Anomalous specific heat in high-density QED and QCD

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Long-range quasi-static gauge-boson interactions lead to anomalous (non-Fermi-liquid) behavior of the specific heat in the low-temperature limit of an electron or quark gas with a leading $T \ln T^{-1}$ term. We obtain perturbative results beyond the leading log approximation and find that dynamical screening gives rise to a low-temperature series involving also anomalous fractional powers $T^{(3+2n)/3}$. We determine their coefficients in perturbation theory up to and including order $T^{7/3}$ and compare with exact numerical results obtained in the large-$N_f$ limit of QED and QCD.

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It has been established long ago in the context of a nonrelativistic electron gas that the only weakly screened low-frequency transverse gauge-boson interactions lead to a qualitative deviation from Fermi liquid behavior. A particular consequence of this is the appearance of an anomalous contribution to the low-temperature limit of entropy and specific heat proportional to $\alpha T \ln T^{-1}$, but it was argued that the effect would be probably too small for experimental detection.

More recently, it has been realized that analogous non-Fermi-liquid behavior in ultradegenerate QCD is of central importance to the magnitude of the gap in color superconductivity, and it has been pointed out that the anomalous contributions to the low-temperature specific heat may be of interest in astrophysical systems such as neutron or protoneutron stars, if they involve a normal (non-superconducting) degenerate quark matter component.

So far only the coefficient of the $\alpha T \ln T^{-1}$ term in the specific heat has been determined (with Ref. correcting the result of Ref. by a factor of 4), but not the complete argument of the leading logarithm. While the existence of the $T \ln T^{-1}$ term implies that there is a temperature range where the entropy or the specific heat exceeds the ideal-gas value, without knowledge of the constants “under the log” it is impossible to give numerical values for the required temperatures.

Furthermore, a quantitative understanding of these anomalous contributions is also of interest with regard to the recent progress made in high-order perturbative calculations of the pressure (free energy) of QCD at nonzero temperature and chemical potential, where it has been found that dimensional reduction techniques work remarkably well except for a narrow strip in the $T$-$\mu$-plane around the $T = 0$ line.

In the present Letter we report the results of a calculation of the low-temperature entropy and specific heat for ultradegenerate QED and QCD which goes beyond the leading log approximation. Besides completing the leading logarithm, we find that for $T/\mu \ll g \ll 1$, where $g$ is either the strong or the electromagnetic coupling constant, the higher terms of the low-temperature series involve also anomalous fractional powers $T^{(3+2n)/3}$, and we give their coefficients through order $T^{7/3}$.

Our starting point is an expression for the thermodynamic potential of QED and QCD...
which becomes exact in the limit of large flavor number $N_f$, and which at finite $N_f$ has an error of order $g^4$ in the regime $T/\mu \ll g$,

\[
P = NN_f \left( \frac{\mu^4}{12\pi^2} + \frac{\mu^2 T^2}{6} + \frac{7\pi^2 T^4}{180} \right) - N_g \int \frac{d^3q}{(2\pi)^3} \int_0^\infty \frac{dq_0}{\pi} \\
\times \left[ 2 \left( [n_b + \frac{1}{2}] \text{Im} \ln D_T^{-1} - \frac{1}{2} \text{Im} \ln D_{\text{vac}}^{-1} \right) + \left( [n_b + \frac{1}{2}] \text{Im} \ln \frac{D_L^{-1}}{q^2 - q_0^2} - \frac{1}{2} \text{Im} \ln \frac{D_{\text{vac}}^{-1}}{q^2 - q_0^2} \right) \right] + O(g^4 \mu^4), \quad (T/\mu \ll g)
\]

where $N = 3$, $N_g = 8$ for QCD, and both equal to one for QED. $D_T$ and $D_L$ are the spatially transverse and longitudinal gauge boson propagators at finite temperature $T$ and (electron or quark) chemical potential $\mu$ obtained by Dyson-resumming one-loop fermion loops, and $D_{\text{vac}}$ is the corresponding quantity at zero temperature and chemical potential.

Nonanalytic terms in the low-temperature expansion arise from the contribution

\[
\frac{P_{T,n_b}}{N_g} = - \int \frac{d^3q}{(2\pi)^3} \int_0^\infty \frac{dq_0}{\pi} 2n_b \text{Im} \ln D_T^{-1}.
\]

The bosonic distribution function $n_b = 1/\left[ \exp(q_0/T) - 1 \right]$ restricts the $q_0$ integration to $q_0 \lesssim T$ and $T$ is assumed to be the smallest mass scale in the problem. Consistently dropping contributions proportional to $T^4$ in the pressure ($T^3$ in the entropy), which for $T/\mu \ll g$ are beyond our perturbative accuracy, it turns out that we only need the $T \to 0$ limit of the inverse propagator $D_T^{-1}$, and only the lowest orders in $q_0/q$ and $q_0/\mu$:

\[
\text{Re } D_T^{-1} = q^2 \left( 1 + O(g_{\text{eff}}^2) \right) + \left( \frac{g_{\text{eff}}^2 \mu^2}{\pi^2 q^2} - 1 + O(g_{\text{eff}}^2 q^0) + O(g_{\text{eff}}^2 q^2/\mu^2) \right) q_0^2 + O(g_{\text{eff}}^4 q_0^4),
\]

\[
\text{Im } D_T^{-1} = - \frac{g_{\text{eff}} q_0}{48\pi q^3} \left( q^2 - q_0^2 \right) \left( 12\mu^2 + 3q^2 + q_0^2 \right) \theta(2\mu - q)
\]

where $g_{\text{eff}}^2 = g^2 N_f$ for QED and $g^2 N_f/2$ for QCD.

Keeping only the leading terms in the limit $q_0 \to 0$ gives

\[
\text{Im } \ln D_T^{-1} \simeq \text{arctan} \frac{-g_{\text{eff}}^2 (4\mu^2 + q^2) q_0 \theta(2\mu - q)}{16\pi q^3}.
\]

Inserting this approximation into Eq. (2) leads to the integral

\[
\int_0^{2\mu} dq q^2 \text{arctan} \frac{q_0 (4\mu^2 + q^2)}{q^3} \simeq \frac{4\mu^2}{3} \frac{\ln (2\mu/q_0 + 5/2)}{q_0} + O(q_0^{5/3}).
\]
Performing the $q_0$ integration then gives the following contribution to the entropy $S = \partial P/\partial T$ (per unit volume):

$$\frac{S_{T,n_b}}{N_g} = \frac{g_{\text{eff}}^2 \mu^2 T}{36\pi^2} \left( \ln \frac{32\pi \mu}{g_{\text{eff}}^2 T} + 1 + \gamma_E - \frac{6}{\pi^2} \zeta'(2) \right) + O(T^{5/3}). \quad (7)$$

While this reproduces the coefficient of the anomalous $T \ln T^{-1}$ term reported in [3], the coefficient under the logarithm as well as the suppressed $O(T^{5/3})$-contribution are still incomplete.

To complete the term linear in $T$, one has to perform an exactly analogous calculation of the longitudinal contribution, which involves

$$\text{Im} \ln D_L^{-1} \simeq \frac{g_{\text{eff}}^2 (4\mu^2 - q^2)^2 q_0 \theta(2\mu - q)/\sqrt{8\pi q}}{q^2 + (g_{\text{eff}}^2 \mu^2)/\pi^2}, \quad (8)$$

and when inserted into (11) contributes

$$\frac{S_{L,n_b}}{N_g} = \frac{g_{\text{eff}}^2 \mu^2 T}{24\pi^2} \left( \ln \frac{g_{\text{eff}}^2}{4\pi^2} + 1 \right) + O(g_{\text{eff}}^4) + O(T^3). \quad (9)$$

Finally, the remaining parts of (11), which do not involve the bosonic distribution function, yield

$$\frac{S_{\text{non}-n_b}}{N_g} = -\frac{g_{\text{eff}}^2}{8\pi^2} \mu^2 T. \quad (10)$$

The latter contribution matches exactly the one from the standard perturbative result to order $g^2$ [11], while the contributions (7) and (9) depend on having $T/\mu \ll g$. In this region, all of the contributions listed so far are negligible compared to the zero-temperature contribution $\sim g^4 \mu^4$ in the pressure (which is only partially included in (11)). However, by considering instead the entropy (and further below the specific heat), the above contributions become the dominant ones.

Adding them all together, we obtain

$$\frac{S - S_0}{N_g} = \frac{g_{\text{eff}}^2 \mu^2 T}{36\pi^2} \left( \ln \frac{4g_{\text{eff}}\mu}{\pi^2 T} - 2 + \gamma_E - \frac{6}{\pi^2} \zeta'(2) \right) + c_{5/3} T^{5/3} + c_{7/3} T^{7/3} + O(T^3), \quad (11)$$

where $S_0$ is the ideal-gas value of the entropy per unit volume.

To also obtain completely the coefficients of the terms in the low-temperature expansion which involve fractional powers of $T$ we need to include more terms of (3) and (4) than those kept in (11). A lengthy calculation, whose details will be discussed elsewhere, gives

$$c_{5/3} = -\frac{8 \cdot 2^{2/3} \Gamma\left(\frac{8}{9}\right) \zeta\left(\frac{8}{9}\right)}{9\sqrt{3} \pi^{11/3}} (g_{\text{eff}} \mu)^{4/3}, \quad \text{(12)}$$

$$c_{7/3} = \frac{80 \cdot 2^{1/3} \Gamma\left(\frac{10}{3}\right) \zeta\left(\frac{10}{3}\right)}{27\sqrt{3} \pi^{13/3}} (g_{\text{eff}} \mu)^{2/3}. \quad \text{(13)}$$
FIG. 1: Transverse \( n_b \)-contribution to the interaction part of the low-temperature entropy density in the large-\( N_f \) limit for the three values \( g^2_{\text{eff}} = 1, 4, 9 \). The heavy dots give the exact numerical results; the full, dashed, and dash-dotted lines correspond to our perturbative result up to and including the \( T \ln T^{-1} \), \( T^{5/3} \), and \( T^{7/3} \) contributions.

Setting \( T/\mu \sim g^{1+\delta}_{\text{eff}} \) with \( \delta > 0 \), one finds that the terms in the expansion (11) correspond to the orders \( g^{3+\delta}_{\text{eff}} \ln(c/g_{\text{eff}}) \), \( g^{3+(5/3)\delta}_{\text{eff}} \), and \( g^{3+(7/3)\delta}_{\text{eff}} \), respectively, with a truncation error of the order \( g^{3+3\delta}_{\text{eff}} \). Hence, the expansion parameter in this low-temperature series is \( T/(g_{\text{eff}}\mu) \), which is also the scaleless parameter appearing in the argument of the leading logarithm (remarkably however only after the transverse and the longitudinal contributions have been added together). The combination \( g_{\text{eff}}\mu \) is the scale of the Debye mass at high chemical potential, whose leading-order value is \( m_D = g_{\text{eff}}\mu/\pi \). In fact, the calculation of the coefficients in (11) required keeping the leading-order “hard-dense-loop” (HDL) part of the gauge boson propagator [12, 13], in particular the dynamic screening in (4), but also a HDL correction to the real part of the transverse self energy in (3). The above calculation is therefore in a certain sense another application of HDL resummation [13], which thus turns out to be necessary also for a perturbative treatment of the low-temperature regime \( T/\mu \ll g \).

As a check on our result and also as a test of its convergence properties, we compare the anomalous transverse contributions \( S_{T,n_b} \) with those of the exactly (albeit only numerically) solvable large-\( N_f \) limit [10] in Fig. 1. We find good convergence to the exact result as long as \( T/\mu \lesssim g_{\text{eff}}/(2\pi^2) \). This is also the region where the complete large-\( N_f \) result for the low-temperature entropy [10] has the anomalous property of exceeding the ideal-gas value.

Our results do not, however, seem to agree with the results of Ref. [7] which recently questioned the presence of a term \( \propto \alpha T \ln T^{-1} \). The (renormalization group resummed) result reported therein rather corresponds to a leading nonanalytic \( \alpha T^3 \ln T \) term when expanded out perturbatively, which is in fact the type of nonanalytic terms that appear already in regular Fermi-liquids [14].

For potential phenomenological applications in astrophysical systems, the specific heat \( C_v \) at constant volume and number density is of more direct interest than the entropy density that we have calculated so far. The former (per unit volume) is given by [15]

\[
C_v = T \left\{ \left( \frac{\partial S}{\partial T} \right)_\mu - \left( \frac{\partial N}{\partial T} \right)_\mu \left( \frac{\partial N}{\partial \mu} \right)_T^{-1} \right\},
\]

(14)
FIG. 2: The perturbative result for the specific heat, normalized to the ideal-gas value, to order \( T^{5/3} \) and \( T^{7/3} \) (lower and upper curves, respectively) for two particular values of \( \alpha_s \) in two-flavor QCD (chosen for comparability to Ref. [7]) and \( g_{\text{eff}} \approx 0.303 \) for QED. The deviation of the QED result from the ideal-gas value is enlarged by a factor of 20, and the plot terminates where the expansion parameter \( (\pi^2 T)/(g_{\text{eff}} \mu) \approx 1 \).

where \( \mathcal{N} \) is the number density, but to the order of accuracy of our expansions, \( C_v \) can be simply obtained as the logarithmic derivative of the entropy:

\[
C_v = T \left( \frac{\partial S}{\partial T} \right)_\mu + O(T^3). \tag{15}
\]

Explicitly, the result is

\[
\frac{C_v - C_v^0}{N_g} = \frac{g_{\text{eff}}^2 \mu^2 T}{36 \pi^2} \left( \ln \frac{4 g_{\text{eff}} \mu}{\pi^2 T} - 3 + \gamma_E - \frac{6}{\pi^2} \zeta'(2) \right) + \frac{5}{3} c_{5/3} T^{5/3} + \frac{7}{3} c_{7/3} T^{7/3} + O(T^3). \tag{16}
\]

with \( C_v^0 = N N_f \mu^2 T/3 + O(T^3) \), and \( c_{5/3}, c_{7/3} \) given by Eqs. [12], [13].

For illustrative purposes, we evaluate the ratio of \( C_v \), as given by (16) to the ideal-gas value \( C_v^0 \) for QCD with two massless quark flavors in Fig. 2 using alternatively two values for \( \alpha_s \) which have been used also in Ref. [7] and which correspond to one-loop running couplings with renormalization point 0.5 GeV (full line) and 1 GeV (dashed line). The shaded bands shown are limited from below and above by the results to order \( T^{5/3} \) and \( T^{7/3} \), respectively, and thus indicate the quality of the low-temperature expansion. One may interpret these results as roughly corresponding to QCD with a quark chemical potential of 0.5 GeV and the total variation corresponding to different renormalization schemes with minimal subtraction scale varied between \( \mu \) and \( 2\mu \). The critical temperature for the color superconducting phase transition may be anywhere between 6 and 60 MeV [16], so the range \( T/\mu \geq 0.012 \) in Fig. 2 might correspond to normal quark matter. While it is certainly questionable to apply perturbative results for \( \alpha_s \gtrsim 0.65 \), Fig. 2 suggests that the anomalous feature of an excess of the specific heat over its ideal-gas value may possibly come into play in astrophysical situations, in particular in the cooling of (proto-)neutron stars [17, 18].
contrasted with the ordinary perturbative estimate for $C_v/C_v^0$ based on the well-known exchange term $\propto g^2$ (which, as we have shown, requires $T/\mu \gg g$). The latter would predict $C_v/C_v^0 \lesssim 0.6$ for $\alpha_s \gtrsim 0.65$.

For completeness, we also give the numerical results corresponding to QED, where $g_{\text{eff}} \approx 0.303$. Here the range of temperature, where the specific heat exceeds the ideal-gas value, and the deviations from the latter, are much smaller (the deviations from the ideal-gas value have been enlarged by a factor of 20 in Fig. 2 to make them more visible).

To summarize, we have presented a quantitative evaluation of the leading contributions to the entropy and specific heat of high-density QCD and QED in the regime $T/\mu \ll g \ll 1$, which is dominated by non-Fermi-liquid behavior. While the effect remains small in QED, it seems conceivable that the anomalous terms in the specific heat play a noticeable role in the thermodynamics of a normal quark matter component of neutron or proto-neutron stars.

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