Effective lattice action for the configurations smeared by the Wilson flow

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We investigate a trajectory for the Wilson flow in the theory space. For this purpose, we determine the coefficient of the plaquette and rectangular terms in the action for the configurations defined by the solution of the Wilson flow. The demon method regarded as one of the inverse Monte Carlo methods is used for the determination of them. Starting from the conventional Wilson plaquette action of quenched QCD, we find that the coefficient of the plaquette grows while that of the rectangular tends to negative with the development of the flow as the known improved actions. We also find that the trajectory forms a straight line in the two-coupling theory space.

Subject Index B01, B32, B38
1 Introduction

The lattice gauge theory is one of the most powerful tools for investigating the quantum field theory. However, it is not suitable for studying quantities which are sensitive to the cutoff: the topological charge, energy momentum tensor and so on. The Wilson flow is considered to be a remedy for such issues [1–9]. It is regarded as a continuous stout smearing and one can measure such observables even on the discretized spacetime [10, 11]. In Ref. [10], M. Lüscher has studied the effective action for the configuration at finite $\hat{t} = t/a^2$ in the context of the transformation in the field space and given an analytic form of it. The purpose of this paper is to determine it numerically by the so-called demon method.

The demon method has been proposed by M. Creutz [12] and improved by M. Hasenbusch et al. [13], which enables us to determine effective couplings from a given configuration. There has been various applications of this method. For example, investigating the couplings induced by the fermionic determinant [14], studying the renormalization group of the $SU(3)$ lattice gauge theory [15], and describing the $SU(3)$ lattice gauge theory in terms of effective Polyakov-loop models [16]. Using the method, one can investigate effective actions without any perturbative approximation.

A similar work to ours has been done by QCD-TARO collaboration [17, 18]. They have studied the renormalization trajectory for quenched QCD in the two-coupling theory space performing the blocking of the link variables [19] and using the Schwinger-Dyson equation method [20] to compute the effective coupling of the action for the blocked configurations. As mentioned above, since the Wilson flow is regarded as a continuous smearing, we compare our effective action with the ones investigated in Ref. [17, 18]. As we will see later, our effective action shows the same tendency as the known improved action, however it has different pictures.

In this work, we determine a trajectory for the Wilson flow in the theory space using the demon method. We choose $\beta = 6.0$ Wilson plaquette action for configuration generation and apply the Wilson flow. The range of the flow time is taken to $0 \leq \sqrt{8\hat{t}} \lesssim 1.3$. In order to evaluate finite size effects, our lattice sizes are taken to $\hat{L}^4 = (L/a)^4 = 4^4, 8^4 \text{ and } 16^4$. We find that the coefficient of the plaquette grows while that of the rectangular tends to negative with the flow time as the known improved actions. We also find that the trajectory forms a straight line in the two-coupling theory space.

This paper is organized as follows. In Sec. 2 we briefly review the demon method and explain our strategy to obtain the effective action. A numerical setup and results are shown in Sec. 3. Finally, a conclusion, discussions and future perspectives are given in Sec. 4.
In this section, we review the demon method \cite{12, 13} and how we utilize it for the determination of the effective action for configurations which are continuously smeared by the Wilson flow equation \cite{1}:

\[
\dot{\bar{V}}_{\mu}(x; t) = -g_0^2 \{ \partial_{x,\mu} S_W(\bar{V}) \} \bar{V}_{\mu}(x; t), \quad \bar{V}_{\mu}(x; t)|_{t=0} = U_{\mu}(x),
\]

where \( S_W(U) \) is defined by

\[
S_W(U) = \beta \sum_{x, \mu < \nu} \left[ 1 - \frac{1}{3} \text{Re} \text{Tr} \ W_{\mu\nu}^{1\times 1}(x) \right], \quad \beta = \frac{6}{g_0^2},
\]

\[
W_{\mu\nu}^{1\times 1}(x) = U_{\mu}(x)U_{\nu}(x + \mu)U^\dagger_{\mu}(x + \nu)U^\dagger_{\nu}(x).
\]

For the definition of the effective action, let us consider an expectation value of any operator \( O[\bar{V}_t(U)] \)

\[
\langle O \rangle_t = \frac{1}{Z} \int DU O[\bar{V}_t(U)] e^{-S_W[U]},
\]

where

\[
Z = \int DU e^{-S_W[U]},
\]

and \( \bar{V}_t \equiv \{ \bar{V}_{\mu}(x; t) \} \). The effective action is defined by inserting the delta function and changing the integration variables from \( U \) to \( V_t \) as follows.

\[
\langle O \rangle_t = \frac{1}{Z} \int DU \left[ \int DV_t \delta(V_t - \bar{V}_t(U)) \right] O[\bar{V}_t(U)] e^{-S_W[U]}
\]

\[
= \frac{1}{Z} \int DU D\bar{V}_t \delta(V_t - \bar{V}_t(U)) O[V_t] e^{-S_W[U]}
\]

\[
\equiv \frac{1}{Z_{\text{eff}}} \int DV_t O[V_t] e^{-S_{\text{eff}}[V_t]},
\]

where

\[
Z_{\text{eff}} = \int DV_t e^{-S_{\text{eff}}[V_t]}.
\]

\( S_{\text{eff}} \) can be regarded as the “improved” action since the discretization effects for observables are reduced at finite \( \hat{t} \).
2.1 The demon method

Here, we briefly review the demon method which enables us to determine the couplings of the action from a given configuration.

Let us consider a configuration whose distribution is the Boltzmann weight

$$P(U) \propto e^{-\beta S[U]},$$

(7)

where $\beta$ is a priori unknown coupling and to be determined by the demon method.

Now we introduce an extra degree of freedom “demon” to the system and consider a microcanonical partition function of a joint system,

$$Z_{\text{mic}} = \sum_U \sum_{E_d} \delta(S[U] + E_d - E_0),$$

(8)

where $E_0$ is an initially determined total energy of the joint system, and $E_d$ is the energy carried by the demon. The demon energy is restricted within the range $E_{\text{min}} < E_d < E_{\text{max}}$ which can be chosen suitably. Hereafter we set $E_{\text{min}} = -E_{\text{max}}$ for simplicity. Starting from a given configuration, we update the system keeping $S[U] + E_d$ constant. The updating process is described as follows. First, a new configuration $U'$ is proposed. To keep the total energy constant, the new demon energy is given by

$$E_d' = E_d - S[U'] + S[U].$$

(9)

When $E_d'$ is in the allowed region $[-E_{\text{max}}:E_{\text{max}}]$, the new configuration and the new demon energy are accepted, otherwise they are rejected and the configuration and the demon energy remain to be same. If the probability of the change is symmetric in $\{U, E_d\}$ and $\{U', E_d'\}$, a generated sequence of combined configurations is distributed according to a uniform distribution.

When the degrees of freedom of $U$ are sufficiently large, a given configuration behaves as a heat bath to thermalize the demon. Therefore standard statistical mechanics arguments show that $E_d$ is distributed with

$$P(E_d) \propto e^{-\beta E_d},$$

(10)

in the thermodynamic limit. The average of $E_d$ is given by

$$\langle E_d \rangle = \frac{1}{Z} \int_{-E_{\text{max}}}^{E_{\text{max}}} dE_d \ E_d \ e^{-\beta E_d} = \frac{1}{\beta} \left[ 1 - \frac{\beta E_{\text{max}}}{\tanh(\beta E_{\text{max}})} \right],$$

(11)

where

$$Z = \int_{-E_{\text{max}}}^{E_{\text{max}}} dE_d \ e^{-\beta E_d}.$$  

(12)

Finally, $\beta$ is obtained by measuring $\langle E_d \rangle$ and solving Eq. (11).
The extension to a multi-coupling system is straightforward. Suppose that the action is parametrized by

\[ S[U] = \sum_i \beta_i S_i[U], \]

where \( S_i[U] \) is the interaction term and \( \beta_i \) is the corresponding coupling. Now we introduce demons \( E_d^i \) for each coupling. During the microcanonical update, a proposed configuration and demon energies are accepted only if all demon energies are in the allowed region. Here we take the common allowed region \([ -E_{\text{max}} : E_{\text{max}} ]\) to all couplings for simplicity. \( \beta_i \) is obtained by solving

\[ \langle E_d^i \rangle = \frac{1}{\beta_i} \left[ 1 - \frac{\beta_i E_{\text{max}}}{\tanh(\beta_i E_{\text{max}})} \right]. \]

2.2 Our strategy

In this subsection, we explain our strategy to obtain the effective action defined by Eq. (5). We assume that \( S_{\text{eff}} \) can be parametrized by the form of Eq. (13), where \( S_i[U] \) denotes several types of Wilson loops such as the plaquette, rectangular, chair, sofa, etc. The simulation is implemented as the following steps.

1. Evolve a given configuration by integrating the Wilson flow equation Eq. (1) from 0 to \( \hat{t} \).
2. Perform the microcanonical updates for the joint system using the configuration at \( \hat{t} \) according to the procedure described in Subsec. 2.1.
3. After a sufficiently large number of microcanonical updates, take an average of the demon energy and obtain \( \beta_i \) by solving Eq. (14).
4. Replace the configuration at \( \hat{t} = 0 \) by a new statistically independent one. This is done by the HMC updates of the original system at \( \hat{t} = 0 \). In this step the demon degrees of freedom are frozen.
5. Go back to the step (1) with the new configuration. The initial demons’ energies in the step (2) are given by the averages of the previous run.

The flowchart of the above algorithm is shown schematically in Fig. 1. By taking averages of \( \beta_i \)s obtained for each gauge configuration, we finally determine the couplings of the effective action. The step (4) and (5) are needed to suppress the systematic errors coming from the finiteness of the volume [13].
3 Numerical setup and results

3.1 Numerical setup

Here, we introduce our numerical setup. We employ the Wilson plaquette action defined by Eq. (2) with \( \beta = 6.0 \) for configuration generations. Configurations are generated by the HMC algorithm. Our lattice sizes are \((L/a)^4 = \hat{L}^4 = 4^4, 8^4, \) and \(16^4\).

In this work, we implement the demon method with the truncation

\[
S_{\text{eff}}(U) = \beta_{\text{plaq}} \sum_{x,\mu<\nu} \left[ 1 - \frac{1}{3} \text{ReTr} \ W_{\mu\nu}^{1\times 1}(x) \right] + \beta_{\text{rect}} \sum_{x,\mu\not=\nu} \left[ 1 - \frac{1}{3} \text{ReTr} \ R_{\mu\nu}^{1\times 2}(x) \right],
\]

where \( R_{\mu\nu}^{1\times 2}(x) \) is the rectangular loop defined by

\[
R_{\mu\nu}^{1\times 2}(x) = U_\mu(x)U_\mu(x+\mu)U_\nu(x+2\mu)U_\mu^\dagger(x+\nu+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x).
\]

Each gauge configuration is separated by 10 trajectories of the HMC. The total number of measurements is 500, 500 and 250 for \( \hat{L}^4 = 4^4, 8^4, \) and \(16^4\) respectively. The autocorrelation times defined by the average of the plaquette of each configuration are around 4.6, 11 and 11 for \( \hat{L}^4 = 4^4, 8^4 \) and \(16^4\).

Our measurements are performed as follows. All of the initial demon energies are set to be 0 and \( E_{\text{max}} \) is set to be 5.0. We perform 10000 microcanonical updates of the joint system with a given trial configuration for the thermalization of the demon energies. First 2000 updates are discarded and the averages of the demon energies over the last 8000 updates
are taken. These averages are used as the initial demon energies for the next microcanonical updates. Following steps are repeated during the measurements. After a replacement of the gauge configuration, we always perform the 10000, 5000 and 5000 microcanonical updates for $\hat{L}^4 = 4^4$, $8^4$ and $16^4$. First 2000, 1000 and 1000 updates are discarded and we take the averages of the demon energies over the last 8000, 4000 and 4000 updates for $\hat{L}^4 = 4^4$, $8^4$ and $16^4$ respectively. The above procedures are implemented for each flow time. The range of the flow time is taken to $0 \leq \sqrt{8\hat{t}} \lesssim 1.3$.

3.2 Results

We show the results of the flow time dependence of the effective action in this subsection. The values of $\beta_{\text{plaq}}$ and $\beta_{\text{rect}}$ for $0 \leq \hat{t} \leq 0.20$ are listed in Table. We examine the consistency between the effective action for $\hat{t} = 0$ and the initial action. Since our initial couplings $(\beta_{\text{plaq}}, \beta_{\text{rect}}) = (6.0,0.0)$ are reconstructed within 2 sigma, the demon method works well.

The results show that the value of $\beta_{\text{plaq}}$ grows with $\hat{t}$ (Fig. 2). We expect that this is because the Wilson flow has a smoothing effect on the gauge field and lowers the value of the Wilson action as $\hat{t}$ increases [10].

On the other hand, the value of $\beta_{\text{rect}}$ tends to a negative region and decreases with $\hat{t}$ (Fig. 2). This fact seems reasonable since the negativeness of $\beta_{\text{rect}}$ is a common feature of the known improved actions such as the Symanzik [21–23], Iwasaki [24, 25] and DBW2 action [18]. Note that our scheme is not based on any perturbative analysis and therefore defines the non-perturbatively “improved” action. Moreover, the action may be systematically improved by adding conceivable Wilson loops to the ansatz of $S_{\text{eff}}$.

$\beta_{\text{plaq}}$ and $\beta_{\text{rect}}$ are plotted in Fig. 2, where the horizontal axis indicates $\sqrt{8\hat{t}}$, which is the effective range of smearing the link variables. Since the values of $\beta_{\text{plaq}}$ and $\beta_{\text{rect}}$ for each $\hat{L}$ at a fixed $\hat{t}$ coincide with each other, finite volume effects turn out to be irrelevant.

$\beta_{\text{plaq}}$ and $\beta_{\text{rect}}$ for $\hat{L}^4 = 16^4$ are fitted in the forms

$$\beta_{\text{plaq}}(\hat{t}) = Ae^{8B\hat{t}}, \quad \beta_{\text{rect}}(\hat{t}) = C(1 - e^{8D\hat{t}}),$$

yielding the numerical results

$$A = 6.040(47), \quad B = 1.906(7), \quad C = 1.010(47), \quad D = 1.911(35),$$

and plotted with the curved lines respectively in Fig. 2. This result shows that the $\hat{t}$ dependence of the exponent is common to these two couplings within the errors.

We also plot the flow of the effective action in the two-coupling theory space in Fig. 3. Fig. 4 shows the plot around the origin of Fig. 3. Reflecting the fact that two couplings have
the common exponent as the function of $\hat{t}$, the flow of the effective action forms a non-trivial straight line (Fig. 3):

$$\beta_{\text{plaq}} + 5.931(31) \beta_{\text{rect}} = 6.216(121),$$

(19)
determined by a numerical fitting for $\hat{L}^4 = 16^4$ with $\chi^2/\text{d.o.f.} = 0.857(371)$.

4 Conclusion and Discussions

We have investigated a trajectory for the Wilson flow in the theory space by the demon method. The effective action is truncated such that it has the plaquette and rectangular terms. We have measured the flow time dependence of $\beta_{\text{plaq}}$ and $\beta_{\text{rect}}$, and found that $\beta_{\text{plaq}}$ increases while $\beta_{\text{rect}}$ tends to the negative region with $\hat{t}$. As shown in Fig. 3 and Fig. 4, we have found that the trajectory forms the straight line in two-coupling theory space, although we do not know whether this is just an accident or not. Since the value of $\beta$ has been fixed throughout this work, it should be examined whether this fact is universal to all values of $\beta$. A trajectory in a wider parameter space has to be investigated by adding more Wilson loops to the ansatz as a next step of this work. It is also interesting to study how the trajectory is changed when we employ other actions for the gradient flow.

As noted in Sec. 1, one can measure the quantities which are sensitive to the cutoff by using the Wilson flow. From this fact, our effective action is expected to be an “improved”

| $\hat{t}$ | $\beta_{\text{plaq}}$ | $\beta_{\text{rect}}$ | $\beta_{\text{plaq}}$ | $\beta_{\text{rect}}$ | $\beta_{\text{plaq}}$ | $\beta_{\text{rect}}$ |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.00     | 5.972(23)       | 0.009(9)        | 6.072(38)       | -0.024(13)      | 5.932(66)       | 0.030(24)       |
| 0.02     | 8.116(44)       | -0.363(12)      | 8.368(75)       | -0.387(17)      | 8.087(58)       | -0.329(25)      |
| 0.04     | 10.856(59)      | -0.837(17)      | 11.334(49)      | -0.895(20)      | 11.088(70)      | -0.817(30)      |
| 0.06     | 14.741(83)      | -1.490(19)      | 15.512(63)      | -1.602(24)      | 15.120(141)     | -1.440(69)      |
| 0.08     | 19.789(86)      | -2.329(21)      | 21.204(82)      | -2.565(30)      | 20.749(148)     | -2.408(41)      |
| 0.10     | 26.671(189)     | -3.477(39)      | 28.637(200)     | -3.791(72)      | 28.660(243)     | -3.705(56)      |
| 0.12     | 35.881(190)     | -5.000(34)      | 37.32(117)      | -4.622(701)     | 37.622(841)     | -5.045(347)     |
| 0.14     | 46.793(619)     | -6.682(152)     | 51.208(848)     | -7.207(523)     | 51.535(289)     | -7.692(75)      |
| 0.16     | 64.23(600)      | -9.635(127)     | 66.03(287)      | -8.48(205)      | 68.986(557)     | -10.711(132)    |
| 0.18     | 81.79(260)      | -11.88(101)     | 90.41(251)      | -12.86(185)     | 90.930(801)     | -14.408(207)    |
| 0.20     | 112.06(373)     | -16.83(101)     | 122.56(99)      | -19.663(216)    | 106.69(776)     | -11.92(613)     |

| $\hat{L}^4$ | $\hat{L}^4 = 4^4$ | $\hat{L}^4 = 8^4$ | $\hat{L}^4 = 16^4$ |
|-------------|-------------------|-------------------|-------------------|
| $\beta_{\text{plaq}}$ | 5.972(23) | 6.072(38) | 5.932(66) |
| $\beta_{\text{rect}}$ | 0.009(9) | -0.024(13) | 0.030(24) |

Table 1 The values of $\beta_{\text{plaq}}$ and $\beta_{\text{rect}}$ at $0 \leq \hat{t} \leq 0.20$ for $\hat{L}^4 = 4^4, 8^4$ and $16^4$. 
action which reduces the discretization effects. Actually, we have found that the flow time
dependence of $\beta_{\text{plaq}}$ and $\beta_{\text{rect}}$ shows the same tendency as in the already known improved
actions [18, 21–25]. In order to confirm this, the cutoff sensitive quantities, such as the
topological charge and the energy momentum tensor, should be studied using our effective
action. It is an interesting question whether such quantities are well-measured by tuning
only two parameters in the action. The restoration of the rotational symmetry should also
be checked by measuring, for example, the difference between the on-axis and off-axis Wilson
loops [17, 18]. The advantages of our effective action are as follows. First, $S_{\text{eff}}$ itself is totally
well-defined without any ambiguity such as truncations in blocking and projections of the
link variables. Second, it does not require any perturbative analysis. Finally, we have one
tunable parameter: the flow time.

Related to the improvement of the lattice action, lattice artifacts could be reduced by
the Wilson flow due to its smearing property. There is a well-known spurious UV fixed
point caused by lattice artifacts when we have the adjoint plaquette term or the fermionic
determinant [14]. The demon method combined with the Wilson flow, which is explained in
Subsec. 2.2 can also be applied to this issue. Namely, one can check the issue by measuring the
coupling of the adjoint plaquette term of configurations generated by the fundamental-adjoint
mixed action.
Fig. 3 The flow of the effective action in two-coupling theory space. The value of $\hat{t}$ becomes larger along the arrow. The dotted line represents our numerical fit for $\hat{L}^4 = 16^4$. Other lines represent the Symanzik [21–23], Iwasaki [24, 25] and DBW2 action [18]. We take $-\beta_{\text{rect}}/\beta_{\text{plaq}} = 0.05$, 0.09073 and 0.11481 for the Symanzik, Iwasaki and DBW2 action respectively.

Fig. 4 The same figure as Fig. 3 but around the origin.
Since the Wilson flow is similar to a continuous “block spin transformation”, it may be used to define the exact renormalization group of the lattice gauge theories. However, the trajectory for the Wilson flow in the theory space travels in the opposite direction to the renormalization trajectory investigated by QCD-TARO collaboration [17, 18]. Therefore, our effective action cannot be regarded as an action obtained after a renormalization group transformation. This is because the Wilson flow itself is just a “blocking” and does not contain a rescaling, which is necessary for the renormalization group transformation. Just replacing the blocking procedure [19] used in Ref. [17, 18] by the Wilson flow is considered as a way to construct a renormalization group scheme via the Wilson flow. This is also an interesting subject related to our work.

Finally, we comment on two types of systematic errors in this work.

One stems from the truncation of the effective action. This can be reduced by adding possible Wilson loops to our truncated action. Since the configurations at large $\hat{t}$ picks up the information of the original configurations at widely separated points, larger Wilson loops should be required with increasing the flow time. Therefore our truncation may be reasonable only in the small $\hat{t}$ regime.

Another comes from the demon method itself. As noted in Subsec. 3.2, the Wilson flow lowers the value of the Wilson action as $\hat{t}$ increases. Since the gauge configurations are changed randomly during the microcanonical updates, most of attempts raise the value of $S_W$ and are rejected. Therefore it is hard to determine a large-valued $\beta_i$ precisely in the demon method. This error is unavoidable as long as using the demon method, however there is another way to determine the couplings from a given configuration using the Schwinger-Dyson equation method [20].

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