Stable fluctuations of iterated perturbed random walks
in intermediate generations of a general branching process tree

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Abstract. Consider a general branching process, a.k.a. Crump–Mode–Jagers process, generated by a perturbed random walk \( \eta_1, \xi_1 + \eta_2, \xi_2 + \eta_3, \ldots \), where \( (\xi_1, \eta_1), (\xi_2, \eta_2), \ldots \) are independent identically distributed random vectors with arbitrarily dependent positive components. Denote by \( N_j(t) \) the number of the \( j \)th generation individuals with birth times \( \leq t \). Assume that \( j = j(t) \to \infty \) and \( j(t) = o(t^a) \) as \( t \to \infty \) for some explicitly given \( a > 0 \) (to be specified in the paper). The corresponding \( j \)th generation belongs to the set of intermediate generations. We provide sufficient conditions under which the finite-dimensional distributions of the process \( (N_{j(t)}(u))_{u>0} \), properly normalized and centered, converge weakly to those of an integral functional of a stable Lévy process with finite mean.

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1 Introduction and main result

1.1 Definition and motivation

Let \( (\xi_1, \eta_1), (\xi_2, \eta_2), \ldots \) be independent copies of an \( \mathbb{R}^2 \)-valued random vector \((\xi, \eta)\) with arbitrarily dependent components. Denote by \( (S_i)_{i \geq 0} \) the zero-delayed standard random walk with increments \( \xi_i \) for \( i \in \mathbb{N} \), that is, \( S_0 := 0 \) and \( S_i := \xi_1 + \cdots + \xi_i \) for \( i \in \mathbb{N} \). Define

\[ T_i := S_{i-1} + \eta_i, \quad i \in \mathbb{N}. \]

The sequence \( T := (T_i)_{i \in \mathbb{N}} \) is called a perturbed random walk (PRW). The so-defined PRW is a nontrivial generalization of the standard random walk. Apart from being an interesting object of investigation, see, for instance, [2, 3, 19], The PRW is known to be an important ingredient of perpetuities [11], the Bernoulli sieve [1, 10], \( G/G/\infty \)-queues [14], a perturbed branching random walk [4], to name a few. A detailed exposition of various properties of PRW and its applications can be found in [13].

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Also, when investigating that one of the following two conditions holds.

We write \( \Rightarrow, \rightarrow, \) and \( \xrightarrow{f.d.d.} \) to denote weak convergence of probability measures in a function space, weak convergence of one-dimensional and finite-dimensional distributions, respectively. The following result is a combination of Theorems 3.1 and 3.2 in [16]; see also Section 3 in [8] for an earlier weaker version. Put

\[
V_j(t) := \mathbb{E} N_j(t), \quad j \in \mathbb{N}, \quad t \geq 0.
\]

**Proposition 1.** Assume that \( \sigma^2 = \text{Var}\, \xi \in (0, \infty) \) and \( \mathbb{E} \eta < \infty \). Let \( j = j(t) \) be any positive integer-valued function satisfying \( j(t) \to \infty \) and \( j(t) = o(t^{1/2}) \) as \( t \to \infty \). Then, as \( t \to \infty \),

\[
\left( \left\lfloor \frac{j(t)}{\sigma^2 m^{2j(t)u}} \right\rfloor - 1 \right) \left( N_{j(t)u}(t) - V_{j(t)u}(t) \right) \xrightarrow{f.d.d.} \int_0^u e^{-uy} \, dS_2(y),
\]

where \( m := \mathbb{E} \xi < \infty, \) and \( S_2 := (S_2(v))_{v \geq 0} \) is a standard Brownian motion.

In this paper, we intend to prove a counterpart of Proposition 1 under the assumptions that \( m < \infty, \sigma^2 = \infty, \) and the distribution of \( \xi \) belongs to the domain of attraction of a stable distribution. More precisely, we assume that one of the following two conditions holds.

**CONDITION RW I.** \( \sigma^2 = \infty \) and, for some \( \ell \) slowly varying at infinity,

\[
\mathbb{E} (\xi^2 1_{\{\xi \leq t\}}) \sim \ell(t), \quad t \to \infty,
\]

in which case the distribution of \( \xi \) belongs to the domain of attraction of a normal distribution.

**CONDITION RW II.** For some \( \alpha \in (1, 2) \) and some \( \ell \) slowly varying at infinity,

\[
\mathbb{P}\{\xi > t\} \sim t^{-\alpha} \ell(t), \quad t \to \infty,
\]

in which case the distribution of \( \xi \) belongs to the domain of attraction of an \( \alpha \)-stable distribution.
Assume that (1.2) holds with \( \alpha = 1 \). There exists a slowly varying function \( \ell \) for which \( m < \infty \). Thus, in principle, this situation could have also been considered. However, we do not treat the case \( \alpha = 1 \), for it is technically more complicated than the others and does not shed any new light on weak convergence we are interested in.

We will write \( N \) for \( N_1 \), that is, \( N(t) := \sum_{i \geq 1} 1_{\{T_i \leq t\}} \) for \( t \geq 0 \). Denote by \( D := D[0, \infty) \) the Skorokhod space of right-continuous functions defined on \([0, \infty)\) with finite limits from the left at positive points. For later needs, we recall the following functional limit theorems, obtained in Theorem 3.2 of [1], for the process \((N(ty))_{y \geq 0}\) as \( t \to \infty \): under the additional assumption \( E\eta^a < \infty \) for some \( a > 0 \), for a diverging function \( c_\alpha \), to be specified below,

\[
\lim_{y \to 0, t \to \infty} \frac{\left( N(ty) - m^{-1} \int_0^{ty} \mathbb{P}\{\eta \leq x\} \, dx \right)}{m^{-(\alpha+1)/\alpha} c_\alpha(t)} = \left( S_\alpha(y) \right)_{y \geq 0}, \quad t \to \infty. \tag{1.3}
\]

Here

- under Condition RW I, \( \alpha = 2 \), \( S_2 \) is a standard Brownian motion, \( c_2 \) is a positive function satisfying \( \lim_{t \to \infty} t\ell(c_2(t))/c_2^2(t) = 1 \), and the convergence takes place in the \( J_1 \)-topology on \( D \);
- under Condition RW II, \( S_\alpha := (S_\alpha(u))_{u \geq 0} \) is a spectrally negative \( \alpha \)-stable Lévy process such that \( S_\alpha(1) \) has the characteristic function

\[
E \exp(i z S_\alpha(1)) = \exp \left\{ -|z|^\alpha \Gamma(1 - \alpha) \left( \cos \frac{\pi \alpha}{2} + \operatorname{sgn}(z) \sin \frac{\pi \alpha}{2} \right) \right\}, \quad z \in \mathbb{R}, \tag{1.4}
\]

where \( \Gamma \) is Euler’s gamma function, \( c_\alpha \) is a positive function satisfying \( \lim_{t \to \infty} t\ell(c_\alpha(t))/c_\alpha^2(t) = 1 \), and the convergence takes place in the \( M_1 \)-topology on \( D \).

Comprehensive information concerning the \( J_1 \)- and \( M_1 \)-convergence on \( D \) can be found in the monographs [5, 18] and [21], respectively.

We recall that \( E\xi^\gamma < \infty \) for all \( \gamma \in (0, \alpha) \) whenever either Condition RW I or RW II holds. Further, \( E\xi^2 = \infty \) under Condition RW I. In contrast, \( E\xi^\alpha \) may be finite or infinite under Condition RW II. After this discussion, we are ready to state assumptions on the distribution of \( \eta \).

**CONDITION PERT(\( \gamma \)).** If \( \alpha \in (1, 2) \) and \( E\xi^\alpha < \infty \), then we set \( \gamma := \alpha \) and assume that

\[
E(\eta \wedge t) = O(t^{2-\gamma}), \quad t \to \infty. \tag{1.5}
\]

If \( \alpha \in (1, 2) \) and \( E\xi^\alpha = \infty \), then we assume that (1.5) holds for some \( \gamma \in (2 - 1/\alpha, \alpha) \).

Here is our main result.

**Theorem 1.** Assume that \( E\xi^2 = \infty \), that the distribution of \( \xi \) belongs to the domain of attraction of an \( \alpha \)-stable distribution, \( \alpha \in (1, 2) \), and that Condition PERT(\( \gamma \)) holds. Let \( j(t) \) be any positive integer-valued function satisfying \( j(t) \to \infty \) and \( j(t) = o(t^{(\gamma-1)/2}) \) as \( t \to \infty \). Then, as \( t \to \infty \),

\[
\left( \frac{\left| j(t) \right| - 1}{t|j(t)|^{\alpha+1}} c_\alpha(t/j(t)) \left( N[j(t)u](t) - V[j(t)u](t) \right) \right)_{u > 0} \xrightarrow{\text{f.d.d.}} \left( \int_{(0, \infty)} e^{-uy} \, dS_\alpha(y) \right)_{u > 0},
\]

where \( m = E\xi < \infty \), \( S_2 \) is a standard Brownian motion, and \( S_\alpha \) is a spectrally negative \( \alpha \)-stable Lévy process with characteristic function (1.4).

**Remark 1.** We can check that the inequality \( E\eta^{\gamma-1} < \infty \) ensures Condition PERT(\( \gamma \)) and that this condition guarantees that \( E\eta^{\gamma-1-\delta} < \infty \) for any \( \delta \in (0, \gamma - 1) \). The latter means that under the assumptions of Theorem 1, relation (1.3) holds.

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Remark 2. The limit process in Theorem 1, which we denote by \( L_\alpha \), is in fact defined as the result of integration by parts:

\[
L_\alpha(u) = u \int_0^\infty e^{-uy} S_\alpha(y) \, dy, \quad u > 0.
\]

We can check that this definition produces the same process as an alternative definition appearing in Theorem 1, in which \( L_\alpha \) is understood as the stochastic integral with respect to a semimartingale. Note that the process \( L_\alpha \) is a.s. continuous and self-similar with negative index \(-1/\alpha\), i.e., for any \( a > 0 \), \( r \in \mathbb{N} \), and \( 0 < u_1 < \cdots < u_r < \infty \), the vector \((L_\alpha(au_1), \ldots, L_\alpha(au_r))\) has the same distribution as \( a^{-1/\alpha}(L_\alpha(u_1), \ldots, L_\alpha(u_r))\).

Assume that \( \alpha \in (1, 2) \). The process \( S_\alpha \) that describes the limit fluctuations of \( N_1 = N \) (in the first generation) is a.s. discontinuous. The structure of the process \( L_\alpha \) indicates that the limit fluctuations of \( N_j \) (in the intermediate generations \( j \)) are driven by two factors: (i) the fluctuations of the input process \( N_1 \) governed by \( S_\alpha \) and (ii) the renewal structure of the tree reflected in the function \( u \mapsto e^{-uy} \). Furthermore, we see that the renewal structure of the tree makes the limit \( L_\alpha \) continuous, thereby smoothing out the fluctuations of the input process.

The remainder of the paper is structured as follows. Some auxiliary results are stated and proved in Section 2. The proof of Theorem 1 is given in Section 3.

2 Auxiliary results

The Lebesgue–Stieltjes convolution of functions \( r, s : [0, \infty) \to [0, \infty) \) of locally bounded variation is given by

\[
(r * s)(t) = \int_{[0, t]} r(t - y) \, ds(y) = \int_{[0, t]} s(t - y) \, dr(y), \quad t \geq 0.
\]

We write \( r^{*(j)} \) for the \( j \)-fold Lebesgue–Stieltjes convolution of \( r \) with itself.

We proceed by recalling an extended version of Proposition 3.1 in [7]. Inequality (2.1) is not a part of the cited result but is contained in its proof.

Lemma 1. Let \( f : \mathbb{R} \to [0, \infty) \) be a nondecreasing right-continuous function vanishing on the negative half-line and satisfying

\[
f(t) = at + O(t^\beta), \quad t \to \infty,
\]

for some \( a > 0 \) and \( \beta \in [0, 1) \). Then, for some constant \( C \geq 1 \),

\[
\left| f^{*(j)}(t) - \frac{a^j t^j}{j!} \right| \leq \sum_{i=0}^{j-1} \binom{j}{i} \frac{a^i C^{j-i} (t+1)^{\beta(j-i)+i}}{i!}, \quad j \in \mathbb{N}, \ t \geq 0.
\]

In particular, for any integer-valued function \( j = j(t) \) satisfying \( j(t) = o(t^{(1-\beta)/2}) \) as \( t \to \infty \),

\[
f^{*(j)}(t) \sim \frac{a^j t^j}{j!}, \quad t \to \infty.
\]

Of principal importance for what follows is the decomposition

\[
N_j(t) = \sum_{k \geq 1} N_{j-1}^{(k)}(t - T_k) 1_{\{T_k \leq t\}}, \quad j \geq 2, \ t \geq 0.
\]
where $N_{j-1}^{(r)}(t)$ is the number of successors in the $j$th generation with birth times within $[T_r, t + T_r]$ of the first generation individual with birth time $T_r$. In what follows, we write $V$ for $V_1$. Note that (2.2) entails $E N_j(t) = V_j(t) = V^{*j}(t)$ for $j \in \mathbb{N}$ and $t \geq 0$.

Corollary 1 is our important technical tool to be used in all subsequent proofs.

**Corollary 1.** Assume that the assumptions of Theorem 1 hold. Then, for some constant $C \geq 1$,

$$V_j(t) - \frac{t^j}{j!m^j} \leq \sum_{i=0}^{j-1} \binom{j}{i} C^{j-i}(t+1)^{(2-\gamma)(j-i)+i} \frac{i!m^i}{i^2 m^i}, \quad j \in \mathbb{N}, \ t \geq 0, \quad (2.3)$$

with the same $\gamma$ as in Condition PERT$(\gamma)$. In particular, for any integer-valued function $j = j(t)$ satisfying $j(t) = o(t^{(\gamma-1)/2})$ as $t \to \infty$,

$$V_j(t) \sim \frac{t^j}{j!m^j}, \quad t \to \infty. \quad (2.4)$$

Let $j \in \mathbb{N}$ and $s \geq 0$ satisfy $(s + 1)^{\gamma-1} \geq 2Cm^2$. Then, for $1 \leq k \leq j$,

$$V_k(s) \leq \frac{2(s + 1)^k}{k!m^k}, \quad (2.5)$$

and

$$\sum_{i=0}^{k-1} \binom{k}{i} C^{k-i}(s + 1)^{(2-\gamma)(k-i)+i} \frac{i!m^i}{(i+1)!m^{i+1}} \leq \frac{2C(s + 1)^{k+2-\gamma}}{(k-1)!m^k}. \quad (2.7)$$

**Proof.** We will show that the function $V$ satisfies the assumptions of Lemma 1 with $a = m^{-1}$ and $\beta = 2 - \gamma$. Then (2.3) and (2.4) are an immediate consequence of Lemma 1.

Recall that $S_0 = 0$ and $S_i = \xi_1 + \ldots + \xi_i$ for $i \in \mathbb{N}$. Let $S_0^*$ be a random variable with distribution

$$\mathbb{P}\{S_0^* \in \mathrm{d}x\} = m^{-1} \mathbb{P}\{\xi > x\} \mathbb{1}_{(0,\infty)}(x) \mathrm{d}x.$$

Then, according to formula (2) in [9],

$$U(t) - m^{-1}t = \int_{[0,t]} \mathbb{P}\{S_0^* > t - y\} \mathrm{d}U(y), \quad t \geq 0,$$

where $U(t) := \sum_{i \geq 0} \mathbb{P}\{S_i \leq t\}$ for $t \geq 0$, that is, $U$ is the renewal function of $(S_i)_{i \in \mathbb{N}_0}$. Since the assumption $E \xi^2 = \infty$ is equivalent to $ES^* = \infty$, we conclude that

$$U(t) - m^{-1}t \sim m^{-1} \int_0^t \mathbb{P}\{S_0^* > y\} \mathrm{d}y, \quad t \to \infty,$$

by Theorem 4 in [20].

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Assume that $\mathbb{E}\xi^\alpha < \infty$. Then $\gamma = \alpha \in (1, 2)$, $\mathbb{E}(S_0^\alpha)^{\alpha-1} < \infty$, and, by Markov’s inequality,

$$
\int_0^t \mathbb{P}\{S_0^\alpha > y\} \, dy \leq (2 - \alpha)^{-1}\mathbb{E}(S_0^\alpha)^{\alpha-1}t^{2-\alpha}.
$$

This, together with Condition PERT($\gamma$), which reads $\mathbb{E}(\eta \wedge t) = O(t^{2-\alpha})$, entails

$$
V(t) - m^{-1}t = \int_{[0,t]} (U(t - y) - m^{-1}(t - y)) \, d\mathbb{P}\{\eta \leq y\} - m^{-1}\mathbb{E}(\eta \wedge t)
= O(t^{2-\alpha}), \quad t \to \infty.
$$

Assume that $\mathbb{E}\xi^\alpha = \infty$. Since $\mathbb{E}\xi^\gamma < \infty$ and hence $\mathbb{E}(S_0^\gamma)^{\gamma-1} < \infty$ for all $\gamma_1 \in (0, \alpha)$, the same reasoning as above leads to the conclusion

$$
\int_{[0,t]} (U(t - y) - m^{-1}(t - y)) \, d\mathbb{P}\{\eta \leq y\} = O(t^{2-\gamma}), \quad t \to \infty.
$$

In conjunction with (1.5), this yields $V(t) - m^{-1}t = O(t^{2-\gamma})$. In particular, there exists a constant $c > 0$ such that

$$
|V(t) - m^{-1}t| \leq c(t + 1)^{2-\gamma}, \quad t \geq 0. \quad (2.8)
$$

Next, we prove (2.5). According to (2.3), it suffices to check that

$$
\sum_{i=0}^{k-1} \binom{k}{i} \frac{C^{k-i}(s+1)^{(2-\gamma)(k-i)+i}}{i!m^i} \leq \frac{(s+1)^k}{k!m^k}, \quad 1 \leq k \leq j, \quad (s+1)^{\gamma-1} \geq 2Cmj^2.
$$

Using

$$
\binom{k}{i} \leq \frac{k!}{i!} \leq k^{k-i} \quad (2.9)
$$

and

$$
(s+1)^k = (s+1)^{(2-\gamma)k}(s+1)^{(\gamma-1)k}, \quad (2.10)
$$

this follows from

$$
\frac{k!m^k}{(s+1)^k} \sum_{i=0}^{k-1} \binom{k}{i} \frac{C^{k-i}(s+1)^{(2-\gamma)(k-i)+i}}{i!m^i}
= \sum_{i=0}^{k-1} \binom{k}{i} \frac{k!}{i!} \left( \frac{Cm}{(s+1)^{\gamma-1}} \right)^{k-i} \leq \sum_{i=0}^{k-1} \left( \frac{Cm^2}{(s+1)^{\gamma-1}} \right)^{k-i}
\leq \sum_{i=0}^{k-1} \left( \frac{Cm^2}{2Cmj^2} \right)^{k-i} = \sum_{i=1}^{k} \left( \frac{k^2}{2j^2} \right)^i \leq \sum_{i=1}^{\infty} 2^{-i} = 1 \quad (2.11)
$$

because $k \leq j$. 

Now we pass to the proof of (2.6). Invoking once again (2.9) and (2.10), we arrive at

$$\frac{(k-1)!m^{k-1}}{k(s+1)^{k+1-\gamma}} \sum_{i=0}^{k-2} \binom{k}{i} \frac{C^{k-i}(s+1)^{(2-\gamma)(k-i)+i}}{i!m^i}$$

$$= \frac{(s+1)^{\gamma-1}k^{-2}}{m^{k^2}} \sum_{i=0}^{k-2} \binom{k}{i} \frac{k!}{i!} \left( \frac{Cm}{(s+1)^{\gamma-1}} \right)^{k-i} \leq \frac{(s+1)^{\gamma-1}k^{-2}}{m^{k^2}} \sum_{i=0}^{k-2} \left( \frac{Cm^2}{(s+1)^{\gamma-1}} \right)^{k-i}$$

$$\leq \frac{(s+1)^{-\gamma-1}}{m^{k^2}} \sum_{i=2}^k \left( \frac{Cm^2}{(s+1)^{\gamma-1}} \right)^i = \frac{m(Ck)^2}{(s+1)^{\gamma-1}} \left( 1 - \frac{Cm^2}{(s+1)^{\gamma-1}} \right)^{-1} \leq C,$$

and (2.6) follows. The proof of (2.7) is analogous and hence omitted. The proof of the corollary is complete. \(\square\)

Lemma 2 will be used in the proof of relation (3.2).

**Lemma 2.** Let \(u > 0\) be fixed. Under the assumptions of Theorem 1,

$$\lim_{t \to \infty} \frac{(j(t)u) - 1)!m^{[j(t)u]-1}}{t^{[j(t)u]-1}V_{[j(t)u]-1}} \left( t \left( 1 - \frac{y}{j} \right) \right) = e^{-uy} \quad (2.12)$$

for each fixed \(y \geq 0\), and

$$\lim_{T \to \infty} \limsup_{t \to \infty} \frac{(j(t)u) - 1)!m^{[j(t)u]}}{t^{[j(t)u]-1}c(y)} \int_{T(t)j(t)}^{t \alpha}(y) \, d\alpha(-V_{[j(t)u]-1}(t - y)) = 0. \quad (2.13)$$

**Proof.** For notational simplicity, we only treat the case \(u = 1\). We first prove (2.12). According to (2.3),

$$\left| V_j(t) - \frac{t^j}{m^j} \right| \leq g_j(t), \quad j \in \mathbb{N}, \ t \geq 0,$$

where

$$g_j(t) := \sum_{i=0}^{j-1} \binom{j}{i} \frac{C^{j-i}(t+1)^{(2-\gamma)(j-i)+i}}{m^i!}, \quad j \in \mathbb{N}, \ t \geq 0.$$ 

It suffices to prove that, for each fixed \(y > 0\),

$$\lim_{t \to \infty} \frac{(j-1)!m^{j-1} \left( t(1 - y) \right)^{j-1}}{t^{j-1}m^{j-1}(j-1)!} = e^{-y}$$

and

$$\lim_{t \to \infty} \frac{(j-1)!m^{j-1}}{t^{j-1}g_j(t)} = 0. \quad (2.14)$$

The first of these is immediate. To prove the second, we first recall that \(j(t) = o(t^{(\gamma-1)/2})\) as \(t \to \infty\). Hence, for \(t\) large enough, \(Cm(j-1)^2/(t+1)^{\gamma-1} \leq 1/2\), say. Write, for such \(t\), with the help of (2.11),

$$\frac{(j-1)!m^{j-1}}{t^{j-1}g_j(t)} \leq \sum_{i=0}^{j-2} \left( \frac{Cm(j-1)^2}{(t+1)^{\gamma-1}} \right)^{j-1-i} \leq \frac{Cm(j-1)^2}{(t+1)^{\gamma-1}} \left( 1 - \frac{Cm(j-1)^2}{(t+1)^{\gamma-1}} \right)^{-1}.$$
Since
\[ \frac{(j - 1)^2}{t(1 - \frac{y}{j}) + 1^{\gamma - 1}} \sim \frac{j^2}{t^{\gamma - 1}} \to 0, \quad t \to \infty, \]
the last inequality entails (2.14).

Next, we intend to prove (2.13). The function \( c_\alpha \) is regularly varying at infinity of index \( 1/\alpha \); see, for instance, Lemma 6.1.3 in [13]. By Theorem 1.8.3 in [6] and its proof there exists an asymptotically equivalent
\[ \lim_{t \to \infty} \]

Since
\[ \int_{(Tt/j,t]} c_\alpha(y) d_y(-V_{j-1}(t-y)) = V_{j-1}\left(t\left(1 - \frac{T}{j}\right)\right) c_\alpha\left(\frac{T t}{j}\right) + \int_{Tt/j}^t V_{j-1}(t-y)c'_\alpha(y) dy. \]
In view of (2.12),
\[ \lim_{t \to \infty} \frac{(j - 1)!m^j}{j! c_\alpha\left(\frac{T t}{j}\right)} V_{j-1}\left(t\left(1 - \frac{T}{j}\right)\right) c_\alpha\left(\frac{T t}{j}\right) = m T^{1/\alpha} e^{-T}. \]
The right-hand side converges to 0 as \( T \to \infty \). Using (2.3), we obtain
\[ \frac{(j - 1)!m^j}{j! c_\alpha\left(\frac{T t}{j}\right)} \int_{Tt/j}^t V_{j-1}(t-y)c'_\alpha(y) dy \]
\[ \leq \frac{m}{j! c_\alpha\left(\frac{T t}{j}\right)} \int_{Tt/j}^t (t-y)^{j-1} dc_\alpha(y) \]
\[ + \frac{(j - 1)!m^j}{j! c_\alpha\left(\frac{T t}{j}\right)} \sum_{i=0}^{j-2} \binom{j - 1}{i} \frac{C^{j-1-i}}{m^i} \int_{Tt/j}^t (t + 1 - y)^{(2-\gamma)(j-1-i)+i} c'_\alpha(y) dy \]
\[ =: a_j(t) + b_j(t). \]
Since \( \lim_{t \to \infty} (c_\alpha(ty/j)/c_\alpha(t/j)) = y^{1/\alpha} \) for each \( y > 0 \), we infer
\[ a_j(t) = \frac{m}{c_\alpha\left(\frac{T t}{j}\right)} \int_{T}^j \left(1 - \frac{y}{j}\right)^{j-1} d_y c_\alpha\left(\frac{ty}{j}\right) \leq \frac{m}{c_\alpha\left(\frac{T t}{j}\right)} \int_{T}^j e^{-(j-1)y/j} d_y c_\alpha\left(\frac{ty}{j}\right) \]
\[ \leq \frac{m}{c_\alpha\left(\frac{T t}{j}\right)} \int_{T}^j e^{-y/2} d_y c_\alpha\left(\frac{ty}{j}\right) \to \frac{m}{\alpha} \int_{T}^{\infty} e^{-y/2} y^{1/\alpha - 1} dy, \quad t \to \infty. \]
Here the limit relation is justified by the continuity theorem for Laplace–Stieltjes transforms. The right-hand side of the last centered formula converges to 0 as \( T \to \infty \). We claim that \( \lim_{t \to \infty} b_j(t) = 0 \). To prove this, we first observe that

\[
\frac{1}{c_\alpha(T_j/T)} \int_{T_j}^{t} (t+1-y)^{(2-\gamma)(j-1)\alpha} c_\alpha'(y) \, dy
\]

\[
\leq \int_{T_j}^{t} (t+1-y)^{(2-\gamma)(j-1)\alpha} \, dy = \frac{c_j(t(1-T_j)+1)\gamma}{(2-\gamma)(j-1)\alpha} \left( \frac{1}{T_j} \right) + 1
\]

where the first inequality follows from the fact that \( c_\alpha \) is nonincreasing, and the last inequality holds for \( t \) so large that \( T_j/t \geq 1 \) and, as a consequence, \( t(1-T_j)/j + 1 \leq t \). Further, in view of (2.15),

\[
\lim_{t \to \infty} \frac{\gamma^j c_\alpha(T_j/T)}{c_\alpha(T_j/T)} = \alpha^{-1}T^{1/\alpha-1}.
\]

Hence, for large \( t \) and some constant \( A(T) > 0 \),

\[
\frac{\gamma^j c_\alpha(T_j/T)}{c_\alpha(T_j/T)} \leq A(T).
\]

With these at hand, we infer, for large \( t \),

\[
b_j(t) \leq \frac{\gamma^j c_\alpha(T_j/T)}{c_\alpha(T_j/T)} \frac{1}{T_j} \sum_{i=0}^{j-2} \binom{j-1}{i} \frac{(2-\gamma)(j-1)\alpha + i+1}{(i+1)!m^i} \leq A(T) \sum_{i=0}^{j-2} \left( \frac{cm^j}{T_j^{\gamma-1}} \right)^{j-1-i} \rightarrow 0, \quad t \to \infty.
\]

We have used (2.9) for the second inequality. \( \Box \)

3 Proof of Theorem 1

3.1 Preparation

We will use a decomposition of \( N_j - V_j \) into “martingale” and “shot-noise” parts obtained with the help of (2.2):

\[
N_j(t) - V_j(t) = \left( \sum_{k \geq 1} (N_{j-1}^{(k)}(t-T_k) - V_{j-1}(t-T_k)) 1_{\{T_k \leq t\}} \right) + \left( \sum_{k \geq 1} V_{j-1}(t-T_k) 1_{\{T_k \leq t\}} - V_j(t) \right), \quad j \geq 2, \quad t \geq 0.
\]
We will prove that, as \( t \to \infty \),
\[
\left( \frac{|j(t)| - 1}{m\left(\frac{c_\alpha}{\nu}\right)} \sum_{k \geq 1} \left( N_{\nu(j(t)) - 1}^{(k)}(t) - V\left|_{j(t)-1}(t-T_k)\right|_1 \right) \right)_{u \geq 0} \xrightarrow{f.d.d.} \left( \Theta(u) \right)_{u \geq 0},
\]
(3.1)
where \( \Theta(u) := 0 \) for \( u > 0 \), and
\[
\left( \frac{|j(t)| - 1}{m\left(\frac{c_\alpha}{\nu}\right)} \sum_{k \geq 1} \left( V\left|_{j(t)-1}(t-T_k)\right|_1 - V\left|_{j(t)}(t)\right|_1 \right) \right)_{u \geq 0} \xrightarrow{f.d.d.} \left( \int_0^{\infty} e^{-uy} dS_\alpha(y) \right)_{u \geq 0},
\]
(3.2)
thereby showing that the asymptotics in focus is driven by the “shot-noise” part, whereas the contribution of the “martingale” part is negligible.

We start with several preparatory results needed for the proof of (3.2). Lemma 3 is a version of limit relation (1.3) with different centering.

**Lemma 3.** Under the assumptions and notation of Theorem 1, as \( t \to \infty \),
\[
\left( \frac{N(ut) - V(ut)}{m^{-1}(\nu + 1)\alpha c_\alpha(t)} \right)_{u \geq 0} \Rightarrow \left( S_\alpha(u) \right)_{u \geq 0}
\]
(3.3)
in the \( J_1 \)-topology on \( D \) if \( \alpha = 2 \) and in the \( M_1 \)-topology on \( D \) if \( \alpha \in (1, 2) \).

**Proof.** Put \( \nu(t) := \# \{ k \in \mathbb{N}_0: S_k \leq t \} \) for \( t \geq 0 \), so that \( U(t) = \mathbb{E}\nu(t) \). According to Wald’s identity, \( U(t) = m^{-1}\mathbb{E}S_\nu(t) \geq m^{-1}t \) for \( t \geq 0 \). It is shown in the proof of Corollary 1 (see a few lines preceding (2.8)) that
\[
U(t) - m^{-1}t = O\left( t^{2-\gamma} \right), \quad t \to \infty.
\]
(3.4)
As a consequence,
\[
0 \leq V(t) - m^{-1} \int_0^t \mathbb{P}\{ \eta \leq y \} dy = \int_{[0,t]} (U(t) - m^{-1}(t - y) ) d\mathbb{P}\{ \eta \leq y \}
\]
\[
= O\left( t^{2-\gamma} \right), \quad t \to \infty.
\]
Hence relation (3.3) follows from (1.3), which holds according to Remark 1, if we show that
\[
\lim_{t \to \infty} \frac{t^{2-\gamma}}{c_\alpha(t)} = 0.
\]
(3.5)
To prove (3.5), recall that the function \( c_\alpha \) is regularly varying at infinity of index \( 1/\alpha \) and that \( \gamma \) appearing in Condition PERT(\( \gamma \)) satisfies \( \gamma \in (2 - 1/\alpha, \alpha] \). Thus \( 2 - \gamma < 1/\alpha \). This justifies (3.5) and completes the proof of Lemma 3. \( \square \)

**Lemma 4.** Under the assumptions and notation of Theorem 1,
\[
\lim_{t \to \infty} \mathbb{E}\frac{|N(t) - V(t)|}{c_\alpha(t)} = m^{-(\alpha+1)/\alpha}\mathbb{E}|S_\alpha(1)|.
\]
Proof. Putting \( u = 1 \) in (3.3) yields

\[
\frac{N(t) - V(t)}{m^{-\gamma(\alpha+1)/\alpha c_\alpha(t)}} \to S_\alpha(1), \quad t \to \infty.
\]

(3.6)

Fix any \( r \in (1, \alpha) \). Suppose we can show that

\[
E|N(t) - V(t)|^r = O\left((c_\alpha(t))^r\right), \quad t \to \infty.
\]

(3.7)

Then the family \( ((N(t) - V(t))/c_\alpha(t))_{t \geq 1} \) is uniformly integrable. This, together with (3.6), is sufficient for completing the proof.

Proof of (3.7). We will use the decomposition

\[
N(t) - V(t) = \sum_{k \geq 0} \left(1_{\{\eta_{k+1} \leq t\}} - G(t - S_k)\right) + \int_{[0, t]} G(t - x) \, d\left(\nu(x) - U(x)\right),
\]

where \( G(x) := P\{\eta \leq x\}, \nu(x) = \#\{k \in \mathbb{N}_0: S_k \leq x\} \) and \( U(x) = E\nu(x) \) for \( x \geq 0 \). In view of

\[
|x + y|^r \leq 2^{r-1}(|x|^r + |y|^r), \quad x, y \in \mathbb{R},
\]

it suffices to check that

\[
E\left|\sum_{k \geq 0} \left(1_{\{\eta_{k+1} \leq t\}} - G(t - S_k)\right)^r\right| = O\left((c_\alpha(t))^r\right), \quad t \to \infty,
\]

(3.8)

and

\[
D(t) := E\left|\int_{[0, t]} G(t - x) \, d\left(\nu(x) - U(x)\right)^r\right| = O\left((c_\alpha(t))^r\right), \quad t \to \infty.
\]

(3.9)

We first prove (3.8). By Jensen’s inequality, \( E|X|^r \leq (E X^2)^{r/2} \) for any real-valued random variable \( X \). Thus (3.8) follows if we can check that

\[
E\left(\sum_{k \geq 0} \left(1_{\{\eta_{k+1} \leq t\}} - G(t - S_k)\right)^2\right) = O\left((c_\alpha(t))^2\right), \quad t \to \infty.
\]

In fact, we will prove even more, namely, that the right-hand side is \( O(c_\alpha(t)) \). The last expectation is equal to

\[
\int_{[0, t]} G(t - x) \, dU(x) \leq \int_{[0, t]} (1 - G(t - x)) \, dU(x) \sim m^{-1} E(\eta \wedge t), \quad t \to \infty,
\]

where the asymptotic relation is secured by Theorem 4 in [20]. Recall that the function \( c_\alpha \) is regularly varying at infinity of index \( 1/\alpha \). According to Condition \( \text{PERT}(\gamma) \) and (3.5),

\[
\lim_{t \to \infty} \frac{E(\eta \wedge t)}{c_\alpha(t)} = 0,
\]

which proves (3.8).
Next, we intend to prove (3.9). As has already been mentioned in the proof of Lemma 2, we can assume that \( c_\alpha \) is a nondecreasing function. Integration by parts in (3.9) followed by an application of Jensen’s inequality yields
\[
D(t) = E \left| \int_{[0,t]} (\nu(t-x) - U(t-x)) \, dG(x) \right|^r \leq \int_{[0,t]} E \left| \nu(t-x) - U(t-x) \right|^r \, dG(x).
\]
By Theorems 1.1 and 1.4 in [15]
\[
\lim_{t \to \infty} E |\nu(t) - m^{-1}t|^r = \lim_{t \to \infty} E |S_\alpha(1)|^r < \infty. \tag{3.10}
\]
Recalling (3.4) and (3.5), we conclude that
\[
\lim_{t \to \infty} \frac{U(t) - m^{-1}t}{c_\alpha(t)} = 0.
\]
This, together with (3.10), shows that
\[
\lim_{t \to \infty} \frac{E |\nu(t) - U(t)|^r}{(m^{-1}(\alpha + 1)/c_\alpha(t))^r} = \lim_{t \to \infty} E |S_\alpha(1)|^r < \infty.
\]
Modifying \( c_\alpha \) if needed in the right vicinity of 0, we infer \( E |\nu(t) - m^{-1}t|^r \leq A(c_\alpha(t))^r \) for some constant \( A > 0 \) and all \( t \geq 0 \). With this at hand,
\[
D(t) \leq \int_{[0,t]} E \left| \nu(t-x) - U(t-x) \right|^r \, dG(x) \leq A \int_{[0,t]} (c_\alpha(t-x))^r \, dG(x) = O((c_\alpha(t))^r), \quad t \to \infty,
\]
where we have used the monotonicity of \( c_\alpha \) for the last equality. \( \square \)

Lemma 5 is a slight reformulation of Lemma A.5 in [12].

**Lemma 5.** Let \( 0 \leq a < b < \infty \), and let, for each \( n \in \mathbb{N} \), \( y_n : [0, \infty) \to [0, \infty) \) be a right-continuous bounded nondecreasing function. Assume that \( \lim_{n \to \infty} x_n = x \) in the \( J_1 \)- or \( M_1 \)-topology on \( D \) and that, for each \( t \geq 0 \), \( \lim_{n \to \infty} y_n(t) = y(t) \), where \( y : [0, \infty) \to [0, \infty) \) is a bounded continuous function. Then
\[
\lim_{n \to \infty} \int_{[a,b]} x_n(t) \, dy_n(t) = \int_{[a,b]} x(t) \, dy(t).
\]

The integrals appearing in the lemma exist as Lebesgue–Stieltjes integrals and are finite because the functions \( x_n \in D \) and \( x \in D \) are measurable and locally bounded; see [5, p. 122].

We are ready to prove (3.1) and (3.2).

### 3.2 Proof of (3.1)

This proof proceeds along the lines of the proof of Theorem 3.1 in [16].

For \( j \in \mathbb{N} \) and \( t \geq 0 \), put \( D_j(t) := \text{Var} N_j(t) \) and
\[
I_j(t) := E \left( \sum_{r \geq 1} V_{j-1}(t-T_r)1_{\{T_r \leq t\}} - V_j(t) \right)^2
\]
with the convention that \( V_0(t) = 1 \) for \( t \geq 0 \). Our starting point is the following recursive formula, which is a consequence of (2.2): for \( j \geq 2 \) and \( t \geq 0 \),

\[
D_j(t) = E \left( \sum_{r \geq 1} (N_{j-1}^{(r)}(t - T_r) - V_{j-1}(t - T_r)) 1_{\{T_r \leq t\}} \right)^2 + E \left( \sum_{r \geq 1} V_{j-1}(t - T_r) 1_{\{T_r \leq t\}} - V_j(t) \right)^2
\]

\[
= \int_{[0,t]} D_{j-1}(t - y) \, dV(y) + I_j(t). \tag{3.11}
\]

Starting with \( D_1(t) = I_1(t) \) and iterating (3.11), we obtain

\[
\int_{[0,t]} D_{j-1}(t - y) \, dV(y) = \sum_{k=1}^{j-1} \int_{[0,t]} I_k(t - y) \, dV_{j-k}(y), \quad j \geq 2, \ t \geq 0. \tag{3.12}
\]

Our purpose is to show that whenever \( j = j(t) \rightarrow \infty \) and \( j(t) = o(t^{(\gamma - 1)/2}) \) as \( t \rightarrow \infty \),

\[
\int_{[0,t]} D_{j-1}(t - y) \, dV(y) = O\left( \frac{t^{2j-\gamma}}{(j-2)!(j-1)!!m^{2j-2}} \right), \quad t \rightarrow \infty. \tag{3.13}
\]

We proceed in two steps. First, we show that \( I_j \) is upper bounded by a nonnegative nondecreasing function \( h_j \), say, and that the corresponding inequality is valid for all nonnegative arguments. This leads by (3.12) to a useful inequality for \( D_j \), which holds for all nonnegative arguments. Second, we derive an upper bound for both \( h_j \) and \( D_j \), which is valid for large arguments.

**Step 1.** Throughout this step, we tacitly assume that both \( j \in \mathbb{N} \) and \( t \geq 0 \) are arbitrary. We start with

\[
E \sum_{r \geq 1} \sum_{1 \leq i < r} V_{j-1}(t - T_i) 1_{\{T_i \leq t\}} V_{j-1}(t - T_r) 1_{\{T_r \leq t\}}
\]

\[
\leq E \sum_{i \geq 1} \sum_{r \geq 1} V_{j-1}(t - T_i) 1_{\{T_i \leq t\}} \left( V_{j-1}(t - \eta_{i+1} - S_i) 1_{\{\eta_{i+1} < t - S_i\}} + V_{j-1}(t - \eta_{i+2} - S_i) 1_{\{\eta_{i+2} < t - S_i\}} + \cdots \right) \left( \xi_k, \eta_k, \xi_k \leq t - S_i \right) 1_{\{S_i \leq t\}}
\]

\[
= E \sum_{i \geq 1} V_{j-1}(t - T_i) 1_{\{T_i \leq t\}} V_j(t - S_i) 1_{\{S_i \leq t\}} \leq E \sum_{i \geq 0} V_{j-1}(t - S_i) V_j(t - S_i) 1_{\{S_i \leq t\}}.
\]

Hence

\[
I_j(t) = E \sum_{r \geq 1} V_{j-1}^2(t - T_r) 1_{\{T_r \leq t\}} + 2E \sum_{r \geq 1} \sum_{1 \leq i < r} V_{j-1}(t - T_i) 1_{\{T_i \leq t\}} V_{j-1}(t - T_r) 1_{\{T_r \leq t\}} - V_j^2(t)
\]

\[
\leq V_{j-1}(t) E \sum_{r \geq 1} V_{j-1}(t - T_r) 1_{\{T_r \leq t\}} + 2 \int_{[0,t]} V_{j-1}(t - y) V_j(t - y) \, dU(y) - V_j^2(t)
\]

\[
= V_{j-1}(t) V_j(t) + 2 \int_{[0,t]} V_{j-1}(t - y) V_j(t - y) \, dU(y) - V_j^2(t). \tag{3.14}
\]
Put $\bar{U}(t) := \sum_{i \geq 1} \mathbb{P}\{S_i \leq t\}$ for $t \geq 0$. Using $\bar{U}(t) = U(t) - 1$ for $t \geq 0$ and (3.4), we conclude that there exists a constant $\tilde{c} > 0$ such that $|\bar{U}(t) - m^{-1}t| \leq \tilde{c}(t + 1)^{2-\gamma}$ for all $t \geq 0$. With this at hand, integration by parts yields

$$
\int_{[0,t]} V_{j-1}(t-y)V_j(t-y) \, dU(y)
= V_{j-1}(t)V_j(t) + \int_{[0,t]} V_{j-1}(t-y)V_j(t-y) \, d\bar{U}(y)
= V_{j-1}(t)V_j(t) + \int_{[0,t]} \bar{U}(t-y) \, d(V_{j-1}(y)V_j(y))
\leq (\tilde{c}(t + 1)^{2-\gamma} + 1)V_{j-1}(t)V_j(t) + m^{-1} \int_{0}^{t} V_{j-1}(y)V_j(y) \, dy,
$$

whence by (3.14)

$$
I_j(t) \leq (2\tilde{c}(t + 1)^{2-\gamma} + 3)V_{j-1}(t)V_j(t) + 2m^{-1} \int_{0}^{t} V_{j-1}(y)V_j(y) \, dy - V_j^2(t)
\leq (2\tilde{c} + 3)(t + 1)^{2-\gamma}V_{j-1}(t)V_j(t) + 2m^{-1} \int_{0}^{t} V_{j-1}(y)V_j(y) \, dy - V_j^2(t).
$$

Invoking (2.3) yields

$$
2m^{-1} \int_{0}^{t} V_{j-1}(y)V_j(y) \, dy
\leq 2m^{-1} \int_{0}^{t} \left( \frac{y^{j-1}}{(j-1)!m^j} + \sum_{i=0}^{j-2} \binom{j-1}{i} \frac{C^{j-1-i}(y+1)^{2-\gamma(j-1-i)+i}}{i!m^i} \right) \times \left( \frac{y^{j-i}}{j!m^j} + \sum_{i=0}^{j-1} \binom{j}{i} \frac{C^{j-i}(y+1)^{2-\gamma(j-i)+i}}{i!m^i} \right) \, dy
\leq \frac{t^{2j}}{(j)!^2m^{2j+1}} + 2 \frac{(t + 1)^{j+1}}{j!m^{j+1}} \sum_{i=0}^{j-2} \binom{j-1}{i} \frac{C^{j-1-i}(t+1)^{2-\gamma(j-1-i)+i}}{(2-\gamma)(j-1-i) + j + 1 + i)!m^i}
+ 2 \frac{(t + 1)^{j}}{(j-1)!m^j} \sum_{i=0}^{j-1} \binom{j}{i} \frac{C^{j-i}(t+1)^{2-\gamma(j-i)+i}}{(2-\gamma)(j-i) + j + i)!m^i}
+ 2 \left( \sum_{i=0}^{j-2} \binom{j-1}{i} \frac{C^{j-1-i}(t+1)^{2-\gamma(j-1-i)+i}}{i!m^i} + \frac{t-1}{j-1} \right) \frac{t^{j-1}}{j} \sum_{i=0}^{j-1} \binom{j}{i} \frac{C^{j-i}(y+1)^{2-\gamma(j-i)+i}}{i!m^{i+1}} \, dy.
$$
Appealing to (2.3) once again, we obtain

$$V_j^2(t) - \frac{t^{2j}}{(j!)^2m^2j} = \left(V_j(t) + \frac{t^j}{j!m^j}\right)\left(V_j(t) - \frac{t^j}{j!m^j}\right)$$

$$\geq -\left(V_j(t) + \frac{t^j}{j!m^j}\right)\sum_{i=0}^{j-1} \left(\frac{j^j}{i!m^i}\right) \frac{C^{j-i}(t+1)^{(2-\gamma)(j-i)+i}}{(i+1)!m^{i+1}}$$

$$= -\left(V_j(t) + \frac{t^j}{j!m^j}\right)g_j(t) := -\tilde{g}_j(t).$$

Note that both $\tilde{f}_j$ and $\tilde{g}_j$ are nonnegative nondecreasing functions. Summarizing, we have

$$I_j(t) \leq (2\tilde{c} + 3)(t + 1)^{2-\gamma}V_j^2(t)V_j(t) + \tilde{f}_j(t) + \tilde{g}_j(t) =: \tilde{h}_j(t). \quad (3.15)$$

Since $\tilde{h}_j$ is a nondecreasing function, we further infer

$$D_{j-1}(t) = \sum_{k=1}^{j-1} \int_{[0,t]} I_k(t - y) dV_{j-k-1}(y)$$

$$\leq \tilde{h}_{j-1}(t) + \sum_{k=1}^{j-2} \tilde{h}_k(t)V_{j-k-1}(t), \quad j \geq 2, \ t \geq 0.$$

**Step 2.** Fix now $j \in \mathbb{N}$ and $s \geq 0$ satisfying $(s + 1)^{\gamma-1} \geq 2Cm_j$ and let $1 \leq k \leq j$, where $C$ is the same as in (2.3). Throughout this step, we tacitly assume that all formulae hold for this range of parameters.

By (2.5) we have

$$V_{k-1}(s)V_k(s) \leq \frac{4(s + 1)^{2k-1}}{(k - 1)!k!m^{2k-1}} \leq \frac{4(s + 1)^{2k-1}}{((k - 1)!)^2m^{2k-1}}.$$

Next, we show that

$$\tilde{f}_k(s) \leq \frac{12C(s + 1)^{2k+1-\gamma}}{((k - 1)!)^2m^{2k-1}}.$$
Indeed, according to (2.6),
\[
2 \frac{(s + 1)^{k+1}}{(k + 1)!m^{k+1}} \sum_{i=0}^{k-2} \binom{k-1}{i} \frac{C^{k-1-i}(s + 1)^{(2-\gamma)(k-1-i)+i}}{i!m^i} \\
\leq 2 \frac{(s + 1)^{k+1}}{(k + 1)!m^{k+1}} \frac{2C(k-1)(s + 1)^{k-\gamma}}{(k - 2)!m^{k-2}} \leq 4C(s + 1)^{2k+1-\gamma}/((k - 1)!)^2m^{2k-1}.
\]

Analogously,
\[
2 \frac{(s + 1)^{k}}{k!m^{k}} \sum_{i=0}^{k-1} \binom{k}{i} \frac{C^{k-i}(s + 1)^{(2-\gamma)(k-i)+i}}{i!m^i} \leq 4C(s + 1)^{2k+1-\gamma}/((k - 1)!)^2m^{2k-1}.
\]

Finally, the third summand in the definition of $\tilde{f}_k$ can be treated as follows:
\[
\begin{align*}
2 \left( \sum_{i=0}^{k-2} \binom{k-1}{i} \frac{C^{k-1-i}(s + 1)^{(2-\gamma)(k-1-i)+i}}{i!m^i} \right) \left( \sum_{i=0}^{k-1} \binom{k}{i} \frac{C^{k-i}(s + 1)^{(2-\gamma)(k-i)+i+1}}{(i+1)!m^{i+1}} \right) \\
\leq 2 \frac{2C(k-1)(s + 1)^{k-\gamma}}{(k - 2)!m^{k-2}} \frac{2C(s + 1)^{k+2-\gamma}}{(k - 1)!m^{k}} = \frac{8C^2(s + 1)^{2k+2-2\gamma}}{((k - 2)!)^2m^{2k-2}} \leq \frac{4C(s + 1)^{2k+1-\gamma}}{((k - 1)!)^2m^{2k-1}}.
\end{align*}
\]

Here we have used (2.6) and (2.7) to bound the first and second factors, respectively, and the inequality $(s + 1)^{\gamma-1} \geq 2Cm(k - 1)^2$ for the last passage. Finally,
\[
\tilde{g}_k(s) \leq \frac{6C(s + 1)^{2k+1-\gamma}}{((k - 1)!)^2m^{2k-1}}
\]

by (2.5) and (2.6). Summarizing, we have shown that
\[
\tilde{h}_k(s) \leq \frac{A(s + 1)^{2k+1-\gamma}}{((k - 1)!)^2m^{2k-1}},\tag{3.16}
\]

where $A := 12 + 8\delta + 18C$.

Further, for $s$ satisfying $(s + 1)^{\gamma-1} \geq 2\max(c, 1)m^2 =: a_j$, where $c$ is as given in (2.8), we obtain
\[
D_{j-1}(s) \leq \tilde{h}_{j-1}(s) + \sum_{k=1}^{j-2} \tilde{h}_k(s)V_{j-k-1}(s) \\
\leq \frac{A(s + 1)^{2j-1-\gamma}}{((j - 2)!)^2m^{2j-3}} + 2A \sum_{k=1}^{j-2} \frac{(s + 1)^{j+k-\gamma}}{(j - k - 1)!(k - 1)!^2m^{j+k-2}} \\
= \frac{A(s + 1)^{2j-1-\gamma}}{((j - 2)!)^2m^{2j-3}} \left( 1 + 2 \sum_{k=1}^{j-2} \frac{(j - 2)!}{k - 1}(k - 1)!(s + 1)^{j-k-1} \right) \\
\leq \frac{A(s + 1)^{2j-1-\gamma}}{((j - 2)!)^2m^{2j-3}} \left( 1 + 2 \frac{m^2}{s + 1} \left( 1 - \frac{m^2}{s + 1} \right)^{-1} \right) \leq \frac{3A(s + 1)^{2j-1-\gamma}}{((j - 2)!)^2m^{2j-3}}.\tag{3.17}
\]

Here the first inequality is just formula (3.15), the second inequality is implied by (2.5) and (3.16), and the third inequality is justified by (2.9).
Assume now that \( j = j(t) \to \infty \) and \( j(t) = o(t^{(\gamma - 1)/2}) \) as \( t \to \infty \), so that \( t \geq a_j \) for \( t \) large enough. We intend to prove (3.13). To this end, we write

\[
\int_{[0, t]} D_{j-1}(t - y) \, dV(y) = \int_{[0, t-a_j]} D_{j-1}(t - y) \, dV(y) + \int_{(t-a_j, t]} D_{j-1}(t - y) \, dV(y)
\]

\[
\leq \frac{3A}{((j-2)!)^2 m^{2j-3}} \int_{[0, t]} (t + 1 - y)^{2j-1-\gamma} \, dV(y) + \left( \max_{s \in [0, a_j]} D_{j-1}(s) \right) U(a_j)
\]

\[
\leq \frac{3A(t+1)^{2j-\gamma}}{((j-2)!)^2 (2j-\gamma) m^{2j-2}} + \frac{3A(c+1)^{2j+1-2\gamma}}{((j-2)!)^2 m^{2j-3}} + \left( \max_{s \in [0, a_j]} D_{j-1}(s) \right) U(a_j),
\]

where we used (3.17) and the inequality \( V(x + y) - V(x) \leq U(y) \) for \( x, y \in \mathbb{R} \) (for the proof, see formula (40) in [7]) for the first inequality and integration by parts together with (2.8) for the second. The asymptotic relation

\[
\frac{3A(t+1)^{2j+1-2\gamma}}{((j-2)!)^2 m^{2j-3}} = o\left( \frac{t^{2j-\gamma}}{(j-2)! (j-1)! m^{2j-2}} \right), \quad t \to \infty,
\]

is a consequence of \( j(t) = o(t^{\gamma - 1}) \) as \( t \to \infty \). Using (2.5) and (3.17) for the second inequality below, we further obtain

\[
\left( \max_{s \in [0, a_j]} D_{j-1}(s) \right) U(a_j) \leq (D_{j-1}(a_j) + V_{j-1}^2(a_j)) U(a_j)
\]

\[
\leq \left( \frac{3A(a_j+1)^{2j-1-\gamma}}{((j-2)!)^2 m^{2j-3}} + \frac{4(a_j+1)^{2j-2}}{((j-1)!)^2 m^{2j-2}} \right) U(a_j).
\]

By the elementary renewal theorem, with \( a = 2 \max(c, 1)m \),

\[
\frac{(j-2)! (j-1)! m^{2j-2}}{t^{2j-\gamma}} \frac{(a_j+1)^{2j-2}}{(j-1)! m^{2j-2}} U(a_j) \sim \left( \frac{a_j^2}{t} \right)^{2j-\gamma} a_j \frac{1}{m (a_j^2)^{2-\gamma}} \to 0, \quad t \to \infty,
\]

because \( \lim_{t \to \infty} j^b (a_j^2/t)^{2j-2} = 0 \) for any \( b > 0 \). The last two limit relations hold whenever \( j(t) = o(t^{1/2}) \) and particularly under the assumption \( j(t) = o(t^{(\gamma - 1)/2}) \). Analogously,

\[
\frac{(j-2)! (j-1)! m^{2j-2}}{t^{2j-\gamma}} \frac{(a_j+1)^{2j-1-\gamma}}{(j-2)! m^{2j-3}} U(a_j) \sim \left( \frac{a_j^2}{t} \right)^{2j-\gamma} j \to 0, \quad t \to \infty.
\]

Thus

\[
\left( \max_{s \in [0, a_j]} D_{j-1}(s) \right) U(a_j) = o\left( \frac{t^{2j-\gamma}}{(j-2)! (j-1)! m^{2j-2}} \right),
\]

and (3.13) follows.

According to the Cramér–Wold device and Markov inequality, relation (3.1) follows if we can show that, for any fixed \( u > 0 \),

\[
\lim_{t \to \infty} \left( \left( \frac{[(j(t)u] - 1)!}{t^2 (j(t)u)} - 2c_2^2 (\alpha_j(t)) \right) \mathbb{E}(N_{[j(t)u]}(t) - \sum_{r \geq 1} V_{[j(t)u]}(t - T_r) 1_{\{T_r \leq t\})}^2 = 0. \quad (3.18)
\]

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In view of (3.11) and (3.13), the left-hand side is the big $O$ of
\[ \frac{j^{l^2-\gamma}}{c_\alpha(\frac{t}{j})} = \frac{j^2}{t^\gamma} c_\alpha(\frac{t}{j}). \]

The first factor on the right-hand side is $o(1)$ by assumption. We claim that
\[ \lim_{x \to \infty} x^{-1} c_\alpha^2(x) = \infty, \tag{3.19} \]
so that the second factor on the right-hand side also is $o(1)$, which proves (3.18).

To check (3.19), recall that the function $c_\alpha$ is regularly varying at infinity of index $1/\alpha$, which particularly entails
\[ \lim_{x \to \infty} c_\alpha(x) = \infty. \tag{3.20} \]

In the case $\alpha \in (1, 2)$ the regular variation implies (3.19). Assume now that $\alpha = 2$. Then $c_2$ satisfies $\ell(c_2(x)) \sim x^{-1} c_2^2(x)$ as $x \to \infty$, where $\ell$ is a slowly varying diverging to infinity function; see (1.1). Recalling (3.20), we infer (3.19) with $\alpha = 2$.

### 3.3 Proof of (3.2)

In what follows, we write $j$ for $j(t)$. According to the Cramér–Wold device, it suffices to show that for any $r \in \mathbb{N}$, any real $\alpha_1, \ldots, \alpha_r$, and any $0 < u_1 < \cdots < u_r < \infty$, as $t \to \infty$,
\[ \sum_{i=1}^{r} \alpha_i \left(\lfloor ju_i \rfloor - 1\right)! m^{\lfloor ju_i \rfloor + 1/\alpha} Z(j u_i, t) \frac{d}{d t} \sum_{i=1}^{r} \alpha_i u_i \int_{0}^{\infty} S_\alpha(y) e^{-u_i y} d y, \]

where
\[ Z(j u, t) := \sum_{k \geq 1} V_{[ju] - 1}(t - T_k) 1_{\{T_k \leq t\}} - V_{[ju]}(t), \quad u > 0. \]

For any $u, T > 0$ and $t$ sufficiently large,
\[ \frac{(\lfloor ju \rfloor - 1)! m^{\lfloor ju \rfloor + 1/\alpha} Z(j u, t)}{t^{\lfloor ju \rfloor - 1} c_\alpha(\frac{t}{j})} \]
\[ = \frac{(\lfloor ju \rfloor - 1)! m^{\lfloor ju \rfloor + 1/\alpha}}{t^{\lfloor ju \rfloor - 1} c_\alpha(\frac{t}{j})} \int_{[0, t]} V_{[ju] - 1}(t - y) d (N(y) - V(y)) \]
\[ = \frac{(\lfloor ju \rfloor - 1)! m^{\lfloor ju \rfloor - 1}}{t^{\lfloor ju \rfloor - 1} c_\alpha(\frac{t}{j})} \int_{[0, T]} \frac{N(u t)}{m^{-(\alpha + 1)/\alpha} c_\alpha(\frac{t}{j})} d y \left( -V_{[ju] - 1} \left( t \left( 1 - \frac{y}{j} \right) \right) \right) \]
\[ + \frac{(\lfloor ju \rfloor - 1)! m^{\lfloor ju \rfloor + 1/\alpha}}{t^{\lfloor ju \rfloor - 1} c_\alpha(\frac{t}{j})} \int_{(T t/j, t]} (N(y) - V(y)) d y \left( -V_{[ju] - 1}(t - y) \right). \]

By Lemma 3
\[ \left( \frac{N(u t)}{m^{-(\alpha + 1)/\alpha} c_\alpha(\frac{t}{j})} \right)_{u \geq 0} \Rightarrow (S_\alpha(u))_{u \geq 0} \]
in the $J_1$-topology on $D$ if $\alpha = 2$ and in the $M_1$-topology on $D$ if $\alpha \in (1, 2)$. Here we have used the assumption $t/j(t) \to \infty$. By Skorokhod’s representation theorem there exist versions $\hat{N}_t$ and $\hat{S}_\alpha$ of $((N(ut/j) - V(ut/j)) / (m^{-\alpha}/c_\alpha(t/j)))_{u \geq 0}$ and $S_\alpha$, respectively, such that

$$\lim_{t \to \infty} \hat{N}_t(y) = \hat{S}_\alpha(y) \quad \text{a.s.}$$

(3.21)

in the $J_1$-topology on $D$ if $\alpha = 2$ and in the $M_1$-topology on $D$ if $\alpha \in (1, 2)$. In view of (2.12),

$$\lim_{t \to \infty} \frac{\lfloor j u \rfloor - 1!m_{\lfloor j u \rfloor - 1}}{t_{\lfloor j u \rfloor - 1}} V_{\lfloor j u \rfloor - 1}\left(t\left(1 - \frac{y}{j}\right)\right) = e^{-uy}, \quad t \to \infty,$$

for each fixed $y \geq 0$. By Lemma 5 this in combination with (3.21) yields

$$\lim_{t \to \infty} \sum_{i=1}^r \alpha_i \frac{\lfloor j u_i \rfloor - 1!m_{\lfloor j u_i \rfloor - 1}}{t_{\lfloor j u_i \rfloor - 1}} \int_0^T \hat{N}_t(y) \, d\hat{S}_\alpha(y) e^{-uy} \, dy \quad \text{a.s.,}$$

and therefore

$$\sum_{i=1}^r \alpha_i \frac{\lfloor j u_i \rfloor - 1!m_{\lfloor j u_i \rfloor - 1}}{t_{\lfloor j u_i \rfloor - 1}} \int_{[0, T]} N(\tfrac{u(t)}{j}) - V(\tfrac{u(t)}{j}) \, d\hat{S}_\alpha(y) e^{-uy} \, dy$$

$$= \sum_{i=1}^r \alpha_i u_i \int_0^T S_\alpha(y) e^{-uy} \, dy$$

as $t \to \infty$. Since

$$\lim_{T \to \infty} \sum_{i=1}^r \alpha_i u_i \int_0^T S_\alpha(y) e^{-uy} \, dy = \sum_{i=1}^r \alpha_i u_i \int_0^\infty S_\alpha(y) e^{-uy} \, dy \quad \text{a.s.,}$$

we are left with proving that

$$\lim_{T \to \infty} \limsup_{t \to \infty} P\left\{\sum_{i=1}^r \alpha_i \frac{\lfloor j u_i \rfloor - 1!m_{\lfloor j u_i \rfloor + 1}/c_\alpha(t/j)}{t_{\lfloor j u_i \rfloor - 1}c_\alpha(t/j)} \int_{[Tj/i, t]} (N(y) - V(y)) \, d\hat{S}_\alpha(y) e^{-uy} \, dy \right\} > \varepsilon = 0$$

for all $\varepsilon > 0$. By Lemma 4, $E|N(y) - V(y)| \sim m^{-(\alpha+1)/\alpha} E|S_\alpha(1)/c_\alpha(y)$ as $y \to \infty$. With this at hand, the last limit relation follows from Markov’s inequality and (2.13). The proof of (3.2) is complete.

**Remark 3.** We do not know whether the weak convergence of the finite-dimensional distributions in Theorem 1 can be strengthened to the weak convergence on the Skorokhod space $D(0, \infty)$ equipped with the $J_1$-topology. To ensure this, it suffices to show that relations (3.1) and (3.2) hold in the sense of weak convergence on $D(0, \infty)$. It is not a big deal to prove the required strengthening of (3.2). However, we were able neither prove nor disprove the aforementioned functional version of (3.1).
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