The edge-connectivity of vertex-transitive hypergraphs

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Abstract
A graph or hypergraph is said to be vertex-transitive if its automorphism group acts transitively upon its vertices. A classic theorem of Mader asserts that every connected vertex-transitive graph is maximally edge-connected. We generalise this result to hypergraphs and show that every connected linear uniform vertex-transitive hypergraph is maximally edge-connected. We also show that if we relax either the linear or uniform conditions in this generalisation, then we can construct examples of vertex-transitive hypergraphs which are not maximally edge-connected.

KEYWORDS
connectivity, hypergraphs, vertex-transitivity

MATHEMATICAL SUBJECT CLASSIFICATION
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1 | INTRODUCTION

A graph or hypergraph is connected if there is a path connecting each pair of vertices, where a path is a sequence of alternating incident vertices and edges without repetition. A cut set of edges in a graph or hypergraph is a set of edges whose deletion renders the graph or hypergraph disconnected. The edge-connectivity of a graph or hypergraph $H$ is the size of a minimum cut set of edges and is denoted $\kappa'(H)$. For a graph or hypergraph $H$, $\delta(H)$ is the minimum degree...
among the vertices and \( \Delta(H) \) is the maximum degree among the vertices, where the degree of a vertex is the number of edges incident with it.

In [9], Whitney observes that, for a graph \( G \), \( \kappa'(G) \) never exceeds \( \delta(G) \), a result which extends naturally to hypergraphs. This bound is in fact tight and a graph or hypergraph \( H \) which satisfies \( \kappa'(H) = \delta(H) \) is said to be maximally edge-connected. Hellwig and Volkmann list several sufficient conditions for graphs to be maximally edge-connected in their 2008 survey [5].

The subject of connectivity in hypergraphs has been developing recently with results like those in [1, 3, 6]. In [1], Bahmanian and Šajna study various connectivity properties in hypergraphs with an emphasis on cut sets of edges and vertices. In [3], Dewar, Pike, and Proos consider both vertex and edge-connectivity in hypergraphs with additional details on the computational complexity of these problems. In [6], Jami and Szigeti investigate the edge-connectivity of permutation hypergraphs. Dankelmann and Meierling extend several well-known sufficient conditions for graphs to be maximally edge-connected to the realm of hypergraphs in [2]. Tong and Shan continue this work with more extensions from graphs to hypergraphs in [8]. Zhao and Meng present sufficient conditions for linear uniform hypergraphs to be maximally edge-connected that generalise results from graphs in [10]. These three papers were primarily focused on the properties of distance and girth.

In this paper, we investigate the edge-connectivity of vertex-transitive hypergraphs which are linear and uniform. A graph or hypergraph \( H \) is said to be vertex-transitive if, for any two vertices \( u \) and \( v \) of \( V(H) \), there exists some automorphism \( \phi \) of \( H \) such that \( \phi(u) = v \). Note that any vertex-transitive graph or hypergraph must also be regular, and so \( \delta(H) = \Delta(H) \). A linear hypergraph is one in which any pair of vertices is contained in at most one edge. A uniform hypergraph is one in which each edge has the same cardinality; moreover, if each edge has cardinality \( k \), then we say that the hypergraph is \( k \)-uniform.

A classic result of Mader establishes the edge-connectivity of vertex-transitive graphs.

**Theorem 1** (Mader [7]). Let \( G \) be a vertex-transitive and connected graph. Then \( G \) is maximally edge-connected.

Our main result is a generalisation of Mader’s Theorem (Theorem 1) to linear uniform hypergraphs. In particular, we show the following:

**Theorem 2.** Let \( H \) be a linear \( k \)-uniform hypergraph with \( k \geq 3 \). If \( H \) is vertex-transitive and connected, then \( H \) is maximally edge-connected.

In Section 2 we demonstrate the existence of vertex-transitive hypergraphs which fail to be maximally edge-connected when we relax either the uniformity or linearity conditions of Theorem 2. In Section 3 we present the proof of Theorem 2.

### 2 | NONUNIFORM AND NONLINEAR HYPERGRAPHS

In this section, we present two examples of vertex-transitive hypergraphs which are not maximally edge-connected. Both examples meet all of the criteria of the hypothesis of Theorem 2 except for linearity in the first case and uniformity in the second.
2.1 Uniform but nonlinear hypergraphs

For \( k \geq 3 \), let \( H \) be the complete \( k \)-uniform hypergraph on \( n \) vertices, that is, \( V(H) \) consists of \( n \) vertices and \( E(H) \) is equal to the set of all \( k \)-subsets of \( V(H) \). Then \( H \) is a connected \( k \)-uniform hypergraph which is simple but nonlinear, where a simple hypergraph is one with no repeated edges and no loops. For any two vertices \( u \) and \( v \), there exists an automorphism \( \phi \) such that \( \phi(u) = v, \phi(v) = u \) and \( \phi(w) = w \) for any other vertex \( w \). Therefore \( H \) is also vertex-transitive.

Now let \( H_1, H_2, \ldots, H_k \) be distinct copies of \( H \), each with its own vertex set \( V(H_i) = \{ v_i \} \). Take \( H^* \) to be the union of these copies along with \( n \) edges of the form \( E_v = \{(v_1, 1), (v_2, 2), \ldots, (v_k, k)\} \) (one for each vertex \( v \in V(H) \)). Then \( H^* \) is a connected \( k \)-uniform hypergraph which is simple but nonlinear.

Now we must verify that \( H^* \) is vertex-transitive. For any two vertices within the same copy of \( H \), we can find an automorphism \( \phi \) of \( H^* \) similar to the ones described for \( H \); for example, to map \( u(1) \) to \( v(1) \), use the map \( \phi(u) = v, \phi(v) = u \) and \( \phi(w) = w \) when \( w \notin \{u, v\} \).

For any two vertices within an edge of the form \( E_v \), simply take an automorphism \( \psi \) of \( H^* \) which exchanges the two corresponding copies of \( H \) and fixes the rest; for example, to map \( (v_1, 1) \) to \( (v_1, 2) \), use the map \( \psi(u) = v, \psi(v) = u \) and \( \psi(w) = w \) when \( i \notin \{1, 2\} \).

Finally, for any two vertices in general, we may take a composition (if needed) of the two types of automorphisms we have just described. Therefore, \( H^* \) is a vertex-transitive hypergraph. However, so long as \( n \geq k + 2 \) and \( k \geq 3 \),

\[
\kappa'(H^*) \leq n < \left(\begin{array}{c} n-1 \\ k-1 \end{array}\right) + 1 = \Delta(H) + 1 = \Delta(H^*)
\]

and so \( H^* \) is not maximally edge-connected.

2.2 Linear but nonuniform hypergraphs

To construct an example of a vertex-transitive hypergraph that is linear but nonuniform, we rely on a well-known example from combinatorial designs, a finite affine plane. A finite affine plane of order \( n \) is a set of \( n^2 + n \) lines on \( n^2 \) points such that each line contains \( n \) points and each point lies on \( n + 1 \) lines. Additionally, each pair of points lies on a unique line and the lines of an affine plane can be partitioned into \( n + 1 \) equivalence classes under the equivalence relation of parallelism; we will refer to these classes as parallel classes. We give a direct construction of a finite affine plane of prime order as follows.

Let \( k \) be an odd prime and form a \( k \) by \( k \) array \( A \) such that the entry in row \( i \) and column \( j \) is \( a_{ij} = (i - 1)k + j \), where \( i, j \in \{1, 2, \ldots, k\} \).

Let the first parallel class \( \Pi_0 \) be the set of all rows of \( A \), that is,
\[ \Pi_0 = \{ \{1, \ldots, k\}, \{k+1, \ldots, 2k\}, \ldots, \{(k-1)k+1, \ldots, k^2\}\}. \]

For each \(i = 1, \ldots, k\), form the lines of parallel class \(\Pi_i\) by selecting a point from row 1 and \(k-1\) other points, one from each subsequent row, such that each subsequent point is located \((i - 1)\) cells to the right of the last point (wrapping around if necessary). Repeat this process for each point in row 1 to form all \(k\) lines of parallel class \(\Pi_i\). Precisely, \(\Pi_i\) is the collection of lines \(\mathcal{B}_{ij}\), with \(j = 1, 2, \ldots, k\) such that each line is a set of points \(\mathcal{B}_{ij} = \{tk + s \mid t \in \{0, 1, \ldots, k-1\}\}\), where \(s\) is the unique integer between 1 and \(k\) inclusive for which \(s \equiv (i-1)t + j \pmod{k}\).

Now let \(H\) be the \(k\)-uniform hypergraph with vertex set \(V(H) = \{1, 2, \ldots, k^2\}\) and edge set \(E(H) = \bigcup_{i=1}^{k} \Pi_i\). Note that we have intentionally left out the class \(\Pi_0\). To verify that \(H\) is vertex-transitive, let \(x\) and \(y\) be two vertices of \(H\). Find the parallel class among \(\Pi_0, \Pi_1, \ldots, \Pi_k\) which contains the pair \(xy\) in a line together and write the lines of this class in order as a permutation \(\sigma\). For example, if \(x\) and \(y\) are both contained in the line \(\mathcal{B}_{ij}\), then
\[
\sigma = (ii, ii, ii, ii, iii, iii, iii, iii, \ldots, i, ii, ii, ii, ii, i, i, i),
\]
where \(L_i, L_ii, \ldots, L_{i,k}\) is a list of the points of \(\mathcal{B}_{i,\ell}\) written in a fixed order as a permutation. Note that one of \(\sigma, \sigma^2, \ldots, \sigma^{k-1}\) is an automorphism in \(H\) which maps \(x\) to \(y\) (and preserves the parallel classes).

Now take a copy of \(H\) (denoted \(H'\)) on the vertex set \(\{1', 2', \ldots, (k^2)\}\) with edges corresponding to those of \(H\). Using the parallel class \(\Pi_0\), form \(k\) additional edges of size \(k^2\) as follows. For each \(i \in \{1, 2, \ldots, k\}\), let \(e_i\) be the edge containing the \(k\) vertices of the \(i\)th row of \(A\) along with the corresponding vertices in \(H'\). In particular, for each \(i \in \{1, 2, \ldots, k\}\),
\[
e_i = \{(i-1)k+1, (i-1)k+2, \ldots, (i-1)k+k, ((i-1)k+1)', ((i-1)k+2)', \ldots, ((i-1)k+k)\}'.
\]

Then take the union of \(H, H'\), and the \(k\) edges of the form \(e_i\), each of size \(k^2\), to form the hypergraph \(H^*\). Note that \(H^*\) is a connected linear nonuniform hypergraph with edges of sizes \(k\) and \(2k\). By composing the automorphisms described for \(H\) with the automorphism which maps each vertex of \(H\) to its copy in \(H'\), we can verify that \(H^*\) is also vertex-transitive. However the edge-connectivity \(\kappa'(H^*) = k\) whereas the degree \(\Delta(H^*) = k + 1\), and so \(H^*\) is not maximally edge-connected.

3 | A GENERALISATION OF MADER’S THEOREM

Let \(H\) be a hypergraph with vertex set \(V(H)\). For \(Y \subseteq V(H)\), we let \(\delta(Y)\) denote the set of edges in \(H\) in which each edge has at least one vertex in \(Y\) and at least one vertex in \(V \setminus Y\). A key part of the proof of our main theorem is the following lemma.

**Lemma 3.** Let \(H\) be a \(k\)-uniform hypergraph and \(X, Y \subseteq V(H)\). Then
\[
|\delta(X \cup Y)| + |\delta(X \cap Y)| \leq |\delta(X)| + |\delta(Y)|.
\]

**Proof.** In a Venn diagram of two (possibly intersecting) sets, there are four distinct regions. For our subsets \(X\) and \(Y\), these are \(X \setminus Y, Y \setminus X, X \cap Y\) and \((X \cup Y)^C\). Any edges that contain vertices in more than one of these regions will contribute to the values of \(|\delta(X \cup Y)| + |\delta(X \cap Y)|\) and \(|\delta(X)| + |\delta(Y)|\).
When \( k = 2 \), we have \( \binom{4}{2} = 6 \) pairs of regions and hence, six types of relevant edges which may exist. By checking each pair of regions, we see that \( |\partial(X)| + |\partial(Y)| \) accounts for all of the edges of \( |\partial(X \cup Y)| + |\partial(X \cap Y)| \) but counts any edges with vertices in both \( X \setminus Y \) and \( Y \setminus X \) twice, whereas \( |\partial(X \cup Y)| + |\partial(X \cap Y)| \) does not count these edges at all.

When \( k = 3 \), we have \( \binom{4}{3} = 4 \) additional types of possible edges. Then \( |\partial(X)| + |\partial(Y)| \) accounts for all of the edges of \( |\partial(X \cup Y)| + |\partial(X \cap Y)| \) but counts any edges with vertices in both \( X \setminus Y \) and \( Y \setminus X \) twice, whereas \( |\partial(X \cup Y)| + |\partial(X \cap Y)| \) counts these edges at most once.

When \( k \geq 4 \), there is only one additional type of possible edge, one that contains vertices from all four regions. Such edges are contained in each of \( \partial(X) \), \( \partial(Y) \), \( \partial(X \cup Y) \) and \( \partial(X \cap Y) \), and so they are counted twice by both \( |\partial(X)| + |\partial(Y)| \) and \( |\partial(X \cup Y)| + |\partial(X \cap Y)| \).

We now proceed with the proof of our main result. Note that the examples detailed in Section 2 imply the necessity of the linear and uniform conditions in the statement of this result.

**Theorem 2.** Let \( H \) be a linear \( k \)-uniform hypergraph with \( k \geq 3 \). If \( H \) is vertex-transitive and connected, then \( H \) is maximally edge-connected.

**Proof.** Since \( \chi'(H) \leq \Delta(H) \), it suffices to show that \( \chi'(H) \geq \Delta(H) \). Choose a proper nonempty subset \( X \subset V(H) \) such that

(i) \( |\partial(X)| \) is minimum and
(ii) \( |X| \) is minimum (subject to (i)).

Note that by condition (i), \( |\partial(X)| = \chi'(H) \), so it suffices to show that \( |\partial(X)| \geq \Delta(H) \). By definition \( \partial(X) = \partial(V(H) \setminus X) \), so condition (ii) implies that \( |X| \leq \frac{1}{2} |V(H)| \). In [4] such a set \( X \) is referred to as an edge atom.

Now suppose there exists \( \phi \in \text{Aut}(H) \) such that \( \emptyset \neq X \cap \phi(X) \neq X \). Then by Lemma 3,

\[
|\partial(X \cup \phi(X))| + |\partial(X \cap \phi(X))| \leq |\partial(X)| + |\partial(\phi(X))| = 2|\partial(X)|.
\]

If \( |\partial(X \cup \phi(X))| < |\partial(X)| \) then the set \( X \cup \phi(X) \) contradicts our choice of \( X \) by condition (i). Otherwise, \( |\partial(X \cap \phi(X))| \leq |\partial(X)| \), but then \( X \cap \phi(X) \) contradicts our choice of \( X \) by condition (i) or (ii). Therefore, for every \( \phi \in \text{Aut}(H) \), either \( X \cap \phi(X) = X \) or \( X \cap \phi(X) = \emptyset \). For this reason, we say that \( X \) is a block of imprimitivity (for more information on this terminology, see [4]). This proof so far has loosely followed the proof of Mader’s Theorem found in [4], however, to proceed from here we must make use of original techniques.

Now, for \( Y \subset V(H) \), we let \( \partial_i(Y) \) denote the set of edges in \( H \) in which each edge has exactly \( i \) vertices in \( Y \) and \( k - i \) vertices in \( V \setminus Y \). Note that \( \partial(Y) = \bigcup_{i=1}^{k-1} \partial_i(Y) \). For any \( x \in X \) and \( 1 \leq i \leq k \), let \( a_i \) be the number of neighbours of \( x \) in \( X \) which occur in edges of \( \partial_i(X) \). Similarly, let \( b_i \) be the number of neighbours of \( x \) in \( V \setminus X \) which occur in edges
of $\delta(X)$. Since $X$ is a block of imprimitivity, the values of $a_i$ and $b_i$ for $1 \leq i \leq k$ do not depend on the choice of $x \in X$.

If $|X| = 1$, then $|\delta(X)| = \Delta(H)$, so from now on we assume $|X| \geq 2$. Let $x, y \in X$ and note that

$$|\delta(X \setminus \{y\})| = |\delta(X)| + \frac{a_k}{k-1} - \frac{b_1}{k-1}.$$ 

So, if $a_k \leq b_1$ then $X \setminus \{y\}$ contradicts our choice of $X$. Otherwise we assume $a_k > b_1$ which implies $\delta_k(X)$ is nonempty.

For the remainder of the proof, we will refer to an edge contained in the set $\delta_k(X)$ as a $\partial_k$-edge. If $|X| = k$ then $X$ is simply a single $\partial_k$-edge. Then by linearity, the only edges of $\delta(X)$ are $\partial_1$-edges and by vertex transitivity, $|\delta(X)| = k(\Delta(H) - 1)$. Now $|\delta(X)| = k(\Delta(H) - 1)$ is strictly greater than $|\delta(\{x\})| = \Delta(H)$ as long as $k \geq 3$ and $\Delta(H) \geq 2$. But this is easy to confirm as a connected hypergraph $H'$ with $\Delta(H') = 1$ would be a single edge of $k$ vertices. So $\{x\}$ contradicts our choice of $X$ by condition (ii). Hence $|X|$ must be strictly greater than $k$.

Now since $a_k \neq 0$ and $X$ is a block of imprimitivity, every vertex of $X$ must be incident with at least one $\partial_k$-edge. Then the collection of $\partial_k$-edges is either a collection of nonintersecting edges or a collection of edges in which each vertex of $X$ lies at the intersection of at least two of these edges. In the first case, there must be paths in $H$ connecting the disjoint $\partial_k$-edges. But then any one of the $\partial_k$-edges would be a better choice for our set $X$ by condition (ii).

Therefore, we know that each vertex of $X$ lies at the intersection of at least two $\partial_k$-edges. For $x \in X$, let $r_x$ be the number of $\partial_k$-edges within $X$ which contain $x$. Observe that $r_x = \frac{a_k}{k-1}$ and so $r_x$ does not depend on our choice of $x$. So we will simply use $r$ to denote the number of $\partial_k$-edges within $X$ which contain any given vertex of $X$. Observe that the degree of $H$, $\Delta(H)$, must be strictly greater than $r$, since otherwise every neighbour of any vertex in $X$ must also be a vertex of $X$ and therefore either $H$ is disconnected or $X = V(H)$.

In addition, we note that $\Delta(H)$ must be strictly greater than $|\delta(X)|$, since otherwise $\kappa'(H) = |\delta(X)| = \Delta(H)$. Also $|\delta(X)| \geq \frac{|X|(\Delta(H) - r)}{k-1}$, since the edges of $\partial(X)$ can be shared by at most $k - 1$ vertices of $X$. Therefore,

$$\Delta(H) > \frac{|X|(\Delta(H) - r)}{k-1},$$

rearranging for $|X|$ gives a strict upper bound

$$|X| < \frac{\Delta(H)(k-1)}{\Delta(H) - r}.$$

Observe that $X$ contains the vertex $x$ and at least $r(k - 1)$ other vertices. So

$$\frac{\Delta(H)(k-1)}{\Delta(H) - r} > |X| \geq 1 + r(k - 1).$$
This implies \( \Delta(H)(k-1) > (\Delta(H) - r) + (\Delta(H) - r)r(k-1) \) and since \( \Delta(H) - r > 0 \), we have \( \Delta(H)(k-1) > (\Delta(H) - r)r(k-1) \). Dividing both sides by \( k - 1 \neq 0 \) we have \( \Delta(H) > (\Delta(H) - r)r \).

Now \( \Delta(H) > (\Delta(H) - r)r \) rearranges to \( r^2 > \Delta(H)(r - 1) \). To make the arithmetic easier, let \( d \) be the difference \( \Delta(H) - r \), and note that \( d \) is a positive integer. Substitute \( dr + r \) for \( \Delta(H) \) and continue

\[
\begin{align*}
r^2 &> (d + r)(r - 1) \\
\Rightarrow r^2 &> r^2 + dr - d - r \\
\Rightarrow 0 &> dr - d - r \\
\Rightarrow d &> r(d - 1).
\end{align*}
\]

If \( d > 1 \) then \( r < \frac{d}{d-1} \), a ratio of two consecutive positive integers, so \( 1 < r < \frac{d}{d-1} < 2 \) which implies \( r = 1 \). This means that each vertex of \( X \) is incident with a single \( \partial_k \)-edge of \( X \). But we previously established that each vertex of \( X \) lies at the intersection of at least two \( \partial_k \)-edges, a contradiction.

Finally, if \( d = 1 \), then each vertex is incident with a single boundary edge. Recall the lower bound \( |X| \geq 1 + r(k - 1) \). Replacing \( r \) with \( \Delta(H) - d = \Delta(H) - 1 \), we get \( |X| \geq 1 + (\Delta(H) - 1)(k - 1) \), which is strictly greater than \( (\Delta(H) - 1)(k - 1) \). So we have \( \frac{|X|}{k-1} > \Delta(H) - 1 \). Observe that \( |\partial(X)| \geq \frac{|X|}{k-1} \) since boundary edges take up vertices of \( X \) at most \( k - 1 \) at a time. Therefore \( \frac{|X|}{k-1} > \Delta(H) - 1 \) implies \( |\partial(X)| > \Delta(H) - 1 \) and so \( \kappa'(H) = |\partial(X)| \geq \Delta(H) \).

\[\square\]

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