HARDY-LITTLEWOOD-SOBOLEV INEQUALITIES ON COMPACT RIEMANNIAN MANIFOLDS AND APPLICATIONS

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Abstract. In this paper we extend Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds for dimension $n \neq 2$. As one application, we solve a generalized Yamabe problem on locally conformally flat manifolds via a new designed energy functional and a new variational approach. Even for the classic Yamabe problem on locally conformally flat manifolds, our approach provides a new and relatively simpler solution.

1. Introduction

Curvature equations involving high order derivatives (including $Q$–curvature equations) and fully nonlinear curvature equations (such as $\sigma_k$ operators of Schouten tensor) have been extensively studied in the past decade, and have broad applications in the study of global geometry and topology. See, e.g. [4], [14], [1], [28], [13], [15], [7] and references therein. All these differential operators, such as Paneitz operators with even powers and $\sigma_k$ operators of Schouten tensor, are introduced as a locally defined operators.

Recently, there have been some interesting results concerning the fractional Yamabe problem, as well as the fractional prescribing curvature problem, see, e.g. [12], [10], [11], [22]–[25] and references therein. In these studies the notion for the globally defined fractional Paneitz operator $P_\alpha$ (via an integral operator), which is introduced in [12], is used and has a direct link to singular integral operators (see Caffarelli and Silvestre [3] for a new viewpoint of fractional Laplacian operator).

Motivated by the globally defined fractional Paneitz operator, as well as the study of sharp Sobolev inequality with negative power by W. Chen, et al [5], Yang and Zhu [31], Hang and Yang [18], Ni and Zhu [31]–[33], Hang [16], etc. we started to investigate the general extension of Hardy-Littlewood-Sobolev (HLS) inequality. In Dou and Zhu [8], we established the HLS inequality on the upper half space, and outline the rough idea on the extension of HLS on general manifolds; In Dou and Zhu [9], a surprising reversed HLS inequality was obtained when the differential order is higher than the dimension. In Zhu [35], a more general prescribing curvature equation on $\mathbb{S}^n$ was introduced and the existence result for antipodally symmetric function was obtained; in particular, the reversed HLS inequality was first used in the study of curvature equations with negative critical Sobolev exponents. In the same paper, a more general Yamabe type problem was also introduced for general compact Riemannian manifolds. In this paper we shall extend the classic HLS inequality as well as the reversed HLS inequality on compact Riemannian manifolds and provide solution to the general Yamabe problem on locally conformally flat manifolds.
Proposition 1.1. In a conformal normal coordinates centered at \( x \in M \), introduce the following integral operator:

\[
I_\alpha f(x) = \int_{M^n} \frac{f(y)}{|x - y|^\alpha} dV_y,
\]

We first have the following HLS inequality on \((M^n, g)\) for \( \alpha < n \):

**Proposition 1.1.** Assume that \( \alpha \in (0, n) \), \( 1 < p < \frac{n}{\alpha} \) and \( q \) is given by

\[
\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n},
\]

then there is an optimal positive constant \( C(\alpha, p, M^n, g) \), such that

\[
||I_\alpha f||_{L^q(M^n)} \leq C(\alpha, p, M^n, g)||f||_{L^p(M^n)}.
\]

holds for all \( f \in L^p(M^n) \). Moreover, for \( 1 \leq r < q \), operator \( I_\alpha : L^p(M^n) \rightarrow L^r(M^n) \) is a compact embedding.

Proposition 1.1 seems to be a known fact. Since we cannot find the proof in literatures, we will outline the proof in this paper.

For \( \alpha > n \), we have the following reversed HLS inequality for nonnegative functions.

**Theorem 1.2.** Assume that \( \alpha > n \geq 1 \), \( 1 > p > \frac{n}{\alpha} \) and \( q \) is given by \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \), then there is an optimal positive constant \( C(\alpha, p, M^n, g) \), such that

\[
||I_\alpha f||_{L^q(M^n)} \geq C(\alpha, p, M^n, g)||f||_{L^p(M^n)}.
\]

holds for all nonnegative \( f \in L^p(M^n) \).

One of the main motivations for obtaining the above embedding theorems comes from the study of curvature equations, including the following generalized Yamabe problem, introduced in Zhu [35]:

For a given compact Riemannian manifold \((M^n, g_0)\) \((n \neq 2)\) with positive scalar curvature and a positive parameter \( \alpha \neq n + 2k \) for \( k = 0, 1, \ldots \), let \( G^0_x(y) = n(n - 2)\omega_n \Gamma^{0y}_{x}(y) \), where \( \Gamma^{0y}_{x}(y) \) is the Green’s function with pole at \( x \) for the conformal Laplacian operator \(-\Delta_{g_0} + \frac{n-2}{n-2} R_{g_0} \), \( \omega_n \) is the volume of the unit ball. In a conformal normal coordinates centered at \( x \), \( G^0_x(y) = |y|^{2-n} + A + O(|y|) \). The \( \alpha \)-curvature \( Q_{\alpha, g} \) under the conformal metric \( g = \phi^{-4/(n-\alpha)} g_0 \) is defined as a function implicitly given by

\[
u(x) = \int_{M^n} |G^0_x(y)|^{\frac{4}{n-\alpha}} Q_{\alpha, g}(y) u^{\frac{n-\alpha}{4}}(y) dV_{g_0}.
\]

It is clear that \( Q_{\alpha, g} \), up to a constant multiplier, is the classic scalar curvature for \( \alpha = 2 \).

Let

\[
I_{M^n, g, \alpha}(f) = \int_{M^n} |G^0_x(y)|^{\frac{4}{n-\alpha}} f(y) dV_g.
\]

It was showed in [35] that \( I_{M^n, g, \alpha}(f) \) has the conformal covariance property. Similar to the Yamabe problem, one may ask [35]: for a given compact Riemannian manifold \((M^n, g_0)\), is there a conformal metric \( g = u^{4/(n-\alpha)} g_0 \) such that \( Q_{\alpha, g} = \text{constant} \)?
We shall solve this problem on any locally conformally flat manifold with positive scalar curvature, based on the positive mass theorem.

Theorem 1.3. For a given compact locally conformally flat manifold \((M^n, g)\) \((n \neq 2)\) with positive scalar curvature, there always exists a conformal metric \(g_\alpha = u^{4/(n-\alpha)}g\) such that \(\alpha\)– curvature \(Q_{\alpha, g_\alpha}\) is a constant.

From now on in this paper, we always assume the compact manifold \((M^n, g)\) under consideration has positive scalar curvature.

The traditional approach to solve the classic Yamabe problem is to seek the minimizer to the Sobolev quotient energy:

\[
J_2(u) = \frac{\int_{M^n} |\nabla u|^2 + \frac{n-2}{4(n-1)} R_{g_0} u^2 dV_{g_0}}{\|u\|_{L^{2n/(n-2)}(M^n)}}.
\]

Unfortunately, for fractional order \(\alpha\), such an energy functional is hard to find. To prove the above theorem, we design the following energy functional for positive functions:

\[
J_{g, \alpha}(u) := \frac{\int_{M^n} \int_{M^n} u(x)u(y)[G^g_\alpha(y)]^{n\alpha} dV_g(x)dV_g(y)}{\|u\|_{L^{2n/(n+\alpha)}(M^n)}}.
\]

The above functional was successfully used in [35] to solve a prescribing curvature problem on \(\mathbb{S}^n\) with negative exponent (in the case of \(\alpha > n\)). In this paper, we will show that it can also be used to solve Yamabe type problems.

For \(\alpha < n\), we consider the supremum

\[
Y_{\alpha}(M^n, g) := \sup_{u \in C^0(M^n) \setminus \{0\}, u \geq 0} J_{g, \alpha}(u).
\]

Similar to the proof of Proposition 1.1, one can show that \(Y_{\alpha}(M^n, g) < \infty\) (see remark 2.2 below). Moreover, it follows from Lieb’s classic result [29], that the supremum on the standard sphere \((\mathbb{S}^n, g_0)\) or on flat plane \((\mathbb{R}^n, g_E)\) is given by

\[
Y_{\alpha}(\mathbb{S}^n, g_0) = \pi ^{\frac{n}{2\alpha}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)} \left\{ \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma(n)} \right\} ^{-\frac{n}{2\alpha}}
\]

and the corresponding extremal functions on \((\mathbb{R}^n, g_E)\) are \(f(x) = (1 + |x|^2)^{-\frac{n}{2\alpha}}\) and its conformal equivalent class:

\[
f_{\epsilon, x_0}(x) = \epsilon^{-\frac{n+\alpha}{2}} f\left(\frac{x-x_0}{\epsilon}\right) = \epsilon^{-\frac{\alpha}{2} - \frac{n}{2}} \left( \frac{\epsilon}{\epsilon^2 + |x-x_0|^2} \right)^{\frac{n+\alpha}{2}}
\]
problem. We recently learned from Hang and Yang that such approach was also used in their recent work \[19\] for $Q$-curvature problem ($\alpha = 4$ in their case).

Parallel to the case of $\alpha < n$, for $\alpha > n$, we consider the infimum

$$Y_\alpha(M^n, g) := \inf_{u \in C^0(M^n)} \{ J_{g,\alpha}(u) \}. \quad (1.10)$$

It follows from Theorem 1.2 that $Y_\alpha(M^n, g) > 0$. And, it follows from the sharp reversed HLS inequality \[9\] that the infimum on the standard sphere or flat plane is given by \(1.8\) and the corresponding extremal functions on \((\mathbb{R}^n, g_E)\) are given by \(1.9\). Again, we will first show that $Y_\alpha(M^n, g) \leq Y_\alpha(\mathbb{S}^n, g_0)$, and for a locally conformally flat manifold $(M^n, g)$ with positive scalar curvature, equality holds iff $(M^n, g)$ is conformally equivalent to $(\mathbb{S}^n, g_0)$. We then show that the strict inequality yields the existence of the minimizer through a new blowup analysis. It is interesting to point out that the local blowup analysis does not work due to the lack of local Sobolev inequality for $\alpha > n$.

This paper is organized as follows. In Section 2, we deal with the case of $\alpha < n$. Based on the Marcinkiewicz interpolation theorem, we prove the roughly HLS inequality \(1.2\) on $(M^n, g)$ and the compactness of embedding for subcritical exponent. We then establish an $\epsilon$-level sharp HLS inequality on any general compact manifold and complete the proof of Theorem 1.3 for $\alpha < n$. In Section 3, we deal with the case of $\alpha > n$. The analog $\epsilon$-level inequality is not known. Instead, a new blow up analysis enables us to show that there is at most one blow up point for a minimizing sequence. Energy condition will be used to eliminate the case of single blow up point for the manifold not conformally equivalent to the standard sphere $(\mathbb{S}^n, g_0)$.

2. Case of $\alpha < n$

In this section, we first prove Proposition 1.1. We then analyze the sharp constant and derive Aubin type $\epsilon$-level sharp HLS inequality. Using such a sharp inequality, we finally prove Theorem 1.3 for $\alpha < n$.

2.1. Roughly HLS inequality on Manifolds. To prove Proposition 1.1 we need the following Young’s inequality on manifolds.

**Lemma 2.1.** For a given compact manifold $(M^n, g)$, define

$$g * h(x) = \int_{M^n} g(y)h(|y - x|_g)dV_y.$$  

There is a constant $C > 0$, such that

$$||g * h||_{L^r} \leq C||g||_{L^p} \cdot ||h||_{L^p},$$

where $p, q, r \in (1, \infty)$ and satisfy

$$1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}.$$  

The proof is similar to the classic Young inequality in $\mathbb{R}^n$. See, e.g. Lieb and Loss \[30\]. It is worthy of pointing out that $g * h(x)$ may not equal to $h * g(x)$ for $x \in M^n$.  


Proof of Proposition 1.1 The proof is quite standard. Similar proof appeared, e.g. in Hang, Yan and Wang [17] (proof of Proposition 2.1 there). To prove (1.2), we only need to show that there is a constant $C > 0$, such that for any $\lambda > 0$,

$$m\{x \in M^n: |I_\alpha f| > \lambda\} \leq C\left(\frac{\|f\|_{L^p}}{\lambda}\right)^q. \quad (2.1)$$

Inequality (2.1) follows from the above inequality via the classical Marcinkiewicz interpolation theorem.

For any $\gamma > 0$, define

$$I^1_\alpha f(x) = \int_{|y-x| \leq \gamma} \frac{f(y)}{|y-x|^{n-\alpha}} dy,$$

and

$$I^2_\alpha f(x) = \int_{|y-x| > \gamma} \frac{f(y)}{|y-x|^{n-\alpha}} dy.$$

Thus, for any $\tau > 0$,

$$m\{x : I_\alpha f(x) > 2\tau\} \leq m\{x : I^1_\alpha f(x) > \tau\} + m\{x : I^2_\alpha f(x) > \tau\}. \quad (2.2)$$

We note that it suffices to prove inequality (2.1) with $2\tau$ in place of $\tau$ in the left side of the inequality, and we can further assume $\|f\|_{L^p} = 1$.

From Young inequality (Lemma 2.1), we have

$$\|I^1_\alpha f\|_{L^p} \leq C \int_{|y| \leq \gamma} \frac{1}{|y|^{n-\alpha}} dV_y \cdot \|f\|_{L^p} = C\gamma^\alpha.$$

Thus

$$m\{x : I^1_\alpha f(x) > \tau\} \leq \frac{\|I^1_\alpha f\|_{L^p}}{\tau^p} \leq C\gamma^\alpha \cdot \tau^{-p}.$$

On the other hand, Young inequality implies

$$\|I^2_\alpha f\|_{L^\infty} \leq C \left( \int_{|y| > \gamma} \frac{1}{|y|^{n-\alpha}} dV_y \right)^{1/p'} \cdot \|f\|_{L^p} = C_1 \gamma^{-n/q}.\tag{2.1}$$

Choose $\gamma$ so that $C_1 \gamma^{-n/q} = \tau$. Then $m\{x : I^2_\alpha f(x) > \tau\} = 0$, and

$$m\{x : I^2_\alpha f(x) > \tau\} \leq C\gamma^\alpha \cdot \tau^{-p} = C_2 \tau^{-q}.$$

(2.1) follows from the above easily.

For any $r \in (1, q)$, we will show the embedding is a compact. This shall be a known fact since the compact embedding is a local property, and for a bounded domain $\Omega \subset \mathbb{R}^n$, $L^r(\Omega) \subset \subset W^{\alpha,p}(\Omega)$ is compact, see, for example, [9]. We only outline the proof here.

Let $(\Omega_i, \phi_i)_{i=1}^N$ be a finite covering of $M^n$, with each $\Omega_i$ being homeomorphic to the unite ball $B_i(0)$ in $\mathbb{R}^n$. Let $\{\alpha_i\}_{i=1}^N$ be a $C^\infty$ partition of unity subordinate to the covering $\{\Omega_i\}_{i=1}^N$.

Let $\{f_m\}_{m=1}^\infty$ be a bounded sequence in $L^p(M^n)$, then for each fixed $i = 1, 2, \cdots, N$, there exists a subsequence $\{\alpha_i I_\alpha f_{m_j}\}$ which is precompact in $L^r(\Omega_i)$ due to the compact embedding result on bounded domain in $\mathbb{R}^n$.

Choosing a diagonal subsequence $\{I_\alpha f_{m_j}\}$, such that $\alpha_i I_\alpha f_{m_j}$ is precompact in $L^r(\Omega_i)$ for all $i = 1, \cdots, N$, we then know that $\{I_\alpha f_{m_j}\}$ is precompact in $L^r(M^n)$, following from Minkowski inequality

$$\|I_\alpha f_{m_j} - I_\alpha f_{m_i}\|_{L^r(M^n)} \leq \sum_{i=1}^N \|\alpha_i I_\alpha f_{m_j} - \alpha_i I_\alpha f_{m_i}\|_{L^r(M^n)} \to 0.$$
We hereby complete the proof of Proposition 1.1.

**Remark 2.2.** It is quite clear that a similar argument to the above leads to: for \( q \) satisfying (1.1), there is a positive constant \( C > 0 \), such that

\[
||I_{M^n,g,\alpha}f||_{L^q(M^n)} \leq C||f||_{L^p(M^n)}.
\] 

(2.3) holds for all \( f \in L^p(M^n) \). Moreover, for \( 1 \leq r < q \), operator \( I_{M^n,g,\alpha} : L^p(M^n) \to L^r(M^n) \) is a compact embedding.

2.2. Sharp constant and the generalized Yamabe problem.

2.2.1. Best constant. We first give a lower bound estimate for the optimal constant \( Y_\alpha(M^n, g) \).

**Proposition 2.3.**

\[ \xi_\alpha \geq Y_\alpha(S^n, g_0), \]

where

\[ \xi_\alpha := \sup_{f \in C^0(M^n) \setminus \{0\}} \frac{\int_{M^n \times M^n} f(x)f(y)|x-y|^{-n} dV_g(x)dV_g(y)}{||f||_{L^{2n/(n+\alpha)}(M^n)}^2}. \]

**Proof.** For small positive constant \( \lambda > 0 \), recall that \( f_\lambda(x) \) is given in (1.9). Take

\[ \tilde{f} = \begin{cases} f_\lambda(x), & \text{in } B_\delta(0), \\ 0, & \text{in } \mathbb{R}^n \setminus B_\delta(0), \end{cases} \]

where \( \delta > 0 \) is a fixed constant to be determined later. Then, for small enough \( \lambda \), \( \tilde{f} \in L^{2n/(n+\alpha)}(\mathbb{R}^n) \) and

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} \tilde{f}(x)\tilde{f}(y)|x-y|^{-n} dxdy \\
= \int_{\mathbb{R}^n \times \mathbb{R}^n} f_\lambda(x)f_\lambda(y)|x-y|^{-n} dxdy \\
- 2 \int_{\mathbb{R}^n \times (\mathbb{R}^n \setminus B_\delta(0))} f_\lambda(x)f_\lambda(y)|x-y|^{-n} dxdy \\
+ \int_{(\mathbb{R}^n \setminus B_\delta(0)) \times (\mathbb{R}^n \setminus B_\delta(0))} f_\lambda(x)f_\lambda(y)|x-y|^{-n} dxdy \\
= Y_\alpha(S^n, g_0)||f_\lambda||_{L^{2n/(n+\alpha)}(\mathbb{R}^n)}^2 - I + II, \]

(2.4)

where

\[ I = 2 \int_{\mathbb{R}^n \times (\mathbb{R}^n \setminus B_\delta(0))} f_\lambda(x)f_\lambda(y)|x-y|^{-n} dxdy, \]

\[ II = \int_{(\mathbb{R}^n \setminus B_\delta(0)) \times (\mathbb{R}^n \setminus B_\delta(0))} f_\lambda(x)f_\lambda(y)|x-y|^{-n} dxdy. \]

Note (see, e.g. [29] or [27])

\[
\int_{\mathbb{R}^n} f_\lambda(x)|x-y|^{-n} dx = Bf_{\lambda}^{\frac{n-\alpha}{n}}(y),
\]

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On the other hand, from HLS inequality, we know that $II$ can be estimated as

$$I = C \int_{\mathbb{R}^n \setminus B(0)} |f|^{2/\nu} \, dx$$

$$= C \int_{\delta}^{+\infty} \left( \frac{\lambda}{\lambda^2 + r^2} \right)^{n-1} \, dr$$

$$= C \int_{\delta}^{+\infty} (1 + t^2)^{-n/2} \, dt = O\left( \frac{\delta}{\lambda} \right)^{-n}, \quad \text{as } \lambda \to 0.$$  

On the other hand, from HLS inequality, we know that $II$ can be estimated as

$$II \leq Y_\alpha(S^n, g_0) \|f\|_{L^{2n/(n+\alpha)}(\mathbb{R}^n \setminus B(0))} \leq C \left( \frac{\delta}{\lambda} \right)^{-n}.$$

So, for small enough $\lambda$,

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \delta(x) \delta(y) |x - y|^{\alpha-n} \, dxdy \geq Y_\alpha(S^n, g_0) - C \left( \frac{\delta}{\lambda} \right)^{-n}. \quad (2.5)$$

For any given point $P \in M^n$, choose a neighbourhood $\Omega_P \subset M^n$ so that for $\delta > 0$ small enough, in a normal coordinate, $exp(B(\delta)) \subset \Omega_P$ and

$$(1 - \epsilon)I \leq g(x) \leq (1 + \epsilon)I, \quad \forall x \in B(\delta).$$

Thus,

$$(1 - \epsilon)|x - y| \leq |x - y|_g \leq (1 + \epsilon)|x - y|, \quad \forall x, y \in B(\delta).$$

In the normal coordinates with respect to the center $P \in M^n$, let

$$v(x) = \begin{cases} f_\lambda(\exp^{-1}(x)), & \text{in } exp(B(\delta)) \\ 0, & \text{in } M^n \setminus exp(B(\delta)). \end{cases}$$

Then

$$\int_{M^n} |v|^{2n/(n+\alpha)} \, dV \leq (1 + \epsilon)^\frac{\alpha}{2} \int_{B(\delta)} |f_\lambda(x)|^{2n/(n+\alpha)} \, dx,$$

$$\int_{M^n} \int_{M^n} v(x)v(y) |x - y|^{\alpha-n} \, dV_x dV_y = \int_{B(\delta)} \int_{B(\delta)} v(x)v(y) \sqrt{\det g(x) \det g(y)} \, dxdy$$

$$\geq \int_{B(\delta)} \int_{B(\delta)} \frac{f_\lambda(x)f_\lambda(y)}{(1 + \epsilon)^{n-\alpha}} |x - y|^{\alpha-n} \, dxdy$$

$$= \frac{(1 - \epsilon)^n}{(1 + \epsilon)^{n-\alpha}} \int_{B(\delta)} \int_{B(\delta)} f_\lambda(x)f_\lambda(y) \frac{|x - y|^{\alpha-n}}{|x - y|^{n-\alpha}} \, dxdy. \quad (2.6)$$

Thus

$$\xi_\alpha \geq \frac{1}{(1 + \epsilon)^{n-\alpha}} \int_{B(\delta)} \int_{B(\delta)} f_\lambda(x)f_\lambda(y) |x - y|^{\alpha-n} \, dxdy$$

$$\geq \frac{(1 - \epsilon)^n}{(1 + \epsilon)^{n-\alpha}} \int_{B(\delta)} \int_{B(\delta)} f_\lambda(x)f_\lambda(y) |x - y|^{\alpha-n} \, dxdy$$

$$\geq \frac{(1 - \epsilon)^{n-1}}{(1 + \epsilon)^{\frac{n}{2} + \alpha}} \left( Y_\alpha(S^n, g_0) - C \left( \frac{\delta}{\lambda} \right)^{-n} \right).$$

Sending $\epsilon$ and $\lambda$ to 0, we obtain the estimate. 

With a slight modification of the above proof, we have
Corollary 2.4. For $\alpha < n$,

$$Y_\alpha(M^n, g) \geq Y_\alpha(S^n, g_0).$$

Similar to Aubin’s approach for solving Yamabe problem, we will establish an $\epsilon$-level sharp Hardy-Littlewood-Sobolev inequality for solving general curvature equations.

For $\alpha \in (0, n)$, $p > 1$ and $q$ satisfying (1.1), define

$$N_{\alpha, p} = \sup_{f \in L^p(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} f(x) f(y) |x - y|^{\alpha - n} dxdy}{\|f\|_{L^p(\mathbb{R}^n)}^2}.$$

Proposition 2.5 ($\epsilon$-Level Inequality). For $\alpha \in (0, n)$, $p > 1$, let $q$ be given by (1.1). For any given $\epsilon > 0$, there is a constant $C(\epsilon) > 0$, such that

$$\|I_\alpha f\|_{L^q(M^n)}^p \leq (N_{\alpha, p} + \epsilon) \|f\|_{L^p(M^n)}^p + C(\epsilon) \|I_{\alpha + 1} f\|_{L^q(M^n)}^p \quad (2.7)$$

holds for all $f \in L^p(M^n)$.

Proof. We only need to prove (2.7) for nonnegative function $f \in C(M^n)$.

For fixed $\epsilon > 0$, let $\{\eta_i, \epsilon\}_{i=1}^k$ be a partition of the unit covering, such that $0 \leq \eta_i, \epsilon \leq 1$ for all $i = 1, \ldots, k$ and $\sum_{i=1}^k \eta_i, \epsilon = 1$, and for all $i = 1, \ldots, k$,

$$\|I_\alpha (\eta_i, \epsilon f)\|_{L^q(\text{supp}(\eta_i, \epsilon))} \leq (N_{\alpha, p} + \epsilon) \|\eta_i, \epsilon f\|_{L^p(\text{supp}(\eta_i, \epsilon))}. \quad (2.8)$$

Thus

$$\|I_\alpha f\|_{L^q(M^n)}^p = \|(I_\alpha f)^p\|_{L^{q/p}(M^n)}$$

$$= \left( \sum_{i=1}^k \eta_i, \epsilon (I_\alpha f)^p \right)_{L^{q/p}(M^n)} \leq \sum_{i=1}^k \|\eta_i, \epsilon (I_\alpha f)^p\|_{L^{q/p}(\text{supp}(\eta_i, \epsilon))}$$

$$= \sum_{i=1}^k \|\eta_i, \epsilon I_\alpha f\|_{L^q(\text{supp}(\eta_i, \epsilon))}^p$$

$$\leq \sum_{i=1}^k \left( \|I_\alpha (\eta_i, \epsilon f)\|_{L^q(\text{supp}(\eta_i, \epsilon))} \right)^p + \|\eta_i, \epsilon I_\alpha f - I_\alpha (\eta_i, \epsilon f)\|_{L^q(\text{supp}(\eta_i, \epsilon))}^p. \quad (2.9)$$

For fixed $i$,

$$\|\eta_i, \epsilon I_\alpha f - I_\alpha (\eta_i, \epsilon f)\|_{L^q(\text{supp}(\eta_i, \epsilon))}^q$$

$$= \int_{\text{supp}(\eta_i, \epsilon)} \int_{M^n} |\eta_i, \epsilon(x) - \eta_i, \epsilon(y)||f(y)||x - y||^{\alpha - n} dV_y dV_x$$

$$\leq (\max |\nabla \eta_i, \epsilon|) \int_{\text{supp}(\eta_i, \epsilon)} \int_{M^n} f(y) \text{supp}\eta_i, \epsilon |x - y|^{\alpha - n} dV_y dV_x$$

$$\leq C(\max |\nabla \eta_i, \epsilon|) \cdot (1 + \epsilon)^q \|I_{\alpha + 1} f\|_{L^q(M^n)}^q. \quad (2.10)$$
Combining (2.8) and (2.10) into (2.9), we have
\[
\|I_\alpha f\|_{L^q(M^n)}^p \leq \sum_{i=1}^k [(N_{\alpha,p} + \epsilon)\|\eta_{i,\epsilon} f\|_{L^p(\text{supp}\{\eta_{i,\epsilon}\})} + C\|I_{\alpha+1} f\|_{L^q(M^n)}]^{p - 1}
\leq \sum_{i=1}^k (N_{\alpha,p} + \epsilon)^p \cdot (1 + \epsilon)\|\eta_{i,\epsilon} f\|_{L^p(M^n)}^p + C(\epsilon)\|I_{\alpha+1} f\|_{L^q(M^n)}^p
= (N_{\alpha,p} + \epsilon)^p \cdot (1 + \epsilon)\|f\|_{L^p(M^n)}^p + C(\epsilon)\|I_{\alpha+1} f\|_{L^q(M^n)}^p.
\]

It is obvious that a similar $\epsilon$-level inequality also holds for operator $J_{M^n,g,\alpha}$.

**Corollary 2.6.** For $\alpha \in (0,n)$, $p > 1$, let $q$ be given by (1.1). For any given $\epsilon > 0$, there is a constant $C(\epsilon) > 0$, such that
\[
\|I_{M^n,g,\alpha} f\|_{L^q(M^n)}^p \leq (N_{\alpha,p} + \epsilon)^p \|f\|_{L^p(M^n)}^p + C(\epsilon)\|I_{M^n,g,\alpha+1} f\|_{L^q(M^n)}^p
\] (2.11)
holds for all $f \in L^p(M^n)$.

Based on the $\epsilon$-level sharp HLS inequality, we can establish the criterion for the existence of maximizer to the following quotient energy.

**Proposition 2.7.** If
\[
\xi_{\alpha,p} := \sup_{f \in L^p(M^n) \setminus \{0\}} \left[ \frac{\int_{M^n \times M^n} f(x)f(y)|x-y|^{2-p}dV_g(x)dV_g(y)}{\|f\|_{L^p(M^n)}^p} \right] > N_{\alpha,p},
\]
then the supremum $\xi_{\alpha,p}$ is attained.

**Proof.** Let $q$ be given by (1.1). Choose a maximizing sequence $\{f_i\}_{i=1}^{+\infty} \subset L^p(M^n)$ such that $\|I_{\alpha} f_i\|_{L^q(M^n)} = 1$. Without loss of generality, we can also assume that $f_i \geq 0$.

Claim: there exists a subsequence (still denoted as $\{f_i\}$) and $f_* \in L^p(M^n)$ such that
\[
\begin{align*}
    f_i \rightharpoonup f_* & \quad \text{weakly in } L^p(M^n), \\
    I_{\alpha} f_i \rightharpoonup I_{\alpha} f_* & \quad \text{weakly in } L^q(M^n), \\
    I_{\alpha+1} f_i \to I_{\alpha+1} f_* & \quad \text{strongly in } L^q(M^n).
\end{align*}
\]
In fact, from Hölder inequality, we know $\|f_i\|_{L^q(M^n)} \leq 1/\xi_{\alpha,p} + 1$ for large $i$, thus $\{f_i\}$ is a bounded sequence in $L^p(M^n)$. So, there exists a subsequence (still denoted as $\{f_i\}$) and a function $f_* \in L^p(M^n)$ such that
\[
    f_i \rightharpoonup f_* \quad \text{weakly in } L^p(M^n).
\]
Meanwhile, for any $g \in L^{q'}(M^n)$, we have $I_\alpha g \in L^{p'}(M^n)$ and
\[
    \|I_{\alpha+1} g\|_{L^{p'}(M^n)} \leq C(M^n)\|I_{\alpha} g\|_{L^{p'}(M^n)} < +\infty,
\]
where $q'$ is the conjugate of $q$ and $p'$ is the conjugate of $p$. So,
\[
    < I_{\alpha} f_i - I_{\alpha} f_*, g > = < f_i - f_*, I_{\alpha} g > \to 0 \quad \text{as } i \to +\infty,
\]
and
\[
    < I_{\alpha+1} f_i - I_{\alpha+1} f_*, g > = < f_i - f_*, I_{\alpha+1} g > \to 0 \quad \text{as } i \to +\infty.
\]
Combining the compactness of $\{I_{\alpha+1} f_i\}$ concluded from Proposition (1.1) we have
\[
I_{\alpha+1} f_i \to I_{\alpha+1} f_* \quad \text{strongly in } L^q(M^n).
\]
Applying Brezis-Lieb Lemma [2], we have
\[
\|f_i\|_{L^p(M^n)} - \|f_i - f_*\|_{L^p(M^n)} - \|f_*\|_{L^p(M^n)} = o(1),
\]
\[
1 - \|I_\alpha f_i - I_\alpha f_*\|_{L^\infty(M^n)} - \|I_\alpha f_*\|_{L^\infty(M^n)} = o(1).
\]
Also note:
\[
\xi_{\alpha,p} := \frac{(I_\alpha f_i, f_i)}{\|f_i\|_{L^p(M^n)^2}} + o(1) \leq \frac{1}{\|f_i\|_{L^p(M^n)}} + o(1).
\]
Thus,
\[
\frac{1}{\xi_{\alpha,p}} \geq \|f_i\|_{L^p(M^n)} + o(1)
\]
\[
= \|f_i - f_*\|_{L^p(M^n)} + \|f_*\|_{L^p(M^n)} + o(1)
\]
\[
\geq \frac{1}{(N_{\alpha,p} + \epsilon)^p} \|I_\alpha f_i - I_\alpha f_*\|_{L^p(M^n)} + \frac{1}{\xi_{\alpha,p}} \|I_\alpha f_*\|_{L^p(M^n)}
\]
\[
- \frac{C(\epsilon)}{(N_{\alpha,p} + \epsilon)^p} \|I_\alpha f_i - I_\alpha f_*\|_{L^p(M^n)}
\]
\[
+ \frac{1}{\xi_{\alpha,p}} \left( \|I_\alpha f_i - I_\alpha f_*\|_{L^\infty(M^n)} + \|I_\alpha f_*\|_{L^\infty(M^n)} \right)
\]
\[
- \frac{C(\epsilon)}{(N_{\alpha,p} + \epsilon)^p} \|I_\alpha f_i - I_\alpha f_*\|_{L^\infty(M^n)} + o(1)
\]
\[
\geq \left( \frac{1}{(N_{\alpha,p} + \epsilon)^p} - \frac{1}{\xi_{\alpha,p}^p} \right) \|I_\alpha f_i - I_\alpha f_*\|_{L^\infty(M^n)}
\]
\[
+ \frac{1}{\xi_{\alpha,p}} \left( \|I_\alpha f_i - I_\alpha f_*\|_{L^\infty(M^n)} + \|I_\alpha f_*\|_{L^\infty(M^n)} \right) + o(1)
\]
\[
= \left( \frac{1}{(N_{\alpha,p} + \epsilon)^p} - \frac{1}{\xi_{\alpha,p}^p} \right) \|I_\alpha f_i - I_\alpha f_*\|_{L^\infty(M^n)} + \frac{1}{\xi_{\alpha,p}^p} + o(1).
\]
So
\[
\lim_{i \to +\infty} \|I_\alpha f_i - I_\alpha f\|_{L^\infty(M^n)} = 0.
\]
On the other hand,
\[
\|f_*\|_{L^p(M^n)} \leq \liminf_{i \to +\infty} \|f_i\|_{L^p(M^n)},
\]
we thus know
\[
\lim_{i \to +\infty} \frac{(I_\alpha f_i, f_i)}{\|f_i\|_{L^p(M^n)^2}} \leq \frac{(I_\alpha f_* f_*)}{\|f_*\|_{L^p(M^n)^2}}
\]
So \(f_* \in L^p(M^n)\) is a maximizer. \(\square\)

Similarly, based on Corollary 2.6, we can obtain the following

**Corollary 2.8.** If
\[
\xi_{\alpha,p,G} := \sup_{f \in L^p(M^n) \setminus \{0\}} \frac{\left| \int_{M^n \times M^n} f(x)f(y)|G_\alpha^g(y)| \frac{2^n}{2^n} dV_g(x)dV_g(y) \right|}{\|f\|_{L^p(M^n)}^2} > N_{\alpha,p},
\]
then the supremum \(\xi_{\alpha,p,G}\) is attained.

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2.2.2. Generalized Yamabe problem. We shall prove Theorem 1.3 for \( \alpha < n \) in this subsection. Due to Corollary 2.8, we only need to prove

**Proposition 2.9.** If \((M^n, g)\) is locally conformally flat, but not conformally equivalent to the standard sphere \((\mathbb{S}^n, g_0)\), then for \( \alpha < n \), \( Y_\alpha(M^n, g) > Y_\alpha(\mathbb{S}^n, g_0) \).

From now on in this subsection, we will assume that \((M^n, g)\) is a locally conformally flat manifold. We need the following expansion for Green’s function of conformal Laplacian operator near its singular point (Lemma 6.4 in [26], here we use the same notations).

**Lemma 2.10.** Let \((M^n, g)\) be a locally conformally flat manifold \((n \neq 2)\). In conformal normal coordinates \(\{x^i\} \) at \(x\), \(G_\alpha^2(y)\) has an asymptotic expansion

\[
G_\alpha^2(y) = r^{2-n} + A + O''(r), \quad \forall \ y \in B_{\delta_0}(x) \tag{2.12}
\]

where \(A \geq 0 \) is a constant.

**Proof of Proposition 2.9** Let \( P \in M^n \) be a fixed point. In a conformal normal coordinate around \( P \), \( G_\alpha^2(y) \) satisfies (2.12). Further, since the manifold is not conformally equivalent to the standard sphere, \( A > 0 \) by the positive mass theorem.

For simplicity, we denote \( B_\delta(P) \) as \( B_\delta \).

For small enough \( \delta > 0 \), choose a test function

\[
u = \begin{cases} f_\delta(x), & B_\delta, \\ 0, & M^n \setminus B_\delta. \end{cases} \tag{2.13}
\]

Then by a similar argument to Proposition 2.3, we can obtain

\[
Y_\alpha(M^n, g) \geq J_{g, \alpha}(\nu) \geq Y_\alpha(\mathbb{S}^n, g_0) - C \left( \frac{\delta}{\lambda} \right)^{-n} + A \cdot \frac{\int_{B_\delta \times B_\delta} |x - y|^{\alpha-2} f_\lambda(x) f_\lambda(y) \, dx \, dy}{\|\nu\|^2_{L^{n/\alpha} (M^n)}} \tag{2.14}
\]

Since

\[
\int_{B_\delta \times B_\delta} |x - y|^{\alpha-2} f_\lambda(x) f_\lambda(y) \, dx \, dy = \lambda^{-(n+\alpha)} \int_{B_\delta \times B_\delta} |x - y|^{\alpha-2} \left( 1 + \frac{|x|^2}{\lambda^2} \right)^{-\frac{\alpha+\alpha}{2}} \left( 1 + \frac{|y|^2}{\lambda^2} \right)^{-\frac{\alpha+\alpha}{2}} \, dx \, dy \tag{2.15}
\]

\[
= \lambda^{n-2} \int_{B_{\delta/\lambda} \times B_{\delta/\lambda}} |u - v|^{\alpha-2} (1 + |u|^2)^{-\frac{\alpha+\alpha}{2}} (1 + |v|^2)^{-\frac{\alpha+\alpha}{2}} \, du \, dv \geq C_1 \lambda^{n-2},
\]

thus

\[
- C \left( \frac{\delta}{\lambda} \right)^{-n} + A \cdot \frac{\int_{B_\delta \times B_\delta} |x - y|^{\alpha-2} f_\lambda(x) f_\lambda(y) \, dx \, dy}{\|\nu\|^2_{L^{n/\alpha} (M^n)}} \tag{2.16}
\]

\[
\geq - C \left( \frac{\delta}{\lambda} \right)^{-n} + C_2 A \lambda^{n-2} = \lambda^{n-2} (C_2 A - C \lambda^2 \delta^{-n}) > 0
\]
by choosing \( \lambda \) much smaller than \( \delta \). We hereby obtain
\[
Y_\alpha(M^n, g) > Y_\alpha(S^n, g_0).
\]

(2.17)

3. CASE OF \( \alpha > n \)

We first establish the reversed HLS inequality on general compact Riemannian manifolds.

3.1. Reversed HLS inequality on Manifolds. We need the following two lemmas.

**Lemma 3.1** (Conversed Young’s Inequality). There is a constant \( C > 0 \), such that
\[
\|g \ast h\|_{L^r} \geq C\|g\|_{L^p} \cdot \|h\|_{L^p}
\]
where \( p \in (0, 1) \), \( q, r < 0 \) and satisfying
\[
1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}.
\]

This can be proved in a similar way to that for Conversed Young’s Inequality in \( \mathbb{R}^n \). We skip details here.

For a given measurable function \( f(x) \) on \( M^n \) and \( 0 < p < +\infty \), the weak \( L^p \) norm of \( f(x) \) is defined by
\[
\|f\|_{L^p_w} = \inf\{A > 0 : \text{meas}\{|f(x)| > t\} \cdot t^p \leq A^p\},
\]
For \( p < 0 \), the norm is defined as
\[
\|f\|_{L^p_w} = \sup\{A > 0 : \text{meas}\{|f(x)| < t\} \cdot t^p \leq A^p\}.
\]

Thus, for \( p < 0 \),
\[
\|f\|_{L^p_w}^p = \inf\{B > 0 : \text{meas}\{|f(x)| < t\} \cdot t^p \leq B\}.
\]

Let \( T : L^p(M^n) \to L^q(M^n) \) be a linear operator. We recall that for \( 0 < p, q < +\infty \), operator \( T \) is called the weak type \((p, q)\) if there exists a constant \( C(p, q) > 0 \) such that for all \( f \in L^p(M^n) \)
\[
\text{meas}\{x : |Tf(x)| > \tau\} \leq (C(p, q)\frac{\|f\|_{L^p}}{\tau})^q, \quad \forall \tau > 0.
\]

For \( q < 0 < p < 1 \), we say operator \( T \) is of the weak type \((p, q)\), if there exists a constant \( C(p, q) > 0 \), such that for all \( f \in L^p(M^n) \),
\[
\text{meas}\{x : |Tf(x)| < \tau\} \leq (C(p, q)\frac{\|f\|_{L^p}}{\tau})^q, \quad \forall \tau > 0.
\]

**Lemma 3.2** (Marcinkiewicz type interpolation theorem). Let \( T \) be a linear operator which maps any nonnegative function to a nonnegative function. For a pair of numbers \((p_1, q_1), (p_2, q_2)\) satisfying \( q_i < 0 < p_i < 1 \), \( i = 1, 2 \), \( p_1 < p_2 \) and \( q_1 < q_2 \), if \( T \) is weak type \((p_1, q_1)\) and \((p_2, q_2)\) for all nonnegative functions, then for any \( \theta \in (0, 1) \), and
\[
\frac{1}{p} = \frac{1}{p_1} - \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{\theta}{q_2}.
\]

\( T \) is reversed strong type \((p, q)\) for all nonnegative functions, that is,
\[
\|Tf\|_{L^q} \geq C\|f\|_{L^p}, \quad \forall f \in L^p \quad \text{and} \quad f \geq 0,
\]
for some constant \( C = C(p_1, p_2, q_1, q_2) > 0 \).
The proof of Lemma 3.2 is almost identical to that for the same inequality in $\mathbb{R}^n$, see Dou and Zhu [9].

**Proof of Theorem 1.2** The proof is quite standard. We shall follow the proof of reversed Hardy-Littlewood-Sobolev inequality, given in Dou and Zhu [9] (proof of Proposition 2.3 there). To prove (1.3), we only need to show that there is a constant $C > 0$, such that for any $\lambda > 0$,

$$m\{x \in M^n : |I_\alpha f| < \lambda\} \leq C \frac{\|f\|_{L^p}}{\lambda^q}. \quad (3.1)$$

Inequality (3.3) follows from the above inequality via the above Marcinkiewicz type interpolation theorem (Lemma 3.2).

For any $\gamma > 0$, define

$$I^{\lambda}_\alpha f(x) = \int_{|y - x| \leq \gamma} \frac{f(y)}{|x - y|^{n - \alpha}} dy,$$

and

$$I^2_\alpha f(x) = \int_{|y - x| > \gamma} \frac{f(y)}{|x - y|^{n - \alpha}} dy.$$

Thus, for any $\tau > 0$,

$$m\{x : I_\alpha f(x) < 2\tau\} \leq m\{x : I^{\lambda}_\alpha f(x) < \tau\} + m\{x : I^2_\alpha f(x) < \tau\}. \quad (3.2)$$

We note that it suffices to prove inequality (3.1) with $2\tau$ in place of $\tau$ in the left side of the inequality, and we can further assume $\|f\|_{L^p} = 1$. From Conversed Young’s inequality (Lemma 3.1), we have

$$\|I^{\lambda}_\alpha f\|_{L^{r_1}} \geq C \left( \int_{M^n} \left( \frac{\chi_\gamma(y)}{|y|^{n-\alpha}} \right)^{t_1} dy \right)^{\frac{1}{t_1}} \|f\|_{L^p} =: D_1,$$

where $\frac{1}{p} + \frac{1}{t_1} = 1 + \frac{1}{r_1}$ with $t_1 \in (\frac{n}{n - \alpha}, 0)$, $r_1 < 0$, $\chi_\gamma(x) = 1$ for $|x| \leq \gamma$ and $\chi_\gamma(x) = 0$ for $|x| > \gamma$, and

$$D_1 = \left( \int_{M^n} \left( \frac{\chi_\gamma(y)}{|y|^{n-\alpha}} \right)^{t_1} dy \right)^{\frac{1}{t_1}} = C_1(n, \alpha) \gamma^{\frac{n - (n-\alpha)t_1}{t_1}}.$$

Thus

$$m\{x : I^{\lambda}_\alpha f(x) < \tau\} \leq \frac{\|I^{\lambda}_\alpha f\|_{L^{r_1}}^{r_1}}{\tau^{r_1}} \leq C_2(n, \alpha) \gamma^{\frac{r_1(n - (n-\alpha)t_1)}{t_1}}. \quad (3.3)$$

On the other hand, Conversed Young’s inequality implies

$$\|I^2_\alpha f\|_{L^{r_2}} \geq C \left( \int_{M^n} \left( \frac{1 - \chi_\gamma(y)}{|y|^{n-\alpha}} \right)^{t_2} dy \right)^{\frac{1}{t_2}} \|f\|_{L^p} =: D_2,$$

where $\frac{1}{p} + \frac{1}{t_2} = 1 + \frac{1}{r_2}$ with $t_2 < \frac{n}{n - \alpha}$, $r_2 < 0$ and

$$D_2 = \left( \int_{M^n} \left( \frac{1 - \chi_\gamma(y)}{|y|^{n-\alpha}} \right)^{t_2} dy \right)^{\frac{1}{t_2}} = C_3(n, \alpha) \gamma^{\frac{n - (n-\alpha)t_2}{t_2}}.$$

It follows that $r_1 < \frac{np}{n - \alpha p} < r_2$ and

$$m\{x : I^2_\alpha f(x) < \tau\} \leq \frac{\|I^2_\alpha f\|_{L^{r_2}}^{r_2}}{\tau^{r_2}} \leq C_4(n, \alpha) \gamma^{\frac{r_2(n - (n-\alpha)t_2)}{t_2}}. \quad (3.4)$$
Proof. Take Best constant.

3.2.1. Proposition 3.3.

If \( \alpha > n \), then

\[
\inf_{f \in L^{2n/(n+\alpha)}(M^n)} \left| \int_{M^n \times M^n} f(x)f(y)|G^2_{B_f}(y)|^{\frac{2}{2-n}}dV_f(x)dV_f(y) \right|.
\]

We first give an upper bound estimate for the optimal constant \( Y_\alpha(M^n, g) \).

Proposition 3.3. If \( \alpha > n \), then

\[
Y_\alpha(M^n, g) \leq Y_\alpha(S^n, g_0).
\]

Proof. Take

\[
f(y) = \begin{cases} f_\lambda(x), & \text{in } B_\delta(0), \\ 0, & \text{in } \mathbb{R}^n \setminus B_\delta(0), \end{cases}
\]

where \( \delta > 0 \) is a fixed constant to be determined later. Then, for small enough \( \lambda \), \( \tilde{f} \in L^n(\mathbb{R}^n) \) and

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} \tilde{f}(x)\tilde{f}(y)|x-y|^{\alpha-n}dxdy
\]

\[
= \int_{\mathbb{R}^n \times \mathbb{R}^n} f_\lambda(x)f_\lambda(y)|x-y|^{\alpha-n}dxdy
\]

\[
- \int_{\mathbb{R}^n \times (\mathbb{R}^n \setminus B_\delta(0))} f_\lambda(x)f_\lambda(y)|x-y|^{\alpha-n}dxdy
\]

\[
- \int_{B_\delta(0) \times (\mathbb{R}^n \setminus B_\delta(0))} f_\lambda(x)f_\lambda(y)|x-y|^{\alpha-n}dxdy
\]

\[
=Y_\alpha(S^n, g_0)\|f_\lambda\|^2_{L^{2n/(n+\alpha)}(\mathbb{R}^n)} - I - II,
\]

where

\[
I = \int_{\mathbb{R}^n \times (\mathbb{R}^n \setminus B_\delta(0))} f_\lambda(x)f_\lambda(y)|x-y|^{\alpha-n}dxdy,
\]

\[
II = \int_{B_\delta(0) \times (\mathbb{R}^n \setminus B_\delta(0))} f_\lambda(x)f_\lambda(y)|x-y|^{\alpha-n}dxdy.
\]

Note (see [9])

\[
\int_{\mathbb{R}^n} f_\lambda(x)|x-y|^{\alpha-n}dx = Bf_\lambda^{\frac{\alpha-n}{n}}(y),
\]
where \( B = \pi^{n/2} \frac{\Gamma(\alpha/2)}{\Gamma((n+\alpha)/2)} \). We have
\[
I = C \int_{\mathbb{R}^n \setminus B_{\delta}(0)} |f_\lambda|^{2n/(n+\alpha)} \, dx \\
= C \int_{\delta}^{+\infty} \left( \frac{\lambda}{\lambda^2 + r^2} \right)^n r^{n-1} \, dr \\
= C \int_{\delta}^{+\infty} (1 + t^2)^{-n} t^{n-1} \, dt = O(\frac{\delta}{\lambda})^{-n}, \quad \text{as } \lambda \to 0.
\]

On the other hand, from the reversed HLS inequality, we know that \( II \) can be estimated as
\[
II \geq C \|f_\lambda\|_{L^2/(n+\alpha)(\mathbb{R}^n \setminus B_{\delta}(0))} \|f_\lambda\|_{L^2/(n+\alpha)(\mathbb{R}^n \setminus B_{\delta}(0))} = O\left(\frac{\delta}{\lambda}\right)^{\frac{n+\alpha}{n}} \quad \text{as } \lambda \to 0.
\]

Note that \( \frac{n+\alpha}{n} > 1 \). We have
\[
\|f_\lambda\|_{L^2/(n+\alpha)(\mathbb{R}^n)}^2 = \left( \int_{B_{\delta}(0)} |f_\lambda|^{2n/(n+\alpha)} \, dx + \int_{\mathbb{R}^n \setminus B_{\delta}(0)} |f_\lambda|^{2n/(n+\alpha)} \, dx \right)^{(n+\alpha)/n} \\
\leq \|f_\lambda\|_{L^2/(n+\alpha)(B_{\delta}(0))}^2 + C \|f_\lambda\|^2_{L^2/(n+\alpha)(B_{\delta}(0))} \|f_\lambda\|^p_{L^2/(n+\alpha)(\mathbb{R}^n \setminus B_{\delta}(0))}.
\]

So, for small enough \( \lambda \),
\[
\frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} \tilde{f}(x) \tilde{f}(y)|x-y|^{-n} \, dx \, dy}{\|f\|_{L^p(\mathbb{R}^n)}^2} \leq Y_\alpha(S^n, g_0) + C(\frac{\delta}{\lambda})^{-n}. \quad (3.6)
\]

The rest of the argument can be carried out in the same way as in the proof of Proposition 2.3.

To prove Theorem 3.3 for \( \alpha > n \), we first prove

**Proposition 3.4.** If \( Y_\alpha(M^n, g) < Y_\alpha(S^n, g_0) \), then the infimum is attained.

The proof will base on a new blowup analysis. For subcritical power \( p \in (0, \frac{2n}{n+\alpha}) \), we consider the infimum
\[
Y_{\alpha,p}(M^n, g) := \inf_{u \in C^0(M^n) \setminus \{0\}, u \geq 0} J_{g,\alpha,p}(u). \quad (3.7)
\]

where
\[
J_{g,\alpha,p}(u) := \int_{M^n} \int_{M^n} u(x)u(y)|G_g^p(y)|^{\frac{2n}{n+\alpha}} V_g(x) V_g(y) \frac{dV_g(x) \, dV_g(y)}{\|u\|_{L^p(M^n)}^2}.
\]

**Lemma 3.5.** For subcritical power \( p \in (0, \frac{2n}{n+\alpha}) \), infimum \( Y_{\alpha,p}(M^n, g) \) is attained.

**Proof.** The lemma could be proved via establishing certain compactness embedding for \( \alpha > n \), which is not known. To circumnavigate this difficulty, we here use a new blowup type argument. The main difference between our new blowup analysis with the traditional one is that: our argument is a global one since we do not have a local Sobolev type inequality (the classic concentration compactness, as well as Nash-Moser iteration are not available).

For fixed \( p \in (0, \frac{2n}{n+\alpha}) \), let \( u_i \) be a minimizing positive sequence of \( Y_{\alpha,p}(M^n, g) \) with \( \|u\|_{L^p(M^n)}^2 = 1 \). Easy to see that, up to further subsequence, \( u_i \to u_\ast \in L^p(M^n) \) pointwise.

We consider two cases:
Case 1: There are at least two points on $M^n$, say $x_0$, $x_1$ and a universal positive constant $C > 0$, such that, for any $r > 0$, there is a subsequence of $u_i$, satisfying

$$
\lim_{i \to \infty} \int_{B_r(x_0)} u_i^p > C; \quad \lim_{i \to \infty} \int_{B_r(x_1)} u_i^p > C.
$$

Note $p < 1$. The above inequality implies:

$$
\lim_{i \to \infty} \int_{B_r(x_0)} u_i > 0; \quad \lim_{i \to \infty} \int_{B_r(x_1)} u_i > 0. \quad (3.8)
$$

Denote

$$
I_{g,\alpha} u_i(x) = \int_{M^n} u_i(y) |G^n_{\alpha}(y)|^{\frac{2-n}{2n}} dV(y).
$$

We then know, due to (3.8) that there is a universal positive constant $C > 0$, such that

$$
I_{g,\alpha} u_i(\xi) \geq C \quad \text{for all} \quad \xi \in M^n. \quad (3.10)
$$

On the other hand, if $\text{meas} \{ \xi \in M^n : I_{g,\alpha} u_i(\xi) \to \infty \text{ as } i \to \infty \} = \text{vol}(M^n)$, then we have, using (3.8) that $H_{\alpha,R}(u_i, u_i) \to \infty$, which contradicts the assumption that $u_i$ is a minimizing sequence. Thus $I_{g,\alpha} u_i(\xi)$ stays uniformly bounded in a set with positive measure. This implies: there is a constant $C_1 > 0$, such that

$$
\int_{M^n} u_i(\xi) dS_{\xi} \leq C_1. \quad (3.11)
$$

From (3.11) we know that sequence $\{I_{g,\alpha} u_i(\xi)\}_{i=1}^{\infty}$ is uniformly bounded and equiv-

ous continuous on $M^n$. Up to a subsequence, $I_{g,\alpha} u_i(x) \to L(x) \in C(M^n)$.

Using Fatou Lemma and the reversed Hardy-Littlewood-Sobolev inequality (see Dou and Zhu [9]), we have, up to a further subsequence, that, for any positive integer $m > 0$,

$$
0 \geq \left( \lim_{i \to \infty} \int_{M^n} |I_{g,\alpha} u_i - I_{g,\alpha} u_i + m|^{2n/(n-\alpha)} \right)^{(n-\alpha)/2n} \geq C \left( \lim_{i \to \infty} \| u_i - u_i + m \|^{2n/(n-\alpha)} \right)^{(n-\alpha)/2n}.
$$

Thus $\| u_i - u_i + m \|_{L^{2n/(n-\alpha)}} \to 0$. This implies $\| u_i - u_* \|_{L^{2n/(n-\alpha)}} \to 0$. Thus the infimum is achieved by $u_*$. Easy to see that $u_* > 0$ every where on $M^n$.

We are left to rule out the following case.

Case 2. Single point blow up point: There is only one point $x_0 \in M^n$, such that for any $r > 0$, there is a subsequence of $u_i$, such that

$$
\lim_{i \to \infty} \int_{B_r(x_0)} u_i^p = 1. \quad (3.12)
$$

It follows from (3.12) and Hölder inequality that

$$
\lim_{i \to \infty} \int_{B_r(x_0)} u_i^{\frac{2n}{n-\alpha}} \to \infty.
$$

On the other hand, $\| I_{g,\alpha} u_i(x) \|_{L^{2n/(n-\alpha)}}$ is bounded, thus $\| u_i \|_{L^{2n/(n-\alpha)}}$ is bounded via the reversed HLS inequality. Contradiction. Thus case 2 cannot happen.

\[\square\]
Lemma 3.5 yields that the infimum \( Y_{g,p}(M^n, g) \) is attained. Let \( u_p \) be a minimizer such that \( \|u\|_{L^p(M^n)} = 1 \). It can be proved that \( u_p \) is smooth function (see, for example, [27], or [8]). To complete the proof of Proposition 3.4, we discuss two cases.

**Case 1:** There are at least two points on \( M^n \), say \( x_0, x_1 \) and a universal positive constant \( C > 0 \), such that, for any \( r > 0 \), there is a subsequence of \( u_p \), satisfying

\[
\lim_{p \to \frac{n}{2n+\alpha}} \int_{B_r(x_0)} u_p^p > C; \quad \lim_{p \to \frac{n}{2n+\alpha}} \int_{B_r(x_1)} u_p^p > C.
\]

Note \( p < 1 \). The above inequality implies:

\[
\lim_{p \to \frac{n}{2n+\alpha}} \int_{B_r(x_0)} u_p > 0; \quad \lim_{p \to \frac{n}{2n+\alpha}} \int_{B_r(x_1)} u_p > 0.
\]

We then know, due to (3.13) that there is a universal positive constant \( C_4 > 0 \), such that

\[ I_{g,a}u_p(\xi) \geq C_4 \quad \text{for all} \quad \xi \in M^n. \]  \hspace{1cm} (3.14)

On the other hand, if \( \text{meas}\{\xi \in M^n : I_{g,a}u_p(\xi) \to \infty \text{ as } i \to \infty\} = \text{vol}(M^n) \), then we have, using (3.13), that \( H_{g,a}(u_p, u_p) \to \infty \), which contradicts the assumption that \( u_p \) is a minimizing sequence. Thus \( I_{g,a}u_p(\xi) \) stays uniformly bounded in a set with positive measure. This implies: there is a constant \( C_5 > 0 \), such that

\[ \int_{M^n} u_p(\xi) dS_\xi \leq C_5. \]  \hspace{1cm} (3.15)

From (3.15) we know that sequence \( \{I_{g,a}u_p(\xi)\}_{i=1}^\infty \) is uniformly bounded and equi-continuous on \( M^n \). Up to a subsequence, \( I_{a,R}u_p(x) \to L(x) \in C(M^n) \).

Using Fatou Lemma and the reversed Hardy-Littlewood-Sobolev inequality (see Dou and Zhu [9]), we have, up to a further subsequence, that,

\[
0 \geq \lim_{p \to \frac{n}{2n+\alpha}} \int_{M^n} |I_{g,a}u_{p_1} - I_{g,a}u_{p_2}|^{2n/(n-\alpha)}(n-\alpha)/2n
\]

\[
\geq C \left( \lim_{p \to \frac{n}{2n+\alpha}} \|u_{p_1} - u_{p_2}\|_{L^{2n/(n+\alpha)}}^{2n/(n+\alpha)}(n-\alpha)/2n \right).
\]

Thus \( \|u_i - u_j\|_{L^{2n/(n+\alpha)}} \to 0 \). This implies \( \|u_i - u_o\|_{L^{2n/(n+\alpha)}} \to 0 \) for some \( u_o \). Thus, up to a further subsequence, \( u_i \to u_o \geq 0 \) almost everywhere. Dominate convergence theorem yields that \( \|u_o\|_{L^{2n/(n+\alpha)}} = 1 \). It follows, via Fatou Lemma, that \( \lim_{i,j \to \infty} H_{a,R}(u_i, u_j) \geq H_{a,R}(u_o, u_o) \). Thus the infimum is achieved by \( f_0 \geq 0 \).

Using energy condition, we will rule out

**Case 2:** Single point blow up point: There is only one point \( x_0 \in M^n \), such that for any \( r > 0 \), there is a subsequence of \( u_p \), such that

\[ \lim_{p \to \frac{n}{2n+\alpha}} \int_{B_r(x_0)} u_p^p = 1. \]  \hspace{1cm} (3.16)

It follows (3.16) that

\[ \lim_{p \to \frac{n}{2n+\alpha}} \int_{B_r(x_0)} u_p > 0. \]

If there is another point \( x_1 \neq x_0 \), such that for small \( r > 0 \),

\[ \lim_{p \to \frac{n}{2n+\alpha}} \int_{B_r(x_1)} u_p > 0. \]
We then again can obtain the existence of minimizer using the above argument. Finally, if for any \( x \neq x_0 \), such that for small \( r < \text{dist}(x, x_0) \),

\[
\lim_{p \to \frac{2-n}{n+\alpha}} \int_{B_r(x)} u_p = 0,
\]

we shall show that in this case \( Y_\alpha(M^n, g) \geq Y_\alpha(\mathbb{S}^n, g_0) \) which contradicts to the energy constraint \( Y_\alpha(M^n, g) < Y_\alpha(\mathbb{S}^n, g_0) \).

In fact, from the assumption of one blowup point \( (3.16) \) and \( (3.17) \) (also notice that \( \alpha > n \)), we know that for small enough \( r > 0 \),

\[
\lim_{p \to \frac{2-n}{n+\alpha}} J_{g, \alpha, p}(u_p) = \lim_{p \to \frac{2-n}{n+\alpha}} \left| \frac{\int_{B_r(x)} \int_{B_r(x)} u_p(x) u_p(y) |G^g_x(y)|^\frac{2-n}{2} dV_g(x) dV_g(y)}{\|u_p\|^2_{L^p(B_r(x))}} \right|
\]

Since \( M^n \) is locally conformally flat, we know

\[
\lim_{p \to \frac{2-n}{n+\alpha}} \frac{\int_{B_r(x)} \int_{B_r(x)} u_p(x) u_p(y) |G^g_x(y)|^{\frac{2-n}{2}} dV_g(x) dV_g(y)}{\|u_p\|^2_{L^p(B_r(x))}} = \lim_{p \to \frac{2-n}{n+\alpha}} \frac{\int_{B_r(x)} \int_{B_r(x)} u_p(x) u_p(y) |G^g_x(y)|^{\frac{2-n}{2}} dV_g(x) dV_g(y)}{\|u_p\|^2_{L^p(B_r(x))}} \]

\[
\geq \inf_{u \in L^{2n/(n+\alpha)}(\mathbb{R}^n) \setminus \{0\}, u \geq 0} \frac{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x) u(y) |x-y|^{\alpha-n} dxdy}{\|u\|^2_{L^{2n/(n+\alpha)}(\mathbb{R}^n)}} \]

\[
= Y_\alpha(\mathbb{S}^n, g_0),
\]

where \( S_R \in \mathbb{R}^n \) is the image of \( B_r(x_0) \in M^n \) under a conformal map from a local chart containing \( B_r(x_0) \) to \( \mathbb{R}^n \).

We hereby complete the proof of Proposition 3.4.

To complete the proof Theorem 1.3 for \( \alpha > n \), we are left to show

**Proposition 3.6.** If \( (M^n, g) \) is locally conformally flat, but not conformally equivalent to the standard sphere \( (\mathbb{S}^n, g_0) \), then for \( \alpha > n \), \( Y_\alpha(M^n, g) < Y_\alpha(\mathbb{S}^n, g_0) \).

**Proof.** Let \( P \in M^n \) be a fixed point. In a conformal normal coordinate around \( P \), \( G^g_x(y) \) satisfies \( (2.12) \). Further, since the manifold is not conformally equivalent to the standard sphere, \( A > 0 \) by the positive mass theorem.

Since \( \alpha > n > 2 \), we know that there exist two positive constants \( \delta_0, A_0 \) such that

\[
(G_x(y))^{\frac{2-n}{2}} \leq |x-y|_g^{\alpha-n} - A_0 |x-y|_g^{\alpha-2}, \quad \forall \ x, y \in B_{\delta_0}(P).
\]

In the sequel of the proof, we denote \( B_\delta(P) \) as \( B_\delta \).

For any fixed \( \delta \in (0, \delta_0) \), take a specific test function as

\[
u = \begin{cases} f_\lambda(x), & B_\delta, \\ 0, & M^n \setminus B_\delta. \end{cases}
\]
Similar to the computation in the proof of Proposition 2.3, we can obtain
\[ Y_\alpha(M^n, g) \leq J_{g,\alpha}(u) \]
\[ \leq Y_\alpha(S^n, g_0) + C \left( \frac{\delta}{\lambda} \right)^{-n} \left( 1 + \frac{|x|^2}{\lambda^2} \right)^{-\frac{n+\alpha}{2}} \left( 1 + \frac{|y|^2}{\lambda^2} \right)^{-\frac{n+\alpha}{2}} \]
\[ \leq Y_\alpha(S^n, g_0) + C \left( \frac{\delta}{\lambda} \right)^{-n} \left( 1 + \frac{|x|^2}{\lambda^2} \right)^{-\frac{n+\alpha}{2}} \left( 1 + \frac{|y|^2}{\lambda^2} \right)^{-\frac{n+\alpha}{2}} \]
\[ \leq Y_\alpha(S^n, g_0) + C \left( \frac{\delta}{\lambda} \right)^{-n} \left( 1 + \frac{|x|^2}{\lambda^2} \right)^{-\frac{n+\alpha}{2}} \left( 1 + \frac{|y|^2}{\lambda^2} \right)^{-\frac{n+\alpha}{2}} \]
\[ \geq C_2 \lambda^{n-2}, \]
then
\[ - C \left( \frac{\delta}{\lambda} \right)^{-n} + A_0 \frac{\int_{B_\delta \times B_\delta} |x - y|^\alpha f_\lambda(x) f_\lambda(y) dxdy}{\|u\|^2_{L^{2n/(n+\alpha)}(M^n)}} \]
\[ \geq - C \left( \frac{\delta}{\lambda} \right)^{-n} + C_2 A_0 \lambda^{n-2} = \lambda^{n-2} (C_2 A_0 - C_\lambda^2 \delta^{-n}) > 0 \]
by choosing \( \lambda \) much smaller than \( \delta \). Therefore, we deduce that
\[ Y_\alpha(M^n, g) < Y_\alpha(S^n, g_0). \]

**Remark 3.7.** Due to the lack of local sharp inequality for the case of \( \alpha > n \), it is not clear what is the form for the Aubin type \( \epsilon \)-inequality. It is also very interesting to analyze the blowup behavior of solutions to the equations with negative power, since the concentration compactness principle does not hold, and the classical Nash-Moser type iteration does not work neither due to the lack of local sharp inequality.

**Remark 3.8.** While we are working on this paper, M. Zhu was informed by F. Hang and P. Yang of their recent work on \( Q \)-curvature problem on 3 manifolds [19], where their estimates relies on the crucial sharp Sobolev inequality originally proved by Yang and Zhu [34]. It seems that their argument is hard to be extended for operator with fractional order. The recent discovery of the reversed sharp Hardy-Littlewood-Sobolev inequality [9] is the foundation for our current work for the case of \( \alpha > n \). A unified approach for the Nirenberg problem for \( \alpha < n \) was given in a recent paper [24].

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