An analytic approximate solution of the SIR model

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Abstract

The SIR(D) epidemiological model is defined through transcendental equations not solvable by elementary functions. In the present paper those equations are successfully replaced by approximate ones, whose solutions are given explicitly in terms of elementary functions, namely, piece-wisely, generalized logistic functions: they unveil a useful feature, that in fact is also owned by the (numerical) solutions of the exact equations.

Keywords:
SIR epidemic model, Kermack-McKendrick model, epidemic dynamics, approximate analytic solution.

1. Introduction

The SIR model [1–6] is a simple compartmental model of infectious diseases developed by Kermack and McKendrick [1] in 1927. It considers three compartments:

S, the set of susceptible individuals;
I, the set of the infectious (or currently positive) individuals, who have been infected and are capable of infecting susceptible individuals;
R, the set of the removed individuals, namely people who recovered (healed, H subset) from the disease or deceased (D subset), the former assumed to remain immune afterwards.

The SIR model does not consider at all the sub-compartments H and D; instead the SIRD model simply assumes them to constitute a partition of R, fractionally fixed over time, so that, actually compared to the SIR model, nothing substantially changes in the dynamics of the epidemic progression.

It is assumed that births and non-epidemic-related deaths can be neglected in

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the epidemic timescale and that the incubation period is negligible too. Indicating with letters not in bold the cardinality of each of the compartments, it is taken

\[ S(t_0) + I(t_0) + R(t_0) = N , \]  

(1)

where \( t_0 \) is an initial time; then the time evolution of the SIR(D) model is defined by the following system of non-linear first order differential equations:

\[
\begin{align*}
\frac{dS}{dt} &= -\beta \frac{I}{N} S , \\
\frac{dI}{dt} &= \beta \frac{S}{N} I - \gamma I , \\
\frac{dR}{dt} &= \gamma I .
\end{align*}
\]  

(2a) (2b) (2c)

Clearly

\[
\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0 ,
\]  

(3)

so that

\[ S(t) + I(t) + R(t) = S(t_0) + I(t_0) + R(t_0) = N . \]  

(4)

Usually \( R(t_0) = 0 \).

Both the parameters \( \beta \) and \( \gamma \) have dimension of a frequency. \( \gamma \) in eq(2c) is the fractional removal rate \((1/I)(dR/dt)\) of individuals from the infectious compartment. \( SI \) in eq(2a) is understood as the number of possible contacts among the infectious and the susceptible individuals, so that \( \beta/N \) is the fractional decrease rate \(-((SI)^{-1}(dS/dt))\) of the number of individuals in the susceptible compartment; correspondingly \( \beta/N \) is the fractional increment rate of the number of infected individuals, determining the increment rate in the infectious compartment \( I \), after subtraction of the rate of people entering the removed compartment \( R \): this is in fact what eq(2b) states.

It is obvious that for the epidemic to spread, the increment rate of the newly infectious individuals must be higher then the increment rate of the newly removed individuals. Thus, introducing the so called basic reproduction ratio \( \alpha \) (quite often denoted as \( R_0 \)), namely

\[ \alpha = R_0 = \frac{\beta}{\gamma} , \]  

(5)

and dividing eq(2a) by eq(2c), it must be

\[ \frac{-dS}{dR} = \alpha \frac{S}{N} > 1 . \]  

(6)

As a matter of fact, re-writing eq(2b) as

\[ \frac{dI}{dt} = \gamma I \left( \alpha \frac{S}{N} - 1 \right) , \]  

(7)
condition \[6\] is seen to be equivalent to require the time derivative of \( I \) to be positive, or \( I \) itself increasing. By the way, since all the functions in the model are defined positive, from eq.\[2A\] it is seen that \( S \) is monotonic decreasing, tending to zero; then from eq.\[7\] it follows that necessarily the number of infectious individuals must decrease after reaching a maximum and tend to zero even before the susceptible compartment may get empty.

Taking eq.\[6\] at the very beginning of the epidemic, when \( R = 0 \) and \( I \ll S \), thus \( S \approx N \), one understands that the meaning of \( \alpha \) is the number of newly infectious individuals while just one infectious individual gets removed. Of course the average number of new infections from an infectious individual strongly influences the basic reproduction ratio, but it is not exactly the basic reproduction ratio itself.

It is convenient to introduce the following non-dimensional variable and functions:

\[
x := \gamma t, \quad s(x) := \frac{S(t)}{N}, \quad i(x) := \frac{I(t)}{N}, \quad r(x) := \frac{R(t)}{N},
\]

Then the basic equations of the SIR(D) model are written as

\[
\begin{align*}
\frac{ds}{dx}(x) &= -\alpha i(x) s(x) \quad (9a) \\
\frac{di}{dx}(x) &= i(x)(\alpha s(x) - 1) \quad (9b) \\
\frac{dr}{dx}(x) &= i(x) \quad (9c)
\end{align*}
\]

with

\[
s(x) + i(x) + r(x) = s(x_0) + i(x_0) + r(x_0) = 1, \tag{10}
\]

and

\[
s_0 := s(x_0), \quad i_0 := i(x_0), \quad r_0 := r(x_0) \equiv 0. \tag{11}
\]

As is well known, the solutions of the equations of the SIR(D) model depend completely on the basic reproduction number (and the initial conditions), while of course \( \beta \) (indeed not \( \gamma \) !) gives the time scale.

From eq.\[9a\] and then eq.\[9c\] one gets

\[
s(x) = s_0 \exp \left\{ -\alpha \int_{x_0}^{x} d\xi \, i(\xi) \right\} = s_0 e^{-\alpha r(x)}. \tag{12}
\]

Using this in \[9b\] one easily finds then

\[
\frac{di}{dx}(x) = -s_0 \frac{d}{dx} e^{-\alpha r(x)} - \frac{dr}{dx},
\]

whence

\[
i(x) = 1 - s_0 e^{-\alpha r(x)} - r(x). \tag{13}
\]

Using this in \[9c\] again, one gets

\[
\frac{dr}{dx}(x) = 1 - s_0 e^{-\alpha r(x)} - r(x). \tag{14}
\]
This is a transcendental equation, whose solutions one cannot give explicitly in closed analytic form by elementary functions. So in the sequel it is replaced by an approximate one, whose solutions are given explicitly. Of course, once \( r(x) \) is given, \( i(x) \) comes from eq.13 and \( s(x) \) from eq.12.

2. Getting the key differential equation

The functions \( s(x) \), \( i(x) \) and \( r(x) \) are defined positive, so \( s(x) \) must be monotonic decreasing according to eq.9a and \( r(x) \) monotonic increasing according to eq.9c. It is assumed

\[
\alpha > \frac{1}{s_0} > 1,
\]

that is the condition for an epidemic to trigger, according to the short discussion above. Due to eq.9b, the function \( i(x) \) starts growing to a maximum which is reached at a time \( t_M = x_M/\gamma \) such that

\[
\alpha s(x_M) = 1;
\]

then asymptotically it decreases to zero. Consequently, for eq.9c the bounded monotonically increasing function \( r(x) \) must exhibit a point of inflection at \( t_M \), after which it bends, increasing slower and slower, finally flattening to some limiting value

\[
r_\infty \equiv r(+\infty) \leq 1.
\]

So one must have

\[
0 = \lim_{x \to +\infty} \frac{dr}{dx}(x) = 1 - s_0 e^{-\alpha r_\infty} - r_\infty,
\]

thus getting a transcendental equation for \( r_\infty \).

Conveniently for the following developments, a new function is introduced, namely

\[
w(x) = 1 - s_0 e^{-\alpha r(x)},
\]

in terms of which eq.14 is re-written as

\[
\frac{dw}{dx} = F[w],
\]

\[
F[w] := (1 - w) [\epsilon + \alpha w + \ln(1 - w)] ,
\]

\[
\epsilon = -\ln(s_0) = -\ln(1 - i_0).
\]

Clearly

\[
\hat{w} := \lim_{x \to +\infty} w(x) = 1 - s_0 e^{-\alpha r_\infty} = r_\infty
\]

must be solution of the equation

\[
F[\hat{w}] = 0,
\]

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for eq.18 and the fact that
\[ \frac{dw}{dx} = s_0 \alpha e^{-\alpha r(x)} \frac{dr}{dx}, \]
so that
\[ \frac{dw}{dx} = 0 \iff \frac{dr}{dx} = 0. \]
The functional \( F[w] \) is null in \( w = 1 \), but \( \hat{w} \) cannot be 1 because \( 0 \leq r(x) \leq 1 \) and \( s_0 \) is not null (see eq.19); thus \( \hat{w} \) must be solution of the equation
\[ \epsilon + \alpha \hat{w} + \ln(1 - \hat{w}) = 0, \tag{23} \]
which is nothing but eq.18, as can be easily verified. The ordinary eq.23 is transcendental and is to be solved numerically; the interval \([0, \hat{w}]\) is the range of \( w(x) \) as \( x \) runs from \( x_0 \) to \( +\infty \).

The functional \( F[w] \) starts and remains negative from \( w = 0 \), until it reaches the point of inflection \( w_{\text{fl}} \), given by
\[ w_{\text{fl}} = 1 - \frac{1}{2\alpha}; \tag{25} \]

then it becomes positive: thus \( F[w] \) starts and remains concave until \( w = w_{\text{fl}} \);

then becomes convex. Of course, in an interval around its inflection point, \( F[w] \) is nearly straight. Fig.1 shows how \( \hat{w} \) and \( w_{\text{fl}} \) vary as a function of \( \alpha \): for \( \alpha < \alpha_{cr} \simeq 1.75 \) one has \( \hat{w} < w_{\text{fl}} \) and consequently \( F[w] \) is always concave in the

\[ \alpha_{cr} = 1.75527 \]
the domain $[0, \hat{w}]$: otherwise it changes from concave to convex after $w = w_{su}$. It is worth noting that as $\alpha$ increases, $\hat{w}$ (together with $w_{su}$) approaches more and more the limiting value 1, namely a region where the log term in $F[w]$ becomes important: this fact is relevant here because such log term, with its argument approaching zero, rises complications in searching for an effective approximation.

3. Approximating the key differential equation

$$\beta = 0.25$$

$$\alpha = 2.5 \quad \alpha = 8.3$$

Figure 2: Examples of the two main cases.

The idea is to approximate $F[w]$ by few stretches of up to second order polynomials, joining continuously each other with the first derivative. Then in each stretch the obtained approximate differential equation becomes analytically and explicitly solvable by a generalized logistic function. For $w \ll 1$, it is taken

$$(1 - w) \ln(1 - w) \approx -w \left(1 - \frac{1}{2}w\right), \quad (26)$$

so that

$$\frac{dw}{dx} \approx \epsilon + (\alpha - 1 - \epsilon) w - \left(\alpha - \frac{1}{2}\right) w^2 := F^{(1)}[w]. \quad (27)$$

Fig. 2 shows on the left, in red, this $F^{(1)}[w]$ segment against $F[w]$ (black curve) for $\alpha = 2.74$ and (consequently) $\hat{w} \simeq 0.92$, extending to its maximum point, which is rather close to the maximum of $F[w]$. Clearly $F^{(1)}[w]$ is a parabola with axis along the ordinate line, so that the maximum is its vertex.

Denoting by $w_1^{(1)}$ and $w_2^{(1)}$ the roots of $F^{(1)}[w]$, one can write

$$F^{(1)}[w] = -A \left(w - w_1^{(1)}\right) \left(w - w_2^{(1)}\right), \quad (28a)$$

$$A := \alpha - \frac{1}{2}, \quad (28b)$$
with
\[ w_{1/2}^{(1)} = \frac{\alpha - 1 - \epsilon \pm \sqrt{(\alpha - 1 - \epsilon)^2 + 2(2\alpha - 1)\epsilon}}{2\alpha - 1}. \] (29)

The vertex is located in
\[ w_M = \frac{w_1^{(1)} + w_2^{(1)}}{2}. \] (30)

A new parabola is chosen as the second approximation stretch, tangent to \( F[w] \) on its descending side, with axis along the ordinates and the vertex coincident with that of the first segment \( F^{(1)}[w] \):
\[ F^{(2)}[w] = -Z^* (w - w_1^{(2)}) (w - w_2^{(2)}), \] (31a)
\[ \frac{w_1^{(1)} + w_2^{(1)}}{2} = w_M = \frac{w_1^{(2)} + w_2^{(2)}}{2}, \] (31b)
\[ -A (w_M - w_1^{(1)}) (w_M - w_2^{(1)}) = -Z^* (w_M - w_1^{(2)}) (w_M - w_2^{(2)}), \] (31c)
\[ F^{(2)}[w] = F[w], \] (31d)
\[ \frac{\delta F^{(2)}}{\delta w} [w(x)] = \frac{\delta F}{\delta w} [w(x)]. \] (31e)

Equations 31b and 31c impose that the two stretches have in common their vertexes, located in \( w = w_M \); the system of the last two equations states the conditions for \( F^{(2)}[w] \) to be tangent to \( F[w] \). It is convenient expressing \( Z^* \) in terms of the unknown tangency point \( w^* \) using eq. 31e, so that then one solves eq. 31d for \( w^* \).

Introducing
\[ \delta w^{(1)} := \frac{w_1^{(1)} - w_2^{(1)}}{2}, \] (32a)
\[ \delta w^{(2)} := \frac{w_1^{(2)} - w_2^{(2)}}{2}, \] (32b)

due to eq. 31c one can write
\[ Z^* (\delta w^{(2)})^2 = A (\delta w^{(1)})^2, \] (33)
while from eq. 31e and eq. 31d one has
\[ (1 - w^*) [\epsilon + \alpha w^* + \ln(1 - w)] = -Z^* (w - w_M)^2 + A (\delta w^{(1)})^2, \] (34a)
\[ Z^* = \frac{1 + \epsilon + 2\alpha w^* + \ln(1 - w^*) - \alpha}{2(w^* - w_M)}. \] (34b)

Using this expression for \( Z^* \) in eq. 34a, one obtains a transcendental ordinary equation for \( w^* \), to be solved numerically:
\[ 2\epsilon + (\alpha - \epsilon - 1)w_M - 2A(\delta w^{(1)})^2 + (\alpha - \epsilon - 2\alpha w_M + 1) w^* + (2 - w^* - w_M) \ln(1 - w^*) = 0. \] (35)
Using $w^*$ so obtained, one gets $Z^*$ from eq. 34b and finally $w_1^{(2)}$ and $w_2^{(2)}$ via eq. 33 and eq. 31b. In fig. 2 on the left, the second segment for $\alpha = 2.6$ is shown in blue, extending from $w_M$ to the point of tangency of the successive approximation segment still to be chosen.

![Figure 3: $w_1^{(2)}$ and $\hat{w}$ as functions of $\alpha$.](image)

With reference to the discussion at the end of Section 2, it should be noted that $F[w]$ remains concave up to $w = \hat{w}$ when $\alpha \leq \alpha_{cr}$, while it happens that the root $w_1^{(2)}$ of $F^{(2)}[w]$ (see fig. 3) remains very close to $\hat{w}$: this suggests in that range of $\alpha$ values replacing the above $F^{(2)}[w]$ by a different arc of parabola $f^{(2)}[w]$, keeping its vertex in common with $F^{(1)}[w]$ as $F^{(2)}[w]$ does, but just ending in $\hat{w}$, thus imposing the constraint $w_1^{(2)} = \hat{w}$ instead of the tangency to $F[w]$.

Then for $\alpha \leq \alpha_{cr}$

\begin{align*}
  f^{(2)}[w] &= -Z \left( w - w_1^{(f)} \right) \left( w - w_2^{(f)} \right), \\
  \frac{w_1^{(1)} + w_2^{(1)}}{2} &= w_M = \frac{w_1^{(f)} + w_2^{(f)}}{2}, \\
  -A \left( w_M - w_1^{(1)} \right) \left( w_M - w_2^{(1)} \right) &= -Z \left( w_M - w_1^{(f)} \right) \left( w_M - w_2^{(f)} \right), \\
  Z &= \left( \alpha - \frac{1}{2} \right) \left( \frac{\delta w^{(f)}}{\hat{w} - w_M} \right)^2, \\
  w_1^{(f)} &= \hat{w}, \quad w_2^{(f)} = 2w_M - \hat{w}, \quad \delta w^{(f)} = \hat{w} - w_M. 
\end{align*}

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For $\alpha_{\alpha} < \alpha \leq 6$ $F[w]$ is almost always concave, ending roughly as a straight line when approaching $\hat{w}$. In this range of $\alpha$’s one keeps $F^{(2)}[w]$ and completes the approximation through a new parabola, requiring it to be tangent to $F^{(2)}[w]$ and to reach $\hat{w}$ along the tangent to $F[w]$ in $\hat{w}$; an alternative is the ray tangent to $F^{(2)}[w]$, extending from the point of tangency to $\hat{w}$. The latter is settled by

$$L[w] := -2 \psi Z^* (w - \hat{w}), \quad (37a)$$

$$\implies \left\{ L[w] - F^{(2)}[w] = 0 \land \Delta \left( L[w] - F^{(2)}[w] \right) = 0 \right\}. \quad (37b)$$

Here $\Delta \left( L[w] - F^{(2)}[w] \right)$ is the discriminant of the second order algebraic equation $L[w] - F^{(2)}[w] = 0$, set to zero to assure $L[w]$ to be tangent to $F^{(2)}[w]$. The appropriate solution for $u$ is

$$u_- = \hat{w} - w_M - \sqrt{(\hat{w} - w_M)^2 - (\delta w^{(2)})^2}. \quad (38)$$

The problem with this approximation is that, looking for instance at the function $r(x)$ obtained from $w(x)$, it gets unacceptably overestimated in the region where it bends to reach the asymptotic value as $x \to +\infty$: this is because $L[w]$ necessarily remains below $F[w]$ due to the concavity of the latter.

The quadratic alternative is defined by

$$F^{(3)}[w] := -2 \lambda (w - \hat{w}) + \sigma (w - \hat{w})^2, \quad (39a)$$

$$\lambda = \left( F^{(2)}[w] \right)'_{w=\hat{w}} = \frac{1 - \alpha (1 - \hat{w})}{2}, \quad (39b)$$

$$\implies \left\{ F^{(3)}[w] - F^{(2)}[w] = 0 \land \Delta \left( F^{(3)}[w] - F^{(2)}[w] \right) = 0 \right\}. \quad (39c)$$

where “prime” stands for derivative and $\Delta \left( F^{(3)}[w] - F^{(2)}[w] \right)$ is the discriminant of the second order algebraic equation $F^{(3)}[w] - F^{(2)}[w] = 0$, set to zero so to assure $F^{(3)}[w]$ to be tangent to $F^{(2)}[w]$. In this case, however, with respect to using $L[w]$, one has the opposite effect on $r(x)$, because the given choice for $\lambda$ forces $F^{(3)}[w]$ to stay somewhat above $F[w]$. The solution is to keep the quadratic alternative, but replacing the previous value of $\lambda$ by a compromise one, defined through

$$\lambda^* := \tan \left( \arctan(-2 \lambda) \right) + \tan \left( \frac{\arctan(-2 \lambda) - \arctan(-2 u_- Z^*)}{2} \right). \quad (40)$$

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Then the parameter $\sigma$ in (39a) is set by means of the condition (39c):

$$\sigma = \frac{Z^* h - g^2}{2 \hat{w} g - h - Z^* \hat{w}^2}, \quad (41a)$$

$$g = Z^* w_M + \lambda^*,$$

$$h = Z^* w_1^{(2)} w_2^{(2)} + 2 \lambda^* \hat{w}, \quad (41b)$$

$$w^o = \frac{\sigma \hat{w} + g}{\sigma + Z^*}, \quad (41c)$$

where $w^o$ is the tangency point of $F^{(3)}[w]$ to $F^{(2)}[w]$.

So, for $\alpha_{cr} < \alpha <= 6$ the third and last approximation segment is given by (39a), with $\lambda$ replaced by $\lambda^*$, extending from $w^o$ to $\hat{w}$.

For $w > 6$ the convexity trait of $F[w]$, following the almost straight stretch around $w_{flx}$, gets more and more included in the domain $[0, \hat{w}]$, because $\hat{w}$ increases with $\alpha$. Then, the solution adopted is to introduce a linear segment $T[w]$ parallel to the tangent in $w_{flx}$ to $F[w]$ and tangent to $F^{(2)}[w]$ in a point that will be denoted $\hat{w}$; this linear segment will be continued by a new parabola $F^{(4)}[w]$, which is similar to $F^{(3)}[w]$, thus ending in $\hat{w}$, but tangent to $T[w]$.

Namely

$$T[w] := -2 \hat{f} w + \hat{l}, \quad (42a)$$

$$-2 \hat{f} := F'[w] |_{w=w_{flx}} = \ln(2\alpha) - \alpha - \epsilon, \quad (42b)$$

$$\exists' \left\{ T[w] - F^{(2)}[w] = 0 \wedge \Delta (T[w] - F^{(2)}[w]) = 0 \right\}, \quad (42c)$$

giving

$$\hat{l} = Z^* \left[ \hat{w}^2 - w_M^2 + (\delta w^{(2)})^2 \right], \quad (43a)$$

$$\hat{w} = w_M + \frac{\hat{f}}{Z^*}. \quad (43b)$$

Then the already mentioned $F^{(4)}[w]$ approximation stretch is constrained to end in $\hat{w}$ and to be tangent to $T[w]$ in a point $w^u$ chosen by trial and error optimization:

$$F^{(4)}[w] := -2 \lambda^u (w - \hat{w}) + \sigma^u (w - \hat{w})^2, \quad (44a)$$

$$w^u := (1-z) w_{flx} + z \hat{w}, \quad z = 0.575, \quad (44b)$$

$$\exists' \left\{ F^{(4)}[w] - T[w] = 0 \wedge \Delta (F^{(4)}[w] - T[w]) = 0 \right\}, \quad (44c)$$

giving

$$\lambda^u = \hat{f} + \frac{2 \hat{w} \hat{f} - \hat{l}}{w^u - \hat{w}}, \quad (45a)$$

$$\sigma^u = \frac{2 \hat{w} \hat{f} - \hat{l}}{(w^u - \hat{w})^2}. \quad (45b)$$
4. The approximate analytic solution

For each of the above approximation segments, a differential equations is defined of the type

\[ \frac{dw}{dx}(x) = F[w(x)], \]  

where \( F[w] \) is one of \( F^{(i)}[w] \) \((i = 1, 2, 3, 4)\) or \( f^{(2)}[w] \) or \( T[w] \), with given \( \alpha \) and \( \beta \) parameters (or \( \beta \) and \( \gamma \)) and initial conditions. For \( F[w] = F^{(i)}[w] \), from the definition in eq. [19] the initial condition is \( w(x_0) = 1 - s_0 = i_0 \) \( (x_0 = 0 \) without loss of generality), while for each of the remaining approximation segments it is given by the value of the respective preceding segment at the junction point. Since \( F[w] \) is at most a second order polynomial, eq. [46] is indeed quite trivially solved, giving a generalized logistic function.

For \( F[w] = F^{(1)}[w] \):

\[
F^{(1)}[w](x) = \frac{w^{(1)}_1 + w^{(1)}_2 k e^{-x/\gamma \tau_1}}{1 + k e^{-x/\gamma \tau_1}},
\]

\[
k = \frac{w^{(1)}_1 - i_0}{i_0 - w^{(1)}_2}, \quad \tau_1 = \frac{1}{\gamma \cdot \alpha - \frac{1}{2} (w^{(1)}_1 - w^{(1)}_2)}, \quad (47a)
\]

For \( F[w] = f^{(2)}[w] \), thus \( \alpha \leq \alpha_{cr} \):

\[
F^{(2)}[w](x) = \hat{\varphi} e^{-\frac{(x - x^*)}{\gamma \tau_2}}
\]

\[
x^* = \gamma \tau_2 \ln \left(k \right) \ \ \varphi \ w^{(1)}(x_M) = w_M, \quad \tau_2 = \frac{\delta w^{(2)}}{w^{(1)}} \tau_1 > \tau_1. \quad (48b)
\]

For \( F[w] = T[w] \), thus \( \alpha > \alpha_{cr} \):

\[
F^{(3)}[w](x) = \hat{\varphi} e^{-\frac{(x - x^*)}{\gamma \tau_3}}
\]

\[
x^* = \gamma \tau_3 \ln \left(k \right) \ \ \varphi \ w^{(1)}(x_M) = w_M, \quad \tau_3 = \frac{\delta w^{(3)}}{w^{(1)}} \tau_1 > \tau_1. \quad (49b)
\]

For \( F[w] = F^{(3)}[w] \), thus \( \alpha_{cr} < \alpha \leq 6 \):

\[
F^{(3)}[w](x) = \hat{\varphi} e^{-\frac{(x - x^*)}{\gamma \tau_3}}
\]

\[
x^* = \gamma \tau_3 \ln \left(k \right) \ \ \varphi \ w^{(1)}(x_M) = w_M, \quad \tau_3 = \frac{\delta w^{(3)}}{w^{(1)}} \tau_1 > \tau_1. \quad (50b)
\]
For $\mathcal{F}[w] = T[w]$ , thus $\alpha > 6$ (see [42] and [43]):

$$w^{(T)}(x) = \frac{1}{2f} \left[ \hat{I} - (\hat{I} - 2\hat{\bar{w}}\hat{\bar{f}}) e^{-(x-\bar{x})/\gamma\hat{\bar{f}}} \right],$$

(51a)

$$\hat{\bar{f}} = \frac{1}{2f} \gamma, \quad \bar{x} = \gamma x + \gamma \tau_2 \ln \left( \frac{\hat{\bar{w}} - w^{(2)}}{w^{(2)}_1 - \hat{\bar{w}}} \right) \hat{\bar{f}}' w^{(T)}(\bar{x}) = \hat{\bar{w}}.$$  (51b)

Finally for $\mathcal{F}[w] = F^{(4)}[w]$ thus $\alpha > 6$:

$$w^{(4)}(x) = \frac{\hat{\bar{w}} - (\hat{\bar{w}} + 2\lambda^u/\sigma^u) \phi^u e^{-(x-x^u)/\gamma\tau_4}}{1 - \phi^u e^{-(x-x^u)/\gamma\tau_4}},$$

(52a)

$$x^u = \gamma \bar{x} + \gamma \hat{\bar{f}} \ln \left( \frac{\hat{\bar{f}} - 2\hat{\bar{f}} \hat{\bar{w}}}{\hat{\bar{f}} - 2\hat{\bar{f}} w^u} \right) \hat{\bar{f}}' w^{(T)}(x^u) = w^u,$$

(52b)

$$\phi^u = \frac{\hat{\bar{w}} - w^u}{\hat{\bar{w}} - w^u + \frac{2\lambda^u}{\sigma^u}}, \quad \tau_4 = \frac{1}{2 \frac{\lambda^u}{\gamma}}.$$  

(52c)

It is convenient to introduce

$$\hat{\gamma}(t) := r(\gamma t), \quad \hat{i}(t) := i(\gamma t), \quad \hat{s}(t) := s(\gamma t), \quad \hat{w}(t) := w(\gamma t), \quad \text{etc.},$$

(53)

Then, from eq.19 one has

$$\hat{\gamma}(t) = \frac{1}{\alpha} \ln \frac{1 - i_0}{1 - w(\gamma t)},$$

(54)

so that

$$\hat{i}(t) = \frac{d\hat{\gamma}}{dt}(t) = \frac{1}{\alpha} \left[ \frac{1}{1 - w(x)} \frac{dw}{dx}(x) \right]_{x=\gamma t}.$$  

On the other hand eq.20 implies

$$\frac{1}{1 - w} \frac{dw}{dx} = \alpha w - \ln \frac{1 - i_0}{1 - w},$$

and consequently (see eq.54)

$$\hat{i}(t) = \left[ w(x) - \frac{1}{\alpha} \ln \frac{1 - i_0}{1 - w(x)} \right]_{x=\gamma t} = \hat{w}(t) - \hat{\gamma}(t).$$

(55)

Finally, of course, due to [10]:

$$\hat{s}(t) = 1 - \hat{i}(t) - \hat{\gamma}(t) = 1 - \hat{w}(t).$$  

(56)
$$\beta = 0.25$$

\[ \alpha = 1.6 \quad \alpha = 2.5 \quad \alpha = 4.5 \quad \alpha = 8.3 \]

Figure 4: Comparison of “exact” numerical solutions and approximate solutions for the SIRD model.

In the case of the SIRD model one defines

\[
\begin{align*}
\dot{r} &= \dot{h} + \dot{d}, \\
\gamma &\rightarrow \gamma + \mu \quad \text{so that} \quad \dot{h} = \frac{\gamma}{\gamma + \mu} \dot{r} \quad \text{and} \quad \dot{d} = \frac{\mu}{\gamma + \mu} \dot{r}.
\end{align*}
\]

Figure 4 shows a comparison between the numerical “exact” solutions of the SIRD model and the approximate solutions of this work with $\beta = 0.25$ and $\alpha = 1.6, 2.5, 4.5, 7.1, 10.0$.

Imitating a formal expression typical of computing languages\footnote{(a \leq b) \ ? \ c = f : \ otherwise \ c = g} the result for $w$
can be summarized as follows:

for $\alpha \leq \alpha_c$ 
\[ \tilde{w}(t) = (t \leq t_m) ? \tilde{w}^{(1)}(t) : \tilde{w}^{(1)}(t) \]  \hspace{1cm} (58a)

for $\alpha_c < \alpha \leq 6$ 
\[ \tilde{w}(t) = (t \leq t_m) ? \tilde{w}^{(1)}(t) : \left( (t \leq t^*) ? \tilde{w}^{(2)}(t) : \tilde{w}^{(3)}(t) \right) \]  \hspace{1cm} (58b)

for $\alpha > 6$ 
\[ \tilde{w}(t) = (t \leq t_m) ? \tilde{w}^{(1)}(t) : \left( (t \leq \tilde{t}) ? \tilde{w}^{(2)}(t) : \left( (t \leq t^u) ? \tilde{w}^{(3)}(t) : \tilde{w}^{(4)}(t) \right) \right) \]  \hspace{1cm} (58c)

Similarly for $\tilde{s}(t), \tilde{y}(t), \tilde{h}(t)$ and $\tilde{d}(t)$.

5. A useful feature

The equation of the first approximation segment can be re-written as

\[ \frac{1}{w^{(1)}_2} \frac{dw^{(1)}}{dx} = -\frac{A}{w^{(1)}_2} (w^{(1)}_1 - w^{(1)}_2 - \delta w^{(1)}_2) (w^{(1)}_1 - w^{(1)}_2) \]  \hspace{1cm} (59)

Using the explicit solution eq.47, one has

\[ w^{(1)}_1 - w^{(1)}_2 = \frac{2 \delta w^{(1)}_1}{\left[ 1 + k e^{-(x-x_0)/\gamma \tau_1} \right]^2} \]  \hspace{1cm} (60)

and consequently

\[ \frac{1}{w^{(1)}_2} \frac{dw^{(1)}_1}{dx} = 4A k \left( \delta w^{(1)}_1 \right)^2 \frac{e^{-(x-x_0)/\gamma \tau_1}}{w^{(1)}_2} \left[ 1 + \frac{w^{(1)}_2}{w^{(1)}_1} k e^{-(x-x_0)/\gamma \tau_1} \right]^2 \]  \hspace{1cm} (61)
Typically
\[ \frac{w_1^{(1)}}{w_2^{(1)}} \ll 1 \quad \text{and} \quad \frac{w_1^{(2)}}{w_2^{(2)}} \ll 1, \]  
but anyway with \( t - t_0 \) greater then some \( \tau \)'s, in the end one can write
\[ \ln \left( \frac{1}{(w_2^{(1)})^2} \frac{d w_1^{(1)}}{dt} \right) (t) \simeq \ln(4A\gamma k) - \frac{t - t_0}{\tau_1}. \]  
Analogous results hold for all the approximation stretches in the different \( \alpha \) intervals as summarized in eqs. \ref{eq:approximation_stretches} for instance, with \( t - t^* \) greater enough then \( \tau_3 \), one has
\[ \ln \left( \frac{1}{(w_2^{(3)})^2} \frac{d w_1^{(3)}}{dt} \right) (t) \simeq \ln \left[ 4\sigma \gamma \phi^* \left( \frac{2\lambda^*}{\sigma \phi} \right)^2 \right] - \frac{t - t^*}{\tau_3}. \]  
These piecewise linear behaviors can be seen in fig. \ref{fig:behavior} for \( \alpha = 2.6 \). The plot on the left shows the numerical solution of the exact equation, compared with the corresponding approximate analytic solution: it is worth recalling (see eq. \ref{eq:approximation_w}) that \( w(x) = r(x) + i(x) \), so that \( w \) is directly related to the data. The plot on the right shows that the function \( \tilde{r}(t) \) of the \textit{removed} individuals exhibits an analogous behavior: since in the SIRD model the \( \tilde{d}(t) \) function is a fraction of \( \tilde{r}(t) \), then one has the analogous behavior for the function of the deceased individuals. Fig. \ref{fig:deceased_data} refers to the data of the deceased individuals during the winter-

![Graph](image.png)

\textbf{Figure 6:}

\textit{First closures in Lombardy and Veneto}
determined by the parameters $\alpha$ and $\beta$ (besides the initial conditions) and so is the angle between such straight segments: consequently one can compare that angle with the theoretically predicted one and argue about the effects of social measures to reduce the pandemic, of course within the trustworthiness of the model.

6. Conclusions

In this paper the equations of the SIR(D) epidemiological model are replaced by approximate ones, whose solutions are totally defined by the basic reproduction ratio $\alpha$ and the fractional removal rate $\gamma$ of individuals from the infectious compartment (alternatively by $\beta = \gamma / \alpha$). These solutions are chains of two or three or four generalized logistic functions, depending on the value of $\alpha$ only; they are summarized in eq.s 58.

In practice, to get them one does:

- solve numerically the transcendental ordinary eq.23 to get $\hat{w}$;
- use eq.29 and eq.s47 to get $w^{(1)}(x)$ as in eq.47;
- for $\alpha \leq \alpha_{cr}$ use eq.30, eq.36d and 36c to get $w^{(1)}(x)$ as in eq.48;
- for $\alpha > \alpha_{cr}$ use eq.35, eq.34b, eq.30, eq.32b, eq.33 and eq.s49 to get $w^{(2)}(x)$ as in eq.49;
- for $\alpha_{cr} < \alpha <= 6$ use eq.40, eq.41a and eq.41b, eq.41c and finally eq.s50 to get $w^{(3)}(x)$ as in eq.50;
- for $\alpha > 6$ use eq.42b, eq.s43 and eq.s51 to get $w^{(4)}(x)$ as in eq.51;
- for $\alpha > 6$ use eq.25, eq.s44b, eq.s45 and eq.s52 to get $w^{(4)}(x)$ as in eq.52;
- eventually use eq.54, eq.55, eq.56, eq.57.

Having such explicit solution would help, for instance, to study the data through easy fits.

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