Hamiltonian Tetrahedralizations with Steiner Points

Francisco Escalona∗  Ruy Fabila-Monroy†  Jorge Urrutia †

May 3, 2014

Abstract

Let $S$ be a set of $n$ points in 3-dimensional space. A tetrahedralization $T$ of $S$ is a set of interior disjoint tetrahedra with vertices on $S$, not containing points of $S$ in their interior, and such that their union is the convex hull of $S$. Given $T$, $D_T$ is defined as the graph having as vertex set the tetrahedra of $T$, two of which are adjacent if they share a face. We say that $T$ is Hamiltonian if $D_T$ has a Hamiltonian path. Let $m$ be the number of convex hull vertices of $S$. We prove that by adding at most $\left\lfloor \frac{m^2 - 2}{2} \right\rfloor$ Steiner points to interior of the convex hull of $S$, we can obtain a point set that admits a Hamiltonian tetrahedralization. An $O(m^2) + O(n \log n)$ time algorithm to obtain these points is given. We also show that all point sets with at most 20 convex hull points admit a Hamiltonian tetrahedralization without the addition of any Steiner points. Finally we exhibit a set of 84 points that does not admit a Hamiltonian tetrahedralization in which all tetrahedra share a vertex.

1 Introduction

All point sets considered throughout this paper will be in general position in $\mathbb{R}^2$ and $\mathbb{R}^3$. This are point sets such that: in $\mathbb{R}^2$ not three of its elements are colinear and in $\mathbb{R}^3$ not four of its elements are coplanar.

Let $S$ be a set of $n$ points in $\mathbb{R}^3$. A tetrahedralization $T$ of $S$ is a set of tetrahedra with vertices in $S$, such that:

1. Their union is the convex hull of $S$.

2. The tetrahedra only intersect at points, lines or faces.

3. The tetrahedra do not contain points of $S$ in their interior.
Given $\mathcal{T}$, we define the dual graph of $\mathcal{T}$, $D_{\mathcal{T}}$ to be the graph whose vertices are the elements of $\mathcal{T}$, two of which are adjacent if they share a common face.

In a similar way a triangulation of a point set of points $S$ in the plane is set of interior disjoint triangles with vertices on $S$, not containing points of $S$ and such that their union is the convex hull of $S$.

Again, the dual graph $D_{\mathcal{T}}$ of $\mathcal{T}$ is the graph having the elements of $\mathcal{T}$ as vertices, two of which are adjacent if they share an edge.

A Hamiltonian path (cycle) of a graph $G$ is a path (cycle) spanning all the vertices of $G$. If $D_{\mathcal{T}}$ contains a Hamiltonian path or cycle, we say that $\mathcal{T}$ is a Hamiltonian tetrahedralization (or triangulation, if $S$ is in the plane).

The problem of finding a Hamiltonian triangulation for a given point set in the plane, has been settled in both the existential and algorithmic sense: every set of $n$ points in the plane admits a Hamiltonian triangulation and it can be computed in time $O(n \log n)$ [1, 7].

The Hamiltonian Tetrahedralization Problem [1], Problem 29 in [6], has been a long standing problem in Computational Geometry. It is not known if every point set in general position in $\mathbb{R}^3$ admits a Hamiltonian tetrahedralization. It is conjectured in [1], that the problem of finding such a tetrahedralization is $NP$-hard for arbitrary point sets.

Hamiltonian triangulations were initially studied, among other reasons, because they speed the rate at which a triangulation can be sent to a graphic processor for rendering ([1], [3]). A similar speed up applies for tetrahedralizations.

In this paper we study the problem of computing Hamiltonian tetrahedralizations by adding Steiner points.

We call the points in the interior of the convex hull of $S$, interior points and the points on the boundary of the convex hull, exterior points Let $m$ and $m'$ be the number of exterior and interior points of $S$ respectively; we denote the convex hull of $S$ by Conv($S$).

Our main result is thus:

**Theorem 1** Let $S$ be a set of $n$ points in general position with $m$ exterior points in $\mathbb{R}^3$. Then we need to add at most $\left\lfloor \frac{m-2}{2} \right\rfloor$ Steiner points to the interior of Conv($S$), so that the resulting point set admits a Hamiltonian tetrahedralization. Moreover this tetrahedralization can be found in time $O(m^2) + O(n \log n)$

We note that the Steiner points are added as interior points of $S$, it is not hard to see that if this requirement is dropped a Hamiltonian tetrahedralization can be found by adding only two Steiner points.

The paper is organized as follows:

In Section 2 we present an algorithm that adds at most $\left\lfloor \frac{m-2}{2} \right\rfloor$ Steiner points to the interior of Conv($S$). Our algorithm produces a Hamiltonian tetrahedralization. The overall complexity of the algorithm is $O(m^2) + O(n \log n)$. We consider its complexity and implementation issues in Section 3. In Section 4 we study the dual graph of the convex hull of $S$. We show that all point sets with at most 20 exterior points admit a Hamiltonian path tetrahedralization. In
the same section a lower bound on the number of Steiner points our algorithm might add is given. With the same techniques we improve on the result of [5] and exhibit a point set of 84 elements that does not admit a Hamiltonian pulling tetrahedralization (a pulling tetrahedralization is a tetrahedralization in which all tetrahedra share a point). Finally in Section 5 a summary of results is given and new directions offered.

2 The algorithm

In this section we sketch an algorithm that achieves Theorem 1. The main idea of our algorithm is to first add a point to $S$ to obtain a tetrahedralization such that its dual graph can be partitioned into cycles. Steiner points are then inserted to join existing cycles. We continue this process until the cycle partition consists of just one cycle. This final cycle is a Hamiltonian cycle in the dual graph of the final tetrahedralization.

The first step is to remove the interior points and those exterior points of degree 3 (that is, points adjacent to 3 other points in the boundary of $\text{Conv}(S)$). We can do this in view of the following:

**Lemma 2** If the set of exterior points of $S$ admit a Hamiltonian tetrahedralization, so does $S$.

**Proof.** Consider an interior point $x$ of $S$ and suppose $S - \{x\}$ admits a Hamiltonian tetrahedralization $T$. Let $\tau$ be the unique tetrahedron of $T$ that contains $x$ in its interior. If we remove $\tau$ from $T$ and add the four tetrahedra induced by the faces of $\tau$ with $x$, we obtain a tetrahedralization of $S$ and the Hamiltonian cycle of $D_T$ can be extended to a Hamiltonian cycle of the new tetrahedralization. Applying this process recursively the result follows. $\square$

Assume thus that $S$ does not have interior points.

**Lemma 3** Let $x$ be an exterior point of $S$ of degree 3. If $S - \{x\}$ admits a Hamiltonian tetrahedralization, then so does $S$.

**Proof.** Suppose $S - \{x\}$ admits a Hamiltonian tetrahedralization $T$. The three convex hull vertices of $S$ adjacent to $x$ form a face $F$ of the boundary of $\text{Conv}(S - \{x\})$. Let $\tau_1$ be the only tetrahedron of $T$ that contains $F$ as a face and let $\tau_2$ be the tetrahedron induced by $x$ and $F$. Clearly $\tau_1 \cup \tau_2$ is convex. If we remove $\tau_1$ and $\tau_2$ from $T$ and replace them with the three tetrahedra induced by the faces of $\tau_1$ (except $F$) and $x$, we obtain a tetrahedralization $T'$ of $S$. The Hamiltonian cycle of $D_T$ can now be extended to a Hamiltonian cycle of $D_{T'}$. $\square$

Assume now that $S$ does not contain exterior points of degree 3.

We insert a point $p_0$ in the interior of $\text{Conv}(S)$ and join every face of the boundary of $\text{Conv}(S)$ to $p_0$, forming a tetrahedralization $T$ of $S \cup \{p_0\}$.
Let $G$ be the graph induced by the 1-skeleton of the boundary of $\text{Conv}(S)$, that is, the graph whose vertex set consists of the exterior points of $S$ and whose edges are the edges of the boundary of $\text{Conv}(S)$. It is easy to see that both $G$ and its dual graph are planar and 3-connected. By construction, the dual graph of $G$ is isomorphic to $D_T$. Since every face of $G$ is a triangle, $D_T$ is a regular graph of degree 3.

To obtain the initial partition, we use a theorem of Petersen [11] that states that every 2-connected cubic graph contains a perfect matching. Since $D_T$ is 3-connected, in particular it is 2-connected and therefore contains a perfect matching $M$. If we remove the edges of $M$ from $D_T$, we obtain a regular graph of degree 2. This subgraph of $D_T$ is the initial cycle partition.

2.1 Joining cycles

Consider two disjoint cycles, $C_1$ and $C_2$, in our cycle partition of $D_T$, and suppose that there is an edge $e$ of $D_T$ that has its end points $\tau_1$ and $\tau_2$ in $C_1$ and $C_2$ respectively. Since $\tau_1$ and $\tau_2$ are tetrahedra in $T$, $e$ corresponds to a shared face $F$ of $\tau_1$ and $\tau_2$.

The join operation consists of adding a point $p$ to the interior of $\tau_1$ so that the line segment joining the point $q$ in $\tau_2$ opposite to $F$ in $\tau_2$ intersects $F$. We now remove $\tau_1$ and $\tau_2$ and replace them by the six tetrahedra induced by the faces of $\tau_1$, $\tau_2$ and $p$ (except $F$) as shown in Figure 2.

$C_1$ and $C_2$ are joined into a cycle passing the tetrahedra of $C_1 \cup C_2 - \{\tau_1, \tau_2\}$ plus the six new tetrahedra containing $p$ as a vertex (see Figure 2).

We repeat this process until a single cycle is obtained. We will show in the next section that the number of Steiner points we need to insert before a Hamiltonian cycle is reached is at most $\left\lfloor \frac{n-2}{2} \right\rfloor$.

3 Complexity and implementation.

In this section we will analyze the running time and implementation issues of the algorithm sketched in Section 2.
Suppose now that $S$ is a point set with $n$ points in $\mathbb{R}^3$ with $m$ exterior points and $m'$ interior points, $m + m' = n$. We first calculate the convex hull of $S$ in $O(n \log n)$, and then remove the interior points of $S$.

Next, we remove the exterior vertices of degree 3. This can be done in $O(m)$ by using a priority queue with all exterior points of degree 3. Each time one is removed, the degree of its neighbors is checked and if necessary they are added to the queue.

Adding the first Steiner point $p_0$ and tetrahedralizing as in the previous section takes time $O(m)$.

The complexity of finding the initial cycle partition described at the end of Section 2 is that of finding a perfect matching in $G$. In a graph with $|V|$ vertices and $|E|$ edges, a perfect matching can be found in time $O(|E|\sqrt{|V|})$ [10]. Since we are dealing with a cubic graph, we have $|E| = \frac{3}{2}|V|$. Thus we can find the initial cycle cover in $O(\frac{3}{2}m\sqrt{m}) = O(m \frac{3}{2})$ time.

Once we have the initial cycle cover, we return the vertices that we removed. This is done before the join operations in order to take advantage of the structure of the tetrahedralization and return the exterior points of degree 3 and interior points efficiently. Using the fact that $D_T$ is a planar graph, the interior points and exterior points of degree 3 can be added using point location at a cost of $O(\log m)$ per point. The exterior points of degree 3 are added first and the interior points afterwards. As these points are returned, the initial cycle partition is updated as in Lemma 2 and Theorem 3.

We have to be careful about the order in which the interior points are added. Suppose we have a tetrahedra $\tau$ which contains $k$ interior points that remain to be added, and that we return one of these points. When we retetrahedralize the point set, $\tau$ would be split into 4 new tetrahedra. We have to guarantee that each of these tetrahedra receives a linear fraction of the points in $\tau$, for otherwise the iterative process could take as much as $O(k^2)$. That is, we need a splitter vertex (see [2]). Such a vertex can be found in time $O(k)$, thus ensuring a total of $O(m' \log m)$ running time.
Finally we proceed to merge the set of cycles obtained so far into a single cycle as in Subsection 2.1. Each time we join two cycles, we insert one Steiner point. Since $G$ has $m$ vertices, the number of faces of $G$ is $2m - 4$, and since all the cycles obtained have at least four vertices, the initial cycle partition contains at most $\left\lfloor \frac{2m - 4}{4} \right\rfloor$ elements. Thus the number of Steiner points required is at most $\left\lfloor \frac{m - 2}{2} \right\rfloor$. Since there are $O(m)$ edges in $G$ the merging of the cycles can be done in time $O(m)$. The overall complexity of the algorithm is thus $O(m^3) + O(n \log n)$. From this and all previous observations we obtain Theorem 1.

4 3-connected cubic planar graphs

To conclude the paper we study tetrahedralizations of point sets in terms of the dual graph of the 1-skeleton of their convex hull. The 1-skeleton of the convex hull of $S$ is the graph having as vertices the exterior points of $S$, two of which are adjacent if they are joined by an edge in the boundary of $\text{Conv}(S)$. In particular we prove that every point set of at most 20 exterior points admits a Hamiltonian tetrahedralization. Also a set of 84 points that does not contain a Hamiltonian pulling tetrahedralization is shown. Improving therefore on the result of [5]. The techniques employed, allow us to give a lower bound on the number of Steiner points our algorithm might add.

The convex hull of a point set in $\mathbb{R}^3$ is a convex polyhedron of triangular faces. It is known that the dual graph of such a polyhedron is a 3-connected cubic planar graph (3CCP). The converse is also true, that is every 3CCP graph can be realized as the dual graph of some polyhedron and therefore as the dual graph of the 1-skeleton of the convex hull of a point set in convex position [12].

3CCP graphs are uniquely embeddable in the plane [15]. This in particular means that faces of a 3CPP graph are defined regardless of any particular embedding. Now, given a polyhedron $\mathcal{P}$, take any point $q$ in the interior of a face and do a stereo-graphic projection to a plane not containing $\mathcal{P}$ and such that any line segment joining a point of the plane and $q$ cuts $\mathcal{P}$ in its interior. This yields a planar embedding of the dual graph $D(\mathcal{P})$ of $\mathcal{P}$. $D(\mathcal{P})$ is the graph having as vertices the faces of $\mathcal{P}$ two of the adjacent if they share an edge. Note that in this embedding: faces of $D(\mathcal{P})$ correspond to all the faces of $\mathcal{P}$ containing a given point and all the faces of $\mathcal{P}$ containing a point correspond to faces of $D(\mathcal{P})$. Since all embeddings are essentially unique we may assume that this is always the case for any embedding of $D(\mathcal{P})$.

3CCP graphs were once conjectured to be Hamiltonian by Tait [13], until a 3CCP non-Hamiltonian graph of 46 vertices was found by Tutte [14]. Nevertheless using an exhaustive computer search it has been shown that the smallest non-Hamiltonian 3CCP graphs have 38 vertices [9], in other words all 3CCP graphs of at most 36 vertices have a Hamiltonian cycle. We use this fact to show that all point sets of at most 20 elements admit a Hamiltonian tetrahedralization.

At this point it should be stressed that we are making no assumption on the existence of interior points. Indeed if the dual graph of the convex hull of a
point set contains a Hamiltonian cycle then joining all the vertices of the convex hull to an interior point would yield a tetrahedralization whose dual graph is isomorphic to the dual graph of the and thus would contain a Hamiltonian cycle.

**Theorem 4** Let $G$ be a 3CCP Hamiltonian graph. There exists a face $F$ of $G$ so that $G - F$ contains a Hamiltonian path.

**Proof.** Consider a planar embedding of $G$ and a Hamiltonian cycle $C$ of $G$. Define the distance of two vertices as the minimum length of the two paths joining them in $C$. Take any edge $e$ in $G$ not in $C$, joining vertices two vertices $x$ and $y$, whose distance in $C$ is minimum. The path $\Gamma$ in $C$ joining $x$ and $y$, realizing this distance together with $e$ forms a face $F$; since any other edge would join vertices at a strictly less distance. Now $C - \Gamma$ is a Hamiltonian path of $G - F$. □

Take any vertex $p$ of $P$ and consider the tetrahedralization $T_p$ formed by joining all other vertices of $P$ to $p$. Such tetrahedralizations are known in the literature as “pulling” tetrahedralizations. Let $F_p$ be the corresponding face of $p$ in $D(P)$. $T_p$ is isomorphic to $D(P) - F_p$. Note that if $D(P)$ is Hamiltonian, Theorem 4 implies the existence of a point $p$ such that $T_p$ is Hamiltonian.

By Euler’s formula, a 3CCP graph on $n$ vertices has $\frac{n+4}{2}$ faces. Since all 3CCP graphs of 36 or less vertices are Hamiltonian we have:

**Corollary 5** Every point set in $\mathbb{R}^3$ in general position of at most 20 points admits a Hamiltonian path (“pulling”) tetrahedralization.

Although it is a simple observation, Theorem 4 serves as a bridge between 3CCP graphs and point sets in space. For example it is known that 3CCP graphs of at most 176 vertices and face size at most 6 are Hamiltonian. For point sets this implies that all sets of at most 90 vertices and with vertices of degree at most 6 in its convex hull admit a Hamiltonian tetrahedralizations. See [8] for various similar results on 3CCP graphs. Also a well known conjecture on 3CCP graphs states that all bipartite 3CCP graphs are Hamiltonian [4].

Recently, point sets of 92 with no Hamiltonian path pulling tetrahedralizations have been shown to exist [5]. We improve on this previous result and exhibit a set with less points (84) without a pulling Hamiltonian path tetrahedralizations. Our construction also enable us to find lower bounds on the number of Steiner points added by the algorithm presented in section 2 and 3.

### 4.1 Blowing up vertices

In this section we introduce an operation that will allow us to replace any vertex in a 3CCP graph with another 3CCP graph, so that the resulting graph is again a 3CCP graph. Using this operation we will construct 3CPP graphs with certain desired properties.

Let $G$ and $H$ be 3CCP graphs and $v$ be any vertex of $H$. We may assume that $H$ is embedded in the plane so that $v$ is a vertex on the exterior face. Let
$N_H(v) = \{v_1, v_2, v_3\}$ be the neighbourhood of $v$. Remove $v$ from $H$ and add a path of 4 new vertices $(v'_1, v'_2, w, v'_3)$. Join $v_i$ with $v'_i$; call the resulting graph $H'$, see Figure 3.

Although $H'$ is no longer cubic, it can however be used to replace any vertex $u$ of $G$: Let $N_G(u) = \{u_1, u_2, u_3\}$ with its elements in a given order; Remove $u$ from $G$ and replace it with $H'$; join $v'_i$ with $u'_i$. The resulting graph $G'$ is a 3CCP graph. By choosing an adequate order on the elements of $N_G(u)$, we can place $H$ in a particular face of $G$, see Figure 4.

We use this property to place a non Hamiltonian 3CCP graph $H$ on each face of $K_4$ (which is a 3CCP graph). Call the resulting graph $G$. The faces of $K_4$ isolate each non Hamiltonian graph. Assume that after a removal of a face of $G$ there exists a Hamiltonian path on $G$. Since there are 4 copies of $H$ on $G$ two of them may contain an endpoint of the path, one more may contain the face that was removed. In the remaining copy of $H$ the Hamiltonian path enters and leaves $H$. This is a contradiction since from this path we could derive a Hamiltonian cycle in $H$, see Figure 5.

For $H$ in this construction we may use the smallest non Hamiltonian 3CCP graph on 38 vertices. After each insertion of $H$ into $K_4$, we remove a vertex and add 41 new vertices. In total $G$ has $41 \times 4 = 164$ vertices. Therefore there
exists a polyhedron of \( \frac{164+4}{2} = 84 \) points in \( \mathbb{R}^3 \), whose dual graph is isomorphic to \( G \). By the observations at the beginning of this section, there is no pulling Hamiltonian tetrahedralization of this polyhedron.

Using the same technique we can construct a 3CCP graph that needs a linear number of disjoint cycles to cover its vertices. Take any 3CCP graph and replace every vertex with a copy of the smallest non Hamiltonian graph \( H \) of 38 vertices. In total per vertex 41 new vertices are added. It is easy to see that any partition of the vertices into cycles in this new graph would need at least a cycle per copy of \( H \). Therefore since our algorithm adds one Steiner point per cycle, there are point sets for which our algorithm adds at least \( \frac{m}{41} \) Steiner points.

This gives a lower bound on the number of Steiner points our algorithm might need to find a Hamiltonian tetrahedralization of a given point set.

## 5 Conclusions

In this paper we considered the problem of computing Hamiltonian Tetrahedralizations of point sets in 3-space by adding Steiner points. An algorithm was detailed to do so for points sets with \( n \) points and \( m \) exterior points in time \( O(m^2) + O(n \log n) \). Our algorithm adds at most \( \lceil \frac{m-2}{2} \rceil \) Steiner points.

It seems natural that there must be a compromise between number of Steiner points added and the running time of the algorithm employed to do so. A natural question would be to ask: What is the least number of points that need
to be added while maintaining an efficient algorithm? We conjecture that a sublinear number of Steiner points cannot be added to obtain a Hamiltonian tetrahedralization in polynomial time.

Regarding the implementation of the algorithm, the algorithms we used as a subroutines are not straightforward to program. Simpler algorithms would also be desirable.

On the combinatorial side of the problem, it would be interesting to prove that a sublinear number of Steiner points suffice to obtain a Hamiltonian Tetrahedralization. Note that the current conjecture is that actually none are needed. This weakening of the conjecture, nevertheless is worth studying.

Finally we also showed that point sets with at most 20 points always admit a Hamiltonian tetrahedralization. Actually this tetrahedralization is a pulling tetrahedralization. We also improved the previous upper bound of 92 to 84 on point sets without a Hamiltonian pulling tetrahedralization. This gap remains to be closed.

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