Pluricapacity and approximation numbers
of composition operators

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Abstract. For suitable bounded hyperconvex sets \( \Omega \) in \( \mathbb{C}^N \), in particular the ball or the polydisk, we give estimates for the approximation numbers of composition operators \( C_\varphi: H^2(\Omega) \to H^2(\Omega) \) when \( \varphi(\Omega) \) is relatively compact in \( \Omega \), involving the Monge-Ampère capacity of \( \varphi(\Omega) \).

Key-words: approximation numbers; composition operator; Hardy space; hyperconvex domain; Monge-Ampère capacity; pluricapacity; pluripotential theory; Zakharyuta conjecture

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1 Introduction

Let \( D \) be the unit disk in \( \mathbb{C} \), \( H^2(D) \) the corresponding Hardy space, \( \varphi \) a non-constant analytic self-map of \( D \) and \( C_\varphi: H^2(D) \to H^2(D) \) the associated composition operator. In [40], we proved a formula connecting the approximation numbers \( a_n(C_\varphi) \) of \( C_\varphi \), and the Green capacity of the image \( \varphi(D) \) in \( D \), namely, when \( \overline{\varphi(D)} \subset D \), we have:

\[
\beta(C_\varphi) := \lim_{n \to \infty} [a_n(C_\varphi)]^{1/n} = \exp \left( -\frac{1}{\text{Cap} \left( \varphi(D) \right)} \right),
\]

where \( \text{Cap} \left( \varphi(D) \right) \) is the Green capacity of \( \varphi(D) \).

A non-trivial consequence of that formula was the following:

\[
\|\varphi\|_\infty = 1 \implies a_n(C_\varphi) \geq \delta e^{-n\varepsilon_n} \quad \text{where } \varepsilon_n \to 0_+.
\]

In other terms, as soon as \( \|\varphi\|_\infty = 1 \), we cannot hope better for the numbers \( a_n(C_\varphi) \) than a subexponential decay, however slowly \( \varepsilon_n \) tends to 0.

In [41], we pursued that line of investigation in dimension \( N \geq 2 \), namely on \( H^2(D^N) \), and showed that in some cases the implication (1.2) still holds ([41, Theorem 3.1]):

\[
\|\varphi\|_\infty = 1 \implies a_n(C_\varphi) \geq \delta e^{-n^{1/N}\varepsilon_n} \quad \text{where } \varepsilon_n \to 0_+.
\]
(the substitution of \( n \) by \( n^{1/N} \) is mandatory as shown by the results of \cite{30}).

We show in this paper that, in general, for non-degenerate symbols, we have similar formulas to \cite{30} at our disposal for the parameters:

\[
\beta_N^-(C_\varphi) = \lim \inf_{n \to \infty} \{ a_n (C_\varphi) \}^{1/n} \quad \text{and} \quad \beta_N^+(C_\varphi) = \lim \sup_{n \to \infty} \{ a_n (C_\varphi) \}^{1/n}.
\]

These bounds are given in terms of the Monge-Ampère (or Bedford-Taylor) capacity of \( \varphi(\mathbb{D}^N) \) in \( \mathbb{D}^N \), a notion which is the natural multidimensional extension of the Green capacity when the dimension \( N \) is \( \geq 2 \) (\cite[Theorem 6.4]{30}). We show that we have \( \beta_N^-(C_\varphi) = \beta_N^+(C_\varphi) \) for well-behaved symbols.

\section{Notations and background}

\subsection{Complex analysis}

Let \( \Omega \) be a domain in \( \mathbb{C}^N \); a function \( u : \Omega \to \mathbb{R} \cup \{-\infty \} \) is said plurisubharmonic (psh) if it is u.s.c. and if for every complex line \( L = \{ a + zw : z \in \mathbb{C} \} \) \((a \in \Omega, w \in \mathbb{C}^N)\), the function \( z \mapsto u(a + zw) \) is subharmonic in \( \Omega \cap L \). We denote \( \mathcal{PSH}(\Omega) \) the set of plurisubharmonic functions in \( \Omega \). If \( f : \Omega \to \mathbb{C} \) is holomorphic, then \( \log |f| \) and \( |f|^\alpha, \alpha > 0 \), are psh. Every real-valued convex function is psh (convex functions are those whose composition with all \( \mathbb{R} \)-linear isomorphisms are subharmonic, though plurisubharmonic functions are those whose composition with all \( \mathbb{C} \)-linear isomorphisms are subharmonic; see \cite[Theorem 2.9.12]{30}).

Let \( dd^c = 2i\partial\bar{\partial} \), and \((dd^c)^N = dd^c \wedge \cdots \wedge dd^c \) (\( N \) times). When \( u \in \mathcal{PSH}(\Omega) \cap C^2(\Omega) \), we have:

\[
(dd^c u)^N = 4^N N! \det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) d\lambda_{2N}(z),
\]

where \( d\lambda_{2N}(z) = (i/2)^N dz_1 \wedge \bar{dz}_1 \wedge \cdots \wedge dz_N \wedge \bar{dz}_N \) is the usual volume in \( \mathbb{C}^N \). In general, the current \((dd^c u)^N\) can be defined for all locally bounded \( u \in \mathcal{PSH}(\Omega) \) and is actually a positive measure on \( \Omega \) (\cite{30}).

Given \( p_1, \ldots, p_J \in \Omega \), the pluricomplex Green function with poles \( p_1, \ldots, p_J \) and weights \( c_1, \ldots, c_N > 0 \) is defined as:

\[
g(z) = g(z, p_1, \ldots, p_J) = \sup \{ v(z) : v \in \mathcal{PSH}(\Omega), v \leq 0 \text{ and } v(z) \leq c_j \log |z - p_j| + O(1), \forall j = 1, \ldots, J \}.
\]

In particular, for \( J = 1 \) and \( p_1 = a, c_1 = 1 \), \( g(\cdot, a) \) is the pluricomplex Green function of \( \Omega \) with pole \( a \in \Omega \). If \( 0 \in \Omega \) and \( a = 0 \), we denote it by \( g_0 \) and call it the pluricomplex Green function of \( \Omega \); hence:

\[
g_a(z) = g(z, a) = \sup \{ u(z) : u \in \mathcal{PSH}(\Omega), u \leq 0 \text{ and } u(z) \leq \log |z - a| + O(1) \}.
\]
Let $\Omega$ be an open subset of $\mathbb{C}^N$. A continuous function $\rho: \Omega \to \mathbb{R}$ is an exhaustion function if there exists $a \in (-\infty, +\infty)$ such that $\rho(z) < a$ for all $z \in \Omega$, and the set $\Omega_c = \{ z \in \Omega ; \rho(z) < c \}$ is relatively compact in $\Omega$ for every $c < a$.

A domain $\Omega$ in $\mathbb{C}^N$ is said hyperconvex if there exists a continuous psh exhaustion function $\rho: \Omega \to (-\infty, 0)$ (see [30, p. 80]). We may of course replace the upper bound 0 by any other real number. Without this upper bound, $\Omega$ is said pseudocovex.

Let $\Omega$ be a hyperconvex domain, with negative continuous psh exhaustion function $\rho$ and $\mu_{\rho,r}$ the associated Demailly-Monge-Ampère measures, defined as:

$$
\mu_{u,r} = (dd^c u_r)^N = \mathbb{I}_{\Omega \setminus B_{\Omega,u}(r)}(dd^c u)^N,
$$

for $r < 0$, where $u_r = \max(u, r)$ and:

$$
B_{\Omega,u}(r) = \{ z \in \Omega ; u(z) < r \}.
$$

The nonnegative measure $\mu_{u,r}$ is supported by $S_{\Omega,u}(r) := \{ z \in \Omega ; u(z) = r \}$.

If:

$$
\int_{\Omega} (dd^c \rho)^N < \infty,
$$

these measures, considered as measures on $\Omega^\ast$, weak-* converge, as $r$ goes to 0, to a positive measure $\mu = \mu_{\Omega,\rho}$ supported by $\partial \Omega$ and with total mass $\int_{\Omega} (dd^c \rho)^N$ ([16, Théorème 3.1], or [30, Lemma 6.5.10]).

For the pluricomplex Green function $g_a$ with pole $a$, we have $(dd^c g_a)^N = (2\pi)^N \delta_a$ ([16, Théorème 4.3]) and $g_a(a) = -\infty$, so $a \in B_{\Omega,g_a}(r)$ for every $r < 0$ and $\mathbb{I}_{\Omega \setminus B_{\Omega,g_a}(r)}(dd^c g_a)^N = 0$. Hence the Demailly-Monge-Ampère measure $\mu_{g_a,r}$ is equal to $(dd^c (g_a)_r)^N$. By [31, Lemma 1], we have $(1/|r|) (dd^c (g_a)_r)^N = u_{g_a,(r),\Omega}$, the relative extremal function of $B_{\Omega,g_a}(r) = \{ z \in \Omega ; g_a(z) \leq r \}$ in $\Omega$ (see [32] for the definition), and this measure is supported, not only by $S_{\Omega,g_a}(r)$, but merely by the Shilov boundary of $B_{\Omega,g_a}(r)$ (see Section 2.2.1 for the definition).

Since $(dd^c g_a)^N = (2\pi)^N \delta_a$ has mass $(2\pi)^N < \infty$, these measures weak-* converge, as $r$ goes to 0, to a positive measure $\mu = \mu_{\Omega,g_a}$ supported by $\partial \Omega$ with mass $(2\pi)^N$. Demailly ([16, Définition 5.2] call the measure $\frac{1}{(2\pi)^N} \mu_{\Omega,g_a}$ the pluriharmonic measure of $a$. When $\Omega$ is balanced ($az \in \Omega$ for every $z \in \Omega$ and $|a| = 1$), the support of this pluriharmonic measure is the Shilov boundary of $\Omega$ ([51] very end of the paper]).

A bounded symmetric domain of $\mathbb{C}^N$ is a bounded open and convex subset $\Omega$ of $\mathbb{C}^N$ which is circled ($az \in \Omega$ for $z \in \Omega$ and $|a| \leq 1$) and such that for every point $a \in \Omega$, there is an involutive bi-holomorphic map $\gamma: \Omega \to \Omega$ such that $a$ is an isolated fixed point of $\gamma$ (equivalently, $\gamma(a) = a$ and $\gamma'(a) = -id$: see [52, Proposition 3.1.1]). For this definition, see [13, Définition 16 and Théorème 17], or [14, Definition 5 and Theorem 4]. Note that the convexity is automatic.
2.2 Hardy spaces on hyperconvex domains

2.2.1 Hardy spaces on bounded symmetric domains

We begin by defining the Hardy space on a bounded symmetric domain, because this is easier.

The Shilov boundary (also called the Bergman-Shilov boundary or the distinguished boundary) $\partial_S \Omega$ of a bounded domain $\Omega$ is the smallest closed set $F \subseteq \partial \Omega$ such that $\sup_{z \in F} |f(z)| = \sup_{z \in \Omega} |f(z)|$ for every function $f$ holomorphic in $\Omega$ (because every function $f \in A(\Omega)$ can be approximated by $f_z$ with $f_z(z) = f(z_0 + (1-\varepsilon)z)$, where $z_0 \in \Omega$ is given; see [20] pp. 152–154).

The Shilov boundary of the ball $B_N$ is equal to its topological boundary, but the Shilov boundary of the bidisk is $\partial_S \mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| = |z_2| = 1\}$, whereas, its usual boundary $\partial \mathbb{D}^2$ is $T \times \mathbb{D}$; for the unit ball $B_N$, the Shilov boundary is equal to the usual boundary $S^{N-1}$ (see [13] § 4.1). Another example of a bounded symmetric domain, in $\mathbb{C}^3$, is the set $\Omega = \{(z_1, z_2, z_3) \in \mathbb{C}^3; |z_1|^2 + |z_2|^2 < 1, |z_3| < 1\}$ and its Shilov boundary is $\partial_S \Omega = \{(z_1, z_2, z_3) ; |z_1|^2 + |z_2|^2 = 1, |z_3| = 1\}$. For $p \geq q$, the matrix $A$ is in the topological boundary of $M(p,q)$ if and only if $|A| = 1$, but $A$ is in the Shilov boundary if and only if $A^*A = I_q$; therefore the two boundaries coincide if and only if $q = 1$, i.e. $\Omega = B_N$ (see [14] Example 2, p. 30).

Equivalently (see [24] Corollary 9, or [13] Theorem 33, [14] Theorem 10), $\partial_S \Omega$ is the set of the extreme points of the convex set $\Omega$.

The Shilov boundary $\partial_S \Omega$ is invariant by the group $\Gamma$ of automorphisms of $\Omega$ and the subgroup $\Gamma_0 = \{ \gamma \in \Gamma ; \gamma(0) = 0 \}$ act transitively on $\partial_S \Omega$ (see [22]). A theorem of H. Cartan states that the elements of $\Gamma_0$ are linear trans-
formations of $\mathbb{C}^N$ and commute with the rotations (see [24, Theorem 1] or [26, Proposition 2.1.8]). It follows that the Shilov boundary of a bounded symmetric domain $\Omega$ coincides with its topological boundary only for $\Omega = \mathbb{B}_N$ (see [35, p. 572] or [36, p. 367]); in particular the open unit ball of $\mathbb{C}^N$ for the norm $\| \cdot \|_p$, $1 < p < \infty$, is never a bounded symmetric domain, unless $p = 2$.

The unique $\Gamma_0$-invariant probability measure $\sigma$ on $\partial S\Omega$ is the normalized surface area (see [22]). Then the Hardy space $H^2(\Omega)$ is the space of all complex-valued holomorphic functions $f$ on $\Omega$ such that:

$$
\|f\|_{H^2(\Omega)} := \left( \sup_{0 < r < 1} \int_{\partial S\Omega} |f(r\xi)|^2 d\sigma(\xi) \right)^{1/2}
$$

is finite (see [22] and [23]). It is known that the integrals in this formula are non-decreasing as $r$ increases to $1$, so we can replace the supremum by a limit.

The same definition can be given when $\Omega$ is a bounded complete Reinhardt domain (see [1]).

The space $H^2(\Omega)$ is a Hilbert space (see [22, Theorem 5]) and for every $z \in \Omega$, the evaluation map $f \in H^2(\Omega) \mapsto f(z)$ is uniformly bounded on compacts subsets of $\Omega$, by a depending only on that compact set, and of $\Omega$ ([22, Lemma 3]).

For every $f \in H^2(\Omega)$, there exists a boundary values function $f^*$ such that $\|f_r - f^*\|_{L^2(\partial S\Omega)} \to 0$ as $r \to 1$, where $f_r(z) = f(rz)$ ([1, Theorem 3]), and the map $f \in H^2(\Omega) \mapsto f^* \in L^2(\partial S\Omega)$ is an isometric embedding ([22, Theorem 6]).

### 2.2.2 Hardy spaces on hyperconvex domains

For hyperconvex domains, the definition of Hardy spaces is more involved. It was done by E. Poletsky and M. Stessin ([47, Theorem 6]). Those domains are associated to a continuous negative $psh$ exhaustion function $u$ on $\Omega$ and the definition of the Hardy spaces uses the Demailly-Monge-Ampère measures. The space $H^2_u(\Omega)$ is the space of all holomorphic functions $f : \Omega \to \mathbb{C}$ such that:

$$
\sup_{r < 0} \int_{S_{0,u}(\Omega)} |f|^2 d\mu_{u,r} < \infty
$$

and its norm is defined by:

$$
\|f\|_{H^2_u(\Omega)} = \sup_{r < 0} \left( \frac{1}{(2\pi)^N} \int_{S_{0,u}(\Omega)} |f|^2 d\mu_{u,r} \right)^{1/2}.
$$

We can replace the supremum by a limit since the integrals are non-decreasing as $r$ increases to $0$ ([16, Corollaire 1.9]).

The space $H^\infty(\Omega)$ of bounded holomorphic functions in $\Omega$ is contained in $H^2_u(\Omega)$ (see [17], remark before Lemma 3.4).

These spaces $H^2_u(\Omega)$ are Hilbert spaces ([17, Theorem 4.1]), but depends on the exhaustion function $u$ (even when $N = 1$: see for instance [19]). Nevertheless, they all coincide, with equivalent norms, for the functions $u$ for which the measure $(dd^c u)^N$ is compactly supported ([17, Lemma 3.4]); this is the case
when $u(z) = g(z, a)$ is the pluricomplex Green function with pole $a \in \Omega$ (because then $(dd^cu)^N = (2\pi)^N \delta_a$; see [10, Théorème 4.3], or [30, Theorem 6.3.6]).

When $\Omega$ is the ball $B_N$ and $u(z) = \log \|z\|_2$, then $(dd^cu)^N = C \delta_0$ and $\mu_{u_r} = (2\pi)^N d\sigma_t$, where $d\sigma_t$ is the normalized surface area on the sphere of radius $t := e^z$ (see [47, Section 4] or [17, Example 3.3]). When $\Omega$ is the polydisk $D^N$ and $u(z) = \log \|z\|_\infty$, then $(dd^cu)^N = (2\pi)^N \delta_0$ ([18, Corollary 5.4]) and $\frac{1}{(2\pi)^N} \mu_{u_r}$ is the Lebesgue measure of the torus $rT^N$ (see [17, Example 3.10]).

Note that in [17] and [18], the operator $d^c$ is defined as $\frac{1}{2\pi}(\overline{\partial} - \partial)$ instead of $i(\overline{\partial} - \partial)$, as usually used.

In these two cases, the Hardy spaces are the same as the usual ones (see [2, Remark 5.2.1]).

In the sequel, we only consider the exhausting function $u = g_{\Omega}$; hence we will write $B_{\Omega}(r)$, $S_{\Omega}(r)$ and $H^2(\Omega)$ instead of $B_{\Omega,u}(r)$, $S_{\Omega,u}(r)$ and $H^2_{g_{\Omega}}(\Omega)$.

The two notions of Hardy spaces for a bounded symmetric domain are the same:

**Proposition 2.1.** Let $\Omega$ be a bounded symmetric domain in $\mathbb{C}^N$. Then the Hardy space $H^2(\Omega)$ coincides with the subspace of the Poletski-Stessin Hardy space $H_{g_{\Omega}}(\Omega)$, with equality of the norms.

**Proof.** First let us note that if $\| \cdot \|$ is the norm whose open unit ball is $\Omega$, then $g_{\Omega}(z) = \log \|z\|$ (see [7, Proposition 3.3.2]).

Let $\mu_{\Omega}$ be the measure which is the $s$-weak limit of the Demayll-Monge-Ampère measures $\mu_r = (dd^c g_{\Omega})_r^N$. We saw that it is supported by $\partial S_{\Omega}$. By the remark made in [10, pp. 536-537], since the automorphisms of $\Omega$ continuously extend on $\partial \Omega$, the measure $\mu_{\Omega}$ is $\Gamma$-invariant. By unicity, the harmonic measure $\overline{\mu}_{\Omega} = (2\pi)^{-N} \mu_{\Omega}$ at $0$ hence coincides with the normalized area measure on $\partial S_{\Omega}$.

We have, for $r: \Omega \to \mathbb{C}$ holomorphic and $0 < s < 1$:

$$\int_{\partial S_{\Omega}} |f(sz)|^2 d\overline{\mu}_{\Omega}(z) = \int_{\partial S_{\Omega}} |f(sz)|^2 d\overline{\mu}_{\Omega}(z) = \lim_{r \to 0} \frac{1}{(2\pi)^N} \int_{S_{\Omega}(r)} |f(sz)|^2 d\mu_r(z),$$

because $z \mapsto |f(sz)|^2$ is continuous on $\overline{\Omega}$. Now, since $g_{\Omega}(z) = \log \|z\|$, we have $S_{r,s}(r) = e^{r-s} \partial \Omega$ and $(g_{\Omega})_r(z) + t = (g_{\Omega})_{r+t}(sz)$; hence $\mu_r(sA) = \mu_{r+t}(A)$ for every Borel subset $A$ of $\partial \Omega$, where $t = \log s$. It follows that:

$$\int_{S_{\Omega}(r)} |f(sz)|^2 d\mu_r(z) = \int_{S_{\Omega}(r+t)} |f(z)|^2 d\mu_{r+t}(z).$$

By letting $r$ and $t$ going to $0$, we get:

$$\|f\|^2_{H^2(\Omega)} = \lim_{r,t \to 0} \frac{1}{(2\pi)^N} \int_{S_{\Omega}(r+t)} |f(z)|^2 d\mu_{r+t}(z) = \|f\|^2_{H^2_{g_{\Omega}}},$$

hence $f \in H^2(\Omega)$ if and only if $f \in H^2_{g_{\Omega}}(\Omega)$, with the same norms.

We have ([17, Theorem 3.6]):
Proposition 2.2 (Poletsky-Stessin). For every \( z \in \Omega \), the evaluation map \( f \in H^2(\Omega) \to f(z) \) is uniformly bounded on compacts subsets of \( \Omega \), by a constant depending only on that compact set, and of \( \Omega \).

Hence \( H^2(\Omega) \) has a reproducing kernel, defined by:

\[
f(a) = \langle f, K_a \rangle, \quad \text{for } f \in H^2(\Omega),
\]

and for each \( r < 0 \):

\[
L_r := \sup_{a \in B_0(r)} \| K_a \|_2 < \infty.
\]

2.3 Composition operators

A Schur map, associated with the bounded hyperconvex domain \( \Omega \), is a non-constant analytic map of \( \Omega \) into itself. It is said non degenerate if its Jacobian is not identically null. It is equivalent to saying that \( \phi' : \mathbb{C}^N \to \mathbb{C}^N \) is an invertible linear map for at least one point \( a \in \Omega \). In [4], we used the terminology truly \( N \)-dimensional. Then, by the implicit function theorem, this is equivalent to saying that \( \phi(\Omega) \) has non-void interior. We say that the Schur map \( \phi \) is a symbol if it defines a bounded composition operator \( C_\phi : H^2(\Omega) \to H^2(\Omega) \) by \( C_\phi(f) = f \circ \phi \).

Let us recall that although any Schur function generates a bounded composition operator on \( H^2(\mathbb{D}) \), this is no longer the case on \( H^2(\mathbb{B}_N) \) as soon as \( N \geq 2 \), as shown for example by the Schur map \( \psi(\mathbf{z}_1, \mathbf{z}_2) = (\mathbf{z}_1, \mathbf{z}_1) \). Indeed (see [3]), if say \( N = 2 \), taking \( f(\mathbf{z}) = \sum_{j=0}^n z_1^j z_2^{n-j} \), we see that:

\[
\|f\|_2 = \sqrt{n+1} \quad \text{while} \quad \|C_\phi f\|_2 = \|(n+1)z_1^n\|_2 = n + 1.
\]

The same phenomenon occurs on \( H^2(\mathbb{B}_N) \) (see also [11], [12], and [15]; see also [47]).

2.4 \( s \)-numbers of operators on a Hilbert space

We begin by recalling a few operator-theoretic facts. Let \( H \) be a Hilbert space. The approximation numbers \( a_n(T) = a_n \) of an operator \( T : H \to H \) are defined as:

\[
a_n = \inf_{\rank R < n} \| T - R \|, \quad n = 1, 2, \ldots
\]

The operator \( T \) is compact if and only if \( \lim_{n \to \infty} a_n(T) = 0 \).

According to a result of Allahverdiev [10, p. 155], \( a_n = s_n \), the \( n \)-th singular number of \( T \), i.e. the \( n \)-th eigenvalue of \( |T| := \sqrt{T^*T} \) when those eigenvalues are rearranged in non-increasing order.

The \( n \)-th width \( d_n(K) \) of a subset \( K \) of a Banach space \( Y \) measures the defect of flatness of \( K \) and is by definition:

\[
d_n(K) = \inf_{\dim E < n} \left[ \sup_{f \in E} \sup_{E \subset K} \dist(f, E) \right],
\]
where \( E \) runs over all subspaces of \( Y \) with dimension < \( n \) and where \( \text{dist} (f, E) \) denotes the distance of \( f \) to \( E \). If \( T: X \to Y \) is an operator between Banach spaces, the \( n \)-th Kolmogorov number \( d_n(T) \) of \( T \) is the \( n \)-th-width in \( Y \) of \( T(B_X) \) where \( B_X \) is the closed unit ball of \( X \), namely:

\[
d_n(T) = \inf_{\dim E < n} \left( \sup_{f \in B_X} \text{dist} (Tf, E) \right).
\]

In the case where \( X = Y = H \), a Hilbert space, we have:

\[
a_n(T) = d_n(T) \quad \text{for all } n \geq 1,
\]

and (2.7) the following alternative definition of \( a_n(T) \):

\[
a_n(T) = \inf_{\dim E < n} \left( \sup_{f \in B_H} \text{dist} (Tf, TE) \right).
\]

In this work, we use, for an operator \( T: H \to H \), the following notation:

\[
\beta^N(T) = \liminf_{n \to \infty} [a_{nN}(T)]^{1/n}
\]

and:

\[
\beta^N(T) = \limsup_{n \to \infty} [a_{nN}(T)]^{1/n}.
\]

When these two quantities are equal, we write them \( \beta_N(T) \).

3 Pluripotential theory

3.1 Monge-Ampère capacity

Let \( K \) be a compact subset of an open subset \( \Omega \) of \( \mathbb{C}^N \). The Monge-Ampère capacity of \( K \) has been defined by Bedford and Taylor ([5]; see also [30, Part II, Chapter 1]) as:

\[
\text{Cap} (K) = \sup \left\{ \int_K (dd^c u)^N ; u \in \mathcal{PSH}(\Omega) \text{ and } 0 \leq u \leq 1 \text{ on } \Omega \right\}.
\]

When \( \Omega \) is bounded and hyperconvex, we have a more convenient formula ([5, Proposition 5.3], [30, Proposition 4.6.1]):

\[
\text{Cap} (K) = \int_{\Omega} (dd^c u_K^*)^N = \int_K (dd^c u_K^*)^N,
\]

(the positive measure \( (dd^c u_K^*)^N \) is supported by \( K \); actually by \( \partial K \); see [17, Properties 8.1 (c)]), where \( u_K = u_{K,\Omega} \) is the relative extremal function of \( K \), defined, for any subset \( E \subseteq \Omega \), as:

\[
u_{E,\Omega} = \sup \{ v \in \mathcal{PSH}(\Omega) ; v \leq 0 \text{ and } v \leq -1 \text{ on } E \},
\]

where \( H \) is a Hilbert space, we have:

\[
a_n(T) = d_n(T) \quad \text{for all } n \geq 1,
\]

and (2.7) the following alternative definition of \( a_n(T) \):

\[
a_n(T) = \inf_{\dim E < n} \left( \sup_{f \in B_H} \text{dist} (Tf, T(E)) \right).
\]

In this work, we use, for an operator \( T: H \to H \), the following notation:

\[
\beta^N(T) = \liminf_{n \to \infty} [a_{nN}(T)]^{1/n}
\]

and:

\[
\beta^N(T) = \limsup_{n \to \infty} [a_{nN}(T)]^{1/n}.
\]

When these two quantities are equal, we write them \( \beta_N(T) \).
and $u^*_E,\Omega$ is its upper semi-continuous regularization:

$$u^*_E,\Omega(z) = \limsup_{\zeta \to z} u_{E,\Omega}(\zeta), \quad z \in \Omega,$$

called the \textit{regularized relative extremal function} of $E$.

For an open subset $\omega$ of $\Omega$, its capacity is defined as:

$$\text{Cap} (\omega) = \sup \{ \text{Cap} (K) ; \ K \text{ is a compact subset of } \omega \}.$$ 

When $\bar{\omega} \subset \Omega$ is a compact subset of $\Omega$, we have ([5] equation (6.2), [30] Corollary 4.6.2):

(3.3) \hspace{2cm} \text{Cap} (\omega) = \int_{\Omega} (dd^c u_{\omega})^N.

The \textit{outer capacity} of a subset $E \subseteq \Omega$ is:

$$\text{Cap}^*(E) = \inf \{ \text{Cap} (\omega) ; \ \omega \supseteq E \text{ and } \omega \text{ open} \}.$$ 

If $\Omega$ is hyperconvex and $E$ relatively compact in $\Omega$, then ([30] Proposition 4.7.2):

$$\text{Cap}^*(E) = \int_{\Omega} (dd^c u^*_E,\Omega)^N.$$

\textbf{Remark.} A. Zeriahi ([57]) pointed out to us the following result.

\textbf{Proposition 3.1.} \hspace{0.5cm} \textit{Let $K$ be a compact subset of $\Omega$. Then:}

$$\text{Cap} (K) = \text{Cap} (\partial K).$$

\textit{Proof.} Of course $u_K \leq u_{\partial K}$ since $\partial K \subseteq K$. Conversely, let $v \in \mathcal{PSH}(\Omega)$ non-positive such that $v \leq -1$ on $\partial K$. By the maximum principle (see [30] Corollary 2.9.6), we get that $v \leq -1$ on $K$. Hence $v \leq u_K$. Taking the supremum over all those $v$, we obtain $u_{\partial K} \leq u_K$, and therefore $u_{\partial K} = u_K$.

By ([57]), it follows that:

(3.4) \hspace{2cm} \text{Cap} (K) = \int_{\Omega} (dd^c u^*_K)^N = \int_{\Omega} (dd^c u^*_\partial K)^N = \text{Cap} (\partial K). \hspace{1cm} \square

\subsection*{3.2 Regular sets}

Let $E \subseteq \mathbb{C}^N$ be bounded. Recall that the polynomial convex hull of $E$ is:

$$\hat{E} = \{ z \in \mathbb{C} ; \ |P(z)| \leq \sup_E |P| \text{ for every polynomial } P \}.$$ 

A point $a \in \hat{E}$ is called \textit{regular} if $u^*_E,\Omega(a) = -1$ for an open set $\Omega \supseteq \hat{E}$ (note that we always have $u_{E,\Omega} = u_{E,\Omega} = -1$ on the interior of $E$; see [17] Properties 8.1 (c))). The set $E$ is said to be \textit{regular} if all points of $\hat{E}$ are regular.
The pluricomplex Green function of $E$, also called the $L$-extremal function of $E$, is defined, for $z \in \mathbb{C}^N$, as:

$$V_E(z) = \sup\{v(z) : v \in \mathcal{L}, \quad v \leq 0 \text{ on } E\},$$

where $\mathcal{L}$ is the Lelong class of all functions $v \in \mathcal{PSH}(\mathbb{C}^N)$ such that, for some constant $C > 0$:

$$v(z) \leq C + \log(1 + |z|) \quad \text{for all } z \in \mathbb{C}^N.$$

A point $a \in \hat{E}$ is called $L$-regular if $V_E^*(a) = 0$, where $V_E^*$ is the upper semi-continuous regularization of $V_E$. The set $E$ is $L$-regular if all points of $\hat{E}$ are $L$-regular.

By [28, Proposition 2.2] (see also [30, Proposition 5.3.3, and Corollary 5.3.4]), for $E$ bounded and non pluripolar, and $\Omega$ a bounded open neighbourhood of $\hat{E}$, we have:

$$(3.5) \quad m(u_{E,\Omega} + 1) \leq V_E \leq M(u_{E,\Omega} + 1)$$

for some positive constants $m, M$. Hence the regularity of $a \in \hat{E}$ is equivalent to its $L$-regularity.

Recall that $E$ is pluripolar if there exists an open set $\Omega$ containing $E$ and $v \in \mathcal{PSH}(\Omega)$ such that $E \subseteq \{v = -\infty\}$. This is equivalent to say that there exists a hyperconvex domain $\Omega$ of $\mathbb{C}^N$ containing $E$ such that $u_{E,\Omega} \equiv 0$ (see [30, Corollary 4.7.3 and Theorem 4.7.5]). By Josefson’s theorem ([30, Theorem 4.7.4]), $E$ is pluripolar if and only if there exists $v \in \mathcal{PSH}(\mathbb{C}^N)$ such that $E \subseteq \{v = -\infty\}$. Recall also that $E$ is pluripolar if and only if its outer capacity $\text{Cap}^*(E)$ is null ([30, Theorem 4.7.5]).

When $\Omega$ is hyperconvex and $E$ is compact, non pluripolar, the regularity of $E$ implies that $u_{E,\Omega}$ and $V_E$ are continuous, on $\Omega$ and $\mathbb{C}^N$ respectively ([30, Proposition 4.5.3 and Corollary 5.1.4]). Conversely, if $u_{E,\Omega}$ is continuous, for some hyperconvex neighbourhood $\Omega$ of $E$, then $u_{E,\Omega}(z) = -1$ for all $z \in \hat{E}$; hence $V_E(z) = 0$ for all $z \in E$, by (3.5); but $V_E = V_{\hat{E}}$ when $E$ is compact ([30, Theorem 5.1.7]), so $V_E(z) = 0$ for all $z \in \hat{E}$; by (3.5) again, we obtain that $u_{E,\Omega}(z) = -1$ for all $z \in \hat{E}$; therefore $E$ is regular. In the same way, the continuity of $V_E$ implies the regularity of $E$. These results are due to Siciak ([30, Proposition 6.1 and Proposition 6.2]).

Every closed ball $B = B(a, r)$ of an arbitrary norm $\| \cdot \|$ on $\mathbb{C}^N$ is regular since its $L$-extremal function is:

$$V_B(z) = \log^+ (\|z - a\|/r)$$

([50, p. 179, § 2.6]).

### 3.3 Zakharyuta’s formula

We will need a formula that Zakharyuta, in order to solve a problem raised by Kolmogorov, proved, conditionally to a conjecture, called Zakharyuta’s conjecture, on the uniform approximation of the relative extremal function $u_{K,\Omega}$.
by pluricomplex Green functions. This conjecture has been proved by Nivoche (\[15\], Theorem A), in a more general setting that we state below:

**Theorem 3.2** (Nivoche). Let $K$ be a regular compact subset of a bounded hyperconvex domain $\Omega$ of $\mathbb{C}^N$. Then for every $\varepsilon > 0$ and $\delta$ small enough, there exists a pluricomplex Green function $g$ on $\Omega$ with a finite number of logarithmic poles such that:

1) the poles of $g$ lie in $W = \{z \in \Omega; \ u_K(z) < -1 + \delta\};$
2) we have, for every $z \in \Omega \setminus W$:

$$(1 + \varepsilon)g(z) \leq u_K(z) \leq (1 - \varepsilon)g(z).$$

In order to state Zakharyuta’s formula, we need some additional notations.

Let $K$ be a compact subset of $\Omega$ with non-empty interior, and $A_K$ the set of restrictions to $K$ of those functions that are analytic and bounded by 1, i.e. those functions belonging to the unit ball $B_{H^\infty(\Omega)}$ of the space $H^\infty(\Omega)$ of the bounded analytic functions in $\Omega$, considered as a subset of the space $\mathcal{C}(K)$ of complex functions defined on $K$, equipped with the sup-norm on $K$.

Let $d_n(A_K)$ be the $n$th-width of $A_K$ in $\mathcal{C}(K)$, namely:

(3.6) $$d_n(A_K) = \inf_L \left[ \sup_{f \in A_K} \text{dist}(f, L) \right],$$

where $L$ runs over all $k$-dimensional subspaces of $\mathcal{C}(K)$, with $k < n$.

Equivalently, $d_n(A_K)$ is the $n$th-Kolmogorov number of the natural injection $J$ of $H^\infty(\Omega)$ into $\mathcal{C}(K)$ (recall that $K$ has non-empty interior). It is convenient to set, as in \[55\]:

(3.7) $$\tau_N(K) = \frac{1}{(2\pi)^N} \text{Cap}(K)$$

and:

(3.8) $$\Gamma_N(K) = \exp \left[ - \left( \frac{N!}{\tau_N(K)} \right)^{1/N} \right],$$

i.e.:

(3.9) $$\Gamma_N(K) = \exp \left[ - 2\pi \left( \frac{N!}{\text{Cap}(K)} \right)^{1/N} \right].$$

Observe that $\text{Cap}(K) > 0$ since we assumed that $K$ has non-empty interior. Now, we have (\[55\], Theorem 5.6); see also \[59\], Theorem 5 or \[54\], pages 30–32, for a detailed proof):

**Theorem 3.3** (Zakharyuta-Nivoche). Let $\Omega$ be a bounded hyperconvex domain and $K$ a regular compact subset of $\Omega$ with non-empty interior, which is holomorphically convex in $\Omega$ (i.e. $K = \overline{K}_\Omega$). Then:

(3.10) $$- \log d_n(A_K) \sim \left( \frac{N!}{\tau_N(K)} \right)^{1/N} n^{1/N}.$$
Here $\tilde{K}_\Omega$ is the holomorphic convex hull of $K$ in $\Omega$, that is:

$$\tilde{K}_\Omega = \{z \in \Omega; |f(z)| \leq \sup_K |f| \text{ for every } f \in \mathcal{O}(\Omega)\},$$

where $\mathcal{O}(\Omega)$ is the set of all functions holomorphic in $\Omega$.

Relying on that theorem, which may be seen as the extension of a result of Erokhin, proved in 1958 (see [19]; see also Widom [53] which proved a more general result, with a different proof), to dimension $N > 1$, and as a result on the approximation of functions, we will give an application to the study of approximation numbers of a composition operator on $H^2(\Omega)$ for a bounded symmetric domain of $\mathbb{C}^N$.

4 The spectral radius type formula

In [41, Section 6.2], we proved the following result.

**Theorem 4.1.** Let $\varphi: \mathbb{D}^N \rightarrow \mathbb{D}^N$ be given by $\varphi(z_1, \ldots, z_N) = (r_1 z_1, \ldots, r_N z_N)$ where $0 < r_j < 1$. Then:

$$\beta_N(C_\varphi) = \Gamma_N[\varphi(\mathbb{D}^N)] = \Gamma_N[\varphi(\mathbb{D}^N)].$$

The proof was simple, based on result of Blocki [8] on the Monge-Ampère capacity of a cartesian product, and on the estimation, when $A \rightarrow \infty$, of the number $\nu_A$ of $N$-tuples $\alpha = (\alpha_1, \ldots, \alpha_N)$ of non-negative integers $\alpha_j$ such that $\sum_{j=1}^N \alpha_j \sigma_j \leq A$, where the numbers $\sigma_j > 0$ are fixed. The estimation was:

$$\nu_A \sim \frac{A^N}{N! \sigma_1 \cdots \sigma_N}.$$  \hspace{1cm} (4.1)

As J. F. Burnol pointed out to us, this is a consequence of the following elementary fact. Let $\lambda_N$ be the Lebesgue measure on $\mathbb{R}^N$, and let $E$ be a compact subset of $\mathbb{R}^N$ such that $\lambda_N(\partial E) = 0$. Then:

$$\lambda_N(E) = \lim_{A \rightarrow \infty} A^{-N}|(A \times E) \cap \mathbb{Z}^N|.$$  

Then, just take $E = \{(x_1, \ldots, x_N); x_j \geq 0 \text{ and } \sum_{j=1}^N x_j \sigma_j \leq 1\}$.

In any case, this lets us suspect that the formula of Theorem 4.1 holds in much more general cases. This is not quite true, as evidenced by our counterexample of [41, Theorem 5.12]. Nevertheless, in good cases, this formula holds, as we will see in the next sections.

In remaining of this section, we consider functions $\varphi: \Omega \rightarrow \Omega$ such that $\varphi(\Omega) \subseteq \Omega$. If $\rho$ is an exhaustion function for $\Omega$, there is some $R_0 < 0$ such that $\varphi(\Omega) \subseteq B_\Omega(R_0)$, and that implies that $C_\varphi$ maps $H^2(\Omega)$ into itself and is a compact operator (see [14], Theorem 8.3), since, with their notations, for $r > R_0$, we have $T(r) = \emptyset$ and hence $\delta_\varphi(r) = 0$.
4.1 Minoration

Recall that every hyperconvex domain $\Omega$ is pseudoconvex. By H. Cartan-Thullen and Oka-Bremermann-Norguet theorems, being pseudoconvex is equivalent to being a domain of holomorphy, and equivalent to being holomorphically convex (meaning that if $K$ is a compact subset in $\Omega$, then its holomorphic hull $\tilde{K}$ is also contained in $\Omega$): see [33 Corollaire 7.7]. Now (see [32 Chapter 5, Exercise 11]), a domain of holomorphy $\Omega$ is said a Runge domain if every holomorphic function in $\Omega$ can be approximated uniformly on its compact subsets by polynomials, and that is equivalent to saying that the polynomial hull and the holomorphic hull of every compact subset of $\Omega$ agree. By [32 Chapter 5, Exercise 13], every circled domain (in particular every bounded symmetric domain) is a Runge domain.

Definition 4.2. A hyperconvex domain $\Omega$ is said strongly regular if there exists a continuous psh exhaustion function $\rho$ such that all the sub-level sets:

$\Omega_c = \{ z \in \Omega ; \rho(z) < c \}$

($c < 0$) have a regular closure.

For example, every bounded symmetric domain $\Omega$ is strongly regular since if $\| \cdot \|$ is the associated norm, its sub-level sets $\Omega_c$ (with $\rho(z) = \log \| z \|$) are the open balls $B(0, e^c)$, and the closed balls are regular, as said above.

Theorem 4.3. Let $\Omega$ be a strongly regular bounded hyperconvex and Runge domain in $\mathbb{C}^N$, and let $\varphi : \Omega \to \Omega$ be an analytic function such that $\varphi(\Omega) \subseteq \Omega$, and which is non-degenerate. Then:

\[
\Gamma_N[\varphi(\Omega)] \leq \beta_N(C_\varphi). 
\]

(4.2)

Recall that if $\Omega$ is a domain in $\mathbb{C}^N$, a holomorphic function $\varphi : \Omega \to \mathbb{C}^M$ ($M \leq N$) is non-degenerate if there exists $a \in \Omega$ such that $\text{rank}_a \varphi = M$. Then $\varphi(\Omega)$ has a non-empty interior.

Proof. Let $(r_j)_{j \geq 1}$ be an increasing sequence of negative numbers tending to 0. The set $H_j = \Omega | r_j$ is a regular compact subset of $\Omega$, with non-void interior (hence non pluripolar). Let $\tilde{H}_j$ its polynomial convex hull; this compact set is contained in $\Omega$, since $\Omega$ being a Runge domain, we have $\tilde{H}_j = \tilde{H}_j$, and since $\tilde{H}_j \subseteq \Omega$, because $\Omega$ is holomorphically convex (being hyperconvex). Moreover $\tilde{H}_j$ is regular since $V_E = V_\tilde{E}$ for every compact subset of $\mathbb{C}^N$ (5.1 Corollary 4.14)).

Let $K_j = \varphi(\tilde{H}_j)$ and let $G$ be a subspace of $H^2(\Omega)$ with dimension $< n^N$.

The set $K_j$ is regular because of the following result (see [30 Theorem 5.3.9], [47] top of page 40], [29 Theorem 1.3], or [41] Theorem 4], with a detailed proof).

Theorem 4.4 (Pleśniak). Let $E$ be a compact, polynomially convex, regular and non pluripolar, subset of $\mathbb{C}^N$. Then if $\Omega$ is a hyperconvex domain such that $E \subseteq \Omega$ and if $\varphi : \Omega \to \mathbb{C}^N$ is a non-degenerate holomorphic function, the set $\varphi(E)$ is regular.
As before, the polynomial convex hull $\hat{K}_j$ of $K_j$ is contained in $\Omega$ and is also regular. Since $\varphi$ is non-degenerate, $K_j$ has a non-void interior; hence $\hat{K}_j$ also.

We can hence use Zakharyuta’s formula (Theorem 2.3) for the compact set $\hat{K}_j$. By Zakharyuta’s formula, for $0 < \varepsilon < 1$, there is $n_\varepsilon \geq 1$ such that, for $n \geq n_\varepsilon$:

$$d_n(A_{\hat{K}_j}) \geq \exp \left[ - (1 + \varepsilon) (2\pi) n \left( \frac{N!}{\text{Cap}(\hat{K}_j)} \right)^{1/N} \right].$$

Hence, there exists $f \in B_{H^\infty} \subseteq B_{H^2}$ such that, for all $g \in G$:

$$\|g - f\|_{C(\hat{K}_j)} \geq (1 - \varepsilon) \exp \left[ - (1 + \varepsilon) (2\pi) n \left( \frac{N!}{\text{Cap}(\hat{K}_j)} \right)^{1/N} \right].$$

Since $\hat{K}_j = \hat{K}_j$ and, by definition $\| \cdot \|_{C(\hat{K}_j)} = \| \cdot \|_{C(K_j)}$, we have:

$$\|g - f\|_{C(\hat{K}_j)} = \|g - f\|_{C(K_j)} = \|C\varphi(g) - C\varphi(f)\|_{C(\hat{K}_j)}.$$

Equivalently, since, by definition $\| \cdot \|_{C(\hat{K}_j)} = \| \cdot \|_{C(H_j)}$, we have, for all $g \in G$:

$$\|C\varphi(g) - C\varphi(f)\|_{C(H_j)} \geq (1 - \varepsilon) \exp \left[ - (1 + \varepsilon) (2\pi) n \left( \frac{N!}{\text{Cap}(\hat{K}_j)} \right)^{1/N} \right].$$

This implies, thanks to (2.8), that, for all $g \in G$:

$$\|C\varphi(g) - C\varphi(f)\|_{H^2(\Omega)} \geq L_{\tau_j}^{-1}(1 - \varepsilon) \exp \left[ - (1 + \varepsilon) (2\pi) n \left( \frac{N!}{\text{Cap}(\hat{K}_j)} \right)^{1/N} \right].$$

Using (2.8), we get, since the subspace $G$ is arbitrary:

$$a_{nN}(C\varphi) \geq L_{\tau_j}^{-1}(1 - \varepsilon) \exp \left[ - (1 + \varepsilon) (2\pi) n \left( \frac{N!}{\text{Cap}(\hat{K}_j)} \right)^{1/N} \right].$$

Taking the $n$th-roots and passing to the limit, we obtain:

$$\beta_{\infty}(C\varphi) \geq \exp \left[ - (1 + \varepsilon) (2\pi) \left( \frac{N!}{\text{Cap}(\hat{K}_j)} \right)^{1/N} \right] \cdot$$

and then, letting $\varepsilon$ go to 0:

$$\beta_{\infty}(C\varphi) \geq \exp \left[ - (2\pi) \left( \frac{N!}{\text{Cap}(\hat{K}_j)} \right)^{1/N} \right] = \Gamma_{\infty}(\hat{K}_j).$$

Now, the sequence $(\hat{K}_j)_{j \geq 1}$ is increasing and $\bigcup_{j \geq 1} \hat{K}_j = \varphi(\Omega)$; hence, by Theorem 8.2 (8.3), we have $\text{Cap}(\hat{K}_j) \to \text{Cap}(\bigcup_{j \geq 1} \hat{K}_j) \geq \text{Cap}(\varphi(\Omega))$, so:

$$\beta_{\infty}(C\varphi) \geq \Gamma_{\infty}[\varphi(\Omega)],$$

and the proof of Theorem 4.3 is finished.
4.2 Majorization

For the majorization, we assume different hypotheses on the domain $\Omega$. Nevertheless these assumptions agree with that of Theorem 4.3 when $\Omega$ is a bounded symmetric domain.

4.2.1 Preliminaries

Recall that a domain $\Omega$ of $\mathbb{C}^N$ is a Reinhardt domain (resp. complete Reinhardt domain) if $z = (z_1, \ldots, z_N) \in \Omega$ implies that $(\zeta_1 z_1, \ldots, \zeta_N z_N) \in \Omega$ for all complex numbers $\zeta_1, \ldots, \zeta_N$ of modulus 1 (resp. of modulus $\leq 1$). A complete bounded Reinhardt domain is hyperconvex if and only if $\log j_\Omega$ is psh and continuous in $\mathbb{C}^N \setminus \{0\}$, where $j_\Omega$ is the Minkowski functional of $\Omega$ (see Exercise following Proposition 3.3.3). In general, the Minkowski functional $j_\Omega$ of a bounded complete Reinhardt domain $\Omega$ is usc and log $j_\Omega$ is psh if and only if $\Omega$ is pseudoconvex ([7, Theorem 1.4.8]). Other conditions for a bounded complete Reinhardt domain to being hyperconvex can found in [34, Theorem 3.10].

For a bounded hyperconvex and complete Reinhardt domain $\Omega$, its pluricomplex Green function with pole 0 is $\eta_\Omega(z) = \log j_\Omega(z)$, where $j_\Omega$ is the Minkowski functional of $\Omega$ ([7, Proposition 3.3.2]), and $S_\Omega(r) = e^{r} \partial \Omega$. Since $\partial \Omega$ is in particular invariant by the pluri-rotations $z = (z_1, \ldots, z_N) \mapsto (e^{i \theta_1 z_1}, \ldots, e^{i \theta_N z_N})$, with $\theta_1, \ldots, \theta_N \in \mathbb{R}$, the harmonic measure $\mu_\Omega$ at 0 (see the proof of Proposition 2.1) is also invariant by the pluri-rotations (note that it is supported by the Shilov boundary of $\Omega$: see [31, very end of the paper]). We have, as in the proof of Proposition 2.1, for $f \in H^2(\Omega)$:

$$\sup_{0 < s < 1} \int_{\partial \Omega} |f(sz)|^2 d\mu_\Omega(z) = \|f\|_{H^2(\Omega)}^2 < \infty.$$ 

Since $\mu_\Omega$ is in particular invariant by the rotations $z \mapsto e^{i \theta} z$, $\theta \in \mathbb{R}$, there exists, by [31, Theorem 3], a function $f^* \in L^2(\partial \Omega, \mu_\Omega)$ such that:

$$\int_{\partial \Omega} |f(sz) - f^*(z)|^2 d\mu_\Omega(z) \to 0 \text{ as } s \to 1.$$ 

It ensues that the map $f \in H^2(\Omega) \to f^* \in L^2(\partial \Omega, \mu_\Omega)$ is an isometric embedding (in fact, $f^*$ is the radial limit of $f$: see [21, Lemma 2]). Therefore, we can consider $H^2(\Omega)$ as a complemented subspace of $L^2(\partial \Omega, \mu_\Omega)$, and we call $P$ the orthogonal projection of $L^2(\partial \Omega, \mu_\Omega)$ onto $H^2(\Omega)$.

Every holomorphic function $f$ in a Reinhardt domain $\Omega$ containing 0 (in particular if $\Omega$ is a complete Reinhardt domain) has a power series expansion about 0:

$$f(z) = \sum_{\alpha} b_\alpha z^\alpha$$

which converges normally on compact subsets of $\Omega$ ([32, Proposition 2.3.14]). Recall that if $z = (z_1, \ldots, z_N)$ and $\alpha = (\alpha_1, \ldots, \alpha_N)$, then $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and $\alpha! = \alpha_1! \cdots \alpha_N!$.

We have:
**Proposition 4.5.** Let $\Omega$ be a bounded hyperconvex and complete Reinhardt domain, and set $e_\alpha(z) = z^\alpha$. Then the system $(e_\alpha)_\alpha$ is orthogonal in $H^2(\Omega)$.

**Proof.** We use the fact that the level sets $S(r)$ and the Demailly-Monge-Ampère measures $\mu_r = (dd^c(y_0))_r^N$ are pluri-rotation invariant. For $\alpha \neq \beta$, we choose $\theta_1, \ldots, \theta_N \in \mathbb{R}$ such that $1, (\theta_1/2\pi), \ldots, (\theta_N/2\pi)$ are rationally independent. Then $\exp [i(\sum_{j=1}^N (\alpha_j - \beta_j)\theta_j)] \neq 1$. Hence, as in [25, p. 78], we have, making the change of variables $z = (e^{i\theta_1}w_1, \ldots, e^{i\theta_N}w_N)$:

$$\int_{S(r)} z^\alpha \overline{z}^\beta d\mu_r(z) = \exp \left[i\left(\sum_{j=1}^N (\alpha_j - \beta_j)\theta_j\right)\right] \int_{S(r)} w^\alpha \overline{w}^\beta d\mu_r(w),$$

which implies that:

$$\int_{S(r)} z^\alpha \overline{z}^\beta d\mu_r(z) = 0,$$

and hence:

$$(z^\alpha | z^\beta) := \lim_{r \to 0} \int_{S(r)} z^\alpha \overline{z}^\beta d\mu_r(z) = 0. \quad \square$$

For the polydisk, we have $\|e_\alpha\|_{H^2(\mathbb{D}^N)} = 1$, and for the ball (see [18, Proposition 1.4.9]):

$$\|e_\alpha\|^2_{H^2(\mathbb{B}_N)} = \frac{(N - 1)! \alpha!}{(N + 1 - |\alpha|)!}.$$

**Definition 4.6.** We say that $\Omega$ is a good complete Reinhardt domain if, for some positive constant $C_N$ and some positive integer $c$, we have, for all $p \geq 0$:

$$\sum_{|\alpha| = p} \frac{|z^\alpha|^2}{\|e_\alpha\|^2_{H^2(\Omega)}} \leq C_N p^c N |j_\Omega(z)|^{2p},$$

where $j_\Omega$ is the Minkowski functional of $\Omega$.

**Examples**

1. The polydisk $\mathbb{D}^N$ is a good Reinhardt domain because $\|e_\alpha\|_{H^2(\mathbb{D}^N)} = 1$, $|z^\alpha| \leq |z|^{|\alpha|}_\infty$, and the number of indices $\alpha$ such that $|\alpha| = p$ is $\binom{N - 1 + p}{p} \leq C_N p^N$ (see [33, p. 498] or [37, pp. 213–214]).

2. The ball $\mathbb{B}_N$ is a good Reinhardt domain. In fact, observe that:

$$\frac{(N - 1 + p)!}{(N - 1)!} = p! \frac{(p + 1)(p + 2) \cdots (p + N - 1)}{1 \times 2 \times \cdots \times (N - 1)} \leq p! (p + 1)^{N - 1} \leq p! (p + 1)^N;$$

hence:

$$\sum_{|\alpha| = p} \frac{|z^\alpha|^2}{\|e_\alpha\|^2_{H^2(\mathbb{B}_N)}} = \sum_{|\alpha| = p} |z^\alpha|^2 \frac{(N - 1 + |\alpha|)!}{(N - 1)! \alpha!} \leq (p + 1)^N \sum_{|\alpha| = p} \frac{|\alpha|!}{\alpha!} |z_1|^{2\alpha_1} \cdots |z_N|^{2\alpha_N}$$

$$= (p + 1)^N (|z_1|^2 + \cdots + |z_N|^2)^p,$$
by the multinomial formula, so:
\[
\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|^2_{H^2(\mathbb{B}_N)}} \leq (p + 1)^N\|z\|^2_2 \leq 2^N p^N \|z\|^2_2.
\]

3. More generally, if \( \Omega = \mathbb{B}_{l_1} \times \cdots \times \mathbb{B}_{l_m}, l_1 + \cdots + l_m = N, \) is a product of balls, we have, writing \( \alpha = (\beta_1, \ldots, \beta_m), \) where each \( \beta_j \) is an \( l_j \)-tuple:
\[
\|e_\alpha\|^2_{H^2(\Omega)} = \int_{\mathbb{S}_{l_1} \times \cdots \times \mathbb{S}_{l_2}} |u_1^{\beta_1}|^2 \cdots |u_m^{\beta_m}|^2 d\sigma_1(u_1) \cdots d\sigma_m(u_m)
\]
\[
= \prod_{j=1}^m \frac{(l_j - 1)! \beta_j!}{(l_j - 1 + |\beta_j|)!},
\]
and, writing \( z = (z_1, \ldots, z_m), \) with \( z_j \in \mathbb{B}_{l_j}; \)
\[
\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|^2_{H^2(\Omega)}} \leq \sum_{p_1+\cdots+p_m=p} \prod_{j=1}^m (p_j + 1)^{l_j} |z_j|^{2p_j}
\]
\[
\leq C_m p^m (p + 1)^{l_1 + \cdots + l_m} [j_\Omega(z)]^{2(p_1 + \cdots + p_m)},
\]
since \( j_\Omega(z) = \max\{\|z_1\|_2, \ldots, \|z_m\|_2\}. \) Hence:
\[
\sum_{|\alpha|=p} \frac{|z^\alpha|^2}{\|e_\alpha\|^2_{H^2(\Omega)}} \leq C N p^{2N} [j_\Omega(z)]^{2p}.
\]

4.2.2 The result

**Theorem 4.7.** Let \( \Omega \) be a bounded hyperconvex domain which is a good complete Reinhardt domain in \( C^N, \) and let \( \varphi: \Omega \to \Omega \) be an analytic function such that \( \varphi(\Omega) \subseteq \Omega. \) Then, for every compact subset \( K \supseteq \varphi(\Omega) \) of \( \Omega \) with non void interior, we have:
\[
\beta_N^+ (C_\varphi) \leq \Gamma_N (K).
\]

In particular, if \( \varphi \) is moreover non-degenerate, we have:
\[
\beta_N^+ (C_\varphi) \leq \Gamma_N \left[ \varphi(\Omega) \right].
\]

The last assertion holds because \( \varphi(\Omega) \) is open if \( \varphi \) is non-degenerate.

**Corollary 4.8.** Let \( \Omega \) be a good complete bounded symmetric domain in \( C^N, \) and \( \varphi: \Omega \to \Omega \) a non-degenerate analytic map such that \( \varphi(\Omega) \subseteq \Omega. \) Then:
\[
\Gamma_N \left[ \varphi(\Omega) \right] \leq \beta_N^+ (C_\varphi) \leq \beta_N^+ (C_\varphi) \leq \Gamma_N \left[ \varphi(\Omega) \right].
\]

For the proof of Theorem 4.7, we will use the following result ([56, Proposition 6.1]), which do not need any regularity condition on the compact set (because it may be written as a decreasing sequence of regular compact sets).
Proposition 4.9 (Zakharyuta). If $K$ is any compact subset of a bounded hyperconvex domain $\Omega$ of $\mathbb{C}^N$ with non-empty interior, we have:

$$\limsup_{n \to \infty} \log \frac{d_n(A_K)}{n^{1/N}} \leq \left( \frac{N!}{\tau_N(K)} \right)^{1/N}.$$ 

Proof of Theorem 4.7. In the sequel we write $\| \cdot \|_{H^2}$ for $\| \cdot \|_{H^2(\Omega)}$. We set:

$$\Lambda_N = \limsup_{n \to \infty} [d_n(A_K)]^{n-1/N}.$$ 

Changing $n$ into $n^N$, Proposition 4.9 means that for every $\varepsilon > 0$, there exists, for $n$ large enough, an $(n^N - 1)$-dimensional subspace $F$ of $C(K)$ such that, for any $g \in H^\infty(\Omega)$, there exists $h \in F$ such that:

$$(4.5) \quad \| g - h \|_{C(K)} \leq (1 + \varepsilon)^n \Lambda_N^n \| g \|_{\infty}.$$ 

Let $l$ be an integer to be adjusted later, and

$$f(z) = \sum_{|\alpha| \leq l} b_{\alpha} z^\alpha \in H^2(\Omega) \quad \text{with} \quad \| f \|_{H^2} \leq 1.$$ 

By Proposition 4.5 we have:

$$\| f \|_{H^2}^2 = \sum_{\alpha} |b_{\alpha}|^2 \| e_\alpha \|_{H^2}^2.$$ 

We set:

$$g(z) = \sum_{|\alpha| \leq l} b_{\alpha} z^\alpha.$$ 

By the Cauchy-Schwarz inequality:

$$|g(z)|^2 \leq \left( \sum_{|\alpha| \leq l} |b_{\alpha}|^2 \| e_\alpha \|_{H^2}^2 \right) \left( \sum_{|\alpha| \leq l} |z^\alpha|^2 \| e_\alpha \|_{H^2}^2 \right) \leq \sum_{|\alpha| \leq l} |z^\alpha|^2 \| e_\alpha \|_{H^2}^2.$$ 

Since $\Omega$ is a good complete Reinhardt domain and since $j_\Omega(z) < 1$ for $z \in \Omega$, we have:

$$|g(z)|^2 \leq \sum_{p=0}^l p^c N^p [j_\Omega(z)]^{2p} \leq (l + 1)^{cN+1}.$$ 

It follows from (4.5) that there exists $h \in F$ such that:

$$\| g - h \|_{C(K)} \leq (1 + \varepsilon)^n \Lambda_N^n (l + 1)^{(cN+1)/2}.$$ 

Since $C_\varphi f(z) - C_\varphi g(z) = f(\varphi(z)) - g(\varphi(z))$ and $\varphi(\Omega) \subseteq K$, we have $\| C_\varphi f - C_\varphi g \|_{\infty} \leq \| f - g \|_{C(K)}$; therefore:

$$(4.6) \quad \| g \circ \varphi - h \circ \varphi \|_{H^2} \leq \| g \circ \varphi - h \circ \varphi \|_{\infty} \leq \| g - h \|_{C(K)} \leq (1 + \varepsilon)^n \Lambda_N^n (l + 1)^{(cN+1)/2}.$$
Now, the subspace $\bar{F}$ formed by functions $v \circ \varphi$, for $v \in F$, can be viewed as a subspace of $L^\infty(\partial \Omega, \tilde{\mu}_1) \subseteq L^2(\partial \Omega, \tilde{\mu}_1)$ (indeed, since $v$ is continuous, we can write $(v \circ \varphi)^* = v \circ \varphi^*$, where $\varphi^*$ denotes the almost everywhere existing radial limits of $\varphi(rz)$, which belong to $K$). Let finally $E = P(\bar{F}) \subseteq H^2(\Omega)$ where $P : L^2(\partial \Omega, \tilde{\mu}_1) \to H^2(\Omega)$ is the orthogonal projection. This is a subspace of $H^2(\Omega)$ with dimension $< n^N$, and we have dist $(C_\varphi g, E) \leq \| g \circ \varphi - P(h \circ \varphi) \|_{H^2}$; hence, by (4.6):

$$\text{dist} (C_\varphi g, E) \leq (1 + \varepsilon)^n \Lambda_N^n (l + 1)^{(cN+1)/2}.$$

Now, the same calculations give that:

$$|f(z) - g(z)|^2 \leq \sum_{p \geq l} p^{cN}[j_\Omega(z)]^{2p};$$

hence, for some positive constant $M_N$:

$$|f(z) - g(z)| \leq M_N (l + 1)^{(cN+1)/2} \frac{|j_\Omega(z)|^l}{(1 - |j_\Omega(z)|^2)^{(cN+1)/2}},$$

by using the following lemma, whose proof is postponed.

**Lemma 4.10.** For every non-negative integer $m$, there exists a positive constant $A_m$ such that, for all integers $l \geq 0$ and all $0 < x < 1$, we have:

$$\sum_{p \geq l} p^m x^p \leq A_m l^m \frac{x^l}{(1-x)^{m+1}}.$$ 

Since $K$ is a compact subset of $\Omega$, there is a positive number $r_0 < 1$ such that $j_\Omega(z) \leq r_0$ for $z \in K$. Since $C_\varphi f(z) - C_\varphi g(z) = f(\varphi(z)) - g(\varphi(z))$ and $\varphi(\Omega) \subseteq K$, we have $\| C_\varphi f - C_\varphi g \|_\infty \leq \| f - g \|_{C(K)}$, and we get:

$$\| C_\varphi f - C_\varphi g \|_{H^2} \leq \| C_\varphi f - C_\varphi g \|_\infty \leq M_N (l + 1)^{(cN+1)/2} \frac{r_0^l}{(1 - r_0^2)^{(cN+1)/2}}.$$

Now, (4.7) and (4.8) give:

$$\text{dist} (C_\varphi f, E) \leq M_N (l + 1)^{(cN+1)/2} \left( \frac{r_0^l}{(1 - r_0^2)^{(cN+1)/2}} + (1 + \varepsilon)^n \Lambda_N^n \right).$$

It ensues, thanks to (2.7), that:

$$[\alpha_n(C_\varphi)]^{1/n} \leq [M_N (l + 1)^{(cN+1)/2}]^{1/n} \left( \frac{r_0^l}{(1 - r_0^2)^{(cN+1)/2n}} + (1 + \varepsilon) \Lambda_N \right).$$

Taking now for $l$ the integer part of $n \log n$, and passing to the upper limit as $n \to \infty$, we obtain (since $l/n \to \infty$ and $(\log l)/n \to 0$):

$$\beta_n(C_\varphi) \leq (1 + \varepsilon) \Lambda_N,$$

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and therefore, since \( \varepsilon > 0 \) is arbitrary:

\[
\beta^+_N(C, \varphi) \leq \Lambda_N.
\]

That ends the proof, by using Proposition 4.9. \qed

**Proof of Lemma 4.10.** We make the proof by induction on \( m \). We set:

\[
S_m = \sum_{p \geq l} p^m x^p
\]

The result is obvious for \( m = 0 \), with \( A_0 = 1 \), since then \( S_0 = \sum_{p \geq l} x^p = \frac{x^l}{1 - x} \).

Let us assume that it holds till \( m = m - 1 \) and prove it for \( m \). We observe that, since \( p^m - (p - 1)^m \leq mp^{m-1} \), we have:

\[
(1 - x)S_m = \sum_{p \geq l} p^m x^p - \sum_{p \geq l} p^m x^{p+1} = \sum_{p \geq l} p^m x^p - \sum_{p \geq l+1} (p - 1)^m x^p
\]

\[
= \sum_{p \geq l+1} (p^m - (p - 1)^m) x^p + l^m x^l \leq \sum_{p \geq l+1} mp^{m-1} x^p + l^m x^l
\]

\[
\leq \sum_{p \geq l} mp^{m-1} x^p + l^m x^l \leq mA_{m-1} l^m \frac{x^l}{(1 - x)^m} + l^m x^l
\]

\[
\leq (mA_{m-1} + 1) l^m \frac{x^l}{(1 - x)^m},
\]

giving the result, with \( A_m = mA_{m-1} + 1 \). \qed

### 4.3 Equality

**Proposition 4.11.** Let \( \Omega \) be a bounded hyperconvex domain and \( \omega \) a relatively compact open subset of \( \Omega \). Assume that:

\[
(4.9) \quad \text{For every } a \in \partial \omega, \text{ except on a pluripolar set } E \subseteq \partial \omega, \text{ there exists } z_0 \in \omega \text{ such that the open segment } (z_0, a) \text{ is contained in } \omega.
\]

Then:

\[
\text{Cap } (\overline{\omega}) = \text{Cap } (\omega).
\]

In particular, if \( \varphi: \Omega \to \Omega \) a non-degenerate holomorphic map such that \( \overline{\varphi(\Omega)} \subseteq \Omega \) and \( \omega = \varphi(\Omega) \) satisfies (4.9), we have:

\[
\text{Cap } [\varphi(\Omega)] = \text{Cap } [\overline{\varphi(\Omega)}].
\]

Before proving Proposition 4.11, let us give an example of such a situation.

**Proposition 4.12.** Let \( \Omega \) be a bounded hyperconvex domain with \( C^1 \) boundary. Let \( U \) be an open neighbourhood of \( \overline{\Omega} \) and \( \varphi: U \to \mathbb{C}^N \) be a non-degenerate holomorphic function such that \( \overline{\varphi(\Omega)} \subseteq \Omega \). Then the condition (4.9) is satisfied.
Proof. Let \( \omega = \varphi(\Omega) \).

We may assume that \( U \) is connected, hyperconvex and bounded. Let \( B_\varphi \) be the set of points \( z \in U \) such that the complex Jacobian \( J_\varphi \) is null. Since \( J_\varphi \) is holomorphic in \( \Omega \), we have \( \log |J_\varphi| \in \mathcal{PSH}(U) \) and hence (see [31 proof of Lemma 10.2]):

\[
B_\varphi = \{ z \in U : J_\varphi(z) = 0 \} = \{ z \in U : \log |J_\varphi(z)| = -\infty \}
\]

is pluripolar. Therefore (see [5 Theorem 6.9]), \( \text{Cap}(B_\varphi, U) = 0 \). It follows (see [5 page 2, line -8]) that:

\[
\text{Cap}(\varphi(B_\varphi), \Omega) = 0.
\]

Proof of Proposition 4.11. Let \( a \in \partial \omega \) and \( L \) be a complex line containing \((z_0, a)\); we have \( a \in \omega \cap L \). Assume now that this point \( a \) is a fine ("effilé") point of \( \omega \), i.e. that there exists \( u \in \mathcal{PSH}(V) \) for \( V \) a neighbourhood of \( a \), such that:

\[
\limsup_{z \to a, z \in \omega} u(z) < u(a).
\]

By definition, the restriction \( \bar{u} \) of \( u \) to \( \omega \cap L \) is subharmonic and we keep the inequality:

\[
\limsup_{z \to a, z \in \omega \cap L} \bar{u}(z) < \bar{u}(a) = u(a).
\]

That means that \( a \) is a fine point of \( \omega \cap L \). But \( a \in \omega \cap L \) and \( \omega \cap L \) is connected, so this is not possible, by [40 Lemma 2.4]. Hence no point of \( \partial \omega \setminus E \) is fine.

Let now \( \omega^f \) be the closure of \( \omega \) for the fine topology (i.e. the coarsest topology on \( U \) for which all the functions in \( \mathcal{PSH}(U) \) are continuous; it is known: see [6, comment after Theorem 2.3], that it is the trace on \( U \) of the fine topology on \( \mathbb{C}^N \)). It is also known (see [34 Corollary 4.8.10]) that \( \omega^f \) is the set of points of \( \varphi \) which are not fine. By the above reasoning, we thus have:

\[
\varphi \setminus \omega^f \subseteq E.
\]

Since \( \text{Cap}(E) = 0 \), we have:

\[
\text{Cap}(\varphi \setminus \omega^f) = 0,
\]

and it follows that:

\[
\text{Cap}(\varphi) = \text{Cap}(\omega \cup (\varphi \setminus \omega^f)) \leq \text{Cap}(\omega^f) + \text{Cap}(\varphi \setminus \omega^f) = \text{Cap}(\omega^f),
\]

and hence \( \text{Cap}(\omega^f) = \text{Cap}(\varphi) \).

But, since, by definition, the \( psh \) functions are continuous for the fine topology, it is clear, that the relative extremal functions \( u_{\omega, \Omega} \) and \( u_{\omega^f, \Omega} \) are equal; hence we have, by [30 Proposition 4.7.2]:

\[
\text{Cap}(\omega) = \int_{\Omega} (dd^c u_{\omega, \Omega}^*)^N = \int_{\Omega} (dd^c u_{\omega^f, \Omega}^*)^N = \text{Cap}(\omega^f).
\]

Hence \( \text{Cap}(\omega) = \text{Cap}(\varphi) \). \( \square \)
4.4 Consequences of the spectral radius type formula

Theorem 4.3 has the following consequence.

Proposition 4.13. Let \( \Omega \) be a regular bounded symmetric domain in \( \mathbb{C}^N \), and let \( \varphi: \Omega \to \Omega \) be a non-degenerate analytic function inducing a bounded composition operator \( C_\varphi \) on \( H^2(\Omega) \).

Then, if \( \text{Cap}[\varphi(\Omega)] = \infty \), we have \( \beta_N(C_\varphi) = 1 \).

In other words, if, for some constants \( C, c > 0 \), we have \( a_n(C_\varphi) \leq Ce^{-cn^{1/N}} \) for all \( n \geq 1 \), then \( \text{Cap}[\varphi(\Omega)] < \infty \).

As a corollary, we can give a new proof of [11, Theorem 3.1].

Corollary 4.14. Let \( \tau: \mathbb{D} \to \mathbb{D} \) be an analytic map such that \( \|\tau\|_\infty = 1 \) and \( \psi: \mathbb{D}^{N-1} \to \mathbb{D}^{N-1} \) such that the map \( \varphi: \mathbb{D}^N \to \mathbb{D}^N \), defined as:

\[
\varphi(z_1, z_2, \ldots, z_N) = (\tau(z_1), \psi(z_2, \ldots, z_N)),
\]
is non-degenerate. Then \( \beta_N(C_\varphi) = 1 \).

Proof. Since the map \( \varphi \) is non-degenerate, \( \psi \) is also non-degenerate. Hence (see [11, Proposition 2] \( \psi(\mathbb{D}^{N-1}) \) is not pluripolar, i.e. \( \text{Cap}_{N-1}[\psi(\mathbb{D}^{N-1})] > 0 \). On the other hand, it follows from [10, Theorem 3.13 and Theorem 3.14] that \( \text{Cap}_1[\tau(\mathbb{D})] = +\infty \). Then, by [8, Theorem 3], we have:

\[
\text{Cap}_N[\varphi(\mathbb{D}^N)] = \text{Cap}_N[\tau(\mathbb{D}) \times \psi(\mathbb{D}^{N-1})] = \text{Cap}_N[\tau(\mathbb{D})] \times \text{Cap}_{N-1}[\psi(\mathbb{D}^{N-1})] = +\infty.
\]

It follows from Proposition 4.13 that \( \beta_N(C_\varphi) = 1 \).

Proof of Proposition 4.13. If \( R: H^2(\Omega) \to H^2(\Omega) \) is a finite-rank operator, we set, for \( t < 0 \):

\[
(R_t f)(w) = (Rf)(e^t w), \quad f \in H^2(\Omega).
\]

Then the rank of the operator \( R_t \) is less or equal to that of \( R \).

Recall that if \( \|\cdot\| \) is the norm whose unit ball is \( \Omega \), then the pluricomplex Green function of \( \Omega \) is \( g_\Omega(z) = \log \|z\| \), and hence the level set \( S(r) \) is the sphere \( S(0, e^r) = e^r \partial \Omega \) for this norm. Since:

\[
\int_{S(r)} |f[\varphi(e^t w)] - (Rf)(e^t w)|^2 d\mu_r(w) = \int_{S(r+t)} |f[\varphi(z)] - (Rf)(z)|^2 d\mu_{r+t}(z),
\]
we have, setting \( \varphi_t(w) = \varphi(e^t w) \):

\[
\|C_{\varphi_t}(f) - R_t(f)\|_{H^2} \leq \|C_{\varphi}(f) - R(f)\|_{H^2}.
\]

It follows that \( a_n(C_{\varphi_t}) \leq a_n(C_{\varphi}) \) for every \( n \geq 1 \). Therefore \( \beta_N(C_{\varphi_t}) \leq \beta_N(C_{\varphi}) \).
By Theorem 4.3, we have:
\[
\exp \left[ -2\pi \left( \frac{N!}{\text{Cap} [\varphi_t(\Omega)]} \right)^{1/N} \right] \leq \beta_N(C_{\varphi_t}).
\]
Since \( \varphi_t(\Omega) = \varphi(e^t\Omega) \) increases to \( \varphi(\Omega) \) as \( t \uparrow 0 \), we have (see [30, Corollary 4.7.11]):
\[
\text{Cap} [\varphi(\Omega)] = \lim_{t \to 0} \text{Cap} [\varphi_t(\Omega)].
\]
As \( \text{Cap} [\varphi(\Omega)] = \infty \), we get:
\[
\beta_N(C_{\varphi}) \geq \limsup_{t \to 0} \beta_N(C_{\varphi_t}) = 1.
\]

Remark 1. In [41, Theorem 5.12], we construct a non-degenerate analytic function \( \varphi : \mathbb{D}^2 \to \mathbb{D}^2 \) such that \( \varphi(\mathbb{D}^2) \cap \partial \mathbb{D}^2 \neq \emptyset \) and for which \( \beta^+_{\mathbb{D}^2}(C_{\varphi}) < 1 \). We hence have \( \text{Cap} [\varphi(\mathbb{D}^2)] < \infty \).

Remark 2. The capacity cannot tend to infinity too fast when the compact set approaches the boundary of \( \Omega \); in fact, we have the following result, that we state for the ball, but which holds more generally.

**Proposition 4.15.** For every compact set \( K \) of \( \mathbb{B}_N \), we have, for some constant \( C_N \):
\[
\text{Cap} (K) \leq \frac{C_N}{(\text{dist} (K, \mathbb{S}_N))^N}.
\]

**Proof.** We know that:
\[
\text{Cap} (K) = \int_{\mathbb{B}_N} (dd^c u^*_K)^N.
\]
Let \( \rho(z) = |z|^2 - 1 \) and \( a_K := \min_{z \in K} [-\rho(z)] = -\max_{z \in K} \rho(z) \). Then \( \rho \) is in \( \mathcal{PSH} \) and is non-positive. Since \( a_K > 0 \), the function:
\[
v(z) = \frac{\rho(z)}{a_K}
\]
is in \( \mathcal{PSH} \), non-positive on \( \mathbb{B}_N \), and \( v \leq -1 \) on \( K \). Hence \( v \leq u_K \leq u^*_K \).

Since \( v(w) = 0 \) for all \( w \in \mathbb{S}_N \) and (see [5, Proposition 6.2 (iv)], or [30, Proposition 4.5.2]):
\[
\lim_{z \to w} u^*_K(z) = 0,
\]
for all \( w \in \mathbb{S}_N \), the comparison theorem of Bedford and Taylor ([5, Theorem 4.1]; [20, Theorem 3.7.1] gives, since \( v \leq u^*_K \) and \( v, u^*_K \in \mathcal{PSH} \):
\[
\int_{\mathbb{B}_N} (dd^c u^*_K)^N \leq \int_{\mathbb{B}_N} (dd^c v)^N = \frac{1}{a_K} \int_{\mathbb{B}_N} (dd^c \rho)^N.
\]
As \( (dd^c \rho)^N = 4^N N! d\lambda_{2N} \), we get, with \( C_N := 4^N N! \lambda_{2N}(\mathbb{B}_N) \):
\[
\text{Cap} (K) \leq \frac{C_N}{a_K^N}.
\]
That ends the proof since:

\[ a_K = \min_{z \in K} (1 - |z|^2) \geq \min_{z \in K} (1 - |z|) = \text{dist} (K, S_N) \]

We have assumed that the symbol \( \varphi \) is non-degenerate. For a degenerate symbol \( \varphi \), we have:

**Proposition 4.16.** Let \( \Omega \) be a bounded hyperconvex and good complete Reinhardt domain in \( \mathbb{C}^N \), and let \( \varphi: \Omega \to \Omega \) be an analytic function such that \( \varphi(\Omega) \subseteq \Omega \) is pluripolar. Then \( \beta_N(\mathcal{C}_\varphi) = 0 \).

Recall that \( \varphi(\Omega) \) is pluripolar when \( \varphi \) is degenerate (see [44, Proposition 2]); its closure is also pluripolar if it satisfies the condition (4.9).

**Proof.** Let \( K = \varphi(\Omega) \). By hypothesis, we have \( \text{Cap} (K) = 0 \). For every \( \varepsilon > 0 \), let \( K_\varepsilon = \{ z \in \Omega ; \text{dist} (z, K) \leq \varepsilon \} \). By Theorem 4.7, we have \( \beta_N^+(\mathcal{C}_\varphi) \leq \Gamma_N(K_\varepsilon) \). As \( \lim_{\varepsilon \to 0} \text{Cap} (K_\varepsilon) = \text{Cap} (K) = 0 \) ([30, Proposition 4.7.1(iv)]), we get \( \beta_N(\mathcal{C}_\varphi) = 0 \).

**Remark 1.** In [41, Section 4], we construct a degenerate symbol \( \varphi \) on the bi-disk \( D^2 \), defined by \( \varphi(z_1, z_2) = (\lambda_\theta(z_1), \lambda_\theta(z_1)) \), where \( \lambda_\theta \) is a lens map, for which \( \beta^-(\mathcal{C}_\varphi) > 0 \). For this function \( \varphi(D^2) \cap \partial D^2 \neq \emptyset \) and hence \( \varphi(D^2) \) is not a compact subset of \( D^2 \).

**Remark 2.** In the one dimensional case, for any (non constant) analytic map \( \varphi: D \to D \), the parameter \( \beta(\mathcal{C}_\varphi) = \beta_1(\mathcal{C}_\varphi) \) is determined by its range \( \varphi(D) \), as shown by the formula:

\[ \beta(\mathcal{C}_\varphi) = e^{-1/\text{Cap} [\varphi(D)]} \]

proved in [40]. This is no longer true in dimension \( N \geq 2 \). In [42], we construct pairs of (degenerate) symbols \( \varphi_1, \varphi_2 : D^2 \to D^2 \), such that \( \varphi_1(D^2) = \varphi_2(D^2) \) and:

1) \( C_{\varphi_1} \) is not bounded, but \( C_{\varphi_2} \) is compact, and even \( \beta_2(C_{\varphi_2}) = 0 \);

2) \( C_{\varphi_1} \) is bounded but not compact, so \( \beta_2(C_{\varphi_1}) = 1 \), and \( C_{\varphi_2} \) is compact, with \( \beta_2(C_{\varphi_2}) = 0 \);

3) \( C_{\varphi_1} \) is compact, with \( 0 < \beta_2(C_{\varphi_1}) < 1 \), and \( C_{\varphi_2} \) is compact, with \( \beta_2(C_{\varphi_2}) = 0 \).

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