Acceleration of the universe with a simple trigonometric potential

Narayan Banerjee 1
Sudipta Das 2
Relativity and Cosmology Research Centre,
Department of Physics, Jadavpur University,
Calcutta - 700 032,
India

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Abstract

In this paper, we investigate the quintessence model with a minimally coupled scalar field in the context of recent supernovae observations. By choosing a particular form of the deceleration parameter $q$, which gives an early deceleration and late time acceleration for dust dominated model, we show that this sign flip in $q$ can be obtained by a simple trigonometric potential. The early matter dominated model expands with $q = 1/2$ as desired and enters a negative $q$ phase quite late during the evolution.

1 INTRODUCTION

Over the last few years, there are growing evidences in favour of the scenario that the universe at present is expanding with an acceleration. The supernovae project [1] and also the Maxima [2] and Boomerang [3] data on cosmic microwave background (CMB) strongly suggest this acceleration. The very recent WMAP data [4] also seem to confirm this. This result is indeed counter-intuitive, as gravity holds matter together and it might be expected that in the absence of any exotic field, the universe should be decelerating. As this acceleration could be brought about by an effective negative pressure, the first choice of candidate for this ‘dark energy’ had been the ‘cosmological constant’ or a time varying cosmological parameter $\Lambda(t)$. Due to well-known reasons, $\Lambda$ has fallen from grace (for an excellent up-to-date review, see [5],[6]). A scalar field with a positive definite potential can indeed give rise to an effective negative pressure if the potential term dominates over the kinetic term. The pressure to density ratio for the scalar field, as required by the supernovae observations, is given as $w_\phi < -\frac{2}{3}$ (See ref [7] and references therein). This source of energy is called the quintessence matter (Q-matter). In this context, non-minimally coupled scalar fields had been investigated thoroughly to check if they could drive an accelerated expansion [8]. Brans-Dicke’s scalar field appears to generate sufficient acceleration

1E-mail: narayan@juphys.ernet.in
2E-mail:dassudiptadas@rediffmail.com
in the matter era, but it has its problems in the earlier evolution [9]. A viscous fluid along with a Q-matter could also be a useful candidate and this appears to solve the coincidence problem also [10]. This coincidence problem, i.e., why the Q-matter dominates only recently, was tackled in the so-called tracker solutions [11] where the scalar field energy density runs parallel to the matter energy density from below through the evolution and gets to dominate only during later stages. Very recently Chaplygin gas, which has a nonlinear contribution of the energy density to the dynamics of the model, has also been invoked [12]. Most of these models do exhibit an accelerated expansion in the matter-dominated regime.

It deserves mention that the same model should have a deceleration in the early phase of matter era in order to provide a perfect ambience for structure formation. Furthermore, the accelerated phase is perhaps only a very recent one. There are observational evidences too that beyond a certain value of the redshift \((z \sim 1.7)\), our universe had been going through a decelerated expansion [13]. This indication is indeed reassuring, as the formation of structure in the universe is better supported by a decelerating model. This is because local inhomogeneities will grow and become stable from the seeds of density fluctuation only if the force field is attractive.

Amendola [14] has argued that all the required structure formation and other relevant observations regarding the supernovae could well be explained even if the alleged acceleration of universe started quite a long time back, even beyond \(z = 5\). However, this work also shows that the model requires both an accelerated and a decelerated phase of expansion. But a more recent work by Padmanabhan and Roychowdhury [15] shows a striking result. It indicates that if we take the complete data set, i.e., acceleration up to a certain \(z\) and deceleration beyond that (i.e., for higher \(z\)), then only this conclusion of the change of signature of the deceleration parameter holds. On the other hand, the individual data sets of the high and low redshift supernovae may well be consistent with a decelerating universe without any ‘dark energy’.

So indeed we are in need of some form of fields which governs the dynamics in such a way that the deceleration parameter becomes negative well into the matter era. One such Q-matter had been given by Sen and Sethi [16] where they include a potential which is a ‘double exponential’ of the scalar field. They obtained a scale factor which is a sine hyperbolic function of time in the matter dominated regime. The deceleration parameter \((q)\) indeed has a sign flip and with a little fine-tuning, the scale factor can grow during the early stages as \(t^{2/3}\) which is indeed the usual solution for the Einstein equations for a flat FRW spacetime for pressureless dust.

In the present work, we adopt the following strategy. We choose a form of \(q\) as a function of the scale factor \(a\) so that it has the desired property of a signature flip. Then with this input, the scalar field and the required potential are found out. It turns out that a fairly simple trigonometric potential does the needful. The origin of the scalar potential, however, cannot be indicated. Surely this is not the ideal way to find out the dynamics of the universe, as here the dynamics is assumed and then the fields are found out without any reference to the origin of the field. But in the absence of more rigorous ways, this kind of investigations collectively might finally indicate towards the path where one really has to search. This ‘reverse’ way of investigations had earlier been used extensively by Ellis and Madsen [17] for finding out the potential driving inflation, i.e., an accelerated phase of the universe at a very early stage of its evolution.
2 Results

For a spatially flat Robertson-Walker spacetime

\[ ds^2 = dt^2 - a^2(t)[dr^2 + r^2d\Omega^2], \]

(1)

the deceleration parameter \( q \) is given by

\[ q = -\frac{\ddot{a}a}{\dot{a}^2} \]

(2)

where \( a \) is the scale factor of the universe and is a function of the cosmic time ‘\( t \)’ alone.

In order to get a model consistent with observations, one needs an expanding universe, i.e, a positive Hubble parameter \( H = \frac{\dot{a}}{a} \) throughout the evolution, but a deceleration parameter \( q \), unlike being a positive constant throughout the matter era at \( q = 1/2 \) as believed until the recent observations, should be a function of the scale factor ( or that of \( t \) ). Furthermore, this functional dependence should be such that \( q \) undergoes a transition from its positive phase to a negative one in the matter dominated period itself. It is thus imperative that the scale factor cannot have a simple power-law behaviour. If \( a \sim t^n \), the universe will have an accelerated or a decelerated expansion for \( n > 1 \) or \( n < 1 \) respectively throughout the period.

In the quest for a varying \( q \) consistent with observations, in the same line as that floated by Ellis and Madsen, we propose the relation

\[ q = -\frac{\ddot{a}a}{\dot{a}^2/a^2} = -1 - \frac{pa^p}{1 + a^p}, \]

(3)

where \( p \) is a constant. It is found that for a certain range of negative values of \( p \), this works remarkably well.

The equation (3) integrates to yield

\[ H = \frac{\dot{a}}{a} = A(1 + a^p) \]

(4)

where \( A \) is an arbitrary constant of integration. \( A \) is taken to be positive, which ensures the positivity of the Hubble parameter (the expansion of the universe is never denied!) irrespective of the signature or value of the constant \( p \).

It is found that for values of \( p \) between -2 and -1, the model shows exactly the behaviour which is desired (as shown in Figure 1).

In what follows, we work out the problem completely for \( p = -3/2 \), for which the model works with a non minimally coupled scalar field with a potential expressed as a simple trigonometric function of the scalar field.

As the interest is in a matter dominated universe, the fluid is taken in the form of a pressureless dust. The Einstein equations for the space-time given by equation (1) are,

\[ 3\frac{\dot{a}^2}{a^2} = \rho + \frac{1}{2}\dot{\phi}^2 + V(\phi), \]

(5)

\[ 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -\frac{1}{2}\ddot{\phi}^2 + V(\phi) \]

(6)
where $\rho$ is the density of matter, $\phi$ is the scalar field and $V(\phi)$ is a scalar potential.

The wave equation for the scalar field is

$$\ddot{\phi} + \frac{3}{a} \dot{\phi} + V'(\phi) = 0$$  \hspace{1cm} (7)

In all equations an overhead dot implies differentiation w.r.t. time and a prime is that w.r.t. the scalar field $\phi$. The matter conservation equation, which can in fact be obtained from these three equations in view of the Bianchi identity, yields

$$\rho = \frac{\rho_0}{a^3}$$  \hspace{1cm} (8)

$\rho_0$ being a constant. So we have three equations to solve for four unknowns.

We assume the deceleration parameter as given in equation (3), which can be integrated twice to give $H = \dot{a}/a$ as in equation (4) and the scale factor as

$$a = [e^{-Ap_t} - 1]^{-\frac{1}{p}}$$  \hspace{1cm} (9)

Now, the system of equations (5), (6) and (7) is closed with the assumption of equation (3) or equivalently equation (4). So in order to solve the system completely, the parameter ‘$p$’ should have a fixed value. We choose $p = -\frac{3}{2}$, as it yields $q = 0.5$ for a very low value of $\frac{a}{a_0}$, where $a_0$ is the present value of the scale factor. The physical motivation for choosing this value of $q$ for early matter dominated epoch is that for a spatially flat FRW model with $p = 0$ without any Q-matter indeed has $q = 0.5$ and that the transition from radiation to matter dominated epoch for this value of $q$ is well studied [18].

With $p = -\frac{3}{2}$, equations (5) and (6) are used to eliminate $V(\phi)$, and $\dot{\phi}$ can be calculated to be

$$\dot{\phi} = \sqrt{3}Aa^{-\frac{3}{2}} = \frac{\sqrt{3}A}{[e^{\frac{3A}{2}t} - 1]^\frac{1}{2}}$$  \hspace{1cm} (10)

The scalar field is found out by integrating equation (10) as,

$$\phi = \frac{4}{\sqrt{3}}tan^{-1}(e^{3At/2} - 1)^\frac{1}{2},$$  \hspace{1cm} (11)
The potential $V(\phi)$ can also be calculated from equations (5) and (6) first as a function of time and by the use of equation (11) as a function of $\phi$ as,

$$V(\phi) = \frac{9A^2}{2} \cot^2\left(\frac{\sqrt{3} \phi}{4}\right) + 3A^2 \quad (12)$$

Now one has the complete set of the solutions, $a = a(t)$, $\phi = \phi(t)$, $\rho = \rho(t)$ and $V = V(\phi)$ for $p = -3/2$. The solutions, when plugged in the field equations, namely (5), (6) and (7), satisfy all of them provided

$$\rho_0 = 3A^2. \quad (13)$$

From equations (5) and (6), we note that the contribution from the quintessence field $\phi$ towards the density and effective pressure are given as

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (14)$$

and

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) \quad (15)$$

respectively. From these, one can write down the expressions for the dimensionless density parameters $\Omega_m = \frac{\rho_m}{3H^2}$ and $\Omega_\phi = \frac{\rho_\phi}{3H^2}$ respectively for the visible matter and the $Q$-matter.

With $p = -3/2$, the present model yields

$$\Omega_m = \frac{(1 + z)^3}{[1 + (1 + z)^{-3/2}]^2}, \quad (16)$$

and

$$\Omega_\phi = 1 - \Omega_m, \quad (17)$$

where $z$ is the redshift parameter given by

$$1 + z = \frac{a_0}{a}, \quad (18)$$

$a_0$ being the present value of the scale factor. $\Omega_{m0}$, the present value of $\Omega_m$, comes out to be 0.25 and $\Omega_{\phi0} = 0.75$. These values are well within the constraints of $0.2 \leq \Omega_{m0} \leq 0.8$ [19],[5]. Figure 2 shows that $\Omega_m$ increases with $z$, i.e, decreases with the evolution of the universe. $\Omega_\phi$ starts dominating over $\Omega_m$ roughly at $z = 0.8$. At the earlier epoch, i.e, at high $z$, $\Omega_\phi$ is very small, allowing a conducive matching onto the radiation era for the perfect ambience for nucleosynthesis.

Unlike Newtonian gravity, general relativity ensures that the pressure also contributes in driving the acceleration of the model. From equations (5) and (6), one has

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_t + 3p_t) \quad (19)$$

where $\rho_t$ and $p_t$ are the total effective density and pressure respectively. If the pressure is connected with the density by the relation

$$p = w\rho, \quad (20)$$
the model will accelerate ( $\ddot{a} > 0$ ) only if

$$w_t = \frac{p_m + p_\phi}{\rho_m + \rho_\phi} < -\frac{1}{3}.$$  \hspace{1cm} (21)

The subscripts ‘t’, ‘m’ and ‘$\phi$’ stand for total, normal fluid distribution and the Q-matter $\phi$ respectively. For a matter dominated universe, $p_m = 0$ and hence $w_m = 0$. This particular model gives

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{1 + (1 + z)^{\frac{2}{3}}}{-1 - 2(1 + z)^{\frac{2}{3}}}$$ \hspace{1cm} (22)

$$w_t = \frac{p_\phi}{\rho_\phi + \rho_m} = -\left[1 + (1 + z)^{\frac{2}{3}}\right]^{-1}$$ \hspace{1cm} (23)

The evolution of $w_t$ versus $z$ and $w_\phi$ versus $z$ are shown in figure 3, which indicates that $w_t$ attains the required value of $-\frac{1}{3}$ or less only close to $z = 0.5$. Beyond that, $w_t$ is less negative, and the universe still decelerates.

The value of $w_{\phi0}$, i.e, the value of the equation of state parameter for the scalar field at $z = 0$ as given by the present model and as indicated by figure 3(b) is definitely within the constraint range [19].

We also plot the rate of change of $w_t$ against $z$ (as shown in figure 4). It shows that $\frac{dw_t}{dz}$ is still negative at the present epoch, but the magnitude of $\frac{dw_t}{dz}$ is decreasing.

The solution for the scale factor is good enough to allow the density contrast to grow favourably for the formation of large scale structure. Figure (5) shows the growth of linearized density perturbation in this model and evidently indicates that it grows linearly with the scale factor during later stages as expected for the matter dominated epoch [18].

In the absence of the final form of the quintessence matter, search for the relevant form of potential will continue and the present investigation is one of them. In view of the high degree of non-linearity of Einstein’s equations, exact solutions always play a vital role as piecewise solutions have the problem of proper matching at different interfaces. The present model shows that inspite

Figure 2: Plot of $\Omega$ vs. $z$. 

Figure 3: Evolution of $w_t$ and $w_\phi$ versus $z$. 

Figure 4: Rate of change of $w_t$ against $z$. 

Figure 5: Growth of linearized density perturbation.
Figure 3: Plot of (a) $w_t$ vs. $z$ and (b) $w_\phi$ vs. $z$.

Figure 4: Plot of $\frac{dw_\phi}{dz}$ vs. $z$ for spatially flat dust dominated R-W model.

Figure 5: Plot of density contrast $D$ vs. $a$ where $D = \frac{\rho - \bar{\rho}}{\bar{\rho}}$.

of the severe constraints imposed by observations, one can still find an exact FRW model which gives values for the relevant parameters like $q$, $w$, $\Omega_\phi$, $\Omega_m$ etc. safely within the range given by
observations. It also has the merit of having a single analytical expression for \( q = q(a) \), which gracefully transits from its positive phase to the negative one and adds to the list of quintessence potentials that serve the purpose of modelling a presently accelerating universe [20]. Definitely the model has problems, particularly that of fine tuning, but in fact, all quintessence models have some such problems.

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