Extremal sizes of subspace partitions

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Abstract A subspace partition \( \Pi \) of \( V = V(n, q) \) is a collection of subspaces of \( V \) such that each 1-dimensional subspace of \( V \) is in exactly one subspace of \( \Pi \). The size of \( \Pi \) is the number of its subspaces. Let \( \sigma_q(n, t) \) denote the minimum size of a subspace partition of \( V \) in which the largest subspace has dimension \( t \), and let \( \rho_q(n, t) \) denote the maximum size of a subspace partition of \( V \) in which the smallest subspace has dimension \( t \). In this article, we determine the values of \( \sigma_q(n, t) \) and \( \rho_q(n, t) \) for all positive integers \( n \) and \( t \). Furthermore, we prove that if \( n \geq 2t \), then the minimum size of a maximal partial \( t \)-spread in \( V(n + t - 1, q) \) is \( \sigma_q(n, t) \).

Keywords Subspace partition · Vector space partitions · Partial \( t \)-spreads

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1 Introduction

Let \( V = V(n, q) \) denote a vector space of dimension \( n \) over a finite field with \( q \) elements. A subspace partition \( \Pi \) of \( V \) is a collection of subspaces of \( V \) such that each 1-dimensional subspace of \( V \) is in exactly one subspace of \( \Pi \). A subspace partition \( \Pi \) is also called a vector space partition (or simply a partition) of \( V \). There is a rich literature about vector space partitions, see e.g., \([1,3,5,15,24]\) and the references therein.

The size of \( \Pi \) is the number of its subspaces. Let \( \sigma_q(n, t) \) denote the minimum size of a subspace partition of \( V \) in which the largest subspace has dimension \( t \), and let \( \rho_q(n, t) \) denote the maximum size of a subspace partition of \( V \) in which the smallest subspace has dimension \( t \). The purpose of this study is to find these numbers. Since \( \sigma_q(n, n) = \rho_q(n, n) = 1 \), and \( \sigma_q(n, 1) = \rho_q(n, 1) = (q^n - 1)/(q - 1) \), we will focus on the case \( 1 < t < n \). Moreover, if \( t \) divides \( n \), then \( \sigma_q(n, t) = \rho_q(n, t) \) is the size of a \( t \)-spread in \( V \), i.e., a subspace partition of \( V \) in which all the subspaces have dimension \( t \).

We will prove the following theorem:

**Theorem 1** Let \( n, k, t, \text{ and } r \) be integers such that \( 0 \leq r < t, k \geq 2, \text{ and } n = kt + r \). Then

\[
\rho_q(n, t) = q^{t+r} \sum_{i=0}^{k-2} q^{it} + 1,
\]

and if furthermore \( 1 \leq r < t \), then

\[
\sigma_q(n, t) = q^{t+r} \sum_{i=0}^{k-2} q^{it} + q^{\left\lceil \frac{kt}{2} \right\rceil} + 1.
\]

This theorem improves a result of Beutelspacher \([2]\) who in 1980 proved that

\[
\sigma_q(n, t) \geq q^{\left\lfloor \frac{n}{2} \right\rfloor} + 1.
\]

We must also remark that the last two authors of this paper recently found the value of \( \sigma_q(2t + 1, t) \), see \([22]\). They used some equations for subspace partitions derived by the first two authors in \([21]\). Furthermore, our derivation of the value of \( \sigma_q(n, t) \) uses arguments quite similar to those used in \([22]\).

After some preliminary results in Sect. 2, we will prove our theorem in Sect. 3 and Sect. 4. Finally, in Section 5, we combine our result on \( \sigma_q(n, t) \) with a construction of P. Goovaerts \([14]\) to show that the minimum size of a maximal partial \( t \)-spread in \( V(n + t - 1, q) \) is \( \sigma_q(n, t) \) for any integer \( n \geq 2t \).

2 Preliminary results

Let \( \Pi \) be a subspace partition of \( V = V(n, q) \), \( n \geq 2 \), with \( m_i \) subspaces of dimension \( i, 1 \leq i \leq n - 1 \). Let \( H \) be any hyperplane, i.e., any \( (n - 1) \)-dimensional subspace of \( V \), and let \( b_i \leq m_i \) be the number of subspaces of \( \Pi \) that are contained in \( H \). We say that \((m_{n-1}, \ldots, m_1)\) is the type of \( \Pi \) and \( b = (b_{n-1}, \ldots, b_1) \) is the type of the hyperplane \( H \) (with respect to \( \Pi \)). Let \( s_b \) denote the number of hyperplanes in \( V \) of type \( b \) and define the set

\[ B = \{ b : s_b > 0 \}. \]
For $1 \leq i \leq n$, let
\[ \theta_i = \frac{q^i - 1}{q - 1} \]
denote the number of 1-dimensional subspaces in an $i$-space; then
\[ h_q(n, i) = \max \{0, \theta_{n-i}\} \]
denotes the number of hyperplanes containing a given $i$-dimensional subspace. The following two lemmas were derived in [21].

**Lemma 1** Let $\Pi$ be a subspace partition of $V = V(n, q)$ of type $(m_{n-1}, \ldots, m_1)$ and let $b = (b_{n-1}, \ldots, b_1)$ be the type of the hyperplane $H$ with respect to $\Pi$. Let $s_b$ denote the number of hyperplanes in $V$ with type $b$. Assume furthermore that $\Pi$ contains a subspace of dimension $d$ and a subspace of dimension $d'$, with $1 \leq d, d' \leq n - 2$. Then

(i) \[ \sum_{b \in B} s_b = \frac{q^n - 1}{q - 1} = h_q(n, 0), \]
(ii) \[ \sum_{b \in B} b_d s_b = m_d h_q(n, d), \]
(iii) \[ \sum_{b \in B} \left(\frac{b_d}{2}\right) s_b = \left(\frac{m_d}{2}\right) h_q(n, 2d), \]
(iv) \[ \sum_{b \in B} b_d b_{d'} s_b = m_d m_{d'} h_q(n, d + d'). \]

**Lemma 2** Let $\Pi$ be a subspace partition of $V = V(n, q)$ and let $(b_{n-1}, \ldots, b_1)$ be the type of the hyperplane $H$ with respect to $\Pi$. Then the number of subspaces in $\Pi$ is
\[ |\Pi| = 1 + \sum_{i=1}^{n-1} b_i q^i. \]

We will also use the following lemma due to Herzog and Schönheim [18] and independently Beutelspacher [1] and Bu [5].

**Lemma 3** Let $n$ and $d$ be integers such that $1 \leq d \leq n/2$. Then $V = V(n, q)$ admits a partition with one subspace of dimension $n - d$ and $q^{n-d}$ subspaces of dimension $d$.

For $n = kt + r$, $0 \leq r < t$, and $k \geq 2$, let
\[ \ell = q^r \sum_{i=0}^{k-2} q^{it}. \]

The following proposition is an immediate consequence of Lemma 3.

**Proposition 1** Let $n, k, t,$ and $r$ be integers such that $0 \leq r < t$, $k \geq 2$, and $n = kt + r$. Then $V = V(n, q)$ admits a partition $\Pi_m$ of size
\[ |\Pi_m| = \ell q^l + 1, \]
consisting of $\ell q^l$ subspaces of dimension $t$ and one subspace of dimension $t + r$. If furthermore, $1 \leq r < t$, then $V$ admits a partition $\Pi_M$ of size
\[ |\Pi_M| = \ell q^l + q^{\left\lceil \frac{t+r}{2} \right\rceil} + 1, \]
consisting of $\ell q^l$ subspaces of dimension $t, q^{\left\lceil (t+r)/2 \right\rceil}$ subspaces of dimension $\left\lfloor (t+r)/2 \right\rfloor$ and one subspace of dimension $\left\lceil (t+r)/2 \right\rceil$. 

\[ \text{Springer} \]
We close this section by giving three relations that will be frequently used. They follow easily from the definitions of $\ell$ and the function $\theta_i$; the third is an immediate consequence of the first two:

$$\theta_{n-t} - \theta_r = \ell \theta_t, \quad (2)$$

$$\theta_{a+b} - \theta_b = q^b \theta_a, \quad (3)$$

$$\theta_n - \ell q^t \theta_t = \theta_{t+r}. \quad (4)$$

3 The minimum size

In this section we will find $\sigma_q(n, t)$, as indicated in Theorem 1. We will need the following lemma, which may be of independent interest.

**Lemma 4** Let $n, k, t,$ and $r$ be integers such that $k \geq 2$, $1 \leq r < t$, and $n = kt + r$. Let $\Pi$ be a subspace partition of $V = V(n, q)$ with no subspace of dimension higher than $t$. Assume furthermore that $\Pi$ contains a subspace of dimension $t$ and a subspace of dimension $d$, with $0 \leq d < t$. Then

$$|\Pi| \geq q^{t+i} \sum_{i=0}^{k-2} q^{it} + q^d + 1.$$  

**Proof** Let $\Pi$ be a subspace partition of $V$ containing subspaces of dimension $t$ and $d$ with $t > d$. Since there exist subspaces of dimensions $t$ and $d$ in $\Pi$, we have $m_t > 0$ and $m_d > 0$. So it follows from Lemma 1(iv) that

$$\sum_{b \in B} b_t b_d s_b = m_t m_d \theta_{n-t-d} \neq 0. \quad (5)$$

Additionally,

$$\sum_{b \in B} b_t b_d s_b = \sum_{b \in B} b_t b_d s_b + \sum_{b \in B} b_t b_d s_b.$$  

If

$$\sum_{b \in B, b_t \geq \ell} b_t b_d s_b \neq 0,$$

then there exists $b \in B$ such that $b_t \geq \ell$, $b_d \geq 1$, and $s_b \geq 1$. In this case, Lemma 2 yields

$$|\Pi| = \sum_{i=1}^{n-1} b_i q^i + 1 \geq b_t q^t + b_d q^d + 1 \geq \ell q^t + q^d + 1,$$

and the lemma follows. So we may assume that $\sum_{b \in B, b_t \geq \ell} b_t b_d s_b = 0$. This assumption, combined with (5) and Lemma 1(iv), yields

$$(\ell - 1) m_d \theta_{n-d} = \sum_{b \in B} (\ell - 1) \cdot b_d s_b$$

$$= \sum_{b \in B, 0 \leq b_t \leq \ell - 1} (\ell - 1) \cdot b_d s_b + \sum_{b \in B, b_t \geq \ell} (\ell - 1) \cdot b_d s_b$$
\[ \sum_{b \in B} b_t \cdot b_d s_b + 0 \geq \sum_{0 \leq b_t \leq \ell - 1} b_t \cdot b_d s_b + \sum_{b_t \geq \ell} b_t \cdot b_d s_b \]

\[ = \sum_{b \in B} b_t b_d s_b \]

\[ = m_t m_d \theta_{n-t-d} \tag{6} \]

Since \( m_d > 0 \), dividing both sides of (6) by \( m_d \) yields

\[ m_t \leq \frac{(\ell - 1) \theta_{n-d}}{\theta_{n-t-d}}. \]

We now show that this implies that

\[ m_t \leq (\ell - 1) q^t + q^d. \tag{7} \]

From (3) we obtain that \( \theta_{n-d} = \theta_t + q^t \theta_{n-d-t} \), and hence it remains to prove that

\[ \frac{(\ell - 1) \theta_t}{\theta_{n-d-t}} \leq q^d. \]

This fact follows from Eqs. 2, 3 and 4:

\[ q^d \theta_{n-d-t} - \ell \theta_t + \theta_t = \theta_{n-t} - \theta_d - \theta_{n-t} + \theta_t + \theta_t = \theta_t + \theta_t - \theta_d, \]

as \( \theta_t > \theta_d \).

Note that \( \Pi \) is the disjoint union of \( \mathcal{A} = \{ W \in \Pi : \dim(W) = t \} \) and \( \mathcal{B} = \{ W \in \Pi : \dim(W) \leq t - 1 \} \). By counting the 1-dimensional subspaces not taken up by \( \mathcal{A} \), we can bound the size of \( \mathcal{B} \) by

\[ |B| \geq \frac{\theta_n - |A| \cdot \theta_t}{\theta_{t-1}}. \]

Since \( |\mathcal{A}| = m_t \), we obtain from (7) that

\[ |\Pi| = |\mathcal{A}| + |B| \geq m_t + \frac{\theta_n - m_t \cdot \theta_t}{\theta_{t-1}} \geq \frac{\theta_n - (\ell q^t - q^t + q^d)(\theta_t - \theta_{t-1})}{\theta_{t-1}}. \tag{8} \]

By using Eq. 4, the above inequality can be further simplified

\[ |\Pi| \geq \ell q^t + q^d + \frac{\theta_t r + q^t (\theta_t - \theta_{t-1}) - q^d \theta_t}{\theta_{t-1}} > \ell q^t + q^d + \frac{q^t (\theta_t - \theta_{t-1}) - q^d \theta_t}{\theta_{t-1}}. \]

As furthermore,

\[ q^t (\theta_t - \theta_{t-1}) = q^{2t-1} > q^d \theta_t, \]

we finally obtain

\[ |\Pi| \geq \ell q^t + q^d + 1. \]

This concludes the proof of the lemma.

We now prove that under the assumptions of Theorem 1, \( \sigma_q(n, t) = \ell q^t + q^\lceil \frac{t+1}{2} \rceil + 1. \)
Proof Let \( \Pi \) be a subspace partition of \( V = V(n, q) \) in which the largest subspace has dimension \( t \). Let \( \beta = \lceil (t + r)/2 \rceil \). If there is a subspace of dimension \( d \) in \( \Pi \) with \( \beta \leq d < t \), then by Lemma 4

\[
|\Pi| \geq \ell q^t + q^d + 1 \geq \ell q^t + q^\beta + 1.
\]  

(9)

It remains to consider the case where every subspace in \( \Pi \) has either dimension \( t \) or a dimension less than or equal to \( \beta - 1 \).

If there exists a hyperplane \( H \) of type \( b \) with \( b_t \geq \ell + 1 \), then by Lemma 2

\[
|\Pi| = \sum_{i=1}^{n-1} b_i q^i + 1 \geq (\ell + 1)q^t + 1 \geq \ell q^t + q^\beta + 1,
\]  

(10)

where the last inequality holds since \( \beta \leq t \).

So now assume that if \( s_b \neq 0 \) then \( b_t \leq \ell \). Then Lemma 1(ii) yields

\[
m_t \theta_{n-t} = \sum_{b \in B} b_t s_b \leq \ell \cdot \sum_{b \in B} s_b = \ell \cdot \theta_n.
\]  

(11)

From (2), we derive \( \ell \theta_t < \theta_{n-t} \). By combining this inequality with (3), we obtain

\[
\ell \theta_n = \ell (q^t \theta_{n-t} + \theta_t) = \ell q^t \theta_{n-t} + \ell \theta_t < \ell q^t \theta_{n-t} + \theta_{n-t}.
\]

Consequently, (11) yields

\[
m_t \leq \ell q^t + 1.
\]  

(12)

Note that \( \Pi \) is the disjoint union of \( \mathcal{A} = \{ W \in \Pi : \dim(W) = t \} \) and \( \mathcal{B} = \{ W \in \Pi : \dim(W) \leq \beta - 1 \} \). By Eq. 12 and since \( m_t \) is an integer, we may assume that \( m_t \leq \ell q^t \). So by using Eq. 3, we obtain that

\[
|\Pi| = |\mathcal{A}| + |\mathcal{B}| \geq m_t + \frac{\theta_n - m_t \cdot \theta_t}{\theta_{\beta-1}} = \frac{\theta_n - m_t (\theta_t - \theta_{\beta-1})}{\theta_{\beta-1}} \geq \frac{\theta_n - \ell q^t (\theta_t - \theta_{\beta-1})}{\theta_{\beta-1}},
\]

and hence from (4), and the fact that \( \theta_{\beta-1} (q^\beta + 1) \leq \theta_{t+r} \), we conclude that

\[
|\Pi| \geq \ell q^t + \frac{\theta_n - \ell q^t \theta_t}{\theta_{\beta-1}} = \ell q^t + \frac{\theta_t + r}{\theta_{\beta-1}} \geq \ell q^t + q^\beta + 1.
\]  

(13)

Summarizing the distinct cases we have considered, we thus obtain

\[
\sigma_q(n, t) \geq \ell q^t + q^\beta + 1.
\]  

(14)

Finally, by using Proposition 1 we may conclude that

\[
\sigma_q(n, t) = \ell q^t + q^\beta + 1.
\]

\( \square \)

4 The maximum size

In this section we now prove that under the assumptions of Theorem 1, \( \rho_q(n, t) = \ell q^t + 1 \).
Proof Let $\Pi$ be a subspace partition of $V = V(n, q)$ in which the smallest subspace has dimension $t$. Suppose $|\Pi| > \ell q^t + 1$. The type of $\Pi$ is $(m_{n-1}, \ldots, m_1, 0, \ldots, 0)$. Let $H$ be any hyperplane of $V$, and let $(b_{n-1}, \ldots, b_1, 0, \ldots, 0)$ be the type of $H$ with respect to $\Pi$. Then by Lemma 2, we have

$$|\Pi| = 1 + \sum_{i=t}^{n-1} b_i q^i = 1 + q^t \sum_{i=t}^{n-1} b_i q^{i-t}.$$ 

Thus, $|\Pi| \equiv 1 \pmod{q^t}$, and by our above assumption on $|\Pi|$, we have $|\Pi| \geq \ell q^t + q^t + 1$. As the dimension of each member of $\Pi$ is at least $t$, we may use relation (4) and the fact that $(q^t + 1)\theta_t = \theta_{2t}$ to conclude that

$$\theta_n \geq |\Pi| \theta_t \geq (\ell q^t + q^t + 1)\theta_t = \theta_n - \theta_{t+r} + \theta_{2t},$$

which is a contradiction as $\theta_{2t} > \theta_{t+r}$. Thus $|\Pi| \leq \ell q^t + 1$. Since $\Pi$ is an arbitrary partition, we obtain

$$\rho_q(n, t) \leq \ell q^t + 1.$$  

Hence, from Proposition 1 now follows that

$$\rho_q(n, t) = \ell q^t + 1.$$  

5 Application to maximal partial $t$-spreads

A partial $t$-spread of $V = V(n, q)$ is a collection $S = \{W_1, \ldots, W_k\}$ of $t$-dimensional subspaces of $V$ such that $W_i \cap W_j = \{0\}$ for $i \neq j$. The size of $S$ is its cardinality $|S|$. If $V = \bigcup_{W \in S} W$, then $S$ is called a $t$-spread. A partial $t$-spread is called maximal if it cannot be extended to a larger one. Maximal partial $t$-spreads have been extensively studied, see e.g. [4,9,12,14,16,19,20]. They can be used to construct error-correcting codes [6,8], orthogonal arrays [7,10], and recently factorial designs [23].

We let $\tau_q(n, t)$ denote the minimum number of subspaces in any maximal partial $t$-spread of $V(n, q)$. A maximal partial $t$-spread $S$ of $V(n, q)$ such that $|S| = \tau_q(n, t)$, is called a minimum size maximal partial $t$-spread. Let $n$ and $t$ be fixed integers and let $k$ and $r$ be the unique integers defined by $n = kt + r$ and $0 \leq r < t$. Beutelspacher [1] showed that if $r = 0$ and $k \geq 2$, then

$$\tau_q(n + t - 1, t) = \sigma_q(n, t) = \frac{q^{kt} - 1}{q^t - 1}.$$ 

For $r > 0$, P. Govaerts [14] proved several results related to the number $\tau_q(n + t - 1, t)$. In particular, he provided the following upper bound for $\tau_q(n + t - 1, t)$.

Lemma 5 (Govaerts [14]) Let $n$ and $t > 1$ be integers such that $n \geq 2t$. Then there exist (see page 610 in [14] for a construction) maximal partial $t$-spreads of $V(n + t - 1, q)$ of size $\sigma_q(n, t)$. Consequently, $\tau_q(n + t - 1, t) \leq \sigma_q(n, t)$.

We will prove the following theorem.

Theorem 2 Let $n$ and $t > 1$ be integers such that $n \geq 2t$. Then $\tau_q(n + t - 1, t) = \sigma_q(n, t)$.  

\[ \square \]
The method employed to prove Theorem 2 will be the same as was used in [22] to prove τ_q(3t, t) = σ_q(2t + 1, t). In particular, we will use Theorem 1 in Sect. 1. We first introduce the relevant definitions and a useful Lemma due to Govaerts [14]. A set of points \( B \), i.e., 1-spaces of \( V \), is called a blocking set with respect to the \( t \)-spaces of \( V \) if \( W \cap B \neq \{0\} \) for any \( t \)-space \( W \) in \( V \). Note that any \((n - t + 1)\)-dimensional subspace of \( V \) is a blocking set with respect to the \( t \)-spaces of \( V \). Such blocking sets are called trivial. The following lemma follows from the results of Govaerts (see Case 2, page 612 in [14]).

**Lemma 6** (Govaerts [14]) Let \( n \) and \( t > 1 \) be integers such that \( n \geq 2t \). If \( S \) is a minimum size maximal partial \( t \)-spread of \( V(n, q) \), then \( \bigcup_{W \in S} W \) contains a trivial blocking set.

In the proof of Theorem 2 we will also use the following proposition.

**Proposition 2** Let \( d, d' \), and \( n \) be integers such that \( 0 < d' < d \leq n/2 \). Then

\[
\sigma_q(n, d) < \sigma_q(n, d') .
\]

**Proof** We will prove that \( \sigma_q(n, t) < \sigma_q(n, t - 1) \) holds, for \( 1 < t \leq n/2 \).

If \( t \) divides \( n \), then \( \sigma_q(n, t) = \theta_n/\theta_t \). Consequently, by Theorem 1 and with the use of Eq. 4, we note that it is always true that

\[
\theta_n/\theta_t \leq \sigma_q(n, t) < \frac{\theta_n}{\theta_t} + q^\beta ,
\]

where \( 0 \leq r = n - kt < t \) and \( \beta = \lceil (t + r)/2 \rceil \). As \( \theta_t > q\theta_{t-1} \) and \( q^\beta < \theta_n/\theta_t \), we thus get

\[
\sigma_q(n, t) < 2\frac{\theta_n}{\theta_t} \leq q\frac{\theta_n}{\theta_t} < \frac{\theta_n}{\theta_{t-1}} \leq \sigma_q(n, t - 1) .
\]

\[\square\]

**Proof [Theorem 2]** By Lemma 5, we have \( \tau_q(n + t - 1, t) \leq \sigma_q(n, t) \). So, it remains to show that

\[
\tau_q(n + t - 1, t) \geq \sigma_q(n, t) .
\]

Let \( S \) be a minimum size maximal partial \( t \)-spread in \( V(n + t - 1, q) \). Then by Lemma 6, \( A = \bigcup_{W \in S} W \) contains a trivial blocking set. In other words, there exists an \( n \)-dimensional subspace \( B \subseteq A \). Let

\[
\Pi_S = \{W \cap B : W \in S\}.
\]

Since \( B \) is a blocking set with respect to \( t \)-spaces, we have \( W \cap B \neq \{0\} \) for any \( W \in S \). Thus, \( \Pi_S \) is a subspace partition of \( B \cong V(n, q) \) containing subspaces of dimensions at most \( t \). If \( \Pi_S \) contains a \( t \)-subspace, then it follows from Theorem 1 and the minimality of \( S \) that

\[
\tau_q(n + t - 1, t) = |S| = |\Pi_S| \geq \sigma_q(n, t) .
\]

If \( \Pi_S \) does not contain any \( t \)-subspace, then each subspace in \( \Pi_S \) has dimension at most \( t - 1 \) (and contains at most \( \theta_{t-1} \)-dimensional subspaces). So the theorem now follows from the fact that the function \( \sigma_q(n, t) \) is antimonotone in \( t \) by Proposition 2. \[\square\]
6 Some remarks

We note that the type of a subspace partition that has the maximum size is not always unique. Indeed, $V(8, 2)$ (see [11]) has a partition with one subspace of dimension five and 32 subspaces of dimension three, as well as a partition with three subspaces of dimension four and 30 subspaces of dimension three. We do not know whether the type of a subspace partition of minimum size is unique.

Let $P$ be a subspace partition of $V = V(n, q)$ consisting of $n_i$ subspaces of dimension $d_i$, for $1 \leq i \leq k$. Let us assume that $d_1 < d_2 < \cdots < d_k$ (and $n_1 n_2 \cdots n_k \neq 0$). In [17] a lower bound on $n_1$ was given as a function of $q, d_1$ and $d_2$, and, as easily verified from that result, it is always true that $n_1 \geq \sigma_q(d_2, d_1)$. Working on the results of this paper has given us many indications that the following conjecture holds.

**Conjecture 1** Let $P$ be a subspace partition of $V(n, q)$ with $n_i > 0$ subspaces of dimension $d_i$, $1 \leq i \leq k$, and where $d_1 < \cdots < d_k$. Then, for any integer $j$, $1 \leq j < k$, we have

$$n_1 + \cdots + n_j \geq \sigma_q(d_{j+1}, d_j).$$

Finally, let us remark that for $n \leq 2t - 1$, the problem of determining the minimum size $\tau_q(n + t - 1, t)$ of a maximal partial $t$-spread in $V(n + t - 1, q)$ is still open. For $t = 2$ and $n = 3$, the following lower bound was achieved by Glynn [13]:

$$\tau_q(4, 2) \geq 2q,$$

while the following two upper bounds are due to Gács and Szönyi [12]:

$$\tau_q(4, 2) \leq (2 \log_2 q + 1)q + 1, \quad \text{if } q \text{ odd},$$

and

$$\tau_q(4, 2) \leq (6.1 \ln q + 1)q + 1, \quad \text{if } q > q_0 \text{ even}.$$

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