Weyl Fermions in a Linear Class of Gödel-Type Space-Time Backgrounds

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Abstract—We study Weyl fermions in a linear class of topologically trivial Gödel-type geometries in Einstein’s general relativity. We solve the Weyl equation and evaluate in detail the energy eigenvalues and the corresponding wave functions.

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1. INTRODUCTION

The particle motion is commonly described using either the Klein–Gordon or the Dirac equation [1, 2], depending on the particle spin. Spin-zero particles like mesons and other bosons are described by the Klein–Gordon equation, and spin-half particles such as the electron are described satisfactorily by the Dirac equation. The Dirac and Klein–Gordon equations have been of interest for theoretical physicists in many branches of physics [3, 4]. Exact solutions of the Klein–Gordon and the Dirac equation play an important role in relativistic quantum Physics as well as in various physical applications including those in nuclear and high energy physics [5, 6].

Several studies of the physical problems involving Gödel-type space-times have been developed in the recent years. Recently, a large number of issues related to rotating Gödel solutions in general relativity as well as alternative theories of gravitation have been studied (e.g., [7–16]). The relativistic quantum dynamics of spin-zero and spin-half particles have been investigated in Gödel-type space-times in relativistic quantum mechanics by many authors. For example, Figueiredo et al. [17] investigated scalar and spin-1/2 particles in Gödel space-times with positive, negative and zero curvatures. The relationship between the Klein–Gordon solution in a class of Gödel solutions in general relativity with Landau levels in curved spaces were investigated in Refs. [18, 19]. This analogy was also observed by Das et al. [20] studying the quantum dynamics of a scalar particle in Som–Raychaudhuri space-time and compared the results with the Landau levels in flat space. In [21], the relativistic quantum dynamics of a scalar quantum particle in Som-Raychaudhury space-time were investigated. In [23], a scalar quantum particle in Gödel-type metrics with a cosmic string passing through the axis were investigated. In [23], a solution of the Weyl equation in a non-stationary Gödel-type cosmological model were obtained, and also the Weyl equation in a family of Gödel-type metrics was studied. In [24], the Klein–Gordon equation for a particle confined in two concentric thin shells in Gödel, Kerr–Newman and FRW space-times in the presence of topological defects were studied. In [25], solutions of the photon equation in a stationary Gödel and Gödel-type space-times were obtained.

In this article, we investigate the relativistic quantum dynamics of fermions, without topological defects, in a class of flat or linear Gödel-type metric. The purpose of this contribution is to investigate the influence of the curvature and rotation of Gödel-type metrics in quantum dynamics of Weyl fermions. In [26], quantum dynamics of Dirac fermions in a Gödel-type geometry of positive, negative and flat curvature with topological defects and torsion were investigated. In [27], the relativistic quantum dynamics of a massless fermion in the presence of topological defects in a class of Gödel-type metrics were investigated. They solved the Weyl equation in the Som–Raychaudhury, spherical and hyperbolic background metrics pierced by topological defects. In [28], the relativistic quantum dynamics of a spin–0 scalar particle in a linear class of topologically trivial Gödel-type space-time was investigated. In [29], linear and Coulomb confinement of a scalar particle in a linear class of topologically trivial Gödel-type space-time was investigated. In [30], the Dirac equation in a linear class of topologically trivial Gödel-type space-time backgrounds was investigated. In [31], a spin–0 system of the DKP equation in a linear class of topologically trivial Gödel-type space-time backgrounds
was investigated. In [32], the relativistic quantum dynamics of a spin-0 system of the DP oscillator in a linear class of Gödel-type space-time was studied.

The results obtained for the present case of a Weyl fermion in a linear class of Gödel-type metrics in Einstein’s relativity theory are quite different from the previous results obtained for a Dirac fermion with torsion in [26], and for Weyl fermions with a topological defect in [27]. The generalized KG oscillator with a position-dependent mass in a linear or flat class of Gödel-type space-times were studied in [33]. In the present system, the model we present is described by Weyl fermions, where the Fermi velocity plays the role of the speed of light in this effective theory.

The study carried out in this article can be used to investigate the influence of rotation and curvature in the condensed matter systems described by massless fermions. The approach applied in this paper can be used to investigate the Hall effect as studied in spherical droplets [34] with rotation in the absence of topological defects. We claim that the studies of this problem in the present paper can be used to investigate the influence of rotation in Gödel-type background metrics. In [13], the influence of rotation in fullerene molecules was investigated, where, in this model, the rotation is introduced via a three-dimensional Gödel-type metric. In this paper, we analyze the relativistic quantum dynamics of a massless fermion in a linear class of Gödel-type metrics. We find the eigenvalues of energy and observe their similarity with the Landau levels for a massless spin-1/2 particle on the same space-time geometries. The possibility of a zero mode for the eigenvalues of the Weyl spinor is discussed, and the physical implications are analyzed.

We will introduce the Dirac equation in Weyl representation on a curved background. Following the theory of spinors in curved space-time [35–37], one can write the equations for a massless spin-half field in the following way:

\[ i\gamma^\mu(x)\nabla_\mu\Psi = 0 \quad (\hbar = 1 = c), \]

\[ (1 + \gamma^0)\Psi = 0, \]  

where \( \gamma^\mu(x) = e^{(a)}_\mu \gamma^{(a)} \) are the gamma matrices in Weyl representation, and \( \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \).

The Dirac matrices will be

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \]  

where \( I \) is a 2 × 2 unit matrix, \( 0 \) is the zero matrix, and \( \sigma^i \) are the Pauli matrices given by

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

\[ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

2. WEYL FERMIONS IN A LINEAR CLASS OF GÖDEL-TYPE SPACE-TIME

Consider the following stationary space-time [38] (see also [16, 28–33]) in the coordinates \( (x^0 = t, x^1 = x, x^2 = y, x^3 = z) \):

\[ ds^2 = -dt^2 + dx^2 + (1 - \alpha_0^2 x^2) dy^2 - 2\alpha_0 x dt dy + dz^2, \]  

where \( \alpha_0 > 0 \) is a real number. The ranges of the coordinates are

\[ -\infty < t < \infty, \quad -\infty < x < \infty, \]

\[ -\infty < y < \infty, \quad -\infty < z < \infty. \]  

The determinant of the metric tensor \( \det g = -1 \), and the metric tensor for the space-time (5) is

\[ g_{\mu\nu}(x) = \begin{pmatrix} -1 & 0 & -\alpha_0 x & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha_0 x & 0 & (1 - \alpha_0^2 x^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]  

with its inverse

\[ g^{\mu\nu}(x) = \begin{pmatrix} -1 & 0 & -\alpha_0 x & 0 \\ -\alpha_0 x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]  

Using the definition of \( e_{\mu\nu}^{(a)} \) and \( e_\mu^{(a)} \), we have

\[ e_\mu^{(a)}(x) = \begin{pmatrix} 1 & 0 & \alpha_0 x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]  

\[ e_{\mu\nu}^{(a)}(x) = \begin{pmatrix} 1 & 0 & -\alpha_0 x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]  

which must satisfy

\[ g_{\mu\nu}(x) = e_\mu^{(a)}(x)e_\nu^{(b)}(x)\eta^{(a)(b)}, \]  

where \( \eta^{(a)(b)} \) is the Pauli matrices given by

\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

\[ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
where $\eta_{(a)(b)} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski flat space metric.

The spin connections can be determined using the Christoffel symbols $\Gamma_{\nu\sigma}^\mu$ given in [30] with the definition

$$\omega_{\mu(a)(b)}(x) = \eta_{(a)(c)} e_{\nu}^{(c)} e_{(b)}^{\tau} \Gamma_{\tau\mu}^{\nu} - \eta_{(a)(c)} e_{\nu}^{(c)} \partial_{\mu} e_{(b)}^{\nu},$$

and these are

$$\omega_t(x) = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\alpha_0}{2} & 0 \\ 0 & -\frac{\alpha_0}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

$$\omega_x(x) = -\frac{3\alpha_0}{2} \left( \begin{array}{cccc} \alpha_0 x & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

$$\omega_y(x) = \frac{\alpha_0}{2} \left( \begin{array}{cccc} 0 & -\frac{\alpha_0}{2} & 0 & 0 \\ 0 & 0 & \alpha_0 x & 0 \\ 0 & -\alpha_0^2 x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

$$\omega_z(x) = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

With Eq. (11), we can use the chiral representation [39], so that the spinorial connection is defined by

$$\Gamma_{\mu} = \frac{1}{8} \omega_{\mu(a)(b)}(x)[\sigma^a, \sigma^b],$$

and these are

$$\Gamma_t(x) = \frac{i\alpha_0}{4} \sigma^3,$$

$$\Gamma_x(x) = 0,$$

$$\Gamma_y(x) = \frac{i\alpha_0^2 x}{4} \sigma^3,$$

$$\Gamma_z(x) = 0.$$

The generalized Pauli matrices $\sigma^\mu(x) = e^\mu_{(a)}(x)\sigma^a$ in curved space-time are

$$\sigma^t(x) = \sigma^0 - \alpha_0 x \sigma^2,$$

$$\sigma^x(x) = \sigma^1,$$

$$\sigma^y(x) = \sigma^2,$$

$$\sigma^z(x) = \sigma^3.$$  (15)

The Weyl equation assumes the form

$$i\sigma^\mu(\partial_{\mu} + \Gamma_{\mu})\Psi = 0.$$  (16)

With the generalized Dirac matrices and spinorial connections derived above, we have

$$\sigma^\mu(x)\Gamma_{\mu}(x) = \frac{i\alpha_0}{4} \sigma^3.$$  (17)

Since the metric (5) is independent of $t, y, z$, suppose the wave function to be

$$\Psi_0(t, x, y, z) = e^{i(-Et + ty + k z)} \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}. $$  (18)

From Eq. (16) we arrive at the following equations:

$$i[\sigma^0 \partial_t + \sigma^2(\partial_y - \alpha_0 x \partial_t) + \sigma^1 \partial_x + \sigma^3 \partial_z]\Psi = \frac{\alpha_0}{4} \sigma^3 \Psi.$$  (19)

Now we carry out the transformation

$$\Psi = e^{-i\frac{\alpha_0}{4}(z-z_0)} \Psi_0$$  (20)

in Eq. (19), and we get

$$i\sigma^0 \partial_t \Psi_0 + i\sigma^2(\partial_y - \alpha_0 x \partial_t)\Psi_0$$

$$+ i\sigma^1 \partial_x \Psi_0 + i\sigma^3 \partial_z \Psi_0 = 0.$$  (21)

It is possible to rewrite the Weyl equation (21) in the following way:

$$i \frac{\partial \Psi_0}{\partial t} = \frac{\sigma^1 \hat{\pi}_x + \sigma^2 \hat{\pi}_y + \sigma^3 \hat{\pi}_z}{1 - \sigma^2 \alpha_0 x} \Psi_0 = \hat{H} \Psi_0,$$  (22)

where $\hat{H}$ is the Hamiltonian of the Weyl particle, and the conjugated momentum is given by

$$\hat{\pi}_x = -i \frac{\partial}{\partial x},$$

$$\hat{\pi}_y = -i \frac{\partial}{\partial y},$$

$$\hat{\pi}_z = -i \frac{\partial}{\partial z}.$$  (23)

To solve Eq. (21), we must substitute the ansatz (18) and the Pauli matrices (4). Then we obtain two coupled differential equations,

$$[E - k] \psi_1 = -i[(\alpha_0 E x + l) \psi_2 + \psi'_2],$$  (24)

$$[E + k] \psi_2 = i[(\alpha_0 E x + l) \psi_1 - \psi'_1].$$  (25)

We are capable of converting these two first-order differential equations to the second-order differential equation

$$\frac{d^2 \psi_i}{dx^2} + [\lambda_s - \beta^2 x^2 - \eta x] \psi_i = 0,$$  (26)
where
\[ \lambda_s = E^2 - \alpha_0 E s - k^2 - l^2, \]
\[ \beta = \alpha_0 E, \]
\[ \eta = 2\beta / s, \quad s = \pm 1. \]  \hspace{1cm} (27)

Note that \( \psi_1(x) \) and \( \psi_2(x) \) are the wave functions of \( \sigma^3 \) with the eigenvalues \( \pm 1 \), and so we can write \( \psi_s = (\psi_+, \psi_-)^T \) with \( \sigma^3 \psi_s = s\psi_s, \ s = \pm 1 \).

Substituting the new variable \( r = \sqrt{\beta} x \) into Eq. (26), we get
\[ \psi''_i(r) + \left[ \frac{\lambda_s}{\beta} - r^2 - \frac{\eta}{\beta^{3/2}} r \right] \psi_i(r) = 0. \]  \hspace{1cm} (28)

The asymptotic behavior of the possible solution to Eq. (28) is to be determined for \( r \to 0 \) and \( r \to \infty \). These conditions are necessary since the wave functions must be well-behaved in these limits, and thus bounded states of energy eigenvalues can be obtained. Let us impose the requirement that the function \( \psi_i(r) \) should vanish as \( r \to 0 \) and \( r \to \infty \). Let the solution be given by
\[ \psi_i(r) = r^A e^{-(Br+Dr^2)} H(r). \]  \hspace{1cm} (29)

Substituting the solution (29) into the equation (28), we get
\[ H''(r) + \left[ \frac{2A}{r} - 2B - 4Dr \right] H'(r) + \left[ \frac{A^2 - A}{r^2} - \frac{2AB}{r} - 4AD - 2D \right. \]
\[ \left. + \frac{\lambda_s}{\beta} + B^2 + \left( 4BD - \frac{\eta}{\beta^{3/2}} \right) r \right]
\[ + (4D^2 - 1) r^2 \right] H(r) = 0. \]  \hspace{1cm} (30)

Equating the coefficients of \( r^{-2}, r, r^2 \) to zero in the above differential equation, we get
\[ A^2 - A = 0 \quad \Rightarrow \quad A = 1, \quad A \neq 0, \]
\[ 4BD - \frac{\eta}{\beta^{3/2}} = 0 \quad \Rightarrow \quad B = \frac{1}{2} \frac{\eta}{4D \beta^{3/2}}, \]
\[ 4D^2 - 1 = 0 \quad \Rightarrow \quad D = \frac{1}{2}. \]  \hspace{1cm} (31)

With these the above relations, Eq. (30) can be expressed as
\[ H''(r) + \left[ \frac{\gamma}{r} - \zeta - \delta r \right] H'(r) \]
\[ + \left[ - \frac{\eta}{r} + \theta \right] H(r) = 0, \]  \hspace{1cm} (32)

where
\[ \gamma = 2A, \]
\[ \zeta = 2B, \]
\[ \delta = 4D, \]
\[ q = 2AB, \]
\[ \theta = B^2 + \frac{\lambda_s}{\beta} - 4AD - 2D. \]  \hspace{1cm} (33)

Equation (32) is the biconfluent Heun differential equation, and \( H(r) \) are Heun polynomials.

Equation (32) can be easily solved by using the Frobenius method as follows:
\[ H(r) = \sum_{i=0}^{\infty} c_i r^i. \]  \hspace{1cm} (34)

Substituting (34) into Eq. (32), we get the following recurrence relation for the coefficients:
\[ c_{n+2} = \frac{1}{(n + 2)(n + 1 + \gamma)} \left\{ q + \zeta(n + 1) \right\} c_{n+1} \]
\[ - (\theta - 2n) c_n \].  \hspace{1cm} (35)

And the first two coefficients are
\[ c_1 = \frac{q}{\gamma} c_0, \]
\[ c_2 = \frac{1}{2(1 + \gamma)} \left[ (q + \zeta) c_1 - \theta c_0 \right]. \]  \hspace{1cm} (36)

The power series becomes a polynomial of degree \( n \) by imposing the following two conditions:
\[ c_{n+1} = 0, \quad (\theta - 2n) = 0, \quad n = 1, 2, \ldots \]  \hspace{1cm} (37)

Using the above energy quantization condition, we get the following eigenvalue equation:
\[ B^2 + \frac{\lambda_s}{\beta} - 4AD - 2D = 2n. \]  \hspace{1cm} (38)

Substituting \( A, B, D \) into the eigenvalue equation, we get the following energy eigenvalues:
\[ E_n = \frac{1}{2} \left[ \alpha_0(2n + 3 + s) \right. \]
\[ \pm \sqrt{\left( \alpha_0^2(2n + 3 + s)^2 + 4k^2 \right)} \]
\[ \left. = \Omega(2n + 3 + s) \pm \sqrt{\Omega^2(2n + 3 + s)^2 + k^2} \right]. \]  \hspace{1cm} (39)

The corresponding wave functions are
\[ \psi_{in}(r) = r \exp \left[ - \frac{|l|}{\sqrt{2\Omega E_n}} r \right] e^{-r^2/2H(r)}, \]  \hspace{1cm} (40)

where \( l = \pm 1, \pm 2, \ldots \in Z \) is an integer.
If we set the constant $k = 0$, we get the following energy levels from (39):

$$E_n = 2Ω(2n + 3 + s), \quad (41)$$

which is similar to the energy eigenvalues obtained in [30] (see Eq. (46) in [30] with $κ = 0$ there).

Now we evaluate the individual energy levels and wave functions one by one by imposing the additional recurrence condition $c_{n+1} = 0$, and others are in the same way. For example, at $n = 1$ we have $c_2 = 0$, which implies from Eq. (36)

$$E_1 = \sqrt{2l \alpha_0}, \quad (42)$$

de the ground state energy level. The corresponding ground state wave function is given by

$$\psi_{i1}(r) = r \exp \left[-\frac{[l]}{\sqrt{2\Omega E_1}} r \right] e^{-r^2/2H(r)}. \quad (43)$$

3. CONCLUSIONS

In this paper, we have studied Weyl fermions in a Gōdel-type space–time background. We have solved the Weyl equation in a linear class of Gōdel-type space–times and obtained the relativistic energy eigenvalues (39) and the wave function (40).

The energy levels obtained for this class of space–times has the same properties as the energy levels obtained in the case of quantum dynamics of Dirac fermions in the same space–time geometries in the theory of relativity [30]. Note that the rotation of Gōdel space–times introduces a contribution in the Weyl energy levels. We claim the analogy between the energy levels for Weyl fermions in a linear class of Gōdel-type metrics and the Landau levels in curved spaces that can be used to investigate the Weyl semimetals [40]. The systems investigated here can be used to describe condensed matter systems described by massless fermions in curved geometries under the influence of rotation [13, 26, 41–43].

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