Dilaton Stabilization in (A)dS Spacetime with Compactified Dimensions

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(Received March 7, 2003)

We investigate dilaton stabilization in a higher-dimensional theory. The background geometry is based on an eleven-dimensional Kaluza-Klein/supergravity model, which is assumed to be a product of four-dimensional de Sitter ($dS_4$) spacetime and a seven sphere ($S^7$). The dilaton potential has a local minimum resulting from contributions of the cosmological constant, the curvature of the internal spacetime and quantum effects of the background scalar, vector, spinor, and tensor fields. The dilaton settles down to the local minimum, and the scale of the extra dimensions eventually become time independent. Our four-dimensional universe evolves from $dS_4$ into $AdS_4$ after stabilization of the extra dimension.

§1. Introduction

In a unified description of gravity and gauge interactions, the extra dimension plays a very important role in providing a consistent quantum theory of all known interactions. One of the ideas for this unification is that four-dimensional gauge interactions arise from isometries of the higher-dimensional gravity. In the general strategy usually employed, it is considered that obvious difference between the usual four dimensions and the extra dimension could result from a process of spontaneous breakdown of the vacuum symmetry, which is often called “spontaneous compactification”\textsuperscript{1} of the extra dimensions. This process has been actively investigated in connection with supergravity.\textsuperscript{2,3}

However, the modern version of the unification scheme is much different from the conventional Kaluza-Klein approach.\textsuperscript{4,5} Such a modification is motivated by several factors. First, we cannot naturally understand the hierarchy involving the electroweak scale and the fundamental scale in particle physics. Second, it has been shown that the dimensionally reduced bare action directly obtained from higher-dimensional gravity cannot give rise to a stable compactification. A new compactification mechanism, recently proposed by Randall and Sundrum (RS),\textsuperscript{6} has been studied intensively. This is different from the conventional Kaluza-Klein picture, in which the gauge and matter field are assumed to be confined on the “brane,” while the gravity and moduli exist in the “bulk.” However, because the RS prescription at present cannot be applied directly to more than five dimensions, the Kaluza-Klein description is still considered useful. In fact, Kaluza-Klein supergravity theories\textsuperscript{7} are regarded as low energy effective version of M-theory. The extra dimensions in the Kaluza-Klein description are assumed to be microscopic, with a size much smaller

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than the scale of four dimensions. Constraints on the smallness of the extra dimensions have been obtained with the accelerator experiments. Specifically, the upper bound on this has found in this way is well known to be approximately $10^{-16}$ cm. The extra dimensions must be then not only small but also almost static. The gauge coupling constant is in general related to the ratio of the higher-dimensional gravitational constant and the Kaluza-Klein mass.\(^8\) The Kaluza-Klein mass depends on the size of extra dimensions. Hence the time dependence of the extra dimensions induces that of the gauge coupling constants. At present, the time variation of gauge coupling constants is constrained by the measurement of the quasar absorption line,\(^9\)–\(^12\) the cosmic microwave background (CMB),\(^13\),\(^14\) and primordial nucleosynthesis,\(^15\) according to which their change is at least very small after nucleosynthesis.\(^16\) We thus see that the stabilization of the dilaton is a crucial issue.

The process of spontaneous compactification generates a degree of freedom of the dilaton field, which is characterized by the scale of the extra dimensions in lower dimensions. This implies that the dynamics of the extra dimensions are governed by the dilaton potential. Then, in order to stabilize the extra dimensions, the dilaton potential must have a minimum. Unfortunately, in a system with $D$-dimensional ($D > 4$) gravity (or gravity plus a scalar) and a cosmological constant, classically the dilaton potential has no local minimum in a naive compactification, such as the product of the four-dimensional spacetime and a $(D - 4)$ sphere or torus.

Recently, Carroll et al. proposed a dilaton stabilization mechanism in a classical system\(^17\). The essence of their idea is to consider a combination of background matter fields. The dilaton potential in their model consists of the background energy-momentum tensor, which contains the curvature of the extra dimensions, a cosmological constant and a Yang-Mills field, which is wrapped around the extra dimensions. Positive curvature effects are contributed to the attractive force for the dilaton, while the cosmological constant and the Yang-Mills field strength form a repulsive force,\(^*\) so that the dilaton is stabilized by the balance of these forces. Torii and Shiromizu also studied stabilization of the Freund-Rubin type of compactification with a $p$-form gauge field strength.\(^18\)

In this paper, taking into account quantum effects on a compactified spacetime, we consider the dynamics of the extra dimension in the four-dimensional de Sitter spacetime ($dS_4$). First, we consider the system consisting of scalar, vector, spinor and tensor fields and a cosmological constant in eleven dimensions. The eleven-dimensional space is assumed to break up into a four-dimensional de Sitter space and a curved compact seven-dimensional manifold. The energy-momentum tensor in this model consists of the matter and its quantum effects. This quantum correction is often called the Casimir effect.\(^19\),\(^20\) The quantum effective potential associated with the matter fields is similar power to the part of dilaton potential arising from the curvature and the cosmological constant.\(^21\) However, their signs are opposite, so that a local minimum is created by their combination. Taking account of this

\(^*\) The magnetic strength is sort of $U(1)$ fiber bundle. The strength of the repulsive force for the extra dimension increases because the force arising from the magnetic strength is proportional to the flux density.
result, we investigate the stabilization in the bosonic part of the eleven-dimensional supergravity model. The energy-momentum tensor possesses not only classical fields but also the contribution of the 1-loop quantum corrections in graviton, because the physics associated with the extra dimension in the Kaluza-Klein theories is comparable to or not too much larger than the Planck length. Therefore, the quantum effects of moduli are presumably of some importance, and these effects are able to contain Einstein-Hilbert action, which is assumed to be the low-energy effective action of the M-theory.  

To evaluate the functional determinant in this calculation of the 1-loop quantum effect, we use the generalized zeta function, which is the sum of the operator eigenvalues. Because the de Sitter spacetime has a unique Euclidean section which is an $S^4$, we can assume that the grand state of this system is the product $S^4 \times S^7$ of a four-dimensional sphere and a seven-dimensional sphere. We will see that the zeta function regularization method is useful. Finally, we show that the Casimir effect produces a potential minimum and moduli field is stabilized.

The plan of this paper is as follows. A brief description of $D$-dimensional Kaluza-Klein gravity is given in §2. In §3, we calculate the effective action due to one loop of the scalar, vector, spinor and tensor fields in eleven dimensions, which is a product of four-dimensional de Sitter spacetime and $S^7$. We use these results in §4 to carry out an analysis of the stability of the dilaton in eleven-dimensional supergravity model. Section 5 contains our conclusions.

§2. Extra dimension classical dynamics in a $D$-dimensional model

2.1. Gravity plus scalar field system

First, as a simple example, we review the classical dynamics of a dilaton field in a $D(D > 4)$-dimensional model. The background geometry is given by the product space of four-dimensional spacetime and the $(D - 4)$-dimensional sphere $S^d$. The four-dimensional metric is not specified here. We obtain a massive mode of the dilaton after the background spacetime is compactified on $S^d$. This model is simple but important for studying the dynamics of dilaton fields.

We consider the following $D$-dimensional Einstein-Hilbert action with a cosmological constant:

$$I_{EH} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-\bar{g}} (\bar{R} - 2\bar{\Lambda}).$$  

Here, $\kappa$ is a positive constant, $\bar{R}$ is the $D$-dimensional Ricci scalar, and $\bar{\Lambda}$ is the cosmological constant. We consider the case of a positive cosmological constant, $\Lambda > 0$. The line element is now assumed to take the form

$$ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu + b^2 \Omega^{(d)}_{ij} dx^i dx^j,$$

where $\hat{g}_{\mu\nu}$ is a four-dimensional metric depending only on the four-dimensional coordinates $(x^\mu)$ ($\mu = 0, 1, 2, 3$), and $\Omega^{(d)}_{ij} dx^i dx^j$ is a line element of unit $d$-dimensional sphere. The dilaton field, $b$, which is a so-called “radion”, characterizes the scale of the extra dimension and is assumed to depend only on the four-dimensional co-
ordinate $x^\mu$. Substituting the metric (2.2) into the action (2.1), we obtain a four-dimensional action

$$I_{EH} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\hat{g}} \left( \frac{b}{b_0} \right)^d \left[ \hat{R} + Kd(d-1)\hat{g}^{\mu\nu}(\partial_\mu \ln b)(\partial_\nu \ln b) ight. \\
+ \left. d(d-1)\hat{g}^{\mu\nu} \partial_\mu \ln b \partial_\nu \ln b \right] ,$$  \hspace{1cm} (2.3)

where $\kappa$ is a positive constant defined by $\kappa^2 = \bar{\kappa}^2/(2d b_0^2 \pi)$, $K (K > 0)$ is the curvature parameter of $S^d$, and $\hat{R}$ is the Ricci scalar of the four-dimensional metric tensor $\hat{g}_{\mu\nu}$. Because we consider the case in which the scale $b$ of the compactification is a time-dependent variable, this action is different from the ordinary four-dimensional Einstein-Hilbert action. For this reason, we carry out the following conformal transformation in order to obtain a useful expression:

$$\hat{g}_{\mu\nu} = \left( \frac{b}{b_0} \right)^{-d} g_{\mu\nu} .$$ \hspace{1cm} (2.4)

Note that if we take $b_0$ to be the initial value of $b$, then the conformal factor is initially unity. After the conformal transformation, which changes the $D$-dimensional line element to

$$\hat{g}_{MN}dx^Mdx^N = \left( \frac{b}{b_0} \right)^{-d} g_{\mu\nu}dx^\mu dx^\nu + b^2 \Omega_{ij}^{(d)} dx^i dx^j ,$$ \hspace{1cm} (2.5)

we obtain

$$I_{EH} = \int d^4x \sqrt{-\hat{g}} \left[ \frac{1}{2\kappa^2} \hat{R} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - U_0(\sigma) \right] ,$$ \hspace{1cm} (2.6)

where $\hat{R}$ is the Ricci scalar of the four-dimensional metric tensor $g_{\mu\nu}$, the field $\sigma$ is defined by

$$\sigma = \sigma_0 \ln \left( \frac{b}{b_0} \right) , \hspace{1cm} \sigma_0 = \sqrt{\frac{d(d+2)}{2\kappa^2}} ,$$ \hspace{1cm} (2.7)

and the potential $U_0(\sigma)$ of $\sigma$ is

$$U_0(\sigma) = \frac{\bar{\Lambda}}{\kappa^2} e^{-d\sigma/\sigma_0} - \frac{K(d-1)}{2\kappa^2 b_0^2} e^{-(d+2)\sigma/\sigma_0} .$$ \hspace{1cm} (2.8)

Unfortunately, it can be easily confirmed that the dilaton field $\sigma$ [i.e. $b(t)$] potential $U_0(\sigma)$ does not possess a local minimum. This implies that there is no stable compactification on $S^d$ if the model is modified no further.

### 2.2. Gravity, the cosmological constant and the gauge field system

Next, we consider the contribution of the gauge field, in addition to the dilaton field and the cosmological constant. This system has been investigated by Carroll et al.\(^{17}\) Although they considered the six-dimensional spacetime, motivated by the work of Cremmer and Scherk\(^{23}\) and Sundrum\(^{24}\) we consider the general $D$-dimensional case. The ansatz here for the line element is given by (2.5). The starting point of this model is the Einstein-Hilbert action with the matter term

$$I_{EH} = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-\hat{g}} (\hat{R} - 2\Lambda) + \mathcal{L}_m ,$$ \hspace{1cm} (2.9)
where $L_{m}$ is the matter Lagrangian density. Using the line element (2.2) and the conformal transformation (2.4), we obtain the four-dimensional action as

$$I_{EH} = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_{\mu} \sigma \partial_{\nu} \sigma - V(\sigma) \right], \quad (2.10)$$

where the dilaton potential $V(\sigma)$ is written

$$V(\sigma) = \frac{1}{\kappa^2} \left\{ -\frac{Kd(d-1)}{b_0^2} e^{-(d+2)\sigma/\sigma_0} + e^{-d\sigma/\sigma_0} (A - L_{m}) \right\}. \quad (2.11)$$

We introduce the electromagnetic fields $F_{\mu\nu}$ as matter fields. The explicit form of $F_{\mu\nu}$ is the Freund-Rubin type, 25)

$$F_{ij} = F_{ji} = (\epsilon_{ij} F_0) \sqrt{\Omega},$$

where $i$ and $j$ run over the extra dimensions. The moduli potential is

$$V(\sigma) = \frac{1}{\kappa^2} \left\{ -\frac{Kd(d-1)}{b_0^2} e^{-(d+2)\sigma/\sigma_0} + e^{-d\sigma/\sigma_0} A + f_0^2 e^{-(d+4)\sigma/\sigma_0} \right\}, \quad (2.12)$$

where $f_0^2 = F_0^2 \Omega/2$. We search for values of the parameters $A$ and $f_0$ such that there exists a local minimum of the potential $V(\sigma)$. Following the approach of Carroll et al., 17) we choose the parameters as

$$\frac{Kd(d-1)}{4\kappa^2 b_0^2} = \frac{A}{\kappa^2} = \frac{f_0^2}{\kappa^2} = A, \quad (2.13)$$

where $A$ is constant. The dilaton potential $V(\sigma)$ has a local minimum at $b = 0$. The dilaton acquires a mass $m_\sigma = d^2 V/db^2|_{b=b_0} \propto 1/b_0^2$, and the energy scale at the maximum of the potential is $V(\kappa^2 \sigma) \simeq A$. As remarked above, the dynamics of the extra dimension are to introduce an additional massive dilaton field into the four-dimensional theory. Although the extra dimension is stabilized by the contribution of the background matter contents (i.e. the repulsive force of the magnetic flux and the cosmological constant, and the attractive force of the curvature of the extra dimension), the state $b = 0$ is not a global minimum, so that there is the possibility that the dilaton will tunnel through the potential barrier. Because the dilaton in that case runs away to infinity, there arises the problem of extra dimensions expanding to infinity. 26)

§3. Quantum effect in the D-dimensional Kaluza-Klein model

It is known that an attractive force exists between uncharged superconductivity plates as a consequence of the quantum mechanical vacuum fluctuations of the electromagnetic field. This is called the Casimir effect. 27) Such a quantum correction is expected to arise in Kaluza-Klein theory, because a boundary condition is imposed on the quantum field in the direction of the extra dimension. Although the extra dimension is stabilized by the contribution of the background matter contents (i.e. the repulsive force of the magnetic flux and the cosmological constant, and the attractive force of the curvature of the extra dimension), the state $b = 0$ is not a global minimum, so that there is the possibility that the dilaton will tunnel through the potential barrier. Because the dilaton in that case runs away to infinity, there arises the problem of extra dimensions expanding to infinity. 26)
spacetime \((dS_n)\) and the \(d\)-dimensional sphere \((S^d)\). To carry out the 1-loop evaluation of the effective potential, the gauge-fixing and ghost terms are introduced. Then, we explain a convenient regularization based on the \(\zeta\) function. As concrete examples of the method, calculations of the effective potential in the product space of four-dimensional Minkowski spacetime and the \(d\)-sphere \((M_4 \times S^d)\) and in \(M_4 \times S^n \times S^d\) have been presented previously by several authors.\(^{29-31}\)

3.1. Dimensional reduction

As a simple example, we consider the action of the Einstein-Hilbert and massless scalar field \(\bar{\phi}\) in \(D (= n + d)\) dimensions,

\[
I[b, \bar{\phi}] = I_{EH} + I_S, \tag{3.1}
\]

where \(I_{EH}\) is the Einstein-Hilbert action (2.1), and

\[
I_S = -\frac{1}{2} \int d^Dx \sqrt{-\hat{g}} \hat{g}^{MN} \partial_M \bar{\phi} \partial_N \bar{\phi} \tag{3.2}
\]

is a massless Klein-Gordon field. The background geometry is the \(n\)-dimensional de Sitter spacetime with \(S^n\). The \(D\)-dimensional line element here has the same form as (2.4). The metric \(\hat{g}_{\mu\nu}\) in (2.4) initially represents an \(n\)-dimensional de Sitter spacetime and depends only on the \(n\)-dimensional coordinates \(\{x^\mu\}\). The scalar field \(\bar{\phi}\) is expanded as

\[
\bar{\phi} = b_0^{-d/2} \sum_{l,m} \phi_{lm}(x^\mu) Y_{lm}^{(d)}(x^i), \tag{3.3}
\]

where the constant \(b_0\) is the initial value of \(b\). In this expression, the \(Y_{lm}^{(d)}\) are real harmonics on the \(d\)-sphere satisfying

\[
\frac{1}{\sqrt{\Omega^{(d)}}} \partial_i \left( \sqrt{\Omega^{(d)}} \Omega^{(d)ij} \partial_j Y_{lm}^{(d)} \right) + l(l + d - 1) Y_{lm}^{(d)} = 0, \tag{3.4}
\]

\[
\int d^d x \sqrt{\Omega^{(d)}} Y_{lm}^{(d)} Y_{l'm'}^{(d)} = \delta_{ll'} \delta_{mm'}, \tag{3.5}
\]

and \(\phi_{lm}(x^\mu)\) is a real function depending only on the \(n\)-dimensional coordinates \(\{x^\mu\}\). The massless scalar field action in this case is given by

\[
I_S = -\frac{1}{2} \sum_{l,m} \int d^n x \sqrt{-\hat{g}} e^{\frac{b_0^2}{4}} \left[ \hat{g}^{\mu\nu} \partial_\mu \phi_{lm} \partial_\nu \phi_{lm} + \frac{l(l + d - 1)}{b_0^2} \phi_{lm}^2 \right]. \tag{3.6}
\]

The massless scalars in \(D\) dimensions acquire mass through spontaneous compactification. The massive scalar field action (3.6) is quite different from the usual \(n\)-dimensional form of that action, because the dilaton field couples to the kinetic term of the scalar field \(\phi_{lm}\). Carrying out the conformal transformation

\[
\hat{g}_{\mu\nu} = \left( \frac{b}{b_0} \right)^{-2d/(n-2)} g_{\mu\nu}, \tag{3.7}
\]
where $g_{\mu\nu}$ is the metric on $dS_n$ and $b_0$ is chosen as the initial value of $b$, we obtain

$$I_S = -\frac{1}{2} \sum_{l,m} \int d^n x \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi_{lm} \partial_\nu \phi_{lm} + M_\phi^2 \phi_{lm}^2 \right],$$  \hspace{1cm} (3.8)

where the mass $M_\phi^2$ of the $n$-dimensional scalar field $\phi_{lm}$ is given by

$$M_\phi^2 = \frac{l(l + d - 1)}{b_0^2} e^{-2(d+n-2)\sigma/(n-2)\sigma_0},$$  \hspace{1cm} (3.9)

$R$ is the Ricci scalar on the $dS_n$ background, and the dilaton field $\sigma$ is also defined by Eq. (2.7).

3.2. The quantum correction

The calculation of the effective potential is carried out using the path integral method. The fields are split into a classical part, $\phi_{lm,c}$, and a quantum part, $\delta \phi_{lm}$. The action is then expanded with respect to the quantum fields around an arbitrary classical background field, up to second order, to generate all 1-loop diagrams.

In the path integral approach to quantum field theory, the amplitude is given by the expression

$$Z = \int \mathcal{D}\phi_{lm} \exp \left( iI_t[b, \phi_{lm}] \right),$$  \hspace{1cm} (3.10)

where $\mathcal{D}\phi_{lm}$ is the integration measure on the space of scalar fields, and $I_t[b, \phi_{lm}]$ is the total action. The action can be expanded in the neighborhood of these classical background fields as

$$I_t[b, \phi_{lm}] = I[b, \phi_{lm,c}] + I_q[b, \delta \phi_{lm}] + O \left( (\delta \phi_{lm})^3 \right),$$  \hspace{1cm} (3.11)

where $\phi_{lm} = \phi_{lm,c} + \delta \phi_{lm}$. The action $I_t[b, \phi_{lm,c}]$ is quadratic in $\delta \phi_{lm}$, and no terms linear in $\phi_{lm}$ appear, because such terms can be eliminated by using the classical equations of motion. In the 1-loop approximation, we ignore all terms of higher than quadratic order in the expansion of

$$\ln Z = iI[b, \phi_{lm,c}] + \ln \left\{ \int \mathcal{D}\delta \phi_{lm} \exp \left( iI_q[b, \delta \phi_{lm}] \right) \right\}.$$  \hspace{1cm} (3.12)

We note that the integrals here are ill-defined, because the operators in this equation are unbounded in the $dS_n$ spacetime with Lorentz signature. We need to perform a Wick rotation in order to make this expression well-defined. Doing so, we obtain

$$\ln Z = -I_E[b, \phi_{lm,c}] + \ln \left\{ \int \mathcal{D}\delta \phi_{lm} \exp \left( -I_q[b, \delta \phi_{lm}] \right) \right\},$$  \hspace{1cm} (3.13)

where $I_E$ is the Euclidean action, which is expressed as

$$I_E[b, \phi_{lm}] = \sum_{l,m} \int d^n x \sqrt{-g} \frac{1}{2} \left\{ \partial_\mu \phi_{lm} \partial^\mu \phi_{lm} + M_\phi^2 \phi_{lm}^2 \right\}.$$  \hspace{1cm} (3.14)
Using the assumption $\phi_{lm} = \phi_{lm}^c + \delta\phi_{lm}$, we can integrate the kinetic term in the action by parts, obtaining

$$
I_E[b, \delta\phi_{lm}] = \frac{1}{2} \int d^n x \sqrt{-g} \delta\phi_{lm} \left\{ -\Box_{(n)} + M^2_{\phi} \right\} \delta\phi_{lm},
$$

(3.15)

where $\Box_{(n)}$ denotes the Laplacian in the $n$-dimensional de Sitter spacetime.

The effective potential $V_{\text{eff}}$ is defined through the relation

$$\exp \left( \int d^n x V_{\text{eff}} \right) = \int \mathcal{D}\phi_{lm} \exp \left( -I_{\text{qE}}[b, \delta\phi_{lm}] \right) \equiv \left\{ \text{det} (\mu M) \right\}^{-\frac{1}{2}},
$$

(3.16)

where $I_{\text{qE}}$ is the total Euclidean action and $\mu$ is a normalization constant with dimension of mass that comes from the Euclidean path integral. Using Eqs. (3.13) and (3.16), we find that the 1-loop effective potential is

$$V_{\text{eff}}(\sigma) = V_0(\sigma) - \frac{1}{2\Omega_{\text{vol}}} \ln \text{det} \left\{ \mu^{-2} \left( \Box_{(n)} - M^2_{\phi} \right) \right\},
$$

(3.17)

where $\Box_{(n)}$ is the $n$-dimensional Laplace-Beltrami operator and $V_0(\sigma)$ is given by

$$V_0(\sigma) = \frac{\Lambda}{\kappa^2} e^{-\frac{2d}{(n-2)} \frac{\sigma}{\sigma_0}} - \frac{K d (d-1)}{2 \kappa^2 b_0^2} e^{\left( \frac{2d}{n-2} + 2 \right) \frac{\sigma}{\sigma_0}},
$$

(3.18)

with $\Omega_{\text{vol}}$ the volume of the $n$-dimensional de Sitter spacetime.

### 3.3. Zeta function regularization

Now we need to evaluate the functional determinant appearing in the expression for the effective potential on a background manifold in $dS_n \times S^d$. We apply the standard technique of zeta function regularization in order to deal with the ultraviolet divergence with respect to the Kaluza-Klein mode. Many authors have previously used this method to study the Casimir effect for the spacetime $M^4 \times S^d$. 21), 29) - 31)

We now define the functional determinants in terms of the generalized zeta function, which is the sum of operator eigenvalues, as

$$\zeta_{n \times d}(s) \equiv \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} D(l) D(l') \left\{ \frac{\lambda(l')}{a^2} + \left( \frac{b_0}{b} \right)^{2d/(n-2)} \frac{A(l)}{b_0^2} \right\}^{-s},
$$

(3.19)

where $a$ and $b$ are the scales of $dS_n$ and $S^d$, $\lambda(l')$ and $A(l)$ are eigenvalues of the scalar field $\phi$ in $dS_n$ and $S^d$, and $d(l')$ and $D(l)$ are degeneracies of $dS_n$ and $S^d$, respectively. This expansion is well-defined and converges for $\text{Re}(z) > (n + d)/2$.

Using this function, the effective potential (3.17) is written

$$V_{\text{eff}}(\sigma) = V_0(\sigma) - \frac{1}{2\Omega_{\text{vol}}} \left\{ \zeta'_{n \times d}(0) + \zeta_{n \times d}(0) \ln(\mu^2 a^2) \right\},
$$

(3.20)

where, in order to get the second term, we have used the relation

$$\text{det} (\mu M) = \mu^\zeta(0) \text{det} M.
$$

(3.21)
From this point, we drop the \( n \times d \) subscript on \( \zeta \). Our task is to calculate \( \zeta(s) \) and to continue it analytically to \( s = 0 \). (\( \zeta \) is regular at \( s = 0 \).) Note that the logarithmic term proportional to \( \ln(\mu^2 a^2) \) is ill-defined if a rescaling of \( \mu \) adds a linear combination of \( \zeta(0) \) to it. It has been pointed out by several authors\(^{31}, 32 \) that the finite term in the quantum effect depends on the regularity technique used. Although those authors were not able to calculate \( \zeta(0) \) for a technical reason, we evaluate the terms \( \zeta'(0) \) and \( \zeta(0) \) explicitly. (see Appendices A, B and C.)

3.4. The effective potential for \( dS_4 \times S^7 \)

In this subsection, we compute the effective potential of the dilaton field coming from quantum effects of the background scalar, spinor, vector and tensor fields in a background geometry that is the direct product of a four-dimensional de Sitter spacetime with a seven-dimensional sphere.

3.4.1. Scalar field

The zeta function \( \zeta \) can be evaluated if there is a formula for the eigenvalue of the operator in de Sitter spacetime that is a four-dimensional hyperboloid of constant curvature. It has a unique Euclidean section which is a four-sphere \( S^4 \) of radius \( a \).

The curvature tensors are

\[
R_{\mu\nu\rho\sigma} = \frac{k}{a^2} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}), \quad R_{\mu\nu} = \frac{3k}{a^2} g_{\mu\nu}, \quad R = \frac{12k}{a^2}, \quad (3.22)
\]

where \( k \) is the curvature parameter for \( dS_4 \) and is taken as \( k = 1 \). Fortunately, the degeneracy \( d_4(l) \) and the eigenvalue \( \lambda_4(l) \) of the massless scalar field in \( dS_4 \) spacetime are known. They are given by

\[
d_4(l) = \frac{1}{6}(l+1)(l+4)(2l+3), \quad \lambda_4(l) = \frac{1}{a^2} l(l+3), \quad l = 0, 1, 2, \cdots, \quad (3.23)
\]

where \( a \) is the radius of the four-sphere. It is necessary to regularize \( \zeta(s) \) in order to evaluate the effective potential \( V_{\text{eff}} \) because \( \zeta(s) \) has a divergence resulting from the infinite mode sum. We use the prescription of the zeta function regularization given by Kikkawa et al.\(^{29} \) In Appendix A, we give the details of the calculation of the zeta function in the massive scalar field case. To simplify the calculation, it is assumed that the scale of de Sitter spacetime, \( a \), is much larger than that of the compactification scale, \( b \). Under this assumption, \( V_{\text{eff}} \) is evaluated by numerical integration of several terms appearing in \( \zeta'(0) \) and \( \zeta(0) \). We can see that the terms proportional to \( (a/b_0)^4 \) in Eq. (A.19) are dominant when the condition \( a \gg b_0 \) is satisfied. Our final result for the scalar potential to 1-loop order is then

\[
V_{\text{eff}}(\sigma) = V_0(\sigma) - \frac{1}{2\Omega_{\text{vol}}} \left\{ \zeta'(0) + \zeta(0) \ln(\mu^2 a^2) \right\}
= \frac{A}{k^2} e^{-\frac{\sigma}{\sqrt{b_0}}} - \frac{21K}{k^2 b_0^2} e^{-\frac{9\sigma}{\sqrt{b_0}}} + \frac{2.61083 \times 10^{-2}}{b_0^4} e^{-18\frac{\sigma}{\sqrt{b_0}}}, \quad (3.24)
\]

where \( V_0(\sigma) \) is given by Eq. (3.18).
Dirac Spinor field

Next, we calculate the quantum effect associated with the massless Dirac spinor field $\psi$ on $dS_4 \times S^7$ with the action

$$ I_\psi = i \int d^{11}x \sqrt{-g} \bar{\psi} \gamma^M \nabla_M \psi. \quad (3.25) $$

The eleven-dimensional gamma matrix $\bar{\gamma}$ is given by

$$ \bar{\gamma}^\mu = \gamma^\mu \otimes 1 , \quad \bar{\gamma}^i = \gamma^5 \otimes \gamma^i , \quad \left( \bar{\gamma}^5 \right)^2 = 1 , \quad \{ \bar{\gamma}^M , \bar{\gamma}^N \} = 2g^{MN} , \quad (3.26) $$

where the $\gamma^\mu (\mu = 0, 1, 2, 3)$ are the Dirac matrices in $dS_4$, while the $\gamma^i (i = 4, 5, \cdots, 10)$ are those in $S^7$. The spinor representation of $O(1,3 + 7)$ is a direct product of the spinor representation of $O(1,3)$ and $O(7)$. The Dirac spinor field $\psi$ is expanded in spinor harmonics, in analogy to the scalar field:

$$ \bar{\psi}(x^\mu, x^i) = b_0^{-7/2} \sum_{l,m} \psi_{lm}(x^\mu) Y_{lm}^{(7)} (x^i) . \quad (3.27) $$

Here, the constant $b_0$ is the initial value of $b$, and $\psi_{lm}(x^\mu)$ is the Dirac spinor field in the four-dimensional spacetime. In Eq. (3.27), $Y_{lm}^{(7)}$ (where $l = 1, 2, \cdots, m$ denotes a set of six numbers that are needed in order for the set of all $Y_{lm}^{(7)}$ to be a complete set of $L^2$ functions on $S^7$) are real spinor harmonics on $S^7$ satisfying

$$ i \bar{\gamma}^i \nabla_i Y_{lm}^{(7)} \psi_{lm} = A_{\psi} Y_{lm}^{(7)} , \quad (3.28) $$

$$ \int d^7x \sqrt{\Omega^{(7)}} \psi_{lm}^{(7)} \psi_{lm}' = \delta_{ll} \delta_{mm'} , \quad (3.29) $$

and $\psi_{lm}(x^\mu)$ is a real function depending only on the four-dimensional coordinates $x^\mu$. Here, $\bar{\gamma}^i \nabla_i$ is the Dirac operator on the unit seven sphere $S^7$, and $A_{\psi}(l)$ denotes an eigenvalue for the Dirac spinor field $\psi$. Using the relation for the eleven-dimensional Dirac operator $\bar{\gamma}^M \nabla_M = \bar{\gamma}^\mu \nabla_\mu \otimes 1 + \gamma^5 \otimes \bar{\gamma}^i \nabla_i$ and the conformal transformation (2.4), we obtain the four-dimensional effective action

$$ I (\psi_{lm}, \bar{\psi}_{lm}) \sum_{l,m} \int d^4x \sqrt{-g} e^{-7\sigma/\sigma_0} \bar{\psi}_{lm} \left( i \gamma^\mu \nabla_\mu + A_{\psi} \gamma^5 \right) \psi_{lm} . \quad (3.30) $$

The partition function $Z$ for the massless Dirac spinor field on $dS_4$ is

$$ Z = \int D\psi_{lm} D\bar{\psi}_{lm} \exp \left\{ -i S (\psi_{lm}, \bar{\psi}_{lm}) \right\} , \quad (3.31) $$

where $D\psi_{lm}$ and $D\bar{\psi}_{lm}$ are the functional measures over the spinor field $\psi_{lm}$ and its Dirac adjoint field $\bar{\psi}_{lm}$, respectively. Using the definition of a Gaussian functional for anti-commuting fields Eq. (3.31) become

$$ Z = \det \left\{ \mu^{-1} e^{-7\sigma_0} \left( i \gamma^\mu \nabla_\mu + A_{\psi}(l) \gamma^5 \right) \right\} . \quad (3.32) $$
Then, we obtain
\[
\ln Z = \ln \det \left\{ \nu^{-1} \left( i\gamma^\mu \nabla_\mu + e^{-\frac{7\sigma_0}{\sigma_0}} A_\psi(l) \gamma^5 \right) \right\}
\]
\[
= \frac{1}{2} \ln \det \left[ \nu^{-2} \left\{ - (\gamma^\mu \nabla_\mu)^2 + e^{-14\frac{\sigma_0}{\sigma_0}} (A_\psi(l))^2 \right\} \right],
\]
where the four-dimensional Dirac operator \((\gamma^\mu \nabla_\mu)^2\) is given by \(^{34},^{35}\)
\[
(\gamma^\mu \nabla_\mu)^2 = \Box + \frac{1}{4} R(4),
\]
and \(\Box\) denotes the four-dimensional Laplace-Bertlami operator. Following the pro-
cedure of §§3.2 and 3.3, the effective potential must satisfy the relation
\[
V_{\text{eff}}(\sigma) = V_0(\sigma) - \frac{1}{4 \Omega_{\text{vol}}} \left\{ \zeta'(0) + \zeta(0) \ln(\nu^2 a^2) \right\},
\]
where \(\zeta(0)\), which is defined in Appendix B, is the generalized zeta function for the
Dirac spinor field and \(V_0(\sigma)\) is expressed by Eq. (3.18). We can see that under the
assumption \(a \gg b_0\), the dominant term in \(\zeta'(0)\) is proportional to \(e^{-18\sigma/\sigma_0}\). Using
the calculation method presented in Appendix B, the effective potential to 1-loop
order is finally given by
\[
V_{\text{eff}}(\sigma) = \frac{A}{\kappa^2} e^{-\frac{7\sigma}{\sigma_0}} - \frac{21K}{\kappa^2 b_0^2} e^{-\frac{9\sigma}{\sigma_0}} - \frac{1.258212 \times 10^{-6}}{b_0^4} e^{-\frac{18\sigma}{\sigma_0}}.
\]

3.4.3. \(U(1)\) gauge field

We now compute the quantum correction for the vector field \(A_M\) in eleven di-
mensions. We take the eleven-dimensional action for \(U(1)\) gauge field to be
\[
I_{U(1)} = \frac{1}{4} \int d^{11}x \sqrt{-g} F_{MN} F^{MN},
\]
where \(F_{MN} = \nabla_M A_N - \nabla_N A_M\). In order to perform the dimensional reduction for
the \(U(1)\) field action in \(dS_4 \times S^d\) spacetime, it is convenient to expand in the vector
harmonics on \(S^7\) as
\[
\tilde{A}_M dx^M = b_0^{-\frac{d}{2}} \sum_{l, m} \left[ A_{\mu lm}^{(4)} Y_{lm}^{(7)} dx^\mu + \left\{ A_{(T) lm}^{(4)} \left( V_{(T) lm}^{(7)} \right) + A_{(L) lm}^{(4)} \left( V_{(L) lm}^{(7)} \right) \right\} dx^i \right],
\]
where \(A_{\mu lm}^{(4)}, A_{(T) lm}^{(4)}\) and \(A_{(L) lm}^{(4)}\) depend only on the four-dimensional coordinates \(x^\mu\),
and \(Y_{lm}^{(7)}, V_{(T) lm}^{(7)}\) and \(V_{(L) lm}^{(7)}\) are the scalar harmonics, transverse vector harmonics
and longitudinal vector harmonics, respectively. [see the Appendix B of Ref. 36)] for
definitions and properties of there harmonics.) The summations are taken over values
of \(l\) satisfying \(l \geq 0\) for the scalar harmonics and \(l \geq 1\) for the vector harmonics. This
decomposition is unique and orthogonal. As \(A_{(L) lm}^{(4)}\) represents the gauge degrees of
freedom, we eliminate the longitudinal mode for $A_M$ after the gauge fixing. [see also
the appendix in the Ref. 36].] By substituting the expansion (3.38) into the action
(3.37) and using the conformal transformation (2.4), we obtain the four-dimensional
effective action

$$I_{U(1)} = I_{(T)} + I_{(V)},$$

$$I_{(T)} = -\frac{1}{2} \int dx^4 \sqrt{-g} \left[ e^{-2\sigma_0} g^{\mu\nu} \partial_\mu A_{(T)} \partial_\nu A_{(T)} + e^{-11\sigma_0} M_{(T)}^2 A_{(T)}^2 \right],$$

$$I_{(V)} = -\int dx^4 \sqrt{-g} \left[ \frac{1}{4} e^{-7\sigma_0} g^{\mu\rho} g^{\nu\sigma} F_{\mu\rho} F_{\nu\sigma} + \frac{1}{2} e^{-2\sigma_0} M_{(V)}^2 g^{\mu\nu} A_\mu A_\nu \right].$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The terms containing $M_{(T)}^2$ and $M_{(V)}^2$ in the
$U(1)$ gauge field action, given by

$$M_{(T)}^2 = \frac{l(l+6)}{b_0^2} + 5$$

$$M_{(V)}^2 = \frac{l(l+6)}{b_0^2},$$

denote the mass of the four-dimensional scalar field $A_{(T)}$ and the vector field $A_\mu$,
respectively. The vector field $A_\mu$ can be decomposed with Hodge decomposition,
which is the same as $A_M$. We expand the field $A_\mu$ in terms of harmonics in the
four-dimensional de Sitter spacetime of constant curvature as

$$A_\mu^{(4)} dx^\mu = \left( A_{(T)\mu}^{(4)} + A_{(L)\mu}^{(4)} \right) dx^\mu.$$  (3.41)

Here, we consider a gauge fixing for the four-dimensional $U(1)$ field $A_\mu^{(4)}$. A simple
and natural gauge choice for the quantization of this system is the Lorentz gauge,
$\nabla_\mu A_\mu = 0$. The total action (3.39) can be affected by the addition of the gauge
fixing action

$$\delta I = \frac{1}{2} \alpha \int d^4x \sqrt{-g} (\nabla_\mu A^\mu)^2,$$  (3.42)

where $\alpha$ is a positive constant. The De Witt-Faddeev-Popov ghost factor $\Delta$ is defined by

$$\Delta \int D\eta \exp(-\delta I) = 1,$$  (3.43)

where $D\eta$ is the measure on the gauge group. The ghost factor must be independent
of the gauge field. We can define the elements of the gauge group as $\eta = \exp(i\omega(x))$.

The field $\omega(x)$ is written in terms of orthogonal scalar eigenfunctions as $\omega(x) = \sum_{n=0}^{\infty} \omega_n \chi_n(x)$. These elements transform the gauge fields $A_\mu$ into $A_\mu + \partial_\mu \omega(x)$. The
action is invariant under this $U(1)$ gauge transformation. The measure of the $U(1)$
gauge group $D\eta$ is then written

$$D\eta = \prod_{n=1}^{\infty} \mu^2 d\omega_n.$$  (3.44)

Using these gauge elements, the ghost factor $\Delta$ is explicitly expressed as

$$\Delta = \mu^{-2} a^{-2} \det \left( \frac{1}{2} \mu^{-1} a^2 Q \right).$$  (3.45)
where the above action, we find the functional $Z$ where we ignore terms of higher than more than quadratic order in the field. From where we have absorbed a numerical constant into $\mu$ kinetic terms of quantum correction to vector field due to the coupling dilaton. After integrating the in the path integral is expressed by

\[
Z = \Delta \int \mathcal{D}A^{(4)}_{(T)} \mathcal{D}A^{(4)}_{(T) \mu} \mathcal{D}q \exp \left\{ - \left( I \left[ A^{(4)}_{(T)}, A^{(4)}_{(T) \mu} \right] + \delta I \right) \right\} ,
\]  
(3.46)

Using the action given in Eqs. (3.39) and (3.42), we compute the quantum correction for the $U(1)$ gauge field. However, this action is not useful for the calculation of quantum correction to vector field due to the coupling dilaton. After integrating the

\[
A^{(4)}_{(T) \mu}
\]

for scalar field $U^{(1)}$, and $A^{(4)}_{(T) \mu}$ by parts, the actions are given by

\[
I_{(V)} = - \sum_{l, m} \frac{1}{2} \int d^4x \sqrt{-g} A^{(4) \mu}_{(T) lm} \left\{ \frac{1}{2} (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) e^{7 \frac{\phi}{\sigma_0}} 
- e^{7 \frac{\phi}{\sigma_0}} \Delta_{\mu\nu} + g_{\mu\nu} e^{-2 \frac{\phi}{\sigma_0}} M^2_{(V)} \right\} A^{(4) \nu}_{(T) lm} ,
\]  
(3.47)

\[
I_{(T)} = - \sum_{l, m} \frac{1}{2} \int d^4x \sqrt{-g} A^{(4) \mu}_{(T) lm} \left\{ \frac{1}{2} e^{-2 \frac{\phi}{\sigma_0}} - e^{-2 \frac{\phi}{\sigma_0}} \Box + M^2_{(T)} \right\} A^{(4) \nu}_{(T) lm} ,
\]  
(3.48)

where $\Delta_{\mu\nu} = g_{\mu\nu} \Box + R_{\mu\nu}$. The total action can be expressed by

\[
I + \delta I = \frac{1}{2} \int d^4x \left[ A^{(4) \mu}_{(L) lm} (\alpha \nabla_{\mu} \nabla_{\nu}) A^{(4) \nu}_{(L) lm} 
+ A^{(4) \mu}_{(T) lm} \left\{ \frac{1}{2} (g_{\mu\nu} \Box - \nabla_{\mu} \nabla_{\nu}) e^{-7 \frac{\phi}{\sigma_0}} - e^{-7 \frac{\phi}{\sigma_0}} \Delta_{\mu\nu} + g_{\mu\nu} e^{-2 \frac{\phi}{\sigma_0}} M^2_{(V)} \right\} A^{(4) \nu}_{(T) lm} 
+ A^{(4) \mu}_{(T) lm} \left\{ \frac{1}{2} e^{-2 \frac{\phi}{\sigma_0}} - e^{-2 \frac{\phi}{\sigma_0}} \Box + M^2_{(T)} \right\} A^{(4) \nu}_{(T) lm} \right) ,
\]  
(3.49)

where we ignore terms of higher than more than quadratic order in the field. From the above action, we find the functional $Z$ to be

\[
Z = \mu^{-2} a^{-2} \det \left( \alpha^{1/2} \mu^2 Q_{(L)} \right) \det \left( \mu^2 Q_{(T)} \right)^{1/2} \det \left( \mu^2 Q_{(S)} \right)^{1/2} \det \left( \alpha \mu^2 Q_{(L)} \right)^{-1/2} 
= \mu^{-2} a^{-2} \det \left( \mu^2 Q_{(L)} \right)^{1/2} \det \left( \mu^2 Q_{(T)} \right)^{1/2} \det \left( \mu^2 Q_{(S)} \right)^{1/2} ,
\]  
(3.50)

where we have absorbed a numerical constant into $\mu$, and $Q_{(S)}$ denotes the operator for scalar field $A_{(T)}$, and $Q_{(T)}$ and $Q_{(L)}$ denote the operators for vector fields $A_{(T) \mu}$ and $A_{(L) \mu}$, respectively. Note that $a$ has cancelled out in $z$. Using the zeta function regularization, we can compute the effective potential for the vector field in the $dS_4 \times S^7$ background. We do not consider the contribution from $A^{(4) \mu}_{(L) lm}$, because it is not dominant term in the effective potential.

(i) 1-loop quantum correction from $A^{(4) \mu}_{(T)}$

First, we consider the field $A^{(4) \mu}_{(T)}$. This is a scalar field on $dS_4$ [see Eq. (3.38)]. We define the generalized zeta function

\[
\zeta_T(z) \equiv \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} D_{(T)}(l) d_{(T)}(l') \left\{ e^{-2 \frac{\phi}{\sigma_0}} \lambda_{(T)}^2 (l') a^{-2} + A_{(T)}^2 (l) e^{-11 \frac{\phi}{\sigma_0}} \right\}^{-z} ,
\]  
(3.51)
where \(d_{(T)}(l')\) and \(D_{(T)}(l)\), given by

\[
D_{(T)}(l) = \frac{(2l + 5)(l + 4)!}{720l}, \quad d_{(T)}(l') = \frac{(2l + 3)(l + 2)!}{6l},
\]

(3.52)
denote the degeneracies of \(A_{(T)}^{(4)}\) in \(dS^4\) and \(S^7\), respectively, and \(A^2_{(T)}(l)\) and \(\lambda^2_{(T)}(l')\), given by

\[
A^2_{(T)}(l) = \frac{l(l + 6) + 5}{b_0^2}, \quad \lambda^2_{(T)}(l') = \frac{1}{2}l(l + 3),
\]

(3.53)
denote the eigenvalues for \(A_{(T)}^{(4)}\) in \(dS^4\) and \(S^7\), respectively. Using the same strategy as in the case of the scalar and spinor fields, we can regularize the \(\zeta\) function for the vector field. Then, we can compute \(\zeta'_T(0)\) and \(\zeta_T(0)\) and pick up the dominant term in \(\zeta'_T(0)\) and \(\zeta_T(0)\). In this way, we find that the dominant term in the effective potential for \(A_{(T)}^{(4)}\) is proportional to \(b^{-18}\) for \(A_{(T)}^{(4)}\).

(ii) 1-loop calculation for \(A_{(T)}^{(4)}\)

Next, we compute the quantum effect from \(A_{(T)}^{(4)}\). This consists of the vector field components in \(dS^4\) [see Eq. (3.38)]. We define the zeta function as

\[
\zeta_V(z) = \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} D_{(V)}(l) d_{(V)}(l') \left\{ e^{-\frac{7}{\sigma_0} \lambda^2_{(V)}(l') a^{-2}} + A^2_{(V)}(l) b^{-2} \right\}^{-z},
\]

(3.54)
where \(D_{(V)}(l)\) and \(d_{(V)}(l)\), given by

\[
D_{(V)}(l) = \frac{(2l + 5)(l + 4)!}{5l}, \quad d_{(V)}(l') = \frac{(2l + 3)(l + 2)!}{3l},
\]

(3.55)
denote the degeneracies of the vector fields in \(S^7\) and \(dS^4\), and \(A^2_{(V)}(l)\) and \(\lambda^2_{(V)}(l')\), given by

\[
A^2_{(V)}(l) = \frac{1}{2}l(l + 6), \quad \lambda^2_{(V)}(l') = l(l + 3) - \frac{5}{2},
\]

(3.56)
denote the eigenvalues for the vector fields in \(S^7\) and \(dS^4\), respectively. Using a formula in Appendix C, we see that \(\zeta_V(0)\) is also vanishing, and the dominate term in \(\zeta'_V(0)\) is proportional to \(b^{-18}\). Finally, we obtain the quantum correction for the \(U(1)\) vector field as

\[
V_{\text{eff}}(\sigma) = V_0(\sigma) - \frac{1}{2\Omega_{\text{vol}}} \left[ \{\zeta'_T(0) + \zeta'_V(0)\} + \{\zeta_T(0) + \zeta_V(0)\} \ln(\mu^2a^2) \right]
\]

\[
= \frac{1}{\kappa^2} e^{-\frac{7}{\sigma_0}} - \frac{21K}{\kappa^2 b_0^2} e^{-9\frac{\sigma}{\sigma_0}} + \frac{2.1974 \times 10^{-3}}{b_0^4} e^{-18\frac{\sigma}{\sigma_0}},
\]

(3.57)
where \(V_0(\sigma)\) is given by Eq. (3.18).
3.4.4. Tensor field

Next, we calculate the quantum correction of the gravitational field. Although a full quantum theory of gravity does not exist, the quantum gravitational effect at the 1-loop level is important for the Kaluza-Klein theories, because the scale of the internal spacetime in the Kaluza-Klein picture is assumed to be not far larger than the Planck length.

We consider a gravitational perturbation \( h_{MN} \) around a background metric \( \bar{g}_{MN}^{(0)} \), which we specify below:

\[
\bar{g}_{MN} = \bar{g}_{MN}^{(0)} + h_{MN}.
\]

First, by substituting Eq. (3.58) into Eq. (2.1), we obtain the perturbed Einstein-Hilbert action as

\[
I_{EH} = \frac{1}{2k_0^2} \int d^4x \sqrt{-\bar{g}^{(0)}} \left[ \bar{R}^{(0)} - 2\bar{\Lambda} - h^{MN} \left( \bar{R}_{MN}^{(0)} - \frac{1}{2} \bar{R}^{(0)} \bar{g}_{MN}^{(0)} + \bar{\Lambda} \bar{g}_{MN}^{(0)} \right) 
+ \frac{1}{8} \left( h^2 - 2h_{MN} \bar{h}_{MN}^{(0)} \right) \bar{R}^{(0)} + \frac{1}{2} \left( 2h_{MM'}h_{M'} - h_{MN}^{(0)} \right) \bar{R}_{MN}^{(0)} 
+ \frac{1}{4} \left( h_{MN}^{(0)} ; M' \right) \left( 2h_{M':N}^{(0)} - h_{MN}^{(0)} ; M' \right) + h_{MN} \left( h_{MN}^{(0)} - 2h_{MN}^{(0)} \right) \right] 
- \frac{1}{4} \left( h^2 - 2h_{MN} \bar{h}_{MN}^{(0)} \right) + O(h^3),
\]

where \( ; \) denotes the covariant derivative compatible with \( g_{MN}^{(0)} \), and \( \bar{R}_{MN} \) and \( \bar{R}^{(0)} \) are the Ricci tensor and scalar constructed from \( \bar{g}_{MN}^{(0)} \). For the background geometry \( dS_4 \times S^7 \), we compactify this action on the seven-dimensional sphere \( S^7 \).

The gravitational perturbation \( h_{MN} \) in this particular background can be expanded in harmonics on \( S^7 \) as

\[
h_{MN}dx^Mdx^N = \sum_{l,m} \left[ h_{\mu\nu}^{lm}Y_{lm}dx^\mu dx^\nu + 2\{h_{(T)\mu}^{lm}(V_{(T)lm})_i + h_{(L)\mu}^{lm}(V_{(L)lm})_i\}dx^\mu dx^i \right.
+ \{h_{(T)ij}^{lm}(T_{(T)lm}) + h_{(LT)ij}^{lm}(T_{(LT)lm})_i + h_{(LL)ij}^{lm}(T_{(LL)lm})_i \}
+ \{h_{(Y)ij}^{lm}(T_{(Y)lm})_i \}dx^i dx^j \right],
\]

where the \( Y_{lm} \) are the scalar harmonic functions, \( V_{(T)lm} \) and \( V_{(L)lm} \) are the vector harmonics, and \( T_{(T)lm} \), \( T_{(LT)lm} \), and \( T_{(LL)lm} \) are the tensor harmonics. Here, the coefficients \( h_{\mu\nu}^{lm} \), \( h_{(T)\mu}^{lm} \), \( h_{(L)\mu}^{lm} \), \( h_{(T)ij}^{lm} \), \( h_{(LT)ij}^{lm} \), \( h_{(LL)ij}^{lm} \), and \( h_{(Y)ij}^{lm} \) depend only on the four-dimensional coordinates \( x^\mu \), while the harmonics depend only on the coordinates \( x^i \) on \( S^7 \).

Although the above expression of \( h_{MN} \) includes many terms, some of them represent degrees of freedom of coordinate transformations. In fact, it is shown in appendix of Ref. 36) that, after gauge-fixing and redefining \( g_{\mu\nu} \) and \( b \), the perturbation \( h_{MN} \) can be expressed as

\[
h_{MN}dx^Mdx^N = \sum_{l,m} \left[ h_{\mu\nu}^{lm}Y_{lm}dx^\mu dx^\nu + 2h_{(T)\mu}^{lm}(V_{(T)lm})_i dx^\mu dx^i \right]
+ \{h_{(T)ij}^{lm}(T_{(T)lm}) + h_{(LT)ij}^{lm}(T_{(LT)lm})_i + h_{(LL)ij}^{lm}(T_{(LL)lm})_i \}
+ \{h_{(Y)ij}^{lm}(T_{(Y)lm})_i \}dx^i dx^j ,
\]
\[ + \left\{ h_{(T)lm}^{(T)}(T_{(T)lm})_{ij} + h_{(Y)lm}^{(Y)}(T_{(Y)lm})_{ij} \right\} dx^i dx^j, \] (3.61)

where the summations are taken over values \( l \) satisfying \( l \geq 1 \) for the scalar and vector harmonics and \( l \geq 2 \) for the tensor harmonics.

Finally, by substituting this expression into Eq. (3.59), we obtain the action

\[ I_{EH} = I^{(0)} + I^{(1)} + I^{(2)} + O(h^3), \] (3.62)

where

\[ I^{(0)} = \int d^4x \sqrt{-g^{(0)}} \left[ \frac{1}{2\kappa^2} R^{(0)} - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - U_0(\sigma) \right], \] (3.63)

\[ I^{(2)} = \sum_{l,m} \int d^4x \sqrt{-g^{(0)}} \left( L^{(T)}_{lm} + L^{(Y)}_{lm} \right), \] (3.64)

and \( I^{(1)} \) is linear in \( h \). When the total action of the system is considered, \( I^{(1)} \) should be canceled by other linear terms in the total action, because of the equations of motion. Finally, the Lagrangian densities \( L^{(T,V,Y)}_{lm} \) are given by

\[ L^{(T)}_{lm} = -\frac{1}{2} e^{-4\sigma/\sigma_0} g^{\mu\nu} \partial_\mu \chi^{lm} \partial_\nu \chi^{lm} - \frac{1}{2} M^2(\chi) \chi^{lm} \chi^{lm}, \] (3.65)

\[ L^{(V)}_{lm} = L^{(V)}_{lm} [h^{lm}_{(T)}, \chi^{lm}], \] (3.66)

\[ L^{(Y)}_{lm} = L^{(Y)}_{lm} [h^{lm}_{\mu\nu}, h^{lm}_{Y}], \] (3.67)

where

\[ M^2(\chi) \equiv e^{-13\sigma/\sigma_0} \{ l(l + 6) + 30 \} b_0^{-2} + e^{-4\sigma/\sigma_0} \left[ 11 \nabla^2 \left( \frac{\sigma}{\sigma_0} \right) - \frac{55}{2} \left\{ \nabla \left( \frac{\sigma}{\sigma_0} \right) \right\}^2 + \left\{ R^{(0)} - 2 e^{-7\sigma/\sigma_0} \bar{\Lambda} \right\} \right], \] (3.68)

\[ \chi^{lm} \equiv \sqrt{\frac{b_0^{-4}}{2d^2 + 2\pi \kappa^2}} h^{lm}_{(T)}. \quad (l \geq 2) \] (3.69)

[Since hereafter we analyze \( h^{lm}_{(T)} \) (or \( \chi^{lm} \)) only, we have not included the explicit forms of \( L^{(V)}_{lm} \) and \( L^{(Y)}_{lm} \) here.] Hence, up to the second order with respect to the Kaluza-Klein modes, \( h^{lm}_{(T)} \) (or \( \chi^{lm} \)) is decoupled from all other Kaluza-Klein modes.

In this paper, for simplicity, we investigate the quantum effects of \( \chi^{lm} \) (or \( h^{lm}_{(T)} \)) only.

Hereafter, we investigate the dynamics of the field \( \chi^{lm} \) coupled to the dilaton field \( \sigma \). These fields are described by the action

\[ I = I_\sigma + I_\chi, \] (3.70)

where

\[ I_\sigma = -\int d^4x \sqrt{-\bar{g}} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma + U_0(\sigma) \right], \]

\[ I_\chi = -\frac{1}{2} \int d^4x \sqrt{-\bar{g}} \left[ e^{-4\sigma/\sigma_0} g^{\mu\nu} \partial_\mu \chi^{lm} \partial_\nu \chi^{lm} + A^{(T)}_{lm} \chi^{lm} \chi^{lm} \right]. \] (3.71)
Here, we mention that in the total action there may be other terms that are second order in $\chi_{lm}$.

Next, we rewrite (3.71) in a useful form in order to analyze the quantum correction due to the coupling of the dilaton to the kinetic term of $\chi$. After the some calculations, we obtain

$$I_\chi = -\frac{1}{2} \int d^4x \sqrt{-\bar{g}(0)} \chi_{lm} \left[ \frac{1}{2} \Box \left( e^{-4\phi_0} - e^{-4\phi_0} \Box + M_\chi^2 \right) \right] \chi_{lm} .$$

(3.72)

As in the case of the scalar field, we define the zeta function in order to compute the 1-loop quantum correction as

$$\zeta_\chi(s) = \sum_{l=2}^{\infty} \sum_{l'=0}^{\infty} D_\chi(l) d_\chi(l') \left[ \frac{\lambda_\chi(l')}{a^2} e^{-\frac{4\phi_0}{\sigma_0}} + e^{-\frac{13\phi_0}{\sigma_0}} \Lambda_\chi^2 \right] ,$$

(3.73)

where $D_\chi(l)$ and $d_\chi(l')$ are given by

$$D_\chi(l) = \frac{40(l-1)(l+2)(l+3)^2(l+4)(l+7)}{7!} ,$$

(3.74)

$$d_\chi(l') = \frac{1}{6} (l+1)(l+4)(2l+3) ,$$

(3.75)

and $\lambda_\chi(l')$ and $\Lambda_\chi(l)$ are expressed as

$$\Lambda_\chi^2(l) = \frac{l(l+6)+30}{b_0^2} + e^{9\phi_0} \left\{ \frac{12}{a^2} - 2A e^{-7\phi_0} \right\} ,$$

(3.76)

$$\lambda_\chi^2(l') = \frac{1}{2} (l+3) .$$

(3.77)

Using the results given in Appendix A, we can compute the values of $\zeta_\chi(s)$ and $\zeta_\chi'(s)$ at $s = 0$. Under the assumption that $a \gg b_0$, most integrals in $\zeta_\chi(s)$ can be calculated analytically, or at least they can be put into simple forms, that can be computed numerically. After tedious calculations, we obtain the expression

$$V_{eff}(\sigma) = \frac{\bar{A}}{\kappa^2} e^{-\frac{7\phi_0}{\sigma_0}} - \frac{21}{\kappa^2 b_0^2} e^{-9\phi_0} + \frac{5.93731 \times 10^{-5}}{b_0^4} e^{-18\phi_0} .$$

(3.78)

3.5. Quantum correction in $AdS_4 \times S^7$

As seen in the last section, the dilaton potential energy is negative at the minimum. Then, we have to calculate the quantum correction in the $AdS_4 \times S^7$ background and derive the effective potential.

Because the procedure to obtain the effective potential in $AdS_4 \times S^7$, is almost the same as that in $dS_4 \times S^7$, except for the zeta functions, we do not repeat it here. Instead, showing how to regularize the zeta functions in $AdS_4 \times S^7$ in Appendix D or Ref. 39), we just summarize our results as follows.

The effective potentials for bosonic fields (the scalar field $\bar{\phi}$, the $U(1)$ gauge field $A_M$, and scalar mode of gravitational field $h^{lm}_{(T)}$) are given by

$$V_{eff}(\sigma) = \frac{\bar{A}}{\kappa^2} e^{-\frac{7\phi_0}{\sigma_0}} - \frac{10}{\kappa^2 b_0^2} e^{-9\phi_0} + \frac{D_c}{b_0^4} e^{-18\phi_0} ,$$

(3.79)
where $D_c$ is given by

\begin{align*}
D_c &= 6.03329 \times 10^{-3} \quad \text{for a scalar field}, \\
&= -7.21715 \times 10^{-6} \quad \text{for a Dirac spinor field}, \\
&= 9.97116 \times 10^{-3} \quad \text{for a gauge field}, \\
&= 8.21713 \times 10^{-5} \quad \text{for a gravitational field}.
\end{align*}

(3.80)

3.6. **Dilaton dynamics and stabilization of $S^7$**

We expect that the effective potential can be used to judge the stability of the solution with respect to the spacetime independent dilaton or contractions of the compact manifold. If the extra dimensions ($S^7$) are stable, the dilaton effective potential must possess a minimum or at least a local minimum. From (3.24), (3.36), (3.57) and (3.78), the effective potential in $dS_4 \times S^7$ is given by

\[
V_{\text{eff}}(\sigma) = \frac{\bar{\Lambda}}{\kappa^2} e^{-7 \sigma / \sigma_0} - \frac{21}{\kappa^2 b_0^2} e^{-9 \sigma / \sigma_0} + C e^{-18 \sigma / \sigma_0},
\]

(3.81)

where $C$ is given by

\[
C = N_S C_S + N_F C_F + N_V C_V + C_\chi,
\]

(3.82)

with $N_S$, $N_F$, and $N_V$ the numbers of the scalar, Dirac spinor and vector field.

![Graph](image)

Fig. 1. The dilaton effective potential in Eq. (3.81). Note that the local minimum of the dilaton potential is $b = 72.4691 l_p$. Here $M_{4p}$ denotes the four-dimensional Planck mass.
As shown in Fig. 1, we find the a minimum of the potential at $72.4691l_p$ if we take $N_S = N_V = N_F = 10^5, A = 2.62723 \times 10^{-3} M_P^2$, and $b_0 = 100 l_p$, where $M_p$ and $l_p$ denote the Planck mass and Planck scale, respectively. We have also calculated the back reaction due to changing the background geometry. Because the effective potential is zero at $b = 0.873448 b_0$, the background geometry is the four-dimensional Minkowski spacetime $M_4$. Thus we need to calculate the quantum correction in the $M_4$ fixed background. However, the effective potential calculated in $M_4$ is also negative at this minimum point. That is, the effective potential immediately becomes negative. Then, we have to calculate the quantum correction in the negative cosmological constant. We obtain the effective potential derived in the spacetime with negative cosmological constant,\cite{39} which is the potential minimum at $b = 0.669486 b_0$.

Note that the geometry is $AdS_4$ at the local minimum of the potential.

Finally, we discuss the difference between $M_4 \times S^7$ and $(A)dS_4 \times S^7$. The effective potential of the scalar and spinor fields in $M_4 \times S^d$ was calculated by Candelas and Weinberg\cite{21} to be

$$V_{\text{eff}}(\sigma) = U(\sigma) + \frac{f_M}{b_0^2} e^{-2(d+2)\frac{\sigma}{\sigma_0}},$$

(3.83)

where $f_M$ is a numerical coefficient. Similarly, the effective potential in $dS_4 \times S^d$ is given by

$$V_{\text{eff}}(\sigma) = U(\sigma) + \frac{f_d}{b_0^2} e^{-2(d+2)\frac{\sigma}{\sigma_0}}.$$  

(3.84)

Thus, we see that there is no difference between the two cases. This is the reason why the condition $a \gg b_0$ in $dS_4 \times S^d$ is assumed in the calculation of the effective potential. The full result is the given by

$$V_{\text{eff}}(\sigma) = U(\sigma) + \frac{f_1}{b_0^4} e^{-2(d+2)\frac{\sigma}{\sigma_0}} + \frac{f_2}{a^2 b_0^2} e^{-(d+2)\frac{\sigma}{\sigma_0}}.$$  

(3.85)

Because we make the assumption

$$\left(\frac{b_0}{a}\right) 10^{-8} \ll 1 \quad \rightarrow \quad \frac{f_2}{a^2 b_0^2} \ll \frac{f_1}{b_0^4}, \quad (f_1 \sim f_2) \quad (3.86)$$

the effective potential in $dS_4 \times S^d$ is given by Eq. (3.85). In the Minkowski limit $a \to \infty$, the effective potential in the $dS_4 \times S^d$ is equal to the form of the effective potential in $M_4 \times S^7$. The effective action and effective potential depend on the parameter $b_0$ (the initial value of $b$). This is the reason why we have calculated the quantum correction of the various fields in the $dS_4 \times S^7$ fixed background. The four-dimensional spacetime for $b = b_0$ becomes $dS_4$. Then, the quantum correction for $b \neq b_0$ generally cannot be calculated, because it is difficult to solve the field equation in the unknown spacetime $\hat{g}_{\mu \nu}$. By the rescaling the matter field, however, the calculation of the effective potential can be performed with the fixed background $dS_4 \times S^7$. Therefore, the final results depend on the parameter $b_0$. 


§4. Dilaton stabilization with eleven-dimensional supergravity

In this section, we consider dilaton stabilization in an eleven-dimensional supergravity model. This model is considered to be a possible theory to provide a low energy description of M-theory. We consider the model in which the background geometry is a product of $dS_4$ and $S^7$ and assume that the radius of $S^7$ is microscopically small. Although the supersymmetry is broken in $dS_4$, this causes no problem from the cosmological point of view.

We consider the bosonic part of the action for $N = 1$ supergravity in eleven-dimensional spacetime,

\[ I = \int d^{11}x \sqrt{-g} \left\{ \frac{1}{2 \kappa^2} \bar{R} - \frac{1}{48} F_{MNPQ} F^{MNPQ} \right\} + \frac{\sqrt{2}}{6 \cdot (4!)^2} \epsilon^{M_1 \cdots M_9} F_{M_1 \cdots M_4} F_{M_5 \cdots M_8} A_{M_9 \cdots M_{11}} \right\}, \tag{4.1} \]

where $\epsilon^{M_1 \cdots M_9}$ is the Levi-Civita symbol. The background line element is assumed to take the form given in (2.2). We assume that the gauge field strength has the Freund-Rubin form, proportional to Levi-Civita tensor in four-dimensional spacetime:

\[ F_{M_1 \cdots M_4} = \begin{cases} \left( f / \sqrt{\Omega} \right) \epsilon_{M_1 \cdots M_4}, & M_1 = \mu_1, \cdots, M_4 = \mu_4 \\ 0, & \text{otherwise} \end{cases} \tag{4.2} \]

\[ F_{\mu_1 \mu_2 \mu_3 \mu_4} F^{\mu_1 \mu_2 \mu_3 \mu_4} \equiv 96 \Lambda. \tag{4.3} \]

Here, $f$ is the constant and $(\mu_1, \cdots, \mu_4) = 0, \cdots, 3$. Then, the action is given by

\[ I = \int d^{11}x \sqrt{-g} \left( \frac{1}{2 \kappa^2} \bar{R} - 2 \Lambda \right), \tag{4.4} \]

which has the same form as (2.1). Now we consider the gravitational perturbation (3.58) and quantum fluctuations of the scalar mode of $h_{MN}$. The procedure for calculation of the quantum correction is the same as (D) in §3.4. Therefore the dilaton potential is given by (3.78). The local minimum for the dilaton potential is located at $b \simeq 0.03 b_0$. Although the potential energy of the dilaton is initially positive, the dilaton settle down to the minimum of its potential, which is a negative value.

§5. Conclusion

In this paper, we have investigated the stabilization of the extra dimension in a higher-dimensional theory. In particular, we have focused on the eleven-dimensional Kaluza-Klein and supergravity. As the source of the stabilization, we have considered a quantum effect associated with background matter field. We have studied the stability for the compactification, which is the product spacetime $dS_4 \times S^7$. We have considered the case that the scale of $dS_4$ is far larger than that of the extra dimension and demonstrated the compactification model, in which the dilaton potential has a
local minimum with respect to the quantum correction of the matter fields. The part coming from the quantum effects in the dilaton effective potential is dominant for $b \to l_p$, while the cosmological constant in it is dominant for large $b$. The dilaton field in our model does not evolve forever but, rather, settles down to its potential minimum. Consequently, the extra dimension is asymptotically almost time independent in $dS_4$ spacetime, and its scale is $30 \ l_p$. Furthermore, we have found that the dilaton potential energy is negative at its minimum. Because the dilaton potential energy is the effective cosmological constant in four dimensions, the $dS_4$ spacetime is transformed to a spacetime with negative cosmological constant after the dilaton has settled down at the local minimum of the dilaton potential. Our model describes the situation in which a geometrical change has occurred over the whole spacetime, because the bulk cosmological constant $\Lambda$ and the dilaton field are homogeneous. This mechanism is applicable to a model in which the background geometry changes from $dS_5$ to $AdS_5$. We will discuss the dynamics of this change in another paper.\(^{39)}\)

In the very early universe, the scale of $dS_4$ may have been on the same orders as that of $S^d$. Therefore, the dynamics of $b$ and the behavior of its effective potential are cosmologically very interesting. However, the integral in $\zeta(s)$ cannot be naively calculated in the case $a \simeq b_0$. This point will be considered in a future work.

Acknowledgements

We would like to thank K. Maeda, T. Kubota, O. Yasuda and N. Sakai for valuable comments. We would also like to thank M. Sakagami and J. Soda for continuing encouragement and useful comments.

Appendix A

Zeta Function Regularization for a Scalar Field

In this appendix, we present a method to regularize the zeta function for an odd total number of dimensions that has the product geometry $dS_n \times S^d$. As mentioned in §3.4, this geometry is identical to $S^n \times S^d$ in the Euclidean spacetime after performing the Wick rotation in $dS^n$. It can easily seen that this method is applicable for an even number of dimensions.\(^{*)}\) These formulations are similar to those proposed by Kikkawa et al.\(^{29)}\) We employ the method and notation used by them. First, we define a generalized zeta function for the scalar fields in $S^n \times S^d$ as

\[
\zeta(s) = \sum_{l'=0}^{\infty} \sum_{l=0}^{\infty} \frac{(l'+n-2)! \ (l+d-2)! \ (2l'+n-1) \ (2l+d-1)}{(n-1)! \ (d-1)! \ l'! \ l!}
\]

\(^{*)}\) Strictly speaking, in an even number of dimensions, a conformal anomaly arises, and the arguments become more complicated.
where we assume that \( n \) is an even number and \( d \) is an odd number. Using \( D = (d - 1)/2 \) and \( N = n/2 \) instead of \( d \) and \( n \) and the running variables \( L = l + D \) and \( L' = l' + N \), we rewrite (A.1) as

\[
\zeta(s) = \sum_{L' = N}^{\infty} D_n \left( L' - \frac{1}{2} \right) \sum_{L = D} \frac{A_n \left( L' - \frac{1}{2} \right)}{a^2} + \frac{A_d(L)}{b_0^2} \left( \frac{b_0}{b} \right)^{d+2} \right)^{-s},
\]

(A.2)

where the variables are given by

\[
D_n \left( L' - \frac{1}{2} \right) = \frac{2L' - 1}{(2N - 1)!} \left\{ \left( L' - \frac{1}{2} \right)^2 - \left( N - \frac{3}{2} \right)^2 \right\} \cdots \left\{ \left( L' - \frac{1}{2} \right)^2 - \left( \frac{1}{2} \right)^2 \right\},
\]

\[
D_d(L) = \frac{2L^2}{(2D)!} \{ L^2 - (D - 1)^2 \} \cdots \{ L^2 - 1 \},
\]

\[
A_n \left( L' - \frac{1}{2} \right) = \left( L' - \frac{1}{2} \right)^2 - \left( N - \frac{1}{2} \right)^2,
\]

\[
A_d(L) = L^2 - D^2.
\]

(A.3)

Now, we replace the infinite mode sum over \( L' \) by complex integration. The generalized zeta function (A.2) is then

\[
\zeta(s) = (a)^{2s} \sum_{L = D}^{\infty} D_d(L) \frac{-i}{2} \int_{C_1} dz \tan(\pi z) D_n(z) \times \left\{ z^2 + \left( \frac{a}{b_0} \right)^2 \frac{b_0}{b} \right\}^{d+2} \left( N - \frac{1}{2} \right)^2 \right\}^{-s},
\]

(A.4)

where the contour \( C_1 \) in the complex plane is showed in Fig. 2. In order to avoid the singularity of \( z \) in the integrand, it is convenient to divide the sum over \( L \) into two parts,

\[
\zeta(s) = Z(s) + W(s),
\]

(A.5)

where

\[
Z(s) = (a)^{2s} \sum_{L = D}^{L_0} D_d(L) \frac{-i}{2} \int_{C_1} dz \tan(\pi z) D_n(z) \left( z^2 - A_L^2 \right)^{-s},
\]

\[
W(s) = (a)^{2s} \sum_{L = L_0 + 1}^{\infty} D_d(L) \frac{-i}{2} \int_{C_1} dz \tan(\pi z) D_n(z) \left( z^2 + B_L^2 \right)^{-s},
\]

(A.6)

with

\[
A_L^2 = \left( N - \frac{1}{2} \right)^2 - \left( \frac{a}{b_0} \right)^2 \frac{b_0}{b} A_d(L),
\]

\[
B_L^2 = -A_L^2.
\]

(A.7)
Here, $A_L^2$ and $B_L^2$ are positive for $D \leq L < L_0$ and $L_0 \leq L$, respectively. $L_0$ is defined as the largest integer smaller than or equal to

$$L_0 = \left\lfloor \sqrt{D^2 + \left(\frac{b_0}{a}\right)^2 \left(\frac{b}{b_0}\right)^{d+2} \left(N - \frac{1}{2}\right)^2} \right\rfloor. \quad (A.8)$$

We consider the function $Z(s)$, which is a sum over the $L$ covers from $D$ to $L_0$. There are branch points at $z = \pm A_L$. Now, we replace the contour $C_1$ so as to move in a direction parallel to the imaginary axis in order to deal with the poles of $\tan(\pi z)$ (see Fig. 2). Because the displacing contour $C_2$ is defined as running above the cut associated with $z = \pm A_L$, the result for $Z(s)$ is given by

$$Z(s) = (a)^{2s} \sum_{L=D}^{L_0} D_d(L) \times \left\{ \frac{1}{2} \left( e^{-i\pi s} + e^{i\pi s} \right) \int_0^\infty dx \ i \ D_n(ix) \left(x^2 + A_L^2\right)^{-s} \tanh(\pi x) \right. \right.$$

$$- \left. i \frac{1}{2} e^{i\pi s} \int_0^{A_L} dx \ \tan(\pi x - i\epsilon) D_n(x) \left(A_L^2 - x^2\right)^{-s} \right.$$  

$$+ \left. i \frac{1}{2} e^{-i\pi s} \int_0^{A_L} dx \ \tan(\pi x + i\epsilon) D_n(x) \left(A_L^2 - x^2\right)^{-s} \right\}, \quad (A.9)$$

where $D_n(ix)$ is the polynomial with coefficients $r_{nk}$:

$$D_n(ix) = i(-1)^{N-1} \frac{2x}{(2N-1)!} \left\{ x^2 + \left(N - \frac{3}{2}\right) \right\} \cdots \left\{ x^2 + \left(\frac{1}{2}\right)^2 \right\}$$
\[ \equiv i(-1)^{N-1} \sum_{k=0}^{N-1} r_{Nk} x^{2k+1}. \] (A.10)

The first term in Eq. (A.9) comes from the integral along the imaginary axis. The second and third terms in Eq. (A.9) are the contributions from the contour along the cut on the real axis. Since we use the relation
\[ \tanh(\pi x) = 1 - \frac{2}{e^{2\pi x} + 1}, \] (A.11)
and substitute it into the first term of (A.9), the function \( Z(s) \) is

\[
Z(s) = -(a)^{2s} \sum_{L=L_0}^{L_a} D_d(L) \times \left\{ \cos(\pi s) \frac{1}{\Gamma(s)} (-1)^{N-1} \frac{1}{2} \sum_{k=0}^{N-1} f_{Nk} \left( A^2_L \right)^{k+1-s} \Gamma(k+1) \Gamma(s-k-1) \\
+ \cos(\pi s) \int_0^\infty dx \ iD_n(ix) \left( x^2 + A^2_L \right)^{-s} \frac{2}{e^{2\pi x} - 1} \\
+ \sin(\pi s) \int_0^{A_L} dx D_n(x) \left( A^2_L - x^2 \right)^{-s} \tan(\pi x) \right\}. \] (A.12)

We note that third term in Eq. (A.12) does not contribute to \( \zeta(0) \).

Next, we consider the function \( W(s) \). The branch points in the integrand are on the imaginary axis at \( z = \pm iB_L \). Replacing the contour \( C_1 \) by \( C_3 \), we obtain

\[
W(s) = (a)^{2s} \sum_{L=L_0}^{L_a} D_d(L) \int_0^\infty dx \ iD_n(ix) \tan(\pi x) \left( B^2_L - x^2 \right)^{-s} \\
= (a)^{2s} \sum_{L=L_0}^{L_a} D_d(L) \left[ \frac{1}{\Gamma(s)} (-1)^{N-1} \frac{1}{2} \sum_{p=0}^{N-1} r_{Np} \left( B^2_L \right)^{p+1-s} \right. \\
\times \left\{ \frac{\Gamma(s-p-1) \Gamma(-s+1)}{\Gamma(-p)} \cos(\pi s) + \frac{\Gamma(p+1) \Gamma(-s+1)}{\Gamma(2+p-s)} \right\} \\
- \cos(\pi s) \int_{B_L}^\infty dx \ iD_n(ix) \left( x^2 - B^2_L \right)^{-s} \frac{2}{e^{2\pi x} + 1} \\
- \int_0^{B_L} dx \ iD_n(ix) \left( B^2_L - x^2 \right)^{-s} \frac{2}{e^{2\pi x} + 1} \right], \] (A.13)

where the contour \( C_3 \) in the complex plane is showed in Fig. 3. Note that the function \( W(s) \) includes an infinite sum over \( L \). As far as the second term of Eq. (A.13) is concerned, the infinite sum over \( L \) is convergent, due to the fact that integral decreases exponentially as \( L \) goes to infinity. For the first term in Eq. (A.13),
\[ Z \]

\[ C_1 \]

\[ C_3 \]

\[ \text{Fig. 3. The contour } C_1 \text{ in (A.12) replaced by the contour } C_3. \text{ The path of } C_3 \text{ runs parallel to the imaginary axis.} \]

However, we must regularize the summation

\[
\sum_{L=L_0+1}^{\infty} D_d(L) \left( B_L^2 \right)^{p+1-s} = \left\{ \left( \frac{a}{b_0} \right)^2 \left( \frac{b}{b_0} \right)^{d+2} \right\}^{p-s+1} \sum_{L=L_0+1}^{\infty} D_d(L) \left( L^2 - C^2 \right)^{-\beta},
\]

(A.14)

with

\[
C^2 = D^2 + \left( \frac{b_0}{a} \right)^2 \left( \frac{b}{b_0} \right)^{d+2} \left( N - \frac{1}{2} \right)^2,
\]

(A.15)

where \( \beta = s - p - 1 \). Again replacing the infinite sum with a complex integral, we obtain

\[
S = \sum_{L=L_0+1}^{\infty} D_d(L) \left( L^2 - C^2 \right)^{-\beta} = -\frac{1}{2i} \int_{C_4} dz D_d(z) \left( z^2 - C^2 \right)^{-\beta} \cot(\pi z),
\]

(A.16)

where the sum is taken over half integers and the contour \( C_4 \) is that depicted in Fig. 4. By replacing the contour \( C_4 \) by \( C_5 \), we obtain

\[
S = -\cos(\pi \beta) \sum_{L=D}^{L_0} D_d(L) \left( C^2 - L^2 \right)^{-\beta} + \sin(\pi \beta) \left\{ \int_{0}^{\infty} dx \ D_d(ix) \left( x^2 + C^2 \right)^{-\beta} \left( \frac{2}{e^{2\pi x} - 1} \right) + \int_{i}^{C} dx \ D_d(x) \left( C^2 - x^2 \right)^{-\beta} \cot(\pi x) \right\}
\]
Fig. 4. The contour $C_4$ in (A.16) replaced by the contour $C_5$. The pole of $\cos(\pi z)$ exists at $z = L_0$, which is a smaller value than that at the cut of the branch point.

$$\left.\begin{aligned} &+(-1)^D \frac{1}{2} \sum_{q=0}^{D-1} r Dq \left( C^2 \right)^{q-\beta+\frac{1}{2}} \frac{\Gamma \left( q + \frac{3}{2} \right) \Gamma \left( \beta - q - \frac{3}{2} \right)}{\Gamma (\beta)} \right) , \\
\end{aligned}\right\}, \quad (A.17)$$

where the function $D_d(ix)$ is defined as

$$D_d(ix) = (-1)^D \frac{2x^2}{(2D)!} \left\{ x^2 + (D - 1)^2 \right\} \cdots \left\{ x^2 + 1^2 \right\} \equiv (-1)^D \sum_{q=0}^{D-1} r Dq x^{2q+2} , \quad (A.18)$$

$P$ stands for the principal value integral, and the coefficients $jDq$ in $D_d$ are defined in Eq. (A.18). Substituting (A.18) into the first term of Eq. (A.13), we obtain an expression for $W(s)$. Finally, the generalized $\zeta$ function is found to be

$$\zeta(s) = (a)^{2s} \sum_{L=D}^{L_0} D_d(L)$$

$$\times \left[ \cos(\pi s) \left\{ \frac{1}{\Gamma(s)}(-1)^N \frac{1}{2} \sum_{k=0}^{N-1} r_{Nk} \left( A_L^2 \right)^{k-s+1} \Gamma(k+1) \Gamma(s-k-1) \right. \right.$$

$$- \left. \int_0^\infty dx \ i D_n(ix) \left( x^2 + A_L^2 \right)^{-s} \frac{2}{e^{2\pi x} + 1} \right]$$

$$- \sin(\pi s) \int_0^{A_L} dx D(x) \left( A_L^2 - x^2 \right)^{-s} \tan(\pi x)$$
\[-(a)^2 s \sum_{L=L_0+1}^{\infty} D_d(L) \left\{ \int_{B_L}^{\infty} dx \, i \, D_n(ix) \left( x^2 - \frac{B_L^2}{L^2} \right)^{-s} - \sum_{L=0}^{\infty} D_d(L) \left\{ \int_0^B dx \, i \, D_n(ix) \left( B_L^2 - x^2 \right)^{-s} \cos(\pi s) \right\} \right. \\
+ \int_0^\infty dx \, i \, D_n(ix) \left( B_L^2 - x^2 \right)^{-s} \frac{2}{e^{2\pi x} + 1} \\
+ (a)^2 s (-1)^{N_1} \frac{1}{2} \sum_{p=0}^{N_1} r_{N_p} \left\{ \left( \frac{a}{b_0} \right)^2 \left( \frac{b_0}{b} \right)^{d+2} \right\}^{p-s+1} \\
\times \left\{ \frac{\Gamma(s-p-1) \Gamma(-s+1)}{\Gamma(-p)} \cos(\pi s) + \frac{\Gamma(p+1) \Gamma(-s+1)}{\Gamma(2+p-s)} \right\} \times S. \quad \text{(A.19)} \]

**Appendix B**

**Zeta Function Regularization for a Spinor Field**

In this appendix, we present a method to regularize the zeta function for the Dirac spinor field in an odd total number of dimensions. It can easily be seen that this method is able to apply the case of an even number of dimensions. We define the generalized zeta function for the Dirac spinor fields in $S^n \times S^d$ as

\[
\zeta(s) = \sum_{L'=n/2}^{\infty} \sum_{L=d/2}^{\infty} (L' + n - 1)! (l + d - 1)! (n-1)! (d-1)! (n-2)! (d-1)! \left\{ \frac{(L' + \frac{n}{2})^2 - \frac{n(n-1)}{4}}{a^2} + \frac{(l + \frac{d}{2})^2}{b^2} \right\}^{-s}, 
\]

(B.1)

where $n$ is an even number and $d$ is an odd number. Using the running variables $L = l + d/2$ and $L' = l' + n/2$, we rewrite (B.1) as

\[
\zeta(s) = \sum_{L'=n/2}^{\infty} D_n(L') \sum_{L=d/2}^{\infty} D_d(L) \left\{ \frac{L'^2 - \frac{n(n-1)}{4}}{a^2} + \frac{L^2}{b^2} \right\}^{-s}, \quad \text{(B.2)}
\]

where the eigenvalues and degeneracies are now given by

\[
D_n(L') = \frac{L'}{(n-1)!} \left\{ L'^2 - \left( \frac{n}{2} - 1 \right)^2 \right\} \cdots \left( L'^2 - 1 \right), \\
D_d(L) = \frac{1}{(d-1)!} \left\{ L^2 - \left( \frac{d}{2} - 1 \right)^2 \right\} \cdots \left( L^2 - \left( \frac{1}{2} \right)^2 \right). \quad \text{(B.3)}
\]

Next, replacing the infinite mode sum over $L'$ by complex integration, the generalized zeta function is

\[
\zeta(s) = (a)^2 s \sum_{L=d/2}^{\infty} D_d(L) \frac{-i}{2} \int_{F_1} dz \tan(\pi z) D_n(z) \left\{ z^2 + \left( \frac{a}{b} \right)^2 L^2 - \frac{n(n-1)}{4} \right\}^{-s}, \quad \text{(B.4)}
\]

where the contour $F_1$ in the complex plane is showed in Fig. 5. We divide the sum over $L$ into two parts, (A.5) and (A.6), with

\[
A_L^2 = \frac{n(n-1)}{4} - \left( \frac{a}{b} \right)^2 L^2, 
\]
Fig. 5. The contour $F_1$ in Eq. (B.4) is replaced by the contour $F_2$. Note that the contour $F_2$ avoids the branch points at $z = \pm A_L$.

$$B_L^2 = -A_L^2,$$  \hspace{1cm} (B.5)

where $A_L^2$ and $B_L^2$ are positive for $D \leq L < L_0$ and $L_0 \leq L$, respectively. The number $L_0$ is defined as the largest integer smaller than or equal to

$$\sqrt{\frac{n(n-1)}{4} \left(\frac{b}{a}\right)^2}.$$  \hspace{1cm} (B.6)

We consider a function $Z(s)$ which is a sum over the $L$ covers from $d/2$ to $L_0$. There is a branch point in integrand at $z = \pm A_L$. Now, we replace the contour $F_1$ so that it is parallel to the imaginary axis, in order to deal with the poles of $\tan(\pi z)$. The displacing contour $F_2$ is defined as running above the cut associated with $z = \pm A_L$.

We then obtain an expression for $Z(s)$ by carrying out the same calculation as in Appendix A:

$$Z(s) = (a)^{2s} \sum_{L=d/2}^{L_0} D_d(L)$$

$$\times \left\{ \cos(\pi s) \frac{1}{\Gamma(s)} (-1)^{n/2} \frac{1}{2} \sum_{k=0}^{n/2-1} f_{nk} \left( A_L^2 \right)^{k+1-s} \Gamma(k+1)\Gamma(s-k-1) \right. $$

$$+ \cos(\pi s) \int_0^{\infty} dx \ i \ D_n(ix) \left( x^2 + A_L^2 \right)^{-s} \frac{2}{e^{2\pi x} - 1}$$

$$\left. + \sin(\pi s) \int_0^{-A_L} dx \ D_n(x) \left( A_L^2 - x^2 \right)^{-s} \tan(\pi x) \right\}.$$  \hspace{1cm} (B.7)
Here, $D_n(ix)$ is a polynomial with coefficients $f_{Nk}$:

$$D_n(ix) = i(-1)^{n/2-1} \frac{x}{(n-1)!} \left\{ x^2 + \left( \frac{n}{2} - 1 \right)^2 \right\} \cdots \left\{ x^2 + 1 \right\}$$

$$\equiv i(-1)^{n/2-1} \sum_{k=0}^{n/2-1} f_{nk} x^{2k+1}. \quad (B.8)$$

Next, we calculate the function $W(s)$. The branch points in the integrand are on the imaginary axis at $z = \pm iB_L$. Replacing the contour $F_1$ by $F_3$, we obtain

$$W(s) = (a)^{2s} \sum_{L=L_0+1}^{\infty} D_d(L) \left[ \cos(\pi s) \int_{B_L}^{\infty} dx \ iD_n(ix) \left( x^2 + B_L^2 \right)^{-s} \frac{-2}{e^{2\pi x} + 1} \right.$$  
$$+ \int_{0}^{B_L} dx \ iD_n(ix) \left( B_L^2 - x^2 \right)^{-s} \frac{-2}{e^{2\pi x} + 1}$$  
$$+ \frac{1}{2}(-1)^{n/2} \sum_{p=0}^{n/2-1} f_{np} \left( B_L^2 \right)^{p+1-s} \left\{ \frac{\Gamma(1-s)\Gamma(s-p-1)}{\Gamma(-p)} \cos(\pi s) \right.$$  
$$\left. + \frac{\Gamma(1-s)\Gamma(p+1)}{\Gamma(-s+p+2)} \right\}. \quad (B.9)$$

The function $W(s)$ contains an infinite sum over $L$. As far as the first term of Eq. (B.9) is concerned, the infinite sum over $L$ is convergent, due to the fact that the integral decreases exponentially as $L$ goes to infinity. Contrastingly, for the first
term in (B.9), we must regularize the summation. First we consider the sum

$$\sum_{L=L_0+1}^{\infty} D_d(L) \left( B_L^2 \right)^{p+1-s} = \left( \frac{a}{b} \right)^{2(p-s+1)} \sum_{L=L_0+1}^{\infty} D_d(L) \left( L^2 - C^2 \right)^{-\beta}, \quad (B.10)$$

with

$$C^2 = \frac{n(n-1)}{4} \left( \frac{b}{a} \right)^2,$$  \quad (B.11)

where $\beta = s - p - 1$. Then, using Eq. (B.9), the function $W(s)$ is given by

$$W(s) = (a)^{2s} \sum_{L=L_0+1}^{\infty} D_d(L) \left[ \cos(\pi s) \int_{B_L}^{\infty} dx \, i D_n(ix) \left( x^2 + B_L^2 \right)^{-s} \frac{-2}{e^{2\pi x} + 1} \right] + \int_{0}^{B_L} dx \, i D_n(ix) \left( B_L^2 - x^2 \right)^{-s} \frac{-2}{e^{2\pi x} + 1} \right] + (a)^{2s} \frac{1}{2} \frac{(-1)^{n/2}}{\Gamma(1-s)\Gamma(s-p-1)} \cos(\pi s) \sum_{p=0}^{n/2-1} C_{np} \left( \frac{a}{b} \right)^{2(p+1-s)} \left\{ \frac{\Gamma(1-s)\Gamma(s-p-1)}{\Gamma(-p)} \right\} + \frac{\Gamma(1-s)\Gamma(p+1)}{\Gamma(-s+p+2)} \right\} \sum_{L=L_0+1}^{\infty} D_d(L) \left( L^2 - C^2 \right)^{p+1-s}. \quad (B.12)$$

For the last term in Eq. (B.12), we replace the sum by the following integration:

$$S \equiv \sum_{L=L_0+1}^{\infty} D_d(L) \left( L^2 - C^2 \right)^{-\beta} = \frac{-1}{2i} \int_{F_4} dz D_d(z) \left( z^2 - C^2 \right)^{-\beta} \cot(\pi z). \quad (B.13)$$

The contour is the same as that for Eq. (A.15) in Appendix A. By replacing the contour $F_4$ by $F_5$, we obtain

$$S = -\sin(\pi \beta) \left\{ \int_{0}^{\infty} dx \, D_d(ix) \left( x^2 + C^2 \right)^{-\beta} \left( \frac{2}{e^{2\pi x} - 1} \right) \right\} + P \int_{0}^{C} dx \, D_d(x) \left( C^2 - x^2 \right)^{-\beta} \cot(\pi x) \right\} + (-1)^{d/2} \frac{1}{2} \sum_{q=0}^{d/2-1} f_{dq} \left( C^2 \right)^{q-\beta+\frac{1}{2}} \frac{\Gamma \left( q + \frac{3}{2} \right) \Gamma \left( \beta - q - \frac{3}{2} \right)}{\Gamma(\beta)}, \quad (B.14)$$

where $\beta = s - r - 1$, the function $D_d(ix)$ is defined as

$$D_d(ix) = (-1)^{(d-1)/2} \frac{2x^2}{d!} \left\{ x^2 + \left( \frac{d}{2} - 1 \right)^2 \right\} \cdots \left\{ x^2 + \left( \frac{1}{2} \right)^2 \right\} \equiv (-1)^{(d-1)/2} \sum_{q=0}^{(d-1)/2-1} f_{dq} x^{2q}, \quad (B.15)$$

$P$ stands for the principal value integral, and the coefficients $f_{dq}$ in $D_d$ are defined in Eq. (B.15). Substituting (B.14) into the last term of Eq. (B.12), we obtain an
expression for $W(s)$. Note that the value of $S$ at $s = 0$ is zero and thus does not contribute to $\zeta(0)$. Then, $W(s)$, is given by

$$W(s) = (a)^{2s} \sum_{L=L_0+1}^{\infty} D_d(L) \left[ \cos(\pi s) \int_{B_L}^{\infty} dx \ iD_n(ix) \left( x^2 + B_L^2 \right)^{-s} \frac{-2}{e^{2\pi x} + 1} \right.$$  
$$+ \int_{0}^{B_L} dx \ iD_n(ix) \left( B_L^2 - x^2 \right)^{-s} \frac{-2}{e^{2\pi x} + 1} \bigg]$$  
$$+ (a)^{2s} \frac{1}{2} (-1)^{n/2} \sum_{p=0}^{n/2-1} f_{np} \left( \frac{a}{b} \right)^{2(p+1-s)} \left\{ \frac{\Gamma(1-s)\Gamma(s-p-1)}{\Gamma(-p)} \cos(\pi s) \right.$$  
$$+ \frac{\Gamma(1-s)\Gamma(p+1)}{\Gamma(-s+p+2)} \right\} \times S.$$  

(B-16)

Finally, we find that the generalized $\zeta$ function is expressed as

$$\zeta(s) = (a)^{2s} \sum_{L=d/2}^{L_0} D_d(L)$$  
$$\times \left\{ \cos(\pi s) \frac{1}{\Gamma(s)} (-1)^{n/2} \left( \frac{1}{2} \right) \sum_{k=0}^{n/2-1} f_{nk} \left( A_L^2 \right)^{k+1-s} \Gamma(k+1) \Gamma(s-k-1) \right.$$  
$$+ \cos(\pi s) \int_{0}^{\infty} dx \ iD_n(ix) \left( x^2 + A_L^2 \right)^{-s} \frac{2}{e^{2\pi x} - 1} \right\} \times S.$$
\[\begin{align*}
+ \sin(\pi s) \int_0^{A_L} dx D_n(x) \left(A_L^2 - x^2\right)^{-s} \tan(\pi x) \\
+ (a)^{2s} \sum_{L=L_0+1}^{\infty} D_d(L) \left[\cos(\pi s) \int_{B_L}^{\infty} dx iD_n(ix) \left(x^2 + B_L^2\right)^{-s} \frac{-2}{e^{2\pi x} + 1} \right. \\
+ \left. \int_0^{B_L} dx iD_n(ix) \left(B_L^2 - x^2\right) \frac{-2}{e^{2\pi x} + 1}\right] \\
+ (a)^{2s} \frac{1}{2} \left(-1\right)^{n/2} \sum_{\mu=0}^{n/2-1} f_{np} \left(\frac{a}{b_0}\right)^{p+1-2s} \left\{\frac{\Gamma(1-s)\Gamma(s-p-1)}{\Gamma(-p)} \cos(\pi s) \right. \\
+ \left. \frac{\Gamma(1-s)\Gamma(p+1)}{\Gamma(-s+p+2)}\right\} \times S.
\end{align*}\]

**Appendix C**

**Zeta Function Regularization for a Vector Field**

In this appendix, we present a method to regularize the zeta function for a vector field in an odd total number of dimensions. We consider the case of the field \(A_{(T)\mu}\). The quantity \(A_{(T)\mu}\) can be calculated in a similar way, and this method can be applied to the case of an even number of dimensions also. We define a generalized zeta function for the vector fields \(A_{(T)\mu}\) in \(S^n \times S^d\) as

\[
\zeta(s) = \sum_{l'=1}^{\infty} \sum_{l=0}^{\infty} \frac{(l' + n - 3)!}{(l' + n - 1)!} \frac{(l + d - 2)!}{(l + d - 1)!} \left(\frac{a^2}{\frac{a^2}{b_0} + \frac{l(l + d - 1)}{b_0^2}}\right)^{d+2} \left(\frac{b}{b_0}\right)^{-s},
\]

where \(n\) is an even number and \(d\) is an odd number. Using \(N = n/2\) and \(D = (d-1)/2\) instead of \(n\) and \(d\) and running variables \(L' = l' + N\) and \(L = l + D\), we rewrite (C.1) as

\[
\zeta(s) = \sum_{L' = N}^{\infty} D_n \left(L' - \frac{1}{2}\right) \sum_{L = D}^{\infty} D_d(L) \left\{\frac{A_n \left(L' - \frac{1}{2}\right)}{a^2} + \frac{A_d(L)}{b_0^2} \left(b_0\right)^{d+2}\right\}^{-s},
\]

where the eigenvalues and degeneracies are now given by

\[
D_n \left(L' - \frac{1}{2}\right) = \frac{2L' - 1}{(2N - 1)!} \left\{\left(L' - \frac{1}{2}\right)^2 - \left(N - \frac{1}{2}\right)^2\right\} \\
\times \left\{\left(L' - \frac{1}{2}\right)^2 - \left(N - \frac{5}{2}\right)^2\right\} \cdots \left\{\left(L' - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\},
\]

\[
D_d(L) = \frac{2L^2}{(2D)!} \left\{L^2 - (D - 1)^2\right\} \cdots \left\{L^2 - 1\right\},
\]
Fig. 8. The contour $V_1$ in Eq. (C.4) is replaced by the contour $V_2$. Note that the contour $V_2$ avoids the branch points at $z = \pm A_L$.

$$
\Lambda_n \left( L' - \frac{1}{2} \right) = \left( L' - \frac{1}{2} \right)^2 - \left( N - \frac{1}{2} \right)^2 - N - \frac{1}{2},
$$

$$
\Lambda_d(L) = L^2 - D^2. \quad (C.3)
$$

Next, replacing the infinite mode sum over $L'$ by complex integration, the generalized zeta function is

$$
\zeta(s) = (a)^{2s} \sum_{L=D}^{\infty} D_d(L) \frac{-1}{2i} \int_{V_1} dz \tan(\pi z) D_n(z)
\times \left\{ z^2 + \left( \frac{a}{b_0} \right)^2 \left( \frac{b_0}{b} \right)^{d+2} A_d(L) - \left( N - \frac{1}{2} \right)^2 - N - \frac{1}{2} \right\}^{-s}, \quad (C.4)
$$

where the contour $V_1$ in the complex plane is showed in Fig. 8. In order to avoid the singularity of $z$ in the integrand, it is convenient to divide the sum over $L$ into two parts as

$$
\zeta(s) = Z(s) + W(s), \quad (C.5)
$$

where

$$
Z(s) = (a)^{2s} \sum_{L=D}^{L_0} D_d(L) \frac{-i}{2} \int_{V_1} dz \tan(\pi z) D_n(z) \left( z^2 - A_L^2 \right)^{-s},
$$

$$
W(s) = (a)^{2s} \sum_{L=L_0+1}^{\infty} D_d(L) \frac{-i}{2} \int_{V_1} dz \tan(\pi z) D_n(z) \left( z^2 + B_L^2 \right)^{-s}, \quad (C.6)
$$
with
\[
A_L^2 = \left( N - \frac{1}{2} \right)^2 + N + \frac{1}{2} - \left( \frac{a}{b_0} \right)^2 \left( \frac{b_0}{b} \right)^{d+2} A_d(L),
\]
\[
B_L^2 = -A_L^2(T) L.
\]

Here, \( A_L^2 \) and \( B_L^2 \) are positive for \( D \leq L < L_0 \) and \( L_0 \leq L \), respectively. \( L_0 \) is defined as the largest integer smaller than or equal to
\[
\left[ \frac{1}{2} \left( N - \frac{1}{2} \right)^2 + N + \frac{1}{2} \right]^{1/2}.
\]

We consider the function \( Z(s) \) which is sum over the \( L \) covers from \( D \) to \( L_0 \). There are branch points at \( z = \pm A_L \). Now, we replace the contour \( V_1 \) so that it is along to the imaginary axis in order to deal with the poles of \( \cot(\pi z) \). Since the displacing contour \( V_2 \) is defined as running above the cut associated with \( z = \pm A_L \), the result of \( Z(s) \) is given by
\[
Z(s) = (a)^{2s} \sum_{L=D}^{L_0} D_d(L) \left\{ \frac{1}{2} \left( e^{-i\pi s} + e^{i\pi s} \right) \int_0^\infty dx \ i D_n(ix) \left( x^2 + A_L^2 \right)^{-s} \tanh(\pi x) \right.
\]
\[
- \frac{1}{2} e^{i\pi s} \int_0^{A_L} dx \ \pi (x - i\epsilon) D_n(x) \left( A_L^2 - x^2 \right)^{-s}
\]
\[
- \frac{1}{2} e^{-i\pi s} \int_0^{A_L} dx \ \pi (x + i\epsilon) D_n(x) \left( A_L^2 - x^2 \right)^{-s} \right\},
\]
where \( D_n(ix) \) is a polynomial with coefficients \( j_{Nk} \):
\[
D_n(ix) = i(-1)^{N-1} \frac{2x}{(2N-1)!} \left\{ x^2 + \left( \frac{N - 1}{2} \right)^2 \right\}
\]
\[
\times \left\{ x^2 + \left( \frac{N - 3}{2} \right)^2 \right\} \cdots \left\{ x^2 + \left( \frac{1}{2} \right)^2 \right\}
\]
\[
\equiv i(-1)^{N-1} \sum_{k=1}^{N-1} j_{Nk} x^{2k+1}.
\]

The first term in Eq. (C.9) comes from the integral along the imaginary axis. The second and third terms in Eq. (C.9) are the contributions from the contour along the cut on the real axis. Since we use the relation
\[
\tanh(\pi x) = 1 - \frac{2}{e^{2\pi x} + 1},
\]
and we substitute this into the first term, the function \( Z(s) \) is
\[
Z(s) = - (a)^{2s} \sum_{L=D}^{L_0} D_d(L)
\]
Fig. 9. The contour $V_1$ in (C.13) replaced by the contour $V_3$. The path of $V_3$ runs parallel to the imaginary axis.

\[
\begin{align*}
&\times \left\{ \cos(\pi s) \frac{1}{\Gamma(s)} (-1)^{N/2} \sum_{k=1}^{N-1} j_{Nk} \left( A_L^2 \right)^{k+1-s} \Gamma(k+1) \Gamma(s-k-1) \\
&+ \cos(\pi s) \int_0^\infty dx \, i \, D_n(ix) \left( x^2 + A_L^2 \right)^{-s} \frac{2}{e^{2\pi x} - 1} \\
&+ \sin(\pi s) \int_0^{A_L} dx D_n(x) \left( A_L^2 - x^2 \right)^{-s} \tan(\pi x) \right\}. \quad (C.12)
\end{align*}
\]

We note that first term in Eq. (C.12) does not contribute to $\zeta(s)$.

We next consider the function $W(s)$. The branch points in integrand are on the imaginary axis at $z = \pm B_L$. Replacing the contour $V_1$ by $V_3$, we obtain

\[
W(s) = - \sum_{L=L_0+1}^{\infty} D_d(L)(a)^{2s} \cos(\pi s) \times \left\{ \frac{1}{\Gamma(s)} (-1)^{N/2} \sum_{p=0}^{N-1} j_{Np} \left( B_L^2 \right)^{p+1-s} \Gamma(s-p-1) \Gamma(p+1) \\
+ \int_0^{A_L} dx \, i \, D_n(ix) \left( x^2 - B_L^2 \right)^{-s} \frac{2}{e^{2\pi x} + 1} \right\}. \quad (C.13)
\]

The function $W(s)$ is contains an infinite sum over $L$. As far as the first term of Eq. (C.13) is concerned, the infinite sum over $L$ is convergent, due to the fact that the integral decreases exponentially as $L$ goes to infinity. For the second term in
Eq. (C.13), however, we must regularize the summation
\[ \sum_{L=L_0+1}^{\infty} D_d(L) \left( B_{L}^{2} \right)^{p+1-s} = \left\{ \left( \frac{a}{b_0} \right)^2 \left( \frac{b_0}{b} \right)^{d+2} \right\}^{p-s+1} \sum_{L=L_0+1}^{\infty} D_d(L) \left( L^2 - C^2 \right)^{-\beta}, \]
with
\[ C^2 = D^2 + \left( \frac{b_0}{a} \right)^2 \left( \frac{b}{b_0} \right)^{d+2} \left\{ \left( N - \frac{1}{2} \right)^2 + N + \frac{1}{2} \right\}, \]
where \( \beta = s - p - 1 \). We again replace the infinite sum with a complex integral and thereby obtain, the integral is
\[ S = \sum_{L=L_0+1}^{\infty} D_d(L) \left( L^2 - C^2 \right)^{-\beta} = \frac{-1}{2i} \int_{C_3} dz D_d(z) \left( z^2 - C^2 \right)^{-\beta} \cot(\pi z), \]
where the sum is taken over half integers, and the contour \( V_4 \) is depicted in Fig. 9.

The function \( D_d(ix) \) is defined as
\[ D_d(ix) = (-1)^D \frac{2x^2}{(2D)!} \left\{ x^2 + (D - 1)^2 \right\} \cdots \left\{ x^2 + 1^2 \right\} \]
\[ \equiv (-1)^D \sum_{q=0}^{D-1} j_{Dq} x^{2q+2}. \]
Replacing the contour \( V_4 \) by \( V_5 \), we obtain
\[ S = -\cos(\pi \beta) \sum_{L=D}^{L_0} D_d(L) \left( C^2 - L^2 \right)^{-\beta} \]
\[ + \sin(\pi \beta) \left\{ \int_0^{\infty} dx \ i \ D_n(ix) \left( x^2 + C^2 \right)^{-\beta} \frac{1}{e^{2\pi x} - 1} \right\} \]
\[ + \int_0^{C} dx \ D_n(x) \left( C^2 - x^2 \right)^{-\beta} \cot(\pi x) \]
\[ + (-1)^D \frac{1}{2} \sum_{q=0}^{D-1} j_{Dq} \left( C^2 \right)^{q-\beta+\frac{3}{2}} \left( \frac{\Gamma \left( q + \frac{3}{2} \right) \Gamma \left( \beta - q - \frac{3}{2} \right)}{\Gamma(\beta)} \right), \]
where \( P \) stands for the principal value integral, and the coefficients \( j_{Dq} \) in \( D_d \) are defined in Eq. (C.17). Substituting Eq. (C.18) into the second term of Eq. (C.13), we obtain an expression for \( W_s \). Finally, the generalized \( \zeta \) function is found to be
\[ \zeta(s) = (a)^{2s} \sum_{L=D}^{L_0} D_d(L) \]
\[ \times \left[ \cos(\pi s) \left\{ \frac{1}{\Gamma(s)} (-1)^\frac{1}{2} \sum_{k=0}^{\infty} j_{Nk} \left( A_k^2 \right)^{k-s+1} \Gamma(k+1) \Gamma(s-k-1) \right\} \right. \]
\[ \left. - \int_0^{\infty} dx \ i \ D_n(ix) \left( x^2 + A_k^2 \right)^{-s} \frac{2}{e^{2\pi x} + 1} \right] \]
Fig. 10. The contour $V_4$ in (C.18) replaced by the contour $V_5$. The pole of $\cos(\pi z)$ exists at $z = L_0$ which is smaller value than that at the cut of the branch point.

\begin{align*}
-\sin(\pi s) & \left[ \int_0^{A_L} dx D(x) \left( A_L^2 - x^2 \right)^{-s} \tan(\pi x) \right] \\
-(a)^{2s} \sum_{L=L_0+1}^{\infty} & D_d(L) \cos(\pi s) \left\{ \int_{B_L}^{\infty} dx i D_n(ix) \left( x^2 - B_L^2 \right)^{-s} \frac{2}{e^{2\pi x} + 1} \right\} \\
+ \cos(\pi s)(a)^{2s} & \Gamma(-s + 1)(-1)^{N-1/2} \sum_{p=0}^{N-1} j_{Np} \left\{ \left( \frac{a}{b_0} \right)^2 \left( \frac{b_0}{b} \right)^{d+2} \right\}^{p-s+1} \\
\times \frac{\Gamma(s - p - 1)}{\Gamma(-p)} & S. \\
\end{align*}

\textbf{Appendix D}  

\textit{Zeta Function Regularization of One-Loop Effective Potential in $AdS_n \times S^d$}

In this appendix, we present a method to calculate the zeta function regularization for a scalar field in the product spacetime $AdS_n \times S^d$. The Euclidean section for the $AdS_n$ spacetime is the $n$-dimensional hyperbolic space $H^n$. The calculation of the zeta function for $AdS_n$ is discussed in Ref. 40). We extend the calculation technique given them to the zeta function for $AdS_n \times S^d$. On a compact Euclidean section, the zeta function is given by

\begin{equation}
\zeta_\phi(s) = \sum_{l=0}^{\infty} D_l \, A_l^{-s},
\end{equation}
where $\Lambda_l$ is the discrete eigenvalue of the Laplace-Beltrami operator and $D_l$ is the degeneracy of the eigenvalue. The calculation of the zeta function on $S^d$ is performed using the well-known spectrum of the Laplace-Beltrami operator on $S^d$. The zeta function for a noncompact manifold, however, is not the same as that in the compact case. In the homogeneous $n$-dimensional hyperbolic space $H^n$, the zeta function takes the form \[ \zeta_\phi(s) = \int_0^\infty d\lambda \mu(\lambda) \Lambda(\lambda)^{-s}, \] (D.2)

where $\Lambda(\lambda)$ is the eigenvalue of the Laplace-Beltrami operator on $H^n$, $\lambda$ is the real parameter that labels the continuous spectrum, and $\mu(\lambda)$ is the spectrum, function (or Plancherel measure) on $H^n$ corresponding to the discrete degeneracy on $S^d$. The spectrum function for the scalar and spinor fields on $H^n$ is calculated in Refs. 41) and 42). Also, the spectrum function for the transverse-traceless tensor field on $H^n$ is given in Ref. 43).

Here, we consider the zeta function regularization of the one-loop effective potential for scalar field in $AdS_n \times S^d$. This regularization scheme can be extended for other fields. Define the generalized zeta function in $H^n \times S^d$ for the scalar field as

\[
\zeta_\phi(s) = \sum_{l=0}^{\infty} \frac{(l+d-2)!}{(d-1)!} \frac{(2l+d-1)}{l!} \int_0^\infty d\lambda \mu(\lambda) \\
\times \left\{ \lambda^2 + \left(\frac{(n-1)/2}{a^2}\right)^2 + \frac{l(l+d-1)}{b_0^2} e^{-2d/(n-2)+2/\kappa \sigma} \right\}^{-s},
\] (D.3)

For even dimensions, the Plancherel measure $\mu(\lambda)$ is given by \[ \mu(\lambda) = \frac{\pi \lambda \tanh(\pi \lambda)}{2^{2(n-2)} \{\Gamma(n/2)\}^2} \prod_{j=1/2}^{(n-3)/2} \left( \lambda^2 + j^2 \right). \] (D.4)

Then, using $D = (d-1)/2$ and $N = (n-1)/2$ instead of $d$ and $n$ and the running variables $L = l + D$, we rewrite (D.3) as

\[
\zeta_\phi(s) = \sum_{L=D}^{\infty} D\phi(L) \int_0^\infty d\lambda \mu(\lambda) \left( \frac{\lambda^2 + N^2}{a^2} + M_\phi^2 \right)^{-s},
\] (D.5)

where the $D\phi(L)$ and $A\phi(L)$ are given by

\[
D\phi(L) = \frac{2L^2}{(2D)!} \left\{ L^2 - (D-1)^2 \right\} \cdots \left\{ L^2 - 1 \right\},
\]

\[
A\phi(L) = L^2 - D^2,
\]

\[
M_\phi^2 = \frac{A\phi(L)}{b_0^2} e^{-2d/(n-2)+2/\kappa \sigma}.
\] (D.6)

Next, integrating Eq. (D.5) over $\lambda$, we find the expression

\[
\zeta_\phi(s) = \frac{\pi}{2^{2n-1} \{\Gamma(2N+1)\}^2} \frac{a^{2s}}{\Gamma(s)} \sum_{L=D}^{\infty} D\phi(L) \]

where $\Lambda_{lD}$ is the discrete eigenvalue of the Laplace-Beltrami operator and $D_l$ is the degeneracy of the eigenvalue. The calculation of the zeta function on $S^d$ is performed using the well-known spectrum of the Laplace-Beltrami operator on $S^d$. The zeta function for a noncompact manifold, however, is not the same as that in the compact case. In the homogeneous $n$-dimensional hyperbolic space $H^n$, the zeta function takes the form \[ \zeta_\phi(s) = \int_0^\infty d\lambda \mu(\lambda) \Lambda(\lambda)^{-s}, \] (D.2)

where $\Lambda(\lambda)$ is the eigenvalue of the Laplace-Beltrami operator on $H^n$, $\lambda$ is the real parameter that labels the continuous spectrum, and $\mu(\lambda)$ is the spectrum, function (or Plancherel measure) on $H^n$ corresponding to the discrete degeneracy on $S^d$. The spectrum function for the scalar and spinor fields on $H^n$ is calculated in Refs. 41) and 42). Also, the spectrum function for the transverse-traceless tensor field on $H^n$ is given in Ref. 43).

Here, we consider the zeta function regularization of the one-loop effective potential for scalar field in $AdS_n \times S^d$. This regularization scheme can be extended for other fields. Define the generalized zeta function in $H^n \times S^d$ for the scalar field as

\[
\zeta_\phi(s) = \sum_{l=0}^{\infty} \frac{(l+d-2)!}{(d-1)!} \frac{(2l+d-1)}{l!} \int_0^\infty d\lambda \mu(\lambda) \\
\times \left\{ \lambda^2 + \left(\frac{(n-1)/2}{a^2}\right)^2 + \frac{l(l+d-1)}{b_0^2} e^{-2d/(n-2)+2/\kappa \sigma} \right\}^{-s},
\] (D.3)

For even dimensions, the Plancherel measure $\mu(\lambda)$ is given by \[ \mu(\lambda) = \frac{\pi \lambda \tanh(\pi \lambda)}{2^{2(n-2)} \{\Gamma(n/2)\}^2} \prod_{j=1/2}^{(n-3)/2} \left( \lambda^2 + j^2 \right). \] (D.4)

Then, using $D = (d-1)/2$ and $N = (n-1)/2$ instead of $d$ and $n$ and the running variables $L = l + D$, we rewrite (D.3) as

\[
\zeta_\phi(s) = \sum_{L=D}^{\infty} D\phi(L) \int_0^\infty d\lambda \mu(\lambda) \left( \frac{\lambda^2 + N^2}{a^2} + M_\phi^2 \right)^{-s},
\] (D.5)

where the $D\phi(L)$ and $A\phi(L)$ are given by

\[
D\phi(L) = \frac{2L^2}{(2D)!} \left\{ L^2 - (D-1)^2 \right\} \cdots \left\{ L^2 - 1 \right\},
\]

\[
A\phi(L) = L^2 - D^2,
\]

\[
M_\phi^2 = \frac{A\phi(L)}{b_0^2} e^{-2d/(n-2)+2/\kappa \sigma}.
\] (D.6)

Next, integrating Eq. (D.5) over $\lambda$, we find the expression

\[
\zeta_\phi(s) = \frac{\pi}{2^{2n-1} \{\Gamma(2N+1)\}^2} \frac{a^{2s}}{\Gamma(s)} \sum_{L=D}^{\infty} D\phi(L) \]
To regularize the mode sum in Eq. (D.7), we replace the infinite sum over $S_A$ where

\[
\sum_{j=1/2}^{\infty} \left( \lambda^2 + j^2 \right) \left( \frac{2\lambda}{(e^{2\pi\lambda} + 1)(\lambda^2 + N^2 + a^2M^2)} \right) , \tag{D.7}
\]

where we have used the identity

\[
\tanh(\pi\lambda) = 1 - \frac{2}{e^{2\pi\lambda} + 1}. \tag{D.8}
\]

To regularize the mode sum in Eq. (D.7), we replace the infinite sum over $L$ by complex integration. The generalized zeta function is then

\[
\zeta_\phi(s) = \pi \frac{i}{2^{2n-1} \{\Gamma(2N + 1)\}^2 \Gamma(s)} \left[ \frac{i}{4(s-1)(s-2)} \left\{ \left( \frac{a}{b_0} \right)^2 e^{-2d/(n-2)+2}\kappa_\sigma \right\}^{s+2} \right.

\times \int_{S_1} dz \, D_\phi(z) \cot(\pi z) \left( z^2 - A_L^2 \right)^{-s+2}

\left. + \frac{i}{2(s-1)} \left\{ \left( \frac{a}{b_0} \right)^2 e^{-2d/(n-2)+2}\kappa_\sigma \right\}^{s+1} \right]

\times \int_{S_1} dz \, D_\phi(z) \cot(\pi z) \left( z^2 - A_L^2 \right)^{-s+1}

- i \int_0^\infty d\lambda \prod_{j=1/2}^{(n-3)/2} \left( \lambda^2 + j^2 \right) \frac{1}{e^{2\pi\lambda} + 1} \int_{S_3} dz \, \cot(\pi z) \left( z^2 - B_L^2 \right)^{-s} \right], \tag{D.9}
\]

where $A_L^2$ and $B_L^2$ are given by

\[
A_L^2 = D^2 - N^2 \left( \frac{b_0}{a} \right)^2 e^{2d/(n-2)+2}\kappa_\sigma \partial, \tag{D.10}
\]

\[
B_L^2 = D^2 - \left( N^2 + j^2 \right) \left( \frac{b_0}{a} \right)^2 e^{2d/(n-2)+2}\kappa_\sigma \partial.
\]

and the contours $S_1$ and $S_3$ in the complex plane are showed in Figs. 11 and 12. Note that there are two branch points, $z = \pm A_L$, in the integration. We introduce the functions $\bar{\zeta}_k(s)$ and $\hat{\zeta}_k(s)$ as

\[
\bar{\zeta}_k(s) = \frac{\pi^2}{2^{2n-1} \{\Gamma(2N + 1)\}^2 \Gamma(s)} \left[ \frac{i}{4(s-1)(s-2)} \left\{ \left( \frac{a}{b_0} \right)^2 e^{-2d/(n-2)+2}\kappa_\sigma \right\}^{s+k} \right.

\times \int_{S_1} dz \, D_\phi(z) \cot(\pi z) \left( z^2 - A_L^2 \right)^{-s+k}

\left. \right] , \tag{D.11}
\]

\[
\hat{\zeta}_k(s) = \frac{\pi^2}{2^{2n-1} \{\Gamma(2N + 1)\}^2 \Gamma(s)} \int_0^\infty d\lambda \prod_{j=1/2}^{(n-3)/2} \left( \lambda^2 + j^2 \right) \frac{1}{e^{2\pi\lambda} + 1}

\times \int_{S_3} dz \, \cot(\pi z) \left( z^2 - B_L^2 \right)^{-s+k}.
\]
Fig. 11. The contour $S_1$ in Eq. (D.9) is replaced by the contour $S_2$. Note that the contour $S_2$ avoids the branch points at $z = \pm A_L$.

Using the above definition, $\zeta_\phi(s)$ is then expressed as

$$
\zeta_\phi(s) = \tilde{\zeta}_2(s) + 2(s - 2) \tilde{\zeta}_1(s) - \tilde{\zeta}_0(s). 
$$

(D.12)

Now, we move a contour $S_1$ to a parallel line along the imaginary axis, in order to deal with the poles of $\cot(\pi z)$ in (D.11) (see Fig. 11). The contour $S_2$ is replaced with lines passing just above the cuts associated with $z = \pm A_L$. $\tilde{\zeta}(s)$ is then given by

$$
\tilde{\zeta}_k(s) = \frac{\pi a^{2s}}{2^{2n-1} \{\Gamma(2N + 1)\}^2 \Gamma(s)} \left\{ \left( \frac{a}{b_0} \right)^2 e^{-[2d/(n-2)+2]k_\sigma \sigma} \right\}^{-s+k} \sin \{\pi (s - k)\}
$$

$$
\times \frac{1}{2(s - 1)(s - 2)} \left\{ \int_0^\infty dx ~ D_\phi(ix) \left( x^2 + A_L^2 \right)^{-s+k} \cot(\pi x)
$$

$$
- \int_0^{A_L} dx ~ \cot(\pi x) ~ D_\phi(x) \left( A_L^2 - x^2 \right)^{-s+k} \right\}, 
$$

(D.13)

where $D_\phi(ix)$ is a polynomial with coefficients $r_{NK}$:

$$
D_\phi(ix) = (-1)^D \frac{2x^2}{(2D)!} \left\{ x^2 + (D - 1)^2 \right\} \cdots \left\{ x^2 + 1 \right\}
$$

$$
\equiv (-1)^D \sum_{p=0}^{D-1} r_{NP} x^{2p+2}. 
$$

(D.14)

The first term in Eq. (D.11) comes from the integral along the imaginary axis, and the second term in Eq. (D.11) is the contribution from the contours along the cuts.
Fig. 12. The contour $S_3$ in Eq. (D.9) is replaced by the contour $S_4$. Note that the contour $S_4$ avoids the branch points at $z = \pm B_L$.

on the real axis. Substituting the relation

$$\coth(\pi x) = 1 + \frac{2}{e^{2\pi x} - 1}$$  \hspace{1cm} (D.15)$$

into the first term of (D.13), the function $\tilde{\zeta}_k(s)$ is finally given by

$$\tilde{\zeta}_k(s) = \frac{\pi a^{2s}}{2^{2n-1} \{ \Gamma(2N + 1) \}^2 \Gamma(s)} \left\{ \left( \frac{a}{b_0} \right)^2 e^{-[\sigma/(n-2)+2] \kappa \sigma} \right\}^{-s+k} \sin \{ \pi (s - k) \}
\times \frac{1}{2(s-1)(s-2)} \left\{ \frac{1}{2} \sum_{p=0}^{D-1} r_N p \left( A_L^2 \right)^{p+k+3/2-s} \frac{\Gamma(p+\frac{3}{2}) \Gamma(s-p-k-\frac{3}{2})}{\Gamma(s-k)} \right\}
- \int_0^\infty dx D_\phi(ix) \left( x^2 + A_L^2 \right)^{-s+k} \frac{2}{e^{2\pi x} - 1}
+ \int_0^{A_L} dx D_\phi(x) \left( A_L^2 - x^2 \right)^{-s+k} \cot(\pi x) \right\}.  \hspace{1cm} (D.16)$$

Also, the contour $S_3$ is replaced with lines passing just above the cuts associated with $z = \pm B_L$ (see Fig. 12). Using the procedure for calculating $\hat{\zeta}_k(s)$, the function $\hat{\zeta}_k(s)$ is then given by

$$\hat{\zeta}_k(s) = \frac{\pi a^{2s}}{2^{2n-1} \{ \Gamma(2N + 1) \}^2 \Gamma(s)} \left\{ \left( \frac{a}{b_0} \right)^2 e^{-[\sigma/(n-2)+2] \kappa \sigma} \right\}^{-s+k} \sin \{ \pi (s - k) \}
\times \int_0^\infty d\lambda \prod_{j=1/2}^{(n-3)/2} \left( \lambda^2 + j^2 \right) \frac{1}{e^{2\pi \lambda} + 1}$$
\[
\times \left\{ \frac{1}{2} \sum_{p=0}^{D-1} \mathcal{T}_{NP} (B_{L}^{2})^{p+k+3/2-s} \frac{\Gamma \left( p + \frac{3}{2} \right) \Gamma \left( s - p - k - \frac{3}{2} \right)}{\Gamma (s-k)} \right. \\
- \int_{0}^{\infty} dx \, D_{\phi} (ix) \left( x^{2} + B_{L}^{2} \right)^{-s+k} \frac{2}{e^{2\pi x} - 1} \\
+ \int_{0}^{B_{L}} dx D_{\phi} (x) \left( B_{L}^{2} - x^{2} \right)^{-s+k} \cot (\pi x) \right\}.
\]

(D-17)

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