Research article

A decent three term conjugate gradient method with global convergence properties for large scale unconstrained optimization problems

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Abstract: The conjugate gradient (CG) method is a method to solve unconstrained optimization problems. Moreover CG method can be applied in medical science, industry, neural network, and many others. In this paper a new three term CG method is proposed. The new CG formula is constructed based on DL and WYL CG formulas to be non-negative and inherits the properties of HS formula. The new modification satisfies the convergence properties and the sufficient descent property. The numerical results show that the new modification is more efficient than DL, WYL, and CG-Descent formulas. We use more than 200 functions from CUTEst library to compare the results between these methods in term of number of iterations, function evaluations, gradient evaluations, and CPU time.

Keywords: conjugate gradient method; inexact line search; global convergence
AMS Subject Classifications: 49M37, 65K05, 90C3

1. Introduction

The conjugate gradient (CG) method is a method to solve large scale unconstrained optimization problems, we consider the following problem

$$\min f(x), \ x \in \mathbb{R}^n,$$

(1)
where \( f: \mathbb{R}^n \to \mathbb{R} \) is a continuous and differentiable function and the gradient is available. The CG method generates a sequence \( x_k \) as follows:

\[
x_{k+1} = x_k + \alpha_k d_k, \quad k = 1, 2, \ldots,
\]

(2)

where \( x_k \) is the current point (iteration) and \( \alpha_k > 0 \) is a steplength. The search direction \( d_k \) of the CG method is defined as follows:

\[
d_k = \begin{cases} 
-g_k, & \text{if } k = 1, \\
-g_k + \beta_k d_{k-1}, & \text{if } k \geq 2,
\end{cases}
\]

(3)

where \( g_k = g(x_k) = \nabla f \) and \( \beta_k \) is known as the CG formula.

To compute the steplength normally we use the strong Wolfe-Powell (SWP) [1,2] line search is defined as follows:

\[
f(x_k + \alpha_k d_k) \leq f(x_k) + \delta \alpha_k g_k^T d_k,
\]

(4)

and

\[
|g(x_k + \alpha_k d_k)^T d_k| \leq \sigma |g_k^T d_k|,
\]

(5)

where \( 0 < \delta < \sigma < 1 \).

The SWP line search is a strong version of the weak Wolfe-Powell (WWP) line search; the latter is given by (4) and

\[
g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k
\]

(6)

The most famous classical formulas of CG methods Hestenes-Stiefel (HS) [3], Polak-Ribiere-Polyak (PRP) [4], Liu and Storey (LS) [5], Fletcher-Reeves (FR) [6], Fletcher (CD) [7], Dai and Yuan (DY) [8], are as follows:

\[
\beta_{k}^{\text{HS}} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}}, \quad \beta_{k}^{\text{PRP}} = \frac{g_k^T y_{k-1}}{\|g_{k-1}\|^2}, \quad \beta_{k}^{\text{LS}} = \frac{g_k^T y_{k-1}}{d_{k-1}^T g_{k-1}}
\]

\[
\beta_{k}^{\text{FR}} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2}, \quad \beta_{k}^{\text{CD}} = \frac{\|g_k\|^2}{d_{k-1}^T g_{k-1}}, \quad \beta_{k}^{\text{DY}} = \frac{\|g_k\|^2}{d_{k-1}^T g_{k-1}}
\]

where \( y_{k-1} = g_k - s_{k-1} \).

Polak and Ribièere [4] proven that PRP method with exact line search is globally converge. In the other hand Powell [9] proposed an example show that there exists a function does not global convergence even if the exact line search is employed with PRP formula. Powell suggests using non-negative value of PRP method. Gilbert and Nocedal [10] proved that if \( \beta_{k}^{\text{PRP+}} = \max\{0, \beta_{k}^{\text{PRP}}\} \)
with the WWP line search is employed and the sufficient descent condition (See Eq (13)) is satisfied, and then $\beta_{k}^{\text{PRP}+}$ is globally convergent.

Zoutendijk [11] show the global convergence of FR formula with CG method and the exact line search. Al-Baali [12] show that the CG method with the FR coefficient is globally convergent when $\sigma \leq 1/2$, and SWP is employed.

Dai and Liao [13] proposed the following conjugacy condition

$$ d_k^T y_{k-1} = -t g_k^T s_{k-1}, $$

(7)

where $s_{k-1} = x_k - x_{k-1}$, and $t \geq 0$. In the case of $t = 0$, Eq (8) becomes the classical conjugacy condition. By using (2) and (7), [10] proposed the following CG formula

$$ \beta_{k}^{DL} = \frac{g_k^T y_{k-1}}{d_{k-1}^T y_{k-1}} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} = \beta_{k}^{HS} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}. $$

(8)

However, $\beta_{k}^{DL}$ inherits the same problem as $\beta_{k}^{PRP}$ and $\beta_{k}^{HS}$ i.e., $\beta_{k}^{DL}$ is not non-negative in general. Thus [10] replaced Eq (8) by

$$ \beta_{k}^{DL+c} = \max \{ \beta_{k}^{HS}, 0 \} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}. $$

Hager and Zhang [14,15] presented a modified CG parameter that satisfies the descent property for any inexact line search with $g_k^T d_k \leq -(7/8) \| g_k \|^2$. This new version of CG method is globally convergent whenever the line search satisfies the (WP) line search. This formula is given as follows:

$$ \beta_{k}^{HZ} = \max \{ \beta_{k}^{N}, \eta_k \} $$

(9)

where $\beta_{k}^{N} = \frac{1}{d_k^T y_k} (y_k - 2d_k y_k) (d_k^T y_k) g_k - \eta_k = -\frac{1}{\| d_k \| \min \{ \eta, \| g_k \| \} }$, and $\eta > 0$ is a constant. Notes that if $t = 2 \| y_k \|^2 / s_k^T y_k$, then $\beta_{k}^{N} = \beta_{k}^{DY}$.

In 2006, Wei et al. [16], gave a new positive CG method, which is quite similar to original PRP method which has a global convergence under exact and inexact line search that is,

$$ \beta_{k}^{\text{PP}} = \frac{g_k^T (g_k - \frac{\| g_k \|}{\| g_{k-1} \|} g_{k-1})}{\| g_{k-1} \|^2}, $$

where $y_{k-1} = g_k - g_{k-1}$. In 2016, Alhawarat et al. [17] presented the following formula

$$ \beta_{k}^{\text{AZPP}} = \begin{cases} \frac{\| g_k \|^2 - \mu_k \| g_k^T g_{k-1} \|}{\| g_{k-1} \|^2}, & \text{if } \| g_k \|^2 > \mu_k \| g_k^T g_{k-1} \|, \\ 0, & \text{otherwise,} \end{cases} $$

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where \( \| \| \) represents the Euclidean norm. And \( \mu_k \) is defined as follows:

\[
\mu_k = \frac{\| x_k - x_{k-1} \|}{\| y_k \|}.
\]

Kaelo et al. [18] proposed the following CG formula

\[
\beta_{k-1} = \begin{cases} \frac{\| g_k \|^2 - g_k^T g_{k-1}}{\| g_k \|^2} , & \text{If } 0 < g_k^T g_{k-1} < \| g_k \|^2 \\ \max \{ d_{k-1}^T y_{k-1} - g_k^T d_{k-1} \} , & \text{otherwise} \end{cases}
\]

Yao et al. [19] proposed three terms of CG with a new choice of \( t \) as follows:

\[
d_{k+1} = -g_{k+1} + \left( g_k^T y_k - t_k g_{k+1}^T s_k \right) y_k + t_k s_{k+1}^T d_k y_k.
\]

Based on the SWP line search, Yao et al. [19] selected \( t_k \) to satisfy the descent condition as follows:

\[
t_k > \frac{\| y_k \|^2}{y_k^T s_k}.
\]

Yao et al. [19] also proposed a theorem stating that if \( t_k \) is close to \( \frac{\| y_k \|^2}{y_k^T s_k} \), then the search direction results in a zigzag search path. Therefore, they selected the following choice for \( t_k \):

\[
t_k = 1 + 2 \frac{\| y_k \|^2}{y_k^T s_k}.
\]

For more about CG method and its application, the reader can refer to the following references [20–24].

2. The new formula and the algorithm

Since \( \beta^{DL}_{k} \) method in (8) has negative values for some times similar to \( \beta^{HS}_{k} \) and \( \beta^{PRP}_{k} \), we construct new method depend on \( \beta^ {WYL}_{k} \) and \( \beta^ {DL}_{k} \) to be nonnegative and inherits the advantages of \( \beta^ {HS}_{k} \), \( \beta^ {WYL}_{k} \), and \( \beta^ {DL}_{k} \). The new method constructed as follows:

\[
\beta^ {DL-WYL}_{k} = \frac{g_k^T (g_k - \frac{\| g_k \|}{\| g_{k-1} \|} g_{k-1})}{d_{k-1}^T y_{k-1}} - t_k \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}
\]

(10)

Algorithm 1 shows that the steps to find the optimal solution of optimization function.
Algorithm 1. The steps of CG methods with Eq (10) to obtain the optimum method.

**Step 1:** Provide a starting point \( x_1 \). Set the initial search direction \( d_1 = -g_1 \). Let \( k = 1 \).

**Step 2:** If a stopping criteria is satisfied, then stop.

**Step 3:** Compute \( d_k \) based on (2) with (10).

**Step 4:** Compute \( \alpha_k \) using (4) and (5).

**Step 5:** Update \( x_{k+1} \) based on (1).

**Step 6:** Set \( k := k + 1 \) and go to Step 2.

### 3. Global convergence analysis of the CG algorithm with the coefficient \( \beta_{k}^{DL-WL} \)

To establish the convergence properties of the new formula, the following assumption is required.

**Assumption 1.**

A. The level set \( \Omega = \{ x | f(x) \leq f(x_i) \} \) is bounded, that is, a positive constant \( \tau \) exists such that

\[
\|x\| \leq \tau, \quad \forall x \in \Omega.
\]

B. In some neighbourhood \( Q \) of \( \Omega \), \( f \) is continuously differentiable, and its gradient is Lipschitz continuous; that is, for all \( x, y \in Q \), there exists a constant \( L > 0 \) such that

\[
\|g(x) - g(y)\| \leq L\|x - y\| \tag{11}
\]

This assumption implies that there exists a positive constant \( B \) such that

\[
\|g(u)\| \leq B, \quad \forall u \in N.
\]

The descent condition (downhill condition)

\[
g_k^T d_k < 0, \quad \forall k \geq 1, \tag{12}
\]

is useful in the study of CG method and serves important rule in the proof of global convergence analysis. Abaali [12] modified (12) to the following form and used it to prove the FR method

\[
g_k^T d_k \leq -c\|g_k\|^2, \quad \forall k \geq 1 \tag{13}
\]

where \( c \in (0, 1) \). Equation (14) is the sufficient descent condition. Note that the general form of the sufficient descent condition is (14) with \( c > 0 \). Moreover, using (13) is better than (12) since we can control the quantity of \( g_k^T d_k \) by using \( \|g_k\|^2 \).

#### 3.1. Global convergence for \( \beta_{k}^{DL-WL} \) with the SWP line search

The following theorem demonstrates that \( \beta_{k}^{DL-WL} \) ensures that the sufficient descent condition (13) is satisfied with the SWP line search.
Theorem 3.1. Let the sequences \( \{g_k\} \) and \( \{d_k\} \) are generated using (1), (2), and (10), where \( \alpha_k \) is computed by the SWP line search (4) and (5). If \( \sigma \in (0, \frac{1}{4}) \), then the sufficient descent condition (14) holds.

Proof. By Multiplying (2) by \( g_k^T \), we obtain
\[
g_k^T d_k = g_k^T (-g_k + \beta_k d_{k-1}) = -\|g_k\|^2 + \beta_k g_k^T d_{k-1}
\] (14)
Substitute \( \beta_k \) instead of \( \beta_k \)
\[
g_k^T d_k = g_k^T (-g_k^T + \beta_k d_{k-1}) = -\|g_k\|^2 + \left( g_k^T - \frac{g_k}{\|g_k\|} \frac{g_k}{\|g_k\|} - \frac{t g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_k^T d_{k-1} \] (15)
Then we have the following two cases:

Case 1.
\[
g_k^T g_{k-1} > 0.
\]
Then
\[
g_k^T d_k = -\|g_k\|^2 + \left( \frac{g_k}{d_{k-1}^T y_{k-1}} - \frac{t g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \right) g_k^T d_{k-1} \]
\[
g_k^T d_k = -\|g_k\|^2 + \frac{g_k}{d_{k-1}^T y_{k-1}} g_k^T d_{k-1} - \frac{t g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} g_k^T d_{k-1} \]
\[
g_k^T d_k \leq -\|g_k\|^2 - \frac{\|g_k\|^2}{(\sigma - 1) g_k^T d_{k-1}} - \frac{t \alpha_k}{d_{k-1}^T y_{k-1}} \|g_k^T d_{k-1}\|^2 \]
\[
g_k^T d_k \leq -\|g_k\|^2 - \frac{\|g_k\|^2}{(\sigma - 1) g_k^T d_{k-1}} - \frac{t \alpha_k}{d_{k-1}^T y_{k-1}} \|g_k^T d_{k-1}\|^2 \]
Since
\[
d_{k-1}^T y_{k-1} > 0
\]
\[
g_k^T d_k \leq -\|g_k\|^2 - \sigma \|g_k\|^2 \]
Divide both sides by \( \|g_k\|^2 \)
\[
\frac{g_k^T d_k}{\|g_k\|^2} \leq -1 - \frac{\sigma}{(\sigma - 1)} = -(1 + \frac{\sigma}{\sigma - 1}).
\]

Let \( c = (1 + \frac{\sigma}{\sigma - 1}) \), thus we obtain the result

\[
g_k^T d_k \leq -c\|g_k\|^2.
\]

**Case 2.**

\[
g_k^T g_{k-1} < 0.
\]

Then

\[
g_k^T d_k \leq -\|g_k\|^2 + \left(\frac{2\|g_k\|^2}{d_{k-1}^T y_{k-1}} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}}\right)g_k^T d_{k-1}
\]

\[
g_k^T d_k = -\|g_k\|^2 + \frac{2\|g_k\|^2}{d_{k-1}^T y_{k-1}} g_k^T d_{k-1} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} g_k^T d_{k-1}
\]

\[
g_k^T d_k \leq -\|g_k\|^2 - \frac{2\|g_k\|^2}{(\sigma - 1)g_{k-1}^T d_{k-1}} \alpha_k \|g_{k-1}^T d_{k-1}\|^2 - t \frac{\alpha_k \|g_{k-1}^T d_{k-1}\|^2}{d_{k-1}^T y_{k-1}}
\]

Since

\[
d_{k-1}^T y_{k-1} > 0
\]

\[
g_k^T d_k \leq -\|g_k\|^2 - \frac{2\|g_k\|^2}{(\sigma - 1)}.
\]

Divide both sides by \( \|g_k\|^2 \)

\[
\frac{g_k^T d_k}{\|g_k\|^2} \leq -1 - \frac{2\sigma}{(\sigma - 1)} = -(1 + \frac{2\sigma}{\sigma - 1}).
\]

Let \( c = (1 + \frac{2\sigma}{\sigma - 1}) \), then if \( \sigma \leq \frac{1}{4} \) we obtain

\[
g_k^T d_k \leq -c\|g_k\|^2.
\]

The following lemma, which is referred to as the Zoutendijk condition [11], is useful for analysing the global convergence property of the CG method.
Lemma 3.1. Consider CG method with SWP line search and \( g_k^T d_{k-1} < 0 \) then the CG formula \( \beta_k^{DL-WYL} \) in non-negative.

Proof. By using SWP line search we obtain that

\[ d_{k-1}^T y_{k-1} \geq 0. \]

By using Theorem 1 we have

\[ -t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} > 0. \]

Since

\[ g_k^T s_{k-1} - \left( \frac{g_k^T}{g_k^T g_{k-1}} \right) g_k^T y_{k-1}, \]

\[ -t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \geq -t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \geq 0. \]

Thus we obtain the result.

Lemma 3.2. Let Assumption 1 be holds. Consider any CG method in the form (1), (2), and \( \alpha_k \) satisfies the WWP line search (5) and (6), in which the search direction is descent. Then, the following condition holds:

\[ \sum_{k=0}^{\infty} (g_k^T d_k)^2 < \infty \]  

(16)

3.2. Global convergence for \( \beta_k^{HS} \) with the SWP line search

The following property, which is referred to as Property*, was presented by Gilbert and Nocedal in [10]. This property is useful to obtain the global convergence properties of CG methods related to PRP or HS family. The property is given as follows:

Property* Consider a method of the form (1) and (2). Assume that

\[ 0 < \gamma \leq \|g_k\| \leq \bar{g} \]  

(17)

for all \( k \geq 1 \). Then, the CG method has Property* if there exist constants \( b > 1 \) and \( \lambda > 0 \) such that for all \( k \) whereby for \( |\beta_k| \leq b \) and \( \|s_k\| \leq \lambda \) , we obtain \( |\beta_k| \leq \frac{1}{2b} \).

Lemma 3.3. Consider the CG method of the form (1), and (2) with the new formula \( \beta_k^{DL-WYL} \). If Eq (17) holds true then \( \beta_k^{DL-WYL} \) has Property*.

Proof. Set \( b = \frac{2\bar{g}^2 + t\bar{g} \beta}{c(1-\sigma)\gamma^2} \geq 1 \), and \( \lambda = \frac{c(1-\sigma)\gamma^2}{L(L+1)\lambda \bar{g}} \). Using (10) and (17)
\[ |\beta_k^{\text{DL-WYL}}| = \frac{g_k^T (g_k - \|g_k\| g_{k-1})}{d_{k-1}^T y_{k-1}} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \]

\[ \leq \frac{\|g_k\|(\|g_k\| + \|g_{k-1}\|) + t\|g_k\|B}{c(1-\sigma\|g_{k-1}\|^2) \leq \frac{2\bar{\gamma}^2 + t\bar{\gamma}B}{c(1-\sigma)\gamma^2} = b > 1. \]

To obtain \( |\beta_k^{\text{DL-WYL}}| \leq 1/2b \),

\[ |\beta_k^{\text{DL-WYL}}| \leq \frac{g_k^T (g_k - \|g_k\| g_{k-1})}{d_{k-1}^T y_{k-1}} - t \frac{g_k^T s_{k-1}}{d_{k-1}^T y_{k-1}} \]

\[ \leq \frac{\|g_k\|(\|g_k - g_{k-1}\|) + \|g_{k-1}\| + t\|s_{k-1}\|)}{d_{k-1}^T y_{k-1}} \leq \frac{\|g_k\|(\|g_k - g_{k-1}\| + \|s_{k-1}\|)}{d_{k-1}^T y_{k-1}} \leq \frac{\|g_k\|(L\lambda + L\lambda + t\lambda)}{c(1-\sigma)\gamma^2} = \frac{\lambda\bar{\gamma}(2L + t)}{c(1-\sigma)\gamma^2} .\]

Thus we obtain

\[ |\beta_k^{\text{DL-WYL}}| \leq \frac{1}{2b} . \]

The forthcoming lemmas correspond to Lemmas 4.1 and 4.2 in [10].

**Lemma 3.4.** Let Assumption 1 holds. Let the sequences \( \{g_k\} \) and \( \{d_k\} \) are generated by Algorithm 1 in which \( \alpha_k \) is computed by the WWP line search in which the sufficient descent condition (16) holds, and assume that the method has Property*. Suppose that (17) holds. Then there exists \( \lambda > 0 \) such that for any \( \Delta \in N \) and any index \( k_0 \), there exist an index \( k > k_0 \) that satisfies

\[ |\kappa^\lambda_{k,\Delta}| = \frac{\lambda}{2}, \]

where \( \kappa^\lambda_{i,\Delta} = \{i \in N : k \leq i \leq k + \Delta - 1, \|y_i\| > \lambda\} \), \( N \) denotes the set of positive integers, and \( |\kappa^\lambda_{i,\Delta}| \) denotes the number of elements in \( \kappa^\lambda_{i,\Delta} \).

**Lemma 3.5.** Let Assumption 1 holds. Let the sequences \( \{g_k\} \) and \( \{d_k\} \) are generated by Algorithm 1 in which \( \alpha_k \) is computed by the WWP line search and the sufficient descent condition (13) holds. If \( \beta_k \geq 0 \) and (17) hold, then \( d_k \neq 0 \) and
\[
\sum_{k=0}^{\infty} \left\| u_{k+1} - u_k \right\|^2 < \infty, \text{ where } u_k = \frac{d_k}{\|d_k\|}.
\]

From Lemmas 3.1 and 3.3–3.5, the global convergence of Algorithm 1 with the WWP line search can be established in a manner that is similar to that of Theorem 4.3 in [10]; therefore, the proof of the following theorem is omitted.

**Theorem 3.2.** Let the sequences \( \{g_k\} \) and \( \{d_k\} \) be generated by (1) and (2) with the CG formula \( \beta_k^{DL-WYL} \) and the step size satisfies (4) and (6). If Lemmas 3.1 and 3.3–3.5 are true, then

\[
\liminf_{k \to \infty} \|g_k\| = 0.
\]

4. **Numerical results and discussion**

To analyse the efficiency of the new formula, we selected several test problems in Table 1 from CUTEr [25]. We performed a comparison with other CG coefficients, including CG-Descent, DY, and WYL coefficients based on the CPU time, number of iterations, number of function evaluations, and number of gradient evaluations. In Table 1 we define the following abbreviations as follows:

- No. Iter: Number of iterations;
- No. fun.ev: Number of function evaluations;
- No. grad.ev: Number of gradient evaluations;
- CPU time: central processing unit time in seconds.

We employed the SWP line search with \( \delta = 0.01 \) and \( \sigma = 0.1 \) for all algorithms except CG-Descent we use approximate WWP line search. The norm of the gradient was employed as the stopping criterion, specifically, \( \|g_k\| \leq 10^{-6} \) for all algorithms. The host computer is AMD A4-7210 APU Radeon R3 Graphics where the installed memory is 4 GB with operating system Ubuntu 20.04.2.0 LTS. The results are shown in Figures 1–4 in which a performance measure introduced by Dolan and More [26] was employed.

As shown from Figure 1 which present the number of iterations, we note \( \beta_k^{DL-WYL} \) strongly out perform all methods. Figure 2 presents the CPU time, we note that \( \beta_k^{DL-WYL} \) also outperform WYL, DL, and CG-Descent methods. From Figure 3 which present the number and gradient evaluations we note that the new method \( \beta_k^{DL-WYL} \) is completive with CG-Descent in terms of number of gradient evaluations since the later use approximation Wolfe-Powell line search for more about this line search the reader can refer to [14,15] in the other hand we can note that \( \beta_k^{DL-WYL} \) out perform all other methods in terms of gradient evaluations since we SWP line search for WYL, DL, and \( \beta_k^{DL-WYL} \).

Figure 4 presents the number of function evaluations we note that the new modification outperform all methods.
| function | DIM | No. iter DL-WYL | No. fun.ev DL-WYL | No. grad.ev DL-WYL | CPU-time DL-WYL | No. iter CG-Descent | No. Fun.ev CG-Descent | No. grad.ev CG | CPU-time CG-Descent | No. iter WYL | No. fun.ev WYL | No. grad.ev WYL | CPU-time WYL | No. iter DL | No. fun.ev DL | No. grad.ev DL | CPU-time DL |
|----------|-----|---------------|-----------------|------------------|----------------|-------------------|---------------------|-------------|------------------|---------------|----------------|----------------|-------------|-------------|----------------|----------------|-------------|
| AKIVA    | 2   | 8             | 20              | 15               | 0.02           | 10                | 21                  | 11           | 0.02             | 2             | 8             | 20             | 0.02        | 8           | 20             | 15             | 0.02        |
| ALLINITU | 4   | 9             | 25              | 18               | 0.02           | 12                | 29                  | 18           | 0.02             | 9             | 25             | 18             | 0.02        | 9           | 25             | 18             | 0.03        |
| ARGLINC  | 200 | 5             | 67              | 67               | 0.02           | 5                 | 13                  | 13           | 0.02             | 1             | 3             | 2              | 0.02        | 5           | 73             | 72             | 0.09        |
| ARGLINC  | 200 | 5             | 96              | 94               | 0.02           | 11                | 106                 | 110          | 0.02             | 5             | 79             | 78             | 0.02        | 5           | 79             | 78             | 0.06        |
| ARWHEAD  | 5000| 6             | 16              | 12               | 0.03           | 7                 | 15                  | 8            | 0.02             | 7             | 16             | 12             | 0.02        | 6           | 16             | 12             | 0.02        |
| BARD     | 3   | 12            | 32              | 22               | 0.02           | 16                | 33                  | 17           | 0.02             | 12            | 32             | 22             | 0.02        | 12          | 32             | 22             | 0.02        |
| BDEXP    | 5000| 2             | 7               | 7                | 0.02           | 5                 | 11                  | 6            | 0.02             | 2             | 7             | 7              | 0.02        | 2           | 7              | 7              | 0.02        |
| BDQRTIC  | 5000| 198           | 434             | 389              | 0.66           | 136               | 273                 | 237          | 0.52             | 140           | 318            | 280            | 0.47        | 168         | 363            | 359            | 0.63        |
| BEALE    | 2   | 11            | 33              | 26               | 0.02           | 15                | 31                  | 16           | 0.02             | 11            | 33             | 26             | 0.02        | 11          | 33             | 26             | 0.02        |
| BIGGS3   | 6   | 79            | 207             | 144              | 0.02           | 110               | 231                 | 125          | 0.02             | 79            | 207            | 144            | 0.02        | 79          | 207            | 144            | 0.02        |
| BIGGS5   | 6   | 79            | 207             | 144              | 0.02           | 110               | 231                 | 125          | 0.02             | 79            | 207            | 144            | 0.02        | 79          | 207            | 144            | 0.02        |
| BIGGS6   | 6   | 24            | 64              | 44               | 0.02           | 27                | 57                  | 31           | 0.02             | 24            | 64             | 44             | 0.02        | 24          | 64             | 44             | 0.02        |
| BIGGBS1  | 5000| 2500          | 2507            | 4995             | 2.37           | 2500              | 2507                | 4995         | 2.47             | 2500          | 2507           | 4995            | 2.66        | 8328        | 8335           | 16651          | 10.86       |
| BOX2     | 3   | 10            | 23              | 14               | 0.02           | 11                | 24                  | 13           | 0.02             | 10            | 23             | 14             | 0.02        | 10          | 23             | 14             | 0.02        |
| BOX2     | 3   | 10            | 23              | 14               | 0.02           | 11                | 24                  | 13           | 0.02             | 10            | 23             | 14             | 0.02        | 10          | 23             | 14             | 0.02        |
| BOX      | 10000| 7             | 25              | 21               | 0.09           | 8                 | 25                  | 19           | 0.08             | 7             | 24             | 20             | 0.08        | 7           | 25             | 21             | 0.09        |
| BRKMCC   | 2   | 5             | 11              | 6                | 0.02           | 5                 | 11                  | 6            | 0.02             | 5             | 11             | 6              | 0.02        | 5           | 11             | 6              | 0.02        |
| BROYDNBDSL | 10  | 25            | 57              | 34               | 0.02           | 25                | 51                  | 26           | 0.02             | 25            | 57             | 34             | 0.02        | 25          | 57             | 34             | 0.02        |
| BROWNAL  | 200 | 3             | 11              | 9                | 0.03           | 9                 | 25                  | 18           | 0.02             | 9             | 26             | 20             | 0.02        | 10          | 29             | 21             | 0.02        |
| BROWNBS  | 2   | 10            | 24              | 18               | 0.02           | 13                | 26                  | 15           | 0.02             | 10            | 24             | 18             | 0.02        | 10          | 24             | 18             | 0.02        |

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| function   | DIM | No. | No.       | No.       | CPU-time | No. | No.       | CPU-time | No. | No.       | CPU-time | No. | No.       | CPU-time | No. | No.       | CPU-time |
|------------|-----|-----|-----------|-----------|----------|-----|-----------|----------|-----|-----------|----------|-----|-----------|----------|-----|-----------|----------|
|            |     |     | fun.ev    | grad.ev   | DL-WYL   |     | DL-WYL   |     |     | DL-WYL   |     |     | DL-WYL   |     |     | DL-WYL   |     |
| DL-WYL     |     |     |           |           |          |     |          |           |     |          |           |     |          |           |     |          |           |
|            |     |     | fun.ev    | grad.ev   | DL-WYL   |     | DL-WYL   |     |     | DL-WYL   |     |     | DL-WYL   |     |     | DL-WYL   |     |
| DL-WYL     |     |     |           |           |          |     |          |           |     |          |           |     |          |           |     |          |           |
|            |     |     | fun.ev    | grad.ev   | DL-WYL   |     | DL-WYL   |     |     | DL-WYL   |     |     | DL-WYL   |     |     | DL-WYL   |     |
| DL-WYL     |     |     |           |           |          |     |          |           |     |          |           |     |          |           |     |          |           |
| BROWNDEN   | 4   | 16  | 38        | 31        | 0.02     | 16  | 31        | 0.02     | 16  | 38        | 0.02     | 16  | 38        | 0.02     | 16  | 38        | 0.02   |
| BROYDN7D   | 5000| 58  | 106       | 80        | 0.34     | 1411| 2810      | 0.22     | 60  | 111       | 0.28     | 149 | 317       | 0.55     | 138 | 149       | 0.36   |
| BRYBND     | 5000| 72  | 170       | 107       | 0.25     | 85  | 174       | 0.28     | 38  | 103       | 0.2     | 149 | 317       | 0.55     | 112 | 149       | 0.36   |
| CAMEL6     | 2   | 6   | 22        | 18        | 0.02     | 13  | 34        | 22       | 0.02 | 6         | 22       | 18   | 6         | 22       | 18   | 6         | 22     |
| CHNROSNB   | 50  | 282 | 572       | 303       | 0.37     | 287 | 564       | 0.02     | 266 | 542       | 0.02     | 1009| 1998      | 0.01     | 1800| 1180      | 0.01   |
| CLIFF      | 2   | 10  | 46        | 39        | 0.02     | 18  | 70        | 0.02     | 10  | 46        | 0.02     | 10  | 46        | 0.02     | 39   | 39        | 0.01   |
| COSINE     | 10000| 12  | 51        | 42        | 0.02     | 11  | 39        | 0.19     | 12  | 54        | 0.19     | 12  | 54        | 0.19     | 43   | 43        | 0.2    |
| CUBE       | 2   | 17  | 48        | 34        | 0.02     | 32  | 77        | 0.02     | 17  | 48        | 0.02     | 17  | 48        | 0.02     | 34   | 34        | 0.02   |
| CURLY10    | 10000| 51546| 72517     | 82155     | 184.3    | 47808| 67294     | 171.25   | 58045| 78735     | 95434    | 213.64| 68087     | 88635    | 1E+05| 240.64    |        |
| CURLY20    | 10000| 74486| 98857     | 1E+05     | 402     | 66587| 89245     | 377.8    | 78064| 1E+05     | 437.5    | 88068| 1E+05     | 540.5    |        |          |        |
| CURLY30    | 10000| 84928| 1E+05     | 1E+05     | 588.5   | 79030| 102516    | 635.36   | 83528| 1E+05     | 619.27   | 93324| 1E+05     | 712.27   |        |          |        |
| DENSCHNA   | 2   | 6   | 16        | 12        | 0.02     | 9   | 19        | 0.02     | 6   | 16        | 0.02     | 6   | 16        | 0.02     | 12   | 12        | 0.02   |
| DENSCHNB   | 2   | 6   | 18        | 15        | 0.02     | 7   | 15        | 0.02     | 6   | 18        | 0.02     | 6   | 18        | 0.02     | 15   | 15        | 0.02   |
| DENSCHNC   | 2   | 11  | 36        | 31        | 0.02     | 12  | 26        | 0.02     | 11  | 36        | 0.02     | 11  | 36        | 0.02     | 31   | 31        | 0.02   |
| DENSCHND   | 3   | 14  | 46        | 40        | 0.02     | 47  | 98        | 0.02     | 14  | 46        | 0.02     | 14  | 46        | 0.02     | 40   | 40        | 0.02   |
| DENSCHNE   | 3   | 12  | 43        | 38        | 0.02     | 18  | 49        | 0.02     | 12  | 43        | 0.02     | 12  | 43        | 0.02     | 38   | 38        | 0.02   |
| DENSCHNF   | 2   | 9   | 31        | 26        | 0.02     | 8   | 17        | 0.02     | 9   | 31        | 0.02     | 9   | 31        | 0.02     | 26   | 26        | 0.02   |
| DIXMAANA   | 3000| 7   | 17        | 12        | 0.02     | 7   | 15        | 0.02     | 6   | 16        | 0.02     | 6   | 15        | 0.02     | 11   | 11        | 0.02   |
| DIXMAANB   | 3000| 6   | 16        | 12        | 0.02     | 6   | 13        | 0.02     | 6   | 15        | 0.02     | 6   | 15        | 0.02     | 11   | 11        | 0.02   |
| DIXMAANC   | 3000| 6   | 14        | 9         | 0.02     | 6   | 13        | 0.02     | 6   | 14        | 0.02     | 6   | 14        | 0.02     | 9    | 9         | 0.02   |
| DIXMAAND   | 3000| 6   | 15        | 11        | 0.02     | 7   | 15        | 0.02     | 6   | 15        | 0.02     | 7   | 17        | 0.02     | 12   | 12        | 0.02   |
| DIXMAANE   | 3000| 248 | 272       | 482       | 0.22     | 222 | 239       | 0.23     | 265 | 289       | 0.3     | 394 | 428       | 0.5      | 764 | 764       | 0.5    |
| DIXMAANF   | 3000| 98  | 201       | 106       | 0.09     | 161 | 323       | 0.17     | 146 | 297       | 0.14     | 247 | 499       | 0.27     |       | 255       | 0.27   |

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| function     | DIM | No. | No. | CPU- | No. | No. | CPU- | No. | No. | CPU- | No. | No. | CPU- |
|--------------|-----|-----|-----|------|-----|-----|------|-----|-----|------|-----|-----|------|
|              | Iter| fun.ev | grad.ev | time | CG | Fun.ev | grad.ev | CG | Descent | DESC | WYL | WYL | DESC |
| DL- WYL      | DL- WYL | WYL | WYL | WYL | WYL | WYL | WYL | WYL | WYL | WYL | WYL | WYL | WYL |
| DIXMAANG     | 3000 | 170 | 345 | 178 | 0.16 | 157 | 315 | 158 | 0.12 | 171 | 347 | 179 | 0.14 | 348 | 701 | 356 | 0.38 |
| DIXMAANH     | 3000 | 175 | 355 | 183 | 0.16 | 173 | 347 | 174 | 0.2  | 172 | 351 | 183 | 0.2  | 332 | 671 | 343 | 0.45 |
| DIXMAANI     | 3000 | 3215 | 3311 | 6344 | 0.66 | 3856 | 3926 | 7644 | 4.09 | 3115 | 3205 | 6150 | 3.33 | 3522 | 3623 | 6953 | 4.66 |
| DIXMAANJ     | 3000 | 332 | 669 | 340 | 0.28 | 327 | 655 | 328 | 0.25 | 360 | 723 | 365 | 0.36 | 476 | 957 | 486 | 0.56 |
| DIXMAANK     | 3000 | 304 | 613 | 312 | 0.25 | 283 | 567 | 284 | 0.22 | 309 | 622 | 316 | 0.25 | 425 | 854 | 432 | 0.49 |
| DIXMAANL     | 3000 | 249 | 505 | 260 | 0.23 | 237 | 475 | 238 | 0.2  | 250 | 507 | 261 | 0.2  | 320 | 647 | 331 | 0.3  |
| DIXMAANP     | 3000 | 618 | 1241 | 626 | 0.5  | 686 | 1373 | 687 | 0.5  | 620 | 1244 | 627 | 0.52 | 857 | 1721 | 872 | 0.89 |
| DIXON3DQ     | 10000 | 10000 | 10007 | 19995 | 19.48 | 10000 | 10007 | 19995 | 19.48 | 10000 | 10007 | 19995 | 19.48 | 15258 | 15265 | 30511 | 37.63 |
| DFTL         | 2   | 0.02 | 82 | 917 | 880 | 0.02 | 75 | 1163 | 1148 | 0.02 | 75 | 1163 | 1148 | 0.02 |
| DQDRTIC      | 5000 | 5    | 11 | 6   | 0.02 | 5    | 11 | 6   | 0.02 | 15 | 32 | 18 | 0.02 | 15 | 32 | 18 | 0.02 |
| ECKERLE4LSIF | 3   | 7    | 4  | 0.03 | 3    | 7   | 4  | 0.02 | 2    | 6    | 4  | 0.02 | 2    | 6    | 4  | 0.02 |
| EDENSCHE     | 2000 | 26   | 56 | 50  | 0.02 | 26   | 52 | 38  | 0.02 | 27   | 62 | 53 | 0.08 | 27   | 66 | 54 | 0.03 |
| EG2          | 1000 | 3    | 8  | 5   | 0.02 | 5    | 11 | 6   | 0.02 | 3    | 13 | 10 | 0.02 | 3    | 13 | 10 | 0.02 |
| EIGENALS     | 2550 | 9318 | 16870 | 11105 | 156.5 | 10083 | 18020 | 12244 | 172.67 | 7876 | 13863 | 9786 | 139.67 | 9534 | 18450 | 18540 | 185.64 |
| EIGENBLS     | 2550 | 30617 | 61246 | 30632 | 453 | 15301 | 30603 | 15302 | 225.23 | 16792 | 33591 | 16800 | 251.25 | 22540 | 45340 | 24700 | 350.43 |
| EIGENCLS     | 2652 | 10020 | 18835 | 11248 | 168.4 | 10136 | 19292 | 11118 | 167.52 | 9928 | 18702 | 1106 | 164.84 | 13450 | 26740 | 18450 | 203.45 |
| ELATVIDU     | 2   | 11   | 25 | 15  | 0.02 | 11   | 25 | 15 | 0.02 | 8    | 32 | 29 | 0.02 | 8    | 32 | 29 | 0.02 |
| ENGVAL1      | 5000 | 25   | 49 | 42  | 0.06 | 27   | 50 | 36  | 0.06 | 23   | 48 | 40 | 0.06 | 21   | 48 | 37 | 0.06 |
| ENGVAL2      | 2   | 26   | 73 | 55  | 0.02 | 26   | 61 | 37  | 0.02 | 26   | 73 | 55 | 0.02 | 26   | 73 | 55 | 0.02 |
| ENSOLS       | 9   | 23   | 45 | 26  | 0.02 | 23   | 45 | 26  | 0.02 | 22   | 47 | 27 | 0.02 | 22   | 47 | 27 | 0.02 |
| ExpFt        | 2   | 9    | 29 | 22  | 0.02 | 13   | 29 | 16  | 0.02 | 9    | 29 | 22 | 0.02 | 9    | 29 | 22 | 0.02 |
| EXTROSNB     | 1000 | 1130 | 2671 | 1686 | 0.41 | 3808 | 7759 | 3982 | 1.05 | 2636 | 5854 | 3402 | 0.83 | 7182 | 12662 | 10680 | 2.39 |
| exp2         | 2   | 9    | 18 | 10  | 0.02 | 8    | 17 | 9   | 0.02 | 7    | 16 | 9  | 0.02 | 7    | 16 | 9  | 0.02 |

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| function     | DIM | No. | No. | CPU-time | No. | CPU-time | No. | CPU-time | No. | CPU-time | No. | CPU-time | No. | CPU-time | No. | CPU-time | No. | CPU-time | No. | CPU-time | No. | CPU-time |
|--------------|-----|-----|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|
|              |     |     |     | Iter     |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | fun.ev   |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | grad.ev  |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
|              |     |     |     | DL-      | WYL |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |     |          |
| function         | DIM | No. Iter DL-WYL | No. fun.ev DL-WYL | No. CPU-time DL-WYL | No. grad.ev DL-WYL | No. CG DESC DL-WYL | No. CPU-timegrad.ev DL-WYL | No. fun.ev grad.ev DL-WYL | No. CG DESC grad.ev DL-WYL | No. CPU-timegrad.ev DL-WYL | No. fun.ev grad.ev DL-WYL | No. CG DESC grad.ev DL-WYL | No. CPU-timegrad.ev DL-WYL | No. fun.ev grad.ev DL-WYL | No. CG DESC grad.ev DL-WYL | No. CPU-timegrad.ev DL-WYL |
|------------------|-----|-----------------|-------------------|---------------------|-------------------|------------------|----------------------|---------------------|----------------------|----------------------|---------------------|----------------------|----------------------|---------------------|----------------------|----------------------|
| HIMMELBF         | 4   | 23              | 59                | 46                  | 0.02              | 26               | 60                   | 36                  | 0.02                 | 23                   | 59                  | 46                   | 0.02                 | 23                   | 59                  | 46                   | 0.02                 |
| HIMMELBG         | 2   | 7               | 22                | 17                  | 0.02              | 8                | 20                   | 13                  | 0.02                 | 7                    | 22                  | 17                   | 0.02                 | 7                    | 22                  | 17                   | 0.02                 |
| HIMMELBH         | 2   | 5               | 13                | 9                   | 0.02              | 7                | 16                   | 9                   | 0.02                 | 5                    | 13                  | 9                    | 0.02                 | 5                    | 13                  | 9                    | 0.02                 |
| HUMPS            | 2   | 45              | 223               | 202                 | 0.02              | 52               | 186                  | 146                 | 0.02                 | 45                   | 223                 | 202                 | 0.02                 | 45                   | 223                 | 202                 | 0.02                 |
| HYDCAR6LS.SIF    | 29  | 53              | 107               | 54                  | 0.09              | 14401            | 29028                | 14875               | 0.45                 | 923866               | 9E+05               | 2E+06               | 39.11                | 1001                 | 2027                | 1174                | 0.03                 |
| INDEF            | 5000| 1               | 46                | 147                 | 0.36              | 1                | 46                   | 147                 | 0.44                 | 1                    | 46                  | 147                 | 0.41                 | 1                    | 46                  | 147                 | 0.42                 |
| INTEQNELS.SIF    | 12  | 6               | 13                | 7                   | 0.02              | 6                | 13                   | 7                   | 0.02                 | 6                    | 13                  | 7                   | 0.02                 | 6                    | 13                  | 7                   | 0.02                 |
| JENSMP           | 2   | 12              | 47                | 41                  | 0.02              | 15               | 33                   | 22                  | 0.02                 | 12                   | 47                  | 41                   | 0.02                 | 12                   | 47                  | 41                   | 0.02                 |
| JIMACK           | 3549| 8331            | 16664             | 8333                | 1172              | 8314             | 16629                | 8315                | 1169.4               | 8305                 | 16612               | 8307                | 1167.42              | 11978               | 23971               | 12235               | 1732.9               |
| JUDGE            | 2   | 9               | 24                | 18                  | 0.02              | 10               | 23                   | 13                  | 0.02                 | 9                    | 24                  | 18                   | 0.02                 | 9                    | 24                  | 18                   | 0.02                 |
| KOWOSB           | 4   | 16              | 46                | 32                  | 0.02              | 17               | 39                   | 23                  | 0.02                 | 16                   | 46                  | 32                   | 0.02                 | 16                   | 46                  | 32                   | 0.02                 |
| KSSLS            | 1000| 6               | 19                | 16                  | 0.47              | 10               | 25                   | 16                  | 0.58                 | 5                    | 18                  | 16                   | 0.47                 | 6                    | 19                  | 16                   | 0.55                 |
| LANCZOS1LS       | 6   | 61              | 177               | 135                 | 0.02              | 148              | 325                  | 181                 | 0.02                 | 61                   | 177                 | 135                 | 0.02                 | 61                   | 177                 | 135                 | 0.02                 |
| LANCZOS2LS       | 6   | 60              | 169               | 125                 | 0.02              | 169              | 379                  | 215                 | 0.02                 | 60                   | 169                 | 125                 | 0.02                 | 60                   | 169                 | 125                 | 0.02                 |
| LANCZOS3LS       | 6   | 61              | 164               | 118                 | 0.02              | 179              | 392                  | 219                 | 0.02                 | 61                   | 164                 | 118                 | 0.02                 | 61                   | 164                 | 118                 | 0.02                 |
| LIARWHD          | 5000| 15              | 46                | 34                  | 0.08              | 21               | 45                   | 25                  | 0.06                 | 16                   | 47                  | 37                   | 0.05                 | 15                   | 41                  | 31                   | 0.03                 |
| LOGHAIKY         | 2   | 26              | 196               | 179                 | 0.02              | 27               | 81                   | 58                  | 0.02                 | 26                   | 196                 | 179                 | 0.02                 | 26                   | 196                 | 179                 | 0.02                 |
| LSC1LS           | 3   | 31              | 108               | 89                  | 0.02              | 36               | 101                  | 71                  | 0.02                 | 31                   | 108                 | 89                   | 0.02                 | 31                   | 108                 | 89                   | 0.02                 |
| LSC2LS           | 3   | 37              | 106               | 86                  | 0.02              | 54               | 119                  | 67                  | 0.02                 | 37                   | 106                 | 86                   | 0.02                 | 37                   | 106                 | 86                   | 0.02                 |
| LUKSAN11LS       | 100 | 928             | 1887              | 962                 | 0.03              | 955              | 1912                 | 957                 | 0.03                 | 926                 | 1894               | 973                 | 0.03                 | 2434                | 5355               | 3048                | 0.13                 |
| LUKSAN12LS       | 98  | 162             | 352               | 263                 | 0.02              | 160              | 302                  | 233                 | 0.02                 | 154                 | 340                 | 268                 | 0.02                 | 252                 | 529                 | 407                 | 0.01                 |
| LUKSAN13LS       | 98  | 82              | 168               | 128                 | 0.02              | 84               | 158                  | 121                 | 0.02                 | 85                   | 171                 | 142                 | 0.02                 | 142                 | 279                 | 243                 | 0.02                 |
| LUKSAN14LS       | 98  | 151             | 325               | 211                 | 0.02              | 98               | 122                  | 156                 | 0.02                 | 170                 | 370                 | 247                 | 0.02                 | 157                 | 313                 | 201                 | 0.02                 |

*Continued on next page*
| function         | DIM | No. Iter | No. fun.ev | No. grad.ev | CPU-time DL- | No. Iter | No. fun.ev | No. grad.ev | CPU-time DL- | No. Iter | No. fun.ev | No. grad.ev | CPU-time DL- | No. Iter | No. fun.ev | No. grad.ev | CPU-time DL- | No. Iter | No. fun.ev | No. grad.ev | CPU-time DL- |
|------------------|-----|----------|------------|-------------|--------------|-----------|------------|-------------|--------------|-----------|------------|-------------|--------------|-----------|------------|-------------|--------------|-----------|------------|-------------|--------------|
| LUKSAN15LS       | 100 | 25       | 57         | 42          | 0.02         | 28        | 59         | 44          | 0.02         | 26        | 57         | 42          | 0.02         | 27        | 60         | 45          | 0.02         |
| LUKSAN16LS       | 100 | 28       | 57         | 41          | 0.02         | 31        | 57         | 38          | 0.02         | 28        | 58         | 42          | 0.02         | 35        | 72         | 53          | 0.02         |
| MANCINO          | 100 | 12       | 30         | 19          | 0.09         | 11        | 23         | 12          | 0.06         | 11        | 23         | 12          | 0.06         | 11        | 23         | 12          | 0.06         |
| MARATOSB         | 2   | 589      | 2885       | 2585        | 0.02         | 1145      | 3657       | 2779        | 0.02         | 589       | 2885       | 2585        | 0.02         | 589       | 2885       | 2585        | 0.02         |
| MEXHAT           | 2   | 14       | 59         | 55          | 0.02         | 20        | 56         | 39          | 0.02         | 14        | 59         | 55          | 0.02         | 14        | 59         | 55          | 0.02         |
| MEYER3           | 3   | 19       | 76         | 63          | 0.02         | 19        | 67         | 52          | 0.02         | 19        | 76         | 63          | 0.02         | 19        | 76         | 63          | 0.02         |
| MGH09LS          | 4   | 25       | 82         | 72          | 0.02         | 57        | 137        | 86          | 0.02         | 25        | 82         | 72          | 0.02         | 25        | 82         | 72          | 0.02         |
| MGH10LS          | 3   | 1082     | 4052       | 4968        | 0.02         | 1134      | 4464       | 5357        | 0.03         | 1082      | 4052       | 4968        | 0.03         | 1082      | 4052       | 4968        | 0.02         |
| MGH10SLS         | 3   | 19       | 112        | 102         | 0.02         | 146       | 505        | 401         | 0.03         | 19        | 112        | 102         | 0.03         | 19        | 112        | 102         | 0.03         |
| MGH17LS          | 5   | 84       | 323        | 265         | 0.02         | 228       | 564        | 363         | 0.02         | 19        | 112        | 102         | 0.02         | 84        | 323        | 365         | 0.02         |
| MISRA1BLS.SIF    | 2   | 26       | 113        | 101         | 0.02         | 35        | 139        | 117         | 0.02         | 84        | 323        | 265         | 0.02         | 26        | 113        | 101         | 0.02         |
| MISRA1CLS.SIF    | 2   | 26       | 145        | 121         | 0.02         | 26        | 110        | 91          | 0.02         | 26        | 145        | 121         | 0.02         | 26        | 145        | 121         | 0.02         |
| MISRA1DLS.SIF    | 2   | 22       | 90         | 84          | 0.02         | 24        | 74         | 75          | 0.02         | 22        | 90         | 84          | 0.02         | 22        | 90         | 84          | 0.02         |
| MODBEALE.SIF     | 20000 | 47     | 108       | 65          | 1.22        | 517       | 1157       | 1010        | 13.17        | 158       | 340        | 218         | 3.53         | 224       | 473        | 304         | 4.89         |
| MOREBV           | 5000 | 161     | 168       | 317         | 0.31        | 161       | 168        | 317         | 0.42         | 161       | 168        | 317         | 0.3          | 117       | 124        | 229         | 0.23         |
| MSQRTALS         | 1024 | 2889    | 5787      | 2900        | 8.36        | 2905      | 5815       | 2911        | 8.89        | 2845      | 5699       | 2856        | 8.5          | 8953      | 17316      | 9581        | 28.81        |
| MSQRTBLS         | 1024 | 2354    | 4336      | 2744        | 7.48        | 2280      | 4525       | 2326        | 6.91        | 2359      | 4726       | 23694       | 7.63         | 5786      | 11558      | 5818        | 17.72        |
| NCB20            | 5010 | 1186    | 2748      | 1757        | 14.72       | 879       | 1511       | 1463        | 11.38       | 975       | 2265       | 1484        | 12.5         | 11026     | 20505      | 15341       | 129.2        |
| NELSONLS         | 3    | 1101    | 5415      | 7690        | 0.23        | 1118      | 5692       | 7331        | 0.17        | 1101      | 5415       | 7690        | 0.25         | 1101      | 5415       | 7690        | 0.23         |
| NONCVXU2         | 5000 | 6980    | 13302     | 7644        | 16.25       | 6610      | 12833      | 6999        | 15          | 7100      | 13344      | 7966        | 16.56       | 54585     | 94397      | 84907       | 182.92       |
| NONDIA           | 5000 | 7       | 25         | 19          | 0.01        | 7         | 25         | 20          | 0.03        | 7         | 25         | 19          | 0.03         | 7         | 25         | 19          | 0.03         |
| NONDQUAR         | 5000 | 2354    | 4770      | 2454        | 2.92        | 1942      | 3888       | 1947        | 2.3         | 2843      | 5758       | 2959        | 3.28         | 2349      | 4787       | 2488        | 2.83         |
| OSBORNEA         | 5    | 82      | 230       | 174         | 0.02        | 94        | 213        | 124         | 0.02        | 82        | 230        | 174         | 0.02         | 82        | 230        | 174         | 0.02         |

Continued on next page
| function     | DIM | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. | CPU- |
|--------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|
|              | Iter| fun.ev | grad.ev | CPU-time | No. iter | No. | Fun.ev | grad.ev | CPU-time | No. Iter | No. | Fun.ev | grad.ev | CPU-time | No. Iter | No. | CPU-time |
| OSBORNEB    | 11  | 57   | 134 | 84  | 0.02 | 62  | 127  | 65  | 0.02 | 57    | 134 | 84  | 0.02 | 57    | 134 | 84  | 0.02 |
| OSCIPATH    | 10  | 3E+05 | 8E+05 | 5E+05 | 2.19 | 3E+05 | 670953 | 4E+05 | 1.91 | 295029 | 8E+05 | 5E+05 | 2.3  | 3E+05 | 8E+05 | 5E+05 | 2.42 |
| PALMER1C    | 8   | 12   | 27  | 28  | 0.02 | 11  | 26   | 26  | 0.02 | 12    | 27  | 28  | 0.02 | 12    | 27  | 28  | 0.02 |
| PALMER1D    | 7   | 10   | 24  | 23  | 0.02 | 11  | 25   | 25  | 0.02 | 10    | 24  | 23  | 0.02 | 10    | 24  | 23  | 0.02 |
| PALMER2C    | 8   | 11   | 21  | 22  | 0.02 | 11  | 21   | 21  | 0.02 | 11    | 21  | 22  | 0.02 | 11    | 21  | 22  | 0.02 |
| PALMER3C    | 8   | 11   | 21  | 21  | 0.02 | 11  | 20   | 20  | 0.02 | 11    | 21  | 21  | 0.02 | 11    | 21  | 21  | 0.02 |
| PALMER4C    | 8   | 11   | 21  | 21  | 0.02 | 11  | 20   | 20  | 0.02 | 11    | 21  | 21  | 0.02 | 11    | 21  | 21  | 0.02 |
| PALMER5C    | 6   | 6    | 13  | 7   | 0.02 | 6   | 13   | 7   | 0.02 | 6     | 13  | 7   | 0.02 | 6     | 13  | 7   | 0.02 |
| PALMER6C    | 8   | 11   | 24  | 24  | 0.02 | 11  | 24   | 24  | 0.02 | 11    | 24  | 24  | 0.02 | 11    | 24  | 24  | 0.02 |
| PALMER7C    | 8   | 11   | 20  | 20  | 0.02 | 11  | 20   | 20  | 0.02 | 11    | 20  | 20  | 0.02 | 11    | 20  | 20  | 0.02 |
| PALMER8C    | 8   | 11   | 19  | 19  | 0.02 | 11  | 18   | 17  | 0.02 | 11    | 19  | 19  | 0.02 | 11    | 19  | 19  | 0.02 |
| PARKCH       | 15  | 263  | 592 | 361 | 9.31 | 672 | 1385 | 1128 | 29  | 740   | 1560 | 1359 | 34.86 | 412  | 982 | 611 | 16.02 |
| PENALTY1     | 1000| 20   | 63  | 50  | 0.02 | 28  | 69   | 44  | 0.02 | 14    | 51  | 43  | 0.02 | 14    | 51  | 43  | 0.02 |
| PENALTY2     | 200 | 189  | 223 | 359 | 0.03 | 191 | 221  | 354 | 0.03 | 192   | 226 | 365 | 0.03 | 337  | 480 | 758 | 0.06 |
| PENALTY3     | 200 | 24   | 77  | 60  | 0.45 | 99  | 285  | 219 | 1.74 | 80    | 295 | 247 | 1.94 | 102  | 346 | 290 | 2.19 |
| PENALTY3     | 200 | 24   | 77  | 60  | 0.5  | 99  | 285  | 219 | 1.75 | 80    | 295 | 247 | 1.92 | 102  | 346 | 290 | 2.22 |
| POWELLBSLS   | 2   | 50   | 211 | 234 | 0.02 | 61  | 247  | 246 | 0.02 | 50    | 211 | 234 | 0.02 | 50    | 211 | 234 | 0.02 |
| POWELLSG     | 5000| 33   | 89  | 64  | 0.06 | 26  | 53   | 27  | 0.03 | 27    | 68  | 47  | 0.03 | 36    | 92  | 65  | 0.05 |
| POWER        | 10000| 357 | 731 | 382 | 0.58 | 372 | 754  | 384 | 0.58 | 359   | 734 | 384 | 0.61 | 356  | 733 | 391 | 0.58 |
| POWERSUM     | 4   | 4    | 10  | 6   | 0.02 | 5   | 11   | 6   | 0.02 | 4     | 10  | 6   | 0.02 | 4     | 10  | 6   | 0.02 |
| PRICE3       | 2   | 10   | 25  | 17  | 0.02 | 11  | 23   | 12  | 0.02 | 10    | 25  | 17  | 0.02 | 10    | 25  | 17  | 0.02 |
| PRICE4       | 2   | 9    | 30  | 23  | 0.02 | 16  | 32   | 18  | 0.02 | 9     | 30  | 23  | 0.02 | 9     | 30  | 23  | 0.02 |

Continued on next page
| function      | DIM | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. | No. |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
|               | Iter| fun.ev | grad.ev | CPU-time | No. iter | No. Fun.ev | No. grad.ev | CPU-time | No. Iter | No. fun.ev | No. grad.ev | CPU-time | No. Iter | No. DL | CPU-time |
|               | DL-WYL | WYL | DL-WYL | WYL | DL-WYL | WYL | DL-WYL | WYL | DL-WYL | WYL | DL-WYL | WYL | DL-WYL | WYL | DL-WYL |
| QING          | 100 | 66 | 140 | 76 | 0.02 | 68 | 135 | 84 | 0.02 | 68 | 136 | 86 | 0.02 | 85 | 179 | 96 | 0.02 |
| QUARTC        | 5000 | 24 | 72 | 57 | 0.05 | 17 | 37 | 21 | 0.02 | 15 | 32 | 18 | 0.02 | 15 | 32 | 18 | 0.02 |
| RAT43LS.SIF   | 4 | 44 | 156 | 122 | 0.02 | 54 | 145 | 97 | 0.02 | 44 | 156 | 122 | 0.02 | 44 | 156 | 122 | 0.02 |
| RECIPELS.SIF  | 3 | 16 | 49 | 38 | 0.02 | 27 | 58 | 32 | 0.02 | 16 | 49 | 38 | 0.02 | 16 | 49 | 38 | 0.02 |
| ROSENBRO     | 2 | 28 | 84 | 65 | 0.02 | 34 | 77 | 44 | 0.02 | 28 | 84 | 65 | 0.02 | 28 | 84 | 65 | 0.02 |
| ROSENBRTU.SIF| 2 | 37 | 175 | 153 | 0.02 | 49 | 156 | 113 | 0.02 | 37 | 175 | 153 | 0.02 | 37 | 175 | 153 | 0.02 |
| S308          | 2 | 7 | 21 | 17 | 0.02 | 8 | 19 | 12 | 0.02 | 7 | 21 | 17 | 0.02 | 7 | 21 | 17 | 0.02 |
| SCHMVETT      | 5000 | 42 | 73 | 59 | 0.19 | 43 | 73 | 60 | 0.02 | 40 | 71 | 53 | 0.17 | 59 | 103 | 88 | 0.28 |
| SENSORS       | 100 | 26 | 67 | 48 | 0.47 | 21 | 50 | 34 | 0.02 | 27 | 70 | 49 | 0.41 | 24 | 71 | 53 | 0.47 |
| SINEVAL       | 2 | 46 | 181 | 153 | 0.02 | 64 | 144 | 88 | 0.02 | 46 | 181 | 153 | 0.02 | 46 | 181 | 153 | 0.02 |
| SINQUAD       | 5000 | 14 | 52 | 46 | 0.09 | 14 | 40 | 33 | 0.08 | 13 | 41 | 33 | 0.06 | 13 | 46 | 38 | 0.09 |
| SISER         | 2 | 5 | 19 | 19 | 0.02 | 6 | 18 | 14 | 0.02 | 5 | 19 | 19 | 0.02 | 5 | 19 | 19 | 0.02 |
| SNAIL         | 2 | 61 | 251 | 211 | 0.02 | 100 | 230 | 132 | 0.02 | 61 | 251 | 211 | 0.02 | 61 | 251 | 211 | 0.02 |
| SPMSRRTLS     | 4999 | 390 | 744 | 461 | 0.94 | 203 | 412 | 210 | 0.61 | 391 | 747 | 463 | 0.87 | 310 | 633 | 327 | 0.81 |
| SROSENBRO     | 5000 | 9 | 23 | 15 | 0.02 | 11 | 23 | 12 | 0.02 | 9 | 23 | 15 | 0.02 | 9 | 23 | 15 | 0.02 |
| SSI           | 3 | 307 | 1162 | 990 | 0.02 | 345 | 948 | 657 | 0.02 | 307 | 1162 | 990 | 0.02 | 307 | 1162 | 990 | 0.02 |
| STREG         | 4 | 60 | 218 | 180 | 0.02 | 96 | 224 | 139 | 0.02 | 60 | 218 | 180 | 0.02 | 60 | 218 | 180 | 0.02 |
| STRATEC       | 10 | 170 | 419 | 283 | 6.61 | 462 | 1043 | 796 | 19 | 170 | 419 | 283 | 6.83 | 170 | 419 | 283 | 6.3 |
| STRTCHDV.SIF  | 10 | 12 | 38 | 32 | 0.02 | 16 | 35 | 20 | 0.02 | 12 | 38 | 32 | 0.02 | 12 | 38 | 32 | 0.02 |
| TESTQUAD      | 5000 | 1573 | 1580 | 3141 | 1.27 | 1577 | 1584 | 3149 | 1.28 | 1573 | 1580 | 3141 | 1.25 | 20325 | 20361 | 40674 | 21.61 |
| THURBERLS     | 7 | 105 | 259 | 216 | 0.02 | 102 | 232 | 175 | 0.02 | 105 | 259 | 216 | 0.02 | 105 | 259 | 216 | 0.02 |
| TOINTGOR      | 50 | 115 | 213 | 146 | 0.02 | 135 | 233 | 174 | 0.02 | 124 | 222 | 164 | 0.02 | 192 | 348 | 270 | 0.02 |
| TOINTGSS      | 5000 | 4 | 9 | 5 | 0.02 | 4 | 9 | 5 | 0.02 | 4 | 9 | 5 | 0.02 | 4 | 9 | 5 | 0.02 |

Continued on next page
| function | DIM | No. | No. | CPU- | No. | No. | CPU- | No. | No. | CPU- | No. | No. | No. | CPU- |
|----------|-----|-----|-----|------|-----|-----|------|-----|-----|------|-----|-----|-----|------|
|          |     |     |     | Iter | fun.ev | grad.ev | time | CG- | Fun.ev | grad.ev | Descent | CG | Descent | WYL |
| TQINTPSP | 50  | 159 | 339 | 361  | 0.02  | 143  | 279 | 182 | 0.02  | 117  | 258  | 196 | 0.02  | 145 | 313  |
| TOINTQOR | 50  | 29  | 36  | 53   | 0.02  | 29   | 36  | 53  | 0.02  | 29   | 36   | 53  | 0.02  | 49  | 56   |
| TQUARTIC | 5000| 12  | 44  | 36   | 0.02  | 14   | 40  | 27  | 0.02  | 12   | 43   | 35  | 0.02  | 11  | 41   |
| TRIDIA   | 5000| 781 | 788 | 1557 | 0.87  | 782  | 7889| 1559| 0.89  | 782  | 789  | 1559| 0.91  | 4699| 4721 |
| TRIGON1  | 10  | 19  | 41  | 22   | 0.02  | 22   | 45  | 23  | 0.02  | 19   | 41   | 22  | 0.02  | 19  | 41   |
| TRIGON2  | 10  | 22  | 57  | 43   | 0.02  | 26   | 52  | 28  | 0.02  | 22   | 57   | 43  | 0.02  | 22  | 57   |
| VANDANMSLS. | 22 | 5   | 11  | 6    | 0.02  | 5    | 12  | 7   | 0.02  | 5    | 11   | 6   | 0.02  | 5   | 11   |
| VARDIM   | 200 | 10  | 24  | 19   | 0.02  | 10   | 21  | 11  | 0.02  | 9    | 20   | 15  | 0.02  | 9   | 20   |
| VAREIVGL | 50  | 24  | 51  | 29   | 0.02  | 23   | 47  | 21  | 0.02  | 24   | 51   | 29  | 0.02  | 28  | 71   |
| VESUVIALS | 8  | 1262| 1954| 3155 | 1.22  | 1519 | 2317| 4111| 1.56  | 1262 | 1954 | 3155| 1.45  | 1262| 1954 |
| VESUVIOLS | 8  | 79  | 211 | 173  | 0.09  | 80   | 180 | 131 | 0.06  | 79   | 211  | 173 | 0.14  | 79  | 211  |
| VIBRBEAM | 8   | 98  | 255 | 174  | 0.02  | 138  | 323 | 199 | 0.02  | 98   | 255  | 174 | 0.02  | 98  | 255  |
| WAYSEA1  | 2   | 11  | 55  | 50   | 0.02  | 18   | 39  | 22  | 0.02  | 11   | 55   | 50  | 0.02  | 11  | 55   |
| WAYSEA2  | 2   | 9   | 28  | 23   | 0.02  | 31   | 68  | 39  | 0.02  | 9    | 28   | 23  | 0.02  | 9   | 28   |
| WOODS    | 4000| 22  | 52  | 33   | 0.02  | 22   | 51  | 30  | 0.03  | 24   | 63   | 45  | 0.03  | 24  | 62   |
| YATP1CLS | 1E+05| 17  | 51  | 39   | 7.72  | 23   | 53  | 31  | 6.13  | 17   | 50   | 38  | 7.31  | 17  | 48   |
| YATP2CLS | 1E+05| 7   | 28  | 23   | 2.66  | 8    | 23  | 17  | 2.09  | 7    | 29   | 24  | 2.8   | 7   | 27   |
| YFITU    | 3   | 68  | 208 | 167  | 0.02  | 84   | 197 | 118 | 0.02  | 68   | 208  | 167 | 0.02  | 68  | 208  |
| ZANGWIL2 | 2   | 1   | 3   | 2    | 0.02  | 1    | 3   | 2   | 0.02  | 1    | 3    | 2   | 0.02  | 1   | 3    |
Figure 1. Performance measure based on the number of iterations.

Figure 2. Performance measure based on the CPU time.
Figure 3. Performance measure based on the gradient evaluations.

Figure 4. Performance measure based on the function evaluations.
5. Conclusions

In this paper, we propose a new three term CG method based on WYL and DL CG methods. We have also proved that the CG algorithm which employs the new coefficient has global convergence properties with the sufficient descent condition when SWP line search is employed. In numerical result part we show that the new three term CG method is more efficient than recent CG methods such as WYL, DL, and CG-Descent methods.

Availability of data and material

The data available inside the paper.

Acknowledgements

The authors are grateful for all these supported and improve our paper, also we would like to thank the University of Malaysia Terengganu (UMT) for funding this paper.

Conflict of interest

The authors declare that they have no competing interests.

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