The effects of disturbance on quantum speed limit

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Quantum theory sets a bound on the minimal time evolution between initial and target states. This bound is called as quantum speed limit time. It is used to quantify maximal speed of quantum evolution, in the sense that, if quantum speed limit time decreases the speed of quantum evolution will be faster. In this work, we study the quantum speed limit time of a quantum state in the presence of disturbance effects of environment. We use the model which is provided by Masashi Ban Phys. Rev. A 99, 012116 (2019). In this model two quantum systems $\mathcal{A}$ and $\mathcal{S}$ interact with environment sequentially. At first, quantum system $\mathcal{A}$ interacts with the environment $E$ as an auxiliary system then quantum system $\mathcal{S}$ interacts with disturbed environment immediately. In this work, we consider dephasing coupling with two types of environment with different spectral density: Ohmic and Lorentzian spectral density. We observe that as a result of the interaction of quantum system $\mathcal{A}$ with the environment, non-Markovian effects will appear in the dynamics of quantum system $\mathcal{S}$. Given the fact that quantum speed limit time reduces due to non-Markovian effects, we show that disturbance effects of environment reduce the quantum speed limit time.

I. INTRODUCTION

Quantum speed limit ($QSL$) determines the maximal evolution velocity of a quantum system. It sets a bound on the minimum evolution time needed for a quantum state of a closed and open quantum system to evolve from initial state to target state. It has many important applications in the field of quantum physics ranging from, quantum metrology [1], computation [2], communication [3] to non-equilibrium quantum thermodynamics [4] and quantum optimal control [5]. In Ref. [6], Mandelstam and Tamm (MT) provided a new insight on energy-time uncertainty relation. They showed that, the $QSL$ time $\tau_{QSL}$ for the dynamics of closed quantum system is given by

$$\tau \geq \frac{\pi \hbar}{2 \Delta E},$$  \hspace{1cm} (1)

where $\Delta E = \sqrt{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2}$ is the inverse of the variation of energy of the initial state and $\hat{H}$ is time-independent Hamiltonian describing the dynamics of quantum system. Also in Ref. [7], Margolus and Levitin (ML) introduced the ($QSL$) time for the dynamics of closed quantum systems based on the mean energy $E = \langle \hat{H} \rangle$ due to the ground state as

$$\tau \geq \frac{\pi \hbar}{2 E}.$$  \hspace{1cm} (2)

Considering and combining the results of (MT) and (ML), the ($QSL$) time for closed quantum systems and orthogonal states can be expressed as

$$\tau \geq \tau_{QSL} = \max\left\{ \frac{\pi \hbar}{2 \Delta E}, \frac{\pi \hbar}{2 E} \right\}.$$ \hspace{1cm} (3)

Due to the fact that the Hamiltonian is the generator of unitary evolution, it is reasonable to express the ($QSL$) time based on the initial energy of the system. In Refs.[8–13], the ($QSL$) time for closed quantum systems and orthogonal states is generalized to non-orthogonal states and driven systems.

In the real world, the interaction of the system with its surrounding, is inevitable. In this context, the theory of open quantum systems is used to examine such systems [14]. Due to the direction of information flow, the dynamics of open quantum systems can be classified into Markovian and non-Markovian quantum evolution. In Markovian dynamics the information only flow from the system to the environment, i.e the system smoothly loses its information. For non-Markovian dynamics the information flow-back from the environment to the system in some moments during quantum evolution.

In recent years, the ($QSL$) for open quantum systems has been widely studied. For open quantum systems, ($QSL$) has characterized using quantum Fisher information [15, 16], Bures angle [17], relative purity [18, 19] and other proper distance measures [20–24]. In Ref. [18] del Campo et al. showed that when the master equation has the Lindblad form the relative purity bound of ($QSL$) is similar to the MT bound. For pure initial state, Deffner and Lutz employed Bures angle to introduce ($QSL$) and they have defined a unique bound that covers (MT) and (ML) bound [17].

In addition to all these attempts to study the ($QSL$) for the open quantum systems, various studies have also been conducted in areas related to the ($QSL$) in open quantum systems such as the connection between initial state and

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quantum speed limit [25–27], relativistic quantum speed limit [28, 29], Quantum speed limit based on alternative fidelity [30], quantum speed limit in non-equilibrium environment [31], quantum speed limit for multipartite open quantum systems [32].

In this work, we consider an interesting case of open quantum systems in which the quantum system interacts with disturbed environment. We review the model of two quantum systems that interact with environment sequentially. First, one of the quantum systems interacts with initial non-disturbed environment in a finite time and disturbs the environment, afterwards the second system interacts with the disturbed environment [33]. In this work we consider the case in which the two quantum system interacts with common bosonic environment through a dephasing coupling consecutively. The quantum evolution of the second quantum system can be non-Markovian. It is due to the disturbance of the environment caused by interacting of the first quantum system with initial environment. Therefore, it can be concluded that, even if the evolution of the first quantum system is Markovian, the dynamics of the second quantum system can be non-Markovian. We will quantify the degree of non-Markovianity stems from disturbance in terms of environment Hamiltonian. We consider environments with Ohmic and Lorentzian spectral density. We also quantify the degree of non-Markovianity stems from disturbance in terms of environment Hamiltonian. We conclude and summary of this work is given in Sec.IV.

II. DYNAMICS OF OPEN QUANTUM SYSTEM INTERACTING WITH DISTURBED ENVIRONMENT

Here, we consider two quantum systems \( A \) and \( S \). At first, system \( A \) interacts with the environment \( E \) as an auxiliary system from time \( t_0 \) to time \( t_1 \) and leads to disturbances in the environment. Then system \( S \) interacts with disturbed environment from time \( t_2 \) to time \( t_3 \). For that time ordering from \( t_0 \) to \( t_3 \), the inequality holds in the form \( t_0 < t_1 \leq t_2 < t_3 \). If the time interval \( t_2 - t_1 \) between the end of the interaction of system \( A \) with the environment and the start of the interaction of system \( S \) with the disturbed environment is greater than the correlation time of the environment, then the disturbance effects will be ignored.

Now we supposed that the two quantum system \( A \) and \( S \) are two-level systems. We also assume that, they interact with bosonic environment through a dephasing model. Thus Hamiltonian of the two qubit quantum system and interaction Hamiltonian are given by 

\[
H_i = \frac{i}{\hbar} \omega_i \sigma_i^z / 2 \quad \text{and} \quad H_{AE} = h/2 \sigma_i^z \otimes \mathcal{E} \quad \text{respectively},
\]

where \( \mathcal{E} = \sum_k g_k (a_k + a_k^\dagger) \) is environmental operator, \( i = A, S \) and \( \sigma_i^z \) is the z-component of the Pauli operator, \( a_k(a_k^\dagger) \) is an annihilation(creation) operator of the \( k \)-th environmental oscillator with angular frequency \( \omega_k \). Hamiltonian of the environment \( \mathcal{E} \) given in the form

\[
H_{\mathcal{E}} = \sum_k \hbar \omega_k a_k^\dagger a_k.
\]

The whole system consists of \( A, S \), and \( \mathcal{E} \), from \( t_0 \) to \( t_1 \) are evolved through Hamiltonian \( H_A + H_{\mathcal{E}} + H_{AE} \) and from \( t_2 \) to \( t_3 \) are evolved through Hamiltonian \( H_S + H_{\mathcal{E}} + H_{SE} \). Given that the disturbance effects are important to us, here we assume that \( t_1 = t_2 \).

The evaluated state of the whole system is given by

\[
\rho_{A S E}^{\text{out}} = e^{-i (L_S + L_{AE} + L_{SE})(t_3 - t_2)} e^{-i (L_A + L_{AE} + L_{SE})(t_1 - t_0)} \rho_{A S E}^{\text{in}},
\]

where \( L_{\mathcal{O}}(\square) = -i/\hbar \{ h_{\mathcal{O}}, \square \} \) with \( \mathcal{O} \in \{ A, S, E, AE, SE \} \) is Liouvillian superoperator [34, 35].

We assume that there exist no initial correlation between quantum systems \( A \) and \( S \), i.e. \( \rho_{A S}^{\text{in}} = \rho_A^{\text{in}} \otimes \rho_S^{\text{in}} \). Let us consider the initial state of two quantum systems \( A \) and \( S \) as

\[
\rho_A^{\text{in}} = \rho_{ee}^{A} | e \rangle \langle e | + \rho_{gg}^{A} | g \rangle \langle g | + \rho_{eg}^{A} | g \rangle \langle e | + \rho_{ge}^{A} | e \rangle \langle g | \quad \text{(5)}
\]

\[
\rho_S^{\text{in}} = \rho_{ee}^{S} | e \rangle \langle e | + \rho_{gg}^{S} | g \rangle \langle g | + \rho_{eg}^{S} | g \rangle \langle e | + \rho_{ge}^{S} | e \rangle \langle g | \quad \text{(6)}
\]

where \( \sigma_z | e \rangle = | e \rangle \) and \( \sigma_z | g \rangle = -| g \rangle \). Using the method outlined in Ref. [33], one can derive the reduced time evolution of two quantum systems \( A \) and \( S \) as

\[
\rho_A(t) = \rho_{ee}^{A} | e \rangle \langle e | + \rho_{gg}^{A} | g \rangle \langle g | + \rho_{ge}^{A} e^{-i \omega_a (t - t_0)} | g \rangle \langle e | + \rho_{eg}^{A} e^{-i \omega_a (t - t_0)} | e \rangle \langle g |,
\]

\( 0 \leq t \leq t_1 \),

\[
\rho_S(t) = \rho_{ee}^{S} | e \rangle \langle e | + \rho_{gg}^{S} | g \rangle \langle g | + \rho_{eg}^{S} f(t) | e \rangle \langle g | + \rho_{ge}^{S} f(t) | g \rangle \langle e |,
\]

\( 0 \leq t \leq t_3 \),

with the case there exist no disturbance effects. The conclusion and summary of this work is given in Sec.IV.
with

\[ f(t) = \left[ \cos \mathcal{G}(t, t_2; t_1, t_0) - i \langle \sigma^z \rangle \sin \mathcal{G}(t, t_2; t_1, t_0) \right] \times e^{i \omega_c (t - t_2 - \mathcal{G}(t, t_2)} \]

where \( \langle \sigma^z \rangle = \text{tr}_A (\rho_A |0 \rangle \langle 0 |) \) and

\[ \mathcal{G}_R, \mathcal{I}(t, t_k) = \int_{t_k}^t d\tau' \int_{t_k}^\tau d\tau' \mathcal{C}_R, \mathcal{I}(\tau - \tau'), \quad k = 0, 2 \]

\[ \mathcal{G}_R, \mathcal{I}(t, t_2; t_1, t_0) = \int_{t_2}^{t_1} d\tau' \int_{t_0}^{t_1} d\tau' \mathcal{C}_R, \mathcal{I}(\tau - \tau'), \]

where \( \mathcal{C}_R(\tau - \tau') \) and \( \mathcal{C}_I(\tau - \tau') \) are the real and imaginary parts of the two-time correlation function

\[ C(\tau - \tau') = \mathcal{C}_R(\tau - \tau') + i \mathcal{C}_I(\tau - \tau') \]

\[ = \langle \mathcal{E}(\tau | t_0) \mathcal{E}(\tau' | t_0) \rangle \mathcal{E} \]

\[ = \sum_k g_k^2 \left\{ (\bar{N}_k + 1) e^{-i \omega_k (\tau - \tau')} + \bar{N}_k e^{i \omega_k (\tau - \tau')} \right\} \]

where \( \langle \cdot \rangle \mathcal{E} = \text{tr}_E [\langle \cdot \rangle \rho_E] \), \( \bar{N}_k = \langle a_k^\dagger a_k \rangle \mathcal{E} \) and

\[ \mathcal{E}(\tau | t_0) = e^{i \mathcal{H}_E (\tau - t_0) / \hbar} \mathcal{E} e^{-i \mathcal{H}_E (\tau - t_0) / \hbar} \]

\[ = \sum_k g_k (e^{-i \omega_k (\tau - t_0) / \hbar} a_k^\dagger + e^{i \omega_k (\tau - t_0) / \hbar} a_k) \]

In general, for dephasing coupling with bosonic environment we have

\[ \mathcal{G}(t, t_m) = \mathcal{G}_R(t, t_m) + i \mathcal{G}_I(t, t_m) \]

\[ = \sum_k \frac{g_k^2}{\omega_k} (1 - e^{-i \omega_k (t - t_m)} + i \omega_k (t - t_m)) = \int_0^\infty d\omega J(\omega) \frac{1 - e^{-i \omega (t - t_m)} + i \omega (t - t_m)}{\omega^2}, \quad m = 0, 2 \]

\[ \mathcal{G}_I(t, t_2; t_1, t_0) = \psi(t - t_0) - \psi(t - t_1) - \psi(t - t_2) + \psi(t_2 - t_1), \]

where \( J(\omega) \) is the spectral density of the environment.

In the following we consider dephasing model with two types of environment and different spectral density: Ohmic and Lorentzian spectral density.

### A. Dephasing model with Ohmic spectral density

In this work, it is assumed that two quantum system interacts with a common bosonic environment through dephasing coupling. Let us consider the case in which the environment is initially in the ground state i.e. \( \bar{N}_k = 0 \) and it has Ohmic spectral density

\[ J(\omega) = \eta \frac{\omega^s}{\omega_c^{s+1}} \exp(-\frac{\omega}{\omega_c}), \]

where \( \omega_c \) is the cutoff frequency, \( s \) is an Ohmicity parameter and \( \eta \) is a dimensionless coupling constant. From Eqs. 14, 15 and 16, we obtain

\[ \mathcal{G}_R(t, t_m) = \eta \Gamma |s - 1| \left( 1 - \frac{\cos[(s - 1) \arctan(\omega_c(t - t_m))] \omega_c(t - t_m)}{[1 + (\omega_c(t - t_m))^{s+1}]^{s+1}} \right), \quad m = 0, 2, \]

\[ \mathcal{G}_I(t, t_2; t_1, t_0) = \psi(t - t_0) - \psi(t - t_1) - \psi(t_2 - t_0) + \psi(t_2 - t_1), \]

For sub Ohmic \((s < 1)\) and super Ohmic \((s > 1)\) environment, where

\[ \psi(t) = \eta \Gamma |s - 1| \left( \frac{\sin[(s - 1) \arctan(\omega_c t)]}{[1 + \omega_c t^{s+1}]^{s+1}} \right). \]

For Ohmic \((s = 1)\) environment we have

\[ \mathcal{G}_R(t, t_m) = \frac{1}{2} \eta \ln[1 + (\omega_c(t - t_m))^2], \]

\[ m = 0, 2, \]

\[ \mathcal{G}_I(t, t_2; t_1, t_0) = \psi(t - t_0) - \psi(t - t_1) - \psi(t_2 - t_0) + \psi(t_2 - t_1), \]
where
\[ \psi(t) = \eta \arctan[\omega, t]. \tag{22} \]

From hereafter, we set \( t \) for the time elapsed from time \( t_2 \) at which quantum system \( S \) starts to interact with environment, i.e. we set \( (t-t_2) \rightarrow t \). Thus from Eqs. 8 and 9, the dynamics of reduced quantum system \( S \) can be written as
\[ \rho_S(t) = \rho_{ee}^s |e\rangle\langle e| + \rho_{gg}^s |g\rangle\langle g| \tag{23} \]
\[ + \rho_{eg}^s f(t) |e\rangle\langle g| + \rho_{ge}^s f^*(t) |e\rangle\langle e|, \quad 0 \leq t \leq t_3, \]
with
\[ f(t) = \{ \cos G(t,t_2;t_1,t_0) - i |s^2\rangle \sin G(t,t_2;t_1,t_0) \} \times e^{i \omega_s(t) - G_R(t)}. \tag{24} \]

For the case in which there is not exist disturbance effect, we have
\[ f(t) = e^{i \omega_s(t) - G_R(t)}. \tag{25} \]

\[ G_R(t,t_m) = \frac{\gamma}{2\lambda} \frac{1}{1 + \left(\frac{2 \lambda}{\gamma}\right)^2} \left( \lambda(t-t_m) - \frac{1 - \left(\frac{\delta}{\lambda}\right)^2}{1 + \left(\frac{4 \lambda}{\gamma}\right)^2} (1 - e^{\lambda(t-t_m)} \cos \delta t) - \frac{2 \delta}{1 + \left(\frac{4 \lambda}{\gamma}\right)^2} \sin \delta t \right), \tag{27} \]
\[ G_I(t,t_2;t_1,t_0) = \psi(t-t_0) - \psi(t-t_1) - \psi(t_2-t_0) + \psi(t_2-t_1), \tag{28} \]

where
\[ \psi(t) = \frac{\gamma}{2\lambda} \frac{e^{\lambda t}}{1 + \left(\frac{2 \lambda}{\gamma}\right)^2} \left( 1 - \left(\frac{\delta}{\lambda}\right)^2 \sin \delta t + \frac{2 \delta}{1 + \left(\frac{4 \lambda}{\gamma}\right)^2} \cos \delta t \right). \tag{29} \]

We chose \( t \) for the time elapsed from time \( t_2 \) at which quantum system \( S \) begin to interaction with environment, i.e. we set \( (t-t_2) \rightarrow t \). Thus from Eqs. 8 and 9, the dynamics of reduced quantum system \( S \) can be written as
\[ \rho_S(t) = \rho_{ee}^s |e\rangle\langle e| + \rho_{gg}^s |g\rangle\langle g| \tag{30} \]
\[ + \rho_{eg}^s f(t) |e\rangle\langle g| + \rho_{ge}^s f^*(t) |e\rangle\langle e|, \quad 0 \leq t \leq t_3, \]
with
\[ f(t) = \{ \cos G(t,t_2;t_1,t_0) - i |s^2\rangle \sin G(t,t_2;t_1,t_0) \} \times e^{i \omega_s(t) - G_R(t)}. \tag{31} \]

For the case in which there is not exist disturbance effect, we have
\[ f(t) = e^{i \omega_s(t) - G_R(t)}. \tag{32} \]

In the following, we study the non-Markovianity of the quantum evolution of quantum system \( S \), which is caused by the disturbance of environment.

B. Dephasing model with Lorentzian spectral density

As an another dephasing model, let us consider the case in which the environment is initially in the ground state i.e. \( N_k = 0 \) and it has Lorentzian spectral density spectral density
\[ J(\omega) = \frac{\gamma}{2\pi} \frac{\lambda^2}{(\omega - \delta)^2 + \lambda^2}, \tag{26} \]

where \( \lambda \) defines the spectral width of the coupling, \( \gamma \) is coupling constant and \( \delta \) is the frequency of the oscillator supported by the environment. From Eqs. 14, 15 and 16, we obtain

\[ G_R(t,t_0) = \psi(t-t_0) - \psi(t-t_1) - \psi(t_2-t_0) + \psi(t_2-t_1), \tag{28} \]

C. Non-Markovianity due to disturbance

We first review some basic notions of the theory of open quantum systems. The dynamical map is divisible if it can be written as two completely positive and trace preserving (CPTP) maps \( \phi_t = \phi_{t,t_0} \phi_{t_0,0} \quad \forall \quad t_0 \leq t \). So, the dynamical map is non-divisible if there exist times \( t_p \) at which \( \phi_{t,t_p} \) is not (CPTP). In general, the most important common character of all non-Markovianity measures is that they are introduced based on the non-monotonic time evolution of certain quantities when the divisibility property of (CPTP) maps is violated. We should point out that the inverse statement is not true. Actually, there exist non-divisible dynamical maps that certain quantity shows monotonic behaviour under them. In this work, we focus on non-Markovianity measure which is founded base on the measure of quantum coherence. Quantum coherence is a power full resource in quantum information theory. In recent years significant measure are introduced to quantify the quantum coherence, such as relative entropy of coherence [36], trace norm of coherence [37] and \( l_1 \) norm of coherence [36]. \( l_1 \)-norm of quantum coherence for a quantum state with the density matrixis \( \rho \) is
FIG. 1: The scheme for dynamics of two quantum systems $A$ and $S$. Quantum system $A$ interacts with the environment from time $t_0$ to time $t_1$, then system $S$ interacts with disturbed environment from time $t_2$ to time $t_3$.

From Eq. 34, one concluded that when initial state of the quantum system $A$ is maximally coherent state i.e. $\langle \sigma_z^A \rangle = 0$, disturbance effect has its maximum value while for $\langle \sigma_z^A \rangle = \pm 1$ disturbance effect has its minimum value. When quantum dynamical map is incoherent completely positive trace preserving (ICTPT), $l_1$-norm of coherence decreases monotonically. For non-monotonic behavior of $l_1$-norm of coherence, one conclude that the dynamical map is non-divisible and quantum evolution is non-Markovian. In Ref. [38], the authors proposed a measure based on $l_1$-norm of coherence to quantify the degree of non-Markovianity as

$$N = \max_{\rho_S(0) \in \{|\psi_2\rangle\}} \int_{\mathbb{R}} \frac{dC_{l_1}(\rho_S(t))}{dt} dt,$$

where the optimization is done over the set of all maximally coherent states $|\psi_2\rangle = \frac{1}{\sqrt{2}} \sum_{i=1}^{2} e^{i\varphi_i} |i\rangle$ and $\varphi_i \in [0, 2\pi)$.

1. Non-Markovianity when environment have Ohmic spectral density

Now, we want to investigate how disturbance of environment with ohmic spectral density affects on non-Markovianity. For dephasing model with Ohmic spectral density without disturbance the dynamics is non-Markovian for $s > 2$ while it is Markovian for $s \leq 2$. We concentrate on this range in which the dynamic is Markovian without disturbance. In Fig. 2, non-Markovianity is plotted as function of coupling parameter for sub-Ohmic environment with $s = 0.5$. As can be seen disturbance leads to non-Markovianity for $\eta \geq 3.6$, however when the disturbance effect is ignored and environment is in equilibrium the degree of non-Markovianity vanishes. In Fig. 3, non-Markovianity is plotted as function of coupling parameter for Ohmic environment with $s = 1$. As can be seen disturbance leads to non-Markovianity for $\eta \geq 4$, however when the disturbance effect is ignored and environment is in equilibrium the degree of non-Markovianity vanishes. In Fig. 4, non-Markovianity is plotted as function of coupling parameter for super-Ohmic environment with $s = 2$. As can be seen disturbance leads to non-Markovianity for $\eta \geq 2.8$, however when the disturbance effect is ignored and environment is in equilibrium the degree of non-Markovianity vanishes.

2. Non-Markovianity when environment have Lorentzian spectral density

In this part we investigate non-Markovianity for dephasing model with Lorentzian spectral density. In Fig. 5 non-Markovianity is plotted as a function of coupling parameter $\gamma$. As can be seen in the presence of disturbance dynamic is non-Markovian for $\gamma \geq 6.2$ while in the absence of disturbance the degree of non-Markovianity is defined as [36]

$$C_{l_1}(\rho) = \sum_{i,j: i \neq j} |\rho_{ij}|,$$

where $\rho_{ij}$'s are the off-diagonal elements of density matrix. When second quantum system $S$ interacts with disturbed environment, $l_1$-norm of coherence is changed as

$$C_{l_1}(\rho_S(t)) = 2|\rho_{eg}| (\cos^2 G_{I}(t,t_2; t_1,t_0) + \langle \sigma_z^a \rangle^2 \sin^2 G_{I}(t,t_2; t_1,t_0))^\frac{1}{2}.$$
FIG. 2: (color online) Non-Markovianity as a function of coupling parameter $\eta$ for sub-Ohmic environment with $s=0.5$. Blue (circle line) shows the degree of non-Markovianity in the presence of disturbance $\langle \sigma_a^x \rangle = 0.05$. Red (circle line) shows the degree of non-Markovianity without disturbance.

FIG. 3: (color online) Non-Markovianity as a function of coupling parameter $\eta$ for Ohmic environment with $s=1$. Blue (circle line) shows the degree of non-Markovianity in the presence of disturbance $\langle \sigma_a^x \rangle = 0.05$. Red (circle line) shows the degree of non-Markovianity without disturbance.

FIG. 4: (color online) Non-Markovianity as a function of coupling parameter $\eta$ for super-Ohmic environment with $s=2$. Blue (circle line) shows the degree of non-Markovianity in the presence of disturbance $\langle \sigma_a^x \rangle = 0.05$. Red (circle line) shows the degree of non-Markovianity without disturbance.

FIG. 5: (color online) Non-Markovianity as a function of coupling parameter $\gamma$ for Lorentzian environment with $\lambda = 1$ and $\delta = 1$. Blue (circle line) shows the degree of non-Markovianity in the presence of disturbance $\langle \sigma_a^x \rangle = 0.05$. Red (circle line) shows the degree of non-Markovianity without disturbance.

FIG. 6: (color online) Non-Markovianity as a function of coupling parameter $\eta$ for sub-Ohmic environment with $s=0.5$. Blue (circle line) shows the degree of non-Markovianity in the presence of disturbance $\langle \sigma_a^x \rangle = 0.05$. Red (circle line) shows the degree of non-Markovianity without disturbance.

In this part we investigate non-Markovianity for dephasing model with Lorentzian spectral density. In Fig. 6 non-Markovianity is plotted as a function of $\Delta$. As can be seen in the presence of disturbance dynamic is non-Markovian around $\Delta \approx 1$ due to disturbance of the environment and $\Delta \geq 3.6$ because of pure environmental effects. In the absence of disturbance effect dynamic is non-Markovian for $\Delta \geq 3.6$ due to environmental effects.

III. QUANTUM SPEED LIMIT TIME FOR ARBITRARY INITIAL STATE

Here we consider the dynamics of open quantum systems. At time $t$ the evaluated state of the open quantum system is represented by $\rho_t$. The dynamics of such a quantum system can be described by the time-dependent master equations of the form $\dot{\rho}_t = L_t(\rho_t)$, where $L_t$ is the generator of the evolution [14]. Now we want to calculate the minimum time it takes for a system to evolve from state $\rho_\tau$ to state $\rho_{\tau + \tau_D}$, where $\tau_D$ is the driving time. This minimum time is called quantum speed limit time $\tau_{QSL}$ (QSLT). One should use a suitable distance measure to characterize quantum speed limit time.
Ref. [19] Zhang et al. have used relative purity as the distance measure to quantify quantum speed limit time $\tau_{QSL}$. The important point about their quantum speed limit time is that, it can be used for both mixed and pure initial states. Relative purity $R(\tau)$ between initial state $\rho_i$ and evolved state $\rho_{\tau+\tau_D}$ can be written as

$$R(\tau+\tau_D) = \frac{tr(\rho_i \rho_{\tau+\tau_D})}{tr(\rho_i^2)}.$$  

Following the methodology presented in Ref. [19], one can obtain the (ML) bound of quantum speed limit time for dynamics of open quantum system as

$$\tau \geq \frac{|R(\tau+\tau_D) - 1| tr(\rho_{\tau}^2)}{\sum_{i=1}^{n} \sigma_i \rho_i},$$  

where $\sigma_i$ and $\rho_i$ are the singular values of $\dot{\rho}_i$ and $\rho_\tau$, respectively, $\mathcal{B} = \frac{1}{\tau_D} \int_\tau^{\tau+\tau_D} B dt$. Doing an analogous procedure, (MT) bound of $QSL$ time for dynamics of open quantum system can be derive as

$$\tau \geq \frac{|R(\tau+\tau_D) - 1| tr(\rho_{\tau}^2)}{\sum_{i=1}^{n} \sigma_i^2}.$$  

(37)

Considering these two bound, one can define the unified bound as

$$\tau_{QSL} = \max\{\frac{1}{\sum_{i=1}^{n} \sigma_i \rho_i}, \frac{1}{\sum_{i=1}^{n} \sigma_i^2}\} \times |R(\tau+\tau_D) - 1| tr(\rho_{\tau}^2).$$  

(38)

For dephasing coupling, $QSL$ time for dynamics of quantum system $S$ can be written as [19]

$$\tau_{QSL} = \frac{C_{2i}(\rho_S(0)) \int |f(\tau)| f(\tau+\tau_D) - f^2(\tau)| d\tau}{\int_{\tau_D}^{\tau+\tau_D} |f(t)| dt}.$$  

(39)

In Fig. 7, $(QSL)$ time is plotted as a function of initial time parameter $\tau$ for dephasing model with sub-Ohmic environment $s = 0.5$ and driving time $\tau_D = 1$. As can be seen in the presence of disturbance effects $(QSL)$ time is shorter than $(QSL)$ time when there exist no disturbance effects. Actually, It is due to the fact that existence of disturbance leads to non-Markovian quantum evolution, hence the quantum evolution is faster than the case in which the environment be in equilibrium. In Fig. 8, $(QSL)$ time is plotted as a function of initial time parameter $\tau$ for dephasing model with Ohmic environment $s = 1$ and driving time $\tau_D = 1$. As can be seen in the presence of disturbance effects $(QSL)$ time is shorter than $(QSL)$ time when there exist no disturbance effects. Actually, It is due to the fact that existence of disturbance leads to non-Markovian quantum evolution, hence the quantum evolution is faster than the case in which the environment be in equilibrium. In Fig. 9, $(QSL)$ time is plotted as a function of initial time parameter $\tau$ for dephasing model with super-Ohmic environment $s = 2$. 

FIG. 6: (color online) Non-Markovianity as a function of $\Delta$ for Lorentzian environment with $\lambda = 1$ and $\gamma = 1$. Blue (circle line) shows the degree of non-Markovianity in the presence of disturbance $\langle \sigma^2 \rangle = 0.05$. Red (circle line) shows the degree of non-Markovianity without disturbance.

FIG. 7: (color online) quantum speed limit time as a function of initial time parameter $\tau$ for dephasing model with sub-Ohmic environment $s = 0.5$ and driving time $\tau_D = 1$.

FIG. 8: (color online) quantum speed limit time as a function of initial time parameter $\tau$ for dephasing model with Ohmic environment $s = 1$.

FIG. 9: (color online) quantum speed limit time as a function of initial time parameter $\tau$ for dephasing model with super-Ohmic environment $s = 2$. 

FIG. 8: (color online) quantum speed limit time as a function of initial time parameter $\tau$ for dephasing model with Ohmic environment $s = 1$.
leads to non-Markovian quantum evolution, hence the quantum evolution is faster than the case in which the environment be in equilibrium. In Fig. 10, (QSL) time is plotted as a function of initial time parameter \( \tau \) for dephasing model with Lorentzian spectral density. In order to show the effects of initial time parameter \( \tau \) for dephasing model with Ohmic environment \( s = 2 \).

![Graph](image1)

**FIG. 9:** (color online) quantum speed limit time as a function of initial time parameter \( \tau \) for dephasing model with super-Ohmic environment \( s = 2 \).

![Graph](image2)

**FIG. 10:** (color online) quantum speed limit time as a function of initial time parameter \( \tau \) for dephasing model with Lorentzian spectral density with \( \gamma = 10 \) and \( \Delta = 1 \). As can be seen in the presence of disturbance effects (QSL) time is shorter than (QSL) time when there exist no disturbance effects. Actually, It is due to the fact that existence of disturbance leads to non-Markovian quantum evolution, hence the quantum evolution is faster than the case in which the environment be in equilibrium.

**IV. SUMMARY AND CONCLUSION**

In this work we considered the dephasing model in which two quantum systems \( A \) and \( S \) interacts with environment sequentially. The environment is disturbed by the interaction of first quantum system \( A \) with environment. Then quantum system \( S \) interacts with environment which has been disturbed. Note that, If the time interval between the beginning of the interaction of quantum system \( S \) and the end of the interaction of quantum system \( A \) with environment is greater than correlation time of the environment then environment returns to the equilibrium before interacting with the quantum system \( S \). According to Eq. 34, one can concluded that parameter \( \langle \sigma^z_0 \rangle \) defines the amount of disturbance in environment. In general, the coherence of the firs quantum system \( A \) quantify the power of disturbance of the environment. In the sense that if initial state of the quantum system \( A \) is maximally coherent then disturbance has its maximum value and and vice versa. In this paper, we studied the effects of disturbance on the (QSL) time. Here, two types of environment are considered with Ohmic and Lorentzian spectral density. We showed that the disturbance of the environment leads to the non-Markovian quantum evolution of quantum system \( S \). In Ref. [17], authors show that non-Markovian effects reduce the (QSL) time. In confirmation of their result, we showed that in the presence of disturbance effects the quantum speed limit time is shorter than that in which there is no disturbance.

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