Finite-Energy Spectral-Weight Distributions of a 1D Correlated Metal

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We derive general closed-form analytical expressions for the finite-energy one- and two-electron spectral-weight distributions of an one-dimensional correlated metal with on-site electronic repulsion. Our results also provide general expressions for the one- and two-atom spectral functions of a correlated quantum system of cold fermionic atoms in a one-dimensional optical lattice with on-site atomic repulsion. In the limit of zero spin density our spectrum-function expressions provide the correct zero-spin density results. Our results reveal the dominant non-perturbative microscopic many-particle mechanisms behind the exotic spectral properties observed in quasi-one-dimensional metals and correlated systems of cold fermionic atoms in one-dimensional optical lattices.

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I. INTRODUCTION

The main goal of this paper is to provide a detailed derivation of the one-electron and two-electron spectral-weight distributions recently used in Refs. [1, 2, 3] in the unusual spectral properties of low-dimensional metals. Moreover, our results will be used elsewhere in the study of the spectral properties of the new quantum systems described by cold fermionic atoms in optical lattices: following the experimental studies of strongly correlated quantum systems of ultra cold bosonic atoms [4], new experiments involving cold fermionic atoms (such as $^6\text{Li}$) in an optical lattice formed by interfering laser fields are in progress [5].

We profit from symmetries and properties specific to the metallic phase of the 1D Hubbard model [6] to derive closed-form analytical expressions for the finite-energy one- and two-electron spectral-weight distributions. The model also describes cold fermionic atoms in optical lattices, provided that the electrons are replaced by these atoms. The recently developed quantum optical tools for manipulating atoms can be used to realize a 1D lattice or any other lattice described by the Hubbard model [3, 7, 8]. Such correlated quantum systems have applications in quantum computing [7]. Our study involves the use of the holon and spinon description of Ref. [9] and of the general pseudofermion dynamical theory introduced in Ref. [10]: we are able to explicitly calculate the pseudfermion spectral weights for the metallic phase. The pseudofermion description was introduced in Ref. [11] and is such that the pseudofermions which describe the elementary excitations are freely propagating, and the information about the interactions is encoded in their overall phase shifts [12]. The method used here is a generalization for all values of the on-site repulsion $U$ of that used in Refs. [13, 14] for $U \to \infty$. For finite values of $U$ there were no closed-form analytical expressions for the finite-energy spectral-weight distributions, in contrast to simpler models [15].

Our general closed-form analytical expressions for the one- and two-electron spectral functions are controlled by the dominant microscopic processes which are behind nearly the whole spectral weight for all values of the momentum $k$, energy $\omega$, and on-site repulsion $U$. In addition to other spectral features, we explicitly derive the spectral-weight distribution expressions in the vicinity of the the singular and edge branch lines used in the study of the photoemission spectra and phase diagram of quasi-1D metals in Refs. [1, 2, 3, 16]. The preliminary results of those papers confirm that the spectral-weight distributions derived here describe quantitatively, for the whole energy band width, the unusual spectral properties observed in such materials.

The paper is organized as follows: In Sec. [11] we introduce the model and provide basic information about the pseudofermion description needed for our studies. The specific classes of elementary microscopic processes which generate the one and two-electron spectral weight of the metallic phase is the subject of Sec. [11]. In Sec. [14] we calculate the general expressions for the finite-energy spectral-weight distributions of the metallic phase. The characterization of the dominant processes behind such distributions and the derivation of the closed-form analytical

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expressions generated by these processes for the same distributions is the subject of Sec. \( \text{Sec} \) Finally, the concluding remarks are presented in Sec. \( \text{Sec} \)

### II. THE MODEL, THE WEIGHT DISTRIBUTIONS, ROTATED ELECTRONS, AND THE HOLON, SPINON, AND PSEUDOFERMION DESCRIPTION

The 1D Hubbard model reads,

\[
\hat{H} = -t \sum_{j, \sigma} [c_{j, \sigma}^\dagger c_{j+1, \sigma} + \text{h.c.}] + U \sum_j \hat{n}_{j, \uparrow} \hat{n}_{j, \downarrow},
\]

where \( c_{j, \sigma}^\dagger \) and \( c_{j, \sigma} \) are spin-projection \( \sigma = \uparrow, \downarrow \) electron operators at site \( j = 1, 2, ..., N_a \) and \( \hat{n}_{j, \sigma} = c_{j, \sigma}^\dagger c_{j, \sigma} \). The model describes \( N_\uparrow \) spin-up electrons and \( N_\downarrow \) spin-down electrons in a chain of \( N_a \) sites. We use periodic boundary conditions and units such that the Planck constant and electronic lattice constant are one. We denote the electronic number by \( N = N_\uparrow + N_\downarrow \). The number of lattice sites \( N_a \) is even and very large and in the units used here the chain length reads \( L = N_a \). We consider electronic densities \( n = n_\uparrow + n_\downarrow \) and spin densities \( m = n_\uparrow - n_\downarrow \) in the range \( 0 < n < 1 \) and \( 0 < m < n \), respectively, where \( n_\sigma = N_\sigma / L = N_\sigma / N_a \). In the thermodynamic limit the Fermi momenta read \( k_F = \pi n/2 \) and \( k_{F, \sigma} = \pi n_\sigma \). We denote the electronic charge by \( -e \).

We consider the following general finite-\( N \)-electron spectral-weight distribution,

\[
B_N^l(k, \omega) = \sum_f |\langle f| \hat{O}^l_N(k)|GS\rangle|^2 \delta \left( \omega - l[E_f - E_{GS}] \right); \quad \omega > 0; \quad l = \pm 1.
\]

Here the general \( N \)-electron operators \( \hat{O}^l_N(k) \equiv \hat{O}^l_N(k) \) and \( \hat{O}^{-l}_N(k) \equiv \hat{O}^l_N(k) \) carry momentum \( k \), the \( f \) summation runs over the excited energy eigenstates, the energy \( E_f \) corresponds to these states, and \( E_{GS} \) is the ground-state energy. The local operator \( \hat{O}^l_{N,j} \equiv \hat{O}^l_{N,j} \) or \( \hat{O}^{-l}_{N,j} \equiv \hat{O}^{-l}_{N,j} \) is related to the corresponding momentum-representation operator \( \hat{O}^l_N(k) \) of Eq. \( \text{(2)} \) by a Fourier transform. (Examples of such \( N \)-electron operators are given in Ref. \( \text{Ref} \).

For simplicity, we use in expression \( \text{(2)} \) a momentum extended scheme such that \( k \in (-\infty, +\infty) \), yet it is a simple exercise to obtain the corresponding spectral function expressions for the first Brillouin zone. The general spectral-weight distribution \( \text{(2)} \) obeys the following sum rule,

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \int_0^{\omega_{\infty}} d\omega B_N^l(k, \omega) = \sum_{j=1}^{N_a} \langle GS| \hat{O}^{-l}_{N,j} \hat{O}^l_{N,j}\rangle (GS); \quad l = \pm 1.
\]

When \( \hat{O}^l_{N,j} \) is hermitian one has that \( \hat{O}^l_{N,j} = \hat{O}^{-l}_{N,j} \). In this case the convention is that \( l = +1 \) and thus the energy \( \omega \) is positive.

The concept of a rotated electron plays a key role in the pseudofermion description. Concerning its relation to the holons, spinons, and \( c_0 \) pseudofermions whose occupancy configurations describe the energy eigenstates of the model \( \text{[1]} \), see Ref. \( \text{[1]} \). The holons have \( \eta \)-spin 1/2, \( \eta \)-spin projections \( \pm 1/2 \), charge \( \pm 2e \), and spin zero. The spinons have spin 1/2, spin projections \( \pm 1/2 \), and no charge degrees of freedom. Throughout this paper we denote such holons and spinons according to their value of \( \eta \)-spin projections \( \pm 1/2 \) and spin projections \( \pm 1/2 \), respectively. Furthermore, the \( c_0 \) pseudofermion is a composite quantum object of a chargeon of charge \( -e \) and an antichargeon of charge \( +e \). It has no \( \eta \)-spin and no spin degrees of freedom. The \( 2\nu \)-holon (and \( 2\nu \)-spinon) composite \( c\nu \) pseudofermions (and \( s\nu \) pseudofermions) are \( \eta \)-spin and spin zero (and spin zero) quantum objects (and with no charge degrees of freedom). These composite quantum objects contain an equal number \( \nu \) of \(-1/2\) holons and \(+1/2\) holons (and \(-1/2\) spinons and \(+1/2\) spinons). The \( \pm 1/2 \) holons (and \( \pm 1/2 \) spinons) which are not associated with \( 2\nu \)-holon \( c\nu \) pseudofermions (and \( 2\nu \)-spinon \( s\nu \) pseudofermions) are the \( \pm 1/2 \) Yang holons (and \( \pm 1/2 \) HL spinons). We denote the numbers of \( \alpha \nu \) pseudofermions and \( \alpha \nu \) pseudofermion holes by \( N_{\alpha \nu} \) and \( N_{\alpha \nu}^h \), respectively, where \( \alpha = c, s \) and \( \nu = 0, 1, 2, \ldots \) for \( \alpha = c \) and \( \nu = 1, 2, \ldots \) for \( \alpha = s \). (The value of \( N_{\alpha \nu}^h \) is given in Eqs. (B7) and (B8) of Ref. \( \text{[9]} \)). Moreover, \( L_{\alpha, \pm 1/2} \) denotes the number of \( \pm 1/2 \) Yang holons (\( \alpha = c \)) or \( \pm 1/2 \) HL spinons (\( \alpha = s \)).

Often in this paper we use the notation \( \alpha \nu \neq 0, s1 \) branches, which refers to all \( \alpha \nu \) branches except the \( c0 \) and \( s1 \) branches. Moreover, the summations (and products) \( \sum_{\alpha \nu} \sum_{\alpha \nu = 0, s1} \) and \( \sum_{\alpha \nu \neq 0, s1} \) (and \( \prod_{\alpha \nu} \), \( \prod_{\alpha \nu = 0, s1} \), and \( \prod_{\alpha \nu \neq 0, s1} \)) run over all \( \alpha \nu \) branches with finite \( \alpha \nu \) pseudofermion occupancy in the corresponding state or subspace,
Moreover, for that state the limiting bare-momentum values of both the excited energy eigenstates originated from it by creation, annihilation, and particle-hole processes involving the generation of a finite number of active pseudofermion scattering centers, $-1/2$ Yang holons, and $-1/2$ HL spinons plus a vanishing or small density of low-energy and small-momentum $\alpha\nu = c_0, s_1$ pseudofermion particle-hole processes. All these processes are defined in Ref. \[12\].

The $\alpha\nu$-pseudofermion discrete canonical-momentum values have a functional character and read, $\bar{q}_j = q_j + Q_{\alpha\nu}^j(q_j)/L = [2\pi/L]I^\nu_j + Q_{\alpha\nu}^j(q_j)/L$ where $j = 1, 2, ..., N_{\alpha\nu}^*, N_{\alpha\nu}^* = N_{\alpha\nu} + N_{\alpha\nu}^r$. In this paper we use the designation functional for all quantities whose value depends on the energy eigenstates bare-momentum occupancy configurations. According to the studies of Ref. \[12\], $Q_{\alpha\nu}^j(q_j)/2$ is a $\alpha\nu$ pseudofermion scattering phase shift given by,

$$Q_{\alpha\nu}^j(q_j)/2 = \pi \sum_{\alpha'\nu'} \sum_{j'=1}^{N_{\alpha'\nu'}^*} \Phi_{\alpha\nu, \alpha'\nu'}(q_j, q_{j'}) \Delta N_{\alpha'\nu'}(q_{j'}) ; \quad j = 1, 2, ..., N_{\alpha\nu}^*.$$ \(4\)

Here $\Delta N_{\alpha\nu}(q_j) = \Delta N_{\alpha\nu}^*(q_j)$ is the bare-momentum distribution function deviation $\Delta N_{\alpha\nu}(q_j) = N_{\alpha\nu}(q_j) - N_{\alpha\nu}^0(q_j)$ corresponding to the excited energy eigenstate. (The ground-state densely-packed bare-momentum distribution function $N_{\alpha\nu}(q_j)$ is defined in Eqs. (C.1)-(C.3) of Ref. \[9\].) This deviation is expressed in terms of the bare-momentum $q_j = [2\pi I^\nu_j]/L$, which is carried by the $\alpha\nu$ pseudo-particles, where $I^\nu_j$ are the quantum numbers provided by the Bethe-ansatz solution \[4\]. Although the $\alpha\nu$ pseudo-particles carry bare-momentum $q_j$, one can also label the corresponding $\alpha\nu$ pseudofermions by such a bare-momentum. When we refer to the pseudofermion bare-momentum $q_j$ we mean that $\bar{q}_j$ is the bare-momentum value that corresponds to the pseudofermion canonical momentum $\bar{q}_j = q_j + Q_{\alpha\nu}^j(q_j)/L$.

In contrast to the $\alpha\nu$ pseudofermions, the $\alpha\nu$ pseudofermions have no residual-interaction energy terms. Instead, under the ground-state - excited-energy-eigenstate transitions the $\alpha\nu$ pseudofermions and $\alpha\nu$ pseudofermion holes undergo elementary scattering events with the $\alpha'\nu'$ pseudofermions and $\alpha'\nu'$ pseudofermion holes created in these transitions \[12\]. This leads to the elementary two-pseudofermion phase shifts $\pi \Phi_{\alpha\nu, \alpha'\nu'}(q_{j, q_{j'}})$ on the right-hand side of the overall scattering phase shift \[11\], defined by Eqs. (15) and (A1)-(A13) of Ref. \[11\]. Moreover, the overall $\alpha\nu$ pseudofermion or hole phase shift,

$$Q_{\alpha\nu}(q_j)/2 = Q_{\alpha\nu}^0/2 + Q_{\alpha\nu}^j(q_j)/2,$$ \(5\)

plays an important role in the pseudofermion theory \[12\]. Here $Q_{\alpha\nu}(q_j)/L$ gives the shift in the discrete canonical-momentum value $\bar{q}_j$ that arises due to the transition from the ground state to an excited energy eigenstate and $Q_{\alpha\nu}^j/2$ can have the values $Q_{\alpha\nu}^j/2 = 0, \pm \pi/2$ and is defined in Eq. (11) of Ref. \[11\]. In this paper we use boundary conditions such that $Q_{\alpha\nu}^j/2 = 0, -\sgn(k)\pi/2$, where $k$ is the excited-state momentum relative to that of the initial ground state. Here we assume that for the latter state $N/2$ and $N$ are odd and even numbers, respectively. $Q_{\alpha\nu}^j/L$ gives the shift in the discrete bare-momentum value $q_j$ that arises as a result of the same transition. In reference \[12\] it is confirmed that $Q_{\alpha\nu}(q_j)/2$, Eq. \(4\), is a $\alpha\nu$ pseudofermion or hole overall phase shift associated with the ground-state - excited-energy-eigenstate transition. Its scattering part is the functional $Q_{\alpha\nu}^j(q_j)/2$ given in Eq. \(4\). (In contrast to the pseudofermions, the $-1/2$ Yang holons and $-1/2$ HL spinons are scattering-less objects \[12\].) The $\alpha\nu$ pseudofermion creation and annihilation operators $f_{\bar{q}_j, \alpha\nu}$ and $f_{\bar{q}_j, \alpha\nu}$, respectively, have exotic anticommutation relations, provided in Ref. \[11\]. These anticommutators involve the overall phase shifts \(5\) and play a key role in the spectral properties.

For the ground state the pseudofermion, $-1/2$ Yang holon, and $-1/2$ HL spinon numbers are given by $N_{c_0} = N, N_{s_1} = N, N_{\alpha\nu} = L_c - 1/2 = L_s - 1/2 = 0$ for $\alpha\nu \neq c_0, s_1$. We call $N_{c_0}^0$ and $N_{s_1}^0$ the ground-state $c_0$ and $s_1$ pseudofermion numbers, respectively. The ground-state $\alpha\nu = c_0, s_1$ bare-momentum distribution functions are such that there is pseudofermion occupancy for $|q| \leq q_{\alpha\nu}^0$ and unoccupancy for $q_{\alpha\nu}^0 < |q| \leq q_{\alpha\nu}^*$, where in the thermodynamic limit the Fermi-point values are given by,

$$q_{F,c_0}^0 = 2k_F ; \quad q_{F,s_1}^0 = k_F.$$ \(6\)

Moreover, for that state the limiting bare-momentum values of both the $\alpha\nu = c_0, s_1$ and $\alpha\nu \neq c_0, s_1$ bands read,

$$q_{\alpha\nu}^0 = \pi ; \quad q_{\alpha\nu}^* = k_F^\uparrow ; \quad q_{\alpha\nu}^{\nu} = [\pi - 2k_F], \quad \nu > 0 ; \quad q_{\alpha\nu}^0 = [k_F^\uparrow - k_F^\downarrow] , \quad \nu > 1.$$ \(7\)
Within the thermodynamic limit, when in this paper we refer to creation (or annihilation) of $\alpha \nu = c, 0$, s1 pseudofermions with bare-momentum values at the Fermi points, we mean that such a process leads to a densely packed bare-momentum pseudofermion distribution for the final state, where the created (or annihilated) pseudofermions occupy neighboring discrete bare-momentum values $\pm q^0_{\alpha \nu, 1} + 2j\pi /L$ such that $j = \pm 1, \pm 2, ...$ (or $j = 0, \mp 1, \mp 2, ...$). Similarly, when we refer to creation of $\alpha \nu \neq c, 0$, s1 pseudofermions at the limiting bare-momentum values, we mean that such a process leads to a densely packed bare-momentum pseudofermion hole distribution for the final state, where the created pseudofermions occupy neighboring discrete bare-momentum values $\pm q^0_{\alpha \nu, 1} + 2j\pi /L$ such that $j = 0, \mp 1, \mp 2, ...$. As further discussed below, the $c\nu$ pseudofermions of limiting bare-momentum value $q = \pm q^0_{\nu, 0} = \pm [k_{F \uparrow} - k_{F \downarrow}]$ and $s\nu$ pseudofermions of limiting bare-momentum value $q = \pm q^0_{0} = \pm [k_{F \uparrow} - k_{F \downarrow}]$ can be described in terms of $2\nu$ independent holons and $2\nu$ independent spinons, respectively.

Our study of the metallic phase of the model involves several numbers and number deviations. $N^{{ph, NF}}_{\alpha \nu}$ is the number of finite-momentum and finite-energy $\alpha \nu = c, 0$, s1 pseudofermion particle-hole processes. The quantum number $\iota = \text{sgn}(q)1 = \pm 1$ refers to the right pseudfermion movers (i = +1) and left pseudfermion movers (i = -1) and $\Delta N^{{F}}_{\alpha \nu, \pm 1}$ is the deviation in the number of $\alpha \nu$ pseudofermions at the right (+1) and left (-1) Fermi points. In turn, the deviation in the number of $\alpha \nu = c, 0$, s1 pseudofermions created or annihilated away from these points is denoted by $\Delta N^{{NF}}_{\alpha \nu}$. The actual number of $\alpha \nu$ pseudofermions created or annihilated at the right (+1) and left (-1) Fermi points is denoted by $\Delta N^{{0, F}}_{\alpha \nu, \pm 1}$. It is such that $\Delta N^{{F}}_{\alpha \nu, \pm 1} = \Delta N^{{0, F}}_{\alpha \nu, \pm 1} + \iota Q^0_{\alpha \nu} /2\pi$, where $Q^0_{\alpha \nu} /2$ is the scattering-less phase shift on the right-hand side of Eq. (4). Furthermore, $N^{{F}}_{\alpha \nu, \iota}$ refers to the $\alpha \nu \neq c, 0$, s1 branches and is the number of $\alpha \nu$ pseudofermions of limiting bare momentum $q = \iota q^0_{\nu, 0}$ such that $\iota = \pm 1$. The number of $\alpha \nu$ pseudofermions created away from the limiting bare-momentum values is called $N^{{NF}}_{\alpha \nu}$.

### III. SPECTRAL-WEIGHT ELEMENTARY PROCESSES FOR DENSITIES $0 < n < 1$ AND $0 < m < n$

In this section we study the pseudofermion occupancy configurations of the PS energy eigenstates. As in reference we classify the pseudofermion elementary processes that generate the PS from the ground state into three types:

(A) Finite-energy and finite-momentum elementary $c, 0$ and s1 pseudofermion processes away from the corresponding Fermi points involving creation or annihilation of a finite number of pseudofermions plus creation of $\alpha \nu \neq c, 0$, s1 pseudofermions with bare-momentum values different from the limiting bare-momentum values $\pm q^0_{\nu, 1}$

(B) Zero-energy and finite-momentum processes that change the number of $c, 0$ and s1 pseudofermions at the $\iota = \text{sgn}(q)1 = +1$ right and $\iota = \text{sgn}(q)1 = -1$ left $c, 0$ and s1 Fermi points - these processes transform the ground-state densely packed bare-momentum occupancy configuration into an excited-state densely packed bare-momentum occupancy configuration. Furthermore, creation of a finite number of independent $-1/2$ holons and independent $-1/2$ spinons, including $-1/2$ Yang holons, $-1/2$ HL spinons, and $-1/2$ holons and $-1/2$ spinons associated with $c\nu$ pseudofermions of limiting bare momentum $q = \pm q^0_{\nu, 1} = \pm [k_{F \uparrow} - k_{F \downarrow}]$ and $s\nu$ pseudofermions of limiting bare momentum $q = \pm q^0_{\nu, 0} = \pm [k_{F \uparrow} - k_{F \downarrow}]$, respectively.

(C) Low-energy and small-momentum elementary $c, 0$ and s1 pseudofermion particle-hole processes in the vicinity of the $\iota = \text{sgn}(q)1 = +1$ right and $\iota = \text{sgn}(q)1 = -1$ left $c, 0$ and s1 Fermi points, relative to the excited-state $\alpha \nu = c, 0$, s1 pseudofermion densely packed bare-momentum occupancy configurations generated by the above elementary processes (B).

It is found in reference that the invariance under the electron - rotated-electron unitary transformation of the $\alpha \nu$ pseudofermions created at limiting bare momentum $q = \pm q^0_{\nu, 1}$ and belonging to $\alpha \nu \neq c, 0$, s1 branches implies that each of such $c\nu$ pseudofermions (and $s\nu$ pseudofermions) separates into $2\nu$ independent holons and $2\nu$ independent spinons and a $c\nu$ and $s\nu$ FP scattering center. By independent holons and spinons it is meant those which remain invariant under the electron - rotated-electron unitary transformation. (The Yang holons and HL spinons are also independent holons and spinons, respectively.) The above designation FP stands for Fermi points. Indeed, it is found in the same reference that the $c, 0$ and s1 pseudofermion and pseudofermion hole scatterers feel the created $c\nu$ and $s\nu$ FP scattering centers as being $c\nu$ (and $c, 0$ and s1) pseudofermion scattering centers at the Fermi points. Therefore, creation of $\alpha \nu$ pseudofermions of limiting bare momentum $q = \pm q^0_{\nu, 1}$ and belonging to $\alpha \nu \neq c, 0$, s1 branches is included above in the elementary processes (B).

In addition to the PS, the following subspaces play an important role in our studies:

A CPHS ensemble subspace is a subspace of a canonical ensemble subspace with fixed $N_T$ and $N_z$ electronic numbers, which is spanned by all energy eigenstates with fixed values for the $-1/2$ Yang holon number $L_{c, -1/2, -1/2}$ HL spinon
number \( L_{s,-1/2} \), \( c_0 \) pseudofermion number \( N_{c^0,0} \), and for the sets of \( \alpha \nu \neq c_0 \) pseudofermion numbers \( \{N_{\alpha \nu}\} \) and \( \{N_{\alpha \nu}\} \) corresponding to the \( \nu = 1, 2, \ldots \) branches. Here CPHS stands for \( c_0 \) pseudofermion, holon, and spinon.

A \( J \)-CPHS ensemble subspace is a subspace of a CPHS ensemble subspace spanned by the excited energy eigenstates with the same values for the numbers \( N_{c^0}^{phNF}, N_{s^0}^{phNF}, \Delta N_{c^0,+1}, \Delta N_{c^0,-1}, \Delta N_{s^0,+1}, \Delta N_{s^0,-1} \), and sets of numbers \( \{N_{\alpha \nu}^{F}\} \) and \( \{N_{\alpha \nu}^{F}\} \) for the \( \alpha \nu \neq c_0 \), \( s_1 \) branches with finite pseudofermion occupancy in the CPHS ensemble subspace;

A \( J \)-CPHS subspace is a \( J \)-CPHS ensemble subspace whose numbers \( N_{c_0}^{phNF} \) and \( N_{s_1}^{phNF} \) have finite values;

A reduced \( J \)-CPHS subspace is a subspace of a \( J \)-CPHS subspace spanned by all the energy eigenstates of the latter subspace which are generated from the ground state by elementary processes (A) and (B) only.

The CPHS ensemble subspace dimension is given by,

\[
D_{CPHS} = \left( \frac{N_0}{N_{c^0}} \right) \left( N_0 - N_{s^1} - 2 \sum_{\nu' > 1} N_{s^\nu'} \right) \prod_{\nu > 0} \left( \frac{N_{c^\nu}}{N_{c^\nu}} \right) \prod_{\nu' > 1} \left( \frac{N_{s^\nu'}}{N_{s^\nu'}} \right). 
\]

It is a product of the number of occupancy configurations of the \( c_0 \) pseudofermions, \( s_1 \) pseudofermions, and \( \alpha \nu \neq c_0 \), \( s_1 \) pseudofermions belonging to branches with CPHS ensemble subspace finite occupancy. Once the ground-state numbers \( N_{c^0}^0 = N_0^0 \) and \( N_{s^1}^0 = N_1^0 \) are known and the corresponding excited-state values of these numbers are given by \( N_{c^0} = N_{c^0}^0 + \Delta N_{c^0} \) and \( N_{s^1} = N_{s^1}^0 + \Delta N_{s^1} \), one can define a CPHS ensemble subspace by the set of deviation number values \( \Delta N_{c^0} \) and \( \Delta N_{s^1} \) and set of number values \( L_{c,-1/2}, L_{s,-1/2} \), and \( \{N_{\alpha \nu}\} \) for the \( \alpha \nu \neq c_0 \), \( s_1 \) branches.

Since the PS excited energy eigenstates involve the creation of none or at least a finite number of \( \alpha \nu \) pseudofermions belonging to the \( \alpha \nu \neq c_0 \), \( s_1 \) branches such that \( \Delta N_{\alpha \nu} = N_{\alpha \nu} \), all the \( \left( \frac{N_{\alpha \nu}^*}{N_{\alpha \nu}} \right) \) CPHS ensemble subspace occupancy configurations of each of the latter \( \alpha \nu \) bands correspond to PS excited energy eigenstates. The same applies to the CPHS ensemble subspace \( c_0 \) (and \( s_1 \)) pseudofermion occupancy configurations provided that \( n = 1 \) (and \( m = 0 \)) for the initial ground state. Indeed, for such a density there are no \( c_0 \) (and \( s_1 \)) pseudofermion holes for that state, and again all excited-state CPHS ensemble subspace configurations can be reached by a finite number of \( c_0 \) (and \( s_1 \)) pseudofermion processes. However, for densities \( 0 < n < 1 \) and \( 0 < m < n \) the ground state has both "particles" and "holes" in the \( c_0 \) and \( s_1 \) pseudofermion bands, whose numbers are such that \( N_{\alpha \nu} \rightarrow \infty \) and \( N_{\alpha \nu}^h \rightarrow \infty \) as \( N_{\alpha} \rightarrow \infty \).

Thus, reaching from the ground state many of the excited-state CPHS ensemble subspace \( c_0 \) and \( s_1 \) pseudofermion occupancy configurations involves the thermodynamic limit an infinite number of finite-momentum and finite-energy pseudofermion processes. It follows that for the metallic phase at finite spin density only the \( J \)-CPHS subspaces of each CPHS ensemble subspace are contained in the PS. Indeed, \( J \)-CPHS ensemble subspaces with \( N_{c^0}^{phNF} \) and/or \( N_{s_1}^{phNF} \) infinite, do not belong to the PS and have no finite overlap with the one- and two-electron excitations.

A. GENERAL DISTRIBUTION-FUNCTION DEVIATIONS FOR DENSITIES \( 0 < n < 1 \) AND \( 0 < m < n \)

A \( J \)-CPHS subspace is spanned by the excited energy eigenstates contained in the corresponding reduced \( J \)-CPHS subspace and by all excited energy eigenstates generated from the former states by the elementary processes (C). Here we study the pseudofermion occupancy configurations of the excited energy eigenstates that span a \( J \)-CPHS subspace. There is a particular type of reduced \( J \)-CPHS subspace which we call point-subspace. A point-subspace is spanned by a single excited energy eigenstate whose deviation numbers and numbers are such that,

\[
\Delta N_{\alpha \nu} = \Delta N_{\alpha \nu}^{F} = \sum_{i = \pm 1} \Delta N_{\alpha \nu, i}, \quad \alpha \nu = c_0, s_1; \quad N_{\alpha \nu} = N_{\alpha \nu}^{F} = \sum_{i = \pm 1} N_{\alpha \nu, i}, \quad \alpha \nu \neq c_0, s_1. 
\]

The number deviation \( \Delta N_{\alpha \nu}^{F} \) obeys the relation \( \Delta N_{\alpha \nu} = \Delta N_{\alpha \nu}^{NF} + \Delta N_{\alpha \nu}^{F} \), where \( \Delta N_{\alpha \nu}^{NF} \) and \( \Delta N_{\alpha \nu}^{F} \) refer to the \( c_0 \) and \( s_1 \) pseudofermion number deviations generated by the elementary processes (A) and (B), respectively. For the \( \alpha \nu = c_0, s_1 \) branches, the deviation number \( \Delta N_{\alpha \nu, i}^{F} \) is related to the current number deviation, \( \Delta I_{\alpha \nu}^{F} = \frac{1}{2} \sum_{i = \pm 1}(i) \Delta N_{\alpha \nu, i}^{F} \) such that \( \Delta N_{\alpha \nu, i}^{F} = \Delta N_{\alpha \nu, i}^{F}/2 + i \Delta I_{\alpha \nu}^{F} \). Furthermore, for the \( \alpha \nu \neq c_0, s_1 \) branches the number \( N_{\alpha \nu, i}^{F} \) is associated with the current number \( I_{\alpha \nu}^{F} = \frac{1}{2} \sum_{i = \pm 1}(i) N_{\alpha \nu, i}^{F}, \quad i \Delta I_{\alpha \nu}^{F} \). The excited energy eigenstates that span a reduced \( J \)-CPHS subspace have well-defined \( c_0 \) and \( s_1 \) pseudofermion
right ($\iota = +1$) and left ($\iota = -1$) bare-momentum and canonical-momentum Fermi points,

$$q_{Fav,\iota} = q_{Fav}^0 + \Delta q_{Fav,\iota}; \quad \Delta q_{Fav,\iota} = \frac{2\pi}{L} \Delta N_{av,\iota}^F; \quad \alpha\nu = c0, s1; \quad \iota = \pm 1,$$

and

$$\bar{q}_{Fav,\iota} = q_{Fav}^0 + \Delta \bar{q}_{Fav,\iota}; \quad \Delta \bar{q}_{Fav,\iota} = \frac{2\pi}{L} \Delta N_{av,\iota}^F + \frac{Q_{av}(t q_{Fav}^0)}{2\pi}; \quad \alpha\nu = c0, s1; \quad \iota = \pm 1,$$

respectively, where $q_{Fav}^0$ is the initial ground-state Fermi-point value given in Eq. \(10\). Here $Q_{av}(q)/2$ is the $\alpha\nu$ pseudofermion overall scattering phase shift \(11\). We call $2\Delta_{av}$ the square of the $\alpha\nu, \iota$ Fermi-point value shift in units of $2\pi/L$,

$$2\Delta_{av}^c \equiv \left( \frac{\Delta q_{Fav,\iota}}{2\pi/L} \right)^2 = \left( \frac{\iota \Delta N_{av,\iota}^0 + \frac{Q_{av}(t q_{Fav}^0)}{2\pi}}{2\pi} \right)^2 = \left( \frac{\iota \Delta N_{av,\iota}^F + \frac{Q_{av}(t q_{Fav}^0)}{2\pi}}{2\pi} \right)^2; \quad \alpha\nu = c0, s1; \quad \iota = \pm 1.$$  \(12\)

This quantity plays a key-role in the spectral properties, as discussed in Secs. IV and V. It has the form given in Eq. \ref{eq12} for $N$-electron spectral functions leading to deviations in the electronic numbers, $\Delta N$, such that $|\Delta N| = 0, 1, 2$. Our analysis is limited to such spectral functions. Our studies of these functions involve the use of specific expressions for the pseudofermion bare-momentum distribution-function deviation functions. The $c0$ and $s1$ pseudofermion bare-momentum distribution-function deviations of the excited energy eigenstates that span a J-CPHS subspace have the following general form,

$$\Delta N_{av}(q) = \Delta N_{av}^N(q) + \Delta N_{av}^F(q) + \Delta N_{av}^{phF}(q); \quad \alpha\nu = c0, s1.$$  \(13\)

The bare-momentum distribution-function deviation $\Delta N_{av}^N(q)$ corresponds to the above elementary processes (A) and is given by,

$$\Delta N_{av}^N(q) = \text{sgn}(\Delta N_{av}^N) \frac{2\pi}{L} \sum_{i=1}^{|\Delta N_{av}^N|} [\delta(q - q_i) + \frac{2\pi}{L} \sum_{j=1}^{N_{phF}} [\delta(q - q_j) - \delta(q - q'_j)]]; \quad \alpha\nu = c0, s1,$$

where $\text{sgn}(\Delta N_{av}^N) = \text{sgn}(\Delta N_{av})$ and $q_i$ with $i = 1, \ldots, |\Delta N_{av}^N|$ gives the set of bare-momentum values associated with creation ($\Delta N_{av}^N > 0$) or annihilation ($\Delta N_{av}^N < 0$) of $\alpha\nu$ pseudofermions away from the Fermi points. Moreover, $q_j$ and $q'_j$ with $j = 1, \ldots, N_{phF}$ denote the bare-momentum values associated with the particles and holes, respectively, of the finite-energy and finite-momentum $\alpha\nu$ pseudofermion particle-hole processes.

The bare-momentum distribution function deviation $\Delta N_{av}^F(q)$ results from the above elementary processes (B) and reads,

$$\Delta N_{av}^F(q) = \frac{2\pi}{L} \delta(|q| - q_{Fav}^0) \Delta N_{av,\iota}^F \delta_{\iota 1, \text{sgn}(q) 1}; \quad \alpha\nu = c0, s1.$$  \(14\)

Finally, the bare-momentum distribution function deviation $\Delta N_{av}^{phF}(q)$ is associated with the elementary processes (C) and is given by,

$$\Delta N_{av}^{phF}(q) = \frac{2\pi}{L} \sum_{i=1}^{N_{phF}} \sum_{j=1}^{N_{phF}} [\delta(q - q_{j}) - \delta(q - q'_j)]; \quad \alpha\nu = c0, s1, \quad \iota = \text{sgn}(q) 1.$$  \(15\)

In the latter equation $q_{j}$ and $q'_{j}$, with $j = 1, \ldots, N_{phF}$, denote the bare-momentum values associated with the "particles" and "holes", respectively, in the vicinity of the Fermi point of Eq. \(10\) corresponding to the $\alpha\nu, \iota$ subbranch index value $\iota = \text{sgn}(q) 1 = \pm 1$. Such bare-momentum values are ordered as $\iota q_1, < \iota q_2, < \ldots < \iota q_{N_{phF}}$ and $\iota q'_{1}, < \iota q'_{2}, < \ldots < \iota q'_{N_{phF}}$ for $\alpha\nu = c0, s1$ and $\iota = \pm 1$. Once the elementary processes (C) generate the J-CPHS
subspaces from the corresponding reduced subspaces, the set of "particle" and "hole" bare-momentum values \( \{ q_i \} \) and \( \{ q'_i \} \) are such that \( q = q_i \) and \( q = q'_i \) was unoccupied and occupied in the corresponding initial excited energy eigenstate belonging to the latter subspace, respectively. The number \( N_{αν}^{EF} \) is such that, \( N_{αν}^{EF} \leq m_{αν} \), where \( m_{αν} \) is the number of \( αν \), \( s1 \) pseudofermion elementary particle-hole processes of momentum \( \ell [2π/L] \) involved in the deviation \( ΔN_{αν}^{EF}(q) \) of Eq. \([16]\). Different such deviations may correspond to the same number \( m_{αν} \).

The \( αν \neq c0 \), \( s1 \) branches have no finite pseudofermion occupancy for the ground state. The \( αν \neq c0 \), \( s1 \) pseudofermion occupancy configurations of the excited energy eigenstates which span a J-CHPS subspace can be defined by the values of the sets of numbers \( \{ N_{αν}^{EF} \} \) associated with the independent holons or spinons and the \( αν \) FP scattering centers and the following bare-momentum distribution-function deviations,

\[
ΔN_{αν}^{NF}(q) = \frac{2π}{L} ∑_{i=1}^{N_{αν}^{NF}} δ(q - q'_i) ; \quad αν \neq c0, s1 .
\]  

Here \( q'_i \), with \( i = 1, ..., N_{αν}^{NF} \), gives the set of bare-momentum values of the pseudofermions created by the elementary processes (A) away from the limiting bare-momentum values \( ±q_{αν}' \).

### B. THE ENERGY AND MOMENTUM SPECTRA FOR DENSITIES 0 < n < 1 AND 0 < m < n

Our studies of Secs. IV and V require an analysis of the energy and momentum spectra beyond that of Ref. \([10]\). The pseudofermion dispersions \( ε_{αν}(q) \) determine the form of the energy excitation spectrum of the weight distributions studied in these sections. These energy bands are plotted in Figs. 6-9 of Ref. \([17]\) as a function of \( ǫ \).

Here the two-pseudofermion phase shifts \( \tilde{Φ} \) are given in Eqs. \([A3]\) and \([A4]\) of Appendix A, \( Q \) is the parameter defined by Eq. \([A6]\) of that Appendix, and \( μ = μ(n, m, U/t) \) and \( H = H(n, m, U/t) \) can be expressed in terms of the same phase shifts as,

\[
μ = \frac{U}{2} + 2t \cos Q - t ∫_{-Q}^{+Q} dk [2Φ_{c0, c0}(k, Q) + Φ_{c0, s1}(k, B)] \sin k ; \quad μ0H = -t ∫_{-Q}^{+Q} dk Φ_{c0, s1}(k, B) \sin k .
\]  

The ground-state rapidity functions \( k^0(q) \), \( Λ^0_{cf}(q) \), and \( Λ^0_{sm}(q) \) appearing on the right-hand side of Eqs. \([A1]-[A2]\) of Appendix A. (Analytical expressions for the rapidity functions \( k^0(q) \), \( Λ^0_{cf}(q) \), and \( Λ^0_{sm}(q) \) as \( m \to 0 \) and both \( U/t \to 0 \) and \( U/t >> 1 \) are provided in Ref. \([16]\).)

The zero-energy, \( 2μν \)-energy, and \( 2μνH \)-energy levels of the pseudofermion energy bands \([A5]-[A6]\) relative to the ground-state energy are such that,

\[
ε_{c0}(±2k_F) = ε_{s1}(±k_F) = 0 ; \quad ε_{c0}(±π - 2k_F) = 2μν , \quad ν > 0 ; \quad ε_{sν}(±[k_F↑ - k_F↓]) = 2μνH , \quad ν > 1 ;
\]

\[
epsilon_{c0}^0(±[π - 2k_F]) = 0 , \quad ν > 0 ; \quad ε_{sν}^0(±[k_F↑ - k_F↓]) = 0 , \quad ν > 1 .
\]  

\]
Creation onto the ground state of one \( \nu \nu \) pseudofermion or \( \nu \sigma \) pseudofermion belonging to \( \nu > 0 \) and \( \nu > 1 \) branches, respectively, is a finite-energy process whose minimal energy value is given by \( \epsilon_{\alpha\nu}(\pm|2kF|) = 2\nu\mu \) or \( \epsilon_{\sigma\nu}(0) = 2\nu\mu H + \epsilon_{\sigma\nu}^0(0) \), respectively. Creation onto the ground state of one independent \(-1/2\) holon or independent \(-1/2\) spinon requires energy \( 2\mu \) or \( 2\nu\mu H \), respectively. In the limit \( n \to 1 \) (and \( m \to 0 \)) both the energy and bare-momentum band widths of the dispersions \( \epsilon_{\alpha\nu}(q) \) for \( \nu > 0 \) (and \( \epsilon_{\sigma\nu}(q) \) for \( \nu > 1 \)) vanish. Moreover, while the energy band width \( 4t \) and bare-momentum band width \( 2\pi \) of the dispersion \( \epsilon_{\sigma\nu}(q) \) remain the same for all values of \( U/t \), the energy band width of the dispersions of all other pseudofermion branches vanishes as \( U/t \to \infty \). This effect follows from the localized character reached by the \( \alpha\nu \) pseudofermions of \( \nu = 1, 2, ..., \) branches as \( U/t \to \infty \). In that limit the \( \alpha\nu \neq c0 \) pseudofermions and \( \nu\nu \) pseudofermions separate into \( 2\nu \) independent holons and spinons, respectively.

The group velocity \( v_{\alpha\nu}(q) \) and the light velocity \( v_{\alpha} \) are given by,

\[
v_{\alpha\nu}(q) = \frac{\partial \epsilon_{\alpha\nu}(q)}{\partial q}, \quad \text{all branches}; \quad v_{\alpha\nu} = v_{\alpha\nu}(q_{F\alpha\nu}^0), \quad \alpha\nu = c0, s1,
\]

and play an important role in the spectral-weight distribution expressions obtained in Sec. V.

The \( \alpha\nu \)-branch finite-energy spectrum associated with the elementary processes (A) corresponds to the reduced J-CPHS subspaces and reads,

\[
\Delta E_{\alpha\nu}(\{q_i\}, \{q_j\}, \{q'_j\}) = \text{sgn}(\Delta N_{\alpha\nu}^N) \sum_{i=1}^{N_{\alpha\nu}^N} \epsilon_{\alpha\nu}(q_i) + \sum_{j=1}^{N_{\alpha\nu}^N} \left[ \epsilon_{\alpha\nu}(q_j) - \epsilon_{\alpha\nu}(q'_j) \right]; \quad \alpha\nu = c0, s1;
\]

\[
\Delta E_{\alpha\nu}(\{q'_i\}) = \sum_{i=1}^{N_{\alpha\nu}^N} \epsilon_{\alpha\nu}(q'_i); \quad \alpha\nu \neq c0, s1,
\]

where the pseudofermion energy bands are given in Eqs. (18)-(20).

The finite-momentum spectrum has contributions both from the elementary processes (A) and (B) and also corresponds to the reduced J-CPHS subspaces. It is given by,

\[
\Delta P_{\alpha\nu}(\{q_i\}, \{q_j\}, \{q'_j\}) = \Delta P_{\alpha\nu}^F + \Delta P_{\alpha\nu}^{NF}(\{q_i\}, \{q_j\}, \{q'_j\}); \quad \alpha\nu = c0, s1;
\]

\[
\Delta P_{\nu\nu}(\{q'_i\}) = k_{\nu\nu} + \Delta P_{\nu\nu}^{NF}(\{q'_i\}); \quad \nu\nu \neq c0; \quad \Delta P_{\nu\nu}(\{q'_i\}) = \sum_{i=1}^{N_{\nu\nu}^F} q'_i; \quad \nu\nu \neq s1.
\]

Here \( \Delta P_{\alpha\nu}^F \) is given in Eq. (67) of Ref. [11],

\[
\Delta P_{\alpha\nu}^{NF}(\{q_i\}, \{q_j\}, \{q'_j\}) = \text{sgn}(\Delta N_{\alpha\nu}^N) \sum_{i=1}^{N_{\alpha\nu}^N} q_i + \sum_{j=1}^{N_{\alpha\nu}^N} [q_j - q'_j]; \quad \alpha\nu = c0, s1,
\]

\[
\Delta P_{\nu\nu}^{NF}(\{q'_i\}) = -\sum_{i=1}^{N_{\nu\nu}^NF} q'_i; \quad k_{\nu\nu} = (1+\nu)\pi N_{\nu\nu}^{NF}; \quad \nu\nu \neq c0,
\]

and \( q_{F\alpha\nu}^0 \) is the \( c0 \) and \( s1 \) pseudofermion ground-state Fermi-point value given in Eq. (6). The general finite-energy and finite-momentum spectrum generated by the elementary processes (A) and (B) reads,

\[
\Delta E = \Delta E(\{q_i\}, \{q'_i\}, \{q_j\}, \{q'_j\}) = \sum_{\alpha\nu=c0, s1} \Delta E_{\alpha\nu}(\{q_i\}, \{q_j\}, \{q'_j\}) + \sum_{\alpha\nu\neq c0, s1} \Delta E_{\alpha\nu}(\{q'_i\}) + \sum_{\alpha=c, s} E_\alpha,
\]

and
\[ \Delta P = \Delta P ([q_i, \{ q'_i \}, \{ q_j \}, \{ q'_j \}]) = P_0 + \sum_{\alpha = c, s} \Delta P^{NF}_{\alpha} ([q_i, \{ q_j \}, \{ q'_j \}]) \]

\[ + \sum_{\nu \neq 0} \Delta P^{NF}_{\nu} ([q'_i]) + \sum_{\nu \neq s} \Delta P_{sv} ([q'_i]), \]

where

\[ P_0 = k_0^F + \sum_{\nu \neq 0} k_{c\nu} + \sum_{\alpha = c, s} P_{\alpha} ; \quad k_0^F = \sum_{\alpha = 0, s} \Delta P^{F}_{\alpha}, \]

and the energy \( E_\alpha \) and momentum \( P_\nu \) correspond to the independent holons (\( \alpha = c \)) and spinons (\( \alpha = s \)) and are given in Eqs. (66) and (67) of Ref. \[10\], respectively. Both the spectra \[28\] and \[29\] are associated with reduced J-CPHS subspaces.

Generation of the J-CPHS subspaces from the corresponding reduced subspaces involves the elementary processes (C). Given the linear \( \alpha \nu = c0, s1 \) pseudofermion energy dispersion near the Fermi points, these processes lead to small momentum and energy values such that

\[ k' = \sum_{\alpha \nu = 0, s1} \sum_{\nu = \pm 1} \frac{2\pi}{L} m_{\alpha \nu, \nu} ; \quad \omega' = \sum_{\alpha \nu = 0, s1} \sum_{\nu = \pm 1} \frac{2\pi}{L} v_{\alpha \nu} m_{\alpha \nu, \nu}. \]

Here \( m_{\alpha \nu, \nu} \) is the number of elementary \( \alpha \nu \) pseudofermion particle-hole processes of momentum \( \pm 2\pi/L \) defined above and \( v_{\alpha \nu} \) is the \( \alpha \nu = c0, s1 \) pseudofermion light velocity given in Eq. \[29\]. Thus, the elementary processes (C) generate a set of excited energy eigenstates with energy and momentum given by those of the initial reduced-subspace state, Eqs. \[24\] and \[26\], respectively, plus the small energy and momentum given in Eq. \[30\]. According to the results of Ref. \[11\], the elementary processes (C) have a non-interacting character in terms of \( \alpha \nu = c0, s1 \) pseudoparticles. In turn, for the remaining low-energy excitations such pseudoparticles have residual interactions, in contrast to the corresponding \( \alpha \nu = c0, s1 \) pseudofermons. This pseudoparticle non-interacting character implies that the energy spectrum of Eq. \[29\] remains linear in \( m_{\alpha \nu, \nu} \) for small finite values of \( m_{\alpha \nu, \nu}/N_a \) as \( N_a \to \infty \).

The energy \( \omega_0 \) for transitions from the ground state to a point-subspace plays an important role in the spectral-weight distributions. Such an energy is that of the excited energy eigenstate whose deviation numbers are given in Eq. \[30\] and reads,

\[ \omega_0 = \sum_{\alpha = c, s} E_\alpha = 2\mu M_{c, \nu} + 2\mu_0 H [M_{s, \nu} - N_{s1}], \]

where \( N_{s1} = N_{s1}^0 + \Delta N_{s1} \). We note that each CPHS ensemble subspace contains at least one point-subspace and that all point-subspaces of a CPHS ensemble subspace have the same energy, given in Eq. \[30\]. Furthermore, the smallest excitation energy for transitions from the ground state to a CPHS ensemble subspace reads,

\[ E^{0}_{CPHS} = \omega_0 + \sum_{\nu \neq s1} \epsilon_{\nu}^0 (0) N_{s\nu}, \]

where \( \omega_0 \) is the energy \[32\] of the point-subspace(s) contained in the CPHS ensemble subspace and \( \epsilon_{\nu}^0 (0) < 0 \) for \( m > 0 \) and \( \epsilon_{\nu}^0 (0) \to 0 \) as \( m \to 0 \). Thus, for \( m \to 0 \), one has that \( E^{0}_{CPHS} = \omega_0 = 2\mu M_{c, \nu} \). In this case the energy value \( E^{0}_{CPHS} = \omega_0 = 0 \) corresponds to a \( M_{\nu} = 0 \) CPHS ensemble subspace, \( E^{0}_{CPHS} = \omega_0 = 2\mu \) to the \( M_{\nu} = 1 \) first-Hubbard band CPHS ensemble subspace, \( E^{0}_{CPHS} = \omega_0 = 4\mu \) to the \( M_{\nu} = 2 \) second-Hubbard band CPHS ensemble subspace, and so on. We recall that the number of \(-1/2\) holons, \( M_{\nu} = 1/2 \), equals that of rotated-electron doubly occupied sites \([9]\).

The excited energy eigenstate that spans a point-subspace has energy \( \omega = \omega_0 \) and momentum \( k = k_0 \) relative to the initial ground state, where \( k_0 \) is given by,

\[ k_0 = k_0^F + \sum_{\alpha = c, s} P_{\alpha} = \pi [L_{c, \nu} + 1] + \sum_{\nu = 1}^{\infty} \nu N_{\nu}^F + 4k_0 \left[ \Delta J_{c0}^F + \sum_{\nu = 1}^{\infty} J_{c\nu}^F + \sum_{\nu = 2}^{\infty} J_{s\nu}^F \right] + 2k_{F} \left[ \Delta J_{s1}^F - 2 \sum_{\nu = 2}^{\infty} J_{s\nu}^F \right]. \]
Here \( k_0^F \) is provided in Eq. \((40)\). A property of the one- and two-electron spectral-weight distributions is that for small values of the energy deviation \((\omega - \omega_0)\) and \(m \to 0\) there is spectral weight only for values of momentum \(k\) such that \((k - ik_0)\) is also small. While the momentum spectrum \(\pi \left[ L_c, -1/2 + \sum_{\nu=1}^\infty \nu N_{c\nu}^F \right]\) is additive in the momentum \(\pi\) of each independent \(-1/2\) holon, the momenta \(4k_F \left[ \Delta J_{c0}^F + \sum_{\nu=1}^\infty J_{c\nu}^F + \sum_{\nu=2}^\infty J_{\nu\nu}^F \right] \) and \(2k_F \left[ \Delta J_{c1}^F - 2 \sum_{\nu=2}^\infty J_{\nu\nu}^F \right]\) refer to the \(c0\) and \(s1\) pseudofermion \textit{Fermi points} current number deviations. Note that although the current numbers \(J_{c\nu}^F\) and \(J_{\nu\nu}^F\) arise from the \(c\nu\) and \(s\nu\) \(1\) \(\text{FP}\) scattering centers, they lead to momentum contributions associated with the \(c0\) \textit{Fermi points} and \(c0\) and \(s1\) \textit{Fermi points}, respectively. These current numbers correspond to phase shifts whose scattering centers are felt by the \(c0\) and \(s1\) pseudofermions has being at the \(\alpha\nu = c0, s1\) \textit{Fermi points} \((12)\). Point-subspaces contained in the same CPHS ensemble subspace have the same energy but different momentum values. For a given CPHS ensemble subspace, we call \textit{first point-subspaces} and \textit{second point-subspaces} the subspaces with the momentum \((34)\) given by \(k_0 = \pm |k_0|\) which correspond to the minimum value of \(|k_0|\) and the second smallest value, respectively.

### C. THE ELEMENTARY PROCESSES AND THE ACTIVE SCATTERING CENTERS

Active scattering centers are those which contribute to the scattering phase shift \((1)\). We find below that only the elementary processes \((A)\) and \((B)\) generate active scattering centers. For the excited energy eigenstates described by the bare-momentum distribution function deviations given in Eqs. \((13)-\(17)\) the scattering phase shift \((1)\) can be written as,

\[
Q_{\alpha\nu}^F(q)/2 = Q_{\alpha\nu}^{(NF)}(q)/2 + Q_{\alpha\nu}^F(q)/2 .
\]

Here the phase shifts \(Q_{\alpha\nu}^{(NF)}(q)/2\) and \(Q_{\alpha\nu}^F(q)/2\) result from active scattering centers generated by the elementary processes \((A)\) and \((B)\), respectively. The spectral properties are mainly controlled by the scattering phase shifts \(Q_{\alpha\nu}^{(nf)}(q_{F\alpha\nu})/2\) of the \(\alpha\nu = c0, s1\) pseudofermion and hole scatterers of bare momentum values \(q \approx \pm q_{F\alpha\nu}^0\). The scattering phase shift \(Q_{\alpha\nu}^{(NF)}(q_{F\alpha\nu})/2\), where \(\alpha\nu = c0, s1\) and \(\iota = \pm 1\), can be written as,

\[
Q_{\alpha\nu}^{(NF)}(q_{F\alpha\nu}, \{q_i\}, \{q'_{i}\}) = \pi \left\{ \sum_{\alpha'\nu' = c0, s1} \left( \text{sgn}(\Delta N_{c\nu'}/(q_{F\alpha\nu}, q_i) \right) \right\} \left| \Delta N_{c\nu'}/(q_{F\alpha\nu}, q_i) \right| + \sum_{j=1}^{N_{c\nu'}/(q_{F\alpha\nu}, q_i)} \left[ \Phi_{\alpha\nu, \alpha'\nu'}(q_{F\alpha\nu}, q_j) - \Phi_{\alpha\nu, \alpha'\nu'}(q_{F\alpha\nu}', q_j) \right] \right\} + \sum_{\alpha'\nu' = c0, s1} \sum_{i=1}^{N_{c\nu'}/(q_{F\alpha\nu}, q_i)} \Phi_{\alpha\nu, \alpha'\nu'}(q_{F\alpha\nu}, q_i) .
\]

In turn, the phase shift \(Q_{\alpha\nu}^F(q_{F\alpha\nu})/2\) was derived in Ref. \((12)\) and reads,

\[
Q_{\alpha\nu}^F(q_{F\alpha\nu})/2 = \pi \left( \xi_{\alpha\nu, c0}/2 \frac{\Delta N_{c0}^F}{2} + \xi_{\alpha\nu, s1}/2 \frac{\Delta N_{s1}^F}{2} - \xi_{\alpha\nu, c0}/2 \frac{\Delta N_{c0}^F}{2} - \xi_{\alpha\nu, s1}/2 \frac{\Delta N_{s1}^F}{2} \right) + \xi_{\alpha\nu, c0}/2 \left[ \Delta J_{s1}^F - 2 \sum_{\nu=2} J_{\nu\nu}^F \right] .
\]

Here the \textit{Fermi-point} parameters \(\xi_{\alpha\nu, \alpha'\nu'}^1\) are symmetrical \((j = 0)\) and antisymmetrical \((j = 1)\) combinations in the \(\alpha'\nu' = c0, s1\) pseudofermion scattering centers right and left \textit{Fermi-point} values of the elementary \(\alpha\nu = c0, s1\) two-pseudofermion phase shifts in units of \(\pi\) given by,

\[
\xi_{\alpha\nu, \alpha'\nu'}^j = \delta_{\alpha, \alpha'} \delta_{\nu, \nu'} + \sum_{i=\pm 1} \left( \xi_{\alpha\nu, \alpha'\nu'}^j(q_{F\alpha\nu}, q_{F\alpha'\nu'}) \right) ; \quad j = 0, 1 ; \quad \alpha\nu = c0, s1 ; \quad \alpha'\nu' = c0, s1.
\]

The parameters \((38)\) appear in the spectral-function exponents derived in Secs. IV and V. In the limit \(m \to 0\), they are given by \(\xi_{c0, c0}^0 = 1/\xi_0, \xi_{c0, s1}^0 = 0, \xi_{s1, c0}^0 = -1/\sqrt{2}, \xi_{s1, s1}^0 = \sqrt{2}, \xi_{c0, c0}^1 = \xi_0, \xi_{c0, s1}^1 = \xi_0/2, \xi_{s1, c0}^1 = 0, \) and \(\xi_{s1, s1}^1 = 1/\sqrt{2}\). Here \(\xi_0\) is the parameter defined in Eq. \((74)\) of Ref. \((18)\) and in the text above that equation. It is
such that \( \xi_0 \to \sqrt{2} \) and \( \xi_0 \to 1 \) as \( U/t \to 0 \) and \( U/t \to \infty \), respectively. As further discussed in Ref. \cite{12}, the form of the general phase-shift expression \cite{57} confirms that although the current numbers \( J^F_{\alpha \nu} \) and \( J^s_{\alpha \nu} \) are associated with the \( cv \neq c0 \) and \( sv \neq s1 \) FP scattering centers, the \( c0 \) and \( s1 \) pseudofermions feel such centers as being at the \( c0 \) Fermi points and \( c0 \) and \( s1 \) Fermi points, respectively.

The following functional plays a major role in the spectral properties,

\[
\xi_0 = \xi_0(\{q_i\}, \{q'_i\}, \{q\}, \{q'_j\}) = 2\Delta_{c0} + 2\Delta_{s1}; \quad 2\Delta_{\alpha \nu} = 2\Delta_{\alpha \nu}^{+1} + 2\Delta_{\alpha \nu}^{-1}; \quad \alpha \nu = c0, s1. \tag{39}
\]

Moreover, by use of of the general expressions \cite{56} and \cite{57} one finds that the four functionals \( 2\Delta_{\alpha \nu} \), Eq. \cite{12}, can be written as follows,

\[
2\Delta_{\alpha \nu} = 2\Delta_{\alpha \nu}^{0}(\{q_i\}, \{q'_i\}, \{q\}, \{q'_j\}) = \left( \frac{Q^{\Phi(NF)}_{\alpha \nu}(\{q'_j\})}{2\pi} \right) \left( \xi_{\alpha \nu}^{0} + \xi_{\alpha \nu}^{s1} \right), \tag{40}
\]

where the phase shift \( Q^{\Phi(NF)}_{\alpha \nu}(\{q'_j\}) \) is that of Eq. \cite{36}.

Analysis of the above general phase-shift expressions confirms that the \( c0 \) and \( s1 \) elementary processes (C) associated with the bare-momentum distribution function deviations \( \Delta N^{phF}_{\alpha \nu}(q) \) of Eq. \cite{10} do not generate active scattering centers. Let us consider a “hole” and “particle” created at bare-momentum values \( q' \) and \( q \) such that \( \Delta_{\alpha \nu} = 2\Delta_{\alpha \nu}^{+1} + 2\Delta_{\alpha \nu}^{-1} \) where \( \Delta_{\alpha \nu} = 2\Delta_{\alpha \nu}^{+1} + 2\Delta_{\alpha \nu}^{-1} \) is a finite integer number, such a deviation does not contribute to the scattering phase shifts. Therefore, up to first order in \( 1/L \), the scattering phase-shift contributions from the deviations involving the summations of the \( \Delta N^{phF}_{\alpha \nu}(q) \) “particle” and “hole” pairs of the bare-momentum distribution function deviation \( \Delta N^{phF}_{\alpha \nu}(q) \) do not contribute to the scattering phase shifts. The overall scattering phase shift \( \Phi_{\alpha \nu}^{q}(q)/L \) such that \( j > 1 \) have no physical significance \cite{11}, we thus conclude that \( Q^{\Phi}_{\alpha \nu}(q)/L = 0 \).

Once all the \( \Delta N^{phF}_{\alpha \nu}(q) \) “particle” and “hole” pairs of the bare-momentum distribution function deviation \( \Delta N^{phF}_{\alpha \nu}(q) \) of Eq. \cite{10} are such that \( \eta_1 - \eta_2 = \frac{2\pi}{Nf} \), where \( Nf = \ell_1, \ell_2, ... \) is a finite integer number, such deviation does not contribute to the scattering phase shifts. Therefore, up to first order in \( 1/L \), the scattering phase-shift contributions from the deviations involving the summations of the \( \Delta N^{phF}_{\alpha \nu}(q) \) “particle” and “hole” pairs of the bare-momentum distribution function deviation \( \Delta N^{phF}_{\alpha \nu}(q) \) are exactly canceled by those involving the \( \Delta N^{phF}_{\alpha \nu}(q) \) “particle” and “hole” pairs of the bare-momentum distribution function deviation \( \Delta N^{phF}_{\alpha \nu}(q) \) which are not active scattering centers.

IV. FINITE-ENERGY WEIGHT DISTRIBUTIONS FOR DENSITIES \( 0 < n < 1 \) AND \( 0 < m < n \)

Our starting point for the derivation of closed-form analytical expressions for the one- and two-electron weight distributions obtained in the ensuing section is the general \( \mathcal{N} \)-electron spectral function given in Eq. \cite{36} of Ref. \cite{10},

\[
B^l_{\mathcal{N}}(k, \omega) = \sum_{i=0}^{\infty} c_i^l \sum \{\Delta N_{\alpha \nu}; \{L_{\alpha \nu}; 1/2\}; \{N^{phF}_{\alpha \nu}; \{N^{\Delta F}_{\alpha \nu, \downarrow}; \{N^{\Delta F}_{\alpha \nu, \uparrow}\}\}\}; \quad c_i^l = 1, \quad l = \pm 1, \tag{41}
\]

where for the present case of densities \( 0 < n < 1 \) and \( 0 < m < n \) the summation over the numbers \( N^{phF}_{\alpha \nu} \) is limited to finite values and thus refers to reduced J-CPHS subspaces only. On the right-hand side of Eq. \cite{14}, \( c_i^l \) is the constant of the operator expressions given in Eqs. \cite{32}-\cite{34} of Ref. \cite{10} such that \( c_i^l \to 0 \) as \( U/t \to \infty \) for \( i > 0 \) and the function \( B^l_{\mathcal{N}}(k, \omega) \) is defined in Eq. \cite{63} of that reference. Let us rewrite the \( \alpha \nu \) pseudofermion spectral functions introduced in Eq. \cite{63} of Ref. \cite{14} as follows,
Moreover, here we have explicitly written the summation \( \prod_i \) \( \frac{1}{N_a} \left( \sum_{q_j = -q_{0,\alpha \nu}^0}^{q_{0,\alpha \nu}^0} \frac{1}{N_a} \sum_{q'_j = -q_{0,\alpha \nu}^0}^{q_{0,\alpha \nu}^0} \right) \delta(\omega - l\Delta E_{\alpha \nu}) \delta_{k,\ell \Delta P_{\alpha \nu}}; \quad \alpha \nu = c0, s1; \)

\[ B^{1,NF,i}_{\alpha \nu}(k, \omega) = N_a \left\{ \left( \Theta(\Delta N^{NF}_{\alpha \nu}) \frac{|\Delta N^{NF}_{\alpha \nu}|}{\prod_i} \left[ \frac{1}{N_a} \left( \sum_{q_j = -q_{0,\alpha \nu}^0}^{q_{0,\alpha \nu}^0} + \sum_{q'_j = -q_{0,\alpha \nu}^0}^{q_{0,\alpha \nu}^0} \right) \right] \right\} + \Theta(-\Delta N^{NF}_{\alpha \nu}) \frac{|\Delta N^{NF}_{\alpha \nu}|}{\prod_i} \left[ \frac{1}{N_a} \left( \sum_{q_j = -q_{0,\alpha \nu}^0}^{q_{0,\alpha \nu}^0} \right) \right] \right\} \delta(\omega - l\Delta E_{\alpha \nu}) \delta_{k,\ell \Delta P_{\alpha \nu}}; \quad \alpha \nu \neq c0, s1; \quad l = \pm 1; \quad i = 0, 1, 2, \ldots, \] (42)

where the energy and momentum spectra are given in Eqs. (24) and (25), respectively. On the right-hand side of Eq. (22) and in all expressions given below the \( \Theta \) function is such that \( \Theta(x) = 0 \) for \( x < 0 \) and \( \Theta(x) = 1 \) for \( x \geq 0 \). Moreover, here we have explicitly written the summation \( \sum_i \) \( -(\text{CPSH} - \alpha \nu - (A)) \) of expression (63) of Ref. 10, which runs over the reduced J-CPSH subspace \( \alpha \nu \) pseudofermion occupancy configurations generated by the elementary processes (A). The number of such occupancy configurations is given by \( \prod_{\alpha \nu} N_{\alpha \nu} \), where for the \( \alpha \nu = c0 \), s1 branches the dimension \( D_{\alpha \nu} \) reads,

\[ D_{\alpha \nu} = \left\{ \Theta(\text{sgn}(\Delta N^{NF}_{\alpha \nu})) \left( N^{h,\alpha \nu}_{N^{NF}_{\alpha \nu}} \right) + \Theta(-\text{sgn}(\Delta N^{NF}_{\alpha \nu})) \left( -N^{\omega,\alpha \nu}_{N^{NF}_{\alpha \nu}} \right) \right\} \left( N^{h,\alpha \nu}_{N^{NF}_{\alpha \nu}} \right) \left( N^{\omega,\alpha \nu}_{N^{NF}_{\alpha \nu}} \right), \] (43)

and is given in Eq. (64) of Ref. 10 for the \( \alpha \nu \neq c0, s1 \) branches. We recall that Eq. (43) refers to densities \( 0 < n < 1 \) and \( 0 < m < n \) and thus the numbers \( N_{\alpha \nu}, N^{h,\alpha \nu}, N^{\omega,\alpha \nu}, N^s_{\alpha \nu} \) are such that \( N_{\alpha \nu} \rightarrow \infty, N^{h,\alpha \nu} \rightarrow \infty, N^{\omega,\alpha \nu} \rightarrow \infty, N^s_{\alpha \nu} \rightarrow \infty \), whereas for the PS excited energy eigenstates the numbers \( \Delta N^{NF}_{\alpha \nu}, \Delta N^{h,\alpha \nu}, \Delta N^{\omega,\alpha \nu} \) have finite values. Thus, in expression (43) we have not considered small finite corrections to the numbers \( \alpha \nu = c0, s1 \), and \( N^s_{\alpha \nu} \), which lead to higher order corrections that vanish in the thermodynamic limit. Moreover, in expression (43) we could treat the bare-momentum summations as independent for each created or annihilated pseudofermion. It follows that for \( N_{\alpha \nu} \rightarrow \infty \), exclusion of the occupancy configurations where two or several created or annihilated pseudofermions meet at the same discrete bare-momentum value leads to vanishing corrections to expression (42).

From use of the spectral function expression (22) in the second expression of Eq. (68) of Ref. 11, we arrive to the following expression for the function \( B^{1,i}(k, \omega) \) appearing on the right-hand side of Eq. (41),

\[ B^{1,i}(k, \omega) = \left\{ \prod_{\alpha \nu = c0, s1} \left[ \Theta(\Delta N^{NF}_{\alpha \nu}) \frac{|\Delta N^{NF}_{\alpha \nu}|}{\prod_i} \left[ \frac{1}{N_a} \left( \sum_{q_j = -q_{0,\alpha \nu}^0}^{q_{0,\alpha \nu}^0} + \sum_{q'_j = -q_{0,\alpha \nu}^0}^{q_{0,\alpha \nu}^0} \right) \right] \right\} + \Theta(-\Delta N^{NF}_{\alpha \nu}) \frac{|\Delta N^{NF}_{\alpha \nu}|}{\prod_i} \left[ \frac{1}{N_a} \left( \sum_{q_j = -q_{0,\alpha \nu}^0}^{q_{0,\alpha \nu}^0} \right) \right] \right\} \delta(\omega - l\Delta E_{\alpha \nu}) \delta_{k,\ell \Delta P_{\alpha \nu}}; \quad \alpha \nu \neq c0, s1; \quad l = \pm 1; \quad i = 0, 1, 2, \ldots, \] (44)

Here the general energy functional \( \Delta E = \Delta E(q_i, q'_i, q_j, q'_j) \) and momentum functional \( \Delta P = \Delta P(q_i, q'_i, q_j, q'_j) \) are given in Eqs. (28) and (29), respectively, and in the thermodynamic limit the value of the small positive energy \( \Omega \), which corresponds to the energy range of the elementary processes (C), is controlled by the functional \( \Theta_\Omega \), Eq. (39), of the corresponding initial reduced-subspace state. (The energy \( \Omega \) vanishes when \( \Theta_\Omega \rightarrow 0 \) and the weight distribution becomes \( \delta \)-function like, as further discussed in Sec. V.) In numerical studies of the spectral-function expressions derived in this paper the small energy \( \Omega \) is treated as a parameter whose value for fixed finite values of \( n, m \), and \( U/t \) is determined by imposing the exact sum rule (43) in the end of the calculations. The bare-momentum summations of expression (44) run over all \( \prod_{\alpha \nu} D_{\alpha \nu} \) occupancy configurations generated by the elementary processes (A). For the whole \( (k, \omega) \)-plane and except for \( k \) and \( \omega \) values such that \( \omega \approx i \nu \), \( k \approx k_0 \) is provided in Eq. (49), and \( k_0 \) in Eq. (44), the function \( B^{1,i}(\Delta \omega, \nu) \) on right-hand side of Eq. (44) is the convolution of the \( c0 \) and \( s1 \) spectral functions given in Eq. (62) of Ref. 10. These functions read,
\[ B_{Q_{\alpha \nu}}^{i}(k', \omega') = \sum_{J-\text{CPHS} - \alpha \nu - (C)} |\langle 0 | F_{J-\text{GS}, \alpha \nu} F_{P-h, \alpha \nu} F_{\text{GS}, \alpha \nu} | 0 \rangle|^2 \delta \left( \omega' - \frac{2\pi}{L} v_{\alpha \nu} m_{\alpha \nu} I \right) \delta_{k', l} \sum_{l} \delta_{m_{\alpha \nu} I} \]  

In this equation \( i = 0, 1, 2, \ldots, l = \pm 1, \Delta \nu = (\nu - l \Delta E), \Delta k = (k - l \Delta P) \), the summation \( \sum_{J-\text{CPHS} - \alpha \nu - (C)} \) runs over the J-CPHS ensemble subspace \( \alpha \nu = c0, s1 \) pseudofermion occupancy configurations generated by the elementary processes (C), the matrix-element generators are given in Appendix B, and \( m_{\alpha \nu} = \sum_{i=\pm 1} m_{\alpha \nu, i} \). The identity of the first and second expressions of Eq. (45) follows from the structure of the spectrum of Eq. (44), which is generated by the elementary processes (C). However, we note that the validity of the second expression and of the spectral-function expressions given below is limited to electronic densities such that \( v_{c0} > v_{s1} \). We emphasize that this inequality holds for all electronic densities \( n \) of the metallic phase except for a small domain in the vicinity of \( n = 1 \). The velocity \( v \) appearing in the argument of the function \( f \) plays an important role in our study and is given by,

\[ v = \frac{\Delta \omega}{\Delta k} = \left( \frac{\omega - l \Delta E}{k - l \Delta P} \right); \quad \text{sgn}(v) = \text{sgn}(\Delta k) l; \quad |v| > v_{s1}. \]  

The inequality \( |v| > v_{s1} \) and the theta-functions appearing on the right-hand side of Eq. (44) follow again from the structure of the spectrum of Eq. (43).

Each excited energy eigenstate generated from the ground state by the elementary processes (A) and (B) corresponds to one point, \( (l \Delta P, l \Delta E) \), of the finite-weight \( (k, \omega) \)-plane region associated with such a subspace. The set of all such points generated by the bare-momentum summations on the right-hand side of Eq. (44) span a well-defined finite-weight \( (k, \omega) \)-plane point-like, line-like, or surface-like domain. From the reduced J-CPHS subspace excited energy eigenstates corresponding to points \( (l \Delta P, l \Delta E) \) in the vicinity of the \( (k, \omega) \)-plane point that the spectral-function expression (45) refers to, the elementary processes (C) generate excited energy eigenstates whose momentum and energy relative to the initial ground state is \( k \) and \( \omega \), respectively. Thus, the momentum \( l \Delta P \) and energy \( l \Delta E \) of the reduced J-CPHS subspace excited energy eigenstates relative to the initial ground state are such that \( (\omega - l \Delta E) \) and \( (k - l \Delta P) \) are small and \( |v| = |(\omega - l \Delta E)/(k - l \Delta P)| \) obeys the inequality \( |v| > v_{s1} \).

The \( \alpha \nu = c0, s1 \) pseudofermion spectral function \( B_{Q_{\alpha \nu}}^{i}(k', \omega') \) of Eq. (45) can be expressed as the convolution (with an extra pre-factor of \( N_{\alpha} / 2 \)) of the two following \( \alpha \nu, \iota \) spectral functions for the \( \iota = +1 \) and \( \iota = -1 \) sub-branches, respectively,

\[ B_{Q_{\alpha \nu}}^{i, i}(k', \omega') = \sum_{J-\text{CPHS} - \alpha \nu, \iota - (C)} |\langle 0 | F_{J-\text{GS}, \alpha \nu, \iota} F_{P-h, \alpha \nu, \iota} F_{\text{GS}, \alpha \nu, \iota} | 0 \rangle|^2 \delta \left( \omega' - \frac{2\pi}{L} v_{\alpha \nu} m_{\alpha \nu, \iota} I \right) \delta_{k', l}. \]  

where \( \alpha \nu = c0, s1 \) and \( l = \pm 1 \). Both the function \( B_{Q_{\alpha \nu}}^{i, i}(k', \omega') \) of Eq. (45) and the function (46) refer to the subspace spanned by the tower of states generated by the elementary processes (C) from each initial reduced J-CPHS subspace excited state. The summation \( \sum_{J-\text{CPHS} - \alpha \nu, \iota - (C)} \) on the right-hand side of Eq. (47) runs over all \( \alpha \nu = c0, s1 \) pseudofermion occupancy configurations generated by the small-momentum and low-energy elementary processes (C) for the \( \alpha \nu \) pseudofermions belonging to the \( \iota = \text{sgn}(q) = \pm 1 \) sub-branch.

The contributions of the elementary processes (A) to the matrix-element overlaps of the above spectral functions are easy to evaluate, and lead to the matrix-element expressions given in Eq. (59) of Ref. 10. After such contributions are accounted for, the problem is reduced to the overlap of the \( c0 \) and \( s1 \) pseudofermion probability amplitudes provided in Eq. (60) of that reference. That amplitude is associated with the matrix element \( |\langle 0 | F_{J-\text{GS}, \alpha \nu} F_{P-h, \alpha \nu} F_{\text{GS}, \alpha \nu} | 0 \rangle|^2 \) given in Eq. (60) of Ref. 10 and the corresponding \( \alpha \nu = c0, s1 \) pseudofermion spectral function \( B_{Q_{\alpha \nu}}^{i, i}(k', \omega') \) of Eq. (45). The generators appearing in the matrix elements given in Eqs. (59) and (60) of Ref. 10 are expressed in terms of the sets of canonical-momentum and bare-momentum values of the distribution function deviations \( \alpha \nu \) in Eqs. (61) and (65) of Appendix B. In contrast to the matrix elements of Eq. (59) of that reference, the evaluation of the probability amplitude \( |\langle 0 | F_{J-\text{GS}, \alpha \nu} F_{P-h, \alpha \nu} F_{\text{GS}, \alpha \nu} | 0 \rangle|^2 \) is a complex problem, which we address in the ensuing section. The states \( F_{\text{GS}, \alpha \nu} | 0 \rangle \) and \( F_{P-h, \alpha \nu} F_{\text{GS}, \alpha \nu} | 0 \rangle \) involved in that probability amplitude describe \( N_{\alpha \nu} \) and
\[ \Delta N^F_{\alpha \nu} \] pseudofermions, whose discrete canonical-momentum values are those of the ground-state and excited-state J-CPHS subspace, respectively. Also the state \( F^j_{J-\text{GS,} \alpha \nu} |0\rangle \) describes \( N^0_{\alpha \nu} + \Delta N^F_{\alpha \nu} \) pseudofermions, whose discrete canonical-momentum values are those of the excited-state reduced J-CPHS subspace. Both the states \( F^j_{J-\text{GS,} \alpha \nu} |0\rangle \) and \( F^j_{J-\text{GS,} \alpha \nu} |0\rangle \) refer to densely packed canonical-momentum occupancy configurations. The probability amplitude \( A^{(0,0)}_{\alpha \nu} \equiv |\langle 0| F_{J-\text{GS,} \alpha \nu} F^j_{J-\text{GS,} \alpha \nu} |0\rangle|^2 \) associated with such canonical-momentum densely packed configurations gives the \( \alpha \nu \) pseudofermion spectral-function lowest-peak weight. By use of the general \( \alpha \nu = 0, 1 \) pseudofermion determinants given in Eqs. (71) and (72) of Ref. [10], we find the following expression for such a weight,

\[
A^{(0,0)}_{\alpha \nu} = \frac{1}{N^0_{\alpha \nu}} ^{2N^0_{\alpha \nu} + \Delta N^F_{\alpha \nu}} \prod_{j=1}^{N^*_{\alpha \nu}} \sin^2 \left( \frac{N^0_{\alpha \nu}(q_j)[Q_{\alpha \nu}(q_j) - \pi] + \pi}{2} \right) \prod_{j=1}^{N^*_{\alpha \nu} - 1} \sin \left( \frac{\pi j}{N^*_{\alpha \nu}} \right) \]

\[
\times \prod_{i=1}^{N^*_{\alpha \nu}} \prod_{j=1}^{N^*_{\alpha \nu}} \theta(j-i) \times \sin^2 \left( \frac{N^0_{\alpha \nu}(q_j)N^0_{\alpha \nu}(q_i)[Q_{\alpha \nu}(q_i) - Q_{\alpha \nu}(q_j) + 2\pi(j-i) - \pi N^*_{\alpha \nu} + \pi N^*_{\alpha \nu}] + N^*_{\alpha \nu}}{2N^0_{\alpha \nu}} \right) \times \prod_{i=1}^{N^*_{\alpha \nu}} \prod_{j=1}^{N^*_{\alpha \nu}} \sin^2 \left( \frac{N^0_{\alpha \nu}(q_i)N^0_{\alpha \nu}(q_j)[Q_{\alpha \nu}(q_j) - Q_{\alpha \nu}(q_i) + 2\pi(j-i) - \pi N^*_{\alpha \nu} + \pi N^*_{\alpha \nu}] + N^*_{\alpha \nu}}{2N^0_{\alpha \nu}} \right) ; \quad \alpha \nu = 0, 1 , \quad \text{(48)}
\]

where \( N^0_{\alpha \nu}(q) = N^0_{\alpha \nu}(q) + \Delta N^F_{\alpha \nu}(q) \). \( N^0_{\alpha \nu}(q) \) is the ground-state bare-momentum distribution function provided in Eqs. (C.1)-(C.3) of Ref. [6], and the deviation \( \Delta N^F_{\alpha \nu}(q) \) is given in Eq. [15]. Moreover, we find that in the thermodynamic limit the lowest-peak weight [18] has the following approximate behavior,

\[
A^{(0,0)}_{\alpha \nu} \approx \prod_{i=\pm 1} A^{(0,0)}_{\alpha \nu, i} \left[ 1 + \mathcal{O} \left( \frac{1}{N^0_{\alpha \nu}} \right) \right] ; \quad A^{(0,0)}_{\alpha \nu, i} = \frac{f_{\alpha \nu, i}^0}{S^0_{\alpha \nu} S^0_{\alpha \nu}} \left| N_{\alpha \nu}^0 \right|^{1/2 + 2\Delta^i_{\alpha \nu}} ; \quad \alpha \nu = 0, 1 ; \quad i = \pm 1 . \quad \text{(49)}
\]

Here \( 2\Delta^i_{\alpha \nu} \) is the functional given in Eq. [10] and \( f_{\alpha \nu, i} \) reads,

\[
f_{\alpha \nu, i} = \sqrt{f(Q_{\alpha \nu}(q_0^\nu) + \text{sgn}(k)\pi)} ; \quad f_{\alpha \nu} = \prod_{i=\pm 1} f_{\alpha \nu, i} ; \quad \alpha \nu = 0, 1 , \quad \text{(50)}
\]

where \( k \) is the excited-state momentum relative to the initial ground state, \( f(Q) = f(-Q) \) is the function defined in Ref. [14], which appears on the right-hand side of Eq. (24) of that reference, and \( f_{\alpha \nu} \) appears in spectral-function expressions introduced below. Moreover, \( S^0_{\alpha \nu} \) is a \( n, m \) and \( U/t \) dependent constant such that \( S^0_{\alpha \nu} S^0_{\alpha \nu} \rightarrow 1 \) both for \( U/t \rightarrow 0 \) and for \( U/t \rightarrow \infty \) and \( m \rightarrow 0 \). (From Ref. [14] we learn that \( S^0_{\alpha \nu} \rightarrow \sin(\pi n) \) for \( U/t \rightarrow 0 \) and \( m \rightarrow 0 \), and thus \( S^0_{\alpha \nu} \rightarrow 1/\sin(\pi n) \) in such a limit.)

The momentum and energy dependence of the spectral-function expression [14] is controlled by the convolution function, Eq. [15], of the \( \alpha \nu = 0, 1 \) pseudofermion spectral functions \( B^i_{\alpha \nu, \nu} (k', \omega') \) given in the same equation. The main mechanism of such a control is the exotic matrix-element overlap associated with the \( \alpha \nu = 0, 1 \) pseudofermion elementary processes (C), which generate \( \alpha \nu = 0, 1 \) pseudofermion "particles" and "holes" in the vicinity of the Fermi points. Although such "particles" and "holes" are not active scattering centers, they are active scatterers whose overall phase shifts originate the orthogonality catastrophe leading to the unusual overlaps of the matrix elements \( \langle 0| F_{J-\text{GS,} \alpha \nu} F_{p-h, \nu} F^j_{J-\text{GS,} \alpha \nu} |0\rangle \). Interestingly, only the overall phase shifts of the scatterers which are not active scattering centers and are generated by the elementary processes (C) contribute to the relative weights associated with such an orthogonality catastrophe. On the other hand, the value of the corresponding overall phase shifts is solely determined by the occupancy configurations of the initial reduced J-CPHS space states generated by the elementary processes (A) and (B) from the ground state. Indeed, the latter processes generate the active scattering centers that determine the value of the overall phase shifts of the "particles" and "holes" created by the elementary processes (C). Therefore, the unusual spectral properties result from the interplay of the elementary processes (A) and (B), which generate the active scattering centers, with the elementary processes (C), which generate the active scatterers, as far as the overall phase shift associated with the relative weights and the corresponding orthogonality catastrophe that controls these properties are concerned. All the \( \alpha \nu = 0, 1 \) pseudofermions with bare momentum away from the
Fermi points are also scatterers, but their overall phase shifts only contribute to the lowest-peak weight probability amplitude $A_{αν}^{(0,0)}$ given in Eq. (48).

Next, let us calculate the convolution function and spectral functions given in Eqs. (45) and (47). The expressions of these functions fully define the general spectral functions (41) and (44). The canonical-momentum occupancy configuration of the state $F_{J,Gs,αν}^{(0)}$ associated with the lowest-peak weight probability amplitude $A_{αν}^{(0,0)}$ is densely packed and thus such that $(m_{αν}+1, m_{αν}-1) = (0, 0)$. Following the form of the momentum and energy spectra provided in Eq. (41), the summation on the right-hand side of the $B_{Q,αν}^{(0)}(k', ω')$ expression of Eq. (45) (and Eq. (47)) over all occupancy configurations generated by the elementary $αν = c0$, s1 pseudofermion particle-hole processes (C) with the same energy and momentum, is equivalent to consider such a summation for all processes with the same value of $m_{αν}, i = \lfloor L/4πv_{αν}\rfloor(ω' + v_{αν}k')$ (and $m_{αν}, i = \lfloor L/2πv_{αν}\rfloorω' = \lfloor L/2πk'\rfloor$). By performing that summation we reach the relative weight $A_{αν}^{(0,0)}/[0|F_{J,Gs,αν}F_{p-h,αν} F_{l1gs,αν}^{(1)}(0)]^2$ corresponding to excited energy eigenstates with $N_{αν}^{ph}$ $αν$ pseudofermion particle-hole processes. In Appendix B we consider states generated by $N_{αν}^{ph} αν = c0, s1$ pseudofermion finite-momentum and finite-energy particle-hole processes generated by the elementary processes (A). Instead, it is such that for $N_{αν} → ∞$ its values only contribute to the $αν$ pseudofermion relative weights when the sum rule $N_{αν}^{ph} = \sum_{αν, i = ±1} N_{αν,F}^{ph}$ is obeyed.) For $N_{αν} → ∞$ only the relative weights associated with the small-momentum and low-energy $αν = c0, s1$ pseudofermion elementary particle-hole processes (C) in the vicinity of the Fermi points $±q_{αν}^{0}$ lead to finite contributions to the $αν = c0, s1$ pseudofermion spectral function $B_{Q,αν}^{(1)}(k', ω')$ of Eq. (45). Each of the excited energy eigenstates described by the numbers $(m_{αν}+1, m_{αν}-1)$ can be multiply degenerate. Therefore, we need to sum up the relative weights $a_{αν}^{[N_{αν}^{ph}]}$, Eq. (B6) of Appendix B, for all the $αν$ pseudofermion particle-hole configurations with the same values of $(m_{αν}+1, m_{αν}-1)$. Fortunately, for these processes, the general relative weight expressions given in Eqs. (B6)-(B9) of Appendix B simplify. This follows in part from the symmetry found in Sec. III that all the excited energy eigenstates generated by the elementary processes (C) from a given initial reduced-subspace state have the same value for the four functionals $2Δ_{αν}$ defined in Eq. (10). We find the following general expression for the relative weights in the tower of excited energy eigenstates,

$$a_{αν}(m_{αν}+1, m_{αν}-1) = \prod_{i = ±1} a_{αν, i}(m_{αν}, i) \left[1 + O\left(\ln N_{αν}/N_{αν}\right)\right]; \quad αν = c0, s1. \quad (51)$$

Here $a_{αν, i}(m_{αν}, i)$ is the relative weight associated of the spectral function (47). It reads,

$$a_{αν, i}(m_{αν}, i) = Θ(m_{αν}, i) \prod_{j=1}^{m_{αν}, i} \frac{(2Δ_{αν} + j - 1)}{j} = Θ(m_{αν}, i) \frac{Γ(m_{αν} + 2Δ_{αν})}{Γ(m_{αν} + 1)Γ(2Δ_{αν})}; \quad αν = c0, s1; \quad i = ±1, \quad (52)$$

where $Γ(x)$ is the usual gamma function. (While, by construction, the integer $m_{αν, i}$ is such that $m_{αν, i} ≥ 0$, for later use it is convenient to include the theta-function in the relative weight general expression (52).) It follows from Eq. (52) that,

$$a_{αν, i}(1) = 2Δ_{αν}^i; \quad αν = c0, s1; \quad i = ±1, \quad (53)$$

what reveals that the functional (12) is the relative weight associated with the $m_{αν, i} = 1$ peak of the $αν, i$ pseudofermion spectral function (47). Moreover,

$$a_{αν}(1, 0) = 2Δ_{αν}^1; \quad a_{αν}(0, 1) = 2Δ_{αν}^{-1}; \quad αν = c0, s1. \quad (54)$$

The function (52) has the following asymptotic behavior,

$$a_{αν, i}(m_{αν}, i) ≈ Θ(m_{αν}, i) \frac{1}{Γ(2Δ_{αν})} \left(\frac{m_{αν}}{2Δ_{αν} - 1}\right)^{2Δ_{αν} - 1}; \quad 2Δ_{αν}^i ≠ 0; \quad αν = c0, s1; \quad i = ±1. \quad (55)$$

Use of the above results in the pseudofermion spectral functions given in Eqs. (45) and (47) leads to the following expressions for these functions,
\[ B_{Qav}^{i,i}(k', \omega') = \sum_{m_{av},+1, m_{av},-1} A_{av}^{i}(0) a_{av}(m_{av},+1, m_{av},-1) \delta(\omega' - \frac{2\pi}{L} v_{av} \sum_{i=\pm 1} m_{av,i}) \delta_{k', i, \omega'} \sum_{i=\pm 1} m_{av,i} \]

\[ \frac{L}{4\pi v_{av}} \prod_{i=\pm 1} a_{av,i}(\frac{[\omega' + t v_{av} k']}{4\pi v_{av}/L}); \quad \alpha\nu = c_0, s_1, \]

and

\[ B_{Qav}^{i'}(k', \omega') = \sum_{m_{av},i} A_{av}^{i}(0) a_{av,\nu}(m_{av},i) \delta(\omega' - \frac{2\pi}{L} v_{av} m_{av,i}) \delta_{k', \omega'} \]

\[ = \frac{L}{2\pi v_{av}} A_{av}^{i}(0) a_{av,\nu}(\frac{\omega'}{2\pi v_{av}/L}) \delta_{k', \omega'} \quad \alpha\nu = c_0, s_1; \quad \iota = \pm 1, \]

respectively. In order to distinguish the momentum and energy values of these pseudofermion spectral functions from those of the spectral function \( k, \omega \), in Eqs. 15, 17, 56, and 57, we denote the former momentum and energy values by \( k' \) and \( \omega' \), respectively. It follows from Eq. 155 that for \( 2\Delta_{av} \neq 0 \) the spectral-function expressions 56 and 57 have the following asymptotic behavior,

\[ B_{Qav}^{i,i}(k', \omega') \approx \frac{f_{av}}{4\pi v_{av} S_{av}^{0}} \prod_{i=\pm 1} \frac{1}{\Gamma(2\Delta_{av})} \Theta(l[\omega' + t v_{av} k']) \left( \frac{[\omega' + t v_{av} k']}{4\pi v_{av} S_{av}^{0}} \right)^{2\Delta_{av} - 1}; \quad \alpha\nu = c_0, s_1; \]

\[ B_{Qav}^{i,i}(k', \omega') \approx \sqrt{\frac{\bar{\omega}}{v_{av} S_{av}^{0}}} \Gamma(2\Delta_{av}) \Theta(l[\omega' \left( \frac{\omega'}{2\pi v_{av} S_{av}^{0}} \right)^{2\Delta_{av} - 1} \delta_{k', \omega'}; \quad \alpha\nu = c_0, s_1; \quad \iota = \pm 1, \]

for small finite values of \( l[\omega' + t v_{av} k'] \) and \( \omega' \), respectively.

Let us consider the convolution function 155 for vanishing energy values of the order of 1/L. For instance, the six smallest discrete energy values read \( l\Delta \omega = 0, l\Delta \omega = [2\pi/L]v_{s1}, l\Delta \omega = [2\pi/L]v_{c0}, l\Delta \omega = [4\pi/L]v_{s1}, l\Delta \omega = [2\pi/L]v_{c0} + v_{s1} \), and \( l\Delta \omega = [4\pi/L]v_{s1} \) such that \( l\Delta \omega \leq [4\pi/L]v_{c0} \). For vanishing energy values we define that convolution function as \( \bar{B}_{Qav}(\Delta, v) = \bar{B}_{Qav}(\Delta, \Delta k) \), where by use of the first expression of Eq. 155, we find,

\[ \bar{B}_{Qav}(\Delta, \Delta k) = 2\pi \left( \frac{1}{N_{av}} \right)^{1/2} D_{0} \delta(\Delta \omega) \delta(\Delta k); \quad \delta_{0} = \prod_{\alpha,\nu=0,1} \frac{S_{av}^{0} f_{av}}{(S_{av}^{0})^{2\delta_{av}}}; \quad l\Delta \omega = 0; \quad l\Delta k = 0; \]

\[ \bar{B}_{Qav}(\Delta, \Delta k) = 2\pi \left( \frac{1}{N_{av}} \right)^{1/2} D_{0} \alpha_{0} \delta(l(\Delta \omega - \omega(C))) \delta(l(\Delta k - k(C)));
\]

\[ l\Delta \omega = \omega(C); \quad v = \frac{\omega(C)}{k(C)}. \]

Here \( \zeta_{0} \) and \( 2\Delta_{av} \) are the functions given in Eq. 89, and for the above five energy values such that \( l\Delta \omega > 0 \) the relative weight \( \alpha_{0} \) reads: \( a_{0} = 2\Delta_{av} \) for \( \omega(C) = [2\pi/L]v_{s0}, k(C) = \iota [2\pi/L] \) and \( v = \iota v_{av}; \alpha_{0} = 2\Delta_{\bar{C}} \sqrt{2\Delta_{av}} \) for \( \omega(C) = [4\pi/L]v_{s0}, k(C) = 0, \) and \( 1/\nu \); \( a_{0} = 2\Delta_{\bar{C}} \sqrt{2\Delta_{av}} \) for \( \omega(C) = [2\pi/L]v_{s1}, k(C) = \iota \) \([2\pi/L]\), and \( v = \iota [v_{s0} + v_{s1}]/2 \) where we introduced the index \( \bar{\alpha} \) \( \bar{\omega} \) such that \( \bar{\omega} = s1, \bar{s} = c0; \alpha_{0} = 2\Delta_{av} \sqrt{2\Delta_{av}} \) for \( \omega(C) = [2\pi/L]v_{s0} + v_{s1}, k(C) = 0, \) and \( 1/\nu \); \( a_{0} = 2\Delta_{\bar{C}} \sqrt{2\Delta_{av}} + 1/2 \) for \( \omega(C) = [4\pi/L]v_{s0}, k(C) = \iota \) \([2\pi/L]\), and \( v = \iota v_{av} \). All these expressions refer to \( \alpha\nu = c_0, s_1 \) and \( \iota = \pm 1 \). Similar \( 2\Delta_{av} \) expressions can be derived for larger energy values of order of 1/L.

In turn, for small finite values of \( l\Delta \omega \) we use the \( \alpha\nu \) pseudofermion spectral function expression 81 and the variables \( x = \omega'/\Delta \omega \) and \( y = \text{sgn}(v)k' [v_{av}, \omega] / \Delta \omega \) where \( \Delta \omega = (\omega - \Delta E) \) in the second expression of Eq. 19, and find,

\[ \bar{B}_{Qav}(\Delta, \Delta k) \approx \frac{A_{av}^{(0)} A_{av}^{(0)}}{2\pi v_{av}} \left( \frac{l\Delta \omega}{4\pi v_{av}} \right)^{2} \int_{0}^{\Delta \omega} dx \int_{-1}^{1} dy \sum_{i=\pm 1} a_{av,i} \left( \frac{l\Delta \omega}{4\pi v_{av}} \right) \left[ \frac{v_{av} + v_{s1}}{v_{s0}} (x + iy) \right] \]

\[ \times \left( \frac{l\Delta \omega}{4\pi v_{av}} \right) \left[ \frac{v_{av} + v_{s1}}{v_{s0}} (1 - x + iy) \right]; \quad i = 0, 1, 2, ..., l = \pm 1. \]

The expressions given in Eqs. 159 and 160 combined with Eqs. 41 and 44 provide the desired general spectral function expression, except for \( k \) and \( \omega \) values such that \( \omega \approx \iota v_{av}(k - \Delta) + \omega_{0} \), where \( \alpha\nu = c_0, s_1 \) and \( \iota = \pm 1 \).
In the ensuing section we use the function \(\zeta\) to derive closed-form analytical expressions for the finite-energy one- and two-electron spectral-weight distributions.

For small finite values of \(|\Delta \omega| = |\omega - \Delta E|\), use of the relative weight asymptotic expression \(\zeta_{\alpha}\) in Eq. (40) for the whole \(x\) and \(y\) integration domains leads to the following asymptotic behavior for the convolution function \(\Phi\):

\[
\Phi_{\alpha,\beta}(l_{\omega}, v) \approx \frac{F_{0}(1/v)}{4\pi \sqrt{v_{c0} v_{x1}}} \Theta(l_{\Delta \omega}) \left( \frac{l_{\Delta \omega}}{4\pi \sqrt{v_{c0} v_{x1}}} \right)^{-2+\zeta_{0}}; \quad i = 0, 1, 2, ..., \quad l = \pm 1.
\]

Here \(\zeta_{0}\) is the functional given in Eq. (39) and the function \(F_{0}(z)\) reads,

\[
F_{0}(z) = 2D_{0} \sqrt{v_{x1}} v_{c0} \int_{0}^{1} dx \int_{-1}^{1} dy \prod_{i=\pm 1} \Theta\left(1 - x + \text{sgn}(z) \xi \left[ v_{x1} |z| - \frac{v_{c0}}{v_{x1}} y \right] \right) \Theta\left(1 + x + \text{sgn}(z) \xi y \right) \Gamma(2\Delta_{c0}) \Gamma(2\Delta_{s1})^{-1} \left( \frac{v_{x1}}{v_{c0}} |z| - \frac{v_{c0}}{v_{x1}} y \right)^{2\Delta_{s1}-1},
\]

where \(2\Delta_{s1}\) is the functional given in Eq. (60) and \(D_{0}\) is defined in Eq. (45). Note that both the values of the functional \(\zeta_{0} = \zeta_{0}(\{q_{i}\}, \{q'_{i}\}, \{q_{j}\}, \{q'_{j}\})\) defined in Eq. (39) and of the function \(F_{0}(z) = C_{0}(z, \{q_{i}\}, \{q'_{i}\}, \{q_{j}\}, \{q'_{j}\})\) of Eq. (42) depend on the set of bare-momentum values \(\{q_{i}\}, \{q'_{i}\}, \{q_{j}\}, \{q'_{j}\}\) which define the active-scattering-centers occupancy configurations of the initial reduced-subspace excited energy eigenstates generated by the elementary processes (A). Such a dependence follows from the functional character of the quantity \(2\Delta_{\alpha\nu} = 2\Delta_{\alpha\nu}(\{q_{i}\}, \{q'_{i}\}, \{q_{j}\}, \{q'_{j}\})\) defined by Eq. (30). The dependence of the value of the latter functional on the set of bare-momentum values \(\{q_{i}\}, \{q'_{i}\}, \{q_{j}\}, \{q'_{j}\}\) occurs through the scattering phase shift \(Q_{\alpha\nu}(\{q_{i}\}, \{q'_{i}\}, \{q_{j}\}, \{q'_{j}\})/2\) appearing in Eq. (40), which is defined in Eq. (60). Thus, we should keep in mind that the values of the functionals \(2\Delta_{\alpha\nu}\) and \(\zeta_{0}\) appearing in the spectral-function expressions derived below are different for each pseudofermion bare-momentum occupancy configuration defining a specific excited energy eigenstate generated from the ground state by the elementary processes (A) and (B).

## V. WEIGHT DISTRIBUTION EXPRESSIONS GENERATED BY THE DOMINANT PROCESSES

In this section we derive analytical closed-form expressions for the one- and two-electron weight distributions associated with the function \(\zeta\) for \((k, \omega)\)-plane regions whose weight is generated by the dominant processes. These processes are characterized below in terms of rotated-electron processes. Our study includes the derivation of the weight distribution expressions valid for \(k\) and \(\omega\) values such that \(\omega \approx \xi v_{x1}(k-\xi l_{\zeta_{0}}) + l_{\zeta_{0}}\), where \(\omega = \epsilon_{0}\), \(s_{1}\) and \(\xi = \pm 1\). Some of the analytical expressions found below refer to the vicinity of the weight distribution singularities and edges. Importantly, the spectral features observed in real experiments correspond to such singularities [2, 13, 14].

### A. THE DOMINANT PROCESSES AND THE FUNCTIONAL \(\zeta_{0}(k, \omega)\)

The spectral weight distribution associated with the terms of the general finite-energy spectral function \(\zeta\) of index \(i > 0\) decreases very rapidly for increasing values of the number \(i\) of extra rotated-electron pairs. Indeed, for the excited energy eigenstates associated with the spectral-function contributions of increasing \(i\) value the functional \(\zeta_{0}\) of Eq. (59) also has increasingly larger values. Typically, the contributions of orders \(i\) larger than \(i = 1\) are beyond numerical measurability, once the spectral weight associated with the \(i = 0\) term of the spectral function \(\zeta\) corresponds in general to over 99% of the whole spectral weight. (This is confirmed for the one-electron spectral function in Ref. [14].) Thus, for practical applications it is in general enough to consider the elementary processes associated with the \(i = 0\) term only.

The smallness of the weight associated with the \(i > 0\) terms of the spectral function \(\zeta\) results from the form of the function \(\zeta\). Indeed, for increasing values of the index \(i\) the expression of the operator \(\Theta^{l_{\kappa,\alpha\nu}}_{i,\zeta_{0}}\) associated with the operator \(\Theta^{l_{\kappa,\alpha\nu}}_{i,\zeta_{0}}\), Eq. (58) of Ref. [10], has in terms of rotated-electron operators an increasing number of rotated-electron particle-hole pair operators. Thus, for increasing values of \(i\) application of such an operator onto the ground state produces elementary processes (A) and (B) which generate an increasing number of active scattering centers. Since the value of the functional \(\zeta_{0}\) given in Eq. (60) is in general an increasing function of the number of
generated active scattering centers, it follows that the $i > 0$ contributions to the spectral function are very small. Below it is confirmed that most singularities of the latter function occur at $(k, \omega)$-plane isolated points and lines, are of power-law form, and are controlled by processes which generate excited states whose $\zeta_0$ values are smaller than two and one, respectively. Thus, a quite good approximation corresponds to replacing the general spectral-function expression \(\mathcal{B}_i\) by its $i = 0$ term. The excited-energy-eigenstate deviations generated by the dominant processes associated with the $i = 0$ operator $\hat{\Theta}_{N_0,j}$ on the right-hand side of Eq. (32) of Ref. \[10\] obey the two selection rules given in Eq. (41) of the same reference. The numbers $\bar{N}_1$ and $\bar{N}_0$ in that selection rule are those of Eq. (30) of Ref. \[10\] specific to the corresponding operator $\hat{\Theta}_{N,j}$ defined in Eqs. (27) and (28) of the same reference. Furthermore, all excited energy eigenstates whose deviations do not obey that selection rule are generated by the $i > 0$ operators on the right-hand side of the latter equation. It follows that a good approximation for the spectral function consists in using expression \(\mathcal{B}_i\) for $i = 0$ and the summation on the right-hand side of Eq. (41) over the J-CPHS subspaces whose number deviations obey the sum rules (18), (19), (39), and (43) of Ref. \[10\] and selection rules given in Eqs. (21) and (41) of the same reference.

Moreover, for the one- and two-electron spectral functions nearly the whole weight is concentrated in the $(k, \omega)$-plane regions associated with creation by the elementary processes (A) of none, one, and two active scattering centers away from the $\omega = 0$, $s_1$ Fermi points and $\omega \neq 0$, $s_1$ limiting bare-momentum values. Concerning singular weight features, we find below that for finite values of the on-site repulsion $U$ only exceptionally and for some spectral functions and isolated $(k, \omega)$-plane points these features are of $\delta$-function type. In general, such singularities are instead of power-law shape. Power-law spectral-function behavior occurs for $(k, \omega)$-plane regions in the vicinity of points or lines corresponding to different reduced J-CPHS subspaces than that associated with such regions. In order to describe such an effect it is convenient to associate each $(k, \omega)$-plane point with the minimum value of the functional $\zeta_0$, Eq. (39), of the excited energy eigenstates generated by the elementary processes (A) and (B) that correspond to such a point. We thus introduce the functional,

$$\zeta_0(k, \omega) \equiv \min \zeta_0.$$  

In general, for the same regions of the $(k, \omega)$-plane the spectral function \(\mathcal{B}_i\) has contributions from different functions $B^{i}\mathcal{B}(k, \omega)$, associated with different J-CPHS subspaces of the same CPHS ensemble subspace. The value of the functional \(\zeta_0\) corresponds to the smallest value of $\zeta_0$ of all states associated with the $(k, \omega)$-plane point into consideration. For two-dimensional $(k, \omega)$-plane regions associated with a given reduced J-CPHS subspace, the value of the functional $\zeta_0$, Eq. (39), is in general a continuous and smooth function of $k$ and $\omega$. Therefore, in that case the value of the functional $\zeta_0(k, \omega)$ is also in general a continuous and smooth function of $k$ and $\omega$. Importantly, the spectral weight-power features correspond to the points or lines where $\zeta_0(k, \omega)$ shows discontinuities. (The inverse is not always true.) Typically, such discontinuities occur in the border lines of the $(k, \omega)$-plane domains associated with the reduced J-CPHS subspaces. Such subspaces can correspond to a $(k, \omega)$-plane isolated point, line, or two-dimensional domain. In general, the corresponding value of the the functional $\zeta_0$, Eq. (39), is smallest for the point-subspaces whose deviation numbers and numbers are given in Eq. (19). It is also in general smaller for the line-like reduced J-CPHS subspaces than for the subspaces corresponding to a $(k, \omega)$-plane two-dimensional domain. Therefore, when the reduced J-CPHS subspace $(k, \omega)$-plane domains correspond to isolated points or lines, the discontinuities of the value of the functional \(\zeta_0\) occur at these points or lines. It also shows discontinuities at the $(k, \omega)$-plane border lines defining the limits of the reduced J-CPHS subspace two-dimensional domains.

The reason why the spectral function has power-law behavior in the vicinity of the $(k, \omega)$-plane isolated points and lines, where the value of the functional \(\zeta_0\) has discontinuities, is the small energy gap between these regions and the point or line corresponding to the discontinuities. The value of the functional \(\zeta_0\) is in general different at the point or line than in its vicinity. Moreover, in general that value is smaller for the former line or point than in its proximity. Thus, the small gap gives the minimal excitation energy of the tower states associated with the $(k, \omega)$-plane point where one is calculating the spectral function and which are generated by the elementary processes (C) relative to that of the initial states corresponding to the point or line where the value of the functional \(\zeta_0\) has a discontinuity. We recall that all states generated by the elementary processes (C) have the same value for the functional $\zeta_0$, Eq. (39), as the initial state, which corresponds to the point or line where the value of the functional \(\zeta_0\) has a discontinuity. The occurrence of the small gap implies that the corresponding convolution function \(\mathcal{B}_i\) has the power-law asymptotic form \(\mathcal{B}_1\). Note that such a power-law behavior is controlled by the exponent $-2 + \zeta_0$, which is negative provided that $\zeta_0 < 2$. Furthermore, once the value of the functional \(\zeta_0\) is smaller for these excited states than for states generated by elementary processes (C) from initial states corresponding to $(k, \omega)$-plane points located closer to the point where we are calculating the spectral function, the spectral function itself has a power-law shape, as confirmed below.

In turn, if in the vicinity of a $(k, \omega)$-plane point the value of the functional \(\zeta_0\) has no discontinuities, the spectral function expression \(\mathcal{B}_i\) at that point involves integrations over gapless contributions from transitions generated by
the elementary processes (C) from initial states corresponding to a small two-dimensional \((k, \omega)\)-plane domain just below \((l = +1)\) or above \((l = -1)\) the point. Once the value of the functional \(\mathcal{Z}_0(k, \omega)\) is in this case a continuous and smooth function of \(k\) and \(\omega\), its value remains the same in the infinitesimal vicinity of the point. In this case the spectral function has no power-law behavior, as confirmed below. The lack of such a behavior follows from the gapless character of the excitations contributing to the spectral function at that point. Below we consider the spectral-weight distribution behavior generated by creation of one, two active scattering centers away from the \(\alpha \nu = c_0, s_1\) Fermi points and \(\alpha \nu \neq c_0, s_1\) limiting bare-momentum values. For one- and two-electron spectral functions the weight generated by processes involving the creation of more than two active pseudofermion or hole scattering centers away from the \(\alpha \nu = c_0, s_1\) Fermi points and \(\alpha \nu \neq c_0, s_1\) limiting bare-momentum values is very small and can in general be neglected.

Our results reveal that the functional \((63)\) plays a major role in the control of the spectral weight distribution. Increasing the number of created active pseudofermion and pseudofermion hole scattering centers increases in general the value of that functional. Moreover, the summations in expression \((64)\) over the states generated by the elementary processes (A) further increase the value of the power-law exponent \(-2 + \zeta_0\) appearing in the convolution function \((61)\). Indeed, these summations add integer numbers to the exponent \(-2 + \zeta_0\): creation of each active scattering center adds 1 to that exponent, as confirmed below. A central point is that most spectral weight singularities are of power-law type and correspond to the discontinuities of the value of the functional \((63)\) in the \((k, \omega)\)-plane. Also the weight associated with creation of two or more active scattering centers away from the \(\alpha \nu = c_0, s_1\) Fermi points and \(\alpha \nu \neq c_0, s_1\) limiting bare-momentum values is controlled by that functional. Furthermore, the same functional controls the exceptional occurrence of \(\delta\)-function singularities: such singularities correspond to the \((k, \omega)\)-plane points where its value vanishes.

Each spectral-weight distribution \((44)\) corresponds to a J-CPHS subspace. We start by considering the spectral-weight distributions generated by creation of two active pseudofermions and/or pseudofermion holes scattering centers away from the \(\alpha \nu = c_0, s_1\) Fermi points and \(\alpha \nu \neq c_0, s_1\) limiting bare-momentum values. Thereafter, we consider spectral-weight distributions \((44)\) associated with contributions from processes involving creation of none and one active scattering center away from the \(\alpha \nu = c_0, s_1\) Fermi points and \(\alpha \nu \neq c_0, s_1\) limiting bare-momentum values. We find below that the latter processes lead to important point- and line-like singular spectral-weight features.

### B. TWO-DIMENSIONAL \((k, \omega)\)-PLANE REGIONS WHERE THE SPECTRAL WEIGHT IS CONTROLLED BY TWO ACTIVE-SCATTERING-CENTER CREATION

The parametric equations that define the two-dimensional \((k, \omega)\)-plane domains generated by creation of two active scattering centers away from the \(\alpha \nu = c_0, s_1\) Fermi points and \(\alpha \nu \neq c_0, s_1\) limiting bare-momentum values are of the general form,

\[
\begin{align*}
\kappa &= l[k_0 + c_1 q + c_1' q'] ; \\
\omega &= \omega_{\alpha \nu, \alpha' \nu'}(q, q') = [\omega_0 + c_1 \epsilon_{\alpha \nu}(q) + c_1' \epsilon_{\alpha' \nu'}(q')] , \quad \alpha \nu = c_0, s_1 , \quad \alpha' \nu' = c_0, s_1 ; \\
\kappa &= l[k_0 + c_1 q + k_{\omega \nu} - q'] ; \\
\omega &= \omega_{\alpha \nu, \alpha' \nu'}(q, q') = [\omega_0 + c_1 \epsilon_{\alpha \nu}(q) + \epsilon_{\nu'}(q')] , \quad \alpha \nu = c_0, s_1 , \quad \nu' > 0 ; \\
\kappa &= l[k_0 + c_1 q + q'] ; \\
\omega &= \omega_{\alpha \nu, \alpha' \nu'}(q, q') = [\omega_0 + c_1 \epsilon_{\alpha \nu}(q) + \epsilon_{\nu'}(q')] , \quad \alpha \nu = c_0, s_1 , \quad \nu' > 1 .
\end{align*}
\]

Here the index \(l = \pm 1\) is that of the corresponding spectral function \(B_{\alpha \nu}^l(k, \omega)\) of Eq. \((63)\), the momentum \(k_{\omega \nu}\), energy \(\omega_0\), and momentum \(k_0\) are given in Eqs. \((27)\), \((32)\), and \((34)\), respectively, and the constant \(c_1\) \([and c_1']\) is such that \(c_1 = +1\) \([and c_1 = -1]\) for creation of a \(\alpha \nu\) pseudofermion \([and \alpha \nu\) pseudofermion hole]. For simplicity, in Eq. \((65)\) we considered that the two created active scattering centers are both \(\alpha \nu = c_0, s_1\) pseudofermions and/or holes, a \(\alpha \nu = c_0, s_1\) pseudofermion or hole and a \(\nu \neq c_0, s_1\) pseudofermion, and a \(\alpha \nu = c_0, s_1\) pseudofermion or hole and a \(\nu \neq c_0, s_1\) pseudofermion. Generalization to other active-scattering-center choices is straightforward.

The summation over the excited energy eigenstates generated by the elementary processes (A) which contribute to the spectral weight at the \((k, \omega)\)-plane point defined by Eq. \((65)\) is performed in Appendix B. The momentum and energy of these states relative to the ground state is close to \(k\) and \(\omega\), respectively. These contributions occur through generation from these states, by the elementary processes (C), of new excited energy eigenstates whose momentum and energy relative to the initial ground state are precisely \(k\) and \(\omega\), respectively. Since the former excited energy eigenstates generated by the elementary processes (A) have very close momentum and energy, except for vanishing
contributions of order $1/N_{m}$ the corresponding value of the functional $\Omega$ is the same. For finite values of that functional it is found in Appendix B that the spectral function has following general form,

$$B_{N}^{l}(k, \omega) \approx \frac{1}{\pi C_{c} C_{s}} \left[ \int_{-1/\nu_{1}}^{1/\nu_{1}} dz F_{0}(z) \right] \frac{\Omega}{4 \pi \sqrt{\nu_{0} \nu_{1}}} \left( \frac{\zeta_{0}(q, q')}{\zeta_{0}(q, q') \sqrt{\nu_{0} \nu_{1}}} - \frac{\sqrt{\nu_{0} \nu_{1}}}{|v_{\alpha \nu}(q) - v_{\alpha' \nu'}(q')|} \right); \quad l = \pm 1. \quad (66)$$

While this expression and all other spectral-weight distribution expressions obtained in this paper correspond to densities $0 < n < 1$ and $0 < m < n$, taking the limit $m \to 0$ in such expressions leads to the correct $m = 0$ results. The expression (66) is not valid in the vicinity of isolated $(k, \omega)$-plane points whose weight is associated with point-subspaces and in the proximity of the branch lines studied below. In the vicinity of these points and lines one must consider the contributions from processes involving the creation of none and one active scattering center away from the $\alpha \nu = c_{0}, s_{1}$ Fermi points and $\alpha \nu \neq c_{0}, s_{1}$ limiting bare-momentum values, respectively. On the right-hand side of Eq. (66), $\zeta_{0} = \zeta_{0}(q, q')$ is the value of the general functional $\zeta_{0}$ specific to the two active-scattering-center excitation under consideration, $F_{0}(z)$ is the function (22), and the small but finite energy $\Omega$ corresponds to the range of the elementary processes (C). We recall that $\Omega$ is treated as a parameter whose value for fixed finite values of $n, m$, and $U/t$ is determined by imposing the exact sum rule (3) to the full spectral-function expression in the end of the calculations. $\Omega$ also appears in the spectral-function contributions from none and one active scattering center creation away from the $\alpha \nu = c_{0}, s_{1}$ Fermi points and $\alpha \nu \neq c_{0}, s_{1}$ limiting bare-momentum values, as confirmed below. The general spectral-function expression (66) is valid for finite values of the functional $\zeta_{0}$. For regions of the $(k, \omega)$-plane where the value of that functional vanishes the energy $\Omega$ is such that $\Omega \to 0$ and the spectral-function expression has a different form, as discussed below.

The $k$ and $\omega$ dependence of the weight-distribution expression (66) occurs mainly through the $q$ and $q'$ dependence of the value of the functional $\zeta_{0}(q, q')$ and through the absolute value of $v_{\alpha \nu}(q) - v_{\alpha' \nu'}(q')$. The created active-scattering-center bare momentum values $q$ and $q'$ are related to the weight-distribution value of $k$ and $\omega$ by Eq. (65). We emphasize that there are two types of $(k, \omega)$-plane points: (i) those associated with one pair of $(q, q')$ values and (ii) those corresponding to two different pairs of such bare-momentum values. In the case (ii) the spectral function at the $(k, \omega)$-plane point is given by the sum of two terms of the general form given in Eq. (66). The shape of the two-dimensional sub-domains of type (i) and (ii) depends on the specific form of the parametric equations (65).

The two-active-scattering-center $(k, \omega)$-plane two-dimensional domain is bounded by well defined lines. Also the two sub-domains (i) and (ii) are bounded by well defined lines. An important general property is that the part of the limiting line of the two-dimensional domain which is also a limiting line for the sub-domain (ii) [this excludes the limiting line between the sub-domains (i) and (ii)] is such that $v_{\alpha \nu}(q) = v_{\alpha' \nu'}(q')$. We call such a line border line. The shape $\omega = \omega_{BL}(k)$ of a border line is defined by the following parametric equations,

$$\omega_{BL}(k) = l[\omega_{0} + c_{1} \epsilon_{\alpha \nu}(q) + c'_{1} \epsilon_{\alpha' \nu'}(q')] \delta_{v_{\alpha \nu}(q), v_{\alpha' \nu'}(q')}; \quad k = lk_{\alpha \nu, \alpha' \nu'}(q, q') \delta_{v_{\alpha \nu}(q), v_{\alpha' \nu'}(q')} \quad (67)$$

According to expression (66), the weight distribution has singular behavior at that line. Note that such an expression defines the $k$ and $\omega$ dependence of the weight distribution for the regions just below ($l = +1$) or above ($l = -1$) the border line. We recall that the border line corresponds to a discontinuity in the values of the functional $\zeta_{0}$. Thus, the weight distribution has power-law behavior in the region just above ($l = +1$) or below ($l = -1$) the border line. The $k$ and $\omega$ dependence of the weight distribution in that region is not given by Eq. (66). The study of that dependence proceeds much as for the branch lines considered below and will be carried out for specific weight distributions elsewhere. The same applies to the region just above ($l = +1$) or below ($l = -1$) the remaining limiting lines of the two-dimensional two-active-scattering-center $(k, \omega)$-plane domain.

We emphasize that depending on the specific weight distribution, the dominant processes can correspond to different choices of active-scattering-center pairs. Thus, it can occur that the same $(k, \omega)$-plane region involves contributions from several expressions of general form given in Eq. (66). In that case the weight distribution at a $(k, \omega)$-plane point is given by the sum of several terms of the general form provided in that equation. Moreover, one must also add the spectral features generated by creation of none and one active scattering center away from the $\alpha \nu = c_{0}, s_{1}$ Fermi points and $\alpha \nu \neq c_{0}, s_{1}$ limiting bare-momentum values, which we derive below.
C. THE POINT-LIKE AND LINE-LIKE SPECTRAL-WEIGHT DISTRIBUTION FEATURES

The smallest reduced subspaces are point-subspaces whose deviation numbers and numbers are given in Eq. \ref{eq:69}. Analysis of the general spectral-function expression \ref{eq:11} for \( i = 0 \) reveals that the most divergent singular power-law spectral features appear in the vicinity of the \((k, \omega)\)-plane points, \((\kappa_0, \omega_0)\), corresponding to such subspaces. Such a weight results from processes that do not involve creation away from the \( \alpha \nu = c_0, s_1 \) Fermi points and \( \alpha \nu \neq c_0, s_1 \) limiting bare-momentum values of active scattering centers. For \((k, \omega)\)-plane regions in the vicinity of the corresponding \((\kappa_0, \omega_0)\) points the dominant contributions to the general expression \ref{eq:24} are generated by elementary processes \( (C) \) whose initial state corresponds to the point-subspace. Here \( \omega_0 \) and \( \kappa_0 \) are given in Eqs. \ref{eq:32} and \ref{eq:34}, respectively. Such contributions lead to \( B_N^i(k, \omega) \approx \Theta(\Omega - l[\omega - \omega_0]) \dot{B}^i,0(\omega - \omega_0, [\omega - \omega_0]/(k - \kappa_0)) \) and thus the weight distribution has the following power-law behavior,

\[
B_N^i(k, \omega) \approx \frac{F_0(1/v)}{4\pi \sqrt{v_0/v_1} C_0 C_s} \Theta \left( \Omega - l[\omega - \omega_0] \right) \Theta \left( l[\omega - \omega_0] \right)^{-2+\zeta_0}, \tag{68}
\]

where \( l = \pm 1 \) and the velocity \( v = [\omega - \omega_0]/[k - \kappa_0] \) is always such that \( |v| > v_{s1} \) and \( v \neq \pm v_{c0} \). Depending on the specific point-subspace and weight distribution, the domain, of \( z = 1/v \) values corresponding to the small finite spectral-weight \((k, \omega)\)-plane region where the expression \ref{eq:68} is valid in general bounded by two of the four values \(-1/v_{c0}, 1/v_{c0}, -1/v_{s1}, 1/v_{s1}\). This expression refers to small finite values of \([\omega - \omega_0]\), the functional \( \zeta_0 \) and the function \( F_0(z) \) are given in Eqs. \ref{eq:35} and \ref{eq:39}, respectively, and the value of the corresponding functionals \( 2\Delta^2_{\pm 1} \) whose general expression is given in Eq. \ref{eq:10} is that of the initial point-subspace. It is not valid for \( k \) and \( \omega \) values such that \( \approx l \nu \omega_0/(k - \kappa_0) + \omega_0 \), where \( \alpha \nu = c_0, s_1, l = \pm 1 \). The expression for such \( k \) and \( \omega \) values is given below.) When the exponent \( -2 + \zeta_0 \) is negative, the weight distribution \ref{eq:68} has a singularity at the \((\kappa_0, \omega_0)\) point. In general, the value of that exponent increases for increasing value of \([k_0] \). Thus, it is larger for the spectral function expression in the vicinity of the \((\kappa_0, \omega_0)\) points corresponding to the second point-subspaces than in the vicinity of the points associated with the first point-subspaces.

For the small \((k, \omega)\)-plane region in the vicinity of the point \((\kappa_0, \omega_0)\), where the expression \ref{eq:68} is valid, the contributions from functions of form \ref{eq:11} associated with J-CPHS subspaces other than the point-subspace are very small. Therefore, in that region the spectral function \ref{eq:11} has for \( i = 0 \) approximately the form given in Eq. \ref{eq:68}.

Let us now consider the contributions from processes involving the creation of a single active scattering center away from the \( \alpha \nu = c_0, s_1 \) Fermi points and \( \alpha \nu \neq c_0, s_1 \) limiting bare-momentum values. These contributions lead to the next more divergent singular features, which arise in the vicinity of branch lines. In this case the energy eigenstates which span a reduced J-CPHS subspace are associated with a \((k, \omega)\)-plane line that corresponds to a discontinuity in the values of the functional \ref{eq:68}. By changing the bare momentum \( q \) of the active \( \alpha \nu \) pseudofermion or pseudofermion hole scattering center, one generates a branch line in the \((k, \omega)\)-plane. Often one of the end points (or both end points) of such a line coincides with a \((\kappa_0, \omega_0)\) point associated with a point-subspace. For each CPHS ensemble subspace, at least one of the end points of the most divergent of such branch lines coincides with a first point-subspace point \((\kappa_0, \omega_0)\). The active \( \alpha \nu \) pseudofermion (and \( \alpha \nu \) pseudofermion hole) scattering center can belong to any branch (to the \( \alpha \nu = c_0, s_1 \) branches). A \( \alpha \nu = c_0, s_1 \) branch line is generated by creation of the \( \alpha \nu \) pseudofermion (or \( \alpha \nu \) pseudofermion hole) for all available bare-momentum values in the domain \( |q| \in [q_{\alpha \nu}^0, g_{\alpha \nu}^0] \) (or \( |q| \in [0, q_{\alpha \nu}^0] \)).

The parametric equations that define the \((k, \omega)\)-plane points belonging to a \( \alpha \nu \) pseudofermion (or \( \alpha \nu \) pseudofermion hole for the \( \alpha \nu = c_0, s_1 \) bands) branch line is of the general form,

\[
k = lk_{\alpha \nu}(q) = l[k_0 + c_1 q]; \quad \omega = \omega_{\alpha \nu}(q) = l[\omega_0 + c_1 \epsilon_{\alpha \nu}(q)], \quad \alpha \nu = c_0, s_1;
k = lk_{\alpha \nu}(q) = l[k_0 + k_{\alpha \nu} - q]; \quad \omega = \omega_{\alpha \nu}(q) = l[\omega_0 + \epsilon_{\alpha \nu}(q)], \quad \alpha \nu \neq c_0;nk = lk_{\alpha \nu}(q) = l[k_0 + q]; \quad \omega = \omega_{\alpha \nu}(q) = l[\omega_0 + \epsilon_{\alpha \nu}(q)], \quad \alpha \nu \neq s_1. \tag{69}
\]

Here the index \( l = \pm 1 \) is that of the corresponding spectral function \( B_N^i(k, \omega) \) of Eq. \ref{eq:11}, the momentum \( k_{\alpha \nu} \), energy \( \omega_0 \), and momentum \( k_0 \) are given in Eqs. \ref{eq:24}, \ref{eq:32}, and \ref{eq:34}, respectively, and the constant \( c_1 \) is such that \( c_1 = +1 \) (and \( c_1 = -1 \) for creation of a \( \alpha \nu \) pseudofermion (and a \( \alpha \nu \) pseudofermion hole).

Let us consider a \((k, \omega)\)-plane point located just above \( (l = +1) \) or below \( (l = -1) \) the branch line whose momentum \( k \) obeys the relation given in Eq. \ref{eq:29} and the energy \( \omega \) is such that \( l(\omega - \omega_{\alpha \nu}(q)) \) is small and positive. The weight-distribution expression at that point is controlled by the elementary processes \( (C) \), which generate from the initial excited energy eigenstates corresponding to the line a set of tower states whose momentum and energy relative to the ground state are precisely \( k \) and \( \omega \). By performing the summations of the general expression \ref{eq:14} over the initial excited energy eigenstates generated from the ground state by the elementary processes \( (A) \) which correspond to line points in the vicinity of such a point, in Appendix B we derive the following spectral-weight distribution expression,
\[ B'_N(k, \omega) \approx \Theta \left( \Omega - l[|\omega - l\omega_{\alpha\nu}(q)|] \right) \Theta \left( l[|\omega - l\omega_{\alpha\nu}(q)|] \right) C_{\alpha\nu}(q) \left( \frac{l[|\omega - l\omega_{\alpha\nu}(q)|]}{4\pi \sqrt{v_{\alpha\nu} v_{\alpha\nu}}} \right)^{-1 + \zeta_0(q)}; \quad \zeta_0(q) > 0, \quad (70) \]

where

\[ C_{\alpha\nu}(q) = \frac{\text{sgn}(q) c_{\alpha\nu}}{2\pi C_c C_s} \left\{ \Theta \left( v_{\alpha\nu} |1 - l[|\omega - l\omega_{\alpha\nu}(q)|] - |\nu_{\alpha\nu}(q)|\right) \right\} \int_{-\frac{v_{\alpha\nu}(q) c_{\alpha\nu}}{v_{\alpha\nu}}}^{\frac{v_{\alpha\nu}(q) c_{\alpha\nu}}{v_{\alpha\nu}}} \Theta \left( \frac{1}{|\nu_{\alpha\nu}(q)|} \left| 1 - l[|\omega - l\omega_{\alpha\nu}(q)|] - |\nu_{\alpha\nu}(q)| \right| \right) dz \]

\[ + \Theta \left( \nu_{\alpha\nu}(q) - v_{\alpha\nu} |1 - l[|\omega - l\omega_{\alpha\nu}(q)|] - |\nu_{\alpha\nu}(q)|\right) \int_{-\frac{v_{\alpha\nu}(q) c_{\alpha\nu}}{v_{\alpha\nu}}}^{\frac{v_{\alpha\nu}(q) c_{\alpha\nu}}{v_{\alpha\nu}}} \Theta \left( \frac{1}{|\nu_{\alpha\nu}(q)|} \left| 1 - l[|\omega - l\omega_{\alpha\nu}(q)|] - |\nu_{\alpha\nu}(q)| \right| \right) dz \left\{ \frac{1}{1 - z v_{\alpha\nu}(q)} \zeta_0(q) \right\}; \quad (71) \]

This expression does not apply to \( k \) and \( \omega \) values such that \( \omega \approx l v_{\alpha\nu}(k - l k_0) + l \omega_0 \), where \( \alpha\nu = c_0, s_1 \) and \( l = \pm 1 \). The functional \( \zeta_0 = \zeta_0(q) \) appearing in expression (71) is given in Eqs. (59) and \( F_0(z) \) is the function (62). Furthermore, \( c_{\alpha\nu} = 1 \) for \( \alpha\nu = c_0 \), \( s_\nu \), \( c_{1\nu} = -1 \) for \( \alpha\nu = c_0 \neq c_0 \).

The unity added to the exponent \(-2 + \zeta_0 \) of the convolution function (61) and spectral function expression (65) to reach the branch-line exponent \(-1 + \zeta_0(q) \) of expression (70), stems from the summations performed in Appendix B over the suitable initial excited energy eigenstates generated by the elementary processes (A). For a singular \( \alpha\nu \) branch line the corresponding power-law exponent \(-1 + \zeta_0(q) \) is negative. Since when \(-1 + \zeta_0(q) < 0 \) the weight distribution shows singularities at the branch line, we expect that in this case spectral peaks will be observed in experiments. This is confirmed for one-electron removal in Ref. (18) for both the TTF and TCNQ spectral dispersions of the quasi-1D organic compound TTF-TCNQ, following the preliminary results of Refs. (2, 16) for the TCNQ stacks of molecules. When the region above \((l = +1)\) or below \((l = -1)\) the branch line is not contained in a \((k, \omega)\)-plane domain associated with a two-active-scattering-center reduced J-CPHS subspace, the expression (70) describes the spectral-function behavior when \(-1 + \zeta_0(q) < 0 \), whereas \( 0 < -1 + \zeta_0(q) < 1 \) is a sign of near absence of spectral weight. In contrast, if that region is contained in such a domain, the expression (70) describes a spectral-function weight edge when \( 0 < -1 + \zeta_0(q) < 1 \). Finally, \(-1 + \zeta_0(q) > 1 \) is always a sign of near absence of spectral weight in the vicinity of the line.

For momentum values \( k \) and energies \( \omega \) such that \( \omega \approx l v_{\alpha\nu}(k - l k_0) + l \omega_0 \), where \( \alpha\nu = c_0, s_1 \) and \( l = \pm 1 \), the spectral function corresponds to the vicinity of a \( \alpha\nu = c_0 \), \( s_1 \) branch line end point. However, in this case the expression of the spectral function in the vicinity of the branch-line is not that of Eq. (70). This results from a resonance effect: the branch line group velocity \( v_{\alpha\nu}(q) \) equals the velocity \( l v_{\alpha\nu} \) associated with the \( \alpha\nu, l \) pseudofermion particle-hole excitation sub-branch generated by the elementary processes (C). Moreover, the \( \alpha\nu = c_0 \), \( s_1 \) branch line is in this limiting situation generated by these elementary processes. Indeed, one can consider that in a first step the branch-line \( \alpha\nu = c_0 \), \( s_1 \) pseudofermion or hole is created at the \( q = l q_{F\alpha\nu}^0 \) Fermi point by the elementary processes (B). This first step corresponds to a given reduced J-CPHS subspace. One then accesses the corresponding J-CPHS subspace by moving the \( \alpha\nu = c_0 \), \( s_1 \) pseudofermion or hole along its energy and momentum dispersion. This second small-momentum and low-energy process does not involve creation of any quantum object and thus corresponds to an elementary process (C). When moving the \( \alpha\nu = c_0 \), \( s_1 \) pseudofermion or hole in the vicinity of the \( q = l q_{F\alpha\nu}^0 \) Fermi point, its energy dispersion is linear in the momentum. It follows that for these values of \( k \) and \( \omega \) the elementary processes (C) replace the elementary processes (A) in what the generation of the \( \alpha\nu \) branch line is concerned. While the elementary \( \alpha\nu = c_0 \), \( s_1 \) pseudofermion particle-hole processes (C) generate the branch line, the elementary \( \bar{\alpha}\bar{\nu} \) pseudofermion particle-hole processes (C) play the usual role of these processes in the general convolution function (45). (As in Sec. IV, here we have used the index \( \bar{\alpha}\bar{\nu} \) such that \( \bar{\alpha} = s_1 \) and \( \bar{\nu} = c_0 \).) It follows that for values of \( k \) and \( \omega \) obeying the relation \( \omega \approx l v_{\alpha\nu}(k - l k_0) + l \omega_0 \), where \( l = \pm 1 \), the convolution function associated with the \( \alpha\nu = c_0 \), \( s_1 \) spectral function is given by,

\[ B^l_{\alpha\nu, l}(\Delta \omega, v) \approx \frac{l}{2\pi} \int_0^{\Delta \omega} d\omega' \int_{\omega'/v_{\alpha\nu} - \Delta q_{line}}^{\omega'/v_{\alpha\nu}} dk' B^l_{0, Q_{\alpha\nu}}(\Delta \omega/v' - k', \Delta \omega - \omega') B^l_{0, Q_{\alpha\nu}}(k', \omega') \; ; \quad l = \pm 1, \quad (72) \]

where \( \Delta q_{line} = l[2\pi/L] x \) such that \( 0 < x < 1 \) is the effective momentum width of the branch line. Here we consider the more general case of the functional \( 2\Delta_{\alpha\nu}^0 \) having values in the domain \( 0 < 2\Delta_{\alpha\nu}^0 < 1 \) and define \( x \) as \( x = [1/N_{\alpha}]^{1/2\Delta_{\alpha\nu}^0 - 1} \). The introduction of the branch-line momentum effective width \( \Delta q_{line} \) provides a good approximation for the value for the power-law multiplicative constant given below in Eq. (72). In turn, the power-law \( k \) and \( \omega \) dependence obtained below is exact. The function (72) differs from the general convolution function (45).
in the momentum integration domain. Indeed, now the role of one of the elementary \(\alpha\nu = c_0, s_1\) pseudofermion particle-hole processes (C) is to generate the \(\alpha\nu\) branch line.

By use of the same methods as for the general spectral function expression (70) in the vicinity of a branch line, in Appendix B it is found that for \(k\) and \(\omega\) values such that \(\omega \approx \nu v_{\alpha\nu}(k - l k_0) + l \omega_0\) the spectral-weight distribution expression is given by,

\[
B_{\alpha\nu}^t(k, \omega) \approx \Theta\left(\Omega - l(\omega - l[\omega_0 + \nu v_{\alpha\nu}(k - l k_0)])\right) \Theta\left(l(\omega - l[\omega_0 + \nu v_{\alpha\nu}(k - l k_0)])\right) \\
\times C_{\alpha\nu, t}\left(\frac{l(\omega - l[\omega_0 + \nu v_{\alpha\nu}(k - l k_0)])}{4\pi \sqrt{c_0 s_0}}\right)^{-1 + \zeta_{\alpha\nu, t}}; \quad \alpha\nu = c_0, s_1; \quad \nu = \pm 1; \quad l = \pm 1,
\]

(73)

for \(\zeta_{\alpha\nu, t}^0 > 0\), where the exponent \(\zeta_{\alpha\nu, t}^0\) is given in Eq. (B20) of Appendix B,

\[
C_{\alpha\nu, t} = \frac{\mu_1}{2\pi c_c c_s} \left\{ \theta\left(v_{\alpha\nu} - \bar{v}_{\alpha\nu}[1 - \frac{l(\omega - l[\omega_0 + \nu v_{\alpha\nu}(k - l k_0)])}{\Omega}]\right) \int \frac{dz}{v_{\alpha\nu}} \frac{F_{\alpha\nu, t}(z)}{(1 - \nu z v_{\alpha\nu})^{\zeta_{\alpha\nu, t}}} \right\},
\]

(74)

and the function \(F_{\alpha\nu, t}(z)\) is provided in Eq. (B21) of that Appendix. This weight-distribution expression refers to the proximity of a \(\alpha\nu = c_0, s_1\) pseudofermion or hole branch-line end point and differs from the corresponding general spectral function expression given in Eq. (40). The latter expression applies in the vicinity of the \(\alpha\nu = c_0, s_1\) branch lines when the branch-line group velocity \(v_{\alpha\nu}(q)\) is such that \(v_{\alpha\nu}(q) \neq \nu v_{\alpha\nu}\) for \(\nu = \pm 1\). As \(v_{\alpha\nu}(q) \to \nu v_{\alpha\nu}\), the spectral function \(k\) and \(\omega\) values correspond to the vicinity of a branch-line end point and thus it is given by expression (73) instead of (70). Moreover, in the proximity of the \((l k_0, l \omega_0)\) point but for values of \(k\) and \(\omega\) corresponding to the region where the expression (76) is valid, the spectral function is given by that expression. There are very small cross-over regions between the \((k, \omega)\)-plane regions where these different expressions apply.

D. THE \(\delta\)-FUNCTION SINGULARITIES

For some spectral-weight distributions there are isolated \((k, \omega)\)-plane points, \((l k_0, l \omega_0)\), associated with point-subspaces such that \(\Delta N_{\alpha\nu} = 0\) for \(\alpha\nu = c_0, s_1\), \(N_{\alpha\nu} = 0\) for \(\alpha\nu \neq c_0, s_1\), and \(L_{\nu, -1/2} > 0\) and/or \(L_{\nu, -1/2} > 0\). These point-subspaces correspond to \((l k_0, l \omega_0)\) points such that,

\[
k_0 = \pi L_{\nu, -1/2}; \quad \omega_0 = \left[2\mu L_{\nu, -1/2} + 2\mu_0 H L_{\nu, -1/2}\right], \quad L_{\nu, -1/2} \quad \text{and/or} \quad L_{\nu, -1/2} > 0,
\]

(75)

where the energies \(2\mu\) and \(2\mu_0 H\) are given in Eq. (21). Interestingly, the four functionals \(2\Delta_{\alpha\nu}^t\) of Eq. (40) and thus the value of the functional (63) vanish at these points for all values of \(U/t\). In this case the expression of the general operator \(\hat{\Theta}_{\nu, k}^t\) defined in Eq. (27) of Ref. 10 does not include any pseudofermion operators, since no pseudofermions are created nor annihilated under the ground-state - excited-energy transition. Moreover, the elementary processes\(C\) give no contribution, since when the four functionals \(2\Delta_{\alpha\nu}^t\) of Eq. (40) vanish the energy \(\Omega\) which defines the energy range of these processes also vanishes. In this case the operator \(\hat{\Theta}_{\nu, k}^t\) is such that in the matrix element \(\langle f; CPHS.L|\hat{\Theta}_{\nu, k}^t|GS\rangle\) of Eq. (26) of Ref. 10 the energy eigenstate \(|f; CPHS.L\rangle\) is the ground state itself. Thus, for such point subspaces the matrix elements derived in Sec. V of Ref. 10 must be replaced by \(\langle GS|\hat{\Theta}_{\nu, k}^t|GS\rangle\) and the spectral function has at the corresponding \((k, \omega)\)-plane points the following \(\delta\)-function behavior,

\[
B_{\alpha\nu}^t(k, \omega) = \frac{\langle GS|\hat{\Theta}_{\nu, k}^t|GS\rangle^2}{C_c C_s} \delta(\omega - \omega_0) \delta_{k, k_0}; \quad l = \pm 1.
\]

(76)

The \(\delta\)-peak singularities of Eq. (76) correspond to isolated \((l k_0, l \omega_0)\) points and occur for all finite values of \(U/t\). Another example of \(\delta\)-peak singularities occurs for the one-electron spectral-weight distributions. Such a behavior occurs for some \(c_0\) or \(s_1\) branch lines (63), in whose vicinity the spectral function has the form (70) for \(U/t > 0\), and
for \( U/t \to 0 \) the value of the functional \( \Xi_0(k, \omega) = \Xi_0(k, \omega_{av}(q)) \to 0 \). Also the overall phase shifts \( Q_{av}(i\eta_{Fav})/2 \) behave as \( Q_{av}(i\eta_{Fav})/2 \to 0 \) for \( \alpha \nu = c0 \) or \( \alpha \nu = s1 \) and \( \iota = \pm 1 \). In this case the spectral-function expressions derived in this paper remain valid, provided that one considers in these expressions the limit \( 2\Delta_{sv} \to 0 \) for the four functionals given in Eq. (1). The energy \( \Omega \) which defines the energy range of the elementary processes (C) behaves in this case as \( \Omega \to 0 \) for \( \Xi_0(k, \omega) \to 0 \) and, therefore, the weight distribution becomes \( \delta \)-function like. The function \( \Xi_0 \) is such that \( f_{av} = \sqrt{f(-\pi)} \to 1 \) as \( Q_{av}(i\eta_{Fav}) \to 0 \). Indeed, \( f(Q) = f(-Q) \) is the function on the right-hand side of Eq. (24) of Ref. [14] such that \( f(\pi) = f(-\pi) = 1 \). Moreover, \( S_{\alpha 0} S_{0 \nu} \to 1 \) as \( U/t \to 0 \) and thus the \( \delta \)-peak weight \( 2\pi(1/N_{\alpha})\delta D_{0} \) of Eq. (59) is such that \( (1/N_{\alpha})\delta D_{0} \to 2\pi \) as \( U/t \to 0 \), once for the above branch lines also \( \zeta_0 \to 0 \). It follows that in this case the convolution function \( \Xi_0 \) has not the form given in Eq. (61). Instead, as \( U/t \to 0 \) it is given by the first expression of Eq. (65) and the asymptotic expression (70) is not valid for \( U/t \to 0 \). By performing calculations similar to those of Appendix B but with the convolution function given by the first expression of Eq. (61), instead of its asymptotic expression (61), one arrives to the following expression for the spectral-weight distribution at the branch line,

\[
B_{SC}(k, \omega) = \frac{D_0}{C_c C_s} \int dq' \left\{ \frac{1}{N_\alpha} \right\} \delta(\omega - l\omega_{av}(q')) \delta\left(k - lk_{av}(q')\right)
\]

where the functions \( k_{av}(q) \) and \( \omega_{av}(q) \) are defined in Eq. (63) and the \( q' \) integration is over the branch line points such that \( \zeta_0(q') \to 0 \).

For two-electron spectral-weight distributions a similar behavior occurs for the weight generated by two active-scattering-center creation away from the \( \alpha \nu = c0 \), s1 Fermi points and \( \alpha \nu \neq c0 \), s1 limiting-momentum values in two-dimensional regions of the \((k, \omega)\)-plane where \( \Xi_0(k, \omega) \to 0 \). Again, for such regions the convolution function \( \Xi_0 \) has not the form given in Eq. (61) but that corresponding to the first expression of Eq. (65). Thus, as \( \Xi_0(k, \omega) \to 0 \) the expression (66) is not valid and instead the spectral-weight distribution has the following behavior,

\[
B_{SC}(k, \omega) = \frac{D_0}{2\pi C_c C_s} \int dq \int dq' \left\{ \frac{1}{N_{\alpha}} \right\} \delta(\omega - l\omega_{av, \alpha'\nu'}(q, q')) \delta\left(k - lk_{av, \alpha'\nu'}(q, q')\right)
\]

Here the functions \( k_{av, \alpha'\nu'}(q, q') \) and \( \omega_{av, \alpha'\nu'}(q, q') \) are defined in Eq. (65), the bare-momentum integrations correspond to the two-dimensional regions of the \((k, \omega)\)-plane where \( \zeta_0(q, q') \to 0 \), and the function \( q = q(k, q') \) is uniquely defined by the parametric equation \( k = lk_{av, \alpha'\nu'}(q, q') \).

An example of the type of singularity (76) is the \( \delta \)-peak which appears in the spin-singlet Cooper pair \( l = 1 \) addition spectral function at the point \((k_0, \omega_0) = (\pi, 2\mu)\), as a result of the \( \eta \)-pairing mechanism [21]. One-electron spectral-function singularities of the type (77) are studied in Ref. [16]. It is found in that reference that some of the \( c0 \) and s1 pseudofermion branch lines behave as in Eq. (77) for \( U/t \to 0 \) with \( C_c = C_s = 1 \), where the correct \( U/t \to 0 \) non-interacting one-electron spectrum is recovered. The two-electron weight distribution described by expression (76) is valid when \( U/t \to 0 \) with \( C_c = C_s = 1 \). Interestingly, for the dynamical structure factor it is also valid when \( U/t \to \infty \), whereas such a spectral function is that of the spin-less fermions which describe the charge degrees of freedom in that limit. There is a qualitative difference between the spectral-function behavior described by Eq. (76) and that described by Eqs. (77) and (78): the former behavior is valid at isolated points for finite value of \( U/t \), whereas the latter behavior is valid for \( U/t \to 0 \) or \( U/t \to \infty \) only. Moreover, in the case of expressions (77) and (78), as \( \zeta_0 \to 0 \) all the spectral weight becomes located onto the \((k, \omega)\)-plane regions where \( \Xi_0(k, \omega) \to 0 \).

Finally, for the \( n = 1 \) Mott-Hubbard insulator phase there are other approaches to the study of finite-energy spectral-weight distributions for finite values of \( U/t \). A method for the description of the weight distribution in the vicinity of specific spectral features was proposed in Ref. [21]. Also for \( n = 1 \) but for very small values of \( U/t \) the 1D Hubbard model can be mapped onto the sine-Gordon model. In this limit the form factor approach was used in the study of the zero-momentum two-electron charge spectral-weight distribution associated with the frequency-dependent optical conductivity [22]. In general, the form factor approach does not apply to the 1D Hubbard model.
VI. DISCUSSION AND CONCLUDING REMARKS

In this paper we derived general closed-form analytical expressions for the finite-energy one- and two-electron spectral-weight distributions of a 1D correlated metal with on-site electronic repulsion. Our results also provide general expressions for the one- and two-atom spectral functions of a correlated quantum system of cold fermionic atoms in a 1D optical lattice with on-site atomic repulsion. In order to solve that problem, we addressed the issue of the dominant spectral-weight processes. That allowed the derivation of the basic weight-distribution pieces corresponding to surface-like, line-like, and point-like spectral features, which involved the introduction of the concepts of a border line and a singular and edge branch line. Our study reveals that the functional \( \mathcal{F}_\alpha \) plays a key role in the control of the spectral-weight distribution. Indeed, most singularities are of power-law type and correspond to discontinuities of the value such a functional in the \((k, \omega)\)-plane. Moreover, that functional also controls the exceptional occurrence of \( \delta \)-function singularities, which correspond to the \((k, \omega)\)-plane points where its value vanishes.

For the one- and two-electron problems considered in this paper, the spectral-function expressions generated by processes associated with excited J-CPHS subspaces with none, one, and two active scattering centers away from the \( \alpha \nu = c \alpha, s \) Fermi points and \( \alpha \nu \neq c \alpha, s \) limiting bare-momentum values describe nearly the whole \((k, \omega)\)-plane weight distribution. In addition to other spectral features, here we presented a detailed derivation of the spectral-weight distribution expressions used in the studies of Refs. \cite{11, 23}. These expressions are used in Ref. \cite{27} in the study of both the spectral-weight distributions observed for the TTF and TCNQ stacks of molecules in the quasi-1D organic compound TTF-TCNQ, following the preliminary predictions of Refs. \cite{1, 2, 16} for the TCNQ spectral dispersions. Interestingly, the studies of Ref. \cite{27} find quantitative agreement with the observed spectral features for the whole experimental energy band width. Recently, the dynamical density matrix renormalization group method was used in the study of the one-electron spectral function of the 1D Hubbard model \cite{23} for electronic densities \( n < 1 \). These numerical studies found spectral features consistent with those of our spectral-weight distribution expressions. Moreover, the results of Ref. \cite{27} are consistent with the phase diagram observed in the (TMTTF)\(_2\)X and (TMTSF)\(_2\)X series of organic compounds and explain the absence of superconductive phases in TTF-TCNQ. Other applications of our finite-energy spectral-weight-distribution expressions to several materials, correlated quantum systems, and spectral functions are in progress. This includes the use of our theoretical results in the two-atom spectral-weight distributions measured in 1D optical lattices \cite{27}. Finally, it is shown elsewhere \cite{24} that the general spectral-weight distribution expressions introduced in this paper reproduce the low-energy behavior obtained by conformal-field theory \cite{25}.

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APPENDIX A: GROUND-STATE RAPIDITY FUNCTIONS

In this Appendix we introduce simple expressions for the inverse of the ground-state rapidity functions \( k_0(q) \), \( \Lambda_0(c \alpha, q) \), and \( \Lambda_0(s \nu, q) \) that appear in the pseudofermion energy band expressions \cite{15, 16}. These expressions are derived by solution of the integral equations obtained by introducing in Eqs. (13)-(16) of Ref. \cite{8} the ground-state distribution functions given in Eqs. (C1)-(C3) of the same reference. The solution of these equations can be written in terms of the inverse functions of \( k_0(q) \) and \( \Lambda_0(\nu, q) \) for \( \nu > 0 \), which we call \( q_0(k) \) and \( q_{0 \nu}(\Lambda) \), respectively. Here \( k \) and \( \Lambda \) are the rapidity-momentum coordinate and rapidity coordinate, respectively. These are odd functions such that \( q_0(k) = -q_0(-k) \) and \( q_{0 \nu}(\Lambda) = -q_{0 \nu}(-\Lambda) \). The domains of the rapidity-momentum coordinate \( k \) and rapidity coordinate \( \Lambda \) are such that \( -\pi \leq k \leq +\pi \) and \( -\infty \leq \Lambda \leq \infty \), respectively. It follows that \( k_0(\pm \pi) = \pm \pi, \quad \Lambda_0(\pm \pi) = 0, \quad \text{and} \quad \Lambda_0(\pm k_{F1}, \nu) = \pm \infty \). Such relations are equivalent to \( q_0^0(\pm \pi) = \pm \pi, \quad q_{0 \nu}^0(\pm \infty) = \pm k_{F1}, \quad q_{0 \nu}^0(\pm 0) = \pm [\nu - 2 k_{F1}] \) for \( \nu \neq 0 \), and \( q_{0 \nu}^0(\pm \infty) = \pm [k_{F1} - k_{F1}] \) for \( s \nu \neq s 1 \). By suitable manipulation of Eqs.
where the two-pseudofermion phase shifts $\tilde{\Phi}$ are such that,

$$q_{c}^{0}(k) = k + \int_{-Q}^{+Q} dk' \tilde{\Phi}_{c0,c0}(k', k); \quad q_{c\nu}^{0}(\Lambda) = \int_{-Q}^{+Q} dk' \tilde{\Phi}_{c0,\nu}(k', \Lambda); \quad \nu = 1, 2, \ldots, \quad (A1)$$

$$q_{c\nu}^{0}(\Lambda) = 2 \text{Re} \{ \arcsin \left( \Lambda + i \nu U / 4t \right) \} - \int_{-Q}^{+Q} dk' \tilde{\Phi}_{c0,\nu}(k', \Lambda); \quad \nu = 1, 2, \ldots, \quad (A2)$$

where the two-pseudofermion phase shifts $\tilde{\Phi}$ are such that,

$$\tilde{\Phi}_{c0,c0}(k, k') = \tilde{\Phi}_{c0,c0} \left( \frac{4t \sin k}{U}, \frac{4t \sin k'}{U} \right); \quad \tilde{\Phi}_{c0,\nu}(k, \Lambda') = \tilde{\Phi}_{c0,\nu} \left( \frac{4t \sin k}{U}, \frac{4t \Lambda'}{U} \right), \quad (A3)$$

$$\tilde{\Phi}_{\nu\nu, c0}(\Lambda, k') = \tilde{\Phi}_{\nu\nu, c0} \left( \frac{4t \Lambda}{U}, \frac{4t \sin k'}{U} \right); \quad \tilde{\Phi}_{\nu\nu, \nu}(\Lambda, \Lambda') = \tilde{\Phi}_{\nu\nu, \nu} \left( \frac{4t \Lambda}{U}, \frac{4t \Lambda'}{U} \right), \quad (A4)$$

and the phase shifts $\tilde{\Phi}_{\nu\nu', \nu'}(r, r')$ are defined by the integral equations (A1)-(A13) of Ref. [11]. Moreover, the parameters $Q \equiv k_{F}^{0}(2k_{F})$ and $B \equiv A_{N}^{\nu}(k_{F})$ are such that $q_{c}^{0}(\pm Q) = \pm 2k_{F}$ and $q_{s1}(\pm B) = \pm k_{F}$ and are self-consistently defined by the solution of the following equations,

$$2k_{F} = Q + \int_{-Q}^{+Q} dk \tilde{\Phi}_{c0,c0}(k, Q); \quad k_{F} = \int_{-Q}^{+Q} dk \tilde{\Phi}_{c0,s1}(k, B). \quad (A5)$$

**APPENDIX B: SPECTRAL-WEIGHT DISTRIBUTIONS AND RELATED QUANTITIES**

In this Appendix we provide complementary information about the evaluation of the finite-energy spectral-weight distribution expressions studied in this paper. We start by expressing the generators appearing in the matrix elements given in Eqs. (59) and (60) of Ref. [10] in terms of the sets of canonical-momentum and bare-momentum values of the corresponding distribution function deviations provided in Eqs. [15]–[17]. For the generators associated with $\alpha \nu = c0$, $s1$ pseudofermion occupancy configurations this leads to,

$$F_{J-GS, \alpha \nu}^{\dagger} = \prod_{j=1}^{N_{p1}^{NF}} f_{\tilde{q}_{j}, \alpha \nu}^{\dagger} \prod_{j=1}^{\Delta N_{p1}^{NF} \nu} \left[ \Theta \left( \text{sgn}(\Delta N_{p1}^{NF}) \right) f_{\tilde{q}_{j}, \alpha \nu}^{\dagger} \right], \quad (B1)$$

$$F_{J-GS, \alpha \nu}^{\dagger} = \prod_{j=1}^{\Delta N_{p1}^{NF} \nu} \left[ f_{\tilde{q}_{j}, \alpha \nu}^{\dagger} + \Theta \left( \text{sgn}(\Delta N_{p1}^{NF}) \right) f_{\tilde{q}_{j}, \alpha \nu}^{\dagger} \right], \quad (B2)$$

$$F_{J-GS, \alpha \nu}^{\dagger} = \prod_{q_{j}=0}^{\Delta q_{F\nu \alpha}^{\nu} + \Delta q_{F\alpha \nu}^{\nu}} f_{\tilde{q}_{j}, \alpha \nu}^{\dagger}, \quad (B3)$$

$$F_{J-GS, \alpha \nu}^{\dagger} = \prod_{q_{j}=0}^{\Delta q_{F\nu \alpha}^{\nu} + \Delta q_{F\alpha \nu}^{\nu}} f_{\tilde{q}_{j}, \alpha \nu}^{\dagger}, \quad (B4)$$

For the generator corresponding to $\alpha \nu \neq c0$, $s1$ pseudofermion occupancy configurations we find,

$$F_{NF, \alpha \nu}^{\dagger} = \prod_{i=1}^{N_{\alpha \nu}} f_{\tilde{q}_{i}, \alpha \nu}^{\dagger}; \quad \alpha \nu \neq c0, s1. \quad (B5)$$
The operators \(A_{\nu}^{(0,0)}\) and \(B_{\nu}^{(0,0)}\) generate the elementary processes (A), whereas the operator \(C_{\nu}^{(0,0)}\) generates the elementary processes (C). The operators \(F_{\nu,1}^{(1)}\) and \(F_{\nu,1}^{(1)}\) create the whole \(\nu\) pseudofermion Fermi sea, including the Fermi point shifts generated by the elementary processes (B), for the discrete canonical-momentum values of the ground state and reduced-subspace excited state, respectively. In the second expression of both Eqs. \(B_{\nu}^{(0,0)}\) and \(C_{\nu}^{(0,0)}\) the products refer to the \(\alpha \nu = 0, 1\) pseudofermions belonging to the \(\iota = \text{sgn}(q) 1\) sub-branch and \(\Delta q_{\alpha \nu, 1}\) and \(\Delta q_{\alpha \nu, 1}\) are the bare-momentum and canonical-momentum shifts provided in Eqs. \(10\) and \(11\), respectively.

The weight of the peak corresponding to the \(N_{\nu,0}^{ph} \alpha \nu = 0, 1\) pseudofermion particle-hole excitation mentioned in Sec. IV reads,

\[
A_{\nu}^{N_{\nu,0}^{ph}} = A_{\nu}^{(0,0) N_{\nu,0}^{ph}} ; \quad A_{\nu}^{N_{\nu,1}^{ph}} = A_{\nu}^{(0,1) N_{\nu,1}^{ph}}(q_1, \ldots, q_{N_{\nu,1}^{ph}}, q_1', \ldots, q_{N_{\nu,1}^{ph}}'); \quad \alpha \nu = 0, 1 . \tag{B6}
\]

Here \(A_{\nu}^{(0,0)}\) is the lowest-peak weight given in Eq. \(B3\), \(a_{\nu}^{N_{\nu,0}^{ph}}\) is the relative weight, and \(q_1, \ldots, q_{N_{\nu,0}^{ph}}^\prime\) and \(q_1', \ldots, q_{N_{\nu,0}^{ph}}'^\prime\) are the corresponding set of \(\alpha \nu = 0, 1\) pseudofermion "particle" and "hole" bare-momentum values, respectively. By use of the method of Ref. \(14\), we find that for \(N_{\nu,0}^{ph} = 1\) the relative weight \(B6\) is given by,

\[
a_{\nu}^{[1]}(q_1, q_1') = \frac{g(q_1)}{g(q_1') \sin^2(\frac{q_1 - q_1'}{2})} ; \quad \alpha \nu = 0, 1 . \tag{B7}
\]

The function \(g(q)\) appearing on the right-hand side of Eq. \(B7\) reads,

\[
g(q) = \prod_{q_j = q_{\nu,0,1}} \prod_{q_j' = q_{\nu,0,1}} \sin^2\left(\frac{|q_j - q_j'|}{2}\right) ; \quad \alpha \nu = 0, 1 , \tag{B8}
\]

where the correspondence between \(q_j\) and \(q_j'\) is defined by the equation \(q_j = q_1 + Q_{\nu}^{\Phi}(q_j)/L\) with \(Q_{\nu}^{\Phi}(q_j)/2\) given in Eq. \(4\), \(q_{\nu,0,1}^\pm 1\) and \(q_{\nu,0,1}^\pm 1\) are the Fermi points provided in Eqs. \(10\) and \(11\), respectively, and the corresponding products run over the discrete ground-state bare-momentum and excited-state canonical-momentum values, except for \(q_j' = q_j\) and \(q_j'^\prime = q_j\). Similarly, for \(N_{\nu,1}^{ph} = 2\) we find,

\[
a_{\nu}^{[2]}(q_1, q_2, q_1', q_2') = \frac{g(q_1)g(q_2)}{g(q_1')g(q_2')} \prod_{q_j = q_{\nu,0,1}} \prod_{q_j' = q_{\nu,0,1}} \sin^2\left(\frac{|q_j - q_j'|}{2}\right) \sin^2\left(\frac{|q_j - q_j'|}{2}\right) \sin^2\left(\frac{|q_j - q_j'|}{2}\right) \sin^2\left(\frac{|q_j - q_j'|}{2}\right) ; \tag{B9}
\]

where \(\alpha \nu = 0, 1\). The expression of the function \(B9\) is similar for increasing number of \(\alpha \nu = 0, 1\) pseudofermion particle-hole processes, but since it is longer, for simplicity we do not give it here.

We proceed by deriving the spectral-function expression \(60\) associated with creation of two active scattering centers away from the \(\alpha \nu = 0, 1\) Fermi points and \(\alpha \nu \neq 0, 1\) limiting bare-momentum values by the elementary processes (A). For simplicity, we consider that the two created active pseudofermion and/or hole scattering centers belong to the \(\alpha \nu = 0, 1\) branches. Generalization to active scattering centers associated with \(\alpha \nu \neq 0, 1\) branches is straightforward. In order to arrive to the specific form of the general \(i = 0\) spectral function expression \(14\) for the excitations associated with the latter processes, we start by considering excited energy eigenstates generated by the elementary processes (A) whose energy \(\Delta E\) and momentum \(\Delta P\) relative to the initial ground state have values such that \([w - l\Delta E] \land [k - l\Delta P]\) are small and \(0 < [w - l\Delta E] < \Omega\) and \(-1/v_{s1} < 1/v < +1/v_{s1}\). Here \(k\) and \(\omega\) are the momentum and energy values provided in Eq. \(28\), \(v\) is the velocity defined in Eq. \(26\), and the corresponding energy \(\Delta E = \Delta E(q + \Delta q, \nu' + \Delta \nu')\) and momentum \(\Delta P = \Delta P(q + \Delta q, \nu' + \Delta \nu')\) are the general functionals given in Eqs. \(25\) and \(29\), respectively, which for the present case read,

\[
\Delta P = [k_0 + c_1 (q + \Delta q) + c_1' (q' + \Delta q')]; \quad \Delta E = [w_0 + c_1 \epsilon_{\nu}(q + \Delta q) + c_1' \epsilon_{\nu'}(q' + \Delta q')]\tag{B10}\]

where the bare-momentum deviations \(\Delta q\) and \(\Delta q'\) relative to the bare momentum values of Eq. \(25\) are small.

The excited energy eigenstates which contribute to the weight-distribution expression \(60\) have momentum \(k\) and energy \(\omega\) relative to the initial ground state. Such states are generated by the elementary processes (C) from the states whose momentum and energy spectra are given in Eq. \(B10\). In order to reach the momentum \(k\) and energy \(\omega\)
\[ \omega \text{ given in Eq. (59), the elementary processes (C) generate excitations whose momentum and energy are given by } [k - l\Delta P] \text{ and } [\omega - l\Delta E], \text{ respectively. From use of Eqs. (60) and (110) we find,} \]

\[
[k - l\Delta P] = -l[c_1 \Delta q + c'_1 \Delta q'] ; \quad [\omega - l\Delta E] = -l[c_1 v_{\alpha\nu}(q) \Delta q + c'_1 v_{\alpha'\nu'}(q') \Delta q'] ,
\]

where \( \alpha\nu = c_0, s_1 \) and \( \alpha'\nu' = c_0, s_1 \). Use of these expressions leads to,

\[
\Delta q = -l[\omega - l\Delta E] \frac{c_1[1 - v_{\alpha'\nu'}(q') z]}{[v_{\alpha\nu}(q) - v_{\alpha'\nu'}(q')]}; \quad \Delta q' = l[\omega - l\Delta E] \frac{c'_1[1 - v_{\alpha\nu}(q) z]}{[v_{\alpha\nu}(q) - v_{\alpha'\nu'}(q')]}; \quad z \equiv 1/v.
\]

For small but finite values of \( [\omega - l\Delta E] \) we can use the asymptotic expression (111) in the general \( i = 0 \) spectral function expression (11), which corresponds to the dominant contribution to the spectral function (11) for \( i = 0 \). We thus find,

\[
B'_{X'}(k, \omega) \approx \frac{1}{4\pi^2 C_s C_s} \int dq'' \int dq''' \Theta \left( \Omega - l[\omega - l\Delta E] \right) \Theta \left( l[\omega - l\Delta E] \right)
\times \frac{F_0(l[k - l\Delta P]/[\omega - l\Delta E])}{4\pi \sqrt{v_{\alpha\nu} v_{s_1}}} \left( \frac{l[\omega - l\Delta E]}{\sqrt{v_{\alpha\nu} v_{s_1}}} \right)^{-1 + 2\zeta_0(q, q')} ; \quad l = \pm 1.
\]

Here the \( q'' = q + \Delta q \) and \( q''' = q' + \Delta q' \) integration domains correspond to the excitation domains such that \( 0 < [\omega - l\Delta E] < \Omega \) and \(-1/v_{s_1} < 1/v < +1/v_{s_1}, \zeta_0(q, q') \) is the functional (39), and \( F_0(z) \) is given in Eq. (22). By use of Eq. (112), we perform a transformation from \( q'' \) and \( q''' \) to the integration variables \( \omega' = [\omega - l\Delta E] \) and \( v \), what leads to,

\[
B'_{X'}(k, \omega) \approx \frac{1}{4\pi^2 C_s C_s} \int dq' \int dq \Theta \left( \Omega - l[\omega - l\Delta E] \right) \Theta \left( l[\omega - l\Delta E] \right)
\times \frac{F_0(\frac{l[\omega' - q]}{\sqrt{v_{\alpha\nu} v_{s_1}}})}{4\pi \sqrt{v_{\alpha\nu} v_{s_1}}} \left( \frac{l[\omega' - q]}{\sqrt{v_{\alpha\nu} v_{s_1}}} \right)^{-1 + 2\zeta_0(q, q')} ; \quad l = \pm 1.
\]

Finally, by performing the \( \omega' \) integration we reach the general expression (60).

Our next goal is the derivation of the branch-line expression defined by Eqs. (74) and (75). For simplicity, let us consider a \( \alpha\nu = c_0, s_1 \) branch line. The expressions correspond to \( k = l[k_0 + c_1 q] \) and \( \omega \approx l[k_0 + c_1 \epsilon_{\alpha\nu}(q)] \) with the \( (k, \omega) \)-plane point being located in the vicinity of \((l = +1) \) or below \((l = -1) \) the branch line. In this case the initial state integrations of Eq. (44) correspond to points belonging to a branch-line segment associated with \( q'' = q + \Delta q \) values such that \( \Delta q \) is both negative and positive values. The elementary processes (C) generate excited states of momentum \( k' = (\Delta P + k') \) and energy \( \omega = (\Delta E + \omega') \) relative to the initial ground state. The general energy spectrum \( \Delta P \) and momentum spectrum \( \Delta E \) of the corresponding states of the reduced subspace is given in Eqs. (28) and (44), respectively. In the present case these spectra are functions of the bare momentum \( q'' = q + \Delta q \) and read,

\[
\Delta E(q + \Delta q) = \omega_0 + c_1 \epsilon_{\alpha\nu}(q) + \Delta q c_1 v_{\alpha\nu}(q) ; \quad \Delta P(q + \Delta q) = k_0 + c_1 q + c_1 \Delta q .
\]

The velocity \( v_{\alpha\nu}(q) \) is then given by,

\[
v = v_{\alpha\nu}(q) - c_1 \frac{l(\omega - l[\omega_0 + c_1 \epsilon_{\alpha\nu}(q)])}{\Delta q} ; \quad \Delta q = -c_1 \frac{l(\omega - l[\omega_0 + c_1 \epsilon_{\alpha\nu}(q)])}{v - v_{\alpha\nu}(q)},
\]

where we also provided the corresponding value of \( \Delta q \) as a function of \( v \).

For \( \text{sgn}(v) = -\text{sgn}(q) \) one has that \( v \in (\infty, -v_{s_1}) \) if \( \text{sgn}(q) = 1 = +1 \) and \( v \in (v_{s_1}, \infty) \) when \( \text{sgn}(q) = -1 \). By use of Eq. (110) we then find that the limiting values \(-\text{sgn}(q) \infty \) and \(-\text{sgn}(q) v_{s_1} \) correspond to \( \Delta q = 0 \) and \( \Delta q = \text{sgn}(q) c_1 l(\omega - l[\omega_0 + c_1 \epsilon_{\alpha\nu}(q)])/[v_{s_1} + v_{\alpha\nu}(q)] \), respectively. In turn, for \( \text{sgn}(v) = \text{sgn}(q) \) there are two cases. When \( |v_{\alpha\nu}(q)| \leq v_{s_1} [1 - l(\omega - l[\omega_0 + c_1 \epsilon_{\alpha\nu}(q)])]/\Omega \) we find that \( v \in (-\infty, -v_{s_1}) \) for \( \text{sgn}(q) = 1 = -1 \) and \( v \in (v_{s_1}, \infty) \) for \( \text{sgn}(q) = -1 \). In contrast, if \( |v_{\alpha\nu}(q)| \geq v_{s_1} [1 - l(\omega - l[\omega_0 + c_1 \epsilon_{\alpha\nu}(q)])]/\Omega \) we find that \( v \in (-\infty, -|v_{\alpha\nu}(q)|) [1 - l(\omega - l[\omega_0 + c_1 \epsilon_{\alpha\nu}(q)])]/\Omega \) for \( \text{sgn}(q) = 1 = -1 \) and \( v \in (|v_{\alpha\nu}(q)|) [1 - l(\omega - l[\omega_0 + c_1 \epsilon_{\alpha\nu}(q)])]/\Omega, \infty \) for \( \text{sgn}(q) = -1 \). The limiting values \( v_{\alpha\nu}(q) \), \( v_{\alpha\nu}(q) v_{s_1} \), and \( v_{\alpha\nu}(q)/(1 - l(\omega - l[\omega_0 + c_1 \epsilon_{\alpha\nu}(q)])) \) correspond to \( \Delta q = 0, \Delta q = -\text{sgn}(q) c_1 l(\omega - l[\omega_0 + c_1 \epsilon_{\alpha\nu}(q)])/[v_{s_1} - v_{\alpha\nu}(q)], \) and \( \Delta q = -c_1 \Omega [1 - l(\omega - l[\omega_0 + c_1 \epsilon_{\alpha\nu}(q)])]/v_{\alpha\nu}(q) \), respectively. By use of the general spectral function expressions (11) and (44) for \( i = 0 \) we then arrive to,
$$B^i_N(k, \omega) \approx \frac{\text{sgn}(q)}{2\pi C_c C_s} \left\{ \Theta(v_{s1} [1 - \frac{l(\omega - l[\omega_0 + c_1 \epsilon_{av}(q)])}{\Omega}] - |v_{av}(q)|) \right\}^{q + \text{sgn}(q)} c_1 \left( \frac{[\omega - (l[\omega_0 + c_1 \epsilon_{av}(q)])]}{v_{s1} + |v_{av}(q)|} \right) dq'$$

$$+ \theta \left( |v_{av}(q)| - v_{s1} [1 - \frac{l(\omega - l[\omega_0 + c_1 \epsilon_{av}(q)])}{\Omega}] \right) \int_{q - \text{sgn}(q) c_1 \left( \frac{[\omega - (l[\omega_0 + c_1 \epsilon_{av}(q)])]}{v_{s1} + |v_{av}(q)|} \right)}^{+ \text{sgn}(q) c_1 \left( \frac{[\omega - (l[\omega_0 + c_1 \epsilon_{av}(q)])]}{v_{s1} + |v_{av}(q)|} \right)} dq' \right\}$$

$$\times \tilde{B}^{i,0} \left( \omega - l[\omega_0 + c_1 \epsilon_{av}(q)] + \Delta q c_1 v_{av}(q), v_{av}(q) - c_1 \left( \frac{l[\omega - l[\omega_0 + c_1 \epsilon_{av}(q)]]}{\Delta q} \right) \right) ; \ l = \pm 1. \quad (B17)$$

Next, we change the integration variable from \( q' \) to \( v \) and reach the following spectral-function expression,

$$B^i_N(k, \omega) \approx \frac{\text{sgn}(q)}{2\pi C_c C_s} l(\omega - l[\omega_0 + c_1 \epsilon_{av}(q)]) \int_{-\text{sgn}(q) v_{s1}}^{-\text{sgn}(q) v_{s1}} \frac{dv}{\text{sgn}(q) \infty} + \Theta(v_{s1} [1 - \frac{l(\omega - l[\omega_0 + c_1 \epsilon_{av}(q)])}{\Omega}] - |v_{av}(q)|) \int_{\text{sgn}(q) v_{s1}}^{\text{sgn}(q) v_{s1}} \frac{dv}{\text{sgn}(q) \infty}$$

$$\times \tilde{B}^{i,0} \left( \omega - l[\omega_0 + c_1 \epsilon_{av}(q)] + \Delta q c_1 \frac{v}{v_{av}(q)}, v_{av}(q) - c_1 \left( \frac{l[\omega - l[\omega_0 + c_1 \epsilon_{av}(q)]]}{\Delta q} \right) \right) ; \ l = \pm 1. \quad (B18)$$

Changing the integration variable from \( v \) to \( z = 1/v \) and using of the asymptotic expression \( 61 \) of Eq. \( 61 \), leads straightforwardly to expressions \( 61 \) and \( 62 \). (The theta functions added to the expression \( 61 \) define the energy range where it is valid.) Similar procedures lead to these expressions for the \( \alpha \nu \neq \alpha \nu \), \( s \) branches provided one uses the corresponding parametric equations given in Eq. \( 62 \).

The derivation of expressions \( 63 \) and \( 64 \), which refer to \( k \) and \( \omega \) values in the vicinity of the end points of a \( \alpha \nu = \alpha \nu \), \( s \) branch line such that \( \omega \approx \iota v_{av}(k - l k_0) + \iota \omega_0 \), follows the same steps as for the general branch-line spectral-function expression. When \( \Delta \omega \) is small but finite the convolution function \( 66 \) can be written as follows,

$$\tilde{B}^{i,0}_{\alpha \nu, \iota} (\Delta \omega, v) \approx \frac{1}{N_\alpha} \frac{F^0_{\alpha \nu, \iota} (1/v)}{4\pi \sqrt{v_{s1} v_{s1}}} \Theta(l \Delta \omega) \left( \frac{l \Delta \omega}{4\pi \sqrt{v_{s1} v_{s1}}} \right)^{-2 + \zeta^0_{\alpha \nu, \iota}} ; \ \alpha \nu = \alpha \nu, s1 ; \ \iota = \pm 1 ; \ l = \pm 1, \quad (B19)$$

where we used the relation \( l \Delta q_{\text{line}}/2\pi l^2 \Delta q_{\text{line}} = 1/N_\alpha \). Here the functional \( \zeta^0_{\alpha \nu, \iota} \) is given by,

$$\zeta^0_{\alpha \nu, \iota} = 2\Delta^\iota_{\alpha \nu} + 2\Delta^\iota_{\alpha \nu} + 2\Delta^{-1}_{\alpha \nu} = \zeta^0_{\alpha \nu} - 2\Delta^{-1}_{\alpha \nu} ; \ \alpha \nu = \alpha \nu, s1 ; \ \iota = \pm 1, \quad (B20)$$

and the function \( F^0_{\alpha \nu, \iota} (z) \) reads,

$$F^0_{\alpha \nu, \iota} (z) = \frac{D_0}{4 \Delta^\iota_{\alpha \nu}} \left[ \prod_{\alpha \nu' = 0, s1 \ i' = \pm 1} \frac{1}{\Gamma(2\Delta^\iota_{\alpha \nu'})} \right] \int_0^1 dx \left\{ \prod_{\iota' = \pm 1} \Theta(1 - x + \iota'' \nu_{av} z - \iota'' \nu_{av} x) \right\} \left( \frac{\nu_{av}}{\nu_{av}(x)} \right)^{2\Delta^\iota_{\alpha \nu} - 1} \left( \frac{\nu_{av}}{\nu_{av}(x)} \right)^{2\Delta^\iota_{\alpha \nu} - 1} \cdot \quad (B21)$$

Furthermore, use of the same procedures as above but with the use of the functions \( 66 \), \( 67 \), and \( 68 \) instead of \( 63 \), \( 64 \), and \( 65 \), respectively, leads straightforwardly to expressions \( 69 \) and \( 70 \). The main difference is that the factor \( 1/N_\alpha \) which multiplies the \( q' \) integration factor \( L/2\pi \) to give \( 1/2\pi \) on the right-hand side of Eq. \( 71 \) arises from the state summation of Eq. \( 41 \), whereas in the case of the constant \( 66 \) it comes from expression \( 71 \).

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