EMERGENT DYNAMICS IN THE INTERACTIONS OF CUCKER-SMALE ENSEMBLES

SEUNG-YEAL HA
Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 08826, Korea

DONGNAM KO, YINGLONG ZHANG* AND XIONGTAO ZHANG
Department of Mathematical Sciences
Seoul National University, Seoul 08826, Korea

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Abstract. Merging and separation of flocking groups are often observed in our natural complex systems. In this paper, we employ the Cucker-Smale particle model to model such merging and separation phenomena. For definiteness, we consider the interaction of two homogeneous Cucker-Smale ensembles and present several sufficient frameworks for mono-cluster flocking, bi-cluster flocking and partial flocking in terms of coupling strength, communication weight, and initial configurations.

1. Introduction. The terminology “flocking” employed in this paper has a universal meaning representing some collective phenomena, in which self-propelled individuals using only limited environmental information and simple rules organize into an ordered motion [35]. For example, the aggregation of bacteria, flocking of birds, swarming of fish and herding of sheep correspond to flocking phenomena [4, 15, 16, 17, 18, 35, 36, 37]. It has been extensively studied in the literature [3, 6, 7, 8, 16, 17, 18] owing to possible applications to mobile and sensor networks, and in the control of robots and unmanned aerial vehicles [27, 30, 32]. After the pioneering works [26, 38] of Winfree and Kuramoto several decades ago, many phenomenological agent-based models have been proposed and studied analytically and numerically. Among others, our main interest lies on the second-order particle system proposed by Cucker and Smale [14]. This model resembles Newton’s equations for a many-body interaction system for point particles. Let \((x_i, v_i) \in \mathbb{R}^{2d}\) be the

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* Corresponding author: Yinglong Zhang.
phase-space coordinate of the $i$-th Cucker-Smale (C-S) flocking agent. Then, the dynamics of C-S ensemble is governed by the Cauchy problem:

$$\dot{x}_i = v_i, \quad t > 0, \quad i = 1, \cdots, N,$$

$$\dot{v}_i = \frac{K}{N} \sum_{j=1}^{N} \psi(||x_j - x_i||)(v_j - v_i),$$

$$(x_i, v_i)(0) = (x_{i0}, v_{i0}),$$

where $K$ is a positive coupling strength and $\psi$ is the communication weight satisfying the positivity, boundedness, continuity, and monotonicity conditions: there exists a positive constant $\psi_\infty$ such that

$$\psi_\infty \geq \psi(r) > 0, \quad r \geq 0, \quad \psi(\cdot) \in \text{Lip}(\mathbb{R}_+) \quad \text{and} \quad (\psi(r_2) - \psi(r_1))(r_2 - r_1) \leq 0, \quad r_1, r_2 \geq 0.$$  

In this paper, we are addressing the following situation. Suppose the situation where two homogeneous ensembles of C-S particles are interacting in the whole space. Then, what will happen asymptotically after they begin to interact? Do they form a single ensemble moving together? Or do they diverge as a separate flocking ensemble after the initial mixing? etc. We can think of several possible scenarios in this situation. Thus, our main interest is to provide several sufficient frameworks leading to the aforementioned possible scenarios. To fix the idea, let $\mathcal{G}_1 = \{(x_{1i}, v_{1i})\}_{i=1}^{N_1}$ and $\mathcal{G}_2 = \{(x_{2j}, v_{2j})\}_{j=1}^{N_2}$ be two homogeneous C-S ensembles. In the sequel, we call them subsystem $\mathcal{G}_1$ and subsystem $\mathcal{G}_2$, respectively. Here the adjective “homogeneous” means that each C-S particle in the same subsystem has the same mass, so that each particle is indistinguishable. Let $(x_{ai}(t), v_{ai}(t)) \in \mathbb{R}^{2d}$ be the phase-space coordinate of the $i$th Cucker-Smale flocking agent in group $\mathcal{G}_a$.

Consider the interacting Cucker-Smale flocking system:

$$\dot{x}_{1i} = v_{1i}, \quad \dot{x}_{2j} = v_{2j}, \quad t > 0, \quad i = 1, 2, \cdots, N_1, \quad j = 1, 2, \cdots, N_2,$$

$$\dot{v}_{1i} = \frac{K_1}{N_1} \sum_{k=1}^{N_1} \psi_1(||x_{1k} - x_{1i}||)(v_{1k} - v_{1i}) + \frac{K_d}{N_2} \sum_{k=1}^{N_2} \psi_d(||x_{2k} - x_{1i}||)(v_{2k} - v_{1i}),$$

$$\dot{v}_{2j} = \frac{K_2}{N_2} \sum_{k=1}^{N_2} \psi_2(||x_{2k} - x_{2j}||)(v_{2k} - v_{2j}) + \frac{K_d}{N_1} \sum_{k=1}^{N_1} \psi_d(||x_{1k} - x_{2j}||)(v_{1k} - v_{2j}),$$

where $K_1, K_2,$ and $K_d$ are nonnegative intra-system and inter-system coupling strengths, and the communication weight $\psi_a : \mathbb{R}_+ \to \mathbb{R}$ is Lipschitz continuous and satisfies the following conditions:

$$0 < \psi_\alpha(s) \leq \psi_\alpha(0) = 1 < +\infty, \quad \psi_\alpha(s) \in L^1(\mathbb{R}_+), \quad \alpha = 1, 2, d,$$

$$(\psi_\alpha(s_2) - \psi_\alpha(s_1))(s_2 - s_1) \leq 0, \quad s_1, s_2 \in \mathbb{R}_+. \quad (1.4)$$

Note that if we turn off inter-system coupling strength $K_d = 0$, then system (1.3) becomes the collection of two C-S models. The well-posedness of system (1.3) - (1.4) is obvious owing to the standard Cauchy-Lipschitz theory of ordinary differential equations.

The main novelty of this paper is threefold. First, we present a sufficient framework for a mono-cluster flocking to the combined system (1.3)-(1.4). It turns out that the key factor for the emergence of mono-cluster flocking is basically dependent on the inter-system coupling strength $K_d$. For a large inter-system coupling
strength \(K_d\), the combined system leads to mono-cluster flocking for any nonnegative intra-system coupling strengths \(K_1\) and \(K_2\) (Theorem 3.3) for some admissible class of initial configurations. Second, we deal with a sufficient framework for the bi-cluster flocking of subsystems \(G_1\) and \(G_2\). In this case, the inter-system coupling strength should be small, but the intra-system coupling strength should be large. We quantify this plausible guess by providing explicit lower and upper bounds for \(K_\alpha\) and \(K_d\) in terms of initial configuration only. Third, we present a sufficient framework for a partial flocking. More precisely, we present the conditions for local flocking of subsystem \(G_1\).

The rest of this paper is organized as follows. In Section 2, we give the propagation of velocity moments and previous results on the flocking formations to the single ensemble of C-S particles, review some relevant results, and discuss the difference between our results presented in this paper. In Section 3, we study a sufficient framework leading to the formation of mono-cluster flocking of two ensembles. In Section 4, we present a sufficient framework leading to bi-cluster flocking, i.e., each ensemble flocks together, but the whole combined ensemble does not flock. In Section 5, we present a sufficient framework for the partial flocking. Under our framework, only one of the ensembles flocks, whereas the other ensemble does not flock. Finally, Section 7 is devoted to a brief summary of our paper.

**Notation.** Throughout the paper, we use a superscript to denote the component of a vector, e.g., \(x := (x^1, \ldots, x^d) \in \mathbb{R}^d\). Subscripts are used to represent the ordering of particles. For vectors \(x, v \in \mathbb{R}^d\), its \(\ell_2\)-norm and the inner product are defined as follows:

\[
\|x\| := \left( \sum_{i=1}^{d} (x^i)^2 \right)^{\frac{1}{2}}, \quad \langle x, v \rangle := \sum_{i=1}^{d} x^i v^i,
\]

where \(x^i\) and \(v^i\) are the \(i\)th components of \(x\) and \(v\), respectively.

2. **Preliminaries.** In this section, we briefly review the flocking theorems for the C-S model (1.1) and present estimates on the propagation of velocity moments.

2.1. **The Cucker-Smale model.** In this subsection, we briefly review available flocking estimates for the emergence of mono-cluster flocking (global flocking) for the C-S model in (1.1)-(1.2). Mono-cluster flocking formation for system (1.1) was first studied by Cucker and Smale [14] for a special ansatz of \(\psi_{cs}(s) = (1 + s^2)^{-\frac{\beta}{2}}, \quad \beta \geq 0\). They showed that mono-cluster flocking occurs for any initial data for the case of long-range communication \(\beta \in [0, 1)\), whereas for the short-range case, they showed that mono-cluster flocking is possible for initial configurations close to the flocking state using the self-bounding argument. Later, Cucker and Smale’s results were further refined and generalized to several physical settings, e.g., stochastic perturbations [2, 13, 19, 22], bonding force and formation control [31], collision avoidance [1, 11], effect of informed agent [12], nonlinear friction [21], relation with mechanical model [24], network effects [28, 29, 34], kinetic and fluid description [20, 23, 25], etc. So far, the most refined flocking estimates on the mono-cluster formation for (1.1) can be summarized as follows. We first define a mixed norm:

\[
||x||_\infty := \max_{1 \leq i \leq N} ||x_i||, \quad ||v||_\infty := \max_{1 \leq i \leq N} ||v_i||,
\]

where \(|| \cdot ||\) is the standard \(\ell^2\)-norm in \(\mathbb{R}^d\).
Theorem 2.1. [1, 23] Let \((x, v)\) be a solution to (1.1)-(1.2) with initial data \((x_0, v_0)\) satisfying the following conditions:

\[
\|x_0\|_\infty > 0, \quad \|v_0\| < K \int_{\|x_0\|}^\infty \psi(2r) dr. \quad (2.1)
\]

Then there exists a positive number \(x_M\) such that

\[
\sup_{t \geq 0} \|x(t)\| \leq x_M, \quad \|v(t)\| \leq \|v_0\| e^{-\psi(2x_M)t}, \quad t \geq 0.
\]

Remark 2.2. 1. Note that Theorem 2.1 yields a sufficient condition for mono-cluster flocking. The condition (2.1) is a sufficient and necessary condition for mono-cluster flocking only for \(N = 2\).

2. For a small coupling strength \(K \ll 1\), bi-cluster and multi-cluster flocking can emerge from the initial configurations, depending on the geometry of the data. Recently, in Refs. [9, 10], it has been shown that local flocking, in particular bi-cluster flocking, can emerge from some well-prepared configurations close to bi-cluster configurations.

2.2. Propagation of velocity moments. In this subsection, we study the temporal evolution of the normalized first and second velocity moments. For this, we set

\[
m_{1,1} := \frac{1}{N_1} \sum_{i=1}^{N_1} v_{1i}, \quad m_{1,2} := \frac{1}{N_2} \sum_{i=1}^{N_2} v_{2i},
\]

\[
m_{2,1} := \frac{1}{N_1} \sum_{i=1}^{N_1} \|v_{1i}\|^2, \quad m_{2,2} := \frac{1}{N_2} \sum_{i=1}^{N_2} \|v_{2i}\|^2,
\]

\[
M_1 := m_{1,1} + m_{1,2}, \quad M_2 := m_{2,1} + m_{2,2}.
\]

Then, \(m_{i,j}\) satisfies the following estimates.

Lemma 2.3. Let \((x, v)\) be a global solution of the coupled system (1.3). Then, we have

\[
(i) \quad \frac{dm_{1,1}}{dt} = \frac{K_d}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|x_{2k} - x_{1i}\|) (v_{2k} - v_{1i}).
\]

\[
(ii) \quad \frac{dm_{1,2}}{dt} = -\frac{K_d}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|x_{1i} - x_{2k}\|) (v_{2k} - v_{1i}).
\]

\[
(iii) \quad \frac{dm_{2,1}}{dt} = -\frac{K_d}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|x_{1k} - x_{1i}\|) \|v_{1k} - v_{1i}\|^2
\]

\[
+ \frac{2K_d}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|x_{2k} - x_{1i}\|) \langle v_{1i}, v_{2k} - v_{1i} \rangle.
\]

\[
(iv) \quad \frac{dm_{2,2}}{dt} = -\frac{K_d}{N_2^2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|x_{2k} - x_{2i}\|) \|v_{2k} - v_{2i}\|^2
\]

\[
- \frac{2K_d}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|x_{1i} - x_{2k}\|) \langle v_{2k}, v_{2k} - v_{1i} \rangle.
\]

Proof. In the sequel, we only prove the assertions (i) and (iii). The estimates for (ii) and (iv) can be treated similarly.
(i) We use the symmetry of $\psi_1$, $l = 1, 2$ in the transformation

$(i, k) \iff (k, i)$

to obtain

$$\frac{dm_{1,1}}{dt} = \frac{1}{N_1} \sum_{i=1}^{N_1} \dot{\psi}_{1i} = \frac{K_1}{N_1^2} \sum_{k=1}^{N_1} \sum_{i=1}^{N_1} \psi_1(\|x_{1k} - x_{1i}\|)(v_{1k} - v_{1i})$$

$$+ \frac{K_d}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|x_{2k} - x_{1i}\|)(v_{2k} - v_{1i})$$

$$= \frac{K_d}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|x_{2k} - x_{1i}\|)(v_{2k} - v_{1i}).$$

(iii) We take an inner product $(1.3)_2$ with $2\psi_1$ to find

$$\frac{dm_{2,1}}{dt} = \frac{2}{N_1} \sum_{i=1}^{N_1} (\psi_{1i}, \dot{\psi}_{1i})$$

$$= -\frac{K_1}{N_1^2} \sum_{k=1}^{N_1} \sum_{i=1}^{N_1} \psi_1(\|x_{1k} - x_{1i}\|)\|v_{1k} - v_{1i}\|^2$$

$$+ \frac{2K_d}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|x_{2k} - x_{1i}\|)\langle v_{1i}, v_{2k} - v_{1i} \rangle.$$
Note that the dynamics of \((x, v)\). Then, we have
\[
M_1(t) = M_1(0), \quad M_2(t) \leq M_2(0), \quad t \geq 0.
\]

3. Emergence of mono-cluster flocking. In this section, we study a sufficient condition for the mono-cluster flocking in the interaction of two homogeneous C-S ensembles. We will see that the inter-ensemble coupling strength \(K_d\) will play a key role in the mono-cluster flocking estimates as long as the intra-ensemble coupling strengths \(K_1\) and \(K_2\) are nonnegative.

3.1. Lyapunov functionals. In this subsection, we introduce nonlinear functionals measuring the formation of mono-cluster flocking for system (1.3). In order to study the global flocking, we introduce the global averages and fluctuations around them:
\[
x_c := \frac{1}{2} \left( \frac{1}{N_1} \sum_{i=1}^{N_1} x_{1i} + \frac{1}{N_2} \sum_{j=1}^{N_2} x_{2j} \right), \quad v_c := \frac{1}{2} \left( \frac{1}{N_1} \sum_{i=1}^{N_1} v_{1i} + \frac{1}{N_2} \sum_{j=1}^{N_2} v_{2j} \right),
\]
\[
\hat{x}_{\alpha i} := x_{\alpha i} - x_c, \quad \hat{v}_{\alpha i} := v_{\alpha i} - v_c, \quad \alpha = 1, 2.
\]

Then \((x_c, v_c)\) and \((\hat{x}_\alpha, \hat{v}_\alpha)\) satisfy
\[
\dot{x}_c = v_c, \quad \dot{v}_c = 0, \quad \dot{\hat{x}}_{1i} = \hat{v}_{1i}, \quad \dot{\hat{v}}_{1i} = \hat{v}_{2j}, \quad t > 0,
\]
\[
\dot{\hat{v}}_{1i} = \frac{K_1}{N_1} \sum_{k=1}^{N_1} \psi_1(\|\hat{x}_{1k} - \hat{x}_{1i}\|) (\hat{v}_{1k} - \hat{v}_{1i}) + \frac{K_d}{N_2} \sum_{k=1}^{N_2} \psi_d(\|\hat{x}_{2k} - \hat{x}_{1i}\|) (\hat{v}_{2k} - \hat{v}_{1i}) + \frac{K_d}{N_1} \sum_{k=1}^{N_1} \psi_d(\|\hat{x}_{1k} - \hat{x}_{2j}\|) (\hat{v}_{1k} - \hat{v}_{2j}).
\]
\[
\dot{\hat{v}}_{2j} = \frac{K_2}{N_2} \sum_{k=1}^{N_2} \psi_2(\|\hat{x}_{2k} - \hat{x}_{2j}\|) (\hat{v}_{2k} - \hat{v}_{2j}) + \frac{K_d}{N_1} \sum_{k=1}^{N_1} \psi_d(\|\hat{x}_{1k} - \hat{x}_{2j}\|) (\hat{v}_{1k} - \hat{v}_{2j}).
\]

Note that the dynamics of \((x_c, v_c)\) and \((\hat{x}_\alpha, \hat{v}_\alpha)\) are coupled except for \(N_1 = N_2\).

We now define Lyapunov functionals \(\mathcal{X}\) and \(\mathcal{V}\) as the weighted \(l^2\)-norms:
\[
\mathcal{X} := \left( \frac{1}{N_1} \sum_{i=1}^{N_1} \|\hat{x}_{1i}\|^2 + \frac{1}{N_2} \sum_{j=1}^{N_2} \|\hat{x}_{2j}\|^2 \right)^\frac{1}{2},
\]
\[
\mathcal{V} := \left( \frac{1}{N_1} \sum_{i=1}^{N_1} \|\hat{v}_{1i}\|^2 + \frac{1}{N_2} \sum_{j=1}^{N_2} \|\hat{v}_{2j}\|^2 \right)^\frac{1}{2}.
\]

Note that \(\mathcal{X}\) and \(\mathcal{V}\) measure the deviations from the global averages, and it is easy to see that the functional \(\mathcal{X}\) and \(\mathcal{V}\) are Lipschitz continuous in \(t\), so it is differentiable.
for almost all $t \in (0, \infty)$. Before we proceed to the flocking estimate, we recall the definition of mono-cluster flocking as follows.

**Definition 3.1.** Let $(x, v)$ be a global solution to (1.3). We call the subsystems $G_1$ and $G_2$ exhibit a time-asymptotic mono-cluster flocking if $\mathcal{X}$ and $\mathcal{V}$ satisfy

$$\sup_{0 \leq t < \infty} \mathcal{X}(t) < \infty, \quad \lim_{t \to \infty} \mathcal{V}(t) = 0.$$

### 3.2. Temporal evolution of the Lyapunov functionals

In this subsection, we derive a system of dissipative differential inequalities (SDDI) for $(\mathcal{X}, \mathcal{V})$ in (3.2).

Note that, in the following sections, we let $x := (x_1, x_2)$, $v := (v_1, v_2)$, and $N_1 + N_2 := N$.

**Proposition 3.2.** Let $(x, v)$ be a global solution to (1.3) with

$$K_\alpha \geq 0, \quad \alpha = 1, 2, \quad K_d > 0.$$

Then, the Lyapunov functionals defined in (3.2) satisfy

$$\frac{d\mathcal{X}}{dt} \leq \mathcal{V}, \quad \frac{d\mathcal{V}}{dt} \leq -K_d \psi_d(\sqrt{2N\mathcal{X}})\mathcal{V}, \quad a.e. \ t \in (0, \infty).$$

**Proof.** (i) (Derivation of the first inequality): By the definition of $\mathcal{X}$ and the Cauchy inequality, we have

$$\left| \frac{d\mathcal{X}^2}{dt} \right| = \left| \frac{2}{N_1} \sum_{i=1}^{N_1} \langle \dot{x}_{1i}, \dot{v}_{1i} \rangle + \frac{2}{N_2} \sum_{j=1}^{N_2} \langle \dot{x}_{2j}, \dot{v}_{2j} \rangle \right|$$

$$\leq 2 \left( \frac{1}{N_1} \sum_{i=1}^{N_1} \| \dot{x}_{1i} \|^2 \right)^{\frac{1}{2}} \left( \frac{1}{N_1} \sum_{i=1}^{N_1} \| \dot{v}_{1i} \|^2 \right)^{\frac{1}{2}}$$

$$+ \frac{2}{N_2} \sum_{j=1}^{N_2} \| \dot{x}_{2j} \| \| \dot{v}_{2j} \|$$

$$\leq 2 \left[ \frac{1}{N_1} \sum_{i=1}^{N_1} \| \dot{x}_{1i} \|^2 \right] \left[ \frac{1}{N_2} \sum_{j=1}^{N_2} \| \dot{v}_{2j} \|^2 \right]^{\frac{1}{2}}$$

$$= 2 \mathcal{X} \mathcal{V}.$$

This yields the desired first inequality.

(ii) (Derivation of the second inequality): We multiply (3.1) by $2\dot{v}_{1i}$, and (3.1) by $2\dot{v}_{2j}$, and sum the results together, and then using similar calculations to the proof of Lemma 2.3 (ii), we have

$$\frac{d\mathcal{V}^2}{dt} = \frac{2}{N_1} \sum_{i=1}^{N_1} \langle \dot{v}_{1i}, \dot{v}_{1i} \rangle + \frac{2}{N_2} \sum_{j=1}^{N_2} \langle \dot{v}_{2j}, \dot{v}_{2j} \rangle$$

$$= -K_1 \frac{N_1}{N_1} \sum_{i=1}^{N_1} \sum_{k=1}^{N_1} \psi_1(||\dot{x}_{1k} - \dot{x}_{1i}||) \|\dot{v}_{1k} - \dot{v}_{1i}\|^2$$

$$- K_2 \frac{N_2}{N_2} \sum_{k=1}^{N_2} \sum_{j=1}^{N_2} \psi_2(||\dot{x}_{2k} - \dot{x}_{2j}||) \|\dot{v}_{2k} - \dot{v}_{2j}\|^2$$

$$- 2K_\alpha \frac{N_1 N_2}{N_1 N_2} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \psi_3(||\dot{x}_{2i} - \dot{x}_{1j}||) \|\dot{v}_{2i} - \dot{v}_{1j}\|^2$$
Proof. where we used

Suppose that initial data $x_0$, $v_0$ are given and the intra- and inter-ensemble coupling strengths $K_\alpha$ and $K_d$ satisfy the following conditions:

$$K_\alpha \geq 0, \quad \alpha = 1, 2, \quad |\dot{x}_{2k} - \dot{x}_{1i}|^2 \leq 2N\lambda^2.$$  

On the other hand, note that

$$\frac{1}{N_1} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|\dot{x}_{2k} - \dot{x}_{1i}\|)\|\dot{v}_{2k} - \dot{v}_{1i}\|^2$$

$$\leq - \frac{2K_d}{N_1 N_2} \psi_d(\sqrt{2N\lambda}) \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \|\dot{v}_{2k} - \dot{v}_{1i}\|^2$$

$$= - \frac{2K_d}{N_1} \psi_d(\sqrt{2N\lambda}) \sum_{i=1}^{N_1} \|\dot{v}_{1i}\|^2 - \frac{2K_d}{N_2} \psi_d(\sqrt{2N\lambda}) \sum_{k=1}^{N_2} \|\dot{v}_{2k}\|^2$$

$$+ 4K_d \psi_d(\sqrt{2N\lambda}) \left( \frac{1}{N_1} \sum_{i=1}^{N_1} \dot{v}_{1i} \right) \cdot \left( \frac{1}{N_2} \sum_{k=1}^{N_2} \dot{v}_{2k} \right),$$

where we used

$$K_\alpha \geq 0, \quad \alpha = 1, 2, \quad |\dot{x}_{2k} - \dot{x}_{1i}|^2 \leq 2N\lambda^2.$$  

Then, we substitute the relation $\frac{1}{N_1} \sum_{k=1}^{N_2} \dot{v}_{2k} = - \frac{1}{N_1} \sum_{i=1}^{N_1} \dot{v}_{1i}$ into (3.3) to get

$$\frac{d\gamma^2}{dt} \leq - \frac{2K_d}{N_1} \psi_d(\sqrt{2N\lambda}) \sum_{i=1}^{N_1} \|\dot{v}_{1i}\|^2 - \frac{2K_d}{N_2} \psi_d(\sqrt{2N\lambda}) \sum_{k=1}^{N_2} \|\dot{v}_{2k}\|^2$$

$$- \frac{4K_d}{N_1} \psi_d(\sqrt{2N\lambda}) \sum_{i=1}^{N_1} \|\dot{v}_{1i}\|^2$$

$$\leq - \frac{2K_d}{N_1} \psi_d(\sqrt{2N\lambda}) \sum_{i=1}^{N_1} \|\dot{v}_{1i}\|^2 - \frac{2K_d}{N_2} \psi_d(\sqrt{2N\lambda}) \sum_{k=1}^{N_2} \|\dot{v}_{2k}\|^2$$

$$= - 2K_d \psi_d(\sqrt{2N\lambda}) \gamma^2.$$  

This yields the desired differential inequality for $\gamma$. \hfill \Box

3.3. Emergence of a mono-cluster flocking. In this subsection, we provide the proof of the emergence of mono-cluster flocking using the SDDI in Proposition 3.2. We now present our first main result.

**Theorem 3.3.** Suppose that initial data $(x_0, v_0)$ are given and the intra- and inter-ensemble coupling strengths $K_\alpha$ and $K_d$ satisfy the following conditions:

$$K_\alpha \geq 0, \quad \alpha = 1, 2, \quad K_d > \frac{V_0}{\int_{x_0}^{\infty} \psi_d(\sqrt{2N\lambda}) \sqrt{x} \, dx}.$$  

Then, for global solution $(x, v)$ to (1.3), there exists a positive constant $x_{1M}$ such that

$$\sup_{0 \leq t < \infty} x_1(t) \leq x_{1M}, \quad \gamma(t) \leq V_0 e^{-K_d \psi_d(\sqrt{2N\lambda} t)} \sqrt{x}, \quad t \in [0, \infty).$$

**Proof.** Step A (Existence of $x_{1M}$). It follows from Proposition 3.2 that we have

$$\left| \frac{dx}{dt} \right| \leq \gamma, \quad \frac{d\gamma}{dt} \leq -K_d \psi_d(\sqrt{2N\lambda}) \gamma, \quad \text{a.e. } t \in (0, \infty). \quad (3.5)$$
We now define a Lyapunov functional $L_0$ following [23]:

$$L_0(t) := V(t) + K_d \int_0^{X(t)} \psi_d(\sqrt{2N}x)dx, \quad t \in (0, \infty). \quad (3.6)$$

Then, we use (3.5) and (3.6) to obtain

$$\frac{dL_0}{dt} = \frac{dV}{dt} + K_d \psi_d(\sqrt{2N}X) \frac{dX}{dt} \leq -K_d \psi_d(X) \left( V - \frac{dX}{dt} \right) \leq 0.$$  

This yields

$$L_0(t) \leq L_0(0), \quad t \in (0, \infty),$$  

or equivalently

$$V(t) + K_d \int_{X_0}^{X(t)} \psi_d(\sqrt{2N}\xi)d\xi \leq V_0, \quad t \in (0, \infty).$$

In particular, this yields

$$K_d \int_{X_0}^{X(t)} \psi_d(\sqrt{2N}\xi)d\xi \leq V_0, \quad t \in (0, \infty). \quad (3.7)$$

We set

$$F(\beta) := K_d \int_{X_0}^{\beta} \psi_d(\sqrt{2N}\xi)d\xi, \quad \beta \geq 0.$$  

Then, $F(\beta)$ is a continuous and increasing function of $\beta$, and by assumption (3.4), we have

$$0 = F(X_0) < V_0 < \lim_{\beta \to \infty} F(\beta).$$

Hence, by the intermediate value theorem, we can choose the largest value of $x_{1M}$ such that

$$K_d \int_{X_0}^{x_{1M}} \psi_d(\sqrt{2N}\xi)d\xi = V_0.$$  

Then, we claim

$$\sup_{0 \leq t < \infty} \mathcal{X}(t) \leq x_{1M}. \quad (3.8)$$

**Proof of claim (3.8).** Suppose not, i.e., there exists $t_* \in (0, \infty)$ such that

$$\mathcal{X}(t_*) > x_{1M}.$$  

Then, for such $\mathcal{X}(t_*)$, we have

$$K_d \int_{X_0}^{\mathcal{X}(t_*)} \psi_d(\sqrt{2N}\xi)d\xi > K_d \int_{X_0}^{x_{1M}} \psi_d(\sqrt{2N}\xi)d\xi = V_0,$$

which is contradictory to (3.7).

**Step B (Exponential decay of $V$).** We use (3.8) and the non-increasing property of $\psi_d$ to obtain

$$\frac{dV(t)}{dt} \leq -K_d \psi_d(\sqrt{2N}\mathcal{X})V \leq -K_d \psi_d(\sqrt{2N}x_{1M})V(t), \quad \text{a.e. } t \in (0, \infty).$$

This yields the desired result. \qed
Remark 3.4. 1. Note that in (1.4), we assume that the communication weights $\psi_\alpha$ is assumed to be Lipschitz continuous to guarantee the global well-posedness of the coupled system (1.3). However, in the proofs of Theorem 3.3, Theorem 4.2 and Theorem 5.1, we only need $\psi_\alpha$ to be integrable; while in Corollary 4.4 and Corollary 5.2, we need the boundedness of $\psi_\alpha$ to guarantee the existence of the finite time $T_0$. Thus, in principle our flocking estimates can be done for the coupled particle system (1.3) and its kinetic counterpart with singular communication weights [5, 23, 33] in a priori settings. However, we leave this issue for future work.

2. The condition (3.4) on the lower bound for $K_d$ implies that, as $V_0$ increases or $X_0$ increases, the lower bound for $K_d$ increases. This is what we can expect to happen.

3. Consider the system with a bi-partite interaction, i.e., there is no intra-ensemble interaction, i.e., $K_1 = K_2 = 0$: for $i = 1, 2, \cdots, N_1$, $j = 1, 2, \cdots, N_2$,

$$\dot{x}_{1i} = v_{1i}, \quad \dot{x}_{2j} = v_{2j}, \quad t > 0,$$

$$\dot{v}_{1i} = \frac{K_d}{N_2} \sum_{k=1}^{N_2} \psi_d(||x_{2k} - x_{1i}||)(v_{2k} - v_{1i}),$$

$$\dot{v}_{2j} = \frac{K_d}{N_1} \sum_{k=1}^{N_1} \psi_d(||x_{1k} - x_{2j}||)(v_{1k} - v_{2j}).$$

Then, the result of Theorem 3.3 yields that, as long as the inter-ensemble coupling strength $K_d$ is sufficiently large, we still have mono-cluster flocking for the initial configuration. This is a rather counterintuitive result.

In the following two sections, we study the formation of bi-cluster and multi-cluster flocking.

4. Emergence of bi-cluster flocking. In this section, we study the dynamics of system (1.3) in a small inter-coupling regime $K_d \ll 1$. In this regime, we present sufficient conditions where each sub-ensemble $G_1$ and $G_2$ flock by themselves, but there is no mono-cluster flocking. Note that, for a large inter-ensemble coupling regime $K_d > \frac{V_0}{\int_{X_0} \psi_d(\sqrt{2N_x})dx}$, we have a mono-cluster flocking wherein two sub-ensembles flock together independent of the detailed geometry of the initial configurations.

4.1. Description of the main results. In this subsection, we briefly discuss our main results on the formation of bi-cluster flocking. Since we have bi-cluster flocking asymptotics in mind, we introduce local ensemble averages and local fluctuations around them: for $\alpha = 1, 2$, we set

$$\dot{x}_{\alpha c} := \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} x_{\alpha i}, \quad \dot{v}_{\alpha c} := \frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} v_{\alpha i}, \quad \dot{x}_{\alpha i} := x_{\alpha i} - x_{\alpha c},$$

$$\dot{v}_{\alpha i} := v_{\alpha i} - v_{\alpha c}, \quad \alpha := \left(\frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} ||x_{\alpha i}||^2\right)^{\frac{1}{2}}, \quad \dot{\alpha} := \left(\frac{1}{N_\alpha} \sum_{i=1}^{N_\alpha} ||\dot{x}_{\alpha i}||^2\right)^{\frac{1}{2}},$$

$$||\dot{x}_{\alpha}||_{\infty} := \sup_{1 \leq i \leq N_\alpha} ||\dot{x}_{\alpha i}||, \quad ||\dot{v}_{\alpha}||_{\infty} := \sup_{1 \leq i \leq N_\alpha} ||\dot{v}_{\alpha i}||.$$
Here we use the same notation for the local fluctuations as for the global fluctuations in Section 3 for notational simplicity. Then, it is easy to see that
\[ \sum_{i=1}^{N_1} x_{\alpha i} = 0, \quad \sum_{i=1}^{N_2} \dot{x}_{\alpha i} = 0, \quad \alpha = 1, 2. \]
And then \((x_{\alpha c}, v_{\alpha c})\) and \((\dot{x}_{\alpha c}, \dot{v}_{\alpha c})\) satisfy
\[ \dot{x}_{1i} = v_{1i}, \quad \dot{x}_{2i} = v_{2i}, \quad \dot{x}_{1i} = \dot{v}_{1i}, \quad \dot{x}_{2i} = \dot{v}_{2i}, \quad t > 0, \]
\[ v_{1c} = \frac{K_d}{N_1 N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(||x_{2k} - x_{1i}||)(v_{2k} - v_{1i}), \]
\[ \dot{v}_{1c} = -v_{1c} + \frac{K_1}{N_1} \sum_{j=1}^{N_1} \psi_1(||\dot{x}_{1j} - \dot{x}_{1i}||)(\dot{v}_{1j} - \dot{v}_{1i}) \]
\[ + \frac{K_d}{N_2} \sum_{k=1}^{N_2} \psi_d(||x_{2k} - x_{1i}||)(v_{2k} - v_{1i}), \]
\[ \dot{v}_{2c} = -v_{2c} + \frac{K_2}{N_2} \sum_{k=1}^{N_2} \psi_2(||\dot{x}_{2k} - \dot{x}_{2j}||)(\dot{v}_{2k} - \dot{v}_{2j}) \]
\[ + \frac{K_d}{N_1} \sum_{i=1}^{N_1} \psi_d(||x_{1k} - x_{2j}||)(v_{1k} - v_{2j}). \]

**Definition 4.1.** Let \((x, v)\) be a global solution to the coupled system (1.3).
1. The subsystem \(G_i\) exhibits a time-asymptotic flocking if and only if
   \[ \sup_{0 \leq t < \infty} \mathcal{X}_i(t) < \infty, \quad \lim_{t \to \infty} \mathcal{V}_i(t) = 0, \quad i = 1, 2. \]
2. The whole system \((G_1, G_2)\) exhibits a time-asymptotic bi-cluster flocking if and only if both subsystems \(G_1\) and \(G_2\) exhibit a time-asymptotic flocking, but the whole system does not exhibit a time-asymptotic mono-cluster flocking.
3. The whole system \((G_1, G_2)\) exhibits a time-asymptotic partial flocking if and only if only one of \(G_1\) and \(G_2\) exhibits a time-asymptotic flocking, but the other does not.

Our main results on the emergence of bi-cluster flocking can be summarized as follows.

**Theorem 4.2.** Suppose that the following framework \((\mathcal{F}_A)\) holds for the initial data \((x_0, v_0)\) to system (1.3).
\[ \bullet \text{\((\mathcal{F}_A)1\): (Restriction on initial configurations)} \]
\[ \lambda_0 := \frac{1}{2} ||v_{2c}(0) - v_{1c}(0)|| > 0, \]
\[ \max_{1 \leq i \leq N_1} ||v_{1i}(0) - v_{1c}(0)|| \leq \frac{1}{4} \lambda_0, \quad \max_{1 \leq k \leq N_2} ||v_{2k}(0) - v_{2c}(0)|| \leq \frac{1}{4} \lambda_0, \]
\[ \min_{1 \leq i \leq N_1} \left\{ (x_{2i}(0) - x_{1i}(0)) \cdot (v_{2c}(0) - v_{1c}(0)) \right\} \geq 0. \]
\[ K_\alpha > \frac{\nu_\alpha(0) + K_\delta \sqrt{2NM_2(0)} \int_0^\infty \psi_d(x)dx}{\int_{X_\alpha(0)} \psi_\alpha(\sqrt{2N_\alpha x})dx}, \quad 0 \leq K_d < \frac{\lambda^2}{12 \sqrt{2NM_2(0)} \int_0^\infty \psi_d(x)dx}. \]

Then, the whole system \((G_1, G_2)\) exhibits a time-asymptotic bi-cluster flocking. More precisely, for the solution \((x, v)\) to system (1.3) with initial data \((x_0, v_0)\), there exist positive constants \(x_0^\infty\) and \(C_\alpha\), \(\alpha = 1, 2\) that depend only on the initial data and \(\psi\) such that

\[ \sup_{0 \leq t < \infty} \nu_\alpha(t) \leq x_0^\infty, \quad \nu_\alpha(t) \leq C_\alpha \max \left\{ e^{-\frac{\nu_\alpha(x_0(x_0^\infty)^\alpha)t}{2}}, \psi_\alpha \left( \frac{\lambda_0}{2} t \right) \right\}, \]

\[ \min_{0 \leq t < \infty, i, k} \| x_{2k}(t) - v_i(t) \| \geq \lambda_0, \quad \min_{0 \leq t < \infty, i, k} \| x_{1i}(t) - v_1c(t) \| \geq \lambda_0 t, \quad t \in [0, \infty). \]

**Remark 4.3.** 1. The last geometric condition \(\min_{1 \leq i \leq N_1} \min_{1 \leq k \leq N_2} \{(x_{2k}(0) - x_{1i}(0)) \cdot (v_{2c}(0) - v_{1c}(0))\} \geq 0\) means that the particles in different groups depart each other initially. Actually, this geometric condition is not that crucial for the validity of Theorem 4.1 as can be seen in Corollary 4.4. This condition will be attained in a finite time for proper coupling strengths, even if we begin with initial data that do not satisfy this condition.

2. The smallness condition on \(K_d\) is needed to prevent mono-cluster flocking, whereas the largeness condition on \(K_\alpha\) is needed to enable flocking of each subsystem.

In fact, we can get rid of the condition \(\min_{1 \leq i \leq N_1} \min_{1 \leq k \leq N_2} \{(x_{2k}(0) - x_{1i}(0)) \cdot (v_{2c}(0) - v_{1c}(0))\} \geq 0\). For this, we set

\[ T_0 := \frac{1}{\lambda^2_0} \max_{1 \leq i \leq N_1} \max_{1 \leq k \leq N_2} |(x_{2k}(0) - x_{1i}(0)) \cdot (v_{2c}(0) - v_{1c}(0))|. \]

**Corollary 4.4.** Suppose that the following framework \((\mathcal{F}_A)\) holds for the initial data \((x_0, v_0)\) to system (1.3).

- **(F_A)1:** (Restriction on initial configurations)
  \[ \lambda_0 := \frac{1}{2} \| v_{2c}(0) - v_{1c}(0) \| > 0, \quad \max_{1 \leq i \leq N_1} \| v_{1i}(0) - v_{1c}(0) \| \leq \frac{1}{4} \lambda_0, \quad \max_{1 \leq k \leq N_2} \| v_{2k}(0) - v_{2c}(0) \| \leq \frac{1}{4} \lambda_0. \]

- **(F_A)2:** (Restriction on coupling strengths): for \(\alpha = 1, 2\),
  \[ 0 \leq K_d < \min \left\{ \frac{\lambda_0}{16 \sqrt{2NM_2(0)} T_0}, \frac{\lambda_0^2}{24 \sqrt{2NM_2(0)} \int_0^\infty \psi_d(x)dx}, \frac{\lambda_0^2}{2(D(x_1(0), x_2(0))) + \sqrt{2NM_2(0) T_0} \sqrt{2NM_2(0)}} \right\}; \]
  \[ K_\alpha > \frac{P_\alpha(0) + K_\delta \sqrt{2NM_2(0)} \int_0^\infty \psi_d(x)dx}{\int_{R_\alpha(0)} \psi_\alpha(\sqrt{2N_\alpha x})dx}, \quad \alpha = 1, 2. \]
where we have used some quantities that only depend on the initial data:

\[ D(x_1(0), x_2(0)) := \max_{i,k} \|x_{2k}(0) - x_{1i}(0)\|, \]

\[ P_\alpha(0) := \mathcal{V}_\alpha(0) + \frac{\lambda_0}{16}, \quad \alpha = 1, 2, \quad R_\alpha(0) := \mathcal{X}_\alpha(0) + P_\alpha(0)T_0. \]  

(4.2)

Then, the whole system \((G_1, G_2)\) exhibits a time-asymptotic bi-cluster flocking. More precisely, for the solution \((x, v)\) to system (1.3) with initial data \((x_0, v_0)\), there exist \(x_\alpha^\infty\) and \(C_\alpha\), \(\alpha = 1, 2\), that only depend on the initial data and \(\psi\) such that

\[ \sup_{T_0 \leq t < \infty} \mathcal{X}_\alpha(t) \leq x_\alpha^\infty, \quad \mathcal{V}_\alpha(t) \leq C_\alpha \max \left\{ e^{-\frac{K_\alpha\psi_d(\sqrt{2N_\alpha}x_\alpha^\infty)(t-T_0)}{2}}, \psi_d\left(\frac{\lambda_0(t-T_0)}{4}\right) \right\}, \]

\[ \inf_{i,k, 0 \leq t < \infty} \|v_{2k}(t) - v_{1i}(t)\| \geq \frac{\lambda_0}{2}, \quad \min_{i,k} \|x_{2k}(t) - x_{1i}(t)\| \geq \frac{\lambda_0}{2}t, \quad t \in [T_0, \infty). \]

**Remark 4.5.** Note that the conditions imposed on the coupling strengths are explicitly computable from initial data.

### 4.2. Proof of Theorem 4.2

In this subsection, we present a proof of Theorem 4.2 on the formation of bi-cluster flockings resulting from the interaction of two C-S ensembles in the low inter-coupling regime \(K_d \ll 1\).

**Proposition 4.6.** Suppose that the coupling strengths satisfy

\[ K_\alpha \geq 0, \quad \alpha = 1, 2, \quad K_d \geq 0, \]

and let \((x, v)\) be a global solution to (1.3). Then, for \(\alpha = 1, 2\), we have

\[ (i) \quad \left\| \frac{dv_{\alpha c}}{dt} \right\| < \sqrt{2NM_\alpha(0)}\psi_d, \quad \text{a.e.} \ t \in (0, \infty), \]

\[ (ii) \quad \frac{d\mathcal{X}_\alpha}{dt} \leq \mathcal{V}_\alpha, \quad \frac{d\mathcal{V}_\alpha}{dt} \leq -K_\alpha\psi_d(\sqrt{2N_\alpha}\mathcal{X}_\alpha)\mathcal{V}_\alpha + K_d\sqrt{2NM_\alpha(0)}\psi_d, \]

where \(\psi_d\) is the time-dependent maximal communication weight between distinct ensembles:

\[ \psi_d(t) := \max_{1 \leq i \leq N_1} \psi_d(\|x_{2k}(t) - x_{1i}(t)\|) \geq 0. \]

**Proof.** Since the estimates for subsystem \(G_2\) are the same as for subsystem \(G_1\), we only treat estimates for \(\alpha = 1\).

(i) Note that

\[ 2NM_2(t) \geq 2N\left(\frac{1}{N_1}v_{11}^2(t) + \frac{1}{N_2}v_{21}^2(t)\right) > 2(v_{11}^2(t) + v_{21}^2(t)) \geq \|v_{21}(t) - v_{11}(t)\|^2. \]

Thus we can obtain

\[ \|v_{2k}(t) - v_{1i}(t)\| < \sqrt{2NM_2(t)} \leq \sqrt{2NM_2(0)}. \]

We use the above relation and (4.1) to obtain

\[ \left\| \frac{dv_{1c}}{dt} \right\| = \frac{K_d}{N_1N_2} \sum_{k=1}^{N_2} \sum_{i=1}^{N_1} \psi_d(\|x_{2k} - x_{1i}\|)\|v_{2k} - v_{1i}\| \]

\[ < K_d\sqrt{2NM_2(0)}\psi_d(t). \]
We only prove respect to \( i \)

Suppose that the coupling strengths satisfy

\[ (ii) \quad \text{Since the first inequality can be proved similarly with Proposition 3.2, here we only prove the second one. We multiply (4.1) by } 2\mathbf{v}_{1i} \text{ and sum the results with respect to } i \text{ to obtain} \]

\[
\frac{dV}{dt} = \frac{2}{N_1} \sum_{i=1}^{N_1} \langle \dot{\mathbf{v}}_{1i}, \dot{\mathbf{v}}_{1i} \rangle 
\]

\[
= -\frac{2}{N_1} \sum_{i=1}^{N_1} \langle \dot{\mathbf{v}}_{1i}, \dot{\mathbf{v}}_{1c} \rangle - K_1 \sum_{k=1}^{N_1} \sum_{i=1}^{N_1} \psi_1(\|\mathbf{x}_{1k} - \mathbf{x}_{11i}\|) \|\dot{\mathbf{v}}_{1k} - \dot{\mathbf{v}}_{1i}\|^2 
\]

\[
+ 2K_d \sum_{k=1}^{N_1} \sum_{i=1}^{N_1} \psi_d(\|\mathbf{x}_{2k} - \mathbf{x}_{11i}\|) \langle \dot{\mathbf{v}}_{1i}, \mathbf{v}_{2k} - \mathbf{v}_{1i} \rangle 
\]

\[
\leq - \frac{K_1}{N_1} \psi_1(\sqrt{2N_1} \mathbf{x}_1) \sum_{k=1}^{N_1} \sum_{i=1}^{N_1} \|\dot{\mathbf{v}}_{1k} - \dot{\mathbf{v}}_{1i}\|^2 + 2K_d \sqrt{2NM_2(0)} \psi_{dM} \mathcal{V}_1. 
\]

On the other hand, note that

\[
\sum_{i=1}^{N_1} \sum_{k=1}^{N_1} \|\dot{\mathbf{v}}_{1k} - \dot{\mathbf{v}}_{1i}\|^2 = \sum_{i=1}^{N_1} \sum_{k=1}^{N_1} \left( \|\dot{\mathbf{v}}_{1k}\|^2 + \|\dot{\mathbf{v}}_{1j}\|^2 - 2\mathbf{v}_{1k} \cdot \dot{\mathbf{v}}_{1i} \right) 
\]

\[
= 2N_1 \sum_{k=1}^{N_1} \|\dot{\mathbf{v}}_{1i}\|^2 = 2N^2 \mathcal{V}_1^2. 
\]

We now combine all estimates in (4.3) and (4.4) to obtain

\[
\frac{dV}{dt} \leq -K_1 \psi_1(\sqrt{2N_1} \mathbf{x}_1) \mathcal{V}_1 + K_d \sqrt{2NM_2(0)} \psi_{dM}. 
\]

Similarly, we have

\[
\frac{dV}{dt} \leq -K_2 \psi_2(\sqrt{2N_2} \mathbf{x}_2) \mathcal{V}_2 + K_d \sqrt{2NM_2(0)} \psi_{dM}. 
\]

**Proposition 4.7.** Suppose that the coupling strengths satisfy

\[ K_\alpha \geq 0, \quad \alpha = 1, 2, \quad K_d \geq 0, \]

and let \((\mathbf{x}, \mathbf{v})\) be a global solution to (1.3). Then, for \( \alpha = 1, 2 \), we have

\[ \|\hat{\mathbf{v}}_\alpha(t_2)\|_\infty - \|\hat{\mathbf{v}}_\alpha(t_1)\|_\infty \leq 2K_d \sqrt{2NM_2(0)} \int_{t_1}^{t_2} \psi_{dM}(t) dt, \quad \text{for all } 0 \leq t_1 \leq t_2 < \infty. \]

**Proof.** We only prove \( \alpha = 1 \) and the case \( \alpha = 2 \) can be treated similarly. We set

\[ F(t) = 2K_d \sqrt{2NM_2(0)} \psi_{dM}(t). \]

We claim that for any \( t_1 \in [0, \infty) \), there exists \( \Delta t > 0 \) such that

\[ \|\hat{\mathbf{v}}_\alpha(t_2)\|_\infty - \|\hat{\mathbf{v}}_\alpha(t_1)\|_\infty \leq \int_{t_1}^{t_2} F(t) dt, \quad \forall t_2 \in (t_1, t_1 + \Delta t]. \]

(4.5)

**Proof of claim (4.5).** Now we take an arbitrary \( t_1 \in [0, \infty) \). Set

\[ I_{t_1} := \{ 1 \leq j \leq N_1 \mid \|\hat{\mathbf{v}}_{1j}(t_1)\| = \|\hat{\mathbf{v}}_1(t_1)\| \}. \]
For any $j \in I_t$, we have
\[
\frac{d\|\hat{\mathbf{v}}_{1j}(t)\|^2}{dt}_{t=t_t} = \left\{ -2\hat{\mathbf{v}}_{1j} \cdot \mathbf{v}_{1c} + \frac{2K_1}{N_1} \sum_{k=1}^{N_1} \psi_1(\|\hat{x}_{1k}(t) - \hat{x}_{1j}(t)\|)\langle\hat{\mathbf{v}}_{1j}(t), \mathbf{v}_{1k}(t) - \hat{\mathbf{v}}_{1j}(t)\rangle \right. \\
+ \frac{2K_0}{N_2} \sum_{k=1}^{N_2} \psi_2(\|x_{2k}(t) - x_{1j}(t)\|)\langle\hat{\mathbf{v}}_{1j}(t), \mathbf{v}_{2k}(t) - \mathbf{v}_{1j}(t)\rangle \right\}_{t=t_t}.
\]

We use $\langle\hat{\mathbf{v}}_{1j}(t_1), \mathbf{v}_{1k}(t_1) - \hat{\mathbf{v}}_{1j}(t_1)\rangle \leq 0$ and the estimate of $\|\hat{v}_{1c}\|$ in Proposition 4.6 to find
\[
\frac{d\|\hat{\mathbf{v}}_{1j}(t)\|}{dt}_{t=t_t} < F(t_1).
\]

We use the continuity to get that, there exists $\Delta t > 0$ such that for any $t \in [t_1, t_1 + \Delta t]$, any $j \in I_t$, and any $i \in \{1, \cdots, N\} \setminus I_t$, it holds that
\[
\frac{d\|\hat{\mathbf{v}}_{1j}(t)\|}{dt} < F(t) \quad \text{and} \quad \|\hat{\mathbf{v}}_{1i}(t)\| < \|\hat{\mathbf{v}}_{1j}(t)\|.
\]

Thus, for any $t_2 \in [t_1, t_1 + \Delta t]$, there exists $j \in I_t$ such that $\|\hat{\mathbf{v}}_{1j}(t_2)\| = \|\hat{\mathbf{v}}_{1j}(t_2)\|$. Hence, we have
\[
\|\hat{\mathbf{v}}_{1j}(t_2)\|_\infty - \|\hat{\mathbf{v}}_{1j}(t_1)\|_\infty = \|\hat{\mathbf{v}}_{1j}(t_2)\| - \|\hat{\mathbf{v}}_{1j}(t_1)\| \\
= \int_{t_1}^{t_2} \frac{d\|\hat{\mathbf{v}}_{1j}(t)\|}{dt} dt < \int_{t_1}^{t_2} F(t) dt, \quad \forall t_2 \in (t_1, t_1 + \Delta t].
\]

Thus, the claim (4.5) holds. Now we set
\[
T^2 := \sup\{t \in (t_1, \infty) \mid \|\hat{\mathbf{v}}_{1}(s)\|_\infty - \|\hat{\mathbf{v}}_{1}(t_1)\|_\infty \leq \int_{t_1}^{s} F(t) dt, \forall s \in [t_1, t]\}.
\]

Now we claim
\[
T^2 = \infty.
\]

Otherwise, we assume $T^2 < \infty$. Then
\[
\|\hat{\mathbf{v}}_{1}(T^2)\|_\infty - \|\hat{\mathbf{v}}_{1}(t_1)\|_\infty = \int_{t_1}^{T^2} F(t) dt.
\]

We use claim (4.5) to know there exists $\Delta t > 0$ such that
\[
\|\hat{\mathbf{v}}_{1}(s)\|_\infty - \|\hat{\mathbf{v}}_{1}(T^2)\|_\infty \leq \int_{T^2}^{s} F(t) dt, \quad \text{for all} \ s \in (T^2, T^2 + \Delta t].
\]

Thus, we can have
\[
\|\hat{\mathbf{v}}_{1}(s)\|_\infty - \|\hat{\mathbf{v}}_{1}(t_1)\|_\infty \leq (\int_{T^2}^{s} + \int_{T^2}^{T^2}) F(t) dt = \int_{t_1}^{s} F(t) dt, \text{for all} \ s \in [t_1, T^2 + \Delta t].
\]

This contradicts the definition of $T^2$. Thus we obtain $T^2 = \infty$. The conclusion follows.

In the following two subsections, we proceed to prove Theorem 4.2 as follows.
We now take a minimum over Lemma 4.8. Let \( \psi \) fying (1.3).

\[
\inf_{0 \leq t < \infty, i, k} \| v_{2k}(t) - v_{1i}(t) \| \geq \lambda_0, \quad \min_{i, k} \| x_{2k}(t) - x_{1i}(t) \| \geq \lambda_0 t.
\]

- **Step A (Local-in-time estimate).** We will show that each sub-ensemble \( G_\alpha \) satisfies the flocking estimates for some finite time \( T \):

\[
\sup_{0 \leq t < T} X_\alpha(t) < x_\alpha^\infty, \quad \forall \alpha(t) \leq C_\alpha \max \left\{ e^{-\frac{\kappa_\alpha \psi_\alpha(t)}{2}}, \psi_d \left( \frac{\lambda_0 t}{2} \right) \right\},
\]

\( t \in [0, T) \),

\[
\min_{i, k} \| x_{2k}(t) - x_{1i}(t) \| \geq \lambda_0 t.
\]

- **Step B (Continuation to the whole time interval).** We will show that time \( T \) in Step A can be chosen to be infinity.

**4.2.1. Local-in-time flocking estimates.** To find the time interval where all desired flocking estimates hold, we set

\[
\epsilon_{1,2}^0 := \frac{v_{2c}(0) - v_{1c}(0)}{\|v_{2c}(0) - v_{1c}(0)\|}, \quad T_1^*: = \sup \mathcal{T}_1,
\]

\[
\mathcal{T}_1 := \left\{ T \in [0, +\infty) \mid \min_{i, k} ((v_{2k}(t) - v_{1i}(t)) \cdot \epsilon_{1,2}^0) > \lambda_0, \text{ for all } t \in [0, T] \right\}.
\]

We first show that \( T_1^* \) exists and is positive.

**Lemma 4.8.** Let \((x, v)\) be a global solution to (1.3) with initial data \((x_0, u_0)\) satisfying \((FA)\) in Theorem 4.2. Then, we have

\[
T_1^* > 0 \quad \text{and} \quad \psi_{dM}(t) \leq \psi_d \left( \lambda_0 t \right), \quad \text{for all } t \in [0, T_1^*).
\]

**Proof.** (i) We first show that \( T^* > 0 \).

\[
\frac{(v_{2k}(0) - v_{1i}(0)) \cdot \epsilon_{1,2}^0}{\|v_{2c}(0) - v_{1c}(0)\|} = \left( \frac{v_{2c}(0) - v_{1c}(0) - \dot{v}_{1i}(0) + \dot{v}_{2k}(0) \cdot (v_{2c}(0) - v_{1c}(0))}{\|v_{2c}(0) - v_{1c}(0)\|} \right) \left( v_{2c}(0) - v_{1c}(0) \right) \geq \frac{3 \lambda_0}{2} > \lambda_0.
\]

We now take a minimum over \( i \) and \( k \) to obtain

\[
\min_{i, k} \left\{ (v_{2k}(0) - v_{1i}(0)) \cdot \epsilon_{1,2}^0 \right\} > \lambda_0.
\]

Then, by the continuity, there exists \( \delta > 0 \) such that

\[
\min_{i, k} \left\{ (v_{2k}(t) - v_{1i}(t)) \cdot \epsilon_{1,2}^0 \right\} > \lambda_0, \quad t \in [0, \delta), \quad \text{i.e.,} \quad \delta \in \mathcal{T}_1.
\]

Hence \( T_1^* \geq \delta > 0 \).

(ii) For all \( t \in [0, T_1^*] \), we have

\[
\| x_{2k}(t) - x_{1i}(t) \| \geq (x_{2k}(t) - x_{1i}(t)) \cdot \epsilon_{1,2}^0
\]

\[
= (x_{2k}(0) - x_{1i}(0)) \cdot \epsilon_{1,2}^0 + \int_0^t (v_{2k}(s) - v_{1i}(s)) \cdot \epsilon_{1,2}^0 ds \geq \lambda_0 t.
\]

Thus, by the non-increasing property of \( \psi_d \), we have

\[
\psi_{dM}(t) \leq \psi_d \left( \lambda_0 t \right), \quad \text{for all } t \in [0, T_1^*).
\]

\[\square\]
Lemma 4.9. (Flocking estimate in $[0, T^*_1]$) Suppose that the initial data $(x_0, v_0)$ satisfy $(\mathcal{F}_A 1)$ and the coupling strengths satisfy

$$K_\alpha > \frac{\mathcal{V}_\alpha(0) + \frac{K_d \sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_d(x)dx}{\int_{|x_\alpha(0)|}^{\infty} \psi_\alpha(\sqrt{2N_\alpha x})dx}, \quad \alpha = 1, 2, \quad K_d \geq 0.$$ 

Then, for the solution $(x, v)$ to system (1.3) with initial data $(x_0, v_0)$, there exist positive constants $x_\alpha^\infty$ and $C_\alpha$ independent of time $t$ such that

$$\sup_{0 \leq t < T_1^*} \mathcal{X}_\alpha(t) < x_\alpha^\infty, \quad \mathcal{V}_\alpha(t) \leq C_\alpha \max \left\{ e^{-\frac{K_\alpha \mathcal{X}_\alpha((\sqrt{2N_\alpha x})^t)}{t}} \psi_d \left( \frac{\lambda_0 t}{2} \right) \right\}, \quad t \in [0, T_1^*].$$

Proof. (i) (Existence of an upper bound $x_\alpha^\infty$): We fix $\alpha \in \{1, 2\}$ and define a Lyapunov functional $\mathcal{L}_{1\alpha}$:

$$\mathcal{L}_{1\alpha}(t) := \mathcal{V}_\alpha(t) + K_\alpha \int_{|x_\alpha(0)|}^{\mathcal{X}_\alpha(t)} \psi_\alpha(\sqrt{2N_\alpha x})dx.$$ 

It follows from Proposition 4.6 and Lemma 4.8 that

$$\frac{d\mathcal{L}_{1\alpha}(t)}{dt} = \frac{d}{dt} \mathcal{V}_\alpha(t) + K_\alpha \psi_\alpha(\sqrt{2N_\alpha x_\alpha(t)}) \frac{d}{dt} x_\alpha(t) \leq -K_\alpha \psi_\alpha(\sqrt{2N_\alpha x_\alpha(t)}) \left( \mathcal{V}_\alpha(t) - \frac{d}{dt} x_\alpha(t) \right) + K_d \sqrt{2NM_2(0)} \psi_d$$

$$\leq K_d \sqrt{2NM_2(0)} \psi_d(\lambda_0 t), \quad t \in [0, T_1^*).$$

We integrate the aforementioned relation to obtain

$$\mathcal{V}_\alpha(t) + K_\alpha \int_{|x_\alpha(0)|}^{\mathcal{X}_\alpha(t)} \psi_\alpha(\sqrt{2N_\alpha x})dx \leq \mathcal{V}_\alpha(0) + \frac{K_d \sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_d(x)dx.$$ 

In particular, this yields

$$K_\alpha \int_{|x_\alpha(0)|}^{\mathcal{X}_\alpha(t)} \psi_\alpha(\sqrt{2N_\alpha x})dx \leq \mathcal{V}_\alpha(0) + \frac{K_d \sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_d(x)dx, \quad t \in [0, T_1^*).$$

On the other hand, the assumption on $K_\alpha$ implies

$$\mathcal{V}_\alpha(0) + \frac{K_d \sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_d(x)dx < K_\alpha \int_{|x_\alpha(0)|}^{\infty} \psi_\alpha(\sqrt{2N_\alpha x})dx.$$ 

We use (4.7) and (4.8) to see the existence of a solution to the following equation:

$$K_\alpha \int_{|x_\alpha(0)|}^{x_\alpha^\infty} \psi_\alpha(\sqrt{2N_\alpha x})dx = \mathcal{V}_\alpha(0) + \frac{K_d \sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_d(x)dx.$$ 

We set $x_\alpha^\infty$ to be the largest positive value. Then, by the same argument employed in Theorem 3.3, we have

$$\mathcal{X}_\alpha(t) \leq x_\alpha^\infty, \quad \forall \quad t \in [0, T_1^*).$$

(ii) (Decay estimate of $\|\dot{\mathcal{V}}_\alpha(t)\|_\infty$): By the estimate (ii) in Proposition 4.6, we have

$$\frac{d\mathcal{V}_\alpha}{dt} \leq -K_\alpha \psi_\alpha(\sqrt{2N_\alpha x_\alpha^\infty}) \mathcal{V}_\alpha + K_d \sqrt{2NM_2(0)} \psi_d(\lambda_0 t).$$

We now apply Lemma A.1 in Appendix A with

$$a := K_\alpha \psi_\alpha(\sqrt{2N_\alpha x_\alpha^\infty}), \quad f := K_d \sqrt{2NM_2(0)} \psi_d(\lambda_0 t).$$
to find the desired flocking estimate:

\[ V_\alpha(t) \leq V_\alpha(0)e^{-K_\alpha \psi_e(\sqrt{2N \alpha \sigma^c})t} + \frac{K_d \sqrt{2NM_2(0)}}{K_\alpha \psi_e(\sqrt{2N \alpha \sigma^c})} \left[ e^{-\frac{K_\alpha \psi_e(\sqrt{2N \alpha \sigma^c})t}{2}} + \psi_e \left( \frac{\lambda_0 t}{2} \right) \right]. \]

As a final step, we are now ready to complete the proof of Theorem 4.2 by showing that \( T^*_1 = \infty \).

4.2.2. Step B: \( T^*_1 = \infty \): Suppose that initial data \((x_0, v_0)\) and coupling strengths satisfy the framework \((F_A)\). We claim

\[ T^*_1 = \infty. \] (4.9)

Proof of claim (4.9). Suppose not, i.e., \( 0 < T^*_1 < \infty \). Then, it follows from the definition of \( T^* \) in (4.6) that

\[ (v_{2k}(T^*_1) - v_{1i}(T^*_1)) \cdot e_0 = \lambda_0. \] (4.10)

On the other hand, we use Proposition 4.6 and Proposition 4.7 to obtain: for \( \alpha = 1, 2 \),

\[ \|v_{\alpha c}(T^*_1) - v_{\alpha c}(0)\| \leq \frac{K_d \sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_e(x)dx, \] (4.11)

\[ \|\dot{v}_{\alpha}(T^*_1)\|_\infty \leq \|\dot{v}_{\alpha}(0)\|_\infty + \frac{2K_d \sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_e(x)dx. \]

We use assumptions on the initial data \( K_d \) and (4.11) to derive the following relation:

\[ (v_{2k}(T^*_1) - v_{1i}(T^*_1)) \cdot e_0 \]
\[ \geq \|v_{2c}(0) - v_{1c}(0)\| - \|v_{1c}(T^*_1) - v_{1c}(0)\| - \|v_{2c}(T^*_1) - v_{2c}(0)\| - \|\dot{v}_{1c}(T^*_1)\|_\infty \]
\[ \geq 2\lambda_0 - \|v_{1c}(0)\|_\infty - \|\dot{v}_{2c}(0)\|_\infty - \frac{6K_d \sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_e(s)ds \]
\[ > \lambda_0. \]

This contradicts relation (4.10). Thus \( T^*_1 = \infty \).

4.3. Proof of Corollary 4.4. In this subsection, we provide the proof of Corollary 4.4 under the following condition:

\[ K_d \ll 1. \]

The main idea is to show that there exists a positive finite time such that all the position difference along the average velocity difference will be positive, i.e., for some finite time \( T_0 \in (0, \infty) \),

\[ \min_{1 \leq k \leq N_2} \{ \langle x_{2k}(T_0) - x_{1i}(T_0) \rangle \cdot (v_{2c}(T_0) - v_{1c}(T_0)) \} \geq 0. \]

This is plausible because for \( K_d = 0 \), two C-S subsystems are independent so that the quantity \( \langle x_{2k}(t) - x_{1i}(t) \rangle \cdot (v_{2c}(t) - v_{1c}(t)) \) will grow in time. Then, after \( t = T_0 \), we can restart the C-S flow (1.3) with a new initial data \((x_0(T_0), v_0(T_0))\) and apply the result of Theorem 4.2 to derive the desired bi-cluster estimates. For notational simplicity, we set

\[ \Delta_{1i,2k}(t) := \langle x_{2k}(t) - x_{1i}(t) \rangle \cdot (v_{2c}(t) - v_{1c}(t)), \quad 1 \leq i \leq N_1, 1 \leq k \leq N_2. \]
In the sequel, we will show that
\[ \min_{1 \leq i \leq N_1} \min_{1 \leq k \leq N_2} \Delta_{1t,2k}(T_0) \geq 0. \tag{4.12} \]
Then, we can apply the first assertion in Theorem 4.2 for the solution starting from \( t = T_0 \). To verify (4.12), we need several a priori estimates. By posterior calculation, we set
\[ T_0 := \frac{1}{\lambda_0^2} \max_{1 \leq i \leq N_1} \max_{1 \leq k \leq N_2} |\Delta_{1t,2k}(0)| \geq 0. \tag{4.13} \]
In the sequel, we will show that \( T_0 \) defined in (4.13) satisfies the estimate (4.12).

**Lemma 4.10.** Suppose that the initial data \((x_0, v_0)\) satisfy \((\mathcal{F}_{A1})\) and that the inter-subsystem coupling strength \(K_d\) is sufficiently small in the sense that
\[ K_\alpha \geq 0, \quad \alpha = 1, 2, \quad 0 \leq K_d < \frac{\lambda_0}{16 \sqrt{2NM(0)T_0}}. \]
Then, for the solution \((x, v)\) to system (1.3) with initial data \((x_0, v_0)\), we have: for \( t \in [0, T_0] \),
\[
(i) \quad \|v_{2c}(t) - v_{1c}(t)\| \geq \|v_{2c}(0) - v_{1c}(0)\| - \frac{\lambda_0}{8} \geq \frac{15\lambda_0}{8}, \\
(ii) \quad \|\dot{\nu}_\alpha(t)\| \leq \|\dot{\nu}_\alpha(0)\| + \frac{\lambda_0}{8} \leq \frac{3\lambda_0}{8}, \quad \alpha = 1, 2.
\]

**Proof.** (i) It follows from Proposition 4.6 (i) that we have
\[ \left\| \frac{d(v_{2c} - v_{1c})}{dt} \right\| \leq 2K_d \sqrt{2N}M(0)\psi_d M \leq 2K_d \sqrt{2N}M(0). \]
This and the assumption on \(K_d\) imply
\[
\|v_{2c}(t) - v_{1c}(t)\| \geq \|v_{2c}(0) - v_{1c}(0)\| - 2K_d \sqrt{2N}M(0)T_0 \geq \|v_{2c}(0) - v_{1c}(0)\| - \frac{\lambda_0}{8} \geq \frac{15\lambda_0}{8}, \quad t \in [0, T_0].
\]
(ii) It follows from Proposition 4.7 that we have for: \( \alpha = 1, 2 \),
\[ \|\dot{v}_\alpha(t)\| \leq \|\dot{v}_\alpha(0)\| + 2K_d \sqrt{2N}M(0)T_0 \leq \|\dot{v}_1(0)\| + \frac{\lambda_0}{8} \leq \frac{3\lambda_0}{8}, \quad t \in [0, T_0]. \]

**Remark 4.11.** By Proposition 4.6(iii), we can see that
\[ \frac{d\mathcal{V}_\alpha}{dt} \leq K_d \sqrt{2N}M(0)\psi_d(t) \leq K_d \sqrt{2N}M(0), \quad \alpha = 1, 2. \]
Thus, we have
\[ \mathcal{V}_\alpha(t) \leq \mathcal{V}_\alpha(0) + K_d \sqrt{2N}M(0)T_0 \leq \mathcal{V}_\alpha(0) + \frac{\lambda_0}{16}, \quad t \in [0, T_0]. \]

**Proposition 4.12.** Suppose that the initial data satisfy \((\mathcal{F}_{A1})\) and that the inter-group coupling strength \(K_d\) is sufficiently small:
\[ K_\alpha \geq 0, \quad \alpha = 1, 2, \]
\[ K_d < \min \left\{ \frac{\lambda_0}{16 \sqrt{2N}M(0)T_0}, \frac{\lambda_0^2}{2(D(x_1(0), x_2(0)) + \sqrt{2N}M(0)T_0 \sqrt{2N}M(0))} \right\}, \]
Lemma 4.14. Keep the assumptions of Proposition 4.12 and let $t \in [0, T_0]$.

Proof. We first claim that
\[
\frac{d}{dt} \Delta_{1i,2k}(t) \geq \lambda_0^2, \quad t \in [0, T_0].
\]  
(4.14)

Proof of claim (4.14). We use the relation
\[
\|v_{2k}(t) - v_{1i}(t)\| \leq \sqrt{2NM_2(0)},
\]
to obtain
\[
\|x_{2k}(t) - x_{1i}(t)\| \leq \|x_{2k}(0) - x_{1i}(0)\| + \sqrt{2NM_2(0)}T_0, \quad t \in [0, T_0].
\]

Then, we use the condition on $K_d$ to obtain
\[
\frac{d}{dt} \Delta_{1i,2k}(t) = (v_{2k}(t) - v_{1i}(t)) \cdot (v_{1c}(t) - v_{1c}(t)) + (x_{1i}(t) - x_{2k}(t)) \cdot (\dot{v}_{2c}(t) - v_{1c}(t)) 
\geq \|v_{2k}(t) - v_{1c}(t)\| \|v_{2c}(t) - v_{1c}(t)\| - \|\dot{v}_{1i}(t)\| - \|v_{2k}(t)\| 
- \|x_{2k}(0) - x_{1i}(0)\| + \sqrt{2NM_2(0)}T_0 \geq \frac{135\lambda_0^2}{64} - \lambda_0^2 > \lambda_0^2.
\]

We now integrate the aforementioned relation in $t$ from $t = 0$ to $t = T_0$ to obtain
\[
\Delta_{1i,2k}(t) \geq \Delta_{1i,2k}(0) + \lambda_0^2 t, \quad t \in [0, T_0].
\]

This and the definition of $T_0$ in (4.13) imply
\[
\Delta_{1i,2k}(T_0) \geq \Delta_{1i,2k}(0) + \lambda_0^2 T_0 \geq 0.
\]

We now take an infimum over $i, k$ to obtain the desired result.

Remark 4.13. Note that the intra-coupling strengths $K_{\alpha}$, $\alpha = 1, 2$, do not play any role in the mixing phase.

4.3.2. Emergence of bi-cluster flocking. In this part, we finally provide the proof of Corollary 4.4. We set
\[
e_{1,2}(T_0) := \frac{v_{2c}(T_0) - v_{1c}(T_0)}{\|v_{2c}(T_0) - v_{1c}(T_0)\|},
\]
\[
\bar{T}_1^* := \sup \{T \in [T_0, +\infty) \mid \min_{i,k}\{(v_{1i}(t) - v_{2k}(t)) \cdot e_{1,2}(T_0)\} > \frac{\lambda_0}{2},
\]
for all $t \in [T_0, T)$.

Lemma 4.14. Keep the assumptions of Proposition 4.12 and let $(x, v)$ be the global solution to (1.3) with initial data $(x_0, v_0)$. Then the following hold:
(i) $(v_{2k}(T_0) - v_{1i}(T_0)) \cdot e_{1,2}(T_0) > \frac{\lambda_0}{2}$ and $\bar{T}_1^* > T_0$,
(ii) $\psi_{dM}(t) \leq \psi_d(\frac{\lambda_0}{2}(t - T_0)), \quad t \in [T_0, \bar{T}_1^*].$
Proof. It follows from Lemma 4.10 that
\[ \| \mathbf{v}_{2k}(T_0) - \mathbf{v}_{1e}(T_0) \| \geq \frac{15\lambda_0}{8} \]
and
\[ \| \hat{\mathbf{v}}_1(T_0) \|_\infty + \| \hat{\mathbf{v}}_2(T_0) \|_\infty \leq \| \hat{\mathbf{v}}_1(0) \|_\infty + \| \hat{\mathbf{v}}_2(0) \|_\infty + \frac{\lambda_0}{4} \leq \frac{3\lambda_0}{4}. \]
This yields
\[
(\mathbf{v}_{2k}(T_0) - \mathbf{v}_{1e}(T_0)) \cdot e_{1,2}(T_0)
\]
\[
= (\mathbf{v}_{2k}(T_0) - \mathbf{v}_{1e}(T_0) - \hat{\mathbf{v}}_1(T_0) + \hat{\mathbf{v}}_{2k}(T_0)) \cdot e_{1,2}(T_0)
\]
\[ \geq \| \hat{\mathbf{v}}_1(T_0) \|_\infty - \| \hat{\mathbf{v}}_2(T_0) \|_\infty \]
\[ \geq \frac{9\lambda_0}{8} > \frac{\lambda_0}{2}. \]
Thus we have \( \tilde{T}_1 > T_0 \). For the second estimate of \( \psi_{dM}(t) \), we use the same argument as in Lemma 4.8. \( \square \)

**Lemma 4.15.** (Flocking estimate in \([T_0, \tilde{T}_1] \)) Suppose that the initial data \((x_0, \mathbf{v}_0)\) satisfy \( \mathcal{F}_A1 \) and the coupling strengths satisfy
\[
K_d < \min \left\{ \frac{\lambda_0}{16\sqrt{2NM_2(0)T_0}}, \frac{\lambda_0^2}{2(D(x_0(0), x_2(0)) + \sqrt{2NM_2(0)T_0})^2} \right\},
\]
\[
K_\alpha > \frac{\lambda_0}{\alpha \int_{\lambda_0/(\psi_d(\sqrt{2N\alpha}))\chi_{[0,T_0]} dx}} \int_{\lambda_0/(\psi_d(\sqrt{2N\alpha}))\chi_{[0,T_0]} dx} \psi_d(\sqrt{2N\alpha} dx), \quad \alpha = 1, 2,
\]
let \((x, \mathbf{v})\) be the solution to the coupled system (1.3) with initial data \((x_0, \mathbf{v}_0)\). Then, there exist positive constants \(x_\alpha^\infty\) and \(C_{\alpha, T_0}\) independent of time \(t\) such that
\[
\sup_{T_0 \leq t < \tilde{T}_1} \mathcal{X}_\alpha(t) \leq x_\alpha^\infty, \quad t \in [T_0, \tilde{T}_1),
\]
\[
\mathcal{V}_\alpha(t) \leq C_{\alpha, T_0} \max \left\{ e^{-\frac{K_d\psi_d(\sqrt{2NM_2(0)})(t-T_0)}{2}}, \psi_d(\frac{\lambda_0(t-T_0)}{4}) \right\}.
\]

**Proof.** Since the proof is almost the same as in Lemma 4.9, we omit it here. \( \square \)

We next provide the proof of Corollary 4.4 by showing that \( \tilde{T}_1 = \infty \). Suppose that initial data \((x_0, \mathbf{v}_0)\) and the coupling strengths satisfy the framework \( \mathcal{F}_A \).

\( \diamond \) Procedure A: (The conditions on \( K_d \) in Corollary 4.4 imply the conditions on \( K_\alpha \) in Lemma 4.15).

Suppose that \( K_\alpha \) satisfies
\[
K_\alpha > \frac{P_\alpha(0) + \frac{K_d\psi_d(\sqrt{2NM_2(0)})}{\lambda_0} \int_{R_{\alpha}(0)} \psi_d(\sqrt{2N\alpha} dx)}{\int_{R_{\alpha}(0)} \psi_d(\sqrt{2N\alpha} dx)}, \quad \alpha = 1, 2,
\]
(4.16)
On the other hand, we use Corollary 2.5, Proposition 4.6, and Remark 4.11 to see that
\[
\mathcal{V}_{\alpha}(T_0) \leq \mathcal{V}_{\alpha}(0) + \frac{\lambda_0}{16} = P_\alpha(0) \quad t \in [0, T_0],
\]
\[
\mathcal{X}_{\alpha}(T_0) = \mathcal{X}_{\alpha}(0) + \int_0^{T_0} \mathcal{V}_{\alpha}(s) ds \leq \mathcal{X}_{\alpha}(0) + P_\alpha(0)T_0 = R_\alpha(0).
\]
Thus, we have
\[
\mathcal{V}_{\alpha}(T_0) \leq P_\alpha(0), \quad \int_{\mathcal{X}_{\alpha}(T_0)} \psi_d(\sqrt{2N\alpha} dx) \geq \int_{R_{\alpha}(0)} \psi_d(\sqrt{2N\alpha} dx).
\]
(4.17)
We combine (4.16) and (4.17) to recover the condition in Lemma 4.15:

\[
K_\alpha > \frac{\psi_\alpha(T_0) + \frac{K_d\sqrt{2NM_2(0)}}{\lambda_0}}{\int_{x_\alpha(T_0)} \psi_\alpha(\sqrt{2N_\alpha x})dx}, \quad \alpha = 1, 2.
\]

○ Procedure B: We claim that 

\[ \tilde{T}_1^* = \infty. \]

By Lemma 4.14, we have \( \tilde{T}_1^* > T_0 \). Suppose that \( \tilde{T}_1^* < \infty \). Then, by the definition of \( \tilde{T}_1^* \) in (4.15), we have

\[
(v_{2k}(\tilde{T}_1^*) - v_{i1}(\tilde{T}_1^*)) \cdot e_{1,2}(T_0) = \frac{\lambda_0}{2}.
\]

By the same arguments in Section 4.2.2, we have: for \( \alpha = 1, 2 \),

\[
\|v_{\alpha c}(\tilde{T}_1^*) - v_{\alpha c}(T_0)\| \leq \frac{2K_d\sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_d(x)dx,
\]

\[
\|\dot{v}_\alpha(\tilde{T}_1^*)\|_\infty \leq \|\dot{v}_\alpha(T_0)\|_\infty + \frac{4K_d\sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_d(x)dx.
\]

Then, straightforward calculations using Lemma 4.10 and the assumption of \( K_d \) imply

\[
(v_{2k}(\tilde{T}_1^*) - v_{i1}(\tilde{T}_1^*)) \cdot e_{1,2}(T_0) \\
\geq \|v_{2c}(T_0) - v_{1c}(T_0)\| - \|v_{1c}(\tilde{T}_1^*) - v_{1c}(T_0)\| - \|v_{2c}(\tilde{T}_1^*) - v_{2c}(T_0)\| \\
- \|\dot{v}_1(\tilde{T}_1^*)\|_\infty - \|\dot{v}_2(\tilde{T}_1^*)\|_\infty \\
\geq \frac{9\lambda_0}{8} - \frac{12K_d\sqrt{2NM_2(0)}}{\lambda_0} \int_0^\infty \psi_d(x)dx > \frac{\lambda_0}{2}.
\]

Thus, we prove that \( \tilde{T}_1^* = \infty \). We now apply Lemma 4.15 with \( \tilde{T}_1^* = \infty \) to obtain the desired flocking estimates. This completes the proof of Corollary 4.4.

5. Emergence of partial flocking. In this section, we continue to study the dynamics of system (1.3) in a small inter-system coupling. In the previous section, we see that if the intra-subsystem coupling strengths \( K_\alpha \) are sufficiently large and the inter-subsystem coupling strengths \( K_d \) is small, then each subsystem evolves to the flocking state so that the total system exhibits a bi-cluster flocking. In this section, we consider the case in which exactly one of the intra-system coupling strengths is sufficiently large, but the other is not. To fix the idea, we assume that \( K_1 \gg 1 \) and \( K_2 \ll 1 \).

5.1. Statement of the main results. In this subsection, we briefly discuss the main results for the emergence of partial flocking (see Definition 4.1) for some class of initial configurations under the following situation:

\[ K_1 \gg 1, \quad K_2 \ll 1, \quad K_d \ll 1. \]

In this case, subsystem \( G_1 \) flocks, but the other subsystem, \( G_2 \), does not. More precisely, our result is as follows.

Theorem 5.1. Suppose that the following framework \( (F_B) \) holds for the initial data \( (x_0, v_0) \) to system (1.3).

• \( (F_B1) \): (Restriction on initial configurations)

\[
\max_{1 \leq k \leq N_1} \|v_{1k}(0) - v_{1c}(0)\| < \frac{1}{4} \min_{1 \leq k \leq N_2} \|v_{2k}(0) - v_{1c}(0)\|,
\]

\[
\max_{1 \leq k \leq N_2} \|v_{2k}(0) - v_{1c}(0)\| < \frac{1}{4} \min_{1 \leq k \leq N_1} \|v_{1k}(0) - v_{1c}(0)\|,
\]

\[
\max_{1 \leq k \leq N_1} \|v_{1k}(0) - v_{1c}(0)\| < \frac{1}{4} \min_{1 \leq k \leq N_2} \|v_{2k}(0) - v_{1c}(0)\|.
\]
For this, we define
\[ \min_{1 \leq i \leq N_1} \{ (x_{2k}(0) - x_{i1}(0)) \cdot (v_{2k}(0) - v_{i1}(0)) \} \geq 0 \quad \text{and} \]
\[ \min_{1 \leq i \neq k \leq N_2} \{ (x_{2k}(0) - x_{2i}(0)) \cdot (v_{2k}(0) - v_{2i}(0)) \} \geq 0. \]

- \((F_2)\): (Restriction on coupling strengths):

\[
0 \leq K_d < \frac{\Lambda_0 \min \{ \mu_0, \frac{\Lambda_0}{2} \} }{4 \sqrt{2NM_2(0)} \int_0^\infty \psi_d(x) dx},
\]
\[
0 \leq K_2 < \frac{\mu_0 \min \{ \mu_0, \frac{\mu_0}{2} \} }{4 \sqrt{2NM_2(0)} \int_0^\infty \psi_2(x) dx},
\]

\[
K_1 > \frac{V_1(0) + \frac{K_4 \sqrt{2NM_2(0)}}{\Lambda_0} \int_0^\infty \psi_d(x) dx}{\int_{\mathcal{X}_1(0)} \psi_1(\sqrt{2N_1} x) dx},
\]

where positive constants \( \Lambda_0 \) and \( \mu_0 \) are given by the following relations:

\[
\Lambda_0 := \frac{1}{2} \min_{1 \leq k \leq N_2} \| v_{2k}(0) - v_{i1}(0) \| \quad \text{and} \quad \mu_0 := \frac{1}{2} \min_{1 \leq i \neq k \leq N_2} \| v_{2k}(0) - v_{2i}(0) \|.
\]

Then, the subsystem \( G_1 \) and \( G_2 \) exhibit a time-asymptotic partial flocking. More precisely, for the solution \((x, v)\) to system (1.3) with initial data \((x_0, v_0)\), there exist \( \bar{x}_1 \) and \( \bar{C}_1 \) that only depend on the initial data and \( \psi_1 \) such that

\[
\sup_{0 \leq t < \infty} \mathcal{X}_1(t) < \bar{x}_1 \]

\[
\inf_{0 \leq i \leq \infty} \| v_{2k}(t) - v_{2i}(t) \| \geq \mu_0, \quad \min_{1 \leq i \neq k \leq N_2} \| x_{2k}(t) - x_{2i}(t) \| \geq \mu_0 t, \quad t \in [0, \infty)
\]

As a corollary of Theorem 5.1, we get rid of the assumptions \( \min_{1 \leq i \neq k \leq N_2} \{ (x_{2k}(0) - x_{i1}(0)) \cdot (v_{2k}(0) - v_{i1}(0)) \} \geq 0 \) and \( \min_{1 \leq i \neq k \leq N_2} \{ (x_{2k}(0) - x_{2i}(0)) \cdot (v_{2k}(0) - v_{2i}(0)) \} \geq 0 \).

For this, we define

\[
T_1 := \frac{1}{\Lambda_0^2} \max_{1 \leq k \leq N_2} \| (x_{2k}(0) - x_{i1}(0)) \cdot (v_{2k}(0) - v_{i1}(0)) \|,
\]

\[
T_2 := \frac{1}{\mu_0^2} \max_{1 \leq i \neq k \leq N_2} \| (x_{2k}(0) - x_{2i}(0)) \cdot (v_{2k}(0) - v_{2i}(0)) \|,
\]

\[
T_0 := \max\{ T_1, T_2 \} \geq 0.
\]

**Corollary 5.2.** Suppose that the following framework \((F_B)\) holds for the initial data \((x_0, v_0)\) to system (1.3).

- \((F_B1)\): (Restriction on initial configurations)

\[
\max_{1 \leq i \leq N_1} \| v_{i1}(0) - v_{i1}(0) \| < \frac{1}{4} \min_{1 \leq k \leq N_2} \| v_{2k}(0) - v_{1c}(0) \|,
\]

\[
\| v_{2k}(0) - v_{2i}(0) \|, \quad \text{for} \ i \neq k.
\]

- \((F_B2)\): (Restriction on coupling strengths):

\[
0 \leq K_d < \min \left\{ \frac{\min \{ \Lambda_0, \mu_0 \} }{16 \sqrt{2NM_2(0)} T_0}, \frac{\Lambda_0 \min \{ \mu_0, \frac{\Lambda_0}{2} \} }{8 \sqrt{2NM_2(0)} \int_0^\infty \psi_d(x) dx} \right\},
\]

\[
0 \leq K_2 < \min \left\{ \frac{\min \{ \Lambda_0, \mu_0 \} }{16 \sqrt{2NM_2(0)} T_0}, \frac{\mu_0 \min \{ \mu_0, \frac{\mu_0}{2} \} }{8 \sqrt{2NM_2(0)} \int_0^\infty \psi_2(x) dx} \right\},
\]
Proof of Theorem 5.1.

Proposition 5.3.

5.1. For this, we first remind about a similar result on the SDDI.

Proof.\( \quad \)

where \( P_1(0) \) and \( R_1(0) \) are defined in (4.2). Then, the subsystems \( G_1 \) and \( G_2 \) exhibit a time-asymptotic partial flocking. More precisely, for the solution \((x,v)\) to system (1.3) with initial data \((x_0,v_0)\), the following estimates hold: there exist \( \bar{x}_1 > 0 \) and \( T_0 \) such that, for some positive constant \( \mu \),

\[
\sup_{0 \leq t < \infty} X_1(t) < \bar{x}_1, \quad \forall \quad \forall
\]

In the following two subsections, we present the proof of the aforementioned two results.

5.2. Proof of Theorem 5.1. In this subsection, we present a proof of Theorem 5.1. For this, we first remind about a similar result on the SDDI.

Proposition 5.3. Suppose that the coupling strengths satisfy

\[ K_{\alpha} \geq 0, \quad \alpha = 1, 2, \quad K_d \geq 0, \]

and let \((x,v)\) be a global solution to (1.3). Then, for \( \alpha = 1, 2 \), we have

\[ \| \frac{d^2 x_{\alpha}}{dt^2} \| < K_d \sqrt{2NM_2(0)} \psi_{dM}, \ a.e. \ t \in (0, \infty), \]

\[ \| \frac{d^2 v_k(t)}{dt^2} \| \leq K_d \sqrt{2NM_2(0)} \psi_{2M}(t) + K_d \sqrt{2NM_2(0)} \psi_{dM}(t), \]

\[ \frac{dX_1}{dt} \leq V_1, \quad \frac{dV_1}{dt} \leq -K_1 \psi_1(\sqrt{2N_1 X_1} V_1) + K_d \sqrt{2NM_2(0)} \psi_{dM}, \]

\[ \| \dot{v}_\alpha(0) \| - \| \dot{v}_\alpha(t) \| \leq 2K_d \sqrt{2NM_2(0)} \int_{t_1}^{t_2} \psi_{dM}(t) \, dt, \quad 0 \leq t_1 \leq t_2 < \infty. \]

where \( \psi_{dM} \) and \( \psi_{2M} \) are given by the following relations:

\[ \psi_{dM} := \max_{1 \leq i \leq N_1} \psi_d(\| x_{2i} - x_{1i} \|), \quad \psi_{2M} := \max_{1 \leq i \neq j \leq N_2} \psi_2(\| x_{2k} - x_{2j} \|). \quad (5.1) \]

Proof. It is an analogue of the proof of Proposition 4.6 and Proposition 4.7. \( \Box \)

In the following two subsections, we proceed to prove Theorem 5.1 as follows.

• Step A (Local-in-time estimate). We will show that, for some finite time \( T \), subsystem \( G_1 \) satisfies the flocking estimate, but subsystem \( G_2 \) does not:

\[ \sup_{0 \leq t < T} X_1(t) < \bar{x}_1, \quad V_1(t) \leq C_1 \max \left\{ e^{-K_1 \psi_1(\sqrt{2N_1 X_1} V_1) t}, \psi_d \left( \frac{\Lambda_0 t}{2} \right) \right\}, \quad t \in [0,T), \]

\[ \min_{i \neq k} \| x_{2k}(t) - x_{2i}(t) \| \geq C t, \quad \text{for some positive constant } C. \]

• Step B (Continuation to the whole time interval). We will show that the time \( T \) in Step A can be chosen to be infinity.
5.2.1. **Step A (Local-in-time flocking estimates).** In this part, we show that the flocking estimates hold at least locally in time. For this, we set
\[ e_{2k,1}^0 := \frac{v_{2k}(0) - v_{1c}(0)}{\|v_{2k}(0) - v_{1c}(0)\|}, \quad e_{2k,2i}^0 := \frac{v_{2k}(0) - v_{2i}(0)}{\|v_{2k}(0) - v_{2i}(0)\|}, \]
\[ T_0^* := \sup \left\{ T \in [0, \infty) \mid \min_{i,k}(v_{2k}(t) - v_{1i}(t)) \cdot e_{2k,1}^0 > \Lambda_0, \forall t \in [0, T) \right\}, \]
\[ \hat{T}_0^* := \sup \left\{ T \in [0, T_0^*) \mid \min_{i\neq k}(v_{2k}(t) - v_{2i}(t)) \cdot e_{2k,2i}^0 > \mu_0, \forall t \in [0, T) \right\}. \]

**Lemma 5.4.** Suppose that initial data \((x_0, v_0)\) satisfy the following relations:
\[ \max_{1 \leq i \leq N_1} \|v_{1i}(0) - v_{1c}(0)\| < \frac{1}{4} \min_{1 \leq k \leq N_2} \|v_{2k}(0) - v_{1c}(0)\|, \quad v_{2k}(0) \neq v_{2i}(0), \text{ for } i \neq k. \]
Then, we have
\[ T_0^* > 0 \quad \text{and} \quad \hat{T}_0^* > 0. \]

**Proof.** We use assumptions to see
\[ (v_{2k}(0) - v_{1i}(0)) \cdot e_{2k,1}^0 = (v_{2k}(0) - v_{1c}(0) - \hat{v}_{1i}(0)) \cdot e_{2k,1}^0 \]
\[ \geq \|v_{2k}(0) - v_{1c}(0)\| - \|\hat{v}_{1i}(0)\| \geq \frac{3\Lambda_0}{2} > \Lambda_0, \]
\[ (v_{2k}(0) - v_{2i}(0)) \cdot e_{2k,2i}^0 = \|v_{2k}(0) - v_{2i}(0)\| \geq 2\mu_0 > \mu_0. \]

Then, by the continuity argument, we have \( T_0^* > 0 \) and \( \hat{T}_0^* > 0 \).

**Lemma 5.5.** Suppose that initial data \((x_0, v_0)\) satisfy \((F_B1)\). Let \(\psi_{2M}\) and \(\psi_{dM}\) be the functions defined in (5.1). Then, they satisfy
\[ \psi_{2M}(t) \leq \psi_2(\mu_0 t), \quad \psi_{dM}(t) \leq \psi_d(\Lambda_0 t), \quad t \in [0, \hat{T}_0^*). \]

**Proof.** For \( t \in [0, T_0^*] \) and \( i \neq k \), we have
\[ \|x_{2k}(t) - x_{2i}(t)\| \geq (x_{2k}(t) - x_{2i}(t)) \cdot e_{2k,2i}^0 \]
\[ = (x_{2k}(0) - x_{2i}(0)) \cdot e_{2k,2i}^0 + \int_0^t (v_{2k}(s) - v_{2i}(s)) \cdot e_{2k,2i}^0 ds \]
\[ \geq \mu_0 t. \]

Similarly, we have
\[ \|x_{2k}(t) - x_{1i}(t)\| \geq \Lambda_0 t. \]

Thus, by the non-increasing property of \(\psi_2\) and \(\psi_d\), we have the desired estimates.

**Lemma 5.6.** Suppose that initial data \((x_0, v_0)\) satisfy \((F_B1)\), then the following estimate holds: for \( t \in [0, T_0^*] \),
\[ \|v_{2i}(t) - v_{2i}(0)\| \leq \|v_{2i}(0)\| + \frac{K_2 \sqrt{2NM_2(0)}}{\mu_0} \int_0^t \psi_2(x) dx + \frac{K_d \sqrt{2NM_2(0)}}{\Lambda_0} \int_0^t \psi_d(x) dx, \]
where \( \|v_{2i}(t) - v_{2i}(0)\| := \max_{1 \leq i \leq N_2} \|v_{2i}(t) - v_{2i}(0)\|. \)

**Proof.** It follows from Proposition 5.4 that, for \( i \in \{1, \ldots, N_2\}, \)
\[ \|v_{2i}(t) - v_{2i}(0)\| \leq \frac{K_2 \sqrt{2NM_2(0)}}{\mu_0} \int_0^t \psi_2(x) dx + \frac{K_d \sqrt{2NM_2(0)}}{\Lambda_0} \int_0^t \psi_d(x) dx \]
Lemma 5.7. Suppose that the initial data \((x_0, v_0)\) satisfy \((F_B1)\) and the coupling strengths \(K_2\) and \(K_d\) satisfy

\[ 0 \leq K_d < \frac{\Lambda_0 \mu_0}{4 \sqrt{2NM_2(0)} \int_0^\infty \psi_d(x)dx} \quad \text{and} \quad 0 \leq K_2 < \frac{\mu_0^2}{4 \sqrt{2NM_2(0)} \int_0^\infty \psi_2(x)dx}. \]

Then, we have \(T_0^* = T_0^*\).

Proof. It follows from Lemma 5.4 that we have \(\hat{T}_0^* > 0\). We now assume that

\[ \hat{T}_0^* < T_0^*. \]

Then, the definition of \(\hat{T}_0^*\) implies

\[ (v_{2k}(\hat{T}_0^*) - v_{2i}(\hat{T}_0^*)) \cdot e_{2k,2i}^0 = \mu_0. \]  \hspace{1cm} (5.3)

On the other hand, by the initial assumption and Lemma 5.6, for \(t \in [0, \hat{T}_0^*]\) and \(i \neq k\)

\[ (v_{2k}(t) - v_{2i}(t)) \cdot e_{2k,2i}^0 \geq \|v_{2k}(0) - v_{2i}(0)\| - \|v_{2k}(t) - v_{2k}(0)\| - \|v_{2i}(t) - v_{2i}(0)\| \]

\[ \geq 2\mu_0 - \left( \frac{2K_2}{\mu_0} \int_0^\infty \psi_2(x)dx + \frac{2K_d}{\Lambda_0} \int_0^\infty \psi_d(x)dx \right) \sqrt{2NM_2(0)} \]

\[ > 2\mu_0 - \mu_0 = \mu_0. \]

The last inequality is from the assumptions of \(K_2\) and \(K_d\). This gives a contradiction to (5.3). Hence we obtain \(\hat{T}_0^* = T_0^*\). \(\square\)

Lemma 5.8. Keep the assumption of Lemma 5.7, we have

(i) \(\|v_{1c}(t) - v_{1c}(0)\| \leq \frac{K_d \sqrt{2NM_2(0)}}{\Lambda_0} \int_0^\infty \psi_d(x)dx, t \in [0, \hat{T}_0^*]\).

(ii) \(\|\dot{v}_1(t)\| \leq \|\dot{v}_1(0)\| + \frac{2K_d \sqrt{2NM_2(0)}}{\Lambda_0} \int_0^\infty \psi_d(x)dx, t \in [0, \hat{T}_0^*]\).

Proof. The estimates follow directly from the Proposition 5.4. \(\square\)

Lemma 5.9. (Local-in-time flocking estimate) Suppose that the initial data \((x_0, v_0)\) satisfy \((F_B1)\) and the coupling strengths satisfy

\[ 0 \leq K_d < \frac{\mu_0 \Lambda_0}{4 \sqrt{2NM_2(0)} \int_0^\infty \psi_d(x)dx}, \quad 0 \leq K_2 < \frac{\mu_0^2}{4 \sqrt{2NM_2(0)} \int_0^\infty \psi_2(x)dx}, \quad K_1 > \frac{\Lambda_1(0) + \int_{X_1(0)} \psi_1(\sqrt{2N_1}x)dx}{\int_{X_1(0)} \psi_1(\sqrt{2N_1}x)dx}. \]

Then, for the solution \((x, v)\) to system (1.3) with initial data \((x_0, v_0)\), there exist positive constants \(\tilde{x}_1^0\) and \(C_1\) independent of time \(t\) such that

\[ \sup_{0 \leq t < \infty} X_1(t) < \tilde{x}_1^0, \quad V_1(t) \leq C_1 \max \left\{ e^{-\frac{K_1 \tau_1(\sqrt{2N_1}t t_1)}{2}}, \psi_d \left( \frac{\Lambda_0 t}{2} \right) \right\}, \quad \text{as } t \to \infty, \]

\[ \inf_{0 \leq t < \infty} \|v_{2k}(t) - v_{2i}(t)\| \geq \mu_0, \quad \min_{1 \leq i \neq k \leq N_2} \|x_{2k}(t) - x_{2i}(t)\| \geq \mu_0 t, \quad t \in [0, \hat{T}_0^*]. \]
Proof. (i) (Existence of $\bar{x}^\infty_1$): Define a Lyapunov functional $L_2$:

$$L_2(t) := \mathcal{V}_1(t) + K_1 \int_{X_1(t)} \psi_1(\sqrt{2N_1}x)dx.$$ 

It follows from Proposition 5.3 and Lemma 5.5 that we have

$$\frac{dL_2(t)}{dt} = -K_1 \psi_1(\sqrt{2N_1}x_1(t)) \frac{d}{dt}x_1(t) \leq K_1 \psi_1(\sqrt{2N_1}x_1(t)) (\mathcal{V}_1(t) - \frac{d}{dt}x_1(t)) + K_d \sqrt{2NM_2(0)} \psi_{dM} \leq K_d \sqrt{2NM_2(0)} \psi_d \left(\Lambda_0 t\right), \quad t \in [0, T^*_0).$$

We integrate the aforementioned relation to obtain

$$\mathcal{V}_1(t) + K_1 \int_{X_1(0)}^{X_1(t)} \psi_1(\sqrt{2N_1}x)dx \leq \mathcal{V}_1(0) + \frac{K_d \sqrt{2NM_2(0)}}{\Lambda_0} \int_0^\infty \psi_d(x)dx.$$ 

In particular, this yields

$$K_1 \int_{X_1(0)}^{X_1(t)} \psi_1(\sqrt{2N_1}x)dx \leq \mathcal{V}_1(0) + \frac{K_d \sqrt{2NM_2(0)}}{\Lambda_0} \int_0^\infty \psi_d(x)dx, \quad t \in [0, T^*_0). \tag{5.4}$$

On the other hand, the condition on $K_1$ implies

$$\mathcal{V}_1(0) + \frac{K_d \sqrt{2NM_2(0)}}{\Lambda_0} \int_0^\infty \psi_d(x)dx < K_1 \int_{X_1(0)}^{\bar{x}_1^\infty} \psi_1(\sqrt{2N_1}x)dx.$$ 

We set $\bar{x}_1^\infty$ to be a positive number satisfying the following relation:

$$K_1 \int_{X_1(0)}^{\bar{x}_1^\infty} \psi_1(\sqrt{2N_1}x)dx = \mathcal{V}_1(0) + \frac{K_d \sqrt{2NM_2(0)}}{\Lambda_0} \int_0^\infty \psi_d(x)dx. \tag{5.5}$$

Then, by using (5.4) and (5.5), we have

$$\sup_{0 \leq t < T^*_0} X_1(t) < \bar{x}_1^\infty, \quad t \in [0, T^*_0].$$

(ii) (Decay estimate of $\mathcal{V}_1$): It follows from Proposition 5.4 and the result of (i) that we have

$$\frac{d\mathcal{V}_1}{dt} \leq -K_1 \psi_1(\sqrt{2N_1}\bar{x}^\infty_1) \mathcal{V}_1 + K_d \sqrt{2NM_2(0)} \psi_{dM}.$$ 

This yields the desired decay estimate of $\mathcal{V}_1$.

The two remaining estimates are direct results of Lemma 5.5 and Lemma 5.7. \qed

5.2.2. Step B: $T^*_0 = \infty$. In this part, we complete the proof of Theorem 5.1. Suppose that the initial data and coupling strength satisfy the framework $(\mathcal{F}_B)$ in Theorem 5.1. Then, it follows from Lemma 5.4 that we have

$$T^*_0 > 0.$$ 

Suppose that $T^*_0 < \infty$. Then, by the definition of $T^*_0$ in (5.2), we have

$$(v_{2k}(T^*_0) - v_{1k}(T^*_0)) \cdot e^0_{2k,1} = \Lambda_0. \tag{5.6}$$

It follows from Lemma 5.5-Lemma 5.8 that we have

$$(v_{2k}(T^*_0) - v_{1k}(T^*_0)) \cdot e^0_{2k,1} \geq \|v_{2k}(0) - v_{1c}(0)\| - \|v_{1k}(T^*_0)\| - \|v_{2k}(T^*_0) - v_{2k}(0)\| - \|v_{1c}(T^*_0) - v_{1c}(0)\| \geq \|v_{2k}(0) - v_{1c}(0)\| - \|v_{1c}(0)\|.$$
\[- \frac{K_2 \sqrt{2NM_2(0)}}{\mu_0} \int_0^\infty \psi_2(x)dx - \frac{4K_d \sqrt{2NM_2(0)}}{\Lambda_0} \int_0^\infty \psi_d(x)dx > \Lambda_0,
\]

where we have used the initial assumption and the assumptions of \(K_2, K_d\) to get the last inequality. This gives a contradiction to (5.6). Thus \(T_0^* = \infty\). Now we apply Lemma 5.9 with \(T_0^* = \infty\) to get the desired estimates and complete the proof of Theorem 5.1.

5.3. Proof of Corollary 5.2. In this subsection, we provide the proof of Corollary 5.2. As in Section 4.2, we will show that there exists a positive time \(T_0\) such that

\[
\min_{1 \leq i \leq N_1, 1 \leq k \leq N_2} \{(x_{2k}(T_0) - x_{1i}(T_0)) \cdot (\psi_{2k}(T_0) - \psi_{1c}(T_0))\} \geq 0, \quad \text{and}
\]

\[
\min_{1 \leq i \neq k \leq N_2} \{(x_{2k}(T_0) - x_{2i}(T_0)) \cdot (\psi_{2k}(T_0) - \psi_{2i}(T_0))\} \geq 0.
\]

For this, we introduce the following auxiliary functions:

\[
\bar{\Delta}_{2k,1i}(t) := (x_{2k}(t) - x_{1i}(t)) \cdot (v_{2k}(t) - v_{1c}(t)), \quad \text{for all} \ 1 \leq i \leq N_1, 1 \leq k \leq N_2,
\]

\[
\bar{\Delta}_{ki}(t) := (x_{2k}(t) - x_{2i}(t)) \cdot (v_{2k}(t) - v_{2i}(t)), \quad \text{for all} \ 1 \leq i \neq k \leq N_2,
\]

and we set

\[
T_1 := \frac{1}{\Lambda_0} \max_{1 \leq i \leq N_1} |\bar{\Delta}_{2k,1i}(0)|, \quad T_2 := \frac{1}{\mu_0} \max_{1 \leq i \neq k \leq N_2} |\bar{\Delta}_{ki}(0)|, \quad T_0 := \max\{T_1, T_2\} \geq 0.
\]

(5.7)

Lemma 5.10. Suppose that the initial data \((x_0, v_0)\) satisfy (F-B1) and the coupling strengths \(K_2\) and \(K_d\) satisfy

\[
0 \leq K_2 < \frac{\min\{\Lambda_0, \mu_0\}}{16 \sqrt{2NM_2(0)}T_0} \quad \text{and} \quad 0 \leq K_d < \frac{\min\{\Lambda_0, \mu_0\}}{16 \sqrt{2NM_2(0)}T_0}.
\]

Then, we have, for \(t \in [0, T_0]\)

\[
\|\dot{\psi}_1(t)\|_\infty \leq \|\dot{\psi}_1(0)\|_\infty + \frac{\Lambda_0}{8} \leq \frac{5\Lambda_0}{8},
\]

\[
\|v_{2k}(t) - v_{1c}(t)\| \geq \|v_{2k}(0) - v_{1c}(0)\| - \frac{3\Lambda_0}{16} \geq \frac{29\Lambda_0}{16},
\]

\[
\|v_{2k}(t) - v_{2i}(t)\| \geq \|v_{2k}(0) - v_{2i}(0)\| - \frac{\mu_0}{4} \geq \frac{7\mu_0}{4}, \quad \text{for all} \ i \neq k.
\]

Proof. (i) It follows from Proposition 5.3 that

\[
\|\dot{\psi}_1(t)\|_\infty \leq \|\dot{\psi}_1(0)\|_\infty + 2K_d \sqrt{2NM_2(0)}T_0 \leq \|\dot{\psi}_1(0)\|_\infty + \frac{\Lambda_0}{8} \leq \frac{5\Lambda_0}{8},
\]

where we have used the assumption of \(K_d\).

(ii) It follows from Proposition 5.3 that

\[
\left\| \frac{d\psi_{1c}(t)}{dt} \right\| \leq K_d \sqrt{2NM_2(0)} \psi_{dM}(t) \leq K_d \sqrt{2NM_2(0)},
\]

\[
\left\| \frac{d\psi_{2k}(t)}{dt} \right\| \leq K_2 \sqrt{2NM_2(0)} \psi_{2M}(t) + K_d \sqrt{2NM_2(0)} \psi_{dM}(t) \leq K_2 \sqrt{2NM_2(0)} + K_d \sqrt{2NM_2(0)}.
\]

Thus, we have

\[
\left\| \frac{d(v_{2k}(t) - v_{1c}(t))}{dt} \right\| \leq K_2 \sqrt{2NM_2(0)} + 2K_d \sqrt{2NM_2(0)}.
\]
This implies
\[
\|v_{2k}(t) - v_{1c}(t)\| \geq \|v_{2k}(0) - v_{1c}(0)\| - K_2\sqrt{2NM_2(0)}t_0 - 2K_d\sqrt{2NM_2(0)}t_0 \\
\geq \|v_{2k}(0) - v_{1c}(0)\| - \frac{3\Lambda_0}{16} \geq \frac{29\Lambda_0}{16},
\]
where we have used the assumptions of $K_2$ and $K_d$.

(iii) By (ii), we have
\[
\left\| \frac{d(v_{2k}(t) - v_{2i}(t))}{dt} \right\| \leq 2K_2\sqrt{2NM_2(0)} + 2K_d\sqrt{2NM_2(0)}.
\]
Thus, we have
\[
\|v_{2k}(t) - v_{2i}(t)\| \geq \|v_{2k}(0) - v_{2i}(0)\| - 2K_2\sqrt{2NM_2(0)}t_0 - 2K_d\sqrt{2NM_2(0)}t_0 \\
\geq \|v_{2k}(0) - v_{2i}(0)\| - \frac{1}{4}\mu_0 \geq \frac{7\mu_0}{4}.
\]

\[
\text{Proposition 5.11. Suppose that the initial data } (x_0, u_0) \text{ satisfy } (F_B1) \text{ and the coupling strengths } K_2 \text{ and } K_d \text{ are sufficiently small:}
\]
\[
0 \leq K_2 < \min \left\{ \frac{\min\{\Lambda_0, \mu_0\}}{16\sqrt{2NM_2(0)}T_0} \cdot \min\{\Lambda_2^0, \mu_2^0\} \cdot \min\{\Lambda_2^0, \mu_2^0\} \cdot \frac{\min\{D(x_0, x_0), D(x_0, x_i)\} + \sqrt{2NM_2(0)}T_0}{4\sqrt{2NM_2(0)}} \right\};
\]
\[
0 \leq K_d < \min \left\{ \frac{\min\{\Lambda_0, \mu_0\}}{16\sqrt{2NM_2(0)}T_0} \cdot \min\{\Lambda_2^0, \mu_2^0\} \cdot \min\{\Lambda_2^0, \mu_2^0\} \cdot \frac{\min\{D(x_0, x_0), D(x_0, x_i)\} + \sqrt{2NM_2(0)}T_0}{4\sqrt{2NM_2(0)}} \right\}.
\]
Then, we have
\[
\min_{1 \leq k \leq N_1} \Delta_{2k, 1i}(T_0) \geq 0, \quad \min_{1 \neq k \leq N_2} \Delta_{k, i}(T_0) \geq 0.
\]

Proof. We first claim that
\[
\frac{d}{dt}\Delta_{2k, 1i}(t) > \Lambda_0^2 \quad \text{and} \quad \frac{d}{dt}\Delta_{k, i}(t) > \mu_0^2, \quad t \in [0, T_0]
\]
(5.8)

Proof of the claim (5.8). (i) For $t \in [0, T_0]$, we use
\[
\|v_{2k}(t) - v_{1i}(t)\| \leq \sqrt{2NM_2(0)}
\]
to obtain
\[
\|x_{2k}(t) - x_{1i}(t)\| \leq \|x_{2k}(0) - x_{1i}(0)\| + \sqrt{2NM_2(0)}T_0, \quad t \in [0, T_0].
\]
Then, we have
\[
\frac{d}{dt}\Delta_{2k, 1i}(t)
\]
\[
= (v_{2k}(t) - v_{1i}(t)) \cdot (v_{2k}(t) - v_{1c}(t)) + (x_{2k}(t) - x_{1i}(t)) \cdot (v_{2k}(t) - v_{1c}(t))
\geq \|v_{2k}(t) - v_{1c}(t)\| \left( \|v_{2k}(t) - v_{1c}(t)\| - \|v_{1i}(t)\| \right)
\geq \left( \|x_{2k}(0) - x_{1i}(0)\| + \sqrt{2NM_2(0)}T_0 \right) (K_2 + 2K_d) \sqrt{2NM_2(0)}
\geq \frac{551\Lambda_0^2}{256} - \frac{3\Lambda_0^2}{4} > \Lambda_0^2,
\]
where we have used the assumptions of $K_2$ and $K_d$.

(ii) For $t \in [0, T_0]$ and $i \neq k$, we have
\[
\|x_{2k}(t) - x_{2i}(t)\| \leq \|x_{2k}(0) - x_{2i}(0)\| + \sqrt{2NM_2(0)}T_0.
\]
Thus, we can show
\[
\frac{d}{dt} \tilde{\Delta}_{ki}(t) = \|v_{2k}(t) - v_{2i}(t)\|^2 + (x_{2k}(t) - x_{2i}(t)) \cdot (\dot{v}_{2k}(t) - \dot{v}_{2i}(t)) \\
\geq \|v_{2k}(t) - v_{2i}(t)\|^2 - (\|x_{2k}(0) - x_{2i}(0)\| + \sqrt{2N_2M_2(0)T_0}) \cdot (2K_2 + 2K_d)\sqrt{2N_2M_2(0)} \\
\geq \frac{49}{16} \mu_0^2 - \mu_0^2 > \mu_0^2.
\]

It follows from the claim and the definition of the relation of \(T_0\) in (5.7) that we have
\[
\tilde{\Delta}_{2k,1i}(T_0) \geq \tilde{\Delta}_{2k,1i}(0) + \Lambda_0^2 T_0 \\
= \tilde{\Delta}_{2k,1i}(0) + |\tilde{\Delta}_{2k,1i}(0)| \geq 0, \\
\tilde{\Delta}_{ki}(T_0) \geq \tilde{\Delta}_{ki}(0) + \mu_0^2 T_0 \\
= \tilde{\Delta}_{ki}(0) + |\tilde{\Delta}_{ki}(0)| \geq 0.
\]
These yield the desired results. \(\square\)

### 5.3.1. Emergence of partial flocking

In this part, we present a proof of Corollary 5.2. For this, we set
\[
e_{2k,1}(T_0) := \frac{v_{2k}(T_0) - v_{1i}(T_0)}{\|v_{2k}(T_0) - v_{1i}(T_0)\|}, \quad e_{2k,2i}(T_0) := \frac{v_{2k}(T_0) - v_{2i}(T_0)}{\|v_{2k}(T_0) - v_{2i}(T_0)\|}, \\
T_1^\infty := \sup \left\{ T \in [T_0, \infty) \mid \min_{i,k} \{v_{2k}(t) - v_{1i}(t)\} \cdot e_{2k,1}(T_0) > \frac{\Lambda_0}{2}, t \in [0, T) \right\}, \\
\hat{T}_1^\infty := \sup \left\{ T \in [T_0, T_1^\infty) \mid \min_{i \neq k} \{v_{2k}(t) - v_{1i}(t)\} \cdot e_{2k,2i}(T_0) > \frac{\mu_0}{2}, t \in [0, T) \right\}.
\]

**Lemma 5.12.** Keep the assumptions of Lemma 5.10 and let \(T_1^\infty\) and \(\hat{T}_1^\infty\) be the positive values defined in (5.9). Then, we have
\[
T_1^\infty > T_0 \quad \text{and} \quad \hat{T}_1^\infty > T_0.
\]

**Proof.** By Lemma 5.10, we have
\[
(v_{2k}(T_0) - v_{1i}(T_0)) \cdot e_{2k,1}(T_0) \\
\geq \|v_{2k}(T_0) - v_{1i}(T_0)\| \cdot \|v_{2k}(T_0) - v_{1i}(T_0)\| \geq \frac{19\Lambda_0}{16} > \frac{\Lambda_0}{2}, \\
(v_{2k}(T_0) - v_{2i}(T_0)) \cdot e_{2k,2i}(T_0) = \|v_{2k}(T_0) - v_{2i}(T_0)\| \geq \frac{\mu_0}{4} > \frac{\mu_0}{2}.
\]
Thus we have \(T_1^\infty > T_0\) and \(\hat{T}_1^\infty > T_0\). \(\square\)

**Lemma 5.13.** Keep the assumptions of Proposition 5.11, then the following estimates hold:

(i) \(\psi_{2M}(t) \leq \psi_{2}\left(\frac{\mu_0}{2}(t - T_0)\right)\) and \(\psi_{dM}(t) \leq \psi_{d}\left(\frac{\Lambda_0}{2}(t - T_0)\right), t \in [T_0, \hat{T}_1^\infty).\)

(ii) \(\|v_2(t) - v_2(T_0)\| \leq \max_{1 \leq i \leq N_2} \|v_{2i}(t) - v_{2i}(T_0)\| \leq \frac{2K_2\sqrt{2NM_2(0)}}{\mu_0} \int_0^\infty \psi_2(x)dx + \frac{2K_d\sqrt{2NM_2(0)}}{\Lambda_0} \int_0^\infty \psi_d(x)dx, t \in [T_0, \hat{T}_1^\infty).\)
Proof. (i) For $t \in [T_0, \hat{T}_1^\infty]$ and for all $1 \leq i \neq k \leq N_2$, we have
\[
\|\mathbf{x}_{2k}(t) - \mathbf{x}_{2i}(t)\| \\
\geq (\mathbf{x}_{2k}(t) - \mathbf{x}_{2i}(t)) \cdot e_{2k,2i}(T_0) \\
= (\mathbf{x}_{2k}(T_0) - \mathbf{x}_{2i}(T_0)) \cdot e_{2k,2i} + \int_{T_0}^t (\mathbf{v}_{2k}(s) - \mathbf{v}_{2i}(s)) \cdot e_{2k,2i}(T_0) ds \\
\geq \frac{\mu_0}{2} (t - T_0).
\]
Similarly, we can obtain
\[
\|\mathbf{x}_{2k}(t) - \mathbf{x}_{1i}(t)\| \geq \frac{\lambda_0}{2} (t - T_0).
\]
Then, by the non-increasing property of $\psi_2(t)$ and $\psi_d$, we have
\[
\psi_2(\|\mathbf{v}_{2k}(t) - \mathbf{v}_{2i}(t)\|) \leq \psi_2 \left( \frac{\mu_0}{2} (t - T_0) \right),
\]
\[
\psi_d(\|\mathbf{x}_{2k}(t) - \mathbf{x}_{1i}(t)\|) \leq \psi_d \left( \frac{\lambda_0}{2} (t - T_0) \right).
\]
(ii) We use Proposition 5.3 to derive the desired estimate.

\[\square\]

**Lemma 5.14.** Suppose that the initial data $(\mathbf{x}_0, \mathbf{v}_0)$ satisfy (F1) and the coupling strengths $K_2$ and $K_d$ satisfy
\[
0 \leq K_2 < \min \left\{ \frac{\min\{\Lambda_0, \mu_0\}}{16 \sqrt{2N M_2(0) T_0}} \sqrt{8N M_2(0)} \int_0^\infty \psi_2(x) dx, \frac{\min\{\lambda_0^2, \mu_0^2\}}{4 (D(\mathbf{x}_1(0), \mathbf{x}_2(0)) + \sqrt{2N M_2(0) T_0}) \sqrt{2N M_2(0)}} \right\},
\]
\[
0 \leq K_d < \min \left\{ \frac{\min\{\Lambda_0, \mu_0\}}{16 \sqrt{2N M_2(0) T_0}} \sqrt{8N M_2(0)} \int_0^\infty \psi_d(x) dx, \frac{\min\{\lambda_0^2, \mu_0^2\}}{4 (D(\mathbf{x}_1(0), \mathbf{x}_2(0)) + \sqrt{2N M_2(0) T_0}) \sqrt{2N M_2(0)}} \right\}.
\]
Then, we have
\[
\hat{T}_1^\infty = T_1^\infty.
\]

**Proof.** It follows from Lemma 5.12 that we have $\hat{T}_1^\infty > T_0$. Assume $\hat{T}_1^\infty < T_1^\infty$. Then, by the definition of $T_1^\infty$, we have
\[
(\mathbf{v}_{2k}(\hat{T}_1^\infty) - \mathbf{v}_{2i}(\hat{T}_1^\infty)) \cdot e_{2k,2i}(T_0) = \frac{\mu_0}{2}.
\]
Then by Lemma 5.10 and Lemma 5.13, for $t \in [T_0, \hat{T}_1^\infty]$ and $i \neq k$
\[
(\mathbf{v}_{2k}(t) - \mathbf{v}_{2i}(t)) \cdot e_{2k,2i}(T_0) \\
\geq \|\mathbf{v}_{2k}(T_0) - \mathbf{v}_{2i}(T_0)\| - \|\mathbf{v}_{2k}(t) - \mathbf{v}_{2k}(T_0)\| - \|\mathbf{v}_{2i}(t) - \mathbf{v}_{2i}(T_0)\| \\
\geq \frac{\mu_0}{4} - \left( \frac{4 K_2}{\mu_0} \int_0^\infty \psi_2(s) ds + \frac{4 K_d}{\lambda_0} \int_0^\infty \psi_d(s) ds \right) \sqrt{2N M_2(0)} \\
> \frac{\mu_0}{4} - \mu_0 > \frac{\mu_0}{2}.
\]
This gives a contradiction to (5.10). Hence we have $\hat{T}_1^\infty = T_1^\infty$.

\[\square\]
Lemma 5.15. Keep the assumption of Lemma 5.14, then the following estimates hold:

(i) \( \|v_{1c}(t) - v_{1c}(T_0)\| \leq \frac{2K_d\sqrt{2NM_2(0)}}{\Lambda_0} \int_{0}^{\infty} \psi_2(x)dx, \quad t \in [T_0, T_1^{\infty}) \).

(ii) \( \|\dot{v}_1(t)\|_{\infty} \leq \frac{4K_d\sqrt{2NM_2(0)}}{\Lambda_0} \int_{0}^{\infty} \psi_2(x)dx, \quad t \in [T_0, T_1^{\infty}) \).

Proof. It is a direct consequence of Proposition 5.3 and Lemma 5.14.

Lemma 5.16. Suppose that the initial data \((x_0, v_0)\) satisfy \((\mathcal{F}_B)\) and the coupling strengths satisfy

\[
0 \leq K_2 < \min\left\{ \frac{\min\{\Lambda_0, \mu_0\}}{16\sqrt{2NM_2(0)T_0}}, \frac{\mu_0^2}{8\sqrt{2NM_2(0)}\int_{0}^{\infty} \psi_2(x)dx}, \frac{\min\{\Lambda_0^2, \mu_0^2\}}{4(D(x_0(0), x_2(0)) + \sqrt{2NM_2(0)T_0})\sqrt{2NM_2(0)}} \right\},
\]

\[
0 \leq K_d < \min\left\{ \frac{\min\{\Lambda_0, \mu_0\}}{16\sqrt{2NM_2(0)T_0}}, \frac{\Lambda_0\mu_0}{8\sqrt{2NM_2(0)}\int_{0}^{\infty} \psi_2(x)dx}, \frac{\min\{\Lambda_0^2, \mu_0^2\}}{4(D(x_0(0), x_2(0)) + \sqrt{2NM_2(0)T_0})\sqrt{2NM_2(0)}} \right\},
\]

\[
K_1 > \frac{V_1(T_0) + \frac{K_d\sqrt{2NM_2(0)}\int_{0}^{\infty} \psi_2(x)dx}{\Lambda_0}}{\int_{T_0}^{\infty} \psi_1(\sqrt{2NM_2x})dx}.
\]

Then, for the solution \((x, v)\) to system (1.3) with initial data \((x_0, v_0)\), there exist positive constants \(\tilde{x}_1^\infty\) and \(C_2\) independent of time \(t\) such that

\[
\sup_{0 \leq t < T_0} x_1(t) < \tilde{x}_1^\infty, \quad V_1(t) \leq C_2 \max \left\{ e^{-\frac{\kappa_1\psi_2(\sqrt{2NM_2x})(t-T_0)}{2}}, \psi_2\left(\frac{\lambda_0}{4}(t-T_0)\right) \right\},
\]

\[
\inf_{0 \leq t < \infty} \|v_{2k}(t) - v_{2k}(t)\| \geq \frac{\mu_0}{2}, \quad \min_{1 \leq i \neq k \leq N_2} \|x_{2k}(t) - x_{2k}(t)\| \geq \frac{\mu_0}{2} t, \quad t \in [T_0, T_1^{\infty}).
\]

Proof. Since the proof is an analogue of the proof of Lemma 5.9, we omit it here.

5.3.2. Step B: \(T_1^{\infty} = \infty\). We next provide the proof of Corollary 5.2 by showing that \(T_1^{\infty} = \infty\). Suppose that initial data \((x_0, v_0)\) and the coupling strengths satisfy the framework \((\mathcal{F}_B)\).

\(\circ\) Procedure A: (The conditions on \(K_1\) in Corollary 5.2 imply the conditions on \(K_1\) in Lemma 5.16). This is exactly the same as in Section 4.

\(\circ\) Procedure B: We claim

\[
T_1^{\infty} = \infty. \tag{5.11}
\]

Proof of claim (5.11). It follows from Lemma 5.12 that we have \(T_1^{\infty} > T_0\). Suppose that

\[
T_1^{\infty} < \infty.
\]

Then, by the definition of \(T_1^{\infty}\), we have

\[
(v_{2k}(T_1^{\infty}) - v_{11}(T_1^{\infty})) \cdot e_{2k,1}(T_0) = \frac{\Lambda_0}{2}.
\]

Then, we have

\[
(v_{2k}(T_1^{\infty}) - v_{11}(T_1^{\infty})) \cdot e_{2k,1}(T_0)
\]
\[ \geq \|v_{2k}(T_0) - v_{1c}(T_0)\| - \|\tilde{v}_{1c}(T_0)\| - \|v_{2k}(T_1^\infty) - v_{2k}(T_0)\| \\
\geq 19\Lambda_0 \frac{16}{2} - \frac{2K_2\sqrt{2N M(0)}}{\mu_0} \int_0^\infty \psi_2(x)dx - \frac{8K_d\sqrt{2N M(0)}}{\Lambda_0} \int_0^\infty \psi_d(x)dx \]

This is contradictory to (5.12). Thus, we have \( T_1^\infty = \infty \). Finally, we apply Lemma 5.16 to obtain the conclusion of Corollary 5.2.

6. Conclusion. In this paper, we have presented the analytical results on the asymptotic emergent dynamics in the interaction of two homogeneous C-S ensembles. The coupling between two homogeneous C-S ensembles is controlled by the strength \( K_d \) and the communication weight function \( \psi_d(\cdot) \). As we can imagine, the asymptotic pictures can be classified into three categories (mono-cluster, bi-cluster, and multi-cluster flockings). For each of the distinct asymptotic states, we have presented sufficient frameworks that lead to the aforementioned asymptotic states. Our proposed sufficient frameworks are formulated in terms of coupling strengths, depending on the initial configurations. In a large inter-subsystem coupling regime, mono-cluster flocking emerges regardless of the intra-coupling strength as long as it is nonnegative. If we turn off intra-system coupling strength \( K_1 = K_2 = 0 \) and take an inter-system coupling strength that is sufficiently large, mono-cluster flocking can occur asymptotically. In contrast, for a small inter-coupling regime, bi-cluster and multi-cluster flockings can emerge asymptotically, depending on the size of the intra-subsystem coupling strength.

Appendix A. Gronwall-type inequalities. In this section, we present a Gronwall-type lemma that have been used in the proof of Theorem 4.2 and Theorem 5.1 for the reader’s convenience.

Lemma A.1. [9] Let \( y : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\} \) be a differentiable function satisfying

\[ y' \leq -ay + f, \quad t > 0, \quad y(0) = y_0, \]

where \( a \) is a positive constant and \( f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R} \) is a continuous function decaying to zero as its argument goes to infinity. Then, \( y \) satisfies

\[ y(t) \leq \frac{1}{a} \max_{s \in [t/2,t]} |f(s)| + y_0 e^{-at} + \frac{\|f\|_{L^\infty}}{a} e^{-\frac{at}{2}} , \quad t \geq 0. \]

Proof. Note that \( y \) satisfies

\[ y' + ay \leq f. \]

We multiply the aforementioned differential inequality by \( e^{at} \) and integrate the resulting relation from \( s = 0 \) to \( s = t \) to find

\[ e^{at}y - y_0 \leq \int_0^t f(\tau)e^{a\tau}d\tau \]

\[ = \int_0^\frac{t}{2} f(\tau)e^{a\tau}d\tau + \int_\frac{t}{2}^t f(\tau)e^{a\tau}d\tau \]

\[ \leq \|f\|_{L^\infty} \int_0^\frac{t}{2} e^{a\tau}d\tau + \max_{\tau \in [\frac{t}{2},t]} |f(\tau)| \int_\frac{t}{2}^t e^{a\tau}d\tau \]

\[ \leq \|f\|_{L^\infty} \frac{e^{\frac{at}{2}} - 1}{a} + \frac{1}{a} \max_{\tau \in [\frac{t}{2},t]} |f(\tau)| \left(e^{at} - e^{\frac{at}{2}}\right). \]
Hence,
\[ y(t) \leq \frac{1}{a} \max_{\tau \in \left[\frac{t}{2}, t\right]} |f(\tau)| + \left( y_0 - \frac{\|f\|_{L^\infty}}{a} \right) e^{-at} + \left( \frac{\|f\|_{L^\infty}}{a} - \frac{1}{a} \max_{\tau \in \left[\frac{t}{2}, t\right]} |f(\tau)| \right) e^{-\frac{at}{2}}. \]

Therefore, for \( t \geq 0 \),
\[ y(t) \leq \frac{1}{a} \max_{\tau \in \left[\frac{t}{2}, t\right]} |f(\tau)| + y_0 e^{-at} + \frac{\|f\|_{L^\infty}}{a} e^{-\frac{at}{2}}. \]

\[ \square \]

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E-mail address: skeeper@snu.ac.kr
E-mail address: pyeong@snu.ac.kr
E-mail address: yinglongzhang@amss.ac.cn
E-mail address: zhangxiongtao1987@gmail.com