Differentially Private Multi-Party Data Release for Linear Regression

Ruihan Wu∗1, Xin Yang2, Yuanshun Yao2, Jiankai Sun2, Tianyi Liu2, Kilian Q. Weinberger1, Chong Wang2

1Cornell University, USA
2ByteDance Inc., USA
∗Work done as an intern at ByteDance Inc.

Abstract

Differentially Private (DP) data release is a promising technique to disseminate data without compromising the privacy of data subjects. However the majority of prior work has focused on scenarios where a single party owns all the data. In this paper we focus on the multi-party setting, where different stakeholders own disjoint sets of attributes belonging to the same group of data subjects. Within the context of linear regression that allow all parties to train models on the complete data without the ability to infer private attributes or identities of individuals, we start with directly applying Gaussian mechanism and show it has the small eigenvalue problem. We further propose our novel method and prove it asymptotically converges to the optimal (non-private) solutions with increasing dataset size. We substantiate the theoretical results through experiments on both artificial and real-world datasets.

1 INTRODUCTION

The machine learning community has greatly benefited from open and public datasets [Chapelle and Chang, 2011, Real et al., 2017, Fast and Horvitz, 2017, Kong et al., 2020]. Unfortunately the privacy concern of data release significantly limits the feasibility of sharing many rich and useful datasets to the public, especially in privacy-sensitive domains like health care, finance, and government etc. This restriction considerably slows down the research in those areas as well as the general machine learning research given many of today’s algorithms are data-hungry. Recently, legal and moral concerns on protecting individual privacy become even greater. Most countries have imposed strict regulations on the usage and release of sensitive data, e.g. CCPA [Legislation, 2018], HIPPA [Act, 1996] and GDPR [Parliament and of the European Union, 2016]. The tension between protecting privacy and promoting research drives the community as well as many ML practitioners into a dilemma.

Differential privacy (DP) [Dwork, 2011, Dwork et al., 2006, 2014, Sheffet, 2017, Lee et al., 2019, Xu et al., 2017, Kenthapadi et al., 2012] is shown to be a promising direction to release datasets while protecting individual privacy. DP provides a formal definition of privacy to regulate the trade-off between two conflicting goals: protecting sensitive information and maintaining data utility. In a DP data release mechanism, the shared dataset is a function of the aggregate of all private samples and the DP guarantees regulate how difficult for anyone to infer the attributes or identity of any individual sample. With high probability, the public data would be barely affected if any single sample were replaced.

Despite the ongoing progress of DP data release, the majority of the prior work mainly focuses on the single-party setting which assumes there is only one party that would release datasets to the public. However in many real-world scenarios, there exist multiple parties who own data relevant to each other and want to collectively share the data as a
whole to the public. For example, in health care domain, some patients may visit multiple clinics for specialized treatments (Figure 1), and each clinic only has access to its own attributes (e.g. blood test and CT images) collected from the patients. For the same set of patients, attributes combined from all clinics can be more useful to train models. In general, the multi-party setting assumes multiple parties own disjoint sets of attributes (features or labels) belonging to the same group of data subjects (e.g. patients).

One straightforward approach to release data in a multi-party setting is combining data from all parties in a centralized way (e.g. one of the data owners or a third-party), and then releasing it using a private single-party data release approach. However, in a privacy-sensitive organization like a clinic, sending data to another party is prohibited by policy. An alternative approach is to let each party individually release its own data to the public through adding sample-wise Gaussian noise, and then ML practitioners can combine the data together to train models. However the resulting models trained on the data combined in this way would show a significantly lower utility compared to the models trained on non-private data (confirmed by experiments in Section 5).

In summary, we study DP data release in multi-party setting where parties share attributes of the same data subjects publicly through a DP mechanism. It protects the privacy of all data subjects and can be accessed by the public, including any party involved. To this end, we propose the following two differentially private algorithms, both based on Gaussian DP Mechanism [Dwork et al., 2014] within the context of linear regression. First, in De-biased Gaussian Mechanism for Ordinary Least Squares (DGM-OLS), each party adds Gaussian noise directly to its data. The learner with the public data is able to remove a calculated bias from the Hessian matrix. However, we show that bias removal brings the small eigenvalue problem. Hence, we propose the second method Random Mixing prior to Gaussian Mechanism for Ordinary Least Squares (RMGM-OLS).

A random Bernoulli projection matrix is shared to all parties, and each party uses it to project its data along sample-wise dimension before adding Gaussian noise. We prove that both algorithms are guaranteed to produce solutions that asymptotically converge to the optimal solutions (i.e. non-private) as the dataset size increases. Through extensive experiments on both synthetic and real-world datasets, we show the latter method achieves the theoretical claims and outperforms the first method that naively adapts Gaussian mechanism.

2 PRELIMINARY

A sequence \( \{X_n\} \) of random variables in \( \mathbb{R}^d \) is defined to converge in probability towards the random variable \( X \) if

\[
\lim_{n \to \infty} \mathbb{P}[\|X_n - X\| > \beta] = 0.
\]

The norm notation \( \|\cdot\| \) denotes \( \ell^2 \) norm in our paper. We denote this convergence as \( \text{plim}_{n \to \infty} X_n = X \).

Differential privacy (DP; [Dwork et al., 2006, 2014]) is a quantifiable and rigorous privacy framework, which is formally defined as follows.

**Definition 1** ((\( \varepsilon, \delta \))-differential privacy). A randomized mechanism \( M : D \to \mathbb{R} \) with domain \( D \) and range \( \mathbb{R} \) satisfies \((\varepsilon, \delta)\)-differential privacy if for any two adjacent datasets \( D, D' \in D \), which differ at exactly one data point, and for any subset of outputs \( S \subseteq \mathbb{R} \), it holds that

\[
\mathbb{P}[M(D) \in S] \leq e^{\varepsilon} \cdot \mathbb{P}[M(D') \in S] + \delta.
\]

**Lemma 1** (Gaussian mechanism). For any deterministic real-valued function \( f : D \to \mathbb{R}^m \) with sensitivity \( S_f \), we can define a randomized function by adding Gaussian noise to \( f \):

\[
f_{\text{dp}}(D) := f(D) + R,
\]

where \( R \) is sampled from a multivariate normal distribution \( \mathcal{N}(0, S_f^2 \sigma^2 \cdot I) \). When \( \sigma \geq \sqrt{2 \log(1.25/\delta)} \), \( f_{\text{dp}} \) is \((\varepsilon, \delta)\)-differentially private for \( 0 < \varepsilon \leq 1 \) and \( \delta > 0 \).

To simplify notations, we define \( \sigma_{\varepsilon, \delta} := \sqrt{2 \log(1.25/\delta)} \).

**Johnson-Lindenstrauss lemma (JL; [Johnson and Lindenstrauss, 1984, Achlioptas, 2003])** is a technique to compress a set of vectors \( S = \{v_1, \ldots, v_l\} \) with dimension \( d \) to a lower dimension space \( k < d \). With a proper selection \( k \), it is able to approximately preserve the inner product between any two vectors in the set \( S \) with high probability.

**Lemma 2** (JL Lemma for inner-product preserving (Bernoulli)). Suppose \( S \) is an arbitrary set of \( l \) points in \( \mathbb{R}^d \) and suppose \( s \) is an upper bound for the maximum \( \ell^2 \)-norm for vectors in \( S \). Let \( B \) be a \( k \times d \) random matrix, where \( B_{ij} \) are independent random variables taking value from 1 or \(-1 \) with probability 1/2 respectively. With the probability at least \( 1 - (l + 1)^2 \exp \left(-k \left( \frac{s^2}{4} - \frac{s^4}{6} \right) \right) \), \( \forall u, v \in S \), we have

\[
\frac{u^\top v}{s^2} - 4\beta \leq \frac{(Bu/\sqrt{k})^\top}{s^2} \left( Bv/\sqrt{k} \right) \leq \frac{u^\top v}{s^2} + 4\beta.
\]
3 NOTATION AND PROBLEM SETUP

Notations. Denote $D^j, j = 1, \cdots, m$, as data matrices for $m$ parties, where $D^j \in \mathbb{R}^{n \times d_j}$ and $m \geq 2$. They are aligned by the same set of subjects but have different attributes and they have the same number of samples. Define $D = [D^1, \cdots, D^m] \in \mathbb{R}^{nx(d+1)}$ as the collection of all datasets, where $d = d_1 + \cdots + d_m - 1$. We define $d$ by subtracting 1 from the total number of attributes because one column is label which we need to treat separately. Denote $D_i$ as the $i$-th row of $D$, we make the following assumption on data distribution:

**Assumption 1.** $D_i, i = 1, \cdots, n$, are i.i.d sampled from an underlying distribution $\mathcal{P}$ over $\mathbb{R}^{d+1}$.

Dataset release algorithm. A private multi-party data release algorithm needs to protect both inter-party and intra-party communications. The general workflow of our proposed algorithms is designed as the following:

1. Pre-generate random variable $B$. The pre-generated one or more random variables will be shared among parties.
2. Privatize the dataset locally with the algorithm $\mathcal{A}^{\text{priv}}$. Each party applies the same privatizing algorithm $\mathcal{A}^{\text{priv}}$ that takes the local dataset $D^j \in \mathbb{R}^{n \times d_j}$ and the random matrix $B$ as the inputs and then outputs $k$ (predefined) “encrypted” samples $(D^{\text{pub}})^j := \mathcal{A}^{\text{priv}}(D^j; B) \in \mathbb{R}^{k \times d_j}$.
3. Release the dataset. All parties jointly release $D^{\text{pub}} := [(D^{\text{pub}})^1, \cdots, (D^{\text{pub}})^m] \in \mathbb{R}^{k \times (d+1)}$ to the public.

Note that we need to specially design random variable $B$ and the privatizing algorithm $\mathcal{A}^{\text{priv}}$, which we will introduce in the next section. In addition, the random variable $B$ allows the dependencies between the randomized output from all parties, which can be utilized to guarantee the final utility.

Privacy constraint. Since the public will observe the released dataset $D^{\text{pub}}$, for each $j \in [m], (D^{\text{pub}})^j$ should not leak the information of the private dataset $D^j$. Formally we require $\forall j \in [m], \mathcal{A}^{\text{priv}}(D^j; B)$ is differentially private, where two neighbouring datasets $D^j$ and $(D^j)'$ differ at one row (sample).

However the multi-party setting requires more than the above guarantee because each party $j' \neq j$ not only observes $D^j$ but also the shared random variable $B$. Thus we need to further require that given $B$, each party $j$ cannot infer information about other private datasets $D^{j'}$. In terms of differential privacy, it is required that condition on $B$ for any possible sample value $I$, $\mathcal{A}^{\text{priv}}(D^j; B)$ is $(\varepsilon, \delta)$-differentially private, i.e. for any two neighbouring datasets $D^{j}$ and $(D^j)'$, and $B$, we have

$$\mathbb{P}(\mathcal{A}^{\text{priv}}(D^j; B)|B) \leq e^\varepsilon \cdot \mathbb{P}(\left(\mathcal{A}^{\text{priv}}\left((D^j)'; B)\right)\right| B) + \delta.$$

**Utility target.** We aim to guarantee the performance of arbitrary linear regression task (arbitrarily selected label and features) on the joint released dataset $[D^1, \cdots, D^m]$. Out of the notation simplicity, we assume the label in the linear regression task is the last attribute, and the features are the rest of the attributes. Under this assumption, the joint private dataset $D$ can be written as $[X, Y]$, where $X \in \mathbb{R}^{n \times d}$ is the private feature matrix and $Y \in \mathbb{R}^{n}$ is the private label vector. Similarly the public dataset $D^{\text{pub}}$ can be written as $[X^{\text{pub}}, Y^{\text{pub}}]$, where $X^{\text{pub}} \in \mathbb{R}^{k \times d}$ and $Y^{\text{pub}} \in \mathbb{R}^k$.

We define the loss function by the expected squared loss:

$$L(w; \mathcal{P}) = \mathbb{E}_{(x, y) \sim \mathcal{P}} \left[ (w^T x - y)^2 \right], \quad (1)$$

where the data point is sampled from the distribution $\mathcal{P}$ in [Assumption 1]. We make two more assumptions for the distribution $\mathcal{P}$: the standard normalization and the no perfect multicollinearity assumption. The latter is common in the literature of linear regression [Farrar and Glauber 1967, Chatterjee and Hadi 2006].

**Assumption 2.** The absolute values of all attributes $|D_{ij}|$ are bounded by 1.

**Assumption 3.** $\mathbb{E}_{(x, y) \sim \mathcal{P}} [xx^T]$ is positive definite.

Under Assumption 3 derived by setting $\nabla_w L(w; \mathcal{P}) = 0$, the optimal solution $w^*$ to the loss in Equation 1 has the following explicit form:

$$w^* = \left( \mathbb{E}_{(x, y) \sim \mathcal{P}} [xx^T] \right)^{-1} \mathbb{E}_{(x, y) \sim \mathcal{P}} [x \cdot y].$$

The utility target (for the trained linear regression model) is determined by our release algorithm $(B, \mathcal{A}^{\text{priv}})$. For a given public dataset $D^{\text{pub}}$ released by our algorithms, we define our utility target as the existence of a training algorithm $\mathcal{A}^{\text{lr}}$ that achieves the asymptotic property for the trained model weights $\hat{w}_n := \mathcal{A}^{\text{lr}}(D^{\text{pub}})$ as the dataset size $n \rightarrow \infty$. The asymptotic property is commonly studied in differential privacy [Chaudhuri and Hsu, 2011, Bassily et al., 2014, Feldman et al., 2020] and we restate it as follows: $w_n$ converges to $w^*$ in probability as the size of dataset $n$ increases, i.e. $\forall \beta > 0$, $\lim_{n \rightarrow \infty} \mathbb{P} [||\hat{w}_n - w^*|| > \beta] = 0$. The randomness from the above property comes from data sampling $\mathcal{P}$, dataset release algorithm $(B, \mathcal{A}^{\text{priv}})$, and the training algorithm $\mathcal{A}^{\text{lr}}$.

4 METHODOLOGY

We now describe our data release algorithms which both satisfy the differential privacy and yield asymptotically optimal solutions to the linear regression task. We start with the first algorithm *De-biased Gaussian Mechanism for Ordinary Least Squares (DGM-OLS)*, which directly applies Gaussian mechanism when releasing the data and then de-biases
Algorithm 1 DGM-OLS

Dataset Release
1: Input: \( D = [D^1, \ldots, D^m] \), \( \varepsilon, \delta \).
2: for \( j = 1, \ldots, m \) do
3: \( \) The party \( j \) computes \( (D^{dgm})^j := D^j + R^j \), where \( R^j \in \mathbb{R}^{n \times d} \) is a random Gaussian matrix and elements in \( R^j \) are i.i.d sampled from \( \mathcal{N} \left( 0, 4d_{\max} \sigma^2_{\varepsilon, \delta} \right) \).
4: end for
5: Return: \( D^{dgm} = [ (D^{dgm})^1, \ldots, (D^{dgm})^m ] \).

Training Algorithm
1: Input: \( D^{dgm} \), \( \varepsilon, \delta \).
2: \( [X^{dgm}, Y^{dgm}] = D^{dgm} \).
3: Compute the de-biased Hessian matrix \( \hat{H}^{dgm}_n := \frac{1}{n} (X^{dgm})^\top X^{dgm} - 4d_{\max} \sigma^2_{\varepsilon, \delta} \cdot I \).
4: \( \hat{w}_{dgm}^n := \left( \hat{H}^{dgm}_n \right)^{-1} \left( \frac{1}{n} (X^{dgm})^\top Y^{dgm} \right) \).
5: Return: \( \hat{w}_{dgm}^n \).

4.1 DE-BIASED GAUSSIAN MECHANISM (DGM-OLS)

The De-biased Gaussian Mechanism for Ordinary Least Squares (DGM-OLS) includes the dataset release algorithm and the corresponding training algorithm. Algorithm 1 shows the overview and we will introduce them next.

Dataset release algorithm. Each party directly applies Gaussian mechanism to their own dataset \( D^j \) (\( j = 1, \ldots, m \)) to satisfy the differential privacy. Consider two neighboring data matrices \( D^j \) and \( (D^j)^\prime \) differing at exactly one row with the row index \( i \). Implied by Assumption 2 we can compute the sensitivity of the data matrix \( D^j \): 

\[ \left\| D^j - (D^j)^\prime \right\| = \left\| D^j_i - (D^j_i)^\prime \right\| \leq 2\sqrt{d_j} \leq 2d_{\max}. \]

Then each party independently adds a Gaussian noise \( R^j \) to \( D^j \). Entries in \( R^j \) are i.i.d sampled from Gaussian distribution \( \mathcal{N}(0, 4d_{\max} \sigma^2_{\varepsilon, \delta}) \).

The dataset release algorithm meets the privacy constraints in section 3. No random matrix \( B \) is shared among different parties. Lemma 1 guarantees that \( (D^{dgm})^j \) is \( (\varepsilon, \delta) \)-differentially private w.r.t. \( D^j \) for any \( 0 < \varepsilon \leq 1, \delta > 0 \).

Training algorithm. Given the dataset released through the above algorithm, there exists an asymptotic linear regression solution. Denote the feature matrix and the label vector of the private and public joint dataset as \( [X, Y] = D \) and \( [X^{dgm}, Y^{dgm}] = D^{dgm} \). Further define \( R := [R^1, \ldots, R^m] \in \mathbb{R}^{n \times (d+1)} \) and split \( R \) into \( R_X \) and \( R_Y \) representing the additive noise to \( X \) and \( Y \) respectively.

Consider the ordinary least square solution for the public data \( X^{dgm} \) and \( Y^{dgm} \), whose explicit form is:

\[ \left( (X^{dgm})^\top X^{dgm} \right) (X^{dgm})^\top Y^{dgm}. \]

Compared with our target solution \( w^* = (E_{(x,y)\sim P}[xx^\top])^{-1} E_{(x,y)\sim P}[x \cdot y] \), we can prove that \( \lim_{n \to \infty} \frac{1}{n} (X^{dgm})^\top Y^{dgm} = E_{(x,y)\sim P}[x \cdot y] \) by the concentration of bounded random variables and multivariate normal distribution. Nevertheless, there is a gap between \( \lim_{n \to \infty} \frac{1}{n} (X^{dgm})^\top X^{dgm} \) and \( E_{(x,y)\sim P}[xx^\top] \):

\[ \lim_{n \to \infty} \frac{1}{n} (X^{dgm})^\top X^{dgm} = \lim_{n \to \infty} \frac{1}{n} (X^T X + X^T R_X + R_X^T X + R_X^T R_X) = E_{(x,y)\sim P}[xx^\top] + 4d_{\max} \sigma^2_{\varepsilon, \delta} \cdot I, \]

where the last equation again holds by the concentration of bounded random variables and multivariate normal distribution. To reduce the bias \( 4d_{\max} \sigma^2_{\varepsilon, \delta} \cdot I \), we can revise the solution computation in Equation 2 to \( \hat{w}_{dgm}^n \) defined as:

\[ \left( \frac{1}{n} (X^{dgm})^\top X^{dgm} - 4d_{\max} \sigma^2_{\varepsilon, \delta} \cdot I \right)^{-1} \left( \frac{1}{n} (X^{dgm})^\top Y^{dgm} \right). \]

The first term is estimated for the inverse of the Hessian matrix \( E_{(x,y)\sim P}[xx^\top] \), which we denote as \( (\hat{H}^{dgm})^{-1} \). The asymptotic optimality for the solution \( \hat{w}_{dgm}^n \) is implied by the theorem below and the proof is in the Appendix.

Theorem 1. When \( \beta \leq c \) for some variable \( c \) that is dependent of \( \sigma_{\varepsilon, \delta}, d, \) and \( P \), but is independent of \( n \),

\[ P \left[ \left\| \hat{w}_{dgm}^n - w^* \right\| > \beta \right] < \exp \left( -\tilde{O} \left( \frac{\beta^2 n}{\sigma_{\varepsilon, \delta} d^2 \max} \right) \right). \]

Problem of small eigenvalues. The expectation of \( \hat{H}^{dgm} \) is a positive definite matrix given Assumption 3 but the sample of \( \hat{H}^{dgm} \) itself is not guaranteed. With a certain probability, it has small eigenvalues that might lead to explosion when computing its inverse. In our experiments (section 5), we find that \( \hat{H}^{dgm} \) suffers from the small eigenvalues even if \( n \) is as large as \( 10^6 \). As a result, the model utility is much more inferior than what is guaranteed theoretically. This motivates us to design the second algorithm.
we can design the data release stage properly, so that the
Algorithm 2 RMGM-OLS

Dataset Release
1: **Input:** $D = [D^1, \ldots, D^m], \varepsilon, \delta, k.$
2: The first party pre-generates an $k \times n$ random matrix
   $B$ where all entries in $B$ are i.i.d. sampled from the
   distribution with probability $1/2$ for $1$ and $1/2$ for $-1$.
   Then first particle sends the random matrix sample $B$ to
   all parties.
3: for $j = 1, \ldots, m$ do
4: The party $j$ computes $(D_{\text{rmgm}})^j := BD^j/\sqrt{k} + R^j,$
   where $R^j$ is a $k \times d_j$ random matrix and all elements
   in $R^j$ are i.i.d. sampled from the multivariate normal
distribution $N \left(0, 4d_{\text{max}} \sigma^2_{\varepsilon, \delta}\right).$
5: end for
6: **Return:** $D_{\text{rmgm}} := \left[(D_{\text{rmgm}})^1, \ldots, (D_{\text{rmgm}})^m\right].$

Training Algorithm
1: **Input:** $D_{\text{rmgm}}, \varepsilon, \delta$
2: $[X_{\text{rmgm}}, Y_{\text{rmgm}}] = D_{\text{rmgm}}$
3: Compute the ordinary least square solution
   $\hat{w}_{n_{\text{rmgm}}} := \left((X_{\text{rmgm}})^\top X_{\text{rmgm}}\right)^{-1} (X_{\text{rmgm}})^\top Y_{\text{rmgm}}.$
4: **Return:** $\hat{w}_{n_{\text{rmgm}}}.$

4.2 RANDOM MIXING PRIOR TO GAUSSIAN MECHANISM (RMGM-OLS)

In previous method’s data release stage, when we directly
add the Gaussian additive noise $R$ to the data, in order to
guarantee DP, the norm of the noise needed has to be the
same order (in $n$) as the norm of the data matrix $D.$ Both $D$
and $R$ have norm in $\Theta(\sqrt{n}).$ Thus later in the training stage,
the additive noise $R$ when compared to the data matrix $X$
would not diminish as $n \to \infty$ and we have to subtract
$4d_{\text{max}} \sigma^2_{\varepsilon, \delta} \cdot I$ from $(X_{\text{rmgm}})^\top X_{\text{rmgm}}$ to remove this additive
noise in order to obtain the optimal model weights. This
subtraction is the problematic part that brings training insta-
ibility (small eigenvalues in the Hessian matrix).

Instead, we can avoid such subtraction in the training stage
by imposing a smaller noise in the data release stage. If we
can design the data release stage properly, so that the
additive noise has relatively smaller order in $n$ than $D,$ in
the later training stage, the learner would no longer need the
problematic de-biasing step.

Algorithm 2 shows the full details of Random Mixing
prior to Gaussian Mechanism for Ordinary Least Squares
(RMGM-OLS). We now explain the design of data release
and training algorithm based on the above insights.

Dataset release algorithm. Suppose $b$ is an
$n$-dimensional vector in $\{-1, 1\}^n.$ For any two neighbouring
datasets $D^i$ and $(D^i)^\prime$ that are different at row index $i,$ the
sensitivity of $b^\top D^i$ is

$$\|b^\top D^i - b^\top (D^i)^\prime\| = \|D^i - (D^i)^\prime\| \leq 2\sqrt{d_i} \leq 2\sqrt{d_{\text{max}}}.$$  
Moreover, when $B \in \{-1, 1\}^{k \times n},$ $BD^j/\sqrt{k}$ has sensitiv-
ity $2\sqrt{d_{\text{max}}}$ as well.

We now introduce the data release algorithm. Suppose all
parties are sharing a random matrix $B \in \{-1, 1\}^{k \times n},$
where all elements in $B$ are i.i.d. sampled from the dis-
tribution with probability $1/2$ for $1$ and $1/2$ for $-1.$ Then we
define the local computation for each party $j$:

$$(D_{\text{rmgm}})^j := BD^j/\sqrt{k} + R^j,$$
where $R^j$ is a $k \times d_j$ random matrix and all elements in $R^j$
are i.i.d. sampled from the multivariate normal distribution
$N \left(0, 4d_{\text{max}} \sigma^2_{\varepsilon, \delta}\right).$ Gaussian mechanism guarantees for any
fixed $B \in \{-1, 1\}^{k \times n},$ $(D_{\text{rmgm}})^j$ is $(\varepsilon, \delta)$-differentially private w.r.t. the dataset $D^j$ for $0 < \varepsilon, \delta > 0.$

Importantly, now the additive noise $R^j$ is relatively small
than $BD^j/\sqrt{k}.$ The order of $\|R^j\|$ is $\Theta(k)$ while the order of
$\|BD^j/\sqrt{k}\| \approx \|D^j\| = \Theta(n)$ (by JL Lemma). If we set
$k = o(n),$ the additive noise compared to the original data
matrix $D$ will diminish as $n \to \infty.$ This implies that the
standard ordinary least square solution to the public dataset
$[X_{\text{rmgm}}, Y_{\text{rmgm}}]$ would converge to the optimal solution $w^*$
without special subtraction.

Training algorithm. Given the feature matrix $X_{\text{rmgm}}$ and
the label vector $Y_{\text{rmgm}}$ from the released dataset, we show
that the vanilla ordinary least square solution

$$\hat{w}_{n_{\text{rmgm}}} := \left((X_{\text{rmgm}})^\top X_{\text{rmgm}}\right)^{-1} (X_{\text{rmgm}})^\top Y_{\text{rmgm}}$$
is asymptotically optimal, i.e. $\text{plim}_{n \to \infty}\hat{w}_{n_{\text{rmgm}}} = w^*.$

To prove the above asymptotic optimality, we show
$\text{plim}_{n \to \infty}(X_{\text{rmgm}})^\top X_{\text{rmgm}} = \mathbb{E}_{(x,y) \sim P}[xx^\top]$ and
$\text{plim}_{n \to \infty}(X_{\text{rmgm}})^\top Y_{\text{rmgm}} = \mathbb{E}_{(x,y) \sim P}[x \cdot y]$ respectively,
and together they prove the optimality.

Define $R = [R^1, \ldots, R^m] \in \mathbb{R}^{k \times (d+1)}$ and split $R$ into $R_X$ and $R_Y$ representing the additive
noises to $BX/\sqrt{k}$ and $BY/\sqrt{k}$ respectively. Because
$\text{plim}_{n \to \infty} \frac{1}{n} X^\top X = \mathbb{E}_{(x,y) \sim P}[xx^\top]$, it is sufficient to show
$\text{plim}_{n \to \infty} \frac{1}{n} (X_{\text{rmgm}})^\top X_{\text{rmgm}} = \frac{1}{n} X^\top X.$ Now, we decompose $\frac{1}{n} (X_{\text{rmgm}})^\top X_{\text{rmgm}} - \frac{1}{n} X^\top X$ as below:

$$\frac{1}{n} \left( X^\top B^\top \frac{B}{k} X - X^\top X \right) + \frac{1}{n} \left( X^\top B^\top \frac{B}{k} R_X + R_X^\top \frac{B}{k} X \right) + \frac{1}{n} R_X^\top R_X.$$

Lemma 2

We informally show how each term converges to $0$ as $n \to \infty$:
1. $\frac{1}{n} \left( X^\top B^\top \frac{B}{k} X - X^\top X \right).$ If $k \to \infty$ as $n \to \infty,$ the
   convergence is directly implied by Lemma 2.
2. \( \frac{1}{n} \left( X^\top \frac{B_k}{\sqrt{k}} R_X + R_X^\top \frac{B_k}{\sqrt{k}} X \right) \). Properties of normal distribution guarantees the approximation \( \frac{B_k}{\sqrt{k}} R_X \approx R_X' \), where \( R_X' \in \mathbb{R}^{n \times d} \) is a Gaussian matrix with \( N\left(0, 4d_{\text{max}}\sigma_{\varepsilon, \delta}^2 \right) \). Then \( \left\| \frac{1}{n} X^\top \frac{B_k}{\sqrt{k}} R_X \right\| \approx \left\| \frac{1}{n} X^\top R_X^\top \right\| = O \left( \frac{1}{\sqrt{n}} \right) \).

3. \( \frac{1}{n} R_X^\top R_X \). If \( k \to \infty \) as \( n \to \infty \), \( \left\| \frac{1}{n} X^\top R_X R_X \right\| \) will converge to \( 4d_{\text{max}}\sigma_{\varepsilon, \delta}^2 I \). On the other hand, when \( k = o(n) \), \( \frac{k}{n} \cdot 4d_{\text{max}}\sigma_{\varepsilon, \delta}^2 I \) will converge to 0 as \( n \to \infty \).

Notice that the above convergence relies on the proper selection of \( k \). There exists a trade-off: larger \( k \) leads to better convergence rate of the first term, but worse rate for the diminishing of additive noise – the third term. The following theorem shows the exact asymptotic rate:

**Theorem 2.** When \( \beta \leq c \) for some variable \( c \) that is dependent of \( d \) and \( P \), but independent of \( \sigma_{\varepsilon, \delta} \), \( n \), we have

\[
\Pr \left[ \left\| \hat{w}^\text{bgm}_n - w^* \right\| > \beta \right] < \exp \left( -\frac{\beta^2}{2d^2} \left[ \frac{n\beta}{d} \cdot \frac{n}{n_{\text{max}}\sigma_{\varepsilon, \delta}^2} \right] \right) + O(1) .
\]

If we choose \( k = O \left( \frac{n^{1/2}d^2}{d_{\text{max}}\sigma_{\varepsilon, \delta}} \right) \), then

\[
\Pr \left[ \left\| \hat{w}^\text{bgm}_n - w^* \right\| > \beta \right] < \exp \left( -\frac{n^{1/2}d^2}{d_{\text{max}}^{1/2}\sigma_{\varepsilon, \delta}^{1/2}} \cdot O \left( \min \{1, \beta\} \right) + O(1) \right) .
\]

In the theorem, \( k \) is selected to balance \( \frac{k\beta^2}{d^2} \) and \( \frac{n\beta}{d_{\text{max}}\sigma_{\varepsilon, \delta}^2} \). To achieve the optimal rate for \( f(\beta) \) with any fixed \( \beta \), the optimal \( k \) is chosen as \( O \left( \frac{n^{1/2}d^2}{d_{\text{max}}^{1/2}\sigma_{\varepsilon, \delta}^{1/2}} \right) \).

**Comparison with DGM-OLS.** The near-zero eigenvalue issue is solved since \( (X^\text{bgm})^\top X^\text{bgm} \geq 0 \) holds naturally by its definition. Moreover, although the convergence rate of \( n \) is sacrificed, the orders in \( d, d_{\text{max}} \) and \( \sigma_{\varepsilon, \delta} \) are much improved. In section 5, we show that the RMGM-OLS outperforms DGM-OLS on both synthetic datasets even when \( n \) is as large as \( 3 \times 10^6 \).

## 5 EXPERIMENTAL EVALUATION

In this section, we evaluate DGM-OLS and RMGM-OLS on both synthetic and real-world datasets. Our experiments on synthetic dataset are designed to verify the theoretical asymptotic results in section 4 by increasing the training set size \( n \). We further justify the algorithm performance on five real-world datasets, four from UCI Machine Learning Repository\footnote{https://archive-beta.ics.uci.edu/ml/datasets} [Dua and Graff 2017] and one from kaggle.

### 5.1 EXPERIMENT SET-UP

**Algorithm set-up.** We evaluate both DGM-OLS and RMGM-OLS. For \( k \) in RMGM-OLS, we set \( k = \frac{n}{\sigma_{\varepsilon, \delta}} \) in synthetic dataset experiments and select the best \( k \) from \( \{10^2, 3 \times 10^2, 10^3, 3 \times 10^3, 10^4\} \) in real-world dataset experiments. Because of the numerical instability of computing Hessian inverse mentioned early, we add small \( \lambda \cdot I \) with \( \lambda = 10^{-5} \) to all Hessian matrices.

**Baseline.** In addition, we consider the following baselines to help qualify the performance of proposed algorithms.

- **OLS:** The explicit solution for linear regression given training data \( (X,Y) \) and serves as the performance’s upper bound for private algorithms, i.e. non-private solution.
- **Biased Gaussian mechanism (BGM-OLS):** The same data release algorithm in DGM-OLS, but has a different training algorithm. Given a released dataset \( (X^\text{bgm}, Y^\text{bgm}) \) by Gaussian mechanism, BGM-OLS outputs the vanilla ordinary least square solution \( \hat{w}^\text{bgm}_n = (X^\text{bgm})^\top (X^\text{bgm})^{-1} (X^\text{bgm})^\top Y^\text{bgm} \). In other words, it is DGM-OLS without training debiasing.

**Evaluation metric.** In the experiments on synthetic datasets, we estimate the probability of the \( \ell^2 \) distance between the model weights \( \hat{w}_n \) from each algorithm or baseline and the ground truth model weight \( w^* \):

\[
\Pr \left[ \left\| \hat{w}_n - w^* \right\| > \beta \right] .
\]

We also evaluate the expectation of the \( \ell^2 \) distance between weights for different algorithms:

\[
\mathbb{E} \left[ \| \hat{w}_n - w^* \| \right] .
\]

If an algorithm is asymptotically optimal, we can see both \( \Pr \left[ \| \hat{w}_n - w^* \| > \beta \right] \) and \( \mathbb{E} \| \hat{w}_n - w^* \| \) converge to 0 when \( n \) increases.

For the experiments on real-world datasets, we evaluate learned models \( w_n \) by the mean squared loss on the test set.

### 5.2 EVALUATION ON SYNTHETIC DATASETS

**Data generation.** We define the feature dimension \( d = 10 \). Each weight value of the ground truth linear model \( w^* \) is independently sampled from uniform distribution between \(-1/d \) and \( 1/d \). A single data point \( (x, y) \) is sampled as the following: each feature value in \( x \) is independently sampled from a uniform distribution between \(-1 \) and \( 1 \); label \( y \) is computed as \((w^*)^\top x \). Two assumptions for the data distribution \( P \), Assumption 2 and Assumption 3, can be verified. Moreover, we set 6 parties in total, 5 of which have 2 attributes and the remaining one has 1 attribute.
Figure 2: $P[∥\hat{w}_n - w^*∥ > \beta]$ and $E[∥\hat{w}_n - w^*∥]$ as dataset size $n$ increases for different algorithms when $\varepsilon = 1.0, 0.3, 0.1$. For all pairs of $(\varepsilon, \beta)$ except two most extreme cases $(0.3, 0.1)$ and $(0.1, 0.1)$, RMGM-OLS shows asymptotic tendencies $\lim_{n \to \infty} P[∥\hat{w}_{rmgm}^n - w^*∥ > \beta] = 0$. DGM-OLS does not show such tendencies even when training set size $n$ is as large as $3 \times 10^6$.

Results. We vary the training set size $n \in \{10^4, 3 \times 10^4, 10^5, 3 \times 10^5, 10^6, 3 \times 10^6\}$ and privacy budget $\varepsilon \in \{1, 0.3, 0.1\}$ with fixed $\delta = 10^{-5}$. We estimate the $P[∥\hat{w}_n - w^*∥ > \beta]$ and $E[∥\hat{w}_n - w^*∥]$ for different algorithms with 1000 random seeds. Figure 2 shows how $P[∥\hat{w}_n - w^*∥ > \beta]$ and $E[∥\hat{w}_n - w^*∥]$ of each algorithm change when training set size $n$ increases.

Regarding two baselines, $P[∥\hat{w}_n - w^*∥ > \beta]$ of OLS solutions, without any private constraint, are close to the ground truth $w^*$ under all $\beta$ with probability 0. Nonetheless, $P[∥\hat{w}_n - w^*∥ > \beta]$ of BGM-OLS keeps mostly unchanged as $n$ increases. Especially, $P[∥\hat{w}_n - w^*∥ > 0.1]$ stays at 1 for all $n$. Such results are expected in BGM-OLS’s convergence: $\lim_{n \to \infty} \frac{1}{n} (X^{bgm})^T X^{bgm} = \mathbb{E}_x [xx^T] + 4d_{max}\sigma^2_{\epsilon, \delta} \cdot I$, which introduces a non-diminishing bias $4d_{max}\sigma^2_{\epsilon, \delta} \cdot I$.

Next, we compare DGM-OLS and RMGM-OLS. RMGM-OLS outperforms DGM-OLS at both the convergence of probability $P[∥\hat{w}_n - w^*∥ > \beta]$ (the first three figures in Figure 2) and the expected distance $E[∥\hat{w}_n - w^*∥]$ (the last figure in Figure 2). RMGM-OLS shows the asymptotic tendencies in all values of $\beta$ when $\varepsilon = 1.0$. Although DGM-OLS has better rate at $n$ than RMGM-OLS theoretically, $n = 3 \times 10^6$ is not large enough to show the asymptotic
tendencies for DGM-OLS.

DGM-OLS is even much worse than BGM-OLS, which is almost random guess. It is caused by the small eigenvalue issue discussed in section 4. To illustrate it, Figure 3(a) shows the scatter plot, where the x-axis is minimum eigenvalues of the Hessian matrix $H_n$ and y-axis is the distance between our solutions and the optimal solution $\|w_n - w^*\|$. Each point is processed by a different random seed for DGM-OLS and BGM-OLS when $n = 10^6$ and $\epsilon = 1.0$. $\|w_n - w^*\|$ and the minimum absolute eigenvalues of $H_n$ have a strong positive correlation. With a certain probability, the minimum eigenvalue of DGM-OLS is smaller than $10^{-2}$ and corresponding $\|w_n - w^*\|$ is larger than 10.

Overall RMGM-OLS has the best empirical performance across various settings of $\epsilon$ and $n$ on the synthetic data, as its asymptotically optimality is verified and it consistently outperforms two other private algorithms when $n$ is large enough. Though DGM-OLS seems to have stronger theoretical guarantee in the aspect of rate in $n$, its poor empirical performance comes from two aspects: 1. small eigenvalues occur due to the design of the training algorithm; 2. extremely large $n$ is necessary to show the asymptotic optimality due to the worse rates of $d$, $d_{\text{max}}$ and $\sigma_{\epsilon, \delta}$.

5.3 EVALUATION ON REAL WORLD DATASETS

Dataset. We experiment with five datasets:

- **Insurance** [Lantz 2019]: predicting the insurance premium from features including age, bmi, expenses, etc.
- **Bike** [Fanæe-T and Gama 2014]: predicting the count of rental bikes from features such as season, holiday, etc.
- **Superconductor** [Hamidieh 2018]: predicting critical temperature from chemical features.
- **GPU** [Ballester-Ripoll et al. 2019, Nugteren and Cordoneau 2015]: predicting Running time for multiplying two $1048 \times 1048$ matrices using a GPU OpenCL SGEMM kernel with varying parameters.
- **Music Song** [Bertin-Mahieux et al. 2011]: predicting the release year of a song from audio features.

We split the original dataset into train and test by the ratio 4 : 1. The number of training data $n$, the number of features $d$ and the number of parties are listed in Table 1. The attributes are evenly distributed among parties. All features and labels are normalized into $[0, 1]$.

Results. For each dataset, we evaluate OLS and three differentially private algorithms by the mean squared loss on the test split. Table 1 shows the results for $\epsilon \in \{0.1, 0.3, 1.0\}$ and $\delta = 10^{-5}$. We can check that the loss of DGM-OLS is usually much larger than others and RMGM-OLS achieves the lowest losses for most cases (12 out of 15). Moreover, Figure 3(b) shows that DGM-OLS has the small eigenvalue problem as well in the real world dataset experiments. These results are consistent with the results on synthetic dataset. We therefore recommend RMGM-OLS as a practical solution to privately release the dataset and build the linear regression models.

6 RELATED WORK

Differentially private dataset release. Many recent works [Sheffet 2017, Gondara and Wang 2020, Xie et al. 2018, Jordon et al. 2018, Lee et al. 2019, Xu et al. 2017, Kenthapadi et al. 2012] study the differentially private data release algorithms. However, those algorithms either only serve for data release from a single-party [Sheffet 2017, Gondara and Wang 2020], or focus on the feature dimension reduction or empirical improvement [Lee et al. 2019, Xu et al. 2017, Kenthapadi et al. 2012], which is orthogonal to the study of asymptotic optimality w.r.t. dataset size. In Sheffet [2017] and Gondara and Wang [2020], the random Gaussian projection matrices in their method contribute to the differential privacy guarantee, hence the sharing of projection matrix would violate the privacy guarantee between parties. Nevertheless, without sharing this projection matrix, the utility cannot be guaranteed anymore. In Xie et al. 2018 and Jordon et al. 2018, they train a differentially private GAN. However, it is not obvious to rigorously privately share data information during their training when each party holds different attributes but same instances. Lee et al. 2019 proposes a random mixing method and also analyzes the linear model. However, the way they mix only works for realizable linear data. It is not able to be extended to the general linear regression and the asymptotic optimality guarantee. Xu et al. 2017 and Kenthapadi et al. 2012 focus on the feature dimension reduction, which is orthogonal to the study of asymptotic optimality w.r.t. dataset size.

Asymptotically optimal differentially private convex optimization. A large amount of work study differentially private optimization for convex problems [Bassily et al. 2014, 2019, Feldman et al. 2020] or particularly for linear regression [Sheffet 2017, Kasiviswanathan et al. 2011, Chaudhuri and Hsu 2012]. They mainly differ from our work in the sense that their goal is to release the final model while ours is to release the dataset.

Linear regression in vertical federated learning. Linear regression is a fundamental machine learning task. Hall et al. 2011, Nikolaenko et al. 2013, Gascón et al. 2017 studying linear regression over vertically partitioned datasets based on secure multi-party computation. However, cryptographic protocols such as Homomorphic Encryption [Hall et al. 2011, Nikolaenko et al. 2013] and garbled circuits [Nikolaenko et al. 2013, Gascón et al. 2017] lead to heavy overhead on computation and communication. From this aspect, DP-based techniques are more practical.
Table 1: Mean squared losses on real world datasets. RMGM-OLS achieves the lowest losses in most settings (12 out of 15).

7 CONCLUSION

We propose and analyze two differentially private algorithms under multi-party setting for linear regression, and theoretically both of them are asymptotically optimal with increasing dataset size. Empirically, RMGM-OLS has the best performance on both synthetic datasets and real-world datasets, while extremely large training set size $n$ is necessary for DGM-OLS. We hope our work can bring more attention to the need for multi-party data release algorithms and we believe that ML practitioners would benefit from such effort in the era of privacy.

Future work. We focus on linear regression only, and one future direction is to extend our algorithm to classification, e.g. logistic regression, while achieving the same asymptotic optimality. In addition, we assume different parties own the same set of data subjects. Another future direction is to relax this assumption: the set of subjects owned by different parties might be slightly different.

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