\[ C^{1,\alpha} \text{-Regularity of energy minimizing maps from a 2-dimentional domain into a Finsler space} \]

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Abstract
We show \( C^{1,\alpha} \)-regularity for energy minimizing maps from a 2-dimensional Riemannian manifold into a Finsler space \((\mathbb{R}^n, F)\) with a Finsler structure \(F(u, X)\).

1 Introduction
Let \( N \) be an \( n \)-dimensional \( C^\infty \)-manifold and \( TN \) its tangent bundle. We write each point in \( TN \) as \((u, X)\) with \( u \in N \) and \( X \in T_u N \). We put
\[ TN \setminus 0 := \{(u, X) \in TN ; X \neq 0\} . \]
\( TN \setminus 0 \) is called the slit tangent bundle of \( N \). A Finsler structure of \( N \) is a function \( F: TN \to [0, \infty) \) with the following properties:

(F-1) **Regularity:** \( F \in C^\infty(TN \setminus 0) \).

(F-2) **Positive homogeneity:** \( F(u, \lambda X) = \lambda F(u, X) \) for all \( \lambda \geq 0 \).

(F-3) **Convexity:** The Hessian matrix of \( F^2 \) with respect to \( X \)
\[ (f_{ij}(u, X)) = \left( \frac{1}{2} \frac{\partial^2 F^2(u, X)}{\partial X^i \partial X^j} \right) \]

is positive definite at every point \((u, X) \in TN \setminus 0\).

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We call the pair \((N, F)\) a Finsler manifold, and \((f_{ij})\) the fundamental tensor of \((N, F)\). Since \(F\) is positively homogeneous of degree 1, we can see that the coefficients of the fundamental tensor are positively homogeneous of degree 0;

\[ f_{ij}(u, \lambda X) = f_{ij}(u, X), \quad \lambda > 0. \]  

Moreover, since \(F^2\) is homogeneous of degree 2, using Euler’s theorem for homogeneous functions, we have

\[ F^2(u, X) = f_{ij}(u, X)X^iX^j. \]  

For maps between Finsler manifolds P. Centore defined the energy density by using of the integral mean on the indicatrix of each point on the source manifold. According to his definition we define the energy density \(e_C(u)\) of a map \(u\) from a Riemannian into a Finsler manifold as follows. Let \((M, g)\) be a smooth Riemannian \(m\)-manifold and \((N, F)\) a Finsler \(n\)-manifold. Let \(I_x M\) be the indicatrix of \(g\) at \(x \in M\), namely,

\[ I_x M := \{\xi \in T_x M; \|\xi\|_g \leq 1\}. \]

For a \(C^1\)-map \(u : M \to N\) and a domain \(\Omega \subset M\), we define the energy density \(e_C(u)(x)\) of \(u\) at \(x \in M\) and the energy on \(\Omega\) \(E_C(u; \Omega)\) by

\[ e_C(u)(x) := \int_{I_x M} (u^*F)^2(\xi)d\xi = \frac{1}{\int_{I_x M} d\xi} \int_{I_x M} (u^*F)^2(\xi)d\xi \]  

\[ E_C(u; \Omega) := \int_{\Omega} e_C(u)(x)d\mu. \]

Here and in the sequel, \(\bar{\int}\) denotes the integral mean, \(u^*F\) the pull-back of \(F\) by \(u\), and \(d\mu\) the measure deduced from \(g\). We call (weak) solutions of the Euler-Lagrange equation of the energy (weakly) harmonic maps.

Concerning harmonic maps from a Finsler manifold into a Riemannian manifold, see, for example, H. von der Mosel and S. Winklmann.

Let us take an orthonormal frame \(\{e_\alpha\}\) for the tangent bundle \(TM\) of \(M\), given in local coordinates by

\[ e_\alpha = \eta_\alpha^\kappa(x) \frac{\partial}{\partial x^\kappa}, \quad 1 \leq \alpha \leq m. \]

Using \(\{e_\alpha\}\), we identify each \(I_x M\) at \(x \in M\) with the unit Euclidean \(m\)-ball \(B^m\). Then, by virtue of the identity

\[ g^{\kappa\nu}(x) = \eta_\alpha^\kappa(x)\delta^{\alpha\beta}\eta_\beta^\nu(x), \]

we can write \(E_C\) as

\[ E_C(u; \Omega) = \int_{\Omega} \left( \frac{1}{|B^m|} \int_{B^m} f_{ij}(u(x), du_x(\xi)) \xi^i \xi^j d\xi \right) \eta_\alpha^\kappa \eta_\beta^\nu D_\alpha u^i D_\beta u^j \sqrt{g} dx, \]  

\[ 1. \]
where $D_{\alpha}u^i = \partial u^i / \partial x^\alpha$ and $g = \det(g_{\alpha\beta})$. (cf. [8].) Although the terms in parentheses are not defined at points $x$ where $du_x = 0$, we can define them to be arbitrary numbers without changing the values of the integrands $(....)_{\kappa\nu}^{\alpha\beta}D_\alpha u^i D_\beta u^j$, because the integrands are equal to 0, being independent on the values of $f_{ij}$ when $du_x = 0$. So, here and in the sequel, we regard $f_{ij}(u, X)$ as being defined also for $X = 0$.

As in [9], let us put

$$E_{ij}^{\alpha\beta}(x, u, p) = \left( \frac{1}{|B^m|} \int_{B^m} f_{ij}(u(x), p \xi) \xi^\alpha \xi^\beta d \xi \right)^{\eta^\alpha_{\kappa}(x)\eta^\beta_{\nu}(x) \sqrt{g(x)}}. \tag{1.6}$$

Then, we can write

$$E_C(u; \Omega) = \int_{\Omega} E_{ij}^{\alpha\beta}(x, u, Du) D_{\alpha} u^i D_{\beta} u^j dx. \tag{1.7}$$

In case that $m = \dim(M) = 2$, the Hölder continuity of a energy minimizing map is shown in [9]. For a energy minimizing map between Riemannian manifolds, or more generally for a minimizer $u$ of a quadratic functional

$$\int A_{ij}^{\alpha\beta}(x, u) D_{\alpha} u^i D_{\beta} u^j dx$$

with smooth coefficients $A_{ij}^{\alpha\beta}(x, u)$, once the Hölder continuity of $u$ has been shown, we see that the coefficients $A_{ij}^{\alpha\beta}(x, u(x))$ are Hölder continuous, and therefore we can show the $C^{1,\alpha}$-regularity of $u$ by virtue of Schauder-type estimate. Then, inductively we get higher regularity. In contrast, if the target manifold is a Finsler manifold, the Höder continuity of $u$ does not imply the continuity of the coefficients $E_{ij}^{\alpha\beta}(x, u(x), Du(x))$. So, if we want to obtain $C^{1,\alpha}$-regularity of a minimizer, we have to show it directly.

In differential geometric setting, usually one assumes $C^\infty$-regularity on the metric as (F-1). However, to get $C^{0,\alpha}$- or $C^{1,\alpha}$-regularity for energy minimizing maps, it is enough to employ the following conditions instead of (F-1)

(F-1a) There exists a concave increasing function $\omega : [0, \infty) \rightarrow [0, \infty)$ with

$$\lim_{t \rightarrow +0} \omega(t) = 0$$

such that

$$|F^2(u, X) - F^2(v, X)| \leq \omega(|u - v|^2)|X|^2 \tag{1.8}$$

holds for any $u, v \in \mathbb{R}^n$ and $X \in \mathbb{R}^n$.

(F-1b) $F(u, X)$ is twice differentiable in $X$ for every $(u, X) \in T\mathbb{R}^n \setminus 0$.

On the other hand, about convexity we need the following uniformly convexity condition which is stronger than (F-3).
There exist positive constants $\lambda < \Lambda$ for which
\[
\lambda |\xi|^2 \leq f_{ij}(u, X)\xi^i\xi^j = \frac{1}{2} \frac{\partial^2 F^2(u, X)}{\partial X^i \partial X^j} \xi^i\xi^j \leq \Lambda |\xi|^2
\] (1.9)
holds for any $u, v \in \mathbb{R}^n$ and $(X, \xi) \in (\mathbb{R}^n \setminus 0) \times \mathbb{R}^n$.

The main result of this paper is as follows.

**Theorem 1.1.** Let $(M, g)$ a 2-dimensional smooth Riemannian manifold, $\Omega \subset M$ a bounded domain with smooth boundary $\partial \Omega$ and $(\mathbb{R}^n, F)$ a Finsler space with the Finsler structure $F$ satisfying (F-1a), (F-1b), (F-2) and (F-3a). Let $u \in H^{1,2}(\Omega, \mathbb{R}^n)$ be an energy minimizing map in the class $H^{1,2}_0(\Omega, \mathbb{R}^n) := \{v \in H^{1,2}(\Omega, \mathbb{R}^n) : v - \phi \in H^{1,2}_0(\Omega, \mathbb{R}^n)\}$. Then $u \in C^{1,\alpha}(\Omega) \cap C^{0,\beta}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and any $\beta \in (0, 1)$.

## 2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we prepare the following higher integrability results of minimizers which can be deduced easily from [7, Lemma 1] as mentioned in [9].

**Lemma 2.1 ([9, Remark 5.3]).** Let $(M, g)$ be a smooth Riemannian $m$-manifold and $\Omega \subset M$ a bounded domain with smooth boundary $\partial \Omega$ and $(\mathbb{R}^n, F)$ a Finsler space with the Finsler structure $F$ satisfying (1.9). Suppose that $\phi \in H^{1,p}(\Omega, \mathbb{R}^n)$ for some $p > 2$. Let $u \in H^{1,2}(\Omega, \mathbb{R}^n)$ be an energy minimizing map in the class $H^{1,2}_\phi(\Omega, \mathbb{R}^n)$. Then, there exists a positive number $q_0 > 2$ such that for every $q \in (2, q_0)$, the estimate
\[
\int_\Omega |Du|^q dx \leq C \int_\Omega |D\phi|^q dx
\] (2.1)
holds.

Now, using several estimates which are obtained in [9], we can show the main result of this paper. In [9] the author supposed that
\[
A(x, u, p) = E_{ij}^\alpha(x, u, p)p^i_p^j
\]
is in the class $C^{1,1}(\mathcal{X}) \cap C^3(\mathcal{X}')$, where
\[
\mathcal{X} = \Omega \times \mathbb{R}^n \times \mathbb{R}^{mn} \text{ and } \mathcal{X}' = \Omega \times \mathbb{R}^n \times (\mathbb{R}^{mn} \setminus \{0\}).
\]
However, it is clearly superfluous to obtain $C^{0,\alpha}$-regularity of the minimizer. In fact, it is easy to see that every proof in [9] can be carried assuming on the regularity of $A(x, u, p)$ only that
(i) $A(x, u, p)$ is in the class $C^{1,1}(\mathcal{X})$ and twice differentiable in $p$ at every $(x, u, p) \in \mathcal{X}$.

(ii) There exists a concave increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\lim_{t\to 0} \omega(t) = 0$ such that

$$|A(x, u, p) - A(y, v, p)| \leq \omega(|x - y|^2 + |u - v|^2)|p|^2,$$

holds for all $x, y \in \Omega, u, v \in \mathbb{R}^n$ and $p \in \mathbb{R}^{mn} \setminus 0$.

Therefore, all results in [9] hold under the assumptions in Theorem 1.1 in the present paper.

If $u : \Omega \subset M \to \mathbb{R}^n$ minimizes the energy functional on $\Omega$, then $u$ minimizes it on every sub-domain of $\Omega$. On the other hand, the regularity is a local property. So, it is suffices to study the regularity problem on a domain $\Omega \subset \mathbb{R}^m$.

**Proof of Theorem 1.1.** First, we show that $u \in C^{0,\beta}(\Omega)$ for any $\beta \in (0, 1)$.

We use the following notation as in [9]. For $x \in \Omega$ and $R > 0$ we put

$$Q(x, R) := \{y \in \mathbb{R}^m ; |y^\alpha - x^\alpha| < R, \alpha = 1, \ldots, m\}. \tag{2.2}$$

For $x_0 \in \partial \Omega$ we always choose local coordinates so that for sufficiently small $R_0 > 0$

$$Q(x_0, R_0) \cap \Omega \subset \mathbb{R}^m_+ = \{x \in \mathbb{R}^m ; x^m > 0\},$$
$$Q(x_0, R_0) \cap \partial \Omega \subset \{x \in \mathbb{R}^m ; x^m = 0\},$$

and put for $0 < R < R_0$

$$Q^+(x_0, R) := Q(x_0, R) \cap \{x \in \mathbb{R}^m ; x^m > 0\}. \tag{2.3}$$

Sometimes we write also

$$\Omega(x, R) := \{y \in \Omega ; |y^\alpha - x^\alpha| < R, \alpha = 1, \ldots, m\}, \tag{2.4}$$

for general $x \in \Omega$ and $R > 0$.

From [9] (5.9)], when $x_0$ is an interior point and $Q(x_0, 2r) \subset \subset \Omega$, we have for any $\delta \in (0, 1)$

$$\int_{Q^+(x_0,p)} |Du|^2 dx \leq C \left\{ \left( \frac{\rho}{r} \right)^{2-\delta} + \tilde{\omega}(r^2 + \int_{Q(x_0,2r)} |Du|^2 dx) \right\} \int_{Q^+(x_0,2r)} |Du|^2 dx, \tag{2.5}$$

where $\tilde{\omega} = \omega(q^{-2})$ for some $q > 2$. For a boundary point $x_0$, assuming that $\phi \in H^{1,s}(s > m = 2)$, from [9] (5.10)], we have for any $\delta \in (0, 1)$

$$\int_{Q^+(x_0,p)} |Du|^2 dx \leq C \left\{ \left( \frac{\rho}{r} \right)^{2-\delta} + \tilde{\omega}(r^2 + \int_{Q^+(x_0,2r)} |Du|^2 dx) \right\} \int_{Q^+(x_0,2r)} |Du|^2 dx \tag{2.6}$$

$$+ C(\phi)r^\gamma,$$
where γ = 2(1 − 2/s) > 0. Since we are assuming that φ ∈ H1,∞, we can take γ = 2 − ε for any ε > 0.

Let us choose δ so that 2 − ε < 2 − δ. Proceeding as in [4, pp.317–318], we can deduce from (2.5) and (2.6) that
\[
\int_{Q} (x_0, \rho) |Du|^2 dx \leq M_1 (\rho^2 - \varepsilon \int_{Q} (x_0, r) |Du|^2 dx + M_2 \rho^2 - \varepsilon) \quad \text{for } x_0 \in \Omega,
\]
(2.7)
and
\[
\int_{Q^+} (x_0, \rho) |Du|^2 dx \leq M_1 (\rho^2 - \varepsilon \int_{Q^+} (x_0, r) |Du|^2 dx + M_2 \rho^2 - \varepsilon) \quad \text{for } x \in \partial \Omega,
\]
(2.8)
for sufficiently small r > 0 and ρ ∈ (0, r), where M_1 and M_2 are constants depending on g, F, Ω and φ. Here, we used also the fact that
\[
\lim_{r_0 \to 0} \left\{ r_0^2 + \int_{\Omega(x_0, 2r_0)} |Du|^2 dx \right\} = 0
\]
(2.9)
holds for any x_0 ∈ Ω.

Now, proceeding as in [4, pp.318–319], we can have that for any ε ∈ (0, 1) there exists a positive constant M such that
\[
\int_{\Omega(x_0, \rho)} |Du|^2 dx \leq \rho^{2-\varepsilon} M,
\]
(2.10)
for any x_0 ∈ Ω. So, putting 2β = 2 − ε, by Morrey’s Dirichlet growth theorem, we see that u ∈ C^{0,\beta}(\Omega).

Let us show C^{1,\alpha}-regularity of u, proceeding as in [4]. For a cube Q_0 = Q(x_0, R) ⊂⊂ Ω, we consider the following frozen functional A^0 defined by
\[
A^0(v) = \int_{Q_0} E_{ij}^{\alpha \beta} (x_0, u_R, Dv) D_{\alpha i} v^i D_{\beta j} v^j dx,
\]
(2.11)
where
\[
u_R = \int_{Q_0} u dx.
\]
Let v be a minimizer of A^0 in the class
\[
\{v \in H^{1,2}(Q_0) : v - u \in H^{1,2}_0(Q_0) \}.
\]
Since u ∈ H^{1,q} for every q ∈ (2, q_0) for some q_0 > 2 by Lemma 2.1 using Lemma 2.1 for v, we see that there exists a positive number q_1 > 2 such that for every q ∈ (2, q_1) there holds
\[
\int_{Q_0} |Dv|^q dx \leq \int_{Q_0} |Du|^q dx.
\]
(2.12)
Moreover, as in [9], by using of difference quotient method, we can see that $v \in H^{2,2}$ and that $Dv$ satisfies a system of uniformly elliptic equations weakly. So, for any $Q(x, r) \subset Q_0$, $Dv$ satisfies the Caccioppoli inequality,

$$
\int_{Q(x, r/2)} |D^2v|^2 dy \leq \frac{C}{r^2} \int_{Q(x, r)} |D - (Dv)_r|^2 dy, \quad (2.13)
$$

and $D^2v$ satisfies reverse Hölder inequalities with increasing supports due to Giaquinta-Modica (cf. [3, p.299, Theorem 3]),

$$
\left( \int_{Q(x, r/2)} |D^2v|^q dy \right)^{1/q} \leq C \left( \int_{Q(x, r)} |D^2v|^2 dx \right)^{1/2}, \quad (2.14)
$$

for every $q \in (2, q_2)$ for some $q_2 > 2$.

Since we are considering 2-dimensional case, the Sobolev-Morrey imbedding theorem (cf. [4, Theorem 3.11]) yields that $v \in C^{1,\delta}$ for $\delta = 1 - (2/q)$. Moreover, we have for $\rho \in (0, R/4)$

$$
\left\{ \rho^{-2-2\delta} \int_{Q(x_0,\rho)} |Dv - (Dv)_\rho|^2 dx \right\}^{1/2} \leq \sup_{Q(x_0, R/4)} \frac{|Dv(x) - Dv(y)|}{|x - y|^{\delta}} \leq C \|D^2v\|_{L^2(Q(x_0, R/4)).} \quad (2.15)
$$

For the last inequality, we used Morrey-type inequality.

Combining (2.15), (2.14) and (2.13), we obtain

$$
\left\{ \rho^{-2-2\delta} \int_{Q(x_0,\rho)} |Dv - (Dv)_\rho|^2 dx \right\}^{1/2} \leq CR^{q-1}\|D^2v\|_{L^2(Q(x_0, R/2))} \leq \left( R^{-2-2\delta} \int_{Q(x_0, R)} |Dv - (Dv)_R|^2 dx \right)^{1/2}. \quad (2.16)
$$
Putting $w = u - v$, we obtain

$$
\int_{Q(x_0, \rho)} |Du - (Du)_\rho|^2 dx \\
\leq \int_{Q(x_0, \rho)} |Du - (Dv)_\rho|^2 dx \\
\leq \int_{Q(x_0, \rho)} |Dv - (Dv)_\rho|^2 dx + \int_{Q(x_0, \rho)} |Dw|^2 dx \\
\leq C \left( \frac{B}{R} \right)^{2+2\delta} \int_{Q(x_0, R)} |Dv - (Dv)_R|^2 dx + C \int_{Q(x_0, \rho)} |Dw|^2 dx \\
\leq C \left( \frac{B}{R} \right)^{2+2\delta} \int_{Q(x_0, R)} |Dv - (Du)_R|^2 dx + C \int_{Q(x_0, \rho)} |Dw|^2 dx \\
\leq C \left( \frac{B}{R} \right)^{2+2\delta} \int_{Q(x_0, R)} |Du - (Du)_R|^2 dx + C \int_{Q(x_0, R)} |Dw|^2 dx. \quad (2.17)
$$

Let us estimate $\int |Dw|^2 dx$. Proceeding as in [9, pp.1967-1968], it is easy to see that

$$
\int_{Q(x_0, R)} |Dw|^2 dx \leq C \left[ \int_{Q(x_0, R)} \omega \left( (|x - x_0|^2 + |u - u_R|^2) |Du|^2 dx \\
+ \int_{Q(x_0, R)} \omega \left( (|x - x_0|^2 + |v - u_R|^2) |Dv|^2 dx \right) \right] \\
=: I + II. \quad (2.18)
$$

Using Jensen’s inequality, Hölder’s inequality and reverse Hölder inequality, we can estimate $I$ as follows.

$$
I \leq C \left( \int_{Q(x_0, R)} \omega^{q/(q-2)} dx \right)^{(q-2)/q} \left( \int_{Q(x_0, R)} |Du|^q \right)^{2/q} \\
\leq C \left( \int_{Q(x_0, R)} \omega dx \right)^{(q-2)/q} R^{m(q-2)/q} \left( \int_{Q(x_0, R)} |Du|^q dx \right)^{2/q} \\
\leq C \left( \omega \left( \int_{Q(x_0, R)} (|x - x_0|^2 + |u - u_R|^2) dx \right) \right)^{(q-2)/q} R^{m(q-2)/q} R^{2m/q} \\
\cdot \left( \int_{Q(x_0, R)} |Du|^q dx \right)^{2/q} \\
\leq C \left( \omega \left( \int_{Q(x_0, R)} (R^2 + |u - u_R|^2) dx \right) \right)^{(q-2)/q} \int_{Q(x_0, 2R)} |Du|^2 dx. \quad (2.19)
$$

Here we used the boundedness of $\omega$. By virtue of (2.12), we can estimate $II$
similarly and get

\[
II \leq C \left( \int_{Q(x_0, R)} \omega^{q/(q-2)} dx \right)^{(q-2)/q} \left( \int_{Q(x_0, R)} |Du|^q dx \right)^{2/q}
\]

\[
\leq C \left( \omega \left( \int_{Q(x_0, R)} \left( R^2 + |v - u_R|^2 \right) dx \right) \right)^{(q-2)/q} \left( \int_{Q(x_0, R)} |Du|^q dx \right)^{2/q}
\]

\[
\leq C \left( \omega \left( C \int_{Q(x_0, R)} \left( R^2 + |u - u_R|^2 + |v - u|^2 \right) dx \right) \right)^{\frac{q-2}{q}} \int_{Q(x_0, R)} |Du|^2 dx. \tag{2.20}
\]

Let us estimate the ingredients in \( \omega \). Using Sobolev’s inequality (cf. [4, p.103]), we can see that for \( 2_\ast = 2m/(m+2) \)

\[
\int_{Q(x_0, R)} |u - u_R|^2 dx \leq CR^{-m} \left( \int_{Q(x_0, R)} |Du|^2 dx \right)^{2/2_\ast}
\]

\[
\leq CR^{-m} \left( \int_{Q(x_0, R)} 1^{2/(2-2_\ast)} dx \right)^{2-2_\ast} \left( \int_{Q(x_0, R)} |Du|^2 dx \right)
\]

\[
\leq CR^{-m+2m-2_\ast} \left( \int_{Q(x_0, R)} |Du|^2 dx \right)
\]

Since we are assuming that \( m = 2 \), we have \( 2_\ast = 1 \). Thus, the above estimate together with (2.20) gives for every \( \varepsilon \in (0,1) \) the following estimate

\[
\int_{Q(x_0, R)} |u - u_R|^2 dx \leq C \int_{Q(x_0, R)} |Du|^2 dx \leq CR^{2-\varepsilon}. \tag{2.21}
\]

We can see also that

\[
\int_{Q(x_0, R)} |u - v|^2 dx
\]

\[
\leq C \int_{Q(x_0, R)} (|Du|^2 + |Dv|^2)
\]

\[
\leq C \int_{Q(x_0, R)} |Du|^2 dx \leq CR^{2-\varepsilon}. \tag{2.22}
\]

Since we can assume that \( R \leq 1 \), we see that the ingredient in \( \omega \) can be estimated by \( CR^{2-\varepsilon} \) for every \( \varepsilon \in (0,1) \).

Using the assumption that \( \omega(t) \leq Ct^\sigma \) for some \( \sigma \in (0,1) \), we obtain

\[
\omega(\ldots) \leq CR^{\sigma(2-\varepsilon)}, \tag{2.23}
\]

So, we can estimate \( \omega^{(q-2)/q} \int |Du|^2 dx \) in (2.19) and (2.20) as

\[
\omega^{(q-2)/q} \int_{Q(x_0, 2R)} |Du|^2 dx \leq CR^{(2-\varepsilon) \{ 1 + \sigma(q-2)/q \}}, \tag{2.24}
\]
where we used (2.10) again. Now, take $\varepsilon \in (0,1)$ sufficiently small so that
\[(2 - \varepsilon)\left(1 + \sigma \cdot \frac{2 - q}{q}\right) > 2,\]
and put
\[\gamma := (2 - \varepsilon)\left(1 + \sigma \cdot \frac{2 - q}{q}\right) - 2 > 0.\]  
(2.25)

Combining (2.19), (2.20), (2.24) and (2.25), we get
\[
\int_{Q(x_0, R)} |Dw|^2 dx \leq CR^{2+\gamma}.
\]  
(2.26)

Now, substituting the above inequality into (2.17), we obtain
\[
\int_{Q(x_0, \rho)} |Du - (Du)_\rho|^2 dx 
\leq C\left(\frac{\rho}{R}\right)^{2+\delta} \int_{Q(x_0, R)} |Du - (Du)_R|^2 dx + CR^{2+\gamma}.\]
(2.27)

Using well known lemma (cf. [2, Lemma 2.2], we conclude that
\[
\int_{Q(x_0, \rho)} |Du - (Du)_\rho|^2 dx \leq C\rho^{2+\alpha}
\]  
(2.28)

with $\alpha = \min\{\delta, \gamma/2\}$ for every $Q(x_0, 2\rho) \subset \Omega$, and hence $Du \in C^\alpha(\Omega)$.

Remark 2.2. The perfect dominance functions treated by S. Hildebrandt and H. von der Mosel in [3, 4] have the structure similar to that of the energy density $e_e$. So, some of their results are valid for weakly harmonic maps in 2-dimensional case. More precisely, for the case that $F(u, X)$ is continuously differentiable in $u$, once the Hölder continuity of a weakly harmonic map have shown, we can get its $C^{1,\alpha}$-regularity proceeding exactly as in the fourth section of [4]. On the other hand, in this paper, we prove $C^{1,\alpha}$-regularity using the minimality without assuming the differentiability of $F(u, X)$ with respect to $u$.

We should mention also that in [4] the minimality is not necessary to get $C^{1,\alpha}$-regularity for Hölder continuous weak solutions of the Euler-Lagrange equation of a perfect dominance function. However, in both of [4] and this paper, the minimality is necessary to get the Hölder continuity.

References

[1] P. Centore. Finsler Laplacians and minimal-energy maps. *Internat. J. Math.*, 11(1):1–13, 2000.

[2] M. Giaquinta and E. Giusti. Differentiability of minima of nondifferentiable functionals. *Invent. Math.*, 72(2):285–298, 1983.
[3] M. Giaquinta, G. Modica, and J. Souček. *Cartesian currents in the calculus of variations*. II, volume 38 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998. Variational integrals.

[4] E. Giusti. *Direct methods in the calculus of variations*. World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[5] S. Hildebrandt and H. von der Mosel. Plateau’s problem for parametric double integrals. I. Existence and regularity in the interior. *Comm. Pure Appl. Math.*, 56(7):926–955, 2003. Dedicated to the memory of Jürgen K. Moser.

[6] S. Hildebrandt and H. von der Mosel. Plateau’s problem for parametric double integrals. II. Regularity at the boundary. *J. Reine Angew. Math.*, 565:207–233, 2003.

[7] J. Jost and M. Meier. Boundary regularity for minima of certain quadratic functionals. *Math. Ann.*, 262(4):549–561, 1983.

[8] S. Nishikawa. Harmonic maps of Finsler manifolds. In *Topics in differential geometry*, pages 207–247. Ed. Acad. Române, Bucharest, 2008.

[9] A. Tachikawa. Partial regularity results up to the boundary for harmonic maps into a Finsler manifold. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1953–1970, 2009.

[10] H. von der Mosel and S. Winklmann. On weakly harmonic maps from Finsler to Riemannian manifolds. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(1):39–57, 2009.