Holographic Entanglement and Causal Shadow in Time-Dependent Janus Black Hole

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Abstract

We holographically compute an inter-boundary entanglement entropy in a time-dependent two-sided black hole which was constructed in [1] by applying time-dependent Janus deformation to BTZ black hole. The black hole contains “causal shadow region” which is causally disconnected from both the conformal boundaries. We find that the Janus deformation results in an earlier phase transition between the extremal surfaces and that the phase transition disappears when the causal shadow is sufficiently large.

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1 Introduction

The relation between entanglement and black hole interior has been paid much attentions recently \cite{2, 3, 4, 5}. For eternal AdS black holes it was discussed that the time-evolution of holographic entanglement entropy \cite{6, 7, 8} along a particular time-slice in which the black hole looks time-dependent can capture some information of interior of it, if we take appropriate subsystems which are the disjoint union of a subsystem in the original CFT and that in its thermofield doubled copy of the original CFT \cite{9}. The resulting entanglement entropy with this choice of the subsystem grows linearly in time, and after a certain critical time, it saturates to twice of the thermal entropy of the black hole. The linear growth of the entanglement entropy diagnoses the growth of size the wormhole inside of the eternal black hole. The dual CFT interpretation of this behavior of entanglement entropy in terms of the global quench process \cite{10} was also pointed out.

More general two-sided black holes can have even richer interior structures. For example, similar inter-boundary entanglement entropies on charged or rotating black holes, which have vertically extended Penrose diagrams, were investigated in \cite{11, 12}.

Another interesting class of black holes, which we will focus on in this paper, have so called “causal shadow region”, which is causally inaccessible from both the boundaries. We refer \cite{13, 14} in which its implications to holographic entanglement entropy are discussed. Asymptotically AdS black holes with a causal shadow region can be constructed for example from the eternal AdS black holes by sending shock waves from one of the boundaries \cite{15, 16, 17}. We also refer a dual 2d CFT computation \cite{18}.

Black holes with a causal shadow region can also be constructed in a different way. In \cite{1}, as a solution of the Einstein-scalar theory, they discovered a three dimensional black hole geometry with a causal shadow, which is called the three dimensional time-dependent Janus black hole.\footnote{There is also a static type of Janus deformation of BTZ black hole \cite{19}.} In the bulk, this solution turns on nontrivial dilaton configuration to the eternal BTZ black hole. In the viewpoint of the dual CFT, it amounts to change the coupling constant of the original CFT while keeping that of the thermofield doubled copy of the original CFT fixed \cite{1}. The Hamiltonian of the original CFT is altered by it and the resulting CFT state was determined in \cite{20}, which gives a natural extension of the usual eternal AdS black hole/thermofield double state correspondence \cite{21, 22}. The proposal was checked by computing a one point function both on the CFT and gravity sides \cite{1, 20}. A natural question is how the information of the causal shadow of the Janus black hole is encoded in the time-evolution of the holographic entanglement entropies.
In this paper we study the time-evolution of such an inter-boundary holographic entanglement entropy in the 3D Janus black hole. As in the BTZ black hole, there are two candidate extremal surfaces in the bulk, whose area gives entanglement entropy. One of the extremal surfaces, which we call connected surface, connects two asymptotic boundaries and probe the black hole interior. The other extremal surface, which we call disconnected surface, localizes near each of asymptotic boundaries.

In the BTZ black hole, the area of connected surface is initially dominant in the holographic entanglement entropy, but after a certain "critical time" \( t_c \) the disconnected surface become dominant and the entanglement entropy thermalizes.

In the Janus black hole, when the deformation from the BTZ black hole is not so large, we find a similar behavior but with the critical time \( t_c \) shorter than that of the BTZ black hole. Roughly speaking, this is because the deformation makes the wormhole region longer and results in a longer connected surface. In addition, we also find that with a sufficiently large deformation, the disconnected surface is always dominant, and that the holographic entanglement entropy is already proportional to the size of the subsystem from the initial time. This means that the entanglement entropy of this subsystem does not probe the black hole interior.

This paper is organized as follows. In Section 2, we review properties of the 3D Janus black hole with emphasis on the difference between this black hole and BTZ black hole. In Section 3, we compute the area of extremal surfaces with appropriate boundary conditions in the black hole. In Section 4 we discuss the time-evolution of the holographic entanglement entropy. We conclude this paper in Section 5.

2 Properties of 3-dimensional Janus Black Hole

Here we summarize the properties of the 3-dimensional time-dependent Janus black hole with emphasis on its causal structure and dual CFT interpretation.

2.1 The 3D Janus metric

2.1.1 Time-dependent Janus deformation of BTZ metric

The metric of the Janus black hole with its horizon radius \( Lr_0 \) is given by

\[
ds^2 = L^2 \frac{d\mu^2 - d\tau^2 + r_0^2 \cos^2 \tau d\theta^2}{g(\mu)^2},
\] (2.1)
where the only dimensionful quantity is the AdS radius $L$. The conformal factor $g(\mu)$ is defined as

$$g(\mu) = \frac{\text{cn}(\kappa_+\mu, k^2)}{\kappa_+ \text{dn}(\kappa_+\mu, k^2)}$$

$$\kappa_\pm := \sqrt{1 \pm \sqrt{1 - 2\gamma^2}}$$

$$k := \frac{\kappa_-}{\kappa_+}. \quad (2.2)$$

This metric is a 1-parameter generalization of the BTZ black hole metric by “Janus deformation parameter” $0 \leq \gamma < 1/\sqrt{2}$. When $\gamma = 0$, this $g(\mu)$ becomes $\cos \mu$ and then the metric reduces to the BTZ metric, with its inverse temperature

$$\beta = \frac{2\pi}{r_0}, \quad (2.3)$$

in the unit of the AdS radius $L$. The conformal boundaries $g(\mu) = 0$ are located at $\mu = \pm \mu_0$, where $\mu_0 := K(k^2)/\kappa_+$ and $K(k^2)$ is the complete elliptic integral of the 1st kind: $K(k^2) := \int_0^{\pi/2} d\theta/\sqrt{1 - k^2 \sin^2 \theta}$.  

2.1.2 Dual CFT coordinate $(t, \theta)$ and UV cutoff $\epsilon_{\text{CFT}}$

In applying AdS/CFT techniques, another time coordinate $\tanh r_0 t := \sin \tau$ is useful, because the flat metric $-dt^2 + d\theta^2$ of the dual CFT becomes manifest:

$$ds^2 = L^2 \left[ dy^2 + \frac{r_0^2}{g(y)^2 \cosh^2 r_0 t} (-dt^2 + d\theta^2) \right]. \quad (2.4)$$

Here we have introduced another radial coordinate $y$ : $\tanh y = \text{sn}(\kappa_+ \mu, k^2)$, measuring the proper length $dy = d\mu/g(\mu)$, and we have rewritten $g(\mu)$ as

$$g(y) := g(\mu(y)) = \frac{1}{\kappa_+ \sqrt{(1 - k^2) \cosh^2 y + k^2}} = \sqrt{\frac{2}{1 + \sqrt{1 - 2\gamma^2 \cosh 2y}}}. \quad (2.5)$$

In this coordinate $y$, the origin $\mu = 0$ corresponds to $y = 0$ and the conformal boundaries $\mu = \pm \mu_0$ are located at $y \to \pm \infty$.

Near the conformal boundaries $y \to \pm \infty$, the metric (2.4) approaches to AdS in the Poincaré coordinate

$$ds^2 = L^2 dz^2 - dt^2 + d\theta^2 + O(z) \quad (2.6)$$

by the following identification

$$z := \frac{2}{\sqrt{1 - 2\gamma^2} r_0} e^{-|y| \cosh r_0 t}. \quad (2.7)$$
Hence the CFT UV cutoff $\epsilon_{\text{CFT}}$ is given as

$$\epsilon_{\text{CFT}} = \frac{2}{\sqrt{1 - 2\gamma^2 r_0}} e^{-y_\infty \cosh r_0 t_\infty},$$

where $y_\infty (\gg 1)$ is a bulk volume regulator and $t_\infty$ is the time $t$ in the CFT at $y = \pm y_\infty$.

### 2.1.3 As a solution of Einstein-scalar theory

This geometry is a solution of the three dimensional Einstein-scalar system

$$S = \frac{1}{16\pi G} \int d^3 x \sqrt{g} \left( R - g^{ab} \partial_a \phi \partial_b \phi + \frac{2}{L^2} \right),$$

with the scalar field configuration

$$\phi = \phi_0 + \sqrt{2} \left( \tanh^{-1}(k \sn(k_+ \mu, k^2)) + \log \sqrt{1 - k^2} \right)
= \phi_0 + \sqrt{2} \left( \tanh^{-1}(k \tan y) + \log \sqrt{1 - k^2} \right).$$

Note that the scalar field value $\phi_+ := \phi(y = \infty)$ on the right boundary is different from the one $\phi_- := \phi(y = -\infty)$ on the left boundary by

$$\phi_+ - \phi_- = 2\sqrt{2} \tanh^{-1} k = \sqrt{2} \tanh^{-1} \sqrt{2}\gamma.$$

The three dimensional system can be embedded in type IIB supergravity in ten dimensions with an appropriate ansatz [1]. At that time the scalar field $\phi$ is identified with the dilaton, and hence the boundary values $\phi_\pm$ are related to the coupling constants of the dual conformal field theories $g_\pm$ [1]. Although the difference between $\phi_+$ and $\phi_-$ (2.11) becomes very large when we take $\gamma$ to be very near to $1/\sqrt{2}$, we can also take $\phi_0$ to be negatively large so that the classical gravity does not break down. In terms of the dual CFT, it requires that the theory is weakly coupled in the sense of the Yang-Mills couplings whereas it is strongly coupled in the viewpoint of the 'tHooft couplings, as usual.

### 2.2 Main differences from BTZ black hole

#### 2.2.1 Causal shadow region

By using the conformally flat $(\mu, \tau)$ coordinate (2.1), one can draw the Penrose diagram of the time-dependent Janus black hole geometry (see Figure 1). The length of the horizontal line of the diagram is longer than that of BTZ, because the width $2\mu_0 = 2K(k^2)/\kappa_+$ between the two conformal boundaries monotonically increases with the deformation parameter $\gamma$.  

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As a consequence, unlike the BTZ geometry ($\gamma = 0$), the 3D Janus black hole geometry ($\gamma > 0$) has a finite region causally disconnected from the both conformal boundaries $\mu = \pm \mu_0$. Such regions are sometimes called “causal shadow” [13, 14]. It is an interesting question how the dual CFT encodes information on causal shadow regions. As a first step to answer this question, we will compute holographic entanglement entropies in the Janus black hole in the next two sections, because holographic entanglement entropies can be affected by the inside of the causal shadow.

![Penrose diagram of the 3D time-dependent Janus black hole](image)

Figure 1: Penrose diagram of the 3D time-dependent Janus black hole. The two conformal boundaries are located at $\mu = \pm \mu_0$ (thick lines), and the diagram is a wide rectangle because $\mu_0 \geq \pi/2$. The future and past event horizons are drawn; the blue or red line represents the future or past event horizon which intersects with the right hand side boundary, respectively. The yellow shaded region corresponds to the “causal shadow” region, which is causally disconnected from both boundaries. The apparent horizons (green line) in the time-slices $\tau = \text{const.}$ are located inside the future event horizon.
2.2.2 Time-dependence

Unlike the BTZ metric ($\gamma = 0$), the Janus metric ($\gamma > 0$) is time-dependent: there is no timelike Killing vector. As a result, its apparent horizon

$$\tan \tau = -\frac{d}{d\mu} \log g(\mu) \tag{2.12}$$

in the time-slices $\tau = \text{const.}$ becomes different from the event horizon $\tau - \pi/2 = \mu - \mu_0$ (See Figure 1).

2.3 The CFT interpretation of the Janus black hole

When $\gamma = 0$, the Janus black hole reduces to the ordinary eternal BTZ black hole, which is dual to the thermofield double state \[21, 22\]

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{\beta}{4} E_n} |E_n^+\rangle |E_n^-\rangle. \tag{2.13}$$

The inverse temperature $\beta$ is given by (2.3).

If we turn on the parameter $\gamma$, the Hamiltonian $H_+$ on the right boundary and $H_-$ on the left boundary becomes different. This is because the dilaton $\phi$ in type IIB supergravity takes different values $\phi_+$ and $\phi_-$ on the two boundaries as (2.11) and so the gauge couplings also become different. Hence it is natural to conjecture \[20\] that

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_{(m,n)} e^{-\frac{\beta}{4} (E_n + E_m)} \langle E_m^+ | E_n^- \rangle |E_m^+\rangle |E_n^-\rangle. \tag{2.14}$$

This conjecture has passed some nontrivial checks. For example, the one point function of the Lagrangian density was computed both on gravity and CFT sides, which agrees up to the second order in $\gamma$ \[20\].

3 Calculation of Holographic Entanglement Entropy

In this section we compute a holographic entanglement entropy on the 3D Janus black hole geometry to study entanglement between left and right CFT. We take our subsystem $A$ to be the disjoint union of two intervals $-\theta_\infty \leq \theta \leq \theta_\infty$ in the left and right CFT respectively at a fixed time $t = t_\infty$ (see Figure 2).
3.1 Covariant holographic entanglement entropy

It has been conjectured [8] that for a given bulk geometry, the entanglement entropy of the dual CFT state is given by the area of the extremal surface in the bulk which are anchored to \( \partial A \) in the conformal boundary,

\[
S_A = \text{ext} \frac{A(\gamma_A)}{4G_N},
\]

where \( G_N \) is the three dimensional Newtonian constant. The extrema is chosen among the surfaces \( \gamma_A \) which are homologous to the subsystem \( A \) and satisfy \( \partial A = \partial \gamma_A \). If there are several extremal surfaces, we should choose the one with minimum area among them.

In the current setup with the subsystem \( A = \{ (\pm y_{\infty}, t_{\infty}, \theta); -\theta_{\infty} \leq \theta \leq \theta_{\infty} \} \) in the Janus black hole geometry (2.4), the extremal surface can take two types of topologies (see Figure 2), “connected phase” and “disconnected phase”, like usual BTZ black holes [9]. The disconnected type consists of two geodesics which start from and end at the same boundary (see Figure 2 (a)); starting from \((\pm y_{\infty}, t_{\infty}, -\theta_{\infty})\), turning around at \((\pm y_{\ast}, t_{\ast}, 0)\) and ending at \((\pm y_{\infty}, t_{\infty}, \theta_{\infty})\). The connected type consists of two geodesics which connect the two boundaries (see Figure 2 (b)); starting from \((y_{\infty}, t_{\infty}, \pm \theta_{\infty})\) and ending at \((-y_{\infty}, t_{\infty}, \pm \theta_{\infty})\).

\[\]

Figure 2: The subsystem \( A \) (two red lines) is taken as two disjoint intervals of the same length \( \Delta \theta = 2\theta_{\infty} \) in the right and left boundary (two black squares). The extremal surface \( \gamma_A \) (blue lines) has two phases: disconnected phase (a) and connected phase (b).

In the following, we will obtain and solve differential equations for each type of extremal
surfaces. Identifying the area functional

$$A[t(y), \theta(y)] = L \int dy \sqrt{1 + \frac{r_0^2}{\tilde{g}(y)^2 \cosh^2 r_0 t} (-l^2 + \dot{\theta}^2)}$$

(3.16)

with a classical action for dynamical variables $t(y)$ and $\theta(y)$ as for “time” $y$, this problem reduces to just an Euler-Lagrange problem. Here the dot ($\cdot$) represents the “time” derivative $d/dy$. We will see that the disconnected surface, as well as the connected one, can penetrate the event horizon, and both of their areas are dependent on the boundary time $t_\infty$. The phase transition between these two types will be discussed in Section 4.

### 3.2 Extremal areas in connected phase

For connected surfaces, the area functional is extremized when $\theta = \text{const.} (= \pm \theta_\infty)$. Then the “action” (3.16) becomes

$$A[t(y)]/L = \int_{-y_\infty}^{y_\infty} dy \sqrt{1 - \frac{r_0^2 l^2}{\tilde{g}(y)^2 \cosh^2 r_0 t}},$$

(3.17)

for each of the two pieces of the surface ($\theta = \pm \theta_\infty$). Here we introduce $y_\infty$ to regulate the divergence of the integral. This functional has one conserved charge $E$:

$$E := \frac{\delta A/L}{\delta \dot{t}} = \frac{-r_0^2 \dot{t}}{\tilde{g}(y) \cosh r_0 t \sqrt{\tilde{g}(y)^2 \cosh^2 r_0 t - r_0^2 l^2}} \Leftrightarrow \dot{t} = \frac{-E \tilde{g}(y)^2 \cosh^2 r_0 t}{\sqrt{r_0^2 + E^2 \tilde{g}(y)^2 \cosh^2 r_0 t}},$$

(3.18)

associated to its $t$-translation symmetry. But this charge $E$ vanishes, because $\dot{t}$ cannot change its sign and we have the boundary condition $\int_{-y_\infty}^{y_\infty} \dot{t} \, dy = t_\infty - t_\infty = 0$. In the result, the total area of the connected extremal surface can be explicitly calculated as

$$A_c(t_\infty, \theta_\infty)/L = 2 \times 2y_\infty = 4 \log \frac{2 \cosh r_0 t_\infty}{r_0 \epsilon_{\text{CFT}}} - \log(1 - 2\gamma^2).$$

(3.19)

To derive this we used the relation between $y_\infty$ and CFT cutoff $\epsilon_{\text{CFT}}$ (2.8). For later purposes, it is convenient to define the notion of “renormalized” area which is given by

$$A_c^{(\text{ren})}/L \equiv A_c(t_\infty, \theta_\infty)/L + 4 \log \epsilon_{\text{CFT}} = 4 \log \frac{2 \cosh r_0 t_\infty}{r_0} - \log(1 - 2\gamma^2).$$

(3.20)
This area becomes arbitrarily large in $\gamma^2 \to \frac{1}{2}$ limit. This illuminates the fact that the length of the wormhole behind the Janus black hole becomes infinitely long in this limit.

### 3.3 How to calculate extremal areas in disconnected phase

In this subsection we consider the area of the disconnected surfaces as a function of $(t_\infty, \theta_\infty)$. The disconnected surfaces consist of two disjoint geodesics, one of which is localizing in $y > 0$ and the other is in $y < 0$ respectively. Below we mainly focus on the $y > 0$ part of the surfaces, because the $y < 0$ part can be identified with $y > 0$ part by the parity transformation $y \to -y$. There exist returning points $y = \pm y_*$, at which $\dot{\theta}$ diverges. Their locations $(\pm y_*, t_*)$ are determined by the boundary conditions at $(\pm y_\infty, t_\infty, \pm \theta_\infty)$. The area of these surfaces are given by an integral which involves the returning point $y = y_*$. To see $(t_\infty, \theta_\infty)$-dependence of the disconnected surface area, first one need to find an expression of $y_*$ as a function of $(t_\infty, \theta_\infty)$, and then substitute it into the area integral. In Section 3.3.1, we will explain how to do this.

#### 3.3.1 Solving the equation of motion

One of the Euler-Lagrange equations is a second order differential equation of $t(y)$, which can be solved in the following way.

The action (3.16) has one conserved charge $J$, 

$$
J : = \frac{\delta A/L}{\delta \theta} = \frac{1}{\tilde{g}(y) \cosh r_0 t} \frac{r_0^2 \dot{\theta}}{\sqrt{\tilde{g}(y)^2 \cosh^2 r_0 t + r_0^2 (-\dot{t}^2 + \dot{\theta}^2)}}
$$

(3.21)

associated to its $\theta$ translation symmetry. This charge $J$ can be expressed by the returning point location $(y_*, t_*)$ as $J = r_0(\tilde{g}(y_*) \cosh r_0 t_*)^{-1}$ because $\dot{\theta}$ in (3.21) diverges at the returning point. With the aid of this constant charge $J$, the equation of motion for $t(y)$ can be rewritten into an equation for $t(\theta)$ without any $\tilde{g}(y)$ dependence:

$$
\frac{d}{dy} \frac{\delta A}{\delta t} - \frac{\delta A}{\delta t} = 0 \Leftrightarrow \frac{d}{dy} \left( J \frac{\dot{t}}{\dot{\theta}} \right) = J r_0 \frac{-\dot{t}^2 + \dot{\theta}^2}{\dot{\theta}} \tanh r_0 t 
$$

(3.22)

$$
\Leftrightarrow \frac{d^2 t}{d\theta^2} = r_0 \left[ 1 - \left( \frac{dt}{d\theta} \right)^2 \right] \tanh r_0 t, \quad (3.23)
$$

whose general solution is given by $\sinh r_0 t = \sinh A \cosh r_0 (\theta + B)$ with some constants $A, B$. These constants $A, B$ are determined by geometrical conditions $\theta|_{y=y_*} = 0$ and $dt/d\theta|_{y=y_*} = 0$.
as
\[
\sinh r_0 t = \sinh r_0 t_* \cosh r_0 \theta. \tag{3.24}
\]

This relation allows us to erase \( \theta \) in (3.21), yielding a 1st order differential equation of \( t \):
\[
t = \frac{\cosh r_0 t}{r_0 \tilde{g}(y_*) \cosh r_0 t_*} \sqrt{\frac{\cosh^2 r_0 t - \cosh^2 r_0 t_*}{1 - (\tilde{g}(y)/\tilde{g}(y_*))^2}}, \tag{3.25}
\]
which has a unique solution
\[
\sqrt{1 - \frac{\sinh^2 r_0 t_*}{\sinh^2 r_0 t}} (= \tanh \theta) = \cosh r_0 t_* \tanh \left[ \int_{y_*}^{y_*} dy \frac{\tilde{g}(y)^2}{\sqrt{\tilde{g}(y_*)^2 - \tilde{g}(y)^2}} \right], \tag{3.26}
\]
with an initial condition \( t(y_*) = t_* \). This expression gives the unique solution \( (t(y), \theta(y)) \) of the equations of motion, in terms of the returning point location \((y_*, t_*)\).

### 3.3.2 Returning point \((y_*, t_*)\)

The boundary condition \( (t(y_\infty) = t_\infty, \theta(y_\infty) = \pm \theta_\infty) \) determines \( t_* \) by (3.24) as
\[
\sinh r_0 t_* = \frac{\sinh r_0 t_\infty}{\cosh r_0 \theta_\infty}, \tag{3.27}
\]
and \( y_* \) by (3.24) and (3.26) as
\[
\sinh \left[ \int_{y_*}^{y_*} dy \frac{\tilde{g}(y)^2}{\sqrt{\tilde{g}(y_*)^2 - \tilde{g}(y)^2}} \right] = \frac{\sinh r_0 \theta_\infty}{\cosh r_0 t_\infty}. \tag{3.28}
\]

### 3.3.3 Extremal surface area

Plugging (3.24) and (3.25) into the definition of the surface area (3.16), we obtain the disconnected extremal surface area
\[
A_{dc}(t_\infty, \theta_\infty)/L = 4 \int_{y_*}^{y_\infty} dy \frac{\tilde{g}(y_*)}{\sqrt{\tilde{g}(y_*)^2 - \tilde{g}(y)^2}}, \tag{3.29}
\]
as a function of \((t_\infty, \theta_\infty)\), with \( y_* \) implicitly determined by \((t_\infty, \theta_\infty)\) through (3.28).

Note that this area has a UV divergence \(-4 \log \epsilon_{\text{CFT}}\), because \( \tilde{g}(y) \to 0 \) at each boundary and
\[
A_{dc}/L \to 4 \int_{y_*}^{y_\infty} dy \sim 4y_\infty = 4 \log \frac{2 \cosh r_0 t_\infty}{\sqrt{1 - 2 \gamma_\text{CFT}}}. \tag{3.30}
\]
Figure 3: How deeply the extremal surfaces in the disconnected phase can go inside the Janus black hole (with $\gamma^2 = 0.3$ in the figure). The shaded orange region represents where the extremal surfaces can pass through. The extremal surfaces can go beyond the event horizon (blue line), but cannot go beyond the apparent horizon (green line).

This UV divergence can be renormalized as

$$A_{dc}^{\text{(ren)}} / L \equiv A_{dc} / L + 4 \log \epsilon_{\text{CFT}}$$

$$= A_{dc} / L - \log(1 - 2\gamma^2) + 4 \log \frac{2 \cosh r_0 t_\infty}{r_0} - 4 y_\infty.$$ (3.31)

### 3.4 Some limits of extremal surface areas in disconnected phase

It is difficult to calculate the area of the generic disconnected surface. In this subsection we address some limits in which this area is explicitly calculable. In Sections 3.4.1 and 3.4.2, we compute the disconnected surface area $A_{dc}^{\text{(ren)}}(t_\infty, \theta_\infty)$ with a large subsystem ($\theta_\infty \gg r_0^{-1}$), at an early time ($t_\infty \ll \theta_\infty$) and at a late time ($t_\infty \gg \theta_\infty$), respectively. In Section 3.4.3 we compute the area in the small $\gamma$ limit.
3.4.1 Early time for large subsystem ($\theta_\infty \gg t_\infty$)

In this parameter region, the returning point $(y_*, t_*)$ is near to the origin $(0, 0)$, which can be seen as follows. The $t_*$ is determined by (3.27) as

$$r_0 t_* \simeq 2 e^{-r_0 \theta_\infty} \sinh r_0 t_\infty \quad (\ll 1), \quad (3.32)$$

and the $y_*$ is determined by (3.28) as

$$r_0 \theta_\infty - \log \cosh r_0 t_\infty \simeq \int_{y_*}^{y_\infty} dy \frac{\tilde{g}(y)^2}{\sqrt{\tilde{g}(y)^2 - g(y)^2}} \quad (\gg 1). \quad (3.33)$$

This means $y_* \ll 1$, because the left hand side of (3.33) is large while the integral of the right hand side is a monotonically decreasing function of $y_*$, diverging at $y_* \to 0$. In fact, the right hand side integral can be evaluated as

$$r_0 \theta_\infty - \log \cosh r_0 t_\infty \simeq - \frac{1}{\sqrt{\kappa^2_+ - \kappa^2_-}} \log \left[ \frac{\kappa_+ + \sqrt{\kappa^2_+ - \kappa^2_-}}{4} y_* \right], \quad (3.34)$$

around $y_* \to 0$, by changing integration variable from $y$ to $z := \tanh y$.

By solving this for $y_*$ and plugging it into (3.31), we obtain the renormalized area $A_{dc}^{(\text{ren})}(t_\infty, \theta_\infty)$ as a function of $(t_\infty, \theta_\infty)$. This can be carried out by evaluating the $y$-integral in (3.29) in a similar way to above, as

$$\int_{y_*}^{y_\infty} dy \left[ \frac{\tilde{g}(y_*)}{\sqrt{\tilde{g}(y_*)^2 - g(y)^2}} - 1 \right] \simeq - \frac{\kappa_+}{\sqrt{\kappa^2_+ - \kappa^2_-}} \log \left[ \frac{\kappa_+ + \sqrt{\kappa^2_+ - \kappa^2_-}}{2} y_* \right] - \log \left[ \frac{\kappa_+ + \sqrt{\kappa^2_+ - \kappa^2_-}}{2} r_0 \theta_\infty \right], \quad (3.35)$$

around $y_* \to 0$. Then $y_*$ is deleted easily, yielding

$$A_{dc}^{(\text{ren})}(t_\infty, \theta_\infty)/L \simeq 4 \kappa_+ r_0 \theta_\infty + 4(1 - \kappa_+) \log \cosh r_0 t_\infty - 4 \log \left[ \frac{\kappa_+ + \sqrt{\kappa^2_+ - \kappa^2_-}}{2} r_0 \theta_\infty \right]. \quad (3.36)$$

Note that the area linearly grows with both $t_\infty$ and $\theta_\infty$ with different coefficients, when $\theta_\infty \gg t_\infty \gg r_0^{-1}$.

3.4.2 Late time for large subsystem ($t_\infty \gg \theta_\infty \gg r_0^{-1}$)

In this parameter region, (3.27) and (3.28) lead to

$$2 e^{-r_0 t_*} \simeq \int_{y_*}^{y_\infty} \frac{\tilde{g}(y)^2 dy}{\sqrt{\tilde{g}(y)^2 - g(y)^2}} \simeq e^{r_0 (\theta_\infty - t_\infty)} \quad (\ll 1), \quad (3.37)$$
where we used \( \sinh x \simeq \cosh x \simeq e^x / 2 \) for \( x \gg 1 \) and \( \sinh x \simeq x \) for \( x \ll 1 \). This in turn implies \( y_* \gg 1 \), therefore the integrals in (3.37) and (3.29) can be approximated as
\[
\int_{y_*}^{y_\infty} \frac{\tilde{g}(y)^2 dy}{\sqrt{\tilde{g}(y)^2 - \tilde{g}(y_*)^2}} \simeq \frac{2}{\sqrt{1 - 2\gamma^2}} e^{-y_*}, \tag{3.38}
\]
\[
\int_{y_*}^{y_\infty} \frac{\tilde{g}(y_*) dy}{\sqrt{\tilde{g}(y)^2 - \tilde{g}(y_*)^2}} \simeq y_\infty - y_* + \log 2, \tag{3.39}
\]
respectively, where we also used \( y_\infty - y_* \gg 1 \). By substituting (3.39) into (3.29), and erasing \( y_\infty, y_* \) by (2.8), (3.37) and (3.38), we obtain the asymptotic form of the renormalized area \( A_{dc}^{(\text{ren})} \) (3.31) as
\[
A_{dc}^{(\text{ren})}(t_\infty, \theta_\infty)/L \simeq 4(r_0 \theta_\infty - \log r_0), \tag{3.40}
\]
which does not depend on either of \( \gamma \) or \( t_\infty \). Then in particular, it coincides with the BTZ \((\gamma^2 = 0)\) result.

### 3.4.3 Up to the lowest order of \( \gamma^2 \)

So far, we have determined the early \( t_\infty \ll \theta_\infty \) and the late time \( \theta_\infty \ll t_\infty \) behavior of disconnected surface area for general \( \gamma \). However it is difficult to see the behavior in the intermediate region \( t_\infty \sim \theta_\infty \). Knowing the behavior in this region is important to see when the phase transition between the disconnected and connected surfaces happen, as we will discuss in the next section. Therefore we compute the disconnected surface area perturbatively with regards to \( \gamma^2 \) in this subsection. By expanding (3.28) up to \( \gamma^2 \) order as well as area integral (3.29), we get
\[
A_{dc}^{(\text{ren})}/L = 4 \log \left( \frac{2}{r_0} \sinh r_0 \theta \right) - \left( \frac{3F^2 + 2}{2\sqrt{1 + F^2}} \coth^{-1} \left( \sqrt{1 + F^2} \right) - \frac{3}{2} \right) \gamma^2 + \mathcal{O}(\gamma^4), \tag{3.41}
\]
where
\[
F(t, \theta) = \frac{\cosh r_0 t}{\sinh r_0 \theta}. \tag{3.42}
\]
The detail of the calculation is explained in Appendix A. Note that when \( \gamma = 0 \), it reduces to usual thermal result. In the early and late time limits, it reproduces the results in the previous section, (3.36) and (3.40) respectively.
4 Time-Evolution of Entanglement Entropy and Phase Transition

Here we discuss some properties of the time-evolution of the holographic entanglement entropy. Since there are two extremal surfaces in the bulk geometry, holographic entanglement entropy $S_A$ is given by choosing minimum area among them,

$$S_A = \frac{1}{4G_N} \min \{A_c, A_{dc}\},$$

(4.43)

where $A_c$ and $A_{dc}$ are given by (3.19) and (3.29) respectively. As we will see below, the behavior of this $S_A$ significantly differs depending on the value of the deformation parameter $\gamma$.

$\gamma = 0$ When $\gamma = 0$, the spacetime becomes BTZ black hole, and does not contain any causal shadow region. The area of the connected/disconnected surface is given by

$$A_c(t_\infty, \theta_\infty)/L = 4 \log \left( \frac{2 \cosh r_0 t_\infty}{r_0 \epsilon_{CFT}} \right), \quad A_{dc}(t_\infty, \theta_\infty)/L = 4 \log \left( \frac{2 \sinh r_0 \theta_\infty}{r_0 \epsilon_{CFT}} \right).$$

(4.44)

For $r_0 \theta_\infty \gg 1$, the disconnected surface is chosen in early times because $A_c < A_{dc}$. The corresponding entanglement entropy $S_A$ linearly grows in time, and at $t_\infty = t_c \approx \theta_\infty$, it suddenly stops to grow and becomes a constant, which is twice of the thermal entropy.

On the CFT side, the same behavior can also be observed including the sharp phase transition, since the time-scale of the transition is given by $\beta$ [9] and now $r_0 \theta \gg 1$ implies $t_c \gg \beta$. Furthermore, the initial entanglement entropy at $t_\infty = 0$ can be accounted by the contribution from the boundary of $A$ (4 points). They are intuitively understood by using so-called quasi-particle picture [10]. In this picture we assume that pair creations of entangled quasi-particle pairs happen at each spatial point at the initial time, and they propagate in the speed of light. When one of the particle is located inside of the subsystem and the other is located outside, the quasi particle can contribute to entanglement entropy. This picture explains linear growth and saturation of the entanglement entropy.

$0 < \gamma^2 \ll \frac{1}{2}$ When $0 < \gamma^2 \ll \frac{1}{2}$, the story is quite similar to the BTZ case, $\gamma = 0$. The entanglement entropy grows up until $t_\infty = t_c \approx \theta_\infty$, when the areas of the two surfaces become equal to each other and a “first-order” phase transition takes place. At that time, the growth rate of the entanglement entropy suddenly decreases discontinuously, but does not immediately become zero. The entanglement entropy continues to grow very slowly and
gets saturated into a constant which is independent from $\gamma$. Hence the final value is same as that of BTZ black hole in particular.

Another important difference from the BTZ black hole case is that the initial entanglement entropy includes an additional nonzero term ($-\frac{L}{4G_N} \log(1 - 2\gamma^2)$). This nonzero value can be regarded as a kind of boundary entropy which is the contribution of defects in the system [23] (see also [24] for holographic realization). Note that in our system the defect is localized along the Euclidean time direction.

$$A(t, \theta = 5)$$

Figure 4: The time $t$ dependence of the extremal surface area $A$ for a subsystem $\theta = 5$, in the disconnected phase (black dotted line, numerically obtained) and in the connected phase (gray line). The phase transition from the connected phase to the disconnected phase occurs at their intersection point $t = t_c$. The disconnected phase surface area $A_{dc}$ is well approximated by the early time approximation (3.36) (orange line) in early time, and well approximated by the late time approximation (3.40) (green line) in late time. The whole time-dependence of $A_{dc}$ is qualitatively reproduced by the calculation (3.41) up to $O(\gamma^2)$ (blue line).

Determining $t_c$ analytically for generic $\gamma$ and $\theta_\infty$ is difficult, because one need to evaluate the disconnected surface area (3.29) around $t_\infty \sim \theta_\infty$. Here we evaluate it perturbatively around $\gamma = 0$ up to the second order. The detail of the calculation is given in Appendix A.
By equating (3.19) and (3.41), we obtain

\[ t_c \simeq \theta_\infty - 2.058 \gamma^2 + \mathcal{O}(\gamma^4). \]  

(4.45)

Note that the coefficient of \( \gamma^2 \) does not depend on size of the subsystem \( \theta_\infty \) or \( r_0 \).

We can also solve the equations of motion for the disconnected extremal surface numerically, to calculate the accurate time-dependence of \( A_{dc} \). The result is plotted in Figure 4, together with the \( \gamma^2 \)-perturbation, the early time and late time approximations discussed in the last section. The figure shows that the \( \gamma^2 \)-perturbation gives quite a good approximation around \( t_c \).

\( \gamma^2 \to \frac{1}{2} \) When \( \gamma^2 \) is very close to \( \frac{1}{2} \), the time-evolution of the entanglement entropy does not exhibit a phase transition for large range of \( \theta_\infty \). The minimal value \( \theta_c \) of \( \theta_\infty \) necessary for the phase transition to happen is determined by solving

\[ A_{dc}(t_\infty = 0, \theta_\infty = \theta_c, \gamma^2) = A_c(t_\infty = 0, \gamma^2). \]  

(4.46)

By using the early-time expression (3.36) for the left hand side, we can solve this equation as²

\[ \theta_c \simeq \frac{1}{2\sqrt{2r_0}} (- \log (1 - 2\gamma^2) - 2 \log 2). \]  

(4.47)

When \( \theta \leq \theta_c \), the phase corresponding to the disconnected surface is realized from the starting time \( t_\infty = 0 \). Furthermore, the initial entanglement entropy is proportional to the size of the subsystem (\( \propto \theta \)), which can be also seen by using the early-time approximation. This is one of the very peculiar point in the \( \gamma^2 \to \frac{1}{2} \) limit.

In summary, although the time-evolution of the holographic entanglement entropy in the Janus black hole is similar to that of BTZ black hole, there are some significant differences. First, the growth rate of the holographic entanglement entropy remains positive even after the phase transition, whereas the HEE for BTZ black hole is constant (i.e., the growth rate is zero) after \( t_c \). Second, the introduction of the parameter \( \gamma \) makes the connected surface less easy to realize. It in turn brings a result that the transition time \( t_c \) becomes earlier. Accordingly, the “critical value” \( \theta_c \) for the subsystem size increases as \( \gamma^2 \) grows and approaches to \( \frac{1}{2} \). Third, there are nonzero initial entanglement entropy in general. In particular, when \( \gamma^2 \) is very close to \( \frac{1}{2} \), it is proportional to the size of the subsystem even for relatively large \( \theta_\infty \).

²We dropped subleading terms for \( 1 - 2\gamma^2 \), because in (3.36) we already used \( r_0\theta \gg 1 \) approximation which in turn implies \( 1 - 2\gamma^2 \ll 1 \) approximation here.
Figure 5: The $\gamma^2$ dependence of the transition time $t_c$ of a subsystem $\theta = 3$. The green dots are obtained by calculating the disconnected phase surface area $A_{dc}$ numerically. The approx line (orange line) is obtained by substituting the disconnected phase surface area $A_{dc}$ (3.41) calculated up to $O(\gamma^2)$. The transition time $t_c$ decreases with $-\log(1/2 - \gamma^2)$ almost linearly, and the connected phase disappears with sufficiently large $\gamma^2$.

These results are hard to caught by the intuition of the quasi-particle picture, in contrast to the BTZ results.

It is also interesting to see the time-evolution of the mutual information, which is defined by

$$ I(A; B) = S(A) + S(B) - S(A \cup B), $$

which measures the entanglement between $A$ and $B$. Here we take the subsystem $A$ to be an interval $-\theta_{\infty} < \theta < \theta_{\infty}$ in the right CFT, and $B$ to be the same interval in the left CFT. The original subsystem we have been considering is the union of them. Therefore the $I(A; B)$ vanishes in the disconnected phase, $t_{\infty} \geq t_c$. For BTZ black holes this critical time is given by half of the size of the each subsystem $t_{c(BTZ)} = \theta_{\infty}$ in the high temperature limit. In [15], they considered the perturbation of BTZ black holes by a shock wave sent from one boundary. They found that the critical time becomes shorter by so called scrambling time. Here we see that the inclusion of the $\gamma$-deformation also leads to earlier critical times. The main difference between our case and theirs is that the deviation of the critical time from the BTZ value $t_c - t_{c(BTZ)}$ is proportional to the inverse temperature $\beta$ in their case, while our case is not.
5 Conclusions

In this paper we considered a three dimensional, time-dependent two-sided black hole (Janus black hole) which can be regarded as a one parameter generalization of the BTZ black hole. This black hole contains long wormhole region which is causally disconnected from the conformal boundaries. The black hole is conjectured to be dual of the CFT state (2.14). The question here is how the information of the long wormhole region is encoded in the dual CFT state.

As a first step to answer this question, we calculated the time-evolution of holographic entanglement entropy $S_A$ of the black hole, where the subsystem $A$ is the disjoint union of the region in the original CFT and the region in the thermofield double. To do this, we considered the area of two (disconnected and connected) extremal surfaces in the black hole.

For BTZ black hole, the connected surface is the dominant surface in early time, then after the critical time $t_c$ which is proportional to the size of the subsystem, the disconnected surface becomes dominant. Although holographic entanglement entropy of the Janus black hole shares many similarity to this, there are two notable differences. First of all, we showed that the critical time is shorter than that of the BTZ black hole. Intuitively, this is because the Janus black hole has a longer wormhole region, therefore the length of the connected surface becomes longer than that of the BTZ black hole. We computed this critical time up second order of the deformation parameter $\gamma$ from the BTZ. Secondly we found that for fixed size of the subsystem, when $\gamma^2$ gets sufficiently close to $1/2$ where the length of wormhole becomes infinitely long, the disconnected surface is always dominant.

In Figure 3, we numerically plotted the region where the disconnected surface can arrive, and we found a barrier outside the apparent horizon. It implies that, after the phase transition, the black hole interior region that the entanglement entropy can probe is rather limited. It is especially a strong restriction in the case above, i.e. when $\gamma^2$ is close to $1/2$ and we fix the size of the subsystem.

In [13], it was shown that if we take subsystem $A$ to be the total space of the left CFT, the extremal surface which computes holographic entanglement entropy has to localize inside of causal shadow. This property is necessary for entanglement entropy to respect the CFT causality. We can easily check this in the Janus black hole, because for this subsystem the extremal surface is located at $(\mu, \tau) = (0, 0)$ in the coordinate (2.1).

There are several outlooks for this work. It would be interesting to calculate the entanglement entropy from dual CFT side. One possibility is considering free fermion system [25],
where the explicit form of the twist operator is known [26]. Figure 3 seems to show that the disconnected surface can not penetrate the apparent horizon of the Janus black hole, and it would be interesting to prove this directly as in [27].

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A The $\gamma$-expansion of Holographic Entanglement Entropy

In this section, we compute the entanglement entropy and phase transition time in the leading order of $\gamma$-expansion.

Expansion of $y_*$

First we consider (3.28). The integrand in the left hand side is expanded as

$$\frac{\tilde{g}(y)^2}{\sqrt{\tilde{g}(y_*)^2 - \tilde{g}(y)^2}} = \frac{\text{sech}^2 y}{\sqrt{\text{sech}^2 y_* - \text{sech}^2 y}} + \frac{\text{sech}^2 y(2 - \text{sech}^2 y + \text{sech}^2 y_*)}{4\sqrt{\text{sech}^2 y_* - \text{sech}^2 y}} \gamma^2 + O(\gamma^4),$$

(A.49)
\[ \int_{y_*}^{y_* \infty} dy \frac{\tilde{g}(y)^2}{\sqrt{\tilde{g}(y_*)^2 - \tilde{g}(y)^2}} = \left[ \tanh^{-1} \left( \frac{\cosh y_* \sinh y}{\sqrt{\cosh^2 y_* - \cosh^2 y}} \right) \right]_{y_*}^{y_* \infty} y^2 + \mathcal{O}(\gamma^4) \]

By substituting this into (3.28), we obtain
\[ \sinh r_0 \theta / \cosh r_0 t = \sinh \left[ \tanh^{-1}(\text{sech } y_*) + \left( \frac{3 \cosh^2 y_* + 1}{8 \cosh^2 y_*} \tanh^{-1}(\text{sech } y_* + \frac{1}{8 \cosh y_*}) \right) \gamma^2 + \mathcal{O}(\gamma^4) \right] \]

leading to
\[ \sinh y_* = F \left[ 1 + \left( \frac{3 F^2 + 4}{8 \sqrt{1 + F^2}} \coth^{-1} \left( \sqrt{1 + F^2} \right) + \frac{1}{8} \right) \gamma^2 + \mathcal{O}(\gamma^4) \right], \quad (A.52) \]
where
\[ F(t, \theta) = \frac{\cosh r_0 t}{\sinh r_0 \theta}. \quad (A.53) \]

**Disconnected surface area**

On the other hand, form (3.29) and (3.31), \( \gamma^2 \)-expansion gives
\[ A_{\text{dc}}^{(\text{ren})} / L = 4 \log \left( \frac{2 \cosh r_0 t}{r_0 \sinh y_*} + (2 + \text{sech } y_* \tanh^{-1}(\text{sech } y_*)) \right) \gamma^2 + \mathcal{O}(\gamma^4). \quad (A.54) \]

By using (A.52) above, this results in
\[ A_{\text{dc}}^{(\text{ren})} / L = 4 \log \left( \frac{2}{r_0} \sinh r_0 \theta \right) - \left( \frac{3 F^2 + 2}{2 \sqrt{1 + F^2}} \coth^{-1} \left( \sqrt{1 + F^2} \right) - \frac{3}{2} \right) \gamma^2 + \mathcal{O}(\gamma^4). \quad (A.55) \]
Phase Transition

Phase Transition time for a fixed value of $\theta$ can be computed from

$$A_{dc}^{(ren)} = A_c^{(ren)},$$ 

(A.56)

by using (3.20) and (A.55). It is solved by $t = t_c$, where

$$t_c = t_c^{(0)} + t_c^{(1)} \gamma^2 + \mathcal{O}(\gamma^4),$$ \hspace{1cm} (A.57)

$$r_0 t_c^{(0)} = \cosh^{-1}(\sinh r_0 \theta),$$ \hspace{1cm} (A.58)

$$r_0 t_c^{(1)} = - \left( \frac{1}{2} + \frac{5}{2\sqrt{2}} \coth^{-1}(\sqrt{2}) \right) \frac{\sinh r_0 \theta}{\sqrt{\sinh^2 r_0 \theta - 1}} \approx -2.058 \times \frac{\sinh r_0 \theta}{\sqrt{\sinh^2 r_0 \theta - 1}}.$$ \hspace{1cm} (A.59)

Then, in particular, in large $\theta$ limit ($\theta \gg r_0^{-1}$), we obtain

$$t_c \approx \theta - 2.058 \gamma^2 + \mathcal{O}(\gamma^4).$$ \hspace{1cm} (A.60)

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