Best Approximation Pair of Two Linear Varieties via an (In)Equality
by (Fan-Todd) Beesack

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Abstract. The closest point of a linear variety to an external point is found by using the equality case of an Ostrowski’s type inequality. This point is given in closed form as the quotient of a (formal) and a (scalar) Gram determinant. Then, the best approximation pair of points onto two linear varieties is given, as well as characterization of this pair of best approximation points.

1 Introduction

In this paper, we answer an implicit open question by Ky Fan and John Todd [3, page 63]. We give a determinantal formula for the point where the inequality of the above referred to authors turns into equality, thusly obtaining the point of least norm of the intersection of certain hyperplanes. We present a result, in terms of Gram determinants, for the minimum distance from a certain linear variety to the origin of coordinates [Proposition 2.1]. We note that this formula generalizes the one Mitrinovic [7, 8] has given in the case of two equations. This best approximation problem was dealt with in [10], where the centre of (degenerate) hyperquadrics plays a decisive rôle. In [10], no answer in closed form was given.

In this paper, we give a new proof of Beesack’s inequality ([1, Theorem 1]; [9, Theorem 1.7]), by following arguments used in [3, page 63, Lemma].

The Beesack’s formula [Theorem 3.1] gives the point of a general linear variety closest to the origin of the coordinates. We extend the formula of Beesack [1, Theorem 1] in order to get the nearest point of a linear variety to an external point, in IRn. When extending Theorem 3.1, we obtain the projection of an external point onto a general linear variety [Proposition 4.1]. This Proposition 4.1 is used for getting the best approximation points of two linear varieties [Proposition 5.1]. Also a characterization of the best approximation pair of two linear varieties is presented [Proposition 5.2].

Our context is the Euclidean space IRn, endowed with the standard unit basis

( e_1, e_2, \ldots, e_n )
and the ordinary inner product
\[ \vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n, \]
where
\[ \vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n = (a_1, a_2, \ldots, a_n). \]

The Euclidean norm \( \| \vec{a} \| = +\sqrt{\vec{a} \cdot \vec{a}} \) is used and the Gram determinant is
\[ G(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_r) = \det \begin{bmatrix} \vec{p}_1 \cdot \vec{p}_1 & \vec{p}_1 \cdot \vec{p}_2 & \cdots & \vec{p}_1 \cdot \vec{p}_r \\ \vec{p}_2 \cdot \vec{p}_1 & \vec{p}_2 \cdot \vec{p}_2 & \cdots & \vec{p}_2 \cdot \vec{p}_r \\ \vdots & \vdots & \ddots & \vdots \\ \vec{p}_r \cdot \vec{p}_1 & \vec{p}_r \cdot \vec{p}_2 & \cdots & \vec{p}_r \cdot \vec{p}_r \end{bmatrix}, \ 1 \leq r \leq n. \] (1)

It is well known that \( G(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_r) \geq 0 \) and \( G(\vec{p}_1, \vec{p}_2, \ldots, \vec{p}_r) = 0 \) if and only if the vectors \( \vec{p}_1, \vec{p}_2, \ldots, \vec{p}_r \) are linearly dependent. See, for example, [2 page 132].

Some abuse of notation, authorized by adequate isomorphisms, is to be declared, notably the identification of point, vector, ordered set, column-matrix.

This paper is organized in seven sections. In Section 2, we present and prove a result, Proposition 2.1, which answers an open question of Fan and Todd and make a remark concerning a formula of Mitrinovic. Section 3 is dedicated to a generalization of Proposition 2.1, this meaning that we study the projection of the origin onto a general linear variety. In Section 4, we deal with the projection of an external point onto a general linear variety. In the Section 5, we treat the distance between two disjoint linear varieties. We get and characterize the best two points, one on each linear variety, that are the extremities of the straight line segment that materializes the distance between the two linear varieties. An illustrative numerical example is presented in Section 6. Finally, in Section 7, we draw some conclusions.

## 2 The minimum norm vector of a certain linear variety

In this section, we state the Proposition 2.1, which solves an old open question of Fan and Todd. The proof makes use of a result of the mentioned authors.

The next result [3 page 63, Lemma] gives the radius of the sphere tangent to a certain linear variety, as the quotient of two Gram determinants.

**Theorem 2.1** Let \( \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m \) be \( m \) linearly independent vectors in \( \mathbb{R}^n \), \( 2 \leq m \leq n \). If a vector \( \vec{x} \in \mathbb{R}^n \) varies under the conditions
\[ \vec{a}_i \cdot \vec{x} = 0, \quad \text{with} \ 1 \leq i \leq m - 1 \]
\[ \vec{a}_m \cdot \vec{x} = 1, \]
then
\[ \vec{x} \cdot \vec{x} \geq \frac{G(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{m-1})}{G(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{m-1}, \vec{a}_m)}. \] (3)
Furthermore, the minimum value is obtained if and only if \( \vec{x} \) is a linear combination of \( \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m \).

For the sake of completeness and for later use in the proof of our Proposition 2.1., we present here, essentially, the proof given by Fan and Todd [3 page 63, Lemma].
Proof: For the vector \( \mathbf{x} \) satisfying conditions (2), we have

\[
G(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m, \mathbf{x}) = -G(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{m-1}) + (\mathbf{x} \cdot \mathbf{x}) G(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m) \geq 0.
\]

Hence

\[
\mathbf{x} \cdot \mathbf{x} \geq \frac{G(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{m-1})}{G(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{m-1}, \mathbf{a}_m)}
\]

By hypothesis, the vectors \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \) are linearly independent, so

\[
G(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m, \mathbf{x}) = 0
\]

if and only if \( \mathbf{x} \) is a linear combination of \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \). It follows that

\[
\mathbf{x} \cdot \mathbf{x} = \frac{G(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{m-1})}{G(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{m-1}, \mathbf{a}_m)}
\]

if and only if the vector \( \mathbf{x} \) is of the form \( \mathbf{x} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_m \mathbf{a}_m \). \( \blacksquare \)

Now we are in a position for stating the equality case. A determinantal formula for the closest vector to the origin lying in a certain linear variety is given.

Proposition 2.1 1. Let \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \) be linearly independent vectors in \( \mathbb{R}^n \). The minimum Euclidean norm vector in \( \mathbb{R}^n \) satisfying the equations

\[
\begin{align*}
\mathbf{a}_1 \cdot \mathbf{x} &= 0 \\
\mathbf{a}_2 \cdot \mathbf{x} &= 0 \\
&\quad \vdots \\
\mathbf{a}_{m-1} \cdot \mathbf{x} &= 0 \\
\mathbf{a}_m \cdot \mathbf{x} &= 1
\end{align*}
\]

is given by

\[
\mathbf{x} = \begin{vmatrix}
\mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_1 \cdot \mathbf{a}_{m-1} & \mathbf{a}_1 \cdot \mathbf{a}_m \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{a}_{m-1} \cdot \mathbf{a}_1 & \mathbf{a}_{m-1} \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_{m-1} \cdot \mathbf{a}_{m-1} & \mathbf{a}_{m-1} \cdot \mathbf{a}_m \\
\mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_2 \cdot \mathbf{a}_{m-1} & \mathbf{a}_2 \cdot \mathbf{a}_m \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{a}_m \cdot \mathbf{a}_1 & \mathbf{a}_m \cdot \mathbf{a}_2 & \cdots & \mathbf{a}_m \cdot \mathbf{a}_{m-1} & \mathbf{a}_m \cdot \mathbf{a}_m
\end{vmatrix}
\]

where the determinant in the numerator is to be expanded by the last row, in order to yield a linear combination of the vectors \( \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \).

2. Furthermore,

\[
\| \mathbf{x} \|^2 = \mathbf{x} \cdot \mathbf{x} = \frac{G(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{m-1})}{G(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{m-1}, \mathbf{a}_m)}.
\]

Proof: Part 1. We look for the scalars \( \alpha_1, \alpha_2, \ldots, \alpha_m \), such that the vector \( \mathbf{x} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_m \mathbf{a}_m \) satisfies the conditions (5).
For that end, we solve the system

\[
\begin{bmatrix}
\vec{a}_1 & \vec{a}_1 & \vec{a}_1 & \cdots & \vec{a}_1 & \vec{a}_{m-1} & \vec{a}_1 & \vec{a}_m \\
\vec{a}_2 & \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_2 & \vec{a}_{m-1} & \vec{a}_2 & \vec{a}_m \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vec{a}_{m-1} & \vec{a}_1 & \vec{a}_{m-1} & \cdots & \vec{a}_{m-1} & \vec{a}_{m-2} & \vec{a}_{m-1} & \vec{a}_m \\
\vec{a}_m & \vec{a}_1 & \vec{a}_m & \cdots & \vec{a}_m & \vec{a}_{m-1} & \vec{a}_m & \vec{a}_m \\
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_m \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1 \\
\end{bmatrix}
\]

As the vectors \(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m\) are, by hypothesis, linearly independent, the determinant of the matrix of the above system, which is the Gram determinant

\[G(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{m-1}, \vec{a}_m),\]

is non null.

So, by the Cramer’s Rule, we have

\[\alpha_i = \frac{G_i}{G}, \quad \alpha_i \vec{a}_i = \frac{G_i}{G} \vec{a}_i := \frac{\vec{G}_i}{G},\]

where

\[G = G(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{m-1}, \vec{a}_m),\]

\[G_i = \frac{G_i}{G}, \quad \alpha_i \vec{a}_i = \frac{G_i}{G} \vec{a}_i := \frac{\vec{G}_i}{G},\]

and the symbolic determinant

\[\vec{G}_i = \begin{bmatrix}
\vec{a}_1 & \vec{a}_1 & \vec{a}_1 & \cdots & \vec{a}_1 & \vec{a}_{m-1} & \vec{a}_1 & \vec{a}_m \\
\vec{a}_2 & \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_2 & \vec{a}_{m-1} & \vec{a}_2 & \vec{a}_m \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vec{a}_{m-1} & \vec{a}_1 & \vec{a}_{m-1} & \cdots & \vec{a}_{m-1} & \vec{a}_{m-2} & \vec{a}_{m-1} & \vec{a}_m \\
\vec{a}_m & \vec{a}_1 & \vec{a}_m & \cdots & \vec{a}_m & \vec{a}_{m-1} & \vec{a}_m & \vec{a}_m \\
\end{bmatrix}.
\]

We get, using these notations and rearranging in a suitable manner the terms
of the determinants,

\[ \vec{s} = \sum_{i=1}^{m} a_i \vec{a}_i = \begin{vmatrix} \vec{a}_1 \cdot \vec{a}_1 & \vec{a}_1 \cdot \vec{a}_2 & \cdots & \vec{a}_1 \cdot \vec{a}_{m-1} & \vec{a}_1 \cdot \vec{a}_m \\ \vec{a}_2 \cdot \vec{a}_1 & \vec{a}_2 \cdot \vec{a}_2 & \cdots & \vec{a}_2 \cdot \vec{a}_{m-1} & \vec{a}_2 \cdot \vec{a}_m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vec{a}_{m-1} \cdot \vec{a}_1 & \vec{a}_{m-1} \cdot \vec{a}_2 & \cdots & \vec{a}_{m-1} \cdot \vec{a}_{m-1} & \vec{a}_{m-1} \cdot \vec{a}_m \\ \vec{a}_m \cdot \vec{a}_1 & \vec{a}_m \cdot \vec{a}_2 & \cdots & \vec{a}_m \cdot \vec{a}_{m-1} & \vec{a}_m \cdot \vec{a}_m \end{vmatrix} \]

Part 2. It is just sufficient to use [4], in order to obtain \( \| \vec{s} \|^2 \).

For computational purposes, we notice that, in the numerator of (5), the coefficients of the vectors \( \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m \) are the co-factors of the elements in the last row of the matrix

\[
\begin{bmatrix}
\vec{a}_1 \cdot \vec{a}_1 & \vec{a}_1 \cdot \vec{a}_2 & \cdots & \vec{a}_1 \cdot \vec{a}_{m-1} & \vec{a}_1 \cdot \vec{a}_m \\
\vec{a}_2 \cdot \vec{a}_1 & \vec{a}_2 \cdot \vec{a}_2 & \cdots & \vec{a}_2 \cdot \vec{a}_{m-1} & \vec{a}_2 \cdot \vec{a}_m \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vec{a}_{m-1} \cdot \vec{a}_1 & \vec{a}_{m-1} \cdot \vec{a}_2 & \cdots & \vec{a}_{m-1} \cdot \vec{a}_{m-1} & \vec{a}_{m-1} \cdot \vec{a}_m \\
1 & 1 & \cdots & 1 & 1
\end{bmatrix}
\]

Remark 2.1 The particular case of Mitrinovic

The determinantal formula [6], given in §1 of Proposition 2.1, for the least norm vector of the given linear variety is a generalization of the formula of Mitrinovic [4, page 67] and [8, page 93]

\[
x_k = \frac{b_k \sum_{i=1}^{p} a_i^2 - a_k \sum_{i=1}^{p} a_i b_i}{\left( \sum_{i=1}^{p} a_i^2 \right) \left( \sum_{i=1}^{p} b_i^2 \right) - \left( \sum_{i=1}^{p} a_i b_i \right)^2}, \quad k = 1, 2, \ldots, p,
\]

where \( (a_1, a_2, \ldots, a_p) \) and \( (b_1, b_2, \ldots, b_p) \) are two non proportional sequences of real numbers satisfying

\[
\sum_{i=1}^{p} a_i = 0 \quad \text{and} \quad \sum_{i=1}^{p} b_i = 1.
\]

3 The minimum norm vector of a general linear variety

Here we treat the projection of the origin of the coordinates onto a general linear variety, so extending Proposition 2.1. The point where the sphere centered at the origin is tangent to any linear variety is given in closed form by next relation [8]. This result has been obtained in a different form and by another approach in [1].
Theorem 3.1 (II) Let \( \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m \) be linearly independent vectors in \( \mathbb{R}^n \), with \( m \geq 2 \). The minimum Euclidean norm vector in \( \mathbb{R}^n \) satisfying the equations

\[
\begin{align*}
\vec{a}_1 \cdot \vec{x} &= c_1 \\
\vec{a}_2 \cdot \vec{x} &= c_2 \\
&\vdots \\
\vec{a}_{m-1} \cdot \vec{x} &= c_{m-1} \\
\vec{a}_m \cdot \vec{x} &= c_m,
\end{align*}
\]

with, at least, one non zero \( c_i, i=1,\ldots,m \), is given by the relation

\[
\vec{s} = \begin{bmatrix}
\vec{a}_1^T \quad \vec{a}_2^T \quad \cdots \quad \vec{a}_{m-1}^T \quad \vec{a}_m^T \\
\vdots \\
\vec{a}_{m-1}^T \quad \vec{a}_m^T \\
\vec{a}_m^T
\end{bmatrix},
\]

where

\[
\vec{a}_i = \vec{s}_i - \frac{c_i}{c_m} \vec{a}_m, \quad i = 1, \ldots, m - 1,
\]

and

\[
\vec{a}_m = \frac{1}{c_m} \vec{a}_m.
\]

Furthermore,

\[
\| \vec{s} \|^2 = \vec{s} \cdot \vec{s}^T = \frac{G(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{m-1})}{G(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{m-1}, \vec{a}_m)}.
\]

Proof: Performing elementary matrix operations, we turn into the form \( \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^T \) the last column of the augmented matrix of the system \( \square \)

\[
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} & c_1 \\
a_{21} & a_{22} & \cdots & a_{2n-1} & a_{2n} & c_2 \\
& \vdots \\
a_{m-11} & a_{m-12} & \cdots & a_{m-1n-1} & a_{m-1n} & c_{m-1} \\
a_{m1} & a_{m2} & \cdots & a_{mn-1} & a_{mn} & c_m
\end{bmatrix},
\]

where \( \vec{a}_i = (a_{i1}, a_{i2}, \ldots, a_{in-1}, a_{in}) \).

4 Projection of a point onto a linear variety

For dealing with this problem by taking into account the result of the preceding section, we use the fact that Euclidean distance is preserved under translations.
We are given a linear variety \( V \) and an external point \( Q \). We perform a translation towards the origin \( O \) of the coordinates: the pair \((Q, V)\) turns into the pair \((O, V')\). We, then, apply Theorem 3.1 to the pair \((O, V')\). Finally, we undo the performed translation: we go back from the origin \( O \) to the point \( Q \). We state the following

**Proposition 4.1** Let \( \vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m \) be linearly independent vectors in \( \mathbb{R}^n \), with \( m \geq 2 \). Then:

1. The projection \( S \) of the external point \( Q := \vec{q} = (q_1, q_2, \ldots, q_n) \) onto the linear variety \( V \) defined by

\[
\begin{align*}
\vec{a}_1 \cdot \vec{x} &= c_1 \\
\vec{a}_2 \cdot \vec{x} &= c_2 \\
&\vdots \\
\vec{a}_{m-1} \cdot \vec{x} &= c_{m-1} \\
\vec{a}_m \cdot \vec{x} &= c_m,
\end{align*}
\]

(12)

with, at least, one non zero \( c_i, i = 1, \ldots, m \), is given by

\[
S := \vec{s} = \vec{s}' + \vec{q},
\]

(13)

where

\[
\vec{s}' = \begin{bmatrix}
\vec{a}_1 \cdot \vec{x} & \vec{a}_1 \cdot \vec{x} & \ldots & \vec{a}_1 \cdot \vec{x} \\
\vec{a}_2 \cdot \vec{x} & \vec{a}_2 \cdot \vec{x} & \ldots & \vec{a}_2 \cdot \vec{x} \\
& \vdots & \ddots & \vdots \\
\vec{a}_{m-1} \cdot \vec{x} & \vec{a}_{m-1} \cdot \vec{x} & \ldots & \vec{a}_{m-1} \cdot \vec{x} \\
\vec{a}_m \cdot \vec{x} & \vec{a}_m \cdot \vec{x} & \ldots & \vec{a}_m \cdot \vec{x}
\end{bmatrix}
\]

(14)

with

\[
\vec{a}'_i = \vec{a}_i - \frac{c_i}{c_m} \vec{a}_m, \quad i = 1, \ldots, m - 1,
\]

(15)

and

\[
c'_i = c_i - \vec{a}_i \cdot \vec{q}.
\]

(16)

2. For the distance, we have

\[
d^2(Q, V) = d^2(O, V') = \left\| \vec{s}' \right\|^2 = \frac{G(\vec{a}_{1}, \vec{a}_{2}, \ldots, \vec{a}_{m-1})}{G(\vec{a}_{1}, \vec{a}_{2}, \ldots, \vec{a}_{m-1}, \vec{a}_{m})}.
\]

(17)

**Proof:**
1. We perform a translation towards the origin of the coordinates, of the pair \((Q, V)\) in order to get the pair \((O, V')\). We have

\[
\vec{x} = \vec{r} + \vec{Q} = \vec{r} - \vec{q}.
\]

Replacing, in equations (12), \(\vec{x}\) with \(\vec{x} + \vec{q}\), we get

\[
\begin{align*}
\vec{a}_1 \cdot \vec{x} & = c'_1 \\
\vec{a}_2 \cdot \vec{x} & = c'_2 \\
& \vdots \\
\vec{a}_{m-1} \cdot \vec{x} & = c'_{m-1} \\
\vec{a}_m \cdot \vec{x} & = c'_m,
\end{align*}
\]

with, at least, one non-zero \(c'_i\) and \(c'_i = c_i - \vec{a}_i \cdot \vec{q}\).

Now, by using relations (8), (9), (10) and (11), we obtain the relations (14), (15) and (16).

Finally, undoing the translation, we have

\[
\vec{s} = \vec{s}'' + \vec{q}.
\]

2. The Euclidean distance is translation invariant:

\[
d^2(Q, V) = d^2(Q, S) = d^2(O, V') = \|\vec{s}''\|^2 = \frac{G(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{m-1})}{G(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{m-1}, \vec{a}_m)}.
\]

\[
\blacksquare
\]

5. Distance between two linear varieties

In this section, we deal with the interesting problem of finding the best approximation pair of points of two given disjoint and non-parallel linear varieties \(V_1\) and \(V_2\). In other words, we are looking for the point \(S_1\) on the linear variety \(V_1\) and the point \(S_2\) on the linear variety \(V_2\) such that the vector \(S_1S_2\) is, to within a signal, the shortest one linking the referred to linear varieties. Here, the main tool is the Proposition 4.1. This result is applied twice, just bearing in mind that, in the present case, the external point is either the generic point \(G_{V_1} := \vec{g}_{V_1}\) of the linear variety \(V_1\) or the generic point \(G_{V_2} := \vec{g}_{V_2}\) of the linear variety \(V_2\).

Some notation is in order, for the sake of simplicity of the statement of our next result. We write the vector \(\vec{f} \in \mathbb{R}^n\) the following manner:

\[
\vec{f} = (f_1, f_2, \ldots, f_h, f_{h+1}, f_{h+2}, \ldots, f_n) := (f_1, f_2, \ldots, f_h, \vec{f}) \in \mathbb{R}^h \times \mathbb{R}^{n-h}.
\]

We state the main result of this paper

**Proposition 5.1** Let us consider two disjoint and non-parallel linear varieties \(V_1\) and \(V_2\) given, respectively, by
linear varieties \( V \) in \( \mathbb{R}^n \) and with, at least, one non zero scalar \( c_i \), \( i = 1, \ldots, m_1, m_1 \geq 2 \), and

\[
\begin{align*}
V_1 := & \{ \overrightarrow{a}_1 \cdot \overrightarrow{x} = c_1, \\
& \overrightarrow{a}_2 \cdot \overrightarrow{x} = c_2, \\
& \ldots \}
\end{align*}
\]

where \( \overrightarrow{a}_1, \overrightarrow{a}_2, \ldots, \overrightarrow{a}_{m_1} \) are linearly independent vectors in \( \mathbb{R}^n \) and with, at least, one non zero scalar \( c_i \), \( i = 1, \ldots, m_1 \), \( m_1 \geq 2 \), and

\[
\begin{align*}
V_2 := & \{ \overrightarrow{b}_1 \cdot \overrightarrow{y} = d_1, \\
& \overrightarrow{b}_2 \cdot \overrightarrow{y} = d_2, \\
& \ldots \}
\end{align*}
\]

where \( \overrightarrow{b}_1, \overrightarrow{b}_2, \ldots, \overrightarrow{b}_{m_2} \) are linearly independent vectors in \( \mathbb{R}^n \) and with, at least, one non zero scalar \( d_i \), \( i = 1, \ldots, m_2 \), \( m_2 \geq 2 \).

Let us denote \( \overrightarrow{x} = (x_1, \ldots, x_{m_1}, x_{m_1+1}, \ldots, x_n) \in V_1 \) as \( \overrightarrow{x} = (x_1, \ldots, x_{m_1}, \overrightarrow{\xi}) \in \mathbb{R}^{m_1} \times \mathbb{R}^{n-m_1} \) and \( \overrightarrow{y} = (y_1, \ldots, y_{m_2}, y_{m_2+1}, \ldots, y_n) \in V_2 \) as \( \overrightarrow{y} = (y_1, \ldots, y_{m_2}, \overrightarrow{\eta}) \in \mathbb{R}^{m_2} \times \mathbb{R}^{n-m_2} \).

Let us denote by \([S_1, S_2]\) the shortest straight line segment connecting the two linear varieties \( V_1 \) and \( V_2 \).

Then

1. The points \( S_1 \in V_1 \) and \( S_2 \in V_2 \) are obtained through the unique solution of the overdetermined consistent system of linear algebraic equations

\[
\begin{align*}
S_1 \left( \overrightarrow{\eta} \right) &= G_{V_1} \left( \overrightarrow{\xi} \right), \\
S_2 \left( \overrightarrow{\xi} \right) &= G_{V_2} \left( \overrightarrow{\eta} \right)
\end{align*}
\]

where:

(i) \( G_{V_1} \left( \overrightarrow{\xi} \right) = G_{V_1} \left( x_{m_1+1}, x_{m_1+2}, \ldots, x_n \right) \) and \( G_{V_2} \left( \overrightarrow{\eta} \right) = G_{V_2} \left( y_{m_2+1}, y_{m_2+2}, \ldots, y_n \right) \)

are the generic points of, respectively, the linear varieties \( V_1 \) and \( V_2 \);

(ii) \( S_1 \left( \overrightarrow{\eta} \right) := S_1 \left( y_{m_2+1}, y_{m_2+2}, \ldots, y_n \right) \)

and

\( S_2 \left( \overrightarrow{\xi} \right) := S_2 \left( x_{m_1+1}, x_{m_1+2}, \ldots, x_n \right) \)

are given, respectively, by

\( S_1 \left( \overrightarrow{\eta} \right) = S_1^* \left( \overrightarrow{\eta} \right) + G_{V_2} \left( \overrightarrow{\eta} \right) \)

and

\( S_2 \left( \overrightarrow{\xi} \right) = S_2^* \left( \overrightarrow{\xi} \right) + G_{V_1} \left( \overrightarrow{\xi} \right) \);

and where:
(iii) $S_i''(\overrightarrow{\eta})$ is given by

\[
\overrightarrow{s}_1''(\overrightarrow{\eta}) := S_i''(\overrightarrow{\eta}) = \begin{vmatrix}
\overrightarrow{a}_1 \cdot \overrightarrow{a}_1 & \cdots & \overrightarrow{a}_1 \cdot \overrightarrow{a}_{m_1} \\
\vdots & \ddots & \vdots \\
\overrightarrow{a}_{m_1} \cdot \overrightarrow{a}_1 & \cdots & \overrightarrow{a}_{m_1} \cdot \overrightarrow{a}_{m_1}
\end{vmatrix}
\]

being

\[
\overrightarrow{a}_i = \overrightarrow{a}_i - \frac{c_i}{c_m} \overrightarrow{a}_m, \quad i = 1, \ldots, m_1 - 1,
\]

\[
\overrightarrow{a}_m = \frac{1}{c_m} \overrightarrow{a}_m
\]

with, at least, one non zero $c_i = c_i - \overrightarrow{a}_i \cdot \overrightarrow{g}_{v_2}$, $i = 1, \ldots, m_1$, and

\[
\overrightarrow{s}_2''(\overrightarrow{\xi}) := S_i''(\overrightarrow{\xi}) = \begin{vmatrix}
\overrightarrow{b}_1 \cdot \overrightarrow{b}_1 & \cdots & \overrightarrow{b}_1 \cdot \overrightarrow{b}_{m_2} \\
\vdots & \ddots & \vdots \\
\overrightarrow{b}_{m_2} \cdot \overrightarrow{b}_1 & \cdots & \overrightarrow{b}_{m_2} \cdot \overrightarrow{b}_{m_2}
\end{vmatrix}
\]

being

\[
\overrightarrow{b}_i = \overrightarrow{b}_i - \frac{d_i}{d_{m_2}} \overrightarrow{b}_{m_2}, \quad i = 1, \ldots, m_2 - 1,
\]

\[
\overrightarrow{b}_{m_2} = \frac{1}{d_{m_2}} \overrightarrow{b}_{m_2}
\]

with, at least, one non zero $d_i = d_i - \overrightarrow{b}_i \cdot \overrightarrow{g}_{v_1}$, $i = 1, \ldots, m_2$.

2. The distance $d(V_1, V_2)$ between the two linear varieties is given by

\[
d(V_1, V_2) = \left\| \overrightarrow{s}_1'' \right\|
\]

**Proof:** Essentially the proof consists on dealing once at a time with the two linear varieties $V_1$ and $V_2$:

1. finding the generic point of each linear variety;

2. applying the Proposition\[11\]

In the following way:
(i) The generic points

From the underdetermined system (19), we can, without loss of generality, assume that the generic point \( G_V : \vec{g}_V \) depends on the \( n-m_1+1 \) parameters \( x_{m_1+1}, x_{m_1+2}, \ldots, x_n \).

We write

\[
G_V = G_V (\vec{\zeta}) = \begin{bmatrix}
    x_1(x_{m_1+1}, \ldots, x_n) \\
    \vdots \\
    x_{m_1}(x_{m_1+1}, \ldots, x_n) \\
    x_{m_1+1} \\
    \vdots \\
    x_n
\end{bmatrix} := \vec{g}_V. \tag{24}
\]

Similarly, we write for the generic point \( G_{V_2} = G_{V_2} (\vec{\eta}) \) of the linear variety \( V_2 \):

\[
G_{V_2} = G_{V_2} (\vec{\eta}) = \begin{bmatrix}
    y_1(y_{m_2+1}, \ldots, y_n) \\
    \vdots \\
    y_{m_2}(y_{m_2+1}, \ldots, y_n) \\
    y_{m_2+1} \\
    \vdots \\
    y_n
\end{bmatrix} := \vec{g}_{V_2}. \tag{25}
\]

(ii) The application of the Proposition 4.1

(a) Concerning the pair \((G_{V_2}, V_1)\), we get

\[
S'_1(y_{m_2+1}, y_{m_2+2}, \ldots, y_n) = \begin{bmatrix}
    \vec{a}_{1} \cdot \vec{a}_{1} & \cdots & \vec{a}_{1} \cdot \vec{a}_{m_1-1} & \vec{a}_{1} \cdot \vec{a}_{m_1} \\
    \vdots & \ddots & \vdots & \vdots \\
    \vec{a}_{m_1-1} \cdot \vec{a}_{1} & \cdots & \vec{a}_{m_1-1} \cdot \vec{a}_{m_1-1} & \vec{a}_{m_1-1} \cdot \vec{a}_{m_1} \\
    \vec{a}_{1} & \cdots & \vec{a}_{m_1} \\
    \vdots & \ddots & \vdots & \vdots \\
    \vec{a}_{m_1-1} \cdot \vec{a}_{m_1-1} & \cdots & \vec{a}_{m_1-1} \cdot \vec{a}_{m_1-1} & \vec{a}_{m_1-1} \cdot \vec{a}_{m_1} \\
    \vec{a}_{m_1-1} \cdot \vec{a}_{1} & \cdots & \vec{a}_{m_1-1} \cdot \vec{a}_{m_1-1} & \vec{a}_{m_1-1} \cdot \vec{a}_{m_1} \\
    \vec{a}_{m_1} \cdot \vec{a}_{1} & \cdots & \vec{a}_{m_1} \cdot \vec{a}_{m_1} & \vec{a}_{m_1} \cdot \vec{a}_{m_1}
\end{bmatrix}
\]

where

\[
\vec{a}_{i} := \frac{a_i'}{c_{m_1}}, \quad i = 1, \ldots, m_1 - 1,
\]

\[
\vec{a}_{m_1} = \frac{1}{c_{m_1}} \vec{a}_{m_1}
\]

with, at least, one non zero \( c_i' = c_i - \vec{a}_i \cdot \vec{g}_{V_2}, \ i = 1, \ldots, m_1 \).
For the pair \((G_{V_1}, V_2)\), we get

\[
S_2''(x_{m_1+1}, x_{m_1+2}, \ldots, x_n) = \begin{pmatrix}
\vec{b}_1 \bullet \vec{b}_1^* & \cdots & b_1 \bullet b_{m_2-1}^* & \vec{b}_1 \bullet \vec{b}_{m_2}^*

\vdots & \ddots & \vdots & \vdots \\
\vec{b}_{m_2-1} \bullet \vec{b}_1 & \cdots & \vec{b}_{m_2-1} \bullet \vec{b}_{m_2-1}^* & \vec{b}_{m_2-1} \bullet \vec{b}_{m_2}^*

\end{pmatrix}
\]

where

\[
\vec{b}_{m_2}^* = \vec{b}_i - \frac{d_i'}{d_{m_2}'} \vec{b}_{m_2}, \quad i = 1, \ldots, m_2 - 1,
\]

with, at least, one non zero \(d_i' = d_i - \vec{b}_i \bullet \vec{g}_i, \quad i = 1, \ldots, m_2\).

Essentially, the points \(S_1''\) and \(S_2''\) result from translations of the pairs \((G_{V_2}, V_1)\) and \((G_{V_1}, V_2)\). Undoing the translations, follows

\[
S_1 = S_1'' + G_{V_2}
\]
\[
S_2 = S_2'' + G_{V_1}.
\]

We must get the unique solution of the overdetermined system

\[
\begin{align*}
S_1(y_{m_2+1}, y_{m_2+2}, \ldots, y_n) &= G_{V_1}(x_{m_1+1}, x_{m_1+2}, \ldots, x_n) \\
S_2(x_{m_1+1}, x_{m_1+2}, \ldots, x_n) &= G_{V_2}(y_{m_2+1}, y_{m_2+2}, \ldots, y_n)
\end{align*}
\]

of 2n equations and the \((n - m_1 + 1) + (n - m_2 + 1)\) indeterminates

\[x_{m_1+1}, x_{m_1+2}, \ldots, x_n, y_{m_2+1}, y_{m_2+2}, \ldots, y_n.\]

This system is consistent and has the unique solution

\[
(x^*_1, x^*_{m_1+1}, \ldots, x^*_n, y^*_{m_2+1}, y^*_{m_2+2}, \ldots, y^*_n).
\]

Hence we obtain

\[
S_1 = G_{V_1} = G_{V_1} \left( \vec{x}^* \right) = \begin{pmatrix}
x_1(x^*_{m_1+1}, \ldots, x^*_n) \\
x_2(x^*_{m_1+1}, \ldots, x^*_n) \\
\vdots \\
x_{m_1+1}(x^*_{m_1+1}, \ldots, x^*_n)
\end{pmatrix} = \begin{pmatrix}
x_1^* \\
x_2^* \\
\vdots \\
x_{m_1+1}^*
\end{pmatrix}
\]

and

\[
S_2 = G_{V_2} = G_{V_2} \left( \vec{y}^* \right) = \begin{pmatrix}
y_1(y^*_{m_2+1}, \ldots, y^*_n) \\
y_2(y^*_{m_2+1}, \ldots, y^*_n) \\
\vdots \\
y_{m_2+1}(y^*_{m_2+1}, \ldots, y^*_n)
\end{pmatrix} = \begin{pmatrix}
y_1^* \\
y_2^* \\
\vdots \\
y_{m_2+1}^*
\end{pmatrix}.
\]
Some attention must be paid to the formulas (22) and (23). In fact, Remark 5.1

\[ s'_1 = s'_1(\vec{\eta}) = \frac{1}{A} \sum_{i=1}^{m_1} A_i a'_i \tag{27} \]

where \( A, A_i, i = 1, \ldots, m_1 \) are higher-degree polynomials in several variables \( y_{m_2+1}, y_{m_2+2}, \ldots, y_n \) and

\[ s'_2 = s'_2(\vec{\xi}) = \frac{1}{B} \sum_{j=1}^{m_2} B_j b'_j \tag{28} \]

where \( B, B_j, j = 1, \ldots, m_2 \) are higher-degree polynomials in several variables \( x_{m_1+1}, x_{m_1+2}, \ldots, x_n \).

However, from (27) and (28) we have

\[ s'_1 = s'_1(\vec{\eta}) = \sum_{i=1}^{n} L_{1i}(\vec{\eta}) e'_i \tag{29} \]

where \( L_{1i}, i = 1, \ldots, n, \) are first degree polynomials in the variables \( y_{m_2+1}, y_{m_2+2}, \ldots, y_n \) and

\[ s'_2 = s'_2(\vec{\xi}) = \sum_{i=1}^{n} L_{2i}(\vec{\xi}) e'_i \tag{30} \]

where \( L_{2i}, i = 1, \ldots, n, \) are first degree polynomials in the variables \( x_{m_1+1}, x_{m_1+2}, \ldots, x_n \).

This question is worth a longer explanation. As follows:

By performing the mentioned convenient translations on the systems (19) and (20), we obtain two systems where the right hand sides are vectors whose entries are linear expressions in the parameters that are coordinates of the vectors \( \vec{G}_{V_1} \) and \( \vec{G}_{V_2} \). By using arguments involving the uniqueness of (least squares) solution of a linear system by using the Moore-Penrose inverse, we assert that the solutions of the afore referred to systems are given in terms of such parameters. The best solution in the least squares sense of the system \( A\vec{x} = \vec{b} \) is given [1, page 439] by \( \vec{x} = A^\dagger \vec{b} \), where \( A^\dagger \) stands for the Moore-Penrose inverse of matrix \( A \). In our case, \( A^\dagger \) is a constant matrix, so \( \vec{x} \) depends on the parameters in vector \( \vec{b} \).

Hence,

\[ \vec{s}'_1 = \begin{bmatrix} L_1(y_{m_2+1}, y_{m_2+2}, \ldots, y_n) \\ L_2(y_{m_2+1}, y_{m_2+2}, \ldots, y_n) \\ \vdots \\ L_n(y_{m_2+1}, y_{m_2+2}, \ldots, y_n) \end{bmatrix} \]

and

\[ \vec{s}'_2 = \begin{bmatrix} L_1(x_{m_1+1}, x_{m_1+2}, \ldots, x_n) \\ L_2(x_{m_1+1}, x_{m_1+2}, \ldots, x_n) \\ \vdots \\ L_n(x_{m_1+1}, x_{m_1+2}, \ldots, x_n) \end{bmatrix} \]
For the sake of clarity, we synthesize:

**Scholium** Regarding the given linear varieties and without loss of generality, we can write

\[
V_1 = \begin{cases}
  x_1(x_{m_1+1}, \ldots, x_n) \\
  \vdots \\
  x_{m_1}(x_{m_1+1}, \ldots, x_n) \\
  x_{m_1+1} \\
  \vdots \\
  x_n
\end{cases} : (x_{m_1+1}, \ldots, x_n) \in \mathbb{R}^{n-m_1}
\]

and

\[
V_2 = \begin{cases}
  y_1(y_{m_2+1}, \ldots, y_n) \\
  \vdots \\
  y_{m_2}(y_{m_2+1}, \ldots, y_n) \\
  y_{m_2+1} \\
  \vdots \\
  y_n
\end{cases} : (y_{m_2+1}, \ldots, y_n) \in \mathbb{R}^{n-m_2}
\]

Hence we may write

\[
S_1 = G_{V_1}^* = \begin{bmatrix}
  x_1(x_{m_1+1}, \ldots, x_n) \\
  \vdots \\
  x_{m_1}(x_{m_1+1}, \ldots, x_n) \\
  x_{m_1+1} \\
  \vdots \\
  x_n
\end{bmatrix} = \begin{bmatrix}
  x_1^* \\
  \vdots \\
  x_{m_1}^* \\
  x_{m_1+1}^* \\
  \vdots \\
  x_n^*
\end{bmatrix}
\]

and

\[
S_2 = G_{V_2}^* = \begin{bmatrix}
  y_1(y_{m_2+1}, \ldots, y_n) \\
  \vdots \\
  y_{m_2}(y_{m_2+1}, \ldots, y_n) \\
  y_{m_2+1} \\
  \vdots \\
  y_n
\end{bmatrix} = \begin{bmatrix}
  y_1^* \\
  \vdots \\
  y_{m_2}^* \\
  y_{m_2+1}^* \\
  \vdots \\
  y_n^*
\end{bmatrix},
\]

where

\[
(x_{m_1+1}^*, x_{m_1+2}^*, \ldots, x_n^*, y_{m_2+1}^*, y_{m_2+2}^*, \ldots, y_n^*)
\]

is the unique solution of the overdetermined system \([26]\).

A classical projection theorem \([5]\) page 64, Theorem 1 \([4]\) page 45, Théorème 2.2.5 \([2]\) page 64, Exercise 2) concerning the case of a point and a linear variety, leads us to a result on the projection vector connecting two linear varieties. It is a characterization of the pair of best approximation points, that may be useful when testing the accuracy of numerical examples.

**Proposition 5.2** Let \(V_1\) and \(V_2\) be two non-parallel linear varieties: \(V_1 = P_1 + M_1\) and \(V_2 = P_2 + M_2\), where \(M_1\) and \(M_2\) are subspaces of \(\mathbb{R}^n\) and \(P_1\) and \(P_2\) are fixed points in \(\mathbb{R}^n\). Then, the unique points \(S_1 \in V_1\) and \(S_2 \in V_2\) form a best approximation pair \((S_1, S_2)\) of the linear varieties \(V_1\) and \(V_2\) if and only if the two vectors whose extremities are \(S_1\) and \(S_2\) are orthogonal simultaneously to the subspaces \(M_1\) and \(M_2\).
Proof: We need just two facts: the definition of a vector orthogonal to a set of \( \mathbb{R}^n \) where a vector is said to be orthogonal to set if it is orthogonal to each vector of the set; and a projection theorem, where it is stated that the projection vector is orthogonal to the unique subspace associated to the given linear variety and not to the linear variety itself \([5, \text{page 64, Theorem 1}] [4, \text{page 45, Théorème 2.2.5}] [2, \text{page 64, Exercise 2}]\).

We have:

1. \( S_2 = \overrightarrow{s_2} \) is the projection of \( S_1 := \overrightarrow{s_1} \) onto the linear variety \( V_2 \): hence \( \overrightarrow{S_1 S_2} \) is orthogonal to the subspace \( \mathcal{M}_2 \);

2. \( S_1 = \overrightarrow{s_1} \) is the projection of \( S_2 := \overrightarrow{s_2} \) onto the linear variety \( V_1 \): hence \( \overrightarrow{S_1 S_2} \) is orthogonal to the subspace \( \mathcal{M}_1 \).

\[ \blacksquare \]

Notice that the vector \( \overrightarrow{S_1 S_2} \) is not orthogonal either to the linear varieties \( V_1 \) or \( V_2 \).

Finally, we have a result concerning the separating hyperplanes \([2, \text{pages 105-106}]\) and the smallest sphere tangent to the two linear varieties simultaneously.

**Corollary** The smallest sphere \( S \) tangent to the linear varieties \( V_1 \) and \( V_2 \) is given by

\[
S = \left\{ \overrightarrow{x} \in \mathbb{R}^n : \left\| \overrightarrow{x} - \frac{\overrightarrow{s_1} + \overrightarrow{s_2}}{2} \right\| = \left\| \frac{\overrightarrow{s_1} - \overrightarrow{s_2}}{2} \right\| \right\}
\]

and the supporting hyperplanes are

\[
H_i = \left\{ \overrightarrow{x} \in \mathbb{R}^n : \left( \overrightarrow{s_1} - \overrightarrow{s_2} \right) \cdot \left( \overrightarrow{x} - \overrightarrow{s_i} \right) = 0 \right\}, \quad i = 1, 2.
\]

6 Illustrative numerical example

We are given two linear varieties. We exhibit the best two approximation points — one point on each linear variety — and show that the vector \( \overrightarrow{S_1 S_2} \) is orthogonal to both the subspace \( \mathcal{M}_1 \) and the subspace \( \mathcal{M}_2 \) associated to the linear varieties \( V_1 \) and \( V_2 \), respectively, but not to the linear varieties themselves.

Let the two linear varieties \( V_1 \) and \( V_2 \) be defined as follows

\[
V_1 := \left\{ \overrightarrow{a_1} \cdot \overrightarrow{x} = 1 \right\}
\]

\[
V_2 := \left\{ \overrightarrow{b_1} \cdot \overrightarrow{y} = -10 \right\}
\]

\[
\overrightarrow{a_2} \cdot \overrightarrow{x} = 2, \quad \overrightarrow{b_2} \cdot \overrightarrow{y} = -20 \quad \overrightarrow{b_3} \cdot \overrightarrow{y} = 3,
\]

with \( \overrightarrow{a_1} = (1, -1, -2, 1, 1) \) and \( \overrightarrow{a_2} = (1, 1, -4, 1, 2) \);

\[
V_2 := \left\{ \overrightarrow{b_1} \cdot \overrightarrow{y} = -10 \right\}
\]

\[
\overrightarrow{b_2} \cdot \overrightarrow{y} = -20 \quad \overrightarrow{b_3} \cdot \overrightarrow{y} = 3,
\]

with \( \overrightarrow{b_1} = (1, -1, -2, 1, 1) \), \( \overrightarrow{b_2} = (-1, 1, -4, 1, 2) \) and \( \overrightarrow{b_3} = (1, 1, -4, -1, 3) \).

* Concerning the Proposition 5.1.

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(I) The generic points $G_{V_1}$ and $G_{V_2}$ of the linear varieties $V_1$ and $V_2$ are

$$G_{V_1} = \begin{bmatrix} \frac{3}{2} + 3x_3 - x_4 - \frac{3}{2}x_5 \\ \frac{1}{2} + x_3 - \frac{1}{2}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

and

$$G_{V_2} = \begin{bmatrix} \frac{23}{2} + y_4 - \frac{1}{2}y_5 \\ \frac{23}{2} + \frac{1}{2}y_4 - \frac{1}{2}y_5 \\ 5 + \frac{1}{3}y_4 + \frac{1}{3}y_5 \\ y_4 \\ y_5 \end{bmatrix}.$$ 

(II) We perform a translation along the vector $\overrightarrow{G_{V_2}}(y_4, y_5) = O - G_{V_2}(y_4, y_5)$; the linear variety $V'_1$ is obtained by replacing $x$ in the relation (31) with $x + \overrightarrow{G_{V_2}}$:

$$V'_1 := \begin{cases} \overrightarrow{a_1} \cdot \overrightarrow{x} = 11 \\ \overrightarrow{a_2} \cdot \overrightarrow{x} = -1 - 2y_4 + y_5 \end{cases},$$

getting

$$S'_1(y_4, y_5) = \begin{bmatrix} 15 + \frac{2}{7}y_4 - \frac{1}{2}y_5 \\ - \frac{11}{6} - \frac{5}{6}y_4 + \frac{10}{6}y_5 \\ - \frac{4}{21} + \frac{20}{63}y_4 - \frac{10}{63}y_5 \\ \frac{15}{7} + \frac{2}{21}y_4 - \frac{1}{2}y_5 \\ \frac{2}{21} - \frac{10}{63}y_4 + \frac{5}{63}y_5 \end{bmatrix}.$$ 

and

$$S_1(y_4, y_5) = S'_1 + G_{V_2} = \begin{bmatrix} \frac{101}{7} + \frac{23}{21}y_4 - \frac{23}{21}y_5 \\ \frac{221}{22} + \frac{16}{63}y_4 - \frac{25}{126}y_5 \\ \frac{101}{21} + \frac{44}{63}y_4 + \frac{44}{126}y_5 \\ \frac{15}{7} + \frac{22}{21}y_4 - \frac{1}{2}y_5 \\ \frac{2}{21} - \frac{10}{63}y_4 + \frac{68}{63}y_5 \end{bmatrix}.$$ 

Mutatis mutandis:

(III) We perform a translation along the vector $\overrightarrow{G_{V_1}}(x_3, x_4, x_5) = O - G_{V_1}(x_3, x_4, x_5)$; the linear variety $V'_2$ is obtained by replacing $y$ in the relation (32) with $y + \overrightarrow{G_{V_1}}$:

$$V'_2 := \begin{cases} \overrightarrow{b_1} \cdot \overrightarrow{y} = -11 \\ \overrightarrow{b_2} \cdot \overrightarrow{y} = -19 + 6x_3 - 2x_4 - 3x_5 \\ \overrightarrow{b_3} \cdot \overrightarrow{y} = 1 + 2x_4 - x_5 \end{cases},$$

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(IV) Solving the system

\[
S_2'(x_3, x_4, x_5) = \begin{bmatrix}
633 & 366 & 192 & 148 & 209 \\
35 & 15 & 1 & 15 & 35 \\
674 & 150 & 41 & 159 & 418 \\
-137 & 240 & -191 & 120 & 418 \\
172 & 42 & 70 & 7 & 209 \\
\end{bmatrix}
\]

and

\[
S_2(x_3, x_4, x_5) = S_2' + G_{V_1} = \begin{bmatrix}
1803 & 89 & 831 & 418 & 5 \\
89 & 319 & 418 & 17 & 418 \\
674 & 59 & 418 & 159 & 418 \\
-137 & 240 & 18 & 418 & 418 \\
172 & 42 & 70 & 418 & 418 \\
\end{bmatrix}
\]

we obtain

\[
x_3^* = \frac{837}{848}, \quad x_4^* = \frac{4765}{848}, \quad x_5^* = \frac{1489}{424}, \quad y_4^* = \frac{5550}{209}, \quad y_5^* = \frac{453}{212}.
\]

Hence, using

\[
S_1 = G_{V_1}^* = \begin{bmatrix}
\frac{3}{2} + 3x_3^* - x_4^* - \frac{3}{2}x_5^* \\
\frac{1}{2} + x_3^* - \frac{1}{2}x_5^* \\
x_3^* \\
x_4^* \\
x_5^* \\
\end{bmatrix}
\]

and

\[
S_2 = G_{V_2}^* = \begin{bmatrix}
\frac{23}{2} + y_4^* - \frac{1}{2}y_5^* \\
\frac{33}{2} + \frac{1}{2}y_4^* - \frac{1}{2}y_5^* \\
5 + \frac{1}{2}y_4^* + \frac{1}{2}y_5^* \\
y_4^* \\
y_5^* \\
\end{bmatrix}
\]

we, finally, obtain

\[
S_1 = \begin{bmatrix}
\frac{22}{19} \\
\frac{57}{232} \\
\frac{837}{848} \\
\frac{4765}{848} \\
\frac{1489}{424} \\
\end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix}
\frac{55}{19} \\
\frac{469}{232} \\
\frac{3169}{232} \\
\frac{3560}{209} \\
\frac{453}{212} \\
\end{bmatrix}.
\]

The distance between the two varieties is given by

\[
d(V_1, V_2) = \|S_1S_2\| = \frac{2174}{559}.
\]
** Concerning the Proposition 5.2.

Let us consider

\[ V_1 = P_1 + M_1 := \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3x_3 - x_4 - \frac{3}{2}x_5 \\ x_3 - \frac{1}{2}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} : x_3, x_4, x_5 \in \mathbb{R} \]

and

\[ V_2 = P_2 + M_2 := \begin{bmatrix} \frac{23}{2} \\ 5 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} y_4 - \frac{1}{2}y_5 \\ \frac{1}{3}y_4 - \frac{1}{2}y_5 \\ \frac{1}{3}y_4 + \frac{1}{2}y_5 \\ y_4 \\ y_5 \end{bmatrix} : y_4, y_5 \in \mathbb{R} \].

\( \alpha_1 \) The vector \( \overrightarrow{S_1S_2} \) is orthogonal to the unique subspace \( M_1 \) associated to the linear variety \( V_1 \). Consider the arbitrarily fixed vector

\[ \overrightarrow{v_1} = \begin{bmatrix} 3x_3 - x_4 - \frac{3}{2}x_5 \\ x_3 - \frac{1}{2}x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in M_1. \]

We have

\[ \overrightarrow{S_1S_2} \cdot \overrightarrow{v_1} = 0. \]

\( \alpha_2 \) The vector \( \overrightarrow{S_1S_2} \) is orthogonal to the unique subspace \( M_2 \) associated to the linear variety \( V_2 \). Consider the arbitrarily fixed vector

\[ \overrightarrow{v_2} = \begin{bmatrix} y_4 - \frac{1}{2}y_5 \\ \frac{1}{3}y_4 - \frac{1}{2}y_5 \\ \frac{1}{3}y_4 + \frac{1}{2}y_5 \\ y_4 \\ y_5 \end{bmatrix} \in M_2. \]

We have

\[ \overrightarrow{S_1S_2} \cdot \overrightarrow{v_2} = 0. \]

\( \beta_1 \) The vector \( \overrightarrow{S_1S_2} \) is not orthogonal to the linear variety \( V_1 \). Take the fixed vector

\[ \overrightarrow{u_1} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \in V_1. \]

We have \( \overrightarrow{S_1S_2} \cdot \overrightarrow{u_1} = -1.3750 \neq 0. \)
7 Conclusions

In this paper we presented a determinantal formula for the point satisfying the equality condition in an inequality by Fan and Todd (we answered the implicit old open question in [3, page 63]: to get a closed form for the minimum norm vector of the given linear variety). In a previous paper [10], we got, by using the center of convenient hyperquadrics, the point where the inequality (5) turns into the equality (6).

Here, we also restated a determinantal formula for the point of tangency between a sphere and any linear variety.

Furthermore, we obtained the projection of an external point onto a linear variety as a quotient of two determinants. Subsequently and consequently this result was extended for getting the best approximation pair of two disjoint and non parallel linear varieties. A characterization of this pair of best approximation points is offered.

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