Minimal Models from W-Constrained Hierarchies via the Kontsevich–Miwa Transform

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Abstract
A direct relation between the conformal formalism for 2d-quantum gravity and the W-constrained KP hierarchy is found, without the need to invoke intermediate matrix model technology. The Kontsevich-Miwa transform of the KP hierarchy is used to establish an identification between W constraints on the KP tau function and decoupling equations corresponding to Virasoro null vectors. The Kontsevich-Miwa transform maps the W(l)-constrained KP hierarchy to the (p′, p) minimal model, with the tau function being given by the correlator of a product of (dressed) (l, 1) (or (1, l)) operators, provided the Miwa parameter ni and the free parameter (an abstract bc spin) present in the constraints are expressed through the ratio p′/p and the level l.
1 Introduction

Matrix models, which appeared as a ‘discretized’ approach to 2d quantum gravity [1, 2, 3], have resulted at the continuum level in integrable hierarchies subjected to Virasoro [4, 5] and possibly higher W [5, 6] constraints. However, their relation with the Liouville theory, at least with that described in the formalism of [8, 7], seems, although hardly disputable in principle, somewhat obscure. In a recent paper [9] the Kontsevich-Miwa transform was used to establish a relation between the Virasoro constraints imposed on the tau functions of the KP hierarchy, and the decoupling equation corresponding to the null vector at level 2 in minimal conformal field theories extended by a scalar current [8]. Moreover, the field content of the David-Distler-Kawai formalism for 2d quantum gravity was recovered in this way, with the extra scalar playing a rôle similar to that of the Liouville field in the formalism of [8, 7]. The essence of the method based on the Kontsevich-Miwa transform was therefore to relate the Virasoro-constrained KP hierarchy and 2d quantum gravity, bypassing the matrix model technology.

Now the question arises as to whether the higher W constraints on the KP hierarchy are amenable to the above scheme, and whether a bridge can be established between matter interacting with the continuum DDK quantum gravity and the W-constrained KP hierarchies.

In this letter we give a positive answer to this question, based on the results obtained for the W(3) constraints in addition to the previous results for the W(2) (Virasoro) case.

By analogy with the level-2/Virasoro case, one could expect that under the Kontsevich-Miwa parametrization of the KP times, the higher (l ≥ 3) W constraints \( W_l^{(l)} \tau = 0, n \geq -l+1, \) would lead to higher-level (l ≥ 3) null vector decoupling equations. This argument, however, is fraught with an unpleasant “paradox”: the algebra of the W(l) constraints contains the lower W(l') constraints \( W_{n'}^{(l')} \), l' < l as well, which apparently should then give rise, by the same mechanism, to decoupling equations at the lower levels, thereby leading to an over-determined system. Fortunately, this does not happen; it turns out that a single level-l decoupling equation corresponds to the complete set of W(l) constraints (including the Virasoro ones) \( W_n^{(l)} \tau = 0, n \geq -l' + 1, \) 2 ≤ l' ≤ l, while these do not lead back to lower-level decoupling equations. More precisely, the Kontsevich-Miwa transform maps the W(l)-constrained KP hierarchy to the \((p', p)\) minimal model, with the tau function being given by the correlator of a product of (dressed) \((l, 1)\) (or \((1, l)\)) operators, provided (i) the Miwa parameter \( n_i \) satisfies

\[
n_i^2 = (l - 1)^2 \frac{p'}{2p},
\]

and (ii) the free parameter (an abstract bc spin) \( J \) present in the constraints is determined from

\[
2J - 1 = \frac{l - 1}{n_i} - \frac{2n_i}{l - 1}.
\]

Although we have obtained the complete results only for the W(3) case (in addition to W(2) = Virasoro), it is hard to believe that so simple a result, the W — decoupling correspondence,

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1 The effect of the Kontsevich-Miwa transformation was to replace the time parameters of the hierarchy with the spectral parameter, or, in different words, to create a world-sheet out of the time parameters.

2 A relation between Virasoro null vectors and W algebras has been noted in a different approach in [10] and, more recently, in [24].
following after rather tedious calculations, could be a mere coincidence and would not allow generalizations to $W^{(\geq 4)}$ algebras.

We would like to stress that, rather than taking the decoupling equation in its naïve form, one should first ‘dress’ the theory by tensoring it with an extra $U(1)$ current. The extra degrees of freedom which thus appear are then killed by projecting onto a special form of the null state. Therefore, although the null vector in the matter sector is the same, the specific form of the equation for correlation functions changes.

We begin in the next section by introducing the KP hierarchy and the notion of Kontsevich-Miwa transform. Then, in Sect. 3, we introduce the dressed decoupling equation. For purely technical reasons, we describe in this paper the coincidence between $W$ constraints and decoupling equations proceeding in the direction from the decoupling equations to the $W$ constraints: we show in Sect. 4 how to ‘unkontsevich’ conformal field-theoretic data into the $W^{(3)}$ constraints. It will be clear from the derivation that (and how) one can reverse the argument and thus establish equivalence of the two sets of data, the $W^{(3)}$-constrained KP hierarchy and (appropriately dressed) minimal models.

2 The KP hierarchy and the Kontsevich-Miwa transform

We start with the KP hierarchy \cite{11}. It can be described either in an evolutionary form, as an infinite set of equations on (coefficients of) a pseudodifferential operator, or as (Hirota) bilinear relations on the tau function. The tau-functional description may be considered as having the more direct relevance to ‘physics’, being related to the partition function, while the evolutionary form has all the usual advantages due to the introduction of a spectral parameter and the associated wave function. The wave function and the adjoint wave function depend on the spectral parameter $z$ via $\exp(\pm \xi(t, z))\tau(t \mp [z^{-1}])/\tau(t)$, where $\xi(t, z) = \sum_{r \geq 1} t_r z^r$ and

$$t \pm [z^{-1}] = \left(t_1 \pm z^{-1}, t_2 \pm \frac{1}{2} z^{-2}, t_3 \pm \frac{1}{3} z^{-3}, \ldots \right). \quad (2.1)$$

Here $t = (x \equiv t_1, t_2, t_3, \ldots)$ are the time parameters of the hierarchy. This form of the $z$ dependence is ‘generalized’ by the following presentation for the $t_r$, known as the Miwa parametrization \cite{12, 13}:

$$t_r = \frac{1}{r} \sum_j n_j z_j^{-r}, \quad r \geq 1 \quad (2.2)$$

where $\{z_j\}$ is a set of points on the complex plane and the Miwa parameters $n_j$ are integer classically; we will need, however, to continue off the integer values. As we will see, the Miwa parameters acquire the rôle of $U(1)$ charges w.r.t. the scalar field.

By the Kontsevich-Miwa transform we will understand the dependence, via eq.(2.2), of $t_r$ on the $z_j$. However, it is important that the presentation of the form (2.2) be considered for arbitrary $n_j$ (while, on the other hand, Kontsevich has originally used a parametrization of this type for all the $n_j$ equal \cite{14}, see also \cite{15-18}.

As was noted in the Introduction, an important result of the development of matrix models was the discovery of an ‘integrable’ counterpart of the $2d$ gravity + matter theories, in the
form of Virasoro-constrained hierarchies. Moreover, the fact that appropriately constrained hierarchies provide a description of 2d gravity coupled to matter, may be promoted to a first-principle, in which case it is natural to consider on an equal footing various constraints more general than the Virasoro ones, including those whose derivation from a specific matrix model is not known. This applies, first of all, to the extension of Virasoro constraints to W-ones, and also to the introduction of a parameter into the Virasoro generators acting on the tau function \[19\]. This parameter \(J\), which would have parametrized the central charge as \(2(6J^2 - 6J + 1)\), had the \(L_{n \leq -2}\) generators been present as well, can be thought of as the ‘spin’ (dimension) of an abstract bc system underlying the Virasoro generators, \(\sum_{n \in \mathbb{Z}} L_n z^{-n-2} \sim (1 - J)\partial bc - Jb\partial c\). We repeat that although \(J\)-dependent Virasoro constraints may not have been derived from a matrix model, for us all of them are equally good starting points. The corresponding constraints, \(L_n \tau = 0, \ n \geq -1\),

\[
L_{p > 0} = \frac{1}{2} \sum_{s=1}^{p-1} \frac{\partial^2}{\partial t_{p-s} \partial t_s} + \sum_{s \geq 1} s t_s \frac{\partial}{\partial t_{p+s}} + \left( J - \frac{1}{2} \right) (p + 1) \frac{\partial}{\partial t_p}
\]

\[
L_0 = \sum_{s \geq 1} s t_s \frac{\partial}{\partial t_s}
\]

\[
L_{-1} = \sum_{s \geq 1} (s + 1) t_{s+1} \frac{\partial}{\partial t_s}
\]

were shown in [19] to take, after the Kontsevich-Miwa transform, the form of a decoupling equation corresponding to the level-2 null vector in the \((p', p)\) minimal model determined by \(p'/p = 2n_i^2\), provided the Miwa parameter \(n_i\) was determined from

\[
\frac{1}{n_i} - 2n_i = 2J - 1 \equiv Q \quad \text{(Virasoro)}.
\]

Our task in this paper is to extend this correspondence to include W generators. Let us start therefore with the \(W^{(3)}\) constraints imposed on the tau function in a way similar to (2.4); we will write out the constraints explicitly later, see (4.7). Now, by virtue of the Kontsevich-Miwa transform (2.2), the tau function becomes a function \(\tau\{z_j\}\) of the \(z_j\). We assume for it the ansatz\[3\]

\[
\tau\{z_j\} = \lim_{n \to \infty} \langle \Psi(z_1) \ldots \Psi(z_n) \rangle
\]

with \(\langle \ldots \rangle\) and \(\Psi\) being, respectively, the chiral correlation function and a (primary) field operator in a conformal field theory on the \(z\) plane. The operator and the theory itself should be specified according to the fact that the tau function satisfies the \(W^{(3)}\) constraints. The ansatz will be justified by the fact that, as we are going to show, imposing these constraints amounts to the condition that \(\Psi\) be a ‘31’ operator in a \((p', p)\) minimal model [20, 21] for \(p'/p\) determined by the precise form of the constraints. That is, we will find the relation \(p'/p = \frac{2}{n_i} n_i^2\), where \(n_i\) is the Miwa parameter which in its own turn is related to a ‘spin’ \(J\), read off from the Virasoro part of the constraints, by

\[
\frac{2}{n_i} - n_i = 2J - 1 \quad \text{(W\(^{(3)}\))}.
\]

\[3\]Clearly, there have to be infinitely many points \(z_j\) in the Kontsevich-Miwa transform in order for the times \(t_r\) to be independent.
As was noted above, we find it more convenient to present in this paper the derivation of the correspondence W constraints ↔ decoupling equations in the inverse direction: we start from Ψ being the ‘31’ operator and then show how to ‘unkontsevich’ it into the W(3) constraints. We therefore proceed in the next section with introducing the necessary conformal field-theoretic ingredients.

3 ‘Dressing’ the decoupling equation

Now we introduce the second ingredient: null-vector decoupling equations in minimal models of conformal field theory. To the KP hierarchy we will return at the very end.

In addition to the energy–momentum tensor \( T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \), consider a \( U(1) \) current \( j(z) = \sum_{n \in \mathbb{Z}} j_n z^{-n-1} \). The commutation relations read, in the standard manner,
\[
\begin{align*}
[j_m, j_n] &= m \delta_{m+n,0} \\
[L_m, L_n] &= (m-n) L_{m+n} + \frac{d+1}{12} (m^3 - m) \delta_{m+n,0} \\
[L_m, j_n] &= -nj_{m+n}.
\end{align*}
\] (3.1)

The central charge is parametrized as \( d + 1 \), with 1 being the \( U(1) \) contribution.

Now, let Ψ be a primary field with conformal dimension ∆ and \( U(1) \) charge \( q \). We are interested in null vectors at level 3 [20]. It is straightforward to see that a null vector will result from the action on \( |\Psi\rangle \) of the following operator:
\[
\delta L_{-3} - 2L_{-2} L_{-1} + \frac{1}{\delta+1} L_{-1}^3 + \frac{\delta+1+3q^2}{\delta+1} j_{-1}^2 L_{-1} + \frac{2\delta-1}{\delta+1} j_{-2} L_{-1} + 2q j_{-1} L_{-2} - \frac{3q}{\delta+1} j_{-1} L_{-2}^2 = \frac{\delta^2 + \delta + 2dq^2 - q^2}{\delta+1} j_{1} j_{-2} - q \frac{\delta+1+q^2}{\delta+1} j_{1}^2 - \frac{\delta (\delta-1)}{\delta+1} j_{-3}
\] (3.2)

where
\[
\delta = \Delta - \frac{q^2}{2},
\] (3.3)

provided
\[
\delta = \frac{7-d \mp \sqrt{(1-d)(25-d)}}{6}.
\] (3.4)

(It is understood that for the \((p',p)\) minimal model, \( d = 1 - \frac{6(p'-p)^2}{p'p} \).)

The operator (3.2) allows too much arbitrariness, as the values of ∆ (or δ, which is the matter dimension of Ψ) and \( q^2 \) cannot be fixed separately. However, the Kontsevich-Miwa transform suggests a more special form of the operator (3.2), which can be arrived at as follows. One first converts the expression of the null vector into the decoupling equation for correlation functions of the form
\[
\langle \Psi (z_i) \prod_{j \neq i} O_j (z_j) \rangle.
\] (3.5)
where, obviously, at least one operator insertion must be that of $\Psi$. (From now on, the corresponding insertion point $z_i$ will therefore be singled out from the rest of the $z_j$. The other insertions, of dimensions $\Delta_j$ and $U(1)$ charges $q_j$, may or may not coincide with $\Psi$).

Then, analysing the decoupling equation corresponding to (3.2), one finds that the terms coming from $j^3_1$ cannot be recovered from the Kontsevich-Miwa transform of $W$ constraints on the KP hierarchy\(^5\). To kill these terms, we have to choose

$$\delta = -1 - q^2 .$$

Further, with this condition satisfied, there are terms in the decoupling equation of the form

$$\sum_{j, k, l \neq i} q_j (z_j - z_i)(z_k - z_i)(z_l - z_i) \left\{ -\frac{1}{q} \partial^2 \partial z_j - 3 q_j \partial^2 \partial z_i \right\} \left\langle \Psi(z_i) \prod_{j \neq i} O_j(z_j) \right\rangle = 0 ,$$

which are not acceptable either. Thus, similarly to the Virasoro case\(^6\), we restrict ourselves to a subsector of those operators whose dimensions and $U(1)$ charges satisfy

$$\Delta_j = -\frac{1}{2}(q^2 + 2)\frac{q_j}{q} .$$

This implies fixing a dressing prescription: the extra scalar $\phi(z) \sim f^2 j$ enters vertex operators with an exponent depending on that of the matter part.

As a result of conditions (3.6) and (3.8), the decoupling equation takes a very simple form

$$\left\{ -\frac{1}{q^2} \partial^3 \partial z_i^2 + \sum_{j \neq i} \frac{1 + q^2}{(z_j - z_i)^2} \left( \frac{\partial}{\partial z_j} - \frac{q_j}{q} \frac{\partial}{\partial z_i} \right) + \sum_{j \neq i, k \neq i} \frac{1}{(z_j - z_i)(z_k - z_i)} \left( 2 \frac{\partial^2}{\partial z_j \partial z_i} - 3 \frac{q_j}{q} \frac{\partial^2}{\partial z_i^2} \right) + 2 \sum_{j \neq i, k \neq i} \frac{q_k}{(z_j - z_i)(z_k - z_i)} \right\} \left\langle \Psi(z_i) \prod_{j \neq i} O_j(z_j) \right\rangle = 0 .$$

In the chosen normalization ( + sign on the RHS of the first equation in (3.1)), the $q$’s must be imaginary ($q^2 < 0$), so let us define

$$\left\{ q_j = \sqrt{-1} n_j, \quad j \neq i \\
q = \sqrt{-1} n_i \right\}$$

We claim that these are the $n$’s from the Kontsevich-Miwa transform! In the next section we show how to ‘unkontsevich’ eq. (3.9) into the full set of $W^{(3)}$ constraints.

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\(^4\)Dimensions refer to those as evaluated in the combined theory, by operator product expansions with the energy–momentum tensor $T(z)$. On the other hand, separating away the current contribution by writing $L_n = L_n + (\text{Sugawara})_n$, would leave us with the matter sector in which the dimensions are found from the energy–momentum tensor composed of the $l$’s.

\(^5\)The terms

$$\sum_{j \neq i} \sum_{k \neq i} \sum_{l \neq i} \frac{q_j q_k q_l}{(z_j - z_i)(z_k - z_i)(z_l - z_i)} \left\langle \Psi(z_i) \prod_{j \neq i} O_j(z_j) \right\rangle ,$$

corresponding to $j^3_1$, are not in the image of the Kontsevich-Miwa transform of the $W$ constraints.
Note that in a similar analysis of the level-2/Virasoro case [4], one obtains at this stage a
generalized master equation (cf. [22])

\[
- \frac{1}{2n_i^2} \frac{\partial^2}{\partial z_i^2} + \frac{1}{n_i} \sum_{j \neq i} \frac{1}{z_j - z_i} \left( n_j \frac{\partial}{\partial z_i} - n_i \frac{\partial}{\partial z_j} \right) \left\langle \bar{\Psi}(z_i) \prod_{j \neq i} \mathcal{O}_j(z_j) \right\rangle = 0
\]

(3.11)
in which \( \bar{\Psi} \) is the ‘21’ operator. The null vectors in the matter sector, underlying each one of these
equations, are of course the same as for the respective standard decoupling equations [20].
However, the specific form that the decoupling equation takes in the presence of the current, is
crucial for the derivation of \( W \) generators.

4 From the decoupling equation to the \( W^{(3)} \) constraints

Having arrived in the last section at the decoupling operator

\[- n_i \mathcal{W} \equiv \]

\[
\frac{1}{n_i^2} \frac{\partial^3}{\partial z_i^3} + \sum_{j \neq i} \frac{1 - n_i^2}{(z_j - z_i)^2} \left( \frac{\partial}{\partial z_j} - \frac{n_j}{n_i} \frac{\partial}{\partial z_i} \right) + \sum_{j \neq i} \frac{1}{z_j - z_i} \left( 2 \frac{\partial^2}{\partial z_j \partial z_i} - 3 \frac{n_j}{n_i} \frac{\partial^2}{\partial z_i^2} \right) \]

(4.1)

\[-2 \sum_{j \neq i} \sum_{k \neq i} \frac{n_k}{(z_k - z_i)(z_j - z_i)} \left( n_i \frac{\partial}{\partial z_j} - n_j \frac{\partial}{\partial z_i} \right) .\]

let us now interpret the \( z_j \) and \( n_j \) as the ingredients of the Kontsevich-Miwa transform (2.2),
and see whether the operator \( \mathcal{W} \) can indeed be rewritten in terms of the time parameters \( t_r \).
This is by no means automatic, and in particular would not be true had we not imposed the
restrictions (3.6) and (3.8)!

A trivial part will be to express \( \partial/\partial z \)’s in terms of \( \partial/\partial t \)’s. The non-trivial part is to get
rid of the \( z_j \) themselves, as eq. (2.2) does not allow this to be done straightforwardly. As an
example let us take the second term from (4.1). Using (2.2) to express derivatives w.r.t. the
\( z_j \), we find

\[
\sum_{j \neq i} \frac{1 - n_i^2}{(z_j - z_i)^2} \left( \frac{\partial}{\partial z_j} - \frac{n_j}{n_i} \frac{\partial}{\partial z_i} \right) = \sum_{j \neq i} \frac{1 - n_i^2}{z_j - z_i} n_j \sum_{r \geq 1} \frac{z_i^{-r-1} - z_j^{-r-1}}{z_j - z_i} \frac{\partial}{\partial t_r}
\]

\[
= -(1 - n_i^2) \sum_{p \geq 1} \sum_{q \geq 1} (p + 1) z_i^{-p-3} q t_q \frac{\partial}{\partial t_{p+q}} \]

\[
+(1 - n_i^2) \sum_{p \geq 1} \frac{1}{2} (p + 1) (p + 2) z_i^{-p-3} \frac{\partial}{\partial t_p} + (1 - n_i^2) \sum_{j \neq i} \frac{n_j}{z_j - z_i} \sum_{p \geq 1} z_i^{-p-2} (p + 1) \frac{\partial}{\partial t_p} .
\]

(4.2)

Similarly, in the last term in (4.1) we can also divide by \( z_j - z_i \) in the following way:

\[
-2 \sum_{k \neq i} \frac{n_k}{z_k - z_i} \sum_{j \neq i} n_j \sum_{r \geq 1} \frac{z_i^{-r-1} - z_j^{-r-1}}{z_j - z_i} \frac{\partial}{\partial t_r}
\]

\[
= -2n_i \sum_{j \neq i} \frac{n_j}{z_j - z_i} \sum_{p \geq 1} \sum_{s \geq 1} \frac{s t_s z_i^{-p-2}}{p + s} \frac{\partial}{\partial t_{p+s}} + 2n_i \sum_{j \neq i} \frac{n_j}{z_j - z_i} \sum_{p \geq 1} (p + 1) z_i^{-p-2} \frac{\partial}{\partial t_p}
\]

(4.3)
Computing in a similar manner the terms with the second and the third derivatives and collecting everything together, we get

\[
\mathcal{W} = \sum_{q,r,s \geq 1} z_i^{-q-r-s-3} \frac{\partial^3}{\partial t_q \partial t_r \partial t_s} + \left( \frac{3}{n_i} - 2n_i \right) \sum_{r,s \geq 1} z_i^{-r-s-3}(r+1) \frac{\partial^2}{\partial t_r \partial t_s} + \left( \frac{1}{n_i^2} + \frac{n_i^2}{2} - \frac{1}{2} \right) \sum_{r \geq 1} (r+1)(r+2) z_i^{-r-3} \frac{\partial}{\partial t_r} + \frac{1 - n_i^2}{n_i} \sum_{p \geq 1} \sum_{q \geq 1} z_i^{-p-3}(p+2) q t_q \frac{\partial}{\partial t_{p+q}} + 2 \sum_{p \geq 0} z_i^{-p-3} \sum_{s=-1}^{p-1} \sum_{q \geq 1} q t_q \frac{\partial^2}{\partial t_{p-s} \partial t_{q+s}} + 2 \sum_{j \neq i} \frac{n_j}{n_i} \sum_{p \geq 1} \sum_{r \geq 1} z_j z_i \sum_{p \geq 1} \sum_{r \geq 1} L_p \left( z_i^{-p-2} - z_j^{-p-2} \right)
\]

with

\[
L_{p \geq 1} = \frac{1}{2} \sum_{s=1}^{p-1} \frac{\partial^2}{\partial t_{p-s} \partial t_s} + \sum_{s \geq 1} \frac{n_j}{n_i} \sum_{p \geq 1} \sum_{r \geq 1} z_j z_i \sum_{p \geq 1} \sum_{r \geq 1} L_p \left( z_i^{-p-2} - z_j^{-p-2} \right)
\]

and \( L_0 \) and \( L_{-1} \) are the same as in (2.4). Notice that the Virasoro constraints (4.5) are different from the ones in (2.4), according to the value of \( J \) in terms of the Miwa parameter (assuming it is the same). The last term has been added to and subtracted from the RHS of (4.4). With the combination \((z_i^{-p-2} - z_j^{-p-2})/(z_j - z_i)\) we proceed as above, and in this way we finally arrive at,

\[
0 = \mathcal{W}_\tau = \sum_{p \geq -2} z_i^{-p-3} \mathcal{W}_p^{(3)} \tau + 2 \sum_{j \neq i} \frac{n_j}{n_i} \sum_{p \geq -2} \sum_{r \geq 1} L_p \left( z_j^{-p-2} \right)
\]

with

\[
\begin{align*}
\mathcal{W}_{-2} &= 2 \sum_{r \geq 1} r t_r L_{r-2} \\
\mathcal{W}_{-1} &= 2 \sum_{r \geq 1} r t_r L_{r-1} - 2n_i L_{-1} + \frac{1}{n_i} \sum_{r \geq 2} r t_r \frac{\partial}{\partial t_{r-1}} \\
\mathcal{W}_0 &= 2 \sum_{r \geq 1} r t_r L_r - 4n_i L_0 + 2 \sum_{r \geq 1} r t_r \frac{\partial}{\partial t_r} + 2 \sum_{r \geq 2} r t_r \frac{\partial^2}{\partial t_{1} \partial t_{r-1}} \\
\mathcal{W}_{p \geq 1} &= 2 \sum_{r \geq 1} r t_r L_{r+p} - 2(p+2)n_i L_p + (p+2) \sum_{r \geq 1} r t_r \frac{\partial}{\partial t_{r+p}} \\
&\quad + 2 \sum_{s=1}^{p-1} \sum_{r \geq 1} r t_r \frac{\partial^2}{\partial t_{p-s} \partial t_{r+s}} + (p+1)(p+2) \left( \frac{1}{n_i^2} + \frac{n_i^2}{2} - \frac{1}{2} \right) \frac{\partial}{\partial t_p} + (3/n_i - 2n_i) \sum_{r \geq 1} (r+1) \frac{\partial^2}{\partial t_r \partial t_{p-r}} + \sum_{q \geq 1} \frac{n_j}{n_i} \sum_{p \geq 1} \sum_{q \geq 1} \frac{n_j}{n_i} \sum_{r \geq 1} \sum_{s \geq 1} \frac{\partial^3}{\partial t_{r} \partial t_{p-r} \partial t_{s}}
\end{align*}
\]
The final step consists in demanding that the $W^{(3)}$ and the $L$ generators annihilate the tau function separately:

$$L_n \tau = 0, \quad n \geq -1, \quad W_n^{(3)} \tau = 0, \quad n \geq -2. \quad (4.8)$$

Thus we have recovered the full set of $W^{(3)}$ constraints, (i.e. $W_n^{(3)}$ and $L_n$) with the coefficients in front of the different terms depending on $n_i$ and therefore, via eq. (2.7), on the corresponding minimal model. Recall also that, according to (3.6), (3.8), (3.4) and (3.10), the Miwa parameter $n_i$ is equal to the cosmological constant

$$n_i = \sigma \alpha_\pm \equiv \sigma \left( -\frac{1}{2} \sqrt{\frac{25 - d}{3}} \pm \frac{1}{2} \sqrt{\frac{1 - d}{3}} \right), \quad (4.9)$$

where $\sigma^2 = 1$ and the upper/lower sign corresponds to that in (3.4). Stated differently, for the $(p', p)$ minimal model we have $n_i^2 = 2p'/p$.

Conversely, starting from the full set of $W^{(3)}$ constraints, one constructs the combination

$$\Delta_j = \frac{1}{2} \sigma Q n_j \quad (\text{however unnatural it may seem from the point of view of the } t\text{-variables})$$

and then, fixing the coefficients as explained and inverting the previous steps, one arrives at the decoupling equation. This proves the equivalence between the $W^{(3)}$ constraints and the level-3 decoupling equation in the $(p', p)$ minimal model.

A remark is in order concerning why the Virasoro part of the derived constraints does not lead to the level-2 decoupling equations as was the case in ref. [9]. The point is that the possibility to transform the Virasoro constraints into a decoupling equation depends crucially on the relation between the ‘spin’ $J$ present in the Virasoro generators, and the value of the Miwa parameter $n_i$ (recall that the $i$-th point, being the position of the insertion of $\Psi = \Psi_{31}$, plays a special rôle). However, the spin $J$ as read off from the generators (4.5), eq. (2.7) \cite{9}, is not the same as required by (2.5) and therefore the generators (4.5) do not allow a transformation into the $z$-variables.

## 5 Dressing prescription and generalizations

The dressing prescription (3.8) corresponding to $W^{(3)}$ constraints, can be rewritten as $\Delta_j = \pm \frac{1}{2} \sigma Q n_j$ and in this form it coincides with the dressing prescription derived from the Virasoro constraints on the KP hierarchy \cite{4}. Therefore the sector singled out from the matter $\otimes U(1)$ theory by the dressing prescription is the same for the two cases. Using another ‘universal’ relation, $\delta_j = \Delta_j = \frac{1}{2} \sigma Q n_j$, which gives the matter dimensions by subtracting away the Sugawara part, we arrive at

$$n_j^2 \pm \sigma Q n_j - 2\delta_j = 0. \quad (5.1)$$

\footnote{According to the form of the dependence on $z_i$ and $z_j$ in (4.6); in particular, the combination $\sum_{i \neq j} \frac{u_{ij}}{z_j - z_i}$ cannot be rewritten in terms of the time parameters, while all the rest of eq. (4.6) is expressed through the times, so that the two terms (those involving $W^{(3)}$ and $L$ respectively) are ‘linear independent’, hence the vanishing of each one.}

\footnote{Note that it follows from (2.7) that $J = \frac{1}{2} = (\pm) \frac{1}{2} \sqrt{(1 - d)/3} \equiv \frac{1}{2} (\pm) Q_m$, where $Q_m$ is the matter background charge, which therefore coincides with $Q \equiv 2J - 1$.}
The value of $n_j$ found from here differs from the DDK dressing by the cosmological constant $\alpha_\pm = \frac{1}{2}(-Q_L \pm Q)$.

A general pattern which generalizes the two lowest cases, $W^{(2)}$ and $W^{(3)}$, can be formulated as follows: To the $W^{(l)}$ constraints on the KP hierarchy we associate the decoupling equation corresponding to the irreducible module built on the (dressed) $(l,1)$ (or $(1,l)$) state in the $(p',p)$ minimal model. The value of $p'/p$ is optional, and depends on the ‘spin’ $J$ entering in the Virasoro part of the $W^{(l)}$ constraints:

$$\frac{p'}{p} = 1 + \frac{Q^2}{4} + \frac{Q}{4}\sqrt{Q^2 + 8}, \quad Q \equiv 2J - 1. \quad (5.2)$$

According to the standard formulae, the primary field $\Psi_{l1}$ has dimension (in the matter sector)

$$\delta = \frac{4 + Q^2 + Q\sqrt{Q^2 + 8}}{16}(l^2 - 1) + \frac{1 - l}{2}. \quad (5.3)$$

The dressing of $\Psi_{l1}$ is specified by its $U(1)$ charge $q = \sqrt{-1}n_i$ with $n_i$, which at the same time is the Miwa parameter, determined from

$$\delta = \frac{l + 1}{l - 1}n_i^2 + \frac{1 - l}{2} \quad (5.4)$$

(eq. (3.4) is recovered for $l = 3$). This gives for the ‘bulk’ dimension of the dressed $\Psi_{l1}$ operator:

$$\Delta = \delta - \frac{n_i^2}{2} = \frac{n_i^2}{l - 1} + \frac{1 - l}{2}. \quad (5.5)$$

The dimensions and $U(1)$ charges of all the other operators satisfy

$$\Delta_j = \Delta\frac{n_j}{n_i} = \left(\frac{n_i}{l - 1} - \frac{l - 1}{2n_i}\right)n_j. \quad (5.6)$$

With $n_i$ determined from the above formulae as

$$n_i = (1 - l)\frac{Q + \sqrt{Q^2 + 8}}{4}, \quad (5.7)$$

we find the relation between the Miwa parameter $n_i$ and the spin $J$ present in the Virasoro part of the $W^{(l)}$ constraints:

$$\frac{l - 1}{n_i} - \frac{2n_i}{l - 1} = 2J - 1 \quad (W^{(l)}). \quad (5.8)$$

For the $(1,l)$ operator the formulae would receive the obvious sign modifications.

To conclude let us note that beyond level 4 it would be difficult to give a direct derivation of the correspondence between $W^{(l)}$ constraints and the $(l,1)$ decoupling equations, so one has to rely on more general arguments (cf. however, ref. [23]). Also, it would be interesting to understand the rôle in this picture of the $W$-algebra null vectors, as well as of the decoupling equations other than those considered in [23]. It is also unclear how integer values of $p'$ and $p$ could be singled out in the KP formalism. These problems are under investigation and will be the subject of a future publication.

Acknowledgements. We would like to thank A. Berkovich, V. Fock, E. Kiritsis and W. Lerche for useful discussions.
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