Poisson Equivalence over a Symplectic Leaf

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Abstract

We study the equivalence of Poisson structures around a given symplectic leaf of nonzero dimension. Some criteria of Poisson equivalence are derived from a homotopy argument for coupling Poisson structures. In the case when the transverse Lie algebra of the symplectic leaf is semisimple of compact type, we show that an obstruction to the linearizability is the cohomology class of a Casimir 2-cocycle. This allows us to obtain a semilocal analog of the Conn linearization theorem and to clarify examples of nonlinearizable Poisson structures due to [DW].

1 Introduction

This work is devoted to the problem of classification of Poisson structures near a given symplectic leaf of nonzero dimension.

Let \((M, \Psi)\) be a Poisson manifold with Poisson tensor \(\Psi\). According to the local splitting theorem [We], in a neighborhood of each point \(m \in M\), \(\Psi\) is equivalent to the direct product of a nondegenerate Poisson structure and a transverse Poisson structure vanishing at \(m\). The linearization of the transverse factor at \(m\) leads to the \textit{linearized transverse Poisson structure} \(\Lambda_m\) which leaves naturally on the normal space \(E_m\) to the symplectic leaf through \(m\). This structure is uniquely determined by the \textit{transverse Lie Algebra}.

*Research was partially supported by CONACYT, grant 43208, and by the Russian Foundation for Basic Research, grant 02-01-00952.
algebra $\mathfrak{g}_m$ of $m$. If $m$ is a singular point (a zero-dimensional leaf), $\Psi(m) = 0$, then $E_m = T_m M$ and the linearized transverse Poisson structure $\Lambda_m$ gives a linear approximation to $\Psi$ at $m$. In this case, one can state the linearization problem $\Psi$ which consists of determining whether the original Poisson structure is locally equivalent to its linear approximation. In formal, $C^\infty$, and analytic settings, this problem was studied in $\mathcal{A}r$, $\mathcal{W}e_1$, $\mathcal{C}o_1$, $\mathcal{C}o_2$, $\mathcal{D}u_1$. In particular, it was shown in $\mathcal{W}e_1$ that if the transverse Lie algebra $\mathfrak{g}_m$ is semisimple, then there exists a formal linearization. Later, for analytic Poisson structures, this fact was proved in $\mathcal{C}o_1$. In the smooth case, the local linearization theorem due to $\mathcal{C}o_2$ says that the Poisson structure $\Psi$ vanishing at $m$ is locally isomorphic to the linear approximation $\Lambda_m$ if $\mathfrak{g}_m$ is a semisimple Lie algebra of compact type. Further developments and generalizations of these results where obtained in $\mathcal{W}e_3$, $\mathcal{D}u_2$, $\mathcal{F}e\mathcal{M}$, $\mathcal{D}Z$, $\mathcal{Z}u_1$, $\mathcal{Z}u_2$. A review of recent results on the local linearization problem can be found in $\mathcal{F}e\mathcal{M}$.

We are interested in the linearization problem, or more generally, normal forms for the Poisson structure $\Psi$ in the semilocal context, around a symplectic leaf $B$. A more interesting situation occurs when the symplectic foliation nearby $B$ is singular. The Poisson topology of neighborhoods of (singular) symplectic leaves has been studied in several papers $\mathcal{G}i\mathcal{G}o$, $\mathcal{F}e_1$, $\mathcal{F}e_2$. The recent work $\mathcal{C}r\mathcal{F}e$ devoted is to rigidity and flexibility phenomena in Poisson geometry.

Our approach is based on the notion of a coupling Poisson structure $\mathcal{V}o_1$, $\mathcal{V}a_2$. The coupling procedure for symplectic structures was introduced in $\mathcal{S}t$ and further developed in $\mathcal{G}L\mathcal{S}$. The Poisson coupling can be defined on fibered spaces $\mathcal{V}o_1$ and foliated manifolds $\mathcal{V}a_2$ as well. The key observation is that a given Poisson structure $\Psi$ can be realized as a coupling Poisson structure on the normal bundle $E = TM/TB$ of the leaf. Using a diffeomorphism $f : E \to M$ identical on $B$, one can move from $M$ to the total space $E$. The result is the Poisson tensor $\Pi = f^* \Psi$ on $E$ with the same symplectic leaf $B$ (as the zero section). Then, in a neighborhood of $B$ in $E$ the Poisson tensor $\Pi$ induces an intrinsic Ehresmann connection $\Gamma$ corresponding to the splitting

$$TE = \mathbb{H} \oplus \mathbb{V},$$

where $\mathbb{V}$ is the vertical subbundle of $\pi : E \to B$ and the horizontal subbundle
\( \mathcal{H} \) is uniquely determined by \( \Pi \),

\[
\mathcal{H} = \Pi^* (\text{Ann } \mathcal{V}).
\]  

(1.2)

Here \( \Pi^* : T^* E \to TE \) is the bundle morphism induced by \( \Pi \) \text{ Ann } \mathcal{V} \) is the annihilator of \( \mathcal{V} \). Furthermore, the bivector field \( \Pi \) has the following decomposition with respect to (1.1):

\[
\Pi = \Pi_H + \Pi_V,
\]  

(1.3)

where \( \Pi_H \) and \( \Pi_V \) are the horizontal and vertical bivector fields, respectively. In general, the decomposition of \( \Pi \) involves a bivector field of degree \((1, 1)\). The vertical part \( \Pi_V \) is a Poisson tensor with the property: the symplectic leaf of \( \Pi_V \) through each point \( m \in E \) belongs to the fiber \( E_\xi \) over \( \xi = \pi(m) \). Such Poisson tensors on fibered (or foliated) spaces are called fiber-tangent (or leaf-tangent) Poisson structures [Va2]. The restriction of \( \Pi_V \) to each fiber \( E_\xi \) vanishes at 0 and defines the transverse Poisson structure of \( \xi \). The horizontal part \( \Pi_H \) is not a Poisson tensor in general. The Jacobi identity for \( \Pi_H \) is equivalent to the zero curvature condition for \( \Gamma \). Because of the property \( \mathcal{H}|_B = TB \), in a neighborhood of \( B \), the horizontal bivector field \( \Pi_H \) is uniquely defined, by a nondegenerate \( \mathcal{C}^\infty(E) \)-valued 2-form \( F \) on \( B \), which is called a coupling form. This form can be viewed as a “perturbation” of the symplectic structure of the leaf \( B \). Thus, the Poisson tensor is defined by the triple \((\Pi_V, \Gamma, F)\) and formula (1.3) can be interpreted as a result of coupling the form \( F \) and the fiber-tangent Poisson structure \( \Pi_V \) via the connection \( \Gamma \). We call \( \Pi \) a coupling Poisson tensor associated with geometric data \((\Pi_V, \Gamma, F)\). The Jacobi identity for \( \Pi \) is equivalent to some “integrability” conditions for \((\Pi_V, \Gamma, F)\) which have a natural geometric sense. Thus, given some integrable geometric data we can reconstruct the coupling Poisson tensor.

To define a linearized Poisson structure of \( \Pi \) at \( B \), first, one can linearize the geometric data \((\Pi_V, \Gamma, F)\) and then construct a “new” coupling Poisson tensor. The main point here is that the linearization procedure for \((\Pi_V, \Gamma, F)\) preserves the integrability conditions. Thus, we get the linearized geometric data \((\Lambda, \Gamma^{(1)}, F^{(1)})\) consisting of a linearized transverse Poisson structure \( \Lambda \), a homogeneous (linear) connection \( \Gamma^{(1)} \) on \( E \) and a linearized coupling form \( F^{(1)} \). We define the linearized Poisson structure of \( \Pi \) at \( B \) as the coupling Poisson tensor \( \Pi^{(1)} \) associated with the data \((\Lambda, \Gamma^{(1)}, F^{(1)})\). The Poisson tensor \( \Pi^{(1)} \) is independent of the choice of a diffeomorphism \( \mathbf{f} \) up to an
isomorphism and hence defines an intrinsic infinitesimal characteristic of the original Poisson structure $\Psi$ at $B$. Actually, as was shown in $[\text{Vo}_1, \text{Vo}_3]$, $\Pi^{(1)}$ is uniquely determined by the transitive Lie algebroid of the leaf $B$. A derivation of $\Pi^{(1)}$ in the context of the Lie algebroid approach, can also be found in $[\text{Vo}_4]$.

Now, one can state the semilocal linearization problem saying that the Poisson structure $\Psi$ is linearizable at $B$ if $\Pi$ and $\Pi^{(1)}$ are isomorphic by a diffeomorphism $\phi: E \rightarrow E$ from an appropriate class. We assume that $\phi$ belongs to the group of diffeomorphisms on $E$ satisfying the conditions $\phi|_B = \text{id}_B$ and $d_B\phi|_E = \text{id}_E$. One can expect that there is a semilocal analog of the Conn linearization theorem in the case when the transverse Lie algebra $\mathfrak{g}$ of the leaf $B$ is semisimple of compact type. But, according to an important observation due to [DW], there are nonlinearizable Poisson structures even if $\mathfrak{g}$ is semisimple of compact type. The corresponding examples are derived in the class of Poisson structures called Casimir-weighted products [DW]. This result rises the question on describing the corresponding obstructions to the linearizability. Here we make an attempt to answer this question.

Let $\text{Casim}_B(E)$ be the space of Casimir functions of the linearized transverse Poisson structure $\Lambda$ vanishing on $B$.

**Claim 1.1.** There exists an intrinsic coboundary operator

$$\partial_0 : \Omega^k(B) \otimes \text{Casim}_B(E) \rightarrow \Omega^{k+1}(B) \otimes \text{Casim}_B(E)$$

associated to $(\Psi, B)$.

One can define $\partial_0$ in terms of the exterior covariant derivative $\partial^{\Gamma^{(1)}}$ of a homogeneous connection $\Gamma^{(1)}$, but this definition is independent of $\Gamma^{(1)}$ and the operator $\partial_0$ is rather attributed to the (singular) symplectic foliation of the linearized transverse Poisson structure $\Lambda$. Next, we can associated to the pair $(\Pi, \Pi^{(1)})$ a $\partial_0$-cocycle $C = C_{\Pi,\Pi^{(1)}} \in \Omega^k(B) \otimes \text{Casim}_B(E)$, called a Casimir 2-cocycle. The 2-form $C_{\Pi,\Pi^{(1)}}$ is computed in terms of the geometric data $(\Pi_V, \Gamma, F)$ and $(\Lambda, \Gamma^{(1)}, F^{(1)})$ and is well defined under the assumption on the triviality of the first reduced Poisson cohomology space of $\Lambda$. In particular, this assumption holds in the case when the transverse Lie algebra $\mathfrak{g}$ is semisimple of compact type. We show that the $\partial_0$-cohomology class $[C_{\Pi,\Pi^{(1)}}]$ is an invariant of the leaf $B$. Moreover, using a Casimir 2-cocycle $C = C_{\Pi,\Pi^{(1)}}$, we can define the deformed coupling form $F + C_{\Pi,\Pi^{(1)}}$ and the
deformed linearized Poisson structure $\Pi_C^{(1)}$ as the coupling Poisson tensor associated with the data $(\Lambda, \Gamma^{(1)}, F^{(1)} + C_{\Pi,\Pi^{(1)}})$. Assuming that the transverse Lie algebra $\mathfrak{g}$ of the leaf $B$ is semisimple of compact type, we formulate our main results as follows.

**Claim 1.2.** (Normal Form Theorem). The germs at $B$ of the Poisson structures $\Psi$ and $\Pi_C^{(1)}$ are isomorphic.

A similar result to this statement was formulated in [Br] but without determining the deformed coupling form.

**Claim 1.3.** (Semilocal Linearization Theorem). The Poisson structure $\Psi$ is linearizable at $B$ if and only if the $\partial_0$-cohomology class of the leaf $B$ is zero,

$$[C_{\Pi,\Pi^{(1)}}] = 0.$$

These results are based on the following observation [Vo1]: a homotopy for coupling Poisson structures implies the Poisson equivalence. Here we give a further development of this thesis for general families of Poisson tensors.

We also refer to the work [CrFe], where one can find a conjecture on the linearization around a compact symplectic leaf $B$, formulated in terms of the integrability of the transitive Lie algebroid of $B$.

The paper is organized as follows.

In Sections 2.1–2.3 we give main definitions and formulate some useful technical results in Propositions 2.11 and 2.13. In Section 2.4, we study the relationship between a general homotopy for coupling structures and the Poisson equivalence. Here the main results are formulated in Theorems 2.18 and 2.19. In Section 2.5, we introduce the notion of a relative Casimir 2-cocycle of two coupling Poisson tensors and discuss the corresponding properties. In Section 3, using results of Section 2, we derive some criteria for the equivalence of Poisson structures near a given symplectic leaf. First, in Theorem 3.2, we show that the equivalence of coupling Poisson tensors imply the equivalence of the corresponding vertical parts. Sufficient conditions for the Poisson equivalence are presented in Theorems 3.4 and 3.8 and Proposition 3.9. We show also (Theorem 3.11) that the cohomology class of the Casimir 2-cocycle can be viewed as obstruction to the equivalence of coupling Poisson tensors. Section 4 is devoted to the semilocal linearization problem. The main results are presented in Theorems 4.12 and 4.14.

**Acknowledgement.** I am grateful M. V. Karasev for helpful and stimulating discussions during the preparation of the text.
2 Poisson coupling

In this section we first recall some basic properties of coupling Poisson structures on fibered spaces \([\mathbb{V}_0]\). For generalizations of the coupling procedure to foliated manifolds and Jacobi manifolds, we refer to \([\mathbb{V}_2]\). We introduced a homotopy equivalence for coupling Poisson tensors and formulate some useful technical results. Finally, we introduce the notion of the Casimir 2-cocycle of two coupling Poisson tensors which will play an important role in formulating the main results in Sections 3 and 4.

We shall perform all computations in coordinates. A free-coordinate approach can be found in \([\mathbb{V}_2]\).

2.1 Preliminaries

Let us fix some notation and recall the basic definitions. Let \(\pi : E \rightarrow B\) be a fiber bundle over a base \(B\). By \(\chi^k(E)\) and \(\Omega^k(E)\) we will denote the space of antisymmetric \(k\)-tensor fields and \(k\)-forms on the total space \(E\). In particular, \(\mathcal{X}(E) = \chi^1(E)\) is the space of smooth vector fields on \(E\). Let \(\mathbb{V} = \ker d\pi \subset TE\) be the vertical subbundle. The elements of \(\chi^k(E)\), tangent to \(\mathbb{V}\), form the subspace of vertical \(k\)-tensor fields denoted by \(\chi^k\mathbb{V}(E) = \text{Sect}(\bigwedge^k \mathbb{V})\). A \(k\)-form on \(E\) is said to be horizontal if it annihilates the vertical subbundle. We denote the subspace of horizontal \(k\)-forms by \(\Omega^k_H(E)\).

Suppose we are given an Ehresmann connection \([\mathbb{G}\mathbb{H}\mathbb{V}]\) on \(E\), that is, a smooth splitting

\[
TE = \mathbb{H} \oplus \mathbb{V}
\]

which is given by a subbundle \(\mathbb{H} \subset TE\) called horizontal. The \(k\)-vector fields tangent to \(\mathbb{H}\) are called horizontal and form the subspace \(\chi^k_H(E) \approx \text{Sect}(\bigwedge^k \mathbb{H})\). In particular, \(\mathcal{X}_H(E) = \chi^1_H(E)\) is the space of horizontal vector fields. For every vector field \(X\) on \(E\) we have the decomposition \(X = X_H + X_V\) onto the horizontal and vertical components \(X_H\) and \(X_V\), respectively. More generally,

\[
\chi^k(E) = \bigoplus_{i+j=k} \chi^i_H(E) \wedge \chi^j_V(E),
\]

where an element of \(\chi^i_H(E) \wedge \chi^j_V(E)\) is said to be a multivector field of degree \((i,j)\).
Let $\Gamma \in \Omega^1(E, V)$ be the connection form of (2.1), $\Gamma(X) \overset{\text{def}}{=} X_V$. Thus, $H = \ker \Gamma$. The curvature form $\text{Curv}^{\Gamma} \in \Omega^2(B, \text{End}(V))$ is given by

$$\text{Curv}^{\Gamma}(u_1, u_2) = \text{hor}^{\Gamma}([u_1, u_2]) - [\text{hor}^{\Gamma}(u_1), \text{hor}^{\Gamma}(u_2)].$$

Here $\text{hor}^{\Gamma}(u)$ is the $\Gamma$-horizontal lift of a vector field $u \in \mathcal{X}(B)$.

Consider the tensor product $\Omega^k(B) \otimes C^\infty(E)$ over $C^\infty(B)$. The $\Gamma$-covariant exterior derivative $\partial^{\Gamma} : \Omega^k(B) \otimes C^\infty(E) \rightarrow \Omega^{k+1}(B) \otimes C^\infty(E)$ is defined as

$$(\partial^{\Gamma} \Theta)(u_0, u_1, \ldots, u_k) = \sum_{i=0}^{k} (-1)^i L_{\text{hor}(u_i)} \Theta(u_0, u_1, \ldots, \hat{u}_i, \ldots, u_k)$$

$$+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \Theta([u_i, u_j], u_0, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_k).$$

We have the identity

$$(\partial^{\Gamma} \Theta)^2(u_0, u_1, \ldots, u_k)$$

$$= \sum_{0 \leq i < j \leq k+1} (-1)^{i+j} \Theta([u_i, u_j], u_0, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_{k+1})$$

which says that the coboundary condition $(\partial^{\Gamma})^2 = 0$ holds if and only if $\text{Curv}^{\Gamma} = 0$. This is the integrability condition for the horizontal plane distribution.

One can assign to every $\Theta \in \Omega^k(B) \otimes C^\infty(E)$ a horizontal $k$-form $\pi^* \Theta$ uniquely determined by the condition

$$(\pi^* \Theta)(\text{hor}^{\Gamma}(u_1), \ldots, \text{hor}^{\Gamma}(u_k)) = \Theta(u_1, \ldots, u_k)$$

for any $u_1, \ldots, u_k \in \mathcal{X}(B)$. This correspondence is independent of the choice of a connection.

Consider a (local) coordinate system $(\xi, x) = (\xi^i, x^\sigma)$ on the total space $E$, where $(\xi^i)$ are coordinates on the base $B$ and $(x^\sigma)$ are coordinates along the fibers of $E$. In coordinates, we have

$$\Gamma = \Gamma^\nu \otimes \frac{\partial}{\partial x^\nu}, \quad \Gamma^\nu = dx^\nu + \Gamma^\nu_i(\xi, x)d\xi^i.$$

Here and throughout the text, the summation over repeated indices will be understood. Locally, the horizontal subbundle is generated by the vector fields

$$\text{hor}^\Gamma_i \overset{\text{def}}{=} \text{hor}^{\Gamma} \left( \frac{\partial}{\partial \xi^i} \right) = \frac{\partial}{\partial \xi^i} - \Gamma^\nu_i(\xi, x) \frac{\partial}{\partial x^\nu}.$$
Moreover, the curvature form can be viewed as a 2-form on $B$ with values in the space of vertical vector fields,

$$\text{Curv}^\Gamma = \frac{1}{2} \text{Curv}^\sigma_{ij}(\xi, x) d\xi^i \wedge d\xi^j \otimes \frac{\partial}{\partial x^\sigma}.$$  

Notice that

$$[\text{hor}_i, \text{hor}_j] = - \text{Curv}^\sigma_{ij} \frac{\partial}{\partial x^\sigma} \in \mathcal{X}_V(E), \quad (2.4)$$

and

$$[\text{hor}_i, \frac{\partial}{\partial x^\sigma}] = \frac{\partial \Gamma^\nu_i}{\partial x^\sigma} \frac{\partial}{\partial x^\nu} \in \mathcal{X}_V(E). \quad (2.5)$$

Recall some properties of the Schouten bracket for multivector fields on $E$ (see, for example [KM, Va1]). Let $\Pi \in \chi^2(E)$ be a bivector field and $\Pi^\Gamma : T^*E \to TE$ be the induced bundle morphism, $\langle \beta, \Pi^\Gamma(\alpha) \rangle = \Pi(\alpha, \beta)$. For any $G \in C^\infty(E)$ and $W \in \mathcal{X}(E)$, we have $[G, \Phi] = \Pi^\Gamma dG$ and $[W, \Pi] = L_W \Pi$, where $L_W$ is the Lie derivative,

$$L_W \Pi(\alpha, \beta) = L_{\Pi^\Gamma(\alpha)} \beta(W) - L_{\Pi^\Gamma(\beta)} \alpha(W) + L_W (\Pi(\alpha, \beta)). \quad (2.6)$$

Moreover, we also need the following identities:

$$[W, Z_1 \wedge Z_2] = [W, Z_2] \wedge Z_1 - [W, Z_1] \wedge Z_2 \quad (2.7)$$

and

$$[Z_1 \wedge Z_2, \Pi] = Z_2 \wedge L_{Z_1} \Pi - Z_1 \wedge L_{Z_2} \Pi \quad (2.8)$$

for $W, Z_1, Z_2 \in \mathcal{X}(E)$.

**The Jacobi identity.** Fix an Ehresmann connection $\Gamma$. Suppose we are given a bivector field $\Pi \in \chi^2(E)$ which has the following representation with respect to decomposition (2.1):

$$\Pi = \Pi_H + \Pi_V. \quad (2.9)$$

Here $\Pi_H \in \chi^2_H(E)$ and $\Pi_V \in \chi^2_V(E)$ are horizontal and vertical bivector fields, respectively,

$$\Pi_H = \frac{1}{2} \Pi_H^{ij}(\xi, x) \text{hor}_i \wedge \text{hor}_j, \quad \Pi_V = \frac{1}{2} \Pi_V^{\alpha\beta}(\xi, x) \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}. \quad (2.10)$$

Let $\mathcal{V}^0 \overset{\text{def}}{=} \text{Ann}(\mathcal{V}) \subset T^*E$ be the annihilator of the vertical subbundle $\mathcal{V}$. Notice that (2.9) holds if for every horizontal 1-form $\beta$ the image $\Pi^\Gamma(\beta)$ is a horizontal vector field,

$$\Pi^\Gamma(\mathcal{V}^0) \subseteq \mathbb{H}. \quad (2.11)$$
Proposition 2.1. The Jacobi identity for the bivector field $\Pi$,

$$[\Pi, \Pi] = 0$$

is equivalent to the following relations:

$$(\alpha, \beta, \gamma)$$

$$\mathcal{S} \Pi^\alpha_V \frac{\partial \Pi^\beta V}{\partial x^\sigma} = 0,$$  

$$\Pi^i_H L_{\text{hor}} \Pi_V = 0,$$  

$$\mathcal{S} \Pi^k_H L_{\text{hor}} (\Pi^i_H) = 0,$$  

$$\Pi^i_H \text{Curv}^\sigma_{i'j'} \Pi^j_H = \Pi^\sigma V \frac{\partial}{\partial x^\sigma} (\Pi^i_H).$$

Here $\mathcal{S}$ denotes the cyclic sum and $L_{\text{hor}}$ is the Lie derivative along $\text{hor}$. 

Proof. Using relations (2.4)–(2.8), we compute the Schouten bracket $[\Pi, \Pi]$ by parts.

(i) The “horizontal–horizontal” part:

$$[\Pi_H, \Pi_H] = \frac{1}{4} [\Pi^i_H \text{hor}_i \wedge \text{hor}_j, \Pi^j_H \text{hor}_i \wedge \text{hor}_j]$$

$$- \Pi^k_H L_{\text{hor}} (\Pi^i_H) \text{hor}_i \wedge \text{hor}_j \wedge \text{hor}_k$$

$$- \Pi^i_H \text{Curv}^\sigma_{i'j'} \Pi^j_H \frac{\partial}{\partial x^\sigma} \wedge \text{hor}_j \wedge \text{hor}_j'.$$

(ii) The “horizontal–vertical” part:

$$2[\Pi_H, \Pi_V] = \frac{1}{2} \left[ \Pi^i_H \text{hor}_i \wedge \text{hor}_j, \Pi^\sigma V \frac{\partial}{\partial x^\sigma} \wedge \frac{\partial}{\partial x^\sigma'} \right]$$

$$= \Pi^\sigma V \frac{\partial \Pi^i_H}{\partial x^\sigma} \frac{\partial}{\partial x^\sigma'} \wedge \text{hor}_j \wedge \text{hor}_j' - \frac{1}{2} \Pi^i_H \text{hor}_i \wedge L_{\text{hor}, \Pi_V}.$$

Here

$$L_{\text{hor}} \Pi_V = \frac{1}{2} \left( L_{\text{hor}}, \Pi^\alpha V + \Pi^\alpha V \frac{\partial \Gamma^\beta_j}{\partial x^\sigma} - \Pi^\beta V \frac{\partial \Gamma^\alpha_j}{\partial x^\sigma} \right) \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}.$$
(iii) The “vertical–vertical” part:

\[
[\Pi_V, \Pi_V] = \frac{1}{4} \left[ \Pi_V^{\sigma \rho} \frac{\partial}{\partial x^\sigma} \wedge \frac{\partial}{\partial x^\rho}, \Pi_V^{\alpha \beta} \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta} \right]
\]

\[
= \Pi_V^{\nu \sigma} \frac{\partial}{\partial x^\nu} \frac{\partial}{\partial x^\sigma} \wedge \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}.
\]

It follows from here that the Schouten bracket \([\Pi, \Pi]\) is the sum of 3-vector fields of degrees \((3, 0)\), \((2, 1)\), \((1, 2)\), and \((0, 3)\). Vanishing of these terms leads to relations (2.13)–(2.16).

\[ \square \]

**Corollary 2.2.** The Jacobi identity for \(\Pi\) in (2.9) implies that its vertical part \(\Pi_V\) is a Poisson tensor.

**Remark 2.3.** In \([\text{Va}_2]\), a Poisson tensor satisfying condition (2.11) is called almost coupling.

### 2.2 Coupling Poisson tensors

A bivector field \(\Pi \in \chi^2(E)\) is said to be **horizontally nondegenerate** on \(E\) if for every \(m \in E\) the image \(\Pi^\sharp_H(\mathbb{V}_m^0)\) is a complementary subspace of \(\mathbb{V}_m\) in \(T_mE\). In this case, \(\Pi\) induces the Ehresmann connection \(\Gamma\) corresponding to the horizontal subbundle

\[ \mathbb{H} \overset{\text{def}}{=} \Pi^\sharp(\mathbb{V}^0). \] (2.17)

This implies that \(\Pi(\alpha, \beta) = 0\) for all \(\alpha \in \text{Sect}(\mathbb{H}^0)\) and \(\beta \in \text{Sect}(\mathbb{V}^0)\). Consequently, \(\Pi\) has representation (2.9) with respect to connection (2.17). Then, the horizontal part \(\Pi_H\) is nondegenerate in the sense that

\[ \ker \Pi^\sharp_H|_{\mathbb{V}_m} = \{0\}, \] (2.18)

or, equivalently, \(\Pi^\sharp_H(\mathbb{V}_m^0) = \mathbb{H}_m\). It follows from here that there exists a unique 2-form \(F \in \Omega^2(B) \otimes C^\infty(E)\) defined by the condition

\[ \Pi_H(\beta_1, \beta_2) = (\pi^*F)(\Pi^\sharp_H(\beta_1), \Pi^\sharp_H(\beta_2)) \] (2.19)

for any \(\beta_1, \beta_2 \in \text{Sect}(\mathbb{V}^0)\). It is clear that

\[ \pi^*F|_{\mathbb{V}_m} \text{ is nondegenerate} \] (2.20)
for every $m \in E$. In coordinates,

$$\pi^* F = F_{ij}(\xi, x)d\xi^i \wedge d\xi^j,$$

where $\Pi^i_H(\xi, x)F_{sj}(\xi, x) = -\delta^i_j$ and

$$\det(F_{ij}(\xi, x)) \neq 0.$$  \hfill (2.21)

Thus, the horizontally nondegenerate bivector field $\Pi$ induces the triple $(\Pi_V, \Gamma, F)$ consisting of the vertical bivector field $\Pi_V$, the Ehresmann connection $\Gamma$ in (2.17) and the 2-form $F$ defined by (2.19). The triple $(\Pi_V, \Gamma, F)$ will be called the geometric data of $\Pi$. This correspondence is one-to-one. Conversely, for a given $(\Pi_V, \Gamma, F)$, where $F$ satisfies the nondegeneracy condition (2.20), the corresponding horizontally nondegenerate bivector field $\Pi$ is defined by

$$\Pi = -\frac{1}{2} F^{ij}(\xi, x) \text{hor}_i \wedge \text{hor}_j + \frac{1}{2} \Pi^\alpha_\beta(\xi, x) \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}. \hfill (2.22)$$

Here, $F^{is}(\xi, x)F_{sj}(\xi, x) = \delta^i_j$.

Taking into account (2.21), we deduce the following result straightforwardly from Proposition 2.1.

**Proposition 2.4.** A horizontally nondegenerate bivector field $\Pi$ is a Poisson tensor if and only if the geometric data $(\Pi_V, \Gamma, F)$ satisfy the following conditions

$$[\Pi_V, \Pi_V] = 0, \hfill (2.23)$$
$$L_{\text{hor}_v} \Pi_V = 0, \hfill (2.24)$$
$$\partial^F F = 0, \hfill (2.25)$$
$$\text{Curv}^\Gamma (v, u) = (\Pi_V)^s d(F(v, u)) \hfill (2.26)$$

for any $v, u \in X(B)$.

By (2.16) and (2.21) we also get the following fact.

**Corollary 2.5.** The Jacobi identity for the horizontal component $\Pi_H$ of $\Pi$ is equivalent to the zero curvature condition,

$$\text{Curv}^\Gamma = 0. \hfill (2.27)$$

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The horizontally nondegenerate Poisson tensor $\Pi$ in (2.22) is called a coupling Poisson structure associated with geometric data $(\Pi_V, \Gamma, F)$. The 2-form $F$ will also be called a coupling form. Formula (2.22) is the result of coupling $F$ and $\Pi_V$ via the connection $\Gamma$.

The pairwise Poisson brackets of coordinate functions relative to $\Pi$ are written in terms of geometric data as follows:

\[
\{\xi^i, \xi^j\}_\Pi = -F_{ij}(\xi, x),
\]

\[
\{\xi^i, x^\sigma\}_\Pi = F^i_s(\xi, x)\Gamma^s_\sigma(\xi, x),
\]

\[
\{x^\alpha, x^\beta\}_\Pi = \Pi^\alpha_\beta V(\xi, x) - \Gamma^i_\alpha(\xi, x)F^i_j(\xi, x)\Gamma^j_\beta(\xi, x).
\]

**Example 2.6.** Let $E = B \times N$ is the product of a symplectic manifold $(B, \omega)$ and a Poisson manifold $N$ with Poisson tensor $\Phi$. Let $p_1$ and $p_2$ be the canonical projections to the first and the second factors, respectively. We can think of $E$ as a trivial fiber bundle over $B$ with $\pi = p_1$. Then the direct product Poisson structure on $B \times N$ is a coupling Poisson structure associated with the following data: $(\Pi_V, \mathbb{H} = \ker p_2, F = \omega \otimes 1)$. Here $\Pi_V$ is a vertical bivector field on $B \times N$ which is uniquely determined by the condition: $\Pi_V(p_2^*\alpha, p_2^*\beta) = p_2^*(\Phi(\alpha, \beta))$ for every $\alpha, \beta \in \Omega^1(N)$. In this case, $\Gamma$ is a flat connection with trivial holonomy. By the local splitting theorem [We1], locally, each Poisson manifold has this form.

**Example 2.7.** Every Poisson manifold in a neighborhood of a symplectic leaf is realized as a coupling Poisson structure on the normal bundle (see Section 4).

More examples can be found in [Vo1, Vo4, Va2, Va3, DW].

**Fiber-tangent Poisson structures.** By (2.23) the vertical part of a coupling Poisson tensor is also Poisson. Following the terminology introduced in [Va2], a vertical Poisson tensor $\Upsilon$ on the total space $E$ will be called a fiber-tangent Poisson structure. Such a Poisson structure can be uniquely characterized by the property: for ever $m \in E$ the symplectic leaf of $\Upsilon$ through $m$ belongs to the fiber $E_{\pi(m)}$. The restriction $\Upsilon_\xi = \Upsilon|_{E_\xi}$ is well defined and gives a Poisson tensor on the fiber $E_\xi$ which varies smoothly with $\xi \in B$. Thus, one can think of $\pi : E \to B$ as a bundle of Poisson manifolds with fiberwise Poisson structure $\Upsilon_\xi$. If $\Upsilon$ is the vertical part of a coupling Poisson tensor $\Pi$, $\Upsilon = \Pi_V$, then condition (2.24) implies that the parallel transport operator of the connection $\Gamma$ preserves the fiberwise
Poisson structure on $E$ induced by $\Upsilon$. Such a connection is called a Poisson connection.

**Definition 2.8.** A fiber-tangent Poisson structure $\Upsilon$ is said to be *locally trivial* if in a neighborhood of each point in $E$ there exists a coordinate system $(\xi, x) = (\xi^i, x^\sigma)$ such that $\Upsilon$ has the form

$$\Upsilon = \frac{1}{2} \Upsilon^\alpha_\beta (x) \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}. \quad (2.31)$$

Here $(\xi^i)$ are coordinates on the base $B$ and $(x^\sigma)$ are coordinates on the fibers of $E$.

In the next section, we show that a typical feature of the fiber-tangent Poisson structure $\Upsilon$ which comes from a coupling Poisson tensor is that $\Upsilon$ is locally trivial. The local triviality property allows us to transfer some properties of $\Upsilon$ from the fiber to the total space using the partition of unity argument.

**Fiber preserving transformations.** Suppose we are given a coupling tensor $\Pi$ on $E$ associated with geometric data $(\Pi_V, \Gamma, F)$.

**Proposition 2.9.** Let $g$ be a fiber preserving diffeomorphism. Then the push-forward $\tilde{\Pi} = g_* \Pi$ is a coupling Poisson tensor associated to the geometric data

$$\tilde{\Pi}_V = g_* \Pi_V, \quad \tilde{\Gamma} = g_* \Gamma, \quad \tilde{F} = g_* F. \quad (2.32)$$

**Proof.** Consider the bivector field $\tilde{\Pi} = g_* \Pi$ and the corresponding distribution $\tilde{\mathbb{H}}$ defined by (2.17),

$$\tilde{\mathbb{H}}_{g(m)} = (d_m g) \circ \Pi_m^* \circ (d_m g)^*(\mathbb{V}^0_{g(m)}) \quad (2.33)$$

for every $m \in N$. Here $d_m g : T_mE \to T_{g(m)}E$ is the tangent map of $g$. On the other hand, consider the push-forward $g_* \mathbb{H}$ of $\mathbb{H}$ by $g$,

$$(g_* \mathbb{H})_{g(m)} = (d_m g) \mathbb{H}_m. \quad (2.34)$$

By assumption, $g$ is fiber preserving, that is, $dg$ preserves the vertical subbundle,

$$(d_m g) \mathbb{V}_m = \mathbb{V}_{g(m)}$$

and hence $(d_m g)^*(\mathbb{V}^0_{g(m)}) = \mathbb{V}^0_m$. Comparing (2.33) and (2.34) leads to

$$\tilde{\mathbb{H}} = g_* \mathbb{H}.$$
Thus, $\tilde{H}$ is complementary to $V$ and hence $\tilde{\Pi}$ is a coupling Poisson tensor. For the connection form we have $\tilde{\Gamma}((d_m g) X) = (d_m g) \Gamma(X)$ for $X \in T_m E$. Other two identities in (2.32) are evident.

It follows that decomposition (2.9) is stable under a fiber preserving diffeomorphism $g$,

$$g_*(\Pi_H + \Pi_V) = (g_*)_H + (g_*)_V. \quad (2.35)$$

In general, this is not true.

### 2.3 Infinitesimal Poisson automorphisms

Let $\Pi$ be a coupling Poisson tensor associated with geometric data $(\Pi_V, \Gamma, F)$. Our goal is to describe infinitesimal Poisson automorphisms (Poisson vector fields) of $\Pi$ in terms of the geometric data.

We start with some useful technical formulas. Let

$$X = X^i(\xi, x) \text{ hor}_i \quad (2.36)$$

be a horizontal vector field. Denote $i_X F \in \Omega^1(B) \otimes C^\infty(N)$ defined by $\pi^*(i_X F) = i_X(\pi^* F)$, or locally, $i_X F = X^s(\xi, x) F_{sj}(\xi, x) \otimes d\xi^j$.

**Lemma 2.10.** For any $v_1, v_2 \in \mathcal{X}(B)$, we have

$$L_X (\pi^* F)(\text{hor}(v_1), \text{hor}(v_2)) = \partial^F(i_X F)(v_1, v_2). \quad (2.37)$$

**Proof.** In components, condition (2.25) for $F$ reads

$$\mathbf{S}_{(ijs)} L_{\text{hor}_i} F_{js} = 0.$$

Applying this identity and using the definition of $\partial^F$, we get

$$L_X (F)(\text{hor}_m, \text{hor}_{m'}) = L_X F_{mm'} - F_{sm} L_{\text{hor}_m} X^s + F_{sm'} L_{\text{hor}_m} X^s$$

$$= L_{\text{hor}_m}(i_X F)_{m'} - L_{\text{hor}_m}(i_X F)_m$$

$$= (\partial^F(i_X F))_{mm'}.$$


Proposition 2.11. The Lie derivative of the coupling Poisson tensor $\Pi$ along a horizontal vector field $X$ is given by the formula

$$L_X \Pi = \frac{1}{2} F^{im} [(\partial^F(i_X F))_{nm}] F^{mj} \text{hor}_i \wedge \text{hor}_j$$

$$\quad - F^{si} \text{hor}_i \wedge \Pi_V^i (d(X^m F_{ms})).$$

(2.38)

Proof. It follows from (2.4) and (2.26) that

$$[\text{hor}_s, \text{hor}_m] = - \text{Curv}_{sm}^\sigma \frac{\partial}{\partial x^\sigma} = - \Pi_V^i (dF_{sm}).$$

Then,

$$L_{\text{hor}_s} \Pi_H = - \frac{1}{2} (L_{\text{hor}_s} F^{ij}) \text{hor}_i \wedge \text{hor}_j - F^{mi} \text{hor}_i \wedge \Pi_V^i (dF_{sm})$$

and

$$L_X \Pi_H = \frac{1}{2} F^{im} (L_X F_{mm'} - F_{sm} L_{\text{hor}_m} X^s + F_{sm'} L_{\text{hor}_m} X^{s'}) F^{mj} \text{hor}_i \wedge \text{hor}_j$$

$$\quad - X^s F^{mi} \text{hor}_i \wedge \Pi_V^i (dF_{sm}) - \text{hor}_i \wedge (\Pi_V^i dX^i)$$

From here, by using (2.37), we deduce (2.38). \hfill $\square$

Denote by Casim($E, \Pi_V$) the space of Casimir functions of the vertical Poisson structure $\Pi_V$. It is clear that $\pi^* C^\infty(B) \subset \text{Casim}(E, \Pi_V)$.

Corollary 2.12. A horizontal vector $X$ is an infinitesimal Poisson automorphism of $\Pi$ if and only if

$$L_X (\pi^* F)|_H = 0,$$

(2.39)

and

$$i_X F \in \Omega^1(B) \otimes \text{Casim}(E, \Pi_V).$$

(2.40)

Condition (2.39) says that the Lie derivative of the horizontal 2-form $\pi^* F$ along $X$ is a vertical form.

Furthermore, using (2.24), we get the formula

$$L_X \Pi_V = - \text{hor}_i \wedge (\Pi_V^i dX^i)$$

(2.41)

which says that $X$ is an infinitesimal Poisson automorphism of $\Pi_V$ if and only if the coefficients $X^i$ in (2.36) are (local) Casimir functions.
Now, let
\[ Y = Y^\sigma(\xi, x) \frac{\partial}{\partial x^\sigma} \]
be a vertical vector field. By straightforward calculation we get the formula
\[
L_Y \Pi = \frac{1}{2} F^m_{im}[L_Y F_{mn}] F^{n'j} \text{hor}_i \wedge \text{hor}_j \tag{2.42}
\]
\[ - F^{si}[\text{hor}_s, Y]^\sigma \text{hor}_i \wedge \frac{\partial}{\partial x^\sigma} + L_Y \Pi_V, \]
which gives the following criterion.

**Proposition 2.13.** A vertical vector field \( Y \) is an infinitesimal Poisson automorphism of \( \Pi \) if and only if
\[
L_Y \Pi_V = 0, \tag{2.43}
\]
\[
L_Y(\pi^* F) = 0, \tag{2.44}
\]
\[
[\text{hor}(u), Y] = 0 \tag{2.45}
\]
for every \( u \in \mathcal{X}(B) \).

Remark that condition (2.45) means that the horizontal subbundle \( \mathbb{H} \) is invariant under the flow of \( Y \).

### 2.4 Poisson homotopy

Suppose we are given a smooth 1-parameter family \( \{\Pi_t\}_{t \in [0,1]} \) of coupling Poisson tensors on \( E \). For every \( t \), \( \Pi_t \) is a coupling Poisson tensor associated with geometric data \( (\Pi_t)_V, \Gamma_t, F_t \) where the vertical bivector field \( (\Pi_t)_V \), the Ehresmann connection \( \Gamma_t \), and the 2-form \( F_t \) vary smoothly with \( t \). In coordinates,
\[
\Pi_t = -\frac{1}{2} F^{ij}_t(\xi, x) \text{hor}_i \wedge \text{hor}_j + \frac{1}{2} (\Pi_t)_{1\beta}^\alpha(\xi, x) \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}. \tag{2.46}
\]
Here we denote \( \text{hor}_i^t = \text{hor}^{\Gamma_t}(\frac{\partial}{\partial x^i}) \) and \( F^{is}_t F^{tj}_s = \delta^j_i \).

Let \( Z_t \) be a time-dependent vector field on \( E \). Then we have the decomposition
\[
Z_t = X_t + Y_t, \tag{2.47}
\]
where $X_t$ and $Y_t$ are the horizontal and vertical components with respect to the connection $\Gamma^t$,

$$X_t = X^i_t(\xi, x) \text{hor}_i^t, \quad Y_t = Y^\sigma_t(\xi, x) \partial_{x^\sigma}.$$  \hspace{1cm} (2.48)

Let $\partial^{F^t}$ be the covariant exterior differential (2.2) associated with $\Gamma^t$.

**Proposition 2.14.** A time-dependent vector field $Z_t = X_t + Y_t$ is a solution of the equation

$$L_{Z_t} \Pi_t + \frac{\partial \Pi_t}{\partial t} = 0$$  \hspace{1cm} (2.49)

if and only if the components $X_t$ and $Y_t$ satisfy the relations

$$L_{Y_t} \Pi_V + \frac{\partial \Pi_V}{\partial t} = 0,$$  \hspace{1cm} (2.50)

$$\partial^{F^t}(i_{X_t} F^t) + L_{Y_t} F^t + \frac{\partial F^t}{\partial t} = 0,$$  \hspace{1cm} (2.51)

$$(\Pi^t)_{V}^\sharp d(i_{X_t} F^t)_s + [\text{hor}_s^t, Y_t] - \frac{\partial \text{hor}_s^t}{\partial t} = 0.$$  \hspace{1cm} (2.52)

**Proof.** First, the straightforward computations give

$$\frac{\partial \Pi_t}{\partial t} = \frac{1}{2} F_t^{im} \left( \frac{\partial F_t^{mm'}}{\partial t} \right) F_t^{mj} \text{hor}_i^t \wedge \text{hor}_j^t$$

$$- F_t^{ij} \frac{\partial \text{hor}_i^t}{\partial t} \wedge \text{hor}_j^t + \frac{1}{2} \frac{\partial}{\partial t} (\Pi^t)_V^\alpha \partial_x^\alpha \wedge \partial_x^\beta.$$

By formulas (2.38), (2.41) and (2.42) we get

$$L_{Z_t} \Pi_t = \frac{1}{2} F_t^{im} \left[ (\partial^{F^t}(i_{X_t} F^t))_{mm'} + L_{Y_t} F_t^{mm'} \right] F_t^{mj} \text{hor}_i^t \wedge \text{hor}_j^t$$

$$- F_t^{si} \text{hor}_i^t \wedge (\Pi^t)_V^\sharp (d(X_m^t F^t_{ms})) - F_t^{si} [\text{hor}_s^t, Y_t] \text{hor}_i^t \wedge \frac{\partial}{\partial x_\sigma} + L_{Y_t} \Pi_V.$$

Finally, putting these relations into (2.49) and using that

$$\frac{\partial \text{hor}_s^t}{\partial t}$$

is a vertical vector field,

we see that vanishing the terms of degrees (0, 2), (2, 0), and (1, 1) leads to Eqs. (2.50)–(2.52). \hfill \Box
A time-dependent vector field $Z_t$ satisfying (2.49) is said to be an infinitesimal generator of the family $\{\Pi_t\}$ of coupling Poisson tensors on $E$.

**Definition 2.15.** We say that two coupling Poisson tensors $\Pi'$ and $\Pi''$ on $E$ are homotopic if there exists a smooth family $\{\Pi_t\}_{t \in [0,1]}$ of coupling Poisson tensors on $E$ which admits an infinitesimal generator and joints $\Pi'$ and $\Pi''$, $\Pi_0 = \Pi'$, $\Pi_1 = \Pi''$.

Assume that $Z_t$ is an infinitesimal generator of $\{\Pi_t\}$. Let $\Phi_t$ be the flow of $Z_t$,

$$\frac{d\Phi_t}{dt} = Z_t \circ \Phi_t, \quad \Phi_0 = \text{id}.$$  \hspace{1cm} (2.53)

Suppose that for every $t \in [0,1]$ the flow $\Phi_t$ is well defined on an open domain $N_{\text{flow}} \subseteq E$ independent of $t$. Then, we have

$$\frac{d}{dt}(\Phi_t^*\Pi_t) = \Phi_t^*\left(L_{Z_t}\Pi_t + \frac{\partial\Pi_t}{\partial t}\right) = 0$$  \hspace{1cm} (2.54)

and hence

$$\Phi_t^*\Pi_t = \Pi_0 \quad \text{on} \quad N_{\text{flow}}.$$  \hspace{1cm} (2.55)

Thus, the family $\{\Pi_t\}$ is generated by the flow $\Phi_t$ and the “initial” coupling Poisson tensor.

The next question is to formulate some criteria for the existence of $Z_t$ in terms of the geometric data of $\Pi_t$ analyzing Eqs. (2.50)–(2.52).

**Sufficient conditions for the existence of $Z_t$.** Here we assume that we are given a smooth 1-parameter family $\{\Pi_t\}_{t \in [0,1]}$ of coupling Poisson tensors such that the vertical part of $\Pi_t$ is independent of the parameter $t$,

$$(\Pi_t)_V = \Upsilon \quad \forall t \in [0,1],$$  \hspace{1cm} (2.56)

where $\Upsilon$ is a fiber-tangent Poisson structure. Thus, for every $t$, $\Pi_t$ is a coupling Poisson tensor associated with geometric data $(\Upsilon, \Gamma^t, F^t)$.

Let us introduce the following notation. Denote by $\{,\}_\Upsilon$ the Poisson bracket corresponding to the fiber-tangent Poisson structure $\Upsilon$. One can associate to any $Q \in \Omega^1(B) \otimes C^\infty(E)$ and $\Theta \in \Omega^k(B) \otimes C^\infty(E)$ an element $\{Q \wedge \Theta\}_\Upsilon \in \Omega^{k+1}(B) \otimes C^\infty(E)$ defined by

$$\{Q \wedge \Theta\}_\Upsilon(u_0, u_1, \ldots, u_k) \overset{\text{def}}{=} \sum_{i=0}^{k} (-1)^i \{Q(u_i), \Theta(u_0, u_1, \ldots, \hat{u}_i, \ldots, u_k)\}_\Upsilon.$$  \hspace{1cm} (2.57)

In particular,
• if $\Theta \in Q \in \Omega^1(B) \otimes C^\infty(\mathcal{N})$,
  \[
  \{Q \wedge \Theta\}_\tau(u_0, u_1) = \{Q(u_0), \Theta(u_1)\}_\tau - \{Q(u_1), \Theta(u_0)\}_\tau; \tag{2.58}
  \]

• if $\Theta \in \Omega^2(B) \otimes C^\infty(\mathcal{Y})$,
  \[
  \{Q \wedge \Theta\}_\tau(u_0, u_1, u_2) = \mathcal{S}_{(u_0, u_1, u_2)} \{Q(u_0), \Theta(u_1, u_2)\}_\tau. \tag{2.59}
  \]

Assume that the data $(\Gamma^t, F^t)$ satisfy the following condition: there exists a smooth family of 1-forms $Q^t \in \Omega^1(B) \otimes C^\infty(E)$ such that

\[
\text{hor}^\Gamma_t(u) = \text{hor}^\Gamma_0(u) + \mathcal{Y}^s dQ^t(u), \tag{2.60}
\]

\[
F^t = F^0 - (\partial^\Gamma_0 Q^t + \frac{1}{2} \{Q^t \wedge Q^t\}_\tau) \tag{2.61}
\]

for every $t \in [0, 1]$ and $u \in \mathcal{X}(B)$.

By (2.24), $\text{hor}^\Gamma_t(u) - \text{hor}^\Gamma_0(u)$ is a vertical Poisson vector field of $\mathcal{Y}$. Condition (2.60) says that this difference is a Hamiltonian vector field with Hamiltonian $Q^t(u)$ which must satisfy the compatibility condition

\[
Q^0_0(u) \in \text{Casim}(E, \mathcal{Y}). \tag{2.62}
\]

Furthermore, condition (2.60) implies that the exterior covariant derivatives $\partial^\Gamma_t$ and $\partial^\Gamma_0$ are related by the formula

\[
\partial^\Gamma_t \Theta = \partial^\Gamma_0 \Theta + \{Q^t \wedge \Theta\}_\tau. \tag{2.63}
\]

In particular,

\[
\partial^\Gamma_t Q^t = \partial^\Gamma_0 Q^t + \{Q^t \wedge Q^t\}_\tau. \tag{2.64}
\]

It follows that (2.61) can be rewritten in the form

\[
F^t = F^0 - \left(\partial^\Gamma_t Q^t - \frac{1}{2} \{Q^t \wedge Q^t\}_\tau\right). \tag{2.65}
\]

Proposition 2.16. Under assumptions (2.60), (2.61), a time-dependent vector field $Z_t = X_t + Y_t$ is an infinitesimal generator of the family $\{\Pi_t\}$ if and only if $X_t$ and $Y_t$ satisfy the equations

\[
L_{Y_t} \mathcal{Y} = 0, \tag{2.66}
\]

\[
\partial^\Gamma_t \left(\frac{\partial Q^t}{\partial t} - i_{X_t} F^t\right) - L_{Y_t} F^t = 0, \tag{2.67}
\]

\[
\mathcal{Y}^s d\left(\frac{\partial Q^t(u)}{\partial t} - i_{X_t} F^t(u)\right) + [Y_t, \text{hor}^\Gamma_t(u)] = 0. \tag{2.68}
\]
Proof. By (2.61) and (2.63) we deduce
\[
\frac{\partial F^t}{\partial t} = -\left[\partial^\tau\left(\frac{\partial Q^t}{\partial t}\right) + \{Q^t \wedge \frac{\partial Q^t}{\partial t}\}_\tau\right] = -\partial^\tau\left(\frac{\partial Q^t}{\partial t}\right).
\]
Condition (2.60) implies
\[
\frac{\partial \text{hor}^i}{\partial t} = \Upsilon^i dQ^t.
\]
Putting these relations into (2.51) and (2.52) gives (2.66) and (2.67).

Corollary 2.17. We have the variation of parameter formula for \( \Pi_t \)
\[
\frac{\partial \Pi_t}{\partial t} = -\frac{1}{2} F^m_t \left(\frac{\partial Q^t}{\partial t}\right)^m_{mm'} F^m_j h^j_i \text{hor}^i \wedge \text{hor}^j + F^i_{ij} \text{hor}^j \wedge \Upsilon^i \left(\frac{d}{\partial t} \left(\frac{\partial Q^t}{\partial t}\right)\right).
\]

Now, choosing \( Y_t = 0 \) in (2.66), (2.67), we arrive at the key observation.

Theorem 2.18. Let \( \{\Pi_t\} \) be a family of coupling Poisson structures associated with geometric data \((\Upsilon, \Gamma_t, F^t)\). If conditions (2.60), (2.61) hold, then \( \{\Pi_t\} \) admits the infinitesimal generator \( X_t = X_t^i(\xi, x) \text{hor}^i_t \) which is uniquely determined by
\[
\text{hor}^i_t F^t = \frac{\partial Q^t}{\partial t}.
\]
Remark that \( X_t \) is well defined by (2.70) because of the nondegeneracy property (2.20) for \( F^t \).

Let \( Y_t = \Upsilon^i dh_t \) be the vertical Hamiltonian time-dependent vector relative to \( \Upsilon \) with a time-dependent function \( h_t \in C^\infty(N) \). By (2.24), the horizontal lift \( \text{hor}^{\Gamma_t}(u) \) is a vertical Poisson vector field of \( \Upsilon \) and hence
\[
[\text{hor}^{\Gamma_t}(u), \Upsilon^i dh_t] = \Upsilon^i d(L_{\text{hor}^{\Gamma_t}(u)} h_t).
\]
It follows from (2.65)–(2.67), that under conditions (2.60), (2.61), finding solutions of (2.49) in the form \( Z_t = X_t + \Upsilon^i dh_t \) is reduced to solving the following two equations:
\[
\partial^\tau\left(\frac{\partial Q^t}{\partial t} - i_{X_t} F^t\right) = \{h_t, F^t\}_\tau,
\]
\[
\Upsilon^i d\left(\frac{\partial Q^t(u)}{\partial t} - i_{X_t} F^t(u) - L_{\text{hor}^{\Gamma_t}(u)} h_t\right) = 0.
\]

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Necessary conditions. Theorem 2.18 says that conditions (2.60), (2.61) are sufficient for the existence of an infinitesimal generator of \( \{ \Pi_t \} \). Now we assume that only (2.60) holds. In general, the implication (2.60) \( \implies \) (2.61) is not true. By (2.26) one can conclude that (2.61) holds only modulo an element in \( \Omega^2(B) \otimes \text{Casim}(E, \Upsilon) \). The next statement shows that condition (2.61) is necessary for the existence on an infinitesimal generator of \( \{ \Pi_t \} \) in the class of vector fields of the form \( Z_t = X_t + \Upsilon^z \text{dh}_t \).

**Theorem 2.19.** Let \( \{ \Pi_t \} \) be a family of coupling Poisson structures associated with geometric data \( (\Upsilon, \Gamma^t, F^t) \). Assume that

(a) condition (2.60) holds for a certain family \( \{ Q^t \} \);
(b) \( \{ \Pi_t \} \) admits an infinitesimal generator of the form

\[
Z_t = X_t + \Upsilon^z \text{dh}_t,
\]  

(2.74)

for \( h^t \in C^\infty(E) \).

Then, the family \( \{ Q^t \} \) can be chosen such that (2.61) holds.

**Proof.** Suppose \( \{ Q^t \} \) and an infinitesimal generator \( Z_t \) in (2.74) are fixed. Remark that \( \{ Q^t \} \) is unique up to the transformation \( Q^t \to Q^t + K^t \), where \( K^t \in \Omega^1(B) \otimes \text{Casim}(E, \Upsilon) \). Our point is to choose \( K^t \) to satisfy (2.61). By Proposition 2.14, \( Z_t \) must satisfy Eqs. (2.51) and (2.52). First, let us consider (2.52) for \( Y^t = \Upsilon^z \text{dh}_t \). Using the identity \( (\partial \Gamma^t \text{dh}_t)(u) = L_{\text{hor}^t(u)}h^t \forall u \in X(B) \), by (2.71) we deduce

\[
[\text{hor}^t_s, Y^t] = \Upsilon^z d(\partial^r h^t)_s,
\]

where \( \partial^r h^t \in \Omega^1(B) \otimes C^\infty(E) \). Putting this and (2.68) into (2.52) gives

\[
\Upsilon^z d\left( (i_{X^t F^t})_s + (\partial^r h^t)_s - \frac{\partial Q^t_s}{\partial t} \right) = 0.
\]

Consequently

\[
(i_{X^t F^t}) = \frac{\partial Q^t}{\partial t} - \partial^r h^t + \beta^t
\]

for a certain \( \beta^t \in \Omega^1(B) \otimes \text{Casim}(E, \Upsilon) \). Applying the operator \( \partial^r \) to both sides of this equality, we have

\[
\partial^r (i_{X^t F^t}) = \partial^r \left( \frac{\partial Q^t}{\partial t} - \partial^r h^t + \beta^t \right)
\]

(2.75)

\[
= \partial^r \left( \frac{\partial Q^t}{\partial t} \right) - (\partial^r)^2 h^t + \partial^r \beta^t
\]
Next, consider Eq. (2.51). Remark that we have the identity (see (2.79):

\[ L_Y F^t = \{ h^t, F^t \} \_\_Y = (\partial^F)^2 h^t. \]  

(2.76)

By using (2.75) and (2.76), we rewrite (2.51) in the form

\[ \partial^F t \left( \frac{\partial Q^t}{\partial t} \right) + \partial^F t \beta^t + \frac{\partial F^t}{\partial t} = 0. \]  

(2.77)

Now taking into account (2.63) and that \( \partial^F t \beta^t = \partial^F 0 \beta^t \), from (2.77) we deduce

\[ \frac{\partial F^t}{\partial t} = -\left( \partial^F 0 \left( \frac{\partial Q^t}{\partial t} \right) + \left\{ Q^t, \frac{\partial Q^t}{\partial t} \right\} \right) - \partial^F 0 \beta^t \]

\[ = -\frac{\partial}{\partial t} \left[ \partial^F 0 Q^t + \frac{1}{2} \left\{ Q^t, Q^t \right\}_Y \right] - \partial^F 0 \beta^t. \]

Finally, integrating of this identity gives

\[ F^t = F_0 - \left[ \partial^F 0 Q^t + \frac{1}{2} \left\{ Q^t, Q^t \right\}_Y \right] - \partial^F 0 \left( \int_0^t \beta^t d\tau \right). \]  

(2.78)

Thus, after the transformation

\[ Q^t \mapsto Q^t + \int_0^t \beta^t dt, \]

(2.61) is satisfied. \( \square \)

### 2.5 Relative Casimir 2-cocycles

Denote by Coup\((E, Y)\) the set of all coupling Poisson tensors \( \Pi \) with fixed vertical part, \( \Pi_Y = Y \). Here we introduce an invariant of a pair of bivector fields in Coup\((E, Y)\) which takes values in the space \( \Omega^2(B) \otimes \text{Casim}(E, Y) \). This invariant is well defined for coupling Poisson tensors whose geometric data belong to the same equivalence class.

Pick \( \Pi \in \text{Coup}_B(E, Y) \) associated with geometric data \((Y, \Gamma, F)\). Consider the covariant exterior derivative \( \partial^F \). By (2.3) and (2.26) we have the identity

\[(\partial^F)^2 \Theta(u_0, u_1, \ldots, u_i, \ldots, u_j, \ldots, u_{k+1}) = \sum_{0 \leq i < j \leq k+1} (-1)^{i+j} \left\{ F(u_i, u_j), \Theta(u_0, u_1, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_{k+1}) \right\}_Y. \]  

(2.79)
We also need the following formulas which can be easily derived from definitions. For any $Q \in \Omega^1(B) \otimes C^\infty(E)$ we have

\[
(\partial^F)^2 Q = \{Q \wedge F\}_\Gamma, \quad (2.80)
\]
\[
\partial^F \{Q \wedge Q\}_\Gamma = -2\{Q \wedge \partial^F Q\}_\Gamma, \quad (2.81)
\]
\[
\{Q \wedge \{Q \wedge Q\}_\Gamma\}_\Gamma = 0. \quad (2.82)
\]

Note that the Lie derivative $L_{\text{hor}}$ preserves the subspace $\text{Casim}(E, \Upsilon) \subset C^\infty(E)$. This means that the restriction of $\partial^F$ to $\Omega^k(B) \otimes \text{Casim}(E, \Upsilon)$ is well defined. We will denote this restriction by

\[
\partial^F_0 : \Omega^k(B) \otimes \text{Casim}(E, \Upsilon) \to \Omega^{k+1}(B) \otimes \text{Casim}(E, \Upsilon). \quad (2.83)
\]

It follows from (2.79) that $\partial^F_0$ is a coboundary operator,

\[
(\partial^F_0)^2 = 0.
\]

Now let us take another coupling Poisson tensor $\check{\Pi} \in \text{Coup}(E, \Upsilon)$ associated with geometric data $(\Upsilon, \check{\Gamma}, \check{F})$. Assume that the pair $(\check{\Pi}, \Pi)$ satisfies the condition: there exists $Q \in \Omega^1(B) \otimes C^\infty(E)$ such that

\[
\text{hor}^{\check{\Gamma}}(u) = \text{hor}^\Gamma(u) + \Upsilon^i dQ(u) \quad (2.84)
\]

for $u \in \mathfrak{X}(B)$. Then, for every $\Theta \in \Omega^k(B) \otimes C^\infty(E)$

\[
\partial^{\check{\Gamma}} \Theta = \partial^F \Theta + \{Q \wedge \Theta\}_\Gamma. \quad (2.85)
\]

This implies

\[
\partial^F_0 = \partial^{\check{\Gamma}}_0. \quad (2.86)
\]

Let us associate to the pair $(\check{\Pi}, \Pi)$ the following 2-form

\[
C = C_{\check{\Pi}} \overset{\text{def}}{=} \check{F} - F + \left(\partial^F Q + \frac{1}{2}\{Q \wedge Q\}_\Gamma\right). \quad (2.87)
\]

Then,

\[
C \in \Omega^2(B) \otimes \text{Casim}(E, \Upsilon). \quad (2.88)
\]

Indeed, by (2.81) we get

\[
\text{Curv}^{\check{\Gamma}} = \text{Curv}^\Gamma - \Upsilon^i d(\partial^F Q + \frac{1}{2}\{Q \wedge Q\}_\Gamma).
\]
Comparing this with
\[ \text{Curv} \tilde{F} = \Upsilon^* \tilde{d} \tilde{F} \quad \text{and} \quad \text{Curv} F = \Upsilon^* dF \]
leads to (2.85).

**Proposition 2.20.** The 2-form \( C \) in (2.87) is a 2-cocycle,
\[ \partial_0^F C = 0. \]  
(2.89)

**Proof.** It follows from (2.25), (2.84) that
\[ \partial^F \tilde{F} = (\partial^F - \partial \tilde{F}) \tilde{F} - \{Q \wedge \tilde{F}\}_\Upsilon. \]
On the other hand, by using (2.82) and (2.88) we derive
\[
\{Q \wedge \tilde{F}\}_\Upsilon = \{Q \wedge F\}_\Upsilon + \{Q \wedge C\}_\Upsilon - \{Q \wedge \partial^F Q\}_\Upsilon - \frac{1}{2} \{Q \wedge \{Q \wedge Q\}\}_\Upsilon \\
= \{Q \wedge F\}_\Upsilon - \{Q \wedge \partial^F Q\}_\Upsilon.
\]
Thus,
\[ \partial^F \tilde{F} = -\{Q \wedge \partial^F Q\}_\Upsilon + \{Q \wedge F\}_\Upsilon. \]
Finally, taking into account (2.80),(2.81), we get
\[
\partial^F C = \partial^F \tilde{F} + (\partial^F)^2 Q + \frac{1}{2} \partial^F \{Q \wedge Q\}_\Upsilon \\
= -\{Q \wedge F\}_\Upsilon + \{Q \wedge \partial^F Q\}_\Upsilon + (\partial^F)^2 Q + \frac{1}{2} \partial^F \{Q \wedge Q\}_\Upsilon \\
= 0.
\]

\[ \square \]

If (2.84) holds, then we shall write \( \tilde{\Pi} \sim_Q \Pi \). It is clear that condition (2.84) defines an equivalence relation for the elements of \( \text{Coup}(E, \Upsilon) \).

**Definition 2.21.** For any \( \tilde{\Pi}, \Pi \in \text{Coup}(E, \Upsilon) \) such that \( \tilde{\Pi} \sim_Q \Pi \), the 2-form \( C = C_{\Pi\Pi} \) in (2.87) will be called the Casimir 2-cocycle of the pair \( (\tilde{\Pi}, \Pi) \).

Recall that \( Q \) is uniquely determined by (2.84) up to the transformation
\[ Q \mapsto Q + K, \]
for arbitrary $K \in \Omega^1(B) \otimes \text{Casim}(E, \Upsilon)$. Then $C$ is transformed by the rule

$$C \mapsto C + \partial^0_0 K,$$

which preserves the $\partial^0_0$-cohomology class $[C] = [C_{\Pi\Pi}]$ of $C = C_{\Pi\Pi}$. Thus, $[C_{\Pi\Pi}]$ is an invariant of the pair $(\Pi, \Pi)$. Here are some properties.

**Proposition 2.22.** For any coupling Poisson tensors $\Pi, \Pi', \Pi''$ from the same equivalence class, the following identities hold:

\begin{align}
[C_{\Pi\Pi}] &= -[C_{\Pi\Pi'}], \quad (2.90) \\
[C_{\Pi\Pi}] + [C_{\Pi\Pi''}] + [C_{\Pi\Pi'\Pi}] &= 0. \quad (2.91)
\end{align}

**Proof.** Below we perform computations modulo the elements in $\Omega^1(B) \otimes \text{Casim}(E, \Upsilon)$. By (2.85), we have

$$C_{\Pi\Pi'} = F - F' - \left( \partial^\Gamma Q - \frac{1}{2} \{Q \wedge Q\}_\Upsilon \right)$$

$$= F - F' - \left( \partial^\Gamma Q + \frac{1}{2} \{Q \wedge Q\}_\Upsilon \right)$$

$$= -C_{\Pi\Pi''}.$$

Moreover, (2.84) implies

$$Q^\Gamma'' = Q^\Gamma'' + Q^\Gamma$$

and then

$$C_{\Pi\Pi''} \overset{\text{def}}{=} F'' - F + (\partial^\Gamma Q^\Gamma'' + \frac{1}{2} \{Q^\Gamma'' \wedge Q^\Gamma''\}_\Upsilon$$

$$= (F'' - F') + (F' - F) + \partial^\Gamma Q^\Gamma'' + \partial^\Gamma Q^\Gamma + \frac{1}{2} \{Q^\Gamma'' \wedge Q^\Gamma''\}_\Upsilon + \frac{1}{2} \{Q^\Gamma \wedge Q^\Gamma\}_\Upsilon + \{Q^\Gamma'' \wedge Q^\Gamma''\}_\Upsilon.$$

Finally, using again (2.85),

$$\partial^\Gamma Q^\Gamma'' = \partial^\Gamma Q^\Gamma'' - \{Q^\Gamma \wedge Q^\Gamma''\}_\Upsilon,$$

we derive (2.91). \qed
Let $\text{Ham}(E, \Upsilon)$ be the Lie algebra of Hamiltonian vector fields of the fiber-tangent Poisson tensor $\Upsilon$. Denote by $\text{Poiss}_V(E, \Upsilon)$ the Lie algebra of all vertical Poisson vector fields of $\Upsilon$. Then $\text{Ham}(E, \Upsilon)$ is an ideal of $\text{Poiss}_V(E, \Upsilon)$. Introduce the first reduced Poisson cohomology space as

$$H^1_V(E, \Upsilon) = \frac{\text{Poiss}_V(E, \Upsilon)}{\text{Ham}(E, \Upsilon)}.$$  

(2.92)

The following statement is evident.

**Proposition 2.23.** Assume that

$$H^1_V(E, \Upsilon) = \{0\}. \quad (2.93)$$

Then for arbitrary $\tilde{\Pi}, \Pi \in \text{Coup}(E, \Upsilon)$ there exists a $Q \in \Omega^1(B) \otimes C^\infty(E)$ such that $\tilde{\Pi} \sim_Q \Pi$, and hence the Casimir cocycle $[C_{\tilde{\Pi} \Pi}]$ is well defined.

Under condition (2.93), one can associated to $\Upsilon$ an intrinsic coboundary operator

$$\partial_0 : \Omega^k(B) \otimes \text{Casim}(E, \Upsilon) \to \Omega^{k+1}(B) \otimes \text{Casim}(E, \Upsilon),$$

(2.94)

given by $\partial_0 = \partial_0^\Gamma$, where $\Gamma$ is a Poisson connection on $(E, \Gamma)$. It is clear that this definition is independent of $\Gamma$. The properties of $\partial_0$ are controlled by the (singular) symplectic foliation of $\Upsilon$.

**Inner Poisson automorphisms.** Denote by $\text{Aut}(E, \Upsilon)$ the group of all fiber preserving Poisson automorphisms of $(E, \Upsilon)$. By $\text{Inn}(E, \Upsilon)$ we denote the subgroup of $\text{Aut}(E, \Upsilon)$ corresponding to inner Poisson automorphisms. Recall that a diffeomorphism $g : E \to E$ is called an inner Poisson automorphism if there exists a smooth family of Hamiltonian functions $h_t : E \to \mathbb{R}$ such that the flow $g_t$ of the time-dependent Hamiltonian vector field $\mathcal{V}^t(h^t)$ joints $g$ with the identity map, $g_0 = \text{id}$ and $g_1 = g$.

By Proposition 2.9, the groups $\text{Aut}(E, \Upsilon)$ and $\text{Inn}(E, \Upsilon)$ act naturally on $\text{Coup}(E, \Upsilon)$.

**Proposition 2.24.** Let $\tilde{\Pi}, \Pi \in \text{Coup}(E, \Upsilon)$. If $\tilde{\Pi} \sim_Q \Pi$, then for every $g \in \text{Inn}(E, \Upsilon)$, the push-forward $g_*\Pi \in \text{Coup}(E, \Upsilon)$ is a coupling Poisson tensor associated with geometric data $(\Upsilon, g_*\Gamma, g_*F)$ and such that

$$\tilde{\Pi} \sim_Q g_*\Pi,$$  

(2.95)

$$[C_{\Pi, \tilde{\Pi}}] = [C_{g_*\Pi, \Pi}]$$  

(2.96)

for a certain $Q^1 \in \Omega^1(B) \otimes C^\infty(E)$.  

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Proof. We use the flow $g_t$ of the time-dependent Hamiltonian vector field $V^s d(h^t)$ to define a family of coupling Poisson tensors $\{\Pi_t\}_{t \in [0,1]}$ by

$$\Pi_t = (g_t)_* \Pi.$$  \hfill (2.97)

By Proposition 2.9, the corresponding geometric data are

$$(\Upsilon, \Gamma^t, F^t) = (\Upsilon, (g_t)_* \Gamma, (g_t)_* F),$$

where $(\Upsilon, \Gamma, F)$ are geometric data of $\Pi$. Then, $\Pi_1 = \tilde{\Pi}$ and $\Pi_0 = \Pi$. Taking into account (2.71) for $\text{hor} \Gamma^t(u)$, we get

$$\text{hor} \Gamma^t(u) - \text{hor} \Gamma(u) = \int_0^t [\text{hor} \Gamma(u), V^s d(h^\tau)] d\tau = V^s d \left( \int_0^t L_{\text{hor} \Gamma(u)} h^\tau d\tau \right).$$

Thus

$$\text{hor} \Gamma^t(u) = \text{hor} \Gamma^t(u) + V^s dQ^t,$$

where

$$Q^t = Q + \int_0^t L_{\text{hor} \Gamma(u)} h^\tau d\tau.$$

In particular, at $t = 1$ this gives (2.95).

Now applying Theorem 2.19 to family (2.97) and $Z_t = V^s d(h^t)$ shows that $\beta^t = 0$ in (2.78) and hence $C_{\Pi_t, \Pi} = 0$. From here and property (2.91) we deduce (2.96).

\begin{corollary}
The orbit of the action of $\text{Inn}(E, \Upsilon)$ through every $\Pi \in \text{Coup}(E, \Upsilon)$ belongs to the equivalence class of $\Pi$. Moreover,

$$C_{g^* \Pi, \Pi} = 0$$

for every $g \in \text{Inn}(E, \Upsilon)$.
\end{corollary}

\section{Poisson equivalence}

Here, using the results of previous sections, we obtain some criteria for the equivalence of Poisson structures in the neighborhood of an entire common symplectic leaf.
Let \( \pi : E \to B \) be a vector bundle over a connected base \( B \) equipped with symplectic structure \( \omega \). We shall identify \( B \) with the image of the zero section.

**Proposition 3.1.** Let \( \Pi \) be a Poisson tensor on \( E \) with property:

\[
(B, \omega) \text{ is a symplectic leaf of } \Pi. \tag{3.1}
\]

Then, there exists a neighborhood \( N \) of \( B \) in \( E \) such that \( \Pi \) is a coupling Poisson tensor on \( N \) associated to geometric data \( (\Pi_V, \Gamma, F) \) satisfying the following conditions:

\[
\begin{align*}
\text{rank} \Pi_V &= 0 \quad \text{on } B, \tag{3.2} \\
\text{hor}^\Gamma(u)|_B &= u, \tag{3.3} \\
\pi^*F|_B &= \omega \tag{3.4}
\end{align*}
\]

for every \( u \in \mathcal{X}(B) \).

**Proof.** We have to show that \( \Pi \) is horizontally nondegenerate in a neighborhood of \( B \). Let \( V \subset TE \) be the vertical subbundle and \( V^0 \subset T^*E \) be the annihilator of \( V \). Consider the plane distribution on \( E \) associated with \( \Pi \):

\[
\mathbb{H}_m = \Pi^*_m(V^0_m), \quad m \in E. \tag{3.5}
\]

For every \( \xi \in B \) we have the decomposition

\[
T_\xi E = T_\xi B \oplus E_\xi,
\]

where \( V_\xi = E_\xi \). Then,

\[
T^*_\xi E = (T_\xi B)^0 \oplus V^0_\xi.
\]

On the other hand, it follows from (3.1) that

\[
\Pi^*_\xi(T^*_\xi E) = T_\xi B,
\]

which says that \( \Pi^*_\xi((T_\xi B)^0) = \{0\} \). Consequently,

\[
\mathbb{H}_\xi = T_\xi B \tag{3.6}
\]

for every \( \xi \in B \). Thus, \( \mathbb{H} \) is a complementary of \( V \) at every point of \( B \). This means that there exists an open neighborhood \( N \) of \( B \) in \( E \) such that

\[
T_N E = \mathbb{H} \oplus V. \tag{3.7}
\]

Properties (3.3) and (3.4) follow from (3.1) and (3.6). Remark that \( \text{rank } \Pi = \text{rank } \Pi_H + \text{rank } \Pi_V \). But \( \text{rank } \Pi = \text{rank } \Pi_H = \text{dim } B \) at \( \xi \in B \). This proves (3.2). \[\square\]
Denote by $\text{Coup}_B(E)$ the set of germs at $B$ of all Poisson structures $\Pi$ on $E$ satisfying condition (3.1). Let $\phi$ be a diffeomorphism between two neighborhoods of $B$ which is identical on $B$, $\phi|_B = \text{id}_B$. Then, for every $\Pi \in \text{Coup}_B(E)$ we have $\phi_*\Pi \in \text{Coup}_B(E)$.

We are interested in the equivalence between the coupling Poisson tensors in $\text{Coup}_B(E)$ in the following class of diffeomorphisms identical on $B$.

Denote by $\text{Diff}^0_B(E)$ the group of germs at $B$ of diffeomorphisms $\phi$ satisfying the condition: in a neighborhood of $B$ in $E$ the exists a time-dependent vector field $Z_t$ ($t \in [0, 1]$) such that

$$Z_t|_B = 0$$

(3.8)

and the flow $\Phi_t$ of $Z_t$ joins $\phi$ and the identity map, $\Phi_1 = \phi$, $\Phi_0 = \text{id}$.

Introduce also the subgroup $\text{End}^0_B(E)$ in $\text{Diff}^0_B(E)$ consisting of germs of diffeomorphisms $g$ satisfying the condition: then there exists a time-dependent vertical vector field $Y_t$ such that

$$Y_t|_B = 0$$

(3.9)

and the flow of $Y_t$ joins $g$ and the identity map. In particular, every $g \in \text{End}^0_B(E)$ is a fiber preserving diffeomorphism identical on $B$, $\pi \circ g = \pi$ and $g|_B = \text{id}_B$.

**Theorem 3.2.** Let $\Pi, \tilde{\Pi} \in \text{Coup}_B(E)$ be two coupling Poisson tensors associated with geometric data $(\Pi_V, \Gamma, F)$ and $(\tilde{\Pi}_V, \tilde{\Gamma}, \tilde{F})$, respectively. If $\Pi$ and $\tilde{\Pi}$ are isomorphic by a diffeomorphism $\phi \in \text{Diff}^0_B(E)$,

$$\phi^*\tilde{\Pi} = \Pi,$$

(3.10)

then the vertical Poisson structures $\Pi_V$ and $\tilde{\Pi}_V$ are isomorphic by a $g \in \text{End}^0_B(E)$,

$$g^*\tilde{\Pi}_V = \Pi_V.$$  

(3.11)

**Proof.** Let $Z_t$ be the time-dependent vector field generating the diffeomorphism $\phi$. By (3.8) there exists a neighborhood $N_{\text{flow}}$ of $B$ in $E$ independent of $t$ such that the flow $\Phi_t$ of $Z_t$ is well defined on $N_{\text{flow}}$, for all $t \in [0, 1]$, $\Phi_t : N_{\text{flow}} \to E$. It is clear that

$$\Phi_t|_B = \text{id}_B,$$

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\( \Phi_1 = \phi \) and \( \Phi_0 = \text{id} \). For every \( t \in [0,1] \), one can define the Poisson tensor
\[
\Pi_t = (\Phi_t)_* \Pi
\]
satisfying (3.1). Then, by Proposition 3.1, \( \Pi_t \) is a coupling Poisson tensor associated with geometric data \( (\Pi_t^V, \Gamma^t, F^t) \) varying smoothly with \( t \). In particular,
\[
(\Pi_0^V, \Gamma^0, F^0) = (\Pi_V, \Gamma, F)
\]
and
\[
(\Pi_1^V, \Gamma^1, F^1) = (\tilde{\Pi}_V, \tilde{\Gamma}, \tilde{F}).
\]
Thus, \( Z_t \) is an infinitesimal generator of the smooth family of coupling Poisson structures \( \{\Pi_t\}_{t \in [0,1]} \), that is, \( Z_t \) satisfies (2.50). For every \( t \), we have the decomposition
\[
Z_t = X_t + Y_t  \tag{3.12}
\]
into the horizontal \( X_t \) and vertical \( Y_t \) parts with respect to the connection \( \Gamma^t \). It follows from (2.50) and Proposition 2.14 that the time-dependent vertical vector field \( Y_t \) must satisfy the equation
\[
L_{Y_t} \Pi_t^V + \frac{\partial \Pi_t^V}{\partial t} = 0.
\]
Furthermore, by (3.8) and (3.12) we have
\[
Y_t|_B = 0 \quad \forall t.
\]
Shrinking the neighborhood \( N_{\text{flow}} \), we can arrange that the flow \( g_t \) of \( Y_t \) is well defined on \( N_{\text{flow}} \). Then, \( g_t \Phi_t = \Phi \) and hence
\[
g^*\tilde{\Phi} = \Phi, \quad g = g_t|_{t=1}.
\]

If (3.10) holds for a certain \( \phi \in \text{Diff}^0_B(E) \), then we say that the coupling Poisson structures \( \Pi \) and \( \tilde{\Pi} \) are equivalent. Moreover, we can introduce the equivalence relation for fiber-tangent Poisson structures on \( E \) saying that two fiber-tangent Poisson structures vanishing on \( B \) are equivalent if they are isomorphic by a diffeomorphism in \( \text{End}^0_B(E) \). Then, Theorem 3.2 can be reformulated in the following manner: the vertical components of equivalent coupling Poisson tensors in \( \text{Coup}_B(E) \) are also equivalent.

Combining the Local Splitting Theorem and Theorem 3.2, we arrive at the fact that the vertical part of every coupling Poisson tensor in \( \text{Coup}_B(E) \) is locally trivial (in the sense of Definition 2.8).
Proposition 3.3. Let $\Pi \in \mathfrak{Coup}_B(E)$ be a coupling Poisson structure, and let $\Pi_V$ be the vertical part. For every $\xi \in B$ there exist a neighborhood $U$ of $\xi$ in $B$ with a trivialization $E_U \cong U \times E_\xi$ such that $\Pi_V|_{E_\xi}$ is isomorphic to the direct product of the zero Poisson structure and $\Pi_V|_{E_\xi}$ by a diffeomorphism $\mathfrak{End}_B^0(E_U)$.

Note that the data $(\Gamma, F)$ are not preserved under the action of $\text{Diff}_B^0(E)$ in general (compare with Proposition 2.9).

Resuming, we conclude that, according to Theorem 3.2, the problem on the semilocal equivalence of Poisson structures in the class $\mathfrak{Coup}_B(E)$ is reduced to the study of coupling Poisson tensors with fixed vertical part.

3.1 Sufficient criteria

Denote by $\mathfrak{Coup}_B(E, \Upsilon)$ the set of germs at $B$ of all coupling Poisson tensors $\Pi$ on $E$ with fixed vertical part, $\Pi_V = \Upsilon$, which satisfy (3.1). Then,

$$\text{rank } \Upsilon = 0 \quad \text{on } B. \quad (3.13)$$

Denote by $\mathfrak{F}_B$ the space of all germs at $B$ of all smooth functions on vanishing at $B$. Introduce also the subspace $\text{Casim}_B(E, \Upsilon)$ of $\mathcal{C}^\infty_B(E)$ consisting of all Casimir functions of $\Upsilon$ vanishing at $B$. Then, we have a natural identification $\text{Casim}_B(E, \Upsilon)/\pi^*\mathcal{C}^\infty(B)$. The corresponding space of germs at $B$ of functions in $\text{Casim}_B(E, \Upsilon)$ will be denoted by $\text{Casim}_B$.

Let $\Pi \in \mathfrak{Coup}_B(E, \Upsilon)$ be a coupling Poisson tensor associated with geometric data $(\Upsilon, \Gamma, F)$. By (3.3) it is clear that the restriction of the covariant exterior derivative $\partial^F$ (2.2) to $\Omega^k(B) \otimes \text{Casim}_B$ is well defined. Thus, we have the coboundary operator

$$\partial^F_0 : \Omega^k(B) \otimes \text{Casim}_B \to \Omega^{k+1}(B) \otimes \text{Casim}_B. \quad (3.14)$$

Let $\bar{\Pi} \in \mathfrak{Coup}_B(E, \bar{\Upsilon})$ be another coupling Poisson tensor associated with geometric data $(\bar{\Upsilon}, \bar{\Gamma}, \bar{F})$. Recall that we write $\bar{\Pi} \sim_Q \Pi$ if there exists $Q \in \Omega^1(B) \otimes \mathfrak{F}_B$ satisfying (2.60). This implies $\partial^F_0 = \partial^\bar{F}_0$. By (3.4) we have

$$\pi^*(\bar{F} - F)|_B = 0. \quad (3.15)$$

If $\bar{\Pi} \sim_Q \Pi$, then formula (2.87) defines the Casimir 2-cocycle associated to the pair $(\bar{\Pi}, \Pi)$:

$$C = C_{\Pi, \bar{\Pi}} \in \Omega^2(B) \otimes \text{Casim}_B, \quad \partial^F_0 C = 0. \quad (3.16)$$
Recall that $Q$ is uniquely determined by (2.84) up to the transformation $Q \mapsto Q + K$, for arbitrary $K \in \Omega^1(B) \otimes \mathfrak{cas}_B$. Then $C$ is transformed by the rule $C \mapsto C + \partial^*_F K$ preserving the $\partial^*_F$-cohomology class $[C]$.

We start with the following result which was stated in [Vo1].

**Theorem 3.4.** Let $\Pi, \tilde{\Pi} \in \mathfrak{Coup}_B(E, \Upsilon)$ be two coupling Poisson tensors such that

\[ \tilde{\Pi} \sim_Q \Pi \quad (3.17) \]

and

\[ [C_{\tilde{\Pi}, \Pi}] = 0. \quad (3.18) \]

Then, $\Pi$ and $\tilde{\Pi}$ are isomorphic by a diffeomorphism $\phi \in \mathfrak{Diff}^0_B(E)$,

\[ \phi^*\tilde{\Pi} = \Pi. \quad (3.19) \]

The proof is divided in few steps. According to the results of Section 2, it suffices to find a Poisson homotopy.

**Step I.** Let $(\Upsilon, \Gamma, F)$ and $(\tilde{\Upsilon}, \tilde{\Gamma}, \tilde{F})$ be the geometric data of $\Pi$ and $\tilde{\Pi}$, respectively. By conditions (3.17), (3.18), one can choose $Q \in \Omega^1(B) \otimes \mathfrak{f}_B$ such that

\[ \text{hor} \tilde{F}(u) = \text{hor} \Gamma(u) + \Upsilon^i dQ(u), \quad (3.20) \]

\[ \tilde{F} = F - (\partial^*_F Q + \frac{t}{2} \{Q \wedge Q\}_\Upsilon). \quad (3.21) \]

Define

\[ \Gamma^t \overset{\text{def}}{=} \Gamma - t \Upsilon^i dQ \quad (3.22) \]

and

\[ F^t \overset{\text{def}}{=} F - (t \partial^*_F Q + \frac{t^2}{2} \{Q \wedge Q\}_\Upsilon). \quad (3.23) \]

**Lemma 3.5.** For any $t \in [0, 1]$, the data $(\Upsilon, \Gamma^t, F^t)$ satisfy relations (2.23)—(2.26).

**Proof.** It follows from (2.63) that

\[ \partial^*_F \Theta = \partial^* \Theta + t \{Q \wedge \Theta\}_\Upsilon. \]
From here and by using (2.80)–(2.82), we get
\[
\partial^{\Gamma_t} F = \partial F_t + t\{Q \wedge F_t\}_\Upsilon
\]
\[
= t[-(\partial^F)^2 Q + \{Q \wedge F\}_\Upsilon] - t^2 \left[\frac{1}{2} \partial^F \{Q \wedge Q\}_\Upsilon + \{Q \wedge \partial^F Q\}_\Upsilon\right]
\]
\[
- \frac{t^3}{2} \{Q \wedge \{Q \wedge Q\}_\Pi\}_\Pi = 0.
\]
Next,
\[
L_{\text{hot}^{\Gamma_t}(u)} \Upsilon = L_{\text{hot}^\Gamma(u)} \Upsilon + tL_{\partial^\Upsilon dQ(u)} \Upsilon = 0.
\]
Finally, using (2.71), it is easy to verify that the curvature 2-form of \(\Gamma_t\) satisfies (2.26) precisely for \(F^t\) in (3.23).

Now, we remark that \(F^t\) satisfies the nondegeneracy condition in a neighborhood of \(B\) in \(E\) for all \(t \in [0,1]\). Thus, for every \(t\), the triple \((\Upsilon, \Gamma^t, F^t)\) defines the coupling Poisson tensor \(\Pi_t \in \text{Coup}_B(E, \Upsilon)\).

**Step II.** The family \(\{\Pi_t\}\) satisfies (2.60) and (2.61) for \(Q^t = tQ\). Moreover, \(\Pi_1 = \tilde{\Pi}\) and \(\Pi_0 = \Pi\). In other words, \(\Pi_t \approx Q^t \Pi_0\) and \([C_{\Pi_t \Pi_0}] = 0\) for every \(t\). Define the time-dependent vector field \(X_t = X^i_t(\xi, x) \text{ hot}^\Gamma\) by
\[
i_{X_t} F^t = Q.
\]
Then,
\[
X_t|_B = 0
\]
and hence for every \(t \in [0,1]\), the flow \(\Phi_t\) of \(X_t\) is well defined in a neighborhood \(N_{\text{flow}}\) of \(B\) in \(E\). According to Theorem 2.18, the time-1 flow \(\phi = \Phi_1\) gives an isomorphism in (3.19). This completes the proof of the theorem. \(\square\)

**Remark 3.6.** It follows from (2.41), that the flow \(\Phi_t\) does not preserve the fiber-tangent Poisson structure \(\Upsilon\), in general.

**Deformations of the coupling form.** Let \(\Pi, \tilde{\Pi}\) be two coupling Poisson structures satisfying (3.17). Then the Casimir 2-cocycle \(C = C_{\Pi, \tilde{\Pi}}\) vanishes on \(B\),
\[
\pi^* C|_B = 0.
\]
Thus, the 2-form \(F + C \in \Omega^2(B) \otimes \mathfrak{g}_B\) satisfies (3.4) and the nondegeneracy condition in a neighborhood of \(B\) in \(E\). Moreover, it is easy to see that the triple \((\Upsilon, \Gamma, F + C)\) satisfies the relations (2.23)–(2.26).
Proposition 3.7. Every pair $\Pi, \tilde{\Pi} \in \text{Coup}_B(E, \Upsilon)$ such that $\tilde{\Pi} \sim_Q \Pi$ induces the coupling Poisson tensor $\Pi_C \in \text{Coup}_B(E, \Upsilon)$ associated with the geometric data $(\Upsilon, \Gamma, F + C)$.

We will call $F + C$ the deformed coupling form and $\Pi_C$ the deformed coupling Poisson tensor. It is clear that the relative Casimir 2-cocycle corresponding to $\tilde{\Pi}$ and $\Pi_C$ is zero. Applying Theorem 3.4 to $\tilde{\Pi}$ and $\Pi_C$, we get the following result.

Theorem 3.8. Let $\Pi, \tilde{\Pi} \in \text{Coup}_B(E, \Upsilon)$ be two coupling Poisson tensors such that $\tilde{\Pi} \sim_Q \Pi$. Let $C = C_{\Pi\Pi}$ be the Casimir 2-cocycle and $\Pi_C$ the deformed coupling Poisson tensor associated with geometric data $(\Upsilon, \Gamma, F + C)$. Then, $\Pi$ and $\Pi_C$ are isomorphic under the time-1 flow of the time-dependent vector field $X_t$ defined by

$$i_{X_t}(F^t + C) = Q. \quad (3.26)$$

Equivalent vertical parts. Now suppose we are given a second fiber-tangent Poisson structure $\tilde{\Upsilon}$ vanishing at $B$. Consider the corresponding class of coupling Poisson tensors $\text{Coup}_B(E, \tilde{\Upsilon})$. Let $\Pi \in \text{Coup}_B(E, \Upsilon)$ and $\tilde{\Pi} \in \text{Coup}_B(E, \tilde{\Upsilon})$ be two coupling Poisson tensors. Then, $(B, \omega)$ is a common symplectic leaf of $\Pi$ and $\tilde{\Pi}$. Assume that $\Upsilon$ and $\tilde{\Upsilon}$ are equivalent,

$$g^*\tilde{\Upsilon} = \Upsilon \quad (3.27)$$

for some $g \in \text{End}_B^0(E)$. Then, by Proposition 2.9, $g^*\tilde{\Pi}$ is a coupling Poisson tensor associated with geometric data $(\Upsilon, g^*\tilde{\Gamma}, g^*\tilde{F})$. It is clear that $g^*\tilde{\Pi} \in \text{Coup}_B(E, \Upsilon)$. Assume that

$$g^*\tilde{\Pi} \sim_Q \Pi \quad (3.28)$$

for a certain $Q \in \Omega^1(B) \otimes \mathfrak{g}_B$. Then, one can define the Casimir 2-cocycle $C_g = C_{g^*\tilde{\Pi}, \Pi}$ and the deformed coupling tensor $\Pi_{C_g}$ associated to $g^*\tilde{\Pi}$ and $\Pi$.

As a consequence of Theorem 3.8 we get the following result.

Proposition 3.9. Under assumptions (3.27), (3.28), the coupling Poisson tensors $\tilde{\Pi}$ and $\Pi_{C_g}$ are equivalent.

Let us look at the dependence of the Casimir 2-cocycle $C_g$ on the choice of $g$ in (3.27).

Let $\text{Aut}_B^0(E, \Upsilon) \subset \text{Diff}_B^0(E)$ be the connected component of the group of germs at $B$ of Poisson automorphisms of $\Upsilon$. Denote by $\text{Inn}_B(E, \Upsilon)$ the group of germs at $B$ of inner automorphisms of $\Upsilon$ which are identical on $B$. 34
The diffeomorphism $g$ is uniquely determined by (3.27) up to the multiplication by an element $\varphi \in Aut_B^0(E, \Upsilon)$. Assume that the fiber-tangent Poisson structure $\Upsilon$ possesses the property

$$Aut^0_B(E, \Upsilon) = Inn_B((E, \Upsilon)).$$

(3.29)

**Proposition 3.10.** The $\partial^\Gamma_0$-cohomology class of the Casimir 2-cocycle $C_g = C_{g^*\tilde{\Pi}, \tilde{\Pi}}$ is independent of the choice of $g$ in (3.27).

This assertion follows from Proposition 2.24.

Thus, under assumption (3.29) and the equivalence of $\Upsilon$ and $\tilde{\Upsilon}$, the cohomology class $[C_g^*\tilde{\Pi}, \Pi]$ is an invariant of two coupling Poisson tensors $\Pi \in Coup_B(E, \Upsilon)$ and $\tilde{\Pi} \in Coup_B(E, \tilde{\Upsilon})$.

### 3.2 Cohomological obstructions

Here we show that the cohomology class of the relative Casimir 2-cocycle can be viewed as an obstruction to the Poisson equivalence in the following special case.

Consider again the set $Coup_B(E, \Upsilon)$ of germs at $B$ of all coupling Poisson tensors $\Pi$ on $E$ with fixed vertical part $\Upsilon$ and satisfying (3.1).

For every $\xi \in B$, the fiber $E_\xi$ carries the Poisson structure $\Upsilon_\xi = \Upsilon|_\xi$ vanishing at the origin $0$, $\Upsilon_\xi = 0$ at 0. Since $\Upsilon$ is locally trivial, the germ of $\Upsilon_\xi$ at 0 is independent of $\xi$ up to an isomorphism. Fix $\xi^0 \in B$ and make the following assumption on $(E_{\xi^0}, \Upsilon_{\xi^0})$. Consider the Lichnerowicz complex $D_k = [\Upsilon_{\xi^0}, \cdot] : \chi^k(E_{\xi^0}) \to \chi^{k+1}(E_{\xi^0})$ induced by the Schouten bracket on $E_{\xi^0}$ [KM, Va]. In particular,

$$D_0 f = [\Upsilon_{\xi^0}, f] \equiv \Upsilon_{\xi^0}^* df,$$

$$D_1 w = [\Upsilon_{\xi^0}, w] = L_w \Upsilon_{\xi^0}$$

for $f \in \chi^0(E_{\xi^0}) = C^\infty(E_{\xi^0})$ and for $w \in \chi^1(E_{\xi^0}) = \chi^1(E_{\xi^0})$. We assume that there exists a neighborhood $O$ of 0 in $E_{\xi^0}$ and linear operators

$$H_0 : \chi^1(\bar{O}) \to \chi^0(\bar{O})$$

and

$$H_1 : \chi^2(\bar{O}) \to \chi^1(\bar{O})$$

satisfying the homotopy condition

$$D_0 \circ H_0 + H_1 \circ D_1 = id.$$  

(3.30)
This assumption guarantees that every smooth family of Poisson vector fields on $\tilde{O}$ is a family of Hamiltonian vector fields, where the parameter-dependent Hamiltonian smoothly varies with a parameter. Then, by the standard partition of unity argument and the local triviality of $\gamma$ we deduce that property (2.92) holds in a neighborhood of $B$ in $E$. In particular, for any $\Pi, \tilde{\Pi} \in \mathfrak{Coup}_B(E, \gamma)$ there exists $Q \in \Omega^1(B) \otimes \mathfrak{g}_B$ such that $\tilde{\Pi} \sim_Q \Pi$ and the Casimir 2-cocycle $C_{\tilde{\Pi}, \Pi}$ is well defined with respect to the intrinsic coboundary operator $\partial_0$ be associated with $\gamma$. Remark also that the above assumption implies (3.29).

Now we assume that we have the set $\mathfrak{Coup}_B(E, \tilde{\gamma})$ corresponding to another fiber-tangent Poisson structure $\tilde{\gamma}$. If $\gamma$ and $\tilde{\gamma}$ are equivalent, then by Proposition 2.24 for any $\Pi \in \mathfrak{Coup}_B(E, \gamma)$ and $\tilde{\Pi} \in \mathfrak{Coup}_B(E, \tilde{\gamma})$ the $\partial_0$-cohomology class $[C_{g^*, \Pi}, \Pi]$ is independent of the choice of $g$ in (3.27). Then we can put

$$[C_{\tilde{\Pi}, \Pi}] \overset{\text{def}}{=} [C_{g^*, \Pi}, \Pi].$$

We arrive at the main observation.

**Theorem 3.11.** The coupling Poisson tensors

$$\Pi \in \mathfrak{Coup}_B(E, \gamma) \quad \text{and} \quad \tilde{\Pi} \in \mathfrak{Coup}_B(E, \tilde{\gamma})$$

are isomorphic by a $\phi \in \mathfrak{Diff}_0^0(E)$ if and only if

(a) the germs at $B$ of $\gamma$ and $\tilde{\gamma}$ are equivalent, and

(b) the $\partial_0$-cohomology class of $(\tilde{\Pi}, \Pi)$ is trivial,

$$[C_{\tilde{\Pi}, \Pi}] = 0.$$  

The sufficiency follows from Proposition 3.9. The necessity part is a consequence of Theorem 2.19.

**Example 3.12.** If $E_{\mathfrak{g}^*} = \mathfrak{g}^*$ is the dual of a semisimple Lie algebra $\mathfrak{g}$ of compact type and $\tilde{\gamma}^{\mathfrak{g}^*}$ is the Lie–Poisson bracket on $\mathfrak{g}^*$, then, as shown in [Co2], for a closed ball centered at 0 there exist homotopy operators in (3.30).

### 4 Linearizability and normal forms

Here we give some results on the linearization of a Poisson structure at a given (singular) symplectic leaf. First, we recall a coordinate derivation of
the linearized Poisson structure of the leaf $\mathcal{V}_2$. An invariant definition related to the notion of the Lie algebroid of a symplectic leaf $\mathcal{W}_2$ can be found in $\mathcal{V}_1 \mathcal{V}_2$.

4.1 Linearized Poisson structures

Let $(M, \Psi)$ be a Poisson manifold and $(B, \omega)$ be a symplectic leaf. We will assume that $B$ is an embedded submanifold of $M$. Consider the normal bundle $\pi : E \to B$, $E = TB/M/\mathcal{T}B$.

As usual, to study Poisson geometry around the leaf $B$, we can move from $M$ to the fibered space (the total space $E$ of $\pi$) by means of an exponential map. Consider the decomposition $T_B E = TB \oplus E$ (4.1)

and denote by $\tau_b : T_b E \to E$ the projection along $TB$. We say that a diffeomorphism $f : E \to M$ from the total space onto a neighborhood of the leaf $B$ in $M$ is an exponential map if $f|_B = id_B$ and $\nu_b \circ df = \tau_b$ for every $b \in B$. Here $\nu_b : T_b M \to E_b$ is the natural projection. An exponential map exists always because of the tubular neighborhood theorem [LMr].

Pick an exponential map $f$ and consider the pull-back $\Pi = f^* \Psi$ via $f$. It is clear that $(B, \omega)$ (as the zero section) is a symplectic leaf of the Poisson tensor $\Pi$. Then by Proposition 3.1, $\Pi$ is coupling Poisson tensor on $E$ associated with geometric data $(\Pi_V, \Gamma, F)$ satisfying conditions (3.2)–(3.4).

Let $(\xi, x) = (\xi^i, x^\sigma)$ be a (local) coordinate system on $E$, where $(\xi^i)$ are coordinates on $B$ and $(x^\sigma)$ are affine coordinates along the fibers, so that locally, $B = \{x = 0\}$. Then,

$$\Pi = \Pi_H + \Pi_V = -\frac{1}{2} F^{ij}(\xi, x) \text{hor}_i \wedge \text{hor}_j + \frac{1}{2} \Pi_V^{\alpha\beta}(\xi, x) \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta},$$

where $\text{hor}_i = \frac{\partial}{\partial \xi^i} - \Gamma^i_j(\xi, x) \frac{\partial}{\partial x^j}$. In coordinates, conditions (3.2)–(3.4) read

$$\Pi_V^{\alpha\beta}(\xi, 0) = 0,$$

$$\Gamma^i_j(\xi, 0) = 0$$

$$F^{ij}(\xi, 0) = \omega^{ij}(\xi),$$
Here
\[ \omega = \frac{1}{2} \omega_{ij}(\xi) d\xi^i \wedge d\xi^j. \]

**Linearization of** \( \Pi_V \). The Taylor expansion at \( x = 0 \) for \( \Pi_V \) gives
\[ \Pi_V = \Lambda + O_2, \]
where
\[ \Lambda = \frac{1}{2} \lambda_{\alpha\beta}^{\gamma} x^\gamma \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta} \quad (4.6) \]
and
\[ \lambda_{\alpha\beta}^{\gamma}(\xi) \overset{\text{def}}{=} \frac{\partial}{\partial x^\sigma} \Pi_{\alpha\beta}^{\gamma}(\xi, 0). \]

It is clear that \( \Lambda \) is a global bivector field on \( E \). Linearizing the Jacobi identity for \( \Pi_V \) leads to \([\Lambda, \Lambda] = 0\). Moreover, \( \Lambda \) is independent of the choice of an exponential map. Thus, \( \Lambda \) is an intrinsic fiber-tangent Poisson structure on \( E \) of the leaf \( B \). The restriction of \( \Lambda \) to each fiber \( E_\xi \) gives the Lie–Poisson structure called the **linearized transverse Poisson structure** of \( B \) \([\text{We}1]\). This bundle of Lie–Poisson manifolds is locally trivial with typical fiber \( g^* \), where \( g \) is the **transverse Lie algebra** of the leaf \( B \).

**Linearization of** \( \Gamma \). Consider the connection \( \Gamma \),
\[ \Gamma = (dx^\sigma + \Gamma_\sigma^\nu(\xi, x) d\xi^\nu) \otimes \frac{\partial}{\partial x^\sigma}. \]
By (4.4) we have
\[ \Gamma_\sigma^\nu(\xi, x) = \theta_\nu^\sigma(\xi) x^\nu + O(x^2). \]
Here the coefficients \( \theta_\nu^\sigma(\xi) \) are defined in terms of the Poisson bracket of \( \Pi \) as follows
\[ \theta_\nu^\sigma(\xi) = \frac{\partial \Gamma_\sigma^\nu(\xi, 0)}{\partial x^\nu} = \omega_{ij}(\xi) \left[ \frac{\partial}{\partial x^\nu} \{\xi^j, x^\sigma\}_\Pi \right]_{x=0}. \]
Then,
\[ \Gamma^{(1)} = (dx^\sigma + \theta_\nu^\sigma(\xi) x^\nu d\xi^\nu) \otimes \frac{\partial}{\partial x^\sigma} \quad (4.7) \]
is an **homogeneous** Ehresmann connection on \( E \), in the sense that the Lie derivative along the horizontal lift \( \text{hor}^{(1)}(u) \) preserves the space of fiberwise linear functions on \( E \). The linearization of (2.24) leads to
\[ L_{\text{hor}^{(1)}(u)} \Lambda = 0. \quad (4.8) \]
One can associate to \( \Gamma^{(1)} \) the linear connection \( \nabla \) on \( E \) determined by the matrix-valued connection form \( \theta = (\theta^\sigma_{\nu}(\xi)d\xi^\nu) \). In terms of \( \nabla \), condition (4.9) means that the bivector field \( \Lambda \) is \( \nabla \)-covariantly constant, \( \nabla \Lambda = 0 \). A linear connection satisfying this property is called a Lie–Poisson connection on \( (E, \Lambda) \).

**Linearization of \( F \).** In coordinates

\[
\pi^* F = \frac{1}{2} F_{ij}(\xi, x)d\xi^i \wedge d\xi^j.
\]

Taking into account (4.5), we get

\[
F_{ij}(\xi, x) = \omega_{ij}(\xi) - \mathcal{R}_{ij\sigma}(\xi)x^\sigma + O(x^2),
\]

where

\[
\mathcal{R}_{ij\sigma}(\xi) = -\omega_{ii}(\xi) \left[ \frac{\partial}{\partial x^\sigma} \{\xi^i, \xi^j\}_\Pi \right]_{x=0} \omega_{j'}(\xi).
\]

We define \( F^{(1)} \in \Omega^1(B) \otimes C^\infty(E) \) by

\[
\pi^* F^{(1)} = \frac{1}{2} F^{(1)}_{ij}(\xi, x)d\xi^i \wedge d\xi^j \tag{4.9}
\]

with

\[
F^{(1)}_{ij}(\xi, x) \overset{\text{def}}{=} \omega_{ij}(\xi) - \mathcal{R}_{ij\sigma}(\xi)x^\sigma. \tag{4.10}
\]

It follows from (2.25) and (2.26) that

\[
\partial F^{(1)} F^{(1)} = 0 \tag{4.11}
\]

and

\[
\text{Curv}^{\Gamma^{(1)}}(u_1, u_2) = \Lambda^d dF^{(1)}(u_1, u_2). \tag{4.12}
\]

Now we can define the following bivector field

\[
\Pi^{(1)} \overset{\text{def}}{=} \Pi^{(1)}_H + \Lambda \tag{4.13}
\]

\[
= -\frac{1}{2} (F^{(1)})_{ij}(\xi, x) \text{hor}_i \wedge \text{hor}_j + \frac{1}{2} \lambda^\alpha_{\gamma\beta}(\xi)x^\gamma \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}
\]

where \( (F^{(1)})_{js} F^{(1)}_{st} = \delta_s^t \) and

\[
\text{hor}_i = \text{hor}^{\Gamma^{(1)}}_i = \frac{\partial}{\partial \xi^i} - \theta^\sigma_{\nu}(\xi)x^\nu \frac{\partial}{\partial x^\sigma}.
\]
The bivector $\Pi^{(1)}$ is well defined on the following neighborhood of $B$:

$$N^{(1)} = \{ (\xi, x) \in E \mid \det[F^{(1)}_{ij}(\xi, x)] \neq 0 \}.$$  

Taking into account (4.8), (4.11), and (4.12), by Proposition 2.4 we get the following fact.

**Proposition 4.1.** Bivector field $\Pi^{(1)}$ in (4.13) is the coupling Poisson tensor associated to the geometric data $(\Lambda, \Gamma^{(1)}, F^{(1)})$,

$$\Pi^{(1)} \in \mathfrak{coup}_B(E, \Lambda).$$

In particular, $(B, \omega)$ is a symplectic leaf of $\Pi^{(1)}$ and

$$\Pi = \Pi^{(1)} + O_2,$$

that is, $\Pi^{(1)}$ is a *first approximation* to $\Pi$ at $B$.

Let $E^*$ be the dual bundle of $E$ called the co-normal bundle of the leaf $B$. Then, $E^*$ is a bundle of Lie algebras with typical fiber $\mathfrak{g}$. Using (4.10), one can introduce the vector-valued 2-form $\mathcal{R} \in \Omega^2(B, \text{Sect}(E^*))$, locally given by

$$\mathcal{R} = \frac{1}{2} \mathcal{R}_{ij\sigma}(\xi) d\xi^i \wedge d\xi^j \otimes e^\sigma(\xi).$$

(4.14)

Here $\{e^\sigma\}$ is the basis of local sections of $E^*$ corresponding to the coordinates $x^\sigma$ in (4.10). Let $\text{Curv}^\nabla \in \Omega^2(B, \text{End}(E^*))$ be the curvature form of the linear connection $\nabla$, $\text{Curv}^\nabla(u_1, u_2) = [\nabla u_1, \nabla u_2] - \nabla_{[u_1, u_2]}$. Then, we have

$$\text{Curv}^{\Pi^{(1)}}(u_1, u_2) = \langle \text{Curv}^\nabla(u_1, u_2)x, \frac{\partial}{\partial x} \rangle.$$

It follows from here that equality (4.12) can be rewritten in the form

$$\text{Curv}^\nabla = - \text{ad}^* \circ \mathcal{R},$$

(4.15)

where $\text{ad}^*$ is the coadjoint operator on the fibers of $E$. Thus, the coupling Poisson tensor $\Pi^{(1)}$ is determined by the data $(\Lambda, \nabla, \mathcal{R})$.

**Remark 4.2.** One can define the Poisson tensor $\Pi^{(1)}$ starting with the transitive Lie algebroid $T^*_B E$ of the symplectic leaf $\text{We}_2$. Then, $\Pi^{(1)}$ is completely determined by the choice of a connection of the Lie algebroid $\text{Ku}_1 \text{Mz}_2$. The 2-form $\mathcal{R}$ is just the curvature of this connection.

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Varying the exponential map. Suppose we are given two exponential maps $f : E \to M$ and $\tilde{f} : E \to M$. Consider the corresponding coupling Poisson tensors $\Pi = f^*\Psi$ and $\tilde{\Pi} = \tilde{f}^*\Psi$. Then $\Pi = \phi^*\tilde{\Pi}$, where $\phi = \tilde{f}^{-1} \circ f \in \mathcal{D}iff^0_B(E)$. Indeed, by the definition of the exponential map, we have $\phi|_B = id_B$ and

$$d_e\phi|_E = id_E$$

for every $\xi \in B$. Recall that we have the decomposition (4.1). Using the dilation $m_t : E_\xi \to E_\xi$ by the factor $t \in [0,1]$, $m_t(x) = t \cdot x$, one can define $\Phi_t = m_t^{-1} \circ \phi \circ m_t$. It follows that $\{\Phi_t\}_{t \in [0,1]}$ is a smooth family of diffeomorphisms with properties: $\Phi_0 = id$, $\Phi_1 = \phi$ and $\Phi_t|_B = id_B$. Then, $\Phi_t$ is the flow of the time-dependent vector field $Z_t = \frac{d\Phi_t}{dt} \circ \Phi_t^{-1}$ vanishing at $B$.

From here and Theorem 3.2 we get the following fact.

Proposition 4.3. The vertical part $\Pi_V$ of the coupling Poisson tensor $\Pi = f^*\Psi$ is independent of the choice of an exponential map $f$ up to an isomorphism in $\mathfrak{En}^0_B(E)$.

Now consider the linearized Poisson structures $\Pi^{(1)}$ and $\tilde{\Pi}^{(1)}$ associated to the geometric data $(\Lambda, \Gamma^{(1)}, F^{(1)})$ and $(\Lambda, \tilde{\Gamma}^{(1)}, \tilde{F}^{(1)})$, respectively. Consider the decompositions

$$TB = TB \oplus L = TB \oplus \tilde{L},$$

where

$$L = d_Bf(E) \quad \text{and} \quad \tilde{L} = d_B\tilde{f}(E).$$

Let $l : TB \to \tilde{L}$ be the projection along $TB$. Pick a basis of (local) sections $\{n_\sigma(\xi)\}$ of $L$. Then, we have

$$l_\xi(n_\sigma(\xi)) = n_\sigma(\xi) + u_\sigma(\xi),$$

where $u_\sigma(\xi) \in T_\xi B$. Define $Q^{(1)} \in \Omega^1(B) \otimes C^\infty(E)$ by

$$Q^{(1)} \equiv -(i_{\nu,\omega})x^\nu. \quad (4.16)$$

Let $\partial^{(1)}$ be the exterior covariant derivative associated to $\Gamma^{(1)}$.

Proposition 4.4. The linear connections $\Gamma^{(1)}$ and $\tilde{\Gamma}^{(1)}$ are related by

$$\text{hor}^{(1)}(u) = \text{hor}^{(1)}(u) + \Lambda^\sharp dQ^{(1)}(u), \quad (4.17)$$

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where $Q^{(1)}$ is given by (4.16). The corresponding coupling 2-forms $F^{(1)}$ and $	ilde{F}^{(1)}$ satisfy

$$
\tilde{F}^{(1)} = F^{(1)} - \left( \partial^{\Gamma^{(1)}} Q^{(1)} + \frac{1}{2} (Q^{(1)} \wedge Q^{(1)}) \Lambda \right).
$$

(4.18)

This implies that for $\Pi^{(1)}$, $\bar{\Pi}^{(1)} \in \mathfrak{Coup}_B(E, \Lambda)$ we have that $\bar{\Pi}^{(1)} \sim_{Q^{(1)}} \Pi^{(1)}$ and the relative Casimir cohomology class is trivial,

$$
[C_{\bar{\Pi}^{(1)}\Pi^{(1)}}] = 0.
$$

Applying Theorem 3.4 leads to the following consequence.

**Corollary 4.5.** The germs at $B$ of the coupling Poisson structures $\Pi^{(1)}$ and $\bar{\Pi}^{(1)}$ are isomorphic by a diffeomorphism in $\text{Diff}_B^0(E)$.

Thus, $\Pi^{(1)}$ is independent of the choice of an exponential map up to isomorphism. We call $\Pi^{(1)}$ the *linearized Poisson structure* of the original Poisson structure $\Psi$ at a given symplectic leaf $(B, \omega)$.

Another important consequence of Proposition 4.4 is that the Poisson structure $\Psi$ with a given symplectic leaf $B$ inherits an *intrinsinc coboundary operator*,

$$
\partial_0 : \Omega^k(B) \otimes \mathfrak{Casim}_B(E) \rightarrow \Omega^{k+1}(B) \otimes \mathfrak{Casim}_B(E),
$$

(4.19)

where $\mathfrak{Casim}_B(E)$ is the space of germs at $B$ of Casimir functions of $\Lambda$ vanishing on $B$. We can put $\partial_0 = \partial_0^{(1)}$, but this definition is independent of the connection $\Gamma^{(1)}$ because of (4.17).

### 4.2 Semilocal linearization problem

The linearized Poisson structure of the symplectic leaf is a natural candidate for setting of the semilocal linearization problem.

**Definition 4.6.** The Poisson structure $\Psi$ is said to be *linearizable* at the symplectic leaf $(B, \omega)$ if there exists an exponential map $f : E \rightarrow M$ such that the germs at $B$ of the pull-back Poisson structure $\Pi = f^*\Psi$ and the linearized Poisson structure $\Pi^{(1)}$ are isomorphic by a diffeomorphism in $\text{Diff}_B^0(E)$. Respectively, $\Psi$ is *transversally linearizable* at $B$ if the germs of $\Pi_V$ and $\Lambda$ are isomorphic by a diffeomorphism in $\mathfrak{End}^0_B(E)$. 

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It follows from Proposition 4.3 and Corollary 4.5 that this definition is independent of the choice of the exponential map. Remark also that this definition agrees with the zero-dimensional case when \( \dim B = 0 \).

As a consequence of Theorem 3.2 we get the following result.

**Proposition 4.7.** The linearizability implies the transversal linearizability.

In fact, the linearizability of \( \Pi_V \) at \( B \) is equivalent to the linearizability of the transverse Poisson structure of a point in \( B \). This statement can be established in various ways (see, for example, [Br]). Our arguments are based on the local splitting theorem. Let \( g \) be the transverse Lie algebra of the symplectic leaf \( B \). Recall that the bundle \( E \) of Lie–Poisson manifolds associated to the linearized transverse Poisson structure \( \Lambda \) is locally trivial with typical fiber \( g^* \). In particular, the restriction \( \Lambda_\xi = \Lambda|_{E_\xi} \) is isomorphic to the Lie–Poisson structure of \( g^* \). Consider also the restriction \( (\Pi_V)_\xi \) of \( \Pi_V \) to the fiber \( E_\xi \).

**Proposition 4.8.** Fix a point \( \xi^0 \in B \). Then the following assertions are equivalent:

(a) \( (\Pi_V)_{\xi^0} \) and \( \Lambda_{\xi^0} \) are isomorphic by a (local) diffeomorphism \( \psi \) such that

\[
\psi(0) = 0 \quad \text{and} \quad d_0\psi = \text{id};
\]

(b) the germs at \( B \) of \( \Pi_V \) and \( \Lambda \) are isomorphic by a diffeomorphism in \( \mathfrak{End}_B^0(E) \).

**Proof.** The first part \((b) \implies (a)\) is evident. To prove the implication \((a) \implies (b)\) we consider the following family of fiber-tangent Poisson structures

\[
\Upsilon_t = t\xi^0;\Pi_V
\]

with \( \Upsilon_0 = \Lambda \) and \( \Upsilon_1 = \Pi_V \). It suffices to show that there exists a time-dependent vector field \( Y_t \) vanishing at \( B \) and satisfying the equation

\[
L_{Y_t}\Upsilon_t + \frac{\partial\Upsilon_t}{\partial t} = 0. \tag{4.20}
\]

First, let us solve Eq. (4.20) locally. By Proposition 3.3, in a neighborhood of each point \( \xi^0 \in B \) in \( E \) there exists a coordinate system \((\xi^\alpha, x^\beta)\) such that

\[
\Upsilon_t = \frac{1}{2t}\Pi_V^{\alpha\beta}(tx) \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}.
\]
Then, by condition (a) one can choose a local solution of (4.20) in the form \( Y_t = Y_t(x) \frac{\partial}{\partial x} \). Finally, using a partition of unity on \( B \), we can splice together local solutions to a global one.

**Flat linearized transverse Poisson structures.** We say that \( \Lambda \) is flat if there exists a flat Lie–Poisson connection \( \nabla^{\text{flat}} \) on \((E, \Lambda)\), \( \text{Curv} \nabla^{\text{flat}} = 0 \). Locally, \( \Lambda \) is flat because of the local triviality property. Given a flat connection \( \nabla^{\text{flat}} \) one can define the following coupling Poisson tensor

\[
\Pi^{\text{flat}} = \Pi^{\text{flat}}_H + \Lambda = -\frac{1}{2} \omega^{ij}(\xi) \frac{\partial}{\partial \xi^i} \wedge \frac{\partial}{\partial \xi^j} + \frac{1}{2} \lambda_{\gamma}^{\alpha\beta} x^\gamma \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta}.
\]

(4.21)

Here \((\xi^i, x^\sigma)\) are a coordinate system on \( E \) such that the coordinates \( x^\sigma \) along the fibers correspond to a \( \nabla^{\text{flat}} \)-parallel basis \( \{e_\sigma\} \) of sections of \( E \), \( \nabla^{\text{flat}} e_\sigma = 0 \). Moreover, \( \lambda^{\alpha\beta}_\gamma = \text{const} \) are the structure constants relative to the dual basis \( \{e^\sigma\} \). The coupling form \( F^{\text{flat}} \) of \( \Pi^{\text{flat}} \) is given by \( F^{\text{flat}} = \omega \otimes 1 \).

The horizontal part \( \Pi^{\text{flat}}_H \) is the lifting of the nondegenerate Poisson structure on the leaf \( B \) via the connection \( \nabla^{\text{flat}} \).

**Proposition 4.9.** Let \( \Pi^{(1)} \) be a linearized Poisson tensor associated with data \((\Lambda, \nabla, \mathcal{R})\). Assume that the following conditions hold:

(a) \( \Lambda \) is flat;

(b) the center and the first cohomology group of \( \mathfrak{g} \) are trivial,

\[
Z(\mathfrak{g}) = \{0\},
\]

(4.22)

\[
H^1(\mathfrak{g}, \mathfrak{g}) = \{0\}.
\]

(4.23)

Then, \( \Pi^{(1)} \) and \( \Pi^{\text{flat}} \) are isomorphic by a diffeomorphism in \( \text{Diff}_B^0(E) \). If, in addition to the conditions (a), and (b), the holonomy group of the connection \( \nabla^{\text{flat}} \) is trivial, then the normal bundle \( E \) is trivial and \( \Pi^{(1)} \) is equivalent to the direct product Poisson structure on \( E = B \times \mathfrak{g}^* \).

The proof of this statement is based on Theorem 3.4 and the observation that conditions (4.23), (4.23) imply properties (4.17), (4.18) for the connections \( \nabla \) and \( \nabla^{\text{flat}} \) (for more details, see [Vo4]).

Remark that (4.22), (4.23) hold automatically in the case when \( \mathfrak{g} \) is semisimple. Some examples, where \( \Lambda \) is not flat, can be found in [IKV].
4.3 Main results

Suppose we start with a Poisson manifold \((M, \Psi)\) and an embedded symplectic leaf \((B, \omega)\). Let \((\mathfrak{g}, \Lambda, \partial_0)\) be the intrinsic data of the leaf consisting of the transverse Lie algebra \(\mathfrak{g}\), linearized transverse Poisson structure \(\Lambda\), and the coboundary operator \(\partial_0\) in (4.19).

Assume that \(\mathfrak{g}\) is semisimple of compact type. Fix an exponential map \(f\).

Consider the pull-back Poisson structure \(\Pi = f^*\Psi \in \mathfrak{Coup}_B(E, \Pi_V)\) and the corresponding linearized Poisson structure \(\Pi^{(1)} \in \mathfrak{Coup}_B(E, \Lambda)\). Let \((\Pi_V, \Gamma, F)\) and \((\Lambda, \Gamma^{(1)}, F^{(1)})\) be the corresponding geometric data.

By the local linearization theorem due to [Co2] and Proposition 4.8 we deduce the following fact.

**Proposition 4.10.** \(\Psi\) is transversally linearizable at \(B\), that is, there exists a diffeomorphism such that \(g \in \mathfrak{End}^0_B(E)\)

\[ g^*\Pi_V = \Lambda, \quad (4.24) \]

Pick \(g\) in (4.24) and consider \(g^*\Pi \in \mathfrak{Coup}_B(E, \Lambda)\) associated with \((\Lambda, g^*\Gamma, g^*F)\). By assumption, the typical fiber of \((E, \Lambda)\) is the Lie–Poisson structure of the dual \(\mathfrak{g}^*\) of the semisimple Lie algebra of compact type. Recall that, according to [Co2], in each closed ball of 0 in \(\mathfrak{g}^*\) there exist a homotopy operators in (3.30) and hence conditions (2.92), (3.29) hold. Thus, there exists \(Q \in \Omega^1(B) \otimes \mathfrak{F}_B\) such that

\[ \text{hor}^g \Gamma^1(u) = \text{hor}^\Gamma(u) + \Lambda^\sharp dQ(u). \quad (4.25) \]

Now, one can define the Casimir 2-cocycle \(C \in \Omega^2(B) \otimes \mathfrak{Casim}_B(E)\) by

\[ C \equiv g^*F - F^{(1)} + \left( \partial_0 Q + \frac{1}{2} \{Q \wedge Q\}_\Lambda \right). \quad (4.26) \]

**Proposition 4.11.** The \(\partial_0\)-cohomology class \([C]\) of the Casimir 2-cocycle \(C\) (4.26) is independent of the choice of \(g\) in (4.24), \(Q\) in (4.25) and an exponential map \(f\). Moreover,

\[ \pi^*C = C_{ij}(\xi, x)d\xi^i \wedge d\xi^j, \]

where the coefficients \(C_{ij}\) have zero of the second order at \(x = 0\),

\[ C_{ij} = O(x^2). \]
Proof. The independence of $[C]$ of the choice of $g$ follows straightforwardly from Proposition 2.24. Now, let $\Pi = f^*\Psi$ and $\tilde{\Pi} = \tilde{f}^*\Psi$ be coupling Poisson tensors corresponding to exponential maps $f$ and $\tilde{f}$, respectively. Without of loss of generality, by Propositions 4.3 and 4.10, one can assume that $\Pi_V = \tilde{\Pi}_V = \Lambda$. Since $\phi^*\Pi = \Pi$, for $\phi = \tilde{f}^{-1} \circ f \in \text{Diff}^0_B(E)$, Theorem 3.11 implies that $[C_{\Pi\Pi}] = 0$. Moreover, $[C_{\Pi(0)\Pi(0)}] = 0$. Using property (2.91), we get $[C_{\Pi\tilde{\Pi}}] = [C_{\Pi(1)\Pi(1)}] = [C_{\Pi\Pi}]$.

We shall call $[C]$ the $\partial_0$-cohomology class of the leaf $B$.

Using the 2-cocycle $C$, we define the deformed coupling form $F^{(1)} + C \in \Omega^2(B) \otimes \mathfrak{g}_B$. Consider the coupling Poisson tensor $\Pi^{(1)}_C \in \text{Coup}_B(E, \Lambda)$ associated to the geometric data $(\Lambda, \Gamma^{(1)}, F^{(1)} + C)$.

By Theorem 3.8, Proposition 3.9, and Proposition 4.10, we deduce the normal form theorem.

**Theorem 4.12.** The Poisson tensors $\Pi$ and $\Pi^{(1)}_C$ are isomorphic by a diffeomorphism $\phi \in \text{Diff}^0_B(E)$,

$$\phi^*\Pi = \Pi^{(1)}_C. \quad (4.27)$$

It is clear that the germ of the original Poisson structure $\Psi$ at $B \subset M$ is also isomorphic to $\Pi^{(1)}_C$.

**Remark 4.13.** A similar result to Theorem 4.12 was formulated in [Br] without specifying the coupling form.

Now applying Theorem 3.11, we get the semilocal linearizability theorem which can be considered as a generalization of the local linearization theorem [Co2].

**Theorem 4.14.** The Poisson structure $\Psi$ is linearizable at $B$ if and only if the $\partial_0$-cohomology class of the leaf $B$ is zero,

$$[C] = 0. \quad (4.28)$$

**Corollary 4.15.** If the second $\partial_0$-cohomology space is trivial, then every Poisson structure in $\text{Coup}_B(E, \Lambda)$ is linearizable.

**Corollary 4.16.** Let $\Pi' \in \text{Coup}_B(E, \Lambda)$ be a linearizable coupling Poisson tensor associated with geometric data $(\Lambda, \Gamma', F')$. Let $C' \in \Omega^2(B) \otimes \text{Casim}_B(E)$ be a $\partial_0$-cocycle. Consider the deformed coupling Poisson tensor $\Pi'_{-C'}$ associated to the data $(\Lambda, \Gamma', F' - C')$. If $[C'] \neq 0$, then $\Pi'_{-C'}$ is nonlinearizable.
It is of interest to give a geometric interpretation of the triviality of the second $\partial_0$-cohomology space.

Remark 4.17. In a linearization conjecture due to [CrFe], an integrability property of the transitive Lie algebroid $T_B^*E$ appears as a sufficient condition for the linearizability in the case when $B$ is compact.

As a consequence of Proposition 4.9 and Theorem 4.14 we get the following semilocal analog of the local splitting theorem.

Theorem 4.18. Assume that the linearized transverse Poisson structure $\Lambda$ is globally trivial, that is, $\Lambda$ admits a flat Lie–Poisson connection with trivial holonomy. If (4.28) holds, then, the germ of $\Psi$ at $B$ is isomorphic to the direct product Poisson structure on $E = B \times g^*$.

Remark 4.19. Some versions of the semilocal splitting theorem and corresponding counterexamples can also be found in [Fe].

The natural question is to give some examples where condition (4.28) does not hold and hence the Poisson structure is nonlinearizable. First such examples of nonlinearizable Poisson structures at a symplectic leaf of nonzero dimension were given in [DW]. These structures are constructed by using a special class of Poisson structures called Casimir-weighted products. We illustrate our results with the following example.

Example 4.20. Let $B$ be an orientable 2-surface with symplectic form

$$\omega = ds \wedge d\tau.$$ 

Consider the direct product $E = B \times \mathbb{R}^3$ as a vector bundle over $B$, where $\pi$ is the canonical projection on the first factor. Let $x = (x_1, x_2, x_3)$ be the Euclidean coordinates on $\mathbb{R}^3$. Suppose we are given a vector-valued 1-form $\varepsilon$ on $B$:

$$\varepsilon = \varepsilon^{(1)} ds + \varepsilon^{(2)} d\tau,$$

where $\varepsilon^{(1)}, \varepsilon^{(2)} \in \mathbb{R}^3$. Fix $c > 0$. Then, one can associate to the pair $(\varepsilon, c)$ the Poisson tensor $\Pi$ on $E$ which is defined by the following bracket relations:

$$\{s, \tau\} = \frac{1}{F(x)},$$

$$\{s, x_\alpha\} = \frac{1}{F(x)} \epsilon_{\alpha\beta\gamma} x_\beta \varepsilon^{(2)}_\gamma, \quad \{\tau, x_\alpha\} = -\frac{1}{F(x)} \epsilon_{\alpha\beta\gamma} x_\beta \varepsilon^{(1)}_\gamma,$$

$$\{x_\alpha, x_\beta\} = \epsilon_{\alpha\beta\gamma} x_\gamma + \frac{1}{F(x)} \epsilon_{\alpha\sigma\gamma} \epsilon_{\beta\sigma'\gamma'} [\varepsilon^{(1)}_\sigma \varepsilon^{(2)}_{\sigma'} - \varepsilon^{(1)}_{\sigma'} \varepsilon^{(2)}_\sigma] x_\gamma x_{\gamma'}.$$
Here $\epsilon_{ijk}$ is the completely antisymmetric Levi-Civita tensor,

$$F(x) = 1 - \langle x, a \rangle + c\|x\|^2$$

and

$$a = g^{(1)} \times g^{(2)}.$$

If

$$\|a\|^2 < 4c,$$

then the brackets (4.29)–(4.30) are well defined on the entire space $E$. It is clear that $(B, \omega)$ is the symplectic leaf of this bracket. Using formulas (2.28)–(2.30), we compute the corresponding geometric data:

$$\Pi_V = \frac{1}{2} \epsilon_{\alpha\beta\gamma} x_\gamma \frac{\partial}{\partial x^\alpha} \wedge \frac{\partial}{\partial x^\beta};$$  \hfill (4.32)

$$\Gamma = dx - x \times g,$$  \hfill (4.33)

$$\pi^* F = F(x) ds \wedge d\tau.$$  \hfill (4.34)

The curvature form of $\Gamma$ is

$$Curv^{\Gamma} = \langle x \times a, \frac{\partial}{\partial x} \rangle \otimes ds \wedge d\tau.$$  \hfill (4.35)

The geometric data of the linearized Poisson structure $\Pi^{(1)}$ are given by

$$\Lambda = \Pi_V,$$  \hfill (4.36)

$$\Gamma^{(1)} = \Gamma,$$  \hfill (4.37)

$$\pi^* F^{(1)} = F^{(1)}(x) ds \wedge d\tau,$$  \hfill (4.38)

where

$$F^{(1)}(x) = 1 - \langle x, a \rangle.$$  \hfill (4.39)

One can recognize formula (4.32) as the Lie–Poisson structure of the dual $so^*(3)$ of the Lie algebra $so(3)$. Thus, the transverse Lie algebra of $B$ is $so(3)$. It is clear that $\Lambda$ is globally trivial. Remark that if $a = 0$, then $\Pi$ is the Casimir-weighted product [DW]. Now, comparing formulas (4.34) and (4.37), we compute the Casimir 2-cocycle $C$,

$$\pi^* C = c\|x\|^2 ds \wedge d\tau.$$  \hfill (4.39)
The space of the Casimir functions of $so^*(3)$ is naturally identified with $C^\infty([0, \infty))$. Under this identification, the coboundary operator is $\partial_0 = d_B \otimes 1$, where $d_B$ is the exterior differential on $B$. We conclude that if $B$ is compact, for example, $B$ is a sphere or a torus, then $[\omega] \neq 0$ and hence $[C] \neq 0$. By Theorem 4.14, in this case, the Poisson structure $\Pi$ is nonlinearizable. On the other hand, if $\pi_2(B) = 0$, for example, $B = S^1 \times \mathbb{R}$ is the 2-cylinder, then $[C] = 0$. In this case, $\Pi$ is linearizable and equivalent to the direct product Poisson structure on $B \times so^*(3)$.

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