On the weak solutions for the MHD systems with controllable total energy and cross helicity

Changxing Miao\textsuperscript{1} \* and Weikui Ye\textsuperscript{2} \dag

\textsuperscript{1}Institute of Applied Physics and Computational Mathematics, P.O. Box 8009, Beijing 100088, P. R. China
\textsuperscript{2}School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China

Abstract

In this paper, we prove the non-uniqueness of three-dimensional magneto-hydrodynamic (MHD) system in $C([0, T]; L^2(\mathbb{T}^3))$ for any initial data in $H^{\bar{\beta}}(\mathbb{T}^3)$ ($\bar{\beta} > 0$), by exhibiting that the total energy and the cross helicity can be controlled in a given positive time interval. Our results extend the non-uniqueness results of the ideal MHD system to the viscous and resistive MHD system. Different from the ideal MHD system, the dissipative effect in the viscous and resistive MHD system prevents the nonlinear term from balancing the stress error ($\dot{R}_q, \dot{M}_q$) as doing in [4]. We introduce the box flows and construct the perturbation consisting in seven different kinds of flows in convex integral scheme, which ensures that the iteration works and yields the non-uniqueness.

Keywords: convex integral iteration, the MHD system, weak solutions, cross helicity, non-uniqueness.

Mathematics Subject Classification: 35A02, 35D30, 35Q30, 76D05, 76W05.

Contents

1 Introduction 2

2 Outline of the convex integration scheme 6

2.1 Parameters and the iterative process 6

2.2 The proof sketch of Proposition 2.1 8

\*email: miao changxing@iapcm.ac.cn
\dag email: 904817751@qq.com

arXiv:2208.08311v3 [math.AP] 22 Apr 2024
1 Introduction

In this paper, we consider the Cauchy problem of the following 3D MHD equations:

\[
\begin{aligned}
\partial_t v - \nu_1 \Delta v + \text{div}(v \otimes v) + \nabla p &= \text{div}(b \otimes b), \\
\rho \partial_b - \nu_2 \Delta b + \text{div}(v \otimes b) &= \text{div}(b \otimes v), \\
\text{div} \, v &= 0, \quad \text{div} \, b = 0, \\
(v, b)|_{t=0} &= (v^{in}, b^{in}),
\end{aligned}
\]  

(1.1)

where \(v(t, x)\) is the fluid velocity, \(b(t, x)\) is the magnetic fields and \(p(t, x)\) is the scalar pressure. \(\nu_1\) and \(\nu_2\) are the viscous and resistive coefficients, respectively. We call (1.1) the viscous and resistive MHD system when \(\nu_1, \nu_2 > 0\). For a given initial data \((v^{in}, b^{in}) \in H^\beta(\mathbb{T}^3)\) \(1\) with \(\beta > 0\), we construct a weak solution of (1.1) with controllable total energy and cross helicity, which implies the non-uniqueness of weak solutions in \(C([0, T]; L^2(\mathbb{T}^3))\).

\(^1\)Here and throughout the paper, we denote \(\mathbb{T}^3 = [0, 1]^3\) and \((f, g) \in X \times X\) by \((f, g) \in X\)
To begin with, let us introduce the definition of weak solutions of (1.1).

**Definition 1.1.** Let \((v^{in}, b^{in}) \in L^2(T^3)\). We say that \((v, b) \in C([0, T]; L^2(T^3))\) is a weak solution to (1.1), if \(\text{div} \, v = \text{div} \, b = 0\) in the weak sense, and for all divergence-free test functions \(\phi \in C^\infty_0([0, T) \times T^3)\),

\[
\int_0^T \int_{T^3} (\partial_t - \nu_1 \Delta) \phi v + \nabla \phi : (v \otimes v - b \otimes b) \, dx \, dt = - \int_{T^3} v^{in} \phi(0, x) \, dx, \tag{1.2}
\]

\[
\int_0^T \int_{T^3} (\partial_t - \nu_2 \Delta) \phi b + \nabla \phi : (v \otimes b - b \otimes v) \, dx \, dt = - \int_{T^3} b^{in} \phi(0, x) \, dx. \tag{1.3}
\]

When \(\nu_1 = \nu_2 = 0\), (1.1) becomes the ideal MHD system:

\[
\begin{align*}
\partial_t v + \text{div}(v \otimes v) + \nabla p &= \text{div}(b \otimes b), \\
\partial_t b + \text{div}(v \otimes b) &= \text{div}(b \otimes v), \\
\text{div} v &= 0, \quad \text{div} b = 0,
\end{align*}
\tag{1.4}
\]

For the smooth solutions to (1.4), they possess a number of physical invariants:

- The total energy: \(e(t) = \int_{T^3} (|v(t, x)|^2 + |b(t, x)|^2) \, dx\);
- The cross helicity: \(h_{v,b}(t) = \int_{T^3} (v(t, x) \cdot b(t, x)) \, dx\);
- The magnetic helicity: \(h_{b,b}(t) = \int_{T^3} (A(t, x) \cdot b(t, x)) \, dx\),

where \(A\) is a periodic vector field with zero mean satisfying \(\text{curl} \, A = b\).

In 1949, Lars Onsager conjectured that the Hölder exponent threshold for the energy conservation of weak solutions of the Euler equations is \(1/3\). Since then, many mathematicians are devoted to proving Onsager conjecture on the Euler equations and there have been a flood of papers with this problem [7, 11, 15, 19, 22, 23, 26]. In recent years, Onsager-type conjectures on the ideal MHD equations which possess several physical invariants have caught researchers’ interest and some progress has been made on related issues. For instance, in [2, 9, 20], the magnetic helicity conservation for the 3D ideal MHD was proved in the critical space \(L^3_{t,x}\). Later, Faraco-Lindberg-Székelyhidi [18] showed that the \(L^3_{t,x}\) integrability condition for the magnetic helicity conservation is sharp. The cross helicity and total energy are conservative when the weak solutions \((v, b) \in L^3_{t,x} B^\alpha_{3,\infty}\) with \(\alpha > \frac{1}{3}\), but whether they are conservative for \(\alpha \leq \frac{1}{3}\) or not is still an open problem. Throughout current literatures, whether the solution satisfies the physical invariants or not plays a key role in studying the non-uniqueness problems. In [17], Faraco-Lindberg-Székelyhidi constructed non-trivial weak solutions with compact support in space \(L^\infty_{t,x}\). Beekie-Buckmaster-Vicol [4] constructed distributional solutions in \(C^1 L^2_{x}\) breaking the magnetic helicity conservation, and showed the non-uniqueness of weak solutions.

System (1.4) with \(b \equiv 0\) becomes the famous Euler equations. Many authors are devoted to the study of the non-uniqueness issue on Euler equations. In the pioneering paper [22], De Lellis-Székelyhidi developed the convex integration scheme and constructed the weak solutions in \(L^\infty_{t,x}\) with compact support to the 3D Euler equations, see also [23]. After that, there have been a series of results on non-uniqueness of weak...
solutions. The Onsager conjecture was finally solved in $C_{x,t}^3$ ($0 < \beta < 1/3$) by Isett \cite{19}, and by Buckmaster-De Lellis-Székelyhidi-Vicol \cite{6} for admissible weak solutions. Very recently, Daneri \cite{12}, subsequently in \cite{14} and Runa \cite{13} considered non-uniqueness problem by constructing wild initial data which is $L^2(T^3)$-dense, while Rosa and Haffter \cite{15} also showed that any smooth initial data gives rise to uncountably many solutions.

For the incompressible Navier-Stokes equations (1.1) with $b \equiv 0$, there have many results on the non-uniqueness problems. Buckmaster-Vicol in \cite{8} made the first important break-through by making use of a $L^{2}_x$-based intermittent convex integration scheme. Subsequently, Buckmaster, Colombo and Vicol \cite{5} showed that the wild solutions can be generated by $H^3$ initial data. Recently, another non-uniqueness result based on Serrin condition for the Navier-Stokes equations was proved by Cheskidov-Luo \cite{10}, which shows the sharpness of the Ladyzhenskaya-Prodi-Serrin criteria $2/p + 4/q \leq 1$ at the endpoint $(p,q) = (2,\infty)$.

In \cite{1}, Albritton, Brué and Colombo proved the non-uniqueness of the Leray-Hopf solutions with a special force by skillfully constructing a “background” solution which is unstable for the Navier-Stokes dynamics in similarity variables. For the 3D hyper-viscous NSE, Luo-Titi \cite{25} also proved the non-uniqueness results, whenever the exponent of viscosity is less than the Lions exponent $5/4$.

For the viscous and resistive MHD system, the existence of Leray-Hopf solutions to the MHD equations was proved by Wu \cite{28}. In \cite{24}, Li, Zeng and Zhang proved the non-uniqueness of weak solutions in $H^{1}_{\epsilon}$, where $\epsilon$ sufficiently small. However, the uniqueness of Leray-Hopf solutions is unsolved, even in $C_{t}L^{2}_{x} \cap L^{1}_{t}H^{1}_{x}$ is still open. In \cite{8}, the non-uniqueness result for the Navier-Stokes equations also imply the non-uniqueness of the viscous and resistive MHD system with trivial magnetic field in $C_{t}L^{2}_{x}$. One natural problem is whether the viscous and resistive MHD system with non-trivial magnetic fields in $C_{t}L^{2}_{x}$ is unique or not. In this paper, we solve this problem by showing the non-uniqueness of (1.1) with $\nu_1, \nu_2 > 0$. Now we are in position to state the main result.

**Theorem 1.2.** A weak solution $(v, b)$ of the viscous and resistive MHD system in $C([0,T];L^2(T^3))$ is non-unique if $(v, b)$ has at least one interval of regularity. Moreover, there exist non-Leray-Hopf weak solutions $(v, b)$ in $C([0,T];L^2(T^3))$.

**Remark 1.3.** For the ideal MHD system (1.4) with non-trivial magnetic fields, Beekie-Buckmaster-Vicol in \cite{4} proved the non-uniqueness for the weak solutions in $C_{t}L^{2}_{x}$. However, for the viscous and resistive MHD system (1.1), the uniqueness for solutions in $C_{t}L^{2}_{x}$ is still unsolved. Theorem 1.2 solves this problem and extends the non-uniqueness results of the ideal MHD system to the viscous and resistive MHD system.

Compared with the ideal MHD system, the dissipative effect prevents the nonlinear term from balancing the stress error $(\tilde{R}_q, \tilde{M}_q)$ as doing in \cite{4}. This leads to the major difficulty in convex integral iteration in $C_{t}L^{2}_{x}$. A nature choice is using 3D box type flows instead of the Mikado flows in convex integral iteration. However, these 3D box type flows do not have enough freedom on the oscillation directions in the velocity and magnetic flows, which will give rise to additional errors in the oscillation terms. Inspired by \cite{4,8,10}, we construct “temporal flows” and “Inverse traveling wave flows” to eliminate these extra errors, which help
us construct a weak solution by combining with the principal flows. Moreover, we construct the so-called “Initial flows” and “Helicity flows” to achieve

\[(v(0, x), b(0, x)) = (v^{in}, b^{in}),\] and \[\int_{\mathbb{T}^3} (|v|^2 + |b|^2) \, dx = e(t), \quad \text{and} \quad \int_{\mathbb{T}^3} v \cdot b \, dx = h(t), \quad t \in [1, T],\]
which yields the non-uniqueness of the weak solution.

We now present a main theorem, which immediately implies Theorem 1.2 by showing that the total energy and the cross helicity can be controlled in a given positive time interval:

**Theorem 1.4 (Main theorem).** Let \( T, \bar{\beta} > 0 \) and \((v^{in}, b^{in}) \in H^{\bar{\beta}}(\mathbb{T}^3)\). For fixed \( \delta_2 > 0 \), assume that there exists two smooth functions \( e(t), h(t) \) satisfying

\[
\frac{\delta_2}{2} \leq e(t) - \int_{\mathbb{T}^3} (|v^{in}|^2 + |b^{in}|^2) \, dx \leq \frac{3\delta_2}{4}, \quad t \in [\frac{1}{2}, T] \tag{1.5}
\]

and

\[
\frac{\delta_2}{200} \leq h(t) - \int_{\mathbb{T}^3} v^{in} \cdot b^{in} \, dx \leq \frac{\delta_2}{50}, \quad t \in [\frac{1}{2}, T]. \tag{1.6}
\]

Then there exists a weak solution \((v, b) \in C([0, T]; L^2(\mathbb{T}^3))\) to the viscous and resistive MHD system with initial data \((v^{in}, b^{in})\). Moreover, we have

\[\int_{\mathbb{T}^3} (|v|^2 + |b|^2) \, dx = e(t) \quad \text{and} \quad \int_{\mathbb{T}^3} v \cdot b \, dx = h(t), \quad t \in [1, T] \]

where \( h(t) := h_{v,b}(t) \) denotes the cross helicity.

**Remark 1.5.** For a given \((v^{in}, b^{in}) \in H^{\bar{\beta}}\), one can choose infinitely many functions \( e(t), h(t) \) satisfying (1.5) and (1.6), which implies the non-uniqueness of weak solutions. Moreover, we will prove that \((v, b) \in C([0, T]; H^s(\mathbb{T}^3))\) with \( 0 < s < \bar{\beta} \) in Section 2.2.

For the ideal MHD system (1.4), one can obtain a similar result after a simple modification to the proof of Theorem 1.4.

**Theorem 1.6.** Let \( T, \bar{\beta} > 0 \) and \((v^{in}, b^{in}) \in H^{\bar{\beta}}(\mathbb{T}^3)\). Then there exist infinitely many smooth functions \( e(t), h(t) \) associated with a weak solution \((v, b) \in C([0, T]; L^2(\mathbb{T}^3))\) to the ideal MHD system (1.4) with initial data \((v^{in}, b^{in})\). Moreover, we have

\[\int_{\mathbb{T}^3} (|v|^2 + |b|^2) \, dx = e(t) \quad \text{and} \quad \int_{\mathbb{T}^3} v \cdot b \, dx = h(t), \quad t \in [1, T].\]

**Remark 1.7.** Theorem 1.6 shows that all initial data in \( H^{\bar{\beta}} \) \((\forall \bar{\beta} > 0)\) may generate non unique weak solutions by choosing different total energy \( e(t) \) or cross helicity \( h(t) \). For weak solutions with non-conservative magnetic helicity, one can see [4,18,24] for more details.

As a matter of fact, authors in [4] constructed solutions in \( C_t H^s_{x} \hookrightarrow C_t L^2_x \) which breaks the conservative law of magnetic helicity. In view of Taylor’s conjecture, these weak solutions cannot be the weak ideal limits of Leray-Hopf weak solutions. Mathematically, Taylor’s conjecture is stated as follows:
Theorem 1.8 (Taylor’s conjecture [16,27]). Suppose that \((v, b) \in L^\infty([0, T]; L^2(T^3))\) is a weak ideal limit of sequence of Leray-Hoff weak solutions of the viscous and resistive MHD system, then the magnetic helicity is conservative.

Fortunately, combining Theorem 1.4 with Theorem 1.6 we can prove that the weak solutions constructed in [4] can be a vanishing viscosity and resistivity limit of the weak solutions to (1.1), which is similar to Theorem 1.3 in [8].

Corollary 1.9. Suppose that \((v, b) \in C([0, T]; H^\epsilon(T^3))\) is a weak solution of (1.4). Then, there exist \(0 < \epsilon' \ll \epsilon\) and a sequence of weak solutions \((v^{\nu_n}, b^{\nu_n}) \in C([0, T]; H^\epsilon'(T^3))\) to the viscous and resistive MHD system such that,

\[ (v^{\nu_n}, b^{\nu_n}) \to (v, b) \text{ strongly in } C_t L_x^2, \text{ as } \nu_n \to 0, \]

where \(\nu_n = (\nu_{1,n}, \nu_{2,n})\).

2 Outline of the convex integration scheme

In this paper, it suffices to prove Theorem 1.4 for (1.1) with \(\nu_1, \nu_2 > 0\). Without loss of generality, we set \(\nu_1 = \nu_2 = 1\).

2.1 Parameters and the iterative process

Set \(\bar{\beta} < 1\). If \((v^{in}, b^{in})\) is sufficiently smooth, we still have \((v^{in}, b^{in}) \in H^1 \subset H^{\bar{\beta}}\). We choose \(b = 2^{16[\bar{\beta}^{-1/2}]}, \beta = \frac{\bar{\beta}}{b}, \alpha\) to be a small constant depending on \(b, \beta, \bar{\beta}\) such that \(0 < \alpha \leq \min\{\frac{1}{b^4}, \frac{\beta}{b^3}\}\), and \(a \in \mathbb{N}^+\) to be a large number depending on \(b, \beta, \bar{\beta}, \alpha\) and the initial data. We define

\[ \lambda_q := a^{(b^q)}, \quad \delta_q := \lambda_q^{3\beta} \lambda_q^{-2\beta}, \quad q \in \mathbb{N}^+. \]

For \(q = 1, 2, \delta_q\) is a large number which could bound the \(L^2\) norm of initial data by choosing \(a\) sufficiently large. For \(q \geq 3, \delta_q\) is small and tends to zero as \(q \to \infty\).

Firstly, we choose two smooth functions \(e : [1/2, T] \to [0, \infty), h : [1/2, T] \to (-\infty, \infty)\) such that

\[ \frac{\delta_2}{2} \leq e(t) - \int_{T^3} (|v^{in}|^2 + |b^{in}|^2) \, dx \leq \frac{3\delta_2}{4}, \]

\[ \frac{\delta_2}{200} \leq h(t) - \int_{T^3} v^{in} \cdot b^{in} \, dx \leq \frac{\delta_2}{150}. \]

Secondly, adopting strategy of convex integration scheme, we consider a modification of (1.1) with stress tensor error \((\tilde{R}_q, \tilde{M}_q)\). Assume that \(\psi_\epsilon := \frac{1}{\epsilon} \tilde{\psi}(\frac{x}{\epsilon})\) stands for a sequence of standard mollifiers, where \(\tilde{\psi}\) is a
non-negative radial bump function. Let \((v_q, b_q, p_q, \hat{R}_q, \hat{M}_q)\) solve
\[
\begin{align*}
\partial_t v_q - \Delta v_q + \text{div}(v_q \otimes v_q) + \nabla p_q &= \text{div}(b_q \otimes b_q) + \text{div} \hat{R}_q, \\
\partial_t b_q - \Delta b_q + \text{div}(v_q \otimes b_q) &= \text{div}(b_q \otimes v_q) + \text{div} \hat{M}_q, \\
\nabla \cdot v_q &= 0, \quad \nabla \cdot b_q = 0, \\
(v_q, b_q)_{t=0} &= (v^i, b^i) = (v^i, b^i)_{i,j=1}^3/(\psi_{\ell_{q-1}}, b^i_{\psi^j_{\ell_{q-1}}}).
\end{align*}
\tag{2.4}
\]
where \(\ell_{q-1} := \lambda_{q-1}^{-6}, \quad v \otimes b := (v^j b_j)_{i,j=1}^3\), and vector \text{div}M denotes the divergence of a 2-tensor \(M = (M_{ij})_{i,j=1}^3\) with components:
\[
\text{div}(M)_i := \partial_j M_{ji}.
\]
In particular, \(\text{div}(v \otimes b) = (v \cdot \nabla)b\) if \(\text{div} v = 0\). The magnetic stress \(\hat{M}_q\) is required to be an anti-symmetric matrix. And the Reynolds stress \(\hat{R}_q\) is a symmetric, trace-free 3 \times 3 matrix.
\[
\hat{M}_q = -\hat{M}_q^T, \quad \hat{R}_q = \hat{R}_q^T, \quad \text{Tr} \hat{R}_q = \sum_{i=1}^3 (\hat{R}_q)_{ii} = 0.
\tag{2.5}
\]
The estimates we propagate inductively are:
\[
\|(v_q, b_q)\|_{L^2} \leq C_0 \sum_{i=1}^q \delta_i^{1/2},
\tag{2.6}
\]
\[
\|(v_q, b_q)\|_{H^3} \leq \lambda_q^5,
\tag{2.7}
\]
\[
\|(\hat{R}_q, \hat{M}_q)\|_{L^1} \leq \delta_{q+1} \lambda_q^{-4\alpha},
\tag{2.8}
\]
\[
(v_q(0, x), b_q(0, x)) = (v^i \psi_{\ell_{q-1}}, b^i \psi_{\ell_{q-1}}),
\tag{2.9}
\]
\[
t \in [1 - \tau_{q-1}, T] \implies \frac{1}{2} \delta_{q+1} \leq e(t) - \int_{\mathbb{T}^3} |v_q|^2 + |b_q|^2 \, dx \leq \delta_{q+1},
\tag{2.10}
\]
\[
t \in [1 - \tau_{q-1}, T] \implies \frac{\delta_{q+1}}{100} \leq h(t) - \int_{\mathbb{T}^3} v_q \cdot b_q \, dx \leq \frac{\delta_{q+1}}{100},
\tag{2.11}
\]
where \(\ell_q := \lambda_q^{-6}, \quad \tau_q := \ell_q^3\) and \(C_0 := 600\). By the definition of \(\delta_q\), one can easily deduce that \(\sum_{i=1}^\infty \delta_i\) converges to a finite number. Moreover, we restrict the error of the cross helicity to be much smaller than the energy error in the iterative procedure, which is used to reduce the impact on the energy error, see Section 4.4.

**Proposition 2.1.** Let \((v^i, b^i) \in H^3(\mathbb{T}^3)\) with \(0 < \beta < 1\). Assume that \((v_q, b_q, p_q, \hat{M}_q, \hat{R}_q)\) solves (2.4) and satisfies (2.6)-(2.9), and \(e(t), h(t)\) are any smooth functions satisfying (2.10)-(2.11), then there exists a solution \((v_{q+1}, b_{q+1}, p_{q+1}, \hat{R}_{q+1}, \hat{M}_{q+1})\), satisfying (2.4), (2.6)-(2.11) with \(q\) replaced by \(q + 1\), and such that
\[
\|(v_{q+1} - v_q, b_{q+1} - b_q)\|_{L^2} \leq C_0 \delta_{q+1}^{1/2}.
\tag{2.12}
\]

**Notations:** Throughout this paper, we set that
\[
v \otimes b := v \otimes b - \frac{1}{3} \text{Tr}(v \otimes b) \text{Id}, \quad \mathbb{P}_H := \text{Id} - \nabla \text{div}, \quad \mathbb{P}_\gamma f(x) = f(x) - \int_{\mathbb{T}^3} f(z) \, dz,
\]
where \(\mathbb{P}_H \mathbb{P}_\gamma f(x) := f(x) - \int_{\mathbb{T}^3} f(z) \, dz\).
where $v \otimes b$ is a trace-free matrix and $\mathcal{A}(z) := \sum_{l \in \mathbb{Z}^3/\{0\}} \frac{f_l}{z^l} e^{itz}$ for any mean free function $f$. We have $\partial_t f_l(z) = \sum_{l \in \mathbb{Z}^3/\{0\}} \frac{f_l}{z^l} e^{itz}$.

Next, we prove that Proposition 2.1 implies Theorem 4. To start the iteration, we define $(v_1, b_1, p_1, \hat{R}_1, \hat{M}_1)$ by

$$v_1(x, t) := e^{\Delta} v^i \* \psi_{t_0}, \quad b_1(x, t) := e^{\Delta} b^i \* \psi_{t_0}, \quad p_1(x, t) := |v_1|^2 - |b_1|^2,$$

$$\hat{R}_1(x, t) := v_1 \hat{\otimes} b_1 - b_1 \hat{\otimes} b_1, \quad \hat{M}_1(x, t) := v_1 \otimes b_1 - b_1 \otimes v_1.$$ 

It is easy to verify that $(v_1, b_1, p_1, \hat{R}_1, \hat{M}_1)$ solves (2.4). In addition, letting $a, b$ be sufficiently large, we can guarantee that

$$\|(\hat{R}_1, \hat{M}_1)\|_{L^1} \leq \|(v^i, b^i)\|_{L^2}^2 \leq \delta_2 \lambda_1^{-40\alpha},$$

$$\|(e^{\Delta v}^i, e^{\Delta b}^i)\|_{L^2} \leq \|(v^i, b^i)\|_{L^2} \leq \delta_2^{1/2} < \delta_1^{1/2},$$

$$\|(v^i \* \psi_{t_0}, b^i \* \psi_{t_0})\|_{H^3} \leq \delta_2^{1/2} \lambda_0^5 \leq \lambda_1^5.$$

For arbitrary smooth functions $e : [1/2, T] \to [0, \infty)$, $h : [1/2, T] \to (-\infty, \infty)$ satisfying the estimates (2.2), (2.3), it is easy to verify that (2.10)–(2.11) for $q = 1$.

Then, making use of Proposition 2.1 inductively, we obtain a $L^2$ convergent sequence of functions $(v_q, b_q) \to (v, b)$ which solves (1.1), with $\|v\|_{L^2} + \|b\|_{L^2} = e(t)$ and $\int_\mathbb{T} v \cdot b \, dx = h(t)$ for all $t \in [1, T]$. A standard argument shows that $(v, b) \in C([0, T]; L^2(T^3))$, see [6] for more details. Moreover, from (2.7) and (2.12), there exists $0 < \epsilon \ll \beta$ such that $\{(v_q, b_q)\}$ is also a Cauchy sequence in $C_t H^\epsilon_x$ by interpolation. Thus, we obtain $(v, b) \in C_t H^\epsilon_x$.

The remainder of the paper is devoted to the proof of Proposition 2.1.

### 2.2 The proof sketch of Proposition 2.1

Starting from a solution $(v_q, b_q, p_q, \bar{R}_q, \bar{M}_q)$ satisfying the estimates as in Proposition 2.1, the broad scheme of the iteration is as follows.

1. We defined $(v_{t_q}, b_{t_q}, p_{t_q}, \bar{R}_{t_q}, \bar{M}_{t_q})$ by mollification, and it is standard in convex integration schemes.

2. We define a family of exact solutions $(v_t, b_t)_{t \geq 0}$ to MHD by exactly solving the MHD system with initial data $(v_1, b_1)_{t = t_0} = (v_{t_q}(t_l), b_{t_q}(t_l))$, where $t_l = l \tau_q$ defines an evenly spaced partition of $[0, T]$.

3. These solutions are glued together by a partition of unity, leading to the tuple $(\bar{v}_q, \bar{b}_q, \bar{p}_q, \bar{R}_q, \bar{M}_q)$. The stress error terms are zero when $t \in J_l$, $t \geq 0$, see [5,6].

4. We define $(v_{q+1}, b_{q+1}) = (\bar{v}_q + w_{q+1}, \bar{b}_q + d_{q+1})$ by constructing a perturbation $(w_{q+1}, d_{q+1})$.

5. Finally, we prove that the inductive estimates (2.6)–(2.11) hold with $q$ replaced by $q + 1$.

Step 4 is moderately involved and we breaks it into the following sub-steps:
1. For times $t \geq 1$, we use the ‘squiggling’ cutoffs $\eta_l$ from \cite{6,21} that allow energy to be added at such times, even outside the support of $(\hat{R}_q, \hat{M}_q)$, while cancelling a large part of $(\hat{R}_q, \hat{M}_q)$.

2. For times $t < 1$, we instead employ the straight cutoffs introduced in \cite{5,19}. This ensures that $(\tilde{v}_q, \tilde{b}_q)|_{t=0} = (v^{in} \ast \psi_{t_{q-1}} \ast \psi_{t_q}, b^{in} \ast \psi_{t_{q-1}} \ast \psi_{t_q})$.

3. Then we construct the seven parts of the perturbation by using the “box flows”.

- Principal flows: $(w^{(p)}_{q+1}, d^{(p)}_{q+1})$ plays a role in cancelling the Reynolds and magnetic stresses $(\hat{R}_q, \hat{M}_q)$, while this would lead to some extra errors.

- Temporal flows: $(w^{(t)}_{q+1}, d^{(t)}_{q+1})$ is used to cancel the extra errors which stem from $\text{div}(\phi_k \bar{k})$, where $\phi_k$ is a traveling-wave.

- Inverse traveling wave flows: $(w^{(v)}_{q+1}, d^{(v)}_{q+1})$ is used to cancel the extra errors produced by $\text{div}(\phi_k \bar{k})$, where $\phi_k$ does not depend on $t$.

- Heat conduction flows: $(w^{(\Delta)}_{q+1}, \Delta_{q+1})$ is used to cancel the extra errors producing by the inverse traveling wave flows $(w^{(v)}_{q+1}, d^{(v)}_{q+1})$.

- Corrector flows: $(w^{(c)}_{q+1}, d^{(c)}_{q+1})$ is introduced to correct principal perturbation $(w^{(p)}_{q+1}, d^{(p)}_{q+1})$ to enforce the incompressibility condition.

- Initial flows: $(w^{(s)}_{q+1}, d^{(s)}_{q+1})$ can ensure that $(v_{q+1}, b_{q+1})|_{t=0} = (v^{in} \ast \psi_{t_q}, b^{in} \ast \psi_{t_q})$. It should be noted that the above five types of flows are zero when $t = 0$.

- Helicity flows: $(w^{(h)}_{q+1}, d^{(h)}_{q+1})$ makes the cross helicity satisfy (2.11) at $q+1$ level.

It is noteworthy that the first five flows are enough to produce a weak solution $(v, b)$ for \cite{11}, initial flows and helicity flows are used to control the helicity and show the non-uniqueness.

\section{Preliminary preparation of iteration}

In this section, we provide some preliminary preparation from $(v_q, b_q)$ to $(\tilde{v}_q, \tilde{b}_q)$, and it is essentially a modification as in \cite{6,8,21}. For the sake of completeness, we briefly review relevant results in the process of constructing $(\tilde{v}_q, \tilde{b}_q)$ and readers can refer to \cite{6,8,21} for the more details. In Section 4, we will construct $(\tilde{v}_q, \tilde{b}_q) \rightarrow (v_{q+1}, b_{q+1})$ to complete the iteration, which is the key ingredient of this paper.

\subsection{Mollification}

Let $\ell_q := \lambda_q^{-6}$, and we define the functions $v_{\ell_q}, b_{\ell_q}$ and $\hat{R}_{\ell_q}, \hat{M}_{\ell_q}$ as follows:

\begin{align*}
v_{\ell_q} &:= v_q \ast \psi_{\ell_q}, & \hat{R}_{\ell_q} &:= \hat{R}_q \ast \psi_{\ell_q} - (v_q \otimes v_q) \ast \psi_{\ell_q} + v_{\ell_q} \otimes v_{\ell_q} + (b_q \otimes b_q) \ast \psi_{\ell_q} - b_{\ell_q} \otimes b_{\ell_q}, \\
b_{\ell_q} &:= b_q \ast \psi_{\ell_q}, & \hat{M}_{\ell_q} &:= \hat{M}_q \ast \psi_{\ell_q} - (v_q \otimes b_q) \ast \psi_{\ell_q} + v_{\ell_q} \otimes b_{\ell_q} + (b_q \otimes v_q) \ast \psi_{\ell_q} - b_{\ell_q} \otimes v_{\ell_q}.
\end{align*}
Then, \((v_{\ell q}, b_{\ell q}, p_{\ell q}, \hat{R}_{\ell q}, \hat{M}_{\ell q})\) satisfies the following equations

\[
\begin{aligned}
&\partial_t v_{\ell q} - \Delta v_{\ell q} + \text{div}(v_{\ell q} \otimes v_{\ell q}) + \nabla p_{\ell q} = \text{div}(b_{\ell q} \otimes b_{\ell q}) + \text{div} \hat{R}_{\ell q}, \\
&\partial_t b_{\ell q} - \Delta b_{\ell q} + \text{div}(v_{\ell q} \otimes b_{\ell q}) = \text{div}(b_{\ell q} \otimes v_{\ell q}) + \text{div} \hat{M}_{\ell q}, \\
&\nabla \cdot v_{\ell q} = 0, \quad \nabla \cdot b_{\ell q} = 0, \\
&v_{\ell q}|_{t=0} = \nu \cdot v_{\ell q-1} * \psi_{\ell q}, \quad b_{\ell q}|_{t=0} = \nu \cdot b_{\ell q-1} * \psi_{\ell q},
\end{aligned}
\]  

(3.3)

where \(p_{\ell q} := p_{q} * \psi_{\ell q} - |v_q|^2 + |v_{\ell q}|^2 + |b_q|^2 - |b_{\ell q}|^2\), and we have used the identity \(\text{div}(fI_{3 \times 3}) = \nabla f\) for a scalar field \(f\). A simple computation shows the following mollification estimates:

**Proposition 3.1** (Estimates for mollified functions \([6]\)). For any \(N \geq 0\), we have \(^2\)

\[
\begin{align*}
&\|v_{\ell q} - v_q\|_{L^2} + \|b_{\ell q} - b_q\|_{L^2} \lesssim \delta_q + 1 \ell_q^{3\alpha}, \\
&\|v_{\ell q}\|_{H^{N+3}} + \|b_{\ell q}\|_{H^{N+3}} \lesssim \delta_q + 1 \ell_q^{N-1}, \\
&\|\hat{R}_{\ell q}\|_{W^{N,1}} + \|\hat{M}_{\ell q}\|_{W^{N,1}} \lesssim \delta_q + 1 \ell_q^{3\alpha-N}, \\
&\int_{\mathbb{T}^3} (|v_q|^2 - |v_{\ell q}|^2) \, dx + \int_{\mathbb{T}^3} (|b_q|^2 - |b_{\ell q}|^2) \, dx \lesssim \delta_q + 1 \ell_q^{3\alpha}, \\
&\int_{\mathbb{T}^3} v_q \cdot b_q - v_{\ell q} \cdot b_{\ell q} \, dx \lesssim \delta_q + 1 \ell_q^{3\alpha}.
\end{align*}
\]

\[\text{(3.4) - (3.8)}\]

### 3.2 Classical exact flows

We define \(\tau_q\) and the sequence of initial times \(t_l \ (l \in \mathbb{N})\) by

\[
\tau_q := \ell_q^3 \ll \|(v_{\ell q}, b_{\ell q})\|^{-1}_{H^{3/2 + \alpha}}, \quad t_l := l \tau_q,
\]

(3.9)

and \((v_l, b_l)\) denotes the unique strong solution to the following MHD system on \([t_l, t_{l+2}]\):

\[
\begin{aligned}
&\partial_t v_l - \Delta v_l + \text{div}(v_l \otimes v_l) + \nabla p_l = \text{div}(b_l \otimes b_l), \\
&\partial_t b_l - \Delta b_l + \text{div}(v_l \otimes b_l) = \text{div}(b_l \otimes v_l), \\
&\text{div} v_l = 0, \quad \text{div} b_l = 0, \\
&v_l|_{t=t_l} = v_{\ell q}(\cdot, t_l), \quad b_l|_{t=t_l} = b_{\ell q}(\cdot, t_l).
\end{aligned}
\]

(3.10)

**Proposition 3.2** (Estimates for classical solutions to MHD \([28]\)). Let \((v_0, b_0) \in H^{N_0} \) with \(N_0 \geq 3\), and \(\text{div} \, v_0 = \text{div} \, b_0 = 0\). Then there exists a unique local solution \((v, b)\) to \((3.1)\) with \(v_1 = v_2 = 1\) satisfying

\[
\|(v(\cdot), b(\cdot), t))\|_{H^N} \lesssim \|(v_0, b_0)\|_{H^N}, \quad N \in [\frac{5}{2}, N_0],
\]

where the local lifespan \(T = \frac{\|(v_0\|_{H^{5/2 + \alpha}} + \|b_0\|_{H^{5/2 + \alpha}}}{c}\) for some universal \(c > 0\).

According to Proposition 3.2, the solvability of the Cauchy problem \((3.10)\) on \([t_l, t_{l+2}]\) can be stated as:

\(^2\)Throughout this paper, we use the notation \(x \lesssim y\) to denote \(x \leq Cy\), for a universal constant \(C\) that may be different from line to line, and \(x \ll y\) to denote that \(x\) is much less than \(y\).
Corollary 3.3. System \((3.10)\) possesses a unique local solution \((v_t, b_t)\) in \([t_1, t_{1+}]\) such that
\[
\|v(t, t)|L^2 \lesssim \|v(t, b\|L^2, \quad \forall N \geq 0, \quad \forall N \geq 0, \quad (3.12)
\]
where \((v_t - v_{\ell_0}, b_t - b_{\ell_0})\) has zero mean.

Proof. \((3.11)-(3.12)\) could be obtained by Proposition 3.2. We want to prove (3.13). Let \((v, b) := (v_t - v_{\ell_0}, b_t - b_{\ell_0})\), we have
\[
\begin{aligned}
\partial_t v - \Delta v + \text{div}(v_t \otimes v + v \otimes v_{\ell_0}) + \nabla(p_t - p_{\ell_0}) &= \text{div}(b_t \otimes b + b \otimes b_{\ell_0}) + \text{div} \tilde{R}_{\ell_0}, \\
\partial_t b - \Delta b + \text{div}(v_t \otimes b + b \otimes v_{\ell_0}) &= \text{div}(b_t \otimes v + b \otimes v_{\ell_0}) + \text{div} \tilde{M}_{\ell_0}, \\
v|_{t=t_1} = 0, \quad b|_{t=t_1} = 0.
\end{aligned}
\]
When \(N = 0\), using the causal estimates in Besov space \([3]\) on \([t_1, t_{1+}]\), we deduce that
\[
\|(v, b)\|_{L^\infty_t B^{2,1}_2 \cap L^1_t B^{2,1}_2} \lesssim \|(v_t \otimes v + v \otimes v_{\ell_0})\|_{L^1_t B^{2,1}_2} + \|\text{div}(b_t \otimes b + b \otimes b_{\ell_0})\|_{L^1_t B^{2,1}_2}
\]
where we use the fact that \(\|\tilde{R}_{\ell_0}, \tilde{M}_{\ell_0}\|_{B^{2,1}_2} \lesssim \|\tilde{R}_{\ell_0}, \tilde{M}_{\ell_0}\|_{W^{5/2,1}} \lesssim \delta_{q+1} \ell_q^{-5/2+\alpha}.\)

When \(N \geq 1\), similarly we deduce that
\[
\|(v, b)\|_{L^\infty_t B^{2,1}_2 \cap L^1_t B^{N/2+1}} \lesssim \|(v_t \otimes v + v \otimes v_{\ell_0})\|_{L^1_t B^{N/2+1}} + \|\text{div}(b_t \otimes b + b \otimes b_{\ell_0})\|_{L^1_t B^{N/2+1}}
\]
(3.15) and (3.16) imply (3.13).

3.3 Gluing flows

Define the intervals \(I_l, J_l (l \geq 0)\) by
\[
I_l := \left[t_l + \frac{\tau_l}{3}, t_l + \frac{2\tau_l}{3}\right],
\]
(3.17)
Here \( \rho \) denotes the smallest number so that
\[
[0, T] \subseteq J_0 \cup I_0 \cup J_1 \cup I_1 \cup \cdots \cup J_{N_q} \cup I_{N_q},
\]
i.e.
\[
N_q := \sup \left\{ l \geq 0 : (J_l \cup I_l) \cap [0, T] \neq \emptyset \right\} \leq \left\lfloor \frac{T}{\tau_q} \right\rfloor.
\]
For \( N \geq 0 \), let \( \{\chi_l\}_{l=1}^{N_q} \) be a partition of unity such that
\[
\sum_{l=1}^{N_q} \chi_l(t) = 1, \quad t \in [-\frac{\tau_q}{3}, T + \frac{\tau_q}{3}],
\]
where
\[
\text{supp } \chi_l = I_{l-1} \cup J_l \cup I_l, \quad \chi_l|_{I_l} = 1, \quad \|\partial^N \chi_l\|_{C^2} \leq \tau_q^{-N}, \quad (N_q - 1 \geq l \geq 2), \quad (3.19)
\]
\[
\text{supp } \chi_1 = J_0 \cup I_0 \cup J_1 \cup I_1, \quad \chi_1|_{[0, t]} = 1, \quad \|\partial^N \chi_1\|_{C^2} \leq \tau_q^{-N}, \quad (3.20)
\]
\[
\text{supp } \chi_{N_q} = I_{N_q-1} \cup J_{N_q} \cup I_{N_q}, \quad \chi_{N_q}|_{[t, T]} = 1, \quad \|\partial^N \chi_{N_q}\|_{C^2} \leq \tau_q^{-N}. \quad (3.21)
\]
In particular, for \( |l - j| \geq 2 \), \( \text{supp } \chi_l \cap \text{supp } \chi_j = \emptyset \). Then we define the glued velocity, magnetic fields and pressure \((\bar{v}_q, \bar{b}_q, \bar{p}_q)\) by
\[
\bar{v}_q(x, t) := \sum_{l=0}^{N_q-1} \chi_{l+1}(t) v_l(x, t), \quad (3.22)
\]
\[
\bar{b}_q(x, t) := \sum_{l=0}^{N_q-1} \chi_{l+1}(t) b_l(x, t), \quad (3.23)
\]
\[
\bar{p}_q(x, t) := \sum_{l=0}^{N_q-1} \chi_{l+1}(t) p_l(x, t). \quad (3.24)
\]
One can deduce that
\[
(\bar{v}_q(0, x), \bar{b}_q(0, x)) = (v_{t_q}(0, x), b_{t_q}(0, x)) = (v^{in} * \psi_{\ell_q-1} * \psi_{\ell_q}, b^{in} * \psi_{\ell_q-1} * \psi_{\ell_q}).
\]
Furthermore, \((\bar{v}_q, \bar{b}_q)\) solves the following system for \( t \in [0, T] \):
\[
\begin{cases}
\partial_t \bar{v}_q - \Delta \bar{v}_q + \text{div}(\bar{v}_q \otimes \bar{v}_q) + \nabla \bar{p}_q = \text{div}(\bar{b}_q \otimes \bar{b}_q) + \text{div} \hat{R}_q, \\
\partial_t \bar{b}_q - \Delta \bar{b}_q + \text{div}(\bar{v}_q \otimes \bar{b}_q) = \text{div}(\bar{b}_q \otimes \bar{v}_q) + \text{div} \hat{M}_q, \\
\text{div } \bar{v}_q = 0, \quad \text{div } \bar{b}_q = 0, \\
\bar{v}_q|_{t=0} = v_{t_q}(\cdot, 0), \quad \bar{b}_q|_{t=0} = b_{t_q}(\cdot, 0). \quad (3.25)
\end{cases}
\]
Here \((\hat{R}_q, \hat{M}_q, \bar{p}_q)\) is defined as follows:
\[
\hat{R}_q := \sum_{l=0}^{N_q} \partial_t \chi_l \hat{R} (v_l - v_{l+1}) - \chi_l (1 - \chi_l)(v_l - v_{l+1}) \otimes (v_l - v_{l+1})
\]
\[
\hat{M}_q := \sum_{l=0}^{N_q} \partial_t \chi_l \hat{M} (v_l - v_{l+1}) - \chi_l (1 - \chi_l)(v_l - v_{l+1}) \otimes (v_l - v_{l+1})
\]
\[
\bar{p}_q := \sum_{l=0}^{N_q} \partial_t \chi_l \hat{p} (v_l - v_{l+1}) - \chi_l (1 - \chi_l)(v_l - v_{l+1})
\]
Proposition 3.7 in this section. We define the index

\[ N_q^0 := \left\lfloor \frac{1}{\tau_q} \right\rfloor - 2 \in \mathbb{N}^+ , \]

where we have used the inverse divergence operators \( \mathcal{R} \) and \( \mathcal{R}_a \) in Section A.2. It is easy to check that \( \nu_l - \nu_{l+1}, b_l - b_{l+1} \) and \( \tilde{\nu}_q \) have zero mean, \( \tilde{M}_q \) is anti-symmetric, \( \tilde{R}_q \) is symmetric and trace-free.

Combining Corollary 3.3 with the estimates of solutions for the heat equation in periodic Besov spaces [3], we deduce the following two propositions after some computations:

**Proposition 3.4** (Estimates for \( (\tilde{\nu}_q - \nu_{\ell_q}, \tilde{b}_q - b_{\ell_q}) \)). For all \( t \in [0, T] \) and \( N \geq 0 \), we have

\[
\| (\tilde{\nu}_q, \tilde{b}_q) \|_{H^{3+N}} \lesssim \lambda^5 q^{-N} ,
\]

\[
\| (\tilde{\nu}_q - \nu_{\ell_q}, \tilde{b}_q - b_{\ell_q}) \|_{L^2} \lesssim \delta_t^{1/2} q^{-\alpha} ,
\]

\[
\| (\tilde{\nu}_q - \nu_{\ell_q}, \tilde{b}_q - b_{\ell_q}) \|_{H^N} \lesssim \tau_q \delta_{q+1}^{-N-5/2+\alpha} .
\]

**Proposition 3.5** (Estimates for \( (\mathcal{R}(\nu_l - \nu_{l+1}), \mathcal{R}_a(b_l - b_{l+1})) \)). For all \( t \in [0, T] \) and \( N \geq 0 \), we have

\[
\| (\mathcal{R}(\nu_l - \nu_{l+1}), \mathcal{R}_a(b_l - b_{l+1})) \|_{W^{N,1}} \lesssim \tau_q \delta_{q+1}^{-N+\alpha} .
\]

**Proposition 3.6** (Estimates for \( (\tilde{R}_q, \tilde{M}_q) \)). For all \( t \in [0, T] \) and \( N \geq 0 \), we have

\[
\| (\tilde{R}_q, \tilde{M}_q) \|_{W^{N,1}} \lesssim \delta_{q+1}^{-N+\alpha} ,
\]

\[
\| (\partial_\nu \tilde{R}_q, \partial_{\nu} \tilde{M}_q) \|_{W^{N,1}} \lesssim \delta_{q+1}^{-N+\alpha} .
\]

**Proposition 3.7** (Gaps between energy and helicity). For all \( t \in [0, T] \), we have

\[
\int_{T^3} \left( |\tilde{\nu}_q|^2 + |\tilde{b}_q|^2 - |\nu_{\ell_q}|^2 - |b_{\ell_q}|^2 \right) dx \lesssim \delta_{q+1}^{2\alpha} ,
\]

\[
\int_{T^3} (\tilde{\nu}_q \cdot \tilde{b}_q - \nu_{\ell_q} \cdot b_{\ell_q}) dx \lesssim \delta_{q+1}^{2\alpha} .
\]

### 3.4 Cutoffs

#### 3.4.1 Space-time cutoffs

To control the energy without impacting the initial data, we construct the following space-time cutoffs \( \eta_l \) in this section. We define the index

\[ N_q^0 := \left\lfloor \frac{1}{\tau_q} \right\rfloor - 2 \in \mathbb{N}^+ , \]
and denote by \( \{ \eta_l \}_{l \geq 1} \) the cutoffs such that:

\[
\eta_l(x, t) := \begin{cases} 
\tilde{\eta}_l(t) & 1 \leq l < N_0^q, \\
\tilde{\eta}_l(x, t) & N_0^q \leq l \leq N_q. 
\end{cases}
\] (3.37)

We define \( \tilde{\eta}_l \) as in [5,19] as follows: Let \( \tilde{\eta}_1 \in C_c^\infty(J_1 \cup I_1 \cup J_2; [0, 1]) \) satisfy

\[
\text{supp} \tilde{\eta}_1 = I_1 + \left[ -\frac{\tau_q}{6}, \frac{\tau_q}{6} \right] = \left[ \frac{7\tau_q}{6}, \frac{11\tau_q}{6} \right],
\]

be identically 1 on \( I_1 \), and possess the following estimates for \( N \geq 0 \):

\[
\| \partial^n_t \tilde{\eta}_1 \|_{C^m_t} \lesssim \tau_q^{-N}.
\]

We set \( \tilde{\eta}_l(t) := \tilde{\eta}_1(t - t_{l-1}) \) for \( 1 \leq l < N_0^q \).

Next, we give the definition of \( \tilde{\eta}_l(t, x) \) as in [6,21]. Let \( \varepsilon \in (0, 1/3) \), \( \varepsilon_0 \ll 1 \). For \( N_0^q \leq l \leq N_q \), letting \( \phi \) be a bump function such that \( \text{supp} \phi \subset [-1, 1] \) and \( \phi = 1 \) in \( [-\frac{1}{2}, \frac{1}{2}] \), we define

\[
I_1' := I_l + \left[ -\frac{(1 - \varepsilon)\tau_q}{3}, \frac{(1 - \varepsilon)\tau_q}{3} \right] = \left[ l\tau_q + \frac{\varepsilon\tau_q}{3}, l\tau_q + \frac{3 - \varepsilon)\tau_q}{3} \right],
\]

\[
I_1'' := \left\{ (x, t + 2\varepsilon\tau_q \sin(2\pi x_1)) : x \in \mathbb{T}^3, t \in I_1' \right\} \subset \mathbb{T}^3 \times \mathbb{R},
\]

\[
\tilde{\eta}_l(x, t) := \frac{1}{\varepsilon_0^q} \int_{I_1''} \phi \left( \frac{|x - y|}{\varepsilon_0 \tau_q} \right) \phi \left( \frac{|t - s|}{\varepsilon_0 \tau_q} \right) dy \, ds.
\]

Figure 1: The support of a single \( \tilde{\eta}_l \). For each time \( t \in [t_l, t_{l+1}] \), the integral \( \int_0^1 \tilde{\eta}_l \, dx_1 > c_\eta \approx 1/4 \). Furthermore, \( \{ \text{supp} \tilde{\eta}_l \}_{l \geq N_0^q} \) are pairwise disjoint sets. Figure from [21].

From the above discussion, we can obtain the following lemma:

**Lemma 3.8** ([6,21]). For all \( l = 1, \ldots, N_q \), The functions \( \{ \eta_l \}_{l \geq 1} \) satisfy

1. \( \eta_l \in C_c^\infty(\mathbb{T}^3 \times (J_l \cup I_l \cup J_{l+1}); [0, 1]) \) with:

\[
\| \partial^n_t \eta_l \|_{L^\infty_x C^m_t} \lesssim_{n,m} \tau_q^{-n}, \quad n, m \geq 0.
\] (3.38)
2. $\eta(\cdot, t) \equiv 1$ for $t \in I_t$.

3. $\text{supp } \eta_i \cap \text{supp } \eta_j = \emptyset$ if $l \neq j$.

4. For all $t \in [t_{N^0_q}, T]$, we have
   \[ c_\eta \leq \sum_{l=0}^{N_q} \int_{T_3} \eta_l^2(x, t) \, dx \leq 1 \]
   for a fixed positive constant $c_\eta \approx \frac{1}{4}$ independent of $q$.

5. For all $1 \leq l < N^0_q$, $\eta_l$ only depends on $t$, and $\text{supp } \eta_l \subset \left[\frac{7}{8} \tau_q + (l - 1)\tau_q, \frac{11}{8} \tau_q + (l - 1)\tau_q\right]$.

### 3.5 Helicity gap

First, for $t \in [0, T]$ we set new helicity gap such that
\[
    h_q(t) := \frac{1}{3} \left( h(t) - \int_{T_3} \tilde{\eta}_q \cdot \tilde{\eta}_q \, dx - \frac{\tilde{\delta}_{q+1}}{200} \right). 
\]

We deduce by Proposition 3.1 and Proposition 3.7 that $h_q(t)$ is strictly positive in $[1 - \tau_{q-1}, T]$ and satisfies
\[
    \frac{1}{400} \tilde{\delta}_{q+1} \leq 3h_q(t) \leq \frac{1}{90} \tilde{\delta}_{q+1}, \quad \forall t \in [1 - \tau_{q-1}, T]. 
\]

Next, we define a function $\eta_{-1} := \eta_{-1}(t) \in C^\infty_c([0, t_{N^0_q+1}); [0, 1])$ such that
\[
    \eta_{-1} \equiv 1, \quad 0 \leq t \leq t_{N^0_q}, 
\]
and satisfies
\[
    \sup_t |\partial^N_t \eta_{-1}(t)| \lesssim \tau_q^{-N}, \quad N \geq 0. 
\]

Note that $t_{N^0_q+1} = \tau_q(\lfloor \tau_q^{-1} \rfloor - 1) \leq 1 - \tau_q$, then $t \geq 1 - \tau_q$ implies $\eta_{-1}(t) = 0$. The following figure shows the supports relationship between $\eta_l$ and $\eta_{-1}$ visually:

![Figure 2](image-url)
Then, we modify the energy gap \( h_q(t) \) by setting

\[
h_{b,q}(t) := \frac{\delta_{q+1}}{300} N(t) + \frac{h_q(t)(1-N(t))}{\tau_{q-1}(t) + \sum_{\ell=1}^{N_q} \eta_{q+\ell}^2(x,t) \, dx},
\]

where \( N \) is a smooth cut-off function that is equal to 1 for \( t < 1 - \tau_{q-1} \) and 0 for \( t > 1 - \tau_q \) and satisfies

\[
\partial_t N \lessapprox \tau_{q-1}^{-N} \lessapprox \tau_q^{-N}.
\]

Using Lemma 3.8, one can easily deduce that \( \frac{\delta_{q+1}}{2400} \leq h_{b,q}(t) \leq \frac{1}{50} \delta_{q+1} \) for each \( t \in [0,T] \), which implies that \( h_{b,q} \) is not much different from the original helicity gap in (2.11).

4 Perturbation

In this section, we construct the perturbation \((w_{q+1}, d_{q+1})\) to iterate \((\tilde{v}_q, \tilde{b}_q) \to (v_{q+1}, b_{q+1})\), which is the key ingredient of this paper.

4.1 Stresses associated with the MHD system

Let \((v_{q+1}, b_{q+1})) = (\tilde{v}_q + w_{q+1}, \tilde{b}_q + d_{q+1})\). From (3.25), we obtain the following MHD system with new Reynolds and magnetic stresses \((\tilde{R}_{q+1}, \tilde{M}_{q+1})\):

\[
\begin{align*}
\partial_t v_{q+1} - \Delta v_{q+1} + \text{div}(v_{q+1} \otimes v_{q+1} - b_{q+1} \otimes b_{q+1}) &= \text{div} \tilde{R}_{q+1} - \nabla p_{q+1}, \\
\partial_t b_{q+1} - \Delta b_{q+1} + \text{div}(v_{q+1} \otimes b_{q+1} - b_{q+1} \otimes v_{q+1}) &= \text{div} \tilde{M}_{q+1}, \\
\end{align*}
\]

where

\[
p_{q+1} := \bar{p}_q(x,t) - \frac{1}{3} \text{Tr}[w_{q+1} \otimes \bar{v}_q + \bar{v}_q \otimes w_{q+1} - d_{q+1} \otimes \bar{b}_q - \bar{b}_q \otimes d_{q+1}] - P_v,
\]

\[
\tilde{R}_{q+1} := R[\partial_t w_{q+1} - \Delta w_{q+1}] + [w_{q+1} \otimes \bar{v}_q + \bar{v}_q \otimes w_{q+1} - d_{q+1} \otimes \bar{b}_q - \bar{b}_q \otimes d_{q+1}]
\]

\[
+ R[\text{div}(w_{q+1} \otimes w_{q+1} - d_{q+1} \otimes d_{q+1}) + \text{div} \tilde{R}_q - \nabla P_v]
\]

\[
:= R_{\text{lin}}^{\text{q+1}} + R_{\text{osc}}^{\text{q+1}},
\]

\[
\tilde{M}_{q+1} := R[a[\partial_t d_{q+1} - \Delta d_{q+1}] + [w_{q+1} \otimes \bar{b}_q + \bar{v}_q \otimes d_{q+1} - \bar{b}_q \otimes w_{q+1} - d_{q+1} \otimes \bar{v}_q]
\]

\[
+ R[a \mathbb{P}_H[\text{div}(w_{q+1} \otimes d_{q+1} - d_{q+1} \otimes w_{q+1}) + \text{div} \tilde{M}_q]]
\]

\[
:= M_{\text{lin}}^{\text{q+1}} + M_{\text{osc}}^{\text{q+1}}.
\]

Here we have used the fact that \( \mathbb{P}_H \text{ div } A = \text{ div } A \) for any anti-symmetric matrix \( A \). The definition of \( P_v \) can be seen in (5.29).

Before introducing the perturbation \((w_{q+1}, d_{q+1})\), we provide two useful tools: “geometric lemmas” and “box flows”.

4.2 Two geometric lemmas

**Lemma 4.1** (First Geometric Lemma). Let \( B_{e_b}(0) \) be a ball of radius \( e_b \) centered at 0 in the space of \( 3 \times 3 \) skew-symmetric matrices. There exists a set \( \Lambda_b \subset S^2 \cap Q^3 \) that consists of vectors \( k \) with associated
orthogonal basis \((k, \bar{k}, \tilde{k})\), \(\epsilon_b > 0\) and smooth positive functions \(a_{b,k} : B_{\epsilon_b}(0) \to \mathbb{R}\), such that for \(M \in B_{\epsilon_b}(0)\)

\[
M = \sum_{k \in \Lambda_b} a_{b,k}(M)(\bar{k} \otimes \tilde{k} - \bar{k} \otimes \tilde{k}).
\]

**Lemma 4.2** (Second Geometric Lemma [4]). Let \(B_{\epsilon_b}(\text{Id})\) be a ball of radius \(\epsilon_b\) centered at \(\text{Id}\) in the space of \(3 \times 3\) symmetric matrices. There exists a set \(\Lambda_v \subset S^2 \cap \mathbb{Q}^3\) that consists of vectors \(k\) with associated orthogonal basis \((k, \bar{k}, \tilde{k})\), \(\epsilon_v > 0\) and smooth positive functions \(a_{v,k} : B_{\epsilon_v}(\text{Id}) \to \mathbb{R}\), such that for \(R_u \in B_{\epsilon_v}(\text{Id})\)

\[
R = \sum_{k \in \Lambda_v} a_{v,k}(R)\bar{k} \otimes \tilde{k}.
\]

To eliminate the helicity errors, we need another set \(\Lambda_s\) which does not interact with \(\Lambda_v, \Lambda_b\), see Appendix A.2 for more details. For simplicity, we let \(\Lambda := \Lambda_b \cup \Lambda_v \cup \Lambda_s\).

### 4.3 Box flows

In this section, let \(\Phi : \mathbb{R} \to \mathbb{R}\) be a smooth cutoff function supported on the interval \([0, \frac{1}{8}]\). Assume that

\[
\phi := \frac{d^2}{dx^2} \Phi.
\]

We define the stretching transformation as

\[
r^{-\frac{2}{3}}\phi(N_{\Lambda}r^{-1}x),
\]

where \(r^{-1}\) is a positive integer number and \(N_{\Lambda}\) is a large number such that \(N_{\Lambda}k, N_{\Lambda}\bar{k}, N_{\Lambda}\tilde{k} \in \mathbb{Z}^3\). We periodize it as \(\phi_r(x)\) on \([0, 1]\).

Next, for larger positive integer numbers \(r^{-1}, \bar{r}^{-1}, \tilde{r}^{-1}\), \(\mu\) and \(\sigma\), we set

\[
\phi_k(x) := \phi_r(\sigma k \cdot x), \quad \phi_\bar{k}(x, t) := \phi_r(\sigma k \cdot x + \sigma \mu t), \quad \phi_\tilde{k}(x) := \phi_r(\sigma \tilde{k} \cdot x).
\]

Then we define a set of functions \(\{\phi_{k,\bar{k},\tilde{k}}\}_{k \in \Lambda} : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}\) by

\[
\phi_{k,\bar{k},\tilde{k}}(x, t) := \phi_k(x-x_k)\phi_\bar{k}(x-x_k, t)\phi_\tilde{k}(x-x_k), \quad k \in \Lambda,
\]

where \(x_k \in \mathbb{R}^3\) are shifts which guarantee that

\[
\text{supp } \phi_{k,\bar{k},\tilde{k}} \cap \text{supp } \phi_{k',\bar{k}',\tilde{k}'} = \emptyset, \quad \text{if} \quad k', k \in \Lambda, \quad k \neq k'. \tag{4.1}
\]

There exist such shifts \(x_k\) by the fact that \(r, \bar{r}, \tilde{r} \ll 1\). (4.1) makes sense since \(\phi_{k,\bar{k},\tilde{k}}\) supports in some small 3D boxes. Readers can refer to [10][14][24] for this technique. In the rest of this paper, we still denote \(\phi_k(x-x_k), \phi_\bar{k}(x-x_k, t)\) and \(\phi_\tilde{k}(x-x_k)\) by \(\phi_k(x), \phi_\bar{k}(x, t)\) and \(\phi_\tilde{k}(x)\), respectively.

Now, setting \(\sigma = \lambda_{q+1}^{\frac{m}{2}}, \mu = \lambda_{q+1}^{\frac{m}{4}}\) and \(r = \bar{r} = \lambda_{q+1}^{\frac{m}{8}}, \tilde{r} = \lambda_{q+1}^{\frac{m}{16}}\). One can easily verify that \(\phi_{k,\bar{k},\tilde{k}}\) has zero mean and we can deduce the following proposition:
Proposition 4.3. For $p \in [1, \infty]$, we have
\[ \| \phi_{k,\bar{k},k} \|_{L^p} \leq \frac{24}{p+1} \lambda_{q+1}^{\frac{1}{2}} \lambda_{q+1}^{\frac{1}{2}-\frac{1}{p}} \| \psi \|_{L^p}. \]
Moreover, we have $\| \phi_{k,\bar{k},k} \|_{L^2} = 1$ after normalization.

Finally, let $\Psi \in C^\infty(T)$ and $\psi = \frac{d^2}{dx^2} \Psi$. Set $\psi_k := \psi(\lambda_{q+1} N_k x)$ and normalize it such that $\| \psi_k \|_{L^2} = 1$.

One can deduce that
\[ \psi_k = \lambda_{q+1}^{-2} N^{-2} \Delta[\Psi(\lambda_{q+1} N_k x)] := \lambda_{q+1}^{-2} N^{-2} \Delta \Psi_k, \quad k \in \Lambda. \]

It is easy to verify that $\psi_k \phi_{k,\bar{k},k}$ is also a $C^\infty(T^3)$ function supported in 3D boxes, and we call $\psi_k \phi_{k,\bar{k},k}$ or $\psi_k \phi_{k,\bar{k},k}$ “box flows” throughout this paper.

4.4 Construction of perturbation

Now, we introduce the following seven types of perturbation:

1. To cancel the errors of the cross helicity, we construct a so-called “helicity flow” $(u^{(h)}_{q+1}, d^{(h)}_{q+1})$. We are in position to define
\[ u^{(h)}_{q+1} = d^{(h)}_{q+1} := \sum_{t, k \in \Lambda} \eta_t h^{1/2} \psi_k \phi_{k,\bar{k},k}. \]

where $h_{b,q}(t)$ comes from (4.43).

2. We aim to construct the principal corrector $(u^{(p)}_{q+1}, d^{(p)}_{q+1})$ via geometric lemmas. Note that the geometric lemmas are valid for anti-symmetric matrices perturbed near 0 matrix and symmetric matrices perturbed near Id matrix, we need the following smooth function $\chi$ introduced in [4][25] such that the stresses are pointwise small. More precisely, let $\chi : [0, \infty) \rightarrow \mathbb{R}^+ \setminus \{0\}$ be a smooth function satisfying
\[ \chi(z) = \begin{cases} 1, & 0 \leq z \leq 2, \\ z, & z \geq 4. \end{cases} \]

To cancel the magnetic stress $\tilde{M}_q$, we construct the principal correctors $u^{(pb)}_{q+1}$ and $d^{(pb)}_{q+1}$. Letting $\chi_b := \chi \left( \frac{M_b}{\bar{b}_q + 1 \ell_q^{a/3}} \right)$ and $M_b := -\tilde{M}_q$, $\rho_{b,q} := \chi_b \delta_q + 1 \ell_q^{a/3}$, we set that
\[ u^{(pb)}_{q+1} := \sum_{t, k \in \Lambda} \eta_t \rho_{b,q}^{1/2} a_{k,\bar{k}} (\frac{M_b}{\rho_{b,q}}) \psi_k \phi_{k,\bar{k},k}, \]
\[ d^{(pb)}_{q+1} := \sum_{t, k \in \Lambda} \eta_t \rho_{b,q}^{1/2} a_{k,\bar{k}} (\frac{M_b}{\rho_{b,q}}) \psi_k \phi_{k,\bar{k},k}. \]
where \( E(t) := \int_{\Omega} \left( |u_{q+1}^{(ph)}|^2 + |d_{q+1}^{(ph)}|^2 + |u_{q+1}^{(h)}|^2 + |d_{q+1}^{(h)}|^2 \right) \, dx \). One can easily deduce that \( E(t) \leq \delta_{q+1}/10 \).

Indeed, from the definition of \( \phi_{k,k,k} \) and \( \psi_k \), we deduce by Lemma A.1 that

\[
|E(t)| \leq \left( \sum_{i=1}^{n_i} \eta_i \right) \frac{\rho(t)^{1+8(t)}}{\eta_i(t) + \sum_{i=1}^{n_i} \eta_i \chi_i(x,t) \, dx},
\]

where \( \chi_i := \chi \left( \frac{R_{i\over E_{q+1}}}{{E_{q+1}}} \right) \) and \( R_u := \hat{R}_q - \sum_{l,k} \ell_{q+1}^2 \left( \hat{k} \otimes \hat{k} - \hat{k} \otimes \hat{k} \right) \).

We firstly show that \( \rho_{q,0}(t) \) is well-defined. When \( t \in I_i \cup J_i \) with \( i \leq N_q^0 \), we have \( \eta_{-1}(t) = 1 \), \( \rho_{q,0}(t) \) is well-defined. When \( t \in I_i \) with \( i \geq N_q^0 \), we have

\[
\int_{\Omega} \eta_i^2 \chi_i(x,t) \, dx = \int_{\Omega} \chi_i \, dx = \int_{\Omega} \chi \left( \frac{R_{i\over E_{q+1}}}{{E_{q+1}}} \right) \, dx \geq \int \left( \frac{R_u}{{E_{q+1}}^{1/4}} \right)^2 \, dx \geq 12, \quad \frac{1}{2},
\]

where we use that

\[
\left| m \left\{ x \left| \left( \frac{R_u}{{E_{q+1}}} \right)^2 \geq 2 \right. \right\} \right| = \int \left( \frac{R_u}{{E_{q+1}}^{1/4}} \right)^2 \, dx \leq \frac{\left\| R_u \right\|_{L^2}^2}{{E_{q+1}}^{1/4}} \leq \frac{1}{2}.
\]

When \( t \in J_i \) with \( i \geq N_q^0 \), we conclude that

\[
\hat{R}_q = \hat{M}_q = 0 \quad \text{and} \quad \left( \frac{M_u}{{E_{q+1}}} \right) \left( \frac{R_u}{{E_{q+1}}} \right) \leq 2.
\]

One could easily deduce that \( \chi_{b} = \chi = 1 \). Therefore,

\[
\int_{\Omega} \eta_i^2 \chi_i(x,t) \, dx = \frac{1}{2},
\]

Combining the definition of \( \eta_{-1} \) with the above estimates shows that

\[
\eta_{-1}(t) + \frac{N_q}{\sum_{i=1}^{N_q} \int_{\Omega} \eta_i^2 \chi_i(x,t) \, dx} \geq \begin{cases} 1, & 0 \leq t \leq T_{N_q}, \\ \frac{1}{4}, & T_{N_q} \leq t \leq T. \end{cases}
\]

Therefore, we prove that \( \rho_{q,0} \) is well-defined. Recalling the definitions of \( \eta_{-1}(t), \eta(t) \) and the fact that

\[
1 - \tau_{q-1} \leq t_{N_q} < t_{N_{q+1}} \leq 1 - \tau_q \leq t_{N_{q+2}} \leq 1,
\]

we obtain that

\[
\frac{\delta_{q+1}}{10} \leq \rho_{q,0}(t) \leq 10\delta_{q+1}, \quad \forall t \in [0,T].
\]

Now, setting \( \rho_{v,q} := \rho_{q,0} \chi \), we construct the principal corrector \( w_{q+1}^{(pu)} \) such that

\[
w_{q+1}^{(pu)} := \sum_{l,k \in \Lambda_v} \eta_l \mu_{v,q}^{1/2} a_{v,k} \left( \hat{I} - \frac{R_u}{\rho_{v,q}} \right) \psi_k \phi_{k,k,k}, \quad \bar{K} := \sum_{l,k \in \Lambda_v} a_{v,l,k} \psi_k \phi_{k,k,k}.
\]

(14.14)
Combining with (4.5) and (4.6), we show the principal correctors $w_{q+1}^{(p)}$ and $a_{q+1}^{(p)}$ such that

$$
\begin{align*}
    w_{q+1}^{(p)} &:= \sum_{l,k} a_{v,l,k} \phi_k \tilde{v}_{l,k} + \sum_{l,k} a_{b,l,k} \phi_k \tilde{K} = w_{q+1}^{(pu)} + w_{q+1}^{(ph)},
    \\
a_{q+1}^{(p)} &:= \sum_{l,k} a_{b,l,k} \phi_k \tilde{K}.
\end{align*}
$$

(4.15)

Collecting Proposition 3.4–Proposition 3.7 and using the same method as in the proof of Lemma 4.1–Lemma 4.4 in [8], we obtain:

**Proposition 4.4.** For $N \geq 0$, we have

$$
\begin{align*}
    \|(a_{v,l,k}, a_{b,l,k})\|_{L^2} &\lesssim \ell_q^{1/2}, \quad (4.16) \\
    \|(a_{v,l,k}, a_{b,l,k})\|_{L^\infty} &\lesssim \ell_q^{3+5N}, \quad (4.17) \\
    \|(a_{v,l,k}, a_{b,l,k})\|_{H^N} &\lesssim \ell_q^{1/2-5N}, \quad (4.18) \\
    \|\partial_t (a_{v,l,k}, a_{b,l,k})\|_{H^N} &\lesssim \ell_q^{1/2-5N}. \quad (4.19)
\end{align*}
$$

We emphasize that $(a_{v,l,k}, a_{b,l,k})$ oscillates at a frequency $\ell_q^{-5}$, which have relatively small contribution comparing to the “box flows”. Indeed, the “box flows” oscillate at a much higher frequency $\lambda_{q+1}^{3/5} \gg \ell_q^{-5}$ by taking the parameter $b$ sufficiently large in $\lambda_{q+1}$.

3. Because the supports of the “box flows” are much smaller than the supports of the Mikado flows, we choose the “box flows” instead of Mikado flows. However, this choice gives rise to extra errors in the oscillation terms, since they are not divergence-free. To overcome this difficulty, we introduce temporal type flows $w_{q+1}^{(t)}, d_{q+1}^{(t)}$ and inverse traveling wave flows $w_{q+1}^{(v)}, d_{q+1}^{(v)}$, which help us eliminate the extra errors. Now, let us define the temporal flows $w_{q+1}^{(t)}$ and $d_{q+1}^{(t)}$ by

$$
\begin{align*}
    w_{q+1}^{(t)} &:= -P_{\lambda_{q+1}^{2}} \sum_{l,k} \frac{1}{\mu} a_{v,l,k}^2 \phi_k^2, \quad (4.20) \\
    d_{q+1}^{(t)} &:= -P_{\lambda_{q+1}^{2}} \sum_{l,k} \frac{1}{\mu} a_{b,l,k}^2 \phi_k^2.
\end{align*}
$$

(4.21)

4. We construct the inverse traveling wave flows $w_{q+1}^{(v)}, d_{q+1}^{(v)}$ such that

$$
\begin{align*}
    w_{q+1}^{(v)} &:= (\mu \sigma)^{-1} P_{\lambda_{q+1}^{2}} \sum_{l,k} a_{v,l,k}^2 \text{div} (\partial_{t}^{-1} P_{\lambda_{q+1}^{2}} \phi_k^2 \phi_k^2 \hat{\kappa} \otimes \hat{\kappa}), \\
    d_{q+1}^{(v)} &:= (\mu \sigma)^{-1} P_{\lambda_{q+1}^{2}} \sum_{l,k} a_{b,l,k}^2 \text{div} (\partial_{t}^{-1} P_{\lambda_{q+1}^{2}} \phi_k^2 \phi_k^2 \hat{\kappa} \otimes \hat{\kappa}).
\end{align*}
$$

(4.22)

(4.23)

where $\partial_{t}^{-1} P_{\lambda_{q+1}^{2}} \phi_k^2 := \int_{0}^{\sigma(\tilde{k}(x-u)+\mu t)} (\phi_k^2(z) - 1) \, dz$. Since $\phi_k^2(z) - 1$ has zero mean, we have $|\partial_{t}^{-1} P_{\lambda_{q+1}^{2}}(\phi_k^2)| \leq 2$.

5. In order to match the inverse traveling wave flows $w_{q+1}^{(v)}, d_{q+1}^{(v)}$, we need to construct the following heat conduction flows $w_{q+1}^{\Delta}, d_{q+1}^{\Delta}$:

$$
\begin{align*}
    w_{q+1}^{\Delta} &:= (\mu \sigma)^{-1} P_{\lambda_{q+1}^{2}} \sum_{l,k} a_{b,l,k}^2 \int_{0}^{t} e^{(t-\tau)\Delta} \Delta \text{div} (\partial_{t}^{-1} P_{\lambda_{q+1}^{2}} \phi_k^2 \phi_k^2 \hat{\kappa} \otimes \hat{\kappa}) \, d\tau.
\end{align*}
$$

(4.24)
Firstly, noting that $\psi$ rectors are divergence-free and have zero mean. Hence, by (4.2) we define the following "initial flows":

$$w^{(c)}_{q+1} := \frac{1}{h_{b,q}} \nabla a_{b,c} \times F_{k} + \frac{1}{\lambda_{q+1}} \nabla a_{v,c,2} \times F_{k}. \quad (4.26)$$

This equality leads to

$$w^{(p)}_{q+1} + w^{(h)}_{q+1} + w^{(c)}_{q+1} = \sum_{l, k \in \Lambda} \text{curl} \left( \frac{a_{v,c,1}}{\lambda_{q+2}} F_{k} + \frac{a_{v,c,2}}{\lambda_{q+1}} F_{k} \right)$$

by

$$\text{curl}(fW) = \nabla f \times W + f \text{ curl} W.$$

Similarly, letting

$$a_{b,c} = (a_{b,l,k} 1_{k \in \Lambda_{n}} + \eta h_{b,q}^{1/2} 1_{k \in \Lambda_{n}}) \phi_{k,\tilde{k},\tilde{k}};$$

we define

$$d^{(c)}_{q+1} := \sum_{l, k \in \Lambda} \frac{1}{\lambda_{q+1}} \nabla a_{b,c} \times F_{k}. \quad (4.27)$$

One deduces that

$$w^{(c)}_{q+1} + w^{(p)}_{q+1} + w^{(h)}_{q+1} \quad \text{and} \quad d^{(c)}_{q+1} + d^{(p)}_{q+1} + d^{(h)}_{q+1}$$

are divergence-free and have zero mean.

(7) Finally, we define the following "initial flows":

$$u^{(s)}_{q+1} := e^{i(t-\tau)} (u^{in} * \psi_{q} - u^{in} * \psi_{q-1} * \psi_{\ell_{q}}), \quad (4.28)$$
\[ \delta^{(s)}_{q+1} := \epsilon^{(s)}(\bar{b}^{\text{in}} \ast \psi_{q} - \bar{b}^{\text{in}} \ast \psi_{q-1} \ast \psi_{q}). \] (4.29)

From the definition of \( \eta \) in (3.37), one obtains that

\[
(w_{q+1}^{(h)} + w_{q+1}^{(p)} + w_{q+1}^{(t)} + w_{q+1}^{(v)} + w_{q+1}^{(c)})(0, x) = 0, \quad (d_{q+1}^{(h)} + d_{q+1}^{(p)} + d_{q+1}^{(t)} + d_{q+1}^{(v)} + d_{q+1}^{(c)})(0, x) = 0.
\]

Hence, (4.28) and (4.29) guarantee that \( v_{q+1}(0, x) = v^{\text{in}} \ast \psi_{q} \) and \( b_{q+1}(0, x) = \bar{b}^{\text{in}} \ast \psi_{q} \). This fact implies that (2.9) holds with \( q \) replaced by \( q+1 \). Moreover, one can easily deduce that \( w_{q+1}^{(s)} \) and \( d_{q+1}^{(s)} \) are divergence-free and have zero mean.

To sum up, we construct

\[
w_{q+1} := w_{q+1}^{(h)} + w_{q+1}^{(p)} + w_{q+1}^{(t)} + w_{q+1}^{(v)} + w_{q+1}^{(c)},
\]

\[
d_{q+1} := d_{q+1}^{(h)} + d_{q+1}^{(p)} + d_{q+1}^{(t)} + d_{q+1}^{(v)} + d_{q+1}^{(c)},
\]

which help us finish the iteration from \((\bar{v}_{q}, \bar{b}_{q})\) to \((v_{q+1}, b_{q+1})\).

Now, we show that (2.6) and (2.7) hold at \( q+1 \) level.

**Proposition 4.5** (Estimates for \( w_{q+1} \) and \( d_{q+1} \)).

\[
\|\left( w_{q+1}^{(h)}, d_{q+1}^{(h)} \right) \|_{L^2} + \frac{1}{\lambda_{q+1}} \|\left( w_{q+1}^{(p)}, d_{q+1}^{(p)} \right) \|_{H^3} \leq \frac{1}{H} \delta_{q+1}^{1/2},
\]

(4.32)

\[
\|\left( w_{q+1}^{(t)}, d_{q+1}^{(t)} \right) \|_{L^2} + \frac{1}{\lambda_{q+1}} \|\left( w_{q+1}^{(t)}, d_{q+1}^{(t)} \right) \|_{H^3} \leq 100 \delta_{q+1}^{1/2}
\]

(4.33)

\[
\|\left( w_{q+1}^{(v)}, d_{q+1}^{(v)} \right) \|_{L^2} + \frac{1}{\lambda_{q+1}} \|\left( w_{q+1}^{(v)}, d_{q+1}^{(v)} \right) \|_{H^3} \leq \lambda_{q+1}^{-50} \delta_{q+2},
\]

(4.34)

\[
\|\left( w_{q+1}^{(c)}, d_{q+1}^{(c)} \right) \|_{L^2} + \frac{1}{\lambda_{q+1}} \|\left( w_{q+1}^{(c)}, d_{q+1}^{(c)} \right) \|_{H^3} \leq \lambda_{q+1}^{-50} \delta_{q+2},
\]

(4.35)

\[
\|\left( w_{q+1}^{(s)}, d_{q+1}^{(s)} \right) \|_{L^2} + \frac{1}{\lambda_{q+1}} \|\left( w_{q+1}^{(s)}, d_{q+1}^{(s)} \right) \|_{H^3} \leq \lambda_{q+1}^{-50} \delta_{q+2},
\]

(4.36)

\[
\|\left( w_{q+1}^{(c)}, d_{q+1}^{(c)} \right) \|_{L^2} + \frac{1}{\lambda_{q+1}} \|\left( w_{q+1}^{(c)}, d_{q+1}^{(c)} \right) \|_{H^3} \leq \lambda_{q+1}^{-50} \delta_{q+2},
\]

(4.37)

\[
\|\left( w_{q+1}^{(s)}, d_{q+1}^{(s)} \right) \|_{L^2} + \frac{1}{\lambda_{q+1}} \|\left( w_{q+1}^{(s)}, d_{q+1}^{(s)} \right) \|_{H^3} \leq \lambda_{q+1}^{-50} \delta_{q+2},
\]

(4.38)

\[
\|\left( w_{q+1}^{(s)}, d_{q+1}^{(s)} \right) \|_{L^2} + \frac{1}{\lambda_{q+1}} \|\left( w_{q+1}^{(s)}, d_{q+1}^{(s)} \right) \|_{H^3} \leq \lambda_{q+1}^{-50} \delta_{q+2},
\]

(4.39)

**Proof.** Firstly, from the definition of \( \phi_{k, \tilde{k}, \tilde{k}} \) and \( \psi_{k} \), we deduce by Proposition 4.3, Proposition 4.4 and Lemma A.1 that

\[
\|\left( w_{q+1}^{(h)}, d_{q+1}^{(h)} \right) \|_{L^2} \leq \|\eta b_{q} \partial_{q} \psi_{k} \phi_{k, \tilde{k}, \tilde{k}} \|_{L^2} \lesssim \|\eta b_{q} \|_{L^2} \|\phi_{k, \tilde{k}, \tilde{k}} \|_{L^2} \leq \frac{\delta_{q+1}^{1/2}}{20},
\]

(4.40)

and

\[
\|\left( w_{q+1}^{(h)}, d_{q+1}^{(h)} \right) \|_{H^3} \leq \lambda_{q+1}^{5} \frac{\delta_{q+1}^{1/2}}{20}.
\]

(4.41)

Hence, (4.40) and (4.41) imply (4.32). In the same way as deriving (4.40) and (4.41), we obtain (4.33)–(4.36). For example, to prove (4.33), recall that:

\[
w_{q+1}^{(v)} := (\mu \sigma)^{-1} \bar{P}_{H} \bar{P}_{>0} \sum_{l, \tilde{k} \in \Lambda_{q}} a_{l, \tilde{k}}^{2} \partial_{l} \partial_{\tilde{k}} \phi_{q}^{2} \phi_{k}^{2} \phi_{\tilde{k}}^{2} \phi_{\tilde{k}}^{2}
\]

(4.42)
where $\partial_t^{-1}\mathbb{P}_{>0}\phi_k^2 := \int_0^{\sigma(k(x-x_k)+\mu t)} (\phi_k^2(z) - 1) \, dz$. Since $\phi_k^2(z) - 1$ has zero mean, we have $|\partial_t^{-1}\mathbb{P}_{>0}(\phi_k^2)| \leq 2$.

We deduce that

$$
|w_{q+1}^{(v)}|_{L^2} \lesssim (\mu \sigma)^{-1} \|\phi_{b,l,k}\|_{L^\infty}(\partial_t^{-1}\mathbb{P}_{>0}\phi_k^2) \|\phi_k^2\|_{D}(\partial_t^{-1}\mathbb{P}_{>0}\phi_k^2) \|\phi_k^2\|_{D}(\partial_t^{-1}\mathbb{P}_{>0}\phi_k^2)
$$

Similarly, noting that $\text{div} \parallel \phi_k \parallel = 1 + \frac{1}{\lambda_{q+1}} \|w_{q+1}^{(v)}\|_{H^3} \leq \lambda_{q+1}^5 \delta_{q+2}$. Similarly for $d_{q+1}^{(v)}$, one can easily deduce (4.35).

Secondly, we aim to prove (4.37). For simplicity, we denote $w^\Delta_{q+1}$ by

$$
w^\Delta_{q+1} := (\mu \sigma)^{-1} \mathbb{P}_{H_{>0}} \sum_{l,k,l_k} a_{b,l,k}^2 \int_0^t e^{(t-\tau)\Delta} g(x,t) \, d\tau
$$

where $h(\sigma_k \cdot x, \sigma_k \cdot x) = \phi_k^2 \phi_k^2 = \phi_k^2 (\sigma_k \cdot (x-x_k)) \phi_k^2 (\sigma_k \cdot (x-x_k))$ and

$$
g(t,x) := \text{div} ((\partial_t^{-1}\mathbb{P}_{>0}\phi_k^2) \phi_k^2 \phi_k^2 \sigma_k \cdot \phi_k^2)
$$

Using the estimates of solution for the heat equation in [3], we deduce that for $0 < \alpha \leq \min\{\frac{1}{P}, \frac{\beta}{32}\}$,

$$
\|\int_0^t e^{(t-\tau)\Delta} \Delta g(\tau) d\tau\|_{L^\infty L^2} \leq \|g\|_{L^\infty H^\alpha} \lesssim \lambda_{q+1}^5 \sigma^{-1} \frac{1}{2} - \frac{3}{2}.
$$

Similarly, noting that $\text{div} (h(\sigma_k \cdot x, \sigma_k \cdot x) \phi_k^2 \sigma_k^2) = \sigma (2h)(\sigma_k \cdot x, \sigma_k \cdot x) \phi_k^2$, we deduce that

$$
\|\int_0^t e^{(t-\tau)\Delta} \text{div}(h(\sigma_k \cdot x, \sigma_k \cdot x) \phi_k^2 \sigma_k^2) \phi_k^2 \|_{L^\infty L^2} \lesssim \sigma \|\frac{\partial h}{\Delta} (\sigma_k \cdot x, \sigma_k \cdot x) \phi_k^2 \|_{L^\infty H^\alpha},
$$

where we have used the fact that

$$
(2h)(\sigma_k \cdot x, \sigma_k \cdot x) = \sigma^{-2} \Delta [\frac{\partial h}{\Delta} (\sigma_k \cdot x, \sigma_k \cdot x)], (k, \tilde{k}, k) \in \Lambda.
$$

Since $h(\cdot, \cdot) \in C^\infty (T^2)$ for fixed $q$ and $k \perp \tilde{k}$, we have for $\alpha > 0$,

$$
\|\frac{\partial h}{\Delta} (\sigma_k \cdot x, \sigma_k \cdot x)\|_{L^2(T^2)} = \|\frac{\partial h}{\Delta} (\cdot, \cdot)\|_{L^2(T^2)} \lesssim \|h\|_{W^{1,1}(T^2)}
$$

and

$$
\|\frac{\partial h}{\Delta} (\sigma_k \cdot x, \sigma_k \cdot x)\|_{H^1(T^2)} \lesssim \sigma \|h\|_{L^2(T^2)} \lesssim \sigma \|h\|_{W^{1,1}(T^2)}.
$$

Plugging the above two estimates into (4.46) yields that

$$
\|\int_0^t e^{(t-\tau)\Delta} \text{div}(h(\sigma_k \cdot x, \sigma_k \cdot x) \phi_k^2 \sigma_k^2) \phi_k^2 \|_{L^\infty L^2} \lesssim \sigma^{-1} \|\frac{\partial h}{\Delta} (\sigma_k \cdot x, \sigma_k \cdot x)\|_{L^2(T^2)} \|\frac{\partial h}{\Delta} (\sigma_k \cdot x, \sigma_k \cdot x)\|_{H^1(T^2)} \lesssim \sigma^{-1} \|h\|_{W^{1,1}(T^2)} \lesssim \sigma^{-1} \|h\|_{W^{1,1}(T^2)}.
$$
Combining (4.45) with (4.47), we have
\[
\|u_{q+1}^\Delta\|_{L^\infty L^2} \lesssim \ell_q^{-2} \lambda_{q+1}^\Delta (\mu^{-1} r^{-1/2} \bar{r}^{-3/2} + \sigma^{-1}) \lesssim \ell_q^{-6} \lambda_{q+1}^\Delta (\lambda_{q+1}^{-\frac{17}{16}} + \frac{17}{16} + \frac{3}{5} + 2\alpha) \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2},
\]
where \(\sigma = \lambda_{q+1}^{\frac{17}{16}}, \mu = \lambda_{q+1}^{\frac{14}{16}}\) and \(r = \bar{r} = \lambda_{q+1}^{-\frac{5}{16}}, \bar{r} = \lambda_{q+1}^{-\frac{5}{16}}\). A similar calculation also yields that
\[
\|d_{q+1}^\Delta\|_{L^2} \lesssim \ell_q^{-6} \lambda_{q+1}^\Delta (\mu^{-1} r^{-1/2} \bar{r}^{-3/2} + \sigma^{-1}) \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2}
\]
and
\[
\|(w_{q+1}^\Delta, d_{q+1}^\Delta)\|_{H^3} \lesssim \lambda_{q+1}^{50\alpha} \delta_{q+2}.
\]
This completes the proof of (4.37).

Finally, we turn to prove (4.38). Recalling that
\[
\beta = \frac{\bar{\beta}}{b^4}, b = 2^{16[\bar{\beta}^{-1/2}]}, \quad \alpha \leq \min \left\{ \frac{1}{b^6}, \frac{\beta}{b^3} \right\},
\]
we have
\[
\ell_{q-1}^\beta \lesssim \lambda_{q-1}^{\frac{1}{1-2b^4}} \lesssim \lambda_{q-1}^{-2b^4 - 50\alpha b^2}.
\]

For any \((v^{(s)}, b^{(s)}) \in (H^{\bar{\beta}}, H^{\bar{\beta}}), \beta > \beta\), we have
\[
\|(w_{q+1}^{(s)}, d_{q+1}^{(s)})\|_{L^2} \lesssim \|(v^{(s)} \psi_{q} - v^{(s)} \psi_{q-1} \psi_{q}, b^{(s)} \psi_{q} - b^{(s)} \psi_{q-1} \psi_{q})\|_{L^2} \lesssim \ell_{q-1}^{\bar{\beta}} \|(v^{(s)}, b^{(s)})\|_{H^\beta} \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2},
\]
and
\[
\|(w_{q+1}^{(s)}, d_{q+1}^{(s)})\|_{H^3} \lesssim \|(v^{(s)}, b^{(s)})\|_{L^2} \ell_{q}^{-\bar{\beta}} \lesssim \lambda_{q+1}^{50\alpha} \delta_{q+2}.
\]

Hence, we derive (4.38). Collecting the estimates (4.32), (4.38), one obtains (4.39).

Remark 4.6. Proposition 4.5 tells us that \(w_{q+1}^{(c)}, w_{q+1}^{(s)}, w_{q+1}^{(t)}, w_{q+1}^{(r)}, d_{q+1}^{(c)}, d_{q+1}^{(s)}, d_{q+1}^{(t)}, d_{q+1}^{(r)}\) are small such that
\[
\|(w_{q+1}^{(c)}, w_{q+1}^{(s)}, w_{q+1}^{(t)}, w_{q+1}^{(r)}, d_{q+1}^{(c)}, d_{q+1}^{(s)}, d_{q+1}^{(t)}, d_{q+1}^{(r)})\|_{L^2} \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2}.
\]
So one can omit these terms in estimating the linear errors or the oscillation errors.

## 5 Estimates of the stresses associated with the MHD system

**Proposition 5.1 (Estimate for \(P_{q+1}^{\text{lin}}\) and \(M_{q+1}^{\text{lin}}\)).**

\[
\|P_{q+1}^{\text{lin}} - \mathcal{R}[(\partial_t w_{q+1}^{(t)} + \partial_x w_{q+1}^{(v)}) + (\partial_t w_{q+1}^{\Delta} - \Delta w_{q+1}^{\Delta})]\|_{L^1} \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2},
\]
\[
\|M_{q+1}^{\text{lin}} - \mathcal{R}[(\partial_t d_{q+1}^{(t)} + \partial_x d_{q+1}^{(v)}) + (\partial_t d_{q+1}^{\Delta} - \Delta d_{q+1}^{\Delta})]\|_{L^1} \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2}.
\]
Proof. We deduce by Proposition 4.3–4.5 and Lemma A.1 that

\[
\|w_{q+1} \otimes \tilde{v}_q + \hat{v}_q \otimes w_{q+1}\|_{L^1} \\
\leq \|\tilde{v}_q \otimes (w^{(c)}_{q+1} + w^{(v)}_{q+1} + \Delta w^{(t)}_{q+1} + w^{(s)}_{q+1}) + (w^{(c)}_{q+1} + w^{(v)}_{q+1} + w^{(t)}_{q+1} + w^{(s)}_{q+1}) \otimes \hat{v}_q\|_{L^1} \\
+ \|(w^{(p)}_{q+1} + w^{(h)}_{q+1}) \otimes \hat{v}_q + \hat{v}_q \otimes (w^{(p)}_{q+1} + w^{(h)}_{q+1})\|_{L^1} \\
\leq \lambda_{q+1}^{-50\alpha} \delta_{q+2} + \|\tilde{v}_q\|_{L^2} a_{v,l,k} + a_{b,l,k} + \hat{b}_{b,q}^{(k)} \|\delta_{k,k}\|_{L^1} \\
\leq \lambda_{q+1}^{-50\alpha} \delta_{q+2} + \delta_{q+1}^{1/2} \lambda_{q+1}^{-\frac{23}{16} + \alpha} \\
\leq \lambda_{q+1}^{-50\alpha} \delta_{q+2}, \quad (5.3)
\]

and

\[
\|d_{q+1} \otimes \hat{b}_q + \hat{b}_q \otimes d_{q+1}\|_{L^1} \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2}. \quad (5.4)
\]

Since \(\partial_t w^{(s)}_{q+1} - \Delta w^{(s)}_{q+1} = 0\), we obtain by Proposition 4.3–4.5 and Lemma A.2 that

\[
\|\mathcal{R}[(\partial_t w^{(t)}_{q+1} - \partial_t w^{(v)}_{q+1} - \partial_t w^{(s)}_{q+1} - \partial_k w^{(t)}_{q+1}) - (\Delta w^{(t)}_{q+1} - \Delta w^{(s)}_{q+1})]\|_{L^1} \\
\leq \|\mathcal{R}[(\partial_t w^{(p)}_{q+1} + w^{(c)}_{q+1} + w^{(h)}_{q+1})\|_{L^1} + \|\mathcal{R}[(\Delta w^{(p)}_{q+1} + w^{(c)}_{q+1} + w^{(h)}_{q+1})\|_{L^1} \\
\leq \delta_{q+1}^{-15} \mu \lambda_{q+1}^{-1/2} \lambda_{q+1}^{-1} \delta_{q+1}^{1/2} + (\delta_{q+1}^{-15} \lambda_{q+1}^{-1/2} \sigma \delta_{q+1}^{-1} + \delta_{q+1}^{-15} \mu^{-1} \sigma \delta_{q+1}^{-1}) \\
\leq \delta_{q+1}^{-15} \lambda_{q+1}^{-1} \delta_{q+1}^{1/2} \sigma + \delta_{q+1}^{-15} \lambda_{q+1}^{-1} \delta_{q+1}^{1/2} \\
\leq \delta_{q+1}^{-15} \lambda_{q+1}^{-1} \delta_{q+1}^{1/2} \sigma + \delta_{q+1}^{-15} \lambda_{q+1}^{-1} \delta_{q+1}^{1/2} \\
\leq \lambda_{q+1}^{-50\alpha} \delta_{q+2}. \quad (5.5)
\]

Collecting (5.3)–(5.5) yields (5.1). In the same way, we can prove (5.2). Thus, we complete the proof of Proposition 5.1. \(\square\)

**Proposition 5.2** (Estimate for \(M_{q+1}^{osc}\)).

\[
\|M_{q+1}^{osc} + \mathcal{R}_a[(\partial_t d_{q+1}^{(t)} + \partial_t d_{q+1}^{(v)}) + (\partial_t d_{q+1}^{(s)} - \Delta d_{q+1}^{(s)} - \Delta d_{q+1}^{(t)})]\|_{L^1} \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2}.
\]

Proof. Firstly, a direct computation shows that

\[
\text{div}[M_{q+1}^{osc} + \mathcal{R}_a[(\partial_t d_{q+1}^{(t)} + \partial_t d_{q+1}^{(v)}) + (\partial_t d_{q+1}^{(s)} - \Delta d_{q+1}^{(s)} - \Delta d_{q+1}^{(t)})]] \\
= \mathbb{P}_H \text{div}[w^{(p)}_{q+1} \otimes d^{(p)}_{q+1} - d^{(p)}_{q+1} \otimes w^{(p)}_{q+1} + M_{q+1,0} + \hat{M}_q] + (\partial_t d^{(t)}_{q+1} + \partial_t d^{(s)}_{q+1}) + (\partial_t d^{(t)}_{q+1} - \Delta d^{(t)}_{q+1} - \Delta d^{(s)}_{q+1}) \\
= \mathbb{P}_H \text{div}[w^{(p)}_{q+1} \otimes d^{(p)}_{q+1} - d^{(p)}_{q+1} \otimes w^{(p)}_{q+1} + \hat{M}_q] + \text{div} M_{q+1,0} + (\partial_t d^{(t)}_{q+1} + \partial_t d^{(s)}_{q+1}) + (\partial_t d^{(t)}_{q+1} - \Delta d^{(t)}_{q+1} - \Delta d^{(s)}_{q+1}),
\]

where \(M_{q+1,0}\) is anti-symmetric such that

\[
M_{q+1,0} := (w_{q+1} - w^{(p)}_{q+1} - w^{(h)}_{q+1}) \otimes (d_{q+1} - d^{(p)}_{q+1} - d^{(h)}_{q+1}) + (w^{(p)}_{q+1} + w^{(h)}_{q+1}) \otimes (d_{q+1} - d^{(p)}_{q+1} - d^{(h)}_{q+1}) \\
+ (w_{q+1} - w^{(p)}_{q+1} - w^{(h)}_{q+1}) \otimes (d^{(p)}_{q+1} + d^{(h)}_{q+1}) - (d_{q+1} - d^{(p)}_{q+1} - d^{(h)}_{q+1}) \otimes (w_{q+1} - w^{(p)}_{q+1} - w^{(h)}_{q+1}) \\
- (d_{q+1} - d^{(p)}_{q+1} - d^{(h)}_{q+1}) \otimes (w^{(p)}_{q+1} + w^{(h)}_{q+1}) - (d^{(p)}_{q+1} + d^{(h)}_{q+1}) \otimes (w_{q+1} - w^{(p)}_{q+1} - w^{(h)}_{q+1}). \quad (5.7)
\]
With aid of (4.1), we have $d_{q+1}^{(p)} \otimes w_{q+1}^{(pm)} = w_{q+1}^{(pm)} \otimes d_{q+1}^{(p)} = w_{q+1}^{(b)} \otimes d_{q+1}^{(p)} = d_{q+1}^{(b)} \otimes w_{q+1}^{(p)} = 0$. We can easily deduce by Proposition 4.5 that

$$||M_{low,0}||_{L^1} \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2}. \quad (5.8)$$

Secondly, recalling the definitions of $d_{q+1}^{(v)}$, $d_{q+1}^{(\Delta)}$ in (4.23) and (4.25), we obtain that

$$\partial_t d_{q+1}^{\Delta} - \Delta d_{q+1}^{\Delta} = \partial_t d_{q+1}^{(v)}$$

$$= \left\{ (\mu \sigma)^{-1} F_{H \geq 0} \sum_{l,k \in \Lambda_h} (\partial_t - \Delta)(a_{b,l,k}^2) \int_0^T e^{(t-\tau)\Delta} \mathrm{div} \left( (\partial_t^{-1} F_{0 \geq 0} \phi_k^2 \phi_k^2 \otimes \bar{k}) \right) \right\}$$

$$+ F_{H \geq 0} \sum_{l,k \in \Lambda_h} (\partial_t - \Delta)(a_{b,l,k}^2) \int_0^T e^{(t-\tau)\Delta} \mathrm{div} \left( (\partial_t^{-1} F_{0 \geq 0} \phi_k^2 \phi_k^2 \otimes \bar{k}) \right) \right\}$$

$$- (\mu \sigma)^{-1} F_{H \geq 0} \sum_{l,k \in \Lambda_h} \Delta(a_{b,l,k}^2) \mathrm{div} \left( (\partial_t^{-1} F_{0 \geq 0} \phi_k^2 \phi_k^2 \otimes \bar{k}) \right)$$

$$- (\mu \sigma)^{-1} F_{H \geq 0} \sum_{l,k \in \Lambda_h} \Delta(a_{b,l,k}^2) \mathrm{div} \left( (\partial_t^{-1} F_{0 \geq 0} \phi_k^2 \phi_k^2 \otimes \bar{k}) \right)$$

$$- F_{H \geq 0} \sum_{l,k \in \Lambda_h} \Delta(a_{b,l,k}^2) \mathrm{div} \left( (\partial_t^{-1} F_{0 \geq 0} \phi_k^2 \phi_k^2 \otimes \bar{k}) \right)$$

$$= \mathrm{div} M_{low,1} + \mathrm{div} M_{high,1}. \quad (5.9)$$

On one hand hand, it’s easily to deduce that

$$||R_a [(\mu \sigma)^{-1} F_{H \geq 0} \sum_{l,k \in \Lambda_h} (\partial_t - \Delta)(a_{b,l,k}^2) \int_0^T e^{(t-\tau)\Delta} \mathrm{div} \left( (\partial_t^{-1} F_{0 \geq 0} \phi_k^2 \phi_k^2 \otimes \bar{k}) \right) \right)||_{L^1} \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2}. \quad (5.10)$$

On the other hand, using the Leibniz rule

$$\partial_x f \cdot \partial_x g = \partial_x (\partial_x f \cdot g) - \partial_x^2 f \cdot g,$$

one can easily deduce that

$$||R_a F_{H \geq 0} (\partial_x f \cdot \partial_x g)||_{L^1} \lesssim ||(\partial_x f \cdot g)||_{L^{1+\alpha}} + \||\partial_x^2 f \cdot g||_{L^{1+\alpha}}.$$

We deduce that

$$||R_a [(\mu \sigma)^{-1} F_{H \geq 0} \sum_{l,k \in \Lambda_h} \sum_{i=1}^3 (\partial_x a_{b,l,k}^2) \partial_x \int_0^T e^{(t-\tau)\Delta} \mathrm{div} \left( (\partial_t^{-1} F_{0 \geq 0} \phi_k^2 \phi_k^2 \otimes \bar{k}) \right) \right)||_{L^1} \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2}. \quad (5.10)$$
\[ + \mathbb{P}_H \mathbb{P}_{\geq 0} \sum_{l,k \in \Lambda } \sum_{i=1}^3 \partial_x \left( a_{b,l,k}^2 \partial_x \right) \partial_x, \int_0^t e^{(t-s)\Delta} \text{div} \left( \phi_k^2 \phi_k^2 \chi \otimes \chi \right) d\tau \]

\[ + (\mu)^{-1} \mathbb{P}_H \mathbb{P}_{\geq 0} \sum_{l,k \in \Lambda } \sum_{i=1}^3 \partial_x \left( a_{b,l,k}^2 \partial_x \right) \partial_x, \text{div} \left( \left( \partial_{l}^{-1} \mathbb{P}_{\geq 0} \phi_k^2 \phi_k^2 \chi \otimes \chi \right) \right) \right]\n\text{d}L, \leq \lambda_{q+1}^{-50\alpha} \delta_{q+2}. \tag{5.11} \]

Combining (5.9) with (5.10)–(5.11), we obtain

\[ \| M_{low,1} \|_{L^1} \leq \lambda_{q+1}^{-50\alpha} \delta_{q+2}. \tag{5.12} \]

Thirdly, using the definitions of \( \eta, \Lambda_v, \Lambda_b \) and \( \phi_{k,k,k} \), we have

\[ \sum_{l,k \in \Lambda_v} a_{v,l,k} \psi_k \phi_{k,k,k} + \sum_{l,k \in \Lambda_b} a_{b,l,k} \psi_k \phi_{k,k,k} = 0, \tag{5.13} \]

and

\[ \sum_{l,k \in \Lambda_b} a_{b,l,k} \psi_k \phi_{k,k,k} = \sum_{l,k \in \Lambda_b} a_{b,l,k}^2 \psi_k^2 \phi_k^2 \phi_k^2. \tag{5.14} \]

By \( \text{div} = \text{div} \mathbb{P}_{\geq 0} = \mathbb{P}_{\geq 0} \text{div} \), we show that

\[ \mathbb{P}_H \text{div} \left[ w_{q+1}^{(p)} \otimes d_{q+1}^{(p)} \right] + (\partial_t d_{q+1}^{(t)} + \partial_t d_{q+1}^{(v)} + \text{div} M_{high,1}) \]

\[ = \mathbb{P}_H \text{div} \left[ \sum_{l,k \in \Lambda_b} a_{b,l,k}^2 \psi_k \phi_{k,k,k} \right] + (\partial_t d_{q+1}^{(t)} + \partial_t d_{q+1}^{(v)} + \text{div} M_{high,1}) \]

\[ = \mathbb{P}_H \text{div} \left[ \sum_{l,k \in \Lambda_b} a_{b,l,k} \left( \frac{\phi_k^2 \phi_k^2 \phi_k^2}{\phi_k^2 \phi_k^2 \phi_k^2} \right) \right] + (\partial_t d_{q+1}^{(t)} + \partial_t d_{q+1}^{(v)} + \text{div} M_{high,1}) \]

\[ = \left\{ \mathbb{P}_H \mathbb{P}_{\geq 0} \sum_{l,k \in \Lambda_b} a_{b,l,k}^2 \psi_k \phi_{k,k,k} \phi_k^2 \phi_k^2 \right\} \]

\[ := \text{div} M_{low,2} + \text{div} M_{low,3} + \text{div} M_{low,4}, \tag{5.15} \]

where we have used Lemma 4.1 in the third equality. Combining (5.6), (5.9) and (5.15) shows

\[ \text{div} \left[ M_{low,1} + M_{low,2} + M_{low,3} + M_{low,4} \right] = \text{div} \left( M_{low,0} + M_{low,1} + M_{low,2} + M_{low,3} + M_{low,4} \right). \tag{5.16} \]

We begin to estimate \( M_{low,2} \). Using Lemma A.2 and Proposition 4.4, we deduce that

\[ \| M_{low,2} \|_{L^1} \leq \lambda_{q+1}^{-50\alpha} \delta_{q+2}. \tag{5.17} \]
Next, we consider $M_{\text{low},3}$. Recalling that
\[
\partial_t d_{q+1}^{(t)} = -\frac{1}{\mu} P_H \mathbb{P}_{\mathbb{P}^0} \sum_{l,k \in \Lambda_h} \left\{ \partial_t (a_{b,l,k}^2)(\phi_k^2 \phi_k^2 \phi_k^2 \bar{k}) + a_{b,l,k}^2 \partial_t \phi_k^2 \cdot \phi_k^2 \phi_k^2 \bar{k} \right\},
\]
by
\[
\partial_t [\phi_k^2(x,t) \cdot \phi_k^2 \phi_k^2 \bar{k}] = \mu \text{ div}[\phi_k^2 \phi_k^2 \phi_k^2 \bar{k}],
\]
we have
\[
M_{\text{low},3} = R_a P_H \mathbb{P}_{\mathbb{P}^0} \sum_{l,k \in \Lambda_h} a_{b,l,k}^2 \text{ div}[\phi_k^2 \phi_k^2 \phi_k^2 \bar{k}] + R_a \partial_t d_{q+1}^{(t)}
\]
\[
= -R_a P_H \mathbb{P}_{\mathbb{P}^0} \sum_{l,k \in \Lambda_h} \mu^{-1} \partial_t (a_{b,l,k}^2)(\phi_k^2 \phi_k^2 \phi_k^2 \bar{k}).
\]
So we can easily deduce that
\[
\|M_{\text{low},3}\|_{L^1} \lesssim \mu^{-1} \| \partial_t (a_{b,l,k}^2) \|_{L^\infty} \| \phi_k^2 \phi_k^2 \phi_k^2 \|_{L^{1+\alpha}} \ll \lambda_{q+1}^{-50a} \delta_{q+2}.
\]
(5.18)

Finally, we need to estimate $M_{\text{low},4}$. Using the definition of $M_{\text{high},1}$ in [5.9] and $d_{q+1}^{(v)}$ in [4.23], we deduce that
\[
M_{\text{low},4} = R_a P_H \mathbb{P}_{\mathbb{P}^0} \sum_{l,k \in \Lambda_h} a_{b,l,k}^2 \text{ div}[\mathbb{P} > 0 (\phi_k^2 + 1) \cdot \phi_k^2 \phi_k^2 \bar{k}] + \partial_t d_{q+1}^{(v)} + M_{\text{high},1}
\]
\[
= R_a P_H \mathbb{P}_{\mathbb{P}^0} \sum_{l,k \in \Lambda_h} a_{b,l,k}^2 \text{ div}[\mathbb{P} > 0 (\phi_k^2) \phi_k^2 \phi_k^2 \bar{k}] + \partial_t d_{q+1}^{(v)}
\]
\[
= (\mu \sigma)^{-1} R_a P_H \mathbb{P}_{\mathbb{P}^0} \sum_{l,k \in \Lambda_h} \partial_t (a_{b,l,k}^2) \text{ div}[\partial_t^{-1} \mathbb{P} > 0 \phi_k^2 \phi_k^2 \phi_k^2 \bar{k}]
\]
\[
= (\mu \sigma)^{-1} R_a P_H \mathbb{P}_{\mathbb{P}^0} \sum_{l,k \in \Lambda_h} (\partial_t a_{b,l,k}^2) (\partial_t^{-1} \mathbb{P} > 0 \phi_k^2 \phi_k^2 \phi_k^2 \bar{k}).
\]
(5.20)

One deduces by Proposition 4.4 that
\[
\|M_{\text{low},4}\|_{L^1} \leq (\mu \sigma)^{-1} \| \partial_t (a_{b,l,k}^2) \|_{L^\infty} \| \partial_t^{-1} \mathbb{P} > 0 \phi_k^2 \phi_k^2 \phi_k^2 \|_{L^{1+\alpha}}
\]
\[
\lesssim \epsilon_q^{-15} \lambda_{q+1}^{2a} \mu^{-1} \ll \lambda_{q+1}^{-50a} \delta_{q+2}.
\]
(5.21)

Putting [5.8], [5.12], [5.17], [5.19], and [5.21] into [5.16], we obtain
\[
\| M_{q+1}^{\text{osc}} + R_a [(\partial_t d_{q+1}^{(t)} + \partial_t d_{q+1}^{(v)}) + (\partial_t a_{q+1}^\Delta - \Delta d_{q+1}^{(v)})] \|_{L^1} \ll \lambda_{q+1}^{-50a} \delta_{q+2}.
\]
This completes the proof of Proposition 5.2

**Proposition 5.3** (Estimate for $R_{q+1}^{\text{osc}}$).
\[
\| R_{q+1}^{\text{osc}} + R [(\partial_t w_{q+1}^{(t)} + \partial_t w_{q+1}^{(v)}) + (\partial_t a_{q+1}^\Delta - \Delta a_{q+1}^\Delta - \Delta w_{q+1}^{(v)})] \|_{L^1} \lesssim \lambda_{q+1}^{-50a} \delta_{q+2},
\]
where $P_v$ is defined in [5.29].

28
Proof. Firstly, since \( w_{q+1}^{(h)} = d_{q+1}^{(h)} \) and \( w_{q+1}^{(pa)} \otimes w_{q+1}^{(pb)} = w_{q+1}^{(h)} \otimes w_{q+1}^{(p)} = d_{q+1}^{(h)} \otimes d_{q+1}^{(p)} = 0 \), we have

\[
 w_{q+1} \otimes w_{q+1} - d_{q+1} \otimes d_{q+1} = w_{q+1}^{(p)} \otimes w_{q+1}^{(p)} - d_{q+1}^{(p)} \otimes d_{q+1}^{(p)} + R_{\text{low},0},
\]

where

\[
 R_{\text{low},0} := (w_{q+1} - w_{q+1}^{(p)}) \otimes (w_{q+1} - w_{q+1}^{(h)}) + (w_{q+1}^{(p)} - d_{q+1}^{(p)}) \otimes (d_{q+1} - d_{q+1}^{(h)})
\]

We deduce by Proposition 4.5 that

\[
 \| R_{\text{div}} R_{\text{low},0} \|_{L^1} \leq \lambda_{q+1}^{-500} \delta_{q+2}.
\]

Secondly, straightforward calculations show

\[
 \partial_t w_{q+1}^{(v)} = \Delta w_{q+1}^{(v)} - \Delta w_{q+1}^{(h)}
\]

\[
 = \left\{ (\mu \sigma)^{-1} \mathbb{P}_H \mathbb{P}_{>0} \sum_{l,k \in \Lambda_b} (\partial_t - \Delta)(a_{l,k}^2) \int_0^t e^{(t-s)\Delta} \Delta \mathrm{div} \left((\partial_t - \Delta)(\mathbb{P}_{>0} \phi_k^2) \phi_k^2 \phi_{\tilde{k}}^2 \otimes \tilde{k} \right) d\tau 
\]

\[
 + \mathbb{P}_H \mathbb{P}_{>0} \sum_{l,k \in \Lambda_b} (\partial_t - \Delta)(a_{l,k}^2) \int_0^t e^{(t-s)\Delta} \Delta \mathrm{div} \left((\partial_t - \Delta)(\mathbb{P}_{>0} \phi_k^2) \phi_k^2 \phi_{\tilde{k}}^2 \otimes \tilde{k} \right) d\tau 
\]

\[
 - (\mu \sigma)^{-1} \mathbb{P}_H \mathbb{P}_{>0} \sum_{l,k \in \Lambda_b} \Delta(a_{l,k}^2) \mathrm{div} \left((\partial_t - \Delta)(\mathbb{P}_{>0} \phi_k^2) \phi_k^2 \phi_{\tilde{k}}^2 \otimes \tilde{k} \right)
\]

\[
 - (\mu \sigma)^{-1} \mathbb{P}_H \mathbb{P}_{>0} \sum_{l,k \in \Lambda_b} \sum_{i=1}^3 \partial_{x_i}(a_{l,k}^2) \partial_{x_i} \int_0^t e^{(t-s)\Delta} \Delta \mathrm{div} \left((\partial_t - \Delta)(\mathbb{P}_{>0} \phi_k^2) \phi_k^2 \phi_{\tilde{k}}^2 \otimes \tilde{k} \right) d\tau 
\]

\[
 - \mathbb{P}_H \mathbb{P}_{>0} \sum_{l,k \in \Lambda_b} \sum_{i=1}^3 \partial_{x_i}(a_{l,k}^2) \partial_{x_i} \left(\Delta(a_{l,k}^2) \mathrm{div} \left((\partial_t - \Delta)(\mathbb{P}_{>0} \phi_k^2) \phi_k^2 \phi_{\tilde{k}}^2 \otimes \tilde{k} \right) \right)
\]

\[
 = \mathrm{div} R_{\text{low},1} + \mathrm{div} R_{\text{high},1}.
\]

On one hand, it’s easy to deduce that

\[
 \left\| \mathbb{R}\left[ (\mu \sigma)^{-1} \mathbb{P}_H \mathbb{P}_{>0} \sum_{l,k \in \Lambda_b} (\partial_t - \Delta)(a_{l,k}^2) \int_0^t e^{(t-s)\Delta} \Delta \mathrm{div} \left((\partial_t - \Delta)(\mathbb{P}_{>0} \phi_k^2) \phi_k^2 \phi_{\tilde{k}}^2 \otimes \tilde{k} \right) d\tau \right] \right\|_{L^1} \leq \lambda_{q+1}^{-500} \delta_{q+2}.
\]

On the other hand, using the Leibniz rule

\[
 \partial_t f \cdot \partial_{x_i} g = \partial_{x_i}(\partial_t f \cdot g) - \partial_{x_i}^2 f \cdot g.
\]
one can easily deduce that
\[ \| \mathcal{R}^p \|_{L^1} \lesssim \| (\partial_x, f) \|_{L^{1+\alpha}} + \| (\partial_x^2, f \cdot g) \|_{L^{1+\alpha}}. \]

We deduce that
\[
\begin{align*}
\| \mathcal{R} \left[ (\mu a)^{-1} \mathcal{P}_H \mathcal{P} \sum_{l,k \in \Lambda_h} \sum_{i=1}^3 \partial_{x_i}(a_{l,k}) \partial_{x_i} \left( \int_0^t e^{(t-t')\Delta} \text{div} \left( \left( \partial_t^{-1} \mathcal{P}_H \mathcal{P} \phi_k^2 \phi_k^2 \phi_k^2 k \otimes \bar{k} \right) \right) d\tau \right) \right] + \mathcal{P}_H \mathcal{P} \sum_{l,k \in \Lambda_h} \sum_{i=1}^3 \partial_{x_i}(a_{l,k}) \partial_{x_i} \left( \int_0^t e^{(t-t')\Delta} \text{div} \left( \phi_k^2 \phi_k^2 \phi_k^2 k \otimes \bar{k} \right) d\tau \right)
\end{align*}
\]
\[
\| \mathcal{R} \left[ (\mu a)^{-1} \mathcal{P}_H \mathcal{P} \sum_{l,k \in \Lambda_h} \sum_{i=1}^3 \partial_{x_i}(a_{l,k}) \partial_{x_i} \left( \int_0^t e^{(t-t')\Delta} \text{div} \left( \phi_k^2 \phi_k^2 \phi_k^2 k \otimes \bar{k} \right) d\tau \right) \right] \|_{L^1} \lesssim \lambda_{\gamma+1}^{50\alpha} \delta_{\gamma+2}. \tag{5.27}
\]

Combining (5.26) with (5.27), we obtain
\[
\| R_{\text{low},1} \|_{L^1} \lesssim \lambda_{\gamma+1}^{50\alpha} \delta_{\gamma+2}. \tag{5.28}
\]

Thirdly, we deduce that
\[
\begin{align*}
\text{div} (u^{(p)}_{q+1} \otimes w^{(p)}_{q+1} - d^{(p)}_{q+1} \otimes d^{(p)}_{q+1} + \tilde{R}_q) + (\partial_t u^{(t)}_{q+1} + \partial_t w^{(t)}_{q+1} + \text{div} R_{\text{high},1})
\end{align*}
\]
\[
\begin{align*}
&\text{div} \left[ \sum_{l,k \in \Lambda_h} a_{l,k}^2 \psi_k^2 (\phi_k^2 \phi_k^2 \phi_k^2) (k \otimes \bar{k}) \right] + (\partial_t u^{(t)}_{q+1} + \partial_t w^{(t)}_{q+1} + \text{div} R_{\text{high},1})
\end{align*}
\]
\[
\begin{align*}
&\text{div} \left[ \sum_{l,k \in \Lambda_h} a_{l,k}^2 \psi_k^2 (\phi_k^2 \phi_k^2 \phi_k^2) (k \otimes \bar{k}) \right] - \sum_{l,k \in \Lambda_h} a_{l,k}^2 \psi_k^2 (\phi_k^2 \phi_k^2 \phi_k^2) (k \otimes \bar{k})
\end{align*}
\]
\[
\begin{align*}
&\text{div} \left[ \sum_{l,k \in \Lambda_h} a_{l,k}^2 \psi_k^2 (\phi_k^2 \phi_k^2 \phi_k^2) (k \otimes \bar{k}) \right] + \text{div} R_{\text{high},1}
\end{align*}
\]
\[
\begin{align*}
&\text{div} \left[ \sum_{l,k \in \Lambda_h} a_{l,k}^2 \psi_k^2 (\phi_k^2 \phi_k^2 \phi_k^2) (k \otimes \bar{k}) \right] + \text{div} R_{\text{high},1}
\end{align*}
\]
\[
\begin{align*}
&\text{div} \left[ \sum_{l,k \in \Lambda_h} a_{l,k}^2 \psi_k^2 (\phi_k^2 \phi_k^2 \phi_k^2) (k \otimes \bar{k}) \right] + \text{div} R_{\text{high},1}
\end{align*}
\]
\[ \text{Owing to div } = \text{div} \]

Next, we need to rewrite

\[ \text{we rewrite} \]

Definition of \( \eta_{l}^{2} \)

\[ \text{where we use Lemma \ref{lem:4.2} with } R_{\nu} = \tilde{R}_{q} - \sum_{k,\ell \in \Lambda_{\nu}} a_{\nu,\ell,k}(\bar{k} \otimes \bar{k}) \text{ in the second equality. Thus, by the definition of } R_{q+1}^{\text{oscc}}, \text{ we deduce} \]

\[ \text{div} \left[ \rho_{q+1}^{\text{oscc}} + \mathcal{R} \left[ (\partial_{t}w_{q+1}^{(t)} + \partial_{t}w_{q+1}^{(v)} + (\partial_{t}w_{q+1}^{\Delta} - \Delta w_{q+1}^{\Delta} - \Delta w_{q+1}^{(v)}) \right) \right] = \text{div} \left[ R_{\nu,0} + R_{\nu,1} + \mathcal{R} \text{div}(R_{\nu,2}) + \mathcal{R} \mathcal{P}_{H} \text{div}(R_{\nu,3} + R_{\nu,4}) \right]. \]  

(5.30)

We need to estimate \( \mathcal{R} \text{div} R_{\nu,2} \), \( \mathcal{R} \mathcal{P}_{H} \text{div} R_{\nu,3} \) and \( \mathcal{R} \mathcal{P}_{H} \text{div} R_{\nu,4} \), respectively.

To begin with, we estimate \( \mathcal{R} \text{div} R_{\nu,2} \). Thanks to Lemma \ref{lem:3.2} and Proposition \ref{prop:4.4}, one obtains that

\[ \| \mathcal{R} \text{div} R_{\nu,2} \|_{L^{1}} \lesssim \ell_{q}^{-15} \sigma_{q}^{-1} \lambda_{q+1}^{-1+\alpha} + \ell_{q}^{-15} \sigma_{q}^{-1} \lambda_{q+1}^{\alpha} \ll \lambda_{q+1}^{-50} \delta_{q+2}. \]  

(5.31)

Next, we need to \( \mathcal{R} \mathcal{P}_{H} \text{div} R_{\nu,3} \). Recalling that

\[ \partial_{t}w_{q+1}^{(t)} = \mathcal{P}_{H} \mathcal{P}_{\nu,0} \sum_{l,k \in \Lambda_{\nu}} \frac{1}{\mu} \partial_{t}(a_{\nu,l,k}^{2}(\phi_{k}^{2} \phi_{k}^{2} \phi_{k}^{2})) \bar{k} \mathcal{P}_{H} \mathcal{P}_{\nu,0} \sum_{l,k \in \Lambda_{\nu}} \frac{1}{\mu} \partial_{t}(a_{\nu,l,k}^{2}(\phi_{k}^{2} \phi_{k}^{2} \phi_{k}^{2})) \bar{k} - \mathcal{P}_{H} \mathcal{P}_{\nu,0} \sum_{l,k \in \Lambda_{\nu}} a_{\nu,l,k}^{2}(\phi_{k}^{2} \phi_{k}^{2} \phi_{k}^{2}) \bar{k} \mathcal{P}_{H} \mathcal{P}_{\nu,0} \sum_{l,k \in \Lambda_{\nu}} a_{\nu,l,k}^{2}(\phi_{k}^{2} \phi_{k}^{2} \phi_{k}^{2}) \bar{k}. \]  

(5.32)

Owing to \( \text{div} = \mathcal{P}_{\nu,0} = \mathcal{P}_{\nu,0} \text{div} \), we have

\[ \mathcal{R} \mathcal{P}_{H} \text{div} R_{\nu,3} = \mathcal{R} \mathcal{P}_{H} \mathcal{P}_{\nu,0} \sum_{l,k \in \Lambda_{\nu}} (a_{\nu,l,k}^{2} \text{div}(\phi_{k}^{2} \phi_{k}^{2} \phi_{k}^{2})) \bar{k} \mathcal{P}_{H} \mathcal{P}_{\nu,0} \sum_{l,k \in \Lambda_{\nu}} (a_{\nu,l,k}^{2} \text{div}(\phi_{k}^{2} \phi_{k}^{2} \phi_{k}^{2})) \bar{k} + \mathcal{R} \partial_{t}w_{q+1}^{(t)} \]

\[ = - \mathcal{R} \mathcal{P}_{H} \mathcal{P}_{\nu,0} \sum_{l,k \in \Lambda_{\nu}} \frac{1}{\mu} \partial_{t}(a_{\nu,l,k}^{2}(\phi_{k}^{2} \phi_{k}^{2} \phi_{k}^{2})) \bar{k} \mathcal{P}_{H} \mathcal{P}_{\nu,0} \sum_{l,k \in \Lambda_{\nu}} \frac{1}{\mu} \partial_{t}(a_{\nu,l,k}^{2}(\phi_{k}^{2} \phi_{k}^{2} \phi_{k}^{2})) \bar{k}. \]  

(5.33)

Hence, one can easily deduce that

\[ \| \mathcal{R} \mathcal{P}_{H} \text{div} R_{\nu,3} \|_{L^{1}} \lesssim \mu^{-1} ||\partial_{t}(a_{\nu,l,k}^{2}, a_{\nu,l,k}^{2})||_{L^{-\infty}} \| \phi_{k}^{2} \phi_{k}^{2} \phi_{k}^{2} \|_{L^{1}} \lesssim \lambda_{q+1}^{-50} \delta_{q+2}. \]  

(5.34)

Finally, we turn to estimate \( \mathcal{R} \mathcal{P}_{H} \text{div} R_{\nu,4} \). Using the definition of \( R_{\nu,1} \) in (5.25) and \( w_{q+1}^{(v)} \) in (4.22), we rewrite \( \mathcal{R} \mathcal{P}_{H} \text{div} R_{\nu,4} \) as

\[ \mathcal{R} \mathcal{P}_{H} \text{div} R_{\nu,4} = \mathcal{R} \mathcal{P}_{H} \mathcal{P}_{\nu,0} \sum_{l,k \in \Lambda_{\nu}} -a_{\nu,l,k}^{2} \text{div}[\mathcal{P}_{\nu,0} \phi_{k}^{2} + 1] \phi_{k}^{2} \phi_{k}^{2} \bar{k} + \partial_{t}w_{q+1}^{(v)} + \text{div} R_{\nu,1} \]

\[ = \mathcal{R} \mathcal{P}_{H} \mathcal{P}_{\nu,0} \sum_{l,k \in \Lambda_{\nu}} -a_{\nu,l,k}^{2} \text{div}[\mathcal{P}_{\nu,0} \phi_{k}^{2} \phi_{k}^{2} \bar{k}] + \partial_{t}w_{q+1}^{(v)} \]

(5.35)
\[ (\mu \sigma)^{-1} \mathcal{R}_{H} \mathbb{P}_{>0} \sum_{l,k \in A_b} \partial_t (a_{b,l,k}^2) \text{div} [ (\partial_t^{-1} \mathbb{P}_{>0} \phi_k^2) \phi_k^2 (\bar{k} \otimes \bar{k}) ] \]

\[ = (\mu \sigma)^{-1} \mathcal{R}_{H} \mathbb{P}_{>0} \sum_{l,k \in A_b} (a_{b,l,k}^2) (\partial_t^{-1} \mathbb{P}_{>0} \phi_k^2) \phi_k^2 \text{div} [ \phi_k^2 (\bar{k} \otimes \bar{k}) ] . \] (5.35)

One deduces by Proposition 4.4 that

\[ \| \mathcal{R}_{H} \text{div} R_{\text{low},4} \|_{L^1} \leq (\mu \sigma)^{-1} \| \partial_t (a_{b,l,k}^2) \|_{L^\infty} (\| (\partial_t^{-1} \mathbb{P}_{>0} \phi_k^2) \phi_k^2 \text{div} [ \phi_k^2 (\bar{k} \otimes \bar{k}) ] \|_{L^{1+\alpha}} \leq \epsilon_q^{-15} \lambda_{q+1}^{2\alpha} \mu^{-1} \lambda^{-1} \]

\[ \ll \lambda_{q+1}^{-50\alpha} \delta_{q+2} . \] (5.36)

Plugging (5.24), (5.28), (5.31), (5.34) and (5.36) into (5.30), we obtain

\[ \| R_{q+1}^{\text{osc}} + \mathcal{R} [ (\partial_t u_{q+1}^{(t)} + \partial_t u_{q+1}^{(v)}) + (\partial_t w_{q+1}^{\Delta} - \Delta w_{q+1}^{\Delta} - \Delta u_{q+1}^{(v)}) ] \|_{L^1} \ll \lambda_{q+1}^{-50\alpha} \delta_{q+2} . \]

This completes the proof of Proposition 5.3.

Collecting Proposition 5.1–Proposition 5.3, we obtain that

\[ \| M_{q+1} \|_{L^1} \leq \| M_{\text{osc}} + \mathcal{R} [ (\partial_t d_{q+1}^{(t)} + \partial_t d_{q+1}^{(v)}) + (\partial_t d_{q+1}^{\Delta} - \Delta d_{q+1}^{\Delta} - \Delta d_{q+1}^{(v)}) ] \|_{L^1} \]

\[ \| M_{q+1}^{\text{lin}} - \mathcal{R} [ (\partial_t d_{q+1}^{(t)} + \partial_t d_{q+1}^{(v)}) + (\partial_t d_{q+1}^{\Delta} - \Delta d_{q+1}^{\Delta} - \Delta d_{q+1}^{(v)}) ] \|_{L^1} \]

\[ \leq \lambda_{q+1}^{-40\alpha} \delta_{q+2} , \]

and

\[ \| \bar{R}_{q+1} \|_{L^1} \leq \| R_{q+1}^{\text{osc}} + \mathcal{R} [ (\partial_t u_{q+1}^{(t)} + \partial_t u_{q+1}^{(v)}) + (\partial_t w_{q+1}^{\Delta} - \Delta w_{q+1}^{\Delta} - \Delta u_{q+1}^{(v)}) ] \|_{L^1} \]

\[ + \| R_{q+1}^{\text{lin}} - \mathcal{R} [ (\partial_t u_{q+1}^{(t)} + \partial_t u_{q+1}^{(v)}) + (\partial_t w_{q+1}^{\Delta} - \Delta w_{q+1}^{\Delta} - \Delta u_{q+1}^{(v)}) ] \|_{L^1} \]

\[ \leq \lambda_{q+1}^{-40\alpha} \delta_{q+2} . \]

This fact shows that (2.8) holds by replacing \( q \) with \( q + 1 \).

### 6 Energy iteration

In this section, we show that (2.10) and (2.11) hold with \( q \) replaced by \( q + 1 \).

**Proposition 6.1 (Energy estimate).** For all \( t \in [1 - \tau_q, T] \), we have

\[ \left| e(t) - \int_{\Omega^3} (|v_{q+1}|^2 + |b_{q+1}|^2) \, dx - \frac{\delta_{q+2}}{2} \right| \leq \delta_{q+2} \frac{100}{2} . \]

**Proof.** For \( t \in [1 - \tau_q, T] \), the total energy error can be rewritten as follows:

\[ e(t) - \int_{\Omega^3} (|v_{q+1}|^2 + |b_{q+1}|^2) \, dx - \frac{\delta_{q+2}}{2} \]

\[ = [e(t) - \int_{\Omega^3} (|\bar{v}|^2 + |\bar{b}|^2) \, dx - \frac{\delta_{q+2}}{2}] - \int_{\Omega^3} (|w_{q+1}|^2 + |d_{q+1}|^2) \, dx - 2 \int_{\Omega^3} (w_{q+1} \cdot \bar{v} + d_{q+1} \cdot \bar{b}) \, dx \]

32
\begin{align*}
&= e(t) - \int_{T_3} (|\tilde{v}_q|^2 + |\tilde{b}_q|^2) \, dx - \delta_{q+2} + e_{\text{low},0} \\
&\quad - \int_{T_3} (w_{q+1}^{(p)} + w_{q+1}^{(h)}) \cdot (w_{q+1}^{(p)} + w_{q+1}^{(h)}) + (d_{q+1}^{(p)} + d_{q+1}^{(h)}) \cdot (d_{q+1}^{(p)} + d_{q+1}^{(h)}) \, dx, \\&\quad \text{(6.1)}
\end{align*}

where

\begin{align*}
e_{\text{low},0} &:= - \int_{T_3} (w_{q+1}^{(p)} + w_{q+1}^{(h)}) \cdot (w_{q+1}^{(p)} + w_{q+1}^{(h)}) + (d_{q+1}^{(p)} + d_{q+1}^{(h)}) \cdot (d_{q+1}^{(p)} + d_{q+1}^{(h)}) \, dx \\
&\quad + (w_{q+1}^{(p)} + w_{q+1}^{(h)}) \cdot (d_{q+1}^{(p)} + d_{q+1}^{(h)}) + (w_{q+1}^{(p)} + w_{q+1}^{(h)}) \cdot (d_{q+1}^{(p)} + d_{q+1}^{(h)}) \, dx \\
&\quad + (w_{q+1}^{(p)} + w_{q+1}^{(h)}) \cdot (d_{q+1}^{(p)} + d_{q+1}^{(h)}) + (w_{q+1}^{(p)} + w_{q+1}^{(h)}) \cdot (d_{q+1}^{(p)} + d_{q+1}^{(h)}) \, dx \\
&\quad - 2 \int_{T_3} w_{q+1} \cdot \tilde{v}_q + d_{q+1} \cdot \tilde{b}_q \, dx. \quad \text{(6.4)}
\end{align*}

For the last term on the right-hand side of equality \textbf{(6.1)}, using \(w_{q+1}^{(p)} \cdot w_{q+1}^{(h)} = w_{q+1}^{(p)} \cdot w_{q+1}^{(h)} = d_{q+1}^{(p)} \cdot d_{q+1}^{(h)} = 0\) and \(\text{Tr} \, \hat{R}_q = 0\), we have

\begin{align*}
&\int_{T_3} (w_{q+1}^{(p)} + w_{q+1}^{(h)}) \cdot (w_{q+1}^{(p)} + w_{q+1}^{(h)}) + (d_{q+1}^{(p)} + d_{q+1}^{(h)}) \cdot (d_{q+1}^{(p)} + d_{q+1}^{(h)}) \, dx \\
&= \int_{T_3} \text{Tr}(w_{q+1}^{(p)} \otimes w_{q+1}^{(p)}) + (|w_{q+1}^{(p)}|^2 + |d_{q+1}^{(p)}|^2 + |d_{q+1}^{(h)}|^2 + |w_{q+1}^{(h)}|^2) \, dx \\
&= \int_{T_3} \text{Tr} \left[ \sum_l \eta_l^2 \rho_{\nu, q} \sum_{k \in \Lambda_v} a_{\nu, k}^2 \phi_{\nu, k}^2 \phi_{\nu, k}^2 \right] \, dx \\
&\quad + \int_{T_3} (|w_{q+1}^{(p)}|^2 + |d_{q+1}^{(p)}|^2 + |d_{q+1}^{(h)}|^2 + |w_{q+1}^{(h)}|^2) \, dx \\
&= \int_{T_3} \text{Tr} \left[ \sum_l \eta_l^2 (\rho_{\nu, q} \text{Id} - \hat{R}_q) \right] \, dx + \int_{T_3} (|w_{q+1}^{(p)}|^2 + |d_{q+1}^{(p)}|^2 + |d_{q+1}^{(h)}|^2 + |w_{q+1}^{(h)}|^2) \, dx \\
&\quad + \int_{T_3} \text{Tr} \left[ \sum_{l,k \in \Lambda_v} \eta_l^2 \rho_{\nu, q} a_{\nu, k}^2 [\mathbb{P}_{>0}(\psi_{\nu, k}^2) + \mathbb{P}_{>0}(\psi_{\nu, k}^2)] \phi_{\nu, k}^2 \phi_{\nu, k}^2 \right] \, dx \\
&= 3 \int_{T_3} \sum_l \eta_l^2 \rho_{\nu, q} \, dx + \int_{T_3} (|w_{q+1}^{(p)}|^2 + |d_{q+1}^{(p)}|^2 + |d_{q+1}^{(h)}|^2 + |w_{q+1}^{(h)}|^2) \, dx \\
&\quad + \int_{T_3} \text{Tr} \left[ \sum_{l,k \in \Lambda_v} \eta_l^2 \rho_{\nu, q} [\mathbb{P}_{>0}(\psi_{\nu, k}^2) + \mathbb{P}_{>0}(\psi_{\nu, k}^2)] \phi_{\nu, k}^2 \phi_{\nu, k}^2 \right] \, dx. \quad \text{(6.5)}
\end{align*}

Since \(\eta_{-1}(t) = N(t) = 0\) for \(t \in [1 - \tau_q, T]\), we rewrite \(\rho_{\nu, q}\) as

\begin{align*}
\rho_{\nu, q}(t) &= \chi_{\nu} \left( \delta_{q+1} N(t) + \frac{\rho_q(t)(1 - N(t))}{\eta_{-1}(t) + \sum_{l=1}^{N_q} \int_{T_3} \eta_l^2 \chi_{\nu}(x, t) \, dx} \right) \\
&= \frac{\chi_{\nu} \rho_q(t)}{\sum_{l=1}^{N_q} \int_{T_3} \eta_l^2 \chi_{\nu}(x, t) \, dx}. \quad \text{(6.6)}
\end{align*}

Recalling the definition of \(\rho_q(t)\) in \textbf{(4.7)}, we deduce that

\begin{align*}
3 \int_{T_3} \sum_l \eta_l^2 \rho_{\nu, q} \, dx + \int_{T_3} (|w_{q+1}^{(p)}|^2 + |d_{q+1}^{(p)}|^2 + |d_{q+1}^{(h)}|^2 + |w_{q+1}^{(h)}|^2) \, dx \\
&= 3 \rho_q(t) + E(t) = e(t) - \int_{T_3} |\tilde{v}_q|^2 + |\tilde{b}_q|^2 \, dx - \frac{\delta_{q+2}}{2}. \quad \text{(6.7)}
\end{align*}
\[ e(t) - \int_{T^3} |\tilde{v}_q|^2 + |\tilde{b}_q|^2 \, dx - \frac{\delta_{q+2}}{2} + \text{Tr} \int_{T^3} \sum_{l,k \in A_v} a_{v,l,k}^2 \mathbb{P}_{>0}(\psi_k^2)(\phi_{k,k,k}^2) + \mathbb{P}_{>0}(\phi_{k,k,k}^2)|\tilde{k} \otimes \tilde{k} \, dx. \]  

(6.8)

Next, using

\[ |\int_{T^3} f \mathbb{P}_{>g} \, dx| = |\int_{T^3} |\nabla|^L f |\nabla|^{-L} \mathbb{P}_{>g} \, dx| \leq c^{-L} \|g\|_{L^2} \|f\|_{H^L}. \]

with \( L \) sufficiently large, we obtain

\[ |\text{Tr} \int_{T^3} \sum_{l,k \in A_v} a_{v,l,k}^2 \mathbb{P}_{>0}(\psi_k^2)(\phi_{k,k,k}^2) + \mathbb{P}_{>0}(\phi_{k,k,k}^2)|\tilde{k} \otimes \tilde{k} \, dx| \]

\[ \lesssim \lambda_{q+1}^{\frac{L-1}{L+1}} \lambda_{q+1}^{-L} + \epsilon_q^{-5L} \lambda_{q+1}^{-\frac{L-1}{L+1}} \ll \frac{1}{10000} \delta_{q+2} \]  

(6.9)

and

\[ |\int_{T^3} w_{q+1} \cdot \tilde{v}_q + d_{q+1} \cdot \tilde{b}_q \, dx| \lesssim \lambda_{q+1}^{-5\alpha} \delta_{q+2} \ll \frac{1}{10000} \delta_{q+2}. \]  

(6.10)

From Remark 4.6 we deduce that

\[ |\int_{T^3} (w_{q+1}^{(p)} + w_{q+1}^{(h)}) \cdot (u_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} + w_{q+1}^{(s)}) + (d_{q+1}^{(p)} + d_{q+1}^{(h)}) \cdot (d_{q+1}^{(p)} + d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(s)}) \, dx| \]

\[ \lesssim \lambda_{q+1}^{-5\alpha} \delta_{q+2} \ll \frac{1}{10000} \delta_{q+2}. \]  

(6.11)

Combining the above inequality with (6.10), we show that

\[ |e_{\text{low}}| \leq \frac{1}{2000} \delta_{q+2}. \]  

(6.12)

Finally, putting (6.8), (6.9) and (6.12) into (6.1), we have

\[ |e(t) - \frac{\delta_{q+2}}{2} - \int_{T^3} (|v_{q+1}|^2 + |b_{q+1}|^2) \, dx| \]

\[ = \left| e(t) - \int_{T^3} (|\tilde{v}_q|^2 + |\tilde{b}_q|^2) \, dx - \frac{\delta_{q+2}}{2} \right| \]

\[ - \int_{T^3} (w_{q+1}^{(p)} + w_{q+1}^{(h)}) \cdot (u_{q+1}^{(p)} + w_{q+1}^{(c)} + w_{q+1}^{(t)} + w_{q+1}^{(s)}) + (d_{q+1}^{(p)} + d_{q+1}^{(h)}) \cdot (d_{q+1}^{(p)} + d_{q+1}^{(c)} + d_{q+1}^{(t)} + d_{q+1}^{(s)}) \, dx + e_{\text{low}} \]

\[ = \left| e(t) - \int_{T^3} (|\tilde{v}_q|^2 + |\tilde{b}_q|^2) \, dx - \frac{\delta_{q+2}}{2} \right| \]

\[ - \left| e(t) - \int_{T^3} (|\tilde{v}_q|^2 + |\tilde{b}_q|^2) \, dx - \frac{\delta_{q+2}}{2} \right| - \int_{T^3} \text{Tr} \left( \sum_{l,k \in A_v} a_{v,l,k}^2 \mathbb{P}_{>0}(\phi_{k,k,k}^2)|\tilde{k} \otimes \tilde{k} \, dx \right) + e_{\text{low}} \]

\[ \lesssim \int_{T^3} \text{Tr} \left( \sum_{l,k \in A_v} a_{v,l,k}^2 \mathbb{P}_{>0}(\phi_{k,k,k}^2)|\tilde{k} \otimes \tilde{k} \, dx \right) + e_{\text{low}} \]

\[ \lesssim \frac{1}{1000} \delta_{q+2}, \quad \forall t \in [1 - \tau_q, T]. \]  

(6.13)

This completes the proof of Proposition 6.1.
Proposition 6.2 (Helicity estimate). For all $t \in [1 - \tau_q, T]$, we have
\[
\left| h(t) - \int_{T^3} v_{q+1} \cdot b_{q+1} \, dx - \frac{\delta_{q+2}}{200} \right| \leq \frac{\delta_{q+2}}{1000}.
\]

Proof. Similar to the proof of Proposition 6.1, it suffices to control the helicity for $t \in [1 - \tau_q, T]$. Due to
\[
h_{b,q}(t) = \frac{h_q(t)}{\sum_{l=0}^{N_q} \int_{T^3} \eta_l^2(x,t) \, dx}, \quad t \in [1 - \tau_q, T],
\]
we rewrite that
\[
\begin{align*}
&h(t) - \int_{T^3} v_{q+1} \cdot b_{q+1} \, dx - \frac{\delta_{q+2}}{200} \\
&= \left[ h(t) - \int_{T^3} \bar{v}_q \cdot \bar{b}_q \, dx - \frac{\delta_{q+2}}{200} \right] - \int_{T^3} w_{q+1} \cdot d_{q+1} \, dx - \int_{T^3} w_{q+1} \cdot \bar{b}_q + \bar{v}_q \cdot d_{q+1} \, dx \\
&:= [h(t) - \int_{T^3} \bar{v}_q \cdot \bar{b}_q \, dx - \frac{\delta_{q+2}}{200}] - \int_{T^3} w_{q+1}^{(h)} \cdot d_{q+1}^{(h)} \, dx - \int_{T^3} w_{q+1}^{(h)} \cdot d_{q+1}^{(p)} + h_{low,0} \\
&= [h(t) - \int_{T^3} \bar{v}_q \cdot \bar{b}_q \, dx - \frac{\delta_{q+2}}{200}] - \int_{T^3} w_{q+1}^{(h)} \cdot d_{q+1}^{(h)} \, dx - 0 + h_{low,0},
\end{align*}
\]
where
\[
h_{low,0} := - \int_{T^3} (w_{q+1} - w_{q+1}^{(h)} - w_{q+1}^{(p)}) \cdot (d_{q+1} - d_{q+1}^{(h)} - d_{q+1}^{(p)}) + (w_{q+1}^{(h)} + w_{q+1}^{(p)}) \cdot (d_{q+1} - d_{q+1}^{(h)} - d_{q+1}^{(p)}) \\
+ (w_{q+1} - w_{q+1}^{(h)} - w_{q+1}^{(p)}) \cdot (d_{q+1}^{(h)} + d_{q+1}^{(p)}) \, dx.
\]

By the definitions of $w_{q+1}^{(h)}$ and $d_{q+1}^{(h)}$ in (4.3), we obtain that
\[
\begin{align*}
\int_{T^3} w_{q+1}^{(h)} \cdot d_{q+1}^{(h)} \, dx &= \int_{T^3} \sum_{l,k \in \Lambda_z} \eta_l^2 h_{b,q}(t) \psi_l^2 \phi_{k,k,k}^2 \, dx \\
&= h_{b,q} \int_{T^3} \sum_{l,k \in \Lambda_z} \eta_l^2 \, dx + \left\{ h_{b,q} \int_{T^3} \sum_{l,k \in \Lambda_z} \eta_l^2 \psi_l^2 \phi_{k,k,k}^2 \, dx + h_{b,q} \int_{T^3} \sum_{l,k \in \Lambda_z} \eta_l^2 \psi_{l,0}^2 \phi_{k,k,k}^2 \, dx \right\} \\
&:= h_q(t) + h_{low,1}.
\end{align*}
\]

Using the standard integration by parts and Remark 4.6, one deduces that
\[
|h_{low,0}| + |h_{low,1}| \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2}.
\]

Since $h_q(t) = h(t) - \int_{T^3} \bar{v}_q \cdot \bar{b}_q \, dx - \frac{\delta_{q+2}}{200}$, plugging (6.17) and (6.18) into (6.15) yields that
\[
\left| h(t) - \int_{T^3} v_{q+1} \cdot b_{q+1} \, dx - \frac{\delta_{q+2}}{200} \right| \leq |h_{low,0}| + |h_{low,1}| \lesssim \lambda_{q+1}^{-50\alpha} \delta_{q+2} \leq \frac{\delta_{q+2}}{1000}.
\]

Therefore, we complete the proof of the Proposition 6.2. \hfill \Box

Acknowledgements We thank the anonymous referee and the associated editor for their invaluable comments which helped to improve the paper. This work was supported by the National Key Research and Development Program of China (No. 2020YFA0712900) and NSFC Grant 11831004.
## A Appendix

### A.1 Inverse divergence operator

Recalling the following pseudodifferential operator of order $-1$ as in [4]:

$$R v^{kl} = \partial_k \Delta^{-1} v^j + \partial_l \Delta^{-1} v^k - \frac{1}{2} (\delta_{kl} + \partial_k \partial_l \Delta^{-1}) \text{div} \Delta^{-1} v,$$

$$R_a u^{ij} = \epsilon_{ijk}(-\Delta)^{-1} (\text{curl} u)_k,$$

where $\epsilon_{ijk}$ is the Levi-Civita tensor, $i, j, k \in \{1, 2, 3\}$. One can easily verify that $R$ is a matrix-valued right inverse of the divergence operator, such that $\text{div} R v = v$ for mean-free vector fields $v$. In addition, $R v$ is traceless and symmetric. While $R_a$ is also a matrix-valued right inverse of the divergence operator, such that $\text{div} R_a u = u$ for divergence free vector fields $u$. And $R_a u$ is anti-symmetric.

### A.2 Some notations in geometric lemmas

Using the same idea as in [4], we give the definitions of $\Lambda_b$, $\Lambda_v$ and $\Lambda_s$ by

$$\Lambda_b = \left\{ e_1, e_2, e_3, -\frac{4}{5} e_2 - \frac{3}{5} e_3, \frac{3}{5} e_1 + \frac{4}{5} e_2 \right\},$$

$$\Lambda_v = \left\{ \frac{12}{13} e_1 \pm \frac{5}{13} e_2, \frac{5}{13} e_1 \pm \frac{12}{13} e_3, \frac{12}{13} e_2 \pm \frac{5}{13} e_3 \right\},$$

$$\Lambda_s = \left\{ \frac{9}{41} e_1 + \frac{40}{41} e_2 \right\}.$$

For fixed $k \in \Lambda_b \cup \Lambda_v \cup \Lambda_s$, one can obtain a unique triple $(k, \bar{k}, \tilde{k})$. See the following table:

| $k \in \Lambda_b$, $k \perp \bar{k} \perp \tilde{k}$ | $k \in \Lambda_v$, $k \perp \bar{k} \perp \tilde{k}$ | $k \in \Lambda_s$, $k \perp \bar{k} \perp \tilde{k}$ |
|---|---|---|
| $k$ | $\bar{k}$ | $\tilde{k}$ | $k$ | $\bar{k}$ | $\tilde{k}$ | $k$ | $\bar{k}$ | $\tilde{k}$ |
| $e_1$ | $e_2$ | $e_3$ | $\frac{12}{13} e_1 \pm \frac{5}{13} e_2$ | $\frac{5}{13} e_1 \pm \frac{12}{13} e_3$ | $e_3$ | $\frac{9}{41} e_1 + \frac{40}{41} e_2$ | $\frac{40}{41} e_1 - \frac{9}{41} e_2$ | $e_3$ |
| $e_2$ | $e_3$ | $e_1$ | $\frac{5}{13} e_1 \pm \frac{12}{13} e_3$ | $\frac{5}{13} e_1 \pm \frac{12}{13} e_3$ | $e_2$ | | |
| $e_3$ | $e_1$ | $e_2$ | $\frac{12}{13} e_2 \pm \frac{5}{13} e_3$ | $\frac{12}{13} e_2 \pm \frac{5}{13} e_3$ | $e_1$ | | |
| $-\frac{4}{5} e_2 - \frac{3}{5} e_3$ | $-\frac{3}{5} e_2 - \frac{4}{5} e_3$ | $e_1$ | | | | | |
| $\frac{3}{5} e_1 + \frac{4}{5} e_2$ | $\frac{4}{5} e_1 - \frac{3}{5} e_2$ | $e_3$ | | | | | |
A.3 Two useful lemmas

Lemma A.1 ([4],[8]). Fix positive integers \( L, \kappa, \lambda \geq 1 \) such that
\[
\frac{\lambda}{\kappa} \leq \frac{1}{30} \quad \text{and} \quad \frac{\lambda^{L+4}}{\kappa^L} \leq \frac{1}{30}.
\]
Let \( p \in \{1, 2\} \), and \( f \) be a \( T^3 \)-periodic function such that there exists a constant \( C_f \) such that
\[
\| D^j f \|_{L^p} \leq C_f \lambda^j, \quad j \in [1, L + 4].
\]
Moreover, assume that \( g \) is a \((T/\kappa)^3\)-periodic function. Then, we have
\[
\|fg\|_{L^p} \lesssim C_f \|g\|_{L^p},
\]
where the implicit constant is universal.

One can obtain the following Lemma after some modifications in Lemma B.1 in [8].

Lemma A.2. Fixed \( \kappa > \lambda \geq 1 \) and \( p \in (1, 2] \). Assume that there exists an integer \( L \) with \( \kappa^{L-2} > \lambda^L \). Let \( f \) be a \( T^3 \)-periodic function so that there exists a constant \( C_f \) such that
\[
\| D^j f \|_{L^{p_1}} \lesssim \lambda^j \|f\|_{L^{p_1}}, \quad j \in [0, L].
\]
Assume further that \( \int_{T^3} f(x)P_{\geq \kappa} g(x)dx = 0 \) and \( g \) is a \((T/\kappa)^3\)-periodic function. Then, we have
\[
\| \nabla^{-1}(fP_{\geq \kappa}g) \|_{L^p} \lesssim \frac{\|f\|_{L^{p_1}} \|g\|_{L^{p_2}}}{\kappa}, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},
\]
where the implicit constant is universal.

References

[1] D. Albritton, E. Brué, and M. Colombo. Non-uniqueness of leray solutions of the forced Navier-Stokes equations. Ann.of Math., 196(1):415–455, 2022.

[2] H. Aluie. Hydrodynamic and magnetohydrodynamic turbulence: Invariants, cascades and locality. P h.D. thesis, Johns Hopkins University, 2009.

[3] H. Bahouri, J. Y. Chemin, and R. Danchin. Fourier analysis and nonlinear partial differential equations, volume 343 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2011.

[4] R. Beekie, T. Buckmaster, and V. Vicol. Weak solutions of ideal MHD which do not conserve magnetic helicity. Ann. of PDE., 6(1):1–40, 2020.

[5] T. Buckmaster, M. Colombo, and V. Vicol. Wild solutions of the Navier-Stokes equations whose singular sets in time have Hausdorff dimension strictly less than 1. J. Eur. Math. Soc., 24(9):3333–3378, 2021.
[6] T. Buckmaster, C. D. Lellis, L. Székelyhidi, and V. Vicol. Onsager’s conjecture for admissible weak solutions. *Commun. Pure Appl. Math.*, 72(2):229–274, 2019.

[7] T. Buckmaster, N. Masmoudi, M. Novack, and V. Vicol. Non-conservative $H^{1/2-}$ weak solutions of the incompressible 3D euler equations. *arXiv:2101.09278*, 2021., 2022.

[8] T. Buckmaster and V. Vicol. Nonuniqueness of weak solutions to the Navier-Stokes equation. *Ann. of Math.*, 189(2):101–144, 2019.

[9] R. E. Caflisch, I. Klapper, and G. Steele. Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. *Comm. Math. Phys.*, 184(2):443–455, 1997.

[10] A. Cheskidov and X. Luo. Sharp nonuniqueness for the Navier-Stokes equations. *Invent. math.*, 229(3):987–1054, 2022.

[11] P. Constantin, Weinan E, and E. S. Titi. Onsager’s conjecture on the energy conservation for solutions of Euler’s equation. *Comm. Math. Phys.*, 165(1):207–209, 1994.

[12] S. Daneri. Cauchy problem for dissipative Hölder solutions to the incompressible Euler equations. *Comm. Math. Phys.*, 329(2):745–786, 2014.

[13] S. Daneri, E. Runa, and L. Székelyhidi Jr. Non-uniqueness for the Euler equations up to Onsager’s critical exponent. *Ann. of PDE*, 7(1):44, 2021. No 8.

[14] S. Daneri and L. Székelyhidi Jr. Non-uniqueness and $h$-principle for Hölder-continuous weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 224(2):471–514, 2017.

[15] L. De Rosa and S. Haffter. Dimension of the singular set of wild hölder solutions of the incompressible euler equations. *Nonlinearity*, 35(10):5150–5192, 2021.

[16] D. Faraco and S. Lindberg. Proof of Taylor’s conjecture on magnetic helicity conservation. *Comm. Math. Phys.*, 273(2):707–738, 2021.

[17] D. Faraco, S. Lindberg, and L. Székelyhidi Jr. Bounded solutions of ideal MHD with compact support in space-time. *Arch. Ration. Mech. Anal.*, 239(1):51–93, 2021.

[18] D. Faraco, S. Lindberg, and L. Székelyhidi Jr. Magnetic helicity, weak solutions and relaxation of ideal MHD. *arXiv:2109.09106*, 2021.

[19] P. Isett. A proof of Onsager’s conjecture. *Ann. of Math.*, 188(3):871–963, 2018.

[20] E. Kang and J. Lee. Remarks on the magnetic helicity and energy conservation for ideal magneto-hydrodynamics. *Nonlinearity*, 20(11):2681–2689, 2007.
[21] C. Khor, C. Miao, and W. Ye. Infinitely many non-conservative solutions for the three-dimensional Euler equations with arbitrary initial data in $C^{1/3-}$. *arXiv.2204.03344*, 2022.

[22] C. De Lellis and Jr. L. Székelyhidi. The Euler equations as a differential inclusion. *Ann. of Math.*, 170(3):1417–1436, 2009.

[23] C. De Lellis and L. Székelyhidi Jr. On admissibility criteria for weak solutions of the Euler equations. *Arch. Ration. Mech. Anal.*, 195(1):225–260, 2010.

[24] Y. Li, Z. Zeng, and D. Zhang. Non-uniqueness of weak solutions to 3D generalized magnetohydrodynamic equations. *J. Math. Pures. Appl.*, 165:232–285, 2022.

[25] T. Luo and E.S. Titi. Non-uniqueness of weak solutions to hyperviscous Navier-Stokes equations - on sharpness of j.-L. lions exponent. *Calc. Var. Partial Differential Equations.*, 59(3):1–15, 2020.

[26] M. Novack and V. Vicol. An intermittent onsager theorem. *arXiv.2204.03344*, 2022.

[27] J. B. Taylor. Relaxation and magnetic reconnection in plasmas. *Reviews of Modern Physics.*, 58(3):741, 1986.

[28] J. Wu. Generalized MHD equations. *J. Diff. Equ.*, 195(2):284–312, 2003.