Weak-QES extensions of the Calogero model

Y. Brihaye and P. Kosinski

Dep. of Mathematical Physics, University of Mons-Hainaut, Mons, Belgium
Dep. of Theoretical Physics, University of Lodz, Lodz, Poland

Abstract

We construct families of Hamiltonians extending the Calogero model and such that a finite number of eigenvectors can be computed algebraically.

1 Introduction

The notion of quasi exactly solvable equations has many different meanings. Initially it was invented to qualify operators (typically quantum Hamiltonian ones) which, after suitable change of variables and(or) ”gauge rotation”, are equivalent to an element of the enveloping algebra of some Lie algebra, represented by differential operators. In the following we refer to the above property as to algebraic QES property. In opposition some operators can possess a series of values of the coupling constants for which an explicit eigenvector is available without being related to any representation of a Lie algebra. We will qualify this property as ”weak-QES”. In the recent few years the Calogero model (and several of its extensions) have received a considerable new impetus of interest. One of the new results was the construction of a set of variables (the so called $\tau$-variables) in which the N-body Calogero model can be written as an element of the enveloping algebra of $sl(N)$. The
complete integrability of the model was then directly related to the finite represen-
tations of a Lie algebra, following very closely one of the ideas underlying the notion of
quasi exact solvability \([1]\). Some algebraic QES generalisations of the Calogero models were proposed soon after the basic result of \([3]\). Unfortunately these extentions
are all based of an \(sl(2)\) algebra. Basically, only one of the coordinates, for instance
\(\tau_2\), is involved into the additional piece of the potential. The purpose of this note is
to exhibit series of weak-QES hamiltonians, generalizing the Calogero hamiltonian,
some of them depending generically of all the \(\tau\)’s.

2 The Hamiltonian

We are interested in hamiltonians of the form

\[ H = H_c + V \quad (1) \]

\[ H_c = \frac{1}{2}(-\Delta + \omega^2 r^2) + \frac{\nu(\nu - 1)}{2} \sum_{j<k} \left( x_j - x_k \right)^2 \quad (2) \]

with

\[ \Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \quad , \quad r^2 = \sum_{i=1}^{N} x_i^2 \quad (3) \]

and such that the eigenvalue equation

\[ H\psi = E\psi \quad (4) \]

admits eigenvectors of the form

\[ \psi(x) = (\beta(x))^\nu e^{-\frac{1}{2}r^2} e^{P(x)p(x)} \quad , \quad \beta(x) = \Pi_{j<k}(x_j - x_k) \quad , \quad (5) \]

The function \(\beta(x)\) is the Vandermonde determinant of the matrix \(M_{ij} = (x_i)^j\). Moreover we restrict ourselves to polynomial forms of \(V(x), P(x), p(x)\) in \(x \equiv (x_1, x_2, \ldots, x_N)\).

The operator acting on \(p(x)\) will be denoted \(\tilde{H} : \)

\[ \tilde{H}p = Ep \quad (6) \]
In the following we use the variables $\sigma_1, \tau_2, \tau_3, \ldots, \tau_N$ introduced in [3]. In these variables, the laplacian (3) reads [3]

$$\Delta(\tau) = N \frac{\partial^2}{\partial \sigma_1^2} + \sum_{j,k=2}^{N} A_{jk} \frac{\partial^2}{\partial \tau_j \partial \tau_k} + \sum_{j=2}^{N} B_j \frac{\partial}{\partial \tau_j} \quad (7)$$

$$A_{jk} = \frac{(N - j + 1)(k - 1)}{N} \tau_{j-1} \tau_{k-1}$$

$$B_j = -\frac{(N - i + 2)(N - i + 1)}{N} \tau_{i-2} \quad (8)$$

For the manipulation of a generic (translation-invariant) change of basis, the relevant formula reads

$$\sum_k \frac{\partial w}{\partial x_k} \frac{\partial}{\partial x_k} = \sum_{j,k=2}^{N} A_{jk} \frac{\partial w}{\partial \tau_j \partial \tau_k} \quad (10)$$

The following identities are also useful:

$$\beta^{-\nu} \left( -\frac{1}{2} \Delta(x) + \sum_{j<k} \frac{\nu(\nu - 1)}{(x_j - x_k)^2} \right) \beta^\nu = -\frac{1}{2} \Delta(\tau) + \frac{\nu}{2} \sum_{j=2}^{N} (N - j + 2)(N - j + 1) \tau_{j-2} \frac{\partial}{\partial \tau_j} \quad (11)$$

$$e^{\frac{\omega r^2}{2}} \left( -\frac{1}{2} \Delta(x) + \frac{\omega^2}{2} r^2 \right) e^{-\frac{\omega r^2}{2}} = -\frac{1}{2} \Delta(\tau) + \omega \sum_{j=2}^{N} j \tau_j \frac{\partial}{\partial \tau_j} + \frac{N}{2} \omega \quad ; \quad (12)$$

they allow to handle respectively the repulsive and the harmonic parts of the potential in (III).

We have attempted to construct the polynomial potentials $V(\tau_2, \tau_3, \ldots, \tau_N)$ such that the operator (II) is equivalent (after a change of basis and with the variables $\tau$) to an operator preserving one vector space of the form

$$\mathcal{P}(N, n) = \text{span}\{\tau_2^{n_2} \tau_3^{n_3} \ldots \tau_N^{n_N} \mid n_2 + n_3 + \ldots + n_N \leq n\} \quad (13)$$

Our calculations indicate that, for $N = 2, 3, 4, 5$, the Calogero hamiltonian (corresponding to $V = 0$ in (III)) is the only one to possess such a property. After this negative result, we investigate alternative possibilities of algebraic solutions by imposing weaker requirements. Four types of situation have been considered which presented in the next section.
3 Weak-QES Hamiltonians

3.1 Type 1

We set \( N = 3 \); as previously stated \( V, P, p \) are then polynomials in \( \tau_2, \tau_3 \). Use of the identities of Sec. 2 leads to the operator \( \tilde{H} \) acting on \( p(\tau_2, \tau_3) \):

\[
\tilde{H} = \tau_2 \frac{\partial^2}{\partial \tau_2^2} + 3\tau_3 \frac{\partial^2}{\partial \tau_2 \partial \tau_3} - \frac{1}{3} \tau_2 \frac{\partial}{\partial \tau_2^3} + \frac{\partial}{\partial \tau_2}
+ (\frac{\partial P}{\partial \tau_2})(2\tau_2 \frac{\partial}{\partial \tau_2} + 3\tau_3 \frac{\partial}{\partial \tau_3})
+ (\frac{\partial P}{\partial \tau_3})(3\tau_3 \frac{\partial}{\partial \tau_2} - \frac{2}{3} \tau_2 \frac{\partial}{\partial \tau_3})
+ \omega(2\tau_2 \frac{\partial}{\partial \tau_2} + 3\tau_3 \frac{\partial}{\partial \tau_3})
+ 3\nu \frac{\partial}{\partial \tau_2} + V_{eff}
\]  

(14)

where we define

\[
V_{eff} \equiv V - \frac{1}{2} e^{-P} \Delta e^P
\]

(15)

We have constructed the solutions of this equation for particular values of the degrees of \( P \) and of \( p \).

\( P \) is of degree four

We assume \( P(x) \) to be at most quartic in \( x \), i.e.

\[
P = \frac{c}{2} \tau_2^2 + b \tau_2 + d \tau_3
\]  

(16)

Choosing for \( p \) a polynomial of global degree \( n \) in \( \tau_2, \tau_3 \), a careful power counting in (19), (14), reveals that polynomial solutions can exist only if \( V_{eff} \) is the form

\[
V_{eff} = v_1 \tau_2 + v_0
\]  

(17)

\( (v_0 \) accounts for the eigenvalue of \( \tilde{H} \), i.e. \( v_0 \equiv -E \)). Moreover there are generically \( \frac{(n+2)(n+3)}{2} - 1 \) algebraic equations to be solved and \( \frac{(n+1)(n+2)}{2} + 1 \) parameters (including the parameters of \( p \) and the two defining \( V_{eff} \)). We solved explicitly these equations for the first few values of \( n \).
Let $n = 1$, then
\[ p = \tau_2 + c_1 \tau_3 + c_0 \]  
(18)

Two types of solutions are found after some algebra.

**Solution (a)**
\[
\begin{align*}
    c_1 &= d = 0, \\
    c_0 &= \frac{2b + 2\omega + v_0}{2c}, \\
    v_0 &= -b - w \pm \sqrt{\omega^2 - 2c - 6\nu + 2b\omega + b^2}
\end{align*}
\]  
(19)

This solution does not depend on $\tau_3$ and is therefore not generic. It depends only on $\tau_2$ and it has two possible eigenvalues of the energy ($E \equiv -v_0$). It is a particular case of the QES extension of the Calogero model constructed in [3]. The potential depends on four parameters $\omega, \nu, b, c$. Such solutions can be constructed for any values of $n$. The corresponding Hamiltonian is an element of the enveloping algebra of sl(2) realized on the space of polynomials of degree at most $n$ in $\tau_2$.

**Solution (b)**
\[
\begin{align*}
    c_1 &= \frac{-3c}{2d}, \\
    c_0 &= \frac{2d^2 - \omega c - bc}{3c^2}, \\
    v_1 &= -3c, \\
    v_0 &= \frac{1}{c}(2d^2 - 3\omega c - 3bc)
\end{align*}
\]  
(20)

which has to be supplemented by one condition on $\omega, \nu, b, c, d$:
\[ 3b^2c^2 + 6bc^2 \omega - 8bcd^2 + 9c^3 \nu + 3c^3 + 3c^2 \omega^2 - 8cd^2 \omega + 4d^4 = 0 \]  
(21)

Unlike the "solution (a)", this solution non trivially depends on $\tau_2$ and $\tau_3$. The potential is parametrized by $\omega, \nu, b, c, d$ constrained by (21). One can in principle solve (21) with respect to one of the constants and obtain a finite number of potentials each admitting one algebraic eigenvector. This result is similar in spirit to a QES-type II equation [4].

Repeating the above calculation for $n = 2$, i.e.
\[ p = \tau_2^2 + c_1 \tau_2 \tau_3 + c_3 \tau_3^2 + c_2 \tau_2 + c_4 \tau_3 + c_5 \]  
(22)
As a counterpart of solution (b), we find the following expressions

\[ v_1 = -6c, \quad v_0 = \frac{2d^2 - 3b \ c - 3c \ \omega}{c} \]  \hspace{1cm} (23)

\[ c_1 = -3 \frac{c}{d}, \quad c_2 = \frac{32d^4 - 16b \ c \ d^2 - 9c^3 - 16d^2 \omega}{24c^2 \ d^2} \]  \hspace{1cm} (24)

\[ c_3 = 9c^2 \ d^2, \quad c_4 = \frac{b \ c + c \ \omega - 2d^2}{c \ d} \]  \hspace{1cm} (25)

\[ c_5 = (16b^2 \ c^2 \ d^2 + 9b \ c^4 + 32b \ c^2 \ d^2 \ \omega - 48b \ c \ d^4 + 9c^4 \omega + 36c^3 \ d^2 \ \nu + 15c^3 \ d^2 + 16c^2 \ d^2 \ \omega^2 - 48c \ d^4 \omega + 32d^6)/(36c^4 \ d^2) \]  \hspace{1cm} (26)

which fix the parameters of \( p \) and \( V_{\text{eff}} \) in terms of the coupling constant \( \omega, \nu, b, c, d \).

Two supplementary relations, analog to (21), have to be imposed of these coupling constants

\[ 0 = 24b^2 c^2 + 48bc^2 \omega - 64bcd^2 + 72c^3 \nu + 105c^3 + 24c^2 \omega^2 - 64cd^2 \omega + 32d^4 \]  \hspace{1cm} (27)

\[ 0 = -192b^5 c^3 d^2 - 108b^2 c^5 - 576b^2 c^3 d^2 \omega + 704b^2 c^2 d^4 + 216bc^5 \omega - 576bc^4 d^2 \nu - 156bc^4 d^2 - 576bc^3 d^2 \omega^2 + 1408bc^2 d^4 \omega - 768bcd^6 - 81c^6 \nu - 27c^6 - 108c^5 \omega^2 - 576c^4 d^2 \omega \nu + 156c^4 d^2 \omega + 576c^3 d^4 \nu + 216c^3 d^4 - 192c^3 d^2 \omega^3 + 704c^3 d^4 \omega^2 - 768c d^6 \omega + 256d^8 \]  \hspace{1cm} (28)

So that we end up with a three-parameters family of weak-QES potentials.

**P is of degree six**

We also considered the case of a polynomial \( P \) of degree at most six in \( x \), i.e.

\[ P = \frac{c}{2} \tau_2^2 + b \tau_3 + d \tau_2 + \frac{c}{3} \tau_3^3 + \frac{b}{2} \tau_2^2 + \frac{d}{2} \tau_2 \tau_3 \]  \hspace{1cm} (29)
and found that \( V_{\text{eff}} \) has to be of the form

\[
V_{\text{eff}} = v_0 + v_1 \tau_2 + v_2 \tau_3 + v_3 \tau_2^2
\]

i.e. at most quartic in the variables \( x_i \). Solving the equations (30) for \( n = 3 \), we checked that a four-parameter family of weak-QES potentials exists. The form of the constraints on the parameters rapidly becomes cumbersome when the degree of \( P \) increases.

### 3.2 Type 2

When we demand the functions \( V, P, p \) in (1)-(5) to depend on the variable \( \tau_2 \) only, the operator \( \tilde{H} \) takes the form

\[
\tilde{H} = \tau_2 \frac{\partial^2}{\partial \tau_2^2} + (\tau_2 P' + 2\omega \tau_2 + \frac{N-1}{2}(1 + N\nu)) \frac{\partial}{\partial \tau_2} \\
+ (2\omega \tau_2 + \frac{N-1}{2}(1 + N\nu))P' + \tau_2(P'' + (P')^2) + V(\tau_2)
\]

(31)

where \( P' = \frac{dP}{d\tau_2} \), etc.

The case \( P = -\frac{a}{2} \tau_2^2 - b \tau_2 \) corresponds to the QES extension of \([3]\); the potential \( V \) is then of third degree in \( \tau_2 \). However weak-QES potentials of higher degree can be constructed. Let indeed \( V \) be of degree \( 2\delta + 1 \), then a power counting reveals that \( P \) has to be of degree \( \delta + 1 \) and that the algebraic equation (30) can be fulfilled with a polynomial \( p(\tau_2) \) provided a number of \( \delta \) conditions among the \( 2\delta + 1 \) coupling constants entering in the potential are satisfied. This results into families of weak-QES potentials depending of \( \delta + 1 \) parameters.

### 3.3 Type 3

We have reconsidered the QES Hamiltonian of \([4]\) and tried to generalize it by following the approach presented in Sec.2. The hamiltonian has the form \([7]\)

\[
H = -\Delta + V + g \sum_{i,j=1,i \neq j}^{N} \left( \frac{1}{(x_i - x_j)^2} - \frac{1}{(x_i + x_j)^2} \right).
\]

(32)
We look for the most general change of function $e^P$ and potential $V$ such that

\[ \tilde{H} = e^P H e^{-P} |_{\xi_i}, \quad <qquad \xi_i \equiv x_i^2 \]  

(33)

is an element of the enveloping algebra of the Lie algebra $sl(N)$ in the representation given by Eq.(16) of [3]. After an algebra, we find a four parameters family of potentials with

\[ e^{-P} = (\Pi_{i,j=1,i<j}^N (\xi_i - \xi_j))^\alpha (\Pi_{j=1}^N \xi_j)^\beta \exp\left\{ -\frac{a}{4} \sum_{j=1}^N \xi_j^2 - \frac{b}{2} \sum_{j=1}^N \xi_j \right\} \]  

(34)

and

\[ V = \sum_{i=1}^N \left( \frac{g'^2}{4\xi_i} + (b^2 - a(4n + 4\alpha(N - 1) + 4\beta + 3))\xi_i + 2ab\xi_i^2 + a^2\xi_i^3 \right) \]  

(35)

with $g = \alpha(\alpha - 1), g' = 2\beta(2\beta - 1)$. This is slightly more general than the potential given in [7] since the coupling constants $g$ and $g'$ are here independent.

### 3.4 Type 4

Finally we present a result which directly generalizes to $N$ dimensions the famous one-dimensional sextic QES potential [1, 2]. In this purpose we consider

\[ H = -\frac{1}{2}\Delta + V_6(x) \]  

(36)

and assume that the even function $V_6(x)$ contain powers of degree at most six in $x_j$. The most general change of function $e^P$ such that the operator

\[ \tilde{H} = e^P H e^{-P} |_{t_i}, \quad t_i = x_i^2 \]  

(37)

is an element of the enveloping algebra of $sl(N)$ (in the representation given by Eq.(16) of [4]) is given by the $2N + 1$-parameters function

\[ P = \alpha T^2 + \sum_{a=1}^N p_a t_a + \sum_{a=1}^N q_a \log t_a, \quad T = \sum_{i=1}^N t_i \]  

(38)

where $\alpha, p_a, q_a$ are constants. Correspondingly the most general QES potential reads

\[ V_6 = 4 \sum_i t_i (\frac{\partial P}{\partial t_i})^2 + 4 \sum_i t_i \frac{\partial^2 P}{\partial t_i^2} + 2 \sum_i \frac{\partial P}{\partial t_i} - 16\alpha n T \]  

(39)
For $N = 2$, the 2-body polynomial potential of $\mathbb{R}^8$ is then recovered as the special case $q_1 = q_2 = 0$.

Summarizing, we have analyzed the rational Calogero model from the point of view of the notion of quasi exact solvability. We found, at least for the small number of particles, that it is unique in the following sense: it is exactly solvable but has no exactly or quasi-exactly solvable translational-invariant extension except those following from an $\text{sl}(2)$ structure. By relaxing the notion of quasi-exact solvability (weak QES) we were able to find a number of extensions of the Calogero model. They are characterized by the existence of analytical expressions for some levels without hidden symmetry behind.

Acknowledgments. This work was supported by the Belgian F.N.R.S. and carried out under the grant no. 2 P03B 076 10 of the Polish Government.

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