The curvature of convex sum of metrics and applications

Leonardo F. Cavenaghi and Llohan D. Sperança

Abstract. In this paper we explore the curvature of convex sums of Riemannian metrics $g(t) = (1-t)g_0 + tg_1$, and study whether such a deformation can increase the curvature of an immersed totally geodesic flat tori. We obtain necessary and sufficient conditions for $g(t)$ to have average positive $r$-order variation ($r \geq 2$) along the torus and apply them for paths joining $g_0$ to known classical deformations.

1. Introduction

There are only few known theorems concerning manifolds with positive sectional curvature. For instance, in the compact setting, Synge’s Theorem asserts that:

**Theorem 1.1 (Synge’s Theorem).** Let $(M, g)$ be a compact manifold with positive sectional curvature. Then,

1. If $M$ is even-dimensional and orientable, then it is simply-connected,
2. If $M$ is odd-dimensional, then it is orientable.

On the other hand, the Soul Conjecture (proved by Perelman [Per94]) implies that every complete non-compact positively curved manifold is diffeomorphic to the Euclidean space.

Apart from the previous obstructions theorems, there is no result distinguishing the class of closed simply-connected manifolds with non-negative sectional curvature from that of positive sectional curvature. Consequently, at least in this setting, one one might expect to perform some kind of metric deformation on a non-negatively curved manifold and obtain a positively curved metric.

However, there is a great discrepancy between the amount of examples in these classes (see for instance [Zil14, Zil07]) which shows that, even with the knowledge of a reasonable large range of metric deformations, there is no systematic way to approach the problem.
In many examples of non-negatively curved manifolds, such as $S^n \times S^m$, the flat planes are concentrated in totally geodesic tori. Moreover, [Wil02] presents another class of such examples that \textit{a priori} do not admit metrics of positive sectional curvature. These are particularly interesting since they are counter-examples to the so called \textit{Deformation Conjecture}:

\textbf{Conjecture 1.2.} Suppose that $(M, g)$ is a complete manifold with non-negative sectional curvature with a point $p \in M$ whose two-planes have positive sectional curvature. Then $M$ carries a metric of positive sectional curvature.

In this paper we study the curvature behavior of metrics joined to $g_0$ in a convex way. To motivate its aim, recall that Strake [Str87, Proposition 4.3, p. 641] proves that no metric deformation increases the average curvature of a totally geodesic flat tori in first order. Here we take advantage of a new (to the author’s knowledge) formula for the curvature of a convex sum to study the average of its higher order variations.

The idea is to search for a convex combination (on a small neighbourhood of $g_0$), $g(t) := (1-t)g_0 + tg_1$, such that the sectional curvature along the torus increases in average, i.e,
\[
\int_{T^2} \frac{d^r}{dt^r} R_0(X, Y, Y, X) > 0, \ r \geq 2. \tag{1}
\]

When equation (1) holds we say that $g_0$ has \textit{average positive $r$-order variation}.

Motivated by the fact that one can reach every other metric by a convex sum with $g_0$, we give an easily computable method to verify if $g(t)$ has average positive variation. Precisely, define
\[
S^P_r(X, Y) := g_1(P(1-P)^{r-2}\mathcal{D}_X X, \mathcal{D}_Y Y) - g_1(P(1-P)^{r-2}\mathcal{D}^1_X Y, \mathcal{D}^1_X Y), \tag{2}
\]
where $g_0(PX, Y) = g_1(X, Y)$ and $\mathcal{D}_X Y = \nabla^1_X Y - \nabla^0_X Y$. We prove:

\textbf{Theorem A.} Suppose that $T^2$ is a totally geodesic flat tori in $g(t)$ and denote by $R_t$ the Riemann curvature tensor of $g(t)$. Assume that $R'_0(X, Y, Y, X) = 0$ for every $X, Y \in T\mathbb{T}$. Then,
\[
\int_{T^2} \frac{d^r}{dt^r} R_0(X, Y, Y, X) > 0 \iff \int_{T^2} S^P_r(X, Y) < 0, \ r \geq 2. \tag{3}
\]

The hypothesis of $R'_0(X, Y, Y, X) = 0$ is implied by Strake’s result whenever the path joining the metrics is curvature non-decreasing. Therefore, to our aims, it is a natural supposition. Our expectation is that Theorem A might point out to metric deformations that are effective in improving the curvature of flat planes.

We also studied condition (3) for paths joining a fixed Riemannian metric $g_0$ to several of its classical deformations. Let us fix some notations, before proceeding. Given a metric foliation $\mathcal{F}$ on $M$ we denote by $X^\mathcal{V}$ the component of $X$ tangent to the leaves and by $X^\mathcal{H}$ the component on the orthogonal complement. Recall that the shape operator of a leaf is defined by $\sigma(X^\mathcal{V}, Y^\mathcal{V}) := \nabla^\mathcal{H}_{X^\mathcal{V}} Y^\mathcal{V}$.
The curvature of convex sum of metrics and applications

The O’Neill tensor, defined as \( A_{X H} Y^H = \frac{1}{2} [X^H, Y^H]^V \) has, for each fixed \( X^H \) and metric \( g \), a dual, \( A_{X H} : V \rightarrow H \),

\[ g(A_{X H} Y^H, V^V) = g(A_{X H} V^V, y^H). \]

The metric foliation is said to be integrable if \( A \equiv 0 \). In this case, if one further assumes that \( F \) is regular, \( M \) is topologically a finite quotient of a product manifold. If \( \sigma \equiv 0 \) we say that the foliation has totally geodesic leaves. An integrable foliation with totally geodesic leaves is locally isometric to a product.

**Theorem B.** Let \((M, g_0)\) be closed Riemannian manifold and \( T^2 \subset M \) an isometrically immersed totally geodesic flat tori. For items (ii) – (iv), assume that \( M \) is equipped with a metric foliation \( F \). Then,

(i) **Conformal change.** If \( g_1 = e^{2f} g_0 \) then

\[ \int_{T^2} \frac{d^r}{dt^r} R_0(X, Y, Y, X) \geq 0, \forall r \geq 2. \]

(ii) **General vertical warping.** Assume that \( X \in H, Y \in V \) and consider a general vertical warping \( g_1 = g_0 \big|_H + e^{2f} g_0 \big|_V \) by a smooth basic function \( f : M \rightarrow \mathbb{R} \). Then

\[ \int_{T^2} \frac{d^r}{dt^r} R_0(X, Y, Y, X) > 0 \iff \int_{T^2} e^{4f} (1 - e^{2f})^{r-2} \| df(X)Y \|^2_0 \neq 0, \forall r > 2. \]

(iii) **Canonical variation.** Let \( g_s \) be the canonical variation of \( g_0 \), i.e, \( g_s = g \big|_H + e^{2s} g \big|_V \), \( s \in (-\infty, \infty) \). Consider \( g_i^s = (1-t)g_0 + tg_s \). Then

\[ \int_{T^2} \frac{d^r}{dt^r} R_0^s(X, Y, Y, X) = 0, \forall s \in (-\infty, \infty), \forall r > 2. \]

(iv) **Cheeger deformations.** Assume that the foliation \( F \) is generated by the orbits of an isometric action by a compact Lie group \( G \) with a biinvariant metric \( Q \). Let \( g_s \) be the Cheeger deformation of \( g_0 \) and let \( O \) be the orbit tensor, i.e, \( g(X^V, Y^V) = Q(OX^V, Y^V) \). Let \( g_i^s = (1-t)g_0 + tg_s \) and \( P_s = (1 + sO)^{-1} \). Then

\[ \lim_{s \rightarrow \infty} \int_{T^2} \frac{d^r}{dt^r} R_0(P_s^{-1}X, P_s^{-1}Y, P_s^{-1}Y, P_s^{-1}X) > 0 \iff \int_{T^2} Q(O^{-1}(\nabla_{OX^V}^Q OX^V), \nabla_{OY^V}^Q OY^V) - Q(O^{-1}(\nabla_{OY^V}^Q OY^V), \nabla_{OY^V}^Q OY^V) < 0, \forall r > 2. \]

Theorem B is particularly interesting since it is a common belief that a metric deformation cannot remove flat planes along a totally geodesic flat torus. However, Theorem B states that almost arbitrary conformal change and/or general vertical warping have positive average r-order variation.

Therefore, in order to possibly construct a metric of positive sectional curvature in a product manifold, we could, for instance, take a warped
product (a particular type of general vertical warping). It is worth pointing out that, even though we were able to increase the curvature at every point in the torus, an arbitrary general vertical warping may produce planes with negative sectional curvature (see [Spe16], for example).

The hope, however, is that given the lack of obstructions considering simply connected manifolds, it is possible to combine some classical deformations subsequently and obtain metrics of positive variation along the torus that may fix the possibly originated directions of negative sectional curvature. The same reasoning applies to combining other different deformations.

Furthermore, it was known that Cheeger deformations do not fix curvature along a totally geodesic tori provided one of the directions is horizontal. Item (iv) extends this result by, somehow, replacing the Cheeger deformation with a straight line between \(g\) and \(g^\infty\).

To compute the sectional curvature of \(g(t)\), we present a new general formula for the convex sum of metrics, simply obtained by considering the immersion

\[
F : (M \times M, (1 - t)g_0 \times tg_1) \to M
\]

\[
F(x) = (x, x)
\]

and recognizing the shape operator associated to \(F\) (see Proposition 2.2).

2. Any order variations of linear convex combinations of metrics

Consider two Riemannian metrics \(g_0, g_1\) in \(M\) and denote

\[
g_0(PX,Y) = g_1(X,Y). \tag{4}
\]

We start with the proof the following general result:

**Proposition 2.1.** The curvature operator \(R_t\) associated to the convex sum \(g_t := (1 - t)g_0 + tg_1\) is given by:

\[
R_t(X,Y,Y,X) = (1-t)R_0(X,Y,Y,X) + tR_1(X,Y,Y,X)
+ t(1-t)g_1 \left( (1-t(1-P))^{-1} \mathcal{D}_X X, \mathcal{D}_Y Y \right)
- t(1-t)g_1 \left( ((1-(1-t-P))^{-1} \mathcal{D}_X Y, \mathcal{D}_X Y \right), \tag{5}
\]

where

\[
\mathcal{D}_X Y = \nabla^1_X Y - \nabla^0_X Y. \tag{6}
\]

We prove it by relying in a more general result about the curvature of graphs. To understand how to relate it with Proposition 2.1, note that there is a natural isometric immersion:

\[
F : (M, g_t) \to (M \times M, (1 - t)g_0 \times tg_1)
\]

\[
x \mapsto (x, x).
\]
We claim that such induced metric is precisely the convex sum \( r \). Then it holds that \( \Xi \) is an embedding whose image is \( \Gamma \). Indeed, such identification is given by the map \( \pi : \Gamma_r \to M \) defined by \((p, r(p)) \mapsto p\).

Let \( M = N, r = \text{Id}_M \) and \( f : M \to [0, 1] \) be a smooth function and consider the Riemannian metrics \((1 - f)g_M, fg'_M\) in \( M \). Since \( \Gamma_r \) is a submanifold of \((M \times M, (1 - f)g_M \times fg'_M)\), it is possible to induce a Riemannian metric in \( M \) via the composition

\[
M \to (M \times M, (1 - f)g_M + fg'_M) \to (M, (1 - f)g_M)
\]

\[
p \mapsto \Xi(p) \mapsto \pi \circ \Xi(p) = p
\]

We claim that such induced metric is precisely the convex sum \((1 - f)g_M + fg'_M\). Indeed,

\[
(\pi \circ \Xi)^*((1 - f)g_M) = \Xi^* \circ \pi^*((1 - f)g_M)
\]

\[
= \Xi^*((1 - f)g_M \times fg'_M)
\]

\[
= (1 - f)g_M + fg'_M,
\]

where the last equality follows since \( \Xi(p) = (p, p) \).

Therefore, with the intention of calculate the curvature of the convex sum \((1 - f)g_M + fg'_M\) it suffices to recognize the shape operator of the immersion \( \Gamma_r \to (M \times M, (1 - f)g_M \times fg'_M) \) being \( r = \text{Id}_M \).

We proceed with a few lemmas. The general procedure can be applied to compute the non-reduced sectional curvature of any graphic. We however
focus in the particular case of \( r = \text{Id}_M : M \to M \) since we are mostly concerned with the Riemannian curvature of a convex sum.

Let \( \text{Id}_M : (M, (1-f)g_M) \mapsto (M, fg'_M) \).

Note that since \( g_M \) e \( g'_M \) are Riemannian metrics in \( M \), then there exists a symmetric and positive-definite tensor \( P \) such that

\[
(1-f)g_M(PX,Y) = fg'_M(X,Y), \quad \forall X,Y \in T_pM, \forall p \in M. \tag{7}
\]

**Lemma 1.** The tangent space \( T_{(p,p)}\Gamma_{\text{Id}_M} \) and the normal space \( \nu_{(p,p)}\Gamma_{\text{Id}_M} \) to a point \((p,p)\in \Gamma_{\text{Id}_M} \) are identified with the tangent spaces to \((M,(1-f)g_M)\) and \((M,fg'_M)\), respectively, according to the isomorphisms

\[
\chi : (T_pM,(1-f)g_M) \to T_{(p,p)}\Gamma_{\text{Id}_M} \quad X \mapsto (X,X)
\]

\[
\chi' : (T_pM,fg'_M) \to \nu_{(p,p)}\Gamma_{\text{Id}_M} \quad X \mapsto (-PX,X)
\]

where \( P \) is given by equation (7). Moreover, the inverse

\[
\chi \times \chi'(X,Y) := \chi(X) + \chi'(Y)
\]

is given by

\[
(\chi \times \chi')^{-1} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} (1+P)^{-1} & P(1+P)^{-1} \\ -(1+P)^{-1} & (1+P)^{-1} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}
\]

**Proof.** Note that \( \chi \) is precisely the derivative of \( \Xi \), being therefore an isomorphism. Moreover, fixed \( p \in M \) and given \( X \in T_pM, Y \in T_{\text{Id}_M(p)}M \) we have

\[
(1-f)g_M \times fg'_M((X,X),(-PY,Y)) = (1-f)g_M(X,-PY) + fg'_M(X,Y) = -(1-f)g_M(X,PY) + fg'_M(X,Y) = -fg'_M(X,Y) + fg'_M(X,Y) = 0,
\]

that is, the images of \( \chi \) e \( \chi' \) are always orthogonal. On the other hand, since \( \chi \) is surjective it follows that the image of \( \chi' \) is contained in \( \nu_{(p,p)}\Gamma_{\text{Id}_M} \). But once \( \dim M \times M = 2m \) and \( \dim \Gamma_{\text{Id}_M} = m \), it follows that \( \dim \nu_{(p,p)}\Gamma_{\text{Id}_M} = m \). Once \( \chi' \) is injective the claim follows.

The final part of the statement follows via a direct computation since \((\chi \times \chi')^{-1} \) is given by

\[
(\chi \times \chi')(X,Y) = \begin{pmatrix} 1 & -P \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X-YP \\ X+Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} + \begin{pmatrix} -PY \\ 0 \end{pmatrix}
\]

\( \square \)
Lemma 2. Let
\[ \Pi : (T_pM, (1-f)g_M) \times (T_pM, fg'_M) \to (T_pM, fg'_M) \]
given by \( \Pi(X,Y) := Y - d\text{Id}_M(X) \in \text{Id}^*_M(TM) e O = (1 + P)^{-1} \). Then,
\[ \text{proj}_{\nu\Gamma_{\text{Id}_M}} = \left( \begin{array}{c} \Pi(PO_X, PO_Y) \\ -\Pi(OX, OY) \end{array} \right), \] (8)
where \( \text{proj}_{\nu\Gamma_{\text{Id}_M}} \) stands for the projection in the normal space to the graphic.

Proof. Note that since \((\chi \times \chi')(1-f)g_M \times fg'_M)\) it follows that for each \(p \in M\), the tangent spaces \((T_pM, (1-f)g_M)\), \((T_pM, fg'_M)\) are orthogonal (this is the content of Lemma 1). Moreover, since \(\chi'() = (\chi \times \chi')(0, \cdot)\) is an isomorphism, the orthogonal projection in the normal space of \(\Gamma_{\text{Id}_M}\) can be written as
\[ \text{proj}_{\nu\Gamma_{\text{Id}_M}}(X,Y) = \chi' \circ \text{proj}_{(T_pM,fg'_M)} \circ (\chi \times \chi'), \] (9)
where \(\text{proj}_{(T_pM,fg'_M)}\) denotes the orthogonal projection (with respect to the metric \(1-f)g_M \times fg'_M\)) in the tangent space to \(\{p\} \times (M, fg'_M)\). In particular, it follows that \(\text{proj}_{\nu\Gamma_{\text{Id}_M}} = \)
\[ \begin{pmatrix} 0 & -P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -P & 0 \end{pmatrix} \begin{pmatrix} (1+P)^{-1} & P(1+P)^{-1} \\ -(1+P)^{-1} & (1+P)^{-1} \end{pmatrix} \]
\[ = \begin{pmatrix} 0 & -P \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -P & 0 \end{pmatrix} \begin{pmatrix} -P(1+P)^{-1} & P(1+P)^{-1} \\ (1+P)^{-1} & -(1+P)^{-1} \end{pmatrix}, \]
that is,
\[ \text{proj}_{\nu\Gamma_{\text{Id}_M}} = \left( \begin{array}{c} P(1+P)^{-1} & -P(1+P)^{-1} \\ -(1+P)^{-1} & (1+P)^{-1} \end{array} \right) = \left( \begin{array}{c} PO \ \\ -O \end{array} \right). \] (10)

Hence,
\[ \begin{pmatrix} PO & -PO \\ -O & O \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} POX - POY \\ -OX + OY \end{pmatrix} = \left( \begin{array}{c} \Pi(PO_X, PO_Y) \\ -\Pi(OX, OY) \end{array} \right), \]
what finishes the proof. \(\square\)

Considering the introduced terminology, Proposition 2.2 follows from Lemma 3.

Lemma 3 (The shape operator of a convex sum). Denote by \(\sigma_{\Gamma_{\text{Id}_M}}\) the shape operator of the graphic \(\Gamma_{\text{Id}_M}\) and define \(d^2\text{Id}_M(X,Y) := \nabla_X Y - \nabla Y X\), where \(\nabla'\) denotes the connection of the Riemannian metric \(fg'_M\) and \(\nabla\) denotes the connection of the Riemannian metric \((1-f)g_M\). Then,
\[ ((1-f)g_M \times fg'_M) \left( \sigma_{\Gamma_{\text{Id}_M}}(d \Xi(X), d \Xi(Y)), \sigma_{\Gamma_{\text{Id}_M}}(d \Xi(X'), d \Xi(Y')) \right) \]
\[ = fg'_M \left( (1+P)^{-1} d^2\text{Id}_M(X,Y), d^2\text{Id}_M(X',Y') \right). \] (11)
Proof. This result follows via a straightforward computation:

\[
\sigma_{\text{id}_M} (d \Xi(X), d \Xi(Y)) := \text{proj}_{\Gamma_{\text{id}_M}} (\nabla_X Y, \nabla_X' Y)
\]

\[
= \left( \Pi \left( PO \nabla_X Y, PO \nabla_X' Y \right) \right)
- \left( \Pi \left( O \nabla_X Y, O \nabla_X' Y \right) \right)
= \left( PO \nabla_X Y - \nabla_X' Y \right)
- \left( O \nabla_X Y + O \nabla_X' Y \right)
= \left( PO d^2 \text{Id}_M(X, Y) \right)
- \left( O d^2 \text{Id}_M(X, Y) \right)
\]

Analogously,

\[
\sigma_{\text{id}_M} (d \Xi(X'), d \Xi(Y')) = \left( PO \nabla_{X'} Y' - PO \nabla_{X'}' Y' \right)
- \left( O \nabla_{X'} Y + O \nabla_{X'}' Y' \right)
= \left( PO d^2 \text{Id}_M(X', Y') \right)
- \left( O d^2 \text{Id}_M(X', Y') \right)
\]

Hence,

\[
((1 - f) g_M \times f g_M) \left( \sigma_{\text{id}_M} (d \Xi(X), d \Xi(Y)), \sigma_{\text{id}_M} (d \Xi(X'), d \Xi(Y')) \right)
= (1 - f) g_M \left( PO d^2 \text{Id}_M(X, Y), PO d^2 \text{Id}_M(X', Y') \right)
+ f g_M \left( O d^2 \text{Id}_M(X, Y), O d^2 \text{Id}_M(X', Y') \right)
= f g_M \left( O d^2 \text{Id}_M(X, Y), O d^2 \text{Id}_M(X', Y') \right)
+ f g_M \left( (1 + P) O d^2 \text{Id}_M(X, Y), O d^2 \text{Id}_M(X', Y') \right)
= f g_M \left( (1 + P) O d^2 \text{Id}_M(X, Y), O d^2 \text{Id}_M(X', Y') \right),
\quad (12)
\]

what finishes the proof. □

2.2. The curvature of a convex sum and any order variations

The proof of Proposition 2.1 follows via a straightforward application of the Gauss–Codazzi immersion formula and Lemma 3. We proceed by expanding (5) as a Taylor series with respect to \( t \).

Lemma 4. Denote by \( \nabla^t \), \( R_t \), \( R^\nabla \nabla_t \) the Riemannian connection and curvature of \( g_t \) and the intrinsic curvature of \( \mathbb{T}^2 \) with the metric induced by \( g_t \). Denote by \( \perp \) the \( g_t \)-orthogonal projection to \( T \mathbb{T}^2 \). Then, for small \( t \),

\[
R_t(X, Y, Y, X) = (1 - t) R_0(X, Y, Y, X) + t R_1(X, Y, Y, X)
+ t \left\{ g_1((\nabla^t_X X)^\perp, (\nabla^t_Y Y)^\perp) - g_1((\nabla^1_X X)^\perp, (\nabla^1_Y Y)^\perp) \right\}
+ t \left\{ g_1((\nabla^1_X X)^\perp, (\nabla^1_Y Y)^\perp) - g_1((\nabla^1_X X)^\perp, (\nabla^1_Y Y)^\perp) \right\}
- t^2 \sum_{n=1}^\infty \left\{ g_1 \left( P(t(1 - P))^{n-1} \nabla^1_X X, \nabla^1_Y Y \right) - g_1 \left( P(t(1 - P))^{n-1} \nabla^1_X X, \nabla^1_Y Y \right) \right\}.
\]

(13)

Proof. Assume that

\[
\|t(1 - P)\|_\infty < 1,
\]
The curvature of convex sum of metrics and applications

then
\[
\left(1 + \frac{t}{1-t}P\right)^{-1} = (1-t) ((1-t) + tP)^{-1}
\]
\[
= (1-t) (1 - t(1 - P))^{-1}
\]
\[
= (1-t) \sum_{n=0}^{\infty} t^n (1 - P)^n
\]
\[
= (1-t) + (1-t) \sum_{n=1}^{\infty} t^n (1 - P)^n. \quad (14)
\]

Furthermore,
\[
t(1-t) \sum_{n=1}^{\infty} t^n (1 - P)^n = \sum_{n=1}^{\infty} t^{n+1} (1 - P)^n - \sum_{n=1}^{\infty} t^{n+2} (1 - P)^n
\]
\[
= \sum_{n=1}^{\infty} t^{n+1} (1 - P)^n - \sum_{m=2}^{\infty} t^{m+1} (1 - P)^{m-1} \quad (15)
\]
\[
= \sum_{n=1}^{\infty} t^{n+1} (1 - P)^{n-1}[(1 - P) - 1] + t^2
\]
\[
= -P \sum_{n=1}^{\infty} t^{n+1} (1 - P)^{n-1} + t^2. \quad (16)
\]

Therefore, combining equations (14) and (16) one gets
\[
t \left(1 + \frac{t}{1-t}P\right)^{-1} = t - P \sum_{n=1}^{\infty} t^{n+1} (1 - P)^{n-1}. \quad (17)
\]

Writing
\[
\nabla^1_X Y = (\nabla^1_X Y)^\perp + (\nabla^1_X Y)^\top,
\]
where \(\perp\) stands for the projection in the orthogonal complement to the tangent to torus, and \(\top\) stands for its orthogonal complement, both with respect to \(g_1\) we conclude the result. \(\square\)

Define
\[
S_r^P(X, Y) := g_1(P(1 - P)^{r-2}(\nabla^1_X X - \nabla^0_X X), \nabla^1_Y Y - \nabla^0_Y Y)
\]
\[- g_1(P(1 - P)^{r-2}(\nabla^1_X Y - \nabla^0_X Y), \nabla^1_Y Y - \nabla^0_Y Y). \quad (18)
\]

Then, one has:

**Theorem 2.3.** Assume that \(\nabla^0_Y Y = \nabla^0_X X = \nabla^0_Y Y = 0\) and denote by
\[
\frac{d}{dt} \bigg|_{t=0} R_t(X,Y,Y,X) := R^0_t(X,Y,Y,X). \quad \text{Then},
\]
\[
\int_{\mathbb{T}^2} \frac{1}{\|X \wedge Y\|^2_1} R'_0(X, Y, Y, X) \\
= \int_{\mathbb{T}^2} \frac{1}{\|X \wedge Y\|^2_1} \left\{ g_1((\nabla^1_X X)^\top, (\nabla^1_Y Y)^\top) - g_1((\nabla^1_X Y)^\top, (\nabla^1_Y Y)^\top) \right\}. \tag{19}
\]
Moreover, if \(R'_0(X, Y, Y, X) = 0\), then
\[
\int_{\mathbb{T}^2} \frac{d^r}{dt^r} R_0(X, Y, Y, X) > 0 \iff \int_{\mathbb{T}^2} S^P_r(X, Y) < 0, \ r \geq 2. \tag{20}
\]

Proof of Theorem 2.3. The result is a straightforward consequence of Lemma 4.

Indeed, observe that
\[
R'_t(X, Y, Y, X) \bigg|_{t=0} = R_1(X, Y, Y, X) + g_1((\nabla^1_X X)^\perp, (\nabla^1_Y Y)^\perp) - g_1((\nabla^1_X Y)^\perp, (\nabla^1_Y Y)^\perp) + g_1((\nabla^1_X Y)^\perp, (\nabla^1_Y Y)^\perp) - g_1((\nabla^1_X Y)^\perp, (\nabla^1_Y Y)^\perp).
\]
Now since \(X, Y\) are tangent to \(\mathbb{T}^2\), the Gauss–Bonnet Theorem implies that
\[
\int_{\mathbb{T}^2} \frac{R'_0(X, Y, Y, X)}{\|X \wedge Y\|^2_1} = \int_{\mathbb{T}^2} \frac{1}{\|X \wedge Y\|^2_1} \left\{ g_1((\nabla^1_X X)^\top, (\nabla^1_Y Y)^\top) - g_1((\nabla^1_X Y)^\top, (\nabla^1_Y Y)^\top) \right\}. \tag{21}
\]
Condition (20) is similarly obtained from Lemma 4. \qed

3. Curvature variations along paths joining a metric to some classical metric deformation

In what follows we consider a closed Riemannian manifold \((M, g_0)\) with a totally geodesic flat torus \(\mathbb{T}^2 \subset M\). We always denote by \(\{X, Y\}\) vectors tangent to \(\mathbb{T}^2\). In addition, we shall assume that \(\nabla^0_X Y = \nabla^0_Y X = \nabla^0_Y Y = 0\) and that \(R'_0(X, Y, Y, X) = 0\).

Proposition 3.1 (Conformal class). If \(g_1 = f g_0 = e^{2h} g_0\) then
\[
\int_{\mathbb{T}^2} \frac{d^r}{dt^r} R_0(X, Y, Y, X) \geq 0, \ \forall r \geq 2. \tag{22}
\]

Proof. If one assumes that there is a function \(f : M \to \mathbb{R}\) such that
\[
g_1 = f g_0,
\]
then
\[
P = f.
\]
By writing \(f = e^{2h}\), we have (see [Wal92, p. 144])
\[
\nabla^f_X Y = D_X Y = dh(X)Y + dh(Y)X - g_0(X, Y) \nabla^0 h.
\]
Hence,
\[ g_0(D_X X, D_Y Y) - \|D_X Y\|^2_{g_0} = -2\|\nabla^0 h^\top \|^2 + \|\nabla^0 h^\bot \|^2. \] (22)

By assuming \( R'_0(X, Y, Y, X) = 0 \) Theorem 2.3 implies that
\[ \int_{\mathbb{T}^2} \| (\nabla^0 \sqrt{f})^\top \|^2_0 = 0, \]
thus \( f \rvert_{\mathbb{T}^2} \) is constant. Moreover,
\[ \int_{\mathbb{T}^2} f(1 - f)^{r-2} \left\{ g_1(\nabla^f_X X, \nabla^f_Y Y) - \|\nabla^f_X Y\|^2 \right\} \]
\[ = 2c(1 - c)^{r-2} \int_{\mathbb{T}^2} \|\nabla^0 h^\bot \|^2_0 \geq 0. \quad \square \]

From now on we assume that \((M, g_0)\) is endowed with some kind of metric foliation \( \mathcal{F} \). Moreover, for the sake of brevity we restrict the analysis to \( r > 2 \) since in this cases \( S^P \) (eq. (2)) is only described by the vertical components of the covariant derivatives in its definition.

**Proposition 3.2 (Canonical variation).** Let \( g_s \) be the canonical variation of \( g_0 \). Consider \( g_t^s = (1 - t)g_0 + tg_s \). Then,
\[ \int_{\mathbb{T}^2} \frac{d^r}{dt^r} R_0^s(X, Y, Y, X) = 0, \quad r \geq 2. \]

**Proof.** Recall that the canonical variation of \( g_0 \) is defined by
\[ g_s(X, Y) = g_0(X^\mathcal{H}, Y^\mathcal{H}) + e^{2s}g_0(X^\mathcal{V}, Y^\mathcal{V}), \quad s \in (-\infty, \infty). \]

Hence, \( P_s|_\mathcal{V} = e^{2s}1 \) and \( P_s|_\mathcal{H} = 1 \). In particular, \( \nabla^s_X Y - \nabla^0_X Y \) is horizontal and \( (1 - P)\nabla^s_X Y = (1 - P)\nabla^0_X Y \). Hence,
\[ (1 - P)\nabla^s_X Y = (\nabla^0_X Y)^\mathcal{V}. \] (23)

Indeed, this can be seen since (see [GW09 Chpater 2, p. 45–47]),
\[ \nabla^s_X Y^\mathcal{H} = \nabla^0_X Y^\mathcal{H}, \] (24)
\[ \nabla^s_X Y^\mathcal{V} = \nabla^0_X Y^\mathcal{V} + (1 - e^{2s})A^0_Y X^\mathcal{V}, \] (25)
\[ \nabla^s_X Y^\mathcal{V} = \nabla^0_X Y^\mathcal{H} + (1 - e^{2s})A^0_Y X^\mathcal{V}, \] (26)
\[ \nabla^s_X Y^\mathcal{V} = \nabla^0_X Y^\mathcal{V} + (e^{2s} - 1)\sigma^0 (X^\mathcal{V}, Y^\mathcal{V}). \] (27)

Therefore, one concludes that for \( r > 2 \),
\[ \int_{\mathbb{T}^2} g_s(P_s(1 - P_s)^{r-2} \nabla^s_X X, \nabla^s_X Y) - g_s(P_s(1 - P_s)^{r-2} \nabla^s_X Y, \nabla^s_X Y) \]
\[ = e^{2(r-1}s(e^{2s} - 1)^{r-2} \left[ \int_{\mathbb{T}^2} g_0 (\nabla^0_X X)^\mathcal{V}, (\nabla^0_X Y)^\mathcal{V}) - \| (\nabla^0_X Y)^\mathcal{V} \|^2_{g_0} \right]. \] (28)

Moreover, since \( \nabla^0_X X = \nabla^0_Y Y = \nabla^0_X Y = 0 \) we conclude the proof. \( \square \)
Proposition 3.3 (General Vertical Warping). Assume that the flat planes \( \{X, Y\} \) concentrated in a totally geodesic torus consists in: a horizontal vector \( X \) and a vertical vector \( Y \). Also assume that \( g_f \) is a general vertical warping of \( g \), i.e.,

\[
g_f = g \mid_H + e^{2f} g \mid_V,
\]

where \( f \) is a basic function. Then,

\[
\int_{T^2} \frac{dr}{dt} R_0(X, Y, Y, X) > 0 \iff \int_{T^2} e^{4f}(1 - e^{2f})^r^{-2} \|df(X)Y\|^2_0 \neq 0, \quad \forall r \geq 2.
\]

Remark. Note that Proposition 3.3 recovers Proposition 3.2 when the tangent vectors \( \{X, Y\} \subset T\mathbb{T}^2 \) are as in its hypothesis.

Proof of Proposition 3.3. Consider \( g^f_t := (1 - t)g_0 + tg_f \). Then \( P_f X^V = e^{2f} X^V \) and \( (1 - P_f)^{-2} \mid_V = (1 - e^{2f})^{-2} \) and hence

\[
P_f(1 - P_f)^{-2} \mid_V = e^{2f}(1 - e^{2f})^{-2}.
\]

Recall that [GW09] Chapter 2, p. 45–47

\[
\nabla^f_{X^H} Y^H = \nabla^0_{X^H} Y^H, \tag{29}
\]

\[
\nabla^f_{X^V} Y^H = \nabla^0_{X^V} Y^H + (1 - e^{2f}) A^0_{X^H} X^V + df(Y^H)X^V, \tag{30}
\]

\[
\nabla^f_{X^H} Y^V = \nabla^0_{X^H} Y^V + (1 - e^{2f}) A^0_{X^V} Y^V + df(X^H)Y^V, \tag{31}
\]

\[
\nabla^f_{X^V} Y^V = \nabla^0_{X^V} Y^V + (e^{2f} - 1) \sigma^0(Y^V, Y^V) - e^{2f}(Y^V, Y^V)_0 \nabla^0 f. \tag{32}
\]

Since we assume that \( X = X^H, Y = Y^V, \|X\|_0 = \|Y\|_0 = 1 \); once \( P_f \mid_H = 1 \) we conclude that for \( r > 2 \)

\[
g_1 \left( P_f(1 - P_f)^{-2} \nabla^f_{X^H} X, \nabla^f_{X^V} Y \right) - g_1 \left( P_f(1 - P_f)^{-2} \nabla^f_{X^H} Y, \nabla^f_{X^V} Y \right) = e^{4f}(1 - e^{2f})^{-2} \left\{ g_0 \left( (\nabla^f_{X^H} X), (\nabla^f_{X^H} Y) \right) - g_0 \left( (\nabla^f_{X^H} Y), (\nabla^f_{X^H} Y) \right) \right\}. \tag{33}
\]

Therefore, equations (29)-(32) imply that

\[
g_1 \left( P_f(1 - P_f)^{-2} \nabla^f_{X^H} X, \nabla^f_{X^V} Y \right) - g_1 \left( P_f(1 - P_f)^{-2} \nabla^f_{X^H} Y, \nabla^f_{X^V} Y \right) = e^{4f}(1 - e^{2f})^{-2} \left\{ g_0 \left( (\nabla^0_{X^H} X), (\nabla^0_{Y^V} Y) \right) - \| (\nabla^0_{X^H} Y) + df(X)Y \|^2_0 \right\}. \tag{34}
\]

Finally, once \( \nabla^0_{X^H} X = \nabla^0_{Y^V} Y = \nabla^0_{X^H} Y = 0 \) one gets

\[
\int_{T^2} \frac{dr}{dt} R_0(X, Y, Y, X) > 0 \iff \int_{T^2} e^{4f}(1 - e^{2f})^{-2} \|df(X)Y\|^2_0 \neq 0,
\]

concluding the proof. \( \square \)
Next we consider the so-called Cheeger deformations. Let \((M, g_0)\) be a Riemannian manifold where a Lie group \(G\) acts by isometries. Let \(Q\) be a bi-invariant metric on \(G\) and \(O\) be the orbit tensor of \(g_0\). That is,
\[
g(X^V, Y^V) = Q(OX^V, Y^V).
\]
The Cheeger deformation of \(g_0\) at time \(s\) is the metric \(g_s\) satisfying
\[
g_t(X, Y) = g_0(P_sX, Y),
\]
where \(P_s|_{\mathcal{H}} = 1\) and \(P_s|_{\mathcal{V}} = (1 + sO)^{-1}\).

For other basics and results on Cheeger deformation, we refer to \([Zil]\).

**Proposition 3.4 (Cheeger deformations).** Let \(g_t^s = (1 - t)g_0 + tg_s\). Then for any \(r > 2\) it holds that
\[
\lim_{s \to \infty} \int_{T^2} \frac{d^r}{dt} R_0(P_s^{-1}X, P_s^{-1}Y, P_s^{-1}Y, P_s^{-1}X) > 0 \iff \\
\int_{T^2} Q(O^{-1}(\nabla^Q_{OX^V}OX^V), \nabla^Q_{OY^V}OY^V) - Q(O^{-1}(\nabla^Q_{OX^V}OY^V), \nabla^Q_{OX^V}OY^V) < 0.
\]

**Proof.** Since \(P_s|_{\mathcal{V}} = (1 + sO)^{-1}\) then
\[
1 - P_s = sO(1 + sO)^{-1}.
\]
Therefore,
\[
P_s^2(1 - P_s)^{-2} = (1 + sO)^{-2}s^{r-2}O^{-2}(1 + sO)^{2-r} = \frac{1}{s^2}(1/s + O)^{-2}O^{-2}(1/s + O)^{2-r}
\]
and hence
\[
s^2P_s^2(1 - P_s)^{-2} \to O^{-2} \text{ as } s \to \infty.
\]
On the one hand, following M"uter \([M87]\), if one considers \(P_s^{-1}X, P_s^{-1}Y\), then
\[
\lim_{s \to \infty} g_0(P_s^2(1 - P_s)^{-2}(\nabla^{P_s^{-1}X}_{P_s^{-1}Y}P_s^{-1}X)^V, (\nabla^{P_s^{-1}Y}_{P_s^{-1}X}P_s^{-1}Y)^V) - g_0 \left( P_s^2(1 - P_s)^{-2}(\nabla^{P_s^{-1}X}_{P_s^{-1}Y}P_s^{-1}X)^V, (\nabla^{P_s^{-1}Y}_{P_s^{-1}X}P_s^{-1}Y)^V \right) = \\
\lim_{s \to \infty} s^2 g_0(\nabla^{P_s^{-1}X}_{P_s^{-1}Y}P_s^{-1}X)^V, (\nabla^{P_s^{-1}Y}_{P_s^{-1}X}P_s^{-1}Y)^V) \cdot (36)
\]

**Claim.**
\[
\lim_{s \to \infty} s^2 \{ g_0(\nabla^{P_s^{-1}X}_{P_s^{-1}Y}P_s^{-1}X)^V, (\nabla^{P_s^{-1}Y}_{P_s^{-1}X}P_s^{-1}Y)^V) \}
\]
\[
= \lim_{s \to \infty} s^2 \left\{ Q(O^{-1}(\nabla^Q_{OX^V}OX^V), \nabla^Q_{OY^V}OY^V) - Q(O^{-1}(\nabla^Q_{OX^V}OY^V), \nabla^Q_{OX^V}OY^V) \right\}.
\]

Furthermore, if one drops \(P_s^{-1}\) in each entry, then for \(r \geq 2\)
\[
\lim_{s \to \infty} g_0( P_s^2(1 - P_s)^{-2}(\nabla^{P_s^{-1}X}_{P_s^{-1}Y}P_s^{-1}X)^V, (\nabla^{P_s^{-1}Y}_{P_s^{-1}X}P_s^{-1}Y)^V ) = 0.
\]

(38)
Therefore, for any \( r > 2 \),
\[
\int_{T^2} \frac{dr}{dt} R_0(X, Y, Y, X) > 0
\]
if, and only if,
\[
\int_{T^2} Q(O^{-1}(\nabla^Q_{OXv} OX^\nu), \nabla^Q_{OYv} OY^\nu) - Q(O^{-1}(\nabla^Q_{OXv} OY^\nu), \nabla^Q_{OYv} OY^\nu) < 0,
\]
what shall concludes the proof after verifying the Claim.

**Proof of Claim.** To prove it, note that according to the Koszul formula, if \( X, Y, Z \in V \), then
\[
2g((\frac{1}{s} + O)^{-1}\nabla_X^s Y, Z) = 2sg(P_s\nabla_X^s Y, Z) = X(g(sP_s Y, Z)) + Y(g(sP_s X, Z)) - Z(g(sP_s X, Y))
+ g([X, Y], sP_s Z) - g([X, Z], sP_s Y) - g([Y, Z], sP_s X). \tag{39}
\]

Taking the limity of \( s \to \infty \) on both sides of equation (39), since \( sP_s \to O^{-1} \), one gets
\[
\lim_{s \to \infty} 2g(O^{-1}\nabla_X^s Y, Z) = X(g(O^{-1} Y, Z)) + Y(g(O^{-1} X, Z)) - Z(g(O^{-1} X, Y))
+ g([X, Y], O^{-1} Z) - g([X, Z], O^{-1} Y) - g([Y, Z], O^{-1} X). \tag{40}
\]
In particular, equations (35) and (40) imply the last assertion in the Claim since equation (40) shows that the vertical component of connection \( \nabla^s \) restricted to vertical vectors is bounded in \( s \).

More interesting, note that since \( g(O^{-1} X, Y) = Q(X, Y) \), then the right-hand-side of equation (40) is precisely \( Q(\nabla^Q_X Y, Z) \), where \( \nabla^Q \) denotes the Levi–Civita connection of \( Q \). Therefore,
\[
\lim_{s \to \infty} \nabla_X^s Y = \nabla^Q_X Y,
\]
proving the Claim.

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The curvature of convex sum of metrics and applications

Leonardo F. Cavenaghi
Department of Mathematics – University of Fribourg, Ch. du musée 23, CH-1700 Fribourg, Switzerland
e-mail: leonardofcavenaghi@gmail.com, leonardo.cavenaghi@unifr.ch

Llohann D. Sperança
Instituto de Ciência e Tecnologia – Unifesp, Avenida Cesare Mansueto Giulio Lattes, 1201, 12247-014, São José dos Campos, SP, Brazil
e-mail: speranca@unifesp.br