SOME REFINEMENTS OF NUMERICAL RADIUS INEQUALITIES

ZAHRA HEYDARBEYGI, MARYAM AMYARI* AND MAHNAZ KHANEHGIR

Abstract. In this paper, we give some refinements for the second inequality in $\frac{1}{2}\|A\| \leq w(A) \leq \|A\|$, where $A \in B(H)$. In particular, if $A$ is hyponormal by refining the Young inequality with the Kantorovich constant $K(\cdot, \cdot)$, we show that $w(A) \leq \frac{1}{2} \inf_{\|x\|=1} \zeta(x) \|A\| + |A^*|\| \leq \frac{1}{2}\|A\| + |A^*|\|$, where $\zeta(x) = K(\frac{\langle |A|x,x \rangle}{\|A^*\|_{\|x,x\|}}, 2)^r$, $r = \min\{\lambda, 1 - \lambda\}$ and $0 \leq \lambda \leq 1$.

We also give a reverse for the classical numerical radius power inequality $w(A^n) \leq w^n(A)$ for any operator $A \in B(H)$ in the case when $n = 2$.

1. Introduction

Suppose that $(H, \langle \cdot, \cdot \rangle)$ is a complex Hilbert space and $B(H)$ denotes the $C^*$-algebra of all bounded linear operators on $H$. For $A \in B(H)$, let $w(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of $A$, respectively. It is well known that $w(\cdot)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm $\|\cdot\|$. In fact, for every $A \in B(H)$,

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \quad (1.1)$$

An important inequality for $w(A)$ is the power inequality stating that

$$w(A^n) \leq w^n(A) \quad (1.2)$$

for each $n \in \mathbb{N}$. Many authors have investigated several inequalities involving numerical radius inequalities, see e.g. [1, 5, 6, 8, 13, 14]. If $x, y \in H$ are arbitrary, then the angle between $x$ and $y$ is defined by

$$\cos \phi_{x,y} = \frac{\text{Re}(x, y)}{\|x\| \|y\|}.$$
or by
\[
\cos \psi_{x,y} = \frac{|\langle x, y \rangle|}{\|x\|\|y\|}.
\]
The following inequality for angles between two vectors was obtained by Kreîn
\[
\phi_{x,z} \leq \phi_{x,y} + \phi_{y,z}
\] (1.3)
for any nonzero elements \(x, y, z \in H\). By using the representation
\[
\psi_{x,y} = \inf_{\lambda,\mu \in \mathbb{C} \setminus \{0\}} \phi_{\lambda x,\mu y} = \inf_{\lambda \in \mathbb{C} \setminus \{0\}} \phi_{\lambda x,y} = \inf_{\mu \in \mathbb{C} \setminus \{0\}} \phi_{x,\mu y}
\] (1.4)
and inequality (1.3), he showed that the following triangle inequality is valid
\[
\psi_{x,y} \leq \psi_{x,z} + \psi_{y,z}
\] (1.5)
for any nonzero elements \(x, y, z \in H\).

In section 2 of this paper, we first introduce some new refinements of numerical radius inequality (1.1) by applying the Kreîn-Lîn triangle inequality (1.3) and obtain a reverse of inequality (1.2) in the case when \(n = 2\). In section 3, we obtain some refinements of inequality (1.1) by applying a refinement of the Young inequality.

2. Some refinements of inequality (1.1) by Kreîn-Lîn triangle inequality

In order to achieve our goals, we need the following lemmas. The first lemma is a simple consequence of the classical Jensen and Young inequalities.

**Lemma 2.1.** [12, Lemma 2.1] Let \(a, b \geq 0\) and \(0 \leq \lambda \leq 1\). Then
\[
a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b \leq [\lambda a^r + (1 - \lambda)b^r]^{\frac{1}{r}},
\]
for any \(r \geq 1\).

The second lemma is a simple consequence of the classical Jensen inequality for convex function \(f(t) = t^r\), where \(r \geq 1\).

**Lemma 2.2.** If \(a\) and \(b\) are nonnegative real numbers, then
\[
(a + b)^r \leq 2^{r-1}(a^r + b^r)
\]
for any \(r \geq 1\).
Lemma 2.3. \[4, \text{Lemma 2.4}\] Suppose that \( x, y \in H \) with \( \|y\| = 1 \). Then
\[
\|x\|^2 - |\langle x, y \rangle|^2 = \inf_{\lambda \in \mathbb{C}} \|x - \lambda y\|^2.
\]

The following lemma is known as a generalized mixed Schwarz inequality.

Lemma 2.4. \[12, \text{Lemma 2.3}\] Let \( A \in B(H) \) and \( x, y \in H \) be two vectors.

(i) If \( 0 \leq \lambda \leq 1 \), then
\[
|\langle Ax, y \rangle|^2 \leq \langle |A|^{2\lambda} x, |A^*|^{2(1-\lambda)} y \rangle.
\]

(ii) If \( f \) and \( g \) are nonnegative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t \), then
\[
|\langle Ax, y \rangle| \leq \|f(|A|)x\|\|g(|A^*|)y\|.
\]

In the next result, we use some ideas of \[3\].

Theorem 2.5. Let \( A \in B(H) \) and \( f, g \) be nonnegative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t \). Then for \( r \geq 1 \)
\[
 w^{2r}(A) \leq \frac{1}{2^r} \left( \|f^2(|A|^2) + g^2(|A^*|^2)\| + 2^r \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|^{2r} \right). \tag{2.1}
\]

Proof. By \[1.5\], we get the inequality (9) of \[2\] as follows
\[
\frac{|\langle x, z \rangle|}{\|x\|\|z\|} \frac{|\langle y, z \rangle|}{\|y\|\|z\|} \leq \frac{|\langle x, y \rangle|}{\|x\|\|y\|} + \sqrt{1 - \frac{|\langle x, z \rangle|^2}{\|x\|^2\|z\|^2}} \sqrt{1 - \frac{|\langle y, z \rangle|^2}{\|y\|^2\|z\|^2}} \tag{2.2}
\]
for any \( x, y, z \in H \setminus \{0\} \).

If we multiply (2.2) by \( \|x\|\|y\|\|z\|^2 \), then we deduce
\[
|\langle x, z \rangle| |\langle y, z \rangle| \leq |\langle x, y \rangle| |\langle z, z \rangle| + \sqrt{\|x\|^2\|z\|^2 - |\langle x, z \rangle|^2} \sqrt{\|y\|^2\|z\|^2 - |\langle y, z \rangle|^2}. \tag{2.3}
\]

Applying Lemma 2.3 for any \( x, y, z \in H \) with \( \|z\| = 1 \), we obtain
\[
|\langle x, z \rangle| |\langle y, z \rangle| \leq |\langle x, y \rangle| + \inf_{\lambda \in \mathbb{C}} \|x - \lambda z\| \inf_{\mu \in \mathbb{C}} \|y - \mu z\|. \tag{2.4}
\]

Put \( x = Az \) and \( y = A^*z \) in (2.4) to get
\[
|\langle Az, z \rangle|^2 \leq |\langle A^2 z, z \rangle| + \inf_{\lambda \in \mathbb{C}} \|Az - \lambda z\| \inf_{\mu \in \mathbb{C}} \|A^*z - \mu z\|
\]
\[
\leq |\langle A^2 z, z \rangle| + \|Az - \lambda z\| \|A^*z - \mu z\|. \tag{2.5}
\]
for any \( z \in H \) with \( \| z \| = 1 \) and \( \lambda, \mu \in \mathbb{C} \).

On the other hand, by applying Lemma 2.4 and the AM-GM inequality, we have

\[
|\langle A^2 z, z \rangle| \leq \| f(|A^2|) z \| \| g(|(A^2)^*|) z \|
= \sqrt{\langle f^2(|A^2|) z, z \rangle \langle g^2(|(A^2)^*|) z, z \rangle}
\leq \frac{1}{2} \left( f^2(|A^2|) + g^2(|(A^2)^*|) \right) z, z). \tag{2.6}
\]

Applying again the AM-GM inequality, we get

\[
\| A z - \lambda z \| \| A^* z - \mu z \| \leq \frac{\| A z - \lambda z \|^2 + \| A^* z - \mu z \|^2}{2}. \tag{2.7}
\]

By combining inequalities (2.5), (2.6) and (2.7), we reach

\[
|\langle A z, z \rangle|^2 \leq \frac{1}{2} \left( (f^2(|A^2|) + g^2(|(A^2)^*|)) z, z \right) + \| A z - \lambda z \|^2 + \| A^* z - \mu z \|^2 \right)^{1/2}
\leq \frac{1}{2} \left( (f^2(|A^2|) + g^2(|(A^2)^*|)) z, z \right)^r + \left( \| A z - \lambda z \|^2 + \| A^* z - \mu z \|^2 \right)^{r/2}
\leq \frac{1}{2} \left( (f^2(|A^2|) + g^2(|(A^2)^*|)) z, z \right)^r + 2^{r-1} \left( \| A z - \lambda z \|^{2r} + \| A^* z - \mu z \|^{2r} \right)^{r/2}
\tag{by Lemma 2.1}
\]

Hence

\[
|\langle A z, z \rangle|^{2r} \leq \frac{1}{2^r} \left( (f^2(|A^2|) + g^2(|(A^2)^*|)) z, z \right)^r + 2^{r-1} \left( \| A z - \lambda z \|^{2r} + \| A^* z - \mu z \|^{2r} \right)^{r/2}
\]

By taking the supremum over \( z \in H \) with \( \| z \| = 1 \), we deduce

\[
w^{2r}(A) \leq \frac{1}{2^r} \left( \| f^2(|A^2|) + g^2(|(A^2)^*|) \|^r + 2^{r-1} \left( \| A - \lambda I \|^{2r} + \| A^* - \mu I \|^{2r} \right) \right)
\]

for any \( \lambda, \mu \in \mathbb{C} \).

Finally, taking the infimum over \( \lambda, \mu \in \mathbb{C} \) in the inequality above and utilizing

\[
\inf_{\mu \in \mathbb{C}} \| A^* - \mu I \| = \inf_{\mu \in \mathbb{C}} \| A - \overline{\mu} I \| = \inf_{\lambda \in \mathbb{C}} \| A - \lambda I \|
\]

we obtain the desired result (2.1). \( \square \)
Remark 2.6. In Theorem 2.5 if we choose \( r = 1, \ f(t) = g(t) = \sqrt{t} \), we get
\[
w^2(A) \leq \frac{1}{2} \left( \|A^2\|^2 + \|A^2\|^2 + 2 \inf_{\lambda \in \mathbb{C}} \|A - \lambda \| I \|^2 \right) .
\]

Now, suppose that \( s > 0 \) such that \( s \leq \sqrt{\|A\|^2 - \frac{1}{2} \|A^2\|^2 + \|A^2\|^2} \), if there is \( \lambda_0 \in \mathbb{C} \) in which
\( \|A - \lambda_0 I\| \leq s \), then \( w(A) \leq \sqrt{\frac{1}{2} \|A^2\|^2 + \|A^2\|^2 + s^2} \leq \|A\| \), that is an improvement
of inequality (1.1) for nonnormal operators.

Recall that if \( A \in M_2(\mathbb{R}) \), then \( \|A\| = \max_{1 \leq i \leq n} \sigma_i \), where \( \sigma_i \)’s are the square root of
of eigenvalues of \( A^*A \), which are called the singular values of \( A \), and \( w(A) \) for matrix of the form
\( A = \begin{bmatrix} a_1 & b \\ 0 & a_2 \end{bmatrix} \) or \( A = \begin{bmatrix} a_1 & 0 \\ b & a_2 \end{bmatrix} \) is defined by
\[
w(A) = \frac{1}{2} |a_1 + a_2| + \frac{1}{2} \sqrt{|a_1 - a_2|^2 + |b|^2},
\]
where \( a_1, a_2, b \in \mathbb{R} \).

Example 2.7. By taking \( A = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \) and \( \lambda_0 = \frac{1}{2} \) in Remark 2.6 we have \( w^2(A) \approx 1.5625, \quad \|A\|^2 \approx 3.2822, \quad \|A - \lambda_0 I\| \approx 0.5201 \) and \( \frac{1}{2} \|A^2\| + \|A^2\|^2 \approx 1.5652 \). If \( s^2 \leq \|A\|^2 - \frac{1}{2} \|A^2\| + \|A^2\|^2 \approx 1.7170 \), then \( s \leq 1.3103 \). Hence our inequality \( w(A) \leq \sqrt{\frac{1}{2} \|A^2\|^2 + \|A^2\|^2 + s^2} \leq \|A\| \) provides an improvement of inequality (1.1).

Remark 2.8. If there exists \( \lambda_0 \in \mathbb{C} \) in which \( \|A - \lambda_0 I\| \leq s \), then by putting \( \lambda = \mu = \lambda_0 \) and
by taking the supremum over \( z \in H \) with \( \|z\| = 1 \) in (2.5), we deduce
\[
w^2(A) - w(A^2) \leq \|A - \lambda_0 I\| \|A^* - \lambda_0 I\| .
\]

Therefore
\[
w^2(A) - w(A^2) \leq s^2 .
\]

Now, if \( \|A - \lambda_0 I\| \leq s \leq \sqrt{\|A\|^2 - w(A^2)} \), we have \( w(A) \leq \sqrt{w(A^2) + s^2} \leq \|A\| \), that is an
improvement of inequality (1.1).

Example 2.9. By taking \( A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \) and \( \lambda_0 = 2.5 \) in Remark 2.8, we have \( w(A^2) \approx 6.4142, \quad \|A\|^2 \approx 10.6054, \quad \|A - \lambda_0 I\| \approx 0.955 \) for \( s \leq \sqrt{\|A\|^2 - w(A^2)} \approx 2.0472 \). Hence
our inequality \( w(A) \leq \sqrt{w(A^2) + s^2} \leq \|A\| \) provides an improvement of inequality (1.1).
Recall that the vector $x \in H$ is orthogonal to $y \in H$ (denote by $x \perp y$), if $\langle x, y \rangle = 0$. Now, an argument similar to the proof of Theorem 2.5 with the aid of Lemma 2.1 and Lemma 2.3 gives the following proposition:

**Proposition 2.10.** Let $x, y, z \in H$ with $\|z\| = 1$ and $\lambda, \mu \in \mathbb{C}$, $a, b > 0$, and $r \geq 1$ such that

$$\|x - \lambda z\| \leq a, \quad \|y - \mu z\| \leq b.$$ 

Then

$$(|\langle x, z \rangle||\langle y, z \rangle| - |\langle x, y \rangle|)^r \leq \frac{a^{2r} + b^{2r}}{2}. \quad (2.8)$$

In particular if $x \perp y$, then

$$(|\langle x, z \rangle||\langle y, z \rangle|)^r \leq \frac{a^{2r} + b^{2r}}{2} \quad (2.9)$$

for any $r \geq 1$.

**Proof.** Since $z$ is a unit vector, from (2.3) we have

$$|\langle x, z \rangle||\langle y, z \rangle| - |\langle x, y \rangle| \leq \sqrt{\|x\|^2 - |\langle x, z \rangle|^2} \sqrt{\|y\|^2 - |\langle y, z \rangle|^2} \leq \frac{1}{2} (\|x\|^2 - |\langle x, z \rangle|^2 + \|y\|^2 - |\langle y, z \rangle|^2) \leq \frac{1}{2} \left( \inf_{\lambda \in \mathbb{C}} \|x - \lambda z\|^2 + \inf_{\mu \in \mathbb{C}} \|y - \mu z\|^2 \right) \leq \frac{1}{2} a^2 + b^2 \leq \left( \frac{a^{2r} + b^{2r}}{2} \right)^{\frac{1}{r}} \quad (by \ Lemma \ 2.3).$$

Hence

$$(|\langle x, z \rangle||\langle y, z \rangle| - |\langle x, y \rangle|)^r \leq \frac{a^{2r} + b^{2r}}{2}.$$

$\square$
Corollary 2.11. Let \( A \in B(H) \) and \( B \) be a nonzero self-adjoint element in \( B(H) \), under assumptions of Proposition 2.10, if we choose, \( x = Az \) and \( y = Bz \) with \( \|z\| = 1 \) in (2.9) yields

\[
(|\langle Az, z \rangle| |\langle Bz, z \rangle|)^r \leq \frac{a^{2r} + b^{2r}}{2}.
\]

By taking supremum over \( z \in H \) with \( \|z\| = 1 \), we get

\[
w^r(A) \leq \frac{a^{2r} + b^{2r}}{2} \|B\|^{-r}
\]

provided \( \|A - \lambda I\| \leq a, \|B - \mu I\| \leq b \) and for any \( r \geq 1 \) and \( a, b > 0 \).

Proposition 2.10 induces several inequalities as special cases, but here we only focus on the case \( r = 1 \), i.e.,

\[
|\langle x, z \rangle| |\langle y, z \rangle| \leq \frac{a^2 + b^2}{2} + |\langle x, y \rangle|, \quad (2.10)
\]

whenever \( \|x - \lambda z\| \leq a, \|y - \mu z\| \leq b \) with \( \|z\| = 1 \) and \( \lambda, \mu \in \mathbb{C} \).

Remark 2.12. Suppose that the assumptions of Proposition 2.10 are still valid.

- As an application of inequality (2.10) the following reverse of inequality (1.2) for \( n = 2 \), i.e., an upper bound for \( w^2(A) - w(A^2) \) can be obtained. In fact, by choosing \( x = Az \) and \( y = A^*z \) with \( \|z\| = 1 \) and taking supremum over \( z \in H \) with \( \|z\| = 1 \), we get

\[
w^2(A) - w(A^2) \leq \frac{a^2 + b^2}{2},
\]

provided \( \|A - \lambda I\| \leq a, \|A^* - \mu I\| \leq b \).

- By choosing \( x = Az \) and \( y = A^{-1}z \) with \( \|z\| = 1 \), in inequality (2.10) and taking supremum over \( z \in H \) with \( \|z\| = 1 \), we have

\[
K(A; z) - 1 \leq \frac{a^2 + b^2}{2},
\]

provided \( \|A - \lambda I\| \leq a, \|A^{-1} - \mu I\| \leq b \), where \( K(A; z) = \langle Az, z \rangle \langle A^{-1}z, z \rangle \), is the Kantorovich functional.
3. SOME REFINEMENTS OF INEQUALITY (1.1) BY USING YOUNG’S INEQUALITY

In this section, we obtain some refinements of inequality (1.1) by applying refinements of the Young inequality. The next lemma is an additive refinement of the scalar Young inequality.

Lemma 3.1. [9, Theorem 2.1] If \( a, b \geq 0 \) and \( 0 \leq \lambda \leq 1 \), then
\[
a^\lambda b^{1-\lambda} + r(\sqrt{a} - \sqrt{b})^2 \leq \lambda a + (1 - \lambda)b,
\]
where \( r = \min\{\lambda, 1 - \lambda\} \).

The main result of this section reads as follows.

Theorem 3.2. If \( A \in B(H) \), \( r = \min\{\lambda, 1 - \lambda\} \), where \( 0 \leq \lambda \leq 1 \), then
\[
w(A) \leq \frac{1 - 2r}{2}\||A| + |A^*|| + 2r\|A\|.
\]

Proof. Let \( x \in H \) be a unit vector. Then we have
\[
|\langle Ax, x \rangle| \leq \sqrt{\|A|x, x\|} \langle |A| |x, x\rangle \quad \text{(by Lemma 2.1)}
\]
\[
= (\|A|x, x\|^{1-\lambda}\langle |A^*| |x, x\rangle^\lambda)^{\frac{1}{2}}(\|A^*| |x, x\rangle^{1-\lambda}\langle |A| |x, x\rangle^\lambda)^{\frac{1}{2}}
\]
\[
\leq \frac{1}{2} \left( \|A|x, x\|^{1-\lambda}\langle |A^*| |x, x\rangle^\lambda + \langle |A^*| |x, x\rangle^{1-\lambda}\langle |A| |x, x\rangle^\lambda \right)
\]
(by AM–GM inequality)
\[
\leq \frac{1}{2} \left( (1 - \lambda)\langle |A| |x, x\rangle + \lambda\langle |A^*| |x, x\rangle - r(\sqrt{\langle |A| |x, x\rangle} - \sqrt{\langle |A^*| |x, x\rangle})^2ight.
\]
\[
+ (1 - \lambda)\langle |A^*| |x, x\rangle + \lambda\langle |A| |x, x\rangle - r(\sqrt{\langle |A^*| |x, x\rangle} - \sqrt{\langle |A| |x, x\rangle})^2 \right)
\]
(by Lemma 3.2)
\[
= \frac{1}{2} \left( (\langle |A| + |A^*|)|x, x\rangle - 2r(\langle |A| + |A^*|)|x, x\rangle) + 4r\sqrt{\langle |A| |x, x\rangle \langle |A^*| |x, x\rangle} \right),
\]
so
\[
|\langle Ax, x \rangle| + r(\langle |A| + |A^*|)|x, x\rangle \leq \frac{1}{2} \left( (\langle |A| + |A^*|)|x, x\rangle + 4r\sqrt{\langle |A| |x, x\rangle \langle |A^*| |x, x\rangle} \right).
\]
By taking supremum over \( x \in H \) with \( \|x\| = 1 \), we deduce
\[
w(A) \leq \frac{1 - 2r}{2}\||A| + |A^*|| + 2r\|A\|.
\]
which is an improvement of inequality (1.1).

Example 3.3. Let \( A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \) be as in Theorem 3.2 and \( r = 0.1 \). Then by straightforward computation, we get \( w(A) \approx 2.2071, \| A \| \approx 2.2882 \) and \( \frac{1}{2} \| |A| + |A^*| \| \approx 2.2518 \). Hence

\[
w(A) \leq \frac{1 - 2r}{2} \| |A| + |A^*| + 2r \| A \| \leq \| A \|,
\]

provides an improvement of inequality (1.1). In fact, \( 2.2071 \leq 2.2590 \leq 2.2882 \).

The following lemma is a multiplicative refinement of the Young inequality with the Kantorovich constant.

Lemma 3.4. \([7, \text{Corollary 3}]\) Let \( a, b > 0 \). Then

\[
(1 - \lambda)a + \lambda b \geq k(h, 2)^r a^{1-\lambda} b^\lambda,
\]

where \( 0 \leq \lambda \leq 1 \), \( r = \min\{\lambda, 1 - \lambda\} \), \( h = \frac{b}{a} \) such that \( K(h, 2) = \frac{(h + 1)^2}{4h} \) for \( h > 0 \), which has properties \( K(h, 2) = K(\frac{1}{h}, 2) \geq 1(h > 0) \) and \( K(h, 2) \) is increasing on \([1, \infty)\) and is decreasing on \((0, 1)\).

In \([10]\), Kittaneh obtained the inequality

\[
w(A) \leq \frac{1}{2} \| |A| + |A^*| \|.
\]

(3.1)

In the following theorem, we improve inequality (3.1) for hyponormal operators. Before proceeding recall that the operator \( A \in B(H) \) is said to be hyponormal if \( A^* A - A A^* \geq 0 \).

Theorem 3.5. If \( A \in B(H) \) is hyponormal, \( r = \min\{\lambda, 1 - \lambda\} \), where \( 0 \leq \lambda \leq 1 \), then

\[
w(A) \leq \frac{1}{\inf_{\|x\|=1} \zeta(x)} \| |A| + |A^*| \|,
\]

where \( \zeta(x) = K(\frac{\|A^* x\|}{\|A x\|}, 2)^r \) is a refinement of inequality (1.1).
Proof. Let $x \in H$ be a unit vector.

$$|\langle Ax, x \rangle| \leq \sqrt{\langle |A|x,x \rangle \langle |A^*|x,x \rangle} \quad \text{(by Lemma 2.4)}$$

$$= \left( \langle |A^*|x,x \rangle^{1-\lambda} \langle |A|x,x \rangle^\lambda \right)^{\frac{1}{2}} \left( \langle |A|x,x \rangle^{1-\lambda} \langle |A^*|x,x \rangle^\lambda \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2} \left( \left( \langle |A^*|x,x \rangle^{1-\lambda} \langle |A|x,x \rangle^\lambda \right) + \left( \langle |A|x,x \rangle^{1-\lambda} \langle |A^*|x,x \rangle^\lambda \right) \right)$$

(by AM–GM inequality)

$$\leq \frac{1}{2} \left( \frac{1}{K\left( \frac{\langle |A^*|x,x \rangle}{\langle |A^*|x,x \rangle}, 2 \right)^r} \left( (1 - \lambda) \langle |A^*|x,x \rangle + \lambda \langle |A|x,x \rangle \right) \right.$$

$$+ \left( \frac{1}{K\left( \frac{\langle |A^*|x,x \rangle}{\langle |A^*|x,x \rangle}, 2 \right)^r} \left( (1 - \lambda) \langle |A|x,x \rangle + \lambda \langle |A^*|x,x \rangle \right) \right)$$

(by Lemma 3.4)

$$\leq \frac{1}{2} \left( \frac{1}{K\left( \frac{\langle |A^*|x,x \rangle}{\langle |A^*|x,x \rangle}, 2 \right)^r} \left( \langle |A^*|x,x \rangle + \langle |A|x,x \rangle \right) \right).$$

Taking supremum over $x \in H$ with $\|x\| = 1$, we have

$$w(A) \leq \frac{1}{\inf_{\|x\|=1} \zeta(x)} \frac{\|A\| + \|A^*\|}{2},$$

where $\zeta(x) = K\left( \frac{\langle |A^*|x,x \rangle}{\langle |A^*|x,x \rangle}, 2 \right)^r$.

Note that $2\langle |A|x,x \rangle \langle |A^*|x,x \rangle \leq \langle |A|x,x \rangle^2 + \langle |A^*|x,x \rangle^2$, so

$$\left( \langle |A|x,x \rangle + \langle |A^*|x,x \rangle \right)^2 \geq 4\langle |A|x,x \rangle \langle |A^*|x,x \rangle.$$ 

Hence

$$\frac{\left( \langle |A|x,x \rangle + \langle |A^*|x,x \rangle \right)^2}{4\langle |A|x,x \rangle \langle |A^*|x,x \rangle} \geq 1.$$

Therefore $K\left( \frac{\langle |A^*|x,x \rangle}{\langle |A^*|x,x \rangle}, 2 \right) \geq 1. \quad \Box$

References

1. M. Boumazgour and A. H. Nabwey, *A note concerning the numerical range of a basic elementary operator*, Ann. Funct. Anal. 7(2016), no. 3, 434–441.
2. S. S. Dragomir, *A note on numerical radius and the Kreǐî-Lin inequality*, RGMIA Research Report Collection, **18** (2015), Article 113.

3. S. S. Dragomir, *A note on new refinements and reverses of Young’s inequality*, Transylv. J. Math. Mech. **8** (2016), no. 1, 46–49.

4. S. S. Dragomir, *Some Grüss type inequalities in inner product Spaces*, JIPAM. J. Inequal. Pure Appl. Math. **4** (2003), no. 2, Article 42, 10 pp.

5. S. S. Dragomir, *Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces*, Tamkang J. Math. **39** (2008), no. 1, 1–7.

6. R. Golla, *On the numerical radius of a quaternionic normal operator*, Adv. Oper. Theory **2** (2017), no. 1, 78–86.

7. M. Fuji, H. Zuo and G. Shi, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal. **4** (2011), no. 4, 551–556.

8. F. Kittaneh and Y. Manasrah, *Improved Young and Heinz inequalities for matrices*, J. Math. Anal. Appl. **361** (2010), no. 1, 262–269.

9. F. Kittaneh and Y. Manasrah, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra **59** (2011), 1031–1037.

10. F. Kittaneh, *A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix*, Studia Math. **158** (2003), no. 1, 11–17.

11. M. G. Kreǐî, *The angular localization of the spectrum of a multiplicative integral in Hilbert space*, (Russian) Funkcional. Anal. i Priložen. **3** (1969), 89–90.

12. M. Satari, M. S. Moslehian and T. Yamazaki, *Some generalized numerical radius inequalities for Hilbert space operators*, Linear Algebra Appl. **470** (2015), 216–227.

13. A. Sheikhosseini, M. S. Moslehian and K. Shebrawi, *Inequalities for generalized Euclidean operator radius via Young’s inequality*, J. Math. Anal. Appl. **445** (2017), no. 2, 1516–1529.
14. A. Zamani, *Some lower bounds for the numerical radius of Hilbert space operators*, Adv. Oper. Theory 2 (2017), no. 2, 98–107.

DEPARTMENT OF MATHEMATICS, MASHHAD BRANCH, ISLAMIC AZAD UNIVERSITY, MASHHAD 91735, IRAN.

*Email address:* zheydarbeygi@yahoo.com
*Email address:* maryam_amyari@yahoo.com and amyari@mshdiau.ac.ir
*Email address:* khanehgir@mshdiau.ac.ir