Geometric Flows and Perelman’s Thermodynamics for Black Ellipsoids in $R^2$ and Einstein Gravity Theories

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Abstract

We study geometric relativistic flow and Ricci soliton equations which (for respective nonholonomic constraints and self-similarity conditions) are equivalent to the gravitational field equations of $R^2$ gravity and/or to the Einstein equations with scalar field in general relativity, GR. Perelman’s functionals are generalized for modified gravity theories, MGTs, which allows to formulate an analogous statistical thermodynamics for geometric flows and Ricci solitons. There are constructed and analyzed generic off–diagonal black ellipsoid, black hole and solitonic exact solutions in MGTs and GR encoding geometric flow evolution scenarios and nonlinear parametric interactions. Such new classes of solutions in MGTs can be with polarized and/or running constants, nonholonomically deformed horizons and/or imbedded self-consistently into solitonic backgrounds. They exist also in GR as generic off–diagonal vacuum configurations with effective cosmological constant and/or mimicking effective scalar field interactions. Finally, we compute Perelman’s energy and entropy for black ellipsoids and evolution solitons in $R^2$ gravity.

Keywords: Relativistic geometric flows, Ricci solitons and modified gravity, off-diagonal exact solutions, black ellipsoids/ holes, Perelman’s thermodynamics of gravitational fields.

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1 Introduction

The Ricci flow theory was formulated mathematically in R. Hamilton’s works beginning 1982 [1, 2, 3]. It became famous after an enormous success related to Grisha Perelman’s proof in 2002-2003 of the Poincaré and Thurston conjectures [4, 5, 6]. There were proposed various constructions and applications of geometric flow methods in theoretical and mathematical particle physics and gravity before and after 1980, see reviews and original results in [7, 8, 9, 10, 11, 12, 13, 14]. A series of problems for developing such directions in physics is related to technical difficulties in constructing exact solutions of relativistic geometric flow equations and understanding their physical meaning and properties in modified gravity theories, MG Ts, and general relativity, GR. Rigorous geometric analysis methods elaborated in mathematical works do not provide an effective tool for investigating physical problems on possible geometric flow evolution scenarios related to modern gravity and cosmology.

The main purpose of this work is to elaborate on the anholonomic frame deformation method, AFDM, (see reviews and original results in refs. [15, 16, 17, 18]), as a geometric method for constructing exact solutions of geometric flow equations and gravitational field
equations in $R^{2}$–gravity. As explicit examples, we shall generalize the black hole solutions considered by A. Kehagias, C. Kounnas, D. Lüst and A. Riotto in [19] and study the conditions when corresponding modifications encode geometric flow evolution scenarios. Such new classes of generic off–diagonal solutions depend, in general, on all spacetime coordinates and on a geometric evolution parameter. They can be with prescribed ellipsoid and/or solitonic symmetries, nonholonomically deformed horizons and physical constants with locally anisotropic polarizations and running on flow parameter. For certain conditions, there are generated solutions of Ricci soliton equations modelling MGT effects and/or off–diagonal interactions in GR.

1.1 Geometric flows in string theory and modified gravity

Geometric flows appear naturally in off–critical string theory via the renormalization–group equations and nonlinear sigma model, $\sigma$–models. Physicists knew independently (perhaps, some years before mathematicians) about certain models with geometric flow equations. Here we cite D. Friedan’s works [7, 8, 9] on nonlinear models in two + epsilon dimensions, $2 + \varepsilon$, published in 1980. Those papers were related to developments of Polyakov’s research [20] on renormalization of the $O(N)$–invariant nonlinear $\sigma$–model, which in a low–temperature regime is dominated by small fluctuations around ordered states.

The standard nonlinear $\sigma$–models consist some special cases when $M$ is a homogeneous space (the quotient $G/H$ of a Lie group $G$ by a compact subgroup $H$) and $g_{ij}$ is some $G$–invariant Riemannian metric on $M$. The renormalization of such models considers techniques used for the standard power counting arguments combined with generalizations of the BRST transformation and the method of quadratic identities. In result, the renormalization group equation for the metric coupling is

$$\ell^{-1} \frac{\partial g_{ij}}{\partial \ell^{-1}} = -\beta_{ij}(\varphi),$$

where the $\beta$-function

$$\beta_{ij}(\varphi^{-1}g) = -\varepsilon(\varphi^{-1}g)_{ij} + \frac{1}{2} \varphi_{ij} R_{klm} R_{jklm} + O(\varphi^2)$$

is a vector field on the infinite dimensional space of Riemannian metrics on $M$. The value $(\varphi^{-1}g)_{ij}$ is a (positive definite) Riemannian metric on $M$, called the metric coupling and $\ell$ is the short distance cutoff (in certain models $\ell^{-1}$ is treated as a temperature like parameter).

Two important results on global properties of above type $\beta$–functions were obtained. When a manifold $M$ is a homogeneous space $G/H$, the $\beta$–function is shown to be a gradient type function for a finite dimensional space of $G$–invariant metric couplings on $M$. If $M$ is a two dimensional compact manifold, the $\beta$–function is shown to be a gradient on the infinite dimensional space of metrics of $M$.

A series of mathematical and physical results were obtained for self-similar configurations defined by equations which are similar to field equations in modern gravity. Such fixed points of Ricci flows are described by (latter called Ricci solitons) equations

$$R_{ij} - \lambda g_{ij} = \nabla_i v_j + \nabla_j v_i, \quad (1)$$

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for $\lambda = \pm 1, 0$. In above formulas, $g_{ij}(x^k), \nabla_i$ and $R_{ij}$ are respectively the symmetric metric field, Levi–Civita connection, LC–connection, and the Ricci tensor. For instance, on a two dimensional, 2-d, Riemannian manifold $M$, the local coordinates can be written $x = \{x^i\}$, for indices $i, j, k... = 1, 2, ..n$ and $v_j(x) \subset TM$ is a vector field defined by sections of tangent bundle $TM$. Equations of type (1) are considered in various MGTs as gravitational field equations for pseudo–Riemannian metrics and generalized connections or for LC–connections.

1.2 Ricci flows of Riemannian metrics on 3–d manifolds

Ricci flows (as a particular but very important example of a geometric evolution theory) are known in mathematics due to Hamilton’s programme on geometric analysis and attempts to prove the Poincaré conjecture. In the "standard" Ricci flow theory on three dimensional, 3-d, Riemannian manifolds, one considers the evolution of a family of metrics $g_{ij}(\chi) = g_{ij}(\chi, x^k)$ of signature (+ + +) parameterized by real parameter $\chi$, with respect to the coordinate base $\partial_i := \partial/\partial x^i$ and $\partial_{\chi} := \partial/\partial \chi$, for $i, j, k = 1, 2, 3$. The R. Hamilton equations where postulated in the form

$$\partial_{\chi} g_{ij} = -2 R_{ij} + \frac{2}{3} \rho g_{ij}, \quad (2)$$

where the normalizing factor $\rho = \int_{Vol} \sqrt{|g|} d^3x \cdot R / \int_{Vol} \sqrt{|g|} d^3x$, \quad (for $R := g^{ij} R_{ij}$ and $g := det |g_{ij}|$), is introduced in order to preserve a 3-d compact volume $Vol$. We can take $\rho = 0$ and consider a zero effective cosmological constant $\lambda = \frac{2}{3} \cdot \rho = 0$ for non–renormalized Ricci flows. In certain sense, such equations consist a generalized nonlinear diffusion equation for a tensor filed $g_{ij}$ because $R_{ij} \simeq \Delta$, where $\Delta$ is the Laplace operator if $g_{ij} \simeq 1 + \delta_{ij}$ for small fluctuations $\delta_{ij}$ of the Euclidean metric and $\chi$ treated as a temperature type parameter. If $\partial_{\chi} g_{ij} = 0$, we obtain the equations for 3–d Einstein spaces with metrics of positive definite signature.

One of the most important results due to G. Perelman is that these equations (2) can be derived as gradient flows (4) from certain Lyapunov type functionals for dynamical systems,

$$\mathcal{F}(g_{ij}, \nabla, f) = \int_{Vol} \sqrt{|g|} d^3x \left( R + |\nabla f|^2 \right) e^{-f}, \quad (3)$$

$$\mathcal{W}(g_{ij}, \nabla, f, \tau) = \int_{Vol} \sqrt{|g|} d^3x \left[ \tau \left( R + |\nabla f|\right)^2 + f - 3 \right] \mu, \quad (4)$$

where the function $\tau = \tau(\chi) > 0, \mu := (4\pi \tau)^{-3/2} e^{-f}$ for $\int_{Vol} \mu \sqrt{|g|} d^3x = 1$. Such functionals are called Perelman’s F–functional and W–entropy.

It should be noted that the W–entropy was used by G. Perelman (4) in order to elaborate a statistical thermodynamics approach to the theory of Ricci flows. There were considered generalizations of such functionals in refs. (10) (11) (12) related to nonholonomic and noncommutative geometric flows and black hole entropy and geometric flows. Certain classes of generic

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*1Our notations are different from those in mathematical books because we follow a system of notations which is useful for constructing generic off–diagonal exact solutions of such equations. For instance, we use "primed" indices and a left "vertical line" label like $R$ in order to emphasize that such values are used for 3–d Riemannian spaces/ hypersurfaces.*

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off–diagonal solutions of 4-d Ricci soliton equations were constructed and studied for nonholonomic dynamical systems. We also emphasize that 3-d extensions of the Ricci soliton equations (1), written in the form

$$R_{ij} - \lambda g_{ij} = \nabla_i v_j + \nabla_j v_i,$$

(5)
can be derived as self–similar fixed configurations of functionals (3), or (4), and Ricci flow evolution equations (2).

1.3 $R^2$ gravity and Ricci solitons

Ricci soliton type equations exist naturally in $R^2$ gravity. Let us consider the equations (61) and (62) from [19] (for well defined conditions, such equations are equivalent to the gravitational field equations in MGT),

$$\overline{R}_{\mu\nu} - \overline{\nabla}_\mu \phi \overline{\nabla}_\nu \phi - 2\varsigma^2 \overline{g}_{\mu\nu} = 0,$$

(6)

$$\overline{\nabla}^2 \phi = 0,$$

(7)

where the non-scale mode $\Phi = \frac{1}{2} e^{\sqrt{2/3}\phi}$ plays the role of a Lagrange multiplier used in conformal (Jordan frames) frames without spacetime derivatives and $\varsigma^2$ is a non-zero cosmological constant. There are considered conformal transforms of the metric

$$g_{\mu\nu} \rightarrow \overline{g}_{\mu\nu} = e^{\sqrt{1/3}\phi} g_{\mu\nu},$$

(8)

with

$$e^{\sqrt{1/3}\phi} = \frac{1}{8\varsigma^2} R \text{ and } \overline{g}_{\mu\nu} = e^{\sqrt{1/3}\phi} g_{\mu\nu} = \frac{R}{8\varsigma^2} g_{\mu\nu}, R \neq 0,$$

where $R$ is the Ricci scalar determined by the metric $g_{\mu\nu}$ and corresponding Levi–Civita, LC, connection. Over-lined values like $\overline{R}_{\mu\nu}$ are determined by $\overline{g}_{\mu\nu}$ and $\overline{\nabla}_\mu$, where local coordinates are labels $u^\mu$ for indices with conventional 3+1 splitting $\alpha, \beta, \mu, \ldots = 1, 2, 3, 4$, when $\alpha = (i, 4)$.

The equations (6) can be considered as gravitational field equations for certain MGTs with $R^2$ terms in Lagrangians and Einstein gravity models with additional massless scalar propagating field $\phi$ and nontrivial cosmological constant $\varsigma^2$. Using respective conformal transforms, such theories can be derived equivalently from the action

$$S = \int \sqrt{|g|} d^4 u \left( \frac{1}{16\varsigma^2} R^2 \right),$$

(9)

and/or

$$S = \int \sqrt{|g|} d^4 u \left( \Phi R - 4\varsigma^2 \Phi^2 \right),$$

(10)

and/or

$$S = \int \sqrt{|g|} d^4 u \left( \frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \varsigma^2 \right),$$

(11)

see respective formulas (40)-(43) in [19].
Theories of type (9) are of great interest in cosmology \[21, 22\] and inflationary scenarios in the early universe \[23, 24, 25\]. We note that in string theory such higher curvatures corrections appear naturally but as infinite series which result in the appearance of ghost-like modes originating both from the square of the Riemann and Weyl tensors. Nevertheless, such a pathology is absent for the $R + R^2$ theory, which is equivalent to standard Einstein gravity with an additional scalar field $\phi$ as in (11). Here it should be emphasized that the action of type (10) is an example of action for two measure theories, TMT, see refs. \[26, 27\].

An alternative interpretation of equations (6) is to consider them as a 4–d generalization of the Ricci solitonic equations (1) and (5). Such a modification is not trivial even we can find solutions of a system

$$\nabla_i v_j + \nabla_j v_i = \nabla_i \phi \nabla_j \phi,$$

where $\nabla_\mu = (\nabla_i, \nabla_4)$, for a 3-d $v_j$ extended to 4-d in order to define a scalar field $\phi(u^\nu)$. For instance, we can take a gradient vector field $v_i = \frac{1}{2} \phi \nabla_i \phi$

We have to perform a relativistic 4-d generalization for metrics of signature $(+++)$ of functionals (3), or (1) which result in geometric evolution equations which are not of parabolic type but correspond to a new class of physically important equations. For stationary configurations (we can consider over-lined or other type values),

$$R_{\alpha\beta} - \lambda g_{\alpha\beta} = \nabla_\alpha v_\beta + \nabla_\beta v_\alpha$$

$$= \nabla_\alpha \phi \nabla_\beta \phi + \frac{1}{2} (\nabla_\alpha \nabla_\beta + \nabla_\beta \nabla_\alpha) \phi.$$

For gradient Ricci solitons, the term $\frac{1}{2} g^{\alpha\beta} (\nabla_\alpha \nabla_\beta + \nabla_\beta \nabla_\alpha) \phi$ does not contribute in the action (11) if there are satisfied the equations (12). In general, terms of type $\nabla_\alpha v_\beta + \nabla_\beta v_\alpha$ are contained in some classes of MGTs, for instance, in Hořava–Lifshitz gravity \[28, 29\] and corresponding Ricci flow anisotropic cosmological models \[13\].

We conclude this subsection with the remark that the gravitational field equations in the $R^2$, TMT, and Einstein gravity theories determine self–similar fixed point configurations (with corresponding Ricci soliton equations) of a relativistic 4-d generalization of standard Ricci flow models of 3-d Riemannian metrics. In certain sense, a large class of MGTs can be reproduced via (nonholonomic) geometric flow evolution scenarios, when modified Perelman’s functionals include (as Ricci solitons) various types of actions for MGTs. Such constructions have a rigorous mathematical and physical motivation if we are able to construct in explicit form certain classes of exact solutions for geometric flow scenarios which model important black hole, cosmological and other type solutions in $R^2$ and Einstein gravity.

The paper is structured as follows. Section 2 is devoted to generalizations and formulation of Perelman’s functionals including $R^2$ gravity in nonholonomic variables. We provide an introduction into the geometry of Lorentz manifolds with nonholonomic 2+2 splitting and adapted physical/ geometric objects. There are derived the equations for relativistic geometric flows and generic off–diagonal Ricci solitons. The concepts of W–entropy and statistical thermodynamics are revised in the context of generalizations for nonholonomic relativistic geometric flows and MGTs.
In section 3, we provide an introduction to the AFDM as a geometric method for constructing exact solutions with coefficients of metrics and generalized connections depending on all spacetime coordinates. We develop this method for modelling relativistic flows, modified Ricci solitons and generic off–diagonal configurations in $R^2$ gravity. It is proved the decoupling property of such systems of nonlinear partial differential equations, PDEs, which allows us to integrate such systems in very general forms. There are considered four classes of such solutions for 1) geometric flows of metric coefficients with non-evolution of nonlinear connection coefficients; 2) nonholonomic Ricci soliton equations; 3) geometric evolution models with running physical constants and/or deformed horizons; geometric flows with nonholonomic vacuum. There are studied the equations for geometric evolution and generating Ricci solitons for Levi-Civita configurations.

In section 4, we construct exact solutions for geometric evolution of black ellipsoids as Ricci solitons and/or solutions in $R^2$ gravity. There are analyzed two classes of generic off–diagonal metrics describing 1) solitonic black ellipsoids and limits to $R^2$ and Einstein gravity theories and 2) geometric flows and solitons for asymptotically de Sitter solutions.

Section 5 is devoted to formulation of W–thermodynamics for black ellipsoids and solitonic flows in $R^2$ gravity. We show how such values can defined for 3–d hypersurface configurations and computed in general form for generic off–diagonal solutions describing geometric flows and Ricci solitons, or $R^2$ gravity and (modified) Einstein equations.

Conclusions are formulated in section 6. We provide some necessary coefficient formulas in Appendix.

2 Perelman’s Functionals & MGTs in Nonholonomic Variables

For elaborating a geometric method of constructing exact solutions for geometric flow, Ricci soliton and gravitational field equations, it is important to formulate such theories in nonholonomic variables with nonlinear connection splitting. A nonholonomic 3+1 splitting is convenient for relativistic generalizations of the Hamilton-Perelman theory, see details in [12]. In another turn, nonholonomic 2+2 splitting is important for decoupling (modified) geometric evolution and gravity equations [15, 16, 17, 18]. In general, we can work with models of geometric evolution of certain classes of exact solutions in gravity theories by considering double 3+1 and 2+2 splitting. In this section, we generalize/ modify Perelman’s functionals for pseudo–Riemannian signatures and in nonholonomic variables and prove the main evolution and gravitational field equations for nonholonomic 2+2 splitting and canonical distortions of linear connection structures.

2.1 Nonholonomic 2 + 2 + ... splitting and adapted geometric objects

We consider a (pseudo) Riemannian manifold $V$ enabled with a conventional 2+2 splitting into horizontal (h) and vertical (v) components defined by a Whitney sum

$$N : TV = hV \oplus vV,$$  \hspace{1cm} (13)
where $TV$ is the tangent bundle. A N–connection structure is determined locally by a corresponding set of coefficients $N^i_a$, when $N = N^i_a(u)dx^i \otimes \partial_a$.

For any h–v–splitting, there are structures of N–adapted local bases, $e_\nu = (e_1, e_a)$, and cobases, $e^\mu = (e^i, e^a)$, when

$$e_\nu = (e_i = \partial / \partial x^i - N^i_a \partial / \partial y^a, \ e_a = \partial / \partial y^a), \quad (14)$$
$$e^\mu = (e^i = dx^i, e^a = dy^a + N^a_i dx^i). \quad (15)$$

Such N–adapted bases are nonholonomic because, in general, there are satisfied relations of type

$$[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha = W^\gamma_{\alpha\beta} e_\gamma, \quad (16)$$

with nontrivial anholonomy coefficients $W^b_a = \partial_b N^i_a$, $W^a = \Omega^a_{ij} = e_j (N^a_i) - e_i (N^a_j)$. We obtain holonomic (integrable) bases if and only if $W^\gamma_{\alpha\beta} = 0$. A manifold $(V, N)$ endowed with a nontrivial structure $W^\gamma_{\alpha\beta}$ is called nonholonomic.

For nonholonomic manifolds, we can consider a class of linear connections which are adapted to the N–connection structure. A distinguished connection, d–connection, $D = (hD, vD)$ on $V$ is such a linear connection which preserves under parallel transport the N–connection splitting. A general linear connection $D$ is not adapted to a chosen h–v–decomposition, i.e. it is not a d–connection (for instance, the Levi–Civita connection in GR is not a d–connection). We do not use boldface symbols for not N–adapted geometric objects. For any d–connection $D$, we can consider as an operator of covariant derivative, $D_X Y$, for a d–vector $Y$ in the direction of a d–vector $X$. With respect to N–adapted frames and tensor products. For instance, a vector $Y(u) \in TV$ can be parameterized as a d–vector, $Y = Y^a e_a = Y^i e_i + Y^a e_a$, or $Y = (hY, vY)$, with $hY = \{Y^i\}$ and $vY = \{Y^a\}$. Similarly, we can determine and compute the coefficients of d–tensors, N–adapted differential forms, d–connections, d–spinors etc.

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1. The local coordinates are labelled $u^\mu = (x^i, y^a)$, (in brief, we $u = (x, y)$), where indices run respectively values of type $i, j, ..., = 1, 2$ and $a, b, ..., = 3, 4$. The cumulative small Greek indices run values $\alpha, \beta, ..., = 1, 2, 3, 4$, where $u^4 = y^4 = t$ is a time like coordinate. An arbitrary local basis is denoted $e^\alpha = (e^i, e^a)$ and the corresponding dual one, co-basis, is $e_\beta = (e_j, e_b)$. We consider that there are always nontrivial frame transforms to corresponding coordinate bases, $\partial_\alpha = (\partial_i, \partial_a)$ for instance, $\partial_\alpha = \partial / \partial x^i$, and cobasis. The values $e_\beta = A^\beta_{\alpha}(u) \partial_\alpha$ and $e^\alpha = A^{\alpha}_{\beta}(u) du^\alpha$, for $du^\alpha = (dx^i, dy^a)$, can be treated as frame (vierbein) transforms. On convenience, we shall use primed, underlined indices etc. The Einstein summation rule on repeating up–low indices will be applied if the contrary will be not stated.

2. We can elaborate a N–adapted differential and integral calculus and a corresponding variational formalism for (modified) gravity theories using N–elongated operators and . The geometric constructions are performed with distinguished objects, in brief, d–objects with coefficients determined with respect to N–adapted (co) frames and their tensor products. For instance, a vector $Y(u) \in TV$ can be parameterized as a d–vector, $Y = Y^a e_a = Y^i e_i + Y^a e_a$, or $Y = (hY, vY)$, with $hY = \{Y^i\}$ and $vY = \{Y^a\}$. Similarly, we can determine and compute the coefficients of d–tensors, N–adapted differential forms, d–connections, d–spinors etc.
The $N$–adapted coefficients are correspondingly labeled using $h$- and $v$–indices,

$$
T = \{ T_{\alpha\beta}^\gamma = (T_{jk}^i, T_{ja}^i, T_{ja}^a, T_{ab}^\alpha, T_{be}^a) \}, \quad Q = \{ Q_{\alpha\beta}^\gamma \}, \quad (17)
$$

$$
R = \{ R_{\alpha\beta\gamma\delta}^\alpha = (R_{hjk}^i, R_{bjk}^a, R_{hja}^i, R_{bja}^a, R_{hba}^i, R_{bea}^a) \},
$$

see explicit formulas in [15, 16, 17, 18]. In Appendix, we provide such formulas for the case of canonical distinguished connection (defined in (20)), see (A.2) and (A.4).

Any metric tensor $g$ on $(\mathbf{V}, N)$ can be parameterized in

$$
g = g_{\alpha\beta} du^\alpha \otimes du^\beta, \quad \text{where} \quad g_{\alpha\beta} = \begin{bmatrix}
g_{ij} + N_i^a N_j^b g_{ab} & N_i^e g_{ae} \\
N_i^e g_{be} & g_{ab}
\end{bmatrix}, \quad (18)
$$

with respect to a dual local coordinate basis $du^\alpha$. Equivalently, we can write a metric as a $d$–tensor ($d$–metric)

$$
g = g_\alpha(u)e^\alpha \otimes e^\beta = g_i(x) dx^i \otimes dx^i + g_a(x, y)e^a \otimes e^a, \quad (19)
$$
in brief, $g = (h, v)$, with respect to a tensor product of $N$–adapted dual frame [15]. A metric $g$ [18] with $N$–coefficients $N_j^e$ is generic off–diagonal if the anholonomy coefficients $W_{\alpha\beta}^\gamma$ [16] are not zero.

For any $d$–metric $g$, we can define two important linear connection structures following such geometric conditions:

$$
\mathbf{g} \rightarrow \begin{cases}
\nabla: & \nabla g = 0; \nabla T = 0, \quad \text{the Levi–Civita connection;}
\\mathbf{D}: & \mathbf{D} g = 0; \ h\mathbf{T} = 0, \ v\mathbf{T} = 0, \quad \text{the canonical d–connection.} \quad (20)
\end{cases}
$$

The LC–connection $\nabla$ can be introduced without any $N$–connection structure but the canonical $d$–connection $\mathbf{D}$ depends generically on a prescribed nonholonomic $h$- and $v$-splitting, see formulas (A.1). In above formulas, $h\mathbf{T}$ and $v\mathbf{T}$ are respective torsion components which vanish on conventional $h$- and $v$–subspaces. Nevertheless, there are nonzero torsion components, $hv\mathbf{T}$, (see coefficients $\tilde{T}_{ja}^i, \tilde{T}_{a}^e$ and $\tilde{T}_{ja}^a$ in (A.2)) with nonzero mixed indices with respect to a $N$–adapted basis [14] and/or [15].

On $\mathbf{V}$, all geometric constructions can be performed equivalently with $\nabla$ and/or $\mathbf{D}$ and related via a canonical distorting relation

$$
\mathbf{D}[\mathbf{g}, N] = \nabla[\mathbf{g}] + \mathbf{Z}[\mathbf{g}, N], \quad (21)
$$

when both linear connections and the distorting tensor $\mathbf{Z}$ are uniquely determined by data $(\mathbf{g}, N)$ as an algebraic combination of coefficients of $\tilde{T}_{\alpha\beta}^\gamma$. The $N$–adapted coefficients for $\mathbf{D}$ and corresponding torsion, $\tilde{T}_{\alpha\beta}^\gamma$ (A.2), Ricci $d$–tensor, $\tilde{R}_{\beta\gamma}$ (A.5), and Einstein $d$–tensor, $\tilde{E}_{\beta\gamma}$ (A.7), can be computed in standard form [15, 16, 17, 18]. The canonical distortion relation (21) defines respective distortion relations of the Rieman, Ricci and Einstein tensors and respective curvature scalars which are uniquely determined by data $(\mathbf{g}, N)$. Any (pseudo) Riemannian geometry can be equivalently formulated using $(\mathbf{g}, \nabla)$ or $(\mathbf{g}, \mathbf{D})$.

The canonical $d$–connection $\mathbf{D}$ has a very important role in elaborating the AFDM of constructing exact solutions in geometric flows and MGTs. It allows to decouple the gravitational and matter field equations with respect to $N$–adapted frames of reference. This is not possible if we work only with $\nabla$. Constructing certain general classes of solutions for $\mathbf{D}$, we can impose at the end the condition $\mathbf{T} = 0$ and extract LC–configurations $\mathbf{D}|_{\mathbf{T}=0} = \nabla$. 

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2.2 Relativistic nonholonomic geometric flows

We can consider a family of 4-d Lorentz nonholonomic manifolds $V(\tau)$ with metrics $g(\tau) = g(\tau, u)$ of signature $(++-)$ and N-connection $N(\tau)$ parameterized by a positive parameter $\tau, 0 \leq \tau \leq \tau_0$. Any $V \subset V(\tau)$ can be enabled with a double nonholonomic 2+2 and 3+1 splitting (see [12] for the geometry of spacetimes enabled with such double distributions). Conventionally, the local coordinates are labeled $u^a = (x^i, y^a) = (x^i, u^t)$ for $i, j, k, ... = 1, 2; a, b, c, ... = 3, 4$; and $\lambda, j, k = 1, 2, 3$. The 3+1 splitting can be chosen in such form that any open region $U \subset V$ is covered by a family of 3-d spacelike hypersurfaces $\Xi_t$ parameterized by a time like parameter $t$. There are two generic different types of geometric flow theories when 1) $\tau(\tau)$ is a re-parametrization of the temperature like parameter used for labeling 4-d Lorentz spacetime configurations and 2) $\tau(t)$ is a time like parameter when 3-d spacelike configurations evolve relativistically on a "redefined" time like coordinate. For simplicity, we shall study in this work only theories of type 1.

For arbitrary frame transforms on 4-d nonholonomic Lorentz manifolds, we can generalize the Perelman’s functionals (3) and (4), respectively, in terms of data $(g(\tau), D(\tau))$. We postulate

\[
\hat{\mathcal{F}} = \int_{t_1}^{t_2} \int_{\Xi_t} e^{-\hat{f}} \sqrt{|g_{\alpha\beta}|} d^4u (\hat{R} + |\hat{D}\hat{f}|^2),
\]

and

\[
\hat{\mathcal{W}} = \int_{t_1}^{t_2} \int_{\Xi_t} \hat{M} \sqrt{|g_{\alpha\beta}|} d^4u [\tau(\hat{R} + |\hat{D}\hat{f}| + |\hat{D}\hat{f}|^2)^2 + \hat{f} - 8],
\]

where the normalizing function $\hat{f}(\tau, u)$ satisfies $\int_{t_1}^{t_2} \int_{\Xi_t} \hat{\mu} \sqrt{|g_{\alpha\beta}|} d^4u = 1$ for $\hat{\mu} = (4\pi)^{-3} e^{-\hat{f}}$, see formula (A.6) for the "hat" scalar curvature. It should be noted that $\hat{\mathcal{W}}$ do not have a character of entropy for pseudo–Riemannian metrics but can be treated as a value characterizing relativistic geometric hydrodynamic flows [12]. We can compute entropy like values of type (4) for any 3+1 splitting with hypersurface fibrations $\Xi_t$.

For 4-d configurations with a corresponding re-definition of the scaling function, $\hat{f} \to f$, we can construct models of geometric evolution with $h$– and $v$–splitting for $D$,

\[
\partial_\tau g_{ij} = -2(\hat{R}_{ij} - \hat{D}_i\phi \hat{D}_j\phi - 2\xi^2 g_{ij}),
\]

\[
\partial_\tau g_{ab} = -2(\hat{R}_{ab} - \hat{D}_a\phi \hat{D}_b\phi - 2\xi^2 g_{ab}),
\]

\[
\hat{R}_{ia} = \hat{R}_{ai} = 0; \hat{R}_{ij} = \hat{R}_{ji}; \hat{R}_{ab} = \hat{R}_{ba};
\]

\[
\hat{D}^2\phi = 0;
\]

\[
\partial_\tau f = -\nabla f + \left|\hat{D}\hat{f}\right|^2 - h\hat{R} - v\hat{R}.
\]

These formulas can be derived from the functionals (22) and/or (23) following a calculus which is similar to that presented in N–adapted form in Ref. [10] [11]. In abstract geometric form, we can apply the strategy elaborated originally for such proofs in [4] for metric compatible connections. The conditions $\hat{R}_{ia} = 0$ and $\hat{R}_{ai} = 0$ are necessary if we want to keep the total metric to be symmetric under Ricci flow evolution. The general relativistic character of 4-d
geometric flow evolution is encoded in operators like \( \hat{\Box} = \hat{D}^\alpha \hat{D}_\alpha \), d-tensor components \( \hat{R}_{ij} \) and \( \hat{R}_{ab} \), theirs scalars \( h \hat{R} = g^{ij} \hat{R}_{ij} \) and \( v \hat{R} = g^{ab} \hat{R}_{ab} \) with data \((g_{ij}, g_{ab}, \hat{D}_\alpha)\).

In order to study geometric flow evolution of solutions of \( R^2 \) and equivalent Einstein – scalar field theories, the \( \hat{F} \) and \( \hat{W} \) functionals can be written

\[
\hat{F} = \int_{t_1}^{t_2} \int_{\Xi} e^{-\hat{T}} \sqrt{|g_{\alpha\beta}|} d^4 u (\hat{R} - (\hat{D} \phi)^2 - 8\xi^2 + |\hat{D} \hat{f}|^2),
\]

and

\[
\hat{W} = \int_{t_1}^{t_2} \int_{\Xi} \mu \sqrt{|g_{\alpha\beta}|} d^4 u [\hat{R} - (\hat{D} \phi)^2 - 8\xi^2 + |h \hat{D} \hat{f}| + |v \hat{D} \hat{f}|]^2 + \hat{f} - 8],
\]

for \( \hat{f} \to \hat{f} \), where \( \hat{f}(\tau, u) \) satisfies \( \int_{t_1}^{t_2} \int_{\Xi} \mu \sqrt{|g_{\alpha\beta}|} d^4 u = 1 \) for \( \mu = (4\pi \tau)^{-3} e^{-\hat{T}} \). These formulas are important for investigation of non–stationary configurations of gravitational fields. We can consider \( \hat{D}|_{\tau=0} = \nabla \), where the LC–connection is determined by the over-lined metric \( \hat{g}_{\mu\nu} \).

### 2.3 Generic off–diagonal Ricci solitons

For self–similar fixed point \( \tau = \tau_0 \) configurations, the equations (24) and (25) transform into relativistic Ricci soliton equations, respectively, into a system of nonholonomically deformed Einstein – scalar field equations

\[
\begin{align*}
\hat{R}_{ij} &= \phi \hat{\nabla}_{ij} + 2\xi^2 g_{ij} \quad (26) \\
\hat{R}_{ab} &= \phi \hat{\nabla}_{ab} + 2\xi^2 g_{ab}, \quad (27) \\
\hat{R}_{ia} &= 0; \quad \hat{R}_{aj} = 0; \\
\hat{D}^2 \phi &= 0,
\end{align*}
\]

where the corresponding \( h \)– and \( v \)–sources are \( \phi \hat{\nabla}_{ij} = \hat{D}_i \phi \hat{D}_j \phi \) and \( \phi \hat{\nabla}_{ab} = \hat{D}_a \phi \hat{D}_b \phi \). In this work, we put the left low label \( b \) for necessary values in order to emphasize that such geometric/ physical objects are computed for certain Ricci soliton configurations with \( \tau = \tau_0 \). Such labels will be omitted if that will not result in ambiguities.

Using N–adapted 2+2 frame and coordinate transforms of the metric and source \( \phi \hat{\nabla}_{\alpha\beta} \),

\[
\begin{align*}
g_{\alpha\beta}(\tau, x^i, t) &= e^{\alpha}_{\alpha}(\tau, x^i, y^a) e^{\beta}_{\beta}(\tau, x^i, y^a) \hat{g}_{\alpha\beta}(\tau, x^i, y^a) \quad \text{and} \\
\phi \hat{\nabla}_{\alpha\beta}(\tau, x^i, t) &= e^{\alpha}_{\alpha}(\tau, x^i, y^a) e^{\beta}_{\beta}(\tau, x^i, y^a) \hat{\nabla}_{\alpha\beta}(\tau, x^i, y^a),
\end{align*}
\]

for a time like coordinate \( y^i = t \) \((i', i, k, k', \ldots = 1, 2 \text{ and } a, a', b, b', \ldots = 3, 4)\), we introduce certain canonical parameterizations which will allow us to decouple and solve the system the parameterize the metric and effective source in certain adapted forms. The generic off–diagonal metric ansatz is taken in the form

\[
\begin{align*}
g &= g_{\alpha\beta} e^\alpha \otimes e^\beta = g_1(\tau, x^k)dx^i \otimes dx^j + \omega^2(\tau, x^k, y^3, t) h_a(\tau, x^k, y^3) e^a \otimes e^a \quad (28) \\
&= q_1(\tau, x^k)dx^i \otimes dx^j + q_3(\tau, x^k, y^3, t) e^3 \otimes e^3 - \hat{N}^2(\tau, x^k, y^3, t) e^4 \otimes e^4, \quad (29) \\
e^3 &= dy^3 + w_i(\tau, x^k, y^3)dx^i, e^4 = dt + n_i(\tau, x^k, y^3)dx^i,
\end{align*}
\]

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This ansatz is a general one for a 4–d metric which can be written as an extension of a 3–d metric \( q_{ij} = \text{diag}(q_i) = (q_1, q_3) \) on a hypersurface \( \hat{\Sigma}_t \) if
\[
q_3 = g_3 = \omega^2 h_3 \quad \text{and} \quad \dot{N}^2(\tau, u) = -\omega^2 h_4 = -g_4,
\]
where \( \dot{N}(\tau, u) \) is the lapse function. It allows a straightforward extension of 3–d ansatz to 4-d configurations by introducing the values \( \dot{\gamma}^2 \) and statistical thermodynamics for geometric flows of off–diagonal solutions depending on all spacetime variables, see details and examples in Refs.

We can work for simplicity with solutions of the Ricci soliton and/or (modified) Einstein equations. To be able to construct solutions in explicit forms, we suppose that the scalar field can be described with respect to N–adapted frames when the exact solutions for \( \omega = 1 \) are with Killing symmetry on \( \partial/\partial t \). For such configurations, there are N–adapted bases when the geometric and physical values do not depend on coordinate \( y^4 = t \). We can work for simplicity with solutions with one Killing symmetry. Technically, it is possible to construct very general classes of generic off–diagonal solutions depending on all spacetime variables, see details and examples in Refs. [15 16 17 18] for "non–Killing" configurations.

### 2.4 W–entropy and statistical thermodynamics for geometric flows and MG Ts

Let us denote by \( \hat{\mathbf{D}} = \hat{\mathbf{D}}_{\hat{\xi}_t} \) the canonical d–connection \( \hat{\mathbf{D}} \) defined on a 3–d hypersurface \( \hat{\Sigma}_t \), when all values depend on temperature like parameter \( \tau(\chi) \). We define also \( \dot{\mathbf{R}} := \dot{R}_{\hat{\xi}_t} \). Using \( (q_i) = (q_1, q_3) \), the Perelman’s functionals parameterized in N–adapted frames as

\[
\tilde{\mathcal{F}}(\tau, \phi) = \int\hat{\Sigma}_t e^{-f} \sqrt{|q_{ij}|} d\bar{x}^3 (\dot{\mathbf{R}} - (\hat{\mathbf{D}}\phi)^2 - 8\xi^2 + |\hat{\mathbf{D}}f|^2),
\]

and
\[
\tilde{\mathcal{W}}(\tau, \phi, \mathbf{f}) = \int\hat{\Sigma}_t \mu \sqrt{|q_{ij}|} d\bar{x}^3 [\tau (\dot{\mathbf{R}} - (\hat{\mathbf{D}}\phi)^2 - 8\xi^2 + |\hat{\mathbf{D}}f| + |\hat{\mathbf{D}}\mathbf{f}|)^2 + f - 6],
\]

where we chose a necessary type scaling function \( f \) which satisfies \( \int_{\hat{\Sigma}_t} \mu \sqrt{|q_{ij}|} d\bar{x}^3 = 1 \) for \( \mu = (4\pi\tau)^{-3} e^{-f}. \) These functionals transform into standard Perelman functionals for 3-d Riemannian metrics on \( \hat{\Sigma}_t \) if \( \hat{\mathbf{D}} \rightarrow \nabla. \)
For any closed hypersurface $\hat{\Xi}_t$, the $W$-entropy $\hat{\mathcal{W}}$ is a Lyapunov type non-decreasing functional which is analogous to minus entropy. We can formulate a statistical thermodynamics model associated to (non) holonomic modified Ricci flow evolution of metrics with local Euclidean structure and nonholonomically modified connections. The constructions in statistical thermodynamics begin with a partition function

$$Z = \exp \left\{ \int_{\hat{\Xi}_t} \mu \sqrt{|q_{ij}|d^3\hat{x}} \left[ -\hat{f} + n \right] \right\}$$

for the conditions stated for definition of (33) and (34). This allows us to compute main thermodynamical values for the Levi-Civita connection, $\nabla$ and $n = 3$.\footnote{We remember that a statistical model can be elaborated for any prescribed partition function $Z = \int \exp(-\ell^{-1}E)d\omega(E)$ considering canonical ensemble at temperature $\ell$. The measure is taken to be the density of states $\omega(E)$. We compute the standard thermodynamical values for the average energy, $\mathcal{E} = \langle E \rangle := -\partial \log Z/\partial(\ell^{-1})$, the entropy $S := \ell^{-1}\langle E \rangle + \log Z$ and the fluctuation $\sigma := \langle (E - \langle E \rangle)^2 \rangle = \partial^2 \log Z/\partial(\ell^{-1})^2$.}

To elaborate the analogous thermodynamics constructions in $N$-adapted form we consider a family of $q_{ij}(\tau)$, with $\partial \tau/\partial \ell^{-1} = -1$, being a real re-parametrization of $\ell^{-1}$. We compute (see similar details in [12, 10])

$$\hat{\mathcal{E}} = -\tau^2 \int_{\hat{\Xi}_t} \mu \sqrt{|q_{ij}|d^3\hat{x}} \left( \hat{R} - (\hat{D}_i\phi)^2 - 8\varsigma^2 + |\hat{D}\hat{f}|^2 - \frac{3}{\tau} \right), \tag{35}$$

$$\hat{\mathcal{S}} = -\int_{\hat{\Xi}_t} \mu \sqrt{|q_{ij}|d^3\hat{x}} \left[ \tau \left( \hat{R} - (\hat{D}_i\phi)^2 - 8\varsigma^2 + |\hat{D}\hat{f}|^2 \right) + \hat{f} - 6 \right],$$

$$\hat{\sigma} = 2\tau^4 \int_{\hat{\Xi}_t} \mu \sqrt{|q_{ij}|d^3\hat{x}} \left[ \hat{R}_{ij} - \hat{D}_i\phi \hat{D}_j\phi + \hat{D}_i \hat{D}_j \hat{f} - (\frac{1}{2\tau} + 2\varsigma^2)q_{ij} \right].$$

These formulas can be considered for 4-d configurations taking the lapse function $\tilde{N} = 1$ for $N$-adapted Gaussian coordinates. In such cases it is more difficult to compute in explicit form the values corresponding to 4-d physically important solutions. They characterize stationary exact solutions in 4-d MGTs and GR if the values $q_{ij}$, $\hat{D}$ and $\phi$ are computed on a closed 3-d hypersurface $\hat{\Xi}_t$.

The formulas (33) provide a thermodynamic characterization of nonholonomic geometric flows and (modified) gravitational field equations, in particular, in $R^2$ gravity and its equivalent Einstein – scalar field formulation. We can re-define the normalization function $\hat{f}$ and flow parameter in such a form with terms

$$-(\hat{D}_i\phi)^2 - 8\varsigma^2 + |\hat{D}\hat{f}|^2 = 0$$

and

$$\hat{D}_i\phi \hat{D}_j\phi + \hat{D}_i \hat{D}_j \hat{f} = 2\varsigma^2 q_{ij}.$$
we can generalize the constructions for 4–d functionals

\[ \tilde{F}(q, \tilde{D}, f) = \int_{t_1}^{t_2} dt \tilde{N}(\tau) \tilde{F}(q, \tilde{D}, f), \] (36)

and

\[ \tilde{W}(q, \tilde{D}, f) = \int_{t_1}^{t_2} dt \tilde{N}(\tau) \tilde{W}(q, \tilde{D}, f(t)), \] (37)

where the lapse function is taken for an exact solution of certain nonholonomic Ricci flow/soliton
and/or (modified) gravitational field equations. To elaborate a 4-d general relativistic thermodynamic formulation is a more difficult task. For instance, we can consider relativistic hydrodynamical type generalizations, see [12]. Nevertheless, we can compute the corresponding average energy, entropy and fluctuations for evolution both on redefined parameter \( \tau \) and on a time like coordinate \( t \) for a time interval from \( t_1 \) to \( t_2 \) of any family of closed hypersurfaces all determined by \( \tilde{D} \),

\[ \tilde{E}(\tau) = \int_{t_1}^{t_2} dt \tilde{N}(\tau) \tilde{E}(\tau), \quad \tilde{S}(\tau) = \int_{t_1}^{t_2} dt \tilde{N}(\tau) \tilde{S}(\tau), \quad \tilde{\Sigma}(\tau) = \int_{t_1}^{t_2} dt \tilde{N}(\tau) \tilde{\sigma}(\tau). \] (38)

Working with distortion formulas [24], we can compute similar values in terms of \( \nabla \),

\[ \nabla E(\tau) = \int_{t_1}^{t_2} dt \tilde{N}(\tau) \nabla E(\tau), \quad \nabla S(\tau) = \int_{t_1}^{t_2} dt \tilde{N}(\tau) \nabla S(\tau), \quad \nabla \Sigma(\tau) = \int_{t_1}^{t_2} dt \tilde{N}(\tau) \nabla \sigma(\tau). \]

We can provide a gravitational thermodynamics interpretation only on 3–d closed hypersurfaces. For more special classes of solutions, we can model the standard black hole thermodynamics by considering 2+1+1 splitting and solutions with horizons and 2-d hypersurface.

3 A Geometric Method for Generating Solutions for Ricci Flows & \( R^2 \) Gravity

We develop a geometric method for integrating 4–d geometric flow and gravitational field equations in MGTs, see reviews of former results and examples in Refs. [15, 16, 17, 18]. Working with nonholonomic (equivalently, anholonomic, i.e. non-integrable) variables is possible to decouple and integrate such systems of nonlinear partial differential equations, PDEs, in certain general forms with generic off–diagonal metrics \( g_{\alpha\beta}(\tau, u^\gamma) \) depending on all spacetime coordinates \( u^\gamma \) and on flow parameter \( \tau \).

3.1 PDEs for off–diagonal geometric flows and Ricci solitons

In this work, the effective scalar field \( \phi \) is subjected to constraints of type \( e_{\alpha} \phi = \overset{0}{\phi}_{\alpha} = const \), which results in \( \tilde{D}^2 \phi = 0 \). We restrict our models to configurations of \( \phi \), which can be
encoded into N–connection coefficients:

\[
e_i \phi = \partial_i \phi - w_i \phi^* - n_i \partial_4 \phi = 0 \phi_i; \partial_3 \phi = 0 \phi_3; \partial_4 \phi = 0 \phi_4;
\]

for \(0 \phi_1 = 0 \phi_2\) and \(0 \phi_3 = 0 \phi_4\).

This results in an additional source \(\phi \tilde{\Upsilon} = \phi \tilde{\Lambda} = const\) and \(\phi \Upsilon = \phi \Lambda = const\). Under geometric flows, it is possible running of such configurations when

\[
e_a \phi(\tau, x^k, y^a) = 0 \phi_a + 0 \phi_a(\tau)
\]

modify the effective h– and v–sources

\[
\phi \tilde{\Upsilon} = \phi \tilde{\Lambda} + \phi \tilde{\Lambda}(\tau) \text{ and } \phi \Upsilon = \phi \Lambda + \phi \Lambda(\tau).
\]

We shall use such effective sources as additional nonholonomic and geometric flow deformations of the evolution and modified gravitational field equations.

### 3.1.1 Geometric flows of d–metric coefficients

Let us consider a set of coefficients \(\alpha_\beta = (\alpha_i, \alpha_3 = 0, \alpha_4)\) determined by a generating function \(\Psi\) when

\[
\alpha_i = h_i^* \partial_\tau \Psi/\Psi, \alpha_3 = h_4^* \Psi^*/\Psi, \gamma = (\ln |h_4|^{3/2}/|h_3|)^* \quad (41)
\]

for \(\Psi := h_i^*/\sqrt{|h_3 h_4|}\). \(\quad (42)\)

Using the ansatz for d–metric \(28\) and sources \(32\), with \(\tau\)–parameter dependencies of coefficients \(31\), and expressing the coefficients \(41\) and related formulas in terms of generating functions like \(42\), we transform the system \(24\) and \(25\) into a system of nonlinear PDEs

\[
\psi_{**} + \psi'' = 2(\tilde{\Upsilon} - \frac{1}{2} \partial_\tau \psi),
\]

\[
\Psi^* h_4^* = 2 h_3 h_4 (\tilde{\Upsilon} - \partial_\tau \ln |\omega^2 h_4|) \Psi,
\]

\[
\partial_\tau \ln |\omega^2 h_4| = \partial_\tau \ln |\omega^2 h_3| = \partial_\tau \ln |N^2|,
\]

\[
\alpha_3 w_i - \alpha_i = 0,
\]

\[
n_i^* + \gamma n_i^* = 0,
\]

\[
e_k \omega = \partial_k \omega + w_k \omega^* + n_k \partial_4 \omega = 0,
\]

\[
e_a \phi = 0 \phi_a = const,
\]

with effective h– and v–sources,

\[
\tilde{\Upsilon} := \tilde{\Upsilon} + \phi \tilde{\Lambda} + 2 \varsigma^2 \text{ and } \Upsilon := \Upsilon + \phi \Lambda + 2 \varsigma^2.
\]

\[5\text{we shall use brief denotations for partial derivatives like } a^* = \partial_1 a, b^* = \partial_2 b, h^* = \partial_3 h \text{ if it will be necessary}\]

\[6\text{see details of such computation in } 16, 17, 18\]
The unknown functions for this system are \( \psi(\tau, x^i), \omega(\tau, x^k, y^3, t), h_0(\tau, x^k, y^3), n_i(\tau, x^k, y^3) \). The first two equations contain possible additional sources determined by other effective polarized cosmological constants or matter fields written as \( \tilde{\Upsilon}(\tau, x^k) \) and \( \Upsilon(\tau, x^k, y^3) \). We omitted the last equation for the rescaling function \( f \) because it can be found at the end when other values are determined by a class of solutions. As a matter of principle, we can work with not normalized geometric flow solutions.

### 3.1.2 Nonholonomic Ricci soliton equations

For stationary configurations with \( \partial_\tau g_{\alpha\beta} = 0 \) and \( \tau = \tau_0 \), the first three equations in the system of nonlinear PDEs (43)–(48) transform into self-similar Ricci soliton equations (26) and (27) which for the off–diagonal ansatz can be written

\[
\begin{align*}
\psi^{**} + \psi'' &= 2 \psi \Upsilon \quad \text{and} \\
\Psi^* \psi h_4 &= 2 \psi h_3 \psi h_4 \psi \Upsilon \psi.
\end{align*}
\]

The equation (51) is just the 2-d Poisson equation which can be solved in general form for any given source \( \psi \Upsilon(x^k) \).

Let us show how we can integrate the system system (42) and (52) for arbitrary source \( \psi \Upsilon(x^k, y^3) \). Here we elaborate a new approach which is different from that considered in [16, 17, 18]. In that work, it was applied the property that such systems are invariant under re-definition of generating function, \( \Psi \leftrightarrow \tilde{\Psi} \), and the effective source, \( (\Upsilon + \phi \Lambda + 2\varsigma^2) \leftrightarrow (\Lambda_0 + \phi \Lambda + 2\varsigma^2) = const, \Lambda_0 \neq 0 \).

For generating off–diagonal locally anisotropic cosmological solutions depending on \( y^3 \), we have to consider generating functions for which \( \Psi^* \neq 0 \). We obtain such a system nonlinear PDEs

\[
\begin{align*}
\psi \Psi^* \psi h_4^* &= 2 \psi h_3 \psi h_4 \psi \Upsilon \psi, \\
\psi \Psi^* \psi w_i - \partial_i \psi \psi &= 0, \\
\psi n_i^{**} + (\ln |\psi h_4|^{3/2}/|\psi h_3|)^* \psi n_i^* &= 0.
\end{align*}
\]

This system for nonholonomic Ricci solitons (51), (52) and (46)–(48) can be solved in very general forms by prescribing \( \psi \psi, \psi \Upsilon \) and \( \psi \Psi \) and integrating the equations "step by step" for a fixed parameter \( \tau_0 \). Introducing the function

\[
q^2 := \epsilon_3 \epsilon_4 \psi h_3 \psi h_4,
\]

for \( \epsilon_{3,4} = \pm 1 \) depending on signature of the metrics, we consider that the system (53) and (54) can be expressed respectively as

\[
\psi \Psi^* \psi h_4^* = 2 \epsilon_3 \epsilon_4 q^2 \psi \Upsilon \psi \quad \text{and} \quad \psi h_4^* = q \psi \Psi.
\]

\(^7\text{Such nonlinear transforms are given by formulas}

\[
(\Lambda_0 + \phi \Lambda + 2\varsigma^2)(\Psi^2)^* = |\Upsilon + \phi \Lambda + 2\varsigma^2|(|\tilde{\Psi}|^2)^*, \quad \text{or} \quad (\Lambda_0 + \phi \Lambda + 2\varsigma^2)\Psi^2 = \tilde{\Psi}^2 |\Upsilon + \phi \Lambda + 2\varsigma^2| - \int dy^3 |\tilde{\Psi}|^2 |\Upsilon|^*.
\]

They can be used for re–definition of generation and source functions and constructing new classes of solutions.
Introducing \( h_4^* \) form the second equation into the first one, we find

\[
q = \frac{\varepsilon_3 \varepsilon_4}{2} \frac{\Psi^*}{b Y}.
\]  

(59)

We can use this value in the second equation of (58) and find

\[
h_4 = h_4^{[0]}(x^k) + \frac{\varepsilon_3 \varepsilon_4}{4} \int dy^3 \frac{(b \Psi^2)^*}{b Y},
\]

(60)

where \( h_4^{[0]}(x^k) \) is an integration function. We compute \( h_3 \) considering (59), (57) and formula (60),

\[
h_3 = \frac{\varepsilon_3 \varepsilon_4}{4h_4} \left( \frac{b \Psi^*}{b Y} \right)^2 = \frac{\varepsilon_3 \varepsilon_4}{4} \left( \frac{h_4^{[0]} + \frac{\varepsilon_3 \varepsilon_4}{4} \int dy^3 \frac{(b \Psi^2)^*}{b Y}}{b Y} \right)^{-1}.
\]

(61)

For a given \( \Psi \), we can solve the linear algebraic equations (55) and express \( w_i = \partial_i \Psi / \Psi^* \).

The second part of N–connection coefficients are found by integrating two times on \( y^3 \) in (56) expressed as

\[
\partial_3 (n_i^*) = - \varepsilon_{i3} \partial_4 (\ln (h_4 | (3/2) | h_3))
\]

for the coefficient \( \gamma \) in (11). The first integration results in \( n_i^* = 2n_i(x^k) | h_3 | h_4 | (3/2) \), for certain integration functions \( 2n_i(x^k) \). Integrating second time on \( y^3 \), including the signature and certain coefficients in integration functions and using formulas (61) and (60), we obtain

\[
\partial_3 (n_i^*) = - \varepsilon_{i3} \partial_4 (\ln (h_4 | (3/2) | h_3))
\]

for the coefficient \( \gamma \) in (11). The first integration results in \( n_i^* = 2n_i(x^k) | h_3 | h_4 | (3/2) \), for certain integration functions \( 2n_i(x^k) \). Integrating second time on \( y^3 \), including the signature and certain coefficients in integration functions and using formulas (61) and (60), we obtain

\[
\text{Putting together all above formulas and writing in explicit form the effective source (50), we obtain the formulas for the coefficients of a d–metric and a N–connection determining a Ricci soliton solution for the system (26) and (27),}
\]

\[
\begin{align*}
g_{\psi}(\tau_0) &= g_{\psi}[\psi, \tilde{\psi}, \tilde{\kappa}, \tilde{\kappa}, \tilde{\kappa}^2] \simeq e^{\psi(x^k)} \text{ as a solution of 2-d Poisson equations (51);} \\
h_{\psi}(\tau_0) &= \frac{\varepsilon_3 \varepsilon_4}{4} \left( h_4^{[0]} + \frac{\varepsilon_3 \varepsilon_4}{4} \int dy^3 \frac{(b \Psi^2)^*}{b Y} \right)^{-1}; \\
h_{\psi}(\tau_0) &= h_4^{[0]}(x^k) + \frac{\varepsilon_3 \varepsilon_4}{4} \int dy^3 \frac{(b \Psi^2)^*}{b Y}; \\
w_{\psi}(\tau_0) &= \partial_\psi \Psi / \Psi^*; \\
n_{\psi}(\tau_0) &= \frac{\varepsilon_3 \varepsilon_4}{2} \frac{\Psi^*}{h_4^{[0]}(x^k) + \frac{\varepsilon_3 \varepsilon_4}{4} \int dy^3 \frac{(b \Psi^2)^*}{b Y}}; \\
w_{\psi}(\tau_0) &= \omega[\psi, \tilde{\psi}, \tilde{\kappa}^2] \text{ is any solution of 1st order system (18).}
\end{align*}
\]
In these formulas, we state that the coefficients \( h_\alpha \) depend functionally on \( \Psi \) and \( \Upsilon \), which (in their turn) may depend on the flow evolution parameter \( \tau \) which is fixed to a value \( \tau_0 \).

We can solve the equations (43) for a nontrivial \( \omega^2 = | h_3 |^{-1} \).

Using coefficients (62), we define such a class of quadratic elements for off–diagonal stationary Ricci solitons with nonholonomically induced torsion (tRs),

\[
\begin{align*}
ds^2_{t\text{Rs}} &= g_{\alpha\beta}(x^k, y^3)du^\alpha du^\beta = e^{\Upsilon}(dx^1)^2 + (dx^2)^2 + \omega^2 e^3\xi_4 \left( \frac{\Psi^*}{\Upsilon} \right)^2 [dy^3 + \partial_3 \frac{\Psi}{\Upsilon} dx^3]^2 \\
+ \omega^2 h_4 [\Psi, \Upsilon] [dt + (n_k + 2\tilde{n}_k \int dy^3 \frac{(\Psi^*)^2}{\Upsilon^2 | h_4 |^{3/2}} dx^3)]^2.
\end{align*}
\]

This class of metrics defines also exact solutions for the canonical d–connection \( \hat{\mathbf{D}} \) in \( R^2 \) gravity with effective scalar field encoded into a nonholonomically polarized vacuum.

### 3.1.3 Geometric evolution with factorized dependence on flow parameter of d–metric and N–connection coefficients

Such classes of solutions are defined by generated functions, \( \psi(\tau, x^k) \) and \( \Psi(\tau, x^k, y^3) \), and effective sources, \( \Upsilon(\tau, x^k, y^3) \) and \( \Upsilon(\tau, x^k, y^3) \), depending in factorized form on flow parameter \( \tau \). For simplicity, we shall analyze stationary solutions with \( \omega = 1 \). We can integrate such equations in explicit form if we consider subclasses of solutions with separation of variables, when

\[
\psi(\tau, x^k) = \\psi(\tau) + \psi(x^k), \quad \Psi = \Psi(\tau) \psi(x^k, y^3),
\]

for \( h_3 = h_3(\tau) \psi h_3(x^k, y^3), h_4 = h_4(\tau) \psi h_4(x^k, y^3) \)

and \( \Upsilon(\tau, x^k) = \Upsilon(\tau) + \psi \Upsilon(x^k), \gamma(\tau, x^k, y^3) = \gamma(\tau) + \psi \gamma(x^k, y^3) \)

see (40). For simplicity, we shall assume in this section a constant value \( \psi \Upsilon(x^k, y^3) = \Upsilon_0 = \gamma_0 + \psi \Upsilon = \gamma_0 = \gamma, \) which is enough to study various classes of geometric flow evolution models. If \( \psi \Upsilon(x^k, y^3) \) is not constant, it is a more difficult task to construct exact solutions in explicit form (see such examples in subsection 3.1.2).

We can solve in explicit form the equations (43)–(47) considering (for simplicity) \( \omega = 1 \) and corresponding factorizations of the generating functions and sources. Via separation of variables, we obtain the system of equations

\[
\begin{align*}
\psi'' + \psi'' &= 2 \psi, \quad \tau \psi'(\tau) = 2 \tau \psi(\tau) + \psi' ; \\
\psi' &= 2 \psi, \quad \gamma(\tau, x^k, y^3) = \gamma(\tau) \psi(x^k, y^3), \gamma(\tau) \psi(x^k, y^3) = \gamma(\tau) \psi \psi + \gamma(\tau) \psi \psi ; \\
\partial_\tau \ln \gamma(\tau) &= \partial_\tau \ln \gamma(\tau) = \partial_\tau \ln \gamma(\tau) , \\
w_i(\tau, x^k, y^3) &= \psi(\tau, x^k, y^3) = \psi(\tau, x^k, y^3) ; \\
n_k(\tau, x^k, y^3) &= n_k + 2n_k \int dy^3 \frac{h_3}{| h_4 |^{3/2}}.
\end{align*}
\]
This system can be integrated in explicit form "step by step" as follows: The first equation in (65) is just the 2-d Poisson equation for $\psi(x^k)$ corresponding to the first line in the solution for Ricci solitons (62). The second equation with dependence on flow parameter can be solved and expressed as

$$e^{-\psi} = A_0 e^{2 \int d\tau \tilde{\Lambda}(\tau)}, A_0 = \text{const},$$

where the integration constant can be taken $A_0 = 1$. Such a solution has physical meaning if $\int d\tau \tilde{\Lambda}(\tau) \leq 0$ for an interval $0 \leq \tau \leq \tau_0$ which correlates possible variation of constants induced by effective scalar fields and effective cosmological constant and other possible matter sources.

We have to solve together both equations (66) with separation of spacetime coordinates and flow parameter. To model the evolution of certain Ricci soliton configurations it is necessary to satisfy the conditions

$$|\perp h_3| = |\perp h_4|$$

for certain integration constants $S_0$ and $S_1$ and $\lambda_1 := (\tilde{\Upsilon}[0])$ and

$$\perp \varepsilon(\tau) := S_0 e^{\lambda_1 \tau} + S_1 e^{\lambda_1 \tau} \int d\tau e^{-\lambda_1 \tau} \left[ \perp \tilde{\Upsilon}(\tau) \right].$$

Such configurations have physical importance if there is an interval $0 \leq \tau \leq \tau_0 \ h_3 \to 1$ for increasing $\tau_0$. For certain deformations of stationary solutions in MGTs, the function $|\perp \varepsilon(\tau)| \ll 1$, has to be found from experimental data. We can express

$$h_3 = |\perp h_3(\tau)|, \ h_4 = |\perp h_4(\tau)|$$

where $\psi h_a$ are taken as $\psi h_a(\tau_0)$ from (62) but with

$$\psi h_3 = \frac{\epsilon_3 \epsilon_4}{4} \frac{\psi [\Psi^*]}{\tilde{\Upsilon}[0]} \left[ \psi \right]^2 \quad \text{and} \quad \psi h_4 = h_4[0](x^k) + \frac{\epsilon_3 \epsilon_4}{4 \tilde{\Upsilon}[0]} \left( \psi \Psi^* \right)^2.$$

Putting together above formulas, we find the d–metric coefficients,

$$g_1(\tau, x^k) = g_2 = e^{-\psi} e^{\psi(x^k)} \quad \text{for} \quad e^{-\psi} = A_0 e^{2 \int d\tau \tilde{\Lambda}(\tau)}, A_0 = \text{const};$$

$$h_3(\tau, x^k, y^3) = \left| \perp h_3(\tau) \right| \left[ \frac{\epsilon_3 \epsilon_4}{4} \frac{\psi [\Psi^*]}{\tilde{\Upsilon}[0]} \left[ \psi \right]^2 \right] \quad \text{for} \quad \perp h_3(\tau) \text{ taken as in (70)};$$

$$h_4(\tau, x^k, y^3) = \left| \perp h_3(\tau) \right| \left[ h_4[0](x^k) + \frac{\epsilon_3 \epsilon_4}{4 \tilde{\Upsilon}[0]} \left( \psi \Psi^* \right)^2 \right].$$

(72)
and N–connection coefficients,

\[ w_i(x^k, y^3) = \partial_i \psi / \psi^*; \]

\[ n_k(\tau, x^i, y^3) = 1n_k(\tau, x^i) + 2n_k(\tau, x^i) \int dy^3 \frac{h_3}{|h_4|^{3/2}} \]

\[ = 1n_k(\tau, x^i) + 2\tilde{n}_k(\tau, x^i) \int dy^3 \left( -\psi^2 \right)^2 \]

\[ = 1n_k(\tau, x^i) + 2\tilde{n}_k(\tau, x^i) \int dy^3 \left( -\psi^2 \right)^2 \left( h_4^{[0]}(x^k) + \frac{\epsilon_3 \epsilon_4}{4\Lambda[0]} (\psi^2)^{-5/2} \right), \]

for certain re-defined integration and generation functions.

In such formulas, the generation functions and sources, the integration functions and constants depend on evolution parameter which determine additional anisotropic polarizations of physical values and running of physical constants. The off–diagonal terms \( w_i(x^k, y^3) \) do not depend on evolution parameter. If we take \( 2n_k = 0 \) and \( 1n_k = n_k(x^k) \), the N–connection coefficients do not depend on geometric evolution parameter being determined by a prescribed Ricci soliton configuration.

The coefficients \((72)\) determine generic off–diagonal quadratic elements for solutions of relativistic geometric flows inducing anisotropic polarizations and running of constants and of Ricci solitons,

\[ ds_{Rs}^2 = g_{\alpha\beta}(\tau, x^k, y^3)du^\alpha du^\beta = e^{2f}d\tau(\tau)e^{1(\psi(x^k))}[(dx^1)^2 + (dx^2)^2] + \]

\[ \{1 + \epsilon(\tau)\} \left\{ \frac{\epsilon_3 \epsilon_4}{4h_4} \frac{\psi^2}{\psi^2(x^k)} \right\} \left[ dy^3 \left( -\psi^2 \right)^2 \left( h_4^{[0]}(x^k) + \frac{\epsilon_3 \epsilon_4}{4\Lambda[0]} (\psi^2)^{-5/2} \right), \right\} \]

In these formulas, the flow evolution is determined by certain parameters and nonholonomic constraints in \( R^2 \) gravity and small corrections with dependence of type \( \epsilon(\tau) \). For this class of solutions, the off–diagonal coefficients are determined by a Ricci solitonic back–ground which became dependent on the evolution parameter \( \tau \) via the vertical part of \( \tau \)–metric and N–connection coefficients. Nevertheless, we can fix such integration fuctions when \( w_i = w_i(x^k, y^3) \) and \( n_k = n_k(x^k) \) and \( 2n_k = 0 \) with N–connection and anholonomy coefficients not subjected to geometric flows. Such a nonholonomic geometric flow evolution is for the canonical \( \partial \)–connection \( \partial \) in \( R^2 \) gravity with effective scalar field encoded into a nonholonomically polarized vacuum.

In explicit form, we generate exact solutions the geometric flow/ Ricci soliton equations for certain prescribed values of \( \psi \) and \( \lambda \), corresponding "prime" constants like \( \Lambda, \phi \Lambda \) and \( \zeta^2 \) and following certain assumptions on initial/boundary/asymptotic conditions, physical arguments on symmetries of solutions, compatibility with observational data etc. Variations of constants \( \phi \Lambda(\tau), \varsigma(\tau) \) etc should be taken from certain observational data which are provided, for instance, in \([30]\).
3.1.4 Geometric flows of effective sources and d–metric and N–connection coefficients

For certain conditions, we can find exact solutions of the geometric flow equations when the d–metric and N–connection coefficients and of generating functions depend in a general form on evolution parameter $\tau$. In the simplest way, we have to impose necessary type constraints on the generating functions and then to compute the corresponding horizontal and vertical effective sources.

Using a necessary effective source $\tilde{\Upsilon}(\tau, x^k)$ constrained to satisfy the conditions

$$\tilde{\Upsilon} + \phi \Lambda_0 + 2\varsigma^2 - \frac{1}{2} \partial_\tau \psi = \tilde{\Lambda}_0(\tau),$$

for an effective $\tilde{\Lambda}_0(\tau)$, we find $\psi(\tau, x^k)$ for (43) as a solution of parametric 2-d Poisson equation,

$$\psi^{**} + \psi''' = 2\tilde{\Lambda}_0(\tau).$$

We can generate a class of solutions of geometric flow equations (44)–(49) for arbitrary $h_4(\tau, x^k, y^3), h_3^* \neq 0$ taken as a generating function if we consider an effective source $\Upsilon(\tau, x^k, y^3)$ determined by the condition

$$\Upsilon + \phi \Lambda + 2\varsigma^2 - \partial_\tau \ln |\omega^2 h_4| = \Lambda_0 \neq 0,$$

where $\Lambda_0$ is an effective cosmological constant. For such a condition, the system of equations (42) and (44) transforms into

$$\sqrt{|h_3|} = \frac{h_4^*}{\Psi \sqrt{|h_4|}} \quad \text{and} \quad h_3 = \frac{\Psi^* h_4^*}{\Psi 2h_4} \Lambda_0,$$

for two un–known functions $h_3(\tau, x^k, y^3)$ and $\Psi(\tau, x^k, y^3)$. Taking the square of the first equation with $h_a = \epsilon_a |h_a|, \epsilon_a = \pm 1$, we compute

$$\Psi^2 = B(\tau, x^k) + \frac{4\epsilon_3 \epsilon_4}{\Lambda_0} h_4 \quad \text{and} \quad (75)$$

$$h_3 = \epsilon_3 \epsilon_4 \frac{h_4^*[B(\tau, x^k) + \frac{2\epsilon_4}{\Lambda_0} h_4]}{h_4^* B(\tau, x^k) + \frac{2\epsilon_4}{\Lambda_0} h_4}, \quad (76)$$

for an integration function $B(\tau, x^k)$.

We can solve the equation (45) if we take $h_3 = h_4$ considering both such values determined by the same generating function $h_4(\tau, x^k, y^3)$. In general, there are similar solutions with $h_3 \neq h_4$ (being involved the formula (76)) but it is a difficult task to solve the mentioned equation for arbitrary $\omega$.

The algebraic equation (46) are solved in general form using the formula (75),

$$w_i(\tau, x^k, y^3) = \frac{\partial_i \Psi}{\Psi^*} = \frac{\partial_i \Psi^2}{\partial_5 (\Psi^2)} = (h_4^*)^{-1} \partial_i [\frac{\Lambda_0}{4\epsilon_3 \epsilon_4} B(\tau, x^k) + h_4)].$$

(77)
We find the complete set of $N$–connection coefficients by integrating two times on $y^3$ in (17) using the condition $h_3 = h_4$,

$$n_k(\tau, x^i, y^3) = n_k(\tau, x^i) + 2\tilde{n}_k(\tau, x^i) \int dy^3 \left(\sqrt{|h_4|}\right)^{-1}. \quad (78)$$

In general, we can use any $\omega(\tau, x^k, y^3)$ as a solution of the equation (49)

$$\partial_k \omega + w_k(\tau, x^i, y^3)\omega^* + n_k(\tau, x^i, y^3)\partial_4 \omega = 0,$$

for coefficients determined by $h_4$ and respective integration functions, see (77) an (78). In particular, we can take $\omega = 1$ and generate solutions for geometric evolution of stationary configurations. As solutions of the equations (49), we can consider distributions of a scalar field subjected to the conditions (39) and (40) resulting in modifications with an effective cosmological constant.

Above formulas determine a quadratic element

$$ds^2_{iRs} = g_{\alpha\beta}(\tau, x^k, y^3) du^\alpha du^\beta = e^{\psi(\tau, x^k)}[(dx^1)^2 + (dx^2)^2] + \omega^2(\tau, x^i, y^3)h_4(\tau, x^i, y^3)$$

$$\left\{ [dy^3 + \frac{\partial_i(\frac{\Lambda_0}{4\epsilon_3\epsilon_4}B(\tau, x^k) + h_4)}{h_4^*}dx^i]^2 + [dt + (1n_k + 2\tilde{n}_k \int dy^3 \sqrt{|h_4|}^{-1})dx^i]^2 \right\}. \quad (79)$$

This class of solutions is a general one with evolution of $N$–connection coefficients and flows of the nonholonomically induced torsion. Such geometric flows may transform one class of Ricci solitons into another one, i.e. a MGT into another MGT, or into a solution in GR (if the final torsion is constrained to be zero). Mutual transforms of classes of (off-) diagonal solutions in GR can be described as some particular examples of such geometric flow evolution models.

### 3.2 Extracting Levi–Civita configurations

The solutions for Ricci solitons (63) and their factorized geometric evolution (73) and non-factorized geometric flow evolution solutions are defined for the canonical $d$–connection $\hat{D}$. There are nontrivial coefficients of nonholonomically induced torsion which can be computed by introducing the coefficients (31) (with fixed flow parameter in the metric anstaz) into (A.1) and (A.2). We have to consider certain restricted classes of parameterizations and nonholonomic constraints on the $d$–metric and $N$–connection coefficients in order to satisfy the zero torsion conditions (A.3) and extract Levi–Civita, LC, configurations. For ansatz (28) with dependence on flow parameter $\tau$, such conditions are equivalent to the system equations (see details in [16, 17, 18])

$$w_i^* = (\partial_i - w_i\partial_3) \ln \sqrt{|h_4[\tau]|}, \quad (\partial_i - w_i\partial_3) \ln \sqrt{|h_3[\tau]|} = 0, \quad (80)$$

$$\partial_i w_j = \partial_j w_i, \quad n_i^*[\tau] = 0, \quad \partial_i n_j[\tau] = \partial_j n_i[\tau],$$

where we denoted in brief, for instance, $h_4[\tau] = h_4(\tau, x^i, y^3)$.

Let us consider, for simplicity, certain classes of solutions with factorized parameterizations of $d$–metrics like (64) which allows to model geometric evolution of self–similar fixed LC–configurations for Ricci solitons.
Any functional $\tilde{\Psi}[\Psi[\tau]]$ satisfies the conditions

$$
e_i \tilde{\Psi}[\tau] = (\partial_i - w_i \partial_3) \tilde{\Psi}[\tau] = \frac{\partial \tilde{\Psi}}{\partial \Psi} (\partial_i - w_i \partial_3) \Psi[\tau] \equiv 0,$$

which follow from (17). We can chose, for instance, $\tilde{\Psi} = \ln \sqrt{h_4[\tau]}$ when $e_i \ln \sqrt{h_4[\tau]} = 0$. If we work with classes of generating functions $\Psi = \tilde{\Psi}(\tau, x^k, y^3)$ for which there are satisfied the integrability conditions

$$(\partial_i \tilde{\Psi}[\tau])^* = \partial_i(\tilde{\Psi}^*[\tau]),$$

we obtain $w_i^*[\tau] = e_i \ln |\tilde{\Psi}^*[\tau]|$. For a given functional dependence $h_3[\Psi[\tau], \Upsilon, \phi \Lambda(\tau), \zeta(\tau)]$ and using $e_i \tilde{\Psi} = 0$, we can express

$$e_i \ln \sqrt{h_3[\tau]} = e_i [\ln |\tilde{\Psi}^*[\tau]| - \ln \sqrt{\Upsilon[\tau]}].$$

In result, $w_i^* = e_i \ln \sqrt{h_3[\tau]}$ if $e_i \ln \sqrt{\Upsilon[\tau]} = 0$. This is possible for any $\Upsilon = \text{const}$, or any effective source expressed as a functional $\Upsilon(x^i, y^3) = \Psi[\tilde{\Psi}, \phi, \Lambda(\tau), \zeta(\tau)]$ with parametric coordinate dependencies.

The conditions that $\partial_i w_j = \partial_j w_i$ can be expressed in conventional form via any function $\tilde{A} = \tilde{A}(\tau, x^k, y^3)$ for which

$$w_i = \tilde{w}_i = \partial_i \tilde{\Psi} / \tilde{\Psi}^* = \partial_i \tilde{A}.$$  

If a functional $\tilde{\Psi}$ is prescribed, we have to solve a system of first order PDEs which allows to find a function $\tilde{A} [\tilde{\Psi}]$. For the second set of N–coefficients, we chose $n_j(\tau, x^k) = \partial_j n(\tau, x^k)$ for a function $n(\tau, x^k)$. As a matter of principle, we can consider running on a geometric flow parameter, like $n(\tau, x^k)$ considering a more generalized class of integration functions.

We can generate off–diagonal torsionless solutions of the Ricci soliton equations and generalized Einstein equations for $R^2$–gravity, with possible polarizations of fundamental constants determined by geometric flows if we chose certain subclasses of generating functions and effective sources in (63) and (73), when

$$\tilde{\Upsilon} = \Upsilon(\tau, x^i, y^3) = \Upsilon[\tilde{\Psi}], w_i = \partial_i \tilde{A}[\tilde{\Psi}[\tau]], n_i = \partial_i n,$$

and the generating function $\Psi = \tilde{\Psi}$ and "associated" $\tilde{A}$ for $N_i^4$–coefficients are subjected to the conditions (81) and (82).

If the generating/effective functions and sources are subjected to the LC–conditions (81)–(83), we obtain such quadratic linear elements:

For Ricci solitons with zero nonholonomic torsion (LCRs) [a particular case of solutions (63)],

$$ds^2_{\text{LCRs}} = g_{\alpha \beta}(x^k, y^3) du^\alpha du^\beta = e^{\psi[(dx^1)^2 + (dx^2)^2]} +$$

$$\omega^2_4 \frac{\epsilon^4}{h_4} (\frac{\tilde{\Psi}^*}{\tilde{\Upsilon}})^2 [dy^3 + \partial_i \tilde{A}[\tilde{\Psi}]) dx^i dx^i]^2 + \omega^2 \gamma h_4 [\tilde{\Psi}, \tilde{\Upsilon}][dt + \partial_k n(x^k) dx^k]^2,$$

where $\gamma \Psi \rightarrow \frac{\tilde{\Psi}}{\tilde{\Upsilon}}$. 

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In the case of geometric flows with zero nonholonomic torsion (i.e. torsionless geometric evolution of type (73)),

\[
\begin{align*}
    ds^2_{LC} &= g_{\alpha\beta}(\tau, x^k, y^3)du^\alpha du^\beta = e^{2\int d\tau \bar{\Lambda}(\tau)e^{\psi(x^k)}}[(dx^1)^2 + (dx^2)^2] + \{1 + \varepsilon(\tau)\} \\
    \{\frac{\varepsilon_4 e^{\psi}}{4(\bar{\Gamma}_{[0]})} \} \bar{h}_4 \left( \bar{\Psi}^* \right)^2 \left[ dy^3 + (\partial_i \bar{A}[\bar{\Psi}])dx^i \right]^2 + \bar{h}_4(x^k, y^3) \left[ dt + \partial_k n(\tau, x^k)dx^k \right]^2 ,
\end{align*}
\]

where the LC–conditions (81)–(83) hold for \( \bar{\Psi} = \bar{\Psi}(\tau, x^k, y^3) \) and \( \bar{\Gamma}_{[0]} = \text{const.} \). The coefficients of these generic off–diagonal metrics also generate exact solutions for geometric flows and Ricci solitons with effective matter source but with zero torsion. We note that the metrics are generic off–diagonal if the anholonomic coefficients \( W_{\alpha\beta}^{\gamma} \) are not zero.

In a similar form, we can model LC geometric evolution of metrics of type (79) [in brief, LCgev], with \( \Psi \rightarrow \bar{\Psi} \) and \( \partial_i \bar{A}[\bar{\Psi}] = \partial_i(\frac{A_{\alpha\beta}B(\tau, x^k)}{h_4})/h_4^* \), when

\[
\begin{align*}
    ds^2_{LCgev} &= g_{\alpha\beta}(\tau, x^k, y^3)du^\alpha du^\beta = \bar{e}^{\psi(\tau, x^k)}[(dx^1)^2 + (dx^2)^2] + \\
    &\omega^2(\tau, x^i, y^3)h_4(x^k, y^3)\left[ dy^3 + \partial_i \bar{A}[\bar{\Psi}(\tau, x^i, y^3)]dx^i \right]^2 + [dt + \partial_k n(\tau, x^k)dx^k] \).
\end{align*}
\]

We can generate subsets of solutions of (85) with N–coefficients which do not depend on flow parameter \( \tau \) but only the d–metric coefficients \( g_i \) and \( h_a \) are functions on \( \tau \) and spacetime coordinates preserving the Killing symmetry on \( \partial_4 = \partial/\partial t \). If \( \Psi \rightarrow \bar{\Psi} \), and \( n = n(x^k) \) in the two last formulas, we can consider that for \( S_0 = S_1 = 0 \) and \( \bar{\Lambda}(\tau) = 0 \), the solutions for geometric evolution transform into a Ricci soliton configuration (84). The LC–configurations with \( \tau \)–dependence describe a self–consistent geometric evolution of LC Ricci solitons for any interval \( 0 \leq \tau \leq \tau_0 \) when the exponential on \( \tau \) terms are not singular.

The class of generic off–diagonal metrics (86) define LC configurations of geometric evolution of exact solutions in \( R^2 \) and/or GR theory. The N–connection structure for such solutions possess a nontrivial dependence on parameter \( \tau \). Such torsionless configurations may mix under geometric evolution different types of Ricci solitons and transform mutually solutions from a MGT into another MGTs, or in GR.

Variations of values \( \bar{\mathcal{T}}(\tau), \phi_\Lambda(\tau), \bar{\mathcal{Y}}, \phi_\Lambda(\tau), \mathrm{c}(\tau) \) etc. have to be taken from observational data [30] (the Dirac’s idea on variation of physical constants is re–considered for modified theories of gravity). We conclude that geometric flow solutions can explain possible locally anisotropic polarizations and running of d–metric and N–connection coefficients and of fundamental physical constants.

### 3.3 Small parametric deformations of off–diagonal solutions for geometric flows and Ricci solitons

It is quite difficult to provide any physical interpretation of general classes of solutions for geometric flows and Ricci solitons constructed in previous subsections. Nevertheless, such theoretical and phenomenological problems can be solved in a more simple form if we consider sub–sets of solutions generated as deformations on a small parameter. It is supposed that in certain limits and/or for special classes of (non) holonomic constraints transform into well
defined and/or known classes of physically important solutions. We emphasize that even for small parameters, the corresponding systems of nonlinear PDEs are generic nonlinear ones with decoupling properties. Mathematically, the solutions can be constructed as exact ones for certain sets of prescribed parameters and generating functions. Small generic off–diagonal deformations of some known (or to certain almost known solutions) are considered in this work only with the aim to understand the physical meaning of some classes of geometric evolution/Ricci soliton solutions with small polarization/running of constants and nonlinear off–diagonal only with the aim to understand the physical meaning of some classes of geometric evolution/Ricci soliton solutions with small polarization/running of constants and nonlinear off–diagonal gravitational interactions in MGTs.

Let us consider a "prime" pseudo–Riemannian metric $\tilde{g} = [\tilde{g}_{ij}, \tilde{h}_a, \tilde{N}_b^j]$, when

$$
\begin{align*}
    ds^2 &= \tilde{g}_{ij}(x^k)(dx^i)^2 + \tilde{h}_a(x^k,y^3)(dy^a)^2(\tilde{e}^a)^2, \\
    \tilde{e}^3 &= dy^3 + \tilde{w}_i(x^k,y^3)dx^i, \quad \tilde{e}^4 = dt + \tilde{n}_i(x^k,y^3)dx^i.
\end{align*}
$$

Such a metric is diagonalizable if there is a coordinate transform $u^a' = u^a(u^\alpha)$ with $\tilde{w}_i = \tilde{n}_i = 0$. In order to construct exact solutions with non–singular coordinate conditions it may be important to work with "formal" off–diagonal parameterizations when the coefficients $\tilde{w}_i(x^k,y^3)$ and/or $\tilde{n}_i(x^k,y^3)$ are not zero but the anholonomy coefficients $\tilde{W}_{\alpha\beta\gamma}(u^\mu) = 0$, see (16).

We suppose that some data $(\tilde{g}_{ij}, \tilde{h}_a)$ may define a diagonal exact solution in MGT or in GR (for instance, a black hole, BH, configuration). Our goal is to study certain small generic off–diagonal parametric deformations into certain target metrics

$$
\begin{align*}
    ds^2 &= \eta_i(x^k)\tilde{g}_i(x^k)(dx^i)^2 + \eta_a(x^k,y^3)\tilde{h}_a(x^k,y^3)(e^a)^2, \\
    e^3 &= dy^3 + w_i\tilde{w}_i(x^k,y^3)dx^i, \quad e^4 = dt + n_i\tilde{n}_i(x^k,y^3)dx^i,
\end{align*}
$$

where the coefficients $(g_a = \eta_a\tilde{g}_a, w_i\tilde{w}_i, n_i\tilde{n}_i)$ define, for instance, a Ricci soliton configuration determined by a class of solutions (62). For certain well–defined conditions, we can express

$$
\begin{align*}
    \eta_i &= 1 + \varepsilon \chi_i(x^k,y^3), \quad \eta_a = 1 + \varepsilon \chi_a(x^k,y^3) \quad \text{and} \\
    w_i &\eta_i = 1 + \varepsilon w_i\chi_i(x^k,y^3), \quad n_i \eta_i = 1 + \varepsilon n_i\chi_i(x^k,y^3),
\end{align*}
$$

for a small parameter $0 \leq \varepsilon \ll 1$, when (88) transforms into (87) for $\varepsilon \to 0$ and $w_i = n_i = 0$. In general, there are not smooth limits from such nonholonomic deformations which can be satisfied for arbitrary generation and integration functions, integration constants and general (effective) sources. The goal of this subsection is to analyze such conditions when $\varepsilon$–deformations with nontrivial $N$–connection coefficients can be related to new classes of solutions of geometric flow and/or Ricci soliton equations.

### 3.3.1 $\varepsilon$–deformations for stationary Ricci solitons

Let us denote nonholonomic $\varepsilon$–deformations of certain prime d–metric (87) into a target one (88) with polarizations (89) in the form $\tilde{g} \to \varepsilon g = (\varepsilon g_{ij}, \varepsilon h_a, \varepsilon N_b^j)$. The goal of this subsection
is to compute the formulas for \(\varepsilon\)-deformations of prime d-metrics resulting in solutions of type (63), or (84), for \(\gamma\omega = 1\).

Deformations of \(h\)-components are characterized by

\[
\varepsilon g_i = \hat{g}_i(1 + \varepsilon \chi_i) = e^{\psi(x^k)}
\]

being a solution of (51). For \(\gamma\psi = 0\psi(x^k) + \varepsilon \psi(x^k)\) and \(\hat{\nu}_3 = \hat{\nu}_3(x^k) + \nu_3(x^k)\), we compute the deformation polarization functions

\[
\chi_i = e^{\psi(x^k)} \frac{\hat{\nu}_i}{\hat{\nu}_i}.
\]

In this formula, the generating and source functions are solutions of

\[
\hat{\nu}_3 = \int dy^3 \left( \hat{\Psi}^2 \right) = (1 + \varepsilon \chi_3) \hat{g}_3; \quad \hat{\nu}_4 = \int dy^3 \left( \hat{\Psi}^2 \right) = (1 + \varepsilon \chi_4) \hat{g}_4.
\]

Parameterizing the generation function

\[
\hat{\nu}_3 = \hat{\Psi}(x^k, y^3)[1 + \varepsilon \chi(x^k, y^3)],
\]

we introduce this value in (91). We obtain

\[
\chi_4 = \frac{\varepsilon_3 e_4}{4 \hat{g}_4} \int dy^3 \left( \hat{\Psi}^2 \right) = \frac{\varepsilon_3 e_4}{4 \hat{g}_4} \int dy^3 \left( \hat{\Psi}^2 \right) = 4\varepsilon_3 e_4 (\hat{g}_4 - h_4^0).
\]

Such formulas show that we can compute \(\chi_4\) for any deformation \(\chi\) from a 2-hypersurface \(y^3 = y^3(x^k)\) defined in non-explicit form from \(\hat{\Psi} = \hat{\Psi}(x^k, y^3)\) when the integration function \(h_4^0(x^k)\), the prime value \(\hat{g}_4(x^k)\) and the fraction \(\left( \hat{\Psi}^2 \right) / \hat{\nu}_3\) satisfy the condition (93).

We can find the formula for hypersurface \(\hat{\Psi}(x^k, y^3)\) by finding the value of \(\hat{\nu}_3\). Introducing (92) into (90), we get

\[
\chi_3 = 2(\chi + \frac{\hat{\Psi}}{\hat{\Psi}} \chi^*) - \chi_4 = 2(\chi + \frac{\hat{\Psi}}{\hat{\Psi}} \chi^*) - \frac{\varepsilon_3 e_4}{4 \hat{g}_4} \int dy^3 \left( \hat{\Psi}^2 \right) = 2(\chi + \frac{\hat{\Psi}}{\hat{\Psi}} \chi^*) - \frac{\varepsilon_3 e_4}{4 \hat{g}_4} \int dy^3 \left( \hat{\Psi}^2 \right)
\]

which allows to compute \(\chi_3\) for any data \(\left( \hat{\Psi}, \hat{g}_4, \chi \right)\). The formula for a compatible source is

\[
\hat{\nu}_3 = \pm \frac{\hat{\Psi}^*}{2 \sqrt{|\hat{g}_3 h_4^0|}},
\]

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which transforms (93) into a 2-d hypersurface formula \( y^3 = y^3(x) \) defined in non-explicit form from
\[
\int dy^3 \tilde{\Psi} = \pm (h^{[0]}_4 - \tilde{g}_4)/\sqrt{|\tilde{g}_3 h^{[0]}_4|}.
\]

The \( \varepsilon \)-deformations of \( d \)-metric and \( N \)-connection coefficients \( \gamma w_i(\tau_0) = \partial_i \gamma \Psi/\gamma \Psi^* \) for nontrivial \( \tilde{w}_i(\tau_0) = \partial_i \tilde{\Psi}/\tilde{\Psi}^* \) are found following formulas (92) and (89),
\[
\tilde{w}_i = \frac{\partial_i (\gamma \tilde{\Psi})}{\gamma \tilde{\Psi}} - \frac{(\gamma \tilde{\Psi})^*}{\gamma \tilde{\Psi}^*},
\]
where there is not summation on index \( i \). In a similar way, one computes the deformation on the \( n \)-coefficients (we omit such details).

Summarizing above formulas, we obtain such coefficients for \( \varepsilon \)-deformations of a prime metric (87) into a target Ricci soliton stationary metric:
\[
\varepsilon g_i(\tau_0) = \tilde{g}_i[1 + \varepsilon \chi_i] = [1 + \varepsilon \varepsilon^0 \psi^0_{\hat{\gamma}}/\hat{\gamma} \hat{g}_i] \tilde{g}_i \text{ as a solution of 2-d Poisson equations (51)};
\]
\[
\varepsilon h_3(\tau_0) = [1 + \varepsilon \chi_3] \tilde{g}_3 = [1 + \varepsilon \left( 2(\chi + \frac{\hat{\Psi}}{\hat{\Psi}^*} \chi^*) - \frac{\varepsilon_3 \epsilon_4}{4 \tilde{g}_4} \int dy^3 \left( \frac{\left( \Psi^2 \right)^*}{\tilde{\Psi}^*} \right) \right) \tilde{g}_3;
\]
\[
\varepsilon h_4(\tau_0) = [1 + \varepsilon \chi_4] \tilde{g}_4 = \left[ 1 + \varepsilon \frac{\varepsilon_3 \epsilon_4}{4 \tilde{g}_4} \int dy^3 \left( \frac{\left( \Psi^2 \right)^*}{\tilde{\Psi}^*} \right) \right] \tilde{g}_4;
\]
\[
\varepsilon w_i(\tau_0) = [1 + \varepsilon \varepsilon \chi_i] \tilde{w}_i = [1 + \varepsilon \left( \frac{\partial_i (\gamma \tilde{\Psi})}{\gamma \tilde{\Psi}} - \frac{(\gamma \tilde{\Psi})^*}{\gamma \tilde{\Psi}^*} \right) \tilde{w}_i;
\]
\[
\varepsilon n_i(\tau_0) = [1 + \varepsilon \varepsilon \chi_i] \tilde{n}_i = [1 + \varepsilon \left( \frac{\partial_i (\gamma \tilde{\Psi})}{\gamma \tilde{\Psi}} - \frac{(\gamma \tilde{\Psi})^*}{\gamma \tilde{\Psi}^*} \right) \tilde{n}_i;
\]
where \( \tilde{n}_i(x) \) is a re-defined integration function including contributions from the prime metric. The corresponding quadratic element
\[
d s_{2 \varepsilon Rs}^2 = \varepsilon \varepsilon g_{\alpha \beta}(x, y^3) du^\alpha du^\beta = \varepsilon g_i(x^3) [(dx^1)^2 + (dx^2)^2] + \varepsilon h_3(x, y^3) [dy^3 + \varepsilon w_i(x^3, y^3) dx^3]^2 + \varepsilon h_4(x, y^3) [dt + \varepsilon n_k(x^3, y^3) dx^k]^2.
\]

3.3.2 Geometric flow evolutions of \( \varepsilon \)-deformed stationary Ricci solitons

Introducing the data (96) into (73), we find the quadratic element
\[
ds_{2 \varepsilon Rs}^2 = g_{\alpha \beta}(\tau, x^3, y^3) du^\alpha du^\beta = e^2 f d\tau \delta(\tau) \varepsilon g_i(x^3) [(dx^1)^2 + (dx^2)^2] + [1 + \varepsilon (\tau)]^2 \left\{ \frac{\varepsilon_3 \epsilon_4}{4} \varepsilon h_4(x^3, y^3) \left( \frac{\varepsilon \Psi^*}{\tilde{\Psi}^*} \right)^2 \left[ dy^3 + \varepsilon w_i dx^3 \right]^2 + \varepsilon h_4(x^3, y^3) \left[ dt + \varepsilon n_k dx^k \right]^2 \right\}.
\]

For simplicity, we do not linearize on \( \varepsilon \) in \( \varepsilon \tilde{\Psi}^*/h_4 \), which is determined by any generating function \( \chi(x^3, y^3) \) and corresponding integration functions.
4 Geometric Evolution of Black Ellipsoids for Ricci Solitons and $R^2$ Gravity

The goal of this section is to construct explicit examples of stationary geometric flow and Ricci soliton exact solutions of type (63), (73), (84) and (85) which for certain classes of nonholonomic constraints and well-defined limits transform into black hole solutions for $R^2$ gravity [19]. We construct also a model of geometric evolution of $R^2$ black holes into 3-d KdP solitonic (from Kadomtsev–Petviashvili, references and geometric methods in [31, 32, 15, 33, 34, 18]) configurations for non–factorizable solutions.

4.1 Ricci solitonic black ellipsoids and limits to black hole solutions in $R^2$ and GR theories

The techniques of $\varepsilon$–deformations outlined in section 3.3 is applied for off–diagonal generalizations of "prime" black hole solutions.

4.1.1 Prime and target metrics

Let us consider a "prime" metric

$$ds^2 = \hat{g}_{\alpha^i\beta^j}(x^k)du^\alpha du^\beta = (1 - \frac{M}{r} + \frac{K}{r^2})^{-1}dr^2 + r^2d\theta^2 + r^2\sin\theta d\varphi^2 - (1 - \frac{M}{r} + \frac{K}{r^2})dt^2$$

$$= \hat{g}_V(dx^V)^2 + \hat{g}_2(x^V)(dx^2)^2 + \hat{h}_3(x^V,x^2)(dy^3)^2 + \hat{h}_4(x^V)(dy^4)^2,$$

for some constants $M$ and $K$, where

$$x^i(r) = \int dr(1 - \frac{M}{r} + \frac{K}{r^2})^{-1/2}, x^2 = \theta, y^3 = \varphi, y^4 = t;$$

$$\hat{g}_V = 1, \hat{g}_2(x^V) = r^2(x^V), \hat{h}_3 = r^2(x^V)\sin(x^2), \hat{h}_4 = -(1 - \frac{M}{r(x^V)} + \frac{K}{r^2(x^V)})$$

are defined on certain carts on an open region $U \subset V$, where $x^V(r)$ allows to find $r(x^V)$ in a unique form. This metric was studied as a spherical symmetric vacuum solution in $R^2$ gravity [19] (in our approach, of the Ricci soliton equations (8)). Such a solution does not exist if $R = 0$ (for LC-configurations) because it is not allowed by transforms (8).

We consider a coordinate transform $u^\alpha = u^\alpha(u^\alpha)$ with $\varphi = \varphi(y^3,x^k)$ and $t = t(y^4,x^k)$. In such cases,

$$d\varphi = \frac{\partial\varphi}{\partial y^a}[dy^a + (\partial_3\varphi)^{-1}(\partial_k\varphi)dx^k] \text{ and } dt = \frac{\partial t}{\partial y^i}[dy^i + (\partial_4t)^{-1}(\partial_k t)dx^k]$$

for $\partial_i\varphi = \partial\varphi/\partial x^i$ and $\partial_a\varphi = \partial\varphi/\partial y^a$. Choosing

$$\hat{w}_i = \partial_i \hat{\Psi}/\hat{\Psi}^* = (\partial_3\varphi)^{-1}(\partial_i\varphi) \text{ and } \hat{n}_i = \partial_i n(x^k) = (\partial_4t)^{-1}(\partial_i t),$$

29
for any \( \hat{\Psi} \) (91), we express (98) as

\[
ds^2 = \hat{g}_1(x^1)'^2 + \hat{g}_2(x^1)'(dx^2)^2 + \hat{g}_3(dy^3 + \hat{w}_i(x^k)dx^i)^2 + \hat{g}_4(x^k(x^k'))(dy^4 + \hat{n}_i(x^k)dx^i)^2, \tag{99}
\]

for \( \hat{g}_3(x^k(x^k')) = (\partial_\tau \phi)^2 x^2(\tau') \sin(x^2) \) and \( \hat{g}_4(x^k(x^k')) = -(\partial_\tau t)^2 (1 - \frac{M}{r} + \frac{K}{r^2}). \tag{100} \)

This is a "formal" off–diagonal metric of type (87) with nontrivial values \( \hat{h}_a, \hat{w}_i \) and \( \hat{n}_i \), but \( \hat{W}^\alpha_{\beta i} (w^a) = 0 \), see (10). Using such an ansatz, we can apply the AFDM with \( \varepsilon \)-deformation of geometric/ physical objects and physical parameters as we described in subsection 3.3.

Our goal is to show how the metric (98) and/or (99) can be off–diagonally deformed into certain classes of "target" new solutions of type (84) and (85) with ellipsoidal configurations which have a well–defined physical interpretation. In a similar way, we can consider solutions with nonholonomically induced torsion if \( \hat{W}^\alpha_{\beta i} (w^a) \neq 0 \). The condition \( R \neq 0 \) together with \( \hat{R} \neq 0 \), see (A.6), can be preserved for such solutions. Here we note that geometric flows with \( \hat{R} \neq 0 \) may allow evolution of solutions via \( R = 0 \) by transforming one class of Ricci soliton solutions for \( R^2 \) gravity into another one. We consider target metrics (88) with \( \varepsilon \)-deformations (89) resulting in solutions of geometric flow or Ricci soliton equations,

\[
ds^2 = \left[ 1 + \varepsilon \chi_i (\tau, x^k, y^3) \right] \hat{g}_i(x^k)(dx^i)^2 + \left[ 1 + \varepsilon \chi_a (\tau, x^k, y^3) \right] \hat{g}_a(x^k, y^3)(e^a)^2, \tag{101}
\]

\[
e^3 = dy^3 + \left[ 1 + \varepsilon \chi_i (\tau, x^k, y^3) \right] \hat{w}_i(x^k, y^3)dx^i, e^4 = dy^4 + \left[ 1 + \varepsilon \chi_i (\tau, x^k, y^3) \right] \hat{n}_i(x^k, y^3)dx^i.
\]

For such target metrics, we can fix \( \tau = \tau_0 \) in order to generate new Ricci soliton configurations or to consider factorizations of type (64). The generating function \( \Psi = \hat{\Psi} \) and \( N^a \)-coefficients are subjected to the conditions (81) and (82) for a function \( \hat{A} \) determined as a solution of \( \partial_\tau \hat{\Psi} = (\partial_\tau \hat{A}) \hat{\Psi}^* \). In this quadratic element, there are also considered the so–called polarization functions \( \eta_\alpha (\tau, x^k, y^a) \approx 1 + \varepsilon \chi_\alpha (\tau, x^k, y^a) \) which can be used for computing small parametric deformation effects from a prime metric. If for a class of solutions there are smooth limits \( \varepsilon \rightarrow 0 \) and \( N^a_i \rightarrow 0 \), we obtain that \( g_{\alpha \beta} \rightarrow \hat{g}_{\alpha \beta} \). For general nonlinear generic off–diagonal geometric evolution and/or gravitational interactions such limits with \( \varepsilon \rightarrow 0 \) do not exist. Nevertheless, it is important to study subclasses of solutions with smooth configurations for \( \varepsilon \rightarrow 0 \) and \( N^a_i \rightarrow 0 \) because it is more "easy" to provide certain physical interpretation for such metrics.

Finally, we note that we can impose additional constraints on (101) in order to model, for instance, geometric evolution of Ricci solitons with a \( \tau = \tau_0 \) fixed N–connection structure when \( w_\chi_i = \hat{w}_\chi_i (x^k, y^3) \) and \( n_\chi_i = \hat{n}_\chi_i (x^k, y^3) \). We chose \( \hat{\Psi} = \hat{\Psi}(x^k, y^3)[1 + \varepsilon \chi(x^k, y^3)] \) as in (22) and consider

\[
ds^2 = g_{\alpha \beta}(\tau, x^k, y^3)du^\alpha du^\beta = [1 + 2 \int d\tau \hat{\Lambda}(\tau)] \left[ 1 + \varepsilon \chi_i (\tau, x^k, y^3) \right] \hat{g}_i(x^k)[(dx^1)^2 + (dx^2)^2] + \left[ 1 + \varepsilon(\tau) \right] \left[ 1 + \varepsilon \chi_3(x^k, y^3) \right] \hat{g}_3(x^k, y^3) \left[ dy^3 + \left( 1 + \varepsilon \chi_1(x^k, y^3) \right) \hat{w}_i(x^k, y^3)dx^i \right]^2 + \left[ 1 + \varepsilon \chi_4(x^k, y^3) \right] \hat{g}_4(x^k, y^3) \left[ dy^4 + \left( 1 + \varepsilon \chi_2(x^k, y^3) \right) \hat{w}_i(x^k, y^3)dx^i \right]^2. \tag{102}
\]

We consider \( \tau \)-depending terms \( 2 \int d\tau \hat{\Lambda}(\tau) \) and \( \varepsilon(\tau) \) to be of order \( \varepsilon \) introducing additional dependencies of physical constants and polarization functions on flow parameter.
4.1.2 Black ellipsoids in $R^2$ gravity as Ricci solitons

Let us model an $\varepsilon$–deformation of (98) into an ellipsoidal Ricci soliton configuration when

$$h_{4'} = -(1 - \frac{M}{r} + K \frac{r^2}{r^2}) \left[ 1 - \varepsilon \frac{M}{r} \cos(\omega_0 \phi + \varphi_0) \right]$$

$$= \hat{h}_{4'}(x') \left[ 1 + \varepsilon \frac{M}{r} (\hat{h}_{4'})^{-1} \cos(\omega_0 \phi + \varphi_0) \right] \approx \left[ 1 - \hat{M} (\phi) + K \right]$$

(103)

for some constant values $\omega_0 \phi + \varphi_0$ and anisotropically polarized mass

$$\hat{M} (\phi) = M [1 + \varepsilon \cos(\omega_0 \phi + \varphi_0)].$$

(105)

We obtain a zero value of $h_{4'}$, i.e. the effective horizons for (104), if $r_+(\phi) = \frac{M}{2} \pm \frac{M}{2} \sqrt{1 - \frac{4K}{M^2}}$.

In the linear approximations on $\varepsilon$ and $K$, we write

$$r_+ \approx \frac{M}{1 - \varepsilon \cos(\omega_0 \phi + \varphi_0)},$$

which is just the parametric equation of an ellipse with radial parameter $\hat{r}_+ = M$ and eccentricity $\varepsilon$.

Following formulas (95) for $\hat{h}_4(\tau_0) \approx h_{4'}$ from (103), we can identify up to coordinate transforms

$$-\frac{1}{4} \int dy^3 (\hat{\Psi}^2 \chi)^* \frac{\hat{\Psi}}{\hat{\Psi}^*} = \frac{M}{r} \cos(\omega_0 \phi + \varphi_0).$$

For $\hat{\Psi} = const$, we find the polarization function for ellipsoidal configurations

$$\chi = \hat{\chi} = 4 \frac{M}{r} \hat{\Psi}^{-2} \cos(\omega_0 \phi + \varphi_0).$$

Introducing $\hat{\chi}$ into formulas for d–metric coefficients (95), we compute for ellipsoid deformations of (98),

$$\hat{g}_1 (\tau_0) = \hat{g}_1 [1 + \varepsilon \chi_i] = [1 + \varepsilon \hat{g}_1 \psi^*]\hat{g}_1 \hat{\Psi}$$

$$\hat{g}_3 (\tau_0) = [1 + \varepsilon \hat{g}_3 \chi_3] \hat{g}_3 = \left[ 1 + \varepsilon \left( 2 (\hat{\chi} + \hat{\Psi}^* \hat{\chi}^*) + \frac{1}{4\hat{g}_4} \frac{\hat{\Psi}^2 \hat{\chi}}{\hat{\Psi}^*} \right) \right] \hat{g}_3;$$

$$\hat{g}_4 (\tau_0) = [1 + \varepsilon \hat{g}_4 \chi_4] \hat{g}_4 = \left[ 1 - \varepsilon \frac{1}{4\hat{g}_4} \frac{\hat{\Psi}^2 \hat{\chi}}{\hat{\Psi}^*} \right] \hat{g}_4;$$

(106)

$$\hat{w}_i (\tau_0) = [1 + \varepsilon \hat{w}_i \chi_i] \hat{w}_i = \left[ 1 + \varepsilon \left( \frac{\partial_i (\hat{\chi} \hat{\Psi})}{\partial_i \hat{\Psi}} - (\hat{\chi} \hat{\Psi}^*) \right) \right] \hat{w}_i;$$

$$\hat{n}_i (\tau_0) = [1 + \varepsilon \hat{n}_i \chi_i] \hat{n}_i = \left[ 1 + \varepsilon \hat{n}_i \int dy^3 \left( \hat{\chi} + \frac{\hat{\Psi}^* \hat{\chi}^*}{\hat{\Psi}^*} \hat{\Psi}^* - \frac{5}{8} \frac{\varepsilon_3 \hat{\chi}_4 (\hat{\Psi}^2 \hat{\chi})^*}{\hat{\Psi}^*} \right) \right] \hat{n}_i,$$
where $\tilde{n}_i(x^k)$ is a re-defined integration function including contributions from the prime metric. Re-defining the coordinates, the corresponding quadratic element can be written in the form

$$ds^2_{\text{etRs}} = \frac{e}{y} g_{\alpha\beta}(x^k, \varphi) du^\alpha du^\beta = \frac{e}{y} g_i(x^k) \left[(dx^1)^2 + (dx^2)^2 \right] +$$

$$\frac{e}{y} h_3(x^k, \varphi) \left[ d\varphi + \frac{e}{y} w_i(x^k, \varphi) dx^i \right]^2 + \frac{e}{y} h_4(x^k, \varphi) \left[ dt + \frac{e}{y} n_k(x^k, \varphi) dx^k \right]^2,$$

where the coefficients are given by formulas (106). We can impose additional constraints in order to extract stationary LC-configurations as we considered in (84). Such solutions with rotoid deformations were studied in [18, 17] for certain MGTs and higher dimension, see references therein. For small values of $\varepsilon$ and well defined asymptotic conditions, the metrics of type (107) define black ellipsoid configurations which are stable. In this section, such solutions were derived as generic off-diagonal Ricci solitons.

Finally, we emphasize that black ellipsoids exist also in $R^2$ gravity as it is encoded into effective source $\gamma \Upsilon$ determined by constants of such a theory, or for GR with effective scalar field. The limit $\varepsilon \to 0$ is not allowed because in such cases $R \to 0$, as it was found in [19]. We conclude that Ricci solitons with spherical symmetry do not exist in $R^2$ gravity but deformations to rotoid configurations are allowed in such a theory.

4.1.3 Geometric evolution of black ellipsoid Ricci solitons

Geometric flows of ellipsoidal Ricci solitonic configurations with factorized $\tau$-evolution can be described by metrics of type (102). We have to use a set of coefficients (106) for a fixed self-similar configuration. The corresponding quadratic line element is

$$ds^2 = g_{\alpha\beta}(\tau, x^k) du^\alpha du^\beta = [1 + 2 \int d\tau \tilde{\Lambda}(\tau)] \frac{e}{y} g_i(x^k) \left[(dx^1)^2 + (dx^2)^2 \right] +$$

$$[1 + \perp \varepsilon(\tau)] \left\{ \frac{e}{y} h_3(x^k, \varphi) \left[ d\varphi + \frac{e}{y} w_i(x^k, \varphi) dx^i \right]^2 + \frac{e}{y} h_4(x^k, \varphi)\left[ dt + \frac{e}{y} n_k(x^k, \varphi) dx^k \right]^2 \right\}.$$  

The explicit computation of the conditions of vanishing of the time-time coefficient, $[1 + \perp \varepsilon(\tau)] \frac{e}{y} h_4(x^k, \varphi) = 0$, emphasizes two physical effects:

1. There are a running on $\tau$ mass (induced by geometric flows) $\perp M(\tau) = M(1 + \perp \varepsilon(\tau))$ with an effective locally anisotropic mass $\star M = M(1 + \varepsilon \cos(\omega_0 \varphi + \varphi_0) + \perp \varepsilon(\tau))$ containing contributions both from evolution of geometric objects and generic off-diagonal deformations.

2. We can compute horizon deformations and modifications determined by additional $\tau$- and $\varphi$-depending ellipsoid deformations $r_+(\tau, \varphi, \varepsilon) \simeq M_\perp(\tau)/1 - \varepsilon \cos(\omega_0 \varphi + \varphi_0)$.

In general, $N$-adapted geometric flow evolution results in running and anisotropic polarization of physical constants, horizon deformations and locally anisotropic polarizations of $d$-metric and $N$-connection coefficients.
4.2 Ricci flows and solitons for asymptotically de Sitter solutions

Asymptotically de Sitter solutions with spherical symmetry for $R^2$ gravity were studied in [19]. Generic off–diagonal ellipsoid–solitonic deformations of similar Kerr Sen black holes were constructed in [35]. Combining the results and methods of the mentioned works, we can construct exact solutions for ellipsoidal de Sitter Ricci solitons in MGTs and geometric flow evolution of such theories and corresponding classes of solutions.

The metric

$$ds^2 = \frac{3\lambda}{2\xi^2} \left\{ (1 - \frac{M}{r} - \lambda r^2)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin \theta d\varphi^2 - (1 - \frac{M}{r} - \lambda r^2) dt^2 \right\}$$ (108)

for

$$e^{\sqrt{1/3} \phi} = \frac{3\lambda}{2\xi^2} = \frac{1}{8\xi^2} R$$ and $\tilde{g}_{\mu\nu} = e^{\sqrt{1/3} \phi} g_{\mu\nu} = \frac{R}{8\xi^2} g_{\mu\nu}, R \neq 0,$

define an exact solution with spherical symmetry in $R^2$ gravity,

$$\mathcal{R}_{\mu\nu} = 2\xi^2 \tilde{g}_{\mu\nu}.$$ 

The asymptotically de Sitter solutions with $\lambda > 0$ and $R \neq 0$ correspond to equations Re–defining the coordinates,

$$\tilde{x}^i r = \sqrt{\frac{3\lambda}{2\xi^2}} \int dr (1 - \frac{M}{r} - \lambda r^2)^{-1/2}, \tilde{x}^2 = \theta, y^3 = \varphi, y^4 = t;$$

$$\tilde{g}_{\mu
u} = 1, \tilde{g}_{\nu
u}(\tilde{x}^i) = r^2(\tilde{x}^i), \tilde{h}_{3
u} = r^2(\tilde{x}^i) \sin(x^i), \tilde{h}_{4
u} = -(1 - \frac{M}{r(\tilde{x}^i)} + \lambda r^2(\tilde{x}^i)),$$

the metric (108) is written as a "prime" metric

$$ds^2 = \tilde{g}_{\mu
u}(\tilde{x}^k) du^\alpha du^\beta = \tilde{g}_{\alpha
u}((d\tilde{x}^i)^2 + \tilde{g}_{\nu
u}(\tilde{x}^i)(d\tilde{x}^j)^2 + \tilde{h}_{\nu
u}(\tilde{x}^i, \tilde{x}^j)(dy^k)^2 + \tilde{h}_{\nu
u}(\tilde{x}^i)(dy^j)^2,$$

for some constants $M$ and $\lambda$ and $u^\alpha = (\tilde{x}^k, y^a).$ To construct a "formal" off–diagonal metric of type (87) with nontrivial values $\tilde{h}_s, \tilde{\omega}_s$ and $\tilde{n}_s,$ but $W^s_{\beta\gamma}(\tilde{u}^s) = 0,$ see (16), we consider a coordinate transform $u^\alpha = u^\alpha(u^a)$ with $\varphi = \varphi(y^3, \tilde{x}^k)$ and $t = t(y^4, \tilde{x}^k).$ For such transforms,

$$d\varphi = \frac{\partial \varphi}{\partial y^3} dy^3 + (\partial_3 \varphi)^{-1}(\tilde{\partial}_k \varphi) d\tilde{x}^k$$

and $dt = \frac{\partial t}{\partial y^4} dy^4 + (\partial_4 t)^{-1}(\tilde{\partial}_t t) d\tilde{x}^k$

for $\tilde{\partial}_i \varphi = \partial \varphi/\partial \tilde{x}^i$ and $\partial_\varphi \varphi = \partial \varphi/\partial y^a.$ Choosing

$$\tilde{\omega}_i = \tilde{\partial}_i \Psi / \Psi^* = (\partial_3 \varphi)^{-1}(\tilde{\partial}_i \varphi)$$

and $\tilde{n}_i = \tilde{\partial}_i n(x^k) = (\partial_4 t)^{-1}(\tilde{\partial}_i t),$ for any $\tilde{\Psi}$ (94), we express (108) as

$$ds^2 = \tilde{g}_{\alpha
u}((d\tilde{x}^i)^2 + \tilde{g}_{\nu
u}(\tilde{x}^i)(d\tilde{x}^j)^2 + \tilde{h}_{\nu
u}(\tilde{x}^i) dy^k)^2 + \tilde{g}_{\nu
u}(\tilde{x}^k) dy^k]^2 + \tilde{g}_{\nu
u}(\tilde{x}^k) dy^k)^2], (109)$$

for $\tilde{g}_{\alpha
u}(\tilde{x}^k) = (\partial_3 \varphi)^2 r^2(\tilde{x}^i) \sin(\tilde{x}^i)$ and $\tilde{g}_{\nu
u}(\tilde{x}^k) = -(\partial_t t)^2(1 - \frac{M}{r} + \lambda r^2).$ (110)

The prime d–metric (109) allows us to apply the AFDM and construct $\varepsilon$–deformation of geometric/ physical objects and physical parameters as we considered in details in subsections 33 and 44.
4.2.1 Asymptotically de Sitter black ellipsoids in $R^2$ gravity as Ricci solitons

For $\hat{g}_4 = \hat{L}_4(x^{1'}) = (1 - \frac{M}{r} + \lambda r^2)$ and $(\partial_4)^2 = 1$ and anisotropically polarized mass $\widetilde{M}(\varphi) = M[1 + \varepsilon \cos(\omega_0 \varphi + \varphi_0)]$, we obtain

$$s h_4 = -(1 - \frac{M}{r} + \lambda r^2)[1 - \varepsilon \frac{M \cos(\omega_0 \varphi + \varphi_0)}{r} - \frac{M M}{r^3} + \lambda r^2]$$

$$= \hat{L}_4(x^{1'}) \left[1 - \varepsilon \frac{M}{r}(\hat{L}_4)^{-1} \cos(\omega_0 \varphi + \varphi_0)\right] \approx - \left[1 - \frac{\widetilde{M}(\varphi)}{r} + \lambda r^2\right]$$

The parametric equation of an ellipse with radial parameter $\hat{r}_+ = M$ and eccentricity $\varepsilon$,

$$r_+ \approx \frac{M}{1 - \varepsilon \cos(\omega_0 \varphi + \varphi_0)},$$

can be determined in a simple way for $\lambda = 0$. We have to find solutions of a third order algebraic equation in order to determine possible horizons for nontrivial $\lambda$.

We construct ellipsoidal deformations of d–metric (109) if

$$\chi = s \chi = 4 \frac{M}{r} \zeta^2 \Psi^2 \cos(\omega_0 \varphi + \varphi_0),$$

when the former value $\zeta$ is substituted into $2 \Psi^2$. Following the same method as in section 4.1 but for $s \chi$ used for d–metric coefficients (95), we compute

$$s g_i(\tau_0) = \hat{g}_i[1 + \varepsilon \chi_i] = [1 + \varepsilon \hat{g}_i \Psi^2 \cos(\omega_0 \varphi + \varphi_0)]$$

$$s h_3(\tau_0) = [1 + \varepsilon s \chi_3] \hat{g}_3 = \left[1 + \varepsilon \left(2 \left(s \chi + \frac{\hat{\Psi}}{\hat{\Psi}^*} s \chi^* \right) + \frac{1}{8 \hat{\Psi}^2} \hat{\Psi}^2 s \chi \right) \right] \hat{g}_3;$$

$$s h_4(\tau_0) = [1 + \varepsilon s \chi_4] \hat{g}_4 = \left[1 - \varepsilon \frac{1}{8 \hat{\Psi}^2 \hat{g}_4} \hat{\Psi}^2 s \chi \right] \hat{g}_4;$$

$$s w_i(\tau_0) = [1 + \varepsilon w_i \chi_i] \hat{w}_i = \left[1 + \varepsilon \left(\frac{\partial_i (s \chi \hat{\Psi})}{\partial_i \hat{\Psi}} - \frac{(s \chi \hat{\Psi})^*}{\hat{\Psi}^*} \right) \right] \hat{w}_i;$$

$$s n_i(\tau_0) = [1 + \varepsilon n_i \chi_i] \hat{n}_i = \left[1 + \varepsilon \hat{n}_i \int dy^3 \left(c \chi + \frac{\hat{\Psi}}{\hat{\Psi}^*} c \chi^* + \frac{5}{16 \hat{\Psi}^2} \hat{\Psi}^2 s \chi^* \right) \right] \hat{n}_i,$$

where $\hat{n}_i(x^0)$ is a re-defined integration function including contributions from the prime metric (111). The generating functions $s \chi$ and $\hat{g}_i$ can be determined for an ellipsoid configuration induced by the effective cosmological constant $\zeta^2$ in $R^2$ gravity.

The solutions for stationary generic off–diagonal Ricci solitons (111) encode also the data for a black hole metric $[\hat{g}_4, \hat{g}_3, \hat{w}_i, \hat{n}_i]$ with a prime generating function $\hat{\Psi}$ fixed by a 2-d hypersurface (91). This reflects the fact that we parameterize the ellipsoid small deformations in N–adapted form.
4.2.2 Geometric evolution of asymptotically de Sitter black ellipsoid Ricci solitons

The corresponding quadratic element with $\varepsilon$–deformations and factorized $\tau$–evolution are computed as in (102),

$$ds^2 = g_{\alpha\beta}(\tau, \bar{x}^k, \varphi) du^\alpha du^\beta = [1 + 2 \int d\tau \vec{\Lambda}(\tau)] s g_i(\bar{x}^k) [(d\bar{x})^2] + [1 + \varepsilon(\tau)]$$

is determined by the coefficients $g_{\alpha\beta}$ (111). The evolution of such self–similar ellipsoidal configurations is characterized by locally anisotropic polarizations and running of physical constants. For instance, the effective mass modifications are parameterized in a form similar as for black ellipsoids considered in previous subsection when $\varepsilon M = M(1 + \varepsilon \cos(\omega_0 \varphi + \varphi_0) + \varepsilon(\tau)$. Such values have to be defined from experimental data.

LC–configurations can be extracted by additional nonholonomic constraints as we described in previous sections. This is also an issue for experimental verifications of MGTS and possible limits to GR and equivalent modelling.

4.3 Geometric evolution as 3-d KdV configurations

In a different context, the geometric evolution of certain black hole/ ellipsoid and/or Ricci soliton configurations can be characterized by solitonic wave solutions which provide examples of generic nonlinear evolution models.

Let us consider the class of metrics (79) when, for simplicity, $\omega = 1$. We generate families of 3-d solitonic wave equation of Kadomtsev–Petviashvili (KP) type, see details in [31, 32, 33, 34, 35], if it is taken as a generating function any $h_4(\tau, x^1, y^3) = h(\tau, x^1, y^3)$ being a solution of

$$\pm \partial_1^2 h + (\partial_\tau h + hh^* + \varepsilon h^{**})^* = 0. \quad (112)$$

The so–called dispersionless limit is characterized by $\varepsilon \to 0$ and corresponding Burgers’ equation $\partial_\tau h + hh^* = 0$. Integrating above equation on $y^3$, we obtain

$$\partial_\tau h_4 = -h_4 h_4^* - \varepsilon h^{**} \mp \int dy^3 \partial_1^2 h_4.$$

Substituting this value in (74), we construct an effective solitonic source

$$\Upsilon = \Lambda_0 - \phi \Lambda - 2s^2 - hh^* - \varepsilon h^{**} \mp \int dy^3 \partial_1^2 h.$$

(113)

Having a solution $h_4(\tau, x^1, y^3)$, we compute

$$\Psi^2 = B(\tau, x^1) - \frac{4}{\Lambda_0} h_4^2$$

and

$$h_3 = \frac{h_4^2}{h_4[B(\tau, x^1) - \frac{4}{\Lambda_0} h_4]}$$

8in a similar form, we can consider solution of any 3-d solitonic equations, for instance, of generalized sine–Gordon ones
for an integration function $B(\tau, x^1)$. For simplicity, we can take $h_3 = h_4 = h$ and solve \([15]\).

The next step is to use the algebraic equation \([46]\) and find a solution of type \([75]\),

$$w_1(\tau, x^1, y^3) = \frac{\partial \Psi}{\Psi^*} = \frac{\partial \Psi^2}{\partial_3(\Psi^2)} = (h^*)^{-1}\partial_t[\frac{\Lambda_0}{4} B(\tau, x^1) + h], w_2 = 0,$$

when $\Psi = \Psi(\tau, x^1, y^3)$. Integrating two times on $y^3$ in \([47]\) and using the condition $h_3 = h_4$, we obtain

$$n_k(\tau, x^1, y^3) = 1 n_k(\tau, x^1) + 2 \widetilde{n}_k(\tau, x^1) \int dy^3 (\sqrt{|h|})^{-1}.$$

Summarizing the results in this subsection, we constructed a 3-d KdP solitonic quadratic element

$$ds_{KdP}^2 = G_{\alpha\beta}(\tau, x^1, y^3)du^\alpha du^\beta = e^{\psi(\tau, x^k)}[(dx^1)^2 + (dx^2)^2] + h(\tau, x^1, y^3)$$

\{ $dy^3 + \frac{\partial_t(-\frac{\Lambda_0}{4} B(\tau, x^1) + h)}{h^*} dx^1)^2 + [dt + (1 n_k(x^1) + 2 \widetilde{n}_k(x^1) \int dy^3 \sqrt{|h|})^{-1}] dx^1 \}.$

This class of solutions possesses two Killing vectors, $\partial_2$ and $\partial_4$. Nevertheless, this defines a model with a quite general evolution of N-connection coefficients and flows of the nonholonomically induced torsion. Such stationary on time metrics are generic off-diagonal and can be characterized by solitonic symmetries and derived solitonic hierarchies, see details in Refs. \[33, 34, 35\].

5 W–thermodynamics for Black Ellipsoids and Solitonic Flows in \(R^2\) Gravity

In this work, we constructed generic off-diagonal stationary solutions of geometric flow and Ricci soliton equations modeling nonlinear evolution and interactions in MGTs and GR. Using the W–entropy \([34]\), we can elaborate a statistical thermodynamics model characterizing both the spacetime geometric evolution and fixed parameter 3-d configurations embedded in 4-d relativistic spacetimes.

Any d-metric can be parameterized in the form \([31]\). For 3-d thermodynamical values, we obtain

$$\hat{\mathcal{E}} = \hat{\tau}^2 \int_{\hat{\Xi}_t} \hat{\mu} \sqrt{|q_1 q_2 q_3|} d\hat{x}^3 \left( |\hat{\mathcal{R}} + |\hat{\mathcal{D}} \hat{f}|^2 - \frac{3}{\hat{\tau}} \right),$$

$$\hat{\mathcal{S}} = \int_{\hat{\Xi}_t} \hat{\mu} \sqrt{|q_1 q_2 q_3|} d\hat{x}^3 \left[ \hat{\tau} \left( |\hat{\mathcal{R}} + |\hat{\mathcal{D}} \hat{f}|^2 \right) + \hat{f} - 6 \right],$$

$$\hat{\sigma} = -2 \hat{\tau}^4 \int_{\hat{\Xi}_t} \hat{\mu} \sqrt{|q_1 q_2 q_3|} d\hat{x}^3 \left[ |\hat{\mathcal{R}}_{ij} + \hat{\mathcal{D}_i} \hat{\mathcal{D}_j} \hat{f} - \frac{1}{2 \hat{\tau}} q_{ij}|^2 \right],$$

up to any parametric function $\hat{\tau}(\tau)$ in $\hat{\mu} = (4\hat{\pi} \hat{\tau})^{-3} e^{-\hat{f}}$ with any $\hat{\tau}(\tau)$ for $\partial \hat{\tau}/\partial \tau = -1$ and $\tau > 0$. Taking respective 3-d coefficients of a d-metric \([63]\), or \([73]\), or \([86]\) for any solution of
type ellipsoidal deformed black hole solutions (106), or (111), or a KdP evolution model (114), and prescribing a closed 3-d hypersurface $\Xi_0$, we can compute such values for any effective source (50).

The vertical conformal factor $\omega(\tau, x^k, y^3, t)$ in (115) depends (in general, for non-stationary solutions) on a time like coordinate $t$. In such cases, we have to consider relativistic evolution models and integrate additionally on a time interval in order to compute for $\hat{\mathcal{E}}$, $\hat{\mathcal{S}}$, and $\hat{\sigma}$ some values of type (58). Such constructions are elaborated for relativistic hydrodynamical geometric models in [12]. In this work, for simplicity we shall consider stationary solutions with $\omega = 1$. We have to fix an explicit $N$–adapted system of reference and scaling function $\tilde{f}$ in order to find certain explicit values for corresponding average energy, entropy and fluctuations for evolution on a time like parameter $t$. For explicit examples, we can decide if certain solutions with effective Lorentz-Ricci soliton source and/or with contributions from additional MGT sources may be more convenient thermodynamically than other configurations.

5.1 Perelman’s energy and entropy for stationary Ricci solitons and their factorized geometric evolution

Stating a configuration with $\tilde{f} = 0$ and $\tilde{D}\tilde{f} = 0$, we compute the values $\hat{\mathcal{E}}$ and $\hat{\mathcal{S}}$ from (115) (for simplicity, we omit more cumbersome computations for $\hat{\sigma}$) for a $d$–metric of type (73). From effective Einstein equations $\hat{\mathbf{R}}_{\alpha\beta} = \mathbf{Y}_{\alpha\beta}$ with effective $N$–adapted source

$$\mathbf{Y}_{\alpha\beta} = \text{diag}[\sim\Psi(\tau, x^k) := \Psi(\tau, x^k) + \phi\Lambda(\tau) + 2\varsigma^2(\tau)]$$

and $\Psi(\tau, x^k, y^3) := \Psi(\tau, x^k, y^3) + \phi\Lambda(\tau) + 2\varsigma^2(\tau)$,

we find $\hat{\mathcal{E}} = \sim\Psi + \frac{1}{2}\mathcal{Y}$. We have (see relevant formulas (62))

$$q_1 = q_2 = e^\int d\tau\Lambda(\tau) e^{\imath\psi(x^k)}, q_3 = -\{1 + \frac{1}{2} \varepsilon(\tau)\} \frac{1}{b\h_4} \left( \frac{\imath\Psi^*}{\Psi} \right)^2,$$

for $b\h_4 = h_4^0(x^k) - \frac{1}{4} \int dy^3 \left( \frac{\imath\Psi^*}{\Psi} \right)$,

when for Ricci soliton, $Rs$, evolution

$$RsQ(\tau, x^k, y^3) := \sqrt{|q_1q_2|} = \left( 1 + 2 \int d\tau\Lambda(\tau) + \frac{1}{2} \varepsilon(\tau) \right) \frac{e^{\imath\psi(x^k)}}{2\Psi^*} \frac{1}{b\h_4},$$

is considered for small values $|2 \int d\tau\Lambda(\tau)|, |\varepsilon(\tau)| \ll 1$. Introducing such data in in respective formulas in (115) for redefined flow parameter, we obtain

$$\hat{\mathcal{E}} = \tau^2 \int_{\Xi_0} \frac{dx^1dx^2dy^3}{(4\pi\tau)^3} RsQ(\tau, x^k, y^3) \left[ \sim\Psi(\tau, x^k) + \frac{3}{2} \Psi(\tau, x^k, y^3) - \frac{3}{\tau} \right],$$

$$\hat{\mathcal{S}} = \int_{\Xi_0} \frac{dx^1dx^2dy^3}{(4\pi\tau)^3} RsQ(\tau, x^k, y^3) \left[ \tau \left( \sim\Psi(\tau, x^k) + \frac{3}{2} \Psi(\tau, x^k, y^3) - 6 \right) \right].$$
In explicit form, such values can be computed if we prescribe corresponding generating and integration functions, integration constants and fix a closed 3-d hypersurface. For a fixed $\tau_0$, these formulas can be used for determining gravitational thermodynamic values of Ricci solitons.

5.2 Non-factorized thermodynamic configurations for N–adapted effective sources

Similar formulas can be considered for the class of solutions of geometric evolution equations (86) with

$q_1 = q_2 = e^{\psi(\tau, x^k)}, q_3 = h_4(\tau, x^i, y^3),$

for arbitrary generating function

$\Psi(\tau) = \Lambda_0 + \partial_\tau \ln |h_4(\tau, x^i, y^3)| + \phi(\Lambda(\tau) + 2\varsigma^2(\tau))$

when related to source $\Upsilon + \phi \Lambda + 2\varsigma^2 - \partial_\tau \ln |h_4| = \Lambda_0 \neq 0$, see (74) for $\omega = 1$. We obtain

$Q(\tau, x^k, y^3) := \sqrt{|q_1 q_2 q_3|} = e^{\psi(\tau, x^k)} \sqrt{|h_4|}$

and

$\hat{R} = \Upsilon + \frac{3}{2} \Upsilon = \Upsilon(\tau, x^k) + \frac{1}{2} [\Lambda_0 + \partial_\tau \ln |h_4(\tau, x^i, y^3)| + \phi(\Lambda(\tau) + 2\varsigma^2(\tau))]

The thermodynamic values are

$\hat{E} = \tau^2 \int_{\Omega_0} \frac{dx^1 dx^2 dy^3}{(4\pi \tau)^3} e^{\psi(\tau, x^k)} \sqrt{|h_4|}$

and

$\hat{S} = \int_{\Omega_0} \frac{dx^1 dx^2 dy^3}{(4\pi \tau)^3} e^{\psi(\tau, x^k)} \sqrt{|h_4|}$

We can compute such values in explicit form for any generating functions $h_4(\tau, x^i, y^3)$ and $\psi(\tau, x^k)$ and above mentioned sources.

5.3 W–energy and W–entropy for black ellipsoids and solitons in $R^2$ gravity

5.3.1 Thermodynamic values for asymptotic de Sitter black ellipsoids

We use the d–metric coefficients $[111]$ constructed as $\varepsilon$–deformations of the prime black hole solution (108) for generating function $s\chi = \frac{M}{r\Psi^2(\tau, \theta, \varphi)} \varsigma^2 \cos(\omega(\varphi + \varphi_0)$, when $\hat{R} = 6\varsigma^2$. Parameterizing

$q_i = [1 + \varepsilon \frac{\partial}{\partial \varepsilon} \frac{s\chi}{\Psi^2(\tau, \theta, \varphi)}] \frac{\chi^2}{2} \frac{\partial}{\partial \varepsilon}, q_3 = \left[1 + \varepsilon \left(2(s\chi + \frac{\Psi}{\Psi^2(\tau, \theta, \varphi)} s\chi) + \frac{1}{8\varsigma^2} \frac{\partial}{\partial \varepsilon} \frac{\chi^2}{2} \frac{\partial}{\partial \varepsilon} s\chi^2\right)\right] \frac{\chi^2}{2}.$

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we find
\[ ds Q(x^k, y^3) := \sqrt{|q_1 q_2 q_3|} = \sqrt{|\hat{g}_1 \hat{g}_2 \hat{g}_3|} [1 + \varepsilon(e^{\psi}_1 \psi_1 \hat{g}_1 + e^{\psi}_2 \psi_2 \hat{g}_2 + 4\varepsilon^2 + e^{\psi}_3 \psi_3 \hat{g}_3)] +
\varepsilon (\hat{\chi} + \frac{\Psi^* s \chi^*}{16\varepsilon^2 \hat{g}_4 \hat{\Psi}^2 s \chi})],
\]
for \( \sqrt{|\hat{g}_1 \hat{g}_2 \hat{g}_3|} = r^2(|\hat{x}| \sin \hat{x}^2(\theta)). \) Introducing such values in (115), we get
\[ \mathcal{E} = \tau_0 \int_{\Xi_0} \frac{dx^1 dx^2 dy^3}{(4\pi\tau_0)^3} ds Q(x^k, y^3) \left[ 6\varepsilon^2 - \frac{3}{\tau_0} \right], \quad \tilde{S} = \int_{\Xi_0} \frac{dx^1 dx^2 dy^3}{(4\pi\tau_0)^3} R_s Q(x^k, y^3) \left[ 6\varepsilon^2 \tau_0 - 6 \right]. \]

For \( \varepsilon \to 0, \) \( ds Q \to \sqrt{|\hat{g}_1 \hat{g}_2 \hat{g}_3|}. \) We can chose such \( \tau_0 \) and \( \hat{\Psi}^2(r, \theta, \varphi) \) which would allow to relate such values to those of Hawking-Bekenstein black hole thermodynamics. Nevertheless, it should be emphasized that Perelman’s thermodynamics for 3-d hypersurfaces is different from the standard black hole thermodynamics determined by 2-d surface geometries.

### 5.3.2 Thermodynamic values for 3-d soliton KdV evolution

A stationary geometric flow evolution thermodynamics can be associated also to 3-d soliton KdV flows of type (114), \( q_1 = q_2 = e^{\psi(\tau, x^k)}, q_3 = h(\tau, x^1, y^3), \) for \( h \) being a solution of KdV equation (112). The related source is
\[ \Upsilon = \Lambda_0 - \phi \Lambda - 2\varepsilon^2 - hh^* - ch^{***} + \int dy^3 \partial_{11}^2 h. \]

In result, we compute
\[ \text{KdV} Q(\tau, x^k, y^3) := \sqrt{|q_1 q_2 q_3|} = e^{\psi(\tau, x^k)} \sqrt{|h|} \text{ and}
\]
\[ \tilde{R} = \sim \Upsilon + \frac{1}{2} \Upsilon = \frac{3}{2} \Lambda_0 - \phi \Lambda - 2\varepsilon^2 - hh^* - ch^{***} + \int dy^3 \partial_{11}^2 h. \]

The thermodynamic values are
\[ \tilde{\mathcal{E}} = \tau^2 \int_{\Xi_0} \frac{dx^1 dx^2 dy^3}{(4\pi\tau)^3} e^{\psi(\tau, x^k)} \sqrt{|h|} \left\{ \frac{3}{2} \Lambda_0 - \phi \Lambda - 2\varepsilon^2 - hh^* - ch^{***} + \int dy^3 \partial_{11}^2 h - \frac{3}{\tau} \right\}, \]
\[ \tilde{\mathcal{S}} = \int_{\Xi_0} \frac{dx^1 dx^2 dy^3}{(4\pi\tau)^3} e^{\psi(\tau, x^k)} \sqrt{|h|} \left\{ \tau \left( \frac{3}{2} \Lambda_0 - \phi \Lambda - 2\varepsilon^2 - hh^* - ch^{***} + \int dy^3 \partial_{11}^2 h \right) - 6 \right\}. \]

It is obvious that certain parametric 3-d solitonic waves can be not admissible as physical solutions if they result in negative effective thermodynamics energy and/or entropy.

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6 Discussion and Conclusions

We studied a model of relativistic geometric flow theory which for self–similar stationary configurations defines Ricci solitons modelling modified $R^2$ gravity theories. Although the Lorentz signature changes substantially the physical character of geometric evolution which in such cases is not governed by a nonlinear diffusion operator with modified Laplacian (but by nonlinear generalizations of d ’Alambert operator), such models seem to be more realistic and important for research in modified gravity theories, MGTs, and understanding generic off–diagonal interactions in GR. Our key idea was to define such nonholonomic variables when the generalized geometric evolution (and Ricci soliton) equations decouple in very general forms. In certain sense, the bulk of MGTs can be modelled geometrically by a corresponding nonholonomic Ricci soliton configuration.

Applying the anholonomic frame deformation method, AFDM, very general classes of exact solutions of generalized R. Hamilton and modified Ricci soliton equations can be constructed. Such solutions are, in general, with nontrivial torsion structure and depend on all spacetime coordinates via corresponding classes of generating and integration functions, generalized effective sources, integration parameters etc. Metrics are generic off–diagonal and the nonlinear and linear connections can be nonholonomically constrained in order to extract Levi-Civita, LC, configurations. This geometric method of generating exact solutions allow to integrate in very general forms different nonlinear systems of PDEs for geometric flow evolution and MGTs, string and brane models with nonholonomic / noncommutative / supersymmetric variables, see reviews of results in [11, 16, 17, 18, 34, 35, 36].

Mathematically, one has not been elaborated yet necessary methods of geometric analysis for Lorentzian manifolds with pseudo-Euclidean signature and for non-Riemannian manifolds (for instance, with nontrivial torsion structure and/or endowed with additional distributions of Lagrange densities for gravitational and matter fields defining MGTs and nonholonomic GR models). In result, it is not possible at present to elaborate a mathematical rigorous theory of relativistic/ supersymmetric / nonholonomic geometric flows like it was possible for for Riemannian manifolds. Nevertheless, we can study a number of applications and possible physically important effects for various types of relativistic and MGTs modifications using exact solutions generated following the AFDM. For certain nonholonomic configurations, we can solve the Cauchy problem, or satisfy certain boundary/ asymptotic conditions, analyse the necessary criteria for gravitational (nonlinear) diffusion, consider noncommutative interactions, topological changing etc.

Positively, we can apply methods of standard Ricci flow theory for 3+1 splitting. Such constructions were considered, for instance, in the super-renormalizable versions of Hořava-Lifshitz gravity, with Ricci–Cotton flows, focusing on Bianchi cosmological models, see [13], for study low dimensional Ricci flow equations etc. [10, 11, 12]. The AFDM allows to construct generic off–diagonal solutions in MGTs of arbitrary dimension [18]. This geometric method can be developed for finding solutions of geometric flow equations by considering additional dependencies on evolution parameter. Even, in general, the parametric dependence and relativistic evolution of generalized Ricci flow models may change the type of corresponding nonlinear PDE (for instance, locally parabolic systems can be transformed into certain hyperbolic ones etc.)
we can investigate and understand main properties of such nonlinear systems working with
nonholonomic variables which allows to find exact solutions.

One should be emphasized here that the AFDM works effectively, and the resulting solutions
admit certain realistic physical interpretation, if we consider auxiliary linear connections
with nonholonomically induced torsion all determined by certain off-diagonal deformations of
physically important solutions (like black holes, wormholes, locally anisotropic cosmological
models etc. [15, 16, 17, 18]). This way we work with very general ansatz for metrics and con-
nections when the corresponding geometric evolution / gravitational field modified equations
can be integrated in certain general forms. The bulk of exact solutions constructed by other
authors were obtained for much "simple" ansatz with diagonalizable metrics when coefficients
depend on one spacelike/time like coordinates and the corresponding effective Einstein equa-
tions transforms into a nonlinear system of ordinary differential equations, ODEs. Even such
an approach with ansatz of high symmetry offers certain possibilities to construct exact and
very important astrophysical and cosmological solutions for some special classes of systems of
nonlinear PDEs, it is very restrictive comparing to the AFDM. Transforming a system of PDEs
into a system of ODEs for special ansatz, we cut from the very beginning the possibility to
find exact solutions with generic off-diagonal metrics depending on 3-4 and extra dimension
variables. For researchers on physical mathematics, there is a very important question: Shall
we really modify the GR theory or preserve the physical paradigm by considering generic non-
linear off-diagonal solutions which for certain conditions mimic MGTs effects and provide a
theoretical explanation of observable data in modern acceleration cosmology?

MGTs can be treated alternatively as some nonholonomic Ricci soliton configurations of
relativistic geometric flow models. Various classes of exact solutions for corresponding evolution
/ self-similar equations can be related to important physically solutions, and their off-diagonal
deformations, via certain locally anisotropic polarization functions and variation of constants.
This may provide a theoretical background for recent experimental and phenomenological work
on variation of constants [30]. In another turn, observational data in modern cosmology and
related research on MGTs and dark energy and dark matter physics may serve as certain crucial
indication how a realistic geometric flow theories can be developed in relativistic and physically
motivated forms. In result, we addressed the issue how the $R^2$ gravity (which is of great interest
for physicists beginning original cosmological papers [21, 22]) can be involved into a realistic
generic flow scenarios and realized as a nonholonomic Ricci flow model.

As a toy model for testing our constructions on physically motivated geometric flow and
Ricci soliton models we chosen the black hole solutions for $R^2$ gravity [19]. Generalizations of
such classes of solutions can be obtained by applying the AFDM to modified R. Hamilton and
Ricci soliton equations written in nonholonomic variables. For small parametric deformations,
we can construct stationary black ellipsoid configurations when the "eccentricity" is related
to possible locally anisotropic polarization and/or running of physical constants. It should
be noted that black ellipsoids have spheroidal topology and, in consequence, such objects are
not prohibited by black hole uniqueness theorems in GR. They positively exist in $R^2$ gravity
and other modifications, see [17, 18, 34, 35, 36]. Vacuum black hole solutions of Kerr type
are not admitted in certain $R^2$ models for the Levi-Civita connection, but such solutions can
be obtained for a nontrivial cosmological constant, nonholonomic deformations of connection
structures, off–diagonal modifications of metrics, contributions from geometric flows etc.

The AFDM allows us to integrate systems of nonlinear PDEs (for modified geometric flow and gravity theories, in particular, in $R^2$) in very general forms without small parametric limits to well known classes of exact solutions with very special symmetries. It is not clear what physical importance may have such general classes of solutions. We provided some examples for the cases when nontrivial vacuum configurations and polarizations of effective cosmological constants in $R^2$ gravity are determined by 3–d solitonic waves, for instance, of KdV type \[31, 32, 33, 34, 35\]. Such new types of solutions have a well defined physical interpretation as nonlinear solitonic waves for gravitational and matter field interactions.

Our approach to geometric flows and MGTs is based on generalizations of Perelman’s functionals reformulated in nonholonomic variables. Such functionals for the LC–connection and 3-d Riemannian metrics played a crucial role in the proof of the Poincaré conjecture. The so–called W–functional is a Lyapunov type functional which play the role of effective entropy which was used for formulating an analogous statistical thermodynamics characterizing Ricci flows. Geometrically, it is possible to generalize the constructions for various types of gravity theories, for generalized connections and new physical objects but the Lorentz signature does not allow to treat directly the W–functional as an entropy one. We have to consider additional nonholonomic 3+1 and 2+2 decompositions and, in general, to elaborate models of locally anisotropic relativistic geometric flow by analogy to relativistic hydrodynamics and relativistic kinetics theories, as we discuss in \[12\]. For stationary configurations in different MGTs realized as nonholonomic Ricci solitons, the Perelman’s functionals can be determined almost in a standard way on 3-d spacelike hypersurfaces. This is very important because nonholonomic versions of W–functionals provide a thermodynamic interpretation to various classes of generalize off–diagonal solutions in such theories (like black ellipsoids / holes, wormholes etc.). The standard Hawking-Bekenshtein black hole thermodynamics is based on 2-d hypersurface gravity which is not applicable for more general classes of solutions in MGTs. One of the goals of this work was to show in explicit form how to compute Perelman’s thermodynamical energy and entropy for black ellipsoid and KdV solitons in $R^2$ gravity.

Finally, another interesting problem is the application of MGTs (in particular, of $R^2$ gravity) in order to test physically viable supersymmetric generalizations of geometric flows and supergravity models. We have a self–consistent variant of noncommutative geometric flow theory in the A. Connes approach, see \[11\] with generalized Perelman’s functionals, nonholonomic Dirac operators and spectral triples. Such noncommutative Ricci flow models can be elaborated for other approaches to noncommutative geometry. There is a number of formulations of modified supergravity and superstring theories which do not allow to elaborate an unified model of supergeometric flows. Mathematically, the problem is also less clear because different groups of mathematicians work with different definitions of supermanifolds \[37\]. In order to study possible indications from modern gravity and cosmology how a supersymmetric modification of geometric flow theory could be physically motivated, we plan to apply and develop the results of this work and paper \[38\] in \[39\] (a research on supersymmetric Ricci flows and $R^2$ inflation from scale invariant supergravity).

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A Some Formulas for N-adapted 2+2 splitting

For convenience, we summarize in this Appendix some important N-adapted coefficient formulas (see details and proofs in Refs. [15, 16, 17, 18]).

The N-adapted coefficients of the canonical d-connection \( \tilde{\Gamma} = \{ \tilde{\Gamma}_{i}^{\alpha \beta} = (\tilde{L}_{ij}^{\alpha}, \tilde{L}_{bk}^{\alpha}, \tilde{C}_{jc}^{\alpha}, \tilde{C}_{bc}^{\alpha}) \} \) are

\[
\begin{align*}
\tilde{L}_{ij}^{\alpha} &= \frac{1}{2} g^{ir} (e_{k} g_{jr} + e_{j} g_{kr} - e_{r} g_{jk}) , \quad \tilde{L}_{bk}^{\alpha} = e_{b} (N_{k}^{\alpha}) + \frac{1}{2} h^{ac} (e_{k} h_{bc} - h_{dc} e_{b} N_{k}^{d} - h_{db} e_{c} N_{k}^{d}) , \\
\tilde{C}_{jc}^{\alpha} &= \frac{1}{2} g^{jk} e_{c} g_{jk} , \quad \tilde{C}_{bc}^{\alpha} = \frac{1}{2} h^{ad} (e_{c} h_{bd} + e_{c} h_{cd} - e_{c} h_{bd}) .
\end{align*}
\]

(A.1)

The nonholonomically induced torsion \( \tilde{T} = \{ \tilde{T}_{i}^{\alpha \beta} \} \) of (A.1) satisfy the conditions \( \tilde{T}_{i}^{\alpha \beta} = 0 \) and \( \tilde{T}_{i}^{\alpha \beta} = 0 \), but with nontrivial h-v-coefficients

\[
\tilde{\Gamma}^{\alpha \beta} = \tilde{\Gamma}^{\alpha \beta} = \Gamma^{\alpha \beta} .
\]

(A.2)

We can consider N-splitting with zero nonholonomically induced d-torsion, when \( \tilde{T}_{i}^{\alpha \beta} = 0 \), i.e.

\[
\tilde{C}_{jb} = 0 , \quad \Omega_{ji}^{\alpha} = 0 \quad \text{and} \quad \tilde{L}_{aj} = e_{a} (N_{j}^{c}) .
\]

(A.3)

These conditions follow from formulas (A.1) and (A.2). If the Levi-Civita conditions, LC-conditions, (A.3) are satisfied, we obtain that in N-adapted frames (14) and (15) \( \tilde{T}_{i}^{\alpha \beta} = 0 \) and \( \tilde{T}_{i}^{\alpha \beta} = \Gamma_{i}^{\alpha \beta} \). Here we note that the definition and the frame/coordinate transformation laws of a d-connection are different from that of a "usual" linear connection (for instance, \( \tilde{D} \neq \nabla \)), we can impose additional conditions on coefficients \( (g_{a \beta} , N_{j}^{c}) \) which allow us to generate LC-configurations.

The curvature \( \tilde{R} = \{ \tilde{R}_{i}^{\alpha \beta \gamma} \} \) of the canonical d-connection \( \tilde{D} \) is characterized by six groups of N-adapted coefficients,

\[
\begin{align*}
\tilde{R}_{i}^{h_{j} k} &= e_{k} \tilde{L}_{h j}^{i} - e_{j} \tilde{L}_{hk}^{i} + \tilde{L}_{h}^{m}_{j k} \tilde{L}_{h m}^{i} - \tilde{L}_{mk}^{i} \tilde{L}_{h_{j} k}^{i} - \tilde{C}_{ha}^{i} \Omega_{kj}^{i} , \\
\tilde{R}_{i}^{b_{j} k} &= e_{k} \tilde{L}_{bk}^{i} - e_{j} \tilde{L}_{bk}^{i} + \tilde{L}_{bk}^{i} \tilde{L}_{b}^{i} - \tilde{L}_{b}^{i} \tilde{L}_{bk}^{i} - \tilde{C}_{bc}^{i} \Omega_{kj}^{i} , \\
\tilde{R}_{j}^{k a} &= e_{a} \tilde{L}_{jk}^{i} - \tilde{D}_{k} \tilde{C}_{ja}^{i} + \tilde{C}_{ja}^{i} \tilde{D}_{k}^{a} , \quad \tilde{R}_{b k}^{c} = e_{a} \tilde{L}_{bk}^{c} - \tilde{D}_{k} \tilde{C}_{ba}^{c} + \tilde{C}_{ba}^{c} \tilde{D}_{k}^{c} , \\
\tilde{R}_{j b c}^{a} &= e_{a} \tilde{C}_{jb}^{i} - e_{b} \tilde{C}_{jc}^{i} + \tilde{C}_{jb}^{i} \tilde{C}_{jc}^{i} - \tilde{C}_{jc}^{i} \tilde{C}_{jb}^{i} , \quad \tilde{R}_{b c d}^{a} = e_{a} \tilde{C}_{bc}^{d} - e_{c} \tilde{C}_{bd}^{a} + \tilde{C}_{bd}^{a} \tilde{C}_{c d}^{a} - \tilde{C}_{bd}^{a} \tilde{C}_{c d}^{a} .
\end{align*}
\]

(A.4)

The Ricci d-tensor \( \tilde{R}_{a \beta} := \tilde{R}_{i}^{\alpha \beta \gamma} \) of \( \tilde{D} \) is defined by standard formulas and characterized by four groups of N-adapted coefficients

\[
\tilde{R}_{a \beta} = \{ \tilde{R}_{i j}^{a} = \tilde{R}_{i j k}^{a} , \quad \tilde{R}_{a i} = - \tilde{R}_{i k a} , \quad \tilde{R}_{a i} = - \tilde{R}_{a i b} , \quad \tilde{R}_{a b} = \tilde{R}_{a b c} \} .
\]

(A.5)

The corresponding scalar curvature \( \tilde{R} \) of \( \tilde{D} \) is also a usual one when by definition

\[
\tilde{R} := g_{a \beta}^{\gamma} \tilde{R}_{a \beta} = g^{ij} \tilde{R}_{ij} + g^{ab} \tilde{R}_{ab} .
\]

(A.6)
Now, we can define and compute the Einstein tensor \( \hat{E}_{\alpha\beta} \) of \( \hat{D} \),

\[
\hat{E}_{\alpha\beta} := \hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \hat{R}.
\] (A.7)

This d–tensor is different from that for the Levi–Civita connection \( \nabla \), but related via distortion relation depending only on \( g_{\alpha\beta} \) and \( N_i^j \) to the Einstein tensor \( E_{\alpha\beta} \) computed for data \( (g_{\alpha\beta}, \nabla) \).

Using formulas (21), we can compute distortions of connections, torsions and curvatures, Ricci and Einstein tensors and, respective, scalars.

The N–adapted coefficients \( \hat{\Gamma}_{\gamma\alpha\beta} \) of \( \hat{D} \) are equal to the coefficients \( \Gamma_{\gamma\alpha\beta} \) of \( \nabla \), both sets computed with respect to N–adapted frames (14) and (15), if and only if there are satisfied the conditions \( \hat{L}^c_{aj} = e_a(N^c_j) \), \( \hat{C}^i_{jb} = 0 \) and \( \Omega^a_{ji} = 0 \). In such a case, all N–adapted coefficients of the torsion \( \hat{T}_{\gamma\alpha\beta} \) (A.2) and the distortion d–tensor \( \hat{Z}_{\gamma\alpha\beta} \) are zero.

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