Analysis of Regularized Learning for Linear-functional Data in Banach Spaces

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Abstract

In this article, we study the whole theory of regularized learning for linear-functional data in Banach spaces including representer theorems, pseudo-approximation theorems, and convergence theorems. The input training data are composed of linear functionals in the predual space of the Banach space to represent the discrete local information of multimodel data and multiscale models. The training data and the multi-loss functions are used to compute the empirical risks to approximate the expected risks, and the regularized learning is to minimize the regularized empirical risks over the Banach spaces. The exact solutions of the original problems are approximated globally by the regularized learning even if the original problems are unknown or unformulated. In the convergence theorems, we show the convergence of the approximate solutions to the exact solutions by the weak* topology of the Banach space. Moreover, the theorems of the regularized learning are applied to solve many problems of machine learning such as support vector machines and artificial neural networks.

Keywords: Regularized learning, linear-functional data, Banach space, weak* topology, reproducing kernel

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1. Introduction

In many problems of machine learning, we originally try to find an exact solution to minimize the excepted risks or errors over a Banach space such
as Optimization (3.1). The exact solution will give the best decision rule to take the appropriate action. Unfortunately, the expected risks are usually unknown or unformulated and thus it is too difficult to directly find the exact solution by the original problem. In many mathematical models of physics and engineering, the observed data are collected from the original problem such that the large scale data are used to approximate the exact solution. Now the regularized learning provides an efficient tool to construct the approximate solutions. The regularized learning plays a major role in disciplines such as statistical learning, regression analysis, approximation theory, inverse problems, and signal processing. The regularized learning has many applications in various areas of engineering, computer science, psychology, intelligent medicine, and economic decision making. The regularized learning discussed here is to find an approximate solution to minimize the regularized empirical risks which are the sum of the empirical risks and the regularization terms such as Optimization (3.4). The motivation of the regularized learning is the computation of the empirical risks by finite many data and simple loss functions as in Equation (3.2) and the approximation of the expected risks by the empirical risks such as the pointwise convergence in Equation (3.3).

The theory of the regularized learning already achieves a success in reproducing kernel Hilbert spaces (RKHS) such as support vector machines in [17]. Recently, the learning theory is extended to reproducing kernel Banach space (RKBS) in [22, 25]. By the classical methods, the regularized learning is only used to analyze the classical input data composed of regular vectors. The reproduction of RKHS and RKBS guarantees that the classical input data can be equivalently transferred to the point evaluation functions. In our papers [10, 23, 24], we therefore look at a generalized concept of linear-functional data to represent the discrete local information of multimodel data and multiscale models. As in Equation (2.4), the generalized input data are composed of linear functionals. As in [10 Theorem 3.1], we already generalize the representer theorem in a Banach space, which exists a predual space, to show that the regularized learning is also a feasible tool to analyze the linear-functional data. Our main idea is that the regularized learning is locally interpretable by the linear-functional data and the exact solution is approximated globally by the regularized learning. In this article, we will study the whole theory of the regularized learning by the weak* topology including the representer theorems, pseudo-approximation theorems, and convergence theorems. The convergence theorems guarantee that the approximate solutions converge to the exact solutions and the representer theorems or pseudo-approximation
Theorems guarantee that the approximate solutions can be equivalently or approximately computed by the finite-dimensional optimization.

Many people are interested in the constructions and applications of the regularized learning. Now there are only a few papers [3, 16] to discuss the convergence of the regularized learning for the classical data, but the convergence analysis of the regularized learning for the linear-functional data is still lack of study in Banach spaces. In Section 4.3, we will discuss the convergence theorems of the approximate solutions in Banach spaces. The predual space guarantees the existence of the weak* topology of the Banach space such that the empirical risks and the regularization terms can be connected by the weak* topology. Moreover, even when the uniform convergence or Γ-convergence of the empirical risks to the expected risks may not hold true, the weak* compactness provides another way to prove the weak* convergence of the approximate solutions to the exact solution. Different from the classical proof of the convergence of learning theory, the assumptions of the independently and identically distributed data are not needed here in the proof of the weak* convergence.

Remark 1.1. The formulas of the regularized learning are similar to the inverse problems while their details are not the same. For example, the regularized learning are usually computed by the finite many discrete data, but
the inverse problems focus on the continuous problems such as the integral equations. We will complete the proof of the convergence theorems by the properties of the weak* topology. The regularized learning can be viewed as the interdisciplinary field of approximation theory, optimization theory, and regularization theory.

As in Examples 6.1 and 6.2, the classical input data can be equivalently transferred to the linear-functional data composed of the point evaluation functions. As in Example 6.3, the linear-functional data can represent different kinds of data induced by partial differential equations. In Theorem 4.2 and Corollary 4.3, we apply the techniques in [2, 18] to redo the generalized representer theorems in [10] to cover the multi-loss functions. This shows that the approximate solutions can be equivalently computed by the finite-dimensional optimizations such as support vector machines. Under the universal approximation, Theorem 4.5 also shows that the approximate solution can be approximately computed by the finite-dimensional optimizations such as artificial neural networks. Under Conditions (I) and (II) in Section 4.3, the convergence analysis of the approximate solutions is shown in Theorems 4.6 and 4.7. Theorem 4.8 further gives a special way to choose the adaptive regularization parameters to show the convergence of the approximate solutions. As in Example 6.5, the technique of regularization is necessary for the ill-posed problems even if Conditions (I) and (II) hold true. We finally complete the proof of all theorems of the regularized learning in Section 5.

2. Notations and Preliminaries

In this section we introduce several notions and results of functional analysis as well as some properties of Banach spaces which are frequently used throughout the article. We here only look at the real-valued cases.

2.1. Banach Spaces and Predual Spaces

Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}^*$ be the dual space of $\mathcal{B}$. Thus $\mathcal{B}^*$ is the collection of all bounded linear functionals on $\mathcal{B}$. We denote that $\|\cdot\|$ and $\|\cdot\|_*$ are the norms of $\mathcal{B}$ and $\mathcal{B}^*$, respectively, and $\langle \cdot, \cdot \rangle$ is the dual bilinear product of $\mathcal{B}$ and $\mathcal{B}^*$. Let $B_\mathcal{B}$ be the closed unit ball of $\mathcal{B}$ and let $S_\mathcal{B}$ be the unit sphere of $\mathcal{B}$. Thus $rB_\mathcal{B} = \{ f \in \mathcal{B} : \|f\| \leq r \}$ and $rS_\mathcal{B} = \{ f \in \mathcal{B} : \|f\| = r \}$ for any $r > 0$. Let $\partial \|\cdot\|$ be the subdifferential of the norm $\|\cdot\|$. Thus the
characterizing subdifferential of norm as in [26, Corollary 2.4.16] shows that
\[
\partial \| \cdot \| (f) = \begin{cases} 
\{ \xi^* \in S_{\mathcal{B}^*} : \| f \| = \langle f, \xi^* \rangle \}, & \text{if } f \in \mathcal{B} \setminus \{0\}, \\
\text{otherwise.} & 
\end{cases}
\] (2.1)

Let \( \mathcal{A} \) be any nonempty subset of \( \mathcal{B} \). We denote that the dual cone of \( \mathcal{A} \) is \( \mathcal{A}^+ := \{ \xi^* \in \mathcal{B}^* : \langle f, \xi^* \rangle \geq 0 \text{ for all } f \in \mathcal{A} \} \) and the orthogonal space of \( \mathcal{A} \) is \( \mathcal{A}^\perp := \{ \xi^* \in \mathcal{B}^* : \langle f, \xi^* \rangle = 0 \text{ for all } f \in \mathcal{A} \} \). Specially, if \( \mathcal{A} \) is a closed subspace of \( \mathcal{B} \), then \( \mathcal{A}^+ = \mathcal{A}^\perp = \mathcal{A} \). Let \( \chi_\mathcal{A} \) be the indicator function of \( \mathcal{A} \), that is, \( \chi_\mathcal{A}(f) := 0 \) if \( f \in \mathcal{A} \) and \( \chi_\mathcal{A}(f) := \infty \) otherwise. As in [26, Section 2.3], if \( \mathcal{A} \) is a closed affine space of \( X \) and \( f \in \mathcal{A} \), then
\[- \partial \chi_\mathcal{A}(f) = (\text{cone } (\mathcal{A} - f))^+ = (\mathcal{A} - f)^\perp. \] (2.2)

Moreover, the dual bilinear product \( \langle \cdot, \cdot \rangle \) is extended to
\[
\langle f, \xi^* \rangle := (\langle f, \xi^*_1 \rangle, \langle f, \xi^*_2 \rangle, \ldots, \langle f, \xi^*_N \rangle),
\]
where \( f \in \mathcal{B} \) and \( \xi^* = (\xi^*_1, \xi^*_2, \ldots, \xi^*_N) \) composed of \( \xi^*_1, \xi^*_2, \ldots, \xi^*_N \in \mathcal{B}^* \). We say that \( \ker(\xi^*) \) is the kernel of \( \xi^* \), that is,
\[
\ker(\xi^*) := \{ f \in \mathcal{B} : \langle f, \xi^* \rangle = 0 \}.
\]
Thus \( \ker(\xi^*) \) is a closed linear subspace of \( \mathcal{B} \) and
\[
\ker(\xi^*)^\perp = \text{span} \{ \xi^*_1, \xi^*_2, \ldots, \xi^*_N \}. \] (2.3)

As in [5, Definition 2.2.27], we say that \( \mathcal{B}_s \) is a predual space of \( \mathcal{B} \) if \( \mathcal{B}_s \) is a subspace of \( \mathcal{B}_s^* \) such that the dual space of \( \mathcal{B}_s \) is isometrically isomorphic to \( \mathcal{B} \), that is, \( (\mathcal{B}_s)^* \cong \mathcal{B} \). For example, if \( \mathcal{X} \) is a normed space and \( \mathcal{B} = \mathcal{X}^* \), then \( \mathcal{B} \) exists a predual space \( \mathcal{B}_s \cong \mathcal{X} \). Specially, if \( \mathcal{B} \) is reflexive, then \( \mathcal{B}_s = \mathcal{B}^* \). The predual space \( \mathcal{B}_s \) assures that the weak* topology of \( \mathcal{B} \) is introduced by the topologizing family \( \mathcal{B}_s \) and all weak* properties hold true on \( \mathcal{B} \). We say that a net \( (f_\alpha) \subseteq \mathcal{B} \) weakly* converges to an element \( x \in \mathcal{B} \) if and only if \( \lim_{\alpha} \langle f_\alpha, \xi^* \rangle = \langle f, \xi^* \rangle \) for all \( \xi^* \in \mathcal{B}_s \). We will discuss the weak* convergence in \( \mathcal{B} \) by the weak* topology \( \sigma(\mathcal{B}, \mathcal{B}_s) \). For one example, the Riesz–Markov theorem assures that the bounded total variation space \( \mathcal{M}(\mathbb{R}^d) \) of regular countably additive Borel measures on \( \mathbb{R}^d \) exists a predual space \( \left( \mathcal{M}(\mathbb{R}^d) \right)_* \cong C_0(\mathbb{R}^d) \) as in [19, Generalized Total Variation]. For another example, if \( \Omega \) is a compact domain, then [5, Theorem 6.4.1] assures that
C(Ω) exists a predual space if and only if Ω is hyper-Stonean. In this article, we assume that \( B \) always exists a predual space \( B^* \) for convenience.

The Banach-Alaoglu theorem assures that \( rB_B \) is weakly* compact. Thus \( rB_B \) with respect to the relative weak* topology is a compact Hausdorff space. In this article, we will look at a special continuity on \( rB_B \). Let a function \( R \in \mathbb{R}^B \). The \( R \) is weakly* lower semi-continuous or weakly* continuous on \( rB_B \) if and only if

\[
R(f_0) \leq \liminf_{\alpha} R(f_\alpha) \quad \text{or} \quad R(f_0) = \lim_{\alpha} R(f_\alpha),
\]

for any weakly* convergent net \((f_\alpha) \subseteq rB_B\) to an element \( f_0 \in rB_B \). Obviously, if \( R \) is weakly* lower semi-continuous or weakly* continuous, then \( R \) is weakly* lower semi-continuous or weakly* continuous on \( rB_B \) for all \( r > 0 \).

For example, the norm function \( \|\cdot\| \) is weakly* lower semi-continuous and the linear functional \( \xi^* \in B^* \) is weakly* continuous, thus \( \|\cdot\| \) is weakly* lower semi-continuous on \( rB_B \) for all \( r > 0 \) and \( \xi^* \) is weakly* continuous on \( rB_B \) for all \( r > 0 \). Moreover, we denote that \( C(rB_B) \) is the collection of all continuous functions on \( rB_B \) with respect to the relative weak* topology. Thus \( R \) is weakly* continuous on \( rB_B \) if and only if \( R \in C(rB_B) \). To simplify the notations, the restriction \( R|_{rB_B} \) is rewritten as \( R \). Let a set \( \mathfrak{F} \subseteq \mathbb{R}^B \). The \( \mathfrak{F} \) is weakly* equicontinuous on \( rB_B \) if and only if

\[
\limsup_{\alpha} \sup_{R \in \mathfrak{F}} |R(f_0) - R(f_\alpha)| = 0,
\]

for any weakly* convergent net \((f_\alpha) \subseteq rB_B\) to an element \( f_0 \in rB_B \). Now we look at a special example of \( \mathfrak{F} \).

**Lemma 2.1.** If \( \mathfrak{F} \subseteq B^* \), then the following are equivalent.

(a) The \( \mathfrak{F} \) is weakly* equicontinuous on \( B_B \).

(b) The \( \mathfrak{F} \) is weakly* equicontinuous on \( rB_B \) for all \( r > 0 \).

(c) The \( \mathfrak{F} \) is relatively compact in \( B^* \).

Obviously, Lemma 2.1 (b) is also equivalent that \( \mathfrak{F} \) is weakly* equicontinuous on all bounded subset of \( B \).

**Remark 2.1.** In this article, we will mix the notations of machine (statistical) learning in [10, 22] and topological vector spaces in [12, 26]. We mainly use the same symbol system in [10], specially, the notation of input and output data is denoted as \((\xi^*, y)\). As in [12, Definitions 2.1.8 and 2.1.26], we use the theory of net and subnet to study the weak* properties of Banach spaces.
2.2. Motivations of Linear-functional Data

For the classical machine learning of regressions and classifications, we will approximate the exact solution by the classical input data

\[ x_1, x_2, \ldots, x_N \in \Omega, \]

where \( \Omega \) is a subset of \( \mathbb{R}^d \). If we solve the learning problems in \( \mathcal{H}_K(\Omega) \), then the representer theorem in RKHS in [17] assures that the approximate solution is a linear combination of the kernel basis

\[ K(\cdot, x_1), K(\cdot, x_2), \ldots, K(\cdot, x_N) \in \mathcal{H}_K(\Omega), \]

where \( \mathcal{H}_K(\Omega) \) is a RKHS and \( K : \Omega \times \Omega \to \mathbb{R} \) is its reproducing kernel. The proof of the representer theorem in RKHS is based on the reproduction of RKHS which shows that \( x_1, x_2, \ldots, x_N \) can be equivalently transferred to

\[ \delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_N} \in (\mathcal{H}_K(\Omega))^* \cong \mathcal{H}_K(\Omega), \]

where \( \delta_x \) is a point evaluation function for a \( x \in \Omega \), that is, \( \langle f, \delta_x \rangle = f(x) \) for any \( f \in \mathbb{R}^{\Omega} \). Recently, the representer theorem in RKBS in [22] guarantees that the approximate solution in \( \mathcal{B}_K^p(\Omega) \) is also based on the reproduction of RKBS which shows that \( x_1, x_2, \ldots, x_N \) can be equivalently transferred to

\[ \delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_N} \in (\mathcal{B}_K^p(\Omega))^* \cong \mathcal{B}_K^q(\Omega), \]

where \( \mathcal{B}_K^p(\Omega) \) is a \( p \)-norm RKBS for \( 1 \leq p < \infty \) and \( p, q \) is a pair of conjugate exponents, specially, \( \mathcal{B}_K^2(\Omega) = \mathcal{H}_K(\Omega) \) and \( \mathcal{B}_K^1(\Omega) \) is nonreflexive, nonstrictly convex, and nonsmooth. Thus this gives us an idea to discuss the learning problems by linear-functional data

\[ \xi_1^*, \xi_2^*, \ldots, \xi_N^* \in \mathcal{B}_v. \]

Thus the linear-functional data can represent the discretization of integral equations and differential equations, for example \( \langle f, \xi^* \rangle := \int_\Omega f(x)\mu(x)dx \) or \( \langle f, \xi^* \rangle := \partial f(x) \) as in Examples 6.2 or 6.3. This shows that the linear-functional data can be used to represent the local information of different models to construct the global approximate solutions by regularized learning, specially, the combination of the algorithms of finite difference, finite element, and finite volume in one system. More details of the linear-functional data and the similar concepts are mentioned in [10 18 20 23 24].
We denote that $Y$ is the collection of all output elements dependent of the different learning problems, for example, $Y := \mathbb{R}$ for regressions and $Y := \{\pm 1\}$ for classifications. We here mainly discuss the deterministic data. Actually the output data can be extended to the stochastic data, more precisely, $Y$ is the collection of random variables with discrete or continuous distributions such as in Example 6.2.

In this article, we look at training data $(\xi^*_1, y_1), (\xi^*_2, y_2), \ldots, (\xi^*_{N_n}, y_{N_n}) \in \mathcal{B}_* \times Y$, for all $n \in \mathbb{N}$, (2.4) where $N_n \in \mathbb{N}$. Usually $N_n \to \infty$ when $n \to \infty$. For each nth approximate step, we collect $N_n$ observations which may be partial duplicate in different steps. Let

$\xi^*_n := (\xi^*_1, \xi^*_2, \ldots, \xi^*_{N_n})$, $y_n := (y^*_1, y^*_2, \ldots, y^*_{N_n})$, for all $n \in \mathbb{N}$.

Thus Equation (2.4) is rewritten as

$(\xi^*_n, y_n) \in \mathcal{B}^{N_n}_* \times Y^{N_n}$, for all $n \in \mathbb{N}$, (2.5)

where $\mathcal{B}^{N_n}_* := \bigotimes_{k=1}^{N_n} \mathcal{B}_*$ and $Y^{N_n} := \bigotimes_{k=1}^{N_n} Y$. The collection of all training data is denoted as

$\mathcal{D} := \{(\xi^*_n, y_n) : n \in \mathbb{N}\}$.

Next, the collection of all input data is denoted as

$\mathcal{F}_D := \{\xi^*_{nk} : k_n \in \mathbb{N}, n \in \mathbb{N}\}$,

where $N_N := \{1, 2, \ldots, N\}$ for any $N \in \mathbb{N}$. Usually $\mathcal{F}_D$ is an infinite countable subset of $\mathcal{B}_*$. Based on the construction of $\mathcal{F}_D$, we will discuss a special condition of $\mathcal{F}_D$ dependent of the norm of $\mathcal{B}$ to verify the convergence of the approximate solutions to the exact solutions such as the weak* equicontinuity of $\mathcal{F}_D$ on $\mathcal{B}_R$ which is equivalent to the relative compactness of $\mathcal{F}_D$ in $\mathcal{B}_*$ by Lemma 2.1. Now we illustrate that the classical input data composed of point evaluation functions cover the special condition. Let $C^{0,\vartheta}(\Omega)$ be a Hölder continuous space for an exponent $\vartheta > 0$. According to [22, Chapter 4], $\mathcal{B}^{p}_K(\Omega)$ is embedded into $C^{0,\vartheta}(\Omega)$ for many reproducing kernels.

**Lemma 2.2.** Suppose that $\Omega$ is a compact set and $\delta_x \in \mathcal{B}_*$ for all $x \in \Omega$. If $\mathcal{B}$ is embedded into $C^{0,\vartheta}(\Omega)$, then $\{\delta_{x_n} : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{B}_*$ for any $\{x_n : n \in \mathbb{N}\} \subseteq \Omega$. 8
Remark 2.2. It is easy to check that $\mathcal{F}_D$ is equicontinuous when $\mathcal{F}_D$ is uniformly bounded. But the equicontinuity is unusable to verify the weak* convergence. Usually the weak* equicontinuity is an important condition for the proof of the weak* convergence. Unfortunately, the weak* equicontinuity is too strong to check for the real data because a weakly* convergent subnet may be divergent or may be even unbounded. We here consider a weaker condition which is easy to check in many learning problems. For the classical data, Lemma 2.2 guarantees that the classical $\mathcal{F}_D$ satisfies the relative compactness in $\mathcal{B}_*$ which is equivalent to the weak* equicontinuity on $\mathcal{B}_B$. This shows that the relative compactness of $\mathcal{F}_D$ in $\mathcal{B}_*$ is a nature condition. Moreover, if $\mathcal{F}_D$ is weakly* equicontinuous or bounded weakly* equicontinuous, then $\mathcal{F}_D$ is weakly* equicontinuous on $\mathcal{B}_B$, and if $\mathcal{F}_D$ is weakly* equicontinuous on $\mathcal{B}_B$, then $\mathcal{F}_D$ is equicontinuous. The weak* equicontinuity of $\mathcal{F}_D$ on $\mathcal{B}_B$ can be roughly seen as the intermediate condition. The intermediate condition of the linear-functional data further implies the special weak* equicontinuity of the empirical risk functions such as Condition (II') in Section 4.3 for the proof of the convergence theorems.

2.3. Extensions of Loss Functions

By the classical machine learning, we usually use one loss function $L : \mathcal{B}_* \times Y \times \mathbb{R} \to [0, \infty)$ to compute the empirical risks, for example,

$$\text{risk} = \frac{1}{N_n} \sum_{k=1}^{N_n} L(\xi^*_nk, y_{nk}, \langle f, \xi^*_nk \rangle), \quad \text{for} \quad f \in \mathcal{B}. \quad (2.6)$$

We will combine different kinds of data from different models to construct new learning algorithms. According to the complex models, we can compute the empirical risks by different loss functions, for example,

$$\text{risk} = \frac{1}{M_n} \sum_{j=1}^{M_n} \frac{1}{N_n^j} \sum_{k=1}^{N_n^j} L_n^j(\xi^j_{nk}, y^j_{nk}, \langle f, \xi^j_{nk} \rangle), \quad \text{for} \quad f \in \mathcal{B}, \quad (2.7)$$

where $L_n^j$ is a different loss function related to a different kind of the data $(\xi^j_{n1}, y^j_{n1}), (\xi^j_{n2}, y^j_{n2}), \ldots, (\xi^j_{nN_n^j}, y^j_{nN_n^j})$ for $j = 1, 2, \ldots, M_n$.

Now we extend the concept of loss functions to cover the standard formulas in Equation (2.6), the complex formulas in Equation (2.7), and another
general formulas. For each $n$th approximate step, we define a multi-loss function as

$$L_n : \mathcal{B}_N^* \times Y^N \times \mathbb{R}^N \to [0, \infty),$$

(2.8)

where $N_n$ is the number of the data $(\xi_1^*, y_n)$. For example of the classical machine learning, we have

$$L_n (\xi^*, y, t) = \frac{1}{N_n} \sum_{k=1}^{N_n} L (\xi_k^*, y_k, t_k), \quad \text{for } \xi^* \in \mathcal{B}_N^*, \ y \in Y^N, \ t \in \mathbb{R}^N,$$

(2.9)

where $\xi^* = (\xi^*_1, \xi^*_2, \cdots, \xi^*_N)$, $y = (y_1, y_2, \cdots, y_N)$, and $t = (t_1, t_2, \cdots, t_N)$. We say that $L_n$ is a lower semi-continuous, continuous, or convex multi-loss function if $L_n (\xi^*, y, \cdot)$ is lower semi-continuous, continuous, or convex for all $\xi^* \in \mathcal{B}_N^*$ and all $y \in Y^N$. We say that $L_n$ is a local Lipschitz continuous multi-loss function if for any $\theta > 0$, there exists a $C_n^\theta > 0$ such that

$$\sup_{\xi^* \in \mathcal{B}_N^*, y \in Y^N} \left| L_n (\xi^*, y, t_1) - L_n (\xi^*, y, t_2) \right| \leq C_n^\theta \| t_1 - t_2 \|_\infty,$$

for all $t_1, t_2 \in [-\theta, \theta]^N$. The collection of all multi-loss functions is denoted as

$$\mathcal{L} := \{ L_n : n \in \mathbb{N} \}.$$

We further say that $\mathcal{L}$ is uniformly local Lipschitz continuous if $\sup_{n \in \mathbb{N}} C_n^\theta < \infty$ for all $\theta$. The local Lipschitz continuity is a usual condition of loss functions similar to [17, Definition 2.18] for support vector machines.

Lemma 2.3. If $L_n$ has the formula as in Equation (2.9), then the following hold.

(a) If $L$ is a lower semi-continuous, continuous, convex, or local Lipschitz continuous loss function, then $L_n$ is a lower semi-continuous, continuous, convex, or local Lipschitz continuous multi-loss function.

(b) If $L$ is a local Lipschitz continuous loss function, then $\mathcal{L}$ is uniformly local Lipschitz continuous.

3. Regularized Learning in Banach Spaces

In many learning problems, our original goal is to find an exact solution $f^0$ to minimize the expected risks (errors) over a Banach space $\mathcal{B}$, that is,

$$R(f^0) = \inf_{f \in \mathcal{B}} R(f),$$

(3.1)
where $R : \mathcal{B} \to [0, \infty)$ is an expected risk function. Let $\mathcal{S}^0(\mathcal{B})$ be the collection of all minimizers of Optimization (3.1). Thus $f^0 \in \mathcal{S}^0(\mathcal{B})$. Unfortunately, $R$ is usually unknown or unformulated and it is difficult to directly solve Optimization (3.1). To resolve the abstract problem, we approximate the exact solution by the training data.

**Remark 3.1.** In practical applications, any minimizer of Optimization (3.1) is a feasible solution. Thus the assumption of the singleton of $\mathcal{S}^0(\mathcal{B})$ is not needed. If $\mathcal{B}$ is a dense subspace of a topological vector space $\mathcal{X}$ and $R$ is continuously extended on $\mathcal{X}$, then Optimization (3.1) can be seen as an equivalent problem of the minimization of $R$ over $\mathcal{X}$. Thus we can assume the existence of $f^0$ in the following discussions. In this article, we only focus on the minimization of $R$ over $\mathcal{B}$ which exists a predual space $\mathcal{B}^*$ such that the weak* topology of $\mathcal{B}$ is always well-defined.

### 3.1. Regularized Learning for Linear-functional Data

For each $n$th approximate step, we use the training data $(\xi^*_n, y_n)$ in Equation (2.5) and the multi-loss function $L_n$ in Equation (2.8) to construct an empirical risk function $R_n$, that is,

$$R_n(f) := L_n(\xi^*_n, y_n, \langle f, \xi^*_n \rangle), \quad \text{for } f \in \mathcal{B}. \tag{3.2}$$

Thus the empirical risks are computable by the training data and the multi-loss functions. The output $y_n$ can be seen as the values of the exact solution at the input $\xi^*_n$ without or with noises, for example of noiseless data, $y_n = \langle f^0, \xi^*_n \rangle$ for regressions and $y_n = \text{sign}(\langle f^0, \xi^*_n \rangle)$ for classifications. Roughly speaking, the approximate solution driven by the training data can be seen as the equivalent element of the normal vector of some decision rule. The multi-loss functions are usually simple and they can be nonsmooth and nonconvex.

In the same manner as the classical learning theory, the empirical risks will be used to approximate the expected risks, and the empirical risks will be seen as the explicitly computable discretization of the expected risks. Therefore, we choose $D$ and $L$ such that $R_n$ converges pointwise to $R$ when $n \to \infty$, that is,

$$R(f) = \lim_{n \to \infty} R_n(f), \quad \text{for all } f \in \mathcal{B}. \tag{3.3}$$

The pointwise convergence is a weak condition which also holds for many other practical problems even if $R_n$ is used to approximate a simplify form.
of R. In learning theory, the feasibility of the regularized learning comes from the approximation law of the large numbers of the data. Thus we will find the approximate solutions by the empirical risk functions based on Equation (3.3). To verify the generalization and avoid the overfitting, we find an approximate solution $f_\lambda^n$ to minimize the regularized empirical risks over $\mathcal{B}$, that is,

$$R_n(f_\lambda^n) + \lambda \Phi(\|f_\lambda^n\|) = \inf_{f \in \mathcal{B}} R_n(f) + \lambda \Phi(\|f\|),$$

(3.4)

where $\lambda > 0$ is a regularization parameter and $\Phi : [0, \infty) \to [0, \infty)$ is a continuous and strictly increasing function such that $\Phi(r) \to \infty$ when $r \to \infty$. Let $S_\lambda^n(\mathcal{B})$ be the collections of all minimizers of Optimization (3.4). Since $\Phi$ is fixed and unfocused, we do not index $S_\lambda^n(\mathcal{B})$ for $\Phi$. Thus $f_\lambda^n \in S_\lambda^n(\mathcal{B})$. A special example of the regularized learning is the classical binary classifier for the hinge loss, that is

$$\inf_{f \in \mathcal{H}_K(\mathbb{R}^d)} \frac{1}{N} \sum_{k=1}^N \max \{0, 1 - y_k f(x_k)\} + \lambda \|f\|^2,$$

where the binary data $\{(x_k, y_k) : k \in \mathbb{N}_N\} \subseteq \mathbb{R}^d \times \{\pm 1\}$.

**Remark 3.2.** By the same technique of the classical regularization, we here discuss the simple form of the regularization terms only dependent of the regularization parameters and the norms of the Banach spaces. There may be many other formulas of the regularization terms to lessen the chance of overfitting. For example, the regularization term $\lambda \Phi(\|f\|)$ can be extended by a general form $\Phi : \mathcal{B} \times [0, \infty)^M \to [0, \infty)$ which satisfies that:

- For all $\lambda \in \mathbb{R}_+^M$, $\Phi(f_1, \lambda) > \Phi(f_2, \lambda)$ if and only if $\|f_1\| > \|f_2\|$, $\Phi(f, \lambda) \to \infty$ when $\|f\| \to \infty$, and $\Phi(\cdot, \lambda)$ is a weakly* lower semi-continuous function.

- For all $f \in \mathcal{B}$, $\Phi(f, 0) = 0$, $\Phi(f, \lambda)/\|\lambda\|_2 \to \Phi(\|f\|)$ when $\lambda \to 0$, and $\Phi(f, \cdot)$ is a concave function.

By a similar argument, the same conclusions still hold true for the extended $\tilde{\Phi}(f, \lambda)$.

Optimization (3.4) is the main formula of the regularized learning. Clearly, the regularized learning is based on the techniques of regularization and $R_n$ is its key factor. Now we discuss the properties of $R_n$. 
Proposition 3.1. (a) Then
\[ R_n(f) = R_n(f + h), \quad \text{for all } f \in \mathcal{B} \text{ and all } h \in \ker (\xi_n^*). \]

(b) If \( L_n \) is a lower semi-continuous, continuous, or convex multi-loss function, then \( R_n \) is weakly* lower semi-continuous, weakly* continuous, or convex.

(c) If \( \mathcal{F}_\mathcal{D} \) is relatively compact in \( \mathcal{B}_* \) and \( \mathcal{L} \) is uniformly local Lipschitz continuous, then \( \{ R_n : n \in \mathbb{N} \} \) is weakly* equicontinuous on \( rB_B \) for all \( r > 0 \).

(d) Suppose that all conditions of (c) hold true. If \( R_n \) converges pointwise to \( R \) when \( n \to \infty \), then
\[ \lim_{n \to \infty} \sup_{f \in rB_B} |R(f) - R_n(f)| = 0, \quad \text{for all } r > 0. \]

(e) Suppose that all conditions of (c) hold true. Let \( (f_\alpha) \subseteq \mathcal{B} \) be any weakly* convergent bounded net to an element \( f_0 \) and \( (\lambda_\alpha) \subseteq \mathbb{R}_+ \) be any net. If
\[ \lim_{\alpha} \frac{1}{\lambda_\alpha} \sup_{\xi^* \in \mathcal{F}_\mathcal{D}} |\langle f_0 - f_\alpha, \xi^* \rangle| = 0, \]
then
\[ \lim_{\alpha} \frac{1}{\lambda_\alpha} \sup_{n \in \mathbb{N}} |R_n(f_0) - R_n(f_\alpha)| = 0. \]

In the next section, we will discuss the representer theorems, pseudo-approximation theorems, and convergence theorems of the approximate solutions. Generally speaking, the convergence theorems guarantee that the exact solutions of the original problems are approximated by the regularized learning, and the representer theorems or pseudo-approximation theorems guarantee that the approximate solutions are equivalently or approximately computed by the finite-dimensional optimizations.

3.2. Approximate Approximation

Let \( \mathcal{U}_m \) be a subset of \( \mathcal{B} \) such that there exists an surjection \( \Gamma_m \) from \( \Omega_m \) onto \( \mathcal{U}_m \), where \( \Omega_m \subseteq \mathbb{R}^m \) and \( m \in \mathbb{N} \). For examples, \( \mathcal{U}_m \) is a \( m \) dimensional subspace and \( \mathcal{U}_m \) is a collection of artificial neural networks with \( m \) weights and bias while it may not be a linear space. Now we look at an approximately
approximate solution \( f_{nm}^\lambda \) to minimize the regularized empirical risks over \( U_m \), that is,
\[
R_n(f_{nm}^\lambda) + \lambda \Phi(\|f_{nm}^\lambda\|) = \inf_{f \in U_m} R_n(f) + \lambda \Phi(\|f\|).
\] (3.5)

Let \( S_n^\lambda(U_m) \) be the collections of all minimizers of Optimization (3.5). Thus \( f_{nm}^\lambda \in S_n^\lambda(U_m) \).

According to Remark 4.1 if \( \mathcal{B}_s \) is smooth, then the representer theorems guarantee that there exists a \( \mathcal{U}_{N_n} \) such that \( f_{nm}^\lambda \) is equivalently solved by Optimization (3.5), for examples of support vector machines in RKHS and RKBS. In many learning problems, we will construct the fast algorithms by the special \( \mathcal{U}_m \). In this case, \( f_{nm}^\lambda \) may not be in any \( \mathcal{U}_m \). Thus we introduce another condition of \( \mathcal{U}_m \) such as universal approximation. With the condition of universal approximation, the pseudo-approximation theorems guarantee that \( f_{nm}^\lambda \) is approximately solved by Optimization (3.5), for examples of artificial neural networks.

Moreover, we can solve a minimizer \( w_m \) from the finite-dimensional optimization
\[
\inf_{w \in \Omega_m} L_m(\xi_n^*, y_n^*, \langle \Gamma_m(w), \xi_n^* \rangle) + \lambda \Phi(\|\Gamma_m(w)\|),
\] (3.6)
and thus \( f_{nm}^\lambda = \Gamma_m(w_m) \). By the surjection from \( \Omega_m \) onto \( \mathcal{U}_m \), Optimization (3.6) can be seen as an equivalent transformation of Optimization (3.5).

### 3.3. Applications to Composite Algorithms

In the real world, there may be many different algorithms to solve one problem. The theory of the regularized learning provides a simple way to combine different kinds of models and data in one system. Now we will illustrate the idea by a general example.

Let \( \mathcal{B}_b^w \subseteq \mathbb{R}^{\Omega} \) be two different Banach spaces and \( \|\cdot\|_b, \|\cdot\|_w \) be their norms, respectively. We assume that the exact solution \( f^0 \in \mathcal{B}_b \cap \mathcal{B}_w \) is solved by the both abstract models
\[
\inf_{f \in \mathcal{B}_b} R_b(f), \quad \inf_{f \in \mathcal{B}_w} R_w(f),
\]
where \( R_b : \mathcal{B}_b \to [0, \infty) \) and \( R_w : \mathcal{B}_w \to [0, \infty) \) are two different expected risk functions. Moreover, \( R_b, R_w \) are approximated by the empirical risk functions \( R_{b,n}, R_{w,n} \), respectively. As in Equation (3.2), \( R_{b,n}, R_{w,n} \) are constructed by the different training data \( (\xi_n^b, y_n^b), (\xi_n^w, y_n^w) \) and multi-loss functions \( L_n^b, L_n^w \).
respectively. Thus $f^0$ is solved approximately by the both regualairzed learning

$$\inf_{f \in B^b_n} R^b_n(f) + \lambda \|f\|_b, \quad \inf_{f \in B^w_n} R^w_n(f) + \lambda \|f\|_w.$$  

Let $B := B^b \cap B^w$ be a normed space endowed with the composite norm, that is, $\|f\| := \|f\|_b + \|f\|_w$ for $f \in B^b \cap B^w$. Obviously $B$ is a Banach space. Since $B^b \cap B^w \cong (B^b)^* \cap (B^w)^* \cong (B^b_+ + B^w_+)^*$, we have $B^b_+ + B^w_+ \cong (B^b \cap B^w)_+$. Thus $B$ exists a predual space $B^b_+ + B^w_+$, and if $\xi^* \in B^b_*$ or $\xi^* \in B^w_*$, then $\xi^* \in B_*$. This shows that the weak* convergence in $\sigma(B, B_*)$ implies the weak* convergence in $\sigma(B^b, B^b_*)$ and $\sigma(B^w, B^w_*)$. We look at the composite models, that is,

$$R := R^b + R^w, \quad R_n := R^b_n + R^w_n.$$  

Thus $f^0$ is a minimizer of the minimization of $R$ over $B$, and $f^0$ is solved approximately by the composite regularized learning which is to minimize $R_n + \lambda \|\cdot\|$ over $B$. If $R^b_n, R^w_n$ both satisfy the conditions of the representer theorems, pseudo-approximation theorems, and convergence theorems in Section 4, then it is easy to check that $R_n$ satisfies the same conditions. Therefore, the representer theorems, approximation theory, and convergence theorems hold true for the composite algorithms if they hold true for the both algorithms.

Let $T^b_n, \lambda := R^b_n + \lambda \|\cdot\|_b$ and $T^w_n, \lambda := R^w_n + \lambda \|\cdot\|_w$. If $R^b_n$ and $R^w_n$ are weakly* lower semi-continuous and convex, then the minimizers of the minimizations of $R^b_n + \lambda \|\cdot\|_b$ and $R^w_n + \lambda \|\cdot\|_w$ over $B^b$ and $B^w$ can be estimated by the proximal gradient method, respectively, that is,

$$g_{k+1} \in \text{prox}_{\theta T^b_n, \lambda} (g_k) \quad \text{or} \quad g_{k+1} \in \text{prox}_{\theta T^w_n, \lambda} (g_k), \quad k = 0, 1, \ldots,$$

where prox represents the proximal operator and $\theta > 0$ is a step size. Under the same hypotheses, the minimizer of the minimization of $R_n + \lambda \|\cdot\|$ over $B$ can be estimated by the Douglas-Rachford splitting method, that is,

$$h^b_k \in \text{prox}_{\theta T^b_n, \lambda} (g_k), \quad h^w_k \in \text{prox}_{\theta T^w_n, \lambda} (2h^b_k - g_k), \quad g_{k+1} := g_k + \sigma (h^w_k - h^b_k), \quad k = 0, 1, \ldots,$$

where $\sigma > 0$ is a relaxation parameter. This shows that the composite algorithm can be written as an alternating iterative system composed of the both algorithms. In our future works, we will construct the composite algorithms to combine the data-driven and model-driven methods.
4. Theorems of Regularized Learning

In the following theorems, the conditions of \((\xi_n^*, y_n)\) and \(\mathbb{L}_n\) are consistent with Proposition 3.1 and we will study the properties of \(f^0\), \(f_n^\lambda\), and \(f_{nm}^\lambda\) including their existence, representation, and convergence.

4.1. Representer Theorems

We redo the generalized representer theorems in Banach spaces in [10] by the techniques of the unifying representer theorems for convex loss in [18].

Lemma 4.1. If \(\mathbb{L}_n\) is a lower semi-continuous multi-loss function, then \(S_\lambda^n (B)\) is nonempty and weakly* compact.

Specially, if \(\mathbb{L}_n\) is a convex multi-loss function which implies that \(R_n\) is convex by Proposition 3.1 (b), then \(S_\lambda^n (B)\) is also convex. Moreover, if \(\mathbb{L}_n\) is a convex multi-loss function and \(\Phi\) is strictly convex, then \(S_\lambda^n (B)\) has one element at most.

Theorem 4.2. If \(\mathbb{L}_n\) is a lower semi-continuous multi-loss function, then for any \(f_n^\lambda \in S_\lambda^n (B)\), there exists a parameter \(c_n \in \mathbb{R}^{N_n}\) such that

\[
\begin{align*}
(\text{i}) \ & c_n \cdot \xi_n^* \in \partial \| \cdot \| (f_n^\lambda), \quad (\text{ii}) \ & c_n \cdot \langle f_n^\lambda, \xi_n^* \rangle = \| f_n^\lambda \|.
\end{align*}
\]

The “\(\cdot\)” represents the dot product, for example, \(c \cdot \xi^* = \sum_{k=1}^N c_k \xi_k^*\), where \(c := (c_1, c_2, \cdots, c_N)\) and \(\xi^* := (\xi_1^*, \xi_2^*, \cdots, \xi_N^*)\).

Remark 4.1. The parameter \(c_n\) in Theorem 4.2 is dependent of the different minimizer \(f_n^\lambda\). If \(B_\ast\) is smooth, then [12, Proposition 5.4.2] guarantees that \(S_\lambda^n (B) = S_\lambda^n (U_{N_n})\), where \(U_{N_n}\) is the collection of all \(f \in B\) which holds \(c \cdot \xi_n^* \in \partial \| \cdot \| (f)\) for a \(c \in \mathbb{R}^{N_n}\). Specially, if \(B\) is a Hilbert space, then \(f_n^\lambda \cong \| f_n^\lambda \| c_n \cdot \xi_n^*\). If \(f_n^\lambda \neq 0\), then the characterizing subdifferential of norm in Equation (2.1) shows that \(f_n^\lambda/\| f_n^\lambda \| \in \partial \| \cdot \| (c_n \cdot \xi_n^*)\). Let \(E_n(a) := \{ \langle f, \xi_n^* \rangle : f \in \partial \| \cdot \| (a \cdot \xi_n^*)\}\) for any \(a \in \mathbb{R}^{N_n}\). Thus \(f_n^\lambda\) can be solved as follows:

\[
\begin{align*}
\text{(i)} \ & \text{Solving } r_n, v_n, a_n \text{ from the finite-dimensional optimization } \\
& \inf_{a \in \mathbb{R}^{N_n}} \mathbb{L}_n (\xi_n^*, y_n, r \cdot v) + \lambda \Phi (r) \text{ subject to } r \geq 0, v \in E_n(a). \\
\text{(ii)} \ & \text{Choosing } g_n \in \partial \| \cdot \| (a_n \cdot \xi_n^*) \text{ subject to } \langle g_n, \xi_n^* \rangle = v_n.
\end{align*}
\]
(iii) Taking $f_n^\lambda := r_ng_n$.

Since $c_n = a_n/\|a_n \cdot \xi_n^*\|$ and $r_n = c_n \cdot \langle f_n^\lambda, \xi_n^* \rangle$, we also have $\|a_n \cdot \xi_n^*\|_* = a_n \cdot v_n$. If $\mathbb{L}_n$ is a linear combination of convex loss functions as in Equation (2.9) and $\mathcal{B}$ is a 1-norm RKBS, the proximity algorithm can be used to solve $c_n$ and $f_n^\lambda$ as in [21, Theorem 64]. We will discuss many other numerical algorithms for the special loss functions and Banach spaces such as alternating direction methods of multipliers and composite optimization methods in our next paper.

If $\mathbb{L}_n$ is a convex multi-loss function, then the Krein-Milman theorem assures that $\mathcal{S}_n^\lambda(\mathcal{B})$ is the closed convex hull of its extreme points. If $\mathcal{B}$ is strictly convex, then for any $f_n^\lambda \neq 0$, the element $f_n^\lambda/\|f_n^\lambda\|$ is an extreme point of $B_{\mathcal{B}}$. Next, we will study that there exists a $\tilde{f}_n^\lambda$ such that it is a linear combination of finite many extreme points of $B_{\mathcal{B}}$.

**Corollary 4.3.** If $\mathbb{L}_n$ is a lower semi-continuous multi-loss function, then for any $f_n^\lambda \in \mathcal{S}_n^\lambda(\mathcal{B})$, there exists a $\tilde{f}_n^\lambda \in \mathcal{S}_n^\lambda(\mathcal{B})$ such that

(i) $\tilde{f}_n^\lambda = b_n \cdot e_n$,  
(ii) $\langle \tilde{f}_n^\lambda, \xi_n^* \rangle = \langle f_n^\lambda, \xi_n^* \rangle$,

where $M_n \leq N_n$, the parameter $b_n \in \mathbb{R}^{M_n}$, and $e_n := (e_{n1}, e_{n2}, \ldots, e_{nM_n})$ composes of the extreme points $e_{n1}, e_{n2}, \ldots, e_{nM_n}$ of $B_{\mathcal{B}}$.

The generalized representer theorems assure that the approximate solutions are computed by the numerical algorithms of the equivalent finite-dimensional optimizations. We can also check that the generalized representer theorems cover the classical representer theorems in RKHS and RKBS. More details of the representer theorems are mentioned in [10, 17, 18, 19, 21, 22, 25].

4.2. Pseudo-approximation Theorems

In the beginning, we look at the existence of $f^\lambda_{nm}$ solved by Optimization (3.5).

**Lemma 4.4.** Suppose that $\mathcal{U}_m$ is weakly* closed. If $\mathbb{L}_n$ is a lower semi-continuous multi-loss function, then $\mathcal{S}_n^\lambda(\mathcal{U}_m)$ is nonempty and weakly* compact.
Remark 4.2. If $U_m$ is not weakly* closed, then Optimization (3.5) or (3.6) may not exist a minimizer. Substituted $U_m$ into the weak* closure of $U_m$, we can reconstruct Optimization (3.5) to guarantee the existence of $f^A_{nm}$ for the substitution. Therefore, we can still solve Optimization (3.6) approximately to find the estimators of $f^A_{nm}$ by the calculus of variations even if Optimization (3.6) does not exist a minimizer.

We say that $\{U_m : m \in \mathbb{N}\}$ satisfies an universal approximation in $\mathcal{B}$ if for any $f \in \mathcal{B}$ and any $\epsilon > 0$, there exists a $g_m \in U_m$ such that $\|f - g_m\| \leq \epsilon$. The universal approximation also implies that if for any $f \in \mathcal{B}$, there exists a sequence $(g_m)$ in an increasing order $(m_k)$ such that $g_{m_k} \in U_{m_k}$ for all $k \in \mathbb{N}$ and $\|f - g_{m_k}\| \to 0$ when $k \to \infty$. This shows that the universal approximation for $U_m$ is the same as the classical universal approximation for reproducing kernels in [14]. Incidentally, $S^A_n(U_m)$ may have many elements. For each $m \in \mathbb{N}$, we actually need one minimizer from $S^A_n(U_m)$ to approximate $f^A_n$. Thus $f^A_{nm}$ will be an arbitrarily chosen element when $m$ is given, and the choices of $f^A_{nm}$ will not affect the conclusions of the pseudo-approximation theorems. Let $(f^A_{nm})$ be a net with a directed set $\mathbb{N}$ in its usual order, that is, $m_1 \succeq m_2$ if $m_1 \geq m_2$.

**Theorem 4.5.** Suppose that $U_m$ is weakly* closed for all $m \in \mathbb{N}$. Further suppose that $\cap_{m \in \mathbb{N}} U_m$ is nonempty and $\{U_m : m \in \mathbb{N}\}$ satisfies an universal approximation in $\mathcal{B}$. If $\mathbb{L}_n$ is a continuous multi-loss function, then there exists a weakly* convergent bounded subnet $(f^A_{nm_a})$ of $(f^A_{nm})$ to an element $f^A_n \in S^A_n(\mathcal{B})$ such that

$$(i) \quad R_n(f^A_n) + \lambda \Phi(\|f^A_n\|) = \lim_{\alpha} R_n(f^A_{nm_a}) + \lambda \Phi(\|f^A_{nm_a}\|),$$

and

$$(ii) \quad \|f^A_n\| = \lim_{\alpha} \|f^A_{nm_a}\|.$$
(I) $R_n$ converges pointwise to $R$ when $n \to \infty$,

(II) $\mathfrak{F}_D$ is relatively compact in $\mathcal{B}$, and $\mathcal{L}$ is uniformly local Lipschitz continuous.

Condition (I) is dependent of $R$ while Condition (II) is independent of $R$. Roughly speaking, Condition (I) is seen as the assumption of natural science and Condition (II) is seen as the assumption of data. In practical applications, Condition (I) can be checked by the test functions and Condition (II) can be checked by the linear-functional data and loss functions even if $R$ is unknown or unformulated. According to Proposition 3.1 (c), Condition (II) implies that

(II') $\{R_n : n \in \mathbb{N}\}$ is weakly* equicontinuous on $rB_B$ for all $r > 0$.

By a similar argument in Remark 2.2, Condition (II') can be seen as the intermediate condition of equicontinuity and weak* equicontinuity. On substituting Condition (II) into (II'), the convergence theorems still hold true by the argument in Remark 5.2.

Since the local Lipschitz continuity of $\mathbb{L}_n$ implies the lower semi-continuity of $\mathbb{L}_n$, Lemma 4.1 assures that $\mathcal{S}^\lambda_n(\mathcal{B})$ is nonempty for all $n \in \mathbb{N}$ and all $\lambda > 0$. Incidentally, $\mathcal{S}^\lambda_n(\mathcal{B})$ may have many elements. For each $n \in \mathbb{N}$ and each $\lambda > 0$, we actually need one minimizer from $\mathcal{S}^\lambda_n(\mathcal{B})$ to approximate $f^0$. Thus $f^\lambda_n$ will be an arbitrarily chosen element when $n, \lambda$ are given, and the choices of $f^\lambda_n$ will not affect the conclusions of the convergence theorems. Let $(f^\lambda_n)$ be a net with a directed set $\mathbb{N} \times \mathbb{R}_+$ such that $(n_1, \lambda_1) \succeq (n_2, \lambda_2)$ if $n_1 \geq n_2$ and $\lambda_1 \leq \lambda_2$. Thus $\lim_{(n,\lambda)} (n, \lambda) = (\infty, 0)$.

**Theorem 4.6.** Suppose that Conditions (I) and (II) hold true. If $\mathcal{S}^0 \{\mathcal{B}\}$ is nonempty, then there exists a weakly* convergent bounded subnet $(f^\lambda_n^\alpha)$ of $(f^\lambda_n)$ to an element $f^0 \in \mathcal{S}^0 \{\mathcal{B}\}$ such that

\[
\begin{align*}
(i) \ R(f^0) &= \lim_{\alpha} R(f^\lambda_n^\alpha), \quad (ii) \ \|f^0\| &= \lim_{\alpha} \|f^\lambda_n^\alpha\|, \\
and \quad (iii) \ \|f^0\| &= \inf \left\{\|\tilde{f}^0\| : \tilde{f}^0 \in \mathcal{S}^0 \{\mathcal{B}\}\right\}.
\end{align*}
\]

**Remark 4.3.** Lemma 5.6 also assures that

\[
R(f^0) = \lim_{\alpha} R_n (f^\lambda_n^\alpha).
\]
By Lemma 2.1, it is easily seen that
\[ \lim_{\alpha} \sup_{\xi^* \in F} \langle f^0 - f^\lambda_n, \xi^* \rangle = 0, \]
where \( F \) is any relatively compact set in \( B_* \). The (iii) also shows that \( f^0 \) is the minimum-norm minimizer of Optimization (3.1). This indicates that the “best” approximate solution could be computed by the regularized learning.

Next, we consider any weakly* convergent bounded subnet of \((f^\lambda_n)\).

**Theorem 4.7.** Suppose that Conditions (I) and (II) hold true. If \((f^\lambda_n)\) is a weakly* convergent bounded subnet of \((f^\lambda_n)\) to an element \( f^0 \), then \( f^0 \in S^0(B) \) such that
\[ (i) \quad R(f^0) = \lim_{\alpha} R(f^\lambda_n). \]
Moreover, if
\[ \lim_{\alpha} \frac{1}{\lambda_n} \sup_{\xi^* \in F_D} \langle f^0 - f^\lambda_n, \xi^* \rangle = 0, \quad (4.1) \]
then
\[ (ii) \quad \| f^0 \| = \lim_{\alpha} \| f^\lambda_n \|. \]

**Remark 4.4.** If \( B_* \) is complete, then any weakly* convergent sequence of \((f^\lambda_n)\) is bounded and thus the condition of boundedness in Theorem 4.7 is not needed when the subnet in Theorem 4.7 is a sequence. Since
\[ \lim_{\alpha} \sup_{\xi^* \in F_D} \langle f^0 - f^\lambda_n, \xi^* \rangle = 0, \quad \lim_{\alpha} \lambda_n = 0, \]
Equation (4.1) shows that the (ii) holds true when the regularization parameters are not faster decreasing than the errors at the data.

As in the proof of Theorems 4.6 and 4.7, if \( B_* \) is separable or \( B \) is reflexive, then \( B_B \) is weakly* sequentially compact and thus the weakly* convergent subnets discussed above can be substituted into the weakly* convergent sequences. Finally, we look at a special sequence \((f^\lambda_n)\). We will study the adaptive regularization parameter \( \lambda \) with the nice property of Optimization (3.1). If \( R(f^0) = 0 \), then the pointwise convergence in Equation (3.3) shows that \( R_n(f^0) \to 0 \) when \( n \to \infty \), and thus there exists a decrease sequence \((\lambda_n)\) to 0 such that \( \lambda_n^{-1} R_n(f^0) \to 0 \) when \( n \to \infty \). Obviously, the sequence \((f^\lambda_n)\) is a subnet of \((f^\lambda_n)\).
Theorem 4.8. Suppose that Conditions (I) and (II) hold true. If $\mathcal{S}^0(B) = \{f^0\}$ and $(\lambda_n)$ is a decrease sequence to 0 such that
\[
\lim_{n \to \infty} \frac{R_n(f^0)}{\lambda_n} = 0,
\] (4.2)
then $(f_{\lambda_n}^n)$ is a weakly* convergent bounded sequence to $f^0$ such that
\[
(i) \ R(f^0) = \lim_{n \to \infty} R(f_{\lambda_n}^n), \quad (ii) \ \|f^0\| = \lim_{n \to \infty} \|f_{\lambda_n}^n\|.
\]

Remark 4.5. If $B$ is a reflexive Radon-Riesz space, then Theorem 4.8 also assures that
\[
\lim_{n \to \infty} \|f^0 - f_{\lambda_n}^n\| = 0.
\]
Equation (4.2) implies that $R_n(f^0) \to 0$ when $n \to \infty$ and thus $R(f^0) = 0$.

In the proof of Theorem 4.8, the condition of Equation (4.2) is only used to guarantee the boundedness of $(f_{\lambda_n}^n)$ and thus these condition is not needed if $(f_{\lambda_n}^n)$ is bounded.

5. Proof of Theorems

In the beginning, we prove Lemmas 2.1, 2.2, 2.3, and 5.1. In this article, some lemmas may be already done in another monographs, but they are not the same exactly as in the references. Due to the strict discussions, we will prove all lemmas by ourselves. We mention again that $C(B_B)$ is a continuous function space defined on $B_B$ with respect to the relative weak* topology.

Proof of Lemma 2.1. Since $\mathfrak{F}$ are linear functionals, it is obvious that (a) and (b) are equivalent. Next, if we prove that (a) and (c) are equivalent, then the proof is complete. Since $\|\xi^*\|_* = \sup_{f \in B_B} |\langle f, \xi^* \rangle|$ for $\xi^* \in \mathfrak{F}$, $\mathfrak{F}$ is relatively compact in $B_*$ if and only if $\mathfrak{F}$ is relatively compact in $C(B_B)$. Moreover, $\mathfrak{F}$ is weakly* equicontinuous on $B_B$ if and only if $\mathfrak{F}$ is equicontinuous in $C(B_B)$. If $\mathfrak{F}$ is equicontinuous in $C(B_B)$, [15, Corollary 4.1] assures that $\mathfrak{F}$ is bounded in $C(B_B)$ by its linearity. Thus the Arzelá-Ascoli theorem assures that $\mathfrak{F}$ is equicontinuous in $C(B_B)$ if and only if $\mathfrak{F}$ is relatively compact in $C(B_B)$. This shows that (a) and (c) are equivalent. \hfill $\square$

Proof of Lemma 2.2. By Lemma 2.1, we only need to prove that $\{\delta_{x_n} : n \in \mathbb{N}\}$ is weakly* equicontinuous on $B_B$. We take any weak* convergent net $(f_{\alpha}) \subseteq B_B$ to an element $f_0 \in B_B$. By the embedding property, we have
\[
|\langle f, \delta_{x} - \delta_{z} \rangle| = |f(x) - f(z)| \leq \|f\|_{C^0(\Omega)} \|x - z\|_2^0 \leq C\|f\| \|x - z\|_2^0, \quad (5.1)
\]
for any $x, z \in \Omega$ and any $f \in \mathcal{B}$, where the constant $C > 0$ is independent of $x, z, f$. Next, we take any $\epsilon > 0$. Let $\theta := (\epsilon/(3C))^{1/\vartheta}$. The collection of closed balls $z + \theta B_{\mathbb{R}^d}$ for all $z \in \Omega$ forms a cover of $\Omega$. Since $\Omega$ is compact, this cover has a finite subcover $z_k + \theta B_{\mathbb{R}^d}$ for $k \in \mathbb{N}_N$. For any $n \in \mathbb{N}$, there exists a $k_n \in \mathbb{N}_N$ such that $x_n - z_{k_n} \leq \theta B_{\mathbb{R}^d}$ and $f_0, f_\alpha \in \mathcal{B}$, Equation (5.1) shows that

\[
\langle f_0, \delta x_n - \delta z_{kn} \rangle \leq C \theta \leq \frac{\epsilon}{3}, \quad \langle f_\alpha, \delta x_n - \delta z_{kn} \rangle \leq C \theta \leq \frac{\epsilon}{3}.
\]  

(5.2)

Moreover, since $\langle f_0, \delta z_k \rangle = \lim_\alpha \langle f_\alpha, \delta z_k \rangle$ for all $k \in \mathbb{N}_N$, there exists a $\gamma$ such that

\[
\langle f_\alpha - f_0, \delta z_k \rangle \leq \frac{\epsilon}{3} \quad \text{for all } k \in \mathbb{N}_N, \quad \text{when } \alpha \geq \gamma.
\]  

(5.3)

We conclude from Equations (5.2) and (5.3) that

\[
\limsup_{\alpha \rightarrow n \in \mathbb{N}} \langle f_0 - f_\alpha, \delta x_n \rangle = 0.
\]

This shows that $\langle f_0 - f_\alpha, \delta x_n \rangle$ is relatively compact in $\mathcal{B}_s$. Therefore, Lemma 2.1 assures that $\{\delta x_n : n \in \mathbb{N}\}$ is relatively compact in $\mathcal{B}_s$. □

**Proof of Lemma 2.3.** Since $L_n(\xi_n, y_n, \cdot)$ is a linear combination of $L(\xi_1, y_1, \cdot), \ L(\xi_2, y_2, \cdot), \ldots, \ L(\xi_N, y_N, \cdot)$, the proof of (a) is straightforward. As easy computation shows that $C_n = C_\theta$ for all $n \in \mathbb{N}$, where $C_\theta$ is the local Lipschitz constant of $L$ for any $\theta > 0$. Thus (b) holds true. □

It is obvious that $\Phi(r_1) < \Phi(r_2)$ if and only if $r_1 < r_2$ for any $r_1, r_2 \in [0, \infty)$.  

**Lemma 5.1.** For any net $(r_\alpha) \subseteq [0, \infty)$ and $r_0 \in [0, \infty)$, $\Phi(r_0) = \lim_\alpha \Phi(r_\alpha)$ if and only if $r_0 = \lim_\alpha r_\alpha$.  

**Proof.** Since $\Phi$ is strictly increasing and continuous, the inverse of $\Phi$ exists and $\Phi^{-1}$ is also strictly increasing and continuous. Thus the proof is straightforward. □
5.1. Proof of Properties of $R_n$

Now we prove Proposition 3.1 when $R_n$ has the formula as in Equation (3.2).

**Proof of Proposition 3.1.** Since

$$L_n(\xi^*_n, y_n, \langle f, \xi^*_n \rangle) = L_n(\xi^*_n, y_n, \langle f + h, \xi^*_n \rangle),$$
for all $f \in B$ and all $h \in \ker(\xi^*_n)$, the (a) holds true.

Next, we prove the (b). If $L_n$ is a lower semi-continuous multi-loss function, then $L_n(\xi^*_n, y_n, \cdot)$ is lower semi-continuous; hence it is easy to check that $R_n$ is weakly* lower-semi continuous. Similar arguments apply to the cases of continuity and convexity.

Moreover, we prove the (c). Let $(f_\alpha) \subseteq rB_B$ be any weakly* convergent net to an element $f_0 \in rB_B$ for a $r > 0$. Since $\mathfrak{F}_D$ is relatively compact in $B_\alpha$, there exists a $\varrho > 0$ such that $\|\xi^*\|_{r} \leq \varrho$ for all $\xi^* \in \mathfrak{F}_D$. Thus we have

$$|\langle f_0, \xi^* \rangle| \leq \|\xi^*\|_{r}\|f_0\| \leq \varrho r, \quad |\langle f_\alpha, \xi^* \rangle| \leq \|\xi^*\|_{r}\|f_\alpha\| \leq \varrho,$$
for all $\xi^* \in \mathfrak{F}_D$. We take $\theta := \varrho r$. Thus

$$\|\langle f_0, \xi^*_n \rangle\|_{\infty} \leq \theta, \quad \|\langle f_\alpha, \xi^*_n \rangle\|_{\infty} \leq \theta, \quad \text{for all } n \in \mathbb{N}.$$

Moreover, since $L$ is uniformly local Lipschitz continuous, there exists a constant $C_\theta := \sup_{n \in \mathbb{N}} C^\theta_n$ such that

$$|L_n(\xi^*_n, y_n, \langle f_0, \xi^*_n \rangle) - L_n(\xi^*_n, y_n, \langle f_\alpha, \xi^*_n \rangle)| \leq C_\theta \|\langle f_0, \xi^*_n \rangle - \langle f_\alpha, \xi^*_n \rangle\|_{\infty},$$
for all $(\xi^*_n, y_n) \in D$, where $C^\theta_n$ is the local Lipschitz constant of $L_n$. Thus

$$|R_n(x_0) - R_n(x_\alpha)| \leq C_\theta \|\langle f_0, \xi^*_n \rangle - \langle f_\alpha, \xi^*_n \rangle\|_{\infty}. \quad (5.4)$$

Since $\mathfrak{F}_D$ is weakly* equicontinuous on $(f_\alpha) \subseteq rB_B$ by Lemma 2.1, we have

$$\limsup_{\alpha \to n} |R_n(f_0) - R_n(f_\alpha)| \leq C_\theta \limsup_{\alpha \to \xi^*_n} |\langle f_0, \xi^*_n \rangle - \langle f_\alpha, \xi^*_n \rangle| = 0.$$

This shows that $\{R_n : n \in \mathbb{N}\}$ is weakly* equicontinuous on $rB_B$.

Now we prove the (d). According to the (c), $\{R_n : n \in \mathbb{N}\}$ is also equicontinuous in $C(rB_B)$. By the pointwise convergence in Equation (3.3), the Arzelá-Ascoli theorem assures that $R_n$ uniformly converges to $R$ on $rB_B$.

Finally, we prove the (e). According to Equation (5.4), we also have

$$\lim_{\alpha \to n} \frac{1}{\lambda_\alpha} \sup_{x_\alpha} |R_n(f_0) - R_n(f_\alpha)| \leq C_\theta \lim_{\alpha \to \xi^*_n} \frac{1}{\lambda_\alpha} \sup_{\xi^*_n} |\langle f_0 - f_\alpha, \xi^*_n \rangle| = 0.$$

$\square$
5.2. Proof of Representer Theorems

In this subsection, we fix $n, \lambda$ in the proof of Lemma 4.1, Theorem 4.2, and Corollary 4.3. Since $\mathbb{L}_n$ is a lower semi-continuous multi-loss function, Proposition 3.1 (b) assures that $R_n$ is weakly* lower semi-continuous.

**Proof of Lemma 4.1.** We first prove that $S^\lambda_n (B)$ is nonempty. Since $\Phi(r) \rightarrow \infty$ when $r \rightarrow \infty$, we have $R_n(f) + \lambda \Phi (\|f\|) \rightarrow \infty$ when $\|f\| \rightarrow \infty$. Thus there exists a $r > 0$ such that

$$\inf_{f \in B} R_n(f) + \lambda \Phi (\|f\|) = \inf_{f \in rB} R_n(f) + \lambda \Phi (\|f\|).$$

Since $rB$ is weakly* compact and $R_n + \lambda \Phi (\|\cdot\|)$ is weakly* lower semi-continuous, the Weierstrass extreme value theorem assures that $R_n + \lambda \Phi (\|\cdot\|)$ attains a global minimum on $rB$ and thus on $B$. Therefore $S^\lambda_n (B) \neq \emptyset$.

Next, for any $f \in S^\lambda_n (B)$, we have

$$\lambda \Phi (\|f\|) \leq R_n(f) + \lambda \Phi (\|f\|) \leq R_n(0) + \lambda \Phi (0),$$

and thus

$$\|f\| \leq \Phi^{-1} \left( \frac{R_n(0)}{\lambda} + \Phi(0) \right).$$

This shows that $S^\lambda_n (B)$ is bounded. Therefore, if we prove that $S^\lambda_n (B)$ is weakly* closed, then the Banach-Alaoglu theorem assures that $S^\lambda_n (B)$ is weakly* compact. Let $(f_\alpha) \subseteq S^\lambda_n (B)$ be any weakly* convergent net to an element $f_0 \in B$. Thus

$$\liminf_{\alpha} R_n(f_\alpha) + \lambda \Phi (\|f_\alpha\|) = \inf_{f \in B} R_n(f) + \lambda \Phi (\|f\|).$$

Since $R_n + \lambda \Phi (\|\cdot\|)$ is weakly* lower semi-continuous, we have

$$R_n(f_0) + \lambda \Phi (\|f_0\|) \leq \liminf_{\alpha} R_n(f_\alpha) + \lambda \Phi (\|f_\alpha\|).$$

Thus

$$R_n(f_0) + \lambda \Phi (\|f_0\|) \leq \inf_{f \in B} R_n(f) + \lambda \Phi (\|f\|).$$

This shows that $f_0 \in S^\lambda_n (B)$. Therefore $S^\lambda_n (B)$ is weakly* closed. \qed
Next, we will prove Theorem 4.2 by Lemmas 5.2 and 5.3. For any \( t \in \mathbb{R}^n \), we denote an affine space

\[
\mathcal{A}(t) := \{ f \in \mathcal{B} : \langle f, \xi^*_n \rangle = t \},
\]

where \( \xi^*_n \) is given in Theorem 4.2. We look at the optimization

\[
\inf_{f \in \mathcal{A}(t)} \| f \|. \tag{5.5}
\]

Let \( \mathcal{I}(t) \) be the collection of all minimizers of Optimization (5.5). Obviously, if \( \mathcal{A}(t) \neq \emptyset \), then \( \mathcal{I}(t) \neq \emptyset \).

**Lemma 5.2.** If \( \mathcal{A}(t) \) is nonempty for a fixed \( t \in \mathbb{R}^n \), then for any \( f_t \in \mathcal{I}(t) \), there exists a \( c_n \in \mathbb{R}^n \) such that

\[
\begin{align*}
(i) \quad c_n \cdot \xi^*_n &\in \partial \| f \| (f_t), \\
(ii) \quad c_n \cdot \langle f_t, \xi^*_n \rangle &= \| f_t \|. 
\end{align*}
\]

**Proof.** We take any \( f_t \in \mathcal{I}(t) \). If \( f_t = 0 \), then 0 \( \in \partial \| f \| (f_t) \). The proof is straightforward by \( c_n := 0 \). Next, we assume that \( f_t \neq 0 \). Thus \( f_t \in \mathcal{A}(t) \). If we prove that the (i) holds true, then Equation (2.1) shows that the (ii) holds true. Thus the proof is completed by showing that an element of \( \partial \| f \| (f_t) \) is a linear combination of \( \xi^*_1, \xi^*_2, \ldots, \xi^*_N \). Since \( \mathcal{A}(t) - f_t = ker (\xi^*_n) \), Equation (2.3) shows that

\[
(\mathcal{A}(t) - f_t)^\perp = \text{span} \{ \xi^*_1, \xi^*_2, \ldots, \xi^*_N \}. \tag{5.6}
\]

We conclude from Equations (2.2) and (5.6) that

\[
-\partial \chi_{\mathcal{A}(t)} (f_t) = \text{span} \{ \xi^*_1, \xi^*_2, \ldots, \xi^*_N \}. 
\]

Therefore the Pshenichnyi-Rockafellar theorem ([26 Theorem 2.9.1]) assures that

\[
\partial \| f \| (f_t) \cap \text{span} \{ \xi^*_1, \xi^*_2, \ldots, \xi^*_N \} = \partial \| f \| (f_t) \cap (-\partial \chi_{\mathcal{A}(t)} (f_t)) \neq \emptyset.
\]

\[\square\]

**Lemma 5.3.** If \( f_0 \in S^\lambda_n (\mathcal{B}) \) and \( t := \langle f_0, \xi^*_n \rangle \), then \( f_0 \in \mathcal{I}(t) \) and \( \mathcal{I}(t) \subseteq S^\lambda_n (\mathcal{B}) \).
Thus, we assume that $f \in \mathcal{B}$. We conclude from Equations (5.7) and (5.9) that

$$R_n (f_t) = R_n (f_t + h) = R_n (f_0).$$

Since $f_0 \in \mathcal{S}_n^\lambda (\mathcal{B})$, we have

$$R_n (f_0) = R_n (f_t + \lambda \Phi (\|f_0\|)) \leq R_n (f_t) + \lambda \Phi (\|f_t\|).$$

Subtracting Equations (5.7) from (5.9), we have $\Phi (\|f_0\|) \leq \Phi (\|f_t\|)$ and thus $\|f_0\| \leq \|f_t\|$ by Lemma 5.1. This also shows that

$$\|f_0\| = \|f_t\|.$$  \hspace{1cm} (5.9)

We conclude from Equations (5.7) and (5.9) that $R_n (f_t) + \lambda \Phi (\|f_t\|) = R_n (f_0) + \lambda \Phi (\|f_0\|)$, hence that $f_t \in \mathcal{S}_n^\lambda (\mathcal{B})$, and finally that $\mathcal{I}(t) \subseteq \mathcal{S}_n^\lambda (\mathcal{B})$.

**Proof of Theorem 4.2.** Lemma 4.1 first assures that $\mathcal{S}_n^\lambda (\mathcal{B})$ is nonempty. We take any $f_n^\lambda \in \mathcal{S}_n^\lambda (\mathcal{B})$. Thus Lemma 5.3 assures that $f_n^\lambda \in \mathcal{I}(t)$, where $t := \langle f_n^\lambda, \xi_n^* \rangle$. Obviously $\mathcal{A}(t) \neq \emptyset$. Therefore Lemma 5.2 also assures that the (i) and (ii) hold true.

Finally, we will complete the proof of Corollary 4.3 by the techniques of [2, Theorem 3.1].

**Proof of Corollary 4.3.** If $f_n^\lambda = 0$, then we take $\tilde{f}_n^\lambda := f_n^\lambda$. Next, we assume that $f_n^\lambda \neq 0$. Let $t := \langle f_n^\lambda, \xi_n^* \rangle$ and $v := t/\|f_n^\lambda\|$. By Lemma 5.3, $f_n^\lambda \in \mathcal{I}(t)$ and thus $\mathcal{I}_n (v) = B_{\mathcal{B}} \cap \mathcal{A}(v) \neq \emptyset$. Let $f_0$ be an extreme point of $\mathcal{I}(v)$. Thus $\|f_n^\lambda\| \langle f_0, \xi_n^* \rangle = \langle f_n^\lambda, \xi_n^* \rangle$. Moreover, since the codimension of the affine space $\mathcal{I}(v)$ is at most $N_n$ and the convex set $B_{\mathcal{B}}$ is linear closed and linear bounded, [6, Main Theorem] assures that $f_0$ is a convex combination of at most $N_n + 1$ extreme points of $B_{\mathcal{B}}$. Thus $f_0$ is a linear combination of at most $N_n$ extreme points of $B_{\mathcal{B}}$. We take $\tilde{f}_n^\lambda := \|f_n^\lambda\| f_0$. Therefore $\tilde{f}_n^\lambda$ is a linear combination of the same extreme points of $B_{\mathcal{B}}$ as $f_0$. By Lemma 5.3, $\tilde{f}_n^\lambda \in \mathcal{I}(v) \subseteq \mathcal{S}_n^\lambda (\mathcal{B})$. The proof is complete.

**Remark 5.1.** In [10, Theorem 3.1], we deduce the generalized representer theorems for the loss functions of which the range is $[0, \infty]$. To simplify the proof of the convergence theorems, the range of $L_n$ is restrictive to $[0, \infty)$ here consistently. If the range of $\mathbb{L}_n$ is extended to $[0, \infty]$, then the above proof of the representer theorems still hold true.
5.3. Proof of Pseudo-approximation Theorems

In this subsection, we also fix $n, \lambda$ in the proof of Lemma 4.4 and Theorem 4.5.

**Proof of Lemma 4.4.** Since $U_m$ is weakly* closed, $U_m \cap rB_B$ is weakly* compact for any $r > 0$. Thus a slight change in the proof of Lemma 4.1 shows that the conclusions of Lemma 4.4 hold true.

**Lemma 5.4.** If $\cap_{m \in \mathbb{N}} U_m$ is nonempty, then there exists a $r > 0$ such that $\|f_{nm}^\lambda\| \leq r$ for all $m \in \mathbb{N}$.

**Proof.** We take a $h \in \cap_{m \in \mathbb{N}} U_m$. Since $f_{nm}^\lambda \in S_n^\lambda (U_m)$, we have

$$R_n(f_{nm}^\lambda) + \lambda \Phi(\|f_{nm}^\lambda\|) \leq R_n(h) + \lambda \Phi(\|h\|).$$

Thus

$$\|f_{nm}^\lambda\| \leq \Phi^{-1} \left( \frac{R_n(h) + 1}{\lambda} + \Phi(h) \right).$$

**Proof of Theorem 4.5.** Lemma 4.1 assures that $S_n^\lambda (B)$ is nonempty. Thus we take a $g_0 \in S_n^\lambda (B)$. The universal approximation assures that there exists a sequence $(g_{mk})$ in an increasing order $(m_k)$ such that

$$g_{mk} \in U_{mk}, \quad \text{for all } k \in \mathbb{N},$$

and

$$\lim_{k \to \infty} \|g_0 - g_{mk}\| = 0. \quad (5.10)$$

Since $L_n$ is a continuous multi-loss function, $R_n$ is weakly* continuous. Thus Equation (5.10) shows that

$$R_n(g_0) + \lambda \Phi(\|g_0\|) = \lim_{k \to \infty} R_n(g_{mk}) + \lambda \Phi(\|g_{mk}\|). \quad (5.11)$$

Next, Lemma 4.4 assures that $S_n^\lambda (U_m)$ is nonempty for all $m \in \mathbb{N}$. Thus $(f_{nm}^\lambda)$ is well-defined. Let $(f_{nm_k}^\lambda)$ be a net with a directed set $\mathbb{N}$ in its usual order, that is, $k_1 \geq k_2$ if $k_1 \geq k_2$. Thus $(f_{nm_k}^\lambda)$ is a subnet of $(f_{nm}^\lambda)$. Since $(f_{nm_k}^\lambda) \subseteq rB_B$ by Lemma 5.4, the Banach-Alaoglu theorem assures that there exists a weakly* convergent subnet $(f_{nm_{k_0}}^\lambda)$ of $(f_{nm_k}^\lambda)$ to an element $f_0 \in rB_B$. 

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Clearly \((f_{nmk}^\lambda)\) is also a subnet of \((f_{nm}^\lambda)\) and \(\lim_\alpha k_\alpha = \infty\). Moreover, we also have

\[
R_n (f_0) + \lambda \Phi (\|f_0\|) \leq \liminf_\alpha R_n (f_{nmk_\alpha}^\lambda) + \lambda \Phi (\|f_{nmk_\alpha}^\lambda\|). 
\]

Now we will prove that \(f_0 \in S_\lambda^n (B)\). Since \(g_{m_{k_\alpha}} \in U_{m_{k_\alpha}}\), we have

\[
\liminf_\alpha R_n (f_{nmk_\alpha}^\lambda) + \lambda \Phi(\|f_{nmk_\alpha}^\lambda\|) \leq \liminf_\alpha R_n (g_{m_{k_\alpha}}) + \lambda \Phi (\|g_{m_{k_\alpha}}\|). 
\]

Thus

\[
R_n (f_0) + \lambda \Phi (\|f_0\|) \leq \liminf_\alpha R_n (g_{m_{k_\alpha}}) + \lambda \Phi (\|g_{m_{k_\alpha}}\|). \tag{5.12} 
\]

We conclude from Equations (5.11) and (5.12) that

\[
R_n (f_0) + \lambda \Phi (\|f_0\|) \leq \inf_{f \in B} R_n (f) + \lambda \Phi (\|f\|), 
\]

hence that \(f_0\) is a minimizer of Optimization (3.4). Since

\[
R_n (g_0) = \lim_\alpha R_n (g_{m_{k_\alpha}}), \quad \Phi (\|g_0\|) = \lim_\alpha \Phi (\|g_{m_{k_\alpha}}\|), 
\]

and

\[
R_n (f_0) = \lim_\alpha R_n (f_{nmk_\alpha}^\lambda), \tag{5.13} 
\]

we have

\[
\lim_\alpha R_n (g_{m_{k_\alpha}}) - R_n (f_{nmk_\alpha}) + \lambda \Phi (\|g_{m_{k_\alpha}}\|) = R_n (g_0) - R_n (f_0) + \lambda \Phi (\|g_0\|). 
\]

It is easy to check that

\[
R_n (g_0) - R_n (f_0) + \lambda \Phi (\|g_0\|) = \lambda \Phi (\|f_0\|). 
\]

Thus

\[
\lim_\alpha R_n (g_{m_{k_\alpha}}) - R_n (f_{nmk_\alpha}) + \lambda \Phi (\|g_{m_{k_\alpha}}\|) = \lambda \Phi (\|f_0\|). 
\]

Since \(f_{nmk_\alpha} \in S_\lambda^n (U_{m_{k_\alpha}})\) and \(g_{m_{k_\alpha}} \in U_{m_{k_\alpha}}\), we have

\[
\lambda \limsup_\alpha \Phi (\|f_{nmk_\alpha}^\lambda\|) \leq \limsup_\alpha R_n (g_{m_{k_\alpha}}) - R_n (f_{nmk_\alpha}) + \lambda \Phi (\|g_{m_{k_\alpha}}\|). 
\]
Thus
\[ \lambda \limsup_{\alpha} \Phi(\|f_{nmk\alpha}^{\lambda}\|) \leq \lambda \Phi(\|f_0\|). \]

It is also easy to check that
\[ \Phi(\|f_0\|) \leq \liminf_{\alpha} \Phi(\|f_{nmk\alpha}^{\lambda}\|). \]

Therefore
\[ \Phi(\|f_0\|) = \lim_{\alpha} \Phi(\|f_{nmk\alpha}^{\lambda}\|). \quad (5.14) \]

Combining Equations (5.13) and (5.14), we have
\[ R_n(f_0) + \lambda \Phi(\|f_0\|) = \lim_{\alpha} R_n(f_{nmk\alpha}^{\lambda}) + \lambda \Phi(\|f_{nmk\alpha}^{\lambda}\|), \]

and Lemma 5.1 assures that
\[ \|f_0\| = \lim_{\alpha} \|f_{nmk\alpha}^{\lambda}\|. \]

Substituting \( f_{nmk\alpha}^{\lambda} \) and \( f_0 \) into \( f_{nmk\alpha}^{\lambda} \) and \( f_n^{\lambda} \), the (i) and (ii) hold true. \( \square \)

5.4. Proof of Convergence Theorems

In the beginning of the proof, we assume that Conditions (I) and (II) always hold true in this subsection. This shows that \( L_n \) is a continuous multi-loss function for any \( n \in \mathbb{N} \). Thus Lemma 4.1 assures that \( S_{\alpha}^{\lambda}(B) \) is nonempty for any \( n \in \mathbb{N} \) and any \( \lambda > 0 \). In the following, for any given \( n, \lambda \), we only choose a fixed \( f_n^{\lambda} \) from \( S_{\alpha}^{\lambda}(B) \).

**Lemma 5.5.** For any \( r > 0 \), \( R \) is weakly* continuous on \( rB \) and \( R_n \) uniformly converges to \( R \) on \( rB \) when \( n \to \infty \).

**Proof.** According to Conditions (I) and (II), analysis similar to that in the proof of Proposition 3.1 (d) shows that \( R \in C(rB) \) and
\[ \sup_{f \in rB} |R(f) - R_n(f)| \to 0, \quad \text{when} \ n \to \infty. \]

\( \square \)
Remark 5.2. By Proposition 5.2, the uniform convergence guarantees that \( R_n \) \( \Gamma \)-converges to \( R \) on \( rB \) when \( n \to \infty \) for any \( r > 0 \). Some lemmas can be proved by the techniques of the \( \Gamma \)-convergence such as Lemma 5.9. But, all convergent results of \( f_\alpha \) cannot be proved directly by the fundamental theorems of the \( \Gamma \)-convergence. To reduce the different concepts in the proof, the \( \Gamma \)-convergence is not discussed here. Even though Condition (II’) is weaker than Condition (II), we can still prove Lemma 5.5 when substituted into Condition (II’). Roughly speaking, we replace the strong condition of uniform convergence to the weak condition of pointwise convergence with the additional condition of the linear-functional data and the loss functions which can be checked independent of the original problems.

Lemmas 5.6 and 5.7 will be often used in the following proof. We first prove Lemma 5.6 by Lemma 5.5.

**Lemma 5.6.** Let \( f_0 \in rB \) and \( (n_\alpha, f_\alpha) \subseteq N \times rB \) be a net for a \( r > 0 \). Suppose that \( (n_\alpha) \) is also a subnet of \( (n) \), where \( (n) \) is a net with the directed set \( \mathbb{N} \) in its usual order. If \( (f_\alpha) \) weakly* converges to \( f_0 \), then

\[
R(f_0) = \lim_{\alpha} R_{n_\alpha}(f_\alpha).
\]

**Proof.** Since \( \lim_\alpha n_\alpha = \lim_n n = \infty \), Lemma 5.5 assures that

\[
\lim_\alpha |R(f_0) - R(f_\alpha)| = 0,
\]

and

\[
\lim_{\alpha} \sup_{\beta \in (\alpha)} |R(f_\beta) - R_{n_\alpha}(f_\beta)| = 0.
\]

We conclude from Equations (5.15) and (5.16) that

\[
R(f_0) = \lim_{\alpha} R(f_\alpha) = \lim_{\alpha} R_{n_\alpha}(f_\alpha).
\]

\(
\square
\)

**Lemma 5.7.** Let \( T \in \mathbb{R}^B \) and \((T_\alpha, f_\alpha) \subseteq \mathbb{R}^B \times B \) be a net. Suppose that \( T_\alpha \) converges pointwise to \( T \) on \( B \), \( f_\alpha \) is a minimizer of \( T_\alpha \) over \( B \) for all \( \alpha \), and the limitation of \( T_\alpha(f_\alpha) \) exists. If

\[
\inf_{f \in B} T(f) \leq \lim_{\alpha} T_\alpha(f_\alpha),
\]

then

\[
\inf_{f \in B} T(f) = \lim_{\alpha} T_\alpha(f_\alpha).
\]
Proof. We will complete the proof by contradiction. Let

$$\tau := \lim_{\alpha} T_\alpha (f_\alpha).$$

(5.17)

We assume that

$$\inf_{f \in B} T(f) < \tau.$$

Thus there exists a $$\rho > 0$$ such that

$$\inf_{f \in B} T(f) < \tau - 3\rho.$$

Moreover, there exists a $$g_\rho \in B$$ such that

$$T(g_\rho) < \inf_{f \in B} T(f) + \rho.$$ 

Therefore

$$T(g_\rho) < \tau - 2\rho.$$ 

This shows that

$$\lim_{\alpha} T_\alpha(g_\rho) = T_\beta(g_\rho) < \tau - 2\rho,$$

and finally that there exists a $$\gamma_1$$ such that

$$T_\alpha(g_\rho) < \tau - 2\rho, \quad \text{when } \alpha \geq \gamma_1.$$ 

(5.18)

Next, Equation (5.17) shows that there exists a $$\gamma_2$$ such that

$$\tau - \rho < T_\alpha (f_\alpha), \quad \text{when } \alpha \geq \gamma_2.$$ 

(5.19)

We take a $$\beta$$ such that $$\beta \geq \gamma_1$$ and $$\beta \geq \gamma_2$$. We conclude from Equations (5.18) and (5.19) that

$$T_\beta(g_\rho) < T_\beta(f_\beta) - \rho,$$

hence that

$$T_\beta(g_\rho) < T_\beta(f_\beta) = \inf_{f \in B} T_\beta(f).$$

This contradicts the fact of the infimum; therefore we must reject the assumption of the strict inequality. Thus the equality holds true.

The proof of Theorem 4.6 will be divided into three steps including Lemmas 5.9, 5.12, and 5.13. Thus their notations are consistent. In the proof of Lemmas 5.8 and 5.9 we fix $$\lambda > 0.$$

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Lemma 5.8. There exists a $N_{\lambda} \in \mathbb{N}$ such that $\|f_k^\lambda\| \leq r_\lambda$ for all $k \geq N_{\lambda}$, where $r_\lambda := \Phi^{-1}((R(0) + 1)/\lambda + \Phi(0))$.

Proof. Since $R_k(0) \to R(0)$ when $k \to \infty$, there exists a $N_{\lambda} \in \mathbb{N}$ such that $R_k(0) \leq R(0) + 1$ when $k \geq N_{\lambda}$. We conclude from $f_k^\lambda \in S_k^\lambda(\mathcal{B})$ that

$$\lambda \Phi(\|f_k^\lambda\|) \leq R_k(f_k^\lambda) + \lambda \Phi(\|f_k^\lambda\|) \leq R_k(0) + \lambda \Phi(0),$$

hence that

$$\|f_k^\lambda\| \leq \Phi^{-1}\left(\frac{R(0) + 1}{\lambda} + \Phi(0)\right),$$

and finally that $f_k^\lambda \in r_\lambda B_{\mathcal{B}}$ for all $k \geq N_{\lambda}$. 

Let $(f_k^\lambda)$ be a net with the directed set $\{k \in \mathbb{N} : k \geq N_{\lambda}\}$ in its usual order, that is, $k_1 \geq k_2$ if $k_1 \geq k_2$. Thus $\lim_k k = \infty$. Obviously, the directed set of $(f_k^\lambda)$ is dependent of the fixed $\lambda$. Let $S^\lambda(\mathcal{B})$ be the collection of all minimizers of the optimization

$$\inf_{f \in \mathcal{B}} R(f) + \lambda \Phi(\|f\|). \quad (5.20)$$

Lemma 5.9. There exists a weakly* convergent subnet $(f_{k_\alpha}^\lambda)$ of $(f_k^\lambda)$ to an element $f^\lambda \in S^\lambda(\mathcal{B})$ such that

(i) $R(f^\lambda) = \lim_{\alpha} R(f_{k_\alpha}^\lambda)$, (ii) $\|f^\lambda\| = \lim_{\alpha} \|f_{k_\alpha}^\lambda\|.$

Proof. Lemma 5.8 assures that $(f_k^\lambda) \subseteq r_\lambda B_{\mathcal{B}}$. Thus the Banach-Alaoglu theorem assures that there exists a weakly* convergent subnet $(f_{k_\alpha}^\lambda)$ of $(f_k^\lambda)$ to an element $f^\lambda \in r_\lambda B_{\mathcal{B}}$. This shows that $\lim_\alpha k_\alpha = \lim_k k = \infty$.

Next, we prove the (i) and (ii). Lemma 5.5 assures that the (i) holds true. Since $x_{k_\alpha}^\lambda \in S_{k_\alpha}^\lambda(\mathcal{B})$, we have

$$R_{k_\alpha}(f_{k_\alpha}^\lambda) + \lambda \Phi(\|f_{k_\alpha}^\lambda\|) \leq R_{k_\alpha}(f^\lambda) + \lambda \Phi(\|f^\lambda\|).$$

This shows that

$$\limsup_{\alpha} R_{k_\alpha}(f_{k_\alpha}^\lambda) + \lambda \Phi(\|f_{k_\alpha}^\lambda\|) \leq \limsup_{\alpha} R_{k_\alpha}(f^\lambda) + \lambda \Phi(\|f^\lambda\|). \quad (5.21)$$

Substituting $f_0$ and $(f_\alpha)$ into $f^\lambda$ and $(f_{k_\alpha}^\lambda)$, Lemma 5.6 assures that

$$R(f^\lambda) = \lim_\alpha R_{k_\alpha}(f_{k_\alpha}^\lambda). \quad (5.22)$$

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Moreover, Condition (I) shows that
\[ R(f^\lambda) = \lim_{\alpha} R_{k_\alpha}(f^\lambda). \] (5.23)

Subtracting Equations (5.22) and (5.23) from Equation (5.21), we have
\[ \limsup_{\alpha} \Phi(\|f^\lambda_k\|) \leq \Phi(\|f^\lambda\|). \]

It is easy to check that
\[ \Phi(\|f^\lambda\|) \leq \liminf_{\alpha} \Phi(\|x^\lambda_{k_\alpha}\|). \]

Thus
\[ \Phi(\|f^\lambda\|) = \lim_{\alpha} \Phi(\|f^\lambda_k\|). \] (5.24)

This shows that the (ii) holds true by Lemma 5.1.

Finally, we prove that \( f^\lambda \in S^\lambda(B) \). Combing Equations (5.22) and (5.24), we have
\[
\inf_{f \in B} R(f) + \lambda \Phi(\|f\|) \leq R(f^\lambda) + \lambda \Phi(\|f^\lambda\|) = \lim_{\alpha} R_{k_\alpha}(f^\lambda_k) + \lambda \Phi(\|f^\lambda_k\|).
\]

Therefore, Lemma 5.7 assures that
\[
\inf_{f \in B} R(f) + \lambda \Phi(\|f\|) = \lim_{\alpha} R_{k_\alpha}(f^\lambda_k) + \lambda \Phi(\|f^\lambda_k\|),
\]
on substituting \( T, T_\alpha, \) and \( f_\alpha \) into \( R + \lambda \Phi(\|\cdot\|), R_{k_\alpha} + \lambda \Phi(\|\cdot\|), \) and \( f^\lambda_{k_\alpha} \).

Let the function
\[ \Psi(\lambda) := \inf_{f \in B} R(f) + \lambda \Phi(\|f\|), \quad \text{for } \lambda \in [0, \infty). \]

Thus \( \Psi(0) = \inf_{f \in B} R(f) \). Since Lemma 5.9 assures that \( f^\lambda \in S^\lambda(B) \) for all \( \lambda > 0 \), we have
\[ \Psi(\lambda) = R(f^\lambda) + \lambda \Phi(\|f^\lambda\|), \quad \text{for all } \lambda > 0. \] (5.25)

**Lemma 5.10.** The \( \Psi \) is concave, continuous, and increasing.

**Proof.** Let \( \psi_f(\lambda) := R(f) + \lambda \Phi(\|f\|) \) for \( \lambda \geq 0 \). Thus \( \Psi = \inf_{f \in B} \psi_f \). Since \( \psi_f \) is affine linear and increasing for any fixed \( f \in B \), \( \inf_{f \in B} \psi_f \) is concave, continuous, and increasing. \( \Box \)
Lemma 5.11. If $S^0(B)$ is nonempty, then for any $\tilde{f}^0 \in S^0(B)$, we have
\[ \|f^\lambda\| \leq \|\tilde{f}^0\|, \quad \text{for all } \lambda > 0. \]

Proof. We take any $\tilde{f}^0 \in S^0(B)$. Assume on the contrary that there exist a $\lambda > 0$ such that
\[ \|\tilde{f}^0\| < \|f^\lambda\|. \]
Since $R(\tilde{f}^0) \leq R(f^\lambda)$ and $\Phi(\|\tilde{f}^0\|) < \Phi(\|f^\lambda\|)$, we have
\[ R(\tilde{f}^0) + \lambda \Phi(\|\tilde{f}^0\|) < R(f^\lambda) + \lambda \Phi(\|f^\lambda\|). \]
This contradicts the fact that $f^\lambda \in S^\lambda(B)$ and therefore we must reject the assumption. \qed

Let $(f^\lambda)$ be a net with the directed set $\mathbb{R}_+$ in its inverse order, that is, $\lambda_1 \geq \lambda_2$ if $\lambda_1 \leq \lambda_2$. Thus $\lim \lambda = 0$.

Lemma 5.12. There exists a weakly* convergent subnet $(f^{\lambda_\alpha})$ of $(f^\lambda)$ to an element $f_0 \in S^0(B)$ such that
\begin{enumerate}[label=(i),itemsep=0pt]
\item $R(f_0) = \lim_{\alpha} R(f^{\lambda_\alpha})$,
\item $\|f_0\| = \lim_{\alpha} \|f^{\lambda_\alpha}\|.$
\end{enumerate}

Proof. Since $S^0(B) \neq \emptyset$, Lemma 5.11 assures that there exists a $r_0 > 0$ such that $(f^\lambda) \subseteq r_0B_B$. Thus the Banach-Alaoglu theorem assures that there exists a weakly* convergent subnet $(f^{\lambda_\alpha})$ of $(f^\lambda)$ to an element $f_0 \in r_0B_B$. This shows that $\lim_{\alpha} \lambda_\alpha = \lim_{\lambda} \lambda = 0$.

Next, we prove that $f_0 \in S^0(B)$. Lemma 5.5 assures that
\[ R(f_0) = \lim_{\alpha} R(f^{\lambda_\alpha}). \quad (5.26) \]
Combining Equations (5.25) and (5.26), we have
\[ R(f_0) \leq \liminf_{\alpha} R(f^{\lambda_\alpha}) + \lambda_\alpha \Phi(\|f^{\lambda_\alpha}\|) = \liminf_{\alpha} \Psi(\lambda_\alpha). \]
Moreover, Lemma 5.10 assures that
\[ \lim_{\alpha} \Psi(\lambda_\alpha) = \lim_{\alpha} \Psi(\lambda_\alpha) = \Psi(0) = \inf_{f \in B} R(f). \]
Thus
\[ R(f_0) \leq \inf_{f \in B} R(f). \]
This shows that $R(f_0) = \inf_{f \in B} R(f)$.

Finally, we prove the (i) and (ii). Equation (5.26) shows that the (i) holds true. It is easy to check that

$$\|f_0\| \leq \liminf_{\alpha} \|x^{\lambda_\alpha}\|.$$  

Lemma 5.11 assures that

$$\limsup_{\alpha} \|f^{\lambda_\alpha}\| \leq \|f_0\|.$$  

Thus the (ii) holds true.

**Lemma 5.13.** If $S^0(B)$ is nonempty, then there exists a $f_0 \in S^0(B)$ and for any $\epsilon > 0$, there exists a $f^{\lambda_{n_\epsilon}} \in S^{\lambda_{n_\epsilon}}(B)$ for a $n_\epsilon \in \mathbb{N}$ and a $\lambda_\epsilon > 0$ such that

(i) $|R(f_0) - R(f^{\lambda_{n_\epsilon}})| < \epsilon$,  
(ii) $\|f_0\| - \|f^{\lambda_{n_\epsilon}}\| < \epsilon$.

**Proof.** Lemma 5.12 assures that there exists a $f_0 \in S^0(B)$ and for any $\epsilon > 0$, there exists a $f^{\lambda_{n_\epsilon}} \in (f^{\lambda_{n_\epsilon}})$ given in the proof of Lemma 5.12, that is, $f^{\lambda_{n_\epsilon}} \in S^{\lambda_{n_\epsilon}}(B)$ for a $\lambda_\epsilon > 0$, such that

$$|R(f_0) - R(f^{\lambda_{n_\epsilon}})| < \frac{\epsilon}{2}, \quad \|f_0\| - \|f^{\lambda_{n_\epsilon}}\| < \frac{\epsilon}{2}. \quad (5.27)$$

Next, for the fixed $\epsilon$ and $\lambda_\epsilon$, Lemma 5.9 assures that there exists a $f^{\lambda_{n_\epsilon}} \in (f^{\lambda_{n_\epsilon}})$ given in the proof of Lemma 5.9, that is, $f^{\lambda_{n_\epsilon}} \in S^{\lambda_{n_\epsilon}}(B)$ for a $n_\epsilon \in \mathbb{N}$, such that

$$|R(f^{\lambda_{n_\epsilon}}) - R(f^{\lambda_{n_\epsilon}})| < \frac{\epsilon}{2}, \quad \|f^{\lambda_{n_\epsilon}}\| - \|f^{\lambda_{n_\epsilon}}\| < \frac{\epsilon}{2} \quad (5.28)$$

Combining Equations (5.27) and (5.28), we complete the proof.

**Remark 5.3.** In Lemma 5.13, $f_0$ is independent of $\epsilon$, and $n_\epsilon, \lambda_\epsilon$ are dependent of $\epsilon$. The proof above gives more, namely we choose the pair $(n_\epsilon, \lambda_\epsilon)$ for each $\epsilon > 0$ such that $n_{\epsilon_1} \geq n_{\epsilon_2}, \lambda_{\epsilon_1} \leq \lambda_{\epsilon_2}$ when $\epsilon_1 \leq \epsilon_2$ and $n_\epsilon \to \infty, \lambda_\epsilon \to 0$ when $\epsilon \to 0$ in the proof of Theorem 4.6.

**Lemma 5.14.** If $S^0(B)$ is nonempty, then for any $\tilde{f}^0 \in S^0(B)$, we have

$$\|f_0\| \leq \|\tilde{f}^0\|,$$

where $f_0$ is given in Lemma 5.13.
Proof. Assume on the contrary that \( \| \tilde{f}^0 \| < \| f_0 \| \). Let \( \epsilon := \| f_0 \| - \| \tilde{f}^0 \| \). As in the proof of the (ii) of Lemma 5.13, there exists a \( \lambda_\epsilon > 0 \) such that \( \| f_0 \| - \| f^{\lambda_\epsilon} \| < \epsilon \). Thus \( \| \tilde{f}^0 \| = \| f_0 \| - \epsilon < \| f^{\lambda_\epsilon} \| \). Since \( \tilde{x}^0 \in S^0 (B) \), Lemma 5.11 assures that \( \| f^{\lambda_\epsilon} \| \leq \| \tilde{f}^0 \| \). Therefore it has a contradiction and we must reject the assumption. The proof is complete. \( \square \)

Now we will prove Theorem 4.6 by Lemma 5.13 and Lemma 5.14. Let \( (f^{\lambda_\epsilon}_{n_\epsilon}) \) be a net with a directed set \( \{ \epsilon \in \mathbb{R}_+ : \epsilon \leq 1 \} \) in its inverse order, that is, \( \epsilon_1 \geq \epsilon_2 \) if \( \epsilon_1 \leq \epsilon_2 \). Thus \( \lim \epsilon = 0 \). According to Remark 5.3, \( (f^{\lambda_\epsilon}_{n_\epsilon}) \) is a subnet of \( (f^{\lambda_\epsilon}_{n_\epsilon}) \) and \( \lim \epsilon (n_\epsilon, \lambda_\epsilon) = (\infty, 0) \).

Proof of Theorem 4.6. The (ii) of Lemma 5.13 assures that \( \| f^{\lambda_\epsilon}_{n_\epsilon} \| \leq \| f_0 \| + 1 \) for all \( \epsilon \leq 1 \). Let \( r_0 := \| f_0 \| + 1 \). Thus \( (f^{\lambda_\epsilon}_{n_\epsilon}) \subseteq r_0 B_S \) and the Banach-Alaoglu theorem assures that there exists a weakly* convergent subnet \( (f^{\lambda_\epsilon}_{n_\epsilon}) \) to an element \( f^0 \in r_0 B_S \). This shows that \( \lim \epsilon \alpha = \lim_\epsilon = 0 \) and \( (f^{\lambda_\epsilon}_{n_\epsilon}) \) is also a subnet of \( (f^{\lambda_\epsilon}_{n_\epsilon}) \) such that \( \lim_\alpha n_\epsilon = \infty \) and \( \lim_\alpha \lambda_\epsilon = 0 \).

Next, we prove that \( f^0 \in S^0 (B) \). We take any \( \delta > 0 \). Lemma 5.5 assures that there exists a \( \gamma_1 \) such that

\[
| R(f^0) - R(f^{\lambda_\epsilon}_{m_\epsilon}) | \leq \frac{\delta}{2}, \quad \text{when } \alpha \geq \gamma_1.
\]

Moreover, Lemma 5.13 assures that there exists a \( \gamma_2 \) such that

\[
| R(f_0) - R(f^{\lambda_\epsilon}_{m_\epsilon}) | \leq \epsilon_\alpha \leq \frac{\delta}{2}, \quad \text{when } \alpha \geq \gamma_2.
\]

Let \( \gamma_3 \) such that \( \gamma_3 \geq \gamma_1 \) and \( \gamma_3 \geq \gamma_2 \). Thus

\[
| R(f^0) - R(f_0) | \leq | R(f^0) - R(f^{\lambda_\epsilon}_{m_\epsilon}) | + | R(f^{\lambda_\epsilon}_{n_\epsilon}) - R(f_0) | \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.
\]

This shows that

\[
R(f^0) = R(f_0) = \inf_{f \in B} R(f).
\]

Now we prove the (i), (ii), and (iii). Lemma 5.5 assures that the (i) holds true. The (ii) of Lemma 5.13 assures that

\[
\lim_{\alpha} \| f_0 \| - \| f^{\lambda_\epsilon}_{n_\epsilon} \| \leq \lim_{\alpha} \epsilon_\alpha = 0.
\]
Thus
\[ \lim_{\alpha} \| f^{\lambda_{\alpha}}_{n_{\alpha}} \| = \| f_0 \|. \]

It is easy to check that
\[ \| f^0 \| \leq \liminf_{\alpha} \| f^{\lambda_{\alpha}}_{n_{\alpha}} \|. \]

Therefore
\[ \| f^0 \| \leq \| f_0 \|. \]

Lemma 5.14 also assures that
\[ \| f_0 \| \leq \| f^0 \|. \]

Thus
\[ \| f^0 \| = \| f_0 \| = \lim_{\alpha} \| f^{\lambda_{\alpha}}_{n_{\alpha}} \|. \]

This shows that the (ii) holds true. Finally, Lemma 5.14 assures that the
(iii) holds true. \qed

Next, we will prove Theorem 4.7 by the similar argument of Theorem 4.6.

**Proof of Theorem 4.7.** We first prove that \( f^0 \in \mathcal{S}^0 (B) \). Substituting \( f_0 \) and \((f_{n_{\alpha}})\) into \( f^0 \) and \((f^{\lambda_{\alpha}}_{n_{\alpha}})\), Lemma 5.6 assures that
\[ R(f^0) = \lim_{\alpha} R_{n_{\alpha}} (f^{\lambda_{\alpha}}_{n_{\alpha}}). \] (5.29)

Since \((f^{\lambda_{\alpha}}_{n_{\alpha}})\) is bounded, there exists a \( r_0 > 0 \) such that \((f^{\lambda_{\alpha}}_{n_{\alpha}}) \subseteq r_0 B_B\). Thus
\[ \lim_{\alpha} (n_{\alpha}, \lambda_{\alpha}) = \lim_{(n, \lambda)} (n, \lambda) = (\infty, 0). \]
It is easy to check that
\[ \lim_{\alpha} \lambda_{\alpha} \Phi (\| f^{\lambda_{\alpha}}_{n_{\alpha}} \|) \leq \lim_{\alpha} \lambda_{\alpha} \Phi (r_0), \]
and thus
\[ \lim_{\alpha} \lambda_{\alpha} \Phi (\| f^{\lambda_{\alpha}}_{n_{\alpha}} \|) = 0. \] (5.30)

Combining Equations (5.29) and (5.30), we have
\[ \inf_{f \in \mathcal{B}} R(f) \leq R(f^0) = \lim_{\alpha} R_{n_{\alpha}} (f^{\lambda_{\alpha}}_{n_{\alpha}}) + \lambda_{\alpha} \Phi (\| f^{\lambda_{\alpha}}_{n_{\alpha}} \|). \]

Therefore Lemma 5.7 assures that
\[ \inf_{f \in \mathcal{B}} R(f) = \lim_{\alpha} R_{n_{\alpha}} (f^{\lambda_{\alpha}}_{n_{\alpha}}) + \lambda_{\alpha} \Phi (\| f^{\lambda_{\alpha}}_{n_{\alpha}} \|) = R(f^0), \]

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on substituting $T$, $T_\alpha$, and $f_\alpha$ into $R_n \alpha + \lambda_\alpha \Phi (\| \cdot \|)$, and $f_n^{\lambda_\alpha}$.

Finally, we prove the (i) and (ii). Lemma 5.5 assures that the (i) holds true. Since $f_n^{\lambda_\alpha} \in S_n^{\lambda_\alpha} (\mathcal{B})$, we have
\[
R_n \alpha (f_n^{\lambda_\alpha}) + \lambda_\alpha \Phi (\| f_n^{\lambda_\alpha} \|) \leq R_n \alpha (f^0) + \lambda_\alpha \Phi (\| f^0 \|).
\]
Thus
\[
\Phi (\| f_n^{\lambda_\alpha} \|) \leq \frac{R_n \alpha (f^0) - R_n \alpha (f_n^{\lambda_\alpha})}{\lambda_\alpha} + \Phi (\| f^0 \|).
\]
This shows that
\[
\limsup_{\alpha} \Phi (\| f_n^{\lambda_\alpha} \|) \leq \limsup_{\alpha} \frac{R_n \alpha (f^0) - R_n \alpha (f_n^{\lambda_\alpha})}{\lambda_\alpha} + \Phi (\| f^0 \|) \text{ (5.31)}
\]
By the additional condition in Equation (4.1), Proposition 3.1 (e) assures that
\[
\lim_{\alpha} \frac{|R_n \alpha (f^0) - R_n \alpha (f_n^{\lambda_\alpha})|}{\lambda_\alpha} = 0 \text{ (5.32)}
\]
Subtracting Equations (5.32) from (5.31), we have
\[
\limsup_{\alpha} \Phi (\| f_n^{\lambda_\alpha} \|) \leq \Phi (\| f^0 \|) \text{ (5.33)}
\]
It is also easy to check that
\[
\Phi (\| f^0 \|) \leq \liminf_{\alpha} \Phi (\| f_n^{\lambda_\alpha} \|) \text{ (5.34)}
\]
This shows that
\[
\Phi (\| f^0 \|) = \lim_{\alpha} \Phi (\| f_n^{\lambda_\alpha} \|) \text{ (5.35)}
\]
Thus the (ii) holds true by Lemma 5.1.

Finally, we prove Theorem 4.8 by Theorem 4.7.

**Proof of Theorem 4.8** We first prove that $(f_n^{\lambda_\alpha})$ is bounded. Since $f_n^{\lambda_\alpha} \in S_n^{\lambda_\alpha} (\mathcal{B})$, we have
\[
\Phi (\| f_n^{\lambda_\alpha} \|) \leq \frac{R_n (f^0) - R_n (f_n^{\lambda_\alpha})}{\lambda_n} + \Phi (\| f^0 \|) \leq \frac{R_n (f^0)}{\lambda_n} + \Phi (\| f^0 \|). \text{ (5.33)}
\]
Combining Equation (5.33) and the additional condition in Equation (4.2), we have
\[
\limsup_{n \to \infty} \Phi (\| f_n^{\lambda_\alpha} \|) \leq \Phi (\| f^0 \|). \text{ (5.34)}
\]
Thus the strictly increasing of $\Phi$ shows that

$$\limsup_{n \to \infty} \| f_n^{\lambda_n} \| \leq \| f^0 \|. \quad (5.34)$$

Therefore, there exists a $r_0 > 0$ such that $(f_n^{\lambda_n}) \subseteq r_0 B_B$.

Next, we prove that $(f_n^{\lambda_n})$ is a weakly* convergent sequence to $f^0$. If we prove the statement, then Lemma 5.5 assures that the (i) holds true. The Banach-Alaoglu theorem assures that there exists a weakly* convergent subnet $(f_n^{\lambda_{n_{\alpha}}})$ of $(f_n^{\lambda_n})$ to an element $f_0 \in r_0 B_B$. It is obvious that $(f_n^{\lambda_{n_{\alpha}}})$ is also a subnet of $(f_n^{\lambda_n})$. Thus Theorem 4.7 assures that $f_0 \in S^0 (B) = \{ f^0 \}$. This shows that

$$f^0 = f_0 = w^* - \lim_{\alpha} x_{n_{\alpha}}^{\lambda_{n_{\alpha}}}. $$

Thus Lemma 5.5 assures that

$$R(f^0) = \lim_{\alpha} (f_n^{\lambda_{n_{\alpha}}}).$$

It is easy to check that

$$\| f^0 \| \leq \liminf_{\alpha} \| f_n^{\lambda_{n_{\alpha}}} \|. $$

Since $\lim_{n} n_{\alpha} = \lim_{n} n = \infty$, Equation (5.34) also shows that

$$\limsup_{\alpha} \| f_n^{\lambda_{n_{\alpha}}} \| \leq \| f^0 \|. $$

Thus

$$\| f^0 \| = \lim_{\alpha} \| f_n^{\lambda_{n_{\alpha}}} \|. $$

Applying the same argument to all subnets of $(f_n^{\lambda_n})$, we assert that

$$f^0 = w^* - \lim_{n \to \infty} f_n^{\lambda_n}, \quad \| f^0 \| = \lim_{n \to \infty} \| f_n^{\lambda_n} \|. $$

6. Examples of Regularized Learning

In the beginning, we look at an example of noisy data.
Example 6.1. Let $\Omega := [0, 1]$ and the Gaussian kernel

$$K(x, z) := \exp \left( -\theta^2 \left| x - z \right|^2 \right), \quad \text{for } x, z \in \Omega,$$

where the shape parameter $\theta > 0$. As in [22] Section 4.4, the RKBS $\mathcal{B}_K^1(\Omega)$ exists a predual space $(\mathcal{B}_K^1(\Omega))_* \cong B_{\infty}^1(\Omega)$ and $\mathcal{B}_K^1(\Omega)$ is embedded into the RKHS $\mathcal{H}_K(\Omega)$. As in [3] Example 5.8, $\mathcal{H}_K(\Omega)$ is embedded into the Sobolev space $\mathcal{H}^j(\Omega)$ of any order $j$. Thus the Sobolev embedding theorem assures that $\mathcal{B}_K^1(\Omega)$ is embedded into $C^{0,1}(\Omega)$.

We study the expected risk function

$$R(f) := \int_{\Omega} |f(x) - f^0(x)| \omega(x) dx, \quad \text{for } f \in \mathcal{B}_K^1(\Omega),$$

where the unknown function $f^0 \in \mathcal{B}_K^1(\Omega)$ and the positive weight $\omega \in C(\Omega)$. This shows that $R(f^0) = 0$ and $f^0$ is the unique minimizer of $R$ over $\mathcal{B}_K^1(\Omega)$, that is, $S^0(\mathcal{B}_K^1(\Omega)) = \{ f^0 \}$.

Next, we have the points $x_{nk} \in ((k - 1)/n, k/n)$ and the noise values $y_{nk} \in \mathbb{R}$ such that $|f^0(x_{nk}) - y_{nk}| \leq \zeta_{nk}$ for $k \in \mathbb{N}_n$ and $n \in \mathbb{N}$, where $\zeta_{nk} > 0$ and $\max_{k \in \mathbb{N}_n} \zeta_{nk} \to 0$ when $n \to \infty$. Thus there exists a $C > 0$ such that

$$\sup_{x \in \Omega} |f^0(x)| + \sup_{n \in \mathbb{N}} \max_{k \in \mathbb{N}_n} \zeta_{nk} \leq C. \quad \text{This shows that } f^0(x_{nk}), y_{nk} \in [-C, C].$$

By the reproducing property of $\mathcal{B}_K^1(\Omega)$, the classical noisy data

$$(x_{n1}, y_{n1}), (x_{n2}, y_{n2}), \ldots, (x_{nn}, y_{nn}) \in \Omega \times [-C, C], \quad \text{for all } n \in \mathbb{N},$$

are equivalently transferred to the data

$$(\delta_{x_{n1}}, y_{n1}), (\delta_{x_{n2}}, y_{n2}), \ldots, (\delta_{x_{nn}}, y_{nn}) \in \mathcal{B}_K^\infty(\Omega) \times [-C, C], \quad \text{for all } n \in \mathbb{N}.$$

Let

$$\xi_n := (\delta_{x_{n1}}, \delta_{x_{n2}}, \ldots, \delta_{x_{nn}}), \quad y_n := (y_{n1}, y_{n2}, \ldots, y_{nn}), \quad \text{for all } n \in \mathbb{N}.$$

Thus $\mathfrak{F}_D = \{ \delta_{x_{nk}} : k_n \in \mathbb{N}_n, n \in \mathbb{N} \}$ and Lemma 2.2 assures that $\mathfrak{F}_D$ is relatively compact in $\mathcal{B}_K^\infty(\Omega)$. Let the multi-loss function

$$L_n(\xi^*, y, t) := \frac{1}{n} \sum_{k=1}^n L(\xi^*_k, y_k, t_k), \quad \text{for } \xi^* \in \left( \mathcal{B}_K^\infty(\Omega) \right)^n, \quad y \in [-C, C]^n, \quad t \in \mathbb{R}^n,$$

where $n \in \mathbb{N}$ and the absolute loss

$$L(\xi^*, y, t) := \begin{cases} \left| t - y \right| \omega(x), & \text{if } \xi^* = \delta_x \text{ for } x \in \Omega \text{ and } y \in [-C, C], \\ 0, & \text{otherwise}, \end{cases}$$

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for $\xi^* \in \mathcal{B}_K^\infty(\Omega)$, $y \in [-C, C]$, $t \in \mathbb{R}$. Since $\omega$ is bounded on $\Omega$, $L$ is a local Lipschitz continuous loss function. Thus Lemma 2.3 (b) assures that $\mathcal{L}$ is uniformly local Lipschitz continuous. This shows that Condition (II) holds true.

Moreover, the empirical risk function is written as

$$R_n(f) = \frac{1}{n} \sum_{k=1}^{n} L(\delta_{x_{nk}}, y_{nk}, \langle f, \delta_{x_{nk}} \rangle) = \frac{1}{n} \sum_{k=1}^{n} |f(x_{nk}) - y_{nk}| \omega(x_{nk}),$$

for $f \in \mathcal{B}_K^1(\Omega)$. Thus the continuity of $|f - f^0| \omega$ and the boundedness of $\omega$ guarantee that $R_n$ converges pointwise to $R$ when $n \to \infty$. This shows that Condition (I) holds true.

Let $\Phi(r) := r$ for $r \in [0, \infty)$. Since the extreme point of $B_{B_k^1}(\Omega)$ has the formula as $\sqrt{\vartheta}\varphi$, Corollary 4.3 assures that there exists an approximate solution $f^\lambda_n \in \mathcal{S}_n^\lambda(B_K^1(\Omega))$ such that $f^\lambda_n$ is a linear combination of

$$\sqrt{\vartheta_1}\varphi_1, \sqrt{\vartheta_2}\varphi_2, \ldots, \sqrt{\vartheta_{M_n}}\varphi_{M_n},$$

where $M_n \leq N_n$ and $(\vartheta, \varphi)$ is a pair of the eigenvalue and eigenfunction of $K$.

Finally, we choose a decrease sequence $(\lambda_n)$ to $0$ such that

$$\lim_{n \to \infty} \frac{\max_{k \in \mathbb{N}_n} \zeta_{nk}}{\lambda_n} = 0.$$}

Since

$$R_n(f^0) = \frac{1}{n} \sum_{k=1}^{n} |f^0(x_{nk}) - y_{nk}| \omega(x_{nk}) \leq 2 \sup_{x \in \Omega} \omega(x) \max_{k \in \mathbb{N}_n} \zeta_{nk},$$

we have

$$\lim_{n \to \infty} \frac{R_n(f^0)}{\lambda_n} \leq 2 \sup_{x \in \Omega} \omega(x) \lim_{n \to \infty} \frac{\max_{k \in \mathbb{N}_n} \zeta_{nk}}{\lambda_n} = 0.$$

Thus Theorem 4.8 assures that $(f_n^\lambda)$ is a weakly* convergent bounded sequence to $f^0$. This shows that $f_n^\lambda$ converges pointwise to $f^0$ when $n \to \infty$. This result is consistent with the variational characterization of Tikhonov regularization.
In this article, we focus on the deterministic convergence. Actually the pointwise convergence as in Equation (3.3) can be extended to the stochastic convergence such as convergence almost surely or convergence in probability. In the same manner we can check the same conclusions of the convergence theorems for stochastic data. Now we look at a binary classification for stochastic data.

**Example 6.2.** Let $\Omega := [0, 1]^2$ and the min kernel

$$K(x, z) := \min \{v_1, w_1\} \min \{v_2, w_2\}, \text{ for } x = (v_1, v_2), z = (w_1, w_2) \in \Omega.$$ 

As in [22, Section 4.3], the RKBS $B^p_K(\Omega)$ is a reflexive Banach space and its dual space is isometrically isomorphic to the RKBS $B^q_K(\Omega)$, where $1 < p \leq 2$ and $q = p/(p - 1)$. As in [9, Example 5.1], the RKHS $H_K(\Omega)$ is equivalent to the Sobolev space $H^{1,1}_{mix}(\Omega)$ of order 1, 1. Since $B^p_K(\Omega)$ is embedded into $H_K(\Omega)$, the Sobolev embedding theorem assures that $B^p_K(\Omega)$ is embedded into $C^{0,1/2}(\Omega)$.

Given a probability distribution $\mathbb{P}$ on $\Omega \times \{\pm 1\}$, we study the expected risk function

$$R(f) := \int_{\Omega \times \{\pm 1\}} \max \{0, 1 - yf(x)\} \, d\mathbb{P}, \quad \text{for } f \in B^p_K(\Omega).$$

Thus the binary classifier is constructed by the minimization of $R$ over $B^p_K(\Omega)$.

Next, we look at the stochastic data $(x_n, y_n) \sim i.i.d. (x, y)$ for all $n \in \mathbb{N}$. By the reproducing property of $B^p_K(\Omega)$, the classical data $(x_n, y_n)$ are equivalently transferred to the data $(\delta_{x_n}, y_n)$. Let

$$\xi_n := (\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_n}), \quad y_n := (y_1, y_2, \ldots, y_n), \quad \text{for all } n \in \mathbb{N}.$$ 

Thus $\mathcal{F}_D = \{\delta_{x_n} : n \in \mathbb{N}\}$ and Lemma 2.2 assures that $\mathcal{F}_D$ is relatively compact in $B^q_K(\Omega)$. Let the multi-loss function

$$\mathbb{L}_n(\xi^*, y, t) := \frac{1}{n} \sum_{k=1}^n L(\xi^*_k, y_k, t_k), \quad \text{for } \xi^* \in (B^q_K(\Omega))^n, \ y \in \{\pm 1\}^n, \ t \in \mathbb{R}^n,$$

where $n \in \mathbb{N}$ and the hinge loss

$$L(\xi^*, y, t) := \begin{cases} \max \{0, 1 - yt\}, & \text{if } \xi^* = \delta_x \text{ for } x \in \Omega \text{ and } y \in \{\pm 1\}, \\ 0, & \text{otherwise}, \end{cases}$$

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for $\xi^* \in \mathcal{B}_K^q(\Omega)$, $y \in \{\pm 1\}$, $t \in \mathbb{R}$. Since $L$ is a local Lipschitz continuous loss function, Lemma 2.3 (b) assures that $L$ is uniformly local Lipschitz continuous. This shows that Condition (II) holds true.

Moreover, since $L(\delta_x, y_n, \langle f, \delta_x \rangle) \overset{\text{P}}{\sim} \text{i.i.d. max } \{0, 1 - yf(x)\}$, the strong law of large numbers shows that

$$R(f) = \lim_{n \to \infty} \frac{1}{n} L(\delta_x, y_n, \langle f, \delta_x \rangle) = \lim_{n \to \infty} R_n(f)$$

almost surely, for all $f \in \mathcal{B}^p_K(\Omega)$. This shows that Condition (I) holds true almost surely.

Let $\Phi(r) := r^p$ for $r \in [0, \infty)$. If $p = \frac{2l}{2l - 1}$ for a $l \in \mathbb{N}$, then Theorem 4.2 assures that there exists the unique approximate solution $f^{\lambda}_n \in S^{\lambda}_n(\mathcal{B}^p_K(\Omega))$ such that $
abla \|\cdot\| f^{\lambda}_n$ is a singleton including the unique element which is equivalent to a linear combination of $K(x_1, \cdot), K(x_2, \cdot), \ldots, K(x_n, \cdot)$ and thus $f^{\lambda}_n$ is a linear combination of the multi-kernel basis

$$K_{2l}\left(\cdot, x_{i_1}, x_{i_2}, \ldots, x_{i_{2l-1}}\right), \text{ for } i_1, i_2, \ldots, i_{2l-1} \in \mathbb{N},$$

which is computed by the eigenvalues and eigenfunctions of $K$, as in the proof of [22, Theorem 5.10].

In the same manner as in Section 5.4, we exclude the null set of the unconvergent $R_n$ to prove that Theorems 4.6, 4.7, and 4.8 still hold true almost surely. Since $\mathcal{B}^p_K(\Omega)$ is separable and complete, the weakly* convergent bounded subnet of $(f^{\lambda}_n)$ in Theorem 4.7 can be interchanged to a weakly* convergent sequence $(f^{\lambda}_j)$. Since $\text{span } \{\delta_x : x \in \Omega\}$ is dense in $\mathcal{B}^p_K(\Omega)$, $(f_j)$ weakly* converges to $f_0$ when $j \to \infty$ if and only if $(f_j)$ converges pointwise to $f_0$ when $j \to \infty$, where $(f_j)$ is a sequence of $\mathcal{B}^p_K(\Omega)$ and $f_0 \in \mathcal{B}^p_K(\Omega)$. In practical applications, if we check that a stochastic sequence $(f^{\lambda}_{n_j})$ converges pointwise to a deterministic $f^0 \in \mathcal{B}^p_K(\Omega)$ when $j \to \infty$ almost surely, then Theorem 4.7 assures that $f^0 \in S^0(\mathcal{B}^p_K(\Omega))$ and thus the binary classifier is constructed by sign $(f^0(x))$ or sign $(f^0_{n_j}(x))$ approximately for $x \in \Omega$.

Remark 6.1. For the binary classification, we can still obtain the same results as in Example 6.2 when the hinge loss is interchanged to many other loss functions such as truncated least squares loss and logistic loss, and even nonconvex loss functions. By Zhang’s inequality in [17, Theorem 2.31], the exact binary classifier is the Bayes classifier $\text{sign } (2\omega(x) - 1)$, where $\omega(x) := \mathbb{P}(y = 1|x) \text{ for } x \in \Omega$. Unfortunately $\omega$ is usually unknown. Since $C_0(\mathbb{R}^d)$ is a predual space of $\mathcal{M}(\mathbb{R}^d)$, we will use the techniques of total variations to
construct the approximate classifiers by the nonconvex and nonsmooth loss when $2\omega - 1 \in \mathcal{M}(\mathbb{R}^d)$ in our next paper.

Next, we look at a special example of two kinds of linear-functional data induced by a Poisson equation.

**Example 6.3.** Let $\Omega := [0, 1]^2$ and the Sobolev kernel

$$K(x, z) := \theta^{j-1}\|x - z\|_2^{j-1}K_{1-j}(\theta\|x - z\|_2), \quad \text{for } x, z \in \Omega,$$

where $\theta > 0$, $j \geq 4$, and $K_{1-j}$ is the modified Bessel function of the second kind of order $1-j$. The RKHS $\mathcal{H}_K(\Omega)$ is a reflexive Banach space. As in [8, Example 5.7], $\mathcal{H}_K(\Omega)$ is equivalent to the Sobolev space $\mathcal{H}^j(\Omega)$ of order $j$. By the Sobolev embedding theorem, $\mathcal{H}_K(\Omega)$ is embedded into $C^{2,1}(\Omega)$ and $C^{0,1}(\partial \Omega)$, respectively.

We study a Poisson equation with Dirichlet boundary, that is,

$$\Delta u = h \text{ in } \Omega, \quad u = g \text{ on } \partial \Omega,$$

where $h \in C(\Omega)$ and $g \in C(\partial \Omega)$ are given such that the Poisson equation exists the unique solution $f^0 \in \mathcal{H}^j(\Omega) \cong \mathcal{H}_K(\Omega)$. Let the expected risk function

$$R(f) := \frac{1}{2} \int_{\Omega} |\Delta f(x) - h(x)|^2 \, dx + \frac{1}{2} \int_{\partial \Omega} |f(z) - g(z)|^2 \, dS, \quad \text{for } f \in \mathcal{H}_K(\Omega).$$

Thus $R(f^0) = 0$ and $\mathcal{S}^0(\mathcal{H}_K(\Omega)) = \{f^0\}$.

Next, we look at two kinds of linear-functional data induced by the Poisson equation, that is,

$$(\delta_{x_{11}} \circ \Delta, v_{n1}), (\delta_{x_{n2}} \circ \Delta, v_{n2}), \ldots, (\delta_{x_{nn}} \circ \Delta, v_{nn}),$$

and

$$(\delta_{z_{n1}}, b_{n1}), (\delta_{z_{n2}}, b_{n2}), \ldots, (\delta_{z_{nn}}, b_{nn}),$$

where $v_{nk} := h(x_{nk})$, $x_{nk}$ are the Halton points in $\Omega$, $b_{nl} := g(z_{nl})$, and $z_{nl}$ are the uniform grid points on $\partial \Omega$. Let

$$\xi_n^k := (\delta_{x_{11}} \circ \Delta, \delta_{x_{n1}} \circ \Delta, \cdots, \delta_{x_{nn}} \circ \Delta, \delta_{z_{n1}}, \delta_{z_{n2}}, \cdots, \delta_{z_{nn}}),$$

and

$$y_n := (v_{n1}, v_{n2}, \cdots, v_{nn}, b_{n1}, b_{n2}, \cdots, b_{nn}).$$
for all \( n \in \mathbb{N} \). Thus \( \mathcal{F} = \{ \delta_{x_{kn}} \circ \Delta, \delta_{z_{ln}} : k_n \in \mathbb{N}, l_n \in \mathbb{N}, n \in \mathbb{N} \} \). Since Halton points are deterministic, the linear-functional data are also deterministic. If we choose Sobol points, then the linear-functional data become stochastic. Analysis similar to that in the proof of Lemma 2.2 shows that \( \mathcal{F} \) is relatively compact in \( \mathcal{H}_K(\Omega) \). Let the multi-loss function

\[
\mathbb{L}_n (\xi^*, y, t) := \frac{1}{2n^2} \sum_{k=1}^{n^2} L^1 (\xi^*_k, y_k, t_k) + \frac{1}{2n} \sum_{l=1}^{n} L^2 (\xi^*_{n+l}, y_{n+l}, t_{n+l}),
\]

for \( \xi^* \in (\mathcal{H}_K(\Omega))^{n^2+n} \), \( y \in \mathbb{R}^{n^2+n} \), \( t \in \mathbb{R}^{n^2+n} \), where the square losses

\[
L^1 (\xi^*, y, t) := \begin{cases} |t - y|^2, & \text{if } \xi^* = \delta_x \circ \Delta \text{ for } x \in \Omega \text{ and } y \in \text{range}(h), \\ 0, & \text{otherwise}, \end{cases}
\]

and

\[
L^2 (\xi^*, y, t) := \begin{cases} |t - y|^2, & \text{if } \xi^* = \delta_z \text{ for } z \in \partial\Omega \text{ and } y \in \text{range}(g), \\ 0, & \text{otherwise}, \end{cases}
\]

for \( \xi^* \in \mathcal{H}_K(\Omega) \), \( y \in \mathbb{R} \), \( t \in \mathbb{R} \). Since \( h, g \) are bounded on \( \Omega \) and \( \partial\Omega \), respectively, \( L^1, L^2 \) are both local Lipschitz continuous loss functions. In the same manner as Lemma 2.3 (b) \( \mathcal{L} \) is uniformly local Lipschitz continuous. This shows that Condition (II) holds true.

Since

\[
R_n(f) = \frac{1}{2n^2} \sum_{k=1}^{n^2} |\Delta f (x_{nk}) - h (x_{nk})| + \frac{1}{2n} \sum_{l=1}^{n} |f (z_{nl}) - g (z_{nl})|,
\]

for \( f \in \mathcal{H}_K(\Omega) \), the Koksma–Hlawka inequality as in quasi-Monte Carlo methods shows that \( R_n \) converges pointwise to \( R \) when \( n \to \infty \). This shows that Condition (I) holds true.

Let \( \Phi(r) := r^2 \) for \( r \in [0, \infty) \). The reproducing property of \( \mathcal{H}_K(\Omega) \) shows that \( \delta_x \circ \Delta \cong \Delta_x K(x, \cdot) \) and \( \delta_z \cong K(z, \cdot) \) for \( x \in \Omega \) and \( z \in \partial\Omega \). Thus Theorem 4.2 assures that there exists the unique approximate solution \( f_\lambda^* \in \mathcal{S}_n^\lambda (\mathcal{H}_K(\Omega)) \) such that \( f_\lambda^* \) is a linear combination of

\[
\Delta_x K (x_{n1}, \cdot), \Delta_x K (x_{n2}, \cdot), \ldots, \Delta_x K (x_{nn}, \cdot),
\]

and

\[
K (z_{n1}, \cdot), K (z_{n2}, \cdot), \ldots, K (z_{nn}, \cdot).
\]
Finally, we can check that \( R_n(f^0) = 0 \) for all \( n \in \mathbb{N} \). Therefore, for any decrease sequence \( (\lambda_n) \) to 0, Theorem 4.8 assures that \( f_n^{\lambda_n} \) weakly* converges to \( f^0 \) and \( \|f_n^{\lambda_n}\| \) converges to \( \|f^0\| \) when \( n \to \infty \). Since \( \mathcal{H}_K(\Omega) \) is a Radon-Riesz space, we have \( \|f^0 - f_n^{\lambda_n}\| \to 0 \) when \( n \to \infty \). This shows that
\[
\lim_{n \to \infty} \sup_{x \in \Omega} \left| \partial^\vartheta f^0(x) - \partial^\vartheta f_n^{\lambda_n}(x) \right| \to 0, \quad \text{for any } |\vartheta| \leq 2.
\]

In Examples 6.1, 6.2, and 6.3, we use the reproducing kernels to solve the approximate solutions directly by the representer theorems. Now we look at a popular example of artificial neural networks. Under the universal approximation of multi-layer neural networks, we will use the sigmoid functions to construct the approximate solutions approximately by the pseudo-approximation theorems.

**Example 6.4.** Let \( \Omega := [-1,1]^d \). It is obvious that \( L_1(\Omega) \) is a predual space of \( L_\infty(\Omega) \). Let \( \mathcal{U}_m \) be the collection of all multi-layer neural networks with \( m \) coefficients, that is,
\[
f_m(x) = W_2 \sigma(W_1 x + b_1) + b_2, \quad \text{for } x \in \Omega,
\]
where \( \sigma \) is a continuous sigmoid function, \( W_1, W_2 \) are weight matrixes, and \( b_1, b_2 \) are bias vectors as in [1, Section 6.4]. Thus \( \mathcal{U}_m \subseteq L_\infty(\Omega) \) and there exists a surjection \( \Gamma_m \) from \( \mathbb{R}^m \) onto \( \mathcal{U}_m \). It is easy to check that \( \mathcal{U}_m \) is weakly* closed if the coefficients are bounded. Since \( \{\mathcal{U}_m : m \in \mathbb{N}\} \) satisfies the universal approximation in \( C(\Omega) \) as in [4, 13], \( \{\mathcal{U}_m : m \in \mathbb{N}\} \) also satisfies the universal approximation in \( L_\infty(\Omega) \).

Moreover \( \langle f, \xi^* \rangle = \int_{\Omega} f(x) \xi^*(x) dx \) for \( f \in L_\infty(\Omega) \) and \( \xi^* \in L_1(\Omega) \). Thus the input data of standard sigmoid neural networks or convolutional neural networks can be equivalently transferred to the linear-functional data \( \xi_n^* \in (L_1(\Omega))^N_n \), for example of mollifiers and convolutions. As in Lemma 2.2, we can show that \( \mathfrak{F}_D \) is relatively compact for many kinds of the classical data, for example of digital images. The multi-loss function \( \mathbb{L}_n \) is represented by the classical loss function \( L \) as in Equation (2.9). It is simple to choose the local Lipschitz continuous loss functions to construct the algorithms of multi-layer neural networks. Thus Lemma 2.3 (b) assures that \( L \) is uniformly local Lipschitz continuous. This shows that Condition (II) holds true.

For the multi-layer neural networks, \( R \) is usually unknown or unformulated, but \( R_n \) is computed by \( \xi_n \) and \( \mathbb{L}_n \). In the convergence theorems, we only need to check the pointwise convergence for \( R \). In the numerical sense,
Condition (I) can be seen to hold true if the convergence of $R_n(f)$ exists for
different test function $f \in L_\infty(\Omega)$.

Therefore, the pseudo-approximation theorems and convergence theorems
hold true for many kinds of multi-layer neural networks. By the theory of the
regularized learning, we will study the composite algorithms of multi-layer
neural networks and support vector machines to solve partial differential
equations adaptively in our next work.

Finally, we look at a simple example of an ill-posed problem in the Eu-
clidean space $\mathbb{R}^2$ equipped with the 2-norm.

**Example 6.5.** Since $\mathbb{R}^2$ is a finite-dimensional Hilbert space, the weak*
topology of $\mathbb{R}^2$ is equal to the norm topology of $\mathbb{R}^2$ and $\langle f, \xi^* \rangle = f \cdot \xi^*$ for
$f, \xi^* \in \mathbb{R}^2$. Let

$$A := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_n := \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix} \quad \text{for } n \in \mathbb{N}, \quad b := \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ 

Now we discuss the linear equation $A c = b$. Let the expected risk function
$R(f) := \|A f - b\|_2^2$ for $f \in \mathbb{R}^2$. Thus $\mathcal{S}^0(\mathbb{R}^2)$ is the collection of all least-
squared solutions of $A c = b$ and the best-approximate solution of $A c = b$ is
the minimum-norm element of $\mathcal{S}^0(\mathbb{R}^2)$ as in [7, Definition 2.1].

We look at the special data

$$\xi_n := (\xi^*_n, \xi^*_n, \xi^*_n), \quad y_n := (1, 1, 1), \quad \text{for all } n \in \mathbb{N}.$$ 

where

$$\xi^*_n := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^*_n := \begin{pmatrix} 0 \\ 1/n \end{pmatrix}, \quad \text{and } \xi^*_n := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

It is easy to check that $\mathcal{F}_D$ is relatively compact in $\mathbb{R}^2$. Let the multi-loss function

$$\mathbb{L}_n (\xi^*, y, t) := \frac{1}{3} \sum_{k=1}^3 L (\xi^*_k, y_k, t_k), \quad \text{for } \xi^* \in \mathbb{R}^{3 \times 2}, \ y \in \mathbb{R}^3, \ t \in \mathbb{R}^3,$$

where $n \in \mathbb{N}$ and

$$L (\xi^*, y, t) := \begin{cases} 3 |t - y|^2, & \text{if } y \in \{1\}, \\ 0, & \text{otherwise}, \end{cases}$$

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for $\xi^* \in \mathbb{R}^2$, $y \in \mathbb{R}$, $t \in \mathbb{R}$. Since $L$ is a local Lipschitz continuous loss function, Lemma 2.3 (b) assures that $L$ is uniformly local Lipschitz continuous. This shows that Condition (II) holds true.

Moreover, $R_n(f) = \|A_n f - b\|_2^2$ for $f \in \mathbb{R}^2$. Thus $R_n$ converges pointwise to $R$ when $n \to \infty$. This shows that Condition (I) holds true. [7, Theorem 2.5] assures that the best-approximate solution of $A c = b$ is

$$f^0 := A^\dagger b = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

and the best-approximate solution of $A_n c = b$ is

$$f_n := A_n^\dagger b = \begin{pmatrix} 1 \\ 1/n \end{pmatrix},$$

where $A^\dagger$ and $A_n^\dagger$ are the pseudo inverses of $A$ and $A_n$, respectively. Thus $f_n$ is also a minimizer of $R_n$ over $\mathbb{R}^2$. But $f_n$ is not convergent when $n \to \infty$. This shows that $f_n$ is not a well-posed approximate solution of $A c = b$ for any $n \in \mathbb{N}$.

Let $\Phi(r) := r^2$ for $r \in [0, \infty)$. Thus [7, Theorem 5.1] assures that there exists the unique approximate solution $f_n^\lambda \in S_n^\lambda (\mathbb{R}^2)$ which is written as

$$f_n^\lambda = (A_n^T A_n + \lambda I)^{-1} A_n^T b = \begin{pmatrix} 1 \\ n/(1 + \lambda n^2) \end{pmatrix},$$

where $I$ is an identity matrix. By Theorem 4.6, $f_n^\lambda$ can become a well-posed approximate solution for a $n \in \mathbb{N}$ and a $\lambda > 0$. For example, if we choose $\lambda_n := 1/\sqrt{n}$, then $f_n^{\lambda_n} \to f^0$ when $n \to \infty$. However, if we choose $\lambda_n := 1/n^2$, then $f_n^{\lambda_n}$ is not convergent when $n \to \infty$. This shows that the approximate solution may not be well-posed for any pair $n, \lambda$ even when $n \to \infty$ and $\lambda \to 0$.

**Remark 6.2.** In Example 6.5, it is easily seen that the empirical risk functions are not equicontinuous while they are equicontinuous on all bounded subset. Even though we may not solve the ill-posed problems directly by the empirical risks, the regularized learning provides another way to find the well-posed approximate solutions. Example 6.5 also shows that the choice of $\lambda$ affects the convergence of the approximate solutions of the ill-posed problems. As in Theorem 4.8, we will try to discuss the open problem to choose the adaptive $\lambda$ to the special learning problems by the weak* topology.
7. Final Remark

In this article, the theory of Banach spaces is the foundation of the regularized learning to analyze the linear-functional data. By the techniques of weak* topology, we complete the proof of the representer theorems, pseudo-approximation theorems, and convergence theorems of the regularized learning. My philosophical idea is to approximate the unknown rules by the explicit models and observed data based on the theory of the regularized learning. The intention of the regularized learning is to solve an approximate solution which is efficiently computable by the machine. The work of the regularized learning provides another road to study the computational learning theory including:

- the interpretability in approximation theory,
- the nonconvexity and nonsmoothness in optimization theory,
- the generalization and overfitting in regularization theory.

In my opinion, the exact solutions of the original problems are approximated globally by the learning algorithms which are locally interpretable by the linear-functional data. Based on the theorems of the regularized learning, the existence and convergence of the approximate solutions can be guaranteed by the nonconvex and nonsmooth loss functions and thus the techniques of the nonconvex and nonsmooth optimization will be used to construct the iterative algorithms. Since the weak* compactness is motivated in the proof of the weak* convergence of the approximate solutions, the regularization terms are successful to guarantee the generalization and avoid the overfitting. The properties of the weak* convergence also indicate that the convergent rates and error bounds of the approximate solutions and exact solutions could be observed at the given linear-functional data.

Moreover, the representations of the regularized learning are similar to the formulas of inverse problems. By the classical methods of inverse problems, the condition of the uniform convergence is usually needed in the proof for the convergent analysis. However, it is too strong to directly check the uniform convergence in many learning problems because the original problem is usually unknown or unformulated. Actually the pointwise convergence is weaker than the uniform convergence. Moreover, the pointwise convergence of the empirical risks can be checked by the test functions in the practical applications or assumed by modeling in the sense of natural science. Therefore, for
the convergence theorems, we mainly discuss the condition of the pointwise convergence such as Condition (I). Unfortunately, the pointwise convergence is not a sufficient condition to verify the convergence of the approximate solutions to the exact solutions. Thus we have a natural question whether the strong convergence condition can be replaced to the weak convergence condition with the additional checkable condition to verify the convergence. Our idea is that the weak condition and the additional condition can be checked individually. Since the linear-functional data are always known, the additional condition of the linear-functional data is feasible to check as in Lemma 2.2. By the construction of the linear-functional data, the relative compactness is a natural additional condition of the linear-functional data as in Remark 2.2. The additional condition of the loss functions is straightforward because the loss functions are simple. Clearly the additional condition of the linear-functional data and the loss functions is independent of the original problem such as Condition (II). The additional condition further implies the special weak* equicontinuity of the empirical risks as in Proposition 3.1 (c). As in Section 5.4, the weak condition and the additional condition are sufficient to prove the convergence theorems.

Specially, it is well known that the data-driven methods are usually used to analyze the black-box models and the model-driven methods are usually used to analyze the white-box models. For a special example, the regularized learning gives a new method to construct the composite algorithms to combine the black-box and white-box models in one system. Our original idea is inspired by the eastern philosophy such as the golden mean and the Tai Chi diagram. In our current researches, we discuss the composite algorithms of the black-box and white-box models such as the support vector machines, artificial neural networks, and decision trees for the big data analysis in education and medicine by the theorems of the regularized learning.

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