figure 6.) \(\phi^X_T(M^a)\) is the shadowed part, \(M^a\) is the part below \(f^{-1}(a)\) and \(h(M^a)\) is the part below \(h(f^{-1}(a))\). By an isotopy along the flows generated by \(X\), we can construct a homeomorphism \(\psi : h(M^a) \to M^a\) such that \(\psi|_{\phi^X_T(M^a)} = \text{Id}\) and \(\psi(h(M^a) - \phi^X_T(M^a)) = M^a - \phi^X_T(M^a)\). Then \(\tilde{h} = \psi \circ h\) is the desired homeomorphism.
\end{proof}

\begin{proof}[Proof of Theorem 9.3]
The CW structure of \(K\) is obvious.

By Theorem 9.1 \(i : K^a \hookrightarrow M^a\) is a homotopy equivalence for any regular value \(a\). Thus, it’s straightforward to check that \(i : K \to M\) is a weak homotopy equivalence, i.e. \(i\) induces the isomorphisms between homotopy groups. Since \(M\) carries a triangulation, by Whitehead’s Theorem, \(i\) is a homotopy equivalence.
\end{proof}

\begin{proof}[Proof of Theorem 9.3]
There are two proofs.

First, duplicate the proof of \cite[thm. 3.9]{21}. Certainly, the local triviality of the vector field \(X\) is assumed in \cite{21}. However, the only reason for making this assumption is that the (2) of Theorem 8.1 was proved under it in \cite{21}. In this
paper, this orientation formula is true even if we drop this assumption. Thus, the first proof is valid.

Second, reduce it to the case of a locally trivial vector field $Y$. The map $h_*$ in Theorem 6.7 induces an isomorphism between $C_*(K^a(X))$ and $C_*(K^a(Y))$. By Lemma 8.4 $h_*$ preserves the orientation of $M(p, q; X)$. Since this statement is true for $C_*(K^a(Y))$, the second proof is complete.

□

Appendix A.

In this appendix, we shall prove Lemma 8.2. Suppose $M$ is an $n$ dimensional manifold. Suppose $L$ is a connected and closed $k$ dimensional submanifold of $M$. Let $U$ be a closed tubular neighborhood of $L$ such that $U$ is diffeomorphic to a closed disk bundle over $L$ via the exponential map. Let $i : L \rightarrow U$ be the inclusion and $\pi : U \rightarrow L$ be the smooth projection. Clearly, $i$ and $\pi$ are proper. Thus $\pi^* : H^k_c(L) \rightarrow H^k_c(U)$ and $i^* : H^k_c(U) \rightarrow H^k_c(L)$ are isomorphisms and they are a pair of inverses, where $H^*_c$ is the cohomology with compact support. Furthermore, $H^k_c(L) \cong \mathbb{Z}$, its generator is an orientation of $L$.

Define $H^a_c(U, U - L) = \lim_{K \subseteq L} H^a(U, U - K)$, where $K$ is compact. We can prove the inclusion $H^n(U, U - \{x\}) \rightarrow H^a_c(U, U - L)$ is an isomorphism for any $x \in L$.

Suppose $\alpha \in H^k_c(L)$ and the Thom class $\beta \in H^{n-k}(U, U - L)$ represent the orientation and the normal orientation of $L$ respectively. Suppose the orientation and the normal orientation define the orientation of $M$.

Lemma A.1. The following cup product homomorphism is an isomorphism.

$$H^k_c(U) \otimes H^{n-k}(U, U - L) \xrightarrow{\cup} H^a_c(U, U - L).$$

Furthermore, for all $x \in L$, via the isomorphism $H^n(U, U - \{x\}) \rightarrow H^a_c(U, U - L)$, we get $\pi^* \alpha \cup \beta \in H^a_c(U, U - L)$ represents the orientation of $M$ in $H^n(U, U - \{x\})$. 

Figure 6. Construction of $\psi$
Proof. For any $x \in L$, we have a commutative diagram

$$
\begin{array}{c}
H^k(U, U - \pi^{-1}(x)) \otimes H^{n-k}(U, U - L) \overset{\cup}{\longrightarrow} H^n(U, U - \{x\}) \\
\cong \\
H^k(U) \otimes H^{n-k}(U, U - L) \overset{\cup}{\longrightarrow} H^n(U, U - L).
\end{array}
$$

Here the vertical maps are induced by inclusions and are isomorphisms. The horizontal ones are given by cup product pairings. By excision and the basic property of Thom class, we can localize the argument near $x$. However, the disk bundle near $x$ has a product structure. Now apply Künneth Formula to the upper horizontal map, which completes the proof. \hfill \Box

Proof of Lemma 8.2. It suffices to prove the special case of that $L_i$ is connected.

Let $U_2$ be a closed tubular neighborhood of $L_2$ with the smooth projection $\pi_2 : U_2 \rightarrow L_2$. Let $\alpha_2 \in H^k_0(L_2)$ be the orientation of $L_2$, by the above lemma, we have $\pi_2^* \alpha_2 \cup \beta_2|_{U_2} = \gamma_2 \in H^n_0(U_2, U_2 - L_2)$ represents the orientation of $M_2$ on $L_2$. Here $\beta_2|_{U_2}$ is the image of $\beta_2$ under the inclusion $H^{n-k}(M_2, M_2 - L_2) \rightarrow H^{n-k}(U_2, U_2 - L_2)$. It is the restriction of $\beta_2$ to $U_2$.

Let $U'_1 = h^{-1}(U_2)$. Choose a closed tubular neighborhood $U_1$ of $L_1$ such that $U_1 \subseteq \text{Int} U'_1$ and $\pi_1 : U_1 \rightarrow L_1$ is a smooth projection. By the above lemma again, we have the following isomorphism

$$
H^k_0(U_1) \otimes H^{n-k}(U_1, U_1 - L_1) \overset{\cup}{\cong} H^n_0(U_1, U_1 - L_1),
$$

and

(A.1) 

$$
\pi_1^* \alpha_1 \cup \beta_1|_{U_1} = \gamma_1
$$

represents the orientation of $M_1$ on $L_1$, where $\beta_1|_{U_1}$ is the restriction of $\beta_1$ to $U_1$.

Consider the following commutative diagram:

$$
\begin{array}{c}
H^k_0(U_2) \overset{h^*}{\longrightarrow} H^k_0(U'_1) \overset{i_2^*}{\longrightarrow} H^k_0(U_1) \\
\downarrow \quad \quad \downarrow \\
H^k_0(L_2) \overset{h}{\longrightarrow} H^k_0(L_1),
\end{array}
$$

where, $i_1$, $i_2$, $j$ and $\iota$ are inclusions. Since $h^* \pi_2^* \alpha_2 \cup h^* \beta_2|_{U_2} = h^* \gamma_2$, we have $\iota^* h^* \pi_2^* \alpha_2 \cup \iota^* h^* \beta_2|_{U_2} = \iota^* h^* \gamma_2$. Since $h$ preserves the orientation of $M_1$ and the Thom class, we have $\iota^* h^* \gamma_2 = \gamma_1$ and $\iota^* h^* \beta_2|_{U_2} = \beta_1|_{U_1}$. Thus

(A.2) 

$$
\iota^* h^* \pi_2^* \alpha_2 \cup \beta_1|_{U_1} = \gamma_1.
$$

Since the cup product pairing above is an isomorphism, by (A.1) and (A.2), we infer $\iota^* h^* \pi_2^* \alpha_2 = \pi_1^* \alpha_1$. So we have

$$
\alpha_1 = i_1^* \pi_1^* \alpha_1 = i_1^* \iota^* h^* \pi_2^* \alpha_2 = h^* i_2^* \pi_2^* \alpha_2 = h^* \alpha_2.
$$

This completes the proof. \hfill \Box
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