A metal-insulator transition as a quantum glass problem

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We discuss a recent mapping of the Anderson-Mott metal-insulator transition onto a random field magnet problem. The most important new idea introduced is to describe the metal-insulator transition in terms of an order parameter expansion rather than in terms of soft modes via a nonlinear sigma model. For spatial dimensions $d > d_c^+ = 6$ a mean field theory gives the exact critical exponents. For $d = 6 - \varepsilon$ the critical exponents are identical to those for a random field Ising model. Dangerous irrelevant quantum fluctuations modify Wegner’s scaling law relating the conductivity exponent to the correlation or localization length exponent. This invalidates the bound $s \geq 2/3$ for the conductivity exponent $s$ in $d = 3$. We also argue that activated scaling might be relevant for describing the AMT in three-dimensional systems.

I. INTRODUCTION

Metal-insulator transitions of purely electronic origin, i.e., those for which the structure of the ionic background does not play a role, are commonly divided into two categories. In one category the transition is triggered by electronic correlations, or interactions, and in the other it is driven by disorder. The first case is known as a Mott transition, and the second one as an Anderson transition. It is believed that for many real metal-insulator transitions both correlations and disorder are relevant. The resulting quantum phase transition, which carries aspects of both types of transitions, we call an Anderson-Mott transition (AMT).

Until very recently virtually all approaches studied the AMT only in the vicinity of two dimensions by generalizing Wegner’s theory for the Anderson transition. Renormalization-group methods lead to a critical fixed point in $d = 2 + \varepsilon$ dimensions, and standard critical behavior with power-law scaling was found. However, the framework of these theories does not allow for an order parameter (OP) description of the AMT, and does not lead to a simple Landau or mean-field theory. As a result, the physics driving the AMT remains relatively obscure in this approach, compared to standard theories for other phase transitions. An alternative line of attack has recently been explored by the present authors. We have shown that an OP description of the AMT is possible with the tunneling density of states (DOS) as the OP. A simple Landau theory then yields the exact critical exponents, above the upper critical dimension, $d_c^+ = 6$. In this respect the AMT is conceptually simpler than the Anderson transition, which has no known simple OP description, and whose upper critical dimension may be infinite.

One of the most far-reaching implications of our approach is that the AMT is in some respects similar to magnetic transitions in random fields. Qualitatively, this can be understood as follows. Consider a model of an interacting disordered electron gas. In terms of anticommuting Grassmann fields, $\bar{\psi}$ and $\psi$, the action can be written

$$ S = S_k + S_{\text{dis}} + S_{\text{int}} \ ,$$

with,

$$ S_k = - \sum \sigma \int dx \bar{\psi}_\sigma(x) \left[ \partial_\tau - \frac{\nabla^2}{2m} - \mu \right] \psi_\sigma(x) \ ,$$

the kinetic or free part of $S$,

$$ S_{\text{dis}} = - \sum \sigma \int dx \ u(x) \bar{\psi}_\sigma(x) \psi_\sigma(x) \ ,$$

the disorder part of $S$, and
\[ S_{\text{int}} = -\frac{\Gamma}{2} \sum_{\sigma_1,\sigma_2} \int dx \psi_{\sigma_1}(x) \bar{\psi}_{\sigma_2}(x) \psi_{\sigma_2}(x) \psi_{\sigma_1}(x) \quad , \]  

(1.2c)

denoting the interaction part of \( S \). In these equations, \( x = (x, \tau) \) with \( \tau \) denoting imaginary time, \( \int dx \equiv \int dx \int_0^{1/T} d\tau, m \) is the electron mass, \( \mu \) is the chemical potential, \( \sigma \) is a spin label, and for simplicity we have assumed an instantaneous point-like electron-electron interaction with strength \( \Gamma \). \( u(x) \) is a random potential which represents the disorder. For simplicity we also assume \( u(x) \) to be \( \delta \)-correlated, and to obey a Gaussian distribution with second moment

\[ \{ u(x)u(y) \} = \frac{1}{2\pi N_F \tau_{\text{el}}} \delta(x-y) \quad , \]  

(1.3)

where the braces denote the disorder average, \( N_F \) is the bare DOS per spin at the Fermi energy, and \( \tau_{\text{el}} \) is the bare elastic mean-free time. For future use we write \( S_{\text{dis}} \) as,

\[ S_{\text{dis}} = -\sum_{n,\sigma} \int dx u(x) \bar{\psi}_{\sigma,n}(x) \psi_{\sigma,n}(x) \quad , \]  

(1.4)

where a Matsubara frequency decomposition of \( \bar{\psi}(\tau) \) and \( \psi(\tau) \) has been used.

As mentioned above, the most obvious candidate for an OP for the AMT is the single particle DOS, \( N \), at the Fermi level. In terms of Grassmann variables this quantity is proportional to the zero-frequency limit of the expectation value of the variable \( \bar{\psi}\psi \):

\[ N = \text{Im} N(i\omega_n \to 0 + i0) \quad , \]  

(1.5a)

with,

\[ N(i\omega_n) = \frac{-1}{2\pi N_F} \sum_{\sigma} \langle \bar{\psi}_{\sigma,n}(x) \psi_{\sigma,n}(x) \rangle \]  

(1.5b)

where we have normalized the DOS by \( 2N_F \), and the brackets denote an expectation value with respect to the action \( S \). Note that the so defined DOS is actually a local DOS, i.e. it depends on \( x \). Examining Eqs. (1.4) and (1.5), we see that the local OP for the AMT couples linearly to the random potential, and depending on the sign of \( u(x) \) it will favor either an increasing or a decreasing DOS. Similarly, \( S_{\text{int}} \sim -\Gamma N^2 \), i.e. \( S_{\text{int}} \) always favors a decreasing DOS. We conclude that the interaction term in general frustrates the disorder term, just like in a random field (RF) magnet problem.

This conclusion has a number of important implications. For example, if conventional scaling exists at the AMT, then one expects hyperscaling to be violated due to a dangerous irrelevant variable (DIV), as it is in RF magnets. As a consequence of this, we argue below that Wegner's scaling law relating the conductivity exponent \( s \) to the correlation length exponent \( \nu \) is modified. Furthermore, if the AMT shares all of the features known to be induced in magnets by a random field, then one would expect glasslike features and unconventional or activated scaling similar to what has been predicted and observed in classical RF magnets.

The plan of this paper is as follows. In Section II we give a sketch of our order-parameter theory of the AMT. An explicit scaling theory near \( d = 6 \) is constructed. In the first part of Section III we give a general scaling theory of the AMT, assuming it is a conventional phase transition. In the second part of this section we review some aspects of an activated scaling theory for the AMT. We conclude in Section IV with a short discussion.

II. FORMALISM AND MEAN FIELD THEORY

A. Formalism

Here we briefly review the formalism we have used to show that at least near \( d = 6 \), the AMT and the magnetic transition in a RF Ising model have many features in common. For details we refer to two recent papers.

Our starting point is the nonlinear sigma model (NL\( \sigma \)M) that has been used to describe the AMT near two dimensions. The solution procedure we use near the upper critical dimension is closely analogous to the treatment of the \( O(n) \) symmetric NL\( \sigma \)M in the limit of large \( n \). The NL\( \sigma \)M for the AMT is derived from Eq. (1.1) by assuming that all of the relevant physics can be expressed in terms of fluctuations of the particle number density, the spin density,
can be expanded in a restricted spin-quaterion basis, respectively. For simplicity we restrict ourselves to the particle-hole degrees of freedom. For this case the matrix elements

\[ \alpha \]

frequency labels, \( n \) and \( m \), and two replica labels, \( \tilde{\tau} \) with \( \tilde{\omega} \) in the quaternion basis, and

\[ \{ \tilde{\tau}, \tilde{\omega} \} \]

quaternion degrees of freedom describing the particle-hole (\( \tilde{\tau} \) with \( \tilde{\omega} \)) channels, respectively, and \( \Omega \) in the spin basis (\( \sigma \) and \( \beta \). The matrix elements are spin quaternions, with the quaternion degrees of freedom describing the particle-hole (\( \tilde{\tau} \sim \tilde{\psi} \tilde{\psi} \)) and particle-particle (\( \tilde{\tau} \sim \tilde{\psi} \tilde{\psi} \)) channels, respectively. For simplicity we restrict ourselves to the particle-hole degrees of freedom. For this case the matrix elements can be expanded in a restricted spin-quaterion basis,

\[ \tilde{Q}^\alpha_\beta = \sum_{r=0,3} \sum_{i=0}^3 i \tilde{Q}^\alpha_\beta (\tilde{\tau} \otimes s_i) \]

(2.2)

with \( \tau_{0,1,2,3} \) the quaternion basis, and \( s_{0,1,2,3} \) the spin basis (\( s_{1,2,3} = i \sigma_{1,2,3} \) with \( \sigma_{1,2,3} \) the Pauli matrices). The matrix \( \tilde{Q} \) is subject to the constraints

\[ \tilde{Q}^2 = 1 \]

(2.3a)

\[ tr \tilde{Q} = 0 \]

(2.3b)

\[ \tilde{Q}^+ = C^T \tilde{Q}^T C = \tilde{Q} \]

(2.3c)

where \( C = i \tau_3 \otimes s_2 \).

In Eq. (2.1a), \( G = 2/\pi \sigma \), with \( \sigma \) the bare conductivity, is a measure of the disorder, and \( H = \pi N_F/2 \) is a frequency coupling parameter. \( K_s \) and \( K_t \) are bare interaction amplitudes in the spin singlet and spin triplet channels, respectively, and \( \Omega_{nm}^{\alpha\beta} = \delta_{nm} \delta_{\alpha\beta} \omega_{n} \tau_0 \otimes s_0 \), with \( \omega_n = 2\pi \tau t_n \), is a bosonic frequency matrix. Notice that \( K_s < 0 \) for repulsive interactions.

The correlation functions of the \( \tilde{Q} \) determine the physical quantities. Correlations of \( \tilde{Q}_{nm} \) with \( nm < 0 \) determine the soft particle-hole modes associated with charge, spin and heat diffusion, while the DOS is determined by \( < \tilde{Q}_{nm}^\alpha_\beta > \), i.e., \( \tilde{Q}_{nm} \) with \( nm > 0 \). It is therefore convenient to separate \( \tilde{Q} \) into blocks.

\[ \tilde{Q}_{nm}^{\alpha\beta} = \Theta(nm) Q_{nm}^{\alpha\beta} + \Theta(n)\Theta(-m) q_{nm}^{\alpha\beta} + \Theta(-n)\Theta(m) (q^\dagger)_{nm}^{\alpha\beta} \]

(2.4)

Normally a NL\( \sigma \)M is treated by integrating out the massive modes, i.e., the \( Q_{nm} \), to obtain an effective theory for the massless modes, which are here the diffusion processes described by \( q \) and \( q^\dagger \). However, since our goal is to obtain a field theory for the OP for the AMT, \( Q_{nm} \), we instead integrate out the massless \( q \)-fields here.

Using standard techniques [4], the above program can be carried out. The resulting OP field theory for the AMT is:
Here where terms on the r.h.s. yields the correlation length exponent $\nu$ is the SP value of $\Lambda$. All remaining critical exponents can now be read off Eq. (2.10a). Comparing the first and mean-field solution in the critical region then follows from Eq. (2.5) as,

$$S[Q] = -\frac{1}{2G} \int dx \ tr \left[ (\nabla Q(x))^2 + (\Lambda \langle Q(x) \rangle)^2 \right] + 2H \int dx \ tr \left[ \Omega Q(x) + \frac{u}{2G^2} \int dx \ tr \left[ (1 - f) Q^2(x) \right] \right] - \frac{u}{4G^2} \int dx \ tr Q^4(x) - \frac{v}{4G^2} \int dx \ (tr + Q^2(x)) (tr - Q^2(x))^2 + \cdots ,$$

(2.5)

where $tr_\pm$ denotes ‘half-traces’ that sum over all replica labels but only over positive and negative frequencies, respectively: $tr_+ = \sum_\alpha \sum_{n\geq 0} \cdot tr_- = \sum_\alpha \sum_{n<0} \cdot f = f(\langle \Lambda \rangle)$ is a matrix with elements $i f_{nm} = \delta_{r\alpha} \delta_{\omega} \delta_{nm} f_n$ with $f_n > f_m > 0$ for $|n| < |m|$. $f_n$ is an increasing function of disorder, $G$, and $|K_s|$. $\langle \Lambda \rangle$ in Eq. (2.5) is proportional to $\Omega / <Q>$, and $u$ and $v$ are finite constants, at least for $d > 4$. In giving Eq. (2.5) we have neglected terms that can be shown to be renormalization group (RG) irrelevant near the AMT.

**B. Mean-field Theory**

Here we construct a mean-field or saddle-point (SP) solution of Eq. (2.3)\(2.4\) We look for solutions, $Q_{sp}$, that are spatially uniform and satisfy,

$$i r (Q_{sp})\alpha^\beta = \delta_{r\alpha} \delta_{\omega} \delta_{nm} N_{n}^{(0)} ,$$

(2.6)

where the subscript $(0)$ denotes the SP approximation. The replica, frequency, and spin-quaternion structures in Eq. (2.3) are due to the fact that $\langle i Q^3 \rangle$ has these properties, and that in the mean-field approximation averages are replaced by the corresponding SP values.

In the zero-frequency limit, the SP equation of state obtained from Eq. (2.6) is,

$$\left( N_{n=0}^{(0)} \right)^2 = 1 - f_{n=0}(\langle \Lambda \rangle) = \ell^{(0)} ,$$

(2.7a)

or

$$N_{n=0}^{(0)} = \left( \ell^{(0)} \right)^{\frac{1}{2}} .$$

(2.7b)

Here $\ell^{(0)}$ is the mean-field value of the distance from the critical point, $t$. Equation (2.7b) yields the mean-field value for the critical exponent $\beta$,

$$\beta = 1/2 .$$

(2.8)

To obtain the remaining mean-field critical exponents we expand $Q$ about its expectation value, which is proportional to $N$,

$$i r Q_{nm}^{\alpha\beta} = \delta_{r\alpha} \delta_{\omega} \delta_{nm} N_{n} + \sqrt{2G} i r \varphi_{nm}^{\alpha\beta} ,$$

(2.9)

where the factor of $\sqrt{2G}$ has been inserted for convenience. The action $S_G$ governing Gaussian fluctuations about the mean-field solution in the critical region then follows from Eq. (2.4) as,

$$S_G[\varphi] = -\int dx \ tr \left[ (\nabla \varphi(x))^2 + \ell^{(0)} \varphi^2(x) + \frac{2u}{G} \ell^{(0)} \varphi^2(x) \right] - \frac{v}{2G} \ell^{(0)} \int dx \ (tr + \varphi(x)) (tr - \varphi(x)) + O(\varphi^3) .$$

(2.10a)

Here

$$\ell^{(0)} = 2GH \omega_n / N_{n}^{(0)} ,$$

(2.10b)

is the SP value of $\Lambda$. All remaining critical exponents can now be read off Eq. (2.10a). Comparing the first and third terms on the r.h.s. yields the correlation length exponent $\nu = 1/2$. With $\ell^{(0)} \sim \omega / Q$, the first and second term gives the dynamical exponent $z = 3$. Finally, the $\varphi - \varphi$ correlation function near the transition has the standard
Ornstein-Zernike form, which yields the critical exponents $\gamma = 1$ and $\eta = 0$. We thus have standard mean-field values for all static exponents,

$$\beta = \nu = 1/2 \quad , \quad \gamma = 1 \quad , \quad \eta = 0 \quad , \quad \delta = 3 \quad ,$$

and for the dynamical exponent we have,

$$z = 3 \quad .$$

Inspection of Eq. (2.10a) shows that the AMT saddle point is a local minimum and therefore stable.

It is also possible to determine the critical behavior of the transport coefficients by directly computing the $q-q$ correlation functions and identifying the change, spin and heat diffusion coefficients ($D_c, D_s, D_h$). Near the mean-field AMT, all three of these coefficients behave in the same way and vanish like the OP,

$$D_a \sim N_{n=0/GH} \quad ,$$

with $a = c, s, h$.

The next step in the standard approach for describing any continuous phase transition is to introduce RG ideas.

In the problem considered here, application of the RG method accomplishes three things. First, it generates all additional terms in the action that are consistent with the symmetry of the problem. Second, it enables us to prove that there exists an upper critical dimension, $d^c +$, above which mean-field theory for the critical exponents is exact. Third, it enables us to do an $\varepsilon$-expansion below $d^c +$. We begin by noting that Eq. (2.5) does not have the RF term we argued for in the Introduction. A Wilson-type RG procedure generates this term as well as others. It has the form,

$$S_{RF} = \frac{\Lambda}{2} \int \! dx \sum_{i=\pm} (\text{tr}_i \varphi(x))^2 \quad .$$

In terms of the original fermion action, this contribution arises from a RF term of the form,

$$S_{RF} = \sum_{n, \sigma} \int \! dx \, h_n(x) \bar{\psi}_{\sigma,n}(x) \psi_{\sigma,n}(x) \quad ,$$

where $h_n(x)$ is a random field with

$$\{h_n(x) \ h_m(x)\} = \theta(nm) \frac{\Lambda}{4G} \delta(x - y) \quad .$$

All other terms generated by the renormalization process are irrelevant near the upper critical dimension, $d^c +$.

Standard arguments imply that such a RF term yields $d^c + = 6$ instead of the usual $d^c + = 4$. The same arguments also prove that the mean-field critical behavior quoted above is the exact critical behavior for $d > d^c + = 6$. For $d = 6 - \varepsilon$, an $\varepsilon$-expansion of the critical exponents is possible. The main idea is that under renormalization, the disorder $\Delta$ scales to infinity while $u$, the coefficient of the quartic term in Eq. (2.5), scales to zero such that their product

$$g = u \Delta \quad ,$$

scales to a stable fixed point value that is of $O(\varepsilon)$. In all of the other flow equations only the product $g$ appears so that a stable critical fixed point is obtained. To first order in $\varepsilon = 6 - d$, the resulting critical exponents are

$$\nu = \frac{1}{2} + \frac{\varepsilon}{12} + O(\varepsilon^2) \quad ,$$

$$\gamma = \frac{1}{2} - \frac{\varepsilon}{6} + O(\varepsilon^2) \quad ,$$

$$\delta = 3 + \varepsilon + O(\varepsilon^2) \quad ,$$

$$\eta = 0 + O(\varepsilon^2) \quad ,$$

$$z = 3 - \frac{\varepsilon}{2} + O(\varepsilon^2) \quad .$$

We finally mention that the RG flow properties, $\Delta \to \infty, u \to 0, g \sim O(\varepsilon)$, have an interesting physical interpretation. $\Delta$ represents the disorder, while $u$ is a measure of the importance of quantum fluctuations about the SP. Because $u$ determines physical quantities like the order parameter, this implies that quantum fluctuations are dangerously irrelevant near the OP driven AMT. This in turn modifies the standard hyperscaling equalities.
III. SCALING DESCRIPTIONS OF THE ANDERSON-MOTT TRANSITION

Based on the known or suspected behavior of the random field Ising model, there are two distinct scaling scenarios one can imagine for the AMT. The first is a conventional one that takes into account in a general way the dangerous irrelevant variable discussed in Section II. The second, strikingly different one is new in the context of metal-insulator transitions, and is called the activated scaling scenario.

A. Conventional scaling description of the AMT

There are standard ways to construct conventional scaling descriptions of either classical or quantum \((T = 0)\) phase transitions. For the random field like transition considered here, one of the most important features is the presence of a dangerous irrelevant variable (DIV), namely \(u\). Suppose that \(u\) is characterized by an exponent \(\theta\), defined so that

\[
 u(b) \sim b^{-\theta}.
\]

One-loop perturbation theory gives \(\theta = 2 + O(\varepsilon)\), but here we keep \(\theta\) general. This adds a third independent exponent to the usual two independent static exponents. In addition, there is the dynamical scaling exponent \(z\). For the case considered here it turns out that \(z\) is not independent, but rather it is equal to the scale dimension of the field conjugate to the OP. This is due to the fact that RF fluctuations are much more important than quantum fluctuations. The dominance of RF fluctuations compared to either thermal or quantum fluctuations is a general feature of RF problems. The net result, confirmed explicitly near \(d = 6\), is,

\[
z = y_h = \delta \beta / \nu.
\]  

As for the classical RF problem, the DIV \(u\), changes in all scaling relations to \(d - \theta\). For example, near the transition the OP obeys a scaling or homogeneity relation,

\[
 N(t, \Omega) = b^{-\beta/\nu} N(b^{1/\nu} t, b^z \Omega) ,
\]  

with \(\beta\) related to \(\nu\) and \(\eta\) by the usual scaling law, but with \(d \to d - \theta\) due to the violation of hyperscaling by the DIV,

\[
 \beta = \frac{\nu}{2} (d - \theta - 2 + \eta) .
\]  

Similarly, the exponents \(\delta\) and \(\gamma\) are given by,

\[
 \delta = (d - \theta + 2 - \eta) \nu / 2 \beta ,
\]

\[
 \gamma = \nu (2 - \eta) .
\]

Next we consider the transport coefficients. The charge, spin, or heat diffusion coefficients, which we denote collectively by \(D\), all scale like a length squared times a frequency, so that

\[
 D(t, \Omega) = b^{z - 2} D(t b^{1/\nu}, \Omega b^z) = t^\nu (z - 2) D(1, \Omega / t^\nu) .
\]  

Denoting the static exponent for the diffusion coefficient by \(s_D\), defined by \(D(t, \Omega = 0) \sim t^{s_D}\), we have found,

\[
 s_D = \nu (z - 2) = \beta - \nu \eta = \frac{\nu}{2} (d - 2 - \theta - \eta) .
\]  

The behavior of the electrical conductivity \(\sigma\), which is related to the charge diffusion coefficient by means of an Einstein relation, \(\sigma = D_c \partial n / \partial \mu\), depends on the behavior of \(\partial n / \partial \mu\). If \(\partial n / \partial \mu\) has a constant contribution at the AMT, then \(\sigma \sim t^s\) vanishes as \(D_c\), so that

\[
 s = s_D = \frac{\nu}{2} (d - 2 - \theta - \eta) .
\]

Dimensionally, however, all of the thermodynamic susceptibilities scale like an inverse volume times a time, which implies a singular part \((\partial n / \partial \mu)_s\) of \(\partial n / \partial \mu\) that scales like,

\[
 (\partial n / \partial \mu)_s (t, T) = b^{-d+\theta+z} (\partial n / \partial \mu)_s (tb^{1/\nu}, T b^z) .
\]
If there is no constant, analytic, background term, then Eqs. (3.4) and (3.6) give,

\[ s = \nu(d - 2 - \theta) \]  \hspace{1cm} (3.7)

In either case, Wegner’s scaling law \( s = \nu(d - 2) \) [12], which previously had been believed to hold for the AMT as well as for the Anderson transition, is violated, unless Eq. (3.5) holds and \( \theta = 2 - d - \eta \). Finally, we note that Eqs. (3.5) and (3.7) are identical if \( \eta = \theta + 2 - d \), and that this result is consistent with Wegner scaling apart from the replacement \( d \rightarrow d - \theta \), and with \( \partial n / \partial \mu \) being noncritical across the AMT. However, Eq. (3.21) shows that this result is not consistent with a vanishing OP unless the theory has multiple dynamical scaling exponents.

### B. Activated scaling description of the AMT

An important characteristic of a glass transition is the occurrence of extremely long time scales. While critical slowing down at an ordinary characteristic means that the critical time scale grows as a power of the correlation length, \( \tau \sim \xi^{\nu} \) with \( \nu \) the dynamical scaling exponent, at a glass transition the critical time scale grows exponentially with \( \xi \),

\[ \ln(\tau / \tau_0) \sim \xi^{\psi} \]  \hspace{1cm} (3.8)

with \( \tau_0 \) a microscopic time scale, and \( \psi \) a generalized dynamical scaling exponent. Effectively, Eq. (3.8) implies \( z = \infty \). As a result of such extreme slowing down, the system’s equilibrium behavior near the transition becomes inaccessible for all practical purposes. That is, realizable experimental time scales are not sufficient to reach equilibrium, and one says that the system falls out of equilibrium. It has been proposed[13] that the phase transition in classical RF magnets is of this type, and there are experimental observations that seem to corroborate this suggestion.

Here we speculate that the analogy between RF magnets and the AMT leads to such ‘activated’ scaling for the AMT as well. For this quantum phase transition one expects time and inverse temperature to show the same scaling behavior, irrespective of whether the critical slowing down follows an ordinary power law, or Eq. (3.8). Quantum mechanics thus makes it very difficult to observe the static scaling behavior, since it requires exponentially small temperatures. Thus from Eq. (3.8) we see that static zero temperature scaling will be observed only if

\[ T < T_0 \exp \left( -\xi^{\psi} \right) \]  \hspace{1cm} (3.9)

with \( T_0 \) some microscopic temperature scale on the order of the Fermi temperature \( \sim T_F \). This is potentially a crucial point in the interpretation of experimental data.

Activated scaling, as described by Eq. (3.8), follows from a barrier picture of the system’s free energy landscape. The physical idea we have in mind is that while a repulsive electron-electron interaction always leads to a decrease in the local DOS, the random potential can in general lead to an increase in the local DOS as well. The competition between these two effects leads to frustration and to, for example, large insulating clusters within the metallic phase. Delocalizing these large clusters requires energy barriers to be overcome, which are assumed to grow like \( \xi^{\psi} \) as the AMT is approached. A further notion of the barrier model is that the frequency or temperature argument of the scaling function is expected to be \( \ln(\tau / \tau_0) / \ln(T_0 / T) \), rather than \( \tau T \) as in, for example, Eq. (3.2) and (3.4). The reason is that one expects a very broad distribution of energy barriers. The natural, self-averaging, variable is therefore \( \ln \tau \) rather than \( \tau \).

It makes physical sense to assume scaling forms only for self-averaging quantities. For a system with quenched disorder it is known that the free energy is self-averaging, while the partition function is not, and correlation functions in general are not, either. Therefore, all thermodynamic quantities, which can be obtained as partial derivatives of the free energy, are self-averaging. For a general thermodynamic quantity, \( Q \), one therefore expects a homogeneity law[14]

\[ Q(t, T) = b^{-x_Q} F_Q \left( t^{1/\nu} \frac{b^{\psi}}{\ln(T_0 / T)} \right) \]  \hspace{1cm} (3.10)

where \( x_Q \) is the scale dimension of \( Q \), and \( F_Q \) is a scaling function. For example, for the DOS one expects,

\[ N(t, T) = b^{-\beta/\nu} F_N \left( t^{1/\nu} \frac{b^{\psi}}{\ln(T_0 / T)} \right) \]  \hspace{1cm} (3.11)

with \( \beta \) still given by Eq. (3.2b). Alternatively, Eq. (3.11) can be written,
The scaling function $G_N$ is related to the function $F_N$ in Eq. (3.12) by $G_N(x) = F_N(x^{1/\nu}, 1)$, and has the properties $G_N(x \to \infty) \sim x^{\beta/\nu\psi}$, and $G_N(x \to 0) \to \text{const.}$.

Equation (3.12) makes a qualitative prediction that can be used to check experimentally for glassy aspects of the AMT: Measurements of the tunneling DOS very close to the transition should show an anomalously slow temperature dependence, i.e., $N$ should vanish as some power of $\ln T$ rather than as a power of $T$. While in principle this should be straightforward, similar checks for the RF problem have shown that a very large dynamical or temperature range is needed to produce conclusive results.

Other thermodynamic quantities can be considered and are discussed in detail elsewhere. One chief result is the occurrence of a ‘Griffiths phase’, where both the spin susceptibility and the specific heat expansion coefficient are singular away from the AMT.

We conclude this subsection by considering the electrical conductivity. Let $\tilde{\sigma}$ be the unaveraged conductivity, and $\sigma_0$ a suitable conductivity scale, e.g., the Boltzmann conductivity. Since $\tilde{\sigma}$ is directly related to a relaxation time, we expect it not to be self-averaging, while its logarithm should be self-averaging. We define $\ell_\sigma = \log(\sigma_0/\tilde{\sigma}) > 0$ and assume it is self-averaging and that it satisfies,

$$\ell_\sigma(t,T) = b^\psi F_{\sigma} \left( t b^{1/\nu}, \frac{b^\psi}{\ln(T_0/T)} \right) = \ln(t_0/T) \cdot G_{\sigma} \left( t^{\nu\psi} \ln(T_0/T) \right).$$

(3.13)

Notice that the scale dimension of $\ell_\sigma$ is necessarily $\psi$, since $\psi$ characterizes the free energy barriers near the AMT.

As a measure of the conductivity, let us define,

$$\sigma(t,T) \equiv \sigma_0 \exp(-\ell_\sigma).$$

(3.14)

One can argue on physical grounds that $G_{\sigma}(x \to \infty) \sim 1/x$. This yields,

$$\sigma(t, T = 0) \sim \exp(-1/t^{\nu\psi})$$

(3.15a)

and

$$\sigma(t = 0, T) \sim T^{G_{\sigma}(0)}.$$  

(3.15b)

Note that at zero temperature, $\sigma$ vanishes exponentially with $t$, and that at the critical point $\sigma$ vanishes like a nonuniversal power of $T$.

**IV. DISCUSSION**

We conclude by briefly summarizing our order parameter description of the AMT and its relation to the random field magnet problem. We also make a few additional comments on the experimental situation.

The RF nature of the AMT was made plausible in the introduction. In order to derive this result it is necessary to have an OP description of the AMT. In Section II we illustrated how to obtain an OP field theory for the AMT. This is an important advance because an OP description is conceptually simpler, and physically more intuitive, than the standard sigma model description of the AMT. Renormalization of this OP field theory then generates the expected RF structure, which for unknown reasons is not present in the bare theory. The upper critical dimension $d^c_\epsilon$ is found to be $d^c_\epsilon = 6$. For $d > 6$, mean-field theory gives the exact critical behavior, and for $d < 6$, an $\epsilon = 6 - d$ expansion for the critical exponents can be obtained. One of the important results is that hyperscaling is violated at the AMT due to a dangerous irrelevant variable. As a consequence, Wegner’s scaling law near the metal-insulation transition is modified.

In Section III we reviewed two distinct scaling scenarios for the AMT. The first one was a conventional scaling theory, in the presence of a dangerous irrelevant variable. The second one introduced the idea that activated scaling might be relevant near the AMT. Physically, one of the main results in this second approach is that static or zero-temperature scaling is expected to set in only at exponentially low temperatures, and that for practical purposes it is inaccessible close to the AMT.

Electron-electron interactions are necessary in order for the AMT discussed here to exist, since for noninteracting electrons one has an Anderson transition with an uncritical DOS. This point is correctly reflected by the theory.
FIG. 1. Schematic phase diagram in the disorder ($G$) - interaction ($K_s$) plane proposed for a system with $K_t = 0$ and a short-ranged $K_s$. $M$, $AI$, and $AMI$ denote a metal phase, an Anderson insulator, and an Anderson-Mott insulator, respectively. The transition from $M$ to $AI$ is an Anderson transition, while the one from $M$ to $AMI$ is an AMT.

since $f_n$ in Eq. (2.7a) vanishes for noninteracting systems, so that the critical point discussed here is never reached: For $K_{s,t} \to 0$ the critical disorder for the AMT increases without bound, $G_c \to \infty$. This suggests a number of distinct phase transition scenarios. The simplest one is that for sufficiently small interaction constants, or large $G_c$, the AMT discussed here gets preempted by some other transition, such as a pure Anderson transition. This scenario is particularly likely if $K_t = 0$, and if the electron-electron interactions are short ranged, since in this case $K_s$ is irrelevant near the Anderson transition FP, at least near $d = 2$. For this case a likely phase diagram is shown in Fig. 1. A different possibility is that in the above picture the Anderson transition is replaced by an AMT of a different type than the one discussed here, possibly one that is related to the transition studied near $d = 2$ for the case when either $K_s$ and $K_t$ are nonzero, or the electron-electron interaction is of long range.

In Section III.B we suggested that the AMT is a quantum glass transition. Following this notion, our chief results are as follows: (1) The specific heat and spin susceptibilities are singular as $T \to 0$ even in the metallic phase. These results are consistent with existing experiments, and the theory given here provides an alternative to the previous exploration in terms of noninteracting local moments. (2) The DOS is the order parameter for the quantum glass transition. At criticality it is predicted to vanish logarithmically with temperature. (3) The electrical conductivity $\tilde{\sigma}$ is so broadly distributed that it is not a self-averaging quantity, but $\log \tilde{\sigma}$ is both self-averaging and a scaling quantity. This result may be relevant to explain the sample-to-sample fluctuations in the conductivity that are observed in Si:P at low temperature near the AMT.

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