Chains(\mathbb{R}) does not admit a geometrically meaningful properadic homotopy Frobenius algebra structure

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Abstract. The embedding Chains_•(\mathbb{R}) \hookrightarrow Cochains^{1-*}(\mathbb{R}) as the compactly supported cochains might lead one to expect Chains_•(\mathbb{R}) to carry a nonunital commutative Frobenius algebra structure, up to a degree shift and some homotopic weakening of the axioms. We prove that under reasonable “locality” conditions, a cofibrant resolution of the dioperad controlling nonunital shifted-Frobenius algebras does act on Chains_•(\mathbb{R}), and in a homotopically-unique way. But we prove that this action does not extend to a homotopy Frobenius action at the level of properads or props. This gives an example of a geometrically meaningful algebraic structure on homology that does not lift in a geometrically meaningful way to the chain level.

1. Introduction

A basic tenet of algebraic topology is that algebraic structures on the homology of a space should come from algebraic structures at the chain level, with the understanding that equations are weakened to homotopy equivalences. Given some structure on homology, one then can pose the following questions: What is the appropriate weakening from equation to homotopy necessary to define a structure on chains? What is that structure at the chain level?

Here is an unsatisfying answer. Suppose that \( M \) is some space, and let \( H_\bullet(M) \) denote its homology with coefficients in a field. Suppose that \( P \) is a properad (or operad, or dioperad, or \ldots) that acts on \( H_\bullet(M) \), and let \( hP \) denote any cofibrant replacement of \( P \). Let \( C_\bullet(M) \) denote the complex of chains for your favorite chain model. There are myriad ways to choose a deformation retraction of \( C_\bullet(M) \) onto \( H_\bullet(M) \). Any such choice determines, via homotopy transfer theory, an action of \( hP \) on \( C_\bullet(M) \) that induces the action of \( P \) on \( H_\bullet(M) \). The space of choices required to carry out this procedure is contractible, which is to say there is no choice at all.

The reason the above answer is unsatisfying is that it doesn’t lead to any further insight into the topology of \( M \) than what was already available from the action of \( P \) on \( H_\bullet(M) \). Rather, when one asks to lift the \( P \)-action to the chain level, one usually means that the action should be “geometrically meaningful,” preferably with some notion of “locality” built in. For example, the cohomology \( \check{H}^\bullet(M) \) is naturally a commutative algebra; the multiplication corresponds to the diagonal map \( M \hookrightarrow M \times M \). One can look for cochain-level multiplications \( C^\bullet(M) \otimes C^\bullet(M) \to C^\bullet(M) \) that respect some notion of “locality,” and extend such a multiplication to an action by some cofibrant replacement \( h\text{Com} \) of the commutative operad \( \text{Com} \). By using a deformation retraction of \( C^\bullet(M) \) onto \( \check{H}^\bullet(M) \) and some homotopy transfer theory, one can then use the \( h\text{Com} \)-structure on \( C^\bullet(M) \) to build an \( h\text{Com} \)-algebra structure on \( \check{H}^\bullet(M) \). This structure will begin with the commutative multiplication, but include more data, namely the Massey products.

However, as this paper demonstrates, not all algebraic structures on homology lift in a geometric way to the chain level. Define a Frob_{1}-algebra to be a graded commutative Frobenius algebra in which the comultiplication has homological degree 0 but the multiplication has homological degree -1; the main example is the homology \( H_\bullet(S^1) \) of a circle. We will focus on “open and coopen” Frobenius algebras, which need not have unit or counit, and can be infinite-dimensional. It is reasonable to expect the chains \( C_\bullet(\mathbb{R}) \) on the line to support a Frob_{1}-algebra structure: if we take our chain model to be the complex \( C_\bullet(\mathbb{R}) = \Omega^{1-*}_{\text{cpt}}(\mathbb{R}) \) of compactly-supported smooth de Rham forms,
then we get a non-unital degree\((-1)\) multiplication which is strictly \((\text{anti})\)commutative and associative; if we take our chain model to be the complex \(C_\bullet(\mathbb{R}) = \Omega^{\text{cpt},\bullet}_\text{dist}(\mathbb{R})\) of compactly supported distributional \(\text{de Rham forms},\) then we get a degree-0 comultiplication which is strictly cocommutative and coassociative. Indeed, we will prove that any reasonable model of \(C_\bullet(\mathbb{R})\) supports an \(h\text{Frob}_1\)-algebra structure generalizing these (co)multiplications, provided \(h\text{Frob}_1 = h^\text{dist}\text{Frob}_1\) is resolved as a dioperad (meaning only tree-like compositions are used). But we will prove that \(C_\bullet(\mathbb{R})\) does not support any geometrically meaningful action of the properadic (graph-like compositions) cofibrant resolution \(h^{\text{pr}}\text{Frob}_1\). This contradicts the main result of [Wil07].

1.1. Outline. In Section 2 we define the notion of \textit{quasilocality} that we take as a minimal requirement for a chain-level structure to be \textit{geometrically meaningful.} Section 3 recalls the basic theory of dioperads and properads, and introduces our main example, the \((\text{di/pr})\text{operad Frob}_1\) controlling nonunital noncounital commutative Frobenius algebras in which the multiplication has homological degree \(-1\). The Koszulity of \(\text{Frob}_1\) is proven in Section 4, and used to compute a small cofibrant replacement \(\text{shFrob}_1\). The main results are in Section 5: Theorem 5.1 constructs a contractible space of quasilocally translation-invariant actions of the dioperad \(\text{sh}^\text{dist}\text{Frob}_1\) on \(\text{Chains}_\bullet(\mathbb{R})\), and Theorem 5.2 proves that the properad \(\text{sh}^{\text{pr}}\text{Frob}_1\) does not act quasilocally in a way that induces the \((\text{co})\text{multiplication on (co)homology.}

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1.3. Conventions. We work over a ground field of characteristic 0, which we will call \(\mathbb{Q}\), even though the reader may in fact choose to use \(\text{de Rham forms with coefficients in }\mathbb{R}\). We always use homological conventions — the differential has homological degree \(-1\) — and denote the category of chain complexes by \(\text{DGVec}\). We use the usual Koszul sign rules: the canonical isomorphism \(V \otimes W \equiv W \otimes V\), for \(V, W \in \text{DGVec}\), sends \(v \otimes w \mapsto (-1)^{(\deg v)(\deg w)}w \otimes v\) if \(v\) and \(w\) are homogeneous. We let \([n]\) denote the one-dimensional graded vector space satisfying \(\dim [n]_n = 1\) and \(\dim [n]_\bullet = 0\) if \(\bullet \neq n\), and we shift chain complexes by \(V[n] = V \otimes [n]\); note that this introduces signs to formulas involving homogeneous elements. Our primary references for the theory of properads, including their homotopy theory and Koszul duality, are [Val07, MV09a, MV09b].

2. Quasilocality

We fix a model of chains \(C_\bullet(\mathbb{R}^n),\) defined (at least) on \(n\)-dimensional Euclidean space. Many models work, including the complex \(\Omega^{\text{cpt},\bullet}_\text{dist}(\mathbb{R}^n)\) of smooth (resp. distributional) compactly-supported \(\text{de Rham forms},\) and the complex of cellular chains for the cubulation of \(\mathbb{R}^n\) given by slicing along the hyperplanes \(\mathbb{R}^{k-1} \times \{z\} \times \mathbb{R}^{n-k}\) for \(z \in \mathbb{Z}\) and \(k = 1, \ldots, n\). For these models, one has a canonical isomorphism \(C_\bullet(\mathbb{R}^m) \otimes C_\bullet(\mathbb{R}^n) \cong C_\bullet(\mathbb{R}^{m+n})\); the tensor product is the algebraic tensor product if cellular chains are used, but the projective tensor product for the \(\text{de Rham models}.\) Moreover, there is a natural inclusion \(C_\bullet(\mathbb{R}^m) \hookrightarrow C^{m-\bullet}(\mathbb{R}^m)\) of chains as compactly supported cochains. The reader may always check if any favorite chain model works in our construction.

Choose \(f : C_\bullet(\mathbb{R}^m) \rightarrow C_\bullet(\mathbb{R}^n)\) linear and homogeneous. Its \textit{graph} is the corresponding cochain \(\text{graph}(f)\) on \(\mathbb{R}^{m+n}\), for the \(\text{de Rham models}, \text{graph}(f)\) is the integral kernel of \(f\), and for cellular chains \(\text{graph}(f) \in C^\bullet(\mathbb{R}^{m+n})\) simply records the matrix coefficients of \(f\). In particular, it makes sense to talk about the \textit{support} of \(\text{graph}(f)\). (Depending on the \text{de Rham model used, some linear operators may not be represented by integral kernels. But it suffices for our purposes to restrict just to integral operators whose kernels are sufficiently smooth.)

Let \(\text{diag} : \mathbb{R} \hookrightarrow \mathbb{R}^{m+n}\) denote the diagonal embedding. For \(\ell \in \mathbb{R}_{>0}\), let \(B_\ell(\text{diag}(\mathbb{R}))\) denote the closed tubular neighborhood of \(\text{diag}(\mathbb{R})\) of radius \(\ell\).
Definition 2.1. A linear map \( f : C_*(\mathbb{R}^m) \to C_*(\mathbb{R}^n) \), for \( m, n > 0 \), is \( \ell \)-quasilocal if its graph is supported in \( B_\ell(\text{diag}(\mathbb{R})) \), and quasilocal if it is \( \ell \)-quasilocal for some \( \ell > 0 \). It is clear that the boundary \( [\partial, f] = \partial \circ f - (-1)^{\deg f} f \circ \partial \) of \( f \) is \( \ell \)-quasilocal if \( f \) is. The complex of all quasilocal maps \( C_*(\mathbb{R}^m) \to C_*(\mathbb{R}^n) \) is denoted \( \text{QLoc}(m, n) \).

A linear map \( f : C_*(\mathbb{R}^m) \to C_*(\mathbb{R}^n) \), for \( m, n > 0 \), is translation-invariant if it is equivariant for the translation action of \( \mathbb{R} \) (for the de Rham models) or \( \mathbb{Z} \) (for cellular chains) acting on \( \mathbb{R} \). The subcomplex of \( \text{QLoc}(m, n) \) consisting of translation-invariant maps is denoted \( \text{QLoc}^{\text{inv}}(m, n) \).

Quasilocality provides a good chain-level version of “locality” for the purposes of intersection theory, since often one must perturb things slightly to make intersections well-defined. Translation-invariance is a reasonable request for any geometrically-meaningful structure on \( \mathbb{R} \).

An important side effect of quasilocality is that quasilocal maps extend from chains to cochains:

Let \( C^{1-\bullet}(\mathbb{R}) \) denote the complex of cochains corresponding to the chain model \( C_*(\mathbb{R}) \), such that \( C_*(\mathbb{R}) \hookrightarrow C^{1-\bullet}(\mathbb{R}) \) embeds the chains as the compactly supported cochains. Then any \( f \in \text{QLoc}(m, n) \) defines a map \( C^{1-\bullet}(\mathbb{R})^\otimes m \to C^{1-\bullet}(\mathbb{R})^\otimes n \), in addition to the map \( C_*(\mathbb{R})^\otimes m \to C_*(\mathbb{R})^\otimes n \) given in Definition 2.1. These two versions of \( f \) are intertwined by the inclusion \( C_*(\mathbb{R}) \hookrightarrow C^{1-\bullet}(\mathbb{R}) \).

In particular, if \( f \) is \( \partial \)-closed, then it defines a map \( H_\bullet(f) : Q = H_\bullet(\mathbb{R})^\otimes m \to H_\bullet(\mathbb{R})^\otimes n = Q \) and also a map \( H^{1-\bullet}(f) : Q[m] = H^{1-\bullet}(\mathbb{R})^\otimes m \to H^{1-\bullet}(\mathbb{R})^\otimes n = Q[n] \).

Finally, the space of quasilocal maps has a very manageable homology:

Proposition 2.3. The homology of \( \text{QLoc}^{\text{inv}}(m, n) \) is

\[
\dim H_\bullet(\text{QLoc}^{\text{inv}}(m, n), \partial) = \begin{cases} 
1, & \bullet = -m \text{ or } -m + 1 \\
0, & \text{otherwise}. 
\end{cases}
\]

Proof. The neighborhood \( B_\ell(\text{diag}(\mathbb{R})) \) contracts onto \( \text{diag}(\mathbb{R}) \) in a translation-invariant way. Thus the complex of translation-invariant cochains supported in \( B_\ell(\text{diag}(\mathbb{R})) \) has the homology of a circle, and the inclusions as \( \ell \) increases are quasisomorphisms. The degree shift corresponds to the shift required to embed \( C_*(\mathbb{R}^m) \) into \( C_*(\mathbb{R}^m) \). \( \square \)

3. Dioperads and properads

Dioperads were introduced in [Gan03] and properads in [Val07]. Both provide frameworks in which to axiomatize algebraic structures with many-to-many operations. There are many equivalent definitions; we will use the following.

Definition 3.1. Let \( \mathcal{S} \) denote the groupoid of finite sets and bijections. An \( \mathcal{S} \)-bimodule is a functor \( P : \mathcal{S}^{\text{op}} \times \mathcal{S} \to \text{DGVect} \). Thus, the data of an \( \mathcal{S} \)-bimodule is a collection of chain complexes \( P(m, n) \) for \( (m, n) \in \mathbb{N}^2 \), along with, for each \( (m, n) \), an action on \( P(m, n) \) of \( \mathcal{S}_n^{\text{op}} \times \mathcal{S}_m \), where \( \mathcal{S}_n \) denotes the symmetric group on \( n \) letters.

Definition 3.2. A properad is an \( \mathcal{S} \)-bimodule along with, for every tuple of finite sets \( m_1, m_2, n_1, n_2, k \) with \( k \) nonempty, a composition map:

\[
\begin{array}{ccc}
m_2 & \cdots & m_1 \\
\vdots & \ddots & \vdots \\
\cdots & \ddots & \cdots \\
& \ddots & \cdots \\
m_2 & \cdots & m_1 \\
\end{array} : P(m_1, k \sqcup n_1) \otimes P(m_2 \sqcup n_2) \to P(m_1 \sqcup m_2, n_1 \sqcup n_2)
\]

The composition maps should be compatible with the \( \mathcal{S} \)-bimodule structure in the obvious way from the picture. Moreover, we demand an associativity condition, which is actually four conditions for
the types of connected directed acyclic graphs with three vertices:

\[ v_1 \rightarrow v_2 \rightarrow v_3, \quad v_1 \rightarrow v_3 \rightarrow v_2, \quad v_2 \rightarrow v_1 \rightarrow v_3, \quad v_2 \rightarrow v_3 \rightarrow v_1. \]

The reader is invited to spell out the details of the associativity equations; note that for the last two, one must reverse the order of two factors in a tensor product, and this introduces signs.

A dioperad is as above, but the only compositions that are defined are when \( k \) is a set of size 1:

\[ P(\mathbf{m}_1, \{\ast\} \sqcup \mathbf{n}_1) \otimes P(\mathbf{m}_2 \sqcup \{\ast\}, \mathbf{n}_2) \to P(\mathbf{m}_1 \sqcup \mathbf{m}_2, \mathbf{n}_1 \sqcup \mathbf{n}_2) \]

There are associativity axioms for each of the following types of diagrams:

A coproperad has instead a decomposition map \( P(\mathbf{m}_1 \sqcup \mathbf{m}_2, \mathbf{n}_1 \sqcup \mathbf{n}_2) \to P(\mathbf{m}_1, k \sqcup \mathbf{n}_1) \otimes P(\mathbf{m}_2 \sqcup k, \mathbf{n}_2) \) for each tuple \((\mathbf{m}_1, \mathbf{m}_2, \mathbf{n}_2, \mathbf{n}_2, k)\), satisfying coassociativity axioms. A codioperad similarly has decomposition maps whenever \( k = \{\ast\} \).

Our convention will be that (co)(di/pr)operads may be non(co)unital.

**Definition 3.3.** For any \( V \in \text{DGVect} \), the (di/pr)operad \( \text{End}(V) \) satisfies \( \text{End}(V)(\mathbf{m}, \mathbf{n}) = \text{hom}(V^\mathbf{m}, V^\mathbf{n}) \). An action of a (di/pr)operad \( P \) on \( V \) is a homomorphism \( P \to \text{End}(V) \). If \( V \) is equipped with an action of \( P \), then we will call \( V \) a \( P \)-algebra.

The category of (di/pr)operads has a model category structure in which the weak equivalences are the quasiisomorphisms, and the fibrations are the surjections [MV09b, Appendix A]. Abstract nonsense of model categories guarantees that if \( h_1 P \) and \( h_2 P \) are any two cofibrant replacements of the same (di/pr)operad \( P \), then we can turn any action of \( h_1 P \) on \( V \) into an action of \( h_2 P \), and vice versa, and the spaces of choices required to do so are contractible. Thus we are justified in saying that a homotopy \( P \)-action on \( V \) is an action on \( V \) of any cofibrant replacement \( hP \) of \( P \).

There is a forgetful functor from properads to dioperads, whose left adjoint defines the universal enveloping properad of a dioperad. The reader should be aware that these functors are known not to be exact [MV09a, Theorem 47]. For comparison, define a prop to be an \( S \)-bimodule in which composition is defined even when \( k = \emptyset \) (and satisfying some extra commutativity/associativity axioms for disconnected directed acyclic graphs). By [Val07], the forgetful functor from props to properads and its adjoint constructing the universal enveloping prop of a properad are exact. In particular, properadic homotopy actions always extend to propic homotopy actions.

We now list the (di/pr)operads that will be of primary interest:

**Definition 3.4.** It follows from the triangle inequality that \( Q\text{Loc}^\text{inv} \) is a sub-(di/pr)operad of both \( \text{End}(C_*(\mathbb{R})) \) and \( \text{End}(C^{1-\ast}(\mathbb{R})) \). An action of \( P \) on \( C_*(\mathbb{R}) \) or \( C^{1-\ast}(\mathbb{R}) \) is quasilocal and translation invariant if it factors through \( Q\text{Loc}^\text{inv} \).
Definition 3.5. The (di/pr)operad \( \text{Frob}_1 \) of open and coopen 1-shifted commutative Frobenius algebras satisfies:

\[
\dim \text{Frob}_1(m, n) = \begin{cases} 1, & m, n > 0 \text{ and } \bullet = 1 - m, \\ 0, & mn = 0 \text{ or } \bullet \neq 1 - m. \end{cases}
\]

The \( S_n \) action on \( \text{Frob}_1(m, n) \) is trivial, whereas \( S_n \) acts via the sign representation. The composition \( P(m_1, k \sqcup n_1) \otimes P(m_2 \sqcup k, n_2) \to P(m_1 \sqcup m_2, n_1 \sqcup n_2) \) is 0 unless \( k = \{ \ast \} \), in which case it is multiplication by \((-1)^{m_2} \).

Note that the universal enveloping properad of the dioperad \( \text{Frob}_1 \) is the properad \( \text{Frob}_1 \): the fact that composition vanishes whenever \( k = |k| > 0 \) follows from the \( S \)-actions. Thus the notion of “\( \text{Frob}_1 \)-algebra” is unambiguous. However, because the universal enveloping properad functor is not exact, the notion of “homotopy \( \text{Frob}_1 \)-algebra” is ambiguous. Namely, let \( h^{\text{di}} \text{Frob}_1 \) denote a cofibrant replacement of \( \text{Frob}_1 \) in the category of dioperads, and \( h^{\text{pr}} \text{Frob}_1 \) a cofibrant replacement in properads. Since forgetting is not exact, the underlying dioperad of \( h^{\text{pr}} \text{Frob}_1 \) may not be cofibrant, although it does fiber acyclicly over \( \text{Frob}_1 \); the universal enveloping properad of \( h^{\text{di}} \text{Frob}_1 \) will be cofibrant, but may not fiber acyclicly over \( \text{Frob}_1 \). In either case, there is a canonical contractible space of homomorphisms \( h^{\text{di}} \text{Frob}_1 \to h^{\text{pr}} \text{Frob}_1 \), through which any \( h^{\text{pr}} \text{Frob}_1 \)-algebra forgets to an \( h^{\text{di}} \text{Frob}_1 \)-algebra. As we will see from trying to represent each on \( \mathbb{C}_\bullet(\mathbb{R}) \), this forgetting really is a loss of information: not all actions of \( h^{\text{di}} \text{Frob}_1 \) extend to actions of \( h^{\text{pr}} \text{Frob}_1 \).

4. Koszulity of \( \text{Frob}_1 \)

As far as this paper is concerned, the raison d’être of Koszul duality theory is to provide small cofibrant replacements of objects of interest. We will briefly recall enough of the theory for our purposes.

For any \( S \)-bimodule \( T \), we denote by \( \mathcal{F}(T) \) the free (di/pr)operad generated by \( T \). (This is not much of an abuse of notation, as the universal enveloping properad of the free dioperad generated by \( T \) is the free properad generated by \( T \).) We note that \( \mathcal{F}(T) \) is also a co(di/pr)operad. We let \( \mathcal{F}^{(k)}(T) \) denote the sub-\( S \)-bimodule of \( \mathcal{F}(T) \) that transforms with weight \( k \) under the canonical \( \mathbb{Q}^\times \)-action on \( T \). Note that \( \mathcal{F}^{(1)}(T) = T \), and \( \mathcal{F}(T) = \bigoplus_{k \geq 1} \mathcal{F}^{(k)}(T) \).

Definition 4.1. A quadratic (di/pr)operad is a (di/pr)operad presented as \( P = \mathcal{F}(T)/(R) \), where \( T \) is an \( S \)-bimodule and \( R \subseteq \mathcal{F}^{(2)}(T) \) is a sub-\( S \)-bimodule generating the ideal \( (R) \).

The quadratic dual \( P^i \) of a quadratic (di/pr)operad \( P \) is the maximal graded sub-co(di/pr)operad of \( \mathcal{F}(T[1]) \) whose intersection with \( \mathcal{F}^{(2)}(T[1]) \) is precisely \( R[2] \subseteq \mathcal{F}^{(2)}(T[1]) \).

Definition 4.2. Let \( Q \) be any co(di/pr)operad, such that for each \( m_1, m_2, n_1, n_2 \), there are only finitely many \( k \in \mathbb{N} \) for which the decomposition map \( P(m_1 \sqcup m_2, n_1 \sqcup n_2) \to P(m_1, k \sqcup n_1) \otimes P(m_2 \sqcup k, n_2) \) is nonzero. The cobar construction applied to \( Q \) produces the (di/pr)operad \( BQ = \mathcal{F}(Q[-1]) \), with the differential extending the degree-(-1) map \( \sum \text{(decompositions)} : Q[-1] \to \mathcal{F}^{(2)}(Q[-1]) \). Coassociativity is equivalent to the differential squaring to 0.

The (di/pr)operad \( BQ \) is an example of a quasifree (di/pr)operad, which more generally is any dg (di/pr)operad which would be free if one were to forget its differential.

Lemma 4.3. Let \( P \) be a quadratic (di/pr)operad, with generators \( T \) and relations \( R \). Then \( B P^i \) is cofibrant and fibers over \( P \).

Proof. Cofibrancy follows from [MV09b, Corollary 40]. To define the map \( B P^i \to P \), it suffices to define the action on generators \( P^i[-1] \subseteq \mathcal{F}(T[1])[-1] \). We declare that \( T[1][-1] \subseteq P[-1] \) maps by the identity to \( T \subseteq P \), and that all other generators map to 0. To check that this is well-defined, it suffices to check that the derivatives of the generators in \( \mathcal{F}^{(2)}(T[1])[-1] \) get mapped to 0. But
these generators are precisely a copy of $R[1]$, and differentiating and mapping gives the (trivial) image of $R$ in $P = \mathcal{F}(T)/(R)$. □

**Definition 4.4.** A quadratic (di/pr)operad is Koszul if the canonical fibration $BP_i \to P$ from Lemma 4.3 is acyclic, in which case $BP_i$ is a cofibrant replacement of $P$. When $P$ is Koszul, we let $\mathbf{sh}P = BP_i$, and call $\mathbf{sh}P$-algebras strong homotopy $P$-algebras.

For any (di/pr)operad $P$ satisfying some mild finite-dimensionality assumptions, the (di/pr)operad $B((BP^*)^*)$ is always a cofibrant replacement of $P$. (The second dual should be taken relative to the grading induced by the $\mathbb{Q}$ action on the $S$-bimodule $P$.) The point is that $BP_i$ is generally much smaller than $B((BP^*)^*)$, and hence more manageable.

The main result of this section says that $\text{Frob}_1$ is Koszul, as both a dioperad and as a properad:

**Proposition 4.5.** The (di/pr)operad $\text{Frob}_1$ of open and coopen commutative Frobenius algebras has the following quadratic presentation, with respect to which it is Koszul. The generating $S$-bimodule $T$ is spanned by:

\[
\begin{align*}
\begin{array}{c}
\text{homological degree } 0 \\
\text{homological degree } -1
\end{array}
\end{align*}
\]

A basis for the relations $R$ is:

\[
\begin{align*}
&\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{homological degree } 0 \\
\text{homological degree } -1
\end{array}
\end{array}
\end{array}
\end{align*}
\]

**Proof.** The signs arise because the degree-($-1$) multiplication does not make a $\text{Frob}_1$-algebra $V$ into a commutative algebra, but rather makes $V[-1]$ into a commutative algebra. It is clear that these define the dioperad $\text{Frob}_1$ and hence its universal enveloping properad. The properad $\text{Frob}_1$ has no operations with genus, because the composition $\triangledown$ vanishes, as it must transform both trivially and by the sign representation under the $S_2$-action interchanging the two interior edges.

The suboperad of $\text{Frob}_1$ generated by just the multiplication $\triangleleft$ is a shear of the nonunital commutative operad (meaning its representations on $V$ are representations of $\text{Com}$ on $V[1]$), which is known to be Koszul [LV12, Theorem 8.57]; similarly the comultiplication $\triangledown$ generates a copy of the Koszul properad defining nonunital cocommutative coalgebras. The second line of relations, along with the relation $\triangledown = 0$, are together a “replacement rule” in the sense of [Val07, 8.1], and hence $\text{Frob}_1$ is Koszul by [Val07, Proposition 8.4]. □

**Corollary 4.6.** $\text{Frob}_1$ has a cofibrant replacement $\mathbf{sh}\text{Frob}_1$, which is quasifree with generating $S$-bimodule $(\text{Frob}_1)[-1]$. The generating $S$-bimodule $T$ of $\text{Frob}_1$ has a bigrading by $(\# \triangleleft, \# \triangledown)$, and the relations $R$ are homogeneous for this bigrading, hence this bigrading extends to $(\text{Frob}_1)[-1]$, and the differential on $\mathbf{sh}\text{Frob}_1$ preserves this bigrading.

If we are working with dioperads, the piece $(\text{Frob}_1)[-1](m,n)$ with $m$ inputs and $n$ outputs is homogeneous for this bigrading, with $\# \triangleleft = m-1$ and $\# \triangledown = n-1$, and is entirely in homological degree $\# \triangledown - 1 = n - 2$.

If we are working with properads, the piece $(\text{Frob}_1)[-1](m,n)$ with $m$ inputs and $n$ outputs has a grading by genus $\beta$ satisfying $\# \triangleleft = \beta + m - 1$, $\# \triangledown = \beta + n - 1$, and homological degree $= \# \triangledown - 1 = \beta + n - 2$.

**Proof.** The formulas follow from definition-unpacking and elementary combinatorics. □
We conclude this section by describing the generators \((\text{Frob}_1)^i[-1]\) of \(\text{sh Frob}_1\) for small \(#\updownarrow + \#\downarrow\). We will record the representations of the symmetric group using the usual Young tableaux, placed under a generator of that irrep. The \(#\updownarrow + \#\downarrow = 1\) piece of \((\text{Frob}_1)^i[-1]\) is a copy of the generators \(T\), and decomposes as:

The \(#\updownarrow + \#\downarrow = 2\) piece is a copy of \(R\), decomposing as:

In the middle summand, \(2 = \square \oplus \mathbb{1}\) denotes the two-dimensional permutation representation of \(S_2\). We have adopted the following diagrammatic convention: the diagram inside each rectangle is an element of \((\text{Frob}_1)^i \subseteq \mathcal{F}(T[1])\), and the boxed diagram is the corresponding element of \((\text{Frob}_1)^i[-1] \subseteq \mathcal{B}(\text{Frob}_1)^i \subseteq \mathcal{B}(\mathcal{F}(T[1]))\). Later we will stack diagrams; this corresponds to composition in \(\mathcal{B}((\text{Frob}_1)^i) \subseteq \mathcal{B}(\mathcal{F}(T[1]))\).

When \(#\updownarrow + \#\downarrow \geq 3\), the dioperadic and properadic versions of \((\text{Frob}_1)^i[-1]\) diverge. The dioperadic version appears as the genus \(\beta = 0\) direct summand of the properadic version. A linear combination of graphs with \(#\updownarrow + \#\downarrow = 3\) is in \((\text{Frob}_1)^i[-1]\) if and only if its derivative in \(\mathcal{B}(\mathcal{F}(T[1]))\) lands in \(\mathcal{B}((\text{Frob}_1)^i)\). Of the \(#\updownarrow + \#\downarrow = 3\) part of \((\text{Frob}_1)^i[-1]\), we will compute just the summand with genus \(\beta = 1\). Any composition of \(\updownarrow\) and \(\downarrow\) with genus \(\beta \geq 1\) has \(#\updownarrow \geq 1\) and \(#\downarrow \geq 1\). Since \(\downarrow = 0\), the only graphs with \(\beta = 1\) and \((#\updownarrow, #\downarrow) = (1, 2)\) are \(\updownarrow\) and \(\updownarrow\). We claim that each of these is in \((\text{Frob}_1)^i[-1]\). Indeed, one may compute that, in \(\mathcal{B}(\mathcal{F}(T[1]))\),

\[
(*) \quad \partial \left( \begin{array}{c}
\begin{array}{c}
\boxtimes \downarrow
\end{array}
\end{array} \right) = \begin{array}{c}
\begin{array}{c}
\updownarrow
\end{array} - \begin{array}{c}
\downarrow
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\boxtimes 2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array} \in \text{sh Frob}_1,
\]

where \(T\) is the \(S\)-bimodule spanned by \(\{\updownarrow, \downarrow\}\), and therefore \(\downarrow \in (\text{Frob}_1)^i[-1]\). It follows that the \(#\updownarrow + \#\downarrow = 3\) part of the properadic \((\text{Frob}_1)^i[-1]\) is:

\[
(\beta = 0 \text{ part}) \oplus \begin{array}{c}
\begin{array}{c}
\boxtimes 2
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
2 \boxtimes
\end{array}
\end{array}
\]

Finally, we will need later to know the piece of \((\text{Frob}_1)^i[-1]\) with \(#\updownarrow + \#\downarrow = 4\) and genus \(\beta = 2\). Since \(#\updownarrow = \beta + m - 1 \geq \beta\) and \(#\downarrow = \beta + n - 1 \geq \beta\), the only way to have a composition of
and with \((\# \begin{diagram} \rightarrow \end{diagram} + \# \begin{diagram} \rightarrow \end{diagram}, \beta) = (4, 2)\) is if \((\# \begin{diagram} \rightarrow \end{diagram}, \# \begin{diagram} \rightarrow \end{diagram}) = (2, 2)\) and \((m, n) = (1, 1)\). Recalling that \(\hat{\beta} = 0\), the only graph with \((m, n, \beta) = (1, 1, 2)\) is \(\begin{diagram} \rightarrow \end{diagram} + \begin{diagram} \rightarrow \end{diagram}\). In particular, the (anti)symmetry of the \(\begin{diagram} \rightarrow \end{diagram}\) and \(\begin{diagram} \rightarrow \end{diagram}\) implies that this graph is equal up to sign to any of its permutations. Since this graph spans a homogeneous piece of \(\mathcal{F}(T[1])[-1]\), it either is or is not in \((\text{Frob}_1)^i[-1]\), depending on whether its derivative in \(B(\mathcal{F}(T[1]))\) is in \(\text{sh Frob}_1 = B((\text{Frob}_1)^i)\). In fact:

\[
(\dagger) \quad \partial \left( \begin{array}{c}
\begin{diagram} \rightarrow \end{diagram} \\
\begin{diagram} \rightarrow \end{diagram}
\end{array} \right) = \begin{diagram} \rightarrow \end{diagram} + \begin{diagram} \rightarrow \end{diagram} + \frac{1}{3} \begin{diagram} \rightarrow \end{diagram} \in \text{sh Frob}_1.
\]

Therefore in weight \# \(\begin{diagram} \rightarrow \end{diagram} + \# \begin{diagram} \rightarrow \end{diagram} = 4\), \((\text{Frob}_1)^i[-1]\) looks like:

\[
(\beta = 0 \text{ part}) \oplus (\beta = 1 \text{ part}) \oplus \begin{diagram} \rightarrow \end{diagram} \quad (\beta = 1 \text{ part})
\]

5. **The (non)existence of homomorphisms \(\text{sh Frob}_1 \rightarrow Q\text{Loc}^\text{inv}\)**

Let \(\text{sh}^{\text{di}}\text{Frob}_1\) denote the cofibrant replacement as a dioperad of \(\text{Frob}_1\) coming from Corollary 4.6, and let \(\text{sh}^{\text{pr}}\text{Frob}_1\) denote the cofibrant replacement as a properad. There is a canonical inclusion \(\text{sh}^{\text{di}}\text{Frob}_1 \hookrightarrow \text{sh}^{\text{pr}}\text{Frob}_1\), given by including the generators of \(\text{sh}^{\text{di}}\text{Frob}_1\) as the genus \(\beta = 0\) generators of \(\text{sh}^{\text{pr}}\text{Frob}_1\). In this final section, we will prove that the space of quasilocally invariant actions of \(\text{sh}^{\text{di}}\text{Frob}_1\) on \(C_\bullet(\mathbb{R})\) that induce the standard multiplication on \(H^{1-\bullet}(\mathbb{R})\) and the standard comultiplication on \(H_\bullet(\mathbb{R})\) is contractible, but that such a dioperadic action does not lift to a properad homomorphism \(\text{sh}^{\text{pr}}\text{Frob}_1 \rightarrow Q\text{Loc}^\text{inv}\).

**Theorem 5.1.** Recall from Lemma 2.2 that we have two embeddings \(Q\text{Loc} \hookrightarrow \text{End}(C_\bullet(\mathbb{R}))\) and \(Q\text{Loc} \hookrightarrow \text{End}(C^{1-\bullet}(\mathbb{R}))\), and so any \(\partial\)-closed element of \(Q\text{Loc}\) defines actions on both homology and cohomology. Consider the space of maps \(\eta : \text{sh}^{\text{di}}\text{Frob}_1 \rightarrow Q\text{Loc}^{\text{inv}}\) for which \(H_\bullet(\eta(\begin{diagram} \rightarrow \end{diagram})) : H_\bullet(\mathbb{R}) \rightarrow H_\bullet(\mathbb{R})^{\otimes 2} \rightarrow H^{1-\bullet}(\mathbb{R})\) is the standard comultiplication. This space is contractible.

By definition, if \(P\) and \(Q\) are (di/pr)operads, the space of maps \(P \rightarrow Q\) is the simplicial set whose \(k\)-simplices are homomorphisms \(P \rightarrow Q \otimes Q[\Delta^k]\), where \(Q[\Delta^k] = \mathbb{Q}[t_0, \ldots, t_k, \partial t_0, \ldots, \partial t_k] / \langle \sum t_i = 1, \sum \partial t_i = 0 \rangle\) is the commutative dg algebra of polynomial de Rham forms on the \(k\)-dimensional simplex. If \(P\) is cofibrant, then this simplicial set satisfies the Kan horn-filling condition. By convention, a space is contractible if it has the homotopy type of \(\{\ast\}\), which in particular implies that it is nonempty.

To prove Theorem 5.1, we will use the following well-known basic facts of obstruction theory. Let \(P\) be a quasifree (di/pr)operad with a well-ordered set of generators \(f\) (each of which we take, for simplicity, to be homogeneous of homological degree \(\text{deg}(f)\)), and such that for each generator \(f\) of \(P\), \(\partial(f)\) is a composition of generators of \(P\) that are strictly earlier than \(f\) in the well-ordering. Then for any (di/pr)operad \(Q\), one may understand homomorphisms \(\eta : P \rightarrow Q\) inductively. Suppose
that such a homomorphism has been defined on all generators that are strictly earlier than $f$ in the well-ordering. Then $\eta(\partial f) \in Q$ is closed of homological degree $\deg(f) - 1$. The obstruction to defining $f$ is the class of $\eta(\partial f)$ in $H_{t}(Q)$. The first basic fact of obstruction theory is that $\eta(f)$ can be defined, and therefore the induction can be continued, if and only if its obstruction vanishes. In particular, maps $P \to Q$ are easy to construct whenever the homology groups of $Q$ vanish in degrees $\deg(f) - 1$ for generators $f$ of $P$.

The second basic fact of obstruction theory concerns the homotopy groups of the space of maps $P \to Q$. Suppose that the obstruction to defining $\eta(f)$ vanishes. How many choices are there in the definition of $\eta(f)$? Of course, there are exactly as many choices are there are cycles in $Q$ of degree $\deg(f)$. But many of these choices will be connected by a path, i.e. a map $P \to Q \otimes Q[\Delta^1] = Q[t, \partial_t]$. To construct a homomorphism $P \to Q[t, \partial_t]$ is the same as constructing a map $P \to Q$ that depends polynomially on $t$ and satisfies a polynomial ordinary differential equation. Moreover, by the assumptions on $P$, the integral form of this differential equation is a contraction mapping, form which it follows that solving the initial value problem is easy, and the boundary value problem is straightforward to study. For example, it is straightforward to show that if two choices for $\eta(f)$ are connected by a path, then which one is chosen will not affect whether later obstructions vanish.

The conclusion of such an analysis is that, provided the obstruction to defining $\eta(f)$ vanishes, the connected components of the space of choices for $\eta(f)$ is precisely $H_{\deg(f)}(Q)$. A similar analysis more generally shows that $\pi_j\{\text{space of choices for } \eta(f)\} = H_{\deg(f)+j}(Q)$, provided $\pi_i = 0$ for $i < j$, with the convention that $\pi_{-1}(X) = 0 = \{\ast\} = \text{"true"}$ for $X$ nonempty, and $\pi_{-1}(\emptyset) = \emptyset = \text{"false"}$, and that $\pi_j(X) = 0$ for all $X$ and all $j < -1$.

In particular, we may immediately conclude that the space of choices for $\eta(f)$ is contractible whenever $H_{\deg(f)+j}(Q) = 0$ for all $j \geq -1$.

**Proof of Theorem 5.1.** Consider first the generators $\bigotimes$ and $\bigodot$ of $\text{sh}^{\text{di}} Frob_1$. In order for them to induce the standard (co)multiplications on (co)homology, they must each be assigned to translation-invariant Thom forms around $\text{diag}(\mathbb{R}) \subseteq \mathbb{R}^3$. The space of translation-invariant Thom forms is contractible.

For the remaining generators, the homotopy type of the space of choices is contractible. We computed in Proposition 2.3 that $\text{QLoc}^{\text{inv}}(m, n)$ has homology only in degrees $-m$ and $-m+1$. Let $f$ be a generator of $\text{sh}^{\text{di}} Frob_1$; we may suppose without loss of generality that it is homogeneous for the bigrading $(\# \bigotimes, \# \bigodot)$. Then $f$ is in homological degree $\deg(f) = \# \bigodot - 1$ by Corollary 4.6. Thus the obstruction might be nonzero only when $\# \bigodot - 2 = -m$ or $-m+1$. Recalling that $m = \# \bigotimes + 1$, we see that the obstruction automatically vanishes unless $\# \bigotimes + \# \bigodot = 1$ or 2.

We have addressed already the generators for which $\# \bigotimes + \# \bigodot = 1$. For the generators with $\# \bigotimes + \# \bigodot = 2$, the obstruction is the difference between two translation-invariant Thom forms around $\text{diag}(\mathbb{R}) \subseteq \mathbb{R}^4$, and hence vanishes in homology. \hfill \Box

Theorem 5.1 gives an affirmative answer to the question of geometrically lifting the *dioperadic* action of $Frob_1$ on $H_{\bullet}(\mathbb{R})$ to the chain level. The main result of this paper is that the similar question for *properads* has a negative answer.

**Theorem 5.2.** There does not exist a properad morphism $\eta : \text{sh}^{\text{pr}} Frob_1 \to \text{QLoc}^{\text{inv}}$ such that $H_{\bullet}(\eta(\bigodot))$ and $H^{1-\bullet}(\eta(\bigotimes))$ are the standard comultiplication and multiplication.

**Proof.** Let $f$ be a generator of $\text{sh}^{\text{pr}} Frob_1$ with $m$ inputs and $n$ outputs which is homogeneous for the bigrading $(\# \bigotimes, \# \bigodot)$; as in Corollary 4.6, we let $\beta = \# \bigotimes - m + 1 = \# \bigodot - n + 1$ denote its genus. Restricting $\eta$ just to the genus $\beta = 0$ generators determines a map $\text{sh}^{\text{di}} Frob_1 \to \text{QLoc}^{\text{inv}}$, and by Theorem 5.1 the space of such maps is contractible. It then follows from the basic facts

Chains($\mathbb{R}$) does not admit a geometrically meaningful properadic homotopy Frobenius algebra structure
of obstruction theory that the homotopy type of the space of extensions to a map $\text{sh}^{\text{pr}} \text{Frob}_1 \to \text{QLoc}^{\text{inv}}$ is independent of the choices made for the map $\text{sh}^{\text{dR}} \text{Frob}_1 \to \text{QLoc}^{\text{inv}}$.

Simple combinatorics implies moreover that the obstruction automatically vanishes unless $m + n + 2\beta = 3$ or $4$, and that the space of choices is contractible if nonempty. Thus the only generators of $\text{sh}^{\text{pr}} \text{Frob}_1$ beyond those of $\text{sh}^{\text{dR}} \text{Frob}_1$ for which the obstruction might not vanish are permutations of $\text{sh}^{\text{pr}} \text{Frob}_1$. We will calculate their obstructions.

We will focus on the cellular model of chains $C_{\bullet}(\mathbb{R})$. The result for the de Rham models follows by replacing the choices below by smooth approximations thereof.

For the cellular model, there is a 0-cell $c_z$ for each $z \in \mathbb{Z}$, and a 1-cell $c_{z+\frac{1}{2}}$ for each $z \in \mathbb{Z}$, and the boundary operator is $\partial c_{z+\frac{1}{2}} = c_{z+1} - c_z$. We might as well choose the following Thom forms:

\begin{align*}
\begin{cases}
  c_z \mapsto & c_z \otimes c_z, \\
  c_{z+\frac{1}{2}} \mapsto & \frac{1}{2} \left( (c_z + c_{z+1}) \otimes c_{z+\frac{1}{2}} + c_{z+\frac{1}{2}} \otimes (c_z + c_{z+1}) \right),
\end{cases}
\end{align*}

\begin{align*}
\begin{cases}
  c_{z+\frac{1}{2}} \otimes c_{z+\frac{1}{2}} \mapsto & c_{z+1}, \\
  c_z \otimes c_{z+\frac{1}{2}} \mapsto & -\frac{1}{2} c_z, \\
  c_{z+\frac{1}{2}} \otimes c_z \mapsto & \frac{1}{2} c_z.
\end{cases}
\end{align*}

In both lines $z \in \mathbb{Z}$, and in the second line all non-listed pairs $c_x \otimes c_y$ get mapped to 0.

With these choices, $\partial$ sends $c_z \mapsto 0$ when $z \in \mathbb{Z}$, and sends:

\begin{align*}
  c_{z+\frac{1}{2}} \mapsto & \frac{1}{4} \left( (c_{z+1} - c_z) \otimes (c_{z+1} - c_z) \otimes c_{z+\frac{1}{2}} - c_{z+\frac{1}{2}} \otimes (c_{z+1} - c_z) \otimes (c_{z+1} - c_z) \right).
\end{align*}

Any assignment for the generator itself must have this as its derivative, and must also generate the two-dimensional irrep of $S_3$. One choice that works is to send $c_z \mapsto 0$ for $z \in \mathbb{Z}$ and:

\begin{align*}
  c_{z+\frac{1}{2}} \mapsto & \frac{1}{6} (c_{z+1} - c_z) \otimes (c_{z+1} - c_z) \otimes c_{z+\frac{1}{2}} + \frac{1}{12} \left( (c_{z+1} - c_z) \otimes c_{z+\frac{1}{2}} \otimes c_{z+\frac{1}{2}} + c_{z+\frac{1}{2}} \otimes c_{z+\frac{1}{2}} \otimes (c_{z+1} - c_z) \right)
\end{align*}

By our obstruction-theoretic analysis, the particular choice won’t affect whether later obstructions vanish.

A similar computation implies that the generator may be chosen to send

\begin{align*}
  c_z \otimes c_{z+\frac{1}{2}} & \mapsto \pm \frac{1}{4} c_z \otimes c_{z+\frac{1}{2}}, \quad \forall z \in \mathbb{Z}, \\
  \text{all other } c_x \otimes c_y & \mapsto 0.
\end{align*}

We may now compute the action of $\partial$, using equation (*). With our choices, this derivative vanishes on $c_z$ for $z \in \mathbb{Z}$, and sends:

\begin{align*}
  c_{z+\frac{1}{2}} & \mapsto -\frac{1}{12} (c_{z+1} - c_z) \otimes c_{z+\frac{1}{2}} + \frac{1}{12} c_{z+\frac{1}{2}} \otimes (c_{z+1} - c_z).
\end{align*}
Chains(\mathbb{R}) does not admit a geometrically meaningful properadic homotopy Frobenius algebra structure

Since the $S_2$ action is free, we may choose:

\[
: c_z \mapsto 0 \text{ and } c_{z+\frac{1}{2}} \mapsto -\frac{1}{12} c_{z+\frac{1}{2}} \otimes c_{z+\frac{1}{2}}, \quad \forall z \in \mathbb{Z}.
\]

The last generator for which the obstruction might not vanish is 

\[
\begin{pmatrix}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{pmatrix}
\]

Our computations above imply that in equation (†), the first summand on the right hand side acts by \(-\frac{1}{12}\) on 1-chains and by 0 on 0-chains. The second summand acts by \(-\frac{1}{12}\) on 0-chains and by 0 on 1-chains by a similar computation; indeed, we may make all choices symmetrically with respect to reversing inputs and outputs, and simultaneously reversing 0-chains and 1-chains. Finally, is non-zero only on 1-chains, but the image of

\[
\begin{pmatrix}
\begin{array}{c}
\uparrow \quad +
\end{array}
\end{pmatrix}
\]

consists only of 0-chains, and so the last summand in equation (†) vanishes, as it describes a map of homological degree 0.

All together, we see that $\partial \begin{pmatrix}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{pmatrix}$ acts on $C_*(\mathbb{R})$ by multiplication by \(-\frac{1}{12}\). The identity operator is exact only on complexes with trivial homology, and in particular multiplication by \(-\frac{1}{12}\) is not exact in $QLoc$. It therefore obstructs the existence of a homomorphism $sh^{pr}_{Frob_1} \rightarrow QLoc^{inv}$ sending $\begin{pmatrix}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{pmatrix}$ and $\begin{pmatrix}
\begin{array}{c}
\uparrow \quad +
\end{array}
\end{pmatrix}$ to Thom forms.

In fact, our calculations obstruct quasilocal $sh^{pr}_{Frob_1}$ actions even if translation-invariance is dropped (the homology of $QLoc(m,n)$ is one-dimensional concentrated in degree $-m+1$). As we remarked in the introduction, homotopy perturbation theory does construct a non-quasilocal homotopy Frob_1 structure on $C_*(\mathbb{R})$. It necessarily makes a different non-quasilocal choice for the generator $\begin{pmatrix}
\begin{array}{c}
\uparrow \\
\downarrow
\end{array}
\end{pmatrix}$. Up to equivalence, there was only one quasilocal choice for this generator, but many inequivalent non-quasilocal choices.

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