INSTABILITY AND NONORDERING OF LOCALIZED STEADY STATES TO A CLASS OF REACTION-DIFFUSION EQUATIONS IN $\mathbb{R}^N$

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Abstract. We show that the elliptic problem $\Delta u + f(u) = 0$ in $\mathbb{R}^N$, $N \geq 1$, with $f \in C^1(\mathbb{R})$ and $f(0) = 0$ does not have nontrivial stable solutions that decay to zero at infinity, provided that $f$ is nonincreasing near the origin. As a corollary, we can show that any two nontrivial solutions that decay to zero at infinity and are not of opposite sign must intersect each other. We also discuss implications of our result on the existence of monotone heteroclinic solutions to the corresponding reaction-diffusion equation.

1. Introduction

1.1. The setting and known results. We consider the elliptic problem

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^N, \ N \geq 1,$$

with

$$f \in C^1(\mathbb{R}) \text{ and } f(0) = 0.$$  

(1.2)

A solution is called localized if it satisfies

$$u(x) \to 0 \text{ as } |x| \to \infty.$$  

(1.3)

The above problem has been studied extensively for various nonlinearities (see for example [13] and the references therein). A solution is called stable if

$$\int_{\mathbb{R}^N} \left\{ |\nabla \varphi|^2 - f'(u)\varphi^2 \right\} \, dx \geq 0 \ \forall \ \varphi \in C_0^\infty(\mathbb{R}^N),$$

(1.4)

see for instance [10]. Otherwise, it is called unstable.

Let us briefly mention some important rigidity results for stable solutions to (1.1) that hold without any further restrictions on $f$ and without the assumption that $f(0) = 0$. There are no nontrivial stable solutions in the Sobolev space $W^{1,2}(\mathbb{R}^N)$ (see [6, 9]). If $N = 2$, then bounded stable solutions depend only on one variable (possibly after a rotation), see [12, 8]. If $N \leq 10$, there are no nonconstant radially symmetric stable solutions that are bounded (see [6, 18]). If one further assumes that $f \geq 0$ and $N \leq 10$, then there are no nonconstant stable solutions that are bounded from below, see [11]. On the other hand, if
$N \geq 11$ and $f$ is an appropriate nonnegative polynomial, there exist positive, nonconstant localized stable radial solutions (see [6]).

Positive solutions of (1.1)-(1.3), under the assumption (1.2) on $f$, are known to be radially symmetric and decreasing with respect to some point, provided that

$$f'(s) \leq 0 \text{ for small } |s|,$$

(1.5)

(see [14]). Moreover, in this case any two such positive solutions must intersect (see [5, Lem. 3.2]).

If

$$f'(0) < 0,$$

(1.6)

then any localized solution to (1.1) and its gradient must decay exponentially fast as $|x| \to \infty$. Thus, by the above discussion, such solutions must be unstable (see also [17] for the case of radial solutions). In this note, we will show that this property continues to hold under the weaker condition (1.5). We note in passing that it was shown in [19], under the sole assumption (1.2), that there are no nontrivial localized solutions that are minimizers in the sense of Morse (a property that is stronger than stability).

1.2. Our results and methods of proof. Our main result is the following.

**Theorem 1.1.** If $u$ is a stable solution to (1.1) and (1.3) with $f$ satisfying (1.2) and (1.5), then $u \equiv 0$.

Our proof proceeds by showing that all first order partial derivatives of $u$ are identically equal to zero. These satisfy the linearized equation on $u$ and tend to zero at infinity (by standard elliptic estimates). It is a well known fact that the stability of $u$ implies the existence of a positive solution $\Psi$ to the aforementioned linearized equation. If $\liminf_{|x| \to \infty} \Psi > 0$, then the linearized operator satisfies the maximum principle (see [3]), and the assertion of the theorem follows at once. In any case, the existence of $\Psi > 0$ implies that the maximum principle holds in any bounded domain. On the other hand, the assumption (1.5) implies that the maximum principle holds in the exterior of large balls. Remarkably, these two separate properties can be combined to show that the linearized operator satisfies the maximum principle in the whole space, and therefore conclude. This can be shown by adapting an argument from [7] which is based on considering the quotient of the solution over $\Psi$. However, we found it more convenient to argue directly using Serrin’s sweeping principle, in the spirit of the moving plane argument of [14].

As we have already mentioned, for $f$ as in the above theorem, it was shown in [5, Lem. 3.2] that any two positive solutions of (1.1) and (1.3) must intersect each other. This was accomplished by the famous sliding method [2], exploiting that (1.1) is invariant under translations. However, this approach breaks down in the case of sign changing solutions. On the other hand, our Theorem 1.1 can be used
to show that this intersection property also holds for sign changing solutions. More precisely, the following result holds.

**Corollary 1.1.** Let \( f \) satisfy (1.2) and (1.5). If \( u_1 \) and \( u_2 \) are two distinct solutions of (1.1) and (1.3) that do not satisfy \( u_1 \cdot u_2 < 0 \) in \( \mathbb{R}^N \), then \( u_1 - u_2 \) must change sign.

The main idea of the proof is the following. If they were ordered, since both are unstable by Theorem 1.1, we can use a dynamical systems argument to show that there exists a stable solution between them, which is in contradiction to Theorem 1.1. This type of arguments are well known in the case of bounded domains (see [15, 16]). The case of the whole space requires a bit of extra care.

### 1.3. Applications

Let us briefly discuss some interesting implications of Theorem 1.1.

Let \( f \in C^1(0, \infty) \cap C[0, \infty) \) satisfy \( f(0) = 0 \) and (1.5). Assume that there exists a positive solution \( w \) to (1.1) and (1.3). As we have already mentioned, \( w \) has to be radially symmetric and decreasing with respect to some point and there does not exist another such solution below it. Moreover, we know from Theorem 1.1 that \( w \) is unstable. So, there exists a large ball \( B_R \) such that the principal eigenvalue of

\[
- \Delta \psi - f'(w)\psi = \lambda \psi \quad \text{in} \quad B_R; \quad \psi = 0 \quad \text{on} \quad \partial B_R,
\]

(1.7)
is negative (one can take \( B_R \) to include the support of a test function that violates (1.4)). In [5], [13, Sec. 1.4] the existence of such a ball was shown under the stronger condition (1.6) and an approximation argument. Armed with the above information in \( B_R \), the approach in the latter reference applies to establish that the reaction-diffusion equation

\[
u_t = \Delta u + f(u), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R},
\]

admits a heteroclinic solution such that \( u_t < 0 \) and

\[
u \to w \quad \text{as} \quad t \to -\infty; \quad u \to 0 \quad \text{as} \quad t \to +\infty, \quad \text{uniformly in} \quad \mathbb{R}^N.
\]

**Remark 1.1.** It is easy to see that in the scheme of [13] for the construction of the aforementioned heteroclinic solution one can also take as initial condition the function

\[
\min\{w(\cdot + \varepsilon), w\}, \quad \text{where} \quad e = (1, 0, \cdots, 0) \quad \text{and} \quad 0 < |\varepsilon| \ll 1,
\]

(1.8)

(from the aforementioned result of [5], \( w \) must intersect any translate of itself), instead of

\[
w_{\varepsilon} := \begin{cases} 
  w - \varepsilon \Phi, & x \in B_R, \\
  w, & x \in \mathbb{R}^N \setminus B_R,
\end{cases}
\]

(1.9)

where \( \Phi > 0 \) stands for the principal eigenfunction of (1.7) and \( 0 < \varepsilon \ll 1 \), which was used therein. The point is that both are strict weak supersolutions to (1.1). This observation, which implies that \( w \) is a dynamically unstable steady state in
\( L^\infty(\mathbb{R}^N) \) (without any information on the linearized operator), was actually our heuristic motivation behind Theorem 1.1. We point out that the solution of the corresponding Cauchy problem with initial condition as in (1.8) or (1.9) converges to zero as \( t \to +\infty \) (see [5, 13]).

1.4. Outline of the paper. The rest of the paper is devoted to the proofs of Theorem 1.1 and Corollary 1.1.

2. Proofs

2.1. Proof of Theorem 1.1.

Proof. Our goal is to prove that
\[
\partial_{x_i} u = 0 \quad \text{for} \quad i = 1, \cdots, N, \tag{2.1}
\]
from where the assertion of the theorem follows at once. We first note that standard elliptic estimates yield
\[
\partial_{x_i} u \to 0 \quad \text{as} \quad |x| \to \infty. \tag{2.2}
\]
Moreover, each \( \partial_{x_i} u \) satisfies the linearized equation of (1.1) at \( u \).

Since \( u \) is stable, as in [1, Prop. 4.2] or [4, Thm. 1.7], there exists a \( \Psi \in C^2(\mathbb{R}^N) \) such that
\[
-\Delta \Psi - f'(u) \Psi = 0 \quad \text{and} \quad \Psi > 0 \quad \text{in} \quad \mathbb{R}^N.
\]
We will show that (2.1) holds with the use of Serrin’s sweeping principle (see [16, Thm. 2.7.1]). For a fixed \( i \in \{1, \cdots, N\} \), let us consider the set
\[
\Lambda = \{ \lambda \geq 0 : \partial_{x_i} u \leq \mu \Psi \quad \text{in} \quad \mathbb{R}^N \quad \text{for every} \quad \mu \geq \lambda \}.
\]
Our goal is to show that \( \Lambda = [0, \infty) \), which will yield \( \partial_{x_i} u \leq 0 \). We can also apply the same argument, with \( \partial_{x_i} u \) replaced by \( -\partial_{x_i} u \), to obtain \( \partial_{x_i} u \geq 0 \) and therefore conclude.

By virtue of (1.3) and (1.5), there exists an \( R > 0 \) such that
\[
f'(u) \leq 0 \quad \text{for} \quad |x| \geq R. \tag{2.3}
\]
Clearly, there exists a \( \bar{\lambda} > 0 \) such that
\[
\partial_{x_i} u \leq \bar{\lambda} \Psi \quad \text{for} \quad |x| \leq R.
\]
Since both \( \partial_{x_i} u \) and \( \Psi \) satisfy the linearized equation of (1.1) at \( u \), it follows from (2.2), (2.3) and the maximum principle that the above ordering is also valid for \( |x| > R \), i.e. \( \bar{\lambda} \in \Lambda \). Hence, \( \Lambda \) is an interval of the form \( [\tilde{\lambda}, \infty) \) for some \( \tilde{\lambda} \in [0, \bar{\lambda}] \).

It remains to show that \( \tilde{\lambda} = 0 \). To this end, we will argue by contradiction and suppose that \( \tilde{\lambda} > 0 \). From the relation
\[
\partial_{x_i} u \leq \tilde{\lambda} \Psi \quad \text{in} \quad \mathbb{R}^N,
\]
and the strong maximum principle which implies that the above inequality is strict, we infer that there exists a \( \delta \in (0, \tilde{\lambda}/2) \) such that

\[
\partial_x_i u \leq (\tilde{\lambda} - \delta)\Psi \text{ for } |x| \leq R.
\]

Then, as before, we deduce by the maximum principle that the above relation holds in \( \mathbb{R}^N \), which contradicts the minimality of \( \tilde{\lambda} \) and completes the proof of the theorem.

\[\square\]

### 2.2. Proof of Corollary 1.1.

**Proof.** We will argue by contradiction. So, thanks to the strong maximum principle, let us suppose that \( u_1 < u_2 \) in \( \mathbb{R}^N \). Since Theorem 1.1 guarantees that \( u_2 \) is unstable, as we have already explained, there exists a strict weak supersolution \( u_{2,\varepsilon} \) to (1.1) of the form (1.9) such that \( u_1 < u_{2,\varepsilon} \leq u_2 \). Let us consider the solution \( v \) of the Cauchy problem

\[
\begin{cases}
    u_t = \Delta u + f(u), & x \in \mathbb{R}^N, \; t > 0, \\
    u(x, 0) = u_{2,\varepsilon}(x), & x \in \mathbb{R}^N.
\end{cases}
\]

It is well known that \( v_t < 0 \), and thus

\[v(\cdot, t) \to z(\cdot) \text{ as } t \to +\infty, \text{ uniformly in } \mathbb{R}^N,\]

where \( z \) is a steady state such that \( u_1 \leq z < u_{2,\varepsilon} \). Similarly, taking into account that \( z \) cannot be identically equal to zero (recall that \( u_1 \) and \( u_2 \) cannot be of opposite sign), there exists a strict weak lower solution \( z_\varepsilon \) of the form (1.9) such that

\[
z \leq z_\varepsilon < u_{2,\varepsilon}.
\]

On the other hand, the maximum principle yields

\[
z_\varepsilon < v, \; x \in \mathbb{R}^N, \; t > 0.
\]

Letting \( t \to +\infty \) in the above relation gives

\[
z_\varepsilon \leq z, \; x \in \mathbb{R}^N,
\]

which contradicts (2.4) and the fact that \( z_\varepsilon \) is a strict lower solution. \[\square\]

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