Improved nonparametric estimation of the drift in diffusion processes

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Abstract. In this paper, we consider the robust adaptive non parametric estimation problem for the drift coefficient in diffusion processes. An adaptive model selection procedure, based on the improved weighted least square estimates, is proposed. Sharp oracle inequalities for the robust risk have been obtained.

Keywords: Improved estimation, stochastic diffusion process, mean-square accuracy, model selection, sharp oracle inequality.

1 Introduction

Let \((\Omega, \mathcal{F}, (\mathcal{F})_{t \geq 0}, P)\) be a filtered probability space on which the following stochastic differential equation is defined:

\[
y_t = S(y_t) t + w_t, \quad 0 \leq t \leq T,
\]

where \((w_t)_{t \geq 0}\) is a scalar standard Wiener process, the initial value \(y_0\) is some given constant, and \(S(\cdot)\) is an unknown function. The problem is to estimate the function \(S(x), x \in [a, b]\), from the observations \((y_t)_{0 \leq t \leq T}\). The calibration problem for the model (1) is important in various applications. In particular, it appears, when constructing optimal strategies for investor behavior in diffusion financial markets. It is known that the optimal strategy depends on unknown market parameters, in particular, on unknown drift coefficient \(S\). Therefore, in practical financial calculations it is necessary to use statistical estimates for the function \(S\) which are reliable on some fixed time interval \([0, T]\) [6]. Earlier, the problem of non-asymptotic estimation of the parameters of diffusion processes was studied in [9]. Here it was shown that many difficulties of asymptotic estimation of parameters for one-dimensional diffusion processes can be overcome by using a sequential approach. It turns out that the theoretical analysis of successive estimates is simpler than the analysis of classical procedures. In particular, it is possible to calculate non-asymptotic bounds for quadratic risk. Owing to the use of a sequential approach, the problems of non-asymptotic estimation of parameters were studied in [1] for multidimensional diffusion processes and recently in [2] for multidimensional continuous and discrete semimartingales. In [7] a truncated sequential method for estimating the parameters of diffusion processes was developed. Now about nonparametric estimation. A consistent approach to nonparametric criteria for minimax estimation of the drift coefficient in (ergodic) diffusion processes was developed in [3]. In this article, sequential pointwise kernel estimates are considered. For such estimates, non-asymptotic upper bounds of the root-mean-square risk are obtained, and these estimates give the optimal convergence rate as \(T \to \infty\).

This paper deals with the estimating the unknown function \(S(x), a \leq x \leq b\), in the sense of the mean square risk

\[
\mathcal{R}(\hat{S}_T, S) = \mathbb{E}_S ||\hat{S}_T - S||^2, \quad ||S||^2 = \int_a^b S^2(x) dx,
\]

(2)
where \( \hat{S}_T \) is the estimate of \( S \) by observations \((y_t)_{0 \leq t \leq T}, a < b \) are some real numbers. Here \( \mathbb{E}_S \) is the expectation with respect to the distribution \( P_S \) of the random process \((y_t)_{0 \leq t \leq T} \) given the drift function \( S \).

The goal of this paper is to construct an adaptive estimate \( \hat{S}^* \) of the drift coefficient \( S \) in (1) and to show that the quadratic risk of this estimate is less then the one of the estimate proposed in [3], i.e. we construct the improved estimate in the mean square accuracy sense. For this we use the improved estimation approach proposed in [10] and [8] for parametric regression models and recently developted in [11] for a nonparametric estimation problem. Moreover in this paper we consider the estimation problem in adaptive setting, i.e. when the regualry of \( S \) is unknown. For this we use a model selection method proposed in [4]. Such approach provides adaptive solution for the nonparametric estimation through oracle inequalities which give the nonparametric upper bound for the quadratic risk of estimate.

The rest of the paper is organized as follows. In section 2 we reduce the initial problem to an estimation problem in a discrete time nonparametric regression model. In section 3 we construct the improved weighted least square estimates. In section 4 the sharpc nonasymptotic oracle inequality for quadratic risk of model selection procedure is given.

2 Passage to a discrete time regression model

To obtain a reliable estimate of the function \( S \), it is necessary to impose on it certain conditions that are analogous to the periodicity of the deterministic signal in the white noise model [5]. One of the conditions sufficient for this purpose is the assumption that the process \((y_t)_{t \geq 0} \) in (1) returns to any neighborhood of each points \( x \in [a, b] \). As in [3] to get the ergodicity of the process (1) we define the following functional class:

\[
\Sigma_{L,N} = \{ S \in \text{Lip}_L(\mathbb{R}) : |S(N)| \leq L ; \forall |x| \geq N, \exists \dot{S}(x) \in C(\mathbb{R}) \\
\text{such that } -L \leq \inf_{|x| \geq N} \dot{S}(x) \leq \sup_{|x| \geq N} \dot{S}(x) \leq -1/L \},
\]

where \( L > 1, N > |a| + |b|, \dot{S}(x) - \text{derivative } S(x), \)

\[
\text{Lip}_L(\mathbb{R}) = \left\{ f \in C(\mathbb{R}) : \sup_{x,y \in \mathbb{R}} \frac{|f(x) - f(y)|}{|x - y|} \leq L \right\}.
\]

We note that if \( S \in \Sigma_{L,N} \), then there exists an invariant density

\[
q(x) = q_S(x) = \frac{\exp\{2 \int_0^x S(z)zdz\}}{\int_{-\infty}^{+\infty} \exp\{2 \int_0^y S(z)zdz\}dy}.
\]

We note that the functions in \( \Sigma_{L,N} \) are uniformly bounded on \([a, b]\), i.e.

\[
s^* = \sup_{a \leq x \leq b} \sup_{S \in \Sigma_{L,N}} S^2(x) < \infty.
\]

We start with the partition of the interval \([a, b]\) by the points \((x_k)_{1 \leq k \leq n} \), defined as

\[
x_k = a + \frac{k}{n}(b - a),
\]
where $n = n(T)$ is an integer-valued function of $T$ such that

$$n(T) \leq T \quad \text{and} \quad \lim_{T \to \infty} \frac{n(T)}{T} = 1.$$  \hfill (6)

Now at any point $x_k$ we estimate the function $S$ by a sequential kernel estimation. We fix some $0 < t_0 < T$ and put

$$\tau_k = \inf \left\{ t \geq t_0 : \int_{t_0}^{t} Q \left( \frac{y_s - x_k}{h} \right) \xi_s \geq H_k \right\};$$

$$\tilde{S}_k = \frac{1}{H_k} \int_{t_0}^{\tau_k} Q \left( \frac{y_s - x_k}{h} \right) \xi_s,$$  \hfill (7)

where $Q(z) = 1_{|z| \leq 1}$, $1_A$ is an indicator of the set $A$, $h = (b - a)/(2n)$ and $H_k$ is a positive threshold, which will be indicated below. From (1) it is easy to obtain that

$$\tilde{S}_k = S(x_k) + \zeta_k.$$  

The error $\zeta_k$ is represented as a sum of the approximating and stochastic parts, i.e.

$$\zeta_k = B_k + \frac{1}{\sqrt{H_k}} \xi_k, \quad B_k = \frac{1}{H_k} \int_{t_0}^{\tau_k} Q \left( \frac{y_s - x_k}{h} \right) \Delta S(y_s, x_k) \xi_s,$$

where $\Delta S(y, x) = S(y) - S(x)$ and

$$\xi_k = \frac{1}{\sqrt{H_k}} \int_{t_0}^{\tau_k} Q \left( \frac{y_s - x_k}{h} \right) \xi_s.$$  

Taking into account that $S$ is Lipshitz function, we obtain an upper bound for the approximating part as

$$|B_k| \leq Lh.$$  

It is easy to see that random variables $(\xi_k)_{1 \leq k \leq n}$ are independent identically distributed from $\mathcal{N}(0, 1)$. In [3] it is established that an effective kernel estimate of the form (7) has a stochastic part distributed as $\mathcal{N}(0, 2Thq_S(x_k))$, where $q_S(x_k)$ is the ergodic density defined in (4). Therefore, for an effective estimate at each point $x_k$ by the kernel estimate (7), we need to estimate the density (4) from observations $(y_t)_{0 \leq t \leq t_0}$. To this end, we establish that

$$\tilde{q}_T(x_k) = \max \{ \tilde{q}(x_k), \epsilon_T \},$$

where $\epsilon_T$ is positive, $0 < \epsilon_T < 1$,

$$\tilde{q}(x_k) = \frac{1}{2t_0h} \int_{t_0}^{t_0} Q \left( \frac{y_s - x_k}{h} \right) \xi_s.$$  

Now choose the threshold $H_k$ in (7):

$$H_k = (T - t_0)(2\tilde{q}_T(x_k) - \epsilon_T^2)h.$$  

Suppose that the parameters $t_0 = t_0(T)$ and $\epsilon_T$ satisfy the following conditions:
For any $T \geq 32$, 
\[ 16 \leq t_0 \leq T/2 \quad \text{and} \quad \sqrt{2/t_0^{1/8}} \leq \epsilon_T \leq 1. \]

**H2)** \[ \lim_{T \to \infty} t_0(T) = \infty, \quad \lim_{T \to \infty} \epsilon_T = 0, \quad \lim_{T \to \infty} T\epsilon_T/t_0(T) = \infty. \]

**H3)** For any $\nu > 0$ and $m > 0$, 
\[ \lim_{T \to \infty} T\epsilon^m_T = \infty \quad \text{and} \quad \lim_{T \to \infty} T^m e^{-\nu \sqrt{t_0}} = 0. \]

For example, for $T \geq 32$, 
\[ t_0 = \max\{\min\{\ln^4 T, T/2\}, 16\} \quad \text{and} \quad \epsilon_T = \sqrt{2} t_0^{-1/8}. \]

Let 
\[ \Gamma = \left\{ \max_{1 \leq l \leq n} \tau_l \leq T \right\} \quad \text{and} \quad Y_k = \tilde{S}_k 1_\Gamma. \] (8)

Then on the set $\Gamma$ there exists a temporary heteroscedastic regression model
\[ Y_k = S(x_k) + \zeta_k, \quad \zeta_k = \sigma_k \xi_k + \delta_k \] (9)
with $\delta_k = B_k$ and
\[ \sigma^2_k = \frac{1}{(T-t_0)(\tilde{q}_T(x_k) - \epsilon^2_T/2)(b-a)}. \]

It should be noted that from (6) and $H_1$), we get the following upper bound
\[ \max_{1 \leq k \leq n} \sigma^2_k \leq \frac{4}{(b-a)\epsilon_T} = \sigma_\ast \] (10)
for which, by condition $H_3$),
\[ \lim_{T \to \infty} \frac{\sigma_\ast}{T^m} = 0 \quad \text{for any} \quad m > 0. \]

To estimate the $S$ function from the observations of (9) should study some properties of the set $\Gamma$ in (8).

**Proposition 1.** Suppose that the parameters $t_0$ and $\epsilon_T$ satisfy the following conditions: $H_1$) – $H_3$). Then 
\[ \sup_{S \in \Sigma_{L,N}} P_S(\Gamma^c) \leq \Pi_T, \]
where $\lim_{T \to \infty} T^m \Pi_T = 0$ for any $m > 0$.

### 3 Improved estimates

In this section we consider the estimation problem for the model (9). The function $S(\cdot)$ is unknown and has to be estimated from observations $Y_1, \ldots, Y_n$.

The accuracy of any estimator $\hat{S}$ will be measured by the empirical squared error of the form
\[ \| \hat{S} - S \|^2_n = (\hat{S} - S, \hat{S} - S)_n = \frac{b-a}{n} \sum_{i=1}^n (\hat{S}(x_i) - S(x_i))^2. \]
Now we fix a basis \((\phi_j)_{1 \leq j \leq n}\) which is orthonormal for the empirical inner product:

\[
(\phi_i, \phi_j)_n = \frac{b-a}{n} \sum_{l=1}^{n} \phi_i(x_l) \phi_j(x_l) = \text{Kr}_{ij},
\]

where \(\text{Kr}_{ij}\) is Kronecker’s symbol. By making use of this basis we apply the discrete Fourier transformation to (9) and we obtain the Fourier coefficients

\[
\hat{\theta}_{j,n} = \frac{b-a}{n} \sum_{l=1}^{n} Y_l \phi_j(x_l), \quad \theta_{j,n} = \frac{b-a}{n} \sum_{l=1}^{n} S(x_l) \phi_j(x_l).
\]

From (9) it follows directly that these Fourier coefficients satisfy the following equation

\[
\hat{\theta}_{j,n} = \theta_{j,n} + \zeta_{j,n} \quad \text{with} \quad \zeta_{j,n} = \sqrt{\frac{b-a}{n}} \xi_{j,n} + \delta_{j,n},
\]

where

\[
\xi_{j,n} = \sqrt{\frac{b-a}{n} \sum_{l=1}^{n} \sigma_l \xi_l \phi_j(x_l)} \quad \text{and} \quad \delta_{j,n} = \frac{b-a}{n} \sum_{l=1}^{n} \delta_l \phi_j(x_l).
\]

Note that the upper bound (10) and the Bonyakovskii-Cauchy-Schwarz inequality imply that

\[
|\delta_{j,n}| \leq \|\delta\|_n \|\phi_j\|_n = \|\delta\|_n.
\]

We estimate the function \(S\) in (9) on the sieve (5) by the weighted least squares estimator

\[
\hat{S}_\lambda(x_l) = \sum_{j=1}^{n} \lambda(j) \hat{\theta}_{j,n} \phi_j(x_l) 1_F, \quad 1 \leq l \leq n,
\]

where the weight vector \(\lambda = (\lambda(1), \ldots, \lambda(n))\) belongs to some finite set \(\Lambda \subset [0, 1]^n\). We set for any \(a \leq x \leq b\)

\[
\hat{S}_\lambda(x) = \hat{S}_\lambda(x_1) 1_{a \leq x \leq x_1} + \sum_{l=2}^{n} \hat{S}_\lambda(x_l) 1_{x_{l-1} < x \leq x_l}.
\]

Further we suppose that the first \(d \leq n\) components of the weight vector \(\lambda\) are equal to 1, i.e. \(\lambda(j) = 1\) for any \(1 \leq j \leq d\).

We consider a new estimate for the function \(S\) in (9) of the form

\[
S^*_\lambda(x_l) = \sum_{j=1}^{n} \lambda(j) \theta^*_{j,n} \phi_j(x_l) 1_F, \quad 1 \leq l \leq n,
\]

where

\[
\theta^*_{j,n} = \left(1 - \frac{c(d)}{\|\theta_n\|_1 1_{1 \leq j \leq d}}\right) \hat{\theta}_{j,n},
\]

where

\[
c(d) = \frac{(d-1)\sigma^2 \text{L}(b-a)^{1/2}}{n(s^* + \sqrt{d} \sigma_s / n)}, \quad \|\theta_n\|^2 = \sum_{j=1}^{d} \theta^2_{j,n}.
\]
Now we define the estimate for $S$ in (1). We set for any $a \leq x \leq b$

$$S_\lambda^*(x) = S_\lambda^*(x_1)1_{\{a \leq x \leq x_1\}} + \sum_{l=2}^{n} S_\lambda^*(x_l)1_{\{x_{l-1} < x \leq x_l\}}. \quad (12)$$

We denote the difference of quadratic risks of the estimates (12) and (11) as

$$\Delta_n(S) := E_S \|S_\lambda^* - S\|_n^2 - E_S \|\hat{S}_\lambda - S\|_n^2.$$  

The choice of estimate (12) is motivated by the desire to control the quadratic risk.

**Theorem 1.** The estimate (12) outperforms in mean square accuracy the estimate (11), i.e.

$$\sup_{S \in \Sigma_{L,N}} \Delta_n(S) < -c^2(d).$$

### 4 Oracle inequalities

In order to obtain a good estimator, we have to write a rule to choose a weight vector $\lambda \in \Lambda$ in (12). It is obvious, that the best way is to minimize the empirical squared error with respect to $\lambda$:

$$\text{Err}_n(\lambda) = \|S_\lambda^* - S\|_n^2 \to \min.$$  

Making use of (12) and the Fourier transformation of $S$ imply

$$\text{Err}_n(\lambda) = \sum_{j=1}^{n} \lambda^2(j)\theta_{j,n}^2 - 2 \sum_{j=1}^{n} \lambda(j)\theta_j^* \theta_{j,n} + \sum_{j=1}^{n} \theta_{j,n}^2.$$  

Since the coefficient $\theta_{j,n}$ is unknown, we need to replace the term $\theta_j^* \theta_{j,n}$ by some its estimator which we choose as

$$\tilde{\theta}_{j,n} = \hat{\theta}_{j,n} \theta_j^* - \frac{b-a}{n} s_{j,n} \quad \text{with} \quad s_{j,n} = \frac{b-a}{n} \sum_{l=1}^{n} \sigma_l^2 \phi_j^2(x_l).$$  

One has to pay a penalty for this substitution in the empirical squared error. Finally, we define the cost function of the form

$$J_n(\lambda) = \sum_{j=1}^{n} \lambda^2(j)\theta_{j,n}^2 - 2 \sum_{j=1}^{n} \lambda(j)\tilde{\theta}_{j,n} + \rho P_n(\lambda),$$  

where the penalty term is defined as

$$P_n(\lambda) = \frac{b-a}{n} \sum_{j=1}^{n} \lambda^2(j)s_{j,n}$$  

and $0 < \rho < 1$ is some positive constant which will be chosen later. We set

$$\hat{\lambda} = \arg\min_{\lambda \in \Lambda} J_n(\lambda)$$  

and define an estimator of $S$ of the form (11):

$$S^*(x) = S_\hat{\lambda}\lambda^*(x) \quad \text{for} \quad a \leq x \leq b. \quad (13)$$  

Now we obtain the non asymptotic upper bound for the quadratical risk of the estimator (13).
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Theorem 2. Let \( \Lambda \subset \{0,1\}^n \) be any finite set such that the first \( d \leq n \) components of the weight vector \( \lambda \) are equal to 1. Then, for any \( n \geq 3 \) and \( 0 < \rho < 1/6 \), the estimator (13) satisfies the following oracle inequality

\[
E_S \| S^* - S \|_n^2 \leq \frac{1 + 6\rho}{1 - 6\rho} \min_{\lambda \in \Lambda} E_S \| \hat{S}_\lambda - S \|_n^2 + \frac{\Psi_n(\rho)}{n},
\]

where \( \lim_{n \to \infty} \Psi_n(\rho)/n = 0 \).

Now we consider the estimation problem (11) via model (12). We apply the estimating procedure (13) with special weight set introduced in [3] to the regression scheme (9). Denoting \( S^*_\alpha = S^*_\lambda^\alpha \) we set

\[
S^* = S^*_\hat{\alpha} \quad \text{with} \quad \hat{\alpha} = \arg\min_{\alpha \in A} \epsilon J_n(\lambda_\alpha).
\]

We obtain through Theorem 2 the following oracle inequality.

Theorem 3. Assume that \( S \in \Sigma_{L,N} \) and the number of the points \( n = n(T) \) in the model (9) satisfies (6). Then the procedure \( S^* \) satisfies, for any \( T \geq 32 \), the following inequality

\[
\mathcal{R}(S^*, S) \leq \frac{(1 + \rho)^2(1 + 6\rho)}{1 - 6\rho} \min_{\alpha \in A} \mathcal{R}(S^*_\alpha, S) + \frac{\mathcal{B}_T(\rho)}{n},
\]

where \( \lim_{T \to \infty} \mathcal{B}_T(\rho)/n(T) = 0 \).

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