Some models of viscous gas dynamics

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Abstract. In this paper, the problems containing two or more small parameters are investigated. In the analysis of such problems, a diagrammatic method is used to visualize the structure of perturbed flow regions, and to estimate the similarity parameters and mathematical models depending on the limiting transitions. The interaction between the laminar flows and transonic flows is considered in the context of the problem containing a small parameter inverse to the Reynolds number, the Mach number other than 1 and the pressure perturbation amplitude of the initiating viscous-inviscid interaction processes. The results of the flow analysis in a laminar boundary layer under the assumption of discontinuity of boundary conditions, for example, discontinuity of the surface velocity, are presented. For such problems a diagram of possible embedded perturbed flow regions is plotted. In addition, the results of studying the nonstationary processes of interaction between the flow in the boundary layer and the external supersonic flow are discussed.

1. Introduction
It is assumed that at a distance $l$ from the leading edge of a plane surface there is a region of local interactions initiated by a certain perturbation, for example, an obstacle on the surface or a shock wave. It is assumed that the origin of the Cartesian coordinate system coincides with the leading edge of the plate subject to a transonic gas flow. The notation adopted for time, Cartesian coordinates, components of the velocity vector, pressure, density, and dynamic viscosity coefficient, is as follows:

\[ t, x, y, \frac{1}{2} Re, u_x, u_y, \rho, \mu \]

Despite the fact that the Reynolds number $Re = \frac{\rho_x u_x}{\mu}$ is supposed to be large, a laminar flow is considered. This suggestion is justified for the analysis of unsteady problems, including linear stability problems.

2. Methodology
In addition to a small parameter, characterizing the thickness of the boundary layer $\varepsilon = Re^{-1/2}$, the problem contains another two small parameters $\beta = \sqrt{(M^2 - 1)}$ and the amplitude of the pressure perturbation $\Delta p$. In principle, the perturbation can also be characterized by other parameters, such as the characteristic spatial and temporal scales, because, as for a turbulence of the external flow, the scale and spectral composition of the perturbations are of considerable significance. In what follows,
the discussion will be limited to the analysis of regimes, for which the viscous-inviscid interaction processes are important.

3. Results for transonic flows analysis

For further analysis, it is essential to distinguish a three-layer structure in the laminar boundary layer [1-3]. The asymptotic analysis shows that the following relation is satisfied

\[ \Delta p \sim \mathcal{E}^{1/2} \beta^{-1/2} \]  

In the near-wall region, the influence of the forces of viscosity, inertia, and induced pressure has the same order, provided that in the external flow the boundary layer induces linear perturbations. For the subsequent analysis, this relationship is convenient to represent in the following form:

\[ \ln(\Delta p) / \ln \mathcal{E} = f(\ln \beta / \ln \mathcal{E}). \]

Then, relation (1) is represented by the line AM in figure 1.

![Figure 1. Transonic flow regimes diagram](image)

From a comparison between the characteristic times in the outer region and in the near-wall region, it follows that their ratio is

\[ \frac{T}{t} \sim \Delta p^{1/2} \beta^{-2} \]  

Then, the condition of equality of the orders of characteristic times can be represented by the line OK in figure 1. For linear flow regimes in the near-wall region, this estimate is given as

\[ \frac{T}{t} \sim \mathcal{E}^{1/4} \beta^{-9/4} \]  

The times for this case are described by the line KL in figure 1 [4].
It is also important to evaluate the effect of the transonic parameter \( K = \beta^2 / \delta^{2/3} \). For linear and nonlinear flow regimes in the near-wall region, the following relationship leads to a significant effect of the transonic parameter

\[
\Delta p \sim \beta^2 \quad (4)
\]

The last relationship is represented by the OMN line in figure 1.

Thus, using the results obtained above and presented in figure 1, for each limiting procedure, corresponding to the lines in the diagram (or the relation between the small parameters of the problem), there is a corresponding boundary-value problem (or model) and the similarity parameters. It should be noted that the most common models correspond to the intersection points of the lines in the diagram [5].

4. Results for problems with discontinuous boundary conditions
As an example, we consider a flow near the velocity discontinuity region on a plate having a moving section at a distance \( l \) from the leading edge, whose velocity is equal to \( u_w \) (figure 2).

![Flow structure near the point of velocity discontinuity](image)

Figure 2. Flow structure near the point of velocity discontinuity

The notation used for the Cartesian coordinates measured along the surface and normal to it, time, velocity vector components, density, pressure, viscosity coefficient, and total enthalpy is similar to the notation introduced in the previous Section.

We consider the structure of a perturbed stationary flow, for which

\[
u(x < 1,0) = 0, \quad u(x > 1,0) = u_w > 0. \quad (5)
\]

A change in the velocity of the streamlines located near the surface moving with a velocity \( u_u \) at \( x > 1 \), and streamlines with almost zero velocities for \( x < l \), can lead to the formation of a new boundary layer downstream from the point of discontinuity of the boundary condition.

Using the momentum variation equations, one can find the distance, at which the friction in the resulting boundary layer becomes compatible with the friction in the main boundary layer

\[
u_{u} / \varepsilon \sim 1 / \varepsilon, \quad x_1 - u_w^3 \quad (6)
\]

\[
u_{u} / \varepsilon 
\]
The relations obtained can be conveniently depicted graphically in the form of the relationship 
\[ \ln x_i / \ln \varepsilon = f(\ln u_w / \ln \varepsilon) \], and then relation (6) is represented by OB line (figure 3).

Figure 3. Disturbed flows scales as a function of the wall velocity

One can also find the distance \( x_2 \) from the discontinuity point of the boundary condition, at which the thickness and length of the region of nonlinear perturbations have the same order of magnitude and where, in essence, the assumptions of the boundary layer theory are violated

\[ x_2 \sim \varepsilon^2 / u_w \]  
(7)

The relation (7) is represented by the line AB in figure 3.

An estimate of the pressure perturbation allows us to find the distance \( x_3 \), at which the induced pressure gradient exerts a nonlinear effect on the near-wall region of the main boundary layer. Here, it should be noted that a change in the thickness of this region determines the total change in the displacement of the boundary layer

\[ \Delta p \sim u_3^2, \quad x_3 \sim \varepsilon^{4/5} u_w^{-1/5} \]  
(8)

The last of the relations (8) is represented by the line EF in figure 3. In the case of linear perturbation regime, its realization can be associated with greater relative influence of the viscosity forces. The equality of the orders of magnitude of the terms, describing the influence of the forces of viscosity and inertia, leads to an estimate

\[ x_4 \sim \varepsilon^{3/2} \]  
(9)

The relation (9) is described by the line BC in figure 3. For linear interaction modes the following estimate holds true:
In Figure 3, it corresponds to ED line.

The diagram of disturbed flow regions depicted in figure 3 can be used to determine the dimensions of these regions and the nature of the flow, occurring in these regions at given amplitude of the parameter \( u_w \). Thus, the perturbations with the amplitude \( O(\varepsilon^{1/4}) \leq u_w \leq O(1) \) lead to the appearance of the region near the discontinuity with the dimensions determined by the line AB, in which the arising flow is described by the system of the Navier-Stokes equations for an incompressible fluid. The next in length is the region whose longitudinal dimension is determined by the line EF, where the flow is described in the first approximation by the Burgers equation. At the intermediate distances, when the parameters are located in the region between the lines AB and EF, the influence of viscosity on the flow in the region of nonlinear perturbations is insignificant and a compensation interaction mode is realized [7]. The absence of viscous terms in the equations, describing the perturbed flow, necessitates the introduction of a sub-region, in which the influence of viscous forces is of the same order as the influence of inertia forces. At the same time, there is a region with a length defined by the line, in which the effect of viscosity is significant and the surface friction is of the same order of magnitude as the friction at the initial boundary layer. The point E, as noted above, corresponds to the general case, when the nonlinear processes of interaction with the external flow occur in the “free interaction” region [1-3].

When the parameter \( u_w \) changes in the range \( O(\varepsilon^{1/4}) \leq u_w \leq O(1) \) other than the above mentioned range, in which the flow is described by the Navier-Stokes equations, another region appears. Its length is determined by the line BE. In this case, the initiated flow is described by a system of boundary layer equations with a compensation interaction condition. In this region, the skin friction on the surface is comparable with the skin friction in the undisturbed flow. Finally, for \( u_w \sim \varepsilon^{1/2} \), the friction tends to undisturbed value directly in the region where the flow is described by the Navier-Stokes system of equations.

5. Results for unsteady viscous-inviscid interaction

The flow of supersonic viscous gas is considered. Assume that at the distance \( l \) from the leading edge of the plate there is a flap or base region. The flap angle changes with time or the base pressure changes. The analysis is performed for such flows regimes, in which the amplitude of the flap deflection or the amplitude of pulsations of the bottom pressure are small.

Using the estimates that follow from the equations of motion, we can show that if the relation

\[ \Delta \rho \tau \sim \varepsilon \]  

holds true, the regime of nonlinear nonstationary viscous-inviscid interaction is realized. In the case of realization of the linear regime, the following relations are satisfied

\[ \Delta \rho \sim \tau \sim \varepsilon^{1/2} \]  

For clarity, it is useful to represent the asymptotic relations between the obtained parameters in the form of a diagram, as in the previous sections.

Then relation (11) is represented by the line AB, and relation (12) by the line AD. Then the BAD line separates the regions of variation of the parameters, corresponding to the steady-state and nonsteady flow regimes. The EAB line separates the regions of parameter variation corresponding to the linear and nonlinear flow regimes, and the condition determining the boundary between the linear and nonlinear regimes in the case of steady flows, (the EA line was obtained in [7-8]). The region of
variation of the parameters locates to the left of the line AD and its continuation corresponds to stationary modes of viscous-inviscid interaction. In turn, the parameters to the right of this line correspond to the nonstationary flow interaction regimes in the laminar boundary layer with an external supersonic flow. In this case, the AB line separates the regions of parameters, for which the influence of viscosity (regions of nonlinear and linear regimes) is significant or insignificant.

Figure 4. Disturbed flow regimes diagram

6. Conclusions

Thus, the above diagram and also the diagrams obtained in the previous sections have proved to be useful for visualization of possible flow regimes and estimation of the corresponding mathematical models along with similarity parameters.

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