Abstract: Let \( d \in \mathbb{N} \) and let \( D_d \) denote the class of all pairs \((R,M)\) in which \( R = \bigoplus_{n \in \mathbb{N}_0} R_n \) is a Noetherian homogeneous ring with Artinian base ring \( R_0 \) and such that \( M \) is a finitely generated graded \( R \)-module of dimension \( d \). For such a pair \((R,M)\) let \( \mathop{\mathsf{length}}_0 \) denote the (finite) \( R_0 \)-length of the \( n \)-th graded component of the \( i \)-th \( R^+ \)-transform module \( \mathop{\mathsf{length}}_0 \). The cohomology table of a pair \((R,M)\) in \( D_d \) is defined as the family of non-negative integers \( \mathop{\mathsf{length}}_0 \). We say that a subclass \( C \) of \( D_d \) is of finite cohomology if the set \( \{ dM|(R,M)\in C \} \) is finite. A set \( S \subseteq \mathbb{Z} \) is said to bound cohomology, if for each family \((h)\) of non-negative integers, the class \( \{ dM|(R,M)\in C \} \) is finite. Our main result says that this is the case if and only if \( S \) contains a quasi diagonal, that is a set of the form \( \{(i,n)|i=0,\ldots,d-1\} \) with integers \( n_0 > n_1 > \cdots > n_{d-1} \). We draw a number of conclusions of this boundedness criterion.

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BOUNDDEDNESS OF COHOMOLOGY

MARKUS BRODMANN, MARYAM JAHANGIRI, AND CAO HUY LINH

Abstract. Let \(d \in \mathbb{N}\) and let \(\mathcal{D}^d\) denote the class of all pairs \((R, M)\) in which \(R = \bigoplus_{n \in \mathbb{N}_0} R_n\) is a Noetherian homogeneous ring with Artinian base ring \(R_0\) and such that \(M\) is a finitely generated graded \(R\)-module of dimension \(\leq d\).

The cohomology table of a pair \((R, M) \in \mathcal{D}^d\) is defined as the family of non-negative integers \(d_M := \left(d^i_M(n)\right)_{(i, n) \in \mathbb{N} \times \mathbb{Z}}\). We say that a subclass \(\mathcal{C}\) of \(\mathcal{D}^d\) is of finite cohomology if the set \(\{d_M \mid (R, M) \in \mathcal{C}\}\) is finite. A set \(S \subseteq \{0, \cdots, d-1\} \times \mathbb{Z}\) is said to bound cohomology, if for each family \((h^o)_{o \in S}\) of non-negative integers, the class \(\{\{(R, M) \in \mathcal{D}^d \mid d^i_M(n) \leq h^{(i, n)}\} \forall (i, n) \in S\}\) is of finite cohomology. Our main result says that this is the case if and only if \(S\) contains a quasi diagonal, that is a set of the form \(\{(i, n) \mid i = 0, \cdots, d-1\}\) with integers \(n_0 > n_1 > \cdots > n_{d-1}\).

We draw a number of conclusions of this boundedness criterion.

1. Introduction

This paper continues our investigation [6], which was driven by the question "What bounds cohomology of a projective scheme?"

A considerable number of contributions has been given to this theme, mainly under the aspect of bounding some cohomological invariants in term of other invariants (see [1], [2], [3], [4], [7], [8], [9], [11], [12], [13], [15], [16], [17], [18], [19], [21], [22] for example).

Our aim is to start from a different point of view, focussing on the notion of cohomological pattern (s. [5]). So, our main result characterizes those sets \(S \subseteq \{0, \cdots, d-1\} \times \mathbb{Z}\) "which bound cohomology of projective schemes of dimension \(< d\)."

To make this precise, fix a positive integer \(d\) and let \(\mathcal{D}^d\) be the class of all pairs \((R, M)\) in which \(R = \bigoplus_{n \geq 0} R_n\) is a Noetherian homogeneous ring with Artinian base ring \(R_0\) and \(M\) is a finitely generated graded \(R\)-module with \(\text{dim}(M) \leq d\). In this situation let \(R_+ = \bigoplus_{n > 0} R_n\) denote the irrelevant ideal of \(R\).

For each \(i \in \mathbb{N}_0\) consider the graded \(R\)-module \(D^i_{R_+}(M)\), where \(D^i_{R_+}\) denotes the \(i\)-th right derived functor of the \(R_+\)-transform functor \(D_{R_+}((\bullet)) := \lim_{\rightarrow \infty} \text{Hom}_R((R_+)^n, (\bullet))\). In addition, for each \(n \in \mathbb{Z}\) let \(d^i_M(n)\) denote the (finite) \(R_0\)-length of the \(n\)-th graded component \(D^i_{R_+}(M)_n\) of \(D^i_{R_+}(M)\).

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Finally, for \((R, M) \in D^d\) let us consider the so called cohomology table of \((R, M)\), that is the family of non negative integers 

\[d_M := (d_M^i(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}}.\]

A subclass \(C \subseteq D^d\) is said to be of finite cohomology if the set \(\{d_M | (R, M) \in C\}\) is finite. The class \(C\) is said to be of bounded cohomology if the set \(\{d_M^i(n) | (R, M) \in C\}\) is finite for all pairs \((i, n) \in \mathbb{N}_0 \times \mathbb{Z}\). It turns out that these two conditions are both equivalent to the condition that the class \(C\) is of finite cohomology "along some diagonal", e.g. there is some \(n_0 \in \mathbb{Z}\) such that the set \(\triangle_{C,n_0} := \{d_M^i(n_0 - i) | (R, M) \in C, 0 \leq i < d\}\) is finite (s. Theorem 3.5).

So, if one bounds the values of \(d_M^i(n)\) along a "diagonal subset"

\[\{(j, n_0 - j) | j = 0, \ldots, d - 1\} \subseteq \{0, \ldots, d - 1\} \times \mathbb{Z}\]

for an arbitrary integer \(n_0\) one cuts out a subclass \(C \subseteq D^d\) of finite cohomology. Motivated by this observation we say that the subset \(S \subseteq \{0, \ldots, d - 1\} \times \mathbb{Z}\) bounds cohomology in the class \(C \subseteq D^d\) if for each family \((h^\sigma)_{\sigma \in S}\) of non-negative integers \(h^\sigma \in \mathbb{N}_0\) the class

\[\{(R, M) \in C | \forall (i, n) \in S : d_M^i(n) \leq h^{(i,n)}\}\]

is of finite cohomology. Now, we may reformulate our previous result by saying that for arbitrary \(n_0\) the diagonal set \(\{(j, n_0 - j) | j = 0, \ldots, d - 1\}\) bounds cohomology in \(D^d\). It seems rather natural to ask, whether one can characterize the shape of those subsets \(S \subseteq \{0, \ldots, d - 1\} \times \mathbb{Z}\) which bound cohomology in \(D^d\). This is indeed done by our main result (s. Corollary 4.10):

A subset \(S \subseteq \{0, \ldots, d - 1\} \times \mathbb{Z}\) bounds cohomology in \(D^d\) if and only if it contains a quasi-diagonal, that is a set of the form \(\{(i, n_i) | i = 0, \ldots, d - 1\}\) with

\[n_0 > n_1 > \cdots > n_{d-1}.\]

Our next aim is to apply the previous result in order to cut out classes \(C \subseteq D^d\) of finite cohomology by fixing some numerical invariants which are defined on the class \(C\). A finite family \((\mu_i)_{i=1}^r\) of numerical invariants \(\mu_i\) on \(C\) is said to bound cohomology in \(C\) if for all \(n_1, \ldots, n_r \in \mathbb{Z} \cup \{\pm \infty\}\) the class \(\{(R, M) \in C | \mu_i(M) = n_i \text{ for } i = 1, \ldots, r\}\) is of finite cohomology.

We define a numerical invariant \(g : D^d \to \mathbb{N}_0\) by setting \(g(M) := d_M^0(\text{reg}^2(M))\), where \(\text{reg}^2(M)\) denotes the Castelnuovo-Mumford regularity of \(M\) at and above level 2. Then, we show (s. Theorem 5.8):

The pair of invariants \((\text{reg}^2, g)\) bounds cohomology in \(D^d\).

As an application of this we prove (s. Theorem 5.9 and Corollary 5.10)
Fix a polynomial $p \in \mathbb{Q}[t]$ and an integer $r$. Let $\mathcal{C} \subseteq \mathcal{D}^d$ be the class of all pairs $(R, M)$ such that $M$ is a graded submodule of a finitely generated graded $R$-module $N$ with Hilbert polynomial $p_N = p$ and $\text{reg}^2(N) \leq r$. Then $\text{reg}^2$ bounds cohomology in $\mathcal{C}$.

An immediate consequence of this is (s. Corollary 5.11):

Let $(R, N) \in \mathcal{D}^d$, let $r \in \mathbb{Z}$ and let $M$ run through all graded submodules $M \subseteq N$ with $\text{reg}^2(M) \leq r$. Then only finitely many cohomology tables $d_M$ occur.

As applications of this, we generalize two finiteness results of Hoa-Hyry [17] for local cohomology modules of graded ideals in a polynomial ring over a field to graded submodules $M \subseteq N$ for a given pair $(R, N) \in \mathcal{D}^d$ (s. Corollaries 5.13 and 5.14).

In order to translate our results to sheaf cohomology of projective schemes observe that for all $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$ and all pairs $(R, M) \in \mathcal{D}^d$ we have $H^i(X, \mathcal{F}(n)) \cong D^i_{R_+}(M)_n$, where $X := \text{Proj}(R)$ and $\mathcal{F} := \tilde{M}$ is the coherent sheaf of $\mathcal{O}_X$-modules induced by $M$ (see [10, chap. 20] for example).

2. PRELIMINARIES

In this section we recall a few basic facts which shall be used later in our paper.

**Notation 2.1.** Let $R = \oplus_{n \geq 0} R_n$ be a homogeneous Noetherian ring, so that $R$ is positively graded, $R_0$ is Noetherian and $R = R_0[l_0, \cdots, l_r]$ with finitely many elements $l_0, \cdots, l_r \in R_1$. Let $R_+$ denote the irrelevant ideal $\oplus_{n > 0} R_n$ of $R$.

**Reminder 2.2.** (Local cohomology and Castelnuovo-Mumford regularity) (A) Let $i \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}$. By $H^i_{R_+}(\bullet)$ we denote the $i$-th local cohomology functor with respect to $R_+$. Moreover by $D^i_{R_+}(\bullet)$ we denote the $i$-th right derived functor of the ideal transform functor $D^i_{R_+}(\bullet) = \lim_{n \to \infty} \text{Hom}_R((R_+)^n, \bullet)$ with respect to $R_+$.

(B) Let $M := \oplus_{n \in \mathbb{Z}} M_n$ be a graded $R$-module. Keep in mind that in this situation the $R$-modules $H^i_{R_+}(M)$ and $D^i_{R_+}(M)$ carry natural gradings. Moreover we then have a natural exact sequence of graded $R$-modules

$$
(i) \quad 0 \longrightarrow H^0_{R_+}(M) \longrightarrow M \longrightarrow D^0_{R_+}(M) \longrightarrow H^1_{R_+}(M) \longrightarrow 0
$$

and natural isomorphisms of graded $R$-modules

$$
(ii) \quad D^i_{R_+}(M) \cong H^{i+1}_{R_+}(M) \quad \text{for all} \quad i > 0.
$$

(C) If $T$ is a graded $R$-module and $n \in \mathbb{Z}$, we use $T_n$ to denote the $n$-th graded component of $T$. In particular, we define the *beginning* and the *end* of $T$ respectively by
(i) \( \text{beg}(T) := \inf \{ n \in \mathbb{Z} | T_n \neq 0 \} \),

(ii) \( \text{end}(T) := \sup \{ n \in \mathbb{Z} | T_n \neq 0 \} \).

with the standard convention that \( \inf \emptyset = \infty \) and \( \sup \emptyset = -\infty \).

(D) If the graded \( R \)-module \( M \) is finitely generated, the \( R_0 \)-modules \( H^i_{R_+}(M)_n \) are all finitely generated and vanish as well for all \( n \gg 0 \) as for all \( i > \dim(M) \). So, we have

\[-\infty \leq a_i(M) := \text{end}(H^i_{R_+}(M)) < \infty \quad \text{for all } i \geq 0 \]

with \( a_i(M) := -\infty \) for all \( i > \dim(M) \).

If \( k \in \mathbb{N}_0 \), the Castelnuovo-Mumford regularity of \( M \) at and above level \( k \) is defined by

\[ \text{reg}^k(M) := \sup \{ a_i(M) + i \mid i \geq k \} \quad (< \infty). \]

The Castelnuovo-Mumford regularity of \( M \) is defined by \( \text{reg}(M) := \text{reg}^0(M) \).

(E) We also shall use the generating degree of \( M \), which is defined by

\[ \text{gendeg}(M) = \inf \{ n \in \mathbb{Z} \mid M = \Sigma_{m \leq n} RM_m \} \]

If the graded \( R \)-module \( M \) is finitely generated, we have \( \text{gendeg}(M) \leq \text{reg}(M) \).

Reminder 2.3. (Cohomological Hilbert functions) (A) Let \( i \in \mathbb{N}_0 \) and assume that the base ring \( R_0 \) is Artinian. Let \( M \) be a finitely generated graded \( R \)-module. Then, the graded \( R \)-modules \( H^i_{R_+}(M) \) are Artinian. In particular for all \( i \in \mathbb{N}_0 \) and all \( n \in \mathbb{Z} \) we may define the non-negative integers

(i) \( h_M^i(n) := \text{length}_{R_0}(H^i_{R_+}(M)_n) \),

(ii) \( d_M^i(n) := \text{length}_{R_0}(D^i_{R_+}(M)_n) \).

Fix \( i \in \mathbb{N}_0 \). Then the functions

(iii) \( h_M^i : \mathbb{Z} \to \mathbb{N}_0, \quad n \mapsto h_M^i(n) \),

(iv) \( d_M^i : \mathbb{Z} \to \mathbb{N}_0, \quad n \mapsto d_M^i(n) \)

are called the \( i \)-th Cohomological Hilbert functions of the first respectively the second kind of \( M \).

(B) Let \( M \) be a finitely generated graded \( R \)-module and let \( x \in R_1 \). We also write \( \Gamma_{R_+}(M) \) for the \( R_+ \)-torsion submodule of \( M \) which we identify with \( H^0_{R_+}(M) \). By NZD\( _R(M) \) resp. ZD\( _R(M) \) we denote the set of non-zerodivisors resp. of zero divisors of \( R \) with respect to \( M \). The linear form \( x \in R_1 \) is said to be \( (R_+\text{-}) \) filter regular with respect to \( M \) if \( x \in \text{NZD}_R(M/\Gamma_{R_+}(M)) \).
Reminder 2.4. (cf. [6, Definition 5.2]) For \(d \in \mathbb{N}\) let \(D^d\) denote the class of all pairs \((R, M)\) in which \(R = \bigoplus_{n \in \mathbb{N}_0} R_n\) is a Noetherian homogenous ring with Artinian base ring \(R_0\) and \(M = \bigoplus_{n \in \mathbb{Z}} M_n\) is a finitely generated graded \(R\)-module with \(\dim(M) \leq d\).

3. Finiteness and Boundedness of Cohomology

We keep the notations and hypotheses introduced in Section 2.

Definition 3.1. The \textit{cohomology table} of the pair \((R, M) \in D^d\) is the family of non-negative integers

\[d_M := (d^i_M(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}}.\]

Reminder 3.2. (A) According to [5] the \textit{cohomological pattern} \(\mathcal{P}_M\) of the pair \((R, M) \in D^d\) is defined as the set of places at which the cohomology table of \((R, M)\) has a non-zero entry:

\[\mathcal{P}_M := \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid d^i_M(n) \neq 0\}.\]

(B) A set \(P \subseteq \mathbb{N}_0 \times \mathbb{Z}\) is called a \textit{tame combinatorial pattern of width} \(w \in \mathbb{N}_0\) if the following conditions are satisfied:

\[(\pi_1)\quad \exists m, n \in \mathbb{Z} : (0, m), (w, n) \in P;\]

\[(\pi_2)\quad (i, n) \in P \Rightarrow i \leq w;\]

\[(\pi_3)\quad (i, n) \in P \Rightarrow \exists j \leq i : (j, n + i - j + 1) \in P;\]

\[(\pi_4)\quad (i, n) \in P \Rightarrow \exists k \geq i : (k, n + i - k - 1) \in P;\]

\[(\pi_5)\quad i > 0 \Rightarrow \forall n \gg 0 : (i, n) \notin P;\]

\[(\pi_6)\quad \forall i \in \mathbb{N} : (\forall n \ll 0; (i, n) \in P) \text{ or else } (\forall n \ll 0 ; (i, n) \notin P).\]

By [5] we know:

(a) If \((R, M) \in D^d\) with \(\dim(M) = s > 0\), then \(\mathcal{P}_M\) is a tame combinatorial pattern of width \(w = s - 1\).

(b) If \(P\) is a tame combinatorial pattern of width \(w \leq d - 1\), then there is a pair \((R, M) \in D^d\) such that the base ring \(R_0\) is a field and \(P = \mathcal{P}_M\).
By the previous observation, the set of patterns \( \{ P \mid (R, M) \in D \} \) is quite large, and hence so is the set of cohomology tables \( \{ d_M \mid (R, M) \in D \} \). Therefore, one seeks for decompositions \( \bigcup_{i \in I} C_i = D \) of \( D \) into “simpler” subclasses \( C_i \) such that for each \( i \in I \) the set \( \{ d_M \mid (R, M) \in C_i \} \) is finite. Bearing in mind this goal, we define the following concepts:

**Definitions 3.3.** (A) Let \( C \subseteq D \) be a subclass. We say that \( C \) is a subclass of finite cohomology if

\[
\sharp \{ d_M \mid (R, M) \in C \} < \infty.
\]

(B) We say that \( C \subseteq D \) is a subclass of bounded cohomology if

\[
\forall (i, n) \in \mathbb{N}_0 \times \mathbb{Z} : \sharp \{ d^i_M (n) \mid (R, M) \in C \} < \infty.
\]

**Remark 3.4.** (A) Let \( C, D \subseteq D \) be subclasses of \( D \). Then clearly

(a) If \( C \subseteq D \) and \( D \) is of finite cohomology or of bounded cohomology, then so is \( C \) respectively.

(B) If \( r \in \mathbb{Z} \), we have a bijection

\[
\{ d_M \mid (R, M) \in C \} \rightarrow \{ d_M(r) \mid (R, M) \in C \}
\]

given by \( d_M \mapsto d_M(r) \).

Now, we show how the finiteness and boundedness conditions defined above are related.

**Theorem 3.5.** For a subclass \( C \subseteq D \) the following statements are equivalent:

(i) \( C \) is a class of finite cohomology.

(ii) \( C \) is a class of bounded cohomology.

(iii) For each \( n_0 \in \mathbb{Z} \) the set \( \triangle_{C, n_0} := \{ d^i_M (n_0 - i) \mid (R, M) \in C, \ 0 \leq i < d \} \) is finite.

(iv) There is some \( n_0 \in \mathbb{Z} \) such that the set \( \triangle_{C, n_0} \) of statement (iii) is finite.

**Proof.** The implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) are clear from the definitions. To prove the implication (iv) \( \Rightarrow \) (i) fix \( n_0 \in \mathbb{Z} \) and assume that the set \( \triangle_{C, n_0} \) is finite. Then there is some non-negative integer \( h \) such that \( d^i_M (n_0) \leq h \) for all pairs \( (R, M) \in C \) and all \( i \in \{0, \ldots, d - 1 \} \). By [6, Theorem 5.4] it thus follows that the set of functions

\[
\{ d^i_M (n_0) \mid (R, M) \in C, i \in \mathbb{N}_0 \}
\]

is finite. By Remark 3.4 (B) we now may conclude that the class \( C \) is of finite cohomology. \( \square \)

So, by Theorem 3.5 boundedness and finiteness of cohomology are the same for a given class \( C \subseteq D \).
Definition 3.6. Let \( d \in \mathbb{N}_0 \), let \( \mathcal{C} \subseteq \mathcal{D}^d \) and let \( \mathbb{S} \subseteq \{0, \ldots, d-1\} \times \mathbb{Z} \) be a subset. We say that the set \( \mathbb{S} \) bounds cohomology in \( \mathcal{C} \) if for each family \((h^{\sigma})_{\sigma \in \mathbb{S}}\) of non negative integers the class
\[
\{(R, M) \in \mathcal{C} \mid \forall (i, n) \in \mathbb{S} : d_M^i(n) \leq h^{(i,n)}\}
\]
is of finite cohomology.

Remark 3.7. (A) Let \( d \in \mathbb{N}_0 \), let \( \mathcal{C}, \mathcal{D} \subseteq \mathcal{D}^d \) and \( \mathbb{S}, \mathbb{T} \subseteq \{0, \ldots, d-1\} \times \mathbb{Z} \). Then obviously we can say

If \( \mathbb{S} \subseteq \mathbb{T} \) and \( \mathbb{S} \) bounds cohomology in \( \mathcal{C} \), then so does \( \mathbb{T} \).

(B) If \( r \in \mathbb{Z} \), we can form the set \( \mathbb{S}(r) := \{(i, n+r) \mid (i, n) \in \mathbb{S}\} \). In view of the bijection of Remark 3.4 (B) we have

\( \mathbb{S}(r) \) bounds cohomology in \( \mathcal{C}(r) := \{(R, M(r)) \mid (R, M) \in \mathcal{C}\} \) if and only if \( \mathbb{S} \) does in \( \mathcal{C} \).

(C) For all \( s \in \{0, \ldots, d\} \) we set
\[
\mathbb{S}^{<s} := \mathbb{S} \cap (\{0, \ldots, s-1\} \times \mathbb{Z}).
\]
as \( \mathcal{D}^s \subseteq \mathcal{D}^d \) it follows easily:

If \( \mathbb{S} \) bounds cohomology in \( \mathcal{C} \), then \( \mathbb{S}^{<s} \) bounds cohomology in \( \mathcal{D}^s \cap \mathcal{C} \).

Corollary 3.8. Let \( \mathcal{C} \subseteq \mathcal{D}^d \) and \( n \in \mathbb{Z} \). Then, the "\( n \)-th diagonal"
\[
\{(i, n-i) \mid i = 0, \ldots, d-1\}
\]
bounds cohomology in \( \mathcal{C} \).

Proof. This is immediate by Theorem 3.5.

4. QUASI-DIAGONALS

Our first aim is to generalize Corollary 3.8 by showing that not only the diagonals bound cohomology on \( \mathcal{C} \), but rather all “quasi-diagonals”. We shall define below, what such a quasi-diagonal is.

Lemma 4.1. Let \( t \in \{1, \ldots, d\} \), let \( (n_i)_{i=d-t}^{d-1} \) be a sequence of integers such that \( n_{d-1} < \ldots < n_{d-t} \) and let \( \mathcal{C} \subseteq \mathcal{D}^d \) be a class such that the set \( \{d_M^i(n_i) \mid (R, M) \in \mathcal{C}\} \) is finite for all \( i \in \{d-t, \ldots, d-1\} \). Then the set \( \{d_M^i(n) \mid (R, M) \in \mathcal{C}\} \) is finite whenever \( n_i \leq n \) and \( d-t \leq i \leq d-1 \).
Proof. By our hypothesis there is some \( h \in \mathbb{N}_0 \) with \( d^i_M(n_i) \leq h \) for all \( i \in \{d-t, \cdots, d-1\} \) and all pairs \((R, M) \in \mathcal{C}\).

On use of standard reduction arguments we can restrict ourselves to the case where the Artinian base ring \( R_0 \) is local with infinite residue field. Let \((R, M) \in \mathcal{C}\). Replacing \( M \) by \( M/\Gamma_{R_+}(M) \) we may assume that \( M \) is \( R_+ \)-torsion free. Therefore, there exists \( x \in R_1 \cap \text{NZD}(M) \). For each \( i \in \mathbb{N}_0 \) and \( m \in \mathbb{Z} \), the short exact sequence \( 0 \rightarrow M(-1) \rightarrow M \rightarrow M/xM \rightarrow 0 \) induces long exact sequences

\[
(*_{i,m}) \quad D^i_{R_+}(M)_{m-1} \rightarrow D^i_{R_+}(M)_m \rightarrow D^i_{R_+}(M/xM)_m \rightarrow D^{i+1}_{R_+}(M)_{m-1}.
\]

As \( \text{dim}(M/xM) < d \), the sequences \((*_{d-1,m})\) imply that \( d^{i-1}_M(m) \leq d^{d-1}_M(m-1) \) for all \( m \in \mathbb{Z} \). This proves our claim if \( t = 1 \). So, let \( t > 1 \).

Assume inductively that the set \( \{d^i_M(n_i) \mid (R, M) \in \mathcal{C}\} \) is finite whenever \( n_i \leq n \) and \( d-t+1 \leq i \leq d-1 \). It remains to find a family of non-negative integers \( (h_n)_{n \geq n_{d-t}} \) such that \( d^{i-t}_M(n) \leq h_n \) for all \( n \geq n_{d-t} \). Let \( \mathcal{E} \) denote the class of all pairs \((R, M/xM) = (\overline{R}, \overline{M})\) in which \((R, M) \in \mathcal{C}\) and \( x \in R_1 \cap \text{NZD}(M) \). As \( n_i - 1 \geq n_{i+1} \) for all \( i \in \{d-t, \cdots, d-2\} \), the sequences \((*_{i,n_i})\) show that

\[
d^{i-1}_M(n_i) \leq d^{i+1}_M(n_i - 1) + h \text{ for } i \in \{d-t, \cdots, d-2\}.
\]

This means that the set \( \{d^i_M(n_i) \mid (R, \overline{M}) \in \mathcal{E}\} \) is finite whenever \((d-1) - (t-1) \leq i \leq d-2 \). So, by induction the set \( \{d^i_M(n_i) \mid (R, \overline{M}) \in \mathcal{E}\} \) is finite whenever \( n_i \leq n \) and \((d-1) - (t-1) \leq i \leq d-2 \).

In particular there is a family of non-negative integers \( (k_m)_{m \geq n_{d-t}} \) such that \( d^{d-t}_M(M/xM)(m) \leq k_m \) for all \( m \geq n_{d-t} \). Now, for each \( n \geq n_{d-t} \) set \( h_n := h + \sum_{n_{d-t} < m \leq n} k_m \). If we choose \((R, M) \in \mathcal{C}\), the sequences \((*_{d-1,n_i})\) imply that \( d^{i-t}_M(n) \leq h_n \) for all \( n \geq n_{d-t} \).

\[\square\]

**Proposition 4.2.** Let \( (n_i)_{i=0}^{d-1} \) be a sequence of integers such that \( n_{d-1} < \cdots < n_0 \) and let \( \mathcal{C} \subseteq \mathcal{D}^d \). Then the set \( \{(i, n_i) \mid i = 0, \cdots, d-1\} \) bounds cohomology in \( \mathcal{C} \).

Proof. Let \( (h^i)_{i=0}^{d-1} \) be a family of non-negative integers and let \( \mathcal{C}' \) be the class of all pairs \((R, M) \in \mathcal{C}\) such that \( d^i_M(n_i) \leq h^i \) for \( i = 0, \cdots, d-1 \). Then, by Lemma 4.1 the set \( \{d^i_M(n) \mid (R, M) \in \mathcal{C}\} \) is finite, whenever \( n \geq n_i \) and \( 0 \leq i \leq d-1 \). Therefore the set \( \Delta_{\mathcal{C}', n_0} := \{d^i_M(n_0 - i) \mid (R, M) \in \mathcal{C}', 0 \leq i < d\} \) is finite. So, by Theorem 3.5 the class \( \mathcal{C}' \) is of finite cohomology. It follows that \( \{(i, n_i) \mid i = 0, \cdots, d-1\} \) bounds cohomology in \( \mathcal{C} \). \[\square\]

**Definition 4.3.** A set \( \mathbb{T} \subseteq \{0, 1, \cdots, d-1\} \times \mathbb{Z} \) is called a quasi-diagonal if there is a sequence of integers \( (n_i)_{i=0}^{d-1} \) such that \( n_{d-1} < n_{d-2} < \cdots < n_0 \) and

\[
\mathbb{T} = \{(i, n_i) \mid i = 0, \cdots, d-1\}.
\]
Corollary 4.4. Let $\mathcal{S} \subseteq \{0, 1, \cdots, d\} \times \mathbb{Z}$ be a set which contains a quasi-diagonal. Then $\mathcal{S}$ bounds cohomology in each subclass $\mathcal{C} \subseteq \mathcal{D}^d$.

Proof. Clear by Proposition 4.2. \qed

Our next goal is to show that the converse of Corollary 4.4 holds, namely: if a set $\mathcal{S} \subseteq \{0, 1, \cdots, d-1\} \times \mathbb{Z}$ bounds cohomology in $\mathcal{D}^d$, then $\mathcal{S}$ contains a quasi-diagonal.

Reminder 4.5. Let $K$ be a field, let $R = K \oplus R_1 \oplus \cdots$ and $R' = K \oplus R'_1 \oplus \cdots$ be two Noetherian homogeneous $K$-algebras. Let $R \boxtimes K R' := K \oplus (R_1 \otimes R'_1) \oplus (R_2 \otimes R'_2) \oplus \cdots \subseteq R \otimes_K R'$ be the Segre product ring of $R$ and $R'$, a Noetherian homogeneous $K$-algebra. For a graded $R$-module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and a graded $R'$-module $M' = \bigoplus_{n \in \mathbb{Z}} M'_n$ let $M \boxtimes K M' := \bigoplus_{n \in \mathbb{Z}} M_n \otimes_K M'_n \subseteq M \otimes_K M'$ the Segre product module of $M$ and $M'$, a graded $R \boxtimes_K R'$-module. Keep in mind, that the K"unneth relations (for Segre products) yield isomorphism of graded $R \boxtimes_K R'$-modules

$$D^i_{(R \boxtimes_K R')^+}(M \boxtimes K M') \cong \bigoplus_{j=0}^{i} D^j_{R^+}(M) \boxtimes_K D^{i-j}_{R^+}(M')$$

for all $i \in \mathbb{N}_0$ (cf. [23], [14], [20]). \qed

Lemma 4.6. Let $d > 1$ and set $R := K[x_1, \cdots, x_d]$ be a polynomial ring over some infinite field $K$. Let $\mathcal{S} \subseteq \{0, 1, \cdots, d-1\} \times \mathbb{Z}$ such that

1. $\mathcal{S}$ contains no quasi-diagonal,
2. $\mathcal{S} \cap (\{0, \cdots, d-2\} \times \mathbb{Z})$ contains a quasi-diagonal $\{(i, n_i) \mid i = 0, \cdots, d-2\}$ and
3. $\mathcal{S} \cap (\{d-1\} \times \mathbb{Z}) \neq \emptyset$.

Then

(a) $(d-1, n) \notin \mathcal{S}$ for all $n \ll 0$,
(b) There is a family $(M_k)_{k \in \mathbb{N}}$ of finitely generated graded $R$-modules, locally free of rank $\leq ((d-1)!)^2$ on $\text{Proj}(R)$ such that the set $\{d^i_{M_k}(n) \mid k \in \mathbb{N}\}$ is finite for all $(i, n) \in \mathcal{S}$ and

$$\lim_{k \to \infty} d^1_{M_k}(r) = \infty, \text{ where } r := \inf\{n \in \mathbb{Z} \mid (d-1, n) \in \mathcal{S} - 1\}.$$

Proof. For all $i \in \{1, \cdots, d\}$ we write $R^i := K[x_1, \cdots, x_i]$ and $\mathcal{S}^i := \mathcal{S} \cap (\{i\} \times \mathbb{Z})$. Statement (a) follows immediately from our hypotheses on the set $\mathcal{S}$. So, it remains to prove statement (b). After shifting appropriately we may assume that $r = -1$.

By our hypotheses on $\mathcal{S}$ it is clear that $\mathcal{S}^i \neq \emptyset$ for all $i \in \{0, \cdots, d-1\}$. Let

$$\alpha_i := \sup\{n \in \mathbb{Z} \mid (i, n) \in \mathcal{S}^i\}$$

for all $i \in \{0, \cdots, d-1\}$. 

Observe, that diagonals in $\{0, \cdots, d-1\} \times \mathbb{Z}$ are quasi-diagonals. So, the next result generalizes Corollary 3.8.
Then by our hypothesis on $S$ we have $\alpha_i < \infty$ for some $i \in \{1, \cdots, d - 2\}$. Let
\[ s := \min\{i \in \{0, \cdots, d - 2\} \mid \alpha_i < \infty\} \]
and
\[ n_s := \max\{n \in \mathbb{Z} \mid (s, n) \in S^*\}. \]

Now, we may find a quasi-diagonal $\{(i, n_i) \mid i = 0, \cdots, d - 2\}$ in $S \cap (\{0, \cdots, d - 2\} \times \mathbb{Z})$ such that for all $i \in \{s + 1, \cdots, d - 2\}$ we have
\[ n_i = \max\{n < n_{i-1} \mid (i, n) \in S\}. \]

As $S$ contains no quasi-diagonal, we must have $n_{d-2} \leq 0$. For all $m, n \in \mathbb{Z} \cup \{\pm\infty\}$ we write $[m, n[ := \{t \in \mathbb{Z} \mid m < t < n\}$. Using this notation we set
\[ t_{-1} := \infty; \ t_{d-s-1} := -\infty; \ t_i := \max\{d - s - i - 2, n_{i+s}\}, \ \forall i \in \{0, \cdots, d - s - 2\} \]
and write
\[ P := \bigcup_{i=0}^{d-s-1} ([i] \times [t_i, t_{i-1}]). \]

Observe, that by our choice of the pairs $(i, n_i)$ we have
\[(*) \quad \text{if } s \leq i \leq d - 1 \text{ and } (i, n) \in S, \text{ then } (i - s, n) \notin P. \]
Moreover by [5, 2.7] the set $P \subseteq \{0, \cdots, d - s - 1\} \times \mathbb{Z}$ is a minimal combinatorial pattern of width $d - s - 1$. So, by [5, Proposition 4.5], there exists a finitely generated $R^{d-s}$-module $N$, locally free of rank $\leq (d - s - 1)!$ on $\text{Proj}(R^{d-s})$ such that $\mathcal{P}_N = P$.

Now, consider the Segre product ring $S := R^{s+1} \boxtimes_K R^{d-s}$ and for each $k \in \mathbb{N}$ let $M_k$ be the finitely generated graded $S$-module $R^{s+1}(-k) \boxtimes_K N$, which is locally free of rank $\leq (d - 1)!/s!$ on $\text{Proj}(S)$. Observe that
\[ d_{R^{s+1}}^j \equiv 0 \text{ for all } j \neq 0, \text{ and } d_N^l \equiv 0 \text{ for all } l > d - s - 1. \]

Now, we get from the Künneth relations (cf. Reminder 4.5) for all $i \in \{0, \cdots, d - 1\}$ and all $n \in \mathbb{Z}$
\[ d_{M_k}^i(n) = \begin{cases} d_{R^{s+1}}^i(-k + n)d_N^i(n) & \text{for } 0 \leq i < s \\ d_{R^{s+1}}^i(-k + n)d_N^s(n) + d_{R^{s+1}}^s(-k + n)d_N^{i-s}(n) & \text{for } s \leq i \leq d - s - 1, \\ d_{R^{s+1}}^i(-k + n)d_N^{d-s}(n) & \text{for } d - s - 1 < i \leq d - 1. \end{cases} \]

As $P = \mathcal{P}_N$ and in view of $(*)$ we have $d_N^{i-s}(n) = 0$ for all $(i, n) \in S$ with $s \leq i \leq d - 1$. Moreover, for all $n \in \mathbb{Z}$ and all $k \in \mathbb{N}$ we have $d_{R^{s+1}}^0(-k + n) \leq d_{R^{s+1}}^0(n - 1)$. So for all $k \in \mathbb{N}$ and all $(i, n) \in S$ we get
\[ d_{M_k}^i(n) \begin{cases} \leq d_{R^{s+1}}^i(n - 1)d_N^i(n), & \text{for } 0 \leq i \leq d - s - 1, \\ = 0, & \text{if } d - s - 1 < i \leq d - 1. \end{cases} \]

Therefore the set $\{d_{M_k}^i(n) \mid k \in \mathbb{N}\}$ is finite for all $(i, n) \in S$.
Moreover $d_{M_k}^{d-1}(-1) = d_{R^{s+1}}^s(-k-1)d_N^s(-1)$. As $(d-s-1, -1) \in P$ we have $d_{N}^{d-s-1}(-1) > 0$ and hence $d_{R^{s+1}}^s(-k-1) = \binom{k}{s}$ implies that
\[
\lim_{k \to \infty} d_{M_k}^{d-1}(-1) = \infty.
\]

As $\dim(S) = d$, there is a finite injective morphism $R \to S$ of graded rings, which turns $S$ in an $R$-module of rank $(d-1)!/s!(d-s-1)!$. So $M_k$ becomes an $R$-module locally free of rank $\leq [(d-1)!/s!(d-s-1)!][(d-1)!/s!] \leq ((d-1)!)^2$ on $\text{Proj}(R)$. Moreover, by Graded Base Ring Independence of Local Cohomology, we get isomorphisms of graded $R$-modules $D_{S+}^1(M_k) \cong D_{R+}^1(M_k)$ for all $j \in \mathbb{N}_0$. Now, our claim follows easily. \qed

**Definition 4.7.** A class $\mathcal{D} \subseteq \mathcal{D}^d$ is said to be big, if for each $t \in \{1, \ldots, d\}$ there is an infinite field $K$ such that $\mathcal{D}$ contains all pairs $(R, M)$ in which $R$ is the polynomial ring $K[x_1, \ldots, x_t]$. \hfill \bull

**Proposition 4.8.** Let $\mathcal{C} \subseteq \mathcal{D}^d$ be a big class and let $\mathcal{S} \subseteq \{0, \ldots, d-1\} \times \mathbb{Z}$ be a set which bounds cohomology in $\mathcal{C}$. Then $\mathcal{S}$ contains a quasi-diagonal.

**Proof.** There is an infinite field $K$ such that with $R := K[x_1, \ldots, x_d]$ we have $(R, R(-k)) \in \mathcal{C}$ for all $k \in \mathbb{N}$. The set $\{d_{R(-k)}^t(n) \mid k \in \mathbb{N}\}$ is finite for all $(i, n) \in \{0, \ldots, d-2\} \times \mathbb{Z}$ and $\lim_{k \to \infty} d_{R(-k)}^{d-1}(0) = \infty$. It follows that $\mathcal{S}^{d-1} := \mathcal{S} \cap (\{d-1\} \times \mathbb{Z}) \neq \emptyset$. This proves our claim if $d = 1$.

So, let $d > 1$. Clearly $\mathcal{D}^{d-1} \cap \mathcal{C} \subseteq \mathcal{D}^{d-1}$ is a big class and $\mathcal{S}^{(d-1)} \subseteq \mathcal{S} \cap (\{0, \ldots, d-2\} \times \mathbb{Z})$ bounds cohomology in $\mathcal{D}^{d-1} \cap \mathcal{C}$ (s. Remark 3.7 (C)). So, by induction the set $\mathcal{S}^{(d-1)}$ contains a quasi-diagonal. If $\mathcal{S}$ would contain no quasi-diagonal, Lemma 4.6 would imply that for our polynomial ring $R$ there is a class $\mathcal{D}$ of pairs $(R, M) \in \mathcal{D}^d$ which is not of bounded cohomology but such that the set $\{d_M^t(n) \mid (R, M) \in \mathcal{D}\}$ is finite for all $(i, n) \in \mathcal{S}$. As $\mathcal{C}$ is a big class, we have $\mathcal{D} \subseteq \mathcal{C}$, and this would imply the contradiction that $\mathcal{S}$ does not bound cohomology in $\mathcal{C}$. \qed

**Theorem 4.9.** Let $\mathcal{C} \subseteq \mathcal{D}^d$ be a big class and let $\mathcal{S} \subseteq \{0, \ldots, d-1\} \times \mathbb{Z}$. Then $\mathcal{S}$ bounds cohomology in $\mathcal{C}$ if and only if $\mathcal{S}$ contains a quasi-diagonal.

**Proof.** Clear by Corollary 4.4 and Proposition 4.8. \qed

**Corollary 4.10.** The set $\mathcal{S} \subseteq \{0, \ldots, d-1\} \times \mathbb{Z}$ bounds cohomology in $\mathcal{D}^d$ if and only if $\mathcal{S}$ contains a quasi-diagonal.

**Proof.** Clear by Theorem 4.9. \qed
5. Bounding Invariants

In this section we investigate numerical invariants which bound cohomology.

**Definitions 5.1.** (A) (s. [2], [8], [9]). Let $C \subseteq \mathcal{D}^d$ be a subclass. A numerical invariant on the class $C$ is a map

$$\mu : C \to \mathbb{Z} \cup \{\pm \infty\}$$

such that for any two pairs $(R, M), (R, N) \in C$ with $M \cong N$ we have $\mu(R, M) = \mu(R, N)$. We shall write $\mu(M)$ instead of $\mu(R, M)$.

(B) Let $(\mu_i)_{i=1}^r$ be a family of numerical invariants on the subclass $C \subseteq \mathcal{D}^d$. We say that the family $(\mu_i)_{i=1}^r$ bounds cohomology on the class $C$, if for each $(n_1, \cdots, n_r) \in (\mathbb{Z} \cup \{\pm \infty\})^r$ the class

$$\{(R, M) \in C \mid \mu_i(M) = n_i \text{ for all } i \in \{1, \cdots, r\}\}$$

is of bounded cohomology.

(C) A numerical invariant $\mu$ on the class $C \subseteq \mathcal{D}^d$ is said to be finite if $\mu(M) \in \mathbb{Z}$ for all $(R, M) \in C$.

(D) A numerical invariant $\mu$ on the class $C \subseteq \mathcal{D}^d$ is said to be positive if $\mu(M) \geq 0$ for all $(R, M) \in C$.

**Remark 5.2.** (A) If $\mu : C \to \mathbb{Z} \cup \{\pm \infty\}$ is a numerical invariant on the class $C \subseteq \mathcal{D}^d$ and if $\mathcal{D} \subseteq C$, then the restriction $\mu \upharpoonright \mathcal{D} : \mathcal{D} \to \mathbb{Z} \cup \{\pm \infty\}$ is a numerical invariant on the class $\mathcal{D}$. Clearly, if $\mu$ is finite (resp. positive) then so is $\mu \upharpoonright \mathcal{D}$.

(B) If $(\mu_i)_{i=1}^r$ bounds cohomology on the class $C \subseteq \mathcal{D}^d$ and if $\mathcal{D} \subseteq C$, then $(\mu_i \upharpoonright \mathcal{D})_{i=1}^r$ bounds cohomology in $\mathcal{D}$.

(C) A family $(\mu_i)_{i=1}^r$ of positive numerical invariants bounds cohomology in $C$ if and only if for all $(n_1, \cdots, n_r) \in (\mathbb{N}_0 \cup \{\infty\})^r$ the class

$$\{(R, M) \in C \mid \mu_i(M) \leq n_i \text{ for all } i \in \{1, \cdots, r\}\}$$

is of bounded cohomology.

(D) A family $(\mu_i)_{i=1}^r$ of finite positive invariants bounds cohomology on $C$ if and only if the sum invariant $\sum_{i=1}^r \mu_i : C \to \mathbb{N}_0$ bounds cohomology in $C$.

**Remark 5.3.** Let $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. Then, the map

$$d_i^* : \mathcal{D}^d \to \mathbb{N}_0; \quad ((R, M) \mapsto d_i^*(n))$$

is a finite positive numerical invariant on $\mathcal{D}^d$.

**Theorem 5.4.** Let $(n_i)_{i=0}^{d-1}$ be a sequence of integers such that $n_0 > n_1 > n_2 > \cdots > n_{d-1}$. Then the family of numerical invariants $(d_i^*(n_i))_{i=0}^{d-1}$ bounds cohomology in $\mathcal{D}^d$.

**Proof.** Clear by Proposition 4.2. $\square$
**Reminder 5.5.** For each \( k \in \mathbb{N}_0 \) we may define the numerical invariant
\[
\text{reg}^k : D^d \to \mathbb{Z} \cup \{-\infty\}; \quad ((R, M) \mapsto \text{reg}^k(M)).
\]

**Notation 5.6.** For \((R, M) \in D^d\) we set
\[
\varrho(M) := \begin{cases} 
    d_M^0(\text{reg}^2(M)), & \text{if } \dim(M) > 1, \\
    d_M^0(0), & \text{if } \dim(M) \leq 1.
\end{cases}
\]

**Remark 5.7.** (A) If \((R, M) \in D^d\) with \( \dim(M) \leq 1 \), the cohomological Hilbert function \( d_M^0 \) of \( M \) is constant, and this constant is strictly positive if and only if \( M \neq 0 \).

(B) The function \( \varrho : D^d \to \mathbb{N}_0; \quad ((R, M) \mapsto \varrho(M)) \) is a finite positive numerical invariant on \( D^d \).

**Theorem 5.8.** The pair of invariants \((\text{reg}^2, \varrho)\) bounds cohomology in \( D^d \).

**Proof.** Fix \( u, v \in \mathbb{Z} \) and set
\[
C := \{ (R, M) \in D^d \mid \text{reg}^2(M) = u, \varrho(M) = v \}.
\]
If \((R, M) \in C\) we have \( d_M^0(u) = d_M^0(\text{reg}^2(M)) = v \).

Let \( i \in \mathbb{N} \). Then \( u - i = \text{reg}^2(M) - i > a_{i+1}(M) \) and hence \( d_M^i(u - i) = h_{M}^{i+1}(u - i) = 0 \). Therefore \((R, M)\) belongs to the class
\[
D := \{ (R, M) \in D^d \mid d_M^0(u) = v \text{ and } d_M^i(u - i) = 0 \text{ for all } i \in \{1, \ldots, d - 1\} \}.
\]
But according to Theorem 5.4 the class \( D \) is of bounded cohomology. \( \square \)

**Lemma 5.9.** Let \((R, M) \in D^d\) be such that \( \dim(R/p) \neq 1 \) for all \( p \in \text{Ass}_R(M) \). Then
\[
d_M^0(n - 1) \leq \max\{0, d_M^0(n) - 1\} \quad \text{for all } n \in \mathbb{Z}.
\]

**Proof.** For an arbitrary finitely generated graded \( R \)-module \( N \) let
\[
\lambda(N) := \inf\{\text{depth}(N_p) + \text{height}((p + R_+)/p) \mid p \in \text{Spec}(R) \setminus \text{Var}(R_+)\}.
\]
Clearly, for all \( n \in \mathbb{Z} \) we have \( \lambda(N(n)) = \lambda(N) \). So, for all \( n \in \mathbb{Z} \), we get by our hypotheses that \( \lambda(M(n)) = \lambda(M) > 1 \). Now, according to [8, Proposition 4.6] we obtain
\[
d_M^0(n - 1) = d_M^0(n)(-1) \leq \max\{0, d_M^0(n)(0) - 1\} = \max\{0, d_M^0(n) - 1\}.
\]
\( \square \)

**Theorem 5.10.** Let \( r, s \in \mathbb{Z} \) and let \( p \in \mathbb{Q}[t] \) be a polynomial. Let \( C \subseteq D^d \) be the class of all pairs \((R, M) \in D^d\) satisfying the following conditions:
(α) There is a finitely generated graded R-module N with Hilbert polynomial \( p_N = p \) and \( \reg^2(N) \leq r \) such that \( M \subseteq N \).

(β) \( \reg^2(M) \leq s \).

Then, \( C \) is a class of finite cohomology.

**Proof.** Let \( v := \max\{r, s\} \). We first show that for each pair \((R, M) \in C\) we have

\[
\tag{*} \quad \varrho(M) \leq p(v)
\]

and

\[
\tag{**} \quad \dim(M) \leq 1 \text{ or } \reg^2(M) \geq -v - p(v).
\]

So, let \((R, M) \in C\). Then, there is a monomorphism of finitely generated graded R-modules \( M \hookrightarrow N \) such that \( p_N = p \) and \( \reg^2(N) \leq r \leq v \).

Assume first that \( \dim(M) > 1 \). As \( \reg^2(M) \leq v \) we then get

\[
\varrho(M) = d_M^0(\reg^2(M)) \leq d_M^0(v) \leq d_N^0(v) = p_N(v) = p(v).
\]

If \( \dim(M) \leq 1 \), the function \( d_M^0 \) is constant and therefore

\[
\varrho(M) = d_M^0(0) = d_M^0(v) \leq d_N^0(v) = p_N(v) = p(v).
\]

Thus we have proved statement (⋆).

To prove statement (**) we assume that \( \dim(M) > 1 \). Then there is a short exact sequence of finitely generated graded R-modules

\[
0 \longrightarrow H \longrightarrow M \longrightarrow \overline{M} \longrightarrow 0
\]

such that \( \dim(H) \leq 1 \) and \( \text{Ass}_R(\overline{M}) \) does not contain any prime \( p \) with \( \dim(R/p) \leq 1 \). As \( \dim(H) \leq 1 \), we have \( H^i_{R_+}(H) = 0 \) for all \( i > 1 \). Therefore \( H^i_{R_+}(M) \cong H^i_{R_+}(\overline{M}) \) for all \( i > 1 \) and hence \( \reg^2(M) = \reg^2(\overline{M}) \). Moreover by the observation made on \( \text{Ass}_R(\overline{M}) \), we have (s. Lemma 5.9)

\[
d_M^0(n-1) \leq \max\{0, d_M^0(n) - 1\} \text{ for all } n \in \mathbb{Z}.
\]

As \( D^1_{R_+}(H) = H^2_{R_+}(H) = 0 \), we have

\[
d_M^0(v) \leq d_M^0(v) \leq d_N^0(v) = p_N(v) = p(v)
\]

and it follows that

\[
d_M^0(n) = 0 \text{ for all } n \leq -v - p(v) - 1.
\]

One consequence of this is, that \( T := D^0_{R_+}(\overline{M}) \) is a finitely generated R-module. As \( H^i_{R_+}(M) \cong H^i_{R_+}(\overline{M}) \) for all \( i > 1 \), we have \( \reg^2(T) = \reg^2(\overline{M}) = \reg^2(M) \). As \( H^i_{R_+}(T) = 0 \) for \( i = 0, 1 \), we thus get \( \reg^2(M) = \reg(T) \). As \( T_n = 0 \) for all \( n \leq -v - p(v) - 1 \), we finally obtain (s. Reminder 2.2(E))

\[
\reg^2(M) = \reg(T) \geq \text{gendeg}(T) \geq \text{beg}(T) \geq -v - p(v).
\]

This proves statement (**).
Now, we may write
\[ C \subseteq C_{-\infty} \cup \bigcup_{t = -v - p(v)}^{s} C_t, \]
where
\[ C_{-\infty} := \{(R, M) \in D^d \mid \dim(M) \leq 1 \text{ and } \rho(M) \leq p(v)\}, \]
and, for all \( t \in \mathbb{Z} \) with \( -v - p(v) \leq t \leq s \),
\[ C_t := \{(R, M) \in D^d \mid \operatorname{reg}^2(M) = t, \rho(M) \leq p(v)\}. \]

The class \( C_{-\infty} \) clearly is of bounded cohomology.

Now, by Remark 5.2(C) and by Corollary 5.8, each of the classes \( C_t \) is of bounded cohomology. This proves our claim. \( \square \)

**Corollary 5.11.** Let \( r \in \mathbb{Z} \) and let \( p \in \mathbb{Q}[t] \) be a polynomial. Let \( C \subseteq D^d \) be the class of all pairs \((R, M) \in D^d\) satisfying the condition \((\alpha)\) of Theorem 5.10. Then, the invariant \( \operatorname{reg}^2 \) bounds cohomology in the class \( C \).

**Proof.** This is immediate by Theorem 5.10. \( \square \)

**Corollary 5.12.** Let \( r \in \mathbb{Z} \) and let \((R, N) \in D^d\). If \( M \) runs through all graded submodules \( M \subseteq N \) with \( \operatorname{reg}^2(M) \leq r \), only finitely many cohomology tables \( d_M \) and hence only finitely many Hilbert polynomials \( p_M \) occur.

**Proof.** This is clear by Theorem 5.10. \( \square \)

**Corollary 5.13.** Let \( r \in \mathbb{Z} \) and let \((R, N) \in D^d\). If \( M \) runs through all graded submodules of \( N \) with \( \operatorname{reg}^1(M) \leq r \) only finitely many families
\[ (h_i^N(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}} \quad \text{and} \quad (h_i^N(M)(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}} \]
can occur.

**Proof.** Let \( \mathcal{P} \) be the set of all graded submodules \( M \subseteq N \) with \( \operatorname{reg}^1(M) \leq r \).

Now, for each \( M \in \mathcal{P} \) we have the following three relations
\[ d_M^0(n) = h_M^1(n) \text{ for all } i \geq 1 \text{ and all } n \in \mathbb{Z}; \]
\[ \begin{cases} h_M^1(n) \leq d_M^0(n) & \text{for all } n \in \mathbb{Z}; \\ h_M^1(n) = d_M^0(n) & \text{for all } n < \text{beg}(N); \\ h_M^1(n) = 0 & \text{for all } n \geq r \end{cases} \]
and
\[ h_M^0(n) \leq h_N^0(n) \text{ for all } n \in \mathbb{Z}. \]
So, by Corollary 5.12 the set
\[ \mathcal{U} := \{(h_M^i(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}} \mid M \in \mathcal{P}\} \]
is finite.

For each \( M \in \mathcal{P} \) the short exact sequence \( 0 \rightarrow M \rightarrow N \rightarrow N/M \rightarrow 0 \) yields that for all \( n \in \mathbb{Z} \) and all \( i \in \mathbb{N}_0 \)
\[
h^0_{N/M}(n) \leq h^0_N(n) + h^1_M(n), \tag{1}
\]
\[
d^i_{N/M}(n) \leq d^i_N(n) + h^{i+2}_M(n). \tag{2}
\]
By the finiteness of \( \mathcal{U} \) it follows that the set of functions
\[
\mathcal{U}_0 := \{(h^0_{N/M}(n))_{n \in \mathbb{Z}} \mid M \in \mathcal{P}\}
\]
is finite and that the set of cohomology diagonals
\[
\mathcal{W} := \{(d^i_{N/M}(-i))_{i=0}^{d-1} \mid M \in \mathcal{P}\}
\]
is finite.

In view of Theorem [6, Theorem 5.4] the finiteness of \( \mathcal{W} \) implies that the set
\[
\mathcal{U}_1 := \{(d^i_{N/M}(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}} \mid M \in \mathcal{P}\}
\]
is finite. Moreover for all \( M \in \mathcal{P} \) we have
\[
\text{end}(H^1_{R_+}(N/M)) < \text{reg}^1(N/M) \leq \max\{\text{reg}^2(M) - 1, \text{reg}^2(N)\} \leq \max\{r - 1, \text{reg}^1(N)\};
\]
\[
h^1_{N/M}(n) \leq d^0_{N/M}(n) \text{ for all } n \in \mathbb{Z}, \text{ with equality if } n < \text{beg}(N).
\]
As \( d^i_{N/M} \equiv h^{i+1}_{N/M} \) for all \( i > 0 \) the finiteness of \( \mathcal{U}_0 \) and \( \mathcal{U}_1 \) shows that the set
\[
\{(h^i_{N/M}(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}} \mid M \in \mathcal{P}\}
\]
is finite, too. \( \Box \)

**Corollary 5.14.** Assume that \( R \) is a homogeneous Noetherian Cohen-Macaulay ring with Artinian local base ring \( R_0 \). Let \( s \in \mathbb{Z} \) and let \( N \) be a finitely generated graded \( R \)-module. If \( M \) runs trough all graded submodules of \( N \) with \( \text{gendeg}(M) \leq s \) only finitely many families
\[
(h^i_M(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}} \text{ and } (h^i_{N/M}(n))_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}}
\]
may occur.

**Proof.** By [4, Proposition 6.1] we see that \( \text{reg}(M) \) finds an upper bound in terms of \( \text{gendeg}(M) \), \( \text{reg}(N) \), \( \text{reg}(R) \), \( \text{beg}(N) \), \( \text{dim}(R) \), the multiplicity \( e_0(R) \) of \( R \) and the minimal number of homogeneous generators of the \( R \)-module \( N \). Now, we conclude by Corollary 5.13. \( \Box \)

**Remark 5.15.** If we apply Corollary 5.13 in the special case where \( N = R = K[x_1, \cdots, x_r] \) is a polynomial ring over a field, we get back the finiteness result [17, Corollary 14]. Correspondingly, if we apply Corollary 5.14 in this special case, we get back [17, Corollary 20].
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