SIMPLICIAL 2-SPHERES OBTAINED FROM NON-SINGULAR COMPLETE FANS

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ABSTRACT. We prove that a simplicial 2-sphere satisfying a certain condition is the underlying simplicial complex of a 3-dimensional non-singular complete fan. In particular, this implies that any simplicial 2-sphere with \( \leq 18 \) vertices is the underlying simplicial complex of such a fan.

1. Introduction

A rational strongly convex polyhedral cone in \( \mathbb{R}^n \) is a cone \( \sigma \) spanned by finitely many vectors in \( \mathbb{Z}^n \) which does not contain any non-zero linear subspace of \( \mathbb{R}^n \).

A fan in \( \mathbb{R}^n \) is a non-empty collection \( \Delta \) of such cones satisfying the following conditions:

1. If \( \sigma \in \Delta \), then each face of \( \sigma \) is in \( \Delta \);
2. If \( \sigma, \tau \in \Delta \), then \( \sigma \cap \tau \) is a face of each.

A fan \( \Delta \) is non-singular if any cone in \( \Delta \) is spanned by a part of a basis of \( \mathbb{Z}^n \), and complete if \( \bigcup_{\sigma \in \Delta} \sigma = \mathbb{R}^n \).

A toric variety of complex dimension \( n \) is a normal algebraic variety \( X \) over \( \mathbb{C} \) containing \( (\mathbb{C}^*)^n \) as an open dense subset, such that the natural action of \( (\mathbb{C}^*)^n \) on itself extends to an action on \( X \). The category of toric varieties is equivalent to the category of fans (see [3]). A toric variety is smooth if and only if the corresponding fan is non-singular, and compact if and only if the fan is complete.

Given a non-singular fan \( \Delta \) with \( m \) edges spanned by \( v_1, \ldots, v_m \in \mathbb{Z}^n \), we define its underlying simplicial complex as

\[
\{ I \subset \{1, \ldots, m\} \mid \{ v_i \mid i \in I \} \text{ spans a cone in } \Delta \}.
\]

The underlying simplicial complex of an \( n \)-dimensional complete fan is a simplicial \((n-1)\)-sphere, that is, a triangulation of the \((n-1)\)-sphere.

For \( n \geq 4 \), a simplicial \((n-1)\)-sphere is not always the underlying simplicial complex of an \( n \)-dimensional non-singular complete fan (see [2, Corollary 1.23]). On the other hand, successive equivariant blow-ups of \( \mathbb{C}P^2 \) produce non-singular complete fans whose underlying simplicial complexes are all simplicial 1-spheres.

We consider the following problem:

**Problem 1.** Is any simplicial 2-sphere the underlying simplicial complex of a 3-dimensional non-singular complete fan?
No counterexamples to Problem [1] are currently known. In this paper we give a partial affirmative answer to Problem [1]. The degree of a vertex of a simplicial 2-sphere is the number of incident edges.

**Theorem 2.** Let $K$ be a simplicial 2-sphere with $m_K$ vertices. We denote the number of vertices of $K$ with degree $k$ by $p_K(k)$. If $p_K(3) + p_K(4) + 18 \geq m_K$, then $K$ is the underlying simplicial complex of a 3-dimensional non-singular complete fan. In particular, if $m_K \leq 18$, then $K$ is the underlying simplicial complex of such a fan.

The proof is done by reducing a given simplicial 2-sphere to another one in a collection of certain simplicial 2-spheres with minimum degree 5. For each such simplicial 2-sphere, we use a computer to find a non-singular complete fan whose underlying simplicial complex is the simplicial 2-sphere.

The structure of the paper is as follows: In Section 2, we give a complete list of the simplicial 2-spheres with minimum degree 5 up to 18 vertices. In Section 3, we prove Theorem 2.

2. **The simplicial 2-spheres with minimum degree 5 up to 18 vertices**

G. Brinkmann and B. D. McKay calculated the number of combinatorially different simplicial 2-spheres with minimum degree 5 [1]:

| vertices | simplicial 2-spheres | simplicial 2-spheres with min. deg. 5 |
|----------|----------------------|--------------------------------------|
| 4        | 1                    | 0                                    |
| 5        | 1                    | 0                                    |
| 6        | 2                    | 0                                    |
| 7        | 5                    | 0                                    |
| 8        | 14                   | 0                                    |
| 9        | 50                   | 0                                    |
| 10       | 233                  | 0                                    |
| 11       | 1,249                | 0                                    |
| 12       | 7,595                | 1                                    |
| 13       | 49,566               | 0                                    |
| 14       | 339,722              | 1                                    |
| 15       | 2,406,841            | 1                                    |
| 16       | 17,490,241           | 3                                    |
| 17       | 129,664,753          | 4                                    |
| 18       | 977,526,957          | 12                                   |

**Table 1.** The number of simplicial 2-spheres.

**Remark 3.** An $n$-dimensional small cover of a simple $n$-polytope is a closed $n$-manifold $M$ with a locally standard $(\mathbb{Z}_2)^n$-action such that the orbit space $M/(\mathbb{Z}_2)^n$ is the simple polytope. It follows from Steinitz’s theorem that any simplicial 2-sphere is the boundary of a simplicial 3-polytope. The dual of the simplicial 3-polytope is a simple 3-polytope $P$. It follows from the four color theorem that $P$ is the orbit space of a 3-dimensional small cover. A 3-dimensional small cover of $P$ admits a hyperbolic structure if and only if $P$ has no triangles or squares as
facets, that is, the original simplicial 2-sphere has no vertices with degree 3 or 4 \[2\]. Table 1 shows that “most” 3-dimensional small covers do not admit any hyperbolic structure.

We give a complete list of such simplicial 2-spheres up to 18 vertices (see Tables 2 and 3). They are labeled as \( \prod_{k \geq 5} k^{p(k)} \). If there are more than one simplicial 2-spheres with the same label, then we add (i), (ii), ... to the label. Letters and \( \star \) on vertices in Tables 2 and 3 are used in Section 3.

For each simplicial 2-sphere, we consider the subcomplex consisting of the vertices with degree greater than or equal to 6 and the edges whose both endpoints have degree greater than or equal to 6 (red vertices and edges in Tables 2 and 3). These show that all simplicial 2-spheres in Tables 2 and 3 are distinct except 5\(^{12}6^{6}\) (ii) and 5\(^{12}6^{6}\) (iii) (they have the same subcomplex).

Since the subcomplexes of 5\(^{12}6^{6}\) (ii) and 5\(^{12}6^{6}\) (iii) are cycles, each cycle determines two subcomplexes surrounded by the cycle (see Figures 1 and 2). These are clearly distinct.

So all simplicial 2-spheres in Tables 2 and 3 are distinct.

For \( m \leq 18 \), the number of the simplicial 2-spheres with \( m \) vertices in Tables 2 and 3 agrees with the number in Table 1. So this is a complete list of the simplicial 2-spheres with minimum degree 5 up to 18 vertices.

3. Proof of the Theorem 2

Let \( K \) be a simplicial 2-sphere with \( m_K \) vertices.

Lemma 4. If \( K \) is the underlying simplicial complex of a non-singular complete fan, then a simplicial 2-sphere obtained from \( K \) by an operation (i), (ii) or \( C_k \) \( (k \geq 5) \) is also the underlying simplicial complex of such a fan (see Figure 3).
Table 2. The simplicial 2-spheres with minimum degree 5 up to 17 vertices.
Table 3. The simplicial 2-spheres with minimum degree 5 and 18 vertices.
For the operation $C_k$, the degree of the vertex in the center of the diagram is $k$.

**Figure 3.** Operations (i), (ii) and $C_k$.

**Proof.** Suppose that the three vertices of a 2-face of $K$ correspond to edge vectors $v_1, v_2, v_3 \in \mathbb{Z}^3$. Then we have $\det(v_1, v_2, v_3) = 1$. We assign $v_1 + v_2 + v_3$ to the new vertex made by the operation (i). The corresponding fan is non-singular and complete since $\det(v_1, v_2, v_1 + v_2 + v_3) = \det(v_2, v_3, v_1 + v_2 + v_3) = \det(v_3, v_1, v_1 + v_2 + v_3) = 1$. Thus the lemma holds for an operation (i) (see Figure 4).

![Figure 4. An operation (i).](image)

Suppose that $K$ contains a subcomplex in Figure 5 and the vertices correspond to edge vectors $v, v_1, \ldots, v_k \in \mathbb{Z}^3$ as in Figure 6. Then we have $\det(v, v_1, v_3) = \det(v, v_3, v_2) = 1$. We assign $v_2 + v_3$ to the new vertex made by the operation (ii). The corresponding fan is non-singular and complete since $\det(v, v_1, v_2 + v_3) = \det(v, v_2, v_1 + v_3) = \det(v, v_4, v_2 + v_3) = \det(v, v_3, v_2 + v_3) = 1$. Thus the lemma holds for an operation (ii).

![Figure 5. An operation (ii).](image)

Suppose that $K$ contains a subcomplex in Figure 6 and the vertices correspond to edge vectors $v, v_1, \ldots, v_k \in \mathbb{Z}^3$ as in Figure 6. Then we have $\det(v, v_1, v_{i+1}) = 1$ for any $i = 1, \ldots, k$, where $v_{k+1} = v_1$. For each $i = 1, \ldots, k$, we assign $v + v_i$ to the new vertex between $v$ and $v_i$, which is made by the operation $C_k$. The corresponding fan is non-singular and complete since $\det(v, v + v_i, v + v_{i+1}) = \det(v, v + v_{i+1}, v + v_i) = \det(v, v_{i+1}, v + v_{i+1}) = 1$ for any $i = 1, \ldots, k$. Thus the lemma holds for an operation $C_k$. This completes the proof. □
Now we prove Theorem 2 by induction on \( m_K \). The tetrahedron is the only simplicial 2-sphere with 4 vertices, which is the underlying simplicial complex of the fan of \( \mathbb{CP}^3 \). Assume that \( m_K \geq 5 \).

(1) The case where there exists a vertex with degree 3. All adjacent vertices have degree greater than or equal to 4, since, if two vertices with degree 3 are adjacent, then \( K \) must be the tetrahedron, which contradicts \( m_K \geq 5 \). Thus we can perform an inverse operation of (i) and we get a simplicial 2-sphere \( K' \). We see that \( p_{K'}(3) + p_{K'}(4) \geq p_K(3) + p_K(4) - 1 \). So we have \( p_{K'}(3) + p_{K'}(4) + 18 \geq p_K(3) + p_K(4) + 18 - 1 \geq m_K - 1 = m_{K'} \). \( K' \) is the underlying simplicial complex of a non-singular complete fan by the induction hypothesis. Hence \( K \) is also the underlying simplicial complex of such a fan by Lemma 4.

(2) The case where there does not exist a vertex with degree 3 and there exists a vertex with degree 4. Since all adjacent vertices have degree greater than or equal to 4, we can perform an inverse operation of (ii) and we get a simplicial 2-sphere \( K' \). We see that \( p_{K'}(3) + p_{K'}(4) \geq p_K(3) + p_K(4) - 1 \). The same argument as (1) implies that \( K \) is the underlying simplicial complex of a non-singular complete fan.

(3) The case where there does not exist a vertex with degree 3 or 4. The Euler relation implies that \( \sum_{k \geq 3} (6 - k)p_K(k) = 12 \) (see [3, p.190]). This shows that \( K \) must have a vertex with degree 5. Since \( m_K \leq p_K(3) + p_K(4) + 18 = 18 \) by assumption, \( K \) falls into 22 types in Tables 2 and 3. Suppose that \( K \) has a vertex \( v \) with degree \( k \geq 5 \) such that any vertex adjacent to \( v \) has degree 5, and any vertex adjacent to a vertex adjacent to \( v \) has degree greater than or equal to 5. Then we can perform an inverse operation of \( C_k \) and we get a simplicial 2-sphere \( K' \). Since \( m_{K'} = m_K - k < 18 \leq p_{K'}(3) + p_{K'}(4) + 18 \), \( K' \) is the underlying simplicial complex of a non-singular complete fan by the induction hypothesis. Hence \( K \) is also the underlying simplicial complex of such a fan by Lemma 4.

Each of \( 5^{12}, 5^{12}6^5 \) (i) and \( 5^{14}6^27^2 \) (i) has such a vertex for \( k = 5 \); each of \( 5^{13}6^2, 5^{12}6^3, 5^{12}6^4 \) (i), \( 5^{12}6^5 \) (ii) and \( 5^{13}6^47^1 \) (ii) has such a vertex for \( k = 6 \); each of \( 5^{14}7^2, 5^{13}6^37^1 \) and \( 5^{13}6^27^2 \) (ii) has such a vertex for \( k = 7 \); \( 5^{16}8^2 \) has such a vertex for \( k = 8 \) (these vertices are indicated by \( \star \) in Tables 2 and 3). So they are the underlying simplicial complexes of non-singular complete fans.

We show that the rest of simplicial 2-spheres \( 5^{12}6^4 \) (ii), \( 5^{12}6^5 \) (ii), \( 5^{14}6^27^2 \) (iii), \( 5^{13}6^47^1 \) (i) and \( 5^{12}6^6 \) (i)–(vi) are the underlying simplicial complexes of non-singular complete fans with a computer aid. We assign vectors to the vertices as in Table 4.
They determine complete fans and it can be checked that all fans are non-singular by calculation.

| vertex | $5^{14}6^4$ (ii) | $5^{14}6^6$ (ii) | $5^{14}6^27^2$ (iii) | $5^{14}6^47^1$ (i) | $5^{14}6^6$ (i) |
|--------|------------------|------------------|---------------------|------------------|-----------------|
| $a$    | $(1, 0, 0)$      | $(1, 0, 0)$      | $(0, -1, 0)$        |                  |                 |
| $b$    | $(0, 1, 0)$      | $(1, 0, 1)$      | $(1, -1, 0)$        |                  |                 |
| $c$    | $(0, 0, 1)$      | $(2, -1, 1)$     | $(0, -1, 1)$        |                  |                 |
| $d$    | $(-1, 2, -1)$    | $(3, 0, -1)$     | $(-1, -1, 1)$       |                  |                 |
| $e$    | $(0, -1, -1)$    | $(2, 1, -1)$     | $(-1, -1, 0)$       |                  |                 |
| $f$    | $(1, 0, -1)$     | $(1, 1, 0)$      | $(-1, -1, 1)$       |                  |                 |
| $g$    | $(1, -1, 0)$     | $(1, -1, 1)$     | $(0, -1, -1)$       |                  |                 |
| $h$    | $(1, -1, -1)$    | $(2, 0, -1)$     | $(1, 0, 0)$         |                  |                 |
| $i$    | $(-1, 1, 0)$     | $(1, 1, -1)$     | $(0, 0, 1)$         |                  |                 |
| $j$    | $(-1, 1, 0)$     | $(0, 1, 0)$      | $(-1, 0, 1)$        |                  |                 |
| $k$    | $(-1, 1, -1)$    | $(0, 0, 1)$      | $(-1, 0, -1)$       |                  |                 |
| $l$    | $(0, -2, -1)$    | $(0, -1, 1)$     | $(0, 0, -1)$        |                  |                 |
| $m$    | $(1, -1, 1)$     | $(2, -1, 0)$     | $(0, 1, -1)$        |                  |                 |
| $n$    | $(0, -1, 1)$     | $(1, 0, -1)$     | $(1, 1, 0)$         |                  |                 |
| $o$    | $(0, -1, 0)$     | $(0, 1, -1)$     | $(0, 1, 1)$         |                  |                 |
| $p$    | $(0, -2, 1)$     | $(-1, 1, 0)$     | $(-1, 0, 0)$        |                  |                 |
| $q$    | $(-1, 0, 0)$     | $(-1, 1, -1)$    | $(0, 1, 0)$         |                  |                 |
| $r$    | $(-1, 0, 0)$     | $(-1, 0, 0)$     | $(0, 1, 0)$         |                  |                 |

TABLE 4. Assigning vectors to the vertices.

For example, we show that $5^{14}6^27^2$ (iii) is the underlying simplicial complex of a non-singular complete fan. Vectors in Table 4 determine a 3-dimensional complete fan. Its underlying simplicial complex is illustrated in Figure 7 which confirms that
there are no overlaps among the 3-dimensional cones. Calculating determinants, say \(\det(a, b, c) = 1\), we see that every cone is non-singular.

![Diagram](image)

**Figure 7.** \(5^{14}6^{27}2\) (iii).

**Acknowledgement.** The author wishes to thank Professor Mikiya Masuda for his valuable advice and continuing support, and Professors Hiroshi Sato, Tadao Oda and Masanori Ishida for useful comments on Problem 1 in the introduction.

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