On Moduli of G-bundles over Curves for exceptional G

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1. Introduction

Let G be a simple and simply connected complex Lie group, \( \mathfrak{g} \) its Lie algebra. In the following, I remove the restriction "G is of classical type or \( G_2 \)" made on G in the papers of Beauville, Laszlo and myself [L-S],[B-L-S] on the moduli of principal G-bundles on a curve. As I will just “patch” the missing technical points, this note should be seen as an appendix to the above cited papers.

Let \( M_{G,X} \) be the stack of G-bundles on the smooth, connected and projective algebraic curve X of genus \( g \). If \( \rho : G \to \text{SL}_r \) is a representation of G, denote by \( \mathcal{D}_\rho \) the pullback of the determinant bundle [D-N] under the morphism \( M_{G,X} \to M_{\text{SL}_r,X} \) defined by extension of the structure group. Associate to G the number \( d(G) \) and the representation \( \rho(G) \) as follows:

| Type of G | \( A_r \) | \( B_r \) \((r \geq 3)\) | \( C_r \) | \( D_r \) \((r \geq 4)\) | \( E_6 \) | \( E_7 \) | \( E_8 \) | \( F_4 \) | \( G_2 \) |
|-----------|---------|-----------------|---------|-----------------|-------|-------|-------|-------|-------|
| \( d(G) \) | 1       | 2               | 1       | 2               | 6     | 12    | 60    | 6     | 2     |
| \( \rho(G) \) | \( \varpi_1 \) | \( \varpi_1 \) | \( \varpi_1 \) | \( \varpi_1 \) | \( \varpi_6 \) | \( \varpi_7 \) | \( \varpi_8 \) | \( \varpi_4 \) | \( \varpi_1 \) |

**Theorem 1.1.** — There is a line bundle \( \mathcal{L} \) on \( M_{G,X} \) such that \( \text{Pic}(M_{G,X}) \cong \mathbb{Z}_\mathcal{L} \). Moreover we may choose \( \mathcal{L} \) in such a way that \( \mathcal{L}^{\otimes d(G)} = \mathcal{D}_{\rho(G)} \).

The above theorem is proved, for classical G and \( G_2 \), in [L-S] where it also shown that the space of sections \( H^0(M_{G,X}, \mathcal{L}^\ell) \) may be identified to the space of conformal blocks \( B_{G,X}(\ell; p; 0) \) (see (2.2.1) for its definition). Now, once the generator of the Picard group is known in the exceptional cases, this identification is also valid in general, as well what happens when one considers additionally parabolic structures as we did in [L-S] (theorems 1.1 and 1.2).

In fact, as we will see, to prove theorem 1.1 for the exceptional groups it is enough to prove the existence of the 60-th root of \( \mathcal{D}_{\varpi_8} \) on \( M_{E_8,X} \). This will be deduced from the splitting of a certain central extension, which in turn will follow from the fact that \( B_{E_8,X}(1; p; 0) \) is one dimensional in any genus \( g \) as predicted by the Verlinde formula. However, in our particular case we don’t need the Verlinde formula in order to prove the last statement: it will follow directly from the decomposition formulas.

Suppose \( g(X) \geq 2 \). For the coarse moduli spaces \( M_{G,X} \) of semi-stable G-bundles, we will see that the roots of the determinant bundle of theorem 1.1 do only exist on the open subset of regularly stable G-bundles which, as shown in [B-L-S], has as consequence the following:
Theorem 1.2.— Let $G$ be semi-simple and $\tau \in \pi_1(G)$. Then $M^\tau_{G,X}$ is locally factorial if and only if $G$ is special in the sense of Serre.

Note that $\dim H^0(M^\tau_{E_8,X}, \mathcal{L}) = \dim B_{E_8,X}(1; p; 0) = 1$ has the somehow surprising consequence that the stack $M^\tau_{E_8,X}$ and (for $g(X) \geq 2$) the normal variety $M^\tau_{E_8,X}$ have a canonical hypersurface.

I would like to thank C. Teleman for pointing out that a reference I used in a previous version of this paper was incomplete and mention his preprint [T], which contains a different, topological approach to theorem 1.1.

2. Conformal Blocks

(2.1) **Affine Lie algebras.** Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of rank $r$ over $\mathbb{C}$. Let $P$ be the weight lattice, $P_+$ the subset of dominant weights and $(\varpi_i)_{i=1,\ldots,r}$ be the fundamental weights. Given a dominant weight $\lambda$, we denote $L(\lambda)$ the associated simple $\mathfrak{g}$-module with highest weight $\lambda$. Finally $(\cdot, \cdot)$ will be the Cartan-Killing form normalized such that for the highest root $\theta$ we have $(\theta, \theta) = 2$.

Let $L_\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}(z)$ be the loop algebra of $\mathfrak{g}$ and $\hat{L}_\mathfrak{g}$ be the central extension of $L_\mathfrak{g}$ defined by the 2-cocycle $(X \otimes f, Y \otimes g) \mapsto \left( X, Y \right) \text{Res}_0(\theta f g)$.

Fix an integer $\ell$. Call a representation of $\hat{L}_\mathfrak{g}$ of level $\ell$ if the center acts by multiplication by $\ell$. The theory of affine Lie algebras affirms that the irreducible and integrable representations of $\hat{L}_\mathfrak{g}$ are classified by the dominant weights belonging to $P_\ell = \{ \lambda \in P_+ / (\lambda, \theta) \leq \ell \}$. For $\lambda \in P_\ell$, denote $H_\ell(\lambda)$ the associated representation.

(2.2) **Definition of conformal blocks.** Fix an integer (the level) $\ell \geq 0$. Let $(X, p)$ be an $n$-pointed stable curve (we denote $p = (p_1, \ldots, p_n)$) and suppose that the points are labeled by $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_\ell^n$ respectively. Choose a non-singular point $p \in X$ and a local coordinate $z$ at $p$. Let $X^* = X - \{ p \}$ and $L_{Xg}$ be the Lie algebra $\mathfrak{g} \otimes \mathcal{O}(X^*)$. We have a morphism on Lie algebras $L_{Xg} \rightarrow L_\mathfrak{g}$ by associating to $X \otimes f$ the element $X \otimes \hat{f}$, where $\hat{f}$ is the Laurent development of $f$ at $p$. By the residue theorem, the restriction to $L_{Xg}$ of the central extension (2.1.1) splits and we may see $L_{Xg}$ as a Lie subalgebra of $\hat{L}_\mathfrak{g}$. In particular, the $\hat{L}_\mathfrak{g}$-module $H_\ell(0)$ may be seen as a $L_{Xg}$-module. In addition, we may consider the $\mathfrak{g}$-modules $L(\lambda_i)$ as a $L_{Xg}$-modules by evaluation at $p_i$. The vector space of conformal blocks is defined as follows:

\begin{equation}
B_{G,X}(\ell; p; \lambda) = \left[ H_\ell(0) \otimes \mathbb{C} L(\lambda_1) \otimes \mathbb{C} \cdots \otimes \mathbb{C} L(\lambda_n) \right]_{L_{Xg}}
\end{equation}

where $[\cdot]_{L_{Xg}}$ means that we take co-invariants. It is known ([T-U-Y] or [S], 2.5.1) that these vector spaces are finite-dimensional. Important properties are as follows:
a) \( \dim B_{G,P_1}(\ell; p_1; 0) = 1 \)

b) If one adds a non-singular point \( q \in X \), then the spaces \( B_{G,X}(\ell; p; \lambda) \) and \( B_{G,X}(\ell; p, q; \lambda; 0) \) are canonically isomorphic ([S], 2.3.2).

c) Suppose \( X \) is singular in \( c \) and let \( \tilde{X} \to X \) be a partial desingularization of \( c \). Let \( a \) and \( b \) be the points of \( \tilde{X} \) over \( c \). Then there is a canonical isomorphism

\[
\bigoplus_{\mu \in P_{\ell}} B_{G,X}(\ell; p, a, b; \lambda, \mu, \mu^*) \cong B_{G,X}(\ell; p; \lambda)
\]

d) The dimension of \( B_{G,X}(\ell; p; \lambda) \) does not change when \( (X; p) \) varies in the stack of \( n \)-pointed stable curves \( \mathfrak{M}_{g,n} ([\text{T-U-Y}]) \).

(2.3) Application: Consider the case of \( G = E_8 \) and level 1 and remark that \( P_1 \) contains only the trivial representation. In order to calculate \( B_{E_8,X}(\ell; p; 0) \), one reduces to \( P_1 \) with points labeled with the trivial representation using \( c \) and \( d \), then it follows from \( b \) and \( a \) that it is one-dimensional.

3. The Picard group of \( \mathcal{M}_{G,X} \)

(3.1) We recall the description of \( \text{Pic}(\mathcal{M}_{G,X}) \) of [L-S], which uses as main tool the uniformization theorem which I now recall. Let \( L_G \) be the loop group \( G(\mathbb{C}((z))) \), seen as an ind-scheme over \( \mathbb{C} \), \( L^+G \) the sub-group scheme \( G(\mathbb{C}[\lbrack [z]])) \) and \( Q_G = L_G/L^+G \) be the infinite Grassmannian, which is a direct limit of projective integral varieties (loc. cit.). Finally let \( L_XG \) be the sub-ind group \( G(\mathcal{O}(X^*)) \) of \( L_G \). The uniformization theorem ([L-S], 1.3) states that there is a canonical isomorphism of stacks \( L_XG\backslash Q_G \cong \mathcal{M}_{G,X} \) and moreover that \( Q_G \to \mathcal{M}_{G,X} \) is a \( L_XG \)-bundle.

Let \( \text{Pic}_{L_XG}(Q_G) \) be the group of \( L_XG \)-linearized line bundles on \( Q_G \). Recall that a \( L_XG \)-linearization of the line bundle \( L \) on \( Q_G \) is an isomorphism \( m^*L \cong pr_2^*L \), where \( m : L_XG \times Q_G \to Q_G \) is the action of \( L_XG \) on \( Q_G \), satisfying the usual cocycle condition. It follows from the uniformization theorem that

\[
\text{Pic}(\mathcal{M}_{G,X}) \cong \text{Pic}_{L_XG}(Q_G),
\]

hence in order to understand \( \text{Pic}(\mathcal{M}_{G,X}) \) it suffices to understand \( \text{Pic}_{L_XG}(Q_G) \). The Picard group of \( Q_G \) itself is infinite cyclic; let me recall how its positive generator may be defined in terms of central extensions of \( L_G \).

(3.2) If \( H \) is an (infinite) dimensional vector space over \( \mathbb{C} \), we define the \( \mathbb{C} \)-space \( \text{End}(H) \) by \( R \mapsto \text{End}(H \otimes_{\mathbb{C}} R) \), the \( \mathbb{C} \)-group \( \text{GL}(H) \) as the group of its units and \( \text{PGL}(H) \) by \( \text{GL}(H)/\mathbb{G}_m \). The \( \mathbb{C} \)-group \( L_G \) acts on \( L_{\mathfrak{g}} \) by the adjoint action which is extended to \( \hat{L}_{\mathfrak{g}} \) by the following formula:

\[
\text{Ad}(\gamma).(\alpha', s) = (\text{Ad}(\gamma).\alpha', s + \text{Res}_{z=0}(\gamma^{-1}\frac{d}{dz}\gamma, \alpha'))
\]
where $\gamma \in \text{LG}(R)$, $\alpha = (\alpha', s) \in \widehat{\mathfrak{g}}(R)$ and $(,) \text{ is the } R((z))\text{-bilinear extension of the Cartan-Killing form. The main tool we use is that if } \bar{\pi} : \widehat{\mathfrak{g}} \to \text{End}(\mathcal{H}) \text{ is an integral highest weight representation, then for } R \text{ a } \mathbb{C}\text{-algebra and } \gamma \in \text{LG}(R) \text{ there is, locally over } \text{Spec}(R), \text{ an automorphism } u_\gamma \text{ of } \mathcal{H}_R = \mathcal{H} \otimes_\mathbb{C} R, \text{ unique up to } R^*, \text{ such that}

\begin{align*}
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\bar{\pi}(\alpha)} & \mathcal{H} \\
\downarrow u_\gamma & & \downarrow u_\gamma \\
\mathcal{H} & \xrightarrow{\bar{\pi}(\text{Ad}(\gamma) \cdot \alpha)} & \mathcal{H}
\end{array}
\end{align*}

is commutative for any $\alpha \in \widehat{\mathfrak{g}}(R) \ (\text{[L-S], Prop. 4.3}).$

By the above, the representation $\bar{\pi}$ may be “integrated” to a (unique) algebraic projective representation of LG, i.e. that there is a morphism of $\mathbb{C}$-groups $\pi : \text{LG} \to \text{PGL}(\mathcal{H})$ whose derivate coincides with $\bar{\pi}$ up to homothety. Indeed, thanks to the unicity property the automorphisms $u$ associated locally to $\gamma$ glue together to define an element $\pi(\gamma) \in \text{PGL}(\mathcal{H})(R)$ and still because of the unicity property, $\pi$ defines a morphism of $\mathbb{C}$-groups. The assertion on the derivative is consequence of (3.2.1). We apply this to the basic representation $\mathcal{H}_1(0)$ of $\widehat{\mathfrak{g}}$. Consider the central extension

\begin{align*}
(3.2.2) \quad 1 & \longrightarrow G_m \longrightarrow \text{GL}(\mathcal{H}_1(0)) \longrightarrow \text{PGL}(\mathcal{H}_1(0)) \longrightarrow 1.
\end{align*}

The pull back of (3.2.2) to LG defines a central extension to which we refer as the canonical central extension of LG:

\begin{align*}
(3.2.3) \quad 1 & \longrightarrow G_m \longrightarrow \widehat{\text{LG}} \longrightarrow \text{LG} \longrightarrow 1
\end{align*}

A basic fact is that the extension (3.2.3) splits canonically over $L^+G$ ([L-S], 4.9), hence we may define a line bundle on the homogeneous space $Q_G = \widehat{\text{LG}}/L^+G$ via the character $G_m \times L^+G \to G_m$ defined by the first projection. Then this line bundle generates $\text{Pic}(Q_G)$ ([L-S], 4.11); we denote by $\mathcal{O}_{Q_G}(1)$ its dual.

(3.3) By ([L-S], 6.2) the forgetful morphism $\text{Pic}_{L_XG}(Q_G) \to \text{Pic}(Q_G)$ is injective, and moreover (loc. cit., 6.4), the line bundle $\mathcal{O}_{Q_G}(1)$ admits a $L_XG$-linearization if and only if the restriction of the central extension (3.2.3) to $L_XG$ splits. It is shown in [L-S] that this is indeed the case for classical $G$ and $G_2$ by directly constructing line bundles on $M_{G,X}$ which pull back to $\mathcal{O}_{Q_G}(1)$. In one case the existence of the splitting can be proved directly:

**Proposition 3.4.** — The restriction of the central extension (3.2.3) to $L_XG$ splits for $G = E_8$.  

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Proof: Let $H = H_1(0)$. It suffices to show that the representation $\tilde{\pi} : L_X g \to \text{End}(H)$ integrates to an algebraic representation $\pi : L_X G \to \text{GL}(H)$, which in turn will follow from the fact that in the case $\gamma \in L_X G(\mathbb{R})$ we can normalize the automorphism $u_\gamma$ of (3.2.1). Indeed, the commutativity of (3.2.1) shows that coinvariants are mapped to coinvariants under $u_\gamma$. For $g = e_8$, $\ell = 1$ and $\lambda = 0$, we know by (2.3) that these spaces are 1-dimensional, hence we may choose $u_\gamma$ (in a unique way) such that it induces the identity on coinvariants.

Corollary 3.5.— Suppose $G = F_4, E_6, E_7$ or $E_8$. There is a line bundle $L$ on $M_{G,X}$ such that the pullback to $Q_G$ is $O_{Q_G}(1)$.

Proof: For $E_8$, this follows from the above proposition. Now consider the well known tower of natural inclusions

\[(3.5.1) \quad F_4 \xrightarrow{\alpha} E_6 \xrightarrow{\beta} E_7 \xrightarrow{\gamma} E_8.\]

On the level of Picard groups we deduce

\[
\begin{array}{cccccc}
\text{Pic}(Q_{E_8}) & \xrightarrow{\tilde{f}_\alpha^*} & \text{Pic}(Q_{E_7}) & \xrightarrow{\tilde{f}_\beta^*} & \text{Pic}(Q_{E_6}) & \xrightarrow{\tilde{f}_\gamma^*} & \text{Pic}(Q_{F_4}) \\
\pi_{E_8} \downarrow & & \pi_{E_7} \downarrow & & \pi_{E_6} \downarrow & & \pi_{F_4} \\
\text{Pic}(M_{E_8,X}) & \xrightarrow{f_{\alpha}^*} & \text{Pic}(M_{E_7,X}) & \xrightarrow{f_{\beta}^*} & \text{Pic}(M_{E_6,X}) & \xrightarrow{f_{\gamma}^*} & \text{Pic}(M_{F_4,X}) \\
f_{\varpi_8}^* \downarrow & & f_{\varpi_8|F_4}^* \downarrow & & & & \\
\text{Pic}(M_{SL_{248},X}) & & & & & & \\
\end{array}
\]

The Dynkin index of the representation $\varpi_8$ of $E_8$ is 60, and an easy calculation shows that $\varpi_8|F_4 = 14 \mathbb{C} \oplus \varpi_1 \oplus 7 \varpi_4$, hence is equally of Dynkin index 60 ([K-N], [L-S], 2.3). By the Kumar-Narasimhan-Ramanathan lemma ([L-S], 6.8) the determinant bundle $D$ pulls back, to $O_{Q_{E_8}}(60)$ via $\pi_{E_8} \circ f_{\varpi_8}^*$ and to $O_{Q_{F_4}}(60)$ via $\pi_{F_4} \circ f_{\varpi_8|F_4}^*$. If follows that $f_{\alpha}^*, \tilde{f}_\beta$ and $\tilde{f}_\gamma$ are isomorphisms and that the pullback of the line bundle $L$ on $M_{E_8,X}$ under $f_{\alpha}^*$ (resp. $f_{\beta}^* \circ f_{\alpha}^*$, $f_{\gamma}^* \circ f_\beta^* \circ f_{\alpha}^*$) pulls back to $O_{Q_{E_7}}(1)$ (resp. $O_{Q_{E_6}}(1), O_{Q_{F_4}}(1)$).

(3.6) Proof of theorem 1.2: According to ([B-L-S], 13) it remains to prove that $M_{G,X}$ is not locally factorial for $G = F_4, E_6, E_7$ or $E_8$. In order to see this we consider again the tower (3.5.1) with additionally the natural inclusion $\text{Spin}_8 \hookrightarrow F_4$. Again the restriction of the representation $\varpi_8$ of $E_8$ to $\text{Spin}_8$ has Dynkin index 60, hence if the generator of $\text{Pic}(M_{G,X})$ would exist on $M_{G,X}$, then the Pfaffian bundle would exist on $M_{\text{Spin}_8,X}$, which is not the case ([B-L-S], 8.2). But the generators exist on the open subset of regularly stable bundles, as the center of $G$ acts trivially on the fibers by construction (we started with the trivial representation) and then the arguments of ([B-L-S], 13) apply.
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