Solitons from Dressing in an Algebraic Approach to the Constrained KP Hierarchy

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Abstract

The algebraic matrix hierarchy approach based on affine Lie $sl(n)$ algebras leads to a variety of $1+1$ soliton equations. By varying the rank of the underlying $sl(n)$ algebra as well as its gradation in the affine setting, one encompasses the set of the soliton equations of the constrained KP hierarchy.

The soliton solutions are then obtained as elements of the orbits of the dressing transformations constructed in terms of representations of the vertex operators of the affine $sl(n)$ algebras realized in the unconventional gradations. Such soliton solutions exhibit non-trivial dependence on the KdV (odd) time flows and KP (odd and even) time flows which distinguishes them from the conventional structure of the Darboux-Bäcklund Wronskian solutions of the constrained KP hierarchy.
1 Introduction. The Algebraic cKP Model

A large class of $1+1$ soliton equations belongs to the so-called constrained KP (cKP) hierarchy. Some of the most prominent members of this group are the KdV and the nonlinear Schrödinger equations of the AKNS model. The cKP evolution equations possess the familiar Lax pair representations with generally pseudo-differential Lax operators which emerge naturally as reductions of the complete KP hierarchy Lax operators ([1]). Conventionally, the constrained KP hierarchy is obtained from the KP hierarchy by a process of reduction involving the so-called eigenfunctions. The eigenfunctions appear in the constraint relations introducing a functional dependence between initially infinitely many coefficients of the KP Lax operator. This scheme results in the pseudo-differential cKP Lax operator of the type

$$L = L_{K+1} + \sum_{i=1}^{M} \Phi_i \partial^{-1} \Psi_i,$$

where $L_{K+1}$ is the differential operator of the $K+1$-order, while $\Phi_i, \Psi_i$ are the eigenfunctions of $L$. In general, $L$ possesses a finite number of coefficients which enter the soliton equations and depend on all $(t_1, t_2, t_3, \ldots)$ isospectral time flows of the KP hierarchy. In the special $M = 0$ case with the cKP Lax operator being a purely differential operator $L = L_{K+1}$ we encounter dependence on only some of the original time-flows of the KP hierarchy. The most simple example ($K = 1, M = 0$) is the KdV hierarchy with only odd time flows present.

The soliton solutions for the cKP models have been found in ([2]) for the arbitrary $K$ and $M = 1$ case using the Darboux-Bäcklund technique. Generalization to an arbitrary $M$ is simple and was given in ([3]) (see also ([4])). These solutions appeared in the Wronskian form in terms of the eigenfunctions of the “undressed” $L = \partial^{K+1}$ Lax operator.

Here, we will present an alternative algebraic viewpoint of the constrained KP hierarchy. In this setting the algebraic dressing methods will provide new soliton solutions which appear to differ from the conventional form of the Darboux-Backlund Wronskian solutions due to a non-trivial mixing of the KdV-like versus KP time flows. This will be shown explicitly in the example characterized by $K = M = 1$.

In an algebraic approach to the constrained KP hierarchy ([5]) the soliton evolution equations emerge as integrability conditions of the following matrix eigenvalue problem:

$$L \Psi = 0 ; \quad L \equiv (D - A - E) ; \quad D \equiv I \frac{\partial}{\partial x}$$

with the matrix Lax operator $L$ belonging to Kac-Moody algebra $\hat{G} = \hat{sl}(M + K + 1)$. The integrable hierarchy is determined by the choice of gradation of $\hat{G}$. By varying the Kac-Moody algebras together with their gradations one is able to reproduce from the matrix hierarchy of eq. ([9]) the nonlinear evolution equations of the cKP hierarchy.

We will be working with a simple setting in which the matrix $E$ in eq.([11]) has gradation 1 with respect to gradation specified by the vector ([6]):

$$s = (1, 0, \ldots, 0, 1, \ldots, 1)$$

We call this gradation an intermediate gradation as it interpolates between the principal $s_{\text{principal}} = (1, 1, \ldots, 1)$ and the homogeneous one $s_{\text{homogeneous}} = (1, 0, \ldots, 0)$. As it is well-known the Wilson-Drinfeld-Sokolov ([7, 8, 9, 10, 11]) procedure gives, respectively, the (m-)KdV ([10]) and AKNS ([12, 13]) hierarchies in these two limits.
Alternatively, the gradation $s$ can be specified by an operator:

$$Q_s \equiv \sum_{a=1}^{K} \lambda_{M+a} \cdot H^0 + (K + 1)d$$  \hfill (3)

Here $\lambda_i$ are the fundamental weights and $d$ is the standard loop algebra derivation. Correspondingly, $E$ stands for

$$E = \sum_{a=1}^{K} E^{(0)}_{\alpha_{M+a}} + E^{(1)}_{-(\alpha_{M+1}+\cdots+\alpha_{M+K})}$$  \hfill (4)

It is a non-regular (for $M > 0$) and semisimple element of $\hat{G}$.

The matrix $A$ in eq. (5) contains the dynamical variables of the model. $A$ is such that it has gradation zero and is parametrized in terms of the dynamical variables $q_i$, $r_i$, $U_a$ and $\nu$ as follows:

$$A = \sum_{i=1}^{M} (q_i P_i + r_i P_{-i}) + \sum_{a=1}^{K} U_{M+a} (\alpha_{M+a} \cdot H^{(0)}) + \nu \hat{c}$$  \hfill (5)

where $P_{\pm i} = E^{(0)}_{\pm (\alpha_i + \alpha_{i+1} + \cdots + \alpha_M)}$, $i = 1, 2, \ldots, M$, and $\hat{c}$ is a central element of $\hat{G}$.

2 The Heisenberg Subalgebra and the Vertex Operator

For a regular element $E$ in the conventional Drinfeld-Sokolov approach the isospectral flows are associated with the Heisenberg subalgebra which can be identified with Ker $\text{(ad } E)$. Here, due to non-regularity of $E$ the Heisenberg algebra is associated to the center of Ker $\text{(ad } E)$. It consists of the following three separate sets of operators:

1) A homogeneous part of $\hat{sl}(M)$, $i = 1, 2, \ldots, M - 1$

$$K^{(n)}_i = \sum_{p=1}^{i} \frac{p \alpha_p \cdot H^{(n)}}{N_i}; \quad N_i \equiv \sqrt{i(i + 1)}$$  \hfill (6)

2) A principal part of $\hat{sl}(K + 1)$, $a = 1, 2, \ldots, K$

$$A_{a+n(K+1)} = \sum_{i=1}^{K+1-a} E^{(n)}_{\alpha_{i+M+\alpha_{i+M+1}+\cdots+\alpha_{i+M+a-1}+\alpha_{i+M+1}+\cdots+\alpha_{i+M+K-a}}} + \sum_{i=1}^{a} E^{(n+1)}_{-(\alpha_i+\alpha_{i+M+1}+\cdots+\alpha_{i+M+K-a})}$$  \hfill (7)

3) "A border term"

$$A_{n(K+1)}^0 = \sqrt{M + K + 1} M \lambda_M \cdot H^{(n)} - \frac{K}{2} \sqrt{M} \sqrt{M + K + 1} \hat{c} \delta_{n,0}$$  \hfill (8)

These relations provide a parametrization of the Heisenberg subalgebra in terms of elements:

$$b_{N,a} \equiv A_{N=a+n(K+1)}^a, \quad b_{N,0} \equiv A_{N=n(K+1)}^0, \quad b_{N,i} \equiv K^{(N)}_i$$  \hfill (9)
where \( a = 1, 2, \ldots, K, \ i = 1, 2, \ldots, M - 1 \). The Heisenberg subalgebra elements from (9) enter the oscillator algebra relations (we put \( c = 1 \)):

\[
\begin{align*}
[b_{N,a} , b_{N',b}] &= N \delta_{N+N'} \delta_{a,K+1-b} \quad ; \ a, b = 1, 2, \ldots, K \\
[b_{N,0} , b_{N',0}] &= N \delta_{N+N'} \\
[b_{N,i} , b_{N,j}] &= N \delta_{N+N'} \delta_{ij} \quad ; \ i, j = 1, 2, \ldots, M - 1
\end{align*}
\]

Define next the Fubini-Veneziano operators:

\[
Q_{1\leq i\leq M-1}(z) = i \sum_{n=1}^{\infty} \frac{\gamma^{(n)}_{i}}{n} z^{-n} \quad ; \quad Q_{M}(z) = i \sum_{n=1}^{\infty} \frac{\gamma^{(0)}_{n}(K+1)}{n(K+1)} z^{-n(K+1)}
\]

\[
Q_{M+a}(z) = i \sum_{n=0}^{\infty} \frac{\gamma^{(a+n(K+1))}_{n}}{a + n(K+1)} z^{-a+n(K+1)} \quad ; \quad a = 1, 2, \ldots, K
\]

The corresponding conjugated Fubini-Veneziano operators \( Q^\dagger(z) \) are obtained from (13)-(14) by taking into consideration rules \( \gamma^{(n)}_{i} = \gamma^{(n)}_{i} \), \( \gamma^{(a+n(K+1))}_{n} = \gamma^{(a+n(K+1))}_{n} \), \( \gamma^{(0)}_{n}(K+1) = \gamma^{(0)}_{n}(K+1) \), \( \gamma^{(a+n(K+1))}_{n} = \gamma^{(a+n(K+1))}_{n} \) as well as \( z^{\dagger} = z^{-1} \).

In reference (14) we found the step operators of \( sl(M + K + 1) \) associated with the Cartan subalgebra defined by the Heisenberg subalgebra (9). That in turn enabled us to find the corresponding simple root structure for \( sl(M + K + 1) \) with the intermediate grading.

Knowledge of roots and the Fubini-Veneziano operators is all what is needed to write down a compact expression for the general vertex operator in the normal ordered form:

\[
V^{\alpha}(z) \equiv z^{\frac{1}{2}} \sum_{j=1}^{M} (\alpha^{j})^{2} \exp \left( i \alpha^{\dagger} \cdot \vec{Q}^{\dagger}(z) \right) \exp \left( i \alpha^{\dagger} \cdot \vec{q} \right) \exp \left( i \vec{p} \ln z \right) \exp \left( i \vec{\alpha} \cdot \vec{Q}(z) \right)
\]

with \( M + K \)-component root vector \( \vec{\alpha} \) described in (14) and \( M + K \)-component vector \( \vec{Q} \) having components described in (13)-(14). The zero-mode vectors \( \vec{p} \) and \( \vec{q} \) have only first \( M \) components different from zero according to \( \langle \vec{p} \rangle_{i} = p_{i} \theta(M - i) \) and \( \langle \vec{q} \rangle_{i} = q_{i} \theta(M - i) \). They satisfy relations \( [p_{i}, q_{j}] = -i \delta_{ij} \). Furthermore, \( p_{M} \) is equal to \( A^{0}_{0} = 0 \) from expression (8).

An explicit example of the \( sl(3) \) vertex will be given below in section 4.

### 3 The Dressing Technique and the Tau-function

The dressing technique (15) deals with reproducing of the nontrivial part \( E + A \) of the Lax matrix operator from eq. (1) by the gauge transformations involving generators of positive and negative gradings applied to the semisimple element \( E \):

\[
\begin{align*}
E + A &= \Theta E \Theta^{-1} + (\partial_{x} \Theta) \Theta^{-1} \\
E + A &= (B^{-1} \Gamma) E (\Gamma^{-1} B) + (\partial_{x} B^{-1} \Gamma) (\Gamma^{-1} B)
\end{align*}
\]

where \( B^{-1} \Gamma \) contains positive terms and \( \Theta \) is an expansion in the terms of negative grading such that \( \Theta = \exp \left( \sum_{l<0} \theta^{(l)} \right) = 1 + \theta^{-1} + \ldots \). From expressions (16) and (17) we obtain two alternative formulas for the same term \( A \) of grade 0:

\[
A = -\left[ E , \theta^{(-1)} \right] \quad \text{or} \quad A = -B^{-1} \left( \partial_{x} B \right)
\]
The term $\theta^{(-1)}$ of grade $-1$ can be expanded as

$$
\theta^{(-1)} = \sum_{a=M+1}^{M+K} \theta_a^{(-1)} E^{(0)}_{-\alpha_a} + \theta^{(-1)}_{\psi} E^{(-1)}_{\alpha_{M+1} + \ldots + \alpha_{M+K}}
$$

$$
\quad + \sum_{l=1}^{M} \theta_l^{(-1)} E^{(0)}_{-(\alpha_l + \ldots + \alpha_{M+1})} + \sum_{l=1}^{M} \bar{\theta}_l^{(-1)} E^{(-1)}_{\alpha_l + \ldots + \alpha_{M+K}}
$$

(19)

where we included all possible terms of grade $-1$ according to (18).

Therefore

$$
\left[ E, \theta^{(-1)} \right] = \sum_{a=M+1}^{M+K} \theta_a^{(-1)} \alpha_a \cdot H^{(0)} + \theta^{(-1)}_{\psi} \left( -(\alpha_{M+1} + \ldots + \alpha_{M+K}) \cdot H^{(0)} + c \right)
$$

$$
\quad + \sum_{l=1}^{M} \theta_l^{(-1)} \epsilon(\alpha_{M+1} - \alpha_l - \ldots - \alpha_{M+1}) E^{(0)}_{-(\alpha_l + \ldots + \alpha_M)}
$$

$$
\quad + \sum_{l=1}^{M} \bar{\theta}_l^{(-1)} \epsilon(-\alpha_{M+1} - \ldots - \alpha_{M+K}, \alpha_l + \ldots + \alpha_{M+K}) E^{(0)}_{\alpha_l + \ldots + \alpha_M}
$$

(20)

Comparing the last expression with the field content of $A$ as given by (19) we obtain relations for expansions parameters used in (13):

$$
\nu = -\theta^{(-1)}_{\psi}, \quad U_a = -\theta_a^{(-1)} + \theta^{(-1)}_{\psi}, \quad r_l = -\theta_l^{(-1)} \epsilon(\alpha_{M+1}, \alpha_l, \ldots, \alpha_{M+1})
$$

$$
q_l = -\bar{\theta}_l^{(-1)} \epsilon(-\alpha_{M+1} - \ldots - \alpha_{M+K}, \alpha_l + \ldots + \alpha_{M+K})
$$

(21)

We now work with representation of $A$ as given in eq. (18). We split the grade zero element $B$ in a product $B = B_1B_2$ with $B_1$ containing the grade zero $sl(M)$ elements and

$$
B_2 \equiv \exp \sum_{a=1}^{K} \phi_{M+a} \alpha_{M+a} \cdot H^{(0)} + \rho \cdot \hat{c}
$$

(22)

Accordingly, eq. (18) becomes $A = -B_2^{-1}B_1^{-1}(\partial_x B_1)B_2 - B_2^{-1}(\partial_x B_2)$ and $A$ can be rewritten as

$$
A = -\sum_{a=1}^{K} \partial_x \phi_{M+a} \alpha_{M+a} \cdot H^{(0)} - \partial_x \rho \cdot \hat{c} + O(sl(M))
$$

(23)

where $O(sl(M))$ contains all possible terms belonging to the $sl(M)$ algebra.

Comparison with (3) yields

$$
U_{M+a} = -\partial_x \phi_{M+a} \quad; \quad \nu = -\partial_x \rho
$$

(24)

We define a family of the first order differential matrix operators $L_N = \partial/\partial t_N - A_N$, $N = 1, \ldots$. The hierarchy is then formulated in terms of the zero-curvature equations for the Lax operators

$$
\left[ L_N, L_M \right] = 0
$$

(25)
expressing commutativity of the higher time flows. The zero-curvature equations imply the pure gauge solutions for the potentials $A_N$:

$$\mathcal{L}_N = \Psi \frac{\partial}{\partial t_N} \Psi^{-1}$$

(26)

The starting point of the dressing method ([13]) is the vacuum solution $\nu = U = r = q = 0$. The corresponding $L^{(\text{vac})} = D - E$ matrix Lax operator together with higher flows operators $L_N$, $N > 1$ for the vacuum solutions are expected to be recovered via (26) from $\Psi$, which is expressed entirely by the Heisenberg algebra associated with the center of $\text{Ker} (\text{ad } E)$. Explicitly, for our model

$$\Psi = \Psi^{(\text{vac})} = \exp \left( \sum_{N=1}^{\infty} t_N b^{(N)} \right)$$

(27)

with $b_N$ given in (2).

We define the tau-function vectors as:

$$|\tau_0\rangle = \Psi^{(\text{vac})} \phi \Psi^{(\text{vac})}^{-1} |\lambda_0\rangle \quad ; \quad |\tau_{M+a}\rangle = \Psi^{(\text{vac})} \phi \Psi^{(\text{vac})}^{-1} |\lambda_{M+a}\rangle$$

(28)

They are associated with the constant group element $h$ and the highest-weight vectors $|\lambda_0\rangle, |\lambda_{M+a}\rangle$ such that

$$\alpha_{M+a} \cdot H^{(0)} |\lambda_0\rangle = 0 \quad ; \quad \alpha_{M+b} \cdot H^{(0)} |\lambda_{M+a}\rangle = \delta_{a,b} |\lambda_{M+a}\rangle \quad ; \quad a, b = 1, \ldots, K.$$  

(29)

Assuming that $h$ allows the “Gauss” decomposition of $\Psi^{(\text{vac})} \phi \Psi^{(\text{vac})}^{-1}$ in positive, negative and zero grade elements we get for the tau-function vectors from (28) an alternative expression:

$$|\tau_0\rangle = \Theta^{-1} B^{-1} |\lambda_0\rangle \quad ; \quad |\tau_{M+a}\rangle = \Theta^{-1} B^{-1} |\lambda_{M+a}\rangle$$

$$\Theta^{-1} = \left( \Psi^{(\text{vac})} \phi \Psi^{(\text{vac})}^{-1} \right)_{-} \quad ; \quad B^{-1} = \left( \Psi^{(\text{vac})} \phi \Psi^{(\text{vac})}^{-1} \right)_{0}$$

(30)

for $a = 1, \ldots, K$. Like before, we make a splitting $B^{-1} = B_2^{-1} B_1^{-1}$ in (30) and notice that $B_1^{-1} |\lambda_{M+a}\rangle = |\lambda_{M+a}\rangle$ for $B_1$ being an exponential of sl($M$) generators. Inserting $B_2$ from (22) we find

$$|\tau_0\rangle = \Theta^{-1} |\lambda_0\rangle e^{(-\rho)} \quad ; \quad |\tau_{M+a}\rangle = \Theta^{-1} B^{-1} |\lambda_{M+a}\rangle e^{(-\rho - \phi_{M+a})}$$

(31)

Denote

$$\tau_{0}^{(0)} \equiv \exp (-\rho) \quad ; \quad \tau_{M+a}^{(0)} \equiv \exp (-\rho - \phi_{M+a})$$

(32)

Accordingly, expanding $\Theta^{-1}$ as below eq.(17) we find

$$\frac{|\tau_{M+a}\rangle}{\tau_{0}^{(0)}} = \left( 1 - \theta^{(-1)} - \ldots \right) |\lambda_{M+a}\rangle$$

(33)

and similarly for $|\tau_0\rangle/\tau_{0}^{(0)}$. We find by comparing with relation (24) that

$$r_{1} = e^{(\alpha_{M+1}, -\alpha_{1}, \ldots, -\alpha_{M+1})} \langle \lambda_{M+1} | E_{\alpha_{1}, \ldots, +\alpha_{M+1}}^{(0)} | \tau_{M+1} \rangle / \tau_{M+1}^{(0)}$$

(34)

$$q_{1} = e^{(-\alpha_{M+1}, -\alpha_{1}, \ldots, -\alpha_{M+1})} \langle \lambda_{0} | E_{-\alpha_{1}, \ldots, -\alpha_{M+1}}^{(1)} | \tau_{0} \rangle / \tau_{0}^{(0)}$$

(35)

$$U_{M+a} = -\partial_{x} \ln \left( \frac{\tau_{0}^{(0)}}{\tau_{M+a}^{(0)}} \right) \quad ; \quad \nu = -\partial_{x} \ln \left( \tau_{0}^{(0)} \right)$$

(36)
The multi-soliton tau functions are defined in terms of the constant group elements $h$ which are the product of exponentials of eigenvectors of the Heisenberg subalgebra elements

$$h = e^{F_1} e^{F_2} \cdots e^{F_n}, \quad [b_N, F_k] = \omega^{(k)}_N F_k, \quad k = 1, 2, \ldots, n.$$  \hspace{1cm} (37)

As seen from eq. (37) for such group elements the dependence of the tau-vectors upon the times $t_N$ can be made quite explicit

$$|\tau_\alpha\rangle = \prod_{k=1}^n \exp(e^{\sum_N \omega^{(k)}_N t_N F_k}) |\lambda_\alpha\rangle$$  \hspace{1cm} (38)

The multi-soliton solutions are conveniently obtained in terms of representations of the “vertex operator” type where the corresponding eigenvectors are nilpotent.

4 The sl(3) Example: Solitons of the Yajima-Oikawa Hierarchy

We apply the above method to the particular case of $sl(3)$ with $M = K = 1$. From eq. (3) the surviving elements of the Heisenberg subalgebra are in this case:

$$b^{(2n+1)} \equiv b_{N=2n+1, a=1} = A^1_{N=1+n, 2} = E_{\alpha_2}^{(n)} + E_{-\alpha_2}^{(n+1)}$$  \hspace{1cm} (39)

$$b^{(2n)} \equiv b_{N=2n, 0} = \sqrt{3}\lambda_1 \cdot H^{(n)} - \frac{\hat{c}}{2\sqrt{3}} \delta_{n,0}$$  \hspace{1cm} (40)

and they satisfy the usual Heisenberg subalgebra $[b^{(k)}, b^{(k')}] = k\delta_{k+k'}$ for both even and odd $k$.

The structure of eigenvectors of Heisenberg subalgebra facilitates construction of multisoliton solutions according to (37) and (38). In the current example we find that the eigenvectors and their corresponding eigenvalues (in notation of (37)) are

$$E_{\tilde{\alpha}_1} = \sqrt{2} \sum_{n \in \mathbb{Z}} \left[ z^{-2n} E_{\alpha_1}^{(n)} - z^{2n-1} E_{\alpha_1 + \alpha_2}^{(n)} \right] ; \quad \omega^{(2n+1)}_{\tilde{\alpha}_1} = z^{2n+1} ; \quad \omega^{(2n)}_{\tilde{\alpha}_1} = \sqrt{3} z^{2n}$$  \hspace{1cm} (41)

$$E_{\tilde{\alpha}_2} = \sum_{n \in \mathbb{Z}} \left[ z^{-2n-1} \left( E_{\alpha_2}^{(n)} - E_{-\alpha_2}^{(n+1)} \right) + z^{2n} \left( \alpha_2 \cdot H^{(n)} - \frac{\hat{c}}{2} \delta_{n,0} \right) \right]$$  \hspace{1cm} (42)

$$\omega^{(2n+1)}_{\tilde{\alpha}_2} = -2z^{2n+1} ; \quad \omega^{(2n)}_{\tilde{\alpha}_2} = 0$$

$$E_{\tilde{\alpha}_1 + \tilde{\alpha}_2} = \sqrt{2} \sum_{n \in \mathbb{Z}} \left[ z^{-2n} E_{\alpha_1}^{(n)} + z^{2n-1} E_{\alpha_1 + \alpha_2}^{(n)} \right] ; \quad \omega^{(2n+1)}_{\tilde{\alpha}_1 + \tilde{\alpha}_2} = -z^{2n+1} ; \quad \omega^{(2n)}_{\tilde{\alpha}_1 + \tilde{\alpha}_2} = \sqrt{3} z^{2n}$$  \hspace{1cm} (43)

We now realize the above eigenvectors by the nilpotent vertex operators. The construction involves the Fubini-Veneziano operators defined in terms of the Heisenberg elements as in eqs. (13)-(14):

$$Q_1(z) \equiv i \sum_{n \in \mathbb{Z}} \frac{b^{(2n+1)} z^{-2n-1}}{2n+1} ; \quad Q_2(z) \equiv q - i p \ln z + i \sum_{n \neq 0} \frac{b^{(2n)} z^{-2n}}{2n}$$  \hspace{1cm} (44)
where the zero mode momentum \( p = b^{(0)} = \sqrt{3} \lambda_1 \cdot H^{(0)} - \dot{c}/2\sqrt{3} \) satisfies \( [q, p] = i \). The step operators from (11)-(13) are then realized, from the algebra point of view, as vertex operators via:

\[
E_{\tilde{\alpha}_1} \leftrightarrow E_{(1, \sqrt{3})} (z) = \sqrt{2} z^{3/2} : \exp \left( iQ_1 (z) + i\sqrt{3} Q_2 (z) \right) : \quad (45)
\]

\[
E_{\tilde{\alpha}_2} \leftrightarrow E_{(-2,0)} (z) = -\frac{1}{2} : \exp (-2iQ_1 (z)) : e^{i\kappa p} \quad (46)
\]

\[
E_{\tilde{\alpha}_1 + \tilde{\alpha}_2} \leftrightarrow E_{(-1, \sqrt{3})} (z) = \sqrt{2} z^{3/2} : \exp \left( -iQ_1 (z) + i\sqrt{3} Q_2 (z) \right) : \quad (47)
\]

and similarly for the negative root step operators, with change of sign of \( i \) in exponentials. A care has to be exercised in applying this correspondence within the setting of the Fock space as seen from expression (42). Similar consideration applies for the products of Klein factor in eq.(46) and from the character of the subsequent (47). The latter gives rise to the factor \( \exp \sigma \) for \( \lambda \) where

\[
\text{Introduce the notation:}
\]

\[
V_{c_i, d_i}(z) \equiv z^{d_i^2/2} : \exp (i c_i Q_1 (z) + i d_i Q_2 (z)) :
\]

It is not difficult to establish the following correlation function:

\[
\left\langle \lambda_\sigma \right| \Psi^{(\text{vac})} V_{c_i, d_i} (z_1) \cdots V_{c_n, d_n} (z_n) \Psi^{(\text{vac})-1} \left| \lambda_\sigma \right\rangle = \delta_{\sum_{j=1}^n d_j, 0} \exp \sum_{j=1}^n \Gamma_{c_j, d_j} (z_j)
\]

\[
\prod_{j=1}^n z_j^{-(\sigma + 2)/2} \prod_{1 \leq i < j \leq n} \left( z_i - z_j \right) \left( z_i + z_j \right)^{d_i d_j/2} \quad (49)
\]

for \( \sigma = 0, 2 \) and with

\[
\Gamma_{c_j, d_j} (z_j) = \sum_{n=0}^\infty c_j t_{2n+1} z_j^{2n+1} + \sum_{n=1}^\infty d_j t_{2n} z_j^{2n} \quad (50)
\]

When substituting \( V_{c_i, d_i} (z_j) \) by \( E_{c_j, d_j} (z_j) \) one encounters extra phases originating from the Klein factor in eq.(44) and from the character of the \( |\lambda_2\rangle \) vacuum as discussed below eq. (17). The latter gives rise to the factor \( \exp \left( (i\pi/2) \sum_{j=1}^n c_j \right) \) for the \( \lambda_2 \) correlation function as verified on several examples.

Recall that for the problem in hand the Lax matrix operator from (1) with \( A \) from (3) and \( E \) from (4) specifies to

\[
L = D - \begin{pmatrix}
0 & q & 0 \\
r & U_2 & 1 \\
0 & \lambda & -U_2
\end{pmatrix} - \nu \dot{c} \quad (51)
\]

where \( \lambda \) is the usual loop parameter. In terms of the tau-vectors we have from (34)-(30) the following \( n \)-soliton representation of the components of the Lax operator

\[
r = \frac{1}{\tau_2 (0)} \langle \lambda_2 | E_{\alpha_1 + \alpha_2}^{(0)} | \tau_2 \rangle = \frac{1}{\tau_2 (0)} \langle \lambda_2 | E_{\alpha_1 + \alpha_2}^{(0)} \Psi^{(\text{vac})} \prod_{j=1}^n \left( 1 + E_{c_j, d_j} (z_j) \right) \Psi^{(\text{vac})-1} | \lambda_2 \rangle \quad (52)
\]

\[
q = \frac{1}{\tau_0 (0)} \langle \lambda_0 | E_{\alpha_1 + \alpha_2}^{(1)} | \tau_0 \rangle = \frac{1}{\tau_0 (0)} \langle \lambda_0 | E_{\alpha_1 + \alpha_2}^{(1)} \Psi^{(\text{vac})} \prod_{j=1}^n \left( 1 + E_{c_j, d_j} (z_j) \right) \Psi^{(\text{vac})-1} | \lambda_0 \rangle \quad (53)
\]

\[
U_2 = -\partial_x \ln \left( \tau_0 (0) / \tau_2 (0) \right) \quad ; \quad \nu = -\partial_x \ln \left( \tau_0 (0) \right) \quad (54)
\]
where
\[
\tau_{\sigma}^{(0)} = \langle \lambda_{\sigma} | \Psi^{(\text{vac})} \prod_{j=1}^{n} \left( 1 + E_{c_j, d_j}(z_j) \right) \Psi^{(\text{vac})}^{-1} | \lambda_{\sigma} \rangle \quad ; \quad \sigma = 0, 2
\] (55)

Using association between the step operators (11)-(13) and the vertex operators (15)-(17) we can rewrite the step operators appearing in (54) and (53) as
\[
E_{\alpha_1 + \alpha_2}^{(0)} = -\frac{1}{2i\pi} \int dz_0 V_{1,\sqrt{3}}(z_0) \quad ; \quad E_{\alpha_1 - \alpha_2}^{(1)} = \frac{1}{2i\pi} \int dz_0 V_{1,-\sqrt{3}}(z_0)
\] (56)

We now calculate the zero-curvature equations (25) \[ D - E - A, \partial_{\alpha_n} - A_n \] = 0 for the first two non-trivial cases of \( n = 2, 3 \). We expand \( A_n = \sum_{i=0}^{n} A_n(i) \) where the index \( i \) in the parenthesis equals grading with respect to \( Q_s = \lambda_2 \cdot H^0 + 2d \). We choose \( A_1(3) = E_{\alpha_2}^{(1)} + E_{-\alpha_2}^{(2)} \) and \( A_2(2) = \sqrt{3} \lambda_1 \cdot H^{(1)} \) in order to ensure truncation of the expansion. This method yields for the first non-trivial case \( n = 2 \) the evolution equations
\[
0 = \partial_t r + \partial_x^2 r + r \partial_x U_2 - q r^2 - U_2^2 r \quad ; \quad 0 = \partial_t U_2 + \partial_x (r q)
\] (57)
\[
0 = \partial_t q - \partial_x^2 q + q \partial_x U_2 + q r^2 + U_2^2 q
\] (58)
where we defined for simplicity \( t = \sqrt{3}t_2 \). The evolution equations for \( t_3 \) are
\[
0 = \partial_{t_3} U_2 - \frac{1}{4} \partial_x^2 U_2 + \frac{1}{2} \partial_x U_3^2 + \frac{3}{4} \partial_x (r \partial_x q - q \partial_x r)
\] (59)
\[
0 = \partial_{t_3} r - \partial_x^3 r - \frac{3}{2} \partial_x r \partial_x U_2 - \frac{3}{4} r \partial_x^2 U_2 + \frac{3}{2} r U_2 \partial_x U_2 + \frac{3}{2} U_2^2 \partial_x r
\] + \[
\frac{3}{2} q r^2 U_2 + \frac{9}{4} r q \partial_x r - \frac{3}{4} r^2 \partial_x q
\] (60)
\[
0 = \partial_{t_3} q - \partial_x^3 q + \frac{3}{2} \partial_x q \partial_x U_2 + \frac{3}{4} q \partial_x^2 U_2 + \frac{3}{2} q U_2 \partial_x U_2 + \frac{3}{2} U_2^2 \partial_x q
\] - \[
\frac{3}{2} q^2 r U_2 + \frac{9}{4} r q \partial_x q - \frac{3}{4} q^2 \partial_x q
\] (61)
These equations follow also from the conventional Sato equations \( \partial_{t_n} \mathcal{L} = [ \mathcal{L}^{(n/2)}_+ , \mathcal{L} ] \) applied to the scalar cKP Lax operator \( \mathcal{L} = (\partial - U_2)(\partial + U_2 - q \partial^{-1} r) \). We note here that the simple reduction of the matrix Lax operator from (21) yields the scalar spectral problem \( \mathcal{L}_{1,\chi} = \lambda \chi \) with the scalar Lax operator \( \mathcal{L}_1 = (\partial + U_2)(\partial - U_2 - r \partial^{-1} q) \). Both scalar Lax operators are related by a conjugation and the Darboux-Bäcklund transformation: \( \mathcal{L}_1 = (\partial + U_2)\mathcal{L}_+^*(\partial + U_2)^{-1} \).

We now present few examples of the soliton solutions (52)-(55) satisfying the above evolution equations.

1) One-soliton solution, \( n = 1 \). With \( h = (1 + E_{-2,0}(z_1)) \) we recover the standard m-KdV one-soliton configuration with \( r = q = 0 \) and
\[
\tau_0^{(0)} = 1 - \frac{1}{2} e^{(-2x z_1 - 2 t_3 z_1^3)} \quad ; \quad \tau_2^{(0)} = 1 + \frac{1}{2} e^{(-2x z_1 - 2 t_3 z_1^3)}
\] (62)

2) Two-soliton solutions, \( n = 2 \). For \( h = (1 + E_{-2,0}(z_1))(1 + E_{1,\sqrt{3}}(z_2)) \) we find \( \tau_0^{(0)} \) and \( \tau_2^{(0)} \) as in (12) but now with \( q \neq 0 \) and equal to
\[
q = -\sqrt{2} z_2 e^{(t_3 z_2 + t_2 z_2^2 + x z_2)} \left( 1 + \frac{1}{2} e^{(-2x z_1 - 2 t_3 z_1^3)} \right) \left( \frac{z_1 + z_2}{z_1 - z_2} \right) / \tau_0^{(0)}
\] (63)
while \( r = 0 \). Similarly, for \( h = (1 + E_{-2,0}(z_1))(1 + E_{1,-\sqrt{3}}(z_2)) \) we find \( r \neq 0 \) but \( q = 0 \).

For \( h = (1 + E_{1,\sqrt{3}}(z_1))(1 + E_{1,-\sqrt{3}}(z_2)) \) we find that both \( q \neq 0 \) and \( r \neq 0 \):

\[
\tau_{\sigma}^{(0)} = 1 + (-1)^{(\sigma/2)} \frac{1}{2} e^{(-2 x z_1 - 2 t_3 z_3^3)} + (-1)^{(\sigma/2)} 2 z_2^{1+\sigma/2} z_2^{2-\sigma/2} e^{(x z_1 + t z_1^3 + t z_2^3 + t z_3^3 + t z_4^3 + t z_5^3)} ; \quad \sigma = 0, 2 \tag{64}
\]

\[
r = \frac{\sqrt{2} z_2 e^{(-t z_2^3 + x z_2 + t z_3^3)}}{\tau_2^{(0)}} ; \quad q = \sqrt{2} z_1 e^{(t z_1^2 + x z_1 + t z_2^3)} \tag{65}
\]

3) Three-soliton solutions, \( n = 3 \). As an example we take here \( h = (1 + E_{-2,0}(z_1))(1 + E_{1,\sqrt{3}}(z_2))(1 + E_{1,-\sqrt{3}}(z_3)) \). We find

\[
\tau_{\sigma}^{(0)} = 1 + (-1)^{(\sigma/2)} \frac{1}{2} e^{(-2 x z_1 - 2 t_3 z_3^3)} + (-1)^{(\sigma/2)} 2 z_2^{1+\sigma/2} z_2^{2-\sigma/2} e^{(t z_1^3 + t z_2^3 + t z_2^3 + t z_3^3)} ; \quad \sigma = 0, 2 \tag{66}
\]

\[
r = \sqrt{2} z_3 e^{(-t z_3^3 + x z_3 + t z_3^3)} \left(1 + \frac{1}{2} e^{(-2 x z_1 - 2 t_3 z_3^3)} \frac{z_1 + z_3}{z_1 - z_3} \right) / \tau_2^{(0)} \tag{67}
\]

\[
q = \sqrt{2} z_2 e^{(t z_2^3 + x z_2 + t z_3^3)} \left(1 - \frac{1}{2} e^{(-2 x z_1 - 2 t_3 z_3^3)} \frac{z_1 + z_2}{z_1 - z_2} \right) / \tau_0^{(0)} \tag{68}
\]

In the above examples \( U_2 \) and \( \nu \) can be obtained from [54]. We note that we only kept the explicit time dependence on times \( t_n \) with \( n \leq 3 \), which was enough to verify the evolution equations [57]-[81].

The novel feature of the above soliton solutions is that they mix exponentials exp \((\sum_{n=1}^{\infty} t_n z_n^n)\) which represent a typical time dependence for the KP solutions with pure KdV-like time dependence of the type exp \((\sum_{n=0}^{\infty} t_{2n} z_j^{2n+1})\) involving only odd times.

The exception is provided by the pure KP type of solution in eq. [64]-[85], which can also be obtained as a Wronskian arising from the Darboux-Bäcklund transformations.

5 Conclusions

We presented here new soliton solutions for the cKP hierarchy. They exhibit a nontrivial mixing of KP times \( t_n \) with all indices \( n \)’s and KdV-like times \( t_{2n-1} \) with only odd indices. From the point of view of intermediate gradation, such a mixture is indeed very natural. In fact, we are able to obtain pure KdV solutions, pure KP solutions (meaning all times) and the arbitrary mixtures thereof, just by varying the constant group elements \( h \) of the dressing orbit.

We were not able to obtain these solutions from the conventional Darboux-Bäcklund method involving Wronskian representation. Recall, that such Wronskian is given in terms of the eigenfunctions \( f_i \) satisfying \( \partial_n f_i = \partial_x^n f_i \) for all \( n \). This type of time dependence fits
only a subclass of our soliton solutions, namely those which are of the pure KP type (meaning
that all \( t_n \)’s with all \( n \)’s are present without the nontrivial mixture with KdV times). Based
on studying many examples of these solutions, we believe that the Wronskian method works
equally well in these cases.

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