Exact dynamics for fully connected nonlinear networks

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We investigate the dynamics of the discrete nonlinear Schrödinger equation in fully connected networks. For a localized initial condition the exact solution shows the existence of two dynamical transitions as a function of the nonlinearity parameter, a hyperbolic and a trigonometric one. In the latter the network behaves exactly as the corresponding linear one but with a renormalized frequency.

I. INTRODUCTION

Nonlinear dynamics in complex networks incorporates competition of propagation, nonlinearity and bond disorder and may find some applications in complex natural or man-made materials. For a given random network of $N$ sites the limit of fully coupled lattice where each site is connected to every other one with the same strength plays an important role since, for zero nonlinearity, introduces a high degree of degeneracy and thus localization [1]. It is therefore interesting to probe the dynamics in this fully coupled limit when the network is nonlinear. We use the Discrete Nonlinear Schrödinger (DNLS) equation for this study since the latter is a prototypical equation with a large number of applications in condensed matter physics, optics, Bose-Einstein condensation, etc. [2–4].

The DNLS equation in the fully coupled or Mean Field (MF) limit is given by:

\[ i \frac{d\psi_n}{dt} = \varepsilon_n \psi_n + V \sum_{m \neq n} \psi_m - \gamma |\psi_n|^2 \psi_n \]  

(1)

where $\psi_n(t)$ is a complex amplitude, $V$ denotes the constant overlap integral connecting any two sites, $\varepsilon_n$ a local site value and $\gamma$ is a parameter that controls the strength of the nonlinear term. When $t$ is time, the equation describes the dynamics of interacting particles with the nonlinear term representing polaronic effects or interaction of bosons in an optical lattice. When $t$ is a space variable, the equation describes electric wave modes propagating in fibers that include nonlinearity. In the present analysis we take $\varepsilon_n = 0$ while the DNLS norm equal to one, viz. $\sum |\psi_n|^2 = 1$. For notational simplicity we rescale time to $\tau = Vt$ and $\gamma$ to $\chi = \gamma/V$; we note that for $\chi > 0$ we have the defocusing DNLS equation while for $\chi < 0$ we obtain the focusing case.

II. REVIEW OF EARLIER LITERATURE

The problem of the DNLS equation in fully coupled lattices has been addressed in the past [2, 7–10]. Eilbeck et al. [2] analysed the stationary states of the problem for the general $N$-site case. Molina and Tsironis [7] showed that in the case of $N = 3$, i.e. a symmetric trimer, and for a localized initial condition, a transition occurs for $\chi = -6$ where the evolution ceases to be periodic and the averaged probability becomes equal in all three sites. It was also pointed out that no selftrapping occurs for positive nonlinearity values. This dynamical behaviour is analogous, but not identical, to the well known selftrapping transition of the dimer [5, 6]. Andersen and Kenkre [8] solved analytically the symmetric trimer case for localized as well as symmetrically delocalized initial conditions. They wrote explicitly the time evolution at selftrapping ($\chi = -6$), connected it to the stationary states, calculated the averaged occupation probability of the initially populated site and showed that for a given value of the nonlinearity parameter extreme delocalization occurs. They also pointed out that the nature of selftrapping in the trimer was distinct from that in the dimer case. Subsequently, Andersen and Kenkre [9] analysed the fully connected $N$-site problem using similar methods for localized and partially localized initial conditions and showed that a “migratory” transition takes place in some cases. The latter is related to the fact that localization of the initially non-occupied sites may occur for some parameter values. In [9] a general expression in terms of the Weierstrass elliptic function is given for the time evolution of the occupation probability as well as expressions for some special cases. Additionally the time-averaged occupation probability is analysed. Finally, Molina [10] used an extension of the method of ref. [7] and determined explicitly the values of the nonlinearity parameter at which there is a “transition” for arbitrary values of $N$.

From the previous review of the existing literature it is clear that the problem of nonlinearity induced localization in the fully coupled, MF limit, has been analysed thoroughly and many of the features of the solution have been already discovered. However, we believe that a complete dynamical picture has not emerged yet. Thus we will follow the approach of ref. [9], re-derive the complete dynamical solution and use it to focus on the explicit dynamics and not just to the average occupation probability.

III. NONLINEAR LOCALIZATION

In the purely linear case for $\chi = 0$, Eq. (1) reduces to a linear eigenvalue problem that has $N - 1$ fold degenerate eigenvalues [1]. This high degree of degeneracy forces an initially spatially localized state to remain localized executing incomplete oscillations to the other, initially unoccupied, sites. This linear localization at the initially excited site increases with the system size.
A. Nonlinear mean field limit

For the localized initial condition we observe that if we set initially all probability at site one, then due to the symmetry of the lattice, the subsequent evolution of all sites except the initially populated one will be identical; as a result we can cast the original $N$-equation problem to a simpler one consisting of only two equations, viz. [7–10]

\begin{align*}
  i\dot{\psi}_1 &= (N-1)\psi_2 - \chi|\psi_1|^2\psi_1 \\
  i\dot{\psi}_2 &= \psi_1 + (N-2)\psi_2 - \chi|\psi_2|^2\psi_2
\end{align*}

(2) (3)

We now form the invariants $g_2, g_3$:

\begin{align*}
  g_2 &= a_0a_4 - 4a_1a_3 + 3a_2^2 \\
  g_3 &= a_0a_2a_4 + 2a_1a_2a_3 - a_3^2 - a_0a_3^2 - a_2^2a_4
\end{align*}

(12) (13)

The solution is given in terms of the Weierstrass elliptic function $\wp(t, g_2, g_3)$ as follows [11]:

\[ \wp(\tau) = 1 - \frac{24(N-1)}{12\wp(\tau, g_2, g_3) + \chi^2 + 2(N-2)\chi + N^2} \]

(14)

This solution depends only on the nonlinearity parameter $\chi$ and the number of sites $N$ and presents the exact solution of the nonlinear DNLS problem in the MF limit for localized initial condition and arbitrary site number; it is equivalent to the solution in ref. [9] for perfect initial localization on a single site. The Weierstrass function is doubly periodic with periods $2\omega$ and $2\omega'$ where $\omega/\omega'$ is not real. The discriminant $\Delta = g_2^3 - 27g_3^2$ may be expressed as follows

\[ \Delta(N) = -\frac{1}{16}N^2\chi^2g(N)(\chi + 2N) \]

(15)

\[ g(N) = 2(N-2)\chi^3 + 2(N-2)(3N-8)\chi^2 + (6\chi^3 - 53N^2 + 112N - 64)\chi + 2(N-1)N^3 \]

(16)

When the discriminant becomes zero, the Weierstrass solution changes character signalling a change in the physical behaviour of the system. The quantity $\Delta(N)$ has three real roots, denoted hereafter as $\chi_0$, $\chi_c$ and $\chi'_c$ for $\chi_0 = 0$, $g(\chi_c) = 0$ and $\chi'_c = -2N$ respectively.
The root for $\chi_0 = 0$ corresponds to the linear MF problem resulting in $g_2 = N^4/12$, $g_3 = N^6/216$, the Weierstrass function becomes a trigonometric function, viz. $\wp(t, g_2, g_3) = -N^2/12 + N^2/4\sin^2((N\tau/2))$ and the solution for $p(\tau)$ is

$$p(\tau) = p_{\text{lin}}(\tau) = \frac{(N-2)^2 + 4(N-1)\cos(N\tau)}{N^2}$$

(17)
a result that agrees with the linear solutions derived in ref. [1]. We note that in the linear case the system executes oscillations complete, viz. there is always some occupation probability at the origin. In all other geometries the linear system oscillations are incomplete, viz. there is always some occupation probability at the originally populated site. In the limit of very large $N$ there is extreme localization at the initially populated site.

In order to understand the role of $\chi_c$, $\chi_c'$ in the dynamical evolution, it is instructive to address first explicitly the regime for $N \leq 4$; although these cases have been studied in the past, the present exposition furnishes additional results and new intuition.

B. $2 \leq N \leq 4$

In this range we have $\chi_c' \leq \chi_c$ with the equality holding for $N = 4$.

1. $N=2$

In the dimer case we have $\chi_c = 4$ and $\chi_c' = -4$; when $\chi$ assumes one of these two critical values, the invariants become $g_2 = 4/3$, $g_3 = -8/27$, the frequencies $\omega = K(1) = \infty$, $\omega' = iK'(1) = i\pi/2$ and the Weierstrass function becomes a hyperbolic function, viz. $\wp(t, 4/3, -8/27) = 1/3 + 1/\sinh^2 t$ leading to $p(t) = \text{sech}(2t)$. This situation corresponds to the well known selftrapping transition for a localized initial condition where the solution at the transition is expressed as hyperbolic secant [5]. We note that in the dimer case $\chi_c = |\chi_c'|$ and for $-\chi_c < \chi < \chi_c$ the particle executes complete oscillations.

2. $N=3$

In ref. [7] a transition was found occurring at $\chi_0' = -6$, while in refs. [8, 9] the dynamical study was performed. Analysis of the discriminant shows that in addition to the previously identified root $\chi_0' = -2N = -6$ there is also the real root of $g(\chi_c) = 0$ at

$$\chi_c = -\left[\frac{131}{(6614 - 774\sqrt[3]{43})^{1/3}} + \frac{(3307 - 387\sqrt[3]{43})^{1/3}}{3^{2/3}}\right]$$

(18)
or $\chi_c \approx -6.039904249$.

For $\chi = \chi_c' = -6$ the invariants are $g_2 = 3/4$, $g_3 = -1/8$, $\omega$ becomes infinite and the Weierstrass function turns into a hyperbolic function, viz. $\wp(t, 3/4, -1/8) = 1/4 + 3/4|\sinh(\sqrt{3}/4t)|$. The time evolution for $p(\tau)$ becomes

$$p(\tau) = \frac{1}{3} \left[\frac{3 - 4\sin^2(\sqrt{3}/4\tau)}{1 + 4\sin^2(\sqrt{3}/4\tau)}\right]$$

(19)

At long times $p(\tau)_{\text{lim}} \approx -1/3 \approx -0.333$ corresponding to equal probability 1/3 on all three sites and should be compared with $p_{\text{lin}} = -7/9 \approx -0.777$ and $p_{\text{lim}} = 1/9 \approx 0.111$. When $\chi = \chi_c'$ both invariants are positive and the Weierstrass function turns into a simple trigonometric function.

We now describe the dynamics of the trimer as a function of the nonlinearity parameter $\chi$. For $\chi = 0$, $p(\tau)$ performs incomplete oscillations between $p(0) = 1$ and $p_{\text{lin}} = -7/9$. Increasing $\chi$ to positive values augments gradually the effect of localization and the particle stays almost completely at the initial site for large values of the nonlinearity parameter. For negative values of $\chi$, however, the behaviour is markedly different; upon increasing $\chi$ in absolute value, the particle becomes more detrapped and performs oscillations beyond the $p_{\text{lin}}$ limit. This behaviour continues as we further increase $|\chi|$ and at $\chi = -4$ the particle executes complete oscillations to the other sites. Further increase of nonlinearity reduces the amplitude of the oscillatory motion while the oscillation period increases. For $\chi = \chi_c'$ there is a transition to a hyperbolic, non oscillatory behaviour; the particle collapses from complete oscillations to complete equipartition of probability in the three sites (at infinite times). This is a remarkable transition since it localizes again the particle and all motion stops asymptotically [5, 9]. As the value of $|\chi|$ increases further, the trapping tendency increases rapidly and the particle becomes more trapped in the initial site.

As the trimer crosses the value $\chi_c'$ the discriminant becomes zero again with positive invariants; this means that the time evolution simplifies to one with trigonometric evolution. Although this feature is not exceptional in the trimer, it is quite interesting when it appears for $N > 4$; we will comment on this trigonometric transition below.

The transition at $\chi_c'$ is a selftrapping transition in the sense that it forces probability equipartition on all three sites, similar to the corresponding behaviour in the dimer. It is remarkable that nonlinearity first detraps the particle and after reaching a regime of complete oscillatory motion it then forces the particle to selftrapping. As a result nonlinearity first restores the degeneracy that has been lifted by the presence of more than two sites and subsequently acts in way analogous to the dimer case, viz. reaches an equipartition state through a hyperbolic function transition.

3. $N=4$

The tetrahedral configuration obtained for $N = 4$ is rather interesting with $\chi_c = \chi_c' = -8$, $g_2 = g_3 = 0$ and the Weierstrass
function becoming a simple rational function \( p(t, 0, 0) = 1/\tau^2 \)
leading to the solution
\[
p(\tau) = \frac{1 - 2\tau^2}{1 + 4\tau^2} \quad (20)
\]

For long times the solution reaches the value \(-1/2\) while \( p_{lin,M} = -1/2 \) and \(<p_{lin,>}> = 1/4 \). For positive nonlinearities the localization tendency increases while for negative \( \chi \) we have first detrapping, complete oscillations for \( \chi = -6 \) and

\[ P_{\text{asym}} \text{ in } P_{\text{r},N} < \] subsequent asymptotic equipartition at \( \chi = -8 \) with the transition being now algebraic as seen in Eq. (20). Further absolute increase of \( \chi \) produces nonlinear localization. We note that as in the trimer also in the present case at \( \chi = \chi_c \) we have an asymptotic equipartition of the probability on all four sites; in that sense we could use the term “selftrapping” meaning however the asymptotic and irreversible onset of equal probability on all sites.

\[ p(\tau) = 1 - \frac{24(N - 1)}{12c \left[ 1 + \frac{3}{\text{snth}^3(\sqrt{3} \tau)} \right]} + \chi_c^2 + 2(N - 2)\chi_c + N^2 \quad (22) \]

where \( c \) is the value of the double root of the discriminant of the equation \( x^3 - g_2x - g_3 = 0 \) while the other root is equal

\[ \text{C. } N > 4 \]

We observed from the previous three cases that in the dimer the two roots at \( \chi_c \) and \( \chi'_c \) determined two distinct, yet identical in nature, selftrapping transitions. In the trimer \( \chi'_c \) was responsible for the hyperbolic transition, \( \chi_c \) for trigonometric evolution, while the fact that the two roots were equal in the tetramer turned the (unique) transition to an algebraic one. The (negative) roots \( \chi_c \) and \( \chi'_c \) “collide” in the tetramer and we observe that their role changes for \( N > 4 \). Specifically, we find that \( |\chi'_c| > |\chi_c| \), with the algebraic transition occurring at \( \chi_c \) while at \( \chi'_c \) we have a specific trigonometric evolution.

For negative nonlinearities as we increase in absolute value \( \chi \) we have initially detrapping, complete oscillatory motion followed by subsequent reduction of the amplitude of the motion that leads to the hyperbolic transition at \( \chi_c \). This state does not populate equally at long times the sites of the \( N \)-mere as in the cases for \( N = 2, 3, 4 \); the asymptotic population of the initially excited site is now larger than that of the other sites. Further increase of \( |\chi| \) leads to periodic motion with increasing amplitude reduction. As the nonlinearity parameter crosses the value \( \chi'_c = -2N \) the dynamics becomes trigonometric while, at the same time the amplitude of oscillation becomes identical to that of the corresponding linear \( N \)-mere. Thus at \( \chi'_c \) the nonlinear system behaves identically to the corresponding linear one in what regards the site probabilities, nevertheless the evolution is done with different frequency.

Further increase of nonlinearity leads to more localization.

The value of \( \chi_c \) is determined numerically from the solution of the cubic equation \( g(\chi_c) = 0 \); the values obtained coincide with those found in ref. [10]. We observe that as \( N \) increases this root separates from the one at \(-2N\) and asymptotically reaches the value \(-N\). Thus in a very large system we expect to have the hyperbolic selftrapping at nonlinearity

values of order \(-N\) while at \(-2N\) the nonlinear system behaves as the corresponding linear one with renormalized frequency. We also observe that the system reaches a regime of complete oscillations for \( \chi' = -2(N - 1) \) while \( \chi'_c < \chi_c \). For \( N \geq 15 \) the equality for \( \chi' \) does not hold any more and the system begins to execute incomplete oscillations reaching a maximum at \( \chi'_c \leq \chi_c \). Thus, while for \( N < 14 \) the particle first reaches complete delocalization and at a later stage relaxes hyperbolically, for \( N \geq 15 \) the two phenomena occur almost for the same \( \chi \)-value, first reaching a maximum in the delocalization and subsequently, for infinitesimal change in the \( \chi \)-value, hyperbolic-type evolution. In this last regime the maximum excising occurs for \( \chi \) very close to the hyperbolic transition value \( \chi_c \).

The exact expression for the evolution at the trigonometric transition at \( \chi'_c = -2N \) for \( N > 4 \) is given by:

\[
p(\tau) = \frac{4 - 6N + N^2 + 2(N - 2)\cos \left( \sqrt{N(N - 4)}\tau \right)}{N \left[ N - 2 - 2\cos \left( \sqrt{N(N - 4)}\tau \right) \right]} \quad (21)
\]

The oscillation extends from \( p_{\text{max}} = 1 \) to \( p_{\text{min}} = (N^2 - 8N + 8)/N^2 \); these values are identical to the one obtained for the linear MF limit. Thus, in the MF limit of the DNLS equation there are two values of the nonlinearity parameter, viz. \( \chi = 0 \) and \( \chi'_c \) at which an initially localized excitation oscillates trigonometrically with the same amplitude, but with completely different frequencies. In the second case, the finite value of nonlinearity is equivalent to a “polaronic” dressing of the excitation leading to a smaller oscillation frequency compared to the purely linear oscillation at \( \chi = 0 \).

The hyperbolic transition at \( \chi_c \) is given, on the other hand, by

\[
p(\tau) = 1 - \frac{24(N - 1)}{12c \left[ 1 + \frac{3}{\text{snth}^3(\sqrt{3} \tau)} \right]} + \chi_c^2 + 2(N - 2)\chi_c + N^2 \quad (22)
\]
FIG. 1: (Color online) Exact dynamics for fully connected networks: probability $p(\tau)$ of Eq. (14) as a function of the rescaled time $\tau$. Each subfigure is for a network with different size and specific nonlinearity. In all plots the periodic continuous red line presents the motion for localized initial condition and for the specific $N$ and $\chi$ while the dashed orange line the solution of the corresponding linear network. The continuous green line is the solution at $\chi = \chi'_r \equiv -2N$ while the dashed blue line the solution at $\chi = \chi_c'$. The three horizontal lines parallel to the time axis denote the maximum excursion for the corresponding linear network $p_{\text{lin,}M}$ (blue dashed), the equipartition line at $1 - 1/N$ (purple) and the maximum range for the probability $p(\tau)$ at $-1$ (dashed with spaces). In (a) $N = 3$ and $\chi = -4$. We note the restoration of complete oscillations to $-1$ due to nonlinearity and the trigonometric evolution (blue curve) that in the present case is distinct from the purely linear motion (dashed orange curve). (b) $N = 4$ and $\chi = -6$. The green and blue curves collapse onto a single algebraically decaying solution ($\chi_c = \chi'_r = -8$ that relaxes to the equipartition line, while complete periodic motion to the other sites is also observed (red curve reaching $-1$). In this case the maximum linear excursion also reaches the equipartition line. (c) $N = 10$ and $\chi = -18$. The solutions for the roots at $\chi_c \approx -19.333$ and $\chi'_r = -20$ have now changed roles with the former (blue line) designating the hyperbolic transition (to non-equipartition- equipartition is denoted by the dashed purple line) while the latter (green line) forcing the network to pure trigonometric evolution with the same amplitude as for the linear ($\chi = 0$) case. In (d) $N = 15$ and $\chi = -15$ we observe similarly the hyperbolic transition ($\chi_c \approx -27.9772$, blue line) to non-equipartition as well as the two trigonometric evolutions for $\chi = 0$ (dashed orange line) and $\chi'_r = -20$ (green line)

FIG. 2: (Color online) Plot of the values $-\chi_c$ (red) and $-\chi'_r$ (green) as a function of the network size for two regions: Left panel, for $N < 12$ and right panel for $50 \leq N \leq 60$. The black circle designates the "collision" of the two branches at $N = 4$. For large-$N$ $\chi_c \sim -N$ [10], while $\chi'_r = -2N$ for any system size.

IV. CONCLUSIONS

The analysis of the MF model for the DNLS equation with a localized initial condition and arbitrary number of sites $N$ shows that the system is rich and has interesting dynamical behaviour. In the linear case ($\chi = 0$) the presence of the long range bonds leads to localization that increases with $N$ [1]. As focusing nonlinearity is turned on we observe initially a tendency for detrapping, i.e. the particle executes oscillations of larger amplitude that can reach complete or partial ($N \gtrsim 15$) escape from the initially populated site.

The time dependence of the system is typical of elliptic function dependence, viz. periodic oscillatory motion that includes an infinite number of frequencies. This time dependence changes for two specific values of the nonlinearity parameter at $\chi_c$ and $\chi'_r$, respectively. For the former value the time evolution becomes hyperbolic (except in the trimer where the hyperbolic evolution occurs at $\chi'_r$) and periodic motion ceases at long times. In the well studied case of $N = 2$, the state for this critical nonlinearity separates periodic delocalized from localized motion and we have selftrapping. For $N = 3, 4$, while there is no such separation in the hyperbolic transition, there is asymptotic equipartition of probability for this value. For $N > 4$ the hyperbolic solution at $\chi_c$ simply represents an asymptotic state where motion ceases while the initially populated site retains most of the probability, a tendency that increases with $N$. Thus, if by "selftrapping" we mean the separation of delocalized to localized motion then, strictly
speaking, it only occurs in the dimer case. On the other hand we may extend the term to include the case where asymptotic equipartition of probability takes place; in this case the symmetric trimer and tetramer also qualify, but not networks with larger number of sites.

A new feature appears at $\chi = \chi_c$ ($\chi = \chi_c$ in the trimer); the elliptic function evolution becomes trigonometric and the infinite set of frequencies it encompasses collapses to a single one. Additionally, for $N > 4$, the amplitude of the oscillation becomes identical to the corresponding one for $\chi = 0$. Thus the evolution at $\chi_c'$ is effectively linear but with a new frequency equal to $\sqrt{N(N-4)}$ (for $N > 4$) while in the purely linear case the frequency is $N$ \[1\]. If we think that the nonlinear term stems from the coupling with additional degrees of freedom \[12\] then the evolution becomes more sluggish since the particle is dressed and carries with it the additional degrees of freedom. It is remarkable that such a dressing leading to frequency renormalization may occur and still the evolution be trigonometric with the same amplitude as in the purely linear case. In the large-$N$ limit the two frequencies approach one the other.

As the size of the system increases we observe two distinct features, one at $\chi_c \sim -N$ where time-hyperbolic behaviour takes place while at $\chi_c' = -2N$ where we have an effective linearization of the motion. Both features appear after the initial detrapping and retrapping has occurred as a function of $|\chi|$. The onset of nonlinear localization is typically associated with discrete breather solutions that are localized in space and time periodic. The case for $\chi = \chi_c'$ found here would then correspond to a "linear breather", i.e. a localized nonlinear solution with exact trigonometric evolution. It would be interesting to find out if a similar situation can occur in other discrete nonlinear systems as well.

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