Bijective Proofs of Identities from Colored Binary Trees

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Abstract. This note provides bijective proofs of two combinatorial identities involving generalized Catalan number $C_{m,5}(n) = \frac{m}{m+n} \binom{5n+m}{n}$ recently proposed by Sun.

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1. Introduction

Recently, by using generating functions and Lagrange inversion formula, Sun [2] deduced the following identity involving generalized Catalan number $C_{m,5}(n) = \frac{m}{m+n} \binom{5n+m}{n}$, i.e.,

$$
\sum_{p=0}^{[n/4]} \frac{m}{5p+m} \binom{5p+m}{p} \binom{n+p+m-1}{n-4p}
= \sum_{p=0}^{[n/2]} (-1)^p \frac{m}{m+n} \binom{m+n+p-1}{p} \binom{m+2n-2p-1}{n-2p},
$$

(1.1)

which, in the case $m = 1$, leads to

$$
\sum_{p=0}^{[n/4]} \frac{1}{5p+1} \binom{5p+m}{p} \binom{n+p}{n-4p}
= \sum_{p=0}^{[n/2]} (-1)^p \frac{1}{n+1} \binom{n+p}{p} \binom{2n-2p}{n-2p}.
$$

(1.2)

In this note, we give a parity reversing involution on colored binary trees which leads to a combinatorial interpretation of Formula (1.2). We make a simple variation of the bijection between colored ternary trees and binary trees proposed by Sun [2] and find a correspondence between certain class of binary trees and the set of colored 5-ary trees. The generalization of the parity reversing involution and the bijection to forests of colored binary trees and forests of colored 5-ary trees leads to a bijective proof of Formula (1.1).

2. A parity reversing involution on colored binary trees

In this section, we provide a parity reversing involution on a class of colored binary trees. Before introducing the involution, we recall some definitions and notations. Let $B_n$ denote the set of complete binary trees with $n$ internal vertices. Let $B \in B_n$ and $P = v_0v_1 \ldots v_k$ be a path
Let \( f \) be a function and \( B \) is said to be an initial vertex of \( B \) and \( P \) is called the associate path of \( v_0 \). Denote by \( l(v) \) the length of the associate maximal L-path of \( v \). A colored binary tree is a binary tree in which each initial vertex \( v \) is assigned a color \( c(v) \) such that \( 0 \leq c(v) \leq \lfloor l(v)/2 \rfloor \). The color number of a colored binary tree \( B \), denoted by \( c(B) \), is equal to the sum of all the colors of initial vertices of \( B \). A colored binary tree \( B \) with \( c(B) = 1 \) is illustrated in Figure 1.

![Figure 1: A colored binary tree B with c(B) = 1.](image)

Let \( CB_{n,p} = \{ B | B \in CB_n \text{ with } c(B) = p \} \) and \( f_{n,p} \) be its cardinality. Define \( CB_n = \bigcup_{p=0}^{\lfloor n/2 \rfloor} CB_{n,p} \).

Let \( f(x, y) = \sum_{n \geq 0} \sum_{p=0}^{\lfloor n/2 \rfloor} f_{n,p} y^p x^n \) be the ordinary generating function for \( f_{n,p} \) with the assumption \( f(0,0) = 1 \). Given a colored binary tree \( B \in CB_n \), let \( v \) be the root of \( B \). Suppose that \( c(v) = k \), then \( l(v) \geq 2k \). Then the generating function for the number of colored binary trees whose root \( v \) colored by \( k \) is equal to \( \sum_{k \geq 0} y^k (xf(x,y))^{2k} (xf(x,y))^{l(v)-2k} \). Summing over all the possibilities for \( k \geq 0 \), we arrive at

\[
f(x, y) = \frac{1}{(1 - yx^2f^2(x,y))(1 - xf(x,y))}.
\]

By applying Lagrange inversion formula, we get

\[
f^m(x, y) = \sum_{n \geq 0} \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{m}{n+m} \binom{m+n+p-1}{p} \binom{m+2n-2p-1}{n-2p} y^p x^n,
\]

which in the case of \( m = 1 \), reduces to

\[
f(x, y) = \sum_{n \geq 0} \sum_{p=0}^{\lfloor n/2 \rfloor} \frac{1}{n+1} \binom{n+p}{p} \binom{2n-2p}{n-2p} y^p x^n.
\]

Let \( B \in CB_n \) and \( v \) be an initial vertex of \( B \). If \( c(v) = 2k \) and \( l(v) = 4k \) or \( 4k + 1 \) for some nonnegative integer \( k \), then we say \( v \) is a proper initial vertex, otherwise it is said to be improper. A colored binary tree \( B \) is called proper if all the initial vertices are proper; otherwise, \( B \) is said to be improper. Denote by \( CB'_n \) the set of all proper binary trees with \( n \) internal vertices. Define \( ECB_n \) and \( OCB_n \) to be the sets of colored binary trees with \( n \) internal vertices whose color numbers are even and odd, respectively.

**Theorem 2.1** There is a parity reversing involution \( \phi \) on the set of improper colored binary trees with \( n \) internal vertices. Furthermore,

\[
|ECB_n| - |OCB_n| = |CB'_n|.
\]
Step 2. Let $T$ be an improper colored binary trees with $n$ internal vertices with $c(B) = p$. Traverse the binary tree $B$ by depth first search, let $v$ be the first improper vertex traversed. If $c(v)$ is even, then let $\phi(B)$ be a colored binary tree obtained from $B$ by coloring $v$ by $c(v) + 1$. If $c(v)$ is odd, then let $\phi(B)$ be a colored binary tree obtained from $B$ by coloring $v$ by $c(v) - 1$. Obviously, the obtained binary tree $\phi(B)$ are improper colored binary trees in both cases. Furthermore, in the former case the color number of $\phi(B)$ is $p + 1$, while in the latter case, the color number is $p - 1$. Hence $\phi$ is an involution on the set of improper colored binary trees with $n$ internal vertices, which reverses the parity of the color number. Since each initial vertex $v$ of a proper binary tree is colored by an even number, it is clear that $\mathcal{CB}_n' \subseteq \mathcal{ECB}_n$. Hence, we have

$$|\mathcal{ECB}_n| - |OCB_n| = \mathcal{CB}_n'.$$

From Theorem 2.1, we see that the right side summand of Formula (1.2) counts the number of proper colored binary trees, that is,

$$|\mathcal{CB}_n'| = \sum_{p=0}^{[n/2]} (-1)^p \frac{1}{n+1} \binom{n+p}{p} \binom{2n-2p}{n-2p}.$$ 

3. The bijective proof

A (complete) $k$-ary tree is an ordered tree in which each internal vertex has $k$ children. The number of $k$-ary trees with $n$ internal vertices is counted by the Catalan number $C(1,k)(n) = \frac{1}{kn+1} \binom{kn+1}{n}$ [1]. A colored 5-ary tree is a 5-ary tree in which each vertex is assigned a nonnegative integer called color number, denoted by $c_v$. Let $T_{n,p}$ denote the set of colored 5-ary trees with $p$ internal vertices such that the sum of all the color numbers of each tree is $n - 4p$. Denote by $T_n = \bigcup_{p=0}^{[n/4]} T_{n,p}$. Let $B \in \mathcal{CB}_n'$ be a proper binary tree with $n$ internal vertices. Since each initial vertex $v$ of $B$ is colored by $\lfloor l(v)/2 \rfloor$, we can discard all the colors of vertices of $B$.

Now we construct a map $\sigma$ from $T_n$ and $\mathcal{CB}_n'$ as follows:

Step 1. For each vertex $v$ of $T \in T_n$ with color number $c_v = k$, remove the color number and add a path $P = v_1v_2\ldots v_k$ to $v$ such that $v$ is a right child of $v_k$ and $v_1$ is a child of the father of $v$, and annex a left leaf to $v_i$ for $1 \leq i \leq k$. See Figure 2(a) for example.

Step 2. Let $T^*$ be the tree obtained from $T$ by Step 1. For any internal vertex $v$ of $T^*$ which has 5 children, let $T_1, T_2, T_3, T_4, T_5$ be the five subtrees of $v$. Annex a path $P = v_1v_2v_3$ to $v$ such that $v_1$ is a left child of $v$, then take $T_1$ and $T_2$ as the left and right subtree of $v_3$, take $T_3$ as the right subtree of $v_2$, and take $T_4$ as the subtree of $v_1$. See Figure 2(b) for example.

It is clear that the obtained tree $\sigma(T)$ is a proper binary tree with $n$ internal vertices.

Conversely, we can obtain a colored 5-ary tree from a proper binary tree by a similar procedure from binary trees to colored ternary trees given by Sun [2]. We omit the reverse map of $\sigma$ here.

**Theorem 3.2** The map $\sigma$ is a bijection between $T_n$ and $\mathcal{CB}_n'$.

Given a 5-ary tree $T$ with $p$ internal vertices, there are totally $5p + 1$ vertices. Choose $n - 4p$ vertices repeatedly, and define the color of the vertex as the number of times being chosen. Then
Figure 2: The bijection $\sigma$.

there are \(\binom{n+p}{n-4p}\) colored 5-ary trees in $T_n$ obtained from $T$. Since there are \(\frac{1}{5p+1}\binom{5p+1}{p}\) 5-ary trees with $p$ internal vertices, the involution $\phi$ and the bijection $\sigma$ lead to a combinatorial proof of Formula (1.2).

In order to prove Formula (1.1), we consider the forest of colored binary trees $F = (B_1, B_2, \ldots, B_m)$ with $n$ internal vertices and $m$ components where $B_i \in CB_{n_i}$ and $n_1 + n_2 + \ldots + n_m = n$. Define the color number of $F$ as the sum of all the color numbers of $B_i$, where $1 \leq i \leq m$. It is easy to check that the number of forests of colored binary trees with $m$ components and $n$ internal vertices, whose color number equals $p$, is counted by

$$[y^p x^n] f^m(x, y) = \frac{m}{n+m} \binom{m+n+p-1}{p} \binom{m+2n-2p-1}{n-2p}.$$

$F$ is said to be a proper forest if each $B_i$ is proper; otherwise, $F$ is said to be improper. Now we can modify the involution $\phi$ as follows: suppose that $B_k$ be the leftmost improper binary tree, then let $\phi(F)$ be the forest of colored binary tree obtained from $F$ by changing $B_k$ to $\phi(B_k)$. From Theorem 2.1, we see $\phi(F)$ is an improper forest of colored binary trees with $n$ internal vertices and $m$ components. Hence the modified involution $\phi$ is an involution on the set of improper forests of colored binary trees, which reverses the parity of the color numbers of the forests. Hence the right side summand of Formula (1.1) counts the number of proper forests of colored binary trees with $n$ internal vertices and $m$ components.

Let $F = (T_1, T_2, \ldots, T_m)$ be a forest of 5-ary trees such that $T_i \in T_{n_i}$ and $n_1 + n_2 + \ldots + n_m = p$. Define $\sigma(F) = (\sigma(T_1), \sigma(T_2), \ldots, \sigma(T_m))$. From Theorem 3.2, it is clear that $\sigma(F)$ is a proper forest of binary trees. Note that there are totally $m + 5p$ vertices in a forest $F$ of 5-ary trees with $p$ internal vertices and $m$ components, so there are \(\binom{n+p+m-1}{n-4p}\) forests of colored 5-ary trees with $m$ components, $p$ internal vertices and the sum of the color numbers equal to $n - 4p$. Hence the modified involution $\phi$ and the modified bijection $\sigma$ leads to a bijective proof of Formula (1.1).

References

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