Interplay between speed and fidelity in off-resonant quantum-state transfer protocols

Guilherme M. A. Almeida

1 Instituto de Física, Universidade Federal de Alagoas, 57072-900 Maceió, AL, Brazil

(Dated: November 6, 2018)

An arbitrary qubit can be transmitted through a spin chain by perturbatively coupling both communicating parties to it. These so-called weak-coupling models rely on effective Rabi oscillations between the outer spins, yielding nearly maximum fidelity while offering great resilience against disorder with the cost of having long transfer times. Considering that framework, here we address a 1D non-symmetric channel connecting two spins, one placed at each end of it. Given any pattern of nearest-neighbor coupling strengths, we obtain an analytical expression that accounts for the effective long-range interaction between them and study the interplay between transfer time and fidelity. Furthermore, we show that homogeneous channels provide the best speed-fidelity tradeoff.

I. INTRODUCTION

Spin chains have been extensively exploited for many quantum information processing tasks such as the transmission of quantum states [1]–[19] and the creation/distribution of entanglement between distant sites [20–23] (cf. Refs. [27]–[29] for reviews on the subject). As it was put forward in [1], the scheme is based on wiring up different quantum processing units via spin-spin exchange interactions and let them evolve following the natural Hamiltonian dynamics. This means that the Hamiltonian can be engineered in order to avoid dynamical manipulation during the quantum communication task.

In standard single-qubit quantum-state transfer (QST) protocols [1], the sender, say Alice, sends an arbitrary state through the channel and Bob’s only role is to retrieve it at some prescribed time. The interplay between transfer fidelity and speed will be dictated by way the chain is manufactured. Perfect QST can be achieved in fully-engineered chains [2]–[4]. In order to bypass some possible practical issues [31]–[32] concerning tuning the entire set of couplings within the chain, several other schemes were put forward. For instance, by locally adjusting the end bonds of the chain, it is possible to perform ballistic transfer through an arbitrarily long chain [15]. One may also apply strong magnetic fields near the communicating parties in order to energetically detach them from the channel [19]–[21]–[33].

Here, in particular, we deal with a similar class of protocols based on setting weak couplings between the end spins and the rest of the chain (bulk) [7–11]–[18] in order to effectively span a reduced subspace involving the sender and receiver only, up to leading order. That kind of configuration has also been addressed for the sake of generating long-distance entanglement [22–24]–[31]. In the QST context, it entails nearly perfect transfer, though requiring very long times. Taking $a$ as the (coupling) perturbation parameter, the transfer time scales as $O(a^{-2})$ for a channel featuring an even number of sites [8]–[9]–[30]–[31]–[32]–[33]–[34] operating in the Rabi-like (that is, two-level) off-resonant regime, where the frequency of both outer spins does not match any of the natural frequencies of the channel. In this work, we explore in detail the inner workings of that class of QST protocols and discuss the speed-fidelity balance for channels with arbitrary (non-symmetric) couplings. We derive a simple, exact formula that accounts for the end-to-end effective coupling strength as a function of the coupling sequence of the channel, up to second order perturbation theory. By using it, we investigate the speed-fidelity cost for a variety of configurations and show that uniform channels [7] are optimal in respect to time in the weak-coupling regime.

Next, in Sec. II we introduce the XX spin Hamiltonian with arbitrary coupling strengths and work out the perturbation approach in Sec. III. Then, in Sec. IV we derive a relationship between the coupling pattern of the channel and its related QST time in the Rabi-like regime. In Sec. V we discuss the response of the end-to-end correlation amplitude to the channel coupling scheme. Our conclusions are drawn in Sec. VI.

II. SPIN HAMILTONIAN

We consider XX spin-1/2 chains with open boundaries featuring $N+2$ sites, with $N$ (even) being the length of the channel (that is, the bulk of the chain) and the remaining spins acting as the sender/receiver parties which we denote by $S$ and $R$. They are connected, respectively, to each end of the channel, sites 1 and $N$. The Hamiltonian of the system is expressed as $\hat{H} = \hat{H}_{\text{ch}} + \hat{H}_{I}$, with ($\hbar = 1$)

$$\hat{H}_{\text{ch}} = \sum_{i=1}^{N-1} J_i (\hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \hat{\sigma}_i^y \hat{\sigma}_{i+1}^y), \quad (1)$$

$$\hat{H}_{I} = \frac{a_S}{2} (\hat{\sigma}_S^x \hat{\sigma}_1^x + \hat{\sigma}_S^y \hat{\sigma}_1^y) + \frac{a_R}{2} (\hat{\sigma}_R^x \hat{\sigma}_N^x + \hat{\sigma}_R^y \hat{\sigma}_N^y), \quad (2)$$

where $\hat{\sigma}_i^{x,y}$ are the Pauli operators for the $i$-th spin, and $J_i$ and $a_{S,R}$ are nearest-neighbor spin coupling strengths. Herein we define $J = J_{\text{max}} \equiv 1$ as the energy unit. The local magnetic fields are set uniformly across the chain ($\omega \equiv 0$). Since $\hat{H}$ commutes with the total magnetization...
operator, $\hat{H}, \sum, \hat{\sigma}_i^z = 0$, the Hamiltonian is block diagonal with respect to the number of excitations. In this work we only consider single-excitation states which can be expressed by $|j\rangle \equiv |1\rangle|0\rangle$, namely a spin up located at site $j$ ($j = S, 1, 2, \ldots, N, R$) with the remaining ones in their ground state $|0\rangle \equiv |00\ldots0\rangle$.

Basically, here we deal with a generalized version of the weak coupling-based QST, with arbitrary channel couplings $\{J_i\}$ and $a_{S,R} \ll J$ in general. Particularly, our discussion is focused on the off-resonant two-level dynamical regime, whereby spins $S$ and $R$ span their own subspace, up to a second-order approximation, so that QST occurs via Rabi-like oscillations between them $S$.

### III. EFFECTIVE TWO-LEVEL HAMILTONIAN

In general – without specifying any particular coupling scheme for now – the basic QST protocol consists in the transmission of an arbitrary qubit state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ from one point to another. This can be done by initializing the whole chain in $|\psi(0)\rangle = |\phi\rangle_s|0\rangle_1|0\rangle_2 \ldots |0\rangle_N$ and letting it evolve naturally through $|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$. In the ideal scenario, the final output should be as close as possible to $|\psi(\tau)\rangle = |0\rangle_s|0\rangle_1|0\rangle_2 \ldots |\phi\rangle_R$ at some prescribed time $\tau$, so that the receiver can properly retrieve the information.

The corresponding transfer fidelity can thus be evaluated by $F_\phi(\tau) = \langle \phi|\rho_{S,R}(\tau)|\phi\rangle$, where $\rho_{S,R}(\tau)$ is obtained by tracing out the channel spins (1 through $N$), $\rho_{S,R}(\tau) = \text{Tr}_{\text{channel}}(|\psi(\tau)\rangle \langle \psi(\tau)|)$. Now, the figure of merit of the channel may be evaluated by averaging the transmission fidelity $F_\phi$ over all input states $|\phi\rangle$ (i.e., over the Bloch sphere such that $|\alpha|^2 + |\beta|^2 = 1$) which – given the dynamics is restricted to occur in the zero- and one-excitation subspaces – results in $F_\phi(T)$

$$F(t) = \frac{1}{2} + \frac{|f(t)|}{3} + \frac{|f(t)|^2}{6}, \quad (3)$$

where $f(t) \equiv \langle R|e^{-i\hat{H}t}|S\rangle$ is the transition amplitude between spins $S$ and $R$. This quantity ultimately embodies the quality of the transfer.

In the case of Rabi-type QST $[7]$, the goal is to design the channel such that a couple of eigenstates having the form $|\lambda^\pm\rangle \equiv (|S\rangle \pm |R\rangle)/\sqrt{2}$, with corresponding energy gap $\delta \lambda$, is obtained. In this scenario, the transition amplitude takes

$$|f(t)| \simeq C e^{-i\delta \lambda t/2} - e^{i\delta \lambda t/2}$$

$$= C \left| \sin \left( \frac{\delta \lambda t}{2} \right) \right|,$$

where $C \equiv 2|\langle R|\lambda^\pm\rangle\langle \lambda^\pm|S\rangle| \simeq 1$ measures the end-to-end correlation amplitude. The transfer time is thus $\tau = \pi/\delta \lambda$. The energy gap between both dominating eigenstates sets the typical energy scale of the transfer.

While there are many schemes to achieve that kind of behavior $[7, 10, 15, 18]$, our goal here is to find out the optimal combination $\{J_i\}$ which provides the highest possible $|f(t)|$ with maximum $\delta \lambda$ allowed. In order to do so, we use a second-order perturbation approach (cf. Refs. $[8, 9]$) in $a_{S,R}$ so that we can derive $\delta \lambda$ in terms of the spectral resolution of the channel. The effective interaction between states $|S\rangle$ and $|R\rangle$ is accounted by

$$H_{\text{eff}} = \left( h_S J_{\text{eff}} J_{\text{eff}} h_R \right) \quad (5)$$

where $h_{S,R} = \sum_k (|v_{k,1}(N)|^2/\epsilon_k)$, $J_{\text{eff}} = -a_{S,R} \Lambda$, $\Lambda = \sum_k (\epsilon_{k,1}\epsilon_{k,N}/\epsilon_k)$, $v_{k,1} = (1\epsilon_k)$, and $v_{k,N} = \langle N|\epsilon_k\rangle$, with $\epsilon_k$ and $\epsilon_{k,1}$ being, respectively, the eigenstates and eigenvalues of the channel Hamiltonian $[\text{Eq. (1)}]$. Note that the spectrum features particle-hole symmetry, that is $\epsilon_{k,1} = -\epsilon_k$ and $|v_{k,j}| = |v_{-k,j}|$ for every $k$, and thus $h_S = h_R = 0$.

The resulting two-level gap then reads $\delta \lambda = 2J_{\text{eff}}$. According to the above effective description, the time it takes to transfer a quantum state from $S$ to $R$ is $\tau = \pi/(2a^2|\Lambda|)$, with $a = a_S = a_R$ from now on. Since a must very small in order to assure the validity of Hamiltonian $[7]$, we shall expect long transfer times. Given $\{J_i\}$, the appropriate value for $a$ will depend upon the spectral properties of the channel, its size, and the required fidelity outcome. In homogeneous channels, for example, it is enough to set $a \ll J/\sqrt{N}$ $[7]$. There is also the possibility of searching for coupling patterns that maximize $\Lambda$ in order to counterbalance $a$. In the following section, we derive an exact analytical expression for $\Lambda$ as a function of $\{J_i\}$.

### IV. END-TO-END INTERACTION AS A FUNCTION OF THE COUPLING SCHEME OF THE CHANNEL

First, it is convenient to introduce the shorthand notation $\{J_1, J_2, \ldots, J_{N-1}\}$ to represent the coupling sequence for a $N$-site channel (see Fig. $\text{1}$). Recall

![FIG. 1. (Color online) Coupling scheme for the XX spin chain described by Hamiltonian $\text{1}$. Both end spins denoting the communicating parties are connected, at rates $a_S$ and $a_R$, respectively, to a channel featuring an even number of sites $N$ with arbitrary couplings. Up to second-order perturbation theory, both outer spins eventually develop an effective interaction $J_{\text{eff}}$ mediated by the channel through $\Lambda = \Lambda(J_1, J_2, \ldots, J_{N-1})$ [cf. Eq. $\text{4}$], which characterizes its strength.](image-url)
that we consider \( N \) to be even throughout the paper. For instance, \((J_1)\) and \((J_1, J_2, J_3)\) denote a 2- and a 4-site channel, respectively. Their corresponding values of \( \Lambda \) are easily obtained and read \( \Lambda_2 \equiv \Lambda(J_1) = 1/J_1 \) and \( \Lambda_4 \equiv \Lambda(J_1, J_2, J_3) = -J_2/(J_1 J_3) \) (cf. Appendix). The latter outcome is equivalent to that of a dimer with coupling \( \Lambda_2^{-1} \). Now, consider a 6-site chain with weak end couplings, \( J_1, J_5 \ll J \), for instance. From the perturbation theory framework discussed previously, the outer sites should develop an effective coupling of the form \(-J_1 J_5'\), with \( J' = \Lambda(J_2, J_3, J_4) = -J_3/(J_2 J_4) \). Note we can treat the above configuration as a virtual (4-site) chain with coupling sequence \((J_1, \Lambda^{-1}, J_5)\). This yields \( \Lambda_6 \equiv \Lambda(J_1, J_2, J_3, J_4, J_5) = J_2 J_4/(J_1 J_3 J_5) \). Curiously, this very same result should be obtained regardless of the strength of \( J_1 \) and \( J_5 \) (see Appendix \[\text{d} \]) for details). Henceforth, using the \( \Lambda \) values of \( \Lambda_2 \) are easily obtained and read \( \Lambda_2 \equiv \Lambda(J_1, J_2, J_3, J_4, J_5) = J_2 J_4/(J_1 J_3 J_5) \). Curiously, this very same result should be obtained regardless of the strength of \( J_1 \) and \( J_5 \) (see Appendix \[\text{d} \]) for details). Henceforth, using the same reasoning, \( \Lambda \) can be built for arbitrary (even) \( N \) and uniform on-site potential distribution following the recursive relation: \( \Lambda_{M,j} = -\left(\Lambda_{M,j-1} J_{M-j} J_{M+j}\right)^{-1} \) for \( j = 1, 2, \ldots, M-1 \), with \( \Lambda_{M,j} \equiv \Lambda_{M,j-1} J_{M-j+1}, \ldots, J_M J_{M+1}, \ldots, J_{M+j-1}, J_{M+j}, \ldots, J_{M-1} \), and \( \Lambda_{M,0} \equiv \Lambda(J_M) = J_{M+1}^{-1} \). By iterating the above rule over and over until \( j = M-1 \) we get
\[
\Lambda_N \equiv \Lambda_{M, M-1} = (-1)^{M+1} J_2 J_3 J_6 \cdots J_{N-2} J_1 J_3 J_5 J_7 \cdots J_{N-1}^{-1}.
\]
The above equation comprises the key result of this work (a formal derivation of it is provided in Appendix \[\text{d} \]). By weakly switching on the interaction between spins \( S, R \) and the channel, the effective coupling between the outer ends of the chain becomes \( J_{\text{eff}} = -a^2 \Lambda_N \) [cf. Eq. \( \text{(5)} \)] up to second order in \( a \). It is worth stressing that the term \( \Lambda_N \equiv \sum_k v_k v_{k,N}/\epsilon_k \) as shown above in Eq. \( \text{(5)} \) is exact by itself, although it emerges from the effective Hamiltonian \( \text{(5)} \) which, on its side, has been derived using a perturbation approach. As we are about to see, our result displayed in Eq. \( \text{(5)} \) is a handy resource for studying the trade-off between speed and fidelity given any pattern of couplings along the channel.

V. SPEED VERSUS FIDELITY

In the following discussion, we will often treat the whole channel mediating spins \( S \) and \( R \) as a single dimer coupled by \( \Lambda_N^{-1} \) for convenience. In the light of second-order perturbation theory, the full chain is equivalent to a 4-site chain with coupling sequence \((a, \Lambda_N, a)\). This simple picture shows that the outer spins no longer have access to precise information over the number of spins within the channel as well as its exact coupling sequence. All they “see” is \( \Lambda_N \) which, in turn, can be set in infinitely many ways. The fidelity thus scales as \( F \sim 1 - O(a^2 \Lambda_N^2) \) entailing an almost perfect QST at times \( \tau \sim (a^2 |\Lambda_N|)^{-1} \).

It also deserves notice the fact that \( \Lambda_N \) contains useful information about the channel itself. That gives us insight over how large is the gap in the center of the energy band, based on the picture established above. First, note in Eq. \( \text{(3)} \) that if a given coupling \( J \) promotes an increase (or decrease) in \( \Lambda_N \), the next one, \( J_{i+1} \), will do the opposite, and so forth. This curious fact explains many of the properties featured in staggered chains with alternating weak and strong bonds (see e.g., Refs. \[\text{[10, 11, 15, 22]} \]). This kind of configuration is known for providing very high fidelities on the one hand, even for moderately distorted couplings, and very long QST times on the other hand. Indeed, the characteristic gap \( \delta \lambda \) between (wanted) states \("1\) \( \sim (|S| \pm |R|)/\sqrt{2} \) is very sensitive to changes in \( N \) and to the ratio between weak and strong couplings, say \( b \) and \( J = J_{\text{max}} \), respectively. Such behavior emerges very clearly when we set the sequence \((J, b, J, \ldots, b, J)\) for the channel. (Let us take \( M = N/2 \) odd, without loss of generality, only to assure \( \Lambda_N \) positive.) Thereby, \( \Lambda_N \) (and so the effective gap \( \delta \lambda = 2J_{\text{eff}} \)) will decrease abruptly since \( \Lambda_N \sim b^{M-1} \). The above coupling configuration would then improve the fidelity but with the cost of having a transfer about \( \Lambda_N \) times slower when compared with the uniform channel, \( b \to J \), considering the same \( a \).

If one decides to increase \( \Lambda_N \) instead, say, by adjusting the couplings to \((b, J, b, \ldots, J, b)\) such that \( \Lambda_N \sim b^{-M} \), the transfer time would be about \( \Lambda_N \) times faster, but with unavoidable fidelity loss. To better see this, let us set \( a/J = \xi \) for the uniform channel and \( a/J = \xi/\sqrt{\Lambda_N} \) for the modified (staggered) channel, with \( \xi \ll 1 \). Both channels should now perform QST in the same time \( \tau \sim \pi/(2\xi^2 J) \) and we are left with the task of comparing the resulting fidelities.

In order to properly operate in the effective Rabi regime, the staggered channel must satisfy \( \xi \sqrt{\Lambda_N} \ll 1 \). Therefore, departing from the uniform channel (for which \( \Lambda_N = 1/J \), any attempt to increase \( \Lambda_N \) (regardless of the coupling pattern) should bring the channel progressively out of the perturbation regime, for fixed \( \xi \). As an example, in Fig. \[\text{b} \] we plot the end-to-end correlation amplitude \( C \) versus \( \xi \) for various \( \Lambda_N \) values, after exact diagonalization of the full Hamiltonian, Eq. \( \text{(5)} \) for \( N = 30 \) (plus spins \( S \) and \( R \)). Recall that \( C \) primarily indicates how successful the transfer will be, with \( C \approx 1 \) implying \( F \approx 1 \) [cf. Eqs. \( \text{(3)} \) and \( \text{(4)} \)]. Figure \[\text{b} \] clearly shows that the curve for the homogeneous channel bounds all the other configurations, entailing that if the fidelity is to be evaluated in time window \( \sim O(\xi^{-2}) \) – note that, due to the finiteness of \( \xi \), higher-order interactions between sender/receiver and the channel might deviate from \( \tau \) a little bit \[\text{[7, 8]} \] – the former always outperforms and, for decreasing \( b \) (increasing \( \Lambda_N \)), the decay is more critical since the channel now demands smaller \( \xi \) values in order to secure the Rabi-like regime.

From the above analysis we note that for 1D chains with weak end bonds operating in the Rabi-like regime, given a target value for fidelity \( \sim O(1) \), there is no way
to configure the set of couplings of the channel \{J_i\} to achieve a faster QST other than keep them all equal. Despite we have arrived at this conclusion using spatially symmetric (staggered) channels as an example, similar arguments hold for non-symmetric channels as well.

Noting in Eq. (6) that \(J_{\mu} \) and \(J_{\nu} \) with even \(\mu\) and odd \(\nu\) can cancel each other out, a question that naturally arises now is whether any other combination of \(\{J_i\}\) resulting in \(\Lambda_N = 1 / J = 1 / J_{\text{max}}\) would outperform the homogeneous channel, which features the same property. To address this, let us recall our 4-site virtual system \((a, \Lambda_N^{-1}, a)\). Even when we do not have enough information about the full spectral decomposition of the channel (here reduced to a single dimer coupled by \(\Lambda_N^{-1}\)), we do know – from Eq. (6) and condition \(a\Lambda_N \ll 1\) – its tendency to close or open the energy gap in the middle of the band given \(\{J_i\}\). If the coupling pattern is such that \(\Lambda_N = 1 / J_{\text{max}}\) and generated out of a sequence fulfilling \(J_i \leq J_{\text{max}}\), it will shift normal modes towards the band center thereby disturbing \(|\lambda^\pm| \approx (|S| \pm |R|) / \sqrt{2}\).

As an example, in Fig. 3 we show the decay of the end-to-end correlation amplitude versus \(\xi\) for numerous randomly-generated paired up coupling sequences satisfying \(J_i = J_{i+1}\), i.e. \(J_1 = J_2\), \(J_3 = J_4\), and so forth with exception of \(J_M\) (again for \(M = N/2\) odd without loss of generality) which has been singled out to \(J_{\text{max}}\). It clearly shows that none of the samples were able to outperform the regular, uniform channel in terms of speed-fidelity cost, they all sharing the same \(\Lambda_N = 1 / J_{\text{max}}\).

VI. CONCLUSIONS

We have investigated single-qubit QST protocols between two parties weakly coupled to the ends of a \(XX\) spin-1/2 channel with arbitrary (non-symmetric) nearest-neighbor couplings. We obtained an analytical formula that accounts for the effective coupling between spins \(S\) and \(R\) given any set of couplings \(\{J_i\}\). Straightforward analysis showed that uniform channels, \(J_i = J\), offer the best speed-fidelity cost within the Rabi-like (off-resonant) regime.

We would also like to highlight that all the analysis performed in this work was done in the perturbative regime up to second order. Naturally, the overall behavior changes considerably as we depart from that scenario. For example, a lot of theoretical progress has been achieved for chains operating in the ballistic regime [15].

Finally, our results illuminate the inner machinery Rabi-type QST protocols, their physical limitations, and shall also be of interest for studies on long-range entanglement generation schemes [22, 23]. Further extensions of this work could involve non-trivial channel topologies, such as complex networks [35, 37].

VII. ACKNOWLEDGMENTS

We thank A. M. C. Souza for valuable discussions. This work was supported by CNPq (Grant No. 152722/2016-5).

Appendix: Derivation of \(\Lambda_N\)

In this appendix, we carry out a proof of \(\Lambda_N = \sum_k (\nu_{k,1} \nu_{k,N} / \epsilon_k)\) as expressed in Eq. (6). Our starting point is the eigenvalue equation \(H_{\text{ch}} \nu_{k,1} = \epsilon_k \nu_{k,1}\) for the channel’s Hamiltonian \(H_{\text{ch}}\) [Eq. (1)] with coupling...
The above relation yields the following set of equations:

\[
\begin{pmatrix}
0 & J_1 & 0 & \cdots \\
J_1 & 0 & J_2 & \\n0 & J_2 & 0 & J_3 \\
\vdots & \ddots & \ddots & \ddots \\
0 & J_{N-2} & 0 & J_{N-1} \\
\end{pmatrix}
\begin{pmatrix}
v_{k,1} \\
v_{k,2} \\
v_{k,3} \\
\vdots \\
v_{k,N} \\
\end{pmatrix}
=
\begin{pmatrix}
v_{k,1} \\
v_{k,2} \\
v_{k,3} \\
\vdots \\
v_{k,N} \\
\end{pmatrix},
\]

(A.1)

where \( \{v_{k,i}\} \) and \( \{\epsilon_k\} \) are all real valued and \( N \) is even. The above relation yields the following set of equations:

\[
\begin{align*}
J_1 v_{k,2} &= \epsilon_k v_{k,1} \\
J_1 v_{k,1} + J_2 v_{k,3} &= \epsilon_k v_{k,2} \\
J_2 v_{k,2} + J_3 v_{k,4} &= \epsilon_k v_{k,3} \\
\vdots \\
J_{i-1} v_{k,i-1} + J_i v_{k,i+1} &= \epsilon_k v_{k,i} \\
\vdots \\
J_{N-1} v_{k,N-1} &= \epsilon_k v_{k,N}
\end{align*}
\]

(A.2)

from which we get

\[
\frac{v_{k,i+1}}{\epsilon_k} = \frac{v_{k,i}}{J_i} - \frac{J_{i-1} v_{k,i-1}}{J_i \epsilon_k}
\]

(A.3)

Now, as our sole purpose here is to evaluate the sum \( \Lambda = \sum_k (v_{k,1} v_{k,N}/\epsilon_k) \), we may write it as [cf. Eq. (A.3)],

\[
\Lambda_{i+1} = \sum_k \frac{v_{k,1} v_{k,i+1}}{\epsilon_k} = \frac{1}{J_1} \delta_{i,i} - \frac{J_{i-1}}{J_i} \sum_k \frac{v_{k,1} v_{k,i-1}}{\epsilon_k}
\]

(A.4)

which proves Eq. (6).
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