1. Introduction

Recent cosmic microwave background anisotropy (see, e.g., [1, 16]), baryon acoustic oscillation peak length scale (see, e.g., [3, 12]), supernova Type Ia apparent magnitude versus redshift (see, e.g., [5,17]), and Hubble parameter as a function of redshift (see, e.g., [11,13,18]) measurements have small enough statistical error bars to encourage the belief that we will soon be in an era of precision cosmology. Of course, there have also been many earlier measurements, most having larger error bars, that have helped the field develop to the current position. In this paper we use statistical techniques to combine the results of the many earlier measurements, and so derive summary estimates of the corresponding cosmological parameters with much tighter error bars than any individual earlier measurement. We then compare these summary results to more precise recent measurements, largely those from the recent analysis of early Planck space mission cosmic microwave background (CMB) anisotropy data [1]. Using large-angle CMB anisotropy data to measure cosmological parameters is appealing because, once initial conditions and ionization history are established, it is possible to accurately compute cosmological model CMB anisotropy predictions as a function of cosmological parameter values.

Previous CMB anisotropy experiments, such as WMAP\(^1\) and ground-based ones, along with data from other techniques discussed above, have focused attention on a “standard” cosmological model (for detailed discussions see [1,16]). This model, called the \(\Lambda\)CDM model [20], is a spatially-flat cosmological model with a current energy budget dominated by a time-independent dark energy density in the form of Einstein’s cosmological constant, \(\Lambda\), that contributes 68.3\% of the current energy budget, non-relativistic cold dark matter (CDM) is the next largest contributor at 26.7\%, followed by non-relativistic baryonic matter at 4.9\% [1]. For recent reviews see [28,27,26].

A main goal of the Planck mission is to measure cosmological parameters accurately enough to check consistency with the \(\Lambda\)CDM model, as well as to possibly detect deviations. However, it is also of interest to find out if previous estimates of cosmological parameters are consistent with the Planck results. [1], and the references therein, have compared the Planck results to individual earlier measurements, most notably to the results from the WMAP experiment, from which they find small differences. However, it is also of interest to attempt to derive summary estimates for cosmological parameters from the many earlier measurements that are available, and to compare these summary estimates to the Planck results. This is what we do in this paper.

To derive our summary estimates of cosmological parameter values we use the very impressive compilation of data of [9]. We use 582 (of the 637) measurements for the dozen cosmological parameters collected by [9]. These values were published during 1990–2010, and, as estimated by [9], are approximately 60\% of the measurements of the 12 cosmological parameters published during these two decades. The main focus of the [9] paper was to compare earlier and more recent measurements and analyze how measuring techniques and results evolve over time. In our paper we use two statistical techniques, namely weighted mean and median statistics, to find the best-fit summary measured value of each of the 12 cosmological parameters. We then compare our summary values to those found from the Planck data.

In the next section we briefly review the [9] data compilation. Sections 3 and 4 are brief summaries of the weighted mean and median statistics techniques we use to analyze the [9] data. Our analyses and results are described and discussed in Section 5, and we conclude in Section 6.

\(^{1}\) E-mail addresses: sara1990@ksu.edu (S. Crandall), ratra@phys.ksu.edu (B. Ratra).

For more discussions on the use of WMAP data to estimate cosmological parameters see [16].
2. Data compilation

The data we use in our analyses here were compiled by [9]. These data were collected from the abstracts of papers listed on the NASA Astrophysics Data System (ADS). They estimate that by searching abstracts only, about 40% of available measurements were missed. Nevertheless, a great deal of data were collected. [9] searched papers published in a 20 year period (1990–2010) and tabulated 637 measurements. Of the 637 measurements, 582 were listed with a central value and 1σ error bars (these are the data we use in this paper) while 55 were upper or lower limits with no central value.

The 12 cosmological parameters [9] considered are:

1. \( \Omega_m \), the non-relativistic matter density parameter.
2. \( \Omega_\Lambda \), the cosmological constant density parameter.
3. \( h \), the Hubble constant in units of 100 km s\(^{-1}\) Mpc\(^{-1}\).
4. \( \sigma_8 \), the rms amplitude of (linear) density perturbations averaged over 8h\(^{-1}\) Mpc spheres.
5. \( \Omega_b \), the baryonic matter density parameter.
6. \( n \), the primordial spectral index.
7. \( \beta = \Omega_m^{0.6}/b \), where \( b \) is the galaxy bias.
8. \( m_\nu \), the sum of neutrino masses.
9. \( \Gamma = \Omega_m h \).
10. \( \sigma_{m,8}/\sigma_8 \).
11. \( \Omega_k \), the space curvature density parameter.
12. \( \omega_0 \), the dark energy equation of state parameter in a simplified, incomplete, XCDM-like parameterization.

Figs. 1 and 2 show the 12 histograms of the 582 [9] measurements. The histograms for parameters \( \Omega_k \), \( \Omega_m \), \( m_\nu \), and \( n \) have outlying values of 0.7, 39, 2.48 eV, and \(-1.5\), respectively, omitted from their plots, though these values were used in our analyses.

3. Weighted mean statistics

In analyzing data with known errors it is conventional to first consider a weighted mean statistic. This method yields a goodness of fit criterion that can be a valuable diagnostic tool.

The standard formula (see, e.g., [22]) for the weighted mean of cosmological parameter \( q \) is

\[
q_{wm} = \frac{\sum_{i=1}^{N} q_i / \sigma_i^2}{\sum_{i=1}^{N} 1 / \sigma_i^2},
\]

where \( q_i \pm \sigma_i \) are the central values and one standard deviation errors of the \( i = 1, 2, \ldots, N \) measurements. The weighted mean standard deviation of cosmological parameter \( q \) is

\[
\sigma_{wm} = \left( \frac{\sum_{i=1}^{N} 1 / \sigma_i^2} N \right)^{-1/2}.
\]

One can also compute the goodness of fit \( \chi^2 \),

\[
\chi^2 = \frac{1}{N-1} \sum_{i=1}^{N} \frac{(q_i - q_{wm})^2}{\sigma_i^2}.
\]
Fig. 2. Histograms of $\beta$, $m_\nu$, and $\Gamma$ (top row, from left to right), and $\Omega_m^0 e_b$, $\Omega_k$, and $\omega_0$ (bottom row, from left to right). Although used in our analyses, values of 2.48 eV for $m_\nu$ and 0.7 for $\Omega_k$ are not plotted. All of the above plots have a bin size of 0.01.

Since this method assumes Gaussian errors, $\chi$ has expected value unity and error $1/\sqrt{2(N - 1)}$. Hence, the number of standard deviations that $\chi$ deviates from unity is a measure of good-fit and is given as

$$N_\sigma = |\chi - 1|\sqrt{2(N - 1)}.$$  \hspace{1cm} (4)

A large value of $N_\sigma$ could be an indication of unaccounted-for systematic error, the presence of correlations between the measurements, or the invalidity of the Gaussian assumption.

4. Median statistics

The second statistical method we use is median statistics. This method makes fewer assumptions than the weighted mean method, and so can be used in cases when the weighted mean technique cannot. For a detailed description of the median statistics technique see [14]. In summary, if we assume that the given measurements are: (1) statistically independent; and (2) have no systematic error for the data set as a whole (as we also assume for weighted mean statistics), then as the number of measurements, $N$, increases to infinity, the median will reveal itself as a true value. This median is independent of measurement error [14], which is an advantage if the errors are suspect. This is also a disadvantage that results in a larger uncertainty for the median than for the weighted mean, because the information in the error bar is not used.

If (1) is true then any value in the data set has a 50% chance of being above or below the true median value. As described in [14], if $N$ independent measurements $M_i$, where $i = 1, \ldots, N$, are taken then the probability of exactly $n$ measurements being higher (or lower) than the true median is

$$P_n = \frac{2^{-N} N!}{n!(N - n)!}.$$  \hspace{1cm} (5)

It is interesting to note that for large $N$ the expectation value of the distribution width, $\chi$, of the true median is $\langle \chi \rangle = 0.5$, with a standard deviation $(\langle \chi^2 \rangle - \langle \chi \rangle^2)^{1/2} = 1/(4N)^{1/2}$ [14]. Of course, as $N$ increases to infinity, a Gaussian distribution is reached and median statistics recovers the usual standard deviation proportionality to $1/N^{1/2}$.

5. Analysis

Since both weighted mean and median statistics techniques have individual benefits, we analyze the compilation of data for 12 parameters from [9] using both methods. Our results are shown in Table 1. Among other things, the table lists our computed weighted mean and corresponding standard deviation $\sigma_{\text{wm}}$ value for the cosmological parameters, as well as the computed median value and the $1\sigma$ and $2\sigma$ intervals around the median.

Column 5 of Table 1 lists $N_\sigma$, the number of standard deviations the weighted mean goodness-of-fit parameter $\chi$ deviates from unity, see Eq. (4). In all cases $N_\sigma$ is much greater than unity, indicating that the weighted mean results cannot be trusted. In the case of the Hubble constant this is likely due to the fact that the observed error distribution is non-Gaussian, see [6]. Perhaps a similar effect explains the large $N_\sigma$ values for some of the other

---

5 For recent applications of median statistics see, e.g., [25,24,19,4,10].

6 The weighted mean technique also could not be used to combine different $\Omega_m$ measurements [7] or different cosmic microwave background temperature anisotropy observations [22].
parameters here. In any case, for our purpose here, the important point is that the weighted mean technique cannot be used to derive a summary estimate by combining together the different measurements tabulated by [9] for each cosmological parameter.

In a situation like this the median statistic technique can be used to combine together the measurements to derive an effective summary value of the cosmological quantity of interest (e.g., [7, 22]). Column 6 of Table 1 lists the computed medians of the 12 cosmological parameters; the corresponding 1σ and 2σ ranges of these parameters are listed in columns 7 and 8.

The median statistics estimate for the Hubble parameter here, \( h = 0.68^{0.08}_{0.14} \), is consistent with that estimated earlier by [8] from 553 measurements of \( h \) tabulated by Huchra, \( h = 0.68 \pm 0.08 \) (with understandably much tighter error bars as a consequence of the many more measurements than the 124 we have used here). Interestingly, from many fewer \( \Omega_m \) measurements than considered here, [7] determine consistent, but somewhat tighter median statistics constraints on \( \Omega_m \) by discarding the most discrepant, \(-5\%\), of the measurements (those which contribute the most to \( \chi^2 \)).

Also of interest, the median statistics estimates in Table 1 of \( \Omega_m = 0.29 \) and \( \sigma_8 = 0.84 \) result in \( \Omega_m^2 \sigma_8^2 = 0.40 \), which is significantly smaller than the median statistics estimate \( \Omega_m^2 \sigma_8^2 = 0.52 \) listed in Table 1 that was determined directly from the 11 measurements of [9]. On the other hand, \( \Gamma = \Omega_m h \) computed using the median statistics estimates of \( \Omega_m = 0.29 \) and \( h = 0.68 \) is \( \Gamma = 0.20 \), and is in very good agreement with Table 1 median statistics value of \( \Gamma = 0.19 \) from the 17 measurements of [9].

In most cases the median statistics results of Table 1 provide reasonable (2010) summary estimates for the cosmological parameters. The one exception, perhaps, is that for \( h \), which is estimated to be \( h = 0.68 \pm 0.028 \) by [8] from very many more measurements than the 124 used to derive the \( h \) value in Table 1. Perhaps the best current estimate of cosmological parameter values are those determined from the initial cosmic microwave background anisotropy measurements made by the Planck satellite [1]. The last two columns of Table 1 lists the Planck estimates for most of these parameters. Here, the estimated cosmological constrained value and 1σ standard deviation range (with the exception of \( \Omega_k \) and \( \Omega_b \) that have 2σ ranges, and \( m_v \) that has a 2σ upper limit) are listed.

Comparing our computed median results to the recent Planck values, one finds that almost all of the Planck central value results fall within the 1σ range of our median results. One exception is \( \Omega_m^2 \sigma_8^2 \) possibly because of reasons discussed above; our estimates of \( \Omega_m = 0.29 \) and \( \sigma_8 = 0.84 \) result in a \( \Omega_m^2 \sigma_8^2 \) value which is very consistent with the Planck estimate of \( \Omega_m^2 \sigma_8^2 = 0.415 \). The other exception is \( \Omega_b \) which Planck estimates to be \(-1.49 \). Our median statistics 2σ range is \(-1.25 \leq \Omega_b \leq -0.808 \) computed from the 36 measurements of [9], [9] note that the number of measurements for \( \Omega_b \) are still increasing with time, unlike the case for the other parameters. Also, as stated in [1], the Planck+WMAP constraint on \( \Omega_b \) is not very significant. However, when combined with other data tighter constraints result; for instance, including BAO data

Table 1

| Parameter | N\(^a\) | WM\(^b\) | \( \sigma_m^2 \) | \( N_m \)^d | MS\(^e\) | 1σ MS range\(^f\) | 2σ MS range\(^g\) | ECV\(^h\) | 1σ or 2σ range\(^i\) |
|-----------|-------|--------|----------------|--------|------|----------------|----------------|-------|----------------|
| \( \Omega_m \) | 138   | 0.28   | \( 3.8 \times 10^{-4} \) | 140    | 0.29 | (0.21, 0.41) | (0.053, 0.76) | 0.315 | (0.297, 0.331) |
| \( \Omega_{\Lambda} \) | 38    | 0.72   | 9.1 \times 10^{-4} | 30     | 0.72 | (0.63, 0.77) | (0.47, 0.81) | 0.685 | (0.669, 0.703) |
| \( h \) | 124   | 0.63   | 4.3 \times 10^{-4} | 160    | 0.68 | (0.54, 0.76) | (0.41, 0.88) | 0.673 | (0.661, 0.685) |
| \( \sigma_8 \) | 80    | 0.86   | 1.1 \times 10^{-4} | 130    | 0.84 | (0.72, 1.0) | (0.56, 1.3) | 0.829 | (0.817, 0.841) |
| \( n \) | 24    | 0.96   | 9.2 \times 10^{-4} | 41     | 0.98 | (0.94, 1.1) | (1.15, 1.11) | 0.960 | (0.953, 0.968) |
| \( \beta \) | 48    | 0.34   | 2.9 \times 10^{-3} | 87     | 0.52 | (0.39, 0.75) | (0.20, 1.2) | -0.93 | 12 In fact, only around the time of 2010, the Planck temperature power spectrum data as well as WMAP polarization measurements at low multipoles. [1] do not provide a Planck estimate for \( \Gamma \).

\(^a\) Number of measurements.

\(^b\) Weighted mean central value.

\(^c\) Standard deviation of weighted mean.

\(^d\) Number of standard deviations \( \chi \) deviates from unity, Eq. (4).

\(^e\) Median statistics central value.

\(^f\) Median statistics range. In several cases for the 2σ range there were not enough measurements to determine a 2σ lower limit. In these cases, the lowest data point was used to represent the 2σ lower limit. This is the case for \( \Omega_m \), \( \Omega_{\Lambda} \), and \( \Gamma \).

\(^g\) Estimated constrained value using Planck + WMAP (WMAP polarization) data. These are from the last column of Table 2 of [1], except for \( m_v \), \( \Omega_m \), and \( \Omega_b \) which are from the third column of Table 10 in [1]. For \( m_v \), there was no central value listed and so a 2σ upper limit is given.

\(^h\) Values are taken from tables listed in the previous footnote. A 1σ range was given for all parameters except for \( m_v \), \( \Omega_m \), and \( \Omega_b \) where a 2σ upper limit or range is given.

\(^i\) Here we have added in quadrature the errors on \( \Omega_m \) and \( h \) to get the range of \( \Gamma \). To get the range for \( \Omega_m^2 \sigma_8^2 \), we have taken the error on \( \Omega_b^2 \) which is given as 0.62\( \Omega_m^2 \sigma_8^2 \) and added it in quadrature with the error on \( \sigma_b \).
they find \( \omega_0 = -1.13^{+0.24}_{-0.25} \) at 2\( \sigma \), Table 10 of [1]. As such, the estimation of \( \omega_0 \) is an area still under development and so we should not give much weight to the difference in our estimate from that of Planck, and we emphasize that our median statistics value for \( \omega_0 \) is reasonably consistent with the Planck + WMAP estimate.

More provocatively, it is instructive to compare our median statistics central estimates to the 1\( \sigma \) (or 2\( \sigma \)) Planck ranges. As expected, we see that our estimate of \( \Omega_m \) (\( \Omega_\Lambda \)) lies somewhat above (below) the corresponding Planck 1\( \sigma \) range. Our estimates of \( \Omega_b \) and \( \Gamma \) are below the corresponding Planck 1\( \sigma \) ranges. Our estimate of \( n \) is well above the Planck 1\( \sigma \) range, being quite consistent with the simplest scale-invariant spectrum \([15, 21, 29]\) while Planck data strongly favors a non-scale-invariant spectrum, also readily generated by quantum fluctuations during inflation (see, e.g., [23]). And as might have been anticipated, our median statistics central \( \Omega_0^{0.66\sigma_B} \) value is well above the Planck 1\( \sigma \) range.

6. Conclusion

From the measurements compiled by [9], the median statistics technique can be used to compute summary estimates of 12 cosmological parameters. On comparing 11 of these values to those recently estimated by the Planck Collaboration, we find good consistency in ten cases. The exception is the parameter \( \Omega_0^{0.66\sigma_B} \), and it is likely that the Planck estimate of this cosmological parameter is more accurate. We also note that the \( \omega_0 \) estimation is still in its infancy and so one should not give much significance to this current mild discrepancy.

It is very reassuring that summary estimates for a majority of cosmological parameters considered by [9] are very consistent with corresponding values estimated from the almost completely independent Planck + WMAP polarization data. This provides strong support for the idea that we are now converging on a “standard” cosmological model.

Acknowledgements

We are grateful to Rupert Croft for giving us the [9] data and for useful advice. We also thank Omer Farooq for helpful discussions and useful advice. This work was supported in part by DOE grant DEFG03-99EP41093 and NSF grant AST-1109275.

References

[1] P.A.R. Ade, et al., arXiv:1303.5076 [astro-ph.CO], 2013.
[2] A. Barreira, P.P. Avelino, Phys. Rev. D 84 (2011) 083521.
[3] N.G. Busca, et al., Astron. Astrophys. 552 (2013) A96.
[4] E. Calabrese, et al., Phys. Rev. D 86 (2012) 043520.
[5] H. Campbell, et al., Astrophys. J. 763 (2013) 88.
[6] G. Chen, J.R. Gott, B. Ratra, Publ. Astron. Soc. Pac. 115 (2003) 1269.
[7] G. Chen, B. Ratra, Publ. Astron. Soc. Pac. 115 (2003) 1143.
[8] G. Chen, B. Ratra, Publ. Astron. Soc. Pac. 123 (2011) 1127.
[9] R.A.C. Croft, M. Dailey, arXiv:1112.3108 [astro-ph.CO], 2011.
[10] O. Farooq, S. Crandall, B. Ratra, Phys. Lett. B 726 (2013) 72.
[11] O. Farooq, D. Mania, B. Ratra, Astrophys. J. 764 (2013) 138.
[12] O. Farooq, B. Ratra, Phys. Lett. B 723 (2013) 1.
[13] O. Farooq, B. Ratra, Astrophys. J. Lett. 766 (2013) L7.
[14] J.R. Gott, et al., Astrophys. J. 549 (2001) 1.
[15] E.R. Harrison, Phys. Rev. D 1 (1970) 2726.
[16] G. Hinshaw, et al., Astrophys. J. Suppl. Ser. 208 (2013) 19.
[17] K. Liao, Y. Pan, Z.-H. Zhu, Res. Astron. Astrophys. 13 (2013) 159.
[18] M. Moresco, et al., J. Cosmol. Astropart. Phys. 1208 (2012) 006.
[19] M.J. Peca, E.E. Mamajek, E.J. Bubar, Astrophys. J. 746 (2012) 154.
[20] P.J.E. Peebles, Astrophys. J. 284 (1984) 439.
[21] P.J.E. Peebles, J.T. Yu, Astrophys. J. 162 (1970) 815.
[22] S. Podariu, et al., Astrophys. J. 559 (2001) 9.
[23] B. Ratra, Phys. Rev. D 45 (1992) 1913.
[24] R. Rowlands, et al., arXiv:1109.6274 [astro-ph.CO], 2011.
[25] A. Shafieloo, T. Clifton, P. Ferreira, J. Cosmol. Astropart. Phys. 1108 (2011) 017.
[26] J. Solà, arXiv:1306.1527 [gr-qc], 2013.
[27] S. Tsujikawa, arXiv:1304.1961 [gr-qc], 2013.
[28] Y. Wang, in: PoS, DSU 2012, 2012, p. 016.
[29] Ya.B. Zeldovich, Mon. Not. R. Astron. Soc. 160 (1972) 1P.