Nonlinear Realisations of $w_{1+\infty}$

E. Sezgin

Center for Theoretical Physics
Texas A&M University
College Station, TX 77843, U.S.A.

and

K.S. Stelle

The Blackett Laboratory
Imperial College
Prince Consort Road
London SW7 2BZ, U.K.

ABSTRACT

The nonlinear scalar-field realisation of $w_{1+\infty}$ symmetry in $d = 2$ dimensions is studied in analogy to the nonlinear realisation of $d = 4$ conformal symmetry $SO(4,2)$. The $w_{1+\infty}$ realisation is derived from a coset-space construction in which the divisor group is generated by the non-negative modes of the Virasoro algebra, with subsequent application of an infinite set of covariant constraints. The initial doubly-infinite set of Goldstone fields arising in this construction is reduced by the covariant constraints to a singly-infinite set corresponding to the Cartan-subalgebra generators $v^\ell_{-(\ell+1)}$. We derive the transformation rules of this surviving set of fields, finding a triangular structure in which fields transform into themselves or into lower members of the set only. This triangular structure gives rise to finite-component subrealisations, including the standard one for a single scalar. We derive the Maurer-Cartan form and discuss the construction of invariant actions.
1. Introduction

The $w_{1+\infty}$ algebra that has been studied as a higher-spin symmetry algebra in $d = 2$ conformal field theories admits its simplest field-theoretic realisation in terms of a single scalar field $\varphi(x)$. This realisation is necessarily nonlinear because its field content is far too small to support a linear realisation of the algebra. $w_{1+\infty}$ can naturally be viewed as the algebra of symplectic diffeomorphisms of a two-dimensional cylinder $(y, \theta)$; this linear realisation can in turn be viewed as a Poisson bracket algebra of the basis functions on the cylinder

$$v_m^\ell = -iy^{\ell+1}e^{im\theta},$$

with a resulting algebra of differential operators $v_m^\ell = e^{im\theta}(my^{\ell+1}\partial/\partial y + i(\ell + 1)y^{\ell}\partial/\partial\theta)$ given by

$$[v_m^j, v_n^\ell] = [(\ell + 1)m - (j + 1)n]v_{m+n}^{j+\ell} \quad \ell, j \geq -1, \quad -\infty < m, n < \infty.$$  

(2)

For the $w_{\infty}$ algebra, the upper indices are restricted to values $\ell \geq 0$. Both $w_{1+\infty}$ and $w_{\infty}$ contain the Virasoro algebra generated by $v_0^m$.

Since only one of the $d = 2$ worldsheet coordinates of $\varphi(x)$ is involved in the $w_{\infty}$ transformations, this field is effectively a function of only one variable in as far as the realisation is concerned, and is thus insufficient to support a full linear realisation of the algebra (2). The nonlinear realisation on $\varphi$,

$$\delta \varphi = k_\ell (\partial_+ \varphi)^{\ell+1},$$

(3)

with parameters $k_\ell(x^+)$ that are “semilocal”, since they depend on $x^+$ but not $x^-$, may be viewed as arising from a coset-space construction $w_{1+\infty}/w_{\infty}$ [1] in which the coset parameter may be considered to be a function of just the $\theta$ variable on the cylinder, since the basis functions in (1) that are left out of $w_{\infty}$ are independent of $y$. The field $\varphi$ may be identified with this coset parameter provided $x^+$ is identified with $\theta$. Note that $x^-$ here is a “time” unrelated to this chiral algebra.

The derivation of the transformations (3) from the $w_{1+\infty}/w_{\infty}$ coset construction given in [1] follows standard techniques of the theory of nonlinear realisations [2, 3]. An unusual feature of this construction, however, is the fact that the transformations of the Goldstone field $\varphi(x)$ are nonlinear also for the divisor group $w_{\infty}$, and not just for the transformations belonging to the coset. This is due to the fact that the $w_{1+\infty}/w_{\infty}$ coset is non-reductive, i.e. the coset generators do not form a linear representation of the divisor group $w_{\infty}$, as can be seen in (2), since commutators of coset generators with $w_{\infty}$ generators produce results lying mostly in $w_{\infty}$ and not in the coset. The only generators in $w_{\infty}$ with respect to which $\varphi(x)$ actually transforms linearly are the Virasoro generators $v_m^0$, as can be seen in (3).

In this paper, we seek a more detailed understanding of the group-theoretic aspects of the nonlinear realisation (3) by starting from an essentially reductive coset construction in
which we choose the divisor group to be generated by the \( m \geq 0 \) modes of the Virasoro algebra, which we shall denote by \( \text{Vir}^+ \). In this construction, we shall view the Virasoro algebra as a conformal algebra on the circle \( S^1 \), working in a chart where the \( x^+ \) coordinate is itself viewed as a coset parameter associated to the Virasoro generator \( L_{-1} = v_{-1}^0 \). In this realisation, the generators correspond to a Laurent expansion of the basis functions instead of the Fourier expansion used in (1). The Virasoro generators in this basis, which act only on the \( x^+ \) dependence of \( \varphi(x) \) since it is a scalar, are then the differential operators

\[
v_m^0 = -(x^+)^{m+1} \frac{\partial}{\partial x^+}. \tag{4}
\]

The nonsingular generators are those for which \( m \geq -1 \); the corresponding nonsingular Virasoro subalgebra shall be denoted \( \text{Vir}^\uparrow \). This may be extended to a nonsingular subalgebra of \( w_{1+\infty} \), denoted by \( w_{1+\infty}^\uparrow \), that is generated by \( v_m^\ell \) with \( m \geq -\ell - 1 \). Our coset construction will be based on \( w_{1+\infty}^\uparrow / \text{Vir}^+ \). We shall consider this coset to be “essentially reductive” in the sense that the divisor subgroup transformations of all Goldstone fields are linear (i.e. all the coset generators except \( v_{-1}^0 \leftrightarrow x^+ \) form a linear realisation of the divisor subgroup \( \text{Vir}^+ \)).

2. The \( \text{SO}(4,2)/\text{SO}(3,1) \) analogy

In order to set the stage for our later discussion of \( w_{1+\infty} \), we recall first a more familiar non-reductive coset construction: the Minkowski-space realisation of \( d = 4 \) conformal symmetry. The group \( \text{SO}(4,2) \) contains two Poincaré subgroups — the usual one, which we shall denote by \( P \), composed of \( \text{SO}(3,1) \otimes \{P^\mu\} \), i.e. the Lorentz subgroup taken in a semidirect product with the translations \( P^\mu \), and also an unorthodox one, \( P' \), in which the “translational” generators are the proper conformal generators \( K^\mu \). Minkowski space may be realised as the coset \( \text{SO}(4,2)/(P' \times \{D\}) \), leaving the \( P^\mu \) generators in the coset [4]. Owing to the fact that the subgroup \( P' \times \{D\} \) is a maximal subgroup of \( \text{SO}(4,2) \), this coset space is actually compact, and in fact has the global topology \( S^3 \times S^1 \). Accordingly, it is known as compactified Minkowski space \( \mathcal{M}^\# \).

The \( \text{SO}(4,2)/(P' \times \{D\}) \) coset construction yields nonlinear transformations of the coset parameter \( x^\mu \) in the usual fashion via multiplication on the left by an arbitrary element of the group and then factorisation of the result into \((\text{coset element}) \times (\text{divisor group element})\). In this fashion, one recovers the usual Poincaré group transformations together with the proper conformal transformations:

\[
x^\mu \rightarrow x'^\mu = \frac{x^\mu - c^\mu x^2}{1 - 2c \cdot x + c^2 x^2}. \tag{5}
\]

This coset construction is nonreductive since \([K_\mu, P_\nu] = -2i(\eta_{\mu\nu} D + M_{\mu\nu})\), so the coset generators \( P_\nu \) do not form a linear representation of the divisor group. As a result, the
proper conformal transformations of $x^\mu$ generated by the $K_\mu$ are nonlinear even though they belong to the divisor group of our coset.

The conformal transformations of Minkowski space may alternatively be obtained in a way that makes use of a reductive coset-space construction. This will necessarily involve a larger coset than the above non-reductive construction. Since we want to maintain Lorentz covariance, the divisor group must contain the Lorentz group. Accordingly, we consider the coset $SO(4, 2)/SO(3, 1)$, which requires 9 coset parameters. Following Ref. [3] for non-linear realisations of spacetime symmetries, we let the 9 coset parameters be represented by the four $x^\mu \leftrightarrow P_\mu$ and by five Goldstone fields, which are taken to be functions of $x^\mu$: $\varphi(x) \leftrightarrow D$ and $b^\mu(x) \leftrightarrow K^\mu$.

The 9 coset parameters $x^\mu$, $\varphi(x)$ and $b^\mu(x)$ evidently form linear representations of the divisor group $H = SO(3, 1)$. They also form a larger coset than we had in the non-reductive construction above. In this case, however, not all of the coset parameters are really essential, for we may subsequently eliminate the proper conformal Goldstone fields by covariant constraints; this procedure has been called the “inverse Higgs effect” [5, 6].

In deriving the covariant constraints of the inverse Higgs effect, it is appropriate to use Maurer-Cartan forms, starting from a coset element written using the standard exponential parametrisation,

$$k = e^{i x^\mu P_\mu} e^{i \varphi(x) D} e^{i b^\mu(x) K_\mu}. \quad (6)$$

The Maurer-Cartan forms are then given by decomposing the Lie algebra element $k^{-1}dk$ into its various projections in the Lie algebra:

$$\mathcal{P} = k^{-1}dk = i \omega^P_P P_\mu + i \omega^K_K K_\mu + i \omega_D D + i \omega_{H \rho \sigma} M^{\rho \sigma}. \quad (7)$$

The Maurer-Cartan forms $\omega^P_P$, $\omega^K_K$ and $\omega_D^D$ belonging to the coset transform in a standard way according to their $H = SO(3, 1)$ indices, but with field-dependent parameters when transformed by group elements in the coset. The parameters for these field-dependent $H$ transformations are found by left multiplication and repolarisation into a product $k'h'$, where $h' \in H$:

$$g_0 k = k'(g_0; x^\mu, \varphi(x), b^\mu(x)) h'(g_0; x^\mu, \varphi(x), b^\mu(x)); \quad (8)$$

the transformed values of the Goldstone fields and $x^\mu$ are then given by rewriting $k'$ in the form (6),

$$k' = e^{i x'^\mu P_\mu} e^{i \varphi'(x') D} e^{i b'^\mu(x') K_\mu}. \quad (9)$$

The Maurer-Cartan forms can be calculated from (7). In particular, one finds that

$$\omega^P_P = \partial_\mu \varphi - 2 b_\mu. \quad (10)$$

which shows that the independent Goldstone field $b_\mu$ is inessential since we can covariantly impose the constraint

$$\omega^P_P = 0 \Rightarrow b_\mu = \frac{1}{2} \partial_\mu \varphi. \quad (11)$$
The possibility of eliminating the independent $b_\mu$ Goldstone field here stems in part from the fact that there are two cosets that one could take for a nonlinear realisation involving a dilaton Goldstone field: $SO(4,2)/P'$ and $SO(4,2)/SO(3,1)$. In other words, there is a choice as to whether the $K_\mu$ generators belong to the coset or are divided out by being included into the divisor. The constraint (11) returns us to the field content of the nonlinear realisation based on the smaller coset $SO(4,2)/P'$, after having initially started out with the larger coset $SO(4,2)/SO(3,1)$.

Having found, via the inverse Higgs constraint (11), that one may construct realisations of the $SO(4,2)$ symmetry with only a dilaton Goldstone field $\varphi(x)$, one may then ask under what conditions the full $SO(4,2)$ symmetry may be realised without even this Goldstone field, i.e. on the minimal coset, $SO(4,2)/(P' \times \{D\}) = \mathcal{M}^d$. Such a realisation is indeed possible, but since there are no further covariant constraints to impose, realising $SO(4,2)$ symmetry on $\mathcal{M}^d$ requires a new feature: gauge invariance of the action with respect to local $D$ transformations. A well-known example of this situation occurs in classical Yang-Mills theory, which can be formulated in terms of a $SO(4,2)/P'$ realisation, but it then turns out that the $\varphi(x)$ Goldstone field for the $D$ transformations decouples, or “drops out”, of the action — i.e. there is a local $D$ invariance for the system comprising both the Yang-Mills fields and the dilaton field $\varphi(x)$. Consequently, one may view Yang-Mills theory as a realisation of $SO(4,2)$ on its smallest coset space $SO(4,2)/(P' \times \{D\})$. Such occurrences are clearly “accidental” from the standpoint of nonlinear realisation theory — there is no way in which one could adjust the Yang-Mills Lagrangian for the vector gauge field alone in order to achieve this local $D$ invariance had it not been present. Indeed, at the quantum level, this local symmetry is generally lost as a result of the familiar trace anomaly. A standard way of calculating this anomaly in fact is to introduce a dilaton field $\varphi(x)$ and to compute its purely quantum-induced coupling.

This discussion of the $SO(4,2)$ realisations in $d = 4$ will be instructive for our purposes because many of its features have direct analogues in the $w_{1+\infty}$ realisations to which we shall now turn. As mentioned above and as discussed in Ref. [1], the minimal $w_{1+\infty}/w_{\infty}$ coset space that we can use to realise $w_{1+\infty}$ symmetry on scalar fields is also non-reductive. We shall see below that we may reformulate this realisation using an essentially reductive coset space construction followed by an imposition of covariant constraints. Finally, we shall see that the single-scalar realisation of $w_{1+\infty}$ is obtained thanks to the possibility of requiring extra gauge symmetries analogous to the local $D$ symmetry discussed above.

3. $w_{1+\infty}^+/\text{Vir}^+$

Let us now return to the $w_{1+\infty}^+$ algebra given in Eq. (1). We shall concentrate on realisations of this algebra and of its associated group instead of the full $w_{1+\infty}$ because in this way we may pursue more closely the $d = 4$ conformal analogy given in section 2. The realisation of $w_{1+\infty}^+$ given here generalises the realisation of Virasoro symmetry on a
single coordinate $x^+$ used in [7, 8], in which $x^+$ is interpreted as the coordinate of a coset $\text{Vir}^\uparrow/\text{Vir}^+$, i.e. as the coset parameter corresponding to $L_{-1} = v_{-1}^0$. Since the $\text{Vir}^\uparrow/\text{Vir}^+$ coset is itself non-reductive, the divisor-group $\text{Vir}^+$ transformations of $x^+$ are nonlinear, in analogy to (5):

$$\delta x^+ = k_n (x^+)^{n+1}. \tag{12}$$

The restriction to $\text{Vir}^\uparrow$ corresponds to the non-singular Virasoro generators in this realisation, and correspondingly for $w^\uparrow_{1+\infty}$.

Consider now the Maurer-Cartan decomposition for the coset $w^\uparrow_{1+\infty}/\text{Vir}^+$:

$$v^i_m \rightarrow v^0_m \ (m \geq 0) \oplus \{ v^0_{-1}, v^\ell_m; \quad \ell \neq 0, \ m \geq -\ell - 1 \}, \tag{13}$$

where the $v^0_{m\geq0}$ are the $\text{Vir}^+$ generators. In analogy with the case of $SO(4,2)$ discussed earlier, we choose the coset representative

$$k = e^{-x^+v_{-1}^0} \prod_{\ell \neq 0} e^{-\phi^\ell}, \tag{14}$$

where we have used the notation

$$\phi^\ell = \sum_{m=\ell-1}^{\infty} \phi^\ell_m v^\ell_m. \tag{15}$$

Note that we have now chosen to work with antihermitean generators. From (15) one sees that there are infinitely many initial Goldstone fields $\phi^\ell_m$ corresponding to the coset generators $v^\ell_m, \ \ell \neq 0$.

Next we consider the Maurer-Cartan form, which reads

$$\mathcal{P} = k^{-1} dk = \sum_{m \geq 0} E^{-1}_m v^{-1}_m + E^0_{-1} v^0_{-1} + \sum_{m \geq 1 \geq -\ell - 1} E^\ell_m v^\ell_m + \sum_{m \geq 0} \omega^0_m v^0_m, \tag{16}$$

where $E^\ell_m$ and $\omega^0_m$ are all 1-forms, i.e. $E^\ell_m = dx^+E^\ell_{(+)m} + dx^-E^\ell_{(-)m}$, etc. The “vierbein” components $E^{-1}_m$ and $E^\ell_{m \geq 1}$ belong to linear representations of the divisor group $\text{Vir}^+$, and hence will transform homogeneously under $w^\uparrow_{1+\infty}$ transformations, albeit with field-dependent parameters when the transformations are taken from $w^\uparrow_{1+\infty}/\text{Vir}^\uparrow$. This homogeneous transformation property is the benefit that we derive from the essentially reductive structure of the coset $w^\uparrow_{1+\infty}/\text{Vir}^+$.

In the computation of the various components of $\mathcal{P}$, the following formulas [8] will prove useful:

$$e^\phi \beta e^{-\phi} = e^\phi \wedge \beta, \quad e^\phi de^{-\phi} = \left(1 - \frac{e^\phi}{\phi}\right) \wedge d\phi, \tag{17}$$
where we have used the notation

\[ \phi \wedge \beta \equiv [\phi, \beta], \quad \phi^2 \wedge \beta \equiv \phi \wedge \phi \wedge \beta = [\phi, [\phi, \beta]], \text{ etc.} \quad (18) \]

Note that the wedge (\wedge) notation used here denotes an operation involving multiple \( \omega_{1+\infty} \) commutators, and is not to be confused with the exterior product for forms. All equations containing forms in this paper will involve 1-forms without exterior products. Using (17, 18) to evaluate (16), we find

\[
\cdots e^{\phi^{(4)}} \cdots e^{\phi^{(1)}} \wedge \left( -dx^+ v_{-1}^0 - \phi^{(-1) \wedge} dx^+ d\phi^{(-1)} \right) \\
+ \cdots e^{\phi^{(4)}} \cdots e^{\phi^{(2)}} \left( \frac{1 - e^{\phi^{(1)}}}{\phi^{(1)}} \right) \wedge d\phi^{(1)} + \cdots e^{\phi^{(4)}} e^{\phi^{(3)}} \left( \frac{1 - e^{\phi^{(2)}}}{\phi^{(2)}} \right) \wedge d\phi^{(2)} + \cdots \\
= \sum_{m \geq 0} E_{m\ell}^{-1} v_{m-1}^0 + E_{-1}^0 v_{-1}^0 + \sum_{\ell \geq 1} \sum_{m \geq -\ell - 1} E_{m\ell}^{\ell} v_{m \ell}^0 + \sum_{m \geq 0} \omega_0^m v_0^m. \\
\quad (19)
\]

We can now compute the forms \( E_{m\ell}^\ell \) and \( \omega_0^m \) by equating terms proportional to \( v_{m \ell}^0 \) and \( v_0^m \), respectively. We shall refer to the spin label \( \ell \) as the level of a generator. From the (+) component of (19) at level \( \ell = -1 \), we read off the equation

\[
\sum_{m \geq 0} E_{-1}^{-1}(+) v_{m}^{-1} = -\partial_+ \phi^{(-1)} - \phi^{(-1)} \wedge (v_{-1}^0). \quad (20)
\]

We now observe that one can impose the covariant constraint \( E_{(-1)}^{-1} = 0 \). This constraint enables us to solve algebraically for \( \phi_m^{(-1)} \) in terms of \( \partial_+ \) derivatives of \( \phi_0^{-1} \). Specifically, we have

\[
E_{(-1)}^{-1} = 0 \quad \Rightarrow \quad \partial_+ \phi_m^{-1} + (m + 1)\phi_{m+1}^{-1} = 0, \quad m \geq 0. \quad (21)
\]

Note that \( \phi_0^{-1} \) is the only \( \ell = -1 \) Goldstone field that is not eliminated as an independent field by this constraint. Using the \( \ell = -1 \) constraint (21), we find at level \( \ell = 0 \)

\[
E_{(-1)}^0 = -dx^+ \quad (22) \\
\omega_0^0 = 0. \quad (23)
\]

Thus, the constraint (21) has the consequence that all of the connection terms \( \omega_{(-1)}^0 \) vanish.

The remainder of the (+) component of the form \( P \) at levels \( \ell \geq 1 \) transforms homogeneously and thus can also be set to zero. This gives us the maximum number of constraints with which we can eliminate inessential Goldstone fields. Anticipating the answer, we observe that substituting

\[
\partial_+ \phi^{(\ell)} + \phi^{(\ell)} \wedge v_{-1}^0 = 0, \quad \ell \neq 0, \quad (24)
\]
into the (+) component of $\mathcal{P}$ gives rise to pairwise cancellation of all terms except $-dx^+$. Conversely, we can derive (22–24) from the set of covariant constraints

$$\begin{cases} E_{(+)}^{-1} m = 0, & m \geq 0 \\ E_{(+)}^{\ell} m = 0, & \ell \geq 1, \ m \geq -\ell - 1. \end{cases} \quad (25)$$

From (19), we can also compute $E_{(-)}^\ell m$ and $\omega^0_{(-)m}$. The first few levels of the $E_{(-)}^\ell m$ are given by

$$E_{(-)}^{-1} m = -\partial_- \phi^{-1}_m; \quad (26)$$

$$E_{(-)-1}^0 = 2\phi^{-1}_2 \partial_- \phi^{-1}_1; \quad (27)$$

while for the $\omega^0_{(-)m}$ the result is

$$\omega^0_{(-)m} = 2 \sum_{n \geq 1} n\phi^{-1}_{m-n} \partial_- \phi^{-1}_n. \quad (28)$$

We may summarise the results of our inverse-Higgs-effect analysis by the following diagram of the $w_{1,1+\infty}$ generators:

Fig. 1 The generators of $w_{1,1+\infty}$

\[
\begin{array}{cccccccccccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\circ & x & x & x & x & x & x & x & x & x & x & x & x & x & x & \cdots \\
\circ & x & x & x & x & x & x & x & x & x & x & x & x & x & x & \cdots \\
\circ & x & x & x & x & x & x & x & x & x & x & x & x & x & x & \cdots \\
\circ & x & x & x & x & x & x & x & x & x & x & x & x & x & x & \cdots \\
\circ & x & x & x & x & x & x & x & x & x & x & x & x & x & x & \cdots \\
\circ & x & x & x & x & x & x & x & x & x & x & x & x & x & x & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\ell = 2 & \circ & x & x & x & x & x & x & x & x & \cdots \\
\ell = 1 & \circ & x & x & x & x & x & x & x & x & \cdots \\
\ell = 0 & \bullet & \circ & x & x & x & x & x & x & x & \cdots \\
\ell = -1 & \circ & x & x & x & x & x & x & x & x & \cdots \\
\end{array}
\]

In this diagram, the generators corresponding to reducible Goldstone fields eliminable by covariant constraints in the inverse Higgs effect are indicated by $\times$, the irreducible Goldstone fields surviving the inverse Higgs effect are indicated by $\circ$, the $v^0_{-1}$ generator associated to the coordinate $x^+$ is indicated by $\bullet$, and the generators of the divisor subalgebra $\text{Vir}^+$ are indicated by $\Diamond$. 

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4. Transformation rules for the Goldstone fields

We now derive the transformation rules for the surviving Goldstone fields lying on the left edge of the Fig. 1 diagram. Similarly to the $SO(4,2)$ case given in Eq. (8), the action of the group $w_1^+\infty$ on a coset representative, which we shall generically denote by $e^{-\phi(x)}$, is as follows:

$$ge^{-\phi(x)} = e^{-\phi'(x')}h,$$

(29)

where $h$ is an element of the divisor subgroup $Vir^\uparrow$. For infinitesimal transformations, we have

$$e^{\phi(x) + \delta\phi(x)}(1 + \delta g)e^{-\phi(x)} = 1 + \delta h,$$

(30)

where

$$\tilde{\delta}\phi(x) \equiv \phi'(x') - \phi(x)$$

(31)

$$= \phi'(x) + \delta x^+ \partial_+ \phi(x) - \phi(x)$$

$$\equiv \delta \phi(x) + \delta x^+ \partial_+ \phi(x).$$

We use in our derivations a version of the theory of nonlinear realisations adapted specifically to the realisation of spacetime symmetries [3]. This shall give us the Einstein-style transformation $\tilde{\delta}\phi$ directly. However, we shall subsequently view these transformations from an active viewpoint, in which the variation of the field is taken to be the quantity $\delta\phi$ as defined in (31). Thus, the transformations of $x^+$ that would occur in Einstein-style transformations generated by the $v^\ell_m$ will be replaced in the active viewpoint by transport terms. These transport terms will be field-dependent for the $v^\ell_m \geq 1$ but not for the $Vir^+$ generators, as one can see in (4). As defined in the introduction, this is what we mean by an essentially reductive coset construction. Projecting (30) into the coset direction yields the formula (c.f. [9])

$$\left( e^{\phi}\delta e^{-\phi} \right)_{G/H} = \left( e^{\phi} \delta ge^{-\phi} \right)_{G/H}. $$

(32)

Upon the use of the constraint (24), the variation (31) simplifies to

$$\tilde{\delta}\phi^{(\ell)} = \delta\phi^{(\ell)} - \phi^{(\ell)} \wedge \delta x^+ v^0_{-1}. $$

(33)

Substituting this result in (32) we find

$$\ldots e^{\phi^{(4)}} \ldots e^{\phi^{(1)}} \left( - \delta x^+ v^0_{-1} - \phi^{(-1)} \wedge \delta x^+ v^0_{-1} \right) - \ldots e^{\phi^{(4)}} \ldots e^{\phi^{(1)}} \wedge \left( \delta\phi^{(-1)} - \phi^{(-1)} \wedge \delta x^+ v^0_{-1} \right)$$

$$+ \ldots e^{\phi^{(4)}} \ldots e^{\phi^{(2)}} \left( 1 - e^{\phi^{(1)}} \right) \wedge \left( \delta\phi^{(1)} - \phi^{(1)} \wedge \delta x^+ v^0_{-1} \right)$$

$$+ \ldots e^{\phi^{(4)}} e^{\phi^{(3)}} \left( 1 - e^{\phi^{(2)}} \right) \wedge \left( \delta\phi^{(2)} - \phi^{(2)} \wedge \delta x^+ v^0_{-1} \right) + \ldots$$

$$= \ldots e^{\phi^{(4)}} e^{\phi^{(2)}} e^{\phi^{(1)}} e^{\phi^{(-1)}} e^{x^+ v^0_{-1}} \wedge \delta g. $$

(34)
Note that this expression is obtainable from the (+) component of (19) by the replacements $dx^+ \rightarrow \delta x^+$ and $d\phi^{(\ell)} \rightarrow \delta \phi^{(\ell)} = \delta \phi - \phi \wedge \delta x^+ v_0^0$. Just as all the $dx^+$ terms except the $-dx^+$ in the first term in (19) cancel pairwise by virtue of the constraint (22), so do all the $\delta x^+$ terms cancel pairwise in the above equation except the $-\delta x^+$ in the first term. To simplify Eq. (34) further, we calculate the first two levels of “dressing” of $\delta g$, i.e. we calculate $e^{\phi^{-1}} e^{x^+ v_0^0 - 1} \wedge \delta g$. Let us parametrise $\delta g$ as follows

$$\delta g = \sum_{\ell, m} \alpha^\ell_m v^\ell_m, \quad (35)$$

where the $\alpha^\ell_m$ are $x^+$-independent parameters. Consider a spin-$\ell$ transformation with parameter $\alpha^{(\ell)} \equiv \sum_m \alpha^\ell_m v^\ell_m$. We find that, analogously to (20),

$$e^{x^+ v_0^0 - 1} e^{(x^+ + v_0^0 - 1) \delta g} = e^{x^+ v_0^0 - 1} \wedge \alpha^{(\ell)} = \sum_m \beta^\ell_m (x^+) v^\ell_m, \quad (36)$$

where the dressed $x^+$-dependent parameters $\beta^\ell_{m+1}(x^+)$ are

$$\beta^\ell_{m+1}(x^+) = \sum_{p=0}^\infty (-1)^p \frac{(\ell + m + 2 + p)!}{(\ell + m + 2)! p!} (x^+)^p \alpha^\ell_{m+1+p}. \quad (37)$$

It follows that

$$\beta^\ell_{m+1}(x^+) = -\frac{1}{\ell + m + 2} \partial_+ \beta^\ell_m(x^+). \quad (38)$$

Note that this is the same relation as that satisfied by the fields $\phi_{\ell}^m$. Proceeding on to the next level of dressing, with $\phi^{(-1)}$, we evaluate

$$\sum_{\ell m} e^{\phi^{(-1)}} \beta^\ell_m v^\ell_m e^{-\phi^{(-1)}} = \sum_{\ell m} \gamma^\ell_m (\phi^{(-1)}) v^\ell_m, \quad (39)$$

where

$$\gamma^\ell_m = \beta^\ell_m + \sum_{k\{n_k\}} \frac{1}{k!} (\ell + 2)(\ell + 3) \cdots (\ell + k + 1)(n_1\phi_{n_1}^{-1})(n_2\phi_{n_2}^{-1}) \cdots (n_k\phi_{n_k}^{-1}) \beta^{\ell+k}_{m-n_1-n_2-\cdots-n_k}. \quad (40)$$

From this expression, we learn that

$$\gamma^{\ell+1}_{m-n} = \frac{1}{n(\ell + 2)} \frac{\delta \gamma^\ell_m}{\delta \phi^{-1}_{n}}. \quad (41)$$

Hence, for the $\gamma$-parameters corresponding to the left edge of the Fig. 1 diagram, we have the relation

$$\gamma^{-\ell-1}_{-\ell-1} = \frac{1}{(\ell + 1)!} \frac{\delta^{\ell+1} \phi^{-1}_0}{\delta \phi^{\ell+1}_0}. \quad (42)$$
where we have introduced the notation
\[ y \equiv -\partial_0^{-1}. \]  \hfill (43)

Turning back to Eq. (29) and comparing the \( \ell = 0 \) and \( \ell = -1 \) terms on both sides, we learn that
\[
\begin{align*}
\delta x^+ &= -\gamma^0_1, \\
\delta \phi^{-1}_0 &= -\gamma^{(-1)}_1. 
\end{align*} \hfill (44a)
\]

Then, with the notation
\[
\beta^\ell_{-\ell-1}(x^+) \equiv k^\ell(x^+) \hfill (45)
\]
and \( y \) as defined in (43), we find that
\[
\delta x^+ = \sum_{\ell=0}^{\infty} (\ell + 1)k^\ell y^\ell \hfill (46)
\]
and that the transformation rule (44b) can be written as
\[
\delta \phi^{-1}_0 = -\sum_{\ell=-1}^{\infty} k^\ell y^{\ell+1}. \hfill (47)
\]

Equation (47) is precisely the \( w_{1+\infty} \) transformation rule of Eq. (3), after identifying the scalar field \( \varphi \) of Eq. (3) with the field \( (-\phi_0^{-1}) \) here.

Substituting the relations (44) back into Eq. (34), we find that the \( \delta x^+ \) terms as well as the terms proportional to \( \delta \phi^{(-1)} \) on the left-hand side and to \( \gamma^{(-1)} \) on the right-hand side all cancel. We are left with the result
\[
\ldots e^{\phi^{(4)}} \ldots e^{\phi^{(2)}} \left( \frac{1 - e^{\phi^{(1)}}}{\phi^{[1]}} \right) \wedge \delta \phi^{[1]} + \ldots e^{\phi^{(4)}} \left( \frac{1 - e^{\phi^{(2)}}}{\phi^{[2]}} \right) \wedge \delta \phi^{[2]} \\
+ \ldots e^{\phi^{(4)}} \left( \frac{1 - e^{\phi^{(3)}}}{\phi^{[3]}} \right) \wedge \delta \phi^{[3]} + \ldots \\
= \ldots e^{\phi^{(4)}} \ldots e^{\phi^{(2)}} e^{\phi^{[1]}} \wedge (\gamma^{[0]} + \gamma^{[1]} + \gamma^{[2]} \ldots) - \gamma^{[0]}.
\] \hfill (48)

From this formula, we can read off the transformation rules for \( \phi^{(2N+1)} \) and \( \phi^{(2N)} \) as follows:
\[
\begin{align*}
\delta \phi^{(2N+1)} &= -\gamma^{(2N+1)} - e^{\phi^{(1)}} \wedge \gamma^{(2N)} - e^{\phi^{(2)}} e^{\phi^{(1)}} \wedge \gamma^{(2N-1)} \\
&- \ldots - e^{\phi^{(2N+1)}} \ldots e^{\phi^{(1)}} \wedge \gamma^{[0]} + e^{\phi^{(N+1)}} \left( \frac{1 - e^{\phi^{(N)}}}{\phi^{[N]}} \right) \wedge \delta \phi^{[N]} \\
&+ e^{\phi^{(N+2)}} e^{\phi^{(N+1)}} \left( \frac{1 - e^{\phi^{(N-1)}}}{\phi^{[N-1]}} \right) \wedge \delta \phi^{[N-1]} \\
&+ \ldots + e^{\phi^{(2N)}} \ldots e^{\phi^{(2)}} \left( \frac{1 - e^{\phi^{(1)}}}{\phi^{[1]}} \right) \wedge \delta \phi^{[1]},
\end{align*} \hfill (49)
\]
where \( N = 0, 1, 2, \ldots \) and only terms with upper indices summing to \( 2N + 1 \) and parameters \( \gamma^{(0)}, \gamma^{(1)}, \ldots, \gamma^{(2N+1)} \) are to be kept. Similarly, for the transformation rule for \( \phi^{(2N)} \) we find the result

\[
\delta \phi^{(2N)} = -\gamma^{(2N)} - e^{\phi^{(1)}} \wedge \gamma^{(2N-1)} - e^{\phi^{(2)}} \wedge \gamma^{(2N-2)} - \cdots - e^{\phi^{(2N)}} \cdots e^{\phi^{(1)}} \wedge \gamma^{(0)} + \left( \frac{1 - e^{\phi^{(N)}}}{\phi^{(N)}} \right) \wedge \delta \phi^{(N)}
\]

\[
+ e^{\phi^{(N+1)}} e^{\phi^{(N)}} \left( \frac{1 - e^{\phi^{(N-1)}}}{\phi^{(N-1)}} \right) \wedge \delta \phi^{(N-1)}
\]

\[
+ \cdots + e^{\phi^{(2N-1)}} \cdots e^{\phi^{(2)}} \left( \frac{1 - e^{\phi^{(1)}}}{\phi^{(1)}} \right) \wedge \delta \phi^{(1)},
\]

where \( N = 1, 2, 3, \ldots \) and only terms with upper indices summing to \( 2N \) and parameters \( \gamma^{(0)}, \gamma^{(1)}, \ldots, \gamma^{(2N)} \) are to be kept.

We next consider a number of examples that illustrate the use of these formulas and give us the results for the low-lying levels. At level \( \ell = 1 \), (49) yields

\[
\delta \phi^{(1)} = -\gamma^{(1)} - \phi^{(1)} \wedge \gamma^{(0)}.
\]  

(51)

Restricting attention to the independent field \( \phi^{1}_{-2} \) and using the commutation rules (2), we find from (51)

\[
\delta \phi^{1}_{-2} = -\gamma^{1}_{-2} - 2\phi^{1}_{-2} \partial_{+} \gamma^{0}_{-1} + \partial_{+} \phi^{1}_{-2} \gamma^{0}_{-1}.
\]  

(52)

At level \( \ell = 2 \), from (50) we find

\[
\delta \phi^{(2)} = -\gamma^{2} - \frac{1}{2} \phi^{(1)} \wedge \gamma^{(1)} - \phi^{(2)} \wedge \gamma^{(0)},
\]  

(53)

where we have used \( \delta \phi^{(1)} \) as found in (51). Again, restricting this to the left-edge field \( \phi^{2}_{-3} \) and using the commutation relations (2), we find

\[
\delta \phi^{2}_{-3} = -\gamma^{2}_{-3} + \gamma^{0}_{-1} \partial_{+} \phi^{2}_{-3} - 3\partial_{+} \gamma^{0}_{-1} \phi^{2}_{-3} + \gamma^{1}_{-2} \partial_{+} \phi^{1}_{-2} - \partial_{+} \gamma^{1}_{-2} \phi^{1}_{-2}.
\]  

(54)

Note that only fields and parameters corresponding to the left edge of the Fig. 1 diagram occur in these results. For the next two levels the results are

\[
\delta \phi^{[3]} = -\gamma^{[3]} - \phi^{(1)} \wedge \gamma^{(2)} - \frac{1}{6} (\phi^{(1)})^{2} \wedge \gamma^{(1)} - \phi^{(3)} \wedge \gamma^{(0)},
\]

\[
\delta \phi^{[4]} = -\gamma^{[4]} - \phi^{(1)} \wedge \gamma^{(3)} - \frac{1}{2} (\phi^{(2)} + (\phi^{(1)})^{2}) \wedge \gamma^{(2)} - \frac{1}{4} (\phi^{(2)} + \frac{1}{2} (\phi^{(1)})^{2}) \phi^{(1)} \wedge \gamma^{(1)} - \phi^{(4)} \wedge \gamma^{(0)}.
\]  

(55)
5. Other coset realisations

The possibility of eliminating most of the original $\phi^\ell_m$ Goldstone fields by the covariant constraints (24), thus reducing the essential set of Goldstone fields to just the $\phi^-_{-\ell-1}$, suggests that there should be an alternative coset space construction giving these fields directly as the only coset representatives. We shall restrict ourselves here to cosets formed from the generators of $w_{1+\infty}^\uparrow$ as shown in Fig. 1. In order to obtain only coset parameters corresponding to the left edge of Fig. 1, one needs to divide out by a group corresponding to the complement of the left-edge generators in $w_{1+\infty}^\uparrow$. This can be done because these generators, \{$v^-_{-\ell}; \ell \geq -1$\}, close amongst themselves to form a subalgebra which we shall denote $w_{1+\infty}^+$, in analogy to the Virasoro subalgebra Vir$^+$. The coset $w_{1+\infty}^\uparrow/w_{1+\infty}^+$ so obtained provides an alternative non-reductive construction of the Goldstone-field transformation rules of section 4. The structure of this coset construction is summarised in Fig. 2, using the same notation for the coset ($\circ$), divisor subgroup (♦) and spatial coordinate $x^+$ (●) generators as in Fig. 1:

Yet other coset realisations may be constructed by observing, from Eq. (2), that one may also form closed subalgebras of $w_{1+\infty}^\uparrow$ by transferring all but a finite number of left-edge generators into the divisor, which then is generated by \{$w_{1+\infty}^+ \oplus \{v^-_{-\ell-1}; \ell > N\}$\}. The resulting diagram of generators is shown in Fig. 3. The finite-dimensional Goldstone field realisation corresponding to Fig. 3 can also be extracted from the explicit transformation rules for the Goldstone fields given in section 4. The transformation rules (49, 50) have a “triangular” structure: each left-edge Goldstone field of our surviving set transforms into itself and into fields lower down on the left edge of the Fig. 1 diagram, but not into fields higher up on the left edge. Consequently, it is possible to consistently truncate our infinite
set of left-edge Goldstone fields down to the finite set \((\phi_{-\ell-1}; \ell = -1, 1, 2, \ldots, N)\). This finite set then corresponds precisely to the Goldstone fields of the non-reductive coset of Fig. 3.

Fig. 3 A finite-dimensional coset

6. Invariant actions

The canonical way to construct an invariant action with a non-linearly realised symmetry from a reductive coset-space construction is to use the “vierbein” and “connection” components of the Maurer-Cartan form to build a Goldstone-field-dependent Lagrangian that is locally invariant under the divisor group of the coset. In the present case, however, the covariant inverse-Higgs constraints (24) reduce the \((+)\) component of the Maurer-Cartan form (16) completely down to the coordinate differential \((-dx^+)\) — with even the connection terms \(\omega_{(+)m}^0\) eliminated. Any Lagrangian must have an overall \(d = 2\) Lorentz weight equal to zero and therefore must involve both \(\partial_+\) and \(\partial_-\) derivative terms. Thus, the absence of any non-trivial \((+)\) components of the Maurer-Cartan form means that one cannot use the canonical procedure to construct actions.

Nonetheless, if we relax the condition of strict invariance of the Lagrangian and seek only an invariant action, \(i.e.\) if we allow the Lagrangian to transform by a total derivative, then a simple action is ready to hand — constructed from the free scalar Lagrangian for the field \(\phi_{-1}^{-1}\) alone:

\[
\mathcal{L}_0 = \frac{1}{2} \partial_+ \phi_{0}^{-1} \partial_- \phi_{0}^{-1}.
\]
It may be verified that this Lagrangian transforms by a total derivative under the full set of $w_{1+\infty}^\uparrow$ transformations (47):

$$\delta \phi_0^{-1} = - \sum_{\ell=-1}^{\infty} k^\ell (\partial_+ \phi_0^{-1})^{\ell+1}. \quad (57)$$

The field equation $\partial_- \partial_+ \phi = 0$ which follows from (56) is covariant under (57) since it is equivalent to the manifestly covariant set of equations $E_{(-)m\geq1}^{-1} = 0$, where $E_{(-)m}^{-1}$ is given in (26), upon use of the set of manifestly covariant inverse Higgs constraints (21).* By contrast, the existence of the invariant action $\int d^2x L_0$ and its relation to the general scheme of $w_{1+\infty}^\uparrow$ nonlinear realisations that we have been presenting remains something of a mystery. The existence of the action requires firstly a single-Goldstone-scalar realisation of $w_{1+\infty}^\uparrow$ symmetry that we can obtain as the $N = 0$ finite-component realisation of section 5. Even given this, however, we do not have a canonical method of constructing an action, so the existence of (56) is something of an accident. It is worth recalling again the analogy of $d = 4$ conformal symmetry that we developed in section 2. The existence of classical conformally-invariant theories such as Yang-Mills theory is also in a sense accidental. We saw that nonlinear realisations of $SO(4,2)$ would generically require coupling to the dilaton Goldstone field $\phi(x)$. Only in special cases, such as that of Yang-Mills theory, is there a gauge symmetry that causes this field to decouple, leaving a realisation on the minimal coset space for conformal symmetry, i.e. on compactified Minkowski space $\mathcal{M}^2 = SO(4,2)/(P' \times \{D\})$.

In the present case, one may view the existence of the single-scalar Lagrangian (56) as a consequence of similar gauge symmetries, in this case for all the Goldstone fields ($\phi_{-\ell-1}^\ell; \ell \geq 1$) that do not appear in (56). As in the Yang-Mills case, such gauge symmetries at the classical level might be suspected to be vulnerable to anomalies in which the classically-decoupled Goldstone fields could re-couple to the theory at the quantum level. Indeed, in $w_\infty$-gravity theory, which has analogous gauge symmetries, it appears that there are in fact such anomalies. Owing to the factorisation of the classical currents $(\partial_+ \phi)^{\ell+2}$, the $w_\infty$-gravity Lagrangian has an infinite set of “Stueckelberg” symmetries that allow one to gauge away all but the $\ell = 0$ gauge field (when coupling to a single scalar as we are considering here) [1]. At the quantum level, one may arrange for finite renormalisations of the $w_\infty$ currents so as to preserve their symmetries at the quantum level [10] (albeit at the price of having the algebra be renormalised from $w_\infty$ to $W_\infty$), but the Stueckelberg symmetries for the $\ell \geq 1$ gauge fields appear to become anomalous in the process.

Free second-order scalar actions involving the higher left-edge Goldstone fields $\phi_{-\ell-1}^\ell$ cannot be made because they do not have Lorentz weight zero, since the Lorentz weight

---

* We recall that the set of the $E_{(-)m}^{-1}$ is manifestly covariant since it is manifestly covariant under the Vir$^+$ divisor subgroup, and hence transforms homogeneously under the full $w_{1+\infty}^\uparrow$. We note also from (27, 28) that the dynamical equations $E_{(-)m\geq1}^{-1} = 0$ have the property of setting to zero both $E_{(-)1}^0$ and $\omega_{(-)m}^0$. 

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of $\phi_{-\ell-1}$ is $-(\ell + 1)$. One may, however, restore a set of local gauge symmetries and at the same time couple the higher Goldstone fields to currents built from $\phi_0^{-1}$. In order to couple the higher Goldstone fields in this way, one may take the $w_\infty$-gravity action of Ref. [1] but with its fundamental gauge fields replaced by composite fields constructed so as to transform correctly according to standard gauge field transformation rules under the $w_\infty$ transformations.

Since we have not introduced Goldstone fields for any of the Virasoro generators (not counting the coordinate $x^+$, which is the coset parameter for $v_{0,1}^0$, but is not a field), there is no way for us to introduce a composite gauge field for the Virasoro generators $v_{m}^0$. Moreover, the $w_\infty$ gravity action of Ref. [1] has a local $w_\infty$ symmetry but not a full local $w_{1+\infty}$ symmetry, and consequently does not contain a gauge field for the $v_{m}^{-1}$ generators (in fact, no gauge-invariant construction of this type with full local $w_{1+\infty}$ symmetry exists). Accordingly, we shall construct here an action with composite gauge fields for the $v_{m}^{\ell \geq 1}$ generators only. Truncation of the full set of gauge fields $\{A^\ell(x^+, x^-)\}$ for $w_{1+\infty}^\uparrow$ to the set $\{A^{\ell \geq 1}\}$ is consistent for a concurrent restriction of the symmetry group to $w_\infty$, as one may verify directly from the transformation rules for gauge fields [1], under local transformations with parameters $k^\ell(x^+, x^-)$ as given in (45), but now allowed to be $x^-$-dependent also:

$$
\delta A^\ell = \partial_- k^\ell - \sum_{j=0}^{\ell+1} [(j+1)A^j \partial_+ k^{\ell-j} - (\ell-j+1)k^{\ell-j} \partial_+ A^j].
$$

(58)

Given a set of composite gauge fields $A^\ell(\phi(x^+, x^-))$ that transform according to (58), one can then take over the form of the $w_\infty$-gravity Lagrangian from [1], but restricted here to gauge fields with $\ell \geq 1$:

$$
L_\infty = L_0 - \sum_{\ell=1}^{\infty} \frac{1}{\ell + 2} A^\ell(\phi)(-\partial_+ \phi_0^{-1})^{\ell + 2}.
$$

(59)

Given (58), this action will be invariant under global (i.e. $x^-$-independent) $w_{\infty}^\uparrow$ transformations. From its origins in $w_\infty$-gravity, one knows that (59) is in fact also invariant under local $k^\ell(x^+, x^-)$ transformations for $\ell \geq 1$. Using these, one may of course gauge away to

* In this paper, we have chosen to work with a linear realisation of the Vir$^+$ algebra on Goldstone fields, obtaining an essentially reductive construction in the sense explained in the introduction. An alternative procedure would be to take the $v_{m \geq 1}^0$ generators into the coset instead of leaving them in the divisor subalgebra. In that case, the divisor would be generated by $L_0 = v_0^0$ alone and the construction would be strictly reductive, even for the coordinate $x^+ \leftrightarrow v_0^0$. Such a formalism could be obtained from that of the present paper by “unfixing” a local $v_0^0$ symmetry in the action and transformation rules. This would include an $A^0$ composite gauge field among those appearing in (59). In the coset constructions, the equivalent modifications would be obtained by including a factor $e^{-s(0)}$ to the extreme right of all the other coset elements in (14); by placing this factor in this position, one avoids upsetting the covariance of the inverse Higgs constraints (25) for the $\ell \neq 0$ levels. Such a construction may be found in Ref. [7].
zero all of the higher left-edge Goldstone fields $\phi^\ell_\ell$, and hence eliminate all of the “pure-gauge” composite gauge fields $A^{\ell \geq 1}(\phi(x^+, x^-))$. In this special gauge, the Lagrangian (58) must thus reduce back to the free-field Lagrangian (56) for $\phi^0_0$ alone. This is the sense in which local gauge symmetries cause the Goldstone fields $\phi^\ell_\ell$ to decouple classically, just as local scale invariance causes the dilaton to decouple from classical Yang-Mills theory.

The existence of these local gauge symmetries also gives us an alternative way to obtain the form of the Lagrangian (59): one may equivalently start from (56) and perform local $k^{\ell \geq 1}(x^+, x^-)$ transformations, under which the free-field Lagrangian (56) is not invariant, and then turn around and promote the parameters of these local transformations to Goldstone fields $\phi^\ell_{\ell - 1}(x^+, x^-)$.

In order to obtain the explicit forms of the composite gauge fields appearing in (59), we need to solve for the $A^\ell$ in terms of the $\phi^\ell_m$. Following the work of Ref. [6], this may be done by generalising the construction of the Maurer-Cartan form (16) through the incorporation of a set of gauge fields $\{A^m_m(x^-)\}$. Note that by a “dressing” procedure analogous to (36, 37, 45), this doubly-indexed set of functions of a single variable is equivalent to the singly-indexed set of functions of two variables $A^\ell(x^+, x^-)$. As mentioned above, we introduce here gauge fields only for $\ell \geq 1$ and concurrently restrict the symmetry group to $w^{\uparrow \infty}$. With the gauge fields included, the Maurer-Cartan form becomes

$$\hat{\mathcal{P}} = k^{-1}(d + A)k = \sum_{m \geq 0} E_m^{-1}v_m^{-1} + E^0_0v^0_0 + \sum_{\ell \geq 1, \ell \geq -1} \hat{E}_m^{\ell}v_m^\ell + \sum_{m \geq 0} \hat{\omega}_m^0v_m^0, \quad (60)$$

where $A = dx^- \sum_{\ell, m} A^\ell_m(x^-)v_m^\ell$ and the modified components $\hat{E}_m^{\ell \geq 1}$ and $\hat{\omega}_m^0$ now include contributions from the gauge fields. Since the $\hat{E}_m^{\ell \geq 1}$ belong to linear representations of the coset divisor group $\text{Vir}^+$, they will transform homogeneously under global $w^{\uparrow \infty}$ transformations (in fact, they also transform homogeneously under local transformations $k_m^{\ell \geq 1}$). Thus, we may impose the covariant conditions

$$\hat{E}_m^{\ell \geq 1} \big|_{x^+ = 0} = 0 \quad (61)$$

and these may then be solved to obtain the $A_m^{\ell \geq 1}(x^-)$. Note that since $\text{Vir}^+$ is the stability group of the point $x^+ = 0$, the evaluation of (61) at this point is a covariant procedure.

The solution to the set of covariant constraints (61) is obtained, following Ref. [6], by first writing the transformation rules (49, 50) for the Goldstone fields $\phi_m^\ell(x^+ = 0, x^-)$ as

$$\delta \phi_m^\ell \big|_{x^+ = 0} = -\sum_{p, n} \left( F_p^\ell_{m,n}(\phi)\alpha_p^m \right) \big|_{x^+ = 0}, \quad (62)$$

where the nonsingular matrix $F_p^\ell_{m,n}(\phi)$ is a nonlinear functional of the $\phi_m^\ell$ (it may be seen to be nonsingular by noting from (49, 50) that its leading term is the unit matrix $\delta^p\delta_{mn}$).
The solution to (61) is then given by

$$A^\ell_m(\phi(x^-)) = -\sum_{p,n} \left( \mathcal{F}^{-1\ell p}_{mn} \partial_n \phi_m^p \right) \bigg|_{x^+=0}, \quad \ell, p \geq 1 \quad m \geq -\ell - 1; \quad n \geq -p - 1,$$

(63)

where \(\mathcal{F}^{-1\ell p}_{mn}\) is the matrix inverse of \(\mathcal{F}^{\ell p}_{mn}\), whose existence is guaranteed by the nonsingularity of the latter.\footnote{The matrix inverse \(\mathcal{F}^{-1\ell p}_{mn}\) in (63) may be taken simply with respect to the \(\ell, p \geq 1\) submatrix of the full \(w_{1+\infty}^1\) realisation (the full realisation includes also \(\ell, p = -1\)) by virtue of the “triangular” nature of the realisation. This has the consequence that the \(\ell, p \geq 1\) submatrix of \(\mathcal{F}^{-1}\) in the full realisation is identical to the inverse of the \(\ell, p \geq 1\) submatrix of \(\mathcal{F}^{\ell p}_{mn}\).}

After obtaining the \(A^\ell_m(\phi(x^-))\) from (63), the \(x^+\)-dependent gauge fields \(A^\ell(x^+, x^-)\) are then constructed, analogously to (36, 37, 45), by dressing with \(x^+\),

$$A^\ell(x^+, x^-) = \sum_{m=0}^\infty (-x^+)^m A^\ell_{-\ell-1+m}(x^-).$$

(64)

One may verify directly that the composite \(A^\ell(x^+, x^-)\) constructed in this way do transform correctly according to (58) simply by varying, taking into account the fact that (62) forms a realisation of the \(w_{1+\infty}^1\) algebra, as derived in section 4. The resulting Lagrangian (59) involves sums over infinite numbers of Goldstone fields for each of the composite gauge fields, as a result of the matrix inversion in (63), but owing to the nonsingularity of \(\mathcal{F}^{\ell p}_{mn}\) the result can be evaluated to arbitrary order in the \(\phi_m^\ell\).

7. Conclusions

We have derived the general pattern of nonlinear realisations of \(w_{1+\infty}^1\) symmetry. An infinite number of Goldstone fields arises in the general nonlinear realisation, corresponding to the generators \(v^\ell_{-\ell-1}\) lying on the left edge of the diagram of Fig. 1. In general, these may all be expected to be present when the set of Goldstone fields is used to promote a Virasoro-invariant theory to one with \(w_{1+\infty}^1\) invariance. For the pure Goldstone-field Lagrangian, however, there is a classical decoupling of all the higher fields \(\phi^\ell_{-\ell-1}\) for \(\ell \geq 1\), leaving just a free scalar action for \(\phi_0^{-1}\).

The derivation of the Goldstone fields’ transformation laws (49, 50) follows straightforwardly from the theory of nonlinear realisations when one starts from the essentially reductive coset space construction \(w_{1+\infty}^1/\text{Vir}^+\). An infinite set of covariant constraints (25) can subsequently be imposed to eliminate the inessential Goldstone fields. These constraints, however, also remove from the (+) component of the Cartan-Maurer form (7) all combinations of Goldstone fields with which one would normally construct actions by making locally-invariant constructions with respect to the linearly-realised divisor group \(\text{Vir}^+\). Thus, the construction of \(w_{1+\infty}^1\)-invariant actions on a proper group-theoretical basis still remains to be
systematised. One possible approach to a geometrical interpretation of the action would be to view it as a kind of “Chern-Simons” action for $w_{1+\infty}$.

We have dealt in this paper only with realisations of a single chiral copy of $w_{1+\infty}$. Of course, the free scalar action (56) is invariant under both a left-handed as well as a right-handed copy of $w_{1+\infty}$. Under the right-handed $w_{1+\infty}$, the roles of $x^+$ and $x^-$ as spectator coordinate and coset parameter are reversed as compared to the left-handed copy. The simultaneous nonlinear realisation of both copies on a single scalar field $\varphi(x^+, x^-)$ is another subject on which the analysis of the present paper can be extended. A related problem is the geometrical origin of the multi-scalar $w_\infty$ realisations employed in the $w_\infty$ gravity constructions of Ref. [1]. It may prove to be fruitful to view the extra scalars as Goldstone fields for off-diagonal combinations of higher left and right $v^\ell_0$ generators in the fashion of the $W_3$ construction of Ref. [11]. Since one may also view the study of nonlinearity-realised global symmetries as the study of pure-gauge connections, there should also be interesting relations between the present group-theoretic framework and the geometry of $w_\infty$ gravity. For example, topological $w_\infty$ gravity has been interpreted as a theory of flat connections for the $SL(\infty, \mathbb{R})$ “wedge” subalgebra of $w_{1+\infty}$ [12]. In that case, there are also covariant constraints that allow the elimination of all but the set of connections corresponding to the “left-edge” generators in our Fig. 1, suggesting a relation to the present work. Finally, one would also like to have a better understanding of how the realisations described in this paper carry over to the quantum level, especially given the known anomaly structure of $w_\infty$ gravity.

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