Some Remarks on Stable Densities and Operators of Fractional Differentiation

Yuri A. Neretin

Vienna, Preprint ESI 1537 (2004) November 16, 2004

Supported by the Austrian Federal Ministry of Education, Science and Culture
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Some remarks on stable densities and operators of fractional differentiation

NERETIN YURI A.

To A.M. Vershik in his 70 birthday

Let $D(s)$ be a fractional derivation of order $s$. For real $\alpha \neq 0$, we construct an integral operator $A(\alpha)$ in an appropriate functional space such that $A(\alpha)D(s)A(\alpha)^{-1} = D(\alpha s)$ for all $s$. The kernel of the operator $A(\alpha)$ is expressed in terms of a function similar to the stable densities.

0.1. Definition of functions $L_{\alpha, \beta}$. This paper contains several simple observations concerning the special function

$$L_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\alpha n + \beta)}{n!} z^n$$

(0.1)

where $0 < \alpha < 1$, $\text{Re} \beta > 0$, and $z \in \mathbb{C}$. We can also represent this function in the form

$$L_{\alpha, \beta}(z) = \int_{0}^{\infty} t^{\beta-1} \exp(-zt^\alpha - t) \, dt$$

(0.2)

$$L_{\alpha, \beta}(z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \Gamma(\beta - \alpha s) \Gamma(s) z^{-s} \, ds$$

(0.3)

The integrals (0.2), (0.3) also make sense for $\alpha > 1$. The definition of functions $L_{\alpha, \beta}$ is discussed in details below in Section 1.

The function $L_{\alpha, \beta}$ is one of the simplest examples of the so-called $H$-functions (or Fox functions), see [14]. In a strange way, the function $L_{\alpha, \beta}$ has no official name. Obviously, for rational $\alpha = p/q$ the function $L_{\alpha, \beta}$ can be expressed in the terms of higher hypergeometric functions. But for $q > 4$ such expressions do not seem very useful.

0.2. Results of the paper. Integral operators with functions $L_{\alpha, \beta}$ in kernels.

A) We consider the space $\mathcal{K}$ of functions holomorphic in the half-plane $\text{Re} z > 0$, smooth up to the line $\text{Re} z = 0$, and satisfying the following condition

- for each $k > 0$ and $N > 0$ there exists $M$ such that

$$|f^{(k)}(z)| \leq M(1 + |z|)^{-N}$$

(0.4)

We define the operators of fractional differentiation $D_h$ in the space $\mathcal{K}$ by

$$D_h f(z) = \frac{\Gamma(h+1)}{2\pi} \int_{-\infty}^{+\infty} \frac{f(it)}{(-it + z)^{h+1}} \, dt$$
For a positive integer \( n \), we have \( D_n = (-1)^n d^n/dz^n \); the operator \( D_{-n} \) is the indefinite integration iterated \( n \) times. Also \( D_{n+r} = D_n D_r \). See Section 2 below for details.

Next, for \( \alpha > 0 \) we define the kernel

\[
K_\alpha(u, v) = \int_0^\infty \exp(-ux^\alpha - vx) \, dx = -v^{-1}L_\alpha(u/v^\alpha),
\]

and the operator in the space \( \mathcal{K} \) given by

\[
A_\alpha f(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} K_\alpha(-it, v) f(it) \, dt
\]

(0.5)

The operators \( A_\alpha \) form an one-parameter group (see Subsection 3.5),

\[
A_\alpha A_\beta = A_{\alpha \beta};
\]

(0.6)

We also show that they satisfy the property

\[
A_\alpha D_h A_{-1}^{-1} = D_{ah}
\]

(0.7)

**Remark.** Emphasis the following particular cases of (0.7): \( A_{m/n} d^{m/n} A_{m/n}^{-1} = (-1)^{m-n} d^{m/n} \)

\[
A_k \frac{d}{dz} A_k^{-1} = (-1)^k \frac{d^k}{dz^k}
\]

for integer \( m, n, k \).

**Remark.** Also

\[
A_\alpha z A_{\alpha}^{-1} f(z) = \frac{1}{\alpha} D_{1-\alpha}(zf(z))
\]

(0.8)

**Remark.** It seems that (0.7), (0.8) and formula (0.12) below give a possibility to strange transformations of partial differential equations and their solutions.

Further, the operators of dilatation

\[
R_\alpha g(z) = a^{-1} g(z/a), \quad a > 0
\]

satisfy

\[
A_{\alpha}^{-1} R_\alpha A_{\alpha} = R_{\alpha^*}
\]

(0.9)

The generator of the one-parameter group \( R_\alpha \) is \( (z \, dz/dz + 1) \). Hence (0.9) can be written in the form

\[
A_{\alpha}^{-1} \left( \frac{d}{dz} + 1 \right) A_{\alpha} = \alpha \left( \frac{d}{dz} + 1 \right)
\]

**B) In Section 3,** we consider the group \( G \) of operators in \( \mathcal{K} \) generated by the operators \( A_\alpha \), the fractional derivations \( D_n \), and the dilatations \( R_\alpha \). We
observe that $G$ is a 6-dimensional solvable Lie group with 2-dimensional center and kernels of all elements of this group admit simple expressions in the terms of the functions $L_{\alpha, \beta}$ (Theorems 3.1, 3.2).

C) In Section 4, we consider the usual Riemann–Liouville fractional integrations $J_h$ in the space of functions on the half-line $x \geq 0$, see below (4.1). For $0 < \alpha < 1$, we consider the Zolotarev operators [25]–[26] defined by the formula

$$B_\alpha f(x) = \frac{1}{\pi x} \int_0^\infty \text{Im} \left\{ L_{\alpha, 1}(x^{-\alpha} ye^{i\pi \alpha}) \right\} f(y) \, dy$$

(0.10)

We have

$$B_\alpha B_\beta = B_{\alpha \beta}$$

(0.11)

$$B_\alpha J_h = J_{h \alpha} B_\alpha$$

(0.12)

(but we can not represent the identity (0.12) in the form (0.7), since the operators $B_\alpha$ are not invertible).

These operators can be included to a 7-dimensional semigroup of integral operators on the half-line, this semigroup has a 2-dimensional center (Theorem 4.2).

0.3. Some references on functions $L_{\alpha, \beta}$.

1) Barnes in 1906 [2] evaluated asymptotics of several $H$-functions and, in particular, for $L_{\alpha, \beta}$. But it seems that he had no reasons to investigate $L_{\alpha, \beta}$ in details; in the sequel years this function (as far as I know) had not attracted specialists in special functions.

2) The functions

$$W_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\alpha n + \beta)}, \quad \alpha > 0$$

(0.13)

('Wright functions', 'Bessel–Maitland functions') were discussed more, see [24], [1], and references in "Higher transcendental functions" [6] (Section "Mittag-Leffler function"). The functions $L_{\alpha, \beta}(z)$, $W_{\alpha, \beta}(z)$ quite often appear in formulae in similar cases (for instance, see below Subsection 1.5). Another 'relative' of the function $L_{\alpha, \beta}$ is the Mittag-Leffler function $\sum z^n / \Gamma(\alpha n + 1)$, that appear in literature quite often.

3) The functions $L_{\alpha, \beta}$ appear (see Feller [7]) if we solve the Cauchy problem for the partial pseudo-differential equation

$$\left[ \frac{d}{dt} - \frac{d^\alpha}{dx^\alpha} \right] f(x, t) = 0, \quad f(x, 0) = \psi(x)$$

where $d^\alpha/dx^\alpha$ is some fractional derivative. Sometimes it is possible to write

$$f(x, t) = \int K(t, x; y) f(y) \, dy$$

1Apparently, Wright and Maitland are coinciding persons (Edward Maitland Wright). He also coincides with the author of the well-known book Hardy, Wright "An introduction to the number theory".
where the kernel $K$ can be expressed in the terms of the function $\mathbb{L}_{\alpha,\beta}$. The work of Feller generated a wide literature on diffusions generated by pseudodifferential operators.

4) Now, we recall the most important situation, where the functions $\mathbb{L}_{\alpha,\beta}$ arise in a natural way.

Consider a sequence $\xi_j$ of independent random variables and its partial sums $S_n = \xi_1 + \cdots + \xi_n$. Consider the distribution $\mu_n$ of $S_n$. Let us center and normalize $\mu_n$ in some way, $\bar{\mu}_n(t) := \mu_n(a_n t + b_n)$, where $a_n > 0$, $b_n \in \mathbb{R}$ are some constants. Which distributions can appear as limits of sequences $\bar{\mu}_n$? In the most common cases, we obtain a normal (Gauss) distribution. Nevertheless, there are other possible limits ([12], see also [8]), they are named by stable distributions. Densities of these distributions admit a simple expression (0.15) in terms of the functions $\mathbb{L}_{\alpha,\beta}$.

A logical possibility of non-normal distributions in limit theorems of this kind was observed by Cauchy in 1853, see [4]. He claimed that the distribution, whose densities are given by

$$\varphi_\alpha(x) = \int_0^\infty \exp(-x^\alpha) \cos(tx) \, dt \quad (0.14)$$

can appear in limit theorems for sums of independent random variables. Firstly, it was necessary to verify positivity of the functions $\varphi_\alpha$. They are is really positive for $0 < \alpha \leq 2$, but Cauchy could not prove this except several simple cases ($\alpha = 1, 1/2, 2$). In 1922 P. Levy\textsuperscript{2} attracted attention to the problem ([10]), and in 1923 Polya [19] proved positivity of (0.14) for $0 < \alpha < 1$.

After appearance of Kolmogorov–Levy–Hinchin integral representation for infinitely divisible laws, a complete description of stable distributions became solvable problem, the final result is present in the books of P. Levy [11], 1937, and A. Hinchin (another spelling is ’Khintchine’), [9], 1938. The stable densities can be represented in the form

$$p(x; \alpha, \gamma) = \frac{1}{\pi x} \operatorname{Im} \mathbb{L}_{\alpha,1}(x^{-\alpha} e^{i(\gamma-\alpha)\pi/2}) \quad (0.15)$$

where $0 < \alpha < 2$, $\gamma \in \mathbb{R}$, and $|\gamma| < \min(\alpha, 2-\alpha)$ (in this formula, we omit the exceptional and simple case $\alpha = 1$).

It was clear that the integrals of the form (0.2), (0.14) have no expression in terms of classical special functions, but they were important for probabilists and attracted their interest, see [7], [8], [25]. The basic text on this subject is Zolotarev’s book [26], 1986, see also bibliography in this book.

Levy also introduced stable stochastic processes (see [12]). Non-explicitness of stable densities make stable processes difficult for investigations; nevertheless some collection of explicit formulae is known, see Dynkin [5], Neretin [16], Pitman, Yor [18].

\textsuperscript{2}Some references between Cauchy and Levy can be found in [23]. Also, there was work of Holtsmark (1919) on distribution of gravitation force in Universe, see its exposition in [8], [26].
In this paper, the expression (0.15) appears in the formulae (0.10), (1.10). Also the formulae (0.11), (0.6) are variants of the "multiplication theorem for the stable laws" [26], Theorem 3.3.1; there are many other places, where we touch formulae from Zolotarev’s book [26], I do not try to fix all similarities in formulae.

5) The functions \( L_{\alpha,\beta} \) arise in a relatively natural way in the theory of the Laplace transform (the 'operation calculus'), see below. The tables of McLachlan, Humbert, Poli [13], 1950, contain 18 partial cases of the integral transformations defined below; also the transformations (0.10) are contained in Zolotarev [25], and a similar construction with Wright functions is a subject of Agarwal [1].

6) It is known (see [15]) that pseudodifferential equations with constant coefficients of the type

\[
\sum_{k=0}^{n} a_k D_k \alpha = 0
\]

admit explicit analysis. Apparently this phenomena is related to the identities (0.7), (0.12).

0.4. Structure of the paper. In Section 1, we discuss various definitions of the functions \( L_{\alpha,\beta} \), theirs integral representations, and also some integrals containing products of two functions \( L_{\alpha,\beta} \).

In Section 2, we discuss the space \( K \) of holomorphic functions defined above; also we introduce a standard scale \( H_\mu \) of Hilbert spaces of holomorphic functions in a half-plane. The latter spaces are well-known in representation theory of \( SL_2(\mathbb{R}) \).

For our purposes, the space \( K \) and the Hardy space \( H^2 \) are almost sufficient.

In Section 3, we introduce a simple construction in a spirit of the Vilenkin–Klimyk book [22]. We consider the 6-dimensional solvable Lie group of operators

\[
f(x) \mapsto \lambda x^h f(ax^\alpha);
\]

acting in a space of functions on a half-line and consider the image of this group under the Laplace transform. As a result, we obtain a group of continuous operators, whose kernels are expressed in terms of \( L_{\alpha,\beta} \). The most interesting property of these operators is the identity (0.7) given above.

In Section 4, we consider a similar construction. We start from a 7-dimensional semigroup \( (4, 2) \) of operators acting in a space of holomorphic function on half-plane and consider its image under the inverse Laplace transform. As a result, we obtain a semigroup of integral operators acting in an appropriate space of functions on half-line.

1. Some properties of the functions \( L_{\alpha,\beta} \).

1.1. Definition. We define the function \( L_{\alpha,\beta} \) as the Barnes integral

\[
L_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Gamma(s) \Gamma(\beta - \alpha s) z^{-s} ds
\]

(1.1)
We must explain a meaning of elements of this formula.

1) Our indices are in the domain \( \beta \in \mathbb{R}, \alpha \in \mathbb{C} \). Assume also
\[
\beta + \alpha m + n \neq 0 \quad \text{for all } n, m = 0, 1, 2, \ldots \tag{1.2}
\]

2) Our integral is convergent if \(| \arg z | < (1 + \alpha)\pi /2 \).

3) Our integrand has poles at the points
\[s = 0, -1, -2, \ldots \quad \text{and} \quad s = \beta /\alpha, (\beta + 1)/\alpha, (\beta + 2)/\alpha, \ldots \]

Now, we consider two cases \( \beta > 0 \) and \( \beta < 0 \).

First, let \( \beta > 0 \). For \( \beta > 0 \), we can assume that the contour of the integration is the imaginary axis \( i\mathbb{R} \) and we leave the pole \( s = 0 \) on the left side from the contour (denote this contour by \(+0 + i\mathbb{R})\). Otherwise, we consider a contour \( L \) coinciding with \( i\mathbb{R} \) near \( \pm i\infty \) and separating the left series of poles \( (s = -n) \) and the right series \( s = (\beta + n)/\alpha \) of poles. Such contour exists due the condition (1.2).

We also can transform this contour integral to
\[
\int_{L} = \int_{+0 + i\mathbb{R}} - \sum_{n : (\beta + n)/\alpha < 0} \text{res}_{s=(\beta+n)}
\]

Second, let \( \beta < 0 \). Then we consider an arbitrary contour \( L \) coinciding with the imaginary axis near \( \pm i\infty \) and leaving all the poles of the integrand on the left side. If \( \beta > 0 \), then we can choice \( L \) being \(+0 + i\mathbb{R}\).

Remark. For fixed \( \beta, \ z > 0 \), the function \( \mathbb{L}_{\alpha,\beta}(z) \) as a function of the parameter \( \alpha \) is \( C^{\infty} \)-smooth at \( \alpha = 0 \) but it is not real analytic in \( \alpha \) at this point (compare (1.4) and (1.5)). Thus it is not quite clear, is natural to consider \( \mathbb{L}_{\alpha,\beta} \) as one function or as two functions defined for \( \alpha > 0 \) and \( \alpha < 0 \). For local purposes of this paper, the first variant is more convenient.

1.2. Expansion of \( \mathbb{L}_{\alpha,\beta} \) into power series. We write expansions of \( \mathbb{L}_{\alpha,\beta} \) in series applying the standard Barnes method, see [21], [14].

a) Let \( 0 < \alpha < 1 \). Then the integral (1.1) is the sum of residues at the points \( s = -n \), i.e.,
\[
\mathbb{L}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\alpha n + \beta) }{n!} z^n \tag{1.3}
\]
This function is well defined on the whole complex plane \( z \in \mathbb{C} \). Due (1.2), the \( \Gamma \)-functions in numerators have no poles.

b) Let \( \alpha > 1 \). Then (1.1) is the sum of residues at the points \( s = (\beta + n)/\alpha \), i.e.,
\[
\mathbb{L}_{\alpha,\beta}(z) = - \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma\left((n + \beta)/\alpha\right) }{n!} z^{(n+\beta)/\alpha} \tag{1.4}
\]
\[\text{See the integral representation (1.9), we differentiate it in } \alpha \text{ and apply the Lebesgue dominant convergence theorem.}\]
Here we assume \( z^{\nu} = \exp(\nu \ln z) \) and \( \ln z \in \mathbb{R} \) for \( z > 0 \). The series is convergent in the domain \( |\arg z| < \infty \) (i.e., our function is defined on the universal covering surface \( \mathbb{C}^* = \mathbb{C} \setminus 0 \)).

c) For \( \alpha < 0 \), the integral (1.1) is the sum of residues at all the poles, i.e.,

\[
L_{\alpha, \beta}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\alpha n + \beta)}{n!} z^n - \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma((n + \beta)/\alpha)}{n!} z^{(n+\beta)/\alpha} \tag{1.5}
\]

This expression is valid if poles are simple, i.e., \( \beta + \alpha n + m \neq 0 \) for \( n, m = 0, 1, 2, 3, \ldots \). But the points \( \bar{\beta} = -\alpha n - m \) are not really singular, in these cases some of poles of the integrand have order 2, and we must apply a formula for a residue in a non-simple pole (or remove singularities in (1.5)).

The domain of convergence of (1.5) is \( |\arg z| < 1 \).

1.3. A symmetry.

**Lemma 1.1.** a) For \( \alpha > 0, \beta > 0 \),

\[
\mathbb{L}_{\alpha, \beta}(z) = \alpha^{-\beta/\alpha} \mathbb{L}_{1, \alpha, \beta/\alpha}(z^{-1/\alpha}) \tag{1.6}
\]

b) For \( 0 < \alpha < 1 \),

\[
\mathbb{L}_{\alpha, \alpha}(z) = -\frac{1}{\alpha z} [\mathbb{L}_{\alpha, 1}(z) - 1]
\]

c) For \( \alpha > 0 \),

\[
\mathbb{L}_{\alpha, 1}(z) = 1 - \mathbb{L}_{1, \alpha, 1}(z^{-1/\alpha}) \tag{1.7}
\]

**Proof.** a) Substituting \( t = \beta - \alpha s \) to (1.1), we obtain

\[
\mathbb{L}_{\alpha, \beta}(z) = \frac{1}{2 \pi i \alpha} \int_{-\infty}^{+\infty} \Gamma((\beta-t)/\alpha) \Gamma(t) z^{-(\beta-t)/\alpha} dt \tag{1.8}
\]
as it was required.

Statement b) follows from

\[
\mathbb{L}_{\alpha, \alpha}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\alpha n + \alpha)}{n!} z^n = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\alpha n + \alpha)(\alpha n + \alpha)}{(n+1)!} z^n
\]

c) Substitute \( \beta = 1 \) to (1.6) and assume \( \alpha > 1 \). Applying b), we obtain the required statement for \( \alpha > 1 \). But the identity (1.6) is symmetric with respect to the transformation \( \alpha \mapsto 1/\alpha, z \mapsto z^{-1/\alpha} \).

**Remark.** The statement c) is a well-known symmetry in the theory of stable distributions, see [8], (17.6.10), see also [26], Section 2.3.

1.4. Some integral representations of \( \mathbb{L}_{\alpha, \beta} \).

**Lemma 1.2.** Let \( \alpha \in \mathbb{R}, \ Re u > 0, \ Re v > 0, \ Re h > 0 \). Then

\[
\int_{0}^{\infty} x^{h-1} \exp(-ux^{\alpha} - vx) \, dx = v^{-h} \mathbb{L}_{\alpha, h}(u/v^\alpha) \tag{1.9}
\]
Proof. It is very easy to verify this for $\alpha > 0$.

1) for $0 < \alpha < 1$: we expand the factor $\exp(-ux^\alpha)$ in (1.9) in Taylor series and integrate term-wise

$$
\sum_{n=0}^{\infty} \frac{(-u)^n}{n!} \int_0^\infty x^{\alpha n + h - 1} e^{-vx} \, dx = \sum_{n=0}^{\infty} \frac{(-u)^n}{n!} \cdot \frac{\Gamma(\alpha n + h)}{\nu^{\alpha n + h}}
$$

2) Similarly, for $\alpha > 1$, we expand the factor $\exp(-vx)$ into a Taylor series

$$
\sum_{n=0}^{\infty} \frac{(-v)^n}{n!} \int_0^\infty x^{h + n - 1} \exp(-ux^\alpha) \, dx = \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{(-v)^n}{n!} \cdot \frac{\Gamma((h + n)/\alpha)}{u^{(h + n)/\alpha}} = \frac{1}{\alpha} u^{-h/\alpha} \frac{\nu}{\alpha u^{\lambda/\alpha}}
$$

and apply the symmetry (1.6)

3) The case $\alpha < 0$ is not obvious, and we give a calculation that is valid for all $\alpha \in \mathbb{R}$. Consider the space $L^2$ on $\mathbb{R}_+$ with respect to the measure $dx/x$. The left hand side of (1.9) is the $L^2$-inner product of the functions $\Phi_1, \Phi_2$ given by

$$
\Phi_1(x) = \exp(-ux^\alpha); \quad \Phi_2(x) = x^{\alpha} \exp(-vx^\alpha)
$$

The Mellin transform of $\Phi_1$ is

$$
\tilde{\Phi}_1(\lambda) = \int_0^\infty x^{\lambda - 1} \exp(-ux^\alpha) \, dx = \frac{\text{sgn}(\alpha)}{\alpha} \int_0^\infty \exp(-uy) y^{\lambda/\alpha - 1} \, dy = \frac{\text{sgn}(\alpha) \Gamma(\lambda/\alpha)}{\alpha u^{\lambda/\alpha}}
$$

The Mellin transform of $\Phi_2$ is

$$
\tilde{\Phi}_2(\lambda) = \int_0^\infty x^{\lambda + \alpha - 1} \exp(-vx^\alpha) \, dx = v^{-\alpha} \Gamma(\lambda)
$$

By the Plancherel formula for the Mellin transform, we have

$$
\int_0^\infty \frac{\Phi_1(x)\Phi_2(x)}{x} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\Phi}_1(is)\tilde{\Phi}_2(is) \, ds
$$

i.e.,

$$
\int_0^\infty x^{h - 1} \exp(-ux^\alpha - vx) \, dx = \frac{\text{sgn}(\alpha)}{2\pi \alpha u^{\lambda}} \int_{-\infty}^{+\infty} \Gamma(is/\alpha) \Gamma(h + is) u^{-i\alpha} v^i \, ds
$$

Then we introduce the new variable $t = s/\alpha$.

1.5. Integral representations. Variants. Now let $x, y > 0$. Let $\alpha < 1, \theta > 0$. Then

$$
\frac{1}{2i} \int_{-\infty}^{+\infty} \rho^{-\theta} \exp(-p^\alpha x + py) \, dp = -\text{Im} \left[ y^{-\theta} e^{\pi i \theta \mu \alpha} e^{(xy - \mu \alpha \pi i \alpha)} \right] = -\sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n\alpha + \theta)}{n!} x^n y^{-n\alpha - \theta} \sin(n\alpha + \theta)
$$

(1.10)
where the integration is given over the imaginary axis, and \( p^\alpha \) is positive real for \( p > 0 \).

Indeed, we represent the integral in the form

\[
\frac{1}{2i} e^{\pi i \theta/2} \int_0^\infty \exp(-t^\alpha e^{\pi i \alpha/2} + ity) t^{\theta-1} dt - \frac{1}{2i} e^{-\pi i \theta/2} \int_0^\infty \exp(-t^\alpha e^{-\pi i \alpha/2} - ity) t^{\theta-1} dt
\]

and apply (1.9) with \( u = x \exp(\pm \pi i \alpha/2) \), \( v = y \exp(\mp \pi i/2) \).

**Remark.** For \( 0 < \alpha < 1 \) the expression (1.10) is a density of a stable subordinator.

**Remark.** The calculation given above survives for the case \( \alpha < 0 \). The factor \( \exp(-t^\alpha e^{\pi i \alpha/2}) \) is flat at \( t = 0 \) and hence we can consider \( \theta < 0 \). We transform

\[
\gamma(\alpha n + \theta) \sin[(\alpha n + \theta)\pi] = \pi/\Gamma(1 - \theta - \alpha n)
\]

and we reduce (1.10) to the form \( y^{-\theta} \mathcal{W}_{-\alpha,1-\theta}(x/y^\alpha) \), where \( \mathcal{W} \) is the Wright function.

**1.6. Remark.** An \( L^2(\mathbb{R}) \)-inner product. Let \( \alpha > 0 \), \( \beta > 0 \). Let \( x, y > 0 \).

Consider the function

\[
\Psi_{\alpha,\beta,y}(x) := x^{\beta-1} \exp(-yx^\alpha)
\]

By (1.9), its Laplace transform is

\[
\hat{\Psi}_{\alpha,\beta,y}(\xi) = \int_0^\infty x^{\beta-1} \exp\left(-yx^\alpha - \xi x\right) dx = \xi^{-\beta} \mathcal{L}_{\alpha,\beta}(y/\xi^\alpha)
\]

Evaluating the \( L^2(\mathbb{R}) \)-inner product of \( \Psi_{\alpha_1,\beta_1,y_1} \) and \( \Psi_{\alpha_2,\beta_2,y_2} \), we obtain

\[
\int_0^\infty x^{\beta_1-1} \exp(-y_1 x^{\alpha_1}) x^{\beta_2-1} \exp(-y_2 x^{\alpha_2}) dx =
\alpha_1^{-1} y_2^{-(\beta_1+\beta_2-1)/\alpha_2} \mathcal{L}_{\alpha_1/\alpha_2,\beta_1+\beta_2-1/\alpha_2}(y_1 y_2^{-\alpha_1/\alpha_2})
\]

(we substitute \( t = x^{\alpha_2} \) and apply Lemma 1.2). The expression in the right-hand side is symmetric with respect to \( (\alpha_1, \beta_1) \leftrightarrow (\alpha_2, \beta_2) \) by Lemma 1.1.

By the Plancherel formula for the Fourier transform, the same expression can be written in the form

\[
\frac{1}{2\pi} \int_{-\infty}^{+\infty} (it)^{-\beta_1} \mathcal{L}_{\alpha_1,\beta_1}(y_1/(it)^{\alpha_1}) (-it)^{-\beta_2} \mathcal{L}_{\alpha_2,\beta_2}(y_2/(-it)^{\alpha_2}) dt
\]

Thus, the expression (1.14) equals (1.13).

**Remark.** The integral (1.14) looks like a kernel of a product of two integral operators; moreover (1.13) shows that this product has the same form, i.e., we
obtain a family of integral operators closed with respect to multiplication. Below we propose two ways to give a precise sense for this observation; apparently, there are other possibilities.

1.7. Remark. Convolutions. Preserve notation (1.11), (1.12). We have

$$\Psi(x,y)(x) = \Psi(x,y)(z - u)du = \hat{\Psi}(x,y)(z)$$

Hence

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{\Psi}(x,y)(z - u)du = \hat{\Psi}(x,y)(z)$$

2. Spaces of holomorphic functions. Preliminaries

2.1. Spaces $L^2(R_+).$ Fix $\mu > 0.$ Denote by $L^2(R_+)$ the space $L^2$ on the half-line $R_+, x > 0,$ with respect to the weight $\Gamma(\mu)^{-1}x^{\mu-1}dx,$ i.e., the Hilbert space with the inner product

$$\langle f, g \rangle_{[\mu]} := \frac{1}{\Gamma(\mu)} \int_0^{\infty} f(x)g(x)x^{\mu-1}dx$$

For instance,

$$\langle \exp(-zx), \exp(-ux) \rangle_{[\mu]} = (z + u)^{-\mu}$$

for arbitrary complex $u, z$ satisfying $\Re z > 0, \Re u > 0.$

2.2. Hilbert spaces of holomorphic functions on a half-plane. Let $\Pi$ be the right half-plane $\Re z > 0$ on the complex plane. We consider the Hardy space $H^2$ on $\Pi.$ Recall that this space consists of functions holomorphic in the half-plane, whose boundary values on the imaginary axis $\Re z = 0$ exist and are contained in $L^2(R).$

The Hardy space is an element of the following one-parametric scale $H_\mu,$ $\mu > 0,$ of spaces of holomorphic functions. Fix $\mu > 1.$ Consider the space $H_\mu = H_\mu(\Pi)$ consisting of functions $f(z)$ holomorphic in $\Pi$ and satisfying the condition

$$\int_{\Pi} |f(z)|^2(\Re z)^{\mu-2} dz d\bar{z} < \infty$$

where $dz d\bar{z}$ denotes the Lebesgue measure on $\Pi.$

We define an inner product in $H_\mu$ by

$$\langle f, g \rangle_\mu = \frac{\mu - 1}{\pi} \int_{\Pi} f(z)g(z)(\Re z)^{\mu-2} dz d\bar{z}$$

The space $H_\mu$ is a Hilbert space with respect to this inner product.
The reproducing kernel⁴ of this space is
\[ K_{\mu}(z, u) = (z + u)^{-\mu} \]
This means that the function \( \Xi_u(z) \) given by
\[ \Xi_u(z) = (z + u)^{-\mu} \]
satisfies the reproducing property
\[ (f, \Xi_u)_{\mu} = f(u) \quad \text{for all } f \in H_{\mu}. \] (2.3)
In particular,
\[ (\Xi_u, \Xi_w)_{\mu} = (w + u)^{-\mu} \] (2.4)

The space \( H_{\mu} \) can be defined by (2.3), (2.4) without reference to explicit formula (2.2) for the inner product. Indeed, consider an abstract Hilbert space \( H \) with a system of vectors \( \Xi_u \), where \( u \in \Pi \), and assume that their inner products age given by (2.4). Such Hilbert space exists, see formula (2.1). Assume also that linear combinations of \( \Xi_u \) are dense in \( H \). Then for each \( h \in H \) we consider the holomorphic function on \( \Pi \) given by
\[ f_h(u) := (h, \Xi_u)_H \] (2.5)
and thus we have identified our space \( H \) with some space of holomorphic functions on \( \Pi \).

But the last construction survives for arbitrary \( \mu > 0 \) (since the existence of \( H_{\mu} \) is provided by formula (2.1) and this formula is valid for \( \mu > 0 \)).

Remark. For \( \mu = 1 \) we obtain the Hardy space \( \mathcal{H}^2 \).

Remark. For \( 0 < \mu < 1 \), it is possible to write an integral formula for the inner product in \( H_{\mu} \) involving derivatives. But it is more convenient to use the definition (2.3)–(2.4) or to consider the analytic continuation of the integral (2.2) with respect to \( \mu \).

We define the weighted Laplace transform \( \mathcal{L}_\mu \) by
\[ \mathcal{L}_\mu f(z) = \frac{1}{\Gamma(\mu)} \int_0^\infty f(x) \exp(-ux) x^{\mu-1} dx \]
For \( \mu = 1 \) we obtain the usual Laplace transform \( \mathcal{L} = \mathcal{L}_1 \). The following statement is well-known⁵.

**Lemma 2.1.** The weighted Laplace transform is a unitary operator
\[ \mathcal{L}_\mu : L^2_0(\mathbb{R}_+) \rightarrow H_{\mu}(\Pi) \]

**Proof.** Consider a function
\[ \xi_u(x) = \exp(-ux) \]

⁴ For machinery of reproducing kernels, see, for instance, [3], [17].
⁵ The case \( \mu = 1 \) is a Paley–Wiener theorem.
in $L^2_\mu$. Its image under $L_\mu$ is $\Xi_u$. It remains to compare (2.1) and (2.4).

**Remark.** The transform $L_\mu$ is precisely the operator defined by the formula (2.5). Indeed, we can assume $H = L_\mu$, then the corresponding space of functions $f_h$ is $H_\mu$.

### 2.3. Operators in the spaces $H_\mu$.
Recall a standard general trick, apparently discovered by Berezin [3] (formulae (2.6), (2.9) given below are valid in arbitrary Hilbert space defined by a reproducing kernel).

Let $A$ be a bounded operator $H_\mu \to H_\mu$. Define the function

$$M(z,u) = A\Xi_u(z)$$ (2.6)

Then $M(z,u)$ is the *kernel* of the operator $A$. For $\mu > 1$ we can write literally

$$Af(z) = \frac{\mu - 1}{\pi} \int H(z,u)f(u)(\text{Re } u)^{\mu-2}du\,d\mu$$ (2.7)

respectively, for $\mu = 1$,

$$Af(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} M(z,-it)f(it)\,dt$$ (2.8)

For general $\mu > 0$, we can write

$$Af(z) = \langle f, M(z,u) \rangle_\mu$$ (2.9)

the inner product is given in the space $H_\mu$ of functions depending in the variable $u$. Formulae (2.7)–(2.8) are partial cases of this formula. In particular, the integrals (2.7)–(2.8) are convergent for each $z \in \Pi$ and $f \in H_\mu$.

### 2.4. Space of rapidly decreasing functions.
We also consider the space $K_\mu = H^2 \cap S(i\mathbb{R})$, where $S(i\mathbb{R})$ is the Schwartz space (consisting of functions on the imaginary axis rapidly decreasing with all derivatives).

We can say that $K_\mu$ is the space of functions holomorphic in $\text{Re } z > 0$ and continuous in $\text{Re } z > 0$ such that

a) $f(it)$ is $C^\infty$-smooth
b) For each $k > 0$ and $N > 0$ there exists $M$ such that

$$|f^{(k)}(z)| \leq M(1 + |z|)^{-N}$$ (2.10)

Consider the space $S(\mathbb{R}_+)$ consisting of smooth functions $f$ on $[0,\infty)$ such that

a) $f^{(k)}(0) = 0$ for all $k \geq 0$
b) $\lim_{x \to +\infty} f^{(k)}(x)x^N = 0$ for all $N > 0$, $k \geq 0$.

In other words, $S(\mathbb{R}_+)$ is the intersection of the Schwartz space $S(\mathbb{R})$ on $\mathbb{R}$ and the space $L^2(\mathbb{R}_+)$. **Lemma 2.2.** The space $K_\mu$ is the image of $S(\mathbb{R}_+)$ under the Laplace transform.
Proof. Let $f \in S(\mathbb{R}_+)$ Integrating by parts, we obtain
\[
\int_0^\infty f^{(k)}(x)e^{-px}dx = p^k \int_0^\infty f(x)e^{-px}dx
\]
The left-hand side is a bounded function in $p$, looking to the right-hand side, we observe that $(\mathcal{L}f)(p)$ is rapidly decreases for $\text{Re} \, p > 0$.

Conversely, a function $F$ satisfying (2.10) is an element of $\mathcal{H}^2$. Hence $f = \mathcal{L}^{-1}F$ is supported by $\mathbb{R}_+$. Since (2.10) is valid for $z \in i\mathbb{R}$, we have $f \in S(\mathbb{R})$.

2.5. Fractional derivations. We define the operators of fractional differentiation $D_h$ in $\mathcal{K}$ by
\[
D_h f(z) = \frac{\Gamma(h+1)}{2\pi i} \int_{-\infty}^{\infty} f(it) \frac{dt}{(-it + z)^{h+1}}
\]
(2.11)
A branch of $\mu(z, it) = (-it + z)^{h+1}$ is determined from the condition $\mu(x, 0) > 0$ for $x > 0$.

Lemma 2.3. a) $D_h$ is an operator $\mathcal{K} \to \mathcal{K}$ for each $h \in \mathbb{C}$.

b) For integer $n > 0$,
\[
D_n f(z) = (-1)^n \frac{d^n}{dz^n} f(z)
\]

c) For positive integer $m$,
\[
D_{-m} f(z) = \lim_{s \to \infty} \frac{\Gamma(-s+1)}{2\pi i} \int_{-\infty}^{\infty} (-it + z)^{s-1} f(t) \frac{dt}{(-it + z)^{h+1}}
\]
(2.12)
\[
= (-1)^m \int_{-\infty}^{z_1} dz_1 \int_{-\infty}^{z_2} dz_2 \ldots \int_{-\infty}^{z_n} f(z_n) dz_n
\]
d) $D_h D_z = D_{h+z}$

Proof. a) Convergence of the integral for $\text{Re} \, u > 0$ is obvious. Let us show rapid decreasing of $g := D_h f$ at $z \to \infty$.

Let $\text{Re} \, h < 0$. We represent (2.11) as a contour integral
\[
\frac{\Gamma(h+1)}{2\pi i} \int_{-\infty}^{\infty} f(u) \frac{du}{(-u + z)^{h+1}}
\]
Denote $R := |z|$. Then we replace a part $(-iR, +iR)$ of the contour of the integration $i\mathbb{R}$ by the semi-circle $\exp(i\varphi)$, where $\varphi \in (-\pi, \pi)$. Since (2.10), all the 3 summands of the integral rapidly tend to 0 as $R$ tend to $\infty$.

If $\text{Re} \, h \geq 0$, we integrate our expression by parts and obtain
\[
\frac{(-1)^k \Gamma(h-k+1)}{2\pi i} \int_{-\infty}^{\infty} (z-u)^{k-1} f^{(k)}(u) \frac{du}{z-u}
\]
We choose $k > \text{Re}(h+1)$ and repeat the same consideration.
Also,
\[ \frac{d}{dz} D_h f = -D_{h+1} f \]
and this implies rapid decreasing (2.10) of derivatives.

b) This is the Cauchy integral representation for derivatives.

c) First, we give a remark that formally is not necessary. Factor \( \Gamma(1 + h) \)
has a pole for \( h = -m \). Let us show that \( \int \) vanishes at this point. Indeed, we
have the expression
\[ \int_{-i\infty}^{i\infty} (z - u)^{m-1} f(u) \, du \]
We replace a part \((-iR, +iR)\) of the contour of the integration \( iR \) by the semi-
circle \( R \exp(i\varphi) \) as above and tend \( R \) to \( \infty \).

Now give a proof of c). Consider the operator of indefinite integration
\[ If(z) = \int_{-i\infty}^{i\infty} f(u) \, du \]  
(2.13)
Changing the contour as above, we obtain \( If \in \mathcal{K} \).
For \( f \in \mathcal{K} \) we have
\[ \Gamma(1 - s) \int_{-i\infty}^{i\infty} (z - u)^{s-1} f(u) \, du = \]
\[ = (-1)^m \Gamma(1 - s + m) \int_{-i\infty}^{i\infty} (z - u)^{s-m-1} (I^m f)(u) \, du = \]
Now we can substitute \( s = m \).

d) This is valid for \( \text{Re} h_1 = \text{Re} h_2 = 0 \) by the statement b) of following
Lemma 2.4. Then we consider the analytic continuation in \( h \).

Lemma 2.4. a) For \( s \in \mathbb{R} \), the operator \( D_{is} \) is a unitary operator in each
\( H_\mu \). Its kernel (in the sense of \( H_\mu \)) is
\[ \frac{\Gamma(\mu + is)}{\Gamma(is)} (z + \overline{u})^{-\mu - is} \]

b) \( D_{is_1} D_{is_2} = D_{is_1 + is_2} \)

Proof. a) Consider the operator \( U_{is} \) in \( L^2(\mathbb{R}_+) \) given by
\[ U_{is} f(x) = f(x) x^{is} \]
Let us evaluate the kernel of the operator
\[ L_\mu U_{is} L_\mu^{-1} : H_\mu \to H_\mu \]
By (2.6), we must evaluate the function \( L_\mu U_{is} L_\mu^{-1} \Xi_u \). We have
\[ L_\mu^{-1} \Xi_u(x) = \exp(-\overline{x}u), \]
\[ U_{is} L_\mu^{-1} \Xi_u(x) = \exp(-\overline{x}u)x^{is}; \]
\[ L_\mu U_{is} L_\mu^{-1} \Xi_u(z) = (z + \overline{u})^{-\mu - is} \Gamma(\mu + is)/\Gamma(is) \]
For \( \mu = 1 \) our operator coincides with the operator \( D_{is} \) defined above. In fact, all the operators \( L_{\mu}U_{is}L_{\mu}^{-1} : H_{\mu} \rightarrow H_{\mu} \) induce the same operator \( D_{is} \) in \( K \). Indeed, the operator \( L_{\mu}^{-1}L_{\nu} \) is the operator of multiplication by \( x^{\mu-\nu} \), and this operator commutes with \( U_{is} \).

b) corresponds to the identity \( x^{i(s_1+s_2)} = x^{is_1}x^{is_2} \) after the Laplace transform.

3. Operators in spaces of holomorphic functions

3.1. Some operators acting in \( S(\mathbb{R}_+) \). We consider the following one-parameter groups of operators in the space \( S(\mathbb{R}_+) \) (see 2.4).

\[
U_\alpha f(x) = f(x^\alpha), \quad \alpha > 0; \quad (3.1)
\]
\[
V_a(x)f(x) = f(ax), \quad a > 0; \quad (3.2)
\]
\[
W_h f(x) = x^h f(x), \quad h \in \mathbb{C} \quad (3.3)
\]

The last group is a complex one-parameter group, i.e., a real two-parameter group. The infinitesimal generators of these groups are respectively

\[
E_1 f(x) = x \ln x \frac{d}{dx} f(x); \quad E_2 f(x) = x \frac{d}{dx} f(x); \quad E_3 f(x) = (\ln x) f(x)
\]

They satisfy the commutation relations

\[
[E_1, E_2] = -E_2; \quad [E_1, E_3] = E_3; \quad [E_2, E_3] = 1 \quad (3.4)
\]

Thus we obtain a real 6-dimensional Lie algebra \( \mathfrak{g} \) spanned by the operators

\[
E_1, \quad E_2, \quad E_3, \quad iE_3, \quad 1, \quad i
\]

The algebra \( \mathfrak{g} \) is solvable and it contains a two-dimensional center \( \mathbb{R} \cdot 1 + \mathbb{R} \cdot i \).

Also \( \mathfrak{g} \) is a real subalgebra (but not a real form) in 4-dimensional complex Lie algebra

\[
\mathbb{C} \cdot E_1 + \mathbb{C} \cdot E_2 + \mathbb{C} \cdot E_3 + \mathbb{C} \cdot 1
\]

Obviously,

\[
U_\alpha W_h U_\alpha^{-1} = W_{\alpha h}; \quad (3.5)
\]
\[
U_\alpha V_a U_\alpha^{-1} = V_{a^{i/\alpha}} \quad (3.6)
\]

Consider the group \( G \) generated by the one-parameter groups (3.1)–(3.3). General element of this group is an operator of the form

\[
\lambda \cdot R(h, \alpha, a)
\]

where \( \lambda \in \mathbb{C}^* \) and

\[
R(h, \alpha, a)f(x) = x^h f(ax^\alpha) = W_h U_\alpha V_a f(x) \quad (3.7)
\]
The product is given by

\[ R(g, \beta, b)R(h, \alpha, a) = b^h R(g + h\beta, \alpha\beta, ab^\alpha) \]  \hspace{1cm} (3.8)

We also can add the operator

\[ U_1 f(x) = f(1/x) \]

or equivalently we can allow \( \alpha \in \mathbb{R} \setminus 0 \) in (3.1), (3.7), (3.8). Then we obtain a Lie group \( G^a \) of operators containing two connected components; the group \( G \) defined above is the connected component containing 1.

**3.2. Operators in \( L^2_\mu(\mathbb{R}_+) \).** Now, let us fix \( \mu > 0 \). Then the operators

\[ |\alpha|^{1/2}a^{1/2} R((\alpha - \mu)/2 + is, \alpha, a) \]

are unitary in \( L^2_\mu(\mathbb{R}_+) \). Such operators form a 4-dimensional solvable Lie group with an one-dimensional center; denote this group by \( G^a_\mu \).

**Remark.** All other operators \( R(h, \alpha, a) \) are unbounded in \( L^2_\mu(\mathbb{R}) \).

**3.3. Operators in \( H_\mu \).** Let us evaluate the kernel in \( H_\mu \) of the operator \( L_\mu R(h, \alpha, a) L_\mu^{-1} \) using (2.3).

We have

\[ L^{-1}_\mu \Theta_u(x) = \exp(-\pi x); \]
\[ R(h, \alpha, a) L^{-1}_\mu \Theta_u(x) = x^h \exp(-a\pi x^\alpha); \]
\[ L_\mu R(h, \alpha, a) L^{-1}_\mu \Theta_u(z) = \frac{1}{\Gamma(\mu)} \int_0^\infty x^{h+\mu-1} \exp(-a\pi x^\alpha - \pi x) \, dx = \]
\[ = z^{-\mu-b/2} \int_0^1 J_{\mu, h+\mu}(\pi a/z^\alpha) \]

\[ \text{(3.9)} \]
\[ \text{(3.10)} \]

and we obtain (for \( \mu > 1 \)) the integral operator

\[ \tilde{R}(h, \alpha, a) F(z) = \frac{1}{\pi \Gamma(\mu - 1)} \int_0^\infty L_{\alpha, h+\mu}(\pi a/z^\alpha) F(u) \, (\text{Re} u)^{\mu-2} \, du \, d\pi \]

\[ \text{(3.11)} \]

For \( \mu = 1 \) we understand (3.11) as (2.8), and for \( \mu < 1 \) as (2.9).

Formally, our algorithm of evaluating of the kernel is valid only for bounded operators. Thus we proved the following theorem

**Theorem 3.1.** Let

\[ \text{Re } h = \frac{(\alpha - \mu)}{2} \]

Then an operator \( \tilde{R}(h, \alpha, a) \) defined by (3.11) is unitary in \( H_\mu \) up to a scalar factor. The product of two operators \( \tilde{R} \) is given by the formula

\[ \tilde{R}(g, \beta, b) \tilde{R}(h, \alpha, a) = b^h \tilde{R}(g + h\beta, \alpha\beta, ab^\alpha) \]  \hspace{1cm} (3.12)

These operators generate a 4-dimensional solvable Lie group isomorphic to the group \( G^a_\mu \) described in 3.2.
3.4. Operators in the space $K$. The group $G^\circ$ defined in 3.1 acts in the space $\mathcal{S}(\mathbb{R}_+)$. Since the weighted Laplace transform $\mathcal{L}_\mu$ identifies $\mathcal{S}(\mathbb{R}_+)$ and $K$, this group also acts in $K$. For the subgroup $G^\circ_n \subset G^\circ$, the action was constructed in the previous subsection, but formula (3.11) for the kernel almost survives for a general element of $G^\circ$. We consider separately $\alpha > 0$ and $\alpha < 0$.

a) Let $\alpha > 0$. We substitute $y = x^\alpha$ to the expression (3.9) and obtain

\begin{equation}
\frac{1}{[\alpha]} \int_0^\infty y^{(h+\mu)/\alpha - 1} \exp(-a\pi y - zy^{1/\alpha}) \, dy
\end{equation}

(3.13)

If $h$ satisfies the condition

\begin{equation}
(\text{Re} \, h + \mu)/\alpha > 0
\end{equation}

then our integral is convergent (otherwise, we have a non-integrable singularity at 0). The expression (3.13) is the Laplace transform of the function $y^{(h+\mu)/\alpha - 1} \exp(-zy^{1/\alpha})$. Since this function is an element of $L^1$, its Laplace transform is a bounded function in $H$. Hence, for $F \in K$, the integral (3.11) is convergent and is holomorphic in $h$. Thus the formula (3.11) defines the $\mathcal{L}_\mu$-image of $R(h, \alpha, a)$ for all triples $h, \alpha, a$ satisfying (3.14).

Next, we write

\begin{equation}
R(h, \alpha, a) = W_{-n} \circ R(h + n, \alpha, a)
\end{equation}

(3.15)

For sufficiently large $n$, the $\mathcal{L}_\mu$-image of $R(h + n, \alpha, a)$ is defined by formula (3.11); the $\mathcal{L}_\mu$-image of $W_{-n}$ is the iterated indefinite integration (2.12).

b) $\alpha < 0$. We again transform the integral to the form (3.13). Now the integrand is smooth at 0; for convergence at infinity we are need in the condition $(\text{Re} \, h + \mu)/\alpha < 0$. Then we repeat (3.15)\(^6\)

Thus we obtain the following theorem

**THEOREM 3.2.** Fix $\mu > 0$. Let $h \in \mathbb{C}, \alpha \in \mathbb{R} \setminus 0$, and $a > 0$. If $\text{Re} \, h + \mu > 0$, then we define the integral operator $\tilde{R}(h, \alpha, a)$ in $K$ by (3.11). Otherwise, we consider $n$ such that $(\text{Re} \, h + n)/\alpha > 0$ and define

\begin{equation}
\tilde{R}(h, \alpha, a) = (-1)^n I^n \circ \tilde{R}(h + n, \alpha, a)
\end{equation}

there $I$ is the operator of indefinite integration (2.13) Then all these operators are bounded in $K$ and their product is given by (3.12). The group generated by $\tilde{R}(h, \alpha, a)$ is isomorphic to the group $G^\circ$ defined in 3.1.

**REMARK.** For different $\mu$ we obtain the same group of operators in $K$; but identification of this group with $G^\circ$ depends on $\mu$.

3.5. Statements formulated in Introduction. Let $\mu = 1$. The operator $A_\alpha$ given by (0.5) is the $\mathcal{L}_1$-image of $U_\alpha$, the fractional derivation $D_h$ is the $\mathcal{L}_1$-image of $W_h$, and the operator $R_\alpha$ is the $\mathcal{L}_1$-image of $V_\alpha$. Now (0.7), (0.9) follow from (3.5), (3.6).

\(^6\)This is not really necessary, since for $\text{Re} \, u > 0$ integral (3.13) is convergent. But for the case $\mu = 1$ it is pleasant to have an expression for $\text{Re} \, u = 0$. 

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The operator $F \mapsto zF$ is the $\mathcal{L}$-image of $d/dx$, and this implies (0.8).

### 3.6. Hankel type transforms.

The kernel of the operator $\tilde{R}(h, -1, 1)$ is

$$K(z, u) = \text{const} \cdot \int_0^\infty x^{h+\mu-1} \exp(-\pi/x - zx) \, dx$$

This expression is a modified Bessel function of Macdonald, see [6]. The corresponding integral transform is similar to the Hankel transform.

### 3.7. Another group of symmetries.

Now we consider the group of unitary operators in $L^2(\mathbb{R}_+)$ generated by

$$U_\alpha f(x) = |\alpha|^{1/2} f(x^\alpha)x^{(\alpha-1)/2};$$
$$V_\alpha f(x) = a^{1/2} f(ax);$$
$$T_\beta(is)f(x) = \exp(is^\beta)f(x), \quad s \in \mathbb{R}, \beta \in \mathbb{R}$$

This group is infinite dimensional since it contains all the operators having the form

$$f(x) \mapsto f(x) \exp(i \sum_{j=1}^N s_j x^\beta)$$

Obviously, we have

$$U_\alpha T_\beta(is)U^{-1}_\alpha f(x) = T_{\alpha \beta}(is); \quad V_\alpha T_\beta(is)V^{-1}_\alpha = T_{\alpha \beta}(is\beta) \quad (3.16)$$

Consider the image of our group of operators under the standard Laplace transform $\mathcal{L}$. The operators $U_\alpha', V_\alpha'$ are contained in the group $G^\alpha$ and their images $\tilde{U}_\alpha', \tilde{V}_\alpha'$ are described above. The image of $T_\beta(is)$ is the convolution operator

$$\tilde{T}_\beta(is)F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} M(z - u)F(u)du$$

where

$$M(z) = \int_0^\infty \exp(is^\beta - zx) \, dx = \mathbb{L}_{\beta,1}(is/z^\beta)$$

In particular, we obtain the identity

$$\tilde{U}_\alpha \tilde{T}_\beta(is)\tilde{U}^{-1}_\alpha f(x) = \tilde{T}_{\alpha \beta}(is)$$

for our operators with L-kernels.

### 4. Operators in space of functions on half-line

#### 4.1. Spaces of functions.

Consider the space $\mathcal{P}$ consisting of $C^\infty$-functions on half-line $x \geq 0$ satisfying

a) $f^{(k)}(0) = 0$ for all $k \geq 0$

b) $\lim_{x \to +\infty} f(m)(x) \exp(-\varepsilon x) = 0$ for all $\varepsilon > 0$, $m \geq 0$. 

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Consider also the space $\mathcal{F}$, whose elements are functions holomorphic in the half-plane $\Re z > 0$ satisfying the condition
— for each $\varepsilon > 0$, $N > 0$ where exists $C$ such that

$$|f(z)| < C/|z|^{-N} \quad \text{for } \Re z > \varepsilon$$

**Lemma 4.1.** The image of the space $\mathcal{P}$ under the Laplace transform is $\mathcal{F}$.  

**Proof.** a) For $f \in \mathcal{P}$ denote $F = \mathcal{L}f$. Fix small $\delta > 0$.

$$F(p) = \int_0^\infty f(x)e^{-px}dx = \int_0^\infty [f(x)e^{-\delta x}] e^{-(p-\delta)x}dx$$

Since $f(x)e^{-\delta x}$ is an element of the Schwartz space, for each $N$ we have an estimate

$$|F(p)| \leq C|p-\delta|^{-N}$$

for $\Re p \geq \delta$. For $\Re p > 2\delta$, we can write

$$|F(p)| \leq 2^NC|p|^{-N}$$

Thus $F \in \mathcal{F}$.

b) Let $F \in \mathcal{F}$. The inversion formula for $\mathcal{L}$ gives

$$f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{px} F(p) dp = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} F(a + it) dt$$

Since $F(a + it)$ is an element of Schwartz space $S(\mathbb{R})$, the function

$$e^{-ax}f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} F(a + it) dt$$

is an element of $S(\mathbb{R})$. By the Paley–Wiener theorem this function is supported by $\mathbb{R}^+$. Since $a > 0$ is arbitrary, we obtain we required statement.

**4.2. Fractional derivations.** Consider the Riemann–Liouville operators of fractional integration (see [20])

$$J_r f(x) = \frac{1}{\Gamma(r)} \int_0^x f(y)(x-y)^{r-1}dy \quad (4.1)$$

in the space $\mathcal{P}$. The integral is convergent for $r > 0$. For integer positive $r = n$ we have

$$J_r f(x) = \int_0^x dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{n-1}} f(x_n) dx_n$$

For fixed $f$ and $x$, the function $J_r f(x)$ admits a holomorphic continuation to the whole plane $r \in \mathbb{C}$, and for integer negative $r = -n$ we have

$$J_{-n} f(x) = \frac{d^n}{dx^n}f(x)$$
We also have the identity
\[ J_r J_p = J_{r+p} \]
for all \( r, p \in \mathbb{C} \).

Laplace transform \( \mathcal{L} \) identifies the Riemann–Liouville fractional integrations \( J_r \) with the operators in \( \mathcal{F} \) given by
\[ F(z) \mapsto z^{-r} F(z) \]

4.3. Operators. Now we consider the semigroup \( \Gamma \) of operators in \( \mathcal{F} \) consisting of transformations
\[ \lambda Q(\theta, \alpha, a) \]
where \( \lambda \in \mathbb{C}^* \),
\[ Q(\theta, \alpha, a) F(z) = z^\theta F(az^\alpha) \] (4.2)
and the parameters satisfy the conditions
\[ 0 < \alpha < 1, \quad \arg a + \alpha \pi / 2 < \pi / 2, \quad \arg a - \alpha \pi / 2 > -\pi / 2, \quad \theta \in \mathbb{C} \] (4.3)

Remark. The restrictions (4.4) mean that \( z \in \Pi \) implies \( az^\alpha \in \Pi \).

Obviously, we have
\[ Q(\theta', \alpha', a')Q(\theta, \alpha, a) = (a')^\theta Q(\theta' + \theta \alpha' + \alpha a', a(\alpha')^\alpha) \] (4.4)

Thus \( \Gamma \) is a 7-dimensional semigroup with 2-dimensional center.

Remark. The semigroup \( \Gamma \) can be embedded to a 7-dimensional Lie group. The parameters of this group are \( \lambda \in \mathbb{C}^*, \quad a \in \mathbb{C}^*, \quad \theta \in \mathbb{C}, \quad a > 0 \). The multiplication in this group is determined by the formula (4.4), where \( \alpha > 0 \) and \( \theta \), \( a \in \mathbb{C} \). But the corresponding operators (4.2) are not well-defined in the space \( \mathcal{F} \).

The \( \mathcal{L}^{-1} \)-image of the operators \( Q(\theta, \alpha, a) \) is given by the formula
\[ \tilde{Q}(\theta, \alpha, a)f(x) = \int_0^\infty N(x, y)f(y) dy \] (4.5)
where the kernels \( N(x, y) \) are given by
\[ N(x, y) = \frac{1}{2\pi} \int_{-i\infty}^{i\infty} z^\theta \exp(-az^\alpha y + zx) dz \]
For \( \theta > -1 \), we transform this integral in the same way as in 1.5 and obtain
\[ N(x, y) = \frac{1}{2\pi i y^{\theta+1}} \left\{ e^{i(\theta+1)\pi/2} L_{\alpha, \theta+1}(yax^{-\alpha}e^{i\pi\alpha/2}) - e^{-i(\theta+1)\pi/2} L_{\alpha, \theta+1}(yax^{-\alpha}e^{-i\pi\alpha/2}) \right\} \]
For real \( a > 0, \theta > -1 \) we have an expression of the form (1.10).
If $\theta < -1$, we write $z^\theta F(a z^\alpha) = z^{-n} z^{\theta+n} F(a z^\alpha)$ and

$$Q(\theta, \alpha, a) := J_n \circ Q(\theta + n, \alpha, a)$$

(4.6)

for sufficiently large $n$.

Theorem 4.1. For $\alpha, a$ satisfying (4.3), the operators $\tilde{Q}(\theta, \alpha, a)$ defined by (4.5), (4.6) are bounded in $\mathcal{P}$ and their product is given by (4.4).

Formulae (0.11)–(0.12) given in Introduction are particular cases of this statement.

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Math.Phys. Group, Institute of Theoretical and Experimental Physics,
B.Cheremushkinskaya, 25, Moscow 117259
& University of Vienna, Math. Dept., Nordbergstrasse, 15, Vienna 1090, Austria
neretin@mccme.ru