Does scarring prevent ergodicity?

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Classically chaotic systems are ergodic, that is after a long time, any trajectory will be arbitrarily close to any point of the available phase space, filling it uniformly. Using Born’s rules to connect quantum states with probabilities, one might then expect that all chaotic quantum states should be uniformly distributed in phase space. This simplified picture was shaken by the discovery of quantum scarring, where some eigenstates are concentrated along unstable periodic orbits. Despite of that, it is consensus that most eigenstates of chaotic models are indeed ergodic. Our results show instead that all eigenstates of the chaotic Dicke model are actually scarred. Even the most random states of this interacting atom-photon system never occupy more than half of the available phase space. Quantum ergodicity is achieved only as an ensemble property, after temporal averages are performed covering the phase space.

A striking feature of the quantum-classical correspondence not recognized in the early days of the quantum theory is the repercussion that measure-zero structures of the classical phase space may have in the quantum domain. A recent example is the effect of unstable fixed points, that cause the exponentially fast scrambling of quantum information in both integrable and chaotic quantum systems [1–5]. Another better known example is the phenomenon of quantum scarring [6–8]. As a classical system transits from a regular to a chaotic regime as a parameter is varied, periodic orbits that may be present in the phase space change from stable to unstable. These classical unstable periodic orbits can get imprinted in the quantum states as regions of concentrated large amplitudes known as quantum scars. Even though the phase space may be densely filled with unstable periodic orbits, they are still of measure zero, explaining why it took until the works by Gutzwiller [9, 10] for their importance in the quantum chaotic dynamics to be finally recognized.

Quantum scarring was initially observed in the Bunimovich stadium billiard [11] and soon in various other one-body systems [12, 13] giving rise to a new line of research in the field of quantum chaos [8, 14–18]. The recent experimental observation of long-lived oscillations in chains of Rydberg atoms [19], associated with what is now called “many-body quantum scars”, has caused a new wave of fascination with the phenomenon of quantum scarring [20–24]. While the interest in many-body quantum scars lies in their potential as resources to manipulate and store quantum information, a direct relationship between them and possible structures in the classical phase space has not yet been established.

Halfway between one-body and many-body models, one finds systems such as two-dimensional harmonic oscillators and the Dicke model [25], where quantum scars have also been observed [26, 27]. In the first case, the model is not fully chaotic and scarring can be understood as an extension of the regular orbits [28, 29]. The Dicke model, on the other hand, has a region of strong chaos, where the Lyapunov exponents are positive and the level statistics agrees with the predictions from random matrix theory [30]. The model describes a large number of two-level atoms that interact collectively with a quantized radiation field and was first introduced to explain the phenomenon of superradiance [31, 32]. It can be studied experimentally with cavity assisted Raman transitions [33, 34], trapped ions [35, 36], and superconducting circuits [37].

In this work, we investigate the intricate relationship between quantum scarring and phase-space localization in the superradiant phase of the Dicke model. Even though both phenomena are often treated on an equal footing, their connection is rather subtle. Scarring refers to structures that resemble periodic orbits in the phase-space distribution of quantum eigenstates, while phase-space localization implies that a state exhibits a low degree of spreading in the phase space. Here, we demonstrate that scarring does not necessarily imply phase-space localization.

In the systems studied before, scarred eigenstates were not a fraction of the total number of eigenstates. In contrast to that, we show that deep in the chaotic regime of the Dicke model, all eigenstates are scarred. Their phase-space probability distributions always display structures that can be traced back to periodic orbits in the classical limit. Yet, we find both eigenstates that are localized in phase space and eigenstates that are as much spread out as random states, although none of them, including the random states, can cover more than approximately half of the available phase space.

We introduce a new quantity to measure the level of localization of quantum states in the phase space and use it to provide a definition of quantum ergodicity. A quantum state is ergodic if its infinite-time average leads to full delocalization. Under this definition, stationary quantum states are never ergodic, while random states are, and coherent states lie somewhere in between.

Dicke model and chaos

The Hamiltonian of the Dicke model is written as

\[ \hat{H}_D = \omega \hat{a}^\dagger \hat{a} + \omega_0 \hat{J}_z + \frac{2 \gamma}{\sqrt{N}} \hat{J}_z (\hat{a}^\dagger + \hat{a}), \]

(1)
FIG. 1. Projected Husimi distribution \( \tilde{Q}_k(Q,P) \) superposed by periodic orbits from families \( A \) (blue lines in the top panels) and \( B \) (red lines in the bottom panels). Dashed lines mark the mirror image (in \( Q \) and \( q \)) of the periodic orbits drawn with solid lines. The mirror images are also periodic orbits due to the parity symmetry of the Hamiltonian. Lighter colors of the Husimi projections indicate higher concentrations. The values of the energy \( \epsilon_k \) and localization measure \( \mathcal{L}_k \) of each eigenstate \( k \) are indicated in the panels. Energies larger than \(-0.8\) are in the chaotic region. The system size is \( j = 30 \).

where \( h = 1 \). It describes \( N \) two-level atoms with atomic transition frequency \( \omega_0 \) interacting with a single mode of the electromagnetic field with radiation frequency \( \omega \). In the equation above, \( \hat{a} \) (\( \hat{a}^\dagger \)) is the bosonic annihilation (creation) operator of the field mode, \( \hat{J}_{x,y,z} = \frac{1}{2} \sum_{k=1}^{N} \hat{\sigma}^k_{x,y,z} \) are the collective pseudo-spin operators, with \( \hat{\sigma}^k_{x,y,z} \) being the Pauli matrices, and \( \gamma \) is the atom-field coupling strength.

The eigenvalues \( j(j+1) \) of the squared total spin operator \( \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \) specify the different invariant subspaces of the model. We use the symmetric atomic subspace defined by the maximum pseudo-spin \( j = N/2 \), which includes the ground state. When the Dicke model reaches the critical value \( \gamma_c = \sqrt{\omega_0/2} \), it goes from a normal phase \( (\gamma < \gamma_c) \) to a superradiant phase \( (\gamma > \gamma_c) \). Our studies are done in the superradiant phase, \( \gamma = 2 \gamma_c \), and we choose \( \omega = \omega_0 = 1 \). The rescaled energies are denoted by \( \epsilon = E/j \). For the selected parameters, \( \epsilon_{GS} = -2.125 \) is the ground-state energy.

The classical Hamiltonian, \( h_{cl}(x) \) in the coordinates \( x = (q,p;Q,P) \), is obtained by calculating the expectation value of the quantum Hamiltonian under the product of bosonic Glauber and pseudo-spin Bloch coherent states \( |x⟩ = |q,p⟩ \otimes |Q,P⟩ \) (see Methods) and dividing it by \( j \). The effective Planck constant \( h_{eff} = 1/j \) [38] determines the resolution of the quantum states on the four dimensional phase space \( \mathcal{M} \). We are able to work with large system sizes \( (j \sim 100 \) and Hilbert space dimensions \( D \sim 6 \times 10^5 \), due to the use of an efficient basis that guarantees the convergence of a broad range of eigenvalues and eigenstates [39, 40] (see Methods). The Dicke model displays regular and chaotic behavior. For the Hamiltonian parameters selected in this work, the system is in the strong-coupling hard-chaos regime for \( \epsilon > -0.8 \) (see supplementary information).

Scars in all eigenstates

The (unnormalized) Husimi function of a state \( \hat{\rho} \) is defined as \( \tilde{Q}_{\rho}(x) = \langle x | \hat{\rho} | x \rangle \), which is the expectation value of the density matrix over the coherent state \( |x⟩ \) centered at \( x \). This function is used to visualize how the state \( \hat{\rho} \) is distributed in the phase space. Quantum scars are localized around the classical periodic orbits in an energy shell of the phase space. To visualize the scars, we consider the Husimi projection over the classical energy shell at \( \epsilon \),

\[
\tilde{Q}_{\epsilon,\rho}(Q,P) = \int dq dp \delta(\epsilon - h_{cl}(x)) \tilde{Q}_{\rho}(x).
\] (2)

By integrating out the bosonic variables \((q,p)\), the remaining function can be compared with the projection of the periodic orbits on the plane of atomic variables \((Q,P)\).

By calculating the overlap of the eigenstates with tubular phase-space distributions located around classical periodic orbits [8], we selected twelve eigenstates \( \hat{\rho}_k = |E_k⟩⟨E_k| \) scarred by two different families of periodic orbits. In Fig. 1 we plot their Husimi projections \( \tilde{Q}_k = \tilde{Q}_{\epsilon,\rho_k} \) at \( \epsilon_k = E_k/j \) along with the corresponding periodic orbit of each family. Family \( A \) (solid blue line in Fig. 1 (a1)-(a6)) contains the periodic orbits of lowest period of the Dicke model, which emanate from one of the two normal modes around a stable stationary point at the ground-state energy. Family \( B \) (solid red line in Fig. 1 (b1)-(b6)) arises from the other normal mode around the same point. Scarring is clearly visible in all panels of Fig. 1. The quantum states are highly concentrated around the classical periodic orbits. This happens even in the chaotic region of high excitation energy, where the classical dynamics is completely ergodic, as seen in Figs. 1 (a5), (a6), (b5), and (b6).

It is evident from Fig. 1 that the degree of delocalization of the eigenstates in phase space varies. The Husimi distr-

![FIG. 1. Projected Husimi distribution $\tilde{Q}_k(Q,P)$ superposed by periodic orbits from families $A$ (blue lines in the top panels) and $B$ (red lines in the bottom panels). Dashed lines mark the mirror image (in $Q$ and $q$) of the periodic orbits drawn with solid lines. The mirror images are also periodic orbits due to the parity symmetry of the Hamiltonian. Lighter colors of the Husimi projections indicate higher concentrations. The values of the energy $\epsilon_k$ and localization measure $\mathcal{L}_k$ of each eigenstate $k$ are indicated in the panels. Energies larger than $-0.8$ are in the chaotic region. The system size is $j = 30$.](image-url)
bution of the eigenstates in Fig. 1 (a5) and (a6), for instance, is not entirely confined to the two periodic orbits drawn in blue. This contrasts with the high density concentration that the eigenstate in Fig. 1 (a4) shows around the plotted unstable periodic orbits. To quantify these differences, we introduce a measure of the degree of localization of a quantum state $\hat{\rho}$ in the classical energy shell. It is given by

$$\mathcal{L}(\epsilon, \hat{\rho}) = \left( \frac{V(\epsilon)}{N(\epsilon, \hat{\rho})^2} \int_{\mathcal{M}} dx \, \delta(\epsilon - h_{cl}(x)) Q^2_{\epsilon}(x) \right)^{-1},$$

(3)

where $N(\epsilon, \hat{\rho}) = \int_{\mathcal{M}} dx \, \delta(\epsilon - h_{cl}(x)) Q_{\epsilon}(x)$ is a normalization constant and $V(\epsilon) = \int_{\mathcal{M}} dx \, \delta(\epsilon - h_{cl}(x))$ is the phase-space volume of the classical energy shell at $\epsilon$. This measure is an energy-restricted second moment of the Husimi function [41] and can be understood as a linear version of the Wehrl entropy [42]. Since the Husimi function for a pure state $\hat{\rho} = |\psi\rangle \langle \psi|$ is $Q_{\epsilon} = |\langle \psi | x \rangle|^2$, $\mathcal{L}(\epsilon, \hat{\rho})$ is also similar to a participation ratio [43–45] on the overcomplete basis of coherent states, but restricted to a single classical energy shell. The value of $\mathcal{L}(\epsilon, \hat{\rho})$ indicates the fraction of the classical energy shell at $\epsilon$ that is covered by state $\hat{\rho}$. It varies from its minimum value $\mathcal{L}(\epsilon, \hat{\rho}) \sim (2\pi\hbar_{eff})^2 / V(\epsilon)$, which indicates maximum localization, to $\mathcal{L}(\epsilon, \hat{\rho}) = 1$, which implies complete delocalization over the energy shell. The former occurs for coherent states, and the latter happens if $Q_{\epsilon}(x)$ is a constant for all $x \in \mathcal{M}$ at the energy $h_{cl}(x) = \epsilon$, in which case the projection $\hat{Q}_{\epsilon, \hat{\rho}}(Q, P)$ is also constant for the allowed values of $Q$ and $P$.

All eigenstates in Fig. 1 have values of $\mathcal{L}_\epsilon = \mathcal{L}(\epsilon, \hat{\rho}_k)$ below 1/2. For the eigenstates in Fig. 1 (a1), (a2), (a4), (b3), these values are very small, since the eigenstates are almost entirely localized around the plotted periodic orbits. The value of $\mathcal{L}_\epsilon$ is larger in Fig. 1 (a3), because at the center of the diagram, there is an unstable stationary point [2], which produces a one-point scar. The localization measure is larger in Fig. 1 (b1) and (b2) simply because in these cases the phase space is very small. It is also larger for the states in the high energy region in Fig. 1 (a5), (a6), (b5), (b6), because they spread beyond the marked periodic orbits. As the energy increases approaching the chaotic region, more periodic orbits emerge in the classical limit, enhancing the likelihood that single quantum eigenstates get scarred by different periodic orbits. We stress, however, that even for those high-energy states, the drawn periodic orbits cast a bright green shadow that is clearly visible in the Husimi projections.

In the large panel in Fig. 2 (a), we show $\mathcal{L}_\epsilon$ against energy for all eigenstates between $\epsilon_{GS}$ and $\epsilon = 0.06$. This plot is equivalent to a Peres lattice [46] for expectation values of observables used in studies of chaos and thermalization. In the low-energy regular regime, $\mathcal{L}_\epsilon$ is organized along lines that can be classified with quasi-integrals of motion linked with classical periodic orbits [47]. Conversely, as the system enters the chaotic region at higher energies, the distribution of $\mathcal{L}_\epsilon$ becomes dense and looses any order. All the eigenstates in this region have values of $\mathcal{L}_\epsilon$ much lower than 1, mostly clustering below 1/2.

The value $\mathcal{L} \sim 1/2$ marks a limit on the spreading of any pure state in the high energy shells of the phase space. To show this, we build random states $|\mathcal{R}_{\epsilon}\rangle = \sum_k c_k |E_k\rangle$, where $c_k$ are complex random numbers from a Gaussian energy pro-
Coherent states Random state
-0.8 -0.3 0.005 0.004 0.001 -0.6 -0.3 -0.7 -0.3 -0.7 -0.2 -1.0 0.0 -1.0 0.0
... the evolution of the coherent states with large PR, such as those in Figs. 3 (d2)-(g2), and the evolution of the random ensemble, \( \bar{Q}_{\epsilon=-0.5,\rho} \), for the corresponding states (a3)-(h3) with the indicated values of \( \bar{\Sigma}(\epsilon = -0.5, \hat{\rho}) \). In all panels: \( j = 100 \).

file centered at energy \( \epsilon \) (see Methods). In Figs. 2 (r1)-(r4), we plot the projected Husimi distributions \( \bar{Q}_{\epsilon,\hat{\rho}}(Q, P) \) and show the localization measure \( \Sigma(\epsilon, \hat{\rho}) \) for four different random states \( \hat{\rho}_\epsilon = |\mathcal{R}_\epsilon\rangle \langle \mathcal{R}_\epsilon| \) centered at increasing energies \( \epsilon \) between \(-0.6 \) and \(-0.1 \). For all of them \( \bar{\Sigma} \sim 1/2 \), but they do not show structures that resemble closed periodic orbits. Even though random states are not scarred, they only cover approximately half of the energy shell, just as the most delocalized high-energy eigenstates. Hence, this upper bound on the phase-space delocalization within the high energy shells is not due to quantum scarring, but to quantum interference effects.

Surprisingly, and in contrast to the random states, the Husimi projections of all eigenstates in the chaotic region show structures that look like periodic orbits. This is evident from the 22 eigenstates presented in Fig. 2 (s1)-(s22), which are taken at fixed steps of \( k \) with \( \epsilon_k \in [-0.62, 0.06] \), and from various other examples provided in the supplementary information. One sees that deep in the chaotic region of the Dicke model, all quantum eigenstates are scarred with different degrees of localization, ranging from strong localization (\( \bar{\Sigma} \sim 0.1 \)) to states as delocalized as random states (\( \bar{\Sigma} \sim 1/2 \)).

**Quantum Ergodicity**

To quantify how much of the energy shell is visited on average by the evolved state \( \hat{\rho}(t) = e^{-iH_Dt}/\hat{\rho}_\epsilon e^{iH_Dt} \), we consider the infinite-time average [48],

\[
\hat{\rho} = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \hat{\rho}(t),
\]

and compute \( \bar{\Sigma}(\epsilon, \hat{\rho}) = \mathcal{L}(\epsilon, \hat{\rho}) \) with equation (3). If the whole energy shell \( \epsilon \) is homogeneously visited by \( \hat{\rho} \), then \( \bar{\Sigma}(\epsilon, \hat{\rho}) = 1 \). Covering the energy shell homogeneously is exactly what defines ergodicity in the classical limit, so we adopt this as a definition of quantum ergodicity as well. We thus say that a quantum state \( \hat{\rho} \) is ergodic over the energy shell \( \epsilon \) if \( \bar{\Sigma}(\epsilon, \hat{\rho}) = 1 \). According to this definition, all stationary states in the chaotic region of the Dicke model are therefore non-ergodic, since \( \bar{\Sigma}(\epsilon_k, \hat{\rho}_\epsilon) = \mathcal{L}(\epsilon_k, \hat{\rho}_\epsilon) \lesssim 1/2 \), as shown above.

A natural question that follows from this discussion is what happens to non-stationary states, such as coherent states or random states. We study the evolution of initial coherent states \( |\mathcal{R}(0)\rangle = |x_0\rangle = \sum_k c_k |E_k\rangle \) with mean energies \( \epsilon = -0.5 \), which are in the chaotic region. We select both coherent states that are highly localized and delocalized in the eigenbasis, with the degree of delocalization measured by the participation ratio \( P_R = (\sum_k |c_k|^4)^{-1} \). Their energy distributions are shown in Figs. 3 (a1)-(g1). The components of the states with low \( P_R \) are bunched around specific energy levels [Fig. 3 (a1) and (b1)], exhibiting the comb-like structure typical of scarred states [6]. As \( P_R \) increases, the coherent states become more spread in the eigenbasis, looking more similar to the random state \( |\mathcal{R}(0)\rangle = |\mathcal{R}_\epsilon\rangle \) shown in Fig. 3 (h1), whose mean energy is also \( \epsilon = -0.5 \). The evolution of the survival probability, \( S_P(t) = |\langle \Psi(0) | e^{-iH_Dt} |\Psi(0)\rangle|^2 \), for the coherent states with low \( P_R \) leads to large revivals before the saturation of the dynamics, as seen in Figs. 3 (a2), (b2), and (c2). This contrasts with the evolution of the coherent states with large \( P_R \), such as those in Figs. 3 (d2)-(g2), and the evolution of the random

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**FIG. 3.** Energy distribution for coherent states with different \( P_R \) values (a1)-(g1) and for a random state \( |\mathcal{R}_\epsilon\rangle \) with energy width \( \sigma = 0.3 \) (h1). All states are centered at \( \epsilon = -0.5 \) in the chaotic region. Quantum survival probability (gray solid line), its running average (blue solid line) and its asymptotic value (black dashed line) for the corresponding states (a2)-(h2). Projected Husimi distributions of the time-averaged ensemble, \( \bar{Q}_{\epsilon=-0.5,\rho} \) for the corresponding states (a3)-(h3) with the indicated values of \( \bar{\Sigma}(\epsilon = -0.5, \hat{\rho}) \). In all panels: \( j = 100 \).
state in Fig. 3 (h2). In all these cases, Figs. 3 (d2)-(h2), the approach to the asymptotic value of \( S_P(t) \) is much smoother and exhibits the so-called correlation hole, which corresponds to the ramp towards saturation. The correlation hole reflects the presence of correlated eigenvalues and is a quantum signature of classical chaos [49, 50].

In Figs. 3 (a3)-(h3), we plot the projected time-averaged Husimi distributions \( Q_{\epsilon,\hat{\rho}}(Q, P) \) for the states in Figs. 3 (a1)-(h1) and indicate their values of \( \Sigma(\epsilon, \hat{\rho}) \). The random state is indeed ergodic, reaching \( \Sigma \sim 1 \). For the coherent states, \( \Sigma \) increases as \( P_R \) does, but even for the states with the largest values of the participation ratio, \( \Sigma \) is still slightly under 1.

Remarkably, even the coherent states with high \( P_R \), which do not exhibit any comb-like structure in their energy distributions and do not show revivals in the evolution of their survival probability, still identify manifestations of quantum non-ergodicity similar to the one found in this work.

Discussion

The results presented here go beyond what has been known about scarring in quantum billiards and kicked tops, where scarring is rather an exception. For the Dicke model, all eigenstates in the chaotic region are scarred. This ubiquitous scarring leaves an imprint in the time averages of even the most random-like coherent states. For actual random states, on the other hand, the phase space can be fully covered after the time averages, so that quantum ergodicity is recovered in the ensemble sense.

These findings are important in themselves, because the Dicke model is employed to describe experiments with trapped ions and superconducting circuits. We leave it as an open question whether these features are specific to this model or present also in other interacting quantum systems. Lastly, if there are no practical phase-space tools to make use of, as for example, in the case of interacting spin-1/2 models, can one still identify manifestations of quantum non-ergodicity similar to the one found in this work?

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Methods

Classical Hamiltonian. To construct the classical Hamiltonian in a four-dimensional phase space $\mathcal{M}$ with coordinates $x = (q, p; Q, P)$, we use the Glauber-Bloch coherent states $|x\rangle = |q, p\rangle \otimes |Q, P\rangle$ [26, 30, 51–54]. They are tensor products of the bosonic Glauber coherent states $|q, p\rangle = e^{-(q/p)^2} e^{i\sqrt{2}(q+ip)} \alpha^\dagger |0\rangle$ and the pseudo-spin Bloch coherent states $|Q, P\rangle = (1 - Z^2/4)^{1/2} e^{[(Q+ip)/\sqrt{4-Z^2}] J_z |j, -j\rangle}$, where $Z^2 = Q^2 + P^2$, $|0\rangle$ denotes the photon vacuum, $|j, -j\rangle$ designates the state with all atoms in their ground state, and $J_z$ is the raising atomic operator. The classical Hamiltonian is given by

$$h_{cl}(x) = \frac{\omega}{2} (q^2 + p^2) + \frac{\omega_0}{2} Z^2 + 2\gamma Qq \sqrt{1 - Z^2/4} - \omega_0.$$  

(5)

Efficient basis and system sizes. The efficient basis is the Dicke Hamiltonian (1) eigenbasis in the limit $\omega_0 \to 0$, which can be analytically obtained by a displacement of the bosonic operator $\hat{A} = \hat{a} + (2\gamma/\sqrt{N}) \hat{J}_z$ and a rotation of $-\pi/2$ around the y axis of the collective pseudo-spin operators $(\hat{J}_x, \hat{J}_y, \hat{J}_z) \to (\hat{J}_x', \hat{J}_y', -\hat{J}_z')$, where $N = (\hat{A}^\dagger \hat{A})^N/\sqrt{N!} |N = 0; j, m\rangle$, $N$ is the eigenvalue of the modified bosonic number operator $\hat{A}^\dagger \hat{A}$ and $m' = m_x$ is the eigenvalue of the original collective pseudo-spin operator $\hat{J}_x$. The modified bosonic vacuum states in $|N = 0; j, m\rangle = |N = 0\rangle \otimes |j, m\rangle$ are Glauber coherent states $|N = 0\rangle = \sum_{j=0}^{\infty} \sum_{m} \hat{J}_z = \hat{J}_z = \hat{J}_z' = \hat{J}_x' = \hat{J}_y' = 0$. The Hilbert space dimension of this basis is given by $D = (2j + 1)(N_{\text{max}} + 1)$, where $N_{\text{max}}$ designates a cutoff of the modified bosonic subspace.

This basis allows to work with larger values of $j$ by reducing the value of $N_{\text{max}}$ required for convergence of the high-energy eigenstates. With $j = 100$ and $N_{\text{max}} = 300$ ($D = 60501$), we are able to get 30825 converged eigenstates covering the whole energy spectrum up to $\epsilon = 0.853$. Having converged eigenstates in such a high-energy regime would be infeasible with the usual Fock basis for $j = 100$. [40, 55–57]

Husimi projection and localization measure. To compute the Husimi projection given in equation (2) and the localization measure given by equation (3), one has to compute integrals of the form

$$\bar{f}(Q, P) = \int dq \int dp \delta (\epsilon - h_{cl}(x)) f(x),$$  

(7)

where $x = (q, p; Q, P)$ and $f(x)$ is a non-negative function in the phase space.

By using properties of the $\delta$ function, those integrals are reduced to

$$\bar{f}(Q, P) = \int_{p_+}^{p_-} dp \frac{\sum_{q_\pm} f(q_\pm, p; Q, P)}{\sqrt{\Delta(\epsilon, p, Q, P)}},$$  

(8)

where $q_\pm$ are the two solutions in $q$ of the second-degree equation $h_{cl}(q, p; Q, P) = \epsilon$,

$$\Delta(\epsilon, p, Q, P) = \left| \frac{\partial h_{cl}}{\partial q} (q_\pm, p; Q, P) \right|^2$$

$$= 2\omega_0 \left( \frac{\epsilon}{\omega_0} + 1 - \frac{Q^2 + P^2}{2} \right) + 4\gamma^2 Q^2 \left( 1 - \frac{Q^2 + P^2}{4} \right) - \omega^2 p^2,$$

and $p_\pm$ are the two solutions in $p$ of the second-degree equation $\Delta(\epsilon, p, Q, P) = 0$.

Because of the form of the weight $1/\sqrt{\Delta}$, the integral given by equation (8) may be computed efficiently using a Chebyshev-Gauss quadrature method.

In the case that $f(x)$ equals a constant $K \geq 0$ for all $x$ in the energy shell, then $\bar{f}(Q, P) = 4\pi K$ for all allowed $Q, P$.

It is worth noting that a quantum state with a Wigner distribution that is constant in an energy shell, will lead to a Husimi function that needs not to be constant within the same energy shell. This is because the Husimi distribution is the convolution of the Wigner distribution with Gaussian Wigner distributions of the coherent states, which have different energy widths due to the geometry of the energy shells in the phase space. This rather marginal effect may be seen in Fig. 3 (h), where there is a barely visible weak concentration towards the center of the plot causing $\bar{f}$ to be slightly under 1. We stress that this effect is not related to quantum scarring. It is just a manifestation of the phase-space geometry in the Husimi distributions.

Random states. The state $|\mathcal{R}_\lambda\rangle = \sum_k c_k |E_k\rangle$ is built by sampling random numbers $r_k > 0$ from an exponential distribution $\lambda e^{-\lambda x}$ and random phases $\theta_k$ from a uniform distribution in $[0, 2\pi)$. We use

$$c_k = \sqrt{\frac{r_k \rho(E_k)}{\nu(E_k) M}} e^{i\theta_k},$$  

(10)

where $\nu(E)$ is the density of states, $\rho(E)$ is a Gaussian profile of width $j\sigma$ centered at energy $je$, and $M$ ensures normalization. This way, $|\mathcal{R}_\lambda\rangle$ has a defined energy center $\epsilon$ where the Husimi projection and localization measure are calculated. [50, 58]

Data availability

All the data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request.

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Author contributions
SP-C and DV were responsible for most of the calculations and the development of the work. MAB-M, SL-H, LFS and JGH provided the original ideas and shaped the manuscript.

Competing financial interests
The authors declare no competing financial interests.

Additional information
Supplementary information is available for this paper.
Does scarring prevent ergodicity?

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SUPPLEMENTARY INFORMATION

Classical and quantum chaos in the Dicke model

The Dicke model displays regular and chaotic behavior [1–8]. For the parameters selected in the main text, the dynamics are regular up to $\epsilon \approx -1.6$ [7], then there is a mixed region of regularity and chaos up to $\epsilon \approx -0.8$, after which strong chaos sets in. The onset of chaos is illustrated in Fig. 1 for the classical limit (a)-(b) and for the quantum domain (c)-(d).

Figure 1 (a) shows the percentage of chaos defined as the ratio of the number of chaotic initial conditions, determined by the Lyapunov exponent, over the total number of initial conditions for a very large sample. The percentage is presented as a function of the rescaled energy $\epsilon$ and the coupling strength $\gamma$. Following the vertical red dashed line marked at $\gamma = 2\gamma_c$, one sees that energies $\epsilon \sim -0.5$ are already deep in the chaotic region (light color). This is confirmed in Fig. 1 (b), where the Poincaré section for $\epsilon = -0.5$ exhibits hard chaos, that is, all chaotic trajectories cover the entire energy shell densely and have the same positive Lyapunov exponent.

Figure 1 (c) displays the distribution $P(s)$ of the spacings $s$ between nearest-neighboring unfolded energy levels. The eigenvalues of quantum systems whose classical counterparts are chaotic are correlated and repel each other. In this case, $P(s)$ follows the Wigner surmise [9], as indeed seen in Fig. 1 (c).

In Fig. 1 (d), we show the quantum survival probability, $S_P(t) = |\langle \mathcal{R}_\epsilon | e^{-iH_D t} | \mathcal{R}_\epsilon \rangle |^2$ for a Gaussian ensemble of random initial states $|\mathcal{R}_\epsilon \rangle = \sum_k c_k |E_k\rangle$ whose components $|c_k|^2$ were generated through a random sampling (see Methods) and are centered at energy $\epsilon = -0.5$ in the chaotic region [10, 11]. The survival probability of individual random states are shown with gray solid lines, their ensemble average with an orange solid line, the running time average with a blue solid line, which overlaps with a green line that represents an analytical curve derived from the Gaussian orthogonal ensemble of the random matrix theory [10, 11]. The asymptotic value of $S_P(t)$ is shown with a horizontal red dashed line. The green and blue curves exhibit a dip below the saturation value of the quantum survival probability known as correlation hole, which is a dynamical manifestation of spectral correlations. It contains more information than the level spacing distribution $P(s)$, since in addition to short-range correlations, it captures also long-range correlations [10, 12–15].

![Figure 1](image)
All eigenstates exhibit scars

Figure 2 shows the Husimi distributions of 160 eigenstates projected over the $(Q,P)$ plane for $j = 100$. The eigenstates are selected from a list of 16,000 converged eigenstates with energies between $\epsilon_{\text{GS}} = -2.125$ and $\epsilon = 0$, sampled in steps of 100 from $k = 100$ to $k = 16000$. The values of the localization measure $L_k$ are indicated in the panels. States with $k \leq 800$ are in the regular region, those with $800 < k \leq 5600$ in the mixed region, and those with $k > 5600$ are in the region of strong chaos. In all projections, the Husimi distributions display ellipsoidal shapes that can be associated with periodic orbits in the classical limit.

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FIG. 2. Husimi projections $\tilde{Q}_k$ of 160 eigenstates for $j = 100$. The values of $k$ are indicated in the top left of each panel, along with the value of $\Delta k$. The energy range is indicated on the right side of each row of panels.