Existence of weak solutions for inhomogeneous generalized Navier-Stokes equations

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Abstract We prove existence of weak solutions for the fully inhomogeneous, stationary generalized Navier-Stokes equations for shear-thinning fluids. Our proof is based on the theory of pseudomonotone operators and the Lipschitz truncation method, whose application is presented as a general result. Our approach requires a smallness and a regularity assumption on the data; we show that this is inevitable in the framework of pseudomonotone operators.

Keywords Generalized Newtonian fluid, pseudomonotone operator, existence of weak solutions, inhomogeneous problem.

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1. Introduction

Motivated by the equations describing the steady motion of generalized Newtonian fluids we study the following fully inhomogeneous system

\[- \text{div} \mathcal{S}(Dv) + v \cdot \nabla v + \nabla \pi = f, \]
\[\text{div} v = g_1, \]
\[v|_{\partial \Omega} = g_2.\]  

(1.1)

In this setting, \(\mathcal{S}\) is an extra stress tensor with \(p\)-\(\delta\)-structure, \(v\) is the velocity field with its symmetric gradient \(Dv\), \(\pi\) is the pressure, \(f\) is the external force and \(g_1\) and \(g_2\) are data on a sufficiently regular bounded domain \(\Omega \subset \mathbb{R}^d\) of dimension \(d \in \{2, 3\}\).

Since (1.1) leads to a pseudomonotone and coercive operator in the homogeneous case \(g_1 = 0, g_2 = 0\) and \(p > \frac{3d}{d+2}\) (cf. [11]) and in the shear-thickening case \(p > 2\) (cf. [13]), the existence of weak solutions \((v, \pi)\) to (1.1) follows directly from the theory of pseudomonotone operators in these cases. This approach can be adapted to the situation of homogeneous data and very low values of \(p\): if \(g_1 = 0, g_2 = 0\) and \(p \in (\frac{2d}{d+2}, \frac{3d}{d+2}]\), one can construct approximate solutions by the theory of pseudomonotone operators and prove their convergence with the Lipschitz truncation method (cf. [3], [6]). In the case \(p = 2\), we have to deal with the fully inhomogeneous steady Navier-Stokes equations which are studied intensively (cf. [9]) and where the existence of solutions is known under appropriate smallness conditions. In the shear-thinning, inhomogeneous case, i.e. if \(p \in (1, 2)\) and the data \(g_1, g_2\) do not vanish, the coercivity of the elliptic term is weaker than the growth of the convective term, i.e. we are in the supercritical
case. This situation is treated in [1], [16] for \( g_1 = 0 \). In [16] even the case of electrorheological fluids is covered. The result there is based on a nice smallness argument ([16, Lemma 3.2]), which is applied to estimate the convective term. Since we did not understand the application of this lemma in detail, we give a different proof of local coercivity here. Our main result shows the existence of weak solutions of the fully inhomogeneous problem (1.1) in the shear-thinning case under appropriate smallness conditions involving higher regularity of the data.

The paper is organised as follows: by representing the inhomogeneous data by a fixed function \( g \) (Subsection 2.2, [11]), (1.1) turns into a homogeneous problem. We investigate the newly formed elliptic and convective terms in Subsections 2.4 and 2.5. Then we conclude properties and local coercivity of the whole system and prove existence of solutions (Subsection 3.2). In the case \( p \in (\frac{2d}{d+2} - \frac{3d}{d+2}) \) we use the Lipschitz truncation method in order to establish convergence of approximate solutions. This step is presented as an abstract statement, Theorem 2.32, which should fit to more general situations. In contrast to [16], we had to require additional regularity of the data in our proof of local coercivity. We discuss this issue in Subsection 3.3 and prove that the additional regularity assumption is necessary in the framework of pseudomonotone operator theory.

The results presented here are based on the thesis [10] of the first author.

2. Preliminaries

2.1. Notation

We work on a bounded Lipschitz domain \( \Omega \subset \mathbb{R}^d \), \( d \in \{2, 3\} \), with possesses an exterior normal \( \nu \). Points and scalar-valued quantities are written in normal letters whereas vector- and matrix-valued functions, variables and operators are denoted in bold letters. The space of symmetric quadratic matrices is denoted as \( \mathbb{R}^{d \times d}_{\text{sym}} \).

We use standard Lebesgue measure and integration theory. For a ball \( B \), we denote the ball with the same center and the double radius by \( 2B \). The characteristic function of a set \( S \subset \mathbb{R}^d \) is called \( \chi_S \).

We use standard notation for Lebesgue and Sobolev spaces. Due to [7], there exists a well-defined, surjective trace operator \( W^{1,p}(\Omega) \to W^{1-\frac{1}{p}}(\partial \Omega) \) that assigns boundary values to a Sobolev function. We denote by \( L^p_0(\Omega) \) the subspace of \( L^p(\Omega) \) of functions with mean value zero and by \( V_p \) the subspace of \( W^{1,p}_0(\Omega) \) of vector fields with zero boundary values and zero divergence. For a vector-valued function \( v \in W^{1,p}(\Omega) \), the definition of the (weak) gradient field follows the convention \( (\nabla v)_{ij} = \partial_j v_i \) and the symmetric gradient is defined as \( Dv := \frac{1}{2}(\nabla v + \nabla v^\top) \). On \( W^{1,p}_0(\Omega) \) and on \( V_p \), we may work with the symmetric gradient norm \( \|D\|_p \), thanks to Poincaré’s and Korn’s inequalities.

The dual of some Banach space \( X \) is denoted as \( X^* \) and \( \langle \cdot, \cdot \rangle \) denotes their canonical dual pairing. For an exponent \( p \in [1, \infty] \), we define its conjugate exponent \( p' \in [1, \infty] \) via \( \frac{1}{p} + \frac{1}{p'} = 1 \) and use the duality \( L^p(\Omega) = L^{p'}(\Omega)^* \) for
Finally, we define the critical Sobolev exponent $p^* := \frac{pd}{d-p} \in (p, \infty)$ for $p < d$.

2.2. The divergence equation

In order to fulfil the boundary and divergence conditions in (1.1), we follow the usual ansatz $v = u + g$, where $u \in V_p$ and $g \in W^{1,p}(\Omega)$ fulfills the boundary and divergence data, i.e. the vector field $g \in W^{1,p}(\Omega)$ solves

$$
\begin{align*}
\text{div } g &= g_1, \\
g_{\partial \Omega} &= g_2.
\end{align*}
$$

For the corresponding homogeneous system, we have the fundamental result due to Bogovskiǐ (cf. [2], [3], [9]):

**Theorem 2.2** (Bogovskiǐ operator). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with $d \geq 2$ and $p \in (1, \infty)$. Then there exists a linear and bounded operator $B: L^p_0(\Omega) \to W^{1,p}_0(\Omega)$ and a constant $c_{Bog} = c(\Omega, p)$ such that

$$
\begin{align*}
\text{div } B f &= f, \\
\|B f\|_{1,p} &\leq c_{Bog} \|f\|_p
\end{align*}
$$

for all $f \in L^p_0(\Omega)$.

For the inhomogeneous system, we combine Bogovskiǐ’s Theorem and the fact that $W^{1-\frac{1}{p},p}(\partial \Omega)$ is precisely the space of boundary values of $W^{1,p}(\Omega)$-functions:

**Lemma 2.3** (The inhomogeneous divergence equation). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with $d \geq 2$ and $p \in (1, \infty)$. Suppose $g_1 \in L^p(\Omega)$ and $g_2 \in W^{1-\frac{1}{p},p}(\partial \Omega)$ satisfy $\int_\Omega g_1 \, dx = \int_{\partial \Omega} g_2 \cdot \nu \, do$. Then there exists a solution $g \in W^{1,p}(\Omega)$ of problem (2.1) that satisfies

$$
\|g\|_{1,p} \leq c_{\text{lift}} (1 + c_{Bog}) \|g_2\|_{1-\frac{1}{p},p} + c_{Bog} \|g_1\|_p
$$

with constants $c_{\text{lift}}$ and $c_{Bog}$ from the trace lifting and the Bogovskiǐ operator.

**Proof.** Due to [7], there exists a trace lifting $\hat{g} \in W^{1,p}(\Omega)$ of the boundary values $g_2$. By integration by parts, we see that the function $g_1 - \text{div } \hat{g}$ has mean value zero. Thus, we may apply the Bogovskiǐ operator and directly obtain that $g := \hat{g} + B(g_1 - \text{div } \hat{g}) \in W^{1,p}(\Omega)$ solves (2.1). The estimate of $g$ follows from the boundedness of the trace lifting and the Bogovskiǐ operator. \qed

2.3. Local coercivity

We will work with the following notion of local coercivity:

**Definition 2.4** (local coercivity). Let $X$ be a Banach space. An operator $A: X \to X^*$ is called locally coercive with radius $R$ if there exists a positive real number $R$ such that

$$
\langle Ax, x \rangle \geq 0
$$

holds for all $x \in X$ with $\|x\|_X = R$. 

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Local coercivity is precisely the condition that allows to apply Brouwer’s fixed point theorem in order to obtain approximate solutions in the proof of Brézis’ theorem about pseudomonotone operators [18, Thm. 27.A]. Therefore, we get a generalized version of Brézis’ theorem that can be proved along the lines of the standard version. It can also be regarded as a special case of the existence theorem of Hess and Kato [18, Thm. 27.B].

**Theorem 2.5** (Existence theorem for pseudomonotone operators). Let $X$ be a reflexive and separable Banach space and $A: X \to X^*$ be a pseudomonotone, demicontinuous and bounded operator that is locally coercive with radius $R$. Then there exists a solution $u \in X$ of the problem

$$Au = 0$$

that satisfies $\|u\|_{X} \leq R$.

2.4. The extra stress tensor and its induced operator

The stress tensor describes the mechanical properties of the fluid in dependence on the strain rate $Dv$. In Newtonian fluid dynamics, the viscosity is a constant $\kappa \in \mathbb{R}$ which induces the linear operator $-\text{div} S(Dv) = -\kappa \Delta v$ describing the viscous part of the stress tensor. The general situation of non-Newtonian fluids can be modeled in various ways (cf. [15], [4]). Here, we consider the class of fluids with extra stress tensor having $p$-$\delta$-structure. This class includes and generalizes power law fluids, where the constitutive relation is given by

$$S(Dv) = \mu_0 Dv + \mu (\delta + |Dv|)^{p-2} Dv$$

with material constants $p \in (1, \infty)$, $\mu_0, \mu, \delta \geq 0$ (cf. [14]).

**Definition 2.6** (extra stress tensor). An operator $S: \mathbb{R}^{d \times d}_{\text{sym}} \to \mathbb{R}^{d \times d}_{\text{sym}}$ is called an extra stress tensor with $p$-$\delta$-structure if it is continuous, satisfies $S(0) = 0$ and if there exist constants $p \in (1, \infty)$, $\delta \geq 0$ and $C_1(S), C_2(S) > 0$ such that

$$\langle S(A) - S(B), A - B \rangle \geq C_1(S) (\delta + |B| + |A - B|)^{p-2} |A - B|^2,$$

$$|S(A) - S(B)| \leq C_2(S) (\delta + |B| + |A - B|)^{p-2} |A - B|$$

(2.7)

holds for all $A, B \in \mathbb{R}^{d \times d}_{\text{sym}}$. The constants $C_1(S), C_2(S)$ and $p$ are called the characteristics of $S$.

**Lemma 2.8** ([14]). Let $S$ be an extra stress tensor with $p$-$\delta$-structure. Then, it holds

$$\langle S(Dv) - S(Dw), Dv - Dw \rangle \geq C_3(S) \int_{\Omega} \int_0^{Dv - Dw} (|Dw| + \delta + s)^{p-2} s ds dx$$

for $v, w \in W^{1,p}(\Omega)$ with a constant $C_3(S)$ that only depends on the characteristics of $S$. 

4
Since we represented the inhomogeneous data in (1.1) by a fixed function \( g \) and since we want to solve (1.1) by the ansatz \( v = u + g \) with \( u \in V \), we shall work with a shifted version of the viscous stress tensor. Therefore, we define the induced operator \( S : W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega)^* \) via
\[
\langle S(v), \varphi \rangle := \langle S(Dv + Dg), D\varphi \rangle
\]
for \( v, \varphi \in W^{1,p}_0(\Omega) \).

**Lemma 2.10** (Properties of \( S \)). Let \( S \) be an extra stress tensor with \( p\delta \)-structure, \( p \in (1, 2] \) and \( g \in W^{1,p}(\Omega) \). Then the induced operator \( S \) defined in (2.9) is well-defined, bounded and continuous.

**Proof.** Using (2.7) with \( A = Dw \) and \( B = 0 \), we obtain
\[
|S(Dw)|^p \leq C_2(\|Dw\| + \delta)^{p-2} |Dw|^p \leq C_2(\|Dw\| + \delta)^p
\]
and consequently
\[
\|S(Dw)\|_p \leq C_2(\|Dw\| + \delta)^{p-1}
\]
for any \( w \in W^{1,p}(\Omega) \). From this, we deduce that \( S : W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega)^* \) is well-defined and bounded.

In order to prove continuity, let \( v^n \to v \in W^{1,p}_0(\Omega) \) be a convergent sequence. Then, by the Hölder inequality and by (2.7) we get
\[
\|S(v^n) - S(v)\|_{W^{1,p}_0(\Omega)^*} \leq \|S(Dv^n + Dg) - S(Dv + Dg)\|_{p'}
\]
\[
\leq C_2(\|Dv^n + Dg| + |Dv^n - Dv|)^{p-2} |Dv^n - Dv|_{p'}
\]
\[
\leq C_2(\|Dv^n - Dv\|^p_{p'}) \overset{n \to \infty}{\longrightarrow} 0.
\]

Our next goal is to describe coercivity properties of the operator \( S \). For the proof of a good lower bound of \( S \), we prove an auxiliary algebraic result.

**Lemma 2.12.** Let \( a, t \geq 0 \) and \( p \in (1, 2] \). Then it holds
\[
\int_0^t (a + s)^{p-2} s^p ds \geq \frac{1}{p} t^p - ta^{p-1}.
\]

**Proof.** The statement becomes trivial if \( a = 0 \), so we may assume \( a > 0 \). For all \( s \geq 0 \), it holds \( \frac{a}{(a+s)^{2-p}} \leq \frac{a}{a^{2-p}} = a^{p-1} \). We estimate
\[
\frac{s}{(a+s)^{2-p}} = (a+s)^{p-1} - \frac{a}{(a+s)^{2-p}} \geq (a+s)^{p-1} - a^{p-1} \geq s^{p-1} - a^{p-1}
\]
and by integration we obtain the result. 

With this tool, we are able to prove a lower bound for \( S \):
Lemma 2.13 (Lower bound for $S$). For a given extra stress tensor $S$ with $p$-$\delta$-structure, $p \in (1, 2]$, and a function $g \in W^{1,p}(\Omega)$, the induced operator $S: W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega)^*$, defined in (2.9), satisfies the lower bound
\[
\langle S(v), v \rangle \geq C_1(S) \|Dv\|_p^p - \left( C_2(S) + C_3(S) \right) \|Dg\|_p + \delta \|Dv\|_p^{p-1} \|Dv\|_p
\]
for all $v \in W^{1,p}_0(\Omega)$.

Proof. We apply Lemma 2.8 with $v = v + g$ and $w = g$ and Lemma 2.12 to estimate
\[
\langle S(Dv + Dg) - S(Dg), Dv \rangle \geq C_3(S) \int_\Omega \int_0^{|Dv|} (|Dg| + \delta + s)^{p-2} s \, ds \, dx
\]
\[
\geq C_3(S) \int_\Omega \frac{1}{p} |Dv|^p - |Dv| (|Dg| + \delta)^{p-1} \, dx.
\]
This, the Hölder inequality and (2.11) with $w = g$
\[
\langle S(v), v \rangle = \langle S(Dv + Dg) - S(Dg), Dv \rangle + \langle S(Dg), Dv \rangle
\]
\[
\geq \frac{C_1(S)}{p} \|Dv\|_p^p - \left[ C_3(S) \|(|Dg| + \delta)^{p-1}\|_{p'} + \|S(Dg)\|_{p'} \right] \|Dv\|_p
\]
\[
\geq \frac{C_1(S)}{p} \|Dv\|_p^p - \left( C_2(S) + C_3(S) \right) \|Dg\|_p + \delta \|Dv\|_p^{p-1} \|Dv\|_p,
\]
which is the assertion. \qed

In the treatment of the inhomogeneous problem (1.1), we will have to deal with the shifted extra stress tensor $A \mapsto S(A + G)$ for some constant symmetric matrix $G$. In order to get a precise description of the growth behavior of this mapping, we introduce the notion of locally uniform monotonicity:

Definition 2.14 (Locally uniform monotonicity). Let $X$ be a reflexive Banach space and $A: X \to X^*$ an operator. The operator $A$ is called locally uniformly monotone on $X$ if for every $y \in X$ there exists a strictly monotonically increasing function $\rho_y: [0, \infty) \to [0, \infty)$ with $\rho_y(0) = 0$ such that for all $x \in X$ holds
\[
\langle Ax - Ay, x - y \rangle \geq \rho_y(\|x - y\|_X).
\]
(2.15)

By the lower bound (2.7), we obtain that (possibly shifted) extra stress tensors are locally uniformly monotone.

Lemma 2.16. Let $S: \mathbb{R}^{d \times d}_{sym} \to \mathbb{R}^{d \times d}_{sym}$ be an extra stress tensor with $p$-$\delta$-structure and $G \in \mathbb{R}^{d \times d}_{sym}$ be a symmetric matrix. Then the shifted extra stress tensor $A \mapsto S(A + G)$ is a locally uniformly monotone operator on $\mathbb{R}^{d \times d}_{sym}$.

Proof. By (2.7), we obtain for any $A, B \in \mathbb{R}^{d \times d}_{sym}$
\[
\langle S(A + G) - S(B + G), A - B \rangle \geq C_1(S)(\delta + |B + G| + |A - B|)\|A - B\|^2.
\]
For any $\nu$, the function $\rho_{\nu}(t) := C_{1}(\mathcal{S})(\delta + |\nu + G| + t)^{-\frac{3}{2}}t^{2}$ is non-negative, satisfies $\rho_{\nu}(0) = 0$, and it is strictly monotonically increasing since for its derivative it holds

$$\rho'_{\nu}(t) = C_{1}(\mathcal{S})(\delta + |\nu + G| + t)^{-\frac{3}{2}}t^{2}$$

for all $t > 0$. Therefore, it fulfills the requirements from Definition 2.14.

2.5. Properties of the convective term

Since we fixed a function $\nu$ that expresses the inhomogeneous data in (1.1), we shall work with a "shifted" version of the convective term $\langle (\nu + \nu) \cdot \nabla (\nu + \nu), \varphi \rangle$ that is integrable and thus well-defined even for $p > \frac{2d}{d+2}$ and sufficiently regular $\varphi$ and $\nu$. Therefore, we set

$$s = s(p) := \max\left\{ p, \left( \frac{p^*}{2} \right)^* \right\} = \begin{cases} p & \text{if } p > \frac{2d}{d+2}, \\ \left( \frac{p^*}{2} \right)^* & \text{if } p \leq \frac{2d}{d+2} \end{cases}$$

(2.17)

for $p \in \left( \frac{2d}{d+2}, 2 \right)$ and define the convective term $T : V_p \to W_{0}^{1,s}(\Omega)^{\ast}$ via

$$\langle T(\nu), \varphi \rangle := -\langle (\nu + \nu) \otimes (\nu + \nu), D\varphi \rangle - \langle (\nu g)(\nu + \nu), \varphi \rangle$$

(2.18)

for $\nu \in V_p$ and $\varphi \in W_{0}^{1,s}(\Omega)^{\ast}$.

Lemma 2.19 (Properties of the convective term). For $p \in \left( \frac{2d}{d+2}, 2 \right)$ let $s$ be defined in (2.17) and let $q \in \mathbb{R}$ satisfy $q \geq s$ and $q > \left( \frac{p^*}{2} \right)^*$. Then, for any given function $\nu \in W^{1,s}(\Omega)$, the operator $T$ defined in (2.18) is formally equivalent to $\langle (\nu + \nu) \cdot \nabla (\nu + \nu), \varphi \rangle$. It is well-defined and bounded from $V_p$ to $W_{0}^{1,s}(\Omega)^{\ast}$ and also from $V_p$ to $W_{0}^{1,q}(\Omega)^{\ast}$. The operator $T$ is continuous from $V_p$ to $W_{0}^{1,s}(\Omega)^{\ast}$ and strongly continuous from $V_p$ to $W_{0}^{1,q}(\Omega)^{\ast}$. It fulfills the estimate

$$|\langle T(\nu), \nu \rangle| \leq c_{\text{Sob}} \frac{\nu_{\text{Korn}}}{2}(\|Du\|_{s} + \frac{1}{2}\|\nu g\|_{s}) \|Du\|_{p}^{2} + c_{\text{Sob}} (\|g\|_{1,s} + c_{\text{Korn}} \|\nu g\|_{s} \|g\|_{1,s}) \|Du\|_{p}$$

(2.20)

for all $\nu \in V_q$, where $c_{\text{Sob}}$ are Sobolev embedding constants and $c_{\text{Korn}}$ is the constant in the Korn inequality for $\Omega$.

Proof. The formal equivalence follows from a straightforward computation with integration by parts. We abbreviate $q_{1} := \text{div } g \in L^{s}(\Omega)$ and use the continuous Sobolev embeddings $W^{1,s}(\Omega) \hookrightarrow W^{1,\nu}(\Omega) \hookrightarrow L^{\nu}(\Omega)$. The definition of $s$ implies $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} \leq 1$, so both well-definedness of $T(\nu) \in W_{0}^{1,s}(\Omega)^{\ast}$ for $\nu \in V_p$ and boundedness follow by the H"{o}lder inequality.

In view of the continuous embedding $W_{0}^{1,s}(\Omega)^{\ast} \hookrightarrow W_{0}^{1,q}(\Omega)^{\ast}$, we immediately obtain well-definedness and boundedness if $T$ is considered as an operator from $V_p$ to $W_{0}^{1,q}(\Omega)^{\ast}$.

Since $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} < 1$, there is some $\tau < p^*$ such that $\frac{1}{2} + \frac{1}{\tau} + \frac{1}{s} = 1$. Let $\nu^{n} \rightharpoonup \nu$ be a weakly convergent sequence. The Sobolev embedding
\[ W^{1,p}(\Omega) \hookrightarrow L^r(\Omega) \] is compact, so \( u^n \to u \in L^r(\Omega) \) converges strongly. Thus, we estimate

\[
\sup_{\|D\varphi\|_q \leq 1} \left| (u^n + g) \otimes (u^n + g) - (u + g) \otimes (u + g), D\varphi \right|
\]

\[
= \sup_{\|D\varphi\|_q \leq 1} \left| u^n \otimes (u^n - u) + (u^n - u) \otimes u + g \otimes (u^n - u) + (u^n - u) \otimes g, D\varphi \right|
\]

\[
\leq \|u^n\|_{p^*}, \|u^n - u\|_\tau + \|u^n - u\|_\tau \|u\|_{p^*} + 2 \|g\|_{p^*} \|u^n - u\|_\tau \xrightarrow{n \to \infty} 0.
\]

Similarly, we obtain

\[
\sup_{\|D\varphi\|_q \leq 1} \left| \langle g_1(u^n - u), \varphi \rangle \right| \leq C \|g_1\|_s \|u^n - u\|_\tau \xrightarrow{n \to \infty} 0.
\]

Thus, we proved \( T(u^n) \to T(u) \) in \( W^{1,q}_0(\Omega)^* \), i.e. \( T: V_p \to W^{1,q}_0(\Omega)^* \) is strongly continuous.

Analogously, we prove continuity for \( T: V_p \to W^{1,s}_0(\Omega)^* \) using \( u^n \to u \in V_p \) and the continuous embedding \( W^{1,p}(\Omega) \hookrightarrow L^{p'}(\Omega) \).

For the bound (2.20) of \( T \), we use

\[
\langle u \otimes u, Du \rangle = 0 \tag{2.21}
\]

which follows by integration by parts, since \( u \) has zero divergence and zero boundary values. In the same way, we see

\[
\langle u \otimes g, \nabla u \rangle = -\langle u \otimes u, Dg \rangle \tag{2.22}
\]

and

\[
\langle u \otimes g, \nabla u^T \rangle = -\frac{1}{T} \langle g_1 u, u \rangle. \tag{2.23}
\]

Using (2.21), (2.22) and (2.23) in the definition of \( T \), we obtain the following expression for the convective term:

\[
\langle T(u), u \rangle = \langle u \otimes u, Du \rangle + \langle u \otimes g, \nabla u \rangle + \langle u \otimes g, \nabla u^T \rangle + \langle g \otimes g, Du \rangle
\]

\[
+ \langle g_1 u, u \rangle + \langle g_1 g, u \rangle
\]

\[
= -\langle u \otimes u, Dg \rangle + \frac{1}{2} \langle g_1 u, u \rangle + \langle g \otimes g, Du \rangle + \langle g_1 g, u \rangle. \tag{2.24}
\]

In order to estimate this expression, we use the Sobolev embedding \( W^{1,s}(\Omega) \to L^{2p'}(\Omega) \). In fact, in the case \( s \geq d \) this follows directly. If \( \frac{3d}{d+2} < p < 2 \) and \( s < d \), we have \( s^* = p^* > 2p' \), due to a straightforward computation. If \( \frac{2d}{d+1} < p \leq \frac{3d}{d+2} \) and \( s < d \), we get \( s^* \geq \left( \frac{d}{2} \right)^* = \frac{2d}{3d-2d+p} \geq 2p' \). Finally, if \( \frac{2d}{d+2} < p \leq \frac{2d}{d+1} \), then it holds \( s \geq \left( \frac{d}{2} \right)^* \geq d \). Applying the Hölder and the Korn inequality and the embeddings \( W^{1,s}(\Omega) \to L^{2p'}(\Omega) \) and \( W^{1,p}(\Omega) \to L^{p'}(\Omega) \) to (2.24), the claimed estimate follows.
2.6. Lipschitz truncation

In case of a small growth parameter $p$, this means $p \in (\frac{d}{d+2}, \frac{d}{d+3})$, a function $v \in W^{1,p}(\Omega)$ does not have enough integrability to be chosen as a test function in operators like $(v \otimes v, D\varphi)$. Hence, we use sufficiently smooth approximations of the test functions in the limit process of the existence proof, which are given by the Lipschitz truncation method. The existence of Lipschitz truncations is guaranteed by the following result proved in [6], [5], [14]:

**Theorem 2.25** (Lipschitz truncation). Let $\Omega$ be a bounded domain with Lipschitz continuous boundary, let $p \in (1, \infty)$ and let $(v^n)_{n \in \mathbb{N}} \subset W^{1,p}_0(\Omega)$ be a sequence such that $v^n \rightharpoonup 0$ weakly.

Then, for all $j, n \in \mathbb{N}$, there exists a function $v^n_j \in W^{1,\infty}_0(\Omega)$ and a number $\lambda_j^n \in [2^{2j}, 2^{2j+1}]$ such that

\[
\lim_{n \to \infty} \left( \sup_{j \in \mathbb{N}} \|v^n_j\|_\infty \right) = 0,
\]

\[
\|\nabla v^n_j\|_\infty \leq c \lambda_j^n,
\]

\[
\lim_{n \to \infty} \sup_{j \in \mathbb{N}} \left( \lambda_j^n \right)^p \left| \{v^n_j \neq v^n\} \right| \leq c 2^{-j},
\]

\[
\lim_{n \to \infty} \sup_{j \in \mathbb{N}} \left\| \nabla v^n_j \chi(v^n_j \neq v^n) \right\|_p \leq c 2^{-j}
\]

holds with a uniform constant $c = c(d, p, \Omega)$.

Moreover, for fixed $j \in \mathbb{N}$ and $r \in [1, \infty)$, we have

\[
\nabla v^n_j \to 0 \quad \text{in} \quad L^r(\Omega),
\]

\[
\nabla v^n_j \rightharpoonup 0 \quad \text{in} \quad L^\infty(\Omega)
\]

as $n \to \infty$.

The following lemma shows how Lipschitz truncation can be used to get a connection between weak and almost everywhere convergence. Statement and proof are close to [9] Lemma 2.6, only the assumptions on the operator $S$ have been reduced for the reasons discussed in the previous Subsection 2.4

**Lemma 2.28** (Almost everywhere convergence for the Lipschitz truncation method). Let $\Omega$ be a bounded domain, $p \in (1, \infty)$, $(u^n)_{n \in \mathbb{N}} \subset W^{1,p}(\Omega)$ be a weakly convergent sequence with limit $u \in W^{1,p}(\Omega)$. Let $A: \mathbb{R}^{d \times d}_{\text{sym}} \to \mathbb{R}^{d \times d}$ be a locally uniformly monotone operator on $\mathbb{R}^{d \times d}_{\text{sym}}$ such that the induced operator $w \mapsto A(Dw)$ is well-defined and bounded in $W^{1,p}(\Omega) \to L^p(\Omega)$.

Now let $B$ be a ball with $2B \subset \subset \Omega$ and $\xi \in C^\infty_0(\Omega)$ be a cutoff function such that $\chi_B \leq \xi \leq \chi_{2B}$. We set $v^n := \xi(u^n - u)$ and let $v^n_j$ be the Lipschitz truncation of $v^n$ with respect to the domain $2B$ as described in Theorem 2.25.

If we have

\[
\lim_{n \to \infty} \left| \left(A(Du^n) - A(Du), Dv^n_j \right) \right| \leq \delta_j
\]

for all $j \in \mathbb{N}$ and some sequence $(\delta_j)_{j \in \mathbb{N}}$ with $\lim_{j \to \infty} \delta_j = 0$, then a subsequence of $Du^n$ converges to $Du$ almost everywhere in $B$. 


Remark 2.30. In Lemma 2.16 and lemma 2.10 we have seen that for an extra stress tensor $S: \mathbb{R}^{d \times d}_{sym} \to \mathbb{R}^{d \times d}_{sym}$ with $p$-$\delta$-structure and a given vector field $g \in W^{1,p}(\Omega)$ the operator $A(B) := S(B + Dg)$ fulfils the requirements in Lemma 2.28.

Proof of Lemma 2.28. Let $\theta \in (0, 1)$. Making use of the properties of $A$, we obtain strong convergence $[(A(Du^n) - A(Du)) \cdot (Du^n - Du)]^\theta \to 0$ in $L^1(B)$ as $n \to \infty$ along the lines of Lemma 2.6. We switch to a subsequence that converges almost everywhere. By the definition of locally uniform monotonicity, there exists a strictly monotonically increasing function $\rho_\varepsilon: [0, \infty) \to [0, \infty)$ with

$$\langle A(Du^n(x)) - A(Du(x)) \rangle \cdot (Du^n(x) - Du(x)) \geq \rho_\varepsilon(|Du^n(x) - Du(x)|)$$

for all $n \in \mathbb{N}$ and almost every $x \in B$ ($\rho_\varepsilon$ depends on $Du(x)$). Utilizing the almost everywhere convergence of the left-hand side and the non-negativity of the right-hand side, we obtain a subsequence that fulfils $\rho_\varepsilon(|Du^n(x) - Du(x)|) \to 0$ almost everywhere. Thus, it holds $Du^n(x) \to Du(x)$ as $n \to \infty$ for this subsequence and almost every $x \in B$.

By applying a covering argument and taking the diagonal sequence we obtain a global version of Lemma 2.28 (cf. [14, Cor. 3.32]):

Corollary 2.31. Assume that the assumptions of Lemma 2.28 are fulfilled for all balls $B$ with $2B \subseteq \Omega$ (with sequences $\delta_i$ that may depend on the ball $B$). Then $Du^n$ converges to $Du$ almost everywhere on $\Omega$ for a suitable subsequence.

Using the almost everywhere convergence established in Corollary 2.31 we may prove a general statement about the limit process with the Lipschitz truncation method in existence proofs:

Theorem 2.32 (Identification of limits using the Lipschitz truncation method). Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain, $p \in (1, \infty)$ and $g \in W^{1,p}(\Omega)$ be given. Let $A: \mathbb{R}^{d \times d}_{sym} \to \mathbb{R}^{d \times d}_{sym}$ be a continuous, locally uniformly monotone operator on $\mathbb{R}^{d \times d}_{sym}$ such that the induced operator $w \mapsto A(Dw)$ is well-defined and bounded in $W^{1,p}(\Omega) \to L^p(\Omega)$. Let there be an operator $B: V_p \to V^*_s$ for some $s \in [p, \infty)$ and a space $X$ such that $X \hookrightarrow V_p$ embeds continuously and such that $B$ is well-defined as an operator $X \hookrightarrow X^*$. Assume we have a sequence of operators $A_n: X \to X^*$ and solutions $u^n \in X$ to

$$\langle A(Du^n), D\varphi \rangle + \langle B(u^n), \varphi \rangle + \langle A_n(u^n), \varphi \rangle = 0 \quad (2.33)$$

with test functions $\varphi \in X$.

In addition, assume that for some $r \in (1, \infty)$ the embedding $V_r \hookrightarrow X$ is continuous and dense, that $B$ is strongly continuous as an operator $V_p \to W^{1,r}_0(\Omega)^*$ and that we have convergences

$$u^n \rightharpoonup u \quad \text{weakly in } V_p,$$

$$A_n(u^n) \to 0 \quad \text{strongly in } W^{1,r}_0(\Omega)^* \quad (2.34)$$
as \( n \to \infty \).

Then \( u \) is a solution of the limit equation

\[
\langle A(Du), D\varphi \rangle + \langle B(u), \varphi \rangle = 0
\]  

(2.35)

for all \( \varphi \in V_s \).

**Remark 2.36.** The operator \( A \) represents a (possibly shifted) extra stress tensor (cf. Remark 2.30) and \( B \) may be chosen as the convective term. Typical choices for the space \( X \) are \( X = V_q \) or \( X = V_p \cap L^q(\Omega) \) with coercive operators

\[
\langle A_n(v), \varphi \rangle = \langle |Dv|^{q-2} Dv, D\varphi \rangle \quad \text{and} \quad \langle A_n(v), \varphi \rangle = \langle |v|^{p-2} v, \varphi \rangle
\]

respectively.

The inclusions \( X \hookrightarrow V_p, V_s \hookrightarrow V_p \) and \( V_r \hookrightarrow X \) guarantee the well-definedness of \( A_n \) and of the operator which is induced by \( A \).

**Proof of Theorem 2.32.** The proof of Theorem 2.32 follows and generalizes the procedure in [5, 14]. First, we check the assumptions of Lemma 2.28/Corollary 2.31 in order to obtain almost everywhere convergence \( Du^n \to Du \), then we use this to prove (2.35).

As in Lemma 2.28 we let \( B \) be a ball with \( 2B \subset \subset \Omega \) and \( \xi \in C^\infty_0(\Omega) \) be a cutoff function such that \( \chi_B \leq \xi \leq \chi_{2B} \). We set \( v^n := \xi(u^n - u) \) and let \( v^n \) be the Lipschitz truncation of \( v^n \) with respect to the domain \( 2B \).

Since the functions \( v^n \) are in general not divergence-free, we have to introduce correction terms in order to use them as test functions in (2.33). We use the Bogovski˘ı operator \( B : L^q(\Omega) \to W^{1,r}_0(\Omega) \) and set

\[
\psi^n_j := B(\text{div } v^n) \in W^{1,r}_0(\Omega) \quad \text{and} \quad \eta^n_j := v^n_j - \psi^n_j \in V_r.
\]

(2.37)

By (2.27), we get \( \text{div } v^n_j \to 0 \) in \( L^r(\Omega) \) for each \( j \in \mathbb{N} \) as \( n \to \infty \). Since both the divergence and the Bogovski˘ı operator are linear and continuous, we get the convergence

\[
\psi^n_j \to 0 \quad \text{weakly in } W^{1,r}_0(\Omega)
\]

(2.38)

as \( n \to \infty \) for every \( j \in \mathbb{N} \). By a well-known fact, we know \( \text{div } v^n = \text{div } v^n \) on the set \( \{ v^n_j = v^n \} \) (cf. [12]). Thus, we obtain \( \text{div } v^n = \nabla \xi \cdot (u^n - u) \) by the product rule and get \( \text{div } v^n_j = \chi_{\{v^n \neq v^n_j\}} \text{div } v^n_j + \chi_{\{v^n = v^n_j\}} \nabla \xi \cdot (u^n - u) \).

Together with the continuity of the Bogovski˘ı operator and the \( W^{1,\infty}(\Omega) \)-boundedness of the cutoff function \( \xi \), this implies

\[
\|\psi^n_j\|_{1,p} \leq c \|\text{div } v^n_j\|_p \leq c \|\chi_{\{v^n \neq v^n_j\}} \nabla v^n_j\|_p + c(\xi) \|u^n - u\|_p.
\]

(2.39)

Furthermore, due to the assumption \( u^n \to u \) in \( W^{1,p}_0(\Omega) \) and the compact embedding \( W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega) \), we have strong convergence \( u^n \to u \) in \( L^p(\Omega) \).

Applying (2.26) and this strong convergence in (2.39), we obtain

\[
\limsup_{n \to \infty} \|\psi^n_j\|_{1,p} \leq c 2^{-j/\pi}
\]

(2.40)

for all \( j \in \mathbb{N} \).
From (2.27), (2.38) and the compact embedding $W_{0}^{1,r}(\Omega) \hookrightarrow L^{r}(\Omega)$, we conclude

$$\eta_{j}^{n} \rightharpoonup 0 \text{ weakly in } W_{0}^{1,r}(\Omega)$$  \hspace{1cm} (2.41)

for all $j \in \mathbb{N}$ as $n \to \infty$.

Since $B : V_{p} \to W_{0}^{1,r}(\Omega)^{*}$ is strongly continuous and $u^{n} \to u$ in $V_{p}$, we obtain the convergence $B(u^{n}) \to B(u)$ in $W_{0}^{1,r}(\Omega)^{*}$. This and (2.41) imply

$$\lim_{n \to \infty} \langle B(u^{n}), \eta_{j}^{n} \rangle = 0.$$  \hspace{1cm} (2.42)

Similarly, we obtain

$$\lim_{n \to \infty} \langle A_{n}(u^{n}), \eta_{j}^{n} \rangle = 0$$  \hspace{1cm} (2.43)

from (2.34) and (2.41). Furthermore, (2.27) implies $v_{j}^{n} \rightharpoonup 0$ in $W_{0}^{1,p}(\Omega)$ and

$$\lim_{n \to \infty} \langle A(Du^{n}), Dv_{j}^{n} \rangle = 0$$  \hspace{1cm} (2.44)

for all $j \in \mathbb{N}$.

By (2.37) and equation (2.33) we have

$$\langle A(Du^{n}) - A(Du), Dv_{j}^{n} \rangle = -\langle B(u^{n}), \eta_{j}^{n} \rangle - \langle A_{n}(u^{n}), \eta_{j}^{n} \rangle + \langle A(Du^{n}), D\psi_{j}^{n} \rangle - \langle A(Du), Dv_{j}^{n} \rangle.$$

We use the convergences (2.42), (2.43), (2.40) and (2.44) in this identity and obtain

$$\lim_{n \to \infty} \sup_{j} \langle A(Du^{n}) - A(Du), Dv_{j}^{n} \rangle \leq c 2^{-j/p}.$$

Since $2^{-j/p} \to 0$ as $j \to \infty$, we may apply Corollary 2.31 and conclude $Du^{n} \to Du$ almost everywhere in $\Omega$ up to some subsequence. By the continuity of $A$, it follows $A(Du^{n}) \rightharpoonup A(Du)$ almost everywhere in $\Omega$.

By assumption, the mapping $v \mapsto A(Dv)$ defines a bounded operator $W^{1,p}(\Omega) \to L^{p}(\Omega)$, and thus the sequence $(A(Du^{n}))_{n \in \mathbb{N}}$ is bounded. We may extract a weakly convergent subsequence $A(Du^{n}) \to \chi$ in $L^{p}(\Omega)$. The combination of almost everywhere convergence $A(Du^{n}) \to A(Du)$ and weak convergence $A(Du^{n}) \to \chi$ (for some subsequences) implies $A(Du) = \chi$ by a well-known convergence principle (cf. [8]); in particular, it follows

$$A(Du^{n}) \rightharpoonup A(Du) \text{ weakly in } L^{p}(\Omega).$$ \hspace{1cm} (2.45)

We pass to the limit for $n \to \infty$ in (2.33) and use (2.45), the strong continuity of $B$ and (2.34) to obtain

$$0 = \lim_{n \to \infty} \langle A(Du^{n}), D\varphi \rangle + \langle B(u^{n}), \varphi \rangle + \langle A_{n}(u^{n}), \varphi \rangle$$

$$= \langle A(Du), D\varphi \rangle + \langle B(u), \varphi \rangle$$

for $\varphi \in V_{p} \cap V_{r} \cap X = V_{r}$ and therefore, by density, for all $\varphi \in V_{r}$. $\square$
3. Existence of weak solutions

3.1. Smallness condition and main result

As mentioned in the introduction, our ansatz for proving existence requires smallness of the boundary and the divergence data which is necessary for proving local coercivity. In order to formulate a precise smallness condition, we define the following dependent constants:

For a domain \( \Omega \), an extra stress tensor \( \mathcal{S} \) with \( p, \delta \)-structure, \( s = \max \{ p, \left( \frac{p}{\delta} \right) \} \), a functional \( f \in W_0^{1, p}(\Omega)^* \) and a function \( g \in W^{1, s}(\Omega) \) we define

\[
\begin{align*}
G_1 &= \frac{1}{p} C_3(\mathcal{S}), \\
G_2 &= c_{\text{Sob}} c_{\text{Korn}}^2 \left[ \| Dg \|_s + \frac{1}{2} \| \text{div} g \|_s \right], \\
G_3 &= (C_2(\mathcal{S}) + C_3(\mathcal{S})) \| Dg \| + \delta \| p \|_p^{-1} + c_{\text{Sob}} \| g \|_1^2 \\
&+ c_{\text{Sob}} c_{\text{Korn}} \| \text{div} g \|_s \| g \|_1^2 + c_{\text{Korn}} \| f \|_{W_0^{1, p}(\Omega)}
\end{align*}
\]

(3.1)

with constants \( c_{\text{Korn}}, c_{\text{Sob}}, C_2(\mathcal{S}) \) and \( C_3(\mathcal{S}) \) that do only depend on \( \Omega \) and the characteristics of \( \mathcal{S} \).

With these constants, we impose a smallness condition on the data \( g_1 \) and \( g_2 \):

**Assumption 3.2.** We assume that \( g_1 \in L^s(\Omega) \) and \( g_2 \in W^{s-\frac{1}{\delta}, s}(\partial \Omega) \) satisfy the compatibility condition \( \int_{\Omega} g_1 \, dx = \int_{\partial \Omega} g_2 \cdot \nu \) do and that their norms are so small that a solution \( g \in W^{1, s}(\Omega) \) of the corresponding inhomogeneous divergence equation (see Lemma 2.3) satisfies

\[
(2 - p)^{2-p} (p - 1)^{p-1} G_1 \geq G_2^{p-1} G_3^{2-p}
\]

(3.3)

for the constants \( G_1, G_2, G_3 \) from (3.1).

Under that condition, we are able to prove the following existence result:

**Theorem 3.4** (Existence). Let \( \Omega \subset \mathbb{R}^d \) be a bounded Lipschitz domain with \( d \in \{2, 3\} \). Let \( \mathcal{S} \) be an extra stress tensor with \( p, \delta \)-structure, \( p \in \left( \frac{2d}{d+2}, 2 \right) \), \( s := \max \{ p, \left( \frac{p}{\delta} \right) \} \) and \( f \in W_0^{1, p}(\Omega)^* \). For any \( g_1 \in L^s(\Omega) \) and \( g_2 \in W^{s-\frac{1}{\delta}, s}(\partial \Omega) \) that satisfy Assumption 3.2 there exists a weak solution \( (\mathbf{v}, \pi) \in W^{1,p}(\Omega) \times L^s(\Omega) \) of (1.1).

3.2. Existence proof

To get a formulation of (1.1), we use the definitions (2.9) and (2.18) of the operators \( \mathcal{S} \) and \( T \) and define the "full" operator \( \mathbf{P} : V_p \rightarrow W_0^{1, s}(\Omega)^* \) via

\[
\langle \mathbf{P}(\mathbf{v}), \varphi \rangle := \langle \mathcal{S}(D\mathbf{v}) + Dg, D\varphi \rangle - \langle (\mathbf{v} + g) \otimes (\mathbf{v} + g), D\varphi \rangle - \langle (\text{div} g)(\mathbf{v} + g), \varphi \rangle - \langle f, \varphi \rangle
\]

(3.5)

for \( \mathbf{v} \in V_p \) and \( \varphi \in W_0^{1, s}(\Omega) \).

We collect our results on \( \mathcal{S} \) and \( T \) to deduce properties of \( \mathbf{P} \):
**Corollary 3.6.** For \( p, q \) and \( s \) as in Lemma 2.19, the operator \( P \) defined in (3.5) is well-defined, bounded and continuous on \( V_p \to W_0^{1,s}(\Omega)^* \). It is well-defined, bounded, continuous and pseudomonotone on \( V_q \to V_q^* \) and it fulfills the estimate

\[
\langle P(u), u \rangle \geq G_1 \|Du\|_p^p - G_2 \|Du\|_p^2 - G_3 \|Du\|_p^p
\]

for \( u \in V_q \). If furthermore

\[
(2 - p)^{2-p}(p - 1)^{p-1}G_1 \geq G_2^{p-1}G_3^{2-p}, \quad (3.7)
\]

then

\[
\langle P(u), u \rangle \geq 0
\]

holds for all \( u \in V_q \) with \( \|Du\|_p = R := \left[ \frac{G_3}{(2-p)G_1} \right]^{\frac{1}{p-1}} \).

**Proof.** Well-definedness, boundedness and continuity follow similarly to the properties of \( S \) and \( T \) in Lemmas 2.10 and 2.19.

The continuity of \( S \) and its monotonicity, which follows from (2.7), yield that \( S: V_q \to V_q^* \) is pseudomonotone. Lemma 2.19 shows that \( T: V_q \to V_q^* \) is strongly continuous and thus pseudomonotone. Therefore, the sum \( P = S + T - f \) is also pseudomonotone.

By Lemmas 2.13 and 2.19 and by the definition of the constants \( G_1, G_2, G_3 \) in (3.1), we have

\[
\langle P(u), u \rangle \geq \langle S(u), u \rangle - \|T(u), u\| - \langle f, u \rangle
\]

\[
\geq G_1 \|Du\|_p^p - G_2 \|Du\|_p^2 - G_3 \|Du\|_p^p \quad (3.8)
\]

for any \( u \in V_q \).

Now assume that (3.7) holds. It follows \( \left( \frac{(p-1)G_1}{G_2} \right)^{p-1} \geq \left[ \frac{G_3}{(2-p)G_1} \right]^{2-p} \), so we may define \( R := \left[ \frac{G_3}{(2-p)G_1} \right]^{\frac{1}{p-1}} \) and obtain \( \frac{(p-1)G_1}{G_2} \geq R^{2-p} \). Together, we get

\[
G_1 R^p = (p-1)G_1 R^p + (2-p)G_1 R^p \geq G_2 R^2 + G_3 R. \quad (3.9)
\]

We use \( \|Du\|_p = R \), insert (3.9) into (3.8) and obtain the result.

**Remark 3.10.** (i) Note that the dependence of the constants \( G_i, i = 1, 2, 3 \), on \( \|g\|_{1,s} \) stems only from the estimate of the convective term.

(ii) In order to prove \( G_1 R^p - G_2 R^2 - G_3 R \geq 0 \), we have split the positive summand \( G_1 R^p \) into two parts using the weights \( p - 1 \) and \( 2 - p \). By considering the weights as a free parameter, one can show that this choice is optimal.

Now we are ready to complete the proof of Theorem 3.4. Since the proof requires an approximation process only if \( p \in \left( \frac{2d}{d+2}, \frac{3d}{d+2} \right) \), we handle the two cases separately.
Proof of Theorem 3.4 in the case \( p \in \left( \frac{3d}{d+2}, 2 \right) \). In this case we have \( s = p \). By Lemma 2.3, we find a function \( g \in W^{1,p}(\Omega) \) that solves the corresponding inhomogeneous divergence equation (2.1).

We consider the corresponding operator \( P \) defined in (3.5) and prove existence of a function \( u \in V_p \) which satisfies \( P(u) = 0 \). The space \( V_p \) is reflexive and separable as a closed subspace of \( W^{1,p}(\Omega) \). In Corollary 3.6 (with \( q = s = p \)), we proved that \( P \) is well-defined, bounded, continuous and pseudomonotone on \( V_p \to V^* \) and we concluded from assumption (3.3) that there exists a positive number \( R \) such that \( P \) is locally coercive with radius \( R \). So we may apply the main theorem on pseudomonotone operators, Theorem 2.5, to the operator \( P \) on the space \( V_p \) and obtain a weak solution \( u \in V_p \) of \( P(u) = 0 \).

By a standard characterization of weak gradient fields (cf. [17]), this is equivalent to the existence of a pressure \( \pi \) such that \( (u + g, \pi) \in W^{1,p}(\Omega) \times L^p(\Omega) \) solves the original system (1.1).

Proof of Theorem 3.4 in the case \( p \in \left( \frac{3d}{d+2}, \frac{3d}{d+2} \right] \). In this case we have \( s = (\frac{p}{2})' \). By assumption 3.2, there is a function \( g \in W^{1,s}(\Omega) \) which solves the inhomogeneous divergence equation (2.1) and satisfies (3.3). We prove the existence of a function \( u \in V_p \) that solves \( P(u) = 0 \) for the corresponding operator \( P \) from (3.5). For regularization, we choose some \( q > s \) with \( q > 2 \) and consider the symmetric \( q \)-Laplacian \( A : V_q \to V_q^* \) defined via

\[
\langle A(u), \varphi \rangle := \langle |Du|^{q-2} Du, D\varphi \rangle.
\]

The operator \( A \) is well-defined, bounded, continuous and monotone.

We work in the reflexive and separable spaces \( V^n_q \), which are defined as the space \( V_q \) equipped with the equivalent norms \( \|u\|_{q,n} := \max \left\{ n^{\frac{2}{q-2}} \|Du\|_q, \|Du\|_p \right\} \).

For sufficiently large \( n \), we want to establish the existence of solutions \( u^n \in V^n_q \) to the equation

\[
\langle P(u^n), \varphi \rangle + \frac{1}{n} \langle A(u^n), \varphi \rangle = 0 \quad (3.11)
\]

for \( \varphi \in V_q \), which shall approximate a solution of the original equation. The operator \( P \) is pseudomonotone, continuous and bounded by Corollary 3.4 and the same holds for \( A \) and their sum \( P + \frac{1}{n} A \). We prove local coercivity of \( P + \frac{1}{n} A \) with radius \( R := \left[ \frac{G_3}{(2-p)G_1} \right]^{\frac{1}{d-1}} \). So, let \( \|u\|_{q,n} = R \). If \( n^{\frac{2}{q-2}} \|Du\|_q \leq \|Du\|_p \), we have \( \|Du\|_p = \|u\|_{q,n} = R \) and we get \( \langle P(u), u \rangle \geq 0 \) by assumption (3.3) and Corollary 3.6. Otherwise, suppose \( n^{\frac{2}{q-2}} \|Du\|_q > \|Du\|_p \), so \( R = n^{\frac{2}{q-2}} \|Du\|_q \) and \( R > \|Du\|_p \). This, Corollary 3.6 and the Sobolev embedding \( V_q \hookrightarrow V_p \) imply

\[
\langle P(u), u \rangle + \frac{1}{n} \langle A(u), u \rangle \geq G_1 \|Du\|_p^p - G_2 \|Du\|_p^2 - G_3 \|Du\|_p + \frac{1}{n} \|Du\|_q^q > -G_2 R^2 - G_3 R + n^{\frac{2}{q-2}} R^q.
\]

As \( n^{\frac{2}{q-2}} \) grows to infinity, the latter expression becomes positive for any \( R > 0 \) and sufficiently large \( n \).
Thus, the existence Theorem 2.5 gives us solutions $u^n \in V^n_q$ of (3.11) with
\[
\max \left\{ n^{\frac{2}{q-n}} \| Du \|_q, \| Du \|_p \right\} = \| u^n \|_{q,n} \leq R
\]
and the bound $R$ holds uniformly with respect to $n$.

We switch to a weakly convergent (and renamed) subsequence $u^n \rightharpoonup u$ in $V_p$. The bound (3.12) implies $\| n^{-1} A(u^n) \|' = n^{-1} \| Du^n \|_q^{q-1} \leq n^{\frac{2}{q-n}} R^{q-1} \to 0$ as $n \to \infty$.

We apply Theorem 2.32 with the shifted extra stress tensor $A(\cdot) := S(\cdot + Dg)$, $B := T - f : V_p \to V^*_q$, $r := q$, $X := V_q$ and $A_n := n^{-1} A : V_q \to V^*_q$ and obtain that $u$ solves $P(u) = 0$ weakly.

Similarly to the proof in the first case, we obtain a pressure $\pi$ such that the pair $(u + g, \pi) \in W^{1,p}(\Omega) \times L^{s'}(\Omega)$ is a weak solution of (1.1). \hfill \Box

3.3. Less regular data

In Theorem 3.4 we demanded additional regularity of the data: we required $g \in W^{1,s}(\Omega)$ with $s = \left(\frac{p}{q} \right)' > p$ in the case $p \in \left(\frac{2d}{d+2}, \frac{3d}{3d+2}\right)$, while the solution $v$ is sought only in $W^{1,p}(\Omega)$. Thus, we want to discuss whether this assumption is really necessary or if it may be removed, perhaps for the price of more regular test functions. This question only arises if $p \in \left(\frac{2d}{d+2}, \frac{3d}{3d+2}\right)$, since one has $s' = p$ and $g \in W^{1,s}(\Omega) = W^{1,p}(\Omega)$ in the other case.

In the proof of Theorem 3.4 we used the additional regularity of our data to get more convenient estimates of the convective term in Lemma 2.19. Since these estimates are mainly based on the Hölder inequality, one has to use a stronger norm of $u$ if only $g \in W^{1,p}(\Omega)$ is presumed. By (2.24) and the Hölder inequality, one obtains the following result similar to Lemma 2.19.

Lemma 3.13 (Properties of $T$). Let $p \in \left(\frac{2d}{d+2}, \frac{3d}{3d+2}\right)$ and $q$ be so large that it holds both $q > s = \left(\frac{p}{q} \right)'$ and $\frac{1}{q} \geq \frac{1}{p} + \frac{1}{p'}$. For any given function $g \in W^{1,p}(\Omega)$, the operator $T$ has the upper bound
\[
|\langle T(u), u \rangle| \leq c_{\text{Sob}}^2 K_{\text{Korn}}^2 \left( \| Dg \|_p + \frac{1}{2} \| \text{div} g \|_p \right) \| Du \|_p \| Du \|_q
\]
\[
+ c_{\text{Sob}} \left( \| g \|_{1,p}^2 + c_{\text{Korn}} \| \text{div} g \|_p \| g \|_{1,p} \right) \| Du \|_q
\]
for all $u \in V_q$, where $c_{\text{Sob}}$ are Sobolev embedding constants.

This and Lemma 2.13 yield an alternative lower bound of $P$:

Corollary 3.14 (Alternative estimate of $P$). Let $p \in \left(\frac{2d}{d+2}, \frac{3d}{3d+2}\right)$ and $q$ be so large that it holds both $q > s = \left(\frac{p}{q} \right)'$ and $\frac{1}{q} \geq \frac{1}{p} + \frac{1}{p'}$. For any given function $g \in W^{1,p}(\Omega) \setminus \{0\}$, there are constants $F_1, F_2, G_1 \geq 0$ with $F_1, G_1 > 0$ such that it holds
\[
\langle P(u), u \rangle \geq G_1 \| Du \|_p^p - F_1 \| Du \|_q - F_2 \| Du \|_p \| Du \|_q
\]
for all $u \in V_q$. 

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To the authors’ knowledge, a substantial improvement of the estimates in Lemma 3.13 and Corollary 3.14 is not available. So we ask whether the proof of Theorem 3.4 can be modified such that it works out with these estimates.

The main theorem on pseudomonotone operators, Theorem 2.5, which was used to obtain approximate solutions in some smoother space $X = V_p \cap Y$ already gave a priori estimates for these approximate solutions. These a priori bounds were needed to establish a weak accumulation point. The next Lemma shows that it is impossible to obtain approximate solutions which are coming with an a priori bound in $V_p$, by Brouwer’s fixed point theorem/the main theorem on pseudomonotone operators.

**Lemma 3.15** (Limits for the applicability of pseudomonotone operator theory for solving (1.1)). Let $F_1, G_1, R, 1 < p < 2 < q$ be arbitrary, positive constants, $Y \subset L^1(\Omega)$ be a Banach space with norm $\|\cdot\|_Y$ and assume there is no continuous embedding $V_p \hookrightarrow Y$. Define the operators $P_n : V_p \cap Y \rightarrow \mathbb{R}$ via

$$P_n(u) := G_1 \|Du\|_p^p + \frac{1}{n} \|u\|_Y^q - F_1 \|u\|_Y.$$ 

Consider Banach spaces $Y_n := (V_p \cap Y, \|\cdot\|_{Y_n})$ which satisfy $Y_n \hookrightarrow Y$ and $\|D\cdot\|_p \leq \|\cdot\|_{Y_n}$ for all $n \in \mathbb{N}$. Then, for all sufficiently large $n$ there exists a function $u^n \in V_p \cap Y$ with $\|u^n\|_{Y_n} = R$ and

$$P_n(u^n) < 0.$$ 

(3.16)

**Remark 3.17** (Discussion of the assumptions in Lemma 3.15). The fact that the space $V_p \cap Y$, where we seek for approximate solutions, needs to be strictly smoother than $V_p$ is caught up in the assumption that $V_p$ does not embed continuously into $Y$.

The term $\frac{1}{n} \|u\|_Y^q$ in the definition of $P_n$ may be equivalently replaced by any operator that is coercive on $Y$ damped by some factor which is decreasing in $n$.

The existence of embeddings $Y_n \hookrightarrow Y$ is a natural assumption for the intersection of Banach spaces. The requirement $\|D\cdot\|_p \leq \|\cdot\|_{Y_n}$ for all $n$ implies the existence of embeddings $Y_n \hookrightarrow V_p$ and further means that a-priori estimates of the form $\|u^n\|_{Y_n} < c$ imply uniform boundedness in the weaker norm $\|Du^n\|_p < c$ which is a necessary element in the proof of Theorem 3.4.

Typically, one chooses $Y = V_q$ or $Y = L^q(\Omega)$ for some large number $q \in \mathbb{R}$ and works with the weighted sum norm $\|\cdot\|_{Y_n} = \|D\cdot\|_p + \frac{1}{n} \|\cdot\|_Y$. Obviously, the assumptions from Lemma 3.15 are fulfilled for such choices.

**Proof of Lemma 3.15**. Since $Y \subset L^1(\Omega)$, the intersection $V_p \cap Y$ is well-defined. For each $n \in \mathbb{N}$, we define $f(n) := \inf_{u \in Y_n \setminus \{0\}} \frac{\|u\|_{Y_n}}{\|u\|_Y}$. The embedding $Y_n \hookrightarrow Y$ and the assumption on $\|\cdot\|_{Y_n}$ imply that $f(n)$ is strictly positive and that

$$\max \left\{ \|Du\|_p : f(n) \|u\|_Y \right\} \leq R$$

(3.18)

holds for all $u \in Y_n$ with $\|u\|_{Y_n} = R$. 

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We define the constant $c_* := \sup_{u \in Y_n \setminus \{0\}} \|Du\|_{Y_n^*} < \infty$ and use the convention $\frac{c_*}{c_1} = 0$ for any $t \in \mathbb{R}$.

Our first step is the indirect proof of an upper bound on $f$. Therefore, we choose $c_1 \in \mathbb{R}$ so large that $\frac{R^{q-1}}{c_1} - \frac{F_1}{2} < 0$ is satisfied.

**Step 1:** We prove that (3.16) is true for those $n \in \mathbb{N}$ with $f_n^{q-1} \geq c_1 n^{q-1}$.

By (3.18), we obtain

$$\frac{1}{n} \|u\|_y^q - \frac{F_1}{2} \|u\|_y \leq \|u\|_y \left[ \frac{R^{q-1}}{nf_n^{q-1}} - \frac{F_1}{2} \right] \leq \|u\|_y \left[ \frac{R^{q-1}}{c_1} - \frac{F_1}{2} \right] < 0$$

for any $u \in Y_n$ with $\|u\|_{Y_n} = R$. Since there is no embedding $V_p \hookrightarrow Y$, we may find some $u^n \in Y_n$ with $\|u^n\|_{Y_n} = R$ and $G_1 \|Du^n\|_p < \frac{F_1}{2} \|u^n\|_y$. Together, we obtain $G_1 \|Du^n\|_p^q + \frac{1}{n} \|u^n\|_y - F_1 \|u^n\|_y < 0$, which is (3.16).

In the following, we only consider those $n \in \mathbb{N}$ with $f_n^{q-1} < c_1 n^{q-1}$.

**Step 2:** Computation of suitable norms for functions $u^n$ such that (3.16) becomes true.

We choose $c_2 > 1$ so large that $\frac{c_2 R^{q-1}}{F_1} > c_1$ and define the auxiliary functions $t_n : \mathbb{R} \to \mathbb{R}$ via

$$t_n(x) := \frac{c_2}{n} x^{q-1} - F_1.$$

We claim that for all sufficiently large $n$, the equation $t_n(x) = 0$ has a solution $y_n \in \left( \frac{R}{c_*}, \frac{R}{f_n} \right)$. Since the upper bound on $f$ implies $f(n) \to 0$ as $n \to \infty$, it holds $\frac{R}{c_*} < \frac{R}{f_n}$ for sufficiently large $n$ and the interval $\left( \frac{R}{c_*}, \frac{R}{f_n} \right)$ is not empty.

We have $t_n \left( \frac{R}{f_n} \right) = \frac{c_2 R^{q-1}}{F_1} - F_1 < 0$ for sufficiently large $n$ and the definitions of $c_1$ and $c_2$ imply $\frac{c_2 R^{q-1}}{F_1} > c_1 > n f_n^{q-1}$, thus $t_n \left( \frac{R}{f_n} \right) = \frac{c_2 R^{q-1}}{n f_n^{q-1}} - F_1 > 0$.

The existence of zeroes $y_n$ then follows from the mean value theorem.

Right from the definition of $y_n$, we obtain $\frac{1}{n} y_n^q - \frac{1}{c_2} F_1 y_n = 0$ and

$$\left[ 1 - \frac{1}{c_2} \right] F_1 y_n = \left[ 1 - \frac{1}{c_2} \right] F_1 \left[ \frac{y_n^q}{c_2} \right]^{\frac{1}{q-1}} > G_1 R^p$$

for sufficiently large $n$. Together, it follows

$$G_1 R^p + \frac{1}{n} y_n^q - F_1 y_n = G_1 R^p - \left[ 1 - \frac{1}{c_2} \right] F_1 y_n + \frac{1}{n} y_n^q - \frac{1}{c_2} F_1 y_n < 0.$$

Thus, any function $u^n \in V_p \cap Y$ with $\|u^n\|_{Y_n} = R$ and $\|u^n\|_Y = y_n$ satisfies

$$P_n(u^n) \leq G_1 R^p + \frac{1}{n} y_n^q - F_1 y_n < 0.$$

**Step 3:** Construction of functions $u^n \in V_p \cap Y$ with prescribed norms.

It remains to prove that for any $y_n \in \left( \frac{R}{c_*}, \frac{R}{f_n} \right)$, there is a function $u^n \in V_p \cap Y$ with $\|u^n\|_{Y_n} = R$ and $\|u^n\|_Y = y_n$. 

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Assume that $\|u\|_Y > y_n$ for all $u \in Y_n$ with $\|u\|_{Y_n} = R$. Since $y_n \in \left(\frac{R}{R_n}, \frac{R}{f(n)}\right)$, it holds $c_s > \frac{R}{y_n}$. By the definition of $c_s$, this implies the existence of a function $u \in Y_n$ such that $\|Du\|_p > \frac{R}{y_n} \|u\|_Y$. Without loss of generality, we may scale $u$ such that $\|u\|_{Y_n} = R$. We compile these estimates of $u$ and apply (3.18) to obtain

$$R < \frac{R}{y_n} \|u\|_Y < \|Du\|_p \leq R,$$

which is impossible.

Now assume that $\|u\|_Y < y_n$ for all $u \in Y_n$ with $\|u\|_{Y_n} = R$. As above, we obtain $\frac{R}{y_n} > f(n)$. This and the definition of $f(n)$ imply the existence of a function $u \in Y_n$ such that it holds $\|u\|_{Y_n} < \frac{R}{y_n} \|u\|_Y$. We may scale $u$ such that $\|u\|_{Y_n} = R$ and it follows

$$R = \|u\|_{Y_n} < \frac{R}{y_n} \|u\|_Y < R,$$

which is again a contradiction.

Hence, there are $v^n, w^n \in Y_n$ with $\|v^n\|_{Y_n} = \|w^n\|_{Y_n} = R$ and $\|v^n\|_Y \leq y_n \leq \|w^n\|_Y$ for every $n \in \mathbb{N}$. By assumption, the mapping $u \mapsto \|u\|_Y$ is continuous on $Y_n$. We apply the mean value theorem on the (path-connected) sphere $\{\|\cdot\|_{Y_n} = R\}$ and obtain functions $u^n \in V_p \cap Y$ with $\|u^n\|_{Y_n} = R$ and $\|u^n\|_Y = y_n$. In Step 2 we proved that such an element $u^n$ solves (3.16) for all sufficiently large $n$.

Lemma 3.15 shows that, without assuming additional regularity, it is impossible to find a radius $R$ such that local coercivity is fulfilled and Brouwer’s fixed point theorem becomes applicable. Consequently, the authors view the existence proof in [16, Theorem 1.3] with suspicion; in particular, the requirements for Brouwer’s fixed point theorem do not seem to be satisfied in our eyes. We conclude from Lemma 3.15 that it is impossible to modify the proof of existence theorem 3.4 such that it avoids the critical regularity assumption within the framework of pseudomonotone operator theory.

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