**The Garden of Eden theorem for cellular automata on group sets**

Simon Wacker

Institute of Theoretical Informatics, Karlsruhe Institute of Technology, Karlsruhe, Germany

**ABSTRACT**

We prove the Garden of Eden theorem for big-cellular automata with finite set of states and finite neighbourhood on right amenable left homogeneous spaces with finite stabilisers. It states that the global transition function of such an automaton is surjective if and only if it is pre-injective. Pre-Injectivity means that two global configurations that differ at most on a finite subset and have the same image under the global transition function must be identical. The theorem is proven by showing that the global transition function of an automaton as above is surjective if and only if its image has maximal entropy and that its image has maximal entropy if and only if it is pre-injective. Entropy of a subset of global configurations measures the asymptotic growth rate of the number of finite patterns with growing domains that occur in the subset.

The notion of amenability for groups was introduced by John von Neumann in 1929. It generalises the notion of finiteness. A group $G$ is left or right amenable if there is a finitely additive probability measure on $\mathcal{P}(G)$ that is invariant under left and right multiplication respectively. Groups are left amenable if and only if they are right amenable. A group is amenable if it is left or right amenable.

The definitions of left and right amenability generalise to left and right group sets respectively. A left group set $(M, G, \triangleright)$ is left amenable if there is a finitely additive probability measure on $\mathcal{P}(M)$ that is invariant under $\triangleright$. There is in general no natural action on the right that is to a left group action what right multiplication is to left group multiplication. Therefore, for a left group set there is no natural notion of right amenability.

**CONTACT** Simon Wacker simon.wacker@kit.edu

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A transitive left group action \( \triangleright \) of \( G \) on \( M \) induces, for each element \( m_0 \in M \) and each family \( \{ g_{m_0 m} \}_{m \in M} \) of elements in \( G \) such that, for each point \( m \in M \), we have \( g_{m_0 m} \triangleright m_0 = m \), a right quotient set semi-action \( \triangleleft \) of \( G \) on \( M \) with defect \( G_0 \) given by \( m \triangleleft g G_0 = g g_{m_0 m}^{-1} \triangleright m \), where \( G_0 \) is the stabiliser of \( m_0 \) under \( \triangleright \). Each of these right semi-actions is to the left group action what right multiplication is to left group multiplication. They occur in the definition of global transition functions of semi-cellular automata over left homogeneous spaces as defined in [7]. A cell space is a left group set together with choices of \( m_0 \) and \( \{ g_{m_0 m} \}_{m \in M} \).

A cell space \( \mathcal{R} \) is right amenable if there is a finitely additive probability measure on \( \mathcal{P}(M) \) that is semi-invariant under \( \triangleleft \). For example cell spaces with finite sets of cells, abelian groups, and finitely right generated cell spaces with finite stabilisers of sub-exponential growth are right amenable, in particular, quotients of finitely generated groups of sub-exponential growth by finite subgroups acted on by left multiplication. A net of non-empty and finite subsets of \( M \) is a right Følner net if, broadly speaking, these subsets are asymptotically invariant under \( \triangleleft \). A finite subset \( E \) of \( G / G_0 \) and two partitions \( \{ A_e \}_{e \in E} \) and \( \{ B_e \}_{e \in E} \) of \( M \) constitute a right paradoxical decomposition if the map \( _e \triangleleft e \) is injective on \( A_e \) and \( B_e \), and the family \( \{ (A_e \triangleleft e) \cup (B_e \triangleleft e) \}_{e \in E} \) is a partition of \( M \). The Tarski–Følner theorem states that right amenability, the existence of right Følner nets, and the non-existence of right paradoxical decompositions are equivalent. We prove it in [8] for cell spaces with finite stabilisers.

For a right amenable cell space with finite stabilisers we may choose a right Følner net \( \mathcal{F} = \{ F_t \}_{t \in T} \). The entropy of a subset \( X \) of \( Q^M \) with respect to \( \mathcal{F} \), where \( Q \) is a finite set, is, broadly speaking, the asymptotic growth rate of the number of finite patterns with domain \( F_t \) that occur in \( X \). For subsets \( E \) and \( E' \) of \( G / G_0 \), an \( (E, E') \)-tiling is a subset \( T \) of \( M \) such that \( \{ t \triangleleft E \}_{t \in T} \) is pairwise disjoint and \( \{ t \triangleleft E' \}_{t \in T} \) is a cover of \( M \). If for each point \( t \in T \) not all patterns with domain \( t \triangleleft E \) occur in a subset of \( Q^M \), then that subset does not have maximal entropy.

The global transition function \( \Delta \) of a big-cellular automaton with finite set of states and finite neighbourhood over a right amenable cell space with finite stabilisers, as introduced below, is surjective if and only if its image has maximal entropy. Indeed, if \( \Delta \) is surjective, then its image is equal to the set of all global configurations, which has maximal entropy. And, if \( \Delta \) is not surjective, then there is a global configuration that is not in its image; thus, because \( \Delta \) is continuous and \( Q^M \) is compact, where \( Q^M \) is equipped with the prodiscrete topology, there is a finite pattern that does not occur in the global configurations of \( \Delta(Q^M) \); and hence, \( \Delta(Q^M) \) does not have maximal entropy.

The image of the global transition function \( \Delta \) has maximal entropy if and only if it is pre-injective. Indeed, if \( \Delta(Q^M) \) does not have maximal entropy, that is, the asymptotic growth rate of finite patterns in \( \Delta(Q^M) \) is less than the one of \( Q^M \), then there are two distinct finite patterns with the same domain that can be identically extended to global configurations with the same image under \( \Delta \) and thus \( \Delta \) is not pre-injective. And, if \( \Delta \) is not pre-injective, then there are two distinct finite patterns \( p \) and \( p' \) with the same domain that have the same image under a restriction of \( \Delta \); thus, \( \Delta(Q^M) \) is equal to the image of the set \( Y \) of all global configurations in which the pattern \( p \) does not occur at the cells of a tiling, which is chosen such that its cells are far apart with respect to the domain of \( p \); and hence, because the entropy of \( \Delta(Y) \) is less than or equal to the one of \( Y \) and the entropy of \( Y \) is not maximal, the entropy of \( \Delta(Q^M) \) is not maximal.

The previous two paragraphs establish the Garden of Eden theorem, which states that a global transition function as above is surjective if and only if it is pre-injective. This answers a question posed by Sébastien Moriceau at the end of his paper ‘Cellular Automata on a G-Set’ [6]. The Garden of Eden theorem for cellular automata over \( \mathbb{Z}^2 \) is a famous theorem by Edward Forrest Moore and John R. Myhill from 1962 to 1963, see the papers ‘Machine models of self-reproduction’ [4] and ‘The converse of Moore’s Garden-of-Eden theorem’ [5]. The present paper is greatly inspired by the monograph ‘Cellular Automata and Groups’ [1] by Tullio CeccheriniSPSHYPSilberstein and Michel Coornaert.

In Section 2 we introduce \( E \)-interiors, \( E \)-closures, and \( E \)-boundaries of subsets of \( M \), characterise right Følner nets using boundaries, which motivates the definition of right Erling nets and tractability, and present examples that are used throughout the present paper. In Section 3 we introduce \( (E, E') \)-tilings of cell spaces, show their existence, and relate them, interiors, and right Erling nets combinatorially.
In Section 4 we introduce entropies of subsets of \( Q^M \), show that applications of global transition functions to subsets of \( Q^M \) do not increase entropy, and show that subsets of \( Q^M \) that miss a pattern at each cell of a tiling do not have maximal entropy. In Section 5 we prove the Garden of Eden theorem by characterising surjectivity and pre-injectivity by maximality of the entropy of the image. And in Section 6 we construct non-degenerate right amenable left homogeneous spaces.

1. Preliminary notions

A left group set is a triple \( (M, G, \triangleright) \), where \( M \) is a set, \( G \) is a group, and \( \triangleright \) is a map from \( G \times M \) to \( M \), called left group action of \( G \) on \( M \), such that \( G \rightarrow \operatorname{Sym}(M) \), \( g \mapsto (g \triangleright \cdot) \), is a group homomorphism. The action \( \triangleright \) is transitive if \( M \) is non-empty and for each \( m \in M \) the map \( \triangleright m \) is surjective; and free if for each \( m \in M \) the map \( \triangleright m \) is injective. For each \( m \in M \), the set \( G_m \) is the orbit of \( m \), the set \( G_0 = (\triangleright m)^{-1}(m) \) is the stabiliser of \( m \), and, for each \( m' \in M \), the set \( G_{m,m'} = (\triangleright m)^{-1}(m') \) is the transporter of \( m \) to \( m' \).

A left homogeneous space is a left group set \( \mathcal{M} = (M, G, \triangleright) \) such that \( \triangleright \) is transitive. A coordinate system for \( \mathcal{M} \) is a tuple \( K = (m_0, \{g_{m_0, m}\}_{m \in M}) \), where \( m_0 \in M \) and for each \( m \in M \) we have \( g_{m_0, m} \triangleright m_0 = m \). The stabiliser \( G_{m_0} \) is denoted by \( G_0 \). The tuple \( \mathcal{R} = (\mathcal{M}, K) \) is a cell space. The set \( \{gG_0 \mid g \in G\} \) of left cosets of \( G_0 \) in \( G \) is denoted by \( G/G_0 \). The map \( \triangleleft : M \times G/G_0 \rightarrow M, (m, gG_0) \mapsto g_{m_0, m} g^{-1} \triangleright m \) is a right semi-action of \( G/G_0 \) on \( M \) with defect \( G_0 \), which means that

\[
\forall m \in M : m \triangleleft g_0 = m,
\]

\[
\forall m \in M \forall g \in G \exists g_0 \in G_0 : \forall g' \in G/G_0 : m \triangleleft g \cdot g' = (m \triangleleft g_0) \triangleleft g_0 \cdot g'.
\]

It is transitive, which means that the set \( M \) is non-empty and for each \( m \in M \) the map \( m \triangleleft \) is surjective; and free, which means that for each \( m \in M \) the map \( m \triangleleft \) is injective; and semi-commutes with \( \triangleright \), which means that

\[
\forall m \in M \forall g \in G \exists g_0 \in G_0 : \forall g' \in G/G_0 : (m \triangleright m) \triangleleft g' = g \triangleright (m \triangleleft g_0 \cdot g').
\]

The maps \( \iota : M \rightarrow G/G_0, m \mapsto G_{m_0, m}, \) and \( m \triangleleft \) are inverse to each other. Under the identification of \( M \) with \( G/G_0 \) by either of these maps, we have \( \triangleleft : (m, g) \mapsto g_{m_0, m} \triangleright g \). (See [7].)

A left homogeneous space \( \mathcal{M} \) is right amenable if there is a coordinate system \( K \) for \( \mathcal{M} \) and there is a finitely additive probability measure \( \mu \) on \( M \) such that

\[
\forall g \in G/G_0 \forall A \subseteq M : (\triangleleft g) \mid_A \text{ injective} \implies \mu(A \triangleleft g) = \mu(A),
\]

in which case the cell space \( \mathcal{R} = (\mathcal{M}, K) \) is called right amenable. When the stabiliser \( G_0 \) is finite, that is the case if and only if there is a right Følner net in \( \mathcal{R} \) indexed by \( (I, \leq) \), which is a net \( \{F_i\}_{i \in I} \) in \( \{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\} \) such that

\[
\forall g \in G/G_0 : \lim_{i \in I} \frac{|F_i \triangleleft (\triangleleft g)^{-1}(F_i)|}{|F_i|} = 0.
\]

If a net is a right Følner net for one coordinate system, then it is a right Følner net for each coordinate system. In particular, a left homogeneous space \( \mathcal{M} \) with finite stabilisers is right amenable if and only if, for each coordinate system \( K \), the cell space \( (\mathcal{M}, K) \) is right amenable. (See [8].)

A cell space \( \mathcal{R} \) is finitely and symmetrically right generated if there is a finite subset \( S \) of \( G/G_0 \) with \( G_0 \cdot S \subseteq S \) and \( S^{-1} \subseteq S \), where \( S^{-1} = \{g^{-1}g_0 \mid s \in S, g \in g\} \), such that

\[
\forall m \in M \exists k \in \mathbb{N}_0 \exists \{s_i\}_{i \in \{1, 2, \ldots, k\}} \subseteq S \cup S^{-1} : ((m_0 \triangleleft s_1) \triangleleft s_2) \triangleleft \cdots \triangleleft s_k = m.
\]
The S-Cayley graph of $R$ is the symmetric and $2|S|$-regular directed graph $G = (M, \{(m, m \in s) \mid m \in M, \ s \in S\})$, the $S$-metric on $R$ is the geodesic distance $d$ on $G$, and the $S$-length on $R$ is the map $|.| = d(m_0, \_).$ For each cell $m \in M$, each integer $\rho$, and each integer $\theta$, the $S$-ball of radius $\rho$ centred at $m$ is the set

$$B(m, \rho) = \{m' \in M \mid d(m, m') \leq \rho\},$$

the $S$-sphere of radius $\rho$ and thickness $\theta$ centred at $m$ is the set

$$S(m, \rho, \theta) = \{m' \in M \mid \rho - \theta \leq d(m, m') \leq \rho\},$$

the ball $B(m_0, \rho)$ is denoted by $B(\rho)$, the sphere $S(m_0, \rho, \theta)$ by $S(\rho, \theta)$, and the sphere $S(\rho, 0)$ by $S(\rho)$. The $S$-growth rate of $R$ is the limit point $\lim_{\rho \to \infty} \sqrt[\rho]{|B(\rho)|}$. It is equal to $1$ if and only if $R$ has subexponential growth, in which case, when the stabiliser $G_0$ is finite, the cell space $R$ is right amenable and a subsequence of $(B(\rho))_{\rho \in \mathbb{N}}$ is a right Foelner net in $R$. (See [9].)

A semi-cellular automaton is a quadruple $C = (R, Q, N, \delta)$, where $R$ is a cell space; $Q$, called set of states, is a set; $N$, called neighbourhood, is a subset of $G/G_0$ such that $G_0 \cdot N \subseteq N$; and $\delta$, called local transition function, is a map from $Q^N$ to $Q$. A local configuration is a map $\ell \in Q^N$, a global configuration is a map $c \in Q^M$, and a pattern is a map $p \in Q^A$, where $A$ is a subset of $M$. The stabiliser $G_0$ acts on $Q^N$ on the left by $\bullet: G_0 \times Q^N \to Q^N$, $(g_0, \ell) \mapsto [n \mapsto \ell(g_0^{-1} \cdot n)]$, and the group $G$ acts on the set of patterns on the left by

$$\triangleright: G \times \bigcup_{A \subseteq M} Q^A \to \bigcup_{A \subseteq M} Q^A,$$

$$(g, p) \mapsto \left[\begin{array}{c}
g \triangleright \dom(p) \to Q, \\
m \mapsto p(g^{-1} \triangleright m).\end{array}\right]$$

The global transition function of $C$ is the map $\Delta: Q^M \to Q^M, c \mapsto [m \mapsto \delta(n \mapsto c(m \in n))]$.

A cellular automaton is a semi-cellular automaton $C = (R, Q, N, \delta)$ such that the local transition function $\delta$ is $\bullet$-invariant, which means that, for each $g_0 \in G_0$, we have $\delta(g_0 \bullet \_) = \delta(_\_).$. Its global transition function is $\bullet$-equivariant, which means that, for each $g \in G$, we have $\Delta(g \triangleright \_) = g \triangleright \Delta(_\_)$.

A subgroup $H$ of $G$ is $K$-big if the set $\{g_{m_0, m} \mid m \in M\}$ is included in $H$. A big-cellular automaton is a semi-cellular automaton $C = (R, Q, N, \delta)$ such that, for some $K$-big subgroup $H$ of $G$, the local transition function $\delta$ is $\bullet_H$-invariant, which means that, for each $h_0 \in G_0 \cap H$, we have $\delta(h_0 \bullet \_) = \delta(_\_).$. Its global transition function is $\bullet_H$-equivariant, which means that, for each $h \in H$, we have $\Delta(h \triangleright \_) = h \triangleright \Delta(_\_)$. Note that each $K$-big subgroup of $G$ includes the subgroup of $G$ generated by $\{g_{m_0, m} \mid m \in M\}$ and that hence a semi-cellular automaton is a big-cellular automaton if and only if its local transition function is $\bullet_{G_0 \cap \{g_{m_0, m} \mid m \in M\}}$-invariant. (See [7].)

Identify $M$ with $G/G_0$ by $\iota: m \mapsto g_{m_0, m}$. The map

$$\leftarrow: M \times \bigcup_{A \subseteq M} Q^A \to \bigcup_{A \subseteq M} Q^A,$$

$$(m, p) \mapsto \left[\begin{array}{c}
m \iota \dom(p) \to Q, \\
m \iota a \mapsto p(a),\end{array}\right]$$

broadly speaking, maps a point $m$ and a pattern $p$ that is centred at $m_0$ to the corresponding pattern centred at $m$. For each cell $m \in M$, each subset $A$ of $M$, and each pattern $p \in Q^A$, we have $m \leftarrow p = g_{m_0, m} \triangleright p$. It follows that the global transition function $\Delta$ of a big-cellular automaton is $\leftarrow$-equivariant, which means that, for each $m \in M$, we have $\Delta(m \leftarrow \_) = m \leftarrow \Delta(_\_)$.

For each $A \subseteq M$, let $\pi_A: Q^M \to Q^A, c \mapsto c|_A$. 
2. Interiors, closures, and boundaries; right Følner nets and right Erling nets

In this section, let $\mathcal{R} = (\mathcal{M}, \mathcal{K}) = ((M, G, \triangleright), (m_0, \{g_{m_0,m}\}_{m \in M}))$ be a cell space.

In Definition 2.1 we introduce $E$-interiors, $E$-closures, $E$-boundaries, and internal $E$-boundaries of subsets $A$ of $M$, where the subset $E$ of $G/G_0$ determines thickness and shape of the subtraction from, addition to, or boundary of $A$. The Examples 2.2, 2.3, and 2.4 are used throughout the present paper to illustrate new notions. In Lemma 2.5 we show essential properties of interiors, closures, and boundaries. In Lemma 2.8 we define surjective restrictions $\Delta_{X,A}$ of global transition functions to patterns. In Theorem 2.13 we show that right Følner nets are those nets whose components are asymptotically invariant under taking finite boundaries. In Definitions 2.16 and 2.17 we introduce right Erling nets and right tractability, which are weak variants of right Følner nets and right amenability. And in Lemma 2.18 we show that each finitely and symmetrically right generated cell space is right tractable.

Definition 2.1: Let $A$ be a subset of $M$ and let $E$ be a subset of $G/G_0$.

1. The set
   \[ A^{-E} = \{ m \in M \mid m \triangleleft E \subseteq A \} \]
   is called $E$-interior of $A$.

2. The set
   \[ A^{+E} = \{ m \in M \mid (m \triangleleft E) \cap A \neq \emptyset \} \]
   is called $E$-closure of $A$.

3. The set
   \[ \partial_E A = A^{+E} \setminus A^{-E} \]
   is called $E$-boundary of $A$.

4. The set
   \[ \partial_E^{-1} A = A \setminus A^{-E} \]
   is called internal $E$-boundary of $A$.

Remark 1: Let $\mathcal{R}$ be the cell space $((G, G, \cdot), (e_G, \{g \in G\}))$, where $G$ is a group and $e_G$ is its neutral element. Then, $G_0 = \{ e_G \}$ and $\triangleleft = \cdot$. Hence, the notions of $E$-interior, $E$-closure, and $E$-boundary are the same as the ones defined in [1, Section 5.4, Paragraph 2].

Example 2.2 (Lattice): Let $M$ be the two-dimensional integer lattice $\mathbb{Z}^2$, let $S_2$ be the symmetric group on $\{1, 2\}$, let $V$ be the multiplicative group $\{-1, 1\}$, let $\psi$ be the group homomorphism $S_2 \rightarrow \text{Aut}(V^2)$, $\pi \mapsto [(v_1, v_2) \mapsto (v_{\pi(1)}, v_{\pi(2)})]$, let $V^2 \rtimes_{\psi} S_2$ be the outer semidirect product of $S_2$ acting on $V^2$ by $\psi$, let $\psi$ be the group homomorphism $V^2 \times_{\psi} S_2 \rightarrow \text{Aut}(\mathbb{Z}^2)$, $((v_1, v_2), \pi) \mapsto [(t_1, t_2) \mapsto (v_1 \cdot t_{\pi(1)}, v_2 \cdot t_{\pi(2)})]$, let $G = \mathbb{Z}^2 \rtimes_{\psi} (V^2 \times_{\psi} S_2)$ be the outer semidirect product of $V^2 \times_{\psi} S_2$ acting on $\mathbb{Z}^2$ by $\psi$, and let $\triangleright$ be the transitive left group action of $G$ on $M$ by $((t_1, t_2), ((v_1, v_2), \pi), (z_1, z_2)) \mapsto (t_1 + v_1 \cdot z_{\pi(1)}, t_2 + v_2 \cdot z_{\pi(2)})$. The triple $\mathcal{M} = (M, G, \triangleright)$ is a left homogeneous space.

The group $G$ encodes the symmetries of the lattice $M$, more precisely, the component $\mathbb{Z}^2$ encodes the translational symmetries and the component $V^2 \times S_2$ the reflectional and rotational ones that stabilise the origin. For example, the element $((4, 2), ((-1, 1), \text{id}))$ encodes the reflection about the $x$-axis, followed by a translation by $(4, 2)$; the element $((0, 0), ((1, 1), (1 \ 2)))$ encodes the reflection about the line through the origin of slope 1; and the element $((0, 0), ((-1, 1), (1 \ 2)))$ encodes the anticlockwise rotation about the origin through 90, where the permutation $(1 \ 2)$, written in cycle notation, is the transposition that swaps 1 and 2. The symmetry group of $M$ is the group $\{g \triangleright \_ \mid g \in G\}$ under composition, which is isomorphic to $G$.

Let $m_0$ be the origin $(0, 0)$ and, for each point $m \in M$, let $g_{m_0,m}$ be the translation $(m, ((1, 1), \text{id}))$. The tuple $\mathcal{K} = (m_0, \{g_{m_0,m}\}_{m \in M})$ is a coordinate system for $\mathcal{M}$. The stabiliser $G_0$ of $m_0$ under $\triangleright$ is the set
{(0,0)} \times (V^2 \times S_2), the quotient group \(G/G_0\) is isomorphic to the group \(\mathbb{Z}^2\) by \((t, ((v_1, v_2), \pi))G_0 \mapsto t\), and, under this isomorphism, the right quotient group semi-action \(\circ\) of \(G/G_0\) on \(M\) is the map \(M \times \mathbb{Z}^2 \to M, (z, t) \mapsto z + t\).

The cell space \(\mathcal{R} = (\mathcal{M}, K)\) is finitely and symmetrically right generated by \(S = \{(−1, 0), (0, −1), (0, 1), (1, 0)\}\). The \(S\)-metric \(d\), \(S\)-length \(||\), \(S\)-balls \(B(\ldots)\), and \(S\)-spheres \(S(\ldots)\) are restrictions of the corresponding notions on \(\mathbb{R}^2\) with respect to the taxicab metric on \(\mathbb{R}^2\). The balls \(B_{\mathbb{R}^2}(\ldots)\) and spheres \(S_{\mathbb{R}^2}(\ldots)\) induced by the taxicab metric on \(\mathbb{R}^2\) are diamonds, that is, filled and unfilled squares with sides oriented at 45° to the coordinate axes.

Let \(A\) be the ball \(B(2)\). The \((1, 0)\)-interior of \(A\) is the ball \(A − (1, 0) = B((-1, 0), 2)\), which is not included in \(A\); the \((0, 0), (1, 0)\)-interior of \(A\) is the set \(A \cap (A − (1, 0)) = B_{\mathbb{R}^2}((-1/2, 0), 3/2) \cap M\), which is included in \(A\) on the left; the \((-1, 0), (0, 0), (1, 0)\)-interior of \(A\) is the ball \((A + (1, 0)) \cap A \cap (A − (1, 0)) = B(1), which is included in \(A\) in the middle; and the \(E\)-interior of \(A\), where \(E = \{(−1, 0), (0, −1), (0, 0), (1, 0)\} = B(1)\), is also the ball \(B(1)\) (see Figure 1).

The \{(1, 0)\}-closure of \(A\) is the ball \(A − (1, 0) = B((−1, 0), 2), which does not include \(A\); the \((0, 0), (1, 0)\)-closure of \(A\) is the set \(A \cup (A − (1, 0)) = B(2) \cup B((-1, 0), 2), which includes \(A\) on the right; the \((-1, 0), (0, 0), (1, 0)\)-closure of \(A\) is the set \((A + (1, 0)) \cup A \cup (A − (1, 0)) = B((1, 0), 2) \cup B(2) \cup B((-1, 0), 2), which includes \(A\) in the middle; and the \(E\)-interior of \(A\). The \(E\)-boundary of \(A\) is the ball \(B(3)\) (see Figure 1).

The above calculations suggest that the notions of \(E\)-interior, -closure and -boundary behave best if the set \(E\) contains \((0, 0)\) and is invariant under \(G_0\), which we also see in the forthcoming Lemma 2.5.

In general, for each cell \(m \in M\) and for each pair \((\rho, g) \in N_0 \times N_0\), we have \(B(m, \rho)^{-B(g)} = B(m, \rho - g), B(m, \rho)^{+B(g)} = B(m, \rho + g),\) and \(\partial_{B(g)} B(m, \rho) = B(m, \rho + g) \setminus B(m, \rho - g) = S(m, \rho + g, 2b - 1)\).

**Example 2.3 (Tree):** Let \(M\) be the vertices of the \([a, b, a^{-1}, b^{-1}]\)-Cayley graph of the free group \(F_2\) over \([a, b]\), where \(a \neq b\), let \(\varsigma\) be the group automorphism from \(F_2\) to \(F_2\) determined by \(a \mapsto b\) and \(b \mapsto a^{-1}\), let \(R\) be the cyclic group \(\{0, 90, 180, 270\}\) under addition modulo 360, let \(\psi\) be the group homomorphism \(R \to \text{Aut}(F_2), r \mapsto \varsigma^{r/90}\), let \(G = F_2 \rtimes_{\psi} R\) be the outer semidirect product of \(R\) acting on \(F_2\) by \(\psi\), and let \(\sigma\) be the transitive left group action of \(G\) on \(M\) by \((f, r, m) \mapsto f \cdot \psi(r)(m)\). The triple \(\mathcal{M} = (M, G, \sigma)\) is a left homogeneous space.

The group \(G\) encodes some graph automorphisms of \(M\), more precisely, the component \(F_2\) encodes the translational automorphisms and the component \(R\) the rotational ones that stabilise the origin. For example, the element \((ab^{-1}, 90)\) encodes the anticlockwise rotation about the origin through 90°, followed by a translation by \(ab^{-1}\), which is the anticlockwise rotation about \(a\) through 90°; see Figure 2 for further examples. In general, for each vertex \(m \in M\) and each angle \(r \in R\), the anticlockwise rotation about \(m\) through \(r\) is the graph automorphism \(m \cdot \psi(r)(m^{-1} \cdot \_ \_ \) = \(m \cdot \psi(r)(m)^{-1} \cdot \psi(r)(\_ \_ \), which is encoded by \((m, 0) \cdot (e_{F_2}, r) \cdot (m^{-1}, 0) = (m \cdot \psi(r)(m))^{-1}, r\). The map \(g \mapsto g \cdot \_ \_ \ embeds the group \(G\) into the graph-automorphism group of \(M\).

Let \(m_0\) be the neutral element \(e_{F_2}\) of \(F_2\) and, for each vertex \(m \in M\), let \(g_{m_0, m}\) be the translation \((m, 0). The tuple \(K = (m_0, \{g_{m_0, m}\}_{m \in M})\) is a coordinate system for \(\mathcal{M}\). The stabiliser \(G_0\) of \(m_0\) under \(\sigma\) is the set \(\{e_{F_2}\} \times R\), the quotient group \(G/G_0\) is isomorphic to the group \(F_2\) by \((f, r)G_0 \mapsto f\), and, under this isomorphism, the right quotient group semi-action \(\circ\) of \(G/G_0\) on \(M\) is the map \(M \times F_2 \to M, (m, f) \mapsto m \cdot f\).

The cell space \(\mathcal{R} = (\mathcal{M}, K)\) is finitely and symmetrically right generated by \(S = \{a, b, a^{-1}, b^{-1}\}\). The \(S\)-Cayley graph of \(\mathcal{R}\) is equal to the \(S\)-Cayley graph of \(F_2\) and hence the \(S\)-metric \(d\) is identical to the word metric on \(F_2\).

Let \(A\) be the ball \(B(2)\). The \((a)\)-interior of \(A\) is the set \(Aa^{-1}\), the \((a, b)\)-interior of \(A\) is the ball \(Aa^{-1} \cap Ab^{-1} = B(1)\), and the \((a, a^{-1})\)- and \((a, b, a^{-1}, b^{-1})\)-interiors of \(A\) are also the ball \(B(1)\).
In each subfigure, the whole space is $\mathbb{R}^2$, the grid points are elements of $M = \mathbb{Z}^2$, the grid points in the region enclosed by the diamond with solid border are the elements of $A = B(2)$, the grid points in the regions enclosed by diamonds with dashed, dash-dotted, or dotted borders are the elements of the balls that occur in the calculation of the interior, closure, or boundary in Example 2.2, and the dots are the elements of the respective interior, closure, or boundary of $A$, where $E$ is the set $\{(−1, 0), (0, −1), (0, 0), (0, 1), (1, 0)\}$. The subfigures in the first row depict interiors, the ones in the second closures, and the ones in the third boundaries.

(a) $A^−\{(1,0)\}$

(b) $A^−\{(0,0),(1,0)\}$

(c) $A^−\{(-1,0),(0,0),(1,0)\}$

(d) $A^−\varepsilon$

(e) $A^+\{(1,0)\}$

(f) $A^+\{(0,0),(1,0)\}$

(g) $A^+\{(-1,0),(0,0),(1,0)\}$

(h) $A^+\varepsilon$

(i) $\partial_{\{(1,0)\}} A$

(j) $\partial_{\{(0,0),(1,0)\}} A$

(k) $\partial_{\{(-1,0),(0,0),(1,0)\}} A$

(l) $\partial_{\varepsilon} A$

Figure 1. In each subfigure, the whole space is $\mathbb{R}^2$, the grid points are elements of $M = \mathbb{Z}^2$, the grid points in the region enclosed by the diamond with solid border are the elements of $A = B(2)$, the grid points in the regions enclosed by diamonds with dashed, dash-dotted, or dotted borders are the elements of the balls that occur in the calculation of the interior, closure, or boundary in Example 2.2, and the dots are the elements of the respective interior, closure, or boundary of $A$, where $E$ is the set $\{(−1, 0), (0, −1), (0, 0), (0, 1), (1, 0)\}$. The subfigures in the first row depict interiors, the ones in the second closures, and the ones in the third boundaries.

The above calculations suggest that the notions of $E$-interior, -closure and -boundary behave best if the set $E$ is invariant under $G_0$, which we also see in the forthcoming Lemma 2.5.

In general, for each cell $m \in M$ and for each pair $(\rho, \varrho) \in \mathbb{N}_0 \times \mathbb{N}_0$, we have $B(m, \rho)^-B(\varrho) = B(m, \rho - \varrho)$, $B(m, \rho)^+B(\varrho) = B(m, \rho + \varrho)$, and $\partial_{B(\varrho)} B(m, \rho) = B(m, \rho + \varrho) \setminus B(m, \rho - \varrho) = S(m, \rho + \varrho, 2\varrho - 1)$.

Example 2.4 (Sphere): Let $M$ be the Euclidean unit 2-sphere, that is, the surface of the ball of radius 1 in 3-dimensional Euclidean space, and let $G$ be the rotation group. The group $G$ acts transitively but not freely on $M$ on the left by function application, that is, by rotation about the origin. This action is denoted by $\triangleright$. For each point $m \in M$, its orbit is $M$ and its stabiliser is the group of rotations about the line through the origin and itself.

Furthermore, let $m_0$ be the north pole $(0, 0, 1)^T$ of $M$ and, for each point $m \in M$, let $g_{m_0,m}$ be the rotation about an axis in the $(x,y)$-plane that rotates $m_0$ to $m$. The stabiliser $G_0$ of the north pole $m_0$
Figure 2. The same part of the \( \{a, b, a^{-1}, b^{-1}\} \)-Cayley graph of \( F_2 \) is depicted six times, each time the vertex in the centre is the neutral element \( e_{F_2} \), the arrows of the form \( \mapsto \) are translations, rotations, or their compositions, and the different line styles illustrate which vertices are mapped to which vertices.

Under \( \triangleright \) is the group of rotations about the \( z \)-axis. An element \( gG_0 \in G/G_0 \) semi-acts on a point \( m \) on the right by the induced semi-action \( \triangleright \) by first changing the rotation axis of \( g \) such that the new axis stands to the line through the origin and \( m \) as the old one stood to the line through the origin and \( m_0 \), \( g_{m_0,m}gg_{m_0,m}^{-1}m \), and secondly rotating \( m \) as prescribed by this new rotation.

Moreover, let \( A \) be a curved circular disk of radius \( 3\rho \) with the north pole \( m_0 \) at its centre, let \( g \) be the rotation about an axis \( a \) in the \((x, y)\)-plane by \( \rho \) radians, let \( E \) be the set \( \{g_0gG_0 \mid g_0 \in G_0\} \), and, for each point \( m \in M \), let \( E_m \) be the set \( m \triangleright E \). Because \( G_0 \) is the set of rotations about the \( z \)-axis and \( m_0 \triangleright E = g_{m_0,m_0}gg_{m_0,m}m_0 = G_0 \triangleright (g \triangleright m_0) \), the set \( E_{m_0} \) is the boundary of a curved circular disk of radius \( \rho \) with the north pole \( m_0 \) at its centre. And, for each point \( m \in M \), because \( m \triangleright E = g_{m_0,m} \triangleright E_{m_0} \), the set \( E_m \) is the boundary of a curved circular disk of radius \( \rho \) with \( m \) at its centre.

The \( E \)-interior of \( A \) is the curved circular disk of radius \( 2\rho \) with the north pole \( m_0 \) at its centre. The \( E \)-closure of \( A \) is the curved circular disk of radius \( 4\rho \) with the north pole \( m_0 \) at its centre. And the \( E \)-boundary of \( A \) is the annulus bounded by the boundaries of the \( E \)-interior and the \( E \)-closure of \( A \).

Essential properties of and relations between interiors, closures, and boundaries are given in the next lemma.

**Lemma 2.5:** Let \( A \) be a subset of \( M \), let \( \{A_i\}_{i \in I} \) be a family of subsets of \( M \), let \( e \) be an element of \( G/G_0 \), and let \( E \) and \( E' \) be two subsets of \( G/G_0 \). Furthermore, for each element \( e' \in E' \), let \( e' \cdot E \) be the set
In each subfigure, the same part of the \( \{a, b, a^{-1}, b^{-1}\} \)-Cayley graph of \( F_2 \) is depicted, the vertex in the centre is the neutral element \( e_{F_2} \), the vertices that are adjacent to solid edges are the elements of \( A = B(2) \), the dots are the elements of the respective interior, closure, or boundary of \( A \), the squares are the elements of \( A_{a^{-1}} \), and the circles are the elements of \( Ab \). The subfigures in the first row depict interiors, the ones in the second closures, and the ones in the third boundaries.

\[
\begin{align*}
(g' \cdot e &\mid e \in E, g' \in e') \text{, let } E' \cdot E \text{ be the set } \bigcup_{e' \in E} e' \cdot E \left( = \{ g' \cdot e \mid e \in E, e' \in E', g' \in e' \} \right), \text{ and let } (E')^{-1} \text{ be the set } (\{ g' \}^{-1} G_0 \mid e' \in E', g' \in e') .
\end{align*}
\]

1. \( A^{-\{G_0\}} = A, A^{+\{G_0\}} = A, \text{ and } \partial_{\{G_0\}} A = \emptyset. \)
2. \( A^{-\{G_0,e\}} = A \cap (\subseteq e)^{-1}(A), A^{+\{G_0,e\}} = A \cup (\subseteq e)^{-1}(A), \text{ and } \partial_{\{G_0,e\}} A = A \setminus (\subseteq e)^{-1}(A) \cup (\subseteq e)^{-1}(A) \setminus A. \)
3. \( (M \setminus A)^{-E} = M \setminus A^{+E} \text{ and } (M \setminus A)^{+E} = M \setminus A^{-E}. \)
4. \( \bigcup_{i \in I} A_i^{-E} \subseteq (\bigcup_{i \in I} A_i)^{-E} \text{ and } \bigcup_{i \in I} A_i^{+E} = (\bigcup_{i \in I} A_i)^{+E}. \)
5. \( (\bigcap_{i \in I} A_i)^{-E} = \bigcap_{i \in I} A_i^{-E} \text{ and } (\bigcap_{i \in I} A_i)^{+E} \subseteq \bigcap_{i \in I} A_i^{+E}. \)
6. \( \text{Let } E \subseteq E'. \text{ Then, } A^{-E} \supseteq A^{-E'}, A^{+E} \subseteq A^{+E'}, \text{ and } \partial_E A \subseteq \partial_{E'} A. \)
(7) Let $G_0 \subseteq E$. Then, $A^{-E} \subseteq A \subseteq A^{+E}$.

(8) Let $G_0$, $A$, and $E$ be finite. Then, $A^{-E}$, $A^{+E}$, and $\partial_E A$ are finite. More precisely, $|A^{-E}| \leq |G_0| \cdot |A|$ and $|A^{+E}| \leq |G_0| \cdot |A| \cdot |E|$.

(9) Let $g \in G$ and let $G_0 \cdot E \subseteq E$. Then, $g \triangleright A^{-E} = (g \triangleright A)^{-E}$, $g \triangleright A^{+E} = (g \triangleright A)^{+E}$, and $g \triangleright \partial_E A = \partial_E (g \triangleright A)$.

(10) Let $m \in M$, let $G_0 \cdot E \subseteq E$, and let $i: M \rightarrow G/G_0$, $m \mapsto G_{m_0,m}$. Then, $m \trianglelefteq i(A^{-E}) = (m \trianglelefteq i(A))^{-E}$, $m \trianglelefteq i(A^{+E}) = (m \trianglelefteq i(A))^{+E}$, and $m \trianglelefteq i(\partial_E A) = \partial_E (m \trianglelefteq i(A))$.

(11) Let $G_0 \cdot E \subseteq E$. Then, $(A^{-E})^{-E} = A^{-E \cdot E}$ and $(A^{+E})^{+E} = A^{+E \cdot E}$.

(12) Let $G_0 \cdot E \subseteq E$. Then, $(A^{+E})^{-E} = \bigcap_{i \in E} A^{+E}$ and $(A^{-E})^{+E} = \bigcup_{i \in E} A^{-E}$. And, if $(E')^{-1} \subseteq E$, then $A \subseteq (A^{+E})^{-E}$ and $(A^{-E})^{+E} \subseteq A$.

**Proof:**

(1) Because $\subseteq G_0 = \text{id}_M$, this is a direct consequence of Definition 2.1.

(2) Because $(\subseteq G_0)^{-1} (A) = A$, this is a direct consequence of Definition 2.1.

(3) For each $m \in M$,

\[
m \in (M \setminus A)^{-E} \iff m \trianglelefteq E \subseteq M \setminus A
\]

\[
\iff (m \trianglelefteq E) \cap A = \emptyset
\]

\[
\iff m \in M \setminus A^{+E}.
\]

Hence, $(M \setminus A)^{-E} = M \setminus A^{+E}$. Therefore,

\[
(M \setminus A)^{+E} = M \setminus (M \setminus (M \setminus A)^{+E})
\]

\[
= M \setminus (M \setminus (M \setminus A))^{-E}
\]

\[
= M \setminus A^{-E}.
\]

(4) For each $m \in M$,

\[
m \in \bigcup_{i \in I} A_i^{-E} \iff \exists i \in I : m \in A_i^{-E}
\]

\[
\iff \exists i \in I : m \trianglelefteq E \subseteq A_i
\]

\[
\iff m \trianglelefteq E \subseteq \bigcup_{i \in I} A_i
\]

\[
\iff m \in (\bigcup_{i \in I} A_i)^{-E}.
\]

Hence, $\bigcup_{i \in I} A_i^{-E} \subseteq (\bigcup_{i \in I} A_i)^{-E}$.

Moreover, for each $m \in M$,

\[
m \in \bigcup_{i \in I} A_i^{+E} \iff \exists i \in I : m \in A_i^{+E}
\]

\[
\iff \exists i \in I : (m \trianglelefteq E) \cap A_i \neq \emptyset
\]

\[
\iff (m \trianglelefteq E) \cap (\bigcup_{i \in I} A_i) \neq \emptyset
\]

\[
\iff m \in (\bigcup_{i \in I} A_i)^{+E}.
\]

Therefore, $\bigcup_{i \in I} A_i^{+E} = (\bigcup_{i \in I} A_i)^{+E}$. 
(5) According to Item (3) and Item (4),

\[
\left( \bigcap_{i \in I} A_i \right)^{-E} = M \setminus (M \setminus \left( \bigcap_{i \in I} A_i \right)^{-E}) = M \setminus (M \setminus \left( \bigcap_{i \in I} A_i \right)) + E = M \setminus \left( \bigcup_{i \in I} M \setminus A_i \right) + E = M \setminus \left( \bigcup_{i \in I} (M \setminus A_i)^{+E} \right) = M \setminus \left( \bigcup_{i \in I} M \setminus A_i^{-E} \right) = M \setminus \left( \bigcap_{i \in I} A_i^{-E} \right)
\]

and

\[
\left( \bigcap_{i \in I} A_i \right)^{+E} = M \setminus \left( \bigcap_{i \in I} A_i \right) = M \setminus \left( \bigcap_{i \in I} A_i \right)^{-E} = M \setminus \left( \bigcup_{i \in I} M \setminus A_i \right) \subseteq M \setminus \left( \bigcup_{i \in I} M \setminus A_i^{+E} \right) = M \setminus \left( \bigcap_{i \in I} A_i^{+E} \right) = \bigcap_{i \in I} A_i^{+E}.
\]

(6) This is a direct consequence of Definition 2.1.
(7) This is a direct consequence of Definition 2.1.
(8) Let \( e \in E \) and let \( a \in A \) such that \( (\_ \sqsubseteq e)^{-1}(a) \neq \emptyset \). There are \( m \) and \( m' \in M \) such that \( G_{m,0} = e \) and \( m' \sqsubseteq e = a \). For each \( m'' \in M \), we have \( m'' \sqsubseteq e = g_{m,0,m''} \triangleright m \) and hence

\[
m'' \sqsubseteq e = a \iff m'' \sqsubseteq e = m' \sqsubseteq e \iff g_{m,0,m''}^{-1} g_{m,0,m''} \triangleright m = m \iff g_{m,0}^{-1} g_{m,0,m''} \in G_m \iff g_{m,0,m''} \in g_{m,0,m} G_m.
\]
Moreover, for each $m''$ and each $m''' \in M$ with $m'' \neq m'''$, we have $g_{m_0,m''} \neq g_{m_0,m'''}$. Therefore,

$$
|\bigcap (\; g^{-1} \; e) - 1(a) | = | \{ m'' \in M \mid m'' \; g \; e = a \} |
= | \{ m'' \in M \mid g_{m_0,m''} \in g_{m_0,m'''} G_m \} |
\leq | g_{m_0,m} G_m |
= | G_m |
= | G_0 |.
$$

Thus,

$$
\bigcup_{a \in A} (\; g^{-1} \; e) - 1(a) \leq \sum_{a \in A} | (\; g^{-1} \; e) - 1(a) | \leq | G_0 | \cdot | A |.
$$

Hence, because $A^{-E} = \bigcap_{e \in E} \bigcup_{a \in A} (\; g^{-1} \; e) - 1(a)$, we have $| A^{-E} | \leq | G_0 | \cdot | A | < \infty$. And, because $A^{+E} = \bigcup_{e \in E} \bigcup_{a \in A} (\; g^{-1} \; e) - 1(a)$, we have $| A^{+E} | \leq | E | \cdot | G_0 | \cdot | A | < \infty$.

And, because $\partial_E A \subseteq A^{+E}$, we also have $| \partial_E A | < \infty$.

(9) Let $m \in M$. Because $\; g^{-1} \; e \; g \; e$ semi-commutes with $\; g \; e$, there is a $g_0 \in G_0$ such that $(g^{-1} \; g \; e) \; g \; e = g^{-1} \; g \; e (m \; g \; e \; g \; e \; E)$. And, because $G_0 \cdot E \subseteq E$, we have $g_0 \cdot E \subseteq E$ and $g_0^{-1} \cdot E \subseteq E$; hence $E = g_0 g_0^{-1} \cdot E = g_0 \cdot (g_0^{-1} \cdot E) \subseteq g_0 \cdot E$; thus $g_0 \cdot E = E$. Therefore, $(g^{-1} \; g \; e) \; g \; e = g^{-1} \; g \; e (m \; g \; e \; g \; e \; E)$. Thus, for each $m \in M$,

$$
m \in g \; A^{-E} \iff \exists m' \in A^{-E} : g \; A \cdot m' = m
\iff g^{-1} \; g \; e \; m \in A^{-E}
\iff (g^{-1} \; g \; e) \; m \subseteq A
\iff g^{-1} \; g \; e (m \; g \; e \; g \; e \; E) \subseteq A
\iff m \; g \; e \; g \; e \; E \subseteq g \; A
\iff m \in (g \; A)^{-E}.
$$

In conclusion, $g \; A^{-E} = (g \; A)^{-E}$. Moreover, for each $m \in M$,

$$
m \in g \; A^{+E} \iff g^{-1} \; g \; e \; m \in A^{+E}
\iff (g^{-1} \; g \; e) \; m \cap A \neq \emptyset
\iff (g^{-1} \; g \; e) \; (m \; g \; e \; g \; e \; E) \cap A \neq \emptyset
\iff (m \; g \; e \; g \; e \; E) \cap (g \; A) \neq \emptyset
\iff m \in (g \; A)^{+E}.
$$

In conclusion, $g \; A^{+E} = (g \; A)^{+E}$. Ultimately,

$$
g \; A \; \partial_E A = g \; (A^{+E} \; \cap \; A^{-E})
= (g \; A^{+E}) \; \cap \; (g \; A^{-E})
= (g \; A)^{+E} \; \cap \; (g \; A)^{-E}
= \partial_E (g \; A).
$$

(10) According to
Item 9,

\[ m \trianglelefteq \iota(A^{-E}) = g_{m_0,m} \triangleright A^{-E} \]

\[ = (g_{m_0,m} \triangleright A)^{-E} \]

\[ = (m \trianglelefteq \iota(A))^{-E}, \]

and

\[ m \trianglelefteq \iota(A^{+E}) = g_{m_0,m} \triangleright A^{+E} \]

\[ = (g_{m_0,m} \triangleright A)^{+E} \]

\[ = (m \trianglelefteq \iota(A))^{+E}, \]

and

\[ m \trianglelefteq \iota(\partial E A) = g_{m_0,m} \triangleright \partial E A \]

\[ = \partial E (g_{m_0,m} \triangleright A) \]

\[ = \partial E (m \trianglelefteq \iota(A)). \]

(11) For each \( m \in M \), according to [8, Lemma 18], we have \((m \trianglelefteq E') \triangleleft E = m \trianglelefteq E' \cdot E\). Therefore, for each \( m \in M \),

\[ m \in (A^{-E})^{-E'} \iff m \trianglelefteq E' \subseteq A^{-E} \]

\[ \iff (m \trianglelefteq E') \triangleleft E \subseteq A \]

\[ \iff m \trianglelefteq E' \cdot E \subseteq A \]

\[ \iff m \in A^{-E'}^{-E}. \]

In conclusion, \((A^{-E})^{-E'} = A^{-E'}^{-E}\). Moreover, for each \( m \in M \),

\[ m \in (A^{+E})^{+E'} \iff (m \trianglelefteq E') \cap A^{+E} \neq \emptyset \]

\[ \iff \exists e' \in E' : m \trianglelefteq e' \in A^{+E} \]

\[ \iff \exists e' \in E' : ((m \trianglelefteq e') \triangleleft E) \cap A \neq \emptyset \]

\[ \iff ((m \trianglelefteq E') \triangleleft E) \cap A \neq \emptyset \]

\[ \iff (m \trianglelefteq E' \cdot E) \cap A \neq \emptyset \]

\[ \iff m \in A^{+E' \cdot E}. \]

In conclusion, \((A^{+E})^{+E'} = A^{+E' \cdot E}\).

(12) For each \( m \in M \) and each \( e' \in E' \), according to [8, Lemma 18], we have \((m \trianglelefteq e') \triangleleft E = m \trianglelefteq e' \cdot E\). Therefore, for each \( m \in M \),

\[ m \in (A^{+E})^{-E'} \iff m \trianglelefteq E' \subseteq A^{+E} \]

\[ \iff \forall e' \in E' : m \trianglelefteq e' \in A^{+E} \]

\[ \iff \forall e' \in E' : ((m \trianglelefteq e') \triangleleft E) \cap A \neq \emptyset \]

\[ \iff \forall e' \in E' : (m \trianglelefteq e' \cdot E) \cap A \neq \emptyset \]

\[ \iff \forall e' \in E' : m \in A^{+E' \cdot E} \]

\[ \iff m \in \bigcap_{e' \in E'} A^{+e' \cdot E}. \]
In conclusion, \((A^{+E})^{-E'} = \bigcap_{e' \in E'} A^{+e'\cdot E}\). Moreover, for each \(m \in M\), \[
  m \in (A^{-E})^{+E'} \iff (m \in E') \land A^{-E} \neq \emptyset
\]
\[
  \iff \exists e' \in E' : m \in e' \in A^{-E}
\]
\[
  \iff \exists e' \in E' : (m \not{\in} e') \land e' \subseteq A
\]
\[
  \iff \exists e' \in E' : m \in e' \cdot E \subseteq A
\]
\[
  \iff m \in \bigcup_{e' \in E'} A^{-e'\cdot E}.
\]

In conclusion, \((A^{-E})^{+E'} = \bigcup_{e' \in E'} A^{-e'\cdot E}\).

From now on, let \((E')^{-1} \subseteq E\). Then, for each \(e' \in E'\), we have \(G_0 \subseteq e' \cdot E\) and hence, according to (7), we have \(A \subseteq A^{+e'\cdot E}\) and \(A^{-e'\cdot E} \subseteq A\). In conclusion, \(A \subseteq (A^{+E})^{-E'}\) and \((A^{-E})^{+E'} \subseteq A\). \(\Box\)

**Corollary 2.6:** Let \(G_0\) be finite, let \(A\) be a finite subset of \(M\), and let \(g\) be an element of \(G/G_0\). Then, \(|(_{G \subseteq g})^{-1}(A)| \leq |G_0| \cdot |A|\).

**Proof:** This is a direct consequence of Definition 2.1 and Item (8) of Lemma 2.5. \(\Box\)

The restriction \(\Delta_{X,A}^{-}\) of \(\Delta\) given in Lemma 2.8 is well-defined according to the next lemma, which itself holds due to the locality of \(\Delta\).

**Lemma 2.7:** Let \(C = (R, Q, N, \delta)\) be a semi-cellular automaton, let \(\Delta\) be the global transition function of \(\Delta\), let \(c\) and \(c'\) be two global configurations of \(C\), and let \(A\) be a subset of \(M\). If \(c_{\mid A} = c'_{\mid A}\), then \(\Delta(c)_{\mid A^{-N}} = \Delta(c')_{\mid A^{-N}}\).

**Proof:** Let \(c_{\mid A} = c'_{\mid A}\). Furthermore, let \(m \in A^{-N}\). Then, \(m \subseteq N \subseteq A\). Hence, \(\Delta(c)(m) = \Delta(c')(m)\). \(\Box\)

**Lemma 2.8:** Let \(C = (R, Q, N, \delta)\) be a semi-cellular automaton, let \(\Delta\) be the global transition function of \(C\), let \(X\) be a subset of \(Q^M\), and let \(A\) be a subset of \(M\). The map
\[
  \Delta_{X,A}^{-} : \pi_A(X) \to \pi_{A^{-N}}(\Delta(X)),
\]
\[
  p \mapsto \Delta(c)_{\mid A^{-N}}, \text{ where } c \in X \text{ such that } c_{\mid A} = p,
\]
is surjective. The map \(\Delta_{Q^M,A}^{-}\) is denoted by \(\Delta_{A}^{-}\).

**Proof:** Let \(p' \in \pi_{A^{-N}}(\Delta(X))\). Then, there is a \(c' \in \Delta(X)\) such that \(c'_{\mid A^{-N}} = p'\). Moreover, there is a \(c \in X\) such that \(\Delta(c) = c'\). Put \(p = c_{\mid A} \in \pi_A(X)\). Then, \(\Delta_{X,A}(p) = \Delta(c)_{\mid A^{-N}} = c'_{\mid A^{-N}} = p'\). Hence, \(\Delta_{X,A}^{-}\) is surjective. \(\Box\)

The restrictions \(\Delta_{X,A}^{-}\) of global transition functions \(\Delta\) of big-cellular automata are \(\bullet\)-equivariant as stated in the following Lemma.

**Lemma 2.9:** Let \(H\) be a \(K\)-big subgroup of \(G\), let \(C = (R, Q, N, \delta)\) be a semi-cellular automaton such that \(\delta\) is \(\bullet_{G_0} \cap H\)-invariant, let \(\Delta\) be the global transition function of \(C\), let \(X\) be a subset of \(Q^M\), and let \(A\) be a subset of \(M\). Then,
\[
  \forall m \in M \forall p \in \pi_A(X) : \Delta_{m \bullet X,m \bullet A}(m \bullet p) = m \bullet \Delta_{X,A}^{-}(p).
\]

**Proof:** Let \(m \in M\) and let \(p \in \pi_A(X)\). Then, the domain of \(m \bullet p\) is \(m \subseteq A\) and we have \(m \bullet p = g_{m \bullet m} \bullet p \in g_{m \bullet m} \bullet \pi_A(X) = \pi_{g_{m \bullet m} \bullet A}(g_{m \bullet m} \bullet X) = \pi_{m \bullet A}(m \bullet X)\). Hence, the term \(\Delta_{m \bullet X,m \bullet A}(m \bullet p)\) is well-defined. Moreover, by definition of \(p\), there is a \(c \in X\) such that \(c_{\mid A} = p\), and hence \((m \bullet c)_{\mid m \bullet A} = m \bullet p\). Therefore, by definition of \(\Delta_{m \bullet X,m \bullet A}\) and of \(\Delta_{X,A}^{-}\), because \(\Delta\) is \(\bullet_H\)-equivariant and \(g_{m \bullet m} \in H\), we have \(\Delta_{m \bullet X,m \bullet A}(m \bullet p) = \Delta(m \bullet c)_{\mid m \bullet A^{-N}} = \Delta(g_{m \bullet m} \bullet c)_{\mid g_{m \bullet m} \bullet A^{-N}} = g_{m \bullet m} \bullet \Delta(c)_{\mid A^{-N}} = m \bullet \Delta(c)_{\mid A^{-N}} = m \bullet \Delta_{X,A}^{-}(p)\). \(\Box\)
Theorem 2.13 characterises right Følner nets as those nets whose components are asymptotically invariant under taking finite boundaries. Broadly speaking, right Følner nets are by definition those nets whose components are asymptotically invariant under small perturbations and, intuitively, taking finite boundaries is a small perturbation.

In the proof of Theorem 2.13, the upper bound given in Lemma 2.12 is essential, which itself follows from the upper bound given in Corollary 2.6 and the inclusion given in Lemma 2.11, which in turn follows from the equality given in Lemma 2.10.

**Lemma 2.10:** Let $m$ be an element of $M$, and let $g$ be an element of $G/G_0$. There is an element $g \in g$ such that

$$\forall g' \in G/G_0 : (m \trianglelefteq g) \trianglelefteq g' = m \trianglelefteq g \cdot g',$$

in particular, for said $g \in g$, we have $(m \trianglelefteq g) \trianglelefteq g^{-1}G_0 = m$.

**Proof:** There is a $g \in G$ such that $gG_0 = g$. Moreover, because $\trianglelefteq$ is a semi-action with defect $G_0$, there is a $g_0 \in G_0$ such that

$$\forall g' \in G/G_0 : (m \trianglelefteq gG_0) \trianglelefteq g' = m \trianglelefteq g \cdot (g_0^{-1} \cdot g').$$

Because $g \cdot (g_0^{-1} \cdot g') = gg_0^{-1} \cdot g'$ and $gg_0^{-1} \in g$, the statement holds. \qed

**Lemma 2.11:** Let $A$ and $A'$ be two subsets of $M$, and let $g$ and $g'$ be two elements of $G/G_0$. Then, for each element $m \in (_\trianglelefteq g)^{-1}(A) \setminus (_\trianglelefteq g')^{-1}(A')$,

$$m \trianglelefteq g \in \bigcup_{g' \in g} A \setminus (_\trianglelefteq g^{-1} \cdot g')^{-1}(A'),$$

$$m \trianglelefteq g' \in \bigcup_{g' \in g} (_\trianglelefteq g)^{-1}(A) \setminus A'.$$

**Proof:** Let $m \in (_\trianglelefteq g)^{-1}(A) \setminus (_\trianglelefteq g')^{-1}(A')$. Then, $m \trianglelefteq g \in A$ and $m \trianglelefteq g' \notin A'$. According to Lemma 2.10, there is a $g \in g$ and a $g' \in g'$ such that $(m \trianglelefteq g) \trianglelefteq g^{-1} \cdot g' = m \trianglelefteq g' \notin A'$ and $(m \trianglelefteq g') \trianglelefteq (g')^{-1} \cdot g = m \trianglelefteq g \in A$. Hence, $m \trianglelefteq g \notin (_\trianglelefteq g^{-1} \cdot g')^{-1}(A')$ and $m \trianglelefteq g' \notin (_\trianglelefteq (g')^{-1} \cdot g)^{-1}(A)$. Therefore, $m \trianglelefteq g \in A \setminus (_\trianglelefteq g^{-1} \cdot g')^{-1}(A')$ and $m \trianglelefteq g' \in (_\trianglelefteq (g')^{-1} \cdot g)^{-1}(A) \setminus A'$. In conclusion, $m \trianglelefteq g \in \bigcup_{g \in A} A \setminus (_\trianglelefteq g^{-1} \cdot g')^{-1}(A')$ and $m \trianglelefteq g' \in \bigcup_{g' \in g'} (_\trianglelefteq (g')^{-1} \cdot g)^{-1}(A) \setminus A'$. \qed

**Lemma 2.12:** Let $G_0$ be finite, let $F$ and $F'$ be two finite subsets of $M$, and let $g$ and $g'$ be two elements of $G/G_0$. Then,

$$|(_\trianglelefteq g)^{-1}(F) \setminus (_\trianglelefteq g')^{-1}(F')| \leq \left\{ \begin{array}{ll} |G_0|^2 \cdot \max_{g \in g} |F \setminus (_\trianglelefteq g^{-1} \cdot g')^{-1}(F')|, \\ |G_0|^2 \cdot \max_{g' \in g'} |(_\trianglelefteq g)^{-1} \cdot g')^{-1}(F) \setminus F'|. \end{array} \right.$$ 

**Proof:** Put $A = (_\trianglelefteq g)^{-1}(F) \setminus (_\trianglelefteq g')^{-1}(F')$. For each $g \in g$, put $B_g = F \setminus (_\trianglelefteq g^{-1} \cdot g')^{-1}(F')$. For each $g' \in g'$, put $B'_g = (_\trianglelefteq (g')^{-1} \cdot g)^{-1}(F) \setminus F'$.

According to Lemma 2.11, the image of $A$ under $\trianglelefteq g$ is included in $\bigcup_{g \in g} B_g$ and the image of $A$ under $\trianglelefteq g'$ is included in $\bigcup_{g' \in g'} B'_g$. Moreover, for each $m \in M$, according to Corollary 2.6, we have $|(_\trianglelefteq g)^{-1}(m)| \leq |G_0|$ and $|(_\trianglelefteq g')^{-1} m)| \leq |G_0|$. Therefore, because $|g| = |G_0|$, $|A| \leq |G_0| \cdot |\bigcup_{g \in g} B_g| \leq |G_0| \cdot \sum_{g \in g} |B_g| \leq |G_0|^2 \cdot \max_{g \in g} |B_g|$ and analogously $|A| \leq |G_0|^2 \cdot \max_{g' \in g'} |B'_g|$. \qed
A net of non-empty and finite subsets of $M$ is a right Følner net if and only if these subsets are asymptotically invariant under the right semi-action induced by $\vartriangleright$. If the stabiliser of the origin under $\vartriangleright$ is finite, that is the case if and only if those subsets are asymptotically invariant under taking finite boundaries. In other words, if and only if the finite boundaries of the subsets grow much slower than the subsets themselves. This is shown in the next theorem.

**Theorem 2.13:** Let $G_0$ be finite and let $\{F_i\}_{i \in I}$ be a net in $\{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\}$ indexed by $(I, \leq)$. The net $\{F_i\}_{i \in I}$ is a right Følner net in $\mathcal{R}$ if and only if

$$\forall E \subseteq G/G_0 \text{ finite} : \lim_{i \to \infty} \frac{|\partial_E F_i|}{|F_i|} = 0.$$ 

**Proof:** First, let $\{F_i\}_{i \in I}$ be a right Følner net in $\mathcal{R}$. Furthermore, let $E \subseteq G/G_0$ be finite. Moreover, let $i \in I$. For each $e \in E$ and each $e' \in E$, put $A_{i,e,e'} = (\_ \_ \_ e)^{-1}(F_i) \setminus (\_ \_ \_ e')^{-1}(F_i)$. For each $g \in G/G_0$, put $B_{i,g} = F_i \setminus (\_ \_ \_ g)^{-1}(F_i)$. According to Definition 2.1,

$$\partial_E F_i = \left( \bigcup_{e \in E} (\_ \_ \_ e)^{-1}(F_i) \setminus \left( \bigcap_{e' \in E} (\_ \_ \_ e')^{-1}(F_i) \right) \right) = \bigcup_{e,e' \in E} (\_ \_ \_ e)^{-1}(F_i) \setminus (\_ \_ \_ e')^{-1}(F_i) = \bigcup_{e,e' \in E} A_{i,e,e'}.$$ 

Hence, $|\partial_E F_i| \leq \sum_{e,e' \in E} |A_{i,e,e'}|$.

According to Lemma 2.12, we have $|A_{i,e,e'}| \leq |G_0|^2 \cdot \max_{g \in E} |B_{i,g}^{-1} \cdot e'|$. Put $E' = \{ g^{-1} \cdot e' \mid e', e' \in E, g \in G \}$. Because $E$ is finite, $G_0$ is finite, and, for each $e \in E$, we have $|e| = |G_0|$, the set $E'$ is finite. Therefore,

$$\frac{|\partial_E F_i|}{|F_i|} \leq \frac{1}{|F_i|} \sum_{e,e' \in E} |A_{i,e,e'}| \leq \frac{|G_0|^2}{|F_i|} \sum_{e,e' \in E} \max_{g \in E} |B_{i,g}^{-1} \cdot e'| \leq \frac{|G_0|^2 \cdot |E|^2}{|F_i|} \max_{e' \in E'} |B_{i,e'}| \leq |G_0|^2 \cdot \max_{e' \in E'} \frac{|E|^2}{|F_i|} \frac{|F_i \setminus (\_ \_ \_ e)^{-1}(F_i)|}{|F_i|} \to 0.$$ 

In conclusion, $\lim_{i \to \infty} \frac{|\partial_E F_i|}{|F_i|} = 0$.

Secondly, for each finite $E \subseteq G/G_0$, let $i \in I$, let $e \in G/G_0$, and put $E = \{G_0, e\}$. According to Item (2) of Lemma 2.5, we have $F_i \setminus (\_ \_ \_ e)^{-1}(F_i) \subseteq \partial_E F_i$. Therefore,

$$\frac{|F_i \setminus (\_ \_ \_ e)^{-1}(F_i)|}{|F_i|} \leq \frac{|\partial_E F_i|}{|F_i|} \to 0.$$ 

In conclusion, $\{F_i\}_{i \in I}$ is a right Følner net in $\mathcal{R}$. \qed

**Example 2.14 (Lattice):** In the situation of Example 2.2, the sequence $(S(\rho))_{\rho \in N^+}$ grows linearly in size, indeed, $(|S(\rho)|)_{\rho \in N^+} = (4^\rho)_{\rho \in N^+}$; the sequence $(B(\rho))_{\rho \in N^0}$ grows polynomially in size, indeed, $(|B(\rho)|)_{\rho \in N_0} = (\sum_{\rho=0}^\rho |S(\rho)|)_{\rho \in N_0} = (2^\rho \rho + 1 + 1)_{\rho \in N_0}$; the sequence $(\partial_B(1)B(\rho))_{\rho \in N^+}$ grows linearly in size, indeed, $(|\partial_B(1)B(\rho)|)_{\rho \in N^+} = (|B(\rho + 1) \setminus B(\rho - 1)|)_{\rho \in N^+} = (4(2\rho + 1))_{\rho \in N^+}$; in general, for each non-negative integer $\varrho$, we have $|(\partial_B(\varrho)B(\rho))|_{\rho \in N_0} = (4\varrho(2\rho + 1))_{\rho \in N_0}$. It follows that the $S$-growth rate $\lim_{\rho \to \infty} \sqrt[\rho]{|B(\rho)|}$ of $\mathcal{R}$ is equal to 1 and that

$$\forall \varrho \in N_0 : \lim_{\rho \to \infty} \frac{|\partial_B(\varrho)B(\rho)|}{|B(\rho)|} = 0.$$ 

Hence, the cell space $\mathcal{R}$ is right amenable and, according to Theorem 2.13, the sequence $(B(\rho))_{\rho \in N_0}$ is a right Følner net in $\mathcal{R}$. 

Example 2.15 (Tree): In the situation of Example 2.3, the sequence \((S(\rho))_{\rho \in N_+}\) grows exponentially in size, indeed, \(|S(\rho)|_{\rho \in N_+} = (3^\rho + 3^{\rho-1})_{\rho \in N_+}\); the sequence \((B(\rho))_{\rho \in N_0}\) also grows exponentially in size, indeed, \(|B(\rho)|_{\rho \in N_0} = (\sum_{g=0}^{\rho} S(\rho))_{\rho \in N_0} = (2 \cdot 3^\rho - 1)_{\rho \in N_0}\); for each non-negative integer \(\rho\), the sequence \((\partial_{E(\rho)} B(\rho))_{\rho \in N_{\geq \rho}}\) also grows exponentially in size, indeed, \(|\partial_{E(\rho)} B(\rho)|_{\rho \in N_{\geq \rho}} = (|B(\rho + \rho') - B(\rho)|)_{\rho \in N_{\geq \rho}} = (2(3^{\rho+\rho} - 3^{\rho-\rho}))_{\rho \in N_{\geq \rho}}\). It follows that the \(S\)-growth rate \(\lim_{\rho \to \infty} \sqrt[3]{|B(\rho)|}\) of \(\mathcal{R}\) is equal to \(3\) and that
\[
\forall \rho \in N_0: \lim_{\rho \to \infty} \frac{|\partial_{E(\rho)} B(\rho)|}{|B(\rho)|} = 3^\rho - 3^{-\rho} \neq 0.
\]

Hence, the cell space \(\mathcal{R}\) is not right amenable and, according to Theorem 2.13, each subsequence of \((B(\rho))_{\rho \in N_0}\) is not a right Følner net in \(\mathcal{R}\), actually, there is no right Følner net in \(\mathcal{R}\).

In Definition 4.1 we use a net of non-empty and finite subsets of \(M\) to define the entropy of a subset of global configurations. If that net is a right Følner net, then the global transition function of a big-cellular automaton is surjective if and only if the entropy of its image is maximal and the entropy of its image is maximal if and only if it is pre-injective. For the first equivalence it suffices that the net has a weaker property, which we define below; for the second equivalence though that weaker property does not suffice.

Definition 2.16: Let \(\{F_i\}_{i \in I}\) be a net in \(\{F \subseteq M | F \neq \emptyset, F\ finite\}\) indexed by \((I, \leq)\). It is called right Erling net in \(\mathcal{R}\), indexed by \((I, \leq)\) if and only if
\[
\forall E \subseteq G/G_0 \ finite: \limsup_{i \in I} \frac{|\partial_E^{-} F_i|}{|F_i|} < 1.
\]

Remark 2: Regardless of whether \(\{F_i\}_{i \in I}\) is an Erling net or not, the above limit superior is always less than or equal to 1.

Definition 2.17: The cell space \(\mathcal{R}\) is called right tractable if and only if there is a right Erling net in \(\mathcal{R}\).

Remark 3: Because each right Følner net is a right Erling net, each right amenable cell space with finite stabilisers is right tractable.

Lemma 2.18: Let \(\mathcal{R}\) be finitely and symmetrically right generated. It is right tractable. And, for each symmetric and finite right generating set \(S\) of \(\mathcal{R}\) such that \(G_0 \in S\), the sequence \((B(\rho))_{\rho \in N_0}\) is a right Erling net in \(\mathcal{R}\).

Proof: Let \(S\) be a symmetric and finite right generating set of \(\mathcal{R}\) with \(G_0 \in S\), let \(\{F_i\}_{i \in I}\) be the sequence \((B(\rho))_{\rho \in N_0}\), and let \(E\) be a finite subset of \(G/G_0\). There is a non-negative integer \(\rho\) such that \(m_0 \sqsubset E \subseteq B(\rho)\) and there is a subset \(E'\) of \(G/G_0\) such that \(B(\rho) = m_0 \sqsubset E'\). Note that, because \(\sqsubset\) is free, we have \(E \subseteq E'\).

Let \(i \in I\) such that \(i \geq \rho + 1\). We have \(\partial_E^{-} F_i \subseteq \partial_E^{-} F_1\) and \(\partial_E^{-} F_i = F_1 \setminus F_{i-\rho}\). Moreover, because \(G_0 \in S\), for each element \(m \in F_i \setminus F_{i-\rho}\), there is an element \(m' \in F_{i-\rho} \setminus F_{i-\rho-1}\) and there is a family \(\{s_k\}_{k \in \{1, 2, \ldots, \rho\}}\) in \(S\) such that \(((m' \sqsubset s_1 \sqsubset s_2 \sqsubset \cdots) \sqsubset s_\rho = m)\). Thus,
\[
|\partial_E^{-} F_i| = |F_i \setminus F_{i-\rho}| \leq |S|^\rho \cdot |F_{i-\rho} \setminus F_{i-\rho-1}| \leq |S|^\rho \cdot |F_{i-\rho}|.
\]
Furthermore, \(|F_i| = |F_i \setminus F_{i-\rho}| + |F_{i-\rho}| = |\partial_E^{-} F_i| + |F_{i-\rho}|\). Hence,
\[
\frac{|F_i|}{|\partial_E^{-} F_i|} \geq \frac{|F_i|}{|\partial_E^{-} F_i|} = 1 + \frac{|F_{i-\rho}|}{|\partial_E^{-} F_i|} \geq 1 + \frac{|F_{i-\rho}|}{|S|^\rho \cdot |F_{i-\rho}|} = 1 + |S|^{-\rho}.
\]
Therefore,
\[
\limsup_{i \in I} \frac{|\partial_E^{-} F_i|}{|F_i|} \leq \frac{1}{1 + |S|^{-\rho}} < 1.
\]
In conclusion, \( \{F_i\}_{i \in I} \) is a right Erling net and hence \( \mathcal{R} \) is right tractable.

**Example 2.19 (Finitely):** Let \( G \) be a finitely generated group, let \( M \) be the vertices of a Cayley graph of \( G \), and let \( \triangleright \) be the left group action of \( G \) on \( M \) by left multiplication. The cell space \( ((M, G, \triangleright), (e_G, (m)_{m \in M})) \) is finitely and symmetrically right generated and hence right tractable.

**Example 2.20 (Lattice):** In the situation of Example 2.14, for each non-negative integer \( \rho \) and each non-negative integer \( \varrho \) such that \( \rho \geq \varrho \), the interior \( \mathcal{B}(\varrho) \)-boundary of \( \mathcal{B}(\rho) \) is equal to \( \mathcal{B}(\rho) \setminus \mathcal{B}(\rho - \varrho) \) and its cardinality is equal to \( 2\varrho(2\rho - \varrho + 1) \). Hence,

\[
\forall \varrho \in \mathbb{N}_0 : \limsup_{\rho \to \infty} \frac{|\partial_{\mathcal{B}(\varrho)}\mathcal{B}(\rho)|}{|\mathcal{B}(\rho)|} = 0.
\]

Therefore, the sequence \( (\mathcal{B}(\rho))_{\rho \in \mathbb{N}_0} \) is a right Erling net in \( \mathcal{R} \) and hence the cell space \( \mathcal{R} \) is right tractable.

**Example 2.21 (Tree):** In the situation of Example 2.15, for each non-negative integer \( \rho \) and each non-negative integer \( \varrho \) such that \( \rho \geq \varrho \), the interior \( \mathcal{B}(\varrho) \)-boundary of \( \mathcal{B}(\rho) \) is equal to \( \mathcal{B}(\rho) \setminus \mathcal{B}(\rho - \varrho) \) and its cardinality is equal to \( 2 \cdot 3^\rho (1 - 3^{-\varrho}) \). Hence,

\[
\forall \varrho \in \mathbb{N}_0 : \limsup_{\rho \to \infty} \frac{|\partial_{\mathcal{B}(\varrho)}\mathcal{B}(\rho)|}{|\mathcal{B}(\rho)|} = 1 - 3^{-\varrho}.
\]

Therefore, the sequence \( (\mathcal{B}(\rho))_{\rho \in \mathbb{N}_0} \) is a right Erling net in \( \mathcal{R} \) and hence the cell space \( \mathcal{R} \) is right tractable. However, as we have seen in Example 2.15, that sequence is not a right Følner net and that cell space is not right amenable.

### 3. Tilings

In this section, let \( \mathcal{R} = (\mathcal{M}, \mathcal{K}) = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M})) \) be a cell space.

In Definition 3.1 we introduce the notion of \((E, E')\)-tilings. In Theorem 3.5 we show using Zorn’s lemma that, for each subset \( E \) of \( G/G_0 \), there is an \((E, E')\)-tiling. And in Lemma 3.6 we show that, for each \((E, E')\)-tiling with finite sets \( E \) and \( E' \), the net \( \{|T \cap F_i|^{-1}\}_{i \in I} \) is asymptotically not less than \( \{|F_i|\}_{i \in I} \).

**Definition 3.1:** Let \( T \) be a subset of \( M \), and let \( E \) and \( E' \) be two subsets of \( G/G_0 \). The set \( T \) is called \((E, E')\)-tiling of \( \mathcal{R} \) if and only if the family \( \{t \leq E\}_{t \in T} \) is pairwise disjoint and the family \( \{t \leq E'\}_{t \in T} \) is a cover of \( M \).

**Remark 4:** Let \( T \) be an \((E, E')\)-tiling of \( \mathcal{R} \). For each subset \( F \) of \( E \) and each superset \( F' \) of \( E' \) with \( F' \subseteq G/G_0 \), the set \( T \) is an \((F, F')\)-tiling of \( \mathcal{R} \). In particular, the set \( T \) is an \((E, E \cup E')\)-tiling of \( \mathcal{R} \).

**Remark 5:** In the situation of Remark 1, the notion of \((E, E')\)-tiling is the same as the one defined in [1, Section 5.6, Paragraph 2].

**Example 3.2 (Lattice):** In the situation of Example 2.20, let \( E \) be the ball \( \mathcal{B}(1) \), let \( E' \) be the set \( \{m-m' \mid m, m' \in E\} \), which is the ball \( \mathcal{B}(2) \), and let \( T \) be the set \( \{(z_1, z_2) \in M \mid (z_1, z_2) \in 2\mathbb{Z}^2, z_1 + z_2 \in 4\mathbb{Z}\} \). The set \( T \) is an \((E, E')\)-tiling of \( \mathcal{R} \) (see Figure 4).

**Example 3.3 (Tree):** In the situation of Example 2.21, let \( E \) be the ball \( \mathcal{B}(1) \), let \( E' \) be the set \( \{e(e')^{-1} \mid e, e' \in E\} \), which is the ball \( \mathcal{B}(2) \), and let \( T \) be the smallest subset of \( M \) such that \( m_0 \in T \) and, for each element \( t \in T \), each element \( x \in \{a, b, a^{-1}, b^{-1}\} \), and each element \( y \in \{a, b, a^{-1}, b^{-1}\} \), we have \( txy^2 \in T \) if and only if \( |txy^2| = |tx| + 2 \) and \( d(tx, y), t) = 3 \), in other words, if and only if \( t = m_0 \) and \( y \neq x^{-1} \), or \( t \neq m_0 \), the last symbol of the reduced word that represents \( t \) is not \( x^{-1} \), and \( y \neq x^{-1} \), or \( t \neq m_0 \), the last symbol is \( x^{-1} \), and \( y \notin \{x, x^{-1}\} \). The set \( T \) is an \((E, E')\)-tiling of \( \mathcal{R} \), in particular, because \( E \subseteq E' \), it is an \((E, E')\)-tiling of \( \mathcal{R} \) (see Figure 5).
Figure 4. The grid points are elements of $M = \mathbb{Z}^2$; the dots are elements of the tiling $T$; for each element $t \in T$, the grid points in the region enclosed by the diamond with solid border about $t$ is the ball $t + E = B(t, 1)$ and the grid points in the region enclosed by the diamond with dashed border about $t$ is the ball $t + E' = B(t, 2)$.

Figure 5. A part of the $(\{a, b, a^{-1}, b^{-1}\})$-Cayley graph of $F_2$ is depicted, the leftmost dot is the neutral element $e_{F_2}$, the dots are elements of $T$, and the vertices that are adjacent to solid edges are elements of $\bigcup_{t \in T} tE$. The neutral element $e_{F_2}$ is contained in $T$ and, figuratively speaking, if you stand at an element $t \in T$, you take one step in a direction $x \in \{a, b, a^{-1}, b^{-1}\}$ to reach the leaf $tx$ of the cross $tE$ and you take another two steps in a direction $y \in \{a, b, a^{-1}, b^{-1}\}$ that leads away from $e_{F_2}$ and away from $t$, then you reach the element $txy^2$, which is contained in $T$, and all elements of $T$ can be reached in that way.

The set $\{ma^z \mid m \in M, z \in \mathbb{Z}, |ma^z| = |m| + 1\}$ is a $(\{a^{-1}, e_{F_2}, a\}, \{a^{-1}, e_{F_2}, a\})$-tiling of $\mathcal{R}$ (see Figure 6). Note that, for each cell $m \in M$, we have $|ma^z| = |m| + 1$ if and only if the last symbol of the reduced word that represents $m$ is neither $a$ nor $a^{-1}$ but $b$ or $b^{-1}$, and that the set $\{ma^z \mid z \in \mathbb{Z}\}$ contains every third element of the set $\{ma^z \mid z \in \mathbb{Z}\}$, which, in Figure 6, is a horizontal bi-infinite path in the $(\{a, b, a^{-1}, b^{-1}\}$-Cayley graph of $F_2$.

**Example 3.4 (Sphere):** In the situation of Example 2.4, let $E'$ be the set $\{g(g')^{-1}G_0 \mid e, e' \in E, g \in e, g' \in e'\} = \{g_0g_0'g_0^{-1}G_0 \mid g_0, g_0' \in G_0\}$, and, for each point $m \in M$, let $E'_m = m \triangleleft E'$. Because $g^{-1}$ is the rotation about the axis $a$ by $-\rho$ radians, the set $G_0g^{-1} \triangleright m_0$ is equal to $E_{m_0}$ and the set $gG_0g^{-1} \triangleright m_0$ is equal to $E_{g \triangleright m_0}$. Because $m_0 \triangleleft E' = g_{m_0m_0}G_0gG_0g^{-1} \triangleright m_0 = G_0 \triangleright (gG_0g^{-1} \triangleright m_0) = G_0 \triangleright E_{g \triangleright m_0}$, the set $E'_{m_0}$ is the curved circular disk of radius $2\rho$ with the north pole $m_0$ at its centre. And, for each point...
Figure 6. A part of the \(\{a, b, a^{-1}, b^{-1}\}\)-Cayley graph of \(F_2\) is depicted, the largest dot is the neutral element \(e_{F_2}\), the dots are elements of \(T\), and the vertices that are adjacent to solid edges are elements of \(\bigcup_{t \in T} t \cdot E\).

Let \(G\) (see Figure and an upper bound of \(a = \epsilon\), \(\epsilon = \epsilon\). By the definition of internal boundaries, we have \(E\) is depicted, the largest dot is the neutral element \(e_{F_2}\), the dots are elements of \(T\), and the vertices that are adjacent to solid edges are elements of \(\bigcup_{t \in T} t \cdot E\).

Proof Idea 3.7: Moreover, it is preordered by inclusion.

Theorem 3.5: Let \(E\) be a non-empty subset of \(G/G_0\). There is an \((E, E')\)-tiling of \(\mathcal{R}\), where \(E' = \{g(g')^{-1}G_0 \mid e, e' \in E, g \in e, g' \in e'\}\).

Proof: Let \(S = \{S \subseteq M \mid \{s \triangleleft E\}_{s \in S} \text{is pairwise disjoint}\}\). Because \(\{m_0\} \in S\), the set \(S\) is non-empty. Moreover, it is preordered by inclusion.

Let \(C\) be a chain in \((S, \subseteq)\). Then, \(\bigcup_{S \in C} S\) is an element of \(S\) and an upper bound of \(C\). According to Zorn’s lemma, there is a maximal element \(T\) in \(S\). By definition of \(S\), the family \(\{t \triangleleft E\}_{t \in T}\) is pairwise disjoint.

Let \(m \in M\). Because \(T\) is maximal and \(m \triangleleft E\) is non-empty, there is a \(t \in T\) such that \((t \triangleleft E) \cap (m \triangleleft E) \neq \emptyset\). Hence, there are \(e, e' \in E\) such that \(t \triangleleft e = m \triangleleft e'\). According to Lemma 2.10, there is a \(g' \in e'\) such that \((m \triangleleft e') \triangleleft (g')^{-1}G_0 = m\), and there is a \(g \in e\) such that \((t \triangleleft e) \triangleleft (g')^{-1}G_0 = t \triangleleft g(g')^{-1}G_0\). Therefore, \(m = t \triangleleft g(g')^{-1}G_0\). Because \((g(g')^{-1}G_0 \in E', \) we have \(m \in t \triangleleft E'\). Thus, \(\{t \triangleleft E'\}_{t \in T}\) is a cover of \(M\).

In conclusion, \(T\) is an \((E, E')\)-tiling of \(\mathcal{R}\).

Lemma 3.6: Let \(G_0\) be finite, let \(\{F_i\}_{i \in I}\) be a right Erling net in \(\mathcal{R}\) indexed by \((I, \leq)\), let \(E\) and \(E'\) be two finite subsets of \(G/G_0\), and let \(T\) be an \((E, E')\)-tiling of \(\mathcal{R}\). There is a positive real number \(\epsilon\) in \(\mathbb{R}_{>0}\) and there is an index \(i_0 \in I\) such that, for each index \(i \in I\) with \(i \geq i_0\), we have \(|T \cap F_i^{-E}| \geq \epsilon|F_i|\).

Proof Idea 3.7: By the definition of internal boundaries, we have \(|F_i| - |\partial_{E,E} F_i| \leq |F_i^{-E} E|\). And, for nice \(E\) and \(E'\), we have \(F_i^{-E} E = (F_i^{-E})^{-E'}\). And, because \(M = T \triangleleft E'\), we have \(|(F_i^{-E})^{-E'}| \leq |T \cap F_i^{-E}| \cdot |E'|\).
Hence,
\[
\frac{|T \cap F_i^{E_i}|}{|F_i|} \geq 1 - \frac{|\partial_{F_i^{-1}} E_i|}{|F_i|}.
\]

For great enough indices \(i \in I\), the right side is bounded below away from zero.

**Proof:** According to Remark 4, we may suppose, without loss of generality, that \(E \subseteq E'\), \(G_0 \cdot E' \subseteq E'\), and that \((E')^{-1} \subseteq E'\), where \((E')^{-1} = \{(g')^{-1}G_0 \mid e' \in E', g' \in e'\}\).

Let \(i \in I\). Put
\[
T_i = T \cap F_i^{-E} = \{t \in T \mid t \not\trianglelefteq E \subseteq F_i\}
\]
and put
\[
T_i' = T \cap ((F_i^{-E})^{-E})^{+E'} = \{t \in T \mid (t \not\trianglelefteq E') \cap (F_i^{-E})^{-E'} \neq \emptyset\}
\]
(see Figure 8). Because the family \(\{t \not\trianglelefteq E'\}_{t \in T}\) is a cover of \(M\),
\[
(F_i^{-E})^{-E'} = M \cap (F_i^{-E})^{-E'} = \bigcup_{t \in T} (t \not\trianglelefteq E') \cap (F_i^{-E})^{-E'}.
\]

And, for each element \(t \in T \setminus T_i'\), we have \((t \not\trianglelefteq E') \cap (F_i^{-E})^{-E'} = \emptyset\). Hence,
\[
(F_i^{-E})^{-E'} = \bigcup_{t \in T_i'} (t \not\trianglelefteq E') \cap (F_i^{-E})^{-E'}.
\]

And, for each element \(t \in T_i'\), we have \((t \not\trianglelefteq E') \cap (F_i^{-E})^{-E'} \subseteq t \not\trianglelefteq E'.\) Thus,
\[
(F_i^{-E})^{-E'} \subseteq \bigcup_{t \in T_i'} t \not\trianglelefteq E'.
\]
And, according to Item (12) of Lemma 2.5, we have \(((F_i^{-E})^{-E'})^{\trianglelefteq} \subseteq F_i^{-E}\) and hence \(T_i' \subseteq T_i\). Therefore,

\[(F_i^{-E})^{-E'} \subseteq \bigcup_{t \in T_i} t \trianglelefteq E'.\]

And, because \(\trianglelefteq\) is free, for each \(t \in T_i\), we have \(|t \trianglelefteq E'| = |E'|\). Hence,

\[|(F_i^{-E})^{-E'}| \leq |T_i| \cdot |E'|.\]

And, according to Items (6) and (11) of Lemma 2.5, we have \((F_i^{-E})^{-E'} \supseteq (F_i^{-G_0'E})^{-E'} = F_i^{-E''}\), where \(E'' = E' \cdot (G_0 \cdot E) = \{g' \cdot (g_0 \cdot e) \mid e \in E, g_0 \in G_0, e' \in E', g' \in e'\}\). And, because \(\partial_{E''}F_i = F_i \setminus (F_i^{-E''} \cap F_i)\),

\[|F_i^{-E''}| \geq |F_i^{-E''} \cap F_i| = |F_i| - |\partial_{E''}F_i|.\]

Hence,

\[|F_i| - |\partial_{E''}F_i| \leq |T_i| \cdot |E'|.\]

Therefore,

\[\frac{|T_i|}{|F_i|} \geq \frac{1}{|E'|} \cdot \left(1 - \frac{|\partial_{E''}F_i|}{|F_i|}\right).\]

Because \(\{F_i\}_{i \in I}\) is a right Erling net, there is a real number \(\xi \in [0, 1)\) and there is an index \(i_0 \in I\) such that

\[\forall i \in I : \left(i \geq i_0 \implies \frac{|\partial_{E''}F_i|}{|F_i|} \leq \xi\right) .\]

Put \(\varepsilon = (1/|E'|) \cdot (1 - \xi)\). Then, for each \(i \in I\) with \(i \geq i_0\),

\[\frac{|T_i|}{|F_i|} \geq \varepsilon.\]

**Example 3.8 (Lattice):** In the situation of Example 3.2, for each non-negative integer \(\rho\), one can see that \(|T \cap \mathbb{B}(\rho)| = |T \cap \mathbb{B}(4\lfloor \rho/4 \rfloor)| = (2\lfloor \rho/4 \rfloor + 1)^2\) and, for each positive integer \(\rho\), recall that \(\mathbb{B}(\rho)^{-\mathbb{B}(1)} = \mathbb{B}(\rho - 1)\).
Let $\rho$ be a positive integer. The integer $\varrho = 4\lceil \rho/4 \rceil$ is the multiple of 4 such that $\rho \leq \varrho < \rho + 4$. Thus, $\varrho - 1$ is the greatest integer such that $\lfloor (\varrho - 1)/4 \rfloor = \lfloor (\rho - 1)/4 \rfloor$, in particular, $|T \cap \mathbb{B}(\rho - 1)| = |T \cap \mathbb{B}(\varrho - 1)|$. Hence, because $\mathbb{B}(\rho) \subseteq \mathbb{B}(\varrho)$,

$$
\frac{|T \cap \mathbb{B}(\rho - 1)|}{|\mathbb{B}(\rho)|} \geq \frac{|T \cap \mathbb{B}(\varrho - 1)|}{|\mathbb{B}(\varrho)|}.
$$

Moreover, one can show that the sequence

$$
\left( \frac{|T \cap \mathbb{B}(\zeta - 1)|}{|\mathbb{B}(\zeta)|} \right)_{\zeta \in 4^\mathbb{N}_+}
$$

is increasing (and converges to $1/8$). Therefore, because $\varrho \in 4\mathbb{N}_+$,

$$
\frac{|T \cap \mathbb{B}(\rho - 1)|}{|\mathbb{B}(\rho)|} \geq \frac{|T \cap \mathbb{B}(4 - 1)|}{|\mathbb{B}(4)|} = \frac{1}{4^1}.
$$

In conclusion, $|T \cap \mathbb{B}(\rho - 4^{(1)})| \geq (1/41) \cdot |\mathbb{B}((\rho))|$ (actually, for each real number $\varepsilon \in (0, 1/8)$, there is an index $\rho_0 \in \mathbb{N}_0$ such that, for each index $\rho \in \mathbb{N}_0$ with $\rho \geq \rho_0$, we have $|T \cap \mathbb{B}(\rho - 4^{(1)})| \geq \varepsilon \cdot |\mathbb{B}((\rho))|$).

**Example 3.9 (Tree):** In the situation of Example 3.3, by the construction of $T$, we have $m_0 \in T$; moreover, there are four pairwise distinct elements $x_{m_0,1}, x_{m_0,2}, x_{m_0,3},$ and $x_{m_0,4} \in \{a, b, a^{-1}, b^{-1}\}$ and, for each index $j \in \{1, 2, 3, 4\}$, there are three pairwise distinct elements $y_{m_0,1}, y_{m_0,2},$ and $y_{m_0,3} \in \{a, b, a^{-1}, b^{-1}\}$ such that $x_{m_0,1} y_{m_0,1}^2 \in T$ and $|x_{m_0,1} y_{m_0,1}^2| = |x_{m_0,1}| + 2$ (note that $|x_{m_0,1}| + 2 = 3$); furthermore, for each element $t \in T \setminus \{m_0\}$, there are three pairwise distinct elements $x_{t,1}, x_{t,2},$ and $x_{t,3} \in \{a, b, a^{-1}, b^{-1}\}$ such that $|tx_{t,j}| = |t| + 1$, for $j \in \{1, 2, 3\}$, and, for each index $j \in \{1, 2, 3\}$, there are three pairwise distinct elements $y_{t,1}, y_{t,2},$ and $y_{t,3} \in \{a, b, a^{-1}, b^{-1}\}$ such that $|tx_{t,1} y_{t,k}|^2 = |tx_{t,1}| + 2$ (note that $|tx_{t,1}| + 2 = |t| + 3$); and, there is an element $x_{t,4} \in \{a, b, a^{-1}, b^{-1}\}$ such that $|tx_{t,4}| = |t| - 1$, and there are two distinct elements $y_{t,4} \in \{a, b, a^{-1}, b^{-1}\}$ such that $|tx_{t,4} y_{t,4}|^2 = |tx_{t,4}| + 2$ (note that $|tx_{t,4}| + 2 = |t| + 1$); and, the elements $m_0, x_{m_0,1} y_{m_0,1},$ for $j \in \{1, 2, 3, 4\}$ and $k \in \{1, 2, 3\}$, $|

4. Entropies

In this section, let $\mathcal{R} = (\mathcal{M}, \mathcal{K}) = ((M, G, \triangleright), \{m_0, \{g_{m_0,n}\}_{n \in M})$ be a cell space, let $\mathcal{C} = (\mathcal{R}, Q, N, \delta)$ be a semi-cellular automaton, and let $\Delta$ be the global transition function of $\mathcal{C}$ such that the stabiliser $G_0$ of $m_0$ under $\triangleright$, the set $Q$ of states, and the neighbourhood $N$ are finite, and the set $Q$ is non-empty.

In Definition 4.1 we introduce the entropy of a subset $X$ of $Q^M$ with respect to a net $\{F_i\}_{i \in \mathbb{F}}$ of non-empty and finite subsets of $M$, which is the asymptotic growth rate of the number of finite patterns with domain $F_i$ that occur in $X$. In Lemma 4.2 we show that $Q^M$ has entropy $\log |Q|$ and that entropy
is non-decreasing. In Theorem 4.3 we show that applications of global transition functions of semi-cellular automata on subsets of \( Q^M \) do not increase entropy. And in Lemma 4.5 we show that if for each point \( t \) of an \( (E, E') \)-tiling not all patterns with domain \( t \subseteq E \) occur in a subset of \( Q^M \), then that subset has less entropy than \( Q^M \).

**Definition 4.1:** Let \( X \) be a subset of \( Q^M \) and let \( \mathcal{F} = \{ F_i \}_{i \in I} \) be a net in \( \{ F \subseteq M \mid F \neq \emptyset, F \text{ finite} \} \). The non-negative real number or negative infinity

\[
\text{ent}_\mathcal{F}(X) = \limsup_{i \in I} \frac{\log |\pi_{F_i}(X)|}{|F_i|}
\]

is called entropy of \( X \) with respect to \( \mathcal{F} \).

**Remark 6:** In the situation of Remark 1, the notion of entropy is the same as the one defined in [1, Definition 5.7.1].

**Lemma 4.2:** Let \( \mathcal{F} = \{ F_i \}_{i \in I} \) be a net in \( \{ F \subseteq M \mid F \neq \emptyset, F \text{ finite} \} \). Then,

1. \( \text{ent}_\mathcal{F}(Q^M) = \log |Q| \);
2. \( \forall X \subseteq Q^M \forall X' \subseteq Q^M : (X \subseteq X' \implies \text{ent}_\mathcal{F}(X) \leq \text{ent}_\mathcal{F}(X')) \);
3. \( \forall X \subseteq Q^M : \text{ent}_\mathcal{F}(X) \leq \log |Q| \).

**Proof:**

1. For each \( i \in I \), we have \( \pi_{F_i}(Q^M) = Q^{F_i} \) and hence

\[
\frac{\log |\pi_{F_i}(Q^M)|}{|F_i|} = \frac{\log |Q|^{|F_i|}}{|F_i|} = \frac{|F_i| \cdot \log |Q|}{|F_i|} = \log |Q|.
\]

In conclusion, \( \text{ent}_\mathcal{F}(Q^M) = \log |Q| \).

2. Let \( X, X' \subseteq Q^M \) such that \( X \subseteq X' \). For each \( i \in I \), we have \( \pi_{F_i}(X) \subseteq \pi_{F_i}(X') \) and hence, because \( \log \) is non-decreasing, \( \log |\pi_{F_i}(X)| \leq \log |\pi_{F_i}(X')| \). In conclusion, \( \text{ent}_\mathcal{F}(X) \leq \text{ent}_\mathcal{F}(X') \).

3. This is a direct consequence of Items (2) and (1). \( \square \)

Due to the locality of the global transition function \( \Delta \) and the asymptotic invariance of a right Følner net \( \{ F_i \}_{i \in I} \) under taking finite boundaries, the information that flows into \( F_i \) from the boundary under an application of \( \Delta \) is asymptotically negligible and hence applications of \( \Delta \) to subsets of global configurations do not increase entropy.

**Theorem 4.3:** Let \( \mathcal{R} \) be right amenable, let \( \mathcal{F} = \{ F_i \}_{i \in I} \) be a right Følner net in \( \mathcal{R} \) indexed by \( (I, \subseteq) \), and let \( X \) be a subset of \( Q^M \). Then, \( \text{ent}_\mathcal{F}(\Delta(X)) \leq \text{ent}_\mathcal{F}(X) \).

**Proof:** Suppose, without loss of generality, that \( G_0 \in N \).

Let \( i \in I \). According to Lemma 2.8, the map \( \Delta_{X,F_i} : \pi_{F_i}(X) \rightarrow \pi_{F_i}^{-N}(\Delta(X)) \) is surjective. Therefore, \( |\pi_{F_i}^{-N}(\Delta(X))| = |\pi_{F_i}(X)| \).

Because \( G_0 \in N \), according to Item (7) of Lemma 2.5, we have \( F_{i}^{-N} \subseteq F_{i} \). Thus, \( \pi_{F_i}(\Delta(X)) \subseteq \pi_{F_i}^{-N}(\Delta(X)) \times Q^{F_i \setminus F_i^{-N}} \). Hence,

\[
\log |\pi_{F_i}(\Delta(X))| \leq \log |\pi_{F_i}^{-N}(\Delta(X))| + \log |Q^{F_i \setminus F_i^{-N}}|
\]

\[
\leq \log |\pi_{F_i}(X)| + |F_i \setminus F_i^{-N}| \cdot \log |Q|.
\]

Because \( G_0 \in N \), according to Item (7) of Lemma 2.5, we have \( F_i \subseteq F_{i}^{+N} \). Therefore, \( F_i \setminus F_i^{-N} \subseteq F_i^{+N} \setminus F_i^{-N} = \partial_N F_i \). Because \( G_0, F_i, \) and \( N \) are finite, according to Item (8) of Lemma 2.5, the boundary \( \partial_N F_i \) is finite. Hence,

\[
\frac{\log |\pi_{F_i}(\Delta(X))|}{|F_i|} \leq \frac{\log |\pi_{F_i}(X)|}{|F_i|} + \frac{|\partial_N F_i|}{|F_i|} \cdot \log |Q|.
\]
Therefore, because $N$ is finite, according to Theorem 2.13,

$$\text{ent}_\mathcal{F} (\Delta(X)) \leq \limsup_{i \in I} \frac{\log |\pi_{\mathcal{F}_i}(X)|}{|F_i|} + \left( \lim_{i \in I} \frac{\partial_{\mathcal{N}} F_i}{|F_i|} \right) \log |Q| = \text{ent}_\mathcal{F} (X).$$

\[\Box\]

**Example 4.4 (Tree):** In the situation of Example 2.21, the cell space $\mathcal{R}$ is not right amenable and the sequence $\mathcal{F} = (B(\rho))_{\rho \in \mathbb{N}_0}$ is not a right Følner net in $\mathcal{R}$. However, the cell space $\mathcal{R}$ is right tractable and the sequence $\mathcal{F}$ is a right Erling net in $\mathcal{R}$. Nevertheless, for the majority rule over $\mathcal{R}$, there is a subset $X$ of global configurations whose image has greater entropy than $X$ itself, as we show below. Broadly speaking, such a subset exists because the boundaries of components of $\mathcal{F}$ are so big that the information that flows in from them under applications of $\Delta$ is asymptotically significant.

Let $\mathcal{R}$ be the set $\{0, 1\}$, let $N$ be the ball $B(1)$, let $\delta$ be the $\bullet$-invariant map $Q^N \to Q$, $\ell \mapsto 0$, if $\sum_{\rho \in N} \ell(n) \leq \frac{|N|}{2}$, and $\ell \mapsto 1$, otherwise, which is known as majority rule, and let $C$ be the cellular automaton $(\mathcal{R}, Q, N, \delta)$. Furthermore, for each non-negative integer $\rho$, let $Y_\rho$ be the set $\{c \in Q^M \setminus \{0\} \mid c \downarrow_{M \setminus S(\rho)} \equiv 0\}$, let $X_{\rho+1}$ be the set

$$\{y \in Y_{\rho+1} \mid \forall m' \in S(\rho) \exists q \in Q : y \downarrow_{(m' \cdot N) \cap S(\rho+1)} = q\},$$

let $\mathcal{Y}$ be the set $\bigcup_{\rho \in \mathbb{N}_0} Y_\rho$, and let $X$ be the set $\bigcup_{\rho \in \mathbb{N}_0} X_{\rho+1}$.

The global transition function $\Delta$ of $C$ maps $X$ bijectively onto $\mathcal{Y}$; even more, for each non-negative integer $\rho$, it maps $X_{\rho+1}$ bijectively onto $Y_\rho$; more precisely, for each global configuration $x \in X_{\rho+1}$, the global configuration

$$y_x : M \to Q,$$

$$m' \mapsto \begin{cases} 0, & \text{if } m' \in M \setminus S(\rho), \\ x(m), & \text{if } m' \in S(\rho), \text{ where } m \in (m' \cdot N) \cap S(\rho+1), \end{cases}$$

is the unique element of $Y_\rho$ that satisfies $\Delta(x) = y_x$, and, for each global configuration $y \in Y_\rho$, the global configuration

$$x_y : M \to Q,$$

$$m \mapsto \begin{cases} 0, & \text{if } m \in M \setminus S(\rho + 1), \\ y(m'), & \text{if } m \in S(\rho + 1), \text{ where } m' \in S(\rho) \text{ such that } m \in m' \cdot N, \end{cases}$$

is the unique element of $X_{\rho+1}$ that satisfies $\Delta(x_y) = y$ (see Figure 9).

The entropy of $\Delta(X)$ with respect to $\mathcal{F}$ is greater than the entropy of $X$ with respect to $\mathcal{F}$. The reason is that, broadly speaking, the cardinality of $\pi_{B(\rho)}(\Delta(X))$ is approximately $2^{S(\rho)}$, whereas the cardinality of $\pi_{B(\rho)}(X)$ is approximately $2^{S(\rho-1)}$, and the cardinality of $B(\rho)$ is approximately $|S(\rho)|$.

**Proof:** First, we prove that $\Delta(X) = \mathcal{Y}$. Let $y$ be a global configuration of $\mathcal{Y}$ and let $m'$ be a cell of $M$. Then,

$$\Delta(x_y)(m') = \delta(n \mapsto x_y(m' \cdot n)) = \begin{cases} 0, & \text{if } \sum_{n \in N} x_y(m' \cdot n) \leq \frac{5}{2}, \\ 1, & \text{otherwise}. \end{cases}$$

Of the $5$ elements of $m' \cdot N$, the $4$ or $3$ elements of $(m' \cdot N) \cap S(|m'| + 1)$ have the same state in $x_y$, namely $y(m')$. Thus,

$$\sum_{m \in m' \cdot N} x_y(m) \leq \frac{5}{2} \iff y(m') = 0.$$

Hence,

$$\Delta(x_y)(m') = \begin{cases} 0, & \text{if } y(m') = 0, \\ 1, & \text{otherwise,} \end{cases} = y(m').$$
Therefore, $\Delta(x_y) = y$. Moreover, for each $x \in X$, if $\Delta(x) = y$, then $y_x = y$, hence $x = x_{yx} = x_y$, and therefore $x_y$ is unique.

Secondly, we prove that $\text{ent}_x(\Delta(X)) > \text{ent}_x(X)$. Recall that, for each positive integer $i$, we have $|\mathcal{S}(i)| = 3^i + 3^{i-1}$ and $|\mathcal{B}(i)| = 2 \cdot 3^i - 1$.

Let $i$ be an integer such that $i \geq 2$. Then,

$$|\pi_{\mathcal{B}(i)}(X)| = \left| \bigcup_{\rho=0}^{i-1} \pi_{\mathcal{B}(i)}(X_{\rho+1}) \right| \cup \{0\} = \sum_{\rho=0}^{i-1} (2^{|\mathcal{S}(\rho)|} - 1) + 1.$$

Thus, because $i \geq 1$,

$$|\pi_{\mathcal{B}(i)}(X)| \leq \sum_{\rho=0}^{i-1} (2^{|\mathcal{S}(\rho)|} - 1) + 1 = i \cdot 2^{|\mathcal{S}(i-1)|} - i + 1 \leq i \cdot 2^{|\mathcal{S}(i-1)|}.$$

Hence, because $i \geq 2$,

$$\log|\pi_{\mathcal{B}(i)}(X)| \leq \log(i) + |\mathcal{S}(i-1)| \cdot \log(2) = \log(i) + (3^{i-1} + 3^{i-2}) \cdot \log(2).$$

Therefore, because $|\mathcal{B}(i)| \geq 2 \cdot 3^i - 3^i = 3^i$,

$$\frac{\log|\pi_{\mathcal{B}(i)}(X)|}{|\mathcal{B}(i)|} \leq \frac{\log(i) + (3^{i-1} + 3^{i-2}) \cdot \log(2)}{3^i} = \frac{\log(i) + 4 \cdot \log(2)}{3^i}.$$

Thus, $\text{ent}_x(X) \leq (4/9) \cdot \log(2)$.

Let $i$ be an integer such that $i \geq 1$. Then,

$$|\pi_{\mathcal{B}(i)}(Y)| = \left| \bigcup_{\rho \in \mathbb{N}_0} \pi_{\mathcal{B}(i)}(Y_{\rho}) \right| = \sum_{\rho=0}^{i-1} (2^{|\mathcal{S}(\rho)|} - 1) + 1 = \sum_{\rho=0}^{i-1} 2^{|\mathcal{S}(\rho)|} - i.$$

Thus, because $\sum_{\rho=0}^{i-1} 2^{|\mathcal{S}(\rho)|} - i \geq 0$,

$$\log|\pi_{\mathcal{B}(i)}(Y)| \geq \log 2^{|\mathcal{S}(i)|} = |\mathcal{S}(i)| \cdot \log(2) = (3^i + 3^{i-1}) \cdot \log(2).$$
Figure 10. Schematic representation of the set-up of the proof of Lemma 4.5.

Hence, because $|B(i)| \leq 2 \cdot 3^i$,

$$\log|\pi_{B(i)}(\Delta(X))| \geq \frac{3^i + 3^{i-1}}{2 \cdot 3^i} \cdot \log(2) = \frac{6}{9} \cdot \log(2).$$

Therefore, $\text{ent}_E(Y) \geq (6/9) \cdot \log(2)$.

In conclusion,

$$\text{ent}_E(\Delta(X)) = \text{ent}_E(Y) \geq \frac{6}{9} \cdot \log(2) > \frac{4}{9} \cdot \log(2) \geq \text{ent}_E(X). \qed$$

**Lemma 4.5:** Let $\mathcal{R}$ be right tractable, let $\mathcal{F} = \{F_i\}_{i \in I}$ be a right Erling net in $\mathcal{R}$ indexed by $(I, \leq)$, let $Q$ contain at least two elements, let $X$ be a subset of $Q^M$, let $E$ and $E'$ be two non-empty and finite subsets of $G/G_0$, and let $T$ be an $(E, E')$-tiling of $\mathcal{R}$, such that, for each cell $t \in T$, we have $\pi_{\pi \in E}(X) \subseteq Q^t \subseteq E$. Then, $\text{ent}_E(X) < \log|Q|$. 

**Proof:** For each $t \in T$, because $\pi_{t \in E}(X) \subseteq Q^t \subseteq |Q| \geq 2$, and $|t \in E| \geq 1$,

$$|\pi_{t \in E}(X)| \leq |Q^t| - 1 = |Q|^{|t \in E|} - 1 \geq 1.$$

Let $i \in I$. Put $T_i = T \cap F_i^{-E}$ and put $F_i^* = F_i \setminus (\bigcup_{t \in T_i} t \in E)$ (see Figure 10). Because $\bigcup_{t \in T_i} t \in E \subseteq F_i$ and $\{t \in E\}_{t \in T}$ is pairwise disjoint,

$$\pi_{F_i}(X) \subseteq \pi_{F_i^*}(X) \times \prod_{t \in T_i} \pi_{t \in E}(X) \subseteq Q^{F_i^*} \times \prod_{t \in T_i} \pi_{t \in E}(X).$$

Therefore,

$$\log|\pi_{F_i}(X)| \leq \log|Q|^{F_i^*} + \sum_{t \in T_i} \log|\pi_{t \in E}(X)|$$

$$\leq \log|Q|^{F_i^*} + \sum_{t \in T_i} \log(|Q|^{|t \in E|} - 1)$$

$$= |F_i^*| \cdot \log|Q| + \sum_{t \in T_i} \log(|Q|^{|t \in E|} - |Q|^{-|t \in E|})$$

$$= |F_i^*| \cdot \log|Q| + \sum_{t \in T_i} |t \in E| \cdot \log|Q| + \sum_{t \in T_i} \log(1 - |Q|^{-|t \in E|}).$$
Moreover, for each $t \in T_i$, we have $t \trianglelefteq E \subseteq F_i$. Thus,

$$|F_i^\ast| = |F_i| - \sum_{t \in T_i} |t \trianglelefteq E|.$$ 

And, because $\trianglelefteq$ is free, we have $|t \trianglelefteq E| = |E|$. Hence,

$$\log |\pi_{F_i}(X)| \leq |F_i| \cdot \log |Q| + |T_i| \cdot \log (1 - |Q|^{-|E|}).$$

Put $c = -\log (1 - |Q|^{-|E|})$. Because $|Q| \geq 2$ and $|E| \geq 1$, we have $|Q|^{-|E|} \in (0, 1)$ and hence $c > 0$.

According to Lemma 3.6, there are $\varepsilon \in [0, \infty)$ and $l_0 \in \mathbb{N}$ such that, for each $i \in I$ with $i \geq l_0$, we have $|T_i| \geq \varepsilon |F_i|$. Therefore, for each such $i$,

$$\frac{\log |\pi_{F_i}(X)|}{|F_i|} \leq \log |Q| - c\varepsilon.$$ 

In conclusion,

$$\text{ent}_\mathcal{F} (X) = \limsup_{i \in I} \frac{\log |\pi_{F_i}(X)|}{|F_i|} \leq \log |Q| - c\varepsilon$$

$$< \log |Q|. \qedhere$$

**Corollary 4.6:** Let $\mathcal{R}$ be a right tractable, let $\mathcal{F} = \{F_i\}_{i \in I}$ be a right Erling net in $\mathcal{R}$ indexed by $(I, \leq)$, let $Q$ contain at least two elements, let $H$ be a $K$-big subgroup of $G$, let $X$ be a $\triangleright H$-invariant subset of $Q^M$, and let $E$ be a non-empty and finite subset of $G/G_0$, such that $\pi_{m_0 \trianglelefteq E}(X) \subseteq Q^{m_0 \trianglelefteq E}$. Then, $\text{ent}_\mathcal{F} (X) < \log |Q|$. 

**Proof:** According to Theorem 3.5, there is a subset $E'$ of $G/G_0$ and an $(E, E')$-tiling $T$ of $\mathcal{R}$. Because $G_0$ and $E$ are finite, so is $E'$. Let $m \in M$.

Put $h = g_{m_0, m_0}^{-1} g_{m_0, m}^{-1}$. Then, because $H$ is $K$-big, we have $h \in H$. And, $h \triangleright (m \trianglelefteq E) = m_0 \trianglelefteq E$. Hence, because $X$ is $\triangleright H$-invariant,

$$\pi_{m \trianglelefteq E}(X) = \pi_{m_0 \trianglelefteq E}(h^{-1} \triangleright X) = h^{-1} \triangleright \pi_{h \triangleright (m_0 \trianglelefteq E)}(X) = h^{-1} \triangleright \pi_{m_0 \trianglelefteq E}(X).$$

And, because $\pi_{m_0 \trianglelefteq E}(X) \subseteq Q^{m_0 \trianglelefteq E}$,

$$h^{-1} \triangleright \pi_{m_0 \trianglelefteq E}(X) \subseteq h^{-1} \triangleright Q^{m_0 \trianglelefteq E} = q h^{-1} \triangleright (m_0 \trianglelefteq E) = Q^{m_0 \trianglelefteq E}.$$ 

Therefore, $\pi_{m_0 \trianglelefteq E}(X) \subseteq Q^{m_0 \trianglelefteq E}$. In conclusion, according to Lemma 4.5, we have $\text{ent}_\mathcal{F} (X) < \log |Q|$. \qedhere

**Example 4.7 (Necessity of Assumptions):** This example demonstrates that in Lemma 4.5 it is necessary that $\mathcal{F}$, $E$, and $T$ are such that the limit superior of the net $\{|T \cap F_i^\ast|/|F_i|\}_{i \in I}$ is greater than $0$, which, according to Lemma 3.6, is the case if $\mathcal{F}$ is a right Erling net in $\mathcal{R}$.

Let $\mathcal{F} = \{F_i\}_{i \in I}$ be a net in $(F \subseteq M | F \neq \emptyset, F \text{ finite})$ indexed by $(I, \leq)$, let $Q$ contain at least two elements, let $E$ and $E'$ be two non-empty and finite subsets of $G/G_0$, and let $T$ be an $(E, E')$-tiling of $\mathcal{R}$, Moreover, let $q$ be an element of $Q$, let $p$ be the pattern of $Q^{m_0 \trianglelefteq E}$ such that $p \equiv q$, and let $X$ be the subset $Q^M \setminus \bigcup_{t \in T} \{c \in Q^M | c|_{t \trianglelefteq E} = t \triangleright p\}$ of $Q^M$. Then, for each cell $t \in T$, we have $\pi_{t \trianglelefteq E}(X) = Q^{t \trianglelefteq E} \setminus \{t \triangleright p\} \subseteq Q^{t \trianglelefteq E}$.

Let $i$ be an index of $I$. Then, $\pi_{F_i}(X) = Q^{F_i} \setminus \bigcup_{t \in T \cap F_i} Y_t$, where $Y_t = \{p' \in Q^{F_i} | p'|_{t \trianglelefteq E} = t \triangleright p\}$, for $t \in T \cap F_i$. Indeed, we have $\pi_{F_i}(X) \subseteq Q^{F_i} \setminus \bigcup_{t \in T \cap F_i} Y_t$. To show the other inclusion, let $p' \in Q^{F_i} \setminus \bigcup_{t \in T \cap F_i} Y_t$. Then, there is a state $q' \in Q \setminus \{q\}$ and there is a global configuration $c \in Q^M$ such that $c|_{F_i} = p'$ and $c|_{M \setminus F_i} \equiv q'$. Hence, for each $t \in T \cap F_i$, we have $c|_{t \trianglelefteq E} = p'|_{t \trianglelefteq E} \neq t \triangleright p$. 


And, for each $t \in T \setminus F_i^{-E}$, there is an $e \in E$ such that $t \circ e \notin F_i$ and thus, by definition of $c$, we have $c(t \circ e) = q' \neq q$, and hence $c|_t \circ e \neq t \circ p$. Therefore, $c \in X$ and hence $p' = c|_{F_i} \in \pi F_i(X)$. In conclusion, $Q_i^\circ \setminus \bigcup_{t \in T \cap F_i^{-E}} Y_t \subseteq \pi F_i(X)$.

For each subset $S$ of $T \cap F_i^{-E}$, we have $|\bigcap_{s \in S} Y_s| = |Q_i^\circ \setminus \bigcup_{s \in S} s^E| = |Q|^{|F_i|-|S|\cdot |E|}$, which only depends on the cardinality of $S$. Hence, according to a special case of the inclusion-exclusion principle and the binomial formula,

$$|\pi F_i(X)| = \sum_{k=0}^{|T \cap F_i^{-E}|} (-1)^k \binom{|T \cap F_i^{-E}|}{k} |Q|^{|F_i|-k\cdot |E|}$$

$$= |Q|^{|F_i|} \cdot \sum_{k=0}^{|T \cap F_i^{-E}|} \binom{|T \cap F_i^{-E}|}{k} (-1)^{k} (|Q|-|E|)^k$$

Thus,

$$\log |\pi F_i(X)| = \log |Q| + \sum_{k=0}^{|T \cap F_i^{-E}|} \binom{|T \cap F_i^{-E}|}{k} \log (1 - |Q|-|E|)$$

Therefore,

$$\text{ent}_F(X) = \log |Q| + \limsup_{i \in I} \frac{|T \cap F_i^{-E}|}{|F_i|} \cdot \log (1 - |Q|-|E|)$$

Hence, because $\log (1 - |Q|-|E|) < 0$, we have $\text{ent}_F(X) < \log |Q|$ if and only if $\limsup_{i \in I} |T \cap F_i^{-E}| / |F_i| > 0$.

### 5. Gardens of Eden

In this section, let $\mathcal{R} = (\mathcal{M}, \mathcal{K}) = ((M, \mathcal{G}, \triangleright), (m_0, \{g_{m_0,m}\}_{m \in M}))$ be a cell space and let $\mathcal{C} = (\mathcal{R}, Q, N, \delta)$ be a semi-cellular automaton such that the stabiliser $G_0$ of $m_0$ under $\triangleright$, the set $Q$ of states, and the neighbourhood $N$ are finite, and the set $Q$ is non-empty. Furthermore, let $\Delta$ be the global transition function of $\mathcal{C}$.

In Theorem 5.5 we show that if $\Delta$ is not surjective, then the entropy of its image is less than the entropy of $Q^M$. And the converse of that statement obviously holds. In Theorem 5.8 we show that if the entropy of the image of $\Delta$ is less than the entropy of $Q^M$, then $\Delta$ is not pre-injective. And in Theorem 5.14 we show the converse of that statement. These four statements establish the Garden of Eden theorem, see Theorem 5.17.

**Definition 5.1:** Let $c$ and $c'$ be two maps from $M$ to $Q$. The set diff $(c, c') = \{m \in M \mid c(m) \neq c'(m)\}$ is called difference of $c$ and $c'$.

**Definition 5.2:** The map $\Delta$ is called pre-injective if and only if, for each tuple $(c, c') \in Q^M \times Q^M$ such that diff $(c, c')$ is finite and $\Delta(c) = \Delta(c')$, we have $c = c'$.

In the proof of Theorem 5.5, the existence of a Garden of Eden pattern, as stated in Lemma 5.4, is essential, which itself follows from the existence of a Garden of Eden configuration, the compactness of $Q^M$, and the continuity of $\Delta$.

**Definition 5.3:**

1. Let $c : M \to Q$ be a global configuration. It is called **Garden of Eden configuration** if and only if it is not contained in $\Delta(Q^M)$.
2. Let $p : A \to Q$ be a pattern. It is called **Garden of Eden pattern** if and only if, for each global configuration $c \in Q^M$, we have $\Delta(c) |_A \neq p$. 

Remark 7:

(1) The global transition function \( \Delta \) is surjective if and only if there is no Garden of Eden configuration.

(2) If \( p: A \to Q \) is a Garden of Eden pattern, then each global configuration \( c \in Q^M \) with \( c|_A = p \) is a Garden of Eden configuration.

(3) If there is a Garden of Eden pattern, then \( \Delta \) is not surjective.

Lemma 5.4: Let \( \Delta \) not be surjective. There is a Garden of Eden pattern with non-empty and finite domain.

Proof: Because \( \Delta \) is not surjective, there is a Garden of Eden configuration \( c \in Q^M \). Equip \( Q^M \) with the prodiscrete topology. Because \( \Delta \) is continuous, according to [1, Lemma 3.3.2], \( \Delta(Q^M) \) is closed in \( Q^M \). Hence, \( Q^M \setminus \Delta(Q^M) \) is open. Therefore, because \( c \in Q^M \setminus \Delta(Q^M) \), there is a non-empty and finite subset \( F \) of \( M \) such that

\[
\text{Cyl } (c, F) = \{ c' \in Q^M \mid c'|_F = c|_F \} \subseteq Q^M \setminus \Delta(Q^M).
\]

Hence, \( c|_F \) is a Garden of Eden pattern with non-empty and finite domain.

\[ \square \]

Theorem 5.5: Let \( \mathcal{R} \) be right tractable, let \( \mathcal{F} \) be a right Erling net in \( \mathcal{R} \), let \( H \) be a \( K \)-big subgroup of \( G \), let \( \delta \) be \( \bullet G_0 \cap H \)-invariant, let \( Q \) contain at least two elements, and let \( \Delta \) not be surjective. Then, \( \text{ent}_{\mathcal{F}}(\Delta(Q^M)) < \log|Q| \).

Proof: According to Lemma 5.4, there is a Garden of Eden pattern \( p: F \to Q \) with non-empty and finite domain. Let \( E = (m_0 \equiv \_)^{-1}(F) \). Then, \( m_0 \equiv E = F \) and, because \( \equiv \) is free, \( |E| = |F| < \infty \). Because \( p \) is a Garden of Eden pattern, \( p \notin \pi_{m_0 \equiv E}(\Delta(Q^M)) \). Hence, \( \pi_{m_0 \equiv E} \Delta(Q^M) \subseteq Q^{m_0 \equiv E} \). Moreover, according to [7, Item 1 of Theorem 2], the map \( \Delta \) is \( \triangleleft H \)-equivariant. Hence, for each \( h \in H \), we have \( h \triangleleft \Delta(Q^M) = \Delta(h \triangleleft Q^M) = \Delta(Q^M) \). In other words, \( \Delta(Q^M) \) is \( \triangleleft H \)-invariant. Thus, according to Corollary 4.6, we have \( \text{ent}_{\mathcal{F}}(\Delta(Q^M)) < \log|Q| \).

\[ \square \]

Corollary 5.6: Let \( \mathcal{R} \) be right tractable, let \( \mathcal{F} \) be a right Erling net in \( \mathcal{R} \), let \( H \) be a \( K \)-big subgroup of \( G \), let \( \delta \) be \( \bullet G_0 \cap H \)-invariant, and let \( Q \) contain at least two elements. Then, \( \Delta \) is surjective if and only if \( \text{ent}_{\mathcal{F}}(\Delta(Q^M)) = \log|Q| \).

Proof: This is a direct consequence of Theorem 5.5.

\[ \square \]

in the remainder of this section, let \( \mathcal{F} \) be a right Følner net in \( \mathcal{R} \) indexed by \( (l, \leq) \).

In the proof of Theorem 5.8, the fact that enlarging each element of \( \mathcal{F} \) does not increase entropy, as stated in the next lemma, is essential.

Lemma 5.7: Let \( X \) be a subset of \( Q^M \) and let \( E \) be a finite subset of \( G/G_0 \) such that \( G_0 \in E \). Then, \( \text{ent}_{|F_j^+|_{l \in l}}(X) \leq \text{ent}_{\mathcal{F}}(X) \).

Proof: Let \( i \in l \). According to Item (7) of Lemma 2.5, we have \( F_j^+ \subseteq F_i \subseteq F_j^+ \). Hence, \( \pi_{F_j^+}(X) \subseteq \pi_{F_i}(X) \times Q_{F_i}^+ \triangleleft F_i \) and \( F_j^+ \subseteq F_i \subseteq \partial E F_i \). Thus,

\[
\log|\pi_{F_j^+}(X)| \leq \log|\pi_{F_i}(X)| + |F_j^+| \log|Q| \leq \log|\pi_{F_i}(X)| + |\partial E F_i| \log|Q|.
\]

Therefore, according to Theorem 2.13,

\[
\text{ent}_{|F_j^+|_{l \in l}}(X) \leq \limsup_{i \in l} \frac{\log|\pi_{F_i}(X)|}{|F_i|} + \left( \lim_{i \in l} \frac{|\partial E F_i|}{|F_i|} \right) \cdot \log|Q| = \text{ent}_{\mathcal{F}}(X).
\]

\[ \square \]

Theorem 5.8: Let \( \text{ent}_{\mathcal{F}}(\Delta(Q^M)) < \log|Q| \). Then, \( \Delta \) is not pre-injective.
Proof Idea 5.9: The asymptotic growth rate of finite patterns in $\Delta(Q^M)$ is less than the one of $Q^M$. Hence, there are at least two finite patterns that can be identically extended to global configurations that have the same image under $\Delta$. Therefore, $\Delta$ is not pre-injective.

Proof: Suppose, without loss of generality, that $G_0 \in N$. Let $X = \Delta(Q^M)$. According to Lemma 5.7, we have $\forall_{i \in I} \{ \pi_{F_i}^N(x) \leq \operatorname{ent}_\mathcal{F}(X) < \log |Q| \}$. Hence, there is an $i \in I$ such that

$$\frac{\log |\pi_{F_i}^N(x)|}{|F_i|} < \log |Q|.$$  

Thus, $|\pi_{F_i}^N(x)| < |Q|^{|F_i|}$. Furthermore, let $q \in Q$ and let $X' = \{ c \in Q^M \mid c \mid_{M \setminus F_i} \equiv q \}$. Then, $|Q|^{|F_i|} = |X'|$. Hence, $|\pi_{F_i}^N(x)| < |X'|$. Moreover, according to Item (3) of Lemma 2.5, we have $(M \setminus F_i)^{-N} = M \setminus F_i^+$. Hence, for each $(c, c') \in X' \times X'$, according to Lemma 2.7, we have $\Delta(c) \mid_{M \setminus F_i^+} = \Delta(c') \mid_{M \setminus F_i^+}$. Therefore,

$$\Delta(X') = |\pi_{F_i}^N(\Delta(X'))| \leq |\pi_{F_i}^N(\Delta(Q^M))| = |\pi_{F_i}^N(X)| < |X'|.$$

Hence, there are $c, c' \in X'$ such that $c \neq c'$ and $\Delta(c) = \Delta(c')$. Thus, because diff $(c, c') \subseteq F_i$ is finite, the map $\Delta$ is not pre-injective.

Example 5.10 (Muller): In this example we present a cellular automaton on a non-right-amenable but right tractable cell space that, although the image of its global transition function does not have maximal entropy with respect to a right Erling net, is pre-injective. It is Muller's counterexample to Myhill's theorem, see [3, Section 6, p. 55].

Let $G$ be the group with presentation $(x, y, z \mid x^2, y^2, z^2)$ or, equivalently, the 3-fold free product of the cyclic group of order 2 with itself, let $Q$ be the $\mathbb{F}_2$-vector space $(\mathbb{F}_2)^2$, where $\mathbb{F}_2$ is the finite field of order 2, let $N$ be the set $\{x, y, z\}$, let $\delta$ be the map $Q^N \to Q$, $\ell \mapsto (\ell(x) + \ell(y) + \ell(z) - \ell(0)$, where, for each vector $v \in Q$, the first component of $v$ is denoted by $v_1$ and the second by $v_2$. The tuple $\mathcal{R} = ((G, G, \cdot), (e_G, (g)_{g \in G})$ is a cell space and the quadruple $C = (\mathcal{R}, Q, N, \delta)$ is a cellular automaton.

According to Lemma 2.18, the cell space $\mathcal{R}$ is right tractable and the sequence $\mathcal{F} = (\mathbb{F}(\rho))_{\rho \in \mathbb{F}_2}$ is a right Erling net. Moreover, because point evaluation, projection, and addition are linear, the local transition function $\delta$ is linear and hence the global transition function $\Delta$ of $C$ is linear. Furthermore, because the image of $\Delta$ is included in $(\mathbb{F}_2 \times \{0\})^G$, the global transition function $\Delta$ is not surjective. Hence, according to Theorem 5.5, we have $\operatorname{ent}_\mathcal{F}(\Delta(Q^G)) < \log |Q|$. However, as we show now, the global transition function $\Delta$ is pre-injective.

Let $c$ and $c'$ be two global configurations of $Q^G$ such that $\operatorname{diff}(c, c')$ is finite and $\Delta(c) = \Delta(c')$. Then, because $\Delta$ is linear, we have $\Delta(c - c') \equiv 0$. Let $c''$ be the global configuration $c - c'$. Suppose that $c'' \neq 0$. Then, because $c''$ has finite support, there is an element $g \in G$ such that $c''(g) \neq 0$ and $c'' \mid_{M \setminus B(|g|)} \equiv 0$. And, there are two distinct elements $s$ and $s' \in \{x, y, z\}$ such that $g_0, g_{s'} \in \mathbb{S}(|g| + 1)$ (see Figure 11). Hence, because $c'' \mid_{M \setminus B(|g|)} \equiv 0$, we have $c''(g_0) = 0$ and $c''(g_{s'}) = 0$. And, for each element $s'' \in \{s, s'\}$, because $g_{s'} s'' = g$ and $g_{s'} s'' \in \mathbb{S}(|g| + 2)$, for $s'' \in \{x, y, z\}$, $s'' \in \{s, s'\}$,

$$\Delta(c'')(g_{s''})_1 = c''(g_{s''} x_1) + c''(g_{s''} y_2) = \begin{cases} c''(g_{s'' x_1}) + c''(g_{s'' y_2}), & \text{if } s'' = x; \\ c''(g_{s'' x_1}) + c''(g_{s'' y_2}), & \text{if } s'' = y; \\ c''(g_{s'' x_1}) + c''(g_{s'' y_2}), & \text{if } s'' = z. \end{cases}$$

Because $c''(g) \neq 0$, for each element $s'' \in \{s, s'\}$, in two of the three cases we have $\Delta(c'')(g_{s''})_1 = 1$ and hence $\Delta(c'')(g_{s''})_1 = 1$ or $\Delta(c'')(g_{s''})_1 = 1$. Therefore, $\Delta(c'')(g_{s''}) \neq 0$ or $\Delta(c'')(g_{s''}) \neq 0$, which contradicts that $\Delta(c'') \equiv 0$. Thus, contrary to our supposition, we have $c'' \equiv 0$ and hence $c = c'$. In conclusion, the global transition function $\Delta$ is pre-injective.
In the proof of Theorem 5.14, the statement of Lemma 5.13 is essential, which says that if two distinct patterns have the same image and we replace each occurrence of the first by the second in a configuration, we get a new configuration in which the first pattern does not occur and that has the same image as the original one.

**Definition 5.11:** Identify $M$ with $G/G_0$ by $i: m \mapsto g_m$, let $A$ be a subset of $M$, let $p$ be map from $A$ to $Q$, let $c$ be map from $M$ to $Q$, let $m$ be an element of $M$. The pattern $p$ is said to occur at $m$ in $c$ and we write $p \subseteq m \in c$ if and only if $m \triangleleft p = c|_{m \in A}$.

**Lemma 5.12:** Let $A$ be a subset of $M$, and let $E$ and $E'$ be two subsets of $G/G_0$ such that $\{g^{-1} \cdot e' \mid e, e' \in E, g \in e\} \subseteq E'$. Then,

$\forall m \in M : m \triangleleft E \subseteq M \setminus A$ or $m \triangleleft E \subseteq A + E'$.

**Proof:** Let $m \in M$ such that $m \triangleleft E \not\subseteq M \setminus A$. Then, $(m \triangleleft E) \cap A \neq \emptyset$. Hence, there is an $e' \in E$ such that $m \triangleleft e' \in A$. Let $e \in E$. According to Lemma 2.10, there is a $g \in e$ such that $(m \triangleleft e) \triangleleft g^{-1} \cdot e' = m \triangleleft e'$. Because $g^{-1} \cdot e' \in E'$ and $m \triangleleft e' \in A$, we have $(m \triangleleft e) \triangleleft E' \cap A \neq \emptyset$. Thus, $m \triangleleft e \in A + E'$. Therefore, $m \triangleleft E \subseteq A + E'$.

**Lemma 5.13:** Identify $M$ with $G/G_0$ by $i: m \mapsto g_m$, let $H$ be a $K$-big subgroup of $G$, let $\delta$ be $\bullet_{G_0 \cap H}$-invariant, let $A$ be a subset of $M$, let $N'$ be the subset $\{g^{-1} \cdot n' \mid n, n' \in N, g \in n\}$ of $G/G_0$, and let $p$ and $p'$ be two maps from $A + N'$ to $Q$ such that $p|_{A + N'} \cup A = p'|_{A + N'} \cup A$, and $\Delta_{A + N'}(p) = \Delta_{A + N'}(p')$. Furthermore, let $c$ be a map from $M$ to $Q$ and let $S$ be a subset of $M$, such that the family $\{s \triangleleft A + N'\}_{s \in S}$ is pairwise disjoint and, for each cell $s \in S$, we have $p \subseteq s \cdot c$. Put

$$c' = c|_{M \setminus (\bigcup_{s \in S} s \triangleleft A + N')} \times \prod_{s \in S} s \cdot p'.$$

Then, for each cell $s \in S$, we have $p' \subseteq s \cdot c'$, and $\Delta(c) = \Delta(c')$. In particular, if $p \neq p'$, then, for each cell $s \in S$, we have $p \not\subseteq s \cdot c'$.

**Proof:** For each $s \in S$, we have $\text{dom} (s \triangleleft p) = \text{dom} (s \triangleleft p') = s \triangleleft A + N'$. Hence, $c'$ is well-defined. Moreover, for each $s \in S$, we have $(s \triangleleft p)|_{(s \triangleleft A + N') \setminus (s \triangleleft A)} = (s \triangleleft p')|_{(s \triangleleft A + N') \setminus (s \triangleleft A)}$.

Let $m \in M \setminus (\bigcup_{s \in S} s \triangleleft A)$. If $m \in M \setminus (\bigcup_{s \in S} s \triangleleft A + N')$, then $c'(m) = c(m)$. And, if there is an $s \in S$ such that $m \in s \triangleleft A + N$, then, because $m \not\subseteq s \triangleleft A$, we have $c'(m) = (s \triangleleft p')(m) = (s \triangleleft p)(m) = c(m)$. Therefore,

$$c' = c|_{M \setminus (\bigcup_{s \in S} s \triangleleft A + N')} \times \prod_{s \in S} s \cdot (p'|_{A}).$$
Let \( m \in M \).

Case 1: \( m \not\in N \subseteq M \setminus \left( \bigcup_{s \in S} s \subseteq A \right) \). Then, \( c' \upharpoonright_{m \oplus N} = c \upharpoonright_{m \oplus N} \). Hence, \( \Delta(c')(m) = \Delta(c)(m) \).

Case 2: \( m \not\in N \not\subseteq M \setminus \left( \bigcup_{s \in S} s \subseteq A \right) \). Then, there is an \( s \in S \) such that \( m \oplus N \not\subseteq M \setminus (s \subseteq A) \). Thus, according to Lemma 5.12, we have \( m \oplus N \subseteq (s \subseteq A)^{+N} \). Hence, because \( G_0 \cdot N' \subseteq N' \), according to Item (10) of Lemma 2.5, we have \( m \oplus N \subseteq A^{+N} \) and hence \( m \in (s \subseteq A^{+N})^{N} \). Therefore, because \( c \upharpoonright_{s \oplus A^{+N}} = s \circ p, \Delta^{-}_{A^{+N}}(p) = \Delta^{-}_{A^{+N}}(p') \), and \( c' \upharpoonright_{s \oplus A^{+N}} = s \circ p' \), according to Lemma 2.9,

\[
\Delta(c)(m) = \Delta^{-}_{s \oplus A^{+N}}(s \circ p)(m) \\
= (s \circ \Delta^{-}_{A^{+N}}(p))(m) \\
= (s \circ \Delta^{-}_{A^{+N}}(p'))(m) \\
= \Delta^{-}_{s \oplus A^{+N}}(s \circ p')(m) \\
= \Delta(c')(m).
\]

In either case, \( \Delta(c)(m) = \Delta(c')(m) \). Therefore, \( \Delta(c) = \Delta(c') \).

**Theorem 5.14:** Let \( H \) be a \( K \)-big subgroup of \( G \), let \( \delta \) be \( \bullet_{G_0 \cap H} \)-invariant, let \( Q \) contain at least two elements, and let \( \Delta \) not be pre-injective. Then, \( \text{ent}_F(\Delta(Q^M)) < \log|Q| \).

**Proof Idea 5.15:** For \( N' = N^{-1} \cdot N \), there is a subset \( A \subseteq M \) and there are two distinct finite patterns \( p \) and \( p' \) with domain \( A^{+N} \) that have the same image under \( \Delta^{-}_{A^{+N}} \). The set \( Y \) of all global configurations in which \( p \) does not occur at the cells of a tiling has the same image under \( \Delta \) as \( Q^M \), because in a global configuration we may replace occurrences of \( p \) by \( p' \) without changing the image. It follows that \( \text{ent}(\Delta(Q^M)) = \text{ent}(\Delta(Y)) \leq \text{ent}(Y) \); and, because \( Y \) is missing the pattern \( p \) at each cell of a tiling, we also have \( \text{ent}(Y) < \text{ent}(Q^M) = \log|Q| \). In conclusion, \( \text{ent}(\Delta(Q^M)) < \log|Q| \).

**Proof:** Suppose, without loss of generality, that \( G_0 \in N \). Identify \( M \) with \( G/G_0 \) by \( : m \mapsto G_{m_0,m} \).

Because \( \Delta \) is not pre-injective, there are \( c, c' \in Q^M \) such that \( \text{diff}(c, c') \) is finite, \( \Delta(c) = \Delta(c') \), and \( c \neq c' \). Put \( A = \text{diff}(c, c') \), put \( N' = \{g^{-1} \cdot n' | n' \in N, g \in n\} \), put \( E = A^{+N} \), and put \( p = c|_E \) and \( p' = c'|_E \). Because \( \Delta(c) = \Delta(c') \), we have \( \Delta^{-}_{A^{+N}}(p) = \Delta^{-}_{A^{+N}}(p') \).

Because \( A \) is finite and, for each \( n \in N \), we have \( |n| = |G_0| < \infty \), the set \( N' \) is finite. Moreover, \( G_0 \cdot N' \subseteq N' \). According to Item (7) of Lemma 2.5, because \( G_0 \in N' \) and \( A \neq \emptyset \), we have \( E \supseteq A \) and hence \( E \) is non-empty. According to Item (8) of Lemma 2.5, because \( G_0 \in N' \), and \( N' \) are finite, so is \( E \). Because \( E \) is non-empty, according to Theorem 3.5, there is a subset \( E' \) of \( G/G_0 \) and an \((E, E')\)-tiling \( T \) of \( \mathcal{R} \). Because \( G_0 \) and \( E \) are non-empty and finite, so is \( E' \).

Let \( Y = \{y \in Q^M | \forall t \in T : p \not\subseteq t \} y \). For each \( t \in T \), we have \( t \circ p \notin \pi_{t \subseteq E}(Y) \) and therefore \( \pi_{t \subseteq E}(Y) \subseteq Q^{E \ominus E} \). According to Lemma 4.5, we have \( \text{ent}_F(Y) < \log|Q| \). Hence, according to Theorem 4.3, we have \( \text{ent}_F(\Delta(Y)) < \log|Q| \).

Let \( x \in Q^M \). Put \( S = \{t \in T | p \subseteq t \} x \). According to Lemma 5.13, there is an \( x' \in Q^M \) such that \( x' \in Y \) and \( \Delta(x) = \Delta(x') \). Therefore, \( \Delta(Q^M) = \Delta(Y) \). In conclusion, \( \text{ent}_F(\Delta(Q^M)) < \log|Q| \).

**Example 5.16 (Tree):** In the situation of Example 4.4, the global transition function \( \Delta \) is not pre-injective but the entropy \( \text{ent}_F(\Delta(Q^M)) \) is maximal as we show now.

First, let \( c \) and \( c' \) be the two global configurations of \( Q^M \) such that \( c \equiv 0, c'(m_0) = 1 \), and \( c' \upharpoonright_{M \setminus \{m_0\}} = 0 \). Then, \( \text{diff}(c, c') \) is finite and \( \Delta(c) = \Delta(c') \) but \( c \neq c' \). Hence, \( \Delta \) is not pre-injective.

Secondly, let \( c \) be a global configuration of \( Q^M \). And, let \( c' \) be the global configuration of \( Q^M \) such that \( c'(m_0) = 0 \) and, for each cell \( m \in M \) and each generator \( s \in (a, b, a^{-1}, b^{-1}) \) with \( |ms| = |m| + 1 \), we have \( c'(ms) = c(m) \). For each generator \( s \in (a, b, a^{-1}, b^{-1}) \), we have \( |ms| = |m_0| + 1 \); and, for each cell \( m \in M \setminus \{m_0\} \), there are precisely three distinct generators \( s_1, s_2, \) and \( s_3 \in (a, b, a^{-1}, b^{-1}) \) such that \( |ms_k| = |m| + 1 \) for \( k \in \{1, 2, 3\} \). Hence, \( \Delta(c') = c \). Therefore, \( \Delta \) is surjective. In particular, \( \text{ent}_F(\Delta(Q^M)) = \log|Q| \).
Main Theorem 5.17 (Garden of Eden theorem; Edward Forrest Moore, 1962; John R. Myhill, 1963): Let \( \mathcal{M} = (M, G, \triangleright) \) be a right amenable left homogeneous space with finite stabilisers and let \( \Delta \) be the global transition function of a big-cellular automaton over \( \mathcal{M} \) with finite set of states and finite neighbourhood. The map \( \Delta \) is surjective if and only if it is pre-injective.

**Proof:** There is a coordinate system \( \mathcal{K} = (m_0, \{g_{m_0,m}\}_{m \in M}) \) such that there is a big-cellular automaton \( \mathcal{C} = (\mathcal{R}, Q, N, \delta) \) such that \( Q \) and \( N \) are finite and \( \Delta \) is its global transition function. Moreover, because \( \mathcal{C} \) is big, there is a \( K \)-big subgroup \( H \) of \( G \) such that the local transition function \( \delta \) is \( \bullet_{G_0 \cap H} \)-invariant. And, because \( G_0 \) is finite, the cell space \( \mathcal{R} = (\mathcal{M}, \mathcal{K}) \) is right amenable.

Case 1: Case \( |Q| \leq 1 \). If \( |Q| = 0 \), then, because \( |M| \neq 0 \), we have \( |Q^M| = 0 \). And, if \( |Q| = 1 \), then \( |Q^M| = 1 \). In either case, \( \Delta \) is bijective, in particular, surjective and pre-injective.

Case 2: Case \( |Q| \geq 2 \). According to Theorem 5.5 and Item (1) of Lemma 4.2, the map \( \Delta \) is not surjective if and only if \( \text{ent}_{\triangleright} (\Delta(Q^M)) < \log|Q| \). And, according to Theorem 5.8 and Theorem 5.14, we have \( \text{ent}_{\triangleright} (\Delta(Q^M)) < \log|Q| \) if and only if \( \Delta \) is not pre-injective. Hence, \( \Delta \) is not surjective if and only if it is not pre-injective. In conclusion, \( \Delta \) is surjective if and only if it is pre-injective. \( \square \)

**Remark 8:** In the situation of Remark 1, Main Theorem 5.17 is [1, Theorem 5.3.1].

**Example 5.18 (Tree):** The global transition function of the cellular automaton of Example 4.4 is surjective but not pre-injective, which was shown in Example 5.16.

**Example 5.19 (Muller):** The global transition function of the cellular automaton of Example 5.10 is not surjective but pre-injective, which was shown there.

**Example 5.20 (Exclusive Or):** The global transition function of the elementary cellular automaton with Wolfram code 90, whose local transition function combines the states of the left and right neighbours by exclusive or, is 4-to-1 surjective and pre-injective but not injective.

## 6. Construction of non-degenerated left homogeneous spaces

So far we have only seen examples of right amenable left homogeneous spaces with finite stabilisers for which there is a subgroup that acts freely and transitively. The global transition function of a semi-cellular automaton on such a space is essentially the global transition function of a cellular automaton over a group. For those spaces, Main Theorem 5.17 states nothing new. A simple construction of right amenable left homogeneous spaces with finite stabilisers for which there is no subgroup that acts freely and transitively goes like this: Act with the direct product of the automorphism groups of the coloured Cayley graph of a group and a vertex-transitive, finite, and directed non-Cayley graph on the direct product of the vertices of these graphs. Full details and concrete examples of this construction are given below.

**Lemma 6.1:** Let \( \mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0,m}\}_{m \in M})) \) and \( \mathcal{R}' = ((M', G', \triangleright'), (m_0', \{g_{m_0',m'}\}_{m' \in M'})) \) be two cell spaces, and let \( \mathcal{R}'' \) be the cell space \( ((M \times M', G \times G', \triangleright \times \triangleright'), ((m_0, m_0'), \{g_{m_0,m_0', m'}\})_{(m, m') \in M \times M'}) \), where

\[
\triangleright \times \triangleright' : (G \times G') \times (M \times M') \rightarrow M \times M',
\]

\[
((g, g'), (m, m')) \mapsto (g \triangleright m, g' \triangleright' m').
\]

Furthermore, let \( A \) be a subset of \( M \), let \( A' \) be a subset of \( M' \), let \( A'' \) be a subset of \( M \times M' \) such that \( (m \in M | \exists m' \in M' : (m, m') \in A'') = A \), let \( E \) be a subset of \( G/G_0 \), and let \( E' \) be a subset of \( G'/G_0' \). Then,

1. \( A'' = A \times A' \), where

\[
A \times A' : (M \times M') \times ((G \times G')/(G_0 \times G'_0)) \rightarrow M \times M',
\]

\[
((m, m'), (g, g')(G_0 \times G'_0)) \mapsto (m \triangleleft gG_0, m' \triangleleft' g'G'_0).
\]
(2) \((A \times A')^{-E \times E'} = A^{-E} \times (A')^{-E'}, (A \times A')^{+E \times E'} = A^{+E} \times (A')^{+E'}, \) and \(\partial_{E \times E'}(A \times A') = (\partial_{E} A \times (A')^{+E}) \cup (A^{+E} \times \partial_{E} A'),\) where
\[ E \times E' = \{(g, g')(G_0 \times G'_0) \mid gG_0 \in E, g'G'_0 \in E'\}. \]

(3) \((A'')^{-E \times (G'/G'_0)} \subseteq A^{-E} \times M', (A'')^{+E \times (G'/G'_0)} = A^{+E} \times M', \) and \(\partial_{E \times (G'/G'_0)}A'' \supseteq \partial_{E} A \times M',\) where
\[ E \times (G'/G'_0) = \{(g, g')(G_0 \times G'_0) \mid gG_0 \in E, g'G'_0 \in G'/G'_0\}. \]

**Proof:** Note that the stabiliser \(G_0^0\) of \((m_0, m'_0)\) under \(\triangleright \times \triangleright'\) is \(G_0 \times G'_0\).

1. For each \((m, m') \in M \times M'\) and each \((g, g')(G_0 \times G'_0) \in (G \times G')/(G_0 \times G'_0),\)
\[(m, m') \trianglelefteq'' (g, g')(G_0 \times G'_0) = (g_{m_0, m'_0}g'_{m_0, m'_0}) \cdot (g, g') \triangleright \times \triangleright' (m_0, m'_0) = (g_{m_0, m'_0}g \triangleright m_0, g'_{m_0, m'_0}g' \triangleright' m'_0) = (m \trianglelefteq gG_0, m' \trianglelefteq' g'G'_0). \]
Therefore, \(\trianglelefteq'' = \trianglelefteq \times \trianglelefteq'\).

2. Let \((m, m') \in M \times M'.\) Then, \((m, m') \trianglelefteq'' E \times E' = (m \trianglelefteq E) \times (m' \trianglelefteq' E').\) Hence, if \(E \neq 0\) and \(E' \neq 0\), then \((m, m') \trianglelefteq'' E \times E' \subseteq A \times A'\) if and only if \(m \trianglelefteq E \subseteq A\) and \(m' \trianglelefteq' E' \subseteq A'.\) And, \((m, m') \trianglelefteq'' E \times E' \cap (A \times A') \neq 0\) if and only if \(m \trianglelefteq E \cap A \neq 0\) and \(m' \trianglelefteq' E' \cap A' \neq 0\). Therefore, \((A \times A')^{-E \times E'} = A^{-E} \times (A')^{-E'}\) and \((A \times A')^{+E \times E'} = A^{+E} \times (A')^{+E'}\). Note that the first equality holds in the case that \(E \neq 0\) or \(E' \neq 0\), because in that case \((A \times A')^{-E \times E'} = 0\), and \(A^{-E} = 0\) or \((A')^{-E'} = 0\). Moreover,
\[
\partial_{E \times E'}(A \times A') = (A^{+E} \times (A')^{+E'}) \setminus (A^{-E} \times (A')^{-E'})
= ((A^{+E} \setminus A^{-E}) \times (A')^{+E'}) \cup (A^{+E} \times ((A')^{+E'} \setminus (A')^{-E'}))
= (\partial_{E} A \times (A')^{+E'}) \cup (A^{+E} \setminus \partial_{E} A').
\]

3. Let \((m, m') \in M \times M'.\) Then, \((m, m') \trianglelefteq'' E \times (G'/G'_0) = (m \trianglelefteq E) \times M'.\) Hence, if \((m, m') \trianglelefteq'' E \times (G'/G'_0) \subseteq A'',\) then \(m \trianglelefteq E \subseteq A\). And, \((m, m') \trianglelefteq'' E \times (G'/G'_0) \cap A'' \neq 0\) if and only if \(m \trianglelefteq E \cap A \neq 0\). Therefore, \((A'')^{-E \times (G'/G'_0)} \subseteq A^{-E} \times M'\) and \((A'')^{+E \times (G'/G'_0)} = A^{+E} \times M'.\) Moreover,
\[
\partial_{E \times (G'/G'_0)}A'' \supseteq (A^{+E} \times M') \setminus (A^{-E} \times M')
= ((A^{+E} \setminus A^{-E}) \times M') \cup (A^{+E} \times (M' \setminus M'))
= \partial_{E} A \times M'. \quad \square
\]

**Lemma 6.2:** Let \(M = (M, G, \triangleright)\) and \(M' = (M', G', \triangleright')\) be two left homogeneous spaces with finite stabilisers such that \(M'\) is finite, and let \(M''\) be the left homogeneous space \((M \times M', G \times G', \triangleright \times \triangleright')\). The space \(M''\) is right amenable if and only if the space \(M\) is right amenable.

**Proof:** Let \(K = (m_0, \{g_{m_0, m}\}_{m \in M})\) and \(K' = (m'_0, \{g'_{m'_0, m'}\}_{m' \in M'})\) be two coordinate systems for \(M\) and \(M'\) respectively, and let \(K''\) be the coordinate system \((m_0, m'_0), \{(g_{m_0, m}, g'_{m'_0, m'})\}_{(m_0, m'_0) \in M \times M'}\) for \(M''\). Note that, because \(M\) and \(M'\) have finite stabilisers, so has \(M''\).

First, let \(M''\) be right amenable. Then, because \(M''\) has finite stabilisers, there is a right Følner net \(\{F''_i\}_{i \in I}\) in \(R'' = (M'', K'')\). Put \(F_i = \{m \in M' \mid 3m' \in M' : (m, m') \in F''_i\}\). Let \(E\) be a finite subset of \(G/G_0\). Then, according to Item (3) of Lemma 6.1 and because \(F''_i \subseteq F_i \times M',\)
\[
\frac{\|\partial_{E} F_i\|}{|F_i|} \leq \frac{|M'|}{|M|} \cdot \frac{\|\partial_{E \times (G/G_0)} F''_i\|}{|F''_i|} = \frac{\|\partial_{E \times (G/G_0)} F''_i\|}{|F''_i|}.
\]
Hence, because \( \{ F''_i \}_{i \in I} \) is a right Følner net in \( \mathcal{R}'' \), the net \( \{ F_i \}_{i \in I} \) is a right Følner net in \( (\mathcal{M}, K) \). In conclusion, \( \mathcal{M} \) is amenable.

Secondly, let \( \mathcal{M} \) be amenable. Then, because \( \mathcal{M} \) has finite stabilisers, there is a right Følner net \( \{ F_i \}_{i \in I} \) in \( \mathcal{R} = (\mathcal{M}, K) \). Let \( E'' \) be a finite subset of \( (G \times G')/(G \times G')_0 \). Put \( E = \{ gG_0 \mid \exists g' \in G : (g, g')(G \times G')_0 \in E'' \} \) and put \( E' = G'/G'_0 \). Then, according to Item (2) of Lemma 6.1, we have \( \partial_E(F_i \times M') \subseteq \partial_E(F_i \times M') \subseteq \{ (\partial_E F_i) \times M' \} \cup (F_i^E \times (\partial_E M')) = (\partial_E F_i) \times M' \). Hence,

\[
\frac{|\partial_E(F_i \times M')|}{|F_i \times M'|} \leq \frac{|\partial_E F_i| \cdot |M'|}{|F_i| \cdot |M'|} = \frac{|\partial_E F_i|}{|F_i|}.
\]

Therefore, because \( \{ F_i \}_{i \in I} \) is a right Følner net in \( \mathcal{R} \), the net \( \{ F_i \times M' \}_{i \in I} \) is a right Følner net in \( \mathcal{M}'' \). In conclusion, \( \mathcal{M}'' \) is right amenable.

\( \square \)

**Definition 6.3:** Let \( G \) be a directed graph. It is called vertex-transitive if and only if its automorphism group acts transitively on its vertices by function application.

**Remark 9:** Cayley graphs of groups are vertex-transitive.

**Theorem 6.4 (Sabidussi’s Theorem):**

Let \( G \) be a directed graph. It is a Cayley graph if and only if a subgroup of its automorphism group acts freely and transitively on its vertices by function application.

**Proof:** See Proposition 3.1(b) in [2].

**Definition 6.5:** Let \( G = (V, E) \) and \( G' = (V', E') \) be two directed graphs. The graph \( G \boxtimes G' = (V \times V', \{(v_1, v'_1), (v_2, v'_2) \in (V \times V') \times (V \times V') \mid v_1 = v_2 \land (v'_1, v'_2) \in E \lor v'_1 = v'_2 \land (v_1, v_2) \in E \}) \) is called Cartesian product of \( G \) and \( G' \).

**Definition 6.6:** Let \( G \boxtimes G' \) be the Cartesian product of \( G = (V, E) \) and \( G' = (V', E') \), and let \( \triangleright \) and \( \triangleright' \) be the left group actions of \( \text{Aut}(G) \) on \( V \) and of \( \text{Aut}(G') \) on \( V' \) by function application. The direct product \( \text{Aut}(G) \times \text{Aut}(G') \) acts on \( V \times V' \) on the left by

\[
\triangleright \boxtimes \triangleright' : (\text{Aut}(G) \times \text{Aut}(G')) \times (V \times V') \to V \times V',
\]

\[
((\varphi, \varphi'), (v, v')) \mapsto (\varphi \triangleright v, \varphi' \triangleright' v').
\]

**Remark 10:** The map \( \_((\triangleright \boxtimes \triangleright')) : \text{Aut}(G) \times \text{Aut}(G') \to \text{Aut}(G \boxtimes G') \) is an injective group homomorphism.

**Remark 11:** If \( G \) and \( G' \) are vertex-transitive, then the left group action \( \triangleright \boxtimes \triangleright' \) is transitive.

**Lemma 6.7:** Let \( G \boxtimes G' \) be the Cartesian product of \( G = (V, E) \) and \( G' = (V', E') \). There is a subgroup \( H'' \) of \( \text{Aut}(G) \times \text{Aut}(G') \) that acts freely and transitively on \( V \times V' \) by \( \triangleright \quad \triangleright' \) if and only if there is a subgroup \( H \) of \( \text{Aut}(G) \) and there is a subgroup \( H' \) of \( \text{Aut}(G') \) such that \( H \) acts freely and transitively on \( V \) by \( \triangleright \) and \( H' \) acts freely and transitively on \( V' \) by \( \triangleright' \).

**Proof:** First, let \( H'' \) be a subgroup of \( \text{Aut}(G) \times \text{Aut}(G') \) that acts freely and transitively on \( V \times V' \) by \( \triangleright \quad \triangleright' \). Furthermore, let \( v' \) be a vertex of \( V' \). The set \( F = \{ h'' \in H'' \mid h'' \triangleright \triangleright' V \times \{ v' \} \subseteq V \times \{ v' \} \} \) is a subgroup of \( H'' \) and the left group action \((\triangleright \quad \triangleright')_{\mid F \times (V \times \{ v' \}) \to V \times \{ v' \}} \) is free and transitive. The set \( H = \pi_1(F) \) is a subgroup of \( \text{Aut}(G) \), which acts freely and transitively on \( V \) by \( \triangleright \), where \( \pi_1 \) is the projection homomorphism from \( \text{Aut}(G) \times \text{Aut}(G') \) onto \( \text{Aut}(G) \). Analogously, a subgroup \( H' \) of \( \text{Aut}(G') \) can be constructed that acts freely and transitively on \( V' \) by \( \triangleright' \).

Secondly, let \( H \) be a subgroup of \( \text{Aut}(G) \) and let \( H' \) be a subgroup of \( \text{Aut}(G') \) such that \( H \) acts freely and transitively on \( V \) by \( \triangleright \) and \( H' \) acts freely and transitively on \( V' \) by \( \triangleright' \). The direct product \( H'' = H \times H' \) acts freely and transitively on \( V \times V' \) by \( \triangleright \quad \triangleright' \).

**Corollary 6.8:** Let \( G \boxtimes G' \) be the Cartesian product of \( G = (V, E) \) and \( G' = (V', E') \), where \( G \) or \( G' \) is not a Cayley graph. There is no subgroup of \( \text{Aut}(G) \times \text{Aut}(G') \) that acts freely and transitively on \( V \times V' \) by \( \triangleright \quad \triangleright' \).
Proof: This is a direct consequence of Theorem 6.4 and Lemma 6.7.

Example 6.9: Let $G$ be a group, let $S$ be a generating set of $G$, let $G = (V, E)$ be the coloured $S$-Cayley graph of $G$, let $\rhd$ be the left group action of Aut $(G)$ on $V$ by function application, let $G' = (V', E')$ be a vertex-transitive, finite, and directed non-Cayley graph, and let $\triangleright'$ be the left group action of Aut $(G')$ on $V'$ by function application, and let $\mathcal{M}''$ be the left homogeneous space $(V \times V', \text{Aut} (G) \times \text{Aut} (G'), \triangleright \square \triangleright')$.

Note that $V = G$, that $G \simeq \text{Aut} (G)$ by $g \mapsto g \cdot _{\triangleright}$, that under this identification the map $\triangleright$ is the group multiplication $\cdot$ and the map $\triangleright \square \triangleright'$ is $(G \times \text{Aut} (G')) \times (G \times V') \to G \times V', ((g, \psi'), (v, v')) \mapsto (g \cdot v, \psi' (v'))$, and that $(V, \text{Aut} (G), \triangleright)$ is right amenable if and only if $G$ is amenable.

According to Corollary 6.8, there is no subgroup of Aut $(G) \times \text{Aut} (G')$ that acts freely and transitively on $V \times V'$ by $\triangleright \square \triangleright'$. And, according to Lemma 6.2 and the note above, the space $\mathcal{M}''$ is right amenable if and only if the group $G$ is amenable. For example:

(1) If $G$ is the integer lattice $\mathbb{Z}$ and $G'$ is the Petersen graph, then $\mathcal{M}''$ is right amenable.

(2) If $G$ is the free group $F_2$ over $\{a, b\}$, where $a \neq b$ and $G'$ is the Coxeter graph, then $\mathcal{M}''$ is not right amenable.

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ORCID

Simon Wacker http://orcid.org/0000-0002-3409-4418

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