Monoidal 2-Categories: A Review

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Abstract
We review the complete definition of monoidal 2-categories and recover Kapranov and Voevodsky's definition from the algebraic definition of weak 3-category (or tricategory).

1 Introduction
Kapranov and Voevodsky, in their seminal paper [5], introduced monoidal 2-categories. Their definition comprises a list of axioms, the categorical origin of which does not seem transparent.

A monoidal n-category, by the categorical procedure, can be obtained by taking a one-object weak n + 1-category. For instance, a monoidal category is defined as one object weak 2-category (or bicategory). Due to the lack of a complete definition for weak n-categories, n cannot be any arbitrary natural number. However, thanks to the algebraic definition of weak 3-category (or tricategory) introduced by Gordon, Power and Street [2] and Gurski [3], one can define a monoidal 2-category as a tricategory with one object.

After a full list of necessary data, Gurski’s thesis gives two main axioms, namely, Stasheff and unit polytopes: the 2-dimensional correspondence of pentagonal and triangle equations for monoidal categories. However, it does not explicitly spell out all diagrams obtainable by naturality, modification, and (2-)functoriality conditions. Unpacking these conditions, one can recover KV’s axioms, which underlines the goal of this paper.

The recovery procedure needs to consider two main issues. The first one is the difference between tensorators in KV’s version and our version. Kapranov and Voevodsky defined three tensorators which can be obtained by a single tensorator of the tricategory approach. The second difference is an extra piece of data given by KV which can be written based on other data: a 2-morphism $\varepsilon$ between left and right 1-unitor indexed by the unit object, i.e., $r_I, l_I : I \otimes I \rightarrow I$. To show it is redundant, we present a proof sketch in the comparison section 4.2.

Discussion. Baez and Neuchl [1] reviewed the semi-strict definition of monoidal 2-category. Stay, on the other hand, spelt out the definition but without tensorators [9]. Schommer Pries cited Stay [8]. The combination of both constructs the full description of a monoidal 2-category.
Moreover, Stay listed four unit polytopes by alternating the location of the unit object, but only two of them are necessary, namely when the unit object is the second or third object in the polytope. As Gurski proved the other two are corollary of these axioms. Note also Stay’s diagrams, in our case, need to be revised, as in the presence of tensorators, filling 2-morphisms will be modified based on the modified tensor product.

Remark 1. Higher categories and particularity 2-categories have gained attention in topological quantum field theory and topological condensed matter physics, for instance, Kitaev and Kong modelled gapped boundaries in Levin-Wen model by the bicategory of module categories over fusion category of the bulk [6]. The current review tries to be accessible for physicists. Hence, some readers will encounter some parts which may seem excessive, for example, items 3 and 4 in Definition 3.1.

Remark 2. For a concise review of bicategories consult Leinster’s paper [7]. By a 2-category, we mean a strict 2-category. In other words, a bicategory whose associators and unitors are identities.

Outline. The paper is organized in the following way: having in mind both KV’s and Gurski’s definitions, Section 3 presents the definition of monoidal 2-category. So you will see the same structures and diagrams as KV whose origins are explained in Gurski’s tricategory language. We first list the data 3.1, then in Subsection 3.2, we give all conditions under three different groups: Naturality Conditions 3.2.1, Modification Conditions 3.2.2 and Axioms 3.2.3. The diagrams of these subsections are depicted in Appendix A. The paper concludes with Section 4 in which we compare the produced definition with the original definition given by Kapranov and Voevodsky. Appendices present a minimal account of the definitions of 2-functors Appendix B, modification and an explicit procedure for obtaining modification diagrams for pentagonator and 2-unitors Appendix C.

2 Acknowledgments

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3 Definition of Monoidal 2-Categories

Notation. Capital letters, A, B, … are reserved for objects, small letters f, g, … for 1-morphisms and Greek letters α, β, … for 2-morphisms. Horizontal composition of 1- and 2-morphisms is denoted by juxtaposition and vertical composition of 2-morphisms by ⊗. For every object A, the identity 1-morphism or 1-identity is shown by idA, but in the tensor product of idA with 1-morphisms idA ⊗ f, we leave out id and denote it by A ⊗ f. For every 1-morphism f, the identity 2-morphism or 2-identity is represented by 1f. Whenever, it is clear from the context, in the horizontal composition of 1-identities with 2-morphisms, we leave out 1 and use whiskering convention 1f ◦ α = fα.

Convention. Following the style of the stunning figures of Stay’s paper, we color diagrams. Shapes filled by pink are penetrators π, those by blue are
tensorators $\phi$, and those filled by brown are 2-unitors. Unfilled ones are natural 2-isomorphisms.

**Definition 1.** A monoidal 2-category is a 2-category $\mathcal{C}$ with a 2-functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a list of data given in Section 3.1 subject to some conditions presented in Section 3.2.

### 3.1 Data

1. A **unit object** $I \in \text{Obj}(\mathcal{C})$.

2. For every pair of objects $A, B$ in $\text{Obj}(\mathcal{C})$, the **tensor product of objects** is an object $A \otimes B \in \text{Obj}(\mathcal{C})$ denoted by juxtaposition $AB$.

3. For 1-morphisms $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$, the **tensor product of 1-morphisms** defined as $A \otimes C \xrightarrow{f \otimes g} B \otimes D$ exists.

4. For every pair of 2-morphisms, $\alpha : f \Rightarrow h$, $\beta : g \Rightarrow k$, and 1-morphisms $f, h : A \to B$ and $g, k : C \to D$, there exists a **tensor product of 2-morphism** $\alpha \otimes \beta : f \otimes g \Rightarrow h \otimes k$ such that $f \otimes g, h \otimes k \in \text{Hom}(A \otimes C, B \otimes D)$.

5. Functoriality of 2-functor $\otimes$ holds up to natural 2-isomorphisms, meaning, it preserves composition of 1-morphisms and identity 1-morphisms up to natural 2-isomorphisms Appendix B.

- For every quadruple of 1-morphisms $(f, f', g, g')$, such that $f : A \to A'$, $f' : A' \to A''$, $g : B \to B'$, $g' : B' \to B''$, a natural 2-isomorphism called **tensorator**:

$$\phi_{f', g', f, g} : (f' \otimes g') \circ (f \otimes g) \Rightarrow (f' \circ f) \otimes (g' \circ g) \quad (1)$$

In addition to naturality condition, it has to satisfy Equations [24].

Note that KV’s definition introduces three different tensorators which the tricategory approach packs in one single tensorator denoted by $\phi$, for further discussion check Section 4.3.

- For every pair of objects $A, B$, we also need a natural 2-isomorphism:

$$\phi_{A, B} : id_A \otimes id_B \Rightarrow id_{A \otimes B} \quad (2)$$

For our purpose of recovering KV’s definition, we let $\phi_{A, B}$ be identity 2-morphisms. This assumption has a consequence which we elaborate on in Section 4.3.

6. For every triplet of objects $A, B, C$, a natural isomorphism called **1-associator**:

$$a : \otimes(\otimes \times \text{Id}) \to \otimes(\text{Id} \times \otimes)$$

Since $a$ is between two 2-functors, it consists of two pieces of data: a 1-isomorphism indexed by three objects mentioned earlier and a natural 2-isomorphism indexed by at least one 1-morphism. For instance, for $f : A \to A'$ there exists a natural 2-isomorphism $\alpha_{f, B, C}$ subject to the
naturality conditions in the 1-morphism $f$, objects $B$ and $C$, and also Axiom 26.

\[
\begin{array}{ccc}
(A'B)C & \xrightarrow{\alpha_{A',B,C}} & A'(BC) \\
(f \otimes B) \otimes C & \xrightarrow{\alpha_{f,B,C}} & f \otimes (B \otimes C) \\
(AB)C & \xrightarrow{\alpha_{A,B,C}} & A(BC)
\end{array}
\]

(3)

7. For every object $A$, a natural isomorphism called left 1-unitor,

\[l_A : I \otimes - \rightarrow \text{Id} \]

Similar to 1-associators, because $l$ is a natural 2-transformation between two 2-functors $I \otimes -$ and $\text{Id}$, there should be a natural 2-isomorphism indexed by a 1-morphism, $l_f$, which further satisfies the naturality condition and Axiom 26.

\[
\begin{array}{ccc}
A & \xrightarrow{l_f} & A' \\
l_A & \xleftarrow{\Rightarrow} & l_A' \\
IA & \xrightarrow{I \otimes f} & IA'
\end{array}
\]

(4)

8. For every object $A$, a natural isomorphism called right 1-unitor:

\[r : - \otimes I \rightarrow \text{Id} \]

This includes a natural 1-isomorphism $r_A : A \otimes I \rightarrow A$, and a natural 2-isomorphism $r_f$ subject to the naturality conditions and Axiom 26.

\[
\begin{array}{ccc}
A & \xrightarrow{r_f} & A' \\
r_A & \xleftarrow{\Rightarrow} & r_A' \\
AI & \xrightarrow{I \otimes I} & A'\text{I}
\end{array}
\]

(5)

9. For every four objects $A, B, C, D$, there is a modification between composition of 1-associators called pentagonator $\pi_{A,B,C,D}$ shown in Figure 6. Note that $\pi$ is not natural but it should satisfies the modification condition.
10. For every two objects $A, B$, there exist three 2-isomorphisms called 2-unitors Equation 7. They are modifications, hence, subject to the modification condition 28.

\[ \Rightarrow \pi_{A,B,C,D} \]

Remark 3. Before listing the conditions on data, we should notice that the existence of tensorators changes the tensor product of an object with a 2-morphism. We denote the new tensor product with $\hat{\otimes}$. We show a more comprehensible example first, then restate KV’s example Figure 1 for $\pi_{A,B,C,D} \hat{\otimes} E$.

- Assume that there exists a 2-morphism $\alpha : gf \Rightarrow h$, now if we tensor each object from left with an object $A$, the filling 2-morphism will not be modified as $A \otimes \alpha$, but it will be $(A \otimes \alpha) \circ \phi_{A,g,A,f}$.

- Now take the example presented by Kapranov and Voevodsky Figure 1 consider the pentagonator if each object is tensored by an object $E$ from right. One should observe that the filling 2-morphism is $\pi_{A,B,C,D} \otimes E$ which is different from $\gamma = \phi \otimes E$.

Remark 4. As mentioned briefly in the discussion, Stay did not use the modified tensor product. Since the paper assumes tensorators are identity 2-morphisms. However, in the following, we shall work with the modified tensor product when it is necessary.
Figure 1: The missing 1-morphisms in the figure is $g = [(A \otimes a_{B,C,D}) \otimes a_{A,BC,D} \otimes (a_{A,B,C} \otimes D)] \otimes E$, and the missing 2-morphisms are

\[ \alpha = \phi_{A \otimes a_{B,C,D} \otimes a_{A,BC,D} \otimes E}, \gamma = \pi \otimes E, \beta = \phi_{(A \otimes a_{B,C,D}) \otimes a_{A,BC,D} \otimes E}, \xi = \phi_{a_{A,B,CD} \otimes a_{A,B,CD} \otimes E}. \]

3.2 Condition on Data

Each of the data above should further satisfy some conditions either due to naturality or axioms of 2-transformation or modification. However, similar to monoidal categories, in addition to the conditions arising from the nature of the data, monoidal 2-categories are further subject to two axioms called Stasheff and unit polytopes. Due to the coherence theorems, these two conditions are enough and no more conditions are required.

3.2.1 Naturality Conditions

1. Naturality of 1-associators: weakening 2-isomorphisms $\alpha_{f,B,C}$ for 1-associators should be natural in its indices $f, B, C$.

   - Naturality in 1-Morphism $f$: if $\gamma : f \Rightarrow f'$, the cylinder shown in Equation 9 commutes. Similar cylinders when 1-morphism is the first or second index should commute.

   - Naturality in Object $B$: to check the naturality in $B$, assume another object $B'$, 1-morphisms $f : A \to A'$ and $g : B \to B'$, then, the naturality squares of $\alpha$ result in the white squares depicted in the cube. Gluing squares results in the cube.

   - Axiom 26 of naturality: for any composable pair of 1-morphisms $A \xrightarrow{f} A' \xrightarrow{f'} A''$, Diagram 12 commutes.

2. Naturality of 1-unitors: weakening 1-unitors $l_f$ and $r_f$ need to satisfy naturality conditions and Axiom 26. We only present the conditions for the left unitors $l_f$. Corresponding conditions should hold for the right unitors $r_f$. 
• Naturality in 1-Morphism $f$: the left and right 1-unitor are natural, and their natural 2-isomorphisms $l_f$ and $r_f$ are indexed by a 1-morphism. Hence, to check naturality of $l_f$ and $r_f$ we alternate $f$; that is, for $\gamma : f \Rightarrow f'$, the cylinder shown in Figure 13 commutes.

• Axiom 26 of naturality: For any composable pair of 1-morphisms $A \xrightarrow{f} A' \xrightarrow{f'} A''$, the triangle prism shown in Figure 14 commutes.

3. Naturality of tensorators: Observing and finding appropriate diagrams are easier if one tries to alternate indices of KV’s tensorators Section 4.3. We list three classes of them and explicitly mention how you can find these diagrams. One might prefer to check Section 4.3 before proceeding further.

• The first condition is obtained by checking the naturality of tensorator $\otimes_{f,g}$ in $f$ and $g$ (this is the symbol that KV used). If one alternates $f$ or $g$ in $\otimes_{f,g}$ or equally in $\phi^{-1}_{A',g,f,B} \circ \phi_{f,B,A,g}$, for $\gamma : f \Rightarrow f'$, the cylinder in Figure 15 should commute.

• If one alternates one of the 1-morphisms of $\otimes_{g,f,B}$ or in $\phi_{g,B,f,B}$, then for every $\alpha$ and $g$,

\[
\begin{array}{c}
A \\
\downarrow \alpha \\
\bullet
\end{array} \quad \xrightarrow{f} \quad \begin{array}{c}
A' \\
\downarrow \\
A''
\end{array} \quad \xrightarrow{g} 
\]

(8)

Figure 16 commutes.

• Now if we check the naturality of $\otimes_{f',f,B}$ in $f$ and $g$ for $A \xrightarrow{f} A' \xrightarrow{f'} A''$ and $g : B \Rightarrow B'$, the triangle prism commutes Figure 17.

• Axiom 26 for tensorators: For any composable triplet of 1-morphisms $A \xrightarrow{f} A' \xrightarrow{f'} A''$ and an object $B$, Tetrahedron 18 commutes.

3.2.2 Modification Data

Pentagonator and 2-unitors are modification. Hence, the axiom of modification 28 should hold for them. We present the details of obtaining the below diagrams from the modification square in Appendix C.

• 2-Unitors: The modification square for every $f : A \Rightarrow A'$ results in a two dimensional diagram. Gluing the boundaries of the 2-dimensional picture gives us the prism of Figure 19.

• Pentagonator: for every 1-morphism $f : A \Rightarrow A'$ and objects $B, C, D$ the pentagonal prism 20 commutes.

3.2.3 Axioms

• For every five objects, $(A, B, C, D, E)$, Stasheff polytopes Figure 21 should commute.
• For every triplet of objects $A, B, C$, the unit polytopes figures 22 and 23 commute.

**Definition 2.** A monoidal 2-category is **strict** if all weakening 1- and 2-morphisms are identities. It is **semi-strict** if all weakening 1- and 2-morphisms except tensorators $\phi$ are identities.

### 4 Comparison Between Definitions

#### 4.1 Isomorphism or (adjoint)equivalence

Gordon, Power and Street\cite{GP} defined tricategories whose structure 1-morphisms are all equivalences. However, Gurski replaced them with ad joint equivalence. To recover KV’s definition, we make them even stronger, and substitute them with isomorphisms.

#### 4.2 An extra data

KV’s definition included an extra piece of data which is unnecessary and can be obtained by 2-morphisms listed above. They defined a special 2-morphism between the left and right unitors for the unit object,

\[
\begin{array}{ccc}
I \otimes I & \xrightarrow{r_I} & I \\
\Downarrow & & \\
I & \xleftarrow{l_I} & I
\end{array}
\]

such that it satisfies the diagram below,

Although tiresome, one can write $\epsilon$ based on other 2-morphisms by taking the figure given on Page 58 of [1]. The cited figure is used to prove the equality of $r_I$ and $l_I$ for monoidal categories. All inner diagrams can be filled by weakening 2-morphisms to obtain the result. The figure above, regardless of considering $\epsilon$, commutes, which is shown by Gurski in Proposition 4.24 [3].
4.3 Tensorators

Kapranov and Voevodsky enumerated three types of tensorators as data because he did not mention that the tensor product $\otimes$ is a 2-functor at heart. They are detonated as $\otimes_{f,g}$, $\otimes_{f',f, B}$ and $\otimes_{A, g', g}$ and defined in Figure 2.

![Diagram of KV's tensorators](image)

**Figure 2: KV’s tensorators**

We can translate them based on our tensorator $\phi$. For the first case, $\otimes_{f,g} = \phi_{A', f, g, B}^{-1} \otimes \phi_{f, B', A, g}$.

$$(f \otimes B') \circ (A \otimes g) \xrightarrow{\phi_{f', B', A, g}} (f \circ A) \otimes (B' \circ g) = (A' \circ f) \otimes (g \circ B) \xrightarrow{\phi_{A, g'}^{-1} B} (A' \otimes g) \circ (f \otimes B)$$

The second and third KV’s tensorators are labelled by two 1-morphisms and an object, based on $\phi$, they are $\otimes_{f', f, B} = \phi_{f', B, f, B}$ and $\otimes_{A, g', g} = \phi_{A, g', B}$. 

4.4 Consequence of $\phi_{A,B} = 1$

1-Associators and 1-unitors apart from Axiom 26 should further satisfy Axiom 27 which seemingly we have ignored throughout the paper. This axiom is built on the assumption that $id_{A \otimes B}$ and $id_A \otimes id_B$ are isomorphic not equivalent, $id_{A \otimes B} \cong id_A \otimes id_B$. However, to obtain Kapranov and Voevodsky’s definition, here we assume $\phi_{A,B}$ are identities, which results in the proceeding corollaries.

**Corollary 1.** In a monoidal 2-category with the definition given above, 2-unitors are identity 2-morphisms if they are indexed by identity 1-morphisms.

**Proof.** In Equation 27 let $F = - \otimes I$ and $G = Id$, $\sigma_A = r_A$ and $\sigma_f = r_f^{-1}$. We have $r_{id_A} = (1 * \sigma_{A,I})$, which means $r_{id_A} = 1$.

**Corollary 2.** In a monoidal 2-category with the definition given above, 2-associators are identities if at least one of their indices is identity 1-morphism.
Proof. Check Figure 4.4 if $\beta = \phi_{A,B,C} \circ (\phi_{A,B} \otimes C)$ and $\gamma = \phi_{A,BC} \circ (A \otimes \phi_{B,C})$. The blue square is naturality and the filling 2-associator is $(a_{A,B,C} \circ \beta^{-1}) \circ (\gamma \circ a_{A,B,C}) = \alpha_{id_{A},B,C}$. If one lets $\phi_{A,B}$ identity, then 2-associators become identities if at least one of their indices is identity.

Remark 5. Note that since we are working with 2-categories not bicategories, we easily let $a_{A,B,C} \circ id_{AB} = id_{A} \circ a_{A,B,C} = a_{A,B,C}$.

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A Figures

A.1 Diagrams of Naturality

A.1.1 Naturality of 1-Associators

- Naturality of 1-associators in 1-morphism \( f \)

\[
\begin{align*}
A(BC) & \xrightarrow{\gamma \otimes (B \otimes C)} A'(BC) \\
& \xleftarrow{\alpha_{f',B,C}} \\
(AB)C & \xrightarrow{(\gamma \otimes B) \otimes C} (A'BC) \\
& \xrightarrow{\alpha_{f,B,C}} \\
& \xleftarrow{\alpha_{f',B,C}} \\
& \xrightarrow{\alpha_{f,B,C}} \\
& \xleftarrow{\alpha_{f',B,C}} \\
& \xrightarrow{\alpha_{f,B,C}} \\
& \xleftarrow{\alpha_{f',B,C}}
\end{align*}
\]

(9)

- Naturality of 1-associators in Object \( B \) the missing 2-morphisms in the blue squares are

\[
\beta = \phi_{A',g,B,C}^{-1} \circ \phi_{g,c,B,C}^{-1} \circ (f \otimes \phi_{B',c,g,C}) \circ \phi_{f,B',c,A,g} \otimes \phi_{A',g,B,C}
\]

and

\[
\gamma = (\phi_{A',f,B}^{-1} \circ \phi_{f,B',A,g}) \otimes C.
\]

\[
\begin{align*}
(AB)C & \xrightarrow{\alpha_{A,B,C}} A(BC) \\
& \xleftarrow{\alpha_{A,B,C}} \\
& \xrightarrow{\alpha_{A,B,C}} \\
& \xleftarrow{\alpha_{A,B,C}} \\
& \xrightarrow{\alpha_{A,B,C}} \\
& \xleftarrow{\alpha_{A,B,C}} \\
& \xrightarrow{\alpha_{A,B,C}} \\
& \xleftarrow{\alpha_{A,B,C}}
\end{align*}
\]

(10)
\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (A' \otimes (B' \otimes C)) \]

\[ A(BC) \Rightarrow (AB')C \Rightarrow (f \otimes B') \otimes C \Rightarrow (A'B')C \]

\[ (A \otimes g \otimes C) \Rightarrow (f \otimes (B \otimes C)) \Rightarrow (A' \otimes (B \otimes C)) \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B',C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]

\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \Rightarrow (A' \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B,C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]

\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (A' \otimes (B' \otimes C)) \]

\[ A(BC) \Rightarrow (AB')C \Rightarrow (f \otimes B') \otimes C \Rightarrow (A'B')C \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B',C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]

\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \Rightarrow (A' \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B,C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]

\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (A' \otimes (B' \otimes C)) \]

\[ A(BC) \Rightarrow (AB')C \Rightarrow (f \otimes B') \otimes C \Rightarrow (A'B')C \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B',C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]

\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \Rightarrow (A' \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B,C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]

\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (A' \otimes (B' \otimes C)) \]

\[ A(BC) \Rightarrow (AB')C \Rightarrow (f \otimes B') \otimes C \Rightarrow (A'B')C \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B',C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]

\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \Rightarrow (A' \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B,C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]

\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (A' \otimes (B' \otimes C)) \]

\[ A(BC) \Rightarrow (AB')C \Rightarrow (f \otimes B') \otimes C \Rightarrow (A'B')C \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B',C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]

\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \Rightarrow (A' \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B,C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]

\[ (A \otimes (g \otimes C)) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (A' \otimes (B' \otimes C)) \]

\[ A(BC) \Rightarrow (AB')C \Rightarrow (f \otimes B') \otimes C \Rightarrow (A'B')C \]

\[ (AB'C) \Rightarrow (A'BC) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ (A'B'C) \Rightarrow (A'B' \otimes C) \Rightarrow (f \otimes (B' \otimes C)) \Rightarrow (f \otimes (B \otimes C)) \]

\[ \Rightarrow \alpha_{f,B',C} \Rightarrow \alpha_{A,g,C} \Rightarrow \alpha_{A',g,C} \Rightarrow \alpha_{f,B,C} \]
• Axiom 26 of naturality: let $F(A, B, C) = (AB)C$, $G(A, B, C) = A(BC)$, $\sigma_{A,B,C} = a_{A,B,C}$, $\sigma_f = \alpha_{f,B,C}$ and $\kappa = \psi = \phi$, it is straightforward to first obtain the planar diagrams, then from shared boundaries glue them to obtain the 3-dimensional diagram of Kapranov and Voevodsky in Page 219. The missing 2-morphisms in the blue triangles, $\beta = \phi_{f',BC;f,BC}$ and $\gamma = \phi_{f',B,f,\hat{B}} \otimes \hat{C}$.

\[
\begin{array}{c}
\text{(12)} \\
\end{array}
\]

A.1.2 Naturality of 1-Unitors
• Naturality of 1-unitors in 1-morphism $f$: 

\[
\begin{array}{c}
\end{array}
\]
- Axiom for unitor: let $GA = A$, $FA = IA$, $\sigma_A = l_A$, $\sigma_f = l_f$, $\kappa = 1$, $\psi = \phi$. 

\[ (13) \]

\[ (14) \]
A.1.3 Naturality of Tensorors

- Naturality of tensorator 1: The missing 2-morphism is
\[ \beta = \phi_{f',g,f,B}^{-1} \circ \phi_{f,B',A,g} \]
\[ \eta = \phi_{f',g,f,B}^{-1} \circ \phi_{f,B',A,g} \].

- Naturality of tensorator 2:

\[ \xi = \phi_{f',B,f,B}' \]
\[ \beta = \phi_{f',B',f,B}' \]
\[ \gamma = \phi_{f',g,f,B}' \circ \phi_{f,B',A,g} \]
\[ \eta = \phi_{f',g,f,B}' \circ \phi_{f,B',A,g} \].

- Naturality of tensorator 3: The missing 2-morphisms in the figure are
\[ \xi = \phi_{f',B,f,B}' \]
\[ \beta = \phi_{f',B',f,B}' \]
\[ \gamma = \phi_{f',g,f,B}' \circ \phi_{f,B',A,g} \]
\[ \eta = \phi_{f',g,f,B}' \circ \phi_{f,B',A,g} \].
• Axiom 20 of naturality Let $F(A) = AB$. The filling 2-morphisms are 
$\lambda = \phi_{f^\prime,B,f,B}, \beta = \phi_{f^\prime,B,f,f,B}, \gamma = \phi_{f^\prime,B,f,B}, \xi = \phi_{f^\prime,B,f,B}$. 

16
A.2 Diagrams of Modification

A.2.1 2-Unitors

Modification condition for 2-unitors: We only describe the condition on $\eta$, other ones are similar. The missing 2-morphism is $\gamma = \phi_{A',I_B,f,IB}^{-1} \otimes \phi_{f,B,A,l_B}$. 

(18)

(19)
A.2.2 Pentagonator

Modification condition for pentagonator: the 2-isomorphism filling the front square of 3D picture is

\[
\gamma = \phi^{-1}_{A',a,b,c,d,f,(ABCD)} \odot \phi_{f,BCD,A,a,b,c,d}
\]
A.3 Axioms

A.3.1 Stasheff Polytope

Stasheff Polytope for pentagonator: To draw this diagram, we used the coordinates of Stasheff Polytope in Stay's paper [9].
A.3.2 Unit Polytopes

- Unit polytope $I$

\[
\begin{align*}
\alpha_{A,l,B,C} & \quad (A(I)B)C \\
\beta_{A,l,B,C} & \quad (A(I)B)C
\end{align*}
\]
B Definition of 2-Functor

Leinster [7] presented a definition for bifunctor, when source and target are bicategories rather than 2-categories. Since we work with 2-categories, we modify the definition as presented below. We illustrate similar diagrams as Leinster and only add \( = \) instead of isomorphism.

**Definition 3.** A 2-functor \( F : \mathcal{A} \rightarrow \mathcal{B} \) consists of the following data which satisfy the two axioms given below.

- function \( F : \text{obj}(\mathcal{A}) \rightarrow \mathcal{B} \)
- functors \( F_{AB} : \mathcal{A}(A, B) \rightarrow \mathcal{B}(A, B) \)
- natural transformations \( \phi_{g,f} : Fg \circ Ff \rightarrow F(g \circ f) \), \( \phi_A : id_{FA} \rightarrow F(id_A) \)
- **Axiom 1:**

\[
\begin{align*}
(FhFg)Ff & \xrightarrow{\phi \ast 1} F(hg)Ff & \xleftarrow{\phi} F((hg)f) \\
\parallel & \quad \parallel \\
Fh(FgFf) & \xrightarrow{1 \ast \phi} FhF(gf) & \xleftarrow{\phi} F(h(gf)) \\
\end{align*}
\]

(24)

- **Axiom 2:**

\[
\begin{align*}
Ff \circ id_{FA} & \xrightarrow{1 \ast \phi_A} Ff \circ Fid_A & \xleftarrow{\phi} F(f \circ id_A) \\
\parallel & \quad \parallel \\
Ff & \xrightarrow{RF \ast \phi} Ff & \xleftarrow{\phi} F(f) \\
\end{align*}
\]

(25)
**Definition 4.** A transformation $\sigma : F \Rightarrow G$ where $(F, \kappa)$ and $(G, \psi)$ are morphisms, with data 1-morphism $\sigma_A : FA \to GA$ and natural 2-isomorphism $\sigma_f : Gf \circ \sigma_A \Rightarrow \sigma_B \circ Ff$, such that

\[
(GgGf)\sigma_A \xrightarrow{1 \ast \sigma_f} Gg(Gf\sigma_A) \xrightarrow{1 \ast \sigma_f} Gg(\sigma_B Ff) \xrightarrow{\sigma_g \ast 1} (Gg\sigma_B)Ff \xrightarrow{(\sigma_C Fg)f} \sigma_C(FgFf)
\]

and

\[
\begin{array}{c}
\psi \ast 1 \\
G(gf) \circ \sigma_A \\
\end{array}
\begin{array}{c}
\sigma_{gf} \\
G(gf) \circ \sigma_A \circ \sigma_B \\
\end{array}
\begin{array}{c}
G(gf) \circ \sigma_A \circ \sigma_B \\
\end{array}
\begin{array}{c}
1 \ast \kappa \\
(26)
\end{array}
\]

\[
\begin{array}{c}
\psi \ast 1 \\
G(id_A) \sigma_A \\
\end{array}
\begin{array}{c}
\sigma_{id_A} \\
\sigma_{id_A} \circ \sigma_A \\
\end{array}
\begin{array}{c}
\sigma_A F(id_A) \\
\end{array}
\begin{array}{c}
1 \ast \kappa \\
(27)
\end{array}
\]

**C Modification Data**

**Definition 5.** A morphism between two natural morphisms $(\sigma, \tilde{\sigma})$ between two 2-functors is called modification $\Gamma$. For every object $A$, it consists of a 2-morphism $\Gamma_A$, subjects to the modification square (28) for every morphism $f$.

\[
\begin{array}{c}
\sigma_A \\
\Downarrow \Gamma_A \\
\sigma_A \\
\end{array}
\begin{array}{c}
Gf\sigma_A \\
\sigma_f \\
\sigma_B Ff \\
\end{array}
\begin{array}{c}
1_{GF} \ast \Gamma_A \\
\lhd \Gamma_B \ast 1_{Ff} \\
\end{array}
\begin{array}{c}
Gf\tilde{\sigma}_A \\
\tilde{\sigma}_f \\
\sigma_B Ff \\
\end{array}
\begin{array}{c}
(28)
\end{array}
\]

**C.1 Details of Modification Diagram for 2-Unitors**

We only describe the details of our procedure for obtaining the prism for one of 2-unitors, namely, $\mu$, the recipe works for other 2-unitors as well. In square (28) let $G(A, B) = (A\{I\})B$ and $F(A, B) = AB$, $\sigma_{A,B} = (A \otimes l_B)\mu_{aA,B}$ and $\tilde{\sigma}_{A,B} = r_A \otimes B$. Now, we find the edges of Diagram (29) and compose the 2-morphisms.

\[
\begin{array}{c}
Gf\sigma_{A,B} \\
\sigma_f \\
\sigma_B Ff \\
\end{array}
\begin{array}{c}
1_{GF} \ast \mu_{A,B} \\
\lhd \mu_{A,B} \ast 1_{Ff} \\
\end{array}
\begin{array}{c}
Gf\tilde{\sigma}_{A,B} \\
\tilde{\sigma}_f \\
\tilde{\sigma}_B Ff \\
\end{array}
\begin{array}{c}
(29)
\end{array}
\]
• \((\tilde{\sigma}_f \circ (\lambda_{Gf} \ast \mu_{A,B}))\)

\[
\begin{align*}
& (A'I)B \xrightarrow{(f \otimes I) \otimes B} (A'IB) \xrightarrow{r_{A'B}} (A'IB) \xrightarrow{f \otimes B} \Downarrow \varepsilon \xrightarrow{\gamma} A'B \\
\end{align*}
\]

\[
\begin{align*}
& (AI)B \xrightarrow{\sigma_{A'B,C,D} \otimes B} (A'I)B \xrightarrow{f \otimes B} (A'IB) \xrightarrow{\gamma} A'B \\
\end{align*}
\]

\[
\begin{align*}
& (AI)B \xrightarrow{A'I} A'B \xrightarrow{\tilde{\sigma}_f \circ (\lambda_{Gf} \ast \mu_{A,B})} (A'I)B \xrightarrow{f \otimes B} A'B \\
\end{align*}
\]

\[
\begin{align*}
& (AI)B \xrightarrow{\sigma_{A'B,C,D} \otimes B} (A'I)B \xrightarrow{f \otimes B} A'B \\
\end{align*}
\]

C.2 Details of Modification Diagram for Pentagonator

To unpack the modification square, let 2-functors be \(G(A, B, C, D) = ((AB)C)D\), and \(F(A, B, C, D) = A(B(CD))\), natural transformations are \(\sigma_{A,B,C,D} = (A \otimes a_{B,C,D})a_{A,BC,D}(a_{A,B,C} \otimes D)\) and \(\tilde{\sigma}_{A,B,C,D} = a_{A,B,CD}a_{AB,C,D}\).

\[
\begin{align*}
& Gf\sigma_{A,B,C,D} \xrightarrow{\lambda_{Gf} \ast \pi_{A,B,C,D}} Gf\tilde{\sigma}_{A,B,C,D} \\
& \xrightarrow{\sigma_f} \tilde{\sigma}_f \\
& \sigma_{A',B,C,D} Ff \xrightarrow{\lambda_{Gf} \ast \pi_{A,B,C,D}} \tilde{\sigma}_{A',B,C,D} Ff
\end{align*}
\]

(30)
\[ (\pi_f \otimes (\lambda_{gf} * \mu_{A,B})) \]

\[ A(BCD) \xrightarrow{f \otimes (B \otimes (C \otimes D))} A'(BCD) \]

\[ A \otimes a_{B,C,D} \]

\[ A' \otimes a_{B,C,D} \]

\[ A((BC)D) \xrightarrow{f \otimes ((B \otimes C) \otimes D)} A'(BCD) \]

\[ a_{A,B,C,D} \]

\[ a_{A',B,C,D} \otimes D \]

\[ (A(BC))D \xrightarrow{(f \otimes (B \otimes C)) \otimes D} (A'(BC))D \]

\[ a_{A,B,C} \otimes D \]

\[ a_{A',B,C} \otimes D \]

\[ ((AB)CD) \xrightarrow{((f \otimes B) \otimes C) \otimes D} ((A'B)C)D \]

\[ ((A'B)C)D \xrightarrow{a_{A',B,C,D}} (A')((B)CD) \xrightarrow{a_{A',B,C,D}} A'(B(CD)) \]

\[ ((f \otimes B) \otimes (C \otimes D)) \]

\[ ((AB)CD) \xrightarrow{((f \otimes B) \otimes (C \otimes D))} ((A'B)CD) \]

\[ A' \otimes a_{B,C,D} \]

\[ A'(BCD) \]

\[ \pi_{A',B,C,D} \]

\[ a_{A',B,C,D} \]

\[ (A'(BC))D \]

\[ a_{A',B,C} \otimes D \]

\[ 24 \]
\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ A((BC)D) \rightarrow A'((BC)D) \]

\[ (AB)D \rightarrow (A'B)(CD) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ a_{A,B,C,D} \rightarrow \pi_{A',B,C,D} \rightarrow (A'B)(CD) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]

\[ (A(BC))D \]

\[ A((BC)D) \]

\[ A(B(CD)) \rightarrow A'(B(CD)) \]

\[ (AB) \rightarrow (A'B) \]

\[ ((AB)C)D \rightarrow ((A'B)C)D \]

\[ a_{A,B,C,D} \rightarrow a_{A',B,C,D} \rightarrow a_{A,B,C,D} \]

\[ f \otimes (B \otimes (C \otimes D)) \rightarrow f \otimes (B \otimes (C \otimes D)) \]

\[ A \otimes a_{B,C,D} \]

\[ a_{A,B,C,D} \]
\[(A(BC))D \rightarrow A((BC)D)\]

\[\pi_{A,B,C,D} \downarrow\]

\[((AB)C)D \rightarrow (AB)(CD) \rightarrow A(B(CD))\]

\[((A'B)C)D \rightarrow (A'B)(CD) \rightarrow A'(B(CD))\]