Deviation differential equations. Jacobi fields

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Abstract

Given a differential equation on a smooth fibre bundle $Y \to X$, we consider its canonical vertical extension to that, called the deviation equation, on the vertical tangent bundle $VY$ of $Y$. Its solutions are Jacobi fields treated in a very general setting. In particular, the deviation of Euler–Lagrange equations of a Lagrangian $L$ on a fibre bundle $Y$ are the Euler–Lagrange equations of the canonical vertical extension of $L$ onto $VY$. Similarly, covariant Hamilton equations of a Hamiltonian form $H$ are the Hamilton equations of the vertical extension $VH$ of $H$ onto $VY$.

1 Introduction

By a Jacobi field usually is meant a vector field along a geodesic in a pseudo-Riemannian manifold which obeys the geodesic deviation equation.

In a general setting, we consider the deviation equation (8) – (9) and the Jacobi fields of an arbitrary differential equation (3) on a smooth fibre bundle $Y \to X$. In particular, we are concerned with the deviation of the Euler–Lagrange equations (14) and the covariant Hamilton equations (18) on a fibre bundle $Y \to X$.

Let $E \to X$ be a vector bundle. A $E$-valued $r$-order differential operator on $Y$ is a bundle morphism

$$\Delta : J^r Y \to X E$$

over $X$. Given a global zero section $\hat{0}$ of $E \to X$, we treat its inverse image

$$\mathcal{E} = \text{Ker} \Delta = \Delta^{-1}(\hat{0})$$

as a differential equation on $Y$, though it need not be a closed subbundle of $J^r Y$.

Let $(x^\lambda, y^i)$ be bundle coordinates on $Y$, $(x^\lambda, y^i, y^i_\lambda)$ the adapted coordinates on $J^r Y$, and $(x^\lambda, z^A)$ bundle coordinates on $E$. Then the differential equation $\mathcal{E}$ (2) is locally given by equalities

$$\mathcal{E}^A(x^\lambda, y^i_\lambda) = 0.$$
Let $VJ^rY$ and $VE$ be the vertical tangent bundles of fibre bundle $J^rY \to X$ and $E \to X$, respectively. There is the canonical vertical prolongation

$$V\Delta : VJ^rY \overset{\delta}{\to} VE$$

(4)

of the bundle morphism $\Delta$. Due to the canonical isomorphism

$$VJ^rY = J^r(VY), \quad \dot{y}^i_A = (\dot{y}^i)_A,$$

(5)

the bundle morphism (4) is a $VE$-valued $r$-order differential operator

$$V\Delta : J^r(VY) \overset{\Delta}{\to} VE$$

(6)

on the vertical tangent bundle $VY$ of $V \to X$. It is called the vertical extension of the differential operator $\Delta$. Since $VE \to X$ is a vector bundle, the kernel of this operator

$$V\mathcal{E} = \text{Ker} \ V\Delta$$

(7)

defines an $r$-order differential equation on $VY$. With respect to bundle coordinates $(x^\lambda, y^i, \dot{y}^i_A, \ddot{y}^i_A)$ on $J^r(VY)$, the differential equation (7) is locally given by equalities

$$\mathcal{E}^A(x^\lambda, y^i_A) = 0,$$

(8)

$$\partial_V \mathcal{E}^A(x^\lambda, y^i_A) = 0,$$

(9)

$$\partial_V = \dot{y}^i \partial_i + \ddot{y}^i_A \partial_A + \dddot{y}^i_{\lambda\mu} \partial^\lambda \partial^\mu + \cdots,$$

(10)

where $d_V$ is the vertical derivative.

The equation $V\mathcal{E}$ (7) is called the deviation equation (or the variation equation in the terminology of [6]). Its part (8) is the projection of this equation to $J^rY$, and it is equivalent to the original equation (3). Therefore, a solution of the deviation equation (7) is given by a pair $(s, \psi)$ of a solution $s$ of the original differential equation and a section $\psi$ of the pull-back bundle $s^*VY \to X$ which obeys the linear differential equation (9). This section $\psi$ is called the Jacobi field.

In particular, if $Y \to X$ is an affine bundle modelled on a vector bundle $\overline{Y} \to X$, there is the canonical isomorphism

$$VY = Y \oplus \overline{Y},$$

and Jacobi fields $\psi$ are sections of a vector bundle $\overline{Y} \to X$. For instance, if $Y$ is a vector bundle, then $\overline{Y} = Y$. In the case of an affine bundle $Y$, it is readily observed that, given a solution $(s, \psi)$ of the deviation equation $V\mathcal{E}$ (7), the sum $s + \psi$ obeys the original differential equation $\mathcal{E}$ with accuracy to terms linear in $\psi$. Therefore, one can think of Jacobi fields $\psi$ as being deviations of solutions of the original differential equation.

Let us note that any differential equation $\mathcal{E}$ on a fibre bundle can be written in the form (3), and then the equalities (8) – (9) provide the corresponding deviation equation $V\mathcal{E}$. 

2
Turn now to Lagrangian formalism on a fibre bundle $Y \to X$ \cite{2, 6, 10}. We use the fact that any exterior $m$-form $\phi$ on a fibre bundle $Y \to X$ possesses a vertical extension

$$V \phi = \partial_V \phi$$

onto the vertical tangent bundle $VY$ of $Y \to X$. This is the pull-back onto $VY \subset TY$ of its tangent extension $T \phi$ onto $TY$ defined by the equalities

$$T \phi(\tilde{u}_1, \ldots, \tilde{u}_m) = u_{TY} d(\phi(\tilde{u}_1, \ldots, \tilde{u}_m))$$

for any vector fields $u_1, \ldots, u_m$ on $Y$, where $\tilde{u}_a$ are their functorial lift onto $TY$ and $u_{TY}$ is the Liouville vector field on $TY$ \cite{9}.

A $k$-order Lagrangian on a fibre bundle $Y$ is defined as a density

$$L = L(x^{\lambda}, y^i, y^{i\lambda}) d^n x, \quad n = \dim X,$$

on a $k$-order jet manifold $J^kY$. The kernel of the associated Euler–Lagrange operator

$$\delta L = (\partial_i L + \sum_{\Lambda} (-1)^{|\Lambda|} d_{\Lambda} \partial_{\Lambda}^i L) dy^i \wedge d^n x$$

are the $2k$-order Euler–Lagrange equations

$$\partial_i L + \sum_{\Lambda} (-1)^{|\Lambda|} d_{\Lambda} \partial_{\Lambda}^i L = 0$$

on a fibre bundle $Y$.

Let us consider the vertical extension $VL$ \eqref{11} of the Lagrangian $L$ \eqref{12} onto $VJ^kY = J^k(VY)$. It reads

$$VL = \partial_V L d^n x.$$

Therefore, one can think of $VL$ \eqref{15} as being a $k$-order Lagrangian on the vertical tangent bundle. It is easily verified that the Euler–Lagrange operator $\delta VL$ of this Lagrangian $VL$ is the vertical extension $V \delta L$ \eqref{6} of the Euler–Lagrange operator $\delta L$ \eqref{13} of a Lagrangian $L$. Accordingly, the corresponding Euler–Lagrange equations are the deviation of the Euler–Lagrange equations \eqref{14}.

Furthermore, the counterpart of a first order Lagrangian formalism on a fibre bundle $Y \to X$ is polysymplectic Hamiltonian formalism \cite{1, 6} on the Legendre bundle

$$\Pi_Y = V^* Y \otimes (\wedge^n T^* X) \otimes TX = V^* Y \wedge (n-1) \wedge T^* X$$

provided with the holonomic coordinates $(x^{\lambda}, y^i, p^\lambda_i)$, where the fibre coordinates $p^\lambda_i$ possess the transition functions

$$p^\lambda_i = \det \left( \frac{\partial x^\varepsilon}{\partial x'^\mu} \right) \frac{\partial y^j}{\partial y'^i} \frac{\partial x^{\lambda \mu}}{\partial x^{\lambda \mu}} p^\mu_j.$$

The Legendre bundle $\Pi_Y$ \eqref{16} is provided with the polysymplectic form

$$\Omega_Y = dp^\lambda_i \wedge dy^i \wedge \omega \otimes \partial_\lambda.$$
and an exterior Hamiltonian form

\[ H = p^i \lambda dy^i \wedge \omega, \quad \omega = d^n x, \quad \omega_\lambda = \partial_\lambda \omega. \]  

(17)

This Hamiltonian form yields the covariant Hamilton equations

\[ y^i_\lambda = \partial^i_\lambda H, \quad p^i_\lambda = -\partial_i H \]  

(18)
on a fibre bundle \( Y \to X \). A key point is that, due to the canonical isomorphism \( VV^*Y = V^*VY \), the vertical extension \( VH \) (11) of the Hamiltonian form \( H \) (17) is a Hamiltonian form

\[ VH = (\dot{p}^i_\lambda dy^i + p^i_\lambda d\dot{y}^i) \wedge \omega_\lambda - \partial_V H \omega \]
on the Legendre bundle \( \Pi_{VY} \) over the vertical tangent bundle \( VY \), and that the corresponding covariant Hamilton equations are the deviation (7) of the covariant Hamilton equations (18).

For instance, if \( X = \mathbb{R} \), the above mentioned covariant Hamiltonian formalism provides Hamiltonian formalism of non-autonomous mechanics on a fibre bundle \( Y \to \mathbb{R} \) [7]. Its vertical extension has been considered in application to mechanical systems with non-holonomic constraints [3] and completely integrable systems [5]. In particular, one can show that Jacobi fields of a completely integrable Hamiltonian system of \( m \) degrees of freedom make up an extended completely integrable system of \( 2m \) degrees of freedom, where \( m \) additional integrals of motion characterize a relative motion [5].

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