Interacting Topological Insulators with Synthetic Dimensions

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Recent developments of experimental techniques have given us unprecedented opportunities of studying topological insulators in high dimensions, while some of the dimensions are “synthetic”, in the sense that the effective lattice momenta along these synthetic dimensions are controllable periodic tuning parameters. In this work, we study interaction effects on topological insulators with synthetic dimensions. We show that although the free fermion band structure of high dimensional topological insulators can be precisely simulated with the “synthetic techniques”, the generic interactions in these effective synthetic topological insulators are qualitatively different from the local interactions in ordinary condensed matter systems. And we show that these special but generic interactions have unexpected effects on topological insulators, namely they would change (or reduce) the classification of topological insulators differently from the previously extensively studied local interactions.

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I. INTRODUCTION

Ever since the proposal of the analogue of quantum Hall effects in four spatial dimensions,\textsuperscript{14} the topological states of matters in higher dimensions have attracted a great deal of theoretical interests. The classification of free fermion topological insulators (TI) and superconductors (TSC) in all dimensions (the so-called “10-fold way”) was a milestone in our understanding of non-interacting fermionic systems.\textsuperscript{15} Later, a great progress in understanding strongly interacting bosonic states of matter was achieved through the classification and description of bosonic symmetry protected topological states, which can also be generalized to higher dimensions.\textsuperscript{16,17}

Until recently, the study of higher dimensional topological insulators (TI) and its bosonic analogues was of pure theoretical interests only. However, the study of higher dimensional TIs has gained important experimental relevance recently. For example, the four dimensional quantum Hall insulator (or the four dimensional Chern insulator) was successfully constructed experimentally,\textsuperscript{18,19} with two out of its four spatial dimensions “synthetic”. In fact, “synthetic” dimensions are experimentally realized as periodic tuning parameters that can be identified as the corresponding lattice momenta in these dimensions. In general, what such synthetic-dimension techniques directly realize is a Hamiltonian of the form

\[ \hat{H}(\vec{p}) = \sum_{\delta_{i,j}} \hat{H}_{\delta_{i,j}}(\vec{p}) c_{i}^\dagger c_{j}, \]

where \( \vec{p} \) is a \((D - d)\)-dimensional synthetic momentum, and \( i, j \) label the sites on the \( d \)-dimensional physical (optical) lattice. Once we identify \( \vec{p} \) as the lattice momenta in the synthetic dimensions, the entire system can be viewed as a \( D \)-dimensional non-interacting fermion system. Such synthetic construction in principle can give us experimental realization of all classes of non-interacting fermion TI in any dimension. The same perspective, \textit{i.e.} viewing lattice momenta in certain dimensions as tuning parameters, has been used by theorists to connects TIs in different dimensions,\textsuperscript{20} and also to construct topological semimetals\textsuperscript{21,22}

Interestingly, interactions can drastically change the classification of TIs and TSCs with certain symmetries. It was first discovered in Ref.\textsuperscript{23,24} that some TIs non-trivial in the non-interacting limit can be trivialized by interactions, which exemplified the importance of interaction in altering or more precisely “reducing” the classification of TIs. Subsequent studies have shown the same effect of interaction, \textit{i.e.} the reduction of classification, in TIs and TSCs in all dimensions.\textsuperscript{25,26,27,28,29,30,31,32,33,34,35} These conclusions were made under the assumptions of \textit{spatially local} interactions that preserve the crucial symmetries which define the TIs.

In contrast, the generic interaction on a synthetic TI is fundamentally different from that on the ordinary TIs given the “synthetic” nature of the extra dimensions. The interaction must be local in the \( d \)-dimensional real space lattice, and also local in the \( \delta = (D - d) \)-dimensional synthetic momentum space, \textit{i.e.} under adiabatic tuning, at each value of the tuning parameter \( \vec{p} \) the system has an interaction in the \( d \)-dimensional real space labelled by \( \vec{p} \). \( H_{\text{int}}(\vec{p}) \). This allows us to use the “dimensional reduction” procedure of Ref.\textsuperscript{21} even with interactions, which is a method that we will exploit in this work.

A generic interaction \( H_{\text{int}}(\vec{p}) \) in a synthetic TI is no longer completely local in the effective \( D \)-dimensional real space: it is local in the physical dimensions, but nonlocal in the synthetic spatial dimensions. It appears that TIs with a nonlocal interaction is not even definable, since a nonlocal interaction would easily mix the edge states at two opposite boundaries of the TIs. But in the current situation, the interaction is only nonlocal along the synthetic directions in the effective \( D \)-dimensional space. And as long as we are considering boundaries parallel to the \( \delta = (D - d) \)-dimensional synthetic space, namely the \( \delta \)-component synthetic momenta are still conserved at the boundary, the two opposite boundaries will...
not be mixed by the nonlocal interaction (Fig. 1a). This is the most natural choice of boundary in the effective $D$-dimensional synthetic TI, as we simply need to choose the physical boundary of the system while keeping the tuning parameter $\tilde{p}$ periodic. Then one can still study the fate of the edge states at one single boundary, and the effective $D$-dimensional system can be called a nontrivial TI as long as this boundary remains gapless under the interaction.

| $(D, \delta)$ | (1, 0) | (1, 1) | (3, 0) | (3, 1) | (3, 2) |
|---------------|--------|--------|--------|--------|--------|
| Classification, $U(1) \times Z_2^T$ | $Z_4$ | $Z_2$ | $Z_8$ | $Z_4$ | $Z_2$ |

| $(D, \delta)$ | (2, 0) | (2, 1) | (4, 0) | (4, 1) | (4, 2) |
|---------------|--------|--------|--------|--------|--------|
| Classification, $U(1) \times Z_2$ | $Z_4$ | $Z_2$ | $Z_8$ | $Z_4$ | $Z_2$ |

Our main results are the following: (1) The interaction reduced classification for TIs with symmetry $U(1) \times Z_2^T$ at total odd dimensions $D = 2n + 1$ and synthetic dimensions $\delta$ is given by $Z_{2n+2-\delta}$; (2) The interaction reduced classification for non-chiral TIs with symmetry $U(1) \times Z_2$ at total even dimensions $D = 2n$ and synthetic dimensions $\delta$ is given by $Z_{2n+1-\delta}$. We select these symmetries because TIs defined with these symmetries, as we will discuss in the next few sections, will be strongly affected by the interaction. In the tables above, we list the results for $D = 2, 3, 4$ explicitly, which will be discussed in detail in Sec. II[13] and IV[14]. In Secs. V[15] and VI[16] we will discuss the classification reduction of interacting synthetic non-chiral TIs with $U(1) \times Z_2$ symmetry in general even dimensions $D = 2n$ and that of interacting synthetic TIs with $U(1) \times Z_2^T$ symmetry in general odd dimensions $D = 2n + 1$.

II. SYNTHETIC TI WITH $D = 2$

We will start with the example of $D = 2$ synthetic TI, with $d = \delta = 1$, namely one of the two dimensions is synthetic. The synthetic non-interacting Chern insulator with $D = 2$ and $\delta = 1$ has been studied previously[13]. One particular type of two dimensional TIs that we know will be strongly influenced by interaction and reduce its classification is the non-chiral TI with $U(1) \times Z_2$ symmetry[17,18], where physically the $Z_2$ is usually the reflection symmetry about the $z$ axis, which becomes an on-site symmetry in the two dimensional plane. The minimal version of this TI is basically two copies of Chern insulators with opposite Chern numbers, and also opposite eigenvalues ($\pm 1$) under the $Z_2$ symmetry operation. Its 1$d$ edge state has the Hamiltonian

$$H = \int dx \ v \left( \psi_1^\dagger i\partial_x \psi_1 - \psi_2^\dagger i\partial_x \psi_2 \right).$$

(1)

The charge conservation $U(1)$ symmetry and the $Z_2$ symmetry act on the boundary fermions as

$$U(1) : (\psi_1, \psi_2) \to e^{i\theta}(\psi_1, \psi_2),$$

$$Z_2 : (\psi_1, \psi_2) \to (\psi_1, -\psi_2).$$

(2)

One can see that for arbitrary copies of the edge states, any fermion bilinear mass term at the edge will mix left and right moving fermions, and hence break the $Z_2$ symmetry, while any fermion-pairing mass gap would break the $U(1)$ symmetry. Hence without interaction the classification of this TI is $Z_2$. It was shown that under local interaction the classification of this $U(1) \times Z_2$ TI is reduced to $Z_4$. This conclusion has two related implications: (1) Four copies of the one dimensional edge states can be gapped out by local interactions without breaking either $U(1)$ or $Z_2$ symmetry, which was directly shown in Ref. [17]; (2) For four copies of the system, the TI-to-trivial phase transition in the two dimensional bulk can be avoided under interaction, namely the non-interacting TI phase and the trivial phase can be adiabatically connected with interaction while keeping the bulk gap open. This can be understood using the Chalker-Coddington network picture of the bulk topological transition[18], the transition between two phases with the same symmetry and no bulk ground state degeneracy can be interpreted as the percolation of their interface (domain wall). If the interface is fully gapped, then these two phases can be adiabatically connected without closing the bulk gap.

Let us now investigate what if the dimension along the boundary is synthetic. In this case, the boundary Hamiltonian including the interaction will be

$$H = \sum_p H_p, \ H_p = v p \left( \psi_1^\dagger_1 \psi_1^\dagger_2 - \psi_2^\dagger_1 \psi_2^\dagger_2 \right) + H_{int}(p).$$

(3)

Because $p$ is the synthetic momentum, it is actually a tuning parameter, $H_{int}(p)$ is local in the synthetic momentum space, i.e., for each value of $p$ there is an interaction labelled by $p$. Then for each $p$, solving $H_p$ is equivalent to solving a zero dimensional problem with
finite dimensional Hilbert space. The original gapless point \( p = 0 \) becomes a level crossing of two states: one state has \( N_{1,p} = \psi_{1,p}^\dagger \psi_{1,p} = 1, \) \( N_{2,p} = 0 \) and the other state has \( N_{1,p} = 0, \) \( N_{2,p} = 1. \)

If \( H_{\text{int}}(p) \) preserves the \( U(1) \times Z_2 \) symmetry, then for a single copy of Eq. [3] there is no way to avoid the level crossing, because \( H_{\text{int}}(p) \) can always be recombined into a function of \( N_{1,p} \) and \( N_{2,p}, \) and changing the filling of \( N_{1,p} \) and \( N_{2,p} \) will lead to level crossing. But for two copies of the Eq. [3] the story is different. Let us perform a basis transformation of the second copy of the system so that the edge state Hamiltonian reads

\[
H_p = v p \left( \psi_{1,p}^\dagger \tau^z \psi_{1,p} - \psi_{2,p}^\dagger \tau^z \psi_{2,p} \right) + H_{\text{int}}(p).
\] (4)

Our goal is to show that for certain choice of \( H_{\text{int}}(p) \), the ground state in the limit \( p > 0 \) and \( p < 0 \) of Eq. [3] can be connected adiabatically without closing the ground state gap. The following choice of \( H_{\text{int}}(p) \) will suffice:

\[
H_{\text{int}}(p) = J \vec{S}_{p,+} \cdot \vec{S}_{p,-}, \quad \vec{S}_{p,\pm} = \psi_{p,\pm}^\dagger \sigma \psi_{p,\pm},
\] (5)

where \( \sigma \) are three Pauli matrices that act on the fermion index \((1,2)\), and \( \psi_{p,\pm} \) are fermion modes with eigenvalues \( \pm 1 \) of \( \tau^z \). The direct computation of the spectrum of Eq. [4] plotted in Fig. 2 confirms that the interaction \( H_{\text{int}} \) can indeed ensure a finite gap for all values of \( p \) for the Hamiltonian Eq. [4].

This analysis implies that the two dimensional non-chiral TI with \( U(1) \times Z_2 \) symmetry has its classification reduced from \( Z \) to \( Z_2 \) under interaction if one of the spatial dimensions is synthetic, which is different from the \( Z \) to \( Z_4 \) reduction as was discussed previously.

### III. SYNTHETIC TI WITH \( D = 3 \)

Now let us consider the case with \( D = 3 \), and \( \delta = 1 \), namely we are considering an effective \( D = 3 \) dimensional TI with one synthetic dimension. One type of three dimensional TI whose classification is changed by interaction is the TI defined by symmetry \( U(1) \times Z_2^T \). One tight-binding model of this TI is \( \hat{H} = \sum_k c_k^\dagger \hat{H}(\vec{k}) c_k \), and \( H(\vec{k}) \) is

\[
H(\vec{k}) = -t \left( \sum_{i=1}^3 \Gamma^i \sin k_i - \Gamma^4 (h - \sum_{i=1}^3 \cos k_i) \right).
\] (6)

Here, we use the following convention of the Gamma matrices \( \Gamma^1 = \sigma^3, \) \( \Gamma^2 = \sigma^10, \) \( \Gamma^3 = \sigma^22, \) \( \Gamma^4 = \sigma^11, \) \( \Gamma^5 = \sigma^23, \) and \( \sigma^{ab} = \sigma^a \otimes \sigma^b, \) with \( \sigma^0 = 1_{2 \times 2}. \) The anti-unitary time-reversal \( Z_2^T \) symmetry acts on the fermion operators as \( \mathcal{T} : c_k \to i \Gamma^5 c_k^\dagger, \) combined with a complex conjugation. This model is essentially two copies of the topological superconductors \( ^3\text{He-B} \) phase. Note that whether \( \mathcal{T}^2 \) is \( +1 \) or \( -1 \) no longer matters in this case as the sign of \( \mathcal{T}^2 \) can be changed by the \( U(1) \) rotation. In the literatures this \( U(1) \) symmetry is often referred to as the spin \( U(1) \) symmetry since it commutes with the time-reversal (for example, see Ref. [40]).

At the (for example) XY boundary, the system has a 2d gapless Dirac fermion with Hamiltonian

\[
H = \int d^2 x \, v \psi^\dagger (-i \sigma^3 \partial_x - i \sigma^1 \partial_y) \psi,
\] (7)

The time-reversal symmetry acts on the boundary two-component Dirac fermion as

\[
\mathcal{T} : \psi \to i \sigma^y \psi^\dagger.
\] (8)

It is well-known that without any interaction, the classification of this TI is \( Z_2^{30} \), while under ordinary local interactions, the classification of this TI is reduced to \( Z_2^{33} \), i.e. \( 8 \) copies of this TI will be rendered trivial under local interaction, or equivalently the edge state can be gapped by local interaction without developing nonzero expectation value of any fermion bilinear operator, which is forbidden by the \( U(1) \times Z_2^T \).

Now let us investigate what happens if the \( y \)-direction is a synthetic dimension, and consider multiple copies of the boundary Hamiltonian. Then the boundary Hamiltonian at each synthetic momentum \( p \) is

\[
H_p = \int dx \, \sum_{a=1}^N v_a \psi_a^\dagger (-i \sigma^3 \partial_x + \sigma^1 p) \psi_a + H_{\text{int}}(p),
\] (9)

where \( H_{\text{int}}(p) \) is a \( U(1) \times Z_2^T \) symmetry allowed flavor mixing interaction term that is parameterized by \( p \). Notice that the time-reversal symmetry does not mix fermion operators labelled by different momenta, i.e. time-reversal does not mix systems with different parameter \( p \). The entire Hamiltonian Eq. [9] can be viewed as \( N \)-copies of 1d interacting topological insulator with the same \( U(1) \times Z_2^T \) symmetry (which again has \( Z \) classification in the non-interacting limit), tuned close to its transition to the trivial insulator, which corresponds to
$N$ copies of 1$d$ Dirac fermions with Dirac mass $p$. The synthetic momentum $p$ is precisely the Dirac mass that tunes the system across the topological-trivial transition. Let us emphasize again that this does not apply to the ordinary interacting TI if all dimensions are physical dimensions, because different momenta will be mixed by the ordinary local interactions.

Now the problem readily reduces to the interaction effects on the 1$d$ TI with $U(1) \times Z_d^T$ symmetry, which has been studied and well understood. It was shown that the effects on the 1$d$ TI with $U(1) \times Z_d^T$ symmetry, though have $Z$ classification without interaction, is reduced to $Z_4$ classification under local interaction\cite{25,26}. This $Z$ to $Z_4$ reduction under interaction is also consistent with the classification of bosonic symmetry protected topological phases: two copies of the TIs under interaction can be adiabatically connected to the Haldane phase by gapping out the single particle excitations, and it is well-known that two copies of coupled Haldane phases become a trivial phase\cite{30,31}.

The observation above implies that when $N = 4$ in Eq. 4 the phase transition between the two limits $p > 0$ and $p < 0$ can be avoided by turning on interaction, i.e. the two regions in the phase diagrams can be adiabatically connected under interaction without closing the gap, and the original critical point $p = 0$ is rendered gapped and nondegenerate by interaction. This observation leads to the conclusion that the classification of the $D = 3$ TI with $U(1) \times Z_2^T$ symmetry is reduced to $Z_4$ instead of $Z_8$ if one of the three spatial dimensions is synthetic.

If we choose $D = 3$, $\delta = 2$, namely two out of the three directions are synthetic dimensions, and we consider a two dimensional boundary whose both directions are synthetic (let us label them as the $x$ and $y$ directions), then at each $p_x$ and $p_y$ the edge state is a two component complex fermion $\psi_p$. Again the problem at the boundary reduces to solving a zero dimensional system with two tuning parameter $p_x, p_y$. For $N = 2$ copies of the edge states, at every synthetic momentum $\vec{p}$, the free fermion part of the boundary Hamiltonian reads

$$H_B = v \psi_p^\dagger (\sigma^3 p_x + \sigma^1 p_y) \otimes \tau^z \psi_p,$$

where again we have performed a basis transformation for $\tau^z = -1$ component of the edge state. Then the same interaction as Eq. 3 $H_{int}(p) = J \vec{S}_{p,+} \cdot \vec{S}_{p,-}$ can gap out the entire boundary without leading to any ground state degeneracy (see Fig. 3). Hence when there are two synthetic dimensions, the $D = 3$ TI with $U(1) \times Z_2^T$ is reduced to a $Z_2$ classification.

To summarize this section, for a $D = 3$ interacting synthetic TI with $U(1) \times Z_2^T$ symmetry and different choices of $\delta = D - d$, its classification is

$$Z_8, \ (\delta = 0); \ Z_4, \ (\delta = 1); \ Z_2, \ (\delta = 2).$$

One can also discuss the simplest example of $D = 1$ TI with $U(1) \times Z_2^T$ symmetry. If the only dimension is synthetic, namely the lattice momentum along this dimension is actually a tuning parameter, although it is unnatural to discuss edge state of the synthetic dimension, the TI can still be defined as whether there must be a gap closing transition between the TI and the trivial insulator or not. In the non-interacting limit, near the critical point between the TI and trivial insulator, the bulk Hamiltonian takes exactly the same form as Eq. 3 and the time-reversal symmetry acts as $T : \psi_p \rightarrow i \sigma^y \psi_p$. The mass term that tunes the topological-to-trivial transition is $m \psi_p^\dagger \sigma^x \psi_p$. Then one can show that with two copies of the system, an interaction $H_{int}(p)$ similar to Eq. 3 would adiabatically connect the original topologically nontrivial TI to the trivial phase. Thus interaction reduces the classification of the synthetic $D = 1$ TI with $U(1) \times Z_2^T$ symmetry to $Z_2$.

### IV. SYNTHETIC TI WITH $D = 4$

Now we discuss the higher dimensional (total dimension $D = 4$) TI that cannot be realized in lab without using the “synthetic” techniques. First of all, the classification of the $D = 4$ quantum Hall insulator itself is not reduced by interaction, due to the chiral anomaly at its three dimensional boundary, and the anomaly matching condition\cite{32,33,34,35}. Thus we will consider the $D = 4$ non-chiral TI with a $U(1) \times Z_2$ symmetry. Like its $D = 2$ analogue, this TI is simply two copies of the original $D = 4$ quantum Hall state with opposite Chern numbers, and they have eigenvalues $+1$ and $-1$ respectively under the $Z_2$ symmetry operation.

Since this $D = 4$ TI was not discussed much in the past, we will first consider the case where all the dimensions are physical, i.e. $D = d = 4$. The 3$d$ boundary of this TI is two copies of Weyl fermions with opposite chiralities, and also opposite eigenvalues under the $Z_2$ symmetry:

$$H = \int d^3 x \ v \psi_L^\dagger (i \vec{\sigma} \cdot \vec{\partial}) \psi_L - v \psi_R^\dagger (i \vec{\sigma} \cdot \vec{\partial}) \psi_R. \quad (12)$$
The charge conservation $U(1)$ symmetry and the $Z_2$ symmetry act on the boundary fermions as $U(1): \psi_L, \psi_R \rightarrow e^{i\theta} \psi_L, e^{i\theta} \psi_R$, $Z_2: \psi_L \rightarrow \psi_L$, $\psi_R \rightarrow -\psi_R$. With this $U(1) \times Z_2$ symmetry action, this TI has a $Z$ classification without interaction, because any fermion bilinear mass term should either break the $Z_2$ symmetry (which corresponds to the ordinary Dirac mass term that mixes the left and right handed Weyl fermion), or break the $U(1)$ symmetry, which is a fermion pairing term (the so-called Majorana mass term).

To study the interaction effects on the TI, we follow the similar procedure of Ref. [35]. We first couple the left and right moving Majorana fermions both to a complex bosonic order parameter $\phi$:

$$\phi (\psi^T \sigma^y \psi_L + \psi_R^T \sigma^y \psi_R) + H.c. \quad (13)$$

When $\phi$ condenses, i.e. $\langle \phi \rangle \neq 0$, the charge $U(1)$ symmetry is spontaneously broken and the fermions are all gapped. Then we try to restore the $U(1)$ symmetry by proliferation of the vortex loops of the $U(1)$ order parameter $\phi$. Along the vortex loop, by directly solving the Dirac equation with a background vortex configuration of $\phi$, we will find there are non-chiral 1$d$ Majorana fermions described the effective Hamiltonian (Fig. [1])

$$H_{1d} = \int dx \, iv (\chi_L \partial_x \chi_L - \chi_R \partial_x \chi_R). \quad (14)$$

Again the left and right moving Majorana fermions carry eigenvalue $\pm 1$ under the $Z_2$ symmetry respectively.

The 1$d$ Hamiltonian Eq. (14) and its $Z_2$ symmetry are exactly the same as the 1$d$ edge states of 2$d$ $p_x \pm ip_y$ TSC, where the $p_x + ip_y$ and $p_x - ip_y$ superconductors carry eigenvalues $\pm 1$ respectively under the $Z_2$ symmetry. It is known that for eight copies of Eq. (14) a local interaction can gap out Eq. (14) without breaking the $Z_2$ symmetry [31,33,35] or ground state degeneracy. Thus the $U(1) \times Z_2$ symmetry can be restored for eight copies of the 3$d$ edge states Eq. (12) by proliferation of the fully gapped vortex loops. Hence the classification of the $D = 4$ TI described above is reduced from $Z$ to $Z_2$ under local interactions.

Now let’s consider the case with $D = 4$, $d = 3$, namely three out of the four dimensions are physical, while one dimension is synthetic. We will still look at the most natural boundary of the synthetic TI which is parallel to the synthetic dimensions. Then for $N$ copies of the system, at each fixed synthetic momentum $p$, the Hamiltonian for the boundary states reads

$$H_p = \int d^2x \sum_{a=1}^{N} \psi_{a,L}^\dagger (i \sigma^x \partial_x + i \sigma^y \partial_y - p \sigma^z) \psi_{a,L}$$

$$-\psi_{a,R}^\dagger (i \sigma^x \partial_x + i \sigma^y \partial_y - p \sigma^z) \psi_{a,R} + H_{int}(p). \quad (15)$$

The Hamiltonian Eq. (15) can be viewed as the previously discussed two dimensional non-chiral TI with a $U(1) \times Z_2$ symmetry tuned close to the topological-trivial transition point $p = 0$, and $p$ serves as the tuning parameter for this transition. As we have already mentioned before, the classification of this two dimensional non-chiral TI is reduced from $Z$ to $Z_4$ under interaction [32,34]. Hence for $N = 4$, the critical point $p = 0$ can be avoided under interaction without breaking the $Z_2$ symmetry, and the two regions with $p < 0$ and $p > 0$ can be smoothly connected without closing the gap. This implies that the classification of the $D = 4$ and $d = 3$ synthetic TI is reduced to $Z_2$ rather than $Z_8$ under the generic interaction of the synthetic TI.

If we instead choose $D = 4$, $d = 2$ in the first place, namely two out of the four dimensions are synthetic dimensions, then at every synthetic momentum $\bar{p} = (p_z, p_y)$, the Hamiltonian reads

$$H_p = \int d^2x \, \psi_{L}^\dagger (i \sigma^z \partial_x - p_y \sigma^y - p_z \sigma^z) \psi_{L}$$

$$-\psi_{R}^\dagger (i \sigma^z \partial_x - p_y \sigma^y - p_z \sigma^z) \psi_{R}. \quad (16)$$

Now at each $p_z$, the edge state can be viewed as the $D = 2$, $d = 1$ synthetic TI with $U(1) \times Z_2$ symmetry tuned close to the topological-trivial transition, while one of the dimension is a synthetic dimension $p_y$. $p_z$ is precisely the tuning parameter that tunes the two dimensional synthetic system across the transition. As we have argued before, the classification of this $D = 2$ synthetic TI is reduced to $Z_2$ under generic synthetic interaction. This implies that for two copies of Eq. (16) a properly designed $H_{int}(\bar{p})$ would gap out the edge system Eq. (16) without ground state degeneracy. Then the classification reduces from $Z$ to $Z_2$ when $\delta = d = 2$.

To summarize this section, for a $D = 4$ non-chiral interacting synthetic TI with $U(1) \times Z_2$ symmetry and different choices of $\delta = D - d$, its classification is

$$Z_2, (\delta = 0); \quad Z_4, (\delta = 1); \quad Z_2, (\delta = 2). \quad (17)$$

V. SYTHTHC TI IN GENERAL EVEN DIMENSIONS $D = 2n$

In general even spatial dimensions $D = 2n$, we can consider the generalized version of the non-chiral TI with $U(1) \times Z_2$ symmetry discussed in Sec. [11] and [14]. Such TIs in the non-interacting limit can always be constructed by putting the fermions with $Z_2$ eigenvalue $\pm 1$ into a $2n$-dimensional Chern insulator with a non-trivial Chern-Simons response, while simultaneously, putting fermions with $Z_2$ eigenvalue $-1$ also into a Chern insulator but with an opposite Chern-Simons response. In the non-interacting limit, the classification of this type of TI is given by $Z$.

Let us first consider the case with all of the dimensions physical, namely $D = 2n$ and $\delta = 0$. The boundary state of this non-chiral TI can be described by the Hamiltonian

$$H = \int d^{2n-1}x \sum_{k=1}^{2n-1} v (\psi_{L}^\dagger (i \gamma^k \partial_k) \psi_{L} - \psi_{R}^\dagger (i \gamma^k \partial_k) \psi_{R}). \quad (18)$$
where \( \gamma^k \) with \( k = 1, 2, ..., 2n - 1 \) are \( 2^{n-1} \)-dimensional matrices that generate the complex Clifford algebra. The products of all \( \gamma \) matrices \( \prod_{k=1}^{2^{n-1}} \gamma^k \) is a constant. The symmetries act on the boundary fermions as \( U(1) : \psi_L, \psi_R \rightarrow e^{i\theta} \psi_L, e^{i\theta} \psi_R, Z_2 : \psi_L \rightarrow -\psi_L, \psi_R \rightarrow -\psi_R \). The \( U(1) \times Z_2 \) symmetry forbids any fermion bilinear mass terms because all ordinary mass terms break the \( Z_2 \) symmetry. Therefore, this TI has a \( Z \) classification in the non-interacting limit.

In the following, we will show that the classification of \( U(1) \times Z_2 \) non-chiral TIs in \( D = 2n \) spatial dimensions with \( \delta = 0 \) reduces from \( Z \) to \( Z_{2n+1} \), when we consider interactions. To show this, we apply the method of fermion sigma model\(^{22}\). We first consider \( \nu = 2n \) copies of the boundary states described in Eq. \((18)\) rewrite them using Majorana fermions \( \chi \) and couple the Majorana fermions to a \( (2n+2) \)-component dynamical vector field \( n \) via the mass terms:

\[
H_{\chi\nu} = \int d^{2n-1}x \left( \sum_{k=1}^{2n-1} iv\alpha^k \partial_k + \sum_{a=1}^{2n+2} n^a \beta^a \right) \chi.
\]

Here, the set of matrices \( \{ \alpha^1, ..., \alpha^{2n-1}; i\beta^1, ..., i\beta^{2n+2} \} \) generates the real Clifford algebra \( C_{2n-1,2n+2} \) in its \( 2^{2n+1} \)-dimensional representation. All the \( \alpha^i \) are real symmetric matrices while the matrices \( \beta^a \) are imaginary anti-symmetric ones. The \( \alpha^i \) are real symmetric representations on the Majorana fermions \( \chi \) and \( \beta^a \) matrices square to the identity matrix. In this representation, the \( U(1) \) and \( Z_2 \) symmetry actions on the Majorana fermions \( \chi \) are generated by \( i\beta^1 \beta^2 \) and \( \prod_{k=1}^{2^{n-1}} \alpha^i \) respectively. The symmetry transformation of the vector field \( n \) is given by

\[
U(1) : \ (n_1 + in_2) \rightarrow e^{i\theta} (n_1 + in_2); \quad n_a \rightarrow n_a \quad \text{for} \quad a = 3, ..., 2n + 2.
\]

\[
Z_2 : \ n_a \rightarrow -n_a \quad \text{for} \quad \text{all} \ a.
\]

Assuming that the vector field \( n \) is a slowly varying field with unit modulus, we can integrate out the Majorana fermions \( \chi \) and obtain an effective action of \( n \) which can be identified as a \( O(2n+2) \) non-linear sigma model (NL\( \sigma \)M) in \( 2n - 1 \) spatial dimensions with a level-1 (or level-\(-1\)) Wess-Zumino-Witten (WZW) term\(^{23}\). Such an effective action can be identified as that of the boundary state of a bosonic symmetry protected topological (SPT) state with \( U(1) \times Z_2 \) symmetry. The bulk of such a bosonic SPT should be described by a \( O(2n+2) \) NL\( \sigma \)M in \( 2n \) spatial dimensions with a \( 2\pi \Theta \)-term. Now, we can conclude that \( \nu \) copies of the TI whose boundary state is described in Eq. \((18)\) can be adiabatically connected to this bosonic SPT state. This bosonic SPT has a \( Z_2 \) classification because we can consider two copies of the \( O(2n+2) \) NL\( \sigma \)Ms each with a \( 2\pi \Theta \)-term and couple them such that their \( n \) vector fields aligns in the first \( 2n+1 \) components and anti-aligns in the \( 2n+2 \)th component. The coupling effectively merges the two copies into a single \( O(2n+2) \) NL\( \sigma \)Ms which is free of a net \( \Theta \)-term and hence is topologically trivial. Therefore, we can conclude that \( 2\nu \) copies of the boundary state described in Eq. \((18)\) together are topologically equivalent to the boundary state of a trivial state. The classification of the \( U(1) \times Z_2 \) non-chiral TI is hence reduced from \( Z \) in the non-interacting limit to \( Z_{2\nu} = Z_{2n+1} \) in \( D = 2n \) spatial dimensions (with \( \delta = 0 \)) when we take interactions into account.

Now, we consider the case with one of the spatial dimension synthetic, i.e., \( D = 2n, d = 2n - 1 \) and \( \delta = 1 \). We can start with the boundary state Hamiltonian:

\[
H_p = \int d^{2n-2}x \left\{ \psi_L^\dagger \left( \sum_{k=1}^{2n-2} iv\gamma^k \partial_k - p\gamma^{2n-1} \right) \psi_L - \psi_R^\dagger \left( \sum_{k=1}^{2n-2} iv\gamma^k \partial_k - p\gamma^{2n-1} \right) \psi_R \right\}, \tag{21}
\]

where \( p \) represents the synthetic momentum along the synthetic dimension. We notice that this boundary Hamiltonian Eq. \((21)\) can be identified as the bulk Hamiltonian of the \( U(1) \times Z_2 \) symmetric non-chiral TI in \( D' = d' = 2n - 2 \) spatial dimensions tuned to the vicinity of a topological-trivial transition point at \( p = 0 \). As we just discussed, the classification of the \( D' = d' = 2n - 2 \) non-chiral TI is reduced from \( Z \) to \( Z_{2n} \) under interaction. That means when we consider \( 2\nu \) copies of the model Eq. \((21)\) the critical point at \( p = 0 \) can be avoided under interactions without breaking the \( U(1) \times Z_2 \) symmetry. In another word, the classification of the \( U(1) \times Z_2 \) symmetric non-chiral TI in \( D = 2n \) dimensions with \( \delta = 1 \) is reduced to \( Z_{2\nu} \).

For \( \delta \geq 1 \), we can always perform a similar analysis by studying the boundary state Hamiltonians that are counterparts of Eq. \((18)\) and Eq. \((21)\). Interestingly, such boundary state Hamiltonians can always be identified as the bulk Hamiltonians of the \( U(1) \times Z_2 \) symmetric non-chiral TI in \( D' = 2n - 2 \) dimensions with \( \delta' = \delta - 1 \) near the topological-trivial critical point. This identification is done by choosing one of the synthetic momenta as the tuning parameter for the topological-trivial transition while leaving the other \( \delta' = \delta - 1 \) synthetic momenta still representing the synthetic dimensions among the total \( D' = 2n - 2 \) dimensions. With this identification, we can directly conclude that, for general \( \delta \), the classification of the \( U(1) \times Z_2 \) symmetric non-chiral TI becomes \( Z_{2\nu+1-\delta} \).

To summarize this section, for a \( 2n \)-dimensional non-chiral interacting synthetic TI with \( U(1) \times Z_2 \) symmetry, the classification, in the presence of \( \delta \) synthetic dimensions, is given by \( Z_{2\nu+1-\delta} \). This result is consistent with the analysis in Sec. \([\text{II}]\) and \([\text{IV}]\).

VI. SYNTHECTIC TI IN GENERAL ODD DIMENSIONS \( D = 2n + 1 \)

In general odd spatial dimensions \( D = 2n + 1 \), we can consider the generalized version of the TI with \( U(1) \times Z_2^\nu \)
symmetry discussed in Sec. III. In the non-interacting limit, such systems belong to class AIII in the “10-fold way” classification and have a classification of $\mathbb{Z}$.

Let’s first consider the case with all of the dimensions physical, namely $D = d = 2n + 1$ and $\delta = 0$. The boundary state of the $U(1) \times \mathbb{Z}_2^T$ symmetric TI that can generate the whole $\mathbb{Z}$ class can be described by the Hamiltonian

$$H = \int d^{2n}x \sum_{k=1}^{2n} v_k \psi \dagger (i \gamma^k \partial_k) \psi,$$  \hspace{1cm} (22)

where $\{ \gamma^k \}_{k=1}^{2n}$ are a set of $2^n$-dimensional complex matrices that forms the complex Clifford algebra. The symmetries act on the boundary fermions as $U(1) : \psi \rightarrow e^{i\theta} \psi$, $\mathbb{Z}_2^T : \psi \rightarrow (\prod_{k=1}^{2n} \gamma^k) \psi \dagger$. One can check that the $U(1) \times \mathbb{Z}_2^T$ symmetry forbids any fermion bilinear mass terms even when there are multiple copies of such boundary states. Therefore, this TI has a $\mathbb{Z}$ classification in the non-interacting limit.

In the following, we will show that the classification of $U(1) \times \mathbb{Z}_2^T$ TI in $D = 2n + 1$ spatial dimensions with $\delta = 0$ reduces from $\mathbb{Z}$ to $\mathbb{Z}_{2n+2}$ when we take interactions into account. To show this, we will again use the method of fermion sigma model. The following discussion will be largely parallel to Sec. V. We first consider $\nu' = 2^{n+1}$ copies of the boundary states described in Eq. 22, rewrite them using Majorana fermions $\chi$ and couple the Majorana fermions to a $(2n + 3)$-component dynamical vector field $n_a$ via the mass terms

$$H_{\chi \nu'} = \int d^{2n}x \chi^T \left( \sum_{k=1}^{2n} i \alpha^k \partial_k + \sum_{a=1}^{2n+3} n_a \beta^a \right) \chi.$$  \hspace{1cm} (23)

Here, the set of matrices $\{ \alpha^1, ..., \alpha^{2n}; \beta^1, ..., \beta^{2n+3} \}$ generates the real Clifford algebra $\mathbb{C}l_{2n,2n+3}$ in its $2^{2n+2}$-dimensional representation. In this representation, the $U(1)$ symmetry action on the Majorana fermions is generated by $i \beta^1 \beta^2$. The anti-unitary time-reversal symmetry acts as $Z_2^T : \chi \rightarrow (\prod_{k=1}^{2n} \gamma^k) \chi$, combined with a complex conjugation. The symmetry transformation of the vector field $n_a$ is given by

$$U(1) : \begin{cases} (n_1 + in_2) \rightarrow e^{i\theta} (n_1 + in_2); \\ n_a \rightarrow n_a \quad \text{for } a = 3, ..., 2n + 3. \end{cases} \hspace{1cm} (24)$$

As we discussed in Sec. V when the vector field $n_a$ is slowly varying and has a unit modulus, we can integrate out the Majorana fermion $\chi$ and obtain a $O(2n + 3)$ NLSM with a level-1 (or level$-1$) WZW term. Here, the action of $n_a \rightarrow -n_a$ in $O(2n + 3)$ should be identified as the time-reversal symmetry $Z_2^T$. Such an effective action in fact describes the boundary state of a $U(1) \times \mathbb{Z}_2^T$ symmetric bosonic SPT which itself has a $\mathbb{Z}_2$ classification. Therefore, by the same argument given in Sec. V the classification of the $U(1) \times \mathbb{Z}_2^T$ symmetric TI is reduced from $\mathbb{Z}$ in the non-interacting limit to $\mathbb{Z}_{2n'} = \mathbb{Z}_{2n+2}$ in $D = 2n + 1$ spatial dimensions (with $\delta = 0$) when we take interactions into account.

When synthetic dimensions are present, i.e. $\delta \geq 1$, we can identify one of the synthetic momenta as the tuning parameter of a topological-trivial transition, the boundary state Hamiltonian can then be identified as the bulk Hamiltonian of a $U(1) \times \mathbb{Z}_2^T$ TI in the vicinity of the topological-trivial critical point in $D' = 2n - 1$ spatial dimensions and with $\delta' = \delta - 1$ synthetic dimensions. With this identification, we can conclude that the classification of $U(1) \times \mathbb{Z}_2^T$ symmetric interacting TI is given $\mathbb{Z}_{2n+2-\delta}$. To summarize this section, for a $2n + 1$-dimensional interacting synthetic TI with $U(1) \times \mathbb{Z}_2^T$ symmetry, the classification, in the presence of $\delta$ synthetic dimensions, is given by $\mathbb{Z}_{2n+2-\delta}$. This result is consistent with the analysis in Sec. III.

VII. SUMMARY

In this work we have analyzed the interaction effects on the effective TIs simulated with the newly developed synthetic techniques. We demonstrate that unlike ordinary interacting TIs, the interaction causes different classification reduction of the simulated TI, due to the generic while special form of the interaction in systems with synthetic dimensions. We need to point out that the analysis used the fact that the system at every synthetic momentum $\vec{p}$ is a lower dimensional system with the same symmetry as the desired effective $D$-dimensional TI. This analysis no longer naturally applies (although not impossible) if the system involves time-reversal symmetry that does not commute with the charge $U(1)$ symmetry, because in this case time-reversal symmetry would bring $c_k$ to $-c_k$, hence it mixes systems labelled by different parameter $\vec{p}$.

Topological semimetals have also attracted enormous research interests and efforts in the last few years. One can also construct semimetals in lab using the same synthetic techniques, and these semimetals could be vulnerable to interactions, i.e. the bulk of the system can be driven into an insulator with nondegenerate ground state due to interaction. For example, the long wave-length effective Hamiltonians Eq. 7 and Eq. 13 can also be viewed as the bulk Hamiltonian expanded near the gapless momenta of synthetic semimetals with $D = 2$ and $D = 3$, again $p$ is the tuning parameter which is viewed as an extra synthetic momentum. And our previous analysis indicates that for $N = 4$, both cases can be gapped out by interaction $H_{int}(p)$ without leading to ground state degeneracy, and the Dirac semimetal becomes an insulator with nondegenerate ground state.

It has also been shown that under strong interaction the boundary of a nontrivial 3d TI can be a 2d topological order with anomalous quantum number fractionalization. As we explained before, at each parameter $p$,
the edge Hamiltonian Eq. 15 describes an interacting 2d system at the boundary of a $D = 4$ system. Thus under strong interaction it is possible that at each parameter (effective momentum) $p$ the edge is driven into a topological order. This situation could correspond to a very exotic “topological order” at the boundary of the $D = 4$ TI. We will leave this topic for future studies.

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