1. Introduction

Let $G_k$ denote the set of all $k$-dimensional subspaces of an $n$-dimensional vector space. We recall that two elements of $G_k$ are called adjacent if their intersection has dimension $k-1$. The set $G_k$ is point set of a partial linear space, namely a Grassmann space for $1 < k < n-1$ (see Section 5) and a projective space for $k \in \{1, n-1\}$. Two adjacent subspaces are—in the language of partial linear spaces—two distinct collinear points.

W.L. Chow \cite{Chow} determined all bijections of $G_k$ that preserve adjacency in both directions in the year 1949. In this paper we call such a mapping, for short, an $A$-transformation. Disregarding the trivial cases $k = 1$ and $k = n-1$, every $A$-transformation of $G_k$ is induced by a semilinear transformation $V \to V$ or (only when $k = 2n$) by a semilinear transformation of $V$ onto its dual space $V^*$. There is a wealth of related results, and we refer to \cite{2}, \cite{6}, and \cite{9} for further references.

In the present note, we aim at generalizing Chow’s result to product spaces. However, we consider only products of the form $G_k \times G_{n-k}$, where $G_k$ and $G_{n-k}$ stem from the same vector space $V$. Furthermore, for a fixed $k$ we restrict our attention to a certain subset of $G_k \times G_{n-k}$. This subset, say $G$, is formed by all pairs of complementary subspaces. Our definition of an adjacency on $G$ in formula \cite{3} is motivated by the definition of lines in a product of partial linear spaces; cf. e.g. \cite{7}.

One of our main results (Theorem 2) states that Chow’s theorem remains true, mutatis mutandis, for the $A$-transformations of $G$. However, in Theorem 1 we can show even more: Let us say that two elements $(S, U)$ and $(S', U')$ of $G$ are close to each other, if their Hamming distance is 1 or, said differently, if they coincide in precisely one of their components. Then the bijections of $G$ onto itself which preserve this closeness relation in both directions—we call them $C$-transformations of $G$—are precisely the $A$-transformations of $G$. In this way, we obtain for $1 < k < n-1$ two characterizations of the semilinear bijections $V \to V$ and $V \to V^*$ via their action on the set $G$.

Finally, we turn to the following question: What happens to our results if we replace the set $G$ with the entire cartesian product $G_k \times G_{n-k}$? Clearly, the basic notions of adjacency and closeness remain meaningful. We describe all $C$-transformations of $G_k \times G_{n-k}$ in Theorem 3. However, in sharp contrast to Theorem 1 this is a rather trivial task, and the transformations of this kind do not deserve any interest. Then, using a result of A. Naumowicz and K. Prażmowski \cite{7}, we also determine all $A$-transformations of $G_k \times G_{n-k}$ in Theorem 4. Such mappings are closely related with collineations of the underlying partial linear space, and in general they can...
be described in terms of two semilinear bijections, but not in terms of a single semilinear bijection.

Before we close this section, it is worthwhile to mention that the results from [7] could be used to describe the A-transformations of arbitrary finite products of Grassmann spaces, but this is not the topic of the present article.

2. A-transformations and C-transformations

First, we collect our basic assumptions and definitions. Throughout this paper, let $V$ be a $n$-dimensional left vector space over a division ring, $2 \leq n < \infty$. Suppose that $P, T \subset V$ are subspaces. They are said to be incident (in symbols: $P \cap T$) if $P \subset T$ or if $T \subset P$. Note that according to this definition every subspace of $V$ is incident with 0 (the zero subspace) and with $V$. Furthermore, we define

$$P \sim T :\Leftrightarrow \dim P = \dim T = \dim(P \cap T) + 1,$$

where “$\sim$” is to be read as adjacent.

We fix a natural number $k \in \{1, 2, \ldots, n - 1\}$ and adopt the notation

$$\mathcal{G} := \{(S, U) \in \mathcal{G}_k \times \mathcal{G}_{n-k} \mid S + U = V\}.$$

Hence $(S, U) \in \mathcal{G}$ means that $S$ and $U$ are complementary subspaces. On the set $\mathcal{G}$ we define two binary relations: Elements $(S, U)$ and $(S', U')$ of $\mathcal{G}$ are said to be adjacent if

$$(S = S' \text{ and } U \sim U') \text{ or } (S \sim S' \text{ and } U = U').$$

By abuse of notation, this relation on $\mathcal{G}$ will also be denoted by the symbol “$\sim$". Our elements are said to be close to each other (in symbols: $(S, U) \approx (S', U')$) if

$$(S = S' \text{ and } U \neq U') \text{ or } (S \neq S' \text{ and } U = U').$$

According to this definition, any two adjacent elements of $\mathcal{G}$ are close; the converse holds only for $k = 1$ and $k = n - 1$.

We shall establish in Lemma 4 that any two elements $(S, U)$ and $(S', U')$ of $\mathcal{G}$ can be connected by a finite sequence

$$(S, U) = (S_0, U_0) \sim (S_1, U_1) \sim \cdots \sim (S_i, U_i) = (S', U').$$

Consequently, we also have

$$(S, U) = (S_0, U_0) \approx (S_1, U_1) \approx \cdots \approx (S_i, U_i) = (S', U').$$

We refer to the definition of a Plücker space in [2], p. 199, and we point out the (inessential) difference that our relations $\sim$ and $\approx$ are anti-reflexive.

A bijection $f : \mathcal{G} \to \mathcal{G}$ is said to be an adjacency preserving transformation (shortly: an $A$-transformation) if $f$ and $f^{-1}$ transfer adjacent elements of $\mathcal{G}$ to adjacent elements; if $f$ and $f^{-1}$ map close elements of $\mathcal{G}$ to close elements then we say that $f$ is a closeness preserving transformation (shortly: a $C$-transformation).

Example 1. For any two mappings $f' : \mathcal{G}_k \to \mathcal{G}_k$ and $f'' : \mathcal{G}_{n-k} \to \mathcal{G}_{n-k}$ we put

$$f' \times f'' : \mathcal{G}_k \times \mathcal{G}_{n-k} \to \mathcal{G}_k \times \mathcal{G}_{n-k} : (S, U) \mapsto (f'(S), f''(U)).$$

Each semilinear isomorphism $l : V \to V$ induces, for $i = 1, 2, \ldots, n - 1$, bijections

$$G_i(l) : \mathcal{G}_i \to \mathcal{G}_i : S \mapsto l(S).$$
Obviously, the restriction of
\[ G_k(l) \times G_{n-k}(l) \]
to \( G \) is an A-transformation and a C-transformation.

**Example 2.** For any two mappings \( g': G_k \rightarrow G_{n-k} \) and \( g'': G_{n-k} \rightarrow G_k \) we put
\[ g' \times g'': G_k \times G_{n-k} \rightarrow G_k \times G_{n-k} : (S, U) \mapsto (g''(U), g'(S)). \]
Let \( V^* \) denote the dual space of \( V \). Each semilinear isomorphism \( s : V \rightarrow V^* \) induces, for \( i = 1, 2, \ldots, n-1 \), the bijections
\[ D_i(s) : G_i \rightarrow G_{n-i} : S \mapsto (s(S))^\circ, \]
where \((s(S))^\circ\) denotes the annihilator of \( s(S) \). The restriction of
\[ D_k(s) \times D_{n-k}(s) \]
to \( G \) is an A-transformation and a C-transformation. Observe that a necessary and sufficient condition for the existence of such an isomorphism \( s \) is that the underlying division ring admits an anti-automorphism.

**Example 3.** Now suppose that \( n = 2k \). We assume that \( l : V \rightarrow V \) and \( s : V \rightarrow V^* \) are semilinear isomorphisms. The restrictions of
\[ G_k(l) \times G_k(l) \text{ and } D_k(s) \times D_k(s) \]
to \( G \) both are A-transformations and C-transformations.

**Example 4.** Let \( n = 2 \) and \( k = 1 \). Choose an arbitrary bijection \( f : G_1 \rightarrow G_1 \).
Then the restrictions of \( f \times f \) and \( f \times f \) to \( G \) both are A-transformations and C-transformations.

We are now in a position to state our main results:

**Theorem 1.** Every closeness preserving transformation of \( G \) is one of the mappings considered in Examples 1-4. Hence it is an adjacency preserving transformation.

It is trivial that each A-transformation is a C-transformation if \( k = 1 \) or if \( k = n-1 \).
In Section 4 we shall prove this statement for the general case. Thus the following statement holds true.

**Theorem 2.** Every adjacency preserving transformation of \( G \) is one of the mappings considered in Examples 1-4. Hence it is a closeness preserving transformation.

It is clear that our definitions of adjacency and closeness remain meaningful on the entire cartesian product \( G_k \times G_{n-k} \). Also the notions of C- and A-transformation and Examples 1-4 can be carried over accordingly. However, Theorems 1 and 2 do not remain unaltered when \( G \) is replaced with \( G_k \times G_{n-k} \): 

**Example 5.** Let \( f' : G_k \rightarrow G_k \) and \( f'' : G_{n-k} \rightarrow G_{n-k} \) be bijections. Then \( f' \times f'' \) is a C-transformation. Also, if \( g' : G_k \rightarrow G_{n-k} \) and \( g' : G_{n-k} \rightarrow G_k \) are bijections then \( g' \times g'' \) is a C-transformation.

For the sake of completeness, let us state the following rather trivial result:

**Theorem 3.** Every closeness preserving transformation of \( G_k \times G_{n-k} \) is one of the mappings considered in Example 5.
Example 6. If $f' : \mathcal{G}_k \to \mathcal{G}_k$ and $f'' : \mathcal{G}_{n-k} \to \mathcal{G}_{n-k}$ are bijections which preserve adjacency in both directions then $f' \times f''$ is an A-transformation. Also, if $g' : \mathcal{G}_k \to \mathcal{G}_{n-k}$ and $g'' : \mathcal{G}_{n-k} \to \mathcal{G}_k$ are bijections which preserve adjacency in both directions then $g' \times g''$ is an A-transformation.

Suppose that $k = 1$ or $k = n - 1$. Then it suffices to require that the mappings $f'$, $f''$, $g'$ and $g''$ from above are bijections in order to obtain an A-transformation of $\mathcal{G}_k \times \mathcal{G}_{n-k}$.

Provided that $1 < k < n - 1$, we can apply Chow’s theorem ([4] p. 38, [5] p. 81) to describe explicitly the mappings from above.

In the first case we have $f' = G_k(l')$ or $f' = D_k(s')$ (only when $n = 2k$), and $f'' = G_{n-k}(l'')$ or $f'' = D_k(s'')$ (only when $n = 2k$).

In the second case we have $g' = D_k(s')$ or $g' = G_k(l')$ (only when $n = 2k$), and $g'' = D_{n-k}(s'')$ or $g'' = G_k(l'')$ (only when $n = 2k$).

Here $l', l'' : V \to V$ and $s', s'' : V \to V^*$ denote semilinear isomorphisms.

We shall see that the following result is a consequence of [7, Theorem 1.14]:

Theorem 4. Every adjacency preserving transformation of $\mathcal{G}_k \times \mathcal{G}_{n-k}$ is one of the mappings considered in Example 6.

Remark 1. Suppose that the underlying division ring of $V$ is not of characteristic 2. Let $u \in \text{GL}(V)$ be an involution. Then there exist two invariant subspaces $U_+(u)$ and $U_-(u)$ with $V = U_+(u) \oplus U_-(u)$ such that $u(x) = \pm x$ for each $x \in U_{\pm}(u)$. If $\dim U_+(u) = r$ then $\dim U_-(u) = n - r$, and $u$ is called an $(r, n - r)$-involution.

For our fixed $k$ let $J$ be the set of all $(k, n-k)$-involutions. There exists a bijection

\[(14) \quad \gamma : J \to \mathcal{G} : u \mapsto (U_+(u), U_-(u)).\]

Two $(k, n-k)$-involutions $u$ and $v$ are said to be adjacent if the corresponding elements of $\mathcal{G}$ are adjacent. This holds if, and only if, the product of $u$ and $v$ (in any order) is a transvection $\neq 1_V$.

Now let $f : J \to J$ be a bijection which preserves adjacency in both directions. We apply Theorem 2 to the A-transformation $\gamma f \gamma^{-1} : \mathcal{G} \to \mathcal{G}$. If $n > 2$ and $n \neq 2k$ then this last mapping is given as in Example 1 or 2. This means that $f$ can be extended to an automorphism of the group $\text{GL}(V)$ as follows: To each $u \in \text{GL}(V)$ we assign $ulu^{-1}$ or the contragredient of $sus^{-1}$, respectively.

3. Proof of Theorem 4

Our proof of Theorem 4 will be based on several lemmas and the subsequent characterization. In the case $n = 2k$ this statement is a particular case of a result in [3]. The direct analogue of Theorem 5 for buildings can be found in [1, Proposition 4.2].

Theorem 5. Let $1 \leq k \leq n - 1$. Then for any two distinct $S_1, S_2 \in \mathcal{G}_k$ the following two conditions are equivalent:

(a) $S_1$ and $S_2$ are adjacent,
(b) There exists an $S \in \mathcal{G}_k - \{S_1, S_2\}$ such that for all $U \in \mathcal{G}_{n-k}$ the condition $(S, U) \in \mathcal{G}$ implies that $(S_1, U)$ or $(S_2, U)$ belongs to $\mathcal{G}$. 
Proof. (a) ⇒ (b). If $S_1$ and $S_2$ are adjacent then $S_1 \cap S_2 \in G_{k-1}$ and $S_1 + S_2 \in G_{k+1}$. Every $S \in G_k - \{S_1, S_2\}$ satisfying the condition
\begin{equation}
S_1 \cap S_2 \subset S \subset S_1 + S_2
\end{equation}
has the required property, and at least one such $S$ exists.

(b) ⇒ (a). The proof of this implication will be given in several steps. First we show that
\begin{equation}
0 \neq W_1 \subset S_1 \text{ and } 0 \neq W_2 \subset S_2 \Rightarrow (W_1 + W_2) \cap S \neq 0.
\end{equation}
Assume, contrary to (16), that $(W_1 + W_2) \cap S = 0$. Then there exists a complement $U \in G_{n-k}$ of $S$ containing $W_1 + W_2$. By our hypothesis, $U$ is a complement of $S_1$ or $S_2$. This contradicts $W_1 \subset S_1$ and $W_2 \subset S_2$.

Our second assertion is
\begin{equation}
S_1 \cap S_2 \subset S.
\end{equation}
This inclusion is trivial if $S_1 \cap S_2$ is zero. Otherwise, let $P \subset S_1 \cap S_2$ be an arbitrarily chosen 1-dimensional subspace. We apply (15) to $W_1 = W_2 = P$. This shows that $P \cap S \neq 0$. Hence $P \subset S$, as required.

The third step is to show that
\begin{equation}
\dim(S \cap S_1) = \dim(S \cap S_2) = k - 1.
\end{equation}
By symmetry, it suffices to establish that
\begin{equation}
W_1 \cap (S \cap S_1) \neq 0
\end{equation}
for all 2-dimensional subspaces $W_1 \subset S_1$: Let us take a 1-dimensional subspace $P_2 \subset S_2$ such that $P_2 \cap S = 0$. Then (17) implies that $P_2$ is not contained in $S_1$, and for every 2-dimensional subspace $W_1 \subset S_1$ the subspace $W_1 + P_2$ is 3-dimensional. Let $P_1$ and $Q_1$ be distinct 1-dimensional subspaces contained in $W_1$. It follows from (10) that $P_1 + P_2$ and $Q_1 + P_2$ meet $S$ in 1-dimensional subspaces $(\neq P_2)$ which will be denoted by $P$ and $Q$, respectively. As $P_1$ and $Q_1$ are distinct, so are $P$ and $Q$. Therefore $P + Q$ is a 2-dimensional subspace of $S$. Since $W_1$ and $P + Q$ lie in the 3-dimensional subspace $W_1 + P_2$, they have a common 1-dimensional subspace contained in $W_1 \cap S = W_1 \cap (S \cap S_1)$. This proves (18).

Finally, we read off from (17) that
\begin{equation}
S_1 \cap S_2 = (S \cap S_1) \cap (S \cap S_2),
\end{equation}
and we shall finish the proof by showing that this subspace has dimension $k - 1$. By (15) and because of $S_1 \neq S_2$, the dimension of $S_1 \cap S_2$ is either $k - 2$ or $k - 1$. Suppose, to the contrary, that
\begin{equation}
\dim S_1 \cap S_2 = k - 2.
\end{equation}
Then $S \cap S_1$ and $S \cap S_2$ are distinct $(k - 1)$-dimensional subspaces spanning $S$. There exist 1-dimensional subspaces $P_1, P_2$ such that
\begin{equation}
S_i = (S \cap S_i) + P_i
\end{equation}
for $i = 1, 2$. We have $P_1 \neq P_2$ (otherwise (17) would give $P_1 = P_2 \subset S_1 \cap S_2 \subset S$ which is impossible), and (16) guarantees that $(P_1 + P_2) \cap S$ is a 1-dimensional subspace. Then $S_1 + S_2$ is contained in the $(k + 1)$-dimensional subspace $S + P_1$ which, by the dimension formula for subspaces, contradicts (21). □
Lemma 1. If \( l : V \to V \) is a semilinear isomorphism such that \( G_j(l) \) is the identity for at least one \( j \in \{1, 2, \ldots, n - 1\} \) then the same holds for all \( i = 1, 2, \ldots, n - 1 \).

Proof. This is well known. \( \square \)

Lemma 2. Let \( l_i : V \to V \) and \( s_i : V \to V^* \) be semilinear isomorphisms, \( i = 1, 2 \). Then the following assertions hold.

(a) If one of the mappings \( G_k(l_1) \times G_{n-k}(l_2) \) or \( G_k(l_1) \times G_k(l_2) \), when restricted to \( G \), is a C-transformation then \( G_i(l_1) = G_i(l_2) \) for all \( i = 1, 2, \ldots, n - 1 \).

(b) If one of the mappings \( D_k(s_1) \times D_{n-k}(s_2) \) or \( D_k(s_1) \times D_k(s_2) \), when restricted to \( G \), is a C-transformation then \( D_i(s_1) = D_i(s_2) \) for all \( i = 1, 2, \ldots, n - 1 \).

(c) If \( n = 2k > 2 \) then none of the mappings \( G_k(l_1) \times D_k(s_1) \times G_k(l_2), \ G_k(l_1) \times D_k(s_2), \) and \( D_k(s_1) \times G_k(l_2) \) is a C-transformation, when it is restricted to \( G \).

Proof. (a) Let the restriction of \( G_k(l_1) \times G_{n-k}(l_2) \) to \( G \) be a C-transformation. Then \( G_k(l_1) \times G_{n-k}(l_1^{-1}l_2) \) gives also a C-transformation. This means that for each \( U \in G_{n-k} \) the mapping \( G_k(l_1) \) transfers the set of all \( k \)-dimensional subspaces having a non-zero intersection with \( U \), onto the set of all \( k \)-dimensional subspaces having a non-zero intersection with \( l_1^{-1}l_2(U) \). However, \( G_k(l_1) \) is the identity. Thus

\[
(23) \quad l_1^{-1}l_2(U) = U, 
\]

and \( G_{n-k}(l_2l_1^{-1}) \) is the identity. Hence we can apply Lemma 1 to show the assertion in this particular case.

Next, let the restriction of \( G_k(l_1) \times G_k(l_2) \) to \( G \) be a C-transformation. Thus \( n = 2k \) and the assertion follows from the previous case and

\[
(24) \quad G_k(l_1) \times G_k(l_2) = (G_k(l_1) \times G_k(l_1))(G_k(l_1) \times G_k(l_2)). 
\]

(b) can be verified similarly to (a).

(c) Assume, contrary to our hypothesis, that \( G_k(l_1) \times D_k(s_2) \) gives a C-transformation. Hence \( G_k(l_1) \times D_k(s_2) \) is also a C-transformation and, as above, we infer that

\[
(25) \quad D_k(s_2l_1^{-1})(U) = ((s_2l_1^{-1})(U))^\circ = U 
\]

for all \( U \in G_k \). Let \( W \in G_{k-1} \). Then there are subspaces \( U_1, U_2, \ldots, U_{k+1} \in G_k \) such that \( V = \sum_{i=1}^{k+1} U_i \) and \( W = \bigcap_{i=1}^{k+1} U_i \). Consequently,

\[
(26) \quad 0 = (s_2l_1^{-1}(V)) \cap \bigcap_{i=1}^{k+1} ((s_2l_1^{-1})(U_i)) = \bigcap_{i=1}^{k+1} U_i = W 
\]

which implies \( k = 1 \), an absurdity.

The remaining cases can be shown in the same way. \( \square \)

Let us remark that in general the assumption \( n > 2 \) in part (c) of this lemma cannot be dropped. Indeed, if \( n = 2k = 2 \) and if \( K \) is a commutative field then there exists a non-degenerate alternating bilinear form \( b : V \times V \to K \). Hence \( s : V \to V^* : v \mapsto b(v, \cdot) \) is a linear bijection, and \( G_1(l_1) \times D_1(s) \) is the identity on \( G_1 \times G_1 \).
Lemma 3. Let \( n = 2 \), whence \( k = 1 \). Suppose that \( g' : \mathcal{G}_1 \rightarrow \mathcal{G}_1 \) and \( g'' : \mathcal{G}_1 \rightarrow \mathcal{G}_1 \) are bijections such that one of the mappings \( g' \times g'' \) or \( g' \times g'' \), when restricted to \( \mathcal{G} \), is a C-transformation. Then \( g' = g'' \).

Proof. It suffices to discuss the first case, since \( 1_G \times 1_G \) yields a C-transformation. Now we can proceed as in the proof of Lemma 2 (a) in order to establish that the restriction of \( g'^{-1}g'' \) to \( \mathcal{G} \) equals \( 1_G \).

We say that \( \mathcal{X} \subset \mathcal{G} \) is a C-subset if any two distinct elements of \( \mathcal{X} \) are close. (If we consider the graph of the closeness relation on \( \mathcal{G} \) then a C-subset is just a clique, i.e., a complete subgraph.) A C-subset is said to be maximal if it is not properly contained in any C-subset. In order to describe the maximal C-subsets the following notation will be useful. If \( P \) and \( T \) are subspaces of \( V \) then we put

\[
\mathcal{G}(P, T) := \{(S, U) \in \mathcal{G} \mid S \subseteq P \text{ and } U \subseteq T\};
\]

here we use the incidence relation from the beginning of Section 2.

Lemma 4. The maximal C-subsets of \( \mathcal{G} \) are precisely the sets \( \mathcal{G}(S, V) \) with \( S \in \mathcal{G}_k \), and \( \mathcal{G}(V, U) \) with \( U \in \mathcal{G}_{n-k} \).

Proof. Easy verification.

We refer to the sets described in the lemma as maximal C-subsets of first kind and second kind, respectively.

Proof of Theorem 1 (a) Let \( f \) be a C-transformation of \( \mathcal{G} \). Then \( f \) and \( f^{-1} \) map maximal C-subsets to maximal C-subsets. Observe that two maximal C-subsets have a unique common element if, and only if, one of them is of first kind, say \( \mathcal{G}(S, V) \), the other is of second kind, say \( \mathcal{G}(V, U) \), and \( (S, U) \in \mathcal{G} \).

Given \( S, S' \in \mathcal{G}_k \) there exists a subspace \( U \in \mathcal{G}_{n-k} \) such that \( S + U = S' + U = V \). We conclude from

\[
f(\mathcal{G}(S, V)) \cap f(\mathcal{G}(V, U)) = \{f((S, U))\}\]

that \( f(\mathcal{G}(S, V)) \) and \( f(\mathcal{G}(V, U)) \) are maximal C-subsets of different kind. Likewise, \( f(\mathcal{G}(S', V)) \) and \( f(\mathcal{G}(V, U)) \) are of different kind, so that \( f(\mathcal{G}(S, V)) \) and \( f(\mathcal{G}(S', V)) \) are of the same kind.

A similar argument holds for maximal C-subsets of second kind; altogether the action of the C-transformation \( f \) on the set of maximal C-subsets is either type preserving or type interchanging.

(b) Suppose that \( f \) is type preserving. Then there exist bijections

\[
g' : \mathcal{G}_k \rightarrow \mathcal{G}_k \text{ such that } f(\mathcal{G}(S, V)) = \mathcal{G}(g'(S), V) \text{ for all } S \in \mathcal{G}_k, \\
g'' : \mathcal{G}_{n-k} \rightarrow \mathcal{G}_{n-k} \text{ such that } f(\mathcal{G}(V, U)) = \mathcal{G}(V, g''(U)) \text{ for all } U \in \mathcal{G}_{n-k};
\]

thus \( f \) equals the restriction of \( g' \times g'' \) to \( \mathcal{G} \). We distinguish four cases:

Case 1: \( n = 2 \). Hence \( k = 1 \); we deduce from Lemma 3 (a) that \( g' = g'' \), whence \( f \) is given as in Example 4.

Case 2: \( n > 2 \) and \( k = 1 \). Then for each \( U \in \mathcal{G}_{n-1} \) the mapping \( g' \) transfers the set of all 1-dimensional subspaces contained in \( U \) to the set of all 1-dimensional subspaces contained in \( g''(U) \). This means, by the fundamental theorem of projective
geometry, that there exists a semilinear isomorphism \( l' : V \to V \) with \( g' = G_1(l') \). Similarly, \( g'' \) is induced by a semilinear isomorphism \( l'' : V \to V \).

Case 3: \( n > 2 \) and \( k = n - 1 \). By symmetry, this coincides with the previous case.

Case 4: \( n > 2 \) and \( 1 < k < n - 1 \). Then Theorem 5 guarantees that \( g' \) and \( g'' \) are adjacency preserving in both directions; Chow’s theorem ([4, p. 38], [5, p. 81]) says that \( g' \) and \( g'' \) are induced by semilinear isomorphisms. More precisely, we have \( g' = G_k(l') \) with a semilinear bijection \( l' : V \to V \), or \( g' = D_k(s') \) with a semilinear bijection \( s' : V \to V^* \) (only when \( n = 2k \)). A similar description holds for \( g'' \).

In cases 2–4 we infer from Lemma 2(c) that there are only two possibilities:

Case A. \( g' = G_k(l') \) and \( g'' = G_{n-k}(l'') \). Now Lemma 2(a) yields that \( G_i(l') = G_i(l'') \) for all \( i = 1, 2, \ldots, n-1 \), whence \( f \) is the restriction to \( G \) of \( G_k(l') \times G_{n-k}(l'') \); cf. Example 3.

Case B. \( n = 2k \), \( g' = D_k(s') \), and \( g'' = D_k(s'') \). Now Lemma 2(b) yields that \( D_i(s') = D_i(s'') \) for all \( i = 1, 2, \ldots, n-1 \), whence \( f \) is the restriction to \( G \) of \( D_k(s') \times D_k(s'') \); cf. Example 3.

(c) If \( f \) is type interchanging then there exist bijections

\[
\begin{align*}
g' : G_k &\to G_{n-k} \text{ such that } f(G(S, V)) = G(V, g'(S)) \text{ for all } S \in G_k, \\
g'' : G_{n-k} &\to G_k \text{ such that } f(G(V, U)) = G(g''(U), V) \text{ for all } U \in G_{n-k};
\end{align*}
\]

thus \( f \) is the restriction to \( G \) of \( g' \circ g'' \). Now we can proceed, mutatis mutandis, as in (b). So \( f \) is given as in Example 2 or 3.

This completes the proof. \( \square \)

4. Proof of Theorem 2

First, let us introduce the following notion: We say that \( \mathcal{X} \subset G \) is an \( A \)-subset if any two distinct elements of \( \mathcal{X} \) are adjacent. (As before, such a set is just a clique of the graph given by the adjacency relation on \( G \).) An \( A \)-subset is said to be maximal if it is not properly contained in any \( A \)-subset.

If \( k = 1 \) or if \( k = n - 1 \) then an \( A \)-subset is the same as a \( C \)-subset, and Lemma 3 can be applied.

**Lemma 5.** Let \( 1 < k < n - 1 \). Then the maximal \( A \)-subsets of \( G \) are precisely the following sets:

\[
\begin{align*}
(29) &\quad G(S, T) \text{ with } S \in G_k, T \in G_{n-k+1}, \text{ and } S + T = V, \\
(30) &\quad G(S, T) \text{ with } S \in G_k, T \in G_{n-k-1}, \text{ and } S \cap T = \emptyset, \\
(31) &\quad G(T, U) \text{ with } T \in G_{k+1}, U \in G_{n-k}, \text{ and } T + U = V, \\
(32) &\quad G(T, U) \text{ with } T \in G_{k-1}, U \in G_{n-k}, \text{ and } T \cap U = \emptyset.
\end{align*}
\]

**Proof.** From [4, p. 36] we recall the following: Let \( \mathcal{Y} \subset G_i, 1 < i < n - 1 \), be a maximal set of mutually adjacent \( i \)-dimensional subspaces of \( V \). Then there exists a subspace \( T \in G_{k+1} \) such that \( \mathcal{Y} = \{ Y \in G_i \mid Y \perp T \} \).

Suppose now that \( \mathcal{X} \subset G \) is a maximal \( A \)-subset. Clearly, there exists an element \( (S, U) \in \mathcal{X} \). Since \( \mathcal{X} \) is also a \( C \)-subset, we obtain that \( \mathcal{X} \subset G(S, V) \) or that \( \mathcal{X} \subset G(V, U) \).
Let \( X \subset \mathcal{G}(S, V) \). Then the second components of the elements of \( X \) are mutually adjacent elements of \( \mathcal{G}_{n-k} \). Hence, by the above, they all are incident with a subset \( T \in \mathcal{G}_{n-k+1} \). So, due to its maximality, the set \( X \) is given as in \([29]\) or \([30]\).

Similarly, if \( X \subset \mathcal{G}(V, U) \) then \( X \) can be written as in \([31]\) or \([32]\).

Conversely, it is obvious that \([29]\)–\([32]\) define maximal A-subsets.

We shall also make use of the following result:

**Lemma 6.** Any two elements \((S, U)\) and \((S', U')\) of \( \mathcal{G} \) can be connected by a finite sequence which is given as in formula \([33]\). In particular, if \( S = S' \) (or \( U = U' \)) then this sequence can be chosen in such a way that \( S = S_0 = S_1 = \cdots = S_i \) (or \( U = U_0 = U_1 = \cdots = U_i \)).

**Proof.** (a) First, we show the particular case when \((S, U), (S', U') \in \mathcal{G}(S, V)\) with \( S \in \mathcal{G}_k \). We proceed by induction on \( d := (n - k) - \dim(U \cap U') \), the case \( d = 0 \) being trivial.

Let \( d > 0 \). There exists an \((n - k - 1)\)-dimensional subspace \( W \) such that \( U \cap U' \subset W \subset U \). So \( H := W \oplus S \) is a hyperplane of \( V \). It cannot contain \( U' \) because of \((S, U') \in \mathcal{G} \). Thus \( W' := H \cap U' \) has dimension \( n - k - 1 \), and there exists a \( 1 \)-dimensional subspace \( P' \subset U' \) with \( U' = P' \oplus W' \). Consequently, \( P' \not\subset H \) and we obtain

\[
(33) \quad V = P' \oplus H = P' \oplus W \oplus S.
\]

This means that \( U'' := P' \oplus W \) is a complement of \( S \). We have \((S, U) \sim (S, U'')\) and \( (n - k) - \dim(U'' \cap U') = d - 1 \). So the assertion follows from the induction hypothesis, applied to \((S, U'')\) and \((S, U')\).

Similarly, any two elements of \( \mathcal{G}(V, U) \) with \( U \in \mathcal{G}_{n-k} \) can be connected.

(b) Now we consider the general case. Let \((S, U)\) and \((S', U')\) be elements of \( \mathcal{G} \). There exists \( U'' \in \mathcal{G}_{n-k} \) which is complementary to both \( S \) and \( S' \). Then, by (a), there exists a sequence

\[
(34) \quad (S, U) \sim \cdots \sim (S, U'') \sim \cdots \sim (S', U'') \sim \cdots \sim (S', U')
\]

which completes the proof. \( \square \)

The statement in (a) from the above is just a particular case of a more general result on the connectedness of a spine space; cf. \([5\), Proposition 2.9].

**Proof of Theorem** \([2]\) (a) We shall accomplish our task by showing that every A-transformation is a C-transformation. As has been noticed in Section \([4\) this is trivial if \( k = 1 \) or if \( k = n - 1 \). So let \( f \) be an A-transformation of \( \mathcal{G} \) and assume that \( 1 < k < n - 1 \).

(b) We claim that

\[
(35) \quad f(\mathcal{G}(S, V)) \text{ is a maximal C-subset for all } S \in \mathcal{G}_k.
\]

Let us take \( T \in \mathcal{G}_{n-k+1} \) such that \( \mathcal{G}(S, T) \) is a maximal A-subset. Then \( f(\mathcal{G}(S, T)) \) is also a maximal A-subset. According to Lemma \([3\) there are four possible cases.

Case 1: \( f(\mathcal{G}(S, T)) \) is given according to \([29]\). This means \( f(\mathcal{G}(S, T)) = \mathcal{G}(W, Z) \) with \( W \in \mathcal{G}_k \), \( Z \in \mathcal{G}_{n-k+1} \), and \( W + Z = V \). We assert that in this case

\[
(36) \quad f((S, U')) \in \mathcal{G}(W, V) \text{ for all } (S, U') \in \mathcal{G}(S, V).
\]
In order to show this we choose an element \((S, U) \in \mathcal{G}(S, T)\). Clearly, \(f((S, U)) \in \mathcal{G}(W, Z) \subset \mathcal{G}(W, V)\).

First, we suppose that \((S, U)\) and \((S, U')\) are adjacent. Then \(P := U \cap U' \in \mathcal{G}_{n-k-1}\).

We consider the pencil given by \(P\) and \(T\), i.e. the set

\[
\{ X \in \mathcal{G}_{n-k} \mid P \subset X \subset T \}.
\]

It contains at least three elements; precisely one them is not complementary to \(S\). Consequently, the intersection of the maximal A-subsets \(\mathcal{G}(S, T)\) and \(\mathcal{G}(S, P)\) contains more than one element. The same property holds for the intersection of the maximal A-subsets \(f(\mathcal{G}(S, T)) = \mathcal{G}(W, Z)\) and \(f(\mathcal{G}(S, P))\). But this means that \(W\) is the first component of every element of \(f(\mathcal{G}(S, P))\) so that \(f((S, U')) \in \mathcal{G}(W, V)\).

Next, we suppose that \((S, U)\) and \((S, U')\) are arbitrary. By Lemma \(\ref{lemma6}\) \((S, U)\) and \((S, U')\) can be connected by a finite sequence

\[
(S, U) = (S, U_0) \sim (S, U_1) \sim \cdots \sim (S, U_i) = (S, U'),
\]

and the arguments considered above yield that \((38)\) holds.

Since \(f^{-1}\) is adjacency preserving, we can repeat our previous proof, with \(\mathcal{G}(W, Z)\) taking over the role of \(\mathcal{G}(S, T)\). Altogether, this proves

\[
f(\mathcal{G}(S, V)) = \mathcal{G}(W, V).
\]

The remaining cases, i.e., when \(f(\mathcal{G}(S, T))\) is given according to \((30)\), \((31)\), or \((32)\), can be treated similarly, whence \((35)\) holds true.

(c) Dual to (b), it can be shown that \(f(\mathcal{G}(V, U))\) is a maximal C-subset for all \(U \in \mathcal{G}_{n-k}\). Thus \(f\) is a C-transformation. \(\square\)

5. **Proofs of Theorem \(\ref{thm3}\) and Theorem \(\ref{thm4}\)**

In the following proof we use the term **maximal C-subset** just like in Section \(\ref{section3}\).

**Proof of Theorem \(\ref{thm3}\)** Obviosly, each maximal C-subset of \(\mathcal{G}_k \times \mathcal{G}_{n-k}\) has either the form \(\{S\} \times \mathcal{G}_{n-k}\) with \(S \in \mathcal{G}_k\) (first kind) or \(\mathcal{G}_k \times \{U\}\) with \(U \in \mathcal{G}_{n-k}\) (second kind). Distinct maximal C-subsets of the same kind have empty intersection, whereas maximal C-subsets of different kind have a unique common element. So every C-transformation is either type preserving, whence it can be written as \(f' \times f''\), or type interchanging, whence it can be written as \(g' \times g''\). \(\square\)

Let \(1 < k < n - 1\). We shall consider below the following well known **partial linear spaces**: For each \(i = 2, 3, \ldots, n - 2\) the set \(\mathcal{G}_i\) is the point set of the **Segre product** (or **product space**) \((\mathcal{G}_i, \mathcal{L}_i)\); the elements of its line set \(\mathcal{L}_i\) are the pencils

\[
\mathcal{G}_i[P, T] := \{ X \in \mathcal{G}_i \mid P \subset X \subset T \},
\]

where \(P \in \mathcal{G}_{i-1}\), \(T \in \mathcal{G}_{i+1}\), and \(P \subset T\). The **Segre product** (or **product space**) of \((\mathcal{G}_k, \mathcal{L}_k)\) and \((\mathcal{G}_{n-k}, \mathcal{L}_{n-k})\) is the partial linear space with point set

\[
P := \mathcal{G}_k \times \mathcal{G}_{n-k}
\]

and line set

\[
\mathcal{L} := \{ \{S\} \times l \mid S \in \mathcal{G}_k, l \in \mathcal{L}_{n-k} \} \cup \{ m \times \{U\} \mid m \in \mathcal{L}_k, U \in \mathcal{G}_{n-k} \}.
\]

See \cite{ref7} for further details and references.
Proof of Theorem \[2\]

(a) If \(k = 1\) or if \(k = n - 1\) then the assertion follows from Theorem \[3\].

(b) Let \(1 < k < n - 1\). Given a subset \(\mathcal{M} \subset \mathcal{P}\) we put

\[
\mathcal{M}^\perp := \{(S, U) \in \mathcal{P} \mid (S, U) \perp (X, Y) \text{ for all } (X, Y) \in \mathcal{M}\},
\]

where the sign “\(\perp\)" on the right hand side means “adjacent or equal”. Now let \((S, U)\) and \((S, U')\) be adjacent elements of \(\mathcal{P}\). Then

\[
\{(S, U), (S, U')\}^\perp = \{(S, Y) \in \mathcal{P} \mid U \cap U' \subset Y \text{ or } Y \subset U + U'\}
\]

and

\[
\{(S, U), (S, U')\}^{\perp \perp} = \{(S, Y) \in \mathcal{P} \mid U \cap U' \subset Y \subset U + U'\}.
\]

Similarly, if \((S, U)\) and \((S', U)\) are adjacent elements of \(\mathcal{P}\) then

\[
\{(S, U), (S', U)\}^{\perp \perp} = \{(X, U) \in \mathcal{P} \mid S \cap S' \subset X \subset S + S'\}.
\]

Next, suppose that \(g : \mathcal{P} \to \mathcal{P}\) is an \(A\)-transformation. Every line of \((\mathcal{P}, \mathcal{L})\) can be written in the form \(14\) or \(16\), since it contains at least two distinct collinear points or, said differently, two adjacent elements of \(\mathcal{P}\). Thus \(g\) is a collineation of the product space \((\mathcal{P}, \mathcal{L})\). By [7, Theorem 1.14], there are two possibilities:

Case 1. There exist collineations of Grassmann spaces \(f' : \mathcal{G}_k \to \mathcal{G}_k\) and \(f'' : \mathcal{G}_{n-k} \to \mathcal{G}_{n-k}\) such that \(g = f' \times f''\). Clearly, \(f'\) and \(f''\) are adjacency preserving in both directions.

Case 2. There exist collineations of Grassmann spaces \(g' : \mathcal{G}_k \to \mathcal{G}_{n-k}\) and \(g'' : \mathcal{G}_{n-k} \to \mathcal{G}_k\) such that \(g = g' \times g''\). As above, \(g'\) and \(g''\) are adjacency preserving in both directions.

So \(g\) is given as in Example \[6\]. \(\square\)

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