Semigroup techniques for the efficient classical simulation of optical quantum information

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Abstract. A framework to describe a broad class of physical operations (including unitary transformations, dissipation, noise, and measurement) in a quantum optics experiment is given. This framework provides a powerful tool for assessing the capabilities and limitations of performing quantum information processing tasks using current experimental techniques. The Gottesman-Knill theorem is generalized to the infinite-dimensional representations of the group stabilizer formalism and further generalized to include non-invertable semigroup transformations, providing a theorem for the efficient classical simulation of operations within this framework. As a result, we place powerful constraints on obtaining computational speedups using current techniques in quantum optics.

1. Introduction

Information processing using the rules of quantum mechanics may allow tasks that cannot be performed using classical laws [1]. The efficient factorization algorithm of Shor [2] and secure quantum cryptography [3] are two examples. Of the many possible realizations of quantum information processes, optical realizations have the advantage of negligible decoherence: light does not interact with itself, and thus a quantum state of light can be protected from becoming entangled with the environment. Several proposed optical schemes [4, 5, 6, 7] offer significant potential for quantum information processing.

In order to prove theorems regarding the possibilities and limitations of optical quantum computation, one must construct a framework for describing all types of physical processes (unitary transformations, projective measurements, interaction with a reservoir, etc.) that can be used by an experimentalist to perform quantum information processing. Most frameworks currently employed (e.g., [7]) are restricted to describing only unitary transformations. However, such transformations are a subset of all possible physical processes. Non-unitary transformations such as dissipation, noise, and measurement must also be described within a complete framework. The new results of Knill et al [5] show that photon counting measurements allow for operations that are “difficult” with unitary transformations alone; thus, non-unitary processes may be a powerful resource in quantum information processing and must be considered in any framework that attempts to address the capabilities of quantum computation with optics.

In this paper, we show that unitary transformations, measurements and any other physical process can be described in the unified formalism of completely positive (CP) maps. Also, a broad class of these maps which includes linear optics and squeezing transformations, noise processes, amplifiers, and measurements with feedforward that are typical to quantum optics experiments can be described within the framework of
a Gaussian semigroup. This framework allows us to place limitations on the potential power of certain quantum information processing tasks.

One important goal is to identify classes of processes that can be efficiently simulated on a classical computer; such processes cannot possibly be used to provide any form of "quantum speedup". The Gottesman-Knill (GK) theorem [8, 1] for qubits and the CV classical simulatability theorems of Bartlett et al [9, 10] provide valuable tools for assessing the classical complexity of a quantum optical process. It is shown here that semigroup techniques provide a powerful formalism with which one can address issues of classical simulatability. In particular, a classical simulatability result is presented for a general class of quantum optical operations, and thus a no-go theorem for quantum computation with optics is proven using semigroup techniques.

2. Semigroup Description of Gaussian operations

Consider an optical quantum information process involving \( n \) coupled electromagnetic field modes, with each mode described as a quantum harmonic oscillator. The two observables for the (complex) amplitudes of a single field mode serve as canonical operators for this oscillator. A system of \( n \) coupled oscillators, then, carries an irreducible representation of the Heisenberg-Weyl algebra \( \text{hw}(n) \), spanned by the \( 2^n \) canonical operators \( \{q_i, p_i, i = 1, \ldots, n\} \) along with the identity operator \( I \). These operators satisfy the commutation relations 
\[
[q_i, p_j] = i\hbar \delta_{ij} I.
\]
We express the \( 2^n \) canonical operators in the form of a phase space vector \( z \) with components \( z_i = q_i \) and \( z_{n+i} = p_i \) for \( i = 1, \ldots, n \). These operators satisfy 
\[
[z_i, z_j] = i\hbar \Sigma_{ij},
\]
with \( \Sigma \) the skew-symmetric \( 2n \times 2n \) matrix 
\[
\Sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]  
and \( I_n \) the \( n \times n \) identity matrix. For a state \( \rho \) represented as a density matrix, the means of the canonical operators is a vector defined as the expectation values 
\[
\langle z \rangle_\rho = \langle z \rangle,
\]
and the covariance matrix is defined as
\[
\Gamma = \langle (z - \xi)(z - \xi)^\dagger \rangle_\rho - i\Sigma. 
\]

A Gaussian state (a state whose Wigner function is Gaussian and thus possesses a quasiclassical description) is completely characterized by its means and covariance matrix [11]. Coherent states, squeezed states, and position- and momentum-eigenstates are all examples of Gaussian states.

We define \( \mathcal{C}_n \) to be the group of linear transformations of the canonical operators \( \{z_i\} [9] \); this group corresponds to the infinite-dimensional (oscillator) representation of the "Clifford group" employed by Gottesman [5]. For a system of \( n \) oscillators, it is the unitary representation of the group \( \text{ISp}(2n, \mathbb{R}) \) (the inhomogeneous linear symplectic group in \( 2n \) phase space coordinates) [12] which is the semi-direct product of phase-space translations (the Heisenberg-Weyl group \( \text{HW}(n) \)) plus one- and two-mode squeezing (the linear symplectic group \( \text{Sp}(2n, \mathbb{R}) \)). Phase space displacements are generated by Hamiltonians that are linear in the canonical operators; a displacement operator \( X(\alpha) \in \text{HW}(n) \) is defined by a real \( 2n \)-vector \( \alpha \). A symplectic transformation \( M(A) \in \text{Sp}(2n, \mathbb{R}) \), with \( A \) a real matrix satisfying \( A^\dagger \Sigma A = \Sigma \), is generated by a Hamiltonian that is a homogeneous quadratic polynomial in the canonical operators. A general element \( C \in \mathcal{C}_n \) can be expressed as a product 
\[
C(\alpha, A) = X(\alpha)M(A),
\]
and transforms the canonical operators as
\[
C(\alpha, A) : z \to z' = zA + \alpha.
\]
The group $\mathcal{C}_n$ consists of unitary transformations that map Gaussian states to Gaussian states; however, unitary transformations do not describe all physical processes. In the following, we include other (non-unitary) CP maps that correspond to processes such as dissipation or measurement. We define the Gaussian semigroup, denoted $\mathcal{K}_n$, to be the set of Gaussian CP maps [11] on $n$ modes: a Gaussian CP map takes any Gaussian state to a Gaussian state. Because Gaussian CP maps are closed under composition but are not necessarily invertible, they form a semigroup. A general element $T \in \mathcal{K}_n$ is defined by its action on the canonical operators as

$$T(\alpha, A, G) : z \rightarrow z' = zA + \alpha + \eta,$$

where $\alpha$ is a real $2n$-vector, $A$ and $G$ are $2n \times 2n$ real matrices, and $A$ is no longer required to be symplectic. Eq. (4) includes the transformations (3) plus additive noise processes [13] described by quantum stochastic noise operators (the vector $\eta$) with expectation values equal to zero and covariance matrix

$$\langle \eta\eta^\dagger \rangle_{\rho_R} = G - iA^\dagger \Sigma A.$$  

Here, $\rho_R$ is a Gaussian ‘reservoir’ state which, in order to define a CP map, must be chosen such that the noise operators satisfy the quantum uncertainty relations. This condition is satisfied if the noise operators define a positive definite density matrix, which leads to the condition

$$G + i\Sigma - iA^\dagger \Sigma A \geq 0.$$  

The group $\mathcal{C}_n$ is recovered for $G = 0$.

The action of the Gaussian semigroup on the means and covariance matrix is straightforward and given by

$$T(\alpha, A, G) : \begin{cases} \xi \rightarrow \xi' = \xi A + \alpha \\ \Gamma \rightarrow \Gamma' = A^\dagger \Gamma A + G. \end{cases}$$

Because the means and covariance matrix completely define a Gaussian state, the resulting action of the Gaussian semigroup on Gaussian states can be easily calculated via this action.

The Gaussian semigroup $\mathcal{K}_n$ represents a broad framework to describe several important types of processes in a quantum optical circuit. The group $\mathcal{C}_n \subset \mathcal{K}_n$ comprises the unitary transformations describing phase–space displacements and squeezing (both one– and two–mode). Introduction of noise to the circuit (e.g., via linear amplification) is also in $\mathcal{K}_n$. Furthermore, the Gaussian semigroup describes certain measurements in the quantum circuit. These include measurements where the outcome is discarded (thus evolving the system to a mixed state) or retained (where the system follows a specific quantum trajectory defined by the measurement record [14]). Finally, the Gaussian semigroup includes Gaussian CP maps conditioned on the outcome of such measurements. For details and examples of all of these types of Gaussian semigroup transformations, see [10].

3. Classical Simulation of Gaussian Semigroup Processes

Using the framework of the Gaussian semigroup, it is straightforward to prove the classical simulatability result of Bartlett and Sanders [10].

Theorem: Any quantum information process that initiates in a Gaussian state and that performs only Gaussian semigroup maps can be efficiently simulated using a classical computer.
Proof: Recall that any Gaussian state is completely characterized by its means and covariance matrix. For any quantum information process that initiates in a Gaussian state and involves only Gaussian semigroup maps, one can follow the evolution of the means and the covariance matrix rather than the quantum state itself. For a system of \( n \) coupled oscillators, there are \( 2n \) independent means and \( 2n^2 + n \) elements in the (symmetric) covariance matrix; thus, following the evolution of these values requires resources that are polynomial in the number of coupled systems.

\[ QED \]

Because most current experimental techniques in quantum optics are describable by Gaussian semigroup maps, this theorem places a powerful constraint on the capability of achieving quantum computational speedups (tasks that are not efficient on any classical machine) using quantum optics.

4. Conclusions

Semigroup techniques provide a powerful tool for constructing and assessing new quantum information protocols using quantum optics. These techniques have been used to show that algorithms or circuits consisting of only Gaussian semigroup maps can be efficiently simulated on a classical computer, and thus do not provide the ability to perform quantum information processing tasks efficiently that cannot be performed efficiently on a classical machine. Eisert et al. \[15\] use related techniques to show that local Gaussian semigroup transformations are insufficient for distilling entanglement: an important process for quantum communication and distributed quantum computing. Most current quantum optics experiments consist only of Gaussian semigroup transformations; thus, the challenge is to exploit this semigroup to prove new theorems, limitations and possibilities for quantum information processing using optics.

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