On Camassa-Holm equation with self-consistent sources and its solutions

Yehui Huang, Yuqin Yao and Yunbo Zeng
Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, P.R. China

Abstract

Regarded as the integrable generalization of Camassa-Holm (CH) equation, the CH equation with self-consistent sources (CHESCS) is derived. The Lax representation of the CHESCS is presented. The conservation laws for CHESCS are constructed. The peakon solution, N-soliton, N-cuspon, N-positon and N-negaton solutions of CHESCS are obtained by using Darboux transformation and the method of variation of constants.

KEYWORDS: Camassa-Holm equation with self-consistent sources; Lax representation; conservation laws; peakon; soliton; positon; negaton.

1 Introduction

Camassa-Holm (CH) equation, which was implicitly contained in the class of multi-Hamiltonian system introduced by Fuchssteiner and Fokas and explicitly derived as a shallow water wave equation by Camassa and Holm, has the form

\[ u_t + 2\omega u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \]  

(1.1)

where \( u = u(x, t) \) is the fluid velocity in the \( x \) direction and the constant \( 2\omega \) is related to the critical shallow water wave speed. Let \( q = u - u_{xx} + w \), we have the following equivalent equation \( q_t + 2u_x q + uq_x = 0 \).

\[ q_t + 2u_x q + uq_x = 0. \]  

(1.2)

It was shown by Camassa and Holm that this equation shares most of the properties of the integrable system of KdV type. It possesses Lax pair formalism and the bi-hamiltonian structure. When \( w > 0 \), the CH equation has smooth solitary wave solutions. When \( w \to 0 \), these solutions become piecewise smooth and have cusps at their peaks. These kind of solutions are weak solutions of (1.2) with \( \omega = 0 \) and are called "peakons". Since the works of Camassa and Holm, this equation has...
become a well-known example of integrable systems and has been studied from many kinds of views 4–12.

Soliton equations with self-consistent sources (SESCS) have attracted much attention in recent years. They are important integrable models in many fields of physics, such as hydrodynamics, state physics, plasma physics, etc 13–25. For example, the KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves 13. The nonlinear Schrödinger equation with self-consistent sources represents the nonlinear interaction of an electrostatic high-frequency wave with the ion acoustic wave in a two component homogeneous plasma 18. The KP equation with self-consistent sources describes the interaction of a long wave with a short wave packet propagating on the x-y plane at some angle to each other 15. The SESCS were firstly studied by Melnikov 13–15. A systematic way to construct the soliton equations with self-consistent sources and their zero-curvature representations is proposed 21–24. The problem of finding soliton solutions or other specific solutions for SESCS has been considered in the past by many authors 13–25.

The present paper falls in that line of the work on the CH equation concerning with establishing the many facts of its completely integrable character, aiming at the integrable generalization of CH equation by deriving the Camassa-Holm equation with self-consistent sources (CHESCS) and finding its solutions. We first construct the CHESCS by using the approach presented in the reference 21–24. The Lax pair of the CHESCS is obtained, which means that the CHESCS is Lax integrable and can be viewed as integrable generalization of CH equation. Since the CH equation describes shallow water wave and the SESCSs in general describe the interaction of different solitary waves, it is reasonable to speculate on the potential application of CHESCS, that is, CHESCS may describe the interaction of different solitary waves in shallow water. It was pointed out 26,27 that SESCS can be regarded as soliton equations with non-homogeneous terms, and accordingly proposed to look for explicit solutions by using the method of variation of constants. Applying this technique to CHESCS we have been able to find its peakon solution. In order to find other solutions of CHESCS, we consider the reciprocal transformation 28,29, which relates CH equation to an alternative of the associated Camassa-Holm (ACH) equation, and propose the reciprocal transformation, which relates the CHESCS to associated CHESCS (ACHESCS). By using the Darboux transformation (DT), one can find the n-soliton and n-cuspon solution 8,9 as well as positon and negaton solution of alternative ACH equation. Then by means of the method of variation of constants, we can obtain the N-soliton, N-cuspon, N-positon and N-negaton solution for ACHESCS. Finally, using the inverse reciprocal transformation, we obtain the N-soliton, N-cuspon, N-positon and N-negaton solution of CHESCS.

This paper is organized as follows. In section 2, we present how to derive the CHESCS and its Lax representation. In section 3, the conservation laws of the CHESCS are constructed. In section 4, the peakon solution is obtained. In section 5, we consider the reciprocal transformation for CH equation and CHESCS, respectively. In section 6, by using the DT, we find the solution for alternative ACH equation, then by using the method of variation of constants and inverse reciprocal transformation, we obtain the N-soliton, N-cuspon, N-positon and N-negaton solution for CHESCS. In section 7, the conclusion is presented.
2 The CHESCS and its Lax pair

2.1 The CHESCS

The Lax pair for CH equation (1.2) is given by

\[ \varphi_{xx} = \left( \lambda q + \frac{1}{4} \right) \varphi, \]  
\[ \varphi_t = \left( \frac{1}{2 \lambda} - u \right) \varphi_x + \frac{1}{2} u_x \varphi. \]  

(2.1a, 2.1b)

It is not difficult to find that

\[ \frac{\delta \lambda}{\delta q} = -\lambda \varphi^2. \]

(2.2)

The CH equation possesses bi-hamiltonian structure

\[ q_t = -J \frac{\delta H_0}{\delta q} = -K \frac{\delta H_1}{\delta q}, \]

(2.3)

where

\[ K = -\partial^3 + \partial, \]
\[ J = \partial q + q \partial, \]
\[ H_0 = \frac{1}{2} \int u^2 + u^2_x dx, \]
\[ H_1 = \frac{1}{2} \int u^3 + u^2_x dx. \]

According to the approach proposed in the reference 21–24, the CHESCS is defined as follows

\[ q_t = -J \left( \frac{\delta H_0}{\delta q} - 2 \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta q} \right) \]
\[ = -(\partial q + \partial)(u + 2 \sum_{j=1}^{N} \lambda_j \varphi_j^2) \]
\[ = -2qu_x - uq_x + \sum_{j=1}^{N} (-8\lambda_j q \varphi_j \varphi_{jx} - 2\lambda_j q_x \varphi_j^2), \]

(2.4a)

\[ \varphi_{j,xx} = \left( \lambda_j q + \frac{1}{4} \right) \varphi_j, \quad j = 1, \ldots, N, \]

(2.4b)

which has an equivalent form by using (2.4b)

\[ q_t = -2qu_x - uq_x + \sum_{j=1}^{N} [(\varphi_j^2)_x - (\varphi_j^2)_{xxx}], \]

(2.5a)

\[ \varphi_{j,xx} = \left( \lambda_j q + \frac{1}{4} \right) \varphi_j, \quad j = 1, \ldots, N, \]

(2.5b)
2.2 The Lax representation of the CHESCS

Based on the Lax pair of the CH equation (2.1), we may assume the Lax representation of the CHESCS (2.4) or (2.5) has the form

\[ \varphi_{xx} = (\lambda q + \frac{1}{4}) \varphi, \tag{2.6a} \]

\[ \varphi_t = -\frac{1}{2} B_x \varphi + B \varphi_x, \tag{2.6b} \]

\[ B = \frac{1}{2\lambda} - u + \sum_{j=1}^{N} \frac{\alpha_j f(\varphi_j)}{\lambda - \lambda_j} + \sum_{j=1}^{N} \beta_j f(\varphi_j), \tag{2.6c} \]

where \( f(\varphi_j) \) is undetermined function of \( \varphi_j \). The compatibility condition of (2.6a) and (2.6b) gives

\[ \lambda q_t = LB + \lambda(2B_x q + Bq_x), \tag{2.7} \]

where \( L = -\frac{1}{2} \partial^3 + \frac{1}{2} \partial \). Then (2.6) and (2.7) yields

\[ \lambda q_t = -\frac{1}{2} \sum_{j=1}^{N} \frac{\alpha_j}{\lambda - \lambda_j} [f''' \varphi_{3x}^j + 3(f'' \varphi_j - f')(\lambda_j q + \frac{1}{4})\varphi_{jx} + \lambda_j q_x (f' \varphi_j - 2f)] \]

\[ + [-2qu_x - uq_x + \sum_{j=1}^{N} \beta_j (2q \varphi_j f' + q_x f)]\lambda - \frac{1}{2} \sum_{j=1}^{N} \beta_j [f''' \varphi_{3x}^j + (3f'' \varphi_j + f')] \]

\[ \times (\lambda_j q + \frac{1}{4})\varphi_{jx} + \lambda_j f' q_x \varphi_j - f' \varphi_j) + \sum_{j=1}^{N} \alpha_j (q_x f + 2q f' \varphi_{jx}). \tag{2.8} \]

Here \( f' \) denotes the partial derivative of the function \( f \) with respect to the variable \( \varphi_j \). In order to determine \( f, \alpha_j \) and \( \beta_j \), we compare the coefficients of \( \frac{1}{\lambda - \lambda_j} \), \( \lambda \) and \( \lambda^0 \), respectively. We first observe the coefficients of \( \frac{1}{\lambda - \lambda_j} \), then the coefficients of \( \varphi_{jx}^3, \varphi_{jx} \) and other terms gives rise to, respectively

\[ f''' = 0, \quad f'' \varphi_j - f' = 0, \quad f' \varphi_j - 2f = 0, \]

which leads to \( f = b\varphi_j^2 \). Substituting \( f = b\varphi_j^2 \) into the coefficients of \( \lambda \) in (2.8) gives

\[ q_t = -2qu_x - uq_x + 4q \sum_{j=1}^{N} \beta_j b \varphi_j \varphi_{jx} + q_x \sum_{j=1}^{N} \beta_j b \varphi_j^2. \]

Comparing the above equation and (2.4a), we can determine

\[ b = -2, \quad \beta_j = \lambda_j. \]

Substituting \( f = -2\varphi_j^2 \), and \( \beta_j = \lambda_j \) into the coefficients of \( \lambda^0 \) in (2.8), we obtain

\[ \alpha_j = \lambda_j^2. \]
Thus we obtain the Lax pair of the CHESCS (2.5)

\[ \varphi_{xx} = (\frac{1}{4} + \lambda q) \varphi, \]  

(2.9a)

\[ \varphi_t = \frac{u_x}{2} \varphi + (\frac{1}{2\lambda} - u) \varphi_x + 2 \sum_{j=1}^{N} \frac{\lambda \lambda_j \varphi_j}{\lambda - \lambda_j} (\varphi_{jx} \varphi - \varphi_j \varphi_x). \]  

(2.9b)

which means that the CHESCS (2.5) is Lax integrable.

3 The infinite conservation laws of the CHESCS

With the help of the Lax representation of the CHESCS, we could find the conservation laws for the CHESCS by a well-known method. First we assume that \( q, u, \varphi_j \) and its derivatives tend to 0 when \( |x| \to \infty \). Set

\[ \Gamma = \frac{\varphi_x}{\varphi}, \]  

(3.1)

then the identity

\[ \frac{\partial}{\partial t} \left( \frac{\partial \ln \varphi}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial \ln \varphi}{\partial t} \right) \]

together with (2.10) implies that CHESCS has the following conservation law:

\[ \frac{\partial}{\partial t} (\Gamma) = \frac{\partial}{\partial x} (\frac{\varphi_t}{\varphi}) = \frac{\partial}{\partial x} \left( \frac{1}{2} u_x + 2 \sum_{j=1}^{N} \frac{\lambda \lambda_j}{\lambda - \lambda_j} \varphi_j \varphi_{jx} + ((\frac{1}{2\lambda} - u) - 2 \sum_{j=1}^{N} \frac{\lambda \lambda_j}{\lambda - \lambda_j} \varphi_j^2) \Gamma \right) \]  

(3.2)

. Using (2.10a) gives rise to

\[ \Gamma_x = \frac{1}{4} + q \lambda - \Gamma^2. \]  

(3.3)

Let

\[ \Gamma = \sum_{m=0}^{\infty} \mu_m \lambda^{\frac{1-m}{2}}, \]  

(3.4)

then \( \mu_m \) is the density of conservation laws.

Define

\[ \frac{1}{2} u_x + 2 \sum_{j=1}^{N} \frac{\lambda \lambda_j}{\lambda - \lambda_j} \varphi_j \varphi_{jx} + ((\frac{1}{2\lambda} - u) - 2 \sum_{j=1}^{N} \frac{\lambda \lambda_j}{\lambda - \lambda_j} \varphi_j^2) \Gamma = \sum_{m=0}^{\infty} F_m \lambda^{\frac{1-m}{2}} \]  

(3.5)

It is found that the density of the conservation laws \( \mu_m \) and the flux of the conservation laws \( F_m \) satisfy the following recursion relation:

\[ \mu_0 = \sqrt{q}, \]

\[ \mu_1 = -\frac{1}{4} q_x, \]

\[ \mu_2 = \frac{1}{32} \left( \frac{4}{\sqrt{m}} + \frac{m^2}{m^{5/2}} - \left( \frac{4m_x}{m^{3/2}} \right)_x \right), \]

\[ \mu_m = -\mu_{m-1,x} - \sum_{i=1}^{m-1} \mu_i \mu_{m-1-i, \frac{2m}{\mu_0}} \mu_i, \quad m \geq 3, \]  

(3.6)
\[ F_0 = (-u - 2 \sum_{j=1}^{N} \lambda_j \phi_j^2) \sqrt{q}, \]
\[ F_1 = (u + 2 \sum_{j=1}^{N} \lambda_j \phi_j^2) \frac{q_x}{4q} + \frac{1}{2} u_x + 2 \sum_{j=1}^{N} \lambda_j \phi_j \phi_{jx}, \]
\[ F_{2m} = \sum_{i=0}^{m} (-u^{(i)} - 2 \sum_{j=1}^{N} \lambda_j^{i+1} \phi_j^2) \mu_{2m-2i}, \quad m \geq 1, \]
\[ F_{2m+1} = \sum_{i=0}^{m} (u^{(i)} + 2 \sum_{j=1}^{N} \lambda_j^{i+1} \phi_j^2) \mu_{2m-2i+1} + 2 \sum_{j=1}^{N} \lambda_j^{m+1} \phi_j \phi_{jx}, \quad m \geq 1, \quad (3.7) \]

where \( u^{(0)} = u, \quad u^{(1)} = 1, \quad u^{(i)} = 0, \quad i > 1. \)

After some calculations we can find the first few conserved quantities given by \( \mu_0, \mu_2 \) and \( \mu_4 \) are as follows
\[ H_{-1} = \int \sqrt{q} dx, \quad (3.8a) \]
\[ H_{-2} = -\frac{1}{16} \int \left( \frac{4}{\sqrt{q}} + \frac{q_x^2}{q^{5/2}} \right) dx, \quad (3.8b) \]
\[ H_{-3} = - \int \left( \frac{1}{32q^{3/2}} + \frac{5q_x^2}{64q^{7/2}} + \frac{q_{xx}^2}{32q^{7/2}} - \frac{35q_x^4}{512q^{11/2}} \right) dx. \quad (3.8c) \]

The corresponding flux of the conservation laws are
\[ G_{-1} = (-u - 2 \sum_{j=1}^{N} \lambda_j \phi_j^2) \sqrt{q}, \quad (3.9a) \]
\[ G_{-2} = (1 + 2 \sum_{j=1}^{N} \lambda_j \phi_j^2) \sqrt{q} + (u + 2 \sum_{j=1}^{N} \lambda_j \phi_j^2) \left( \frac{4}{\sqrt{q}} + \frac{q_x^2}{q^{5/2}} \right) - \left( \frac{q_x}{4q^{3/2}} \right)_x, \quad (3.9b) \]
\[ G_{-3} = (-u - 2 \sum_{j=1}^{N} \lambda_j \phi_j^2) \left( \frac{1}{32q^{3/2}} + \frac{5q_x^2}{64q^{7/2}} + \frac{q_{xx}^2}{32q^{7/2}} - \frac{35q_x^4}{512q^{11/2}} \right) + \frac{1}{16} \left( 1 + 2 \sum_{j=1}^{N} \lambda_j \phi_j^2 \right) \left( \frac{4}{\sqrt{q}} + \frac{q_x^2}{q^{5/2}} \right) + 2 \sum_{j=1}^{N} \lambda_j^3 \phi_j^3 \sqrt{q}. \quad (3.9c) \]

As the space part of the Lax Pair of the CHESCS is the same as that of CH equation, the densities of the conservation laws of the CHESCS are the same as those of the Camassa-Holm equation \(^1\). As the time part of the Lax pair is different, the fluxes of the conservation laws for CH equation and CHESCS are different.

## 4 One peakon solution of the CHESCS

The CH equation (1.2) has peakon solutions
\[ u = ce^{-(|x-ct+\alpha|)}, \quad (4.1) \]
where \( \alpha \) is an arbitrary constant. The corresponding eigenfunction of (2.1) is

\[
\varphi = \beta e^{-\frac{1}{2} |x-ct+\alpha|},
\]

where \( \beta \) is an arbitrary constant.

Since the CHESCS (2.5) can be considered as the CH equation (1.2) with non-homogeneous terms, we may use the method of variation of constants to find the peakon solution of CHESCS from the peakon solution (4.1) and (4.2). Taking \( \alpha \) and \( \beta \) in (4.1) and (4.2) to be time-dependent \( \alpha(t) \) and \( \beta(t) \) and requiring that

\[
u = ce^{-|x-ct+\alpha(t)|},
\]

\[
\varphi = \beta(t)e^{-\frac{1}{2} |x-ct+\alpha(t)|}
\]

satisfy the CHESCS (2.5) for \( N = 1 \). We find that \( c = \frac{1}{\lambda}, \alpha(t) \) can be an arbitrary function of \( t \) and \( \beta(t) = \sqrt{\alpha'(t)c} \). So we have the one peakon solution for (2.4) with \( N = 1 \), \( \lambda_1 = \lambda = \frac{1}{c} \)

\[
u = ce^{-|x-ct+\alpha(t)|}
\]

\[
\varphi = \sqrt{\alpha'(t)c}e^{-\frac{1}{2} |x-ct+\alpha(t)|}
\]

The one peakon of the CHESCS also has a cusp at its peak, located at \( x = ct - \alpha(t) \). We note that for the one peakon solution of the CH equation, the solution travels with speed \( c \) and has a cusp at its peak of height \( c \), for the CHESCS, the cusp is still at its peak of height \( c \), but the speed \( c - \frac{\alpha(t)}{t} \) of the wave is no longer a constant.

**5 A reciprocal transformation for the CHESCS**

Let \( r = \sqrt{q} \), by the reciprocal transformation \( dy = rdx - urds, \ ds = dt \),

and denoting \( f = r^{-\frac{1}{4}} \phi \), the Lax pair (2.1) of CH equation is transformed to the following system

\[
\phi_{yy} = (\lambda + Q + \frac{1}{4\omega})\phi,
\]

\[
\phi_s = \frac{1}{2\lambda}(r\phi_y - \frac{1}{2}r^2\phi),
\]

where

\[
Q = -\frac{1}{4}\left( \frac{r_y}{r} \right)^2 + \frac{r_{yy}}{2r} + \frac{1}{4r^2} - \frac{1}{4\omega}.
\]

The compatibility condition of (5.1a) and (5.1b) gives an alternative of the associated CH (ACH) equation

\[
Q_s = r_y,
\]

\[
-\frac{1}{4\omega}r_y + \frac{1}{4}r_{yyy} - \frac{1}{2}Qyr - Qr_y = 0.
\]
We now consider the reciprocal transformation for the CHESCS (2.5). (2.5a) gives
\[ r_t = -(ru)_x - 2 \sum_{j=1}^{N} \lambda_j (r \varphi_j^2)_x. \] (5.4)

(5.4) shows that the 1-form
\[ \omega = r dx - (ru + 2 \sum_{j=1}^{N} \lambda_j r \varphi_j^2) dt \] (5.5)
is closed, so we can define a reciprocal transformation \((x, t) \rightarrow (y, s)\) by the relation
\[ dy = r dx - (ru + 2 \sum_{j=1}^{N} \lambda_j r \varphi_j^2) ds, \quad ds = dt, \] (5.6)
and we have
\[ \frac{\partial}{\partial x} = \frac{r}{r} \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial s} - (ru + 2 \sum_{j=1}^{N} \lambda_j r \varphi_j^2) \frac{\partial}{\partial y}. \] (5.7)

Denoting \( \varphi = r^{-\frac{1}{2}} \psi, \varphi_j = r^{-\frac{1}{2}} \psi_j \) and using (5.2), the Lax pair (2.9) of CHESCS (2.5) is correspondingly rewritten as
\[ \psi_{yy} = (\lambda + Q + \frac{1}{4\omega}) \psi, \] (5.8a)
\[ \psi_s = \frac{1}{2\lambda} (r \psi_y - \frac{1}{2} r_y \psi) + 2 \sum_{j=1}^{N} \frac{\lambda_j^2 \psi^2}{\lambda - \lambda_j} (\psi_{jy} \psi - \psi_j \psi_y). \] (5.8b)

The compatibility condition of (5.8a) and (5.8b) leads to an associated CHESCS (ACHESCS)
\[ Q_s = r_y - 8 \sum_{j=1}^{N} \lambda_j^2 \psi_j \psi_{jy}, \] (5.9a)
\[ -\frac{1}{4\omega} r_y + \frac{1}{4} r_{yy} - \frac{1}{2} Q_y r - Q r_y = 0, \] (5.9b)
\[ \psi_{jyy} = (\lambda_j + Q + \frac{1}{4\omega}) \psi_j, \quad j = 1, 2, \ldots, N. \] (5.9c)

The Eqs.(5.9) can be regarded as the Eqs.(5.3) with self-consistent sources. In order to obtain the solutions of the CHESCS (2.5), we have to get the relation of the variables \((y, s)\) and the variables \((x, t)\). From the reciprocal transformation, we have
\[ \frac{\partial x}{\partial y} = \frac{1}{r}, \quad \frac{\partial x}{\partial s} = u + 2 \sum_{j=1}^{N} \lambda_j \varphi_j^2. \] (5.10)

By making use of the compatibility of the above two equations, we have
\[ x(y, s) = \int \frac{1}{r} dy. \] (5.11)
The solutions of the CHESCS (2.5) with respect to the variables \((y,s)\) are given by

\[
q = r^2(y, s), \quad \varphi_j(y, s) = \frac{\psi_j}{\sqrt{r}},
\]

\[
u(y, s) = r^2 - rys + \frac{r_y r_s}{r} - 2rr_y \sum_{j=1}^{N} \lambda_j (\varphi_j^2)_y - 2r^2 \sum_{j=1}^{N} \lambda_j (\varphi_j^2)_{yy} - \omega,
\]

\[
x(y, s) = \int \frac{1}{r} \, dy.
\]

We now prove (5.12b). From \(q = u - u_{xx} + \omega\) and the reciprocal transformation (5.7), we have

\[
u = q + r r_y u_y + r^2 u_{yy} - \omega.
\]

By using the reciprocal transformation (5.7), (5.4) gives rise to

\[
u_y = -\frac{r s}{r^2} - 2 \sum_{j=1}^{N} \lambda_j (\varphi_j^2)_y.
\]

Substituting (5.14) and (5.2) into (5.13) leads to (5.12b).

6 The solutions for the CHESCS

Notice that \(Q = 0, r = \sqrt{\omega}\) is the solution of (5.2) and (5.3). Let the functions \(\phi_0(y, s, \lambda), \Psi_1(y, s, \lambda_1), \cdots, \Psi_n(y, s, \lambda_n)\) be different solutions of (5.1) with \(Q = 0, r = \sqrt{\omega}\) and the corresponding \(\lambda\) and \(\lambda = \lambda_1, \cdots, \lambda_n\), respectively. We construct two Wronskian determinants from these functions

\[
W_1 = W(\Psi_1, \cdots, \Psi_1^{(m_1)}, \Psi_2, \cdots, \Psi_2^{(m_2)}, \cdots, \Psi_n, \cdots, \Psi_n^{(m_n)}),
\]

\[
W_2 = W(\Psi_1, \cdots, \Psi_1^{(m_1)}, \Psi_2, \cdots, \Psi_2^{(m_2)}, \cdots, \Psi_n, \cdots, \Psi_n^{(m_n)}, \phi_0),
\]

where \(m_i \geq 0\) are given numbers and \(\Psi_j^{(i)} = \frac{\partial^i \Psi_j(y,s,\lambda)}{\partial \lambda^i}|_{\lambda=\lambda_j}\).

Based on the generalized Darboux transformation for KdV hierarchy \(30\) and using (5.3a), the following generalized Darboux transformation of (5.1) is valid \(4,30,31\)

\[
Q(y, s) = -2 \frac{\partial^2}{\partial y^2} \log W_1,
\]

\[
r(y, s) = \sqrt{\omega} - 2 \frac{\partial^2}{\partial y \partial s} \log W_1,
\]

\[
\phi(y, s, \lambda) = \frac{W_2}{W_1},
\]

namely \(Q(y, s), r(y, s)\) and \(\phi(y, s, \lambda)\) satisfy (5.1), (5.2) and (5.3).
6.1 The multisoliton solutions

Take $\Psi_i$ and $\Phi_i$ be the solutions of Eq.(5.1) with $Q = 0$, $r = \sqrt{\omega}$ and $\lambda_i = k_i^2 - \frac{1}{4\omega} < 0$, or $4\omega k_i^2 - 1 < 0$, $(0 < k_1 < k_2 < \cdots < k_n)$ as follows

\begin{align*}
\Psi_i &= \cosh \xi_i, \quad i \text{ is an odd number,} \quad (6.3a) \\
\Psi_i &= \sinh \xi_i, \quad i \text{ is an even number.} \quad (6.3b)
\end{align*}

\begin{equation}
\Phi_i = e^{\xi_i} \quad (6.4)
\end{equation}

where henceforth

\begin{equation}
\xi_i = k_i[y + \frac{2\omega^{3/2}s}{4\omega k_i^2 - 1} + \alpha_i]. \quad (6.5)
\end{equation}

By using Darboux transformation (6.2) with $m_1 = \cdots = m_n = 0$, the n-soliton solution $Q(y, s)$ and $r(y, s)$ of (5.3) and the corresponding eigenfunction $\phi_i(y, s, \lambda_i)$ of (5.1) with $\lambda_i = k_i^2 - \frac{1}{4\omega}$ is given by

\begin{align*}
Q(y, s) &= -2[\log W(\Psi_1, \Psi_2, \cdots, \Psi_n)]_{yy}, \quad (6.6a) \\
r(y, s) &= \sqrt{\omega} - 2[\log W(\Psi_1, \Psi_2, \cdots, \Psi_n)]_{ys}, \quad (6.6b) \\
\phi_i(y, s, \lambda_i) &= \frac{W(\Psi_1, \Psi_2, \cdots, \Psi_n, \Phi_i)}{W(\Psi_1, \Psi_2, \cdots, \Psi_n)}. \quad (6.6c)
\end{align*}

When $n = 1$ and $4k_1^2\omega - 1 < 0$, (6.6) gives rise to one soliton solution for (5.3) and the corresponding eigenfunction of (5.1) with $\lambda_1 = k_1^2 - \frac{1}{4\omega}$.

\begin{align*}
Q(y, s) &= -2k_1^2 \text{sech}^2 \xi_1, \quad r(y, s) = \sqrt{\omega} - \frac{4k_1^2 \omega^{3/2} \text{sech}^2 \xi_1}{4k_1^2 \omega - 1}, \quad (6.7a) \\
\phi_1 &= k_1 \text{sech} \xi_1. \quad (6.7b)
\end{align*}

Since Eq.(5.9) can be considered to be Eq.(5.3) with non-homogeneous terms and $\phi_1$ satisfies (5.1a) with $\lambda = \lambda_1$, we may apply the method of variation of constant to find the solutions of the CHECS (5.9) by using the solution (6.7) of ACH equation (5.3) and corresponding eigenfunction.

Taking $\alpha_1$ in (6.5) to be time-dependent functions $\alpha_1(s)$ and requiring that

\begin{align*}
\bar{Q}(y, s) &= -2k_1^2 \text{sech}^2 \bar{\xi}_1, \quad \bar{r}(y, s) = \sqrt{\omega} - \frac{4k_1^2 \omega^{3/2} \text{sech}^2 \bar{\xi}_1}{4k_1^2 \omega - 1}, \quad (6.8a) \\
\bar{\psi}_1 &= \beta_1(s)k_1 \text{sech} \bar{\xi}_1 \quad (6.8b)
\end{align*}

satisfy the system (5.9) for $N = 1$, henceforth, we denote

\begin{equation}
\bar{\xi}_i = k_i[y + \frac{2\omega^{3/2}s}{4\omega k_i^2 - 1} + \alpha_i(s)]. \quad (6.9)
\end{equation}

We find that $\alpha_1(s)$ can be an arbitrary function of $s$ and

\begin{equation}
\beta_1(s) = \frac{2\omega}{1 - 4k_1^2 \omega} \sqrt{2\alpha_1'(s)} \quad (6.10)
\end{equation}
So the one-soliton solution of the CHESCS (2.5) with \( N = 1 \) and \( \lambda_1 = k_1^2 - \frac{1}{4\omega} < 0 \) is obtained with respect to the variables \((y, s)\) from (5.12)

\[
q(y, s) = \omega (1 - \frac{4k_1^2\omega \text{sech}^2 \xi_1}{4k_1^2\omega - 1})^2, \quad (6.11a)
\]

\[
u(y, s) = \frac{8k_1^2\omega^2 \text{sech}^2 \xi_1}{(1 - 4k_1^2\omega)(1 - 4k_1^2\omega + 4k_1^2\omega \text{sech}^2 \xi_1)}, \quad (6.11b)
\]

\[
\varphi_1(y, s) = \frac{2\sqrt{2\alpha_1(s)k_1\omega \text{sech} \xi_1}}{\sqrt{\omega}(1 - 4k_1^2\omega)(1 - 4k_1^2\omega + 4k_1^2\omega \text{sech}^2 \xi_1)}, \quad (6.11c)
\]

\[
x(y, s) = \frac{y}{\sqrt{\omega}} - 2 \ln \frac{1 - 2k_1\sqrt{\omega} \text{tanh} \xi_1}{1 + 2k_1\sqrt{\omega} \text{tanh} \xi_1}. \quad (6.11d)
\]

The requirement \( 4k_1^2\omega - 1 < 0 \) guarantees the nonsingularity of solution (6.11).

In Fig 1, we plot the single soliton solution of \( u \) and \( \varphi_1 \).

![Figure 1. Single soliton solutions for \( u \) and the eigenfunction \( \varphi_1 \) when \( \omega = 0.01, k_1 = 1, \alpha_1(s) = 4s, s = 2 \).](image)

When \( n = 2 \), \( \lambda_1 = k_1^2 - \frac{1}{4\omega} < 0 \), \( \lambda_2 = k_2^2 - \frac{1}{4\omega} < 0 \), we have

\[
\Psi_1 = \cosh \xi_1, \quad \Psi_2 = \sinh \xi_2, \quad (6.12a)
\]

\[
\Phi_1 = e^{\xi_1}, \quad \Phi_2 = e^{\xi_2}, \quad (6.12b)
\]

\[
W_1(\Psi_1, \Psi_2) = k_2 \cosh \xi_2 \cosh \xi_1 - k_1 \sinh \xi_2 \sinh \xi_1, \quad (6.12c)
\]

\[
W_2(\Psi_1, \Psi_2, \Phi_1) = k_2(k_2^2 - k_1^2) \sinh \xi_1, \quad (6.12d)
\]

\[
W_2(\Psi_1, \Psi_2, \Phi_2) = k_1(k_1^2 - k_2^2) \cosh \xi_2, \quad (6.12e)
\]

Then (6.6) with \( n = 2 \) gives rise to two soliton solution for (5.3) and the corresponding eigenfunction of (5.1). In the same way as we did on the one-soliton solution, we can apply the method of variation of constants to get the two soliton solution of the ACHESCS (2.9) which together with (5.12) yields...
to the two soliton solution for CHESCS (2.5) with $N = 2$, $\lambda_1 = k_1^2 - \frac{1}{4\omega}$, $\lambda_2 = k_2^2 - \frac{1}{4\omega}$

$$r(y, s) = \sqrt{\omega} - 2[\log W_1(\Psi_1, \Psi_2)]_{y \xi_i = \tilde{\xi}_i},$$

$$\psi_i = \frac{2\omega\sqrt{(-1)^{(i+1)}2\alpha_i'(s)W_2(\Psi_1, \Psi_2, \Phi_i)}}{(1 - 4k_i^2\omega) \prod_{j \neq i} (k_j^2 - k_i^2)W_1(\Psi_1, \Psi_2)}|_{\xi_i = \tilde{\xi}_i}, i = 1, 2. \quad (6.13a)$$

$$\psi_i = \frac{2\omega\sqrt{(-1)^{(i+1)}2\alpha_i'(s)W_2(\Psi_1, \Psi_2, \Phi_i)}}{(1 - 4k_i^2\omega) \prod_{j \neq i} (k_j^2 - k_i^2)W_1(\Psi_1, \Psi_2)}|_{\xi_i = \tilde{\xi}_i}, i = 1, 2. \quad (6.13b)$$

In Fig 2 we plot the interactions of two soliton solution for $u$ and $\varphi_1, \varphi_2$, which is shown that $u$ is elastic collision.

Figure 2. two soliton solutions for $u$ and the eigenfunction $\varphi_1, \varphi_2$ when $\omega = 0.01$, $k_1 = 2$, $k_2 = 1$, $\alpha_1(s) = 2s$, $\alpha_2(s) = 4s$.

Notice that the soliton solutions of CHESCS contains arbitrary $s$ functions $\alpha_j(s)$. This implies that the insertion of sources into the CH equation may cause the variation of the speed of soliton.
In the same way as in the reference \textsuperscript{27}, we may apply the method of variation of constant to find the N-soliton solution of (2.5) with \( \lambda_i = k_i^2 - \frac{1}{4\omega} > 0 \), \( i = 1, \ldots, N \), from (5.12), where

\[
\begin{align*}
    r(y, s) &= \sqrt{\omega} - 2[logW_1(\Psi_1, \Psi_2, \ldots, \Psi_N)]_{y_s}|_{\xi_i = \bar{\xi}_i}, \quad (6.14a) \\
    \psi_i &= \frac{2\omega\sqrt{(-1)^{i+1}2\alpha'(s)W_2(\Psi_1, \Psi_2, \ldots, \Psi_N, \Phi_i)}}{(1 - 4k_i^2\omega)\sqrt{\prod_{j \neq i}(k_j^2 - k_i^2)W_1(\Psi_1, \Psi_2, \ldots, \Psi_N)}}|_{\xi_i = \bar{\xi}_i} \quad (6.14b)
\end{align*}
\]

### 6.2 The multicuspon solutions

Take \( \Psi_i \) and \( \Phi_i \) be the solutions of Eq.(5.1) when \( Q = 0 \), \( r = \sqrt{\omega} \) and \( \lambda_i = k_i^2 - \frac{1}{4\omega} > 0 \) \((0 < k_1 < k_2 < \cdots < k_n)\), as follows

\[
\begin{align*}
    \Psi_i &= sinh\xi_i, \quad i \text{ is an odd number}, \quad (6.15a) \\
    \Psi_i &= cosh\xi_i, \quad i \text{ is an even number}. \quad (6.15b) \\
    \Phi_i &= e^{\xi_i}, \quad (6.16)
\end{align*}
\]

where \( \xi_i \) is given by (6.5).

The n-cuspon solution \( Q(y, s) \) and \( r(y, s) \) of (5.3) and the corresponding eigenfunction \( \phi_i(y, s, \lambda_i) \) of (5.1) with \( \lambda_i = k_i^2 - \frac{1}{4\omega} \) is given by (6.6).

When \( n = 1 \) and \( 4k_1^2\omega - 1 > 0 \), (6.6) gives rise to one cuspon solution for (5.3) and the corresponding eigenfunction of (5.1) with \( \lambda_1 = k_1^2 - \frac{1}{4\omega} \)

\[
\begin{align*}
    Q(y, s) &= 2k_1^2 csch^2\xi_1, \quad r(y, s) = \sqrt{\omega} + \frac{4k_1^2\omega \frac{3}{2} csch^2\xi_1}{4k_1^2\omega - 1}, \quad (6.17a) \\
    \phi_1 &= -k_1 csch\xi_1. \quad (6.17b)
\end{align*}
\]

Similarly, we may apply the method of variation of constant to find the solutions of the ACHESCS (5.9) by using the solution (6.17) of (5.3) and corresponding eigenfunction. Taking \( \alpha_1 \) in (6.5) to be time-dependent functions \( \alpha_1(s) \) and requiring that

\[
\begin{align*}
    \bar{Q}(y, s) &= 2k_1^2 csch^2\bar{\xi}_1, \quad \bar{r}(y, s) = \sqrt{\omega} + \frac{4k_1^2\omega \frac{3}{2} csch^2\bar{\xi}_1}{4k_1^2\omega - 1}, \quad (6.18a) \\
    \bar{\psi}_1 &= \beta_1(s)k_1 csch\bar{\xi}_1 \quad (6.18b)
\end{align*}
\]

satisfy the system (5.9) for \( N = 1 \), we find that \( \alpha_1(s) \) can be an arbitrary function of \( s \) and

\[
\beta_1(s) = \frac{2\omega}{1 - 4k_1^2\omega} \sqrt{-2\alpha'(s)}. \quad (6.19)
\]

So the one-cuspon solution of the CHESCS (2.5) with \( N = 1 \) and \( \lambda_1 = k_1^2 - \frac{1}{4\omega} > 0 \) is obtained
with respect to the variables \((y, s)\) from (5.12)

\[
q(y, s) = \omega(1 + \frac{4k_1^2 \omega \mathrm{csch}^2 \bar{\xi}_1}{4k_1^2 \omega - 1})^2,
\]

(6.20a)

\[
u(y, s) = \frac{8k_1^2 \omega^2 \mathrm{csch}^2 \bar{\xi}_1}{(1 - 4k_1^2 \omega)(-1 + 4k_1^2 \omega + 4k_1^2 \omega \mathrm{csch}^2 \bar{\xi}_1)},
\]

(6.20b)

\[
\varphi_1(y, s) = \frac{2\sqrt{2\alpha_1'(s)k_1 \omega \mathrm{csch} \bar{\xi}_1}}{\sqrt{\omega}(1 - 4k_1^2 \omega)(-1 + 4k_1^2 \omega + 4k_1^2 \omega \mathrm{csch}^2 \bar{\xi}_1)}
\]

(6.20c)

\[
x(y, s) = \frac{y}{\sqrt{\omega}} + 2 \ln \frac{1 - 2k_1 \sqrt{\omega} \coth \bar{\xi}_1}{1 + 2k_1 \sqrt{\omega} \coth \bar{\xi}_1}
\]

(6.20d)

In Fig 3, we plot the one-cuspon solution of \(u, \varphi_1\).

![Figure 3. Single cuspon solution for \(u\) and the eigenfunction \(\varphi_1\) when \(w = 1, k_1 = 1, \alpha_1(s) = -2s, s = 2\).](image)

Similarly, we can apply the method of variation of constant to find the N-cuspon solution of (2.5) with \(\lambda_i = k_1^2 - \frac{1}{4\omega} < 0, \ i = 1, \cdots, N\) from (5.12), where

\[
r(y, s) = \sqrt{\omega} - 2[\log W_1(\Psi_1, \Psi_2, \cdots, \Psi_N)]_{y_s = \bar{\xi}_i},
\]

(6.21a)

\[
\psi_i = \frac{2\sqrt{(1 - 4k_1^2 \omega)} \prod (k_j^2 - k_i^2) W_1(\Psi_1, \Psi_2, \cdots, \Psi_N, \Phi_i)}{(1 - 4k_1^2 \omega) \prod (k_j^2 - k_i^2) W_1(\Psi_1, \Psi_2, \cdots, \Psi_N)}_{\xi_i = \bar{\xi}_i}
\]

(6.21b)

Further more in the same way, we can find mixed \(k_1\)-soliton-\(k_2\)-cuspon solution for (2.5) with \(N = k_1 + k_2, \lambda_i = k_i^2 - \frac{1}{4\omega} > 0, \ i = 1, \cdots, k_1\) and \(\lambda_i = k_i^2 - \frac{1}{4\omega} < 0, \ i = K_1 + 1, \cdots, k_1 + k_2\), by using (6.6) and (5.12).

### 6.3 The multiposition solutions

Let \(\lambda = -k^2 - \frac{1}{4\omega}, \lambda_i = -k_i^2 - \frac{1}{4\omega}, \ i = 1, \cdots, N\), and take

\[
\Psi_i = \sin \xi_i, \ i \text{ is an odd number},
\]

(6.22a)

\[
\Psi_i = \cos \xi_i, \ i \text{ is an even number}.
\]

(6.22b)
\[ \Phi_i = \cos \xi_i, \quad i \text{ is an odd number}, \]  
(6.23a)

\[ \Phi_i = \sin \xi_i, \quad i \text{ is an even number}. \]  
(6.23b)

where

\[ \xi = k(y - \frac{2\omega^{3/2}s}{4k^2\omega + 1}) + \sum_{i=1}^{N} \prod_{j=1}^{N} \frac{(k - k_j)^2}{k - k_i} \alpha_i \, k - k_i. \]

For \( N = 1 \), we have

\[ \Psi_1 = \sin \xi_1, \quad \Psi_1^{(1)} = \gamma_1 \cos \xi_1, \]  
(6.24a)

\[ \xi_1 = k_1(y - \frac{\sqrt{\omega}s}{2(k_1^2 + \frac{\omega}{4})}). \]  
(6.24b)

\[ \gamma_1 = \frac{\partial \xi}{\partial k} \bigg|_{k=k_1} = \alpha_1 + y + \frac{16k_1^2\omega^{5/2}s}{(1 + 4k_1^2\omega)^2} - \frac{2\omega^{3/2}s}{1 + 4k_1^2\omega}. \]  
(6.24c)

and

\[ W_1(\Psi_1, \Psi_1^{(1)}) = -k_1\gamma_1 + \frac{1}{2} \sin 2\xi_1, \]  
(6.25a)

\[ W_2(\Psi_1, \Psi_1^{(1)}, \Phi_1) = -2k_1^2 \sin \xi_1. \]  
(6.25b)

Then the one-positon solution of (5.3) and the corresponding eigenfunction for (5.1) is given by (6.2) with \( N = 1, m_1 = 1 \),

\[ Q(y, s) = -2[\log W_1]_{yy}, \]  
(6.26a)

\[ r(y, s) = \sqrt{\omega} - 2[\log W_1]_{ys}. \]  
(6.26b)

\[ \psi_1(y, s, \lambda_1) = \beta_1 \frac{W_2}{W_1}, \]  
(6.26c)

where \( \alpha_1 \) and \( \beta_1 \) are arbitrary constants.

\textbf{Figure 4.} one-positon solutions for \( u \) and the eigenfunction \( \varphi_1 \) when \( w = 0.01, \ k_1 = 1, \ \alpha_1(s) = -2s, \ s = 2. \)
By using the method of variation of constants, which means we change \( \alpha_1 \) and \( \beta_1 \) into \( \alpha_1(s) \) and \( \beta_1(s) \), we obtain the one-positon solution for the CHESCS (2.5) with \( N = 1, \lambda_1 = -k_i^2 - \frac{1}{4}\omega \) from (5.12), where

\[
\bar{r}(y, s) = \sqrt{\omega} - 2[\log W_1]_{ys}|_{\gamma_i = \gamma_i}, \quad (6.27a)
\]

\[
\bar{\psi}_1(y, s) = \frac{2\omega\sqrt{-\alpha_1(s)}}{k_i(1 + 4k_i^2\omega)} W_2 \bigg|_{\gamma_i = \gamma_i}, \quad (6.27b)
\]

\[
\gamma_1 = \alpha_1(s) + y + \frac{16k_i^2\omega^{5/2} s}{(1 + 4k_i^2\omega)^2} - \frac{2\omega^{3/2} s}{1 + 4k_i^2\omega}, \quad (6.27c)
\]

where \( \alpha_1(s) \) is an arbitrary function of \( s \).

In Fig 4, we plot the one-positon solution of \( u \) and \( \varphi_1 \).

The positon solution of CHESCS is long-range analogue of soliton and is slowly decreasing, oscillating solution. In the same way we can find N-positon solution for (2.5). For a detailed discussion on positon solution we refer to the reference.

For \( N \), we have

\[
\Psi^{(1)}_i = \gamma_i \cos \xi_i, \quad i \text{ is an odd number,} \quad (6.28a)
\]

\[
\Psi^{(1)}_i = -\gamma_i \sin \xi_i, \quad i \text{ is an even number.} \quad (6.28b)
\]

where

\[
\gamma_i = \frac{\partial \xi}{\partial k} |_{k=k_i} = \prod_{j \neq i} (k_i - k_j)^2 \alpha_i + y + \frac{16k_i^2\omega^{5/2} s}{(1 + 4k_i^2\omega)^2} - \frac{2\omega^{3/2} s}{1 + 4k_i^2\omega}.
\]

We find that

\[
W_1 = W(\Psi_1, \Psi^{(1)}_1, \ldots, \Psi_N, \Psi^{(1)}_N), \quad (6.29a)
\]

\[
\Phi_i = W(\Psi_1, \Psi^{(1)}_1, \ldots, \Psi_N, \Psi^{(1)}_N, \Phi_i), \quad (6.29b)
\]

and the N-positon solution of (5.3) and the corresponding eigenfunction for (5.1) is given by

\[
Q(y, s) = -2[\log W_1]_{yy}, \quad (6.30a)
\]

\[
r(y, s) = \sqrt{\omega} - 2[\log W_1]_{ys}. \quad (6.30b)
\]

\[
\psi_j(y, s, \lambda_i) = \beta_i \frac{W_2}{W_1}, \quad i = 1, \ldots, N, \quad (6.30c)
\]

where \( \alpha_i \) and \( \beta_i \) are arbitrary constants.

By using the method of variation of constants, we obtain the N-positon solution for the CHESCS (2.5) from (5.12), where

\[
\bar{r}(y, s) = \sqrt{\omega} - 2[\log W_1]_{ys}|_{\gamma_i = \gamma_i}, \quad (6.31a)
\]

\[
\bar{\psi}_i(y, s) = \frac{2\omega}{k_i(1 + 4k_i^2\omega) \prod_{j \neq i} (k_j + k_i)} \sqrt{-1} \alpha'_i(s) W_2 \bigg|_{\gamma_i = \gamma_i}, \quad (6.31b)
\]

\[
\gamma_i = \prod_{j \neq i} (k_i - k_j)^2 \alpha_i(s) + y + \frac{16k_i^2\omega^{5/2} s}{(1 + 4k_i^2\omega)^2} - \frac{2\omega^{3/2} s}{1 + 4k_i^2\omega}, \quad (6.31c)
\]

where \( \alpha_i(s) \) are arbitrary functions of \( s \).
6.4 The multinegaton solutions

Let \( \lambda = k^2 - \frac{1}{4\omega} > 0, \lambda_i = k_i^2 - \frac{1}{4\omega} > 0, \quad i = 1, \ldots, N, \) and take

\[
\Psi_i = \sinh \xi_i, \quad i \text{ is an odd number},
\]
\[
\Psi_i = \cosh \xi_i, \quad i \text{ is an even number}.
\]

(6.32a)

(6.32b)

\[
\Phi_i = e^{\xi_i},
\]

(6.33)

where

\[
\xi = k(y + \sqrt{\omega \frac{s}{4(k_1^2 \omega - 1)}}) + \sum_{i=1}^{N} \prod_{j=1}^{N} \left( \frac{k - k_j}{k - k_i} \right)^{\alpha_i} \]

\[
\xi_i = \left. \xi \right|_{k = k_i},
\]

then we have

\[
\Psi_1 = \sinh \xi_1, \quad \Psi_1^{(1)} = \gamma_1 \cosh \xi_1,
\]

(6.34a)

(6.34b)

\[
\gamma_1 = \alpha_1 + y + \frac{-16k_1^2 \omega^{5/2}s}{(4k_1^2 \omega - 1)^2} + \frac{2\omega^{3/2}s}{4k_1^2 \omega - 1},
\]

(6.34c)

and

\[
W_1(\Psi_1, \Psi_1^{(1)}) = -k_1 \gamma_1 + \frac{1}{2} \sinh 2\xi_1,
\]

(6.35a)

\[
W_2(\Psi_1, \Psi_1^{(1)}, \Phi_1) = 2k_1^2 \sinh \xi_1,
\]

(6.35b)

Then the one-negaton solution of (5.3) and the corresponding eigenfunction for (5.1) is given by

\[
Q(y, s) = -2[\log W_1]_{yy},
\]

(6.36a)

\[
r(y, s) = \sqrt{\omega} - 2[\log W_1]_{ys},
\]

(6.36b)

\[
\phi_1(y, s, \lambda_1) = \beta_1 \frac{W_2}{W_1},
\]

(6.36c)

where \( \alpha \) and \( \beta \) are arbitrary constants.

By using the method of variation of constants, we obtain the one-negaton solution for the CHESCS (2.5) with \( N = 1, \lambda_1 = k_1^2 - \frac{1}{4\omega} \) from (5.12), where

\[
\bar{r}(y, s) = \sqrt{\omega} - 2[\log W_1]_{ys} \left|_{\gamma_1 = \bar{\gamma}_1} \right.,
\]

(6.37a)

\[
\bar{\psi}_1(y, s) = \frac{2\omega \sqrt{\alpha_1(s)} W_2}{k_1 (4k_1^2 \omega - 1)} W_1 \left|_{\gamma_1 = \bar{\gamma}_1} \right.,
\]

(6.37b)

\[
\bar{\gamma}_1 = \alpha_1(s) + y + \frac{-16k_1^2 \omega^{5/2}s}{(4k_1^2 \omega - 1)^2} + \frac{2\omega^{3/2}s}{4k_1^2 \omega - 1},
\]

(6.37c)

where \( \alpha(s) \) is an arbitrary function of \( s \).

In Fig 5, we plot the one-negaton solution of \( u \) and \( \varphi_1 \).
Similarly, we can find N-negaton solution for CHESCS (2.5) with \( \lambda_i = k_i^2 - \frac{1}{4\omega}, \quad i = 1, \cdots, N \). For \( N \), we have

\[
\Psi_i^{(1)} = \gamma_i \cosh \xi_i, \quad i \text{ is an odd number},
\]

(6.38a)

\[
\Psi_i^{(1)} = \gamma_i \sinh \xi_i, \quad i \text{ is an even number}.
\]

(6.38b)

where

\[
\gamma_i = \frac{\partial \xi}{\partial k} \bigg|_{k=k_i} = \prod_{j \neq i} (k_i - k_j)^2 \alpha_i + y + \frac{-16k_i^2\omega^{5/2}y}{(4k_i^2\omega - 1)^2} + \frac{2\omega^{3/2}s}{4k_i^2\omega - 1}.
\]

We find that

\[
W_1 = W(\Psi_1, \Psi_1^{(1)}, \cdots, \Psi_N, \Psi_N^{(1)}),
\]

(6.39a)

\[
\phi_i = W(\Psi_1, \Psi_1^{(1)}, \cdots, \Psi_N, \Psi_N^{(1)}, \phi_i),
\]

(6.39b)

and the N-negaton solution of (5.3) and the corresponding eigenfunction for (5.1) is given by

\[
Q(y, s) = -2[\log W_1]_{yy},
\]

(6.40a)

\[
r(y, s) = \sqrt{\omega - 2[\log W_1]_{ys}}.
\]

(6.40b)

\[
\psi_j(y, s, \lambda_i) = \beta_i \frac{W_2}{W_1}, i = 1, \cdots, N,
\]

(6.40c)

where \( \alpha_i \) and \( \beta_i \) are arbitrary constants.

By using the method of variation of constants, we obtain the N-negaton solution for the CHESCS (2.5) from (5.12), where

\[
\bar{r}(y, s) = \sqrt{\omega - 2[\log W_1]_{ys}}|_{\gamma_i = \bar{\gamma}_i},
\]

(6.41a)

\[
\bar{\psi}_i(y, s) = \frac{2\omega}{k_i(4k_i^2\omega - 1)\prod_{j \neq i} (k_j + k_i)} \left[ \frac{\alpha_i(s)}{W_1} \right]_{\gamma_i = \bar{\gamma}_i},
\]

(6.41b)

\[
\bar{\gamma}_i = \prod_{j \neq i} (k_i - k_j)^2 \alpha_i(s) + y + \frac{-16k_i^2\omega^{5/2}s}{(4k_i^2\omega - 1)^2} + \frac{2\omega^{3/2}s}{4k_i^2\omega - 1}.
\]

(6.41c)

where \( \alpha_i(s) \) are arbitrary functions of \( s \).

7 Conclusion

The CHESCS and its Lax representation are derived. Conservation laws are constructed. It is reasonable to speculate on the potential application of CHESCS, that is, CHESCS may describe
the interaction of different solitary waves in shallow water. Since SESCS can be regarded as soliton equations with non-homogeneous terms, we look for explicit solutions by using the method of variation of constants. By considering a reciprocal transformation, which relates CH equation to an alternative of ACH equation, we propose a similar reciprocal transformation, which relates the CHESCS to ACHESCS. By using the Darboux transformation, one can find the n-soliton and n-cuspon solution as well as n-positon and n-negaton solution of alternative ACH equation. Then by means of the method of variation of constants, we can obtain N-soliton, N-cuspon, N-positon and N-negaton solutions of the ACHESCS. Finally, using the inverse reciprocal transformation, we obtain N-soliton, N-cuspon, N-positon and N-negaton solutions of the CHESCS.

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Reference

1. B. Fuchssteiner and A.S. Fokas, Physica D 4, 47 (1981).
2. R. Camassa and D. Holm, Phys. Rev. Lett. 71, 1661 (1993).
3. R. Camassa, D. Holm and J. Hyman, Adv. Appl. Mech. 31, 1 (1994).
4. A. Parker, Proc. R. Soc. Lond. A 460, 2929 (2004).
5. R. S. Johnson, Proc. R. Soc. Lond. A 459, 1687 (2003).
6. Z. J. Qiao, Commun. Math. Phys. 239, 309 (2003).
7. R. Beals, D. H. Sattinger and J. Szmigielski, Adv. Math. 154, 229 (2000).
8. Y. S. Li and J. E. Zhang, Proc. R. Soc. Lond. A 460, 2617 (2004).
9. Y. S. Li, J. Nonlinear Math. Phys. 12, 466 (2005).
10. Z. J. Qiao and G. P. Zhang, EuroPhys. Lett. 73, 657 (2006).
11. H. Holden, J. Hyp. Diff. Equ. 4, 39 (2007).
12. J. Lenells, J. Phys. A: Math. Gen. 38, 869 (2005).
13. V. K. Mel’nikov, Phys. Lett. A 133, 493 (1988).
14. V. K. Mel’nikov, Commun. Math. Phys. 120, 451 (1989).
15. V. K. Mel’nikov, Commun. Math. Phys. 126, 201 (1989).
16. D. J. Kaup, Phys. Rev. Lett. 59, 2063 (1987).
17. J. Leon and A. Latifi, J. Phys. A: Math. Gen. 23, 1385 (1990).
18. C. Claude, A. Latifi and J. Leon, J. Math. Phys. 32, 3321 (1991).
19. M. Nakazawa, E. Yamada and H. Kubota, Phys. Rev. Lett. 66, 2625 (1991).
20. E. V. Doktorov and V. S. Shchesnovich, Phys. Lett. A 207, 153 (1995).
21. Y. B. Zeng, Physica D 73, 171 (1994).
22. Y. B. Zeng, W. X. Ma and Y. J. Shao, J. Math. Phys. 42, 2113 (2001).
23. Y. B. Zeng, Y. J. Shao, W. M. Xue, J. Phys. A: Math. Gen. 36, 5035 (2003).
24. T. Xiao and Y. B. Zeng, J. Phys. A: Math. Gen. 37, 7143 (2004).
25. R. L. Lin, Y. B. Zeng and W. X. Ma, Physica A, 291, 287 (2001).
26. H. X. Wu, Y. B. Zeng and T. Y. Fan, Inverse Problems 24, 1 (2008).
27 H. X. Wu, Y. B. Zeng, X. J. Liu and Y. H. Huang, Solving soliton equations with self-consistent sources by constant variation method, in submission.
28 J. Schiff, Physica D 121, 24 (1998).
29 A. Hone, J. Phys. A 32, L307 (1999).
30 V. B. Matveev and M. A. Salle, Darboux transformations and solitons, Springer-Verlag, 1991.
31 R. Ivanov, Phys. Lett. A, 345, 112 (2005).