BRAID TYPES FORCED BY HOMOCLINIC ORBITS

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Abstract. The complexity of a dynamical system exhibiting a homoclinic orbit is given by the braid types that it forces. In this work we present a method, based in pruning theory, to determine all the braid types forced by an arbitrary homoclinic orbit of an Axiom A diffeomorphism on the 2-disk. Then we apply it for finding the braid types forced by certain infinite families of homoclinic horseshoe orbits.

1. Introduction

Since the Poincaré’s discovery of homoclinic orbits, it is known that dynamical systems with one of these orbits have a very complex behaviour. Such a feature was explained by S. Smale in terms of his celebrated horseshoe map [28]; more precisely, if a surface diffeomorphism $f$ has a homoclinic point then, there exists an invariant set $\Lambda$ where a power of $f$ is conjugated to the shift $\sigma$ defined on the compact space $\{0,1\}^\mathbb{Z}$ of symbol sequences.

To understand how complex is a diffeomorphism containing a periodic or homoclinic orbit, we need the notion of forcing. Let $P$ and $Q$ be two periodic (or homoclinic) orbits of homeomorphisms $f$ and $g$ of the disk $D^2$, respectively. We say that $(P,f)$ and $(Q,g)$ are equivalent if there is an orientation-preserving homeomorphism $h : D^2 \rightarrow D^2$ with $h(P) = Q$ such that $f$ is isotopic to $h^{-1} \circ g \circ h$ relative to $P$. The equivalence class containing $(P,f)$ is called the braid type $bt(P,f)$. When the homeomorphism $f$ is fixed, it will be written $bt(P)$ instead of $bt(P,f)$.

Now we can define the forcing relation between two braid types $\beta$ and $\gamma$. We say that $\beta$ forces $\gamma$, denoted by $\beta \geq_2 \gamma$, if every homeomorphism of $D^2$ which has an orbit with braid type $\beta$, exhibits also an orbit with braid type $\gamma$. Note that we admit that none, one or both braid types $\beta$ and $\gamma$ be a homoclinic one. So we say that $P$ forces $Q$, denoted by $P \geq_2 Q$, if $bt(P) \geq_2 bt(Q)$. When restricted to periodic orbits, Boyland proved that $\geq_2$ is a partial order [3]. In [25] Los also proved it for the special case of the disk and he has concluded that in this particular case the forcing relation on periodic orbits can be extended to homoclinic or heteroclinic orbits in a suitable topology. Thus we can ensure that a diffeomorphism $f$ containing an orbit $P$ is at least as complex as it is restricted to the set $\Sigma_P$ of braid types forced by $P$.

There are several methods for finding the dynamics forced by a homoclinic orbit of a diffeomorphism $f$, using its homoclinic tangle that is a subset of its invariant manifolds. For example, in [6, 8], Collins has constructed suitable surface hyperbolic diffeomorphisms associated to a homoclinic orbit which can be used for approximating the entropy of that orbit as close as we want. In [30, 32], Yamaguchi and Tanikawa have studied the forcing relation of reversible homoclinic horseshoe orbits appearing in area-preserving Hénon maps. In other direction Boyland and Hall have given conditions for which a periodic orbit is isotopy stable relative to a compact set [4], and their result can be used for studying homoclinic orbits.

This is the context where our work is inserted. Thus we will study homoclinic orbits to a fixed point of an Axiom A map $f$, and we will prove that in this particular case the forcing order is related to the non-existence of bigons, that is, simply connected domains, relative to the orbit, which are bounded by two pieces of the invariant manifolds of $f$: a stable and an unstable segment. This is
closely related to the results of Lewowicz and Ures in [23] where similar conclusions were done for proving persistence of periodic orbits of Axiom A maps. Our main result is the following theorem.

**Theorem 1.** Let \( f \) be an Axiom A map on the disk which has a transitive basic set \( K \), and let \( P \) be a periodic orbit or a homoclinic orbit to a fixed point \( p \in K \). If \( f \) does not have bigons relative to \( P \) then \( \Sigma_P = \{ \text{bt}(R) : R \text{ is an orbit included in } K \} \) up to a finite number of boundary periodic points.

Briefly we would like to say which are the elements of the proof of Theorem above. Using a generalization of a Bonatti-Jeandenans’s Theorem [2], it follows that \( f \) is semiconjugated to a generalized pseudoanosov map \( \phi \), defined on the sphere \( S^2 \), by a semiconjugacy which is injective on the set of periodic orbits. By a generalization of a Handel’s theorem [20], if \( g \) is a homeomorphism isotopic to \( \phi \), relative to the homoclinic orbit \( P \), there exists a closed set \( H_g \) such that \( g \) restricted to \( H_g \) is semiconjugated to \( \phi \) on \( S^2 \). Hence every periodic orbit of \( \phi \) is isotopy stable. It implies that all the periodic orbits of \( f \) and their homoclinic and heteroclinic intersections are isotopy stable. Thus, by a Lewowicz-Ures’s Theorem [23], their braid types are forced by \( P \). Although these are the central lines of the proof, in the middle of it we have to be concerned with some details that are not direct, e.g., the fact that the braid types are preserved after the semiconjugacy even when the surfaces, where \( f \) and \( \phi \) are defined, are different.

Just then we present the **pruning method** that is an algorithm that, if finite, could help us to find, given a periodic or homoclinic orbit \( P \), an Axiom A map \( f \) without bigons relative to \( P \). Hence, by Theorem 1, the basic piece of \( f \) has to contain all the orbits whose braid types are forced by \( P \). An important ingredient of this method is the pruning theory introduced by de Carvalho in [10] which is a technique for eliminating dynamics of a surface homeomorphism, but in our context it will be considered its differentiable version for which pruning can be seen as the uncrossing of the invariant manifolds of an Axiom A diffeomorphism in a region called a **pruning domain**. This point of view was inspired by the work of P. Cvitanović [9] where a generic once-folding map is interpreted as a **partial horseshoe**, that is, a map whose dynamics forbids or prunes certain horseshoe orbits.

Some methods, as Bestvina-Handel [1], Los [24], de Carvalho-Hall [12] and Solari-Natiello [29], have been presented for constructing the pseudoanosov map relative to a periodic orbit. The main advantage of them is that they only need the information of the thick map, the train-track or the fatten representative associated to that orbit, whereas our method needs the full knowledge of an Axiom A \( f \) which has an orbit with the same braid type of the orbit under consideration. There are two advantages of our method: (1) since we know which are the orbits of the initial diffeomorphism, we be able of knowing which survive after the pruning process, and (2) the method can be applied to homoclinic ones as well. Unfortunately there is an inconvenient in our treatment: there is not guarantee that the method stops in a finite number of steps.

As an application we will study the forcing relation of homoclinic orbits coming from the Smale horseshoe \( F \). In particular, it will be dealt homoclinic orbits to the fixed point \( 0^\infty \), that is, orbits \( P_w^0 \) whose code in \( \Sigma_2 \) is \( 0^\infty 10w10^\infty \) where \( w \) is a finite word called **decoration**. In [13] de Carvalho and Hall conjectured that the orbits forced by \( P_w^0 \) are those that do not intersect a region \( P_w \) called **pruning region**, and that the forcing relation of periodic orbits depends basically of being able to determine it for homoclinic orbits. In this work we will conclude that the pruning method can be used for finding these pruning regions for **certain** infinite families of decorations \( w \), that is, after a finite number of its application we can eliminate all the bigons relative to those orbits.

Now we would like to explain how the paper is organized. Section 2 is devoted to the proof of Theorem 1. It needs the definition of a generalized pseudoanosov maps on the sphere and the statement of the Bonatti-Jeandenans’s theorem and the Handel’s Theorem for these type of homeomorphisms. In Section 3 we will introduce the pruning method. Section 4 is devoted to the study of homoclinic horseshoe orbits.
2. Homoclinic forcing relation

Here we will define the notions that we are going to use and then we will give the proof of Theorem 1. We assume that the reader has some familiarity with Axiom A maps and pseudoanosov homeomorphisms. Good references for these topics are [2] and [18].

2.1. Generalized pseudoanosov maps. In this section we will present the definition of generalized pseudoanosov homeomorphisms. They were first studied in [11] where were associated to a boundary periodic point. In case (b), since must have a point of f a boundary periodic point. In case (b), since must have a point of f a boundary periodic point.

Definition 2. An homeomorphism φ on S^2 will be called generalized pseudoanosov map associated to a homoclinic orbit Θ = ∪p∈Z{p_j} if φ has a fixed point p such that:
(a) φ restricted to S^2 \ (p) preserves a pair of measured foliations (F^s, µ^s) and (F^u, µ^u) which admits a finite number of (finite or infinite) orbits of n-pronged singularities with n ≥ 3. All the infinite orbits of n-pronged singularities accumulate in p.
(b) φ extends to the elements of {p} ∪ Θ in the following way: φ(p) = p, φ^j(p_0) = p_j

Every p_j is a 1-pronged singularity for φ.
(c) There exists a real number λ > 1 such that

φ(F^s, µ^s) = (F^s, λ^{-1}µ^s) and φ(F^u, µ^u) = (F^u, λµ^u).

(d) The periodic points of φ are dense and it is transitive.

By definition, leaves connecting periodic n-pronged singularities are forbidden, but it is possible to exist leaves connecting two n-prongs with infinite orbital.

2.2. A Bonatti-Jeandenans’s theorem for generalized pseudoanosov maps. Now we can state a generalization of [2, Theorem 8.3.1] by Bonatti-Jeandenans that associates a generalized pseudoanosov homeomorphism to an Axiom A map. Let f be an Axiom A map on the disk D^2 with a basic set K contained in its domain ∆(K), an invariant open region containing K where the dynamics can be explained by the symbol dynamics of K. See the precise definition of ∆(K) in [2]. We will suppose that f can be extended to ∂D^2 such that f = Id on ∂D^2. We will also denote a homoclinic orbit of f to a fixed point p by P = {p_j}. We need the following definition.

Definition 3. A bigon I is a simply connected open domain disjoint from K and bounded by a segment of a stable manifold θ_s ⊂ W^s(p_s) and a segment of an unstable manifold θ_u ⊂ W^u(p_u), where p_s and p_u are boundary periodic points of K. We will say that f have no bigons relative to P if every bigon I of f has a point of P in its boundary or contains an unique point of P.

Remark 4. In Fig. 1 we show the type of bigons that are allowed in Definition 3. All the bigons of f must have a point of P in its boundary. In the case (a), {p} and P are included in K and p is a boundary periodic point. In case (b), since f^i(I) ∩ I = ∅ for all i ∈ Z, it follows that P has the same braid type of the two homoclinic orbits in θ_s ∩ θ_u. Hence without loss of generality we can consider that {p} ∪ P ⊂ K. Case (c) is forbidden by Definition 3.

Since p is fixed, we have f(K) = K. We can also suppose that K is the only non-trivial basic set of f: If there exists another non-trivial basic set K_1 disjoint of P then the domain ∆(K_1) is periodic and disjoint from P. Because our surface is the disk D^2 it follows that ∆(K_1) is topologically a disk less a finite number of points. So the closure of ∆(K_1) is a disk and we can replace f in Orb(∆(K_1)) by an attractor or repeller periodic orbit without modifying f in its exterior. Collapsing these periodic orbits, we can suppose that ∆(K) is D^2 less a finite number
of periodic orbits. The following result is a generalization of [2, Theorem 8.3.1] by Bonatti and Jeandenans.

**Theorem 5.** Let $f$ be an Axiom A map with a transitive basic set $K$ such that $\Delta(K) = D^2$ less a finite number of periodic orbits and let $P$ be a homoclinic orbit to a fixed point $p$. Suppose that $f$ has no bigons relatives to $P$. Then there exists a generalized pseudoanosov homeomorphism $\phi$ on $S^2$ (with possibly a finite number of periodic 1-pronged singularities), a homoclinic orbit $\Theta$ and a continuous surjection $\pi : D^2 \to S^2$ such that $\pi \circ (f|_{D^2}) = \phi \circ \pi$. In fact, the semiconjugacy $\pi$ is injective on the periodic orbits except on the boundary ones.

**(Sketch of the Proof).** The proof is the same as in [2] and we will give some of its details for completeness. The existence of the invariant transversal measures follows as in [2, Proposition 8.2.1] since it does not depend of the presence of bigons. The semiconjugacy is defined considering a partition of the domain $\Delta(K)$ in six types of sets:

1. singletons $\{x \in K : x$ is not a boundary point$\}$,
2. the rectangles bounded by two stable arcs and two unstable arcs of boundary periodic points,
3. arcs $\alpha$ with its end-points in $K$ and $\text{Int}(\alpha) \cap K = \emptyset$, included in an invariant manifold of a non-boundary point,
4. the regions bounded by more than two stable arcs and, consequently, more than two unstable arcs,
5. sets defined by free invariant manifolds of boundary periodic points,
6. the iterates of $I$.

Define an equivalence relation $\sim$ which identifies the points in each of the sets above. Let $S = \Delta(K)/\sim$. Since the partition is $f$-invariant, the quotient map $\phi : S \to S$ is well-defined. The sets in (4) create a finite number of orbits of $n$-pronged singularities, with $n \geq 3$, noting that each orbit is infinite. These ones in (5) could create periodic $1$-prongs or periodic $n$-prongs. Denoting also $p_j$ to $\pi(p_j)$ it follows that $\Theta = \{p_j\}$.

The equivalence classes are simply connected (except this which identifies the boundary $\partial D^2$ to a point) then it follows that the quotient space is a sphere $S^2$. \[\square\]

**Remark 6.** Note that the semiconjugacy of Theorem 5 is injective on the periodic points except on the boundary ones, so it preserves the braid types of periodic orbits except these ones that correspond to boundary points which collapse to repeller or attractor periodic points as this one in Fig. 2 creating a $n$-pronged singularity. Theorem 5 is also valid if $P$ is a periodic orbit of pseudoanosov type, or if $P$ is homoclinic to a periodic point $p$ belonging to a basic set $K$ satisfying $f(K) = K$, but this will not be used here.

Let $f$ be an Axiom A diffeomorphism on the disk which does not have bigons relative to a homoclinic orbit $P$ to a fixed point $p$, and let $\phi$ be the generalized pseudoanosov map obtained from $f$ by Theorem 5. Let $\overline{f} : S^2 \to S^2$ be the homeomorphism obtained from $f$ collapsing the boundary $\partial D^2$ to a point. Noting that $\pi$ is a homotopy it follows by a Epstein’s Theorem [16] that
Figure 2. The semiconjugacy on the boundary points.

\( f \) and \( \phi \) are isotopic relative to \( \{ p \} \cup \Theta \). See [5] for a proof of this result via the hyperbolic geometry of surfaces. Since \( p \) is stable and unstable boundary point of \( f \) it follows that \( p \) is identified with the boundary \( \partial D^2 \) by \( \pi \). It allows us to relate the braid types on \( D^2 \) to the braid types on \( S^2 \). It is known [17] that the \( n \)-braid group of the sphere is the quotient of the braid \( n \) braid group of \( D^2 \) by \( \rho \), the braid showed in Fig. 3 for \( n = 5 \). We can take the first point as being \( p \). After identifying \( \partial D^2 \) to a point it can be seen that \( \rho \) is the identity, thus there exists a bijection between the \( n \)-braids of \( D^2 \), defined by the periodic orbits of \( f \), with the \( n \)-braids of \( S^2 \) defined by periodic orbits of \( \phi \). This relation extends to braid types.

Example 7. Applying Theorem 5 to the Smale horseshoe \( F \) on the disk \( D^2 \) (See Section 4.1 for its description), we obtain the tight horseshoe defined on the sphere \( S^2 \). See Fig. 4. In this case \( p = 0^\infty \) and the generalized pseudoanosov map \( \phi \) has an infinite orbit of a 1-pronged singularity associated to the homoclinic orbit \( \Theta \) which correspond to the symbol sequence \( 0^\infty 10.10^\infty \). See [11] for another construction of the tight horseshoe using the zero-entropy equivalence relation.

Figure 3. The braid \( \rho \) on \( D^2 \).

Figure 4. The tight horseshoe in the sphere \( S^2 \).
2.3. A Handel’s theorem for generalized pseudoanosov maps. Let $M = S^2 \setminus \{(p) \cup \Theta\}$. In this section we will prove a generalization of [20, Theorem 2], by Handel, where global shadowing was used to prove the persistence of the orbits of a pseudoanosov map by isotopies. It says us that the orbits of $\phi$ are isotopy stable on $M$. It will be proved for compactly fixed deformations of $\phi$.

Let $\tilde{M}$ be the universal cover of $M$. Note that the generalized pseudoanosov $\phi$ satisfies the following properties which are also satisfied by pseudoanosov homeomorphisms and were used in the Handel’s proof:

1. The action induced by $\phi$ on the free homotopy classes of $M$ has no periodic orbits. It follows since a non null-homotopic curve $\mathcal{C}$ of $M$ always contains a point of $\Theta$ in its interior and exterior. Since $p_j = \phi^j(p_0)$ and $\lim_{j \to \pm \infty} p_j = p$, $\mathcal{C}$ can not be periodic.

2. There exist an equivariant metric $\tilde{d}$ on the universal cover $\tilde{M}$ of $M$ such that $\tilde{d} = \sqrt{d_u^2 + d_s^2}$ where $d_u : \tilde{M} \times \tilde{M} \to [0, +\infty)$ and $d_s : \tilde{M} \times \tilde{M} \to [0, +\infty)$ are equivariant functions satisfying:

\[
\tilde{d}_u(\tilde{\phi}(x_1), \tilde{\phi}(x_2)) = \lambda \tilde{d}_u(x_1, x_2) \quad \text{and} \quad \tilde{d}_s(\tilde{\phi}^{-1}(x_1), \tilde{\phi}^{-1}(x_2)) = \lambda \tilde{d}_s(x_1, x_2)
\]

for all $\tilde{x}_1, \tilde{x}_2 \in \tilde{M}$ and all lifts $\tilde{\phi}$ of $\phi$. This metric projects to a metric $d$ on $M$.

We will only work with generalized pseudoanosov maps $\phi$ coming from Axiom A ones as in Theorem 5, so we are going to add some properties that these kind of maps satisfy.

3. The metric $d$ can be extended to a finite metric on $S^2$. It follows since $p$ is a stable and unstable periodic point of an Axiom A map and then $p$ and its invariant manifolds looks like Fig. 5(a). After the application of the Bonatti-Jeandenans’s theorem, a neighbourhood of $p$ looks like Fig. 5(b), that is, $p$ is approximated by a sequence of $n$-pronged singularities created by the identifications of Theorem 5. The crucial fact here is that, given a point $q$,

![Figure 5](image)

we can find a sequence of points $q_i$ such that $q_i$ tends to $p$ through the stable (or unstable) manifold with $d(q, q_i) < C_1$ for a finite constant $C_1$, implying that the distance $d(q, p)$ is finite.

4. For all $n \geq 1$, the set $\text{Fix}(\phi^n) \setminus \{p\}$ is compactly included in $M$ with the topology induced by $d$.

By Property (1.4), a homeomorphism $g : M \to M$ isotopic to $\phi$ is always bounded, that is,

\[
\max\{\sup_{x \in M} d(\phi(x), g(x)), \sup_{x \in M} d(\phi^{-1}(x), g^{-1}(x))\} < +\infty.
\]

Even more we can suppose that $g$ is compactly fixed that is $\text{Fix}(g^n) \setminus \{p\}$ is compact in $M$, because if for some $n$ there exists a $x \in \text{Fix}(g^n)$ such that its Nielsen class can be taken arbitrarily close to $p$ then $x$ belongs to a periodic orbit which is collapsible to $p$.

Let $\tilde{g}$ be the unique lift of $g$ which is equivariantly homotopic to $\tilde{\phi}$. We need two definitions.
Definition 8. The \( \phi \)-orbit of \( x \) is globally shadowed by the \( g \)-orbit of \( y \), denote by \( (\phi, x) \sim (g, y) \), if there are lifts \( \tilde{x} \) of \( x \) and \( \tilde{y} \) of \( y \) and a constant \( C \) such that \( d(\tilde{\phi}(\tilde{x}), \tilde{g}^k(\tilde{y})) \leq C \), for all \( k \in \mathbb{Z} \).

Definition 9. Let \( x \) and \( y \) be fixed points of \( \phi^n \) and \( g^n \) respectively. We say that \( (\phi^n, x) \) is Nielsen equivalent to \( (g^n, y) \) if there exist lifts \( \tilde{x} \) and \( \tilde{y} \), and some covering translation \( \delta \) such that \( \phi^n(\tilde{x}) = \delta \tilde{x} \) and \( g^n(\tilde{y}) = \delta \tilde{y} \).

As \( g \) is bounded, there are not obstacles to prove Lemmas 1.7, 2.1 and 2.2 of [20] for generalized pseudoanosov homeomorphisms:

(i) ([20, Lemma 1.7]) If \( x \) is a fixed point of \( \phi \) and \( y \) is a fixed point of \( g^n \), then \( (\phi^n, x) \) is Nielsen equivalent to \( (g^n, y) \) if and only if \( (\phi, x) \sim (g, y) \).

(ii) By property (1.3), no lift of any iterate of \( \phi \) can fix two distinct points. It implies that if \( x_1 \) and \( x_2 \) are distinct fixed points of \( \phi^n \) then \( (\phi^n, x_1) \) and \( (\phi^n, x_2) \) are not Nielsen equivalent.

(iii) Because \( g \) is compactly fixed, it follows by homotopy stability and (1.2) that for every \( \phi \)-periodic point \( x \) with least period \( n \), there exists a \( g \)-periodic \( y \) with least period \( n \) such that \( (\phi, x) \sim (g, y) \). See for instance [27] for homotopic stability on non-compact spaces.

As in [20, Lemma 2.1(ii)] we can prove that \( y \) has least period \( n \).

(iv) As \( g \) is bounded it implies that

\[
\max\{\sup_{\tilde{x} \in \tilde{M}} d(\tilde{\phi}(\tilde{x}), \tilde{g}(\tilde{x})), \sup_{\tilde{x} \in \tilde{M}} d(\tilde{\phi}^{-1}(\tilde{x}), \tilde{g}^{-1}(\tilde{x}))\} < \infty.
\]

So the proof of [20, Lemma 2.2] shows that there is a constant \( C > 0 \) such that \( (\phi, x) \sim (g, y) \) if and only they are globally shadowed with constant \( C \).

(v) If \( x_n \to x \), \( y_n \to y \) and \( (\phi, x_n) \sim (g, y_n) \) then \( (\phi, x) \sim (g, y) \).

Now we can state [20, Theorem 2] for generalized pseudoanosov maps.

Theorem 10. Let \( \phi : M \to M \) be a generalized pseudoanosov homeomorphism and let \( g : M \to M \) be a fixed point Nielsen homeomorphism isotopic to \( \phi \). Then there exists a closed set \( H_g \subset M \) and a surjective map \( \alpha : H_g \to M \) which is homotopic to the inclusion map such that \( \phi \circ \alpha = \alpha \circ g \mid_{H_g} \).

Proof. It follows the same lines of the Handel’s proof. Let

\[ \tilde{H} = \{ \tilde{y} \in \tilde{M} : \text{there exists an } \tilde{x} \text{ with } (\tilde{x}, \tilde{\phi}) \sim (\tilde{y}, \tilde{g}) \}. \]

Note that if \( (\tilde{x}, \phi) \sim (\tilde{y}, \tilde{g}) \) and only if if \( (r \tilde{x}, \tilde{\phi}) \sim (r \tilde{y}, \tilde{g}) \), for all \( r \in \pi(M) \). Then \( r \tilde{H} = \tilde{H} \) for all \( r \in \pi(M) \). Using (v) it can be proved that \( \tilde{H} \) is closed. By (ii) and (iii), for every periodic orbit \( \tilde{y} \in \tilde{Y} \), there exists an unique \( \tilde{x} \) such that \( (\tilde{x}, \tilde{\phi}) \sim (\tilde{y}, \tilde{g}) \), then define \( \tilde{\alpha} : \tilde{H} \to \tilde{M} \) as \( \tilde{\alpha}(\tilde{y}) = \tilde{x} \).

Note that \( \tilde{\alpha}(r \tilde{y}) = r \tilde{\alpha}(\tilde{y}) \) and \( \alpha \circ \tilde{\alpha} = \tilde{\alpha} \circ \tilde{\alpha} \). Projecting \( \tilde{H} \) and \( \alpha \), we obtain a set \( H_g \) and a map \( \alpha : H_g \to M \) satisfying \( \alpha \circ g = \phi \circ \alpha \). Since the periodic points of \( \phi \) are shadowable and dense in \( M \) and \( H_g \) is closed, it follows that \( \alpha(H_g) = M \). \( \square \)

Remark 11. One can suppose that \( y \in H_g \), satisfying \( \alpha(y) = x \), has the braid type of \( x \). In fact, if \( y \) does not have the braid type of \( x \) then its period \( m \) is a multiple of the period \( n \) of \( x \). Since the semiconjugacy is homotopic to the identity, then \( m/n \) elements of the orbit of \( y \) collapse to an element of the orbit of \( x \). See Fig. 6. So in a neighbourhood of \( x \) there exists a curve \( \gamma \) (connecting

![Figure 6. Orbits of x and y with α(y) = x.](image)

some elements of the orbit of \( y \) with non-null index on \( g^n \). So there exists a \( g \)-periodic point \( y' \) of period \( n \). It is clear that \( \text{bt}(y') = \text{bt}(x) \). Thus Theorem 10 claims that the set of all the braid types defined by (periodic or homoclinic) orbits of \( \phi \) is a subset of the braid types defined by any homeomorphism isotopic to \( \phi \).
Proof of Theorem 1. Let $f$ be an Axiom A map without bigons relative to a homoclinic orbit $P$, and let $\overline{f}$ be the map obtained identifying $\partial D^2$ to a point. $\overline{f}$ is isotopic to a generalized map $\phi$ obtained from $f$ by Theorem 5. By Theorem 10 there exists a subset $H_f$ such that $\overline{f}$ restricted to $H_f$ is semiconjugated to $\phi$ on $S^2$.

Now we will prove that $K \subset H_f$. We will follow the same argument of the proof of [23, Theorem 5.3] by Lewowicz and Ures. Take a simply closed curve $\gamma = \gamma_s \cup \gamma_u$ which is bounded by a stable segment and an unstable segment. Now let $q \neq p$ be a fixed point of $f^n$ which belongs to $K$ and let $Q$ be one of its lifts. Since $K$ does not have bigons then $W^u(Q, \overline{f})$ intersects every lift of $\gamma_s$ at most a point. Since $W^u(q, \overline{f})$ intersects infinitely many times $\gamma_s$ then $W^u(Q, \overline{f})$ has exactly two end-points at the boundary of the universal cover of $M$, $S_\infty$. Then [23, Lemma 5.1] proves that $q \in H_f$. By density of periodic points we have that $K \subset H_f$.

Note that every equivalence class in Theorem 5 contains a point of $K$ in its boundary. So every element of $H_f$ is equivalent to some point of $K$. Then $K = H_f$, that is, the braid types defined by the orbits of $K$ are the same as the braid types defined by the orbits in $H_f$. Hence, by Remark 11, the braid types forced by $P$ coincide with the braid types defined by orbits of $K$.

If $P$ is a periodic orbit of an Axiom A map with a transitive basic set $K$ then the same argument proves that the braid types forced by $P$ correspond to the orbits included in $K$. The only difference is that, after the Bonatti-Jeandenans’s semiconjugacy, $P$ becomes a 1-pronged singularity and, since an unstable boundary point of $K$ is collapsible to $\partial D^2$, $\partial D^2$ collapses to a fixed point which is a $n$-pronged singularity.

\[\blacksquare\]

3. A method for eliminating bigons of an Axiom A map

As Theorem 1 claims it, to solve the problem of finding orbits forced by a homoclinic orbit $P$, we just have to find an Axiom A without bigons relative to $P$ and prove that its generalized pseudoanosov Bonatti-Jeandenans homeomorphism associated does not have periodic 1-pronged singularities, except these which are in points of $P$. It can be done using pruning theory.

3.1. Pruning theory. Pruning is a technique introduced by de Carvalho [10] for eliminating bigons of a homeomorphism in a controlled manner, that is, for destroying dynamics contained in the interior of simply connected closed regions that we can define dynamically. Here we will use the differentiable version of pruning that the author, in a joint work with A. de Carvalho, has developed in the forthcoming paper Differentiable pruning and the hyperbolic pruning front conjecture. The main ideas are the following.

Let $f$ be an Axiom A diffeomorphism on a surface $S$ with a basic set $K$. Let $D$ be a simply connected domain bounded by two segments $\theta_s$ and $\theta_u$ with $\theta_s \subset W^s(p_s)$ and $\theta_u \subset W^u(p_u)$ where $p_s$ and $p_u$ are saddle periodic points in $K$.

Definition 12. The domain $D$ is a pruning domain if its boundary satisfies the following property

$$f^n(\theta_s) \cap \text{Int}(D) = \emptyset = f^{-n}(\theta_u) \cap \text{Int}(D), \text{ for all } n \geq 1.$$ 

The differentiable version of the pruning theorem is the following result.

Theorem 13 (Differentiable Pruning Theorem). If $D$ is a pruning domain for $f$ then there exists a diffeomorphism $\psi$, isotopic to $f$, satisfying the following properties:

(i) $\psi$ is an Axiom A map;
(ii) the non-wandering set of $\psi$, $NW(\psi)$, consists of a saddle set $K_\psi$ and (possibly) two periodic orbits (an attractor and a source);
(iii) $\psi$, restricted to $K_\psi$, is semiconjugated (finite-to-one) to $f$, restricted to a subset $K' \subset K$ by a semiconjugacy $\tau: K_\psi \rightarrow K'$ satisfying

$$K' = K \setminus \mathcal{P}(D) = \{q : q \in K \text{ and } \text{Orb}(q) \cap \text{Int}(D) = \emptyset\}$$

where $\mathcal{P}(D) = \bigcup_{n \in \mathbb{Z}} f^n(\text{Int}(D))$; and
Figure 7. A pruning diffeotopy.

By (iii), every point inside Int(D) is wandering for ψ. To prove Theorem 13 we have to construct, if necessary, a DA-map of f which is a similar diffeomorphism to those constructed by Williams [31] for Anosov diffeomorphisms, that is, we have to splitting open the stable manifold of ps and the unstable manifold of pu. Then we define a diffeotopy St inside D as in Fig. 7(a) and compose it with f to obtain a diffeotopy ft = St ◦ f of f. So the diffeomorphism of the pruning theorem will be given by ψ := f1, the end of the diffeotopy. The following observations will be used.

(I) By construction, there are not intersections between invariant manifolds of ψ inside D; furthermore, the unstable invariant manifolds of ψ that are in D are deformations of the invariant manifolds of f by S1, that is, if γu ⊂ D is a segment of an unstable manifold of ψ then γu = S1k(γ′u) where γ′u is a segment of an unstable invariant manifold of f and k is a positive integer. The stable manifolds of ψ satisfy similar properties: If γs is a segment of a stable manifold of ψ inside D then, either γs is equal to a stable segment of f or it is an image by S−1 of a stable segment of f. See Fig. 7(b).

(II) By (iii) and (iv), the set of braid types of ψ is formed by the braid types of the orbits R ⊂ K satisfying that R ∩ Int(D) = ∅.

By these observations, for knowing how the bigons are created or destroyed we just have to concern with the deformation by S1 of the invariant manifolds in D.

3.2. A pruning method. By Theorem 1, to characterize the dynamics forced by a given orbit P we just have to eliminate the bigons relative to P. In this section we will describe how it can be done. In the following suppose that f is an orientation preserving Axiom A map on D2 having a saddle set K. We will prove the following theorem.

Theorem 14. Given an Axiom A diffeomorphism f having a bigon I and an orbit (periodic or homoclinic) P ⊂ K disjoint from I. Then there exists a pruning domain containing I and disjoint from P.

Next subsections are devoted to the proof of Theorem 14.

3.2.1. Finding the maximal domain. Let I be a bigon with ∂I = αs ∪ αu. This section describes how to find the maximal domain, relative to P, which contains I. It will be done finding the maximal rectangle having I as its unique bigon, that is, a set R where the stable and unstable manifolds are vertical and horizontal, respectively.

A segment αu is an u-arc if its end-points are in the stable manifold Ws(q) of some periodic boundary point q ∈ K. By Lemma 2.4.5 of [2], for all u-arc αu included in the unstable manifold of a point of K with end-points in Ws(q), the closed curve, obtained from αu joining its these end-points with a segment of Ws(q) is the boundary of a simply connected closed domain with
interior disjoint from $K$. If we take two such domains $S_1$ and $S_2$, then or $S_1 \subset S_2$ or $S_1 \cap S_2 = \emptyset$. See Fig. 8. Now consider the set of domains which contains $I$ and do not contain any domain disjoint from $I$. This set can be organized by inclusion and let $S$ be the greatest domain in this set. Hence $S$ is bounded by two segments $\gamma_u$ and $\gamma_s$ which have the same end-points $A'$ and $B'$. Since the $u$-arc $\gamma_u$ is an extremal arc then it is included in the stable manifold $W^u(p_u)$ of some $u$-boundary periodic point $p_u$. Let $E$ be the region which has $\gamma_u$ in its boundary and is disjoint from $W^u(K) \cup W^s(K)$. It is clear that $E$ is not a rectangle. See Fig. 8(b).

We want to extend $S$ to find the maximal closed domain $D$, relative to $P$, that intersects $K$. To do it we will find the maximal rectangle $R$ defined by $\gamma_s$. This definition is needed.

**Definition 15.** Two stable segments $\gamma_1$ and $\gamma_2$ are $s$-related if there exists a rectangle non-degenerate whose stable boundary is the union of these two arcs. In the same way, two unstable segments $\gamma_3$ and $\gamma_4$ are $u$-related if there exists a rectangle whose unstable boundary is the union of these two arcs.

Fix the end-point $A'$ of $\gamma_s$ and, if $\gamma$ is a $s$-related segment to $\gamma_s$, denote by $A_\gamma$ the end-point of $\gamma$ which is in the unstable manifold of $A'$ in the rectangle defined by $\gamma_s$ and $\gamma$. Also define $d_u(A', A_\gamma)$ as the distance on $W^u(p_u)$ from $A'$ to $A_\gamma$. Then define the set $T$ of real numbers

$$T := \{t \in \mathbb{R} : \text{there exists a segment } \gamma \text{ s-related to } \gamma_s \text{ with } d_u(A', A_\gamma) = t\}.$$  

**Lemma 16.** $T$ has a well-defined supremum $t_u$.

**Proof.** It is known that $W^u(K)$ has a transverse measure $\nu^u$ which is invariant by holonomy and is called the unstable Margulis measure, and $W^s(K)$ has a transverse measure $\nu^s$ which is invariant by holonomy and is called the stable Margulis measure, such that the product $\nu := \nu^u \cdot \nu^s$ is invariant by $f$. See [2, Section 8.2] for the details. So the area $\nu(K)$ is finite. Let $2a = \nu^s(\gamma_s)$ be the stable measure of $\gamma_s$. If $T$ were unbounded, for each $n \in \mathbb{N}$, we could find a rectangle $\mathcal{R}$ such that $\nu(\mathcal{R}) = 2a \cdot n$. This is a contradiction.

At this point we have two cases.

**Case I.** The first case is when $t_u \in T$. In this case take the stable arc $\theta_s$ related to $\gamma_s$ which makes the supremum and extend $\gamma_u$ to an arc $\theta_u$ such that $\theta_s \cup \theta_u$ bounds a closed region $D$ that is called the maximal domain. See Fig. 9(a).
Case II. The second case is when $t_u \not\in T$, that can happen if there exists a $u$-boundary periodic point $p$ such that it can be defined a rectangle between $\gamma_s$ and a stable segment $\theta_s$ which can be taken arbitrarily close to the stable manifold of $p$. See Fig. 9(b). Hence it is possible to choose a domain $D$ whose boundary consists of two segment $\theta_u \subset W^u(p)$ and $\theta_s \subset W^s(p)$ intersecting in $p$ and another point $q$.

In the two cases we can choose a domain $D$ which is the maximal domain containing $\mathcal{I}$. If $P \cap \text{Int}(D) \neq \emptyset$ then we modify the construction in order to obtain a maximal domain $D$ relative to $P$.

The stable arc $\theta_s$ is included in the stable manifold of some boundary periodic point $p_s$. Let $A$ and $B$ be the end-points of $\theta_s$ and $\theta_u$. Extend $\alpha_u$ until its intersection with $\theta_u$ and denote by $\alpha'_u$ and $\alpha''_u$ these extensions an by $\delta$ the union $\alpha'_u \cup \alpha_u \cup \alpha''_u$. There exists a curve $\theta_c \subset \theta_u$ whose end-points are the same end-points of $\delta$ and such that $\theta_c \cup \delta$ bounded a simply connected domain $Y$ disjoint of $W^u(K)$. See Fig. 9.

**Lemma 17.** There is no boundary periodic points in $\delta$.

**Proof.** Suppose that there is a boundary periodic point $p$ of period $N$ in $\delta$, then $\delta \subset W^u(p)$. Take a stable curve $\delta_s$ close enough to $p$ such that $\delta_s$ and a piece of $\delta$ bound a domain. So the end-points of $\delta_s$ converge to $p$ by backward iterations of $f$. Let $n$ be a positive integer sufficiently large such that the end-points of $f^{-nN}(\delta_s)$ are close to $p$. This is a contradiction with the definition of the domain $D$ since in this case we can not construct a rectangle with the stable segment $f^{-nN}(\delta_s)$ and $\gamma_s$. See Fig. 10.

**Corollary 18.** For all $N > 0$, it holds that $f^N(\delta) \cap \delta = \emptyset$. 

---

**Figure 9.** Maximal domain.

**Figure 10.** Proof of Lemma 17.
Proof. There are two possibilities. If \( f^N(\delta) \subset \delta \) then there exists a \( N \)-periodic point \( p \in \delta \) which has to be a boundary point. It is a contradiction with Lemma 17. If \( \delta \cap f^N(\delta) \neq \emptyset \) then \( \delta \cup f^N(\delta) \) is a boundary of a simply connected domain. This is a contradiction since they are included in the unstable manifold of a boundary point. \( \square \)

3.2.2. **Defining a pruning domain.** If the maximal domain \( D \), relative to \( P \), satisfies pruning conditions then the proof of Theorem 14 is done. If \( D \) does not satisfy these conditions, we will prove that there exists a non trivial pruning domain \( D' \subset D \). The idea is to decrease \( \theta_s \) until to obtain an adequate domain.

**Lemma 19.** If \( f^n(\theta_s) \cap \text{Int}(D) = \emptyset \), for all \( n \geq 1 \) then \( D \) satisfies pruning conditions for \( f \).

**Proof.** By hypothesis \( f^n(\theta_s) \cap \theta_u = \emptyset \) for all \( n \geq 1 \) then \( f^{-n}(\theta_u) \cap \theta_s = \emptyset \) for all \( n \geq 1 \). So if for some \( n \geq 1 \), \( f^{-n}(\theta_u) \cap \text{Int}(D) \neq \emptyset \) then \( f^{-n}(\theta_u) \subset \text{Int}(D) \). This is not possible since \( \gamma_u \subset \theta_u \) is included in the boundary of \( E \) and \( \gamma_u \subset \text{Int}(D) \), which is a contradiction with the definition of \( D \). So \( D \) satisfies the pruning conditions of Definition 12. \( \square \)

By Lemma 19 we just need to study the case when there exist positive integers \( N_i \) such that \( f^{N_i}(\theta_s) \cap \text{Int}(D) \neq \emptyset \).

**Lemma 20.** Let \( D \) be the maximal domain and an integer \( N > 1 \). If \( f^N(\theta_s) \cap \text{Int}(D) \neq \emptyset \) then \( f^N(\theta_s) \cap \theta_u \neq \emptyset \) or \( \alpha_s \subset f^N(\theta_s) \).

**Proof.** If \( f^N(\theta_s) \subset (W^s(K) \cap D) \backslash \gamma_s \), then \( f^N(\theta_s) \) can be continued through the leaves of \( W^u(K) \cap D \). Finding the \( N \)-backward iterate of these continuation, we can see that the rectangle defining \( D \) can be extended to a bigger one. That is a contradiction with the definition of \( D \). The same argument works if \( f^N(\theta_s) \subset \gamma_s \backslash I \). The remaining case is when \( f^N(\theta_s) \cap I \neq \emptyset \) that implies \( \alpha_s \subset f^N(\theta_s) \).

If \( f^N(\theta_s) \) is not included in \( W^s(K) \cap D \) then \( f^N(\theta_s) \cap \theta_u \neq \emptyset \). \( \square \)

**Lemma 21.** There are a finite number of positive integers \( N_1, N_2, ..., N_l \) such that \( f^{N_i}(\theta_s) \cap \text{Int}(D) \neq \emptyset \), for all \( i = 1, ..., l \).

**Proof.** Suppose that there exists a sequence \( \{N_i\}_{i \in \mathbb{N}} \) such that \( f^{N_i}(\theta_s) \cap \text{Int}(D) \neq \emptyset \). By Lemma 20 and since \( \text{diam}(f^{N_i}(\theta_s)) \) goes to 0 when \( i \) goes to \( \infty \), it follows that only a finite number of the \( N_i \) satisfy \( f^{N_i}(\theta_s) \supset \alpha_s \). So we just have to consider the case when \( f^{N_i}(\theta_s) \cap \theta_u \neq \emptyset \). Since \( \theta_u \subset W^s(p_u) \) and \( \lim f^{N_i}(\theta_s) \) goes to \( p_u \) when \( i \) goes to \( \infty \), it follows that \( p_u \in \theta_u \). Hence \( p_u \) is a periodic point in \( W^u(p_u) \), it is only possible if \( p_s = p_u \in \theta_s \cap \theta_u \). But in this case \( f^n(\theta_s) \cap \text{Int}(D) \neq \emptyset \) for a finite number of positive integers. That is a contradiction. \( \square \)

![Figure 11. The N-th iterate of \( \theta_s \).](image)

For the iterates of Lemma 21, we have:

**Lemma 22.** For every \( N_i \) there exists a subdomain \( D_i \subset D \) such that \( \partial D_i = \theta_{s,i} \cup \theta_{u,i} \) where \( \theta_{s,i} \subset \theta_s \), \( f^{N_i}(\theta_{s,i}) \cap \text{Int}(D_i) = \emptyset \) and \( \theta_{u,i} \) is an unstable leaf included in \( D \).
Proof. We have denoted \( 2.\alpha := \nu^\theta(\theta_s) \) where \( \nu^\theta \) is the unstable Margulis measure. Take a point \( q \in \theta_c \). It is possible to define a function \( h : \theta_s \to [-a, a] \), given by \( h(p) := \nu^\theta([p, q]_s) \) where \([p, q]_s \) is the curve included in \( \theta_s \) joining \( p \) and \( q \), and considering \( \nu^\theta([p, q]_s) < 0 \) if \( p \) is to the left of \( q \), and positive if \( p \) is to the right of \( q \).

For \( p \in K \), let \( P_s(p) \) be the smallest projection of \( p \) on \( \theta_s \) along the unstable leaves of \( D \). Note that if \( t \in [-a, a] \) then \( h^{-1}(t) \) contains only one or two points \( p_1(t), p_2(t) \in K \) and 

\[
\begin{align*}
(P_s(f^\theta(p_1(t)))) 
&= h(P_s(f^\theta(p_2(t)))) \quad \text{if} \quad f^\theta(p_2(t)) \in \text{Int}(D) 
\end{align*}
\]

for \( j = 1, 2 \). So there exists a well-defined one-dimensional function \( \xi : [-a, a] \to J \) given by \( \xi(t) = h(P_s(f^\theta(h^{-1}(t)))) \) where \( J \) is an interval that contains \([-a, a]\). By Corollary 18, \( \xi(0) \neq 0 \) and, by the condition \( f^\theta(\theta_s) \cap \text{Int}(D) \neq \emptyset \) and Lemma 20, it follows that \( \xi([-a, a]) \cap [-a, a] \neq \emptyset \). Now consider the biggest interval \( I := [-b, b] \subset [-a, a] \) such that \( \xi(I) \cap I = \emptyset \).

If \( h^{-1}(b) \) has only an element \( p^* \in K \), then the segment \( \theta_{s,i} := [p^*, p^*]_s \) and the unstable leaf \( \theta_{u,i} \) of \( D \) which passes through the end-points of \( \theta_{s,i} \), define a domain \( D_i \subset D \). If \( h^{-1}(b) \) has two elements \( p^*_1, p^*_2 \in K \) then suppose that \( p^*_1 \) is the closest to \( \theta_c \). So the stable segment \( \theta_{s,i} := [p^*_1, p^*_1] \) and the unstable leaf \( \theta_{u,i} \) of \( D \) which passes through the end-points of \( \theta_{s,i} \), define the domain \( D_i \subset D \). See Fig. 12. By condition \( \xi(I) \cap I = \emptyset \), it follows that \( f^\theta(\theta_{s,i}) \cap \text{Int}(D) = \emptyset \). \( \square \)

**Figure 12.** Decreasing the domain \( D \) to another \( D_i \).

With the domains of Lemma 22 we can define a Domain \( D' \) which satisfies at least one property of the Definition 12.

**Corollary 23.** There exists a sub-domain \( D' \subset D \) such that \( I \subset D', \partial D' = \theta'_s \cup \theta'_u \) and

\[
(4) \quad f^n(\theta'_s) \cap \text{Int}(D') = \emptyset, \quad \text{for all} \quad n \geq 1.
\]

**Proof.** Let \( \{D_i\}_{i=1}^l \) be the domains of Lemma 22. Since \( \theta_{s,i} \subset \theta_{s,j} \) or \( \theta_{s,j} \subset \theta_{s,i} \), for all \( i, j \in \{1, \ldots, l\} \), the domains \( \{D_i\}_{i=1}^l \) can be organized by inclusion. Let \( \theta'_{s,i} = \theta_{s,i'} \) be the smallest segment in \( \{\theta_{s,i}\} \), so the domain \( D' := D_{\theta'} \) is the smallest of all of them. Let \( \theta'_u \) be the unstable segment joining the end-points of \( \theta'_{s,i} \). So \( \partial D' = \theta'_s \cup \theta'_u \) and \( f^\theta(\theta'_s) \cap \text{Int}(D) = \emptyset \), for all \( i = 1, \ldots, l \). \( \square \)

Hence for proving that \( D' \) is a pruning domain it is sufficient to prove that

\[
(5) \quad f^{-n}(\theta'_u) \cap \text{Int}(D') = \emptyset, \quad \text{for all} \quad n \geq 1.
\]

This lemma will be used.

**Lemma 24.** \( \theta'_u \) is a segment of the unstable manifold of a \( N_{\nu'} \)-periodic point \( q_u \) which is contained in \( \theta'_u \).

**Proof.** Since \( f \) preserves orientation and \( f^{N_{\nu'}}(D) \) does not contain \( \gamma_u \), it follows that the projection of \( f^{N_{\nu'}}(\theta'_s) \) on \( \theta'_s \) reverses the end-points of \( \theta'_s \), that is, \( \xi \) has a negative slope. Then if \( I := [-b, b] \) is the interval of the proof of Lemma 22 then or \( \xi(-b) = -b \) or \( \xi(b) = b \). Suppose that \( \xi(b) = b \).

If \( h^{-1}(b) \) has two pre-images \( p_1^* \) and \( p_2^* \) then it follows that \( f^{N_{\nu'}}(W_u(p_1^*)) \cap W_u(p_2^*) \neq \emptyset \) and \( f^{N_{\nu'}}(W_u(p_2^*)) \cap W_u(p_1^*) \neq \emptyset \). Then \( f^{2N_{\nu'}}(W_u(p_1^*)) \cap W_u(p_2^*) \neq \emptyset \). By [26, Theorem 1.2], there exists
Lemma 25. For all intersects Since
Proof. has to be point. That is a contradiction since there are not boundary points in \( \text{Int}(D) \).
Now we will prove Property (5).

Pruning Method. What is the domain to be pruned? What are the pruning conditions? How do we construct a pruning diffeomorphism?

4. Forcing on homoclinic horseshoe orbits

In this section we will study certain homoclinic orbits of the Smale horseshoe, one of the most famous diffeomorphism in dynamical systems. We will determine the braid types forced by these orbits exhibiting the sequence of pruning maps (or pruning domains) that are sufficient for eliminating all the bigons relative to these orbits.

4.1. Smale horseshoe. The Smale horseshoe is a well-known Axiom A diffeomorphism \( F \) of the disk \( D^2 \) which acts as in Fig. 13. Its non-wandering dynamics consists of an attracting fixed point in the left semi-circle, and a compact basic set \( K \) included in the regions \( V_0 \) and \( V_1 \). Furthermore, there exists a conjugacy between \( F \), restricted to \( K \), and the shift \( \sigma \), defined on the compact set \( \Sigma_2 = \{0, 1\}^Z \). Thus each point \( p \in K \) is represented by a sequence \( (s_i)_{i \in Z} \) where \( s_i = 0 \), if \( F^i(p) \in V_0 \), and \( s_i = 1 \), if \( F^i(p) \in V_1 \).

In the following sections, a point \( p \in K \) will be always represented by its symbol code. Each point \( p \in K \) has two invariant manifolds \( W^s(p) \) and \( W^u(p) \) that are dense in \( K \) and it will be supposed that they are vertical and horizontal, respectively. If \( p = s_-s_+ \), where \( s_+ \) and \( s_- \) are
unilateral sequences in the compact space $\Sigma^+_2 = \{0, 1\}^\mathbb{N}$, then we can project $p$ using its invariant manifolds until the lowest unstable manifold of $0^\infty$ and the leftmost stable manifold of $0^\infty$. So we can compare two points of $K$ using their positions $s_+$ and $s_-$ in $\Sigma^+_2$ given by the unimodal order $\geq_1$ which relates two symbol sequences by the following rule: If $s = s_0s_1 \cdots$ and $t = t_0t_1 \cdots$ are two sequences in $\Sigma^+_2$ with $s_i = t_i$ for all $i \leq k$ and $s_{k+1} \neq t_{k+1}$, then $s >_1 t$ when:

(O1) $\sum_{i=0}^{k} s_i$ is even and $s_{k+1} > t_{k+1}$, or

(O2) $\sum_{i=0}^{k} s_i$ is odd and $s_{k+1} < t_{k+1}$.

So $s >_1 t$ if $s = t$ or $s >_1 t$. Denote by $\overline{w}$ to the infinite repetition $ww\cdots$ of a word $w$. We will say that a word $w = w_1w_2 \cdots w_M$ is even if $\sum_{i=1}^{M} w_i$ is even; otherwise $w$ is odd. Let $\hat{w} = w_M w_{M-1} \cdots w_1$ be the reversal word of $w$.

Here a horseshoe orbit will be denote by $R$. If $R$ is a periodic orbit then the code of $R$, denoted by $c_R$ is the symbolic representation of $R$ in the unimodal order. When $R$ is not periodic, the code of $R$ can be taken as the symbolic representation of some of its points.

We are only concerned with homoclinic orbits $P = P_0^w$ to $0^\infty$ which have as code $\sigma(\overline{w}) \leq_1 \overline{w}$, for all $i \geq 1$. We need to define two subsets of $\Sigma_2$ as follows:

(a) If $w$ is even then define $P_w$ as the set

$$P_w = \{s_- s_+ : w01^\infty <_1 s_+ <_1 1^\infty, (0\hat{w}) (1\hat{w})^\infty <_1 s_- <_1 (1\hat{w})^\infty\}.$$ 

(b) If $w$ is odd then define $P_w$ as the set:

$$P_w = \{s_- s_+ : w11^\infty <_1 s_+ <_1 1^\infty, (0\hat{w})^\infty <_1 s_- <_1 (1\hat{w}) (0\hat{w})^\infty\}.$$ 

These sets are represented in the symbol plane in Fig. 14 and are called pruning regions associated to $w$. The horizontal lines represent the unstable manifolds and the vertical ones represent the stable manifolds. So the main theorem describes which are the braid types forced by the homoclinic orbit $P_0^w$.

**Theorem 26.** Let $w$ be a maximal decoration. Then $\Sigma_{P_0^w} = \{\text{bt} (R) : R \subset \Sigma_2 \text{ and } R \cap P_w = \emptyset\}$.

From Theorem 26 it is easy to determine the forcing relation between two homoclinic orbits $P_0^w$ and $P_0^{w'}$: if $P_0^{w'}$ does not intersect the pruning region $P_w$ then $P_0^w$ forces $P_0^{w'}$. For example, a consequence of Theorem 26 is the following corollary.

**Corollary 27.** Let $w$ and $w'$ be two maximal decorations satisfying $w \geq_1 w'$ and $\hat{w} \geq_1 \hat{w}'$. Then $P_0^w \geq_2 P_0^{w'}$. 

![Smale horseshoe map](image-url)
The proof of Corollary 27 follows since that, by the conditions $w \geq 1$ and $\tilde{w} \geq 1$, it holds that $P_w \cap P_0 = \emptyset$. Thus, by Theorem 26, $P_w \geq 2 P_0$. This result is closely related to [21, Theorem 4.8] by Holmes and Whitley where it was proved that, for small values of $b > 0$ in the Hénon family $H_{a,b}(x,y) = (a - x^2 - by, x)$, if $\tilde{w}01 \geq 1 \tilde{w}01$ then $P_w$ appears after $P_0$.

Example 28. By Corollary 27, it follows that $\infty0101000010a1100010101001010 \geq 2 \infty010100010b110010101010$ for any words $a$ and $b$ such that $1000010a1100010$ and $100010b110010$ are maximal codes.

To prove Theorem 26 it is needed the following Lemma whose proof is left to the reader.

Lemma 29. Let $w = w_1 \cdots w_M$ be a maximal decoration. Then

(a) If $w$ is even then $w_i w_{i+1} \cdots w_M 010^{\infty} \leq 1 w_010^{\infty}$, for all $i = 2, \cdots, M$.

(b) If $w$ is odd then $w_i w_{i+1} \cdots w_M 110^{\infty} \leq 1 w_110^{\infty}$, for all $i = 2, \cdots, M$.

Proof of Theorem 26. Let $P_w$ be the homoclinic orbit $\infty010w0.10^{\infty}$ where $w$ is maximal and even with length $M$. Note that $F$ has a bigon $I$ at the homoclinic orbit $\infty010.10^{\infty}$, and there exists a maximal domain $D$ bounded by a stable segment of $\infty010.w010^{\infty}$ and an unstable segment of $\infty010w0.1^{\infty}$. Unfortunately $D$ is not a pruning domain, but applying Theorem 14 it is possible to decrease the stable segment of $D$ until to obtain a pruning domain $D'$ which clearly contains $I$. It is not difficult to see that $D'$ is bounded by two segments $\theta_s$ and $\theta_u$ where $\theta_s$ contains $\infty010.w010^{\infty}$ and $\theta_u$ contains the periodic point $(w1)^{\infty}$. So they are defined by the heteroclinic points $q_1 = \infty(w1).w010^{\infty}$ and $q_2 = \infty(w1)w0.w010^{\infty}$, that is $\theta_s \subset W^s(0^{\infty})$ and $\theta_u \subset W^u(\infty(w1))$. The points inside $\Int(D')$ correspond to the symbol sequences of $P_w$. By Lemma 29, it follows that
F^i(\theta_s) is to the left of w010^\infty for all i = 1, \cdots, M−1. Then F^i(\theta_s) \cap \text{Int}(D') = \emptyset for i = 1 \cdots M−1. It is clear that F^i(\theta_s) \cap \text{Int}(D') = \emptyset for all i \geq M. Thus for all i \geq 1, F^i(\theta_s) \cap \text{Int}(D') = \emptyset. A modification of Lemma 29 proves that \text{Orb}(w1^\infty) \cap \text{Int}(D') = \emptyset. Hence the proof of Lemma 19 proves that F^{-n}(\theta_s) \cap \text{Int}(D') = \emptyset for all n \geq 1. Then D' is a pruning domain. Fig. 15(a) shows D' and its (M + 1)-iterate \text{F}^{M+1}(D').

By the Pruning Theorem 13, there exists an Axiom A map \psi, isotopic to F, with a basic set K which is semiconjugated to the set

(6) \{q: q \in \Sigma_2 and \text{Orb}(q) \cap P_w = \emptyset\}

by a semiconjugacy injective on the set of periodic orbits. Note that \psi^{M+1}(0^\infty 10. w010^\infty) = P_0^w and \psi^{M+1}(\theta_u) \cap \theta_u \neq \emptyset. Since the pruning isotopy is supported in D' and the pruning map uncrosses the invariant manifolds inside D' and its iterates (Observation I of Theorem 13), it follows that \psi does not have bigons relative to P_0^w. See Fig. 15(b). By Theorem 1 and (6), the braid type of every orbit that does not intersect D' is forced by P_0^w.

After the semiconjugacy of Theorem 10, the periodic orbit (w1)^\infty becomes a 3-pronged singularity.

If w is odd the proof follows the same lines with minor modifications. □

4.3. Concatenations of NBT decorations. Another group of decorations for which we know their pruning regions are these called concatenations of NBT orbits. To define this kind of orbits we need the notion of NBT orbits introduced in [19] by T. Hall.

Definition 30. To every rational number q in \hat{Q} := \mathbb{Q} \cap (0, 1/2) we associate a symbol code in the following way: If q = \frac{m}{n}, let L_q be the straight line segment joining the origin (0,0) and the point (n,m) in \mathbb{R}^2. Then construct a finite word c_q = s_0s_1s_n by the following rule:

(7) s_i = \begin{cases} 1 & \text{if } L_q \text{ intersects some line } y = k, k \in \mathbb{Z}, \text{ for } x \in (i-1,i+1) \\ 0 & \text{otherwise} \end{cases}

It implies that c_q is palindromic and has the form c_q = 10^{\mu_1} 1^2 0^{\mu_2} 1^2 \cdots 1^2 0^{\mu_m} 1^2 0^{\mu_m} 1. The (n + 2)-periodic orbit \text{P}_q, having c_q 0 or c_q 1 as code, is called a NBT orbit.

The following is the main result of [19] which claims that the Boyland order restricted to NBT orbits coincides with the unimodal order.

Theorem 31 (Hall). Let q, q' \in \hat{Q}. Then

(i) \text{P}_q is quasi-one-dimensional, that is, \text{P}_q \geq 1 R \implies \text{P}_q \geq 2 R.
(ii) q \leq q' \iff \text{P}_q \geq 1 P_{q'} \iff \text{P}_q \geq 2 P_{q'}.

A decoration w is a concatenation of NBT orbits if there exists a finite sequence \{q_i\}_i=1^n of distinct rational numbers in \hat{Q} such that

(8) w = c_{q_1}c_{q_2}0c_{q_3}0 \cdots 0c_{q_n}.

We will give conditions for constructing a pruning diffeomorphism \psi relative to the homoclinic orbit \text{P}_0^w, whose code is \infty 01.0c_{q_1}0c_{q_2}0 \cdots 0c_{q_n}010^\infty, with a well-defined pruning region. At first we have to organize the rational numbers in the real line q_1 > q_2 > \cdots > q_n. By Theorem 31, their codes satisfy c_{q_1} \leq c_{q_2} \leq \cdots \leq c_{q_n}. Denote c_{q_0} = c_{q_{n+1}} = 10^\infty.

Definition 32 (Limiting points). A point C_i = (c_{q_i}, 0c_{q_{i+1}}), with i \in \{1, \cdots, n\}, is called a limiting point if there exists C_j = (c_{q_j}, 0c_{q_{j+1}}), with j > i and c_{q_j} \geq c_{q_i}, such that the region \mathcal{R}_{ij} = \{c_{q_i} < x < 10^\infty, 0c_{q_{j+1}} < y < 10^\infty\} does not contain other point C_k. In this case c_{q_i} is called the successor of c_{q_i}.

It follows that C_1 is always a limiting point and it will be considered that C_{n+1} = (10^\infty, 0c_{q_n}) is a limiting point. The limiting points in Fig. 16(a) are C_1, C_4 and C_5, and in Fig. 16(b) they are C_1, C_2, C_3 and C_5. Denote by \mathbb{L} the set of all the limiting points.
Definition 33 (P-list). We say that \( \{q_i\}_{i=1}^n \) is a \( P \)-list if for every limiting point \( C_i \) and its successor \( C_j \) it holds that the points \( C_k, \) for all \( i < k < j \), are not limiting points. When the successor of \( C_i \) is \( C_{i+1} \), we require that \( C_k, \) for all \( i < k \leq n \), do not be limiting points.

All lists with 2 or 3 elements are \( P \)-lists. The list in Fig. 16(a) is a \( P \)-list, but the list in Fig. 16(b) is not since the successor of \( C_1 \) is \( C_3 \), but \( C_2 \) is also a limiting point.

When \( \{q_i\}_{i=1}^n \) is a \( P \)-list, there exists a pruning region defined as follows.

Definition 34. Let \( \{q_i\}_{i=1}^n \) be a \( P \)-list. For every limiting point \( C_i \) and its successor \( C_j \) define the sets:

\[
P_{i,+} = \{s_- s_+ : c_{q_i} 0 c_{q_{i+1}} \cdots c_{q_n} 0 10^\infty < 1 \text{ s}_+ < 10^\infty \}
\]

and

\[
P_{i,-} = \{s_- s_+ : (0 c_{q_{j-1}} 0 c_{q_{j-2}} \cdots 0 c_{q_i}) (1 c_{q_{j-1}} 0 c_{q_{j-2}} \cdots 0 c_{q_i})^\infty < 1 \text{ s}_- < (1 c_{q_{j-1}} 0 c_{q_{j-2}} \cdots 0 c_{q_i})^\infty \}.
\]

The pruning region associated to \( C_i \) is the set \( P_i = P_{i,+} \cap P_{i,-} \). The pruning region of the list \( \{q_i\}_{i=1}^n \) is the set

\[
P = \bigcup_{C_i \in I} P_i.
\]

Another form of seeing \( P \) is noting that \( P_i \) is the set bounded by a stable segment of the homoclinic point \( \infty 010^\inf_c q_i 0 \cdots 0 c_{q_{j-1}} 0 c_{q_i} \cdots 0 c_{q_n} 010^\infty \) and an unstable segment of the periodic point \( (c_{q_i} 0 \cdots 0 c_{q_{j-1}} 1)^\infty \). Thus we can prove the following result.

Theorem 35. Let \( \{q_i\}_{i=1}^n \) be a \( P \)-list and let \( w = c_{q_1} 0 c_{q_2} \cdots 0 c_{q_n} \) be its associated concatenation. Then \( \Sigma_{P_o} = \{bt(R) : R \subset \Sigma_2 \text{ and } R \cap P = \emptyset\} \).

Proof. As in the case of maximal decorations, there exists a maximal domain \( D \) bounded by a stable segment of \( S_1 = \infty 010^\inf_c q_i 0 \cdots 0 c_{q_n} 010^\infty \) and an unstable segment of its successor

\[
S_j = \infty 010^\inf_c q_i 0 \cdots 0 c_{q_{j-1}} 0 c_{q_j} 0 \cdots 0 c_{q_n} 010^\infty.
\]

Unfortunately \( D \) is not a pruning domain, but applying Theorem 14 one can decrease the stable boundary of \( D \) until to obtain a pruning domain \( D_1 \). It is not difficult to see that \( D_1 \) is bounded by a stable segment \( \theta_s \subset W^s(S_1) \) and an unstable segment \( \theta_u \subset W^u(T_1) \) where \( T_1 = (c_{q_1} 0 c_{q_2} 0 \cdots 0 c_{q_{j-1}} 1)^\infty \). The points inside \( \text{Int}(D_1) \) correspond precisely to the symbol sequences in \( P_1 \). See Fig. 17 for a geometrical explanation of these facts.

To see that \( D_1 \) is a pruning domain note that the end-points of \( \theta_s \) are the heteroclinic points

\[
A_1 = \infty (c_{q_1} 0 c_{q_2} 0 \cdots c_{q_{j-1}} 1) c_{q_i} 0 \cdots 0 c_{q_n} 010^\infty \text{ and }
\]

\[
A_2 = \infty (c_{q_1} 0 c_{q_2} 0 \cdots c_{q_{j-1}} 1) c_{q_i} 0 c_{q_2} 0 \cdots c_{q_{j-1}} 0 c_{q_i} 0 \cdots 0 c_{q_n} 010^\infty.
\]

The iterates of \( A_1 \) and \( A_2 \) can be of the forms:

\[
\cdots 0 l 11,0^\prime, \cdots 0 l 110^\prime, \cdots, \cdots 0 l 110^\prime, \cdots, \cdots 0 l 110^\prime, \cdots, \cdots 0 c_{q_k} 0 c_{q_{k+1}} \cdots \text{ and } \cdots 0 c_{q_k} 0 c_{q_{k+1}} \cdots \text{ with } 1 < k < j.
\]

The four forms to the left are clearly disjoint from \( D_1 \). Since \( (c_{q_{k+1}} 0 c_{q_k}) \) is not a limiting point then the fifth form is disjoint from \( \text{Int}(D_1) \). Thus \( F^n(\theta_s) \cap \text{Int}(D_1) = \emptyset \) for all \( n \geq 1 \). By Lemma 19, it follows that \( D_1 \) is a pruning domain.
Thus the orbits of the pruning diffeomorphism $\psi_1$ associated to $D_1$ are the orbits that do not intersect $\text{Int}(D_1)$. If $M_1$ is the period of $T_1$ then $\psi_1^M_1(A_1)$ is in $\theta_u$, then, by the properties of the pruning diffeotopy, it implies that the bigons are situated to the right of $S_j$.

Since $\{q_i\}$ is a P-list, the orbit of $T_1$ does not intersect the region bounded by a stable segment of $S_j$ and the unstable segments which are between $S_j$ and its successor $S_j'$. So there exists a maximal domain that can be reduced to a pruning domain $D_j$. We can see that $D_j$ is bounded by a stable segment of $S_j$ and an unstable segment of $T_2 = (c_{2/5}0c_{2/7}0\cdots 0c_{q_j-1}1)^\infty$. As in the case of $D_1$, we can prove that $D_j$ is a pruning domain for $\psi_1$. Thus construct a pruning diffeomorphism $\psi_2$. The points inside $\text{Int}(D_j)$ have their symbolic representation in $P_j$.

Proceeding in the same way with all the limiting points we will arrive to a pruning diffeomorphism $\psi$ without bigons relative to $P_w$ and whose orbits are these which do not intersect the pruning region $P = \bigcup_{S_i \in L} P_i$. By Theorem 1, the braid types of all the orbits included in $\Sigma_2 \setminus P$ are forced by $P_w$.

If $q \neq 0^\infty$ is boundary periodic point in $\Sigma_P$ then it collapses to some repeller point $T_k$ which becomes a 3-pronged singularity after the semiconjugacy of Theorem 10.

Remark 36. If $\{q_i\}$ is not a P-list then some iterate of some $T_i$ belongs to a pruning domain $D_k$. It implies that the unstable manifolds in $D_k$ are deformed before of making the pruning isotopy in $D_k$. So we have no more control on these invariant segments and then the proof above can not be implemented.

**Example 37.** Consider the homoclinic orbit $P_w^0$ defined by the P-list $\{2/5, 2/7, 1/3\}$ with code

$$010c_{2/5}0c_{2/7}0c_{1/3}0_{10}^\infty = 0101010101010011001001010_{10}^\infty.$$ 

By Theorem 35, its pruning region is formed by the union of the interior of the following two pruning domains: $D_1$ bounded by a segment of the stable manifold of the limiting point $S_1 = 010_{c_{2/5}0c_{2/7}0c_{1/3}0}^\infty$ and a segment of the unstable manifold of $T_1 = (c_{2/5}0c_{2/7}1)^\infty = (1011010101011)_{10}^\infty$,

and $D_2$ bounded by a stable segment of the limiting point $S_3 = 010_{c_{2/5}0c_{2/7}0c_{1/3}0}^\infty$ and an unstable segment of $T_2 = (c_{1/3}1)^\infty = (10011)_{10}^\infty$. These domains are represented in Fig. 18.

A direct consequence of Theorem 35 is a relation between decorations with the same combinatorics.

**Definition 38.** Two P-lists $\{q_i\}_{i=1}^n$ and $\{q'_i\}_{i=1}^n$ have the same combinatorics if $q_i < q_j \iff q'_i < q'_j$.

So we can prove the following.
Corollary 39. Let \( \{q_i\}_{i=1}^n \) and \( \{q'_i\}_{i=1}^n \) be two \( P \)-lists with the same combinatorics. If \( q_i < q'_i \) for all \( i = 1, \ldots, n \) then
\[
P^c_{q_i} 0^{q_i - c_{q_i}} q_{q_i} \supseteq P^c_{q'_i} 0^{q'_i - c_{q'_i}} q_{q'_i}.
\]

Proof. Note that if \( \{q_i\} \) and \( \{q'_i\} \) have the same combinatorics and \( q_i < q'_i \) then \( c_{q_i} 0_{q_i} < c_{q'_i} 0_{q'_i} \). Hence \( P^c_{q_i} 0^{q_i - c_{q_i} q_{q_i}} \cap P = \emptyset \), where \( P \) is the pruning region of \( P^c_{q_i} 0^{q_i - c_{q_i} q_{q_i}} \). So the conclusion follows from Theorem 35. \( \square \)

4.4. Star homoclinic orbits. In [33] Yamaguchi and Tanikawa have dealt star homoclinic orbits \( P^q_0 \) which have as codes \( ^{\infty}0.c_q0^{\infty} \), with \( q \in \overline{Q} \). These orbits have received this name because their train track types are star [14, 22]. Here we will see that they have well-defined pruning regions. To define it, construct the domain \( D_q \) bounded by a segment \( \theta_u \subset \mathcal{W}^u(\sigma^2(\infty0.c_q0^{\infty})) \) and a segment \( \theta_v \subset \mathcal{W}^u(1^{\infty}) \). Using the properties of \( c_q \), we can prove that \( D_q \) is a pruning domain associated to \( P^q_0 \) and that its pruning diffeomorphism does not have bigons relative to \( P^q_0 \). Note that \( q \geq q' \) if and only if \( D_q \subset D_{q'} \). Hence we have proved two claims:

Claim 1. \( \Sigma P^q_0 = \{ \text{bt}(R) : R \subset \Sigma_2 \text{ and } R \cap \mathcal{W}(D_q) = \emptyset \} \).

Consequence of Claim 1 is the following result found in [33, Theorem 5.2.1].

Claim 2. For star homoclinic orbits, the Boyland order coincides with the order of the rational numbers, that is, \( q \geq q' \iff P^q_0 \supseteq P^{q'}_0 \).

The pruning diffeomorphism \( \psi \) associated to \( D_q \) has a boundary periodic point \( (\hat{c}_q 1)^{\infty} \) of period \( n \), where \( \hat{c}_q \) is \( c_q \) dropping the last two symbols, which collapses, after the semiconjugacy of Theorem 5, to the fixed point \( 1^{\infty} \) creating a \( n \)-pronged singularity.

Example 40. Consider \( q = 2/7 \). Then \( P^{2/7}_0 \) has as code \( ^{\infty}0.c_{2/7}0^{\infty} = ^{\infty}0.10011001^{\infty} \). Its pruning region \( D_{2/7} \) is defined by a stable segment of \( \sigma^2(\infty0.c_{2/7}0^{\infty}) = ^{\infty}0.10011001^{\infty} \) and an unstable segment of the fixed point \( 1^{\infty} \), and is represented in Fig. 19(a) until its fourth iterate. In Fig. 19(b) we have represented the periodic orbits with periods less than 17 whose braid types are forced by \( P^{2/7}_0 \).

Remark 41. In the examples showed in previous subsections, there exists a well-defined pruning region obtained applying Theorem 14 a finite number of times. But it is not true for all decorations. In fact, there are certain decorations \( u \) for which their pruning region is formed by an infinite number of pruning domains and then, in that situation, the pruning method only says that \( \Sigma P^u_0 \) is contained in the set of braid types of the orbits that do not intersect those pruning domains.
Figure 19. (a) Pruning region $D_q$ of the homoclinic orbit $P^{2/7}_{0} = \infty 0.100110010^\infty$.
(b) The set of orbits of periods less than 17 whose braid types are forced by $P^{2/7}_{0}$.

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