Unveiling the dynamics of the universe

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We present two methods for describing, from a pedagogical point of view, the solutions of Einstein’s equations applied to a homogeneous and isotropic universe. In the first method, we define an effective gravitational potential of the universe, and the second method makes use of the fact that we can define a dynamical system of equations. The methods are applied to different cosmological models whose properties are discussed in turn. The ultimate intention is to provide simple examples to be revised in a first Cosmology course for undergraduates.

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I. INTRODUCTION

In any introductory course on Cosmology, some questions undoubtedly arise once the students become familiar with the equations describing the evolution of a homogeneous and isotropic universe. These are: a) which is the universe made of?; b) what is the curvature of the universe? c) how did the universe expand in the past according to the answers in a) and b)?

It is interesting to note that present technology has allowed the humankind to partially answer those questions with some important accuracy[1, 2, 3]. However, such an improvement in our knowledge of the universe will take still some years to reach the textbooks for undergraduates.

In this paper, we briefly review the basic equations of the evolution of the universe, and describe how the latter evolves according to its material contents and curvature. It is not our aim to give an exhaustive study of the gravitational dynamics that arises from the equations of Einstein’s General Relativity (GR), but rather to give some tools to help students to understand the richness of the solutions of the Einstein’s equations in the case of a homogeneous and isotropic universe. In other words, we will focus our attention in the answer to question c) above.

In the same line, our intention is also to provide simple pedagogical exercises, and to solve them with the help of analytical methods which are of widespread use in the specialized literature. The chosen examples in this paper are appropriate, in our opinion, for a first undergraduate course on Cosmology.

We shall not discuss how we can determine the material content of the universe and its true nature through observations, nor how its curvature is measured. We will only mention briefly how such quantities are obtained, and give the reader some interesting references and internet links where more detailed information can be found. However, non-experts will find a comprehensible summary of modern Cosmology in[4], and some interesting questions reviewed and answered in[5, 6].

The present manuscript is organized as follows. In Sec. II we review the metric quantities that describe a homogeneous and isotropic spacetime, and how they are influenced by the material content of the universe through the equations of Einstein’s General Relativity. We describe the types of matter we shall consider, and how each one is classified according to their equation of state.

We shall introduce the so-called density parameters, which measure the relative contribution of each component to the total material content of the universe. These parameters will play a central role in the subsequent calculations. Due to its particular importance, we will take some space to discuss the significance of the curvature’s density parameter.

In Sec. III we describe how the Friedmann equation can be used to define an effective gravitational potential of the universe, so that we can visualize the expansion of the universe in a similar fashion as one describes the motion of a single particle in Classical Mechanics.

In Sec. IV we use again the Friedmann equation and the equations of motion of each single fluid to obtain a set of differential equations that allows us to see the expansion of the universe as the solutions of a dynamical autonomous system. The latter formalism will be used to see whether there is any attractor behavior in the cosmological solutions.

The formalisms developed in the previous two sections is applied to three particular examples in Sec. V. These are: the actual standard cosmological model, also known as the Concordance Model (CM); Einstein’s static model of the universe; and a CM with an arbitrary component of what we shall call dark energy.

Finally, Sec. VI is devoted to conclusions and general comments.

II. MATHEMATICAL BACKGROUND

We start with the so-called Cosmological Principle (CP), first proposed by Einstein, which states that the universe we live in is homogeneous and isotropic in large
scales. By large scales we mean scales much larger than the size of a typical galaxy.

The CP is a working hypothesis, and it is the simplest assumption we can make about the properties of the spacetime of the universe as a whole. The interested reader can find a more detailed discussion on the historical and philosophical relevance of Einstein’s CP in 6, 7.

A. Space geometry

The remarkable thing is that the CP suffices to fix the metric of the spacetime a homogeneous and isotropic universe must have. It is called the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, a metric with constant curvature, whose line element is usually written as (in units with $c = 1$)

$$ds^2 = g_{μν}dx^μdx^ν = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2dΩ^2 \right],$$

where the coordinate $t$ is called the cosmic time, $a(t)$ is the (time-dependent) scale factor, and $k$ is the curvature constant.

The spatial part of the metric is written in terms of comoving coordinates, as they remain fixed as the universe expands. Actually, the physical and comoving distances at a given time $t$ are related through $x_{ph} = a(t)x_{com}$.

Notice that $dΩ$ is the usual spherical solid angle element, and $r$ is a comoving radial coordinate. In general, for a fixed $r$ and $t$, we see that the circumference of a circle is equal to $C = 2πr$, and the surface of a sphere is $S = 4πr^2$. But, the proper radius of these objects, for fixed $θ$ and $φ$, is given by

$$ℓ = \int_0^r \frac{dr}{\sqrt{1 - kr^2}} = \begin{cases} \frac{1}{\sqrt{k}} \sin^{-1}(\sqrt{k}r) & \text{for } k > 0, \\ r & \text{for } k = 0, \\ \frac{1}{\sqrt{-k}} \sinh^{-1}(\sqrt{-k}r) & \text{for } k < 0. \end{cases}$$

As a consequence, we see that for $k > 0$, we recover the usual Euclidean relations $C/ℓ = 2π$ and $S/ℓ^2 = 4π$. However, if the curvature constant is positive $k > 0$, then $C/ℓ < 2π$ and $S/ℓ^2 < 4π$. The opposite happens for a negative curvature $k < 0$, and hence $C/ℓ > 2π$ and $S/ℓ^2 > 4π$.

The expansion of the universe is encoded in the time derivative (which we denote by a dot) of the scale factor, with which one defines the Hubble parameter $H = \dot{a}/a$, named after the astronomer Edwin Hubble. Its importance comes from the fact that nearby galaxies recede with a velocity proportional to their (physical) distance from us, as $v = Hx_p$. 8, 10, 11.

B. Cosmological equations

The equations of motion of the universe are given by Einstein’s GR, which is our current fundamental theory of gravitation. 6, 7, 12, 13, 13, 14. For this, we need to specify the kind of matter the universe is made of, and give its corresponding energy-momentum tensor. The usual choice is that of a homogeneous and isotropic perfect fluid, which is well described and defined only by its energy density $ρ(t)$ and its isotropic pressure $p(t)$.

There are two equations coming from GR, relating the geometry and material contents of the universe, and there is another equation coming from the conservation of the energy-momentum tensor. 6, 9, 12, 13, 14. A heuristic approximations to the cosmological equations see 13, 16. The equations of motion are

$$\dot{H} = -4πG \sum_i (ρ_i + p_i) + \frac{k}{a^2},$$

$$\dot{ρ}_i = -3H (ρ_i + p_i),$$

together with the so called Friedmann (constraint) equation

$$H^2 = \frac{8πG}{3} \sum_i ρ_i - \frac{k}{a^2}.$$ 4

We have allowed for the existence in the universe of more than one perfect fluid; with the explicit restriction that they do not interact with each other except gravitationally. This means that each of the perfect fluids obeys a separate conservation equation, namely Eq. (3b).

C. Types of matter

Only $(n + 1)$ of the above $(n + 2)$ equations are independent, and we need $(n)$ extra equations in order to solve for the $(2n + 1)$ unknowns $a(t), ρ_i(t)$ and $p_i(t)$. The needed equation is what is called an equation of state relating the energy density and pressure of each perfect fluid. We will follow the common wisdom and assume a barotropic equation of state in the form

$$p_i = (γ_i - 1)ρ_i,$$ 5

where $γ_i$ is the equation of state of the $i$-th fluid. In this case, Eq. (3b) integrates to

$$ρ_i = ρ_{i,0}a^{-3γ_i},$$ 6

where $ρ_{i,0}$ is an integration constant. The most usual values of $γ_i$ are:

- Radiation and relativistic particles, $γ_r = 4/3$ and $ρ_r \sim a^{-4}$;
- Pressureless matter (dust or non-relativistic particles), $γ_m = 1$ and $ρ_m \sim a^{-3}$;
- Cosmological constant (Λ or vacuum energy), $γ_Λ = 0$ and $ρ_Λ \sim \text{const}$. 15


y However, it is widely accepted that the allowed range can be $2 \geq \gamma \geq 0$.

Eqs. (3) and (4) shows that the curvature of the universe contributes to the dynamics. Actually, we can think of curvature as a kind of special perfect fluid with $\gamma_k = 2/3$, so that $\rho_k \sim a^{-2}$, as it is indeed the case.

D. Critical energy density and density parameters

The Friedmann equation (1) can be rewritten in the dimensionless form

$$1 = \sum_i \frac{\rho_i}{\rho_c} - \frac{\rho_k}{\rho_c} = \sum_i \Omega_i + \Omega_k.$$  \hspace{1cm} (7)

We have defined in here the critical energy density $\rho_c$ and the density parameters $\Omega_i$ as follows

$$\rho_c = \frac{3H^2}{8\pi G}, \quad \Omega_i = \frac{\rho_i}{\rho_c}, \quad \Omega_k = -\frac{k}{a^2 H^2}. \hspace{1cm} (8)$$

The density parameter for the curvature component $\Omega_k$ is defined together with the negative sign, so that it is on equal footing with respect to the density parameters of the material components $\Omega_i$.

It can be seen from Eq. (7) that if the whole material content of the universe equals the critical energy density, $\sum \rho_i = \rho_c$, then the curvature of the universe should be zero; that is, it should correspond to a flat universe. Likewise, if $\sum_i \rho_i > \rho_c$ ($\sum_i \rho_i < \rho_c$) then the universe has a closed (open) spatial geometry.

The critical energy density depends on the value of the Hubble parameter, and thus is a time-dependent quantity. However, the current value of $\rho_c,0$ can be measured directly from the redshift of nearby galaxies and other objects, see [1] for recent results.

On the other hand, each of the density parameters $\Omega_i$ shows the relative contribution of each type of matter to the critical energy density at any time. The allowed range for any density parameter is then $0 \leq \Omega_i \leq 1$, and $\Omega_i = 1$ means that the evolution of the universe is dominated by the $i$-th fluid. Whenever this happens we usually speak of the $i$-th fluid dominated era in the evolution of the universe.

Also, $\Omega_{k,0}$ would represent the current relative contribution of $\rho_{k,0}$ to the current critical energy density $\rho_{c,0}$. Hereafter, a subscript ‘0’ will denote current values for any quantity.

E. The curvature of the universe

There is a normalization issue about the curvature of the universe we should be careful with. It is often said that the curvature constant can be normalized to take three different values: a) $k = -1$, for an open (Hyperbolic) universe; b) $k = 0$, for a flat (Euclidean) universe; and c) $k = 1$ for a closed (Spherical) universe. For further simplicity, one usually finds the suggestion of normalizing the scale factor too, so that its actual value is $a_0 = 1$.

On the other hand, the curvature contributes to the energy density of the universe, and the product $kr^2$ in metric (1) should be dimensionless. Thus, we have two options.

- **Dimensionless $k$ and $r$, and a scale factor $a(t)$ with dimensions of length.** In this case, we can normalize the curvature constant as mentioned above, but we should notice then that the scale factor cannot be normalized arbitrarily as its actual value is set by the curvature density parameter as $a_0 = (\Omega_{k,0})^{1/2}H_0$.

- **Dimensionless scale factor $a(t)$, $r$ with dimensions of length, and $k$ with dimensions of length $^{-2}$.** Thus, the curvature constant is given by $k = -a_0^2 H_0^2 \Omega_{k,0}$. Without loss of generality, we can choose to normalize the scale factor as $a_0 = 1$.

For convenience, the second option will be used throughout this paper.

The constant of curvature is given, at any time, by $k = -a^2 \Omega_k H^2$. This is an interesting relation, since $d_H \equiv H^{-1}$, which is called the Hubble length, provides us of an estimate of the size of the observable universe at any time $[\Omega_k, 12, 13, 14]$. Hence, $k$ tells us of the deviation of the spatial part of metric (1) from the flat case. For scales $r$ for which $\sqrt{|k|r} = a \sqrt{|\Omega_k|} H r \ll 1$, the universe can be considered to have an Euclidean space geometry.

The current value of the Hubble parameter is $H_0 = 70 \text{km s}^{-1} \text{Mpc}^{-1} [1, 2, 21, 27]$, which implies that the current Hubble distance (an estimation of the current size of our observable universe) is $d_{0,H} \approx 4,300 \text{Mpc}$. Also, the current value of curvature’s density parameter is $\Omega_{0,k} = 0.01 [21]$, which seems to indicate we live in an (slightly) open universe.

Thus, the universe we actually see can be plainly taken as Euclidean, since the scales at which the (current) curvature of the universe could be appreciable in the metric at the present time ($> d_{0,H}/\sqrt{\Omega_{k,0}} \approx 43,000 \text{Mpc}$) are larger than the distance to the farthest object we can observe.

III. THE EFFECTIVE GRAVITATIONAL POTENTIAL OF THE UNIVERSE

We will now present the first method to describe the expansion of the universe according to the type of matter is made of. First, we notice that each energy density can be given in terms of the actual value of its corresponding density parameter as $\rho_i = \Omega_{i,0} \rho_{c,0} a^{-3\gamma_i}$, see Eqs. (3) and (4).
Second, the Friedmann equation \[ \ddot{a}/a + V(a) = \frac{1}{2} \Omega_{k,0}, \tag{9} \]
where now a dot means derivative with respect to the dimensionless time \( \tau = H_0 t \). The cosmic time is normalized in terms of the actual value of the so-called Hubble time \( H_0^{-1} = 14 \text{ Gy} \). The effective gravitational potential \( V(a) \) explicitly reads
\[ V(a) = -\frac{1}{2} \sum_i n \Omega_{i,0} a^{-3\gamma_i + 2}. \tag{10} \]

Eq. (10) resembles the conservation of energy for a particle with “space” coordinate \( a(t) \) and constant energy \((1/2)\Omega_{i,0}^2\). We see that a flat universe has zero total energy, and a closed (open) universe has negative (positive) total energy. Also, we would like to stress out that the actual values of the density parameters are not all independent, but are related through the Friedmann constraint at the present time,
\[ 1 = \sum_i n \Omega_{i,0} + \Omega_{k,0}. \tag{11} \]

The gravitational potential \((10)\) is negative definite, \( V(a) < 0 \), if all of the (actual) density parameters are positive definite, \( \Omega_{i,0} \geq 0 \). This fact may imply in some cases the presence of at least one maximum in the potential. This can be verified by direct calculation of the first and second derivatives,
\[ V'(a) = \frac{1}{2} \sum_i n (3\gamma_i - 2) \Omega_{i,0} a^{-3\gamma_i + 1}, \tag{12a} \]
\[ V''(a) = \frac{1}{a} V'(a) - \frac{3}{2} \sum_i n (3\gamma_i - 2) \gamma_i \Omega_{i,0} a^{-3\gamma_i}. \tag{12b} \]

The existence of a critical point is not a trivial thing, as it marks the value of the scale factor at which \( \ddot{a} = 0 \), since Eq. (12a) is equivalent to the acceleration equation \( \ddot{a} = -V'(a) \). At \( a = a_c \), the acceleration of the universe’s expansion vanishes. Therefore, the existence of a critical point tells us that the universe should have had a decelerated and an accelerated expansion at some stages.

When does a critical point exist? From Eq. (12a) there cannot be a critical point if all of the equations of state \( \gamma_i > 2/3 \); the universe will always accelerate in such case. In other words, it is necessary the presence of at least one perfect fluid with an equation of state with a value less than \( 2/3 \) for the universe to have an accelerated expansion.

Actually, it can be proved in the general case that if there is a critical point \( a_c \) such that \( V'(a_c) = 0 \), then \( V''(a_c) < 0 \). For this, let us assume that \((n-1)\) matter fluids have an equation of state \( \gamma_i > 2/3 \), and that it is only the \( n \)-th fluid which has \( \gamma_n < 2/3 \). Using the condition \( V'(a_c) = 0 \) in Eq. (12a), we can write
\[ V''(a_c) = -\frac{3}{2} \sum_i n (3\gamma_i - 2) (\gamma_i - \gamma_n) \Omega_{i,0} a_c^{-3\gamma_i}. \tag{13} \]

All the terms inside the sum are positive by assumption; therefore, \( V''(a_c) < 0 \). The final result is the same if more than one fluid have an equation of state \( \gamma < 2/3 \). From this we conclude that any critical point corresponds to a maximum.

IV. DYNAMICAL COSMOLOGICAL SYSTEM

In the presentation of the second method, we find convenient to take the density parameters as the dynamical variables themselves. It is then our purpose in this section to show how to write the Einstein equations as a dynamical autonomous system (for a comprehensive reading of such systems see \cite{22}; and \cite{23, 24, 25, 26, 27, 28} for applications in Cosmology).

A. The general case

Let us assume that there are \( n+1 \) perfect fluids present in the universe. This is the most general case, as we have already mentioned that the curvature itself can be seen as a special perfect fluid. The Friedmann constraint helps us to reduce in one the number of independent variables, i.e. it is only necessary to consider \( n \) perfect fluids \cite{41}.

Choosing the \((n+1)\)-th perfect fluid to be absorbed by means of Eq. (17), the Einstein equations for the rest of the \( n \) fluids is given in terms of their corresponding density parameters as
\[ \Omega_j = 3\Omega_j \left[ \sum_{i=1}^n (\gamma_i - \gamma_{n+1}) \Omega_i - (\gamma_j - \gamma_{n+1}) \right], \tag{14} \]
where \( i, j = 1, 2, \ldots, n \), and \( \gamma_{n+1} \) is the barotropic equation of state of the (absorbed) \((n+1)\)-th perfect fluid. The prime denotes derivative with respect to the so-called number of e-foldings \( N \equiv \ln a \). To arrive to Eq. (14), we have also made use of Eq. (18) in the form
\[ 1 + \frac{H}{a} = -\frac{\dot{a}}{H^2 a} = -\frac{3}{2} \sum_i n \Omega_i (\gamma_i - \gamma_{n+1}) - \left( \frac{3}{2} \gamma_{n+1} - 1 \right). \tag{15} \]

1. Exact solution

System (14) may appear redundant and unnecessary at first sight. This is because the behavior of the density parameters as functions of the number of e-foldings \( N \)
can be easily found. By definition (see Eqs. (8)), the exact solutions of Eqs. (14) are
\begin{equation}
\Omega_j = \Omega_{j,0} e^{-3\gamma_j N} \sum_{i=1}^{n+1} \Omega_i e^{-3\gamma_i N},
\end{equation}
where \( j = 1, 2, \ldots, n+1 \). It is not difficult to verify that Eqs. (14) directly follow from Eqs. (7) and (16).

If one only wants to know the evolution of the density parameters, then Eq. (16) suffices to know all of the different stages the universe has gone through during its evolution. See sections below for some explicit examples.

B. Fixed points and stability analysis

However, the interesting thing to note is that Eq. (14) is a dynamical autonomous system of the form \( \mathbf{x}' = \mathbf{f}(\mathbf{x}) \), where \( \mathbf{x} = (\Omega_1, \Omega_2, \ldots, \Omega_n) \). In this respect, Eqs. (14) can be seen as a complementary part of Eq. (9), since the latter can tell us which attractor properties can be found in the dynamical equations (3) and (4).

The critical (fixed) points \( \mathbf{x}_c \) of the dynamical system (14) are found by solving the equations \( \mathbf{f}(\mathbf{x}_c) = 0 \). There are two obvious solutions.

• Trivial \((n+1)\)-th perfect fluid dominated solution\), for which \( \mathbf{x}_c = 0 \), \( \Omega_{n+1} = 1 \), and then the expansion of the universe is driven by the \((n+1)\)-th perfect fluid. Eq. (13) points out that the universe has an accelerated expansion, \( \ddot{a} > 0 \) (decelerated expansion, \( \ddot{a} < 0 \)) if \( \gamma_{n+1} < 2/3 \) (\( \gamma_{n+1} > 2/3 \)).

There are two important remarks we should be aware of at this point.

– If the \((n+1)\)-th perfect fluid is the curvature, then \( \gamma_{n+1} = \gamma_k = 2/3 \), and then the universe expands at a constant rate, i.e., \( \ddot{a} = 0 \). It should be noticed, however, that the existence of this critical point is forbidden by the Friedmann constraint (7) for the case of a closed universe (if all of the density parameters are positive definite).

On the other hand, this trivial solution is permitted for an open universe, and is better known as Milne’s universe (12, 23).

– Milne’s model should be distinguished from the so-called empty static model (12). Eqs. (3) tell us that an empty universe (no perfect fluid present, zero curvature) is indeed permitted, which will remain static, \( a(t) = \text{const} \).

• \( \gamma \)-th perfect fluid dominated solution, for which \( \Omega_j = 1 \), and \( \Omega_{\neq j} = \Omega_\alpha = 0 \); that is, the critical points are \( \mathbf{x}_{c,1} = (1, 0, \ldots, 0) \), \( \mathbf{x}_{c,2} = (0, 1, \ldots, 0) \), etc. Eq. (15) again indicates that the universe will accelerate (decelerate) its expansion if the dominant equation of state is such that \( \gamma_j < 2/3 \) (\( \gamma_j > 2/3 \)), which is the same conclusion we reached at in the previous section.

Once found, the stability of the critical points can be established by a first order perturbation analysis (23), in which one considers a small perturbation \( \mathbf{u} \) in the form \( \mathbf{x} = \mathbf{x}_c + \mathbf{u} \). Hence, Eqs. (14) are linearized in the form \( \mathbf{u}' = \mathbf{M} \mathbf{u} \), where
\begin{equation}
\mathbf{M}_{jl} = \left. \frac{\partial \mathbf{f}_j}{\partial \mathbf{x}_l} \right|_{\mathbf{x}_0},
\end{equation}
are the elements of the perturbation matrix \( \mathbf{M} \).

If the eigenvalues of the matrix \( \mathbf{M} \) have all negative (positive) real parts, then the critical point is called stable (unstable). If neither, it is then called a saddle point.

Back to our case, we have to solve the set of equations (14) for \( \Omega_j = 0 \), and to study the eigenvalues of the corresponding perturbation matrix. Let us suppose that we want to investigate the stability of the fixed point corresponding to the domination of the \( i \)-th perfect fluid, \( \mathbf{x}_{c,i} = (0, 0, \ldots, \Omega_i = 1, \ldots, 0) \).

After careful calculations, the elements of the perturbation matrix are explicitly given by
\begin{equation}
\Pi_{i\neq l}^{n+1} [3 (\gamma_i - \gamma_l) - \omega_l] = 0,
\end{equation}
where \( \delta_{jl} \) is the Kronecker delta, \( \delta_{jl} = 1 \) if \( j = l \), and \( \delta_{jl} = 0 \) otherwise.

The matrix \( \mathbf{M} \) is almost a diagonal matrix, except for the non-zero elements in the row \( j = i \); however, the calculation of its eigenvalues is a simple matter. It can be proved that the eigenvalues \( \omega_i \) are solutions to the algebraic equation
\begin{equation}
\Pi_{i\neq l}^{n+1} [3 (\gamma_i - \gamma_l) - \omega_l] = 0,
\end{equation}
where \( l \) runs through the permitted \((n+1)\) values, except for the particular value \( l = i \), and then there are only \( n \) eigenvalues.

In principle, the trivial critical point should be treated separately, and the eigenvalue equation in this case is
\begin{equation}
\Pi_{i=1}^{n} [3 (\gamma_{n+1} - \gamma_l) - \omega_l] = 0.
\end{equation}
Naively, Eq. (20) seems to be a particular case of Eq. (19), one in which \( i' = n + 1 \). However, this is not formally so, as the \( n+1 \)-th perfect fluid does not appear in Eqs. (14).

Therefore, we come to the following general conclusions.

• The critical point corresponding to the domination of the perfect fluid with the largest equation of state is unstable.

• The critical point corresponding to the domination of the perfect fluid with the smallest equation of state is stable.

• All other critical points are saddle points.

The above statements directly follow from Eqs. (14) and (20), as the positivity or negativity of the perturbation eigenvalues for a particular case depend on the relative value of the dominant equation of state with respect to the others.
V. HOW DOES THE UNIVERSE EXPAND?

In this section, we will work on particular and simple models of the universe, and draw their dynamics according to their material contents and the methods discussed in the previous section.

A. Our universe, or the Concordance Model

We now turn our attention to the so-called concordance model (CM), also known as the ΛCDM (Lambda Cold Dark Matter) model, the model of the universe the cosmological observations altogether seem to favor[1,2,21]. It contains radiation (relativistic particles), matter and vacuum energies in the proportions \( \Omega_r,0 = 10^{-5}, \Omega_m,0 = 0.266, \Omega_\Lambda,0 = 0.732, \) and \( \Omega_k,0 = 0.01, \) respectively.

For numerical purposes, we take \( \Omega_r,0 = 10^{-5}, \Omega_m,0 = 0.3, \Omega_\Lambda,0 = 0.69, \) and \( \Omega_k,0 = 0.01, \) so that the Friedmann constraint is accurately accomplished at the present time, see Eq. (11).

1. Effective gravitational potential

The effective gravitational potential of the CM is

\[
V(a) = -\frac{1}{2} \left( \frac{\Omega_r,0}{a^2} + \frac{\Omega_m,0}{a} + \Omega_\Lambda,0 a^2 \right).
\]  

As the density parameters are positive definite, we see that the gravitational potential has one critical point. As radiation does not contribute significantly, the critical point is approximately at \( a_c \simeq \sqrt{\Omega_m,0}/(2\Omega_\Lambda,0) = 0.6. \) As said before, this critical point is a maximum, as \( V''(a_c) = -\Omega_r,0 a_c^2 - 3\Omega_\Lambda,0 < 0. \) For completeness, we plot \( V(a) \) in Fig. 1 for a CM universe.

The curvature of the universe is so small that one hardly distinguishes our universe from the ‘Flat’ case in Fig. 1 and the maximum of the effective gravitational potential is negative, \( V(a_c) < 0. \) These two facts imply that the CM universe will never stop expanding, had a past decelerating stage, and is currently accelerating its expansion.

Notice that, keeping the radiation and matter present contributions fixed, the larger the vacuum contribution is, the sooner the expansion starts accelerating.

2. Dynamical system

Explicitly, the dynamical system of the CM is

\[
\begin{align*}
\Omega'_r &= \Omega_r (2\Omega_r + \Omega_m - 2\Omega_\Lambda - 2), \quad (22a) \\
\Omega'_m &= \Omega_m (2\Omega_r + \Omega_m - 2\Omega_\Lambda - 1), \quad (22b) \\
\Omega'_\Lambda &= \Omega_\Lambda (2\Omega_r + \Omega_m - 2\Omega_\Lambda + 2). \quad (22c)
\end{align*}
\]

As explained before, the trivial solution is also a solution of the dynamical system (22), but its existence is forbidden on physical grounds because of the Friedmann
constraint.

The equations of state are $\gamma_r > \gamma_m > \gamma_\Lambda$, and thus we conclude that radiation domination is an unstable point, matter domination is a saddle point, and $\Lambda$ domination is the only stable point. The universe will always reach a $\Lambda$ dominated era at some epoch, and will remain in it thereafter. We want to point out that this is true whatever the actual contributions of each component are.

The evolution of the density parameters $\Omega_j$ is also shown in Fig. 1. As discussed before, one can see the radiation, the matter and the $\Lambda$ dominated eras, but the curvature dominated solution is never achieved, even though its contribution is noticeable at the transition between the matter and $\Lambda$ dominated eras.

**B. Einstein’s static universe**

Originally, Einstein considered the universe as static, and he introduced a cosmological constant to allow his equations to have a static solution. That is, Einstein found a solution with \( \dot{a} = \ddot{a} = 0 \). For an interesting discussion on Einstein’s cosmological model see Chs. 2 and 4 in [3].

1. **Effective gravitational potential**

According to Eqs. (17) and (18), Einstein’s static solution corresponds to a universe located at exactly the critical point $V'(a_c) = 0$, for which $V(a_c) = (1/2)\Omega_k,0$. That is, the universe has just the enough total energy to be at the critical point of its effective gravitational potential.

Unfortunately, we have already learned that the critical point of $V(a)$ that Einstein found must be a maximum, and therefore is an unstable point. Hence, the universe should expand or collapse, but cannot remain static! An example of a static universe with density parameters $\Omega_m,0 = 0.3$, $\Omega_{\Lambda,0} = 1.713$, $\Omega_k,0 = -1.013$ is shown in Fig. 2.

Some time after Einstein proposed its static universe, the expansion of the universe was discovered, and Einstein thought he had made a mistake in introducing a cosmological constant to have a static universe; then called the latter his ‘biggest blunder’.

It took a bit longer for cosmologists to recognize that Einstein’s original model is unstable [6, 30], and that the universe can expand even in the presence of a finely-tuned cosmological constant.

2. **Dynamical system**

The dynamical system analysis proposed in a previous section can, in principle, be applied to the Einstein’s static universe. The conclusion is that the critical points are the same as those of the concordance model, see Table II. This is because the existence and nature of the critical points only depend on the equations of state of the material content of the universe.

However, Einstein’s universe is a case we have to deal with carefully, because one cannot define a critical density at the maximum of the potential $V(a_c)$, since the latter this time implies the vanishing of the Hubble parameter $H$.

**C. Dark energy**

Since the discovery of the accelerated expansion of the universe in the observations of Type Ia supernovae [1], cosmologists have been wondering whether the energy responsible for the acceleration is a cosmological constant ($\gamma = 0$, $\rho_\Lambda = \text{const}$), or another exotic kind of matter, usually dubbed dark energy, with an equation of state that lies in the range $2/3 > \gamma_X > 0$ and $\rho_X \neq \text{const}$ (we shall use an $X$ to denote dark energy quantities) [1, 2, 3].

For concreteness, we will next work on the case of a CM universe in which vacuum energy is changed by a dark energy component with a constant equation of state $\gamma_X = 1/3$. It should be warned that this is not the most general case of dark energy, but we consider a constant $\gamma_X$ for pedagogical reasons only.
The effective gravitational potential and the evolution of the density parameters are shown in Fig. 3. As in the standard CM model, there is a maximum in the effective gravitational potential, but the behavior at late times differs from that of the CM. This time the maximum of the potential is at \( a_c \approx \sqrt{\Omega_{m,0}/\Omega_X,0} \approx 0.65 \). That is, it takes longer for this universe to have a dark energy dominated, and then accelerated, stage than in the case of a cosmological constant. This was to be expected, as we mentioned before that a cosmological constant is an extreme case for the equation of state of a perfect fluid.

### TABLE III: Critical points for the dynamical system of a universe containing a dark energy component.

| Domination | \( \Omega_k \) | \( x_0 \) | Stability |
|------------|----------------|------------|-----------|
| Radiation  | 0              | (1, 0, 0)  | Unstable  |
| Matter     | 0              | (0, 1, 0)  | Saddle    |
| \( X \)    | 0              | (0, 0, 1)  | Stable    |

### TABLE IV: Eigenvalues of the perturbation matrix corresponding to the critical points shown in Table III.

| Domination | \( \omega_1 \) | \( \omega_2 \) | \( \omega_3 \) |
|------------|----------------|----------------|----------------|
| Radiation  | 1              | 4              | 3              |
| Matter     | -1             | 3              | 2              |
| \( X \)    | -3             | -2             | -1             |

### VI. FINAL COMMENTS

We stated at the beginning of this manuscript our intention to understand the solutions of Einstein’s equations in the case of a homogeneous and isotropic universe. After all of the mathematical work done in the previous sections, we can extract some general conclusions on how the expansion of the universe depend on its material content and curvature.

Assuming all of the energy densities in the universe are positive definite, the effective gravitational potential is negative definite. Therefore,

1. An open universe will expand for ever, irrespective of the material content and of how large is its negative curvature. If there is a component with an equation of state such that \( \gamma < 2/3 \), an open universe will enter an accelerated stage at late times. Otherwise, it will decelerate but the expansion rate will never vanish.
2. A flat universe will always expand, and will accelerate its expansion if there is a component with an equation of state such that \( \gamma < 2/3 \). Otherwise, the expansion rate will asymptotically vanish (\( \dot{a} \rightarrow 0 \) as \( t \rightarrow \infty \)).

3. A closed universe will have an accelerated expansion if there is a component with an equation of state such that \( \gamma < 2/3 \), and if its curvature is such that \( \Omega_{k,0} < 2V(a_0) \). If at least one of the previous conditions is not accomplished, then the universe will recollapse.

Some other comments are in turn. It was shown how the dynamics of the universe can be studied using two complementary approaches. For instance, the gravitational potential method gave a clear proof of the instability of the Einstein's static universe, whereas the dynamical potential method revealed the stability nature of the different domination stages our universe has passed through. It is our hope that the two methods revised in this paper can help undergraduate students to deal with the expansion of a homogeneous and isotropic universe. Moreover, we would like to stress the fact that such methods are taken seriously for the analysis of a great variety of cosmological models. Already mentioned was the use of the 'gravitational potential' method in \[12\] \[13\]. On the other hand, the use of dynamical systems is widely known in the specialized literature; for a non-exhaustive list of examples, see Refs. \[23\] \[24\] \[25\] \[26\] \[27\] \[28\] and references there in.

In our opinion, it is important to teach undergraduate students different methods and techniques currently used in specialized research. For we cannot say to what extent the methods presented in this work can be used and generalized to other equations in Cosmology; but that we shall know from the work of future cosmologists.

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[35] Nevertheless, the case for \( \gamma < 0 \) cannot be discarded. A fluid with such an equation of state is called phantom energy, see for instance \[55\].
[36] Though the labeling may resemble that of 2-dimensional surfaces, we should keep in mind that the spatial part of metric \[1\] refers to 3-dimensional hypersurfaces.
[37] 1 pc = 3.2 light-years.
[38] The current particle horizon, which is the distance light has traveled since the Big Bang up to date, is of the order of 3.3 \( d_{H,0} \). Notice that this distance is larger than the age of the universe multiplied by the velocity of light, see for instance the discussion on this topic in \[10\].
[39] The equation of motion \[1\] is well known to textbooks,
see for example\cite{6,13}, the detailed discussion in\cite{12}, and the Wikipedia text in\cite{34}. However, its pedagogical properties have not been fully exploited even though Eq. (9) is indeed widely used as a serious research tool in the specialized literature\cite{31}.


d\cite{40} If we think of the density parameters as the coordinates of a $n + 1$-dimensional Cartesian space, the Friedmann equation (7) then forces the universe to 'move' on a particular hyperplane only.