Counting 1-vertex Triangulations
Of Oriented Surfaces

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Abstract

A 1-vertex triangulation of an oriented compact surface $S$ of genus $g$
is an embedded graph $T \subset S$ with a unique vertex such that all
connected components of $S \setminus T$ are triangles (adjacent to exactly 3 edges of $T$).

This paper gives formulas enumerating such triangulations (up to
 equivalence) on an oriented surface of given genus.

Une triangulation à un sommet d’une surface orientée compacte $S$ de
genre $g$ est un graphe $T \subset S$ qui a un unique sommet et dont toutes les
faces (composantes connexes de $S \setminus T$) sont des triangles (incidentes à
trois arêtes de $T$).

Cet article donne des formules permettant d’énumérer ces triangula-
tions.

Introduction

Definition 0.1. A 1-vertex triangulation of an oriented compact surface $S$
of genus $g$ is an embedded graph $T \subset S$ with only one vertex such that all
connected components of $S \setminus T$ are adjacent to exactly 3 edges of $T$ (i.e. are
triangles).

Two such triangulations $T \subset S$ and $T' \subset S'$ are isomorphic (or equivalent)
if there exists an orientation-preserving homeomorphism $\varphi : S \to S'$ such that
$\varphi(T) = T'$.

The aim of this paper is to give formulas for the number of such triangula-
tions (up to equivalence) on an oriented surface of given genus (such triangula-
tions exist in every genus $g \geq 1$).

1-vertex triangulations have several applications:
L. Mosher [11] has constructed a complex whose fundamental group is the mapping class group of an orientable genus $g$ surface. $1-$vertex triangulations appear as the vertices of this complex. There is also a bijection between $1-$vertex triangulations and Eulerian paths in cubic graphs as considered by J. Brenner and R. Lyndon in [4]. They formulated the problem of classification and enumeration of these objects, which is solved in the present paper. Brenner and Lindon conjectured, that the automorphism group of a $1-$vertex triangulation is always cyclic of order 1, 2, 3 or 6. G. Bianchi and R. Cori proved this in a more general form in [2].

Notice that Brenner and Lyndon considered such triangulations from a combinatorial point of view motivated by the study of non-parabolic subgroups in the modular group [4].

Section 1 introduces oriented Wicks forms (cellular decompositions with only one face of oriented surfaces), our main tool. Wicks forms are canonical forms for products of commutators in free groups [12]. Oriented maximal Wicks forms and $1-$vertex triangulations are in bijection. Our main theorem will be expressed in the language of Wicks forms.

Section 2 contains a few facts concerning oriented maximal Wicks forms. Section 3 contains the proof of our main results.

1 Main results

The objects considered in this section are dual to $1$-vertex triangulations. They are slightly easier to handle since they carry some combinatorial structures more immediately.

Definition 1.1. An oriented Wicks form is a cyclic word $w = w_1 w_2 \ldots w_{2l}$ (a cyclic word is the orbit of a linear word under cyclic permutations) in some alphabet $a_1^{\pm 1}, a_2^{\pm 1}, \ldots$ of letters $a_1, a_2, \ldots$ and their inverses $a_1^{-1}, a_2^{-1}, \ldots$ such that

(i) if $a_1^\epsilon$ appears in $w$ (for $\epsilon \in \{\pm 1\}$) then $a_1^{-\epsilon}$ appears exactly once in $w$,

(ii) the word $w$ contains no cyclic factor (subword of cyclically consecutive letters in $w$) of the form $a_i a_{i-1}^{-1}$ or $a_{i}^{-1} a_{i}$ (no cancellation),

(iii) if $a_i^\epsilon a_j^\delta$ is a cyclic factor of $w$ then $a_j^{-\delta} a_i^{-\epsilon}$ is not a cyclic factor of $w$ (substitutions of the form $a_i^\epsilon a_j^\delta \mapsto x$, $a_j^{-\delta} a_i^{-\epsilon} \mapsto x^{-1}$ are impossible).

An oriented Wicks form $w = w_1 w_2 \ldots$ in an alphabet $A$ is isomorphic to $w' = w'_1 w'_2$ in an alphabet $A'$ if there exists a bijection $\varphi : A \rightarrow A'$ with $\varphi(a^{-1}) = \varphi(a)^{-1}$ such that $w'$ and $\varphi(w) = \varphi(w_1) \varphi(w_2) \ldots$ define the same cyclic word.

An oriented Wicks form $w$ is an element of the commutator subgroup when considered as an element in the free group $G$ generated by $a_1, a_2, \ldots$. We define the algebraic genus $g_a(w)$ of $w$ as the least positive integer $g_a$ such that $w$ is a product of $g_a$ commutators in $G$.  

2
The topological genus $g_t(w)$ of an oriented Wicks form $w = w_1 \ldots w_{2e-1}w_{2e}$ is defined as the topological genus of the oriented compact connected surface obtained by labeling and orienting the edges of a $2e$-gon (which we consider as a subset of the oriented plane) according to $w$ and by identifying the edges in the obvious way.

**Proposition 1.1.** The algebraic and the topological genus of an oriented Wicks form coincide (cf. [5, 7]).

We define the genus $g(w)$ of an oriented Wicks form $w$ by $g(w) = g_a(w) = g_t(w)$.

Consider the oriented compact surface $S$ associated to an oriented Wicks form $w = w_1 \ldots w_{2e}$. This surface carries an immerged graph $\Gamma \subset S$ such that $S \setminus \Gamma$ is an open polygon with $2e$ sides (and hence connected and simply connected). Moreover, conditions (ii) and (iii) on Wicks form imply that $\Gamma$ contains no vertices of degree 1 or 2 (or equivalently that the dual graph of $\Gamma \subset S$ contains no faces which are $1$–gons or $2$–gons). This construction works also in the opposite direction: Given a graph $\Gamma \subset S$ with $e$ edges on an oriented compact connected surface $S$ of genus $g$ such that $S \setminus \Gamma$ is connected and simply connected, we get an oriented Wicks form of genus $g$ and length $2e$ by labeling and orienting the edges of $\Gamma$ and by cutting $S$ open along the graph $\Gamma$. The associated oriented Wicks form is defined as the word which appears in this way on the boundary of the resulting polygon with $2e$ sides. We identify henceforth oriented Wicks forms with the associated immerged graphs $\Gamma \subset S$, speaking of vertices and edges of oriented Wicks form.

The formula for the Euler characteristic

$$\chi(S) = 2 - 2g = v - e + 1$$

(where $v$ denotes the number of vertices and $e$ the number of edges in $\Gamma \subset S$) shows that an oriented Wicks form of genus $g$ has at least length $4g$ (the associated graph has then a unique vertex of degree 4 and $2g$ edges) and at most length $6(2g - 1)$ (the associated graph has then $2(2g - 1)$ vertices of degree three and $3(2g - 1)$ edges).

We call an oriented Wicks form of genus $g$ maximal if it has length $6(2g - 1)$. Oriented maximal Wicks forms are dual to 1-vertex triangulations. This can be seen by cutting the oriented surface $S$ along $\Gamma$, hence obtaining a polygon $P$ with $2e$ sides. We draw a star $T$ on $P$ which joins an interior point of $P$ with the midpoints of all its sides. Regluing $P$ we recover $S$ which carries now a 1-vertex triangulation given by $T$ and each 1-vertex triangulation is of this form for some oriented maximal Wicks form (the immerged graphs $T \subset S$ and $\Gamma \subset S$ are dual to each other: faces of $T$ correspond to vertices of $\Gamma$ and vice-versa. Two faces of $T$ share a common edge if and only if the corresponding vertices of $\Gamma$ are adjacent). This construction shows that we can work indifferently with 1-vertex triangulations or with oriented maximal Wicks forms.

Similarly, cellular decompositions of oriented surfaces with one vertex and one face correspond to oriented minimal Wicks forms and were enumerated in [8]. The dual of an oriented minimal Wicks form is again a (generally non-
equivalent) oriented minimal Wicks form and taking duals yields hence an
involution on the set of oriented minimal Wicks forms.

A vertex \( V \) (with oriented edges \( a, b, c \) pointing toward \( V \)) is positive if

\[ w = ab^{-1} \ldots bc^{-1} \ldots ca^{-1} \ldots \] or \( w = ac^{-1} \ldots cb^{-1} \ldots ba^{-1} \ldots \)

and \( V \) is negative if

\[ w = ab^{-1} \ldots ca^{-1} \ldots bc^{-1} \ldots \] or \( w = ac^{-1} \ldots ba^{-1} \ldots ab^{-1} \ldots \).

The automorphism group \( \text{Aut}(w) \) of an oriented Wicks form \( w = w_1 w_2 \ldots w_{2e} \) of length \( 2e \) is the group of all cyclic permutations \( \mu \) of the linear word \( w_1 w_2 \ldots w_{2e} \) such that \( w \) and \( \mu(w) \) are isomorphic linear words (i.e. \( \mu(w) \) is obtained from \( w \) by permuting the letters of the alphabet). The group \( \text{Aut}(w) \) is a subgroup of the cyclic group \( \mathbb{Z}/2e\mathbb{Z} \) acting by cyclic permutations on linear words representing \( w \).

The automorphism group \( \text{Aut}(w) \) of an oriented Wicks form can of course also be described in terms of permutations on the oriented edge set induced by orientation-preserving homeomorphisms of \( S \) leaving \( \Gamma \) invariant. In particular an oriented maximal Wicks form and the associated dual 1-vertex triangulation have isomorphic automorphism groups.

We define the mass \( m(W) \) of a finite set \( W \) of oriented Wicks forms by

\[ m(W) = \sum_{w \in W} \frac{1}{|\text{Aut}(w)|} \] .

Let us introduce the sets

\( W_1^g \): all oriented maximal Wicks forms of genus \( g \) (up to equivalence),

\( W_2^g(r) \subset W_1^g \): all oriented maximal Wicks forms having an automorphism of order \( 2 \) leaving exactly \( r \) edges of \( w \) invariant by reversing their orientation. (This automorphism is the half-turn with respect to the “midpoints” of these edges and exchanges the two adjacent vertices of an invariant edge.)

\( W_3^3(s, t) \subset W_1^g \): all oriented maximal Wicks forms having an automorphism of order \( 3 \) leaving exactly \( s \) positive and \( t \) negative vertices invariant (this automorphism permutes cyclically the edges around an invariant vertex).

\( W_6^g(3r; 2s, 2t) = W_2^g(3r) \cap W_3^3(2s, 2t) \): all oriented maximal Wicks forms having an automorphism \( \gamma \) of order \( 6 \) with \( \gamma^3 \) leaving \( 3r \) edges invariant and \( \gamma^2 \) leaving \( 2s \) positive and \( 2t \) negative vertices invariant (it is useless to consider the set \( W_6^g(r'; s', t') \) defined analogously since \( 3 \) divides \( r' \) and \( 2 \) divides \( s', t' \) if \( W_6^g(r'; s', t') \neq \emptyset \)).
We define now the *masses* of these sets as
\[
\begin{align*}
m_1^g &= \sum_{w \in W_1^g} \frac{1}{|\text{Aut}(w)|}, \\
m_2^g(r) &= \sum_{w \in W_2^g(r)} \frac{1}{|\text{Aut}(w)|}, \\
m_3^g(s, t) &= \sum_{w \in W_3^g(s, t)} \frac{1}{|\text{Aut}(w)|}, \\
m_6^g(3r; 2s, 2t) &= \sum_{w \in W_6^g(3r; 2s, 2t)} \frac{1}{|\text{Aut}(w)|}.
\end{align*}
\]

**Theorem 1.1.**  
(i) The group \( \text{Aut}(w) \) of automorphisms of an oriented maximal Wicks form \( w \) is cyclic of order 1, 2, 3 or 6.

(ii) \( m_1^g = \frac{2}{12} (6g - 5)! / g!(3g - 3)! \).

(iii) \( m_2^g(r) > 0 \) (with \( r \in \mathbb{N} \)) if and only if \( r = \frac{2k+1-r}{4} \in \{0, 1, 2, \ldots\} \) and we have then
\[
m_2^g(r) = \frac{2^2}{2} \int_{-\frac{r}{2}}^{\frac{r}{2}} (6f + 2r - 5)! r! / f!(3f + r - 3)!.
\]

(iv) \( m_3^g(s, t) > 0 \) if and only if \( f = \frac{2g+1-t}{3} \in \{0, 1, 2, \ldots\}, s = 2g + 1 \) (mod 3) and \( t = 2g \) (mod 3) (which follows from the two previous conditions). We have then
\[
m_3^g(s, t) = \frac{2^2}{3} \left[ \frac{3^2}{12} \right] f (6f + 2s + 2t - 5)! s! / f!(3f + s + t - 3)!
\]

(v) \( m_6^g(3r; 2s, 2t) > 0 \) if and only if \( f = \frac{2g+5-3r-4s-4t}{12} \in \{0, 1, 2, \ldots\}, 2s = 2g + 1 \) (mod 3) and \( 2t = 2g \) (mod 3) (follows in fact from the previous conditions). We have then
\[
m_6^g(3r; 2s, 2t) = \frac{2}{6} \left( \frac{6^2}{12} \right) f (6f + 2r + 2s + 2t - 5)! / r!s! / f!(3f + r + s + t - 3)!
\]

Set
\[
\begin{align*}
m_2^g &= \sum_{r \in \mathbb{N}, (2g+1-r)/4 \in \mathbb{N} \cup \{0\}} m_2^g(r), \\
m_3^g &= \sum_{s, t \in \mathbb{N}, (g+1-s-t)/3 \in \mathbb{N} \cup \{0\}, s \equiv 2g+1 \text{ (mod 3)}} m_3^g(s, t), \\
m_6^g &= \sum_{r, s, t \in \mathbb{N}, (2g+5-3r-4s-4t)/12 \in \mathbb{N} \cup \{0\}, 2s \equiv 2g+1 \text{ (mod 3)}} m_6^g(3r; 2s, 2t)
\end{align*}
\]

(all sums are finite) and denote by \( M^g_d \) the number of equivalence classes of oriented maximal genus \( g \) Wicks forms having an automorphism of order \( d \) (i.e. an automorphism group with order divisible by \( d \)).

**Theorem 1.2.** We have
\[
\begin{align*}
M_1^g &= m_1^g + m_2^g + 2m_3^g + 2m_6^g, \\
M_2^g &= 2m_2^g + 4m_6^g, \\
M_3^g &= 3m_3^g + 3m_6^g, \\
M_6^g &= 6m_6^g.
\end{align*}
\]
and \( M_d^g = 0 \) if \( d \) is not a divisor of 6.

The number \( M_d^g \) of this Theorem is of course the number of inequivalent oriented maximal Wicks forms of genus \( g \). The first 15 values \( M_1^1, \ldots, M_{15}^1 \) are displayed in the Table at the end of this paper.

The following result is an immediate consequence of Theorem 1.2.

**Corollary 1.1.** There are exactly
\[
M_d^g \text{ inequivalent Wicks forms with 6 automorphisms,}
\]
\[
M_d^g - M_6^g \text{ inequivalent Wicks forms with 3 automorphisms,}
\]
\[
M_d^g - M_6^g \text{ inequivalent Wicks forms with 2 automorphisms and}
\]
\[
M_d^g - M_2^g - M_3^g + M_6^g \text{ inequivalent Wicks forms without non-trivial automorphisms.}
\]

**Remark.** Computing masses amounts to enumerating pointed objects, i.e. linear words instead of cyclic words in Definition 1.1. Their number is \((12g - 6)m_d^g\), where \( d \) is 1, 2, 3 or 6.

Let us remark that formula (ii) can be obtained from [WL] (formula (9) on page 207 and the formula on the top of page 211) or from [GS] (Theorem 2.1 with \( \lambda = 2^{6g-3} \) and \( \mu = 3^{4g-2} \)). We will reprove it independently. Related objects have also been considered in [HZ].

## 2 Oriented Wicks forms

Let \( V \) be a negative vertex of an oriented maximal Wicks form of genus \( g > 1 \).

There are three possibilities, denoted configurations of type \( \alpha, \beta \) and \( \gamma \) (see Figure 1) for the local configuration at \( V \).

![Figure 1](attachment:image.png)

**Figure 1.**

**Type \( \alpha \).** The vertex \( V \) has only two neighbours which are adjacent to each other. This implies that \( w \) is of the form
\[
w = x_1abcdx^{-1}e^{-1}d^{-1}e^{-1}a^{-1}x_2u_1x^{-1}_2x^{-1}_1u_2
\]
(where \( u_1, u_2 \) are subfactors of \( w \)) and \( w \) is obtained from the maximal oriented Wicks form
\[
w' = xu_1x^{-1}u_2
\]
of genus \( g-1 \) by the substitution \( x \mapsto x_1abcdx^{-1}e^{-1}d^{-1}e^{-1}a^{-1}x_2x^{-1}x^{-1}_2 \) and \( x^{-1} \mapsto x_2x^{-1} \) (this construction is called the \( \alpha \)-construction in [V]).

**Type \( \beta \).** The vertex \( V \) has two non-adjacent neighbours. The word \( w \) is then of the form
\[
w = x_1abca^{-1}x_2u_1y_1db^{-1}c^{-1}d^{-1}y_2u_2
\]
(where perhaps $x_2 = y_1$ or $x_1 = y_2$, see [V] for all the details). The word $w$ is then obtained by a $\beta$-construction from the word $w' = xu_1yu_2$ which is an oriented maximal Wicks form of genus $g - 1$.

Type $\gamma$. The vertex $V$ has three distinct neighbours. We have then

$$w = x_1ab^{-1}y_2u_1z_1ca^{-1}x_2u_2y_1bc^{-1}z_2u_3$$

(some identifications among $x_1$, $y_j$ and $z_k$ may occur, see [V] for all the details) and the word $w$ is obtained by a so-called $\gamma$-construction from the word $w' = xiu_2yu_1zu_3$.

**Definition 2.1.** We call the application which associates to an oriented maximal Wicks form $w$ of genus $g$ with a chosen negative vertex $V$ the oriented maximal Wicks form $w'$ of genus $g - 1$ defined as above the reduction of $w$ with respect to the negative vertex $V$.

An inspection of figure 1 shows that reductions with respect to vertices of type $\alpha$ or $\beta$ are always paired since two doubly adjacent vertices are negative, of the same type ($\alpha$ or $\beta$) and yield the same reductions.

The above constructions of type $\alpha$, $\beta$ and $\gamma$ can be used for a recursive construction of all oriented maximal Wicks forms of genus $g > 1$.

**Definition 2.2.** Consider an oriented maximal Wicks form $w = w_1 \ldots w_{12g-6}$. To any edge $x$ of $w$ we associate a transformation of $w$ called the IH-transformation on the edge $x$. Geometrically, an IH-transformation amounts to contracting the edge $x$ of the graph $\Gamma \subset S$ representing the oriented maximal Wicks form $w$. This creates a vertex of degree 4 which can be split in two different ways (preserving planarity of the graph on $S$) into two adjacent vertices of degree 3: The first way gives back the original Wicks form and the second way results in the IH-transformation. Graphically, an IH-transformation amounts hence to replace a (deformed) letter I (a topological neighbourhood of the edge $x \in \Gamma \subset S$) by a (deformed) letter H.

More formally, one considers the two subfactors $axb$ and $cx^{-1}d$ of the (cyclic) word $w$. Geometric considerations and Definition 1.1 show that $b \neq a^{-1}$, $c \neq b^{-1}$, $d \neq a^{-1}$, $d \neq c^{-1}$ and $(c, d) \neq (a^{-1}, b^{-1})$.

According to the remaining possibilities we consider now the following transformation:

Type 1. $c \neq a^{-1}$ and $d \neq b^{-1}$. This implies that $d^{-1}a^{-1}$ and $b^{-1}c^{-1}$ appear as subfactors in the cyclic word $w$. The IH-transformation on the edge $x$ is then defined by the substitutions

\[
\begin{align*}
axb & \quad \mapsto \quad ab \\
 cx^{-1}d & \quad \mapsto \quad cd \\
 d^{-1}a^{-1} & \quad \mapsto \quad d^{-1}ya^{-1} \\
 b^{-1}c^{-1} & \quad \mapsto \quad b^{-1}y^{-1}c^{-1}
\end{align*}
\]

in the cyclic word $w$.

Type 2a. Suppose $c^{-1} = a$. This implies that $b^{-1}axb$ and $d^{-1}a^{-1}x^{-1}d$ are subfactors of the cyclic word $w$. Define the IH-transformation on the edge $x$ by

\[
\begin{align*}
 b^{-1}axb & \quad \mapsto \quad b^{-1}yab \\
 d^{-1}a^{-1}x^{-1}d & \quad \mapsto \quad d^{-1}y^{-1}a^{-1}d 
\end{align*}
\]
Type 2b. Suppose \( d^{-1} = b \). Then \( axba^{-1} \) and \( cx^{-1}b^{-1}c^{-1} \) are subfactors of the cyclic word \( w \) and we define the IH-transformation on the edge \( x \) by

\[
\begin{align*}
axba^{-1} & \mapsto bya^{-1} \\
 cx^{-1}b^{-1}c^{-1} & \mapsto cb^{-1}y^{-1}c^{-1}.
\end{align*}
\]

**Lemma 2.1.** (i) IH-transformations preserve oriented maximal Wicks forms of genus \( g \).

(ii) Two oriented maximal Wicks forms related by a IH-transformation of type 2 are equivalent.

**Proof.** This results easily by considering the effect of an IH-transformation on the graph \( \Gamma \subset S \). QED

**Proposition 2.1.** An oriented maximal Wicks form of genus \( g \) has exactly \( 2(g - 1) \) positive and \( 2g \) negative vertices.

**Lemma 2.2.** An \( \alpha \) or a \( \beta \) construction increases the number of positive and negative vertices by 2.

The proof is easy.

**Lemma 2.3.** The number of positive or negative vertices is constant under IH-transformations.

**Proof of Lemma 2.3.** The Lemma holds for IH-transformations of type 2 by Lemma 2.1 (ii). Let hence \( w, w' \) be two oriented maximal Wicks forms related by an IH-transformation of type 1 with respect to the edge \( x \) of \( w \) respectively \( y \) of \( w' \). This implies that \( w \) contains the four subfactors

\[
axb, \quad cx^{-1}d, \quad d^{-1}a^{-1}, \quad b^{-1}c^{-1}
\]

and \( w' \) contains the subfactors

\[
ab, \quad cd, \quad d^{-1}ya^{-1}, \quad b^{-1}y^{-1}c^{-1}
\]

in the same cyclic order and they agree everywhere else. It is hence enough to check the lemma for the six possible cyclic orders of the above subfactors.

One case is

\[
\begin{align*}
w &= axbu \ldots cx^{-1}d \ldots d^{-1}a^{-1} \ldots b^{-1}c^{-1} \ldots, \\
w' &= abu \ldots cd \ldots d^{-1}ya^{-1} \ldots b^{-1}y^{-1}c^{-1} \ldots.
\end{align*}
\]

In this case the two vertices of \( w \) incident in \( x \) and the two vertices of \( w' \) incident in \( y \) have opposite signs. All other vertices are not involved in the IH-transformation and keep their sign and the Lemma holds hence in this case.

The five remaining cases are similar and left to the reader. QED

**Proof of Proposition 2.1.** The result is true in genus 1 by inspection (the cyclic word \( a_1a_2a_3a_1^{-1}a_2^{-1}a_3^{-1} \) is the unique oriented maximal Wicks form of genus 1 and has two negative vertices.)

Consider now an oriented maximal Wicks form \( w \) of genus \( g + 1 \). Choose an oriented embedded loop \( \lambda \) of minimal (combinatorial) length in \( \Gamma \).

First case. If \( \lambda \) is of length 2 there are two vertices related by a double edge in \( \Gamma \). This implies that they are negative and of type \( \alpha \) or \( \beta \). The assertion of Proposition 2.1 holds hence for \( w \) by Lemma 2.2 and by induction on \( g \).
Second case. We suppose now that \( \lambda \) is of length \( \geq 3 \). The oriented loop \( \lambda \) turns either left or right at each vertex. If it turns on the same side at two consecutive vertices \( V_i \) and \( V_{i+1} \) the IH-transformation with respect to the edge joining \( V_i \) and \( V_{i+1} \) transforms \( w \) into a form \( w' \) containing a shorter loop. By Lemma 2.2, the oriented maximal Wicks forms \( w \) and \( w' \) have the same number of positive (respectively negative) vertices.

If \( \lambda \) does not contain two consecutive vertices \( V_i \) and \( V_{i+1} \) with the above property (ie. if \( \lambda \) turns first left, then right, then left etc.) choose any edge \( \{V_i, V_{i+1}\} \) in \( \lambda \) and make an IH-transformation with respect to this edge. This produces a form \( w' \) which contains a loop \( \lambda' \) of the same length as \( \lambda \) but turning on the same side at the two consecutive vertices \( V_{i-1}, V_i \) or \( V_{i+1}, V_{i+2} \). By induction on the length of \( \lambda \) we can hence relate \( w \) by a sequence of IH-transformation to an oriented maximal Wicks form \( \hat{w} \) of genus \( g+1 \) containing a loop of length 2 for which the result holds by the first case. The Wicks forms \( w \) and \( \hat{w} \) have of course the same number of positive (respectively negative) vertices by Lemma 2.2. QED

3 Proof of Theorem 1.1

Proof of Theorem 1.1. We prove the corresponding assertions for oriented maximal Wicks forms. The translation in terms of 1-vertex triangulations is immediate.

Let \( w \) be an oriented maximal Wicks form with an automorphism \( \mu \) of order \( d \). Let \( p \) be a prime dividing \( d \). The automorphism \( \mu' = \mu^{d/p} \) is hence of order \( p \). If \( p \neq 3 \) then \( \mu' \) acts without fixed vertices on \( w \) and Proposition 2.1 shows that \( p \) divides the integers \( 2(g-1) \) and \( 2g \) which implies \( p = 2 \). The order \( d \) of \( \mu \) is hence of the form \( d = 2^a3^b \). Repeating the above argument with the prime power \( p = 4 \) shows that \( a \leq 1 \).

All orbits of \( \mu^{2^a} \) on the set of positive (respectively negative) vertices have either \( 3^b \) or \( 3^b - 1 \) elements and this leads to a contradiction if \( b \geq 2 \). This shows that \( d \) divides 6 and proves (i).

Proof of (ii). An element of \( W_1^{g+1} \) (which designs the set of equivalence classes of oriented maximal Wicks forms with genus \( g+1 \)) can be obtained by applying an \( \alpha \), \( \beta \) or \( \gamma \) construction to an element in \( W_1^g \). There are respectively \( 2(6g-3) \), \( 4(6g-3) \) and \( 8(6g-3) + 8(6g-3)(6g-4) + 8(6g-3)^2 \) possibilities for these constructions starting with a given element in \( W_1^g \). On the other hand, Proposition 2.1 shows that we can construct \( 2(g+1) \) oriented maximal Wicks forms in \( W_1^g \) by applying reduction with respect to a negative vertex to a given element in \( W_1^{g+1} \). The numbers of such “augmentations” and “reductions” coincide after weighting with the correct coefficients. These weights have to take care of automorphisms and the fact that type \( \alpha \) and \( \beta \) constructions give rise to 2 negative vertices with the same “inverse”. A careful analysis shows that

\[
4(6g-3) + 8(6g-3)^2 + 8(6g-3) + 16(6g-3) + 8(6g-3) + 8(6g-3) = 2(g+1)m_1^{g+1}
\]
which simplifies to
\[ 2(6g + 1)(6g - 1)(2g - 1)m_i^g = (g + 1)m_i^{g+1} \]
and proves (ii) by induction since the function
\[ g \mapsto 2 - \frac{(6g - 5)!}{12g!(3g - 3)!} \]
satisfies the same recursion and we have equality for \( g = 1 \) (since \( m_i^g = \frac{1}{6} = \frac{1}{12} \)).

Proof of (iii). First case: \( r < 2g + 1 \) and hence \( f = \frac{2g+1-r}{4} \geq 1 \). Let \( w \) be an oriented maximal Wicks form of genus \( g \) with an automorphism \( \mu \) of order 2 reversing the orientation of exactly \( r \) edges. There are \( \frac{6g-3-r}{2} \) orbits of (unoriented) edges not invariant under \( \mu \). Consider the graph obtained by removing all \( \mu \)-invariant edges from the quotient graph \( \Gamma / \mu \). After removing leaves and vertices of degree 2 we get an oriented maximal Wicks form \( \tilde{w} \) with \( \frac{6g-3-r}{2} - r = \frac{3(2g-r-1)}{2} \) edges and hence of genus \( f = \frac{2g+1-r}{4} \geq 1 \) (recall that an oriented maximal Wicks form of genus \( f \) has \( 3(2f-1) \) edges).

More precisely, let \( w \) be represented by the word \( w_1w_2 \ldots w_{6g-3} \). The subword \( w_1w_2 \ldots w_{6g-3} \) contains exactly one representant of each orbit for the action of \( \mu \) on oriented edges. Remove from the word \( w_1 \ldots w_{6g-3} \) all letters \( w_k \) with \( w_k+6g-3 = w_k^{-1} \) (they correspond to edges reversed by \( \mu \)). The resulting word \( w' \) has length \( 6g - 3 - r \) and has the property that if \( w_k \) appears in \( w' \) then either \( w_k^{-1} \) or \( w_k+1+6g-3 \) appears exactly once in \( w' \) also. Replacing \( w_k^{-1} \) by \( w_k^{-1} \) we get a word which satisfies (i) of Definition 2.1. Removing from this word (and of the resulting ones) all cyclic subfactors of the form \( w_kw_k^{-1} \) we get a word \( w'' \) satisfying also condition (ii). Cancel \( w_i \) and its inverse (or \( w_j \) and its inverse) if \( w_iw_j \) and \( w_j^{-1}w_i^{-1} \) both occur as cyclic subfactors. This produces ultimately an oriented maximal Wicks form \( \tilde{w} \). A counting argument shows that it has genus \( f \). (A good way to understand what happens is to write the word \( w \) along two concentric circles related by radial segments indexed by invariant edges, see figure 2 below illustrating the following example).

Example. Consider the oriented maximal Wicks form
\[ w = abceda^{-1}fb^{-1}e^{-1}ghc^{-1}f^{-1}ig^{-1}d^{-1}h^{-1}i^{-1} \]
of genus two. The form \( w \) admits an automorphism \( \mu \) of order two (acting for instance by \( \mu(abceda^{-1}fb^{-1}e^{-1}) = ghc^{-1}f^{-1}ig^{-1}d^{-1}h^{-1}i^{-1} \)). We have \( r = 1 \), \( g = 2 \) and \( f = \frac{4+1}{4} = 1 \) since \( c \) is the unique edge invariant by \( \mu \). The subword \( abceda^{-1}fb^{-1}e^{-1} \) contains exactly one representant of each orbit for the action of \( \mu \) on oriented edges. Removing the unique edge \( c \) reversed by \( \mu \) and replacing \( f \) by \( d^{-1} \) (since \( \mu(f) = d^{-1} \)) we get \( abdea^{-1}d^{-1}b^{-1}e^{-1} \). We cancel \( d \) and its inverse, since both \( bd \) and \( d^{-1}b^{-1} \) occur as cyclic subfactors. The resulting word \( \tilde{w} = abea^{-1}b^{-1}e^{-1} \) is an oriented Wicks form of genus 1.

An oriented maximal Wicks form \( \tilde{w} \) obtained in this way carries an extra structure defined as follows. Write the word \( w \) counterclockwise along two concentric circles related by \( r \) radial segments in such a way that radial segments
correspond to edges invariant under $\mu$ (see figure 2 below for a hopefully explanatory example).

![Diagram of graph](image)

Figure 2.

\[ w = abcdea^{-1}fb^{-1}e^{-1}ghc^{-1}f^{-1}ig^{-1}d^{-1}h^{-1}i^{-1} \]

and the associated word $\tilde{w} = abea^{-1}b^{-1}e^{-1}$.

Given a letter $\tilde{l}$ of $\tilde{w}$ choose a preimage $l$ of $\tilde{l}$. Set $\varphi(\tilde{l}) = 0$ if $l$ and $l^{-1}$ are on the same circle and set $\varphi(\tilde{l}) = 1$ otherwise (we have hence $\varphi(a) = \varphi(b) = 1$ and $\varphi(c) = 0$ for the word $\tilde{w} = abea^{-1}b^{-1}e^{-1}$ of figure 2). One checks then that $\varphi$ is well-defined and satisfies

\[ \varphi(a) + \varphi(b) + \varphi(c) \equiv 0 \pmod{2} \]

whenever $a, b, c$ are 3 edges incident in a common vertex of the graph $\tilde{G}$ (which we identify of course with the word $\tilde{w}$). Such a function is called a $\mathbb{Z}/2\mathbb{Z}$-flow on the graph $\tilde{\Gamma}$.

Conversely, given an oriented maximal Wicks form $\tilde{w}$ of genus $f = \frac{2g+1-r}{4}$ and a $\mathbb{Z}/2\mathbb{Z}$-flow $\varphi$ on its graph $\tilde{\Gamma}$, we can construct

\[ \frac{(12f-6)(12f-2)\cdots(12f-10+4r)}{r!} \]

oriented maximal Wicks forms of genus $g$ having an automorphism $\mu$ of order 2 reversing $r$ edges associated to the pair $(\tilde{w}, \varphi)$. Indeed, we have $(12f-6)$ possibilities to attach the first edge reversed by $\mu$, $(12f-2)$ choices for the second edge and so on. Since there are $r!$ possible orderings of the $\mu$-invariant edges we have to divide by $r!$. Finally, the $\mathbb{Z}/2\mathbb{Z}$ flow shows how to glue together preimages of orbits under $\mu$.

The set of $\mathbb{Z}/2\mathbb{Z}$-flows is a vector space over $\mathbb{Z}/2\mathbb{Z}$ of dimension $2f$. This implies that we have

\[ 2^{2f} \frac{(12f-6)(12f-2)\cdots(12f-10+4r)}{r!} m_1^f = 2 m_2^f(r) \]

(the factor 2 on the right hand side comes from the fact that the Wicks forms contributing to $m_1^f$ are essentially weighted with weight 1 while they have weight
\[ \frac{1}{2} \text{ in } m_2(g). \] This equation is also satisfied by replacing \( m_1^f \) with \( 2 \frac{(6f-5)!}{12f! f!(3f-r-3)!} \) and \( m_2(g) \) with \( \frac{(6f+2r-5)!}{3r! f!(3f+r-3)!} \) (recall that \( g = \frac{4f+r-1}{2} \)) and this proofs (iii) in the first case.

Second case: \( f = 0 \) (the construction of \( \tilde{w} \) as above shows that we cannot have \( f < 0 \)). The idea is the same as in the first case. Here we have to glue a first invariant edge on an empty word (1 possibility) for the second and the third invariant edge we have 2 possibilities, for the forth there are 6 possibilities etc. Since there are no flows on an empty graph we get

\[ 2m_2^g(2g + 1) = 2 \frac{2 \cdot 6 \cdot \cdots (4r - 10)}{r!} \]

which is readily checked.

Proof of (iv). First case: \( t > 0 \). Let \( w \) be an oriented maximal Wicks form having an automorphism of order 3 fixing \( s \) positive and \( t > 0 \) negative vertices. The \( t \) fixed negative vertices give rise to \( t \) possible reductions producing oriented Wicks forms \( w' \) of genus \( g-1 \) invariant under an automorphism of order 3. The parameters of \( w' \) are then \( (t-1, s) \). On the other hand, for any given oriented Wicks form \( w' \) of genus \( g-1 \) with an automorphism of order 3 and parameters \( (t-1, s) \) there are \( 2(2g-3) \) \( \gamma \)-constructions yielding a Wicks form of genus \( g \) with an automorphism of order 3 and parameters \( (s, t) \) (choose the midpoint of any of the \( 6(2g-3) \) oriented edges in \( w' \) and make the \( \gamma \)-construction with respect to its orbit). We have hence

\[ 2(2g-3)m_3^g(t-1, s) = tm_3^g(s, t) \]

which is also satisfied by the righthand side of formula (iii) in Theorem 1.1.

Let us now consider the case \( t = 0 \) (no invariant vertices of negative type). The proof of this case is very similar to the proof of (iii).

We can suppose \( g > 1 \) since there are only two vertices of negative type in genus 1. We consider hence an oriented maximal Wicks form \( w \) of genus \( g \) with an automorphism \( \mu \) of order 3 fixing \( s \) positive and no negative vertices. Since \( \mu \) leaves no edge invariant, there are \( 2g - 3 = 2g - 1 \) orbits of edges. In genus \( g > 1 \), invariant vertices under an automorphism \( \mu \) of order 3 are never adjacent. There are hence \( s \) orbits of edges of \( w \) incident in a vertex fixed under \( \mu \). Removing their orbits from the orbits of edges leaves us with a graph on the orbit space which has \( s \) vertices of degree 2. Removing these vertices of degree 2 yields an oriented maximal Wicks form \( \tilde{w} \) of genus \( f = \frac{2g-1-s}{3} \) (all vertices are of degree 3, there is one face and there are \( 2g - 1 - 2s = 6f - 3 \) edges). The construction of this form is completely analogous to the construction in the proof of (ii). Graphically, one considers 3 concentric circles (indexed by the elements of \( \mathbb{Z}/3\mathbb{Z} \)) connected together by \( s \) “radial edges” (which represent invariant positive vertices together with their edges).

As in the proof of (ii) this form has an extra structure. This extra structure is here a \( \mathbb{Z}/3\mathbb{Z} \)-flow, i.e. an application \( \varphi \) of the set of oriented edges of \( \tilde{w} \) into \( \mathbb{Z}/3\mathbb{Z} \) such that \( \varphi(e) \equiv -\varphi(-e) \pmod{3} \) and \( \varphi(a) + \varphi(b) + \varphi(c) \equiv 0 \pmod{3} \) for three oriented edges \( a, b, c \) pointing toward a common vertex of \( \tilde{w} \).
Conversely, given an oriented maximal Wicks form \( \tilde{w} \) of genus \( f \) together with the above extra structure (a \( \mathbb{Z}/3\mathbb{Z} \)–flow on its graph \( \tilde{\Gamma} \)) there are
\[
\frac{(12f - 6)(12f - 2) \cdots (12f - 10 + 4s)}{s!}
\]
possibilities to “extend” it into an oriented maximal Wicks form \( w \) of genus \( g \) which has an automorphism \( \mu \) of order 3 fixing exactly \( s \) positive and no negative vertices.

Since the set of \( \mathbb{Z}/3\mathbb{Z} \)–flows on \( \tilde{\Gamma} \) is a \( \mathbb{Z}/3\mathbb{Z} \)–vector space of dimension \( 2f \) we get
\[
3^{2f} \frac{(12f - 6)(12f - 2) \cdots (12f - 10 + 4s)}{s!} m_f^1 = 3 m_3^g(s, 0) = 3 m_3^{3f + s - 1}(s, 0) .
\]

A routine calculation shows that this equation is also satisfied with \( m_f^1 \) replaced by \( 2 \frac{(6f - 5)}{3f/(6f - 3)} \) and \( m_3^{3f + s - 1}(s, 0) \) replaced by \( 2 \frac{3}{3} \left( \frac{3}{2} \right)^f (6f + 2s - 5)! \) and this proves (iv) in the case \( f \geq 1 \). The proof for \( f = 0 \) is similar to the analogous proof of (iii).

Proof of (v). We apply again the idea used in the proof of (iii). Let \( w \) be an oriented maximal Wicks form with an automorphism \( \mu \) of order 6. Considering the automorphism \( \mu^3 \) of order 2 and applying the reduction used in the proof of (iii) we get an oriented maximal Wicks form \( \tilde{w} \) of genus \( h = 2s + 1 - 3r \) together with a \( \mathbb{Z}/2\mathbb{Z} \)–flow \( \phi \) on \( \tilde{\Gamma} \). The Wicks form \( \tilde{w} \) is however an element of \( W_h^3 (s, t) \) and has hence an automorphism \( \tilde{\mu} \) of order 3 which leaves \( \phi \) invariant. Analogously to the proof of (iii) we use this data to produce elements in \( W_h^6 (3r; 2s, 2t) \) by making all constructions \( \tilde{\mu} \)–invariant. We must understand the vector space of \( \tilde{\mu} \)–invariant \( \mathbb{Z}/2\mathbb{Z} \)–flows:

**Lemma 3.1.** Let \( \tilde{w} \in W_h^3 (s, t) \) be an oriented maximal Wicks form with an automorphism \( \tilde{\mu} \) of order 3 (having parameters \( s, t \)). The space of \( \tilde{\mu} \)–invariant \( \mathbb{Z}/2\mathbb{Z} \)–flows on \( \tilde{\Gamma} \) is then of dimension \( \frac{h + 1 - s - t}{3} \).

The lemma and a counting argument show then that
\[
3^{(h + 1 - s - t)/3} \frac{(4h - 6)(4h - 2) \cdots (4h - 10 + 4r)}{r!} m_h^3(s, t) = 2 m_3^g(3r; 2s, 2t)
\]
and a routine calculation implies assertion (v).

Proof of Lemma 3.1. Let \( \tilde{\varphi} \) be a \( \tilde{\mu} \)–invariant \( \mathbb{Z}/2\mathbb{Z} \)–flow on \( \tilde{\Gamma} \). We remark that \( \tilde{\varphi}(a) \equiv \tilde{\varphi}(b) \equiv \tilde{\varphi}(c) \equiv 0 \) (mod 2) if \( a, b, c \) are three edges incident in a \( \tilde{\mu} \)–fixed vertex. This shows that all reductions used in the proof of (iv) can also be applied to the flow \( \tilde{\varphi} \) and these constructions are injective on \( \tilde{\mu} \)–invariant \( \mathbb{Z}/2\mathbb{Z} \)–flows.

Theorem 1.1 is proved. QED

**Table.** The number of 1-vertex triangulations or of oriented maximal Wicks
forms in genus 1, \ldots, 15:

|   |     |
|---|-----|
| 1 | 1   |
| 2 | 9   |
| 3 | 1726|
| 4 | 1349005|
| 5 | 2169056374|
| 6 | 5849686966988|
| 7 | 23808202021448662|
| 8 | 136415042681045401661|
| 9 | 1047212810636411989605202|
| 10| 10378926166167927378088819918|
| 11| 129040245485216017874985276329588|
| 12| 196689594180840390142132270340417352|
| 13| 36072568973390464496963227953956789552404|
| 14| 783676560946907841153290887110277871996495020|
| 15| 19903817294929565349602352185144632327980494486370|

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