On 3-Coloring of \((2P_4, C_5)\)-Free Graphs*†

Vít Jelínek¹, Tereza Klímošová¹, Tomáš Masářík¹,²,³, Jana Novotná¹,², and Aneta Pokorná¹

¹Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic
²University of Warsaw, Poland
³Simon Fraser University, Burnaby, Canada
{jelinek, pokorna}@iuuk.mff.cuni.cz
{tereza,masarik,janca}@kam.mff.cuni.cz

Abstract

The 3-coloring of hereditary graph classes has been a deeply-researched problem in the last decade. A hereditary graph class is characterized by a (possibly infinite) list of minimal forbidden induced subgraphs \(H_1, H_2, \ldots\); the graphs in the class are called \((H_1, H_2, \ldots)\)-free. The complexity of 3-coloring is far from being understood, even for classes defined by a few small forbidden induced subgraphs. For \(H\)-free graphs, the complexity is settled for any \(H\) on up to seven vertices. There are only two unsolved cases on eight vertices, namely \(2P_4\) and \(P_8\). For \(P_8\)-free graphs, some partial results are known, but to the best of our knowledge, \(2P_4\)-free graphs have not been explored yet. In this paper, we show that the 3-coloring problem is polynomial-time solvable on \((2P_4, C_5)\)-free graphs.

1 Introduction

Graph coloring is a notoriously known and well-studied concept in both graph theory and theoretical computer science. A \(k\)-coloring of a graph \(G = (V, E)\) is defined as a mapping \(c : V \rightarrow \{1, \ldots, k\}\) which is proper, i.e., it assigns distinct colors to \(u, v \in V\) if \(uv \in E\). The \(k\)-COLORING problem asks whether a given graph admits a \(k\)-coloring. For any \(k \geq 3\), the \(k\)-coloring is a well-known NP-complete problem [27]. We also define a more general \(\text{list-}k\text{-coloring}\) where each vertex \(v\) has a list \(P(v)\) of allowed colors such that \(P(v) \subseteq \{1, \ldots, k\}\). In that case, the coloring function \(c\), in addition to being proper, has to respect the lists, that is, \(c(v) \in P(v)\) for every vertex \(v\).

A graph class is hereditary if it is closed under vertex deletion. It follows that a graph class \(G\) is hereditary if and only if \(G\) can be characterized by a unique (not necessarily finite) set \(H_G\) of minimal forbidden induced subgraphs. Special attention was given to hereditary graph classes where \(H_G\) contains only one or only a very few elements. In such cases, when \(\{H\} = H_G\), or \(\{H_1, H_2, \ldots\} = H_G\), we say that \(G \in G\) is \(H\)-free, or \((H_1, H_2, \ldots)\)-free, respectively. We let \(P_t\) denote the path on \(t\) vertices, and \(C_\ell\) the cycle on \(\ell\) vertices. We let \(\overline{H}\) denote the complement of a graph \(H\). For two graphs \(H_1\) and \(H_2\), we let \(H_1 + H_2\) denote their disjoint union, and we write \(kH\) for the disjoint union of \(k\) copies of a graph \(H\).

*V. Jelínek was supported by project 18-19158S of the Czech Science Foundation. T. Klímošová is supported by the grant no. 19-04113Y of the Czech Science Foundation (GAČR) and the Center for Foundations of Modern Computer Science (Charles Univ. project UNCE/SCI/004). T. Masářík received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme Grant Agreement 714704. He completed a part of this work while he was a postdoc at Simon Fraser University in Canada. J. Novotná and A. Pokorná were supported by SVV-2017-260452 and GAUK 1277018.

†An extended abstract of this paper has been accepted to the proceedings of International Workshop on Graph-Theoretic Concepts in Computer Science (WG) 2021.
In recent years, a lot of attention has been paid to determining the complexity of $k$-coloring of $H$-free graphs. Classical results imply that for every $k \geq 3$, $k$-coloring of $H$-free graphs is NP-complete if $H$ contains a cycle [14] or an induced claw [24, 30]. Hence, it remains to consider the cases where $H$ is a linear forest, i.e., a disjoint union of paths. The situation around complexity of (list) $k$-coloring on $P_t$-free graphs where $k \geq 4$ has been resolved completely. The cases $k = 4, t \geq 7$ and $k \geq 5, t \geq 6$ are NP-complete [25] while cases for $k \geq 1, t = 5$ are polynomial-time solvable [23]. In fact, $k$-coloring is polynomial-time solvable on $sP_t + P_5$-free graphs for any $s \geq 0$ [13]. The borderline case where $k = 4, t = 6$ has been settled recently. There the 4-coloring problem (even the precoloring extension problem with 4 colors) is polynomial-time solvable [33] while the list 4-coloring problem is NP-complete [18]. In 2021, Hajebi, Li, and Spirkl show that 5-coloring on $2P_4$-free graphs is NP-complete [21].

Now, we move our focus towards the complexity of the 3-coloring problem, which was less well understood, in spite of the amount of the research interest it received in the past years. However, a considerable progress has been made in 2020: a quasi-polynomial algorithm running in time $n^{O(\log^3(n))}$ on $n$-vertex $P_t$-free graphs ($t$ is a constant) was shown by Pilipczuk et al. [31], extending a breakthrough of Gartland and Lokstianov [15]. In the realms of polynomiality, Bonomo et al. [1] found a polynomial-time algorithm for $P_7$-free graphs. Klimošová et al. [28] completed the classification of 3-coloring of $H$-free graphs, for any $H$ on up to 7 vertices. These results were subsequently extended to $P_6 + rP_3$-free graphs, for any $r \geq 0$ [5]. There are only two remaining graphs on at most 8 vertices, namely $P_8$ and $2P_4$, for which the complexity of 3-coloring is still unknown.

Algorithms for subclasses of $P_t$-free graphs which avoid one or more additional induced subgraphs, usually cycles, have been studied. They might be a first step in the attempt to settle the case of $P_t$-free graphs. This turned out to be the case for 3-coloring of $P_2$-free graphs (as can be seen from preprints [2, 7, 8] leading to [1]) and 4-coloring of $P_5$-free graphs [6].

Note that the problem of 4-coloring is NP-complete even when some ($P_t$, $C_r$)-free graphs are considered when $t \geq 7$. Hell and Huang [22] and Huang et al. [26] settled many NP-complete cases of this type. These results, in combination with the polynomiality of $P_6$-free case, leave open only the following cases: $(P_7, C_7)$-, $(P_8, C_7)$-, and $(P_7, C_3)$-free graphs, for $7 \leq t \leq 21$.

Chudnovsky and Stacho [11] studied the problem of 3-coloring of $P_k$-free graphs which additionally avoid induced cycles of two distinct lengths; specifically, they consider graphs that are $(P_8, C_3, C_4)$-, $(P_8, C_3, C_5)$-, and $(P_8, C_4, C_5)$-free. For the first two cases, they show that all such graphs are 3-colorable. For the last one, they provide a complete list of 3-critical graphs, i.e., the graphs with no 3-coloring such that all their proper induced subgraphs are 3-colorable. Independently, using a computer search, Goedgebeur and Schaudt [16] showed that there are only finitely many 3-critical ($P_8$, $C_4$)-free graphs. In fact, 3-coloring is polynomial-time solvable on $(P_1, C_4)$-free graphs for any $t \geq 1$ [19].

The situation concerning $2P_4$ or $P_8$ is still far from being determined when two forbidden induced subgraphs are considered: in particular, it is not known whether $(P_8, C_3)$-, $(P_8, C_5)$-, $(2P_4, C_5)$-, or $(2P_4, C_5)$-free graphs can be 3-colored in polynomial time\footnote{First two cases were explicitly mentioned as open in [17] and [32]; the latter two cases are open to the best of our knowledge.}. This is in contrast with the algorithm for $(P_7, C_3)$-free graphs [3] which is considerably simpler than the one for $P_7$-free graphs [1]. Recently, Rojas and Stein [32] approached the problem by showing that for any odd $t \geq 9$, there exists a polynomial-time algorithm that solves the 3-coloring problem in $P_t$-free graphs of odd girth at least $t - 2$. In particular, their result implies that 3-coloring is polynomial-time solvable for $(P_9, C_3, C_5)$-free graphs.

Freshly, a similar question was resolved in the case where, instead of a cycle, a 1-subdivision of $K_{1,s}$ (a star with $s$ leaves), denoted as $SDK_{1,s}$, is considered. Chudnovsky, Spirkl, and Zhong have shown that the class of $(SDK_{1,s}, P_t)$-free graphs is list-3-colorable in polynomial time for any $s, t \geq 1$ [10]. For other related results and history of the problem, please consult a recent survey [17].

In this paper, we resolve one of the remaining open problems mentioned above, which considers
2P₄-free graphs, as we will describe a polynomial-time algorithm for 3-coloring of (2P₄, C₅)-free graphs. To the best of our knowledge, this is a first attempt to attack the 3-coloring of 2P₄-free graphs.

**Theorem 1.1.** The 3-coloring problem is polynomial-time solvable on (2P₄, C₅)-free graphs.

To prove our result, we will make use of some relatively standard techniques. Let ω(G) be the size of the largest clique of graph G. We use a seminal result of Grötschel, Lovász, and Schrijver [20] that shows the k-coloring problem on perfect graphs, i.e., graphs where each induced subgraph G' is ω(G')-colorable, can be solved in polynomial time. Perfect graphs are characterized by the strong perfect graph theorem [9] as the graphs that have neither odd-length induced cycles nor complement of odd-length induced cycles on at least five vertices.

As K₄ and C₇ graphs are not 3-colorable, we can assume that our graph is (2P₄, C₅, C₇, K₄)-free. As K₄ ⊆ C₇ whenever ℓ ≥ 8 and 2P₄ ⊆ C₇ whenever ℓ ≥ 10, it follows that either the graph is perfect, or it contains C₇ or C₉. In the first case, we are done by the aforementioned polynomial-time algorithm. For the latter cases, we divide the analysis into two further subcases. First, we suppose that the graph is (2P₄, C₅, C₇, C₉)-free and therefore it must contain C₉. We analyze this case in Subsection 2.3. Second, we suppose that graph contains C₇ and we analyze this case in Subsection 2.3.

We will exploit the fact that once we find an induced P₄, the vertices that are not adjacent to it must induce a P₄-free graph (also known as cograph). Such graphs were among the first H-free graphs studied, and have many nice properties, e.g., any greedy coloring gives a proper coloring using the least number of colors [4]. We will make use of a slightly stronger statement that handles the list-3-coloring problem.

**Theorem 1.2 ([17]).** The list-3-coloring problem on P₄-free graphs can be solved in polynomial time.

The 3-coloring algorithm that we develop to prove Theorem 1.1 cannot be directly extended to solve the more general list-3-coloring problem, since it uses the 3-coloring algorithm for perfect graphs to deal with graphs avoiding C₇ and C₉. However, apart from this one case, the algorithm works with the more general setting of list-3-coloring. In fact, we use reductions of lists as one of our base techniques. After several branching steps with polynomially many branches and suitable structural reductions of the original graph G, the algorithm will transform a 3-coloring instance of a (2P₄, C₅)-free graph G to a set of polynomially many heavily structured list-3-coloring instances. These structured instances can then be encoded by a 2-SAT formula, whose satisfiability is solvable in linear time [29].

## 2 Proof of Theorem 1.1

We are given a (2P₄, C₅)-free graph G = (V, E), and our goal is to determine whether it is 3-colorable. We will present an algorithm that solves this problem in polynomial time. The algorithm begins by checking that the graph is C₇-free, and that the neighborhood of each vertex induces a bipartite graph, rejecting the instance if the check fails. Note that this check ensures, in particular, that G is K₄-free.

The algorithm then partitions the graph into connected components, solving the 3-coloring problem for each component separately. From now on, we assume that the graph G = (V, E) is connected, C₇-free, and each of its vertices has a bipartite neighborhood.

The basic idea of the algorithm is to choose an initial subgraph N₀ of bounded size, try all possible proper 3-colorings of N₀, and analyze how the precoloring of N₀ affects the possible colorings of the remaining vertices.

We let N₁ denote the vertices in V \ N₀ which are adjacent to at least one vertex of N₀, and we let N₂ be the set V \ (N₀ ∪ N₁). We will use the notation N(x) for the set of neighbors of x in G, and Nᵢ(x) for Nᵢ ∩ N(x).
Our algorithm will iteratively color the vertices of $G$. We will assume that throughout the algorithm, each vertex $v$ has a list $P(v) \subseteq \{1, 2, 3\}$ of available colors. We call $P(v)$ the palette of $v$. The goal is then to find a proper coloring of $G$ in which each vertex is colored by one of its available colors. The problem of deciding the existence of such coloring is known as the list-3-coloring problem, and is a generalization of the 3-coloring problem.

Whenever a vertex $x$ of $G$ is colored by a color $c$ in the course of the algorithm, we immediately remove $c$ from the palette of $x$’s neighbors. Additionally, if the vertex $x$ is not in $N_0$, it is then deleted. The vertices in $N_0$ are kept in $G$ even after they are colored. We then update the list-3-coloring instance using the following basic reductions:

- If a vertex $y$ has only one color $c'$ left in $P(y)$, we color it by the color $c'$ and remove $c'$ from the palettes of its neighbors. If $y \not\in N_0$, we then delete $y$.
- If $P(y)$ is empty for a vertex $y$, the instance of list-3-coloring is rejected.
- If, for a vertex $y \not\in N_0$, the size of $P(y)$ is greater than the degree of $y$, we delete $y$.
- \textbf{Diamond consistency rule:} If $y$ and $y'$ are a pair of nonadjacent vertices such that $P(y) \neq P(y')$, and if $N(y) \cap N(y')$ is not an independent set, then any valid 3-coloring of $G$ must assign the same color to $y$ and $y'$; we therefore replace both $P(y)$ and $P(y')$ with $P(y) \cap P(y')$.
- \textbf{Neighborhood domination rule:} If $y$ and $y'$ are a pair of nonadjacent vertices such that $N(y) \subseteq N(y')$ and $P(y') \subseteq P(y)$, and if $y$ is not in $N_0$, we delete $y$.
- If $G$ has a connected component in which every vertex has at most two available colors, we determine whether the component is colorable by reducing the problem to an instance of 2-SAT. If the component can be colored, we remove it from $G$ and continue, otherwise we reject the whole instance.
- If a connected component of $G$ is $P_4$-free, we solve the list-3-coloring problem for this component by Theorem 1.2. If the component is colorable we remove it, otherwise we reject the whole instance $G$.

It is clear that the rules are correct in the sense that the instance of list-3-coloring produced by a basic reduction is list-3-colorable if and only if the original instance was list-3-colorable. It is also clear that we may determine in polynomial time whether an instance of list-3-coloring (with fixed $N_0$) permits an application of a basic reduction, and perform the basic reduction, if available. Throughout the algorithm, we apply the basic reductions greedily as long as possible, until we reach a situation where none of them is applicable.

The basic reductions by themselves are not sufficient to solve the 3-coloring problem for $G$. Our algorithm will sometimes also need to perform branching, i.e., explore several alternative ways to color a vertex or a set of vertices. Formally, this means that the algorithm reduces a given instance $G$ of list-3-coloring to an equivalent set of instances $\{G_1, \ldots, G_k\}$; here saying that a list-3-coloring instance $G$ is equivalent to a set $\{G_1, \ldots, G_k\}$ of instances means that $G$ has a solution if and only if at least one of $G_1, \ldots, G_k$ has a solution.

In the beginning of the algorithm, we attach to each vertex $v$ of $G$ the list $P(v) = \{1, 2, 3\}$ of available colors, thereby formally transforming it to an instance of list-3-coloring. The algorithm will then try all possible proper 3-colorings of $N_0$, and for each such coloring, apply basic reductions as long as any basic reduction is applicable. If this fails to color all the vertices, more complicated reduction steps and further branching will be performed, to be described later.

Overall, the algorithm will ensure that the initial instance $G$ is eventually reduced to a set of at most polynomially many smaller instances, each of which can be transformed to an equivalent instance of 2-SAT, which then can be solved efficiently.
2.1 The $C_7$-free case

Our choice of $N_0$ will depend on the structure of $G$. More precisely, if $G$ contains an induced copy of $C_7$, we will choose one such copy as $N_0$. This is by far the most challenging case, and we return to it later.

The case when $G$ is $C_7$-free can be handled in a simple way, as we now show.

Proposition 2.1. The 3-coloring problem for a $(2P_4, C_5, C_7)$-free graph $G$ can be solved in polynomial time.

Proof. Recall that we assume that $G$ is $K_4$-free and $C_7$-free; otherwise $G$ would clearly not be 3-colorable. Note that $K_4$-freeness implies that $G$ is $C_k$-free for every $k \geq 8$, and $2P_4$-freeness implies that $G$ is $C_k$-free for every $k \geq 10$.

If $G$ is also $C_9$-free, then it is perfect by the strong perfect graph theorem, and since it is $K_4$-free, it is 3-colorable. Assume then that $G$ contains an induced copy of $C_9$. Fix $N_0$ to be an induced copy of $C_9$ in $G$, and define $N_1$ and $N_2$ accordingly. We will show that for any proper coloring of $N_0$, the basic reductions can solve the resulting list-3-coloring problem.

Fix a 3-coloring of $N_0$, and apply the basic reductions, until none of them is applicable. We claim that this solves the instance completely, i.e., we either color the whole graph, or determine that no coloring exists. For contradiction, suppose that we reached a situation when $G$ still contains uncolored vertices, but no basic reduction is applicable.

It follows that $G$ contains a vertex with three available colors, and this vertex necessarily belongs to $N_2$. In particular, $N_2$ is nonempty, and therefore we may find in $G$ two adjacent vertices $x, y$ with $x \in N_1$ and $y \in N_2$. Recall that $N_0(x)$ is the set of vertices of $N_0$ adjacent to $x$. The vertices of $N_0(x)$ partition the cycle $N_0$ into edge-disjoint arcs, and at least one of these arcs has an odd number of edges. Let $A$ be such an arc of odd length.

If $A$ has length 1, then $x$ is adjacent to two adjacent vertices of $N_0$, hence the color of $x$ is uniquely determined by the coloring of $N_0$ and $x$ should have been deleted. If $A$ has length 3 or 5, then $A \cup \{x\}$ induces a copy of $C_5$ or $C_7$, respectively, which is impossible. Thus, $A$ has length 7 or 9. In such case, we find a copy of $2P_4$ in $G$, where one $P_4$ consists of $y, x$, a vertex $z \in N_0(x)$, and a vertex $w \in A$ adjacent to $z$, while the other $P_4$ is formed by taking four consecutive internal vertices of $A$, each of which is at distance at least two from $z$ and $w$; see Figure 1. In all cases we get a contradiction.

From now on, we assume that the graph $G$ contains an induced $C_7$. We choose one such $C_7$ as $N_0$, and define $N_1$ and $N_2$ accordingly.
2.2 More complicated reductions

Apart from the basic reductions described previously, which we will apply whenever opportunity arises, we will also use more complicated reductions, to be applied in specific situations.

Cut reduction. Suppose $G = (V, E)$ is a connected instance of list-3-coloring. Let $X \subseteq V$ be a vertex cut of $G$, let $C$ be a union of one or more connected components of $G - X$, and let $C_X$ be the subgraph of $G$ induced by $C \cup X$. Suppose further that the following conditions hold.

- $C$ has at least two vertices.
- $X$ is an independent set in $G$.
- All the vertices in $X$ have the same palette, which has size 2.
- For any two vertices $x, x'$ in $X$, we have $N(x) \cap C = N(x') \cap C$.
- The graph $C_X$ is $P_4$-free.

Assume without loss of generality that all the vertices of $X$ have palette equal to $\{1, 2\}$. Let us say that a coloring $c : X \to \{1, 2\}$ of $X$ is feasible for $C$, if it can be extended into a proper 3-coloring of the list-3-coloring instance $C_X$. Note that the feasibility of a given coloring can be determined in polynomial time by Theorem 1.2, because $C_X$ is a cograph.

We distinguish three types of possible colorings of $X$: the \textit{all-1} coloring colors all the vertices of $X$ by the color 1, the \textit{all-2} coloring colors all the vertices of $X$ by color 2, and a \textit{mixed} coloring is a coloring that uses both available colors on $X$. Observe that if $X$ admits at least one mixed coloring feasible for $C$, then every (not necessarily mixed) coloring of $X$ by colors 1 and 2 is feasible for $C$. This is because when we extend a mixed coloring of $X$ to a coloring of $C_X$, all the vertices $y \in C$ must receive the color 3. If such a coloring of $C$ exists, we can combine it with any coloring of $X$ by colors 1 and 2.

The cut reduction of $X$ and $C$ is an operation which reduces $G$ to a smaller, equivalent list-3-coloring instance, determined as follows. We choose an arbitrary mixed coloring $c$ of $X$, and check whether it is feasible for $C$. If it is feasible, we reduce the instance $G$ to $G - C$, leaving the palettes of the remaining vertices unchanged. The new instance is equivalent to the original one, since any proper list-3-coloring of $G - C$ can be extended to a coloring of $G$, because all the colorings of $X$ are feasible for $C$.

If the mixed coloring $c$ is not feasible for $C$, we know that no mixed coloring is feasible. We then test the all-1 and the all-2 coloring for feasibility. If both are feasible, we reduce the instance $G$ by replacing $C$ with a single new vertex $v$, with palette $P(v) = \{1, 2\}$, and connecting $v$ to all the vertices of $X$. Note that the reduced instance is an induced subgraph of the original one. It is easy to see that the reduced instance is equivalent to the original one.

If only one coloring of $X$ is feasible for $C$, we delete $C$, color the vertices of $X$ using the unique feasible coloring, and delete the corresponding color from the palettes of the neighbors of $X$ in $G - C$. If no coloring of $X$ is feasible for $C$, we declare that $G$ is not list-3-colorable.

Neighborhood collapse. Let $G$ be an instance of list-3-coloring, and let $v$ be a vertex of $G$. Suppose that $N(v)$ induces in $G$ a connected bipartite graph with nonempty partite classes $X$ and $Y$. Suppose furthermore that all the vertices of $X$ have the same palette $P_X$, and all the vertices in $Y$ have the same palette $P_Y$. The \textit{neighborhood collapse} of the vertex $v$ is the operation that replaces $X$ and $Y$ by a pair of new vertices $x$ and $y$, adjacent to each other and to $v$, with the property that any vertex of $G - Y$ adjacent to at least one vertex in $X$ will be made adjacent to $x$, and similarly every vertex adjacent to $Y$ in $G - X$ will be adjacent to $y$. We then set $P(x) = P_X$ and $P(y) = P_Y$. Informally speaking, we have collapsed the vertices in $X$ to a single vertex $x$, and similarly for $Y$ and $y$.

It is clear that the collapsed instance is equivalent to the original one. However, since the new instance is not necessarily an induced subgraph of the original one, it might happen, e.g., that a
collapse performed in a $C_5$-free graph will introduce a copy of $C_5$ in the collapsed instance. In our algorithm, we will only perform collapses at a stage when $C_5$-freeness will no longer be needed.

On the other hand, $2P_4$-freeness is preserved by collapses, as we now show.

**Lemma 2.2.** Let $G$ be a $2P_4$-free instance of list-3-coloring in which a neighborhood collapse of a vertex $v$ may be performed, and let $G^*$ be the graph obtained by the collapse. Then $G^*$ is $2P_4$-free.

**Proof.** Suppose $G^*$ contains an induced $2P_4$, and let $P$ and $Q$ be the two nonadjacent copies of $P_4$. Let $x$ and $y$ be the two vertices obtained by collapsing sets $X$ and $Y$, as in the definition of neighborhood collapse. Without loss of generality, $P$ contains the vertex $x$. It follows that $Q$ contains none of $x$, $y$ or $v$, and in particular, $Q$ is also a $P_4$ in $G$.

If the path $P$ contains the edge $xy$, we may ‘lift’ $P$ into the graph $G$ by replacing the vertices $x$ and $y$ by appropriate vertices $x' \in X$ and $y' \in Y$, and by replacing the edge $xy$ by a shortest path from $x'$ to $y'$ in $N(v)$. This transforms $P$ into an induced path $P'$ in $G$ on at least four vertices which is nonadjacent to $Q$. Thus, $G$ also contains a $2P_4$.

Suppose now that $P$ does not contain the edge $xy$, and therefore $y$ is not in $P$. If $x$ is the end-vertex of $P$, say $P = xw_1w_2w_3$, we easily obtain a $2P_4$ in $G$ by simply replacing $x$ by a vertex $x' \in X$ adjacent to $w_1$ in $G$. Suppose then that $x$ is an internal vertex of $P$, say $P = w_1xw_2w_3$. Since we know that $P$ does not contain $y$, we may replace the vertex $w_1$ with $v$ in $P$, knowing that $vxw_2w_3$ is also an induced $P_4$ in $G^*$ nonadjacent to $Q$. By replacing the vertex $x$ by a vertex $x' \in X$ adjacent to $w_2$, we obtain the induced path $vx'w_2w_3$ in $G$ which forms a $2P_4$ together with $Q$. \hfill \qed

### 2.3 Graphs containing $C_7$

We now turn to the most complicated part of our coloring algorithm, which solves the 3-coloring problem for a $(2P_4, C_5)$-free graph $G$ that contains an induced $C_7$. We let $N_0$ be an induced copy of $C_7$ in this graph, and define $N_1$ and $N_2$ accordingly.

We let $v_1, v_2, \ldots, v_8$ denote the vertices of $N_0$, in the order in which they appear on the cycle $N_0$. We evaluate their indices modulo 7, so that, e.g., $v_8 = v_1$.

Fix a proper coloring of $N_0$, and apply the basic reductions to $G$ until no basic reduction is applicable. We now analyze the structure of $G$ at this stage of the algorithm. We again let $N_0(x)$ denote the set of neighbors of $x$ in $N_0$.

**Lemma 2.3.** After fixing the coloring of $N_0$ and applying all available basic reductions, the graph $G$ has the following properties.

- Each vertex $x$ of $N_1$ satisfies either $N_0(x) = \{v_i\}$ for some $i$, or $N_0(x) = \{v_i, v_{i+2}\}$ for some $i$.
- Each induced copy of $P_4$ in $G$ has at most two vertices in $N_2$.
- $G$ is connected.

**Proof.** To prove the first part, use the vertices of $N_0(x)$ to partition the cycle $N_0$ into edge-disjoint arcs. Note that none of these arcs has length 1, since then $x$ would be adjacent to two vertices of distinct colors, and it would have been colored and deleted. Also, none of these arcs can have length 3, since such an arc together with the vertex $x$ would induce a $C_5$ in $G$, contradicting $C_5$-freeness.

On the other hand, at least one of the arcs formed by $N_0(x)$ must have odd length. Thus, there is either an arc of length 7, implying $N_0(x) = \{v_i\}$, or there is an arc of length 5, implying $N_0(x) = \{v_i, v_{i+2}\}$ for some $i$. This proves the first part of the lemma.
To prove the second part, assume that $P$ is an induced copy of $P_4$ in $G$ with at least three vertices in $N_2$. If $P$ is fully contained in $N_2$, then $P$ forms a $2P_4$ together with any $P_4$ contained in $N_0$. Suppose that $P \setminus N_2$ consists of a single vertex $x$, as in Figure 2. Necessarily $x$ is in $N_1$, and by the first part of the lemma, $N_0 \setminus N_0(x)$ contains an induced $P_4$ which forms an induced $2P_4$ with $P$.

To prove the last part of the lemma, note that $N_0$ is connected and therefore contained in a single component of $G$, and if $G$ contained another connected component, then this other component would necessarily be $P_4$-free and would be colored by a basic reduction.

Lemma 2.3 is the last part of the proof that makes use of the $C_5$-freeness of $G$. From now on, we will not need to use the fact that $G$ is $C_5$-free. In particular, we will allow ourselves reduction operations, such as the neighborhood collapse, which do not preserve $C_5$-freeness.

We will assume, without mentioning explicitly, that after performing any modification of the list-3-coloring instance $G$, we always apply basic reductions until no more basic reductions are available.

In the rest of the proof, we use the term top component to refer to a connected component of $N_2$. Observe that every top component is $P_4$-free and therefore has a dominating set of size at most 2 [12]. We say that a top component is relevant, if it contains a vertex $z$ with $|P(z)| = 3$. Note that if $G$ has no relevant top component, then all its vertices have at most two available colors, and the coloring problem can be solved by a single basic reduction.

We will say that a vertex $x$ of $N_1$ is relevant if $x$ is adjacent to a vertex belonging to a relevant top component.

To prove the second part, assume that $P$ is an induced $P_4$ with exactly three vertices in $N_2$. Note that $P$ can look differently, but always contains $x$.

Figure 2: Finding an induced $2P_4$, assuming $P$ is an induced $P_4$ with exactly three vertices in $N_2$. Note that $P$ can look differently, but always contains $x$.

Figure 3: Vertex $x \in N_1$ being a partial neighbor of a top component $C$ and neighboring another top component $C'$ leads to an induced $2P_4$.

Let $x \in N_1$ be a vertex, and let $C$ be a top component. We say that $x$ is a partial neighbor of $C$, if $x$ is adjacent to at least one but not all the vertices of $C$. We say that $x$ is a full neighbor of $C$, if it is adjacent to every vertex of $C$.

Lemma 2.4. Suppose $x \in N_1$ is a partial neighbor of a top component $C$. Then $x$ is not a neighbor of any other top component. Moreover, $|N_0(x)| = 2$.

Proof. Let $y$ and $z$ be two adjacent vertices belonging to $C$, such that $x$ is adjacent to $y$ but not to $z$. Suppose for contradiction that there is a vertex $w \in N_2$ adjacent to $x$ but not belonging to $C$. Then $wxyz$ is a copy of $P_4$ with three vertices in $N_2$, as shown in Figure 3, which contradicts Lemma 2.3. This shows that $x$ is not adjacent to any top component other than $C$.

Suppose now that $N_0(x)$ contains a single vertex $v_i$. Then $v_iyz$ together with $v_{i+2}v_{i+3}v_{i+4}v_{i+5}$ induce a $2P_4$. □
The situation obtained from the assumption that \( N_2(x) \) and \( N_2(y) \) for \( x, y \in R_i \) are not comparable by inclusion. The dash-and-dot line represents an edge which is present in one case (red induced \( 2P_4 \)) and absent in the other (blue induced \( 2P_4 \)).

The situation obtained when the neighborhoods of vertices in \( R_i \) in \( N_2 \) are comparable by inclusion with \( N_2(z) \) being the largest neighborhood. Note that these vertices are full neighbors of their top components.

Figure 4: Illustrations of the situations in the proof of Lemma 2.5.

We will now reduce \( G \) to a set of polynomially many instances in which the set of relevant vertices has special form. We first eliminate the relevant vertices that have only one neighbor in \( N_0 \). Let \( R_i \) be the set of relevant vertices that are adjacent to \( v_i \) and not adjacent to any other vertex of \( N_0 \).

**Lemma 2.5.** For any \( i \in \{1, \ldots, 7\} \), we can reduce \( G \) to an equivalent set of at most two instances, both of which satisfy \( R_i = \emptyset \).

**Proof.** By Lemma 2.4, we know that any vertex \( x \in R_i \) is a full neighbor of each of its adjacent top components.

Let \( x, y \) be two distinct vertices of \( R_i \). We claim that the two sets \( N_2(x) \) and \( N_2(y) \) are comparable by inclusion. To see this, suppose for contradiction that there are vertices \( x' \in N_2(x) \setminus N_2(y) \) and \( y' \in N_2(y) \setminus N_2(x) \). Then we can find in \( G \) a copy of \( 2P_4 \) in which the first \( P_4 \) is \( v_i+2v_i+3v_i+4v_i+5 \), and the second \( P_4 \) is either \( x'x'y' \) (if \( xy \in E(G) \)), or \( x'xv_iy \) (if \( xy \notin E(G) \)); see Figure 4a.

Choose \( z \in R_i \) so that \( N_2(z) \) is as large as possible. In particular, for every \( x \in R_i \), we have \( N_2(x) \subseteq N_2(z) \). We then obtain two instances equivalent to \( G \) by coloring \( z \) by its two available colors. Note that by coloring \( z \), we ensure that all the vertices \( N_2(z) \) have at most two available colors, and since \( z \) is a full neighbor of all its adjacent top components, this ensures that the vertices of \( R_i \) will no longer be relevant after \( z \) has been colored; see Figure 4b for illustration.

From now on, we deal with instances of \( G \) where every relevant vertex has exactly two neighbors in \( N_0 \). Let \( S_i \) be the set of relevant vertices adjacent to \( v_i \).
Figure 5: Considering a pair of vertices \((x, y)\) of type \(\gamma\) for \(x \in S_i, y \in S_{i+3}\), the edge \(x'y'\) must be present, otherwise we obtain an induced \(2P_4\). The other dash-and-dotted edges are not necessarily present, and the vertex \(v_{i+5}\) is adjacent to at most one vertex from \(\{x, y\}\).

Figure 6: Coloring \((b)\) from the proof of Lemma 2.6 for \(k < j\).

Lemma 2.6. For any \(i \in \{1, \ldots, 7\}\), we can reduce \(G\) to an equivalent set of polynomially many instances, each of which satisfies \(S_i = \emptyset\) or \(S_{i+3} = \emptyset\).

Proof. Suppose that the vertices in \(S_i\) have available colors 1 and 2, while the vertices in \(S_{i+3}\) have available colors 2 and 3 (the case when the vertices in \(S_{i+3}\) have the same available colors as the vertices in \(S_i\) is similar and we omit it).

For a pair of vertices \(x \in S_i\) and \(y \in S_{i+3}\), we distinguish the following three possibilities:

\((\alpha)\) \(N_2(x)\) and \(N_2(y)\) are comparable by inclusion,

\((\beta)\) \(x\) is adjacent to \(y\), or

\((\gamma)\) neither of the previous two conditions holds.

We say that the pair \((x, y)\) is of type \(\alpha\) if it satisfies the condition \((\alpha)\) above, and similarly for the other two types. Observe that if the pair \((x, y)\) is of type \(\gamma\), then there exist \(x' \in N_2(x) \setminus N_2(y)\) and \(y' \in N_2(y) \setminus N_2(x)\). Moreover, for any choice of such \(x'\) and \(y'\), the pair \(x'y'\) must be an edge of \(G\), otherwise \(x'xv_{i-1}v_{i+4}\) and \(y'yv_{i+3}v_{i+4}\) would form a copy of \(2P_4\). In particular, \(x'\) and \(y'\) belong to the same top component \(C\), and both \(x\) and \(y\) are partial neighbors of \(C\), as is depicted in Figure 5.

Let \(Z = S_i \cup S_{i+3}\). Let \(m\) be the size of \(Z\), and let us order the vertices of \(Z\) into a sequence \(z_1, z_2, \ldots, z_m\) satisfying \(|N_2(z_1)| \geq |N_2(z_2)| \geq \cdots \geq |N_2(z_m)|\).

We will reduce \(G\) to the set of all the instances that can be constructed by one of the following two rules:

\((a)\) All the vertices in \(Z\) are colored by their available color different from 2 (i.e., the vertices of \(S_i\) are colored by 1, the vertices of \(S_{i+3}\) by 3).

\((b)\) Fix a \(j \in \{1, \ldots, m\}\) and proceed as follows: color the vertices \(z_1, \ldots, z_{j-1}\) by their available color different from 2, and color \(z_j\) by 2. Moreover, if \(z_j\) is a partial neighbor of a top component \(C\), color a dominating set of size two in \(C\), in all the possible ways.
We now verify that in all the colorings described above, after all possible basic reductions are applied, either $S_i$ or $S_{i+3}$ becomes empty. This is clearly the case for the coloring described in (a), in which all the vertices in $S_i \cup S_{i+3}$ will be removed from $G$, so both sets will be empty.

Consider now a coloring described in (b), and assume without loss of generality that $z_j$ is in $S_j$. We claim that after the coloring is performed, there will be no relevant vertex left in $S_{i+3}$. To see this, consider a vertex $z_k \in S_{i+3}$. If $k < j$, then $z_k$ has been colored by the color 3, see Figure 6.

If $k > j$ we distinguish three possibilities depending on the type of the pair $(z_j, z_k)$. If the pair $(z_j, z_k)$ is of type $\alpha$, then $N_2(z_k) \subseteq N_2(z_j)$ (recall that $k > j$ implies $|N_2(z_j)| \geq |N_2(z_k)|$). In particular, all the vertices in any top component adjacent to $z_k$ will only have two available colors (recall that if $z_j$ is a partial neighbor of a top component, we also color a dominating set of this top component, ensuring all its vertices have at most two available colors). Thus, $z_k$ will no longer be relevant. If the pair $(z_j, z_k)$ is of type $\beta$, i.e. $z_j z_k$ is an edge, then $z_k$ has only the color 3 available and can be colored. Finally, suppose $(z_j, z_k)$ is of type $\gamma$. As discussed before, this means both $z_j$ and $z_k$ are partial neighbors of a top component $C$ and have no other neighbors in $N_2$. After the coloring is performed, all the vertices in $C$ will have only two available colors, because we have colored its dominating set of size two. Hence $z_k$ is no longer relevant. We conclude that $S_{i+3}$ becomes empty, as claimed.

It is clear that the coloring rules (a) and (b) admit only polynomially many possible colorings, and that any valid list coloring of $G$ extends one of the partial colorings described in (a) or in (b). Thus, we reduced $G$ to an equivalent set of polynomially many instances.

From now on, assume that we deal with an instance $G$ in which for every $i$, one of the two sets $S_i$ and $S_{i+3}$ is empty. Unless the instance is already completely solved, there must be at least one relevant vertex. Assume without loss of generality that $G$ has a relevant vertex adjacent to $v_1$ and $v_3$. It follows that $S_1$ and $S_3$ are nonempty, and hence $S_4$, $S_5$, $S_6$, and $S_7$ are empty. Moreover, as any relevant vertex is adjacent to a pair of vertices of the form $\{v_i, v_{i+2}\}$, it follows that $S_2$ is empty as well. In particular, every relevant vertex $x$ satisfies $N_0(x) = \{v_1, v_3\}$. It follows that all the relevant vertices have the same palette of size 2; assume without loss of generality that this palette is $\{1, 2\}$.

We will now focus on describing the structure of the subgraph of $G$ induced by the relevant vertices and the relevant top components adjacent to them. Let $R$ denote the set of relevant vertices. Note that the subgraph of $G$ induced by $R \cup N_2$ does not contain $P_4$, otherwise we could use the path $v_1v_3v_5v_7$ to get a $2P_4$ in $G$.

Note also that if two relevant vertices $x$ and $y$ are adjacent, then any common neighbor of $x$ and $y$ must be colored by color 3, thanks to the diamond consistency rule. We thus know that adjacent relevant vertices have no common neighbors outside $N_0$. We may also assume that the graph induced by the relevant vertices is bipartite, otherwise $G$ would clearly not be 3-colorable.
Lemma 2.7. Suppose that $x$ and $y$ are two adjacent relevant vertices. Let us write $X' = N_2(x)$ and $Y' = N_2(y)$. Then there are disjoint sets $X, Y \subseteq R$, with $x \in X$ and $y \in Y$, satisfying these properties:

1. Every vertex in $X \cup Y'$ is adjacent to every vertex in $Y \cup X'$.
2. $X$ and $Y$ are independent sets of $G$.
3. The vertices in $X' \cup Y'$ are only adjacent to vertices in $X \cup Y \cup X' \cup Y'$; in particular, $X' \cup Y'$ induce a top component.

Proof. Consider the subgraph $G[R]$ of $G$ induced by the relevant vertices, and let $C$ be the connected component of $G[R]$ containing $x$ and $y$. Recall that $C$ must be bipartite. We let $X$ and $Y$ be its partite classes containing $x$ and $y$, respectively. Note that $C$ is complete bipartite, otherwise it would contain a $P_4$.

We will now show that all the vertices in $X$ have the same neighbors in $N_2$. Indeed, if we could find a pair of vertices $x_1, x_2 \in X$ and a vertex $x' \in N_2(x_1)$ not adjacent to $x_2$, then $x'x_1y_2$ would induce a $P_4$. It follows that for every $x_1 \in X$ we have $N_2(x_1) = X'$, and similarly for every $y_1 \in Y$ we have $N_2(y_1) = Y'$.

We saw that adjacent relevant vertices have no common neighbors, so $X'$ and $Y'$ are disjoint.

Every vertex in $X'$ must be adjacent to every vertex in $Y'$, for if there were nonadjacent vertices $x' \in X'$ and $y' \in Y'$, then $x'xyy'$ would induce a $P_4$. This proves the first claim of the lemma.

To prove the second claim, observe that $X$ and $Y$ are independent by construction.

To prove the third claim, proceed by contradiction and assume that a vertex $x' \in X' \cup Y'$ is adjacent to a vertex $z$ not belonging to $X \cup Y \cup X' \cup Y'$. We may assume that $x'$ belongs to $X'$. Necessarily, $z$ belongs to $R \cup N_2$, and $zx'xy$ induces a forbidden $P_4$. \hfill \Box

Suppose $G[R]$ contains at least one edge $xy$, and let $X, Y, X', Y'$ be as in the previous lemma. Note that there are only two possible ways to color $G[X \cup Y]$ — either $X$ is colored 1 and $Y$ is colored 2, or vice versa. We can check in polynomial time which of these two colorings can be extended to a valid coloring of $G[X \cup Y \cup X' \cup Y']$. If neither of the two colorings extends, we reject $G$, if only one of the two coloring extends, we color $X \cup Y$ accordingly, and if both colorings extend, we remove the vertices $X' \cup Y'$ from $G$, resulting in a smaller equivalent instance, in which $X \cup Y$ are no longer relevant. Repeating this for every component of $G[R]$ that contains at least one edge, we eventually reduce the problem to an instance in which the relevant vertices form an independent set.

From now on, we assume $R$ is independent in $G$. For a vertex $x \in R$, let $C_2(x)$ denote the set of top components that contain at least one neighbor of $x$.

Lemma 2.8. For any two relevant vertices $x$ and $y$, we either have $C_2(x) = C_2(y)$, or $C_2(x)$ and $C_2(y)$ are disjoint.

Proof. Suppose the lemma fails for some $x$ and $y$. We may then assume that there is a top component $C \in C_2(x) \cap C_2(y)$ and a component $C' \in C_2(x) \setminus C_2(y)$. Since $|C_2(x)| \geq 2$, we know from Lemma 2.4 that $x$ is a full neighbor of all the top components in $C_2(x)$. Choose a vertex $u \in C'$ and a vertex $v \in C \cap N_2(y)$. Then $uxvy$ is a copy of $P_4$ in $R \cup N_2$, which is impossible. \hfill \Box

Let us say that two relevant vertices $x$ and $y$ are equivalent if $C_2(x) = C_2(y)$. As the next step in our algorithm, we will process the equivalence classes one by one, with the aim to reduce the instance $G$ to an equivalent instance in which each relevant vertex is adjacent to a single top component.

Let $x \in R$ be a vertex such that $|C_2(x)| \geq 2$, and let $R_x$ be the equivalence class containing $x$. By Lemma 2.4, each vertex in $R_x$ is a full neighbor of any component in $C_2(x)$, and by Lemma 2.8, no vertex outside of $R_x$ may be adjacent to a relevant top component in $C_2(x)$. Thus, $R_x$ is a vertex cut separating the relevant top components in $C_2(x)$ from the rest of $G$. We may therefore
apply the cut reduction through the vertex cut $R_x$ to reduce $G$ to a smaller instance in which the vertices of $R_x$ are no longer relevant.

We repeat the cut reductions until there is no relevant vertex adjacent to more than one top component. From now on, we deal with instances in which each relevant vertex is adjacent to a unique top component; note that this top component is necessarily relevant.

**Lemma 2.9.** Let $x$ be a relevant vertex, let $C$ be the top component adjacent to $x$, let $R_x$ be the equivalence class of $x$, and let $y \in R_x \cup C$ be a vertex not adjacent to $x$. Then $y$ is adjacent to at least one vertex in $N_2(x)$. Moreover, if $N_2(x)$ induces a disconnected subgraph of $G$, then $y$ is adjacent to all the vertices of $N_2(x)$.

**Proof.** Refer to Figure 8. If $y$ is not adjacent to any vertex of $N_2(x)$, then we can find an induced path with at least four vertices by considering the shortest path from $x$ to $y$ in the graph induced by $C \cup \{x, y\}$. Therefore $y$ has at least one neighbor in $N_2(x)$. Suppose now that $N_2(x)$ is disconnected. If $y$ is not adjacent to all the vertices of $N_2(x)$, then we can find a vertex $u \in N_2(x)$ adjacent to $y$, and a vertex $v \in N_2(x)$ nonadjacent to $y$, in such a way that $u$ and $v$ are in distinct components of $N_2(x)$. Then $yuv$ is an induced $P_4$. \hfill $\square$

Fix now a relevant top component $C$ and let $R$ be set of relevant vertices in $N_1$ adjacent to $C$. Fix a vertex $x \in R$ so that $N_2(x)$ is as large as possible. Let $R_x$ be the equivalence class containing $x$. We distinguish several possibilities, based on the structure of $N_2(x)$.

$N_2(x)$ is disconnected. Suppose first that $N_2(x)$ induces in $G$ a disconnected subgraph. By Lemma 2.9, any vertex in $R_x$ is adjacent to all vertices in $N_2(x)$. By our choice of $x$, this implies that for any $x' \in R_x$ we have $N_2(x') = N_2(x)$. We may therefore apply the cut reduction for the cut $R_x$ that separates $C$ from the rest of $G$, to obtain a smaller instance in which the vertices of $R_x$ are no longer relevant.
$N_2(x)$ is connected, with $\geq 3$ vertices. Now suppose that $N_2(x)$ induces a connected graph, and that $N_2(x)$ has at least three vertices. We now verify that $N_2(x)$ induces a complete bipartite graph, otherwise $C$ contains $P_4$ or $G$ is not 3-colorable. Let $Y$ and $Z$ be the two partite classes of $N_2(x)$. Note that any two vertices $y, y'$ in $Y$ have the same neighbors in $G$: indeed if $u$ were a vertex adjacent to $y$ but not to $y'$, then $uyxy'$ would induce a copy of $P_4$, as depicted in Figure 9. By the same argument, all the vertices in $Z$ have the same neighbors in $G$ as well. Diamond consistency enforces that all the vertices in $Y$ have the same palette, and similarly for $Z$. We may then invoke neighborhood domination to delete from $Y$ all vertices except a single vertex $y$, and do the same with $Z$, reducing $G$ to an equivalent instance in which $N_2(x)$ consists of a single edge.

$N_2(x)$ is a single vertex. Suppose that $N_2(x)$ consists of a single vertex $y$. If $y$ is the only vertex of $C$, then $y$ must have the palette $\{1, 2, 3\}$, otherwise $C$ would not be a relevant component. In such case, we may simply color $y$ with color 3 and delete it, as this does not restrict the possible colorings of $G - y$ in any way. If, on the other hand, $C$ has more than one vertex, it follows from Lemma 2.9 that all the vertices of $R_x$ are adjacent to $y$, and by the choice of $x$, every vertex in $R_x$ is adjacent to $y$ as its only neighbor in $C$. We may then apply cut reduction for the cut $R_x$. In all cases, we obtain a smaller equivalent instance, in which the vertices in $R_x$ are no longer relevant.
$N_2(x)$ is a single edge. The last case to consider deals with the situation when $N_2(x)$ contains exactly two adjacent vertices $u$ and $v$. Assume that $\deg_G(u) \geq \deg_G(v)$. Recall that the set $R$ of relevant vertices is independent. Note that for any vertex $x' \in R_x$, $N_2(x')$ is connected, otherwise Lemma 2.9 implies that $N_2(x')$ is contained in $N_2(x)$, contradicting $N_2(x)$ being a single edge.

We first claim that any vertex $y \in R_x \cup (C - u)$ adjacent to $v$ is also adjacent to $u$. Suppose this is not the case. Then, since $\deg_G(u) \geq \deg_G(v)$, there must also be a vertex $z \in R_x \cup (C - v)$ adjacent to $u$ but not to $v$. If $yz$ is an edge, then $zyvx$ is a copy of $P_4$, and if $yz$ is not an edge, then $zuvy$ is a copy of $P_4$, as shown in Figure 11. In both cases we have a contradiction, establishing the claim. Note that the claim, together with Lemma 2.9, implies that $u$ is adjacent to all the other vertices of $R_x \cup C$.

Next, we show that if $C$ contains a vertex adjacent to both $u$ and $v$, then we may reduce $G$ to a smaller equivalent instance. Suppose $y \in C$ is adjacent to $u$ and $v$. Then $P(y) = P(x) = \{1, 2\}$ by diamond consistency. We now claim that $y$ has no other neighbors in $G$ beyond $u$ and $v$. Suppose that $z \notin \{u, v\}$ is a neighbor of $y$. Then $z$ cannot be adjacent to $v$, since $uzvyz$ would form a $K_4$. Therefore $zyvx$ is a copy of $P_4$, contradicting the assumption that $y$ has no other neighbors in $G$. Then, $P(y) = P(x)$, we may delete $y$ due to neighborhood domination.

From now on, we assume that $u$ and $v$ have no common neighbor in $C$. Recall that $u$ is adjacent to all the other vertices in $C \cup R_x$. We now reduce $G$ to an instance where $C - u$ is an independent set. We already know that $v$ is isolated in $C - u$ by the previous paragraph. Suppose that $C - u$ has a component $D$ with more than one vertex. If $D$ has a vertex $v'$ adjacent to a vertex $x' \in R_x$, we can repeat the reasoning of the previous paragraph with $x'$ and $v'$ in the place of $x$ and $v$, showing that $u$ and $v'$ cannot have any common neighbor in $C$, contradicting the assumption that $D$ has more than one vertex. We can thus conclude that $D$ is not adjacent to any vertex in $R_x$. Then $u$ is a cut-vertex separating $D$ from the rest of $G$. We may test which colorings of $u$ can be extended into $D$ (since $D$ is $P_4$-free, this can be done efficiently), then restrict the palette of $u$ to only the feasible colors, and then delete $D$.

$P(W_1) = \{1, 3\}$ \quad $P(W_2) = \{2, 3\}$

![Figure 12: Situation in which we can apply a neighborhood collapse.](image)

$P(X_1) = P(X_0) = P(X_2) = \{1, 2\}$

![Figure 13: The last case in which each vertex of $C - u$ has palette $\{1, 3\}$ or $\{2, 3\}$. The blue text represents the three possibilities to color $u$ and what colors that implies for other parts of the graph.](image)

We are now left with a situation when $C$ is a star with center $u$, and every vertex of $R_x$ is adjacent to $u$ and to at most one vertex of $C - u$. If there is a vertex $w \in C - u$ adjacent to more than one vertex in $R_x$, it means that the neighborhood of $w$ is a connected bipartite graph (a star with center $u$) to which we may apply neighborhood collapse; see Figure 12.

Suppose now that every vertex $w \in C - u$ has only one neighbor in $R_x$ (if $w$ had no neighbor in $R_x$, it would have degree 1 and we could remove it). If $w$’s palette has 3 colors, we can remove
it, so we may assume that every vertex in \( C - u \) has a palette of size 2. Then \( u \)'s palette must have 3 colors, otherwise \( C \) would not be a relevant component. If a vertex in \( C - u \) has palette \( \{1, 2\} \), then \( u \) must be colored 3 and then \( R_x \) is no longer relevant.

It remains to consider the case when each vertex of \( C - u \) has the palette \( \{1, 3\} \) or \( \{2, 3\} \). Let \( W_1 \) and \( W_2 \) be the sets of vertices of \( C - u \) having palette \( \{1, 3\} \) and \( \{2, 3\} \), respectively. Let \( X_1 \) and \( X_2 \) be the sets of vertices of \( R_x \) that are adjacent to a vertex in \( W_1 \) and \( W_2 \), respectively. Let \( X_0 \) be the set of vertices in \( R_x \) that have no neighbor in \( C - u \). The situation is shown in Figure 13. Let us consider the possible colorings of \( X_0 \). If \( u \) is colored by 3, then the whole set \( W_1 \) is colored by 1, \( W_2 \) is colored by 2, hence \( X_1 \) is colored by 2 and \( X_2 \) by 1, while the vertices in \( X_0 \) can be colored arbitrarily by 1 or 2. On the other hand, if \( u \) receives a color \( \alpha \neq 3 \), then all the vertices in \( R_x \) receive the color \( \beta \in \{1, 2\} \setminus \{\alpha\} \), and the vertices in \( C - u \) can be colored by 3. The set \( R_x \) therefore admits three types of feasible colorings: the all-1 coloring, the all-2 coloring, and any coloring where the set \( X_1 \) is colored by 2 and \( X_2 \) by 1. This set of colorings can be equivalently characterized by the following properties:

- If a vertex in \( X_1 \) is colored by 1, then the whole \( R_x \) receives 1.
- If a vertex in \( X_2 \) is colored by 2, then the whole \( R_x \) is colored by 2.
- All the colors in \( X_1 \) are equal and all the colors in \( X_2 \) are equal.

The above properties can be encoded by a 2-SAT formula whose variables correspond to vertices of \( R_x \).

To summarize, we have shown that a 3-coloring instance \( G \) can be reduced to an equivalent set of polynomially many simpler list-3-coloring instances. The structure of these simpler instances guarantees that for any relevant top component \( C \), we can form a 2-SAT formula describing the colorings of the relevant vertices adjacent to \( C \) that can be extended to a proper coloring of \( C \). Moreover, in the subgraph of \( G \) induced by the vertices not belonging to any relevant top component, each vertex has a palette of size at most two. The colorings of this subgraph can again be encoded by a 2-SAT formula. Such an instance of list-3-coloring then admits a solution if and only if there is a satisfying assignment for the conjunction of the 2-SAT formulas described above. The existence of such an assignment can be found in polynomial time. This completes the proof of Theorem 1.1.

3 Conclusions

We have shown that 3-coloring on \( (2P_4, C_5) \)-free graphs is solvable in polynomial time. As we discussed in the introduction, this approach might serve as a step towards resolving 3-coloring on \( 2P_4 \)-free graphs because it remains to consider \( 2P_4 \)-free graphs containing \( C_5 \).

Apart from the main question above, under more refined scale, the complexity of 3-coloring on \( (2P_4, C_5) \)-free, \( (P_5, C_5) \)-free, or \( (P_k, C_5) \)-free graphs remains unknown. In another direction, it would be interesting to extend our result to the list 3-coloring setting.

Acknowledgements. We acknowledge the comfortable and inspiring atmosphere of the workshop KAMAK 2019 organized by Charles University where part of this work was done.

References

[1] Flavia Bonomo, Maria Chudnovsky, Peter Maceli, Oliver Schaudt, Maya Stein, and Mingxian Zhong. Three-Coloring and List Three-Coloring of Graphs Without Induced Paths on Seven Vertices. Combinatorica, 38(4):779–801, May 2017. doi:10.1007/s00493-017-3553-8.

[2] Flavia Bonomo, Oliver Schaudt, and Maya Stein. 3-colouring graphs without triangles or induced paths on seven vertices, 2014. arXiv:1410.0040v1.
[3] Flavia Bonomo-Braberman, Maria Chudnovsky, Jan Goedgebeur, Peter Maceli, Oliver Schaudt, Maya Stein, and Mingxian Zhong. Better 3-coloring algorithms: excluding a triangle and a seven vertex path. *Theoretical Computer Science*, 2020. doi:10.1016/j.tcs.2020.10.032.

[4] Claude A Christen and Stanley M Selkow. Some perfect coloring properties of graphs. *Journal of Combinatorial Theory, Series B*, 27(1):49–59, August 1979. doi:10.1016/0095-8956(79)90067-4.

[5] Maria Chudnovsky, Shenwei Huang, Sophie Spirkl, and Mingxian Zhong. List 3-Coloring Graphs with No Induced $P_6 + rP_3$. *Algorithmica*, July 2020. doi:10.1007/s00453-020-00754-y.

[6] Maria Chudnovsky, Peter Maceli, Juraj Stacho, and Mingxian Zhong. 4-Coloring $P_6$-Free Graphs with No Induced 5-Cycles. *Journal of Graph Theory*, 84(3):262–285, 2017. doi:10.1002/jgt.22025.

[7] Maria Chudnovsky, Peter Maceli, and Mingxian Zhong. Three-coloring graphs with no induced seven-vertex path I : the triangle-free case, 2014. arXiv:1409.5164.

[8] Maria Chudnovsky, Peter Maceli, and Mingxian Zhong. Three-coloring graphs with no induced seven-vertex path II : using a triangle, 2015. arXiv:1503.03573.

[9] Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas. The strong perfect graph theorem. *Annals of Mathematics*, 164(1):51–229, 2006. URL: http://www.jstor.org/stable/20159988.

[10] Maria Chudnovsky, Sophie Spirkl, and Mingxian Zhong. List-three-coloring $P_t$-free graphs with no induced 1-subdivision of $K_{1,s}$. *Discrete Mathematics*, 343(11):112086, November 2020. doi:10.1016/j.disc.2020.112086.

[11] Maria Chudnovsky and Juraj Stacho. 3-colorable subclasses of $P_8$-free graphs. *SIAM Journal on Discrete Mathematics*, 32(2):1111–1138, January 2018. doi:10.1137/16m1048558.

[12] Derek G. Corneil and Yehoshua Perl. Clustering and domination in perfect graphs. *Discrete Applied Mathematics*, 9(1):27–39, September 1984. doi:10.1016/0166-218x(84)90088-x.

[13] Jean-François Couturier, Petr A. Golovach, Dieter Kratsch, and Daniël Paulusma. List coloring in the absence of a linear forest. *Algorithmica*, 71(1):21–35, April 2013. doi:10.1007/s00453-013-9777-0.

[14] Thomas Emden-Weinert, Stefan Hougardy, and Bernd Kreuter. Uniquely colourable graphs and the hardness of colouring graphs of large girth. *Combinatorics Probability and Computing*, 7(4):375–386, 1998. doi:10.1017/S0963548398003678.

[15] Peter Gartland and Daniel Lokshtanov. Independent set on $P_k$-free graphs in quasi-polynomial time. In 61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020, pages 613–624. IEEE, 2020. doi:10.1109/FOCS46700.2020.00063.

[16] Jan Goedgebeur and Oliver Schaudt. Exhaustive generation of k-critical H-free graphs. *Journal of Graph Theory*, 87(2):188–207, April 2017. doi:10.1002/jgt.22151.

[17] Petr A. Golovach, Matthew Johnson, Daniël Paulusma, and Jian Song. A survey on the computational complexity of coloring graphs with forbidden subgraphs. *Journal of Graph Theory*, 84(4):331–363, March 2016. doi:10.1002/jgt.22028.

[18] Petr A. Golovach, Daniël Paulusma, and Jian Song. Closing complexity gaps for coloring problems on H-free graphs. *Information and Computation*, 237:204–214, October 2014. doi:10.1016/j.ic.2014.02.004.
[19] Petr A. Golovach, Daniël Paulusma, and Jian Song. Coloring graphs without short cycles and long induced paths. *Discrete Applied Mathematics*, 167:107–120, April 2014. doi:10.1016/j.dam.2013.12.008.

[20] Martin Grötschel, László Lovász, and Alexander Schrijver. Polynomial algorithms for perfect graphs. In *Topics on Perfect Graphs*, pages 325–356. Elsevier, 1984. doi:10.1016/s0040-9383(08)72943-8.

[21] Sepehr Hajebi, Yanjia Li, and Sophie Spirkl. List-5-coloring graphs with forbidden induced subgraphs, 2021. arXiv:2105.01787.

[22] Pavol Hell and Shenwei Huang. Complexity of coloring graphs without paths and cycles. *Discrete Applied Mathematics*, 216:211–232, January 2017. doi:10.1016/j.dam.2015.10.024.

[23] Chính T. Hoàng, Marcin Kamiński, Vadim Lozin, Joe Sawada, and Xiao Shu. Deciding k-colorability of $P_5$-free graphs in polynomial time. *Algorithmica*, 57(1):74–81, May 2008. doi:10.1007/s00453-008-9197-8.

[24] Ian Holyer. The NP-completeness of edge-coloring. *SIAM Journal on Computing*, 10(4):718–720, November 1981. doi:10.1137/0210055.

[25] Shenwei Huang. Improved complexity results on k-coloring $P_t$-free graphs. *European Journal of Combinatorics*, 51:336–346, January 2016. doi:10.1016/j.ejc.2015.06.005.

[26] Shenwei Huang, Matthew Johnson, and Daniël Paulusma. Narrowing the complexity gap for colouring ($C_s,P_t$)-free graphs. *The Computer Journal*, 58(11):3074–3088, June 2015. doi:10.1093/comjnl/bxv039.

[27] Richard M. Karp. *Reducibility among Combinatorial Problems*, pages 85–103. Springer US, Boston, MA, 1972. doi:10.1007/978-1-4684-2001-2_9.

[28] Tereza Klimošová, Josef Malík, Tomáš Masařík, Jana Novotná, Daniël Paulusma, and Veronika Slívová. Colouring ($P_r + P_t$)-Free Graphs. *Algorithmica*, 82(7):1833–1858, January 2020. doi:10.1007/s00453-020-00675-w.

[29] Melven R. Krom. The decision problem for a class of first-order formulas in which all disjunctions are binary. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 13(1-2):15–20, 1967. doi:10.1002/malq.19670130104.

[30] Daniel Leven and Zvi Galil. NP completeness of finding the chromatic index of regular graphs. *Journal of Algorithms*, 4(1):35–44, March 1983. doi:10.1016/0196-6774(83)90032-9.

[31] Marcin Pilipczuk, Michał Pilipczuk, and Paweł Rzążewski. Quasi-polynomial-time algorithm for independent set in pt-free graphs via shrinking the space of induced paths. In *Symposium on Simplicity in Algorithms (SOSA)*, pages 204–209. Society for Industrial and Applied Mathematics, January 2021. doi:10.1137/1.9781611976496.23.

[32] Alberto Rojas and Maya Stein. 3-Colouring $P_t$-free graphs without short odd cycles, 2020. arXiv:2008.04845.

[33] Sophie Spirkl, Maria Chudnovsky, and Mingxian Zhong. Four-coloring $P_6$-free graphs. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1239–1256. Society for Industrial and Applied Mathematics, January 2019. doi:10.1137/1.9781611975482.76.