Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions

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Abstract

Estimates on the initial coefficients are obtained for normalized analytic functions in the open unit disk with and its inverse satisfying the conditions that and are both subordinate to a starlike univalent function whose range is symmetric with respect to the real axis. Several related classes of functions are also considered, and connections to earlier known results are made.

Keywords: Univalent functions, bi-univalent functions, bi-starlike functions, bi-convex functions, subordination

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1. Introduction

Let be the class of all analytic functions defined in the unit disk . A domain is convex if the line segment joining any two points in is entirely in . While a domain is starlike with respect to a point if the line segment joining any point of to lies inside . A function is starlike if is a starlike domain with respect to the origin, and convex if . Analytically, is starlike if and only if and convex if and only if , whereas is convex and if and only if and . The classes consisting of starlike and convex functions are denoted by and, respectively. The classes and of starlike and convex functions of order are respectively characterized by and . Various subclasses of starlike and convex functions are often investigated. These functions are typically characterized by the quantity or lying in a certain domain starlike with respect to in the right-half plane. Subordination is useful to unify these subclasses.

An analytic function is subordinate to an analytic function , written , provided there is an analytic function defined on with and . Ma and Minda unified various subclasses of starlike and convex functions for which either of the quantity or is subordinate to a more general superordinate function. For this purpose, they considered an analytic function with positive real part in the unit disk , , and maps onto a region starlike with respect to and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions lying in a certain domain starlike with respect to and symmetric with respect to the real axis. Similarly, the class of Ma-Minda convex functions consists of functions lying in a certain domain convex with respect to and symmetric with respect to the real axis.

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1 + zf''(z)/f'(z) ≤ φ(z). A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f⁻¹ are respectively Ma-Minda starlike or convex. These classes are denoted respectively by \( ST_\sigma(\varphi) \) and \( CV_\sigma(\varphi) \).

Lewin [8] investigated the class \( \sigma \) of bi-univalent functions and obtained the bound for the second coefficient. Several authors have subsequently studied similar problems in this direction (see [10, 6]). Brannan and Taha [5] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike and Taha [5] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike and convex functions and obtained estimates on the initial coefficients. Recently, Srivastava et al. [12] introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients. Several related classes are also considered, and connection to earlier known results are made. The classes introduced in this paper are motivated by the corresponding classes investigated in [2].

2. Coefficient Estimates

In the sequel, it is assumed that \( \varphi \) is an analytic function with positive real part in the unit disk \( \mathbb{D} \), with \( \varphi(0) = 1, \varphi'(0) > 0 \), and \( \varphi(\mathbb{D}) \) is symmetric with respect to the real axis. Such a function has a series expansion of the form

\[
\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad (B_1 > 0).
\]

(2.1)

A function \( f \in A \) with \( \text{Re}(f'(z)) > 0 \) is known to be univalent. This motivates the following class of functions. A function \( f \in \sigma \) is said to be in the class \( \mathcal{H}_\sigma(\varphi) \) if the following subordinations hold:

\[
f'(z) \prec \varphi(z) \quad \text{and} \quad g'(w) \prec \varphi(w), \quad g(w) := f^{-1}(w).
\]

(2.2)

For functions in the class \( \mathcal{H}_\sigma(\varphi) \), the following result is obtained.

**Theorem 2.1.** If \( f \in \mathcal{H}_\sigma(\varphi) \) is given by

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

(2.3)

then

\[
|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{3B_1^2 - 4B_2 + 4B_1}} \quad \text{and} \quad |a_3| \leq \left( \frac{1}{3} + \frac{B_1}{4} \right) B_1.
\]

(2.4)

**Proof.** Let \( f \in \mathcal{H}_\sigma(\varphi) \) and \( g = f^{-1} \). Then there are analytic functions \( u, v : \mathbb{D} \to \mathbb{D} \), with \( u(0) = v(0) = 0 \), satisfying

\[
f'(z) = \varphi(u(z)) \quad \text{and} \quad g'(w) = \varphi(v(w)).
\]

(2.5)

Define the functions \( p_1 \) and \( p_2 \) by

\[
p_1(z) := \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \cdots \quad \text{and} \quad p_2(z) := \frac{1 + v(z)}{1 - v(z)} = 1 + b_1 z + b_2 z^2 + \cdots,
\]

(2.6)

or, equivalently,

\[
u(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2} \left( c_1 z + \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots \right)
\]

(2.7)

and

\[
v(z) = \frac{p_2(z) - 1}{p_2(z) + 1} = \frac{1}{2} \left( b_1 z + \left( b_2 - \frac{b_1^2}{2} \right) z^2 + \cdots \right).
\]

(2.8)
Then $p_1$ and $p_2$ are analytic in $D$ with $p_1(0) = 1 = p_2(0)$. Since $u, v : D \to D$, the functions $p_1$ and $p_2$ have positive real part in $D$, and $|b_i| \leq 2$ and $|c_i| \leq 2$. In view of (2.4), (2.5) and (2.6), clearly

$$f'(z) = \varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) \quad \text{and} \quad g'(w) = \varphi \left( \frac{p_2(w) - 1}{p_2(w) + 1} \right). \quad (2.7)$$

Using (2.5) and (2.6) together with (2.1), it is evident that

$$\varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = 1 + \frac{1}{2} B_1 c_1 z + \left( \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right) z^2 + \cdots \quad (2.8)$$

and

$$\varphi \left( \frac{p_2(w) - 1}{p_2(w) + 1} \right) = 1 + \frac{1}{2} B_1 b_1 w + \left( \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2 \right) w^2 + \cdots. \quad (2.9)$$

Since $f \in \sigma$ has the Maclaurin series given by (2.2), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots.$$ 

Since

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + \cdots \quad \text{and} \quad g'(w) = 1 - 2a_2 w + 3(2a_2^2 - a_3)w^2 + \cdots,$$

it follows from (2.7), (2.8) and (2.9) that

$$2a_2 = \frac{1}{2} B_1 c_1, \quad (2.10)$$

$$3a_3 = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2, \quad (2.11)$$

$$-2a_2 = \frac{1}{2} B_1 b_1 \quad (2.12)$$

and

$$3(2a_2^2 - a_3) = \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2. \quad (2.13)$$

From (2.10) and (2.12), it follows that

$$c_1 = -b_1. \quad (2.14)$$

Now (2.11), (2.13), (2.14) and (2.12) yield

$$a_2^2 = \frac{B_2^2 b_2 + c_2}{4(3B_1^2 - 4B_2 + 4B_1)},$$

which, in view of the well-known inequalities $|b_2| \leq 2$ and $|c_2| \leq 2$ for functions with positive real part, gives us the desired estimate on $|a_2|$ as asserted in (2.3).

By subtracting (2.13) from (2.11), further computations using (2.10) and (2.14) lead to

$$a_3 = \frac{1}{12} B_1 (c_2 - b_2) + \frac{1}{16} B_1^2 c_1^2,$$

and this yields the estimate given in (2.3).
Remark 2.1. For the class of strongly starlike functions, the function $\varphi$ is given by

$$\varphi(z) = \left(\frac{1 + z}{1 - z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots \quad (0 < \gamma \leq 1),$$

which gives $B_1 = 2\gamma$ and $B_2 = 2\gamma^2$. Hence the inequalities in (2.3) reduce to the result in [12, Theorem 1, inequality (2.4), p.3]. In the case

$$\varphi(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} = 1 + 2(1 - \gamma)z + 2(1 - \gamma)z^2 + \cdots ,$$

then $B_1 = B_2 = 2(1 - \gamma)$, and thus the inequalities in (2.3) reduce to the result in [12, Theorem 2, inequality (3.3), p.4].

A function $f \in \sigma$ is said to be in the class $\mathcal{ST}_\sigma(\alpha, \varphi)$, $\alpha \geq 0$, if the following subordinations hold:

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} \prec \varphi(z) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} \prec \varphi(w), \quad g(w) := f^{-1}(w).$$

Note that $\mathcal{ST}_\sigma(\varphi) \equiv \mathcal{ST}_\sigma(0, \varphi)$. For functions in the class $\mathcal{ST}_\sigma(\alpha, \varphi)$, the following coefficient estimates are obtained.

**Theorem 2.2.** Let $f$ given by (2.2) be in the class $\mathcal{ST}_\sigma(\alpha, \varphi)$. Then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|B_1^4(1 + 4\alpha) + (B_1 - B_2)(1 + 2\alpha)^2|}}, \quad (2.15)$$

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{(1 + 4\alpha)}. \quad (2.16)$$

**Proof.** Let $f \in \mathcal{ST}_\sigma(\alpha, \varphi)$. Then there are analytic functions $u, v : \mathbb{D} \rightarrow \mathbb{D}$, with $u(0) = v(0) = 0$, satisfying

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(u(z)) \quad \text{and} \quad \frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} = \varphi(v(w)), \quad (g = f^{-1}). \quad (2.17)$$

Since

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1 + 2\alpha)z + (2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2)z^2 + \cdots$$

and

$$\frac{wg'(w)}{g(w)} + \frac{\alpha w^2 g''(w)}{g(w)} = 1 - (1 + 2\alpha)a_2w + ((3 + 10\alpha)a_2^2 - 2(1 + 3\alpha)a_3)w^2 + \cdots ,$$

then (2.3), (2.9) and (2.17) yield

$$a_2(1 + 2\alpha) = \frac{1}{2} B_1 c_1, \quad (2.18)$$

$$2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2 = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2, \quad (2.19)$$

$$-(1 + 2\alpha)a_2 = \frac{1}{2} B_1 b_1 \quad (2.20)$$

and

$$(3 + 10\alpha)a_2^2 - 2(1 + 3\alpha)a_3 = \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2. \quad (2.21)$$
It follows from (2.18) and (2.20) that
\[ c_1 = -b_1. \]

Equations (2.19), (2.20), (2.21) and (2.22) lead to
\[ a_2^2 = \frac{B_0^2(b_2 + c_2)}{4(B_1^2(1 + 4\alpha) + (B_1 - B_2)(1 + 2\alpha)^2)}, \]
which, in view of the inequalities \(|b_2| \leq 2\) and \(|c_2| \leq 2\) for functions with positive real part, yield
\[ |a_2|^2 \leq \frac{B_0^2}{B_1^2(1 + 4\alpha) + (B_1 - B_2)(1 + 2\alpha)^2}. \]
Since \(B_1 > 0\), the last inequality, upon taking square roots, gives the desired estimate on \(|a_2|\) given in (2.16).

Now, further computations from (2.19), (2.20), (2.21) and (2.22) lead to
\[ a_3 = \frac{(B_1/2)((3 + 10\alpha)c_2 + (1 + 2\alpha)b_2) + b_1^2(1 + 3\alpha)(B_2 - B_1)}{4(1 + 3\alpha)(1 + 4\alpha)}, \]
which, using the inequalities \(|b_1| \leq 2\), \(|b_2| \leq 2\) and \(|c_2| \leq 2\) for functions with positive real part, yields
\[ |a_3| \leq \frac{(B_1/2)(2(3 + 10\alpha) + 2(1 + 2\alpha)) + 4(1 + 3\alpha)(B_2 - B_1)}{4(1 + 3\alpha)(1 + 4\alpha)} = \frac{B_1 + |B_2 - B_1|}{(1 + 4\alpha)}. \]
This completes the proof of the estimate in (2.16).

For \(\alpha = 0\), Theorem 2.2 readily yields the following coefficient estimates for Ma-Minda bi-starlike functions.

**Corollary 2.1.** Let \(f\) given by (2.2) be in the class \(ST_\sigma(\varphi)\). Then
\[ |a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{|B_1^2 + B_1 - B_2|}} \quad \text{and} \quad |a_3| \leq B_1 + |B_2 - B_1|. \]

**Remark 2.2.** For the class of strongly starlike functions, the function \(\varphi\) is given by
\[ \varphi(z) = \left(1 + \frac{z}{1-z}\right)^\gamma = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots \quad (0 < \gamma \leq 1), \]
and so \(B_1 = 2\gamma\) and \(B_2 = 2\gamma^2\). Hence, when \(\alpha = 0\) (bi-starlike function), the inequality in (2.16) reduces to the estimates in [5, Theorem 2.1]. On the other hand, when \(\alpha = 0\) and
\[ \varphi(z) = \frac{1 + (1 - 2\gamma)z}{1-z} = 1 + 2(1 - \gamma)z + 2(1 - \gamma)z^2 + \cdots, \]
then \(B_1 = B_2 = 2(1 - \gamma)\) and thus the inequalities in (2.15) and (2.16) reduce to the estimates in [5, Theorem 3.1].

Next, a function \(f \in \sigma\) belongs to the class \(M_\sigma(\alpha, \varphi)\), \(\alpha \geq 0\), if the following subordinations hold:
\[ (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \]
and
\[ (1 - \alpha)\frac{wg'(w)}{g(w)} + \alpha \left(1 + \frac{wg''(w)}{g'(w)}\right) \prec \varphi(w), \]
g(\(w\)) := \(f^{-1}(w)\). A function in the class \(M_\sigma(\alpha, \varphi)\) is called bi-Mocanu-convex function of Ma-Minda type. This class unifies the classes \(ST_\sigma(\varphi)\) and \(CV_\sigma(\varphi)\).

For functions in the class \(M_\sigma(\alpha, \varphi)\), the following coefficient estimates hold.
Theorem 2.3. Let $f$ given by (2.1) be in the class $\mathcal{M}_\sigma(\alpha, \varphi)$. Then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{(1 + \alpha)|B_1^2 + (1 + \alpha)(B_1 - B_2)|}}$$ \hspace{1cm} (2.23)

and

$$|a_3| \leq \frac{B_1 + |B_2 - B_1|}{1 + \alpha}.$$ \hspace{1cm} (2.24)

Proof. If $f \in \mathcal{M}_\sigma(\alpha, \varphi)$, then there are analytic functions $u, v : \mathbb{D} \rightarrow \mathbb{D}$, with $u(0) = v(0) = 0$, such that

$$(1 - \alpha)zf'(z) + \alpha \left( 1 + zx \right) f'(z) = \varphi(u(z))$$ \hspace{1cm} (2.25)

and

$$(1 - \alpha)wg'(w) + \alpha \left( 1 + \frac{w^2 g''(w)}{g'(w)} \right) = \varphi(v(w)).$$ \hspace{1cm} (2.26)

Since

$$(1 - \alpha)zf'(z) + \alpha \left( 1 + \frac{f''(z)}{f'(z)} \right) = 1 + (1 + \alpha)a_2 z + (2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2)z^2 + \cdots$$

and

$$(1 - \alpha)wg'(w) + \alpha \left( 1 + \frac{w^2 g''(w)}{g'(w)} \right) = 1 - (1 + \alpha)a_2 w + ((3 + 5\alpha)a_2^2 - 2(1 + 2\alpha)a_3)w^2 + \cdots,$$

from (2.28), (2.29), (2.25) and (2.26), it follows that

$$(1 + \alpha)a_2 = \frac{1}{2} B_1 c_1,$$ \hspace{1cm} (2.27)

$$2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2 = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2,$$ \hspace{1cm} (2.28)

$$-(1 + \alpha)a_2 = \frac{1}{2} B_1 b_1$$ \hspace{1cm} (2.29)

and

$$(3 + 5\alpha)a_2^2 - 2(1 + 2\alpha)a_3 = \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2.$$ \hspace{1cm} (2.30)

The equations (2.27) and (2.29) yield

$$c_1 = -b_1.$$ \hspace{1cm} (2.31)

From (2.28), (2.30) and (2.31), it follows that

$$a_2^2 = \frac{B_1^3 (b_2 + c_2)}{4(1 + \alpha)(B_1^2 + (1 + \alpha)(B_1 - B_2))},$$

which yields the desired estimate on $|a_2|$ as described in (2.23).

As in the earlier proofs, use of (2.28), (2.29), (2.30) and (2.31) shows that

$$a_3 = \frac{(B_1/2)((1 + 3\alpha)b_2 + (3 + 5\alpha)c_2) + b_1^2(1 + 2\alpha)(B_2 - B_1)}{4(1 + \alpha)(1 + 2\alpha)},$$

which yields the estimate (2.24). \hfill \square
For $\alpha = 0$, Theorem [2.3] gives the coefficient estimates for Ma-Minda bi-starlike functions, while for $\alpha = 1$, it gives the following estimates for Ma-Minda bi-convex functions.

Corollary 2.2. Let $f$ given by (2.1) be in the class $CV_\sigma(\varphi)$. Then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|2B_1^2 + 2B_1 - 2B_2|}} \text{ and } |a_3| \leq \frac{1}{2} (B_1 + |B_2 - B_1|).$$

Remark 2.3. For $\varphi$ given by

$$\varphi(z) = \frac{1 + (1 - 2\gamma)z}{1 - z} = 1 + 2(1 - \gamma)z + 2(1 - \gamma)z^2 + \cdots,$$
evidently $B_1 = B_2 = 2(1 - \gamma)$, and thus when $\alpha = 1$ (bi-convex functions), the inequalities in (2.23) and (2.24) reduce to a result in [3, Theorem 4.1].

Next, function $f \in \sigma$ is said to be in the class $L_\sigma(\alpha, \varphi)$, $\alpha \geq 0$, if the following subordinations hold:

$$\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} \varphi(z)$$

and

$$\left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} \varphi(w),$$

g(w) := f^{-1}(w). This class also reduces to the classes of Ma-Minda bi-starlike and bi-convex functions. For functions in this class, the following coefficient estimates are obtained.

Theorem 2.4. Let $f$ given by (2.1) be in the class $L_\sigma(\alpha, \varphi)$. Then

$$|a_2| \leq \frac{2B_1 \sqrt{B_1}}{\sqrt{|2(\alpha^2 - 3\alpha + 4)B_1^2 + 4(\alpha - 2)\gamma^2(B_1 - B_2)|}}$$

and

$$|a_3| \leq \frac{2(3 - 2\alpha)(B_1 + |B_1 - B_2|)}{|(3 - 2\alpha)(\alpha^2 - 3\alpha + 4)|}.$$ (2.33)

Proof. Let $f \in L_\sigma(\alpha, \varphi)$. Then there are analytic functions $u, v : D \to D$, with $u(0) = v(0) = 0$, such that

$$\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} = \varphi(u(z))$$ (2.34)

and

$$\left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} = \varphi(v(w)).$$ (2.35)

Since

$$\left( \frac{zf'(z)}{f(z)} \right)^\alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right)^{1-\alpha} = 1 + (2 - \alpha)a_2 z + \left( 2(3 - 2\alpha)a_3 + \frac{(\alpha - 2)^2 - 3(4 - 3\alpha)a_3^2}{2} \right) z^2 + \cdots$$

and

$$\left( \frac{wg'(w)}{g(w)} \right)^\alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right)^{1-\alpha} = 1 - (2 - \alpha)a_2 w + \left( 8(1 - \alpha) + \frac{1}{2}(\alpha + 5)\alpha^2 - 2(3 - 2\alpha)a_3 \right) w^2 + \cdots,$$

from (2.28), (2.29), (2.34) and (2.35), it follows that

$$(2 - \alpha)a_2 = \frac{1}{2} B_1 c_1,$$ (2.36)
\[ 2(3 - 2\alpha)a_3 + \left[ (\alpha - 2)^2 - 3(4 - 3\alpha) \right] \frac{a_2^2}{2} = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2, \]  
(2.37)

\[ -(2 - \alpha)a_2 = \frac{1}{2} B_1 b_1 \]  
(2.38)

and

\[ \left( 8(1 - \alpha) + \frac{1}{2} \alpha(\alpha + 5) \right) a_2^2 - 2(3 - 2\alpha)a_3 = \frac{1}{2} B_1 \left( b_2 - \frac{b_1^2}{2} \right) + \frac{1}{4} B_2 b_1^2. \]  
(2.39)

Now (2.36) and (2.38) clearly yield

\[ c_1 = -b_1. \]  
(2.40)

Equations (2.37), (2.39) and (2.40) lead to

\[ a_2^2 = \frac{B_1^2 (b_2 + c_2)}{2(a^2 - 3\alpha + 4)B_1^2 + 4(\alpha - 2)^2(B_1 - B_2)}, \]
which yields the desired estimate on \( |a_2| \) as asserted in (2.32).

Proceeding similarly as in the earlier proof, using (2.37), (2.38), (2.39) and (2.40), it follows that

\[ a_3 = \frac{(B_1/2) \left( 16(1 - \alpha) + \alpha(\alpha + 5) \right) c_2 + (3(4 - 3\alpha) - (\alpha - 2)^2)b_2 + 2b_1^2(3 - 2\alpha)(B_1 - B_2)}{4(3 - 2\alpha)(a^2 - 3\alpha + 4)}, \]

which yields the estimate (2.33).

\[ \square \]

**Remark 2.4.** The determination of the sharp estimates for the coefficients \(|a_2|\), \(|a_3|\) and for other coefficients of functions belonging to the classes investigated in this paper are open problems. In fact, some estimate (not necessarily sharp) for \(|a_n|\), \((n \geq 4)\) would be interesting.

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