Mass Deformed L-BLG Theory From ABJ Theory

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Abstract: We construct mass deformed $SU(N)$ L-BLG theory together with $U(M - N)_k$ Chern-Simons theory. This mass deformed L-BLG theory is a low energy world volume theory of a stack of $N$ number of M2-brane far away from $\mathbb{C}^4/Z_k$ singularity. We carry out this by defining a special scaling limit of the fields of this theory and simultaneously sending the Chern-Simons level to infinity.

Keywords: Chern-Simons Theory, BLG theory.

*Short term visitor
1. Introduction

Since the last few years there has been a huge interest in constructing three dimensional low energy world volume theories of coincident M2-branes with the aim of finding out the dual boundary gauge theory of a 11-dimensional gravity theory with geometric structure $AdS_4 \times S^7$. With this aim in view Bagger, Lambert \[1, 2, 3\] and Gustavsson \[4, 5\] (BLG) first constructed a Chern-Simons matter theory with $\mathcal{N} = 8$ supersymmetry and $SO(8)$ global symmetry based on 3-algebra. The gauge group of this theory is restricted to $SO(4)$. By complexifying the matter fields, the BLG theory can be rewritten as a Chern-Simons gauge theory with the gauge group $SU(2) \times SU(2)$ generated by ordinary Lie algebra. One of the gauge groups is associated with Chern-Simons level, $k$ and the other, with $-k$. Immediately, it was discovered that the moduli space of BLG theory describes world volume theory of only two coincident M2-branes \[6, 7, 8, 9, 10, 11, 12\]. In order to go beyond this two brane theories, it was suggested in \[10, 13, 14, 15, 16, 17, 18\] to use a 3-algebra with a metric having indefinite Lorentzian signature. The resulting Lorentzian-BLG (L-BLG) theory still preserves $\mathcal{N} = 8$ supersymmetry at the classical level, though its quantum version lacks proper interpretation.

Inspired by BLG theory and subsequent developments, Aharony, Bergman, Jafferis and Maldacena (ABJM) constructed the low energy effective theory for $N$ number of multiple M2-branes sitting at the singularity of the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold \[19\]. Instead of the 3-algebra, this construction is fully based on an ordinary Lie algebra. The gauge group of this theory is $U(N) \times U(N) \times k$ with $\mathcal{N} = 6$ explicit supersymmetry and $SU(4)_R \times U(1)$ global symmetries \[20\]. It has two parameters, $N$ and $k$ which take only integer values. The field content of the theory are $N \times N$ matrices transforming in the bi-fundamental representation of the gauge group. The dual gravity theory is the M theory living on $AdS_4 \times S^7/\mathbb{Z}_k$. Although it seems to be different from that of the original L-BLG theory, this gauge theory...
also admits a 3-algebra interpretation \[21, 22\].

In \[23\] (also in \[24\]), a suggestion was put forward to produce the L-BLG theory from ABJM theory. According to their proposal, L-BLG theory can be obtained from the ABJM theory by sending the Chern-Simons level, \(k\) to infinity and set some of the fields of the theory to zero at the same time so that they decouple. In \[25\], the suggestion was more refined and it was pointed out that the scaling limit alone is not sufficient to produce the L-BLG theory correctly from ABJM theory. One needs to associate an extra ghost multiplet also with ABJM theory which is decoupled from the theory itself but effectively couples with it on redefinition of the fields. With the same spirit in \[26\], the extended L-BLG theory with two pairs of Lorentzian generators is derived by taking a scaling limit of several quiver Chern-Simons theories obtained from different orbifoldings of the ABJM action.

Shortly after ABJM theory, as a further generalization, another model is proposed by Aharony, Bergman, Jafferis\[27\]) named as the ABJ model. This model involves modification of ABJM gauge group \(U(N)_k \times U(N)_{-k}\) to \(U(M)_k \times U(N)_{-k}\) with \(M \geq N\), in the Chern-Simons matter fields kept unchanged. In the gravity picture, the brane construction is equivalent to the low energy \((M - N)\) fractional M2-branes sitting at the \(C^4/Z_k\) singularity in one sector and \(N\) M2-branes freely moving around on the other. The geometric structure of the gravity theory remains same as that of the ABJM theory but with an additional torsion flux that takes values in \(H^4(S^7/Z_k, Z) = Z_k\). Since ABJ theory is slightly more general than ABJM theory it deserves a similar study. More precisely, one should take the same scaling limit as taken in ABJM theory and check whether this theory also reproduces L-BLG theory. With this aim in view, a study is initiated in \[28\] with a redefined scaling limit of fields of the theory. This study was semi-complete, as it concentrated only in the massless sector. However in order to go towards a non-relativistic limit of this theory with some of the supersymmetries of the relativistic theory still preserved, we do require a mass term. The non-relativistic version of the 3-dimension L-BLG theory could be useful to study the strongly coupled condensed matter system. With this motivation, in this paper we extend the study further with the inclusion of mass term in the theory. Although our aim is to produce the mass deformed L-BLG theory in this work, one can also extend this study to construct the full non-relativistic version of the theory which we leave for future.

In this work, we begin with the Lagrangian of the ABJ theory and introduce a suitable supersymmetric mass deformation as in \[23, 29, 30, 31\] and then take the appropriate scaling limit on the fields and at the same time send the Chern-Simons level, \(k\) to infinity. In this process, we get a mass deformed SU\((N)\) L-BLG theory together with \(U(M - N)_k\) Chern-Simons theory. This L-BLG theory is a low energy world volume theory of \(N\) number of M2 branes stack far away from \(C^4/Z_k\) singularity.

The rest of this paper is organized as follows. In section-2, we briefly review the main aspects of the ABJ theory. In section-3, we introduce the maximally supersymmetric mass
deformation term in to the theory. Section-4 is devoted to the scaling limits on the fields of the theory and sending the level $k$ to infinity to finally arrive at the $U(N)$ mass deformed L-BLG theory together with $U(M - N)_k$ Chern-Simons theory. In section-5, we present our conclusions.

2. A Brief Summary of ABJ Theory

In this section we briefly recapitulate few basic facts about the ABJ theory. As mentioned in the introduction, the ABJ theory is the generalized version of the ABJM theory. The gauge group of this theory is $U(M)_k \times U(N)_{-k}$ with $SU(4)$ global symmetry. The quantum consistency demands $|M - N| \geq k$ [27]. Unlike the ABJM theory, which has two parameters, ABJ theory contains one extra parameter due to the addition of a fractional M2 brane. These parameters are $M, N$ and $k$ all of which take integer values. The field content of this theory are:

- two gauge fields, $A_{\mu}^{(L)}$ and $A_{\mu}^{(R)}$ associated with the groups $U(M)$ and $U(N)$ respectively,
- four complex scalar fields, $Y^A (A = 1, 2, 3, 4)$ which are $M \times N$ matrix valued, together with their hermitian conjugates, $Y^A_\dagger$ that are $N \times M$ matrix valued,
- $M \times N$ matrix valued fermions $\psi_A$ and their hermitian conjugates $\psi_A^\dagger$ given by $N \times M$ matrix.

Fields with upper indices, as specified above, transform in the $4$ of $R$ symmetry $SU(4)$ group and those with lower indices transform in the $\bar{4}$ representations. The Lagrangian for this fields has the following form [21]

$$
\mathcal{L} = -\text{Tr}(D^\mu Y_A^\dagger D_\mu Y^A) - i\text{Tr}(\vec{\psi}^A \gamma^\mu D_\mu \psi_A) + V_{\text{pot}} + \mathcal{L}_{\text{CS}}
$$

$$
-2\pi \frac{k}{k} \text{Tr}(Y_A^\dagger Y^B \psi_B - Y^A Y_A^\dagger \psi_B) + 2\pi \frac{k}{k} \text{Tr}(Y_A^\dagger Y_B \psi_B - Y^A Y_A^\dagger \psi_B) + 2\pi \frac{k}{k} \text{Tr}(Y_A^\dagger Y_B Y_C \psi_D - \psi_A^\dagger Y_B Y_C \psi_A).
$$

(2.1)

Where $\mathcal{L}_{\text{CS}}$ given in (2.1) is a Chern-Simons term and $V_{\text{pot}}$ is a sextic scalar potential and they have the following forms

$$
\mathcal{L}_{\text{CS}} = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr}[A_{\mu}^{(L)} \partial_\nu A_{\lambda}^{(L)} + \frac{2i}{3} A_{\mu}^{(L)} A_{\nu}^{(L)} A_{\lambda}^{(L)}] - \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr}[A_{\mu}^{(R)} \partial_\nu A_{\lambda}^{(R)} + \frac{2i}{3} A_{\mu}^{(R)} A_{\nu}^{(R)} A_{\lambda}^{(R)}],
$$

$$
V_{\text{pot}} = \frac{4\pi^2}{3k^2} \text{Tr}[Y_A^\dagger Y_B^\dagger Y_C^\dagger Y_C^\dagger + Y_A^\dagger Y^B Y^C Y_C^\dagger + 6Y_A^\dagger Y_B^\dagger Y_A^\dagger Y_B Y^C].
$$

Further, the covariant derivatives for the scalars are defined as

$$
D_\mu Y^A = \partial_\mu Y^A + i A_{\mu}^{(L)} Y^A - i Y^A A_{\mu}^{(L)}, \quad D_\mu Y_A^\dagger = \partial_\mu Y_A^\dagger - i Y_A^\dagger A_{\mu}^{(L)} + i A_{\mu}^{(R)} Y_A^\dagger,
$$

(2.3)
In what follows, we use the notation as in [32]. The complex scalar matrices of $U_3$-algebra:

$$D_\mu \psi_A = \partial_\mu \psi_A + i A^{(L)}_\mu \psi_A - i \psi_A A^{(R)}_\mu, \quad D_\mu \psi^{\dagger A} = \partial_\mu \psi^{\dagger A} - i \psi^{\dagger A} A^{(L)}_\mu + i A^{(R)}_\mu \psi^{\dagger A}.$$  \hfill (2.4)

The three dimensional gamma matrices are

$$\gamma^0 = i\sigma^2, \quad \gamma^1 = \sigma^1, \quad \gamma^2 = \sigma^3,$$

where the $\sigma$’s are the Pauli matrices.

In [21], the general form for the action for a three dimensional scale-invariant theory with $N = 6$ supersymmetry, $SU(4)$ $R$ symmetry and $U(1)$ global symmetry was found by starting with a 3-algebra in which the triple product is not antisymmetric. The field content is the same as described above with the matter fields now taking values in the 3-algebra:

$$Y^A = T^\alpha Y^A_\alpha \psi_A = T^\alpha \psi_A A_\mu = A_{\mu\alpha\beta} T^\alpha \otimes T^\beta.$$  \hfill (2.5)

In what follows, we use the notation as in [32]. The complex scalar $Y^A$ have explicit form

$$(Y^A)^\alpha_\alpha \in (M,N), \quad (Y^A)^\dagger_\alpha \in (N,M),$$  \hfill (2.6)

where $\alpha = 1, \cdots, M, \hat{\alpha} = 1, \cdots, N$. The gauge field $(A^{(L)}_\mu)^\alpha_\beta$ and $(A^{(R)}_\mu)^\dagger_\beta$ are hermitian matrices of $U(M)$ and $U(N)$ respectively. Then clearly

$$(D_\mu Y^A)^\alpha_\hat{\alpha} = \partial_\mu (Y^A)^\alpha_\hat{\alpha} + i (A^{(L)}_\mu)^\alpha_\hat{\beta} (Y^A)^\dagger_\beta \hat{\alpha} - i (Y^A)^\alpha_\beta (A^{(R)}_\mu)^\dagger_\hat{\beta} \hat{\alpha},$$

$$(D_\mu Y^A)^\dagger_\hat{\alpha} = \partial_\mu (Y^A)^\dagger_\hat{\alpha} - i (Y^A)^\dagger_\beta (A^{(L)}_\mu)^\alpha_\hat{\beta} + i (A^{(R)}_\mu)^\dagger_\hat{\beta} (Y^A)^\dagger_\alpha \hat{\alpha}. $$  \hfill (2.7)

So far we have not introduced the mass term in the theory. However, as mentioned in the introduction, one of the way to get to the non-relativistic limit of ABJ theory is the introduction of suitable mass term which preserves some of the supersymmetries of the original relativistic version. Though we are not going to study the non-relativistic theory in this paper, but we will introduce it in the next section to produce the mass deformed L-BLG theory from the mass deformed ABJ theory. This construction can be useful to construct the non-relativistic version of the theory which we leave for future work.

3. Mass Deformed ABJ Theory

In order to introduce the mass term, we focus on maximally supersymmetric mass deformation. It turns out that the suitable form of this mass term is [29, 30, 31, 33]

$$V_{mass} = \text{Tr}(M^C_A M^B_B Y^A_B Y^A_A) + i \text{Tr}(M^C_A \psi_A B) - \frac{4\pi}{k} \text{Tr}(M^C_B Y^A_A Y^B_A Y^C_A)$$

$$= m^2 \text{Tr}(Y^A_A Y^A_A) + im \text{Tr}(\overline{\psi}_A \psi_a) - im \text{Tr}(\overline{\psi}_A \psi_a')$$

$$- \frac{4\pi}{k} m \text{Tr}(Y^A_A Y^B_B Y^C_C) - Y^A_A Y^B_B Y^C_C).$$  \hfill (3.1)
Here $m$ is the mass parameter and the diagonal form of mass matrix $M^B_B = m \text{ diag}(1, 1, -1, -1)$. The matrix $M^B_A$ satisfy the following constraints

\begin{align*}
M^\dagger = M, \quad \text{Tr} M &= 0 \quad \text{and} \quad M^2 = m^2 \mathbf{1} \\
M^A_B Y^B = m Y^a - m Y^{a'}, \quad M^{A\dagger}_B Y^B = m Y_{a'} - m Y^a \\
M^A_B \psi_A &= m \psi^b - m \psi^{b'}, \quad M^{A\dagger}_B \psi^B = m \bar{\psi} - m \bar{\psi}'
\end{align*}

(3.2)

The introduction of mass term breaks explicitly the original $SU(4) \times U(1)$ R-symmetry and down to $SU(2)_L \times SU(2)_R \times U(1)$ \cite{29, 31, 31}. We set $A = (a, a')$, where $a$ and $a'$ are two $SU(2)$ indices take values 1, 2 and 3, 4 respectively. We used the following convention:

\begin{align*}
X_{[a}X_{b]} &= X_aX_b - X_bX_a.
\end{align*}

In other words we can say that the mass term of the ABJ theory can be introduced in the same way as in the ordinary ABJM theory. On the other hand, our realization is that the gauge group is $U(M) \times U(N)$ makes impact on the structure of vacuum manifold.

4. Scaling Limit of Mass Deformed ABJ Theory

In this section we construct the mass deformed L-BLG theory from the mass deformed ABJ theory following the suggestions given in \cite{23, 25}. Basically we take scaling limit of the fields of the ABJM theory. Therefore as the first step we presume that $A^{(L)}_{\mu}$ takes the following form

\begin{align*}
A^{(L)}_{\mu} &= \begin{pmatrix} A^{(L)}_{1\mu} & A^{(L)}_{12\mu} \\ A^{(L)}_{21\mu} & A^{(L)}_{22\mu} \end{pmatrix}.
\end{align*}

(4.1)

Here $A^{(L)}_{1\mu}$ is $(M - N) \times (M - N)$ matrix. $A^{(L)}_{12\mu}$ and $A^{(L)}_{21\mu}$ are $(M - N) \times N$ and $N \times (M - N)$ matrices respectively. $A^{(L)}_{22\mu}$ is $N \times N$ square matrix. We further introduce the field, $B_{\mu}$ defined as

\begin{align*}
A^{(L)}_{22\mu} = A_{\mu} - \frac{1}{2} B_{\mu}, \quad A^{(R)}_{\mu} = A_{\mu} + \frac{1}{2} B_{\mu}.
\end{align*}

(4.2)

The significance of this notation is that the $B_{\mu}$ is an auxiliary field. To see this note that Chern-Simons term of \cite{22} implies

\begin{align*}
\frac{k}{4\pi} \int d^3 x \varepsilon^{\mu \nu \lambda} [\text{Tr}(A^{(L)}_{\mu} \partial_\nu A^{(L)}_{\lambda}) - \text{Tr}(A^{(R)}_{\mu} \partial_\nu A^{(R)}_{\lambda})] \\
= \frac{k}{4\pi} \int d^3 x \varepsilon^{\mu \nu \lambda} [\text{Tr}(A^{(L)}_{1\mu} \partial_\nu A^{(L)}_{11\lambda}) + 2\text{Tr}(A^{(L)}_{12\mu} \partial_\nu A^{(L)}_{21\lambda}) - \text{Tr}(B_{\mu} (\partial_\nu A_\lambda - \partial_\lambda A_\nu))] \\
\end{align*}

(4.3)

and

\begin{align*}
\frac{k}{4\pi} \frac{2i}{3} \int d^3 x \varepsilon^{\mu \nu \lambda} [\text{Tr}(A^{(L)}_{\mu} A^{(L)}_{\nu} A^{(L)}_{\lambda} - A^{(R)}_{\mu} A^{(R)}_{\nu} A^{(R)}_{\lambda})] \\
= \frac{k}{4\pi} \frac{2i}{3} \int d^3 x \varepsilon^{\mu \nu \lambda} \text{Tr}[A^{(L)}_{1\mu} A^{(L)}_{1\nu} A^{(L)}_{11\lambda} + 3A^{(L)}_{1\mu} A^{(L)}_{2\nu} A^{(L)}_{21\lambda} \\
+ 3A^{(L)}_{2\mu} A^{(L)}_{2\nu} A^{(L)}_{12\lambda} + A^{(L)}_{2\mu} A^{(L)}_{2\nu} A^{(L)}_{22\lambda} - A^{(R)}_{\mu} A^{(R)}_{\nu} A^{(R)}_{\lambda}]
\end{align*}

(4.4)
and together we obtain

\[
\mathcal{L}_{CS} = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr}[A^{(L)}_{11\mu} \partial_\nu A^{(L)}_{11\lambda} + \frac{2i}{3} A^{(L)}_{11\mu} A^{(L)}_{11\nu} A^{(L)}_{11\lambda} \\
+ 2A^{(L)}_{12\mu} \partial_\nu A^{(L)}_{21\lambda} + 2i(A^{(L)}_{11\mu} A^{(L)}_{12\nu} A^{(L)}_{21\lambda} + A^{(L)}_{22\mu} A^{(L)}_{21\nu} A^{(L)}_{12\lambda}) \\
- B_\mu(\partial_\nu A_\lambda - \partial_\lambda A_\nu + i[A_\nu, A_\lambda]) - \frac{i}{6} B_\mu B_\nu B_\lambda].
\]

(4.5)

Since in the above form there is no kinetic term for \(B_\mu\) field, it confirms the claim that \(B_\mu\) is an auxiliary field. Note that the \(N\times N\) matrix gauge field \(A_{22\mu}\) is not decoupled from the massive \(N \times (M - N)\) matrix gauge field \(A_{21\mu}\) which is much needed to construct the mass deformed L-BLG theory on \(N\) number of M2-branes. To decouple this we consider the scaling limit as in undeformed ABJ theory [23]. The form of the scaling limit is

\[
A^{(L)}_{11\mu} = A_{11\mu}, \quad A^{(L)}_{12\mu} = \epsilon^2 \tilde{A}_{12\mu}, \quad A^{(L)}_{21\mu} = \epsilon^2 \tilde{A}_{21\mu}, \\
A^{(L)}_{22\mu} = A_\mu - \frac{1}{2}\epsilon B_\mu, \quad A^{(R)}_\mu = A_\mu + \frac{1}{2}\epsilon B_\mu,
\]

(4.6)

where \(\epsilon\) is the small parameter which controls the scaling limit and finally we take \(\epsilon \rightarrow 0\). Note that \(A_\mu\) and \(B_\mu\) belong to the algebra of \(U(N)\).

We then plug in redefined gauge fields of (4.6) into the Chern-Simons Lagrangian of (4.5) and we get

\[
\mathcal{L}_{CS} = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr}[A^{(L)}_{11\mu} \partial_\nu A^{(L)}_{11\lambda} + \frac{2i}{3} A^{(L)}_{11\mu} A^{(L)}_{11\nu} A^{(L)}_{11\lambda} \\
+ 2\epsilon^4 \tilde{A}_{12\mu} \partial_\nu \tilde{A}_{21\lambda} + 2i\epsilon^4 A^{(L)}_{11\mu} \tilde{A}_{12\nu} A^{(L)}_{21\lambda} + 2i\epsilon^4 A^{(L)}_{22\mu} \tilde{A}_{21\nu} A^{(L)}_{12\lambda}) \\
+ \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \left[ - \epsilon B_\mu(\partial_\nu A_\lambda - \partial_\lambda A_\nu + i[A_\nu, A_\lambda]) + \mathcal{O}(\epsilon^3) \right] \\
\equiv \mathcal{L}^{(1)} + \mathcal{L}^{(2)},
\]

(4.7)

where

\[
\mathcal{L}^{(1)} = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr}[A^{(L)}_{11\mu} \partial_\nu A^{(L)}_{11\rho} + \frac{2i}{3} A^{(L)}_{11\mu} A^{(L)}_{11\nu} A^{(L)}_{11\lambda} \\
+ 2\epsilon^4 A^{(L)}_{11\mu} A^{(L)}_{11\nu} A^{(L)}_{11\lambda}) \\
+ 2\epsilon^4 A^{(L)}_{11\mu} \partial_\nu A^{(L)}_{11\lambda} + 2i\epsilon^4 A^{(L)}_{11\mu} A^{(L)}_{12\nu} A^{(L)}_{21\lambda} + 2i\epsilon^4 A^{(L)}_{22\mu} A^{(L)}_{21\nu} A^{(L)}_{12\lambda})
\]

(4.8)

and

\[
\mathcal{L}^{(2)} = -\frac{k\epsilon}{4\pi} \epsilon^{\mu\nu\lambda} \left[ \text{Tr} B_\mu(\partial_\nu A_\rho - \partial_\rho A_\mu + i[A_\nu, A_\lambda]) - \mathcal{O}(\epsilon^3) \right]
\]

(4.9)

Notice that in order to decouple the massive states we should keep \(k\) unscaled in the first part of the Lagrangian, \(\mathcal{L}^{(1)}\). However, in the second part we should scale \(k = \frac{1}{\epsilon} \tilde{k}\) and keep \(\tilde{k}\) finite in the limit \(\epsilon \rightarrow 0\) and sending \(k\) at infinity. Finally in the limit \(\epsilon \rightarrow 0\) we end up with the following form

\[
\mathcal{L}^{(1)} = \frac{k}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr}(A^{(L)}_{11\mu} \partial_\nu A^{(L)}_{11\lambda} + \frac{2i}{3} A^{(L)}_{11\mu} A^{(L)}_{11\nu} A^{(L)}_{11\lambda}),
\]

\[
\mathcal{L}^{(2)} = -\frac{\tilde{k}}{4\pi} \epsilon^{\mu\nu\lambda} \text{Tr} B_\mu F_{\nu\lambda}, \quad F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu + i[A_\nu, A_\lambda].
\]

(4.10)
We now try to find out the physical interpretation of the above results. The gauge fields of the Lagrangian density, \( \mathcal{L}^{(1)} \) are \((M - N) \times (M - N)\) matrices. Therefore, this Lagrangian should describe \( U(M - N)_k \) Chern–Simons theory living on the world-volume of fractional M2-branes that are localized at the origin of \( \mathbb{C}^4/\mathbf{Z}_k \). On the other hand the fields of the Lagrangian density, \( \mathcal{L}^{(2)} \) are \( N \times N\) matrices. Thus we have \( U(N) \) Chern–Simons gauge theory for the \( N \) number of M2-brane stack far away from the \( \mathbb{C}^4/\mathbf{Z}_k \). For further analysis of this theory we split \( B_\mu \) and \( A_\mu \) gauge fields into \( U(1) \) and \( SU(N) \) parts as

\[
A_\mu = A_\mu^0 I_{N \times N} + \tilde{A}_\mu , \quad \text{Tr} \tilde{A}_\mu = 0 , \\
B_\mu = \epsilon^2 B_\mu^0 I_{N \times N} + \tilde{B}_\mu , \quad \text{Tr} \tilde{B}_\mu = 0 ,
\]

and then rescale the \( U(1) \) part of \( B \) field with \( \epsilon^2 \). \( \mathcal{L}^{(2)} \) then looks like

\[
\mathcal{L}^{(2)} = -\frac{\epsilon^2 \tilde{k} N}{4\pi} \epsilon^{\mu \nu \lambda} B_\mu^0 F_\nu^0 - \frac{\tilde{k}}{4\pi} \epsilon^{\mu \nu \lambda} \text{Tr} \tilde{B}_\mu F_{\nu \lambda} = -\frac{\tilde{k}}{4\pi} \epsilon^{\mu \nu \lambda} \text{Tr} \tilde{B}_\mu F_{\nu \lambda} .
\]

Which is precisely the gauge part of L-BLG theory for the identification of \( \tilde{k} = \pi \).

Inspired by the above result, we now consider the scaling limit of matter fields \( Y^A, \psi_A \) and scale them in the following way

\[
Y^A = \left( \frac{\epsilon^2 Z^A}{\epsilon Y^A_+ I_{N \times N} + \tilde{Y}^A} \right) , \quad \text{Tr} \tilde{Y}^A = 0 , \\
\psi_A = \left( \frac{\epsilon^2 \chi_A}{\epsilon \psi_A_+ I_{N \times N} + \theta_A} \right) , \quad \text{Tr} \theta_A = 0 ,
\]

where \( Z^A, \chi_A \) are \( M - N \times N\) matrices and \( \tilde{Y}^A, \theta_A \) are \( N \times N\) matrices. For analysing the kinetic term of the scalar field we first compute \( D_\mu Y^A \) using the definition of the covariant derivative of \((2,3)\).

\[
D_\mu Y^A = \frac{e^2 \partial_\mu Z^A}{\epsilon} + \frac{i e^2 A_1^{(L)}_\mu Z^A}{\epsilon} - Z e^2 A_\mu^{(R)} + i e^2 A_1^{(L)} \left( \frac{1}{\epsilon} Y^A_+ I_{N \times N} + \tilde{Y}^A \right)
\]

\[
= \left( \frac{1}{\epsilon} \partial_\mu Y^A_+ I_{N \times N} + \partial_\mu \tilde{Y}^A + i \left[ A_\mu, \tilde{Y}^A \right] - i Y^A_+ B_\mu - \frac{i}{2} \epsilon \left\{ B_\mu, \tilde{Y}^A \right\} \right)
\]

\[
\equiv \left( \frac{O(\epsilon)}{(M - N) \times N} \right)
\]

\[
\equiv \left( \frac{1}{\epsilon} \partial_\mu Y^A_+ I_{N \times N} + \tilde{D}_\mu \tilde{Y}^A - \frac{i}{2} \epsilon \left\{ \tilde{B}_\mu, \tilde{Y}^A \right\} \right) .
\]

Here we have defined

\[
\tilde{D}_\mu \tilde{Y}^A = \partial_\mu \tilde{Y}^A + i [A_\mu, \tilde{Y}^A] - i Y^A_+ \tilde{B}_\mu .
\]

In the similar way we find that

\[
D_\mu Y^A_+ = \left( \frac{O(\epsilon)}{N \times (M - N)} \right) \quad \frac{1}{\epsilon} \partial_\mu Y^A_+ I_{N \times N} + (\tilde{D}_\mu \tilde{Y}^A)_+^\dagger + \frac{i}{2} \left\{ \tilde{B}_\mu, \tilde{Y}^A_+ \right\} ,
\]

\[
D_\mu Y^A_+ = \left( \frac{O(\epsilon)}{N \times (M - N)} \right) \quad \frac{1}{\epsilon} \partial_\mu Y^A_+ I_{N \times N} + (\tilde{D}_\mu \tilde{Y}^A)_+^\dagger + \frac{i}{2} \left\{ \tilde{B}_\mu, \tilde{Y}^A_+ \right\} .
\]
where
\[(\bar{D}_\mu Y_A)^\dagger = \partial_\mu \bar{Y}_A^\dagger - i[Y_A^\dagger, A_\mu] + iY_A^\dagger \bar{B}_\mu. \quad (4.17)\]

Using equation (4.14) and (4.13), we rewrite the kinetic term for \(Y_A\) in the following form
\[
\text{Tr}(D_\mu Y_A^\dagger D^\mu Y_A) = \frac{N}{\epsilon^2} \partial_\mu Y_A^\dagger \partial^\mu Y_A + \text{Tr}(\bar{D}_\mu \bar{Y}_A^\dagger \bar{D}^\mu \bar{Y}_A)
- i\partial_\mu Y_A^\dagger \text{Tr}(\bar{B}^\mu \bar{Y}_A) + i\partial_\mu Y_A \text{Tr}(\bar{B}^\mu \bar{Y}_A^\dagger).
\quad (4.18)\]

Note that the first term diverges in the limit \(\epsilon \to 0\). Therefore it seems that the scaling limit is not complete. For this, following the suggestion given in [25], we add an extra ghost term for the bosonic fields in the ABJ Lagrangian. This ghost term is a \(U(1)\) multiplet containing four complex bosonic fields \(U^A\). The form of this ghost term is
\[
\text{Tr}(\partial_\mu U_A^\dagger \partial^\mu U^A).
\quad (4.19)\]

Note that in (4.19) we have considered a “wrong” sign compared to the kinetic term since \(U^A\) is a ghost field. Following the argument as in [25] for the ABJM theory, it is natural to expect that the original ABJ action with similar “ghost” term reduces to L-BLG action. Addition of the ghost, results in an indefinite kinetic-term signature arising of a manifestly definite ABJ action through regular scaling limit. Although the extra ghost term is decoupled at the level of ABJ theory, it is effectively coupled through the following redefinition of the field:
\[
U^A = -\frac{1}{\epsilon} Y_A^\dagger I_{N \times N} + \frac{1}{N} Y_A^\dagger I_{N \times N} \quad (4.20)
\]
in the process when we implement the scaling limit. Note also that we have introduced a new scalar field, \(Y_A^\dagger\) through the redefinition of the field. It will turn out that this new field plays a pivotal role in L-BLG theory. Finally, using (4.20), the kinetic term (4.18) together with the ghost term (4.19) reduces to a finite value in the limit of \(\epsilon \to 0\) and we get:
\[
\text{Tr}(\partial_\mu U_A^\dagger \partial^\mu U^A) - \frac{N}{\epsilon^2} \partial_\mu Y_A^\dagger \partial^\mu Y_A - \text{Tr}(\bar{D}_\mu \bar{Y}_A^\dagger \bar{D}^\mu \bar{Y}_A)
+ i\partial_\mu Y_A^\dagger \text{Tr}(\bar{B}^\mu \bar{Y}_A) - i\partial_\mu Y_A \text{Tr}(\bar{B}^\mu \bar{Y}_A^\dagger)
= -\partial_\mu Y_A^\dagger \partial^\mu Y_A - \partial_\mu Y_A \partial^\mu Y_A^\dagger - \text{Tr}(\bar{D}_\mu \bar{Y}_A^\dagger \bar{D}^\mu \bar{Y}_A)
+ i\partial_\mu Y_A^\dagger \text{Tr}(\bar{B}^\mu \bar{Y}_A) - i\partial_\mu Y_A \text{Tr}(\bar{B}^\mu \bar{Y}_A^\dagger).
\quad (4.21)\]

In order to derive the kinetic term for the L-BLG theory which only contains real scalar field, we first split the complex scalar field of the ABJ theory in to two real scalar fields in the following way
\[
Y_A^\dagger = X_+^{2A-1} + iX_-^{2A}, \quad Y_A^\dagger = X_+^{2A-1} - iX_-^{2A},
\]
\[
\bar{Y}_A^\dagger = -\bar{X}_+^{2A} + i\bar{X}_-^{2A-1}, \quad \bar{Y}_A^\dagger = -\bar{X}_+^{2A} - i\bar{X}_-^{2A-1}.
\quad (4.22)\]

Note that the reality of the scalar fields, \(X_A^\dagger\) implies \(X_A^{\dagger*} = X_A\), \(\bar{X}_A^{\dagger*} = \bar{X}_A\). In the ABJ theory there are four complex fields denoted by gauge indices \(A\) running from 1 to 4. In
our case, we have eight real scalar fields and we denote them by gauge index $I$ which runs from 1 to 8. Finally using these relations we are able to reproduce the exact form of the kinetic term for the scalar filed of the L-BLG theory:

$$-2\partial_\mu X^I_+ \partial^\mu X^I_+ - 2\partial_\mu X^I_- \text{Tr}(\tilde{B}^\mu \tilde{X}^I) - \text{Tr}[D_\mu \tilde{X}^I - \tilde{B}_\mu X^I_+]^2,$$  \hspace{1cm} (4.23)

where we have defined

$$D_\mu \tilde{X}^I = \partial_\mu \tilde{X}^I + i[A_\mu, \tilde{X}^I].$$  \hspace{1cm} (4.24)

We now consider the scaling limit of the kinetic term for fermions. We follow the same procedure as described in the scalar field. We insert (4.6), (4.13) and (4.11) into the kinetic term for fermion and take the limit $\epsilon \to 0$. This finally yields

$$-i\text{Tr}(\bar{\psi}^A \gamma^\mu D_\mu \psi_A) = -\frac{iN}{\epsilon^2} \bar{\psi}^+_A \gamma^\mu \partial_\mu \psi^+_A + i\text{Tr}(\bar{\theta}^A \gamma^\mu \tilde{D}_\mu \theta_A) - \bar{\psi}^+_A \gamma^\mu \text{Tr}(\tilde{B}_\mu \theta_A).$$  \hspace{1cm} (4.25)

As in the case of scalar kinetic term, the first term again diverges in the limit $\epsilon \to 0$. We resolve this issue in a similar spirit as in the scalar case, by adding an extra ghost contribution in the ABJ Lagrangian of the form

$$i\text{Tr}(\bar{V}^A \gamma^\mu \tilde{D}_\mu V_A)).$$  \hspace{1cm} (4.26)

Here $V^A$ is fermionic field and we redefine this as

$$V_A = -\frac{1}{\epsilon} \psi^+_A I_{N \times N} + \frac{\epsilon}{N} \psi^- A I_{N \times N}.$$  \hspace{1cm} (4.27)

Just like in the scalar case, we introduce a new fermion field $\psi^- A$ through the redefinition of the fermionic ghost field. Finally, we find that the kinetic term for fermions with (4.26) taken into account which is now well defined even in the limit $\epsilon \to 0$ and it takes the form

$$-i\text{Tr}(\bar{\psi}^+_A \gamma^\mu \tilde{D}_\mu \psi_A) - \bar{\psi}^+_A \gamma^\mu \text{Tr}(\tilde{B}_\mu \psi_A) - i\bar{\psi}^+_A \gamma^\mu \partial_\mu \psi^+_A - i\bar{\psi}^-_A \gamma^\mu \partial_\mu \psi^- A.$$  \hspace{1cm} (4.28)

In order to get to the form of the L-BLG theory we now write down the complex fermion field as a combination of two real fermion:

$$\bar{\psi}^\pm = \Psi^\pm_{2A-1} + i\Psi^\pm_{2A}, \quad \psi^\pm A = \Psi^\pm_{2A-1} + i\Psi^\pm_{2A},$$

$$\bar{\theta}^A = -\Psi^A_{2A} - i\Psi^A_{2A-1}, \quad \theta_A = -\Psi^A_{2A} + i\Psi^A_{2A-1}.$$  \hspace{1cm} (4.29)

Using the above relations, the expression of equation (4.28) can now be rewritten as

$$-i\text{Tr}(\bar{\Psi}^I \gamma^\mu D_\mu \Psi_I) - 2i\bar{\Psi}^+_I \gamma^\mu \text{Tr}(\tilde{B}_\mu \Psi_I) - 2i\bar{\Psi}^-_I \gamma^\mu \partial_\mu \Psi^+_I.$$  \hspace{1cm} (4.30)

This final expression is exactly the $SO(8)$ invariant fermionic kinetic term of L-BLG model.

Having the analysis on kinetic terms, we now consider the scaling limit in the potential terms in (2.1) and (2.2). In order to take the scaling limit we follow the same approach as in [24]. The extra job in our case, due to our convention, is that we have to explicitly show
that the modes $Z^A$ decouple in the scaling limit. For an example, consider the following term of the bosonic potential (2.2),

$$\frac{1}{k^2} \text{Tr}(Y^B Y^\dagger_B Y^C Y^\dagger_C Y^A Y^\dagger_A).$$

(4.31)

By then using the scaling limit of scalar fields of equation (4.13), we compute

$$\frac{1}{k^2} \text{Tr}(Y^B Y^\dagger_B Y^C Y^\dagger_C Y^A Y^\dagger_A)$$

$$= e^2 \frac{1}{k^2} \text{Tr} \left( \frac{e^A Z^B Z^\dagger_B}{e^A Y^B Y^\dagger_B + e^A \tilde{Y}^B \tilde{Y}^\dagger_B} \frac{e^A Z^B Z^\dagger_B}{e^A Y^B Y^\dagger_B + e^A \tilde{Y}^B \tilde{Y}^\dagger_B} + \frac{1}{e}(Y^B Y^\dagger_B + \tilde{Y}^B \tilde{Y}^\dagger_B) \right)^3$$

$$= e^2 \frac{1}{k^2} \text{Tr} \left( \frac{1}{e} Y^A Y^\dagger_A I_{N \times N} + \frac{1}{e}(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A) + \hat{Y}^A Y^\dagger_A)^3 + O(\epsilon^4). \right)$$

(4.32)

Note that the modes $Z^A$ really decouple in the limit $\epsilon \rightarrow 0$. Further, the final expression is exactly in the same form as in the potential of $U(N) \times U(N)$ ABJM theory and diverges as usual in the limit $\epsilon \rightarrow 0$. Again we can resolve this issue following the analysis of [23]. We write the potential as a sum of $V^{(n)}_B, V_B = \sum_{n=0}^6 V^{(n)}_B$. Where $V^{(n)}_B$ contains $n$ Y_ fields and $(6 - n)$ $\tilde{Y}$ fields and also $V^{(n)}_B$ scales as $\epsilon^2$ in the limit $\epsilon \rightarrow 0$. Then it is obvious that potential terms with $n < 2$ vanish in the limit $\epsilon \rightarrow 0$ and to avoid the divergences the coefficients of the terms, $V^{(n)}_B$ should vanish for $n \geq 3$. Therefore, finally the non-zero contribution comes only from the $V^{(2)}_B$ part of the potential and these contributing terms combine as

$$\frac{4\pi^2}{3k^2} \text{Tr}[Y^A Y^\dagger_A \hat{Y}^B Y^\dagger_B \hat{Y}^C Y^\dagger_C + Y^B Y^\dagger_B \hat{Y}^C Y^\dagger_C \hat{Y}^A Y^\dagger_A + Y^C Y^\dagger_C \hat{Y}^A Y^\dagger_A \hat{Y}^B Y^\dagger_B$$

$$+ (Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)(Y^B Y^\dagger_B + \hat{Y}^B Y^\dagger_B)(Y^C Y^\dagger_C + \hat{Y}^C Y^\dagger_C)(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)Y^B Y^\dagger_B$$

$$+ (Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)(Y^B Y^\dagger_B + \hat{Y}^B Y^\dagger_B)(Y^C Y^\dagger_C + \hat{Y}^C Y^\dagger_C)(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)Y^B Y^\dagger_B$$

$$+ (Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)(Y^B Y^\dagger_B + \hat{Y}^B Y^\dagger_B)(Y^C Y^\dagger_C + \hat{Y}^C Y^\dagger_C)(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)Y^B Y^\dagger_B$$

$$+ (Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)(Y^B Y^\dagger_B + \hat{Y}^B Y^\dagger_B)(Y^C Y^\dagger_C + \hat{Y}^C Y^\dagger_C)(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)Y^B Y^\dagger_B$$

$$+ (Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)(Y^B Y^\dagger_B + \hat{Y}^B Y^\dagger_B)(Y^C Y^\dagger_C + \hat{Y}^C Y^\dagger_C)(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)Y^B Y^\dagger_B$$

$$+ 4(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)(Y^B Y^\dagger_B + \hat{Y}^B Y^\dagger_B)(Y^C Y^\dagger_C + \hat{Y}^C Y^\dagger_C)(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)Y^B Y^\dagger_B$$

$$+ (Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)(Y^B Y^\dagger_B + \hat{Y}^B Y^\dagger_B)(Y^C Y^\dagger_C + \hat{Y}^C Y^\dagger_C)(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)Y^B Y^\dagger_B$$

$$-6(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)(Y^B Y^\dagger_B + \hat{Y}^B Y^\dagger_B)(Y^C Y^\dagger_C + \hat{Y}^C Y^\dagger_C)(Y^A Y^\dagger_A + \hat{Y}^A Y^\dagger_A)Y^B Y^\dagger_B$$

(4.33)

As earlier, by decomposing the complex scalar fields into two real scalar fields by using equation (4.22) we can get to the form of the L-BLG theory. This analysis has already
been done in \cite{23} for ABJM. We are not going to repeat this here but write down the final form of this sextic potential:

\[ V_{pot} = \frac{1}{12} \text{Tr}(X^{J|[X^{J},X^{K}] + X^{[J}[X^{K},X^{I}] + X^{K}[X^{I},X^{J}])^2.} \tag{4.34} \]

In the same way, we can also analyze the potential terms in \cite{23} that contain both fermions and bosons. Let us start with the expression

\[ \frac{2\pi}{k} \text{Tr}(\bar{\psi}^A Y_B^A Y^B - \bar{\psi}_B \bar{\psi} Y^A Y^A). \tag{4.35} \]

Then using the scaling of \cite{13} and \( k = \frac{1}{\epsilon} k \) we obtain that in the limit \( \epsilon \to 0 \) \cite{33} reduces to

\[ \frac{2\pi}{k} \text{Tr}(\bar{\psi}^A Y_B^A Y^B - \bar{\psi}_B \bar{\psi} Y^A Y^A) \to \]

\[ \frac{2\pi}{k^3} \text{Tr}(\bar{\psi}_{+I N \times N} + \epsilon \bar{\theta}) \times (\bar{\psi}_{+B I N \times N} + \epsilon \theta) \times (Y_{+B I N \times N} + \epsilon \bar{Y}^B)(Y_{+A I N \times N} + \epsilon \bar{Y}^A) \]

\[ = \frac{2\pi}{k^3} \text{Tr}(\bar{\psi}_{+A I N \times N} + \epsilon \bar{\theta}) \times (\bar{\psi}_{+B I N \times N} + \epsilon \theta) \times (Y_{+A I N \times N} + \epsilon \bar{Y}^A)(Y_{+B I N \times N} + \epsilon \bar{Y}^B) \]

\[ = \frac{2\pi}{k^3} \text{Tr}(\bar{\psi}^A Y_B^A Y^B + \epsilon \bar{\psi}^A Y_B^A Y^B) + \frac{2\pi}{k^3} \text{Tr}(\cdots). \tag{4.36} \]

The term proportional to \( \frac{1}{\epsilon} \) vanishes due to the fermionic nature of \( \psi \). In the same way we can show that all terms containing \( \bar{\psi}_{+A} \bar{\psi}_{+A} \) vanish. Terms proportional to \( \frac{1}{\epsilon^2} \) vanish due to the fact that \( \text{Tr}(\cdots) \) is zero since it contains either \( \theta \) or \( \bar{Y} \). The coefficient of the term proportional to \( \frac{1}{\epsilon} \) is also zero because this term contains two traceless fields. Therefore the only non-zero terms are proportional to \( \epsilon^0 \). In fact, this analysis is the same as in \cite{23} and again we will not repeat the same here. We just write down the potential term that contains both bosons and fermions. This looks like

\[ V_{scale}^f = \bar{\Psi}^L X^J[Y^J, \Gamma_{IJ} \Psi_L] - \bar{\Psi}^L X^J[X^J, \Gamma_{IJ} \Psi_L] \tag{4.37} \]

The \( \Gamma \) matrices are the \( 8 \times 8 \) matrices and are constructed from the direct product of \( \gamma^\mu \). The form of this gamma matrices is also given in \cite{23}.

After finishing the discussion on undeformed part of the Lagrangian, we now move to the scaling limit of the mass deformed part of the Lagrangian with the aim of getting contribution for the mass deformed L-BLG theory. The mass term we consider here is;

\[ V_{mass} = m^2 \text{Tr}(Y_A^A Y^A) + im \text{Tr}(\bar{\psi} a Y^A_a Y^A_a) - im \text{Tr}(\bar{\psi} a Y^A_a Y^A_a) \]

\[ - \frac{4\pi}{k^2} m \text{Tr}(Y^A a Y^A_b Y^A_{a'}) Y^A_{a'} Y^A_{b'} Y^A_{b'}). \tag{4.38} \]

By performing the scaling limit in the first term we find

\[ V_{sc.mass} = m^2 \text{Tr}(Y_A^A Y^A) = \frac{m^2}{\epsilon^2} NY_A^A Y^A + m^2 \text{Tr}(Y_A^A Y^A). \tag{4.39} \]
We see that in the same way as in case of kinetic term here also we need to add the ghost contribution to cancel the divergence of the scalar mass term. The ghost contribution is of the form

\[ V_{\text{sc.mass}}^{\text{ghost}} = -m^2 \text{Tr}(U_A^U A^A) = -\frac{N}{\epsilon^2} m^2 Y_+^A Y^A + m^2 (Y_+^A Y^A + Y_+^A Y^A). \] (4.40)

Finally mass term with the addition of ghost term looks like

\[ V_{\text{sc.mass}} = m^2 \text{Tr}(\tilde{Y}_+^A Y^A) + m^2 (Y_+^A Y^A + Y_+^A Y^A). \] (4.41)

Using the redefinition of the fields introduced in equation (4.22), we find, \( V_{\text{sc.mass}} \) can be rewritten in the form of scalar mass term of L-BLG theory:

\[
\begin{align*}
& m^2 \text{Tr}(\tilde{Y}_+^A Y^A) + m^2 (Y_+^A Y^A + Y_+^A Y^A) \\
& = 2m^2 (X_{2A-1}^A X_{2A-1}^A + X_{2A}^2 X_{2A}^2) + m^2 \text{Tr}((\tilde{X}_{2A-1}^A \tilde{X}_{2A-1}^A + X_{2A-1}^A \tilde{X}_{2A-1}^A)) \\
& = 2m^2 (X_{4}^A X_{4}^A) + m^2 \text{Tr}(\tilde{X}_I^A \tilde{X}_I^A). \tag{4.42}
\end{align*}
\]

In case of the fermion mass term we find

\[
\begin{align*}
\text{imTr}(\bar{\psi}^a \psi_a) &= \frac{iN}{\epsilon^2} m \bar{\psi}_+^a \psi_+^a + \text{imTr}(\bar{\theta}_a^\theta_a) \\
\text{and the same for } \psi_{a'}. \text{ Clearly again we have to add ghost contribution to the Lagrangian to get a finite contribution. The form of this ghost is}
\end{align*}
\]

\[
\begin{align*}
\mathcal{L}_{\text{m.g.f.}} &= -\text{imTr}(\bar{\psi}^a \psi_a + \bar{\psi}^a \psi_a') \\
& = -\text{imTr} \left[ \frac{N}{\epsilon^2} \bar{\psi}_+^a \psi_+^a + \text{im}(\bar{\psi}_+^a \psi_-^a + \bar{\psi}_-^a \psi_+^a) \right] \\
& + \text{imTr} \left[ \frac{N}{\epsilon^2} \bar{\psi}_+^a \psi_-^a - \text{im}(\bar{\psi}_+^a \psi_-^a + \bar{\psi}_-^a \psi_+^a) \right]. \tag{4.44}
\end{align*}
\]

Clearly, terms proportional to \( \frac{1}{\epsilon^2} \) cancel and we end up with the finite result

\[
V_{f.mass} = \text{imTr}(\bar{\theta}_a^\theta_a - \bar{\theta}_a^\theta_a) + \text{imTr}(\bar{\psi}_+^a \psi_-^a + \bar{\psi}_-^a \psi_+^a) - \text{im}(\bar{\psi}_+^a \psi_-^a + \bar{\psi}_-^a \psi_+^a). \tag{4.45}
\]

As in the previous cases, we would again redefine the \( \psi \) and \( \theta \) fields as in equation (4.23) and finally we are able to write down \( V_{f.mass} \) as the mass term of L-BLG theory.

\[
V_{f.mass} = \text{imTr}(\bar{\Psi}_A \Psi_A - \bar{\Psi}_A^A \Psi_A^A) + 2\text{imTr}(\bar{\Psi}_+^A \Psi_-^A - \bar{\Psi}_+^A \Psi_-^A). \tag{4.46}
\]

Here \( A \) runs from 1 to 4 and \( A' \) runs from 5 to 8.

As the next step we analyze the first term of mass deformed potential with the following expression

\[
\begin{align*}
V_{d.pot} &= \frac{-4\pi}{k} \text{Tr}(Y^a Y^a Y^b Y^b) \\
& = -\frac{4\pi}{k} \frac{N}{\epsilon^2} Y_{[a}^a Y_{+[b}^b Y_{+b]}^b Y_{+a]}^a + \frac{1}{\epsilon^2} \text{Tr}(Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a) \\
& + \text{imTr}((Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a) \\
& + \text{imTr}((Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a Y^a))). \tag{4.47}
\end{align*}
\]
Now the first term vanishes:

\[
Y^a Y^b Y^c + [a + Y^a Y^b Y^c] = Y^a Y^b Y^c + Y^a Y^b Y^c - Y^a Y^b Y^c + [a + Y^a Y^b Y^c] = 0.
\] (4.48)

Let us now analyze the contributions proportional to \( \frac{1}{\epsilon} \)

\[
\text{Tr}(Y^a Y^b Y^c + [a + Y^a Y^b Y^c] + (Y^a Y^b Y^c + [a + Y^a Y^b Y^c] + Y^a Y^b Y^c + [a + Y^a Y^b Y^c]) = \text{Tr}(2Y^a Y^b Y^c + [a + Y^a Y^b Y^c] - Y^a Y^b Y^c + [a + Y^a Y^b Y^c] - Y^a Y^b Y^c + [a + Y^a Y^b Y^c]) = 0 \]
\] (4.49)

using

\[
Y^a Y^b Y^c + [a + Y^a Y^b Y^c] = Y^a Y^b Y^c - Y^a Y^b Y^c + [a + Y^a Y^b Y^c] = Y^a Y^b Y^c - Y^a Y^b Y^c + [a + Y^a Y^b Y^c] = 0.
\] (4.50)

Then we find the finite contribution to the potential;

\[
V_{d, \text{pot}} = -\frac{4 \pi}{k} m \text{Tr}(Y^a Y^b Y^c + [a + Y^a Y^b Y^c] + Y^a Y^b Y^c + [a + Y^a Y^b Y^c] + Y^a Y^b Y^c + [a + Y^a Y^b Y^c])
\]

\[
+ Y^a Y^b Y^c + [a + Y^a Y^b Y^c] + Y^a Y^b Y^c + [a + Y^a Y^b Y^c] + Y^a Y^b Y^c + [a + Y^a Y^b Y^c])
\]

\[
= -\frac{4 \pi}{k} m \text{Tr}(Y^a Y^b Y^c + [a + Y^a Y^b Y^c] + Y^a Y^b Y^c + [a + Y^a Y^b Y^c])
\]

\[
+ Y^a Y^b Y^c + [a + Y^a Y^b Y^c] + Y^a Y^b Y^c + [a + Y^a Y^b Y^c])
\]

\[
(4.51)
\]

We now again redefine the field Y's in the previous way as in equation (4.22) and with these redefinition we can write the above expression as

\[
-4i X^{2a-1} \bar{X}^{2b} \bar{X}^{2a} - 4i X^{2a} \bar{X}^{2b-1} \bar{X}^{2a} - 4i X^{2b} \bar{X}^{2b-1} \bar{X}^{2b-1}
\]

\[
(4.52)
\]

Let us now analyze these expressions. The expressions on the first term of the first line give

\[
-4i X^{2a-1} \text{Tr}(\bar{X}^{2b} [\bar{X}^{2b-1}, \bar{X}^{2a}])
\]

\[
\text{for } a = b : \quad -4i \text{Tr}(\bar{X}^{2b-1} \bar{X}^{2b} - \bar{X}^{2b} \bar{X}^{2b}) = 0,
\]

\[
\text{for } a = 1, b = 2 : \quad -4i X^1 \text{Tr}(\bar{X}^{4} [\bar{X}^{3}, \bar{X}^{2}]),
\]

\[
\text{for } a = 2, b = 1 : \quad -4i X^3 \text{Tr}(\bar{X}^{4} [\bar{X}^{1}, \bar{X}^{2}])
\] (4.53)

and the expression on the last term of the first line gives

\[
-4i X^{2a} \text{Tr}(\bar{X}^{2b-1} [\bar{X}^{2b}, \bar{X}^{2a-1}])
\]

\[
\text{for } a = b : \quad 0
\]

\[
\text{for } a = 1, b = 2 : \quad -4i X^2 \text{Tr}(\bar{X}^{3} [\bar{X}^{4}, \bar{X}^{3}]),
\]

\[
\text{for } a = 2, b = 1 : \quad -4i X^1 \text{Tr}(\bar{X}^{4} [\bar{X}^{2}, \bar{X}^{3}]),
\] (4.54)
On the other hand the expressions on the second line give
\[-4iX^2_+^{-1}\text{Tr}(\tilde{X}^{2b-1}[\tilde{X}^{2a}, \tilde{X}^{2b}]) - 4iX^2_+ \text{Tr}(\tilde{X}^{2b}[\tilde{X}^{2a-1}, \tilde{X}^{2b-1}])\]
for \(a = b = 0\)
for \(a = 1, \ b = 2 : -4iX^1_+ \text{Tr}(\tilde{X}^{3}[\tilde{X}^{2}, \tilde{X}^{4}]) - 4iX^2_+ \text{Tr}(\tilde{X}^{4}[\tilde{X}^{1}, \tilde{X}^{3}])\)
for \(a = 2, \ b = 1 : -4iX^3_+ \text{Tr}(\tilde{X}^{1}[\tilde{X}^{4}, \tilde{X}^{2}]) - 4iX^4_+ \text{Tr}(\tilde{X}^{2}[\tilde{X}^{3}, \tilde{X}^{1}]). \quad (4.55)\]

All together we have
\[-8iX^1_+ \text{Tr}(\tilde{X}^{3}[\tilde{X}^{2}, \tilde{X}^{4}]) - 8iX^2_+ \text{Tr}(\tilde{X}^{4}[\tilde{X}^{1}, \tilde{X}^{3}])\]
\[-8iX^3_+ \text{Tr}(\tilde{X}^{1}[\tilde{X}^{4}, \tilde{X}^{2}]) - 8iX^4_+ \text{Tr}(\tilde{X}^{2}[\tilde{X}^{3}, \tilde{X}^{1}]) \quad (4.56)\]

We can write
\[-6iX^1_+ \text{Tr}(\tilde{X}^{3}[\tilde{X}^{2}, \tilde{X}^{4}]) = i\epsilon_{1324}X^1_+ \text{Tr}(\tilde{X}^{3}[\tilde{X}^{2}, \tilde{X}^{4}]) + i\epsilon_{1342}X^1_+ \text{Tr}(\tilde{X}^{3}[\tilde{X}^{4}, \tilde{X}^{2}])\]
\[+i\epsilon_{1243}X^1_+ \text{Tr}(\tilde{X}^{2}[\tilde{X}^{4}, \tilde{X}^{3}]) + i\epsilon_{1234}X^1_+ \text{Tr}(\tilde{X}^{2}[\tilde{X}^{3}, \tilde{X}^{4}]).\]
\[+i\epsilon_{1234}X^1_+ \text{Tr}(\tilde{X}^{4}[\tilde{X}^{3}, \tilde{X}^{2}]) + i\epsilon_{1342}X^1_+ \text{Tr}(\tilde{X}^{4}[\tilde{X}^{2}, \tilde{X}^{3}]) \quad (4.57)\]

In the same way we can proceed for expressions with primed quantities and in summary, we obtain the scaling limit of the deformed potential in the form
\[V_{d,\text{pot}} = -\frac{16\pi}{3k}m\epsilon_{ABCD}\text{Tr}(X^A_+ \tilde{X}^B[X^C, \tilde{X}^D]) + \frac{16\pi}{3k}m\epsilon_{A'B'C'D'}\text{Tr}(X^A_+ \tilde{X}^B'[X^C', \tilde{X}^D']) \quad (4.58)\]

Finally we would like to sum up all the terms of the mass deformed ABJ Lagrangian with the scaling limit defined and the sum gives us:
\[\mathcal{L}_{\text{BLG}} = -2\partial_\mu X^I_+ \partial^\mu X^I_+ - 2\partial_\mu X^I_+ \text{Tr}(B^\mu X^I) - \text{Tr}(D_\mu X^I - B_\mu X^I)^2\]
\[-i\text{Tr}(\overline{\Psi}_I\gamma^\mu D_\mu \Psi_I) - 2i\overline{\Psi}_I\gamma^\mu \text{Tr}(B_\mu \Psi_I) - 2i\overline{\Psi}_I\gamma^\mu \partial_\mu \Psi_I\]
\[-\frac{1}{4}\epsilon^{\mu\nu\lambda\gamma}\text{Tr}(B_{\mu}F_{\nu\lambda}) + \frac{1}{12}(X^I_+ [X^J, X^K] + X^J_+ [X^K, X^I] + X^K_+ [X^I, X^J])^2\]
\[+\overline{\Psi}_+X^I_+ [X^J, \Gamma_{IJ}\Psi_L] - \overline{\Psi}_L X^I_+ [X^J, \Gamma_{IJ}\Psi_L]\]
\[+2m^2(X^I_+ X^I_+) + m^2 \text{Tr}(X^I X^I) + im\text{Tr} (\overline{\Psi}_A \Psi_A - \overline{\Psi}_A \Psi_A')\]
\[+2im(\overline{\Psi}_+A \Psi_{-A} - \overline{\Psi}_+A \Psi_{-A'}) - \frac{16}{3}im\epsilon_{ABCD}\text{Tr}(X^A_+ X^B[X^C, X^D])\]
\[+ \frac{16}{3}im\epsilon_{A'B'C'D'}\text{Tr}(X^A_+ X^B'[X^{C'}, X^{D'}]) \quad (4.59)\]

where we have done away with the tilde over the fields for notational simplicity. So starting from a mass deformed ABJ theory, we finally arrived at the world volume theory of a stack of \(N\) number of M2-branes by taking the proper scaling limit on the fields and simultaneously sending Chern-Simons level, \(k\) at infinity. These \(N\) number of branes are sitting far away from the \(\mathbb{C}^4/\mathbb{Z}_k\) singularity. The final expression of the Lagrangian matches exactly with the mass deformed L-BLG theory formulated based on 3-algebra.
5. Conclusion

In this paper we have started with the undeformed ABJ theory and then added suitable mass terms which preserve the supersymmetry of the theory. Our motivation to add those mass terms was to obtain the mass deformed L-BLG theory. In our construction we have introduced two new parameters, $\epsilon$ and $\tilde{k}$. These two parameters are related to the Chern-Simons term when $\tilde{k} = \epsilon k$, where we kept $\tilde{k}$ at fixed value $\pi$ by taking $\epsilon \to 0$ at the end of the computation and at the same time sending $k$ to infinity. We have further dropped some of the fields of the ABJ theory due the fact that they are of the order $\epsilon \to 0$. On top of these two new parameters, we also have added an extra ghost term of $U(1)$ multiplet in the ABJ theory. Eventhough the ghost term decoupled at the level of ABJ theory, it is effectively coupled at the level of L-BLG theory by the redefinitions of the fields. Finally we have successfully constructed the mass deformed L-BLG theory. One can extend the present analysis to write down the full non-relativistic ABJ theory and also for L-BLG theory. Further one can take the same scaling limit, as we have taken to produce the mass deformed L-BLG theory, on non-relativistic ABJ theory and check whether this theory also reproduces the non-relativistic L-BLG theory. We wish to return to this problem in future.

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