A Logic of Non-Monotone Inductive Definitions

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Well-known principles of induction include monotone induction and different sorts of non-monotone induction such as inflationary induction, induction over well-founded sets and iterated induction. In this work, we define a logic formalizing induction over well-founded sets and monotone and iterated induction. Just as the principle of positive induction has been formalized in FO(LFP), and the principle of inflationary induction has been formalized in FO(IFP), this paper formalizes the principle of iterated induction in a new logic for Non-Monotone Inductive Definitions (ID-logic). The semantics of the logic is strongly influenced by the well-founded semantics of logic programming.

Our main result concerns the modularity properties of inductive definitions in ID-logic. Specifically, we formulate conditions under which a simultaneous definition $\Delta$ of several relations is logically equivalent to a conjunction of smaller definitions $\Delta_1 \land \cdots \land \Delta_n$ with disjoint sets of defined predicates. The difficulty of the result comes from the fact that predicates $P_i$ and $P_j$ defined in $\Delta_i$ and $\Delta_j$, respectively, may be mutually connected by simultaneous induction. Since logic programming and abductive logic programming under well-founded semantics are proper fragments of our logic, our modularity results are applicable there as well.

Categories and Subject Descriptors: ...

1. INTRODUCTION

This paper fits into a broad project aiming at studying general forms of inductive definitions and their role in diverse fields of mathematics and computer science. Monotone inductive definitions and inductive definability have been studied extensively in mathematical logic [Moschovakis 1974a; Aczel 1977]. The algebraic foundations for monotone induction are laid by Tarski’s fixpoint theory of monotone lattice operators [Tarski 1955]. The notion of inductive definition is the underlying concept in fixpoint logics [Gurevich and Shelah 1986; Dawar and Gurevich 2002] which found its applications in e.g. database theory [Abiteboul et al. 1995] and descriptive complexity theory [Immerman 1999; Ebbinghaus and Flum 1999].
Logics with fixpoint constructs to represent (monotone) inductive and co-inductive definitions play a central role as query and specification languages in the area of verification of dynamic systems using modal temporal logics such as the $\mu$-calculus [Kozen 1983]. Induction axioms have been used successfully in the context of proving properties of protocols using specialised automated reasoning tools [Paulson 1998]. The concept of definitions and definitional knowledge is also fundamental in the area of description logics [Brachman and Levesque 1982], the class of logics that evolved out of semantic networks. Importantly, complexity results in fixpoint logic and logic programming suggest that inductive definitions often combine high expressivity with low complexity. Thus, it appears that the notion of definition and its inductive generalisations emerges as a unifying theme in many areas of mathematics and computational logic. Hence, its study could improve insight in the interrelations between these areas and lead to synergy between them.

In this paper, we are concerned with non-monotone inductive definitions. A familiar example of a non-monotone inductive definition is the definition of the satisfaction relation $\models$ between a truth assignment $I$ and a formula. In case of propositional logic, this relation is defined by induction over the subformula order on formulas:

- $I \models p$ if $p \in I$,
- $I \models \psi \land \phi$ if $I \models \psi$ and $I \models \phi$,
- $I \models \psi \lor \phi$ if $I \models \psi$ or $I \models \phi$,
- $I \models \neg \psi$ if $I \not\models \psi$.

This inductive definition is non-monotone because of its last rule, which adds the pair $(I, \neg \psi)$ to the truth relation if the pair $(I, \psi)$ does not belong to it. This is an example of an inductive definition over a well-founded order. Recently, the authors of [Denecker 1998; Denecker et al. 2001a] investigated certain non-monotone forms of inductive definitions in mathematics and pointed out that semantical studies in the area of logic programming might contribute to a better understanding of such generalised forms of induction. In particular, it was argued that the well-founded semantics of logic programming [Van Gelder et al. 1991] extends monotone induction and formalises and generalises non-monotone forms of induction such as induction over well-founded sets and iterated induction [Feferman 1970; Buchholz et al. 1981]. In [Denecker et al. 2000; Denecker et al. 2004], the well-founded semantics was further generalised into a fixpoint theory of general non-monotone lattice operators. This theory, called approximation theory, generalises Tarski’s theory of fixpoints of monotone lattice operators and provides the algebraic foundation of the principle of iterated induction. Later, it turned out that the same principle is fundamental in an area of artificial intelligence concerned with using logic for knowledge representation — non-monotonic reasoning\footnote{The term “non-monotone” has a different meaning in the context of inductive definitions than in the context of non-monotone reasoning. A logic is non-monotone when adding formulas to a theory may not preserve inferred formulas. A monotone definition is one inducing a monotone operator. In fact, the fragment of monotone inductive definitions in ID-logic is a non-monotone logic.}. In particular, [Denecker et al. 2000; Denecker et al. 2003] demonstrated that the semantics of three major
approaches to non-monotonic reasoning, default logic [Reiter 1980], autoepistemic logic [Moore 1985] and logic programming [Lloyd 1987] are described by approximation theory. Thus, generalised inductive definitions also play a fundamental role in the semantics of knowledge representation formalisms.

In a seminal paper on knowledge representation, Brachman and Levesque [Brachman and Levesque 1982] had observed that definitional knowledge is an important component of human expert knowledge. Motivated by this work, the author of [Denecker 2000] extended classical logic with non-monotone inductive definitions in order to demonstrate that general non-monotone inductive forms of definitions also play an important role in knowledge representation. In this paper, we extend this work. The contributions of the paper are the following:

—We formalize the principle of iterated induction in a new logic for Non-Monotone Inductive Definitions (ID-logic). This logic is an extension of classical logic with non-monotone inductive definitions, and is a generalisation of the logic that was defined in [Denecker 2000].

—We demonstrate that different classes of definitions can be correctly and uniformly formalised in our logic. To achieve this goal, we present an alternative formalisation of these classes in classical first- or second-order logic, and provide an equivalence-preserving transformation from ID-logic to these formalisations.

—We study modularity properties of non-monotone inductive definitions in ID-logic and provide a set of techniques that allow one to break up a big definition into a conjunction of smaller and simpler definitions.

The main result of the paper is a set of formal conditions that guarantee that a simultaneous definition of several predicates can be split up into the conjunction of components of this definition, each component defining some subset of the defined predicates. In addition, our theorems provide conditions under which joining a set of definitions for distinct sets of predicates into one simultaneous definition of all these predicates is equivalence preserving. The problem that we study is similar to that studied in [Verbaeten et al. 2000], but our results are uniformly stronger in the sense that they are proven for a more expressive logic and under more general conditions.

The results are important because modularity is a crucial property in formal verification and knowledge representation [Reichgelt 1991]. For example, it is important to be able to specify a complex dynamic system by describing its components in independent modules which can then be conjoined to form a correct description of the complete system. Thus, the operation of joining modules should preserve the correctness of the component modules. The dual operation of splitting a complex theory into an equivalent set of smaller modules is equally important. It allows one to investigate complex theories by studying its modules independently, and reduces the analysis of the correctness of the complex theory to the much simpler problem of analysing the correctness of its modules.

The paper is structured as follows. In Section 2, we discuss various forms of induction and their formalisations. This discussion provides the intuitions and the motivation for defining the new logic. Section 3.1 introduces some preliminaries from logic and lattice theory. In section 4, we extend classical logic with the generalised non-monotone definitions. In section 6, the modularity of the definition
expressions is investigated. In section 7, we present equivalence-preserving transformations from ID-logic to first-order and second-order logic for different familiar types of definitions. Here, the modularity techniques developed in the previous section are used as a tool to prove correctness of the transformations.

2. FORMAL STUDY OF INDUCTIVE DEFINITIONS

**Mathematical induction** refers to a class of effective construction techniques used in mathematics. There, a set is frequently defined as the limit of a process of iterating some operation. Often, mathematicians describe such a construction by an inductive definition. The core of an inductive definition in mathematics consists of one or more basic rules and a set of inductive rules. Basic rules represent base cases of the induction and add elements to the defined set in an unconditional way; inductive rules add new elements to the set if one can establish the presence or the absence of other elements in the set. The defined set is obtained as the limit of some process of iterated application of these rules.

In this section, we discuss various forms of such inductive definitions and how they are formalised. Then we motivate and preview a new formal logic of definitions. The section is partially based on ideas presented earlier in [Denecker 1998; Denecker et al. 2001a].

2.1 Monotone Inductive Definitions.

In a monotone inductive definition, the presence of new elements in the set depends only on the presence of other elements in the defined set, not on the absence of those. The defined set is the least set closed under application of the rules. Such definitions are frequent in mathematics. A standard example is the transitive closure of a directed graph:

The transitive closure \( T_G \) of a directed graph \( G \) is inductively defined as the set of edges \((x, y)\) satisfying the following rules:

\[ -(x, y) \in T_G \text{ if } (x, y) \in G; \]
\[ -(x, y) \in T_G \text{ if for some vertex } z, (x, z), (z, y) \in T_G. \]

Other typical examples are the the definition of a subgroup generated by a set of group elements, or the definitions of a term, formula, etc. in logic.

Monotone inductive definitions have been studied extensively in mathematics [Moschovakis 1974a; Aczel 1977]. In [Moschovakis 1974a], such a definition is associated with a formula \( \varphi(\bar{x}, X) \). Intuitively, this formula encodes all the conditions under which tuple \( \bar{x} \) belongs to the defined predicate \( X \). The formula \( \varphi(\bar{x}, X) \) must be positive in \( X \), that is no occurrence of \( X \) may appear in the scope of an odd number of occurrences of the negation symbol \( \neg \). For instance, for the transitive closure example above we have:

\[ \varphi_{\text{trans}}(x, y), T_G) := G(x, y) \lor \exists z(T_G(x, z) \land T_G(z, y)). \]

Each disjunct in this formula formally expresses the condition of one of the rules in the informal definition.

Given a structure \( I \) which interprets all constant symbols, the formula \( \varphi(\bar{x}, X) \) characterises an operator \( \Gamma_{\varphi(\bar{x}, X)} \) mapping a relation \( R \) to the relation \( R' \) consisting...
of tuples $\bar{a}$ such that $\varphi(\bar{a}, R)$ is true in $I$. In general, $\Gamma\varphi(\bar{x}, X)$ may have multiple fixpoints, but the fact that $\varphi(\bar{x}, X)$ is a positive formula implies that the operator is monotone and has a least fixpoint, which is the relation inductively defined by $\varphi(\bar{x}, X)$. A logic to represent monotone inductive definitions is the least fixpoint logic $\text{FO}(\text{LFP})$ (see, e.g., [Ebbinghaus and Flum 1999]).

2.2 Inductive Definitions over a Well-Founded Order.

In a non-monotone inductive definition, the presence of new elements in the set depends on the absence of certain elements in the defined set. An example of such a definition is the definition of the truth relation given in the introduction. Let us consider another definition with a similar structure.

The set of even numbers is defined by induction over the standard order $\leq$ on the natural numbers:
- $0$ is an even number;
- $n + 1$ is an even number if $n$ is not an even number.

The definitions of even numbers and of $|=\,$ are examples of inductive definitions over well-founded orders. Such a definition describes the membership of an element in the defined relation in terms of the presence or absence of elements in the defined relation that are strictly smaller with respect to some well-founded (pre-)order. By applying this definition to the minimal elements and then iterating it for higher levels, the defined predicate can be constructed, even if some inductive rules are non-monotone. This type of inductive definitions is fundamentally different from monotone inductive definitions. Indeed, the set defined by a monotone inductive definition can be characterised as the least set closed under the rules. In contrast, a definition over a well-founded order does not characterise a unique least set closed under its rules. For instance, $\{0, 2, 4, 6, \ldots\}$ and $\{0, 1, 3, 5, 7, \ldots\}$ are both minimal sets closed under the above rules.

Using the same representation methodology to represent this inductive definition as in the monotonic case, we would obtain the formula

$$\varphi_{\text{even}}(x, E) := x = 0 \lor \exists y (x = S(y) \land \neg E(y)).$$

In the context of the natural numbers, the operator characterised by this formula is non-monotone and maps any set $S$ of natural numbers to the set consisting of $0$ and all successors of all numbers in the complement of $S$. This is a non-monotone operator which has the set of even numbers as unique fixpoint. In general, the set defined by this type of inductive definitions can be characterised as the unique fixpoint of the operator associated to the definition. This will be formalised in section 7.

Other examples of non-monotone inductive definitions over well-founded orders are given in [Denecker et al. 2001a]. They include a definition of the concept of a rank of an element in a well-founded set (the rank of an element $x$ is the least ordinal strictly larger than the rank of all $y < x$), and a definition of the levels of a monotone operator in the least fixpoint construction. Although induction over a well-founded set is a common principle in mathematics, to our knowledge it has not been studied explicitly in mathematical logic. However, we will argue below that it can be seen as a simple form of iterated induction.
2.3 Inflationary Induction.

In order to extend his theory of inductive definitions to the class of all definitions (monotone and non-monotone), Moschovakis [Moschovakis 1974b] proposed the following approach. The idea is to associate with an arbitrary formula $\varphi(\vec{x}, X)$ (possibly non-positive) the operator $\Gamma'_{\varphi(\vec{x}, X)}$, where

$$\Gamma'_{\varphi(\vec{x}, X)}(R) := \Gamma_{\varphi(\vec{x}, X)}(R) \cup R.$$  

Operator $\Gamma'_{\varphi(\vec{x}, X)}$ is not monotone, but it is inflationary, that is, for every $R$, $R \subseteq \Gamma'_{\varphi(\vec{x}, X)}(R)$. Thus, by iterating this operator starting at the empty relation, an ascending sequence can be constructed. This sequence eventually reaches a fixpoint of $\Gamma'_{\varphi(\vec{x}, X)}$. This fixpoint was later called the inflationary fixpoint, and the corresponding logic $\text{FO(IFP)}$ was introduced [Gurevich and Shelah 1986]. This logic introduces inflationary, and its dual, deflationary, fixpoint constructs. The inflationary fixpoint logic played an important role in descriptive complexity theory and has been used to characterize the complexity class $\text{PTIME}$ [Immerson 1986; Livchak 1983; Vardi 1982].

Inflationary induction and induction over a well-founded order are two different principles. Consider, for example, the definition of the even numbers presented above. The formula $\varphi_{\text{even}}$ is a natural representation of this definition. However, the inflationary fixpoint $[\text{IFP}_{x,E}\varphi_{\text{even}}]^I$ is the set of all natural numbers. Indeed, $\emptyset \cup \Gamma_{\text{even}}(\emptyset) = \mathbb{N}$ and $\mathbb{N} \cup \Gamma_{\text{even}}(\mathbb{N}) = \mathbb{N}$. Even though it is possible to write down a definition of the even numbers using inflationary fixpoints, such an encoding would be neither natural nor direct. It would not reflect the way in which mathematicians express induction over a well-founded order. Since our goal is to formalize the latter sort of induction in a way that reflects the natural rule-based structure in which mathematicians represent such definitions, this paper will not be concerned with inflationary fixpoints. For examples where inflationary and deflationary inductions naturally appear, we address the reader to the work by Grädel and Kreutzer [Grädel and Kreutzer 2003].

2.4 Iterated Inductive Induction.

The basic idea underlying induction is to iterate a basic construction step until a fixpoint is reached. In an iterated induction, this basic construction step itself is a monotone induction. That is, an iterated inductive definition constructs an object as the limit of a sequence of constructive steps, each of which itself is a monotone induction. One can formulate the intuition of the iterated induction of a structure also in the following way. Given a mathematical structure $M_0$ of functions and relations, a positive or monotone inductive definition defines one or more new relations in terms of $M_0$. The definition of these new relations may depend positively or negatively on the relations given in $M_0$. Once the interpretation of the new relations is fixed, $M_0$ can be extended with these, yielding a new extended structure $M_1$. On top of this structure, again new relations may be defined in the similar way as before. The definition of these new predicates may now depend positively or negatively on the relations that were defined in $M_1$. This modular principle can be iterated arbitrarily often, possibly a transfinite number of times. We call this informal principle the principle of Iterated Induction. In general, an
iterated inductive definition must describe, in a finite way, a possibly transfinite sequence of monotone or positive definitions of sets. If the definition of a set depends (positively or negatively) on another defined set, then this other set must be defined in an earlier definition in this sequence.

An example of an iterated inductive definition mentioned in [Denecker et al. 2001a] is the definition of the stable theory of some propositional theory $T$. Basically, this is the standard concept of deductive closure of a propositional theory $T$ under a standard set of inference rules augmented with two additional inference rules:

$$\vdash \psi \text{ and } \nvDash \psi \vdash K\psi.$$  

Note that the second rule is non-monotone. The stable theory of $T$ is a deductively closed modal theory which contains explicit formulas representing whether $T$ “knows” a formula $\psi$ or not. It can be viewed as the set of formulas known by an ideally rational agent with perfect introspection whose base beliefs are represented by $T$.

Let us consider this induction process in more detail. We define the modal nesting depth of a formula $F$ as the length $n$ of the longest sequence $(KF_1, KF_2, \ldots, KF_n)$ such that $F$ contains $KF_1$ and $F_i$ contains $KF_{i+1}$ for each $1 \leq i < n$. The start of the iterated induction is a monotone induction closing $T$ under the propositional logic inference rules. This yields a deductively closed set $T_0$ of propositional formulas of modal nesting depth 0. Next we apply the two modal inference rules to infer modal literals $K\psi$ or $\neg K\psi$, for each propositional formula $\psi$. After computing these literals, we reapply the first step and derive, using the standard inference rules, all logical consequences with modal nesting depth being less or equal to 1. This process can now be iterated for formulas with increasing modal nesting depth. The result of this construction process is the stable theory of $T$ and contains formulas of arbitrary modal nesting depth. It was shown in [Marek 1989] that the stable theory of $T$ is exactly the collection of all modal formulas that are true in the possible world set $W$ consisting of all models of $T$. More precisely, it holds that the stable theory of $T$ is the set of all modal formulas $F$ such that for the collection $W$ of models of $T$ and for each model $M \in W$, it holds that $W, M \models F$.

Iterated Induction is a generalisation of monotone induction. It is also related to induction over a well-founded order. The link is seen if we split up a definition of the latter kind in an infinite number of definitions, each defining a single ground atom, and ordering or stratifying these definitions in a sequence compatible with the well-ordering. For example, even numbers could be defined by the following iterated definition:

\begin{align*}
(0) & \quad 0 \text{ is even} \\
(1) & \quad 1 \text{ is even if } 0 \text{ is not even} \\
(2) & \quad 2 \text{ is even if } 1 \text{ is not even} \\
& \quad \vdots \\
(n+1) & \quad n + 1 \text{ is even if } n \text{ is not even} \\
& \quad \vdots
\end{align*}

\footnote{This stratification corresponds to the notion of local stratification in logic programming [Przymusinski 1988].}
Clearly, the iterated induction described here constructs the set of even numbers. We can thus view an inductive definition in a well-founded set as an iterated inductive definition consisting of a sequence of non-inductive (recursion-free) definitions. Iterated induction is more general than induction over a well-founded set because positive recursion within one level may be involved (as illustrated by the stable theory example).

The logical study of iterated induction was started in [Kreisel 1963] and extended in later studies of so-called Iterated Inductive Definitions (IID) in [Feferman 1970], [Martin-Löf 1971], and [Buchholz et al. 1981]. The IID formalism defined in [Feferman 1970; Buchholz et al. 1981] is a formalism to define sets of natural numbers through iterated induction. To represent an iterated inductive definition of a set \( H \), one associates with each natural number an appropriate level index, an ordinal number. This level index can be understood as the index of the subdefinition which determines whether the number belongs to the defined set or not. The iterated inductive definition is described by a finite parametrised formula \( \varphi(n, x, P, H) \), where \( n \) represents a level index, \( x \) is a natural number, \( P \) is a unary predicate variable with only positive occurrences in \( \varphi \) and ranging over natural numbers, and \( H \) is the defined relation represented as a binary predicate ranging over tuples \((n, x)\) of natural numbers \( x \) and their level indices \( n \). The formula \( \varphi(n, x, P, H) \) encodes that \( n \) is the level index of \( x \), and \( x \) can be derived (using the inductive definition with level index \( n \)) from the set \( P \) and the restriction of \( H \) to tuples with level index \( < n \). Using \( \varphi \), the set \( H \) is characterised by two axioms. The first one expresses that \( H \) is closed under \( \varphi \):

\[
\forall n \forall x \ (\varphi(P(n)/H(n,x)) \rightarrow H(n,x)).
\]

In this formula, \( \varphi(P(n)/H(n,x)) \) (where \( n \) is an arbitrary term) denotes the formula obtained from \( \varphi \) by substituting \( H(n,x) \) for each expression \( P(n) \).

The second axiom is a second-order axiom expressing that for each \( n \), the subset \{ \( x \mid (n, x) \in H \) \} of \( \mathbb{N} \) is the least set of natural numbers closed under \( \varphi \):

\[
\forall n \forall P \ \forall x \ (\varphi \rightarrow P(x)) \rightarrow \forall x \ (H(n,x) \rightarrow P(x)).
\]

As an example, let us encode the non-monotone definition of even numbers in the IID-formalism. It is a definition by induction on the standard order of natural numbers which means that we can take a natural number and its level index to be identical. The formula \( \varphi \) to be inserted in the axioms above is\(^3\):

\[
(n = 0 \land x = 0) \lor \exists y(n = s(y) \land x = s(y) \land \neg H(y, y) \land y < n).
\]

This formula represents that \((n, x)\) can be derived if \( x \) and its level index \( n \) are identical and if \( x = 0 \) or if the predecessor of \( x \) is not even.

2.5 A Preview of ID-Logic

In this paper, we design a logic for formalising several forms of inductive definitions. Just as the principle of Monotone Induction has been formalised in FO(LFP), the principle of Inflationary Induction has been formalized in FO(IFP), the principles

\(^3\)The formula doesn’t contain the predicate variable \( P \) because this is a definition over a well-founded order which does not involve monotone induction.
of Induction over a well-founded order and Iterated Induction are captured by our logic. We call it a Logic for Non-Monotone Inductive Definitions (ID-logic).

The logic is designed as an extension of classical logic with definitions. A definition will be represented as a set of rules of the form:

\[ \forall \bar{x}(P(\bar{x}) \leftarrow \psi), \]

where \( P \) is a relational symbol defined by the definition, and \( \psi \) an arbitrary first-order formula. For example, the non-monotone definition of even numbers will be represented by the set:

\[
\{ \forall x(E(x) \leftarrow x = 0), \\
\forall x(E(s(x)) \leftarrow \neg E(x)) \}
\]

From a representational point of view, this syntax has some interesting features:

— **Rule-based representation.** Formalisations of definitions in ID-logic preserve the rule-based structure of definitions in mathematics. Stated differently, rules in a mathematical definition can be formalised in a modular way by definitional rules in our logic.

— **Uniform formalisation of different types of definitions.** Syntax and semantics of ID-logic is designed for uniform formalisation of non-inductive (recursion-free) definitions, positive or monotone inductive definitions, definitions over well-founded sets and iterated inductive definitions.

— **No explicit level mapping.** A model of an ID-logic definition is constructed following the natural dependency order on defined atoms that is induced by the rules. As a consequence, and contrary to the IID-formalism of the previous section, there is no need to explicitly represent a level mapping of an iterated inductive definition.

— **Simultaneous induction.** Consider for example the following simultaneous inductive definition of even and odd numbers:

\[
\{ \forall x(E(x) \leftarrow x = 0), \\
\forall x(E(s(x)) \leftarrow O(x)), \\
\forall x(O(s(x)) \leftarrow E(x)) \}
\]

— **A logic with second-order variables.** ID-logic allows second-order variables and quantification. As an example, consider the following sentence of ID-logic:

\[ \exists P(\{ \forall x(P(x) \leftarrow x = 0), \\
\forall x(P(s(x)) \leftarrow P(x)) \} \land \forall x P(x)). \]

This axiom, stating that the least set \( P \) containing 0 and closed under the successor operation contains all domain elements, is an ID-logic formalisation of the second-order induction axiom of the natural numbers.

The main differences between ID-logic and the IID-formalism of Section 2.4 are its rule-based nature and the absence of an explicit encoding of a level mapping. The rules of an inductive definition induce an implicit dependency order on the defined atoms. For example, in the definition of even numbers, the rule \( \forall x(E(s(x)) \leftarrow \neg E(x)) \) induces a dependency of each atom \( E[n+1] \) on the atom \( E[n] \). Notice that the transitive closure of this dependency relation corresponds with the standard...
order of the natural numbers, the well-founded order over which the set of even
numbers is defined by this definition. This suggests that the encoding of the level
mapping in the IID-formalism only adds redundant information to the definition.

In ID-logic, the construction of the model of a definition proceeds by following
the implicit dependency order that is induced by the rules. The technique to do this
was developed in logic programming. In [Denecker 1998; Denecker et al. 2001a],
Denecker proposed the thesis that the well-founded semantics of logic programming
[Van Gelder et al. 1991] provides a general and robust formalization of the principle
of iterated induction. In Section 4, we recall this argument and show how the
construction of the well-founded model can be seen as an iterated induction which
follows the natural dependency order induced by the rules.

3. PRELIMINARIES
3.1 Preliminaries from Logic

We begin by fixing notation and terminology for the basic syntactic and semantic
notions related to first- and second-order logic.

We assume an infinite supply of distinct symbols, which are classified as follows:

1. Logical symbols:
   a) Parentheses: (,);
   b) Logical connectives: ∧, ¬;
   c) Existential quantifier: ∃;
   d) Binary equality symbol: = (optional);
   e) Two propositional symbols: t and f.

2. Non-logical symbols:
   a) countably many object symbols. Object symbols are denoted by low-case
      letters;
   b) for each positive integer \( n > 0 \), countably many \( n \)-ary function symbols of
      arity \( n \). Function symbols are denoted by low-case letters;
   c) for each positive integer \( n \), countably many \( n \)-ary relation symbols, also called
      predicate or set symbols of arity \( n \). We use upper-case letters to denote pred-
      icates.

As usual, we identify object symbols with 0-ary function symbols and propositional
symbols with predicate symbols of arity 0.

Remark 3.1. In most parts of this paper, we do not make a formal distinction
between variable and constant symbols. Symbols occurring free in a formula can be
viewed as constants; symbols in the scope of a quantifier can be viewed as variables.
In examples, we tend to quantify over \( x, y, X, Y \), and leave \( c, g, f \) and \( P, Q \) free
and treat them as constants.

We define a vocabulary as any set of non-logical symbols. We denote vocabularies
by \( \tau, \tau_2, \ldots \). We shall denote the set of function symbols of \( \tau \) by \( \tau_{\text{fn}} \), and we use
\( \sigma, \sigma_1, \sigma_2 \) etc., to refer to an arbitrary symbol of the vocabulary. We write \( \bar{\sigma} \) to
denote a sequence of symbols \( (\sigma_1, \sigma_2, \ldots) \) or, depending on the context, simply the
set of symbols \( \{\sigma_1, \sigma_2, \ldots\} \). Likewise, \( \bar{X} \) denotes a sequence or a set of relational
symbols (i.e., set variables or constants), and \( \bar{x} \) is used to denote a sequence or a set of object symbols, etc..

A term is defined inductively as follows:

- an object symbol is a term;
- if \( t_1, \ldots, t_n \) are terms and \( f \) is an \( n \)-ary function symbol, where \( n \geq 1 \), then \( f(t_1, \ldots, t_n) \) is a term.

A formula is defined by the following induction:

- if \( P \) is an \( n \)-ary predicate constant or variable, and \( t_1, \ldots, t_n \) are terms then \( P(t_1, \ldots, t_n) \) is a formula, called an atomic formula or simply an atom;
- if \( \phi, \psi \) are formulas, then so are \( \neg \phi, \phi \land \psi \);
- if \( x \) is a function symbol, \( f \) a function symbol, \( X \) is a predicate symbol and \( \phi \) is a formula, then \( \exists x \phi, \exists f \phi \) and \( \exists X \phi \) are formulas.

A bounded occurrence of symbol \( \sigma \) in formula \( \phi \) is an occurrence of \( \sigma \) in a subformula \( \exists \sigma \phi \) of \( \phi \). A free occurrence of \( \sigma \) in \( \phi \) is an unbounded occurrence. The set of symbols which occur free in \( \phi \) is denoted \( \text{free}(\phi) \). This set can also be defined inductively:

- If \( \phi \) is atomic, say of the form \( A(t_1, \ldots, t_n) \) then the set \( \text{free}(\phi) \) is the set of all object, relational and functional symbols occurring in \( \phi \);
- \( \text{free}(\neg \phi) := \text{free}(\phi) \);
- \( \text{free}(\phi \land \psi) := \text{free}(\phi) \cup \text{free}(\psi) \);
- \( \text{free}(\exists \sigma \phi) := \text{free}(\phi) \setminus \{\sigma\} \).

A relation symbol \( X \) has a negative (positive) occurrence in formula \( F \) if \( X \) has a free occurrence in the scope of an odd (even) number of occurrences of the negation symbol \( \neg \).

A formula \( \phi \) is a formula over vocabulary \( \tau \) if its free symbols belong to \( \tau \) (\( \text{free}(\phi) \subseteq \tau \)). We use \( \text{SO}[\tau] \) to denote the set of all formulas over \( \tau \); and we use \( \text{FO}[\tau] \) to denote the set of first-order formulas over \( \tau \), that is those without quantified predicate or function variables.

We use \( (\phi \lor \psi), (\phi \lor \psi), (\phi \equiv \psi), \forall x \phi, \forall f \phi \) and \( \forall X \phi \), in the standard way, as abbreviations for the formulas \( \neg(\neg \phi \land \neg \psi), \neg(\phi \land \neg \psi), \neg(\phi \land \neg \psi) \land \neg(\psi \land \phi), \neg\exists x (\neg \phi), \neg\exists f (\neg \phi), \neg\exists X (\neg \phi) \), respectively.

Having defined the basic syntactic concepts, we define the semantic concepts. Let \( A \) be a nonempty set. A value for an \( n \)-ary relation (function) symbol \( \sigma \) of vocabulary \( \tau \) in \( A \) is a \( n \)-ary relation (function) in \( A \). A value for a 0-ary function symbol, i.e., an object constant or variable, is an element of the domain \( A \). A value for a 0-ary relation symbol \( Y \) is either \( \emptyset \) or \( \{()\} \), the singleton of the empty tuple.

We identify these two values with \( \text{false} \), respectively \( \text{true} \). The value of the equality symbol is always the identity relation on \( A \). The value of \( \text{t} \) is \( \{()\} \) (true) and the value of \( f \) is \( \emptyset \) (false).

A structure \( I \) for a given vocabulary \( \tau \) (in short, a \( \tau \)-structure) is tuple of a domain \( \text{dom}(I) \), which is a non-empty set, and a mapping of each symbol \( \sigma \) in \( \tau \) to a value \( \sigma^I \) in \( \text{dom}(I) \). If \( \sigma \in \tau \) and \( I \) is a \( \tau \)-structure, we say that \( I \) interprets \( \sigma \).
We also use letters $J$, $K$, $L$, $M$ to denote structures. Given $I$, $\tau_I$ denotes the set of symbols interpreted by $I$.

Let us introduce notation for constructing and modifying structures with a shared domain $A$. Let $I$ be a $\tau$-structure, and $\bar{\sigma}$ be a tuple of symbols not necessarily in $\tau$. Structure $I[\bar{\sigma} : \bar{v}]$ is a $\tau \cup \bar{\sigma}$-structure, which is the same as $I$, except symbols $\bar{\sigma}$ are interpreted by values $\bar{v}$ in $\text{dom}(I)$. Given a $\tau$-structure $I$ and a sub-vocabulary $\tau'$ of $\tau$, the restriction of $I$ to the symbols of $\tau'$ is denoted $I|_{\tau'}$.

Let $t$ be a term, and let $I$ be a structure interpreting each symbol in $t$. We define the denotation $t^I$ of $t$ under $I$ by the usual induction:

- if $t$ is an object symbol $\sigma$, then $t^I$ is $\sigma^I$, the value of $\sigma$ in $I$;
- if $t = f(t_1, \ldots, t_n)$, then $t^I := f^I(t_1^I, \ldots, t_n^I)$.

Next we define the satisfaction or truth relation $\models$. Let $I$ be a structure and let $\phi$ be a formula such that each free symbol in $\phi$ is interpreted by $I$. We define $I \models \phi$ (in words, $\phi$ is true in $I$, or $I$ satisfies $\phi$) by the following standard induction:

- $I \models X(t_1, \ldots, t_n)$ if $(t_1^I, \ldots, t_n^I) \in X^I$;
- $I \models \psi_1 \land \psi_2$ if $I \models \psi_1$ and $I \models \psi_2$;
- $I \models \neg \psi$ if $I \not\models \psi$;
- $I \models \exists \sigma \psi$ if $I \models \psi$ for some value $v$ of $\sigma$ in the domain $\text{dom}(I)$ of $I$, $I[\sigma : v] \models \psi$.

Note that the truth of a formula $\phi$ is only well-defined in a structure interpreting each free symbol of $\phi$. We shall denote the truth value of $\phi$ in $I$ by $\phi^I$, i.e., if $I \models \phi$ then $\phi^I$ is true ($\{\})$ and otherwise, it is false ($\emptyset$).

Sometimes, we wish to investigate the truth value of a formula $\phi$ as a function of the values assigned to a specific tuple of symbols $\bar{\sigma}$. We then call these symbols the parameters of $\phi$ and denote the formula by $\phi(\bar{\sigma})$. Let $I$ be some structure and let $\bar{v}$ be a tuple of values for $\bar{\sigma}$ in the domain $\text{dom}(I)$. We often write $I \models \phi[\bar{v}]$ to denote $I[\bar{\sigma} : \bar{v}] \models \phi$.

Let $X$ be an $n$-ary relation symbol and $\bar{d}$ be an $n$-tuple of elements of some domain $A$. We define a domain atom in $A$ as $X[\bar{d}]$. For $I$ a structure with domain $A$, the value of $X[\bar{d}]$ in $I$ is true if $\bar{d} \in X^I$; otherwise it is false. For a vocabulary $\tau$, we define $\text{At}^\tau_A$ as the set of all domain atoms in domain $A$ over relation symbols in $\tau$.

Suppose we are given a structure $I$ with domain $\text{dom}(I)$, a tuple $\bar{x}$ of $n$ variables and a first-order formula $\phi(\bar{x})$ such that all its free symbols not in $\bar{x}$ are interpreted by $I$. The relation defined by $\phi(\bar{x})$ in the structure $I$ is defined as follows:

$$R := \{ \bar{a} \mid I \models \phi(\bar{a}), \; \bar{a} \in (\text{dom}(I))^n \}.$$  

We call $R$ first-order definable in $I$. In this paper, we study inductive and non-monotone inductive definability. In this context, defined relations are not, in general, first-order definable.

### 3.2 Preliminaries from Set and Lattice Theories

#### 3.2.1 Orders, Lattices, operators and fixpoints

A pre-ordered set is a structured set $(W, \leq)$, where $W$ is an arbitrary set and $\leq$ is a pre-order on $W$, i.e., a reflexive and transitive binary relation. As usual, $x < y$ is a shorthand for $x \leq y \land y \not\leq x$.  

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A **pre-well-founded set** is a pre-ordered set where \( \leq \) is a pre-order such that every non-empty set \( S \subseteq W \) contains a minimal element, i.e., an element \( x \) such that for each \( y \in S \), if \( y \leq x \) then \( x \leq y \). Equivalently, it is a set without infinite descending sequence of elements \( x_0 > x_1 > x_2 > \ldots \).

A **partially ordered set**, or simply poset, is an asymmetric pre-ordered set \( \langle W, \leq \rangle \), i.e., one such that \( x \leq y \) and \( y \leq x \) implies \( x = y \). A **well-founded set** is a pre-well-founded poset.

A **lattice** is a poset \( \langle L, \leq \rangle \) such that every finite set \( S \subseteq L \) has a least upper bound \( \text{ lub}(S) \), the *supremum* of \( S \), and a greatest lower bound \( \text{ glb}(S) \), the *infimum* of \( S \). A lattice \( \langle L, \leq \rangle \) is **complete** if every (not necessarily finite) subset of \( L \) has both a supremum and an infimum. Consequently, a complete lattice has a least element \( (\bot) \) and a greatest element \( (\top) \). An example of a complete lattice is the power set lattice \( \langle \text{ Pow}(A), \subseteq \rangle \) of some set \( A \). For any set \( S \) of elements of this lattice (i.e., for any set \( S \) of subsets of \( A \)), its least upper bound is the union of these elements, \( \text{ lub}(S) = \cup S \). Thus, the greatest element \( \top \) of \( \langle \text{ Pow}(A), \subseteq \rangle \) is \( \cup \text{ Pow}(A) \), which is \( A \).

Similarly, \( \text{ glb}(S) = \cap S \), and the least element \( \bot \) of this lattice is \( \cap \text{ Pow}(A) \), which is \( \emptyset \).

Given a lattice \( \langle L, \leq \rangle \), an operator \( \Gamma : L \rightarrow L \) is **monotone** with respect to \( \leq \) if \( x \leq y \) implies \( \Gamma(x) \leq \Gamma(y) \). Operator \( \Gamma \) is **non-monotone**, if it is not monotone. A **pre-fixpoint** of \( \Gamma \) is a lattice element \( x \) such that \( \Gamma(x) \leq x \). The following theorem was obtained by Tarski in 1939 and is sometimes referred to as the Knaster-Tarski theorem because it improves their earlier joint result. The theorem was published in [Tarski 1955], and it is one of the basic tools to study fixpoints of operators on lattices.

**Theorem 3.2 Existence of a Least Fixpoint.** Every monotone operator over a complete lattice \( \langle W, \leq \rangle \) has a complete lattice of fixpoints (and hence a least fixpoint \( \text{ lfp}(\Gamma) \) and greatest fixpoint \( \text{ gfp}(\Gamma) \)).

This least fixpoint \( \text{ lfp}(\Gamma) \) is the least pre-fixpoint of \( \Gamma \) and is the supremum of the sequence \( (x^\xi)_{\xi} \) which is defined inductively

\[
x^\xi := \Gamma(x^{\xi-}) \quad \text{and} \quad x^{\xi-} := \text{ lub}\{x^\eta : 0 \leq \eta < \xi\}.
\]

Notice that \( x^{\xi-} \) is, by definition, \( \bot \).

An operator \( \Gamma \) is **anti-monotone** if \( x \leq y \) implies \( \Gamma(y) \leq \Gamma(x) \).

**Proposition 3.3.** If \( \Gamma_1 \) and \( \Gamma_2 \) are anti-monotone operators, then \( \Gamma_1 \circ \Gamma_2 \), the composition of \( \Gamma_1 \) and \( \Gamma_2 \), is monotone.

In particular, the square \( \Gamma^2 = \Gamma \circ \Gamma \) of an anti-monotone operator is monotone.

An **oscillating pair** of an operator \( \Gamma \) is a pair \( (x, y) \) such that \( \Gamma(x) = y \) and \( \Gamma(y) = x \). An anti-monotone operator \( \Gamma \) in a complete lattice has a maximal oscillating pair \( (x, y) \), i.e., for any oscillating pair \( (x', y') \), it holds that \( x \leq x' \) and \( y' \leq y \). Since \( (y, x) \) is also an oscillating pair, it follows that \( x \leq y \). Moreover, since each fixpoint \( z \) of \( \Gamma \) corresponds to an oscillating pair \( (z, z) \), it follows that \( x \leq z \leq y \). The maximal oscillating pair \( (x, y) \) of \( \Gamma \) can be constructed by an alternating fixpoint computation. Define four sequences \( (x^\xi), (x^{\xi-}), (y^\xi), (y^{\xi-}) \) by the following transfinite induction:

\[
- \quad x^{\xi-} = \text{ lub}\{x^\eta : \eta < \xi\},
\]

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- $x^\xi = \Gamma(y^{<\xi})$,
- $y^{<\xi} = \text{glb}\{y^n : \eta < \xi\}$,
- $y^\xi = \Gamma(x^{<\xi})$.

Note that $x^{<0} = \bot$ and $y^{<0} = \top$. It can be shown that for each $\xi$, $x^{<\xi} \leq x^\xi \leq y^\xi \leq y^{<\xi}$. The following theorem holds.

**Theorem 3.4.** [Van Gelder 1993] The sequence $(x^\xi)_\xi$ is ascending and its supremum is lfp$(\Gamma^2)$. The sequence $(y^{<\xi})_\xi$ is descending and its infimum is gfp$(\Gamma^2)$. The pair $(\text{lfp}(\Gamma^2), \text{gfp}(\Gamma^2))$ is the maximal oscillating pair of $\Gamma$.

We will use the following simple lemma on lattices.

**Lemma 3.5.** Let $\Gamma_1, \Gamma_2$ be two monotone operators in a lattice with least fixpoints $\text{lfp}(\Gamma_1) = o_1, \text{lfp}(\Gamma_2) = o_2$ respectively.

(a) if $\Gamma_1(x) \leq \Gamma_2(x)$ for each $x \leq o_1$ then $o_1 \leq o_2$;

(b) if $\Gamma_1(x) \leq \Gamma_2(x)$ for each $x \geq o_2$ then $o_1 \leq o_2$.

**Proof.** (a) Define $o_1^\xi$ and $o_1^{<\xi}$ by induction:

- $o_1^{<\xi} := \text{lub}\{o_1^n : \eta < \xi\}$,
- $o_1^\xi := \Gamma(o_1^{<\xi})$.

Then $o_1$ is the limit of the increasing sequence $(o_1^\xi)_\xi$. Moreover, for each $\xi : o_1^\xi \leq o_1$ and $o_1^{<\xi} \leq o_1$.

The proof is by transfinite induction. Obviously $o_1^0 = \Gamma_1(\bot) \leq \Gamma_2(\bot) = o_2^0$. Assume that for each $\eta < \xi$, $o_1^\eta \leq o_2^\eta$. Then also $o_1^{<\xi} \leq o_2^{<\xi}$. Then $o_1^\xi = \Gamma_1(o_1^{<\xi}) \leq \Gamma_2(o_2^{<\xi}) \leq \Gamma_2(o_2^\eta) = o_2^\eta$.

(b) It holds that $o_1$ is the least fixpoint and hence the least pre-fixpoint of $\Gamma_1$. Hence $o_1 = \text{glb}\{x : \Gamma_1(x) \leq x\}$. Since $\Gamma_1(x) \leq \Gamma_2(x)$ for each $x \geq o_2$, it holds that if $x$ is a pre-fixpoint of $\Gamma_2$, then $x$ is also a pre-fixpoint of $\Gamma_1$. Thus we have $\{x : \Gamma_2(x) \leq x\} \subseteq \{x : \Gamma_1(x) \leq x\}$. Since $o_1 = \text{glb}\{x : \Gamma_1(x) \leq x\}$, we have $o_1 \leq o_2$. □

### 3.2.2 Lattice Homomorphisms and Congruences

Let $\langle L, \leq \rangle$ be a complete lattice and let $\equiv$ be an arbitrary equivalence relation (i.e., a reflexive, symmetric and transitive relation) on $L$. For any $x \in L$, we denote its equivalence class $\{y \in L \mid x \equiv y\}$ by $[x]$. The collection of equivalence classes is denoted by $L^\equiv$. The relation $\equiv$ can be extended to tuples: $(x_1, \ldots, x_n) \equiv(y_1, \ldots, y_n)$ if $x_1 \equiv y_1$ and ... and $x_n \equiv y_n$. It is extended to subsets of $L$ by defining for all $S, S' \subseteq L$: $S \equiv S'$ if for each $x \in S$ there exists $x' \in S'$ such that $x \equiv x'$ and vice versa, for each $x' \in S'$ there exists $x \in S$ such that $x \equiv x'$.

An equivalence relation $\equiv$ on $L$ is called a lattice congruence of $\langle L, \leq \rangle$ if for each pair $S, S' \subseteq L$, $S \equiv S'$ implies that $\text{lub}(S) \equiv \text{lub}(S')$ and $\text{glb}(S) \equiv \text{glb}(S')$. We can define a binary relation $\leq$ on $L^\equiv$: for all $S, S' \in L^\equiv$, define $S \leq S'$ if for some $x \in S, y \in S' : x \leq y$. It can be shown easily that if $\equiv$ is a lattice congruence, then the structure $\langle L^\equiv, \leq \rangle$ is a complete lattice.

Let $\langle L, \leq \rangle, \langle L', \leq' \rangle$ be two complete lattices. A mapping $h : L \rightarrow L'$ is called a lattice homomorphism if it is a mapping onto (i.e., $h(L) = L'$), and for each $S \subseteq L$, $h(\text{glb}_\leq(S)) = \text{glb}_{\leq'}(h(S))$ and $h(\text{lub}_\leq(S)) = \text{lub}_{\leq'}(h(S))$. 

ACM Transactions on Computational Logic, Vol. V, No. N, February 2008.
The notions of lattice congruence and lattice homomorphism are strongly related. A homomorphism \( h : L \to L' \) induces a relation \( \cong \) on \( L \) where \( x \cong y \) holds if \( h(x) = h(y) \), for all \( x, y \in L \). The relation \( \cong \) is a lattice congruence of \( \langle L, \leq \rangle \). Moreover, \( \langle L^\cong, \leq \rangle \) and \( \langle L', \leq' \rangle \) are isomorphic. Vice versa, for each lattice congruence \( \cong \), the mapping \( L \to L^\cong \) such that \( x \mapsto [x] \) is a lattice homomorphism.

Let \( h \) be a lattice homomorphism from \( \langle L, \leq \rangle \) to \( \langle L', \leq' \rangle \) and \( \cong \) the induced congruence on \( L \). We say that an operator \( O : L \to L \) preserves \( \cong \) if for all \( x, y \in L \), \( x \cong y \) implies \( O(x) \cong O(y) \). In general, for any operator \( O : L^m \to L^n \), we say that \( O \) preserves \( \cong \) if for any pair of \( \bar{x}, \bar{y} \in L^m \), \( \bar{x} \cong \bar{y} \) implies \( O(\bar{x}) \cong O(\bar{y}) \).

If \( O : L \to L \) preserves \( \cong \) then for any \( x' \in L' \), for any \( x_1, y_1 \in h^{-1}(x'), h(O(x_1)) = h(O(x_2)) \). We then define the homomorphic image \( O^h : L' \to L' \) of \( O \). This operator maps \( x' \in L' \) to \( y' \) iff for each \( x \in h^{-1}(x'), y' = h(O(x)) \). This definition can be extended to operators \( O : L^m \to L^n \).

The following proposition describes relationships between \( O \) and \( O^h \).

**Proposition 3.6.** Let \( O \) be an operator which preserves \( \cong \).

(a) If \( O \) is (anti-)monotone, then \( O^h \) is (anti-)monotone.

(b) If \( O \) is monotone then \( h(lfp(O)) = lfp(O^h) \) and \( h(gfp(O)) = gfp(O^h) \).

(c) If \( O \) is anti-monotone and \( (x, y) \) its maximal oscillating pair then \( (h(x), h(y)) \)
   is the maximal oscillating pair of \( O^h \).

**Proof.** The proof of item (a) is straightforward and is omitted.

(b) The least fixpoint \( lfp(O) \) is the limit of the sequence \( (x^\xi)_{\xi \geq 0} \) which is defined inductively

\[ x^\xi := O(x^{< \xi}), \quad \text{and} \quad x^{< \xi} := \text{lub}\{x^\eta | 0 \leq \eta < \xi\}. \]

The point \( lfp(O^h) \) is the limit of the sequence \( (y^\xi)_{\xi} \) defined similarly using \( O^h \) in the lattice \( L' \). By a straightforward induction, one can show that for each ordinal \( \xi \), \( h(x^\xi) = y^\xi \). Since \( h \) is a lattice homomorphism, \( h(lfp(O)) = h(lub\{x^\xi \mid \xi \geq 0\}) = lub\{h(x^\xi) \mid \xi \geq 0\} = lub\{y^\xi \mid \xi \geq 0\} = lfp(O^h) \). The proof that \( h(gfp(O)) = gfp(O^h) \) is similar.

(c) It is easy to show that \( (O^2)^h \) is \( (O^h)^2 \). Then (c) is a direct consequence of (b) and the fact that the maximal oscillating pair of \( O \) and \( O^h \) are \( (lfp(O^2), gfp(O^2)) \), respectively \( (lfp((O^h)^2), gfp((O^h)^2)) \).

\[ \square \]

**3.2.3 Structure lattices.** The type of lattices that play a central role in this paper are the sets of structures that extend a given structure. For a given vocabulary \( \tau \) and structure \( K_o \) such that \( \tau_{K_o} \subseteq \tau \), define \( S_{K_o}^\tau \) as the set of \( \tau \)-structures that extend \( K_o \), i.e. the set of \( \tau \)-structures \( I \) such that \( I|_{\tau_{K_o}} = K_o \).

For any pair \( I_1, I_2 \) of \( \tau \)-structures, define \( I_1 \equiv I_2 \) if both structures have the same interpreted symbols, the same domain and the same values for all object and function symbols and for each interpreted relation symbol \( X, X^{I_1} \equiv X^{I_2} \).

The structured set \( \langle S_{K_o}^\tau, \equiv \rangle \) is a partial order. In general, it is not a lattice, because elements \( I, J \) giving different interpretation to a function symbol \( f \in \tau \setminus \tau_{K_o} \) have no greatest lowerbound nor least upperbound in \( S_{K_o}^\tau \). However, if \( K_o \)
interprets all function symbols of $\tau$, that is, if $\tau^\mathsf{fn} \subseteq \tau^\mathsf{Ko}$, then $\langle S^\mathsf{Ko}, \sqsubseteq \rangle$ is a complete lattice. Its least element is the structure $\perp^\mathsf{Ko} := \mathsf{Ko}[\bar{X} : \emptyset]$ assigning the empty relations to all symbols $X$ in $\tau \setminus \tau^\mathsf{Ko}$, and its largest element $\top^\mathsf{Ko}$ is the structure assigning the cartesian product $A^n$ to each $n$-ary symbol $X \in \tau \setminus \tau^\mathsf{Ko}$.

The lattice $\langle S^\mathsf{Ko}, \sqsubseteq \rangle$ contains many sublattices. In particular, for any structure $K_0$ extending $\mathsf{Ko}$ such that $\tau^\mathsf{Ko} \subseteq \tau \subseteq \tau^\mathsf{Ko}$, $\langle S^\mathsf{Ko}, \sqsubseteq \rangle$ is a sublattice of $\langle S^\mathsf{Ko}, \sqsubseteq \rangle$.

In this paper, the family of structure lattices and homomorphisms and congruences on them play an important role.

4. ID-LOGIC

In this section, we present an extension of classical logic with non-monotone inductive definitions. This work extends previous work of the authors [Denecker 2000; Ternovskaia 1999].

4.1 Syntax

First, we introduce the notion of a definition. We introduce a new binary connective $\langle$, called the definitional implication. A definition $\Delta$ is a set of rules of the form

$$\forall \bar{x} \ (X(\bar{t}) \leftarrow \varphi) \text{ where } (1)$$

$\bar{x}$ is a tuple of object variables,

$X$ is a predicate symbol (i.e., a predicate constant or variable) of some arity $r$,

$\bar{t}$ is a tuple of terms of length $r$,

$\varphi$ is an arbitrary first-order formula.

The definitional implication $\leftarrow$ must be distinguished from material implication. A rule $\forall \bar{x} \ (X(\bar{t}) \leftarrow \varphi)$ in a definition does not correspond to the disjunction $\forall \bar{x} (X(\bar{t}) \lor \neg \varphi)$, but implies it. Note that in front of rules, we allow only universal quantifiers. In the rule (1), $X(\bar{t})$ is called the head and $\varphi$ is the body of the rule.

Example 4.1. The following expression is a simultaneous definition of the sets of even and odd numbers on the structure of the natural numbers with zero and the successor function:

$$\begin{cases} \forall x \ (E(x) \leftarrow x = 0), \\ \forall x \ (E(s(x)) \leftarrow O(x)), \\ \forall x \ (O(s(x)) \leftarrow E(x)) \end{cases}. \quad (2)$$

Example 4.2. This is the definition of the transitive closure of a directed graph $G$:

$$\begin{cases} \forall x \forall y \ (T(x, y) \leftarrow G(x, y)), \\ \forall x \forall y \ (T(x, y) \leftarrow \exists z \ (T(x, z) \land T(z, y))) \end{cases}. \quad (3)$$

The definitions of bound and free occurrence of a symbol in a formula extend to the case of a rule and a definition $\Delta$. A defined symbol of $\Delta$ is a relation symbol that occurs in the head of at least one rule of $\Delta$; other relation, object and function symbols are called open. In the Example 4.1 above, $E$ and $O$ are defined predicate symbols, and $s$ is an open function symbol. In the Example 4.2, $T$ is a defined predicate symbol, and $G$ is an open predicate symbol. We call $\Delta$ a positive
definition if no defined predicate \( X \) has a negative occurrence in the body of a rule of \( \Delta \). The definitions in Example 4.1 and Example 4.2 are positive.

Let \( \tau \) be a vocabulary interpreting all free symbols of \( \Delta \). The subset of defined symbols of definition \( \Delta \) is denoted \( \tau^d_\Delta \). The set of open symbols of \( \Delta \) in \( \tau \) is denoted \( \tau^o_\Delta \). The sets \( \tau^d_\Delta \) and \( \tau^o_\Delta \) form a partition of \( \tau \), i.e., \( \tau^d_\Delta \cup \tau^o_\Delta = \tau \), and \( \tau^d_\Delta \cap \tau^o_\Delta = \emptyset \).

Now we are ready to define the well-formed formulas of the logic. A well-formed formula of the Logic for Non-Monotone Inductive Definitions, briefly a ID-formula, is defined by the following induction:

— If \( X \) is an \( n \)-ary predicate symbol, and \( t_1, \ldots, t_n \) are terms then \( X(t_1, \ldots, t_n) \) is a formula.

— If \( \Delta \) is a definition then \( \Delta \) is a formula.

— If \( \phi \), \( \psi \) are formulas, then so are \( (\neg \phi) \) and \( (\phi \land \psi) \).

— If \( \phi \) is a formula, then \( \exists \sigma \phi \) is a formula.

The definitions of bound and free occurrence of a symbol in a formula (see Section 3) extend to ID-formulas \( \phi \). We shall denote the set of symbols with free occurrences in \( \phi \) by \( \text{free}(\phi) \).

A formula \( \phi \) is an ID-formula over a vocabulary \( \tau \) if \( \text{free}(\phi) \subseteq \tau \). We use \( \text{SO(ID)}[\tau] \) to denote the set of all formulas of our logic over fixed vocabulary \( \tau \). The first-order fragment \( \text{FO(ID)}[\tau] \) is defined in the same way, except that quantification over set and function symbols is not allowed.

**Example 4.3.** In the structure of the natural numbers, the following formula expresses that \( E \) and \( O \) are respectively the set of even and odd numbers, and that the number 2, which is represented by \( s(0) \), belongs to \( E \).

\[
\begin{cases}
\forall x \ (E(x) \leftarrow x = 0), \\
\forall x \ (E(s(x)) \leftarrow O(x)), \\
\forall x \ (O(s(x)) \leftarrow E(x))
\end{cases}
\land E(s(0)). \tag{4}
\]

**Example 4.4.** The Peano induction axiom is:

\[
\forall P [P(0) \land \forall n (P(n) \supset P(s(n))) \supset \forall n P(n)].
\]

This axiom can be formulated in ID-logic as:

\[
\exists N \left[ \left\{ \forall x \ (N(x) \leftarrow x = 0), \forall x \ (N(s(x)) \leftarrow N(x)) \right\} \land \forall x \ N(x) \right]. \tag{5}
\]

The first conjunct in this formula defines the set variable \( N \) as the set of the natural numbers through the standard induction. The second conjunct expresses that each domain element is a natural number. An equivalent alternative formalisation is:

\[
\forall N \left[ \left\{ \forall x \ (N(x) \leftarrow x = 0), \forall x \ (N(s(x)) \leftarrow N(x)) \right\} \supset \forall x \ N(x) \right]. \tag{6}
\]

The equivalence of axioms (5) and (6) follows from the fact that the defined set is unique. The uniqueness is guaranteed by the semantics we define next.

In the sequel, we use \( T_H \) to denote the ID-theory consisting of axiom (5) and the two other Peano axioms:

\[
\forall n \ (s(n) = 0),
\forall n \forall m \ (s(n) = s(m) \supset n = m).
\]
4.2 Semantics

The exposition below is a synthesis of different approaches to the well-founded semantics, in particular those presented in [Van Gelder 1993; Fitting 2002; Denecker et al. 2001a]. We begin by defining the operator associated with a definition ∆. We shall assume that definitions are finite sets of rules. The theory can easily be extended to the infinite case (using infinitary logic).

Any definition containing multiple rules with the same predicate in the head can be easily transformed into a definition with only one rule per defined predicate.

Example 4.5. The following definition of even numbers

\[
\{ \forall x (E(y) \leftarrow y = 0), \\
\forall x (E(s(s(x))) \leftarrow E(x)) \}
\]

is equivalent to this one:

\[
\{ \forall x (E(y) \leftarrow y = 0 \lor \exists x (y = s(x) \land E(x))) \}
\]

In general, let ∆ be an arbitrary definition with defined relational symbols \( \vec{X} := (X_1, \ldots, X_n) \). For each defined symbol \( X \in \Delta \), we define:

\[
\varphi_X(\vec{x}) := \exists \bar{y}_1 (\bar{x} = t_1 \land \varphi_1) \lor \cdots \lor \exists \bar{y}_m (\bar{x} = t_m \land \varphi_m),
\]

where \( \bar{x} \) is a tuple of new object variables, and \( \forall \bar{y}_1 (X(t_1) \leftarrow \varphi_1), \ldots, \forall \bar{y}_m (X(t_m) \leftarrow \varphi_m) \) are the rules of ∆ with \( X \) in the head. Then ∆ is equivalent to the definition \( \Delta' \) consisting of rules \( \forall \bar{x} (X(\bar{x}) \leftarrow \varphi_X(\bar{x})) \). The formulas \( \varphi_X(\bar{x}) \) play an important role in defining the semantics of definitions.

Let ∆ be a definition over a vocabulary τ.

Definition 4.6 operator \( \Gamma_\Delta \). We introduce a total unary operator \( \Gamma_\Delta : \mathcal{I} \mapsto \mathcal{I} \) where \( \mathcal{I} \) is the class of all τ-structures. We have \( I' = \Gamma_\Delta(I) \) iff

\[-\text{dom}(I) = \text{dom}(I'),
\-\text{for each open symbol } \sigma, \quad \sigma' = \sigma^I \text{ and}
\-\text{for each defined symbol } X \in \tau^d_\Delta,

X' := \{ \bar{a} \mid I \models \varphi_X[\bar{a}] \},
\]

where \( \varphi_X \) is defined by equation (7).

Let \( I_o \) be a structure interpreting the open symbols of ∆ in τ. Lattice \( \langle S^\tau_{I_o}, \sqsubseteq \rangle \) consists of all τ-structures that extend \( I_o \). Operator \( \Gamma_\Delta \) is an operator on this lattice. If ∆ is a positive definition (no negative occurrences of defined symbols in rule bodies), then \( \Gamma_\Delta \) will be monotone. The least fixpoint is the limit of the sequence \( (I^\xi)_{\xi} \) which is defined inductively:

\[
I^\xi := \Gamma_\Delta(I^{<\xi}), \quad \text{and} \quad I^{<\xi} := \bigsqcup \{ I^\eta \mid 0 \leq \eta < \xi \}.
\]

Notice that \( I^{<0} \) is, by definition, the bottom element \( \bot_{I_o} := I_o[\vec{X} : \emptyset] \) in the lattice.

In general, \( \Gamma_\Delta \) is a non-monotone operator with no or multiple minimal fixpoints. Iterating the operator starting from the bottom element may oscillate and never reach a fixpoint, or, when it does reach a fixpoint, this fixpoint may not be the intended fixpoint.
Example 4.7. Consider the following propositional definition:

\[
\Delta_0 := \{ P \leftarrow t, \quad Q \leftarrow \neg P, \quad Q \leftarrow \perp \}.
\]

Formally, structures of \( \Delta_0 \) are mappings of the symbols \( P, Q \) to 0-ary relations. We will represent such a structure in a more traditional way as the set of the propositional symbols that are true (i.e., that are interpreted by \{()\}).

Notice that, in definition \( \Delta_0, Q \) depends on \( P \). In ID-logic this definition is understood as a 2-level iterated inductive definition \((\Delta_{01}, \Delta_{02})\), where

\[
\Delta_{01} := \{ P \leftarrow t \}; \\
\Delta_{02} := \{ Q \leftarrow \neg P, \quad Q \leftarrow \perp \}.
\]

By applying iterated induction, we obtain \{\( P \)\} for the first level, and then \( \emptyset \) for the second. Consequently, the intended model of this definition is \{\( P \)\}. On the other hand, if we iterate the operator \( \Gamma_{\Delta_0} \) from the empty structure, we obtain immediately the fixpoint \{\( P, Q \)\}.

The intuition underlying the semantics is to use definitions to perform iterated induction, while following the implicit dependency order given by the rules. We explain how this intuition is formalised in the well-founded semantics. We compute a converging sequence of pairs \((I^\xi, J^\xi)_{\xi \geq 0}\) of \(\tau\)-structures extending \(I_o\). In each pair, \(I^\xi\) represents a lower bound to the intended model of \(\Delta\) extending \(I_o\); \(J^\xi\) represents an upper bound: domain atoms true in \(I^\xi\) can be derived from the definition; atoms false in \(J^\xi\) cannot be derived; for all atoms false in \(I^\xi\) and true in \(J^\xi\), it is not determined yet whether they can be derived or not. Alternatively, a pair \(I^\xi \subseteq J^\xi\) can be understood as a 3-valued structure defining the truth value of part of the defined domain atoms, namely those domain atoms \(A\) for which \(A^{I^\xi} = A^{J^\xi}\). Thus, the pair \((I^\xi, J^\xi)\) represents approximate information about what can and what cannot be derived from \(\Delta\) in \(I_o\).

The construction process starts with the pair \((\bot_{I_o}, \top_{I_o})\) of the least and largest element in the lattice \(S^T_{I_o}\). This pair obviously consists of a lower and an upper bound of what can be derived from the definition. Assuming we have obtained a pair \((I^\xi, J^\xi)\) of a safe lower and upper bound, we then apply an operation which transforms this pair into a new pair \((I^{\xi+1}, J^{\xi+1})\) with an improved lower and upper bound. By iterating this operation, a sequence \((I^\xi, J^\xi)_{\xi \geq 0}\) of increasing precision is constructed. The sequence of lower bounds \((I^\xi)_{\xi \geq 0}\) is monotonically increasing and has a limit \(I\) (its lub); the sequence of upper bounds \((J^\xi)_{\xi \geq 0}\) is monotonically decreasing and has a limit \(J\) (its glb) such that \(I \subseteq J\). The pair of limits \((I, J)\) is the result of the construction and represents the information that can be derived from \(\Delta\) in the context of the structure \(I_o\). The definition \(\Delta\) properly defines its defined symbols in \(I_o\) if \(I = J\), that is, if for each defined domain atom \(A\), \(A^I = A^J\). If \(I = J\), then we will call \(\Delta\) total in \(I_o\) and \(I\) the extension of \(I_o\) defined by \(\Delta\). If \(I \neq J\), then there will be no extension of \(I_o\) defined by \(\Delta\).

We now explain how a pair \((I^\xi, J^\xi)\) of lower and upperbound is refined into a new pair \((I^{\xi+1}, J^{\xi+1})\). The idea is to compute the new lower bound \(I^{\xi+1}\) and upper bound \(J^{\xi+1}\) by monotone induction using the existing bounds \((I^\xi, J^\xi)\). We cannot
use $\Gamma_\Delta$ for this, due to its non-monotonicity, but there is a way.

In general, defined symbols have positive and negative occurrences in the rule bodies $\varphi_X(\bar{x})$. The negative occurrences are responsible for the non-monotone behaviour of the operator $\Gamma_\Delta$: adding more tuples to the value of a negatively occurring defined symbol in $\varphi_X(\bar{x})$ has an anti-monotone effect on the derived relation and may lead to the derivation of fewer tuples $\bar{a}$ satisfying this formula. Thus we can eliminate the non-monotonicity of $\Gamma_\Delta$ and set up a monotone induction process using $\Delta$ if we fix the value of negative occurrences of defined symbols in rule bodies. Suppose we choose a fixed structure $M$ to evaluate the negative occurrences of defined symbols in rule bodies. We can then perform a monotonic derivation process $\perp_{I_0}, K_1, K_2, \ldots$ in which each $K^{i+1}$ is derived from $\Delta$ by evaluating positive occurrences of defined symbols in each $\varphi_X(\bar{x})$ with respect to $K^i$ and negative occurrences with respect to $M$. This process will be monotone.

We first choose $M$ to be $I^\xi$: negative occurrences of defined symbols are interpreted by the lower bound of what can be derived. Thus, during the derivation process of $\perp_{I_0}, K_1, K_2, \ldots$, we systematically underestimate the truth of negative occurrences of defined predicates. Due to the anti-monotone effect of negative occurrences of defined symbols on what can be derived, in each stage $K^i$, too many atoms may be derived. Consequently, the limit of this derivation process yields an upper bound of what can be derived, and we take it to be our new upper bound $J^{\xi+1}$. Second, we choose $M$ to be $J^\xi$, our best upper bound so far on what can be derived. Thus, during the derivation process $\perp_{I_0}, L_1, L_2, \ldots$, we systematically overestimate the truth of negative occurrences of defined symbols, and in each derivation stage $K^i$, too few atoms are derived. Therefore, the limit of this sequence represents a lower bound of what can be derived and we define it to be $I^{\xi+1}$.

We have constructed our new approximating pair $(I^{\xi+1}, J^{\xi+1})$, by two monotone inductions.

It is now easy to understand in what sense the above construction follows the natural dependency order between domain atoms, induced by the rules of a definition. Assume that at some stage $(I^\xi, J^\xi)$, the truth of a domain atom $A$ has not yet been fixed (i.e. $A^{I^\xi} \neq A^{J^\xi}$), but the truth values of all atoms on which $A$ depends negatively have been derived. In the fixpoint computations $\perp_{I_0} = K_0, K_1, K_2, \ldots$, with limit $J^{\xi+1}$ and $\perp_{I_0} = L_0, L_1, L_2, \ldots$ leading to $I^{\xi+1}$, the structures $K^0$ and $L^0$ evidently coincide on all atoms on which $A$ depends, and this property is preserved during the induction, since the structures $I^\xi$ and $J^\xi$ which are used to evaluate negative occurrences of defined symbols, coincide on all atoms on which $A$ depends negatively. Therefore, the new lower and upperbounds $I^{\xi+1}$ and $J^{\xi+1}$ will coincide also on the value of $A$. Consequently, in this step the truth value of $A$ is obtained.

Now, we will formalise the above concepts. Let $\Delta$ be a definition over vocabulary $\tau$ ($\text{free}(\Delta) \subseteq \tau$). The basis of the construction of the well-founded model is an operator $T_\Delta$ mapping pairs of $\tau$-structures to $\tau$-structures. Given such a pair $(I, J)$, the operator $T_\Delta$ operates like $\Gamma_\Delta$, but evaluates the bodies of the rules in a different way. In particular, it evaluates positive occurrences of defined symbols in rule bodies by $I$, and negative occurrences of defined symbols by $J$.

To formally define this operator, we simply rename the negative occurrences in
rule bodies of $\Delta$. We extend the vocabulary $\tau$ with, for each defined symbol $X$, a new relation symbol $X'$ of the same arity. The extended vocabulary $\tau \cup X'$ will be denoted $\tau'$. Then in each rule body in $\Delta$, we substitute the symbol $X'$ for each negative occurrence of a defined symbol $X$, thus obtaining a new definition $\Delta'$. For example, given the following definition

$$X \text{ has only positive occurrences and a primed symbol } \bar{X}.$$



The definition of $\Delta'$ defines the same predicates as $\Delta$ and its open symbols are those of $\Delta$ augmented with the new primed predicates $X'$. Moreover, a defined symbol $X$ has only positive occurrences and a primed symbol $X'$ only negative occurrences in rule bodies of $\Delta'$. Thus, $\Delta'$ is a positive definition over the vocabulary $\tau'$. As described in the formula (7), with each defined symbol $X$, we construct the formula $\varphi'_X$ using $\Delta'$ instead of $\Delta$. $\varphi'_X$ can be obtained also from $\varphi_X$ by substituting $Y'$ for $Y$ in all negative occurrences of all defined symbols $Y$ in $\varphi_X$.

For any pair of $\tau$-structures $I, J$ which share the same domain, define $I_J$ as the $\tau'$-structure $J[\bar{X} : \bar{X}' : \bar{X}']$. This $I_J$ is a $\tau'$-structure which satisfies the following:

—its domain is the same as the domain of $I$ and $J$,
—each open symbol of $\Delta$ is interpreted by $J$,
—each defined symbol of $\Delta$ is interpreted by $I$,
—the value of each new symbol $X'$ is $X'$, the value of $X$ in $J$.

It is clear that for some defined symbol $X$, evaluating $\varphi'_X$ under $I_J$ simulates the non-standard evaluation of $\varphi_X$ where $J$ is “responsible” for the open and the negative occurrences of the defined predicates, while $I$ is “responsible” for the positive ones.

Let $\Delta$ be a definition over some vocabulary $\tau$.

**Definition 4.8** operator $T_\Delta$. We introduce a partially defined binary operator $T_\Delta : \mathcal{I} \times \mathcal{I} \to \mathcal{I}$, where $\mathcal{I}$ is the class of all $\tau$-structures. The operator is defined on pairs of structures which share the same domain, and is undefined otherwise. We have $\Gamma' = T_\Delta(I, J)$ iff

$-\text{dom}(\Gamma') = \text{dom}(J) = \text{dom}(I)$,

$-\forall \sigma, \sigma' : \sigma' = \sigma_J$ and

$-\forall X \in \tau_\Delta^I$,

$$X'^I := \{ \bar{a} | I_J \models \varphi'_X[\bar{a}] \},$$

where formula $\varphi'_X$ is defined by equation (7) applied to $\Delta'$.

This definition is equivalent to defining $T_\Delta(I, J) := \Gamma_{\Delta'}(I_J)|_\tau$, for any pair of $\tau$-structures $I, J$ such that $\text{dom}(I) = \text{dom}(J)$.

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**Proposition 4.9.** Let $I_o$ be a fixed $\tau^n_\Delta$-structure. In the lattice $S^*_\Delta$, the operator $T_\Delta(I,J)$ is monotone in its first argument, and anti-monotone in its second argument.

**Proof.** Select arbitrary $\tau$-structures $I, I', J, J'$ with the same domain such that $J|_{\tau^n_\Delta} = J'|_{\tau^n_\Delta}, I \subseteq I'$ and $J' \subseteq J$. We need to show that $T_\Delta(I,J) \subseteq T_\Delta(I', J')$. Let $L = T_\Delta(I,J)$ and $L' = T_\Delta(I', J')$.

It holds that $L$ and $L'$ have the same domain and that for each open symbol $\sigma \in \tau^n_\Delta$, $\sigma^L = \sigma^J = \sigma^{J'} = \sigma^{L'}$. So, it suffices to verify that for each defined symbol $X$, $X^L \subseteq X^{L'}$. Let $\bar{a}$ be any element of $X^L$. It holds that $I_J \models \varphi'_X[\bar{a}]$. The structure $I'_J$ assigns the same value to open symbols in $\varphi'_X$, and lesser value to the defined symbols $X'$ which occur negatively in $\varphi'_X$. Consequently, it holds that $I'_J \models \varphi'_X[\bar{a}]$. We find that $\bar{a} \in X^{L'}$. We obtain our proposition. \[\square\]

The next corollary shows a connection between operators $T_\Delta$ and $\Gamma_\Delta$.

**Corollary 4.10.** For any $\tau$-structure $I$, it holds that $T_\Delta(I, I) = \Gamma_\Delta(I)$.

**Proof.** Follows immediately from the fact that $I_I \models \varphi'_X[\bar{a}]$ iff $I \models \varphi'_X[\bar{a}]$. \[\square\]

The proposition has another interesting corollary.

**Corollary 4.11.** Let $I, M, J$ be three $\tau$-extensions of $K_o$ such that $I \subseteq M \subseteq J$. Then it holds that $T_\Delta(I, J) \subseteq \Gamma_\Delta(M) \subseteq T_\Delta(J, I)$.

**Proof.** Since $I \subseteq M \subseteq J$, Proposition 4.9 entails that $T_\Delta(I, J) \subseteq T_\Delta(M, M) = \Gamma_\Delta(M) \subseteq T_\Delta(J, I)$. \[\square\]

This corollary shows that $T_\Delta$ can be used to approximate $\Gamma_\Delta$ over an interval of structures. Indeed, if $(I, J)$ is an approximation of $M$ (i.e., $M \in [I, J]$) then the corollary shows that $(T_\Delta(I, J), T_\Delta(J, I))$ is an approximation of $\Gamma_\Delta(M)$. We shall elaborate on the approximation process in a moment.

Let $J$ be a $\tau$-structure, and $J_o$ its restriction to $\tau^n_\Delta$. The unary operator $M \cdot T_\Delta(I, J)$, often denoted by $T_\Delta(\cdot, J)$, is a monotone operator in the lattice $S^*_\Delta$; and its least fixpoint in this lattice is computed by

$$lfp(T_\Delta(\cdot, J)) := \bigsqcup_\xi E^\xi,$$

where

$$E^\xi := T_\Delta(E^{\leq \xi}, J), \quad E^{\leq \xi} := \bigsqcup_{\eta < \xi} E^\eta.$$

**Definition 4.12 stable operator.** Define the stable operator $ST_\Delta : \mathcal{I} \mapsto \mathcal{I}$ as follows:

$$ST_\Delta(J) := lfp(T_\Delta(\cdot, J)).$$

The operator $T_\Delta(I, J)$ performs one derivation step by interpreting positive occurrences of defined symbols by $I$ and negative occurrences by $J$. The stable operator performs a monotone induction during which negative occurrences of defined predicates $X$ in $\Delta$ are interpreted by the fixed value $X^J$. 

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Example 4.13. We illustrate the stable operator with the definition of Example 4.7:

\[ \Delta_0 = \begin{cases} P \leftarrow t, \\ Q \leftarrow \neg P, \\ Q \leftarrow Q \end{cases}. \]

This definition has no open symbols and is equivalent to the following definition:

\[ \begin{cases} P \leftarrow t, \\ Q \leftarrow \neg P \lor Q \end{cases}. \]

It is straightforward to see that in a propositional definition, the mapping \( ST_{\Delta}(J) \) for any \( J \) is the least fixpoint of the positive definition obtained by substituting each negative occurrence of a defined symbol and each occurrence of an open symbol by its truth value in \( J \). Thus, the stable operator maps the empty structure \( \emptyset \) to the least fixpoint of the definition:

\[ \begin{cases} P \leftarrow t, \\ Q \leftarrow \neg f \lor Q \end{cases}. \]

This yields the structure \( \{ P, Q \} \).

Similarly, the stable operator maps the structure \( \{ P, Q \} \) to the least fixpoint of the definition:

\[ \begin{cases} P \leftarrow t, \\ Q \leftarrow \neg t \lor Q \end{cases}. \]

This yields the structure \( \{ P \} \). Likewise, the stable operator maps \( \{ P \} \) to the least fixpoint of the same definition, and this yields \( \{ P \} \) itself.

Proposition 4.14. Let \( I_0 \) be a fixed \( \tau^\Delta \)-structure. Operator \( ST_{\Delta} \) is anti-monotone on \( S_{I_0}^{\tau} \).

Proof. Let \( I \subseteq J \) be \( \tau \)-extensions of \( I_0 \). To show that \( ST_{\Delta} \) is anti-monotone, it suffices to show that any pre-fixpoint of \( T_{\Delta}(\cdot, I) \) is a pre-fixpoint of \( T_{\Delta}(\cdot, J) \). It will follow then that \( ST_{\Delta}(J) \), the least pre-fixpoint of \( T_{\Delta}(\cdot, J) \), is smaller than \( ST_{\Delta}(I) \), the least pre-fixpoint of \( T_{\Delta}(\cdot, I) \).

Assume that for any \( J' \in S_{I_0}^{\tau} \), \( T_{\Delta}(J', I) \subseteq J' \). Then, by anti-monotonicity of \( T_{\Delta} \) in the second argument (Proposition 4.9), \( T_{\Delta}(J', J) \subseteq T_{\Delta}(J', I) \subseteq J' \). Thus \( J' \) is a pre-fixpoint of \( T_{\Delta}(\cdot, J) \) and this entails that the least (pre-)fixpoint of \( T_{\Delta}(\cdot, J) \) is less than the least (pre-)fixpoint of \( T_{\Delta}(\cdot, I) \). □

Fix some \( \tau^\Delta \)-structure \( I_0 \) with domain \( A \) of the open symbols of \( \Delta \) in \( \tau \).

As is standard for anti-monotone operators on a complete lattice (see Section 3.2), the operator \( ST_{\Delta} \) gives rise to a sequence \( (I^\xi, J^\xi)_{\xi \geq 0} \) in \( S_{I_0}^{\tau} \) defined by

\[ I^\xi := ST_{\Delta}(J^{<\xi}), \quad \text{where} \quad J^{<\xi} := \bigcap_{\eta<\xi} J^{\eta}, \]
\[ J^\xi := ST_{\Delta}(I^{<\xi}), \quad \text{where} \quad I^{<\xi} := \bigcup_{\eta<\xi} I^{\eta}. \]

Notice that \( I^{<0} \) is, by definition, the bottom element \( \perp_{I_0} \) of the lattice, i.e., the structure which assigns \( \emptyset \) to every defined symbol; and \( J^{<0} \) is the top element \( \top_{I_0} \) which assigns Cartesian product \( A^r \) to each \( r \)-ary defined symbol \( X \) of \( \Delta \).
The anti-monotonicity of $\Gamma_\Delta$ implies that the sequence $(I^\xi)_{\xi \geq 0}$ is increasing and $(J^\xi)_{\xi \geq 0}$ is decreasing. Moreover, for each $\xi$, $I^\xi \subseteq J^\xi$. Thus, it holds that the sequence $(I^\xi, J^\xi)_{\xi \geq 0}$ is indeed a sequence of increasingly precise approximations. This sequence has a limit $(I, J)$, which is the maximal oscillating pair of $ST_\Delta$. Equivalently, $I$ and $J$ are fixpoints of the square $ST^2_\Delta$, $\text{lfp}(ST^2_\Delta)$ and $\text{gfp}(ST^2_\Delta)$, respectively.

In the lattice $S^\tau_\Delta$, we define
\[
I^\Delta := \text{lfp}(ST^2_\Delta), \quad \text{and} \quad I^\Delta := \text{gfp}(ST^2_\Delta).
\]
We extend this notation to any structure $L$ which interprets at least $\tau^\circ_\Delta$ and define
\[
L^\Delta := (L|_{\tau^\circ_\Delta})^\Delta, \quad \text{and} \quad L^\Delta := (L|_{\tau^\circ_\Delta})^\Delta.
\]
Note that $L^\Delta$ and $L^\Delta$ agree with $L$ on the open symbols but not necessarily on the defined symbols.

**Definition 4.15 total definition.** Definition $\Delta$ is total in $\tau^\circ_\Delta$-structure $I_\circ$ if $I^\Delta = I^\Delta$. If $\tau_K \subseteq \tau^\circ_\Delta$, we say that $\Delta$ is total in $K$ if $\Delta$ is total in each $\tau^\circ_\Delta$-structure extending $K$. If $\tau^\circ_\Delta \subseteq \tau_K \subseteq \tau$, then we say that $\Delta$ is total in $K$ if $\Delta$ is total in $K|_{\tau^\circ_\Delta}$. We say that a definition $\Delta$ is total if it is total in each $\tau^\circ_\Delta$-structure $I_\circ$.

The aim of a definition is to define its defined symbols. Therefore, a natural quality requirement for a definition is that it is total.

**Definition 4.16 $I$ satisfies $\Delta$.** We say that $\tau$-structure $I$ satisfies $\Delta$, or equivalently, that $\Delta$ is true in $I$ (denoted $I \models \Delta$) if $\Delta$ is total in $I$ and $I^\Delta$ is identical to $I$.

Let $I$ be any structure such that $\tau^\circ_\Delta \subseteq \tau_i \subseteq \tau$.

**Definition 4.17 $\Delta$-extension of $I$.** Let $\Delta$ be total in $I$. Define the $\Delta$-extension of $I$, denoted $I^\Delta$, as $I^\Delta := I^\Delta$ (or, equivalently, $I^\Delta := I^\Delta$). If $\Delta$ is not total in $I$, then $I$ has no $\Delta$-extension.

Note that for any $\tau^\circ_\Delta$-structure $I_\circ$, there is at most one $\Delta$-extension extending $I_\circ$.

**Example 4.18.** We illustrate the iterative process described above with the definition of Example 4.7, which is equivalent to the following definition
\[
\begin{align*}
\{ P & \leftarrow t, \\
Q & \leftarrow \neg P \lor Q \}
\end{align*}
\]
The first pair in this sequence is the least precise pair that approximates all structures:
\[
I^{<0} := \emptyset, \\
J^{<0} := \{ P, Q \}.
\]
To compute the new upper bound $J^0$, we apply the stable operator on $I^{<0}$ which yields, as shown in Example 4.13, $\{ P, Q \}$. To compute $I^0$, the stable operator is applied on $J^{<0} = \{ P, Q \}$ which yields $\{ P \}$. Note that at this moment, $I^0$ and $J^0$ agree on the fact that $P$ is true. So, after this first step, we have derived that $P$ is true.
In the next step, we obtain $I^2 = \{ P \} = J^2$. We derived that $Q$ is false. The next iteration produces exactly the same pair, so, we obtained a fixpoint with identical lower and upper bound $\{ P \}$. The definition $\Delta_0$ is total. Since there are no open symbols, $\{ P \}$ is the unique $\Delta_0$-extension. It coincides with the structure that we obtained in Example 4.7 by applying iterated induction.

**Example 4.19.** Consider the definition:

$$\Delta_{\text{even}} := \left\{ \forall x\ E(x) \leftarrow x = 0, \quad \forall x\ (E(s(x)) \leftarrow \neg E(x)) \right\},$$

which is equivalent to:

$$\left\{ \forall x\ [E(y) \leftarrow y = 0 \lor \exists x\ (y = s(x) \land \neg E(x))] \right\}.$$

We show that in the extension of $\Delta_{\text{even}}$ in the natural numbers, $E$ is interpreted by the set of even numbers. Note that this definition has no positive occurrences of the defined predicate $E$. Therefore, $\Gamma_{\Delta_{\text{even}}}(I,J) = \Gamma_{\Delta_{\text{even}}}(J)$, for all $I,J$ sharing the same domain.

The well-founded model computation starts in the least precise pair extending the natural numbers:

$$E^{I<0} := \emptyset,$$

$$E^{J<0} := \mathbb{N}.$$ 

To compute the new upper bound $J^0$, we apply $\Gamma_{\Delta_{\text{even}}}$ on $I^{<0}$. Since $I^{<0}$ satisfies the body of the rule for each natural number, we obtain the set $\mathbb{N}$ as a new upper bound. As a new lower bound, we derive the singleton $\{0\}$. At this point, we derived that 0 is even.

$$E^{I^0} := \{0\},$$

$$E^{J^0} := \mathbb{N}.$$ 

In the next step, since the upper bound did not change, we derive the same lower bound. When computing the new upper bound, we obtain all natural numbers except 1. This means that we derived that 1 is not even:

$$E^{I^1} := \{0\},$$

$$E^{J^1} := \mathbb{N} \setminus \{1\}.$$ 

In the third step, the upper bound remains unaltered. With respect to the lower bound, we can now derive both 0 and 2:

$$E^{I^2} := \{0, 2\},$$

$$E^{J^2} := \mathbb{N} \setminus \{1\}.$$ 

In the subsequent step, we obtain the same lower bound, but 3 is eliminated from the upper bound.

$$E^{I^3} := \{0, 2\},$$

$$E^{J^3} := \mathbb{N} \setminus \{1, 3\}.$$ 

After iterating this process $\omega$ steps, we obtain the fixpoint:

$$E^{I^\omega} = E^{J^\omega} = \{2n \mid n \in \mathbb{N}\}.$$
Now we are ready to define the satisfaction relation between structures and well-formed formulas of the logic.

**Definition 4.20** \( \phi \) true in structure \( I \). Let \( \phi \) be a ID-formula and \( I \) any structure such that \( \text{free}(\phi) \subseteq \tau_I \).

We define \( I \models \phi \) (in words, \( \phi \) is true in \( I \), or \( I \) satisfies \( \phi \), or \( I \) is a model of \( \phi \)) by the following induction:

\[ \begin{align*}
- I \models X(t_1, \ldots, t_n) & \text{ if } (t_1^I, \ldots, t_n^I) \in X^I; \\
- I \models \psi_1 \land \psi_2 & \text{ if } I \models \psi_1 \text{ and } I \models \psi_2; \\
- I \models \lnot \psi & \text{ if } I \not\models \psi; \\
- I \models \exists \sigma \psi & \text{ if for some value } v \text{ of } \sigma \text{ in the domain } \text{dom}(I) \text{ of } I, I[\sigma : v] \models \psi; \\
- I \models \Delta & \text{ if } I = I^\Delta = I^\Delta^I.
\end{align*} \]

Given an ID-theory \( T \) over \( \tau \), a \( \tau \)-structure \( I \) satisfies \( T \) (is a model of \( T \)) if \( I \) satisfies each \( \phi \in T \). This is denoted by \( I \models T \).

**Example 4.21.** Consider the theory \( T_N \) of Example 4.4. We prove that each model \( I \) of \( T_N \) is isomorphic to the structure of the natural numbers. Let \( I \) be a model of this theory. First, since \( I \) satisfies the first-order Peano axioms, the domain elements \( 0^I, s(0)^I, \ldots, s^n(0)^I, \ldots \) are pair-wise distinct and the set of these domain elements constitutes a subset of \( \text{dom}(I) \), isomorphic to the natural numbers. Therefore, it suffices to show that this set is exactly the domain of \( I \). Since \( I \) satisfies the ID-axiom replacing the induction axiom, there exists a set \( S \subseteq \text{dom}(I) \) such that \( I[N : S] \) satisfies

\[ \left\{ \forall x \ (N(x) \leftarrow x = 0), \quad \forall x \ (N(s(x)) \leftarrow N(x)) \right\} \land \forall x \ N(x). \]

Since \( I[N : S] \) satisfies \( \forall x \ N(x) \), \( S \) must be \( \text{dom}(I) \). As proven later in Theorem 7.3, \( I[N : S] \) satisfies the positive definition in this axiom iff \( S \) is the least set containing \( 0^I \) and closed under \( s^I \). Hence, \( \text{dom}(I) \) is exactly the set \( \{0^I, s(0)^I, \ldots, s^n(0)^I, \ldots \} \).

**Example 4.22.** An ID-theory can contain multiple definitions for the same predicate. A simple illustration is when a natural class is partitioned in subclasses in different ways, depending on the property used. For example, humans can be partitioned in males and females, but also in adults and children, etc. This is modeled by the following formula:

\[ \left\{ \forall x \ (\text{Human}(x) \leftarrow \text{Male}(x)), \quad \forall x \ (\text{Human}(x) \leftarrow \text{Female}(x)) \right\} \land \left\{ \forall x \ (\text{Human}(x) \leftarrow \text{Adult}(x)), \quad \forall x \ (\text{Human}(x) \leftarrow \text{Child}(x)) \right\}. \]

This formula implies that the class humans is the union of the classes males and females, and also of the classes adults and children. The definition

\[ \left\{ \forall x \ (\text{Human}(x) \leftarrow \text{Male}(x) \lor \text{Female}(x)), \quad \forall x \ (\text{Human}(x) \leftarrow \text{Adult}(x) \lor \text{Child}(x)) \right\}. \]

is weaker, in the sense that it does not entail that humans are either males or females.
4.3 Total Definitions

Totality of non-monotone definitions is a fundamental property in our theory of non-monotone induction. In particular, it indicates that a definition is well-constructed, i.e., does not produce undefined atoms.

Example 4.23. Consider the following definition:

\[ \Delta_2 := \{ P \leftarrow \neg P \}. \]

One verifies that the iterated induction yields the limit \((\emptyset, \{P\})\). This definition has no model.

Example 4.24. Consider the definition:

\[ \Delta_3 := \{ P \leftarrow \neg Q, Q \leftarrow \neg P \}. \]

The iterated induction yields the limit \((\emptyset, \{P, Q\})\). The definition has no model.

The next example shows that a (useful) definition which is total in one structure, may not be total in other structures.

Example 4.25. Consider the definition of Example 4.19:

\[ \Delta_{\text{even}} := \{ \forall x (E(x) \leftarrow x = 0), \forall x (E(s(x)) \leftarrow \neg E(x)) \}. \]

Recall that the stable operator of this definition is identical to \(\Gamma_{\Delta_{\text{even}}}\). In Example 4.19, we showed that this definition is total in the structure of the natural numbers. It is not total in many other structures, in particular in those where the successor function contains cycles or infinite descending chains. For example, \(\Delta_{\text{even}}\) is not total in the structure \(I_o\) with domain \(\{0, 1\}\), and \(s^{I_o}(0) = 1, s^{I_o}(1) = 1\). In this structure, the maximal oscillating pair \((I, J)\) of \(\Gamma_{\Delta_{\text{even}}}\) interprets \(E\) as follows:

\[ E^I := \{0\}, \]
\[ E^J := \{0, 1\}. \]

The reason for this oscillation is that in this structure, atom \(E[1]\) depends on \(\neg E[1]\).

Definition \(\Delta_{\text{even}}\) is not total either in the structure \(I_o'\) with domain \(\mathbb{Z}\) and \(s^{I_o'}\) the standard successor function on \(\mathbb{Z}\). In this structure, the maximal oscillating pair \((I, J)\) of \(\Gamma_{\Delta_{\text{even}}}\) interprets \(E\) as follows:

\[ E^I := \{2n \mid n \in \mathbb{N}\}, \]
\[ E^J := \{n \mid n < 0\} \cup \{2n\mid n \in \mathbb{N}\}. \]

Example 4.26. Recall the theory \(T_N\) of Example 4.4,

\[ \exists N \left[ \{ \forall x \ (N(x) \leftarrow x = 0), \forall x \ (N(s(x)) \leftarrow N(x) \} \land \forall x \ N(x) \right], \]
\[ \forall n \ (\neg (s(n) = 0)), \]
\[ \forall n \forall m \ (s(n) = s(m) \rightarrow n = m). \]

and the definition of Example 4.19,

\[ \Delta_{\text{even}} := \{ \forall x \ (E(x) \leftarrow x = 0), \forall x \ (E(s(x)) \leftarrow \neg E(x)) \}. \]
In Example 4.21, we saw that the natural numbers are the unique model of $T_N$ (modulo isomorphism). In Example 4.25, we saw that $\Delta_{\text{even}}$ is total in the natural numbers. Consequently, $\Delta_{\text{even}}$ is total in $T_N$. The theory $T_N \cup \{\Delta_{\text{even}}\}$ is consistent and has one model, the natural numbers and $E$ interpreted by the even numbers.

What is the cause of the non-totality of a definition? In the above examples, the natural dependency order, induced by the rules, contains infinite descending chains in which atoms depend negatively on the same or other atoms. When this happens, the stable operator oscillates between a structure in which all atoms of the chain are false and one in which these atoms are true.

When $\Delta$ is not total in $I_o$. Definition 4.20 states that there is no model that extends $I_o$. To cope with such cases, we might adopt an alternative definition of $\Delta$-extension and define the $\Delta$-extension of $I_o$ as a 3-valued structure. With $(I^{\Delta\uparrow}, I^{\Delta\downarrow})$, a unique three-valued structure corresponds which coincides with $I^{\Delta\downarrow}$ and $I^{\Delta\uparrow}$ on all atoms where $I^{\Delta\downarrow}$ and $I^{\Delta\uparrow}$ agree and is undefined on all atoms where $I^{\Delta\downarrow}$ and $I^{\Delta\uparrow}$ disagree. This is the option that has been taken in the original well-founded semantics of logic programming. In this paper, we will stick to a 2-valued solution and avoid the complexities caused by using three-valued logic.

Notice that we cannot restrict the syntax of the logic to allow total definitions only. Such a restriction would lead to undecidable syntax — there would be no procedure which would decide, for a given formula $\phi$, whether $\phi$ is a well-formed formula of the language. This is because the problem of determining, for a given definition $\Delta$ and structure $I_o$, whether $\Delta$ is total in $I_o$, is undecidable [Schlipf 1995].

For important classes of definitions, it is known that they are total. For example, positive definitions are total in any structure. For other types of definitions, techniques must be developed to prove that they are total. In this paper, we develop such techniques.

5. REDUCTION RELATIONS

In Section 2.5, we mentioned that a definition implicitly induces a dependency relation between atoms and that the well-founded semantics performs iterated induction along this dependency relation, in the sense that the truth assignment to an atom is delayed until enough information about the atoms on which it depends has become available. This shows that the notion of dependency relation induced by a definition is important. In this section, we formalise this intuitive concept by the notion of reduction relation. Intuitively, a reduction relation $\prec$ is a binary relation between domain atoms such that for each defined atom $P[\bar{a}]$, the truth of its defining formula $\varphi_P[\bar{a}]$ depends only on the truth of atoms $Q[\bar{b}] \prec P[\bar{a}]$. In the next paragraphs we formalise what this means.

Let $\tau$ be a vocabulary and $A$ a domain. Recall that $At_A^\tau$ denotes the set of domain atoms over vocabulary $\tau$ in domain $A$. Let $\prec$ be any binary relation on $At_A^\tau$. If $Q[\bar{b}] \prec P[\bar{a}]$, we will say that $P[\bar{a}]$ depends on $Q[\bar{b}]$ (according to $\prec$). The binary relation $\prec$ is derived from $\prec$ in the following way: $Q[\bar{b}] \prec P[\bar{a}]$ iff $Q[\bar{b}] \prec P[\bar{a}] \land P[\bar{a}] \not\prec Q[\bar{b}]$. Intuitively, $P[\bar{a}]$ depends on $Q[\bar{b}]$ but not vice versa.

For any domain atom $P[\bar{a}] \in At_A^\tau$, for any structure $I$ with domain $A$ such that $\tau_I \subseteq \tau$, define $I|_{\prec P[\bar{a}]}$ as the structure $I[X : \bar{R}]$ where $X$ is the set of relation...
symbols in \( \tau_I \) and for each relation symbol \( X \in \bar{X} \), its value is given by

\[
R := \{ \vec{d} \mid I \models X[\vec{d}] \text{ and } X[\vec{d}] \prec P[\vec{a}] \}.
\]

Intuitively, \(|I| \prec_P[\vec{a}] \) falsifies all true atoms \( Q[\vec{b}] \) on which \( P[\vec{a}] \) does not depend. The operation \(| \cdot | \prec_P[\vec{a}] \) is an idempotent operation, that is \(|(|I| \prec_P[\vec{a}])| \prec_P[\vec{a}] = |I| \prec_P[\vec{a}] \).

For any pair \( I, J \) of structures with domain \( A \), we define \( I \equiv \prec_P[\vec{a}] J \) if \(|I| \prec_P[\vec{a}] = |J| \prec_P[\vec{a}] \). When \( I \equiv \prec_P[\vec{a}] J \), then \( I \) and \( J \) interpret the same symbols, assign the same value to all function symbols, and assign the same value to all domain atoms \( Q[\vec{b}] \prec P[\vec{a}] \). We extend this relation to tuples and define \( (I, J) \equiv \prec_P[\vec{a}] (I', J') \) if \( I \equiv \prec_P[\vec{a}] I' \) and \( J \equiv \prec_P[\vec{a}] J' \). Intuitively, \( (I, J) \equiv \prec_P[\vec{a}] (I', J') \) means that \( (I, J) \) and \( (I', J') \) are identical on all atoms on which \( P[\vec{a}] \) depends.

Recall from Section 4 that for any defined symbol \( P \) of \( \Delta \), \( \varphi_P(\bar{x}) \) is obtained by renaming negative occurrences of defined symbols in \( \varphi_P(\bar{x}) \). For each pair of \( \tau \)-structures \( I, J \) with domain \( A \), the associated \( \tau' \)-structure \( I_J \) is the structure \( J[\bar{X} : X', \bar{X}' : X'] \) where \( \bar{X} \) is the collection of defined symbols of \( \Delta \).

Assume a definition \( \Delta \) over \( \tau \) and a structure \( K_o \) with domain \( A \) such that \( \tau_{K_o} \subseteq \tau_{\Delta}^o \).

**Definition 5.1** reduction relation. A binary relation \( \prec \) on \( At_{\Delta}^2 \) is a reduction relation (or briefly, a reduction) of \( \Delta \) in \( K_o \) if for each domain atom \( P[\vec{a}] \) with \( P \) a defined symbol, for all \( \tau \)-structures \( I, J, I', J' \) extending \( K_o \), if \( (I, J) \equiv \prec_P[\vec{a}] (I', J') \) then \( I_J \models \varphi_P[\vec{a}] \) iff \( I'_J \models \varphi_P[\vec{a}] \).

**Example 5.2.** For the following definition

\[
\Delta := \left\{ \forall x \ (E(x) \leftarrow x = 0), \quad \forall x \ (E(s(x)) \leftarrow O(x)), \quad \forall x \ (O(s(x)) \leftarrow E(x)) \right\},
\]

a reduction \( \prec \) in the structure of the natural numbers is the relation represented by the set of tuples:

\[
\{(E[n], O[n + 1]), (O[n], E[n + 1]) \mid n \in \mathbb{N}\}.
\]

Also its transitive closure \( \prec^* \) is a reduction.

It can be easily verified that, e.g. for an atom \( E[n + 1] \), if structures \( I, J \) agree on the atom \( O[n] \prec E[n + 1] \), then \( \Gamma_{\Delta}(I) \) and \( \Gamma_{\Delta}(J) \) will agree on the value of \( E[n + 1] \).

**Example 5.3.** Reduction relations are context dependent. Consider the following propositional definition:

\[
\Delta := \left\{ \begin{array}{l} P \leftarrow Q \land R, \\ Q \leftarrow P \land \neg R \end{array} \right\}.
\]

The relation \( \prec_1 := \{(Q, P), (R, Q), (R, P)\} \) is a reduction relation of \( \Delta \) in the \( \tau_{\Delta}^o \)-structure \( \{R\} \) but not in the \( \tau_{\Delta}^o \)-structure \( \emptyset \). Vice versa, the relation \( \prec_2 := \{(P, Q), (R, Q), (R, P)\} \) is a reduction relation of \( \Delta \) in \( \emptyset \) but not in \( \{R\} \).

Let \( \prec \) be a reduction of \( \Delta \) in \( K_o \).

**Proposition 5.4.** If \( K_o' \) extends \( K_o \), then \( \prec \) is a reduction of \( \Delta \) in \( K_o' \).

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Proposition 5.5. Any superset \( \prec' \) of \( \prec \) is a reduction relation of \( \Delta \) in \( K_\alpha \).

In particular, the transitive closure and the reflexive transitive closure of a reduction are reductions. This stems from the fact that \( I \equiv_{\prec \cdot P[a]} J \) implies \( I \equiv_{\prec P[a]} J \).

As shown by this proposition, a definition \( \Delta \) may have many reduction relations in \( K_\alpha \). The total binary relation \( \prec_\alpha = AT_\mathcal{A}^* \times AT_\mathcal{A}^* \) is always a reduction relation. Since \( I \equiv_{\prec P[a]} J \) iff \( I = J \), the relation \( \prec_\alpha \) trivially satisfies Definition 5.1. It can be seen here that a reduction relation in general overestimates the dependencies between domain atoms in a definition. Only the least reduction relation of a definition reflects the true dependencies. However, as shown in the next example, some definitions do not have a least reduction relation.

Example 5.6. Consider the following definition in the context of the natural numbers:

\[
\Delta := \{ P \leftarrow \exists n \forall m (m > n \supset Q(m)) \}
\]

The predicate \( Q \) is open in this definition. This definition defines \( P \) to be true if there exists a number \( n \) such that \( Q \) contains at least all natural numbers larger than \( n \). It can easily be verified that for each \( n \in \mathbb{N} \), the relation

\[
\prec_n = \{ (Q(m), P) | m > n \}
\]

is a reduction relation of \( \Delta \) in \( \mathbb{N} \). The intersection of these relations is \( \emptyset \), and this is not a reduction relation of \( \Delta \).

We defined the operator \( T_\Delta \) as a map from pairs \( I, J \) of \( \tau \)-structures with shared domain to the interpretation \( J' \) extending \( J|_{\tau_\Delta} \) such that for each defined atom \( P[a], \ P[a]' \) is true iff \( \varphi_P'[a] \) is true in \( I \). We have the following proposition.

Proposition 5.7. Let \( \prec \) be a reduction relation of \( \Delta \) in \( K_\alpha \), \( P[a] \) a domain atom and let \( I, I', J, J' \) be \( \tau \)-structures extending \( K_\alpha \) such that \( (I, J) \equiv_{\prec P[a]} (I', J') \).

(a) If \( P \) is defined then \( P[a]^{T_\Delta(I, J)} = P[a]^{T_\Delta(I', J')} \).

(b) If \( \prec \) is transitive, then \( T_\Delta(I, J) \equiv_{\prec P[a]} T_\Delta(I', J') \).

The condition of item (b) that \( \prec \) should be transitive is not very restrictive since the transitive closure of a reduction is a reduction as well.

Proof. (a) Since \( P \) is a defined predicate of \( \Delta \), \( P[a]^{T_\Delta(I, J)} \) is the truth value of \( \varphi_P'[a] \) in \( I \) and likewise \( P[a]^{T_\Delta(I', J')} \) is the truth value of \( \varphi_P'[a] \) in \( I' \). Since \( \prec \) is a reduction relation, the truth value of this formula is the same in \( I \) as in \( I' \).

(b) Assume that \( \prec \) is transitive. Let \( Q[b] \) be an arbitrary domain atom such that \( Q[b] \prec P[a] \). If \( Q \) is an open predicate of \( \Delta \) then \( Q[b]^{T_\Delta(I, J)} = Q[b]^J = Q[b]^J' = Q[b]^{T_\Delta(I', J')} \). Let \( Q \) be a defined predicate of \( \Delta \). By transitivity of \( \prec \), the set of atoms on which \( Q[b] \) depends is a subset of the set of atoms on which \( P[a] \) depends. This, and the fact that \( (I, J) \equiv_{\prec P[a]} (I', J') \) implies that \( (I, J) \equiv_{\prec Q[b]} (I', J') \). By application of (a) we obtain that \( Q[b]^{T_\Delta(I, J)} = Q[b]^{T_\Delta(I', J')} \).

\( \Box \)

Under the condition that \( \prec \) is a transitive reduction, item (b) of this proposition states that \( T_\Delta \) preserves \( \equiv_{\prec P[a]} \), for each domain atom \( P[a] \). This is a key property.
The reduction relation \(\prec\) defines a collection of lattice congruences \(\cong\) in \(\mathcal{S}_{K_o}^\wedge\), one for each domain atom \(P[\bar{a}]\). The operator \(T_{\Delta}\) is the basic operator in the well-founded model construction. The fact that it preserves the congruences \(\cong\) "propagates" to the stable operator \(ST_{\Delta}\) and to the construction of the well-founded model. This leads to the main theorem of this section.

Let \(\Delta\) be total in \(K_o\) and \(\prec\) a transitive reduction relation of \(\Delta\) in \(K_o\).

**Theorem 5.8.** For \(\tau_{K_o}\)-structures \(I_o, J_o\) extending \(K_o\), \(I_o \cong_{P[\bar{a}]} J_o\) implies \(I_o^\Delta \cong_{P[\bar{a}]} J_o^\Delta\).

In other words, the value of a defined atom \(P[\bar{a}]\) depends only on the open atoms on which \(P[\bar{a}]\) depends according to reduction relation \(\prec\).

Consider the lattice \(\mathcal{S}_{K_o}^\wedge\), which consists of \(\tau\)-structures extending \(K_o\). To prove the theorem, we will show that \(|\cdot|_{P[\bar{a}]\prec}\) is a lattice homomorphism and \(\cong_{P[\bar{a}]\prec}\) the corresponding lattice congruence (confer Section 3.2.2). Since \(T_{\Delta}\) preserves \(\cong_{P[\bar{a}]\prec}\), by application of the basic Proposition 3.6, it will be easy to show that also the stable operator \(ST_{\Delta}\) preserves \(\cong_{P[\bar{a}]\prec}\) and that the statement of the theorem holds.

Let \(\prec\) be a transitive reduction of \(\Delta\) in \(K_o\) and assume that \(\tau_{K_o} \subseteq \tau_{K_o} \subseteq \tau\). Then \(\langle\mathcal{S}_{K_o}^\wedge, \subseteq\rangle\) is a complete lattice. For any domain atom \(P[\bar{a}]\), we define the collection \(\mathcal{S}_{K_o}^\wedge\) as the image of \(\mathcal{S}_{K_o}\) under the mapping \(|\cdot|_{P[\bar{a}]\prec}\).

**Proposition 5.9.** The structure \(\langle\mathcal{S}_{K_o}^\wedge, \subseteq\rangle\) is a complete lattice. The mapping \(|\cdot|_{P[\bar{a}]\prec} : \mathcal{S}_{K_o}^\wedge \to \mathcal{S}_{K_o}^\wedge\) is a lattice homomorphism and its induced lattice congruence is \(\cong_{P[\bar{a}]\prec}\). The operator \(T_{\Delta}\) preserves \(\cong_{P[\bar{a}]\prec}\). If \(K_o\) is idempotent (i.e., \(|K_o|_{P[\bar{a}]\prec} = K_o\)), the homomorphic image of \(T_{\Delta}\) on \(\mathcal{S}_{K_o}^\wedge\) is the operator \(|T_{\Delta}(\cdot, \cdot)|_{P[\bar{a}]\prec}\).

**Proof.** The proposition is straightforward. We prove only the last item. If \(K_o\) is idempotent, then it is easy to see that \(\mathcal{S}_{K_o}^\wedge \subseteq \mathcal{S}_{K_o}\). Since \(T_{\Delta}\) preserves \(\cong_{P[\bar{a}]\prec}\), it has a homomorphic image on \(\mathcal{S}_{K_o}^\wedge\), say \(T\). For all \(I, J \in \mathcal{S}_{K_o}^\wedge\),

\[
T(I, J) = (|I|_{P[\bar{a}]\prec}, |J|_{P[\bar{a}]\prec}) \quad (\text{by idempotence of } |\cdot|_{P[\bar{a}]\prec})
\]

(by definition of \(T\))

\(\square\)

**Remark 5.10.** A condition in Proposition 5.9 is that \(K_o\) is idempotent for \(|\cdot|_{P[\bar{a}]\prec}\). This condition can always be satisfied. Given a reduction \(\prec\) of \(\Delta\) in \(K_o\), there exists an equivalent reduction \(\prec'\) of \(\Delta\) in \(K_o\) such that for all \(P[\bar{a}]\), \(K_o\) is idempotent for \(|\cdot|_{P[\bar{a}]\prec}\).

Define \(\prec' := \prec \cup \{\langle Q[\bar{b}], Q'[\bar{c}]\rangle : Q \in \tau_{K_o} \land Q' \in \tau \setminus \tau_K\}\). In \(\prec'\), each domain atom depends on the same atoms as in \(\prec\) but also on all domain atoms interpreted by \(K_o\). It is easy to see that for all \(P[\bar{a}] \in At^\wedge_{K_o}\):

(a) \(|K_o|_{P[\bar{a}]\prec} = K_o\), and
(b) for all extensions \(I, J\) of \(K_o\), \(I \cong_{P[\bar{a}]\prec} J\) iff \(I \cong_{P[\bar{a}]\prec} J\).

Consequently, if \(\prec\) is a reduction of \(\Delta\) in \(K_o\), then \(\prec'\) is an equivalent reduction of \(\Delta\) in \(K_o\) in which \(K_o\) is idempotent for \(|\cdot|_{P[\bar{a}]\prec}\), for all \(P[\bar{a}] \in At^\wedge_{K_o}\).

Unless explicitly stated otherwise, we assume that \(\tau_{K_o} \subseteq \tau_{K_o} \subseteq \tau_{K_o}\), that \(\prec\) is transitive and that \(K_o\) is idempotent for \(|\cdot|_{P[\bar{a}]\prec}\), for all \(P[\bar{a}] \in At^\wedge_{K_o}\).
Proposition 5.11. For each domain atom $P[\bar{a}]$, the operator $ST_\Delta$ preserves $\equiv_{<P[\bar{a}]}$ in $S^\Delta_K$. Its homomorphic image on $S^\Delta_{K,o}$ is $|ST_\Delta(\cdot)|_{<P[\bar{a}]}$. Moreover, for all $I \in S^\Delta_{K,o}$, $|ST_\Delta(I)|_{<P[\bar{a}]}$ is $lfp(|T_\Delta(\cdot, I)|_{<P[\bar{a}]}$) in the lattice $S^\Delta_{K,o}$.

Proof. Let $I$ be an element of $S^\Delta_{K,o}$. The structure $ST_\Delta(I)$ is the least fixpoint of the operator $T_\Delta(\cdot, I)$ in the sublattice $S^\Delta_{I|\Delta} \subseteq S^\Delta_K$. By Proposition 5.9, the operator preserves $\equiv_{<P[\bar{a}]}$ and, since we assume that $|K_o|_{<P[\bar{a}]} = K_o$, its homomorphic image in the lattice $S^\Delta_{I|\Delta}$ is $|T_\Delta(\cdot, I)|_{<P[\bar{a}]}$. By Proposition 3.6(b), taking the homomorphic image and the least fixpoint of this operator commute. Consequently, we obtain that

$$|ST_\Delta(I)|_{<P[\bar{a}]} = |lfp(T_\Delta(\cdot, I))|_{<P[\bar{a}]} = |lfp(|T_\Delta(\cdot, I)|_{<P[\bar{a}]}|_{<P[\bar{a}]})|_{<P[\bar{a}]}
= |lfp(|T_\Delta(\cdot, I)|_{<P[\bar{a}]})|_{<P[\bar{a}]},$$

( since $I \equiv_{<P[\bar{a}]} I_{<P[\bar{a}]})$

in the image lattice $S^\Delta_{I|\Delta}$.

Assume $I, J \in S^\Delta_{K,o}$ such that $I \equiv_{<P[\bar{a}]} J$. Then $I|_{\Delta} \equiv_{<P[\bar{a}]} J|_{\Delta}$ and the lattices $S^\Delta_{I|\Delta}$ and $S^\Delta_{J|\Delta}$ are identical and the operators $|T_\Delta(\cdot, I)|_{<P[\bar{a}]}$ and $|T_\Delta(\cdot, J)|_{<P[\bar{a}]}$ on this lattice are identical. We obtain that $|ST_\Delta(I)|_{<P[\bar{a}]} = |ST_\Delta(J)|_{<P[\bar{a}]}$ or that $ST_\Delta(I) \equiv_{<P[\bar{a}]} ST_\Delta(J)$.

Now we found that $ST_\Delta$ preserves $\equiv_{<P[\bar{a}]}$ for each domain atom $P[\bar{a}]$, and we can repeat the argument for the construction of the well-founded model.

Proposition 5.12. For each domain atom $P[\bar{a}]$, for all $I_o, J_o \in S^\Delta_{K,o}$, if $I_o \equiv_{<P[\bar{a}]} J_o$ then $I_o|_{\Delta} \equiv_{<P[\bar{a}]} J_o|_{\Delta}$ and $I_o|_{\Delta} \equiv_{<P[\bar{a}]} J_o|_{\Delta}$. Moreover, $|I_o|_{<P[\bar{a}]}|_{<P[\bar{a}]}$ and $|J_o|_{<P[\bar{a}]}|_{<P[\bar{a}]}$ in the lattice $S^\Delta_{I_o|\Delta}$. The proof of this proposition is entirely similar to the proof of Proposition 5.11 and is omitted. The proposition entails that if $\Delta$ is total in $I_o$ and in $J_o$ and $I_o \equiv_{<P[\bar{a}]} J_o$ then $I_o|_{\Delta} \equiv_{<P[\bar{a}]} J_o|_{\Delta}$. This proves Theorem 5.8.

6. Modularity

In this section, we split a definition $\Delta$ into subdefinitions $\{\Delta_1, \Delta_2, \ldots, \Delta_n\}$. We study under what conditions we can guarantee that for structure $I$,

$I \models \Delta$ if and only if $I \models \Delta_1 \land \Delta_2 \land \cdots \land \Delta_n$.

This is the subject of the Modularity theorem.

The Modularity theorem is our main result here. The theorem tells us when we can understand a large definition as a conjunction of component definitions. Frequently, these component definitions have a simpler form — they may be positive definitions or non-recursive definition. Therefore, the ability to decompose definitions without side effects is useful for analyzing large definitions — some properties of large definitions are implied by properties of subdefinitions. Thus, the Modularity theorem is an important tool for simplifying logical formulas with definitions.
From a knowledge representation perspective, problem-free combining and decomposing of definitions is crucial while axiomatizing a complex system. For example, one may write two cycle-free modules of the system, which, when combined, produce a cyclic dependency between its syntactic components. Such a dependency may cause a change in the intended meaning of the original definitions. However, not every syntactic cycle is problematic. If the condition of the Modularity theorem are satisfied, one can guarantee that the composition does not change the intended meaning of the original component definitions.

6.1 Partition of Definitions

Everywhere in this section, we fix a definition $\Delta$ over some vocabulary $\tau$.

**Definition 6.1.** *Partition of definitions.* A partition of definition $\Delta$ is a set $$\{\Delta_1, \ldots, \Delta_n\}, 1 < n,$$ such that $\Delta = \Delta_1 \cup \cdots \cup \Delta_n$, and if defined symbol $P$ appears in the head of a rule of $\Delta_i$, $1 \leq i \leq n$, then all rules of $\Delta$ with $P$ in the head belong to $\Delta_i$.

If $$\{\Delta_1, \ldots, \Delta_n\}$$ is a partition of $\Delta$, then $\bigcup_i \tau^d_{\Delta_i} = \tau^d_{\Delta}$, and $\tau^d_{\Delta_i} \cap \tau^d_{\Delta_j} = \emptyset$ whenever $i \neq j$. Notice that each $\Delta_i$ has some “new” open symbols. For instance, if $P$ is defined in $\Delta$, but not in $\Delta_i$, then it is a new open symbol of $\Delta_i$. Of course, it holds that $\tau = \tau^o_{\Delta} \cup \tau^d_{\Delta} = \tau^o_{\Delta_i} \cup \tau^d_{\Delta_i}, 1 \leq i \leq n$.

The following theorem demonstrates that a model of a definition, is, at the same time, a model of each of its sub-definitions. As a side effect, we demonstrate that the totality of the large definition implies the totality of its sub-definitions.

**Theorem 6.2.** *Decomposition.* Let $\Delta$ be a definition over $\tau$ with partition $(\Delta_1, \ldots, \Delta_n)$. Let $I$ be a $\tau$-structure. If $I \models \Delta$ then $I \models \Delta_1 \land \ldots \land \Delta_n$.

Before proving this theorem, let us consider some of its implications. Let $I$ be a model of $\Delta$. The theorem says that for each $i$, $1 \leq i \leq n$, $\Delta_i$ is total in the restriction $I|_{\tau^o_{\Delta_i}}$ of $I$ to the open symbols of $\Delta_i$ and moreover that $I$ is the $\Delta_i$-extension of $I|_{\tau^o_{\Delta_i}}$. Using the notations of Definition 4.17, this means that $I = I^{\Delta_i} = I^{\Delta_i|}$. We obtain the following corollary.

**Corollary 6.3.** If $I \models \Delta$, then for each $i$, $1 \leq i \leq n$, $\Delta_i$ is total in $I|_{\tau^o_{\Delta_i}}$.

The following example shows that the inverse direction of Theorem 6.2 does not hold in general.

**Example 6.4.** Let $\Delta$, $\Delta_1$, $\Delta_2$ be the following definitions.

$$\Delta := \begin{cases} P \leftarrow Q \bigg\}, \\ Q \leftarrow P \bigg\}, \end{cases}$$

$$\Delta_1 := \{ P \leftarrow Q \},$$

$$\Delta_2 := \{ Q \leftarrow P \}.$$  

Definition $\Delta$ is total, and its unique model is $\emptyset$ in which both $P$ and $Q$ are false. According to Theorem 6.2, $\emptyset$ satisfies $\Delta_1$ and $\Delta_2$. Note that $\{P, Q\}$ is not a model of $\Delta$ and yet, it satisfies $\Delta_1$ and $\Delta_2$. Indeed, $\{P, Q\}$ is the $\Delta_1$-extension of the $\tau^o_{\Delta_1}$-structure $\{Q\}$ and the $\Delta_2$-extension of the $\tau^o_{\Delta_2}$-structure $\{P\}$. 

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To prove the theorem, we shall use two lemmas. The stable operator maps a structure \( I \) to the least fixpoint of \( T_\Delta(\cdot, I) \) in the lattice \( S^\tau_{I,v} \). The first lemma shows that the image of \( I \) is also the least fixpoint of this operator in much larger lattices.

**Lemma 6.5.** Let \( \Delta \) be a definition over \( \tau \) and \( I \) a \( \tau \)-structure. Let \( \tau^o \) be any vocabulary such that \( \tau_n \subseteq \tau^o \subseteq \tau_\Delta \). The least fixpoint of \( T_\Delta(\cdot, I) \) in the lattice \( S^\tau_{I,v} \) is \( ST_\Delta(I) \).

**Proof.** From Proposition 4.9, it follows easily that the operator \( T_\Delta(\cdot, I) \) is a well-defined, monotone operator in the lattice \( S^\tau_{I,v} \). Each structure in the image of \( T_\Delta(\cdot, I) \) belongs to \( S^\tau_{I,v} \subseteq S^\tau_{I,v} \). Thus, the least fixpoint of \( T_\Delta(\cdot, I) \) in \( S^\tau_{I,v} \) belongs to \( S^\tau_{I,v} \) and must be \( ST_\Delta(I) \), the least fixpoint of \( T_\Delta(\cdot, I) \) in \( S^\tau_{I,v} \).

**Lemma 6.6.** Let \( M \) be a fixpoint of the stable operator \( ST_\Delta \) extending \( \tau_\Delta -structure \( I_0 \). Let \( K, L \) be \( \tau \)-structures extending \( I_0 \) and let \( L|_{\tau_\Delta} = M|_{\tau_\Delta} \).

(a) If \( K \subseteq L \) and \( K \subseteq M \) then \( ST_\Delta(L) \subseteq ST_\Delta(K) \).

(b) If \( L \subseteq K \) and \( M \subseteq K \) then \( ST_\Delta(K) \subseteq ST_\Delta(L) \).

**Proof.** Denote \( M_0 := M|_{\tau_\Delta} \). Note that since \( \Delta_i \) is exactly the set of rules of \( \Delta \) defining the predicates of \( \tau_{A_i} \), it holds for each \( I, J \) extending \( I_0 \) that

\[
T_\Delta(I, J)|_{\tau_\Delta} = T_\Delta(I, J)|_{\tau_\Delta}.
\]

Also, by definition of \( T_\Delta \), it holds that

\[
T_\Delta(I, L)|_{\tau_\Delta} = L|_{\tau_\Delta} = M_0.
\]

Let \( K' := ST_\Delta(K) \) and \( L' := ST_\Delta(L) \). The structure \( L' \) is the least fixpoint of the monotone operator \( T_\Delta(\cdot, L) \) in the lattice \( S^\tau_{M_0} \). Using Lemma 6.5, also in the larger lattice \( S^\tau_{I,v} \). The structure \( K' \) is the least fixpoint of the monotone operator \( T_\Delta(\cdot, K) \) in the same lattice \( S^\tau_{I,v} \).

(a) Let \( K \subseteq L \) and \( K \subseteq M \). We show that \( L' \subseteq K' \). Our goal is to show that for each \( I \in S^\tau_{I,v} \) such that \( K' \subseteq I \),

\[
T_\Delta(I, L) \subseteq T_\Delta(I, K).
\]

Then by applying Lemma 3.5(b) in the lattice \( S^\tau_{I,v} \), we will obtain that

\[
lp(T_\Delta(\cdot, L)) \subseteq lp(T_\Delta(\cdot, K)),
\]

or equivalently that \( L' \subseteq K' \). We prove (10) separately for open and defined symbols of \( \Delta_i \).

Because \( K \subseteq M \) and by anti-monotonicity of \( ST_\Delta \), it holds that \( M = ST_\Delta(M) \subseteq ST_\Delta(K) = K' \). Combined with (9), this yields \( T_\Delta(I, L)|_{\tau_\Delta} = M|_{\tau_\Delta} \subseteq K'|_{\tau_\Delta} \).

By monotonicity of \( T_\Delta \) in its first argument, we have that if \( K' \subseteq I \) then \( K' = T_\Delta(K', K) \subseteq T_\Delta(I, K) \). We conclude that

\[
T_\Delta(I, L)|_{\tau_\Delta} \subseteq K'|_{\tau_\Delta} \subseteq T_\Delta(I, K)|_{\tau_\Delta}.
\]

We need to show the same for the defined symbols. Since \( K \subseteq L \), anti-monotonicity of \( T_\Delta \) in the second argument implies that \( T_\Delta(I, L) \subseteq T_\Delta(I, K) \). Using (8), we
obtain that
\[ T_{\Delta_i}(I, L)|_{\tau_{\Delta_i}} = T_{\Delta}(I, L)|_{\tau_{\Delta_i}} \subseteq T_{\Delta}(I, K)|_{\tau_{\Delta_i}}. \tag{12} \]

Statements (11) and (12) give us (10), and we conclude that \( L' \subseteq K' \).

(b) Assume that \( M \subseteq K \) and \( L \subseteq K \). We prove that for each \( I \subseteq K' \), \( T_{\Delta}(I, K) \subseteq T_{\Delta_i}(I, L) \). Then we can apply Lemma 3.5(a) which will yield the desired result that \( K' \subseteq L' \). The proof is similar to that of (a).

First, by anti-monotonicity of \( ST_{\Delta} \), we have that \( K' \subseteq M \). By monotonicity of \( T_{\Delta} \) in its first argument and anti-monotonicity in its second argument, it holds for each \( I \subseteq K' \subseteq M \) that \( T_{\Delta}(I, K) \subseteq T_{\Delta}(I, M) \subseteq T_{\Delta}(M, M) = M \). Using (9), we obtain that
\[ T_{\Delta}(I, K)|_{\tau_{\Delta_i}} \subseteq M|_{\tau_{\Delta_i}} = T_{\Delta_i}(I, L)|_{\tau_{\Delta_i}}. \tag{13} \]

Second, by anti-monotonicity in the second argument it holds that \( T_{\Delta}(I, K) \subseteq T_{\Delta}(I, L) \). Using (8), we find
\[ T_{\Delta}(I, K)|_{\tau_{\Delta_i}} \subseteq T_{\Delta}(I, L)|_{\tau_{\Delta_i}} = T_{\Delta_i}(I, L)|_{\tau_{\Delta_i}}. \tag{14} \]

Statements (13) and (14) yield \( K' \subseteq L' \).

\[ \square \]

Proof. (of Theorem 6.2) Let \( M \) be the \( \Delta \)-extension of the \( \tau_{\Delta_i} \)-structure \( I_0 \), and denote \( M_{\eta} := M|_{\tau_{\Delta_i}} \). Consider the sequences of \( \tau \)-extensions of \( I_0 \), the increasing sequence \( (I^\xi)_{\xi \geq 0} \), and the decreasing sequence \( (J^\xi)_{\xi \geq 0} \). The sequences are determined by operator \( ST_{\Delta} \) in the lattice \( S^*_I \). Since \( \Delta \) is total in \( I_0 \), structure \( M \) is the limit of both sequences. Likewise, for each \( i \), consider two sequences, the increasing sequence \( (I^\xi_i)_{\xi \geq 0} \) and the decreasing sequence \( (J^\xi_i)_{\xi \geq 0} \), determined by the operator \( ST_{\Delta_i} \) in the sublattice \( S^*_I \). They converge to \( M_{\eta} \) and \( M_{\eta} \), respectively.

Recall that
\[ I^\xi := ST_{\Delta}(J^{<\xi}), \quad J^{<\xi} := \bigcap_{\eta < \xi} J^\eta \]
\[ J^\xi := ST_{\Delta}(I^{<\xi}), \quad I^{<\xi} := \bigcup_{\eta < \xi} I^\eta \]

and
\[ I^\xi_i := ST_{\Delta_i}(J_i^{<\xi}), \quad J_i^{<\xi} := \bigcap_{\eta < \xi} J_i^\eta \]
\[ J^\xi_i := ST_{\Delta_i}(I_i^{<\xi}), \quad I_i^{<\xi} := \bigcup_{\eta < \xi} I_i^\eta. \]

We will prove that for each \( \xi \),
\[ I^\xi \subseteq I^\xi_i \subseteq J^\xi_i \subseteq J^\xi. \tag{15} \]

This property allows us to conclude that, since the outer sequences \( (I^\xi)_{\xi \geq 0} \) and \( (J^\xi)_{\xi \geq 0} \) converge to \( M \), the inner sequences \( (I^\xi_i)_{\xi \geq 0} \) and \( (J^\xi_i)_{\xi \geq 0} \) converge to \( M \) as well. Therefore, we obtain that \( \Delta_i \) is total in \( I_i|_{\tau_{\Delta_i}} \) and that structure \( M \) is the unique \( \Delta_i \)-extension of \( I|_{\tau_{\Delta_i}} \). Since \( i \) was arbitrary, we obtain that \( I \models \Delta_1 \land \cdots \land \Delta_n \).

For each \( \xi \), we will prove statement (15) and the following statement:
\[ I_i^{<\xi} \subseteq J_i^{<\xi} \subseteq J^{<\xi}. \tag{16} \]

The two statements are proven by simultaneous induction on \( \xi \).
First we prove the base case of statement (16):

\[ I^{<0} \sqsubseteq I^0_i \sqsubseteq J^{<0}_i \sqsubseteq J^0. \]  

(17)

This is equivalent to

\[ \perp_I \sqsubseteq \perp M \sqsubseteq \top M \sqsubseteq \top I \]

which is straightforward.

Second, assume that for arbitrary \( \xi \geq 0 \), statement (16) holds. We prove that then (15) holds for \( \xi \) as well. Notice that, as a special case, we obtain the base case for (15).

By the construction of the sequences, we have for each \( \xi \) that \( I^{<\xi}_i \sqsubseteq J^{<\xi}_i \). By anti-monotonicity of \( ST_{\Delta} \), we obtain the middle inequality of (15). Similarly, we have that \( I^{<\xi} \sqsubseteq M \sqsubseteq J^{<\xi} \). This, together with the induction hypothesis, entails that the conditions of Lemma 6.6 (a)+(b) are satisfied. Application of the lemma yields that \( I^{\xi} \sqsubseteq I^\xi_i \) and \( J^{\xi}_i \sqsubseteq J^{\xi} \).

Third, assume that for each \( \eta < \xi \),

\[ I^{\eta} \sqsubseteq I^\eta_i \sqsubseteq J^{\eta}_i \sqsubseteq J^{\eta} \]

The standard fixpoint construction of anti-monotone operators guarantees the inner inequality of (16):

\[ I^{<\xi}_i \sqsubseteq J^{<\xi}_i \]

By taking the union and intersection of appropriate sets over all \( \eta < \xi \), we obtain the two outer inequalities of (16):

\[ I^{<\xi} \sqsubseteq I^{<\xi}_i \text{ and } J^{<\xi}_i \sqsubseteq J^{<\xi} \]

This proves the statements (15) and (16) for every \( \xi \) and concludes the proof of the theorem.

6.2 Reduction Partitions

Theorem 6.2 gives one direction of the Modularity theorem. Now our goal is to come up with some condition on the partition of \( \Delta \) so that both directions of the Modularity theorem hold. Recall from Example 6.4 that the other direction does not hold in general.

**Example 6.7.** Recall the definitions in Example 6.4:

\[ \Delta := \{ P \leftarrow Q \} \quad \Delta_1 := \{ P \leftarrow Q \} \quad \Delta_2 := \{ Q \leftarrow P \}. \]

The structure \( \{ P, Q \} \) satisfies \( \Delta_1 \land \Delta_2 \) but not \( \Delta \).

In the example, splitting the definition breaks the circular *dependency* between \( P \) and \( Q \). This causes the broken equivalence between \( \Delta \) and \( \Delta_1 \land \Delta_2 \). The example suggests that splitting a definition will be equivalence preserving if the splitting does not break circular dependencies between atoms. Below we will formalise this notion using the notion of *reduction relation* defined in Section 5.

The following proposition formulates a simple and useful property of (possibly non-transitive) reductions in the context of a partition.
Proposition 6.8. Let \( \{ \Delta_1, \ldots, \Delta_n \} \) be a partition of \( \Delta \).

(a) A relation \( \prec \) is a reduction relation of \( \Delta \) in \( K_\Delta \) iff for each \( i \), \( 1 \leq i \leq n \), \( \prec \) is a reduction relation of \( \Delta_i \) in \( K_\Delta \).

(b) Let for each \( i \), \( 1 \leq i \leq n \), \( \prec \) be reduction relation of \( \Delta_i \) in \( K_\Delta \). Then \( \prec_1 \cup \cdots \cup \prec_n \) is a reduction relation of \( \Delta \) in \( K_\Delta \).

Proof. (a) For each defined predicate \( P \), there is exactly one \( i \) such that \( P \) is defined in \( \Delta_i \) and the formulas \( \varphi_P^\Delta \) defining \( P \) in \( \Delta \) and \( \varphi_P^{\Delta_i} \) defining \( P \) in \( \Delta_i \) are identical. It is obvious then that \( \prec \) is a reduction of \( \Delta \) in \( K_\Delta \) iff \( \prec \) is a reduction of \( \Delta_i \) in \( K_\Delta \), for each \( i \), \( 1 \leq i \leq n \).

(b) If for each \( i \in \{1, \ldots, n\} \), \( \prec_i \) is a reduction of \( \Delta_i \) in \( K_\Delta \), then by Proposition 5.4, \( \prec_1 \cup \cdots \cup \prec_n \) is a reduction relation of each \( \Delta_i \). By (a), \( \prec_1 \cup \cdots \cup \prec_n \) is a reduction relation of \( \Delta \) in \( K_\Delta \).

Recall that a pre-well-order is a reflexive and transitive relation such that every non-empty subset contains a minimal element.

The following definition is crucial for the right-to-left direction of the Modularity theorem. Let \( K_\Delta \) be a structure such that \( \tau_{K_\Delta} \subseteq \tau^\Delta_{\Delta_i} \).

Definition 6.9 reduction partition. Call partition \( \{ \Delta_1, \ldots, \Delta_n \} \) of definition \( \Delta \) a reduction partition of \( \Delta \) in \( K_\Delta \) if there is a reduction pre-well-order \( \prec \) of \( \Delta \) in \( K_\Delta \), and if \( Q[\bar{b}] \prec P[\bar{a}] \) and \( P[\bar{a}] \prec Q[\bar{b}] \), then \( P \) and \( Q \) are both open predicates of \( \Delta \), or they are defined in the same \( \Delta_i \).

Equivalently, if \( P \) and \( Q \) are not defined in the same \( \Delta_i \), then \( Q[\bar{b}] \prec P[\bar{a}] \) iff \( Q[\bar{b}] \prec P[\bar{a}] \). The intuition underlying this definition is that in a reduction partition, if an atom defined in one module depends on an atom defined in another module, then the latter atom is strictly less in the reduction ordering and hence does not depend on the first atom.

In a first step towards proving the second half of the modularity theorem, we prove that if \( \{ \Delta_1, \ldots, \Delta_n \} \) is a reduction partition of \( \Delta \) in a \( \tau^\Delta_{\Delta_i} \)-structure \( I_\Delta \), then the conjunction \( \Delta_1 \wedge \cdots \wedge \Delta_n \) has at most one model extending \( I_\Delta \).

Theorem 6.10. If \( \Delta \) has a reduction partition \( \{ \Delta_1, \ldots, \Delta_n \} \) in a \( \tau^\Delta_{\Delta_i} \)-structure \( I_\Delta \) then \( \Delta_1 \wedge \cdots \wedge \Delta_n \) has at most one model extending \( I_\Delta \).

Proof. Let \( M \) and \( M' \) be models of \( \Delta_1 \wedge \cdots \wedge \Delta_n \). Assume towards contradiction that \( M \) and \( M' \) differ. Let us select a minimal atom \( P[\bar{a}] \) in the reduction pre-well-order \( \prec \) such that \( M \) and \( M' \) disagree on \( P[\bar{a}] \). Assume that \( P[\bar{a}] \) is defined in \( \Delta_i \). Since \( \prec \) is reflexive and \( M \) and \( M' \) disagree on \( P[\bar{a}] \), \( M \not\equiv \ll P[\bar{a}] \) \( M' \). On the other hand, because \( P[\bar{a}] \) is minimal, it holds that \( M \equiv \ll P[\bar{a}] \) \( M' \). It follows that \( M \ll M' \). Moreover, if \( Q \in \tau^\Delta_{\Delta_i} \) then for each atom \( Q[\bar{b}] \), \( Q[\bar{b}] \ll P[\bar{a}] \) iff \( Q[\bar{b}] \ll P[\bar{a}] \). Hence, we have \( M \ll M' \). By Proposition 6.8(a), \( \prec \) is a transitive reduction relation of \( \Delta_i \) in \( I_\Delta \). The condition of Proposition 5.12 holds and we can infer that \( M = (M \ll M') \Delta_i = (M' \ll M') \Delta_i = M' \). We obtain \( M \equiv M' \), a contradiction. \( \square \)
Example 6.11. Consider the definitions from Example 6.4:

\[
\Delta := \{ P \leftarrow Q, \quad Q \leftarrow P \}, \quad \Delta_1 := \{ P \leftarrow Q \}, \quad \Delta_2 := \{ Q \leftarrow P \}.
\]

Each reduction of the definition \(\Delta\) includes tuples \((P, Q)\) and \((Q, P)\). Hence, the partition \((\Delta_1, \Delta_2)\) is not a reduction partition. The formula \(\Delta_1 \land \Delta_2\) has multiple models \(\emptyset\) and \(\{P, Q\}\).

Example 6.12. Consider the partition of the definition from Example 5.2:

\[
\Delta_1 := \{ \forall x (E(x) \leftarrow x = 0) \}, \quad \Delta_2 := \{ \forall x (O(s(x)) \leftarrow E(x)) \}.
\]

The transitive reflexive closure \(\prec^*\) of the reduction of \(\Delta\) presented in Example 5.2 is a well-founded partial order. It holds that \(E[n] \prec^* O[m]\) and \(O[n] \prec^* E[m]\) iff \(n < m\). Consequently, \(\{\Delta_1, \Delta_2\}\) is a reduction partition. The conjunction of \(\Delta_1\) and \(\Delta_2\) has one model.

Example 6.13. Consider the following definitions:

\[
\Delta := \{ P \leftarrow \neg P, \quad Q \leftarrow \neg P \}, \quad \Delta_1 := \{ P \leftarrow \neg P \}, \quad \Delta_2 := \{ Q \leftarrow \neg P \}.
\]

The reflexive closure of the relation \(\{(P, Q)\}\) is a well-order. Clearly, the partition \((\Delta_1, \Delta_2)\) is a reduction partition. The definitions \(\Delta\) and \(\Delta_1\) and the conjunction \(\Delta_1 \land \Delta_2\) are all inconsistent.

In the next step towards proving the second half of the modularity theorem, we prove the totality of a well-behaved definition \(\Delta\) with a reduction partition.

Definition 6.14. A partition \(\{\Delta_1, \ldots, \Delta_n\}\) of definition \(\Delta\) is total in a structure \(K_o\) \((\tau_{K_o} \subseteq \tau^o_{\Delta})\) if each \(\Delta_i\), \(1 \leq i \leq n\), is total in \(K_o\).

Theorem 6.15 Totality. If \(\Delta\) has a total reduction partition \(\{\Delta_1, \ldots, \Delta_n\}\) in a structure \(K_o\) \((\tau_{K_o} \subseteq \tau^o_{\Delta})\) then \(\Delta\) is total in \(K_o\).

Thus, one way to prove that \(\Delta\) is total in \(K_o\) is to prove that it has a reduction partition, and that each definition \(\Delta_i\) in the partition is total in \(K_o\).

To prove this theorem, we need the following lemma.

Lemma 6.16. Let \(\{\Delta_1, \ldots, \Delta_n\}\) be a partition of \(\Delta\) and let \(\prec\) a reduction of \(\Delta\) in \(\tau^o_{\Delta}\)-structure \(I_o\) such that \([I_o]_{\prec P[\bar{a}]} = I_o\) for all \(P[\bar{a}] \in M_i\). Let \(M, M'\) be \(\tau\)-structures extending \(I_o\) such that for some domain atom \(P[\bar{a}]\), \((M, M')\) is an oscillating pair of \([ST\Delta(\cdot)]_{\prec P[\bar{a}]\Delta_{\Delta_i}} = M'|_{\tau^o_{\Delta_j}} = M_{\Delta_i}\). Then for all \(I \in S_{M, o}^{P[\bar{a}]}\), it holds that:

(a) if \(I \subseteq M\), then \(M' \subseteq [ST\Delta_i(I)]_{\prec P[\bar{a}]\Delta}\);
(b) if \(M' \subseteq I\), then \([ST\Delta_i(I)]_{\prec P[\bar{a}]} \subseteq M\).

The lemma has a similar proof as Lemma 6.6.

Proof. By Proposition 5.11, the structure \(M'\) is the least fixpoint of \([T\Delta(\cdot, M)]_{\prec P[\bar{a}]\Delta}\) and \(M\) the least fixpoint of \([T\Delta(\cdot, M')]_{\prec P[\bar{a}]\Delta}\) in the lattice \(S_{I_o}^{P[\bar{a}]}\). Since \(M, M' \in\)
For the defined predicates, for each \( J \),

\( S_{M_o}^{\prec P[a]} \), they are also the least fixpoints of these operators in the sublattice \( S_{M_o}^{\prec P[a]} \).

Let \( I \in S_{M_o}^{\prec P[a]} \) and denote \( I' := |ST_{\Delta_i}(I)|_{\prec P[a]} \). By Proposition 6.8, \( \prec \) is a reduction of \( \Delta_i \). By Proposition 5.11, \( I' \) is the least fixpoint of \(|T_{\Delta_i}(\cdot, I)|_{\prec P[a]} \) in \( S_{M_o}^{\prec P[a]} \).

(a) Let \( I \subseteq M \). We prove that \( M' \subseteq I' \). Our goal is to show that for each \( J \in S_{M_o}^{\prec P[a]} \) such that \( J \subseteq M' \),

\[
|T_{\Delta_i}(J, M)|_{\prec P[a]} \subseteq |T_{\Delta_i}(J, I)|_{\prec P[a]}.
\]  

(18)

Then by Lemma 3.5(a), it will follow that

\[
M' = \text{lfp}(|T_{\Delta_i}(\cdot, M)|_{\prec P[a]}) \subseteq \text{lfp}(|T_{\Delta_i}(\cdot, I)|_{\prec P[a]}) = I'.
\]

We prove (18) separately for open and defined symbols of \( \Delta_i \).

First, let \( J \subseteq M' \). By monotonicity in the first argument, \(|T_{\Delta_i}(J, M)|_{\prec P[a]} \subseteq |T_{\Delta_i}(M', M)|_{\prec P[a]} = M'\). Also, \( T_{\Delta_i}(J, I)|_{\tau_{\Delta_i}} = M_o I = |M_o|_{\prec P[a]} \). Combining these statements, we obtain:

\[
|T_{\Delta_i}(J, I)|_{\tau_{\Delta_i}} \subseteq M' = |T_{\Delta_i}(J, M)|_{\prec P[a]}|_{\tau_{\Delta_i}}.
\]  

(19)

Second, \( T_{\Delta_i} \) is anti-monotone in its second argument, which implies \( T_{\Delta_i}(J, J) \subseteq T_{\Delta_i}(J, I) \), for each \( J \in S_{M_o}^{\prec P[a]} \). Since the operators \( T_{\Delta_i} \) and \( T_{\Delta_i} \) coincide on the defined symbols of \( \Delta_i \), it follows that:

\[
T_{\Delta_i}(J, I)|_{\tau_{\Delta_i}} = T_{\Delta_i}(J, I)|_{\tau_{\Delta_i}}.
\]

By combining these statements, we conclude that:

\[
T_{\Delta_i}(J, M)|_{\tau_{\Delta_i}} \subseteq T_{\Delta_i}(J, I)|_{\tau_{\Delta_i}} = T_{\Delta_i}(J, I)|_{\tau_{\Delta_i}}.
\]

After projection with \(|\cdot|_{\prec P[a]} \), we obtain:

\[
|T_{\Delta_i}(J, M)|_{\prec P[a]}|_{\tau_{\Delta_i}} \subseteq |T_{\Delta_i}(J, I)|_{\prec P[a]}|_{\tau_{\Delta_i}}.
\]  

(20)

The combination of (19) and (20) yields statement (18).

(b) Let \( M' \subseteq I \). We show that for each \( J \in S_{M_o}^{\prec P[a]} \) such that \( M \subseteq J \),

\[
|T_{\Delta_i}(J, I)|_{\prec P[a]} \subseteq |T_{\Delta_i}(J, M')|_{\prec P[a]}.
\]

Then we can apply Lemma 3.5(b) to prove (b).

For the open predicates, if \( M \subseteq J \) then by the same kind of reasoning as in (a),

\[
|T_{\Delta_i}(J, I)|_{\prec P[a]}|_{\tau_{\Delta_i}} = M_o I = |T_{\Delta_i}(M, M')|_{\prec P[a]}|_{\tau_{\Delta_i}} \subseteq |T_{\Delta_i}(J, M')|_{\prec P[a]}|_{\tau_{\Delta_i}}.
\]

For the defined predicates, for each \( J \) it holds that

\[
T_{\Delta_i}(J, I)|_{\tau_{\Delta_i}} = T_{\Delta_i}(J, I)|_{\tau_{\Delta_i}} \subseteq T_{\Delta_i}(J, M')|_{\tau_{\Delta_i}}.
\]

Combining both results, we obtain that \(|T_{\Delta_i}(J, I)|_{\prec P[a]} \subseteq |T_{\Delta_i}(J, M')|_{\prec P[a]}|_{\tau_{\Delta_i}}. \]

PROOF. of Theorem 6.15.

Let \( I_o \) be an arbitrary \( \tau_{\Delta_i} \)-extension of \( K_o \). Then, \( \{\Delta_1, \ldots, \Delta_n\} \) is a total partition of \( \Delta \) in \( I_o \) and, by Proposition 5.4, \( \{\Delta_1, \ldots, \Delta_n\} \) is a reduction partition of \( \Delta \) in

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Let $\prec$ be a pre-well-ordered reduction of $\Delta$ in $I_0$ satisfying the condition of Definition 6.9. We assume idempotence of $I_0$, i.e. $|I_0|_{\prec P[a]} = I_0$ for all $P[a] \in \Delta^*$. This assumption can always be made: see Remark 5.10.

Assume, towards a contradiction, that $\Delta$ is not total in $I_0$. The assumption implies that $I_0^{\Delta_1} \neq I_0^{\Delta_1}$. Let $P[a]$, where $P \in \tau^2_{\Delta_1}$, be a minimal atom in the reduction ordering $\prec$ such that $I_0^{\Delta_1} \not\models P[a]$ and $I_0^{\Delta_1} \models P[a]$. By reflexivity of $\prec$ and our choice of $P[a]$, it holds that

$$I_0^{\Delta_1} \not\models P[a] \implies I_0^{\Delta_1} \models P[a].$$

Because we have a reduction partition, for each atom $Q[b]$ not defined in $\Delta_1$, $Q[b] \prec P[a]$ iff $Q[b] \not\prec P[a]$. Therefore,

$$I_0^{\Delta_1} \models P[a] \implies I_0^{\Delta_1} \not\models P[a].$$

Define $M := |I_0^{\Delta_1}|_{\prec P[a]}$ and $M' := |I_0^{\Delta_1}|_{\prec P[a]}$ and let $M_{oi} := M|_{\tau_{\Delta_i}} = M'|_{\tau_{\Delta_i}} = |(I_0^{\Delta_1}|_{\tau_{\Delta_i}})|_{\prec P[a]}$. Since $I_0$ is idempotent for $| \cdot |_{\prec P[a]}$, $M_{oi}$ is an extension of $I_0$.

The structures $M$ and $M'$ are different in $P[a]$. On the one hand, since $\Delta_i$ is total in $I_0$ and $M_{oi}$ is an extension of $I_0$, it holds that

$$M_{oi}^{\Delta_1} = M_{oi}^{\Delta_1}.$$  

On the other hand, we will prove the following:

$$|M_{oi}^{\Delta_1}|_{\prec P[a]} \subseteq M \subseteq M' \subseteq |M_{oi}^{\Delta_1}|_{\prec P[a]}.$$  

(21)

Since $M \neq M'$, we will obtain the contradiction. The proof of (21) is by induction. By Proposition 5.12, the following equations hold in the lattice $S^{\prec P[a]}_{M_{oi}}$:

$$|M_{oi}^{\Delta_1}|_{\prec P[a]} = \text{gfp}((|ST_{\Delta_1}(\cdot)|_{\prec P[a]})^2)$$

and

$$|M_{oi}^{\Delta_1}|_{\prec P[a]} = \text{gfp}((|ST_{\Delta_1}(\cdot)|_{\prec P[a]})^2).$$

Consider the sequences $(I^i_\xi)_{\xi \geq 0}$ and $(J^i_\xi)_{\xi \geq 0}$ determined by the operator $|ST_{\Delta_1}(\cdot)|_{\prec P[a]}$ in the lattice $S^{\prec P[a]}_{M_{oi}}$. We shall demonstrate that the following holds: for every $\xi$,

$$I^i_\xi \subseteq M \subseteq M' \subseteq J^i_\xi.$$  

(22)

$$I^{\leq \xi} \subseteq M \subseteq M' \subseteq J^{\leq \xi}.$$  

(23)

Statements (22) and (23) are proven by simultaneous transfinite induction on $\xi$.

First, we establish the base case of statement (23):

$$I^{<0} \subseteq M \subseteq M' \subseteq J^{<0}.$$  

This is straightforward since $M, M' \in S^{< P[a]}_{M_{oi}}$ and $I^{<0}$ and $J^{<0}$ are the bottom and top element, respectively, of this lattice.

Second, we show that, for arbitrary $\xi$, if (23) holds then (22) holds. Let us assume that the statement (23) holds for $\xi$. Since $I^{\leq \xi} \subseteq M$, Lemma 6.16(a) implies that $M' \subseteq |ST_{\Delta_1}(I^{\leq \xi})|_{\prec P[a]} = J^{\leq \xi}$. Since $M' \subseteq J^{\leq \xi}$, then by Lemma 6.16(b), $I^{\leq \xi} = |ST_{\Delta_1}(J^{\leq \xi})|_{\prec P[a]} \subseteq M$.  

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Third, it remains to be proven that if for all \( \eta < \xi \), it holds that \( I^\eta_i \sqsubseteq M \sqsubseteq M' \sqsubseteq J^\eta \), then (23) holds for \( \xi \). This is straightforward.
This completes the proof of the theorem. \( \square \)

**Example 6.17.** As seen in Example 6.12, the following partition is a reduction partition of \( \Delta_1 \cup \Delta_2 \) in the natural numbers:

\[
\Delta_1 := \left\{ \forall x \left( E(x) \leftarrow x = 0 \right) \right\}, \quad \Delta_2 := \left\{ \forall x \left( O(s(x)) \leftarrow O(x) \right) \right\}.
\]

Both subdefinitions are non-recursive and positive. Consequently, both are total. So, the conditions of Theorem 6.15 hold. This definition is total in the natural numbers and has a unique model.

**Corollary 6.18 Consistency.** If \( \Delta \) has a total reduction partition \( \{ \Delta_1, \ldots, \Delta_n \} \) in a \( \tau^\Delta \)-structure \( I_o \), then \( \Delta \) and \( \Delta_1 \land \cdots \land \Delta_n \) are consistent and have a model extending \( I_o \).

**Example 6.19.** Recall the definition of Example 6.13:

\[
\Delta := \left\{ P \leftarrow \neg P, \quad Q \leftarrow \neg P \right\}, \quad \Delta_1 := \left\{ P \leftarrow \neg P \right\}, \quad \Delta_2 := \left\{ Q \leftarrow \neg P \right\}.
\]

Although this definition has a reduction partition, it is not consistent. Corollary 6.18 does not hold because \( \Delta_1 \) is not total in any structure.

Now, we are in a position to prove the second direction of the modularity theorem. Let \( \tau^o \sqsubseteq \tau^\Delta \).

**Theorem 6.20.** If \( \Delta \) has a total reduction partition \( \{ \Delta_1, \ldots, \Delta_n \} \) in the \( \tau^o \)-structure \( K_o \), then for any \( \tau \)-structure \( M \) extending \( K_o \), if \( M \models \Delta_1 \land \cdots \land \Delta_n \) then \( M \models \Delta \).

**Proof.** Assume \( M \) extends \( K_o \) and \( M \models \Delta_1 \land \cdots \land \Delta_n \) and let \( I_o = M|_{\tau^\Delta} \). Since \( I_o \) is an extension of \( K_o \), \( \{ \Delta_1, \ldots, \Delta_n \} \) is a total reduction partition of \( \Delta \) in \( I_o \). The conditions of Theorem 6.15 are satisfied. Consequently, \( \Delta \) is total in \( I_o \). The structure \( I_o^\Delta \) is a model of \( \Delta \) and, by Theorem 6.2, of \( \Delta_1 \land \cdots \land \Delta_n \). Since by Theorem 6.10, \( M \) is the unique model of \( \Delta_1 \land \cdots \land \Delta_n \) extending \( I_o \), \( M \) and \( I_o^\Delta \) are identical. \( \square \)

**Theorem 6.21 Modularity.** If \( \{ \Delta_1, \ldots, \Delta_n \} \) is a total reduction partition of \( \Delta \) in \( \tau^o \)-structure \( K_o \), then for any \( \tau \)-structure \( M \) extending \( K_o \),

\[
M \models \Delta \iff M \models \Delta_1 \land \cdots \land \Delta_n.
\]

**Proof.** Combine theorems 6.2 and 6.20. \( \square \)

Another immediate consequence is the following corollary.

**Corollary 6.22.** Let \( T_o \) be a theory over \( \tau^o \) such that for any \( \tau^o \)-model \( M_o \) of \( T_o \), \( \{ \Delta_1, \ldots, \Delta_n \} \) is a total reduction partition of \( \Delta \) in \( M_o \).

Then \( T_o \land \Delta \) and \( T_o \land \Delta_1 \land \cdots \land \Delta_n \) are logically equivalent.
Example 6.23. As seen in Example 6.17, the following partition is a total reduction partition of $\Delta_1 \cup \Delta_2$ in the natural numbers:

$$
\Delta_1 := \left\{ \forall x \left( E(x) \iff x = 0 \right) \right\}, \quad \Delta_2 := \left\{ \forall x \left( O(s(x)) \iff E(x) \right) \right\}.
$$

In ID-logic, the natural numbers are formalised by $T_N$ (Examples 4.4 and 4.21). By Corollary 6.22, the theories $T_N \cup \{\Delta_1 \cup \Delta_2\}$ and $T_N \cup \{\Delta_1 \land \Delta_2\}$ are equivalent.

7. SOME FAMILIAR TYPES OF DEFINITIONS

This section reconsiders the four different types of informal inductive definitions discussed in section 2: non-recursive definitions, positive definitions, definitions over well-founded sets and iterated inductive definitions. We demonstrate that these types of definitions can be correctly and uniformly represented in ID-logic. To this end, we define four formal subclasses of definitions of ID-logic that naturally correspond to the four informal types of inductive definitions and prove theorems to show that the well-founded semantics correctly formalises the meaning of these types of definitions.

7.1 Non-Recursive Definitions.

A first case is that of non-recursive definitions. A definition $\Delta$ is non-recursive if the bodies of the rules do not contain defined predicates.

**Definition 7.1 completion of $\Delta$.** Define the completion of $\Delta$, denoted $\text{comp}(\Delta)$, as the conjunction, for each defined symbol $X$ of $\Delta$, of formulas

$$
\forall \bar{x} \left( X(\bar{x}) \iff \varphi_X[\bar{x}] \right).
$$

The equivalence $\forall \bar{x} \left( X(\bar{x}) \iff \varphi_X[\bar{x}] \right)$ is sometimes referred at as the completed definition of $X$.

**Theorem 7.2.** Let $\Delta$ be a non-recursive definition over $\tau$. Then $\Delta$ is total and a $\tau$-structure $I$ satisfies $\Delta$ iff $I$ satisfies $\text{comp}(\Delta)$.

**Proof.** It is straightforward to show that if $\Delta$ is non-recursive, then for each $\tau_\Delta^I$-structure $I_o$, the operator $T_\Delta$ is constant in the lattice $S_{\Delta_o}^I$ and it maps each pair of $\tau$-structures to the unique structure $I$ such that, for each defined symbol $X$,

$$
X^I = \{ \bar{d} \mid I_o \models \varphi_X[\bar{d}] \}.
$$

This $I$ is the unique model of $\Delta$ and the unique model of $\text{comp}(\Delta)$ in $S_{\Delta_o}^I$. $\blacksquare$

7.2 Positive Definitions.

Let $\Delta$ be a positive definition, defining the symbols $\bar{P}$. Let $\bar{X}$ be a set of new predicate symbols such that for each defined symbol $P_i$, $X_i$ and $P_i$ have the same arity. Define the following formula

$$
\text{PID}(\Delta) := \bigwedge \Delta \land \forall \bar{X} \left( \bigwedge \Delta[\bar{P}/\bar{X}] \supset (\bar{P} \subseteq \bar{X}) \right).
$$

Here, $\bigwedge \Delta$ is the conjunction of formulas obtained by replacing definitional rules with material implications in $\Delta$; $\Delta[\bar{P}/\bar{X}]$ is the definition obtained by substituting $X_i$
for each defined symbol $P_i$ and $\bar{P} \subseteq \bar{X}$ is a shorthand for the formula $(\forall \bar{x} P_1(\bar{x}) \supset X_1(\bar{x})) \land \cdots \land (\forall \bar{x} P_n(\bar{x}) \supset X_n(\bar{x}))$. The formula $PID(\Delta)$ is the standard second-order formula to express that predicates $\bar{P}$ satisfy the positive inductive definition $\Delta$.

Define also $Circ(\Delta; \bar{P}) := \bigwedge \Delta \land \forall \bar{X}(\bigwedge \Delta[\bar{P}/\bar{X}] \land \bar{X} \subseteq \bar{P}) \supset \bar{P} \subseteq \bar{X}$.

This formula is the standard circumscription of $\bigwedge \Delta$ with respect to the defined predicates $\bar{P}$ [Lifschitz 1994].

Theorem 7.3. Let $\Delta$ be a positive definition over $\tau$. Then $\Delta$ is total and for all $\tau$-structures $I$, the following are equivalent:

(a) $I$ is a model of $\Delta$;
(b) $I$ is the least fixpoint of $\Gamma_\Delta$ in the lattice $S_{I^\Delta}$;
(c) $I$ is a model of $PID(\Delta)$;
(d) $I$ is a model of $Circ(\Delta; \bar{P})$.

Proof. In case $\Delta$ is a positive definition, defined symbols have no negative occurrences, so $\Delta$ and $\Delta'$ are identical. Consequently, for any pair of structures $I, J$ in the lattice $S_{I^\Delta}$, it holds that $T_\Delta(I, J) = \Gamma_\Delta(I)$ which does not depend on $J$. Thus, the stable operator $ST_\Delta$ is a constant operator in this lattice and maps any structure $J$ to the least fixpoint of $\Gamma_\Delta$. Thus, it follows that $I_o^{\Delta\uparrow}$ and $I_o^{\Delta\downarrow}$ are identical to the least fixpoint of $\Gamma_\Delta$ in $S_{I_o^\Delta}$. This proves the equivalence of (a) and (b).

The equivalence of (b) and (c) in case of a positive definition is well-known (see e.g. [Aczel 1977]). Finally, the axiom $PID(\Delta)$ expresses that $\bar{P}$ should be the least relations satisfying $\bigwedge \Delta$, while $Circ(\Delta; \bar{P})$ expresses that $\bar{P}$ should be minimal relations satisfying $\bigwedge \Delta$. Both axioms are equivalent, since there is a set of least relations satisfying $\bigwedge \Delta$, and it is the unique set of minimal relations satisfying this formula. □

The theorem is significant since it shows that for positive definitions, the semantics defined here coincides with standard monotone induction. It implies that if $I \models \Delta$ then $I$ is the least structure extending $I_o$ that satisfies the rules of $\Delta$ viewed as a set of first-order implications.

Example 7.4. Consider the formulation of the induction axiom in ID-logic in Example 4.4:

$$\exists N \left[ \left\{ \forall x (N(x) \leftrightarrow x = 0), \forall x (N(s(x)) \leftrightarrow N(x)) \right\} \land \forall x N(x) \right].$$

By Theorem 7.3, it is equivalent to the second-order axiom

$$\exists N \left[ \forall x (N(x) \subseteq x = 0) \land \forall x (N(s(x)) \subseteq N(x)) \land \forall X \left[ \forall x (X(x) \subseteq x = 0) \land \forall x (X(s(x)) \subseteq X(x)) \supset \forall x (N(x) \supset X(x)) \right] \land \forall x N(x) \right].$$

We show that this formula is logically equivalent with the standard induction axiom. The first two conjuncts follow from the last and may be deleted. Using the last
conjunct, the third conjunct can be simplified as follows:
\[
\exists N \left[ \forall X \left( \forall x \left( (X(x) \land x = 0) \land \forall x \left( (X(s(x)) \lor X(x)) \lor \forall x \left( X(x) \right) \right) \right) \right].
\]

Notice that the first element of the conjunction does not depend of \( N \), so the outer existential quantifier can be moved inwards, and the tautological \( \exists N \forall x \) can be removed. We obtain the standard induction axiom:
\[
\forall X \left( \forall x \left( (X(x) \land x = 0) \land \forall x \left( (X(s(x)) \lor X(x)) \lor \forall x \left( X(x) \right) \right) \right) \right).
\]

7.3 Iterated Inductive Definitions

Recall from Section 2 that an iterated inductive definition constructs a set as the limit of a sequence of constructive steps, each of which itself is a monotone induction. Here, we formalise that intuition, and make a connection between this new “formalism” and the representation of iterated inductive definitions in ID-logic.

Let \((\Delta_1, \ldots, \Delta_n)\) be a finite sequence of positive definitions over a vocabulary \( \tau \) such that:

— all definitions define disjunct sets of relation symbols, i.e., \( \tau^d_\Delta \cap \tau^d_\Delta = \emptyset \) for \( i \neq j \);  
— if a relation symbol is defined in some \( \Delta_i \), then it does not occur as an open symbol in \( \Delta_j \), for any \( j < i \).

We call such a sequence an iterated inductive definition and we interpret it as a simple, finite case of an iterated inductive definition.

Let \( \bar{X} \) be the set \( \tau^d_\Delta \cup \cdots \cup \tau^d_\Delta \), i.e., the collection of all symbols defined in at least one definition \( \Delta_i \), \( 1 \leq i \leq n \), and let \( \tau^o \) be the vocabulary \( \tau \setminus \bar{X} \). Select an arbitrary \( \tau^o \)-structure \( I_o \).

We define \( I_o(\Delta_1, \ldots, \Delta_n) \) by induction on \( i \): \( I_o^0 := I_o \) and for each \( i \), \( 1 \leq i \leq n \), \( I_o(\Delta_1, \ldots, \Delta_i) := (I_o(\Delta_1, \ldots, \Delta_{i-1}))^{\Delta_i} \). Note that by Theorem 7.3, \( I_o(\Delta_1, \ldots, \Delta_i) \) is the least fixpoint of the positive definition \( \Delta_i \), extending \( I_o(\Delta_1, \ldots, \Delta_{i-1}) \). The above definition models precisely the process of iterated induction as explained in Section 2. We say that the \( \tau \)-structure \( I_o(\Delta_1, \ldots, \Delta_n) \) is the structure defined by the iterated inductive definition \((\Delta_1, \ldots, \Delta_n)\) in \( I_o \).

Consider the iterated inductive definition \((\Delta_1, \ldots, \Delta_n)\) and the new definition \( \Delta = \Delta_1 \cup \cdots \cup \Delta_n \). It is obvious that \( \tau^d_\Delta \) is equal to \( \tau^o \).

**Theorem 7.5 Iterated Induction.** Let \((\Delta_1, \ldots, \Delta_n)\) be an iterated inductive definition over vocabulary \( \tau \). Definition \( \Delta := \Delta_1 \cup \cdots \cup \Delta_n \) is a total definition, and for all \( \tau \)-structures \( I \) extending a \( \tau^o \)-structure \( I_o \), the following are equivalent:

(a) \( I \) is a model of \( \Delta \);
(b) \( I \) is the structure defined by \((\Delta_1, \ldots, \Delta_n)\) in \( I_o \), i.e., \( I = I_o(\Delta_1, \ldots, \Delta_n) \);
(c) \( I \) satisfies \( PID(\Delta_1) \land \cdots \land PID(\Delta_n) \);

The theorem’s significance is that it shows that the semantics of the logic correctly formalises this type of finite iterated inductive definitions.

Define for each \( i \), \( 0 \leq i \leq n \), \( \tau^i := \tau^o \cup \tau^d_{\Delta_1} \cup \cdots \cup \tau^d_{\Delta_i} \). It is easy to see that \( \tau^0 = \tau^o \) and \( \tau^n = \tau \). Also, it holds that \( \Delta_i \) is a definition over the vocabulary \( \tau^i \), all open symbols in \( \Delta_i \) belong to \( \tau^{i-1} \) and, for any \( \tau^o \)-structure \( I_o \), \( I_o(\Delta_1, \ldots, \Delta_i) \) is a \( \tau^i \)-structure.

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To prove the theorem, we need the following modularity lemma.

**Lemma 7.6.** Let \((\Delta_1, \ldots, \Delta_n)\) be an iterated inductive definition and let \(\Delta := \Delta_1 \cup \ldots \cup \Delta_n\). The definition \(\Delta\) and the conjunction \(\Delta_1 \land \cdots \land \Delta_n\) of definitions are logically equivalent.

**Proof.** Consider an arbitrary \(\tau^2\)-structure \(I_o\) with domain \(A\). We will now prove that \(\{\Delta_1, \ldots, \Delta_n\}\) is a total reduction partition of \(\Delta\) in \(I_o\). Then, by application of Theorem 6.21, we obtain the lemma.

First, by Theorem 7.3, each \(\Delta_i\) is total in \(I_o\). Consequently, the partition \(\{\Delta_1, \ldots, \Delta_n\}\) is total in \(I_o\).

Second, define the following partial order in \(At^A\): for arbitrary domain atoms \(P[\bar{a}]\), \(Q[\bar{b}]\), define \(Q[\bar{b}] \prec P[\bar{a}]\) iff \(P\) is defined in \(\Delta_i\), for some \(i, 1 \leq i \leq n\), and \(Q\) is defined in \(\Delta_i\) for some \(j, 1 \leq j \leq i\).

The relation \(\prec\) is clearly a pre-order. By definition of \(\prec\), it holds that if \(P\) and \(Q\) are not defined in the same \(\Delta_i\) and \(Q[\bar{b}] \prec P[\bar{a}]\), then \(Q[\bar{b}] \not\prec P[\bar{a}]\). We show that for each \(i, 1 \leq i \leq n\), \(\prec\) is a reduction of \(\Delta_i\) in \(I_o\).

Let \(P\) be a defined predicate of \(\Delta_i\). For each domain atom \(P[\bar{a}]\), for all \(\tau\)-structures \(I, J \in S^A_{I_o}\), \(I \equiv_{\tau_P[I]} J\) holds iff \(I|_{\tau_i} = J|_{\tau_i}\). Since \(\varphi_P\) contains only symbols of \(\tau_i\), if \((I, J) \equiv_{\tau_P[I]} (I', J')\) then \(I|_{\tau_P[I]} = J|_{\tau_P[J]}\).

Since \(\prec\) is a reduction of each \(\Delta_i\), 1 \leq i \leq n, Proposition 6.8(a) guarantees that \(\prec\) is a reduction of \(\Delta\).

Combining the above results, we conclude that \(\{\Delta_1, \ldots, \Delta_n\}\) is a total reduction partition of \(\Delta\) in \(I_o\).

**Proof.** of Theorem 7.5.

Let \(I\) be a \(\tau\)-structure extending \(I_o\). The following equivalences hold:

\[ I \models \Delta_1 \land \cdots \land \Delta_n \quad \text{iff} \quad I = I_o(\Delta_1, \ldots, \Delta_n). \]

(Lemma 7.6)

\[ I \models \text{PID}(\Delta_1) \land \cdots \land \text{PID}(\Delta_n) \quad \text{iff} \quad I \models \text{PID}(\Delta_1 \land \cdots \land \Delta_n). \]

(Theorem 7.3)

What remains to be shown is that

\[ I \models \Delta_1 \land \cdots \land \Delta_n \iff I = I_o(\Delta_1, \ldots, \Delta_n). \]

Let \(I\) be any \(\tau\)-structure extending \(I_o\). We show that for each \(i, 0 \leq i \leq n\), \(I \models \Delta_1 \land \cdots \land \Delta_i\) if \(I|_{\tau_i} = I_o(\Delta_1, \ldots, \Delta_i)\). Then for the case \(i = n\), we obtain that \(I \models \Delta_1 \land \cdots \land \Delta_n\) if \(I|_{\tau_n} = I_o(\Delta_1, \ldots, \Delta_n)\) which, since \(\tau^n = \tau\), means that \(I\) and \(I_o(\Delta_1, \ldots, \Delta_n)\) are identical.

The proof is by induction. In the base case \((i = 0)\), the property is trivially satisfied. Assume that the property holds for \(i - 1\). We prove that the equivalence holds for \(i\).

To prove one direction, assume that \(I \models \Delta_1 \land \cdots \land \Delta_i\). Since \(I \models \Delta_i\), it holds that \(I = (I|_{\tau^i})^\Delta_i\). Since all open symbols occurring in \(\Delta_i\) belong to \(\tau^{i-1}\), it is easy to see that \(I|_{\tau^i} = (I|_{\tau^{i-1}})^\Delta_i\). Also, \(I \models \Delta_1 \land \cdots \land \Delta_{i-1}\) and hence, by application of the induction hypothesis, \(I|_{\tau^{i-1}} = I_o(\Delta_1, \ldots, \Delta_{i-1})\). We conclude that \(I|_{\tau^i} = (I|_{\tau^{i-1}})^\Delta_i = I_o(\Delta_1, \ldots, \Delta_{i-1})\).

For the other direction, assume that \(I|_{\tau^i} = I_o(\Delta_1, \ldots, \Delta_i)\). Since \(I_o(\Delta_1, \ldots, \Delta_i) = \)
over, if such a pair, say $(I, J)$, exists. The condition and consequently, $I$ follows that $I$ satisfies also $\Delta_1 \land \ldots \land \Delta_{i-1}$. We obtain that $I \models \Delta_1 \land \ldots \land \Delta_i$. \qed

7.4 Definitions over Well-Founded Order.

We now present a formalisation of the informal concept of a definition over a well-founded order (see section 2) in the framework of ID-logic. Let $\Delta$ be a definition over $\tau$ and $K_\alpha$ a structure such that $\tau_{K_\alpha} \subseteq \tau_\Delta$.

**Definition 7.7** strict reduction relation. A reduction relation $\prec$ of $\Delta$ in $K_\alpha$ is strict if it is a strict well-founded partial order (i.e., an anti-symmetric, transitive binary relation without infinite descending chains).

Hence, a strict reduction $\prec$ has no cycles. If $\Delta$ allows a strict reduction then there are no atoms that depend on themselves.

**Definition 7.8** definition over a well-founded order. We say that $\Delta$ is a definition over the (strict) well-founded order $\prec$ in $K_\alpha$ if $\prec$ is a strict reduction relation of $\Delta$ in $K_\alpha$.

**Theorem 7.9** completion. Suppose $\prec$ is a strict reduction relation of $\Delta$ in $K_\alpha$. The definition $\Delta$ is total in $K_\alpha$ and for any $\tau$-structure $I$ extending $K_\alpha$, $I \models \Delta$ iff $I \models \text{comp}(\Delta)$.

**Proof.** Fix an arbitrary $\tau_\Delta^\alpha$-structure $I_0$ extending $K_\alpha$. We will show that the equality $I_0^{\Delta_1} = I_0^{\Delta_1^\tau} = I_0^\Delta$ holds, and moreover that for any $\tau$-structure $I$ extending $I_0$, $I \models \text{comp}(\Delta)$ iff $I = I_0^\Delta$. Since $I_0$ is arbitrary, we will obtain the proof of the theorem.

We start by showing that there is at most one pair $(I, J)$ in $S(I_0)$ satisfying $T_\Delta(I, J) = I$ and $T_\Delta(J, I) = J$, moreover if such a pair exists then $I = J$.

Suppose that there are two such pairs; i.e., there exist $I, J, I', J' \in S(I_0)$ such that $T_\Delta(I, J) = I$, $T_\Delta(J, I) = J$, $T_\Delta(I', J') = I'$ and $T_\Delta(J', I') = J'$. Let $P[\bar{a}]$ be a minimal atom such that $P[\bar{a}]^I \neq P[\bar{a}]^{I'}$ or $P[\bar{a}]^J \neq P[\bar{a}]^{J'}$. Since $\prec$ is irreflexive, it holds that $I \equiv_{\prec P[\bar{a}]} I'$ and $J \equiv_{\prec P[\bar{a}]} J'$. Hence by Proposition 5.7,

$$P[\bar{a}]^I = P[\bar{a}]^{T_\Delta(I, J)} = P[\bar{a}]^{T_\Delta(I', J')} = P[\bar{a}]^{I'}$$

and

$$P[\bar{a}]^J = P[\bar{a}]^{T_\Delta(J, I)} = P[\bar{a}]^{T_\Delta(J', I')} = P[\bar{a}]^{J'}$$

We obtain a contradiction.

It follows that there can be at most one pair $(I, J)$ satisfying this condition. Moreover, if such a pair, say $(I, J)$, exists then also the symmetric pair $(J, I)$ satisfies the condition and consequently, $I$ and $J$ have to be identical.

Now, the proof of totality follows easily. The pair $(I_0^{\Delta_1}, I_0^{\Delta_1})$ is the maximal oscillating pair of the stable operator. Every oscillating pair $(I, J)$ of the stable operator satisfies $T_\Delta(I, J) = I$ and $T_\Delta(J, I) = J$. By the previous paragraph, it follows that $I_0^{\Delta_1} = I_0^{\Delta_1} = I_0^\Delta$.

We also just proved that $I_0^\Delta$ is the unique structure that extends $I_0$ and satisfies the fixpoint equation $T_\Delta(I, I) = I$. We derive for all $I$ extending $I_0$.
\[ I = I_0 \Delta \]

iff \( I = T_\Delta(I, I) \)

iff \( I = \Gamma_\Delta(I) \) (Corollary 4.11)

iff for each defined domain atom \( P[\bar{a}], P[\bar{a}]^I = \varphi_P[\bar{a}]^I \)

iff \( I \models \text{comp}(\Delta) \).

\[ \square \]

We obtain the following corollary.

**Corollary 7.10.** Suppose a definition \( \Delta \) over \( \tau \) and a theory \( T_\alpha \) over \( \tau^0 \subseteq \tau^\Delta_\alpha \) such that for any model \( K_\alpha \) of \( T_\alpha \), \( \Delta \) is a definition over some well-founded order \( \prec \) in \( K_\alpha \). Then \( T_\alpha \land \Delta \) and \( T_\alpha \land \text{comp}(\Delta) \) are logically equivalent.

**Example 7.11.** Consider the definition \( \Delta \) of Example 4.19:

\[ \Delta_{\text{even}} := \{ \forall x \ (E(x) \leftarrow x = 0), \forall x \ (E(s(x)) \leftarrow \neg E(x)) \}. \]

The transitive closure of the reduction \( \{(E[n], E[n+1]) \mid n \in \mathbb{N}\} \) is a strict reduction of \( \Delta_{\text{even}} \) in the natural numbers. Consequently, in the context of the natural numbers, this definition can be expressed in first-order logic, by \( \text{comp}(\Delta_{\text{even}}) \).

Notice also that \( \text{PID}(\Delta_{\text{even}}) \) is inconsistent in the natural numbers. Indeed, the sets \( \{0, 2, 4, 6, \ldots\} \) and \( \{0, 1, 3, 5, \ldots\} \) are both minimal sets containing 0 and containing \( n+1 \) if \( n \) is not contained. Consequently, there is no least such set.

**Example 7.12.** In this example, we illustrate how an ID-theory can be transformed into an equivalent second-order theory using the techniques that were developed in this paper.

Consider the ID-theory \( T = T_\mathbb{N} \cup \{ \Delta \} \) where \( T_\mathbb{N} \) was defined in Example 4.4 and \( \Delta \) in Example 5.2:

\[ \Delta := \{ \forall x \ (E(x) \leftarrow x = 0), \forall x \ (E(s(x)) \leftarrow O(x)), \forall x \ (O(s(x)) \leftarrow E(x)) \}. \]

In Example 7.4, we showed that the ID-logic induction axiom in \( T_\mathbb{N} \) can be translated into the standard induction axiom and that the unique model of this theory is the set of natural numbers.

To translate \( \Delta \) to classical logic, one can pick among several alternatives.

(1) Since \( \Delta \) is a positive definition, by Theorem 7.3, it can be translated into a second-order induction axiom.

(2) Alternatively, we observe that \( \Delta \) has a strict reduction in the natural numbers.

This is the transitive closure of the relation

\[ \{(E[n], O[n+1]), (O[n], E[n+1]) \mid n \in \mathbb{N}\}. \]

Now we can use Theorem 7.9 to translate \( \Delta \) to the first-order theory \( \text{comp}(\Delta) \).

(3) In Example 6.12, it was shown that \( \Delta \) has a reduction partition in the natural numbers

\[ \Delta_1 := \{ \forall x \ (E(x) \leftarrow x = 0), \forall x \ (E(s(x)) \leftarrow O(x)) \}, \quad \Delta_2 := \{ \forall x \ (O(s(x)) \leftarrow E(x)) \}. \]
Consequently, we can substitute $\Delta_1 \land \Delta_2$ for $\Delta$. Both definitions are non-recursive and, by Theorem 7.2, they are equivalent with $\text{comp}(\Delta_1) \land \text{comp}(\Delta_2)$.

After applying transformations (2) and (3), we obtain the same theory, namely the first-order theory $\text{comp}(\Delta)$ augmented with the second-order induction axiom (and Peano’s disequality axioms).

8. CONCLUSION

Recently, we argued [Denecker 2000; Denecker et al. 2001b] that non-monotone forms of inductive definitions such as iterated inductive definitions and definitions over well-orders, can play a unifying role in logic, AI and knowledge representation, connecting remote areas such as non-monotonic reasoning, logic programming, description logics, deductive databases and fixpoint logics. In this paper, we further substantiated this claim by defining a more general logic integrating classical logic and monotone and non-monotone inductive definitions and investigating its relations to first- and second-order logic and studying its modularity properties.

The main technical theorems here are the Modularity theorem and the theorems translating certain classes of ID-formulas into classical logic formalisations. Problem-free composition is crucial while axiomatizing a complex system. Because definitions in our logic are non-monotone, composing or decomposing definitions is in general not equivalence preserving. However, the conditions we have presented allow one to separate problem-free (de)compositions from those causing change in meaning. We have shown that the Modularity theorem is useful also for analyzing complex definitions — some properties of large definitions are implied by properties of sub-definition. The Modularity theorem is also an important tool for simplifying logical formulas with definitions by translating them into formulas of classical logic.

In [Denecker and Ternovska 2004], we have applied our logic to what has always been the most important test domain of knowledge representation — temporal reasoning. We presented an inductive situation calculus, a formalisation of the situation calculus with ramification as an inductive definition, defining fluents and causality predicates by simultaneous induction in the well-ordered set of situations. An important aspect of our formalisation is that causation rules can be represented in a modular way by rules in an inductive definition. We applied the Modularity theorem to demonstrate its equivalence with a situation calculus axiomatization based on completion and circumscription.

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