HOMOLOGICAL ASPECTS OF PERFECT ALGEBRAS

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Abstract. We investigate homological properties of perfect algebras of prime characteristic. Our principle is as follows: perfect algebras resolve the singularities. For example, we show any module over the ring of absolute integral closure has finite flat dimension. Under some mild conditions, we show any module over this ring has finite projective dimension. We compute weak and global dimensions of perfect rings in a series of nontrivial cases. Some interesting applications are given. In particular, we answer some questions asked by Shimomoto.

1. INTRODUCTION

A commutative ring of characteristic \( p \) is called perfect if the Frobenius map is an isomorphism. For many proposes, the surjectivity is enough. Also, a ring \( A \) of (mixed) characteristic \( p \) is called semiperfect if its mod \( p \) reduction has all of its \( p \)-power roots. Perfect rings get interesting nowadays. Our interest in perfect algebras is as follows: perfect algebras resolve the singularities. Let us recall some important examples of perfect algebras. Let \( R \) be a local domain. The ring of absolute integral closure \( R^+ \) is the integral closure of \( R \) inside an algebraic closure of the field of fractions of \( R \). The symbol \( R^+ \) introduced by Artin in [3], where among other things, he proved \( R^+ \) has only one maximal ideal \( m_{R^+} \), when \((R, m)\) is henselian. By using an idea due to Bhatt and Scholze [12] we show:

Theorem 1.1. Let \((R, m, k)\) be a local complete domain of prime characteristic. Then any \( R^+ \)-module has finite flat dimension. In addition, if \( k \) is of \( \aleph_n \)-cardinality (e.g., \( k \) is countable), then any \( R^+ \)-module has finite projective dimension.

The first part extends a result of Hochster and Aberbach [1] to the full setting. Computing homological dimensions over \( R^+ \) is difficult when \( R \) is of mixed characteristic (even for simple modules). Here is a sample:

Fact 1.2. (See [1] Theorem 3.5) If \( \text{fl. dim}_{R^+}(R^+/m_{R^+}) = \dim(R) \) in all mixed characteristic cases if and only the direct summand conjecture holds in mixed characteristic.

The direct summand conjecture is now a theorem by André’s recent work [2]. The organization of the paper is as follows. In §2, we give a quick review of homological invariants such as weak dimension, global dimension, and depth. In the sequel we will use all of them. A landmark result on these invariants is due to Auslander and Serre: if a ring \((A, m)\) is noetherian and local then

\[
\text{depth}(A) \leq \dim(A) = \text{ara}(m) \leq \mu(m) \leq \text{w. dim}(A) = \text{gl. dim}(A).
\]

2010 Mathematics Subject Classification. Primary 13H05, Secondary 13A35.

Key words and phrases. absolute integral closure; global dimension; prime characteristic method; weak dimension; perfection; (minimal) perfect algebras; regular rings; non-noetherian rings; semiperfect.
Moreover, the equality holds if \( \text{gl. dim}(A) \) is finite (here, \( \text{ara}(m) \) is the minimum number of elements required to generate \( m \) up to radical). The situation is not so simple if \( A \) is not noetherian. For example, the ring of entire functions has infinite Krull dimension and finite global dimension (see [31]). We will use the following result to show certain perfect algebras are not coherent:

**Observation 1.3.** Let \((A, m)\) be a coherent quasilocal ring and of finite weak dimension. Then \( \text{Kdepth}(A) = \text{w. dim}(A) \leq \text{dim}(A) \). In particular, \( \text{w. dim}(A) \leq \text{ara}(m) \).

As an application, we reprove a theorem of Vasconcelos (see Corollary 2.10).

§3 is devoted to computing homological invariants over \( R^\infty \). Following Greenberg [20] and Serre [35], the symbol \( R^\infty \) stands for the perfect closure of a noetherian ring \( R \). As far as we know, Vasconcelos (was the first person who) computed the weak dimension of a very special ring of the form \( R^\infty \), see [39, Example 5.28]. Revisiting [12], we observe that \( \text{gl. dim}(R^\infty) \) is finite. In fact the following stated in [12, Footnote 24] without a proof.

**Corollary 1.4.** Let \((R, m, k)\) be a complete local domain of prime characteristic. Then \( \text{gl. dim}(R^\infty) \leq 2 \text{dim}(R) + 1 \).

This corollary answers [4, Question 9.2] and [4, Question 9.5(ii)]. The structure of free resolutions over \( R^\infty \) is quite mysterious. However for radical ideals, we compute some explicit free resolutions:

**Observation 1.5.** (See Proposition 3.15) For any radical ideal \( a \) of \( R^\infty \), the module \( R^\infty / a \) has a free resolution of countably generated free \( R^\infty \)-modules of length bounded by \( 2 \text{dim}(R) \).

This extends [4, Theorem 1.2(i)] by presenting the bound \( 2 \text{dim}(R) \). The next problem is as follows: What is \( \text{gl. dim}(R^\infty) \)? We answer this in a low-dimensional case, see Corollary 3.14.

Over complete regular local rings that are not field, we show

\[
\text{dim}(R^\infty) + 1 = \text{gl. dim}(R^\infty) > \text{w. dim}(R^\infty) = \text{Kdepth}(R^\infty) = \text{dim}(R^\infty) = \text{ara}(m_{R^\infty}).
\]

Recall that \( R \) is called \( F \)-coherent if \( R^\infty \) is coherent. In §4 we deal with the following questions asked by Shimomoto:

**Question 1.6.** Let \( R \) be a local ring of prime characteristic and \( t \in R \) a non-zero divisor.

i) (See [36, Question 2]) Let \( R/tR \) be \( F \)-coherent. Is \( R \) \( F \)-coherent?

ii) (See [36, Question 3]) Let \( R \) be \( F \)-coherent. Is the Hilbert–Kunz multiplicity of \( R \) rational?

iii) (See [36, Question 1]) Let \( R \) be \( F \)-coherent and \( R \to S \) be flat. What conditions on the fibers required to \( S \) be \( F \)-coherent?

We done Question 1.6(ii) by presenting a perfect ring with zero-divisors such that

\[
\text{dim}(A) = \text{Kdepth}(A) = 1 < \text{w. dim}(A) = 2 < \text{gl. dim}(A) = 3.
\]

In particular, there is “a commutative local ring with finite global dimension and zero divisors.” Also, see [31]. Remark 4.8 resolves Question 1.6(i) by another method. Concerning Question 1.6(ii), we ascend up to the perfection and descend down to \( R \). I.e., we have:
Proposition 1.7. Let \((R, \mathfrak{m}, k)\) be an \(F\)-finite and \(F\)-coherent domain and \(I < R\) be of finite colength. If \(R\) is Cohen-Macaulay, then \(e_{HK}(I, R)\) is rational.

Suppose \(R \to S\) is of finite presentation. Then \(R \to S\) is étale if and only if it is flat and unramified. On the other hand unramified property defined only by the study of fibers. One can drop the finiteness by looking at weakly étale extensions. In particular, the following partially answers Question 1.6(iii): Let \(A\) be a local \(F\)-coherent and \(B\) be weakly étale over \(A\). Then \(B\) is \(F\)-coherent.

In §5, we use the results of §3 to prove Theorem 1.8. §6, deals with desingularization of \(R^*\). As a corollary to the presented results, we show:

Corollary 1.8. Let \(R\) be a complete local domain of mixed characteristic. If \(\dim(R) > 3\) then \(R^*\) is not coherent.

This is the mixed characteristic version of a result of Hochster and Aberbach. In §7, we present situations for which the global dimension of certain perfect algebras depend on the characteristic:

Example 1.9. Let \(R := \mathbb{F}_p[t, t\sqrt{t+1}]\). Then
\[
\text{gl. dim}(R^\infty) = \begin{cases} 3 & \text{if } p \neq 2 \\ 2 & \text{if } p = 2 \end{cases} \quad \text{w. dim}(R^\infty) = \begin{cases} 2 & \text{if } p \neq 2 \\ 1 & \text{if } p = 2 \end{cases}
\]

In §8, we present a simple proof of a miraculous vanishing formula due to Bhatt and Scholze. We drive it from the special case presented in our previous work [4], where it related to the so called telescope conjecture. This has some applications. We finish §8 by the following result:

Observation 1.10. Let \(R \to S\) be a perfectly finitely presented map of perfect \(\mathbb{F}_p\)-algebras. Then \(\text{p. dim}_R(S) < \infty\).

Finiteness of \(\text{fl. dim}_R(S) < \infty\) is a subject of a recent result of Bhatt and Scholze. In §9 we answer a question asked by Shimomoto (see [37, Question 2]):

Example 1.11. (After Kedlaya) Let \(R\) be any 1-dimensional local ring of prime characteristic. Then \(W(R^\infty)\) is not coherent without any regard with respect to coherent property of \(R^\infty\).

Also, the following extends [12, Lemma 7.8] by Bhatt and Scholze:

Remark 1.12. Let \(R\) be perfect and \(Q\) be (not necessarily finitely generated and not necessarily projective) an \(R\)-module of finite projective dimension. Then \(\text{p. dim}_{W(R)}(Q) = \text{p. dim}_R(Q) + 1\).

We emphasize that perfect algebras are almost non-noetherian. This is the main difficulty. Despite of this, perfect closure of a noetherian ring with singularity is a ring of finite global dimension.

2. HOMOLOGICAL INVARIANTS

In this note all rings are commutative. We consider both noetherian and non-noetherian rings. Recall that by \(\text{p. dim}(\cdot)\) (resp. \(\text{fl. dim}(\cdot)\)), we mean the projective dimension (resp. the flat dimension). A quasilocal ring \(A\) is a commutative ring with a unique maximal ideal \(\mathfrak{m}_A\). A local
ring is a noetherian quasilocal ring. By gl. dim(−) we mean the global dimension. Also, w. dim(−) stands for the weak dimension. Recall for any commutative ring A that

\[
\text{w. dim}(A) := \sup \{ \text{fl. dim}(M) : M \text{ is an } A\text{-module} \}
\]

\[
= \sup \{ \text{fl. dim}(A/a) : a \text{ is a finitely generated ideal of } A \}
\] (2.1.1)

In particular, if flat dimension of any finitely generated ideal is bounded by a uniform integer n, then flat dimension any module is bounded by the integer n.

Recall that a set \( \Gamma \) is an ordinal if \( \Gamma \) is totally ordered with respect to inclusion and every element of \( \Gamma \) is a subset of \( \Gamma \). Also, one can see that \( \Omega \) is itself an ordinal number larger than all countable ones, so it is an uncountable set. By \( \aleph_{-1} \) we denote the cardinality of finite sets. By \( \aleph_0 \) we mean the cardinality of the set of all natural numbers. We look at

\[
\Omega := \{ \alpha : \alpha \text{ is a countable ordinal number} \}.
\]

By definition \( \aleph_1 \), is the cardinality of \( \Omega \). Inductively, \( \aleph_n \) can be defined for all \( n \in \mathbb{N} \). A ring is called \( \aleph_n \)-noetherian if each of its ideals can be generated by a set of cardinality bounded by \( \aleph_n \).

So, noetherian rings are exactly \( \aleph_{-1} \)-noetherian rings.

Lemma 2.1. (See the proof of [30, Corollary 2.47]) Let \( a \) be an ideal of a \( \aleph_n \)-noetherian ring \( A \). Then

\[
\text{p. dim}_A(A/a) \leq \text{fl. dim}_A(A/a) + n + 1.
\]

A ring is called coherent, if its finitely generated ideals are finitely presented. The following is a way to show a ring is coherent.

Fact 2.2. (See [18, Theorem 2.3.2]) Any flat direct limit of coherent rings is coherent.

Definition 2.3. The Koszul grade of a finitely generated ideal with a generating set \( \underline{x} \) on a module \( M \) is defined by

\[
\text{K. grade}_A(\underline{x}, M) := \inf \{ i \in \mathbb{N} \cup \{0\} | H^i(\text{Hom}_A(K\bullet(\underline{x}), M)) \neq 0 \}.
\]

For an ideal \( b \) (not necessarily finitely generated), Koszul grade of \( a \) on \( M \) can be defined by

\[
\text{K. grade}_A(b, M) := \sup \{ \text{K. grade}_A(c, M) : c \in \Sigma \} \quad (2.3.1)
\]

where \( \Sigma \) is the family of all finitely generated subideals of \( a \). The notation Kdepth(−) stands for K. grade\(_A(\mathfrak{m}, −)\) where \( (A, \mathfrak{m}) \) is quasilocal.

Remark 2.4. i) (See [27, Page 149]) The classical grade of \( a \) on \( M \), denoted by c. grade\(_A(a, M)\), is defined to the supremum of the lengths of all weak regular sequences on \( M \) contained in \( a \). The polynomial grade of \( a \) on \( M \) is defined by

\[
\text{p. grade}_A(a, M) := \lim_{m \to \infty} \text{c. grade}_A(a[A[t_1, \ldots, t_m], A[t_1, \ldots, t_m] \otimes_A M).
\]

ii) One has \( \text{p. grade}_A(a, −) = \text{K. grade}_A(a, −) \), see e.g. [3], Proposition 2.3).

Let us cite the following basic properties of Koszul grade.

Fact 2.5. Let \( R \) be any ring, \( M \) an \( R \)-module and \( a \) an ideal. The following holds:
i) One has $K.\text{grade}_R(a, M) = K.\text{grade}_R(p, M)$ for some prime ideal $p$ (see [27, Theorem 5.16]).

ii) $K.\text{grade}$ is unique up to radical by [14, Proposition 2.2 (vi)].

iii) (See [14, Proposition 9.1.2(ii)]) If $S \subset R$ containing a system of generators $\underline{x}$ of $a$ then

$$K.\text{grade}_R(a, M) = K.\text{grade}_S(\underline{x}S, M).$$

iv) (See [14, Proposition 9.1.4]) $K.\text{grade}(a, R) = \inf\{K.\text{depth}_R(R_p) : p \in V(a)\}.$

v) (See [14, Theorem 9.1.6]; Buchsbaum-Eisenbud, Northcott) Let

$$F : 0 \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_{j+1} \overset{f_j}{\longrightarrow} F_j \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0,$$

be a complex of finite free $R$-modules and $r_i$ be the expected rank of $f_i.$ Then $F \otimes_R M$ is acyclic if and only if $K.\text{grade}_R(I_{r_i}(f_i), M) \geq i$ for all $i.$

vi) (Auslander-Buchsbaum, Hochster [27, Chap. 6, Theorem 2]) Suppose $F$ in the above item is acyclic and $R$ is quasilocal. Let $N := \text{coker}(f_0).$ Then

$$K.\text{depth}_R(N) + \text{p. dim}_R(N) = K.\text{depth}_R(R).$$

**Theorem 2.6.** Let $(A, m)$ be a coherent quasilocal ring of finite weak dimension. Then $w.\text{dim}(A) = K.\text{depth}(A) \leq \text{dim}(A).$ In particular, $w.\text{dim}(A) \leq \text{ara}(m).$

**Proof.** Let $a \triangleleft A$ be finitely generated. Then $A/a$ is finitely presented and is of finite flat dimension. Finitely present flat modules over quasilocal rings are free. It turns out that

$$\text{fl. \dim}(A/a) = p. \text{dim}(A/a) \quad (\dagger).$$

Thus, $A/a$ has finite free resolution by finitely generated free modules (see [18, Corollary 2.5.2]). By Auslander-Buchsbaum-Hochster Fact [27, viii],

$$\text{fl. \dim}(A/a) = p. \text{dim}(A/a) = K.\text{depth}(A) - K.\text{depth}(A/a) \leq K.\text{depth}(A).$$

From this we deduce that

$$w.\text{dim}(A) \leq K.\text{depth}(A) \quad (\ddagger)$$

**Fact A:** (See [5, Lemma 3.2]) Let $a$ be an ideal of a ring $B$ and $M$ a finitely generated $B$-module. Then

$$K.\text{grade}_B(a, M) \leq \text{ht}_M(a).$$

In view of Fact A, $K.\text{depth}(A) \leq \text{dim}(A).$ Thus, $w.\text{dim}(A) \leq \text{dim}(A).$ To show $w.\text{dim}(A) = K.\text{depth}(A)$ we need to recall the concept of Ext-grade. The Ext grade of $a$ on $-\ddagger$ is defined by

$$E.\text{grade}_A(a, -) := \inf\{i \in \mathbb{N} \cup \{0\} \mid \text{Ext}^i_A(A/a, -) \neq 0\}.$$ 

In general $E.\text{grade}_A(a, -) \neq K.\text{grade}_R(a, -)  \neq K.\text{grade}_A(a, -) \quad (\ddagger)$

However, if $a$ is finitely generated

$$E.\text{grade}_A(a, -) = K.\text{grade}_A(a, -) \quad (\ddagger)$$

\text{let } R := \mathbb{Q}[x_n : n \in \mathbb{N}] / (x_n^n : n \in \mathbb{N}). \text{ Set } a := (x_n : n \in \mathbb{N}). \text{ By [10, Page 367], } K.\text{grade}_R(a, R) \neq E.\text{grade}_R(a, R).\]
(see [5] Proposition 2.3(iii)). Clearly
\[ \text{E. grade}_A(a, -) \leq \text{p. dim}(A/a) \quad (\circ) \]

Let \( \Sigma \) be the family of all finitely generated subideals of \( m \). Thus,
\[
\text{w. dim}(A) \overset{(\dagger)}{\leq} \text{K. grade}(m, A) \quad 2.3 \\
= \sup \{ \text{K. grade}(a, A) : a \in \Sigma \} \\
\overset{(\circ)}{=} \sup \{ \text{E. grade}(a, A) : a \in \Sigma \} \\
\overset{(\circ)}{=} \sup \{ \text{p. dim}(A/a) : a \in \Sigma \} \\
\overset{2.1.1}{=} \text{w. dim}(A).
\]

Thus \( \text{w. dim}(A) = \text{Kdepth}(A) \). In order to show \( \text{w. dim}(A) \leq \text{ara}(m) \) we may assume that \( \ell := \text{ara}(m) < \infty \). Let \( \mathcal{X} := x_1, \ldots, x_\ell \) be such that \( \text{rad}(\mathcal{X}) = m \). In view of Fact 2.5 \( \text{K. grade}(m, A) = \text{K. grade}(\mathcal{X}, A) \). By definition, \( \text{Kdepth}(A) = \text{K. grade}(m, A) = \text{K. grade}(\mathcal{X}, A) \leq \ell \). By the first part, \( \text{w. dim}(A) = \text{Kdepth}(A) \leq \text{ara}(m) \).

In the proof of Theorem 2.6 the following invariant appeared: by very small finitistic dimension we mean
\[ \text{fin}(A) := \sup \{ \text{p. dim}(A/a) : a \text{ is finitely generated and of finite projective dimension} \}. \]

Recall that the classical small finitistic dimension is
\[ \text{fin}(A) := \sup \{ \text{p. dim}(M) : M \text{ is finitely generated and of finite projective dimension} \}. \]

It is easy to find examples with \( \text{fin}(A) \leq \text{fin}(A) \): Let \( A := (\mathbb{F}_2[[X]])^{\infty} \). Then \( \text{fin}(A) = 1 \leq \text{fin}(A) = 2 \). Also there is a situation for which \( \text{w. dim}(A) \leq \text{fin}(A) \) (such a ring is not coherent):

**Example 2.7.** Let \( A \) be the subring of \( C(\mathbb{R}) \) (the ring of all continuous real-valued functions) consisting of piecewise sums of odd roots of polynomials and quotients thereof. By \( [31] \) \( \text{w. dim}(A) = 2 \). Also, Osofsky \( [31] \) presents an element \( f \in A \) such that \( \text{p. dim}(A/fA) = 3 \). So, \( \text{w. dim}(A) \leq \text{fin}(A) \).

**Lemma 2.8.** Let \( (A, m) \) be a coherent quasilocal ring and \( \mathcal{X} := x_1, \ldots, x_d \subset m \) be such that \( \text{K. grade}(\mathcal{X}, A) = d \). Then \( \mathcal{X} \) is a regular sequence.

**Proof.** Let \( 1 \leq i < d \) and set \( a_i := (x_1, \ldots, x_i) \). Since \( A \) is coherent, \( H^i(\text{Hom}_A(K^*_a(A), A)) \) is finitely generated. By using Nakayama's Lemma and an easy induction we see that \( \text{K. grade}_A(a_i, A) = i \). In particular, \( \text{K. grade}_A(x_1, A) = 1 \). Thus, \( x_1 \) is a regular sequence. We note that \( R/x_1R \) is coherent (see [18] Theorem 2.4.1(1))). By using an easy induction on \( d \) we deduce that \( \mathcal{X} \) is a regular sequence. \( \square \)

**Corollary 2.9.** Let \( (A, m) \) be a quasilocal coherent ring such that \( m \) is a radical of a finitely generated ideal with generating set \( \mathcal{X} := x_1, \ldots, x_d \) and \( \text{Kdepth}(A) = d \). Then any permutation of \( \mathcal{X} \) is a regular sequence over \( A \).
Proof. By Fact 2.8 K. grade_A(x, A) = Kdepth(A) = d. In view of Lemma 2.8 x is a regular sequence. Since Koszul homology is invariant under permutation, any permutation of x is a regular sequence. □

By µ(m) we mean the minimal number of elements of A that need to generate m. Here we reprove (and extend) a result of Vasconcelos by a different argument (see [39, Theorem 5.22]):

Corollary 2.10. (Northcott+Vasconcelos) Let (A, m) be a quasilocal ring and m is finitely generated. Then µ(m) ≤ w. dim(A). Suppose in addition that A is coherent and w. dim(A) < ∞. Then m is generated by a regular sequence x of length w. dim(A). Any permutation of x is a regular sequence.

To find maximal regular sequences of different length see [39, Remark 5.23].

Proof. In the light of [28, Theorem 3] we see rank_A/m(m/m) ≤ w. dim(A). Since m is finitely generated and by Nakayama’s lemma, d := µ(m) ≤ w. dim(A). By definition, Kdepth(A) = K. grade(m, A) ≤ µ(m). Suppose that A is coherent and of finite weak dimension. By Theorem 2.6

µ(m) ≤ w. dim(A) = Kdepth(A) ≤ µ(m).

Let x := x_1, . . . , x_d be a generating set of m. Due to Corollary 2.9 any permutation of x is a regular sequence. □

Recall from [11] that a ring is regular if each finitely generated ideal has finite projective dimension. A coherent quasilocal ring is called super regular if its global dimension is finite and equal to its weak dimension. The following is due to Vasconcelos and plays a role in this paper:

Fact 2.11. (See [39, Theorem 5.29]) Let (R, m) be a super regular ring. Then m can be generated by a regular sequence. In particular, m is finitely generated.

The extension A → B is called weakly étale (or absolutely flat) if A → B and B ⊗_A B → B are flat. The following result is due to Olivier:

Fact 2.12. (See [29, Corollary 1]) Let A → B be weakly étale. Then w. dim(B) ≤ w. dim(A).

3. HOMOLOGICAL DIMENSION OVER R

Rings in this section all are of prime characteristic p. Let F : R → R be the Frobenius map. This sends x to x^p. As an easy (but extremely important) fact, F is a ring homomorphism.

Definition 3.1. A ring of prime characteristic is called perfect if the Frobenius map is an isomorphism.

Remark 3.2. For many proposes the surjectivity of the Frobenius map is enough. Let us call such a ring as a semi-perfect ring. Semi-perfect does not imply the perfectness. Note that if a ring is noetherian then any surjective ring-homomorphism is injective (see [26, Ex. 3.6]). But the noetherian assumption is important. Because there are rings such as A such that the Frobenius map over them is surjective but not injective. The point is that semi-perfect rings are (almost) non-noetherian (see the following observation for the explicit examples).
By \( F(R) \), we mean \( R \) as a group equipped with left and right scalar multiplication from \( R \) given by \( a \cdot r \cdot b = ab^p r \), where \( a, b \in R \) and \( r \in F(R) \). Also, \( F^p(-) := (-) \otimes_R F^p(R) \) is the Peskine-Szpiro functor, please see [32].

**Definition 3.3.** (Serre-Greenberg) Recall from [18] that the perfect closure \( R^\infty \) of \( R \) is defined by

\[ R^\infty := \lim_{\rightarrow} \left( R \begin{array}{c} F \to \to R \begin{array}{c} F \end{array} \ldots \right) \).

Since \( F^{\geq 0} \) kills nilpotent elements, \( R^\infty \) is reduced and exists uniquely. This sometimes is called the minimal perfect algebra. Set \( R_{\text{red}} := \frac{R}{\text{rad}(0)} \). In fact, \( R^\infty \) is defined by adjoining to \( R_{\text{red}} \) all \( p \)-power roots of elements of \( R_{\text{red}} \).

**Fact 3.4.** (Greenberg) Let \( f : R \to S \) be a ring of prime characteristic \( p \), there is a ring homomorphism \( f^\infty : R^\infty \to S^\infty \). If \( x \in R^\infty \), then \( x^{p^n} \in R \) for some \( n \). The assignment \( x \mapsto f((x^{p^n})^{1/p^n}) \) defines the well-defined map \( f^\infty \). This makes the perfection as a functor. In fact, Greenberg defined perfection of schemes.

Recall from [36] that a ring is called \( F \)-coherent if its perfect closure is coherent, and we call \( R^\infty \) as a coherent perfect closure of \( R \). Let us collect some elementary properties of perfect algebras that we need.

**Observation 3.5.** Let \((R, m)\) be a quasilocal ring of prime characteristic \( p \). Then

i) Suppose \( R \) is noetherian. Then \( R^\infty \) is noetherian if and only if \( \dim R = 0 \).

ii) If \( R \) is coherent and regular, then \( R^\infty \) is coherent and flat over \( R \).

iii) The coherent assumption in part ii) is really needed.

iv) The class of \( F \)-coherent rings is strictly larger than the class of noetherian regular rings.

v) Product of radical ideals in a perfect algebra is the intersection of them.

vi) Perfect algebras are seminormal.

vii) Perfect algebras are not necessarily normal.

viii) Coherent perfect closure of a complete local domain is normal.

ix) A perfect domain is UFD if and only if it is a field.

x) Any tensor product of perfect algebras is semi-perfect (e.g. has \( p \)-power root) but not necessarily perfect (e.g. roots are not unique). Any tensor product of perfect algebras over a perfect ring is perfect. For example,

\[ (R_1 \otimes_{R_0} R_2)^\infty \cong R_1^\infty \otimes_{R_0^\infty} R_2^\infty. \]

Also, \( F_p \)-endomorphism ring of a perfect algebra has \( p \)-power roots (possibly noncommutative).

xi) Any localization, direct limits, inverse limits and adic-completion of perfect algebras is again perfect. For example, \((R^\infty)_p \cong (R_{F_p/R})^\infty \) for any \( p \in \text{Spec}(R^\infty) \).

xii) If \( A \to B \) is weakly étale, then \( B^{1/p^n} \cong B \otimes_A A^{1/p^n} \). In particular, \( B^\infty \cong B \otimes_A A^\infty \).

xiii) Quotient of a perfect ring by a radical ideal \( r \) is perfect. For example, \( \frac{R^\infty}{P} \cong \frac{(\frac{R}{p^{\infty}})^\infty}{p^\infty} \) for any \( p \in \text{Spec}(R^\infty) \).
Proof. i) If $R$ is zero dimension, then $R_{\text{red}}$ is a field and its perfect closure is again a field. If $\dim R > 0$, we take $x \in \mathfrak{m}$ which is not nil and look at the increasing sequence

$$0 \subseteq (x) \subseteq (x^{1/p}) \subseteq (x^{1/p^2}) \subseteq \cdots \ (\times)$$

ii) By a famous result of Kunz [23] (in the noetherian case), $R^{1/p}$ is flat over $R$. Let us show this in the coherent case as an application of the notion of Koszul grade. We show that $\text{Tor}_i^R(R/a, F(R)) = 0$ for all $i > 0$ and for all finitely generated ideals $a \subset R$. Note that $R/a$ has a free resolution $(F_*, d_*)$ consisting of finitely generated modules, since $R$ is coherent. Then $(F_*, d_*) \otimes_R F(R) = (F_*, d_*^R)$. By $I_t(a_{ij})$ we mean the ideal generated by the $t \times t$ minors of a matrix $(a_{ij})$. Let $r_i$ be the expected rank of $d_*$. Clearly, $r_i$ is the expected rank of $d_*^R$. By Fact 2.5 ii)

$$\text{K.grade}_R(I_t(d_*), R) = \text{K.grade}_R(I_t(d_*^R), R).$$

We apply Fact 2.5 (v) (two times) to deduce that $(F_*, d_*^R)$ is exact. Hence $\text{Tor}_i^R(R/a, F(R)) = 0$. So, $R^{1/p}$ is $R$-flat. Therefore, $R^\infty$ is a flat directed union of coherent rings. In view of Fact 2.4, $R^\infty$ is coherent.

iii) Let $A$ be a non-$F$-coherent ring (such a thing exists, see e.g. Example 4.4 below). We will see in Corollary 5.9 that $R := A^{\infty}$ is regular. By definition, $R^\infty = (A^\infty)^\infty = A^\infty$ which is not coherent.

iv) For example, the ring $\mathbb{F}_2[[X^2, XY, Y^2]]$ is $F$-coherent but not regular.

v) Prove this as an easy (but important) exercise or look at [22, Proposition 2.11].

vi) Such a ring is reduced. In this case seminormality means that if $x \in Q(A)$ (the total quotient ring of $A$) is such that $x^2$ and $x^3$ are in $A$ then $x \in A$. If $p = 2$ or $p = 3$ there is nothing to prove. Let us assume $p > 3$. Then $p - 3 \in 2\mathbb{N}$, e.g., $p = 2r + 3$ for some $r \in \mathbb{N}$. Thus, $x^p \in A$. Since $A$ is perfect, we have $x \in A$ as claim.

vii) Any ring such that its normalization is not purely inseparable extension works. We left the details to the reader.

viii) This is due to Shimomoto (see [36, Theorem 3.8]).

ix) Remark that any UFD satisfies in the ascending chain condition on principal ideals. In view of $(\times)$ we get the claim.

x) Let $A_1$ and $A_2$ be perfect. Let $x := \sum_{i=1}^m x_i \otimes y_i \in A_1 \otimes_R A_2$. Set $y := \sum_{i=1}^m x_i^{1/p} \otimes y_i^{1/p} \in A_1 \otimes_R A_2$. Then $y^p = x$. So, $A_1 \otimes_R A_2$ has $p$-power root. Note that $R$ is not necessarily perfect. But the root is not necessarily unique. For example, $a := 1 \otimes x^{1/p} - x^{1/p} \otimes 1 \in (\mathbb{F}_p[[x]])^\infty \otimes_{\mathbb{F}_p[[x]]} (\mathbb{F}_p[[x]])^\infty$ is nonzero but is $p$-power is zero. So, the Frobenius is not injective.

Now, we assume that $R$ is perfect (i.e. the $p$-roots is unique). The claim follows by the following more general fact: (If $Z \leftarrow X \rightarrow Y$ are perfectoid spaces over a non-Archimedean field, then $Z \times_Y Y$ exists in the category of perfectoid adic spaces. This interesting result is due to Scholze [33, Proposition 6.18]). This implies that

$$\left(A_1 \otimes_R A_2\right)^\infty = A_1^\infty \otimes_R A_2^\infty.$$
Thus, if all of $R$ and $A_1, A_2$ are perfect then all $p$-power roots of $A_1 \otimes_R A_2$ is unique. So, $A_1 \otimes_R A_2$ is perfect.

Let $A := \text{Hom}_{\mathbb{F}_p}(A_1, A_1)$. This is an associative ring. Let $f \in A$. Define $f^{1/p}(a) := (f(a))^{1/p}$. It is easy to see it is additive. Let $r \in \mathbb{F}_p$ and $a \in A$. By Fermat’s little theorem, $r = r^p$. Taking $p$-th root, we have $r^{1/p} = r$. Then

$$f^{1/p}(ra) = (f(ra))^{1/p} = (rf(a))^{1/p} = r^{1/p}f(a)^{1/p} = rf(a)^{1/p}.$$  

Thus, $f^{1/p}$ belongs to $A$ and that $(f^{1/p})^p = f$. So, $A$ has $p$-power root.

xi) This is easy, and we left it to the reader.

xii) This is in [17, Theorem 3.5.13] by Gabber and Ramero.

xiii) Let $r + r' \in A/r$. Define $(r + r')^{1/p} := r^{1/p} + r'$. This is well-defined, because $r$ is a radical ideal. The particular claim is also trivial.

\[ \square \]

**Remark 3.6.** Recall that a GCD domain is an integral domain with the property that any two non-zero elements have a greatest common divisor. This is well-known that any GCD domain is normal.

Homological properties of perfect algebras not only simplified things but also extend them:

**Corollary 3.7.** (Compare with Observation 3.5(viii)) Coherent perfect closure of a local domain $R$ is a GCD domain. In particular, $R^\omega$ is a normal domain.

**Proof.** Recall that $R^\omega$ is quasilocal, coherent and regular (see [4, Theorem 1.2(iii)]). By [18, Corollary 6.2.10] any quasilocal, coherent and regular domain is a GCD domain. By the above remark, $R^\omega$ is normal. \[ \square \]

**Fact 3.8.** (See [12, Proposition 11.31]) Let $R^\omega$ be the perfection of a complete local ring $R$. Then $R^\omega$ has finite global dimension. In fact, flat dimension of any $R$-module is bounded above by some fixed integer $N$. By [12, Remark 11.33], one can choose $N = 2 \dim(R)$.

**Corollary 3.9.** Let $(R, m, k)$ be a complete local domain of prime characteristic. Then $\text{gl. dim}(R^\omega) \leq 2 \dim(R) + 1$.

**Proof.** For each positive integer $n$, set $R_n := \{x \in R^\omega|xp^n \in R\}$. This is easy to see that $R_n$ is noetherian and that $R^\omega = \cup R_n$. Any countable union of noetherian ring is $\aleph_0$-noetherian. We combine Lemma 2.1 along with Fact 3.8 to observe that

$$\text{p. dim}_{R^\omega}(R^\omega/a) \leq \text{fl. dim}_{R^\omega}(R^\omega/a) + 1 \leq 2 \dim(R) + 1.$$  

By Auslander’s local-global-theorem (please see [80, Theorem 2.17]),

$$\text{gl. dim}(R^\omega) = \sup \{\text{p. dim}(R^\omega/a) : a \subset R^\omega\} \leq 2 \dim(R) + 1.$$  

\[ \square \]

*Associative rings that satisfy the polynomial identity $x^{n(x)} = x$ for some $n_x > 1$, are very special: they are commutative. This is an interesting result of Jacobson.
Observation 3.1(xiii) implies that any prime quotient of $R^\infty$ is regular. Such a thing never happens in the local algebra. However, the primeness assumption is important:

Example 3.10. Let $A_0$ be the ring of polynomials with nonnegative rational exponents in an indeterminate $x$ over a field $\mathbb{F}_2$. Let $T$ be the localization of $A_0$ at $(x^\alpha : \alpha > 0)$ and set $A := T/(x^\alpha u : u \text{ is unit}, \alpha > 1)$. In view of [30, Page 53], $A$ has finite global dimension on maximal ideals and $\text{p. dim}(x^{1/2}A) = \infty$.

Theorem 3.11. Let $R$ be a complete local domain of prime characteristic and suppose that its perfect closure is coherent (e.g., $R$ is regular). The following holds:

i) If $R$ is not a field, then $\text{gl. dim}(R^\infty) = \dim(R) + 1$.

ii) Also, $\text{w. dim}(R^\infty) = \dim(R)$.

Proof. i) This is in [6, Proposition 3.4].

ii) One has $\text{w. dim}(R^\infty) \geq \dim(R)$. By i), the weak dimension of $R^\infty$ is finite. In view of Theorem 2.6, $\text{w. dim}(R^\infty) \leq \dim(R^\infty) = \dim(R)$. Thus, $\text{w. dim}(R^\infty) = \dim(R)$.

Second proof of Theorem 3.11(ii). Without loss of the generality we assume that $R$ is not a field. Suppose $\text{w. dim}(R^\infty) \neq \dim(R)$. Then $\dim(R) + 1 \leq \text{w. dim}(R^\infty) \leq \text{gl. dim}(R^\infty)$. Thus, $R^\infty$ is suppr regular. In the light of Fact 2.12 the maximal ideal of $R^\infty$ is finitely generated. But $m_{R^\infty}^p = m_{R^\infty}$. By Nakayama’s lemma, $m_{R^\infty} = 0$. This implies that $R^\infty$ is a field. So, $R$ is a field, a contradiction.

Corollary 3.12. Let $R$ be a complete local $F$-coherent and $S$ be weakly étale. Then $\text{w. dim}(S^\infty) \leq \dim R$.

In the next section we reprove (and extend) this by avoiding Fact 2.12.

Proof. The base change of a flat ring map is flat. This means that $(S \otimes_R R^\infty) \otimes_R S \otimes_R R^\infty \to (S \otimes_R R^\infty)$ and $R^\infty \to S \otimes_R R^\infty$ flat. Thus $R^\infty \to S \otimes_R R^\infty$ is weakly étale. By Fact 2.12 $\text{w. dim}(S \otimes_R R^\infty) \leq \text{w. dim}(R^\infty)$. Since $\text{w. dim}(R^\infty) = \dim R$, we have $\text{w. dim}(S \otimes_R R^\infty) \leq \dim R$. In view of Observation 3.5(xii), $S^\infty \simeq S \otimes_R R^\infty$. So, $\text{w. dim}(S^\infty) \leq \dim R$.

Question 3.13. Let $d > 0$ be any integer and let $e$ be such that $d + 1 \leq e \leq 2d + 1$. Is there a $d$-dimensional local ring $R$ such that $\text{gl. dim}(R^\infty) = e$?

Corollary 3.14. Let $R$ be a 1-dimensional complete local domain of prime characteristic. The following are equivalent:

i) $R^\infty$ is stably coherent.

ii) $\text{gl. dim}(R^\infty) = 2$.

iii) $\text{w. dim}(R^\infty) = 1$.

iv) $R^\infty$ is a valuation ring.

v) $R^\infty$ is the perfect closure of a noetherian regular local ring.

If $R^\infty$ is not coherent, then $\text{gl. dim}(R^\infty) = 3$.

Proof. Suppose first that $R^\infty$ is not coherent. Any integral domain of global dimension less than 3 is coherent, see [18, Theorem 6.3.4]. Thus, $\text{gl. dim}(R^\infty) > 2$. By Corollary 3.9, $\text{gl. dim}(R^\infty) \leq 2 \dim(R) + 1 = 3$. So, $\text{gl. dim}(R^\infty) = 3$. 


i) ⇒ ii) Stably coherent rings are coherent. The claim now follows from Theorem 2.11.

ii) ⇒ iii) Note that 0 < \text{w.dim}(R^\infty) \leq \text{gl.dim}(R^\infty) = 2. Suppose on the contradiction that w. dim(R^\infty) \neq 1. Then w. dim(R^\infty) = 2. In view of [18, Theorem 6.3.4], any integral domain of global dimension less than 3 is coherent. We conclude from Fact 2.11 that m_{R^\infty} is finitely generated. But m_{R^\infty} is not finitely generated, because m_{R^\infty}^p = m_{R^\infty}. This contradiction shows that w. dim(R^\infty) = 1.

iii) ⇒ iv) In view of [18, Corollary 4.2.6] any ring of weak dimension less than 2 is locally a valuation domain. Thus, R^\infty is a valuation ring.

iv) ⇒ v) Any valuation ring is integrally closed. Let \overline{R} be the integral closure of R. Then

$$R \subset \overline{R} = R^\infty.$$ 

On the other hand \overline{R} is local, because R is a 1-dimension complete local ring. Since any 1-dimensional integrally closed local domain is discrete valuation ring (DVR), we get that A := \overline{R} is regular. Clearly, A^\infty = R^\infty. From this we get the claim.

v) ⇒ i) Let A be a DVR such that A^\infty = R^\infty. Since A^{1/p^n} is torsion-free over A it is flat over A. Thus, R^\infty is a flat filtered limit of noetherian rings. In view of Fact 2.2 R^\infty is coherent. Similarly, R^\infty[X] is coherent.

\[\square\]

The structure of free resolutions over R^\infty is quite mysterious. However, for radical ideals we have the following result:

**Proposition 3.15.** Let (R, m) be a Noetherian local domain of prime characteristic. The following holds:

i) For any radical ideal a of R^\infty, the module R^\infty/a has a free resolution of countably generated free R^\infty-modules of length bounded by 2 \text{dim}(R).

ii) For any radical ideal a of R^\infty, the module R^\infty/a has a flat resolution of countably generated flat R^\infty-modules of length bounded by \text{dim}(R).

**Proof.** i): By \((x^\infty)\) we mean that \((x^{1/p^n} : n \in \mathbb{N}_0)\) where \(p := \text{char} R\). Let \(d := \text{dim}(R)\). Let a be a radical ideal of R^\infty and set b := a ∩ R. Clearly, b is radical. By the folklore result of Kronecker, there is a finite sequence \(\underline{a} := a_1, \ldots, a_d\) of elements of R such that \(\sqrt{\underline{a}} = \sqrt{b} = b\). Suppose \(x \in a\). Then \(x^{p^n} \in R \cap a = b\) for some integer m. It yields that \(x^{p^n} = r_1a_1 + \cdots + r_da_d\) for some integer n where \(r_i \in R\). By taking \(p^n\)-th root, \(x = r_1^{1/p^n}a_1^{1/p^n} + \cdots + r_d^{1/p^n}a_d^{1/p^n}\). Therefore, \(x \in \sum_{i=1}^d (a_i^{n})\), i.e., \(a \subset \sum_{i=1}^d (a_i^{n})\). The reverse inclusion is trivial. Any radical ideal of R^\infty is of the form \(\sum_{i=1}^d (a_i^{n})\) for some \(a_1, \ldots, a_d \in R^\infty\). Now we use a trick taken from [4, Lemma 7.7]. Let \(F_0\) be a free R^\infty-module with base \(\{e_n : n \in \mathbb{N}_0\}\). The assignment \(e_n \mapsto x_i^{1/p^n}\) provides a natural epimorphism \(\varphi : F_0 \longrightarrow (x^\infty)\). Let \(F_1\) be a free R^\infty-module with base \(\{f_n : n \in \mathbb{N}_0\}\). Set \(\eta_n := e_n - x_i^{1/p^n}e_{n+1}\). The assignment \(f_n \mapsto \eta_n\) provides a natural epimorphism \(\varphi : F_1 \longrightarrow \ker \varphi\).

Then a free resolution of \(R^\infty/(x_i^{n})\) is given by the following exact complex:

\[
P_1 : 0 \longrightarrow \bigoplus N R^\infty \xrightarrow{X} \bigoplus N R^\infty \xrightarrow{Y} R^\infty \longrightarrow R^\infty/(x_i^{n}) \longrightarrow 0 \quad (*)\]
where the matrices $X$ and $Y$ are defined by:

$$X := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
-x_i^{p-1} & 1 & 0 & 0 & 0 & \cdots \\
0 & -x_i^{p-1} & 1 & 0 & 0 & \cdots \\
0 & 0 & -x_i^{p-1} & 1 & 0 & \cdots \\
0 & 0 & 0 & -x_i^{p-1} & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

and

$$Y := \begin{pmatrix}
x_i & x_i^{1/p} & x_i^{1/p^2} & \cdots 
\end{pmatrix}^t.$$

We apply an easy induction. Set $(\underline{g}^\infty) := \sum_{i=2}^d (x_i^\infty)$. By the induction hypothesis, $P := \otimes_{i=2}^d P^i$ is exact. Recall that

$$H_n(P^1 \otimes_{R^\infty} P) \simeq \text{Tor}_{R^\infty}^n(R^\infty/(x_1^\infty), \frac{R}{(\underline{g}^\infty)}) = 0.$$ 

The ideal $(x_1^\infty)$ is flat. Hence, for each $n > 1$ we have $\text{Tor}_{R^\infty}^n(R^\infty/(x_1^\infty), \frac{R}{(\underline{g}^\infty)}) = 0$. Also,

$$\text{Tor}_{1}^R(R^\infty/(x_1^\infty), \frac{R}{(\underline{g}^\infty)}) \simeq \frac{(x_1^\infty) \cap (\underline{g}^\infty)}{(x_1^\infty)(\underline{g}^\infty)} = 0,$$

by Observation 3.5(v). This is clear that $H_0(P^1 \otimes_{R^\infty} P) \simeq R^\infty/(x_1^\infty) \otimes_{R^\infty} \frac{R}{(\underline{g}^\infty)} \simeq R^\infty/\mathfrak{m}$. Thus, $\otimes_{i=1}^d P^i$ is an explicit free resolution of $R^\infty/\mathfrak{m}$ of length $2d$.

ii): This is similar as above.

$\square$

**Remark 3.16.** The above resolution is not necessarily minimal.

The local assumption is important:

**Example 3.17.** Let $R$ be a noetherian regular ring of prime characteristic and infinite global dimension (such a thing exists). Then $R^\infty$ is of infinite global dimension.

**Proof.** Let $n$ be in $\mathbb{N}$. There is a regular sequence $\underline{x} := x_1, \ldots, x_n$ over $R$. This is also a regular sequence over $R^\infty$. In particular, p. dim$(R^\infty/\underline{x} R^\infty) = n$. So, gl. dim$(R^\infty) = \infty$.

The noetherian assumption is important:

**Example 3.18.** Let $(A, \mathfrak{m}_A)$ be the localization of $\mathbb{F}_p[X_1, \ldots]$ at $(X_1, \ldots)$. Then fl. dim$A$$(A^\infty/mA^\infty) = \infty$.

**Proof.** Let $(A_i, \mathfrak{m}_i) := \mathbb{F}_p[X_1, \ldots, X_i]/(X_{i+1}, \ldots, X_n)$. In view of Observation 3.5(ii) $A^\infty$ is flat over $A$. Hence,

$$\text{Tor}_1^n(A/m_A, A/m_A) \otimes_A A^\infty \simeq \text{Tor}_1^n(A^\infty/mA^\infty, A^\infty/mA^\infty).$$
To conclude its enough to show that $\text{Tor}_i^A(A/m, A/m) \neq 0$ for all $j$. Let $i$ be a non negative integer. Then, $\text{Tor}_i^A(A_i/m_i, A_i/m_i) \neq 0$. By using the rigidity of $\text{Tor}$’s modules (see [29]) we have $\text{Tor}_j^A(A_i/m_i, A_i/m_i) \neq 0$ for any $j \leq i$. Thus, $A_i/m_i \hookrightarrow \text{Tor}_j^A(A_i/m_i, A_i/m_i)$. So,

$$0 \neq \lim_{j \to i} A_i/m_i \hookrightarrow \lim_{j \to i} \text{Tor}_j^A(A_i/m_i, A_i/m_i) \simeq \text{Tor}_i^A(A/m, A/m).$$

$$\square$$

4. AN APPLICATION: SOME QUESTIONS BY SHIMOMOTO

**Question 4.1.** (See [36] Question 2) Let $R$ be local and $t \in R$ a non-zero divisor. If $R/tR$ is $F$-coherent, then is $R$ also $F$-coherent?

**Example 4.2.** (See [12] Footnote 24) Look at the reduced ring $R := F_p[X, Y]/(XY)$. Then $\dim(R/\langle x \rangle) = 2 \dim(R)$. In fact they showed that $\text{Tor}_2^R(R/\langle x \rangle, R/\langle y \rangle) \neq 0$.

Let us give an example of 1-dimensional ring such that $\dim(R^\infty) = 3$. To this end, we recall the following beautiful result of Osofsky (see [30] Proposition 2.36):

**Fact 4.3.** Let $A$ be a quasilocal ring with zero-divisors. Then $\dim(A) \geq 3$ and $\dim(A) \geq 2$.

**Example 4.4.** Look at the reduced local ring $R := \left( F_p[X, Y]/(XY) \right)_{(x,y)}$. Then $\dim(R^\infty) = 3$, $\dim(R^\infty) = 2$ and $\dim(R^\infty) = 1$.

**Proof.** Indeed, let $A := F_p[X, Y]/(XY)$. The lowercase letter will stand for elements of $A$. Set $m_{A^\infty} := (x^{1/n}, y^{1/n} : n \in \mathbb{N})$. It is a maximal ideal of $A$. In view of Example 4.2

$$\text{Tor}_2^A(A^\infty/\langle x \rangle, A^\infty/\langle y \rangle) \neq 0.$$

Since

$$(x, y) \text{Tor}_2^A(A^\infty/\langle x \rangle, A^\infty/\langle y \rangle) = 0,$$

and that $V((x, y)A^\infty) \subset \max(A^\infty)$ we observe that

$$V \left( \text{Ann} \left( \text{Tor}_2^A(A^\infty/\langle x \rangle, A^\infty/\langle y \rangle) \right) \right) = \{ m_{A^\infty} \}.$$

Thus, $\text{Tor}_2^A(A^\infty/\langle x \rangle, A^\infty/\langle y \rangle)_{m_{A^\infty}} \neq 0$. Note that $(A^\infty)_{m_{A^\infty}} \simeq R^\infty$. Against Ext, Tor behaves nicely with respect to the localization over non-noetherian rings, that is

$$0 \neq \text{Tor}_2^A(A^\infty/\langle x \rangle, A^\infty/\langle y \rangle)_{m_{A^\infty}} \simeq \text{Tor}_2^R(R^\infty/\langle x \rangle, R^\infty/\langle y \rangle).$$

In particular, $\dim(R^\infty) \geq 2$ (in fact $\dim(R^\infty) = 2$). Since $R^\infty$ is quasilocal and has zero-divisor, by Fact 4.3 we observe that $\dim(R^\infty) \geq 3$. In turns out that

$$\dim(R^\infty) = 3 = 2 \dim(R) + 1,$$

as claimed. $\square$

We also need the following result:

**Lemma 4.5.** (See [18] Lemma 4.2.3) A quasilocal coherent ring with the property that every principal ideal has finite projective dimension is a domain.
Observation 4.6. Let $R$ be local and zero-dimensional. Then $R$ is $F$-coherent.

Proof. Set $A := R_{\text{red}}$. This is a field. By definition, $R^\infty = A^\infty$. In particular, $R^\infty$ is a field and so coherent.

The following answers Question 4.1.

Corollary 4.7. Let $R := \left( \frac{F_p[X, Y]}{(XY)} \right)_{(x,y)}$ and set $t := x - y$. Then $t$ is not zero-divisor, $R/\langle t \rangle$ is $F$-coherent, and $R$ is not $F$-coherent.

Proof. Clearly, $t \notin \bigcup_{p \in \text{Ass} \, R} p$. We deduce from this that $t$ is not zero-divisor. The following natural isomorphisms

$$R/\langle t \rangle \cong \left( \frac{F_p[X, Y]}{(XY)} \right)_{(x,y)} \cong \left( \frac{F_p[X, Y]}{(XY, X - Y)} \right)_{(x,y)} \cong \left( \frac{F_p[Y]}{(Y^2)} \right)_{(y)}$$

implies that $R/\langle t \rangle$ is an $F$-coherent ring (see the above observation). Combining Lemma 4.5 and Example 4.4, $R$ is not $F$-coherent.

Remark 4.8. In fact, if $R$ is a 1-dimensional integral domain and $t \neq 0$ is any element, then $R/\langle t \rangle$ is $F$-coherent without any regard with respect to $F$-coherent property of $R$. In Example 4.4, we will give examples of 1-dimensional integral domains that are not $F$-coherent.

Let $I \triangleleft A$ be of finite co-length. The Hilbert-Kunz multiplicity of an ideal $I$ is defined by $e_{HK}(I, A) := \lim_{n \to \infty} \left( \frac{(I^n/(A/I))^!}{p^n \dim(A)} \right)$.

Question 4.9. (See [36, Question 3]) Let $A$ be an $F$-coherent local ring. Is $e_{HK}(I, A) \in \mathbb{Q}$?

Let $A$ be any commutative ring with an ideal $a$ with a generating set $a := a_1, \ldots, a_r$. By $H^i_\mathcal{C}(M)$ we mean the $i$-th cohomology of Čech complex of a module $M$ with respect to $a$. This is independent of the choose of the generating set. For simplicity, we denote it by $H^i_\mathcal{C}(M)$. This cohomology theory introduced by Grothendieck in SGA 2 [19].

Discussion 4.10. In the case that $a \triangleleft A$ is finitely generated by generating set $x$, the Čech grade of $a$ on $M$ is defined by $\inf \{ i \in \mathbb{N}_0 | H^i_\mathcal{C}(M) \neq 0 \}$. Denote it by $\mathcal{C}. \text{grade}_A(a, M)$. It is easy to observe that $\mathcal{C}. \text{grade}_A(a, M) = K. \text{grade}_A(a, M)$.

Suppose $(R, m)$ is a reduced and of characteristic $p$. Recall that $R$ is $F$-injective if $F : H^i_m(R) \to H^i_m(R)$ is injective.

Lemma 4.11. Let $(R, m, k)$ be a noetherian local reduced ring of prime characteristic $p$. If $\underline{x} = x_1, \ldots, x_m$ is a regular sequence over $R$ then $\underline{x}$ is a regular sequence over $R^\infty$. Thus, $\text{depth} \ R \leq \text{Kdepth}_{R^\infty}(R^\infty)$. The equality holds if $R$ is $F$-injective or Cohen-Macaulay.

Proof. Let $r \in R^\infty$ be such that $rx_i \in (x_1, \ldots, x_{i-1})R^\infty$. There are $r_j \in R^\infty$ such that $rx_i = \sum_{j=1}^{i-1} r_j x_j$. Recall that $R^\infty = \bigcup R^{1/p^n}$. In particular, there is an $n$ such that $r$ and all of $r_j$ are in $R^{1/p^n}$. Taking $p^n$-th power we have

$$r^{p^n} x_i^{p^n} = \sum_{j=1}^{i-1} r_j^{p^n} x_j^{p^n} \quad (*)$$
Recall from [26, Theorem 16.1] that \( \underline{x}^{m} := x_1^{m}, \ldots, x_m^{m} \) is a regular sequence over \( R \). Since every thing in \((\ast)\) is in \( R \), we have \( r^{m} \in (x_1^{p}, \ldots, x_{i-1}^{p})R \). Taking \( p^{m}-\)th root, we have

\[
 r \in (x_1, \ldots, x_{i-1})R^{1/p^{m}} \subset (x_1, \ldots, x_{i-1})R^{\infty}.
\]

Thus, \( \underline{x} \) is a regular sequence over \( R^{\infty} \). Therefore, \( \text{depth}_{R}(R) \leq \text{depth}_{R^{\infty}}(R^{\infty}) \leq \text{Kdepth}_{R^{\infty}}(R^{\infty}) \).

Suppose now that \( R \) is \( F \)-injective. Since \( R \) is \( F \)-injective, then \( \text{H}^{i}_{m}(R) \to \text{H}^{i}_{m}(R) \) is injective by definition. Since, the local cohomology commutes with direct limit (via Frobenius), \( \text{H}^{i}_{m}(R) \to \text{H}^{i}_{m}(R^{\infty}) \) is injective. Hence, \( C. \text{grade}_{R}(m, R) \geq C. \text{grade}_{R}(m, R^{\infty}) \). Note that Koszul grade is the same as of \( \text{Cech} \) grade. Thus,

\[
\text{depth}_{R}(R) \geq \text{Kdepth}_{R}(R^{\infty}) \quad \quad \text{by (\dag)}
\]

\[
= \text{K. grade}_{R}(m, R^{\infty}) \quad \quad \text{by Definition}
\]

\[
= \text{K. grade}_{R^{\infty}}(mR^{\infty}, R^{\infty}) \quad \quad \text{by Fact 2.5 (iii)}
\]

\[
= \text{Kdepth}_{R^{\infty}}(R^{\infty}) \quad \quad \text{by Fact 2.5 (i)}
\]

\[
\geq \text{depth}_{R}(R) \quad \quad \text{by the first part}
\]

Thus, \( \text{depth}_{R} = \text{Kdepth}_{R}(R^{\infty}) \), as claimed.

Finally, suppose that \( R \) is Cohen-Macaulay. In view of Fact A) in Theorem 2.6 \( \text{Kdepth}_{R}(R^{\infty}) \leq \dim(R^{\infty}). \) By the first part, we have

\[
\dim(R^{\infty}) = \dim R = \text{depth}_{R} \leq \text{Kdepth}_{R}(R^{\infty}) \leq \dim(R^{\infty}).
\]

The proof is now complete. \( \square \)

Remark 4.12. i) In the above lemma, the \( F \)-injectivity assumption is important. There is a non-Cohen-Macaulay \( F \)-coherent ring \( R \), see [36, Example 3.6]. By [36, Theorem 3.11] \( R^{\infty} \) is big Cohen-Macaulay. Thus, \( \text{depth}_{R} < \text{dim}_{R} = \text{depth}(R^{\infty}) = \text{Kdepth}(R^{\infty}) \).

ii) Let \( R \) be a reduced \( F \)-coherent local ring which is a residue class ring of a Gorenstein local ring. It may worth to recall from [36, Corollary 3.16] that \( F \)-injectivity implies Cohen-Macaulayness and \( F \)-rationality.

We need the following result of Seibert:

Fact 4.13. (See [34, Proposition 2(b)]) Let \( C_{M} \) be the class of all finite \( R \)-modules \( P \) such that the length of \( M \otimes P \) is finite. If \( p. \dim M < \infty \) and \( N \in C_{M} \) then there are certain \( b_{i} \in \mathbb{Q} \) such that

\[
\sum_{i=0}^{\dim R} (-1)^{i} \ell(\text{Tor}^{i}(F^{\ast}(M), N)) = \sum_{i=0}^{\dim N} b_{i}p^{i}\cdot
\]

Also, \( R \) is called \( F \)-finite if \( R \) viewed as an \( R \)-module via \( F \) is finite. For example, every ring which is a localization of an affine algebra over a perfect field and every complete local ring with perfect residue field is \( F \)-finite.

Proposition 4.14. Let \((R, m, k)\) be an \( F \)-finite and \( F \)-coherent domain and \( J \subset R \) be of finite colength. If \( R \) is Cohen-Macaulay, then \( e_{HK}(J, R) \) is rational.
Proof. We know that $\text{p. dim}(R^\infty / JR^\infty) < \infty$. Note that $R^\infty / JR^\infty$ is finitely presented. Since $R^\infty$ is coherent, $R^\infty / JR^\infty$ has finite free resolution by finitely generated free modules:

$$
0 \longrightarrow F_N \longrightarrow \ldots \longrightarrow F_{j+1} \overset{f_{j}}{\longrightarrow} F_j \longrightarrow \ldots \longrightarrow F_0 \longrightarrow R^\infty / JR^\infty \longrightarrow 0,
$$

There is an index $i \in \mathbb{N}$ such that all of components of $\{f_j\}$ are in $(R_i, m_{R_i})$. Let $F_i(i)$ be the free $R_i$-module with the same rank as $F_i$. Consider $f_j$ as a matrix over $R_j$, and denote it by $f_j(i)$. Recall that $m$ is finitely generated. Look at the following complex of finite free modules:

$$
(*) \quad 0 \longrightarrow F_N(i) \longrightarrow \ldots \longrightarrow F_{j+1}(i) \overset{f_{j}(i)}{\longrightarrow} F_j(i) \longrightarrow \ldots \longrightarrow F_0(i) \longrightarrow R_i / JR_i \longrightarrow 0.
$$

Note that $(*)$ is a complex. We are going to show that $(*)$ is exact. Let $r_j$ be the expected rank of $f_j$. Recall that $I_i(f_j(i))$ is the ideal generated by $t \times t$ minors of $f_j(i)$. Clearly, $r_j$ is the expected rank of $f_j(i)$.

The localization of Cohen-Macaulay is again Cohen-Macaulay. For each $p \in \text{Spec}(R^\infty)$, we set $P := p \cap R_i$. Then $j \leq \text{K. grade}_{(R^\infty)}(I_i(f_j), R^\infty)$

$$
= \inf\{\text{Kdepth}_{(R^\infty)}(P) : p \in V(I_i(f_j))\}
$$

Fact 2.5(v)

$$
= \inf\{\text{Kdepth}_{((R_i)_p)^\text{co}}((R_i)_p) : P \in V(I_i(f_j(i)))\}
$$

Fact 2.5(iv)

$$
= \inf\{\text{Kdepth}_{(R_i)_p}(R_i) : P \in V(I_i(f_j(i)))\}
$$

Fact 2.5(iv)

$$
= \text{K. grade}_{R_i}(I_i(f_j(i)), R_i)
$$

Lemma 3.11

Again, by applying Fact 2.5(v), $(*)$ is exact. Thus, $R_i / JR_i$ is of finite projective dimension. By Fact 4.13 $e_{HK}(J, R_i) \in \mathbb{Q}$. Since $R$ is $F$-finite, $R_i$ is finitely generated as a module over $R$. Combine this along with [38] Theorem 2.7] we get that

$$
e_{HK}(J, R) = \frac{e_{HK}(J, R_i)|R_i / m_{R_i} : k|}{[Q(R_i) : Q(R)]} \in \mathbb{Q},
$$

where $Q(\cdot)$ stands for the fraction field of an integral domain. The proof is now complete. \qed

The following extends [36] Proposition 3.19 by Shimomoto where he worked with $R \to R^{\text{hens}}$. This may answer [36] Question 1] where he asked conditions on the fibers of a flat extension to ascend the $F$-coherent property:

**Proposition 4.15.** Let $A$ be a local $F$-coherent and $B$ be weakly étale over $A$. Then $B$ is $F$-coherent.

**Proof.** Recall that $A^\infty \to B \otimes_A A^\infty$ is weakly étale. By definition $A^\infty$ is coherent as an $A^\infty$-module. By [29] Proposition, $B \otimes_A A^\infty$ is coherent as an $B \otimes_A A^\infty$-module. In view of Observation 3.5(xiii), $B^\infty \simeq B \otimes_A A^\infty$. Thus, $B^\infty$ coherent. I.e., $B$ is $F$-coherent. \qed

From Proposition 4.15 one can recover Corollary 3.32.
5. HOMOLOGICAL DIMENSION OVER $R^+$

We start by recalling some historical remarks:

**Discussion 5.1.** Hochster proved that $\text{fl. dim}_{R^+}(R^+/m_{R^+}) \leq \dim(R)$, when $R$ is henselian and has residue prime characteristic, see [22 Proposition 2.15]. Hochster and Aberbach extended this by showing that the flat dimension of any radical of a finitely generated ideal has finite flat dimension, please see [I Theorem 3.1]. Also, recall from [4 Theorem 1.1(i)] that $\text{p. dim}_{R^+}(R^+/m_{R^+}) \leq 2 \dim(R)$.

This lead us to ask:

**Conjecture 5.2.** Let $(R, m)$ be a complete local domain of prime characteristic. Then $R^+$ is regular.

In this section we assume the generalized continuum hypothesis that is $2^{\aleph_0} = \aleph_{n+1}$. We will use this only for computing projective dimension (but not for flat dimension). Is it really needed?

**Lemma 5.3.** Let $(R, m, k)$ be a complete local domain of prime characteristic. If $k$ is countable, then $R^+$ is of $\aleph_1$ cardinality.

**Proof.** By Cohen’s Structure theorem, $R$ is a module-finite extension of a complete regular local ring $A$. It is not difficult to see that $A^+ = R^+$ and that the residue field of $A$ is countable. Then without loss of the generality we may assume that $R$ is complete. Again by Cohen’s Structure theorem $R$ is of the form $R := k[[x_1, \ldots, x_d]]$. Any element of $R$ is a formal power series with coefficient taken from $k$. Thus the cardinality of $R$ is the cardinality of $\prod_{\aleph_0} k$. By $|−|$ we mean the cardinality of a set. Hence, $|R| = \aleph_1$. Denote the fraction field of $R$ by $Q$. Since $Q := \{r/s : r \in R, s \neq 0\}$, we observe that $Q$ is the same cardinality as of $R$. So, $|Q| = \aleph_1$. Let $\overline{Q}$ be the algebraic closure of $Q$. We are going to show that $|\overline{Q}| = \aleph_1$. By definition, $\overline{Q}$ determines by the root of polynomial with coefficient in $Q$. Note that $|Q[X]| = \aleph_1$. From this we deduce that $|\overline{Q}| = \aleph_1$. Since $R^+ \subset \overline{Q}$, we get the claim. □

**Proposition 5.4.** Let $(R, m, k)$ be a complete local domain of prime characteristic. The following holds:

i) Any $R^+$-module has a finite flat dimension. In fact, w. dim $(R^*) \leq 2 \dim(R^*)$.

ii) If $k$ is countable, then $R^+$ is regular. In fact, gl. dim $(R^*) \leq 2 \dim(R^*) + 1$.

**Proof.** i) Let $a$ be an ideal of $R^*$. Recall that $R^* = \bigcup R_\gamma$, where $R_\gamma$ is a module-finite extension of $R$. Without loss of the generality we may assume that $R_\gamma$ is complete and local. Indeed, let $\overline{R_\gamma}$ be the integral closure of $R_\gamma$ in its field of fractions. Recall that the integral closure of a complete local domain in its field of fractions is local and complete. Thus $\overline{R_\gamma}$ is a complete local normal domain. To conclude, it remains to recall that $\overline{R_\gamma} \subseteq R^*$. Write $R^* = \bigcup R_\gamma$, where $R_\gamma$ is a complete local domain of dimension equal to $d := \dim(R)$. Note that $R^* \simeq \lim_{\longleftarrow} R_{\gamma}^\infty$. Let $a$ be an ideal of $R^*$. Set $a_\gamma := a \cap R_\gamma^\infty$. One has $\lim_{\longleftarrow} R_{\gamma}^\infty/a_\gamma \simeq R^*/a$. Recall from [15 VI, Exercise 17] that

$$\lim_{\longleftarrow i} \text{Tor}_{R_{\gamma}^\infty}^{R_{\gamma}^\infty}(R^*/a, −) \simeq \text{Tor}_{R^*}^{R^*}(R^*/a, −).$$
Let $j > 2 \dim(R)$. By applying Fact 5.5 we conclude that $\text{Tor}^R_j(R^*/a, -) = 0$. Since $R$ is local, its dimension is finite. Thus, 
\[ \text{fl. dim}(R^*/a) < 2 \dim(R) + 1 < \infty. \]
Due to (2.1.1) any module has finite flat dimension.

ii) Let $a$ be an ideal of $R^*$. In view of Lemma 5.3, $R^*$ is $\mathfrak{n}_0$-noetherian. Keep part i) in mind. In view of Lemma 2.4, we see that $p. \dim_{R^*}(R^*/a) \leq \text{fl. dim}_{R^*}(R^*/a) + 1$. Thus, 
\[ p. \dim_{R^*}(R^*/a) \leq \text{fl. dim}_{R^*}(R^*/a) + 2 \leq 2 \dim(R) + 1 < \infty. \]
By Auslander’s local-global-theorem (please see [30, Theorem 2.17]):
\[ \text{gl. dim}(R^*) = \sup\{p. \dim(R^*/a) : a \triangleleft R^\infty\} \leq 2 \dim(R) + 1. \]

The above bound may not be sharp:

**Example 5.5.** If $R$ is one-dimensional local and complete, then
\[ \text{gl. dim}(R^*) = p. \dim_{R^*}(R^*/m_{R^*}) = 2 = 2 \dim(R) \]
and $w. \dim(R^*) = \dim(R) = 1$.

**Proof.** The first claim is in [4, Theorem 1.1(ii)] where under this assumption we observed that $R^*$ is a valuation domain. So, $w. \dim(R^*) = \dim(R) = 1$. \(\square\)

Since [4, Lemma 4.1] is true via the henselian assumption, we take this opportunity to state its corrected version:

**Lemma 5.6.** (Artin) Let $(R, \mathfrak{m}, k)$ be a henselian (e.g. complete) local domain of prime characteristic. Then $R^*$ is quasilocal.

The henselian assumption is important. Let us analyze this by the help of an explicit example.

**Example 5.7.** (Epstein) It may be $R$ is local but $R^*$ is not quasilocal.

i) Let $R := \mathbb{Z}(p)$, where $p$ is an odd prime number. Let $S := R[\sqrt[p+1]{1}]$. This ring is integral over $R$. In particular, $\dim(S) = \dim(R) = 1$ and that $R^* = S^*$. Look at $\alpha := \sqrt[p+1]{1} - 1$ and $\beta := \alpha + 2$. Of course $\beta - \alpha = 2$ which is a unit of $R$. Hence $\beta - \alpha$ is a unit of $S$. This implies that $\beta$, and $\alpha$ cannot live together in any prime ideal. Neither $\alpha$ nor $\beta$ is a unit of $S$. Indeed, note that $\alpha \beta = p$. Thus, the inverse of $\alpha$ in the real numbers is $\beta/p = \sqrt[p+1]{1} + (1/p)$, which is clearly not in $S$. Similarly, the inverse of $\beta$ in the real numbers is $\alpha/p$, also not in $S$. Therefore, there must be some prime ideal $p \triangleleft S$ that contains $\alpha$, and a different prime ideal $q$ of $S$ that contains $\beta$. Now, note that $R^* = S^*$. By the lying-over property of integrality, there must be prime ideals $p'$ and $q'$ of $R^*$ that contract to $p$, and $q$ respectively. Both of $p'$ and $q'$ must be maximal ideals of $R^*$, because $\dim(R^*) = 1$. So $R^*$ is not quasilocal.

ii) One may ask an example in the prime characteristic case. Let $R := \mathbb{F}_p[x]_{(x)}$. After replacing $p$ in the first item i) by $x$, it is routine to see that $R^*$ is not quasilocal.

Both of the above examples are normal. If the base ring is normal, the following holds:
Fact 5.8. (See [3, Proposition 1.4]) A normal integral domain $A$ is henselian if and only if $A^+$ is quasilocal.

Corollary 5.9. Let $(R, m, k)$ be a complete local domain of prime characteristic. If $k$ is of cardinality $\aleph_n$ (e.g. $k$ is countable), then any ideal of $R^*$ has finite free resolution (by not necessarily finitely generated free modules).

Proof. In view of Lemma 5.6, $R^*$ is quasilocal. Since $k$ is of cardinality $\aleph_n$, see the proof of Lemma 5.3. Let $a$ be an ideal of $R^*$. By the proof of Proposition 5.4, $p.\dim_{R^*}(R^*/a) \leq \dim_{R^*}(R^*/a) + (n + 2) \leq 2 \dim(R) + (n + 2) < \infty$.

By a famous result of Kaplansky, any projective module over a quasilocal ring is free. The proof is now complete. □

6. Failure of coherence $R^*$ in mixed characteristic

Our aim is to understand the higher-dimensional version of the following observation.

Observation 6.1. Let $(R, m, k)$ be a complete regular local ring. If $\dim(R) = 1$, then $R^*$ is a filtered colimit of finitely presented flat $R$-algebras. The transition maps are flat. In particular, $R^*$ is coherent.

Proof. Recall that $R^* = \bigcup R_{\gamma}$, where $R_{\gamma}$ is a module-finite extension of $R$. Without loss of the generality we may assume that $R_{\gamma}$ is normal and local. In particular, $R_{\gamma}$ is DVR. Since torsion-free modules over a DVR are flat, we get the first claim. The particular case follows by Fact 2.2.

Lemma 6.2. (See [26, Theorem 23.1]) Let $\varphi$ be a local map from a regular local ring $(A, m)$ to a Cohen-Macaulay local ring $(B, n)$. Suppose $\dim(A) + \dim(B/mB) = \dim(B)$. Then $\varphi$ is flat.

Proposition 6.3. Let $(R, m, k)$ be a complete regular local ring. If $\dim(R) = 2$, then $R^*$ is a filtered colimit of finitely presented flat $R$-subalgebras. The transition maps are not flat provided $k$ is of positive transcendence degree over $\mathbb{F}_p$.

Proof. Recall that $R^* = \bigcup R_{\gamma}$, where $R_{\gamma}$ is a module-finite extension of $R$. Without loss of the generality we may assume that $R_{\gamma}$ is complete and local. Indeed, let $\overline{R_{\gamma}}$ be the integral closure of $R_{\gamma}$ in its field of fractions. Recall that the integral closure of a complete local domain in its field of fractions is local and complete. Thus $\overline{R_{\gamma}}$ is a complete local normal domain. For simplicity, $R_{\gamma} := \overline{R_{\gamma}}$. We are going to show that $(R, m) \rightarrow (R_{\gamma}, m_{\gamma})$ is flat. Since $R_{\gamma}$ is normal and 2-dimensional, Serre’s characterization of normality implies that $R_{\gamma}$ is Cohen-Macaulay. Note that $R$ is regular, $R_{\gamma}$ is Cohen-Macaulay and that $\dim(R_{\gamma}) = \dim(R) + \dim(\frac{R_{\gamma}}{mR_{\gamma}})$.

In view of Lemma 6.2, $R \rightarrow R_{\gamma}$ is flat. This finishes the proof of first claim.

Suppose $t := \deg(k/\mathbb{F}_p) > 0$ and suppose on the contrary that $R_{\gamma} \rightarrow R_{\beta}$ is flat for all $\gamma < \beta$. Then $R^*$ is a flat filtered limit of noetherian rings. In the light of Fact 2.2, this implies that $R^*$
is coherent. It is shown in [11, Theorem 4.9] that $R^*$ is not coherent (here, we need $t > 0$). This contradiction shows that $R_\gamma \to R_\beta$ is not flat for all cofinal pair $\gamma < \beta$. \hfill \Box

**Corollary 6.4.** Let $(R, m, k)$ be a complete regular local ring. If $\dim(R) = 2$ and of prime characteristic, then $R^*$ is a filtered colimit of finitely presented Cohen-Macaulay $R$-subalgebras. Also, $R^*$ is not a filtered colimit of its regular local subalgebras over $R$ provided $\text{tr. deg}(k/F_p) > 0$.

**Corollary 6.5.** Let $(R, m, k)$ be a complete regular local ring. Then $R^*$ is a filtered colimit of finitely presented Cohen-Macaulay $R$-subalgebras if and only if $R^*$ is filtered colimit of finitely presented flat $R$-subalgebras.

**Proof.** This follows by Lemma 6.2. \hfill \Box

**Theorem 6.6.** Let $R$ be a complete domain having mixed characteristic $p$. If $\dim(R) > 3$ then $R^*$ is not coherent.

What can say when $\dim(R) = 3$?

**Proof.** Let $d := \dim(R)$ and $p := \text{char}(R/m) > 0$. In the light of [2] the direct summand conjecture is now a beautiful theorem. By Fact 1.2 we know that $\text{fl. dim}(R^*/m_{R^*}) = d$. Suppose on the contradiction that $R^*$ is coherent. Let $M$ be a finitely generated $R^*$-module. In view of [18, Corollary 2.5.10] $\dim_{R^*}(M) \leq n$ if and only if $\text{Tor}_{n+1}^{R^*}(M, R^*/m_{R^*}) = 0$. From this we observe that $w. \dim(R^*) = d < \infty$. In view of Theorem 2.6

$$\dim(R) = w. \dim(R^*) = \text{Kdepth}(R^*).$$

Let $\mathbf{x} := p, x_2, \ldots, x_d$ be a system of parameters for $R$. Note that $\text{rad}(\mathbf{x}) = m_{R^*}$. By Corollary 6.9 $\mathbf{x}$ is a regular sequence over $R^*$. Recall from [1, Proposition 3.6] that $R^*$ is not a balanced big Cohen-Macaulay algebra for $R$ (here, we need $d > 3$). This contradiction shows that $R^*$ is not coherent. \hfill \Box

Also, the following left unsolvable:

**Question 6.7.** Suppose $k$ is the algebraic closure of $F_p$. Let $(R, m, k)$ be a 2-dimensional complete normal domain. Is $R^*$ coherent? Specially: Is $I^* = I^+$? (The first one is tight closure and the second one is plus closure. Also, the first question implies the second.)

Brenner proved that $I^* = I^+$ where $R$ is a 2-dimensional standard graded ring over algebraic closure of $F_p$ and for homogeneous ideal $I$. This uses the theory of vector bundles, see [13]. His assumption over $R_0$ is really important.

7. $\text{gl. dim}(R^\infty)$ depends on the characteristic

Let $R$ be one-dimensional local and complete. Then $\text{gl. dim } R^* = 2$ and $w. \dim R^* = 1$ without any regards with respect to the characteristic. Here we show $\text{gl. dim}(R^\infty)$ and $w. \dim(R^\infty)$ depend to the characteristic.

**Lemma 7.1.** Let $p \neq 2$ be a prime integer. Let $R := F_p[x, y]/(y^2 - x^3 - x^2)$. The following holds:

*If the answer is positive, then one can show that $R^*$ is balanced big Cohen-Macaulay. The last one is an open question.*
Proof. First we claim that \( R^\infty \) is not coherent. We follow some lines from \([21]\). Note that \( R \simeq \mathbb{F}_p[t, t\sqrt{T+1}] \). The normalization of \( R \) in its field of fractions is \( \mathbb{F}_p[t,\sqrt{T+1}] \). Suppose on the contrary that \( R^\infty \) is coherent. Over 1-dimensional reduced rings, this is equivalent with the property that \( \overline{R}/R \) is purely inseparable, see \([36]\) Corollary 3.9. Since \( (\sqrt{T+1})^{pn} \notin R \) for each \( n \), the extension \( \overline{R}/R \) is not purely inseparable. This contradiction shows that \( R^\infty \) is not coherent.

i) Recall from \([18]\) Theorem 6.3.4] that a domain of global dimension less than 3 is coherent. By the first paragraph, \( R^\infty \) is not coherent. So, \( \text{gl. dim}(R^\infty) \geq 3 \). Therefore, \( \text{gl. dim}(R^\infty) = 3 \).

ii) By part i) we have \( \text{gl. dim}(R^\infty) = 3 \). Recall from Lemma 2.1 that

\[
3 = \text{gl. dim}(R^\infty) \leq 1 + \text{w. dim}(R^\infty).
\]

So, \( \text{w. dim}(R^\infty) \geq 2 \). Due to \([12]\) Remark 11.33 \( \text{w. dim}(R^\infty) \leq 2 \dim R = 2 \). Thus \( 2 \leq \text{w. dim}(R^\infty) \leq 2 \) as claimed. \( \square \)

Lemma 7.2. Look at the integral domain \( R := \mathbb{F}_2[t, t\sqrt{T+1}] \). Then \( \text{gl. dim}(R^\infty) = 2 \) and \( \text{w. dim}(R^\infty) = 1 \).

Proof. Look at the regular ring \( A := \mathbb{F}_2[t] \). It is easy to see that \( R^\infty = A^\infty \). Since \( A \) is regular and of dimension one, it is easy to observe that \( \text{gl. dim}(R^\infty) = 2 \) and \( \text{w. dim}(R^\infty) = 1 \). \( \square \)

Example 7.3. Let \( R := \mathbb{F}_p[t, t\sqrt{T+1}] \). Then

\[
\text{gl. dim}(R^\infty) = \begin{cases} 
3 & \text{if } p \neq 2 \\
2 & \text{if } p = 2
\end{cases}
\]

Also,

\[
\text{w. dim}(R^\infty) = \begin{cases} 
2 & \text{if } p \neq 2 \\
1 & \text{if } p = 2
\end{cases}
\]

Proof. This is the combination of the above lemmas. \( \square \)

8. Revisiting the Miraculous Formula

We give a simple proof of the following funny fact (see \([12]\) Lemma 3.16]).

Fact 8.1. (Bhatt and Scholze) Let \( B \leftarrow A \rightarrow C \) be a diagram of perfect rings of prime characteristic. Then \( \text{Tor}_i^A(B, C) = 0 \) for all \( i \geq 1 \).

Proof. Let \( \Gamma \) be a generating set of \( B \) as an \( A \)-algebra. Look at perfection of the polynomial ring \( D := (A[x : x \in \Gamma])^\infty \). In particular, the map \( A \to B \) may view as \( A \to D \to B \). There ia an spectral sequence

\[
\text{Tor}_i^D(\text{Tor}_j^A(C, D), B) \Rightarrow \text{Tor}_{i+j}^A(B, C).
\]

Since \( D \) is free as an \( A \)-algebra, the spectral sequence collapses. Set \( E := C \otimes_A D \) which is a perfect algebra by Observation \([355]\). Thus, \( \text{Tor}_i^D(E, B) \simeq \text{Tor}_i^A(B, C) \). We apply the replacement \( E \mapsto C \) and \( D \mapsto A \). Hence, we may assume that \( A \to B \) is surjective. Perfect rings are reduced.
So, $B \simeq A/a$ for some radical ideal $a$. Write $A = \bigcup R_T$, where $R_T$ is a noetherian subring of $A$. Taking perfection, we have $A \simeq \lim\Gamma R_T^\infty$. Since Tor modules behave nicely with respect to direct limits we may assume that $A = R^\infty$ for some noetherian ring $R$. Any radical ideal $a$ of $R^\infty$ is of the form $\sum_{i=1}^m (x_i^m)$ for some $m \in \mathbb{N}$ where $(x^\infty) := (x^{1/p^n} : n \in \mathbb{N})$. Similarly, $C = A/b$ for some radical ideal $b$.

First, we deal with the case that $m = 1$. Clearly, $\text{fl.dim}(\frac{R^\infty}{(x^\infty)}) \leq 1$. Hence, the only crucial Tor$_i$ is Tor$_1$:

$$\text{Tor}_1^{R^\infty}(\frac{R^\infty}{(x^\infty)}, R^\infty/b) \simeq \frac{(x^\infty) \cap b}{(x^\infty)_b}$$

which is zero by Observation 3.5(v). Set $c := \sum_{i=2}^m 2(x_i^{1/p^n} : n \in \mathbb{N})$. By induction we have

$$\text{Tor}_i^{R^\infty}(\frac{R^\infty}{c}, R^\infty/b) = 0 \quad \forall i > 0 \quad (*)$$

There is a change of rings $R^\infty \to \frac{R^\infty}{c} =: S$. Note that $S$ is the perfect closure of $R/(c \cap R)$. Look at the $S$-module $R^\infty/a$ and $R^\infty$-module $R^\infty/b$, there is an spectral sequence

$$\text{Tor}^{R^\infty}_i(\frac{R^\infty}{c}, \frac{R^\infty}{a}) \Rightarrow \text{Tor}^{R^\infty}_{i+j}(\frac{R^\infty}{b}, R^\infty/a).$$

By $(*)$ the spectral sequence collapses. Combine this along with the case $m = 1$ and the natural isomorphism $S/(x_1^\infty) \simeq R^\infty/a$, we have

$$\text{Tor}^{R^\infty}_{i+j}(\frac{R^\infty}{a}, \frac{R^\infty}{b}) \simeq \text{Tor}^{R^\infty}_{i+j}(\text{Tor}^{R^\infty}_0(\frac{R^\infty}{b}, \frac{R^\infty}{c}), \frac{R^\infty}{a}) \simeq \text{Tor}^{R^\infty}_j\left(\frac{S}{b}, S/(x_1^\infty)\right) = 0.$$

The proof is now complete.

Wodzicki has constructed an example of a ring $A$ such that $\text{Jac}(A) \neq 0$ and $\text{Tor}_1^{A^+}(A+/a, A+/b) = 0$. The following extends [4, Remark 4.6].

**Corollary 8.2.** Let $A$ be a domain of prime characteristic. Let $a$ and $b$ be radical ideals of $A^+$. Then $\text{Tor}_{i+1}^{A^+}(A^+/a, A^+/b) = 0$ for all $i \geq 1$.

**Proof.** This is combination of Observation 3.5(xiii) with the miraculous vanishing formula.

This is not true for any ideals:

**Example 8.3.** Let $A$ be a noetherian henselian local domain and let $I$ be a finitely generated proper and nonzero ideal of $A^+$. Then $\text{Tor}_1^{A^+}(A^+/I, A^+/I) \neq 0$.

**Proof.** Recall that $\text{Tor}_{i+1}^{A^+}(A^+/I, A^+/I) \simeq I/I^2$. We need to show $I \neq I^2$. In view of Lemma 5.6, $A^+$ is quasilocal. The claim $I \neq I^2$ follows by Nakayama’s lemma.

However, there is a weak version of rigidity of Tor:

**Example 8.4.** Let $R$ be a noetherian complete local regular ring of prime characteristic and let $I$ and $J$ be ideals of $R$. If $\text{Tor}_j^R(\frac{R^+}{R^+ I}, \frac{R^+}{R^+ J}) = 0$, then $\text{Tor}_j^R(\frac{R^+}{R^+ I}, \frac{R^+}{R^+ J}) = 0$ for all $j > i$.

**Proof.** By [23, 6.7], $R^+$ is flat over $R$. Thus $\text{Tor}_j^R(\frac{R^+}{R^+ I}, \frac{R^+}{R^+ J}) \simeq \text{Tor}_j^R(R/I, R/J) \otimes_R R^+$. The extension $R \to R^+$ is faithful. Hence, $\text{Tor}_j^R(R/I, R/J) = 0$. By the rigidity of Tor-modules over $R$, $\text{Tor}_j^R(R/I, R/J) = 0$ for all $j > i$ (see [4]). So, $\text{Tor}_j^R(\frac{R^+}{R^+ I}, \frac{R^+}{R^+ J}) \simeq \text{Tor}_j^R(R/I, R/J) \otimes_R R^+ = 0$ for all $j > i$. 

□
One may prove the following by the straightforward arguments:

**Corollary 8.5.** Let $R$ be a noetherian regular local ring of prime characteristic. Let $A$ be any perfect algebra over $R^\infty$. If $A$ is finitely presented over $R^\infty$, then $A$ is free.

**Proof.** In the light of the miraculous vanishing formula we see that $\text{Tor}_{>0}^A(R^\infty/m_{R^\infty}, A) = 0$. The claim follows from the following fact:

**Fact A:** (See [13 Theorem 3.1.2]) Let $B$ be a coherent ring and let $J \subseteq \text{Jac}(B)$. If $M$ is a finitely presented $B$-module such that $\text{Tor}_0^B(B/J, M) = 0$, then $\text{p. dim}_B(M) = \text{p. dim}_{B/J}(M/JM)$.

Let $B \leftarrow A \rightarrow C$ be a surjective diagram of perfect rings of prime characteristic. Let us compute $\text{Ext}_{A}^{0}(B, C)$ via an example.

**Example 8.6.** Let $R$ be a 1-dimensional complete local domain of prime characteristic and let $A := R^\ast$. Look at the diagram $R^\ast/m_{R^\ast} =: B \leftarrow A \rightarrow C := R^\ast/m_{R^\ast}$ of perfect rings. Then $\text{Ext}_{A}^{0}(B, C) = 0$ for all $i > 0$.

**Proof.** It is not difficult to see that $\text{p. dim}(R^\ast/m_{R^\ast}) = 2 > 1 = \text{id}(R^\ast/m_{R^\ast})$, see [4 Example 8.2(ii)]. So, the only challenging $\text{Ext}^i$ is $\text{Ext}^1$. We will use the fact that $(A, m_A)$ is a valuation ring. In particular, any finitely generated ideal of $A$ is principal. Let $I = xA$ be any finitely generated ideal of $A$. Then

$$0 \rightarrow A \xrightarrow{x} A \rightarrow A/I \rightarrow 0$$

is a free resolution of $A/I$. By applying $\text{Hom}(-, A/m_A)$ to it we get the exact sequence

$$\text{Hom}_A(A, A/m_A) \xrightarrow{x} \text{Hom}_A(A, A/m_A) \rightarrow \text{Ext}_{A}^{1}(A/I, A/m_A) \rightarrow \text{Ext}_{A}^{1}(A, A/m_A) = 0.$$

Note that $\text{coker}(A/m_A \xrightarrow{x} A/m_A) = A/m_A$ provided $x \in m_A$. Thus,

$$\text{Ext}_{A}^{1}(A/I, A/m_A) = \text{coker} \left( \text{Hom}_A(A, A/m_A) \xrightarrow{x} \text{Hom}_A(A, A/m_A) \right) = A/m_A \quad (*)$$

Let $I = xA$ and $J = yA$ be finitely generated ideals of $A$. Without loss of the generality we may assume that $t = y/x \in A$, since $A$ is a valuation domain. The natural map $A/I \rightarrow A/J \rightarrow 0$ induces the map $\text{Ext}_{A}^{1}(A/I, A/m_A) \rightarrow \text{Ext}_{A}^{1}(A/I, A/m_A)$, a multiplication by $t$. If $I \neq J$ then $t$ is not invertible. So, $t \in m_A$. It turns out that the maps in the inverse system

$$A/m_A \leftarrow A/m_A \leftarrow \cdots$$

are the zero maps. Note that $A/m_A \simeq \varprojlim A/I_i$ where $I_i$ is finitely generated. By [21 Corollary 2.4] $\text{Ext}_{A}^{1}(A/m_A, A/m_A) \simeq \varprojlim \text{Ext}_{A}^{1}(A/I_i, A/m_A)$. We apply this along with $(*)$ to observe that

$$\text{Ext}_{A}^{1}(A/m_A, A/m_A) \simeq \text{Ext}_{A}^{1}(A/m_A, A/m_A) \simeq \varprojlim \text{Ext}_{A}^{1}(A/m_A, A/m_A) \simeq \varprojlim A/m_A \simeq 0.$$

**Example 8.7.** Let $R$ be a 1-dimensional complete local domain of residue prime characteristic and let $A := R^\infty$. Look at the diagram $R^\infty/m_{R^\infty} =: B \leftarrow A \rightarrow C := R^\infty$ of perfect rings. Then $\text{Ext}_{A}^{1}(B, C) = 0$.

*Vanishing of the first tor is enough (see [15 Corollary 2.5.10]). Despite of this, we would like to follow this proof.*
Proof. Let $x \in R$ be the uniformizing element. Recall that a free resolution of $B$ over $A$ is given by the following:

$$0 \rightarrow \bigoplus_{N} R^{\infty} \xrightarrow{X} \bigoplus_{N} R^{\infty} \xrightarrow{Y} R^{\infty} \rightarrow R^{\infty}/m_{R^{\infty}} \rightarrow 0 \quad (\ast)$$

where the matrixes $X$ and $Y$ are defined by:

$$X := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
-x^{-1} & 1 & 0 & 0 & 0 & \cdots \\
0 & -x^{-1} & 1 & 0 & 0 & \cdots \\
0 & 0 & -x^{-1} & 1 & 0 & \cdots \\
0 & 0 & 0 & -x^{-1} & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

and

$$Y := \left( x^{1/p} \ x^{1/p^2} \ \cdots \right)^t.$$

Note that

$$\text{Hom}_{R^{\infty}}\left( \bigoplus_{N} R^{\infty}, R^{\infty} \right) \cong \prod_{N} \text{Hom}_{R^{\infty}}(R^{\infty}, R^{\infty}) \cong \prod_{N} R^{\infty}$$

Anther this identification and after applying $\text{Hom}_{R^{\infty}}(-, R^{\infty})$ to $(\ast)$ we have

$$0 \rightarrow R^{\infty} \xrightarrow{a} \prod_{N} R^{\infty} \xrightarrow{\beta} \prod_{N} R^{\infty} \rightarrow 0 \quad (\ast \ast)$$

where $a$ is the assignment via $a \mapsto (ax^{1/p^n})_{n \in \mathbb{N}_0}$ and $\beta$ assigns the sequence $(a_n)_{n \in \mathbb{N}}$ to the

$$\beta(a_n) := (a_1 - x^{-1/p} a_2, \ldots, a_n - x^{-1/p} a_{n+1}, \ldots).$$

The homology of $(\ast \ast)$ in middle is $\text{Ext}^1_A(B, C)$. Let $(a_n) \in \ker(\beta)$. Then $0 = \beta(a_n)$. One may read as $a_n = x^{1/p^n} a_{n+1}$ for all $n$. Define $a(n) := \sum_{i=1}^{n} \frac{p}{i} a_i$. Iterate this inductively, we have $a_1 = x^{a(n)} a_n$. There is a valuation map $v : A \rightarrow \mathbb{Q}$ such that if $v(a) \geq 1$ then $x | a$. Indeed, $R := V$ is a DVR. Let $v$ be a value map of $V$. Let $r \in R^{\infty}$. Then $r^{p^n} \in R$ for some $n$. The assignment $r \mapsto v(r^{p^n}) / p^n$ defines a normalized value map on $R^{\infty}$. Since $\lim_{n \rightarrow \infty} a(n) = 1$ we observe that $v(a_1) \geq 1$. So, $x | a_1$. Set $a := \frac{a_1}{x} \in A$. Then $a(a) = (a_n)$. From this we observe $\text{im}(a) = \ker(\beta)$. Therefore, $\text{Ext}^1_A(B, C) = 0$.

We close this section by extending a result of Bhatt and Scholze [12] Proposition 11.29. To this end, we recall the following trick of Auslander [5] Proposition 3].

Lemma 8.8. Let $A$ be a ring and let $\Gamma$ be a well-ordered set. Suppose that $\{N_\gamma : \gamma \in \Gamma\}$ is a collection of submodules of an $A$-module $M$ such that $\gamma' \leq \gamma$ implies $N_{\gamma'} \subseteq N_{\gamma}$ and $M = \bigcup_{\gamma \in \Gamma} N_{\gamma}$. Suppose that $p. \text{dim}_A(N_\gamma / \bigcup_{\gamma' < \gamma} N_{\gamma'}) \leq n$ for all $\gamma \in \Gamma$. Then $p. \text{dim}_A(M) \leq n$.

Proposition 8.9. Let $R \rightarrow S$ be a perfectly finitely presented map of perfect $\mathbb{F}_p$-algebras. Then $S$ is of finite projective dimension over $R$. 
Proof. We will show that if $S$ is the perfection of $\frac{R[X_{1}, \ldots, X_{n}]}{(f_{1}, \ldots, f_{n})}$, then $\text{p. dim}_{R}(S) \leq 2m$. Set $T := (R[X])^{\infty}$ and let $M$ be any $R$-module. Also, set $(f_{\infty}) := \sum_{i=1}^{m}(f_{i})^{\infty}$. We look at the base change spectral sequence

$$\text{Ext}_{T}^{i}(T/(f_{\infty}), \text{Ext}_{R}^{j}(T, M)) \Rightarrow \text{Ext}_{R}^{i+j}(T/(f_{\infty}), M).$$

Since $T$ is free over $R$ the spectral sequence collapses and so

$$\text{Ext}_{R}^{m}(T/(f_{\infty}), \text{Hom}_{R}(T, M)) \simeq \text{Ext}_{R}^{m}(T/(f_{\infty}), M).$$

If we show $\text{Ext}_{T}^{i}(T/(f_{\infty}), –) = 0$ for all $i > 2m$, then we get the claim. After replacing $R$ with $T$ we may assume that $n = 0$. We do induction by $m$. First, we assume that $m = 1$. By Lemma 8.2 $\text{p. dim}_{R}(R/(f_{\infty})) \leq 2$. Let $Q^{1}$ be the corresponding free resolution of length two. By using induction on $m$, we will show that $\bigotimes_{i=1}^{m} Q^{1}$ is a free resolution of $R/(x_{i})$ of length at most $2m$. Set $(g_{\infty}) := \sum_{i=2}^{m}(f_{i})$ By the induction hypothesis, $P := \bigotimes_{i=2}^{m} Q^{1}$ is a free resolution of $R/(x_{i})$ of length at most $2m - 2$. Recall that

$$H_{n}(Q^{1} \otimes_{R} P) \simeq \text{Tor}_{n}^{R}(R/(f_{\infty}), \frac{R}{(g_{\infty})}).$$

The ideal $(f_{\infty})$ is flat. Hence, for each $n > 1$ we have $\text{Tor}_{n}^{R}(R/(f_{\infty}), \frac{R}{(g_{\infty})}) = 0$. Also,

i) $\text{Tor}_{1}^{R}(R/(f_{\infty}), \frac{R}{(g_{\infty})}) \simeq \frac{(f_{\infty}) \cap (g_{\infty})}{(1/1) \cap (g_{\infty})} \simeq 0$,

ii) $H_{0}(Q^{1} \otimes_{R} P) \simeq R/(f_{\infty}) \otimes_{R} \frac{R}{(g_{\infty})} \simeq R/(g_{\infty}).$

Thus, $\bigotimes_{i=1}^{m} Q^{1}$ presents a free resolution of $R/(f_{\infty})$ of length $2m$. \[\square\]

9. A GLIMPSE THROUGH MIXED-CHARACTERISTIC RINGS

In his seminal paper Witt proved that the category of perfect quasilocal rings of prime characteristic $p$ is the same as of the category of $p$ torsion-free, $p$-adically complete semiperfect quasilocal rings of mixed characteristic $p$. This result extended by many mathematician. Please see Scholze’s thesis. Shimomoto asked:

Question 9.1. (See [37, Question 2]) Let $A$ be a coherent perfect $\mathbb{F}_{p}$-algebra. Is $W(A)$ coherent?

And he said: “This question seems a bit subtle, because it is known that a power series ring over a valuation ring of Krull dimension greater than one is not coherent.” In fact, by using a $p$-adic modification of the quotation, we answer the question.

Observation 9.2. Let $A$ be a perfect domain of prime characteristic. Then its fraction field $F$ is a perfect field. Conversely, let $F$ be a perfect field of characteristic $p$ equipped with a valuation $v$ and value ring $A$. Then $A$ is a perfect ring.

Proof. For the first assertion, let $x/y \in F$ be nonzero. The elements $x^{1/p^{n}}$ and $y^{1/p^{n}}$ are in $A$. So, $x^{1/p^{n}}/y^{1/p^{n}} \in F$. Its $p^{n}$-power is $x/y$. Thus $F$ is perfect. Conversely, let $F$ be a perfect field of characteristic $p$ equipped with a valuation $v$ and value ring $A$. Note that $A$ consists of elements of nonzero value. Let $a \in A$ be nonzero. Then $a^{1/p^{n}} \in F$, because $F$ is perfect. Since $v(a^{1/p^{n}}) = 1/p^{n}v(a) \geq 0$, we get that $a^{1/p^{n}} \in A$ for all $n > 0$. This shows that $A$ is perfect. \[\square\]
Example 9.3. Let $R$ be any 1-dimensional local ring of prime characteristic. Then $W(R^\infty)$ is not coherent without any regard with respect to coherent property of $R^\infty$.

Proof. Suppose first that $R^\infty$ is coherent. By applying Corollary 3.14 $R^\infty$ is a valuation ring. This is perfect and coherent. By Observation 9.2 $Q(R^\infty)$ is a perfect field. Its value group is the set of rational numbers whose denominator is a $p$-power. In particular, the value group is an strict subgroup of $\mathbb{R}$. Let us follow Kedlaya: In the light of [24 Theorem 1.2] we see that $W(R^\infty)$ is not coherent.

Finally, suppose that $R^\infty$ is not coherent. Suppose on the contradiction that $W(R^\infty)$ is coherent. By Observation 9.2 $Q(R^\infty)$ is a perfect field. Its value group is the set of rational numbers whose denominator is a $p$-power. In particular, the value group is an strict subgroup of $\mathbb{R}$. Let us follow Kedlaya: In the light of [24 Theorem 1.2] we see that $W(R^\infty)$ is not coherent.

Final Fact 9.4. (Auslander-Buchsbaum, P. Kohn, Vasconcelos; see [39 Theorem 5.1]) Let $A \to B$ be a ring homomorphism such that $d := p.\dim_{A}(B) < \infty$ and there is an exact complex $0 \to P_{d} \to \ldots \to P_{0} \to B \to 0$ of finitely generated projective $A$-modules such that $\text{Ext}^{i}_{A}(B, A) = 0$ for all $i < d$. Then for any $B$-module $M$ with $p.\dim_{B}(M) < \infty$ we have $p.\dim_{A}(M) = p.\dim_{B}(M) + p.\dim_{A}(B)$.

The following result extends [12 Lemma 7.8] by Bhatt and Scholze:

Corollary 9.5. Let $R$ be a perfect $\mathbb{F}_{p}$-algebra, and let $Q$ be (not necessarily finitely generated and not necessarily projective) $R$-module of finite projective dimension. Then $p.\dim_{W(R)}(Q) = p.\dim_{R}(Q) + 1$.

Proof. There is an exact sequence $0 \to W(R) \overset{p}{\to} W(R) \to W(R)/pW(R) \to 0$. This implies that $p.\dim_{W(R)}(\frac{W(R)}{pW(R)}) = 1$ and that $\text{Hom}_{W(R)}(\frac{W(R)}{pW(R)}, W(R)) = 0$. In particular, we are in the situation of Fact 9.4. By applying Fact 9.4 we see $p.\dim_{W(R)}(Q) = p.\dim_{R}(Q) + 1$, as claimed.

Let $A$ be a commutative ring and let $p$ be a non-unit prime in $A$. By the Fontaine ring of $A$, we mean

$$\mathbb{E}(A) := \varprojlim (\ldots \xrightarrow{F} A/pA \xrightarrow{F} A/pA).$$

If $\text{nil}(A)^{n} = 0$ for some $n \in \mathbb{N}$, then $\mathbb{E}(A) = \mathbb{E}(A_{\text{red}})$. Also, $\mathbb{E}(A)$ is perfect: the $p^{th}$ root of $(r_{n})$ is $(s_{n})$, where $s_{n} := r_{n+1}$.

Observation 9.6. Here, we show $w.\dim(\mathbb{E}(A))$ has properties both similar to and different from those of $w.\dim(A^{\infty})$ via some examples.

i) (Witt) We look at $A := \mathbb{Z}_{p}$ the ring of $p$-adic integers. By Fermat’s little theorem, the Frobenius map is identity over $\mathbb{F}_{p}$. Then $\mathbb{E}(A) = \varprojlim (\ldots \xrightarrow{\sim} \mathbb{F}_{p} \xrightarrow{\sim} \mathbb{F}_{p}) \simeq \mathbb{F}_{p}$. Thus,

$$w.\dim(\mathbb{E}(A)) = 0 < 1 = w.\dim(A).$$

ii) (This extends [33 Lemma 3.4(iv)] by the same proof) If $A$ is a perfect reduced ring of characteristic $p$, then $\mathbb{E}(A) = \varprojlim (\ldots \xrightarrow{\sim} A \xrightarrow{\sim} A) \simeq A$. Thus,

$$w.\dim(\mathbb{E}(A)) = w.\dim(A).$$
iii) Let us give an example such that \( \text{w. dim}(E(A)) = 0 < n = \text{w. dim}(A) \), where \( n \in \mathbb{N} \cup \{\infty\} \). To this end, let \((A, m, k)\) be a noetherian complete local ring (not necessarily regular) with perfect residue field of characteristic \( p \). Then

\[
E(A) = E(\lim_{i} A/m^i) \simeq \lim_{i} E(A/m^i) \simeq \lim_{i} E((A/m^i)_{\text{red}}) \simeq \lim_{i} E(A/m) \overset{(ii)}{\simeq} \lim_{i} k \simeq k.
\]

Note that weak dimension of \( A \) can be any thing. From this we get the claim.

iv) Let us give an example such that \( \text{w. dim}(E(A)) = 1 < \infty = \text{w. dim}(A) \). To this end, let \( A := \frac{\mathbb{F}_p[[X]]^\infty}{(X)} \). By Corollary 3.15, \( \mathbb{F}_p[[X]]^\infty \) is coherent. In view of [18, Theorem 2.4.1(1)], \( A \) is coherent. If \( A \) were be of finite weak dimension it should be reduced. But \( A \) is not reduced, because \((x^{1/p})^p = x = 0\). Thus \( \text{w. dim}(A) = \infty \). The following completion is the \((X)\)-adic. By definition,

\[
E(A) = \lim_{i} \left( \frac{\mathbb{F}_p[[X]]^\infty}{(X)} \right) \simeq \lim_{i} \left( \frac{\mathbb{F}_p[[X]]^\infty}{(X^{p^{n+1}})} \right) \simeq \left( \mathbb{F}_p[[X]]^\infty \right).
\]

Note that such a completion of a valuation domain is again valuation domain. Thus,

\[
\text{w. dim}(E(A)) = 1 < \infty = \text{w. dim}(A).
\]

v) By a result of Gabber and Ramero [16], if \( A \) is a valuation domain of mixed characteristic then \( E(A) \) is a valuation domain. In particular, \( \text{w. dim}(E(A)) \leq \text{w. dim}(A) = 1 \).

When is \( \text{w. dim}(E(A)) < \infty \)? Of course, this is not true in general:

**Example 9.7.** Let \((R, m_R)\) be the localization of \( \mathbb{F}_p[[X_1, \ldots]] \) at \((X_1, \ldots)\). Let \( A := R^\infty \). In view of Example 3.15 \( \text{w. dim}(A) = \infty \). By Observation 2.3, \( E(A) \simeq A \). Thus, \( \text{w. dim}(E(A)) = \infty \). Also, \( \text{w. dim}(W(E(A))) = \infty \).

**Question 9.8.** (Shimomoto) How can determine \( \text{w. dim}(E(A)) \) in terms of \( \text{w. dim}(A) \)?

Let \( A \) be a mixed characteristic valuation domain. Recall that \( E(A) \) is a valuation ring. So, its weak dimension is one. In view of the following formula

\[
W(E(A))/pW(E(A)) \simeq E(A),
\]

and by applying Fact 2.4, \( \text{w. dim}(W(E(A))) \geq 2 \). Let \( [-]: E(A) \to W(E(A)) \) be the Teichmüller mapping. Let \( x \in E(A) \) be such that its radical is the maximal ideal. By the natural isomorphism we get that \( p, [x] \) is a regular sequence on \( W(R^\infty) \) and that \( \text{rad}(p, [x]) \) is the maximal ideal of \( W(E(A)) \). From this and Fact 2.5 we get that \( \text{Kdepth}(W(E(A))) = 2 \). We have no data about of its Krull dimension (resp. its prime spectrum).

**Acknowledgement.** I would like to thank everyone who help me running this project. I thank Shimomoto for a number of valuable comments and encouragement. Also, Epstein shared his interesting example with us.
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