ON THE STATISTICAL CONVERGENCE OF METRIC-VALUED SEQUENCES

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We study the conditions for the density of a subsequence of a statistically convergent sequence under which this subsequence is also statistically convergent. Some sufficient conditions of this type and almost convergent necessary conditions are obtained in the setting of general metric spaces.

1. Introduction and Definitions

The analysis on metric spaces is now rapidly developed (see [15, 18]). This development is usually based on some generalizations of the notion of differentiability. The generalizations of differentiation involve linear structures by means of embeddings of metric spaces in suitable normed spaces or by the use of geodesics.

A new intrinsic approach to the introduction of smooth structures for general metric spaces was proposed by Martio and Dovgoshey in [10] (see also [1, 3, 4, 7–9]). The approach proposed in [10] is completely based on the convergence of metric-valued sequences but it is not \textit{a priori} clear that the ordinary convergence is the best possible way to obtain smooth structures for arbitrary metric spaces.

The problem of different types of convergence of real- (or complex-) valued divergent sequences goes back to the beginning of the 1800s. Numerous different convergence methods were introduced (Cesaro, Nörlund, weighted mean, Abel et al.) and applied in various branches of mathematics. Almost all convergence methods depend on the algebraic structure of the space. It is clear that, in general, the metric space does not have the algebraic structure. However, the notion of statistical convergence can be readily extended to arbitrary metric spaces and this provides a general framework for the summability in these spaces [13, 21]. Thus, the studies of statistical convergence give a natural foundation for the upbuilding of various tangent spaces to general metric spaces.

The construction of tangent spaces in [3, 4, 7–10] is based on the following fundamental fact: If \((x_n)\) is a convergent sequence in a metric space, then every subsequence \((x_{n(k)})\) of \((x_n)\) is also convergent.

Thus, the convergence of subsequence \((x_{n(k)})\) does not depend on the choice of \((x_{n(k)})\). Unfortunately, this is not the case for the statistically convergent sequences. The applications of statistical convergence to the infinitesimal geometry of metric spaces should be based on the complete understanding of the structure of statistically convergent subsequences.

We study the conditions for the density of a subsequence of statistically convergent sequence under which this subsequence is also statistically convergent. Some sufficient conditions of this type and “almost convergent” necessary conditions are obtained in the setting of general metric spaces.

We now recall the basic definitions. Let \((X, d)\) be a metric space. For convenience, by \(\hat{X}\), we denote the set of all sequences of points from \(X\).

\textbf{Definition 1.1.} A sequence \((x_n) \in \hat{X}\) is called convergent to a point \(a \in X\), \(\lim_{n \to \infty} x_n = a\), if for every \(\epsilon > 0\) there is a number \(n_0 = n_0(\epsilon) \in \mathbb{N}\) such that \(n > n_0\) implies \(d(x_n, a) < \epsilon\).

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Definition 1.2. A metric-valued sequence \( \tilde{x} = (x_n) \in \tilde{X} \) is \( d \)-statistical convergent to \( a \in X \) if

\[
\lim_{n \to \infty} \frac{1}{n} \left| \{ k : k \leq n, d(x_k, a) \geq \epsilon \} \right| = 0
\]

is true for every \( \epsilon > 0 \).

Here and in what follows, \( |B| \) denotes the number of elements of the set \( B \).

The idea of statistical convergence goes back to Zygmund [22]. It was formally introduced by Steinhaus [20] and Fast [11]. In recent years, it becomes an efficient research tool for mathematicians (see, e.g., [5, 6, 12–14, 17]).

Definition 1.3 [11] (dense subset of \( \mathbb{N} \)). A set \( K \subseteq \mathbb{N} \) is called a statistically dense subset of \( \mathbb{N} \) if

\[
\lim_{n \to \infty} \frac{1}{n} |K(n)| = 1,
\]

where \( K(n) = \{ k \in K : k \leq n \} \).

It can be proved that the intersection of two dense subsets is dense. Moreover, it is clear that the supersets of dense sets are also dense. Hence, the family of all dense sets forms a filter on \( \mathbb{N} \). The \( d \)-statistical convergence is simply the convergence in \((X, d)\) with respect to this filter.

Definition 1.4 (dense subsequence). If \( (n(k)) \) is an infinite strictly increasing sequence of natural numbers and \( \tilde{x} = (x_n) \in \tilde{X} \), then we write \( \tilde{x}' = (x_{n(k)}) \) and \( K_{\tilde{x}'} = \{ n(k) : k \in \mathbb{N} \} \). A subsequence \( \tilde{x}' \) is a dense subsequence of \( \tilde{x} \) if \( K_{\tilde{x}'} \) is a dense subset of \( \mathbb{N} \).

In the next definition, we introduce an equivalence relation on the set \( \tilde{X} \).

Definition 1.5. Sequences \( \tilde{x} = (x_n) \in \tilde{X} \) and \( \tilde{y} = (y_n) \in \tilde{X} \) are statistically equivalent, \( \tilde{x} \asymp \tilde{y} \), if there is a statistically dense \( M \subseteq \mathbb{N} \) such that \( x_n = y_n \) for every \( n \in M \).

2. Convergent and Statistically Convergent Sequences

In this section, some basic results on \( d \)-statistical convergence are presented for an arbitrary metric space. In particular, it is shown that there is a one-to-one correspondence between metrizable topologies on \( X \) and the subsets of \( \tilde{X} \) consisting of all statistically convergent sequences.

Let \((X, d)\) be a nonvoid metric space. It is clear that every convergent sequence \((x_n) \in \tilde{X}\) is also \( d \)-statistically convergent. Moreover, all statistically convergent sequences are convergent if and only if \(|X| = 1\). Nevertheless, we have the following result:

Theorem 2.1. Let \((X, d_1)\) and \((X, d_2)\) be two metric spaces with the same underlining set \( X \). Then the following statements are equivalent:

(i) The set of all \( d_1 \)-statistically convergent sequences coincides the set of all \( d_2 \)-statistically convergent sequences.

(ii) The set of all sequences convergent in the space \((X, d_1)\) coincides the set of all sequences convergent in the space \((X, d_2)\).

(iii) The metrics \( d_1 \) and \( d_2 \) induce one and the same topology on \( X \).