Jacobian varieties of reduced tropical curves

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Abstract

On tropical geometry in $\mathbb{R}^2$, the divisor and the Jacobian variety are defined in analogy to algebraic geometry. For study of these objects, it is important to think of the ‘bunch’ of a tropical curve (Figure 1). In this paper, we will show that if the bunch is a bouquet, then the Jacobian is a higher-dimensional torus.

![Figure 1: Bunch of a tropical curve](image)

1 Introduction

In this paper, the affine space $\mathbb{R}^2$ is equipped with interior product $(u_1, u_2) \cdot (v_1, v_2) = u_1v_1 + u_2v_2$ and exterior product $(u_1, u_2) \times (v_1, v_2) = u_1v_2 - v_1u_2$.

Let $C$ be a tropical curve in $\mathbb{R}^2$ (See section 2 for preliminary). The Jacobian variety of $C$ is defined in analogy to algebraic geometry as follows.
**Definition.** The *divisor group* $\text{Div}(C)$ of $C$ is the free abelian group generated by all points of $C$. We define a subgroup

$$\text{Div}^0(C) = \left\{ D = \sum_{P \in C} m_P P \in \text{Div}(C) \mid \deg D := \sum_{P} m_P = 0 \right\}.$$  

Divisors $D, D'$ are *linearly equivalent*, $D \sim D'$, if there are tropical curves $L, L'$ such that

$$\Delta(L) = \Delta(L'),$$

$$D - D' = C \cdot L - C \cdot L',$$

where $\Delta$ denotes the Newton polygon and $C \cdot L$ denotes the stable intersection.

The *Jacobian variety* of $C$ is the residue group

$$\text{Jac}(C) = \text{Div}^0(C)/\sim.$$  

![Diagram](https://via.placeholder.com/150)

**Figure 2:** Tropical elliptic curve

We fix a point $\mathcal{O} \in C$. If $C$ is a tropical elliptic curve, then the ‘bunch’ of $C$ is homeomorphic to $S^1$ (Figure 2), and we have a map

$$\varphi: \text{Bunch}(C) \rightarrow \text{Jac}(C)$$

$$P \mapsto P - \mathcal{O}.$$  

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Vigeland [6] states that $\varphi$ is bijective, which concludes that a tropical elliptic curve has a group structure in a smaller part of it. But the proof in [6] is not complete. He has proved the surjectivity of $\varphi$.

In this paper, we give a complete proof and some generalization of it.

**Definition.** A tropical curve $C$ is reduced if every edge is of weight 1. $C$ is smooth if every vertex is 3-varent and of multiplicity 1. (Hence any smooth tropical curve is reduced.)

**Definition.** An edge $E \subset C$ is tentacle if $C \setminus \text{Int}(E)$ is disconnected. $E$ is a ray if $E$ has only one vertex. We define the bunch of $C$, $\text{Bunch}(C)$, to be the quotient space of $C$ by every tentacle edge and ray contracted.

**Definition.** A bouquet is a topological space $B = \Lambda_1 \cup \cdots \cup \Lambda_g$ with a point $O \in B$ such that

\[
\Lambda_i \approx S^1 \quad (1 \leq i \leq g),
\]

\[
\Lambda_i \cap \Lambda_j = \{O\} \quad (i \neq j).
\]

$O$ is called the center of $B$. (The topological genus of $B$ is $g$.)

**Theorem 1.1.** Let $C$ be a reduced tropical curve in $\mathbb{R}^2$. Suppose that $\text{Bunch}(C)$ is a bouquet of genus $g$, with center $O$ and cycles $\Lambda_1, \ldots, \Lambda_g$. Then the map

\[
\varphi: \Lambda_1 \times \cdots \times \Lambda_g \longrightarrow \text{Jac}(C)
\]

\[
(P_1, \ldots, P_g) \mapsto P_1 + \cdots + P_g - gO
\]

is bijective.

The statement includes that the map $\varphi$ is well-defined, i.e. if $P_1, P_1'$ are on the same tentacle edge or on the same ray, then $P_1 \sim P_1'$.

**Remark 1.2.** Tropical geometry is introduced in three approaches. The first is an algebraic approach (e.g.[4],[1]). If $f \in \tilde{\mathbb{R}}[x,y]$ is a tropical polynomial over the tropical algebra $\tilde{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, then its corner locus is a tropical curve. The second is a valuation-theoretical approach (e.g.[2],[5]). If $K$ is a suitable valuation field, and $V$ is an algebraic curve in $K^2$, then its image by the valuation map $v: K^2 \rightarrow \tilde{\mathbb{R}}^2$ is a tropical curve. This image is expressed as the limit of amoebas of complex curves. And the last is a geometrical approach (e.g.[5]). A 1-dimensional simplicial complex in $\mathbb{R}^2$ (or, a graph in $\mathbb{R}^2$) satisfying some balancing condition is a tropical curve. The equivalence of these definitions is easy to prove (See [2]). In this paper, we take only geometrical approach.
Remark 1.3. In algebraic approach, projective tropical curves are treated as special objects (C is projective if Δ(C) = Δₚ). The above definition of linearly equivalence has another version as follows: \( D \sim_{\text{alg}} D' \) if there are projective tropical curves \( L, L' \) such that

\[
\deg(L) = \deg(L'),
\]

\[
D - D' = C \cdot L - C \cdot L'.
\]

The residue group \( \text{alg Jac}(C) = \text{Div}^0(C)/\sim_{\text{alg}} \) will be called the algebraic Jacobian variety of \( C \). There is a canonical surjection \( \psi: \text{alg Jac}(C) \rightarrow \text{Jac}(C) \). We can think of a map \( \varphi_{\text{alg}}: \text{Bunch}(C) \rightarrow \text{alg Jac}(C) \) instead of \( \varphi: \text{Bunch}(C) \rightarrow \text{Jac}(C) \) (in the case of genus 1). Vigeland \[6\] exactly states that \( \varphi_{\text{alg}} \) is bijective. The injectivity of \( \varphi_{\text{alg}} \) follows from the injectivity of \( \varphi \), but the surjectivity of \( \varphi_{\text{alg}} \) requires extra arguments like \[6\].

\[C\]

\[M\]

\[Q\]

\[P\]

\[L\]

Remark 1.4. One needs to be careful about the ‘degree’ of the tropical curve. Tropical curves \( L, M \) with Newton polygons

\[\Delta(L) = \text{Conv}\{(0, 0), (1, 0), (1, 1)\} \]

\[\Delta(M) = \text{Conv}\{(0, 0), (0, 1), (1, 1)\} \]

are both ‘tropical curves of degree 2’. Figure \[8\] is an example such that \( C \cdot L - C \cdot M = P - Q \). But Theorem \[1.1\] asserts that \( P, Q \) are not linearly equivalent. Note that \( L, M \) are not projective.
2 Preliminary

A primitive vector in \( \mathbb{R}^2 \) is an integral vector \( u = (u_1, u_2) \) such that \( u_1, u_2 \) are relatively prime. Any integral vector \( v \in \mathbb{Z}^2 \) is a primitive vector \( u \) times some natural number \( m \). \( m \) is called the lattice length of \( v \).

Let \( C \) be a 1-dimensional simplicial complex of rational slopes in \( \mathbb{R}^2 \). Each finite edge \( E \subset C \) has two primitive vectors for the directions parallel to \( E \), say \( u, -u \). If the weight \( m \in \mathbb{N} \) of \( E \) is given, we have the weighted primitive vectors \( mu, -mu \) of \( E \). For a vertex \( V \in E \), one of these vectors has the direction from \( V \) to \( E \). We call this, say \( u_E \), the weighted primitive vector of \( E \) starting at \( V \).

If \( E \) is a ray (i.e. an infinite edge), \( E \) has only one weighted primitive vector.

Definition. \( V \) satisfies the balancing condition if the sum of all weighted primitive vectors starting at \( V \) equals 0:

\[
\sum_{E \ni V} u_E = 0.
\]

A tropical curve in \( \mathbb{R}^2 \) is a 1-dimensional weighted simplicial complex of rational slopes such that each vertex satisfies the balancing condition.

Let \( C \) be a tropical curve in \( \mathbb{R}^2 \). Let \( U_1, \ldots, U_r \) be all connected components of \( \mathbb{R}^2 \setminus C \). Let \( N \) be a 1-dimensional simplicial complex with vertex set \( \text{Ver}(N) = \{ w_1, \ldots, w_r \}, w_i \in \mathbb{Z}^2 \).

Definition. \( N \) is a Newton complex of \( C \) if it satisfies the following conditions for any \( i \neq j \).

i) If \( \overline{U_i} \cap \overline{U_j} = \emptyset \), then \([w_i, w_j] \notin \text{Edge}(N)\).

ii) If \( \overline{U_i} \cap \overline{U_j} = E \) for some \( E \in \text{Edge}(C) \), then \([w_i, w_j] \in \text{Edge}(N)\), and \( w_j - w_i \) has lattice length \( \text{wt}(E) \), direction orthogonal to \( E \) from \( U_i \) to \( U_j \).

Proposition 2.1. Let \( C, C_1, C_2 \) be tropical curves in \( \mathbb{R}^2 \).

1) A Newton complex \( \text{Newt}(C) \) of \( C \) exists uniquely up to parallel translation. The convex hull

\[
\Delta(C) = \text{Conv}(\text{Newt}(C))
\]

is called the Newton polygon of \( C \).

2) \( \Delta(C_1 \cup C_2) \) equals the Minkowski sum of \( \Delta(C_1) \) and \( \Delta(C_2) \).
Proof. 1) (See [5], §3.4, if you take an algebraic approach.) Let $w_1 = (0, 0)$, and suppose $w_1, \ldots, w_{k-1}$ are constructed. Rearranging $U_k, \ldots, U_r$, we may assume $U_i \cap U_k = E_k$ for some $i < k$ and some $E_k \in \text{Edge}(C)$. Let $u_k$ be the primitive vector of direction orthogonal to $E_k$ from $U_i$ to $U_k$. Let $w_k$ be the vector
\[ w_k - w_i = \text{wt}(E_k)u_k. \] (1)
If $U_1, \ldots, U_k$ are adjacent at a common vertex $V$, the condition (1) is compatible for $U_k$ and $U_1$, i.e.
\[ w_1 - w_k = \text{wt}(E_1)u_1, \]
where $E_1$ is the boundary of $U_k$ and $U_1$, and $u_1$ is the primitive vector of direction orthogonal to $E_1$ from $U_k$ to $U_1$. This compatibility follows from the balancing condition
\[ \sum_{i=1}^{k} \text{wt}(E_i)u_i = 0. \]
Therefore this construction does not depend on the choice of $U_k$.

2) $\Delta(C_1)$ depends only on the data of rays of $C_1$. Rays of $C_1 \cup C_2$ corresponds to rays of $C_1$ and $C_2$. \hfill \square

Newt$(C)$ can be considered as a dual object of $C$, with correspondence from $U_i$ to $w_i$ (Figure 4). A vertex $V \in C$ corresponds to a polygon $T_V \subset \Delta(C)$ as follows.
i) \( U_i, \ldots, U_k \) are adjacent at \( V \),

ii) \( T_V = \text{Conv}\{w_i, \ldots, w_k\} \).

**Proposition 2.2** (Global balancing condition). Let \( C \) be a tropical curve in \( \mathbb{R}^2 \). Let \( \Lambda \) be a simple closed curve in \( \mathbb{R}^2 \) intersecting edges of \( C \), say \( E_1, \ldots, E_N \), transversely. Then

\[
\sum_{i=1}^{N} u_{E_i} = 0,
\]

where \( u_{E_i} \) is the weighted primitive vector of \( E_i \) starting at the vertex inside \( \Lambda \).

**Proof.** For each vertex \( V_j \in C \) inside \( \Lambda \), the balancing condition

\[
\sum_k u_{j,k} = 0
\]

holds. Thus

\[
\sum_{j,k} u_{j,k} = 0.
\]

On the left hand side, two weighted primitive vectors of the same edge inside \( \Lambda \) are canceled. Thus we have the required equation. \( \square \)

A *tangent vector* \((v, P)\) in \( \mathbb{R}^2 \) is a vector \( v \in \mathbb{R}^2 \) with a starting point \( P \in \mathbb{R}^2 \). We fix a point \( P_0 \in \mathbb{R}^2 \). The *moment* of \((v, P)\) is the exterior product

\[
\text{moment}(v, P) = \overrightarrow{P_0P} \times v.
\]

**Proposition 2.3** (Moment condition). Under the assumption of Proposition 2.2,

\[
\sum_{i=1}^{N} \text{moment}(u_{E_i}, V_{E_i}) = 0,
\]

where \((u_{E_i}, V_{E_i})\) is the weighted primitive tangent vector of \( E_i \) starting at the vertex inside \( \Lambda \). (See Figure 5.)
Proof. For each vertex $V_j \in C$ inside $\Lambda$, the balancing condition

$$\sum_k u_{jk} = 0$$

holds. Thus

$$\sum_{j,k} \overrightarrow{P_0 V_j} \times u_{jk} = 0.$$

On the left hand side, two weighted primitive vectors of the same edge inside $\Lambda$ are canceled as follows.

$$\overrightarrow{P_0 V_j} \times u_{jk} + \overrightarrow{P_0 V_{j'}} \times u_{j'k'} = \overrightarrow{P_0 V_j} \times u_{jk} + \overrightarrow{P_0 V_{j'}} \times (-u_{jk})$$

$$= \overrightarrow{V_j' \times u_{jk}}$$

$$= 0.$$

Thus we have the required equation. \qed

Definition. The *multiplicity* of a vertex $V \in C$ is

$$\text{Mult}(V; C) = 2 \cdot \text{area}(T_V).$$

The *intersection multiplicity* of $V \in C_1 \cap C_2$ is

$$\mu_V = \frac{1}{2} (\text{Mult}(V; C_1 \cup C_2) - \text{Mult}(V; C_1) - \text{Mult}(V; C_2)).$$
(If \( V \) is not a vertex of \( C \), we put \( \text{Mult}(V; C) = 0. \))

The formal sum

\[
C_1 \cdot C_2 = \sum_{V \in C_1 \cap C_2} \mu_V V
\]

is called the *stable intersection* of \( C_1 \) and \( C_2 \).

The stable intersection is characterized as the limit of the transversal intersection (See [3], Theorem 4.3). If \( V \in C_1 \cap C_2 \) is a transversal intersection point, this definition is simplified to

\[
\mu_V = |u_E \times u_F|,
\]

where \( E \subset C_1, F \subset C_2 \) are edges passing through \( V \).

**Theorem 2.4** (Tropical Bezout’s Theorem). Let \( C_1, C_2 \) be tropical curves in \( \mathbb{R}^2 \). Then the following formula holds.

\[
\deg(C_1 \cdot C_2) = \text{area}(\Delta(C_1) + \Delta(C_2)) - \text{area}(\Delta(C_1)) - \text{area}(\Delta(C_2)),
\]

where \( \Delta(C_1) + \Delta(C_2) \) is the Minkowski sum.

**Proof.** This follows from the above definition and Proposition 2.1.

For example, let \( c, d \geq 1 \), and suppose \( \Delta(C_1) = \text{Conv}\{(0,0), (c,0), (0,c)\} \), \( \Delta(C_2) = \text{Conv}\{(0,0), (d,0), (0,d)\} \) (In algebraic approach, \( C_2 \) is said to be *projective* of degree \( d \)). Then \( \Delta(C_1) + \Delta(C_2) = \text{Conv}\{(0,0), (c+d,0), (0,c+d)\} \), and \( \deg(C_1 \cdot C_2) = \frac{1}{2}(c + d)^2 - \frac{1}{2}c^2 - \frac{1}{2}d^2 = cd \).

### 3 Proof of the surjectivity

First we show that \( \varphi \) is well-defined (Lemma 3.4, Lemma 3.5).

**Lemma 3.1.** Let \( u \in \mathbb{Z}^2 \) be a primitive vector. Then for given \( \varepsilon > 0 \), there is a primitive vector \( v \in \mathbb{Z}^2 \) such that

\[
u \times v = 1, \quad |\theta(u) - \theta(v)| < \varepsilon,
\]

where \( \theta(u) \) denotes the angle of \( u \).

This lemma is easy.
Lemma 3.2. Let $E$ be an edge of $C$, and let $P, P', Q, Q'$ be points of $E$ such that $PP' = QQ'$. Then $P' - P \sim Q' - Q$.

Proof. (Figure 6) We may assume that $P', Q$ lie on the interval $[P, Q']$, and that $Q'$ lies in the interior $\text{Int}(E)$. Let $v_1, v_2, v_3$ be primitive vectors such that
\[
|u_E \times v_i| = 1 \quad (i = 1, 2, 3),
\]
\[
\theta(u_E) - \varepsilon < \theta(v_1) < \theta(u_E) < \theta(v_2) < \theta(u_E) + \varepsilon < \theta(v_3).
\]
Then we have a triangle, with vertices $P$ and $R_1, R_2 \in \mathbb{R}^2$, such that
\[
\overrightarrow{PR_i} \text{ has direction } v_i \quad (i = 1, 2),
\]
\[
\overrightarrow{R_1 R_2} \text{ has direction } v_3,
\]
\[
Q' \in [R_1, R_2].
\]
Take $\varepsilon > 0$ small enough so that this triangle is disjoint from $C \setminus E$.

Now we take two tropical curves $L, M$ as follows. $L$ consists of one vertex $P$ and three rays $L_0, L_1, L_2 \subset \mathbb{R}^2$, with $R_1 \in L_1$, $R_2 \in L_2$. $L_0$ is parallel to $E$. $\text{wt}(L_1) = \text{wt}(L_2) = 1$. (The balancing condition at $P$ follows from (2).) $M$ consists of one finite edge $M_0 \subset \mathbb{R}^2$, four rays $M_1, M_2, M_3, M_4 \subset \mathbb{R}^2$, and two vertices $V_1, V_2 \in \mathbb{R}^2$. $M_0$ is parallel to $[R_1, R_2]$, and passes through $Q$. $V_1 \in [P, R_1]$, $V_2 \in [P, R_2]$. For $i = 1, 2$, $M_i$ has vertex $V_i$ and passes through $R_i$. $M_3, M_4$ are parallel to $E$. $\text{wt}(M_0) = \text{wt}(M_1) = \text{wt}(M_2) = 1$.

Move $L$ by $PP'$, and denote it by $L'$. Move $M$ by $QQ'$, and denote it by $M'$. Then we have a relation
\[
(C \cdot L - P) - (C \cdot L' - P') = (C \cdot M - Q) - (C \cdot M' - Q').
\]
Thus $P' - P$ is linearly equivalent to $Q' - Q$. \qed

Corollary 3.3. Let $E$ be any edge, and suppose that all interior points of $E$ are linearly equivalent. Then all points of $E$ are linearly equivalent.

Lemma 3.4. Let $E$ be a ray of $C$. Then all points of $E$ are linearly equivalent.

Proof. (Figure 7, left) Let $P, Q \in \text{Int}(E)$. Take a primitive vector $v$ so that
\[
|u_E \times v| = 1, \quad |\theta(u_E) - \theta(v)| < \varepsilon.
\]
There is a small parallelogram $R_1 R_2 R_3 R_4$ such that

- $\overrightarrow{R_3 R_1}, \overrightarrow{R_4 R_2}$ have direction $v$,
- $\overrightarrow{R_2 R_1}, \overrightarrow{R_4 R_3}$ have direction $w := u_E - v$,
- $P \in [R_1, R_3]$, $Q \in [R_2, R_4]$.

Take $\varepsilon > 0$ small enough so that this parallelogram is disjoint from $C \setminus E$.

Let $M_1$ be a tropical curve, consisting of one vertex $R_1$ and three rays $L_0, L_1, L_2 \subset \mathbb{R}^2$, such that $L_0$ has direction $u_E$, $R_3 \in L_1$, $R_2 \in L_2$, $\operatorname{wt}(L_0) = \operatorname{wt}(L_1) = 1$. Move $M_1$ by $\overrightarrow{R_1 R_3}$, and denote it by $M_i (i = 2, 3, 4)$. Then we have a relation

\[ C \cdot M_1 + C \cdot M_4 - Q = C \cdot M_2 + C \cdot M_3 - P. \]
Thus $P \sim Q$. \hfill \blacksquare

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Figure 7:}
\end{figure}

\textbf{Lemma 3.5.} Let $E$ be a tentacle edge of $C$. Then all points of $E$ are linearly equivalent.

\textit{Proof.} (Figure 7 right) Let $P \in \text{Int}(E)$. Since $C \setminus \text{Int}(E)$ is disconnected, $E$ is a boundary of two unbounded convex open sets $U_1, U_2$ of $\mathbb{R}^2 \setminus C$. For $i = 1, 2$, let $\theta_i$ be the angle of any unbounded direction of $U_i$. For $\delta, \varepsilon > 0$, let

$$W_i = \left\{ P + a + w \mid a, w \in \mathbb{R}^2, |a| < \delta, P + a \in E, \quad |\theta(w) - \theta_i| < \varepsilon \right\},$$

$$W = W_1 \cup W_2.$$ 

Take $\delta, \varepsilon > 0$ small enough so that $W$ intersects $C \setminus E$ only at points of rays of $C$.

Let $v$ be a primitive vector such that $u_E \times v = 1$. Take $w, w' \in \mathbb{Z}^2$ so that

$$w - w' = v,$$

$$|\theta(w) - \theta_1| < \varepsilon,$$

$$|\theta(w') - \theta_1| < \varepsilon.$$ 

Let $L$ be a tropical curve consisting of three rays $L_0, L_1, L_2 \subset \mathbb{R}^2$ such that

$L_1 \cup L_2 \subset W$;
Moving \( L \) on the direction of \( L_0 \), the intersection point \( P \) changes to other point \( Q \in E \), but all other intersection points of \( L \) and \( C \) are stable except for points of rays of \( C \). We have

\[
(u_E \times w)P \sim (u_E \times w)Q.
\]

Similarly,

\[
(u_E \times w')P \sim (u_E \times w')Q.
\]

Thus \( P \sim Q \).

For a proof of Theorem 1.1, the case of genus 1 is essential. If \( \Lambda := \text{Bunch}(C) \approx S^1 \), then we have a map \( \pi: \mathbb{R} \to \Lambda \) with the following properties.

i) \( \pi(0) = O \).

ii) \( \pi \) is increasing with respect to the positive direction of \( S^1 \).

iii) If \( \pi[a, b] \ (a, b \in \mathbb{R}) \) is contained in an edge of primitive vector \( u \), then

\[
\text{length}(\pi[a, b]) = |u|(b - a).
\]

(In other words, \( \pi \) is compatible with the lattice length.)

**Lemma 3.6.** If \( \pi(a) = P, \pi(a') = P', \pi(b) = Q, \pi(b') = Q' \), \( a' - a = b' - b \), then \( P' - P \sim Q' - Q \).

**Proof.** (Figure 5) We may assume that \( a' - a > 0 \) is small enough so that \( P, P' \) lie on the same edge \( E \), and \( Q, Q' \) on \( F \). We may also assume that \( E, F \) are adjacent at a common vertex \( R \), and \( E \neq F \). From Lemma 3.2, we may assume that \( P, P', Q, Q' \) are interior points of edges, and \([P, Q]\) has rational slope.

Let \( L \) be the line passing through \( P, Q' \). Then

\[
E \cdot L = |u_E \times u_L|P,
\]

\[
F \cdot L = |u_F \times u_L|Q.
\]

Let \( P'', Q'' \) be points of \( E, F \) such that

\[
\frac{PP''}{PP'} = \frac{1}{|u_E \times u_L|PP'},
\]

\[
\frac{PP''}{PP'} = \frac{QQ''}{QQ'}.
\]
\[
\overrightarrow{Q'Q''} = \frac{1}{|u_F \times u_L|} \overrightarrow{Q'Q}.
\]

Then
\[
|\overrightarrow{PP''} \times u_L| = \frac{|\overrightarrow{PP''}|}{|u_E|} |u_E \times u_L|
\]
\[
= \frac{|\overrightarrow{PP''}|}{|u_E|}
\]
\[
= \frac{|\overrightarrow{Q'Q''}|}{|u_E|} \quad \text{(because } a' - a = b' - b)\]
\[
= \frac{|\overrightarrow{Q'Q''} \times u_L|}{|u_E|},
\]

which means that \([P'', Q'']\) is parallel to \(L\). Thus
\[
|u_E \times u_L|(P'' - P) \sim |u_F \times u_L|(Q' - Q'').
\]

This means \(P' - P \sim Q' - Q\), from Lemma 3.2.

\[\square\]

**Figure 8:**

**Lemma 3.7** (Interval divisor). *Let \(S, T\) be rays in \(\mathbb{R}^2\) such that \(S \cap C = \emptyset\), \(T \cap C = \emptyset\), \(S \cap T = \emptyset\). Let \(P, P' \in S\), \(Q, Q' \in T\) be points such that \([P, Q]\) is parallel to \([P', Q']\). Then

\[C \cdot [P, Q] \sim C \cdot [P', Q']\]**.
Proof. (Figure 9 left) Let \( L \) be the tropical curve with vertex \( P \), consisting of three rays \( L_0, L_1, L_2 \) such that 
\[
L_1 \subset S, \quad Q \in L_2,
\]
\[
\text{wt}(L_1) = \text{wt}(L_2) = 1.
\]
Let \( R \) be the point such that 
\[
[P, R] \text{ is parallel to } L_0,
\]
\[
[P', R] \text{ is parallel to } [P, Q].
\]
Then we have a triangle \( PP'R \). We may assume that \( |PP'| \) is small enough so that this triangle is disjoint from \( C \). Move \( L \) by \( PR \), and denote it by \( L' \). Then we have a relation 
\[
C \cdot L - C \cdot [P, Q, \infty) = C \cdot L' - C \cdot [P', Q', \infty).
\]
Thus 
\[
C \cdot [P, Q, \infty) \sim C \cdot [P', Q', \infty).
\]
Similarly, 
\[
C \cdot [Q, P, \infty) \sim C \cdot [Q', P', \infty),
\]
\[
C \cdot \text{Line}(P, Q) \sim C \cdot \text{Line}(P', Q').
\]
The statement follows.

Proof of the surjectivity of \( \varphi \). (Figure 9 right) Since the image of \( \Lambda_1 \cup \cdots \cup \Lambda_g \) in \( \text{Bunch}(C) \) is a bouquet, there are \( O_1, \ldots, O_g \sim O \) such that \( O_i \in \Lambda_i \). Because of convexity, there are connected disjoint \( g \) cones \( U_1, \ldots, U_g \subset \mathbb{R}^2 \) with center \( O_1, \ldots, O_g \) such that \( \Lambda_i \subset U_i \). Similarly to the case of genus 1, we have a map \( \pi_i: \mathbb{R} \to \Lambda_i \) for each \( i \). Lemma 3.6 is proved for \( \pi_i \) similarly, only changing \( L \) to an interval divisor \( L \cap U_i \) (Lemma 3.7). Thus we have a homomorphism of abelian groups 
\[
\tilde{\varphi}: \mathbb{R}^g \rightarrow \pi_1 \times \cdots \times \pi_g \rightarrow \text{Jac}(C).
\]
Since \( \text{Jac}(C) \) is generated by \( \{ P_i - O | P_i \in \Lambda_i, 1 \leq i \leq g \} \), \( \tilde{\varphi} \) is surjective.
4 Parameter space of tropical plane curves

Let $L$ be a tropical curve with Newton complex $\mathcal{N}$. Let $V_0 = (b_1, b_2)$ be a fixed vertex of $L$. Let $E_1, \ldots, E_l$ be all finite edges of $L$. Let $a_i$ be the lattice length of $E_i$. Then all tropical curves with Newton complex $\mathcal{N}$ are parametrized by $a_1, \ldots, a_l > 0$ and $b_1, b_2 \in \mathbb{R}$. Let $\mathcal{P}(\mathcal{N}, \mathbb{R}^2) \subset \mathbb{R}^{l+2}$ be the parameter space.

**Proposition 4.1.** $\mathcal{P}(\mathcal{N}, \mathbb{R}^2)$ is connected.

**Proof.** Let $\Gamma_1, \ldots, \Gamma_g$ be all convex cycles of $L$. Let $E_{i(j,1)}, \ldots, E_{i(j,s_j)}$ be all edges of $\Gamma_j$. Let $u_{j,k}$ be the primitive vector of $E_{i(j,k)}$ of positive direction of $S^1$. Then the equation

$$a_{i(j,1)}u_{j,1} + \cdots + a_{i(j,s_j)}u_{j,s_j} = 0$$

is satisfied for any $L \in \mathcal{P}(\mathcal{N}, \mathbb{R}^2)$. Let $H_j \subset \mathbb{R}^{l+2}$ be the linear subspace defined by equation (3). Then

$$\mathcal{P}(\mathcal{N}, \mathbb{R}^2) = \{ (a_1, \ldots, a_l, b_1, b_2) | a_1, \ldots, a_l > 0 \} \cap (H_1 \cap \cdots \cap H_g).$$

Thus $\mathcal{P}(\mathcal{N}, \mathbb{R}^2)$ is a relatively open convex cone in $\mathbb{R}^{l+2}$, which is connected. \hfill \Box

**Definition.** A tropical curve $L'$ is a degeneration of $L$ if $\Delta(L') = \Delta(L)$ and $\text{Newt}(L') \subset \text{Newt}(L)$. 

---

Figure 9:
The set of all degenerations of $L$ is parametrized by $\mathcal{P}(\mathcal{N}, \mathbb{R}^2)$. If a Newton polygon $\Delta$ is fixed, all tropical curves have a common degeneration (that is, a tropical curve consisting of one vertex). All the Newton polygons is countable. Therefore, the space $\mathcal{P}(\mathbb{R}^2)$ of all tropical curves is a disjoint union of countable closed cones in affine spaces.

**Corollary 4.2.** Tropical curves $L, L' \in \mathcal{P}(\mathbb{R}^2)$ lie in the same connected component if and only if $\Delta(L) = \Delta(L')$.

## 5 Proof of the injectivity

Let $\pi: \mathbb{R} \to \Lambda$ be the map defined in section 3. $\Lambda$ is considered as a residue group of $\mathbb{R}$. Let $E_1, \ldots, E_N$ be all edges of $\Lambda$ ordered by the positive direction of $S^1$. Let $\lambda: C \to \Lambda$ be the canonical surjection. For a tropical curve $L \in \mathcal{P}(\mathbb{R}^2)$, we define $\sigma(L) \in \Lambda$ as follows.

$$C \cdot L = P_1 + \cdots + P_r,$$

$$\sigma(L) = \lambda(P_1) + \cdots + \lambda(P_r).$$

**Lemma 5.1.** $\sigma: \mathcal{P}(\mathbb{R}^2) \to \Lambda$ is locally constant.

**Proof.** $\sigma$ is continuous by definition of the stable intersection. Let $\{L_t|0 \leq t \leq 1\}$ be a continuous family of tropical curves with Newton complex $\mathcal{N}$ such that $C$ intersects $L_t$ transversely for any $t$. Then there are points $P_{ijt} \in E_i$, edges $L_{ijt} \subset L_t$, and vectors $u_{ij} \in \mathbb{R}^2$ such that

i) $E_i \cdot L_t = \sum_j P_{ijt},$

ii) $E_i \cap L_{ijt} = P_{ijt},$

iii) $u_{ij}$ is the weighted primitive vector of $L_{ijt}$ starting at the vertex inside $\Lambda$, divided by $\mu_{P_{ijt}}$.

For $L_0$ and $L_1$, we have the moment condition inside $\Lambda$:

$$\sum_{i,j} \text{moment}(u_{ij}, P_{ij0}) = 0,$$

$$\sum_{i,j} \text{moment}(u_{ij}, P_{ij1}) = 0.$$

From these,

$$\sum_{i,j} \left( \overrightarrow{P_{ij0}P_{ij1}} \times u_{ij} \right) = 0.$$
Since $u_{E,i} \times u_{ij} = -1$, this means
\[ \sum_{i,j} (\lambda(P_{ij0}) - \lambda(P_{ij1})) = 0. \]

Thus $\sigma(L_0) = \sigma(L_1)$. \qed

![Figure 10:](image)

**Proof of the injectivity of $\varphi$.** (Figure 10) Suppose
\[
(P_1 + \cdots + P_g) - (Q_1 + \cdots + Q_g) = C \cdot L - C \cdot L',
\]
where $P_i, Q_i \in \Lambda_i \setminus \text{Ver}(\Lambda_i)$,
\[ \Delta(L) = \Delta(L'). \]

Let $\tilde{C}$ be the tropical curve consisting of $\Lambda_1$ and $N$ rays $F_1, \ldots, F_N$. Then $\deg(F_i \cdot L) = \deg(F_i \cdot L')$ because of the tropical Bezout’s theorem. From Corollary 4.2 and Lemma 5.1, we have $\sigma(L) = \sigma(L')$. Thus $P_1 = Q_1$. \qed

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