A Simple Method to Construct Local Equilibrium Function for One Dimensional Lattice Boltzmann Method

Peng Wang\textsuperscript{a,*}, Shiqing Zhang\textsuperscript{a}

\textsuperscript{a}Yangtze Center of Mathematics and College of Mathematics, Sichuan University, Chengdu 610064, People’s Republic of China

Abstract

We have developed a simple method to construct local equilibrium function for one dimensional lattice Boltzmann method (LBM). This new method can make LBM model satisfy compressible flow with a flexible specific-heat ratio. Test cases, including the one dimensional Sod flow and one dimensional Lax flow are presented. Favorable results are obtained using proposed new method, indicating that the proposed method is potentially capable of constructing of the local equilibrium function for one dimensional LBM.

Keywords: Fluid mechanics, Lattice Boltzmann method, Local equilibrium function.

1. Introduction

The lattice Boltzmann method is a mesoscopic-based approach for solving the fluid flow problems at the macroscopic scales. In the LBM, the key issue is the determination of the relaxation time $\tau$ and local equilibrium distribution function $f^{(eq)}$. $\tau$ represents the relaxation time that distribution function $f$ relaxes to the local equilibrium function $f^{(eq)}$. By Chapman-Enskog expansion, relaxation time $\tau$ is linked to the viscosity and the pressure that satisfy the Navier-Stokes equations. $f^{(eq)}$ is local equilibrium function that is constrained by macroscopic conditions. What kind of flow that can be simulated is determined by the form of the $f^{(eq)}$. In this paper we mainly discuss a simple method to construct local equilibrium function $f^{(eq)}$.

In 1993, F. J. Alexander, S. Chen and J. D. Sterling \cite{1} firstly proposed a $f^{(eq)}$ for compressible viscosity flow. In 1994, Chen, etc. \cite{2} developed Alexander’s $f^{(eq)}$ and obtained a model without nonlinearity deviations for recovering Navier-Stokes equations. In 2003, Watari and Tsuchara \cite{3} pointed out that Chen’s model can only be deployed for small viscosity and temperature range. They introduced a global coefficients to construct $f^{(eq)}$ that make the model stable in a larger viscosity and temperature range. For the global coefficients in the Watari’s model, more discrete velocity is required to make the $f^{(eq)}$ get correct form. In 2007, Qu, etc. \cite{4} proposed a circular local equilibrium function. In Qu’s model, the circular local equilibrium function cannot satisfy high order macroscopic statistic conditions that make the model can only recover to Euler equations. 2008, Li, etc. \cite{5} developed Qu’s circular equilibrium function. In Li’s model, the $f^{(eq)}$ satisfy all the macroscopic conditions which make correctly recovering of Navier-Stokes equations is possible. In the development history of $f^{(eq)}$ in LBM, we can find that the macroscopic conditions are crucial for the constructing of the $f^{(eq)}$.

In this work, based on the idea of assignment matrix to assign macroscopic conditions, we obtain a simple method to construct a local equilibrium distribution function for compressible flows. In our method, we do not consider the continuous form of local equilibrium function, and the discrete form of $f^{(eq)}$ is directly derived by assigning the macroscopic conditions, so that $f^{(eq)}$...
do not contain free parameters. For numerical simulation, the differential form of the lattice-Boltzmann equation is solved by the TVD scheme ([13]). Flows with weak and strong shock waves were simulated successfully by the present model.

The paper is organized as follows. In Section 2, we introduce basic macroscopic conditions of LBM. In Section 3, the detail of our method to construct local equilibrium function is described. Firstly, we introduce assignment matrix to assign the non-energy conditions to construct \( f_{\text{eq}} \), then, we use fixed specific rest energy to make the \( f_{\text{eq}} \) satisfy energy conditions, Section 5 presents numerical results and discussions. In the last Section, we make some conclusions.

2. Boltzmann-BGK equation for compressible flow and macroscopic conditions

The standard dynamical theory of mesoscopic model is described by the Boltzmann equation as following:

\[
\frac{\partial f}{\partial t} + \xi \cdot \nabla f = J,
\]

(1)

where \( f \) is the distribution function ([3]). The \( \xi \) is the particle velocity vector, and \( J \) represents the collision term, we can simplify the collision term \( J \) to the following formulation ([2]):

\[
\frac{\partial f}{\partial t} + \xi \cdot \nabla f = \frac{1}{\tau} (f_{\text{eq}} - f),
\]

(2)

where \( \tau \) is the relaxation time and \( f_{\text{eq}} \) is the local equilibrium distribution function.

The moments of \( f_{\text{eq}} \) are

\[
\int f_{\text{eq}} d\xi = \rho,
\]

(3a)

\[
\int f_{\text{eq}} \xi_\alpha d\xi = \rho u_\alpha,
\]

(3b)

\[
\int f_{\text{eq}} \xi_\alpha \xi_\beta d\xi = \rho u_\alpha u_\beta + p \delta_{\alpha\beta},
\]

(3c)

\[
\int f_{\text{eq}} \xi_\alpha \xi_\beta \xi_\gamma d\xi = \rho u_\alpha u_\beta u_\gamma + p (u_\alpha \delta_{\beta\gamma} + u_\beta \delta_{\alpha\gamma} + u_\gamma \delta_{\alpha\beta}),
\]

(3d)

\[
\int f_{\text{eq}} \left( \frac{1}{2} |\xi|^2 + \zeta \right) d\xi = \rho E,
\]

(3e)

\[
\int f_{\text{eq}} \left( \frac{1}{2} |\xi|^2 + \zeta \right) \xi_\alpha d\xi = (\rho E + p) u_\alpha,
\]

(3f)

\[
\int f_{\text{eq}} \left( \frac{1}{2} |\xi|^2 + \zeta \right) \xi_\alpha \xi_\beta d\xi = (\rho E + 2p) u_\alpha u_\beta + p (E + R_g T) \delta_{\alpha\beta},
\]

(3g)

where the \( \zeta \) is the specific rest energy ([10], [9]) and \( R_g \) is gas constant. Consider the connection between mesoscopic variable and macroscopic variable:

\[
\frac{1}{2} \rho c^2 = p,
\]

(4)

\[
\frac{D}{4} \rho c^2 + \rho \zeta = \rho \varepsilon,
\]

(5)

where \( D \) is the spatial dimension, \( c \) is peculiar speed and \( \varepsilon \) is the internal energy. For the polytropic gas, we have

\[
\zeta = [1 - \frac{D}{2} (\gamma - 1)] \varepsilon,
\]

(6)

\[
c = \sqrt{2(\gamma - 1) \varepsilon},
\]

(7)

where \( \gamma \) is the specific heat capacity ratio.
Applying Chapman-Enskog expansion ([6]) and moments (3a) - (3g) to equations (2), we can recover the macroscopic Navier-Stokes equations without body force:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) &= -\nabla p + \mu \Delta \mathbf{u} + (\mu + \mu') \nabla \cdot (\nabla \cdot \mathbf{u}), \\
\frac{\partial (\rho E)}{\partial t} + \nabla \cdot \left[ (\rho E + p) \mathbf{u} \right] &= \nabla \cdot (\lambda \nabla T) + \nabla \cdot (\mathbf{u} \cdot \Pi),
\end{align*}
\]

where \( \Pi = \mu \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T - 2D(\nabla \cdot \mathbf{u})I \right] \), \( \mu \) is viscosity, \( \mu' \) is the bulk viscosity and \( \lambda \) is thermal conductivity. We have

\[ \mu = p \tau, \quad \lambda = (1 + \frac{b^2}{2}) R_g p \tau. \] (8)

3. Construction of discretization local equilibrium function

In the above section, we known that the continuous \( f^{(eq)} \) satisfy the macroscopic condition (3a) - (3g). For the lattice Boltzmann method, we need to known the discrete \( f^{(eq)}_i \) that deploy to the corresponding lattice microscopic velocity. The moments of discrete \( f^{(eq)}_i \) are

\[
\begin{align*}
\sum_i f^{(eq)}(\rho) &= \rho, & (9a) \\
\sum_i f^{(eq)}(\rho \mathbf{u}_\alpha) &= \rho \mathbf{u}_\alpha, & (9b) \\
\sum_i f^{(eq)}(\rho \mathbf{u}_\alpha \mathbf{u}_\beta) &= \rho \mathbf{u}_\alpha \mathbf{u}_\beta + \rho \delta_{\alpha\beta}, & (9c) \\
\sum_i f^{(eq)}(\rho \mathbf{u}_\alpha \mathbf{u}_\beta \mathbf{u}_\gamma) &= \rho \mathbf{u}_\alpha \mathbf{u}_\beta \mathbf{u}_\gamma + p(\mathbf{u}_\gamma \delta_{\alpha\beta} + \mathbf{u}_\beta \delta_{\alpha\gamma} + \mathbf{u}_\alpha \delta_{\beta\gamma}), & (9d) \\
\sum_i f^{(eq)}(\frac{1}{2} |\xi_i|^2 + \zeta) &= \rho E, & (9e) \\
\sum_i f^{(eq)}(\frac{1}{2} |\xi_i|^2 + \zeta) \xi_{i\alpha} &= (\rho E + p) \mathbf{u}_\alpha, & (9f) \\
\sum_i f^{(eq)}(\frac{1}{2} |\xi_i|^2 + \zeta) \xi_{i\alpha} \xi_{i\beta} &= (\rho E + 2p) \mathbf{u}_\alpha \mathbf{u}_\beta + p(E + R_g T) \delta_{\alpha\beta}. & (9g)
\end{align*}
\]

By equations (9a) - (9g), the macroscopic conditions can be classified as non-energy statistic conditions and energy statistic conditions. The non-energy statistic conditions are

\[
S_N = \left( \begin{array}{c}
\rho \\
\rho \mathbf{u}_\alpha \\
\rho \mathbf{u}_\beta \\
\rho \mathbf{u}_\alpha \mathbf{u}_\beta \\
\rho \mathbf{u}_\alpha \mathbf{u}_\gamma + p \\
\vdots \\
\rho \mathbf{u}_\alpha \mathbf{u}_\beta \mathbf{u}_\gamma + p(\mathbf{u}_\gamma \delta_{\alpha\beta} + \mathbf{u}_\beta \delta_{\alpha\gamma} + \mathbf{u}_\alpha \delta_{\beta\gamma})
\end{array} \right).
\] (10)

From equation (9a) - (9g), energy statistic conditions are translations of non-energy statistic conditions at some energy level. Therefore, \( f^{(eq)}_i \) should only conserve the non-energy statistic conditions. For the number of non-energy statistic conditions that we need to conserve, we choose corresponding number of microscopic \( \xi_i \), i.e., the number of lattice microscopic velocity \( \xi_i \) is the
same as the non-energy statistic conditions. The square matrix that made up of moments of microscopic velocity is

\[
V = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\xi_{1\alpha} & \xi_{2\alpha} & \cdots & \xi_{\alpha} \\
\xi_{1\beta} & \xi_{2\beta} & \cdots & \xi_{\beta} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{1\alpha}\xi_{1\beta} & \xi_{2\alpha}\xi_{2\beta} & \cdots & \xi_{\alpha}\xi_{\beta} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{1\alpha}\xi_{1\beta}\xi_{1\gamma} & \xi_{2\alpha}\xi_{2\beta}\xi_{2\gamma} & \cdots & \xi_{\alpha}\xi_{\beta}\xi_{\gamma}
\end{pmatrix}.
\] (11)

Define the \(V^{-1}\) as the assign matrix, where the \(V^{-1}\) is the inverse of \(V\), we have discrete form \(f_i^{(eq)}\) as

\[
\begin{pmatrix}
f_1^{(eq)} \\
f_2^{(eq)} \\
\vdots \\
f_i^{(eq)} \\
\end{pmatrix} = V^{-1} S_N.
\] (12)

By equation (12), we can simply find that the \(V \cdot V^{-1} S_N\) satisfy equation (9a), (9b), (9c) and (9d). However, we know the \(f_i^{(eq)}\) satisfy non-energy conditions, we can not make the \(f_i^{(eq)}\) satisfy the energy macroscopic conditions

\[
S_E = \begin{pmatrix}
\rho E \\
(\rho E + p)u_\alpha \\
\vdots \\
(\rho E + 2p)u_\alpha u_\beta + p(E + R_\gamma T)\delta_{\alpha\beta}
\end{pmatrix}.
\]

For thermal flow, \(\zeta\) is the specific rest energy. In equation (3c), (3i) and (3g), \(\zeta\) is a microscopic variable and \(\zeta\) is unknown that make the total energy \(E\) can not be calculated from the moments of the distribution function \(f_i\). Therefore, the introduction of a fixed \(\zeta_j\) to assign \(f_i^{(eq)}\) is necessary. The introduction of fixed \(\zeta_j\) can also solve the problem of \(f_i^{(eq)}\) which do not satisfy energy conditions. By Qu’s work (9), linearly assigning \(f_i^{(eq)}\) onto two energy levels \(\zeta_1\) and \(\zeta_2\), \(\zeta_1 = 0\) and \(\zeta_2 > e_{max}\) are the minimum and maximum stagnation energy in the whole flow field. We have

\[
f_{i,\zeta_1}^{(eq)} = f_i^{(eq)} \frac{\zeta_2 - \zeta_1}{\zeta_2},
\]

\[
f_{i,\zeta_2}^{(eq)} = f_i^{(eq)} \frac{\zeta}{\zeta_2}.
\]

By equation (9a) - (9g), (11) and (12), we can easily verify that \(f_{i,\zeta_j}^{(eq)}\) satisfy

\[
\sum_j \sum_i f_{i,\zeta_j}^{(eq)} = \rho, \quad (13a)
\]

\[
\sum_j \sum_i f_{i,\zeta_j}^{(eq)} \xi_\alpha = \rho u_\alpha, \quad (13b)
\]

\[
\sum_j \sum_i f_{i,\zeta_j}^{(eq)} \xi_\alpha \xi_\beta = \rho u_\alpha u_\beta + p\delta_{\alpha\beta}, \quad (13c)
\]

\[
\sum_j \sum_i f_{i,\zeta_j}^{(eq)} \xi_\alpha \xi_\beta \xi_\gamma = \rho u_\alpha u_\beta u_\gamma + p(u_\gamma \delta_{\alpha\beta} + u_\beta \delta_{\alpha\gamma} + u_\alpha \delta_{\beta\gamma}), \quad (13d)
\]
implicity of the collision term and got \( \theta \) to the equation (14), and the new distribution function is
\[
\sum_j \sum_i f^{(eq)}_{i,\xi,j} \left( \frac{1}{2} |\xi|^2 + \zeta_j \right) = \rho E, \tag{13e}
\]
\[
\sum_j \sum_i f^{(eq)}_{i,\xi,j} \left( \frac{1}{2} |\xi|^2 + \zeta_j \right) \xi_i = (\rho E + p)u_i, \tag{13f}
\]
\[
\sum_j \sum_i f^{(eq)}_{i,\xi,j} \left( \frac{1}{2} |\xi|^2 + \zeta_j \right) \xi_x = (\rho E + 2p)u_{x_i} + p(E + R_g T)\delta_{x}\beta. \tag{13g}
\]

By this stage, entire \( f^{(eq)}_i \) of the lattice Boltzmann method for compressible flow with a flexible specific-heat ratio is established.

4. Calculation scheme

4.1. Time discrete

For time discrete, we calculates advection term \( \vec{e}_i \cdot \nabla f_i(t, \vec{x}) \Delta t \) as an explicit finite-difference form and the collision term \( \Delta t (f^{(eq)}_i(t, \vec{x}) - f_i(t, \vec{x})) \) as a implicit finite-difference form, and we got
\[
f^{n+1}_i - f^n_i + \Delta t \vec{e}_i \cdot \nabla f^n_i = \Delta t [\theta J^{n+1}_i + (1 - \theta) J^n_i], \tag{14}
\]
where \( x \leq t_x \leq x + \Delta x, J^n_i = \frac{\Delta t}{2} (f^{(eq)}_i(t, \vec{x}) - f_i(t, \vec{x})) \), \( J^{n+1}_i = \frac{\Delta t}{2} (f^{(eq)}_i(t + \Delta t, \vec{x}) - f_i(t + \Delta t, \vec{x})) \), and \( \theta \) represents the degree of implicity, in this paper we set \( \theta \) to be 0.5.

Z.L Guo and T.S Zhao ([5]) introduced a new distribution function to remove the implicity of the equation (14), and the new distribution function is
\[
g_i = f_i + \pi \theta (f_i - f^{(eq)}_i), \tag{15}
\]
where \( \pi = \Delta t/\tau \). Applying this new distribution function to equation (14), we can vanish the implicity of the collision term and got
\[
g^{n+1}_i = -\Delta t \vec{e}_i \cdot \nabla f^n_i + (1 - \pi + \pi \theta) f^n_i + \pi(1 - \theta) f^{(eq), n}, \tag{16}
\]
\[
f^{n+1}_i = \frac{1}{1 + \pi \theta} (g^{n+1}_i + \pi \theta f^{(eq), n+1}).
\]

4.2. Space discrete

For space discrete, we define
\[
f_i(I, J) = f_i(x_I, y_J), \quad f_i(I - 1, J) = f_i(x_I - \Delta x, y_J), \tag{17}
\]
where I and J are node indexes. The second-order TVD scheme is
\[
\frac{\partial (e_{i\alpha} f_i)}{\partial x} = \frac{1}{\Delta t} [F_i(I + 1, J, \frac{1}{2}) - F_i(I - 1, J, \frac{1}{2})], \tag{18}
\]
where \( F_i(I + \frac{1}{2}, J) \) is the numerical flux at the interface of \( x_I + \Delta x/2 \), and is defined as:
\[
F_i(I + \frac{1}{2}, J) = F^L_i(I + \frac{1}{2}, J) + F^R_i(I + \frac{1}{2}, J), \tag{19}
\]
where
\[
F^L_i(I + \frac{1}{2}, J) = F^+_i(I, J) + \frac{1}{2} \text{min mod} (\Delta F^+_i(I + \frac{1}{2}, J), \Delta F^+_i(I - \frac{1}{2}, J)), \tag{20a}
\]
\[
F^R_i(I + \frac{1}{2}, J) = F^-_i(I + 1, J) - \frac{1}{2} \text{min mod} (\Delta F^-_i(I + \frac{1}{2}, J), \Delta F^-_i(I + \frac{3}{2}, J)), \tag{20b}
\]
\[
F^+_i(I, J) = \frac{1}{2} (e_{i\alpha} + |e_{i\alpha}|) f_i(I, J), \tag{20c}
\]
\[
F^-_i(I, J) = \frac{1}{2} (e_{i\alpha} - |e_{i\alpha}|) f_i(I, J), \tag{20d}
\]
and
\[
\Delta F^\pm_i(I + \frac{1}{2}, J) = F^\pm_i(I + 1, J) - F_i^\pm(I, J). \tag{20e}
\]
The function \( \text{min mod} (\ , \ ) \) in (20a) and (20b) is the flux limiter ([13], [11]).
5. Numerical examples

In this section, we will apply the new $f^{(eq)}_i$ to Sod shock tube flow and Lax shock tube flow to test capability of this new method. In numerical computations, the dimensionless form is preferred. There are three independent reference variables for normalization, which are the reference density $\rho_0$, reference length $L_0$, and reference internal energy $e_0$. With the three reference variables, other reference variables and non-dimensional variables can be defined as

$$u_0 = \sqrt{e_0}, t_0 = \frac{L_0}{u_0}, \hat{t} = \frac{t}{t_0}, \hat{x} = \frac{x}{L_0}, \hat{\rho} = \frac{\rho}{\rho_0}, \hat{u} = \frac{u}{u_0}, \hat{e} = \frac{e}{e_0}. \quad (21)$$

For one dimensional LBM, we choose discrete velocity as $e_1 = 1, e_2 = -1, e_3 = 2, e_4 = -2$. By the assignment matrix, we have

$$f^{(eq)}_1 = -\rho \left( \frac{c^2 u_x}{4} + \frac{c^2}{12} + \frac{u_x^3}{6} + \frac{u_x^2}{6} - \frac{2 u_x}{3} - \frac{2}{3} \right)$$
$$f^{(eq)}_2 = -\rho \left( -\frac{c^2 u_x}{4} + \frac{c^2}{12} - \frac{u_x^3}{6} + \frac{u_x^2}{6} + \frac{2 u_x}{3} - \frac{2}{3} \right)$$
$$f^{(eq)}_3 = \rho \left( \frac{c^2 u_x}{8} + \frac{c^2}{12} + \frac{u_x^3}{12} + \frac{u_x^2}{6} - \frac{u_x}{12} - \frac{1}{6} \right)$$
$$f^{(eq)}_4 = \rho \left( -\frac{c^2 u_x}{8} + \frac{c^2}{12} - \frac{u_x^3}{12} + \frac{u_x^2}{6} + \frac{u_x}{12} + \frac{1}{6} \right) \quad (22)$$

5.1. Sod shock tube

First, we use the Sod shock tube to test the proposed model. Sod shock tube is a famous unsteady flow, which includes shock wave. The initial condition is given as

$$\begin{align*}
(\hat{\rho}_L, \hat{u}_L, \hat{e}_L) &= (1, 0, 2.5) \quad (-0.5 < \hat{x} < 0) \\
(\hat{\rho}_R, \hat{u}_R, \hat{e}_R) &= (0.125, 0, 2) \quad (0 < \hat{x} < 0.5)
\end{align*} \quad (23)$$

In this case, we set $\hat{\tau} = 10^{-4}$ and $\hat{C}_2 = 4$. The mesh size is taken as $\Delta \hat{x} = 1/201$ and the time step size is chosen as $\Delta \hat{t} = \hat{\tau}/4$. Before the waves propagate to the two boundary ends, the distribution functions at the boundary can be set as the equilibrium distribution functions computed from the initial value of macroscopic variables. Figure 1 shows the computed density, velocity, pressure, and internal energy profiles (symbols) at $\hat{t} = 0.22$, exact solutions (solid lines) are also displayed in this figure. Clearly, the present results agree excellently well with the exact solution.

5.2. Lax shock tube

The second test case is the Lax shock tube. The initial condition of the problem is given as

$$\begin{align*}
(\hat{\rho}_L, \hat{u}_L, \hat{e}_L) &= (0.445, 0.698, 19.82) \quad (-0.5 < x < 0) \\
(\hat{\rho}_R, \hat{u}_R, \hat{e}_R) &= (0.5, 0, 2.855) \quad (0 < x < 0.5)
\end{align*} \quad (24)$$

We set $\hat{\tau} = 10^{-4}$ and $\hat{C}_2 = 30$. The mesh size and time step size are taken to be the same as those of the Sod shock tube problem. The computed density, velocity, pressure, and internal energy profiles (symbols) at $\hat{t} = 0.14$ are shown and compared with the exact solution (solid lines) in Figure 2. Obviously, the present results are very accurate.
6. Conclusions

In this paper, a simple method to construct local equilibrium function for one dimensional LBM was proposed. To recover the fluid equation, we classify macroscopic conditions as non-energy statistic conditions and energy statistic conditions. For non-energy statistic conditions, we introduce assignment matrix to assign non-energy statistic conditions to obtain the local equilibrium function $f_{i}^{(eq)}$. For energy statistic conditions, we use fixed specific rest energy, to assign $f_{i}^{(eq)}$ to different energy level that make the $f_{i}^{(eq)}$ satisfy energy statistic conditions. Numerical experiments showed that compressible inviscid flows with weak and strong shock waves can be well simulated by the present model.

References

[1] F. J. Alexander, S. Chen and J. D. Sterling, Lattice Boltzmann thermohydrodynamics, *Phys. Rev. E*, 47 (1993), R2249–R2252.

[2] P. L. Bhatnagar, E. P. Gross and M. Krook, A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems, *Phys. Rev.*, 94 (1954), 511–525.

[3] C. Cercignani, *The Boltzmann Equation and Its Applications*, Applied Mathematical Sciences. Springer-Verlag, 1988.
Figure 2: Velocity (left up), density (right up), pressure (left bottom) and internal energy (right bottom) profiles of Lax shock tube.

[4] Y. Chen, H. Ohashi and M. Akiyama, Thermal lattice Bhatnagar-Gross-Krook model without nonlinear deviations in macrodynamic equations, *Phys. Rev. E*, 50 (1994), 2776–2783.

[5] Z. L. Guo and T. S. Zhao, Explicit finite-difference lattice Boltzmann method for curvilinear coordinates, *Phys. Rev. E*, 67 (2003), 066709-1–066709-12.

[6] Y.L. He, Y. Wang and Q. Li, *Lattice Boltzmann Method Theory and Applications*, Science Publishing House, Beijing, 2008.

[7] Q. Li, Y. L. He and Y. J. Gao, Implementation of finite-difference lattice Boltzmann method on general body-fitted curvilinear coordinates, *International Journal of Modern Physics C*, 19(10) (2008), 1581–1595.

[8] Y. H. Qian, Simulating thermohydrodynamics with lattice BGK models, *Journal of Scientific Computing*, 8 (1993), 231–242.

[9] K. Qu, C. Shu and Y. T. Chew, Alternative method to construct equilibrium distribution functions in lattice-Boltzmann method simulation of inviscid compressible flows at high Mach number, *Phys. Rev. E*, 75 (2007), 036706.

[10] C. H. Sun, Lattice-Boltzmann models for high speed flows, *Phys. Rev. E*, 58 (1998), 7283–7287.
[11] Y. Wang, Y. L. He, T. S. Zhao, G. H. Tang and W. Q. Tao, Implicit-explicit finite-difference lattice Boltzmann method for compressible flows, *International Journal of Modern Physics C (IJMPC)*, 18 (2007), 1961–1983.

[12] M. Watari and M. Tsutahara, Two-dimensional thermal model of the finite-difference lattice Boltzmann method with high spatial isotropy, *Phys. Rev. E*, 67 (2003), 036306-1–036306-7.

[13] X. H. Zhang, Non-oscillatory and non-free-parameter dissipation difference scheme, *ACTA Aerodynamica SINICA*, 6 (1988), 143–165.