On a problem of Bharanedhar and Ponnusamy involving planar harmonic mappings

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Abstract. In this paper, we give a negative answer to a problem presented by Bharanedhar and Ponnusamy (Rocky Mountain J. Math. 44: 753–777, 2014) concerning univalency of a class of harmonic mappings. More precisely, we show that for all values of the involved parameter, this class contains a non-univalent function. Furthermore, we discuss criteria for planar harmonic mappings to be close-to-convex. Finally, several results on a new subclass of close-to-convex harmonic mappings, which is motivated by work of Ponnusamy and Sairam Kaliraj (Mediterr. J. Math. 12: 647–665, 2015), are obtained.

1 Introduction

In this paper, we consider univalency criteria for complex-valued harmonic functions \( f \) in the open unit disk \( D \). It is well-known that such functions can be written as \( f = h + \overline{g} \), where \( h \) and \( g \) are analytic functions in \( D \). We call \( h \) the analytic part and \( g \) the co-analytic part of \( f \), respectively. Let \( \mathcal{H} \) be the class of harmonic functions normalized by the conditions \( f(0) = f_z(0) - 1 = 0 \), which have the form

\[
(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k \quad (z \in D).
\]

Since the Jacobian of \( f \) is given by \( |h'|^2 - |g'|^2 \), by Lewy’s theorem (see [10]), it is locally univalent and sense-preserving if and only if \( |g'| < |h'| \), or equivalently, the dilatation \( \omega = g'/h' \) with \( h'(z) \neq 0 \) has the property \( |\omega| < 1 \) in \( D \). The subclass of \( \mathcal{H} \) that are harmonic, univalent and sense-preserving in \( D \) is denoted by \( S_{\mathcal{H}} \). Univalent harmonic functions are also called harmonic mappings.

The classical family \( S \) of analytic univalent and normalized functions in \( \mathbb{D} \) is a subclass of \( S_{\mathcal{H}} \) with \( g(z) \equiv 0 \). The family of all functions \( f \in S_{\mathcal{H}} \) with the additional property that \( f_z(0) = 0 \) is denoted by

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There exists reciprocal transformations between $S_H^0$ and $S_{H}^0$ (see [5, 6]). Observe that the family $S_{H}^0$ is compact and normal, but the family $S_H^0$ is not compact. For recent results involving univalent harmonic mappings, we refer to [1, 2, 4, 7, 8, 13, 16, 20, 22, 23], and the references therein.

A domain $\Omega$ is said to be close-to-convex if $C \setminus \Omega$ can be represented as a union of non-intersecting half-lines. Following the result due to Kaplan [9], an analytic function $F$ is called close-to-convex if there exits a univalent convex analytic function $\phi$ defined in $D$ such that

$$\Re \left( \frac{F'(z)}{\phi'(z)} \right) > 0 \quad (z \in D).$$

Furthermore, a planar harmonic mapping $f : D \to C$ is close-to-convex if it is injective and $f(D)$ is a close-to-convex domain, we denote $C_{H}^0$ by the class of close-to-convex harmonic mappings.

This paper is organised as follows. In Section 2, we give a negative answer to a problem posed by Bharanedhar and Ponnusamy in [3]. In Section 3, we discuss criteria for planar harmonic mappings to be close-to-convex. In Section 4, we study a subclass of close-to-convex harmonic mappings, which is motivated by work of Ponnusamy and Sairam Kaliraj [18]. Coefficient estimates, a growth theorem and a covering theorem, for mappings of this class, are obtained.

## 2 A problem of Bharanedhar and Ponnusamy

Recently, Mocanu [12] proposed the following conjecture involving the univalency of planar harmonic mappings.

**Conjecture 2.1** Let

$$\mathcal{M} = \left\{ f = h + \bar{g} \in \mathcal{H} : g' = zh \text{ and } \Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2} \quad (z \in D) \right\}.$$

Then $\mathcal{M} \subset S_{H}^0$.

By applying the close-to-convexity criterion for analytic functions due to Kaplan [9], Bshouty and Lyzzaik [2] have solved the above conjecture by establishing the following stronger result:

**Theorem A** $\mathcal{M} \subset C_{H}^0$.

Later, Ponnusamy and Sairam Kaliraj [18] Theorem 4.1] generalized Theorem A, under the assumption that the analytic dilatation $\omega$ satisfies the condition

$$\Re \left( \frac{\lambda z \omega'(z)}{1 - \lambda \omega(z)} \right) > -\frac{1}{2}$$

for all $\lambda$ such that $|\lambda| = 1$. In particular, for

$$\omega(z) = \lambda k z^n \quad \left( |\lambda| = 1; 0 < k \leq \frac{1}{2n-1} ; n \in \mathbb{N} := \{1, 2, 3, \ldots \} \right),$$

2
they gave the following result:

**Theorem B** Suppose that \( h \) and \( g \) are analytic in \( D \) such that

\[
\Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2},
\]

and

\[
g'(z) = \lambda kz^n h'(z) \quad (n \in \mathbb{N}; \ |\lambda| = 1; \ 0 < k \leq \frac{1}{2n - 1}).
\]

Then \( f = h + \bar{g} \) is univalent and close-to-convex in \( D \).

Motivated by Theorem B, we introduce the following natural class of close-to-convex harmonic mapping, which will be studied in Section 4. Note that for \( n = 1 \), we have the class \( M(\alpha, \zeta) \), which was studied in [19].

**Definition 2.1** A harmonic mapping \( f = h + \tilde{g} \in \mathcal{H} \) is said to be in the class \( M(\alpha, \zeta, n) \) if \( h \) and \( g \) satisfy the conditions

(2.1) \[
\Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) > \alpha \quad \left( -\frac{1}{2} \leq \alpha < 1 \right),
\]

and

(2.2) \[
g'(z) = \zeta z^n h'(z) \quad \left( \zeta \in \mathbb{C} \text{ with } |\zeta| \leq \frac{1}{2n - 1}; \ n \in \mathbb{N} \right).
\]

In 1995, Ponnusamy and Rajasekaran [15] derived the following univalency criterion for analytic starlike functions.

**Theorem C** Suppose that \( F \) is a normalized analytic function in \( \mathbb{D} \). If \( F \) satisfies the condition

\[
\Re \left( 1 + \frac{zF''(z)}{F'(z)} \right) < \beta \quad \left( 1 < \beta \leq \frac{3}{2} \right).
\]

Then \( F \) is univalent and starlike in \( \mathbb{D} \).

Motivated essentially by Theorems A and C, Bharanedhar and Ponnusamy [3, Problem 1, p. 763] posed the following problem, which we present here in a slightly modified form:

**Problem 2.1** For \( \beta \in (1, 3/2) \), define

\[\mathcal{P}(\beta) = \left\{ f = h + \tilde{g} \in \mathcal{H} : \ g' = zh' \text{ and } \Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) < \beta \quad (z \in \mathbb{D}) \right\}.\]

Determine \( \inf \left\{ \beta \in (1, 3/2) : \mathcal{P}(\beta) \subset S_0^H \right\} \).
Recall the following result of Bshouty and Lyzzaik \cite{2}:

**Theorem D** Suppose that $0 \leq \lambda < 1/2$. Let $f = h + \bar{g}$ be the harmonic polynomial mapping with

$$h(z) = z - \lambda z^2 \quad \text{and} \quad g(z) = \frac{z^2}{2} - \frac{2\lambda z^3}{3}.$$ 

If $0 \leq \lambda \leq 3/10$, then $f$ is univalent in $\mathbb{D}$. But for $3/10 < \lambda < 1/2$, $f$ is not univalent in $\mathbb{D}$.

**Remark 2.2** In view of Theorem C, we see that $\beta$ can be restricted to the value on the interval $(1, 11/8]$, since

$$\sup_{z \in \mathbb{D}} \left\{ \Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) \right\} = \frac{11}{8}$$ 

for

$$h(z) = z - \frac{3}{10} z^2.$$ 

Now, we are ready to give a counterexample, which shows that the for all $\beta \in (1, 11/8]$, the class $P(\beta)$ of Problem 2.1 contains a non-univalent function.

Consider the harmonic function given by $f_\gamma = h + \bar{g} \in \mathcal{H}$, where

$$h(z) = \frac{1}{\gamma} \left[ 1 - (1 - z)^\gamma \right] \quad \left(1 < \gamma \leq \frac{7}{4}\right),$$

and

$$g(z) = \frac{1}{\gamma(1 + \gamma)} \left[ 1 - (1 + \gamma z)(1 - z)^\gamma \right] \quad \left(1 < \gamma \leq \frac{7}{4}\right).$$

Clearly, we have $g' = zh'$. It follows that

$$1 + \frac{zh''(z)}{h'(z)} = \frac{1 - \gamma z}{1 - z},$$

and therefore

$$\Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) < \frac{1 + \gamma}{2} \quad \left(1 < \frac{1 + \gamma}{2} \leq \frac{11}{8}\right).$$

That is,

$$f_\gamma = h + \bar{g} \in \mathcal{P}(1 + \gamma/2) \subset \mathcal{P}(\beta).$$

Now, we shall prove that the function $f_\gamma$ is not univalent in $\mathbb{D}$. It is easy to verify that both the analytic and co-analytic parts of $f_\gamma$ have real coefficients, and thus, $f_\gamma(z) = \overline{f_\gamma(\bar{z})}$ for all $z \in \mathbb{D}$. In particular,

$$\Re \left( f_\gamma \left( re^{i\theta} \right) \right) = \Re \left( f_\gamma \left( re^{-i\theta} \right) \right)$$

for some $r \in (0, 1)$ and $\theta \in (-\pi, 0) \cup (0, \pi)$. It suffices to show that there exists $r_0 \in (0, 1)$ and $\theta_0 \in (-\pi, 0) \cup (0, \pi)$ such that

$$\Im \left( f_\gamma \left( r_0 e^{i\theta_0} \right) \right) = \Im \left( f_\gamma \left( r_0 e^{-i\theta_0} \right) \right) = 0.$$
By noting that
\[ \Im(f(\gamma(z))) = \Im(h(z) - g(z)) = \Im\left(\frac{1 - (1 - z)^{\gamma+1}}{\gamma + 1}\right) = \frac{1 - e^{(\gamma+1)\log(1-z)}}{\gamma + 1}, \]
we see that
\[ \Im(f(\gamma(re^{i\theta}))) = \frac{1 - e^{(\gamma+1)\log(1-re^{i\theta})}}{\gamma + 1} \]
and
\[ -\Im(f(\gamma(re^{-i\theta}))) = e^{(\gamma+1)\log(1-re^{-i\theta})} \frac{1 - e^{(\gamma+1)\log(1-re^{i\theta})}}{\gamma + 1} \sin[(\gamma + 1) \arg(1-re^{i\theta})]. \]

Because
\[ \arg(1-re^{i\theta}) \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right), \]
we see that for each \(1 < \gamma \leq 7/4\), there exists \(r_0 \in (0, 1)\) and \(\theta_0 \in (-\pi, 0) \cup (0, \pi)\) such that
\[ \sin[(\gamma + 1) \arg(1-re^{i\theta})] = 0. \]
It follows that
\[ \Im(f(\gamma(re^{i\theta}))) = \Im(f(\gamma(re^{-i\theta}))) = 0. \]
Therefore, there exists two distinct points \(z_1 = r_0 e^{i\theta_0}\) and \(z_2 = r_0 e^{-i\theta_0}\) in \(D\) such that \(f(\gamma(z_1)) = f(\gamma(z_2))\), which shows that the function \(f(\gamma(z))\) is not univalent in \(D\). Thus, we conclude that conditions given in Problem 2.1 are not satisfied for any \(\beta \in (1, 11/8]\).

The image domain of \(f(\gamma)\) for \(\gamma = 5/4\) is given in Figures 1 and 2 to illustrate our counterexample.

3 Close-to-convexity criteria for planar harmonic mappings

In this section, we present a survey of close-to-convexity criteria for planar harmonic mappings, along with examples and new observations. Define the constant \(K(f, D)\) associated with the harmonic mapping \(f : D \rightarrow \mathbb{C}\) in a simply connected domain \(D\) as follows:
\[ K(f, D) = \inf_{a,b \in D} \left| \frac{f(a) - f(b)}{a - b} \right|. \]
Loosely speaking, the constant \(K(f, D)\) measures the degree of univalency of the harmonic mapping \(f\) in \(D\). Clearly, we see that if \(f\) is not univalent in \(D\), then \(K(f, D) = 0\). By applying the similar method as in [14, Proposition 2.1], we easily get the following assertion associated with harmonic mappings:
Proposition 3.1 Let \( f : D \rightarrow \mathbb{C} \) be a harmonic mapping in the domain \( D \). If \( K(f, D) > 0 \), then \( f \) is a univalent harmonic mapping in \( D \). Conversely, if \( f \) is a univalent harmonic mapping in \( D \) and \( \Omega \subset \overline{\Omega} \subset D \) is a domain strictly contained in \( D \), then \( K(f, \Omega) > 0 \).

The following two lemmas will play crucial roles in the proof of the following result:

Lemma 3.1 (See [14]) Let \( F : D \rightarrow \mathbb{C} \) be a nonconstant analytic function in the convex domain \( D \). If there exists a univalent analytic function \( G : D \rightarrow \mathbb{C} \) such that
\[
|F'(z) - G'(z)| \leq K(G, D) \quad (z \in D),
\]
then the function \( F \) is univalent in \( D \), \( G'(z) = c \in \mathbb{C} \) and \( K(G, D) = |c| \).

Lemma 3.2 (See [5]) If \( h, g \) are analytic in \( \mathbb{D} \) with \( |h'(0)| > |g'(0)| \), and \( h + \lambda g \) is close-to-convex for each \( \lambda (|\lambda| = 1) \), then \( f = h + \overline{g} \) is harmonic close-to-convex in \( \mathbb{D} \).

Theorem 3.3 Let \( f = h + \overline{g} \) be a nonconstant sense-preserving harmonic mapping in the convex domain \( D \), where \( h(0) = g(0) = 0 \), \( h'(0) = 1 \) and \( |g'(0)| < 1 \). If there exists a univalent analytic function \( G : D \rightarrow \mathbb{C} \) such that
\[
|F'_{\lambda}(z) - G'(z)| \leq K(G, D) \quad (z \in D)
\]
holds for each \( \lambda (|\lambda| = 1) \), where \( F_{\lambda}(z) = h(z) + \lambda g(z) \) with \( |\lambda| = 1 \) and \( G'(z) > 0 \), then \( f \) is a close-to-convex harmonic mapping in \( D \).
Proof. Since $G'(z) > 0$, by virtue of Lemma 3.1 and (3.1), we obtain
\[ G'(z) = K(G, D) = \overline{c} > 0, \]
which implies that
\[ (3.2) \quad \left| F'_\lambda(z) - \overline{c} \right| \leq \overline{c} \quad (z \in D). \]
From (3.2), we see that
\[ \Re(h'(z) + \lambda g'(z)) > 0 \quad (z \in D) \]
holds for each $\lambda$ ($|\lambda| = 1$). By Lemma 3.2 we deduce that $f$ is a close-to-convex harmonic mapping in $D$. \hfill \Box

Example 3.4  The function
\[ f(z) = z + \frac{1}{4}z^4 \quad (z \in D) \]
satisfies the conditions of Theorem 3.3 which shows that $f$ is a close-to-convex harmonic mapping (see Figure 3).

Fig. 3: The image of $D$ under the function $f$ of Example 3.4.

In 2012, Ponnusamy and Sairam Kaliraj [16] derived the following criterion for close-to-convex harmonic mappings, which generalized an earlier result obtained by Mocanu [11].

Theorem E  Let $f = h + \overline{g} \in H$. Further, let $\varphi$ be univalent, analytic and convex in $D$. If $f$ satisfies
\[ \Re \left( e^{i\eta} \frac{h'(z)}{\varphi'(z)} \right) > \left| \frac{g'(z)}{\varphi'(z)} \right| \quad \text{for some real } \eta \text{ and for all } z \in D, \]
then $f$ is sense-preserving univalent and close-to-convex in $D$.

By Cauchy inequality and Theorem E, we get the following corollary, which has independent interest.

**Corollary 3.1** Let $f = h + \overline{g}$ be a harmonic mapping in a bounded convex domain $D$, where $h$ and $g$ are analytic functions in $D$ such that $h(0) = g(0) = 0$ and $h'(0) = 1$. Moreover, let $\varphi$ be a convex, univalent and analytic function in $D$ with $\varphi'(0) = 1$. If $h$ and $g$ satisfy the condition

\[
\alpha \Re \left( \frac{h'(z)}{\varphi'(z)} \right) + (1 - \alpha) \Im \left( \frac{h'(z)}{\varphi'(z)} \right) > \left| \frac{g'(z)}{\varphi'(z)} \right| \text{ for some } \alpha \in [0, 1]; \ z \in D,
\]

then $f$ is a close-to-convex harmonic mapping in $D$.

By setting $\varphi(z) = z$ in Corollary 3.1, we get the following result.

**Corollary 3.2** Let $f = h + \overline{g}$ be a harmonic mapping in a bounded convex domain $D$, where $h$ and $g$ are analytic functions in $D$ such that $h(0) = g(0) = 0$, $h'(0) = 1$ and

\[
\alpha \Re (h'(z)) + (1 - \alpha) \Im (h'(z)) > |g'(z)| \text{ for some } \alpha \in [0, 1]; \ z \in D.
\]

Then $f$ is a close-to-convex harmonic mapping in $D$.

**Remark 3.5** For $g(z) \equiv 0$, Corollary 3.2 turns to the result obtained by Kalaj [7]. For $\alpha = 1$ and $g(z) \equiv 0$, Corollary 3.2 reduces to the classical Noshiro-Warschawski-Wolff univalency criterion for analytic functions.

**Theorem 3.6** Let $f = h + \overline{g}$ be a harmonic mapping in $D$, where $h$ and $g$ are analytic functions in $D$ such that $h(0) = g(0) = 0$, $h'(0) = 1$ and $|g'(0)| < 1$. Moreover,

\[
\alpha \Re (h'(z) + \lambda g'(z)) + (1 - \alpha) \Im (h'(z) + \lambda g'(z)) > 0 \quad (\alpha \in [0, 1]; \ z \in D)
\]

holds for each $\lambda$ ($|\lambda| = 1$). Then $f$ is a close-to-convex harmonic mapping in $D$.

**Proof.** Combining Corollary 3.2 with Lemma 3.2, we obtain the desired result of Theorem 3.6.

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**4 The subclass $M(\alpha, \zeta, n)$ of close-to-convex harmonic mappings**

Recall the following lemma, due to Suffridge [21], which will be required in the proof of Theorem 4.2.

**Lemma 4.1** If $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$ satisfies the condition (2.1), then

\[
|a_k| \leq \frac{1}{k!} \prod_{j=2}^{k} (j - 2\alpha) \quad (k \in \mathbb{N} \setminus \{1\}),
\]

with the extremal function given by

\[
h(z) = \int_0^z \frac{dt}{(1 - \delta t)^{2 - 2\alpha}} \quad (|\delta| = 1; \ z \in D).
\]
We now derive the coefficient estimates for the class $\mathcal{M}(\alpha, \zeta, n)$.

**Theorem 4.2** Let $f = h + \bar{g} \in \mathcal{M}(\alpha, \zeta, n)$ be of the form (1.1). Then the coefficients $a_k$ $(k \in \mathbb{N} \setminus \{1\})$ of $h$ satisfy (4.1), moreover, the coefficients $b_k$ $(k = n + 1, n + 2, \cdots; n \in \mathbb{N})$ of $g$ satisfy

\[
|b_{n+1}| \leq \frac{|\zeta|}{n+1} \quad (n \in \mathbb{N}), \quad |b_{k+n}| \leq \frac{|\zeta|}{(k+n)(k-1)!} \prod_{j=2}^{k} (j-2\alpha) \quad (k \in \mathbb{N} \setminus \{1\}; n \in \mathbb{N}).
\]

The bounds are sharp for the extremal function given by

\[
f(z) = \int_{0}^{\zeta} \frac{dt}{(1 - \delta t)^{2-2\alpha}} + \int_{0}^{\zeta} \frac{\zeta t^{m} dt}{(1 - \delta t)^{2-2\alpha}} \quad (|\delta| = 1; z \in \mathbb{D}).
\]

**Proof.** By equating the coefficients of $z^{k+n-1}$ in both sides of (4.2), we see that

\[
(k + n)b_{k+n} = \zeta ka_k \quad (k, n \in \mathbb{N}; a_1 = 1).
\]

In view of Lemma 4.1 and (4.3), we get the desired result of Theorem 4.2.

**Theorem 4.3** Let $f \in \mathcal{M}(\alpha, \zeta, n)$ with $0 \leq \alpha < 1$ and $0 \leq \zeta < \frac{1}{2n-1} \ (n \in \mathbb{N})$. Then

\[
\Phi(r; \alpha, \zeta, n) \leq |f(z)| \leq \Psi(r; \alpha, \zeta, n) \quad (r = |z| < 1),
\]

where

\[
\Phi(r; \alpha, \zeta, n) = \begin{cases} 
\log(1 + r) - \frac{\zeta r^{n+1} F_1(n+1,n+2,2\alpha; -r)}{n+1} & (\alpha = 1/2), \\
\frac{(1+r)^{2n-1}}{2\alpha-1} - \frac{\zeta r^{n+1} F_1(n+1,2-2\alpha,n+2,2\alpha; -r)}{n+1} & (\alpha \neq 1/2),
\end{cases}
\]

and

\[
\Psi(r; \alpha, \zeta, n) = \begin{cases} 
- \log(1 - r) + \frac{\zeta r^{n+1} F_1(n+1,n+2,2\alpha; r)}{n+1} & (\alpha = 1/2), \\
\frac{1-(1-r)^{2n-1}}{2\alpha-1} + \frac{\zeta r^{n+1} F_1(n+1,2-2\alpha,n+2,2\alpha; r)}{n+1} & (\alpha \neq 1/2).
\end{cases}
\]

All these bounds are sharp, the extremal function is $f_{\alpha, \zeta, n} = h_{\alpha} + \bar{g}_{\alpha, \zeta, n}$ or its rotations, where

\[
f_{\alpha, \zeta, n}(z) = \begin{cases} 
- \log(1 - z) + \frac{\zeta r^{n+1} F_1(n+1,n+2,2\alpha; z)}{n+1} & (\alpha = 1/2), \\
\frac{1-(1-z)^{2n-1}}{2\alpha-1} + \frac{\zeta r^{n+1} F_1(n+1,2-2\alpha,n+2,2\alpha; z)}{n+1} & (\alpha \neq 1/2).
\end{cases}
\]

**Proof.** Assume that $f = h + \bar{g} \in \mathcal{M}(\alpha, \zeta, n)$. Also, let $\Gamma$ be the line segment joining 0 and $z$, then

\[
|f(z)| = \left| \int_{\Gamma} \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \zeta} d\zeta \right| \leq \int_{\Gamma} \left| [h'(\xi)]^2 + |g'(\xi)| \right| |d\xi| = \int_{\Gamma} (1 + |\xi|^m) |h'(\xi)| |d\xi|.
\]

Moreover, let $\bar{\Gamma}$ be the preimage under $f$ of the line segment joining 0 and $f(z)$, then we obtain

\[
|f(z)| = \left| \int_{\bar{\Gamma}} \frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \zeta} d\zeta \right| \geq \int_{\bar{\Gamma}} \left| [h'(\xi)]^2 - |g'(\xi)| \right| |d\xi| = \int_{\bar{\Gamma}} (1 - |\xi|^m) |h'(\xi)| |d\xi|.
\]
By observing that $h$ is a convex analytic function of order $\alpha$ ($0 \leq \alpha < 1$), it follows that

$$
\frac{1}{(1+r)^{2(1-\alpha)}} \leq |h'(z)| \leq \frac{1}{(1-r)^{2(1-\alpha)}} \quad (|z| = r < 1).
$$

(4.8)

By virtue of (4.6), (4.7) and (4.8), we see that

$$
\Phi(r; \alpha, \zeta, n) := \int_0^r \frac{(1-|z|^2\rho^n)(1+\rho)^{2(1-\alpha)}}{(1+\rho)^2(1-\alpha)} \leq |f(z)| \leq \int_0^r \frac{(1+|z|^2\rho^n)(1-\rho)^{2(1-\alpha)}}{(1-\rho)^2(1-\alpha)} =: \Psi(r; \alpha, \zeta, n),
$$

which yields the desired inequalities (4.4).

Now, we shall prove the sharpness of the result. We only need to show that $f_{\alpha, \zeta, n}$ defined by (4.5) belongs to the class $M(\alpha, \zeta, n)$ for each $\alpha \in [0, 1)$. Suppose that

$$
h_\alpha(z) = \begin{cases} 
-\log(1-z) & (\alpha = 1/2), \\
\frac{1-(1-\zeta^2\rho^n)}{2\alpha n-1} & (\alpha \neq 1/2).
\end{cases}
$$

Then, we find that $h_\alpha(z)$ satisfies the inequality (2.1) and the relation $g'_{\alpha, \zeta, n}(z) = \xi \xi' h'_\alpha(z)$ for each $\alpha \in [0, 1)$. Moreover, for $0 \leq \alpha < 1$, $0 < \zeta < 1/(2n-1)$ with $n \in \mathbb{N}$, and $0 < r < 1$, it easily to see that

$$
f_{\alpha, \zeta, n}(-r) = -\Phi(r; \alpha, \zeta, n) \quad \text{and} \quad f_{\alpha, \zeta, n}(r) = \Psi(r; \alpha, \zeta, n),
$$

therefore,

$$
|f_{\alpha, \zeta, n}(-r)| = \Phi(r; \alpha, \zeta, n) \quad \text{and} \quad |f_{\alpha, \zeta, n}(r)| = \Psi(r; \alpha, \zeta, n).
$$

This shows that the bounds are sharp.

Next, we consider a covering theorem for functions in the class $M(\alpha, \zeta, n)$.

**Theorem 4.4** Let $f \in M(\alpha, \zeta, n)$ with $0 \leq \alpha < 1$. Then the range $f(\mathbb{D})$ contains the disk

$$
|\omega| < r(\alpha, \zeta, n) = \begin{cases} 
\log 2 - \frac{\zeta z F_1(1, n+1; n+2; -1; n+1)}{n+1} & (\alpha = 1/2), \\
\frac{\zeta z F_1(1, n+1, 2-2\alpha; n+2; -1; n+1)}{2\alpha n-1} & (\alpha \neq 1/2).
\end{cases}
$$

The bounds are sharp for the function $f_{\alpha, \zeta, n} = h_\alpha + \overline{g_{\alpha, \zeta, n}}$ given by (4.5) or its rotations.

**Proof.** By putting $r \rightarrow 1^-$ in the lower bound for $|f(z)|$ in Theorem 4.3, we get the desired result. The sharpness is similar to that of Theorem 4.3, we choose to omit the details.

Now, we consider the area theorem of the mappings belong to the class $M(\alpha, \zeta, n)$. Let us denote $A(f(\mathbb{D}_r))$ by the area of $f(\mathbb{D}_r)$, where $\mathbb{D}_r := r\mathbb{D}$ for $0 < r < 1$.

**Theorem 4.5** Let $f \in M(\alpha, \zeta, n)$ with $0 \leq \alpha < 1$. Then, for $0 < r < 1$, $A(f(\mathbb{D}_r))$ satisfies the inequalities

$$
2\pi \int_0^r \frac{\rho(1-|\zeta|^2\rho^n)}{(1+\rho)^{4(1-\alpha)}} d\rho \leq A(f(\mathbb{D}_r)) \leq 2\pi \int_0^r \frac{\rho(1-|\zeta|^2\rho^n)}{(1-\rho)^{4(1-\alpha)}} d\rho.
$$

(4.9)
Proof. Let \( f = h + \overline{g} \in \mathcal{M}(\alpha, \zeta, n) \). Then for \( 0 < r < 1 \), we see that

\[
\mathcal{A}(f(D_r)) = \iint_{D_r} \left( |h'(z)|^2 - |g'(z)|^2 \right) dx \, dy = \iint_{D_r} \left( 1 - |\zeta|^2 |z|^{2n} \right) |h'(z)|^2 dx \, dy.
\]

By observing that \( h \) is a convex analytic function of order \( \alpha \) \( (0 \leq \alpha < 1) \), in view of (4.8) and (4.10), we obtain the desired inequalities (4.9) of Theorem 4.5.

Remark 4.6 By setting \( n = 1 \) in Theorems 4.2, 4.3, 4.4 and 4.5, we get the corresponding results obtained in [19].

Finally, we discuss the radius of close-to-convexity of a certain class harmonic mappings related to the class \( \mathcal{M}(\alpha, \zeta, n) \).

Theorem 4.7 Suppose that \( f = h + \overline{g} \) satisfy the inequality (2.1) with \( -1/2 < \alpha < 0 \). If \( g'(z) = z^n h'(z) \) with \( n \in \mathbb{N} \setminus \{1\} \), then \( f \) is close-to-convex in the disk

\[
|z| < \sqrt{\frac{1 + 2\alpha}{1 + 2n + 2\alpha}} \quad (n \in \mathbb{N} \setminus \{1\}).
\]

Proof. Suppose that \( F_\lambda(z) = h(z) - \lambda g(z) \) with \( |\lambda| = 1 \). It follows that

\[
\Re \left( 1 + \frac{z F''_\lambda(z)}{F'_\lambda(z)} \right) = \Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) + n \Re \left( \frac{\lambda z^n}{\lambda z^n - 1} \right) = \Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) + n \frac{1}{2} \left( 1 - \frac{1 - |\lambda z^n|^2}{(1 - \lambda z^n)(1 - \lambda z^n)} \right).
\]

For \( z = re^{i\theta} \) \((0 < r < 1)\), we see that

\[
\frac{n}{2} \left( 1 - \frac{1 - |\lambda z^n|^2}{(1 - \lambda z^n)(1 - \lambda z^n)} \right) = \frac{n}{2} \left( 1 - \frac{1 - r^{2n}}{1 + r^{2n} - 2 \Re(\lambda z^n)} \right) \geq - \frac{nr^n}{1 - r^n}.
\]

Thus,

\[
\int_{\theta_1}^{\theta_2} \Re \left( 1 + \frac{z F''_\lambda(z)}{F'_\lambda(z)} \right) d\theta > \int_{\theta_1}^{\theta_2} \left( \alpha - \frac{nr^n}{1 - r^n} \right) \left( \theta_2 - \theta_1 \right) > -\pi \quad (\theta_1 < \theta_2 < \theta_1 + 2\pi)
\]

for

\[
|z| = r < \sqrt{\frac{1 + 2\alpha}{1 + 2n + 2\alpha}} =: r(\alpha, n).
\]

By Lemma 3.2 and Kaplan’s close-to-convexity criterion for analytic functions (see [9]), we deduce that \( f \) is close-to-convex in the disk \( |z| < r(\alpha, n) \).

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