Gradient estimates for the porous medium type equations and fast diffusion type equations on complete noncompact metric measure space with compact boundary

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Abstract

In the paper, we derive Li-Yau gradient estimates and Souplet Zhang type estimates of the following equation

\[ u_t = \Delta_x u^p + \lambda u + A(u), \]

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on complete noncompact metric measure space \((M, g, e^{-\xi}dv_g)\) with compact boundary. We will also give the local gradient estimates of the equation
\[
\Delta_{\xi}(u^p) + \lambda u + A(u) = 0,
\]
on complete noncompact manifold with compact boundary.

Keywords: Gradient estimate; PME equation; Fast diffusion equation;
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1 Introduction

The following equation
\[
u_t = \Delta u^p,
\]
has been studied deeply. When \(p = 1\), it is heat equation, when \(p > 1\) it is called porous medium equation( PME ), when \(p < 1\) it is called fast diffusion equation(FDE). One can find more information of FDE and reference therein in the book [7]. One can find more about the theory of PME and reference therein in the book [27]. Li [12] studied the Porous medium equations on Riemmannian manifold. Huang and Li [13] considered Li-Yau type gradient estimates of porous medium equation for Witten Laplacian. Zhu [34] obtained the Hamilton type gradient estimates for PME equation. Zhu [33] dirived the Hamilton type gradient estimates of fast diffusion equation. However, the above results hold on open manifold. When \((M, g)\) is complete noncompact manifold with boundary, there are few research results in this field. More information about porous medium equation can be found in [22][3, 4, 15, 20, 21, 26] and reference therein. One can refer to [3, 20, 21, 26]and reference therein for the gradient estimates for PME equation along Ricci flow. One can refer to [2, 10, 19, 28]and reference therein about the research of fast diffusion equation. Qiu[22] considered the gradient estimates of the equation
\[
u_t = \Delta_V u^p,
\]
for \(V\)-laplacian which is the gernalization of Witten laplacian. In this paper, we consider the gradient estimates of (1.1) on metric measure space \((M^n, g, e^{-\xi}dv)\) where the metric \(g\) is fixed.

Recently, Kunikawa and Sakurai [17] established Yau and Souplet-Zhang type gradient estimates on Riemannian manifolds with boundary under Dirichlet boundary condition.
Their main method is to use the Reilly formula in [23] and comparison theorem in [16]. Later, Wu et al. [9, 8, 30] considered more general equation and generalized the work [17].

Motivated by Kunikawa and Sakurai’s work, Dung et al. [8] derived gradient estimates for \( f \)-laplacian equation \( \Delta_f u = 0 \) on metric measure space with boundary. Their proof relies on the comparison theorem in [24] and Reilly formula in [9]. Later, Dung and Wu [9] improved the results in [8] by relaxing the Ricci curvature conditions. Furthermore, Fu and Wu [11] studied more general parabolic equation \( u_t = \Delta_x u + au \ln u \) and obtained Hamilton type estimates, generalized Theroem 1.4 in [8].

Inspired by Kunikawa and Sakurai’s work ([17]), Wu et al.’s work ([11]) and Zhao’s work ([6]), we mainly consider the Li-Yau gradient estimates of the following equation

\[
u_t = \Delta_x u^p + \lambda u + A(u), \quad t \in (-\infty, 0],
\]

(1.1)
on metric measure space \((M, g, e^{-\xi}dv_g)\) with compact boundary using the method of [12]. We will also study the Hamilton type gradient estimates of the equation (1.1) using two different auxillary functions. However, the cutoff function is constructed by the distance from the boundary.

Zhao [32] studied the gradient estiamtes of the equation

\[
\Delta_V u^p + \lambda u = 0, \quad p \geq 1.
\]
on complete noncompact manifold. One can refer to [1][14][18][5]on gradient estimates about \( V \)-Laplacian on Riemannian manifolds. Wu et.al [8] has considered the case that \( V = \nabla f, \lambda = 0, A = 0, p = 1 \) on complete noncompact manifold with compact boundary. For generalization of Theorem 1.3 in [8], we will derive the local gradient estimates for

\[
\Delta_x (u^p) + \lambda u + A(u) = 0, \quad p \geq 1,
\]

(1.2)
on complete noncompact metric measure space with compact boundary. In this paper, we will combine the methods in [32], [8], [12], [13], [34] and [33].

2 Preliminary

Throughout the paper, we use the same notation as that in [31] for the study the equation (1.1), let \( u \) solves (1.1), \( f = \log u, \hat{A} = \frac{A(u)}{u} = \hat{A}(f) \). In the proof, we will need the following lemmas.
Lemma 2.1 (c.f. Lemma 2.5 in [29]). Let \((M^n, g, e^{-f}dv)\) be an \(n\)-dimensional complete smooth metric measure space with the compact boundary \(\partial M\). There exists a smooth cut-off function \(\psi = \psi(\rho, t) \equiv \psi(\rho_{\partial M}(x), t)\) supported in \(Q_{R,T}(\partial M)\) and a constant \(C_\epsilon > 0\) depending only on \(0 < \epsilon < 1\) such that

(i) \(0 \leq \psi(\rho, t) \leq 1\) in \(Q_{R,T}(\partial M)\) and \(\psi(\rho, t) = 1\) in \(Q_{R/2,T/2}(\partial M)\).

(ii) \(\psi\) is decreasing as a radial function of parameter \(r\).

(iii) 
\[
\frac{|\psi_t|}{\psi^{1/2}} \leq \frac{C}{T}, \quad |\psi_{\rho}| \leq \frac{C_\epsilon \psi^\epsilon}{R} \quad \text{and} \quad |\psi_{\rho\rho}| \leq \frac{C_\epsilon \psi^\epsilon}{R^2},
\]

where \(C > 0\) is a universal constant.

Lemma 2.2 (c.f. Proposition 2.2 in [9]). Let \((M, g, e^{-\xi}dv)\) be a complete smooth metric measure space with compact boundary \(\partial M\). For any \(u \in C^\infty(M)\), we have

\[
\frac{1}{2} (|\nabla u|^2)_\nu = u_\nu [\Delta_\xi u - \Delta_{\partial M,\xi} (u|_{\partial M}) - H_{\xi} u_\nu] + g_{\partial M} (\nabla_{\partial M} (u|_{\partial M}), \nabla_{\partial M} u_\nu) \quad \text{and} \quad \nabla_{\partial M} (u|_{\partial M}), \nabla_{\partial M} (u|_{\partial M}),
\]

where \(\nu\) is the unit normal vector field of the boundary.

3 Li–Yau type gradient estimate for \(p > 1\)

In the section, we mainly consider the following equation on metric measure space \((M, g, e^{-\xi}dv_g)\),

\[
u = \Delta_\xi u^p + A(u).
\]

Theorem 3.1. Let \((M, g, e^{-\xi}dv_g)\) be an \(n\)-dimensional, complete metric measure space with compact boundary. For \(K, L \geq 0\), we assume \(H_{\partial M,\xi} \geq -L\), where \(H_{\partial M,\xi} = H_{\partial M} + \langle \nabla \xi, \nu \rangle\), \(\nu\) is the unit normal vector field of the boundary, \(LM > p - 1\). Let \(u\) be a positive solution to the heat equation (3.1) on \(Q_{R,T}(\partial M) := B_R(\partial M) \times [-T, 0]\). For \(W > 0\), let us assume \(u < W\). We further assume that \(u\) satisfies the Dirichlet boundary condition (i.e., \(u(\cdot, t)|_{\partial M}\) is constant for each fixed \(t \in [-T, 0]\)), and \(u_{\nu} \geq 0\) and \(u_t \leq \frac{p-1}{p} A(u), u_t = 0\) over \(\partial M \times [-T, 0]\).

\[
\text{Ric}^\xi_{\partial M} \geq -K g.
\]

Then for any \(\alpha > 1\), there exists a positive constant \(\tilde{C} > 0\) depending only on \(n, \alpha\) such
that on $Q_{R/2,T/4}(\partial M)$,

$$
\frac{|\nabla v|^2}{v} - \frac{v_t}{v} \leq \frac{1}{(\frac{1}{\alpha} + (3M + 1)\epsilon M_{p-T})} \left[ \frac{1}{4 (\frac{2(\alpha-1)}{\alpha^2})} \left( \frac{2p}{p-1} M^{rac{1}{2}} \right)^2 \frac{C}{R^2} - \frac{D_1}{\alpha} \right] + \left[ \frac{1}{(\frac{1}{\alpha} + (3M + 1)\epsilon M_{p-T})} \left( \frac{C^2 M^2}{2\epsilon R^2} + \frac{C^2}{4\epsilon T^2} + \frac{C^2}{4\epsilon} (K^2 + L^4) M \right) \right] \\
+ \alpha (p-1) \frac{M}{4p-1} \|\nabla \Delta \xi\|_\infty^2 - \psi^2 \frac{2}{m(p-1)} \left( 1 - \frac{1}{\epsilon} \right) \hat{A}^2 \\
+ \left( \inf_{Q_{R,T}(\partial M)} v \right)^{-1} (LM - p + 1)^2,$$

where $v = \frac{p}{p-1} u^{p-1}$, $M = (p-1) \sup_{Q_{R,T}(\partial M)} v$, $\epsilon$ is a small positive constant such that $\frac{M(3M+1)}{p-1} \epsilon > 0$, $D_1$ and $D_2$ are defined by (3.27).

**Remark 3.1.** In fact, $M$ can be $pW^{p-1}$. When $A(u) = au + bu \log u$, then $A = a + b \log u$ which can be estimated in term of $M$ if $u$ has positive lower bound. Then $D_1$ and $D_2$ can be further simplified. So, our theorem can be applied to this case. One can also consider the other special $A(u)$ to get the corollary.

**Proof.** We let $v = \frac{p}{p-1} u^{p-1}$, $\mathcal{L} = \partial_t - (p-1) v \Delta \xi$. By (1.1), we get that

$$
v_t = pu^{p-2}u_t = pu^{p-2}(\Delta \xi u^p + A(u)) \\
= pu^{p-2}(pu^{p-1}\Delta \xi u + p(p-1)u^{p-2} |\nabla u|^2 - A(u)).
$$

A direct computation shows that

$$
\Delta \xi v = \Delta v + \langle \nabla \xi, \nabla v \rangle = \nabla (pu^{p-2} \nabla u) + \langle \nabla \xi, pu^{p-2} \nabla u \rangle \\
= (pu^{p-2} \nabla \nabla u) + (p - 2)u^{p-3} \nabla u \nabla u + \langle \nabla \xi, pu^{p-2} \nabla u \rangle \\
= pu^{p-2}(\Delta u + \langle \nabla \xi, \nabla u \rangle) + (p - 2)u^{p-3} |\nabla u|^2.
$$

This implies that

$$
(p-1)v \Delta \xi v = pu^{p-1}(pu^{p-2}(\Delta u + \langle \nabla \xi, \nabla u \rangle) + (p - 2)u^{p-3} |\nabla u|^2) \quad (3.3)
$$

By (7.9) and (3.2), we have

$$
\partial_t v = (p-1)v \Delta \xi v + |\nabla v|^2 + (p-1)v \hat{A}(u), p > 1.
$$
Thus
\[ \mathcal{L}v = |\nabla v|^2 + (p - 1)v\hat{A}(u), \quad p > 1, \]
and
\[ \frac{\partial_t v}{v} = (p - 1)\Delta_\xi v + \frac{|\nabla v|^2}{v} + (p - 1)\hat{A}(u). \]
Let \( F = \frac{|\nabla v|^2}{v} - \alpha \frac{\nabla v}{v} - \phi, \alpha > 1 \), in order to compute \( \mathcal{L}(F) \), we need to compute \( \mathcal{L}v_t \) and \( \mathcal{L}(|\nabla v|^2) \).
\[
\mathcal{L}v_t = (\partial_t - (p - 1)v\Delta_\xi)v_t
= ((p - 1)v\Delta_\xi v + |\nabla v|^2 + (p - 1)v\hat{A}(u))_t - (p - 1)v\Delta_\xi v_t
= (p - 1)v_t \Delta_\xi v + 2(\nabla v, \nabla v)_t + (p - 1)v_t \hat{A}(u) + (p - 1)v\hat{A}_u u_t, \tag{3.4}
\]
and
\[
\mathcal{L}(|\nabla v|^2) = 2\nabla v\nabla v_t - 2(p - 1)v\Delta_\xi |\nabla v|^2
\leq 2\nabla v\nabla v_t - (p - 1)v \left( \frac{1}{m}|\Delta_\xi v|^2 + \langle \nabla v, \nabla \Delta_\xi v \rangle + \text{Ric}^m_v(\nabla v, \nabla v) \right)
= 2\left( \nabla v, \nabla \left( (p - 1)v\Delta_\xi v + |\nabla v|^2 + (p - 1)v\hat{A} \right) \right)
- (p - 1) \left( \frac{1}{m}|\Delta_\xi v|^2 + \langle \nabla v, \nabla \Delta_\xi v \rangle + \text{Ric}^m_v(\nabla v, \nabla v) \right)
= 2\langle \nabla v, ((p - 1)v\nabla \Delta_\xi v + \nabla |\nabla v|^2) \rangle + (p - 1)|\nabla v|^2 \hat{A} + (p - 1)v\hat{A}_u \nabla u \nabla v
\]
\[
- 2(p - 1)v \left( \frac{1}{m}|\Delta_\xi v|^2 + \text{Ric}^m_v(\nabla v, \nabla v) \right)
= 2(p - 1)|\nabla v|^2 \Delta_\xi v + 2\langle \nabla v, \nabla |\nabla v|^2 \rangle + (p - 1)|\nabla v|^2 \hat{A} + (p - 1)v\hat{A}_u \nabla u \nabla v
\]
\[
- \frac{2(p - 1)}{m}v|\Delta_\xi v|^2 - 2(p - 1)v \text{Ric}^m_v(\nabla v, \nabla v). \tag{3.5}
\]
Using the formula (see [13, (2.4)]),
\[
\mathcal{L} \left( \frac{f}{g} \right) = \frac{1}{g} \mathcal{L}(f) - \frac{f}{g^2} \mathcal{L}(g) + 2(p - 1)v \nabla \left( \frac{f}{g} \right) \nabla \log g, \quad \forall f, g \in C^\infty(M).
\]
Hence, on the one hand, we have

\[
\begin{align*}
\mathcal{L} \left( \frac{v_t}{v} \right) &= (p - 1)v_t \Delta v + 2 \langle \nabla v, \nabla v_t \rangle + (p - 1)v_t \hat{A}(u) + (p - 1)v \hat{A}_u u_t \\
&= \frac{v_t (|\nabla v|^2)}{v^2} + 2(p - 1)v \nabla \left( \frac{v_t}{v} \right) \nabla \log v \\
&= (p - 1)\frac{v_t \Delta v}{v} + 2 \langle \nabla v, \nabla v_t \rangle - \frac{v_t (|\nabla v|^2)}{v^2} + 2(p - 1)v \left( \nabla \left( \frac{v_t}{v} \right), \nabla \log v \right) \\
&+ \frac{(p - 1)v_t \hat{A}(u) + (p - 1)v \hat{A}_u u_t}{v}.
\end{align*}
\]

On the other hand, we have

\[
\begin{align*}
\mathcal{L} \left( \frac{|\nabla v|^2}{v} \right) &= \frac{1}{v} \mathcal{L}(|\nabla v|^2) - \frac{|\nabla v|^2}{v^2} \mathcal{L}(v) + 2(p - 1)v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla \log v, \\
&\leq \frac{1}{v} \left( 2(p - 1)|\nabla v|^2 \Delta v + 2 \langle \nabla v, \nabla |\nabla v|^2 \rangle + (p - 1)|\nabla v|^2 \hat{A} + (p - 1)v \hat{A}_u \nabla u \nabla v \\
&\quad - \frac{2(p - 1)}{m} v |\Delta v|^2 - 2(p - 1)v \text{Ric}_v^m(\nabla v, \nabla v) \right) \\
&\quad - \frac{|\nabla v|^2}{v^2} \frac{1}{v} (|\nabla v|^2) + 2(p - 1)v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla \log v, \\
&= 2(p - 1)\frac{1}{v} |\nabla v|^2 \Delta v + 2 \frac{1}{v} \langle \nabla v, \nabla |\nabla v|^2 \rangle + \frac{(p - 1)|\nabla v|^2 \hat{A} + (p - 1)v \hat{A}_u \nabla u \nabla v}{v} \\
&\quad - \frac{2(p - 1)}{m} |\Delta v|^2 - 2(p - 1) \text{Ric}_v^m(\nabla v, \nabla v) \\
&\quad - \frac{|\nabla v|^4}{v^2} + 2(p - 1)v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla \log v.
\end{align*}
\]

Recombining, we have

\[
\begin{align*}
\mathcal{L} \left( \frac{|\nabla v|^2}{v} \right) - \alpha \mathcal{L} \left( \frac{v_t}{v} \right) - \alpha' \frac{v_t}{v} - \phi' \\
&\leq 2(p - 1)\frac{1}{v} |\nabla v|^2 \Delta v + 2 \frac{1}{v} \langle \nabla v, \nabla |\nabla v|^2 \rangle \\
&\quad - \frac{2(p - 1)}{m} |\Delta v|^2 - 2(p - 1) \text{Ric}_v^m(\nabla v, \nabla v) \\
&\quad - \frac{|\nabla v|^4}{v^2} + 2(p - 1)v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla \log v.
\end{align*}
\]
\[-\alpha \left( (p - 1) \frac{v_t \Delta \xi v}{v} + 2 \frac{\langle \nabla v, \nabla v_t \rangle}{v} - \frac{v_t (|\nabla v|^2)}{v^2} + 2(p - 1)v \left \langle \nabla \left( \frac{v_t}{v} \right), \nabla \log v \right \rangle \right) \]

\[+ \alpha \frac{(p - 1)v_t \hat{A}(u) + (p - 1)v \hat{A}_u u_t}{v} \]

\[+ \frac{(p - 1)|\nabla v|^2 \hat{A} + (p - 1)v \hat{A}_u \nabla u \nabla v}{v} + \alpha (p - 1) \left \langle \nabla v, \nabla \Delta \xi \right \rangle \]

\[- \alpha' \frac{v_t}{v} - \phi'. \]

So, we have

\[L \left( \frac{|\nabla v|^2}{v} \right) - \alpha L \left( \frac{v_t}{v} \right) - \alpha' \frac{v_t}{v} - \phi' \]

\[\leq 2(p - 1) \frac{1}{v} |\nabla v|^2 \Delta \xi v - \frac{2(p - 1)}{m} |\Delta \xi v|^2 + 2 \frac{1}{v} \left \langle \nabla v, \nabla |\nabla v|^2 \right \rangle \]

\[- 2(p - 1) \operatorname{Ric}_V(v, \nabla v) - \frac{|\nabla v|^4}{v^2} + 2(p - 1)v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla \log v \]

\[- \alpha (p - 1) \frac{v_t \Delta \xi v}{v} - 2\alpha \frac{\langle \nabla v, \nabla v_t \rangle}{v} + \alpha \frac{v_t (|\nabla v|^2)}{v^2} - 2(p - 1)v \left \langle \nabla \left( \frac{v_t}{v} \right), \nabla \log v \right \rangle \] (3.8)

\[+ \alpha \frac{(p - 1)v_t \hat{A}(u) + (p - 1)v \hat{A}_u u_t}{v} \]

\[+ \frac{(p - 1)|\nabla v|^2 \hat{A} + (p - 1)v \hat{A}_u \nabla u \nabla v}{v} \]

\[- \alpha' \frac{v_t}{v} - \phi'. \]

By the formula between (2.7) and (2.8) in [13], we have

\[-2\alpha \frac{\langle \nabla v, \nabla v_t \rangle}{v} + 2 \frac{1}{v} \left \langle \nabla v, \nabla |\nabla v|^2 \right \rangle = 2(F + \phi) \frac{|\nabla v|^2}{v} + 2 \left \langle \nabla v, \nabla F \right \rangle \] (3.9)

and

\[-2\alpha (p - 1)v \left \langle \nabla \left( \frac{v_t}{v} \right), \nabla \log v \right \rangle + 2(p - 1)v \nabla \left( \frac{|\nabla v|^2}{v} \right) \nabla \log v = 2(p - 1) \nabla v \nabla F, \]

(3.10)

By (3.8), we have
\[ \mathcal{L} \left( \frac{|\nabla v|^2}{v} \right) - \alpha \mathcal{L} \left( \frac{v_t}{v} \right) - \frac{\alpha}{v} v_t - \phi' \leq - \frac{2(p-1)}{m} |\Delta_{\xi} v|^2 - 2(p-1) \text{Ric}_V^m(\nabla v, \nabla v) - \alpha v_t \Delta_{\xi} v + \frac{\alpha v_t (|\nabla v|^2)}{v^2} \]

\[ + \left( 2(p-1) \frac{1}{v} |\nabla v|^2 \Delta_{\xi} v - \frac{|\nabla v|^4}{v^2} - \alpha (p-1) \frac{v_t \Delta_{\xi} v}{v} + \alpha \frac{v_t (|\nabla v|^2)}{v^2} \right) \]

\[ + \left( 2p \nabla v \nabla F + 2 \left( \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \frac{|\nabla v|^2}{v} \right) - \alpha' \frac{v_t}{v} - \phi' \right) \]

\[ + \alpha \frac{(p-1)v_t \hat{A}(u) + (p-1)v \hat{A}_u u_t}{v} + \frac{(p-1)|\nabla v|^2 \hat{A} + (p-1)v \hat{A}_u \nabla u \nabla v}{v}. \]

Similar to [13, (2.9)], we also have

\[ \left( 2(p-1) \frac{1}{v} |\nabla v|^2 \Delta_{\xi} v - \frac{|\nabla v|^4}{v^2} - \alpha (p-1) \frac{v_t \Delta_{\xi} v}{v} + \alpha \frac{v_t (|\nabla v|^2)}{v^2} \right) \]

\[ = 2 \frac{1}{v} |\nabla v|^2 \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v} - \hat{A} \right) - \frac{|\nabla v|^4}{v^2} - \alpha \frac{v_t}{v} \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v} - \hat{A} \right) \]

\[ + \alpha \frac{v_t (|\nabla v|^2)}{v^2} \]

\[ = (2\alpha + 2) \frac{v_t}{v} \frac{|\nabla v|^2}{v} - 3 \frac{|\nabla v|^4}{v^2} - \alpha \left( \frac{v_t}{v} \right)^2 + \hat{A} \left( \alpha \frac{v_t}{v} - 2 \frac{|\nabla v|^2}{v} \right). \]
In the end, by (3.11), (3.12), we get

\[
\mathcal{L} \left( \frac{\nabla v}{v} \right) - \alpha \mathcal{L} \left( \frac{v_t}{v} \right) - \alpha' \frac{v_t}{v} - \phi' \\
\leq - \frac{2(p-1)}{m} |\nabla \xi v|^2 - 2(p-1) \text{Ric}_V^m(\nabla v, \nabla v) \\
+ \left( 2\alpha + 2 \right) \frac{v_t}{v} \left( \frac{\nabla v}{v} \right) - 3 \frac{\nabla v}{v}^2 - \alpha \left( \frac{v_t}{v} \right)^2 + \hat{A}(\alpha \frac{v_t}{v} - 2 \frac{|\nabla v|^2}{v}) \\
+ \left( 2p \nabla v \nabla F + 2 \left( \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \right) \frac{|\nabla v|^2}{v} \right) - \alpha' \frac{v_t}{v} - \phi' \\
= - \frac{2(p-1)}{m} |\nabla \xi v|^2 - 2(p-1) \text{Ric}_V^m(\nabla v, \nabla v) \\
+ \left( 2p \nabla v \nabla F - \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right)^2 + (1 - \alpha) \left( \frac{v_t}{v} \right)^2 \right) - \alpha' \frac{v_t}{v} - \phi' \\
+ \frac{\alpha(p-1)v_t \hat{A}(u) + (p-1)v \hat{A}_u u_t}{v} \\
+ \frac{(p-1)|\nabla v|^2 \hat{A} + (p-1)v \hat{A}_u \nabla u \nabla v}{v} + \hat{A}(\alpha \frac{v_t}{v} - 2 \frac{|\nabla v|^2}{v}) \\
+ \alpha(p-1) \left( \nabla v, \nabla \nabla \xi \right) \\
\tag{3.13}
\]

In the sequel of this section, we take \( \phi = 0 \), let \( \tilde{F} = \frac{|\nabla v|^2}{v} - \alpha \frac{\nabla v}{v} \). Let \( G = \psi \tilde{F} \), it is easy to see that

\[
\mathcal{L}(G) = \psi \mathcal{L}(\tilde{F}) + \tilde{F} \mathcal{L}(\psi) - 2(p-1)v \left( \nabla \psi, \nabla \tilde{F} \right). \\
\tag{3.14}
\]

Computing directly, we have

\[
\tilde{F} \mathcal{L}(\psi) = (\partial_t \psi - (p-1)v \Delta \psi) \tilde{F} = \left( \frac{\partial_t \psi \psi}{\psi} - (p-1)v \frac{\Delta \psi}{\psi} \right) G \\
\tag{3.15}
\]

and

\[
- 2(p-1)v \left( \nabla \psi, \nabla \tilde{F} \right) = 2(p-1)v \frac{\nabla \psi, \nabla \psi}{\psi^2} G. \\
\tag{3.16}
\]

Noticing that

\[
\frac{\partial_t v}{v} - \frac{|\nabla v|^2}{v} = (p-1) \Delta \xi v + \hat{A} v.
\]

\[10\]
By (3.13), we infer that

\[
L(\tilde{F}) \leq - \frac{2(p-1)}{m}|\Delta_v v|^2 + \alpha(p-1)2\langle h, \text{Hess} v \rangle - 2(p-1) \text{Ric}^m_v(\nabla v, \nabla v) \\
+ \left[2p \nabla v \nabla F - \left((p-1)\Delta_v v + \hat{A}\right)^2 + (1 - \alpha)\left(\frac{v_t}{v}\right)^2\right] - \alpha'\frac{v_t}{v} \\
+ \alpha(p-1)v_t\hat{A}(u) + (p-1)v\hat{A}_u u_t \\
+ \frac{(p-1)|\nabla|^2\hat{A} + (p-1)v\hat{A}_u \nabla u \nabla v}{v} + \hat{A}\left(\alpha\frac{v_t}{v} - 2\frac{|\nabla v|^2}{v}\right) \\
+ \alpha(p-1)\langle \nabla v, \nabla \Delta \xi \rangle.
\]

(3.17)

As we know,

\[
|\text{Hess} v|^2 \geq \frac{(\Delta_v v)^2}{m} - \frac{1}{m-n}\langle \nabla v, \nabla \phi \rangle^2, m \geq n.
\]

Noticing also that

\[
- \frac{2(p-1)}{m}|\Delta_v v|^2 - \left((p-1)\Delta_v v + \hat{A}\right)^2 \\
= - \frac{1}{m(p-1)}\left(\frac{\partial v}{v} - \frac{|\nabla v|^2}{v} - \hat{A}\right)^2 - \left(\frac{\partial v}{v} - \frac{|\nabla v|^2}{v}\right)^2 \\
\leq - \left(\frac{1}{m(p-1)}(1 - \epsilon) + 1\right)\left(\frac{\partial v}{v} - \frac{|\nabla v|^2}{v}\right)^2 - \left(\frac{1}{m(p-1)}\right)(1 - \frac{1}{\epsilon})\hat{A}^2. \\
\]

(3.18)

\[
= -\frac{1}{\tilde{a}}\left(\frac{1}{\alpha^2}\tilde{F}^2 + \frac{2(\alpha - 1)}{\alpha^2}\tilde{F} |\nabla v|^2 + \left(\frac{\alpha - 1}{\alpha}\right)^2 |\nabla v|^4\right) \\
- \left(\frac{2}{m(p-1)}\right)(1 - \frac{1}{\epsilon})\hat{A}^2.
\]

where \(\tilde{a} = \left(\frac{2}{m(p-1)}(1 - \epsilon) + 1\right)^{-1}\).
Set $M = (p - 1) \sup_{B_{\bar{p}}(x_0) \times [-T, 0]} v$. By (3.17) and $\alpha > 1$, it follows that

$$\psi L(\bar{F}) \leq 2\psi MK \frac{|\nabla v|^2}{v} - \psi \frac{1}{\alpha} \left( \frac{1}{\alpha^2} \bar{F}^2 + \frac{2(\alpha - 1)}{\alpha^2} \bar{F} \frac{|\nabla v|^2}{v} + \frac{1}{\alpha} \frac{|\nabla v|^4}{v^2} \right)$$

$$- \psi \frac{2}{m(p - 1)} (1 - \frac{1}{\epsilon}) \hat{A}^2$$

$$+ \left( 2\psi \frac{p}{(p - 1)^2} M^{\frac{1}{2}} \frac{|\nabla v|}{v^{\frac{1}{2}}} \frac{|\nabla \psi|}{\psi^{\frac{1}{2}}} G + (1 - \alpha) \psi \left( \frac{v t}{v} \right)^2 \right) - \alpha \frac{v t}{v} \psi$$

$$+ \psi \alpha \frac{(p - 1)}{v} \frac{v t \hat{A}(u) + (p - 1)v \hat{A}_u u_t}{v}$$

$$+ \psi \frac{(p - 1)}{v} |\nabla v|^2 \hat{A} + \frac{(p - 1)v \hat{A}_u \nabla u \nabla v}{v} + \psi \hat{A} \left( \frac{v t}{v} - 2 \frac{|\nabla v|^2}{v} \right)$$

$$+ \psi \alpha (p - 1) \left( \nabla v, \nabla \Delta \xi \right).$$

By (3.19), we have

$$0 \leq \mathcal{L}(G) = \psi^2 \mathcal{L}(\bar{F}) + \psi \bar{F} \mathcal{L}(\psi) - 2(p - 1) \psi v \left( \nabla \psi, \nabla \bar{F} \right)$$

$$\leq 2\psi^2 MK \frac{|\nabla v|^2}{v} - \psi^2 \frac{1}{\alpha} \left( \frac{1}{\alpha^2} \bar{F}^2 + \frac{2(\alpha - 1)}{\alpha^2} \bar{F} \frac{|\nabla v|^2}{v} + \frac{1}{\alpha} \frac{|\nabla v|^4}{v^2} \right)$$

$$- \psi^2 \frac{2}{m(p - 1)} (1 - \frac{1}{\epsilon}) \hat{A}^2.$$
at \((x_1, t_1)\), we have

\[
0 \leq \mathcal{L}(G)
\]

\[
\leq 2\psi^2 MK \frac{\lvert \nabla v \rvert^2}{v} - \psi^2 \frac{1}{\tilde{a}^2} \bigg( \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha - 1)}{\alpha^2} \tilde{F} \frac{\lvert \nabla v \rvert^2}{v} + \frac{(\alpha - 1)^2}{\alpha^2} \frac{\lvert \nabla v \rvert^2}{v^2} \bigg) \\
- \psi^2 \left( \frac{2}{m(p-1)} \left( 1 - \frac{1}{\epsilon} \right) \tilde{A}^2 \right) \\
+ \psi^2 \left( 2 \frac{p}{(p-1)^{3/2}} \frac{\sqrt{\psi}}{\lvert \psi \rvert^2} G + (1 - \alpha) \psi \frac{(v_t)^2}{v} \right) \\
+ \psi^2 \frac{(p-1)v\tilde{A} u_t}{v} + \psi^2 (p-1)v\tilde{A} u \frac{\nabla u \nabla v}{v} + (\psi^2 \tilde{A}^2 - \alpha' \psi^2 - (p-1)\tilde{A} \alpha \psi^2) \frac{v_t}{v} \\
+ \left( -2 \tilde{A} \psi^2 + \psi^2 (p-1)\tilde{A} \frac{\lvert \nabla v \rvert^2}{v} + \psi^2 \alpha (p-1) \langle \nabla v, \nabla \Delta \xi \rangle \right) \\
+ \psi \left( \frac{\partial \psi}{\psi} - (p-1) v \frac{\Delta \xi \psi}{\psi} \right) G + \psi 2(p-1)v \frac{\langle \nabla v, \nabla \psi \rangle}{\psi^2} G.
\]

(3.21)

Since

\[
\psi^2 \alpha (p-1) \langle \nabla v, \nabla \Delta \xi \rangle \\
\leq \psi^2 \alpha (p-1) \frac{\lvert \nabla v \rvert^2}{v} + \psi^2 \alpha (p-1) \frac{1}{4p-1} M \frac{\lvert \nabla \Delta \xi \rvert^2}{\lvert \nabla \xi \rvert}. \quad (3.22)
\]

\[
= \psi^2 \alpha (p-1) \frac{\lvert \nabla v \rvert^2}{v} + \psi^2 \alpha (p-1) \frac{1}{4p-1} M \frac{\lvert \nabla \Delta \xi \rvert^2}{\lvert \nabla \xi \rvert}.
\]

Similar to the argument in [8], we estimate the terms on the last line of (3.21) using Lemma 2.1,

\[
\left( -((p-1)v \Delta \xi - \partial_t) \psi + 2(p-1)v \langle \nabla \psi, \nabla \psi \rangle \frac{1}{\psi} \right) G
\]

\[
\leq -((p-1)v (G \partial_t \psi \Delta \xi \rho \partial_t \psi + G \partial^2 \psi) + G \partial_t \psi) + \epsilon \psi G^2 + \frac{C^{3/4}_4}{\epsilon R^4}
\]

\[
\leq M \left( \epsilon \psi G^2 + \frac{(KR + L)^2 \lvert \partial_t \psi \rvert^2}{4 \epsilon} \right) + M \left( \epsilon \psi G^2 + \frac{1 \lvert \partial^2 \psi \rvert^2}{4 \epsilon} \right) + \left( \epsilon \psi F^2 + \frac{1 \lvert \partial \psi \rvert^2}{4 \epsilon} \right)
\]

\[
+ \epsilon M \psi G^2 + M \frac{C^{3/4}_4}{\epsilon R^4}
\]

\[
\leq 3 M \epsilon \psi G^2 + \frac{C^{1/4}_2}{4 \epsilon} \psi \frac{1}{R^4} + \frac{C^2}{4 \epsilon} \frac{1}{T^2} + \frac{C^2}{4 \epsilon} \frac{(KR + L)^2}{R^2} \psi^{1/2}
\]

\[
\leq 3 \epsilon v G^2 + \frac{C^{1/4}_2}{4 \epsilon} \frac{1}{R^4} + \frac{C^2}{4 \epsilon} \frac{1}{T^2} + \frac{C^2}{4 \epsilon} \frac{(KR + L)^2}{R^2}
\]
\[ \leq (3M + 1)\epsilon vG^2 + \frac{C_2}{2\epsilon} \frac{M}{R^4} + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C^2}{4\epsilon} (K^2 + L^4) M, \]

where the constant \( C_2 \) is the same as that in Lemma 2.1.

Since \( u_t = \frac{1}{p}u^{2-p}v_t \), we have

\[ \psi^2 \alpha \frac{(p - 1)v \dot{A}_u u_t}{v} \leq \psi^2 \alpha \frac{1}{p} u^{2-p} M \frac{v_t}{v} \]

\[ \leq \psi^2 \alpha \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2}{p} - \frac{2}{p'}} \left( - \frac{F}{\alpha} + \frac{1}{\alpha} \frac{|\nabla v|^2}{v} - \frac{\phi}{\alpha} \right) \]  

(3.23)

Noticing that \( \nabla u = \frac{1}{p} u^{2-p} \nabla v \), so we have

\[ \psi^2 (p - 1) v \dot{A}_u \frac{\nabla u \nabla v}{v} = \psi^2 (p - 1) v \frac{1}{p} u^{2-p} \frac{|\nabla v|^2}{v} \leq \psi^2 M \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2}{p} - \frac{2}{p'}} \frac{|\nabla v|^2}{v}. \]  

(3.24)

Plugging the above estimates into (3.21), we have

\[ 0 \leq \mathcal{L}(G) = \psi^2 \mathcal{L}(\tilde{F}) + \tilde{F} \mathcal{L}(\psi) - (p - 1) v \left( \nabla \psi, \nabla \tilde{F} \right) \]

\[ \leq 2\psi^2 MK \frac{|\nabla v|^2}{v} - \psi^2 \frac{1}{\alpha} \left( \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha - 1)}{\alpha^2} \tilde{F} \frac{|\nabla v|^2}{v} + \left( \frac{\alpha - 1}{\alpha} \right)^2 \frac{|\nabla v|^4}{v^2} \right) \]

\[ - \psi^2 \frac{2}{m(p - 1)} (1 - \frac{1}{\epsilon}) \dot{A}^2. \]

\[ + \psi^2 \left( \frac{2}{(p - 1)^2} M \frac{1}{p} \frac{|\nabla v|}{v} \frac{|\nabla \psi|}{\psi} G + (1 - \alpha) \psi \left( \frac{v_t}{v} \right)^2 \right) \]

\[ + \psi^2 \alpha \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2}{p} - \frac{2}{p'}} \left( - \frac{F}{\alpha} + \frac{1}{\alpha} \frac{|\nabla v|^2}{v} - \frac{\phi}{\alpha} \right) \]

\[ + \psi^2 M \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2}{p} - \frac{2}{p'}} \frac{|\nabla v|^2}{v} \]

\[ + (\hat{A} \psi^2 - \alpha' \psi^2 - (p - 1) \hat{A} \alpha \psi^2) \left[ - \frac{F}{\alpha} + \frac{1}{\alpha} \frac{|\nabla v|^2}{v} - \frac{\phi}{\alpha} \right] \]

\[ + \left( - 2 \hat{A} \psi^2 + \psi^2 (p - 1) \hat{A} \right) \frac{|\nabla v|^2}{v} \]

\[ + \left[ \psi^2 \alpha (p - 1) \frac{|\nabla v|^2}{v} + \psi^2 \alpha (p - 1) \frac{1}{4p - 1} \frac{M}{\|\nabla \Delta \xi\|_\infty^2} \right] \]

\[ + \left( 3M + 1 \right) \epsilon vG^2 + \frac{C_2}{2\epsilon} \frac{M}{R^4} + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C^2}{4\epsilon} (K^2 + L^4) M. \]  

(3.25)
which can be simplified using the fact that \( \frac{w}{v} = - \frac{F}{\alpha} + \frac{1}{\alpha} \frac{\| \nabla v \|^2}{v} - \frac{\phi}{\alpha} \), we get

\[
0 \leq \mathcal{L}(G) = \psi^2 \mathcal{L}(\tilde{F}) + \tilde{F} \mathcal{L}(\psi) - 2(p-1)v \left\langle \nabla \psi, \nabla \tilde{F} \right\rangle \\
\leq 2\psi^2 MK \frac{\| \nabla v \|^2}{v} - \psi^2 \frac{1}{\alpha} \left( \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha-1)}{\alpha^2} \tilde{F} \frac{\| \nabla v \|^2}{v} + \frac{\alpha-1}{\alpha} \frac{\| \nabla v \|^4}{v^2} \right) \\
- \psi^2 \left( \frac{2}{m(p-1)} \right)(1 - \frac{1}{\epsilon}) \hat{A}^2 + (1 - \alpha) 2\psi \left( \frac{F^2}{\alpha^2} + \frac{1}{\alpha^2} \frac{\| \nabla v \|^4}{v^2} + \frac{\phi^2}{\alpha^2} \right) \\
+ \psi^2 \frac{p}{(p-1)^{\frac{3}{2}}} M^{\frac{1}{2}} \frac{\| \nabla v \|}{v^{\frac{1}{2}}} \frac{|\nabla \psi|}{v} G + D_1 \left[ - \frac{F}{\alpha} + \frac{1}{\alpha} \frac{\| \nabla v \|^2}{v} \right] + D_2 \frac{\| \nabla v \|^2}{v} \\
+ \left( (3M+1) \epsilon v G^2 + \frac{C^2}{2} \frac{M}{R^4} + \frac{C^2}{4} \frac{1}{T^2} + \frac{C^2}{4} \left( K^2 + L^4 \right) M \right) \\
+ \alpha(p-1) \frac{1}{4p-1} \| \nabla \Delta \xi \|_\infty^2,
\]

where

\[
D_1 = \psi^2 (\hat{A} \alpha - \alpha' - (p-1) \hat{A} \alpha + \alpha \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2-p}{p-1}} M) \\
D_2 = \psi^2 \left( -2 \hat{A} + (p-1) \hat{A} + M \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2-p}{p-1}} + \alpha(p-1) \right).
\]

We can further simplify (3.26),

\[
0 \leq \mathcal{L}(G) \leq - \frac{1}{\alpha^2} \frac{1}{\alpha} \frac{G^2}{v} - \psi^2 \frac{1}{\alpha^2} \frac{2(\alpha-1)}{\alpha^2} \tilde{F} \frac{\| \nabla v \|^2}{v} - \psi^2 \frac{1}{\alpha^2} \frac{(\alpha-1)^2}{\alpha} \frac{\| \nabla v \|^4}{v^2} \\
+ \psi^2 \frac{p}{(p-1)^{\frac{3}{2}}} M^{\frac{1}{2}} \frac{\| \nabla v \|}{v^{\frac{1}{2}}} \frac{|\nabla \psi|}{v} G - D_1 \frac{F}{\alpha} + \\
+ \left( D_2 + D_1 \frac{1}{\alpha} + 2\psi^2 MK \right) \frac{\| \nabla v \|^2}{v} \\
+ \left( (3M+1) \epsilon v G^2 + \frac{C^2}{2} \frac{M}{R^4} + \frac{C^2}{4} \frac{1}{T^2} + \frac{C^2}{4} \left( K^2 + L^4 \right) M \right) \\
+ \alpha(p-1) \frac{1}{4p-1} \| \nabla \Delta \xi \|_\infty^2 - \psi^2 \frac{2}{m(p-1)} \left( 1 - \frac{1}{\epsilon} \right) \hat{A}^2.
\]
Similar to the derivation of (2.16) in [13]

\[- \psi \frac{12(\alpha - 1)}{\alpha^2} G \frac{\lVert \nabla v \rVert^2}{v^2} + \frac{2p}{(p - 1)^{3/2}} \psi^{1/2} M^{1/2} \frac{\lVert \nabla \psi \rVert}{\psi^{1/2}} G \leq \frac{1}{4} \left( \frac{12(\alpha - 1)}{\alpha^2} \right) \left( \frac{2p}{(p - 1)^{3/2}} \psi^{1/2} M^{1/2} \frac{\lVert \nabla \psi \rVert}{\psi^{1/2}} \right)^2 \frac{C}{R^2} G, \tag{3.29}\]

and

\[- \psi^2 \frac{1}{\alpha} (\frac{\alpha - 1}{\alpha})^2 \frac{\lVert \nabla v \rVert^4}{v^2} + \left( D_2 + D_1 \frac{1}{\alpha} + 2\psi^2 MK \right) \frac{\lVert \nabla v \rVert^2}{v} \leq \frac{1}{4} \left( \frac{1}{\alpha} \left( \frac{\alpha - 1}{\alpha} \right)^2 \right) \left( D_2 + D_1 \frac{1}{\alpha} + 2\psi^2 MK \right)^2. \tag{3.30}\]

Thus, plugging the above inequality into (3.28), we can get

\[
\frac{1}{a^2} G^2 \leq (3M + 1)\epsilon v C^2 + \frac{C^2}{2\epsilon} M + \frac{C^2}{4\epsilon} T^2 + \frac{C^2}{4\epsilon} (K^2 + L^4) M \\
+ \alpha (p - 1) \frac{1}{4} \frac{M}{p - 1} \lVert \nabla \Delta \xi \rVert^2 - \psi^2 \left( \frac{2}{m(p - 1)} (1 - \frac{1}{\epsilon}) \right) \hat{A}^2 \\
- \frac{D_1}{\alpha} G + \frac{1}{4} \left( \frac{1}{\alpha} \left( \frac{\alpha - 1}{\alpha} \right)^2 \right) \left( \frac{2p}{(p - 1)^{3/2}} \psi^{1/2} M^{1/2} \frac{\lVert \nabla \psi \rVert}{\psi^{1/2}} \right)^2 \frac{C}{R^2} G \\
+ \frac{1}{4} \left( \frac{1}{\alpha} \left( \frac{\alpha - 1}{\alpha} \right)^2 \right) \left( D_2 + D_1 \frac{1}{\alpha} + 2\psi^2 MK \right)^2. \tag{3.31}\]

By the inequality \( ax^2 - bx - c \leq 0 \), we have \( x \leq \frac{b}{a} + \left( \frac{c}{a} \right)^{\frac{1}{2}} \). This implies that

\[
G \leq \frac{1}{\left( \frac{1}{a^2} - (3M + 1)\epsilon v \right)} \left[ \frac{1}{4} \left( \frac{1}{\alpha} \left( \frac{\alpha - 1}{\alpha} \right)^2 \right) \left( \frac{2p}{(p - 1)^{3/2}} \psi^{1/2} M^{1/2} \frac{\lVert \nabla \psi \rVert}{\psi^{1/2}} \right)^2 \frac{C}{R^2} - \frac{D_1}{\alpha} \right] \\
+ \left[ \frac{1}{\left( \frac{1}{a^2} - (3M + 1)\epsilon v \right)} \left( \frac{C^2}{2\epsilon} M + \frac{C^2}{4\epsilon} T^2 + \frac{C^2}{4\epsilon} (K^2 + L^4) M \\
+ \alpha (p - 1) \frac{1}{4} \frac{M}{p - 1} \lVert \nabla \Delta \xi \rVert^2 - \psi^2 \left( \frac{2}{m(p - 1)} (1 - \frac{1}{\epsilon}) \right) \hat{A}^2 \\
+ \frac{1}{4} \left( \frac{1}{\alpha} \left( \frac{\alpha - 1}{\alpha} \right)^2 \right) \left( D_2 + D_1 \frac{1}{\alpha} + 2\psi^2 MK \right)^2 \right]^{\frac{1}{2}}. \]
Case 2: the maximal point \((x_1, t_1) \in \partial M \times [-T, 0]\), on \(\partial M \times [-T, 0]\), we have assumed that
\[
u_t \leq \frac{p-1}{p} A(u), \quad \nu_t = 0, \partial M \times [-T, 0).
\]
So, we have
\[
f_t + q + \hat{A} \leq 0,
\]
at \((x_1, t_1)\), by Hopf maximum principle, we have
\[
F_\nu = (\frac{\left|\nabla v\right|^2}{v})_\nu \geq 0.
\]
Note that \(\left|\nabla v\right|^2 = v^2\), on \(\partial M \times [-T, 0]\), and
\[
\frac{\partial \ell v}{v} = (p-1)\Delta_\xi v + \frac{\left|\nabla v\right|^2}{v} + (p-1)\hat{A}(u)
\]
\[
\left(\frac{\left|\nabla v\right|^2}{v}\right)_\nu = \frac{\left|\nabla v\right|^2}{v} - \frac{\left|\nabla v\right|^2}{v^2} \nu = \frac{\left|\nabla v\right|^2}{v} - \frac{v^2}{v^2}
\]
\[
\frac{v_\nu (\Delta_\xi v - H_\xi v_\nu)}{v} - \frac{v^2}{v^2}
\]
\[
= \frac{v_\nu (\frac{v_t}{v} - (p-1)\hat{A} - \frac{\left|\nabla v\right|^2}{v}) - H_\xi \frac{v^2}{v} - \frac{v^2}{v^2}.
\]
Thus, we have
\[
\frac{v_\nu (\frac{v_t}{v} - (p-1)\hat{A} - \frac{\left|\nabla v\right|^2}{v}) - H_\xi \frac{v^2}{v} - \frac{v^2}{v^2} \geq 0.
\]
Since \(H_\xi \geq -L\), we get
\[
- \frac{1}{(p-1)} v^3 + Lv^2 v_\nu - v^2 \geq 0,
\]
thus, we have
\[
\frac{1}{(p-1)} v^3 - LM \frac{v^2}{(p-1)} + v^2 \leq 0,
\]
so there exists a constant \(C'\) such that
\[
0 < v_\nu \leq LM - p + 1.
\]
(3.32)
So at \((x_1, t_1) \in \partial M \times [-T, 0]\), we have
\[
F \leq (\inf v)^{-1}(LM - p + 1)^2.
\]
(3.33)
Remark 3.2. From the (5.4), we see that if \( L = 0 \), then the maximum point cannot occur on the boundary.

\[ \square \]

4 Li-Yau gradient estimates for \( p < 1 \)

Theorem 4.1. Let \((M, g, e^{-\xi}dv_g)\) be an \( n \)-dimensional, complete metric measure space with compact boundary. For \( K, L \geq 0 \), we assume \( H_{\partial M, \xi} \geq -L \), where \( H_{\partial M, \xi} = H_{\partial M} + \langle \nabla \xi, \nu \rangle \), \( \nu \) is the unit normal vector field of the boundary. Let \( u \) be a positive solution to the heat equation \((1.1)\) on \( Q_{R,T}(\partial M) := B_R(\partial M) \times [-T, 0] \) with \( p \in (1 - \frac{2(1+\epsilon)}{m}, 1) \).

For \( W > 0 \), let us assume \( u < W \). We further assume that \( u \) satisfies the Dirichlet boundary condition (i.e., \( u(\cdot, t)|_{\partial M} \) is constant for each fixed \( t \in [-T, 0] \)), and \( u_t \geq 0 \) and \( u_t \leq \frac{p-1}{p} A(u), u_t = 0 \) over \( \partial M \times [-T, 0] \).

\[ \text{Ric}_\xi^m \geq -Kg. \]

Then for any \( \alpha > 1 \), there exists a positive constant \( \tilde{C} > 0 \) depending only on \( n, \alpha \) such that on \( Q_{R/2,T/4}(\partial M) \),

\[
\frac{|
abla v|^2}{v} - \frac{\alpha v_t}{v} \leq \frac{1}{\Psi} \left[ \frac{D_{11}}{\alpha} - \frac{\tilde{a} \alpha^2 p^2 M}{2\epsilon_1 (1-\tilde{a})(1-\alpha)(1-p)} R^2 \right] \\
+ \left\{ \frac{1}{\Psi} \left[ \frac{C^2 M}{2\epsilon_1 R^4} + \frac{C^2 1}{4\epsilon T^2} + \frac{C^2}{4\epsilon} (K^2 + L^4) M \right] \\
+ \frac{1}{2}(1-\alpha)(1-\alpha - \tilde{a}) \right\}^{1/2} \\
+ \left( \inf_{Q_{R,T}(\partial M)} (-\nu)^{-1} (-LM - p + 1)^2 \right),
\]

where \( \Psi = \frac{1}{-\alpha^2} A(\epsilon_1, \epsilon_2) - (3M + 1) \frac{M}{1-p} \epsilon, v = \frac{M}{p-1} \mu^{p-1}, C' \) determined by inequality (6.1) depends on \( R, \lambda_R, \gamma_R, \alpha, M := (1-p) \sup_{Q_{R,T}(\partial M)} v < \frac{R}{1-p}, \tilde{a} = \left( \frac{2}{m(p-1)} \right)(1+\epsilon) + 1 \)\(^{-1} \), \( \epsilon \) is a small positive constant such that \( \Psi > 0 \). The functions \( D_1 \) and \( D_2 \) are defined by (4.16).

Remark 4.1. One can refer to Remark 3.1.
Proof. Let \( v = \frac{p}{p-1} u^{p-1} \), \( \mathcal{L} = \partial_t - (p-1)v \Delta_\xi \). Then by (7.9) and (3.2), we have

\[
\partial_t v = (p-1)v \Delta_\xi v + |\nabla v|^2 + (p-1)v \hat{A}(u), \quad p < 1,
\]

thus

\[
\mathcal{L}v = |\nabla v|^2 + (p-1)v \hat{A}(u), \quad p < 1,
\]

and

\[
\frac{\partial_t v}{v} = (p-1)\Delta_\xi v + \frac{|\nabla v|^2}{v} + (p-1)\hat{A}(u).
\]

Let

\[
F = \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} - \phi, \quad \alpha > 1
\]

Using the fact that \( p < 1 \), along the same line as that in section 3, it is not hard to see that

\[
\mathcal{L}(F) \geq -\frac{2(p-1)}{m} |\Delta_\xi v|^2 - 2(p-1) \text{Ric}_V^m(\nabla v, \nabla v)
\]

\[
+ \left( 2p v_t \nabla F - \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right)^2 + (1 - \alpha) \left( \frac{v_t}{v} \right)^2 \right) - \alpha' \frac{v_t}{v} - \phi'
\]

\[
+ \alpha \frac{(p-1)v_t \hat{A}(u) + (p-1)v \hat{A}_u u_t}{v} + (p-1) \nabla v \nabla \hat{A} \left( \frac{v_t}{v} - 2 \frac{|\nabla v|^2}{v} \right)
\]

\[
+ \alpha (p-1) \langle \nabla v, \nabla \Delta_\xi \rangle
\]

Thus, we have

\[
\mathcal{L}(-F) \leq \frac{2(p-1)}{m} |\Delta_\xi v|^2 + 2(p-1) \text{Ric}_V^m(\nabla v, \nabla v)
\]

\[
- \left( 2p v_t \nabla F - \left( \frac{v_t}{v} - \frac{|\nabla v|^2}{v} \right)^2 + (1 - \alpha) \left( \frac{v_t}{v} \right)^2 \right) + \alpha \frac{v_t}{v}
\]

\[
- \alpha \frac{(p-1)v_t \hat{A}(u) + (p-1)v \hat{A}_u u_t}{v} - (p-1) \frac{v_t \hat{A} \left( \frac{v_t}{v} - 2 \frac{|\nabla v|^2}{v} \right)}{v}
\]

\[
- \alpha (p-1) \langle \nabla v, \nabla \Delta_\xi \rangle
\]

We take \( \phi = 0 \), let \( \bar{F} = \frac{|\nabla v|^2}{v} - \alpha \frac{v_t}{v} \). Let \( G = -\psi \bar{F} \)

\[
\mathcal{L}(G) = \psi \mathcal{L}(-\bar{F}) - \bar{F} \mathcal{L}(\psi) + 2(p-1)v \left( \nabla \psi, \nabla \bar{F} \right)
\]
However, noticing that
\[-\tilde{F} \mathcal{L}(\psi) = - (\partial_t \psi - (p-1)v \Delta \psi) \tilde{F} = \left( \frac{\partial_t \psi}{\psi} - (p-1)v \frac{\Delta \psi}{\psi} \right) G, \quad (4.4)\]
and
\[2(p-1)v \left\langle \nabla \psi, \nabla \tilde{F} \right\rangle = 2(p-1)v \frac{\langle \nabla \psi, \nabla \psi \rangle}{\psi^2} G. \quad (4.5)\]
Noticing that
\[\frac{\partial_t v}{v} - |\nabla v|^2 = (p-1) \Delta \xi v + \hat{A}. \]
From (4.2), we can get
\[L(-\tilde{F}) \leq \frac{2(p-1)}{m} |\Delta \xi v|^2 + 2(p-1) \text{Ric}^m_v(\nabla v, \nabla v) \]
\[- 2p \nabla v \nabla F - \left( (p-1) \Delta \xi v + \hat{A} \right)^2 + (1 - \alpha) \left( \frac{\nabla_t}{v} \right)^2 \alpha \frac{\nabla_t}{v} \]
\[- \alpha (p-1)v \nabla v \xi (u) + (p-1)v \xi u \xi t \]
\[- \alpha (p-1)(\nabla v, \nabla \Delta \xi). \]
Noticing that
\[\frac{2(p-1)}{m} |\Delta \xi v|^2 + \left( (p-1) \Delta \xi v + \hat{A} \right)^2 \]
\[= \frac{1}{m(p-1)} \left( \frac{\partial_t v}{v} + \frac{|\nabla v|^2}{v} - \hat{A} \right)^2 + \left( \frac{\partial_t v}{v} - \frac{|\nabla v|^2}{v} \right)^2 \]
\[\leq \left( \frac{1}{m(p-1)} (1 + \epsilon) + 1 \right) \left( \frac{\partial_t v}{v} + \frac{|\nabla v|^2}{v} \right)^2 + \frac{1}{m(p-1)} (1 + \frac{1}{\epsilon}) \hat{A}^2 \]
\[= \tilde{a} \left( \frac{\tilde{F}^2}{\alpha^2} + \frac{2(\alpha - 1)}{\alpha^2} \tilde{F} \frac{|\nabla v|^2}{v} + \frac{\alpha - 1}{\alpha}^2 \frac{|\nabla v|^4}{v^2} \right) \]
\[+ \left( \frac{2}{m(p-1)} (1 + \frac{1}{\epsilon}) \hat{A}^2, \right)\]
where \(\tilde{a} = \left( \frac{2}{m(p-1)} (1 + \epsilon) + 1 \right)^{-1}.\)
set $M = (1 - p) \sup_{B_\epsilon(x_0) \times [-T,0]} v$. By (4.6), we have

$$
\psi L(-\tilde{F}) \leq 2\psi M K \frac{|\nabla v|^2}{v} + \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha - 1)}{\alpha^2} \frac{|\nabla v|^2}{v} + \frac{(|\nabla^2 v|^2)}{v^2} + \psi \left( \frac{2}{m(p-1)} \right) (1 - \frac{1}{\epsilon}) \hat{A}^2
$$

$$
+ \psi \left( \frac{2}{m(p-1)} \right) (1 - \frac{1}{\epsilon}) \hat{A}^2
$$

$$
+ \left( 2\psi \frac{p}{(1 - p)^2} M^\frac{1}{2} \frac{|\nabla v|}{|--v|} \frac{|\nabla \psi|}{|\psi|} G - (1 - \alpha) \psi \frac{v_t}{v} \right) + \alpha \frac{v_t}{v} \psi
$$

$$
- \psi \alpha (p - 1) |\nabla v|^2 \hat{A} + (p - 1) v \hat{A}_u \nabla v - \psi \hat{A} \frac{v_t}{v} - 2 \frac{|\nabla v|^2}{v}.$$

Next we use the cut off function $\psi$ in Lemma 2.1, set $G = -\psi \tilde{F}$ we may assume $G$ is positive and attains its maximum point at $(x_1, t_1)$ in $Q_{R,T}(\partial M)$. Firstly, we consider the case that $x_1 \notin \partial M$, by maximum principle, we have

$$
\Delta \xi(G) \leq 0, G_t \leq 0 \quad \text{and} \quad \nabla(GF) = 0,
$$

at $(x_1, t_1)$, we have

By (4.8), we have

$$
0 \leq L(G) = \psi^2 L(-\tilde{F}) - \tilde{F} L(\psi) + 2(p - 1) v \left( \nabla \psi, \nabla \tilde{F} \right)
$$

$$
\leq \left[ 2\psi^2 M K \frac{|\nabla v|^2}{v} + \psi \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha - 1)}{\alpha^2} \frac{|\nabla v|^2}{v} + \frac{(|\nabla^2 v|^2)}{v^2} \right] + \psi \left( \frac{2}{m(p-1)} \right) (1 - \frac{1}{\epsilon}) \hat{A}^2
$$

$$
- \psi^2 \alpha (p - 1) v_t \hat{A}(u) + (p - 1) v \hat{A}_u v_t
$$

$$
- \psi^2 \alpha (p - 1) |\nabla v|^2 \hat{A} + (p - 1) v \hat{A}_u \nabla v - \psi \hat{A} \frac{v_t}{v} - 2 \frac{|\nabla v|^2}{v}.
$$

$$
- \psi^2 \alpha (p - 1) \left( \nabla \psi, \nabla \Delta \xi \right)
$$

$$
+ \psi \left( \frac{\partial \psi}{\psi} - (p - 1) v \frac{\Delta \psi}{\psi} \right) G + 2(p - 1) v \frac{\nabla \psi, \nabla \psi}{\psi} G.
$$

(4.9)
\[ 0 \leq \mathcal{L}(G) = \psi^2 \mathcal{L}(-\tilde{F}) - \tilde{F} \mathcal{L}(\psi) + 2(p-1)v \left\langle \nabla \psi, \nabla \tilde{F} \right\rangle \]

\[ \leq 2\psi^2 MK \frac{\left| \nabla v \right|^2}{v^\alpha} + \psi^2 \frac{1}{a^2} \left( \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha - 1)}{\alpha^2} \tilde{F} \frac{\left| \nabla v \right|^2}{v} + \left( \frac{\alpha - 1}{\alpha} \right)^2 \frac{\left| \nabla v \right|^4}{v^2} \right) \]

\[ + \psi^2 \left( \frac{2}{m(p-1)}(1 + \frac{1}{\epsilon}) \hat{A}^2 \right) \]

\[ - \psi^2 \left( 2 \frac{p}{(1-p)^2} M \frac{1}{v^\alpha} \frac{\left| \nabla v \right|}{v} \frac{\left| \nabla \psi \right|}{v} G + (1 - \alpha) v \left( \frac{\partial \nabla v}{\nabla \psi} + \frac{\nabla v}{\nabla \psi} \right)^2 \right) \]

\[ + \left\{ \psi^2 \alpha \frac{(p-1)v \hat{A} \nabla v}{v} - \psi^2 (p-1) v \hat{A} \frac{\nabla v}{v} + \psi^2 (p-1) \left( -\hat{A} \alpha + \hat{A} \psi^2 \right) \frac{v}{v} \right\} \]

\[ + \psi^2 \alpha (p-1) \left( \nabla \psi, \nabla \Delta \xi \right) \]

\[ + \psi \left( \frac{\partial \nabla \psi}{\psi} - (p-1) v \frac{\Delta \xi}{\psi} \right) G + \psi^2 (p-1) v \left\langle \nabla \psi, \nabla \psi \right\rangle G. \]  

(4.10)

Since

\[ \psi^2 \alpha (p-1) \left( \nabla \psi, \nabla \Delta \xi \right) \]

\[ \leq \psi^2 \alpha (1-p) \frac{\left| \nabla v \right|^2}{v} + \psi^2 \alpha (1-p) \frac{M}{4p-1} \left\| \nabla \Delta \xi \right\|_{\infty}. \]  

(4.11)

Similar to the argument in [8], By Lemma 2.1,

\[ \left( (p-1)v \Delta \xi - \partial_t \psi + 2(p-1)v \nabla \psi, \nabla \psi \right) \frac{1}{\psi} \]

\[ \leq \psi^2 \alpha (p-1)v \left( G \partial_t \psi \Delta \xi \rho_{\theta M} + G \partial^2_t \psi \right) + G \partial_t \psi \right) + \epsilon \psi G^2 + \frac{C^4_{3/4}}{\epsilon R^4} \]

\[ \leq M \left( \epsilon \psi G^2 + \frac{(KR + L)^2}{4} \left| \frac{\partial v}{\psi} \frac{v^2}{\psi} \right| \right) + \left( \psi^2 \alpha (p-1)v \nabla \psi, \nabla \psi \right) \frac{1}{\psi} \]

\[ + \epsilon M \psi G^2 + \frac{C^4_{3/4}}{\epsilon R^4} \]

\[ \leq 3M \epsilon \psi G^2 + \frac{C^2_{3/4}}{4 \epsilon R^4} \psi^{1/2} + \frac{C^2_{3/4}}{4 \epsilon T^2} + \frac{C^2_{3/4}}{4 \epsilon R^4} \frac{(KR + L)^2}{\psi^{1/2}} \]

\[ \leq 3 \epsilon v G^2 + \frac{C^2_{3/4}}{4 \epsilon R^4} \frac{1}{T^2} + \frac{C^2_{3/4}}{4 \epsilon T^2} + \frac{C^2_{3/4}}{4 \epsilon R^4} \frac{(KR + L)^2}{\psi^{1/2}} \]

\[ \leq (3M + 1) \epsilon v G^2 + \frac{C^2_{3/4}}{2 \epsilon R^4} M + \frac{C^2_{3/4}}{4 \epsilon T^2} + \frac{C^2_{3/4}}{4 \epsilon R^4} (K^2 + L^4) M, \]
where the constant $C_4, C$ is the same as that in Lemma 2.1.
Noticing that $u_t = \frac{1}{p}u^{2-p}v_t$, $\nabla u = \frac{1}{p}u^{2-p}\nabla v$, By (3.24),(3.23)

\[-\psi^2 \alpha \frac{(p-1)v\hat{A}_u u_t}{v} \leq \psi^2 \alpha \frac{1}{p}u^{2-p}Mv_t \leq \psi^2 \alpha \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2-p}{p-1}} M \left| -\frac{F}{\alpha} + \frac{1}{\alpha} \frac{|\nabla v|^2}{v} - \frac{\phi}{\alpha} \right|, \tag{4.12} \]

and

\[-\psi^2 (p-1)v\hat{A}_u \frac{\nabla u \nabla v}{v} = \psi^2 (1-p)\frac{1}{p}u^{2-p}\frac{|\nabla v|^2}{v} \leq \psi^2 M \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2-p}{p-1}} \frac{|\nabla v|^2}{v} \tag{4.13} \]

Plugging the above estimates into (4.10), we have

\[0 \leq \mathcal{L}(G) \leq 2\psi^2 MK \frac{|\nabla v|^2}{v} + \psi^2 \frac{1}{\alpha} \left( \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha-1)}{\alpha^2} \tilde{F} \frac{|\nabla v|^2}{v} + \frac{(\alpha-1)^2}{\alpha^2} \frac{|\nabla v|^4}{v^2} \right) \]

\[+ \psi^2 \left( \frac{2}{m(p-1)} + \frac{1}{\epsilon} \hat{A}(u)^2 \right) \]

\[- \psi^2 \left( \frac{2}{(1-p)^{\frac{1}{2}}} M \frac{1}{\psi} \frac{|\nabla v|}{v} \psi G + (1-\alpha)\psi((v_t)^2) \right) \]

\[+ \psi^2 \alpha \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2-p}{p-1}} M \left| -\frac{F}{\alpha} + \frac{1}{\alpha} \frac{|\nabla v|^2}{v} \right| \]

\[+ \psi^2 M \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2-p}{p-1}} \frac{|\nabla v|^2}{v} \]

\[+ (-\hat{A}\alpha \psi^2 + \alpha' \psi^2 - (p-1)\hat{A}\alpha \psi^2) \left( -\frac{F}{\alpha} + \frac{1}{\alpha} \frac{|\nabla v|^2}{v} - \frac{\phi}{\alpha} \right) \]

\[+ \left( 2\hat{A}\psi^2 - \psi^2 (p-1)\hat{A} \right) \frac{|\nabla v|^2}{v} \]

\[+ \left[ \psi^2 \alpha (1-p) \frac{|\nabla v|^2}{v} + \psi^2 \alpha (1-p) \frac{1}{4p-1} \frac{M}{\alpha} \|\nabla \Delta \xi\|_\infty \right] \]

\[+ \left( 3M + 1 \right) \epsilon vG^2 + \frac{C_4^2}{2\epsilon} M + \frac{C_4^2}{4\epsilon} \frac{1}{T} + \frac{C_4^2}{4\epsilon} \left( K^2 + L^4 \right) M \right). \]

Since

\[\frac{v_t}{v} = -\frac{F}{\alpha} + \frac{1}{\alpha} \frac{|\nabla v|^2}{v} - \frac{\phi}{\alpha}. \]
So we get
\[ 0 \leq \mathcal{L}(G) \]
\[ \leq 2\psi^2 MK \frac{\nabla v^2}{v} + \psi^2 \frac{1}{\tilde{a}} \left( \frac{1}{\alpha^2} \tilde{F}^2 + \frac{2(\alpha - 1)}{\alpha^2} \tilde{F} \frac{|\nabla v|^2}{v} + (\frac{\alpha - 1}{\alpha})^2 |\nabla v|^4 \right) \]
\[ + \psi^2 \left( \frac{2}{m(p - 1)} \right) (1 + \frac{1}{\epsilon}) \tilde{A}^2 \]
\[ - (1 - \alpha) \psi \left( \frac{F^2}{\alpha^2} + \frac{1}{\alpha^2} \frac{|\nabla v|^4}{v^2} + \frac{2}{\alpha^2} \tilde{F} \frac{|\nabla v|^2}{v} \right) \]
\[ - \psi^2 \frac{p}{(p - 1)^2} M^2 \frac{|\nabla v||\nabla \psi|}{v^2} G \]
\[ + D_1 \left[ - \frac{F}{\alpha} + \frac{1}{\alpha} \frac{|\nabla v|^2}{v} \right] + + D_2 \frac{|\nabla v|^2}{v} \]
\[ + \left( (3M + 1)\psi G^2 + \frac{C^4_2 M}{2\epsilon R^2} + \frac{C^2}{4\epsilon T^2} + \frac{C^2}{4\epsilon} (K^2 + L^4) M \right) \]
\[ + \alpha(1 - p) \frac{M}{4p - 1} \|\nabla \Delta \xi\|_{\infty}^2, \]

where
\[ D_1 = (- \hat{A} \alpha \psi^2 + \alpha' \psi^2 - (p - 1) \hat{A} \alpha \psi^2 + \psi^2 \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2 - p}{p - 1}} M \) \]
\[ D_2 = \left( 2 \hat{A} \psi^2 - \psi^2 (p - 1) \hat{A} + \left( \alpha(1 - p) \psi^2 M \frac{1}{p} \left( \frac{M}{p} \right)^{\frac{2 - p}{p - 1}} \right) \right) \]

We can simplify (4.15),
\[ 0 \leq \mathcal{L}(G) \leq \left( \frac{1}{\alpha^2} - \frac{2(1 - \alpha)}{\alpha^2} \right) \tilde{G}^2 + \psi^2 \frac{1}{\alpha^2} |\nabla v|^4 \frac{2(\alpha - 1)}{\alpha^2} \tilde{F} \frac{|\nabla v|^2}{v} + \psi^2 \left( \frac{1}{\tilde{a}} (\frac{\alpha - 1}{\alpha})^2 - 2 \right) \frac{1}{\alpha} \tilde{g} \]
\[ + \psi^2 \frac{p}{(1 - p)^2} M^2 \frac{|\nabla v||\nabla \psi|}{v^2} G \]
\[ + D_1 \frac{F}{\alpha} + \left( (3M + 1)\psi G^2 + \frac{C^4_2 M}{2\epsilon R^2} + \frac{C^2}{4\epsilon T^2} + \frac{C^2}{4\epsilon} (K^2 + L^4) M \right) \]
\[ + \alpha(1 - p) \frac{M}{4p - 1} \|\nabla \Delta \xi\|_{\infty}^2 + \psi^2 \frac{2}{m(p - 1)} \left( 1 + \frac{1}{\epsilon} \right) \tilde{A}^2. \]

Similar to the derivation of (2.21) in [13]
\[ \psi^2 (D_1 + D_2 + 2MK) \frac{|\nabla v|^2}{v^2} \]
\[ \leq - \epsilon_1 \psi^2 \frac{1}{\alpha^2} (1 - \alpha) (1 - \alpha - \tilde{a}) \frac{|\nabla v|^4}{v^2} - \frac{1}{\epsilon_1} \tilde{a} \alpha^2 (p - 1)^2 \psi^2 (D_1 + D_2 + 2MK)^2 \frac{2}{(1 - \alpha)(1 - \alpha - \tilde{a})}. \]
and

$$
2 \frac{p}{(1 - p)^2} M^2 \psi^2 G \frac{|\nabla v|}{(-v)^2} \psi^2, \\
\leq - \varepsilon_2 \frac{2}{\bar{a} \alpha^2} (1 - \bar{a})(1 - \alpha) \psi G \frac{|\nabla v|^2}{-v} - \frac{\bar{a} \alpha^2 p^2 M}{2 \varepsilon_2 (1 - \bar{a})(1 - \alpha)(1 - p)} \frac{|\nabla \psi|^2}{\psi} G.
$$

(4.18)

thus, plugging the above inequalities into (3.28), we can get

$$
0 \leq \mathcal{L}(G) \leq \left( \frac{1}{\bar{a} \alpha^2} - \frac{1 - \alpha}{\alpha^2} \right) \mathcal{G}^2 - \frac{1}{\bar{a} \alpha^2} 2 \psi (1 + \varepsilon_2) ((1 - \bar{a})(1 - \alpha) \psi G) \frac{|\nabla v|^2}{-v} \\
+ (1 - \varepsilon_1) \frac{\psi^2}{\bar{a} \alpha^2} (1 - \alpha)(1 - \alpha - \bar{a}) \frac{|\nabla v|^4}{v^2} \\
- D_{11} \frac{G}{\alpha} + \left( (3M + 1) \varepsilon v G^2 + \frac{C^2}{2 \epsilon} \frac{M}{R^4} + \frac{C^2}{4 \epsilon} \frac{1}{T^2} + \frac{C^2}{4 \epsilon} (K^2 + L^4) M \right) \\
+ \alpha (p - 1) \frac{1}{4^p - 1} \left\| \nabla \Delta \xi \right\|_\infty^2 + \psi^2 \left( \frac{2}{m(p - 1)} \right)(1 + \frac{1}{\epsilon}) \hat{A}^2. \\
- \frac{1}{\epsilon_1} \bar{a} \alpha^2 (p - 1)^2 \psi^2 (D_1 + D_2 + 2MK)^2 \\
- \frac{\bar{a} \alpha^2 p^2 M}{2 \varepsilon_2 (1 - \bar{a})(1 - \alpha)(1 - p)} \frac{|\nabla \psi|^2}{\psi} G,
$$

(4.19)

where $D_{11} = (-\hat{A} \alpha \psi^2 + \alpha' \psi^2 - (p - 1) \hat{A} \alpha \psi^2 + \psi^2 \alpha \frac{M}{p} \left( \frac{M}{p} \right)^{\frac{p}{p-1}} M)$. As that in [13, (2.22)], we take $\varepsilon_1, \varepsilon_2$ such that

$$
[1 - \bar{a}(1 - \alpha)] - \frac{(1 + \varepsilon_2)^2 (1 - \bar{a})^2 (1 - \alpha)}{(1 - \varepsilon_1)(1 - \alpha - \bar{a})} := A(\varepsilon_1, \varepsilon_2) > 0.
$$

Similar to the derivation of (2.21) in [13], we deal with the first three terms on the right hand side (4.19), notice that $\bar{a} < 0$, we have

$$
\frac{1}{-\bar{a} \alpha^2} A(\varepsilon_1, \varepsilon_2) \mathcal{G}^2
$$

$$
\leq \frac{D_{11}}{\alpha} G + \left( (3M + 1) \varepsilon v G^2 + \frac{C^2}{2 \epsilon} \frac{M}{R^4} + \frac{C^2}{4 \epsilon} \frac{1}{T^2} + \frac{C^2}{4 \epsilon} (K^2 + L^4) M \right) \\
+ \alpha (p - 1) \frac{1}{4^p - 1} \left\| \nabla \Delta \xi \right\|_\infty^2 - \psi^2 \left( \frac{2}{m(p - 1)} \right)(1 + \frac{1}{\epsilon}) \hat{A}^2. \\
- \frac{1}{\epsilon_1} \bar{a} \alpha^2 (p - 1)^2 \psi^2 (D_1 + D_2 + 2MK)^2 \\
- \frac{\bar{a} \alpha^2 p^2 M}{2 \varepsilon_2 (1 - \bar{a})(1 - \alpha)(1 - p)} \frac{|\nabla \psi|^2}{\psi} G,
$$

(4.20)

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which can be rewritten as

\[
\begin{aligned}
&\leq \left( \frac{D_{11}}{\alpha} - \frac{\tilde{a}a^2 p^2 M}{2\varepsilon_2 (1 - \tilde{a})(1 - \alpha)(1 - p) R^2} C \right) G \\
&+ \left( \frac{C^2}{4\varepsilon} R^4 + \frac{C^2}{4\varepsilon} \frac{1}{T^2} + \frac{C^2}{4\varepsilon} (K^2 + L^4) M \right) \\
&+ \alpha (p - 1) \frac{M}{4p - 1} \| \nabla \Delta \xi \|^2 - \psi^2(\frac{2}{m(p - 1)})(1 - \frac{1}{\varepsilon}) \hat{A}^2 \\
&- \frac{1}{\varepsilon_1} \tilde{a}a^2 (p - 1)^2 \psi^2 (D_1 + D_2 + 2MK)^2)
\end{aligned}
\]

By the inequality \( ax^2 - bx - c \leq 0 \), we have \( x \leq \frac{b}{a} + \left( \frac{c}{a} \right)^{\frac{1}{2}} \). This implies that at \((x_1, t_1)\), we have

\[
G \leq \left[ \frac{D_{11}}{\alpha} - \frac{\tilde{a}a^2 p^2 M}{2\varepsilon_2 (1 - \tilde{a})(1 - \alpha)(1 - p) R^2} C \right] \\
+ \left( \frac{1}{\Psi} \left( \frac{C^2}{4\varepsilon} R^4 + \frac{C^2}{4\varepsilon} \frac{1}{T^2} + \frac{C^2}{4\varepsilon} (K^2 + L^4) M \right) \\
+ \alpha (p - 1) \frac{M}{4p - 1} \| \nabla \Delta \xi \|^2 - \psi^2(\frac{2}{m(p - 1)})(1 - \frac{1}{\varepsilon}) \hat{A}^2 \\
- \frac{1}{\varepsilon_1} \tilde{a}a^2 (p - 1)^2 \psi^2 (D_1 + D_2 + 2MK)^2]) \right]^\frac{1}{2},
\]

where \( \Psi = \frac{1}{\tilde{a}a^2} A(\varepsilon_1, \varepsilon_2) - (3M + 1) \frac{M}{1 - p} \) is positive if \( \varepsilon \) is sufficiently small.

**Case 2:** the maximal point \((x_1, t_1) \in \partial M \times [-T, 0)\). On \( \partial M \times [-T, 0)\), we have assumed that

\[
u \leq \frac{p - 1}{p} A(u), \quad u_t = 0, \partial M \times [-T, 0).
\]

Hence,

\[
f_t + q + \hat{A} \leq 0,
\]

at \((x_1, t_1)\), by Hopf maximum principle, we have

\[
F_\nu = (\frac{1}{\partial v})^2 \geq 0.
\]

Note that

\[
|\nabla v|^2 = v_\nu^2, on \quad \partial M \times [-T, 0),
\]
and
\[ \frac{\partial_t v}{v} = (p - 1)\Delta v + \frac{|\nabla v|^2}{v} + (p - 1)\hat{A}(u). \]

Notice that
\[ \left( \frac{|\nabla v|^2}{v} \right)_v = \frac{(|\nabla v|^2)_v - |\nabla v|^2}{v^2} v_v = \frac{(|\nabla v|^2)_v}{v^2} - \frac{v^2_v}{v^2} v_v = \frac{v_v(\Delta v - H v_v)}{v} - \frac{v^2_v}{v^2}. \]

Thus, we have
\[ \frac{v_v}{(p - 1)v} \left( \frac{v_v}{v} - (p - 1)\hat{A} - \frac{|\nabla v|^2}{v} \right) - H v_v \frac{v^2_v}{v} - \frac{v^2_v}{v^2} \leq 0. \]

Since \( H \leq -L \), we get
\[ -\frac{1}{(p - 1)} v^3_v + vv^2_v L - v^2_v \leq 0. \]

Hence, we have
\[ \frac{1}{1 - p} v^3_v + LM \frac{v^2_v}{1 - p} - v^2_v \leq 0. \]

Thus there exists a constant \( C' \) such that
\[ 0 < v_v \leq 1 - p - LM. \] (4.22)

So at \((x_1, t_1) \in \partial M \times [-T, 0)\), we have
\[ F \leq (\inf(-v))^{-1} \left( \frac{LM}{p - 1} - p + 1 \right)^2 \] (4.23)

**Remark 4.2.** From the (5.4), we see that if \( L = 0 \), then the maximum point cannot occur on the boundary.
5 Souplet-Zhang type estimates for \( p > 1 \)

**Theorem 5.1.** Let \((M, g, e^{-\xi} dv_g)\) be an \(n\)-dimensional complete noncompact metric measure space with compact boundary. For \( K \geq 0 \), we assume \( \text{Ric}_\xi^m \geq -K \) and \( H_{\partial M, \xi} \geq L \). Let \( u \) be a positive solution to the heat equation (1.1) on \( Q_R, T := B_R(\partial M) \times [-T, 0] \) with \( p \in (1, 1 + \frac{1}{\sqrt{2n+1}}) \). For \( A > 0 \), let us assume \( u < A \). We further assume that \( u \) satisfies the Dirichlet boundary condition (i.e., \( u(\cdot, t) |_{\partial M} \) is constant for each fixed \( t \in [-T, 0] \)), and \( u, v \geq 0 \) and \( \partial_t u \leq 0 \) over \( \partial M \times [-T, 0] \). Then there exists a positive constant \( C_n > 0 \) depending only on \( n \) such that on \( Q_{R/2, T/4} \),

\[
\frac{|\nabla v|}{v^{\frac{2}{p}}} \leq \frac{2K(1-p)(p-1)^{\beta-2}M^{2-\beta} + \left((\beta + 2)|\lambda| + \beta |\hat{A}|\right)(p-1)\gamma v^{1-\beta}}{\frac{3}{4} - (3M + 1)\gamma M^{1-\beta} - \frac{3}{4}\epsilon^4 |\nabla \xi|^{\frac{4}{3}}} + \left(\frac{\mathcal{O}}{\frac{3}{4} - (3M + 1)\gamma M^{1-\beta} - \frac{3}{4}\epsilon^4 |\nabla \xi|^{\frac{4}{3}}}\right)^{\frac{1}{2}} + \frac{Lv^{1-\frac{4}{3}}}{\beta + \frac{1}{M}},
\]

where \( v = \frac{p}{p-1} w^{p-1}, M := (p-1) \sup_{Q_{R,T}(\partial M)} v, \beta \) is a constant satisfies \( \beta^2 - \frac{p-2}{p-1} \beta + \frac{n}{2} < 0 \) and

\[
\mathcal{O} = \left(\frac{C^2_4}{2\epsilon} M^4 + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C^2_4}{4\epsilon} (K^2 + L^4) M\right)\gamma M^{1-\beta} + \gamma \left(2 - 2(1-p)\beta\right)^4 M^{4-2\beta} \frac{1}{R^4} + \frac{1}{4\epsilon^4} \left(2 \left(\frac{M}{1-p}\right)^{2-\frac{4}{3}} M^2\right)^4.
\]

**Remark 5.1.** When we take special \( A(u) \) and \( \lambda \), the upper bound can be simplified.

**Proof.** Let \( \omega = \frac{|\nabla v|^2}{v^{\beta}} \), by (1.1), we have

\[
\partial_t v = (p-1) v \Delta \xi v + |\nabla v|^2 + \lambda (p-1) v + (p-1) v \hat{A}(u), p > 1.
\]
Computing directly, we get

\[
\begin{align*}
   w_t &= \frac{(\partial_t |\nabla v|^2)}{v^{\beta}} - \beta \frac{|\nabla v|^2 v_t}{v^{\beta+1}} \\
   &= \frac{2 \left\langle \nabla v, \nabla \left( (p-1)v\Delta_x v + |\nabla v|^2 + \lambda (p-1)v + (p-1)v \hat{A}(u) \right) \right\rangle}{v^{\beta}} \\
   &\quad - \beta \frac{\left| \nabla v \right|^2 \left( (p-1)v\Delta_x v + |\nabla v|^2 + \lambda (p-1)v + (p-1)v \hat{A}(u) \right)}{v^{\beta+1}} \\
   &= \frac{2 \left\langle \nabla v, \nabla \left( (p-1)v\Delta_x v + |\nabla v|^2 \right) \right\rangle}{v^{\beta}} + \frac{2 \left\langle \nabla v, \lambda (p-1)\nabla v + ((p-1)\hat{A}\nabla v + (p-1)v \nabla \hat{A}(u)) \right\rangle}{v^{\beta}} \\
   &\quad - \beta \frac{\left| \nabla v \right|^2 (\lambda (p-1)v + (p-1)v \hat{A})}{v^{\beta+1}}.
\end{align*}
\]

It follows that

\[
\mathcal{L} \omega = ((p-1)v\Delta_x \omega - \partial_t) \omega
\]

\[
= \left( (p-1)v \left( \frac{\Delta_x |\nabla v|^2}{v^{\beta}} - 2(1) \frac{\left\langle \nabla v, \nabla |\nabla v|^2 \right\rangle}{v^{\beta+1}} - \beta \frac{\left| \nabla v \right|^2 \Delta_x v}{v^{\beta+1}} + \beta (1) \frac{\left| \nabla v \right|^4}{v^{\beta+2}} \right) \right)
\]
\[
- \left\{ \frac{2 \left\langle \nabla v, \nabla \left( (p-1)v\Delta_x v + |\nabla v|^2 \right) \right\rangle}{v^{\beta}} + \frac{2 \left\langle \nabla v, \lambda (p-1)\nabla v + ((p-1)\hat{A}\nabla v + (p-1)v \nabla \hat{A}(u)) \right\rangle}{v^{\beta}} \right\}
\]
\[
- \beta \frac{\left| \nabla v \right|^2 (\lambda (p-1)v + (p-1)v \hat{A})}{v^{\beta+1}}
\]
\[
= \left( (p-1)v \frac{\Delta_x |\nabla v|^2}{v^{\beta}} - 2(p-1)v \beta \frac{\left\langle \nabla v, \nabla |\nabla v|^2 \right\rangle}{v^{\beta+1}} - (p-1)v \beta \frac{\left| \nabla v \right|^2 \Delta_x v}{v^{\beta+1}} + (p-1)v \beta (1) \frac{\left| \nabla v \right|^4}{v^{\beta+2}} \right)
\]
\[
- \left\{ \frac{2 \left\langle \nabla v, \nabla \left( (p-1)v\Delta_x v + |\nabla v|^2 \right) \right\rangle}{v^{\beta}} \right\}
\]
\[
- \beta \frac{\left| \nabla v \right|^2 (\lambda (p-1)v + (p-1)v \hat{A})}{v^{\beta+1}}
\]
\[
- \beta \frac{2 \left\langle \nabla v, \lambda (p-1)\nabla v + ((p-1)\hat{A}\nabla v + (p-1)v \nabla \hat{A}(u)) \right\rangle}{v^{\beta}} \right\}.
\]

So, we get
\[
\mathcal{L} \omega = \left( (p-1)v \frac{\Delta \xi |\nabla v|^2}{v^{\beta+1}} - 2(p-1)v \beta \frac{\langle \nabla v, \nabla |\nabla v|^2 \rangle}{v^{\beta+1}} + (p-1)v \beta (\beta + 1) \frac{|\nabla v|^4}{v^{\beta+2}} \right) \\
- 2(p-1) \frac{\langle \nabla v, \nabla (v \Delta \xi v) \rangle}{v^\beta} - 2 \frac{\langle \nabla v, \nabla |\nabla v|^2 \rangle}{v^\beta} \\
+ \beta \frac{|\nabla v|^4}{v^{\beta+1}} \\
- \beta \frac{|\nabla v|^2 \left( \lambda (p-1)v + (p-1)v \hat{A} \right)}{v^{\beta+1}} - 2 \frac{\langle \nabla v, \lambda (p-1) \nabla v + \lambda (p-1) \nabla (v \hat{A} (u)) \rangle}{v^\beta} \\
= \left( (p-1)v \frac{\Delta \xi |\nabla v|^2}{v^{\beta-1}} + (2 - 2(p-1)\beta) \frac{\langle \nabla v, \nabla |\nabla v|^2 \rangle}{v^\beta} \right) \\
+ ((p-1)\beta (\beta + 1) - \beta) \frac{|\nabla v|^4}{v^{\beta+1}} - 2(p-1) \frac{\langle \nabla v, (\nabla \Delta \xi v) \rangle}{v^{\beta-1}} - 2(p-1) \frac{|\nabla v|^2 \Delta \xi v}{v^\beta} \\
+ \Phi_1,
\]

where
\[
\Phi_1 = - \frac{|\nabla v|^2 \left( \lambda (p-1)v + (p-1)v \hat{A} \right)}{v^{\beta+1}} - 2 \frac{\langle \nabla v, \lambda (p-1) \nabla v + \lambda (p-1) \nabla (v \hat{A} (u)) \rangle}{v^\beta}.
\]

By Bochner formula, we have
\[
= \left( (p-1)v \frac{\nabla^2 |v|^2}{v^{\beta-1}} + \text{Ric} \langle \nabla v, \nabla v \rangle + \langle \nabla v, \nabla \Delta \xi v \rangle \right) + (2 - 2(p-1)\beta) \frac{\langle \nabla v, \nabla |\nabla v|^2 \rangle}{v^\beta} \\
+ ((p-1)\beta (\beta + 1) - \beta) \frac{|\nabla v|^4}{v^{\beta+1}} - 2(p-1) \frac{\langle \nabla v, (\nabla \Delta \xi v) \rangle}{v^{\beta-1}} - 2(p-1) \frac{|\nabla v|^2 \Delta \xi v}{v^\beta} \\
+ \Phi_1
\]
\[
= \left( 2(p-1) \frac{|\nabla^2 |v|^2| + \text{Ric} \langle \nabla v, \nabla v \rangle}{v^{\beta-1}} + (2 - 2(p-1)\beta) \langle \nabla \omega, \nabla v \rangle \right) \\
+ \left( (2 - 2(p-1)\beta) \frac{|\nabla v|^4}{v^{\beta+1}} + ((p-1)\beta (\beta + 1) - \beta) \frac{|\nabla v|^4}{v^{\beta+1}} - 2(p-1) \frac{|\nabla v|^2 \Delta \xi v}{v^\beta} \right) + \Phi_1
\]
\[
= \left( 2(p-1) \frac{\text{Ric} \langle \nabla v, \nabla v \rangle}{v^{\beta-1}} + (2 - 2(p-1)\beta) \langle \nabla \omega, \nabla v \rangle \right) \\
+ \left( \beta (2 - 2(p-1)\beta) + (p-1)\beta (\beta + 1) - \beta \right) \frac{|\nabla v|^4}{v^{\beta+1}} - 2(p-1) \frac{|\nabla v|^2 \Delta \xi v}{v^{\beta+1}} + 2(p-1) \frac{|\nabla^2 v|^2}{v^{\beta-1}}
\]
\begin{align*}
+ \beta \lambda (p-1) \frac{|\nabla v|^2}{v^\beta} & - 2\lambda (p-1) \frac{\langle \nabla v, \nabla v \rangle}{v^\beta} \\
\geq \left( 2(p-1) \frac{\operatorname{Ric}^m_{V}(\nabla v, \nabla v)}{v^{\beta-1}} + (2 - 2(p-1) \beta) \langle \langle \nabla \omega, \nabla v \rangle \rangle \\
+ \left( \beta (2 - 2(p-1) \beta) + (p-1) \beta (\beta + 1) - \beta \right) \frac{|\nabla v|^4}{v^{\beta+1}} - 2(p-1) \frac{|\nabla v|^2 \Delta v}{v^\beta} + 2(p-1) \frac{1}{m} \frac{|\Delta v|^2}{v^{\beta-1}} \right) \\
+ \Phi_1. 
\end{align*}

By Young’s inequality, the above formula is

\begin{align*}
= \left( 2(p-1) \frac{\operatorname{Ric}^m_{V}(\nabla v, \nabla v)}{v^{\beta-1}} + (2 + 2(p-1) \beta) \langle \langle \nabla \omega, \nabla v \rangle \rangle \\
+ \left( \beta (2 + 2(p-1) \beta) - \frac{m(p-1)}{2} + (p-1) \beta (\beta + 1) + \beta \right) \frac{|\nabla v|^4}{v^{\beta+1}} - 2(p-1) \frac{\langle V, \nabla v \rangle |\nabla v|^2}{v^{\beta-1}} \\
+ \Phi_1 - 2(p-1) \frac{\langle \nabla f, \nabla v \rangle |\nabla v|^2}{v^{\beta-1}}. 
\end{align*}

That is

\begin{align*}
\mathcal{L} \omega \geq -2K(p-1) v \omega + (2 + 2(p-1) \beta) \langle \langle \nabla \omega, \nabla v \rangle \rangle + \frac{1}{\gamma} v^{\beta-1} \omega^2 \\
+ \left( \beta (\lambda - \hat{A})(p-1) - 2\lambda (p-1) - 2K \right) \omega - 2(p-1) \frac{\langle \nabla f, \nabla v \rangle |\nabla v|^2}{v^{\beta-1}} 
\end{align*}

where

\begin{equation}
\frac{1}{\gamma} = \beta (2 + 2(p-1) \beta) - \frac{m(p-1)}{2} + (p-1) \beta (\beta + 1) + \beta = -(p-1)(\beta^2 - \frac{p-2}{p-1} \beta + \frac{n}{2}),
\end{equation}

which is positive since \( p \in (1, 1 + \frac{1}{\sqrt{2n+1}}) \). One can refer to [34, page 203].

Next we use the cut off function \( \psi \) in Lemma 2.1, set \( G = \psi \omega \) we may assume \( G \) attains its maximum point at \( (x_1, t_1) \) in \( Q_{R,T}(\partial M) \). Firstly, we consider the case that \( x_1 \notin \partial M \), by maximum principle, we have

\[ \Delta \xi (G) \leq 0, G_t \leq 0 \quad \text{and} \quad \nabla (\psi \omega) = 0, \]
at \((x_1, t_1)\), we have
\[
0 \geq \mathcal{L}(\psi \omega) = \psi \mathcal{L}(\omega) + \omega \mathcal{L}(\psi) + 2(1 - p)v \langle \nabla \psi, \nabla \omega \rangle.
\]

Multiplying both sides by \(\psi\), we have
\[
0 \geq \mathcal{L}(\psi \omega) = \psi^2 \mathcal{L}(\omega) + G \mathcal{L}(\psi) + 2(1 - p)v \langle \nabla \psi, \nabla \psi \rangle G.
\]

By Theorem 6.1 in [24] or Theorem 2.1 in [8], we have
\[
\gamma ((1 - p)v \Delta \xi \psi) G \leq (1 - p)vG \left( \frac{(KR + L)^2}{4} \frac{1}{\psi} \right) + 2(1 - p)v \langle \nabla \psi, \nabla \psi \rangle G.
\]

Thus, we have
\[
\left( ((1 - p)v \Delta \xi - \partial_t) \psi + 2(1 - p)v \langle \nabla \psi, \nabla \psi \rangle \frac{1}{\psi} \right) G
\leq (3M + 1)\epsilon \psi G^2 + \frac{C^2}{4\epsilon} \left( \frac{M}{1 - p} \right)^{2 - 2\beta} \frac{M^2}{R^4} \psi^{1/2},
\]

where the constant \(C_{\frac{3}{4}}\) is the same as that in Lemma 2.1.

\[
-\gamma v^{1-\beta} G \partial_t \psi \leq \epsilon \psi G^2 + \frac{1}{4} \frac{\epsilon \psi}{\psi} \leq 3\epsilon \psi G^2 + \frac{C^2}{4\epsilon} \frac{1}{T^2},
\]

where the constant \(C\) is the same as that in Lemma 2.1.

Thus, we have
\[
\gamma 2(1 - p)v^{2-\beta} \langle \nabla \psi, \nabla \psi \rangle \frac{1}{\psi} G \leq \epsilon \psi G^2 + \gamma^2 \frac{C^2}{4\epsilon} \left( \frac{M}{1 - p} \right)^{2 - \beta} M^2 \frac{\psi^{1/2}}{R^4},
\]

where the constant \(C_{\frac{3}{4}}\) is the same as that in Lemma 2.1.
Multiplying both sides by $\gamma v^{1-\beta}$, we get
\[
G^2 \leq (3M + 1)\gamma v^{1-\beta} \epsilon \psi G^2 + \left( \frac{C^2}{4} \frac{M}{2\epsilon R^4} + \frac{C^2}{4} \frac{1}{4\epsilon T^2} + \frac{C^2}{4} \frac{(K^2 + L^4) M}{4\epsilon} \right) \gamma v^{1-\beta} + 2K(1-p)v^{2-\beta}\psi G + \gamma v^{1-\beta} (2 - 2(1-p)\beta) G (\langle \nabla \psi, \nabla v \rangle)
\]
\[
+ 2\gamma v^{1-\beta} \omega^2 (1-p) \left( \frac{\nabla f, \nabla v}{v^{\beta-1}} \right) |\nabla v|^2 - \left( \beta(\lambda - \hat{A})(1-p) - 2\lambda(1-p) - 2K \right) \gamma v^{1-\beta} \psi G.
\]

(5.3)

We can estimate the terms on the right hand side of (5.3) as that in [33, (2.4)-(2.10)] or [34, (2.6)-(2.10)]
\[
2K(p-1)v^{2-\beta}\psi G \leq 2K(p-1)(p-1)^{\beta-2} M^{2-\beta} \psi G,
\]
and
\[
\gamma v^{1-\beta} (2 + 2(p-1)\beta) G (\nabla \psi, \nabla v) \leq \gamma \left( \frac{M}{p-1} \right)^{1-\beta} (2 + 2(p-1)\beta) G |\nabla \psi| |\nabla v| \leq \gamma M^{1-\beta} (2 + 2(p-1)\beta) G |\nabla \psi| \omega \frac{\omega}{v^{\beta}} = \gamma M^{1-\beta} (2 + 2(p-1)\beta) \psi |\nabla \psi| \omega \frac{\omega}{v^{\beta}} \leq \frac{3}{4} G^2 + \gamma^4 (2 + 2(p-1)\beta)^4 M^{2-\beta} \frac{1}{R^4}.
\]

We also have
\[
2\gamma \omega^2 v^{1-\beta} (p-1) \left( \frac{\nabla \xi, \nabla v}{v^{\beta-1}} \right) |\nabla v|^2 = 2\omega^2 v^{2-\beta} (1-p) \left( \frac{\nabla \xi, \nabla v}{v^{\beta}} \right) |\nabla v|^2 = 2\omega^2 v^{2-\beta} (1-p) \langle \nabla \xi, \nabla v \rangle \omega \leq 2\frac{1}{C_1} (\frac{M}{1-p})^{2-\frac{3}{2}\beta} M^2 |\nabla \xi| \psi \frac{1}{\sqrt{\beta}} G^\frac{3}{2}
\]
\[
\leq \frac{3}{4} \epsilon \frac{\psi}{v^{\beta}} |\nabla \xi| \frac{1}{\sqrt{\beta}} G^2 + \frac{1}{4\epsilon C_1} \left( 2(\frac{M}{1-p})^{2-\frac{3}{2}\beta} M^2 \right)^4.
\]

According to the above inequalities, we have
\[
G^2 \leq (3M + 1)\gamma v^{1-\beta} \epsilon \psi G^2 + \left( \frac{C^2}{2\epsilon} \frac{M}{R^4} + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C^2}{4\epsilon} \frac{(K^2 + L^4) M}{4\epsilon} \right) \gamma v^{1-\beta} + 2K(1-p)(p-1)^{\beta-2} M^{2-\beta} \psi G + \frac{1}{4} G^2 + C \gamma (2 - 2(1-p)\beta) M^{4-\beta} \frac{1}{R^4}
\]
\[
+ 2\gamma v^{1-\beta} \omega^2 (1-p) \left( \frac{\nabla f, \nabla v}{v^{\beta-1}} \right) |\nabla v|^2 - \left( \beta(\lambda - \hat{A})(1-p) - 2\lambda(1-p) - 2K \right) \gamma v^{1-\beta} \psi G.
\]

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According to the above inequalities, we have
\[
G^2 \leq (3M + 1)\gamma v^{1-\beta} \psi G^2 + \left( \frac{C^2}{2\epsilon} M^4 + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C^2}{4\epsilon} \left( K^2 + L^4 \right) M \right) \gamma v^{1-\beta} \\
+ 2K(1-p)(p-1)^{2-\beta} M^2 \psi G + \frac{1}{4} G^2 + C\gamma (2 - 2(1-p)\beta) M^{4-2\beta} \frac{1}{R^4} \\
+ \frac{3}{4} \epsilon \frac{1}{2} |V|^\frac{1}{2} \psi^2 G^2 + \frac{1}{4\epsilon^4 C_1^4} \left( 2 \left( \frac{M}{1-p} \right)^2 - \frac{3}{4} \beta M^2 \right)^4 \\
- \left( \beta (\lambda - \hat{A})(1-p) - 2\lambda(1-p) \right) \gamma v^{1-\beta} \psi G.
\]
We can infer that
\[
\left( \frac{3}{4} - (3M + 1)\gamma M^{1-\beta} \epsilon - \frac{3}{4} \epsilon \frac{1}{2} |V|^\frac{1}{2} \right) G^2 \\
\leq \left( \frac{C^2}{2\epsilon} M^4 + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C^2}{4\epsilon} \left( K^2 + L^4 \right) M \right) \gamma M^{1-\beta} \\
+ C\gamma (2 - 2(1-p)\beta) M^{4-2\beta} \frac{1}{R^4} + \frac{1}{4\epsilon^4 C_1^4} \left( 2 \left( \frac{M}{1-p} \right)^2 - \frac{3}{4} \beta M^2 \right)^4 \\
+ \left[ 2K(1-p)(p-1)^{2-\beta} M^{2-\beta} - \left( \beta (\lambda - \hat{A}) - 2\lambda \right) (1-p) \gamma v^{1-\beta} \right] \psi G.
\]
By quadratic formula, we have
\[
G(x_1, t_1) \leq \frac{2K(1-p)(p-1)^{2-\beta} M^{2-\beta} - \left( \beta (\lambda - \hat{A}) - 2\lambda \right) (1-p) \gamma v^{1-\beta}}{\frac{1}{4} - (3M + 1)\gamma M^{1-\beta} \epsilon - \frac{3}{4} \epsilon \frac{1}{2} |V|^\frac{1}{2}} \\
+ \frac{\mathcal{O}}{\left( \frac{1}{4} - (3M + 1)\gamma M^{1-\beta} \epsilon - \frac{3}{4} \epsilon \frac{1}{2} |V|^\frac{1}{2} \right)^{\frac{1}{2}}},
\]
where
\[
\mathcal{O} = \left( \frac{C^2}{2\epsilon} M^4 + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C^2}{4\epsilon} \left( K^2 + L^4 \right) M \right) \gamma M^{1-\beta} \\
+ C\gamma (2 - 2(1-p)\beta) M^{4-2\beta} \frac{1}{R^4} + \frac{1}{4\epsilon^4 C_1^4} \left( 2 \left( \frac{M}{1-p} \right)^2 - \frac{3}{4} \beta M^2 \right)^4.
\]
Secondly, we consider the case that \( x_1 \in \partial M \)
\[
0 \leq \omega_n = -\beta v^{-\beta-1} v_n |\nabla v|^2 + v^{-\beta} (|\nabla v|^2)_n \\
= -\beta v^{-\beta-1} v_n^3 + v^{-\beta} v_n (\Delta v - H_v v) \\
= -\beta v^{-\beta-1} v_n^3 + v^{-\beta} v_n \left( \frac{\partial v}{(1-p)v} + \frac{|\nabla v|^2}{(1-p)v} - \lambda - \hat{A} - H_v v \right).\]
Since
\[(p - 1)\frac{u_t}{u} \leq \lambda + \hat{A}.
\]
Thus, we have
\[\beta v^{-\beta - 1}v^3_n + v^{-\beta}v_n(-\frac{|\nabla v|^2}{(1 - p)v} - L v_n) \leq 0.\]
which yields,
\[(\beta - \frac{1}{(1 - p)v})v^{-\beta - 1}v^3_n - L v^{-\beta}v^2_n \leq 0,\]
which is
\[(\beta - \frac{1}{(1 - p)v})v^{-1}v_n - L \leq 0.\]
So, we have
\[\omega_{\frac{1}{2}} \leq \frac{Lv^{1 - \frac{\beta}{2}}}{\beta + \frac{1}{M}}.\]

6 Souplet-Zhang type estimates for \(p < 1\)

**Theorem 6.1.** Let \((M, g, e^{-\xi}dv_g)\) be an \(n\)-dimensional complete noncompact metric measure space with compact boundary. For \(K \geq 0\), we assume \(\text{Ric}_\xi^n \geq -K\) and \(H_{\partial M, \xi} \geq L\). Let \(u\) be a positive solution to the heat equation (1.1) on \(Q_{R,T}(\partial M) := B_R(\partial M) \times [-T, 0]\) with \(p \in (1 - \frac{2}{n}, 1)\). For \(A > 0\), let us assume \(u < A\). We further assume that \(u\) satisfies the Dirichlet boundary condition (i.e., \(u(\cdot, t)\big|_{\partial M}\) is constant for each fixed \(t \in [-T, 0]\)), and \(u_\nu \geq 0\) and \(\partial_t u \leq 0\) over \(\partial M \times [-T, 0]\). Then there exists a positive constant \(C_n > 0\) depending only on \(n\) such that on \(Q_{R/2,T/4}\),

\[\frac{|\nabla v|}{v^{\frac{\beta}{2}}} \leq \frac{2K(1-p)(p-1)^{\beta - 2}M^{2-\beta} - (\beta(\lambda - \hat{A}) - 2\lambda) (1-p)\gamma v^{1-\beta}}{\frac{3}{4} - (3M + 1)\gamma M^{1-\beta}\epsilon - \frac{\beta}{2}\epsilon^\frac{4}{7}|V|^{\frac{4}{7}}} \]
\[+ \left(\frac{O}{\frac{3}{4} - (3M + 1)\gamma M^{1-\beta}\epsilon - \frac{\beta}{2}\epsilon^\frac{4}{7}|V|^{\frac{4}{7}}}\right)^{\frac{1}{2}} + \frac{Lv^{1 - \frac{\beta}{2}}}{\beta + \frac{1}{M}},\]
where \( v = -\frac{p}{p-1}u^{p-1} \), \( M := (1-p) \sup_{Q_{r,t}(\partial M)} v \), \( \beta \) is an arbitrary constant which satisfies \( \beta^2 - \frac{2-p}{1-p}\beta + \frac{n}{2} < 0 \), \( O \) is defined by

\[
\left( \frac{C^2}{2} \frac{M}{R^4} + \frac{C^2}{4\epsilon} \frac{1}{T^2} + \frac{C^2}{4\epsilon} (K^2 + L^4) \right) \gamma M^{1-\beta} + \frac{\beta}{\gamma} (2 - 2(1-p)\beta)^4 M^{4-2\beta} \left( \frac{1}{R^4} + \frac{1}{4\epsilon^4} \right) \left( \frac{2(1-p)}{1-p} \right)^{2 - \frac{2}{\beta}} M^2.
\]

**Remark 6.1.** When we take special \( A(u) \) and \( \lambda \), the upper bound can be simplified and get the corollary.

**Proof.** In this section, we let \( 1 - \frac{2}{n} < p < 1 \), we set

\[
v = -\frac{p}{p-1}u^{p-1}, \quad \omega = \frac{\|\nabla v\|^2}{v^{\beta}}
\]

Let \( \mathcal{L} \omega = ((1-p)v\Delta - \partial_t)\omega \). thus

\[
\partial_t v = (1-p)v\Delta v - |\nabla v|^2 + \lambda(p-1)v + (p-1)v\hat{A}, p < 1. \tag{6.1}
\]

Now we can do the similar caculations as in section 5 , by (6.1),

\[
w_t = \frac{(\partial_t |\nabla v|^2)}{v^{\beta}} - \beta \frac{|\nabla v|^2 v_t}{v^{\beta+1}} = 2 \frac{\langle \nabla v, \nabla ((1-p)v\Delta v - |\nabla v|^2 + \lambda(p-1)v + (p-1)v\hat{A}) \rangle}{v^{\beta}} - \beta \frac{|\nabla v|^2 ((1-p)v\Delta v - |\nabla v|^2 + \lambda(1-p)v + (1-p)v\hat{A})}{v^{\beta+1}}
\]

\[
= 2 \frac{\langle \nabla v, \nabla ((1-p)v\Delta v - |\nabla v|^2) \rangle}{v^{\beta}} + 2 \frac{\langle \nabla v, \lambda(p-1)\nabla v + (p-1)\nabla (v\hat{A}(u)) \rangle}{v^{\beta}}
\]

\[
- \beta \frac{|\nabla v|^2 ((1-p)v\Delta v - |\nabla v|^2)}{v^{\beta+1}} + \frac{|\nabla v|^2 (\lambda(p-1)v + (p-1)v\hat{A})}{v^{\beta+1}}.
\]

It is easy to see that

\[
w_j = \frac{2v_i v_{ij}}{v^{\beta}} - \beta \frac{v_i^2 v_j}{v^{\beta+1}}. \tag{6.2}
\]
So, we have
\[
\Delta \xi \omega = \Delta w + \langle V, \nabla \omega \rangle = \frac{\Delta \xi |\nabla v|^2}{v^\beta} - 4\beta \frac{\langle \nabla v, \nabla |\nabla v|^2 \rangle}{v^{\beta+1}} - \beta \frac{\nabla v^2 \Delta \xi v}{v^{\beta+1}} + \beta(\beta + 1) \frac{|\nabla v|^4}{v^{\beta+2}}.
\]
Thus, we can infer that
\[
\mathcal{L} \omega = ((1 - p)v \Delta \xi \omega - \partial_t) \omega = (1 - p)v \left( \frac{\Delta \xi |\nabla v|^2}{v^\beta} - 2\beta \frac{\langle \nabla v, \nabla |\nabla v|^2 \rangle}{v^{\beta+1}} - \beta \frac{\nabla v^2 \Delta \xi v}{v^{\beta+1}} + \beta(\beta + 1) \frac{|\nabla v|^4}{v^{\beta+2}} \right)
\]
\[- \left( (1 - p)v \langle \nabla v, \nabla ((1 - p)v \Delta \xi v - |\nabla v|^2) \rangle + 2 \left( \langle \nabla v, \lambda(p - 1) \nabla v + (p - 1) \nabla (v \hat{A}(u)) \rangle \right) \right)
\]
\[- \beta \frac{|\nabla v|^2 ((1 - p)v \Delta \xi v - |\nabla v|^2)}{v^{\beta+1}} \nabla \rho^{\beta+1}
\]
\[- \beta \left( \lambda(p - 1)v + (p - 1)v \hat{A} \right)
\]
\[- \beta \frac{|\nabla v|^4}{v^{\beta+1}} \right) + (1 - p)\frac{\Delta \xi |\nabla v|^2}{v^{\beta-1}} - 2 \left( (1 - p)\frac{\langle \nabla v, \nabla |\nabla v|^2 \rangle}{v^\beta} + (1 - p)\frac{\nabla v^2 \Delta \xi v}{v^{\beta+1}} + \frac{\nabla v^2 \Delta \xi v}{v^\beta} \right)
\]
\[- \beta \frac{|\nabla v|^4}{v^{\beta+1}} \right) + \beta(\lambda + \hat{A})(1 - p) \frac{|\nabla v|^2}{v^\beta} - 2\lambda(p - 1) \frac{\langle \nabla v, \nabla v \rangle}{v^\beta} - \frac{2(p - 1)\nabla (v \hat{A}(u))}{v^\beta}.
\]
By Bochner formula, it is equal to
\[
\left( (1 - p)v \frac{|\nabla^2 v|^2 + \mathcal{R}(\nabla v, \nabla v) + \langle \nabla v, \nabla \Delta \xi v \rangle}{v^{\beta-1}} + (2 - 2(1 - p)\beta) \frac{\langle \nabla v, \nabla |\nabla v|^2 \rangle}{v^\beta} \right)
\]
\[- \beta(\lambda + \hat{A})(1 - p) \frac{|\nabla v|^2}{v^\beta} - 2\lambda(p - 1) \frac{\langle \nabla v, \nabla v \rangle}{v^\beta} - \frac{2(p - 1)\nabla (v \hat{A}(u))}{v^\beta}.
\]

By Bochner formula, it is equal to
\[
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\]
\[- \beta(\lambda + \hat{A})(1 - p) \frac{|\nabla v|^2}{v^\beta} - 2\lambda(p - 1) \frac{\langle \nabla v, \nabla v \rangle}{v^\beta} - \frac{2(p - 1)\nabla (v \hat{A}(u))}{v^\beta}.
\]

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\[
\begin{align*}
&= \left(2(1-p)\frac{|\nabla^2 \nu|^2 + \text{Ric}_v(\nabla \nu, \nabla \nu)}{v^{\beta-1}} + (2 - 2(1-p)\beta) \langle \nabla \omega, \nabla \nu \rangle \right)
\end{align*}
\]

\[
(2 - 2(1-p)\beta) \left( + \beta \frac{|\nabla v|^4}{v^{\beta+1}} + ((1-p)\beta(\beta+1) - \beta) \frac{|\nabla v|^4}{v^{\beta+1}} - 2(1-p)\frac{|\nabla v|^2 \Delta \xi}{v^\beta} \right)
\]

\[
+ \beta(\lambda + \hat{A})(1-p)\frac{|\nabla v|^2}{v^\beta} - 2\lambda(p-1)\frac{\langle \nabla v, \nabla v \rangle}{v^\beta} - 2(p-1)\nabla(v\hat{A}(u))
+ (2 - 2(1-p)\beta) \left( + \beta \frac{|\nabla v|^4}{v^{\beta+1}} + ((1-p)\beta(\beta+1) - \beta) \frac{|\nabla v|^4}{v^{\beta+1}} - 2(1-p)\frac{|\nabla v|^2 \Delta \xi}{v^\beta} \right)
\]

\[
+ \beta(\lambda + \hat{A})(1-p)\frac{|\nabla v|^2}{v^\beta} - 2\lambda(p-1)\frac{\langle \nabla v, \nabla v \rangle}{v^\beta} - 2(p-1)\nabla(v\hat{A}(u))
\]

\[
\geq 2(1-p)\frac{\text{Ric}_\xi(\nabla \nu, \nabla \nu)}{v^{\beta-1}} + (2 - 2(1-p)\beta) \langle \nabla \omega, \nabla \nu \rangle
+ \left( \beta(2 - 2(1-p)\beta) + (1-p)\beta(\beta+1) - \beta \right) \frac{|\nabla v|^4}{v^{\beta+1}} - 2(1-p)\frac{|\nabla v|^2 \Delta \xi}{v^\beta}
\]

\[
+ 2(1-p)\frac{1}{m} \frac{|\Delta v|^2}{v^{\beta-1}} + \Phi_1,
\]

where

\[
\Phi_1 = \beta(\lambda + \hat{A})(1-p)\frac{|\nabla v|^2}{v^\beta} - 2\lambda(p-1)\frac{\langle \nabla v, \nabla v \rangle}{v^\beta} - 2(p-1)\nabla(v\hat{A}(u))
\]
By Young’s inequality, we can get
\[
\begin{align*}
&= \left(2(1-p)\frac{\text{Ric}^m_{\xi}(\nabla v, \nabla v)}{v^{\beta-1}} + (2 - 2(1-p)\beta) \left(\langle \nabla \omega, \nabla v \rangle\right) \right) + \frac{\left|\nabla v\right|^4}{v^{\beta+1}} - 2(1-p)\frac{\langle V, \nabla v \rangle |\nabla v|^2}{v^{\beta-1}} \\
&+ \Phi_1 \\
&\geq -2K(1-p)v\omega + (2 - 2(1-p)\beta) \left(\langle \nabla \omega, \nabla v \rangle\right) \\
&+ \left(\beta(2 - 2(1-p)\beta) - \frac{n(1-p)}{2} + (1-p)\beta(\beta + 1) - \beta\right) v^{\beta-1}\omega^2 \\
&+ \Phi_1 - 2(1-p)\frac{\langle V, \nabla v \rangle |\nabla v|^2}{v^{\beta-1}},
\end{align*}
\]
where we have used the condition that \(\text{Ric}^m_{\xi} = \text{Ric} + \nabla^2 \xi \geq -K\) in the last inequality, we can choose a suitable \(\beta\) such that (see [33])
\[
\frac{1}{\gamma} := \beta(2 - 2(1-p)\beta) - \frac{n(1-p)}{2} + (1-p)\beta(\beta + 1) - \beta = -(1-p)\left(\beta^2 - \frac{2-p}{1-p} + \frac{n}{2}\right),
\]
where we have used the fact that \(p \in \left(1 - \frac{2}{n}, 1\right)\). We can conclude that
\[
\mathcal{L}\omega \geq -2K(1-p)v\omega + (2 - 2(1-p)\beta) \left(\langle \nabla \omega, \nabla v \rangle\right) + \frac{1}{\gamma} \omega^{\beta-1}\omega^2 \\
+ \left(\beta(\lambda - \hat{A})(1-p) - 2\lambda(1-p)\right) \omega - 2(1-p)\frac{\langle \nabla f, \nabla v \rangle |\nabla v|^2}{v^{\beta-1}}.
\]
Next we use the cut off function \(\psi\) in Lemma 2.1, set \(G = \psi\omega\) we may assume \(G\) attains its maximum point at \((x_1, t_1)\) in \(Q_{R,T}(\partial M)\). Firstly, we consider the case that \(x_1 \notin \partial M\), by maximum principle, we have
\[
\Delta_{\xi}(G) \leq 0, G_t \leq 0 \quad \text{and} \quad \nabla(\psi\omega) = 0,
\]
at \((x_1, t_1)\), we have
\[
0 \geq \mathcal{L}(\psi\omega) = \psi\mathcal{L}(\omega) + \omega\mathcal{L}(\psi) + 2(1-p)v \langle \nabla \psi, \nabla \omega \rangle.
\]
Multiplying both sides by \(\psi\), we have
\[
0 \geq \mathcal{L}(\psi\omega) = \psi^2\mathcal{L}(\omega) + G\mathcal{L}(\psi) + 2(1-p)v \frac{\langle \nabla \psi, \nabla \psi \rangle}{\psi} G.
\]
By Theorem 6.1 in [24] or Theorem 2.1 in [8], we have
\[
\gamma ((1-p)v \Delta_{\xi}\psi) G \leq (1-p)vG \left(\partial_{\xi}\psi \Delta_{\xi} \rho_{\partial M} + \partial_{\xi}^2 \psi\right)
\]
Thus, we have

\[
\leq M \left( \epsilon \psi G^2 + \frac{(KR + L)^2 |\partial_v^2|}{4 \epsilon} \right) + M \left( \epsilon \psi G^2 + \frac{1}{4 \epsilon} |\partial_v^2| \right)
\]

\[
\leq 3M \epsilon \psi G^2 + \frac{C_3^2}{4 \epsilon} \psi^{1/2} + \frac{C_3^2 (KR + L)^2}{4 \epsilon} \psi^{1/2}
\]

\[
\leq 3\epsilon \psi G^2 + \frac{C_3^2}{4 \epsilon} \frac{1}{R^4} + \frac{C_3^2 (KR + L)^2}{R^2},
\]

and

\[
-\gamma v^{1-\beta} G \partial_t \psi \leq \epsilon \psi G^2 + \frac{1}{4 \epsilon} |\partial_t \psi|^2 \leq 3 \epsilon \psi G^2 + \frac{C_3^2}{4 \epsilon T^2},
\]

and

\[
\gamma^2 (1 - p) v^{2-\beta} \langle \nabla \psi, \nabla \psi \rangle \frac{1}{\psi} G \leq \epsilon \psi G^2 + \gamma^2 \frac{C_3^2}{4 \epsilon} \frac{M}{1 - p} 2 \psi^{1/2} M^2 \psi^{1/2} R^4. \tag{6.3}
\]

Thus, we have

\[
\left( (1 - p) v \Delta \xi - \partial_t \right) \psi + 2(1 - p) v \langle \nabla \psi, \nabla \psi \rangle \frac{1}{\psi} G \right) \leq (3M + 1) \epsilon \psi G^2 + \frac{C_3^2 M}{2 \epsilon R^4} + \frac{C_3^2}{4 \epsilon T^2} + \frac{C_3^2 (K^2 + L^4)}{4 \epsilon} M,
\]

where we have used the fact that \( \frac{1}{1-f} \leq 1, \frac{f}{1-f} \leq 1, f < 0.\)

\[
\frac{1}{\gamma} v^{\beta-1} G^2 \leq (3M + 1) \epsilon \psi G^2 + \frac{C_3^2 M}{2 \epsilon R^4} + \frac{C_3^2}{4 \epsilon T^2} + \frac{C_3^2 (K^2 + L^4)}{4 \epsilon} M
\]

\[
+ 2K(1 - p) v \psi G + (2 - 2(1 - p) \beta) G \langle \nabla \psi, \nabla v \rangle + 2 \psi^{2(1 - p)} \frac{\langle \nabla f, \nabla v \rangle |\nabla v|^2}{v^{\beta-1}}
\]

\[
- \left( \beta(\lambda - \hat{\lambda})(1 - p) - 2\lambda(1 - p) \right) \psi G.
\]

Multiplying both sides by \( \gamma v^{1-\beta}, \) we get

\[
G^2 \leq (3M + 1) \gamma v^{1-\beta} \epsilon \psi G^2 + \left( \frac{C_3^2 M}{2 \epsilon R^4} + \frac{C_3^2}{4 \epsilon T^2} + \frac{C_3^2 (K^2 + L^4)}{4 \epsilon} M \right) \gamma v^{1-\beta}
\]

\[
+ 2K(1 - p) v^{2-\beta} \psi G + \gamma v^{1-\beta} \left( 2 - 2(1 - p) \beta \right) G \langle \nabla \psi, \nabla v \rangle
\]

\[
+ 2 \gamma v^{1-\beta} \psi^2 (1 - p) \frac{\langle \nabla f, \nabla v \rangle |\nabla v|^2}{v^{\beta-1}} - \left( \beta(\lambda - \hat{\lambda})(1 - p) - 2\lambda(1 - p) - 2K \right) \gamma v^{1-\beta} \psi G
\]

Similar to the derivation of (5.4) (5.5) (5.6), we also have

\[
2K(1 - p) v^{2-\beta} \psi G \leq 2K(1 - p)(1 - p)^{\beta-2} M^{2-\beta} \psi G, \tag{6.4}
\]

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We can infer that
\[ \gamma v^{1 - \beta} (2 - 2(1 - p)\beta) G (\langle \nabla \psi, \nabla v \rangle) \leq \gamma \left( \frac{M}{1 - p} \right)^{1 - \beta} (2 - 2(1 - p)\beta) G |\nabla \psi| |\nabla v| \]
\[ \leq \gamma M^{1 - \beta} (2 - 2(1 - p)\beta) G |\nabla \psi| \omega^{\frac{1}{2}} v^{\frac{2}{2}} \]
\[ = \gamma M^{1 - \beta} (2 - 2(1 - p)\beta) \psi |\nabla \psi| \omega^{\frac{3}{2}} \]
\[ \leq \frac{3}{4} G^2 + \gamma^4 (2 - 2(1 - p)\beta)^4 M^{4 - 2\beta} \frac{1}{R^4}. \]

We also have
\[ 2\gamma \psi^2 v^{1 - \beta} (1 - p) \frac{\langle \nabla \xi, \nabla v \rangle |\nabla v|^2}{\psi^{\beta - 1}} = 2\psi^2 v^{2 - 2\beta} (1 - p) (\nabla \xi, \nabla v) |\nabla v|^2 \]
\[ = 2\psi^2 v^{2 - \beta} (1 - p) (\nabla \xi, \nabla v) \omega \]
\[ \leq 2 \left( \frac{M}{1 - p} \right)^{2 - \beta} \frac{M^2 |\nabla \xi| \psi^\frac{3}{2} G^2}{4} \]
\[ \leq \frac{3}{4} e^\frac{4}{3} |\nabla \xi| \psi^2 G^2 + \frac{1}{4e^4} \left( 2 \left( \frac{M}{1 - p} \right)^{2 - \beta} M^2 \right)^4. \]

According to the above inequalities, we have
\[ G^2 \leq (3M + 1) \gamma v^{1 - \beta} \epsilon G^2 + \left( \frac{C^2}{4} \frac{M}{R^4} + \frac{C^2}{4} \frac{1}{T^2} + \frac{C^2}{4} \frac{K^2 + L^4}{M} \right) \gamma v^{1 - \beta} \]
\[ + 2K(1 - p)(p - 1)^{\beta - 2} M^{2 - \beta} \psi G + \frac{3}{4} G^2 + \gamma^4 (2 - 2(1 - p)\beta)^4 M^{4 - 2\beta} \frac{1}{R^4} \]
\[ + \frac{3}{4} e^\frac{4}{3} |V|^\frac{1}{2} \psi^2 G^2 + \frac{1}{4e^4 C^4} \left( 2 \left( \frac{M}{1 - p} \right)^{2 - \beta} M^2 \right)^4 \]
\[ - \left( \beta(\lambda - \tilde{\lambda})(1 - p) - 2\lambda(1 - p) \right) \gamma v^{1 - \beta} \psi G. \]

We can infer that
\[ \left( \frac{1}{4} - (3M + 1) \gamma M^{1 - \beta} \epsilon - \frac{3}{4} e^\frac{4}{3} |\nabla \xi| \frac{1}{2} \right) G^2 \]
\[ \leq \left( \frac{C^2}{4} \frac{M}{R^4} + \frac{C^2}{4} \frac{1}{T^2} + \frac{C^2}{4} \frac{K^2 + L^4}{M} \right) \gamma M^{1 - \beta} \]
\[ + \gamma^4 (2 - 2(1 - p)\beta)^4 M^{4 - 2\beta} \frac{1}{R^4} + \frac{1}{4e^4} \left( 2 \left( \frac{M}{1 - p} \right)^{2 - \beta} M^2 \right)^4 \]
\[ + 2K(1 - p)(p - 1)^{\beta - 2} M^{2 - \beta} - \left( \beta(\lambda - \tilde{\lambda}) - 2\lambda(1 - p) \right) \gamma v^{1 - \beta} \right) \psi G. \]
We can choose sufficiently small $\epsilon$ which depends on $M, \beta, p$ and the upper bound of $|\nabla \xi|$ such that $\frac{1}{4} - (3M + 1)\gamma M^{1-\beta}\epsilon - \frac{3}{4} \epsilon \frac{\gamma}{3} |\nabla \xi|^{\frac{2}{3}} > 0$. By quadratic formula, we have

$$G(x_1, t_1) \leq \frac{2K(1-p)(1-p)^{2-\beta}M^{2-\beta} + \left((\beta + 2)|\lambda| + \beta |\hat{A}|\right) (1-p) \gamma v^{1-\beta}}{\frac{1}{4} - (3M + 1)\gamma M^{1-\beta}\epsilon - \frac{3}{4} \epsilon \frac{\gamma}{3} |\nabla \xi|^{\frac{2}{3}}}$$

$$+ \left(\frac{\mathcal{O}}{\frac{3}{4} - (3M + 1)\gamma M^{1-\beta}\epsilon - \frac{3}{4} \epsilon \frac{\gamma}{3} |\nabla \xi|^{\frac{2}{3}}}\right)^{\frac{1}{2}},$$

where

$$\mathcal{O} = \left(\frac{C^2}{2\epsilon} M + \frac{C^2}{4\epsilon} + \frac{C^2}{4\epsilon} (K^2 + L^4) M\right) \gamma M^{1-\beta}$$

$$+ \gamma^4 (2 - 2(1-p)\beta)^4 M^{4-2\beta} \frac{1}{R^4} + \frac{1}{4\epsilon^4 C_1^4} \left(2\left(\frac{M}{1-p}\right)^{2-\beta} M^2\right)^4.$$

Secondly, we consider the case that $x_1 \in \partial M$

$$0 \leq \omega_n = -\beta v^{-\beta-1} v_n |\nabla v|^2 + v^{-\beta} (|\nabla v|^2)_n$$

$$= -\beta v^{-\beta-1} v_n^3 + v^{-\beta} v_n (\Delta v - H \xi v_n)$$

$$= -\beta v^{-\beta-1} v_n^3 + v^{-\beta} v_n \left(\frac{\partial v}{1-p} + \frac{|\nabla v|^2}{(1-p)v} - \lambda - \hat{A} - H \xi v_n\right).$$

since

$$(p-1)\frac{u_t}{u} \leq \lambda + \hat{A}.$$}

Thus, we have

$$\beta v^{-\beta-1} v_n^3 + v^{-\beta} v_n (-\frac{|\nabla v|^2}{(1-p)v} - Lv_n) \leq 0,$$

which yields,

$$(\beta - \frac{1}{(1-p)v})v^{-\beta-1} v_n^3 - Lv^{-\beta} v_n^2 \leq 0,$$

which is

$$(\beta - \frac{1}{(1-p)v})v^{-1} v_n - L \leq 0.$$

So, we have

$$\omega^{\frac{1}{3}} \leq \frac{Lv^{1-\frac{2}{3}}}{\beta + \frac{1}{M}}.$$
7 Elliptic estimate

In this section, we study the following equation

\[ \Delta_\xi (u^p) + \lambda u + A(u) = 0, \quad p \geq 1. \quad (7.1) \]

Lemma 7.1 (c.f. (2.1) and (2.2) in [32]). (1) For any \( a, b \in \mathbb{R} \) and \( \alpha > 0 \), we have

\[ (a + b)^2 \geq \frac{a^2}{1 + \alpha} - \frac{b^2}{\alpha}, \quad (7.2) \]

and equality holds if and only if \( b = -\frac{\alpha}{1 + \alpha} a \);

(2) for any \( a, b \in \mathbb{R} \) and \( \delta \in (0, 1) \), we have

\[ (a + b)^2 \leq \frac{a^2}{\delta} + \frac{b^2}{1 - \delta}, \quad (7.3) \]

and equality holds if and only if \( b = \frac{1 - \delta}{\delta} a \).

Let \( \phi = |\nabla u|_u \), we have the following lemma,

Lemma 7.2. Let \( u \) be the solution of (1.2), we have

\[
\Delta_\xi \phi \geq B_1 \phi^3 - B_2(u) \phi^{-1} - \left( K + \frac{1}{p} \hat{A}_u u^{2-p} \right) \phi - 2 \left( p - \frac{1}{(k-1)(1+\beta)} \right) \frac{\langle \nabla u, \nabla \phi \rangle}{u} \\
- \frac{\nabla \lambda}{p} \left( \inf_{B_R(\partial M)} u \right)^{1-p},
\]

where
\[
B_1 = \left( \frac{1}{(k-1)(1+\beta)} - \frac{(p-1)^2}{(k-1)(1+\beta)} - 2\mu(p-1)^2 - \mu(p-2)^2 \right) ,
\]
\[
B_2(u) = \left( \frac{\lambda^2 \left( \inf_{B_R(\partial M)} u \right)^{2-2p}}{(k-1)\beta(1-\delta)p^2} + \frac{\lambda^2 \left( \inf_{B_R(\partial M)} u \right)^{2-2p}}{4\mu p^2} + \frac{\hat{A}_u - \lambda^2 \left( \inf_{B_R(\partial M)} u \right)^{2-2p}}{4\mu p^2} \right).
\]

Proof. By (7.1), we have

\[ pu^{p-1} \Delta_\xi u + p(p-1)u^{p-2}|\nabla u|^2 + \lambda u + A(u) = 0. \quad (7.5) \]

Thus, we get

\[
\Delta_\xi u = -(p-1)\frac{|\nabla u|^2}{u} - \lambda \frac{u^{2-p}}{p} - \frac{A(u)}{pu^{p-1}}, \quad (7.6)
\]
This implies that
\[
\langle \nabla u, \nabla \Delta \xi u \rangle \geq -(p - 1) |\nabla |u|^2 - \left( \frac{\lambda}{p}(2 - p)u^{1-p} + \frac{\nabla \lambda \nabla u}{p} u^{2-p} + \frac{1}{p} \hat{A}_u u^{2-p} + \frac{2-p}{p} \hat{A}u^{1-p} \right) |\nabla u|^2, \tag{7.7}
\]
where we have used the condition \( p \geq 1 \) and Young’s inequality (see [32]). By Bochner formula and (7.7), we get
\[
|\nabla u| \Delta \xi |\nabla u| \geq |\text{Hess } u|^2 - p |\nabla |u|^2 + \text{ Ric }_\xi (\nabla u, \nabla u)
- \left( \frac{\lambda}{p}(2 - p)u^{1-p} + \frac{1}{p} \hat{A}_u u^{2-p} + \frac{2-p}{p} \hat{A}u^{1-p} \right) |\nabla u|^2 - \frac{\nabla \lambda \nabla u}{p} u^{2-p}. \tag{7.8}
\]
Notice that
\[
|\text{Hess } u|^2 - p |\nabla |u|^2 \geq \sum_{i=2}^{n} u_{ii} + \frac{1}{n-1} \left( \sum_{i=2}^{n} u_{ii} \right)^2 - (p - 1) \sum_{j=1}^{n} u_{1j}^2, \tag{7.9}
\]
In the sequel, we let \( V = \nabla \xi \),
\[
- \sum_{i=2}^{n} u_{ii} = -\Delta u + u_{11} = -\Delta f + u_{11} + \langle \nabla f, \nabla u \rangle
= (p - 1) \frac{|\nabla u|^2}{u} + \frac{\lambda}{p} u^{2-p} + u_{11} + V^1 u_1 + \frac{A(u)}{pu^{p-1}} \tag{7.10}
= (p - 1) u_{11}^2 u + \frac{\lambda}{p} u^{2-p} + u_{11} + V^1 u_1 + \frac{A(u)}{pu^{p-1}}.
\]
By using (7.2) twice, we obtain for any \( \alpha = \frac{k-n}{n-1} \) and \( \beta > 0 \), the following inequality
\[
\frac{1}{n-1} \left( \sum_{i=2}^{n} u_{ii} \right)^2 \geq \frac{1}{k-1} \left( (p - 1) \frac{u_{11}^2}{u} + \frac{\lambda}{p} u^{2-p} + u_{11} + V^1 u_1 + \frac{A(u)}{pu^{p-1}} \right)^2
- \frac{1}{k-n} \left( V^1 u_1 \right)^2
\geq \frac{1}{k-1} \left[ \frac{u_{11}^2}{1+\beta} - \frac{1}{\beta} \left( (p - 1) \frac{u_{11}^2}{u} + \frac{\lambda}{p} u^{2-p} + \frac{A(u)}{pu^{p-1}} \right)^2 \right] - \frac{1}{k-n} \left( V^1 u_1 \right)^2, \tag{7.11}
\]
By (7.9), we have
\[
|\text{Hess } u|^2 - p|\nabla|\nabla u|^2 \geq \left[ \frac{1}{(k - 1)(1 + \beta)} - (p - 1) \right] |\nabla|\nabla u|^2 - \frac{(p - 1)^2|\nabla u|^4}{(k - 1)^2\delta_1\delta u^2} \tag{7.12}
\]
\[- \frac{\lambda^2 u^{4-2p}}{(k - 1)\delta_1(1 - \delta)p^2} - \frac{A(u)u^{2-p}}{(k - 1)\beta(1 - \delta_1)p^2} - \frac{1}{k - n} V^b \otimes V^b(\nabla u, \nabla u),
\]
where we have used the inequality (7.3) twice.

Thus, we have
\[
|\nabla u|\Delta_u|\nabla u| \geq \left[ \frac{1}{(k - 1)(1 + \beta)} - (p - 1) \right] |\nabla|\nabla u|^2 - \frac{(p - 1)^2|\nabla u|^4}{(k - 1)^2\delta_1\delta u^2} \tag{7.13}
\]
\[- \frac{\lambda^2 u^{4-2p}}{(k - 1)\delta_1(1 - \delta)p^2} - \frac{A(u)u^{2-p}}{(k - 1)\beta(1 - \delta_1)p^2} + \text{Ric}_\xi(\nabla u, \nabla u) \tag{7.14}
\]
\[- \left( \frac{\lambda}{p}(2 - p)u^{1-p} + \frac{1}{p}\frac{\hat{A}_u u^{2-p}}{2 - p}\frac{\hat{A}_u u^{1-p}}{u} \right) |\nabla u|^2 - \frac{\nabla_\lambda \nabla u}{p}u^{2-p}.
\]

Using
\[
\Delta_u|\nabla u| = u\Delta\phi + 2 \langle \nabla u, \nabla \phi \rangle + \phi\Delta_u u,
\]
and
\[
\Delta_u u = -(p - 1)\frac{|\nabla u|^2}{u} - \frac{A(u)}{p w^{p-1}},
\]
we can infer from (7.13),
\[
\Delta_u \phi = \Delta\phi \frac{|\nabla u|}{u} + 2 \langle \nabla u, \nabla \phi \rangle + \phi \Delta_u u
\]
\[= \Delta\phi \frac{|\nabla u|}{u} + \frac{2}{u} \langle \nabla u, \nabla \phi \rangle + \frac{2}{u^2} \left[ (p - 1)\frac{|\nabla u|^2}{u} \right] \tag{7.15}
\]
\[\geq \frac{1}{u|\nabla u|} \left[ \left( \frac{1}{(k - 1)(1 + \beta)} - (p - 1) \right) |\nabla|\nabla u|^2 - \frac{(p - 1)^2|\nabla u|^4}{(k - 1)^2\delta_1\delta u^2} \tag{7.16}
\]
\[- \frac{\lambda^2 u^{4-2p}}{(k - 1)\delta_1(1 - \delta)p^2} - \frac{A(u)u^{2-p}}{(k - 1)\beta(1 - \delta_1)p^2} + \text{Ric}_\xi(\nabla u, \nabla u) \tag{7.17}
\]
\[- \left( \frac{\lambda}{p}(2 - p)u^{1-p} + \frac{1}{p}\frac{\hat{A}_u u^{2-p}}{2 - p}\frac{\hat{A}_u u^{1-p}}{u} \right) |\nabla u|^2 - \frac{\nabla_\lambda \nabla u}{p}u^{2-p} \right] \tag{7.18}
\]
\[\geq \frac{1}{u|\nabla u|} \left[ \left( \frac{1}{(k - 1)(1 + \beta)} - (p - 1) \right) |\nabla|\nabla u|^2 - \frac{(p - 1)^2|\nabla u|^4}{(k - 1)^2\delta_1\delta u^2} \right] \tag{7.19}
\]
\[- \frac{\lambda^2 u^{4-2p}}{(k - 1)\delta_1(1 - \delta)p^2} - \frac{A(u)u^{2-p}}{(k - 1)\beta(1 - \delta_1)p^2} + \text{Ric}_\xi(\nabla u, \nabla u) \tag{7.20}
\]
\[- \left( \frac{\lambda}{p}(2 - p)u^{1-p} + \frac{1}{p}\frac{\hat{A}_u u^{2-p}}{2 - p}\frac{\hat{A}_u u^{1-p}}{u} \right) |\nabla u|^2 - \frac{\nabla_\lambda \nabla u}{p}u^{2-p} \right] \tag{7.21}
\]
\[- \frac{2}{u} \langle \nabla u, \nabla \phi \rangle + \frac{1}{u^3} \frac{\nabla u^3}{u^3} + \frac{2 - p}{u}\frac{\hat{A}_u u^{1-p}}{p} \phi + \frac{A(u)}{p w^{p-1}} \frac{\nabla u}{u^2} \tag{7.22}
\]
\[
\begin{align*}
&\geq \left[ \frac{1}{(k-1)(1+\beta)} - (p-1) \right] \frac{1}{u| \nabla u|} | \nabla | \nabla u| \| - (p-1)^2 \frac{1}{(k-1)\beta \delta_1 \delta} \phi^3 \\
&\quad - \frac{\lambda^2 u^{2-2p}}{(k-1)\beta(1-\delta)p^2 \phi^{-1}} - \frac{A(u)u^{2-p} \lambda u^{1-p}}{(k-1)\beta(1-\delta)p^2 u | \nabla u|} - K \phi - \left( \frac{\lambda}{p} (2-p)u^{1-p} + \frac{1}{p} \hat{A}_u u^{2-p} + \frac{2-p}{p} \hat{A} u^{1-p} \right) \phi - \left( \frac{\nabla \lambda u}{p} \right) u^{1-p}
\end{align*}
\]

where we have used the formula in [25, page 19]

\[
\frac{2}{u} \langle \nabla u, \nabla \phi \rangle = (2-\epsilon) \frac{\langle \nabla u, \nabla \phi \rangle}{u^2} + \epsilon \frac{\langle \nabla u, | \nabla u| \rangle}{u^2} - \epsilon | \nabla u| \frac{3}{u^3}
\]

\[
\leq (2-\epsilon) \frac{\langle \nabla u, \nabla \phi \rangle}{u} + \epsilon \frac{\langle \nabla u, \nabla u| u \rangle}{u^2} - \epsilon \phi^3,
\]

and

\[
\frac{| \nabla | \nabla u| \|}{u | \nabla u|} \geq 2 \frac{| \nabla u| \cdot | u \nabla u|}{u^2} - | \nabla u| \frac{3}{u^3}.
\]

Take \(\epsilon = 2(\frac{1}{(k-1)(1+\beta)} - (p-1))\), we have

\[
\Delta \xi \phi \geq \left( \frac{1}{(k-1)(1+\beta)} - (p-1)^2 \frac{1}{(k-1)\beta \delta_1 \delta} \right) \phi^3 \\
\qquad - \frac{\lambda^2 u^{2-2p}}{(k-1)\beta(1-\delta)p^2 \phi^{-1}} - \frac{A(u)u^{2-p} \lambda u^{1-p}}{(k-1)\beta(1-\delta)p^2 u | \nabla u|} - K \phi - \left( \frac{\lambda}{p} (2-p)u^{1-p} + \frac{1}{p} \hat{A}_u u^{2-p} + \frac{2-p}{p} \hat{A} u^{1-p} \right) \phi - \left( \frac{\nabla \lambda u}{p} \right) u^{1-p}
\]

By Young’s inequality, we have

\[
\frac{\lambda(p-1)u^{1-p}}{p} \phi \geq -\mu(p-1)^2 \phi^3 - \frac{\lambda^2 u^{2-2p}}{4\mu p^2} \phi^{-1},
\]

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\[
\lambda (p-2)u^{1-p} \phi \geq -\mu (p-2)^2 \phi^3 - \frac{\lambda^2 u^{2-2p}}{4\mu p^2} \phi^{-1},
\]

\[
\frac{|\nabla \lambda|(p-1)u^{1-p}}{p} \phi \geq -\mu (p-1)^2 \phi^3 - \frac{|\nabla \lambda|^2 u^{2-2p}}{4\mu p^2} \phi^{-1},
\]

\[
-\frac{2-p}{p} \dot{A} u^{1-p} \phi + \frac{A(u)}{pu^p} \phi = \frac{p-1}{p} \dot{A} u^{1-p} \phi \geq -\mu (p-1)^2 \phi^3 - \frac{\dot{A}^2 u^{2-2p}}{4\mu p^2} \phi^{-1},
\]

and

\[
-\frac{|\nabla \lambda|}{p} u^{1-p} \geq \frac{|\nabla \lambda|}{p} (\inf_{B_R(\partial M)} u)^{1-p}, \quad (7.18)
\]

and

\[
\frac{A(u)u^{2-p}}{(k-1)\beta(1-\delta_1)p^2 u|\nabla u|} \geq \frac{A(u)u^{-p}}{(k-1)\beta(1-\delta_1)p^2} \phi^{-1}. \quad (7.19)
\]

Using the above formula, we have

\[
\Delta_{\xi} \phi \geq \left( \frac{1}{(k-1)(1+\beta)} - \frac{(p-1)^2}{(k-1)\beta \delta_1 \delta} - 2\mu (p-1)^2 - \mu (p-2)^2 \right) \phi^3 - \left( \frac{\lambda^2 u^{2-2p}}{(k-1)\beta \delta_1 \delta} + \frac{A(u)u^{-p}}{(k-1)\beta(1-\delta_1)p^2} \right) - \frac{\dot{A}^2 u^{2-2p}}{4\mu p^2} + \frac{\lambda^2 u^{2-2p}}{4\mu p^2} \phi^{-1} - \left( K + \frac{1}{p} \dot{A} u^{2-p} \right) \phi - 2 \left( p - \frac{1}{(k-1)(1+\beta)} \right) \frac{\langle \nabla u, \nabla \phi \rangle}{u} - \frac{|\nabla \lambda|}{p} (\inf_{B_R(\partial M)} u)^{1-p}. \quad (7.20)
\]

It is not hard to see that we can take \( \delta \in (0, 1), \delta_1 \in (0, 1), \mu > 0 \) such that

\[
\frac{1}{(k-1)(1+\beta)} - \frac{(p-1)^2}{(k-1)\beta \delta_1 \delta} - 2\mu (p-1)^2 - \mu (p-2)^2 > 0,
\]

we take \( \beta = \sigma, \delta_1 = \frac{1}{2}, \delta = \frac{(1+\sigma)^2}{\sigma} (p-1)^2, \mu = \frac{\sigma}{(k-1)(1+\sigma)^2(p-1)^2}, \) it suffices to prove that

\[
\frac{1}{(k-1)(1+\beta)} - \frac{(p-1)^2}{(k-1)\beta \delta_1 \delta} - 3\mu (p-1)^2 = \frac{\sigma^2 - (1 + \frac{1}{\delta_1})\sigma + 1 - \delta_1}{(1+\sigma)^3(k-1)} > 0,
\]

which is positive if \( \sigma > \frac{1+\frac{2}{\delta_1}+\sqrt{(1+\frac{2}{\delta_1})^2-4(1-\delta_1)}}{2} \). \( \square \)

**Theorem 7.1.** Let \((M, g, e^{-\xi}dv_g)\) be an n-dimensional, complete noncompact metric measure space with compact boundary. For \(K, L \geq 0\), we assume \(\text{Ric}^k \geq -K\) and \(H_\xi \geq -L\), \(L^2 \geq 4p(\frac{\lambda}{p\nu} + \frac{\dot{A}(b)}{p\nu^{b-1}})\). Let \(u : B_R(\partial M) \to (0, \infty)\) be the solution of (1.2) with
Dirichlet boundary condition (i.e. $u = b$ on $\partial M$) and $1 \leq p \leq 1 + \frac{1}{(1 + \sigma)(k - 1)}$, $\sigma > \max\{\frac{1 + \frac{1}{\sigma_1}}{2}, \frac{1}{\sigma_1(k - 1)^2}\}$, $\delta_1 \in (0, 1)$ is a small constant such that $(1 + \frac{1}{\delta_1})^2 - 4(1 - \delta_1) \geq 0$. We assume that its derivative $u_\nu$ in the direction of the outward unit normal vector $\nu$ is nonnegative over $\partial M$. Then we have

$$\frac{|\nabla u|}{u} \leq L + \sqrt{\frac{L^2 - 4p(\frac{\lambda}{\rho\sigma - \nu} + \frac{\hat{A}(b)}{\rho\sigma - \nu})}{2p}} + C'\frac{4}{3}R^{-2}. \tag{7.21}$$

where constant $C'$ which depends on $K, L, \inf_{B_R(\partial M)} u, \sup_{B_R(\partial M)}(\hat{A}_u)$.

**Remark 7.1.** When $A(u) = au \log u$, then $\hat{A}_u = a$.

**Proof.** Let

$$F = (R^2 - \rho^2)\phi.$$

We assume that $F$ attains its maximum at $\bar{x}$. We first consider the case where $\bar{x} \in B_R(\partial M) \setminus \partial M$. Then at $\bar{x}$ we have

$$\nabla F = 0, \Delta \xi F \leq 0,$$

thus, at $\bar{x}$, we have

$$\frac{\Delta \xi \phi}{\phi} - \frac{\Delta \rho^2}{R^2 - \rho^2} - 2\frac{\rho^2}{(R^2 - \rho^2)^2} \leq 0.$$

By (7.4), we get

$$0 \geq \frac{\Delta \phi}{\phi} - \frac{\Delta \rho^2}{R^2 - \rho^2} - 2\frac{\rho^2}{(R^2 - \rho^2)^2} \leq 0.$$

$$\geq B_1\phi^2 - B_2(u)\phi - \left(K + \frac{1}{p} \hat{A}_u u^{2-p} - 4p - \frac{1}{(k - 1)(1 + \beta)}\right)\frac{\rho^2}{(R^2 - \rho^2)^2} \leq 0. \tag{7.22}$$

Thus, we have

$$0 \geq \frac{\Delta \phi}{\phi} - \frac{\Delta \rho^2}{R^2 - \rho^2} - 2\frac{\rho^2}{(R^2 - \rho^2)^2} \leq 0.$$
0 \geq B_1 F^2 - \left( K + \frac{1}{p} \hat{A}_u u^{2-p} \right) (R^2 - \rho_{\partial M}^2)^2
\nonumber
- B_2(u) (R^2 - \rho_{\partial M}^2)^4 F^{-2} - 4 \left( p - \frac{1}{(k-1)(1+\sigma)} \right) F
\nonumber
- \frac{|\nabla \lambda|}{p} \left( \inf_{B_R(\partial M)} u \right)^{1-p} (R^2 - \rho_{\partial M}^2)^2 - \Delta \rho_{\partial M}^2 (R^2 - \rho_{\partial M}^2) - 2|\nabla \rho_{\partial M}^2|^2.  

(7.23)

However, since Ric^\xi \geq -K, H^\xi \geq -L, K, L \geq 0, we have

\nonumber
\Delta \xi \rho_{\partial M}^2 = 2|\nabla \rho_{\partial M}^2|^2 + \rho_{\partial M} \Delta \rho_{\partial M}
\nonumber
\leq 2 + 2\rho (KR + L) \leq (2 + 2KR^2 + 2LR).  

(7.24)

Using this, (7.23) implies that

0 \geq B_1 F^2 - \left( K + \frac{1}{p} \hat{A}_u u^{2-p} \right) R^4
\nonumber
- B_2(u) R^8 F^{-2} - 4 \left( p - \frac{1}{(k-1)(1+\sigma)} \right) F
\nonumber
- \frac{|\nabla \lambda|}{p} \left( \inf_{B_R(\partial M)} u \right)^{1-p} R^4 - (2 + 2KR^2 + 2LR)R^2 - 8R^2.

(7.25)

Thus, we have

0 \geq B_1 F^4 - 4 \left( p - \frac{1}{(k-1)(1+\sigma)} \right) F^3 - B_2(u) R^8
\nonumber
- \left( K + \frac{1}{p} \sup_{B_R(\partial M)} (\hat{A}_u) \left( \inf_{B_R(\partial M)} u \right)^{2-p} \right) R^4 F^2
\nonumber
- \left( \frac{|\nabla \lambda|}{p} \left( \inf_{B_R(\partial M)} u \right)^{1-p} R^4 + (2 + 2KR^2 + 2LR)R^2 + 8R^2 \right) F^2.

(7.26)

Therefore, there exists constant \( C' \) which depends on \( K, L, \inf_{B_R(\partial M)} u, \sup_{B_R(\partial M)} (\hat{A}_u)u \) such that

\frac{3}{4} R^2 \sup_{B_R(x_0)} \frac{\nabla u}{u} \leq C'.

Therefore, we have

\sup_{B_R(x_0)} \frac{\nabla u}{u} \leq C' \frac{4}{3} R^{-2}.

If \( \bar{x} \in \partial M \), By lemma 2.2
0 \leq (\phi^2)_\nu = (|\nabla \log u|^2)_\nu = 2(\log u)_\nu (\Delta_\xi \log u - H_\xi(\log u)_\nu), \quad (7.27)
where we have used the fact that $u = b$, on $\partial M$. A direct computation shows that
\[ \Delta_\xi \log u = \frac{\Delta_\xi u}{u} - \frac{|\nabla u|^2}{u^2} \tag{7.28} \]
and
\[ \Delta_\xi u^p + \lambda u + A(u) = pu^{p-1}\Delta_\xi u + p(p-1)u^{p-2}|\nabla u|^2 + \lambda u + A(u) = 0. \]
Thus, we have
\[ pu^{p-1}\Delta_\xi u + p(p-1)u^{p-2}|\nabla u|^2 + \lambda + \hat{A} = 0, \]
and
\[ \frac{\Delta_\xi u}{u} + (p-1)u^{-1}|\nabla u|^2 - \frac{\lambda}{pu^{p-1}} - \frac{\hat{A}}{pu^{p-1}} = 0. \]
By (7.28), we get that
\[ \Delta_\xi \log u = -(p-1)\frac{|\nabla u|^2}{u^2} - \frac{\lambda}{pu^{p-1}} - \frac{\hat{A}}{pu^{p-1}} = \frac{|\nabla u|^2}{u^2}. \]
Thus, (7.27) can be rewritten as
\[
0 \leq (\phi^2)_\nu = (|\nabla \log u|^2)_\nu \\
= 2(\log u)_\nu \left(-(p-1)\frac{|\nabla u|^2}{u^2} - \frac{\lambda}{pu^{p-1}} - \frac{\hat{A}}{pu^{p-1}} - H_\xi(\log u)_\nu \right).
\]
This yields
\[
(p-1)\frac{|\nabla u|^2}{u^2} + \frac{\lambda}{pu^{p-1}} + \frac{\hat{A}}{pu^{p-1}} + \frac{|\nabla u|^2}{u^2} + H_\xi(\log u)_\nu \leq 0.
\]
Since at $\bar{x} \in \partial M, u = b, H_\xi \geq -L, L \geq 0$, we get
\[
pu_\nu^2 - Lu_\nu \leq -\frac{\lambda}{pu^{p-1}} - \frac{\hat{A}(b)}{pu^{p-1}}.
\]
By quadratic formula, we get
\[
0 \leq \frac{u_\nu}{u} \leq \frac{L + \sqrt{L^2 - 4p(\frac{\lambda}{pu^{p-1}} + \frac{\hat{A}(b)}{pu^{p-1}})}}{2p}.
\]
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