Phase Structure of QED$_3$ at Finite Temperature

I.J.R. Aitchison$^{(a)}$, N. Dorey$^{(b)}$, M. Klein-Kreisler$^{(a)}$ and N.E. Mavromatos$^{(c)}$

$^{(a)}$ Department of Theoretical Physics, 1 Keble Rd, Oxford OX1 3NP, UK.
$^{(b)}$ Theoretical Division T-8, MS B285, Los Alamos National Laboratory,
Los Alamos, NM 87545, USA.
$^{(c)}$ Theory Division, CERN, CH-1211, Geneva 23, Switzerland.

Abstract

Dynamical symmetry breaking in three-dimensional QED with $N$ fermion flavours is considered at finite temperature, in the large $N$ approximation. Using an approximate treatment of the Schwinger-Dyson equation for the fermion self-energy, we find that chiral symmetry is restored above a certain critical temperature which depends itself on $N$. We find that the ratio of the zero-momentum zero-temperature fermion mass to the critical temperature has a large value compared with four-fermion theories, as had been suggested in a previous work with a momentum-independent self-energy. Evidence of a temperature-dependent critical $N$ is shown to appear in this approximation. The phase diagram for spontaneous mass generation in the theory is presented in $T - N$ space.
1 Introduction

Quantum Electrodynamics in (2+1) dimensions (QED$_3$) has attracted considerable interest in the last few years [1] - [6]. One reason for this is that it provides a simple setting for the study of dynamical chiral symmetry breaking which is important for theories such as QCD. Furthermore QED$_3$ also appears to be relevant to some long-wavelength models of certain 2D condensed matter systems, including high-T$_c$ superconductors [8, 9]. At zero temperature a number of studies have shown that chiral symmetry is dynamically broken in QED$_3$. Using the leading order in the $1/N$ expansion of the Schwinger-Dyson (SD) equations Appelquist et al. [2] showed that the theory exhibited a critical behaviour as the number $N$ of fermion flavours approached $N_c = 32/\pi^2$; that is, a fermion mass was dynamically generated only for $N < N_c$. Qualitatively the same behaviour was found by Nash [5], who included $O(1/N^2)$ corrections. As against this, Pennington and collaborators [3], adopting a more general non-perturbative approach to the SD equations, found that the dynamically generated fermion mass decreased exponentially with $N$, vanishing only as $N \rightarrow \infty$ (as originally found by Pisarski [1] using a simplified form of the SD equation). On the other hand, an alternative non-perturbative study by Atkinson et al. [6] suggested that chiral symmetry is indeed unbroken at sufficiently large $N$. We should also mention that Pisarski [7] has used the renormalization group approach and the $\epsilon$ expansion to argue that chiral symmetry remains broken for all $N$, but this result has to be interpreted with some caution as the relevant value of $\epsilon$ is 1. The theory has also been simulated on the lattice [4] and the results appear to be consistent with the existence of a critical $N$ as predicted in the analysis of ref [2]. On the other hand, because Monte-Carlo simulations are performed on lattices of finite size $L$ and typically cannot detect mass scales less than the IR cutoff scale $1/L$, the persistence of an exponentially small fermion mass for large $N$, as suggested in [8], cannot be ruled out.

The extension of the above type of analysis to finite temperature is extremely important, and highly relevant to either application already mentioned. The most obvious question is whether there is a critical temperature $T_c$ above which chiral symmetry is restored. This was first answered affirmatively by Kocic [10], using a very simple approximation to the finite temperature SD equations, in which the entropy of the fermions was not fully taken into account. An improved calculation was made by Dorey and Mavromatos [11], based on the finite temperature Schwinger-Dyson equations, to leading order in $1/N$. These authors found that the ratio $r$ of twice the zero-temperature mass to the critical temperature was approximately independent of $N$ and much larger ($r \approx 10$) than the value obtained in (BCS-like) four fermion theories ($r$ at most 3.5). The latter result could be relevant to an understanding of this ratio in the high-$T_c$ superconductors. In ref [11], however, the (leading order in $1/N$) SD equation for the fermion self-energy function $\Sigma(p)$ was considerably simplified by making the assumption [1] that $\Sigma$ was in fact a constant, independent of $p$. It is clearly necessary to be assured that the results of [11] are essentially independent of this assumption.

In the present paper we extend and complete the analysis of Ref. [11]. We again start from the Schwinger-Dyson equation for $\Sigma(p)$, to leading order in $1/N$, but keep the full momentum dependence of $\Sigma$. Following the approach of [11], and of similar studies of chiral symmetry restoration in four dimensions [12], we further simplify the SD equation by adopting an instantaneous approximation for the kernel. This corresponds to retaining only the part of the kernel
corresponding to the static interaction between charges. We find that the main conclusions of Ref. [11] are indeed robust—in particular, the ratio \( r \) has almost the same value.

A second question concerns the existence of a critical number of flavours \( N_c \), above which chiral symmetry would be restored. Within the \( 1/N \) approach, there is no \( N_c \) if \( \Sigma \) is approximated by a constant [1], as already noted, a result which naturally persists at finite \( T \) in the approximation [11]; but with a \( p \)-dependent \( \Sigma \) we find a (temperature dependent) \( N_c \), analogous to Ref. [2]. We are thus able to present the resulting phase structure of the model, as a function of the variables \( N \) and \( T \): there is a single critical line separating the region of low \((N,T)\) where \( \Sigma \neq 0 \) from the region of high \((N,T)\) where \( \Sigma = 0 \) (see Figure 7 below). As indicated above, we are aware that criticism have been raised [3] concerning the reliability of the \( 1/N \) approach for calculating fermion masses \( \Sigma \) well below the intrinsic scale set by the dimensionful coupling \( \alpha \) (as happens in this case) and it may be that in the exact theory no sharply defined \( N_c \) exists (though it would seem very difficult to establish this conclusively, given that some approximation to the complete set of SD equations has to be made). However, at the very least, the lattice results at \( T = 0 \) strongly indicate the existence of an effective critical region, in which the dynamical fermion mass undergoes a rapid crossover from a regime of large values to one of very small values as \( N \) is increased. We shall take the view here that our \( 1/N \)-based phase diagram should give a qualitatively correct picture in this sense—and that in any case there is merit in obtaining as complete an analysis as possible of one definite model at finite temperature. It would clearly be of interest to investigate the effect, at finite temperature, of the non-perturbative vertex structure advocated in Ref. [3].

2 Momentum-dependent self-energy (gap) equation at finite temperature

The Lagrangian for massless QED\( _3 \) with \( N \) flavours is

\[
\mathcal{L} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} + \bar{\psi}_i (i \partial - e A_\mu) \psi_i \tag{1}
\]

where \( a_\mu \) is the vector potential, \( i = 1, 2, \ldots N \), and a reducible four dimensional Dirac algebra has been chosen so that [1] has a continuous chiral symmetry. The conventions of [1] will be adopted throughout.

The Schwinger-Dyson equation for the fermion propagator at non-zero temperature \( k_B T = \beta^{-1} \) is given by

\[
S_F^{-1}(p_0, P, \beta) = S_F^{(0)-1}(p) - \frac{e}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2k}{(2\pi)^2} \gamma^\mu S_F(k_0, K, \beta) \Delta_{\mu\nu}(q_0, Q, \beta) \Gamma^\nu_{\beta} \tag{2}
\]

where

\[
\begin{align*}
p & = (p_0, P) & P & = |P| & p_0 & = (2m + 1)\pi/\beta \\
k & = (k_0, k) & K & = |K| & k_0 & = (2n + 1)\pi/\beta \\
q & = (q_0, q) & Q & = |Q| & q_0 & = 2(m - n)\pi/\beta
\end{align*}
\]
As stated above, we truncate (2) by working at leading order in $1/N$, in which case $\Gamma^\nu$ is replaced by its bare value $e\gamma^\nu$ and $\Delta_{\mu\nu}$ by the $O(1/N)$ propagator shown in Figure 1, in which the fermions are massless (the massless vacuum polarisation loop already softens the photon propagator [1], and the exact form of this softening does not qualitatively change the behaviour of the fermion propagator [3]). We shall work in Landau gauge and assume that the wave function renormalisation can be neglected to leading order in $1/N$, so that $S^{-1}F = p + \Sigma_m(P,\beta)$ (note however the criticism made of this step by Pennington and Walsh [3]). The trace of equation (2) then yields a closed integral equation for $\Sigma_m$:

$$
\Sigma_m(p) = \frac{\alpha}{N\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2} \Delta(q_0, Q, \beta) \frac{\Sigma_n(K, \beta)}{k^2 + \Sigma_n^2(K, \beta)},
$$

(4)

with

$$
\Delta(q_0, Q, \beta) = \frac{1}{8} T \gamma^\mu \Delta_{\mu\nu}(q_0, Q, \beta) \gamma^\nu
$$

(5)

and $\alpha = Ne^2$. We now follow Ref. [11] in retaining only the $\mu = \nu = 0$ component of $\Delta_{\mu\nu}$ at zero frequency and thus set,

$$
\Delta_{\mu\nu}(q_0, Q, \beta) = \frac{\delta_{\mu0}\delta_{\nu0}}{Q^2 + \Pi_0(Q, \beta)}
$$

(6)

where [11]

$$
\Pi_0(Q, \beta) = 2\frac{\alpha}{\pi \beta} \int_0^1 dx \ln(2 \cosh(qbeta/2 \sqrt{x(1-x))})
$$

(7)

In this approximation $\Sigma_m(p)$ becomes frequency independent and the summation over $n$ in (4) can be performed analytically yielding

$$
\Sigma(P, \beta) = \frac{\alpha}{8N\pi^2} \int d^2k \frac{\Sigma(K, \beta)}{Q^2 + \Pi_0(Q, \beta)} \tanh \frac{\beta}{2} \sqrt{K^2 + \Sigma^2(K, \beta)}.
$$

(8)

The main purpose of this paper is to present the detailed analysis of this temperature- and momentum-dependent gap equation.

3 Numerical procedure and results

Equation (8) involves a two-dimensional integral over $k$, while $\Pi_0$ in (7) involves $Q = \vert p - k \vert$ inside a further integral. The analogous $T = 0$ equation has frequently been simplified [13] by replacing $Q$ by $\max(K, P)$, rendering the angular integral trivial. We do not make this approximation here, but we have found it very convenient to adopt an excellent analytic approximation to (6) (correct to about 1.5% at worst) which is provided by the expression

$$
\Pi_0(Q, \beta) = \frac{\alpha}{8\beta} \left[ Q\beta + \frac{16 \ln 2}{\pi} \exp(-\frac{\pi}{16 \ln 2} Q\beta) \right],
$$

(9)
which incorporates the correct limiting behaviour as either $Q$ or $\beta$ tends to zero or infinity. In particular, as $Q \to 0$,

$$\Pi_0(Q, \beta) \to \frac{2\alpha \ln 2}{\pi \beta}$$  \hspace{2cm} (10)

which exhibits the thermal screening noted before [11].

As regards to the integral over $K = |\mathbf{k}|$ in (8), this of course extends to $K \to \infty$ in principle. For numerical purposes the upper limit would normally be replaced by some cutoff parameter $\Lambda$, chosen to be sufficiently large that further increase of it makes no difference. In the present case however, the dimensionful parameter $\alpha$ provides a natural scale: in particular, Appelquist et al. [2] noted that the integral in the corresponding $T = 0$ equation was rapidly damped for momenta greater than $\alpha$, so that effectively $\Lambda \simeq \alpha$. We have found that the same is true for equation (8) at finite temperature, and thus from now on we shall work with (8) cut off at $\Lambda = \alpha$, and present our results in terms of the scaled quantity $(\Sigma/\alpha)$, momenta being also scaled by $\alpha$.

We have used a numerical algorithm to solve equation (8). An iterative procedure has been followed to find a solution for $\Sigma(P, \beta)$. As usual the algorithm only converged if the input function was sufficiently close to the true solution. The iteration was started by adopting the approximation $Q \simeq \max(K, P)$ and solving the resulting version of (8) for a large value of $\beta$. This solution then provided the input trial function for the true equation (8) at the same low temperature. When the input and the output of the integral equation agreed within a $2\%$ of difference the iterative sequence was stopped, and the output was considered as the solution to equation (8) for the defined temperature. $\beta$ was then incremented downwards in small steps using the solution of (8) for the previous $\beta$ as the input function for the next. In this way we were able to obtain the dependence of $\Sigma$ on $\beta$, as well as on $p$.

The scaled dynamical mass as a function of scaled momentum is shown in Figure 2 for $N = 1$, at various temperatures. We notice that the mass remains constant for a wide range of momenta up to roughly $P \simeq \Sigma(P = 0)$ and then rapidly drops to zero. We also notice that as expected the mass decreases with rising temperature. The rate at which this occurs grows as we approach the critical temperature $T_c$, above which the mass vanishes (and which will depend on $N$). This is illustrated in Figure 3, which shows the zero momentum mass $\Sigma(0, \beta)/\alpha$ as a function of the scaled temperature $k_B T/\alpha$ for $N = 1, 1.5$. The approach of $\Sigma(0, \beta)$ to zero in the vicinity of $T_c$ can be studied by plotting $\ln \Sigma$ versus $\ln(T_c - T)$; we find that $\Sigma \simeq (T_c - T)^x$ where the exponent $x$ depends on $N$ and lies between 0.4 and 0.6 for $N$ between 1 and 2. This is consistent (up to subleading $N$-dependent corrections) with the value $x = 1/2$ characteristic of BCS theory.

Figure 4 shows the scaled mass versus scaled momentum for several values of $N$, at a fixed value of $\beta$. It is clear that $\Sigma$ decreases strongly as $N$ increases from $N = 1$, suggesting that it may vanish for $N$ larger than some critical $N_c$. This possibility is examined in Figure 5, which shows $\Sigma(0, \beta)/\alpha$ versus $N$ for various values of $\beta$. We are not able to follow the $\Sigma/\alpha$ curves much below values of order $10^{-5}$ due to numerical difficulties, but it seems reasonable to conclude that at a given temperature $T$, $\Sigma$ indeed vanishes for $N > N_c$, where $N_c$ depends on $T$. For large $\beta$ (low $T$), $N_c$ approaches a value just greater than 2. As the temperature is raised, $N_c$ decreases. At $T = 0$ Appelquist et al. [2] have found that in the limit $N \to N_c$, the
zero-momentum mass vanishes according to

\[ \frac{\Sigma}{\alpha} \propto \exp \left[ -\frac{2\pi}{\sqrt{N_c/N - 1}} \right]. \]  

(11)

We have explored the possibility of a similar behaviour in our model. As convergence near the critical point is very slow we have had to extrapolate from the calculated values of \( N \) to get the critical value \( N_c \) for the corresponding temperature. Our results are shown in Figure 6 where we have plotted \(-\ln(\Sigma(0, \beta)/\alpha)\) vs. \( 1/\sqrt{N_c/N - 1} \); we observe that for fixed temperature the curves approach straight lines as \( N \) approaches \( N_c \). This leads us to believe that indeed in this region the zero-momentum mass behaves like

\[ \frac{\Sigma(0, T)}{\alpha} \propto \exp \left[ -\frac{C(T)}{\sqrt{N_c(T)/N - 1}} \right]. \]  

(12)

for some temperature-dependent function \( C(T) \) (which is, however, at the temperatures shown in Figure 6, considerably smaller in magnitude than the \( T = 0 \) value of \( 2\pi \) given in (11)).

As mentioned in the previous paragraph, we find that as \( T \to 0 \) \( N_c(T) \) approaches a value just greater than 2. This is, of course, different from the \( N_c \) found in Ref. [2], using an equation which ought to be the zero temperature limit of ours, and hence some further comment is required. In fact, while it is obviously true that the \( T \to 0 \) limit of the full SD equation (3) must be the same as that of Ref. [2], this is not the case after the instantaneous approximation has been made, leading to Eqns. (4) and (8). Nevertheless, Eqn. (4) does reduce as \( T \to 0 \) to an equation of similar form (for small \( \Sigma \)) to that in Ref. [2], but the numerical coefficient in front of the integral is a factor 1.5–2.0 times too small. Effectively this means that in comparing our results with those of Ref. [2] we should take our \( N \) as being roughly equivalent to the \( N \) of Ref. [2] divided by this factor. This is the reason for the discrepancy in the \( N_c(T = 0) \) values.

These results enable us to obtain the phase diagram shown in Figure 7. There is a single critical line, such that for \((N, T)\) below this line \( \Sigma \neq 0 \), and for \((N, T)\) above it \( \Sigma = 0 \). We have only shown the region \( N \geq 1 \), but it seems likely that the line approaches \( N = 0 \) asymptotically as \( T \to \infty \). In this plot we have rescaled the critical line to match the zero-temperature results of Ref. [2], namely \( N_c(T = 0) = 3.2 \).

As mentioned in the Introduction, the dimensionless ratio \( r = 2\Sigma(P=0, T=0)/k_BT_c \) is an important quantity, distinguishing between different mass-generation mechanisms. Our values of \( r \) are shown in Table 1 for \( N = 1, 1.5, 1.7 \), where for convenience we also list the corresponding values of \( \Sigma(P=0, T=0)/\alpha \) and of \( k_BT_c/\alpha \). This table can be directly compared with Table 1 of Ref. [11], which -it will be recalled- was obtained by solving a simplified equation in which the \( P \)-dependence of \( \Sigma \) was neglected. The comparison shows that while our more exact equation (8) yields values of the mass \( \Sigma \) which are about one order of magnitude smaller than those obtained in Ref. [11], the critical temperature \( T_c \) is also correspondingly reduced, so that \( r \) remains with a value of order 10, in agreement with the value found in [11], and also approximately independent of \( N \).

With an eye to the possible relevance to high-\( T_c \) superconductivity, it is natural to wonder about the orders of magnitudes of the quantities appearing in our results, when expressed in physical units. We must emphasize, however, that relatively small changes in the kernel (which is after all only an approximation) can make rather large changes in \( \Sigma \). It is clear that the scale of the model is set by the dimensionful quantity \( \alpha \), which has dimensions of \((\text{length})^{-1}\) or
(energy) in the system in which we have implicitly been working, namely that in which \( \hbar = v = 1 \) with \( v \) the Fermi velocity in the condensed matter case (and that of light for QED\(_3\)). If \( \alpha \) is regarded as a freely adjustable parameter, then from Table 1 estimating \( k_B T_c \sim 10^{-4} \alpha \) for \( N = 2 \) (the value required by our high-\( T_c \) model [9]), we would need to take \( \alpha \sim 80 \text{ eV} \) in order to obtain \( T_c \sim 100^\circ \text{K} \) as is required experimentally. It seems hard to understand how such a large energy could arise naturally. If we readjusted the “effective” \( N - T \) curve so as to agree with \( N_c(T = 0) \) as found in Ref. [9], we would obtain \( k_B T_c \sim 10^{-3} \alpha \) for \( N = 2 \), leading to a required \( \alpha \) of order \( 8 \text{ } \text{eV} \). These different estimates merely underline, of course, the difficulty in making anything other than rather rough order of magnitude calculations as far as numerical values are concerned. A more reliable determination of \( \Sigma \) and \( T_c \) would be obtained from finite temperature Monte-Carlo simulation of lattice QED\(_3\).

A value of \( \alpha \) in the region of a few \( \text{eV} \) is still much larger than typical Heisenberg exchange energies (recall that in our model [8] the gauge field arises in connection with the spin degrees of freedom in the original lattice Hamiltonian). Nevertheless, it is possible to form an estimate of \( e^2 \) in terms of the parameters of the lattice model of [9], which shows that it is effectively enhanced, as follows. The lattice analogue of the fermion kinetic energy is the “hopping term” which enters with coefficient \( t \). If the lattice fermion operators are rescaled by \( t a \) (where \( a \) is the lattice spacing) so as to get the correct dimensions of the fields in the continuum limit, and if space is then rescaled so as to obtain the (Dirac) kinetic energy with unit coefficient, the lattice \( U(1) \) coupling \( g \) becomes effectively replaced by \( g/(ta)^{1/2} \). An estimate of \( ta \) in such models may be obtained by noting that according to Baskaran et al. [1] the maximum doping concentration \( n_{\text{max}} \sim t/U \), where \( U \) is the Hubbard repulsion. Since we may take \( U \sim a^{-1} \) (i.e. \( U \rightarrow \infty \) in the continuum limit) we find \( ta \sim n_{\text{max}} \), which has the empirical value of only a few percent. Assuming that the magnitude of \( g \) is set by the spin magnitude (1/2), and its length scale by the lattice spacing \( a \), we obtain finally for the square of the effective coupling

\[
e^2 \sim \frac{1}{4a} \frac{\hbar v}{n_{\text{max}}}
\]

having reinstated \( \hbar \) and \( v \). The latter quantity can be conveniently found from the relation \( \xi \sim \hbar v/\Sigma \) for the correlation length \( \xi \). Using \( \xi \sim 30\text{Å} \) and \( \Sigma \sim 5k_B T_c \) we find \( v/c \sim 5 \times 10^{-4} \), which gives \( e^2 \sim \text{few eV} \). Thus it is perhaps not impossible that such values could arise within the context of a model such as that of Ref. [9].

Acknowledgements

We thank Mike Pennington for a very useful discussion about the content of the works listed in Ref. [3]. MKK wishes to thank the National University of Mexico, the SERC and Los Alamos National Laboratory for financial support.

References
[1] R. D. Pisarski, *Phys. Rev.* **D29** (1984) 2423.

[2] T. W. Appelquist, M. Bowick, D. Karabali and L. C. R. Wijewardhana, *Phys. Rev.* **D33** (1986) 3704.

T. W. Appelquist, D. Nash and L. C. R. Wijewardhana, *Phys. Rev. Lett.* **60** (1988) 2575.

[3] M. R. Pennington and D. Walsh, *Phys. Lett.* **B253** (1991) 246. See also M. R. Pennington and S. P. Webb, BNL-40886 (January 1988), unpublished and D. Atkinson, P. W. Johnson and M. R. Pennington, BNL-41615 (August 1988), unpublished.

[4] E. Dagotto, A. Kocic and J. B. Kogut, *Phys. Rev. Lett.* **62** (1989) 1083 and *Nucl. Phys.* **B334** (1990) 279.

[5] D. Nash, *Phys. Rev. Lett.* **62** (1989) 3024.

[6] D. Atkinson, P.W. Johnson and P. Maris *Phys. Rev.* **D42** (1990) 602.

[7] R. D. Pisarski, *Phys. Rev.* **D44** (1991) 1866.

[8] A. Kovner and B. Rosenstein, *Phys. Rev.* **B42** (1990) 4748.

N. Dorey and N. E. Mavromatos, *Phys. Lett.* **B250** (1990) 107.

G. W. Semenoff and N. Weiss, *Phys. Lett.* **B250** (1990) 117.

[9] N. Dorey and N. E. Mavromatos, Los Alamos preprint LANL 91-3181 and CERN-TH.6278/91.

[10] A. Kocic, *Phys. Lett.* **B189** (1987) 449.

[11] N. Dorey and N. E. Mavromatos, *Phys. Lett.* **B266** (1991) 163.

[12] A. C. Davis and A. M. Matheson, *Phys. Rev.* **D40** (1989) 2373.

[13] see for example J. B. Kogut, E. Dagotto and A. Kocic *Nucl. Phys.* **B137** (1989) 253.

[14] G. Baskaran, Z. Zou and P. W. Anderson, *Solid State Comm.* **63** (1987) 973.
Tables

| N   | $\Sigma(P = 0, T = 0)/\Lambda$ | 1          | 1.5         | 1.7          |
|-----|--------------------------------|------------|-------------|--------------|
|     | $k_B T_c / \Lambda$            | $3.76 \times 10^{-3}$ | $5.2 \times 10^{-4}$ | $1.90 \times 10^{-4}$ |
|     | $r = 2\Sigma(P = 0, T = 0)/k_B T_c$ | 10.26      | 10.0        | 9.5          |

Table 1: The zero-temperature and zero-momentum fermion mass and the critical temperature at $\alpha/\Lambda = 1$ with the ratio $r$ for $N = 1, 1.5, 1.7$.

Figure captions

Figure 1: The photon propagator to leading order in $1/N$
Figure 2: Scaled dynamical mass $\Sigma/\alpha$ as a function of scaled momentum $p/\alpha$ for $N = 1$, and various scaled (inverse) temperatures $\beta\alpha$. 
Figure 3: Zero-momentum scaled mass versus scaled temperature for $N = 1, 1.5$.
Figure 4: Scaled mass versus scaled momentum for $\beta\alpha = 10^5$ for $N = 1, 1.2, 1.5$.
Figure 5: Zero-momentum scaled mass versus $N$ at various (inverse) temperatures.
Figure 6: Test for the behaviour given by Eqn. (12) near the critical region.
Figure 7: Phase diagram for spontaneous mass generation.