BOREL SUBSYSTEMS AND ERGODIC UNIVERSALITY FOR COMPACT
Z-d-SYSTEMS VIA SPECIFICATION AND BEYOND

NISHANT CHANDGOTIA AND TOM MEYEROVITCH

Abstract. A Borel system \((X, S)\) is “almost Borel universal” if any free Borel dynamical system \((Y, T)\) of strictly lower entropy is isomorphic to a Borel subsystem of \((X, S)\), after removing a null set. We obtain and exploit a new sufficient condition for a topological dynamical system to be almost Borel universal. We use our main result to deduce various conclusions and answer a number of questions. Along with additional results, we prove that a “generic” homeomorphism of a compact manifold of topological dimension at least two can model any ergodic transformation, that non-uniform specification implies almost Borel universality, and that 3-colorings in \(Z^d\) and dimers in \(Z^2\) are almost Borel universal.

1. Introduction and statement of results

In this paper we obtain and exploit a new sufficient condition for a topological dynamical system \((X, S)\) to be universal with respect to embedding in the “almost Borel” category. This means that any free Borel dynamical system of strictly lower entropy is isomorphic to a Borel subsystem of \((X, S)\), after removing a null set (Theorem 5.1). We then derive some applications for this new sufficient condition and use it to answer a number of questions in the interface of measurable, Borel and topological dynamics. A precise statement of our main result, Theorem 5.1 requires some definitions, so we defer it a bit. Let us state some corollaries.

Call a topological \(Z^d\)-system \((X, S)\) fully \(\infty\)-universal if any free measure preserving system can be realized as a fully-supported \(S\)-invariant probability measure on \(X\). A “baby” version of the proof of our main result (Theorem 5.3) provides a sufficient condition for a \(Z^d\)-system \((X, S)\) to be fully \(\infty\)-universal. This implies the following:

Theorem 1.1. Let \(M\) be a compact connected topological manifold (with or without boundary) of dimension \(d \geq 2\). Then there exists a fully \(\infty\)-universal homeomorphism \(h : M \to M\). In fact, for any fully supported, non-atomic probability measure \(\mu \in \text{Prob}(M)\) for which the measure of the boundary is zero, there is a dense \(G_\delta\) set of fully \(\infty\)-universal homeomorphisms in the space of homeomorphisms that preserve \(\mu\).

Catsigeras and Troubetzkoy [10] recently proved that a generic homeomorphism of \(M\) admits an ergodic measure having infinite entropy. By the above result “most” homeomorphisms of \(M\) in fact admit an invariant measure isomorphic to any given free measure-preserving transformation, in particular those having infinite entropy.

Using Theorem 1.1, we can apply an old argument of Lind and Thouvenot [39] to deduce:

Theorem 1.2. If \(k > 1\) then for any free measure preserving \(Z\)-system \((Y, \mu, T)\) there is a homeomorphism \(h\) of \(T_k = \mathbb{R}^k / \mathbb{Z}^k\) that preserves Lebesgue measure, denoted by \(m_{T_k}\), and so that \((Y, \mu, T)\) is isomorphic to \((T_k, m_{T_k}, h)\) as a measure preserving system.

Proof of Theorem 1.2, assuming Theorem 1.1: By Theorem 1.1 there exists a fully \(\infty\)-universal homeomorphism \(\bar{h}\). Let \((Y, \mu, T)\) be a free measure preserving \(Z\)-system. Let \(\nu\) be a fully-supported \(\bar{h}\)-invariant measure on the torus so that \((T_k, \nu, \bar{h})\) is isomorphic to \((Y, \mu, T)\). An old result of Oxtoby and Ulam [42, Corollary 1] implies that there is a homeomorphism of the torus \(g_\nu\) such that the pushforward of \(\nu\) via \(g_\nu\) is the Lebesgue measure. It follows that \(h = g_\nu \circ \bar{h} \circ g_\nu^{-1}\) preserves Lebesgue measure and that \((Y, \mu, T)\) is isomorphic to \((T_k, m_{T_k}, h)\) as a measure preserving system.

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Lind and Thouvenot [39] applied the above argument to prove that any ergodic \((Y, \mu, T)\) that has finite entropy is isomorphic to a Lebesgue measure-preserving homeomorphism of the torus, based on the fact that hyperbolic toral automorphisms are universal. We remark that a fully \(\infty\)-universal homeomorphism of the torus has infinite topological entropy, thus it is not topologically conjugate to a smooth or even Lipschitz homeomorphism of the torus.

Another consequence of our main result is the following:

**Theorem 1.3.** Let \((X, S)\) be a topological \(\mathbb{Z}\)-system that has non-uniform specification. Then for any free Borel \(\mathbb{Z}\)-system \((Y, T)\) whose Gurevich entropy is strictly smaller than the topological entropy of \((X, S)\), there exists a \(T\)-invariant Borel subset \(Y_0 \subset Y\) so that \(Y \setminus Y_0\) is null with respect to any \(T\)-invariant probability measure and a Borel embedding of \((Y_0, T)\) into \((X, S)\). Furthermore, for any \(\mu \in \text{Prob}_*(Y, T)\) we can find a Borel embedding of a \(T\)-invariant \(\mu\)-full subset \(Y_0 \subset Y\) and a Borel embedding of \((Y_0, T)\) into \((X, S)\) so that the push-forward of \(\mu\) has full support in \(X\).

This confirms a conjecture of Quas and Soo [48, Conjecture 1], who proved that non-uniform specification implies ergodic universality under two additional hypothesis: “Asymptotic h-expansiveness” and the “small boundary property”. Benjy Weiss came up with a proof that removes the “asymptotic h-expansiveness” hypothesis. Theorem 1.3 also provides an affirmative answer to a question of Boyle and Buzzi [4, Problem 9.1]. Theorem 1.3 is an immediate consequence of Theorem 5.1 together with Propositions 5.2 and 5.7 below. While preparing this manuscript we learned that David Burguet obtained an independent proof of Theorem 1.3 [8].

Actually, an inspection of our proof shows that under the conditions stated in Theorem 1.3 we can find a Borel embedding of a full subset so that the push-forward of any \(T\)-invariant measure on \(Y\) has full support in \(X\). A topological dynamical system \((X, S)\) that satisfies this is called fully universal in the almost Borel sense. The initial motivation that led us to our main result, concerned certain \(\mathbb{Z}^d\)-systems that do not have specification.

**Theorem 1.4.** The following \(\mathbb{Z}^d\)-subshifts are fully universal in the almost Borel sense:

(i) Proper \(k\)-colorings of \(\mathbb{Z}^d\), for all \(k \geq 3\) and all \(d \geq 1\). (Theorem 8.1)

(ii) Domino tilings in \(d = 2\). (Theorem 9.3)

Theorem 1.4 answers a question by Robinson and Sahin [50] who asked whether proper 3-colorings and domino tilings of \(\mathbb{Z}^2\) are universal. We prove (i) of Theorem 1.4 as a particular case of a more general result about universality for the space of graph-homomorphisms from the standard Cayley graph of \(\mathbb{Z}^d\) to an arbitrary non-bipartite finite graph. This has some consequences for the Borel structure of a graph generated by a finite set of commuting measure preserving transformations: After removing an invariant null set and the periodic points, the Borel chromatic number coincides with a basic spectral invariant and is always equal to 2 or 3 (Corollary 8.16). In very recent work Gao, Jackson, Krohne and Seward announced [22] that in fact the Borel chromatic number of such graphs is at most 3, so there is no need to remove a null set.

Another application of our main result concerns equivariant measurable tiling of free \(\mathbb{Z}^d\)-actions by rectangular shapes:

**Theorem 1.5.** Let \((Y, T)\) be a free Borel \(\mathbb{Z}^d\)-dynamical system and let \(F\) be a set of rectangular shapes in \(\mathbb{Z}^d\) such that the projection of \(F\) onto each of the \(d\) coordinates is a set of intervals in \(\mathbb{Z}\) having coprime lengths. Then after removing a null set, there exists an equivariant measurable map from \(Y\) to the space of tilings of \(\mathbb{Z}^d\) by shapes from \(F\). Furthermore, if the entropy of \((Y, T)\) is sufficiently small (as a function of the set \(F\)), the map can be chosen to be injective.

We prove Theorem 1.5 in Section 9. This is a strengthening of the “\(\mathbb{Z}^d\)-Alpern Lemma” [46, 53]. In particular, from the case when \(F\) consists of rectangular shapes of size two, it follows that the graph associated with a free Borel \(\mathbb{Z}^d\)-dynamical system admits a Borel perfect matching, after removing a null set. Gao, Jackson, Krohne and Seward obtained remarkable results about equivariant tilings of free \(\mathbb{Z}^d\) actions, both in the Borel and in the continuous category [22]. In view of these results, it might be possible to avoid removing a null set in the statement Theorem 1.5, but this goes beyond the scope of this paper.

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1.2. Main idea and sketch of proof. Our proof proceeds by constructing a sequence of “approximate embeddings” that converge pointwise to an embedding on a Borel set which has full measure with respect to any invariant measure. This basic idea goes a long way back. Burton and Rothstein defined a certain notion of $\epsilon$-approximate embedding and used it in conjunction with the Baire category framework to reprove Krieger’s generator Theorem. At this level of generality, this is similar to the basic approach of Quas and Soo from [49]. We do not use the Baire category framework, and instead prove convergence of a sequence of approximate embeddings directly, somewhat along the lines of [48]. The basic difference is that we use a slightly different notion for an “approximate-embedding”. Roughly speaking, a “good approximate embedding” is a map $\rho : Y \to X$ which is:

- “Approximately injective”: Informally, this means that there is a “big subset $Y_0 \subseteq Y$” such that two points in $Y_0$ have “sufficiently close” images under the map if and only if the two points are “very close” (this can be made precise by putting a topological structure on $Y$ or by using finite Borel partitions to indicate “closeness”).

- “Approximately equivariant”: Informally, this means that on a “big part of $Y$” $\rho(T^i(y))$ is “very close” to $S^i(\rho(y))$, as long as $i$ is in a given bounded set (the “largeness” of this set is one of the parameters for the “quality” of the approximate embedding).

The “approximate equivariance” property above can be reformulated by saying that the map is truly equivariant, but the target space is not $X$ itself, but rather $X^{2^d}$, which we think of as the “space of pseudo orbits of the system $(X, S)$”.

Using “Rokhlin towers” (a classical tool in ergodic theory, which we recall later), and a relatively well-known version of the Shannon-McMillan theorem it turns out that it is quite easy to produce “approximate embeddings” which are “arbitrarily good”, and that this does not require any assumptions on the target system $(X, S)$, except that it’s topological entropy has to exceed the entropy of the source system $(Y, T)$.

Our precise definition of an “approximate embedding” appears in Section 6. The actual definition we use is slightly more complicated than the above, in particular because to carry out the full proof we need to assure that a “sufficiently small modification” of an approximate embedding retains it’s good properties. This seemingly minor issue is perhaps the reason why previous proofs for universality needed to assume an additional property called “the small boundary condition”, which roughly speaking means that from a dynamical point of view the space $X$ is in some sense “almost totally disconnected”. Essentially, we overcome this by constructing “approximate embeddings” whose image is already totally disconnected (in the space of pseudo orbits), in a manner which is “tolerant to small perturbations”. Our construction of an approximate embedding is described in Lemma 6.13.

Construction of an initial approximate embedding is the first step. The goal is to produce a sequence of approximate embeddings that converge to a “genuine” embedding on a full set. For this we prove that for systems satisfying our sufficient condition, a “small modification” of given approximate embedding can produce a much better one. This is the main part of the proof. It is carried out in Lemma 6.18. This is where we assume a special property of the system $(X, S)$. Basically, the property we assume is a certain kind of “specification property”: It is possible to shadow a bunch of “sufficiently spaced” orbit segments by a single orbit segment. We do not require the ability of being able to “shadow” any collection of “sufficiently spaced” orbit segments: We only need a “sufficiently big” supply of “good” orbit segments. By “sufficiently big” we roughly mean that these orbit segments are sufficient to “witness enough entropy in the system”. It should be stressed that the collection of points that constitute “good orbit segments” need not be a compact subsystem. For the “full universality” result we actually need them to be a dense subset.
We point out that in contrast to [49] and other works that follow the Burton-Keane-Rothstein-Serafin paradigm, our approach is indifferent to the existence of measures which “locally” maximize the entropy. For some of the systems that motivate our result the question about existence of measures which locally maximize entropy (but not globally) seems to be a subtle issue.

To go from “ergodic universality” (equivalently embedding a set which is co-null with respect to a fixed ergodic measure) to “almost Borel universality” (equivalently embedding a set which is co-null with respect to all ergodic measures), we first check that our “procedure” for constructing an embedding with respect to a given ergodic measure is “Borel” as a function of the given ergodic measure on \((Y, T)\). As observed in [27], it is possible to apply “the embedding procedure” separately on the set of generic points for each ergodic measure and obtain an equivariant Borel function from a full subset of \(Y\) into \(X\) that induces an embedding on a full set. To ensure that the resulting function is actually injective on a full set, we take care so that a pair of points which are generic with respect to distinct ergodic measures will be mapped into different points in \(X\). We do this by making sure that the image of a generic \(y\) also “encodes” the empirical measure associated with \(y\). The part where we go from “ergodic universality” to “almost Borel universality” is described in Section 7.

We have attempted to make our proof reasonably self contained and refrained from using complicated theorems without providing a proof. The main exceptions to this are the \(Z^d\) version of Rokhlin’s lemma, the \(Z^d\) version of the mean ergodic theorem, and the \(Z^d\) version of the Shannon-McMillan theorem (in it’s weak form stating only convergence in measure). For one particular lemma we also use a theorem of Downarowicz and Weiss [16, Theorem 3] (instead of using a slightly longer but self-contained argument).

1.3. Organization of the paper. In Section 2 we recall some background results and introduce some notation and terminology that is used in later sections. In Section 3 there is a formulation and proof for certain variants of the Shannon-McMillan theorem and of the mean ergodic theorem, adapted to a sequence of Rokhlin towers. Section 4 introduces “approximate covering numbers”. This is a certain interpretation of “entropy of a process” in terms of “coding”, and formulates related inequalities. Together Sections 2 to 4 can be considered as background and preparation for the main part of the proof. In Section 5 we introduce flexible sequences and flexible marker sequences and formulate our main result, prove that systems having non-uniform specification satisfy our sufficient condition for universality, and some additional auxiliary results. In Section 6 we prove an ergodic version of Theorem 5.1, and also Theorem 5.3 regarding \(\infty\)-universality. Section 7 contains a proof of our main result (Theorem 5.1). The proofs in Section 7 extend and rely on the previous section. In Section 8 we introduce hom-shifts and prove that they satisfy the assumptions of Theorem 5.1. This includes the case of 3-colorings. In Section 9 we prove the result about equivariant rectangular tilings and universality for dimers in \(Z^2\). In Section 10 we exhibit an example of a fully universal subshift that admits a topological factor which is not universal, providing a negative answer to an old question of Lind and Thouvenot [39]. In Section 11 we show some non-trivial restrictions regarding subshifts that can be continuously embedded in the three colorings of \(Z^2\). This shows that the universality result for the three colorings cannot be deduced by applying the Robinson-Şahin universality result on subsystems. In the last section, we conclude with some further questions.

2. Preliminaries and notation

2.1. Borel and topological \(Z^d\) dynamical systems. In this paper a topological \(Z^d\) dynamical system is an action of \(Z^d\) on a compact metric space by homeomorphisms. Throughout the first few sections \((X, S)\) will denote a topological \(Z^d\) dynamical system. To be precise, \(X\) will be a compact space with a compatible metric \(d_X: X \times X \to \mathbb{R}_+\), and for every \(i \in Z^d\) \(S^i : X \to X\) will be a homeomorphism so that

\[
\forall i, j \in Z^d, S^{i+j} = S^i \circ S^j.
\]

A Borel \(Z^d\) dynamical system is an action of \(Z^d\) on a standard Borel space by Borel automorphisms. Throughout the first few sections \(\mathcal{Y} = (Y, T)\) will denote a Borel \(Z^d\) dynamical system. This means that \(Y\) will be a standard Borel space, that the maps \(T^i: Y \to Y\) are Borel bijections such that \(T^{i+j} = T^i \circ T^j\). We generally assume that \(Y\) is a free Borel system. This means that for any \(y \in Y\) and \(i \in Z^d \setminus \{0\}\) \(T^i(y) \neq y\).

The Borel \(\sigma\)-algebra of \(Y\) will be denoted by \(\text{Borel}(Y)\). When we say that \(A \subset Y\) is measurable without further adjectives, we will mean that \(A \in \text{Borel}(Y)\).
Note that any topological $\mathbb{Z}^d$ system is also a Borel $\mathbb{Z}^d$ dynamical system.

A Borel probability measure $\mu$ is $T$ invariant if $\mu(T^{-i}(A)) = \mu(A)$ for every $i \in \mathbb{Z}^d$ and $A \in \text{Borel}(Y)$. We denote the space of $T$-invariant probability measures on $Y$ by

$$\text{Prob}(Y) = \{\text{$T$-invariant Borel probability measures on $Y$}\}.$$  

We also denote:

$$\text{Prob}_e(Y) = \{\text{ergodic $T$-invariant Borel probability measures on $Y$}\}.$$  

Following [27], we say that a Borel set $Y_0 \subset Y$ is universally null if $\mu(Y_0) = 0$ for all $\mu \in \text{Prob}(Y)$. A set is **full** if its complement is universally null.

### 2.2. Boxes and other subsets in $\mathbb{Z}^d$.  

For $n \in \mathbb{N}$ denote

$$(3) \quad F_n = \{-n, \ldots, n\}^d \subset \mathbb{Z}^d.$$  

Also, for $t \in (0, \infty)$ we denote

$$(4) \quad tF_n = \{-\lfloor tn \rfloor, \ldots, \lfloor tn \rfloor\}^d.$$  

Let $K \subset \mathbb{Z}^d$ be a finite set. We say that $F \subset \mathbb{Z}^d$ is $K$-spaced if

$$(5) \quad (\vec{i} + K) \cap (\vec{j} + K) = \emptyset \quad \text{for every } \vec{i}, \vec{j} \in F \text{ s.t. } \vec{i} \neq \vec{j}.$$  

Later on we will fix a sequence of positive numbers $(\delta_n)_{n=1}^{\infty}$ that tends monotonically to 0. With such a sequence fixed, for integers $n > n_0$ we will denote:

$$(6) \quad S_{n,n_0} = \{A \subset (1 - 2\delta_n)F_n : A \text{ is } (1 + \delta_n)F_{n_0}\text{-spaced}\}.$$  

### 2.3. The space of pseudo-orbits.  

For a compact metric space $X$, the space $X^{\mathbb{Z}^d}$ of $X$-valued functions on $\mathbb{Z}^d$ with the product topology is again a compact metrizable space. For $w \in X^{\mathbb{Z}^d}$ we denote by $w_i$ the value of $w$ at $\vec{i} \in \mathbb{Z}^d$, and for $F \subset \mathbb{Z}^d$ $w|_F \in X^F$ will denote the restriction of $w$ to $F$. The group $\mathbb{Z}^d$ acts on $X^{\mathbb{Z}^d}$ by translations. For $\vec{i} \in \mathbb{Z}^d$ and $w \in X^{\mathbb{Z}^d}$ we write $(S^i(w))_{\vec{j}} = w_{\vec{i} + \vec{j}}$. The resulting dynamical system $(X^{\mathbb{Z}^d}, S)$ is sometimes called the **full shift over $X$**, and the action is called the **shift action**. There is a natural embedding of a topological dynamical system $(X, S)$ into $(X^{\mathbb{Z}^d}, S)$ given by $x \mapsto (S^i(x))_{\vec{i} \in \mathbb{Z}^d}$. In other words, each point in $X$ can be identified with its $S$-**orbit**. This embedding is equivariant with respect to $S$ and the shift. Thus, we identify $X$ with its image in $X^{\mathbb{Z}^d}$ under the orbit map. This justifies the abuse of notation when we denote by $S$ both the shift action $X^{\mathbb{Z}^d}$ and the original action on $X$. In this context we refer to $X^{\mathbb{Z}^d}$ together with the shift action as the space of **pseudo-orbits** for $(X, S)$. This embedding will be useful in the proof of our main result. We will denote the space of pseudo-orbits by

$$\mathcal{X} = (X^{\mathbb{Z}^d}, S).$$  

### 2.4. Bowen metrics and topological entropy.  

For a finite subset $F \subset \mathbb{Z}^d$ and $x, x' \in X$, we denote

$$(7) \quad d^F_X(x, x') = \max_{\vec{i} \in F} \left( d_X(S^i(x), S^i(x')) \right)$$  

For every finite $F \subset \mathbb{Z}^d$, $d^F_X : X \times X \to \mathbb{R}_+$ defines a metric on $X$ that is compatible with the original one. These metrics are known as “Bowen metrics”.

We say that a subset $C \subset X$ is $(\epsilon, F)$-separated if the $d^F_X$-distance between any pair of distinct points in $C$ is greater than $\epsilon$. Let

$$(8) \quad \text{sep}_e(A, F) = \max \{|C| : C \subseteq A \text{ is } (\epsilon, F)\text{-separated} \}$$  

The Bowen metrics can be viewed as restrictions of the **pseudo-metrics** on $X^{\mathbb{Z}^d}$ given by: If $\omega, \omega' \in X^{\mathbb{Z}^d}$ we denote

$$(9) \quad d^F_X(\omega, \omega') = \max_{\vec{i} \in F} d_X(\omega_\vec{i}, \omega'_\vec{i}).$$  

We will also use the above formula when $\omega, \omega' \in X^F$. Similarly, if $x \in X$, and $\omega \in X^{\mathbb{Z}^d}$ or $\omega \in X^F$ we denote

$$(10) \quad d^F_X(\omega, x) = \max_{\vec{i} \in F} d_X(\omega_\vec{i}, S^\vec{i}(x)).$$
This is consistent with the natural embedding of \((X, S)\) into \(\mathcal{X}\).

For \(w \in X^Z\), \(F \subset Z\) and \(A \subset X\), we denote
\[
d^F_X(w, A) = \inf_{x \in A} d^F_X(\omega, x).
\]

Let \(h(X, S)\) denote the topological entropy of \((X, S)\).

Recall that the topological entropy of \((X, S)\) is given by:

\[
h(X, S) = \lim_{n \to \infty} \limsup_{\epsilon \to 0} \frac{1}{|F_n|} \log sep_\epsilon(X, F_n).
\]

2.5. **Non-uniform specification.** We say that \((X, S)\) satisfies non-uniform specification (as in [45]) if there exists a sequence of increasing functions \(g_n : (0, 1) \to (0, \infty)\) so that for every \(\epsilon > 0\), \(g_n(\epsilon) \downarrow 0\) and \(n \uparrow \infty\) and so that for every \(s \in \mathbb{N}\), for every \(\epsilon > 0\) and every pairwise disjoint collection

\[
\{i_1 + (1 + g_{n_1}(\epsilon))F_{i_1}, \ldots, i_s + (1 + g_{n_s}(\epsilon))F_{i_s}\}
\]

and every \(x_1, \ldots, x_s \in X\) there exists \(x \in X\) such that \(d_X^{i_j + F_{i_j}}(x, x_j) < \epsilon\) for all \(1 \leq j \leq s\).

For \(d = 1\) the property we defined above is a “symmetric” version and an easy consequence of the property that Dateyama named almost weak specification [14]. Quas and Soo used Dateyama’s terminology in the context of sufficient conditions for ergodic universality [48, P. 4138]. “Almost weak specification” also goes under the name weak specification property [38]. See [38] for an overview of specification-like properties and historical background.

2.6. **Morphisms for \(Z^d\) dynamical systems.** A morphism between two Borel dynamical systems is a measurable map that intertwines the actions. We denote the collection of morphisms from \(Y\) to \(\mathcal{X}\) by \(Mor(Y, \mathcal{X})\). An injective morphism \(\rho \in Mor(Y, \mathcal{X})\) is a Borel embedding of \((Y, T)\) into \(\mathcal{X}\). A bijective morphism gives a Borel isomorphism, as the inverse is necessarily Borel by Souslin’s theorem.

2.7. **Borel partitions.** By a Borel partition \(P\) of \(Y\) we will mean a partition of \(Y\) into finitely or countably many Borel subsets. We follow the convention of identifying a Borel partition \(P\) of \(Y\) with the function that maps \(y \in Y\) to the unique element of \(P\) that contains \(y\), which we denote by \(P(y)\). A partition \(P\) refines another partition \(Q\) of \(X\) if every partition element \(P \in P\) is contained in some partition element \(Q \in Q\). In this case we write \(Q \preceq P\). The least common refinement of two partitions \(P\) and \(Q\) is given by

\[
P \vee Q = \{P \cap Q : P \in P, Q \in Q\}.
\]

For a Borel partition \(P\) of \(Y\) and a finite subset \(F \subset Z^d\), we write

\[
P^F = \bigvee_{i \in F} T^i(P).
\]

2.8. **Shannon entropy, information and Kolmogorov-Sinai entropy.** The information function \(\mathcal{I}_\mu(P) : Y \to \mathbb{R}_+\) for a measurable partition \(P\) is defined to be

\[
\mathcal{I}_\mu(P) = -\sum_{P \in P} 1_P(y) \log(\mu(P)) = -\log(\mu(P))(y).
\]

More generally, the relative information function of the partition \(P\) given a sub-\(\sigma\)-algebra \(F \subset Borel(Y)\) is given by

\[
\mathcal{I}_\mu(P | F)(y) = -\sum_{P \in P} 1_P(y) \log(\mu(P | F))(y).
\]

The Shannon entropy of a measurable partition \(P\) of \(Y\) with respect to a Borel probability measure \(\mu\) is given by

\[
H_\mu(P) = \int \mathcal{I}_\mu(P) d\mu.
\]

The entropy of a measurable partition \(P\) relative to a sub-\(\sigma\)-algebra \(F \subset Borel(Y)\) with respect to a Borel probability measure \(\mu\) is given by

\[
H_\mu(P | F) = \int \mathcal{I}_\mu(P | F) d\mu.
\]
When $\mathcal{F} = \{\emptyset, Y\}$ is the trivial $\sigma$-algebra, this coincides with the “non-relative” case, For $p \in (0,1)$ we denote
\begin{equation}
\mathcal{H}(p) = p \log \left( \frac{1}{p} \right) + (1 - p) \log \left( \frac{1}{1 - p} \right).
\end{equation}
$\mathcal{H}(p)$ is the Shannon entropy of a two set partition, where the measure of one of the parts is $p$.

If $\mu \in \text{Prob}(Y)$, $\mathcal{P}$ is a finite measurable partition and $\mathcal{F}$ is a $T$-invariant sub-$\sigma$-algebra $\mathcal{F} \subset Borel(Y)$, the Kolmogorov-Sinai of the partition $\mathcal{P}$ relative to $\mathcal{F}$ is given by
\begin{equation}
h_{\mu}(Y, T; \mathcal{P} \mid \mathcal{F}) = \lim_{n \to \infty} \frac{1}{F_n} H_{\mu}(\mathcal{P}_{F_n} \mid \mathcal{F}).
\end{equation}
For the non-relative case we denote:
\begin{equation}
h_{\mu}(Y, T; \mathcal{P}) = \lim_{n \to \infty} \frac{1}{F_n} H_{\mu}(\mathcal{P}_{F_n}).
\end{equation}

The Kolmogorov-Sinai of $(Y, \mu, T)$ (relative to $\mathcal{F}$) is given by:
\begin{equation}
h_{\mu}(Y, T \mid \mathcal{F}) = \sup_{\mathcal{P}} h_{\mu}(Y, T; \mathcal{P} \mid \mathcal{F}),
\end{equation}
where supremum is over all finite measurable partitions $\mathcal{P}$.

2.9. Ergodic universality and Almost Borel universality. We say that a topological $\mathbb{Z}^d$ system $(X, S)$ is $t$-universal in the ergodic sense if every free Borel dynamical system $\mathcal{Y} = (Y, T)$ and $\mu \in \text{Prob}_e(\mathcal{Y})$ such that $h_{\mu}(Y, T) < t$ the ergodic dynamical system $(Y, \mu, T)$ can be realized as an invariant measure on $(X, S)$, in the following sense: There is a Borel $T$-invariant subset $Y_0 \subset Y$ with $\mu(Y \setminus Y_0) = 0$ and $\rho \in \text{Mor}((Y_0, T), (X, S))$ so that $\rho$ is injective on $Y_0$. We say $(X, S)$ has ergodic universality if it is $t$-universal with $t = h(X, S)$. If furthermore under the above conditions we can find $\rho \in \text{Mor}((Y_0, T), (X, S))$ as above so that in addition $\mu(\rho^{-1}(U)) > 0$ for any open subset $U \subset X$, then we say that $(X, S)$ is fully $t$-universal in the ergodic sense. In other words, $(X, S)$ is fully universal if any free ergodic dynamical system $(Y, T, \mu)$ with entropy less than $t$ can be realized as a fully supported invariant measure on $(X, S)$.

An almost Borel embedding of $\mathcal{Y}$ into $(X, S)$ is an injective morphism from a universally full $S$-invariant subset of $Y$ to $(X, S)$. We say that $(X, S)$ is $t$-universal in the almost Borel sense if every free Borel system $\mathcal{Y}$ with $h(\mathcal{Y}) < t$ admits an almost Borel embedding into $(X, S)$. By $h(\mathcal{Y})$ we refer to the Gurevich entropy of $\mathcal{Y}$, given by:
\begin{equation}
h(\mathcal{Y}) = \sup_{\mu \in \text{Prob}(\mathcal{Y})} h_{\mu}(Y, T),
\end{equation}
where $h_{\mu}(Y, T)$ is the Kolmogorov-Sinai entropy of the measure-preserving system $(Y, \mu, T)$.

It follows from the variational principle that the Gurevich entropy coincides with the topological entropy for compact topological dynamical systems. See [40] for a short proof.

2.10. Rokhlin towers. We now recall the notion of Rokhlin towers and certain versions of Rokhlin’s lemma for $\mathbb{Z}^d$ actions. Rokhlin towers are instrumental in the proof of fundamental results in ergodic theory.

From now on $\mathcal{Y} = (Y, T)$ will denote a free Borel $\mathbb{Z}^d$-dynamical system. For $F \subset \mathbb{Z}^d$ and $Z \subset Y$, we will use the notation
\begin{equation}
T^F Z = \bigcup_{i \in F} T^i(Z).
\end{equation}
Given a finite $F \subset \mathbb{Z}^d$, $\mu \in \text{Prob}(\mathcal{Y})$ and $\epsilon > 0$, we say that $Z \in \text{Borel}(Y)$ is the base of an $(F, \epsilon, \mu)$-Rokhlin tower if $(T^F Z)_{i \in F}$ are pairwise disjoint and
\begin{equation}
\mu(T^F Z) > 1 - \epsilon.
\end{equation}

If (18) holds for every $\mu \in \text{Prob}_e(\mathcal{Y})$, we say that $Z$ is the base of an $(F, \epsilon)$-Rokhlin tower for $\mathcal{Y}$.

**Proposition 2.1** (Rokhlin’s lemma for $\mathbb{Z}^d$-actions [31, 41]). For every free Borel $\mathbb{Z}^d$ dynamical system $\mathcal{Y}$, every $\mu \in \text{Prob}(\mathcal{Y})$, $n \in \mathbb{N}$ and $\epsilon > 0$ there exists an $(F_n, \epsilon, \mu)$-Rokhlin tower.
Rokhlin’s lemma for \( \mathbb{Z}\)-actions is classical \([51, 29]\). In \([31]\) a much more general version of Rokhlin’s lemma is obtained for actions of countable amenable groups (see also \([59]\)).

We will need a version for free Borel \( \mathbb{Z}^d\)-actions of Rokhlin’s lemma that works simultaneously for all invariant measures. This is a counterpart of \([23, \text{Proposition 7.9}]\) for actions of \( \mathbb{Z}^d\). The result follows immediately from \([21, \text{Theorem 3.1}]\), which is a much stronger result (with a more involved proof). An alternative proof can be obtained using the techniques developed in \([32, \text{Corollary 2}]\).

**Proposition 2.2.** For every free Borel \( \mathbb{Z}^d \) dynamical system \( Y = (Y, T), \ n \in \mathbb{N} \) and \( \epsilon > 0 \) there exists a Borel subset of \( Y \) which is the base of an \( (F_n, \epsilon, \mu) \) Rokhlin tower for every \( \mu \in \text{Prob}_e(Y) \).

For completeness, in Section 7 we provide a proof of Proposition 2.2, that assumes the measurable version of Rokhlin’s lemma (Proposition 2.1), but is otherwise self-contained.

We will repeatedly use the following simple results:

**Lemma 2.3.** If \( F \subset \mathbb{Z}^d \) is a finite set, \( F' \subset F, Z \subset Y \) is the base of an \( (F, \epsilon, \mu)\)-Rokhlin tower and \( Z' \subset Z \) is measurable then

\[
\mu \left( Y \cap T^{F'} Z' \right) < \epsilon + \frac{|F'|}{|F|} + \mu (Z \cap Z' | Z).
\]

**Proof.** Note that

\[
\mu \left( Y \cap T^{F'} Z' \right) = \mu \left( Y \setminus T^F Z \right) + \mu \left( T^F \setminus T^{F'} Z \right) + \mu \left( T^{F'} \setminus (Z \cap Z') \right),
\]

because the \( F \)-translates of \( Z \) are pairwise disjoint, \( \mu(Z) \leq \frac{1}{|F|} \), so \( \mu(T^F \setminus T^{F'} Z) \leq \frac{|F'|}{|F|} \cdot \mu(Z) \). Again, because the \( F \)-translates of \( Z \) are pairwise disjoint,

\[
\mu \left( T^{F'} \setminus (Z \cap Z') \right) = \sum_{i \in F'} \mu \left( T^i (Z \setminus Z') \right) = \sum_{i \in F'} \mu(Z \cap Z' | Z) \mu(Z) \leq \frac{|F'|}{|F|} \mu(Z \cap Z' | Z) \leq \mu(Z \cap Z' | Z).
\]

Plugging these estimates in (20) we get (19). \( \square \)

**Lemma 2.4.** Suppose that \( \delta, \theta > 0 \) that \( \delta + \theta < 1 \) and \( n < \delta m \). If \( Z_n \) is the base of an \( (F_n, \epsilon_n, \mu) \) tower and that \( Z_m \) is the base of an \( (F_m, \epsilon_m, \mu) \) tower then

\[
\mu(T^{\theta F_n} Z_m | Z_n) \leq (1 - \epsilon_n)^{-1}(\theta + \delta).
\]

**Proof.** By the law of total probability

\[
\mu \left( (\theta + \delta) F_n Z_m \right) \geq \sum_{i \in F_n} \mu \left( T^{(\theta + \delta) F_n Z_m} | T^i Z_n \right) \mu \left( T^i Z_n \right).
\]

Because \( n < \delta m \), for every \( i \in F_n \) we have

\[
T^{\theta + \delta} F_n Z_m \subseteq T^{\theta F_n + F_n} Z_m \subseteq T^{(\theta + \delta) F_n} Z_m.
\]

So

\[
\mu \left( (\theta + \delta) F_n Z_m \right) \geq \mu(Z_m) \sum_{i \in F_n} \mu \left( T^{(\theta + \delta) F_n Z_m} | T^i Z_n \right) = \left| F_n \right| \mu(Z_m) \mu(T^{\theta F_n} Z_m | Z_n).
\]

Now because \( Z_n \) is the base of an \( (F_n, \epsilon_n, \mu) \) tower \( |F_n| \mu(Z_n) > 1 - \epsilon_n \), so

\[
\mu(T^{\theta F_n} Z_m | Z_n) \leq (1 - \epsilon_n)^{-1} \mu \left( (\theta + \delta) F_n Z_m \right) = (1 - \epsilon_n)^{-1} (\theta + \delta) F_n \mu(Z_m),
\]

where in the last equality we used that \( (\theta + \delta) F_n \) are pairwise disjoint because \( Z_m \) is the base of an \( (F_m, \epsilon_m, \mu)\)-tower. Also, \( \mu(Z_m) \) is pairwise disjoint because \( Z_m \) is the base of an \( (F_m, \epsilon_m, \mu)\)-tower. Also, \( \mu(Z_m) < \frac{1}{|F_m|} \). So

\[
\mu(T^{\theta F_n} Z_m | Z_n) \leq (1 - \epsilon_n)^{-1} \left| (\theta + \delta) F_n \right| \left| F_m \right| \leq (1 - \epsilon_n)^{-1} (\theta + \delta).
\]

\( \square \)
3. A Shannon-McMillan theorem and ergodic for averages along Rokhlin towers

In this section we introduce some ergodic theoretic tools that we need to prove the main result. These are variants of the Shannon-McMillan theorem and of the mean ergodic theorem, adapted to a sequence of Rokhlin towers:

**Proposition 3.1.** Fix a sequence of \((F_n, \epsilon_n, \mu)\)-towers with base \(Z_n\) with \(\epsilon_n \downarrow 0\). Let \(\epsilon, \theta > 0\). Then:

(i) For every measurable partition \(P\) with \(H_\mu(P) < \infty\) the following holds:

\[
\lim_{n \to \infty} \mu \left( \left| \frac{1}{|F_n|} I_\mu(P^0F_n | F) - h_\mu(Y, T; P | F) \right| > \epsilon \mid Z_n \right) = 0.
\]

(ii) For every \(f \in L^1(\mu)\) the following holds:

\[
\lim_{n \to \infty} \mu \left( \left| \frac{1}{|F_n|} \sum_{i \in \theta F_n} f \circ T^i - \int f d\mu \right| > \epsilon \mid Z_n \right) = 0.
\]

Variants of Proposition 3.1 can be found in the literature and have been used in particular for ergodic embedding results of the type we are aiming to prove. See for instance [50, Theorem 4.4] and the reference within to Rudolph’s proof for the one dimensional case [52, Theorem 7.15]. The proof here is provided mainly for completeness and for the reader’s convenience.

We will present a proof of Proposition 3.1 along the lines of [52, Theorem 7.15]. We will rely on the following well known relative version of the Shannon-McMillan theorem:

**Proposition 3.2.** (Relative Shannon-McMillan theorem) For every \(\epsilon > 0\)

\[
\lim_{n \to \infty} \mu \left( \left| \frac{1}{|F_n|} I_\mu(P^0F_n | F) - h_\mu(Y, T; P | F) \right| > \epsilon \right) = 0.
\]

As mentioned in [33, Proposition 2.2] it can be obtained by following the proof of Shannon-McMillan theorem (say in [36, Theorem 9.2.5]).

**Proof of Proposition 3.1.** Let us prove (i): We will use the short hand

\[ h = h_\mu(Y, T; P | F) \]

and

\[ I_{\theta, n} = I_\mu(P^0F_n | F), \ n \in \mathbb{N}, \ \theta \in (0, 1). \]

Fix \(\theta \in (0, 1)\) and \(\epsilon > 0\). We will show that for every \(\delta > 0\) there exists \(N \in \mathbb{N}\) such that for all \(n > N\)

\[
\mu \left( \left| \frac{1}{|F_n|} I_{\theta, n} - h \right| > \epsilon \mid Z_n \right) < \delta.
\]

Choose \(\zeta, \gamma > 0\) such that

\[
\zeta < \frac{1}{4d} \frac{\epsilon}{2h + \epsilon \theta}, \ \gamma < \frac{\epsilon}{4}.
\]

Note that \(\zeta < \theta\). From the relative Shannon-McMillan theorem (Proposition 3.2) it follows there exists \(N_1 \in \mathbb{N}\) such that for every \(n > N_1\)

\[
\sup_{\mu(A) > \frac{1}{2} \zeta^d} \mu \left( \left| \frac{1}{|F_n|} I_{\theta, n} - h \right| > \gamma \mid A \right) < \frac{\delta}{2}.
\]

Also, by possibly increasing \(N_1\), we can assume that \(\mu(T^{\zeta}F_n Z_n) > \frac{1}{2} \zeta^d\) for every \(n > N_1\). Choose \(N \in \mathbb{N}\) so that \(N > N_1(\theta - \zeta)^{-1}\). From (27) for any \(n > N\),

\[
\mu \left( \left| \frac{1}{|F_n|} I_{\theta+\zeta, n} - h \right| > \gamma \mid \theta \right) \quad \text{and} \quad \mu \left( \left| \frac{1}{|F_n|} I_{\theta-\zeta, n} - h \right| > \gamma \mid \theta \right) < \frac{\delta}{2}.
\]

Thus there exists \(\tilde{i} \in \zeta F_n\) such that

\[
\mu \left( \left| \frac{1}{|F_n|} I_{\theta+\zeta, n} - h \right| > \gamma \right) \quad \text{and} \quad \mu \left( \left| \frac{1}{|F_n|} I_{\theta-\zeta, n} - h \right| > \gamma \right) < \frac{\delta}{2}.
\]
If \( y \in Y \) and \( i \in \zeta F_n \) satisfy \( \frac{1}{|\theta F_n|} I_{\theta, n}(y) - h \leq \gamma \) then
\[
\frac{1}{|\theta F_n|} I_{\theta, n}(y) - h \leq \frac{(\theta + \zeta)^d}{\theta^d} \frac{1}{|\theta F_n|} I_{\theta + \zeta, n}(T^i(y)) - h \leq \frac{(\theta + \zeta)^d - \theta^d}{\theta^d} h + \frac{(\theta + \zeta)^d \gamma}{\theta^d} \leq \zeta \frac{d(\theta + \zeta)^{d-1}}{\theta^d} h + \frac{(\theta + \zeta)^d \gamma}{\theta^d} < \epsilon,
\]
where in the last inequality we used (26). Similarly if \( y \in Y \) such that \( \frac{1}{|\theta F_n|} I_{\theta - \zeta, n}(T^i(y)) - h \leq \gamma \) then
\[
\frac{1}{|\theta F_n|} I_{\theta, n}(y) - h \geq \frac{(\theta - \zeta)^d}{\theta^d} \frac{1}{|\theta F_n|} I_{\theta - \zeta, n}(T^i(y)) - h \geq \frac{(\theta - \zeta)^d - \theta^d}{\theta^d} h - \frac{(\theta - \zeta)^d \gamma}{\theta^d} > -\epsilon.
\]
It follows that for all \( n > N \),
\[
\mu \left( \frac{1}{|\theta F_n|} I_{\theta, n} - h \right) > \epsilon \mid Z_n \right) < \delta.
\]
Hence (25) holds for all \( n > N \).

The proof of (ii) is almost identical, except for the following changes: By linearity of the integral and the triangle inequality, it is enough to prove (23) assuming \( f \) is non-negative \( f \) and \( \int f d\mu < \infty \). This time we denote
\[
h = \int f d\mu
\]
and
\[
I_{\theta, n} = \sum_{i \in \theta F_n} f \circ T^i \text{ for } \theta \in (0, 1], n \in \mathbb{N}.
\]
By the mean ergodic theorem, with these notations it follows that for every \( \epsilon > 0 \) there exists \( N \) such that for every \( n > N \) (27) holds. From here the proof is identical as part (i) (except that \( h \) and \( I_{\theta, n} \) have been redefined). \( \square \)

4. APPROXIMATE COVERINGS NUMBERS AND RELATIVE APPROXIMATE COVERING NUMBERS FOR PARTITIONS

The Shannon-McMillan theorem interprets the entropy of a process in terms of the number of “symbols per iteration” required to “encode an orbit segment”. We introduce some notation to formalize and exploit the interpretation of entropy in terms of coding. Let \( \mathcal{P} \) be a finite Borel partition of \( Y, \mu \in \text{Prob}_e(Y, T), A \subset Y \) a Borel set and \( \epsilon > 0 \). We define the \( \epsilon \)-covering number of \( A \) with respect to the partition \( \mathcal{P} \) by:
\[
COV_{\mu, \epsilon, \mathcal{P}}(A) = \min \left\{ |\mathcal{G}| : \mathcal{G} \subset \mathcal{P} \text{ and } \mu(\bigcup \mathcal{G} \cap A) \geq (1 - \epsilon)\mu(A) \right\}.
\]

The Shannon-McMillan theorem is roughly equivalent to the statement that for an ergodic system \( (Y, \mu, T) \) and a measurable partition \( \mathcal{P} \) having finite Shannon entropy
\[
\forall \epsilon \in (0, 1) \ h_{\mu}(Y, T; \mathcal{P}) = \lim_{n \to \infty} \frac{1}{F_n} \log \left( COV_{\mu, \epsilon, \mathcal{P}F_n}(A) \right),
\]
for any \( A \in \text{Borel}(Y) \) such that \( \mu(A) > 0 \).

Now let \( \mathcal{Q} \) be another finite Borel partition. The \( \epsilon \)-covering number of \( A \) with respect to the partition \( \mathcal{P} \) relative to \( \mathcal{Q} \) is defined by:
\[
COV_{\mu, \epsilon, \mathcal{P} \mid \mathcal{Q}}(A) = \min \left\{ \max_{\mathcal{Q} \in \mathcal{Q}'} COV_{\mu, \epsilon, \mathcal{P} \cap \mathcal{Q}'}(A) : \mathcal{Q}' \subset \mathcal{Q} \text{ and } \mu(\bigcup \mathcal{Q}' \cap A) \geq (1 - \epsilon)\mu(A) \right\}
\]

This is closely related to relative entropy by the following formula:
\[
\forall \epsilon \in (0, 1) \ h_{\mu}(Y, T; \mathcal{P} \mid \mathcal{Q}^{2d}) = \lim_{n \to \infty} \frac{1}{F_n} \log \left( COV_{\mu, \epsilon, \mathcal{P}F_n \mid \mathcal{Q}F_n}(A) \right),
\]
In the following sections we will often use relative \( \epsilon \)-covering numbers, as a “proxy” to entropy. We are about to prove a few basic lemmas about relative \( \epsilon \)-covering numbers. These statements have closely related well known counterparts in terms of entropy. The following is an elementary auxiliary result in basic probability theory that we include for completeness:
Lemma 4.1. Suppose $0 < \delta < \epsilon < 1$ and that $\nu$ is a probability measure on $Y$, $A \subset Y$ measurable set such that $\nu(A) > 1 - \delta$, and $P$ is a measurable partition. Let:

$$P_\epsilon = \{ P \in P : \nu(P \cap A) \geq (1 - \epsilon)\nu(P) \}.$$  

Then

$$\nu \left( \bigcup P_\epsilon \right) \geq 1 - \frac{\delta}{\epsilon}$$  

Proof. Denote

$$G = \bigcup P_\epsilon \text{ and } B = Y \setminus G = \bigcup (P \setminus P_\epsilon).$$

Then

$$\nu(B \cap A) = \sum_{P \in P \setminus P_\epsilon} \nu(P \cap A) < (1 - \epsilon) \sum_{P \in P \setminus P_\epsilon} \nu(P) = (1 - \epsilon)\nu(B).$$

So

$$1 - \delta < \nu(A) = \nu(G \cap A) + \nu(B \cap A) \leq \nu(G) + (1 - \epsilon)(1 - \nu(G)).$$

It follows that

$$\nu(G) \geq \frac{\epsilon - \delta}{\epsilon} = 1 - \frac{\delta}{\epsilon}. \quad \square$$

Lemma 4.2. Suppose $A \in \text{Borel}(Y)$, $\mu(A) > 0$ and that $P_1, P_2, P_3$ are Borel partitions. For every $0 < \epsilon < 1$ the following inequalities holds:

$$\text{(31)} \quad \text{COV}_{\mu, \epsilon + \epsilon^2, P_1 | P_2 \lor P_3}(A) \leq \text{COV}_{\mu, \epsilon^2, P_1 | P_2}(A)$$

$$\text{(32)} \quad \text{COV}_{\mu, 2\epsilon, P_1 \lor P_2 | P_3}(A) \leq \text{COV}_{\mu, \epsilon, P_1 | P_2}(A) \cdot \text{COV}_{\mu, \epsilon, P_2 | P_3}(A)$$

$$\text{(33)} \quad \text{COV}_{\mu, \epsilon, P_1 | P_3}(A) \leq \text{COV}_{\mu, \epsilon^2/6, P_1 | P_2}(A) \cdot \text{COV}_{\mu, \epsilon^2/6, P_2 | P_3}(A)$$

Inequalities (31), (32) and (33) correspond to the following well-known entropy inequalities respectively:

$$h_\mu(Y, T; P_1 | P_2 \lor P_3) \leq h_\mu(Y, T; P_1 | P_3) + h_\mu(Y, T; P_2 | P_3)$$

and

$$h_\mu(Y, T; P_1 \lor P_2 | P_3) \leq h_\mu(Y, T; P_1 | P_3) + h_\mu(Y, T; P_2 | P_3)$$

The expressions appearing in the “$\epsilon$” parameter have not been fully optimized, and are not very significant for our applications. We only use the fact that we can make the “$\epsilon$-expressions” on the left hand side arbitrary small by making the “$\epsilon$-expressions” on the right hand side of the inequalities sufficiently small.

Proof of Lemma 4.2: By replacing $\mu$ with $\mu(\cdot | A)$ we assume without loss of generality that $\mu(A) = 1$. Let us prove (31). By definition of $\text{COV}_{\mu, \epsilon^2, P_1 | P_2}(A)$, there exists $P_2^{(\epsilon^2)} \subseteq P_2$ such that

$$\mu \left( \bigcup_{P \in P_2 \setminus P_2^{(\epsilon^2)}} P \right) < \epsilon^2,$$

and so that for every $P \in P_2^{(\epsilon^2)}$ we have there exists a subset $G_P^{(\epsilon^2)} \subseteq P_1$ with $|G_P^{(\epsilon^2)}| \leq \text{COV}_{\mu, \epsilon^2, P_1 | P_2}(A)$ and

$$\mu \left( \bigcup_{Q \in G_P^{(\epsilon^2)}} Q \cap P \right) > (1 - \epsilon^2)\mu(P).$$
By Lemma 4.1, for every \( P \in \mathcal{P}_2^{(2)} \) there exists \( \mathcal{P}_3^{(P, \epsilon)} \subset \mathcal{P}_3 \) such that for all \( P' \in \mathcal{P}_3^{(P, \epsilon)} \)

\[
\mu \left( \bigcup_{Q \in \mathcal{G}_P^{(2)}} Q \cap P \cap P' \right) > (1 - \epsilon) \mu(P' \cap P)
\]

and

\[
\mu \left( \bigcup_{P' \in \mathcal{P}_3^{(P, \epsilon)}} P' \cap P \right) > (1 - \epsilon) \mu(P).
\]

Consider the set \( \mathcal{P}_{2,3} \subset \mathcal{P}_2 \vee \mathcal{P}_3 \) given by

\[
\mathcal{P}_{2,3} = \left\{ P \cap P' : P \in \mathcal{P}_2^{(2)}, P' \in \mathcal{P}_3^{(P, \epsilon)} \right\}.
\]

We see that

\[
\mu \left( \bigcup_{(P \cap P') \in (\mathcal{P}_2 \vee \mathcal{P}_3) \setminus \mathcal{P}_{2,3}} P \cap P' \right) \leq \mu \left( \bigcup_{P \in \mathcal{P}_2 \setminus \mathcal{P}_2^{(2)}} P \right) + \sum_{P' \in \mathcal{P}_3^{(P, \epsilon)}} \mu \left( \bigcup_{(P \cap P') \in (\mathcal{P}_3 \setminus \mathcal{P}_3^{(P, \epsilon)})} (P \cap P') \right) \leq \epsilon^2 + \epsilon.
\]

Now if \( P \cap P' \in \mathcal{P}_{2,3} \) then

\[
\text{COV}_{\mu, \epsilon^2 + \epsilon, \mathcal{P}_1}(A \cap P \cap P') \leq \text{COV}_{\mu, \epsilon^2, \mathcal{P}_1}(A \cap P) \leq \text{COV}_{\mu, \epsilon^2, \mathcal{P}_1}(A) \leq \text{COV}_{\mu, \epsilon^2, \mathcal{P}_3}(A).
\]

This proves (31).

Let us prove (32): By definition, there exists subsets \( \mathcal{P}_3^{(1)} \subset \mathcal{P}_3 \) and \( \mathcal{P}_3^{(2)} \subset \mathcal{P}_3 \) such that for \( i = 1, 2 \)

\[
\mu \left( \bigcup_{P \in \mathcal{P}_3 \setminus \mathcal{P}_3^{(i)}} P \right) < \epsilon.
\]

Also for every \( P \in \mathcal{P}_3^{(i)} \) there exists a subset \( \mathcal{G}_P^{(i)} \subset \mathcal{P}_i \) of size at most \( \text{COV}_{\mu, \epsilon, \mathcal{P}_1 \cup \mathcal{P}_3}(A) \) such that the union of the elements of \( \mathcal{G}_P^{(i)} \) cover all but an \( \epsilon \)-fraction of the measure of \( P \). Let \( \mathcal{P}_3^{(1,2)} = \mathcal{P}_3^{(1)} \cap \mathcal{P}_3^{(2)} \). For \( P \in \mathcal{P}_3^{(1,2)} \) define \( \mathcal{G}_P^{(1,2)} \subset \mathcal{P}_1 \vee \mathcal{P}_2 \) by

\[
\mathcal{G}_P^{(1,2)} = \left\{ Q_1 \cap Q_2 : Q_1 \in \mathcal{G}_P^{(1)} \text{ and } Q_2 \in \mathcal{G}_P^{(2)} \right\}.
\]

Then

\[
\mu \left( \bigcup_{P \in \mathcal{P}_3 \setminus \mathcal{P}_3^{(1,2)}} P \right) \leq \mu \left( \bigcup_{P \in \mathcal{P}_3 \setminus \mathcal{P}_3^{(1)}} P \right) + \mu \left( \bigcup_{P \in \mathcal{P}_3 \setminus \mathcal{P}_3^{(2)}} P \right) \leq 2 \epsilon,
\]

and for every \( P \in \mathcal{P}_3^{(1,2)} \)

\[
|\mathcal{G}_P^{(1,2)}| \leq |\mathcal{G}_P^{(1)}| \cdot |\mathcal{G}_P^{(2)}| \leq \text{COV}_{\mu, \epsilon, \mathcal{P}_1 \cup \mathcal{P}_3}(A) \cdot \text{COV}_{\mu, \epsilon, \mathcal{P}_2 \cup \mathcal{P}_3}(A)
\]

\[
\mu \left( P \setminus \bigcup_{Q_1 \cap Q_2 \in \mathcal{G}_P^{(1,2)}} Q_1 \cap Q_2 \right) \leq \mu \left( P \setminus \bigcup_{Q_1 \in \mathcal{G}_P^{(1)}} Q_1 \right) + \mu \left( P \setminus \bigcup_{Q_2 \in \mathcal{G}_P^{(2)}} Q_2 \right) < 2 \epsilon \mu(P).
\]

This proves (32).

Let us prove (33): Denote

\[
N_1 = \text{COV}_{\mu, \epsilon^2/6, \mathcal{P}_2 \cup \mathcal{P}_3}(A), \quad N_2 = \text{COV}_{\mu, \epsilon^2/6, \mathcal{P}_2}(A).
\]

By definition, there exists \( \mathcal{P}_3' \subset \mathcal{P}_3 \) such that

\[
\mu \left( \bigcup_{P \in \mathcal{P}_3 \setminus \mathcal{P}_3'} P \right) < \epsilon^2/6.
\]
and so that for every \( P \in \mathcal{P}_3 \) there exists a subset \( \mathcal{G}_P \subset \mathcal{P}_2 \) with \( |\mathcal{G}_P| \leq N_1 \) and
\[
\mu \left( \bigcup_{Q \in \mathcal{P}_2 \setminus \mathcal{G}_P} Q \cap P \right) < \frac{\epsilon^2}{6} \mu(P).
\]
Similarly, there exists \( \mathcal{P}_2' \subset \mathcal{P}_2 \) such that
\[
\mu \left( \bigcup_{Q \in \mathcal{P}_2 \setminus \mathcal{P}_2'} Q \right) < \frac{\epsilon^2}{6}
\]
and so that for every \( Q \in \mathcal{P}_2' \) there exists a subset \( \mathcal{G}'_Q \subset \mathcal{P}_1 \) with \( |\mathcal{G}'_Q| \leq N_2 \) and
\[
\mu \left( \bigcup_{Q' \in \mathcal{P}_1 \setminus \mathcal{G}'_Q} Q' \cap Q \right) < \frac{\epsilon^2}{6} \mu(Q).
\]
It follows that
\[
\mu \left( \bigcup_{Q \in \mathcal{P}_2 \setminus \mathcal{P}_2'} Q \right) < \frac{\epsilon^2}{6} 
\]
Let
\[
\mathcal{P}_3^* = \left\{ P \in \mathcal{P}_3 : \mu \left( \bigcup_{Q \in \mathcal{P}_2 \setminus \mathcal{P}_2'} \bigcup_{Q' \in \mathcal{P}_1 \setminus \mathcal{G}'_Q} Q' \cap Q \cap P \right) < \frac{\epsilon}{2} \mu(P) \right\}
\]
and
\[
\mathcal{P}_3'' = \left\{ P \in \mathcal{P}_3 : \mu \left( \bigcup_{Q \in \mathcal{P}_2 \setminus \mathcal{P}_2'} Q \cap P \right) < \frac{\epsilon}{2} \mu(P) \right\}
\]
Then by Lemma 4.1
\[
\mu \left( \bigcup_{P \in \mathcal{P}_3 \setminus \mathcal{P}_3^*} P \right) < \frac{\epsilon}{3} \text{ and } \mu \left( \bigcup_{P \in \mathcal{P}_3 \setminus \mathcal{P}_3''} P \right) < \frac{\epsilon}{3}.
\]
For every \( P \in \mathcal{P}_3' \) let
\[
\mathcal{G}_P' = \{ Q' \in \mathcal{P}_1 : \exists Q \in \mathcal{G}_P \cap \mathcal{P}_2' \text{ s.t. } Q' \in \mathcal{G}'_Q \}.
\]
Then
\[
|\mathcal{G}_P'| \leq N_1 \cdot N_2.
\]
Let \( \mathcal{P}_3''' = \mathcal{P}_3^* \cap \mathcal{P}_3' \cap \mathcal{P}_3'' \). Then
\[
\mu \left( \bigcup_{P \in \mathcal{P}_3 \setminus \mathcal{P}_3'''} P \right) < \epsilon.
\]
It follows that for every \( P \in \mathcal{P}_3''' \)
\[
\mu \left( \bigcup_{Q' \in \mathcal{P}_1 \setminus \mathcal{G}_P'} Q' \cap P \right) < \mu \left( \bigcup_{Q \in \mathcal{P}_2 \setminus \mathcal{P}_2'} Q \cap P \right) + \mu \left( \bigcup_{Q \in \mathcal{P}_2 \setminus \mathcal{P}_2'} Q \cap Q' \cap P \right)
\]
Choose \( P \in \mathcal{P}_3''' \). Because \( P \in \mathcal{P}_3''' \subset \mathcal{P}_3^* \), \( \mu \left( \bigcup_{Q \in \mathcal{P}_2 \setminus \mathcal{P}_2'} Q \cap P \right) < \frac{\epsilon \mu(P)}{2} \). Also, because \( P \in \mathcal{P}_3''' \subset \mathcal{P}_3^* \),
\[
\mu \left( \bigcup_{Q \in \mathcal{P}_2 \setminus \mathcal{P}_2'} Q \cap Q' \cap P \right) < \frac{\epsilon}{2} \mu(P).
\]
Thus
\[
\mu \left( \bigcup_{Q \in \mathcal{P}_1 \setminus \mathcal{G}_P''} Q' \cap P \right) < \epsilon \mu(P).
\]
This shows that $COV_{\mu,\epsilon,\mathcal{P}|\mathcal{Q}_n}(A) \leq N_1 \cdot N_2$. □

From now on, let $(\epsilon_n)_{n=1}^\infty$ be a sequence of positive numbers tending to 0 and let $(Z_n)_{n=1}^\infty$ be a sequence of Borel subsets of $Y$ so that for every $n \in \mathbb{N}$ $Z_n \subset Y$ is the base of an $(F_n, \epsilon_n)$-tower for $\mathcal{Y}$.

The following is a manifestation of the Shannon-McMillan theorem for towers (Proposition 3.1), in terms of $\epsilon$-covering numbers:

**Lemma 4.3.** Let $\mathcal{P}$ be a finite Borel partition of $Y$, $\theta \in (0, 1)$ and $\mu \in \text{Prob}_\epsilon(\mathcal{Y})$. For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for every $n > N$

$$COV_{\mu,\epsilon,\mathcal{P}|\mathcal{Q}_n}(Z_n) \leq \exp \left(|\theta F_n| \left(h_\mu(Y, T; \mathcal{P}) + \epsilon \right) \right).$$

**Proof.** For $n \in \mathbb{N}$ and $\epsilon > 0$ and

$$\hat{Z}_{n,\epsilon} = \left\{ y \in Z_n : \frac{1}{|\theta F_n|} \log \left( \mu(\mathcal{P}^{\theta F_n}(y))^{-1} \right) \leq h_\mu(Y, T; \mathcal{P}) + \epsilon \right\}.$$

Choose $y \in \hat{Z}_{n,\epsilon}$ then

$$\mu(\mathcal{P}^{\theta F_n}(y)) \geq e^{-(h_\mu(Y, T; \mathcal{P}) + \epsilon)|\theta F_n|}.$$ It follows that every elements of $\mathcal{P}^{\theta F_n}$ that intersects $\hat{Z}_{n,\epsilon}$ covers at least an $e^{-(h_\mu(Y, T; \mathcal{P}) + \epsilon)|\theta F_n|}$ fraction of $Y$. Thus $\hat{Z}_{n,\epsilon}$ can be covered by at most

$$e^{(h_\mu(Y, T; \mathcal{P}) + \epsilon)|\theta F_n|}$$ elements of $\mathcal{P}^{\theta F_n}$. By Proposition 3.1, if $n$ is sufficiently big then

$$\mu(\hat{Z}_{n,\epsilon}) \geq (1 - \epsilon)\mu(Z_n),$$

This shows that (34) holds. □

We remark that Proposition 3.1 also implies a corresponding lower bound on $COV_{\mu,\epsilon,\mathcal{P}|\mathcal{Q}_n}(Z_n)$. We will not use this in this paper.

Here is a “relative” version of Lemma 4.3:

**Lemma 4.4.** Let $\mathcal{P}, \mathcal{Q}$ be finite Borel partitions of $Y$, $\theta \in (0, 1)$ and $\mu \in \text{Prob}_\epsilon(\mathcal{Y})$. For every $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for every $n > N$

$$COV_{\mu,\epsilon,\mathcal{P}|\mathcal{Q}_n}(Z_n) \leq \exp \left(|\theta F_n| \left(h_\mu(Y, T; \mathcal{P} | \mathcal{Q}^{z^d}) + \epsilon \right) \right).$$

**Proof.** Choose $\epsilon > 0$. For $n \in \mathbb{N}$ let $\mu_{Z_n}(\cdot) = \mu(\cdot | Z_n)$ and

$$\hat{Z}_{n,\epsilon} = \left\{ y \in Z_n : \frac{1}{|\theta F_n|} \log \left( \mu(\mathcal{P}^{\theta F_n}(y) | \mathcal{Q}^{z^d}(y))^{-1} \right) \leq h_\mu(Y, T; \mathcal{P} | \mathcal{Q}^{z^d}) + \epsilon/2 \right\}.$$

We define

$$\mathcal{Q}_n' = \left\{ Q \in \mathcal{Q}_n^{\theta F_n} : \mu_{Z_n}(\hat{Z}_{n,\epsilon} \cap Q) > (1 - \epsilon)\mu_{Z_n}(Q) \right\}.$$ Choose $Q \in \mathcal{Q}_n'$. Then for every $y \in \hat{Z}_{n,\epsilon} \cap Q$, $\mu(\mathcal{P}^{\theta F_n}(y) \cap Q) \geq e^{-(h_\mu(Y, T; \mathcal{P} | \mathcal{Q}^{z^d}) + \epsilon/2)|\theta F_n|} \mu(Q)$. Thus for every $Q \in \mathcal{Q}_n'$ we see that $Q \cap \hat{Z}_{n,\epsilon}$ can be covered by at most $e^{(h_\mu(Y, T; \mathcal{P} | \mathcal{Q}^{z^d}) + \epsilon/2)|\theta F_n|}$ elements of $\mathcal{P}^{\theta F_n}$. This means that

$$COV_{\mu,\epsilon,\mathcal{P}|\mathcal{Q}_n'(Q \cap Z_n)} \leq \exp \left(|\theta F_n| \left(h_\mu(Y, T; \mathcal{P} | \mathcal{Q}^{z^d}) + \epsilon/2 \right) \right).$$

Recall that by definition for $Q \in \mathcal{Q}_n'$ we have

$$\mu(\hat{Z}_{n,\epsilon} \cap \hat{Z}_{n,\epsilon}) > (1 - \epsilon)\mu(Q \cap Z_n).$$
By Proposition 3.1 there exists $N \in \mathbb{N}$ so that for every $n > N$, $\mu_{Z_n}(\tilde{Z}_{n,\epsilon}) > 1 - \epsilon^2$. By Lemma 4.1 applied with $\nu = \mu_{Z_n}$, $A = \tilde{Z}_{n,\epsilon}$, $P = \mathcal{Q}^0F_n$ and $\delta' = \epsilon^2$, we see that for every $n > N$ we have $\mu(\bigcup Q_n \cap Z_n) \geq (1 - \epsilon)\mu(Z_n)$. By definition of the relative $\epsilon$-covering, we see that for $n > N (36)$ holds. □

5. Specification and flexible sequences

We now turn our attention from ergodic theory and measurable dynamics to topological dynamics. Recall that $(X, S)$ is a topological $\mathbb{Z}^d$ dynamical system. This means that $X$ is a compact metric space with a metric $d_X : X \times X \to \mathbb{R}_+$ and for every $i \in \mathbb{Z}^d$, $S^i : X \to X$ is a homeomorphism so that $S^i+j = S^i \circ S^j$ for every $i, j \in \mathbb{Z}^d$.

5.1. Flexible sequences. Let $(\epsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$ be sequences of numbers between 0 and 1, both decreasing to 0, and so that

$$\lim_{n \to \infty} n\delta_n = \infty.$$  

Throughout most of the paper the sequences $(\epsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$ will be fixed in the background. We think of $(\epsilon_n)_{n=1}^{\infty}$ as the “specification precision sequence” and think of $(\delta_n)_{n=1}^{\infty}$ as the “specification gap sequence”.

For $n, m \in \mathbb{N}$, we will use the notation $n \ll m$ to mean that “$m$ is much bigger than $n$”, in some manner that takes into account the sequences $(\epsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$.

Formally it can be defined as follows:

$$n \ll m \text{ iff } \epsilon_m < \frac{1}{10^m d_{n} |F_n|} \cdot \frac{1 - \epsilon_n}{|F_n|}, \quad \delta_m < \frac{1}{10^m d_{n} |F_n|} \cdot \frac{1 - \epsilon_n}{|F_n|} \quad \text{and} \quad n^2 < \frac{\delta_m}{1000 d_{n} m}.$$  

The notation $n \ll m$ obscures the dependence on the sequences $(\epsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$, but it will simplify the notation in later parts of the proof.

Recall that $(X, S)$ is a topological $\mathbb{Z}^d$-dynamical system. Let $C = (C_n)_{n=1}^{\infty} \in (2^X)^\mathbb{N}$ be a sequence of finite subsets of $X$, namely $C_n \subset X$ for every $n$.

Given $k, n \in \mathbb{N}$ with $k \ll n$, an $(n, k)$-specification set is a subset $W \in C_k^K$, where $K \subset (1 - \delta_n)F_n$ is $(1 + \delta_k)F_k$ spaced.

We say that $W$ is an $n$-specification set if it is an $(n, k)$-specification set for some $k \ll n$. Let $\text{Spec}_{n,k}(C)$ denote the collection of $(n, k)$-specification sets for $C$, and let $\text{Spec}_{n}(C) = \bigcup_{k \ll n} \text{Spec}_{n,k}(C)$.

We say that $x \in X$ $n$-shadows $W \in C_k^K$ if

$$d_X^F(S^i(x), W_i) < \frac{1}{4} \epsilon_k$$

for every $i \in K$.

Let $C = (C_n)_{n=1}^{\infty} \in (2^X)^\mathbb{N}$ be a sequence of finite subsets of $X$, namely $C_n \subset X$ for every $n$. We call the sequence $C = (C_n)_{n=1}^{\infty} \in (2^X)^\mathbb{N}$ flexible with respect to $((\epsilon_n)_{n=1}^{\infty}, (\delta_n)_{n=1}^{\infty})$ if for every $k \in \mathbb{N}$, the set $C_k \subset X$ is $(\epsilon_k, F_k)$-separated, and in addition for any $n$-specification set $W \in \text{Spec}_{n}(C)$ there exists $x \in C_n$ that $n$-shadows $W$. Equivalently, $C$ is a flexible sequence if and only if for every $n \in \mathbb{N}$ there exists a function

$$\text{Ext}_n : \text{Spec}_{n}(C) \to C_n,$$

so that

$$\text{Ext}_n(W) \text{ } n\text{-shadows } W \text{ whenever } W \in \text{Spec}_{n}(C).$$

The phrase “$(X, S)$ admits a flexible sequence $C = (C_n)_{n=1}^{\infty} \in (2^X)^\mathbb{N}$" formally means that there exists monotone decreasing sequences $(\epsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$ both tending to 0 so that $C$ is a flexible sequence with respect to those.

Given a flexible sequence $C \in (2^X)^\mathbb{N}$, we say it is a flexible marker sequence (with respect to $(\epsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$) if there exists $\epsilon_* > 0$ such that

$$\forall x \in X \text{ and } n \in \mathbb{N}, \text{ the set } \left\{ i \in \mathbb{Z}^d : d_X^F(S^i(x), C_n) < \epsilon_\ast \right\} \text{ is } (1 - \delta_n)F_n\text{-spaced.}$$

For a flexible sequence $C = (C_n)_{n=1}^{\infty} \in (2^X)^\mathbb{N}$ we define:
\begin{equation}
    h(C) = \lim_{n \to \infty} \frac{1}{|F_n|} \log |C_n|.
\end{equation}
and
\begin{equation}
    h_\ast(C) = \liminf_{n \to \infty} \frac{1}{|F_n|} \log sep_\ast(C_n, F_n).
\end{equation}

The limit in (46) always exists in the extended sense, as we show in Lemma 5.6 below.

We now state the main technical result of our paper:

**Theorem 5.1.** Let \((X, S)\) be a \(\mathbb{Z}^d\)-dynamical system. If \(C = (C_n)_{n=1}^{\infty} \in (2^X)^\mathbb{N}\) is a flexible marker sequence for \((X, S)\) then \((X, S)\) is \(h(C)\)-universal in the almost-Borel sense.

The following result deals with full universality:

**Proposition 5.2.** Under the assumptions of Theorem 5.1, if in addition every \(x \in X\) is an accumulation point of \(\bigcup_{n=1}^{\infty} C_n\), then \((X, S)\) is fully \(h(C)\)-universal.

For systems having a flexible marker sequence \(C\) such that \(h(C) = \infty\) we have the following:

**Theorem 5.3.** Let \((X, S)\) be topological \(\mathbb{Z}^d\)-system with a flexible marker sequence \(C\) such that \(h(C) = \infty\) and \(\bigcup_{k=1}^{\infty} X_k\) is dense for all \(n \in \mathbb{N}\). Let \(Y = (Y, T)\) be a free Borel \(\mathbb{Z}^d\)-system. Then there exists a \(T\)-invariant Borel subset \(Y_0 \subset Y\) so that \(Y \setminus Y_0\) is null with respect to any \(T\)-invariant probability measure and a Borel embedding of \((Y_0, T)\) into \((X, S)\). Furthermore, we can choose a Borel embedding so that the push-forward of any \(T\)-invariant measure on \(Y\) has full support in \(X\).

From now on, assume that \((\epsilon_n)_{n=1}^{\infty}\) and \((\delta_n)_{n=1}^{\infty}\) are fixed, and that \(C = (C_n)_{n=1}^{\infty}\) is a flexible sequence for the \(\mathbb{Z}^d\)-system \((X, S)\).

We also assume that \(Ext_n : Spec_n(C) \to C_n\) satisfies (44).

**Lemma 5.4.** Suppose that \(n \ll m\), that \(K \subset (1 - \delta_m)F_m\) is \((1 + \delta_n)F_n\)-spaced and that \(W, W' \in C_n^K\). Then:
\begin{equation}
    d_{X_n}^{F_n}(Ext_m(W), Ext_m(W')) \geq \frac{1}{2} \max_{i \in K} d_{X_i}^{F_i}(W_i, W'_i),
\end{equation}
with equality iff \(W = W'\), in which case both sides of the inequality are zero. In particular, if \(W \neq W'\),
\begin{equation}
    d_{X_n}^{F_n}(Ext_m(W), Ext_m(W')) > \frac{1}{2} \epsilon_n.
\end{equation}

**Proof.** Suppose \(W, W' \in C_n^K\). If \(W = W'\) then it is trivial that both sides of the inequality in (48) are zero. So suppose \(W \neq W'\). Then there exists \(i \in K\) so that \(W_i \neq W'_i\). Since the elements of \(C_n\) are \((\epsilon_n, F_n)\)-separated it follows that \(d_{X_i}^{F_i}(W_i, W'_i) > \epsilon_n\). Suppose that \(i \in W\) is such that \(d_{X_i}^{F_i}(W_i, W'_i)\) is maximal. We have 
\begin{equation}
    d_{X_i}^{F_i}(Ext_m(W), S^{-i}(W'_i)) < \frac{1}{4} \epsilon_n \quad \text{and} \quad d_{X_i}^{F_i}(Ext_m(W'), S^{-i}(W'_i)) < \frac{1}{4} \epsilon_n.
\end{equation}
By the triangle inequality
\begin{equation}
    d_{X_i}^{F_i}(Ext_m(W), Ext_m(W')) \geq d_{X_i}^{F_i}(W_i, W'_i) - 2 \frac{\epsilon_n}{4} > \frac{1}{2} d_{X_i}^{F_i}(W_i, W'_i),
\end{equation}
where in the last inequality we used that \(n \ll m\) and that \(d_{X_i}^{F_i}(W_i, W'_i) > \epsilon_n\).

**Lemma 5.5.** Suppose \(n \ll m\). Then for any \(0 < \eta\),
\begin{equation}
    sep_{\frac{1}{2} \eta}(C_m, F_m) \geq \left( \text{sep}_{\eta}(C_n, F_n) \right)^{\frac{|(1 - 3\eta m)|F_m|}{|m + 3\eta m|F_m|}}.
\end{equation}
In particular,
\begin{equation}
    sep_{\frac{1}{2} \epsilon_n}(C_m, F_m) \geq |C_n|^{\frac{|(1 - 3\epsilon_n m)|F_m|}{|m + 3\epsilon_n m|F_m|}}.
\end{equation}
Proof. Fix $n \ll m$. Define

$$K = (1 - \delta_m)F_m \cap (2 + 3\delta_n)n\mathbb{Z}^d.$$ 

So $K \subset (1 - \delta_m)F_m$ is $(1 + \delta_n)F_n$-spaced. For $0 < \eta$, let $X_\eta \subset C_n$ be an $(\eta, F_m)$-separated set of maximal cardinality. By Lemma 5.4, the image of $X^K_\eta$ under the function $Ext_m: Spec_m(C) \to C_m$ is $(\frac{1}{2}\eta, F_m)$-separated. This shows that $C_m$ contains an $(\frac{1}{2}\eta, F_m)$-separated set of size at least $|X_\eta||K|$. The inequality (49) follows because

$$|K| \geq \frac{|(1 - 3\delta_m)F_m|}{|(1 + 2\delta_n)F_n|}.$$

Lemma 5.6. Let $C = (C_n)_{n=1}^\infty \in (2^X)^\mathbb{N}$ be a flexible sequence, then the limit in (46) exists. Also, for every $\epsilon > 0$, $h_\epsilon(C) \in [0, +\infty)$ and $h(C) \in [0, \infty]$. Furthermore, we have:

$$h(C) = \lim_{\epsilon \to 0} h_\epsilon(C).$$

Proof. The function $\epsilon \mapsto h_\epsilon(C)$ is monotone non-increasing on $(0, \infty)$ so the limit $\lim_{\epsilon \to 0} h_\epsilon(C)$ exists.

By Lemma 5.5, if $n \ll m$ then

$$\frac{1}{|(1 - 3\delta_m)F_m|} \log \text{sep}_{\epsilon_n}(C_m, F_m) \geq \frac{1}{|(1 + 2\delta_n)F_n|} \log |C_n|.$$

Thus, we have

$$h_\epsilon(C) = \liminf_{m \to \infty} \frac{1}{|F_m|} \log \text{sep}_{\epsilon_n}(C_m, F_m) \geq \frac{1}{|1 + 2\delta_n)F_n|} \log |C_n|.$$

Taking $n \to \infty$ on both sides of the inequality, it follows that

$$\lim_{\epsilon \to 0} h_\epsilon(C) \geq \limsup_{n \to \infty} \frac{1}{|F_n|} \log |C_n|.$$

On the other hand, $\text{sep}_\epsilon(C_m, F_m) \leq |C_m|$ for every $m$ so $h_\epsilon(C) \leq \liminf_{n \to \infty} \frac{1}{|F_n|} \log |C_n|$ for every $\epsilon > 0$. This proves that the limit in (46) exists and (51) holds. \qed

The formula (51) can be viewed as an alternative definition of $h(C)$. In particular, it explains the fact that $h(C) \leq h(X, S)$ for any flexible sequence.

5.2. Non uniform specification implies a flexible marker sequence.

Proposition 5.7. For $d = 1$ if $(X, S)$ has non-uniform specification then it has a flexible marker sequence $C = (C_n)_{n=1}^\infty$ such that $h(C) = h(X, S)$ and $\bigcup_{n=1}^\infty C_n$ is dense in $X$.

The result that non-uniform specification implies universality for $\mathbb{Z}$-actions (Theorem 1.3) follows directly from the combination of Theorem 5.1, Proposition 5.2 and Proposition 5.7.

Remark 5.8. The only part of our proof of Proposition 5.7 that is specific for $d = 1$ is our proof of Lemma 5.9 which uses a result of Downarowicz-Weiss [16]. As an alternative, we could have used a direct proof of the conclusion of Lemma 5.9 by strengthening the positive entropy assumption assuming and replacing it with non-uniform specification. This would work for any $d \geq 1$.

The main issue in our proof of Proposition 5.7 is to extract suitable “markers”. A similar component appears in most proofs of ergodic universality. See for instance the “Marker Lemma” of Quas and Soo [49, Lemma 24].

The following basic “Marker Lemma” says that for a system with positive entropy it is possible find $F_n$-orbit segments that do not overlap each-other and have only trivial self overlaps.

Lemma 5.9. If $(X, S)$ is a topological $\mathbb{Z}$-system with positive topological entropy, then for all sufficiently small $\epsilon_1 > 0$, if $n_1 \in \mathbb{N}$ is sufficiently big there exist $\tilde{x}^{(0)}, \tilde{x}^{(1)}, \tilde{x} \in X$ so that

$$d^{F_n_1 \cap (i + F_n_1)}_X (S^{-i}(\tilde{x}^{(0)}), \tilde{x}^{(1)}) > 2\epsilon_1 \text{ for all } i \in \left\{ \frac{3}{2} F_{n_1} \right\},$$

$$d^{F_n_1 \cap (i + F_n_1)}_X (S^{-i}(\tilde{x}^{(t)}), \tilde{x}^{(t)}) > 2\epsilon_1 \text{ for all } i \in \left\{ \frac{3}{2} F_{n_1} \right\} \setminus \{0\} \text{ and } t \in \{0, 1\},$$

where $\tilde{x}^{(0)}$, $\tilde{x}^{(1)}$, and $\tilde{x}$ are $F_{n_1}$-orbit segments that satisfy $\|S^{-i}(\tilde{x}) - \tilde{x}\| > \epsilon_1 \text{ for all } i \notin \left\{ \frac{3}{2} F_{n_1} \right\}$.

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so that for every $\delta > 0$, Lemma 5.10.

Indeed, a system satisfying the conclusion of this lemma cannot be minimal. In this lemma we still assume $d = 1$, but only so we can apply Lemma 5.9.

**Lemma 5.10.** Suppose $(X, S)$ has non-uniform specification. Then there exists $\epsilon_1 > 0$, $n_1 \in \mathbb{N}$ and $\tilde{x} \in X$ so that for every $\delta > 0$ and every sufficiently big $n \in \mathbb{N}$ there exists $x^{(n)} \in X$ with the following properties:

\[
\text{(55)} \quad d_{X}^{F_n} \left( S^{-i}(x^{(n)}), \tilde{x} \right) > \epsilon_1 \quad \text{for all } i \in \mathbb{Z}.
\]

\[
\text{(56)} \quad d_{X}^{(\tilde{i} + F_n) \cap (\tilde{j} + F_n)} \left( S^{-\tilde{i}}(x^{(n)}), S^{-\tilde{j}}(x^{(n)}) \right) > \epsilon_1 \quad \text{whenever } \tilde{i} - \tilde{j} \in (2 - \delta)F_n.
\]

**Proof.** It is enough to prove for the case $\delta < 1$. Let $\epsilon_1 > 0$, $n_1 \in \mathbb{N}$ and $\tilde{x}^{(0)}, \tilde{x}^{(1)}, \tilde{x} \in X$ satisfy (52), (53) and (54), as in Lemma 5.9, and in addition, assume that $n_1 \in \mathbb{N}$ is sufficiently big so that $g_n(\frac{1}{2}) < \frac{1}{2}$. Choose $\delta n > 3n_1$. Now let $w \in \{0, 1\}^{\mathbb{Z} / 3n\mathbb{Z}}$ be chosen uniformly at random. Let $L = \lceil (2 + 2g_n(\frac{1}{2}))n_1 \rceil$. By almost weak specification, there exists $x^{(n)} \in X$ so that

\[
d_{X}^{L\tilde{i} + F_n} \left( x^{(n)} , S^{-L\tilde{i}}(\tilde{x}^{(0)}) \right) < \epsilon_1 / 2 \text{ if } w(\tilde{i} \mod 3n\mathbb{Z}) = 0
\]

and

\[
d_{X}^{L\tilde{i} + F_n} \left( x^{(n)} , S^{-L\tilde{i}}(\tilde{x}^{(1)}) \right) < \epsilon_1 / 2 \text{ if } w(\tilde{i} \mod 3n\mathbb{Z}) = 1.
\]

Then because $L < 2.5n_1$, the properties (52) and (53) of $\tilde{x}^{(0)}, \tilde{x}^{(1)}$ ensure that

\[
d_{X}^{(\tilde{i} + F_n) \cap (\tilde{j} + F_n)} \left( S^{-\tilde{i}}(x^{(n)}), S^{-\tilde{j}}(x^{(n)}) \right) > \epsilon_1 \quad \text{whenever } \tilde{i} - \tilde{j} \in (2 - \delta)F_n \setminus L\mathbb{Z},
\]

and the property (54) ensures that (55) holds. If $\tilde{i} - \tilde{j} \in (2 - \delta)F_n \cap L\mathbb{Z}$ then still,

\[
d_{X}^{(\tilde{i} + F_n) \cap (\tilde{j} + F_n)} \left( S^{-\tilde{i}}(x^{(n)}), S^{-\tilde{j}}(x^{(n)}) \right) > \epsilon_1 \text{ unless}
\]

\[
w \left( \frac{k - \tilde{i}}{L} \mod 3n\mathbb{Z} \right) = w \left( \frac{k - \tilde{j}}{L} \mod 3n\mathbb{Z} \right) \text{ for all } k \in F_n \subset (\tilde{i} + F_n) \cap (\tilde{j} + F_n), \tilde{k} - \tilde{i} \in L\mathbb{Z}.
\]

For each $\tilde{i}, \tilde{j}$ the probability of this event is less than

\[
2^{-\frac{\epsilon_1}{2} |(\tilde{i} + F_n) \cap (\tilde{j} + F_n)|}
\]

for large enough $n$. Since $\tilde{i} - \tilde{j} \in (2 - \delta)F_n$ this probability is bounded above by

\[
2^{-\frac{\epsilon_1}{2} n}
\]

whenever $n$ is large enough. Thus the probability of this event occurring for at least such choice of $\tilde{i}, \tilde{j}$ is bounded above by

\[
|F_n|^2 \cdot 2^{-\frac{\epsilon_1}{2} n},
\]

which goes to zero as $n$ tends to infinity. Thus there exists $w \in \{0, 1\}^{\mathbb{Z} / 3n\mathbb{Z}}$ so that the corresponding $x^{(n)}$ satisfies (56). \qed
Proof of Proposition 5.7. For each $n \in \mathbb{N}$, let $g_n : (0, +\infty) \to (0, +\infty)$ be a function that witnesses the non-uniform specification of $(X, S)$ as in the definition above. By modifying the functions $g_n : (0, +\infty) \to (0, +\infty)$, we can assume without loss of generality that $g_n : (0, +\infty) \to (0, +\infty)$ is a decreasing function for every $n \in \mathbb{N}$, and that for every fixed $\epsilon > 0$ the sequence $(g_n(\epsilon))_{n=1}^{\infty}$ decreases 0, but slowly enough so that

$$\lim_{n \to \infty} ng_n(\epsilon) = \infty.$$  

We can thus find a sequence $(\epsilon_n)_{n=1}^{\infty}$ of positive numbers decreasing to zero at a slow enough rate such that $g_n(\epsilon_{n}/8)$ tends to zero, $\lim_{n \to \infty} ng_n(\epsilon_{n}/8) = +\infty$ and

$$\lim_{n \to \infty} \frac{1}{|F_n|} \log \text{sep}_{\epsilon_n}(X, F_n) = h(X, S)$$

and $\epsilon_1$ is small enough such that Lemma 5.10 applies to it.

Define the sequence $(\delta_n)_{n=1}^{\infty}$ by $\delta_n = 10 \cdot g_n(\epsilon_{n}/8)$. For $n \in \mathbb{N}$ let $E_n = F_n \setminus (1 - \frac{1}{8}\delta_n)F_n$. Apply Lemma 5.10 to find $n_1 \in \mathbb{N}$, $\tilde{x} \in X$ such that for $n$ large enough there exists $x^{(n)} \in X$ that satisfies (55) and

$$d_{X}^{E_n \cap (i + E_n)} \left(S^{-1}(x^{(n)}), x^{(n)}\right) > \epsilon_1$$

whenever $i \in \left(1 - \frac{1}{8}\delta_nF_n\right).$

For $k \gg n_1$ let

$$\hat{\epsilon}_k := \max\{\epsilon_{k_1} + \epsilon_{k_2} + \ldots + \epsilon_{k_r} : n_1 \ll k_1 \ll k_2 \ll \ldots \ll k_r \ll k\}$$

and

$$\hat{\epsilon}_\infty := \max\{\epsilon_{k_1} + \epsilon_{k_2} + \ldots + \epsilon_{k_r} : n_1 \ll k_1 \ll k_2 \ll \ldots \ll k_r\}$$

By (41) it follows that $\hat{\epsilon}_k < \hat{\epsilon}_\infty < \epsilon_{n_1}$.

Further for $n \gg n_1$ for which $\delta_n < 1/8$, let $X_n \subset X$ consist of all those $x \in X$ that satisfy the following properties:

$$d_{X}^{F_n} (x, x^{(n)}) < \frac{1}{4} \hat{\epsilon}_1.$$  

(59)

$$\forall i \in (1 - \frac{1}{7}\delta_nF_n) \exists j + F_{n_1} \subset i + \frac{1}{64}\delta_nF_n \text{ s.t. } d_{X}^{F_{n_1}} (S^j(x), \tilde{x}) < \frac{1}{8} \epsilon_1 + \hat{\epsilon}_n.$$  

(60)

and empty otherwise. The non-uniform specification property easily implies that $\bigcup_{n=1}^{\infty} X_n$ is dense in $X$, and similarly to the proof of Lemma 5.5 because $\delta_n n \to \infty$ as $n \to \infty$ it follows that for any $\epsilon > 0$

$$\liminf_{n \to \infty} \frac{\log \text{sep}_{\epsilon_n}(X_n, F_n)}{\log \text{sep}_{\epsilon}(X, F_n)} \geq 1.$$  

(61)

Together with (57) this implies

$$\lim_{n \to \infty} \frac{1}{|F_n|} \log \text{sep}_{\epsilon_n}(X_n, F_n) = h(X, S).$$

To check the marker property we will verify that whenever $x, x' \in X_n$ and $i \in (2 - 2\delta_n)F_n \setminus \{0\}$ then

$$d_{X}^{F_n \cap (i + F_n)} \left(S^{-1}(x), x'\right) > \frac{1}{8} \epsilon_1.$$  

(62)

There are two cases to check: For $i \in (1/2\delta_nF_n) \setminus \{0\}$, use the fact that the orbits of both $x$ and $x'$ are $1/4 \epsilon_1$-close to $x^{(n)}$ on $E_n$ in the sense of (59). Along with (58), the marker property in this case follows by the triangle inequality. For $i \in (2 - 2\delta_n)F_n \setminus (1/8\delta_nF_n)$, first note that there exists $i' \in (1 - 1/8\delta_n)F_n$ such that

$$i' + \frac{1}{64}\delta_nF_n \subset (i + E_n) \cap F_n.$$  

In this case (62) follows by (60), (59) and (55) and the triangle inequality.

Fix $\epsilon_* < \frac{1}{4} \epsilon_1$ and let $C_n \subset X_n$ be an $(\epsilon_n, F_n)$-separated set of maximal cardinality. As proved above, we know that it is a marker sequence. We claim that $C = \bigcup_{n=1}^{\infty}$ it is a flexible sequence for $(X, S)$ with respect to the sequences $(\epsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$. Indeed, suppose $k \ll n$, that $K \subset (1 - \delta_n)F_n$ is $(1 + \delta_k)F_k$-spaced and that $W \in C_{k}$.
Let $V \subset \mathbb{Z}$ be a maximal $(1 + g_k(\epsilon_k/8))F_k$-spaced subset so that
\[
V + (1 + g_k(\epsilon_k/8))F_k \subseteq (1 - \frac{1}{7}\delta_n)F_n \setminus (K + (1 + \delta_k)F_k).
\]

Then the non-uniform specification property implies that there exists $x' \in X$ such that
\[
d_{X}^{+,F_k}(x', S^{-i}(W_i)) < \epsilon_k/8 \quad \text{for every } i \in K,
\]
\[
d_{X}^{+,F_n}(x', x^{(n)}) < \epsilon_k/8 \quad \text{and}
\]
\[
d_{X}^{+,F_k}(x', S^{-i}(\tilde{x})) < \epsilon_k/8 \quad \text{for every } i \in V.
\]

Let us check that $x' \in X_n$. The second equation directly implies that (59) holds. So we have that check that (60) holds as well. Let $i \in (1 - \frac{1}{6}\delta_n)F_n$.

There are two possibilities to consider. In the first case it might so happen that for some $k \in K$ we have that $k + (1 + \delta_k)F_k \subseteq i + \frac{1}{64}\delta_n F_n$. Then there exists $j$ so that $j + F_{n_1} \subseteq i + (1 + \delta_k)F_k$ such that
\[
d_{X}^{+,F_{n_1}}(x', S^{-j} \tilde{x}) < 1/8\epsilon_1 + \delta_k + \epsilon_k/8 < 1/8\epsilon_1 + \epsilon_n.
\]

We are left with the case when for all $i' \in K$, $i' + (1 + \delta_k)F_k$ is not contained in $i + \frac{1}{64}\delta_n F_n$. In this case, because $k \ll n$ it follows that $i + \frac{1}{64}\delta_n F_n$ contains a translate of $2(1 + g_k(\epsilon_k/8))F_k$ which is disjoint from $K + (1 + \delta_k)F_k$.

By the maximality property of $V$, there exists $j \in V$ such that $j + (1 + g_k(\epsilon_k/8))F_k \subseteq i + \frac{1}{64}\delta_n F_n$ which proves (60).

Now since $C_n$ is a maximal $(\epsilon_n, F_n)$-separated subset of $X_n$, it is $(\epsilon_n, F_n)$-dense in $X_n$. This means that there exists $x \in C_n$ such that $d_X^{+,F_n}(x, x') < \epsilon_n$. Then $x$, $n$-shadows $W$ (because $\epsilon_n < \frac{1}{10} \epsilon_k$). By (61) it follows that $h(C) = h(X, S)$. Also, because every $C_n$ is $(\epsilon_n, F_n)$-dense in $X_n$ and $\bigcup_{n=1}^{\infty}X_n$ is dense in $X$, it follows that $\bigcup_{n=1}^{\infty}C_n$ is dense in $X$.

\section{6. Ergodic universality via approximate embeddings}

In this section we apply the tools introduced in the previous sections to prove a partial result towards Theorem 5.1. Namely, we prove full ergodic universality for systems admitting a flexible sequence.

Let us list our notational conventions and standing assumptions:

- $Y = (Y, T)$ is a free Borel $\mathbb{Z}^d$-system.
- $(\epsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$ are decreasing sequences of positive numbers tending to 0, and so that $\lim_{n \to \infty} \delta_n = 0$, and $\epsilon_n < \frac{\epsilon}{4}$ for every $n$.
- For every $n \in \mathbb{N}$, $Z_n \subset Y$ is the base of an $(1 + \delta_n)F_n, \epsilon_n)$-tower.
- There is a sequence of finite measurable partitions $(P_k)_{k=1}^{\infty}$ that together generate the $\sigma$-algebra on $Y$, so that $P_k < P_{k+1}$.
- $(X, S)$ is a compact metric topological $\mathbb{Z}^d$-system, and $d_X : X \times X \to \mathbb{R}_+$ is a compatible metric on $X$.
- $\mathcal{X}' = (X^{\mathbb{Z}^d}, S)$ is the space of pseudo-orbits for $(X, S)$.
- $C = (C_n)_{n=1}^{\infty} \in (2^X)^\mathbb{N}$ is a flexible marker sequence for $(X, S)$ with respect to $(\epsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$, in the sense that it satisfies (45).
- We fix an element of $X$ and denote it by $x_0 \in X$.
- Recall that the notation $n \ll m$ intuitively means that $n$ is “much smaller than m” and is formally defined by (41).

As in most of the proofs for ergodic universality, the idea is to construct an embedding of $(Y, T, \mu)$ into $(X, S)$ as a limit of “approximate embeddings” of some sort. In contrast to previous works, the target for our “approximate embeddings” is not $(X, S)$ itself, it is $\mathcal{X}' = (X^{\mathbb{Z}^d}, S)$ “the space of pseudo orbits of $(X,S)$”.

Also, in this section we fix $\mu \in Prob_e(Y, T)$ with $h_{\mu}(Y, T) < h(C)$. Later on, when we prove universality in the “almost-Borel” category we will consider $\mu \in Prob_e(Y, T)$ as a “variable” and pay closer attention to manner that other parameters depend on $\mu$.

Let us introduce a bit more notation and definitions:
Definition 6.1. For $F \subseteq \mathbb{Z}^d$ and $w \in X^F$ let

$$[w] := \left\{ w' \in X^{\mathbb{Z}^d} : w'|_F = w \right\}.$$  

Also, for $x \in X$ let

$$[x]_F := \left\{ w' \in X^{\mathbb{Z}^d} : w'_i = S^i(x) \forall i \in F \right\}.$$  

Definition 6.2. Given $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ we define the following Borel partition of $Y$:

$$\mathcal{P}_\rho := \left\{ \rho^{-1}[\rho(y)]_{\{\emptyset\}} : y \in Y \right\}.$$  

Definition 6.3. We say that $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ is a symbolic morphism if $\rho(y)_0$ takes finitely many values as $y$ ranges over $Y$.

Whenever $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ is a symbolic morphism then $\mathcal{P}_\rho$ is a finite measurable partition. If $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ is a symbolic morphism it follows that the closure of $\rho(Y)$ in $X^{\mathbb{Z}^d}$ is a zero dimensional compact metrizable space, this is our reason for the term “symbolic approximate embedding”.

Recall that $x_* \in X$ is a fixed element of $X$. Here is what we mean by an “approximate embedding”:

Definition 6.4. We say that $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ is $n$-towerable if

$$\forall y \in Z_n, \exists x \in C_n \text{ s.t. } \rho(y)_{\downarrow} = S^i(x) \text{ for all } \bar{i} \in F_n.$$  

and

$$\rho(y)_0 = x_*, \forall y \in Y \setminus T^{F_n} Z_n.$$  

Fix $\mu \in \text{Prob}_x(Y, T)$ and integers $k, n \in \mathbb{N}$ and $\epsilon > 0$. A $(k, n, \epsilon, \mu)$-approximate embedding is an $n$-towerable map $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ such that there exists a Borel set $Z[\rho] \subset Z_n$ satisfying

$$\mu(Z_n \setminus Z[\rho]) \leq \epsilon \mu(Z_n)$$  

Thus there exists a map $\Psi_{k,n} : X \to \mathcal{P}_k^{F_n}$ such that

$$\forall y, y' \in Z[\rho] \text{ if } \rho(y)_{\downarrow} = \rho(y')_{\downarrow} \text{ then } \mathcal{P}_k^{F_n}(y) = \mathcal{P}_k^{F_n}(y').$$  

Remark 6.5. An $n$-towerable map $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ is in particular a symbolic morphism because $\rho(y)_0$ takes values only in $\bigcup_{i \in F_n} T^i C_n \cup \{x_*\}$, which is a finite set.

Remark 6.6. Because the sequence $(\epsilon_k)_{k=1}^\infty$ is decreasing and $\mathcal{P}_j \prec \mathcal{P}_{j+1}$ for all $j \in \mathbb{N}$, whenever $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ is a $(k, n, \epsilon, \mu)$-approximate embedding then it is also a $(k_0, n, \epsilon, \mu)$-approximate embedding for every $k_0 < k$.

Definition 6.7. For $\rho, \tilde{\rho} \in \text{Mor}(\mathcal{Y}, \mathcal{X})$, and $\epsilon > 0$, $n \in \mathbb{N}$ let

$$D_{n,\epsilon}[\rho, \tilde{\rho}] := \left\{ y \in Z_n : d_F^{\epsilon_n}(\rho(y), \tilde{\rho}(y)) \geq \epsilon \right\}.$$  

For future reference, we write the following formula, which is a direct consequence of the definition:

$$\forall n > n_0 \text{ and } \rho, \tilde{\rho} \in \text{Mor}(\mathcal{Y}, \mathcal{X}), \ D_{n_0,\epsilon}[\rho, \tilde{\rho}] \subseteq \left( Y \setminus T^{F_{n_0}} Z_n \right) \cup T^{F_{n_0}}(D_{n,\epsilon}[\rho, \tilde{\rho}]).$$  

Also, for future reference we write the following formula: the value of $\mathcal{P}_k(y)$ is determined by $\tilde{\rho}(y)|_{F_{2n_0}}$.

Lemma 6.8. Let $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ be a $(k, n_0, \epsilon, \mu)$-approximate embedding, and suppose that $\tilde{\rho} \in \text{Mor}(\mathcal{Y}, \mathcal{X})$. Then for every $y \in T^{(1-2\delta_{n_0})F_{n_0}} \left( Z[\rho] \setminus D_{n_0,\epsilon}[\rho, \tilde{\rho}] \right)$, the value of $\mathcal{P}_k(y)$ is determined by $\tilde{\rho}(y)|_{F_{2n_0}}$.  


Proof. Let \( \rho \) be a \((k, n_0, \epsilon, \mu)\)-approximate embedding and \( \tilde{\rho} \in Mor(\mathcal{Y}, \mathcal{X}) \). We will show that there exists a Borel function \( \Phi_{k, n_0} : X^{F_{2n_0}} \to P_k \) so that for every \( \tilde{\rho} \in Mor(\mathcal{Y}, \mathcal{X}) \) the following holds:

\[
\Phi_{k, n_0}(\tilde{\rho}(y) |_{F_{2n_0}}) = P_k(y) \quad \text{for every } y \in T^{(1-2\delta_{n_0})F_{n_0}} \left( Z[\rho] \setminus D_{n_0, \frac{1}{\epsilon + \mu}}[\rho, \tilde{\rho}] \right).
\]

Define a function \( I_{n_0} : X^{F_{2n_0}} \to F_{n_0} \) as follows:

\[
I_{n_0}(w) := \min \left\{ \bar{\iota} \in F_{n_0} : \exists x \in C_{n_0} \text{ s.t. } d^{F_{n_0}}(S^{\bar{\iota}}(w), x) < \epsilon_{n_0}/2 \right\}.
\]

The minimum in the definition of \( I_{n_0} \) above is with respect to some fixed total order on \( F_{n_0} \). If the set is empty we arbitrarily define \( I_{n_0}(w) = 0 \). Suppose \( \bar{\iota} \in (1-2\delta_{n_0})F_{n_0} \) and \( T^{\bar{\iota}}(y) \in Z_{n_0} \) then \( d^{F_{n_0}}(S^{\bar{\iota}}(\rho(y)), C_{n_0}) = 0 \). By the marker property (45) it follows that in this case \( I_{n_0}(\rho(y)) = \bar{\iota} \). We conclude that

\[
T^{I_{n_0}(\rho(y))}(y) \in Z_{n_0} \quad \forall y \in T^{(1-2\delta_{n_0})F_{n_0}} Z_{n_0}.
\]

Furthermore, if \( T^{\bar{\iota}}(y) \in Z_{n_0} \setminus D_{n_0, \frac{1}{\epsilon + \mu}}[\rho, \tilde{\rho}] \) then the same consideration using the marker property (45) also implies that \( I_{n_0}(\tilde{\rho}(y)) = \bar{\iota} \). We conclude that

\[
I_{n_0}(\tilde{\rho}(y)) = I_{n_0}(\rho(y)) \quad \forall y \in T^{(1-2\delta_{n_0})F_{n_0}} (Z_{n_0} \setminus D_{n_0, \frac{1}{\epsilon + \mu}}[\rho, \tilde{\rho}]).
\]

Let \( \Psi_{k, n_0} : X \to P_k^{F_{n_0}} \) satisfy (70). Define \( \Phi_{k, n_0} : X^{F_{2n_0}} \to P_k \) as follows:

\[
\Phi_{k, n_0}(w) = (\Psi_{k, n_0}(\pi_n(S^{I_{n_0}(w)}(w))))_{I_{n_0}(w)},
\]

where \( \pi_n : X^{F_n} \to C_n \) is a Borel function satisfying

\[
d^{F_n}(\pi_n(w), w) = \min_{x \in C_n} d^{F_n}(x, w) \quad \forall w \in X^{F_n}.
\]

By (70) and (76) we have that \( \Phi_{k, n_0}(\rho(y) |_{F_{2n_0}}) = P_k(y) \) whenever \( y \in T^{(1-2\delta_{n_0})F_{n_0}} Z[\rho] \). Also, if \( T^{\bar{\iota}}(y) \in Z[\rho] \setminus D_{n_0, \frac{1}{\epsilon + \mu}}[\rho, \tilde{\rho}] \) for some \( \bar{\iota} \in (1-2\delta_{n_0})F_{n_0} \) then \( \Phi_{k, n_0}(\tilde{\rho}(y)) = \Phi_{k, n_0}(\rho(y)) = \Phi_{k, n_0}(\tilde{\rho}(y)) \). We conclude that (74) holds.

Lemma 6.9. Suppose \( 1 < n < m \) and \( \rho, \tilde{\rho} \in Mor(\mathcal{Y}, \mathcal{X}) \) then:

\[
\mu(D_{n, \epsilon}[\rho, \tilde{\rho}] | Z_{n}) \leq 2\epsilon_m + 8d\delta_m + 2\mu(D_{m, \epsilon}[\rho, \tilde{\rho}] | Z_{m}).
\]

Proof. Let

\[
\tilde{D}_{2n, \epsilon}[\rho, \tilde{\rho}] := \left\{ y \in Y : d^{F_{2n}}(\rho(y), \tilde{\rho}(y)) \geq \epsilon \right\}.
\]

By the law of total probability

\[
\mu\left( \tilde{D}_{2n, \epsilon}[\rho, \tilde{\rho}] \right) \geq \sum_{\bar{\iota} \in F_{n}} \mu\left( \tilde{D}_{2n, \epsilon}[\rho, \tilde{\rho}] | T^{\bar{\iota}}Z_n \right) \mu\left( T^{\bar{\iota}}Z_n \right).
\]

Note that \( T^{\bar{\iota}}D_{n, \epsilon}[\rho, \tilde{\rho}] \subseteq \tilde{D}_{2n, \epsilon}[\rho, \tilde{\rho}] \) for every \( \bar{\iota} \in F_{n} \). So

\[
\mu\left( \tilde{D}_{2n, \epsilon}[\rho, \tilde{\rho}] \right) \geq \mu(Z_n) \sum_{\bar{\iota} \in F_{n}} \mu\left( T^{\bar{\iota}}D_{n, \epsilon}[\rho, \tilde{\rho}] | T^{\bar{\iota}}Z_n \right) = |F_n| \mu(Z_n) \mu(\mu(D_{n, \epsilon}[\rho, \tilde{\rho}] | Z_n)).
\]

Now because \( n \gg 1 \) it follows that \( |F_n| \mu(Z_n) > \frac{1}{2} \), so

\[
\mu(D_{n, \epsilon}[\rho, \tilde{\rho}] | Z_{n}) \leq 2\mu\left( \tilde{D}_{2n, \epsilon}[\rho, \tilde{\rho}] \right).
\]

Again by the law of total probability,

\[
\mu\left( \tilde{D}_{2n, \epsilon}[\rho, \tilde{\rho}] \right) \leq \epsilon_m + 4d\delta_m + \sum_{\bar{\iota} \in (1-\delta_m)F_{n}} \mu(\tilde{D}_{2n, \epsilon}[\rho, \tilde{\rho}] | T^{\bar{\iota}}Z_m) \mu(T^{\bar{\iota}}Z_m).
\]
Now if \( n \ll m \), for every \( \vec{i} \in (1 - \delta_m)F_m \) we have \( \bar{D}_{2n,\epsilon}(\rho, \bar{\rho}) \cap T^n Z_m \subseteq T^n Z_m \). It follows that

\[
\mu \left( \bar{D}_{2n,\epsilon}(\rho, \bar{\rho}) \right) \leq \epsilon_m + 4d \delta_m + \sum_{\vec{i} \in (1 - \delta_m)F_m} \mu(T^n D_{m,\epsilon}(\rho, \bar{\rho}) | T^n Z_m) \mu(T^n Z_m) \leq \\
\leq \epsilon_m + 4d \delta_m + \mu (D_{m,\epsilon}(\rho, \bar{\rho}) | Z_m).
\]

The inequality (79) follows.

The following lemma roughly says that if \( \rho \) is a \((k_0, n_0, \delta, \mu)\)-approximate embedding with \( \delta > 0 \) sufficiently small and \( n_0 \in \mathbb{N} \) sufficiently big, then the log of the approximate covering number of \( \mathcal{P}_{k_0}^{\mathcal{F}_n} \) relative to \( \mathcal{P}_{\rho}^{\mathcal{F}_n} \) on \( Z_n \) is a very small fraction of \( |F_n| \), provided \( n \) is big enough.

**Lemma 6.10.** For any \( \eta > 0 \) and \( k_0 \in \mathbb{N} \) there exist \( N_0 \in \mathbb{N} \) such that for any \((k_0, n_0, \frac{1}{N_0}, \mu)\)-approximate embedding \( \rho \in \text{Mor}(\mathcal{Y}, \mathcal{X}) \) with \( n_0 \geq N_0 \), \( \gamma \in [0, 1) \) and \( \delta > 0 \) there exists \( N \in \mathbb{N} \) so that for every \( n > N \)

\[
\text{COV}_{\mu, \delta, \mathcal{P}_{k_0}^{\mathcal{F}_n} \cap \mathcal{F}_n} (\mathcal{P}_{\rho}^{\mathcal{F}_n}) (Z_n) < e^{\eta |F_n|}.
\]

**Proof.** For \( N_0 \in \mathbb{N} \), denote

\[
\tilde{e}_{N_0} = \epsilon_{N_0} + 6d \delta_{N_0} + \frac{2}{N_0}.
\]

For \( \eta \in (0, 1) \) and \( k_0 \in \mathbb{N} \), let

\[
\bar{N}_R(\eta, k_0) = \inf \{ N_0 \in \mathbb{N} : \mathcal{H}(\tilde{e}_{N_0}) + (\tilde{e}_{N_0} + 12d \delta_{N_0}) \log |\mathcal{P}_{k_0}| < \eta \}.
\]

Then \( \bar{N}_R(\eta, k_0) \in \mathbb{N} \) is well defined because \( \mathcal{H}(\epsilon) \to 0 \) as \( \epsilon \to 0^+ \) and \( \tilde{e}_{N_0}, \delta_{N_0} \to 0 \) as \( N \to \infty \).

Fix \( \eta \in (0, 1) \) and \( k_0 \in \mathbb{N} \). Denote \( N_0 = N_R(\eta, k_0) \). Choose any \( n_0 > N_0 \), \( \gamma \in [0, 1) \), any \((k_0, n_0, \frac{1}{N_0}, \mu)\)-approximate embedding \( \rho \in \text{Mor}(\mathcal{Y}, \mathcal{X}) \) and \( \delta > 0 \).

Denote

\[
G_{n_0} = T^{(1 - 2\delta_{n_0})} Z_{n_0} (\rho) .
\]

Recall that \( Z_{n_0} \) is the base of an \((1 + \delta_{n_0})F_{n_0}, \epsilon_{n_0})\)-tower. Also because \( \rho \) is a \((k_0, n_0, \frac{1}{N_0}, \mu)\)-approximate embedding

\[
\mu(Z_{n_0} \setminus Z | Z_{n_0}) < \frac{1}{N_0}.
\]

Thus by Lemma 2.3

\[
\mu(Y \setminus G_{n_0}) < \tilde{e}_{N_0}.
\]

Let

\[
A_{N_0, n, n_0} = \left\{ y \in Z_n : \sum_{\vec{i} \in F_n \setminus \gamma F_n} 1_{G_{n_0}} (T^\vec{i}(y)) > (1 - \tilde{e}_{N_0}) |F_n \setminus \gamma F_n| \right\}.
\]

We apply the mean ergodic theorem for Rokhlin towers (Proposition 3.1) to deduce that

\[
\lim_{n \to \infty} \mu (A_{N_0, n, n_0} | Z_n) = 1.
\]

More specifically, we apply (23) with \( f = 1_{G_{n_0}} \) twice (with \( n \) and \( \gamma n \)) , taking into account that \( \int f d\mu = \mu(G_{n_0}) > 1 - \tilde{e}_{N_0} \). So there exists \( N \gg n_0 \) so that for every \( n > N \)

\[
\mu (A_{N_0, n, n_0} | Z_n) > 1 - \delta/2.
\]

It is well known and easy to show that for any natural numbers \( k < n \) we have

\[
\frac{1}{n} \log \binom{n}{k} \leq \mathcal{H}(\frac{k}{n}),
\]

where \( \mathcal{H}(\epsilon) \) is given by (14). Thus for every \( n \in \mathbb{N} \)

\[
\left( \frac{|F_n \setminus \gamma F_n|}{\tilde{e}_{N_0} |F_n \setminus \gamma F_n|} \right) \leq e^{\mathcal{H}(\tilde{e}_{N_0}) |F_n \setminus \gamma F_n|}.
\]

Now choose any \( n > N \), so that (85) holds.
By Lemma 6.8, for every \( y \in G_{n_0} \), the value of \( \mathcal{P}_{k_0}(y) \) is determined by \( \rho(y) \mid_{F_{2n_0}} \). This means that

\[
\text{COV}_{\mu_0, \mathcal{P}_{k_0}}|_{\mathcal{P}_{2n_0}^{F_0}}(G_{n_0}) = 1.
\]

Choose \( S \subset F_n \setminus \gamma F_n \). Then (taking into account that \( n \gg n_0 \)) for every \( i \in S \cap (1 - 3\delta_n)F_n \setminus (\gamma + \delta_n)F_n \),

\[
\text{COV}_{\mu_0, T_i^\gamma \mathcal{P}_{k_0}}|_{\mathcal{P}_{\mu_0}^{(1 - 3\delta_n)F_n \setminus \gamma F_n}}(\cap_{i \in S} T^\gamma G_{n_0}) = 1.
\]

It follows (for instance by applying a degenerate easy case of Lemma 4.2 with \( \epsilon = 0 \)) that for any \( S \subset F_n \setminus \gamma F_n \)

\[
\text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(\cap_{i \in S} T^\gamma G_{n_0} \cap Z_n) \leq |\mathcal{P}_{k_0}| [(F_n \setminus \gamma F_n) \setminus S] + |F_n'|.
\]

where:

\[
F_n' \gamma = F_n \setminus (1 - 3\delta_n)F_n \cup ((\gamma + \delta_n)F_n) \setminus \gamma F_n.
\]

Because \( n > N \gg n_0 \) it follows that

\[
|F_n'| \leq 8d\delta_n|F_n|.
\]

Note that

\[
A_{N_0, n, n_0} \subseteq \bigcup_{|S| \geq (1 - \epsilon N_0)|F_n| \setminus \gamma F_n} \left( \bigcap_{i \in S} T^\gamma G_{n_0} \cap Z_n \right),
\]

where the union is over all \( S \subset F_n \setminus \gamma F_n \) such that \( |S| \geq (1 - \epsilon N_0)|F_n| \setminus \gamma F_n| \). From (88) and (89) it follows that

\[
\text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(A_{N_0, n, n_0}) \leq \sum_{|S| \geq (1 - \epsilon N_0)|F_n| \setminus \gamma F_n} |\mathcal{P}_{k_0}| [(F_n \setminus \gamma F_n) \setminus S] + 8d\delta_n|F_n|.
\]

If \( |S| \geq (1 - \epsilon N_0)|F_n| \setminus \gamma F_n \) then

\[
|\mathcal{P}_{k_0}| [(F_n \setminus \gamma F_n) \setminus S] \leq |\mathcal{P}_{k_0}| \epsilon N_0|F_n| \leq e^{\epsilon N_0} \log |\mathcal{P}_{k_0}| |F_n|.
\]

There are \( (\epsilon N_0|F_n| \setminus \gamma F_n) \) summands in the sum in the right hand side of (90). Thus

\[
\text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(A_{N_0, n, n_0}) \leq e^{\epsilon N_0} \log |\mathcal{P}_{k_0}| |F_n|.
\]

By (87) it follows that

\[
\text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(A_{N_0, n, n_0}) \leq e^{H(\epsilon N_0)|F_n| \gamma F_n|}, e^{(\epsilon N_0+8d\delta_n)|F_n| \log |\mathcal{P}_{k_0}|}.
\]

By (85),

\[
\text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(Z_n) \leq \text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(A_{N_0, n, n_0}).
\]

So

\[
\text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(Z_n) \leq e^{H(\epsilon N_0)|F_n| \gamma F_n|} + (\epsilon N_0+8d\delta_n)|F_n| \log |\mathcal{P}_{k_0}| \leq e^{H(\epsilon N_0)+(\epsilon N_0+8d\delta_n) \log |\mathcal{P}_{k_0}|} |F_n|.
\]

Apply (32) of Lemma 4.2 to deduce that

\[
\text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(Z_n) \leq \text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(Z_n) \cdot \text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(Z_n).
\]

Clearly,

\[
\text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(Z_n) \leq |\mathcal{P}_{k_0}| |F_n| \setminus (1 - 2\delta_n)|F_n|.
\]

so

\[
\text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(Z_n) \leq \text{COV}_{\mu_0, \mathcal{P}_{k_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}|_{\mathcal{P}_{\mu_0}^{(1 - 2\delta_n)F_n \setminus \gamma F_n}}(Z_n) \cdot |\mathcal{P}_{k_0}| |F_n| \setminus (1 - 2\delta_n)|F_n|.
\]
Since $n > n_0 > \mathbb{N}_H(\eta, k_0)$ it follows that
\[ (H(\bar{\epsilon}_N) + (\bar{\epsilon}_N + 12d\delta_n) \log |\mathcal{P}_{\mathcal{K}_0}|) |F_n| < \eta. \]
This proves that (80) holds.

Our next goal is to state certain sufficient conditions for a sequence of approximate embeddings converge to a proper embedding.

**Lemma 6.11.** Suppose $(k_j)_{j=1}^{\infty}$ and $(n_j)_{j=1}^{\infty}$ are strictly increasing sequences of natural numbers so that $n_j < n_{j+1}$ for every $j \geq 1$.

Let $(\rho_j)_{j=1}^{\infty}$ be a sequence of morphisms such that $\rho_j$ is a $(k_j, n_j, \frac{1}{2^n}, \mu)$-approximate embedding for every $j$. Let $Y_\infty \subset Y$ be given by:
\[
Y_\infty = \bigcap_{i \in \mathbb{Z}^d} T^{-1} \bigcup_{j=1}^{\infty} T^{(1-2\delta_{n_j})} F_n, \left( Z[\rho_i] \setminus D_{n_i, \frac{2}{\mu} \epsilon_{n_i}}[\rho_t, \rho_{t+1}] \right)
\]
Then:
(i) The limit $\rho(y) = \lim_{t \to \infty} \rho_j(y)$ exists for every $y \in Y_\infty$.
(ii) For every $y \in Y_\infty$ there exist $x \in X$ so that $\rho_j(y) = S^i(x)$ for every $i \in \mathbb{Z}^d$.
(iii) The function $\rho : Y_\infty \to X^{\mathbb{Z}^d}$ is injective.

In other words, on the Borel set $Y_\infty \subset Y$, the sequence $(\rho_j)_{j=1}^{\infty}$ converges pointwise to an equivariant Borel embedding $\rho : Y_\infty \to X$.

**Proof.** For $j \in \mathbb{N}$ let
\[
Y_j = \bigcap_{i \geq j} T^{(1-2\delta_{n_i})} F_n, \left( Z[\rho_i] \setminus D_{n_i, \frac{2}{\mu} \epsilon_{n_i}}[\rho_t, \rho_{t+1}] \right).
\]
If $y \in Y_j$ then
\[
\forall t \geq j \exists \tilde{t} \in (1 - 2\delta_{n_t}) F_{n_t} \text{ s.t. } T^\tilde{t}(y) \in Z[\rho_t] \text{ and } d_X^{F_{n_t}} \left( \rho_t(T^\tilde{t}(y)), \rho_{t+1}(T^\tilde{t}(y)) \right) < \frac{3}{8} \epsilon_{n_t}.
\]
Because $n_t \gg n_s$ whenever $t > s$, it follows that for any $t > s > j$, if $\tilde{t} \in F_{n_s}$ and $\tilde{t} \in (1 - 2\delta_{n_t}) F_{n_t}$ then $F_{n_s} + \tilde{t} \subset F_{n_t} + \tilde{t}$. This implies that if $y \in Y_j$, then for every $t > s > j$
\[
\exists \tilde{t} \in (1 - 2\delta_{n_s}) F_{n_s} \text{ s.t. } T^\tilde{t}(y) \in Z[\rho_s] \text{ and } d_X^{F_{n_s}} \left( \rho_t(T^\tilde{t}(y)), \rho_{t+1}(T^\tilde{t}(y)) \right) < \frac{3}{8} \epsilon_{n_t}.
\]
Because $\epsilon_{n_t+1} < \frac{1}{16} \epsilon_{n_t}$, we have
\[
\sum_{t \geq j} \frac{3}{8} \epsilon_{n_t} \leq \left( \sum_{t=0}^{\infty} \frac{3}{8} 16^{-t} \right) \epsilon_j = \frac{2}{5} \epsilon_j.
\]
So by the triangle inequality and for every $y \in Y_j$ and every $t > s > j$
\[
\exists \tilde{t} \in (1 - 2\delta_{n_s}) F_{n_s} \text{ s.t. } T^\tilde{t}(y) \in Z[\rho_s] \text{ and } d_X^{F_{n_s}} \left( \rho_t(T^\tilde{t}(y)), \rho_{s}(T^\tilde{t}(y)) \right) < \frac{2}{5} \epsilon_{n_s}.
\]
Because $F_{n_j-1} \subset \tilde{t} + F_{n_s}$ this shows
\[
d_X^{F_{n_j-1}} (\rho(y), \rho_s(y)) < \frac{2}{5} \epsilon_{n_s} \text{ for every } y \in Y_j \text{ and } t > s > j.
\]
Because $\lim_{s \to \infty} \epsilon_{n_s} = 0$ and $F_{n_j-1}$ increase to $\mathbb{Z}^d$ as $s \to \infty$, it follows that for every $y \in Y_\infty$ and every $\tilde{t} \in \mathbb{Z}^d$ the sequence $(\rho_j(y)_{j=1}^{\infty})$ is a Cauchy sequence, and thus converges. So the limit $\rho(y) = \lim_{j \to \infty} \rho_j(y)$ exists for every $y \in Y_\infty$. This proves (i). By (92) for every $y \in Y_\infty$ there exists $j \in \mathbb{N}$ such that for every $s > j$ there exists $\tilde{t} \in (1 - 2\delta_{n_s}) F_{n_s}$ so that $T^\tilde{t}(y) \in Z[\rho_s]$ and
\[
d_X^{F_{n_s}} \left( \rho(T^\tilde{t}(y)), \rho_s(T^\tilde{t}(y)) \right) \leq \frac{2}{5} \epsilon_{n_s}.
\]
Since $\rho_s$ is a $(k_s, n_s, \frac{1}{s}, \mu)$ approximate-embedding and $T^s(\vec{F}(y)) \in Z[\rho_s] \subset Z_{n_s}$, and $F_{n_s-1} \subset F_{n_s} + \vec{i}_s$ there exists $x_s \in C_s \subset X$ such that $d_{X,F_{n_s-1}}(\rho(y), x_s) \leq \frac{2}{s} \epsilon_{n_s}$. Using the previous argument, the sequence $(x_s)_{s=1}^\infty$ is a Cauchy sequence so it converges to a point $x \in X$, and so (ii) holds, meaning that $\rho(y)$ is in the image of $X$ for every $y \in Y_\infty$.

We now prove (iii), namely that the function $\rho$ is injective.

Because the sequence of partitions $(P_k)_{k=1}^\infty$ separates points, in order to show that $\rho : Y_\infty \to X^{Z^d}$ is injective, it suffices to show that for every $k \in \mathbb{N}$ there exists a Borel function $\Phi_k : X^{Z^d} \to P_k$ so that

$$\forall y \in Y_\infty \Phi_k(\rho(y)) = P_k(y).$$

By Lemma 6.8, because $\rho_j$ is a $(k_j, n_j, \frac{1}{j}, \mu)$ approximate embedding, for every $s \in \mathbb{N}$ and every $k < k_s$ (using the fact that $\rho \prec \rho_k$, for every $k < k_s$) there exists a Borel function $\Phi_{k,n_s} : X^{Z^d} \to P_k$ so that

$$\Phi_{k,n_s}(\rho(y)) = P_k(y)$$

for every $y \in T(1 - 2\delta_{n_s})F_{n_s} \subset Z[\rho_s] \setminus D_n, \frac{1}{s}\epsilon_{n_s}[\rho_s, \rho]$. We have already concluded that for every $y \in Y_\infty$ there exists $j \in \mathbb{N}$ such that for every $s > j$ there exists $\vec{i}_s \in (1 - 2\delta_{n_s})F_{n_s}$ so that $T^s(\vec{F}(y)) \in Z[\rho_s]$ and (94) holds. But this precisely means that

$$y \in T(1 - 2\delta_{n_s})F_{n_s} \subset Z[\rho_s] \setminus D_n, \frac{1}{s}\epsilon_{n_s}[\rho_s, \rho].$$

We conclude that for every $y \in Y_\infty$ there exists $j \in \mathbb{N}$ such that for every $s > j$, and every $k \leq k_s$ $\Phi_{k,n_s}(y) = P_k(y)$.

It follows that for every $k \in \mathbb{N}$ and every $y \in Y_\infty$ the sequence $(\Phi_{k,n_s}(\rho(y)))_{s=1}^\infty$ stabilizes and we have

$$\lim_{s \to \infty} \Phi_{k,n_s}(\rho(y)) = P_k(y).$$

So we can define a Borel function $\Phi_k : X^{Z^d} \to P_k$ so that $\Phi_k(x) = \lim_{s \to \infty} \Phi_{k,n_s}(X)$ whenever the limit exists and (95) holds. This completes the proof.

\begin{lemma}
Let $(k_j)_{j=1}^\infty$ and $(n_j)_{j=1}^\infty$ be as in Lemma 6.11 above. Further suppose that for every $j > 1$

$$\mu \left( D_{n_j-1}, \frac{1}{s}\epsilon_{n_j-1} [\rho_j, \rho_{j-1}] \mid Z_{n_j-1} \right) < \frac{1}{2^j}.$$ 

Then the set $Y_\infty \subset Y$ given by (91) satisfies $\mu(Y_\infty) = 1$.
\end{lemma}

\begin{proof}
For $t \in \mathbb{N}$ let

$$G_t = T(1 - 2\delta_{n_t})F_{n_t-1} \left( Z[\rho_{t-1}] \setminus D_{n_t-1}, \frac{1}{s}\epsilon_{n_t-1} [\rho_t, \rho_{t-1}] \right).$$

By Lemma 2.3

$$\mu(Y \setminus G_t) \leq \epsilon_{n_t-1} + 6d\delta_{n_t-1} + \mu \left( D_{n_t-1}, \frac{1}{s}\epsilon_{n_t-1} [\rho_t, \rho_{t-1}] \mid Z_{n_t-1} \right) + \mu \left( Z_{n_t-1} \setminus Z[\rho_{t-1}] \mid Z_{n_t-1} \right).$$

Because $\rho_t$ is a $(k_t, n_t, \frac{1}{t}, \mu)$-approximate embedding,

$$\mu \left( Z_{n_t} \setminus Z[\rho_t] \mid Z_{n_t} \right) < \frac{1}{2^t}.$$ 

So using (97) it follows that

$$\mu(Y \setminus G_{t+1}) \leq \epsilon_{n_t} + 6d \delta_{n_t} + \frac{2}{2^t}.$$

It follows that

$$\sum_{t=2}^\infty \mu(Y \setminus G_t) \leq \sum_{t=1}^\infty \left( \epsilon_{n_t} + 6d \delta_{n_t} + \frac{1}{2^{t-1}} \right).$$

By our assumption that $n_t \ll n_{t+1}$ it follows that the series on the right hand side converges. Let $Y'_{\infty} = \bigcup_{t=1}^\infty \bigcap_{t \geq j} G_t$. By the Borel-Cantelli Lemma $\mu(Y'_{\infty}) = 1$. Now $Y_\infty = \bigcap_{t \in \mathbb{N}} T^{-1}Y'_{\infty}$, so it follows that $\mu(Y_\infty) = 1$.
\end{proof}
Given a Borel function $\tilde{\Phi} : Y \rightarrow X$ and $n \in \mathbb{N}$ we define $\rho_{\tilde{\Phi},n} \in Mor(\mathcal{Y}, \mathcal{X})$ by

$$(99) \quad \rho_{\tilde{\Phi},n}(y) := \begin{cases} S_i^{\tilde{\Phi}(y)} & \text{if } \tilde{\Phi}(y) \in F_n \text{ and } y \in T^{-j}Z_n, \\ \infty & \text{if } \tilde{\Phi}(y) \notin F_n. \end{cases}$$

Because $\{T^{-j}Z_n\}_{j \in F_n}$ are pairwise disjoint, $\rho_{\tilde{\Phi},n} \in Mor(\mathcal{Y}, \mathcal{X})$ is well defined. Furthermore, if $\tilde{\Phi}(Z_n) \subset C_n$ then $\rho = \rho_{\tilde{\Phi},n}$ satisfies (66).

The following basic lemma asserts that $(k, n, \epsilon, \mu)$-approximate embeddings exist, as long as $n$ is sufficiently big in terms of the other parameters.

**Lemma 6.13.** For every $k \in \mathbb{N}$ and $\epsilon > 0$ there exists $N$ such that for every $n > N$ there exists a $(k, n, \epsilon, \mu)$-approximate embedding.

**Proof.** Apply Lemma 4.3 with $\mathcal{P} = \mathcal{P}_k$. It follows that for sufficiently large $n$ there exists a subset $\mathcal{G} \subset \mathcal{P}_k$ such that

$$(100) \quad \mu(\mathcal{Z}_n \cap \bigcup \mathcal{G}) \geq (1 - \epsilon) \mu(\mathcal{Z}_n)$$

and

$$(101) \quad |\mathcal{G}| \leq \exp \left( |F_n| \frac{1}{2} (h_\mu(Y, T) + h(\mathcal{C})) \right).$$

Recall that $h_\mu(Y, T) < h(\mathcal{C})$, so if $n$ is sufficiently big then $|\mathcal{Z}_n| > \exp \left(|F_n| \frac{1}{2} (h_\mu(Y, T) + h(\mathcal{C}))\right)$. This implies that

$$(102) \quad |\mathcal{G}| < |\mathcal{Z}_n|$$

Let $\Phi : \mathcal{P}_k \rightarrow C_n$ be a function such that the restriction to $\mathcal{G}$ is injective and $\Phi(\mathcal{G}) \cap \Phi(\mathcal{P}_k \setminus \mathcal{G}) = \emptyset$, and let $\hat{\Phi} : Y \rightarrow C_n$ be given by

$$(103) \quad \hat{\Phi}(y) = \Phi(\mathcal{P}_k^{\mathcal{G}}(y)).$$

Let $\rho = \rho_{\hat{\Phi},n} \in Mor(\mathcal{Y}, \mathcal{X})$ be given by (99). Then $\hat{\Phi}(\mathcal{Z}_n) \subset C_n$ so (66) is satisfied. Because $\Phi |\mathcal{G}$ is injective, if we set $\mathcal{Z}_n[\rho] = \bigcup \mathcal{G}$ then (69) will also be satisfied. This shows that the map $\rho \in Mor(\mathcal{Y}, \mathcal{X})$ is indeed a $(k, n, \epsilon, \mu)$-approximate embedding.

6.1. **The case of infinite entropy.** Our next goal is to prove Theorem 5.3 which states flexibility implies universality for systems with infinite entropy. The proof follows similar pattern to that of the finite entropy case, but it is considerably less involved, so the reader can consider it as a preparation.

**Lemma 6.14.** Suppose $\mathcal{C} = (C_n)_{n=1}^{\infty} \subset X^\mathbb{N}$ is a flexible sequence with $h(\mathcal{C}) = \infty$. For every $k \in \mathbb{N}$ there exists $N_k$ such that for every $n > N_k$ there exists $\rho \in Mor(\mathcal{Y}, \mathcal{X})$ which is a $(k, n, 0, \mu)$-approximate embedding for any $\mu \in \text{Prob}_c(\mathcal{Y})$.

The proof is a simplified version of the proof of Lemma 6.15 below, so we omit it.

**Lemma 6.15.** Suppose $\mathcal{C} = (C_n)_{n=1}^{\infty} \subset X^\mathbb{N}$ is a flexible sequence with $h(\mathcal{C}) = \infty$. For every $k_0, k \in \mathbb{N}$ and $\gamma \in (0, 1)$ there exists $N_{k, \gamma} \in \mathbb{N}$ so that for every $n_0$-torealable $\rho \in Mor(\mathcal{Y}, \mathcal{X})$ with $n_0 \geq N_{k, \gamma}$, $\tilde{x} \in C_{n_0}$ and any $n \gg n_0$, there exists $\hat{\rho} \in Mor(\mathcal{Y}, \mathcal{X})$ which is a $(k, n, 0, \mu)$-approximate embedding for all $\mu \in \text{Prob}_c(\mathcal{Y})$, so that

$$(104) \quad D_{n_0, \delta_{n_0}}[\rho, \hat{\rho}] \subset Y \setminus \mathcal{T}^{(1-2\delta_{n_0})F_n \setminus G_n} Z_n,$$

and so that for any $y \in Z_n$, $d_X(\tilde{x}, \hat{\rho}(y)) < \epsilon_{k_0}$.

**Proof.** Because $h(\mathcal{C}) = \infty$, for any $k \in \mathbb{N}$ and $\gamma \in (0, 1)$ there exists $N_{k, \gamma}$ so that for any $n_0 > N_{k, \gamma}$

$$\frac{1}{(1 + \delta_{n_0})F_{n_0}} \log |C_{n_0}| > \frac{4^d}{\gamma^d} \log |\mathcal{P}_k|.$$
Now fix \( n > N_{k, \gamma} \) and \( n \gg \lceil \gamma^{-1} \eta_0 \rceil \). Let
\[
K_{n, n_0, \gamma} = \frac{1}{2} \gamma F_n \cap \left( \left( 2 + 2\delta_{n_0} \right) n_0 \right) \mathbb{Z}^d \setminus \{0\}.
\]
We have that \( |K_{n, n_0, \gamma}| \geq \frac{|\{ \gamma F_n \}|}{(1 + \delta_{n_0}) n_0 - 1} \). Thus
\[
|C_{n_0}^{K_{n, n_0, \gamma}}| \geq |\mathcal{P}_k^F|.
\]
Then there is an injective map
\[
\phi : \mathcal{P}_k^F \to C_{n_0}^{K_{n, n_0, \gamma}}.
\]
Let \( \phi^{-1} \) denote its left inverse. For \( y \in Y \) set
\[
K_y = \{ \tilde{i} \in (1 - 2\delta_n)F_n \setminus \gamma F_n : T^{\tilde{i}}(y) \in Z_{n_0} \} \cup K_{n, n_0, \gamma},
\]
and \( W_y \in C_{N_k} K_y \) by
\[
(W_y)_{\tilde{i}} = \begin{cases} 
\hat{x} & \text{if } \tilde{i} = 0 \\
\phi \left( \mathcal{P}_{k+1}^{F_{n_0+1}}(y) \right)_{\tilde{i}} & \text{if } \tilde{i} \in K_{n, n_0, \gamma} \\
\rho(T^{\tilde{i}}(y)) & \text{otherwise}.
\end{cases}
\]
Since \( K_y \subset (1 - 2\delta_{n_k+1})F_n \) is \( (1 + \delta_{n_k})F_{n_k} \)-spaced we can define \( \tilde{\phi} : Y \to C_{N_{k+1}} \) by
\[
\tilde{\phi}(y) = \text{Ext}(W_y).
\]
Let \( \hat{\rho} = \rho_{k, n} \in \text{Mor}(Y, X) \) be given by (99). Then \( \hat{\rho} \) is a \((k, n, 0, \mu)\)-approximate embedding because we can take \( Z[\hat{\rho}] = Z_n \) and (104) holds. Also, whenever \( y \in Z_n \), \( d_X(\hat{x}, \hat{\rho}(y)) \leq \epsilon_n \).

**Proof of Theorem 5.3.** We assume that \( C \) is a flexible sequence on \( X \) and that \( h(C) = \infty \). Suppose we have a sequence \( (x_k)_{k=1}^\infty \) so that \( x_k \in C_k \) and \( \{ x_k \}_{k=1}^\infty \) is dense in \( X \). By induction, construct an increasing sequence \( (n_k)_{k=1}^\infty \) of natural numbers so that \( n_1 \gg 1 \) is also big enough to satisfy the conclusion of Lemma 6.14 with \( k = 1 \) and also so that for any \( k \geq 1 \)
\[
n_k \geq N_{k+1,2-(k+1)} \text{ and } n_{k+1} \gg [2^{(k+1)n_k}],
\]
where \( N_{k, \gamma} \) is a number that satisfies the conclusion of Lemma 6.15.

Note that this in particular implies that for any \( k \geq 1 \)
\[
\epsilon_{n_k} + \delta_{n_k} < 2^{-(k+1)}.
\]
By induction, construct sequences \( \rho_k \in \text{Mor}(Y, X) \), \( \hat{x}_k \in C_{n_k} \) so that
\[
d_X(\hat{x}_k, x_k) < \epsilon_k,
\]
and \( \rho_k \) is a \((k, n_k, 0, \mu)\)-approximate embedding and such that
\[
D_{n_k, \hat{x}_k} [\rho_k, \rho_{k+1}] \subseteq E_k,
\]
where
\[
E_k = Y \setminus T^{(1-2\delta_{n_{k+1}})F_{n_{k+1}} \setminus 2^{-(k+10)}F_{n_{k+1}}} Z_{n_{k+1}},
\]
and so that
\[
\forall y \in Z_{n_{k+1}}, d_X(\rho_{k+1}(y), \hat{x}_k) \leq \epsilon_{n_k}.
\]
To obtain \( \rho_{k+1} \) given \( \rho_k \) and \( x_{k+1} \), apply Lemma 6.15. Lemma 6.11 implies that the sequence converges to a Borel embedding of \((Y, T)\) into \((X, S)\), where
\[
Y_\infty = \bigcap_{i \in \mathbb{Z}^d} T^{-i} \bigcup j \geq i \bigcup_{j \geq i} T^{(1-2\delta_{ni})F_{ni}} (Z_i \setminus E_i).
\]
Such sequences can be constructed inductively: To start the induction, apply Lemma 6.13. For the induction step, apply Lemma 6.15 with \( \rho = \rho_k \), and \( n_0 = n_k \), \( n = N_k \), \( \gamma = \frac{1}{2^{k+1}} \). Let us check that (97) holds:

Note that
\[
E_k = \left( Y \setminus T^{(1-2\delta_{n_{k+1}})F_{n_{k+1}}} Z_{n_{k+1}} \right) \cup \left( T^{2(-k+10)F_{n_{k+1}}} Z_{n_{k+1}} \right).
\]
Now
\[ \mu \left( Y \setminus T^{(1-2\delta_{nk+1})F_{nk+1}} Z_{nk+1} | Z_{nk} \right) \leq \frac{1}{\mu(Z_{nk})} \mu \left( Y \setminus T^{(1-2\delta_{nk+1})F_{nk+1}} Z_{nk+1} \right), \]
So Lemma 2.3 implies that
\[ \mu \left( Y \setminus T^{(1-2\delta_{nk+1})F_{nk+1}} Z_{nk+1} | Z_{nk} \right) \leq 2|F_{nk}| (\epsilon_{nk+1} + 6d\delta_{nk+1}) \leq \epsilon_{nk} + \delta_{nk} < 2^{-(k+1)}. \]
By Lemma 2.4,
\[ \mu \left( T^{2^{-2(k+10)}F_{nk+1}} Z_{nk+1} | Z_{nk} \right) \leq 2^{-(k+9)} + \delta_{nk} < 2^{-(k+1)}. \]
Together this implies that
\[ \mu \left( E_k | Z_{nk} \right) < 2^{-k}. \]
Lemma 6.12 now implies that \( \mu(Y_\infty) = 1 \) for any \( \mu \in \text{Prob}_c(\mathcal{Y}) \). To see that the support of \( \mu \circ \rho^{-1} \) is full for any \( \mu \in \text{Prob}_c(\mathcal{Y}) \), note that the assumption that \( \bigcup_{k=1}^{\infty} C_k \) is dense in \( X \) implies that the sequence \( (\tilde{x}_k)_{k=1}^{\infty} \) is also dense in \( X \). The construction implies that for any \( k \in \mathbb{N} \)
\[ \mu(\{ y \in Z_k : d_X(\tilde{x}_k, \rho(y)) < 2\epsilon_k \} | Z_k) > \frac{1}{2}. \]
This implies that the measure \( \mu \circ \rho^{-1} \) charges any open set in \( X \).

Given Theorem 5.3, we can prove the result about full \( \infty \)-universality of a generic homeomorphism of homeomorphisms a manifold \( M \) that preserves a fixed fully supported measure:

**Proof of Theorem 1.1.** Let \( M \) be a finite dimensional compact topological manifold of dimension at least 2 and \( \mu \in \text{Prob}(M) \) a fully supported measure. By a result of Pierre-Antoine Guihèneuf and Thibault Lefeuvre [24, Theorem 3.17] infinite entropy is generic in the space of homeomorphisms that preserve \( \mu \). By more recent result of Guihèneuf and Lefeuvre [25, Corollary 1.4] specification is generic in this space as well. If \( (X,S) \) has specification, then by Proposition 5.7 it has a flexible marker sequence with the same entropy. Theorem 5.3 now implies the result.

A slight modification of the proof of Theorem 5.3 gives the following result about systems admitting a flexible marker sequence:

**Proposition 6.16.** Let \( (X,S) \) be topological \( \mathbb{Z}^d \)-system with a flexible marker sequence \( C \), and let \( \mathcal{Y} = (Y,T) \) be a free \( \mathbb{Z}^d \)-system. Then there exists a \( T \)-invariant Borel subset \( Y_0 \subset Y \) so that \( Y \setminus Y_0 \) is null with respect to any \( T \)-invariant probability measure and a Borel equivariant map from \( Y_0 \) into \( X \). Furthermore, we can choose a Borel embedding so that the push-forward of any \( T \)-invariant measure on \( Y \) has full support in \( X \).

Proposition 6.16 is also an easy consequence of Theorem 5.1 together with the observation that any free Borel system admits free Borel factors of arbitrarily small Gurevich entropy.

### 6.2. The case of finite entropy
In the following subsection we prove that flexibility implies ergodic universality for the more difficult case that \( h(C) < \infty \), that in particular confirms [49, Conjecture 1]. For the finite entropy case we use Lemma 6.18 below as a replacement for Lemma 6.15. The statement and the proof of Lemma 6.18 are similar to those of Lemma 6.15, but slightly more involved.

**Lemma 6.17.** Fix \( 0 < \tilde{h} < h(C) \). Consider the functions
\[ \alpha, \beta : (0,1) \to \mathbb{R} \]
defined by:
\[ \beta(\gamma) = \sqrt{\frac{\tilde{h} + h(C)}{2h(C)}} \gamma. \]
and
\[ \alpha(\gamma) = \frac{1}{2} \min \left\{ \beta(\gamma)^d h(C) - \tilde{h}^d, \frac{1}{10} (\gamma - \beta(\gamma)) \right\}. \]
then for every $\gamma \in (0, 1)$

$$\alpha(\gamma), \beta(\gamma) \in (0, 1),$$

and

$$\gamma > \beta(\gamma) + 10\alpha(\gamma).$$

In addition, there exists $K_0 \in \mathbb{N}$ and a function

$$N_T : (0, 1) \to \mathbb{N}$$

so that for every $\gamma \in (0, 1)$, any $k > K_0$ sufficiently big and every $n > N_T(\gamma)$

$$\gamma > \beta(\gamma) + 10\alpha(\gamma).$$

and

$$\gamma > \beta(\gamma) + 10\alpha(\gamma).$$

Proof. It is clear from (105) that $\beta(\gamma) > 0$ for every $\gamma > 0$. On the other hand, because $\hat{h} < h(C)$ it follows that $\beta(\gamma) < \gamma$ whenever $\gamma > 0$. In particular if $0 < \gamma < 1$ then $0 < \beta(\gamma) < 1$. So for $\gamma \in (0, 1)$ it follows that

$$0 < \beta(\gamma) < \gamma < 1,$$

and in this case by (106)

$$\alpha(\gamma) \leq \frac{1}{10}(\gamma - \beta(\gamma)) < \frac{1}{10} < 1.$$

In addition, by (105),

$$\beta(\gamma)^d h(C) - \hat{h}\gamma^d = \left(\frac{h(C) - \hat{h}}{2}\right)\gamma^d.$$

So $\alpha(\gamma) > 0$ whenever $\gamma > 0$. So we see that $\alpha, \beta : (0, 1) \to (0, 1)$ are well defined. Also, from the definition of $\alpha(\gamma)$ it is clear that (107) holds. Since $\lim_{n \to \infty} \delta_n = 0$, it follows that (108) holds for all sufficiently big $n$. By definition of $\alpha(\gamma)$,

$$\beta(\gamma)^d h(C) > \alpha(\gamma) + \hat{h}\gamma^d.$$

Note that as $n \to \infty$,

$$e^{\alpha(\gamma)|F_n| + \hat{h}\gamma F_n} = e^{\alpha(\gamma)|F_n| + h\gamma^d|F_n| + o(|F_n|)}$$

By definition,

$$\lim_{k \to \infty} \frac{1}{|F_k|} \log |C_k| = h(C).$$

So there exists $K_0 \in \mathbb{N}$ such that for any $k > K_0$

$$\frac{1}{|(1 + \delta_k)F_k|} \log |C_k| > \hat{h}.$$

It follows that (109) and (110) hold for all sufficiently large $n$ and we can choose $N_T(\gamma)$ to be the smallest $N \in \mathbb{N}$ so that (108), (109) and (110) hold for all $n > N$.

The following lemma is a crucial step in the proof of our main result. It says that we can slightly modify a given approximate embedding and get a much better one that is “close” to the original on a “big part of the space”. The “extent of modification required” depends on the “quality” of the original approximate embedding, and goes to zero as the original approximate embedding gets better and better.
Lemma 6.18. For every $\gamma \in (0, 1)$ there exists $k_0, N_0 \in \mathbb{N}$ such that for every $n_0 \geq N_0$ and every $(k_0, n_0, \frac{1}{n_0}, \mu)$-approximate embedding $\rho$, every $k \in \mathbb{N}$ and $\delta > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$ there is a $(k, n, \delta, \mu)$-approximate embedding $\hat{\rho} \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ so that

$$D_{n_0, \frac{1}{n_0}}[\rho, \hat{\rho}] \subset Y \setminus T^{(1-2\delta_n)} \gamma \, F_n \setminus Z_n. \quad (112)$$

Proof. Fix $\gamma \in (0, 1)$. Choose $\hat{h} \in (h_\mu(\mathcal{Y}), h(\mathcal{C}))$.

Because $(\mathcal{P}_k)_{k=1}^\infty$ is an increasing sequence of partitions and $\bigvee_{k=1}^\infty \mathcal{P}_k^\mathcal{E} = \text{Borel}(\mathcal{Y})$, it follows that $\lim_{k \to \infty} h_\mu(\mathcal{Y} \mid \mathcal{P}_k^\mathcal{E}) = 0$. So we can define

$$\overline{k}_0(\mu, \gamma) = \min\{k \in \mathbb{N} : h_\mu(\mathcal{Y} \mid \mathcal{P}_k^\mathcal{E}) < \alpha(\gamma)/8\}, \quad (113)$$

where $\alpha : (0, 1) \to \mathbb{R}$ is given by (106). Let $k_0 = \overline{k}_0(\mu, \gamma)$. We thus have,

$$h_\mu(\mathcal{Y} \mid \mathcal{P}_{k_0}^\mathcal{E}) < \alpha(\gamma)/8. \quad (114)$$

For any $n_0, n \in \mathbb{N}$ with $n > n_0$ recall $S_{n,n_0}$ as defined in (6). Because every element of $S_{n,n_0}$ is a subset of $F_n$ that has cardinality at most $|F_n|/|F_{n_0}|$, for every $n > n_0$

$$\frac{1}{|F_n|}\log |S_{n,n_0}| \leq \frac{1}{|F_n|}\log \left( \frac{|F_n|}{|F_{n_0}|} \right) \leq \mathcal{H} \left( \frac{1}{|F_{n_0}|} \right). \quad (115)$$

In particular, because $\lim_{p \to 0^+} \mathcal{H}(p) = 0$ it follows that

$$\lim_{n_0 \to \infty} \sup_{n > n_0} \frac{1}{|F_n|}\log |S_{n,n_0}| = 0. \quad (116)$$

For $\epsilon > 0$ let

$$\overline{N}_S(\epsilon) = \inf \left\{ N \in \mathbb{N} : \forall n > n_0 > N \ |S_{n,n_0}| < \exp(\epsilon|F_n|) \right\}. \quad (117)$$

The limit (115) shows that $\overline{N}_S : (0, 1) \to \mathbb{N}$ is indeed well defined and finite.

Set

$$N_0 = \max \left\{ \overline{N}_S(\alpha(\gamma)^d h), \overline{N}_R(\alpha(\gamma)/4, k_0), 0 \right\}, \quad (118)$$

where $\alpha : (0, 1) \to (0, 1)$ is given by (106), and $\overline{N}_R(\alpha(\gamma)/4, k_0)$ is the integer obtained by applying Lemma 6.10 with $\eta = \alpha(\gamma)/4$ and $k_0$, and $N_0 \in \mathbb{N}$ is the constant that appears in Lemma 6.17. Explicitly, this means that for every $n_0 > N_0$ , $(k_0, n_0, \frac{1}{n_0}, \mu)$-approximate embedding $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ and any $\delta' > 0$ there exists $N \in \mathbb{N}$ such that for all $n > N$

$$\text{COV}_{\mu, \delta', \mathcal{P}_{k_0}^{F_n \setminus \gamma F_n \setminus F_{n_0}}} (Z_n) < e^{\alpha(\gamma)/4 |F_n|} \quad (119)$$

and also

$$|S_{n,n_0}| < \exp \left( \alpha(\gamma)^d h |F_n| \right), \quad (120)$$

where $S_{n,n_0}$ is defined by (6).

At this point we fix an arbitrary $n_0 > N_0$, $x_0 \in C_{n_0}$, a $(k_0, n_0, \frac{1}{n_0}, \mu)$-approximate embedding $\rho \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ and $\delta \in (0, 1)$. Let $\delta' > 0$ be a positive number much smaller than $\delta$, so that

$$\delta' < 10^{-10} \delta^4. \quad (121)$$

By Lemma 4.3 there exists $N \in \mathbb{N}$ such that for every $n > N$

$$\text{COV}_{\mu, \delta', \mathcal{P}_{k_0}^{F_n \setminus \gamma F_n}} (Z_n) < e^{h |F_n|}, \quad (122)$$

Using the inequality (31) from Lemma 4.2 it follows from (121) that

$$\text{COV}_{\mu, \delta' + \delta, \mathcal{P}_{k_0}^{F_n \setminus \gamma F_n \setminus F_{n_0}}} (Z_n) < e^{h |F_n|}. \quad (123)$$

Choose $k \in \mathbb{N}$. By Lemma 4.4 and (114) there exists $N \in \mathbb{N}$ so that for every $n > N$

$$\text{COV}_{\mu, \delta', \mathcal{P}_{k}^{F_n \setminus \gamma F_n}} (Z_n) < e^{\alpha(\gamma)/4 |F_n|}.$$
Let \( N_T : (0, 1) \to \mathbb{N} \) be the function given by Lemma 6.17. In particular, for every \( n > N_T(\gamma) \) (110) holds. Also, for \( n \gg n_0 \) and \( \theta \in (0, 1) \) let

\[
(124) \quad K_{n,n_0,\theta} = \theta F_n \cap [(2 + 2\delta n_0) n_0] Z^d.
\]

Using (119), it follows that if \( n > \max\{n_0, N_T(\gamma)\} \) then

\[
(125) \quad |C_{n_0}^{K_{n,n_0,\alpha(\gamma)}}| > |S_{n,n_0}|.
\]

Also, by Lemma 6.17, for every \( n > N_T(\gamma) \) we have that (108) and (109) hold.

Let \( N \) be the smallest integer so that \( N > N_T(\gamma) \), \( N \gg n_0 \) and so that for every \( n > N \) (108), (109), (118), (121) and (123) hold, and also

\[
(126) \quad \epsilon_n < \frac{1}{16} \min\{\eta, \epsilon_{n_0}\}.
\]

It follows that (122) and (125) also hold for all \( n > N \).

Now choose any \( n > N \). From (118) and (122) using (32) of Lemma 4.2 it follows that

\[
(127) \quad COV_{\mu, \sqrt{\delta^2 + 2\theta n_0} P_{\frac{4}{n_0}} |P_{\frac{4}{n_0}} |} (Z_n) < \epsilon_n^{\alpha(\gamma)} F_n |\gamma F_n|.
\]

By inequality (33) of Lemma 4.2

\[
COV_{\mu, \delta^2 / 3, \frac{4}{n_0} F_n |P_{\frac{4}{n_0}} |} (Z_n) \leq COV_{\mu, \frac{\delta^2}{6}, \frac{4}{n_0} F_n |P_{\frac{4}{n_0}} |} (Z_n) \cdot COV_{\mu, \frac{\delta^2}{6}, \frac{4}{n_0} F_n |P_{\frac{4}{n_0}} |} (Z_n).
\]

The inequality (120) ensures that \( \delta < 6 \) so that

\[
\sqrt{\delta^2 + 2\theta} < \frac{(\delta/3)^2}{6}.
\]

So using (123) and (127) it follows that

\[
(128) \quad COV_{\mu, \delta / 3, \frac{4}{n_0} F_n |P_{\frac{4}{n_0}} |} (Z_n) < \epsilon_n^{\alpha(\gamma) / 2} |F_n| \gamma F_n|.
\]

This means that there is a set \( \hat{X}_n \subset X(1-\delta F_n \gamma F_n) \) so that

\[
(129) \quad \mu\left(Z_n \setminus \bigcup_{w \in \hat{X}_n} \rho^{-1}([w]) \right) \leq \frac{\delta}{3} \mu(Z_n),
\]

and so that for every \( w \in \hat{X}_n \) there exists a set \( G_w \subset P_{\frac{4}{n_0}} \) such that

\[
(130) \quad |G_w| < \epsilon_n^{\alpha(\gamma) / 2} |F_n| \gamma F_n|.
\]

and

\[
(131) \quad \mu\left(Z_n \cap \rho^{-1}([w]) \setminus \bigcup G_w \right) \leq \frac{\delta}{3} \mu(\rho^{-1}([w]) \cap Z_n).
\]

By (109) and (130) it follows that

\[
(132) \quad \forall w \in \hat{X}_n \quad |G_w| < |\mathcal{C}_{n_0}^{K_{n,n_0,\alpha(\gamma)}}|.
\]

For each \( w \in \hat{X}_n \) let \( \Phi : \mathcal{T}_{\frac{4}{n_0}} \to \mathcal{C}_{n_0}^{K_{n,n_0,\alpha(\gamma)}} \) be a function such that the restriction of \( \Phi \) to \( G_w \) is injective and let \( \Phi^{-1} : \mathcal{C}_{n_0}^{K_{n,n_0,\alpha(\gamma)}} \to \mathcal{T}_{\frac{4}{n_0}} \) be a left inverse on \( G_w \). By this we mean that

\[
(133) \quad \Phi^{-1} \circ \Phi(P) = P \text{ for every } P \in G_w.
\]

Define \( \Upsilon_{n,n_0} : Y \to S_{n,n_0} \) by

\[
(134) \quad \Upsilon_{n,n_0}(y) = \left\{ T \in (1 - 2\delta)n_0 F_n : T^\gamma(y) \in Z_{n_0} \right\}.
\]

The fact that \( \Upsilon_{n,n_0}(y) \in S_{n,n_0} \) follows because \( \{T^{\gamma}Z_{n_0}\}_{T \in (1 + \delta n_0) F_{n_0}} \) are pairwise disjoint. Recall that \( \mathcal{C}_{n_0}^{K_{n,n_0,\alpha(\gamma)}} \) satisfies (125).
Let $\Phi_b : S_{n,n_0} \to C_{n_0}^{K_{n,n_0,\alpha(\gamma)}}$ be an injective function with left inverse $\Phi_b^{-1} : C_{n_0}^{K_{n,n_0,\alpha(\gamma)}} \to S_{n,n_0}$. By this we mean that

\[(135) \quad \Phi_b^{-1} \circ \Phi_b(s) = s \text{ for every } s \in S_{n,n_0}.
\]

Choose $\vec{t}_b$ so that

\[(136) \quad (1 + \delta_n)(\beta(\gamma) + \alpha(\gamma)) \cdot n < \|\vec{t}_b\|_\infty < (1 + \delta_n)(\beta(\gamma) + 2\alpha(\gamma)) \cdot n.
\]

It follows that

\[(137) \quad \left(\vec{t}_b + (1 + \delta_n)F_{[\alpha(\gamma)n]}\right) \cap (1 + \delta_n)F_{[\beta(\gamma)n]} = \emptyset.
\]

By (108)

\[(138) \quad (\vec{t}_b + (1 + \delta_n)F_{[\alpha(\gamma)n]}) \cup (1 + \delta_n)F_{[\beta(\gamma)n]} \subset (1 - 2\delta_n)F_{\gamma n}.
\]

For $y \in Y$, let

\[(139) \quad K_y = Y_{n,n_0}(y) \cap (1 - 2\delta_n)F_n \setminus \gamma F_n
\]

and

\[(140) \quad \mathcal{K}_y = K_y \cup \left(\vec{t}_b + K_{n,n_0,\alpha(\gamma)}\right) \cup K_{n,n_0,\beta(\gamma)}.
\]

Then $\mathcal{K}_y \subset (1 - \delta_n)F_n$ is $(1 + \delta_n)F_{n_0}$-spaced. Let $\mathcal{W}_y \in C_{n_0}$ be given by

\[(141) \quad \mathcal{W}_y = \text{Ext}_{n_0}(\mathcal{W}_y),
\]

and let $\tilde{\rho} = \rho_{\mathcal{W}_y} \in \text{Mor}(\mathcal{Y}, \mathcal{X}')$ be given by (99). If $y \in Z_{n_0}$ whenever $\vec{t} \in (1 - 2\delta_n)F_n \setminus \gamma F_n$ such that $T^{\vec{t}}y \in Z_n$ then

\[(142) \quad \tilde{\rho}(y) \mid_{F_{n_0}} = S^{-\vec{t}}(\tilde{\rho}(T^{\vec{t}}(y))) \mid_{F_{n_0}} = S^{-\vec{t}}(\text{Ext}(T^{\vec{t}}(y))) \mid_{F_{n_0}}.
\]

Also, because $y \in Z_{n_0}$, $-\vec{t} \in Y_{n,n_0}(T^{\vec{t}}(y)) \cap (1 - 2\delta_n)F_n \setminus \gamma F_n$. It follows that $-\vec{t} \in \mathcal{K}_{T^{\vec{t}}(y)}$ and $(\mathcal{W}_{T^{\vec{t}}(y)})_{-\vec{t}} = \rho(y)_{-\vec{t}}$. Thus,

\[(143) \quad d_{X^{n_0}}(\rho(y), \rho(y)) = d_{X^{n_0}}(\rho(y), S^{-\vec{t}}(\text{Ext}(\mathcal{W}_{T^{\vec{t}}(y)}))) < \frac{1}{4} \epsilon_{n_0}.
\]

This implies (112). It remains to show that $\tilde{\rho}$ is a $(k, n, \delta, \mu)$ approximate embedding. Because $\tilde{\rho}$ takes values in $C_n$, it is clear that $\tilde{\rho}$ is $n$-towerable. So to complete the proof we will find a set Borel set $Y_0 \subset Z_n$ so that

\[(144) \quad \forall y \in Y_0 \ P_{K_{n}}^{F_{n}}(y) \text{ is uniquely determined by } \rho(y) \mid_{F_{n}}
\]

and

\[(145) \quad \mu(Z_n \setminus Y_0) < \delta \mu(Z_n).
\]

Let

\[(146) \quad Y_0 = \left\{ y \in Z_n : \exists w \in \bar{X}_n \text{ so that } \rho(y)\mid_{(1 - 2\delta_n)F_n \setminus \gamma F_n} = w \text{ and } \mathcal{P}_{K_{n}}^{F_{n}}(y) \in \mathcal{G}_w \right\}.
\]

Thus,

\[(147) \quad \mu(Z_n \setminus Y_0) \leq \mu(Z_n \setminus \bigcup_{w \in \bar{X}_n} \rho^{-1}([w])) + \sum_{w \in \bar{X}_n} \mu\left(Z_n \cap \rho^{-1}([w]) \setminus \bigcup \mathcal{G}_w\right).
\]

So (142) follows from (129) and (131).
Choose a Borel function $\Pi_{\gamma,n} : X^{2d} \to C_{n_0}^{K_{n_0},\alpha(\gamma)}$ so that
\begin{equation}
\max_{\vec{j} \in K_{n_0},\alpha(\gamma)} d_{X^{2d}}^F(S^{\vec{j}}(w), \Pi_{\gamma,n}(w)_{\vec{j}}) = \min \left\{ \max_{\vec{j} \in K_{n_0},\alpha(\gamma)} d_{X^{2d}}^F(S^{\vec{j}}(w), w'_{\vec{j}}) : w' \in C_{n_0}^{K_{n_0},\alpha(\gamma)} \right\}.
\end{equation}

Because $C_{n_0}$ is $(\epsilon_{n_0}, F_{n_0})$-separated it follows that whenever $w \in X^{2d}$, $w' \in C_{n_0}^{K_{n_0},\alpha(\gamma)}$ and
\[ \max_{\vec{j} \in K_{n_0},\alpha(\gamma)} d_{X^{2d}}^F(S^{\vec{j}}(w), w'_{\vec{j}}) < \frac{1}{2} \epsilon_{n_0}, \]
then $\Pi_{\gamma,n}(w) = w'$. Because $(\Phi_b(\Upsilon_{n,n_0}(y)))_{\vec{i}} = (\overline{W}_y)_{\vec{i}+n_0}$ for every $\vec{i} \in K_{n_0,\alpha(\gamma)}$,
\[ \max_{\vec{j} \in K_{n_0},\alpha(\gamma)} d_{X^{2d}}^F(S^{\vec{j}+\vec{j}}(\text{Ext}_n(\overline{W}_y)), \Phi_b(\Upsilon_{n,n_0}(y))_{\vec{j}}) < \frac{1}{4} \epsilon_{n_0}. \]
It follows that for any $y \in Y_0$
\begin{equation}
\Upsilon_{n,n_0}(y) = \Phi_b^{-1} \left( \Pi_{\gamma,n}(S^{\vec{j}}(\text{Ext}_n(\overline{W}_y))) \right) = \Phi_b^{-1} \left( \Pi_{\gamma,n}(S^{\vec{j}}(\hat{\rho}(\hat{y}))) \right).
\end{equation}
This argument shows that for $y \in Z_n$, we can use the value of $\hat{\rho}(\hat{y}) \mid_{\overline{W}_n+\overline{F}}$ to uniquely recover $\Upsilon_{n,n_0}(y)$ (in a Borel manner). Once we “recovered” $\Upsilon_{n,n_0}(y)$ from $\hat{\rho}(\hat{y}) \mid_{\overline{W}_n+\overline{F}}$ using (112) together with the fact that shows that $\rho$ is a $n_0$-tolerance and that $C_{n_0}$ is $(\epsilon_{n_0}, F_{n_0})$-separated, we can similarly “read-off” $\rho(\hat{y}) \mid_{\overline{W}_n+\overline{F}}$ from $\hat{\rho}(\hat{y}) \mid_{\overline{W}_n+\overline{F}}$ for every $\vec{j} \in K_{n_0,n_0}(y) \cap (1-2\delta_0)F_n \setminus \gamma F_n$. Because $\rho$ is $n_0$-tolerable, for every $y \in Y$, if we know $\rho(y) \mid_{\overline{W}_n+\overline{F}}$ for every $\vec{j} \in K_{n_0,n_0}(y) \cap (1-2\delta_0)F_n \setminus \gamma F_n$ then $\rho(y) \mid_{(1-2\delta_0)F_n \setminus \gamma F_n}$ is trivially determined because $\rho(y)_{\vec{i}} = x_*$ for all remaining $\vec{i} \in (1-2\delta_0)F_n \setminus \gamma F_n$. The above argument explicitly describes a function $\Psi : C_n \to X^{(1-2\delta_0)F_n \setminus \gamma F_n}$ so that for any $y \in Z_n$, $\Psi(\hat{\rho}(\hat{y})) = \rho(y) \mid_{(1-2\delta_0)F_n \setminus \gamma F_n}$.

Now let $\Pi'_{\gamma,n} : X^{2d} \to C_{n_0}^{K_{n_0},\beta(\gamma)}$ be a Borel function that satisfies
\begin{equation}
\max_{\vec{j} \in K_{n_0},\delta(\gamma)} d_{X^{2d}}^F(S^{\vec{j}}(w), \Pi'_{\gamma,n}(w)_{\vec{j}}) = \min \left\{ \max_{\vec{j} \in K_{n_0},\delta(\gamma)} d_{X^{2d}}^F(S^{\vec{j}}(w), w'_{\vec{j}}) : w' \in C_{n_0}^{K_{n_0},\beta(\gamma)} \right\}.
\end{equation}
It follows that for $y \in Y_0$,
\begin{equation}
\mathcal{P}_k^F(y) = \Phi_b^{-1} \left( \Pi'_{\gamma,n}(\hat{\rho}(\hat{y})) \right).
\end{equation}
This shows that (141) holds.

Combining everything we proved so far gives the following:

**Proposition 6.19.** Let $(X, S)$ be a $\mathbb{Z}^d$-dynamical system and $\mathcal{Y} = (Y, T)$ a Borel $\mathbb{Z}^d$-system. If $\mathcal{C} = (C_n)_{n=1}^\infty \in X^\mathbb{N}$ is a flexible sequence, $\mu \in \text{Prob}_n(\mathcal{Y})$ and $h_n(\mathcal{Y}) < h(\mathcal{C})$ then there exists a Borel $T$-invariant $Y_\infty \subset Y$ with $\mu(Y_\infty) = 1$ and an injective equivariant Borel embedding $\rho : Y_0 \to X$.

**Proof.** For every $j$ let $k_j$ and $N_j$ be the numbers obtained as $k_0, N_0$ by applying Lemma 6.18 with $\gamma = \frac{1}{2^{4d+1}}$. We will inductively construct a sequence of natural numbers and $(n_j)_{j=1}^\infty$ with $n_j \geq N_j$ and a sequence $(\rho_j)_{j=1}^\infty \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ so that $n_j \leq n_{j+1}$ and (97) holds for every $j$. To start the induction apply lemma 6.13 with $\epsilon = \frac{1}{N_j}$ and $k = k_1$. Let $n_1 = N_1$. Let $\rho_1 \in \text{Mor}(\mathcal{Y}, \mathcal{X})$ be the resulting $(k_1, n_1, \frac{1}{N_1}, \mu)$-approximate embedding.

For the induction step, suppose $n_j$ and $\rho_j$ have been defined for a fixed $j \in \mathbb{N}$. Apply Lemma 6.18 with $\gamma = \frac{1}{2^{4d+1}}$, $\rho = \rho_j$, $k = k_{j+1}$, $\delta = \frac{1}{N_{j+1}}$. Let $N_j$ be the resulting number $N$. Define $n_{j+1}$ to be the smallest integer greater or equal to $\max\{N_j, n_j+1\}$. Define $n_{j+1}$ to be the smallest integer greater or equal to $\max\{N_j, n_j+1\}$ that also satisfies $n_j \leq n_{j+1}$. Then apply Lemma 6.18 and let $n_{j+1}$ be the resulting $(k_{j+1}, n_{j+1}, \frac{1}{N_{j+1}}, \mu)$-approximate embedding. We have
\[ D_{n_{j+1},n_j} (\rho_j, \rho_{j+1}) \leq Y \setminus T^{(1-2\delta_{n_{j+1}})F_{n_{j+1}} \setminus 2^{-1-j}F_{n_{j+1}} Z_{n_{j+1}}}.
\]
Using Lemma 2.4 it follows that (97) holds.

By Lemma 6.11 and Lemma 6.12 the limit $\rho = \lim_{n \to \infty} \rho_j$ exist on the set $Y_\infty$ given by (91), and satisfies the statement of the lemma.
6.3. Full universality - Realizing ergodic measures with full support. We now prove Proposition 5.2. This follows by a slight modification of Lemma 6.18:

Lemma 6.20. In the statement of Lemma 6.18, for every \( x_0 \in C_{n_0} \) it is possible to arrange that the resulting \((k, n, \delta, \mu)\)-approximate embedding \( \hat{\rho} \in \text{Mor}(Y, X) \) will have the additional property that for some \( i_0 \in F_n \)

\[
\forall y \in T^{-i_0}Z_n, \quad d_X(\hat{\rho}(y), x_0) < \epsilon_{n_0}.
\]

Proof. In the proof of Lemma 6.18 (with the standing assumptions that \( n \) is large enough) given \( \tilde{i}_b \in F_n \) that satisfies (136) and (137), we can find \( \tilde{i}_0 \in F_n \) such that

\[
\tilde{i} + (1 + \delta_{n_0})F_{n_0} \subset (1 - 2\delta_n)F_n
\]

and also so that \( \tilde{i}_0 + (1 + \delta_{n_0})F_{n_0} \) is disjoint from

\[
\left( \tilde{i}_b + K_{n,n_0,n}(\gamma) + (1 + \delta_{n_0})F_{n_0} \right) \cup \left( K_{n,n_0,\beta}(\gamma) + (1 + \delta_{n_0})F_{n_0} \right).
\]

Then proceed exactly as in the proof of Lemma 6.18: Define for \( y \in Y \) \( \overline{W}_y \in \text{Spec}_n(C) \) by (139). Then for each \( y \in Y \) let \( \overline{K}_y \subset (1 - \delta_n)F_n \) be given by (138) as in the proof of Lemma 6.18, and let \( \overline{\mathbf{K}}_y = \overline{K}_y \cup \{ \tilde{i}_0 \} \),

So \( \overline{K}_y \subset (1 - \delta_n)F_n \) is also \((1 + \delta_{n_0})F_{n_0}\)-spaced. Extend \( \overline{W}_y \) to \( \overline{W}_y \in C_{n_0} \) by defining

\[
(\overline{W}_y)_{\tilde{i}} = \begin{cases} \overline{x}_0 & \tilde{i} = \tilde{i}_0 \\ \tilde{i} \in \overline{K}_y & \tilde{i} \in \overline{K}_y. \end{cases}
\]

(149)

\[
\tilde{\phi}(y) = \text{Ext}_n(\overline{W}_y),
\]

and \( \tilde{\rho} = \rho_{\tilde{i},n} \). It follows that the resulting \( \hat{\rho} \in \text{Mor}(X, Y) \) satisfies (148). \( \square \)

Proof of Proposition 5.2. Given \( \mu \in \text{Prob}_+(\mathcal{Y}, T) \) we need to exhibit \( \rho \in \text{Mor}(\mathcal{Y}, \mathcal{X}) \) that induces an embedding of \((Y, T, \mu)\) into \((X, S)\) and such that the closed support of \( \mu \circ \rho^{-1} \) is \( X \).

Let \((x_j)_{j=1}^{\infty}\) be an enumeration of \( \bigcup_{n=1}^{\infty} C_n \), so that the elements of \( C_n \) come before the elements of \( C_m \) whenever \( n < m \). We assume \( x_j \in C_{k_j} \) for every \( j \). For every \( j \in \mathbb{N} \) let \( w_j \in C_{k_j}^{(0)} \) be given by \( w_j(0) = x_j \). Using Lemma 6.18 we can construct a sequence \((\rho_j)_{j=1}^{\infty} \in \text{Mor}(\mathcal{Y}, \mathcal{X})^{(0)} \) as we did in the proof of Proposition 6.19, with the additional feature that for every \( j \in \mathbb{N} \) there exists \( \tilde{j}_j \in (1 - \delta_{n_j})F_{n_j} \) so that for any \( y \in T^{-\tilde{j}_j}Z_{n_j} \), \( d_X(\rho_j(y), x_j) < \epsilon_j \), where \( x_j = \text{Ext}_{n_j}(w_j) \). It follows that there is a Borel set \( Y_0 \subset Y \) with \( \mu(Y_0) = 1 \) so that the limit \( \rho = \lim_{j \to \infty} \rho_j \) exists on \( Y_0 \) and is an injective equivariant Borel embedding \( \rho : Y_0 \to X \), and in addition for every \( j \in \mathbb{N} \),

\[
\mu \left( \left\{ y \in Y : d(\rho(y), x_j) < \frac{1}{2} \epsilon_j \right\} \right) > 0.
\]

By the assumption that every \( x \in X \) is an accumulation point of \( \bigcup_{n=1}^{\infty} C_n \), for every \( x \in X \) and \( \epsilon > 0 \) there exists \( j \in \mathbb{N} \) such that the \((\epsilon, d_X)\)-ball centered at \( x \) contains the \((2\epsilon_j, d_X)\)-ball centered at \( x_j \). This proves that the measure \( \mu \circ \rho^{-1} \) assigns positive measure to every open subset of \( X \). \( \square \)

7. Almost Borel Universality

In this section we prove Theorem 5.1. This follows the basic strategy of the previous section and relies on it. There are additional complications in certain steps: We use the same procedure to construct a converging sequence of approximate embeddings for every measure \( \mu \in \text{Prob}_+(\mathcal{Y}) \), but keep track and make sure that the dependence on the measure \( \mu \) is “Borel measurable”. For “almost all” points \( y \in Y \) that are generic with respect to \( \mu \) we apply the embedding that corresponds to \( \mu \). An extra ingredient that we need is to make sure that generic points corresponding to different measures do not get mapped to the same point in \( X \). This is done by encoding the measure \( \mu \) itself into \( X \) using Lemma 7.8 below.
7.1. Borel setup. It will be convenient for us to use the following (essentially trivial) “universal Borel embedding” result with respect to the shift over \( (\{0,1\}^N)^{\mathbb{Z}^d} \):

**Proposition 7.1.** Any standard Borel dynamical system is isomorphic to a Borel subsystem of the shift over the Cantor set: Let \( \mathcal{Y} = (Y, T) \) be a Borel \( \mathbb{Z}^d \) dynamical system. Then there is a Borel embedding \( \rho : Y \to (\{0,1\}^N)^{\mathbb{Z}^d} \) that is equivariant in the sense that

\[
\rho(T^i(y)) = \rho(y)_i \quad \text{for every} \quad i \in \mathbb{Z}^d, \ y \in Y.
\]

**Proof.** Let \( \mathcal{A} = \{A_1, \ldots, A_n, \ldots\} \subset \text{Borel}(Y) \) be a countable sequence of Borel sets that generate the Borel \( \sigma \)-algebra of \( Y \). Define \( \rho : Y \to (\{0,1\}^N)^{\mathbb{Z}^d} \) by

\[
\rho(y)_i(n) = 1_{A_n}(T^n(y)), \ i \in \mathbb{Z}^d, n \in \mathbb{N}.
\]

Clearly \( \rho \) is a Borel function. Injectivity of \( \rho \) follows because the elements of \( \mathcal{A} \) separate points. \( \square \)

Proposition 7.1 is an easy and well known starting point for “Borel dynamics”. It clearly generalizes verbatim to actions of arbitrary countable groups. For free \( \mathbb{Z} \)-actions, Proposition 7.1 is a direct consequence of the existence of a countable generator. This has a short but non-trivial proof due to Weiss [57].

Using Proposition 7.1 above, from now on we will identify \( Y \) with a shift-invariant Borel subset of \( (\{0,1\}^N)^{\mathbb{Z}^d} \), and assume that the \( \mathbb{Z}^d \)-action \( T \) on \( Y \) is the restriction of the shift on \( (\{0,1\}^N)^{\mathbb{Z}^d} \). Thus \( Y \) inherits the relative topology from \( (\{0,1\}^N)^{\mathbb{Z}^d} \), compatible with it’s Borel structure. We denote by \( C(Y, X) \) the space of continuous functions from \( Y \to X \). The space \( C(Y, X) \) is a standard Borel space. Let \( \text{Mor}_C(Y, X) \) denote the space of continuous morphisms from \( \mathcal{Y} = (Y, T) \) to \( X = (\mathbb{Z}^d, S) \). There is an obvious bijection between \( \text{Mor}_C(Y, X) \) and \( C(Y, X) \) that gives \( \text{Mor}_C(Y, X) \) a standard Borel structure. We denote by \( \text{Clopen}(Y) \) the collection of subsets of \( Y \) obtained by intersecting a clopen subset of \( (\{0,1\}^N)^{\mathbb{Z}^d} \) with \( Y \). This is a countable set (as there are countably many clopen subsets of \( (\{0,1\}^N)^{\mathbb{Z}^d} \)). We consider the space \( \text{Prob}_c(Y) \subset \text{Prob} \left( (\{0,1\}^N)^{\mathbb{Z}^d} \right) \), again with the Borel structure that comes from the weak-\( * \) topology. Clearly, we can choose the Borel embedding of \( Y \) into \( (\{0,1\}^N)^{\mathbb{Z}^d} \) so that with respect to the inherited topology on \( Y \) each of the partitions \( P_k \) consists of clopen sets. We will assume this from now on. This is to ensure continuity of certain functions related to approximate embeddings that appear in the proof. We call a point \( y \in Y \) **generic** if the sequence of probability measures

\[
(150) \quad \frac{1}{|F_n|} \sum_{i \in F_n} \delta_{T^n(y)}
\]

converge in the weak-\( * \) topology to an ergodic measure \( \mu \in \text{Prob}_c(Y) \). Let \( G(Y) \subset Y \) denote the set of generic points for \( \mathcal{Y} \). In this case we refer to the measure

\[
(151) \quad \mu_y = \lim_{n \to \infty} \frac{1}{|F_n|} \sum_{i \in F_n} \delta_{T^n(y)}.
\]

as the **empirical measure** of \( y \). Note that \( Y \) is not assumed to be compact, so for general \( y \in Y \) a limit point of the sequence of measures in (150) need not be supported on \( Y \). We also denote by

\[
(152) \quad \text{emp} : G(Y) \to \text{Prob}_c(Y)
\]

the map that sends a generic point \( y \in G(Y) \) to it’s empirical measure \( \mu_y \). Namely

\[
(153) \quad \text{emp}(y) = \mu_y
\]

For \( \mu \in \text{Prob}_c(Y) \), let

\[
(154) \quad G_\mu(Y) = \text{emp}^{-1}(\{\mu\})
\]

The set \( G_\mu(Y) \) is the collection of \( \mu \)-generic points in \( Y \). We have

\[
G(Y) = \bigcup_{\mu \in \text{Prob}_c(Y, T)} G_\mu(Y, T).
\]

For later reference, we record some Borel measurably results about generic points and empirical measures.
Proposition 7.2.  

(1) The set $G(Y)$ is a Borel subset of $Y$.  

(2) The set $\text{Prob}(Y)$ is a Borel subset of $\text{Prob}\left(\{(0,1)^N\}\right)$.  

(3) The set $\text{Prob}_c(Y)$ is a Borel subset of $\text{Prob}(Y)$.  

(4) For every $\mu \in \text{Prob}_c(Y)$ the set $G_\mu(Y)$ is a Borel subset of $Y$.  

(5) The function $\text{emp} : G(Y) \to \text{Prob}_c(Y)$ is a Borel measurable function.

These statements are all standard and well known, so we omit the proof, referring for instance to the discussion of generic points in [27].

We can now give a short proof of Proposition 2.2 regarding the existence of “Borel” Rokhlin towers:

**Proof of Proposition 2.2.** Choose $n \in \mathbb{N}$ and $\epsilon > 0$. Using the “usual” version of Rokhlin’s lemma (Proposition 2.1), for every $\mu \in \text{Prob}_c(Y,T) \subset \text{Prob}\left(\{(0,1)^N\}\right)$, let $Z'_\mu \subset Y$ be the base an $(F_n, \epsilon/2, \mu)$ Rokhlin tower. By inner regularity of $\mu$, we can find a closed (hence compact) set $Z''_\mu \subset \{(0,1)^N\}$ so that $Z''_\mu \subset Z'_\mu$ and

$$\mu(Z'_\mu \setminus Z''_\mu) < |F_n|^{-1} \epsilon / 2.$$  

Then $Z''_\mu$ is an $(F_n, \epsilon, \mu)$ Rokhlin tower. Because the sets $\{T^{-i}Z''_\mu\}_{i \in F_n}$ are pairwise disjoint compact subsets and the topology of $Y$ has a clopen basis, we can find a clopen set $Z_\mu \subset \{(0,1)^N\}$ that contains $Z''_\mu$ and so that $(T^{-i}Z_\mu)_{i \in F_n}$ are still pairwise disjoint. We conclude that for every $\mu \in \text{Prob}_c(Y,T)$ there exists a clopen set $Z_\mu \subset \{(0,1)^N\}$ that is also the base of an $(F_n, \epsilon, \mu)$ Rokhlin tower. Our next goal is to show that we can furthermore choose $Z_\mu$ as above so that the function $\mu \mapsto Z_\mu$ will be Borel measurable as a function from $\text{Prob}_c(Y)$ to the clopen sets of $\{(0,1)^N\}$. Let $(E_k)_{k=1}^\infty$ be some enumeration of the clopen subsets of $\{(0,1)^N\}$. For every $\mu \in \text{Prob}_c(Y,T)$ let $Z_\mu = E_{k_0}$ if $E_{k_0}$ is the base of an $(F_n, \epsilon, \mu)$ Rokhlin tower and for every $k < k_0$ the clopen set $E_k$ is not the base of an $(F_n, \epsilon, \mu)$ Rokhlin tower. Then using Proposition 7.2 it is not difficult to check that for every $k \in \mathbb{N}$ the set

$$U_k = \{\mu \in \text{Prob}_c(Y) : Z_\mu = E_k\}$$

is a Borel subset of $\text{Prob}_c(Y)$. It follows that the resulting function $\mu \mapsto Z_\mu$ is indeed a Borel measurable function.

Define

$$Z = \bigcup_{k=1}^\infty \text{emp}^{-1}(U_k) \cap E_k,$$

From the fact that the $U_k$’s are Borel subsets of $\text{Prob}_c(Y)$, using Proposition 7.2 again it follows that $Z$ is a Borel subset of $Y$.

One can check that

$$Z = \bigcup_{\mu \in \text{Prob}_c(Y)} (Z_\mu \cap G_\mu(Y)).$$

Thus is clear that for every $\mu \in \text{Prob}_c(Y)$,

$$\mu\left(\bigcup_{i \in F_n} T^{-i}Z\right) = \mu\left(\bigcup_{i \in F_n} T^{-i}Z_\mu\right) \geq 1 - \epsilon.$$  

Thus $Z$ is the base of an $(F_n, \epsilon, \mu)$-Rokhlin tower for every $\mu \in \text{Prob}_c(Y)$.

From now on we will further assume that for every $n \in \mathbb{N}$ the set $Z_n \subset Y$ is clopen in $Y$. We can assume this without any loss of generality by modifying the embedding of $Y$ into the shift over $\{0,1\}^N$.

For $y \in Y$ and $A \subset Y$ define:

$$\bar{d}_Y(y, A) = \limsup_{n \to \infty} \frac{1}{|F_n|} \left| \left\{ i \in F_n : T^i(y) \in A \right\} \right|.$$  

$$\underline{d}_Y(y, A) = \liminf_{n \to \infty} \frac{1}{|F_n|} \left| \left\{ i \in F_n : T^i(y) \in A \right\} \right|.$$
Viewing $\text{Prob}_c(\mathcal{Y})$ as a Borel subset of $\text{Prob}([0,1]^Z)$, it is a standard Borel space. So we can find a sequence $(\mathcal{Q}_n)_{n=1}^{\infty}$ of finite Borel partitions that together generate the Borel $\sigma$-algebra of $\text{Prob}_c(\mathcal{Y})$, and in particular separate points. We further assume that each partition in the sequence refines the previous one and that $|\mathcal{Q}_r| \leq |C_r|$ for every $r$.

7.2. Main lemmas and proof of Theorem 5.1. Our proof of Theorem 5.1 follows a similar structure as the proof of Proposition 6.19 in the previous section. We will use two main lemmas. The easier one is a refinement of Lemma 6.13. It asserts that we can produce a $(k,n,\epsilon,\mu)$-approximate embedding $\rho \in \text{Mor}(\mathcal{Y},\mathcal{X})$, provided that $n$ is sufficiently big, as a Borel function of $k \in \mathbb{N}$ $\mu \in \text{Prob}_c(\mathcal{Y})$ and $\epsilon > 0$:

**Lemma 7.3.** There exists a Borel function

$$N_0 : \text{Prob}_c(\mathcal{Y}) \times \mathbb{N} \times (0,1) \to \mathbb{N},$$

a Borel function

$$\Phi_0 : \text{Prob}_c(\mathcal{Y}) \times \mathbb{N} \times (0,1) \times \mathbb{N} \to \text{Mor}_C(\mathcal{Y},\mathcal{X}),$$

and a Borel function

$$\overline{Z} : \text{Prob}_c(\mathcal{Y}) \times \mathbb{N} \times (0,1) \times \mathbb{N} \to \text{Clopen}(\mathcal{Y}),$$

so that the following holds:

Fix $\mu \in \text{Prob}_c(\mathcal{Y})$, $k,n \in \mathbb{N}$ and $\epsilon > 0$ such that

$$n > N_0(\mu,k,\epsilon).$$

Denote

$$\rho = \Phi_0(\mu,k,\epsilon,n)$$

then $\rho \in \text{Mor}(\mathcal{Y},\mathcal{X})$ is a $(k,n,\epsilon,\mu)$-approximate embedding and we can choose

$$Z_n[\rho] = \overline{Z}(\mu,k,\epsilon,n).$$

The following statement extends Lemma 6.18 in two ways: Firstly, it says that the process of “improving an approximate encoding” described in Lemma 6.18 can be arranged in a Borel manner. Secondly, it says that we can construct the corresponding $(k,n,\delta,\mu)$-approximate embedding in such a way that for a generic $y$, the image $\tilde{\rho}(y) \in X^Z$ will “approximately encode the measure $\mu \in \text{Prob}_c(\mathcal{Y})$”, in the sense that it will be possible to recover $Q_r(\mu)$ from $\tilde{\rho}(y)$, for $r$’s in a certain range.

**Lemma 7.4.** Given functions $q, \tilde{q} : \mathbb{N} \to \mathbb{N}$ so that

$$1 \ll q(1) \ll q(2) \ll q(2) \ll \ldots \ll q(n) \ll \tilde{q}(n) \ll q(n+1) \ll \ldots,$$

there exist Borel functions

$$\begin{align*}
\overline{\Phi} : & \text{Prob}_c(\mathcal{Y}) \times \mathbb{N} \times \mathbb{N} \times (0,1) \times (0,1) \times \text{Mor}_C(\mathcal{Y},\mathcal{X}) \to \text{Mor}_C(\mathcal{Y},X) \\
\tilde{Z} : & \text{Prob}_c(\mathcal{Y}) \times \mathbb{N} \times \mathbb{N} \times (0,1) \times (0,1) \times \text{Mor}_C(\mathcal{Y},\mathcal{X}) \to \text{Clopen}(\mathcal{Y}), \\
\tilde{N} : & \text{Prob}_c(\mathcal{Y}) \times (0,1) \times \text{Mor}_C(\mathcal{Y},\mathcal{X}) \times (0,1) \times \mathbb{N} \to \mathbb{N},
\end{align*}$$

and a partition $\{C_r\}_{Q \in \mathcal{Q}_r}$ for every $r \in \mathbb{N}$, so that whenever $\mu \in \text{Prob}_c(\mathcal{Y})$, $\gamma \in (0,1)$, $\delta, \nu > 0$, $k,k_0,n,n_0 \in \mathbb{N}$ and $\rho \in \text{Mor}_C(\mathcal{Y},\mathcal{X})$ is a continuous $(k_0,n_0,\delta_0,\mu)$-approximate embedding so that

$$\begin{align*}
\delta_0 & \leq \frac{1}{N_0(\mu,\gamma)},
\quad n_0 \geq \tilde{N}_0(\mu,\gamma)
\quad \text{and} \quad
k_0 \geq \tilde{k}_0(\mu,\gamma),
\end{align*}$$

and

$$n \geq \tilde{N}(\mu,\gamma,\rho,\delta,k) \text{ and } n \in \tilde{q}(\mathbb{N}),$$

then

$$\tilde{\rho} = \overline{\Phi}(\mu,k,n,\delta,\gamma,\rho)$$

is a $(k,n,\delta,\mu)$-approximate embedding so that

$$\mu\left(D_{n_0,\tilde{n}_0}[\rho,\tilde{\rho}] \mid Z_{n_0}\right) < \gamma,$$
and whenever $n_0 \ll q(r) \ll n$ we have
\begin{equation}
\mu \left( \left\{ y \in Z_{q(r)} : d_{X}^F(\tilde{r}(y), C_{r, \nu}(\mu)) \geq \frac{3}{8} \epsilon_{q(r)} \right\} \mid Z_{q(r)} \right) < \gamma,
\end{equation}
and we can take
\begin{equation}
Z[\tilde{r}] = \tilde{Z}(\mu, k, n, \delta, \gamma, \rho),
\end{equation}

**Proof of Theorem 5.1 assuming Lemma 7.3 and Lemma 7.4.** We follow a similar scheme as in the proof of Proposition 6.19, this time using Lemma 7.3 instead of Lemma 6.13 and Lemma 7.4 instead of Lemma 6.18.

We fix a rapidly decaying sequence $(\gamma_j)_{j=1}^{\infty}$ by $\gamma_j = \frac{1}{10^{2+j}}$. For every $j \in \mathbb{N}$ and $\mu \in \text{Prob}_c(\mathcal{Y})$, define
\begin{equation}
\overline{\mu}_{j, \nu} = \max\{k_0(\mu, \gamma_{j+1}), k_{j-1, \nu}\}
\end{equation}
and
\begin{equation}
\overline{\mu}_{j, \nu} = \overline{N}(\mu, \gamma_{j+1}),
\end{equation}
where $\overline{N}, \overline{\mu}$ are the Borel functions that appear in the statement of Lemma 7.4. By induction on $j \in \mathbb{N}$ define $n_{j, \nu} \in \mathbb{N}$, $\rho_{j, \nu} \in \text{Mor}_c(\mathcal{Y}, \mathcal{X})$ and $Z_{j, \nu} \in \text{Clopen}(\mathcal{Y})$ for every $\mu \in \text{Prob}_c(\mathcal{Y})$ as follows:

To start the induction, let
\begin{equation}
n_{1, \nu} = N_0(\mu, \overline{\mu}_{1, \nu}, \delta_{1, \nu}),
\end{equation}
\begin{equation}
\rho_{1, \nu} = \Phi_0(\mu, k_{1, \nu}, \delta_{1, \nu}, n_{1, \nu}),
\end{equation}
\begin{equation}
Z_{1, \nu} = \overline{Z}(\mu, \overline{\mu}_{1, \nu}, \delta_{1, \nu}, n_{1, \nu}),
\end{equation}
where $N_0, \Phi_0, \overline{Z}$ are the functions that appear in the statement of Lemma 7.3.

Assume by induction that these have been defined for some $j \geq 1$ and all $\mu \in \text{Prob}_c(\mathcal{Y})$. Define:
\begin{equation}
n_{j+1, \nu} = \overline{N}(\mu, \gamma_{j+1}, \rho_{j, \nu}, \delta_{j+1, \nu}, \overline{\mu}_{j, \nu}),
\end{equation}
\begin{equation}
\rho_{j+1, \nu} = \Phi(\mu, \overline{\mu}_{j+1, \nu}, n_{j+1, \nu}, \delta_{j+1, \nu}, \gamma_{j+1}, \rho_{j, \nu}),
\end{equation}
\begin{equation}
Z_{j+1, \nu} = \overline{Z}(\mu, \overline{\mu}_{j+1, \nu}, n_{j+1, \nu}, \delta_{j+1, \nu}, \gamma_{j+1}, \rho_{j, \nu}),
\end{equation}
where $\overline{N}, \Phi$ and $\overline{Z}$ are the functions that appear in the statement of Lemma 7.4. Then the application of Lemmas 7.3 and 7.4 ensures that for every $\mu \in \text{Prob}_c(\mathcal{Y})$, the function $\rho_{j, \nu} \in \text{Mor}_c(\mathcal{Y}, \mathcal{X})$ is a continuous $(\overline{\mu}_{j, \nu}, n_{j, \nu}, \delta_{j, \nu}, \mu)$-approximate embedding, and that
\begin{equation}
\mu \left( D_{n_{j, \nu}, \overline{\mu}_{j, \nu}}(\rho_{j, \nu}, \rho_{j+1, \nu}) \mid Z_{j, \nu} \right) < \gamma_j.
\end{equation}

Then, by Lemma 6.11, for every $\mu \in \text{Prob}_c(\mathcal{Y})$ the sequence $(\rho_{j, \nu})_{j=1}^{\infty}$ converges pointwise on the set
\begin{equation}
Y_\mu = \bigcap_{t \geq 1} \bigcap_{j \geq 1} T^{-j-1} \bigcap_{j \geq 1} T(1-2^{-j} \nu) F_{n_{t, \nu}} \left( Z_{t, \nu} \setminus D_{n_{t, \nu}, \ov{\nu}_{n_{t, \nu}}}[r, \rho_{t+1, \nu}] \right),
\end{equation}
and on this set the limit is a Borel embedding into $X$. By Lemma 6.12, it follows that $\mu(Y_\mu) = 1$ for all $\mu \in \text{Prob}_c(\mathcal{Y})$. For each $\mu \in \text{Prob}_c(\mathcal{Y})$ let us denote the pointwise limit of $(\rho_{j, \nu})_{j=1}^{\infty}$ by $\rho_\mu : Y_\mu \to X$.

By Lemma 6.9 and (173) it follows that for every $r$ such that $q(r) \ll n_{j, \nu}$
\begin{equation}
\mu \left( D_{q(r), \overline{\mu}_{n_{j, \nu}}}(\rho_{j, \nu}, \rho_{j+1, \nu}) \mid Z_{q(r)} \right) < \frac{1}{10^j}.
\end{equation}
So, using the triangle inequality, it follows that whenever $q(r) \ll n_{j, \nu}$ then
\begin{equation}
\mu \left( D_{q(r), \overline{\mu}_{n_{j, \nu}}}(\rho_{j, \nu}, \rho_\mu) \mid Z_{q(r)} \right) < \frac{1}{10^{j-1}} = \gamma_{j-2}.
\end{equation}
Also, by the construction of $\rho_{j,\mu}$ using Lemma 7.4 if $n_{j-1,\mu} \ll q(r) \ll n_{j,\mu}$ then
\[
\mu \left( \left\{ y \in Y : d_X^{F(r)}(\rho(y), C_{r,Q_j(\mu)}) < \frac{3}{8} \epsilon_q(r) \right\} \right) > 1 - \gamma_j,
\]
where $\{C_{r,Q_j}\}_{Q_j \in Q_r}$ is the partition of $C_{q(r)}$ that appears in the statement of Lemma 7.4.

Using (175), it follows that
\[
\mu \left( \left\{ y \in Y : d_X^{F(r)}(\rho(y), C_{r,Q_j(\mu)}) < \frac{5}{12} \epsilon_q(r) \right\} \right) > 1 - 2\gamma_{j-2}.
\]
Recall the Borel set $G(\mathcal{Y})$ consisting of points of $Y$ which are generic for ergodic probability measures. We will be concerned with the subset of $G(\mathcal{Y})$ on which the maps $\rho_{j,\mu}$ converge:
\[
Y_{\rightarrow} = \{ y \in G(\mathcal{Y}) : \rho_{j,\mu}(y) \text{ converges as } j \to \infty \}.
\]
Since the functions $y \to \rho_{j,\mu}(y)$ is a composition of Borel functions, they are also Borel and hence the set $Y_{\rightarrow}$ is measurable. We have already mentioned that the sequence $(\rho_{j,\mu})_{j=1}^\infty$ converges pointwise on $Y_{\mu}$, so $Y_{\mu} \cap G_{\mu} \subset Y_{\rightarrow}$ for every $\mu \in \text{Prob}_c(\mathcal{Y})$. Consequently $\mu(Y_{\rightarrow}) = 1$ for all $\mu \in \text{Prob}_c(\mathcal{Y})$. Now consider the Borel maps $\rho_j : Y_{\rightarrow} \to X^{2d}$ as $\rho_j(y) = \rho_{j,\mu}(y)$ and its limit $\rho : Y_{\rightarrow} \to X^{2d}$ as $\rho(y) = \lim_{j \to \infty} \rho_j(y)$.

For $r \in \mathbb{N}$ and $Q \in Q_r$, let
\[
A_{r,Q} = \left\{ y \in Y_{\rightarrow} : d_X^{F(r)}(\rho(y), C_{r,Q}) < \frac{1}{2} \epsilon_q(r) \right\},
\]
and
\[
A_r = \left\{ y \in Y_{\rightarrow} : d_X^{F(r)}(\rho(y), C_{r,Q}) < \frac{1}{2} \epsilon_q(r) \right\}.
\]
Then the fact that $C_{q(r)}$ is $(\epsilon_q(r), F_q(r))$-separated implies that $\{A_{r,Q}\}_{Q \in Q_r}$ are pairwise disjoint and $A_r = \bigcup_{Q \in Q_r} A_{r,Q}$. Because $\rho : Y_{\rightarrow} \to X^{2d}$ is a Borel function it follows that $A_r, A_{r,Q} \subset Y_{\rightarrow}$ are Borel sets for $r \in \mathbb{N}$ and $Q \in Q_r$. From (176) it follows that
\[
\lim_{r \to \infty} \mu(A_{r,Q,j}) \cap Z_q(r) = 1 \forall \mu \in \text{Prob}_c(\mathcal{Y}).
\]
Since $\lim_{r \to \infty} |F_q(r)| \cdot \mu(Z_q(r)) = 1$, it follows that $\lim_{r \to \infty} |F_q(r)| \cdot \mu(A_{r,Q,j}) = 1$. The marker property (45) implies that for every $y \in Y$ the set $\{ i \in \mathbb{Z} : \rho(T^iq) \in A_r \}$ is $(1 - \delta_q(r))F_q(r)$-spaced. This means that for every $\mu \in \text{Prob}_c(\mathcal{Y})$ there exist $R \in \mathbb{N}$ so that for every $r > R$
\[
\mu(A_{r,Q,j}) > \max_{Q \in Q_r \setminus Q_{j,\mu}} \mu(A_{r,Q,j}).
\]
Let
\[
G_r = \left\{ y \in Y_{\rightarrow} : \bar{d}_Y(y, A_{r,Q,j}) > \max_{Q \in Q_r \setminus Q_{j,\mu}} \bar{d}_Y(y, A_{r,Q,j}) \right\}.
\]
Then it follows that $G_r \subset Y$ is a Borel set, and that for every $\mu \in \text{Prob}_c(\mathcal{Y})$ there exists $R$ so that $\inf_{r \geq R} \mu(G_r) = 1$. Consequently
\[
G_r \cap emp^{-1}(Q) = \left\{ y \in Y_{\rightarrow} : \bar{d}_Y(y, A_r, Q) > \max_{Q \in Q_r \setminus Q_{r,Q}} \bar{d}_Y(y, A_{r,Q}) \right\} \text{ for } r \in \mathbb{N} \text{ and } Q \in Q_r.
\]
Since the sets $A_{r,Q}$ are measurable with respect to $\rho^{-1}(\text{Borel}(X))$ this shows that for every $r \in \mathbb{N}$, the partition $G_r \cap emp^{-1}(Q_r)$ is also measurable with respect to $\rho^{-1}(\text{Borel}(X))$. Further let
\[
Y_{\infty}' = \bigcup_{R \in \mathbb{N}} \bigcap_{r \geq R} G_r.
\]
It follows that if $y_1, y_2 \in Y_{\infty}'$ and $\mu_{y_1} \neq \mu_{y_2}$ then for some large enough $r$, $Q_r(\mu_{y_1}) \neq Q_r(\mu_{y_2})$ and thus $\rho(y_1) \neq \rho(y_2)$. Clearly $Y_{\infty}' \subset Y$ is a $T$-invariant Borel subset and $\mu(Y_{\infty}') = 1$ for every $\mu \in \text{Prob}_c(\mathcal{Y})$. Denote
\[
\tilde{Y}_\infty = \bigcup_{\mu \in \text{Prob}_c(\mathcal{Y})} (Y_{\mu} \cap G_{\mu}(Y)).
\]
Let us check that $\tilde{Y}_\infty$ is a Borel subset of $Y$. The above formula involves a possibly uncountable union, so this does not directly follow from the observation that $Y_\mu \cap G_\mu(Y)$ is a Borel subset for every $\mu \in \text{Prob}_e(Y)$. To see that $\tilde{Y}_\infty \in \text{Borel}(Y)$ we can use a similar argument used to show that $G(Y) = \bigcup_{\mu \in \text{Prob}_e(Y)} G_\mu(Y)$ is Borel: For $Z \in \text{Clopen}(Y)$, $n, \tilde{n}, j \in \mathbb{N}$ define:

\begin{equation}
P(Z, n, \tilde{n}, j) = \{ \mu \in \text{Prob}_e(Y) : Z_{j, \mu} = Z, n_{j, \mu} = n \text{ and } n_{j+1, \mu} = \tilde{n} \}.
\end{equation}

Then using (174) we can rewrite (180) as follows:

\[
\tilde{Y}_\infty = \bigcap_{t \in \mathbb{Z}} T_{-t}^{-1} \bigcup_{j=1}^{\infty} \bigcup_{n, \tilde{n}, Z} T^{(1-2\lambda_n)} F_n \left( \left( \left( Z \setminus D_{n, \frac{\tilde{n}}{\theta_n}}[\rho_t, \rho_{t+1}] \right) \cap \text{emp}^{-1}(P(Z, n, \tilde{n}, t)) \right) \right).
\]

The right-most union is over $Z \in \text{Clopen}(Y)$ and $n, \tilde{n} \in \mathbb{N}$. Since all the above unions and intersections are countable, it is clear that $\tilde{Y}_\infty$ is a Borel set. Now let

\[Y_\infty = Y'_\infty \cap \tilde{Y}_\infty.\]

Then $Y_\infty \in \text{Borel}(Y)$ is a $T$-invariant set and $\mu(Y_\infty) = 1$ for every $\mu \in \text{Prob}_e(Y)$. Also if $y_1, y_2 \in Y_\infty$ and $\rho(y_1) = \rho(y_2)$ then there exists $\mu \in \text{Prob}_e(Y)$ such that $y_1, y_2 \in G_\mu(Y)$ and so $\rho(y_i) = \rho_\mu(y_i)$ for $i = 1, 2$. Since $\rho_\mu : Y_\mu \rightarrow X$ is injective, it follows that $y_1 = y_2$. It follows that $\rho : Y_\infty \rightarrow X$ is a Borel embedding. $\square$

### 7.3. Completing the proof of Theorem 5.1

Our goal now is to prove Lemma 7.3 and Lemma 7.4 above, in order to complete the proof of Theorem 5.1. We will state and prove a few additional auxiliary lemmas in the process.

**Lemma 7.5.** For $\mu \in \text{Prob}_e(Y)$, $\epsilon > 0$ a finite Borel partition $P$ and $\theta \in (0, 1)$ let $N_\epsilon(\mu, \epsilon, \theta, P)$ denote the smallest integer $N \geq 1$ that satisfies the conclusion of Lemma 4.3 in the sense that for every $n > N (34)$ holds. Then the function $\mu \mapsto N(\mu, \epsilon, \theta, P)$ is Borel measurable. Similarly, if $N(\mu, \epsilon, \theta, P, Q)$ denotes the smallest integer $N \geq 1$ that satisfies the conclusion of Lemma 4.4, then the function $\mu \mapsto N(\mu, \epsilon, \theta, P)$ is Borel measurable.

**Proof.** For any finite Borel partition $P$ any Borel $A \subset Y$ and any $m \in \mathbb{N}$, the set of $\mu \in \text{Prob}_e(Y)$ such that $\text{COV}_{\mu, \epsilon, P} (A) \leq m$ is described by a finite system of inequalities on measures of atoms of the partition $P$ intersected with $A$. This shows that the function $\mu \mapsto \text{COV}_{\mu, \epsilon, P} (A)$ is a Borel function. From this it follows directly that the function $\mu \mapsto N(\mu, \epsilon, \theta, P)$ is also a Borel function. The proof of the second statement is similar. $\square$

**Proof of Lemma 7.3.** The proof of Lemma 6.13 implicitly describes functions $N_0$, $\Phi_0$ and $\overline{Z}$ as in the statement of Lemma 7.3. The idea is to make sure that various arbitrary choices in the proof of Lemma 6.13 can be specified in a “Borel measurable manner”. Here are the details: Define $N_0 : \text{Prob}_e(Y) \times \mathbb{N} \times (0, 1) \rightarrow \mathbb{N}$ by

\[N_0(\mu, k, \epsilon) = \min \left\{ n \in \mathbb{N} : \text{COV}_{\mu, \epsilon, P_k}(Z_n) < |C_n| \right\}.\]

Because

\[h(C) > h(Y, T) \geq h_\mu(Y, T; P_k),\]

it follows from Lemma 4.3 that $N_0(\mu, k, \epsilon)$ is well defined and from Lemma 7.5 above it follows that $N_0 : \text{Prob}_e(Y) \times \mathbb{N} \times (0, 1) \rightarrow \mathbb{N}$ is Borel measurable.

For every $k, n \in \mathbb{N}$ choose an enumeration of the elements of $P_k^{F_n}$

\[F_{k,n} : P_k^{F_n} \rightarrow \{1, \ldots, |P_k^{F_n}|\}.\]

For every $\mu \in \text{Prob}_e(Y)$, $k, n \in \mathbb{N}$ let $<_{\mu, k, n}$ be the linear order on $P_k^{F_n}$ defined by

\[P <_{\mu, k, n} Q \text{ iff } (\mu(P \mid Z_n) > \mu(Q \mid Z_n) \text{ or } F_{k,n}(P) < F_{k,n}(Q) \text{ and } \mu(P \mid Z_n) = \mu(Q \mid Z_n)).\]

Let

\[F_{\mu, k, n} : P_k^{F_n} \rightarrow \{1, \ldots, |P_k^{F_n}|\}\]

be the enumeration of the elements of $P_k^{F_n}$ according to the linear order $<_{\mu, k, n}$. It is routine to check that the map $(\mu, k, n) \mapsto F_{\mu, k, n}$ is also Borel measurable.
Given $\mu \in \text{Prob}_c(\mathcal{Y})$, $k, n \in \mathbb{N}$ such that $n > N_0(\mu, k, \epsilon)$, Define $\mathcal{G} \subset \mathcal{P}_{k}^{F_n}$

$$\mathcal{G} = F_{k, n}^{-1}(\{1, \ldots, \min\{|\mathcal{P}_{k}^{F_n}|, |C_n| - 1\}\}).$$

Then by construction of $N_0(\mu, k, \epsilon)$, as in the proof of Lemma 6.13 $\mathcal{G}$ satisfies (100) and (102).

We specify

$$Z(\mu, k, \epsilon, n) = \bigcup_{P \in \mathcal{G}} P \cap Z_n = \bigcup_{P \in F_{k, n}^{-1}(\{1, \ldots, |\mathcal{G}|\})} P \cap Z_n.$$ 

It follows that $Z : \text{Prob}_c(\mathcal{Y}) \times \mathbb{N} \times (0, 1) \times \mathbb{N} \rightarrow \text{Clopen}(\mathcal{Y})$ is Borel measurable. By (100)

$$\mu(Z(\mu, k, \epsilon, n)) > (1 - \epsilon)\mu(Z_n).$$

Choose some fixed enumeration

$$C_n = \{x_1, \ldots, x_{|C_n|}\}$$

Let $\Phi_{\mu, k, n} : \mathcal{P}_{k}^{F_n} \rightarrow C_n$ be given by

$$\Phi_{\mu, k, n}(P) = \begin{cases} x_{F_{\mu, k, n}(P)} & \text{if } F_{\mu, k, n}(P) \leq |C_n| \\ x_{|C_n|} & \text{otherwise.} \end{cases}$$

Then by definition of $N_0(\mu, k, \epsilon)$, $\Phi_{\mu, k, n}$ is injective on $\mathcal{G}$. Let $\phi : Y \rightarrow X$ be given by

$$\phi = \Phi_{\mu, k, n}(\mathcal{P}_{k}^{F_n}(y)).$$

Then it follows that $\phi \in C(Y, X)$ and $\rho_{\phi, n} \in \text{Mor}_C(\mathcal{Y}, \mathcal{X})$ is a $(k, n, \epsilon, \mu)$-approximate embedding, and we can choose

$$Z_n[\rho_{\phi, n}] = Z(\mu, k, \epsilon, n).$$

In this case define: $\Phi_{0}(\mu, k, \epsilon, n) = \rho_{\phi, n}$. We can clearly extend this definition to a Borel measurable function $\Phi : \text{Prob}_c(\mathcal{Y}) \times \mathbb{N} \times (0, 1) \times \mathbb{N} \rightarrow \text{Mor}_C(\mathcal{Y}, \mathcal{X})$ that satisfies the statement of the lemma.

We now introduce some definitions that will be needed for a technical “Borel” version of Lemma 6.10:

Given $\gamma \in [0, 1)$, $\delta, \eta > 0$ and finite Borel partitions $\mathcal{P}$ and $\mathcal{Q}$ of $Y$ let

$$\tilde{N}_C(\mu, \gamma, \delta, \eta, \mathcal{P}, \mathcal{Q}) = \inf \left\{ N \in \mathbb{N} : \forall n > N, \text{COV}_{\mu, \delta, \mathcal{P}_{F_n} \gamma, \mathcal{Q}_{n}} \left(1 - \delta_n \right) \mathcal{P}_{F_n} \gamma, \mathcal{Q}_{n} \left(\mathcal{Z}_n \right) < e^{\eta |F_n|} \right\}.$$ 

Depending on the parameters $\tilde{N}_C(\mu, \gamma, \delta, \eta, \mathcal{P}, \mathcal{Q})$ could be a finite natural number or $+\infty$. For $A \in \text{Borel}(Y)$ and an $n_0$-towerable map $\rho \in \text{Mor}(Y, \mathcal{X})$ let

$$\tilde{N}_U(\mu, \gamma, \delta, \eta, \mathcal{P}, \rho, A) = \sup \left\{ \tilde{N}_C(\mu, \gamma, \delta, \eta, \mathcal{P}, \rho_{\hat{\rho}}) : \hat{\rho} \in \text{Mor}(Y, \mathcal{X}) \text{ is symbolic and } D_{n_0, \epsilon_{n_0}}[\rho, \hat{\rho}] \subseteq A \right\}.$$ 

**Lemma 7.6.** There exists a Borel function

$$\tilde{N}_R : (0, 1) \times \mathbb{N} \rightarrow \mathbb{N}$$

so that for every $\eta > 0$, and $k_0, n_0 \in \mathbb{N}$ such that $n_0 > \tilde{N}_R(\eta, k_0)$, $\mu \in \text{Prob}_c(\mathcal{Y})$ and any Borel set $A \subset Y$ such that

$$\mu(A \mid Z_{n_0}) < \frac{1}{\tilde{N}_R(\eta, k_0)}.$$ 

any $(k_0, n_0, \frac{1}{\tilde{N}_R(n_0, k_0)}, \mu)$-approximate embedding $\rho \in \text{Mor}(Y, \mathcal{X})$, $\gamma \in [0, 1)$ and $\delta > 0$ we have

$$\tilde{N}_U(\mu, \gamma, \delta, \mathcal{P}_{k_0}, \rho, A) < +\infty.$$ 

Equivalently: There exists $N \in \mathbb{N}$ so that for every $n > N$ and every $\hat{\rho} \in \text{Mor}(Y, \mathcal{X})$ that satisfies

$$D_{n_0, \epsilon_{n_0}}[\rho, \hat{\rho}] \subseteq A$$

we have

$$\text{COV}_{\mu, \delta, \rho_{\hat{\rho}} \mathcal{P}_{F_n} \gamma, \mathcal{Q}_{n+1 - 2\delta_n} \mathcal{P}_{F_n} \gamma, \mathcal{Q}_{n} \left(\mathcal{Z}_n \right)} < e^{\eta |F_n|}.$$
Proof. Although the statement of this lemma is a bit more complicated compared to that of Lemma 6.10, the proof is almost identical. We explain the requirement modifications. As in the proof of Lemma 6.10, define $N_R : (0,1) \times \mathbb{N} \to \mathbb{N}$ by (82), for $\eta \in (0,1)$ and $k_0 \in \mathbb{N}$. It is clear that $N_R : (0,1) \times \mathbb{N} \to \mathbb{N}$ is a Borel function. Fix $\eta \in (0,1)$ and $k_0 \in \mathbb{N}$. Denote $N_0 = N_R(\eta, k_0)$. For $n_0 > N_0$ choose $A \subseteq Y$ with $\mu(A \setminus Z_{n_0}) < \frac{1}{n_0}$, $\rho \in \text{Mor}(Y, \mathcal{X})$ and $\gamma \in [0,1]$ be as in the statement above and $\delta > 0$.

In contrast to the proof of Lemma 6.10 where $G_{n_0}$ has been defined by (83), this time we define $G_{n_0} \subseteq Y$ by

$$G_{n_0} = \mathcal{T}^{(1-2\delta_{n_0})F_{n_0}}(Z[\rho] \setminus A).$$

Then

$$\mu(Z_{n_0} \setminus \{Z[\rho] \setminus A\} | Z_{n_0}) < \mu(Z_{n_0} \setminus Z[\rho] | Z_{n_0}) + \mu(A | Z_{n_0}) < \frac{2}{N_0},$$

so by Lemma 2.3

$$\mu(Y | G_{n_0}) < \epsilon_{n_0} + \left(1 - \frac{2\delta_{n_0}}{1 + \delta_{n_0}}\right) + \frac{2}{N_0}.$$ 

In particular, by definition of $\hat{\epsilon}_{n_0}$ in (81), it follows that $\mu(Y \setminus G_{n_0}) < \hat{\epsilon}_{n_0}$. Let $\hat{\rho} \in \text{Mor}(Y, \mathcal{X})$ satisfy $D_{n_0}[\rho, \hat{\rho}] \subseteq \mathcal{A}$. By Lemma 6.8, for every $y \in G_{n_0}$, the value of $P_{\hat{\rho}}(y)$ is determined by $\rho(y) |_{\mathcal{F}_{2n_0}}$. This means that

$$\text{COV}_{\mu,0,\rho_{n_0}}[\rho_{n_0}^e_{2n_0}(G_{n_0})] = 1.$$ 

From here we proceed exactly as in the proof of Lemma 6.10, replacing $\rho$ by $\hat{\rho}$ throughout the proof.

\[\square\]

Next, we state and prove the following “Borel” refinement of Lemma 6.18:

**Lemma 7.7.** Suppose $(E_m)_{m=1}^\infty$ is a sequence of Borel subsets $E_m \in \text{Borel}(Y)$ and $E_m \subseteq Z_m$ so that

$$\forall \mu \in \text{Prob}_e(Y) \lim_{m \to \infty} \mu(E_m | Z_m) = 0.$$ 

Then there exists Borel functions

$$N_0 : \text{Prob}_e(Y) \times (0,1) \to \mathbb{N},$$

$$\tilde{N} : \text{Prob}_e(Y) \times (0,1) \times \text{Mor}_C(Y, \mathcal{X}) \times (0,1) \times \mathbb{N} \to \mathbb{N},$$

$$\hat{\Phi} : \text{Prob}_e(Y) \times \mathbb{N} \times \mathbb{N} \times (0,1) \times \text{Mor}_C(Y, \mathcal{X}) \times \text{Mor}_C(Y, \mathcal{X}),$$

$$\hat{Z} : \text{Prob}_e(Y) \times \mathbb{N} \times \tilde{N} \times (0,1) \times \text{Mor}_C(Y, \mathcal{X}) \to \text{Clopen}(Y)$$

so that the following holds: Suppose that $\mu \in \text{Prob}_e(Y)$, $\delta_0, \gamma \in (0,1)$, and that $\rho \in \text{Mor}_C(Y, \mathcal{X})$ is a continuous $(k_0, n_0, \delta_0, \mu)$-approximate embedding where

$$k_0 \geq \overline{k}_0(\mu, \gamma),$$

$$n_0 \geq N_0(\mu, \gamma),$$

$$\delta_0 < \frac{1}{N_0(\mu, \gamma)},$$

that $\hat{\rho} \in \text{Mor}_C(Y, \mathcal{X})$ is $\hat{\mu}$-towerable for some $\hat{n} \in \mathbb{N}$ so that $n_0 \leq \hat{n} \ll n$ and satisfies

$$D_{n_0, \hat{n}}[\rho, \hat{\rho}] \subseteq E_{n_0}.$$ 

Fix $\delta \in (0,1)$, $k \in \mathbb{N}$ and let

$$n > \tilde{N}(\mu, \gamma, \rho, \delta, k).$$

Denote

$$\hat{\rho} = \hat{\Phi}(\mu, k, n, \gamma, \delta, \rho) \in \text{Mor}_C(Y, \mathcal{X}).$$

Then

$$\hat{\rho}$$

is a $(k, n, \delta, \mu)$-approximate embedding,

and we can take

$$Z[\hat{\rho}] = \hat{Z}(\mu, k, \gamma, \delta, \rho).$$
and

\[ D_{n, \hat{\xi}}[\hat{\rho}, \hat{\rho}] \subset Y \setminus \left( T^{(1 - 2\delta_n)} F_n \setminus \gamma F_n Z_n \right). \]

Roughly, the lemma asserts that given \( k, n \in \mathbb{N}, \delta, \gamma \in (0, 1), \mu \in \text{Prob}_e(Y) \) and \( \rho \in \text{Mor}(Y, \mathcal{X}) \) that is a “sufficiently good” continuous \((\kappa_0, n_0, \delta_0, \mu, \rho)\)-approximate embedding (as a Borel function of \( \gamma \) and \( \mu \in \text{Prob}_e(Y) \), but independently of \( k, n \) and \( \delta \)), and \( \hat{\rho} \) that is “sufficiently close” to \( \rho \) (in the sense that (188) holds), we can find a \((k, n, \delta, \mu)\)-approximate embedding \( \hat{\rho} \) that is “close” to \( \hat{\rho} \) in the sense that (192) holds, provided that \( n \) is sufficiently big (as a Borel function of \( \gamma, k, \delta, \rho \) and \( \mu \in \text{Prob}_e(Y) \)).

Lemma 6.18 is a particular case of Lemma 7.7 by setting \( \rho = \hat{\rho} \) and \( \hat{n} = n_0 \).

**Proof.** As in the proof of Lemma 7.3, the basic idea is to follow the steps of the proof of Lemma 6.18, taking care to specify all the “choices” so it is clear that the functions constructed are all Borel measurable. Additionally, there is the issue of making the function \( \tilde{\Phi} \) “works properly” even if we replace \( \rho \) by suitable \( \hat{\rho} \).

For this we use the full strength of Lemma 7.6. Here are the details:

Fix a sequence \( \{E_m\}_{m=1}^\infty \) of Borel subsets that satisfies (186). As in the proof of Lemma 6.18, fix \( \hat{\mu} \in (h(Y), h(C)) \). We can take function \( \overline{k}_0 : \text{Prob}_e(Y) \times (0, 1) \to \mathbb{N} \) to be the one given by (113) in the proof of Lemma 6.18, namely, the smallest positive integer \( k \) such that the entropy of \((Y, T, \mu)\) given \( Z_k \) is less than \( \alpha(\gamma)/8 \), where \( \alpha(\gamma) \) is given by (106). The function \( \mu \mapsto h_\mu(Y, T \mid \mathcal{P}^C_k) \) is a Borel function on \( \text{Prob}_e(Y) \). The function \( \alpha : (0, 1) \to \mathbb{R} \) given by (106) is clearly a Borel function. It follows that the function \( \overline{k}_0 : \text{Prob}_e(Y) \times (0, 1) \to \mathbb{N} \) is Borel measurable. Let

\[ \overline{N}_0(\mu, \gamma) = \max \left\{ \overline{N}_S(\alpha(\gamma)^\delta \hat{\mu}), \overline{N}_R(\alpha(\gamma)/4, \overline{k}_0(\mu, \gamma)), K_0(\gamma) \right\}, \]

where \( \overline{N}_S : (0, 1) \to \mathbb{N} \) is given by (116) and \( \overline{N}_R : (0, 1) \times \mathbb{N} \to \mathbb{N} \) is the function implicitly described in the statement of Lemma 6.10 (where the role of \( n_0 \) in Lemma 6.10 is being played by \( N_0, \)) viewed as a function of \( \eta \) and \( k_0 \) and \( K_0 : (0, 1) \to \mathbb{N} \) is the function implicitly defined in Lemma 6.17. Define \( \overline{N}_0 : \text{Prob}_e(Y) \times (0, 1) \to \mathbb{N} \) by

\[ \overline{N}_0(\mu, \gamma) = \min \left\{ n \geq \overline{N}_0(\mu, \gamma) : \mu \left( E_n \mid Z_n \right) < \frac{1}{\overline{N}_R(\alpha(\gamma)/4, \overline{k}_0(\mu, \gamma))} \right\}. \]

This is a well defined natural number by (186). It follows that \( \overline{N}_0 : \text{Prob}_e(Y) \times (0, 1) \to \mathbb{N} \) is a Borel function. Now for \( \mu \in \text{Prob}_e(Y, T) \) we fix \( n_0, \hat{n}, k_0 \) and \( \delta_0 \) as in (187) and \( \rho, \hat{\rho} \in \text{Mor}_C(Y, \mathcal{X}) \) as in the statement of the lemma. Fix \( \delta > 0 \) and let \( \delta' = 10^{-10\delta^4} \). Let \( \tilde{N}(\mu, \gamma, \rho, \delta, k) \) be the smallest natural number \( N \) satisfying \( N > n_0 \) and so that for every \( n > N \) (108), (109) (the role of \( k \) which appears in this inequality is played by \( n_0 \) here), (121), (123) and (126) hold, and in addition

\[ N \geq \tilde{N}(\mu, \gamma, \rho, \delta', \alpha(\gamma)/4, \mathcal{P}_k, \rho, E_{n_0}), \]

where the right hand side is the function defined by (183). The fact that the right hand side is finite follows by Lemma 7.6 because \( \mu \left( E_n \mid Z_n \right) < \frac{1}{\overline{N}_R(\alpha(\gamma)/4, \overline{k}_0(\mu, \gamma))} \) and \( n > \overline{N}_R(\alpha(\gamma)/4, \overline{k}_0(\mu, \gamma)) \). This means that

\[ \text{COV}_{\mu, \delta', P_{\kappa_0} F_n \setminus \gamma F_n} (1 - 2\delta_n) F_n \setminus \gamma F_n (Z_n) < e^{\alpha(\gamma)/4 |F_n|} \]

holds for any \( \hat{n} \)-towerable \( \hat{\rho} \in \text{Mor}_C(Y, \mathcal{X}) \) satisfying (188) with \( \hat{n} \ll n \), (this is the analog of (118) with \( \rho \) replaced with \( \hat{\rho} \)).

It follows that \( \tilde{N} \) is a Borel function for the set of parameters where it has been defined. Extend it to a Borel function

\[ \tilde{N} : \text{Prob}_e(Y) \times (0, 1) \times \text{Mor}_C(Y, \mathcal{X}) \times (0, 1) \times \mathbb{N} \to \mathbb{N}. \]

Let us now describe the functions

\[ \tilde{\Phi} : \text{Prob}_e(Y) \times \mathbb{N} \times \mathbb{N} \times (0, 1) \times (0, 1) \times \text{Mor}_C(Y, \mathcal{X}) \to \text{Mor}_C(Y, \mathcal{X}), \]

and

\[ \tilde{Z} : \text{Prob}_e(Y) \times \mathbb{N} \times \mathbb{N} \times (0, 1) \times (0, 1) \times \text{Mor}_C(Y, \mathcal{X}) \to \text{Clopen}(Y) \]

It is enough to define \( \tilde{\Phi}(\mu, k, n, \gamma, \delta, \rho) \in C(Y, \mathcal{X}) \) and \( \tilde{Z}(\mu, k, n, \gamma, \delta, \rho) \in \text{Clopen}(Y) \) for parameters satisfying the assumptions of the lemma. So assume that \( n \geq \tilde{N}(\mu, \gamma, \rho, \delta, \rho) \in \text{Mor}_C(Y, \mathcal{X}) \) as before, and
that \( \hat{\rho} \in Mor_c(\mathcal{Y}, X) \) is \( \hat{n} \)-towerable for some \( n \leq \hat{n} \leq n \) and satisfies (188). Then, repeating the exact same steps as in the proof of Lemma 6.18, except that this time we use (195) instead of (118), and replace \( \rho \) by \( \hat{\rho} \) throughout, we conclude that

\[
COV_{\mu, \delta/3, P^F_k \mid P^{(1-\delta)n}_\hat{\rho}}(Z_n) < e^{\alpha(\gamma)/2}[F_n \pm \hat{h}[\gamma F_n]].
\]

Let \( K_{n, \hat{n}, \beta(\gamma)} \subset (1 - \delta)n \) be given by (124) and

\[
\hat{X}_n = \left\{ w \in X^{(1-\delta)n} \mid COV_{\ell_{\delta/3}, P^F_k \mid P^{(\gamma F_n)_n}(Z_n)} \leq |C_n^{K_{n, \hat{n}, \beta(\gamma)}}| \right\}.
\]

Then the analog of (129) holds with \( \rho \) replaced by \( \hat{\rho} \). Enumerate the elements of \( C_n^{K_{n, \hat{n}, \beta(\gamma)}} \):

\[
C_n^{K_{n, \hat{n}, \beta(\gamma)}} = \{ w_1, \ldots, w_M \},
\]

where

\[
M = |C_n^{K_{n, \hat{n}, \beta(\gamma)}}|.
\]

As in the proof of Lemma 7.3, for every \( k, n \in \mathbb{N} \)

\[
F_{k, n} : P^F_k \to \{1, \ldots, |P^F_k|\},
\]

be an enumeration of the elements of \( P^F_k \). For every \( \mu \in \text{Prob}_c(\mathcal{Y}) \), \( k, n \in \mathbb{N} \) and \( w \in X^{(1-\delta)n} \) let \( \prec \) be the linear order on \( P^F_k \) define by \( P \prec \mu, k, n, w \) if and only if

\[
\mu(P \mid Z_n \cap \hat{\rho}^{-1}([w])) > \mu(Q \mid Z_n \cap \hat{\rho}^{-1}([w]))
\]

or

\[
F_{k, n}(P) < F_{k, n}(Q) \text{ and } \mu(P \mid Z_n \cap \hat{\rho}^{-1}([w])) = \mu(Q \mid Z_n \cap \hat{\rho}^{-1}([w])).
\]

Let

\[
F_{\mu, k, n, w} : P^F_k \to \{1, \ldots, |P^F_k|\}
\]

be the enumeration of the elements of \( P^F_k \) according to the linear order \( \prec \). As in the proof of Lemma 7.3, \( (\mu, k, n, w) \to F_{\mu, k, n, w} \) is also Borel measurable.

Given \( \mu \in \text{Prob}_c(\mathcal{Y}) \), \( k, n \in \mathbb{N} \) such that \( n > N_0(\mu, k, \epsilon) \), let \( G_w \subset P^F_k \)

\[
G_w = F_{\mu, k, n, w}^{-1}(\{1, \ldots, M, |P^F_k|\}).
\]

For \( w \in \hat{X}_n \), as in the proof of Lemma 6.18, we see that (131) holds with \( \rho \) replaced by \( \hat{\rho} \), and also that (132) holds with \( n_0 \) replaced by \( \hat{n} \). For \( w \in \hat{X}_n \) let \( \Phi_w : P^F_k \to C_n^{K_{n, \hat{n}, \beta(\gamma)}} \) be given by

\[
\Phi_w(P) = \begin{cases} w_{F_{\mu, k, n, w}(P)} & \text{if } F_{\mu, k, n, w}(P) \leq |C_n^{K_{n, \hat{n}, \beta(\gamma)}}| \\ w_{|C_n^{K_{n, \hat{n}, \beta(\gamma)}}|} & \text{otherwise} \end{cases}
\]

Then \( \Phi_w \) is injective on \( G_w \). From here we proceed exactly as in the proof of Lemma 6.18, replacing \( \rho \) with \( \hat{\rho} \) and \( n_0 \) with \( \hat{n} \) in the appropriate places: For \( y \in Y \) define \( \overline{W}_y \in \text{Spec}_c(C) \) by replacing \( \rho \) with \( \hat{\rho} \) and \( n_0 \) with \( \hat{n} \) in (139).

Then let

\[
\hat{\Phi}(\mu, k, n, \gamma, \delta, \hat{\rho}) = \rho_{\hat{\phi}, n},
\]

where \( \hat{\phi} \) is given by (140), and \( \rho_{\hat{\phi}, n} \) is given by (99). Then indeed \( \rho = \rho_{\hat{\phi}, n} \in Mor_c(\mathcal{Y}, X) \), and we can extend \( \hat{\Phi} \) to a Borel function. Let \( Y_0 \subset Z_n \) be the set given by (143). Then it follows that \( Y_0 \) is clopen in \( Y \).

We set \( \hat{Z}_{\mu, k, n, \gamma, \delta, \hat{\rho}} = Y_0 \). From here, the proof concludes exactly as in the proof of Lemma 6.18. \( \square \)

Let us state and prove one more auxiliary lemma:
Proof. Let \( q : \mathbb{N} \to \mathbb{N} \) be an increasing function so that \( q(r) \ll q(r+1) \) for every \( r \in \mathbb{N} \) and \( 1 \ll q(1) \). Then there exists a sequence of Borel functions

\[
\tilde{\Phi}_r : C_r \times \text{Mor}_C(\mathcal{Y}, \mathcal{X}) \to \text{Mor}_C(\mathcal{Y}, \mathcal{X})
\]

with the following property: Suppose \( x \in C_r \) and \( \rho \in \text{Mor}_C(\mathcal{Y}, \mathcal{X}) \) is a continuous \( n \)-towerable with \( r \ll n \ll q(r) \). Denote

\[
\tilde{\rho} = \tilde{\Phi}_r(x, \rho) \in \text{Mor}_C(\mathcal{Y}, \mathcal{X}).
\]

Then \( \tilde{\rho} \) is a \( q(r) \)-towerable continuous morphism so that

\[
D_n, x \in \mathbb{N} [\rho, \tilde{\rho}] \subseteq K_{r,n},
\]

and

\[
\forall y \in Z_{q(r)} \, d^F_X (\tilde{\rho}(y), x) < \frac{11}{40} \epsilon_r.
\]

where:

\[
K_{r,n} = Y \setminus T^{F_{q(n)}} Z_{q(r)}
\]

and

\[
F_{r,n} = (1 - \delta_{q(r)}) F_{q(r)} \setminus (1 + \delta_n) F_{2n}.
\]

\[
\tilde{\Phi}_r : C_r \times \text{Mor}_C(\mathcal{Y}, \mathcal{X}) \to \text{Mor}_C(\mathcal{Y}, \mathcal{X})
\]

(198)

It follows that we can extend the definition of \( \tilde{\Phi}_r \) as above and \( x \in C_r \), be given by \( (W_{r,n})_0 = x \), and let \( \tilde{x} \in C_n \) be given by \( \tilde{x} = \text{Ext}_n(W_{r,n}) \).

For \( y \in Y \) define

\[
K_{y} = \{ \tilde{t} \in \tilde{F}_{r,n} : T^\tilde{\gamma}(y) \in \mathbb{N} \} \cup \{ \tilde{0} \} \subseteq (1 - \delta_n) F_n,
\]

where \( \tilde{F}_{r,n} \subset F_{q(r)} \) is defined by (201). Then \( K_{y} \subset (1 - \delta_{q(r)}) F_{q(r)} \) is \( (1 + \delta_n) F_n \) spaced. Define \( W_{y} \in C_n \) by

\[
(W_{y})_{\tilde{t}} = \begin{cases} 
\tilde{x} & \text{if } \tilde{t} = 0 \\
S^\gamma(\tilde{\rho}(y)) & \text{if } \tilde{t} \in K_{y} \setminus \{0\}.
\end{cases}
\]

Define \( \tilde{\phi} : Y \to X \) by

\[
\tilde{\phi}(y) = \text{Ext}_{q(r)}(W_{y}).
\]

Let

\[
\tilde{\rho} = \rho_{\tilde{\phi}, q(r)}.
\]

It follows that \( \tilde{\rho} \in \text{Mor}_C(\mathcal{Y}, \mathcal{X}) \) is \( q(r) \)-towerable and satisfies (198) and also so that for every \( y \in Y \)

\[
d^F_X (\tilde{\phi}(y), x) < \frac{11}{40} \epsilon_r.
\]

By definition of \( \rho_{\tilde{\phi}, q(r)} \) this implies (199).

For \( \rho \in \text{Mor}_C(\mathcal{Y}, \mathcal{X}) \) as above and \( x \in C_r \), define

\[
\tilde{\Phi}_r(x, \rho) = \rho_{\tilde{\phi}, q(r)}.
\]

It follows that we can extend the definition of \( \tilde{\Phi}_r : C_r \times \text{Mor}_C(\mathcal{Y}, \mathcal{X}) \to \text{Mor}_C(\mathcal{Y}, \mathcal{X}) \) to a Borel function. \( \Box \)

Proof of Lemma 7.4. Let \( q, \tilde{q} : \mathbb{N} \to \mathbb{N} \) be functions that satisfy (157), as in the statement of the lemma. For \( n, n_0 \in \mathbb{N} \) denote:

\[
r_1(n_0) = \min \{ r \in \mathbb{N} : n_0 \ll q(r) \} \quad \text{and} \quad r_2(n) = \max \{ r \in \mathbb{N} : q(r) \ll n \}.
\]

Let \( (\tilde{\Phi}_r)_{r=1}^\infty \) be the functions given by Lemma 7.8.

For \( r \in \mathbb{N} \) let

\[
E_r = \left( Y \setminus T^{(1-\delta_{q(r)}) F_{q(r)}} Z_{q(r)} \right) \cup T^{F_{q(r)}} (K_{r+1,q(r)})
\]
where $K_{r,n}$ is given by (200). For every $n_0 \in \mathbb{N}$ define

$$E_{n_0} = K_{r_1(n_0),n_0} \cup \bigcup_{r=r_1(n_0)}^{\infty} \tilde{E}_r.$$  

By definition of $K_{r,n}$, if $1 \ll n_0$,

$$\mu \left( K_{r_1(n_0),n_0} \mid Z_{n_0} \right) \leq 2(\epsilon_0 + 6d \delta_q + 2^{\frac{\|F_n\|}{\|F_q\|}}) |F_{n_0}| \leq \epsilon_{n_0} + \delta_{n_0},$$

where in the last inequality we used that $n_0 \ll q(r_1(n_0))$.

Note that

$$T^{F_{q(r)}}(K_{r+1,q(r)}) \subseteq (Y \setminus T^{(1-2\delta_q(r+1))F_{q(r+1)}} Z_{q(r+1)}) \cup T^{2(1+\delta_q(r))F_{q(r)} Z_{q(r+1)}}.$$  

An estimate similar to that of (205) shows that for any $r \geq r_1(n_0)$ and $1 \ll k \ll q(r)$ we have

$$\mu \left( \tilde{E}_r \right) < (\epsilon_k + \delta_k) \mu (Z_k).$$

In particular, by union bound it follows that

$$\mu (E_{n_0} \mid Z_{n_0}) \leq 2\epsilon_{n_0} + 2\delta_{n_0}.$$  

This shows that the sequence $(E_m)_{m=1}^{\infty}$ satisfies (186). Let $N_0(\gamma)$ be the smallest $n \in \mathbb{N}$ so that

$$\epsilon_n + \delta_n < \gamma / 4.$$  

It follows that for every $n_0 \geq N_0(\gamma)$,

$$\mu (E_{n_0} \mid Z_{n_0}) < \frac{1}{2} \gamma.$$  

Let $\tilde{N}_0 : \text{Prob}_c(\mathcal{Y}) \times (0,1) \rightarrow \mathbb{N}$ be defined by

$$\tilde{N}_0(\mu, \gamma) = \max \{ \bar{N}_0(\mu, \frac{1}{\gamma}), N_0(\gamma), \frac{1}{\gamma} q(1) \} \text{ for } \gamma \in (0,1), \mu \in \text{Prob}_c(\mathcal{Y}),$$

where $\bar{N}_0 : \text{Prob}_c(\mathcal{Y}) \times (0,1) \rightarrow \mathbb{N}$ is the Borel function obtained by applying Lemma 7.7 with the sequence $(E_m)_{m=1}^{\infty}$ above.

Let $\tilde{k}_0 : \text{Prob}_c(\mathcal{Y}) \times (0,1) \rightarrow \mathbb{N}$ be defined by

$$\tilde{k}_0(\mu, \gamma) = \tilde{k}_0(\mu, \frac{1}{\gamma}) \text{ for } \gamma \in (0,1), \mu \in \text{Prob}_c(\mathcal{Y}),$$

where $\tilde{k}_0 : \text{Prob}_c(\mathcal{Y}) \times (0,1) \rightarrow \mathbb{N}$ is the Borel function obtained by applying Lemma 7.7 with the sequence $(E_m)_{m=1}^{\infty}$ above.

Also, let $\bar{N} : \text{Prob}_c(\mathcal{Y}) \times (0,1) \times \text{Mor}_C(\mathcal{Y}, \mathcal{X}) \times (0,1) \times \mathbb{N} \rightarrow \mathbb{N}$ be obtained by applying Lemma 7.7 with the sequence $(E_m)_{m=1}^{\infty}$ above. Recall that for every $r \in \mathcal{Q}$, $r$ is a finite Borel partition of $\text{Prob}_c(\mathcal{Y})$, and that according to our assumption $|\mathcal{Q}| \leq |C_r|$. For each $r \in \mathbb{N}$, let $\tilde{f}_r : \mathcal{Q}_r \rightarrow C_r$ be an injective function. Recall that for every $r \in \mathbb{N}$ the set $C_r \subset X$ is $(\epsilon_r, F_r)$-separated so we can choose a partition $\{ C_{q(r)} \}_{Q \in \mathcal{Q}}$ of $C_{q(r)}$ so that for every $Q \in \mathcal{Q}_r$

$$\{ x \in C_{q(r)} : d_X^r (x, \tilde{f}_r(Q)) < \frac{1}{2} r \} \subseteq C_{r,Q}.$$

We will now define $\bar{\Phi}(\mu, k, n, \delta, \gamma, \rho)$ assuming that $\rho \in \text{Mor}_C(\mathcal{Y}, X)$ is a continuous $(k_0, n_0, \delta_0, \mu)$-approximate embedding with $k_0, n_0 \in \mathbb{N}$ and $\delta_0$ satisfying (159) and $n \in \mathbb{N}$ satisfying (160). Let $r_1 = r_1(n_0)$, $r_2 = r_2(n)$. If $r_1 > r_2$, let $\hat{\rho} = \rho$ and $\hat{n} = n_0$.

Otherwise for $0 \leq i \leq r_2 - r_1$,

$$\rho_{i+1} = \Phi_{r_1+i} \left( \tilde{f}_{r_1+i}(Q_{r_1+i}(\mu)), \rho_i \right),$$

where $(\Phi_r)_{r=1}^{\infty}$ are the functions given by Lemma 7.8. Also, let $\hat{n} = q(r_2)$, $\hat{\rho} = \rho_{r_2-r_1}$.

The properties of the functions $\Phi_r$ given by Lemma 7.8 ensure that

$$D_{\hat{n}, \hat{\rho} \hat{\mu}, \hat{\rho}_1} (\rho, \rho_1) \subseteq K_{r_1, n_0}. $$
and

\[ (209) \quad \forall r_1 < r \leq r_2 \quad D_{q(r-1), \varepsilon_{q(r-1)}}[\rho_{r-r_1}, \rho_{r-r_1+1}] \subseteq K_{r,q(r-1)}. \]

Applying (72) and (73) together with (208) and (209) it follows that

\[ (210) \quad D_{n_0, \varepsilon_{n_0}}[\rho, \hat{\rho}] \subseteq E_{n_0}. \]

Using (207) it follows that

\[ \mu \left( D_{n_0, \varepsilon_{n_0}}[\rho, \hat{\rho}] | Z_{n_0} \right) < \gamma/2. \]

Let

\[ \tilde{\rho} = \hat{\Phi} \left( \mu, k, n, \frac{1}{8} \gamma, \delta, \hat{\rho} \right), \]

where \( \tilde{\Phi} \) is the function obtained by applying Lemma 7.7 with the sequence \( (E_m)_{m=1}^\infty \) defined as above. Define:

\[ \overline{\Phi}(\mu, k, n, \delta, \gamma, \rho) = \tilde{\rho}, \]

and

\[ \mathcal{Z}(\mu, k, n, \delta, \gamma, \rho) = \tilde{Z} \left( \mu, k, n, \frac{1}{8} \gamma, \delta, \hat{\rho} \right), \]

where \( \tilde{Z} \) is the Borel function described in the statement of Lemma 7.7. This completes the definition of \( \overline{\Phi}(\mu, k, n, \delta, \gamma, \rho) \) and \( \mathcal{Z}(\mu, k, n, \delta, \gamma, \rho) \) for relevant input parameters.

So we assume now that \( \rho \in \text{Mor}_C(\mathcal{Y}, X) \) is a continuous \( (k_0, n_0, \delta_0, \mu) \)-approximate embedding such that (159) and (160) hold. Since \( \hat{\rho} \) was obtained by applying the function \( \Phi \) of Lemma 7.7,

\[ D_{n_0, \varepsilon_{n_0}}[\hat{\rho}, \hat{\rho}] \subseteq Y \setminus \left( T^{(1-2\delta_n)F_n} \setminus F_n Z_n \right). \]

Because \( 1 \ll \hat{n} \ll n \), it follows that

\[ \mu \left( D_{n_0, \varepsilon_{n_0}}[\hat{\rho}, \hat{\rho}] | Z_{\hat{n}} \right) < \gamma/5. \]

Because \( 1 \ll n_0 \ll \hat{n} \) it follows using Lemma 6.9 that

\[ \mu \left( D_{n_0, \varepsilon_{n_0}}[\hat{\rho}, \hat{\rho}] | Z_{n_0} \right) < \gamma/2. \]

Using (73) once again we conclude that (162) holds.

Apply the triangle inequality to conclude that whenever \( n_0 \ll q(r) \ll n \)

\[ \left\{ y \in Z_{q(r)} : d_X^{F_{q(r)}}(\hat{\rho}(y), C_{r, q(r)}) \geq \frac{3}{8} \varepsilon_{q(r)} \right\} \subseteq \]

\[ \subseteq \bigcup_{r'=r+1}^{r_2} \left\{ y \in Z_{q(r')} : d_X^{F_{q(r')}}(\rho_{r'-r_1}(y), \rho_{r'-r_1}(y)) \geq \frac{3}{8} \varepsilon_{q(r')} \right\} \cup \left\{ y \in Z_{q(r)} : d_X^{F_{q(r)}}(\hat{\rho}(y), \hat{\rho}(y)) \geq \frac{3}{8} \varepsilon_{\hat{n}} \right\}. \]

The inequality (163) now follows in a very similar fashion as did (162). We don’t repeat the argument. This completes the proof. \( \square \)

8. Universality for Graph Homomorphisms

The following sections will be concerned with symbolic dynamical systems or subshifts. We briefly recall some notation and basic definitions.

For a finite subset \( \mathcal{A} \) (the “alphabet”), the \( \mathbb{Z}^d \)-full shift over \( \mathcal{A} \) is the \( \mathbb{Z}^d \) dynamical system \( (\mathcal{A}^\mathbb{Z}^d, S) \), where \( S \) is the shift action given by

\[ S^j(x)_i = x_{i+j}. \]

A subshift is a topological subsystem of the full-shift. If \( X \subset \mathcal{A}^\mathbb{Z}^d \) is a subshift, for every finite \( F \subset \mathbb{Z}^d \) let

\[ \mathcal{L}(X, F) = \{ x |_F : x \in X \} \subset \mathcal{A}^F. \]

The elements of \( \mathcal{L}(X, F) \) are called admissible \( F \)-configurations for \( X \).

Let \( X \) be a \( \mathbb{Z}^d \) subshift, and let \( g : \mathbb{N} \to \mathbb{N} \) be a function so that \( \lim_{n \to \infty} \frac{g(n)}{n} = 0. \)
We say that \( C = (\tilde{C}_n)_{n=1}^{\infty} \) with \( \tilde{C}_n \subset \mathcal{L}(X, F_n) \) is a \textit{flexible sequence of patterns} for \( X \) with respect to \( g \) if for every \( k, n \in \mathbb{N} \) any \( F_k + g(k) \)-spaced subset \( K \) contained in \( F_{n-g(n)} \) and \( W \in (\tilde{C}_k)^K \) there exists \( w \in \tilde{C}_n \) so that
\[
S^i(w) |_{F_n} = W(\tilde{i}) \quad \text{for all} \quad \tilde{i} \in K.
\]
We say that \( \tilde{C} \) as above is a \textit{flexible marker sequence of patterns} if in addition
\[
\forall x \in X \text{ and } n \in \mathbb{N}, \text{ the set } \left\{ \tilde{i} \in \mathbb{Z}^d : S^i(x) |_{F_n} \in \tilde{C}_n \right\} \text{ is } F_{n-g(n)} \text{-spaced.}
\]
For a flexible sequence of patterns \( \tilde{C} \) as above let
\[
h(\tilde{C}) = \limsup_{n \to \infty} \frac{1}{|F_n|} \log |C_n|.
\]
It is not difficult to check that any \( \mathbb{Z}^d \) subshift \( X \) that admits a flexible marker sequence of patterns \( \tilde{C} = (\tilde{C}_n)_{n=1}^{\infty} \) also admits a flexible marker sequence \( C = (C_n)_{n=1}^{\infty} \in X^\mathbb{N} \) so that \( h(C) = h(\tilde{C}) \). The idea is that if we choose a suitable function \( \alpha : \mathbb{N} \to \mathbb{N} \) so that \( \lim_{n \to \infty} \alpha(n) = +\infty \) and \( \lim_{n \to \infty} \frac{\alpha(n)}{n} = 0 \), and choose for any \( w \in \tilde{C}_{n+\alpha(n)} \) some \( x(w) \in X \) so that \( x(w) |_{F_{n+\alpha(n)}} = w \), then the sequence \( \tilde{C} = (\tilde{C}_n)_{n=1}^{\infty} \) defined by
\[
C_n = \left\{ x(w) : w \in \tilde{C}_{n+\alpha(n)} \right\}
\]
will be a flexible marker sequence for \( (X, S) \) (with respect to suitable sequences \( (\delta_n)_{n=1}^{\infty} \) and \( (\epsilon_n)_{n=1}^{\infty} \)).

By Theorem 5.1, a \( \mathbb{Z}^d \) subshift \( (X, S) \) that admits a flexible marker sequence of patterns \( \tilde{C} = (\tilde{C}_n)_{n=1}^{\infty} \) is \( h(\tilde{C}) \)-universal in the almost-Borel sense. If furthermore for every \( n \in \mathbb{N} \) and \( W \in \mathcal{L}(X, F_n) \) there exist \( m \geq n \) and \( \tilde{w} \in \tilde{C}_m \) such that \( \tilde{w} |_{F_n} = w \) then \( X \) is fully \( h(\tilde{C}) \)-universal.

Let us also remark that for a slightly modified definition for a “flexible sequence of patterns”, we can show that the existence of a flexible sequence of patterns implies the existence of a flexible sequence of marker patterns of equal entropy. The proof for this is quite similar to that of Proposition 5.7. This slightly modified definition still implies universality and holds for the systems appearing in our main applications below (hom-shifts and rectangular tilings). For the sake of presentation, we choose to bring a direct construction for existence of flexible sequence of marker patterns in the application below rather than proving this general implication.

Let us briefly introduce graph homomorphisms and hom-shifts. For background and more we refer for instance to [6, 7, 11]. In the category of graphs, a \textit{homomorphism} is a function between the vertex sets of two graphs that maps adjacent vertices to adjacent vertices. Note that graphs are one dimensional simplicial complexes and a graph homomorphism is just a simplicial map. If \( \mathcal{G} \) and \( \mathcal{H} \) are graphs, we denote by \( \text{Hom}(\mathcal{G}, \mathcal{H}) \) the set of all graph homomorphisms from \( \mathcal{G} \) to \( \mathcal{H} \). For a finite graph \( \mathcal{H} \) we let \( \text{Hom}(\mathbb{Z}^d, \mathcal{H}) \) denote the space of graph homomorphisms from the standard Cayley graph on \( \mathbb{Z}^d \) to \( \mathcal{H} \). As \( \mathbb{Z}^d \) acts on it’s Cayley graph by graph automorphisms, it also acts on \( \text{Hom}(\mathbb{Z}^d, \mathcal{H}) \). Viewed as a closed shift invariant subset of \( V_{\mathcal{H}}^{\mathbb{Z}^d} \), \( \text{Hom}(\mathbb{Z}^d, \mathcal{H}) \) is a subshift. Here graphs will be undirected and simple in the sense that there is at most one edge between any pair of vertices, but we allow self-loops. Hom-shifts are shift spaces that arise as the space of graph homomorphisms from \( \mathbb{Z}^d \) to a fixed graph \( \mathcal{H} \). For instance, the space proper \( n \)-colorings is the set of graph homomorphisms from \( \mathbb{Z}^d \) to a complete graph on \( n \) vertices.

Equivalently, hom-shifts are nearest neighbor \( \mathbb{Z}^d \) subshifts of finite type that are symmetric with respect to permutations and reflections along the cardinal directions (they are “isotropic and symmetric”). They are very special because of their inherent symmetry, and contain various interesting examples of shift spaces.

Our goal is to prove the following:

\textbf{Theorem 8.1.} If \( \mathcal{H} \) is a finite connected graph which is not bipartite, then the subshift \( \text{Hom}(\mathbb{Z}^d, \mathcal{H}) \) admits a flexible marker sequence of patterns \( \tilde{C} = (\tilde{C}_n)_{n=1}^{\infty} \) with \( h(\tilde{C}) = h(\text{Hom}(\mathbb{Z}^d, \mathcal{H})) \) so that \( \bigcup_{n=1}^{\infty} \tilde{C}_n \) is dense in \( X \). Thus, \( \text{Hom}(\mathbb{Z}^d, \mathcal{H}) \) is universal in the almost Borel sense. Furthermore, it is fully universal.

To prove Theorem 8.1, we will identify a flexible sequence of marker patterns in \( \text{Hom}(\mathbb{Z}^d, \mathcal{H}) \) as follows: We will identify a subset \( F \subset \mathbb{Z}^d \) with the corresponding induced subgraph of the standard Cayley graph
of $\mathbb{Z}^d$. Let $\mathcal{H} = (V_\mathcal{H}, E_\mathcal{H})$ be a finite connected graph. Given neighboring vertices of $v_0, v_1 \in V_\mathcal{H}$ define for $n \geq 1$

\begin{equation}
C_{n}^{(v_0, v_1)} = \left\{ a \in \text{Hom}(F_n, \mathcal{H}) : a_\tau = v_{\text{parity}(\tilde{\tau})} \text{ for all } \tilde{\tau} \in (F_n \setminus F_{n-1}) \right\},
\end{equation}

We assume that $|E_\mathcal{H}| > 1$ (otherwise $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ is either empty or consists of a single point). Because $\mathcal{H}$ is connected this implies that there exists $v_0 \in V_\mathcal{H}$ that is incident in at least two edges. So we can find $v_1, v_2 \in V_\mathcal{H}$ such that $v_1 \neq v_2$ and $(v_0, v_1), (v_0, v_2) \in E_\mathcal{H}$. For $n \geq 1$ let

\begin{equation}
\tilde{C}_{n+1} = \left\{ a \in C_{n+1}^{(v_0, v_1)} : a|_{F_n} \in C_n^{(v_0, v_2)} \right\},
\end{equation}

In other words the patterns in $\tilde{C}_n$ have two layers of “checkerboard boundaries”, so that for each of the two layers there is a different “color” for the odd cites. For this reason when $d \geq 2$ it is not possible for two patterns in $\tilde{C}_n$ to overlap non-trivially except on the most boundary. In other words, for any $x \in X$ the set

\begin{equation}
\left\{ \tilde{\tau} \in \mathbb{Z}^d : S\tilde{\tau}(x) |_{F_{n+1}} \in \tilde{C}_{n+1} \right\}
\end{equation}

is $F_n$-spaced.

Our goal is to show that the sequence $\tilde{C} = (\tilde{C}_n)_{n=1}^\infty$ is a flexible marker sequence of patterns with $h(\tilde{\mathcal{C}}) = h(X, S)$. We will do this by showing that for any neighboring vertices $v_0, v_1$ the sequence $C^{(v_0, v_1)} = (C_n^{(v_0, v_1)})_{n=1}^\infty$ is a flexible sequence of patterns for $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$. It is clear that $|\tilde{C}_{n+1}| = |C_n^{(v_0, v_2)}|$ because every $a \in C_n^{(v_0, v_2)}$ is the restriction of precisely one $\tilde{a} \in \tilde{C}_{n+1}$. In particular $h(\tilde{\mathcal{C}}) = h(C^{(v_0, v_2)})$.

**Proposition 8.2.** Let $\mathcal{H}$ be a connected graph which is not bipartite. For any $(v_0, v_1) \in E_\mathcal{H}$ the sequence $\mathcal{C} = (C_n)_{n=1}^\infty$ given by (216) is flexible marker sequence of patterns in $\text{Hom}(\mathcal{H}, \mathbb{Z}^d)$ and

\begin{equation}
h(\tilde{\mathcal{C}}) = h(\text{Hom}(\mathbb{Z}^d, \mathcal{H})),
\end{equation}

Moreover, for any $a \in \mathcal{C}(\text{Hom}(\mathbb{Z}^d, \mathcal{H}), F_n)$ there exists $w \in \tilde{C}_{(d+1)n+n+2}$ so that $w|_{F_n} = a$.

We prove Proposition 8.2 in several of steps.

**Lemma 8.3.** There exists a graph homomorphism $\tau : \mathbb{Z}^d \to \mathbb{Z}^d$ from the Cayley graph of $\mathbb{Z}^d$ to itself so that:

1. For any $\tilde{\tau} \in \mathbb{Z}^d \setminus \{0\}$, $||\tau(\tilde{\tau})||_1 < ||\tilde{\tau}||_1$.
2. $\tau(0) = \tilde{e}_1$.
3. For any $\tilde{\tau} \in \mathbb{Z}^d$, $\tau(\tilde{\tau})$ is adjacent to $\tilde{\tau}$.
4. For any $n \in \mathbb{N}$, if $\tilde{i}, \tilde{j} \in \mathbb{Z}^d$ are adjacent vertices with $\tilde{i} \not\in F_n$ and $\tilde{j} \in F_n$ then $\tau(\tilde{i}) = \tilde{j}$.

**Proof.** We define $\tau : \mathbb{Z}^d \to \mathbb{Z}^d$ as follows: $\tau(0) = \tilde{e}_1$, and if $\tilde{i} = (i_1, \ldots, i_d) \neq 0$ let

$$\xi(\tilde{i}) = \min\{1 \leq t \leq d : |i_t| = ||\tilde{i}||_\infty\}$$

and let

$$\tau(\tilde{i}) = \begin{cases} \tilde{i} - \tilde{e}_{\xi(\tilde{i})} & \xi(\tilde{i}) > 0 \\ \tilde{i} + \tilde{e}_{\xi(\tilde{i})} & \xi(\tilde{i}) < 0. \end{cases}$$

Suppose $\tilde{i}, \tilde{j} \in \mathbb{Z}^d$ are adjacent vertices then there exists $1 \leq t \leq d$ so that $\tilde{i} = \tilde{j} \pm \tilde{e}_t$. Without loss of generality assume $||\tilde{i}||_1 > ||\tilde{j}||_1$. If $\xi(\tilde{i}) \neq t$ then $||\tilde{j}||_\infty = ||\tilde{i}||_\infty$ and $\xi(\tilde{j}) = \xi(\tilde{i})$ so $\tau(\tilde{i}) = \tau(\tilde{j}) = \tilde{e}_t$. If $\xi(\tilde{i}) = t$ then $\tau(\tilde{i}) = \tilde{j}$, and so $\tau(\tilde{i}) = \tau(\tilde{j}) = \pm \tilde{e}_{\xi(\tilde{i})}$. In either case, we see that $\tau(\tilde{i})$ and $\tau(\tilde{j})$ are adjacent vertices. This shows that $\tau : \mathbb{Z}^d \to \mathbb{Z}^d$ is indeed a graph homomorphism. Properties (1), (2), (3) follow directly from the definition. Let us check property (4): If $\tilde{i}, \tilde{j} \in \mathbb{Z}^d$ are adjacent vertices and in addition $\tilde{i} \not\in F_n$ and $\tilde{j} \in F_n$ then it follows that $\tilde{j} = \tilde{i} \pm \tilde{e}_{\xi(\tilde{i})}$ and $\tilde{j} = \tau(\tilde{i})$.

**Lemma 8.4.** For any $n \in \mathbb{N}$ there exists a graph homomorphism $\tau_n : \mathbb{Z}^d \to \mathbb{Z}^d$ from the Cayley graph of $\mathbb{Z}^d$ to itself so that

1. $\tau_n(\tilde{i}) = \tilde{i}$ for all $\tilde{i} \in F_n$.
2. $\tau_n(\mathbb{Z}^d) = F_n$.
3. If $||\tilde{i}||_1 \geq 2nd$ then either $\tau_n(\tilde{i}) = 0$ or $\tau_n(\tilde{i}) = \tilde{e}_1$, according to the parity of $\tilde{i}$. 

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Proof. Let $\tau : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ be as in the statement of Lemma 8.3. Define $\tau_n : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ as follows:

$$
(219) \quad \tau_n(\vec{t}) = \begin{cases} 
\vec{t} & \vec{t} \in F_n \\
\tau^{2k}(\vec{t}) & \vec{t} \in \tau^{-k}(F_n) \setminus \tau^{-(k-1)}(F_n)
\end{cases}
$$

Let us check that $\tau_n$ is a graph homomorphism: Let us say that $\vec{t} \in \mathbb{Z}^d \setminus F_n$ is of level $k > 0$ if $\vec{t} \in \tau^{-k}(F_n) \setminus \tau^{-(k-1)}(F_n)$ (if $\vec{t} \in F_n$ we say it is of level 0). Suppose $\vec{t}, \vec{j} \in \mathbb{Z}^d$ are adjacent. If both $\vec{t}$ and $\vec{j}$ are of the same level $k > 0$, then $\tau_n(\vec{t}) = \tau^{2k}(\vec{t})$ and $\tau_n(\vec{j}) = \tau^{2k}(\vec{j})$ and so $\tau_n(\vec{t})$ is adjacent to $\tau_n(\vec{j})$ because $\tau^{2k}$ is a graph homomorphism. Otherwise, without loss of generality $\vec{t}$ is of level $k > 0$ and $\vec{j}$ is of level $k - 1$. Since $\tau$ is a graph homomorphism, $\tau^{2k-2}(\vec{t}) \notin F_n$ is adjacent to $\tau^{2k-2}(\vec{j}) \notin F_n$. But by the properties of $\tau$ we have that $\tau^{2k-1}(\vec{t}) = \tau^{2k-2}(\vec{j})$ and consequently $\tau_n(\vec{t}) = \tau^{2k}(\vec{t})$ is adjacent to $\tau_n(\vec{j}) = \tau^{2k}(\vec{j})$. This shows that $\tau_n$ is a graph homomorphism.

It is clear from the definition that $\tau_n(\vec{t}) = \vec{t}$ for $\vec{t} \in F_n$. Because $\tau(F_n) \subseteq F_n$ it follows that $\tau_n(\mathbb{Z}^d) = F_n$. For any $\vec{j} \in F_n$, and $k \geq 2dn$, either $\tau^k(\vec{j}) = \vec{0}$ or $\tau^k(\vec{j}) = \vec{e}_1$. Also, if $||\vec{t}||_1 \geq 2nd$ then there exists $k \geq dn$ so that $\vec{t} \in \tau^{-k}(F_n) \setminus \tau^{-(k-1)}(F_n)$. This show that if $||\vec{t}||_1 \geq 2nd$ then either $\tau_n(\vec{t}) = \vec{0}$ or $\tau_n(\vec{t}) = \vec{e}_1$, according to the parity of $\vec{t}$.

Lemma 8.5. If $H$ is a connected graph which is not bipartite then there exists $N \in \mathbb{N}$ so that for any $(v_0, v_1), (w_0, w_1) \in E_H$, $n \in \mathbb{N}$, $k \geq N$ and $a \in C_{n}^{(v_0, v_1)}$ there exists $\tilde{a} \in C_n^{(w_0, w_1)}$ such that $\tilde{a} \mid F_n = a$.

Proof. The fact that $H$ is connected and not bipartite implies that there exists $N \in \mathbb{N}$ so that for every $n \geq N$ there exists a path of length precisely $n$ between any two vertices of $H$ (a path of length $m$ is a graph homomorphism from $\{0, 1, \ldots, m\}$ to $H$). Suppose $n \in \mathbb{N}$, $k \geq N + 1$, $(v_0, v_1), (w_0, w_1) \in E_H$ and $a \in C_{n}^{(v_0, v_1)}$. We split the proof into two cases depending on the parity of $k$.

Let $k \in \mathbb{N}$ be even. Then we can choose a path $v_0, v_1, v_2, \ldots, v_{k+1}$ such that $v_k = w_0$ and $v_{k+1} = w_1$ and define $\tilde{a} \in C_{n+k}^{(w_0, w_1)}$ by the implicit conditions

$$
\tilde{a} \mid F_n = a,
$$

$$
\tilde{a} \mid F_{n+t} \in C_{n+t}^{(v_0, v_{t+1})} \quad \text{if} \quad 1 \leq t \leq k \quad \text{is even},
$$

$$
\tilde{a} \mid F_{n+t} \in C_{n+t}^{(w_0, w_{t+1})} \quad \text{if} \quad 1 \leq t \leq k \quad \text{is odd}.
$$

The case of odd $k$ is identical, except that we set $v_k = w_1$ and $v_{k+1} = w_0$.

Lemma 8.6. Let $H$ be a connected graph which is not bipartite and $(v_0, v_1) \in E_H$. There exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $k \geq N + d$ and $a \in \text{Hom}(F_n, H)$ there exists $\tilde{a} \in C_{2dn+k}^{(v_0, v_1)}$ for which $\tilde{a} \mid F_n = a$.

Proof. Given $a \in \text{Hom}(F_n, H)$ we can define $\tilde{a} \in \text{Hom}(F_{2dn+k}, H)$ by

$$
(220) \quad \tilde{a} \mid F_{2dn+k} = a_{\tau_n(\vec{t})},
$$

where $\tau_n : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ is the graph homomorphism given by Lemma 8.4. Note that $a_{\tau_n(\vec{t})}$ is well defined for any $\vec{t} \in \mathbb{Z}^d$ because $\tau_n(\vec{t}) \in F_n$. Because it is a composition of graph homomorphisms, $\tilde{a} \in \text{Hom}(F_{2dn+k}, H)$. Because the restriction of $\tau_n$ is the identity it follows that $\tilde{a} \mid F_n = a$. By the last property of $\tau_n : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ in Lemma 8.4 it follows that there exists $(w_0, w_1) \in E_H$ so that for any $\vec{t} \in F_{2dn} \setminus F_{2dn-1}$ either $\tilde{a} \mid \vec{t} = w_0$ or $\tilde{a} \mid \vec{t} = w_1$, according to the parity of $\vec{t}$. In other words, $\tilde{a} \in C_{2dn+k}^{(w_0, w_1)}$ for some $(w_0, w_1) \in E_H$. Let $N$ be the integer given by Lemma 8.5, and suppose $k \geq N$. Apply Lemma 8.5 to find $\tilde{a} \in C_{2dn+k}^{(v_0, v_1)}$ so that $\tilde{a} \mid F_{2dn+k} = \tilde{a}$.

Note that Lemma 8.6 in particular tells us that $\mathcal{L}(\text{Hom}(\mathbb{Z}^d, H), F_n) = \text{Hom}(F_n, H)$.

Lemma 8.7. Let $H$ be a connected graph which is not bipartite. There exists $N \in \mathbb{N}$ so that for any $(v_0, v_1), (w_0, w_1) \in E_H$, $k, n \in \mathbb{N}$, any $F_{k+N}$-spaced subset $K \subset F_n \setminus F_{n-2}$ and any $W \in C_{k}^{(v_0, v_1)}$ there exists $w \in C_{n}^{(w_0, w_1)}$ so that

$$
S^\tau(w) \mid F_n = W(\vec{t}) \quad \text{for all} \quad \vec{t} \in K.
$$
Proof. Let $N \in \mathbb{N}$ be as given Lemma 8.5. Suppose $K \subset F_{n-N-2}$ is $F_{k+N}$-spaced and $W \in (C_{k}^{(v_0,v_1)})^K$. Apply Lemma 8.5 to obtain $\hat{W} \in (\text{Hom}(F_{k+N},\mathcal{H}))^K$ so that for any $\bar{i} \in K$, $\hat{W}(\bar{i}) \in C_{k+N}^{(\text{parity}(\bar{i}),v_0,v_1-\text{parity}(\bar{i}))}$ and $\hat{W}(\bar{i}) \mid F_{k} = W(\bar{i})$. Now define $w \in C_{n}^{(v_0,v_1)}$ as follows:

$$w_{j} = \begin{cases} \hat{W}(\bar{i})_{j-i} & \text{if } \bar{i} \in K \text{ and } j \in F_{k+N} \\ w_{\text{parity}(\bar{i})} & \text{if } j \notin \bigcup_{\bar{i} \in K} (\bar{i} + F_{k+N}) \end{cases}.$$

Then $w \in C_{n}^{(v_0,v_1)}$ has the desired properties.

Given Lemma 8.7 it is easy to prove that $\tilde{C}$ is a flexible sequence of patterns, and the only somewhat non-trivial part is the “entropy estimate” $h(\tilde{C}) = h(\text{Hom}(\mathbb{Z}^d,\mathcal{H}))$. For this purpose it will be convenient to introduce another sequence of patterns:

$$\tilde{C}_n = \left\{ a \in \text{Hom}(F_n,\mathcal{H}) : \text{If } \bar{i}, \bar{j} \in (F_n \setminus F_{n-1}) \text{ and } (\bar{i} - \bar{j}) \in 2\mathbb{Z}^d \text{ then } a_{\bar{i}} = a_{\bar{j}} \right\}.$$

By definition $\tilde{C}_n$ is precisely the set of graph homomorphisms $w \in \text{Hom}(F_n,\mathcal{H})$ such that $w \mid F_{n-F_{n-1}}$ is the composition of some $\hat{w} \in \text{Hom}((0,1)^d,\mathcal{H})$ with natural graph homomorphism $\hat{\bar{i}} \mapsto (\hat{\bar{i}} \text{ mod } 2\mathbb{Z}^d)$ from the standard Cayley graph of $\mathbb{Z}^d$ to the standard Cayley graph of $\mathbb{Z}^d/2\mathbb{Z}^d$. Clearly, $\tilde{C}_n \subset \tilde{C}_n$. We will prove that $h(\tilde{C}) = h(\text{Hom}(\mathbb{Z}^d,\mathcal{H}))$ by proving that $h(\tilde{C}) = h(\tilde{C}) = h(\text{Hom}(\mathbb{Z}^d,\mathcal{H}))$.

The fact that $h(\tilde{C}) = h(\tilde{C})$ is a direct consequence of the following:

**Lemma 8.8.** If $\mathcal{H}$ is a connected graph then for every $n \in \mathbb{N}$, $k \geq 2d$ and $a \in \tilde{C}_n$ there exists $(v_0,v_1) \in E_{\mathcal{H}}$ and $\tilde{a} \in C_{n+k}^{(v_0,v_1)}$ such that

$$\tilde{a} \mid F_n = a.$$

**Proof.** If $a \in \tilde{C}_n$ there exists $a' \in \text{Hom}((0,1)^d,\mathcal{H})$ such that for every $\bar{i} \in F_n \setminus F_{n-1}$ we have $a_{\bar{i}} = a'_{\hat{\bar{i}}}$ where $\hat{\bar{i}} : \mathbb{Z}^d \to (0,1)^d$ is given by $\hat{\bar{i}}(\bar{j}) \equiv \bar{j} \text{ mod } 2\mathbb{Z}^d$. Note that $\hat{\bar{i}}$ is a graph homomorphism. Let $\tau : \mathbb{Z}^d \to \mathbb{Z}^d$ be as in Lemma 8.3. Define $\tilde{a} : \mathbb{Z}^d \to \mathcal{H}$ by

$$\tilde{a}_{\bar{i}} = \begin{cases} a_{\bar{i}} & \bar{i} \in F_n \\ a'_{\tau^{-k}(\bar{i})} & \bar{i} \in \tau^{-k}(F_n) \setminus \tau^{-k-1}(F_n). \end{cases}$$

It follows by the properties of $\tau$ that $\tilde{a} \in \text{Hom}(\mathbb{Z}^d,\mathcal{H})$. Exactly as in the proof of Lemma 8.6 it follows that whenever $k \geq 2d$, $\tilde{a} \mid C_{n+k} = C_{n+k}^{(v_0,v_1)}$, where either $(v_0,v_1) = (a'_{\tilde{a}(v_0)},a'_{\tilde{a}(v_1)})$ or $(v_0,v_1) = (a'_{\tilde{a}(v_0)},a'_{\tilde{a}(v_1)})$, according to the parity of $k$. It is clear from the definition that $\tilde{a} \mid F_n = a$. 

To see that $h(\tilde{C}) = h(\text{Hom}(\mathbb{Z}^d,\mathcal{H}))$ we prove the following:

**Proposition 8.9.** There exists a constant $c > 0$ so that for every $n \in \mathbb{N}$

$$\frac{|\tilde{C}_n|}{|\text{Hom}(F_n,\mathcal{H})|} \geq c^{-n^d-1}.$$  

In particular,

$$h(\tilde{C}) = h(\text{Hom}(\mathbb{Z}^d,\mathcal{H}), S).$$

Proposition 8.9 is essentially the statement that the probability that a uniform random graph homomorphism from the box $F_n \subset \mathbb{Z}^d$ to a finite graph $\mathcal{H}$ takes only two values on the “boundary” of the box $F_n$ is at most exponentially small in the size of the boundary. We include a proof since we did not find one in the literature. As pointed out to us by Yinon Spinka, in the case where $\mathcal{H}$ is the complete graph on $q$ vertices and so $\text{Hom}(\mathbb{Z}^d,\mathcal{H})$ corresponds to $q$-colorings of $\mathbb{Z}^d$, a short proof for Proposition 8.9 follows from the fact that for any finite bipartite graph $G$, the probability that a uniform random $q$-coloring assigns only two colors to a subset $W \subset V_G$ is at least $q^{-|W|}$. For a proof see [20, Lemma 5.1]. For 3-colorings it also follows from [2, Proposition 2.1]. Our proof of Proposition 8.9 is based on an idea that is sometimes called reflection positivity. We mostly follow Biskup [3], with suitable adaptations to our setting.
Let us make a brief detour to introduce a notion of “reflection” and “reflection positivity” for “discrete random fields” (namely, random functions on discrete graphs).

**Definition 8.10.** Let $\mathcal{G} = (V, E)$ be a (discrete, finite or countable) graph. An automorphism $R \in \text{Aut}(\mathcal{G})$ is called a reflection if:

(i) $R$ is an involution: $R^2 = \text{Id}$.

(ii) The complement of the fixed points of $R$ in $V$ has precisely two connected components $V_1, V_2$ that are mapped bijectively onto each other. We refer to these as the sides of the reflection.

A reflection also induces a self map on $A$ elements of $\mu$ is called a reflection if:

**Remark 8.13.**

By successive applications of “reflection positivity” along the $R$ to $\mu$ then the sigma-algebras generated by the restrictions to $A_1$ and $A_2$ are independent conditioned on the sigma-algebra generated by the restriction to $B$.

**Proposition 8.11.** If $\mu \in \text{Prob}(A^V)$ is a Markov random field with respect to $\mathcal{G}$ and $R \in \text{Aut}(\mathcal{G})$ is a reflection that preserves $\mu$, then $\mu$ is reflection positive with respect to $R$.

**Proof.** Let $\mu$ and $R$ be as in the statement of the proposition, let $F \subset V_2$ be the fixed points of $R$, and let $V_1 \subset V_2$ be one of the sides for $R$. Choose $W \subset V_1$, $a \in A^W$ and $b \in A^F$. Because $\mu$ is a Markov random field, and $F$ disconnects $W$ and $R(W)$

$$\mu([a]_W \cap [R(a)]_{R(W)}) = \mu([a]_W) \cdot \mu([R(a)]_{R(W)}) = \mu([a]_W) \cdot \mu([b]_F)^2.$$  

The last equality follows because of invariance of $\mu$ under $R$. Taking expectation over $b \in A^F$ with respect to $\mu$ and applying the Cauchy-Schwarz inequality we conclude that (224) holds.

For $n \in \mathbb{N}$ let $T_n$ denote the Cayley graph of the group $(\mathbb{Z}/2n\mathbb{Z})^d$ with respect to the standard generators; this is a “discrete torus”.

**Lemma 8.12.** There exists a constant $c > 0$ so that for every $n \in \mathbb{N}$,

$$\frac{|\text{Hom}(T_n, \mathcal{H})|}{|\text{Hom}(F_n, \mathcal{H})|} \geq e^{-cn^{d-1}}.$$  

**Proof.** Let $\mu$ denote the uniform measure on $\text{Hom}(F_n, \mathcal{H})$. For all reflections $R$ along coordinate hyperplanes, $\mu$ is an $R$-invariant Markov random field. By Proposition 8.11, it is reflection positive with respect to all such reflections $R$. We can find $a \in \mathcal{V}_H$ such that

$$\mu([a]_{F_n \setminus F_{n-1} \cap (\mathbb{N}^d)}) \geq |\mathcal{H}|^{-2d|\mathbb{Z}/2n\mathbb{Z}^d|}.$$  

By successive applications of “reflection positivity” along the $d$ hyperplanes corresponding to the cardinal directions, we get a pattern $\tilde{a}$ on $F_n \setminus F_{n-1}$ which is periodic, meaning, $\tilde{a}_i = \tilde{a}_{\bar{j}}$ whenever $\tilde{i} - \tilde{j} \in 2n\mathbb{Z}^d$ and so that

$$\mu([\tilde{a}]_{F_n \setminus F_{n-1}}) \geq (\mu([a]_{F_n \setminus F_{n-1} \cap (\mathbb{N}^d)})^{2d} \geq |\mathcal{H}|^{-2d|\mathbb{Z}/2n\mathbb{Z}^d|}.$$  

This gives us the required result because there is a natural bijection between the periodic patterns and elements of $\text{Hom}(T_n, \mathcal{H})$.

**Remark 8.13.** Lemma 8.12 has the following dynamical consequence about $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$: The subshift $\text{Hom}(\mathbb{Z}^d, \mathcal{H})$ has “many” periodic points in the sense that

$$h(\text{Hom}(\mathbb{Z}^d, \mathcal{H}), S) = \lim_{n \to \infty} \frac{\log |P_{2n}(\text{Hom}(\mathbb{Z}^d, \mathcal{H}), S)|}{(2n)^d},$$  

where $P_{2n}(\text{Hom}(\mathbb{Z}^d, \mathcal{H}), S)$ is the set of points in $\text{Hom}(\mathbb{Z}^d, \mathcal{H}), S)$ that are stabilized under the subaction of $(2n\mathbb{Z})^d$. See for instance [18].
The conceptual reason for introducing $\text{Hom}(T_n, \mathcal{H})$ as an auxiliary object is that $T_n$ has additional symmetries coming from reflection and also the action of $(\mathbb{Z}/2n\mathbb{Z})^d$.

**Lemma 8.14.** There exists a constant $c > 0$ so that for every $n \in \mathbb{N}$,

$$|\{a \in \text{Hom}(T_n, \mathcal{H}) : a_{(i,j)} = a_{(i,j)} \text{ whenever } i - j \in (2\mathbb{Z})^{d-1}\}| \geq e^{-cn^{d-1}}.$$  

**Proof.** Let $\nu$ denote the uniform measure on $\text{Hom}(T_n, \mathcal{H})$ and observe that $\nu$ is invariant and reflection positive with respect to reflections of the type $R_{k,r} : T_n \to T_n$ given by

$$R_{r,k}(i) := i - (2i_k - 2r)e_k^r \text{ for } 1 \leq k \leq d \text{ and } 0 \leq r \leq n - 1.$$  

Again we begin with a pattern $a$ on $\{0\} \times \{0, 1\}^{d-1}$, chosen such that

$$\nu([a]_{\{0\} \times \{0, 1\}^{d-1}}) \geq |\mathcal{H}|^{-2^{d-1}}.$$  

Successive reflections of $a$ by

$$R_{1,2}, R_{2,2}, R_{4,2}, \ldots, R_{2^{\log_2 n - 1}, 2}, R_{1,3}, R_{2,3}, R_{4,3}, \ldots, R_{2^{\log_2 n - 1}, 3}, \ldots, R_{2^{\log_2 n - 1}, d}$$

and applying reflection positivity gives us a pattern $\tilde{a}$ on $\{0\} \times [0, n - 1]^d$ such that

$$\tilde{a}_{i} = a_j \text{ whenever } i - j \in (2\mathbb{Z})^d$$

and

$$\nu([\tilde{a}]_{\{0\} \times [0, n - 1]^d}) \geq |\mathcal{H}|^{-(2n)^{d-1}}.$$  

(In fact, we might get a slightly bigger pattern but we do not need it for this proof.) Finally by successively reflecting $\tilde{a}$ ($R_{r,k}$ for $r = 0$ and $2 \leq k \leq d$ in (228)) and applying reflection positivity we get a pattern $a'$ on $\{0\} \times [0, 2n - 1]^d$ such that

$$a'_{i} = a_j \text{ whenever } i - j \in (2\mathbb{Z})^d$$

and

$$\nu([a']_{\{0\} \times [0, 2n - 1]^d}) \geq |\mathcal{H}|^{-(4n)^{d-1}}.$$  

This completes the proof. \qed

We can now complete the proof:

**Proof of Proposition 8.9.** Again, let $\mu$ be the uniform measure on $\text{Hom}(F_n, \mathcal{H})$. The measure $\mu$ is invariant and reflection positive with respect to “diagonal” reflections of the type $R_{k,\pm} : F_n \to F_n$ given by

$$R_{k,\pm}(i) := i - (i_2 e_1 \pm i_2 e_1 + i_2 e_k) \text{ for } 2 \leq k \leq d.$$  

By Lemmas 8.12 and 8.14 we have a pattern $a$ on $\{n\} \times [-n, n]^{d-1}$ such that $a_{\tilde{i}} = a_j$ whenever $\tilde{i} - j \in (2\mathbb{Z})^d$ and

$$\nu([a]_{\{n\} \times [-n, n]^{d-1}}) \geq e^{-Cn^{d-1}}$$

for some constant $C$ independent of $n$. Now we apply successive reflections to $a$. It may so happen that a certain reflections might result in patterns which are larger than the side associated with the next reflection. In this case, we just restrict that pattern to the side which contains $\{n\} \times [-n, n]^{d-1}$ before continuing.

By successive reflections of $a$ along the diagonals by

$$R_{2, +}, R_{3, +}, \ldots, R_{d, +}, R_{2, -}, R_{3, -}, \ldots, R_{d, -}$$

and applying reflection positivity we get a checkerboard boundary pattern $\tilde{a}$ such that

$$\mu([\tilde{a}]_{F_n \setminus F_{n-1}}) \geq e^{-2^{2(d-1)}Cn^{d-1}}.$$  

This completes the proof. \qed
Proof of Proposition 8.2. The result that \( \mathcal{C} \) is a flexible sequence of patterns follows immediately from Lemma 8.7. The fact that it is furthermore a flexible marker sequence of patterns follows from (217). By Lemma 8.8 combined with Lemma 8.5 it follows that there exists \( N \in \mathbb{N} \) such that \( |\hat{C}_n| \leq |\hat{C}_{n+N}| \) so

\[
(229) \quad h(\hat{\mathcal{C}}) \geq \limsup_{n \to \infty} \frac{\log |\hat{C}_n|}{|F_n|}.
\]

As we explained, from Lemma 8.6 it follows that \( \mathcal{L}(\text{Hom}(\mathbb{Z}^d, \mathcal{H}), F_n) = \text{Hom}(F_n, \mathcal{H}) \). By Proposition 8.9 the right hand side of (229) is equal to the topological entropy of the hom-shift \( \text{Hom}(\mathbb{Z}^d, \mathcal{H}) \). The statement beginning with “Moreover” follows by applying Lemma 8.6 together with Lemma 8.5. 

Remark 8.15. If \( \mathcal{H} \) is not bipartite then \( \text{Hom}(\mathbb{Z}^d, \mathcal{H}) \) is not universal because any invariant measure admits a set of measure \( \frac{1}{2} \) which is invariant under the \( (2\mathbb{Z})^d \) sub-action. Nevertheless, in this case it is still true that the \( (2\mathbb{Z})^d \) sub-action on \( \text{Hom}(\mathbb{Z}^d, \mathcal{H}) \) is universal.

As stated in the introduction Theorem 8.1 has consequences related to some problems in “Borel graph theory” discussed for instance in [22] and references within. Given a Borel space \( Y \) and Borel bijections \( T_1, \ldots, T_d : Y \to Y \), let \( \mathcal{G}_{T_1, \ldots, T_d} \) denote the graph on \( Y \) has edges of the form \( (y, T_j^k(y)) \in Y \times Y \) where \( y \in Y \) and \( 1 \leq j \leq d \). The edges of \( \mathcal{G}_{T_1, \ldots, T_d} \) are a Borel subset of \( Y \times Y \). The Borel chromatic number of \( \mathcal{G}_{T_1, \ldots, T_d} \) is the smallest \( k \) so that there exists a Borel function from \( Y \) to \( \{1, \ldots, k\} \) which is a proper coloring of the graph \( \mathcal{G}_{T_1, \ldots, T_d} \).

Corollary 8.16. Let \( T_1, \ldots, T_d : Y \to Y \) are \( d \)-commuting Borel bijections of a standard Borel space \( Y \) that generate a free \( \mathbb{Z}^d \)-action. Then after removing a subset which is null for every Borel probability measure that is invariant with respect to each of the \( T_i \)’s, the Borel chromatic number of \( \mathcal{G}_{T_1, \ldots, T_d} \) is either 2 or 3. It is equal to 2 if and only if there exists a two set partition \( Y_0, Y_1 \) of \( Y \) modulo a null set so that \( T_i(Y_j) = Y_{1-j} \) for \( i = 1, \ldots, d \) and \( j = 0, 1 \).

More generally, Theorem 8.1 implies that for any \( d \)-commuting Borel bijections \( T_1, \ldots, T_d : Y \to Y \) and any connected finite graph \( \mathcal{H} \) that is non-bipartite there exists a Borel map from \( Y \) to the vertices of \( \mathcal{H} \) which is a graph homomorphism on a full subset of \( Y \). In fact, in order to deduce this corollary we do not need to prove that \( h(\hat{\mathcal{C}}) \) is equal to the topological entropy of \( \text{Hom}(\mathbb{Z}^d, \mathcal{H}) \), nor do we need the full strength of Theorem 5.1, as Proposition 6.16 suffices.

9. Universality of dimers and rectangular tilings

In this section we use our main result to prove universality for dimers and more generally for rectangular tilings in \( \mathbb{Z}^d \). Let \( \mathcal{T} \) be a finite collection of finite subsets of \( \mathbb{Z}^d \), which we refer to as prototiles. A \( \mathcal{T} \)-tiling of \( \mathbb{Z}^d \) is a partition of \( \mathbb{Z}^d \) into pairwise disjoint translates of elements of \( \mathcal{T} \). We denote the space of all \( \mathcal{T} \)-tilings by \( \mathcal{X}^\mathcal{T} \) and refer to it as the tiling space corresponding to \( \mathcal{T} \).

In this section we consider rectangular prototiles. To every \( \vec{i} = (i_1, \ldots, i_d) \in \mathbb{N}^d \) we associate the rectangular prototile

\[
(230) \quad T_{\vec{i}} = \{1, \ldots, i_1\} \times \cdots \times \{1, \ldots, i_d\}.
\]

We call a tiling space with rectangular prototiles is called a rectangular tiling shift. Rectangular tilings spaces are naturally in one-to-one correspondence with finite subsets of \( \mathbb{N}^d \). Given a finite subset \( F \subseteq (\mathbb{N})^d \), we denote by \( \mathcal{T}(F) \) the tiling set corresponding to the prototiles

\[
\mathcal{T}(F) = \{T_{\vec{i}} : \vec{i} \in F\}.
\]

We refer to \( \mathcal{X}(F) = \mathcal{X}^{\mathcal{T}(F)} \) as the rectangular tiling shift corresponding to \( F \).

A particularly interesting and well studied instance of a rectangular tiling shift is that of \textit{dimers} or \textit{domino tilings} in \( \mathbb{Z}^d \) where

\[
F = \{\vec{d} + \vec{e}_1, \ldots, \vec{d} + \vec{e}_d\}, \text{ with } \vec{d} = (1, \ldots, 1) = \sum_{t=1}^d \vec{e}_t.
\]

Dimer tilings also correspond to perfect matchings in the standard Cayley graph of \( \mathbb{Z}^d \). There are significant and deep results about dimers in \( \mathbb{Z}^2 \), in particular the topological entropy of the corresponding tiling.
space is known and much more is known about the measure of maximal entropy. We refer for instance to the celebrated result by Cohn-Kenyon-Propp variational principle for domino tilings [13]. More general rectangular tiling problems have been considered by Einsiedler who studied their shift cohomology [17] and by Pak, Sheffer and Tassy who studied some algorithmic aspects of rectangular tilings [43].

Call a set $F \subset \mathbb{N}^d$ coprime if projecting it onto each coordinate yields a coprime set. So $F \subset \mathbb{N}^d$ is coprime if $\gcd\{i_t : t \in F\} = 1$ for every $1 \leq t \leq d$, where $i_t = \vec{i} \cdot \vec{e}_t$ is the projection of $\vec{i}$ onto the $t$th coordinate.

We recall the following result, now known as the $\mathbb{Z}^d$-Alpern Lemma [1]: Let $F \subset \mathbb{N}^d$ be a coprime finite set, and let $(Y, \mu, T)$ be an $\mathbb{F}^d$-system. Then there exists a Borel $T$-invariant subset $Y_0 \subset Y$ with $\mu$-null complement and an equivariant Borel map $\pi : Y_0 \to X_{(F)}$. The above result is due to Alpern proved this for $d = 1$, and to Prikhod’ko [46] and Şahin [53] for arbitrary $d$. They further proved that given any strictly positive probability distribution $(p_T)_{T \in F}$ on $F$ the map $\pi$ can be chosen so that almost surely with respect to $\mu \circ \pi^{-1}$, the proportion of tiles of type $T$ is $p_T$.

**Theorem 9.1.** If $F \subset \mathbb{N}^d$ is coprime and $|F| > 1$ then $X_{(F)}$ admits a flexible marker sequence of patterns and is thus h-universal for some $h > 0$.

The case $|F| = 1$ where $F$ is coprime corresponds to the trivial one point system.

For $n \in \mathbb{N}$ we denote a $\mathbb{Z}^d$-box of side-length $n$ by

$$B_n = \{1, \ldots, n\}^d$$

(we need this notation since the boxes $F_n$ have odd side-lengths, and we will need even ones as well here.)

Roughly speaking, the flexible marker sequence of patterns will consist of perfect tilings of translates of boxes whose side length is divisible by certain integers, and with a specific tiling by a “marker pattern” near the boundary.

An immediate corollary of Theorem 9.1 is the following “almost-Borel” $\mathbb{Z}^d$-Alpern’s Lemma:

**Corollary 9.2.** Let $F \subset \mathbb{N}^d$ be a coprime finite set, and let $(Y, T)$ be a free $\mathbb{Z}^d$ Borel dynamical system. Then there exists a full Borel $T$-invariant subset $Y_0 \subset Y$ and an equivariant Borel map $\pi : Y_0 \to X_{(F)}$.

Theorem 9.1 additionally says that if $(Y, T)$ has sufficiently low entropy it is possible to make the equivariant tiling $\pi : Y_0 \to X_{(F)}$ injective. We remark that although Theorem 9.1 does not directly recover the part of Alpern’s lemma about specifying a probability distribution for the prototiles, it is possible to extract this part of the result by formulating a result about the possibility to control the push forward of a measure $\mu$ when embedding $(Y, T, \mu)$ into a flexible system $(X, S)$.

Notice that for $d = 1$, if $T$ is a prime tile set then $X^T$ is a mixing shift of finite type so Alpern’s Lemma follows from the mixing SFT version of Krieger’s embedding theorem (stated in [37], see [15, Theorem 28.1] for a detailed proof).

For domino tilings in $\mathbb{Z}^2$ we can say a little more:

**Theorem 9.3.** The domino tiling in $\mathbb{Z}^2$ admits a flexible marker sequence of patterns $\tilde{\mathcal{C}}$ such that $h(\tilde{\mathcal{C}})$ is equal to the topological entropy and so that every admissible pattern appears in some element of $\tilde{\mathcal{C}}$. Thus the subshift of domino tilings in $\mathbb{Z}^2$ is fully universal in the almost-Borel sense.

In the case when $(1, \ldots, 1) \in F$ the corresponding space $X_{(F)}$ is strongly irreducible and has dense periodic points. In this case universality of $X_{(F)}$ in the ergodic sense follows from an earlier result of Şahin and Robinson [50]. This is not the case for domino tilings in $\mathbb{Z}^d$ or for more general rectangular tiling spaces. See for instance [55].

**Remark 9.4.** If $F$ is not coprime and $h > 0$, the subshift $X_{(F)}$ fails to be $h$-universal due to periodicity issues, but it is still $h$-universal with respect to a subaction of some finite index subgroup of $\mathbb{Z}^d$. This is similar to the case of hom-shifts associated with bipartite graphs.

One of the ingredients of the proof of Theorem 9.1 is the following lemma.

**Lemma 9.5.** Let $N, n, n', M \in \mathbb{N}$ and $\vec{i} \in \mathbb{Z}^d$ such that the translate of $B_{nM} + FN$ centered at $\vec{i}$ is contained in $B_{(n+n')M}$. Then $B_{(n+n')M} \setminus (\vec{i} + B_{nM})$ can be partitioned into rectangular shapes of which one of the sides is greater than or equal to $N$ and the rest are multiples of $M$. 

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Lemma 9.7. For every $F$, this can be partitioned as required by the induction hypothesis.

Proof. The proof for higher dimensions follows by induction on $d$. For $d = 1$, $B_{(n_1')M} \setminus (i_B + B_{nM})$ is a disjoint union of two intervals of length greater than or equal to $N$, so this gives the required partition.

For the induction step, suppose the result is known for dimensions less than $d$. Extend two opposite faces of $i_B + B_{nM}$ to get a partition of $B_{(n_1'')M}$ into three parts. The parts which do not contain $i_B + B_{nM}$ have one side of length greater than or equal to $N$ and the rest are multiples of $M$. Now the partition element which contains $i_B + B_{nM}$ is the product of a $d - 1$ dimensional instance of the induction with $\{1, \ldots, nM\}$. This can be partitioned as required by the induction hypothesis.

Given a partition as in Lemma 9.5, we will need to tile each such rectangular shape of the partition with elements of our prime tile set. This is given by the following lemma.

Lemma 9.6. Suppose $F \subset \mathbb{N}^d$ is finite and coprime. Let $M$ be the product of all the side lengths of the rectangles in $T_F$. Then each of the following conditions on $i_B \in \mathbb{N}^d$ is sufficient so that $T_F$ can tile $T_i$:

1. $i_B \in MN^d$. Equivalently, all of the side lengths of $T_i$ are positive integer multiples of $M$.
2. There exists $m \in \mathbb{N}$ and $1 \leq t \leq d$ so that $i_B = m e_t + MN^d$. Equivalently, one of its side lengths of $T_i$ is greater than or equal to $M$ and the rest are multiples of $M$.

Furthermore, in the first case the number of possible tilings of $T_i$ is at least $|F|^{M-d|M|F|}$.

Proof. Let $M$ be as above. It is clear that $T_j$ can tile $B_M$ for every $j \in F$ by the obvious grid tiling. Thus, if $i_B \in MN^d$, $T_i$ can be tiled by translates of $B_M$, each of which can be tiled by $T_F$ in at least $|F|$ different ways. It follows that $T_i$ can be tiled by $T_F$, and there are at least $|F|^{M-d|M|F|}$ possible tilings.

Now assume that $i_B \in MN^d + m e_1$, where $m \in \mathbb{N}$. By our assumption that $F$ is coprime,

\[
\gcd \left( \{ j_1 : j \in F \} \right) = 1,
\]

where $j_1 = \bar{j} \cdot \bar{e}_1$ is the first coordinate of $\bar{j}$. Let $i_1 = i_B \cdot \bar{e}_1$. Our assumption is that $i_1 > M$. Thus there exists $c \in \mathbb{N}^F$ so that $i_1 = \sum_{j \in F} c_j j_1$, where again $j_1 = \bar{j} \cdot \bar{e}_1$ is the first coordinate of $\bar{j}$ (this is related to the well-known Diophantine Frobenius problem or the coin problem. For more about this have a look at [5]). For $j \in F$ let $j^* = (c_j j_1, i_2, \ldots, i_d)$, where $i_t = i_B \cdot \bar{e}_t$ is the $t$-th coordinate of $i_B$. By our assumption $i_2, \ldots, i_d \in MN$. It follows that $T_j$ can tile $T_j^*$. By construction, $T_i$ can be tiled by $\{ T_j^* : j \in F \}$ by stacking them next to each other in direction $\bar{e}_1$.

\[
(232) \quad C_n = \{ a \in \mathcal{L}(B_{nM}, X_F) : a \text{ is a perfect } T_F \text{ tiling of } B_{nM} \}
\]

Lemma 9.7. For every $k < n$ and any $B_{kM} + F_M$-separated set $K \subset B_{nM}$ so that the translate of $i_B + B_{kM} + F_M$ is contained in $B_{nM}$. Then for any $w \in C^K_F$ there exists $a \in C_n$ such that $S^a(a)|_{B_{nM}} = w_i$ for all $i \in K$.

Proof. To prove the lemma it is enough to show that for any $k < n$ and any $B_{kM} + F_M$-separated set $K \subset B_{nM}$ so that the translate of $i_B + B_{kM} + F_M$ is contained in $B_{nM}$, there is a perfect tiling of $B_{nM} \setminus \langle K + B_{kM} \rangle$ by the tiles of $T_F$.

The proof will involve an appropriate division of $B_{nM} \setminus \langle K + B_{kM} \rangle$ into rectangular shapes which can be tiled by rectangles in $T_F$ by Lemma 9.6.

1. Let $\mathcal{P}_1$ be the partition of $B_{nM}$ into translates of $B_M$.
2. For all $i_B \in K$, we let $\hat{B}_i$ be the union of all partition elements of $\mathcal{P}_1$ which intersect $i_B + B_{kM} + F_M$.
3. By Lemma 9.5, for every $i_B \in K$, $\hat{B}_i \setminus i_B + B_{kM}$ is partitioned into rectangular shapes of which one of the sides is greater than or equal to $M$ and the rest are multiples of $M$. The partition elements of $\mathcal{P}_1$ give a partition of

\[
B_{nM} \setminus \bigcup_{i_B \in K} \hat{B}_i
\]

into translates of $B_M$. By Lemma 9.6, each of these can be tiled by $T_F$.

This completes the proof. \[\square\]

Now we can proceed towards the proof of the theorem.
Proof of Theorem 9.1. Recall that $C_n$ has been defined by (232). Let $\tilde{C}_n \subset \mathcal{L}(X^T, F_n)$ consist of the patterns $a \in \mathcal{L}(X^T, F_n)$ such that

(1) There exists $\tilde{a} \in C_{M[\frac{2n+1}{M}]}$ so that the “centered” $F_n$ pattern in $\tilde{a}$ is equal to $a$. In other words, $a$ can be extended to a perfect tiling of a slightly bigger box whose side-length is divisible by $M$. If $M[\frac{2n+1}{M}]$ is even, we choose an “approximate center” for $B_{M[\frac{2n+1}{M}]}$.

(2) The restriction of $a$ to the “centered” $B_{M[\frac{2n+1}{M}]}$ of $F_n$ is also a perfect $T_F$ tiling, where the two ‘outermost layers’ of thickness $M$ each tiled by a single tile $T_{\tilde{a}(1)}$ and $T_{\tilde{a}(2)}$ respectively, where $\tilde{a}(1) \neq \tilde{a}(2)$ and $\tilde{a}(1), \tilde{a}(2) \in F$.

By Lemma 9.7 the first property implies that $\tilde{C}_n$ is a flexible sequence of patterns (In general, whenever we have a flexible sequence of patterns “along a subsequence with small gaps” we can obtain a new flexible sequence along the integer). The second property implies the marker property (two such patterns cannot overlap too much).

Now we will prove the universality for the domino tiling shifts.

Proof of Theorem 9.3. Let $D = \{(1, 2), (2, 1)\} \subset \mathbb{N}^2$, and let $X_D$ be the domino tiling shift. As in the proof of the previous theorem we let $M = 4$ be the product of all the side lengths of rectangles in $T_D$. We already proved in the Theorem 9.1 that the sequence $C = (C_n)_{n=1}^\infty$ is a flexible marker sequence of patterns. The additional claim is that $h(C)$ is equal to the topological entropy of $X_D$. This is essentially the statement that the number of perfect matchings of $B_{2n}$ is $e^{h(X_D) + o(n^2)}$ as $n \to \infty$. This follows almost directly from well known results about dimers in $\mathbb{Z}^2$, and in particular from [13, Theorems 4.1 and 10.1]. This seems to be an overkill for what we require. As an alternative, one can use older and more basic classical results by Kastelyn about the the number of tilings of a 2-dimensional discrete torus $\mathbb{Z}^2/(2n\mathbb{Z}^2)$ and of $B_{2n}$, in combination with a “reflection positivity argument” of the sort used in the previous section: Observe that the uniform measure on tilings by dominoses of $B_{4n}$ is “reflection positive” with respect to reflection along the hyperplanes parallel to the coordinate hyperplanes passing through $(2n + \frac{1}{2}, 2n + \frac{1}{2}, \ldots, 2n + \frac{1}{2})$. Thus it follows as in the proof of Lemma 8.12 that

$$\lim_{n \to \infty} \frac{\log (\text{number of perfect matchings of } T_{4n}^2)}{|T_{4n}^2|} = h(X_D).$$

Kasteleyn in [30, the discussion following Equation (27)] further proved that

$$\limsup_{n \to \infty} \frac{\log (\text{number of perfect matchings of } T_{4n}^2)}{|T_{4n}^2|} = \limsup_{n \to \infty} \frac{\log |C_n|}{|B_{4n}|},$$

where $T_{4n}^2 = \mathbb{Z}^2/4n\mathbb{Z}^2$. By these two equations together we conclude that

$$\limsup_{n \to \infty} \frac{\log |C_n|}{|B_{4n}|} = h(X_D).$$

The part about “full universality” follows from [17, Theorem 2.1] which shows that any finite patch of a domino tiling of $\mathbb{Z}^2$ can be extended to a tiling of a square.

10. A Fully Universal System with a Factor which is Not Fully Universal

Lind and Thouvenot [39] asked if a factor of a fully universal dynamical system must also be universal. Here we provide a negative answer by describing a fully-universal $\mathbb{Z}$ subshift that admits a factor which is not universal in the ergodic sense. The construction has some similar features to Haydn’s construction of a subshift with multiple MME’s [26] and also to a related methods of Quas and Şahin [47] that can be adapted to produce a $\mathbb{Z}^2$-SFT with similar properties.

Let $A_{+,0} \subset 2\mathbb{Z} \cap (0, \infty)$ be a finite set of positive even integers. Denote $A_{+1} = A_{+,0} + 1$, $A_{-,0} = -A_{+,0}$, $A_{-,1} = -A_{+1}$. Let $M = |A_{\pm,0}| = |A_{\pm,1}|$, and suppose $M > 4$. Denote:

$$A_\pm = A_{\pm,1} \cup A_{\pm,0}.$$
We now describe a subshift $\hat{X} \subset \mathcal{A}_0^\mathbb{Z}$. The rules for the subshift $X$ are:

1. A negative even integer must be directly followed by a negative odd integer.
2. A negative odd integer must be directly followed by a non-zero even integer.
3. A positive even integer must be directly followed by a positive odd integer.
4. A positive odd integer must be directly followed by non-negative even integer.
5. The symbol 0 must be directly followed by a non-positive even integer.
6. Whenever 0 is directly followed by a sequence of $n_+$ positive integers and then a sequence of $n_-$ negative integers and then another 0, then $n_+ = n_-$. 

In other words, $\hat{X}$ consists of concatenations of blocks of the form 

$$0^k e_1^- o_1^- \ldots e_n^- o_n^- e_1^+ o_1^+ \ldots e_n^+ o_n^+ 0^j,$$

with $e_1^- \ldots, e_n^- \in A_{-0}, o_1^- \ldots, o_n^- \in A_{-1}, e_1^+ \ldots, e_n^+ \in A_{+0}, o_1^+ \ldots, o_n^+ \in A_{+1}$, and their limits. Let $\Phi : \hat{X} \to \mathbb{Z}^{\mathbb{Z}}$ be given by

$$\Phi(x)_i = \min\{x_i, 0\}.$$ 

Let $X' = \Phi(\hat{X})$.

**Proposition 10.1.** The subshift $\hat{X} \subset \mathcal{A}_0^\mathbb{Z}$ is fully universal but the factor $X' = \Phi(\hat{X})$ is not universal.

**Proof.** Let

$$\mathcal{A}_2 = \{n \in \mathbb{N} : 2n \in \mathcal{A}_{+0} \} \cup \{0\}.$$ 

Let $Y \subset \mathcal{A}_2^{\mathbb{Z}}$ be the subshift consisting of strings where the number of symbols between two consecutive occurrences of 0 is divisible by 4. Then $Y$ is a mixing sofic shift. Let $\Psi : \hat{X} \to \mathcal{A}_2^\mathbb{Z}$ be given by

$$\Psi(x)_i = \lfloor |x_i/2| \rfloor.$$ 

One can check that $\Psi(\hat{X}) = Y$. Furthermore, if $y \in Y$ and $y_i = 0$ for some $i \in \mathbb{Z}$, then $y$ has a unique preimage in $\hat{X}$ under $Y$. Let

$$\hat{X}_0 = \{x \in \hat{X} : \exists i \in \mathbb{Z}, x_i = 0\}$$

and

$$Y_0 = \{y \in Y : \exists i \in \mathbb{Z}, y_i = 0\}.$$ 

So $\Psi$ induces a Borel isomorphism between $\hat{X}_0$ and $Y_0$. $Y_0 \subset Y$ and $\hat{X}_0 \subset \hat{X}$ are both dense. Also, $\mu(Y_0) = 1$ for every ergodic fully-supported invariant probability measure on $Y$. Similarly, $\mu(\hat{X}_0) = 1$ for every ergodic fully-supported invariant probability measure on $\hat{X}$. Also, $\Psi : \hat{X} \to Y$ is finite-to-one (actually at most 4 to 1), so $h(\hat{X}, S) = h(Y, S)$. Now since $Y$ is mixing sofic shift, it is fully universal. This shows that every free $\mathbb{Z}$-action with entropy smaller than $h(\hat{X}, S)$ can be realized as an invariant probability measure on $\hat{X}$ that charges every open set, so $\hat{X}$ is fully universal.

Let us prove that $X'$ is not fully universal, note that the set of points $x \in X'$ with no occurrences of 0 is precisely the subshift

$$X'' := \{x \in \mathcal{A}_0^\mathbb{Z} : x_i + x_{i+1} = 1 \pmod{2} \forall i \in \mathbb{Z}\}.$$ 

This is a non-mixing SFT with period 2. It follows that any $\mu \in \text{Prob}_c(X', S)$ with $\mu([0]_0) = 0$ has a non-ergodic square. Also,

$$h(X') \geq h(X'') = \log M.$$ 

Now if $\mu \in \text{Prob}_c(X', S)$ and $\mu([0]_0) > 0$ then

$$\mu([0]_0) \geq \frac{1}{2}.$$ 

So

$$h_\mu(X', S) \leq \frac{1}{2} \log(M) + \log(2).$$
We conclude that no probability preserving system with ergodic square and entropy between $\frac{1}{2} \log(M) + \log(2)$ and $\log(M)$ can be modeled as an invariant measure on $X'$. This concludes the proof. □

11. Mixing properties of subshifts of the three colored chessboard

Let us recall the uniform filling property for $\mathbb{Z}^d$ subshifts [50]:

**Definition 11.1.** A subshift $X \subset \mathcal{A}^{\mathbb{Z}^d}$ has the uniform filling property (UFP) if there exists $M \in \mathbb{N}$ such that for all $x, y \in X$ and $n \in \mathbb{N}$ there exists $z \in X$ such that $z|_{F_n} = x|_{F_n}$ and $z|_{\mathbb{Z}^d \setminus F_{n+M}} = y|_{\mathbb{Z}^d \setminus F_{n+M}}$.

As we mentioned earlier, Robinson and Şahin [50] have shown that if a $\mathbb{Z}^d$ subshift of finite type $X$ has the uniform filling property (UFP) (plus a condition on periodic points) then it is universal.

An early attempt to resolve the universality of the 3-coloring subshift $X_3$ we checked if it could be the case that for every $\epsilon > 0$ there exists a subshift $Y \subset X_3$ with the UFP such that $h(Y') > h(Y) - \epsilon$. The result of [50] could then have been applied to prove that any such $Y$ is universal; this would show that $X_3$ is universal. Such arguments were used by Pavlov to prove universality for subshifts of finite type with “nearly full entropy” [44]. Note that this does not automatically imply full universality. If we allow ourselves to restrict to the subaction of the shift by $(2\mathbb{Z})^d$, this argument works for the $X_3$, and in fact for any mixing hom-shift. Eventually we realized that this approach cannot be used to prove universality for three colorings, because of the following result:

**Proposition 11.2.** Suppose $d \geq 2$ and that $Y$ is a subshift of the 3-colorings subshift in $\mathbb{Z}^d$. Then $Y$ does not have the UFP.

**Remark 11.3.** In response to a question posed by M. Boyle, Quas and Şahin [47] constructed a topologically mixing $\mathbb{Z}^2$ SFT $\overline{X}$ and a number $h_0 \in (0, h(\overline{X}, S))$ such that if $Y \subset \overline{X}$ has UFP then $h(Y, S) < h_0$.

**Proposition 11.2** shows that a somewhat stronger phenomena holds for the 3-coloring subshift.

The main tool we use in this proof is the so called height cocycle or height functions associated to 3-colorings. A (real-valued) cocycle for a $\mathbb{Z}^d$ dynamical system $(X, S)$ is a function $c : X \times \mathbb{Z}^d \to \mathbb{R}$ satisfying the relation

$$c(x, \vec{i} + \vec{j}) = c(x, \vec{i}) + c(S^\vec{i}(x), \vec{j}) \text{ for every } x \in X, \vec{i}, \vec{j} \in \mathbb{Z}^d. \tag{238}$$

Note that (238) implies that $c(x, \vec{0}) = 0$. The height cocycle $c : X_3 \times \mathbb{Z}^2 \to \mathbb{Z}$ is uniquely defined by the following properties:

$$x_i - x_j = c(x, \vec{i}) - c(x, \vec{j}) \mod 3 \text{ whenever } \vec{i} \text{ is adjacent to } \vec{j},$$

and

$$|c(x, \vec{e}_j)| = 1 \text{ for all } x \in X_3 \text{ and } 1 \leq j \leq d.$$

In other words, for every $x \in X_3$, $\vec{x} = (c(x, \vec{i})|_{\mathbb{Z}^2}) \in \mathbb{Z}^{\mathbb{Z}^d}$ is the unique graph homomorphism from the standard Cayley graph of $\mathbb{Z}^d$ to the standard Cayley graph of $\mathbb{Z}$ that satisfies $x_i - x_j = \vec{x}_i - \vec{x}_j \mod 3$ for every $\vec{i} \in \mathbb{Z}^d$ and $\vec{x}_0 = 0$. We will use the following two facts about these cocycles. If for some connected set $A \subset \mathbb{Z}^d$, $x, x' \in X_3$ are such that $x|_A = x'|_A$ then for all $\vec{i}, \vec{j} \in A$

$$c(S^\vec{i}(x), \vec{j} - \vec{i}) = c(S^\vec{i}(x'), \vec{j} - \vec{i}). \tag{239}$$

Lastly

$$|c(x, \vec{i})| \leq \|\vec{i}\|_1 \forall x \in X_3 \text{ and } \vec{i} \in \mathbb{Z}^d. \tag{240}$$

See [12, 19, 54] for details and further references.

**Proof of Proposition 11.2.** Call a subshift $Y \subset X_3$ quasiflat if

$$\sup \{c(y', \vec{i}) - c(y, \vec{i}) : \vec{i} \in \mathbb{Z}^d, y, y' \in Y\} < \infty. \tag{241}$$

We claim that if $Y$ is not quasiflat, then $Y$ does not have UFP. Indeed, suppose that $Y$ is not quasiflat. Fix $M \in \mathbb{N}$. Then there exists $y^{(1)}, y^{(2)} \in Y$, $n \in \mathbb{N}$, $\vec{i}, \vec{j} \in F_n \setminus F_{n-1}$ such that

$$c(S^{\vec{i}}(y^{(1)}), \vec{j} - \vec{i}) - c(S^{\vec{i}}(y^{(2)}), \vec{j} - \vec{i}) > 4M.$$
Now find \( \vec{t}, \vec{j} \in F_{n+M} \setminus F_{n+M-1} \) such that \( \|\vec{t} - \vec{t}\|_1 \leq M \) and \( \|\vec{j} - \vec{j}\|_1 \leq M \). Then it follows from (238) and (240) that
\[
|c(S^2(y^{(k)}), \vec{j} - \vec{t}) - c(S^2(y^{(k)}), \vec{j} - \vec{t})| \leq 2M \text{ for } k = 1, 2.
\]

Thus
\[
c(S^2(y^{(1)}), \vec{j} - \vec{t}) - c(S^2(y^{(2)}), \vec{j} - \vec{t}) > 2M.
\]

Suppose there exists \( y \in Y \) such that \( y |_{F_n} = y^{(1)} |_{F_n} \) and \( y |_{\mathbb{Z}^d \setminus F_{n+M-1}} = y^{(2)} |_{\mathbb{Z}^d \setminus F_{n+M-1}} \). Then by (239)
\[
c(S^2(y), \vec{j} - \vec{t}) = c(S^2(y^{(1)}), \vec{j} - \vec{t})
\]
and
\[
c(S^2(y), \vec{j} - \vec{t}) = c(S^2(y^{(2)}), \vec{j} - \vec{t}).
\]

We conclude that
\[
c(S^2(y), \vec{j} - \vec{t}) - c(S^2(y), \vec{j} - \vec{t}) > 2M,
\]
contradicting (238) and (240). This shows that a subshift \( Y \) that is not quasiflat does not have the UFP.

Now suppose \( Y \subset X_3 \) is quasiflat. We will show that in this case that \( Y \) cannot even be topologically mixing, and in particular does not have UFP.

Since \( Y \) is a quasiflat, for every \( y \in Y \) the map \( \phi_y : \mathbb{Z}^d \to \mathbb{R} \) given by \( \phi_y(\vec{i}) = c(y, \vec{i}) \) is a quasimorphism on the group \( \mathbb{Z}^d \), in the sense that
\[
\sup_{\vec{i}, \vec{j} \in \mathbb{Z}^d} |\phi_y(\vec{i} + \vec{j}) - \phi_y(\vec{i}) - \phi_y(\vec{j})| < \infty.
\]

Then by a well known simple argument the homogenized quasimorphism \([34, 9]\) \( \overline{\phi} : \mathbb{Z}^d \to \mathbb{R} \) given by
\[
\overline{\phi}(\vec{i}) = \lim_{n \to \infty} \frac{\phi(n \cdot \vec{i})}{n}
\]
is a group homomorphism that satisfies \( |\overline{\phi}(\vec{i})| \leq \|\vec{i}\|_1 \). Since \( Y \) is quasiflat it follows that \( \overline{\phi} \) is independent of the choice of \( y \) and
\[
D_Y = \sup_{y \in Y, \vec{i} \in \mathbb{Z}^d} |c(y, \vec{i}) - \overline{\phi}(\vec{i})| < \infty.
\]

We can find \( y \in Y \) and \( \vec{i}_0 \in F_n \) for some \( n \in \mathbb{N} \) such that
\[
|c(y, \vec{i}_0) - \overline{\phi}(\vec{i}_0)| > D_Y - \frac{1}{100}.
\]

Let
\[
L_Y = \{ \vec{j} \in \mathbb{Z}^d : |\overline{\phi}(\vec{j})| \leq 2 \text{ and } \vec{j} \text{ is odd} \}.
\]

Since \( \overline{\phi} \) is a group homomorphism there exists \( \vec{s} \in \mathbb{R}^d \) such that \( \overline{\phi}(\vec{i}) = \langle \vec{i}, \vec{s} \rangle \). By taking integer approximations of the zeros of the inner product \( \langle \cdot, \vec{s} \rangle \) we get that \( L_Y \) is an infinite set. Since \( Y \) is mixing, we get that for large enough \( \vec{j}_0 \in L_Y \), there exists \( z \in Y \) such that
\[
z|_{F_n} = y|_{F_n} \text{ and } S^{\vec{s}_0}(z)|_{F_n} = y|_{F_n}.
\]

Since \( \vec{j}_0 \) is odd it follows that \( |c(z, \vec{j}_0)|, |c(S^{\vec{s}_0}(z), \vec{j}_0)| \geq 3 \). Assume without the loss of generality that
\[
c(z, \vec{i}_0) - \overline{\phi}(\vec{i}_0) > D_Y - \frac{1}{100};
\]
the proof is similar in the other case. Since \( F_n \) is connected we have that either
\[
c(z, \vec{j}_0) = c(S^{\vec{s}_0}(z), \vec{j}_0) \geq 3 \text{ or } c(z, \vec{j}_0) = c(S^{\vec{s}_0}(z), \vec{j}_0) \leq -3.
\]

If the former is true then
\[
c(z, \vec{i}_0 + \vec{j}_0) = c(z, \vec{j}_0) + c(S^{\vec{s}_0}(z), \vec{i}_0) \geq 3 + \overline{\phi}(\vec{i}_0) + D_Y - \frac{1}{100}
\]
while the latter implies
\[
c(S^{\vec{s}_0}(z), \vec{i}_0 - \vec{j}_0) = c(S^{\vec{s}_0}(z), -\vec{j}_0) + c(z, \vec{i}_0) \geq 3 + \overline{\phi}(\vec{i}_0) + D_Y - \frac{1}{100}.
\]
But the choice of $D_Y$ and that $\vec{j}_0 \in L_Y$ shows
\[
c(z, \vec{i}_0 + \vec{j}_0) \leq \overline{\phi}(\vec{i}_0 + \vec{j}_0) + D_Y \leq 2 + D_Y + \overline{\phi}(\vec{i}_0) \text{ and} \\
c(S^n z, \vec{i}_0 - \vec{j}_0) \leq \overline{\phi}(\vec{i}_0 - \vec{j}_0) + D_Y \leq 2 + D_Y + \overline{\phi}(\vec{i}_0)
\]
contradicting both cases. \hfill \Box

12. FURTHER QUESTIONS AND COMMENTS

We conclude with some comments and further questions:

(1) Ergodic vs. almost-Borel universality: Is there a general condition under which ergodic universality of a compact Borel system $(X, S)$ implies almost Borel universality? Specifically: Suppose $(X, S)$ and $(Y, T)$ are compact dynamical systems and that for every $\mu \in \text{Prob}(Y, T)$ the system $(Y, \mu, T)$ can realized as an invariant measure on $(X, S)$. Is there an almost-Borel embedding of $(Y, T)$ into $(X, S)$? It is not difficult to find a counterexample if we require only that $(X, S)$ and $(Y, T)$ be Borel dynamical systems and only assume embedding for ergodic measures.

(2) Borel vs. “almost-Borel” universality: Is it possible to strengthen Theorem 5.1 and prove that systems satisfying the assumptions contain a Borel copy of any free system of sufficiently low entropy? In other words, to what extent is it necessary to disregard a null set? Mike Hochman proved that any Borel $\mathbb{Z}$-system with no invariant probability measure admits a 2-set generator [28], and thus deduced Borel universality for mixing $\mathbb{Z}$-SFTs and more. Mike Hochman and Brandon Seward informed us that they have managed to extend this result to Borel actions of arbitrary countable groups.

(3) Dimers in higher dimensions and rectangular tiling shifts: Are $\mathbb{Z}^d$-dimers fully universal when $d > 2$? We proved that they are $t$-universal for some $t > 0$. It suffices to show that the number of perfect domino tilings of an $F_n$ is $e^{h(F_n) + o(|F_n|)}$ as $n \to \infty$. For this we invoked some classical results based on “hard” computations, in particular on Kastelyn’s formula for the number of perfect matchings of a finite planar graph. Is there a “soft method” to deduce a similar result in greater generality?

(4) Universality for $\mathbb{R}^d$-actions: To what extend do our results and methods apply to $\mathbb{R}^d$ actions? Does specification imply universality for these systems? Quas and Soo obtained results along theses lines for certain $\mathbb{R}$-actions [48]. See also the Kra-Quas-Sahin version of Alpern Lemma for $\mathbb{R}^d$-actions [35].

(5) Does specification imply universality for actions of more general countable groups? In this paper we do not directly deal with action of groups beyond $\mathbb{Z}$, but most of the ergodic theoretic machinery used in our proof (Rokhlin towers, Shannon-McMillan theorem) is available for countable amenable groups. In view of Seward’s version of Krieger generator theorem for arbitrary countable groups [56], it is tempting to ask the question beyond the amenable setting.

(6) Realizing measure preserving actions and Borel actions as continuous actions on a manifold: We now know that any free measure preserving $\mathbb{Z}$-action is isomorphic to some continuous homeomorphism of the 2-torus (as a measure preserving dynamical system, with respect to Lebesgue measure). What about actions of more general groups? For instance, what about $\mathbb{Z}^2$-actions? Actions of the free group? $\mathbb{R}$-flows? (Note that there are no free continuous $\mathbb{R}^{d+1}$-flows on a $d$-dimensional manifold).

(7) Non compact models: Some natural (non-compact) Polish dynamical systems have been shown to be universal. For instance, the space of entire functions on $\mathbb{C}$ is an example of a non-compact Polish $\mathbb{R}^2$ dynamical system that is universal (in the ergodic sense) [58]. Can this be interpreted in the context of our results?

(8) Universality of algebraic actions: Which algebraic actions (continuous actions on a compact group that preserve the group structure) are universal? For $\mathbb{Z}$-actions we know that ergodicity suffices.

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