SCATTERING THEORY WITH UNITARY TWISTS

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Abstract. We study the spectral properties of the Laplace operator associated to a hyperbolic surface in the presence of a unitary representation of the fundamental group. Following the approach by Guillopé and Zworski, we establish a factorization formula for the twisted scattering determinant and describe the behavior of the scattering matrix in a neighborhood of 1/2.

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1. Introduction

We consider a finitely generated Fuchsian group $\Gamma \subset \text{PSL}(2, \mathbb{R})$ and denote the associated hyperbolic surface by $X$. Thus $X = \Gamma \backslash \mathbb{H}$, where $\mathbb{H}$ denotes the hyperbolic upper half-plane and $\text{PSL}(2, \mathbb{R})$ acts via Möbius transformations on $\mathbb{H}$. Throughout this article, we will suppose that $X$ is non-elementary, geometrically finite and of infinite volume. However, we allow that $X$ has orbifold singularities or, equivalently, that $\Gamma$ has torsion.

We further consider a finite-dimensional unitary representation $\chi : \Gamma \rightarrow \text{U}(V)$ on a Hermitian vector space $V$. The representation $\chi$ induces a Hermitian vector orbibundle $E_\chi := \Gamma \backslash (\mathbb{H} \times V) \rightarrow X$ with typical fiber $V$. It is well-known that the (smooth) sections of $E_\chi$ are in bijection with the smooth functions $f : \mathbb{H} \rightarrow V$ that obey the twisting equivariance

$$f(g.z) = \chi(g)f(z), \quad z \in \mathbb{H}, \ g \in \Gamma.$$
See, for example, [DFP, Lemma 3.3] for details. On smooth maps $f : \mathbb{H} \to V$, the hyperbolic Laplacian is given by
\[
\Delta_{\mathbb{H}} f(z) = -\sum_{j=1}^{\dim V} y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(z),
\]
where $z = x + iy \in \mathbb{H}$. Using the identification of twisted functions (see (1)) and sections of $E_\chi$ and the fact that $\chi$ is unitary, the Laplacian $\Delta_{\mathbb{H}}$ gives rise to a non-negative self-adjoint operator $\Delta_{X,\chi}$:
\[
\Delta_{X,\chi} : L^2(X, E_\chi) \to L^2(X, E_\chi).
\]
For $\text{Re } s > 1/2$ and $s \notin [1/2, 1]$, the resolvent of $\Delta_{X,\chi}$ is defined by
\[
R_{X,\chi}(s) := (\Delta_{X,\chi} - s(1 - s))^{-1} : L^2(X, E_\chi) \to L^2(X, E_\chi).
\]
As shown in [DFP, Theorem A], the resolvent $R_{X,\chi}$ admits a meromorphic continuation to $s \in \mathbb{C}$ as an operator $R_{X,\chi}(s) : L^2_{\text{cpt}}(X, E_\chi) \to L^2_{\text{loc}}(X, E_\chi)$.

The poles of $R_{X,\chi}(s)$ are the resonances of $\Delta_{X,\chi}$. The multiplicity of the pole $s \in \mathbb{C}$ is the rank of the residue at $s$.

We consider the scattering matrix, which is a certain operator
\[
S_{X,\chi}(s) : C^\infty(\partial_\infty X, E_\chi) \to C^\infty(\partial_\infty X, E_\chi), \quad s \notin R_{X,\chi} \cup \mathbb{Z}/2,
\]
defined on the boundary of a suitable compactification of $E_\chi$ (see Sections 3 and 5). For each $\psi \in C^\infty(\partial_\infty X, E_\chi)$ there exists $u \in C^\infty(X, E_\chi)$ such that $(\Delta_{X,\chi} - s(1 - s))u = 0$ and
\[
(2s - 1)u \sim \rho_f^{1-s} \rho_c^{-s} \psi + \rho_f^{s} \rho_c^{-1} S_{X,\chi}(s) \psi \quad \text{as } \rho_f \rho_c \to 0,
\]
where $\rho_f$ and $\rho_c$ are the boundary defining functions in the funnel and cusp ends, respectively. Even though the scattering matrix is not trace class, we can define a regularized determinant of $S_{X,\chi}(s)$, that we will call the relative scattering determinant, $\tau_{X,\chi}(s)$.

As the first main result of this article, we prove a factorization of the relative scattering determinant in terms of the Weierstrass product over the resonances.
Theorem A. The scattering determinant admits the factorization
\[ \tau_{X,\chi}(s) = e^{q(s)} \frac{P_{X,\chi}(1-s)}{P_{X,\chi}(s)} \frac{P_{X,\chi}(s)}{P_{X,\chi}(1-s)}, \]
where \( q : \mathbb{C} \to \mathbb{C} \) is a polynomial of degree at most 4.

For \( \dim V = 1 \) and \( \chi = \text{id} \), Theorem A reduces to [GZ97, Proposition 3.7].

This latter result plays a crucial role in the proof of the factorization of the Selberg zeta function by Borthwick–Judge–Perry [BJP05].

We remark that Theorem A implies that the scattering determinant has no pole or zero at \( s = 1/2 \). However, \( s = 1/2 \) might be a resonance. The second main result of this article shows that we are able to describe the behavior of the scattering matrix \( S_{X,\chi}(s) \) in some (small) neighborhood of 1/2. For this, we set
\[ P := \frac{1}{2} \left( S_{X,\chi}(\frac{1}{2}) + \text{id} \right). \]
Then
\[ S_{X,\chi}(s) = -\text{id} + 2P + (2s - 1)T_{X,\chi}(s) \]
with \( T_{X,\chi} \) being an operator family that is holomorphic in a small neighborhood of \( s = 1/2 \).

Theorem B. The operator \( P \) is an orthogonal projection of rank \( m_{X,\chi}(1/2) \) onto the space of elements in \( \mathcal{C}^\infty(\partial_\infty X, E_\chi) \) that are invariant under the map \( S_{X,\chi}(1/2) \).

Structure of this article. In Section 3, we discuss the scattering matrices for the model funnel and the parabolic cylinder. In Section 4, we obtain a decomposition of the resolvent, study the structure of the resolvent close to a resonance and obtain that there are no resonances on the line \( \text{Re}(s) = 1/2 \) except for, maybe, \( s = 1/2 \). In Section 5, we introduce the scattering matrix, the relative scattering determinant and prove Theorems A and B.

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2. Preliminaries and Notation

We let \( X \) and \( E_\chi \) be as above. We denote by \( \langle \cdot, \cdot \rangle_{E_\chi} \) the Hermitian bundle metric on \( E_\chi \) that is induced from the sesquilinear inner product \( \langle \cdot, \cdot \rangle_V \) on \( V \). We denote by \( \langle \cdot, \cdot \rangle_{E_\chi} \) the bilinear metric on \( E_\chi \) corresponding to the bundle metric \( \langle \cdot, \cdot \rangle_{E_\chi} \). We abbreviate the norm \( |v|_{E_\chi} = \sqrt{\langle v, v \rangle_{E_\chi}} \) of any \( v \in E_\chi \) by \( |v| \).

By Selberg’s Lemma [Sel60, Lemma 8], there is a finite cover \( \tilde{X} = \tilde{\Gamma} \backslash \mathbb{H} \) of \( X \) such that the Fuchsian group \( \tilde{\Gamma} \) is a torsion-free subgroup of \( \Gamma \). We denote the pull-back of \( E_\chi \) under the covering map \( \tilde{X} \to X \) by \( \tilde{E} \), which becomes a vector bundle over \( \tilde{X} \). We call an operator \( A \) acting on the sections of \( E_\chi \) a pseudodifferential operator of order \( m \in \mathbb{R} \) if its pull-back, \( \tilde{A} \),
under the map $\tilde{X} \to X$ is a pseudodifferential operator of order $m$, acting on the sections of $\tilde{E}$.

In the case of the 1-sphere $S^1$, pseudodifferential operators have a very simple characterization using Fourier series, which we recall now. To that end let $A: C^\infty(S^1) \to C^\infty(S^1)$ be a continuous linear operator. As proven by McLean [McL91, Theorem 4.4], $A$ is a pseudodifferential operator of order $m \in \mathbb{R}$ if and only if

$$a(x, \xi) := e^{-2\pi i (x, \xi)} A(e^{2\pi i \langle \cdot, \xi \rangle})$$

is a periodic symbol of order $m$. This means that $a \in C^\infty(S^1 \times \mathbb{Z})$, and for all $b, c \in \mathbb{N}_0$, we have

$$|\partial^b_x \triangle^c_\xi a(x, \xi)| \lesssim_{b, c} (\xi)^{m-c}.$$  \hfill (3)

Here, $\triangle_\xi$ denotes the discrete derivative, i.e.,

$$\triangle_\xi a(x, \xi) := u(x, \xi + 1) - u(x, \xi).$$

Further, $\lesssim$ indicates an upper bound with implied constants. More precisely, for any set $Y$ and any functions $a, b: Y \to \mathbb{R}$, we write

$$a \lesssim b \quad \text{or} \quad a(y) \lesssim b(y)$$

if there exists a constant $C > 0$ such that for all $y \in Y$ we have

$$|a(y)| \leq C|b(y)|.$$  \hfill (3)

If the constant, $C$, depends on additional parameters, we indicate the dependence in the subscript.

Let $H$ be a Hilbert space and let $B: H \to H$ be a bounded operator. The non-zero eigenvalues of $(B^*B)^{1/2}$ are called the singular values of the operator $B$. We denote these singular values by $\mu_k(B)$, $k \in \mathbb{N}$, listed in decreasing order.

For $z \in \mathbb{C}$ we define the Japanese bracket $\langle z \rangle := (1 + |z|^2)^{1/2}$. We use the convention to call a function, $f$, meromorphic on an open set $U \subseteq \mathbb{C}$ if there exists a discrete subset, $P$, of $U$ such that $f$, considered as a function, is defined on $U \setminus P$ only, and $f$ is holomorphic on $U \setminus P$ and has poles (of finite order, which might be zero) at the points in $P$.

### 3. The Scattering Matrix for the Model Cylinders

In this section we present the structure of the twisted scattering matrix for the model ends. We discuss the model funnel in Section 3.1 and the model cusp in Section 3.2. The analysis was originally done in [DFP, Section 4]. Here we restrict to presenting the main results only.

#### 3.1. Model funnel

Let $\ell \in (0, \infty)$ and set $\omega := 2\pi/\ell$. We define the hyperbolic cylinder as the quotient $C_\ell := \langle h_\ell \rangle \backslash \mathbb{H}$, where $h_\ell z = e^\ell z$. We may change coordinates via

$$z = e^{\omega^{-1}\phi} \frac{e^r + i}{e^r - i}.$$
to \((r, \phi) \in \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z} \cong C_\ell\) in such a way that the induced metric from the hyperbolic plane becomes

\[
g_{C_\ell}(r, \phi) := dr^2 + \frac{\ell^2}{4\pi^2} \cosh^2 r \, d\phi^2.
\]

We define the model funnel as

\[
F_\ell := \{(r, \phi) \in C_\ell : r > 0\}
\]

with the metric \(g_{F_\ell} := g_{C_\ell}|_{F_\ell}\). The canonical boundary defining function is \(\rho_f(r, \phi) = \cosh(r)^{-1}\).

Taking the boundary defining function as a coordinate function, we may rewrite the funnel metric as

\[
g_{F_\ell}(\rho, \phi) = \rho^{-2} \left( \frac{\ell^2}{4\pi^2} d\phi^2 + \frac{d\rho^2}{1 - \rho^2} \right).
\]

The volume form is

\[
d\mu_{F_\ell} = \frac{\ell}{2\pi} \frac{d\rho \, d\phi}{\sqrt{1 - \rho^2}}.
\]

We also define the metric restricted to the boundary at infinity

\[
g_{\partial_{\infty} F_\ell}(\rho, \partial_\phi) := \rho^2 g_{F_\ell}(\rho, \phi, 0, \partial_\phi)|_{\rho=0} = \frac{\ell^2}{4\pi^2} d\phi^2
\]

and denote the corresponding measure by

\[
d\sigma_{\partial_{\infty} F_\ell} := \frac{\ell}{2\pi} d\phi.
\]

The Laplacian acting on functions \(F_\ell \to \mathbb{C}\) takes the form

\[
\Delta_{F_\ell} = -\rho^2(1 - \rho^2)\partial_\rho^2 + \rho^3 \partial_\rho - \frac{4\pi^2}{\ell^2} \rho^2 \partial_\phi^2.
\]

Let \(\chi : \langle h_\ell \rangle \to U(V)\) be a finite-dimensional unitary representation. As above, we denote by \(\Delta_{C_\ell, \chi}\) the Laplacian acting on sections of the vector bundle \(E_{\chi} = \langle h_\ell \rangle \backslash (\mathbb{H} \times V)\) over \(C_\ell\). The Laplacian \(\Delta_{F_\ell, \chi}\) is the restriction of the Laplacian \(\Delta_{C_\ell, \chi}\) from \(F_\ell\) with Dirichlet boundary conditions at \(r = 0\).

The multiset of its resonances, \(R_{F_\ell, \chi}\), is given by

\[
R_{F_\ell, \chi}(s) : L^2_{\text{cpt}}(F_\ell, E_{\chi}) \to L^2_{\text{loc}}(F_\ell, E_{\chi}),
\]

(6) \(\mathcal{R}_{F_\ell, \chi} := \bigcup_{\lambda \in \text{EV}(\chi(h_\ell))} \bigcup_{p \in \{\pm 1\}} (-1 + 2N_0) + pt^{-1} (\log \lambda + 2\pi i \mathbb{Z})\),

where \(\text{EV}(\chi(h_\ell))\) denotes the multiset of eigenvalues of \(\chi(h_\ell)\). See [DFP, Proposition 4.12].

Let \(\psi \in C^\infty(\overline{F_\ell} \times \overline{F_\ell})\) such that \(\psi\) is supported away from the diagonal and \(s \in \mathbb{C}\) is not a pole of the resolvent \(R_{F_\ell, \chi}\). By [DFP, Proposition 4.12], we have

\[
\psi R_{F_\ell, \chi}(s, \cdot, \cdot) \in (\rho_f \rho_f')^s C^\infty(\overline{F_\ell} \times \overline{F_\ell}, E_{\chi} \boxtimes E'_{\chi}),
\]

(7)
where $E_{\chi} \boxtimes E_{\chi}'$ is the exterior tensor product of $E_{\chi}$ and its dual $E_{\chi}'$ defined by
\[
(E_{\chi} \boxtimes E_{\chi}')_{(x, \varphi)} := (E_{\chi})_{x} \otimes (E_{\chi}')_{\varphi}.
\]
By (7),
\[
E_{F_{t, \chi}}(s; r, \phi, \phi') := \lim_{r' \to \infty} (\rho f(r'))^{-s} R_{F_{t, \chi}}(s; r, \phi, r', \phi')
\]
is well-defined. This allows us to introduce the Poisson operator
\[
E_{F_{t, \chi}}(s) : C^\infty(\partial_{\infty} F_t, E_{\chi}|_{\partial_{\infty} F_t}) \to C^\infty(F_t, E_{\chi}),
\]
\[
(E_{F_{t, \chi}}(s)f)(r, \phi) := \frac{\ell}{2\pi} \int_{0}^{2\pi} E_{F_{t, \chi}}(s; r, \phi, \phi') f(\phi') d\phi'.
\]

We now recall the Fourier expansion of the Poisson operator. Let $(\psi_j)_{j=1}^{\dim V}$ be an eigenbasis of $\chi(h_0)$ with eigenvalues $\lambda_j = e^{2\pi i j\varphi}$, $j = 1, \ldots, \dim V$. Let $\kappa \in \mathbb{R}$ and $s \in \mathbb{C} \setminus (-1 - 2N_0 \pm i\omega \kappa)$. We define
\[
\beta_{\kappa}(s) := \frac{1}{2\Gamma}\left(\frac{s + i\omega \kappa + 1}{2}\right) \Gamma\left(\frac{s - i\omega \kappa + 1}{2}\right).
\]
We recall that the regularized hypergeometric function $F(a, b; c; z)$ is defined for $a, b, c \in \mathbb{C}$ and $z \in \mathbb{C}$, $|z| < 1$, by the power series (see [Olv97, Theorem 9.1])
\[
F(a, b; c; z) := \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(a)\Gamma(b)} \frac{1}{\Gamma(c+n)} \cdot \frac{z^n}{n!}.
\]
For arbitrary $\kappa \in \mathbb{R}$, $s \in \mathbb{C}$ and $r \geq 0$, we define
\[
v_{\kappa}^{\phi}(s; r) := \tanh(r)(\cosh(r))^{-s} F\left(\frac{s + i\omega \kappa + 1}{2}, \frac{s - i\omega \kappa + 1}{2}; \frac{3}{2}; \tanh(r)^2\right).
\]
It was shown in [DFP, Remark 4.13] that
\[
E_{F_{t, \chi}}(s; r, \phi, \phi') \psi_j = \frac{1}{\ell} \sum_{k \in \mathbb{Z}} e^{(k+\varphi_j)(\phi-\phi')} E_{F_{t, \chi}}^k(s; r) \psi_j,
\]
where
\[
E_{F_{t, \chi}}^k(s; r) := \beta_{k+\varphi_j}(s) v_{k+\varphi_j}^{\phi}(s; r) \Gamma(s + \frac{1}{2}).
\]
By [DFP, Lemma 6.15], we have for $\varepsilon \in (0, 1/2)$, $\varphi \in C^\infty_c(F_t)$, there exist $C, c > 0$ such that for all $s \in \mathbb{C}$ with $\Re s > \varepsilon$ we have the estimate
\[
\mu_\varepsilon(\varphi E_{F_{t, \chi}}(s)) \leq e^{C(s) - c\varepsilon}.
\]
Moreover, the scattering matrix
\[
S_{F_{t, \chi}}(s) : C^\infty(\partial_{\infty} F_t, E_{\chi}|_{\partial_{\infty} F_t}) \to C^\infty(\partial_{\infty} F_t, E_{\chi}|_{\partial_{\infty} F_t})
\]
was defined in [DFP, (66)] via the Fourier coefficients of its Schwartz kernel,
\[
S_{F_{t, \chi}}(s; \phi, \phi') \psi_j = \frac{1}{\ell} \sum_{k \in \mathbb{Z}} e^{(k+\varphi_j)(\phi-\phi')} S_{F_{t, \chi}}^k(s) \psi_j,
\]
We note that the terms in \((11)\) combined with \((14)\) shows
that
\[
\frac{\beta_n(s)}{\beta_n(1-s)} = \frac{\Gamma\left(s + \frac{\omega \kappa + 1}{2}\right)}{\Gamma\left(\frac{2 + \omega \kappa}{2}\right)} \times \frac{\Gamma\left(\frac{s - \omega \kappa + 1}{2}\right)}{\Gamma\left(\frac{2 - s - \omega \kappa}{2}\right)}
\sim \sum_{k=0}^{\infty} \frac{|\omega \kappa|^2}{2} k^{2s-1-2k} 2^k \sum_{n=0}^{2k} e^{\pi i (k-n)} G_n(a, b) G_{2k-n}(a, b).
\]
By [DLMF, Eq. 5.11.15], the leading coefficient is given by \(G_0(a, b)^2 = 1\). Combining this with (11), we obtain that
\[
S_{F_i, \chi}(s)_k^j \sim 2^{1-2s} \frac{\Gamma\left(\frac{1}{2} - s\right)}{\Gamma\left(s - \frac{1}{2}\right)} |(k + \vartheta_j)\omega|^{2s-1}.
\]

The full asymptotic expansion now implies that \(S_{F_i, \chi}(s)_k^j\) satisfies the symbol estimates for global pseudodifferential operators on the torus, as stated in (3). Hence,
\[
S_{F_i, \chi}(s) \in \Psi^{2\Re s - 1}(\partial_\infty F_i, E_\chi|_{\partial_\infty F_i}), \quad s \notin R_{F_i, \chi} \cup \left(N_0 + \frac{1}{2}\right).
\]

We define the reduced scattering matrix \(\tilde{S}_{F_i, \chi}(s)\) as follows: we consider the invertible elliptic pseudodifferential operator \(\Lambda \in \Psi^1(\partial_\infty F_i, E_\chi|_{\partial_\infty F_i})\) defined by
\[
\Lambda \psi_j = \sum_{k \in \mathbb{Z}} e^{i(k+\vartheta_j)(\varphi - \varphi')} \langle k \rangle \psi_j,
\]
set
\[
G(s) := \frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(s - \frac{1}{2}\right)} \text{id}_{C^\infty(\partial_\infty F_i, E_\chi|_{\partial_\infty F_i})}
\]
and define
\[
\tilde{S}_{F_i, \chi}(s) := G(s) \Lambda(s) S_{F_i, \chi}(s) \Lambda(1-s)^{-1} G(1-s)^{-1},
\]
for \(s \notin R_{F_i, \chi} \cup (N_0 + 1/2)\). A straightforward calculation shows that the Fourier coefficients of \(\tilde{S}_{F_i, \chi}(s)\) are
\[
\tilde{S}_{F_i, \chi}(s)_k^j = \langle k \rangle^{-2s+1} \frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(s - \frac{1}{2}\right)} (k)^{-2s+1} S_{F_i, \chi}(s)_k^j
\]
\[
= \langle k \rangle^{-2s+1} \frac{(s - \frac{1}{2})^{\beta_k + \vartheta_j(s)}}{\beta_k + \vartheta_j(s)} (1-s)^{-\beta_k + \vartheta_j(s)}. \tag{16}
\]

Since the right-hand side of the last equation is defined for all \(s \notin R_{F_i, \chi}\), the scattering matrix is defined as an operator \(\tilde{S}_{F_i, \chi}(s) \in \Psi^0(\partial_\infty F_i, E_\chi|_{\partial_\infty F_i})\) for \(s \notin R_{F_i, \chi}\). Taking advantage of this property, we can characterize the resonances in terms of the scattering matrix.

**Proposition 3.1.** Let \(s \in \mathbb{C}, \Re s < 1/2\) and let \(m \in \mathbb{N}\). Then the reduced scattering matrix \(\tilde{S}_{F_i, \chi}\) has a pole of rank \(m\) at \(s\) if and only if \(s\) is a resonance of multiplicity \(m\) of \(\Delta_{F_i, \chi}\). In this case, \(m = m_{\chi, \chi}(s)\).

**Proof.** By definition of \(\beta_n\), we have that
\[
\langle k \rangle^{-2s+1} \frac{(s - \frac{1}{2})}{\beta_k + \vartheta_j(s)} (1-s)^{-\beta_k + \vartheta_j(s)}
\]
is holomorphic and non-zero for $\text{Re } s < 1/2$. Therefore, the poles counted with multiplicities of $\tilde{S}_{F,\chi}(s)^{2}_{k}$ are given by the multiset
\[ \bigcup_{\ell \in \{ \pm 1 \}} \left( -(1 + 2N_0) + 2\pi p\ell^{-1}(\vartheta_j + k) \right). \]
Hence, the poles of $\tilde{S}_{F,\chi}(s)$ are given by the multiset (6).

Finally, we recall the singular value estimate for the scattering matrix from [DFP, Lemma 6.14]. For this, we will define functions $d_k : \mathbb{C} \to \mathbb{C}$ for $k \in \mathbb{N}$, which have poles contained in the set of resonances of $\Delta_{F,\chi}$. We set
\[
\begin{align*}
\tilde{R}_0 &:= 1 - 2N_0, \\
R_0 &:= 1 - 2N_0 + i\omega \mathbb{Z} \setminus \{0\}, \\
R_{1/2} &:= 1 - 2N_0 + i\omega \left(\frac{1}{2} + \mathbb{Z}\right), \\
R_{\vartheta} &:= \bigcup_{\ell \in \{ \pm 1 \}} (1 - 2N_0 + i\vartheta(\vartheta + \mathbb{Z})), \quad \vartheta \notin \left\{0, \frac{1}{2}\right\},
\end{align*}
\]
where we denote by $m_{\vartheta}$ the multiplicity of the eigenvalue $\lambda = e^{2\pi i \vartheta}$ of $\chi(h_{\ell})$.
We can assume without loss of generality that $\vartheta \in [0, 1)$. Denote by $d_{\mathbb{C}}$ the Euclidean distance on $\mathbb{C}$. For $k \in \mathbb{N}$ we define $d_k(\vartheta, s)$ as follows: for $\vartheta \in (0, 1) \setminus \{1/2\}$, we set
\[
d_k(\vartheta, s) := \begin{cases} 
  d_{\mathbb{C}}(s, R_{\vartheta})^{-1}, & k \leq m_{\vartheta}, \\
  1, & k > m_{\vartheta}.
\end{cases}
\]
and for $\vartheta \in \{0, 1/2\}$, we set
\[
d_k(\vartheta, s) := \begin{cases} 
  d_{\mathbb{C}}(s, R_{\vartheta})^{-1}, & k \leq 2m_{\vartheta}, \\
  1, & k > 2m_{\vartheta}.
\end{cases}
\]
Moreover, we define the function $\tilde{d}_{k,0}$ by
\[
\tilde{d}_{k,0}(s) := \begin{cases} 
  d_{\mathbb{C}}(s, \tilde{R}_0)^{-2}, & k \leq m_0, \\
  1, & k > m_0.
\end{cases}
\]
Finally, we set
\[
d_k(s) := \tilde{d}_{k,0}(s) \cdot \prod_{\vartheta} d_k(\vartheta, s).
\]
It is shown in [DFP, Lemma 6.14] that for any $\varepsilon \in (0, 1/2)$ there exists $C > 0$ such that for $s \in \mathbb{C}$ with $\text{Re } s < 1/2 - \varepsilon$,
\[
\mu_k(S_{F,\chi}(s)) \leq e^{C(s)} \langle s \rangle^{1 - 2\text{Re } s} \times \begin{cases} 
  d_k(s), & k \leq \max\{m_0, 2m_{\vartheta_j}\}, \\
  k^{2\text{Re } s - 1}, & k > \max\{m_0, 2m_{\vartheta_j}\}.
\end{cases}
\]
3.2. Parabolic cylinders. We now turn to the parabolic cylinder, where the structure of the resolvent is slightly simpler than for the hyperbolic cylinder.

The parabolic cylinder is given by \( C_\infty := (T) \setminus \mathbb{H} \), where \( T.z := z + 1 \). We can choose as fundamental domain the set

\[
\mathcal{F} := \{ x + iy \in \mathbb{H} : x \in (0, 1) \}.
\]

With the coordinates \((\rho, \phi) = (y^{-1}, (2\pi)^{-1}x)\), the induced Riemannian metric reads

\[
g_{C_\infty} = \frac{d\rho^2}{\rho^2} + \rho^2 \frac{d\phi^2}{4\pi^2}
\]

and \( \rho_c(\rho, \phi) = \rho \), where \( \rho_c \) is the canonical boundary defining function. In the \((x, y)\)-coordinates the Laplacian is given by

\[
\Delta_{C_\infty} = -y^2(\partial_x^2 + \partial_y^2).
\]

Let \( \chi : (T) \to U(V) \) be a finite-dimensional unitary representation. We denote by \( E_1(\chi(T)) \) the eigenspace of \( \chi(T) \) for eigenvalue 1, and we set \( n^\chi := \dim E_1(\chi(T)) \).

The meromorphically continued resolvent \( R_{C_\infty, \chi}(s) \) defines a continuous map

\[
\psi R_{C_\infty, \chi}(s) : C_\infty^\infty(C_\infty, E_\chi) \to \rho_c^{s-1}C_\infty^\infty(C_\infty, E_\chi)
\]

provided that \( s \neq 1/2 \), where \( \psi \) is any element of \( C^\infty(C_\infty) \) that is supported away from \( \{ y = 0 \} \). The only pole of \( R_{C_\infty, \chi}(s) \) is at the point \( s = 1/2 \) and its multiplicity is equal to \( n^\chi \).

The integral kernel of the resolvent \( R_{C_\infty, \chi}(s) \) admits a Fourier decomposition. For any \( j \in \{ 1, \ldots, \dim V \} \), the Fourier decomposition of the non-vanishing matrix coefficients \( R_{C_\infty, \chi}(s; z, z')^j \) is given by

\[
R_{C_\infty, \chi}(s; z, z')^j = \sum_{k \in \mathbb{Z}} e^{2\pi i(k+\theta_j)(z-z')} y_{2\pi i(k+\theta_j)}(s; y, y').
\]

Here, the maps \( u_\kappa \) for \( \kappa \in \mathbb{R} \) are defined as follows: for \( \kappa \in \mathbb{R} \setminus \{ 0 \} \), we set

\[
u_\kappa(s; y, y') := \begin{cases} 
\sqrt{yy} I_{\kappa s-1/2}(|\kappa|y) K_{\kappa s-1/2}(|\kappa|y'), & y \leq y', \\
\sqrt{yy} K_{\kappa s-1/2}(|\kappa|y) I_{\kappa s-1/2}(|\kappa|y'), & y > y'.
\end{cases}
\]

where \( I_{\kappa s-1/2} \) and \( K_{\kappa s-1/2} \) is the modified Bessel function of the first and the second kind, respectively (see [Wat66, § 3.7]). Moreover, for \( \kappa = 0 \) and \( s \neq 1/2 \) we set

\[
u_0(s; y, y') := \frac{1}{2s-1} \begin{cases} 
y^s(y')^{1-s}, & y \leq y', \\
y^{1-s}(y')^s, & y > y'.
\end{cases}
\]

The Poisson operator \( E_{C_\infty, \chi}(s) \) is given by

\[
E_{C_\infty, \chi}(s) : C^\infty(\partial_c C_\infty, E_\chi) \to C^\infty(C_\infty, E_\chi)
\]

\[
(E_{C_\infty, \chi}(s)u)(x, y) := \frac{y^s}{2s-1} u(x),
\]

where \( u \in C^\infty(\partial_c C_\infty, E_\chi) \).
where $u \in C^\infty(\partial_c C_\infty) \cong \mathbb{C}^{n_c}$. The Schwartz kernel of $E_{C_{\infty,\chi}}(s)$ is given by

$$E_{C_{\infty,\chi}}(s; x, y, x') = \frac{y'}{2s - 1} \text{id}_{E_1(y(x))}. \tag{22}$$

by the Fourier decomposition in (21). In particular,

$$E_{C_{\infty,\chi}}(s; x, y, x') = E_{C_{\infty,\chi}}(s; y)$$

is independent of $x, x'$.

4. Analysis of the Resolvent

In this section, we discuss fine-structure properties of the resolvent of $\Delta_{X,\chi}$. We start, in Theorem 4.1, with a decomposition of its resolvent into interior and residual terms, which are then discussed separately in more detail. In Section 4.1, we give a description of the resolvent near a resonance. In Section 4.2, we prove that on the line Re$(s) = 1/2$ there are no resonances except for potentially $s = 1/2$. Moreover, in Proposition 4.7, we prove that if the hyperbolic surface $X$ has infinite volume, then $\Delta_{X,\chi}$ has no eigenvalues larger than 1/4.

As in [DFP, Section 3.2.3], we take advantage of the decomposition

$$X = K \sqcup X_f \sqcup X_c,$$

where $K$ is compact and $X_f$ and $X_c$ are finite collections of funnels and cusps, respectively. For $\bullet \in \{f, c\}$ and $r \in [0, \infty)$, we choose a cutoff function $\eta_{\bullet, r} \in C^\infty(X)$ such that

$$\eta_{\bullet, r}(x) = \begin{cases} 1, & \text{if } d(X \setminus X_{\bullet}, x) < r, \\ 0, & \text{if } d(X \setminus X_{\bullet}, x) > r + \frac{1}{2}. \end{cases}$$

We fix $s_0 \in \mathbb{C}$ with sufficiently large real part (such that $s(1 - s)$ is sufficiently far away from the spectrum of $\Delta_{X,\chi}$) and denote by $n_f$ and $n_c$ the number of connected components of $X_f$ and $X_c$, respectively. As in [DFP, Section 5], we set

$$M_i := \eta_{f,2} \eta_{c,2} R_{X_f,\chi}(s_0) \eta_{f,1} \eta_{c,1}, \tag{23}$$

$$M_f(s) := (1 - \eta_{f,0}) R_{X_f,\chi}(s)(1 - \eta_{f,1}), \tag{24}$$

$$M_c(s) := (1 - \eta_{c,0}) R_{X_c,\chi}(s)(1 - \eta_{c,1}), \tag{25}$$

where

$$R_{X_f,\chi}(s) : L^2(X_f, E_\chi) \to L^2(X_f, E_\chi),$$

$$R_{X_f,\chi}(s) := R_{X_{f,1},\chi_1}(s) \oplus \ldots \oplus R_{X_{f,n_f},\chi_{n_f}}(s)$$

and

$$R_{X_c,\chi}(s) : L^2(X_c, E_\chi) \to L^2(X_c, E_\chi),$$

$$R_{X_c,\chi}(s) := R_{X_{c,1},\chi_1}(s) \oplus \ldots \oplus R_{X_{c,n_c},\chi_{n_c}}(s).$$

Further, we set

$$M(s) := M_i + M_f(s) + M_c(s)$$

and, as in [DFP, (85)], we define

$$L(s) := L_i(s) + L_f(s) + L_c(s), \tag{26}$$
where
\[
L_i(s) := -[\Delta_{X,\chi}, \eta_{f,2}\eta_{c,2}]R_{X,\chi}(s_0)\eta_{f,1}\eta_{c,1} + (s(1-s) - s_0(1-s_0))M_i(s_0),
\]
and
\[
L_f(s) := [\Delta_{X,\chi}, \eta_{f,0}]R_{f,\chi}(s)(1 - \eta_{f,1}),
\]
and
\[
L_c(s) := [\Delta_{X,\chi}, \eta_{c,0}]R_{c,\chi}(s)(1 - \eta_{c,1}).
\]
It follows that
\[
(\Delta_{X,\chi} - s(1-s))M(s) = \text{id} - L(s).
\]
It was proven in [DFP, Section 5] that \((\text{id} - L(s))^{-1}\) exists as a meromorphic family in \(s \in \mathbb{C}\).

**Theorem 4.1.** Let \(X = \Gamma \setminus \mathbb{H}\) be geometrically finite and let \(\chi: \Gamma \to U(V)\) be a finite-dimensional unitary representation of \(\Gamma\). For \(s \in \mathbb{C}\) not a pole of neither \(R_{X,\chi}(s)\) nor \(M_f(s)\) nor \(M_c(s)\), the resolvent admits a decomposition
\[
R_{X,\chi}(s) = \tilde{M}_i(s) + M_f(s) + M_c(s) + Q(s),
\]
where
- \(\tilde{M}_i(s)\) is a compactly supported pseudodifferential operator of order \(-2\),
- \(M_f(s)\) and \(M_c(s)\) are as in (24) and (25), respectively, and
- \(Q(s)\) is an integral operator with the integral kernel \(Q(s; \cdot, \cdot)\) satisfying
\[
Q(s; \cdot, \cdot) \in (\rho_f \rho_f')^s(\rho_c \rho_c')^{s-1}C^\infty(\overline{X} \times \overline{X}, E_{\chi} \boxtimes E_{\chi}').
\]

**Remark 4.2.** The product \(\overline{X} \times \overline{X}\) is not a smooth manifold (even in the absence of orbifold points). The reason is that the geodesic boundary at infinity of a cusp end is a single point. Blowing up each parabolic fixed point to a 1-sphere, we obtain a orbifold with boundary \(\overline{X}\). We define smooth functions on \(\overline{X} \times \overline{X}\) as the set of functions that pullback to smooth functions on \(\overline{X} \setminus \overline{X}\).

Note that blowing up a parabolic fixed point amounts to introducing coordinates \((\rho, \phi)\) as in Section 3.2, where \(\{\rho = 0\} \cong \mathbb{S}^1\) is the blowup of the parabolic fixed point.

**Proof of Theorem 4.1.** We set
\[
K(s) := (\text{id} - L(s))^{-1}L(s).
\]
Note that
\[
\text{id} + K(s) = (\text{id} - L(s))^{-1}.
\]
Then (29) implies
\[
R_{X,\chi}(s) = M(s)\left(\text{id} + K(s)\right).
\]
For notational simplicity, define \(\eta_3 := \eta_{f,3}\eta_{c,3}\). We now split \(M(s)K(s)\) as
\[
M(s)K(s) = \eta_3 M(s)K(s)\eta_3 + Q(s),
\]
where

\[ Q(s) := (1 - \eta_3)M(s)K(s)\eta_3 + M(s)K(s)(1 - \eta_3). \]

Moreover, we define \( \tilde{M}_i(s) := M_i + \eta_3 M(s)K(s)\eta_3 \) and note that

\[ R_{X,\chi}(s) = \tilde{M}_i(s) + M_f(s) + M_c(s) + Q(s). \]

We now have to show that \( \tilde{M}_i(s) \) and \( Q(s) \) have the claimed properties.

**Interior term.** The operator \( M_i \) is a compactly supported pseudodifferential operator by definition, so it suffices to show that \( \eta_3 M(s)K(s)\eta_3 \) is a pseudodifferential operator of order at most \(-2\). By (26) we have

\[ \eta_3 L(s) = L(s). \]

Now equation (31) directly implies that

\[ K(s)\eta_3 = (\text{id} - L(s))^{-1}\eta_3 - \eta_3. \]

From \( \eta_3 L(s) = L(s) \), we obtain

\[ (\text{id} + K(s)\eta_3)(\text{id} - L(s)\eta_3) = \text{id}. \]

Consequently,

\[ \text{id} + K(s)\eta_3 = (\text{id} - L(s)\eta_3)^{-1}. \]

for \( s \) close to \( s_0 \). By the identity theorem for holomorphic functions, the equality in (32) is valid for all \( s \in \mathbb{C} \). Formally, we can also obtain this from the geometric series,

\[ \text{id} + K(s)\eta_3 = \text{id} + \sum_{k>0} L(s)^k \eta_3 = (\text{id} - L(s)\eta_3)^{-1}. \]

We have that

\[ L(s)\eta_3 = (s(1 - s) - s_0(1 - s_0)) M_i + \tilde{Q}(s), \]

where \( \tilde{Q}(s) \) is compactly supported and smoothing. Thus, \( L(s)\eta_3 \) is a pseudodifferential operator of order \(-2\) and therefore

\[ K(s)\eta_3 = (\text{id} - L(s)\eta_3)^{-1}L(s)\eta_3 \]

is also a pseudodifferential operator of order \(-2\). By the definition of \( M(s) \), the operator \( \eta_3 M(s) \) is a pseudodifferential operator of order \(-2\) and hence \( \eta_3 M(s)K(s)\eta_3 \) is a pseudodifferential operator of order \(-4\).

**Residual term.** To study the operator \( Q(s) \), we start by considering the operator \( M(s)K(s)(1 - \eta_3) \). We use (30) to show that

\[ K(s) = L(s)(\text{id} + K(s)). \]

Since \( L(s) \) maps to compactly supported smooth sections, we use the explicit calculations for the model resolvents to obtain that, for any \( \varphi \in C^\infty_c(X, E_\chi) \), we have

\[ L(s)^T \varphi \in (\rho_f)^s(\rho_c)^{s-1}C^\infty(\overline{X}, E_\chi). \]

Moreover, the property

\[ L(s)\varphi \in (\rho_f)^s(\rho_c)^{s-1}C^\infty(\overline{X}, E_\chi) \]
implies that the integral kernel \( K(s)(1 - \eta_3)(\cdot, \cdot) \) of the operator \( K(s)(1 - \eta_3) \) satisfies

\[
K(s)(1 - \eta_3)(\cdot, \cdot) \in (\rho_f \rho_c)^\infty (\rho_f')^s (\rho_c')^{s-1} C^\infty(\overline{X} \times \overline{X}, E_\chi \otimes E_\chi')
\]

and is compactly supported in the left-most variable. Using that \( M_t \) is a compactly supported pseudodifferential operator and \( M_f(s) \) and \( M_c(s) \) are given by the model resolvents, we conclude that the integral kernel of the operator \( M(s)K(s)(1 - \eta_3) \) satisfies

\[
M(s)K(s)(1 - \eta_3)(\cdot, \cdot) \in (\rho_f \rho_c)^\infty (\rho_f')^s (\rho_c')^{s-1} C^\infty(\overline{X} \times \overline{X}, E_\chi \otimes_0 E_\chi').
\]

For \((1 - \eta_3)M(s)K(s)\), we use that \( K(s)\eta_3 \) is compactly supported and that the integral kernel of the operator \((1 - \eta_3)M(s)K(s)\eta_3\) satisfies

\[
(1 - \eta_3)M(s)K(s)\eta_3(\cdot, \cdot) \in (\rho_f \rho_c)^\infty (\rho_f')^s (\rho_c')^{s-1} C^\infty(\overline{X} \times \overline{X}, E_\chi \otimes E_\chi').
\]

This proves the theorem. \( \square \)

We will now provide formula for \( Q(s) \) restricted to the boundary that will be useful later on. Let \( \varphi \in C^\infty(\overline{X}, E_\chi) \) such that \( \eta_3 \varphi = 0 \). In this case \( Q(s)\varphi \) simplifies to

\[
Q(s)\varphi = M(s)K(s)\varphi = M(s)(\text{id} - L(s))^{-1}L(s)\varphi.
\]

Using that \( L(s) = \eta_3L(s) \), we obtain

\[
L(s) = \eta_3(\text{id} - L(s)\eta_3)(\text{id} - L(s)\eta_3)^{-1}L(s) = (\text{id} - L(s))\eta_3(\text{id} - L(s)\eta_3)^{-1}L(s).
\]

Hence, we have

\[
Q(s)\varphi = M(s)\eta_3(\text{id} - L(s)\eta_3)^{-1}L(s)\varphi.
\]

4.1. **Resolvent at a Resonance.** Let \( s_0 \in \mathbb{C} \) be a resonance of \( \Delta_{X,\chi} \). As in [DFP, Section 6], we define the multiplicity of the resonance \( s_0 \) as the number

\[
m_{X,\chi}(s_0) := \text{rank} \int_{\gamma_{\varepsilon,s_0}} R_{X,\chi}(s) \, ds,
\]

where \( \varepsilon > 0 \) is chosen such that the path \( \gamma_{\varepsilon,s_0} : [0, 1] \to \mathbb{C} \) with

\[
\gamma_{\varepsilon,s_0}(t) := s_0 + \varepsilon e^{2\pi it}
\]

encloses exactly one resonance (namely \( s_0 \)). We denote the multiset of resonances by

\[
\mathcal{R}_{X,\chi} := \{(s_0, m) \in \mathbb{C} \times \mathbb{N} : s_0 \text{ is a resonance, } m = m_{X,\chi}(s_0)\}
\]

and the multiset of resonances of the model funnel ends by

\[
\mathcal{R}_{X,f,\chi} := \bigcup_{j=1}^{n_f} \mathcal{R}_{X_{f,j},\chi_j},
\]

where the multiset \( \mathcal{R}_{X_{f,j},\chi_j} \) is given as in (6).
In a small neighborhood of the resonance \( s_0 \), the resolvent admits an expansion

\[
R_{X,\chi}(s) = \sum_{j=1}^{p} \frac{A_j(s_0)}{(s(1-s) - s_0(1-s))^j} + H(s, s_0)
\]

for some \( p \in \mathbb{N} \), further referred to as the order of the resonance, where, for \( j = 1, \ldots, p \), the coefficient \( A_j(s_0) \) is a finite rank operator, and the map \( s \mapsto H(s, s_0) \) is holomorphic in a small neighborhood of \( s_0 \).

Now let \( s_0 \neq 1/2 \) and fix \( j = 1, \ldots, p \). We multiply (35) by

\[
(s(1-s) - s_0(1-s))^{j-1}
\]

and integrate both sides along the path \( \gamma_{\varepsilon,s_0} \). We substitute \( \lambda = s(1-s) \) and use \( d\lambda = (1-2s)ds \). The path of the integration changes to

\[
\tilde{\gamma}_{\varepsilon,s_0}(t) = s_0(1-s_0) + (1-2s_0)\varepsilon e^{2\pi it} + \varepsilon^2 e^{4\pi it}.
\]

For \( s_0 \neq 1/2 \) and \( \varepsilon \) small enough, \( \tilde{\gamma}_{\varepsilon,s_0}(t) \) winds around \( s_0(1-s_0) \) once. Applying the Cauchy integration formula, we get

\[
A_j(s_0) = \frac{1}{2\pi i} \int_{\gamma_{\varepsilon,s_0}} (1-2s) (s(1-s) - s_0(1-s))^{j-1} R_{X,\chi}(s) \, ds
\]

for any \( j = 1, \ldots, p \). We note that for \( j = 1 \), the equality (36) was obtained in the proof of [GZ97, Lemma 2.4].

Note that (36) implies that the operator \( A_j(s_0) \) is symmetric. Together with \( m_{X,\chi}(s_0) = \text{rank} A_1(s_0) \), this yields

\[
A_1(s_0) = \sum_{\ell,m=1}^{m_{X,\chi}(s_0)} a_1^{\ell,m}(s_0) \phi_\ell(\phi_m, \cdot),
\]

where \( a_1(s_0) = (a_1^{\ell,m}(s_0))_{\ell,m=1}^{m_{X,\chi}(s_0)} \) is a symmetric invertible matrix and

\[
\phi_j \in \rho^{-N}L^2(X, E_\chi)
\]

for \( j = 1, \ldots, m_{X,\chi}(s_0) \) and \( \text{Re}(s_0) > 1/2 - N \) for any \( N \in \mathbb{N} \). The definition of the resolvent implies that for any \( j = 1, \ldots, p-1, \)

\[
A_{j+1}(s_0) = A_j(s_0)(\Delta_{X,\chi} - s_0(1-s_0))
= (\Delta_{X,\chi} - s_0(1-s_0))A_j(s_0),
\]

\[
A_{p+1}(s_0) = 0.
\]

Therefore, we obtain that

\[
A_k(s_0) = \sum_{\ell,m=1}^{m_{X,\chi}(s_0)} a_k^{\ell,m}(s_0) \phi_\ell(\phi_m, \cdot),
\]

where \( a_k(s_0) := a_1(s_0)d(s_0)^{k-1} \) for \( d(s_0) := a_1(s_0)^{-1}a_2(s_0) \). Note that the matrix \( d(s_0) \) is nilpotent.
4.2. Absence of Poles with $\text{Re } s = 1/2$. In this section, we will show that for $s \in \mathbb{C}$ with $\text{Re } s = 1/2$ there is at most one resonance at $s = 1/2$. This will imply that there are no eigenvalues larger than $1/4$.

The Carleman estimate [Maz91, Theorem (7)] reads in our setting as follows (cf. Borthwick [Bor16, Lemma 7.6]).

**Proposition 4.3.** Let $F_{\ell} \subset C_{\ell} = \langle h_{\ell} \rangle \backslash \mathbb{H}$ be a hyperbolic funnel and let $\chi: \langle h_{\ell} \rangle \rightarrow U(V)$ be a unitary finite-dimensional representation. Denote by $\rho_f$ the boundary defining function of $\partial_{\infty}F_{\ell}$. Let $r_0, k \geq 0$ and suppose that $u \in \mathcal{C}^\infty(X, E_\chi) \cap \mathcal{D}(\Delta_{X, \chi} - s(1 - s))$ is a solution of $(\Delta_{X, \chi} - s(1 - s))u = 0$ for some $s \not\in -\mathbb{N}_0/2$. If

$$k^3 \int_{F_{\ell}} e^{2kr_0} |u|^2 \, d\mu_{F_{\ell}} + k \int_{F_{\ell}} e^{2kr_0} |\nabla \chi u|^2 \, d\mu_{F_{\ell}} \leq C \int_{F_{\ell}} e^{2kr_0} |\Delta_{F_{\ell}, \chi} u|^2 \, d\mu_{F_{\ell}}.$$  

The Carleman estimate implies the following result on unique continuations (see [Bor16, Proposition 7.4] for the untwisted case).

**Proposition 4.4.** Let $X = \Gamma \backslash \mathbb{H}$ be an infinite-volume hyperbolic surface and $\chi: \Gamma \rightarrow U(V)$ be a unitary finite-dimensional representation. Suppose that $u \in \mathcal{C}^\infty(X, E_\chi) \cap \mathcal{D}(\Delta_{X, \chi} - s(1 - s))$ is a solution of $(\Delta_{X, \chi} - s(1 - s))u = 0$ for some $s \not\in -\mathbb{N}_0/2$. If

$$u|_{F_{\ell},j} \in \rho_f^{s+n}\mathcal{C}^\infty(\overline{X_{f,j}}, E_\chi)$$

for some $j = 1, \ldots, n_f$, then $u \equiv 0$.

We adapt the proof of [Bor16, Proposition 7.4] to the twisted case.

**Proof of Proposition 4.4.** Without loss of generality, we assume that $X$ has only one funnel end, that is $n_f = 1$ and $X_f = X_{f,1}$. We prove the proposition in two steps.

**Step 1:** We want to show by induction that

$$u|_{X_f} \in \rho_f^{s+n}\mathcal{C}^\infty(\overline{X_f}, E_\chi), \quad \forall n \in \mathbb{N}.$$ 

The base case is true by (39). Suppose that $u|_{X_f} \in \rho_f^{s+n}\mathcal{C}^\infty(\overline{X_f}, E_\chi)$ for some $n \in \mathbb{N}$. Write $u|_{X_f} = \rho_f^{s+n}v$, where $v \in \mathcal{C}^\infty(\overline{X_f}, E_\chi)$. Using (5) we obtain

$$(\Delta_{X, \chi} - s(1 - s))\rho_f^{s+n}v = n(1 - 2s - n)\rho_f^{s+n}v + O(\rho_f^{s+n+1}).$$

Since $u$ solves $(\Delta_{X, \chi} - s(1 - s))u = 0$, it follows that $v = O(\rho_f)$ under the assumption that $s \not\in -\mathbb{N}_0/2$. Therefore,

$$u|_{X_f} \in \rho_f^{s+n+1}\mathcal{C}^\infty(\overline{X_f}, E_\chi).$$

By induction, we obtain that $u|_{X_f} \in \rho_f^{s}\mathcal{C}^\infty(\overline{X_f}, E_\chi)$.

**Step 2:** We want to show that $u \equiv 0$. Choose $r_0, r_1 \in (0, 1)$ with $r_1 > r_0$ and choose $\eta \in \mathcal{C}^\infty([0, 1])$ such that $\eta(r) = 1$ for $r \leq r_0$ and $\eta(r) = 0$ for $r \geq r_1$. The function $\eta(\rho_f)u|_{X_f}$ satisfies the assumptions of Proposition 4.3. We hence obtain

$$k^3 \int_{X_f} \rho_f^{-2k}\eta(\rho_f)^2 |u|^2 \, d\mu_X \leq C \int_{X_f} \rho_f^{-2k}|\Delta_{X, \chi}\eta(\rho_f)u|^2 \, d\mu_X$$

and suppose that

$$\eta(\rho_f)u|_{X_f} \not\equiv 0.$$
for $k > 0$ large enough. Denote

$$I_1 := (1 + |s(1-s)|^2) \int_{X_f \cap \{\rho_f \leq r_0\}} \rho_f^{-2k}|u|^2 d\mu_X.$$ 

Using the equation $(\Delta_{X,\chi} - s(1-s))u = 0$ and the fact that $\eta(r) = 1$ for $r \leq r_0$, we obtain

$$\frac{I_1 \cdot k^3}{1 + |s(1-s)|^2} = k^3 \int_{X_f \cap \{\rho_f \leq r_0\}} \rho_f^{-2k}|u|^2 d\mu_X$$

$$\leq k^3 \int_{X_f} \rho_f^{-2k} |\eta(\rho_f)|^2 |u|^2 d\mu_X$$

$$\leq C \int_{X_f} \rho_f^{-2k} |\Delta_{X,\chi}\eta(\rho_f)u|^2 d\mu_X$$

$$= C \int_{X_f \cap \{\rho_f \leq r_1\}} \rho_f^{-2k} |\Delta_{X,\chi}\eta(\rho_f)u|^2 d\mu_X$$

$$+ C \int_{X_f \cap \{\rho_f \leq r_0\}} \rho_f^{-2k} |\Delta_{X,\chi}\eta(\rho_f)u|^2 d\mu_X$$

$$\leq C (I_2 + I_3 + I_1),$$

where $C = C(r_0, r_1, \eta) > 0$ and

$$I_2 := (1 + |s(1-s)|^2) \int_{X_f \cap \{\rho_f \leq r_0\}} \rho_f^{-2k}|u|^2 d\mu_X,$$

$$I_3 := \int_{X_f \cap \{\rho_f \leq r_0\}} \rho_f^{-2k} |\nabla_{X,\chi}u|^2 d\mu_X.$$ 

Setting $C' = (1 + |s(1-s)|^2)^{-1}C^{-1}$, we rewrite the above estimate as

$$I_1 \leq (C'k^3 - 1)^{-1}(I_2 + I_3).$$

We estimate $I_2$ and $I_3$ by

$$I_2 + I_3 \leq C'' \int_{r_0}^{r_1} \rho^{-2k-1} d\rho$$

$$= \frac{C''}{2kr_0^{2k}} \left(1 - \left(\frac{r_1}{r_0}\right)^{-2k}\right),$$

for some $C'' > 0$, which depends on $r_0, r_1, s$, and $u$, but is independent of $k$. Therefore we arrive at

$$\int_{X_f \cap \{\rho_f \leq r_0\}} |u|^2 d\mu_X \leq \frac{\int_{r_0}^{r_1} \rho^{-2k} d\rho}{1 + |s(1-s)|^2} I_1$$

$$\leq \frac{C'' \left(1 - \left(\frac{r_1}{r_0}\right)^{-2k}\right)}{2k(1 + |s(1-s)|^2)(C'k^3 - 1)}.$$

Letting $k \to \infty$, we obtain that $||u||_{L^2(X_f \cap \{\rho_f \leq r_0\})} = 0$ and consequently $u = 0$ on $X_f \cap \{\rho_f \leq r_0\}$. By standard uniqueness results of elliptic differential operators, we conclude that $u = 0$ everywhere. \qed
In the case $\Re s = 1/2$ and $s \neq 1/2$, we can prove a better result following [Bor16, Lemma 7.7].

**Proposition 4.5.** Let $X$ and $\chi$ be as above, and let $\Re s = 1/2$ with $s \neq 1/2$. If $u \in C^\infty(X, E_\chi)$ satisfies $u|_{X_{f,j}} \in \rho_f^j C^\infty(X_{f,j}, E_\chi)$ for some $j \in \{1, \ldots, n_f\}$ and

$$(\Delta_{X,\chi} - s(1 - s))u = 0,$$

then $u \equiv 0$.

**Proof.** Without loss of generality, we may suppose that $n_f = 1$ and $X_f = X_{f,1}$. We take local coordinates $(r, \phi) \in \mathbb{R}_+ \times \mathbb{R}/2\pi \mathbb{Z} \cong X_f$. We have that $u(r, \phi + 2\pi) = (\chi(h_f))u(r, \phi)$, where $h_f \in \Gamma$ is the unique (up to inversion) hyperbolic element associated to the funnel end $X_f$ and $\ell \in (0, \infty)$ is the length of the central geodesic of $X_f$ (see [DFP, Section 3.2.3] for details).

Let $\varepsilon > 0$ and let $\psi \in C^\infty(\mathbb{R}_+)$ be real-valued with $\psi(t) = 0$ for $t \leq 1$ and $\psi(t) = 1$ for $t \geq 2$. Set $\psi_\varepsilon \in C^\infty(\mathbb{X})$ with $\psi_\varepsilon(\rho, \theta) = \psi(\rho/\varepsilon)$ for $(\rho, \theta) \in X_f$ and $\psi_\varepsilon = 1$ on $X \setminus X_f$. Since $\Re s = 1/2$, we have that $s(1 - s) \in \mathbb{R}$ and thus

$$0 = \int_X \left( s(1 - s)(\psi_\varepsilon u, u)_{E_\chi} - s(1 - s)(u, \psi_\varepsilon u)_{E_\chi} \right) \, d\mu_X$$

$$= \int_X (\Delta_{X,\chi}, \psi_\varepsilon \cdot \text{id}_{E_\chi})u, u)_{E_\chi} \, d\mu_X$$

$$= \int_{X_f} (\Delta_{X,\chi}, \psi_\varepsilon \cdot \text{id}_{E_\chi})u, u)_{E_\chi} \, d\mu_X.$$

The function $u$ can be written as $u = \rho^s v$, where $v \in C^\infty(X_f, E_\chi)$. By assumption, we have that $\Re s = 1/2$, therefore $|u|^2 = \rho|v|^2$. Writing $x = \rho/\varepsilon$, we obtain, using (5), that

$$(\Delta_{X,\chi}, \psi_\varepsilon \cdot \text{id}_{E_\chi})u, u)_{E_\chi} = -\varepsilon x^2 (x \psi''(x) + 2s \psi'(x))|v(0, \phi)|^2 + O(\varepsilon^2).$$

as $\varepsilon \to 0$. It follows from (4), that the measure $d\mu_X$ restricted to $X_f$ is given by

$$d\mu_X|_{X_f} = \rho^{1-\frac{\ell}{2}} \frac{d\rho \, d\phi}{2\pi \sqrt{1 - \rho^2}}$$

$$= \varepsilon^{-1} x^{-2} \frac{\ell}{2} \frac{dx \, d\phi}{2\pi \sqrt{1 - \varepsilon^2 x^2}}.$$

Therefore we have, as $\varepsilon \to 0$, that

$$\int_{X_f} (\Delta_{X,\chi}, \psi_\varepsilon \cdot \text{id}_{E_\chi})u, u)_{E_\chi} \, d\mu_X$$

$$= -\frac{\ell}{2\pi} \int_1^2 \int_0^{2\pi} (x \psi''(x) + 2s \psi'(x))|v(0, \phi)|^2 \, dx \, d\phi + O(\varepsilon).$$

We calculate that $\int_1^2 \psi'(x) = 1$ and $\int_1^2 x \psi''(x) \, dx = -1$ and therefore

$$\int_{X_f} (\Delta_{X,\chi}, \psi_\varepsilon \cdot \text{id}_{E_\chi})u, u)_{E_\chi} \, d\mu_X = (1 - 2s) \frac{\ell}{2\pi} \int_0^{2\pi} |v(0, \phi)|^2 \, d\phi + O(\varepsilon).$$
as \( \varepsilon \to 0 \). This implies that \( u|_{X_f} \in \rho_f^{s+1}C^\infty(X_f, E_\chi) \). Together with Proposition 4.4 this implies the claim. \( \square \)

Proposition 4.5 implies almost no resonances of the critical line.

**Corollary 4.6.** For \( \text{Re } s = 1/2 \) and \( s \neq 1/2 \), the resolvent \( R_{X,\chi} \) has no pole at \( s \).

**Proof.** By (35), we have that

\[
R_{X,\chi}(s) = \sum_{j=1}^{p} \frac{A_j(s_0)}{(s-1-s_0)(s_0(1-s_0))} + H(s, s_0),
\]

where \( p \in \mathbb{N} \) is the order of the resonance, \( A_j(s_0), j = 1, \ldots, p \) are finite rank operators and \( H(s, s_0) \) is holomorphic in \( s \) near \( s = s_0 \). Let \( \psi \in C_c^\infty(X, E_\chi) \) and write \( u = A_p(s_0)\psi \). By the definition of the resolvent, we have that

\[
(\Delta_{X,\chi} - s_0(1-s_0))u = 0.
\]

By Theorem 4.1, we have that \( u \in \rho_f^s \rho_f^{s-1}C^\infty(X, E_\chi) \). For \( \text{Re } s_0 = 1/2 \) and \( s \neq 1/2 \), Proposition 4.5 implies that \( u = 0 \) and consequently \( A_p = 0 \). This shows that \( R_{X,\chi}(s) \) is holomorphic near \( s_0 \). \( \square \)

**Proposition 4.7.** The Laplacian \( \Delta_{X,\chi} \) has no eigenvalues in the interval \((1/4, \infty)\).

**Proof.** Let \( \lambda \in (1/4, \infty) \) and set \( s := 1/2 + i\sqrt{\lambda - 1/4} \). This implies that \( \lambda = s(1-s) \). Assume that \( \lambda \) is an eigenvalue of \( \Delta_{X,\chi} \), then there exists a function \( u \in L^2(X, E_\chi) \) such that

\[
(\Delta_{X,\chi} - s(1-s))u = 0.
\]

Since \( X \) has infinite volume, there is at least one funnel end, which we will denote by \( X_f \). We choose coordinates \((r, \phi) \in X_f \) as in Section 3.1. Choose \( \psi \in C_c^\infty(X_f, E_\chi|_{X_f}) \) such that \( \text{supp } \psi \subset \{ r \geq 2 \} \). Then we have by (29) that

\[
(\Delta_{X,\chi} - s(1-s))M_f(s)\psi = \psi - L_f(s)\psi.
\]

Let \( \varepsilon > 0 \). Integrating by parts, we have that

\[
\varepsilon(2s - 1 + \varepsilon) \int_{X_f} \langle M_f(s + \varepsilon)\psi, u \rangle_{E_\chi} d\mu_X
\]

\[
= \int_{X_f} \langle (\Delta_{X,\chi} - (s + \varepsilon)(1-s - \varepsilon))M_f(s + \varepsilon)\psi, u \rangle_{E_\chi} d\mu_X
\]

\[
= \int_{X_f} \langle \psi - L_f(s + \varepsilon)\psi, u \rangle_{E_\chi} d\mu_X.
\]

By the Cauchy–Schwarz inequality, we have that

\[
\left| \int_{X_f} \langle M_f(s + \varepsilon)\psi, u \rangle_{E_\chi} d\mu_X \right| \leq \|M_f(s + \varepsilon)\psi\|\|u\|
\]

and using (7) and (24), we obtain that

\[
\rho^{-s-\varepsilon}M_f(s + \varepsilon)\psi \in C^\infty(X_f, E_\chi).
\]
Therefore, we can estimate
\[ \|M_f(s + \varepsilon)\psi\| \leq \sup_{X_f} \rho^{-s-\varepsilon} \|M_f(s + \varepsilon)\psi\| \|\rho^{s+\varepsilon}\|, \]
where the first factor in the right-hand side is bounded by a constant and the second factor is \(O(\varepsilon^{-1/2})\) by a direct calculation. This implies that
\[
\int_{X_f} \langle \psi - L_f(s + \varepsilon)\psi, u \rangle_{E,\chi} d\mu_X = O(\varepsilon^{1/2}) \quad \text{as} \quad \varepsilon \to 0.
\]
By the fundamental lemma of calculus of variations, this implies that
\[
u(z) = (L_f(s) u)(z)
\]
for \(z \in X_f \cap \{r \geq 2\} \). By the definition of \(L_f(s)\), \(\psi\), we have that \(u|_{X_f} \in \rho_f^s C^\infty(X_f, E_\chi)\). Set \(u_0(s) := \rho_f^s u|_{\partial_{\infty} X_f}\). Since \(u \in L^2(X, E_\chi)\) and \(\text{Re} s = 1/2\), it follows that \(u_0(s) \equiv 0\) and therefore
\[
u|_{X_f} \in \rho_f^{s+1} C^\infty(X_f, E_\chi).
\]
Proposition 4.4 now finishes the proof. \(\Box\)

5. Scattering Determinant

In this section, we prove Theorems A and B. We start with introducing the Poisson operator and studying its properties in Section 5.1. In Section 5.2, we define the scattering matrix and show the correspondence of resonances and poles of the scattering matrix for \(\text{Re} s < 1\) and \(s \neq 1/2\).

In Section 5.3, we study the behavior of \(R_{X,\chi}(s)\) near \(s = 1/2\) and prove Theorem B. In Section 5.4, we recall the basics of the Gohberg-Sigal theory and obtain a relation of scattering poles and resonances for \(\text{Re}(s) \leq 1\).

In Section 5.5, we introduce the relative scattering matrix and the relative scattering determinant and, finally, prove Theorem A.

5.1. Poisson Operator. Before we define the scattering matrix, we introduce the Poisson operator, which maps sections \(C^\infty(\partial_{\infty} X_f, E_\chi|_{\partial_{\infty} X_f})\) to solutions of the equation \((\Delta_{X,\chi} - s(1 - s))u = 0\) with prescribed asymptotics at the boundary at infinity. The construction is similar to the one in the untwisted case [GZ97, (2.23)-(2.25)], but in our case the Poisson operator acts on sections of vector bundles and we have to be more careful due to the compactification in the cusp, which depends on the representation \(\chi\).

We recall that the ideal boundary at infinity \(\partial_{\infty} X\) is a disjoint union of circles (representing funnel ends) and points (representing cusp ends) and that we have the decomposition
\[
\partial_{\infty} X = \partial_f X \sqcup \partial_c X.
\]
For \(j \in \{1, \ldots, n_f\}\) and \(s \notin \mathcal{R}_{X,\chi}\) we define the map
\[
E_{X,\chi}^{f,j}(s) : C^\infty(\partial_{\infty} X_f, E_\chi|_{\partial_{\infty} X_f}) \rightarrow C^\infty(X, E_\chi)
\]
by its Schwartz kernel
\[
E_{X,\chi}^{f,j}(s, z, \theta') := (\rho_f^s)^{-s} R_{X,\chi}(s; z, z')|_{X \times \partial_{\infty} X_f}.
\]
The restriction is well-defined by Theorem 4.1 and (7). Similarly, for $j \in \{1, \ldots, n_c\}$ and $s \not\in \mathcal{R}_{X,\chi}$ we define

$$E^c_{X,\chi}(s) : C^\infty(\partial_{\infty}X_{c,j}, E_{X}\big|_{\partial_{\infty}X_{c,j}}) \to C^\infty(X, E_{X}),$$

$$E^{c,j}_{X,\chi}(s, z, \theta') := (\rho_c)^{-s} R_{X,\chi}(s; z, \theta') \big|_{X \times \partial_{\infty}X_{c,j}}.$$  

The restriction is well-defined by Theorem 4.1 and (20). Further, by (33), (28), and (25), the map $E^{c,j}_{X,\chi}(s, z, \theta')$ is independent of $\theta'$ and defines an operator $C^\infty(\partial_{\infty}X_{c,j}, E_{X}\big|_{\partial_{\infty}X_{c,j}}) \to C^\infty(X, E_{X}).$ We denote this two-variable function by $E^{c,j}_{X,\chi}$ as well. We obtain the Poisson operator defined by its Schwartz kernel as follows:

$$E_{X,\chi}(s) : C^\infty(\partial_{\infty}X, E_{X}\big|_{\partial_{\infty}X}) \to C^\infty(X, E_{X}),$$

$$(E_{X,\chi}(s)\psi)(z) := \sum_{j=1}^{n_j} \frac{\ell_j}{2\pi} \int_0^{2\pi} E^{c,j}_{X,\chi}(s, z, \theta') f_j(\theta') d\theta' + \sum_{j=1}^{n_c} E^{c,j}_{X,\chi}(s, z) a_j,$$

where $\psi = (f_1, \ldots, f_{n_j}, a_1, \ldots, a_{n_c}) \in C^\infty(\partial_{\infty}X, E_{X}\big|_{\partial_{\infty}X}).$ The transposed operator

$$E_{X,\chi}(s)^{T} : C^\infty_{c}(X, E_{X}) \to C^\infty(\partial_{\infty}X, E_{X}\big|_{\partial_{\infty}X})$$

is given by

$$E_{X,\chi}(s)^{T} u = (f_1, \ldots, f_{n_j}, a_1, \ldots, a_{n_c}),$$

where

$$f_j(\theta) = \int_X E^{c,j}_{X,\chi}(s, z', \theta) u(z') d\mu_X(z'),$$

$$a_j = \int_X E^{c,j}_{X,\chi}(s, z') u(z') d\mu_X(z').$$

By the same arguments as in the proof of [DFP, Lemma 4.14], we can express the difference of resolvents in terms of the Poisson operator for general hyperbolic surfaces:

**Proposition 5.1.** Let $X = \Gamma \setminus \mathbb{H}$ be a geometrically finite hyperbolic surface and $\chi : \Gamma \to U(V)$ a finite-dimensional unitary representation. For $s \not\in \mathcal{R}_{X,\chi} \cup (1 - \mathcal{R}_{X,\chi})$, we have

$$R_{X,\chi}(s) - R_{X,\chi}(1 - s) = (1 - 2s) E_{X,\chi}(s) E_{X,\chi}(1 - s)^T.$$

**Proof.** We follow the proof of [DFP, Lemma 4.14], but we have to take care of the multiple ends.

We fix a fundamental domain $\mathcal{F} \subset \mathbb{H}$ of $X$. Then the bundle $E_{X} \boxtimes E'_{X}$ is trivial and can be identified with $\text{End}(V)$. We fix $z, w \in \mathcal{F}$. We define the coefficients of $R_{X,\chi}(s; z, w)$ as

$$R_{jk}(s; z, w) := \langle R_{X,\chi}(s; z, w) e_j, e_k \rangle_V.$$

We also set

$$R_{jk}^{T}(s; z, w) := \langle R_{X,\chi}^{T}(s; z, w) e_j, e_k \rangle_V,$$

where $R_{X,\chi}^{T}(s; z, w)$ denotes the Schwartz kernel of the operator $R_{X,\chi}(s)^{T}$. 

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We calculate

\[
R_{jk}(s; z, w) - R_{jk}(1 - s; z, w) \\
= \lim_{\varepsilon \to 0} \int_{\rho(z) > \varepsilon} \sum_{m=1}^{\dim V} \left( R_{jm}(s; z, z') \Delta_{X, \chi} R_{mk}(1 - s; z', w) \\
- \Delta_{X, \chi} R_{jm}(s; z, z') R_{mk}(1 - s; z', w) \right) d\mu_X(z') \\
= \lim_{\varepsilon \to 0} \int_{\rho(z) > \varepsilon} \sum_{m=1}^{\dim V} \left( R_{jm}(s; z, z') \Delta_{X, \chi} R_{km}^T(1 - s; w, z') \\
- \Delta_{X, \chi} R_{jm}(s; z, z') R_{km}^T(1 - s; w, z') \right) d\mu_X(z') \\
= \lim_{\varepsilon \to 0} \int_{\rho(z) = \varepsilon} \sum_{m=1}^{\dim V} \left( - R_{jm}(s; z, z') \partial_{\rho} R_{km}^T(1 - s; w, z') \\
+ \partial_{\rho} R_{jm}(s; z, z') R_{km}^T(1 - s; w, z') \right) d\sigma_X(z').
\]

Here, \( X_\varepsilon := \{ z \in X : \rho(z) = \varepsilon \} \), and \( d\sigma_X \) is the induced measure on \( X_\varepsilon \). If we pick \( \varepsilon > 0 \) sufficiently small, then the area of integration splits into a disjoint union of funnel and cusp ends. Without loss of generality, we suppose that \( X_f = X_{f,\varepsilon} \) and we set \( X_{f,\varepsilon} := X_f \cap X_\varepsilon \). From (4), we see that \( \partial_{\rho} = \rho \partial_{\rho} + O(\rho^2) \). For \( z \in X \) with \( \rho(z) > \varepsilon \) and \( z' \in X_{f,\varepsilon} \), we have that

\[
R_{X, \chi}(s; z, z') = (\rho')^s E_{X, \chi}(s; z, \phi') + O((\rho')^{s+1})
\]

and

\[
\partial_{\rho} R_{X, \chi}(s; z, z') = -\rho' \partial_{\rho} R_{X, \chi}(s; z, z') \\
= -s(\rho')^s E_{X, \chi}(s; z, \phi') + O((\rho')^{s+1}).
\]

Consequently,

\[
- R_{jm}(s; z, z') \partial_{\rho} R_{km}^T(1 - s; w, z') \\
= (1 - s) \varepsilon E_{jm}(s; z, \phi') E_{km}^T(1 - s; w, \phi') + O(\varepsilon^2)
\]

and

\[
\partial_{\rho} R_{jm}(s; z, z') R_{km}^T(1 - s; w, z') \\
= -s \varepsilon E_{jm}(s; z, \phi') E_{km}^T(1 - s; w, \phi') + O(\varepsilon^2).
\]

Moreover, \( d\sigma_X |_{X_{f,\varepsilon}} = (2\pi \varepsilon)^{-1} \ell d\phi = \varepsilon^{-1} d\sigma_{\partial_{\infty} X_f} \), where \( \ell \in (0, \infty) \) is the length of the central geodesic associated to \( X_f \). Therefore, we obtain that

\[
\int_{X_{f,\varepsilon}} \left( - R_{jm}(s; z, z') \partial_{\rho} R_{km}^T(1 - s; w, z') \\
+ \partial_{\rho} R_{jm}(s; z, z') R_{km}^T(1 - s; w, z') \right) d\sigma_X(z') \\
= (1 - 2s) \int_{\partial_{\infty} X_f} \left( E_{jm}(s; z, \phi') E_{km}^T(1 - s; w, \phi') + O(\varepsilon) \right) d\sigma_{\partial_{\infty} X_f}(\phi').
\]

Letting \( \varepsilon \to 0 \) proves the claim for the funnel ends.
For the cusp ends, we also suppose without loss of generality that \( n_c = 1 \) and \( X_c = X_{c,1} \) is a single cusp end. We set \( X_{c,e} := X_c \cap X_e \). We take coordinates \((\rho, \phi) \in \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \cong X_c \) as in Section 3.2 and calculate \( g_{X_c}(\partial_\rho, \partial_\phi) = \rho^{-2} \) and therefore \( \partial_\nu = -\rho^\prime \partial_\rho^\prime \). By the definition of the Poisson operator, we have for \( z \in X \) with \( \rho(z) > \varepsilon \) and \( z' \in X_{c,e} \) and as \( \varepsilon \to 0 \) (hence \( \rho' \to 0 \)),
\[
R_{X,\chi}(s; z, z') = (\rho')^{s-1}E_{X,\chi}(s; z, \phi^\prime) + O((\rho')^s)
\]
and
\[
\partial_\nu R_{X,\chi}(s; z, z') = -\rho^\prime \partial_\rho^\prime R_{X,\chi}(s; z, z')
\]
\[
= -s(\rho')^{s-1}E_{X,\chi}(s; z, \phi^\prime) + O((\rho')^s).
\]
Therefore,
\[
R_{jm}(s; z, z')\partial_\nu R_{km}^T(1 - s; w, z')
\]
\[
= -(\rho')^{s-1}E_{jm}(s; z, \phi)(1 - s)(\rho')^{-s}E_{km}^T(1 - s; w, \phi^\prime)
\]
\[
+ O((\rho')^0)
\]
and
\[
\partial_\nu R_{jm}(s; z, z')R_{km}^T(1 - s; w, z')
\]
\[
= -s(\rho')^{s-1}E_{jm}(s; z, \phi)(\rho')^{-s}E_{km}^T(1 - s; w, \phi^\prime) + O((\rho')^0).
\]
By (19), we have that \( d\sigma_{X_c}|_{X_{c,e}} = \varepsilon \frac{d\phi}{2\pi} \). Using that \( E_{X,\chi}(s; z, \phi^\prime) \) is independent of \( \phi^\prime \), we arrive at
\[
\int_{X_{c,e}} \left( -R_{jm}(s; z, z')\partial_\nu R_{km}^T(1 - s; w, z')
\right.
\]
\[
+ \partial_\nu R_{jm}(s; z, z')R_{km}^T(1 - s; w, z') \right) d\sigma_{X_c}(z')
\]
\[
= (1 - 2s) \frac{1}{2\pi} \int_0^{2\pi} E_{jm}(s; z)E_{km}^T(1 - s; w) d\phi' + O(\varepsilon)
\]
\[
= (1 - 2s)E_{jm}(s; z)E_{km}^T(1 - s; w) + O(\varepsilon)
\]
Taking \( \varepsilon \to 0 \) yields the result. \( \square \)

The Poisson operator \( E_{X,\chi}(s) \) provides generalized eigenfunctions in the following sense.

**Proposition 5.2.** Let \( s \not\in \mathcal{R}_{X,\chi} \). For any \( \psi \in C^\infty(\partial_\infty X, E_{\chi}|_{\partial_\infty X}) \), we have
\begin{align*}
(\Delta_{X,\chi} - s(1 - s))E_{X,\chi}(s)\psi &= 0
\end{align*}
and
\[
E_{X,\chi}(s)\psi \in \rho_j^{-1-s}e^{-c^\infty(X, E_{\chi})} + \rho_j^s e^{s-1}c^\infty(X, E_{\chi}).
\]

If \( s \not\in \mathbb{Z}/2 \), then we have the asymptotics
\begin{align*}
(2s - 1)E_{X,\chi}(s)\psi &\sim \rho_j^{-1-s}e^{-s} \phi_s + \rho_j^s e^{s-1} \phi_s,
\end{align*}
where \( \phi_s \in C^\infty(\partial_\infty X, E_{\chi}|_{\partial_\infty X}) \) depends meromorphically on \( s \in \mathbb{C} \).

**Remark 5.3.** In the case of the model funnel, this result follows directly from (13).
Proof of Proposition 5.2. It is straightforward to see that $E_{X,X}(s) \psi$ solves the equation (41). To obtain (42), we use the result on the structure of the resolvent, Theorem 4.1. We have that

$$E_{X,X}^{c,j}(s; z, \theta') = \lim_{\rho' \to 0} (\rho')^{-s} \left( M_f(s; z, \rho', \theta') + Q(s; z, \rho', \theta') \right)$$

and by (24),

$$\lim_{\rho' \to 0} (\rho')^{-s} M_f(s; z, \rho', \theta') = (1 - \eta_{f,0}) E_{X,f,X}(s; z, \theta'),$$

where $E_{X,f,X}(s)$ is defined by (8). From the asymptotics of $Q(s)$, Theorem 4.1, we obtain

$$E_{X,X}^{c,j} f_j(z) - (1 - \eta_{f,0}) E_{X,f,X}(s) f_j \in \rho_f \rho_c^{s-1} C^\infty(\overline{X}, E_X).$$

For the cusp ends, we have to be more careful, because the compactification at the cusp of the bundle $E_X$ depends on the multiplicity of the eigenvalue 1 of $\chi(\gamma_j)$, where $\gamma_j \in \Gamma$ is a representative of the conjugacy class $[\gamma_j]$, associated to the cusp $X_{c,j}$. Similar to the funnel case, we have

$$E_{X,X}^{c,j}(s; z, \theta') = \lim_{\rho' \to 0} (\rho')^{-s} \left( M_c(s; z, \rho', \theta') + Q(s; z, \rho', \theta') \right).$$

Using the notation of Section 3.2, we have

$$\lim_{\rho' \to 0} (\rho')^{-s} M_c(s; z, \rho', \theta') = (1 - \eta_{c,0}) \frac{\rho_c^{-s}}{2s - 1} \text{id}_{E_1(\chi(\gamma_j))},$$

where $E_1(\chi(\gamma_j))$ is the 1-eigenspace of $\chi(\gamma_j)$. Let

$$\varphi := \eta_3 (\text{id} - L(s) \eta_3)^{-1} [\Delta_{X,X}, \eta_c, 0] \frac{\rho_c^{-s}}{2s - 1} \text{id}_{E_1(\chi(\gamma_j))}.$$

By (33) we have

$$\lim_{\rho' \to 0} (\rho')^{1-s} Q(s; z, \rho', \theta') = (M(s) \varphi)(z).$$

Therefore, we obtain that

$$\lim_{\rho' \to 0} (\rho')^{1-s} Q(s; z, \rho', \theta') \in \rho_f \rho_c^{s-1} C^\infty(\overline{X}, E_X).$$

By the definition of the compactification of $E_X$ at the cusp, we have for $a_j \in C^n c, j$ that

$$E_{X,X}^{c,j} a_j - (1 - \eta_{c,0}) \frac{1}{2s - 1} \rho^{-s} a_j \in \rho_f \rho_c^{s-1} C^\infty(\overline{X}, E_X).$$

\[ \square \]

5.2. Scattering Matrix. The scattering matrix intertwines the asymptotics of solutions of the equation $(\Delta_{X,X} - s(1-s)) u = 0$ as described in Proposition 5.2.

Definition 5.4. For $s \not\in \mathcal{R}_{X,X} \cup \mathbb{Z}/2$, the scattering matrix is given by

$$S_{X,X}(s): C^\infty(\partial_\infty X, E_X|_{\partial_\infty X}) \to C^\infty(\partial_\infty X, E_X|_{\partial_\infty X}),$$

$$S_{X,X}(s): \psi \mapsto \phi_s,$$

where $\phi_s$ is defined by (42).
We observe that

\[(43) \quad S_{X,\chi}(s)^* = S_{X,\chi}(s), \quad S_{X,\chi}(s)^T = S_{X,\chi}(s), \]

where \( S_{X,\chi}(s)^* \) is the adjoint of \( S_{X,\chi}(s) \) with respect to the complex inner product on \( L^2(\partial_\infty X, E_\chi|_{\partial_\infty X}) \) and \( S_{X,\chi}(s)^T \) is the transposed operator.

**Proposition 5.5.** For any \( s \in \mathbb{C}, \ s \notin \mathcal{R}_{X,\chi} \cup (1 - \mathcal{R}_{X,\chi}) \) and any element \( \psi \in \mathcal{C}_c^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}) \), we have

\[
E_{X,\chi}(1 - s) S_{X,\chi}(s) \psi = -E_{X,\chi}(s) \psi, \\
S_{X,\chi}(1 - s) S_{X,\chi}(s) \psi = \psi.
\]

**Proof.** It suffices to prove the statement for \( \text{Re} \, s \leq 1/2, \ s \neq 1/2 \) and \( s \notin \mathcal{R}_{X,\chi} \cup (1 - \mathcal{R}_{X,\chi}) \). By Proposition 5.1,

\[
R_{X,\chi}(s) - R_{X,\chi}(1 - s) = (1 - 2s) E_{X,\chi}(s) E_{X,\chi}(1 - s)^T.
\]

Multiplying this equation from the left with \( \rho_f^{-s} \rho_c^{1-s} \) and restricting to the boundary yields

\[
E_{X,\chi}(s) S_{X,\chi}(1 - s) S_{X,\chi}(s) \psi = -E_{X,\chi}(1 - s) S_{X,\chi}(s) \psi
= E_{X,\chi}(s) \psi.
\]

By (42), \( E_{X,\chi}(s) \) is injective. This proves the claim. \( \square \)

Proposition 5.5 together with Proposition 5.1 implies that

\[(44) \quad R_{X,\chi}(s) - R_{X,\chi}(1 - s) = (1 - 2s) E_{X,\chi}(1 - s) S_{X,\chi}(s) E_{X,\chi}(1 - s)^T. \]

It is convenient to use the identification

\[ \mathcal{C}_c^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}) \cong \mathbb{C}^{n^\chi}, \]

where \( n^\chi = \sum_{j=1}^{n_\chi} n^{\chi_j} \). Using the decomposition into funnel and cusp ends, we can write the scattering matrix as

\[
S_{X,\chi}(s) = \begin{pmatrix} S_{X,\chi}^{ff}(s) & S_{X,\chi}^{fc}(s) \\ S_{X,\chi}^{cf}(s) & S_{X,\chi}^{cc}(s) \end{pmatrix},
\]

where

\[
S_{X,\chi}^{ff}(s) : \mathcal{C}_c^\infty(\partial_f X, E_\chi|_{\partial_f X}) \to \mathcal{C}_c^\infty(\partial_f X, E_\chi|_{\partial_f X}), \\
S_{X,\chi}^{cf}(s) : \mathcal{C}_c^\infty(\partial_f X, E_\chi|_{\partial_f X}) \to \mathbb{C}^{n^\chi}, \\
S_{X,\chi}^{fc}(s) : \mathbb{C}^{n^\chi} \to \mathcal{C}_c^\infty(\partial_f X, E_\chi|_{\partial_f X}), \\
S_{X,\chi}^{cc}(s) : \mathbb{C}^{n^\chi} \to \mathbb{C}^{n^\chi}.
\]

For \( \text{Re} \, s < 1/2 \), we have that

\[(45) \quad S_{X,\chi}(s) = (2s - 1) (\rho_f \rho_f')^{-s} (\rho_c \rho_c')^{1-s} R_{X,\chi}(s; z, z')|_{\partial_\infty X \times \partial_\infty X}. \]

For \( j = 1, \ldots, n_f \) let \( S_{X,f_j,\chi}(s) \) be the scattering matrix for the funnel end \( X_{f,j} \) as described in Section 3.1. The scattering matrix for funnel ends
$S_{X,\chi}(s)$ is diagonal with respect to the decomposition of the boundary $\partial_\infty X$ and given by

$$S_{X,\chi}(s) : C^\infty(\partial_j X, E_\chi|_{\partial_j X}) \to C^\infty(\partial_j X, E_\chi|_{\partial_j X}),$$

$$S_{X,\chi}(s) := S_{X,\chi}(s) + \ldots + S_{X,\chi}(s).$$

As it was already in the case for the resolvent, the scattering matrix $S_{X,\chi}(s)$ is closely related to scattering matrix for the funnel ends, $S_{X,\chi}(s)$.

**Lemma 5.6.** Let $Q^\#(s) : C^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X}) \to C^\infty(\partial_\infty X, E_\chi|_{\partial_\infty X})$ be given by the matrix representation

$$(47) \quad Q^\#(s) = \begin{pmatrix} Q^\#(s)_{\beta} & Q^\#(s)_{\chi} \\ Q^\#(s)_{\chi} & Q^\#(s)_{\beta} \end{pmatrix},$$

where

$$Q^\#(s)_{\beta} = E^X_{X,\chi}(s)(\eta_3 - \eta_{f,1})(\text{id} - L(s))^{-1}[\Delta_{X,\chi}, \eta_{f,0}]E_{X,\chi}(s),$$

$$Q^\#(s)_{\chi} = E^X_{X,\chi}(s)(\eta_3 - \eta_{f,1})(\text{id} - L(s))^{-1}[\Delta_{X,\chi}, \eta_{c,0}]E_{X,\chi}(s),$$

Then the integral kernel of $Q^\#(s)$ is given by

$$Q^\#(s; \omega, \omega') = \lim_{\rho \to 0, \rho' \to 0} (\rho_\beta \rho_\chi)\eta^{-s}(\rho_\beta \rho_\chi)^{-s}Q(s; \rho, \omega, \rho', \omega')$$

for Re $s < 1/2$.

**Proof.** By (33) we have that

$$Q(s)\varphi = M(s)\eta_3(\text{id} - L(s))^{-1}L(s)\varphi$$

for $\varphi \in C^\infty(X, E_\chi)$ with $\eta_3 \varphi = 0$. If $\psi \in C^\infty(X, E_\chi)$ with $\eta_3 \psi = 0$, then we can write

$$(Q(s)\varphi, \psi)_{L^2} = \begin{pmatrix} Q_{\beta,\beta}(s) & Q_{\beta,\chi}(s) \\ Q_{\chi,\beta}(s) & Q_{\chi,\chi}(s) \end{pmatrix} \begin{pmatrix} \varphi_{|X_f} \\ \psi_{|X_c} \end{pmatrix} \begin{pmatrix} \varphi_{|X_f} \\ \psi_{|X_c} \end{pmatrix}.$$  

From the definition of $M(s)$ and $L(s)$, we see that for instance

$$Q_{\beta,\beta}(s) = R_{X_f,\chi}(s)(\eta_3 - \eta_{f,1})(\text{id} - L(s))^{-1}[\Delta_{X,\chi}, \eta_{f,0}]R_{X_f,\chi}(s).$$

Using that the integral kernel of $E_{X_f,\chi}(s)^T$ is given by

$$E_{X_f,\chi}(s)^T(\phi, r', \phi') = \lim_{r \to \infty} \rho_f(r)^{-s}R_{X_f,\chi}(s; r, \phi, r', \phi'),$$

and the integral kernel of $E_{X_f,\chi}(s)$ is given by (8), we obtain that

$$Q^\#(s; \omega, \omega')_{\beta} = \lim_{\rho \to 0, \rho' \to 0} (\rho_\beta \rho_\chi)^{-s}Q^\#(s; \rho, \omega, \rho', \omega').$$

□

**Proposition 5.7.** The two scattering matrices, $S_{X,\chi}(s)$ and $S_{X,\chi}(s)$, are related by

$$(48) \quad S_{X,\chi}(s) = S_{X,\chi}(s) + (2s - 1)Q^\#(s),$$
where $0 : \mathbb{C}^n \to \mathbb{C}^n$ is the zero-map and $Q^\#(s)$ is given by Lemma 5.6. In particular,

$$S^\#_{X,\chi}(s) \in \Psi^{2\Re s-1}(\partial_f X, E_X|_{\partial_f X}), \quad s \not\in R_{X,\chi} \cup (\mathbb{N}_0 + 1/2).$$

**Proof.** For $\Re s < 1/2$, this follows directly from the characterization of the scattering matrix as a limit of the resolvent, (46), Theorem 4.1. For $\Re s \geq 1/2$ we use meromorphic continuation. Note that $Q^\#(s)$ is smoothing and hence a pseudodifferential operator of order $-\infty$. The second part then follows from $S^\#_{X,\chi}(s) = S_{X,\chi}(s) + Q^\#(s)$ and (15). \hfill \Box

**Remark 5.8.** The appearance of the map $0 : \mathbb{C}^n \to \mathbb{C}^n$ in (48) is due to the fact that for $\Re s > 1/2$, we have that

$$\lim_{y \to \infty} \rho_c(y)^{1-s} E_{C_{\infty,\chi}}(s; y) = 0.$$

As in the case of the resolvent, we want to investigate the structure of the scattering matrix near a resonance. For this we consider

$$\phi^\#_\ell \in C^\infty(\partial_{\infty\chi} X, E_X|_{\partial_{\infty\chi} X})$$

defined by

$$\phi^\#_\ell(\omega) := \lim_{\rho \to 0} \rho_c^{-s_0} \rho_c^{1-s_0} \phi_\ell(\rho, \omega),$$

where $\phi_\ell$ is as in (37). Let

$$\Phi^\#(v, w) := \left( \langle \phi^\#_\ell, v \rangle \right)_{\ell = 1, \ldots, m_{X,\chi}(s_0)},$$

where $\langle \cdot, \cdot \rangle$ is the bilinear product on $L^2(\partial_{\infty\chi} X, E_X|_{\partial_{\infty\chi} X})$ defined by

$$\langle u, v \rangle = \int_{\partial_f X} \langle u, v \rangle_{E_X} \, d\sigma_{\partial_f X} + \sum_{j=1}^{n^\chi} u_j v_j.$$

**Lemma 5.9.** Let $s_0 \in \mathbb{C}$ with $\Re s_0 < 1$ and $s_0 \neq 1/2$. The scattering matrix has a pole at $s_0$ if and only if $R_{X,\chi}(s)$ has a pole at $s_0$. In this case we have that

$$S_{X,\chi}(s) = (\Phi^\#)^T E(s, s_0) \left( \sum_{j=1}^{n} (s(1-s) - s_0(1-s_0))^{-k_j} P_j \right) F(s, s_0) \Phi^\#$$

$$+ H^\#(s, s_0),$$

where for some $n, k_j > 0$ with

$$\sum_{j=1}^{n} k_j = m_{X,\chi}(s_0),$$

for each $j \in \{1, \ldots, n\}$ the matrices $P_j$ are rank-1-projections from $\mathbb{C}^{m_{X,\chi}(s_0)}$ to mutually orthogonal subspaces, $E(\cdot, s_0)$ and $F(\cdot, s_0)$ are holomorphically invertible matrices of dimension $m_{X,\chi}(s_0)$, and

$$H^\#(\cdot, s_0) : L^2(\partial_{\infty\chi} X, E_X|_{\partial_{\infty\chi} X}) \to L^2(\partial_{\infty\chi} X, E_X|_{\partial_{\infty\chi} X})$$

is holomorphic near $s = s_0$. 


Proof. Using (46) and (35) we have that

\[ S_{X,\chi}(s) = \sum_{k=1}^{p} \frac{A_k^#(s_0)}{(s(1-s) - s_0(1-s))^k} + H^#(s, s_0) \]

for some (unique) \( p \in \mathbb{N}_0 \) such that \( H^#(\cdot, s_0) \) is holomorphic. For each \( k \in \{1, \ldots, p\} \), the operator \( A_k^#(s_0) \) is determined by the integral kernel

\[ A_k^#(s, \omega, \omega') := (2s_0 - 1) \lim_{\rho \to 0} \lim_{\rho' \to 0} (\rho f \rho'_{\ell})^{-s_0} (\rho e^\ell_{\rho'})^{1-s_0} A_k(s_0, \rho, \omega, \rho', \omega'). \]

Recall from (38) that

\[ A_k(s_0) = \sum_{\ell, m=1}^{m_{X,\chi}(s_0)} a_{\ell,m}(s_0) \phi_{\ell} \otimes \phi_m. \]

This implies

\[ A_k^#(s_0) = \sum_{\ell, m=1}^{m_{X,\chi}(s_0)} \phi_{\ell} \otimes \phi_m \]

\[ = (\Phi^#)^T a_k(s_0) \Phi^#. \]

Above, \( a_k(s_0) \) is as in (38). Recall that \( a_k(s_0) = a_1(s_0)d(s_0)^{k-1} \), where \( d(s_0) \) is nilpotent. Hence, \( S_{X,\chi}(s) \) can be written as

\[ S_{X,\chi}(s) = (\Phi^#)^T a_1(s_0) \left( \sum_{k=0}^{p-1} (s(1-s) - s_0(1-s))^{-(k+1)} d(s_0)^k \right) \Phi^# + H^#(s, s_0) \]

in a sufficiently small neighborhood of \( s_0 \). Denote by \( N_k \) a Jordan block of dimension \( k \) with eigenvalue 0. The Jordan normal form of \( d(s_0) \) is given by

\[ Jd(s_0)J^* = \begin{pmatrix} N_{k_1} & 0 & \cdots & 0 \\ 0 & N_{k_2} & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & N_{k_n} \end{pmatrix}, \]

where \( \sum_{j=1}^{n} k_j = m_{X,\chi}(s_0) \) and \( J \) is unitary. Using linear algebra, we immediately obtain that for each \( j \in \{1, \ldots, n\} \),

\[ \sum_{m=0}^{p-1} x^{-(m+1)} N_{k_j}^m = E_{k_j}(x)(x^{-k_j} P_j + \tilde{P}) F_{k_j}(x), \]

where \( E_{k_j} \) and \( F_{k_j} \) are polynomials in \( x \), and \( P_j, \tilde{P} \) are diagonal matrices, and each \( P_j \) has rank one. Putting \( x = s(1-s) - s_0(1-s_0) \) and applying the argumentation above to every Jordan block, we obtain matrices \( E(s, s_0), F(s, s_0) \) depending polynomially on \( s \) and mutually orthogonal
projections $P_j$ of rank 1 such that

$$
\sum_{k=0}^{p-1} (s(1-s) - s_0(1-s_0))^{-(k+1)} d(s_0)^k = E(s,s_0) \left( \sum_{j=1}^n (s(1-s) - s_0(1-s_0))^{-k_j} P_j \right) F(s,s_0) + \tilde{H}(s,s_0),
$$

where $\tilde{H}(\cdot, s_0)$ is holomorphic. This proves the claim. □

5.3. The Scattering Matrix at $s = 1/2$.

**Lemma 5.10.** The resolvent satisfies

$$
R_{X,\chi}(s) = \frac{1}{2s-1} \sum_{k=1}^{m_{X,\chi}(1/2)} \phi_k(s) \langle \phi_k(s), \cdot \rangle + H(s),
$$

where $H$ is holomorphic near $1/2$, and, for each $k \in \{1, \ldots, m_{X,\chi}(1/2)\}$, the function

$$
\phi_k \in \rho_j^s \rho_c^{s-1} C^\infty(X,E_\chi)
$$

satisfies

$$
(\Delta_{X,\chi} - \frac{1}{4}) \phi_k(\frac{1}{2}) = 0.
$$

**Proof.** We note that $\text{Im}(s^2 - s) = \text{Im}((s - 1/2)^2)$. Let $\psi \in C_c^\infty(X,E_\chi)$. Using the self-adjointness of $\Delta_{X,\chi}$, we obtain the estimate

$$
\left| ((\Delta_{X,\chi} - s(1-s)) u, u)_{L^2} \right| \geq \left| \text{Im} ((\Delta_{X,\chi} - s(1-s)) u, u)_{L^2} \right| = \text{Im}(s^2 - s) \left\| u \right\|_{L^2}^2 = \left| \text{Im} \left( (s - \frac{1}{2})^2 \right) \right| \left\| u \right\|_{L^2}^2.
$$

Therefore, we have

$$
\| R_{X,\chi}(s) \| \leq \left| \text{Im} \left( (s - \frac{1}{2})^2 \right) \right|^{-1}.
$$

Hence, the order of the resonance at $s = 1/2$ is at most 2. This implies that

$$
R_{X,\chi}(s) = \frac{A}{(2s-1)^2} + \frac{B}{2s-1} + h(s),
$$

where $h$ is holomorphic near $1/2$, and $A$ and $B$ are suitable operators, independent of $s$.

Using the resolvent equation, we see that every element $u$ in the range of $A$ and $B$ satisfies $(\Delta_{X,\chi} - 1/4) u = 0$. We note that (51) implies that $A: L^2_{\text{cpt}}(X,E_\chi) \to L^2(X,E_\chi)$. Hence, the range of $A$ consists of eigenfunctions of $\Delta_{X,\chi}$ with eigenvalue $1/4$. By Proposition 4.7 there are no eigenfunctions if $X$ has infinite volume, hence $A = 0$.

By the definition of a multiplicity, we have that rank $B = m_{X,\chi}(1/2)$. Using the decomposition of the resolvent from (37), we can write $R_{X,\chi}(s)$ as
\[(2s - 1)^{-1} B(s) + H(s), \]

where

\[B(s) = \sum_{\ell,m=1}^{m_X,\chi(1/2)} a^{\ell,m}_1(s) \tilde{\phi}_\ell(s) \langle \tilde{\phi}_m(s), \cdot \rangle\]

for some symmetric invertible matrix \(a_1(1/2) = (a^{\ell,m}_1(s_0))_{\ell,m=1}^{m_X,\chi(1/2)}\), \(\tilde{\phi}_k \in \rho_f \rho_c^{-1} C^\infty(X, E_\chi)\) and \(H(s)\) is holomorphic near \(s = 1/2\). Since the resolvent at \(1/2\) is self-adjoint and non-negative, \(a_1(1/2)\) is a positive matrix. Therefore we can find a matrix \((d_{k,l})_{k,l=1}^{m_X,\chi(1/2)}\) such that

\[(52) \quad B(1/2) = \sum_{k=1}^{m_X,\chi(1/2)} \phi_k \langle \phi_k, \cdot \rangle,\]

where \(\phi_k = \sum_{\ell=1}^{m_X,\chi(1/2)} d_{k,\ell} \tilde{\phi}_\ell(1/2)\) for \(k = 1, \ldots, m_X,\chi(1/2)\). We have that \(B(1/2) = B\) and hence

\[\left(\Delta_{X,\chi} - \frac{d}{2}\right) \phi_k(1/2) = 0.\]

**Proof of Theorem B.** From Proposition 5.2 and Definition 5.4, we obtain that

\[(2s - 1) E_{X,\chi}(s) u \sim \rho_f^{1-s} \rho_c^{-s} u + \rho_f^s \rho_c^{-1} S_{X,\chi}(s) u.\]

At first glance, this does not make any sense for \(s = 1/2\), but we will see that \(E_{X,\chi}(s)\) has a simple pole at \(s = 1/2\) and hence \((2s - 1) E_{X,\chi}(s) \neq 0\) for \(s = 1/2\).

By Theorem 4.1, we have the decomposition

\[R_{X,\chi}(s) = \tilde{M}_i(s) + M_f(s) + M_c(s) + Q(s)\]

and we recall that \(\tilde{M}_i(s)\) and \(M_f(s)\) are holomorphic near \(s = 1/2\). We write the remainder term \(Q(s)\) as

\[Q(s) = (2s - 1)^{-1} \tilde{Q} + Q_{\text{hol}}(s),\]

where \(Q_{\text{hol}}\) is holomorphic near \(s = 1/2\). By (22) and (25), the term \(M_c(s)\) is given by

\[(2s - 1) M_c(s) = (1 - \eta_{c,0}) \rho_c^{-1} (\rho_c')^{-s} \text{id}_{\partial_\omega E_\chi}(1 - \eta_{c,1}) + (2s - 1) M_{\text{hol}}^c\]

with \(M_{\text{hol}}^c(s)\) being holomorphic near \(s = 1/2\). If we set

\[\tilde{M}_c = (1 - \eta_{c,0}) (\rho_c \rho_c')^{-1/2} \text{id}_{\partial_\omega E_\chi}(1 - \eta_{c,1}),\]

then we have that

\[R_{X,\chi}(s) = (2s - 1)^{-1} \left( \tilde{Q} + \tilde{M}_c \right) + H(s)\]

and \(H(s)\) is holomorphic near \(s = 1/2\). Recall also that

\[S_{X,\chi}(s) = (S_{X,\chi}(s) \oplus 0) + (2s - 1) Q^#(s),\]

where

\[Q^#(s; \omega, \omega') = (\rho_f \rho_f')^{-s} (\rho_c \rho_c')^{1-s} Q(s; \rho, \omega, \rho', \omega') \big|_{\partial_\omega X \times \partial_\omega X}.\]
Pick $\tilde{Q}^\#$ such that
\[(2s - 1)Q^\#(s) = \tilde{Q}^\# + Q_{\text{hol}}^\#(s),\]
where $Q_{\text{hol}}^\#(s)$ is holomorphic near $s = 1/2$. This implies that
\[
\tilde{Q}^\# = (\rho_f \rho_f')^{-1/2}(\rho_c \rho_c')^{1/2} \tilde{Q}(\rho, \omega, \rho', \omega') \bigg|_{\partial^\infty X \times \partial^\infty X}.
\]
From the Fourier decomposition of $S_{X,\chi}(s)$, we see that $S_{X,\chi}(1/2) = -\text{id}$. This implies that
\[
P := \frac{1}{2} \left( S_{X,\chi}(\frac{1}{2}) + \text{id} \right) = \frac{1}{2} \left( 0 \oplus \text{id}_{\partial_\omega E_\chi} + \tilde{Q}^\# \right)
\]
is a compact operator. Using (43) and Proposition 5.5, we calculate $P^2 = P$ and $P^* = P$.

The residue of the resolvent at $s = 1/2$ is given by
\[
B(\frac{1}{2}) = \tilde{Q} + \tilde{M}_c.
\]
This implies that
\[
(\rho_f \rho_f')^{-1/2}(\rho_c \rho_c')^{1/2} B(\frac{1}{2}) \bigg|_{\partial^\infty X \times \partial^\infty X} = \tilde{Q}^\# + (0 \oplus \text{id}_{\partial_\omega E_\chi}) = 2P.
\]

With $\phi_k$ given by (49), we set
\[
\phi_k^\#(s) := \rho_f^{-s} \rho_c^{1-s} \phi_k(s) \bigg|_{\partial^\infty X},
\]
which defines a function $\phi_k^\#(s) \in C^\infty(\partial^\infty X, E_\chi)$ by (50). We note that $\phi_k^\#(s)$ is holomorphic in $s$ for $s$ close to 1/2. Further, the functions $\phi_k^\#(1/2)$ are linearly independent since, otherwise, a non-trivial linear combination would lead to an $L^2$-integrable solution of the eigenvalue equation in contradiction to $\Delta_{X,\chi}$ having no eigenvalues at $\lambda = 1/4$.

From (52) and (53) we obtain that
\[
(\rho_f \rho_f')^{-1/2}(\rho_c \rho_c')^{1/2} B(\frac{1}{2}) \bigg|_{\partial^\infty X \times \partial^\infty X} = \sum_{k=1}^{m_{X,\chi}(1/2)} \phi_k^\#(\frac{1}{2}) \langle \phi_k^\#(\frac{1}{2}), \cdot \rangle.
\]
Thus the restriction of $B(1/2)$ to the boundary at infinity still has rank $m_{X,\chi}(1/2)$. This finishes the proof. \[\square\]

Remark 5.11. The proof also shows that
\[
S_{X,\chi}(\frac{1}{2}) = -\text{id} + \sum_{k=1}^{m_{X,\chi}(1/2)} \phi_k^\#(\frac{1}{2}) \langle \phi_k^\#(\frac{1}{2}), \cdot \rangle.
\]
5.4. Scattering Poles. Let \( s_0 \in \mathbb{C} \) be a resonance, let \( \varepsilon > 0 \) and let \( \gamma_{s_0, \varepsilon} \) be the path

\[
[0, 1] \ni t \mapsto s_0 + \varepsilon e^{2\pi i t}.
\]

We suppose that \( \varepsilon \) is small enough such that there is no other resonance inside \( \gamma_{s_0, \varepsilon} \) rather than \( s_0 \). Recall that the resonance multiplicity of \( s_0 \) is given as

\[
m_{X, \lambda}(s_0) := \text{rank} \int_{\gamma_{s_0, \varepsilon}} R_{X, \lambda}(t) \, dt, \quad s_0 \neq \frac{1}{2}.
\]

The analogues of resonances for a scattering matrix are scattering poles. The definition of the multiplicity of a scattering pole is more involved.

We start with briefly recalling some definitions from the Gohberg–Sigal theory [GS71]: let \( \mathcal{B} \) be a Banach space and let \( \lambda_0 \in \mathbb{C} \). We further denote by \( \mathfrak{A} \) the algebra of all linear bounded operators from \( \mathcal{B} \) to \( \mathcal{B} \). We denote by \( \mathcal{M}(\lambda_0) \) the germ of \( \mathfrak{A} \)-valued functions that are holomorphic in some punctured neighborhood of \( \lambda_0 \) and have either a pole or a removable singularity at \( \lambda_0 \). In any concrete situation we will pick a suitable neighborhood.

Let \( B \in \mathcal{M}(\lambda_0) \) be holomorphic at least in \( \Omega_B \setminus \{\lambda_0\} \), where \( \Omega_B \) is some open neighborhood of \( \lambda_0 \), and suppose that there exists a function \( \psi: \Omega_B \to \mathcal{B} \) such that \( \psi(\lambda_0) \neq 0 \), the functions \( \psi \) and \( B\psi \) are holomorphic at \( \lambda_0 \); moreover, we suppose that \( B\psi(\lambda_0) = 0 \). We refer to \( \psi(\lambda_0) \) as a root vector and to \( \psi \) as a root function of \( B \) at \( \lambda_0 \). The rank of a root vector \( \psi(\lambda_0) \), further denoted as \( \text{rank}(\psi(\lambda_0)) \), is the maximal order of vanishing of \( B(\lambda)\phi(\lambda) \) at \( \lambda = \lambda_0 \) among all root functions \( \phi \) with \( \phi(\lambda_0) = \psi(\lambda_0) \). If these orders of vanishing are unbounded, we define \( \text{rank}(\psi(\lambda_0)) := \infty \). The set of all root vectors of \( B \) at \( \lambda_0 \) is a vector space. We refer to its closure in \( \mathcal{B} \) as the kernel of \( B(\lambda_0) \) and denote it by \( \ker B(\lambda_0) \). In what follows, we suppose that \( m := \dim \ker B(\lambda_0) < \infty \) and \( \text{rank}(v) < \infty \) for all \( v \in \ker B(\lambda_0) \).

We define a basis, \( \{v^{(1)}, \ldots, v^{(m)}\} \), of \( \ker B(\lambda_0) \) as follows: the rank of \( v^{(1)} \) equals the maximal rank of all root vectors corresponding to \( \lambda_0 \) and the rank of \( v^{(j)} \) for \( j = 2, \ldots, m \) is the maximal rank of root vectors in some direct complement of the span \( \{v^{(1)}, \ldots, v^{(j-1)}\} \). Let \( r_j := \text{rank}(v^{(j)}) \). We set

\[
N_{\lambda_0}(B) := \sum_{j=1}^{m} r_j.
\]

We also recall (see, e.g., [Bor16, Definition 6.6]) that a set of bounded operators \( A(\lambda) \) from \( \mathcal{B} \) to \( \mathcal{B} \), parametrized by \( \lambda \in U \subset \mathbb{C} \), is a finitely meromorphic family if at each point \( \lambda' \in U \), we have a Laurent series representation,

\[
A(\lambda) = \sum_{k=-m}^{\infty} (\lambda - \lambda')^k A_k,
\]

converging (in the operator topology) in some neighborhood of \( \lambda' \), where for \( k < 0 \), the coefficients \( A_k \) are finite rank operators.

The main result of Gohberg–Sigal [GS71, Theorem 2.1] is the following argument principle: Let \( B \in \mathcal{M}(\lambda_0) \) be such that \( B \) is invertible in some neighborhood of \( \lambda_0 \). Suppose that \( B \) and \( B^{-1} \) are finitely meromorphic families of operators in this neighborhood of \( \lambda_0 \). Suppose that all points
inside a sufficiently small contour, $\gamma$, around $\lambda_0$ (except for, maybe, $\lambda_0$ itself) are regular for both $B$ and $B^{-1}$. Additionally, suppose that the non-singular part of $B$ at $\lambda_0$ has index zero. Then

$$N_{\lambda_0}(B) - N_{\lambda_0}(B^{-1}) = \frac{1}{2\pi i} \text{Tr} \int_{\gamma} B(\lambda)^{-1} B'(\lambda) \, d\lambda. \quad (55)$$

If for such $B \in \mathcal{M}(\lambda_0)$ we define

$$M_{\lambda_0}(B) := \frac{1}{2\pi i} \text{Tr} \int_{\gamma} B(\lambda)^{-1} B'(\lambda) \, d\lambda, \quad (56)$$

then for all $B_1, B_2 \in (\lambda_0)$ satisfying the conditions above we have

$$M_{\lambda_0}(B_1 B_2) = M_{\lambda_0}(B_1) + M_{\lambda_0}(B_2). \quad (57)$$

See [GS71, Theorem 5.2].

From (11) and Proposition 5.7, we obtain that $S_{X,\chi}(s)$ has poles of infinite rank at $s = 1/2 + N_0$. Hence, we define the operator

$$G(s) : C^\infty(\partial_\infty X, E_X|_{\partial_\infty X}) \to C^\infty(\partial_\infty X, E_X|_{\partial_\infty X}), \quad G(s) := (\Gamma(s + \frac{1}{2}) \text{id}_{C^\infty(\partial_f X, E_X|_{\partial_f X})} \oplus \text{id}_{C^\infty(\partial_c X, E_X|_{\partial_c X})}.$$

We want to normalize the scattering matrix such that it is a bounded operator for all $s \notin \mathcal{R}_{X,\chi} \cup (1/2 + N_0)$. Denote by $\Lambda_{\partial_f X}$ the square-root of the Laplacian with respect to the bundle metric – or any other invertible elliptic operator $\Lambda_{\partial_f X} \in \Psi^1(\partial_f X, E_X|_{\partial_f X})$. Set

$$\Lambda(s) : C^\infty(\partial_\infty X, E_X|_{\partial_\infty X}) \to C^\infty(\partial_\infty X, E_X|_{\partial_\infty X}), \quad \Lambda(s) = \Lambda_{\partial_f X}^{-s+1/2} \oplus \text{id}_{C^\infty(\partial_c X, E_X|_{\partial_c X})}. \quad (58)$$

Note that $\Lambda(s)$ and $G(s)$ commute, and we have that $\Lambda(1-s)^{-1} = \Lambda(s)$. It follows from Proposition 5.7 that

$$\tilde{S}_{X,\chi}(s) := G(s)\Lambda(s)S_{X,\chi}(s)\Lambda(1-s)^{-1}G(1-s)^{-1}$$

is a meromorphic family of pseudodifferential operators of order 0 with poles of finite rank. Note that both $G(s)$ and $G(1-s)^{-1}$ are invertible away from $s \in \frac{1}{2} \pm \mathbb{N}$. Moreover, we have that

$$\tilde{S}_{X,\chi}(1-s) = \tilde{S}_{X,\chi}(s)^{-1}$$

and that $\tilde{S}(s)$ is a Fredholm operator by Proposition 5.7 and the invertibility of $S_{X,\chi}(s)$. We only have to consider the $S_{X,\chi}^{gf}$, the other entries are finite rank. As for $S_{X,\chi}(s)$, we can write $\tilde{S}_{X,\chi}(s)$ as a $2 \times 2$ matrix,

$$\tilde{S}_{X,\chi}(s) = \begin{pmatrix} S_{X,\chi}^{gf}(s) & \tilde{S}_{X,\chi}^{cf}(s) \\ \tilde{S}_{X,\chi}^{gf}(s) & S_{X,\chi}^{cf}(s) \end{pmatrix}. \quad (59)$$
where
\[
\begin{align*}
\tilde{S}^{cf}_{X,\chi}(s) &:= \frac{\Gamma\left(s + \frac{1}{2}\right)}{\Gamma\left(\frac{3}{2} - s\right)} \Lambda_{\partial_j X}^{-s + 1/2} \tilde{S}^{cf}_{X,\chi}(s) \Lambda_{\partial_j X}^{-s + 1/2}, \\
\tilde{S}^{ec}_{X,\chi}(s) &:= \frac{\Gamma\left(s + 1\right)}{\Gamma\left(\frac{3}{2} - s\right)} \Lambda_{\partial_j X}^{-s + 1/2} \tilde{S}^{ec}_{X,\chi}(s), \\
\tilde{S}^{cc}_{X,\chi}(s) &:= \tilde{S}^{cc}_{X,\chi}(s).
\end{align*}
\]

The multiplicity of a scattering pole \( s_0 \in \mathbb{C} \) is defined as
\[
(60) \quad \nu_{X,\chi}(s_0) := -M_{s_0}(\tilde{S}_{X,\chi}) = -\frac{1}{2\pi i} \text{Tr} \int_\gamma \tilde{S}_{X,\chi}(s)^{-1} \frac{d}{ds} \tilde{S}_{X,\chi}(s) ds.
\]

By (57) it follows that \( \nu_{X,\chi}(s) \) is independent of the specific choice of the operator \( \Lambda_{\partial_j X} \).

**Lemma 5.12.** For \( s_0 \in \mathcal{R}_{X,\chi} \) with \( \text{Re} \, s_0 < 1, s_0 \not= 1/2 \), we have that
\[
N_{1-s_0}(\tilde{S}_{X,\chi}) = N_{1-s_0}(\Lambda S_{X,\chi} \Lambda).
\]

Moreover, for a resonance \( s_0 \in \mathcal{R}_{X,\chi} \) there exists \( n^# > 0 \) and \( k^# \in \mathbb{Z} \), such that we have the decomposition near \( s_0 \in \mathcal{R}_{X,\chi} \),
\[
\Lambda(s) S_{X,\chi}(s) \Lambda(s) = G_1(s) \left( \tilde{P}_0(s) + \sum_{j=1}^{n^#} (s - s_0)^{-k^#} P_j \right) G_2(s),
\]

where \( G_1, G_2 \) are holomorphically invertible near \( s_0 \in \mathcal{R}_{X,\chi} \) and
\[
\tilde{P}_0(s) = \begin{cases} (s - s_0)P_0, & s_0 \in \mathcal{R}_{X,\chi} \cap \left( \frac{1}{2} - \mathbb{N} \right), \\ P_0, & s_0 \in \mathcal{R}_{X,\chi} \setminus \left( \frac{1}{2} - \mathbb{N} \right) \end{cases}
\]

and \( P_0 \) is a projection.

**Proof.** The first part of the statement for \( s_0 \not\in \frac{1}{2} - \mathbb{N} \) follows from (58) and the remark afterwards. Now let us consider \( s_0 \in \frac{1}{2} - \mathbb{N} \) for which we follow [Bor16, Lemma 8.12]. We set
\[
T(s) := \tilde{S}^{cc}(s) - \tilde{S}^{cf}(s) \tilde{S}^{cf}(s)^{-1} \tilde{S}^{cc}(s)
\]
and note that it is well-defined near \( 1 - s_0 \). We can then write
\[
\tilde{S}_X(s) = \begin{pmatrix} \text{id} & 0 \\ \tilde{S}^{cf}(s) \tilde{S}^{cf}(s)^{-1} & \text{id} \end{pmatrix} \begin{pmatrix} \text{id} & 0 \\ 0 & T(s) \end{pmatrix} \begin{pmatrix} \tilde{S}^{cf}(s) & \tilde{S}^{cc}(s) \\ 0 & \text{id} \end{pmatrix}
\]
The first and last factors on the right hand side of the previous equation are both invertible near \( 1 - s_0 \). Together with [GS71, Section 1] this implies that
\[
N_{1-s_0}(\tilde{S}_X) = N_{1-s_0} \left( \begin{pmatrix} \text{id} & 0 \\ 0 & T \end{pmatrix} \right) = N_{1-s_0}(T).
\]
Moreover
\[
\Lambda(s) S_{X,A}(s) \Lambda(s) = \left( \begin{array}{cc}
\Gamma (s + \frac{1}{2}) \tilde{S}^f(s) \tilde{S}^g(s)^{-1} & 0 \\
\frac{1}{\Gamma(s - \frac{1}{2})} & \frac{1}{\Gamma(s - \frac{1}{2})}
\end{array} \right).
\]

We note that the first and third factors of the right-hand side of the equality above are invertible near \( s = 1 - s_0 \). Hence,
\[
N_{1-s_0} (\Lambda S_X \Lambda) = N_{1-s_0} \left( \left( \begin{array}{cc}
\frac{1}{\Gamma(s - \frac{1}{2})} & 0 \\
0 & T(s)
\end{array} \right) \right)
\]

Since \( 1 + s_0 \in \mathbb{N}_0 \), the function \( \Gamma \left( \frac{3}{2} - s \right) \) is singular at \( s = 1 - s_0 \) and hence \( \Gamma \left( \frac{3}{2} - s \right) / \Gamma \left( s + \frac{1}{2} \right) \) is singular as well and thus has no root vectors. Therefore
\[
N_{1-s_0} \left( \left( \begin{array}{cc}
\frac{1}{\Gamma(s - \frac{1}{2})} & 0 \\
0 & T(s)
\end{array} \right) \right) = N_{1-s_0} (T)
\]

which implies \( N_{1-s_0} (\Lambda S_X \Lambda) = N_{1-s_0} (T) \) and proves the result.

The second part of the statement follows from the application of the Gohberg-Sigal Logarithmic Residue Theorem, (55), to \( \Lambda S_X \Lambda \).

We have that
\[
N_{1-s_0} (\Lambda S_X \Lambda) = \sum_{j: k_j^# > 0} k_j^#.
\]

Lemma 5.9 implies that
\[
\sum_{j: k_j^# > 0} k_j^# \leq \sum_{j=1}^n k_j.
\]

**Proposition 5.13** (Relation between scattering poles and resonances). For \( s_0 \in \mathbb{C} \) with \( \Re s_0 \leq 1 \) we have
\[
\nu_{X,A}(s_0) = m_{X,A}(s_0) - m_{X,A}(1 - s_0).
\]

**Proof.** First, we note that \( S_{X,A}(1/2) = \tilde{S}_{X,A}(1/2) \) is unitary, and therefore \( \nu_{X,A}(1/2) = 0 \) by (60). Moreover, \( m_{X,A}(1/2) - m_{X,A}(1 - 1/2) = 0 \), which implies the claimed equality for \( s_0 = 1/2 \). Therefore it suffices to consider a resonance \( s_0 \in \mathbb{C} \) with \( \Re s_0 < 1 \) and \( s_0 \neq 1/2 \). By (55), we have that
\[
\nu_{X,A}(s_0) = -M_{s_0}(\tilde{S}_{X,A}) = N_{1-s_0}(\tilde{S}_{X,A}) - N_{s_0}(\tilde{S}_{X,A}).
\]

It remains to show that \( m_{X,A}(s_0) = N_{1-s_0}(\tilde{S}_{X,A}) \). Note that the inequality \( m_{X,A}(s_0) \geq N_{1-s_0}(\tilde{S}_{X,A}) \) follows from
\[
N_{1-s_0}(\Lambda S_X \Lambda) = \sum_{j: k_j^# > 0} k_j^# \leq \sum_{j=1}^n k_j = m_{X,A}(s_0).
\]

Since the operator \( \Phi^\# \) in Lemma 5.9 might not have full rank, we cannot directly deduce equality. To prove \( m_{X,A}(s_0) \leq N_{1-s_0}(\tilde{S}_{X,A}) \), we have to use (45). Assume that \( s_0(1-s_0) \) does not belong to the discrete spectrum.
of $\Delta_{X,\chi}$. Then we have that $\text{Re } s_0 < 1/2$ and $S_{X,\chi}(s)$ is holomorphic near $1 - s_0$ by definition. Thus, $\mathcal{S}_{X,\chi}(s)$ is holomorphic near $1 - s_0$ and hence $N_{s_0}(\mathcal{S}_{X,\chi}) = 0$, which follows by using that $\mathcal{S}_{X,\chi}(s)\mathcal{S}_{X,\chi}(1 - s) = \text{id}$. By Lemma 5.12 and (45), we have that

$$R_{X,\chi}(s) = R_{X,\chi}(1 - s) + (2s - 1)E_{X,\chi}(1 - s)\Lambda(s)^{-1}G_1(s) \times \left( \tilde{P}_0(s) + \sum_{j=1}^{n^\#} (s - s_0)^{-k_j^\#} P_j \right) G_2(s)\Lambda(s)^{-1}E_{X,\chi}(1 - s)^T.$$

Since all terms except for the factors $(s - s_0)^{-k_j^\#}$ are holomorphic and the $P_j$ have rank 1, we have an upper bound for the rank of the residue $A_1$ of $R_{X,\chi}(s)$ in (35),

$$m_{X,\chi}(s_0) = \text{rank } A_1 \leq \sum_{j: k_j^\# > 0} k_j^\# = N_{1-s_0}(\Lambda S_{X,\chi} A).$$

If $s_0(1-s_0)$ belongs to the discrete spectrum of $\Delta_{X,\chi}$, we consider separately the two cases $\text{Re } s_0 > 1/2$ and $\text{Re } s_0 < 1/2$.

Let $\text{Re } s_0 > 1/2$. The resolvent estimate implies that the order of the resonance at $s_0$ is 1. Straightforward argumentation shows that $A_1$ in (35) is the projection onto the eigenspace. Let $(\phi_i)_{i=1}^{m_{X,\chi}(s_0)}$ be an orthonormal basis of the eigenspace and set

$$\phi_i^\# := \lim_{\rho_i \to 0} \rho_i^{-s} \rho_i^{-1-s} \phi_i \in C^\infty(\partial_\infty X, E_X|_{\partial_\infty X}).$$

The functions $\phi_i^\#$, $i \in \{1, \ldots, m\}$, are linearly independent, by a straightforward contradiction argument using Proposition 4.4. The Laurent expansion of $S_{X,\chi}(s)$ takes the form

$$S_{X,\chi}(s) = -(s - s_0)^{-1} \sum_{i=1}^{m_{X,\chi}(s_0)} \phi_i^\# (\phi_i^\#, \cdot) + H_1(s),$$

where $H_1$ is holomorphic near $s = s_0$. Hence, $\tilde{S}_{X,\chi}^{-1}$ has $m_{X,\chi}(s_0)$ independent root vectors of rank 1 at $s = s_0$.

Let $s_0(1 - s_0) \in \sigma_d(\Delta_{X,\chi})$ with $\text{Re } s_0 < 1/2$. For $i \in \{1, \ldots, m\}$, let $\phi_i$ and $\phi_i^\#$ be as above. Denote the span of $\{\phi_i\}_{i=1}^m$ by $W$. Since $\text{Re } s_0 < 1/2$, we have that $W \subset \rho_s^{1-s_0} \rho_s^{-s} C^\infty(\overline{X}, E_X)$. Using Taylor expansion of $\rho_s^s \rho_c^{1-s}$ as a function of $s(1-s)$ near $s_0(1-s_0)$, we have that

$$\text{ran } A_1(s_0) \subset \sum_{k=0}^{p-1} \rho_s^{s_0} \rho_s^{-1} C^\infty(\overline{X}, E_X).$$

Using the unique continuation again, it follows that $\text{ran } A_1(s_0)$ and $W$ are disjoint. Therefore there exists a decomposition $\rho^{-1}L^2 = W \oplus W'$ with $\text{ran } A_1(s_0) \subset W'$. Denote by $\Pi$ the projection onto $W'$ with $\text{ker } \Pi = W$. We have that $\phi_i \Pi = 0$ and $\Pi A = A$. The Laurent expansion of $R_{X,\chi}(1-s)$ near $s_0$ is given by

$$R_{X,\chi}(s) = (s - s_0)^{-1} R_{-1} + R_{\text{hol}}(s),$$
where $R_{\text{hol}}$ is holomorphic. To calculate the residue, we note that
\[
s(1 - s) - s_0(1 - s_0) = -(s - s_0)(2s_0 - 1 + (s - s_0))
\]
and hence
\[
R_{-1} = - \text{res}_{s=s_0} R_{X,\chi}(s) = (2s_0 - 1)^{-1} \sum_i \phi_k \langle \phi_k, \cdot \rangle.
\]

We define the Laurent expansions
\[
(2s - 1)E_{X,\chi}(1 - s)A(s)^{-1}G_1(s) =: \sum_{l=-1}^{\infty} (s - s_0)^l E_l,
\]
\[
G_2(s)A(s)^{-1}E_{X,\chi}(1 - s)^T =: \sum_{m=-1}^{\infty} (s - s_0)^m F_m.
\]
The principal parts of these Laurent expansions are given by
\[
E_{-1} = \sum_i \phi_i \langle e_i, \cdot \rangle,
\]
\[
F_{-1} = \sum_i f_i \langle \phi_i, \cdot \rangle,
\]
for some $e_i, f_i \in C^\infty(\overline{X}, E_{\chi})$. Consequently,
\[
\Pi R_{-1} = 0, \quad \Pi E_{-1} = 0, \quad \text{and} \quad F_{-1} \Pi^T = 0.
\]
The residue at $s_0$ can be calculated as
\[
A_1(s_0) = \text{res}_{s=s_0} R_{X,\chi} = R_{-1} + \sum_{k^j + l + m = -1} E_l P_j F_m.
\]
Conjugating by $\Pi$ yields
\[
A_1(s_0) = \Pi A_1(s_0) \Pi^T = \sum_{j: k^j > 0} \sum_{l=0}^{k^j-1} \Pi E_l P_j F_{k^j-1-l} \Pi^T.
\]
Hence,
\[
m_{X,\chi}(s_0) = \text{rank} A_1(s_0) \leq \sum_{j: k^j > 0} k^j = N_{1-s_0}(\tilde{S}_{X,\chi}).
\]

5.5. **Relative Scattering Matrix.** The relative scattering matrix, defined by
\[
S_{X,\chi}^{\text{rel}}(s) := \left( S_{X,\chi}(s) - \oplus (-\text{id}) \right) S_{X,\chi}(s),
\]
is a smoothing operator on $C^\infty(\partial_X, E_{\chi}|_{\partial_X})$. Therefore it makes sense to define the relative scattering determinant
\[
\tau_{X,\chi}(s) := \det S_{X,\chi}^{\text{rel}}(s).
\]
The relation (44) implies that
\[
\tau_{X,\chi}(s)\tau_{X,\chi}(1-s) = 1
\]
and thus

\[ |\tau_{X,\chi}(s)| = 1 \quad \text{for } \Re s = \frac{1}{2}. \]

Let

\[ E_2(s) := (1 - s) \exp\left(s + \frac{s^2}{2}\right). \]

By [DFP, Theorem B] and [Boa54, Theorem 2.6.5], the Weierstrass product

\[ P_{X,\chi}(s) := s^{m_{X,\chi}(0)} \prod_{\mu \in R_{X,\chi} \setminus \{0\}} E_2\left(\frac{s}{\mu}\right), \]

is well-defined and holomorphic of order 2.

The Weierstrass product \( P_{X_f,\chi}(s) \) for \( X_f \) is defined analogously, only exchanging \( X \) for \( X_f \) in (65), i.e.,

\[ P_{X_f,\chi}(s) := s^{m_{X_f,\chi}(0)} \prod_{\mu \in R_{X_f,\chi} \setminus \{0\}} E_2\left(\frac{s}{\mu}\right), \]

We recall that \( R_{X_f,\chi} \) is given by (34) and for one funnel end, the resonances are given by (6). As in the untwisted case (see [GZ97, Proposition 2.14]) we prove the following result.

**Proposition 5.14.** The relative scattering determinant admits a factorization

\[ \tau_{X,\chi}(s) = e^{g(s)} \frac{P_{X,\chi}(1 - s)}{P_{X,\chi}(s)} \cdot \frac{P_{X_f,\chi}(s)}{P_{X_f,\chi}(1 - s)}, \]

where \( q: \mathbb{C} \to \mathbb{C} \) is an entire function.

**Proof.** We set

\[ h(s) := \frac{P_{X,\chi}(1 - s)}{P_{X,\chi}(s)} \cdot \frac{P_{X_f,\chi}(s)}{P_{X_f,\chi}(1 - s)} \]

for any \( s \in \mathbb{C} \), for which the map on the right hand side is defined. Then \( h \) is meromorphic on all of \( \mathbb{C} \), as is the map \( \tau_{X,\chi} \). It suffices to show that the zeros and poles of the two maps \( h \) and \( \tau_{X,\chi} \) coincide, including their multiplicities. We first consider \( s \in \mathbb{C} \) with \( \Re s = 1/2 \). If \( s \) is a resonance of \( X \) (or \( X_f \), and hence contributes to the divisor of some of the Weierstrass products in (68), then also \( 1 - s \) is a resonance of \( X \) (or \( X_f \), respectively) with the same multiplicity as \( s \). Therefore the total contribution of \( s \) to the divisor of the quotient of the Weierstrass functions in (68) cancels. Thus, \( h \) does not have a zero or pole at \( s \). From (64) it follows that the same is true for \( \tau_{X,\chi} \).

We consider now \( s \in \mathbb{C} \) with \( \Re s < 1/2 \) and show that the multiplicities of \( s \) as a zero or pole of \( h \) and \( \tau_{X,\chi} \) coincide. Since \( \tau_{X,\chi}(1 - s) = 1/\tau_{X,\chi}(s) \) by (63) as well as \( h(1 - s) = 1/h(s) \), this equality of multiplicities then extends immediately to the right half plane \( \{\Re s > 1/2\} \). We now pick \( \varepsilon > 0 \) such that the ball of radius \( \varepsilon \) around \( s \) contains no zeros of the Weierstrass
products \( P_{X,\chi} \) and \( P_{X,f,\chi} \) except at \( s \). Using the argument principle, it remains to show that
\[
\frac{1}{2\pi i} \int_{\gamma_{s,\epsilon}} \frac{\tau'_{X,\chi}(t)}{\tau_{X,\chi}(t)} \, dt = m_{X,\chi}(1 - s) - m_{X,\chi}(s) + m_{X,f,\chi}(s) - m_{X,f,\chi}(1 - s).
\]
Taking advantage of (62) and (56) we can write the left hand side of (69) as
\[
\frac{1}{2\pi i} \int_{\gamma_{s,\epsilon}} \frac{(\det S_{X,\chi}^\text{rel}(t))'}{(\det S_{X,\chi}^\text{rel}(t))} \, dt
= \frac{1}{2\pi i} \text{Tr} \int_{\gamma_{s,\epsilon}} (S_{X,\chi}^\text{rel}(t))^{-1}(S_{X,\chi}^\text{rel}(t))' \, dt
= M_s(S_{X,\chi}^\text{rel}).
\]
We define the normalized model scattering matrix by
\[
\tilde{S}_{X,f,\chi}(s) := G(s)\Lambda(s)S_{X,f,\chi}(s)\Lambda(1 - s)^{-1}G(1 - s)^{-1}
\]
and obtain, using (61), that
\[
S_{X,\chi}^\text{rel}(s) = G(1 - s)^{-1}\Lambda(s)\tilde{S}_{X,f,\chi}(s)^{-1}\tilde{S}_{X,\chi}(s)\Lambda(s)^{-1}G(1 - s).\]
We recall that \( G(1 - s) \) and \( \Lambda(s) \) are holomorphic for \( \text{Re} \ s < 1/2 \). By (57) we have that
\[
M_s(S_{X,\chi}^\text{rel}) = M_s(\tilde{S}_{X,\chi}) - M_s(\tilde{S}_{X,f,\chi}).
\]
Proposition 5.13 implies
\[
M_s(\tilde{S}_{X,\chi}) = \nu_{X,\chi}(s) = m_{X,\chi}(1 - s) - m_{X,\chi}(s).
\]
Now \( m_{X,f,\chi}(1 - s) = 0 \) since \( \text{Re} \ s < 1/2 \). We note that the equality \( M_s(\tilde{S}_{X,f,\chi}) = -m_{X,f,\chi}(s) \) follows directly from Proposition 3.1, that completes the proof.

To prove Theorem A we have to show that \( q \) is a polynomial of degree at most 4. For this, we need a singular value estimate on the relative scattering matrix. This will give us an estimate on the scattering determinant. We define the set
\[
\mathcal{L}_D^0 := \{ s \in \mathbb{C} : D(s) = 0 \},
\]
where \( D(s) \), as in [DFP, Lemma 6.1], is defined by
\[
D(s) := \det(1 - (L(s)\eta_3)^3).
\]
For \( \delta > 0 \) set
\[
\mathcal{B}(\delta) := B_1(\frac{1}{\delta}) \cup \bigcup_{\zeta \in \mathcal{L}_D^0 \cup \mathcal{R}_{X,\chi,\lambda}(1 - \mathcal{R}_{X,\chi,\lambda})} B_{(\zeta)^{-2+s}}(\zeta),
\]
where \( B_r(z) \) denotes the ball of radius \( r \) around \( z \).

**Lemma 5.15.** For \( \delta > 0 \) large enough there exists \( C > 0 \) and \( c > 0 \) such that for \( s \notin \mathcal{B}(\delta) \) and \( k \in \mathbb{N} \), we have
\[
\mu_k(S_{X,\chi}^\text{rel}(s) - \text{id}) \leq e^{C(s)^{2+s} - ck}.
\]
Proof. By Proposition 5.7, we have the decomposition
\[ S_{X,\chi}(s) = S_{X_f,\chi}(s) \oplus 0 + (2s - 1)Q^\#(s). \]
From (48) and (47), we have that
\[ S_{X,\chi}(s) = \begin{pmatrix} S_{X_f,\chi}(s) + Q^\#(s)^{cf} & S_{X_f,\chi}(s) + Q^\#(s)^{fc} \\ Q^\#(s)^{cf} & Q^\#(s)^{fc} \end{pmatrix} \]
Hence, the matrix coefficients of \( S_{X,\chi}^{rel}(s) \) are given by
\[ S_{X,\chi}^{rel}(s) = \begin{pmatrix} S_{X_f,\chi}^{rel}(s) \\ S_{X_f,\chi}^{rel}(s) \\ S_{X_f,\chi}^{rel}(s) \\ S_{X_f,\chi}^{rel}(s) \end{pmatrix} \]
\[ = \begin{pmatrix} -id + S_{X_f,\chi}(s)^{-1}Q^\#(s)^{cf} & S_{X_f,\chi}(s)^{-1}Q^\#(s)^{fc} \\ -Q^\#(s)^{cf} & -Q^\#(s)^{fc} \end{pmatrix}. \]
By (12), we have that \( S_{X_f,\chi}(s)^{-1}E_{X_f,\chi}(s)T^\prime = -E_{X_f,\chi}(1-s)^T \) and together with (47), we obtain
\[ S_{rel}^{cf}(s) = (2s - 1)E_{X_f,\chi}(1-s)^T((\eta_3 - \eta_1) \times (id - L(s)\eta_3)^{-1}[\Delta_{X_f,\chi}, \eta_0]E_{X_f,\chi}(s), \]
\[ S_{rel}^{fc}(s) = (2s - 1)E_{X_f,\chi}(1-s)^T((\eta_3 - \eta_1) \times (id - L(s)\eta_3)^{-1}[\Delta_{X_f,\chi}, \eta_0]E_{X_f,\chi}(s), \]
\[ S_{rel}^{cf}(s) = (2s - 1)E_{X_f,\chi}(1-s)^T((\eta_3 - \eta_1) \times (id - L(s)\eta_3)^{-1}[\Delta_{X_f,\chi}, \eta_0]E_{X_f,\chi}(s), \]
\[ S_{rel}^{fc}(s) = (2s - 1)E_{X_f,\chi}(1-s)^T((\eta_3 - \eta_1) \times (id - L(s)\eta_3)^{-1}[\Delta_{X_f,\chi}, \eta_0]E_{X_f,\chi}(s), \]
From (22) we obtain for every compactly supported \( A \in \text{Diff}^1(X_c,E_\chi|_{X_c}) \) the bound
\[ \|AE_{X_c,\chi}(s)\| \leq e^{C(s)} \]
for \( s \not\in B_1(1/2) \).
Without loss of generality, we suppose that \( X_f \) is a single funnel, that is contained in the hyperbolic cylinder \( C_{\epsilon} = (h_\epsilon)\setminus H \). If \( Re s > \epsilon > 0 \), we can directly use (10) to estimate the singular values of \( AE_{X_f,\chi}(s) \), where \( A \in \text{Diff}^1(X_f,E_\chi|_{X_f}) \) is compactly supported. For \( Re s < 1/2 - \epsilon \), we use (18) and (10) together with (12), \( E_{X_f,\chi}(s)S_{X_f,\chi}(s)^{-1} = -E_{X_f,\chi}(1-s) \). Hence, for all \( s \in \mathbb{C} \), we obtain the estimate
\[ \mu_k(AE_{X_f,\chi}(s)) \leq \begin{cases} d_k(s)e^{C(s)\log(s)}, & k \leq \max\{m_0,2m_\delta\}, \\ \left(\frac{s}{k}\right)^{2(s)}e^{C(s)-ck}, & k > \max\{m_0,2m_\delta\}, \end{cases} \]
where \( m_\delta \) denotes the multiplicity of the eigenvalue \( \lambda = e^{2\pi i \delta} \) of \( \chi(h_\delta) \), the function \( d_k(s) \) was defined by (17), and \( A \in \text{Diff}^1(X_f,E_\chi) \) is compactly supported. We note that for every \( \delta > 0 \) and \( s \not\in B(\delta) \), we have that
\[ d_0(s) \lesssim s^{2+\delta} \]
due to the fact that there are only finitely many resonances in a ball of radius 1 around \( s \).
The estimate on the determinant $D(s)$ in [DFP, Section 6] implies—as in the untwisted case (see [GZ97, Lemma 3.6])—that for $\delta > 0$ large enough and any

$$s \not\in \bigcup_{\zeta \in \mathcal{L}_0 \cup \mathcal{R}_{X_f,X}} B_{(\zeta)-(2+s)}(\zeta),$$

the following estimate holds for all $\varepsilon > 0$:

$$\| (\text{id} - L(s)\eta_3)^{-1} \|_{L^2(X,E^3_X) \to L^2(X,E^3_X)} \leq e^{C(s)^{2+\varepsilon}}. \quad (73)$$

Using (71), (72), and (73) we obtain

$$\mu_k(S_{X,X}^{\text{rel}}(s)^* \leq e^{C(s)^{2+\varepsilon}}, \quad \bullet \in \{fc, cf, cc\},$$

for $s \not\in \mathcal{B}(\delta)$, where we have used that all matrix components involving cusp terms are finite rank operators. So in particular, $\mu_k(S_{X,X}^{\text{rel}}(s)^* \equiv 0$ for $k > N$ for some $N \in \mathbb{N}$.

For the funnel term, we estimate

$$\mu_k(S_{X,X}^{\text{rel}}(s)^{ff} - \text{id}) \leq \|(\eta_3 - \eta_1)E_{X_f}X(1-s)\| \| (\text{id} - L(s)\eta_3)^{-1} \| \mu_k([\Delta_{X,X}, \eta_0]E_{X_f}X(s)) \leq d_0(1-s)e^{C(1-s)\log(1-s)} e^{C(s)^{2+\varepsilon}} \mu_k([\Delta_{X,X}, \eta_0]E_{X_f}X(s)).$$

From the remark above, we obtain that for $s \not\in \mathcal{B}(\delta)$,

$$\mu_k(S_{X,X}^{\text{rel}}(s)^{ff} - \text{id}) \leq e^{C(s)^{2+\varepsilon}} \mu_k([\Delta_{X,X}, \eta_0]E_{X_f}X(s)).$$

If $k \leq \max\{m_0, 2m_0\}$, then can apply the same argument to obtain that

$$\mu_k(S_{X,X}^{\text{rel}}(s)^{ff} - \text{id}) \leq e^{C(s)^{2+\varepsilon}}.$$

For $k \geq \max\{m_0, 2m_0\}$, we have that

$$\mu_k(S_{X,X}^{\text{rel}}(s)^{ff} - \text{id}) \leq e^{C(s)^{2+\varepsilon}} \left( \frac{s}{k} \right)^{2(s)} e^{C(s) - ck} \leq e^{C(s)^{2+\varepsilon} - ck}.$$

With all these results at our disposal, the proof of Theorem A is analogous to the corresponding statement in the untwisted setting. For the convenience of the reader, we provide the details.

**Proof of Theorem A.** In Proposition 5.14 we established the factorization

$$\tau_{X,X}(s) \cdot \frac{\mathcal{P}_{X,X}(s)\mathcal{P}_{X_f,X}(1-s)}{\mathcal{P}_{X,X}(1-s)\mathcal{P}_{X_f,X}(s)} = e^{\varphi(s)} \quad (74)$$

with $q$ being an entire function. It remains to show that $q$ is polynomial with degree bounded by 4, for which we will take advantage of the Hadamard factorization theorem [Tit58, 8.24]. To that end we let $\varphi : \mathbb{C} \to \mathbb{C}$,

$$\varphi(s) := \tau_{X,X}(s) \cdot \frac{\mathcal{P}_{X,X}(s)\mathcal{P}_{X_f,X}(1-s)}{\mathcal{P}_{X,X}(1-s)\mathcal{P}_{X_f,X}(s)},$$

denote the function on the left hand side of the equation in (74) and note that $\varphi$ is entire and has no zeros (as $q$ is entire). Therefore $e^{\varphi(s)}$ is the
(full) Hadamard factorization of \( \varphi \), and hence \( q \) is polynomial. In order to estimate the degree of \( q \), we now provide a numerical bound on the order of \( \varphi \).

Let \( \delta > 0 \) be as in Lemma 5.15 and set \( \mathcal{B} := \mathcal{B}(\delta) \), where \( \mathcal{B}(\delta) \) is defined in (70). We recall that \( \mathcal{B} \) encloses all zeros of \( \mathcal{P}_{X, \chi} \) and \( \mathcal{P}_{X_f, \chi} \). By [Boa54, Theorem 2.6.5] and the upper bounds on the resonances, [DFP, Remark 4.13] and [DFP, Theorem B], we see that both Weierstrass products \( \mathcal{P}_{X, \chi} \) and \( \mathcal{P}_{X_f, \chi} \) are of order 2. In combination with the minimum modulus theorem [Tit58, 8.71] we obtain that for all \( \varepsilon > 0 \) we have

\[
\log \left| \frac{\mathcal{P}_{X, \chi}(s) \mathcal{P}_{X, \chi}(1-s)}{\mathcal{P}_{X_f, \chi}(1-s) \mathcal{P}_{X_f, \chi}(s)} \right| \lesssim \varepsilon (s)^{2+\varepsilon} \quad \text{for all } s \notin \mathcal{B}.
\]

We may estimate the scattering determinant \( \tau_{X, \chi} \) using [GK69, IV.1.2] and Lemma 5.15 to obtain, for all \( s \notin \mathcal{B} \),

\[
|\tau_{X, \chi}(s)| = |\det (\text{id} + (S_{X, \chi}^{\text{vol}}(s) - \text{id})| \\
\leq \prod_{k=1}^\infty \left(1 + \mu_k(S_{X, \chi}^{\text{vol}}(s) - \text{id})\right) \\
\leq \prod_{k=1}^\infty \left(1 + e^{C(s)2^{+\varepsilon} - ck}\right)
\]

for all \( \varepsilon > 0 \) and suitable \( c, C > 0 \) (possibly depending on \( \varepsilon \)). Choose \( N(s) \in \mathbb{N} \) such that \( cN(s) < C(s)^{2+\varepsilon} < c(N(s) + 1) \). We have that

\[
\log |\tau_{X, \chi}(s)| \leq \sum_{k=1}^{N(s)} \log \left(1 + e^{C(s)2^{+\varepsilon} - ck}\right) \\
= \sum_{k=1}^{N(s)} \log \left(1 + e^{C(s)2^{+\varepsilon} - ck}\right) + \sum_{k=N(s)+1}^\infty \log \left(1 + e^{C(s)2^{+\varepsilon} - ck}\right) \\
\lesssim \varepsilon N(s)(s)^{2+\varepsilon} + \sum_{j=0}^\infty \log \left(1 + e^{-cj}\right) e^{C(s)2^{+\varepsilon} - c(N(s)+1)} \\
\lesssim \varepsilon (s)^{4+2\varepsilon}
\]

Therefore, for every \( \varepsilon > 0 \) and \( s \notin \mathcal{B} \), we obtain \( C > 0 \) such that

(75) \[
\log |\varphi(s)| \lesssim (s)^{4+\varepsilon}.
\]

By [DFP, Theorem B and Proposition 6.2], we have that

\[
\# \{ \zeta \in \mathcal{L}_0^0 \cup \mathcal{R}_{X_f, \chi} \cup (1 - \mathcal{R}_{X, \chi}) : |\zeta| \in (r-1, r) \} \lesssim r^2
\]

for any \( r > 1 \). Hence, we can estimate the area of \( \mathcal{B} \) restricted to the annulus \( \{ r-1 < |z| < r \} \) by

\[
\text{vol}(\mathcal{B} \cap \{ z \in \mathbb{C} : |z| \in (r-1, r) \}) \lesssim r^2 (r-1)^{-2(\delta+2)} = O((r)^{-2\delta-2}), \quad \text{as } r \to \infty.
\]

Hence, taking \( R > 1 \) large enough, for any \( r > R \) and \( s \in \mathbb{C} \) with \( |s| \leq r \), we have the estimate

\[
\log |\varphi(s)| \lesssim (r)^{4+\varepsilon}
\]
by the maximum modulus principle (see for instance [Tit58, 5.1]). Thus, $\varphi$ is of order 4 and hence $q$ is a polynomial of degree at most 4. □

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