Multivariate stochastic volatility using state space models

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Abstract

A Bayesian procedure is developed for multivariate stochastic volatility, using state space models. An autoregressive model for the log-returns is employed. We generalize the inverted Wishart distribution to allow for different correlation structure between the observation and state innovation vectors and we extend the convolution between the Wishart and the multivariate singular beta distribution. A multiplicative model based on the generalized inverted Wishart and multivariate singular beta distributions is proposed for the evolution of the volatility and a flexible sequential volatility updating is employed. The proposed algorithm for the volatility is fast and computationally cheap and it can be used for on-line forecasting. The methods are illustrated with an example consisting of foreign exchange rates data of 8 currencies. The empirical results suggest that time-varying correlations can be estimated efficiently, even in situations of high dimensional data.

Some key words: volatility, multivariate, GARCH, time series, state space model, Bayesian forecasting, dynamic linear model, Kalman filter, generalized Wishart distribution.

1 Introduction

Consider that the $p$-variate time series $\{y_t\}_{t=1,...,N}$ is generated from the multivariate state space model

$$y_t = \theta_t + \Sigma_t^{1/2} \epsilon_t \quad \text{and} \quad \theta_t = \phi \theta_{t-1} + \Omega_t^{1/2} \omega_t,$$

(1)

where the innovations $\{\epsilon_t\}_{t=1,...,N}$ and $\{\omega_t\}_{t=1,...,N}$ are individually and mutually uncorrelated, following the $p$-variate Gaussian distributions $\epsilon_t \sim N_p(0, I_p)$ and $\omega_t \sim N_p(0, I_p)$, for $\Sigma_t^{1/2}$ being the symmetric square root of $\Sigma_t$ (Gupta and Nagar, 1999) and $I_p$ denotes the $p \times p$ identity matrix. Typically, at time $t$, $y_t$ will represent the log-returns of some assets or exchange rates or any other financial time series. $\Sigma_t$ is the volatility matrix at time $t$ and interest is placed on its estimation, while $\Omega_t$ is a non-negative definite matrix. An evolutionary law for $\Sigma_t$ and a density for the initial state $\theta_0$ have to be defined. It is worthwhile to note that several volatility models can be obtained from the formulation of model (1). For example, for $\phi = 1$, $\theta_0 = \theta$ (with probability 1), and $\Omega_t = 0$, one obtains the volatility model $y_t = \theta + \Sigma_t^{1/2} \epsilon_t$. Then, depending on the evolution law for $\Sigma_t$, one can obtain multivariate GARCH (MGARCH) type

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models (Bauwens et al., 2006) or multivariate stochastic volatility (MSV) models (Asai et al., 2006; Maasoumi and McAlleer, 2006).

The purpose of this paper is to develop an estimation procedure that will allow fast and efficient estimation of $\Sigma_t$ and forecasting of $y_t$. Our motivation stems from work on MSV models that experience problems due to the simulation-based estimation procedures they use, see e.g. Uhlig (1997), Aguilar and West (2000), Philipov and Glickman (2006), and references therein. The fast estimation procedures, proposed in this paper, aim to achieve high computational savings (which is necessary in high dimensional data) and yet enjoy the sophistication of stochastic volatility. For this to commence, one needs to define $\Omega_t$ and to propose an evolutionary law for $\Sigma_t$. Since $\theta_t$ is unobserved signal, $\Omega_t$ is suggested to be specified, rather than estimated from the data, as the latter will require to resort to simulation-based estimation techniques (e.g. MCMC or EM algorithm), and this can cause significant delays of the estimation of the volatility. In this paper we adopt Bayesian estimation, for which we aim to specify a prior distribution for $\Sigma_0$. Then it is desirable, in order to develop conjugate analyses that will facilitate fast estimation, to define $\Omega_t$ to be proportional to $\Sigma_t$. Indeed, in the context of a time-invariant volatility $\Sigma_t = \Sigma$, this setting is well known (Harvey, 1989; Durbin and Koopman, 2001; Lütkepohl, 2007). In this setting $\Omega_t = \Omega \Sigma_t$, where $\Omega > 0$ is known. However, this is overly restrictive, as the correlation matrix of $\Sigma_t^{1/2} \epsilon_t$ is the same as the correlation matrix of $\Omega_t^{1/2} \omega_t$. In this paper we define $\Omega_t = \Sigma_t^{1/2} \Omega \Sigma_t^{1/2}$, where now $\Omega$ is a known non-negative definite matrix (later in the paper we explain how this matrix can be specified). This setting clearly encompasses the situation when $\Omega$ is scalar, and it allows more general and more flexible estimation.

For the volatility covariance matrix $\Sigma_t$, we propose a multiplicative stochastic law of its precision $\Sigma_t^{-1}$, i.e.

$$
\Sigma_t^{-1} = k \mathcal{U}(\Sigma_t^{-1-1})' B_t \mathcal{U}(\Sigma_t^{-1-1}), \quad t = 1, \ldots, N,
$$

where $k = \{\delta(1 - p) + p\} \{\delta(2 - p) + p - 1\}$, for a discount factor $0 < \delta < 1$, and $\mathcal{U}(\Sigma_t^{-1})$ denotes the unique upper triangular matrix based on the Choleski decomposition of $\Sigma_t^{-1}$. Here $B_t$ is a $p \times p$ random matrix following the multivariate singular beta distribution $B_t \sim B(m/2, 1/2)$, where $m = \delta(1 - \delta)^{-1} + p - 1$. Some details of this distribution can be found in the appendix (see Lemma 5), but for more details the reader is referred to Uhlig (1994), Díaz-García and Gutiérrez (1997), and Srivastava (2003). In Section 3 it is shown that when $\Omega = I_p$ and with $\mathbb{E}(.)$ denoting expectation, we have $\mathbb{E}(\Sigma_t^{-1}|y^{t-1}) = \mathbb{E}(\Sigma_t^{-1}|y^{t-1})$, while the respective covariance matrix at $t$ is increased of that at time $t - 1$; in this case $\Sigma_t^{-1}$ follows a random walk. When $\Omega$ is a covariance matrix, then the evolution (2) suggests approximately a random-walk type process for $\Sigma_t^{-1}$. The choice of $k$ is made in order to accommodate the above random walk equations. It should be noted that, if $p = 1$, it is $k = 1/\delta$, and (2) is reduced to $\Sigma_t = \delta \Sigma_{t-1} B_t^{-1}$ (Uhlig, 1994; Triantafyllopoulos, 2007). In order to accommodate for the definition of $\Omega_t$, we generalize the Wishart distribution and we extend its convolution with the multivariate singular beta, which was first proved in Uhlig (1994). In order to support conjugate analysis, this generalization is necessary, because of the definition of $\Omega_t$. Finally, at time $t = 0$, $\theta_0$ is assumed to be uncorrelated with $\{\epsilon_t\}_{t=1,2,\ldots,N}$ and $\{\omega_t\}_{t=1,\ldots,N}$ and it is assumed that $\theta_0 \sim N_p(m_0, \Sigma_0^{1/2} P_0 \Sigma_0^{1/2})$, for some known prior mean vector $m_0$ and covariance matrix $P_0$. The scalar constant $\phi$ is assumed known. Compared with existing MGARCH and MSV models, a major advantage of the proposed methodology, is that the likelihood function is provided in closed form. This can facilitate model comparison, but it can also be used as a means for the choice of the parameters, without the need to resort
to numerical methods in order to maximize the likelihood function (more details on this are provided via the data analysis in Section 4).

The remaining of the paper is organized as follows. The following section generalizes the inverted Wishart distribution and discusses some properties of the new distribution. In Section 3 the main algorithm of the volatility is developed and Section 4 analyzes the volatility of foreign exchange rates data. The findings of the paper are summarized in Section 5 and the appendix includes all proofs of arguments in Sections 2 and 3.

2 Generalized inverted Wishart distribution

Let $X \sim IW_p(n, A)$ denote that the matrix $X$ follows an inverted Wishart distribution with $n$ degrees of freedom and with parameter matrix $A$. Given $A$, we use the notation $|A|$ for the determinant of $A$ and the notation $etr(A)$ for the exponent of the trace of $A$. The following theorem introduces a new distribution generalizing the inverted Wishart distribution.

**Theorem 1.** Consider the $p \times p$ random covariance matrix $X$ and denote with $X^{1/2}$ the symmetric square root of $X$. Given $p \times p$ covariance matrices $A$ and $S$ and a positive scalar $n > 2p$, define $Y = X^{1/2}A^{-1}X^{1/2}$ so that $Y$ follows an inverted Wishart distribution $Y \sim IW_p(n, S)$. Then the density function of $X$ is given by

$$p(X) = \frac{|A|^{(n-p-1)/2}|S|^{(n-p-1)/2}}{2^{p(n-p-1)/2}\Gamma_p((n-p-1)/2)|X|^{n/2}}etr(-AX^{-1/2}SX^{-1/2}/2),$$

where $\Gamma_p(.)$ denotes the multivariate gamma function.

The distribution of the above theorem proposes a generalization of the inverted Wishart distribution, since if $A = I_p$ we have $X \sim IW_p(n, S)$ and if $S = I_p$, we have $X \sim IW_p(n, A)$. This is clearly a different generalization of other generalizations of the inverted Wishart distribution, see Dawid and Lauritzen (1993), Brown et al. (1994), Roverato (2002), and Carvalho and West (2007). In the following we refer to the distribution of Theorem 1 as the generalized inverted Wishart distribution, and write $X \sim GIW_p(n, A, S)$. The next result gives some expectations of the GIW distribution.

**Theorem 2.** Let $X \sim GIW_p(n, A, S)$ for some known $n, A$ and $S$. Then we have

(a) $E(X^{1/2}S^{-1}X^{1/2}) = (n - 2p - 2)^{-1}A$; $E(X^{-1/2}SX^{-1/2}) = (n - p - 1)A^{-1}$;

(b) $E|X|^\ell = 2^{-\ell}[\Gamma_p((n - p - 1)/2)]^{-1}\Gamma_p((n - 2\ell - p - 1)/2)|A|^\ell|S|^\ell$,

where $E(.)$ denotes expectation and $0 < \ell < (n - 2p)/2$.

The following property reflects on the symmetry of $A$ and $S$ in the GIW distribution.

**Theorem 3.** If $X \sim GIW_p(n, A, S)$, for some known $n, A$ and $S$, then $X \sim GIW_p(n, S, A)$.

We motivate the estimator $\hat{X}(A, S)$ of $X \sim GIW_p(n, A, S)$ as follows. The estimator should be a symmetric positive definite matrix and for $A$ and $S$ being matrices, one possibility is $\hat{X}(A, S) = kA^{1/2}SA^{1/2}$, for a known constant $k$. This estimator equals the expectation of the inverted Wishart distribution $IW_p(k + 2p + 2, A^{1/2}SA^{1/2})$. Since in general $A^{1/2}SA^{1/2} \neq S^{1/2}AS^{1/2}$, a similar estimator for $X$ can be considered as $\hat{X}^*(A, S) = k^*S^{1/2}AS^{1/2}$, for some constant $k^*$. We propose that the desired estimator for $X$ should satisfy the following requirements:
(1) In the univariate case \((p = 1)\) the estimator should be \(\tilde{X}(A, S) = AS/(n - 4)\);

(2) The estimator should be symmetric in \(A\) and \(S\), i.e. \(\tilde{X}(A, S) = \tilde{X}(S, A)\);

(3) If \(A = I_p\) the estimator should reduce to the expectation from the inverted Wishart density \(IW_p(n, S)\), i.e. \(\tilde{X}(A, S) = (n - 2p - 2)^{-1}S\); If \(S = I_p\) the estimator should reduce to the expectation from the inverted Wishart density \(IW_p(n, A)\), i.e. \(\tilde{X}(A, S) = (n - 2p - 2)^{-1}A\).

Now we propose the estimator

\[
\tilde{X}(A, S) = \frac{1}{2n - 4p - 4} \left( S^{1/2}AS^{1/2} + A^{1/2}SA^{1/2} \right),
\]

for which we can see that (1)-(3) are satisfied.

It is also easy to verify that if \(X \sim GIW_p(n, A, S)\), then the density of \(Y = X^{-1}\) is

\[
p(Y) = \frac{|A|^{(n-p-1)/2}|S|^{(n-p-1)/2}|Y|^{(n-2p-2)/2}}{2^{(n-p-1)/2} \Gamma_p \{(n - p - 1)/2\}} \text{etr}(-AY^{1/2}SY^{1/2}/2).
\]

This distribution generalizes the Wishart distribution; we will say that \(Y\) follows the generalized Wishart distribution with \(n - p - 1\) degrees of freedom, covariance matrices \(A^{-1}\) and \(S^{-1}\), and we will write \(Y \sim GW_p(n - p - 1, A^{-1}, S^{-1})\). It is easy to see that when \(A = I_p\) or \(S = I_p\), the above density reduces to a Wishart density. Again our terminology and notation, should not cause any confusion with other generalizations of the Wishart distribution, proposed in the literature (Letac and Massam, 2004).

The next theorem is a generalization of the convolution of the Wishart and multivariate singular beta distributions (Uhlig, 1994). For some integers \(m, n\), denote with \(B_p(m/2, n/2)\) the multivariate singular beta distribution with \(m\) and \(n\) degrees of freedom. The density of this distribution is given in the appendix (see Lemma \(\text{5}\) and more details can be found in Uhlig (1994), Díaz-García and Gutiérrez (1997), and Srivastava (2003).

**Theorem 4.** Let \(p\) and \(n\) be positive integers and let \(m > p - 1\). Let \(H \sim GW_p(m + n, A, S)\) and \(B \sim B_p(m/2, n/2)\) be independent, where \(A\) and \(S\) are known covariance matrices. Then

\[G \equiv U(H)^tBU(H) \sim GW_p(m, A, S),\]

where \(U(H)\) denotes the upper triangular matrix of the Choleski decomposition of \(H\).

### 3 Estimation

**3.1 The main algorithm**

In this section we consider estimation for model (1), where \(\Sigma_t\) follows the evolution (2). The prior distributions of \(\theta_0|\Sigma_0\) and \(\Sigma_0\) are chosen to be Gaussian and a generalized inverted Wishart respectively, i.e.

\[\theta_0|\Sigma_0 \sim N_p(m_0, \Sigma_0^{1/2}P_0\Sigma_0^{1/2}) \quad \text{and} \quad \Sigma_0 \sim GIW_p(n_0, Q^{-1}, S_0),\]

for some known parameters \(m_0, P_0 = p_0I_p, n_0 > 2p + 2\) and \(S_0\). \(Q\) is the limit of \(Q_{t-1}(1) = P_{t-1} + \Omega + I_p\), where \(P_t\) is a known covariance matrix. The next result shows that the limit of \(P_t\) (and hence the limit of \(Q_{t-1}(1)\)) exist and it provides the value of this limit as a function of \(\phi\) and \(\Omega\).
Theorem 5. If \( P_t = R_t(R_t + I_p)^{-1} \), with \( R_t = \phi^2 P_{t-1} + \Omega \), where \( \Omega \) is a positive definite matrix and considering the prior \( P_0 = p_0 I_p \), for a known constant \( p_0 > 0 \), it is
\[
P = \lim_{t \to \infty} P_t = \frac{1}{2\phi^2} \left\{ (\Omega + (1 - \phi^2)I_p)^2 + 4\Omega \right\}^{1/2} - \Omega - (1 - \phi^2)I_p,
\]
for \( \phi \neq 0 \) and \( P = \Omega(\Omega + I_p)^{-1} \), for \( \phi = 0 \).

This result generalizes relevant limit results for the univariate random walk plus noise model (Anderson and Moore, 1979, page 77; Harvey, 1989, page 119).

Let \( Y \sim t_p(n, m, P) \) denote that the \( p \)-dimensional random vector \( Y \) follows a multivariate Student \( t \) distribution with \( n \) degrees of freedom, mean \( m \) and scale or spread matrix \( P \) (Gupta and Nagar, 1999, Chapter 4). The next result gives an approximate Bayesian algorithm for the posterior distributions of \( \theta_t \) and \( \Sigma_t \) as well as for the one-step forecast distribution of \( y_t \).

Theorem 6. In the multivariate state space model (1) with evolution (2), let the initial priors for \( \theta_0 | \Sigma_0 \) and \( \Sigma_0 \) be specified as in equation (4). The one-step forecast and posterior distributions are approximately given, for each \( 1 \leq t \leq N \), as follows:

(a) One-step forecast at time \( t \): \( \Sigma_t | y^t-1 \sim GIW_p(\delta(1 - \delta)^{-1} + 2p, Q^{-1}, k^{-1}S_{t-1}) \) and \( y_t | y^t-1 \sim t_p(\delta(1 - \delta)^{-1} - k^{-1}S_{t-1}), \) where \( k = (\delta(1 - p) + p)(\delta(2 - p) + p - 1)^{-1} \) and \( \delta, S_{t-1}, m_{t-1} \) are known at time \( t - 1 \).

(b) Posteriors at \( t \): \( \theta_t | \Sigma_t, y^t \sim N_p(m_t, \Sigma_t^{-1}P_t\Sigma_t^{-1}) \)
and \( \Sigma_t | y^t \sim GIW((1 - \delta)^{-1} + 2p, Q^{-1}, S_t) \), with
\[
m_t = m_{t-1} + A_t e_t, \quad P_t = (\phi^2 P_{t-1} + \Omega)(\phi^2 P_{t-1} + \Omega + I_p)^{-1},
\]
e_{t} = y_{t} - m_{t-1}, \quad S_t = k^{-1}S_{t-1} + e_t e_t^t,

where \( A_t = \Sigma_t^{-1}P_t\Sigma_t^{-1} \) is approximated by \( A_t^* = (S_t^*)^{-1}P_t(S_t^*)^{-1} \), with \( S_t^* = \Sigma(Q^{-1}, S_t) \) the estimator of \( \Sigma_t | y^t \) (see equation (4)) and \( Q_{t-1}(1) = P_{t-1} + \Omega + I_p \) being approximated by its limit \( Q = P + \Omega + I_p \), where \( P \) is given by Theorem 5.

From Theorem 6 we have that the one-step forecast vector mean and covariance matrix of \( y_t \) are
\[
y_{t-1}(1) = \mathbb{E}(y_t | y^t-1) = m_{t-1} \quad \text{and} \quad \text{Var}(y_t | y^t-1) = \frac{k^{-1}S_{t-1}}{\delta(1 - \delta)^{-1} - 2} = \frac{(1 - \delta)S_{t-1}}{(3\delta - 2)k},
\]
for \( \delta > 2/3 \).

We note that \( k > 1 \), since \( \delta(1 - p) + p > \delta(2 - p) + p - 1 \), for any \( 0 < \delta < 1 \) and so, if we expand \( S_t \) as
\[
S_t = k^{-t}S_0 + \sum_{i=1}^{t} k^{i-t} e_t e_t^t,
\]
for large \( t \), we can approximate \( S_t \) by \( \sum_{i=1}^{t} k^{i-t} e_t e_t^t \). The observation that \( k^{-1} < 1 \) is important, because otherwise \( S_t \) could tend to infinity.

From Theorem 6 if \( \Omega = I_p \), then \( \Sigma_t | y^t-1 \sim IW_p(\delta(1 - \delta)^{-1} + 2p, k^{-1}Q^{-1}S_{t-1}) \) and \( \Sigma_{t-1} | y^{t-1} \sim IW_p((1 - \delta)^{-1} + 2p, Q^{-1}S_{t-1}) \), where now \( Q \) is a variance. Thus \( \Sigma_t^{-1} | y^{t-1} \sim W_p(\delta(1 - \delta)^{-1} + p - 1, kQS_{t-1}^{-1}) \) and \( \Sigma_{t-1}^{-1} | y^{t-1} \sim W_p((1 - \delta)^{-1} + p - 1, QS_{t-1}^{-1}) \) so that
\[
\mathbb{E}(\Sigma_t^{-1} | y^{t-1}) = \left( \frac{\delta}{1 - \delta} + p - 1 \right) kQS_{t-1}^{-1} = \left( \frac{1}{1 - \delta} + p - 1 \right) QS_{t-1}^{-1} = \mathbb{E}(\Sigma_{t-1}^{-1} | y^{t-1}), \quad (5)
\]
where $\delta > I$ diagonal elements the positive eigenvalues of $\Sigma$ can be set using historical data, but a general guideline suggests the mean absolute deviation (MAD), and the mean one-step forecast error (ME). The priors $-\Sigma_0$ and $k$ with $y \in$ the sense that $\text{Var}(\Sigma^{-1})$ denotes the column stacking operator of $\Sigma^{-1}$. Equations (5) and (6) show that when $\Omega = 1$, $\Sigma^{-1}$ follows a random walk type evolution. When $\Omega$ is a covariance matrix we can see that

$$
\log \text{likelihood function of } \Sigma^{-1} = \log \text{likelihood function of } \Sigma^{-1}
$$

In this section we discuss several performance measures for model (1). We start giving the log-likelihood function of $\Sigma^{-1}$.

**Theorem 7.** In the state space model (1) with evolution (2) denote with $\ell(\Sigma_1, \ldots, \Sigma_N; y^N)$ the log-likelihood function of $\Sigma_1, \ldots, \Sigma_N$, based on data $y^N = \{y_1, \ldots, y_N\}$. Then it is

$$
\ell(\Sigma_1, \ldots, \Sigma_N; y^N) = c - \frac{1}{2} \sum_{t=1}^{N} e_t^T \Sigma^{-1/2} - \frac{2\delta - 1}{1 - \delta} \sum_{t=1}^{N} \log |U(\Sigma^{-1})| - \frac{p}{2} \sum_{t=1}^{N} \log |L_t| - \frac{3\delta - 2}{2(1 - \delta)} \sum_{t=1}^{N} \log |\Sigma_t|,
$$

and

$$
c = -Np \log \pi - \frac{N}{2} \log |Q| - \frac{Np}{2} \log k + N \log \left\{ \Gamma_p \left( \frac{\delta(1 - p) + p}{2(1 - \delta)} \right) / \Gamma_p \left( \frac{\delta(2 - p) + p - 1}{2(1 - \delta)} \right) \right\},
$$

where $\delta > 2/3$, $k = \{\delta(1 - p) + p\} \{\delta(2 - p) + p - 1\}^{-1}$ and $L_t$ is the diagonal matrix with diagonal elements the positive eigenvalues of $I_p - k^{-1} \{U(\Sigma^{-1})\}^{-1} \Sigma^{-1} \{U(\Sigma^{-1})\}^{-1}$, with $\Sigma^{-1} = U(\Sigma^{-1}) U(\Sigma^{-1})$.

The choice of $\delta$, $\Omega$ and the priors $m_0$, $p_0$, and $S_0$ can be done by either maximizing the log-likelihood function or optimizing performance measures, such as the mean of square one-step forecast errors (MSE), the mean of square standardized one-step forecast errors (MSSE), the mean absolute deviation (MAD), and the mean one-step forecast error (ME). The priors can be set using historical data, but a general guideline suggests $m_0 = 0$, $p_0 = 1000$ and
S_0 = I_p. In any case these initial settings are not critical to the performance of the model, especially given plethora of data. It then remains to specify Ω and δ. Given data y_1, \ldots, y_N, the definition of the above mentioned performance measures are

\[ \text{MSE} = \frac{1}{N} \sum_{t=1}^{N} (e_{1t}^2, \ldots, e_{pt}^2), \quad \text{MSSE} = \frac{1}{N} \sum_{t=1}^{N} (u_{1t}^2, \ldots, u_{pt}^2), \]

\[ \text{MAD} = \frac{1}{N} \sum_{t=1}^{N} (\text{mod}(e_{1t}), \ldots, \text{mod}(e_{pt}))', \quad \text{ME} = \frac{1}{N} \sum_{t=1}^{N} (e_{1t}, \ldots, e_{pt})', \]

where \( e_t = (e_{1t}, \ldots, e_{pt})' \) is the one-step forecast error vector, \( \text{mod}(e_{jt}) \) denotes the modulus of \( e_{jt} \) (\( j = 1, \ldots, p \)) and \( u_t = (u_{1t}, \ldots, u_{pt})' \) is the standardized one-step forecast error vector, defined by

\[ u_t = \begin{cases} (1 - \delta)S_t \left( \frac{1}{3\delta - 2} \right)^{1/2} & e_t, \end{cases} \]

so that \( E(u_t|y_t^{-1}) = 0 \) and \( E(u_t u_t'|y_t^{-1}) = I_p \). Thus, if the model is a good fit, it should return \( \text{MSSE} \approx (1, \ldots, 1)' \), \( \text{ME} \approx (0, \ldots, 0)' \), while \( \text{MAD} \) and \( \text{MSE} \) should be as small as possible.

The MSSE is usually preferred to MSE, because it takes into account the forecast covariance matrix of the log-returns. However, since the MSE can be used for comparison of two or more models it is mentioned here. When we look at the performance of a single model the MSSE has the ability to judge the goodness of fit in an effective way. The MAD has a similar performance as the MSE, while the ME is useful if we wish to check how biased is the estimation method (Fildes, 1992).

In order to choose the optimal \( \Omega \) we propose the following search procedure. Since \( \Omega \) has \( p(p + 1)/2 \) distinct elements, for relatively large \( p \) there are many elements in \( \Omega \) to be optimized. One can reduce the dimensionality of this optimization by considering a diagonal choice for \( \Omega \), writing \( \Omega = \text{diag}(w_1, \ldots, w_p) \). Since \( 0 < w_i < \infty \), still a search procedure for the optimal \( w_i \) can be time-consuming. By defining \( Z = \Omega(I_p + \Omega)^{-1} \), we have that \( Z \) is also diagonal and it is \( 0 < Z < I_p \). This means that we can use a grid search procedure to find the optimal value for \( Z \) and then choose \( \Omega = (I_p - Z)^{-1} \). For \( Z = (z_1, \ldots, z_p)' \) we can use \( z_i = 1/10^q, \ldots, (10^q - 1)/10^q \), for \( i = 1, \ldots, p \) and \( q \) a positive integer; for most applications \( q = 2 \) or \( q = 3 \) will suffice. Then we can readily see that \( w_i = z_i/(1 - z_i) \), for \( i = 1, \ldots, p \). We use this search procedure in the example of Section 4.

4 FX data analysis

In this section we consider foreign exchange rates data (FX) of 8 currencies, namely Australian Dollar vs US Dollar (AUD/USD), British Pound vs USD (GBP/USD), Canadian Dollar vs USD (CAD/USD), Dutch Guilder vs USD (DUG/USD), French Franc vs USD (FRF/USD), German DeutschMark vs USD (GDM/USD), Japan Yen vs USD (JPY/USD), and Swiss Franc vs USD (SWF/USD). The data are sampled in daily frequency, from January 1980 to December 1997 and these data are reported in Franses and van Dijk (2000). We form the log-returns vector series \( y_t = (y_{1t}, \ldots, y_{8t})' \), where \( y_{1t} \) is the log-returns of AUD/USD, \ldots, \( y_{8t} \) is the log-returns of SWF/USD; the data are plotted in Figure 6. To specify \( \Omega \) we used the log-likelihood criterion with the search procedure of Section 4. Using \( q = 2 \), an optimal diagonal matrix \( Z \)
Figure 1: FX data; shown are (a) the AUD/USD exchange rate, (b) the GBP/USD rate, (c) the CAD/USD rate, (d) the DUG/USD rate, (e) the FRF/USD rate, (f) the GDM/USD rate, (g) the JPY/USD rate and (h) the SWF/USD rate.

was $Z = \text{diag}(0.44, 0.54, 0.56, 0.87, 0.92, 0.52, 0.99, 0.77)$ and so the diagonal $\Omega$ that maximizes the log-likelihood function is $\Omega = \text{diag}(0.786, 1.174, 1.272, 6.692, 11.500, 1.083, 99.000, 3.348)$, for $\delta = 0.7, \phi = 1, m_0 = 0$ and $S_0 = I_8$. This setting for $\Omega$ reveals a clear benefit as opposed to a setting $\Omega = wI_8$, for a known $w \geq 0$, as we can see that the correlation matrix of $\epsilon_t$ and the correlation matrix of $\omega_t$ are not the same. Indeed, at the posterior estimate $\Sigma_{\epsilon}$ of $\Sigma_{4773}$, we can see that the correlation matrices of $\epsilon_t$ and $\omega_t$ differ significantly, with the latter having larger correlations.

For this data set we observed that larger values of $\delta$ (in particular values of $\delta$ in the range $0.9 \leq \delta \leq 0.99$) can not capture the volatility shocks, returning large values for the MSSE. The log-likelihood function, evaluated at the posterior estimate of $\Sigma$, for the above optimal settings was $-98438.78$. The four performance measures are

$$\text{MSE} = (0.00009, 0.00002, 0.00001, 0.00031, 0.00244, 0.00026, 1.67707, 0.00022)'$$
$$\text{MSSE} = (0.933, 0.911, 0.980, 0.979, 0.948, 0.940, 0.924, 0.916)'$$
$$\text{MAD} = (0.006, 0.003, 0.003, 0.013, 0.036, 0.012, 0.915, 0.011)'$$
$$\text{ME} = (-3.14 \times 10^{-7}, 7.53 \times 10^{-7}, -2.73 \times 10^{-6}, -2.32 \times 10^{-6}, -5.77 \times 10^{-6}, -1.82 \times 10^{-6},$$
$$-4.67 \times 10^{-4}, -2.08 \times 10^{-8})'$$

The MSSE is slightly under $(1, \ldots, 1)'$, which means that the volatilities are slightly overestimated. However, looking at the MSE, MAD, the ME and the log-likelihood function, we
consider this model as acceptable.

For $\Sigma_t = (\sigma_{ij,t})_{i,j=1,...,8}$, Figure 2 shows the posterior volatilities of each of the $\sigma_{ii,t}$ ($i = 1, \ldots, 8$), for the last 774 observations, i.e. from $t = 4000$ until $t = 4773$. Most of the volatilities are small, except for the JPY/USD, but even for small volatilities Figure 2 indicates clearly the highly volatile periods for each exchange rate. Figure 3 shows the posterior correlations of GBP/USD with the other rates. This figure confirms that the correlations are time-varying. By inspecting Figure 3 we observe that GBP/USD is most correlated with DUG/USD, FRF/USD and JPY/USD, while GBP/USD is least correlated (but still significantly correlated) with AUD/USD and CAD/USD.

We note that for this data set, there were 4773 time points and for the 8-dimensional time series $\{y_t\}$, the estimation algorithm (see Theorems 6 and 7), implemented in R, (including the search procedure to maximize the log-likelihood function) took about 8 minutes to run on a Pentium PC.

## 5 Discussion

In this paper we have provided a Bayesian analysis for multivariate stochastic volatility. We propose a generalization of the Wishart and inverted Wishart distributions and we extend the convolution between the Wishart and the multivariate singular beta distributions. This gener-
alization is motivated from the multivariate random walk plus noise model, which innovation vectors are desired to have different correlations. The proposed estimation methodology is delivered in closed form and it is fast and easily implementable, even for high dimensional data. The log-likelihood of the volatility is obtained in closed form and this is an important step forward on multivariate volatility estimation, quoting “The estimation of the canonical SV model and its various extensions was at one time considered difficult since the likelihood function of these models is not easily calculable.” from Chib et al. (2007). The availability of the log-likelihood function in closed form allows more efficient model comparisons, e.g. via sequential likelihood tests or via sequential Bayes’ factors (Salvador and Gargallo, 2004; Triantafyllopoulos, 2006). Moreover, the proposed model develops a fast Bayesian algorithm not depending on simulation-based estimation procedures and not requiring many parameters to be estimated. In the special case where the volatility of state vector is proportional to the volatility of the observation vector, the analysis is exact and the inverse of the volatility matrix follows a Wishart process.

The procedure proposed in this paper attempts to combine the simplicity of non-iterative algorithms with the sophistication of stochastic volatility procedures. Algorithms such as the one developed here, are particularly attractive, because they can model high dimensional
data, with low computational cost, and still they can enjoy the mathematical properties of closed estimation procedures, which aim to address volatility estimation and forecasting for a wide class of financial data.

Appendix

Proof of Theorem 1 Consider the transformation \( Y = X^{1/2}S^{-1}X^{1/2} \). From Olkin and Rubin (1964) the determinant of the Jacobian matrix of \( X \) with respect to \( Y \) is \( J(Y \to X) = J(Y \to X^{1/2}) J(X^{1/2} \to X) = \prod_{i \leq j} (\lambda_i + \lambda_j)(\xi_i + \xi_j)^{-1}, \) where \( \lambda_1, \ldots, \lambda_p \) are the eigenvalues of \( S^{-1/2}X^{1/2}S^{-1/2} \) and \( \xi_1, \ldots, \xi_p \) are the eigenvalues of \( X^{1/2} \). We observe that if \( A = I_p \), then \( p(X) \) is an inverted Wishart distribution, since \( \text{tr}(-X^{-1/2}SX^{-1/2}/2) = \text{tr}(-SX^{-1/2}). \)

The Jacobian \( J(Y \to X) \) does not depend on \( A \) and so we can determine \( J(Y \to X) \) from the special case of \( A = I_p \). With \( A = I_p, X \sim IW_p(n, S) \) and \( Y \sim IW_p(n, I_p) \) and from the transformation \( Y = X^{1/2}S^{-1}X^{1/2} \) we get

\[
p(Y) = \frac{|S|^{(n-p-1)/2} \text{etr}(-Y^{-1}/2) J(Y \to X)}{2(p-n-1)/2 |(n-p-1)/2| S^{n/2}}.
\]

Since \( Y \sim IW_p(n, I_p) \) it must be \( |S|^{-n/2}|S|^{-p-1/2} J(Y \to X) = 1 \) and so \( J(Y \to X) = |S|^{(p-1)/2}. \)

Now, in the general case of a covariance matrix \( A \), we see

\[
\int_{X > 0} p(X) \, dX = \int_{Y > 0} \frac{|A|^{(n-p-1)/2}}{2^{p(n-p-1)/2} \Gamma_p((n-p-1)/2) Y^{n/2}} \text{etr}(-AY^{-1}/2) \, dY = 1,
\]

since \( Y \sim IW_p(n, A) \).

Proof of Theorem 2 First we prove (a). From the proof of Theorem 1 we have that \( Y = X^{1/2}S^{-1}X^{1/2} \sim IW_p(n, A) \) and so \( \mathbb{E}(Y) = (n-2p-2)^{-1} A \) and \( \mathbb{E}(Y^{-1}) = (n-p-1)A^{-1}. \)

Proceeding with (b) we note from the proof of Theorem 1 that for any \( n > 2p \)

\[
\int_{X > 0} |X|^{-n/2} \text{etr}(-AX^{-1/2}SX^{-1/2}/2) \, dX = c^{-1},
\]

where \( c \) is the normalizing constant of the distribution of \( X \). Then

\[
\mathbb{E}|X|^\ell = c \int_{X > 0} |X|^{-(n-2\ell)/2} \text{etr}(-AX^{-1/2}SX^{-1/2}/2) \, dX = \frac{c}{c^*},
\]

where

\[
c^* = \frac{2^{p\ell}|A|^{(n-p-1)/2} |S|^{(n-p-1)/2}}{2^{p(n-p-1)/2} |A|^\ell |S|^{(n-2\ell - p-1)/2}}
\]

and the range of \( \ell \) makes sure that \( n - 2\ell > 2p \). The result follows by eliminating the factor \( 2^{p(n-p-1)/2} \) in the fraction \( c/c^* \).

Proof of Theorem 3 Suppose that \( X \sim GIW_p(n, A, S) \). From the normalizing constant of the density \( f(X) \) of Theorem 1 we can exchange the roles of \( |A| \) and \( |S| \). And from \( \text{tr}(-AX^{-1/2}SX^{-1/2}/2) = \text{tr}(-SX^{-1/2}AX^{-1/2}/2) \) we have that \( X \sim GIW_p(n, S, A). \)
In order to prove Theorem 4 we prove the somewhat more general result in the following lemma.

**Lemma 1.** Let $A_1 \sim W_p(m, I_p)$, $A_2 = \sum_{j=1}^n Y_jY_j'$, with $Y_j \sim N_p(0, I_p)$ and $H \sim GW_p(m + n, A, S)$, where $A_1$, $Y_j$ $(j = 1, \ldots, n)$ and $H$ are independent. Define $C = A_1 + A_2$, $B = \{U(C)^{-1}A_1, U(C)^{-1}\}$, $G = U(H)'BU(U)$ and $D = H^{1/2}AH^{1/2} - G^{1/2}AG^{1/2}$. Then $C \sim W_p(m + n, I_p)$, $G \sim N_p(0, S)$, where $C, G$ and $Z_j$ $(j = 1, \ldots, n)$, are independent.

**Proof.** The proof mimics the proof of Uhlig (1994). Define $Z_j = U(H^{1/2}AH^{1/2})U'(C)^{-1}Y_j$ and note that $D = \sum_{j=1}^n Z_jZ_j'$. From Theorem 4 and from Uhlig (1994), the Jacobian $J(A_1, H, Y_1, \ldots, Y_n \to C, G, Z_1, \ldots, Z_n)$ is $|H|^{-n/2}|C|^{n/2}|A|^{-(p+1)/2}$. Then, the Jacobian function of $A_1, H, A_2$ can be written as

$$p(A_1, H, A_2) = [2^{p(m/2)}\Gamma_p(m/2)]^{-1}etr(-A_1/2)|A_1|^{(m-p-1)/2}$$
$$\times \left[2^{p(m+n)/2}\Gamma_p((m + n)/2) |S|^{(m+n)/2}\right]^{-1} |A|^{(m+n)/2}etr\left(-AH^{1/2}S^{-1}H^{1/2}/2\right)|H|^{(m+n-p-1)/2}$$
$$\times (2\pi)^{-pm/2}etr(-A_2/2)|A|^{-(p+1)}(dA_1)(dH)(dY_1) \cdots (dY_n)$$
$$= \left[2^{p(m+n)/2}\Gamma_p((m + n)/2) |S|^{m/2}\right]^{-1}etr(-C/2)|C|^{(m+n-p-1)/2}$$
$$\times \left[2^{pm/2}\Gamma_p(m/2) |S|^{m/2}\right]^{-1} |A|^{m/2}etr\left(-AG^{1/2}S^{-1}G^{1/2}/2\right)|G|^{(m-p-1)/2}$$
$$\times (2\pi)^{-pm/2}etr(-S^{-1}D/2)|A|^{(n-p-1)/2} = p(C)p(G)p(D),$$

where $A_1 = |C||B|$, $H^{1/2}AH^{1/2} = G^{1/2}AG^{1/2} + D$ and $|H| = |G|/|B|$ are used. □

**Proof of Theorem 4.** The proof is immediate from Lemma 1 after noticing that with the definition of the multivariate singular beta distribution (Uhlig, 1994), $B \sim B_p(m/2, n/2)$. □

Let $A > 0$ denote that the matrix $A$ is positive definite and let $A > B$ denote that the matrices $A > 0$ and $B > 0$ satisfy $A - B > 0$. The following two lemmas are needed in order to prove the limit of Theorem 5.

**Lemma 2.** If the $p \times p$ matrices $A, B > 0$ satisfy $A > B$, then $A^{-1} < B^{-1}$.

The proof of this lemma is given in Horn and Johnson (1999).

**Lemma 3.** If $P_t = R_t(R_t + I_p)^{-1}$, with $R_t = \phi^2 P_{t-1} + \Omega$, where $\Omega$ is a positive definite matrix and $\phi$ is a real number, then the sequence of $p \times p$ positive matrices $\{P_t\}$ is convergent.

**Proof.** First suppose that $\phi = 0$. Then $R_t = \Omega$, for all $t$, and so $P_t = \Omega(\Omega + I_p)^{-1}$, which of course is convergent.

Suppose now that $\phi \neq 0$. It suffices to prove that $\{P_t\}$ is bounded and monotonic. Clearly, $0 \leq P_t$ and since $\phi^2 > 0$ and $\Omega$ is positive definite $0 < P_t$, for all $t > 0$. Since $(R_t + I_p)^{-1} > 0$, $(R_t + I_p - R_t)(R_t + I_p)^{-1} > 0$ $\Rightarrow$ $P_t = R_t(R_t + I_p)^{-1} < I_p$ and so $0 < P_t < I_p$. For the monotonicity it suffices to prove that, if $P_{t-1} > P_{t-2}$ (equivalent $P_{t-1} > P_{t-2}$), then $P_{t-1} > P_{t-2}$ (equivalent $P_{t-1} > P_{t-2}$). From $P_{t-1} > P_{t-2}$ we have $P_{t-1} < P_{t-2}$ $\Rightarrow R_t < R_{t-1}$ $\Rightarrow R_{t-1}^T > P_{t-1}^T > P_{t-2}^T = R_{t-1}^T - R_{t-2}^T > 0$, since $P_{t-1} = (R_t + I_p)R_{t-1} = I_p + R_{t-1}^T$. With an analogous argument we have that, if $P_{t-1} < P_{t-2}$, then $P_{t-1} - P_{t-2} < 0$, from which the monotonicity follows. □
Lemma 4. Let \( \{P_t\} \) be the sequence of Lemma 3 and suppose that \( P_0 = p_0 I_p \), for a known constant \( p_0 > 0 \). Then, with \( \Omega \) as in Lemma 3, the limiting matrix \( P = \lim_{t \to \infty} P_t \) commutes with \( \Omega \).

*Proof.* First we prove that if \( P_{t-1} \) commutes with \( \Omega \), then \( P_t \) also commutes with \( \Omega \). Indeed from \( P_t = (\phi^2 P_{t-1} + \Omega)(\phi^2 P_{t-1} + \Omega + I_p)^{-1} \) we have that \( P_t^{-1} = I_p + (\phi^2 P_{t-1} + \Omega)^{-1} \) and then

\[
P_t^{-1} \Omega^{-1} = \Omega^{-1} + (\phi^2 \Omega P_{t-1} + \Omega^2)^{-1} = \Omega^{-1} + (\phi^2 P_{t-1} \Omega + \Theta^2)^{-1} = \Omega^{-1} P_t^{-1}
\]

which implies that \( \Omega P_t = (P_t^{-1} \Omega^{-1})^{-1} = (\Omega^{-1} P_t^{-1})^{-1} = P_t \Omega \) and so \( P_t \) and \( \Omega \) commute. Because \( P_0 = p_0 I_p \), \( P_0 \) commutes with \( \Omega \) and so by induction it follows that the sequence of matrices \( \{P_t, t \geq 0\} \) commutes with \( \Omega \). Since \( P = \lim_{t \to \infty} P_t \) exists (Lemma 3) we have

\[
P \Omega = \lim_{t \to \infty} (P_t \Omega) = \lim_{t \to \infty} (\Omega P_t) = \Omega P
\]

and so \( P \) commutes with \( \Omega \). \( \Box \)

*Proof of Theorem 5.* From Lemma 3 we have that \( P \) exists and from Lemma 4 we have that \( P \) and \( \Omega \) commute. From \( P_t = (\phi^2 P_{t-1} + \Omega)(P_{t-1} + \Omega + I_p)^{-1} \) we have \( P = (\phi^2 P + \Omega)(\phi^2 P + \Omega + I_p)^{-1} \) from which we get the equation \( P^2 + \phi^2 P(\Omega + I_p - \phi^2 I_p) - \phi^2 \Omega = 0 \). Now since \( P \) and \( \Omega \) commute we can write

\[
P^2 + \phi^2 P(\Omega + I_p - \phi^2 I_p) - \phi^2 \Omega = 0 \Rightarrow P^2 + \frac{1}{2 \phi^2} P(\Omega + (1 - \phi^2)I_p)
\]

\[
= \frac{1}{2 \phi^2}(\Omega + (1 - \phi^2)I_p)P + \frac{1}{4 \phi^4}(\Omega + (1 - \phi^2)I_p)^2 - \frac{1}{4 \phi^4}(\Omega + (1 - \phi^2)I_p)^2 - \Omega = 0
\]

\[
\Rightarrow \left( P + \frac{1}{2 \phi^2}(\Omega + (1 - \phi^2)I_p) \right)^2 = \frac{1}{4 \phi^4}(\Omega + (1 - \phi^2)I_p)^2 + \Omega
\]

\[
\Rightarrow P = \frac{1}{2 \phi^2} \left\{(\Omega + (1 - \phi^2)I_p)^2 + 4 \Omega \right\}^{1/2} - (1 - \phi^2)I_p
\]

after rejecting the negative definite root. \( \Box \)

*Proof of Theorem 6.* The proof is inductive in the distribution of \( \Sigma_t | y^t \). Assume that given \( y^{t-1} \) the distribution of \( \Sigma_{t-1} = \Sigma_{t-1} | y^{t-1} \sim GW((1 - \delta)^{-1} + 2p, Q^{-1}, S_{t-1}) \) and so \( \Sigma_{t-1} | y^{t-1} \sim GW_p((1 - \delta)^{-1} + p - 1, Q, S_{t-1}) \). From the evolution 21 and Theorem 4 we have \( \Sigma_t | y^{t-1} \sim GW_p((1 - \delta)^{-1} + Q, k S_{t-1}) \), which proves that \( \Sigma_t | y^{t-1} \sim GW_p(\delta(1 - \delta)^{-1} + Q, k^{-1} S_{t-1}) \).

From the Kalman filter, conditionally on \( \Sigma_t \), the one-step forecast density of \( y_t \) is

\[
y_t | \Sigma_t, y^{t-1} \sim N_p(m_{t-1}, \Sigma_t^{1/2} Q_{t-1}(1) \Sigma_t^{1/2}) \approx N_p(m_{t-1}, \Sigma_t^{1/2} Q \Sigma_t^{1/2}),
\]

where \( m_{t-1}, Q_{t-1}(1) \) and \( Q \) are as in the theorem.

Given \( y^{t-1} \) the joint distribution of \( y_t \) and \( \Sigma_t \) is

\[
p(y_t, \Sigma_t | y^{t-1}) = p(y_t | \Sigma_t, y^{t-1}) p(\Sigma_t | y^{t-1})
\]

\[
= c_1 \exp\{-Q^{-1} \Sigma_t^{-1/2} (y_t - k^{-1} S_{t-1}) \Sigma_t^{1/2}/2\}/|\Sigma_t|^{(n+1)/2},
\]

(A-1)
where \( n = \delta (1 - \delta)^{-1} + 2p \) and
\[
c_1 = \frac{|k^{-1} S_{t-1}|^{(n-p-1)/2}}{(2\pi)^{p/2} 2^p (n-p-1)! |Q|^{(n-p)/2} \Gamma_p((n-p-1)/2) \Gamma_p((n-p-1)/2)}.
\]

The one-step forecast density of \( y_t \) is
\[
p(y_t|y_t^{-1}) = \int_{\Sigma_t > 0} p(y_t, \Sigma_t|y_t^{-1}) d\Sigma_t
\]
\[
= c_1 \int_{\Sigma_t > 0} \left| \Sigma_t \right|^{-(n+1)/2} \exp\left\{-Q^{-1} \Sigma_t^{-1/2} (e_{t-1} e_{t-1}' + k^{-1} \Sigma_t \Sigma_t^{-1/2}) \right\} d\Sigma_t
\]
\[
= \frac{2p(n-p)/2 \Gamma_p((n-p)/2)}{\Gamma_p((n-2p+p-1)/2)} |k^{-1} S_{t-1}|^{(n-2p+p-1)/2} |e_{t-1} e_{t-1}' + k^{-1} S_{t-1}|^{-(n-p)/2},
\]
and so \( y_t|y_t^{-1} \sim \mathcal{N}(1-\delta)^{-1}, m_{t-1}, k^{-1} S_{t-1} \), as required. This completes (a).

Proceeding with (b) first we derive the distribution of \( \Sigma_t|y^t \). Applying the Bayes’ theorem we have
\[
p(\Sigma_t|y^t) = \frac{p(y_t|\Sigma_t, y_t^{-1}) p(\Sigma_t|y_t^{-1})}{p(y_t|y_t^{-1})}
\]
and from equation (A-1) we have
\[
p(\Sigma_t|y^t) = c_2 |\Sigma_t|^{-n/2} \exp\left\{-Q^{-1} \Sigma_t^{-1/2} S_t \Sigma_t^{-1/2} / 2 \right\}
\]
and
\[
n^* = n + 1 = \frac{\delta}{1 - \delta} + 2p + 1 = \frac{1}{1 - \delta} + 2p,
\]
where \( S_t \) is as in the theorem and the proportionality constant is \( c_2 = c_1/p(y_t|y_t^{-1}) \), not depending on \( \Sigma_t \). Thus \( \Sigma_t|y^t \sim \mathcal{G}(1-\delta)^{-1}, m_{t-1} + k^{-1} S_{t-1} \) as required. Conditionally on \( \Sigma_t \), the distribution of \( \theta_t \) follows directly from application of the Kalman filter and so applying the approximation \( \Sigma_t \approx S_t^* \), with \( S_t^* \) as in the theorem, provides the required posterior distribution of \( \theta_t \).

Before we prove Theorem 7, we give the following lemma.

**Lemma 5.** Suppose that the \( p \times p \) matrix \( B \) follows the singular multivariate beta distribution \( B \sim B_p(m/2, n/2) \), with density
\[
p(B) = \frac{\pi^{(n^2 - pm)/2} \Gamma_p((m+n)/2)}{\Gamma_n(n/2) \Gamma_p(m/2)} |K|^{(n-p-1)/2} |B|^{(m-p-1)/2},
\]
where \( n \) is a positive integer, \( m > p - 1 \), \( I_p - B = H_1 K H_1' \), \( K \) is the diagonal matrix with diagonal elements the positive eigenvalues of \( I_p - B \), and \( H_1 \) is a matrix with orthogonal columns, i.e. \( H_1 H_1' = I_p \). For any non-singular matrix \( A \), the density of \( X = A B^{-1} A' \), is
\[
p(X) = \frac{\pi^{(n^2 - pm)/2} \Gamma_p((m+n)/2)}{\Gamma_n(n/2) \Gamma_p(m/2)} |A|^{n+m-p-1} |L|^{-(p-n+1)/2} |X|^{-(m-p-1)/2},
\]
where \( L \) is the diagonal matrix including the positive eigenvalues of \( I_p - A' X^{-1} A \).
Proof. First note that $X$ is a non-singular matrix and $|B| = |A^2X|^{-1}$. From Díaz-García and Gutiérrez (1997), the Jacobian of $B$ with respect to $X$ is

$$(dB) = |K|^{(p-n+1)/2}|L|^{-(p-n+1)/2}|A|^n(dX),$$

where $K$ is defined as in the theorem. Then from the singular multivariate beta density of $B$ we obtain

$$p(X) = \frac{\pi^{(n^2-pn)/2}}{\Gamma_p(m+n/2)} \frac{\Gamma_p((m+n)/2)}{\Gamma_p(m/2)} \frac{|A|^n|K|^{(n-p-1)/2}|B|^{(m-p-1)/2}}{\Gamma_p|L|^{-(p-n+1)/2}} \times |K|^{(p-n+1)/2}|L|^{-(p-n+1)/2},$$

from which we immediately get the required density of $X$. \qed

Proof of Theorem 7. The likelihood function is

$$L(\Sigma_1, \ldots, \Sigma_N; y^N) = p(y_1|\Sigma_1)p(\Sigma_1|\Sigma_0) \prod_{t=2}^N p(y_t|\Sigma_t, y^{t-1})p(\Sigma_t|\Sigma_{t-1}, y^{t-1})$$

(A-2)

and from the Kalman filter we have $y_t|\Sigma_t, y^{t-1} \sim N_p(m_t-1; \Sigma_t^{-1/2}Q_t^{1/2})$, where $Q_{t-1}(1) \approx Q$. The density $p(\Sigma_t|\Sigma_{t-1}, y^{t-1})$ is the density $p(X)$ of Lemma 2 with $A = k^{-1/2}\{\mathcal{U}(\Sigma_{t-1}^{-1})^{-1}\}'$, $\Sigma_t^{-1} = \mathcal{U}(\Sigma_t^{-1})\mathcal{U}(\Sigma_t^{-1})^{-1}$, $m = \delta(1-\delta)^{-1} + p - 1$ and $n = 1$. The required formula of the log-likelihood function is obtained from (A-2) by taking the logarithm of $L(\Sigma_1, \ldots, \Sigma_N; y^N)$. \qed

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