Towards an effective LQC model for $k = +1$ isotropic cosmologies from spatial discretisations

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Abstract

The closed, spatially isotropic FLRW universe ($k = +1$) is endowed with modifications due to a discrete underlying space-structure. Motivated from Loop Quantum Gravity techniques, the impact of these modifications of the single-graph-sector appearing in the scalar constraint are interpreted as physical quantum gravity effects. We investigate the form the modified scalar constraint for $k = +1$ spacetimes and assume this effective constraint as the generator of dynamics on the reduced isotropic phase space. It transpires that the system still features a classical recollapse with only marginal discreteness corrections. Moreover, first hints at the resolution of initial and final singularities are found.

1 Introduction

A prominent example for solutions of classical general relativity (GR) are the cosmological spacetimes of Friedmann-Lemaître-Robertson-Walker (FLRW) type [1–10]. This is on the one hand due to their precise agreement with observations [11, 12] and on the other hand because they allow to reduce the field-content of GR to finitely many degrees of freedom due to their symmetry. Consequently, they are also often focus of candidate theories of quantum gravity: while a complete quantisation of GR is not available as of today, it is hoped that for these special cases at least partial results can be obtained. Following this spirit, many works concern the isotropic sector of Loop Quantum Gravity (LQG) [13–16], a canonical quantisation of GR expressed in its SU(2)-connection formulation [17–20]. Especially for the case $k = 0$, i.e. spatially flat isotropic spacetimes, a big subfield emerged – called Loop Quantum Cosmology (LQC) [21–25] – where techniques similar to LQG are used to directly quantise the symmetry-reduced phase space of FLRW. Yet its relation to full LQG is as of today unclear but remains an active field of research [26–28]. However, due to the connection formulation of GR on which LQG is based, one has a close connection to the framework of lattice gauge theories (LGT) and could look at discretisations of space similar to the Hamiltonian language of LGT. In the presence of a finite ultra-violet cutoff, the scalar (or Hamiltonian) constraint of GR has to be discretised, i.e. approximated by a function expressed solely in terms of quantities on the lattice such as holonomies and fluxes. While many such discretisations are in general possible, the first one being suitable for quantisation came from Thiemann in his seminal papers [29, 30]. Such a discretisation can be recovered if one takes the expectation value of the scalar constraint operator in coherent states supported on a lattice of the kinematical LQG Hilbert space [31, 32]. Said procedure has shown itself to be especially of interest, if one takes the coherent...
states to be peaked over flat, isotropic cosmology [33–37]: in the case of Thiemann-regularisation one
obtained a function in terms of the parameters of the FLRW-reduced phase space that resembles an
effective scalar constraint which describes the resolution of the initial singularity in terms of a big
bounce on the LQC Hilbert space [38, 39] (see [40, 41] for extensions of this to Bianchi I spacetimes).
The resolutions of singularities via evolution operators in the reduced theory of LQC had already a
long history [42–44], however it used to focus in earlier works on a quantum operator whose discretisa-
did not stem from the full theory, but rather first imposed symmetries of $k = 0$ spacetimes prior
to discretisation. Afterwards, the resulting dynamics differed from the one, one would obtain if one
first discretises the theory and imposes cosmology a posteriori: most notably the “first-reduce-then-
regularising” framework lead to a symmetric bounce, while the bounce was genuinely asymmetric for
the “first-regularise-then-discretise” framework. But since only the later one agrees with the expect-
ation value of the Thiemann-regularisation in suitable coherent states for flat isotropic cosmology,
the question arises how the situation behaves for models involving non-trivial spatial curvature$^1$. In
this paper we will extend therefore the analysis to closed isotropic spacetimes, characterised by $k = +1$.

Few works in the literature have dealt so far with the $k = +1$ sector and those either proposed
modifications of the reduced constraint ad hoc [47] or considered LQC-like quantisations [48–50]. In
the later approach, one again followed – among other simplifications – the philosophy of implementing
symmetries of isotropic curved spacetimes prior to the discretisation process. In this paper we will
refrain from these simplifications and rather ask what properties an effective scalar constraint in line
with the regularisation techniques of Thiemann should have. For this purpose, we will stay purely
classical and base the relation to LQG merely on the kinematical expectation value of the scalar
constraint operator taken in the complexifier-coherent states from Thiemann & Winkler [51–53]. Ac-
cording to several studies [32,37] in leading order of $\hbar$ this expectation value will agree with its classical
discretisation thereby justifying our classical computations. Due to the complexity of the framework
involved, we are forced to truncate this expectation value at finite orders in the lattice spacing in order
to obtain analytical closed formulas.
However, we also want to make comparison to the effective LQC program, where these regularised
functions are taken as effective scalar constraints on some reduced phase space characterised by the
cosmological parameters $^2$. That means, that the modifications due to the discrete nature of spacetime
are interpreted as physical predictions of a theory of quantum gravity! For the purpose of comparison,
we will adopt this point of view inside a toy model as well. To do this, we will require to an additional
assumption into our framework, namely that the dynamics of a discretised lattice theory of finitely
many degrees of freedom driven by a discretised scalar constraint for symmetric initial conditions
is assumed to agree with the dynamics of the symmetry-reduced constraint on the symmetry-reduced
phase space. Under this premise – which is crucial for any effective model – we compute the flow of
the aforementioned truncations of the scalar constraint and study its behaviour in the classical regime.

The organisation of this article is as follows:
In section 2 we repeat the formulation of GR in terms of the Ashtekar-Barbero variables as a SU(2)
gauge theory. We introduce a discretisation of our manifold, in terms of a hyperspherical lattice. The
fundamental building blocks of the continuum theory, i.e. the connection and its conjugate mo-
mementum, are replaced by suitable smearings on the lattice, namely holonomies and fluxes. When
specialising to $k = +1$ cosmology, one can give closed analytical formulas for them along all edges
of the graph. Afterwards, we will turn to $C^\epsilon$, the Thiemann-regularisation of the scalar constraint.
Due to the fact that the holonomies carry an explicit coordinate dependence, evaluating $C^\epsilon$ on
the hyperspherical lattice for $k = +1$ cosmology results in lengthy expressions that are not handleable

$^1$We like to point out that a second regularisation of the scalar-constraint operator exists in the literature [45, 46]
which appears to work well for spatially flat spacetimes and is expected to reproduce the original LQC effective scalar
constraint as expectation values. However, due to limited studies of its semi-classical properties for spatially curved
spacetimes, we refrain from using it for the purpose of this article.

$^2$There is exhaustive literature concerning justification of such procedures, see e.g. [54–56].
analytically. Therefore, we merely present an expansion of $C^\epsilon$ up to $7^{\text{th}}$ order in terms of the regularisation parameter $\epsilon$, i.e. the lattice spacing.

In section 3 we couple the regularised scalar constraint to a free, massless and homogeneous scalar field. The resulting expression is postulated to describe the effective dynamics of a regularised classical system initially found to be in the phase space point derived in section 2. We will then investigate how the approximated dynamics behave as compared to the full theory. Although it transpires that at the current stage a resolution of the initial singularity via a bounce is hinted at, the current order does not yet suffice to guarantee such a behaviour. However, we are able to discuss the remnants of quantum gravity effects at the classical recollapse point.

In section 4 we conclude and discuss future research directions.

2 Connection formulation of general relativity and its discretisation

In the first subsection, the Ashtekar-Barbero formulation of GR [17–20] is reviewed and discretised on a purely classical level. For details we refer to the literature, e.g. [16]. Afterwards, we will investigate a possible incarnation for such a discretisation explicitly in the context of closed, isotropic spacetimes.

2.1 Ashtekar-Barbero formulation on a fixed lattice

One of the major steps towards defining a theory of quantum gravity, was the realisation that the Hamiltonian formulation of GR can be understood as an SU(2) gauge theory of Yang-Mills type: Let $\sigma$ be a 3-dimensional, spatial, orientable, compact $3$ manifold. The Ashtekar-Barbero variables coordinatise the phase space $M$ of an SU(2) Yang-Mills theory, described by an SU(2) connection (gauge potential) $A_a(x) := A^I_a(x)\tau_I : \sigma \rightarrow \text{su}(2)$ and a non-abelian conjugate momenta $E^a(x) := E^I_a(x)\tau_I : \sigma \rightarrow \text{su}(2)$, for which we choose positive orientation (with spatial indices $a, b, \ldots = 1, 2, 3$ and internal $\text{su}(2)$ indices $I, J, \ldots = 1, 2, 3$ and $\tau_I$ being $-i/2$ times the Pauli matrices). The symplectic form on $M$ is given by

$$\omega = \frac{2}{\kappa\beta} \int_\sigma d^3x \, dE^a_I(x) \wedge dA^I_a(x)$$

which leads to the Poisson algebra:

$$\{E^a_J(x), E^b_K(y)\} = \{A^I_a(x), A^K_b(y)\} = 0, \quad \{E^a_K(x), A^I_b(y)\} = \frac{\kappa\beta}{2} \delta^a_b \delta^K_I \delta^{(3)}(x, y)$$

with $\kappa = 16\pi G$ and $\beta \in \mathbb{R} - \{0\}$, the so-called Immirzi parameter. This phase space is subject to the Gauss constraint:

$$G_J = D_aE^a_J = \partial_a E^a_J + \epsilon_{JKL}A^K_aE^a_L = 0$$

The phase space of Ashtekar-Barbero variables becomes equivalent with the phase space of the Hamiltonian framework of GR as long as the Gauss constraint is satisfied.

The other constraints of GR read in this framework:

- The Diffeomorphism (or vector) constraint:

$$D_a = \frac{2}{\kappa\beta} F_{ab}^a E^b_J + D_{a,\text{matter}}$$

Compactness is not a requirement for the general framework to work. However, as we are interested in closed $k = +1$ models in the present paper, we will restrict to compact manifold from the onset.
The Scalar (or Hamiltonian) constraint

\[ C = \frac{1}{\kappa} \left( F^J_{ab} - (1 + \beta^2)K_{ac}e^c_M K_{bd}e^d_N \epsilon_{MNJ} \right) \epsilon_{JKL} \frac{E^b_K E^b_L}{\sqrt{\det(E)}} + C_{\text{matter}} \]  

(2.5)

with \( F_{ab} \) the curvature of the Ashtekar connection and \( K_{ab} \) the extrinsic curvature and the spin connection \( \Gamma^J_a \):

\[ F^J_{ab} := 2 \partial_{[a} A^J_{b]} + e^c_{KL} A^K_a A^L_b, \quad K_{ab} := \beta^{-1} e^J_b (A^J_a - \Gamma^J_a(E)) \]  

(2.6)

The terms \( D_{\text{matter}} \) and \( C_{\text{matter}} \) correspond to the applicable matter content. For the purpose of this section we will set them to zero and discretise only vacuum GR. These quantities will reappear in section 3 where the effective dynamics of a concrete cosmological model is studied in the presence of a free massless scalar field.

A crucial step, before we turn towards discretisation is the possibility to rewrite the scalar constraint (2.5) in such a way that the inverse power of \( \det(E) \) is taken care of. This has important advantages if one goes to the quantisation of the theory and therefore serves as the starting point of modern dynamics in LQG. This transformation is achieved via the first of the famous Thiemann identities \([29,30]\):

\[ \{ V, A^J_a \} = \frac{\kappa \beta}{8} \epsilon_{JKL} \epsilon_{abc} E^b_K E^c_L \sqrt{\det(E)} \]  

(2.7)

\[ \{ \{ V, C_E[1] \}, A^J_a \} = \frac{\kappa \beta^3}{2} K_{ab} e^J_a \]  

(2.8)

where \( V = \int_{\sigma} \sqrt{\det(E)} \) is the volume of the spatial manifold.

Note that the second identity helps to express the function \( K_{ab} \) which is originally a complicated object in terms of the connection \( A \), the momentum \( E \) and its derivatives, as an expression that does not depend on the derivatives anymore.

To stay maximally close the framework of Lattice Gauge Theories the basic variables, to start the quantisation procedure with, is not the connection, but rather its smearing along edges of some graph. Its conjugate momentum will be smeared along surfaces of the dual cell complex, respectively. Let \( \gamma \subset \sigma \) be a graph, that is a collection of edges \( e : [0,1] \rightarrow \sigma \) meeting at most at their endpoints, and such that \( \gamma \) allows for a dual cell complex, i.e. one can associate to each vertex \( \nu \) three faces \( S_e \) such that \( S_e \cap e' = \nu \) and normal to \( e' \) if \( e' \) is in direction \( e \). For each such \( \gamma \) we will now introduce a phase space by considering the collection of discretised phase space variables associated to each edge of \( \gamma \) following the constructions of Lattice Gauge Theory:

Along the edges \( e \) of the lattice, we will compute the holonomies \( h(e) \in \text{SU}(2) \) of the connection i.e. the path ordered exponential

\[ h(e) := \mathcal{P} \exp \left( \int_e dx^a A^J_a(x) \tau_J \right) \]  

(2.9)

where later values are ordered to the right, and the fluxes along the associated surfaces \( S_e \):\(^4\)

\[ E(e) = E_I(e) \tau_I := \int_{S_e} dx^a \wedge dx^b \epsilon_{abc} E^c_I(x) \tau_I \]  

(2.10)

For the purpose of further studies we will consider in this paper families of lattices \( \{ \gamma_\epsilon \}_{\epsilon \in \mathbb{R}} \) parametrised by their lattice spacing \( \epsilon \). The \( \gamma_\epsilon \) are of the form that they are (i) cuboidal (i.e. at each vertex 6 edges

\(^4\)A different construction would be gauge-covariant fluxes. Instead of the smearing used here, one would consider one which transforms feasibly under gauge transformations even for finite graphs. However, for the purpose of this paper we will stick to regular fluxes and refer to \([57]\) for their possible implications.
meet), (ii) can form directed families of subsets amongst each other \( \gamma_\epsilon \subset \gamma_\epsilon \) and (iii) lie dense in \( \sigma \), in the sense that each open neighbourhood will be punctured by \( \gamma_\epsilon \) for \( \epsilon \) small enough.

We will now discretise the scalar constraint, i.e. we will search for a function \( C^\epsilon \) that is completely expressed in terms of the discretised phase space variables of graph \( \gamma_\epsilon \) and such that in the limit \( \epsilon \to 0 \) the original scalar constraint is restored. To be precise, in terms of a smearing against a function \( N \) of compact support, we want:

\[
\lim_{\epsilon \to 0} \sum_{v \in \gamma_\epsilon} N(v) C^\epsilon(v) = \int_\sigma dx^3 N(x) C(x) \tag{2.11}
\]

Of course, there are several possibilities for \( C^\epsilon \) and the discretisation we will study in this paper is the graph-preserving version [31] of the original regularisation of Thiemann [29,30]:

\[
C^\epsilon[N] = C_E^\epsilon[N] + \frac{2^3(1 + \beta^2)}{\kappa^2\beta^2} \sum_v N(v) \sum_{ijk} \epsilon(i, j, k) \times
\]

\[
\times \operatorname{Tr} \left[ h(e_i) \{ h(e_i) \} \{ V, C_E^\epsilon[1] \} \{ h(e_j) \} \{ h^\dagger(e_j) \}, \{ V, C_E^\epsilon[1] \} \{ h(e_k) \} \{ h^\dagger(e_k) \}, V \} \right], \tag{2.12}
\]

\[
C_E^\epsilon[N] = -\frac{1}{2\kappa^2\beta^2} \sum_v N(V) \sum_{ijk} \epsilon(i, j, k) \operatorname{Tr} \left[ (h(\Box^\epsilon_{r,ij}) - h^\dagger(\Box^\epsilon_{r,ij})) h(e_k) \{ h^\dagger(e_k), V \} \right] \tag{2.13}
\]

with \( \epsilon(a, b, c) := \operatorname{sgn}(\det(\dot{\epsilon}^a(0), \dot{\epsilon}^b(0), \dot{\epsilon}^c(0))) = \operatorname{sgn}(a) \operatorname{sgn}(b) \operatorname{sgn}(c) e^{[a][b][c]} \) being a generalised epsilon tensor, which sums over negative indices respecting their sign, too. A similar construction can be carried out for functions whose vanishing is equivalent to the vanishing of the diffeomorphism constraint [31]: Instead of discretising \( D_a \) itself, one can consider \( \tilde{D}_I := E_I^a D_a \) which vanishes iff the vector constraint vanishes due to the non-degeneracy of \( E_I^a \). The form of \( \tilde{D}_I \) is more suited for quantisation due to the fact that its discretisation has no explicit dependence of the regulator. However, since we are interested in isotropic cosmology in the following, and the diffeomorphism constraint vanishes trivially there, we will refrain from considering these expressions explicitly.

The regularised scalar constraint \( C^\epsilon \) is an approximation to the generator of time-gauge-translations in the continuum and we will promote it to the generator of time-translations on \( T^*\mathcal{M}(\gamma_\epsilon) \), the discretised phase space of the graph.

The Poisson algebra of the holonomies and fluxes is given by:

\[
\{ E_I(e), E_J(\bar{e}) \} = \{ h(e), h(\bar{e}) \} = 0 \quad \{ E_I(\bar{e}), h(e) \} = \frac{\beta \kappa}{4} \delta(\bar{e}, e) \tau_I h(e), \tag{2.14}
\]

where \( \delta(\bar{e}, e) \) is one if the curve \( e \) is orthogonal to the surface \( S_{\bar{e}} \), otherwise vanishing. Note that the Poisson brackets of these smeared variables have lost the distributional character of (2.2) and have thus a much more suitable form for a quantization.

## 2.2 The case of isotropic, closed cosmologies

This subsection we apply the above developed framework explicitly to the case of compact spacetimes that are spatially isotropic and homogenous. In order to allow for an isotropic metric, the spatial manifold needs to be of the form \( \sigma \cong S^3 \).

In terms of canonical phase space variables, the spatial metric can be written as the conformal line element:

\[
q = q_{ab} dx^a dx^b = \frac{a(t)^2 \delta_{ab}}{\left(1 + \frac{1}{4} \tau^2 \right)^2} dx^a dx^b \tag{2.15}
\]
with \( \bar{r}^2 = \sum_i (x^i)^2 \) and \( a(t) \) being the scale factor. Upon a change to another set of coordinates

\[
\frac{\bar{r}}{1 + \frac{1}{4} \bar{r}^2} = \sin(r), \quad \text{with } r \in [0, \frac{\pi}{2}],
\]

(2.16)

which from now on we will refer to as hyperspherical ones, we receive: \( \theta \in [0, \pi], \varphi \in [0, 2\pi] \)

\[
q = a(t)^2 \left( dr^2 + \sin^2(r)(d\theta^2 + \sin^2(\theta)d\varphi^2) \right)
\]

(2.17)

Since the space is closed, we can compute the associated finite volume of it:

\[
V = \int_0^{\pi/2} dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \sqrt{\det(q)} = \pi^2 a^3
\]

(2.18)

Starting from this line element, one can immediately compute the connection and its momentum of the Ashtekar-Barbero variables. First, we note that a possible choice of triads (defined by the relation \( q_{ab} = e_a^I \delta_I J e_b^J \)) are:

\[
e_1 = \frac{dr}{a(t)}, \quad e_2 = \frac{d\theta}{\sin(r)a(t)}, \quad e_3 = \frac{d\varphi}{\sin(r)\sin(\theta)a(t)}
\]

(2.19)

Computing the spin connection from the triads, we obtain the following form:

\[
\Gamma^L_a = -\frac{1}{2} \epsilon^{LJK} e^K_b (\partial_b e_a^J - \partial_a e_b^J + e_c^M \partial_b e_c^M) \]

\[
= -\epsilon^{LJK} (\delta_a^2 \delta_J^1 \delta_K^1 \cos(r) + \delta_a^3 \delta_J^3 \delta_K^1 \cos(r) \sin(\theta) + \delta_a^3 \delta_J^3 \delta_K^2 \cos(\theta))
\]

(2.20)

(2.21)

The extrinsic curvature \( K_{ab} \) given by (2.6) is in our case quite simple. Here we do not want the action of any spatial diffeomorphisms, so we set \( N_a \equiv 0 \). In addition the only time dependent quantity in our metric is the scale factor \( a(t) \), it is easy to check that

\[
\dot{q}_{ab} = 2 \frac{\dot{a}(t)}{a(t)} q_{ab}
\]

(2.22)

holds.

We will further pick a comoving frame, i.e. \( N \equiv 1 \). With this knowledge we can calculate the connection and the electric field:

\[
A_1^1 = \beta c, \quad A_2^1 = 0, \quad A_3^1 = \cos(\theta)
\]

\[
A_1^2 = 0, \quad A_2^2 = \beta c \sin(r), \quad A_3^2 = -\cos(r) \sin(\theta)
\]

\[
A_1^3 = 0, \quad A_2^3 = \cos(r), \quad A_3^3 = \beta c \sin(r) \sin(\theta)
\]

(2.23)

\[
E_L^a = p \delta_L^a \left( \delta_1^a \sin^2(r) \sin(\theta) + \delta_2^a \sin(r) \sin(\theta) + \delta_3^a \sin(r) \right).
\]

(2.24)

Here, we have changed the notation

\[
\dot{a}(t) \to c, \quad a(t)^2 \to p
\]

(2.25)

From this point onwards, we will be interested only in families of connection and triads that are parametrized by \( (c, p) \in \mathbb{R}^2 \), i.e. of the form (2.23) and (2.24). These are the classical phase space data that allow an isotropic metric. However, we will not longer employ any relation to \( \dot{a}(t) \), which might no longer hold in the quantum theory, where the evolution is given by the quantum Hamilton equations. In order to perform a symplectic reduction to the submanifold spanned by \( (c, p) \), such an
The hyperspherical lattice is composed of edges along the three coordinates lines $r, \theta, \varphi$. The degenerate points at $r = 0, \frac{\pi}{2}$ and $\theta = 0, \pi$ are the only non-cuboidal points of the lattice. Their valency is directly related to the denseness of the lattice in $\theta$- and/or $\varphi$-direction. We define the number of vertices in each direction $i$ by $N_i$. The lattice spacing $\epsilon$ has to be chosen such that

$$\epsilon_1 = \frac{\pi}{2N_r}, \quad \epsilon_2 = \frac{\pi}{N_\theta}, \quad \epsilon_3 = \frac{2\pi}{N_\varphi} \quad (2.26)$$

With these notations, one can compute for an edge $e_i$ which starts at $e(0) = (r_0, \theta_0, \varphi_0)$ and goes along direction $i$ for the length $\epsilon_i$:

$$h(e_r) = \exp \left[ \epsilon_1 \beta c_1 \right]$$

$$h(e_\theta) = \exp \left[ \epsilon_2 \left( \beta c \sin(r_0) \tau_2 + \cos(r_0) \tau_3 \right) \right]$$

$$h(e_\varphi) = \exp \left[ \epsilon_3 \left( \cos(\theta_0) \tau_1 - \sin(\theta_0) \cos(r_0) \tau_2 + \beta c \sin(r_0) \sin(\theta_0) \tau_3 \right) \right] \quad (2.29)$$

The surface $S_{e_i}$ associate to any each $e_i$ are such that the intersect the edge at its starting point and lie in the $jk$-plane normal to $i$ with boundaries $e(0) = 1 \pm \epsilon_j/2$ and similar for $k$. Therefore:

$$E(e_r) = \int_{\gamma} \int dx^a \wedge dx^b E_{ij} E_{abc} = \int_{\theta_0 - \epsilon_2^{\frac{\epsilon_2}{2}}}^{\theta_0 + \epsilon_2^{\frac{\epsilon_2}{2}}} \int_{\varphi_0 - \epsilon_3^{\frac{\epsilon_3}{2}}}^{\varphi_0 + \epsilon_3^{\frac{\epsilon_3}{2}}} d\theta \int d\varphi \sqrt{P} \sin^2(r_0) \sin(\theta) \frac{1}{\sqrt{P}} \tau_1$$

$$E(e_\theta) = \int_{r_0 - \epsilon_1^{\frac{\epsilon_1}{2}}}^{r_0 + \epsilon_1^{\frac{\epsilon_1}{2}}} \int_{\varphi_0 - \epsilon_3^{\frac{\epsilon_3}{2}}}^{\varphi_0 + \epsilon_3^{\frac{\epsilon_3}{2}}} dr \int d\varphi \sqrt{P} \sin(r) \sin(\theta) \frac{\tau_2}{\sqrt{P} \sin(r)}$$

$$E(e_\varphi) = \int_{r_0 - \epsilon_1^{\frac{\epsilon_1}{2}}}^{r_0 + \epsilon_1^{\frac{\epsilon_1}{2}}} \int d\theta \int d\varphi \sqrt{P} \sin(r) \sin^2(\theta) \frac{\tau_3}{\sqrt{P} \sin(r) \sin(\theta)}$$

Having now complete knowledge of the building blocks available, one is in principle able to compute the values of discretised observables. Exemplarily, we present the computation of the discretised spatial volume, i.e.:

$$V := \sum_{v \in \gamma} V(v), \quad V(v) := \sqrt{\frac{1}{3!} \sum_{\epsilon_i \cap \epsilon_j \cap \epsilon_k = v} \epsilon_1 \epsilon_2 \epsilon_3 E_{1} E_{2} E_{3}} \quad (2.33)$$

This is the regularisation which leads to the Ashtekar-Lewandowski volume upon quantisation in LQG [60].

For the fluxes of the form (2.30) on the hyperspherical lattice, the volume can be exactly computed as:

$$V_0 = \sqrt{P} \left( \frac{\pi}{2\epsilon_1} - 1 \right) \sin \left( \frac{\epsilon_1}{2} \right) \cot \left( \frac{\epsilon_2}{2} \right) \sqrt{2\epsilon_2 \sin \left( \frac{\epsilon_2}{2} \right) \epsilon_2} \quad (2.34)$$
One can check that this indeed has the correct classical limit when removing the regulators $\epsilon \to 0$.

It must be noted that it is a feature of classical, isotropic, closed cosmology, that connection and triad are at all times given by the form (2.23) and (2.24). Once, we go over to the discrete level, this property is not automatically guaranteed and requires further investigation [58, 59].

One realizes that the submanifold $\mathcal{M}$ of the phase space spanned by connections and triads parametrised by $(p, c)$ can be understood as a symplectic reduction. That means, one can naturally endow $\mathcal{M}$ with a symplectic structure coming from $(h(e), E(e))$.

For the case of the continuum geometry this method of symplectic reduction can be easily applied and we repeat the computation for completeness: Let $f_1, f_2$ be functions on the phase space $\mathcal{M}$ of the continuous connection and $X_f$ the Hamiltonian flow generated by $f$, then we can derive:

$$\{f_1, f_2\} = \int d^3 x \sqrt{\gamma} \frac{\delta f_1}{\delta E} \frac{\delta f_2}{\delta A} = \int dE^a \wedge dA^a(X_{f_1}, X_{f_2}) = \int \left( \delta^a_1 \delta^a_1 \sin(r)^2 \sin(\theta) + \delta^a_2 \delta^a_2 \sin(r) \sin(\theta) + \delta^a_3 \delta^a_3 \sin(r) \sin(\theta) \right) \delta c(X_{f_1}, X_{f_2})$$

$$= \frac{3V_0}{\sqrt{p}^3} \{f_1, f_2\}_{(p,c)}.$$

In accordance with the following simplifying assumption, we will also simplify our setting by using this symplectic structure as the reduced phase space of the discretised theory.\(^5\)

To compute the evolution of a system in terms of the finitely many variables $(p, c)$ would be sufficiently simple to be handleable. But in order to justify this, one must ask, what the relation between Poisson brackets in the full theory and at the reduced level is:

$$\{F(h(e), E(e)), G(h(e), E(e))\}_{(h, E)\rightarrow c, p} = \{F(h(e), E(e))|_{h, E\rightarrow c, p}, G(h(e), E(e))|_{h, E\rightarrow c, p}\}_{(c, p)}$$

where $h, E \rightarrow c, p$ refers to the symplectic reduction specified above.

A first, dissatisfactory observation is that both sides are in general not equal. This is the case only for special submanifolds $\mathcal{M}$ and sufficiently adapted functions $F, G$ (one of which typically needs to be invariant with respect to the symmetry of the system). An example for this is the Thiemann regularisation of the Hamiltonian on cubic lattices for isotropic, flat cosmology (see [58] for further details). However, we will in the following assume that symmetric reduction and Poisson brackets do commute in our situation, i.e. the above equation holds with an “=” sign!

On the one hand this allows to simplify the computation of (2.12) drastically and on the other hand this assumption is anyway necessary if one wants to use the resulting expression as an effective Hamiltonian to generate evolution with respect to some scalar field as we will do in the following section.\(^6\) It is this conjecture which claims that the system at a later point of the flow induced by the constraints is still of the form the computed discretisation of $k = +1$ cosmology.

With these assumptions we can simplify the evaluation of the formulas (2.13) and (2.12) for isotropic, closed cosmology. However, the terms involved become quite lengthy and exceed the possibility to be printed as analytical results in a written paper. Therefore, we equate all $\epsilon_i = \epsilon$ and

\[ A^i_\mu(x) = c \omega^i_\mu(x), \quad E^\mu_7(x) = p \tilde{\omega}^\mu_7(x) \quad (2.35) \]

with $\omega$ the Maurer-Cartan form on $\mathfrak{su}(2)$.

Both are different submanifolds and interestingly when reducing of classical GR to these manifolds the evolution stays inside them. Therefore both description are classically equivalent. Whether both (or any of them) are also invariant submanifolds of discretised GR for the evolution produced by the full lattice Hamiltonian remains to be investigated.

For the purpose of this article we stay with the choice (2.23) and (2.24) as in this framework it is easier to compute the regularised Hamiltonian.

\(^5\) We note that in the earlier literature also another submanifold of the phase space was investigated, namely in [48, 49]

\[ \text{with } \omega \text{ the Maurer-Cartan form on } \mathfrak{su}(2). \]

Both are different submanifolds and interestingly when reducing of classical GR to these manifolds the evolution stays inside them. Therefore both description are classically equivalent. Whether both (or any of them) are also invariant submanifolds of discretised GR for the evolution produced by the full lattice Hamiltonian remains to be investigated.

For the purpose of this article we stay with the choice (2.23) and (2.24) as in this framework it is easier to compute the regularised Hamiltonian.

\(^6\) It is worth noting, that – albeit not explicitly stated – variants of this assumption are used in all effective models of LQC type once the evolution due to some reduced effective Hamiltonian is computed.
Several remarks are in order: with some homogeneous lapse function $N$ other. The total scalar constraint studied in this section reads: $C_{\text{tot}}[\mathcal{N}] := N \left( C_{\text{eff}}[\mathcal{N}] + N \frac{\pi^2}{2\sqrt{p^3}} \right)$

with some homogeneous lapse function $N$, and scalar field momentum $\pi_{\Phi}$ (canonically conjugated to $\Phi$, i.e. $\{\pi_{\Phi}, \Phi\} = 1$). Throughout this section we work in natural units, i.e. $\ell_P = \hbar = G = c = 1$.

Several remarks are in order:

- The lapse function $N$ is assumed to be homogeneous to respect the symmetries of the system. Since it only changes the unphysical flow of the scalar constraint and has no physical relevance, we will set $N \equiv 1$ in the following.

- The matter part of the constraint incorporated here carries no further knowledge of the discretisation. This is not completely consistent as the matter Hamiltonian should be discretised as

3 Numerical Analysis of the Approximation of the effective Hamiltonian constraint

In this section, we perform preliminary steps towards investigating the effective scalar constraint for $k = +1$ cosmological spacetimes of the previous section. For that purpose, we will proceed as in [48, 49] by coupling a massless, homogeneous scalar field to the geometry degrees of freedom. This scalar field will serve the role as a physical clock, i.e. since the flow of phase space parameters $(c, p)$ induced by the effective scalar constraint $C_{\text{eff}}[\mathcal{N}]$ is physically meaningless, we will need to compare it with the simultaneous flow of the scalar field $\Phi$ to deduce how physical quantities change with respect to each other. The total scalar constraint studied in this section reads:

$$C_{\text{tot}}[\mathcal{N}] := N C_{\text{eff}}[\mathcal{N}, c, p] + N \frac{\pi^2 \Phi}{2\sqrt{p^3}}$$

The expression corresponding to the effective Hamiltonian of an isotropic, closed Universe, modified by the discreteness corrections, that emerge due to the Thiemann regularisation. In the next section we will analyse these corrections in the regimes where the approximation to seventh order is justified.
As was already stressed in the previous section, we make heavy use of the assumption that computing the flow on the reduced phase space agrees with the reduction of the flow on the full phase space. One should take note that the analytic result of the approximation to $C^\epsilon|_{\hbar,E\rightarrow c,p}$ are also computed using said assumption. (The situation would get even more complicated if one would additionally incorporate the matter field phase space in the continuum and perform a discretisation and reduction of the total scalar constraint afterwards.)

- Under the previous assumption the flow of $C^\epsilon|_{\hbar,E\rightarrow c,p}$ would drive a phase space point parametrized by $(c(0), p(0), \Phi(0), \pi\Phi(0))$ to a different point on the same reduced submanifold, i.e. $(c(t), p(t), \Phi(t), \pi\Phi(t))$ to allow for relational observations. However, in this article we have only access to a power series expansion of $C^\epsilon|_{\hbar,E\rightarrow c,p}$ up to $7^{th}$ order in the regularisation parameter $\epsilon$. This means, we must carefully investigate a priori at which points in the reduced phase space this approximation to $7^{th}$ order is valid, and whether the flow leaves at some point this regime of validity. Points in phase space where the approximation breaks down and any effects found thereon carry no physical relevance.

### 3.1 Preliminary Analysis

We will start our analysis by fixing the free parameters of the model, i.e. $\epsilon$ and $(c(0), p(0), \Phi(0), \pi\Phi(0))$, such that they allow for a sufficiently classical regime. The choice of the regulator $\epsilon$ refers to the lattice spacing with respect to coordinate distance, which we choose earlier to be the same in each direction $r, \theta, \varphi$. We choose our coordinate system such that $r \in [0, \pi/2)$, $\theta \in [0, \pi)$ and $\varphi \in [0, 2\pi)$. The lattice spacing $\epsilon$ as measured in said coordinate distance must therefore be sufficiently smaller than $\pi/2$ in order to allow for an acceptably dense graph. Following (2.26) a possible choice could be $\epsilon = \pi/100$, corresponding to a lattice with $10^6$ many vertices. We point out that this is in contrast to the standard $\mu_0$ scheme of LQC literature, where the regularisation parameter is commonly chosen to be $\mu_0 = 3\sqrt{3}$ (in Planck units) by relating it to the minimal eigenvalue of the area operator in LQG. However, such a value exceeds the coordinate spacing of our manifold and we want to keep its natural value – indicating that we should refrain from choosing the $\mu_0$-regularisation. There is also a strong reason, why such a choice is disfavoured for the current scalar constraint:

We are dealing with $k = +1$ cosmology and want to investigate solutions that feature a classical behaviour. Consequently, we search for those trajectories in phase space that have a recollapse point in the classical regime. A recollapse point in the phase space is defined by lying in the hyperplane $c = 0$. In order for the system to behave classical, the energy density $\rho := \pi^2/(2\pi^2p^3)$ must be a positive value. To see, what kind of restrictions this poses, let us look at the expansion of $C^\epsilon$ up to first order in $\epsilon$:

$$\begin{align*}
C^\epsilon(c = 0, p, \Phi, \pi\Phi) &= \frac{\pi^2}{2p^3/2} - \frac{6\sqrt{p\pi^2}}{\kappa} + \frac{24\sqrt{p\pi\epsilon}}{\kappa} + \mathcal{O}(\epsilon^2)
\end{align*}\quad (3.2)
$$

\footnote{For the purpose of this article, we will not comment on the fact that $\epsilon \in \mathbb{R}$ corresponds to what is called $\mu_0$ scheme in the LQC literature. It is apparent that due to the reference to a fiducial coordinate system, the physical predictions get affected by coordinate effects. Several preliminary proposals exist in the literature of how this can be remedied, e.g. making $\epsilon$ change under scaling as well. The most prominent of them is the $\tilde{\mu}$ -scheme introduced in [43] and frequently used in the literature. However, until to today there is no satisfactory derivation of the $\tilde{\mu}$ scheme from the full theory and therefore we refrain from using it.}

\footnote{In standard GR, this automatically equated to $\dot{\rho} = 0$. Therefore implying a change from an expanding to a collapsing universe.}

In principle, when inverse powers of the volume appear they should also be lifted via a version of Thiemann identities [31] to terms involving Poisson brackets, which upon regularisation have non-trivial contribution. However, to go along these lines would also require a consistent discretisation of the matter degrees of freedom, which has been omitted here as well. Therefore, this system serves merely as a toy model.
Upon imposing vanishing of the constraint, and demanding positivity of $\rho_c$ it follows that: $(6\pi - 24\epsilon) > 0$. In other words, a classical recollapse can only occur if $\epsilon < \pi/4 < \mu_o$.

Of course, this is only for the first order expansion of the constraint – if one considers the 7th order expansion, the terms get more lengthy, however reduce to the requirement: $\epsilon < \epsilon_{\text{max}} = 1.19873$ (which is slightly bigger than $\pi/4 \approx 0.7853$). While this requirement eliminates the possibility of choosing the $\mu_o$ scheme, it does of course not fix a minimal value for $\epsilon$. Thus, we are free to take for the upcoming investigations $\epsilon = \pi/100$.

However, these modifications proportional to $\epsilon$ go with the same power of $p$ as the classical curvature term. This implies that even around the recollapse, where the universe is mostly classical, a possible deviation from standard GR directly proportional to $\epsilon$ could be measured, if one assumes the regularisation $C^{c}_{\text{tot}}$.

For the range of allowed values for the regularisation parameter $\epsilon \in [0, \epsilon_{\text{max}}]$ one can check that the derivations from the reduced scalar constraint of classical GR are always positive. Therefore, in order to satisfy the constraint for fixed $\pi_\Phi$, we will always need a $p_{\text{disc}}$ in the discrete model that is larger than the square of the scale factor for classical general relativity, i.e. $p_{\text{clas}} < p_{\text{disc}}$ at the recollapse.

Such a behaviour was never encountered in LQC before, due to the fact that all modifications proportional to $\epsilon$ appeared as functions of the form $f(\epsilon)$, i.e. in the “limit of late time cosmology” $c \rightarrow 0$ any modification proportional to $\epsilon$ would be damped as well. This model therefore presents a novel feature, that quite generally can happen for any discretisation.

It is interesting to note that the modification due to $\epsilon$ for the scalar constraint at the recollapse, could be absorbed in a redefinition of $\kappa$. E.g. for (3.2) (which only includes linear corrections in $\epsilon$) the dynamics in the classical regime can be equivalently described by the scalar constraint of standard GR with the effective Newton constant: $G_{\text{eff}} := G(1 - 4\epsilon/\pi)$. Comparing measurements of Newton’s constant from cosmology and other methods could therefore provide a further, upper bound for $\epsilon$.

The choice of $p$ and $\pi_\Phi$ are independent of the choice for $\epsilon$ at the recollapse. Via several numerical simulations, we found out that the qualitative behaviour of the phase space trajectories does not change for different values of $\pi_\Phi$. The latter one is a constant of the dynamics, since $\Phi$ is a cyclic variable, i.e. it does not appear in $C^{c}_{\text{tot}}$. Therefore, the absolute value of $\Phi$ is irrelevant as well and merely corresponds to a time shift. Once a value of $\pi_\Phi$ is specified, by imposing the constraint at the recollapse, one can determine $p$ and start numerical simulations.

However, before doing so, we must determine a priori until what point the flow of $C^{c}_{\text{tot}}$ agrees at least qualitatively with the expansions to which we have access. For that purpose, we investigate Hamilton’s equation of motion analytically. First we determine $\dot{c}$, which, after imposing vanishing of the constraint, reads:

$$
\dot{c} = \{C^{c}_{\text{tot}}, c\} = \frac{\kappa\beta}{6\pi^2} \frac{\partial C^{c}_{\text{tot}}}{\partial p} \bigg|_{c=0} = \frac{\kappa\beta}{6\pi^2} \left[ (-3/2) \frac{\pi_\Phi^2}{2p^{3/2}} + (-1/2) \frac{\pi_\Phi^2}{2p^{5/2}} \right] = -\frac{\kappa\beta \pi_\Phi^2}{6\pi^2 p^{5/2}} \quad (3.3)
$$

This implies that $c$ decreases strongly monotonic, since we have chosen positive orientation of the triad (in agreement with positive volume $p > 0$). Especially, the change in $c$ grows the closer the flow drives towards a singularity.

Further, we plot $\dot{p}/\sqrt{p}$ in figure 3.1, where the time derivative is obtained from Hamilton’s equation, i.e.

$$
\dot{p} = \{C^{k}, p\} = \sqrt{p} \ \text{Pol}_k(c, \epsilon, \beta) \quad (3.4)
$$

where $C^k$ denotes the expansion of $C^{c}_{\text{tot}}$ to order $\epsilon^k$. It transpires from figure 3.1 that there exist points in phase space, for which the absolute value of $c$ is so big that it would cause a deviation from the fifth to the seventh order (and, hence, one must assume that shortly thereafter the seventh order deviates from the full, not-approximated evolution as well). As we have seen before $c$ decreases monotonically, consequently it increases monotonically in backwards evolution towards the initial
singularity. Therefore, it will necessarily reach the point where the flow of the approximated constraints drives into these regions of the phase space, where one can no longer trust their predictions. We highlight the deviation between 5th and 7th order exceeds 10% by calling it \( \sigma_f \).

Due to the symmetric form of the constraint (it depends only on natural powers of \( c^2 \)) the same effects appear also in forward evolution.

![Plot of \( c \) vs \( \dot{p}/\sqrt{p} \) as obtained from Hamiltons equation using different scalar constraints \( C_i \) for \( \beta = 0.2375 \) and \( \epsilon = \pi/100 \). In blue dashed, the evolution was obtained from \( C \) of standard GR; in red the 3rd order approximation in \( \epsilon \) of \( C_{\text{tot}}^\epsilon \), in orange the 5th order approximation in \( \epsilon \) of \( C_{\text{tot}}^\epsilon \) and in black the 7th order approximation in \( \epsilon \) of \( C_{\text{tot}}^\epsilon \) are presented. As seen from analytical arguments, \( C_{\text{tot}}^\epsilon \) forced to \( c \) to strictly increase monotonic in backward time evolution. We can deduce that, the crossing through 0 for finite values of \( c \) implies a bounce of the model. Since the blue and orange curves only cross zero once (at the recollapse) both solutions must feature initial singularities. Due to the symmetric behaviour of the constraint in \( c \) the values are point-symmetrically mirrored to negative values of \( c \) describing the flow in forward time evolution.

3.2 Numerical Simulations

Now, we will study the flow of the constraint by numerical methods for observables volume \( v = \pi^2 p^{3/2} \) and energy density \( \rho = \pi^2 \Phi / (2 \pi^2 p^3) \). We stress again that we assume validity of replacing the symplectic structure of the discretised phase space with the reduced Poisson bracket \( \{ p, c \} = \beta \kappa / (6 \pi^2) \).

For the Immirzi parameter we take \( \beta = 0.2375 \) as is custom in LQC literature. As initial data, we pick the recollapse at \( c(0) = 0 \) with the arbitrary choice \( \Phi(0) = 0 \) and determine \( p(0) \) via imposing the respective constraint. It remains therefore to choose the constant of motion \( \pi \Phi \). We will present two cases: (A) for \( \pi_{\Phi,A} = 500 \) and (B) \( \pi_{\Phi,B} = 1.77 \times 10^9 \), and it will transpire that the qualitative behaviour of the phase space trajectory is unaffected by this choice (The later value corresponds to a choice where \( \rho = 10^{-9} \) at the recollapse). Not having access to \( C_{\text{tot}}^\epsilon \) directly, we plot in figure 3.2 and 3.3 the flow of the \( v \) and \( \rho \), deparametrized with respect to scalar field time \( \Phi \), for classical GR constraint and for different approximations of \( C_{\text{tot}}^\epsilon \) namely for 3rd, 5th and 7th order expansion in \( \epsilon \). At the point where 5th and 7th part, the 5th order expansion does not make reliable predictions anymore and therefore it would be premature to trust the behaviour of the 7th order for a long period after the deviation occurs. The scalar field time at which the flow of \( c \) crosses the value \( \sigma_f \) is marked by \( \Phi_f \). One can deduce that close to the initial (and final) singularities a deviation from standard general relativity takes place. To be precise in the early universe we would expect that the expansion was not as strong with respect to \( \Phi \) as predicted by classical GR. However, at the present state it is not possible to deduce whether this model will resolve the initial singularity via a bounce.
Figure 3.2: Case (A) : $\pi_\Phi = 500$. Flow of volume $v = \pi^2 p^{3/2}$ and energy density $\rho = \pi^2_\Phi/(2\pi^2 p^{3/2})$ as driven by various effective constraints. In blue dashed, from C of standard GR; in red the 3rd order approximation in $\epsilon$ of $C^\epsilon_{\text{tot}}$, in orange the 5th order approximation in $\epsilon$ of $C^\epsilon_{\text{tot}}$ and in black the 7th order approximation in $\epsilon$ of $C^\epsilon_{\text{tot}}$ are presented. The approximation of the constraints should not be trusted before the point $\Phi_f$, where 5th and 7th order merge. The initial data were picked at the recollapse at $\Phi(0)$. Due to the symmetry of the constraints, the flow is mirrored in positive $\Phi$ direction.

We shall also comment on possible violations of the approximated constraint: Independent of the
chosen value for $\pi_{\Phi}$ the violation of the constraint will always be satisfied in the regime between recollapse and $\Phi_f$. However, before $\Phi_f$, before the “bounce” occurs the system features an exponential fast contraction – $p$ decreases by 4 orders of magnitude in 0.6 $\Phi$-time. This causes a drastic violation of the constraint in absolute values $C_{\text{tot}} \neq 0$, however compared to the absolute value of the volume of the model this violation is still negligible. A similar situation happens in the LQC cases of cosmological constants that feature a similar exponential contraction/expansion and where only $C_\Lambda/v$ remains small. Nonetheless, we like to stress again that at the present stage no effect beyond $\Phi_f$ can be regarded as physical anyway.

Finally, it is also interesting to note that whether a bounce happens or not is not an intrinsic features that only occurs if the constraint is bounded with respect to $c$ (e.g., as is the case if $c$ appears only inside of trigonometric functions). Instead, also truncations of finite powers of trigonometric functions can cause a bounce (we see that a bounce is predicted for 3rd and 7th order). However, a bounce is also not a unique criterion that always appears as soon as $\epsilon$-corrections a present the 5th order is driven into a singularity – this happens necessarily as also figure 3.1 showed that the corresponding $p$-equation never crosses zero, i.e. there exists no turning point.

3.3 Comparison to earlier closed LQC models

In previous literature, a LQC-like quantisation on a reduced phase space for $k = +1$ models was already proposed. There, an effective constraint was derived from the reduced quantum theory which serves as regularisation of the $k = +1$ version of the scalar constraint [48, 49]. However, a crucial ingredient in this construction was the following identity which is only true for certain coordinates in $k = +1$ spacetime [48]:

$$2K^I_a K^J_b \frac{1}{\beta^2} \epsilon^{IJ} F_{ab}^K + \frac{1}{2} \omega^I_0 \omega^J_0$$

(3.5)

where $^0\omega$ is the Maurer-Cartan form on $su(2)$. Using this simplification which only holds true in this closed isotropic models, a much simpler regularisation of the scalar constraint can be obtained (i.e. it avoids a lengthy regularisation of the Euclidean part). The philosophy behind this procedure can be rephrased as first reducing and then discretising and as one can see these two procedures do not commute (this was already observed in isotropic flat cosmology).

In [49] moreover a further simplification was assumed, namely that the holonomy over any plaquette in the $k = +1$ spacetime has the same functional form in terms of the parameteres $c, p$. This is - of course - not the case in general and also not for the coordinates of $k = +1$ spacetimes used neither here nor in [49]. Therefore, even is one would employ (3.5) before discretisation, afterwards performing the smearing over the whole spatial manifold would lead to a different effective Hamiltonian.

Nonethless, we want to point out that this is no caveat of the methods of [48] and [49]: the authors implement moreover the so-called $\bar{\mu}$-scheme [43] a posteriori by simply replacing $\epsilon \rightarrow \bar{\mu} \propto 1/\sqrt{p}$. Since there is up to today no quantisable regularisation of the Hamiltonian in full general relativity known that features a $\bar{\mu}$-scheme (see [61] for a discssion of the case of flat cosmology and [62, 63] for first steps towards its implementation) any relation to the Thiemann-regularisation is anyway far-fetched. However, the $\bar{\mu}$-scheme resolves an issue regarding the remnant of residual diffeomorphisms. In this sense, if one does not require that the effective Hamiltonian stems from a valid regularisation of the full theory but focuses on solving the rescaling problem, the proposal of [48] and [49] presents a very successful candidate.

4 Conclusion

In this paper we performed first steps towards investigating the dynamics of discretised general relativity modelling a closed, spatially isotropic universe. We proposed as fundamental spatial manifold
a family hyperspherical lattices embedded in $S^3$, on which the metric degrees of freedom are encoded in terms of the holonomies and fluxes of the Ashtekar-Barbero framework on the edges of the lattice. Such a discretisation allows for an approximation of isotropic spatial data (which cannot be exact as the discrete data cannot remain invariant under arbitrary rotations and translations of the $S^3$). The scalar (or Hamiltonian) constraint $C$ of general relativity is discretised, i.e. replaced by a function $C^\epsilon$ expressed in terms of quantities on the lattice which only agrees with $C$ in the limit of infinitely dense lattices. The functional form of $C^\epsilon$ chosen in this paper is the Thiemann regularisation which is custom in LQG, justifying the expectation that our computation could capture certain aspects of a theory of quantum gravity.

Due to the complicated structure involved, we approximate the analytical form of $C^\epsilon$ to finite order in terms of the lattice spacing, i.e. 7th order. With this, we improve earlier work in the literature by not requiring any symmetries of the system prior to discretisation, although we work under the assumption that the reduced dynamics agrees with the dynamics of the graph (i.e. under premise that the flow of $C^\epsilon$ does not kick an initially, approximate isotropic configuration out of its subspace).

The flow of these approximated constraints can then be studied on the reduced level. Due to the approximation involved the evolution can only be trusted up to a certain point: the flow of the 7th order deviates from the 5th order more than 10% at a phase space point called $c_f$, in terms of scalar field time – therefore one should expect deviations from the flow produced from the full $C^\epsilon$ as well. We have put these approximations to the test by coupling them to a massless free scalar field and studying the flow of this cosmological toy model with analytic and numerical tools: all the approximations feature a classical point where $c = 0$, i.e. a classical recollapse point. Indeed, discretisation (or potentially quantum) effects are suppressed at this point to feature largely the known dynamics. However, a small imprint of the discretisation remains that can be absorbed into a rescaling of Newtons coupling constant around the recollapse point. The deviation from the classical constant is directly proportional to the lattice spacing and could therefore in principle be used to find an upper bound for the lattice spacing. Further, since starting from the recollapse point the classical universe is driven towards initial and final singularities, we compared with the evolution of the approximations: clearly a deviation from the classical trajectory is visible, slowing down the descent to the singularities. However, at the present stage the validity of the approximations breaks down before a resolution via the big bounce is certain.

In future work, it will be of interest to increase the resolution around the singularity, e.g. by computing higher order approximations of the effective scalar constraint. It is also an open question, to what degree different graphs as underlying discretisation of the spatial manifold will have impact on the effective dynamics of cosmological solutions. Finally, any serious investigations of the flow requires further work to justify the assumptions of reducing the dynamical evolution to the reduced sector. Promising work, asking for the conditions under which this is possible, is ongoing [58].

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References

[1] A. Friedmann. “Über die Krümmung des Raumes”. Zeitschrift für Physik A 10 (1922)
[2] A. Friedmann. “Die Welt als Raum und Zeit (The World as Space and Time)”. Ostwalds Klassiker der exakten Wissenschaften ISBN 3-8171-3287-5
[3] A. Friedmann. “Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes”. Zeitschrift für Physik A 21 (1924)
[34] E. Alesci, F. Cianfrani. “Quantum Reduced Loop Gravity: Semiclassical limit”. [arXiv:1402.3155] (2014)
[35] E. Alesci, F. Cianfrani. “Loop Quantum Cosmology from Loop Quantum Gravity”. [arXiv:1410.4788] (2014)
[36] A. Dapor, K. Liegener. “Cosmological Effective Hamiltonian from full Loop Quantum Gravity”. Phys. Lett. B 785, 506-510 (2018)
[37] A. Dapor, K. Liegener. “Cosmological Coherent State Expectation Values in LQG I. Isotropic Kinematics”. Class. Quant. Grav. 35, 135011 (2018)
[38] M. Assanioussi, A. Dapor, K. Liegener, T. Pawlowski. “Emergent de Sitter epoch of the quantum Cosmos”. Phys. Rev. Lett. 121 (2018)
[39] M. Assanioussi, A. Dapor, K. Liegener, T. Pawlowski. “Emergent de Sitter epoch of the quantum Cosmos: a detailed analysis”. Phys. Rev. D 100, 084003 (2019)
[40] A. Garcia-Quismondo, G. Mena Marugan. “The Martin-Benito-Mena Marugan-Olmedo prescription for the Dapor-Liegener model of Loop Quantum Cosmology”. Phys. Rev. D 99, 083505 (2019)
[41] A. Garcia-Quismondo, G. Mena Marugan. “Dapor-Liegener formalism of loop quantum cosmology for Bianchi I spacetimes”. [arXiv:1911.09978] (2019)
[42] A. Ashtekar, T. Pawlowski, P. Singh. “Quantum Nature of the Big Bang”. Phys. Rev. Let. 96, 141301 (2006)
[43] A. Ashtekar, T. Pawlowski, P. Singh. “Quantum Nature of the Big Bang: Improved dynamics”. Phys. Rev. D 74, 084003 (2006)
[44] M. Bojowald. “Absence of a Singularity in Loop Quantum Cosmology”. Phys. Rev. Lett. 86, 5227 (2001)
[45] E. Alesci, M. Assanioussi, J. Lewandowski. “A curvature operator for LQG”. Phys. Rev. D, 89, 124017 (2014)
[46] M. Assanioussi, J. Lewandowski, I. Mäkinen. “New scalar constraint operator for loop quantum gravity”. Phys. Rev. D 92, 044042 (2015)
[47] P Singh, A. Toporensky. “Big Crunch Avoidance in k = 1 Semi-Classical Loop Quantum Cosmology”. Phys. Rev. D 69, 104008 (2004)
[48] L. Szulc, W. Kamiński, J Lewandowski. “Closed FRW model in Loop Quantum Cosmology”. Class. Quant. Grav. 24, 2621-2636 (2007)
[49] A. Ashtekar, T. Pawlowski, P. Singh, K. Vandersloot. “Loop quantum cosmology of k=1 FRW models”. Phys. Rev. D 75, 024035 (2007)
[50] J. Mielczarek, O. Hrycyna, M. Szydlowski. “Effective dynamics of the closed loop quantum cosmology”. JCAP 0911:014 (2009)
[51] T. Thiemann. “Gauge Field Theory Coherent States (GCS): I. General Properties”. Class. Quant. Grav. 18 2025-2064 (2001)
[52] T. Thiemann, O. Winkler. “Gauge Field Theory Coherent States (GCS): II. Peakedness Properties”. Class. Quant. Grav. 18 2561-2636 (2001)
[53] T. Thiemann, O. Winkler. “Gauge Field Theory Coherent States (GCS): III. Ehrenfest Theorems”. Class. Quant. Grav. 18 4629-4682 (2001)
[54] V. Taveras. “LQC corrections to the Friedmann equations for a universe with a free scalar field”. Phys. Rev. D 78, 064072 (2008)
[55] M. Bojowald, A. Skirzewski. “Effective theory for the cosmological generation of structure”. Rev. Math. Phys. 18, 713 (2006)
[56] S. Gielen, A. Polaczek. “Generalised effective cosmology from group field theory”. [arXiv:1912.06143] (2019)
[57] T. Thiemann. “Quantum Spin Dynamics (QSD) : VII. Symplectic Structures and Continuum Lattice Formulations of Gauge Field Theories. Class. Quant. Grav. 18, 3293-3338 (2001)
[58] A. Dapor, W. Kamiński, K. Liegener . “Symmetry reduction of Φ-invariant canonical theories”. (to appear)
[59] M. Han, H. Liu. “Effective Dynamics from Coherent State Path Integral of Full Loop Quantum Gravity”. [arXiv:1910.03763] (2019)

[60] A. Ashtekar, J. Lewandowski. “Quantum Theory of Geometry II: Volume operators”. Adv. Theor. Math. Phys. 1, 388-429 (1998)

[61] A. Dapor, K. Liegener, T. Pawłowski. “Challenges in recovering a consistent cosmology from the effective dynamics of loop quantum gravity”. Phys. Rev. D 100, 106016 (2019)

[62] N. Bodendorfer. “An embedding of loop quantum cosmology in (b, v) variables into a full theory context”. Class. Quantum Grav. 33, 125014 (2016)

[63] M. Han, H. Liu. “Improved (µ-Scheme) Effective Dynamics of Full Loop Quantum Gravity”. [arXiv:1912.08668] (2019)