Nonperturbative Renormalization in Light-Cone Quantization

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Abstract

Two approaches to nonperturbative renormalization are discussed for theories quantized on the light cone. One is tailored specifically to a calculation of the dressed-electron state in quantum electrodynamics, where an invariant-mass cutoff is used as a regulator and a Tamm–Dancoff truncation is made to include no more than two photons. The other approach is based on Pauli–Villars regulators and is applied to Yukawa theory and a related soluble model. In both cases discretized light-cone quantization is used to obtain a finite matrix problem that can be solved nonperturbatively.

I. INTRODUCTION

Light-cone quantization [1] has attracted some interest as a means to perform nonperturbative analyses of quantum field theories [2]. There are good reasons to hope that this technique will provide the leverage needed to obtain a qualitative, and perhaps quantitative, connection between quantum chromodynamics (QCD) and the constituent quark model [3]. Given the complexity of QCD, it is useful to first study simpler theories such as quantum electrodynamics (QED) and even models in 1 + 1 spacetime dimensions rather than 3 + 1 dimensions.

Bound-state calculations in QCD$_{3+1}$ and QED$_{3+1}$ require nonperturbative renormalization. Most attempts at such calculations have used Tamm–Dancoff truncations [4] and cutoff-type regularization, which require counterterms that depend on Fock sector [5]. An example of such a calculation is given here for the electron’s anomalous moment [6]. We then explore the practicality of Pauli–Villars regularization [7] as an alternative. In particular, we consider a simple heavy-fermion model abstracted from the Yukawa model.

We define light-cone coordinates by

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\[ x^\pm = t \pm z, \quad x_\perp = (x, y). \]  

Momentum variables are similarly constructed as
\[ p^\pm = E \pm p_z, \quad p_\perp = (p_x, p_y). \]  

The dot product is written
\[ p \cdot x = \frac{1}{2}(p^+ x^- + p^- x^+) - p_\perp \cdot x_\perp. \]  

The time variable is taken to be \( x^+ \), and time evolution of a system is then determined by the conjugate operator \( \mathcal{P}^- \). The energy \( E \) is replaced by the light-cone energy \( p^- \), and the momentum \( p \) by the light-cone momentum \( p \equiv (p^+, p_\perp) \). The light-cone Hamiltonian is
\[ H_{\text{LC}} = \mathcal{P}^+ \mathcal{P}^- - \mathcal{P}_\perp^2, \]  

where \( \mathcal{P}^+ \) and \( \mathcal{P}_\perp \) are momentum operators conjugate to \( x^- \) and \( x_\perp \). The eigenvalue problem is
\[ H_{\text{LC}} \Psi = M^2 \Psi, \quad \mathcal{P} \Psi = P \Psi, \]  

where \( M \) is the mass of the state.

Some of the advantages of light-cone coordinates are the following: They admit the largest possible set of nondynamical generators. In particular, boosts are kinematical. For many theories of massive particles, the perturbative vacuum is the physical vacuum, because \( p_\perp = 0 \) implies that no particle state can contribute to the \( P^+ = 0 \) vacuum. Thus there is no need to compute the vacuum state before computing massive states. Also, well-defined Fock-state expansions exist, with no disconnected vacuum pieces.

Such expansions are written as
\[ \Psi = \sum_n \int [dx]_n [d^2 k_\perp]_n \psi_n(x, k_\perp) |n : xP^+, xP_\perp + k_\perp\rangle, \]  

with \( n \) the number of particles, \( i \) ranging between 1 and \( n \), \( (P^+, P_\perp) \) the total light-cone momentum, and
\[ [dx]_n = 4\pi \delta(1 - \sum_{i=1}^n x_i) \prod_{i=1}^n \frac{dx_i}{4\pi \sqrt{x_i}}, \quad [d^2 k_\perp]_n = 4\pi^2 \delta(\sum_{i=1}^n k_{\perp i}) \prod_{i=1}^n \frac{d^2 k_{\perp i}}{4\pi^2}. \]  

In the Fock basis \( \{|n : p_{\perp i}^+, p_{\perp i}\rangle\} \), \( P^+ \) and \( P_\perp \) are diagonal. The amplitude \( \psi_n \) is interpreted as the wave function of the contribution from states with \( n \) particles.

A common numerical technique is discretized light-cone quantization (DLCQ) \cite{DLCQ}, in which periodic boundary conditions are assigned to bosons and antiperiodic to fermions in a light-cone box \(-L < x^- < L, -L_\perp < x, y < L_\perp\). Integrals are replaced by trapezoidal approximations on a grid: \( p^+ \to \frac{2}{L} n, \quad P_\perp \to \left(\frac{2}{L_\perp} n_x, \frac{2}{L_\perp} n_y\right), \) with \( n \) even for bosons and odd for fermions. The limit \( L \to \infty \) can be exchanged for a limit in terms of the integer \textit{resolution} \( K \equiv \frac{L}{2}P^+ \). The longitudinal momentum fraction \( x = p^+/P^+ \) becomes \( n/K \). \( H_{\text{LC}} \) is independent of \( L \).
Because the $n_i$ are all positive, DLCQ automatically limits the number of particles to no more than $\sim K/2$. The integers $n_x$ and $n_y$ range between limits associated with some maximum integer $N_\perp$ fixed by $L_\perp$ and a cutoff that limits transverse momentum.

To reduce the size of the discrete matrix problem, a Tamm–Dancoff truncation [4] in the number of particles can be applied. This has serious implications for renormalization. These include severe sector dependence of counterterms [5], and, for QED, violation of the Ward identity.

Regularization via cutoffs typically involves limits on the invariant mass. A limit can be placed on the total invariant mass of each Fock state

$$\sum_i \frac{m_i^2 + k_{1i}^2}{x_i} \leq \Lambda^2$$

or on the invariant mass of each particle

$$\frac{m_i^2 + k_{1i}^2}{x_i} \leq \Lambda^2.$$  (1.9)

There can also be a limit on the change in invariant mass across each matrix element of $H_{LC}$ [9]

$$\left| \sum_i \frac{m_i^2 + k_{1i}^2}{x_i} - \sum_j \frac{m_j^2 + k_{1j}^2}{x_j} \right| \leq \Lambda^2.$$  (1.10)

**II. THE ANOMALOUS MOMENT**

The anomalous moment $a_e = F_2(0)$ can be computed from a spin-flip matrix element of the electromagnetic current

$$-\frac{q_1}{2m_e} F_2(q^2) = \frac{1}{2P^+} \langle P + q, \uparrow | J^+(0) | P, \downarrow \rangle$$

in the standard light-cone frame $q = (0, q_\perp/P^+, q_\perp = q_1 \hat{x})$. Brodsky and Drell [10] have given a useful reduction of this matrix element to the form

$$a_e = -2m_e \sum_j e_j \sum_n \int [dx]_n [d^2k_\perp]_n \psi_{n1}^*(x, k_\perp) \sum_{i \neq j} x_i \frac{\partial}{\partial k_{1i}} \psi_{n1}(x, k_\perp),$$  (2.2)

where $e_j$ is the fractional charge of the struck constituent and $x_i = p_i^+ / P^+$. The wave functions $\psi_n$ satisfy coupled integral equations obtained from $H_{LC}\Psi = M^2\Psi$. The QED light-cone Hamiltonian has been given by Tang et al. [11]. However, the bare masses and couplings must be computed from sector dependent renormalization conditions.

Consider the case where there are at most two photons and only one electron. The Fock-state expansion can be written schematically as

$$\Psi = \psi_0|e\rangle + \tilde{\psi}_1|e\gamma\rangle + \tilde{\psi}_2|e\gamma\gamma\rangle.$$  (2.3)
Here $\vec{\psi}_1$ and $\vec{\psi}_2$ are column vectors that contain the amplitudes for individual Fock states with one and two photons, respectively. The eigenvalue problem becomes a coupled set of three integral equations

\begin{align*}
m_0^2 \psi_0 + b_1^\dagger \cdot \vec{\psi}_1 + b_2^\dagger \cdot \vec{\psi}_2 &= M^2 \psi_0, \\
b_1 \psi_0 + A_{11} \vec{\psi}_1 + A_{12} \vec{\psi}_2 &= M^2 \vec{\psi}_1, \\
b_2 \psi_0 + A_{12}^\dagger \vec{\psi}_1 + A_{22} \vec{\psi}_2 &= M^2 \vec{\psi}_2,
\end{align*}

(2.4)

where $m_0$ is the bare electron mass and $b_i^\dagger$ and $A_{ij}$ are integral operators obtained from matrix elements of $H_{LC}$.

The bare electron mass in the one-photon sector is computed from the one-loop self-energy allowed by the two-photon states. We then require that $m_0$ be such that $M^2 = m_e^2$ is an eigenvalue. The second and third equations can be solved for $\vec{\psi}_1/\psi_0$ and $\vec{\psi}_2/\psi_0$. Then the first equation yields $m_0$. Normalization of $\Psi$ fixes the value of $\psi_0$.

The bare coupling for the electron-photon three-point vertex depends on the initial and final momenta of the electron and on the sectors between which the coupling acts. The momentum dependence is present because the amount of momentum available constrains the extent to which higher order corrections can contribute. Similarly, the sector dependence makes itself felt when the number of additional particles in higher-order corrections is restricted. The coupling is fixed by the ratio of the $e\gamma \rightarrow e$ transition matrix element to the bare vertex at zero photon momentum.

In the present calculation we use a Tamm–Dancoff truncation to $\{e, e\gamma, e\gamma\gamma\}$, a nonzero photon mass $m_\gamma = m_e/10$, and a moderate coupling $\alpha = 1/10$. Some results are given elsewhere [6]. When only states with at most one photon and no pairs are retained, one can show that $a_e$ reduces to

\begin{align}
a_e &= \frac{\alpha m_e^2}{\pi^2} \int \frac{dx \, d^2 l_\perp}{x} \frac{\theta(A^2 - (m_e^2 + k_\perp^2)/x - (m_e^2 + k_\perp^2)/(1-x))}{[m_e^2 - (m_e^2 + k_\perp^2)/x - (m_\gamma^2 + k_\perp^2)/(1-x)]^2},
\end{align}

(2.5)

which in the limit of $\Lambda \rightarrow \infty$ becomes [10]

\begin{align}
a_e &= \frac{\alpha}{2\pi} \int_0^1 \frac{2x^2 (1-x) dx}{x^2 + (1-x)(m_\gamma/m_e)^2}.
\end{align}

(2.6)

For $m_\gamma = 0$, this yields the standard Schwinger contribution [12] of $\alpha/2\pi$.

### III. YUKAWA THEORY AT ONE LOOP

As an alternative approach to regularization, we consider Pauli–Villars [7] regularization of the 3 + 1 Yukawa model [13,14]. The one-loop fermion self-energy is proportional to

\begin{align}
I(\mu^2, M^2) \equiv -\frac{1}{\mu^2} \int \frac{dl^+ d^2 l_\perp}{l^+(q^+ - l^+)^2} \frac{(q^+)^2 l_\perp^2 + (2q^+ - l^+)^2 M^2}{M^2 - D_1} \theta(A^2 - D_1),
\end{align}

(3.1)

where $q$ is the fermion momentum, $\mu$ is the boson mass, $M$ is the fermion mass, and
TABLE I. Values of the subtracted integral $I_{\text{sub}}(M^2/\mu^2, \mu_i^2/\mu^2)$ in the limit of infinite cutoff. The Pauli–Villars masses are $\mu_1^2 = 10\mu^2$, $\mu_2^2 = 50\mu^2$ and $\mu_3^2 = 100\mu^2$.

| $M^2$ | 0  | 0.05$\mu^2$ | 0.1$\mu^2$ | 0.2$\mu^2$ |
|-------|----|-------------|-------------|-------------|
| $I_{\text{sub}}$ | -0.064 | 0.70 | 1.37 | 2.70 |

$$D_1 = \frac{\mu^2 + \mu_1^2}{l^+/q^+} + \frac{M^2 + \mu_1^2}{(q^+ - l^+)/q^+}.$$  \hspace{1cm} (3.2)

The boson mass $\mu$ sets the energy scale. When $M^2 = 0$ we obtain

$$I(\mu^2, 0) = \frac{\pi}{\mu^2} \left[ \frac{\Lambda^2}{2} - \frac{\mu^4}{2\Lambda^2} - \mu^2 \ln \left( \frac{\Lambda^2}{\mu^2} \right) \right].$$ \hspace{1cm} (3.3)

In order to maintain $I(\mu^2, M^2) \propto M^2$, three Pauli-Villars bosons are needed:

$$I_{\text{sub}}(\mu^2, M^2, \mu_i^2) = I(\mu^2, M^2) + \sum_{i=1}^{3} C_i I(\mu_i^2, M^2).$$ \hspace{1cm} (3.4)

The $C_i$ are chosen to satisfy

$$1 + \sum_{i=1}^{3} C_i = 0, \quad \mu^2 + \sum_{i=1}^{3} C_i \mu_i^2 = 0, \quad \sum_{i=1}^{3} C_i \mu_i^2 \ln(\mu_i^2/\mu^2) = 0.$$ \hspace{1cm} (3.5)

A DLCQ calculation of $I_{\text{sub}}$ has been done \[16\], with values of 20, 22, and 24 for $K$ and 25 through 30 for $N_\perp$. Modification of the trapezoidal rule, with introduction of unequal weights, is necessary to obtain sufficient accuracy. Each integral in (3.3) was separately extrapolated to infinite $K$ and $N_\perp$ via fits to either $c_0 + a_1/K^3 + b_1/N_\perp^2$ or $c_0 + a_1/K^3 + a_2/K^4 + b_1/N_\perp^2 + b_2/N_\perp^4$. The latter was used for the $\mu_1$ integral. Extrapolation after subtraction is not as accurate. The resulting values of $I_{\text{sub}}$ were extrapolated to infinite cutoff by fits to $a + b/\Lambda^2$. These fully extrapolated values are given in Table I.

The magnitude of the error in each extrapolated integral was found to be $\leq 0.02$ when compared to the analytic result for $M^2 = 0$. This implies an error of $\pm 0.04$ in the $I_{\text{sub}}$ values. The extrapolation in $\Lambda^2$ induces additional uncertainty reflected in the miss of zero by 0.06 for $M^2 = 0$. The values in Table I are consistent with $I_{\text{sub}} \propto M^2$ to within this amount of error.

The number of Fock states required for Pauli–Villars particles is approximately 1.5 times the number for physical states. A listing of counts for two cases is given in Table I. Making $\mu_1$ larger does decrease the number of Pauli-Villars states but this increases the coefficients $C_i$ and thereby amplifies errors in the integrals. Also, with fewer states, the integrals themselves are approximated less accurately.

We could also consider the boson self energy. To lowest order there is a fermion loop contribution

$$\int \frac{dl^+ d^2 l_\perp}{4L^2_\perp} \frac{q^+ (l_\perp^2 + M^2) \theta (\Lambda^2 - D_2)}{l^+ (q^+ - l^+)^2 (\mu^2 - D_2)},$$ \hspace{1cm} (3.6)
where

\[ D_2 \equiv q^+ (M^2 + l_1^2) / [l^+ (q^+ - l^+)] , \]

and a \( \phi^4 \) contribution

\[
\int \frac{d^4l_1^+ d^2l_1^- dk_1^+ d^2k_1^-}{q^+ l^+ (q^+ - l^+ - k^+)} \theta (\Lambda^2 - D_4) / \mu^2 - D_4 ,
\]

where

\[ D_4 \equiv \frac{\mu^2 + l_1^2}{l^+ / q^+} + \frac{\mu^2 + k_1^2}{k^+ / q^+} + \frac{\mu^2 + (l_1^+ + k_1^+)^2}{(q^+ - l^+ - k^+)/q^+} . \]

A Pauli–Villars fermion may be needed.

### IV. A HEAVY-FERMION MODEL

By some severe modifications of the Yukawa Hamiltonian \([17]\) we obtain the following model Hamiltonian:

\[
H_{LC}^{\text{eff}} = M_0^2 \int \frac{dp^+_1 dp^+_2 p^-_1}{16 \pi^3 p^+} \sum_{\sigma} b_{\sigma}^\dagger b_{\sigma} + P^+ \int \frac{dq^+_1 dq^+_2}{16 \pi^3 q^+} \left[ \frac{\mu^2 + q_1^2}{q^+} a_{12}^\dagger a_{12} + \frac{\mu^2 + q_2^2}{q^+} a_1^\dagger a_2 \right] \\
+ g \int \frac{dp_1^+ dp_2^+ p_1^+ p_2^+}{16 \pi^3 p_1^+ p_2^+} \int \frac{dp_3^+ dp_4^+ p_3^+ p_4^+}{16 \pi^3 p_3^+ p_4^+} \sum_{\sigma} b_{\sigma}^\dagger b_{\sigma} \\
\times \left[ a_2^\dagger \delta(p_1 - p_2 + q) + a_2 \delta(p_2 - p_1 + q) + i a_1^\dagger \delta(p_1 - p_2 - q) \right] .
\]

The kinetic energy of the fermion is no longer momentum dependent and only a modified no-flip three-point vertex remains as an interaction. The fermion then acts as a “static” source for the boson. We include one Pauli–Villars field, which will prove sufficient in this case. Similar Hamiltonians, without the Pauli–Villars field, have been considered in equal-time \([18]\) and light-cone coordinates \([19]\).

We write the eigenvector as a Fock-state expansion

\[
\Phi_\sigma = \sqrt{16 \pi^3 p^+} \sum_{n_1} \int \frac{dp_1^+ dp_1^-}{16 \pi^3 p_1^+} \prod_{i=1}^n \int \frac{dq_i^+ dq_i^-}{16 \pi^3 q_i^+} \prod_{j=1}^{n_i} \int \frac{dr_{1j}^+ dr_{1j}^-}{16 \pi^3 r_{1j}^+} \\
\times \delta(p - p - \sum_i q_i - \sum_j r_{ij}) \phi^{(n,n_1)}(q, l_j; p) \frac{1}{\sqrt{n! n_1!}} b_{\sigma}^\dagger \prod_{j} a_{1j}^\dagger a_{1j} |0\rangle ,
\]

where

\[ \Lambda \]

and \( \phi \) model Hamiltonian:

A Pauli–Villars fermion may be needed.

### TABLE II. Number of Fock states used in two typical cases.

| \( \Lambda^2/\mu^2 \) | \( K \) | \( N_\perp \) | Physical boson states | Pauli-Villars boson states |
|-----------------|-----|------|-----------------------|---------------------------|
| 200             | 20  | 25   | 25975                | 22602, 11142, 3305, 37049 |
| 200             | 24  | 30   | 44943                | 39162, 19293, 5695, 64150 |
normalized according to $\Phi^t_\sigma \cdot \Phi_\sigma = 16\pi^3 P^+ \delta(P' - P)$, which yields
\[
1 = \sum_{n,n_1} \prod_i n_i \int dq_i^+ d^2 q_{\perp i} \prod_j n_j \int dr_j^+ d^2 r_{\perp j} \left| \phi^{(n,n_1)}(q_i, r_j; P - \sum_i q_i - \sum_j r_j) \right|^2. \tag{4.3}
\]
For $\Phi_\sigma$ to satisfy the Schrödinger equation (4.3), the amplitudes must satisfy
\[
\left[ M^2 - M_0^2 - \sum_i \frac{\mu_i^2 + q_{\perp i}^2}{z_i} - \sum_j \frac{\mu_j^2 + r_{\perp j}^2}{z_j} \right] \phi^{(n,n_1)} = g \left\{ \sqrt{n + 1} \int \frac{dq_i^+ d^2 q_{\perp i}}{\sqrt{16\pi^3 q_i^+}} \phi^{(n+1,n_1)}(q_i, q_{\perp i}; P) \right.
\]
\[+ \frac{1}{\sqrt{n}} \sum_i \frac{1}{\sqrt{16\pi^3 q_i^+}} \phi^{(n-1,n_1)}(q_i, \ldots, q_{i-1}, q_{i+1}, \ldots, q_i, r_j, P) \]
\[+ i\sqrt{n_1 + 1} \int \frac{dr_j^+ d^2 r_{\perp j}}{\sqrt{16\pi^3 r_j^+}} \phi^{(n,n_1+1)}(q_j, r_j; P) \]
\[+ \frac{1}{\sqrt{n}} \sum_j \frac{1}{\sqrt{16\pi^3 r_j^+}} \phi^{(n,n_1-1)}(q_j, r_1, \ldots, r_{j-1}, r_{j+1}, \ldots, r_{n_1}, P) \right\}. \tag{4.4}
\]
The structure of this coupled set of integral equations is deliberately identical in basic form to the equations considered by Greenberg and Schweber [13]. Therefore, we transcribe their ansatz for a solution to light-cone form
\[
\phi^{(n,n_1)} = \sqrt{Z} \frac{(-g)^n (-i g)^{n_1}}{\sqrt{n! n_1!}} \prod_i \frac{q_i^+}{\sqrt{16\pi^3 q_i^+(\mu_i^2 + q_{\perp i}^2)}} \prod_j \frac{r_j^+}{\sqrt{16\pi^3 r_j^+(\mu_j^2 + r_{\perp j}^2)}}. \tag{4.5}
\]
This does work as a solution if $M_0^2(\mu_i)$ is chosen to satisfy
\[
M^2 - M_0^2 = -\frac{g^2}{16\pi^3} \left\{ \int \frac{dy d^2 q_{\perp}}{\mu^2 + q_{\perp}^2} - \int \frac{dz d^2 r_{\perp}}{\mu_j^2 + r_{\perp}^2} \right\}. \tag{4.6}
\]
From the normalization condition (4.3) we obtain
\[
\frac{1}{Z} = \exp \left\{ \frac{g^2}{16\pi^3} \left[ \int \frac{y dy d^2 q_{\perp}}{(\mu^2 + q_{\perp}^2)^2} + \int \frac{z dz d^2 r_{\perp}}{(\mu_j^2 + r_{\perp}^2)^2} \right] \right\}. \tag{4.7}
\]
The bare mass and wave function renormalization are thus determined as functions of the Pauli–Villars mass.

To fix the coupling we could use the slope of the fermion no-flip form factor, which is related to the transverse size of the dressed fermion. The form factor is most easily evaluated from [10]
\[
F(Q^2) = \frac{1}{2P^+} \langle P + p_\uparrow | J^+(0) | P_\uparrow \rangle \tag{4.8}
\]
\[= \sum_{j} e_j \int 16\pi^3 \delta(1 - \sum_i x_i) \delta(\sum_i k_{\perp i}) \prod_i dx_i d^2 p_{\perp i} \frac{d^2 p_{\perp i}}{16\pi^3}
\times \psi^*_{P + p_\uparrow}(x_i, p_{\perp i}) \psi_{P_\uparrow}(x_i, p_{\perp i}),
\]
where the matrix element has been evaluated in the frame with
\[ P = \left( \frac{M^2}{p^2}, 0_\perp \right), \quad p_\gamma = (0, p_\gamma = 2p_\gamma \cdot P^+, P_\gamma), \quad Q^2 \equiv p_\gamma^2, \]
and
\[ e_j \text{ is the charge of the } j^{\text{th}} \text{ constituent, and} \]
\[ p_{\perp i} = \begin{cases} p_{\perp i} - x_i p_\gamma & i \neq j \\ p_{\perp i} + (1 - x_i) p_\gamma & i = j. \end{cases} \]

A sum over Fock states is understood.

When the fermion is assigned a charge of 1, and the bosons remain neutral, the analytic solution for the amplitudes yields
\[ F(Q^2) = Z \exp \left\{ g^2 \int \frac{dy d^2 q_\perp}{16\pi^3} \frac{\sqrt{y}}{\mu^2 + q_\perp^2 \mu^2 + q_\perp^2} + \text{P-V term} \right\}, \]
with
\[ q_\perp = q_\perp - y p_\perp. \]

From this we find
\[ F'(0) = -g^2 \int \frac{dy d^2 q_\perp}{16\pi^3} \frac{y^3}{(\mu^2 + q_\perp^2)^3} \left[ \frac{2\mu^2}{\mu^2 + q_\perp^2} - 1 \right] + \text{P-V term}. \]

Numerically, the slope is computed from a finite-difference approximation to
\[ F'(0) = \sum_{n, n_1} \prod_i^n \int dq_i^+ d^2 q_{\perp i} \prod_j^{n_1} \int dr_j^+ d^2 r_{\perp j} \]
\[ \times \left[ \left( \sum_i \frac{y_i^2}{4} \nabla_{\perp i}^2 + \sum_j \frac{z_j^2}{4} \nabla_{\perp j}^2 \right) \phi^{(n, n_1)}(q_i, r_j; P - \sum_i q_i - \sum_j r_j) \right]^* \]
\[ \times \phi^{(n, n_1)}(q_i, r_j; P - \sum_i q_i - \sum_j r_j). \]

With the bare parameters determined, we “predict” a value for \( \langle \phi^2(0) \rangle \). For the analytic solution, this expectation value reduces to
\[ \langle \phi^2(0) \rangle = \frac{g^2}{8\pi^2\mu^2} \left[ 1 - \frac{\mu^2}{\Lambda^2} - \frac{\mu^2}{\Lambda^2} \ln \frac{\mu^2}{\Lambda^2} \right]. \]

From a numerical solution it can be computed from a sum similar to the normalization sum
\[ \langle \phi^2(0) \rangle = \sum_{n=1, n_1=0}^n \prod_i^n \int dq_i^+ d^2 q_{\perp i} \prod_j^{n_1} \int dr_j^+ d^2 r_{\perp j} \left( \sum_{k=1}^n \frac{2}{q_k^+ / P^+} \right) \]
\[ \times \left| \phi^{(n, n_1)}(q_i, r_j; P - \sum_i q_i - \sum_j r_j) \right|^2. \]
V. SUMMARY

For the anomalous moment calculation there remain several hurdles. The Tamm–Dancoff truncation results in logarithmically divergent four-point graphs. To deal with these will probably require use of scattering processes, such as Compton scattering [20], to obtain renormalization conditions. Verification of the removal of all logarithms and restoration of symmetries can then be undertaken. Also neglected up to this point have been zero modes, photon modes of zero longitudinal momentum [21]. How they might be included has been indicated by Kalloniatis and Robertson [22].

Additional physics could be included in the calculation by introducing an effective interaction from Z graphs or even putting eee$^+$ states in the basis. In the latter case, photon mass renormalization must be done.

In the Yukawa-model calculations we have learned that Pauli–Villars Fock states increase the basis size by only 150%, which may not be prohibitive. To perform such calculations accurately with a minimal basis size, improvement of ordinary DLCQ, by inclusion of weighting factors, is critical.

We have found a simple 3 + 1 model, related to Yukawa theory, which can be solved analytically. Here we will attempt a nonperturbative numerical solution to further test the use of Pauli–Villars regularization in DLCQ. If successful, we can begin to increase the complexity of the model, eventually reaching the full Yukawa theory.

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