On the Log Partition Function of Ising Model on Stochastic Block Model

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Abstract

A sparse stochastic block model (SBM) with two communities is defined by the community probability \( \pi_0, \pi_1 \), and the connection probability between communities \( a, b \in \{0, 1\} \), namely \( q_{ab} = \frac{2ab}{n} \). When \( q_{ab} \) is constant in \( a, b \), the random graph is simply the Erdős-Rényi random graph. We evaluate the log partition function of the Ising model on sparse SBM with two communities.

As an application, we give consistent parameter estimation of the sparse SBM with two communities in a special case. More specifically, let \( d_0, d_1 \) be the average degree of the two communities, i.e., \( d_0 \overset{def}{=} \pi_0 \alpha_{00} + \pi_1 \alpha_{01} \), \( d_1 \overset{def}{=} \pi_0 \alpha_{10} + \pi_1 \alpha_{11} \). We focus on the regime \( d_0 = d_1 \) (the regime \( d_0 \neq d_1 \) is trivial). In this regime, there exists \( d, \lambda \) and \( r \geq 0 \) with \( \pi_0 = \frac{1}{1+\lambda}, \pi_1 = \frac{r}{1+r}, \alpha_{00} = d(1+\lambda), \alpha_{01} = \alpha_{10} = d(1-\lambda), \alpha_{11} = d(1+\frac{\lambda}{r}) \).

We give a consistent estimator of \( r \) when \( \lambda < 0 \). The estimator of \( \lambda \) given by (Mossel et al., 2015) is valid in the general situation. We also provide a random clustering algorithm which does not require knowledge of parameters and which is positively correlated with the true community label when \( \lambda < 0 \).

Key words: stochastic block model, clustering, parameter estimation, sparsity

AMS 2000 subject classification. Primary 62F12; secondary 90B15

1 Introduction

Stochastic block model (SBM), also known as planted partition model, is one of the most commonly used generative network model. In this model, every node \( i \in V = \{1, 2, \cdots, n\} \) is assigned a latent type (community label) \( \sigma_i \), with probability \( \pi_{\sigma_i} \). Conditioned on node types, the connection between nodes is independent of each other. For every two nodes \( i \) and \( j \), the conditional connection probability is \( q_{\sigma_i, \sigma_j} \), which depends on the types of the two nodes. Denote the SBM defined by \( q, \pi \) as \( SBM(q, \pi) \). When the connection probability is a constant, the model becomes the Erdős-Rényi model \( G(n, q) \). The clustering (community detection) problem is to infer the latent types from the network structure. This is an important problem in many areas such as computer science, social network analysis, statistics, machine learning, biology and image processing (see (Fortunato, 2010) for a thorough introduction). The parameter estimation problem is to estimate model parameters \( \pi_a, q_{ab} \).

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SBM is one of the most popular network models, not only because of its simplicity, but also for the following reasons. First, it well fits a lot of real world data in the following fields: social network (Holland and Leinhardt, 1981; Newman et al., 2002; Robins et al., 2009) (notably, Holland and Leinhardt (1981) first proposed SBM), biology (Robe et al., 2012), gene regulatory network (Schlitt and Brazma, 2007; Pritchard et al., 2006), image processing (Shi and Malik, 2000; Sonka et al., 2008). Second, the model is a nice tool to investigate clustering algorithms from the theoretical perspective. Some early works in this stream are [Dyer and Frieze, 1989; Jerrum and Sorkin, 1998; Condon and Karp, 2001]. Their focus is the algorithmic aspects of the min-bisection problem. Later a vast amount of research is carried out to study and compare the performance of various clustering algorithms on SBM. Roughly speaking, these algorithms can be divided into the following categories. Modularity algorithm (Newman and Girvan, 2004), likelihood algorithm (Bickel and Chen, 2009; Choi et al., 2012; Amini et al., 2013; Celisse et al., 2012), etc., and most importantly, spectral algorithm (Chatterjee et al., 2011; Balakrishnan et al., 2011; Jin, 2015; Sarkar and Bickel, 2013; Krzakala et al., 2013), etc.

Notably, Bickel and Chen (2009) provided a general framework to establish consistency of clustering. It was further extended by [Zhao et al., 2012] to establish consistency of many clustering algorithms in more general models. These algorithms include maximum likelihood estimation and various modularity methods. The technique is largely based on finite covering plus concentration inequality. This line is also followed to establish consistency of spectral clustering (Lei et al., 2015). Although there need the evaluation of the norm of a random matrix, which is more complicated.

1.1 Related work on sparse stochastic block model

In reality, many networks are sparse. For example, Leskovec et al. (2008) found that many large networks with millions of nodes have an average degree less than 20. These networks include, social networks like LinkedIn and MSN Messenger; collaboration networks in movies and on the arXiv (see also (Strogatz, 2001)); and some biological networks.

Despite the vast amount of literature on SBM, most of the literatures focused on dense SBM. Where dense means that the average degree scales with network size and is usually of order at least $\log n$. However, very few is known for sparse SBM. A sparse SBM refers to the SBM with constant level degree, i.e., $q_{ab} = \frac{\alpha}{n}$. Sparse SBM is generally more difficult to handle. For instance, in contrast with dense SBM, consistent clustering is impossible since there exists a constant portion of isolated nodes, and there is no way to identify the community label of an isolated node. Also note that the local structure of the network can not be distinguished from that of a Erdős-Rény model $G(n, \frac{d}{n})$ if the expected degree of each node is $d$. For instance, in such SBM, the degree of the nodes follows the Poisson distribution with mean $d$, which is also the degree distribution in $G(n, \frac{d}{n})$. For this reason, spectral algorithm based on the adjacent matrix $A$ or a constant power of $A$ or modifications of such matrix (say the Laplacian) does not apply to sparse SBM.

In sparse SBM, we say the clustering problem is solvable iff there exists an estimator of the community la-
bel, which is positively correlated to the true community label. Most studies in sparse SBM have been limited to balance case. Decelle et al. (2011) investigated the sparse SBM with two communities and balance parameters i.e., \( q = q^{(b)} = \left( \frac{\alpha}{n}, \frac{\beta}{n} \right), \pi_0^{(b)} = \pi_1^{(b)} = 1/2 \). Based on ideas from statistical physics (cavity method), Decelle et al. (2011) conjectured that clustering in \( SBM(q^{(b)}, \pi^{(b)}) \) is solvable if and only if \( d\lambda^2 > 1 \). On the negative part, Mossel et al. (2015) showed that clustering in \( SBM(q^{(b)}, \pi^{(b)}) \) is not solvable if \( d\lambda^2 < 1 \). The same condition also implies that the model \( SBM(q^{(b)}, \pi^{(b)}) \) and the Erdős-Rény model \( G(n, \frac{d}{n}) \) are contiguous (which implies no consistent estimator of \( q^{(b)} \) exist). On the positive part, Coja-Oghlan (2010) provide a spectral algorithm for clustering. But in their paper, the condition ensuring the positive correlation is stronger than the condition \( d\lambda^2 > 1 \). Finally, Mossel et al. (2013) and Massoulié (2014) independently provide spectral algorithms solving the clustering problem in \( SBM(q^{(b)}, \pi^{(b)}) \) under the condition \( d\lambda^2 > 1 \). Therefore, Mossel et al. (2015), Mossel et al. (2013) and Massoulié (2014) together confirmed the conjecture proposed by Decelle et al. (2011). Recently, Bordenave et al. (2015) deal with the general sparse SBM with arbitrarily many blocks (see also (Abbe and Sandon, 2015)). Their result confirm the "spectral redemption conjecture", which is a generalized version of the conjecture in (Decelle et al., 2011). They prove, based on non-backtracking walks on the graph, that community detection is solvable down to the Kesten-Stigum threshold.

Xu et al. (2014) studied the SBM with edge label. The edge label indicates the type of the connection. For the SBM with edge label, Heimlicher et al. (2012) proposed a conjecture similar to (Decelle et al., 2011). Lelarge et al. (2015), similar to Mossel et al. (2013), confirmed a half of the conjecture. They proved that the condition proposed in (Heimlicher et al., 2012) implies that both consistent parameter estimation and positively correlated clustering are impossible. On the positive part, Xu et al. (2014) proposed a clustering algorithm taking advantage of the edge label. The proof of positive correlation of their algorithm only concerns Chernoff inequality. But the condition ensuring positive correlation is stronger than that proposed by (Heimlicher et al., 2012). It is not known whether the spectral algorithms in (Mossel et al., 2013, Massoulié, 2014) can be adapted to provide a positively correlated clustering algorithm under the mere condition of (Heimlicher et al., 2012). The problem of estimating the distribution of edge label is also unknown.

1.2 Motivation and technique

The technique used in dense SBM can not be directly applied to sparse SBM. In dense SBM, consistent parameter estimation is usually a by product of consistent clustering. But it does not seem that way in sparse SBM. For instance, Mossel et al. (2013) uses the technique of random matrix to estimate the community labels, while Mossel et al. (2015) estimates \( \lambda \) by counting \( k \)-cycles. In dense SBM, the lower bound of estimation error is usually given by information inequality such as Fano’s inequality (Gao et al., 2015). In sparse SBM, second moment method, which yields the results that two models are closed, is used to prove impossibility of parameter estimation (see (Mossel et al., 2015) section 5). There is a good reason to speculate that clustering is not solvable if the SBM is not distinguishable from some Erdős-Rény model \( G(n, \frac{d}{n}) \). Neeman and Netrapalli (2014) recently
obtained a result in this fold. Despite that Bordenave et al. (2015) has solved the community detection problem and the parameter estimation problem for the general sparse SBM down to the Kesten-Stigum threshold, it is not known where exactly the threshold for reconstructibility and distinguishability is (see (Banks et al., 2016) for such results). By far, most of results establishing indistinguishability employ second moment method. Hopefully, calculating the log partition function of the Ising model on SBM provide an alternative approach. Also note that the conditional distribution of $\sigma$ given $G$ is approximately an Ising model when $n$ is large. Therefore it is likely that analysis of the Ising model on a sparse SBM ultimately provide an exact threshold for reconstructibility and distinguishability.

1.3 Outline

Denote the probability of the two communities by $\pi_0, \pi_1$. The connection probability between community $a$ and $b$ is $\frac{\alpha_{ab}}{n}$. Since the graph is undirected, it is required that $\alpha_{00} = \alpha_{11}$. Let $d_0 = \pi_0\alpha_{00} + \pi_1\alpha_{01}, d_1 = \pi_0\alpha_{10} + \pi_1\alpha_{11}$. It is easy to see if $d_0 = d_1 = d$, then there exists $r \geq 0, \lambda$ with $\pi_0 = \frac{1}{1+r}, \pi_1 = \frac{r}{1+r}$, and $q = \left( \frac{\lambda}{n} \frac{\lambda}{n} \frac{\lambda}{n} \frac{\lambda}{n} \right)$. We focus on the regime $d_0 = d_1 = d$. The parameter estimation and community detection in the regime $d_0 \neq d_1$ are trivial. Let $SBM(d, \lambda, r)$ denote the stochastic block model defined by $d, \lambda, r$.

The paper is organized as follows. We show in section 2.1 that $d, \lambda$ can be estimated in the same way (by counting $k$-cycles) as in the balanced SBM. We evaluate the log partition function of an Ising model on graph $G$ in section 2.2. As an application, we propose a consistent estimator of $r$ when $\lambda < 0$ and $d$ being sufficiently large. We provide a random clustering algorithm, which samples $\hat{\sigma}$ according to an appropriate Ising model on $G$, in section 2.3. The clustering algorithm has positive correlation with the true community label when $\lambda < 0$. Section 3 contains proof of lemmas in section 2. Concluding remarks and some further questions are given in section 4.

1.4 Notations

For a given undirected graph $G = (V, E)$ and a node $u \in V$, let $\deg(u)$ denote the degree of $u$ in $G$. For $A, B \subseteq V(G)$, $e_G(A, B) = \{ i, j \} \in E(G) : i \in A, j \in B \}$. For an event $A$, $I(A)$ denote the indicator function of $A$. For two sets $A, B$, $A \Delta B$ denote $A \cup B - A \cap B$. For two sequences of reals $f_n, g_n$ write $f_n \sim g_n$ if $\lim_{n \to \infty} \frac{f_n}{g_n} = 1$; $f_n = \Omega(g_n)$ if $f_n \geq kg_n$ for some positive real $k$. We write $E_{X,Y}[f(X,Y)]$ or $E_X[f(X,Y)]$ ($P_{X,Y}((X,Y) \in Z)$ or $P_X((X,Y) \in Z)$) to denote the expectation (probability) with respect to $X$ conditional on $Y$. Write $E_{X \sim p}$ ($P_{X \sim p}$) to denote the expectation (probability) when the distribution of $X$ is $p$. 

4
2 Main results

Denote both \(d_0\), \(d_1\) by \(d\). Let \(P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \frac{\pi_0}{d_0} & \frac{\pi_1}{d_1} \\ \frac{\pi_0}{d_0} & \frac{\pi_1}{d_1} \end{pmatrix}\). Bear in mind that \(P\) can be regarded as a markov transition matrix. Let \(\lambda = p_{11} + p_{00} - 1\) denote the second large eigenvalue of \(P\). In subsection 2.1 we show that, similar with [Mossel et al., 2015] section 3, by counting \(k\)-cycles for appropriately large \(k\) we can estimate \(\lambda\) consistently provided \(d\lambda^2 > 1\). We give in subsection 2.2 a consistent estimator of \(r\) in the case \(\lambda < 0\); and subsection 2.3 a random clustering algorithm with positive correlation with true labeling in the same case.

2.1 Estimating \(d\), \(\lambda\)

Let \(C_k\) denote the number of cycles of length \(k\). The following proposition says that \(\lambda\) can be consistently estimated by counting \(k\)-cycles.

**Proposition 2.1.** Let \(\hat{d} = \frac{1}{n} \sum_{u \in V} \deg(u)\). Then \(\hat{d}\) is a \(\sqrt{n}\)-consistent estimator of \(d\). If \(d\lambda^2 > 1\), then \(\frac{\lambda}{d}(2kC_k - \hat{d}k)^+\) is a consistent estimator of \(\lambda\).

**Proof.** The \(\sqrt{n}\)-consistency of \(\hat{d}\) is obvious. Prove the second conclusion, we compute the probability that a given sequence of different nodes \(u_1, \cdots, u_k\) forms a cycle. Set \(u_{k+1} = u_1\). Note that

\[
\mathbb{E} \left[ \prod_{i=1}^{k} X_{u_i, u_{i+1}} \right] = \sum_{\sigma \in \{0,1\}^{k+1}, \sigma_{k+1} = \sigma_1} \prod_{i=1}^{k} \frac{\alpha_{\sigma_i, \sigma_{i+1}}}{n} \frac{d^k}{n^k} \sum_{\sigma \in \{0,1\}^{k+1}, \sigma_{k+1} = \sigma_1} \prod_{i=1}^{k} p_{\sigma_i, \sigma_{i+1}}.
\]

Think of \(\sum_{\sigma \in \{0,1\}^{k+1}, \sigma_{k+1} = \sigma_1} \prod_{i=1}^{k} p_{\sigma_i, \sigma_{i+1}}\) as the probability of the following event: a markov chain with transition matrix \(P\) starting at \(h\), arrives at \(h\) after \(k\) steps. Therefore continue (1) we have

\[
\mathbb{E} \left[ \prod_{i=1}^{k} X_{u_i, u_{i+1}} \right] = \frac{d^k}{n^k} \left( (1,0)P^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (0,1)P^k \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \frac{d^k}{n^k} \cdot \text{Tr}[P^k] = \frac{d^k}{n^k} (1 + \lambda^k).
\]

Thus, for \(k = o(\log n)\),

\[
\mathbb{E} [C_k] = \binom{n}{k} \cdot k! \frac{1}{2k} \frac{d^k}{n^k} (1 + \lambda^k) = (1 + O(\frac{1}{n})) \frac{d^k}{2k} (1 + \lambda^k).
\]

Similarly, for \(k = o(\log n)\), \(\text{Var}(C_k) = O(\frac{d^k}{k}(1 + \lambda^k))\). This is given by [Mossel et al., 2015] theorem 3.1 where
\[ E[C_k(C_k-1)\cdots(C_k-m)] \] is calculated. Therefore, if \( d\lambda^2 > 1, k = o(\log n) \), then

\[ 2kC_k - \hat{d}^k = (1 + O(\frac{1}{n}))d\lambda^k + O_p(\frac{d^k}{\sqrt{n}} + \sqrt{k}d^{k/2} + \sqrt{k}(d\lambda)^{k/2}) = (d\lambda)^k(1 + o_p(1)). \]

Thus, if \( d\lambda^2 > 1, k = o(\log n) \), then \( \frac{2kC_k - \hat{d}^k}{d} \) is a consistent estimator of \( \lambda \).

\[ \square \]

## 2.2 Evaluating the log partition function

Let \( SBM(d, \lambda, r, n) \) denote the SBM defined by \( d, \lambda, r \) of size \( n \). Clearly, the SBM defined by \( d, \lambda, r \) and \( d, \lambda, \frac{1}{r} \) are identical. Therefore, without loss of generality, assume \( r \geq 1 \). Also note that \( 1 + r\lambda \geq 0 \) is automatically required since \( \alpha_{00} \geq 0 \).

For any undirected graph \( G \) and \( \sigma \in \{0, 1\}^V(G) \), let \( J(\sigma; G) = \left|\left\{ \{u, v\} : \{u, v\} \in E(G), \sigma(u) = \sigma(v)\right\}\right| \). We evaluate the following log partition function:

\[ Z(\beta, G) = \log \left( \sum_{\sigma \in \{0, 1\}^V(G)} e^{-\beta J(\sigma; G)} \right). \]

To state our main results, we introduce the following symbols. Denote by \( g(z) \) the function

\[ g(z) = \begin{cases} \min\{z - (1 - z) \log(1 - z), (1 + z) \log(1 + z) - z\} & \text{if } 0 < z < 1; \\ \infty & \text{if } z > 1. \end{cases} \]

Let

\[ C(r, \lambda) = \inf_{0 \leq x \leq 1, 0 \leq y \leq 1} \left\{ r\lambda(x - y)^2 + (x + ry - \frac{1+r}{2})^2 + (1 + r)^2 \right\}; \]

\[ y^* = \min\{\frac{r + 1}{2(r + \lambda)}, 1\}, x^* = 0; \]

\[ \varepsilon_0 = g^{-1}\left(\frac{4 \log 2 \cdot (1 + r)^2}{dC(r, \lambda)}\right); \]

\[ \varepsilon_0 = \frac{2(r + 1)^2(\frac{2 \log 2}{\beta d^2} + 2\varepsilon_0)}{\min\{r - 2r\lambda - 1, \frac{\lambda(1+r)^2}{r(1-r)}\}}; \]

\[ \epsilon_1 = \min\left\{ \frac{(1 + r)^2 (\frac{2 \log 2}{\beta d^2} + 2\varepsilon_0)}{|2(r^2 + r\lambda)y^* - r^2 - r|}, \sqrt{\frac{2(1 + r)^2 (\frac{2 \log 2}{\beta d^2} + 2\varepsilon_0)}{r^2 + r\lambda}} \right\}. \]

Condition 2.2.

1. \( 0 < \beta \leq 1; \)

2. \( (r + 1)^2(\frac{2 \log 2}{\beta d^2} + 2\varepsilon_0) < \frac{(1+r)^2}{4} - C(r, \lambda); \)
Remark 2.3. For any \( r, \lambda \) there exists sufficiently large \( d \) and sufficiently small \( \beta \) satisfying the condition (2.2). In the sense \( d \) being large and \( r, \lambda \) being constant, the \( \beta \) we bear in mind satisfy the follows: \( \beta = o(1), \beta d \to \infty, \varepsilon_0 = O(\frac{1}{\sqrt{d}}) = o(1), \varepsilon_0 = O(\frac{1}{\sqrt{d}} + \frac{1}{\beta d}) = o(1), \varepsilon_1 = O(\sqrt{\frac{1}{\beta d}} + \frac{1}{\sqrt{d}}) = o(1). \) To get an intuition of these quantities, the reader is referred to theorem 2.12, lemma 2.11.

Let

\[
C(d, r, \lambda, \beta) = -\frac{\beta d}{(1 + r)^2} \left[ \min \left\{ \frac{(r - 1)(1 - \lambda)}{1 + r}, \frac{|\lambda|(1 + r)^2}{4(r + \lambda)} \right\} - (2\beta + \frac{\log 2}{\beta d} + 12\max\{\varepsilon_0, \varepsilon_1\}) \right].
\]

The following properties of these quantities are needed.

**Proposition 2.4.**

If \( \lambda < 0 \) then:

1. under condition (2.2) item 2, \( \varepsilon_0 \leq x \leq y \) or \( |y - y^*| \geq \varepsilon_1 \leq \frac{1}{2} \) holds implies

\[
\beta \cdot \left\{ \frac{\lambda r(x - y)^2 + (x + ry - \frac{1 + r}{2})^2}{(1 + r)^2} \frac{d}{1 + r} + \frac{d}{4} \right\} \geq \beta \frac{d}{(1 + r)^2} C(r, \lambda) + 2 \log 2 + 2\beta \varepsilon_0 d.
\]

2.

\[
\min \left\{ \frac{x^*}{1 + r} d(1 + r\lambda) + \frac{y^* r}{1 + r} d(1 - \lambda), \frac{1 - x^*}{1 + r} d(1 + r\lambda) + \frac{(1 - y^*) r}{1 + r} d(1 - \lambda) \right\} = \frac{1 + r - y^* r + y^* r \lambda}{1 + r} d,
\]

\[
\min \left\{ \frac{x^*}{1 + r} d(1 - \lambda) + \frac{y^* r}{1 + r} d(1 + \lambda), \frac{1 - x^*}{1 + r} d(1 - \lambda) + \frac{(1 - y^*) r}{1 + r} d(1 + \lambda) \right\} = \frac{y^* (r + \lambda)}{1 + r} d.
\]

3.

\[
C(r, \lambda) = \begin{cases} \frac{r^2 + 2 r \lambda + 1}{2} & \text{if } r \leq 1 - 2\lambda; \\ \frac{(r + 2\lambda)(1 + r)^2}{4(r + \lambda)} & \text{if } r \geq 1 - 2\lambda. \end{cases}
\]

And \( C(r, \lambda) < \frac{(1 + r)^2}{4}. \)

4. \( x^*, y^* \) is a minimizer of \( [r \lambda(x - y)^2 + (x + ry - \frac{1 + r}{2})^2] + \frac{(1 + r)^2}{4} \), and \( y^* = 1 \) if \( r \leq 1 - 2\lambda \).

5. There exists two constants depending on \( \lambda \), namely \( C_1(\lambda), C_2(\lambda) > 0 \), such that for any \( \lambda < 0 \), any \( d \geq C_1(\lambda) \), any \( \frac{2}{\sqrt{d}} \geq \beta \geq \frac{1}{2\sqrt{d}} \) we have:

(a) \( d, r, \lambda, \beta \) satisfy condition (2.2) for all \( r \geq 1 \);

(b) \( r - 1 \geq \frac{C_2(\lambda)}{d^r} \) implies \( C(d, r, \lambda, \frac{1}{\sqrt{d}}) < 0 \) and \( \forall 0 \leq r' < r \) \( (\int_{r'}^r C(d, t, \lambda, \frac{1}{\sqrt{d}}) dt < 0) \).
The proof of proposition \[2.4\] and other lemmas, propositions in this subsection are all delayed to section \[3\]. The following theorem establish the upper derivative of the log partition function with respect to \(r\) when \(\lambda < 0\).

**Theorem 2.5.** Consider this function of \(d, \lambda, r\) and \(\beta\), \(E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right]\).

1. Let \(G_n \sim SBM(d, \lambda, r, n)\), then \(\left| \frac{1}{n} Z(\beta, G_n) - \mathbb{E}[\frac{1}{n} Z(\beta, G_n)] \right| = O_p\left(\frac{1}{\sqrt{n}}\right)\).

2. If \(\lambda < 0\) then under condition \[2.2\]
   \[
   \limsup_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{\delta} \left( E_{G \sim SBM(d, \lambda + \delta, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] - E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] \right) \leq C(d, \lambda, r, \beta).
   \]

3. \(E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right]\) is Lipschitz continuous in \(d, \lambda, r\) and \(\beta\) uniformly in \(n\), i.e.,
   \[
   \limsup_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{\delta} \left| E_{G \sim SBM(d + \delta, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] - E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] \right| = 0.
   \]
   \[
   \limsup_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{\delta} \left| E_{G \sim SBM(d, \lambda + \delta, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] - E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] \right| = 0.
   \]
   \[
   \limsup_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{\delta} \left| E_{G \sim SBM(d, \lambda, r + \delta, n)} \left[ \frac{1}{n} Z(\beta, G) \right] - E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] \right| = 0.
   \]
   \[
   \limsup_{\delta \to 0} \limsup_{n \to \infty} \frac{1}{\delta} \left| E_{G \sim SBM(d, \lambda, r, n + \delta)} \left[ \frac{1}{n} Z(\beta, G) \right] - E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] \right| = 0.
   \]

Before proving theorem \[2.5\] we give a direct application of theorem \[2.5\] providing the following consistent estimator of \(r\).

**Corollary 2.6.** If \(\lambda < 0, d \geq C_1(\lambda), r - 1 \geq \frac{C_2(\lambda)}{d^2}\), then the following estimator of \(r\) is consistent:

\[
\hat{r} = \left( E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] \right)^{-1} \left( \frac{1}{n} E_{G \sim SBM(d, \lambda, r, n)} \left[ Z(\beta, G) \right] \right).
\]

Here \(E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] \) is regarded as a single variable function in \(r\) and \((\cdot)^{-1}\) denote its inverse. Constants \(C_1(\lambda), C_2(\lambda)\) are defined in proposition \[2.4\] conclusion 5.

**Proof.** By theorem \[2.3\] conclusion 2 and proposition \[2.4\] conclusion 5, for all \(r - 1 \geq \frac{C_2(\lambda)}{d^2}, d \geq C_1(\lambda)\), for any \(r' \geq 1\), \(E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right] = E_{G \sim SBM(d, \lambda, r', n)} \left[ \frac{1}{n} Z(\beta, G) \right]\) implies \(r' = r\). So the inverse function is well defined at \(\frac{1}{n} Z(\beta, G_n)\) with large probability. \(\frac{1}{n} Z(\beta, G)\) has fluctuation of order \(\frac{1}{\sqrt{n}}\) by theorem \[2.5\] conclusion 1, so it is closed to its expectation with large probability. Finally the conclusion follows by noting that \(\hat{d}, \hat{\lambda}\) are consistent estimator of \(d, \lambda\) and \(E_{G \sim SBM(d, \lambda, r, n)} \left[ \frac{1}{n} Z(\beta, G) \right]\) is continuous in \(d, \lambda\) by theorem \[2.5\] conclusion 3.

**Proof of theorem \[2.5\].** Conclusion 1 of theorem \[2.5\] follows by concentration inequality such as Azuma’s inequality
and its proof is therefore omitted. Conclusion 3 of theorem 2.5 follows in the same way as conclusion 2 and its proof is therefore omitted. Now we focus on the proof of conclusion 2 of theorem 2.5. Through out the proof, fix $\delta$ to be a sufficiently small positive constant which will be smaller than any other constant whenever necessary.

We prove theorem 2.5 conclusion 2 by evaluating

$$E_{G \sim SBM(d, \lambda, r+\delta, n)}[Z(\beta, G)] - E_{G \sim SBM(d, \lambda, r, n)}[Z(\beta, G)]$$

for small $\delta$. To this end, we adopt the variational method. We firstly construct a graph $\tilde{G}$. Based on $\tilde{G}$, we then inductively construct three sequences of random graph $G_{0,i}, G_{1,i}, G'_{1,i}$ by adding nodes, deleting edges or adding edges. Through these sequences we obtain two graphs $G'_0, G'_1$ (see definition 2.7). We argue that the marginal law of $G'_0$ ($G'_1$) is sufficiently close to $SBM(d, \lambda, r, n)$ ($SBM(d, \lambda, r+\delta, n)$) (see proposition 2.8). Finally we evaluate $E[Z(\beta, G'_0) - Z(\beta, G'_1)]$ by evaluating $E[Z(\beta, G'_{h,i+1}) - Z(\beta, G'_{h,i})]$ (lemma 2.10), $E[Z(\beta, G'_{h,i+1}) - Z(\beta, G'_{h,i})]$ (lemma 2.11), for $h \in \{0, 1\}$.

More specifically, $\tilde{G}$ is generated according to a SBM consisting of two communities $N_0, N_1$ with $|N_0| \approx \frac{n}{1+r}, |N_1| \approx \frac{\delta n}{1+r}$. The within-community connection probability of $N_0, N_1$ are $\frac{d(1+r\lambda)}{n}, \frac{d(1+\lambda)}{n}$ respectively; and the across community connection probability is $\frac{d(1-\lambda)}{n}$. $G'_0$ is constructed by adding $\frac{\delta n}{1+r}$ many nodes to community $N_0$ and connect each new node with every old node according to model $SBM(d, \lambda, r)$. i.e., connect each new node with every old node, say $u$, with probability $\frac{d(1+r\lambda)}{n}$ if $u \in N_0$, probability $\frac{d(1-\lambda)}{n}$ if $u \in N_1$. $G'_1$ is constructed by firstly adding $\frac{\delta n}{1+r}$ many nodes to community $N_1$ and connect each new node with every old node according to model $SBM(d, \lambda, r+\delta)$. i.e., connect each new node with every old node, namely $u$, with probability $\frac{d(1-\lambda)}{n}$ if $u \in N_0$, probability $\frac{d(1+\lambda)}{n}$ if $u \in N_1$. Then adjust the connection probability of $G'_1$ by deleting $\frac{d(1-\lambda)}{n}$ many edges in $e_{G'_1}(N_0, N_0)$ uniformly at random and adding $\frac{d(1+\lambda)}{n}$ many disconnected pairs, $\{u, v\}$, with $u, v \in N_1$ uniformly at random. Note that no new nodes are connected. The precise definition is as follows.

**Definition 2.7.** [Construction of $\tilde{G}, G_{0,i}, G_{1,i}, G'_{1,i}$]

- $\tilde{G}$: $V(\tilde{G}) = N_0 \cup N_1$ where $N_0 = \{1, 2, \cdots, \frac{n}{1+r} - \lfloor \frac{\delta n}{1+r}\rfloor\}$, $N_1 = \{\frac{n}{1+r} - \lfloor \frac{\delta n}{1+r}\rfloor + 1, \cdots, n - \lfloor \frac{\delta n}{1+r}\rfloor\}$;

  $\tilde{G}$ is the following random graph: presence of edges are independent and

  $$\mathbb{P}(\{u, v\} \in E(\tilde{G})) = \begin{cases} \frac{d(1+r\lambda)}{n} & \text{if } u, v \in N_0, \\ \frac{d(1+\lambda)}{n} & \text{if } u, v \in N_1, \\ \frac{d(1-\lambda)}{n} & \text{if } u \in N_0, v \in N_1 \text{ or } u \in N_1, v \in N_0. \end{cases}$$

- $G_{0,0} = G_{1,0} = \tilde{G}$; set $k_i = n - \lfloor \frac{\delta n}{1+r}\rfloor + i$. For $i \leq \lfloor \frac{\delta n}{1+r}\rfloor$,

  $G_{0,i+1} : V(G_{0,i+1}) = V(G_{0,i}) \cup \{k_i\}$; let $X_{0,ui}, u \in V(\tilde{G})$ be independent random variables with $X_{0,ui} \sim Bin(1, \frac{d(1+r\lambda)}{n})$ if $u \in N_0$ and $X_{0,ui} \sim Bin(1, \frac{d(1-\lambda)}{n})$ if $u \in N_1$; $E(G_{0,i+1}) = E(G_{0,i}) \cup \{k_i, u : X_{0,ui} = 1\}$. 

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Proposition 2.8. 

\[
\begin{align*}
\left| \mathbb{E}[ Z(\beta, G_0^l) ] - \mathbb{E}_{G \sim \text{SBM}(d, \lambda, r, n)}[Z(\beta, G)] \right|, \\
\left| \mathbb{E}[ Z(\beta, G_1^l) ] - \mathbb{E}_{G \sim \text{SBM}(d, \lambda, r+\delta, n)}[Z(\beta, G)] \right|
\end{align*}
\]

= \mathcal{O}(\sqrt{n} + \delta^2 n).

The term $\sqrt{n}$ is due to the fluctuation of community size. The term $\delta^2 n$ is due to the approximation error of connection probability. For instance, there is no connection among "new nodes" in $G_1'$, $G_0'$ while the expected number of edges among "new nodes" should be $O(\delta^2 n)$. For instance, the expected number of edges in the two communities in $G_1'$ are $d(1+\lambda)\lambda n - \delta n + O(\delta^2 n), d(1+\lambda)\lambda n + \delta n + O(\delta^2 n)$.

Now we evaluate $\mathbb{E} \left[ Z(\beta, G_{h,i+1}) - Z(\beta, G_{h,i}) \right], \mathbb{E} \left[ Z(\beta, G_{1,i+1}') - Z(\beta, G_{1,i}') \right]$ for $h \in \{0,1\}$. Denote by $\text{IS}(\beta, G)$ the following Ising model on $\{0,1\}^{V(G)}$: 

\[
\mathbb{P}(\sigma) = \frac{e^{-\beta J(\sigma; G)}}{\sum_{\gamma \in \{0,1\}^{V(G)}} e^{-\beta J(\gamma; G)}}.
\]

For $h, l \in \{0,1\}, \sigma \in \{0,1\}^{V(G_{h,i})}$, let $e_{il}^h(\sigma) = |e_{G_{h,i+1}}(k_i, \sigma^{-1}(l))|$. Recall from definition that $k_i$ is the node added into $G_{h,i}$ at step $i$. The key observation is:

\[ Z(\beta, G_{h,i+1}) - Z(\beta, G_{h,i}) = \log \left( \sum_{\sigma \in \{0,1\}^{V(G_{h,i})}} \frac{e^{-\beta J(\sigma; G_{h,i})} - e^{-\beta e_{il}^h(\sigma)} + e^{-\beta e_{il}^h(\sigma)}}{\sum_{\gamma \in \{0,1\}^{V(G_{h,i})}} e^{-\beta J(\gamma; G_{h,i})}} \right) \]

\[ = \log \left( \mathbb{E}_\sigma \text{IS}(\beta, G_{h,i}) \left[ e^{-\beta e_{il}^h(\sigma)} + e^{-\beta e_{il}^h(\sigma)} \right] \right). \]

Here and below, $e_{il}^h(\sigma)$ is short for $e_{il}^h(\sigma)$. Another key point is to take advantage of the convexity of log as
follows:

\[ E_{X,Y} \left[ \log f(X,Y) \right] \leq E_X \left[ \log \left( E_{Y|X} \left[ f(X,Y) \right] \right) \right] \leq \log E_{X,Y} \left[ f(X,Y) \right]. \] (5)

Therefore using (4) (3) we have:

\[ E_{G_{h,i+1}|G_{h,i}} \left[ Z(\beta, G_{h,i+1}) - Z(\beta, G_{h,i}) \right] \]

\[ = E_{G_{h,i+1}|G_{h,i}} \left[ \log \left( E_{\sigma|G_{h,i}} \left[ e^{-\beta e_{i0}^h} + e^{-\beta e_{i1}^h} \right] \right) \right] \]

\[ \leq \log E_{G_{h,i+1,\sigma|G_{h,i}}} \left[ e^{-\beta e_{i0}^h} + e^{-\beta e_{i1}^h} \right]. \] (6)

\[ E_{G_{h,i+1}|G_{h,i}} \left[ Z(\beta, G_{h,i+1}) - Z(\beta, G_{h,i}) \right] \]

\[ \geq E_{G_{h,i+1,\sigma|G_{h,i}}} \left[ \log \left( e^{-\beta e_{i0}^h} + e^{-\beta e_{i1}^h} \right) \right] \]

\[ \geq \log E_{G_{h,i+1,\sigma|G_{h,i}}} \left[ -\beta \min \left\{ e_{i0}^h, e_{i1}^h \right\} \right]. \] (7)

Where \( \sigma|G_{h,i} \) is the Ising model \( IS(\beta, G_{h,i}) \). Note that in the calculation of \( E_{G_{h,i+1,\sigma|G_{h,i}}} \), \( G_{h,i+1} \) conditioned on \( G_{h,i} \) are mutually independent. It is not surprise that we need some properties on the Ising model on \( G_{h,i} \). Let

\[ l = \arg \max_{l \in \{0,1\}} \frac{|\sigma^{-1}(l) \cap N_1|}{|N_1|}; \]

\[ x(\sigma) = \frac{|\sigma^{-1}(\overline{l}) \cap N_0|}{|N_0|}, \quad y(\sigma) = \frac{|\sigma^{-1}(\overline{l}) \cap N_1|}{|N_1|}. \] (8)

We prove that for \( G_{h,i}, G_{1,j}' \), with large probability (with respect to \( G_{h,i}, G_{1,j}' \)): \( y(\sigma) \) ( \( x(\sigma) \) ) is closed to \( y^* \) ( \( x^* \) ) with large probability (with respect to \( \sigma \)). By condition (2) item 2-(d), \( \epsilon_0 \) is well defined and therefore \( \epsilon_0, \epsilon_1 \) are well defined. So the following lemmas make sense.

**Lemma 2.9.** Assume condition [22] holds. If \( \lambda < 0 \), then for any \( i \leq \frac{\delta n}{\lambda + \epsilon n}, j \leq 2 \left( \frac{\delta \lambda}{\lambda + \epsilon n} \right), h \in \{0,1\} \), with probability larger than \( 1 - 4 \cdot 2^{-n} \):

\[ P_{\sigma \sim IS(\beta, G_{h,i})} \left( \epsilon_0 < x(\sigma) \right), \quad P_{\sigma \sim IS(\beta, G_{1,j}')} \left( \epsilon_0 < x(\sigma) \right) \leq 2^{-n}, \]

\[ P_{\sigma \sim IS(\beta, G_{h,i})} \left( \left| y(\sigma) - y^* \right| > \epsilon_1 \right), \quad P_{\sigma \sim IS(\beta, G_{1,j}')} \left( \left| y(\sigma) - y^* \right| > \epsilon_1 \right) \leq 2^{-n}. \]

Combine (9) (7) with lemma (2.9) and after some tedious calculation, we are able to evaluate \( E \left[ Z(\beta, G_{h,i+1}) - Z(\beta, G_{h,i}) \right], E \left[ Z(\beta, G_{1,j}'+i) - Z(\beta, G_{h,i}) \right] \).
Lemma 2.10.

\[
\limsup_{n \to \infty} \sup_{i \leq \left\lfloor \frac{\delta n}{1+r} \right\rfloor} \mathbb{E} \left[ Z(\beta, G_{0,i+1}) - Z(\beta, G_{0,i}) \right] \leq \log 2 + (e^{-\beta} - 1)d \cdot \left[ -2 \max\{\epsilon_0, \epsilon_1\} + \frac{1+r - y^*r + y^*r\lambda}{1+r} \right],
\]

\[ (9) \]

\[
\limsup_{n \to \infty} \sup_{i \leq \left\lfloor \frac{\delta n}{1+r} \right\rfloor} \mathbb{E} \left[ Z(\beta, G_{1,i+1}) - Z(\beta, G_{1,i}) \right] \leq \log 2 + (e^{-\beta} - 1)d \cdot \left[ -2 \max\{\epsilon_0, \epsilon_1\} + \frac{y^*(r + \lambda)}{1+r} \right].
\]

\[ (10) \]

Lemma 2.11.

\[
\limsup_{n \to \infty} \sup_{i \leq \left\lfloor \frac{\delta n}{1+r} \right\rfloor} \mathbb{E} \left[ Z(\beta, G'_{1,j+1}) - Z(\beta, G'_{1,j}) \right] \leq (e^{-\beta} - 1) \left( 1 - 2y^*(1 - y^*) - 2\epsilon_1 \right).
\]

(11)

Now we can prove theorem 2.5 conclusion 2. By lemma 2.10

\[ \mathbb{E} \left[ Z(\beta, G'_0) - Z(\beta, \tilde{G}) \right] \geq -\beta d \cdot \left[ 2 \max\{\epsilon_0, \epsilon_1\} + \frac{1+r - y^*r + y^*r\lambda}{1+r} \right] \cdot \frac{\delta n}{(1+r)^2}. \]

(12)

\[ \mathbb{E} \left[ Z(\beta, G'_{1,0}) - Z(\beta, \tilde{G}) \right] \leq \left[ \log 2 + (e^{-\beta} - 1)d \cdot \left( -2 \max\{\epsilon_0, \epsilon_1\} + \frac{y^*(r + \lambda)}{1+r} \right) \right] \cdot \frac{\delta n}{(1+r)^2}. \]

It is obvious that,

\[ \mathbb{E} \left[ Z(\beta, G'_{1,0}) - Z(\beta, G'_{1,0}) \right] \leq \beta d|\lambda| \frac{\delta n}{2(1+r)^2}. \]

(13)

And by lemma 2.11

\[ \mathbb{E} \left[ Z(\beta, G'_1) - Z(\beta, G'_{1,0}) \right] \leq (e^{-\beta} - 1) \left( 1 - 2y^*(1 - y^*) - 2\epsilon_1 \right) \cdot d|\lambda| \frac{\delta n}{2(1+r)^2}. \]

(14)
In summary of (12) (13) (14),

\[
\mathbb{E}[Z(\beta, G'_0) - Z(\beta, G'_1)] \\
\geq -\beta d \cdot \left( 2 \max\{\epsilon_0, \epsilon_1\} + \frac{1 + r - y^* r + y^* r \lambda}{1 + r} \right) \cdot \frac{\delta n}{(1 + r)^2} \\
- \left[ \log 2 + (e^{-\beta} - 1) d \cdot \left( -2 \max\{\epsilon_0, \epsilon_1\} + \frac{y^*(r + \lambda)}{1 + r} \right) \right] \cdot \frac{\delta n}{(1 + r)^2} \\
- \beta d |\lambda| \frac{\delta n}{2(1 + r)^2} \\
- (e^{-\beta} - 1) \left( 1 - 2y^*(1 - y^*) - 2\epsilon_1 \right) \cdot d |\lambda| \frac{\delta n}{2(1 + r)^2} \\
\geq \frac{\delta dn}{(1 + r)^2} \left( -\beta \cdot \left( \frac{1 + r - y^* r + y^* r \lambda}{1 + r} \cdot \frac{y^*(r + \lambda)}{1 + r} + |\lambda| y^*(1 - y^*) \right) \right. \\
- 2(e^{-\beta} + \beta - 1) - \frac{\log 2}{d} - 4(1 - e^{-\beta} + \beta) \cdot \max\{\epsilon_0, \epsilon_1\} \right).
\]

Intuitively, the dominating term is \(-\beta \cdot \left( \frac{1 + r - y^* r + y^* r \lambda}{1 + r} - \frac{y^*(r + \lambda)}{1 + r} + |\lambda| y^*(1 - y^*) \right)\), which is of order \(\beta\).

It is helpful to recall that \(\beta = o(1), \max\{\epsilon_0, \epsilon_1\} = o(1), \frac{1}{\lambda} = o(1), \beta d >> 1\). The last three terms are of order \(O(\beta^2), O(1/d), o(\beta)\), and are thus ignorable compared to \(\beta\).

The dominating term is,

\[
r \leq 1 - 2\lambda \implies y^* = 1 \implies \frac{1 + r - y^* r + y^* r \lambda}{1 + r} - \frac{y^*(r + \lambda)}{1 + r} + |\lambda| y^*(1 - y^*) = (r - 1)(\lambda - 1) < 0;
\]

\[
r > 1 - 2\lambda \implies y^* = \frac{r + 1}{2(r + \lambda)} \implies \frac{1 + r - y^* r + y^* r \lambda}{1 + r} - \frac{y^*(r + \lambda)}{1 + r} + |\lambda| y^*(1 - y^*) = \frac{\lambda(1 + r)^2}{4(r + \lambda)^2} < 0.
\]

Also note by condition 2.2 item 1, \(e^{-\beta} + \beta - 1 \leq \beta^2\) and \(\beta^2 \leq \beta\). Thus continue (15),

\[
\mathbb{E}[Z(\beta, G'_0) - Z(\beta, G'_1)] \\
\geq \delta n \cdot \frac{\beta d}{(1 + r)^2} \cdot \left( \min\left\{ \frac{(r - 1)(1 - \lambda)}{1 + r}, \frac{|\lambda|(1 + r)^2}{4(r + \lambda)} \right\} - \left( 2\beta + \frac{\log 2}{\beta d} + 12 \max\{\epsilon_0, \epsilon_1\} \right) \right).
\]

The conclusion 2 of theorem 2.5 thus follows.

\[\square\]

2.3 Clustering when \(\lambda < 0\)

Recall that \(\pi_0 = \frac{1}{1 + r}, \pi_1 = \frac{r}{1 + r}\) and \(r \geq 1\). We provide the following random clustering algorithm: given the observed graph \(G_n \sim \text{SBM}(d, \lambda, r, n)\), sample a \(\sigma \sim IS(\sum_{i=0}^{d} G_n)\); let \(l' = \text{arg max}_{l \in \{0, 1\}} |\sigma^{-1}(l)|\); the estimator for the community label is, \(\tau(u) = I(u \in \sigma^{-1}(l'))\). Let \(M_0, M_1\) denote the two communities of \(G_n\) with \(\mathbb{E}[|M_0|] = \pi_0 n\).
Recall from proposition 2.4 conclusion 5 the definition of \( C_1(\lambda) \).

**Theorem 2.12.** If \( \lambda < 0, d \geq C_1(\lambda) \), then the estimator \( \tau(\cdot) \) is positively correlated to the true labeling since

\[
P\left( \left| \frac{\sigma^{-1}(l') \cap M_0}{|M_0|} - \frac{x^*}{1 + r} \right| \leq \epsilon_0, \left| \frac{\sigma^{-1}(l) \cap M_1}{|M_1|} - \frac{y^* r}{1 + r} \right| \leq \epsilon_1 \right) = 1 - e^{-\Omega(n)}.
\]

**Proof.** By proposition 2.4 conclusion 5 and consistency of \( \hat{\tau} \), condition 2.2 holds for \( \beta = \frac{1}{\sqrt{d}}, d, r, \lambda \) with large probability. Therefore, by lemma 2.9

\[
P\left( (\exists l \in \{0, 1\}) \left| \frac{\sigma^{-1}(l) \cap M_0}{|M_0|} - \frac{x^*}{1 + r} \right| \leq \epsilon_0, \left| \frac{\sigma^{-1}(l) \cap M_1}{|M_1|} - \frac{y^* r}{1 + r} \right| \leq \epsilon_1 \right) \geq 1 - 2^{-n+1}.
\]

But for the \( l \in \{0, 1\} \) with \( \left| \frac{\sigma^{-1}(l) \cap M_0}{|M_0|} - x^* \right| \leq \epsilon_0, \left| \frac{\sigma^{-1}(l) \cap M_1}{|M_1|} - y^* \right| \leq \epsilon_1 \), by large deviation theorem for \( |M_1| \), we have that with probability \( 1 - e^{-\Omega(n)} \), \( \frac{\sigma^{-1}(l)}{n} \geq \frac{1}{2} \) since by condition 2.2 item 2-(b) \( \frac{(\epsilon^* - \epsilon_l) r}{1 + r} > \frac{1}{2} \). i.e., with probability \( 1 - e^{-\Omega(n)} \), \( l = l' \). The proof is thus accomplished.

\[
\square
\]

In practice, a variety of techniques are available to sample \( \sigma \sim IS(\beta, G) \), for example the MCMC sampling.

### 3 Proof of lemmas

#### 3.1 Proof of proposition 2.4

Conclusions 2, 3, 4 follow by direct computation. We only give a sketched proof of item 1 and 5. Let \( f(x, y) = [\lambda r(x - y)^2 + (x + ry - \frac{1+r}{2})^2] + \frac{(1+r)^2}{4} \). It suffices to show that the condition for \( x, y \) implies \( f(x, y) - C(r, \lambda) \geq (r + 1)^2 \left( \frac{2 \log^2 \frac{r+1}{2}}{r+1} + 2 \epsilon_0 \right) \).

For the first half of conclusion 1, suppose the minimum of \( f(x, y) \) on \( x \in [\epsilon_0, 1], y \in [\frac{1}{2}, 1] \) is attained at \( x', y' \), then either \( x' \) lies on the boundary of \( [\epsilon_0, 1] \) or \( y' \) lies on the boundary of \( [\frac{1}{2}, 1] \). Therefore it is easy to check that the minimum must be attained at either \( (\epsilon_0, \hat{y}) \) or \( (\frac{1}{2}, \frac{1}{2}) \) for some \( \hat{y} \). Clearly, we only have to deal with the case \( (\epsilon_0, \hat{y}) \) since \( f\left( \frac{1}{2}, \frac{1}{2} \right) \geq C(r, \lambda) + (r + 1)^2 \left( \frac{2 \log^2 \frac{r+1}{2}}{r+1} + 2 \epsilon_0 \right) \) by condition 2.2 item 2-(a). If \( \hat{y} = y^* \) then the conclusion follows by definition of \( \epsilon_0 \left( = 2 \left( \frac{r}{r+1} \right)^2 \frac{2 \log^2 \frac{r+1}{2}}{r+1} + 2 \epsilon_0 \right) \) and the fact \( \frac{\partial}{\partial x} f(0, y^*) > 0 \). \( \frac{\partial^2}{\partial y^2} f(x, y) = 2(1 + r \lambda) > 0. \)

If \( \hat{y} = \frac{1}{2} \) then \( f(\epsilon_0, \hat{y}) \geq f\left( \frac{1}{2}, \frac{1}{2} \right) \geq C(r, \lambda) + (r + 1)^2 \left( \frac{2 \log^2 \frac{r+1}{2}}{r+1} + 2 \epsilon_0 \right) \). If \( \hat{y} \neq y^* \wedge \hat{y} > \frac{1}{2} \) then it must be the case \( \frac{\partial}{\partial y} f(\epsilon_0, \hat{y}) = 0 \wedge \frac{\partial}{\partial y} f(\epsilon_0, y^*) > 0 \) since \( \frac{\partial}{\partial y} f(0, y^*) \leq 0 \) and \( \frac{\partial^2}{\partial y^2} f(x, y) > 0 \). This implies \( \hat{y} \leq y^* \) since \( \frac{\partial^2}{\partial y^2} f(x, y) > 0 \). Moreover,

\[
\frac{\partial}{\partial y} f(\epsilon_0, \hat{y}) - (\hat{y} - y^*) \frac{\partial^2}{\partial y^2} f(x, y) - \epsilon_0 \frac{\partial^2}{\partial y^2} f(x, y) = \frac{\partial}{\partial y} f(0, y^*) \leq 0.
\]

So \( y^* - \hat{y} \leq \frac{\epsilon_0 r(1-\lambda)}{2(1+r \lambda)} \). Also note that for any \( x \in [0, \epsilon_0], y \in [\hat{y}, y^*], \frac{\partial}{\partial y} f(x, y) \leq \epsilon_0 \frac{\partial^2}{\partial y^2} f(x, y) = \epsilon_0 \cdot 2r(1-\lambda) \). Therefore, \( f(\epsilon_0, y^*) - f(\epsilon_0, \hat{y}) \leq (y^* - \hat{y}) \frac{\partial}{\partial y} f(\epsilon_0, y^*) \leq \epsilon_0 \frac{2r^2(1-\lambda)^2}{x^2 + \epsilon_0} \leq (r + 1)^2 \left( \frac{2 \log^2 \frac{r+1}{2}}{r+1} + 2 \epsilon_0 \right) \). (the last inequality...
follows from condition 2.2 item 2-(c)). Therefore by definition of $\epsilon_0$,

$$f(\epsilon_0, \tilde{y}) - C(r, \lambda) \geq \epsilon_0 \frac{\partial}{\partial x} f(0, y^*) - (f(\epsilon_0, y^*) - f(\epsilon_0, \tilde{y})) \geq (r + 1)^2 \left( \frac{2 \log 2}{\beta d} + 2 \varepsilon_0 \right).$$

For the second half of the conclusion 1, note that by condition 2.2 item 2-(b) (which implies $2r(1 - \lambda)(y^* - \epsilon_1) - (r + 1) > 0$) and the fact $\frac{\partial^2}{\partial x \partial y} f(x, y) = 2r(1 - \lambda) > 0$, we have $0 < \frac{\partial}{\partial x} f(0, y^* - \epsilon_1) < \frac{\partial}{\partial x} f(0, y^* + \epsilon_1)$. So by the fact $\frac{\partial^2}{\partial x \partial y} f(x, y), \frac{\partial^2}{\partial x \partial y} f(x, y) > 0$ the minimum of $f(x, y)$ on $x \in [0, 1], |y - y^*| \geq \epsilon_1, y \in [\frac{1}{2}, 1]$ is attained at either of following points: $(0, y^* - \epsilon_1), (\frac{1}{2}, \frac{1}{2})$, and $(0, y^* + \epsilon_1)$ if $y^* + \epsilon_1 \leq 1$ of course. It is clear that by definition of $\epsilon_1$,

$$\frac{1}{2} \epsilon_1^2 \frac{\partial^2}{\partial y^2} f(x, y), \epsilon_1 |\frac{\partial}{\partial y} f(0, y^*)| > (r + 1)^2 \left( \frac{2 \log 2}{\beta d} + 2 \varepsilon_0 \right).$$

Thus, $f(0, y^* - \epsilon_1) \geq C(r, \lambda) + (r + 1)^2 \left( \frac{2 \log 2}{\beta d} + 2 \varepsilon_0 \right)$ and $f(0, y^* + \epsilon_1) \geq C(r, \lambda) + (r + 1)^2 \left( \frac{2 \log 2}{\beta d} + 2 \varepsilon_0 \right)$ if $y^* + \epsilon_1 \leq 1$. By condition 2.2 item 2-(a), $f(\frac{1}{2}, \frac{1}{2}) \geq C(r, \lambda) + (r + 1)^2 \left( \frac{2 \log 2}{\beta d} + 2 \varepsilon_0 \right)$. Thus the second half of conclusion 1 follows.

Now we prove conclusion 5. Note that, for any $\lambda < 0$, the following quantities are bounded away from 0 on $r \in [1, \infty)$: $\min \{ \frac{r + 1}{2(r + \lambda)}, 1 \} - \frac{r + 1}{2r} (\frac{1 + r}{2}) = C(r, \lambda), r^2 + r \lambda, \min \{ r - 2r \lambda - 1, \frac{-\lambda(1 + r)}{r + \lambda} \}, C(r, \lambda)$. Therefore, if $\frac{1}{2 \sqrt{d}} \leq \beta \leq \frac{2}{\sqrt{d}}$, then there exists a constant depending on $\lambda$, namely $C_1(\lambda) > 0$, such that for any $r \geq 1$, any $d \geq C_1(\lambda)$, $d, r, \lambda$ satisfy condition 2.2.

If $\frac{1}{2 \sqrt{d}} \leq \beta \leq \frac{2}{\sqrt{d}}$, then for large $d$, we have, uniformly in $r$, $\epsilon_0 = O(\frac{1}{\sqrt{d}}), \epsilon_0 = O(\frac{1}{\sqrt{d}})$, $\epsilon_1 = O(\frac{1}{d^2})$. Note that when $r$ is close to 1, it is possible that $C(d, r, \lambda, \frac{1}{\sqrt{d}}) > 0$ since $\min \{ \{ \frac{r - 1}{2r} \}, \frac{\sqrt{d}(1 + r)}{d(1 + \lambda)} \} = O(r - 1)$. But obviously, there exists a constant $C_2(\lambda)$ such that for any $d \geq C_1(\lambda)$, any $r - 1 \geq \frac{C_2(\lambda)}{d^2}$:

1. $C(d, \lambda, r, \frac{1}{\sqrt{d}}) < 0$;
2. $\forall 0 \leq r' < r \left( \int_{r'}^r C(d, \lambda, t, \frac{1}{\sqrt{d}}) dt < 0 \right)$.

Thus the conclusion follows.

### 3.2 Proof of proposition 2.8

Let $X_{ij}(Y_{ij}), 1 \leq i < j \leq n$ denote a set of random variables whose joint distribution is the law of $I(\{i, j\} \in E(G_0')), 1 \leq i < j \leq n \ (I(\{i, j\} \in E(G_0'))), 1 \leq i < j \leq n$).

To prove the conclusion for $G_0'$, let $G \sim SBM(d, \lambda, r, n); let Z_{ij} = I(\{i, j\} \in E(G)) and denote by $M_0, M_1$ the two random communities of $G$. Without loss of generality suppose

$$M_1 = \{ \frac{n}{1 + r} - \frac{\delta n}{(1 + r)^2} + 1, \frac{n}{1 + r} - \frac{\delta n}{(1 + r)^2} + 2, \cdots, \frac{n}{1 + r} - \frac{\delta n}{(1 + r)^2} + |M_1| \}.$$

Note that $X_{ij}, i, j \leq n$ are mutually independent. It is easy to see that we can couple $X_{ij}, 1 \leq i < j \leq n$ with $Z_{ij}, 1 \leq i < j \leq n$ in the following way,
• $(X_{ij}, Z_{ij}), i, j \leq n$ are mutually independent;

• if $E[X_{ij}] \leq E[Z_{ij}]$, then $P(Z_{ij} = 1|X_{ij} = 1) = 1, P(Z_{ij} = 1|X_{ij} = 0) = \frac{E[Z_{ij} - X_{ij}]}{1-E[Z_{ij}]}$;

• if $E[X_{ij}] > E[Z_{ij}]$, then $P(X_{ij} = 1|Z_{ij} = 1) = 1, P(Z_{ij} = 1|X_{ij} = 0) = \frac{E[X_{ij} - Z_{ij}]}{1-E[Z_{ij}]}$.

Let $N_{\Delta} = \{n - \lfloor \frac{\delta n}{1+r} \rfloor + 1, \ldots, n\}$. Then we have,

$$E \left[ \sum_{1 \leq i < j \leq n} |X_{ij} - Z_{ij}| \right] \leq 2d \cdot \left| M_1 \right| - \left| N_1 \right| + O(\delta^2 n).$$

Thus,

$$E_{M_0, M_1} \left[ E \left[ \left| Z(\beta, G^0) - Z(\beta, G) \right| \left| M_0, M_1 \right| \right] \right] \leq |\beta| \cdot E_{M_0, M_1} \left[ E \left[ \sum_{1 \leq i < j \leq n} \left| X_{ij} - Z_{ij} \right| \right] \right] \leq O(\delta^2 n) + 2d \cdot E_{M_0, M_1} \left[ \left| M_1 \right| - \left| N_1 \right| \right] = O(\delta^2 n + \sqrt{n}).$$

The conclusion for $G^0_1$ is done.

The proof for $G'_1$ is a little harder because $I(\{i, j\} \in E(G'_1))$ are not mutually independent. Let $G \sim SBM(d, \lambda, r + \lambda, n)$ and $M_0, M_1$ be the two communities as above. The coupling of $Z_{ij}, Y_{ij}$ when $(i, j) \in (N_0 \times N_1) \cup (N_1 \times N_0)$ or $i \in N_{\Delta}$ or $j \in N_{\Delta}$ are the same as previous. In order to couple the rest of $Z_{ij}, Y_{ij}$, consider an auxiliary graph $G''_1$ which is obtained from $G_{1,1}^{\frac{\delta |M_1|}{1+r\lambda}}$ in the following way: delete each edge in $e_{G_{1,1}^{\frac{\delta |M_1|}{1+r\lambda}}}(N_0, N_0)$ independently with probability $\frac{\delta |M_1|}{1+r\lambda}$, add each disconnected pairs in $N_1$ independently to $E(G_{1,1}^{\frac{\delta |M_1|}{1+r\lambda}})$ with probability $\frac{\delta |M_1|}{r \lambda n}$. Let $Y''_{ij} = I(\{i, j\} \in E(G''_1))$. Note that $Y''_{ij}$ are mutually independent. So clearly we can couple $Y''_{ij}, 1 \leq i < j \leq n$ with $Z_{ij}, 1 \leq i < j \leq n$ such that

$$E \left[ \sum_{1 \leq i < j \leq n} \left| Y''_{ij} - Z_{ij} \right| \left| M_0, M_1 \right| \right] \leq 2d \cdot \left| M_1 \right| - \left| N_1 \right| - \left| N_{\Delta} \right| + O(\delta^2 n).$$

Therefore

$$E \left[ \left| Z(\beta, G''_1) - Z(\beta, G) \right| \right] = O(\sqrt{n} + \delta^2 n).$$

Now it remains to couple $Y''_{ij}, Y_{ij}$ for $(i, j) \in (N_0 \times N_0) \cup (N_1 \times N_1)$. We demonstrate the coupling for $Y''_{ij}, Y_{ij}$ with $(i, j) \in N_0 \times N_0$ as follows.

• To generate $Y''_{ij}$, let $Y'' \sim Bin(\left| e_{G_{1,0}^{\frac{\delta |M_1|}{1+r\lambda}}}(N_0, N_0) \right|, \frac{\delta |M_1|}{1+r\lambda})$;

• delete $Y''$ many edges in $e_{G_{1,0}^{\frac{\delta |M_1|}{1+r\lambda}}}(N_0, N_0)$ uniformly at random;

• $Y''_{ij} = 1$ if and only if $\{i, j\} \in e_{G_{1,0}^{\frac{\delta |M_1|}{1+r\lambda}}}(N_0, N_0)$ and $\{i, j\}$ is not deleted.
Clearly

\[ E \left[ \sum_{\{i,j\} \in e_{G_1,0}(N_0, N_0)} |Y_{ij} - Y''_{ij}| \right] = \min \left\{ \left[ \frac{\delta nd}{2(1+r)^2}, |e_{G_1,0}(N_0, N_0)| \right] - Y'' \right\}. \]

However, it is obvious that

\[ E \left[ \left| \min \left\{ \left[ \frac{\delta nd}{2(1+r)^2}, |e_{G_1,0}(N_0, N_0)| \right] - Y'' \right\} \right| \right] = O(\sqrt{n} + \delta^2 n). \]

Thus the proof is accomplished.

### 3.3 Proof of lemma 2.9

We only prove that with probability larger than \( 1 - 2^{-n} \), \( P_{\sigma \sim IS(\beta, G_{h,i})} \left( \epsilon_0 < x(\sigma) \right) < 2^{-n} \). The other conclusions follow in the same fashion.

Firstly we prove a large deviation result for \( J(\sigma; G) \) with \( G \) generated by a SBM. Recall from [3] the definition of \( g(\cdot) \). Fix any \( \sigma \in \{0, 1\}^{V(G)} \), we have for any \( 0 \leq \epsilon \),

\[ P_{G} \left( \left| J(\sigma; G) - E_{G}[J(\sigma; G)] \right| > \epsilon E_{G}[J(\sigma; G)] \right) \leq \exp \left\{ -E_{G}[J(\sigma; G)] \cdot g(\epsilon) \right\}. \] (16)

The proof of [10] follows by standard use of Chernoff inequality and a calculation of \( E_{G}(e^{\theta J(\sigma; G)}) \) as follows. Let \( P(\{u, v\} \in E(G)) = \frac{P_{uv}}{n} \), for any \( \theta \in \mathbb{R} \),

\[ E_{G}[e^{\theta J(\sigma; G)}] = \prod_{u < v \in V(G)} \left( (1 + (e^{\theta} - 1) \frac{P_{uv}}{n}) \right) \leq \exp \left\{ (e^{\theta} - 1) \sum_{u < v \in V(G)} \frac{P_{uv}}{n} \right\} = \exp \left\{ (e^{\theta} - 1) E_{G}[J(\sigma; G)] \right\}. \]
Combine (19) with Borel Cantali’s lemma we have,

\[ P \left( \exists \sigma \in \{0, 1\}^{V(G_{h,i})} \mid J(\sigma; G_{h,i}) - \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] \right) > \epsilon \cdot \left( \sup_{\sigma} \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] \right) \]

\[ \leq 2^n \exp \left\{ - g(\epsilon) \cdot \left( \inf_{\sigma} \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] \right) \right\} . \]  

We take advantage of the following evaluation for \( \frac{1}{n} \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] \) (which clearly follows from the construction of \( G_{h,i} \)):

\[ \left| \frac{1}{n} \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] - \frac{1}{2} \left( \frac{s(\sigma)}{1 + r/y(\sigma)} \right) \left( \begin{array}{cc} d(1 + r \lambda) & d(1 - \lambda) \\ d(1 - \lambda) & d(1 + \frac{r}{y(\sigma)}) \end{array} \right) \right| \]

\[ \leq \frac{1}{n} \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] - [r \lambda (x(\sigma) - y(\sigma))^2 + (x(\sigma) + y(\sigma) - \frac{1 + r}{2})^2] \frac{d}{(1 + r)^2} - \frac{d}{4} \]

\[ \leq \delta d . \]

Thus using (19) (and since \( \delta \) is sufficiently small)

\[ \inf_{\sigma \in \{0, 1\}^{V(G_{h,i})}} \frac{1}{n} \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] \geq \frac{d}{(1 + r)^2} C(r, \lambda) - d \delta \geq \frac{d C(r, \lambda)}{2(1 + r)^2} \]

\[ \sup_{\sigma \in \{0, 1\}^{V(G_{h,i})}} \frac{1}{n} \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] \leq \frac{d}{2} + d \delta \leq \frac{3d}{4} . \]

Substituting (19) into (17),

\[ P \left( \exists \sigma \in \{0, 1\}^{V(G_{h,i})} \mid J(\sigma; G_{h,i}) - \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] \right) > \frac{3 \epsilon}{4} \cdot d n \]

\[ \leq \exp \left\{ \log 2 \cdot n - g(\epsilon) \cdot \frac{d C(r, \lambda)}{2(1 + r)^2} \cdot n \right\} . \]

By condition 2.2 item 2-(d) \( \epsilon_0 \) is well defined. So substituting \( \epsilon \) by \( \epsilon_0 \) in (20) and by definition of \( \epsilon_0 \) we have,

\[ P \left( \exists \sigma \in \{0, 1\}^{V(G_{h,i})} \mid J(\sigma; G_{h,i}) - \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] \right) > \frac{3 \epsilon_0}{4} \cdot d n \]

\[ \leq e^{- \log 2 \cdot n} . \]

By condition 2.2 item 2-(d) \( \epsilon_0 \) is well defined. So substituting \( \epsilon \) by \( \epsilon_0 \) in (20) and by definition of \( \epsilon_0 \) we have,

\[ P \left( \exists \sigma \in \{0, 1\}^{V(G_{h,i})} \mid J(\sigma; G_{h,i}) - \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] \right) > \frac{3 \epsilon_0}{4} \cdot d n \]

\[ \leq e^{- \log 2 \cdot n} . \]

By condition 2.2 item 2-(d) \( \epsilon_0 \) is well defined. So substituting \( \epsilon \) by \( \epsilon_0 \) in (20) and by definition of \( \epsilon_0 \) we have,

\[ P \left( \exists \sigma \in \{0, 1\}^{V(G_{h,i})} \mid J(\sigma; G_{h,i}) - \mathbb{E}_{G_{h,i}}[J(\sigma; G_{h,i})] \right) > \frac{3 \epsilon_0}{4} \cdot d n \]

\[ \leq e^{- \log 2 \cdot n} . \]

Now we can prove lemma 2.9. Clearly for any \( \sigma' \),

\[ P_{\sigma \sim IS(\beta, G_{h,i})} \left( \epsilon_0 < x(\sigma) \right) < \frac{\sum \exp \{-\beta J(\sigma; G_{h,i})\}}{\exp \{-\beta J(\sigma'; G_{h,i})\}} . \]
Set $\sigma'$ to be any element of $\{0,1\}^{V(G_{h,i})}$ satisfying:

$$\left| \mathbb{E}_{G_{h,i}}[J(\sigma';G_{h,i})] - \inf_{0 \leq x,y \leq 1} \left\{ |\lambda (x-y) + (x+ry - \frac{1+r}{2})^2| \frac{d}{(1+r)^2} + \frac{d}{4} \right\} \right| \leq \delta d.$$ 

By (18) such $\sigma'$ exists. By (21), with probability larger than $1 - 2^{-n}$, for all $\sigma$, approximating $J(\sigma;G_{h,i})$ by $\mathbb{E}_{G_{h,i}}[J(\sigma;G_{h,i})]$ introduce an error smaller than $\frac{3\rho d}{4}$. Moreover, by (18), for all $\sigma$ approximating $\mathbb{E}_{G_{h,i}}[J(\sigma;G_{h,i})]$ by $|\lambda (x(\sigma) - y(\sigma)) + (x(\sigma) + ry(\sigma) - \frac{1+r}{2})^2| \frac{d}{(1+r)^2} + \frac{d}{4}$ introduce an error smaller than $\delta d$. Therefore, using (22) we have, for all $i \leq \frac{n}{1 + \epsilon \lambda}$, the following event occurs with probability larger than $1 - 2^{-n}$:

$$\Pr_{\sigma \sim IS(\beta,G_{h,i})} \left( \epsilon_0 < x(\sigma) \right) \leq 2^n \exp \left\{ -\beta n \left( \inf_{x,y} \left\{ |\lambda (x-y) + (x+ry - \frac{1+r}{2})^2| \frac{d}{(1+r)^2} + \frac{d}{4} \right\} - d \delta - d \frac{3\epsilon_0}{4} \right) \right\} \cdot \exp \left\{ \beta n \left( \inf_{0 \leq x,y \leq 1} \left\{ |\lambda (x-y) + (x+ry - \frac{1+r}{2})^2| \frac{d}{(1+r)^2} + \frac{d}{4} \right\} + d \delta + d \frac{3\epsilon_0}{4} \right) \right\} \text{ by proposition (24) conclusion 1 and since } 2\delta < \frac{\epsilon_0}{2} \leq \exp \left\{ \left[ \log 2 + 2 \beta \epsilon_0 d - \left( 2 \beta \epsilon_0 d + 2 \log 2 \right) \right] \cdot n \right\} = 2^{-n}.$$

Similarly, with probability larger than $1 - 2^{-n}$

$$\Pr_{\sigma \sim IS(\beta,G_{h,i})} \left( |y(\sigma) - y^\ast| > \epsilon_1 \right) \leq 2^{-n}.$$

Thus the proof is accomplished.

### 3.4 Proof of lemma 2.10

We demonstrate the proof of (20) by analyzing (9) for $h = 0$. Fix an arbitrary $n,i$, we have to evaluate $\mathbb{E}_{G_0,i+1,\sigma(G_0,i)} \left[ e^{-\beta \epsilon_{i0}} + e^{-\beta \epsilon_{i1}} \right]$. Note that $e_{i}$ is independent whose distribution does not concern $\sigma$. Therefore,

$$\mathbb{E}_{G_0,i+1,\sigma(G_0,i)} \left[ e^{-\beta \epsilon_{i0}} \right] \leq \exp \left\{ (e^{-\beta} - 1) \cdot \left( \frac{|\sigma^{-1}(0) \cap N_0|}{n} \cdot d(1+r\lambda) + \frac{|\sigma^{-1}(0) \cap N_1|}{n} \cdot d(1-\lambda) \right) \right\},$$

$$\mathbb{E}_{G_0,i+1,\sigma(G_0,i)} \left[ e^{-\beta \epsilon_{i1}} \right] \leq \exp \left\{ (e^{-\beta} - 1) \cdot \left( \frac{|\sigma^{-1}(1) \cap N_0|}{n} \cdot d(1+r\lambda) + \frac{|\sigma^{-1}(1) \cap N_1|}{n} \cdot d(1-\lambda) \right) \right\}.$$
Note that,

\[ \left| \frac{1}{n} \sigma^{-1}(h) \cap N \right| - \left| \frac{1}{n+1} \sigma^{-1}(h) \cap N \right|, \quad \left| \frac{1}{n} \sigma^{-1}(h) \cap N \right| - \left| \frac{1}{n+1} \sigma^{-1}(h) \cap N \right| \leq \frac{r \delta}{1+(r+1)} \] (24)

By lemma 2.9, with probability larger than \( 1 - 2^{-n} \), \( \mathbb{P}_{\sigma \sim IS(\beta,G_{n-1})} \{ |x(\sigma) - x^*| \leq \epsilon_0, |y(\sigma) - y^*| \leq \epsilon_1 \} \geq 1 - 2^{-n} \). Therefore continue (23) and approximate \( \frac{1}{n} \sigma^{-1}(h) \cap N \) with \( \frac{1}{n+1} \sigma^{-1}(h) \cap N \) with \( y(r \beta) \) (recall from (3) the definition of \( l, x(\sigma), y(\sigma) \)). By (24), we have that with probability \( 1 - e^{-\Omega(n)} \): for all \( i \leq \left[ \frac{\delta n}{1+(r+1)} \right] \),

\[
\mathbb{E}_{G_{n+1}, \sigma|G_{n+1}} \left[ e^{-\beta \epsilon_0} + e^{-\beta \epsilon_1} \right] \\
\leq \mathbb{E}_{G_{n+1}, \sigma|G_{n+1}} \left[ e^{-\beta \epsilon_0} 1 \cdot 1 + e^{-\beta \epsilon_1} \right] \left| x(\sigma) - x^* \right| \leq \epsilon_0, \left| y(\sigma) - y^* \right| \leq \epsilon_1 \\
+ 2 e^{2d(1-\lambda)} \mathbb{P}_{\sigma \sim IS(\beta,G_{n-1})} \{ |x(\sigma) - x^*| > \epsilon_0 \vee |y(\sigma) - y^*| > \epsilon_1 \} \\
\leq \mathbb{E}_{\sigma|G_{n+1}} \left\{ e^{-\beta} \right\} \left[ - d \delta + \frac{1}{1+r} \frac{1-x(\sigma)}{r} d(1+r) + \frac{1-y(\sigma)}{1+r} d(1-\lambda) \right] \\
+ \mathbb{E}_{\sigma|G_{n+1}} \left\{ e^{-\beta} \right\} \left[ - d \delta + \frac{1-x^*}{1+r} d(1+r) + \frac{1-y^*}{1+r} d(1-\lambda) \right] + e^{-\Omega(n)} \\
\leq \exp \left\{ (e^{-\beta} - 1)(- \max \{ \epsilon_0, \epsilon_1 \} d - \delta d) \right\} \\
\cdot \left\{ \exp \left\{ (e^{-\beta} - 1) \frac{x^*}{1+r} d(1+r) + \frac{y^*}{1+r} d(1-\lambda) \right\} \\
+ \exp \left\{ (e^{-\beta} - 1) \frac{1-x^*}{1+r} d(1+r) + \frac{1-y^*}{1+r} d(1-\lambda) \right\} \right\} + e^{-\Omega(n)} \\
\text{by proposition 2.4 conclusion 2} \\
\leq 2 \exp \left\{ (e^{-\beta} - 1) \left[ - 2 \max \{ \epsilon_0, \epsilon_1 \} + \frac{1+r-y^* x^* + y^*}{1+r} \right] \cdot d \right\}.
\]

Similarly, with probability \( 1 - e^{-\Omega(n)} \): for all \( i \leq \left[ \frac{\delta n}{1+(r+1)} \right] \),

\[
\mathbb{E}_{G_{n+1}, \sigma|G_{n+1}} \left[ e^{-\beta \epsilon_0} + e^{-\beta \epsilon_1} \right] \\
\leq \exp \left\{ (e^{-\beta} - 1)(- \max \{ \epsilon_0, \epsilon_1 \} d - \delta d) \right\} \\
\cdot \left\{ \exp \left\{ (e^{-\beta} - 1) \frac{x^*}{1+r} d(1+r) + \frac{y^*}{1+r} d(1-\lambda) \right\} \\
+ \exp \left\{ (e^{-\beta} - 1) \frac{1-x^*}{1+r} d(1+r) + \frac{1-y^*}{1+r} d(1-\lambda) \right\} \right\} + e^{-\Omega(n)} \\
\text{by proposition 2.4 conclusion 2} \\
\leq 2 \exp \left\{ (e^{-\beta} - 1) \left[ - 2 \max \{ \epsilon_0, \epsilon_1 \} + \frac{1+r-y^* x^* + y^*}{1+r} \right] \cdot d \right\}.
\]
Substitute (26) into (10) we have, with probability $1 - e^{-\Omega(n)}$: for all $i \leq \left[ \frac{\delta n}{1+\rho} \right]$, 

$$
E_{G_0, i+1 \mid G_0, i} \left[ Z(\beta, G_0, i+1) - Z(\beta, G_0, i) \right] 
\leq \log 2 + (e^{-\beta} - 1) \left[ -2 \max \{ \epsilon_0, \epsilon_1 \} + \frac{1 + r - y^*r + y^*r\lambda}{1 + r} \right] \cdot d,
$$

$$
E_{G_1, i+1 \mid G_1, i} \left[ Z(\beta, G_1, i+1) - Z(\beta, G_1, i) \right] 
\leq \log 2 + (e^{-\beta} - 1) \left[ -2 \max \{ \epsilon_0, \epsilon_1 \} + \frac{y^*(r + \lambda)}{1 + r} \right] \cdot d.
$$

Thus the conclusion of (10) follows.

Proving (10) is similar. Using lemma 2.9, proposition 2.4, conclusion 2 and approximating $\frac{\|\sigma^{-1}(G)Nn\|}{n}$ with $n(\sigma)^r$, $\frac{\|\sigma^{-1}(G)Nn\|}{n}$ with $n(\sigma)^r$, in the same way as (25), we have that with probability $1 - e^{-\Omega(n)}$: for all $i \leq \left[ \frac{\delta n}{1+\rho} \right]$, 

$$
E_{G_0, i+1 \mid G_0, i} \left[ \min \{ e^0_{i_0}, e^0_{i_1} \} \right] = E_{G_0, i+1 \mid G_0, i} \left[ \min \{ e^0_{i_1}, e^0_{i_0} \} \right] 
\leq \min \left\{ E_{G_0, i+1 \mid G_0, i} \left[ e^0_{i_0} \right], E_{G_0, i+1 \mid G_0, i} \left[ e^0_{i_1} \right] \right\} 
\leq \left[ 2 \max \{ \epsilon_0, \epsilon_1 \} + \frac{1 + r - y^*r + y^*r\lambda}{1 + r} \right] \cdot d,
$$

$$
E_{G_1, i+1 \mid G_1, i} \left[ \min \{ e^1_{i_0}, e^1_{i_1} \} \right] = E_{G_1, i+1 \mid G_1, i} \left[ \min \{ e^1_{i_1}, e^1_{i_0} \} \right] 
\leq \min \left\{ E_{G_1, i+1 \mid G_1, i} \left[ e^1_{i_0} \right], E_{G_1, i+1 \mid G_1, i} \left[ e^1_{i_1} \right] \right\} 
\leq \left[ 2 \max \{ \epsilon_0, \epsilon_1 \} + \frac{y^*(r + \lambda)}{1 + r} \right] \cdot d.
$$

Substituting (27) into (10), we have that with probability $1 - e^{-\Omega(n)}$: for all $i \leq \left[ \frac{\delta n}{1+\rho} \right]$, 

$$
E_{G_0, i+1 \mid G_0, i} \left[ Z(\beta, G_0, i+1) - Z(\beta, G_0, i) \right] 
\geq - \beta \left[ 2 \max \{ \epsilon_0, \epsilon_1 \} + \frac{1 + r - y^*r + y^*r\lambda}{1 + r} \right] d,
$$

$$
E_{G_1, i+1 \mid G_1, i} \left[ Z(\beta, G_1, i+1) - Z(\beta, G_1, i) \right] 
\geq - \beta \left[ 2 \max \{ \epsilon_0, \epsilon_1 \} + \frac{y^*(r + \lambda)}{1 + r} \right] d.
$$

Thus the conclusion of (10) follows.

### 3.5 Proof of lemma 2.11

Recall that $\{u_j, v_j\}$ denote the edge added at step $j$ in the construction of $G'_i$. Note that, 

$$
Z(\beta, G'_{i,j+1}) - Z(\beta, G'_{i,j}) = \log \left( E_{\sigma \sim IS(\beta,G'_i)} [e^{-\beta I(\sigma(u_j) = \sigma(v_j))}] \right).
$$
Using convexity of \( \log \) as in (6), we have,

\[
E_{G_1',j+1|G_1',j} \left[ Z(\beta, G_1',j+1) - Z(\beta, G_1',j) \right] \leq \log \left( E_{G_1',j+1|\sigma|G_1',j} \left[ e^{-\beta I(\sigma(u_j) = \sigma(v_j))} \right] \right).
\]  

(28)

And in the calculation of \( E_{G_1',j+1|\sigma|G_1',j} \), \( \sigma, G_1',j+1 \) are independent conditional on \( G_1',j \). In another word, \( \{u_j, v_j\} \perp \sigma \).

Thus,

\[
E_{G_1',j+1|\sigma|G_1',j} \left[ e^{-\beta I(\sigma(u_j) = \sigma(v_j))} \right] = E_{\sigma|G_1',j} \left[ E_{G_1',j+1|\sigma, G_1',j} \left[ e^{-\beta I(\sigma(u_j) = \sigma(v_j))} \right] \right]
\]

(29)

\[
= E_{\sigma|G_1',j} \left[ 1 + (e^{-\beta} - 1) \left( y(\sigma)^2 + (1 - y(\sigma))^2 \right) \right].
\]

(30)

By lemma 2.3, for all \( \left[ \frac{d\lambda}{2(1+r)^2} \right] + 1 \leq j \leq 2\left[ \frac{d\lambda}{2(1+r)^2} \right] \), with probability larger than \( 1 - 2^{-n} \):

\[
P_{\sigma \sim IS(\beta,G_1',j)} \left( |y(\sigma) - y^*| > \epsilon_1 \right) \leq 2^{-n}.
\]

Therefore continue (29) we have that with probability \( 1 - e^{-\Omega(n)} \): for all \( \left[ \frac{d\lambda}{2(1+r)^2} \right] + 1 \leq j \leq 2\left[ \frac{d\lambda}{2(1+r)^2} \right] \),

\[
E_{G_1',j+1|\sigma|G_1',j} \left[ e^{-\beta I(\sigma(u_j) = \sigma(v_j))} \right] \leq E_{\sigma|G_1',j} \left[ (e^{-\beta} - 1) \left( y(\sigma)^2 + (1 - y(\sigma))^2 \right) \right]
\]

\[
\leq 1 + (e^{-\beta} - 1) \left( 1 - 2y^*(1 - y^*) - 2\epsilon_1 \right) + 2^{-n}.
\]

Substituting (31) into (28) and using inequality \( \log(1 + x) \leq x \), the conclusion thus follows.

### 4 Concluding remarks

In this paper we evaluate the log partition function of the Ising model on the SBM with two communities. The evaluation yields a consistent estimator of the parameter \( r \). We also provide a random clustering algorithm with positive correlation to the true community label.

### 5 Acknowledgement

We would like to thank Huifeng Peng for helpful conversation on the topic. We would like to thank Jing Zhou for comments on a draft of this work.

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