Nonlinear Dynamics Analysis and Global Stabilization of Underactuated Horizontal Spring-Coupled Two-Link Manipulator

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Received 25 August 2019; Revised 30 January 2020; Accepted 10 February 2020; Published 16 March 2020

1. Introduction

With the improvement of industrial automation, the manipulators have been widely used in many areas of engineering [1–4]. For a manipulator, the damage of actuators is an inevitable problem caused by the wearing of devices and the influence of the surrounding environment. The failure of actuators results in that the manipulator becomes an underactuated mechanical system [5–7], in which the system has less number of actuators than degrees of freedom (DOF). Since an underactuated manipulator consumes less energy, it has a simple structure and is lightweight, and it also has good potential applications in many fields. It is meaningful to study the control of this kind of mechanical system [8–11]. However, this problem is challenging due to the complex inherent nonlinear dynamics and nonholonomic constraints possessed by this kind of system [12].

Over the past few years, a great deal of efforts have been made on the control of underactuated manipulators [13–15]. Among them, a two-link manipulator that has only one actuator is the simplest example. According to whether the manipulator is affected by gravity or not, the underactuated two-link manipulators are divided into two categories: underactuated vertical two-link manipulators (UVTM) and underactuated horizontal two-link manipulators (UHTM). The UVTM moves in a vertical plane [16], which includes Acrobot [17] and Pendubot [18]. Many control strategies have been presented to solve the stabilizing control of the UVTM, e.g., a partial feedback linearization method [19, 20], a fuzzy control method [21], an
energy-based method [22, 23], and an equivalent-input
disturbance (EID) method [24]. In contrast, the UHTM
moves in a horizontal plane and is not affected by the
gravity. It is clear that the UVTM becomes the UHTM in a
weightless environment. Although the UHTM has the same
mechanical configuration as the UVTM, they have quite
different dynamic properties. The absence of gravity makes
the UHTM system possess some unique properties. For
example, any point in the motion space is an equilibrium
point of the UHTM, and the system is not local controllable
and even not small-time local controllable around the
equilibrium point [25]. All these make the control of the
UHTM be a very difficult problem.

In order to solve this control problem, some attempts
have been made to find a gravity substitute for the UHTM.
In [26], a spring was added around the passive joint of the
UHTM, and an underactuated horizontal spring-coupled
two-link manipulator (UHSTM) was constructed (see
Figure 1). The UHSTM uses the spring to create a source of
potential energy. It makes the control of the underactuated
horizontal manipulator be easy to solve. In addition, the cost
of a spring is much lower than an actuator, and the elasticity
of the spring does not disappear in a weightless environ-
ment. Therefore, the UHSTM has a good application
prospect in industrial manufacturing, exploration of outer
space, and other fields.

Currently, there are few results about the control of the
UHSTM system. In order to promote its application in the
engineering areas, it is necessary to analyze the dynamics
and to develop more control methods for this mechanical
system. This is the main motivation of this study. In this
paper, a set of suitable state variables are selected for the
UHSTM system based on its nonlinear dynamics. It changes
the UHSTM into a cascade nonlinear system. And then, the
global stabilization control problem for the cascade system
is concerned. Based on the structural characteristics of the
cascade system, a positive-definite Lyapunov function is
constructed. A control law is designed to guarantee the
closed-loop control system to be Lyapunov stable. After that,
the asymptotical stability of the control system is discussed
by LaSalle’s invariance principle. Moreover, the conditions
that ensure the system to globally converge to the origin are
presented. Finally, numerical examples verified the effect-
iveness of the presented theoretical results.

2. Model and Nonlinear Dynamics Analysis of
the UHSTM System

Figure 1 shows the model of the UHSTM, where \( m_i \), \( L_i \), and
\( J_i \) are the mass, the length, and the moment of inertia of the
i-th link, respectively, \( (i = 1, 2) \); \( L_{c1} \) is the distance from the
spring joint to the COM of the first link, and \( L_{c2} \) is the
distance from the active joint to the COM of the second link;
\( q_i(t) \) is the rotational angle of the i-th link \( (i = 1, 2) \); \( F(t) \) is
the input torque applied on the active joint; and \( k \) is the
elastic coefficient of the spring. Assume that the spring is
fully relaxed when \( q_1(t) = 0 \). It is not difficult to get the
kinetic and potential energy of the system as

\[
K(q, \dot{q}) = \frac{1}{2} D_{11} (q_2) \dot{q}_1^2 (t) + D_{12} (q_2) \dot{q}_1 (t) \dot{q}_2 (t) + \frac{1}{2} D_{22} \dot{q}_2^2 (t),
\]

\[
P(q) = \frac{1}{2} k q_1^2 (t),
\]

where \( q(t) = [q_1(t), q_2(t)]^T \), \( \dot{q}(t) = dq(t)/dt \), and

\[
D_{11} (q_2) = \alpha_1 + \alpha_2 + 2 \alpha_3 \cos q_2(t),
\]

\[
D_{12} (q_2) = D_{21} (q_2) = \alpha_2 + \alpha_3 \cos q_2(t), D_{22} = \alpha_2,
\]

\[
\alpha_1 = m_1 L_{c1}^2 + m_2 L_{c2}^2 + J_1,
\]

\[
\alpha_2 = m_2 L_{c2}^2 + J_2,
\]

\[
\alpha_3 = m_4 L_1 L_{c2}.
\]

We take the Lagrangian of the UHSTM system to be

\[
L(q, \dot{q}) = K(q, \dot{q}) − P(q).
\]

The Euler–Lagrange motion equation of the system is obtained as

\[
\frac{d}{dt} \left[ \frac{\partial L(q, \dot{q})}{\partial \dot{q}_1} \right] − \frac{\partial L(q, \dot{q})}{\partial q_1} = 0,
\]

\[
\frac{d}{dt} \left[ \frac{\partial L(q, \dot{q})}{\partial \dot{q}_2} \right] − \frac{\partial L(q, \dot{q})}{\partial q_2} = F(t),
\]

where \( F(t) \) is the control input applied to variable \( q_2(t) \),
which is for the actuated part. It is equivalent to the fol-
lowing form:

\[
\begin{bmatrix}
D_{11} (q_2) & D_{12} (q_2) & \dot{q}_1 (t) \\
D_{21} (q_2) & D_{22} & \dot{q}_2 (t)
\end{bmatrix}
+ \begin{bmatrix}
k q_1 (t) \\
0
\end{bmatrix}
\begin{bmatrix}
H_1(q, \dot{q}) \\
H_2(q, \dot{q})
\end{bmatrix}
\]

where
\[ H_1(q, \dot{q}) = -\alpha_2 [2\dot{q}_1(t) + \dot{q}_2(t)]\dot{q}_2(t)\sin q_2(t), \]
\[ H_2(q, \dot{q}) = \alpha_1 \dot{q}_2^2(t)\sin q_2(t). \]

It is clear that (3a) and (3b) is a complicated nonlinear system. In order to make the control design of the system be easy to solve, it is necessary to choose a set of state variables to change the system into the state space form. Note that \( q_2(t) \) and \( \dot{q}_2(t) \) are actuated variables of the UHSTM system. As the partial feedback linearization method developed in [20], we choose them to be two state variables of the system. In addition, it follows from (3a) that the first derivative of \( \partial L(q, \dot{q})/\partial \dot{q}_1 \) is equal to \( \partial L(q, \dot{q})/\partial q_1 \). A simple calculation shows that \( \partial L(q, \dot{q})/\partial q_1 \) is not related to \( \dot{q} \). So, we choose it as the third variable of the system. Based on the expression of \( \partial L(q, \dot{q})/\partial q_1 \) and \( D_{11}(q_2) > 0 \), the fourth state variable of the system is chosen to be \( \xi(q_2) \) in order to make the dynamic structure of the state space be simple, where

\[
\xi(q_2) = \int_{\dot{q}_2(q_2)}^{\dot{q}_2(0)} \frac{D_{11}(y) dy}{D_{11}(y)} = \int_{\dot{q}_2(q_2)}^{\dot{q}_2(0)} \frac{\alpha_2 + \alpha_3 \cos \nu}{\alpha_1 + \alpha_3 + 2\alpha_3 \cos \nu} d\nu
\]

\[
= \frac{q_2(t)}{2} + \frac{\alpha_2 - \alpha_1}{\sqrt{(\alpha_1 + \alpha_2)^2 - 4\alpha_3^2}} \cdot \arctan \left( \frac{\alpha_1 + \alpha_2 - 2\alpha_3}{\alpha_1 + \alpha_2 + 2\alpha_3} \tan \frac{q_2(t)}{2} \right).
\]

In summary, the state variables of the UHSTM system are chosen to be

\[
\begin{align*}
z_1(t) &= q_1(t) + \xi(q_2), \\
z_2(t) &= \frac{\partial L(q, \dot{q})}{\partial \dot{q}_1} = D_{11}(q_2)\dot{q}_1(t) + D_{12}(q_2)\dot{q}_2(t), \\
z_3(t) &= q_2(t), \\
z_4(t) &= \dot{q}_2(t).
\end{align*}
\]

Let \( y(t) = [q_1(t), q_2(t), \dot{q}_1(t), \dot{q}_2(t)]^T \) and \( z(t) = [z_1(t), z_2(t), z_3(t), z_4(t)]^T \). It is easy to get

\[
\frac{\partial z(t)}{\partial y(t)} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
D_{12}(q_2) & D_{11}(q_2) & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
= -D_{11}(q_2) < 0.
\]

It means that the transformation from \( y(t) \) to \( z(t) \) is homeomorphic. From (7), we have

\[
\begin{align*}
\dot{z}_1(t) &= \dot{q}_1(t) + \frac{d\xi(q_2)}{dq_2} \dot{q}_2(t) = \frac{D_{11}(q_2)\dot{q}_1(t) + D_{12}(q_2)\dot{q}_2(t)}{D_{11}(q_2)} = \frac{z_2(t)}{D_{11}(z_3)}, \\
\dot{z}_2(t) &= \frac{d}{dt} \left[ \frac{\partial L(q, \dot{q})}{\partial \dot{q}_1} \right] = \frac{\partial L(q, \dot{q})}{\partial \dot{q}_1} = -k\dot{q}_1(t) = -k[z_1(t) - \xi(q_2)], \\
\dot{z}_3(t) &= \dot{q}_2(t) = z_4(t), \\
\dot{z}_4(t) &= \dot{\dot{q}}_2(t).
\end{align*}
\]

These give the state space equation of the UHSTM system as

\[
\begin{align*}
\dot{z}_1(t) &= \frac{z_2(t)}{D_{11}(z_3)}, \\
\dot{z}_2(t) &= -kz_1(t) + k\xi(z_3), \\
\dot{z}_3(t) &= z_4(t), \\
\dot{z}_4(t) &= \tau(t),
\end{align*}
\]

where \( \tau(t) = \ddot{q}_2(t) \) is the control input of system (10). From (4), it is not difficult to get

\[
F(t) = D_{21}(q_2)\ddot{q}_1(t) + D_{22}\ddot{q}_2(t) + H_2(q, \dot{q}), \\
D_{11}(q_2)\ddot{q}_1(t) + D_{12}\ddot{q}_2(t) + H_1(q, \dot{q}) + k\dot{q}_1(t) = 0.
\]

Eliminating the term \( \ddot{q}_1(t) \) from the above equations and considering \( \tau(t) = \ddot{q}_2(t) \) yield

\[
F(t) = H_2(q, \dot{q}) - \frac{D_{21}(q_2)[H_1(q, \dot{q}) + k\dot{q}_1(t)]}{D_{11}(q_2)} + \frac{\Delta(q_2)}{D_{11}(q_2)} \tau(t),
\]

where \( \Delta(q_2) = D_{11}(q_2)D_{22} - D_{12}(q_2)D_{21}(q_2) > 0 \). This is the relationship between the control inputs \( F(t) \) and \( \tau(t) \).

The main concern of this paper is to design the control law \( F(t) \) that globally stabilizes the UHSTM at \( y(t) = 0 \). Note that \( y(t) = 0 \) is equivalent to \( z(t) = 0 \). So, the discussed
stabilizing control objective is achieved so long as system (10) is globally stabilized at \( z(t) = 0 \).

3. Design of Stabilizing Control Law

In this section, the stabilizing control of (10) at \( z(t) = 0 \) is discussed. A control law \( \tau(t) \) is designed by using Lyapunov stability theory. Note that (10) is a cascade nonlinear system. Based on the structural characteristics of this system, a function is constructed to be

\[
V(z) = \frac{\beta_1}{2} W^2(z) + \frac{\beta_2}{2} z_1^2(t) + \frac{\beta_3}{2} z_2^2(t),
\]

where \( \beta_i > 0 \) (i = 1, 2, 3) are constants, and

\[
W(z) = k z_1(t) \left[ \frac{z_1(t)}{2} - \xi(z_3) \right] + \frac{z_3^2(t)}{2D_{12}(z_3)}. \tag{14}
\]

Note that \( V(z) \) is a nonnegative function. It is easy to verify that \( V(z) = 0 \) when \( z(t) = 0 \). In addition, \( V(z) = 0 \) gives \( W(z) = 0 \), \( z_1(t) = 0 \), and \( z_2(t) = 0 \) because \( \beta_i \) (i = 1, 2, 3) are positive constants. In this case, it follows from (14) that

\[
W(z) = \frac{kz_1^2(t)}{2} + \frac{z_3^2(t)}{2D_{11}(0)} = 0. \tag{15}
\]

Equation (15) means that \( z_1(t) = 0 \) and \( z_2(t) = 0 \) since \( k > 0 \) and \( D_{11}(0) > 0 \). So, we get that \( z(t) = 0 \) when \( V(z) = 0 \). In summary, \( V(z) = 0 \) is equivalent to \( z(t) = 0 \). Thus, \( V(z) \) is a typical positive-definite Lyapunov function for system (10). Differentiating \( W(z) \) along (10) gives

\[
\frac{dW(z)}{dt} = \frac{dW(z)}{dz_1} \frac{dz_1}{dt} + \frac{dW(z)}{dz_2} \frac{dz_2}{dt} + \frac{dW(z)}{dz_3} \frac{dz_3}{dt}
\]

\[
= k[z_1(t) - \xi(z_3)] z_1(t) D_{11}(z_3) + \frac{z_3(t)}{D_{11}(z_3)} [-kz_1(t) + k \xi(z_3)]
\]

\[
+ \left[ \alpha_2 z_2^2(t) \sin z_1(t) D_{12}(z_3) - k z_1(t) D_{12}(z_3) \right] z_4(t)
\]

\[
= \Lambda(z) z_4(t), \tag{16}
\]

where

\[
\Lambda(z) = \frac{\alpha_2 z_2^2(t) \sin z_1(t) - k z_1(t) D_{12}(z_3) D_{11}(z_3)}{D_{11}(z_3)}. \tag{17}
\]

It follows from (10), (13), and (16) that

\[
\frac{dV(z)}{dt} = \beta_1 W(z) \frac{dW(z)}{dt} + \beta_2 z_3(t) z_4(t) + \beta_3 z_4(t) \tau(t)
\]

\[
= \left[ \beta_1 W(z) \Lambda(z) + \beta_2 z_3(t) + \beta_3 \tau(t) \right] z_4(t). \tag{18}
\]

The control law is designed to be

\[
\tau(t) = -\frac{\beta_1 W(z) \Lambda(z) - \beta_2 z_3(t) - \gamma z_4(t)}{\beta_3}, \tag{19}
\]

where \( \gamma > 0 \) is a constant. Combining (18) and (19) gives

\[
\frac{dV(z)}{dt} = -\gamma z_4^2(t) \leq 0. \tag{20}
\]

Since \( V(z) \) is positive definite and \( \dot{V}(z) \leq 0 \), closed-loop control systems (10) and (19) are Lyapunov stable.

4. Asymptotical Stability Analysis of the Closed-Loop Control System

In order to get the final stabilizing state of closed-loop systems (10) and (19), the asymptotical stability of the system is analyzed by using LaSalle’s invariance principle. Letting \( \dot{V}(z) = 0 \) gives \( x_1(t) = 0 \). It follows from the third and fourth equations of (10) that \( z_1(t) = z_1^* \) and \( \tau(t) = 0 \), where \( z_1^* \) is a constant. The further stability analysis results are summarized in the following two theorems.

**Theorem 1.** If \( z_1(t) = z_1^* = 0 \), then \( \dot{V}(z) = 0 \) gives \( z(t) = 0 \).

*Proof.* From \( z_1(t) = 0, z_2(t) = 0 \), and (19), it is easy to obtain \( z_1(t) W(t) = 0 \). Thus, \( z_1(t) = 0 \) or \( W(t) = 0 \). If \( z_1(t) = 0 \), then the first equation of (10) gives \( z_2(t) = 0 \). If \( W(t) = 0 \), then combining \( z_1(t) = 0 \), (14), and (15) gives \( z_1(t) = z_2(t) = 0 \). So, \( \dot{V}(z) = 0 \) gives \( z(t) = 0 \). The proof is completed. \( \square \)

**Theorem 2.** If \( z_1(t) = z_1^* \neq 0 \) and \( \alpha_2 \neq \alpha_3 \), then \( \dot{V}(z) = 0 \) gives \( z_2(t) = 0 \), \( z_1(t) = \xi(z_3) \), and

\[
\frac{\beta_1 k \xi^2(z_3) \left[ \alpha_2 + \alpha_3 \cos z_3^* \right]}{2 \left[ \alpha_1 + \alpha_2 + \alpha_3 \cos z_3^* \right]} + \beta_2 z_3^* = 0. \tag{21}
\]

*Proof.* Combining \( z_4(t) = 0 \), \( \tau(t) = 0 \), (17), and (19) yields

\[
\beta_1 W(z) \left[ \alpha_2 z_2^2(t) \sin z_1(t) - k z_1(t) D_{12}(z_3) D_{11}(z_3) \right] + \beta_2 D_{12}^2(z_3) = 0. \tag{22}
\]

Since \( V(z) = 0 \) means that \( V(z) \) equals to a constant, it follows from \( z_3(t) = z_3^* \), \( z_1(t) = 0 \), and (13) that \( W(z) = W^* \), where \( W^* \) is a constant. From \( D_{12}(z_3^*) = 0 \), \( z_1^* = 0 \), and (22), it is easy to get that \( W^* \neq 0 \). As a result, (22) gives

\[
\alpha_2 z_2^2(t) \sin z_3^* - k z_1(t) D_{12}(z_3^*) D_{11}(z_3^*) = C, \tag{23}
\]

where \( C \) is a non-zero constant. Differentiating (23) along (10) yields

\[
2 \alpha_3 \left[ z_1(t) - \xi(z_3^*) \right] \sin z_3^* + D_{12}(z_3^*) z_2(t) = 0. \tag{24}
\]

We continue carrying out the analysis in the following two situations. \( \square \)

**Case 1.** \( z_2(t) \neq 0 \): it follows from (24) that

\[
2 \alpha_3 \left[ z_1(t) - \xi(z_3^*) \right] \sin z_3^* + D_{12}(z_3^*) = 0. \tag{25}
\]

If \( \sin z_3^* = 0 \), then (25) gives \( D_{12}(z_3^*) = 0 \), i.e., \( \alpha_2 + \alpha_3 \cos z_3^* = 0 \). As a result, \( \sin z_3^* = 0 \) and \( \alpha_2 + \alpha_3 \cos z_3^* = 0 \) are contradictory because \( \alpha_2 > 0, \alpha_3 > 0 \), and \( \alpha_2 \neq \alpha_3 \). Thus, it concludes that \( \sin z_3^* \neq 0 \). So, (25) gives...
\[ z_1(t) = \frac{-D_{12}(z'_3)}{2\alpha_3 \sin z'_3} + \xi(z'_3), \tag{26} \]

which is a constant. It follows from the first equation of (10) that \( z_2(t) = 0 \). This is a contradiction.

**Case 2.** \( z_1(t) = 0 \): the first and second equations of (10) give that \( z_1(t) = z'_3 \) and \( z'_3 = \xi(z'_3) \), where \( z'_3 \) is a constant. Substituting \( z_1(t) = 0 \) and \( z'_3 = \xi(z'_3) \) into (14) yields \( W(z) = -k\xi^2(z'_3)/2 \). Combining \( W(z) \) and (22) gives

\[ \frac{\beta_1 k^2 D_{12}(z'_3)^2(z'_3)}{2D_{11}(z'_3)} + \beta_2 z'_3 = 0. \tag{27} \]

It is equivalent to equation (21). The proof is completed.

Based on the analysis results in Theorem 1 and Theorem 2, \( V(z) \equiv 0 \) gives

\[ \Omega = \{z(t) = 0 \text{ or } z(t) = [\xi(z'_3), 0, z'_3, 0]\}, \tag{28} \]

where \( z'_3 \) is the solution of (21). According to LaSalle’s invariance theorem [27], closed-loop control systems (10) and (19) asymptotically converge to the largest invariant set contained in \( \Omega \).

In order to clearly describe the stabilizing state of the control system, it is necessary to solve (21). Note that (21) is a complicated transcendental equation. It is difficult to obtain an analytical expression for \( z'_3 \). To solve this problem, a condition on the control parameters \( \beta_1 \) and \( \beta_2 \) is presented, which ensures (21) has only one solution for \( z'_3 \). The main result is given in the following theorem.

**Theorem 3.** If the control parameters \( \beta_1 \) and \( \beta_2 \) satisfy

\[ \beta_2 > \frac{k^2\alpha_3|\alpha_3 - \alpha_1|(\alpha_3 + \alpha_1)^3\pi^3}{2(\alpha_1 + \alpha_2 - 2\alpha_3)^2}\beta_1, \tag{29} \]

then equation (21) has only one solution \( z'_3 = 0 \). And thus, closed-loop control systems (10) and (19) asymptotically converge to \( z(t) = 0 \).

**Proof.** Note that \( z_3(t) = \eta_1(t) \) stands for the angle position of the second link of the UHSTM (see Figure 1). It is a cyclic variable with a period 2\( \pi \). So, we can assume that \( -\pi \leq z_3(t) < \pi \) (mod 2\( \pi \)) in this paper. Based on (21), an auxiliary function is designed to be

\[ f(x) = \frac{\beta_1 g(x)}{\alpha_1 + \alpha_2 + 2\alpha_3 \cos x} + \beta_2 x, \quad x \in [-\pi, \pi], \tag{30} \]

where

\[ g(x) = \frac{k^3\xi^3(x)|\alpha_3 + \alpha_1 \cos x|}{2} \]

\[ \xi(x) = \int_{0}^{x} \frac{\alpha_3 + \alpha_1 \cos y}{\alpha_1 + \alpha_2 + 2\alpha_3 \cos y} dy. \tag{31} \]

It is not difficult to obtain

\[ |\xi(x)| \leq \frac{\alpha_3 + \alpha_1}{\alpha_1 + \alpha_2 - 2\alpha_3}|x| \leq \frac{(\alpha_3 + \alpha_1)\pi}{\alpha_1 + \alpha_2 - 2\alpha_3}, \tag{32} \]

\[ \frac{dg(x)}{dx} = \frac{k^3\xi^2(x)}{2} \left[ \frac{3(\alpha_3 + \alpha_1 \cos x)^2}{\alpha_1 + \alpha_2 - 2\alpha_3 \cos x} - \alpha_3 \xi(x) \sin x \right]. \tag{33} \]

Combining (30) and (33) yields

\[ \frac{df(x)}{dx} = \beta_1 \frac{(dg(x)/dx)(\alpha_1 + \alpha_2 + 2\alpha_3 \cos x) + 2\alpha_3 g(x) \sin x}{(\alpha_1 + \alpha_2 + 2\alpha_3 \cos x)^2} + \beta_2 \]

\[ = \beta_2 + \frac{\beta_1 k^2}{2} \frac{2(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_1)\xi^3(x) \sin x}{(\alpha_1 + \alpha_2 + 2\alpha_3 \cos x)^2} + \frac{3\xi^2(x)(\alpha_3 + \alpha_1 \cos x)^2}{(\alpha_1 + \alpha_2 + 2\alpha_3 \cos x)^2} \]

\[ \geq \beta_2 + \frac{\beta_1 k^2}{2} \frac{2(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_1)\xi^3(x) \sin x}{(\alpha_1 + \alpha_2 + 2\alpha_3 \cos x)^2}. \tag{34} \]

It follows from (32) that

\[ \left( \frac{\alpha_3 - \alpha_1}{\alpha_1 + \alpha_2 + 2\alpha_3 \cos x} \right)^3 \leq \frac{2(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_1)\pi^3}{(\alpha_1 + \alpha_2 + 2\alpha_3)^2}. \tag{35} \]

Combining (29), (34), and (35) yields \( df(x)/dx > 0 \). It means that \( f(x) \) is a strictly monotonically increasing function during \( x \in [-\pi, \pi] \). By the fact that \( f(0) = 0 \), it is easy to conclude that \( f(x) = 0 \) has only one solution \( x = 0 \). In other words, condition (29) ensures that (21) has only one solution \( z'_3 = 0 \). Substituting \( z'_3 = 0 \) into (28) yields that \( \Omega \) contains only one element \( z(t) = 0 \). So, systems (10) and (19) asymptotically converge to \( z(t) = 0 \). The proof is completed.

From the above statements, we get that the control law \( \tau(t) \) in (19) globally stabilizes system (10) at \( z(t) = 0 \) so as to both conditions, \( \alpha_3 \neq \alpha_3 \) and (29), are satisfied. As a result, the control law \( F(t) \) obtained by substituting \( \tau(t) \) into (12) globally stabilizes the UHSTM at \( y(t) = 0 \) under these conditions.
Table 1: Parameters of the underactuated horizontal spring-coupled two-link manipulator (UHSTM).

| $m_1$ (kg) | $m_2$ (kg) | $L_1$ (m) | $L_2$ (m) | $l_1$ (m) |
|-----------|-----------|-----------|-----------|-----------|
| 1         | 1         | 1         | 2         | 0.5       |
| $J_1$ (kgm$^2$) | $J_2$ (kgm$^2$) | $k$ (N/m) |
| $8.3 \times 10^{-2}$ | 0.33 | 0.1 |

5. Numerical Examples

This section presents numerical examples to verify the effectiveness of the above theoretical results.

The physical parameters of the UHSTM presented in [26] were chosen for simulation. They are listed in Table 1. The control parameters in (19) were chosen to be $\beta_1 = 5$, $\beta_2 = 5$, $\beta_3 = 0.1$, and $\gamma = 1$. A simple calculation gives

$$
\alpha_1 = 1.333, \\
\alpha_2 = 1.33, \\
\alpha_3 = 1,
$$

(36)

$$
k^2 \alpha_2 |\alpha_2 - \alpha_1| (\alpha_2 + \alpha_3)^3 \pi^3 \beta_1 = 0.2296.
$$

So, both conditions, $\alpha_2 \neq \alpha_3$, and (29), are satisfied. The simulation results of the UHSTM with an initial condition,

$$
[q_1 (0), q_2 (0), \dot{q}_1 (0), \dot{q}_2 (0)]^T = \left[ \frac{\pi}{2}, -\pi, 0, 0 \right]^T,
$$

(37)

are shown in Figure 2. Note that the UHSTM is quickly stabilized at the origin. For the sake of comparison, we also carried out the simulation for the UHSTM with the same initial condition as in [26], i.e.,

$$
[q_1 (0), q_2 (0), \dot{q}_1 (0), \dot{q}_2 (0)]^T = [-1, 2, 0, 0]^T.
$$

(38)

The simulation results with (38) show that our method is still effective (see Figure 3). The stabilizing time is less than 1.5 seconds. In contrast, the stabilizing time of the UHSTM in [26] is greater than 5 seconds. This shows the superiority of our presented method.

In order to verify the practicality of our method in a real environment, the simulations were carried out by considering parameter uncertainties (±5% of their nominal values for $m_1$, $L_{c1}$, and $J_1$, $i = 1, 2$), white noises (peak value: ±0.15) in measurements, and the viscous friction disturbances in the joints. The results show that our method is still effective under these conditions. We give the simulation results in Figure 4 for the typical case, where $m_1$, $L_{c1}$, and $J_1$ are 5% smaller than their nominal values; $m_2$, $L_{c2}$, and $J_2$ are 5% larger than their nominal values; the peak value of white noise is ±0.1; and the friction disturbances are $f_{v1} (t) = -0.15 \dot{q}_1 (t)$ and $f_{v2} (t) = -0.25 \dot{q}_1 (t)$ in the spring joint and the active joint, respectively. Note that the UHSTM can still be stabilized at the origin. The robustness of our method is good.
6. Conclusions

This paper concerned the nonlinear dynamics analysis and the global stabilization control of a 2-DOF underactuated system called UHSTM. A set of suitable state variables were constructed that change the system into a cascade nonlinear system. Then, a stabilizing control law was designed for the cascade system based on the Lyapunov theory. After that, the conditions were presented in order to guarantee the global asymptotical stability of the closed-loop control system at the origin. Numerical examples verified the effectiveness of the method. In the future, we will further study how to...
extend the method to stabilize the underactuated horizontal multilink manipulator and other underactuated nonlinear systems.

Data Availability
The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

Acknowledgments
This work was supported in part by the Natural Science Foundation Program of Shandong Province under Grant no. ZR2019KYQ28, the Development Plan of Youth Innovation Team of University in Shandong Province under Grant no. 2019KJN007, and the National Natural Science Foundation of China under Grant nos. 61773193, 61803194, 61877033 and 61903171.

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