Testing $WW\gamma$ vertex in radiative muon decay

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Large numbers of muons will be produced at facilities developed to probe lepton flavor violating process $\mu \rightarrow e\gamma$. We show that by constructing a suitable asymmetry, radiative muon decay $\mu \rightarrow e\gamma\nu_\mu\bar{\nu}_e$ can also be used to test the $WW\gamma$ vertex at such facilities. The process has two missing neutrinos in the final state and on integrating their momenta, the partial differential decay rate shows no radiation-amplitude-zero. We establish, however, that an easily separable part of the normalized differential decay rate, odd under the exchange of photon and electron energies, does have a zero in the case of standard model (SM). This new type of zero has hitherto not been studied in literature. A suitably constructed asymmetry using this fact, enables a sensitive probe for the $WW\gamma$ vertex beyond the SM. With a simplistic analysis, we find that the $C$ and $P$ conserving dimension four $WW\gamma$ vertex can be probed at $O(10^{-2})$ with satisfactory significance level.

I. INTRODUCTION

The $SU(2)_L \otimes U(1)_Y$ theory of electroweak interactions has been tested extensively in last few decades and there is no doubt that it is the correct theory at least up to a TeV scale. This conviction is largely based on the precision measurements at LEP and the consistency of top and Higgs boson masses which could be predicted taking radiative corrections into account. The gauge boson and Higgs boson self interactions are, however, not as well probed either by direct measurement or by radiative corrections and it is possible that some deviations from the standard Model (SM) loop level values might still be seen. To ascertain the validity of SM it is critical that the $WW\gamma$ vertex, which is predicted uniquely in SM, be probed to an accuracy consistent with loop level corrections to it. Several experiments [1-8] have measured parameters that probe the $WW\gamma$ and $WWZ$ vertex, but the accuracy achieved is still insufficient to probe one loop corrections to it within the SM.

In this paper, we have investigated how the $C$ and $P$ conserving dimension four $WW\gamma$ operator can be probed experimentally using radiative muon decays. The vertex factor for this operator is usually denoted by $\kappa_\gamma$ and is uniquely predicted in the SM. At tree level $\kappa_\gamma = 1$ in the SM and the absolute value of the one loop corrections to the tree level values of $\kappa_\gamma$ is restricted to be less than $1.5 \times 10^{-2}$ [9]. However, the current global average $\kappa_\gamma = 0.982 \pm 0.042$ [10] has too large an uncertainty to probe the SM up to one loop accuracy. Of the experimentally measured values of $\kappa_\gamma$, only ATLAS and CMS collaborations use the data for real on-shell photon emission in hadron colliders [1, 2], probing the true magnetic moment of the $W$-boson.

One can expect $\kappa_\gamma$ to deviate from its SM value by only a few percent, hence, we must choose the mode to be studied very carefully. Radiative muon decay $\mu \rightarrow e\gamma\nu_\mu\bar{\nu}_e$ is a promising mode to measure the true magnetic moment (due to real photon in the final state) of the $W$-boson in this regard. At first sight the measurement of $W$-boson gauge coupling using low energy decay process may seem impossible, since the effect is suppressed by two powers of the $W$-boson mass. The process has two missing neutrinos in the final state and on integrating their momenta the partial differential decay rate shows no radiation-amplitude zero [11]. Moreover, the differential decay rate does not show enough sensitivity to a deviation of the $WW\gamma$ vertex from that of the SM. We show, however, that an easily separable part the normalized differential decay rate (odd under the exchange of photon and electron energies) does have a zero in the case of SM. The vanishing of the odd contribution under the exchange of final state electron and photon energies in the decay rate is a new type of zero hitherto not been studied in literature. A suitably constructed asymmetry using this fact enables adequate sensitivity to probe the $WW\gamma$ vertex beyond the SM. We consider a very restricted part of the phase space where the asymmetry is larger than statistical errors for our study. Large number of muons are expected to be produced for COMET [12], MEG [13] and Mu2e [14] collaborations to probe lepton flavor violating processes like $\mu \rightarrow e\gamma$. The radiative muon decay $\mu \rightarrow e\gamma\nu_\mu\bar{\nu}_e$ [15] discussed in this paper is the dominant background process for this case. The large sample of $\mu \rightarrow e\gamma\nu_\mu\bar{\nu}_e$ produced at such facilities make them an ideal environment to probe $WW\gamma$ vertex, with reduced statistical uncertainty, as discussed in this paper. In a simulation using $\eta_1 \equiv \kappa_\gamma - 1 = 0.01$, we find that the asymmetry constructed by us, can probe this $\eta_1$ value with a 3.9$\sigma$ significance.
The rest of the paper is organized as follows. In Sec. II we briefly discuss the decay kinematics and relevant expressions for decay rate. These results are used to construct the observables in Sec. III, where we also explain why a zero in odd amplitude is expected. Section IV deals with the numerical analysis to probe the $WW\gamma$ vertex and finally we conclude in Sec. V.

II. THEORETICAL FRAMEWORK

In this section we briefly discuss the theoretical set up for the radiative muon decay. The radiative muon decay proceeds through three Feynman diagrams, shown in Fig. 1, where the photon in the final state can either arise from any of the initial and final state leptons or the $W$ boson in the propagator. The later process is of our particular interest. We define the four momenta of incoming $\mu^-$, outgoing $e^-$, $\gamma$, $\nu_\mu$, $\bar{\nu}_e$ as $p_\mu$, $p_e$, $p$, $k$ and $k'$, respectively, and the masses of muon, electron and $W$ boson are denoted by $m_\mu$, $m_e$ and $m_W$, respectively. The amplitudes corresponding to these three diagrams (from top to bottom), labeled with subscript 1 to 3, can be expressed as

$$iM_1 = \left(\frac{-ieg^2}{8}\right)\pi(p_\mu)\gamma_\beta(1-\gamma_5)v(k') \left[\frac{g^{\alpha\beta} - \frac{q^\alpha q^\beta}{m_W^2}}{q_1^2 - m_W^2}\right] \gamma_\beta(1-\gamma_5)v(k')e^{i\delta}, \tag{2}$$

$$iM_2 = \left(\frac{-ieg^2}{8}\right)\pi(k)\gamma_\alpha(1-\gamma_5)u(p_m)\left[\frac{g^{\alpha\beta} - \frac{q^\alpha q^\beta}{m_W^2}}{q_2^2 - m_W^2}\right]u(p_m)e^{i\delta}, \tag{3}$$

where $e$ and $g$ are the charge of positron and weak coupling constant, respectively; $q^\alpha_1 = p^\alpha + k^\mu$ and $q^\alpha_2 = p^\mu - k^\mu$. In Eq. (3), $\Gamma_{\rho\sigma\delta}(q_2, q_1, p)$ denotes the effective triple gauge boson vertex for electroweak interaction as shown in Fig. 2.

$$W^\mu_\nu(q_2) \rightarrow W^\nu(q_1) = -ie\Gamma_{\rho\sigma\delta}(q_2, q_1, p)$$

FIG. 2. Feynman rule for effective $WW\gamma$ vertex.

The most general couplings of $W$ to the neutral gauge bosons $\gamma$ and $Z$ can be described by the following effective Lagrangian [16],

$$\mathcal{L}_{\text{eff}}^{V} = -igV^\nu_1^\gamma(W^\mu_\nu W^\mu - W^\mu W^\nu W^\nu) + \kappa_V W^\mu_\nu V^\nu + \frac{\lambda_V}{m_W^2} W^\mu W^\nu W^\nu V^\lambda + if_V W^\mu_\nu W^\nu (\partial^\mu V^\nu + \partial^\nu V^\mu) - if_V e^{\mu\nu\rho\sigma} (W^\nu_\rho W^\mu_\sigma)_{\nu\sigma} + \kappa_V W^\nu_\nu V_{\nu\nu} + \frac{\lambda_V}{m_W^2} W^\mu W^\nu W^\nu V_{\nu\nu}. \tag{4}$$

Here, $V$ corresponds to $\gamma$ or $Z$, $g_1 = e$ and $g_2 = e \cot \theta_W$ where $\theta_W$ is the Weinberg angle. $W^\mu_\nu = \partial_\mu W^\nu - \partial_\nu W^\mu$. $V_{\mu\nu} = \partial_\mu V_{\nu} - \partial_\nu V_{\mu}$. $\tilde{V}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} V^{\rho\sigma}$, $(A\partial_{\mu} B) - (A\partial_{\mu} B)$ and Bjorken-Drell metric is taken as $\epsilon_{0123} = -\epsilon_{0123} = +1$. In the SM, at tree level, $g_1^V = \kappa_V = 1$ and all other coupling parameters are zero.

In the case of radiative muon decay, the vertex with $W$ boson pair and a photon field is involved where among the seven coupling parameters, $f^4_1$, $\tilde{\kappa}$, and $\tilde{\lambda}$, denote the coupling strengths of $CP$ violating interactions in the Lagrangian (in Eq. (4)) and are constrained to be less than $\sim (10^{-4})$ [17] due to the measurements of neutron electric dipole moment in case of direct $CP$ violation. Due to the $CP$ violating nature of these couplings, deviations from the SM contributions are proportional to square of these
couplings and thus are highly suppressed, as compared to CP-conserving contributions. Hence, we neglect the CP violating parameters for the rest of the discussion of the paper. The demand of $C$ and $P$ to be conserved separately in the Lagrangian allows us to choose vanishing $f^0_2$. It is obvious that the muon radiative decay will not be sensitive to the dimension six-operator involving $\lambda_3$, due to an additional $m_W^2$ suppression. The measurement of $\lambda_3$ is possible only at high energy colliders. Hence, we can safely neglect the deviation of $\lambda_3$ from its SM value of zero. Furthermore, the value of the coupling $g_1^e$ is fixed to be unity due to electromagnetic gauge invariance. Thus, in momentum space the $WW\gamma$ vertex can be expressed as

$$\Gamma_{\rho\sigma\delta}(q_2,q_1,p) = g_{\rho\sigma}(q_2 + q_1)i + g_{\rho\sigma}(p - q_1)_\rho - g_{\rho\sigma}(p + q_2)_\sigma + \eta_\gamma(p_\rho g_{\rho\sigma} - p_\sigma g_{\rho\sigma}),$$

where $\eta_\gamma = \kappa_\gamma - 1$ and $q_2, q_1, p$ are the four momenta of incoming $W^-$, outgoing $W^-$ and outgoing photon respectively, as depicted in Fig. 2.

It is apparent from Fig. 1 and Eqs. (1)-(3), that amplitude $(\mathcal{M}_2)$ containing effective vertex $\Gamma_{\rho\sigma\delta}$ is $1/m_W^2$ suppressed as compared to the other two contributions $\mathcal{M}_1$ and $\mathcal{M}_3$. Hence, within the SM, the first two Feynman-diagrams in Fig. 1 are sufficient to study the process. On the other hand only the third diagram is sensitive to $\eta_\gamma$. Thus, in order to retain sensitivity to $\eta_\gamma$ in $\Gamma_{\rho\sigma\delta}$, it is necessary and sufficient to keep contributions up to $O(1/m_W^2)$, in the amplitudes. To achieve this we expand the $W$ boson propagator in the power series of $(q_2^2/m_W^2)$ and $(q_1^2/m_W^2)$ as

$$-i\left[g^{\alpha\beta} - \frac{q_1^{\alpha} q_1^{\beta}}{m_W^2}\right] \approx -i\left[g^{\alpha\beta} + \frac{q_1^{\alpha} q_1^{\beta}}{m_W^2}\right].$$

The total amplitude can be expressed as $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3$ and we calculate differential cross section keeping all the amplitudes up to $O(1/m_W^2)$. Since the neutrinos $\nu_\mu$ and $\bar{\nu}_e$ cannot be observed we integrate the $\nu_\mu$ and $\bar{\nu}_e$ momenta, and define the $\nu_\mu$,$\bar{\nu}_e$ invariant momenta as $q$. As the decay now looks like a 3-body decay it is meaningful to define effective Mandelstam like variables constructed from the invariant momentum square of $e^-$, $\nu_\mu$, $\bar{\nu}_e$ system as $t$ and that of $\gamma\nu_\mu\bar{\nu}_e$ system as $u$. Hence, $(p_e + q)^2 = t$ and $(p_\mu + q)^2 = u$. Notice that, $q^2$ is not a constant for our decay. It is, however, much more convenient to define normalized parameters

$$x_p = \frac{t + u}{2(q^2 + m_\mu^2)},$$
$$y_p = \frac{t - u}{2(q^2 + m_\mu^2)},$$
$$q_p^2 = \frac{q^2}{(q^2 + m_\mu^2)},$$

which can be written in terms of the observable quantities, the photon energy $E_\gamma$, the electron energy $E_e$ and the angle between the electron and photon $\theta$ as follows.

$$x_p = \frac{m_\mu(m_\mu - E_e - E_\gamma)}{2(m_\mu^2 - E_\gamma m_\mu - E_e m_\mu + E_e E_\gamma(1 - \cos \theta))},$$
$$y_p = \frac{m_\mu(E_e - E_\gamma)}{2(m_\mu^2 + E_\gamma m_\mu - E_e m_\mu + E_e E_\gamma(1 - \cos \theta))},$$

$$q_p^2 = \frac{m_\mu^2 - 2E_e m_\mu - 2E_\gamma m_\mu + 2E_e E_\gamma(1 - \cos \theta)}{2(m_\mu^2 - E_\gamma m_\mu - E_e m_\mu + E_e E_\gamma(1 - \cos \theta))}.$$

The parameters of interest for the derivation, $x_p$, $y_p$ and $q_p^2$ can easily be inverted in terms of the observables $E_e$, $E_\gamma$ and $\cos \theta$ as,

$$E_e = \frac{m_\mu}{2}\left(1 - \frac{q_p^2 - x_p^2 + y_p}{1 - q_p^2}\right),$$
$$E_\gamma = \frac{m_\mu}{2}\left(1 - \frac{q_p^2 - x_p^2 - y_p}{1 - q_p^2}\right),$$
$$\cos \theta = \frac{(q_p^2 - x_p^2)^2 + 2x_p y_p - y_p^2 - 1}{(1 - q_p^2 - x_p^2)^2 - y_p^2 - 1}.$$

We notice that replacing $y_p$ by $-y_p$ while keeping $q_p^2$ and $x_p$ unchanged actually results in swapping the energies of photon and electron keeping the angle between them unaltered. This feature will play a very crucial role in defining the observable asymmetry in Sec. III.

We consider only the normalized differential decay rate

$$\frac{\Gamma(x_p, y_p, q_p^2)}{\Gamma} = \frac{1}{\Gamma} \frac{d^3\Gamma}{dq_p^2 dx_p dy_p},$$

where, $\Gamma$ is the total decay width of muon. In terms of these new normalized variables, the phase space for this process is bounded by three surfaces: $q_p^2 = 0$, $x_p = 1/2$ and $(q_p^2 x_p^2 - y_p^2) = 0$. It is easily seen from Eq. (13), the plane $x_p = 1/2$ corresponds to $\theta = 0^\circ$ and the curved surface $(q_p^2 - x_p^2 - y_p^2) = 0$ signifies $\theta = 180^\circ$. The physical region in $q_p^2$, $x_p$ and $y_p$ parameter space is given by,

$$q_p \sqrt{1 - q_p^2} \leq x_p \leq \frac{1}{2},$$
$$|y_p| \leq \frac{1}{2} - q_p^2,$$

$$\left(q_p^2 - q_p^2 x_p^2 - y_p^2\right) \geq 0,$$
$$0 \leq q_p^2 \leq \frac{1}{2}.$$

Form Eq. (7) and Eq. (15), it is clear that both $q_p^2$ and $x_p$ are positive valued functions whereas $y_p$ can have a positive value or a negative value and the physical region allows $y_p$ to have a range symmetric about $y_p = 0$. So, if $(x_p, y_p, q_p^2)$ be a point inside physical region, $(x_p, -y_p, q_p^2)$ will also lie inside the allowed region. This motivates
us to investigate the properties of odd and even part of Π(x_p, y_p, q^2_p) under the variable y_p. In the next section (Sec. III) we construct such an observable as the ratio of odd part in y_p, divided by even part in y_p of Π(x_p, y_p, q^2_p) and demonstrate its heightened sensitivity to η_p.

III. OBSERVABLE AND ASYMMETRY

The ‘odd’ and ‘even’ part Γ_o(x_p, y_p, q^2_p) and Γ_e(x_p, y_p, q^2_p), respectively, of the normalized differential decay rate (Eq. (14)) with respect to y_p are defined as

\[ \Gamma_o(x_p, y_p, q^2_p) = \frac{1}{2} \left[ \Gamma(x_p, y_p, q^2_p) - \Gamma(x_p, -y_p, q^2_p) \right] \approx F_o(x_p, y_p, q^2_p) + \eta_p G_o(x_p, y_p, q^2_p), \quad (16) \]

\[ \Gamma_e(x_p, y_p, q^2_p) = \frac{1}{2} \left[ \Gamma(x_p, y_p, q^2_p) + \Gamma(x_p, -y_p, q^2_p) \right] \approx F_e(x_p, y_p, q^2_p) + \eta_p G_e(x_p, y_p, q^2_p), \quad (17) \]

where the small \( \eta_p \) terms are ignored.

As we have obtained \( \Gamma(x_p, y_p, q^2_p) \) by integrating a positive valued function \( |M|^2 \), it is obvious that both \( \Gamma_o(x_p, y_p, q^2_p) \) and \( \Gamma(x_p, -y_p, q^2_p) \) will be positive. Hence, \( \Gamma_e(x_p, y_p, q^2_p) \), which is proportional to the sum of \( \Gamma_o(x_p, y_p, q^2_p) \) and \( \Gamma(x_p, -y_p, q^2_p) \), as well as \( F_e(x_p, y_p, q^2_p) \), which is \( \eta_p \to 0 \) limit of \( \Gamma_e(x_p, y_p, q^2_p) \), will always be greater than or equal to zero inside the physical region. On the other hand, \( \Gamma_o(x_p, y_p, q^2_p) \), which is proportional to subtraction of two positive quantities, as well as \( F_o(x_p, y_p, q^2_p) \), which is \( \eta_p \to 0 \) limit of \( \Gamma_o(x_p, y_p, q^2_p) \), could be positive, zero or negative inside the allowed region.

We now define an observable, \( R_\eta \), as

\[ R_\eta(x_p, y_p, q^2_p) = \frac{\Gamma_o}{\Gamma_e} \approx \frac{F_o}{F_e} \left[ 1 + \eta_p \left( \frac{G_o}{F_o} - \frac{G_e}{F_e} \right) \right] \quad (18) \]

and the asymmetry, \( A_\eta(x_p, y_p, q^2_p) \), in \( R_\eta \) as

\[ A_\eta(x_p, y_p, q^2_p) = \left( \frac{R_\eta}{R_{SM}} - 1 \right) \approx \eta_p \left( \frac{G_o}{F_o} - \frac{G_e}{F_e} \right) \quad (19) \]

where,

\[ R_{SM} = \frac{\Gamma_o}{\Gamma_e} \bigg|_{\eta_p=0} = \frac{F_o}{F_e} \]

Since, \( F_o \) and \( G_o \) are the zeroth order and first order terms respectively in the expansion of the odd part of \( \Gamma(x_p, y_p, q^2_p) \) with respect to \( \eta_p \) (see Eq. (16)), both of them are expected to be proportional to odd powers of \( y_p \), rendering the ratio \( (G_o/F_o) \) to be finite at \( y_p = 0 \).

We will now show that \( F_o \) i.e. the odd part of SM, has a zero for this mode for all \( q^2_p \). For simplicity, to describe the situation mathematically, we consider only the dominant contributions arising from the first and second Feynman diagrams in Fig. 1. Retaining only relevant terms up to \( O(1/m_t) \), we can write,

\[ F_o \propto y_p h(x_p, y_p, q^2_p) f(x_p, y_p, q^2_p) \quad (20) \]

where,

\[ h = \frac{1 + q^2_p}{(1 - q^2_p)^4(1 - 2x_p)((1 - q^2_p - x_p)^2 - y_p^2)^2}, \quad (21) \]

\[ f = 7 q^8_p - 4(4 - x_p) q^4_p + (11 - 4x_p + 6x^2_p - 6y^2_p) q^4_p - 2 q^2_p (1 - x_p + 8x^2_p - 6x^3_p - 4y^2_p + 2x_p y^2_p) + 3x^4_p - 12x^3_p + x^2_p (11 - 2y^2_p) - x_p (2 - 4y^2_p) - y^2_p (3 + y^2_p) \]

As can be seen from the inequalities in Eq. (15), \( h(x_p, y_p, q^2_p) \) is always positive inside the physical region. Hence, the deciding factor on the sign of \( F_o \) is only \( f(x_p, y_p, q^2_p) \). Now, on \( x_p = 1/2 \) surface, we have

\[ f \left( \frac{1}{2}, y_p, q^2_p \right) = \frac{7}{16} (1 - 2q^2_p)^4 - \frac{3}{2} (1 - 2q^2_p)^2 y^2_p - y^4_p \]

which after using the upper limit of \( |y_p| \) from Eq. (15), implies that

\[ f \left( \frac{1}{2}, y_p, q^2_p \right) \geq 0, \quad (23) \]

\[ \Rightarrow F_o \left| y_p, q^2_p \right| \geq 0, \quad (24) \]

\[ F_o \left| \frac{1}{2}, -|y_p|, q^2_p \right| \leq 0. \quad (25) \]

Similarly for any point on the curved surface \( (q^2_p - q^2_p + x^2_p - y^2_0) = 0 \) denoted as \( C \), we have \( y^2_0 = (q^2_p - q^2_p + x^2_p) \) and hence,

\[ f(x_p, y_p, q^2_p) \bigg|_C = (1 - q^2_p)(1 - 2x_p)^2 (q^2_p - 2x_p). \quad (26) \]

On using the limits of \( x_p \) and \( q^2_p \) from Eq. (15), it can easily be shown that

\[ f(x_p, y_p, q^2_p) \bigg|_C \leq 0, \quad (27) \]

\[ \Rightarrow F_o(x_p, |y_p|, q^2_p) \bigg|_C \leq 0, \quad (28) \]

\[ F_o(x_p, -|y_p|, q^2_p) \bigg|_C \geq 0. \quad (29) \]

We have concluded that \( f(x_p, y_p, q^2_p) \) < 0 along the curve \( C \) and \( f(x_p, y_p, q^2_p) \) > 0 at the other boundary surface \( x_p = 1/2 \). It is obvious therefore that there must be at least one surface within the allowed phase space region where \( f(x_p, y_p, q^2_p) = 0 \). In the first plot of Fig. 3, the blue region signifies \( f(x_p, y_p, q^2_p) < 0 \) and the brown region
FIG. 3. The variations of functions $f(x_p, y_p, q_p^2)$ and $F_o(x_p, y_p, q_p^2)$ are shown in $x_p - y_p$ plane in left and right panel, respectively, where $q_p^2 = 0.01$. The blue line in both the panels indicates one boundary of phase space with $\cos \theta = -1$ or $(q_p^2 - q_o^2 + x_p^2 + y_p^2) = 0$. In the left panel, the blue region signifies negative valued $f(x_p, y_p, q_p^2)$, the brown region symbolizes positive valued $f(x_p, y_p, q_p^2)$ and the black curve indicates $f(x_p, y_p, q_p^2) = 0$. In the right panel, the yellow region signifies negative valued $F_o(x_p, y_p, q_p^2)$, the green region symbolizes positive valued $F_o(x_p, y_p, q_p^2)$ and the red curve indicates $F_o(x_p, y_p, q_p^2) = 0$.

symbolizes $f(x_p, y_p, q_p^2) > 0$ where as the black curve indicates $f(x_p, y_p, q_p^2) = 0$. In the second plot of Fig. 3, the yellow region signifies $F_o(x_p, y_p, q_p^2) < 0$ and the green region symbolizes $F_o(x_p, y_p, q_p^2) > 0$ while the red curve indicates $F_o(x_p, y_p, q_p^2) = 0$.

We have explicitly demonstrated that there exists a surface (besides $y_p = 0$ plane) where $F_o(x_p, y_p, q_p^2) = 0$; we refer to this surface corresponding to the ‘new type of zero’ as “null-surface”. This means that at each point on this surface the differential decay rate $\Gamma(x_p, y_p, q_p^2)$ remains unaltered if we interchange the energies of photon and electron. Hence, $A_0(x_p, y_p, q_p^2)$ diverges on null-surface for any non-zero value of $\eta_p$ and becomes zero everywhere in the phase space for $\eta_p$ being zero. The null-surface divides the phase space into two regions, one where $A_0$ is positive and the other where $A_0$ is negative. For $\eta_p > 0$, $A_0 < 0$ for $x_p$ values smaller than the values indicated by the null-surface, whereas, $A_0 > 0$ for $x_p$ values larger than the values indicated by the null-surface. However, if $\eta_p < 0$, an opposite behavior in the signs of $A_0$ is indicated. This feature can be used to determine the sign of $\eta_p$. To measure the value of $\eta_p$ experimentally, one must average $A_0$ over specified regions of phase space where it could be positive or negative. Such averages are necessitated by the experimental resolutions for $q_p^2$, $x_p$ and $y_p$ and will in general reduce the asymmetry. Hence, it is convenient to use $|A_0|$ as the asymmetry. In the next section (Sec. IV) we probe the feasibility to measure $\eta_p$ using the asymmetry obtained in this section.

IV. SIMULATION AND ANALYSIS

In order to study the sensitivity of muon radiative decay mode we need to include the resolutions for energy of photon, energy of electron and the angle between them. We take them to be 2%, 0.5% and 10 Milli-radian, respectively [18]. As can be seen from Eq. (11)-(13), the resolutions for $x_p$, $y_p$ and $q_p^2$ will also vary at different point in phase space due to the functional form of these parameters. We begin by evaluating the resolutions for $x_p$, $y_p$ and $q_p^2$ for the entire allowed phase space. We find that the resolutions for $x_p$, $y_p$ and $q_p^2$ are always less than 0.01, 0.02 and 0.02 respectively. For simplicity, in our simulation, we take the worst possible scenario and assume constant resolutions for each of $x_p$, $y_p$ and $q_p^2$, corresponding to their largest value of 0.01, 0.02 and 0.02 respectively throughout the entire allowed phase space, which allows us to choose equal size bins. Hence, the phase space region $0 \leq q_p^2 \leq 1/2$, $0 \leq x_p \leq 1/2$, $-1/2 \leq y_p \leq 1/2$ is divided into 25 bins in $q_p^2$ and 50 bins in both $x_p$ and $y_p$ – all equal in size. Among these bins, only 6378 number of bins lie inside the physical phase space region. We next estimate the systematic and statistical error for $|A_0|$ in each of these bins, assuming $\eta_p = 0.01$.

To find the systematic error in $|A_0|$ for a particular $i$-th bin, we evaluate it at 62, 500 equally spaced points in that bin to estimate $|A_0|^2_j$ where $j$ is the index of a point inside the $i$-th bin. However, for the bins near to the boundary of phase space, all of these points will not be inside the physical region and hence, we denote the number of physical points inside $i$-th bin as $n_i$. Now, we can calculate the average of $|A_0|^2_j$ inside a bin, i.e.,

$$\langle |A_0|^2_i \rangle = \frac{1}{n_i} \sum_j |A_0|^2_j,$$

and take this as the asymmetry of that bin. Then we take the systematic error as the average deviation of $|A_0|^2_j$, i.e.,

$$\sigma^{syst}_{i} = \frac{1}{n_i} \sum_j \left| \langle |A_0|^2_i \rangle - |A_0|^2_i \right|.$$

The statistical error for $|A_0|$ in each bin is also estimated by averaging it at the same 62,500 equally spaced points. Note that, while $A_0$ is divergent on the null-surface the average value of $|A_0|$ for the $i$-th bin, i.e. $\langle |A_0|^2_i \rangle$, estimated from Monte Carlo studies is never larger than $10^{-6}$ for any bin. Hence,

$$\sigma^{sta}_{i} = \sqrt{\frac{1 - \langle |A_0|^2_i \rangle^2}{N_i}} \approx \frac{1}{\sqrt{(N_{SM})_i}},$$

where $i$ is the index of the bins and $N_i$ represents the number of events inside $i$-th bin which is almost the same as $(N_{SM})_i$, the number of SM events for the $i$-th bin. We have also assumed that both $A_0$ and the effects of $\eta_p$ on $N_i$ are small and can be ignored. If this were not the case $N_i$ would itself be sensitive to $\eta_p$, contrary to our simulation results. Hence, we simply take the statistical error for all practical purposes to be that in the case of SM events. The number of events in each bin is calculated by taking total number of muons to be $10^{19}$.
The total error in $|A_{\eta}|$ for any particular bin is then given by $\delta|A_{\eta}| = \sqrt{(\sigma_{\eta}^{\text{stat}})^2 + (\sigma_{\eta}^{\text{syst}})^2}$. This error in $|A_{\eta}|$ will affect the measurement of $\eta_\gamma$. Using Eq. (19), we observe that the error in the measurement of $\eta_\gamma$ in each bin as

$$\frac{\delta \eta_\gamma}{\eta_\gamma} = \frac{\delta|A_{\eta}|}{|A_{\eta}|} \quad (30)$$

where $|A_{\eta}| \equiv \langle |A_{\eta}| \rangle$ and we take the theoretical function $\left(G_{\nu e}/F_\nu - G_{\nu e}/F_\nu\right)$ to be free from experimental uncertainties. It is obvious from Eq. (30), that the highest sensitivity is achieved in bins close to the null-surface where $|A_{\eta}|$ is the largest. Hence, we consider only the region along the null-surface by applying a cut $\delta|A_{\eta}|/|A_{\eta}| \leq 10$ to determine $\eta_\gamma$.

In Fig. 4, we depict the bins, which satisfy the above cut, with red dots for different $q^2_\mu$ values, whereas, the green dots signify all the other bins inside the physical region; the purple curve indicates the null-surface where $F_o = 0$ for the corresponding $q^2_\mu$ value. Including only the bins, which satisfy the above cut, for a simulated value of $\eta_\gamma = 0.01$ (at one loop in SM, $|\eta_\gamma| \lesssim 0.015$), we estimate an error of $\delta \eta_\gamma = 2.6 \times 10^{-3}$, implying a 3.9$\sigma$ significance for the measurement. A total of $10^{19}$ muons are aimed for in the long term future. The next-round of experiments are aiming at $10^{18}$ muons /year. This reduces the sensitivity from 3.9$\sigma$ to 1.6$\sigma$. To appreciate the advantage of radiative muon decays in measuring WW$\gamma$ vertex one needs to note that the current global average of $\kappa_{\gamma}$ differs from unity only by 0.4$\sigma$. We note that the significance of the measured value of $\eta_\gamma$ may not be improved by optimizing the chosen cut and binning procedure. However, we refrain from such intricacies as our approach is merely to present a proof of principle.

Finally, we discuss possible sources of inaccuracies in our estimation of uncertainty. Higher order electroweak corrections to the process considered will modify the decay rate and alter $F_o$. While higher order electroweak corrections have not been included in our analysis they have been worked out in detail [19]. However, this is unlikely to affect our analysis technique as we have selected bins to be included in estimating $\eta_\gamma$ purely based on the criterion $\delta|A_{\eta}|/|A_{\eta}| \leq 10$ and not on the location and validity of the null-surface. A possible source of uncertainty that we have ignored in our analysis is the assumption that the muon decays at rest or with known four-momenta. While facilities that produce large numbers of muons are designed to bring the muon to rest, a fraction of them may decay with a finite but unknown 4-momenta, rendering the exact measurement of $q^2_\mu$ inaccurate. This effect can in-principle be considered by including additional systematic errors in $q^2_\mu$.

V. CONCLUSION

In order to probe lepton flavor violating process $\mu \to e\gamma$, facilities that produce large numbers of muons are being designed. We show that radiative muon decay $\mu \to e\gamma\nu_\mu\bar{\nu}_e$ is a promising mode to probe loop level corrections in the SM to the $C$ and $P$ conserving dimension four $WW\gamma$ vertex with good accuracy. The process has two missing neutrinos in the final state and on integrating their momenta the partial differential decay rate removes the well known radiation-amplitude-zero. We show, however, that the normalized differential decay rate, odd under the exchange of photon and electron energies, does have a zero in the case of standard model (SM). This new type of zero had hitherto not been studied in literature. A suitably constructed asymmetry using this fact enables a sensitive probe for the $WW\gamma$ vertex beyond the SM. The large number of muons produced keeps the
statistical error in control for a tiny part of the physical phase space, enabling us to measure $\eta_\gamma = 0.01$ with 3.9σ significance.

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