Remarks on Thermal Strings outside Black Holes

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We discuss the semiclassical approximation to the level density of (super) strings propagating in non-compact coset manifolds $G/H$. We show that the WKB ansatz agrees with heuristic red-shift arguments with respect to the “exact” sigma-model metric, up to some deviations from minimal coupling, parametrized by the dilaton. This approximation is used to study thermal ensembles of free strings in black holes, with the “brick wall” horizon regularization of ’t Hooft. In two dimensions the entropy diverges logarithmically with the horizon thickness, and a local Hagedorn transition occurs in higher dimensional models. We also observe that supersymmetry improves the regularity of strings at the horizon.
1. Introduction

Some interesting proposals have been put forward recently about the role of String Theory in the understanding of black hole thermodynamics and the information problem \[1\]. Traditionally, it was assumed that string dynamics was essential only near space-time singularities. However, as stressed by ’t Hooft, planckian physics could be required to understand horizons, and this clearly opens the applications of String Theory as a particular model of ultra-short dynamics. From a technical point of view, perhaps the simplest problem one can attempt is that of the one loop entropy, which physically corresponds to the total entropy of the Hawking radiation by free fields, and appears as the first quantum correction to the Hawking-Beckenstein entropy in Euclidean Quantum Gravity. This quantity is known to be divergent in Field Theory \[2\],

\[ S_{\text{qu}} \sim \frac{A_H^{(d-2)}}{\epsilon^{d-2}} \left( \frac{T}{T_H} \right)^{d-1} \]  \hspace{1cm} (1.1)

where \( A_H \) is the horizon area, \( \epsilon \) is a short proper-distance cutoff, and \( T_H \) is the Hawking temperature (in two dimensions there is a logarithmic divergence). It is commonly assumed that this divergence is equivalent to the information problem, since it leads to an indefinite storage of information at the horizon. The purpose of a short distance analysis (i.e. strings) is to provide a physical definition of the cutoff \( \epsilon \). There are at least three different derivations of (1.1), which highlight on different aspects of the possible string generalization.

(i) The most basic definition regards \( S_{\text{qu}} \) as the so called “geometric entropy” \[3\]. The entropy comes from the mixed state obtained by tracing over local degrees of freedom in a certain spatial region, and divergences appear associated to the sharp character of the boundary. In the black hole case one traces over modes with support on the horizon at late times. In String Theory, it is intuitively clear that no sharp distinction exists between “inside” and “outside” at the Planck scale, due to the finite size of the string. However, a rigorous construction of the density matrix as a trace over configuration space is extremely difficult, because fixed loops in the target space are off-shell states in String Theory.

(ii) In the euclidean formal construction we derive \( S_{\text{qu}} \) from the euclidean free energy, which in turn is computed as a path integral over the euclidean section of the black hole, with periodic identification modulo \( \beta \). There is a conical singularity at the horizon with deficit angle \( \beta - 2\pi \), responsible for the divergence in this formalism. In String Theory
we are instructed to compute the torus path integral with the euclidean section as target space. Although exact string backgrounds with black hole interpretation are known, they are based on non-compact conformal field theories, which makes explicit computations rather difficult. An expression for the free energy of the two dimensional string black hole at $\beta = 2\pi$ was proposed in ref. [4]. However, it is very unclear how to include the conical singularity while keeping conformal invariance and calculability.

(iii) In a more elementary derivation, one may assume a thermal ensemble as results, for example, from Hartle-Hawking boundary conditions, and directly compute the partition sum. A reflecting boundary condition at the horizon is used to cut off the rising local temperatures, $T/\sqrt{-g_{00}}$ (in momentum space the horizon divergence appears disguised as an infrared divergence [3]). This is the so-called “brick wall” model of ’t Hooft [4]. Clearly, the Dirichlet boundary condition at the horizon is rather naive from the physical point of view. However, it is very appropriate for counting states, because it leads to a well defined thermal ensemble. In the brick wall model, equation (1.1) follows easily from local red-shift arguments. In String Theory such heuristic arguments are suspicious because of the “stretching effect” of [1], which suggest highly non-local string configurations near the horizon, depending on the frame. One of the results of this work is to provide a formal discussion of the red-shift arguments.

In this letter we explore the string analog of (iii), which is the simplest of the three methods (i)-(iii) (It is important to keep in mind though that nothing ensures a priori the equivalence of the three approaches in the string case). The WKB approximation used in [2] for the case of a scalar field in the Schwarzschild geometry is generalized to strings in a large class of backgrounds. This is the subject of the next section. In the second and third sections we apply this approximation to “stringy” black holes.

2. WKB estimates of String Partition sums

Consider the canonical partition function of a free string gas:

$$Z(\beta) = \text{Tr} \mathcal{H} e^{-\beta: H_0:}$$

where $\mathcal{H}$ is the second quantized Hilbert space built over the tree level string spectrum (BRST invariant states), $: H_0 :$ is the free Hamiltonian in this space, and the normal
ordering means that the (finite) vacuum energy has been subtracted. Standard Fock algebra leads to:

\[-\log Z(\beta) = \sum_f (-)^F \text{Tr} \mathcal{H}_f \log (1 - (-)^F e^{-\beta \omega_f}) \quad (2.1)\]

here \(f\) labels the different fields (= physical states) in the string spectrum and \(\mathcal{H}_f\) stands for the first quantized Hilbert space of each field (space-time degrees of freedom). The bosonic or fermionic statistics is incorporated by the sign \((-)^F = \pm 1\). This representation of the one loop thermal free energy is general for any background with a timelike isometry, and is usually termed as the “analog model” because, to this order (no interactions), the string is equivalent to the collection of fields in its spectrum. Using standard zeta function identities, the integral representation of the logarithm, and Poisson resummation, one can easily derive from (2.1) a proper time representation of the free energy as:

\[\beta F(\beta) = -\log Z(\beta) = \beta \int_0^{\infty} \frac{ds}{s^2} \sum_f \Lambda_f(s) \sum_{n\neq 0} (-)^{nF_f} e^{-\frac{\beta^2 n^2}{4\pi s}} \quad (2.2)\]

with a vacuum energy integrand given by the general formula:

\[\Lambda_f(s) = \frac{(-)^{F_f+1}}{2\pi} \sqrt{\frac{s}{2}} \text{Tr} \mathcal{H}_f e^{-\frac{s}{2} \omega_f^2} \quad (2.3)\]

This form of the free energy is appropriate for deriving modular invariant expressions which agree, in all solved flat backgrounds, with the genus one path integral computation [6]. In this case, the proper time integral is restricted to the fundamental domain of the genus one modular group, thus avoiding the ultraviolet region (this different integration domain transforms into (2.2) after summing over the thermal winding modes). Regarding these manipulations, it is important to keep in mind that the vacuum energy has been subtracted (there is no \(n = 0\) term in (2.2)). only the renormalized free energy of the string equals the sum of the field free energies (obviously, this is not so for the vacuum energy since the former is U.V. finite while that of the fields is U.V. divergent). In the brick wall model it is the renormalized free energy the one that shows horizon divergences. The equivalence of the analog model and the path-integral formula for exact black hole backgrounds is an interesting open problem which we shall not address here.

The idea of the semiclassical approximation is to relate the energies \(\omega_f\) to a Schrödinger problem, so that we can evaluate each trace over \(\mathcal{H}_f\) by WKB. The frequencies \(\omega_f\) are
determined by the mass-shell equation of the string ($L_0$ condition). For string models based on current algebras (gauged WZW models), one can introduce the so-called “exact” metric and dilaton backgrounds [7], and write the mass-shell equation as:

$$\left\{ -\frac{1}{e^{-2\Phi}} \sqrt{-g} \partial_\alpha e^{-2\Phi} \sqrt{-g} g^{\alpha\beta} \partial_\beta + M_f^2 \right\} \Psi_f = 0 \quad (2.4)$$

$\Psi_f$ is the vertex operator for the string state $|f\rangle$ and the second order differential operator acts on the zero mode part of $\Psi_f$. The energies $\omega_f$ are defined by $\partial_t \Psi_f = -i\omega_f \Psi_f$, once the “time” coordinate is singled out in the exact background. $M_f$ is the mass of the state, and it comes entirely from the current algebra.

Equation (2.4) is exact for a large class of models. To be as general as possible, let us consider non-compact cosets of Kazama-Suzuki type $G/H$. We have a super Kac-Moody algebra generated by chiral currents $j^a(z)$ and fermions $\psi^a(z)$, $a = 1, ..., \dim G$, with energy-momentum tensor,

$$T(z) = \frac{1}{k + g} \eta_{ab} : (j^a j^b - \psi^a \partial \psi^b) :$$

where $\eta_{ab}$ has Minkowski signature (one time), and $\hat{k} = k + g = k + c_2(G)$ is the level of the corresponding $N = 1$ super-WZW model. To study the mass-shell condition it is convenient to separate the zero mode of the chiral Hamiltonian as,

$$L_0^G = \frac{1}{k + g} j_0^a j_0^b \eta_{ab} + N_B^G + N_F^G \quad (2.5)$$

where $N_{B,F}$ are the bosonic/fermionic oscillator number operators:

$$N_B^G = \frac{2}{k + g} \sum_{n > 0} j_{-n}^a j_n^b \eta_{ab} \quad N_F^G = \frac{2}{k + g} \sum_{r > 0} r \psi_{-r}^a \psi_r^b \eta_{ab}$$

the index $r$ runs over integers in the Ramond sector and half-integers in the Neveu-Schwartz sector. Due to the relations $[L_0, j^a_n] = nj^a_n$ and $[L_0, \psi^b_r] = r\psi^b_{-r}$, $N_B$ and $N_F$ are just mode numbers, $\sum n, \sum r$ when acting on Fock space states of the form $\prod_j j^a_{-n_j} \prod_l \psi^b_{-r_l} |R\rangle$, where $|R\rangle$ is the zero mode or tachyon state. It is labeled by a representation of the group $G$ (in the Ramond sector, it also carries the appropriate representation of the $SO(\dim G)$ Clifford algebra). The zero mode term in (2.5) acts on the tachyon states as the quadratic Casimir in the $R$ representation.

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The mass-shell condition in the coset model is obtained by combining left and right movers,

\[ 0 = (L_0 + \overline{L}_0)^{G/H} - a - \overline{a} = \left[ (L_0 + \overline{L}_0)^G - (L_0 + \overline{L}_0)^H \right]_{\text{zero mode}} + M^2 \]  

(2.6)

where \( a, \overline{a} \) are normal ordering constants (in the standard picture, \( a = 1 \) in the bosonic theory, and \( a = 1/2, 0 \) in the NS and R sector respectively). The mass formula,

\[ M^2 = (N + \overline{N})^G - (N + \overline{N})^H - (a + \overline{a}) \]

must be supplemented with physical state conditions (oscillator “transversality”), plus level matching and GSO constraints. This definition of mass is natural even if the target metric does not approach Minkowski at infinity because, under the current rescaling \( j^a \rightarrow \sqrt{k/2} j^a \), it goes over the usual definition in the flat \( k \rightarrow \infty \) limit. The important point is that the zero mode part of eq. (2.6) can be realized as a second order differential operator, when acting on the space of functions over the \( G/H \) coset manifold. It is just a linear combination of Laplacians over the \( G \) and \( H \) group manifolds with coefficients depending on the Kac-Moody levels (the left and right normalizations may differ in heterotic models). The functions over the coset are \( H \)-invariant: \( (j + \overline{j})_H \Psi = 0 \), and they come labeled by the left-right zero-mode representations \((R, \overline{R})\). Thus, they represent the tachyon part of \( \Psi_f \) and we recover equation (2.4). Examples include \( G_{-k}/H_{-k} \) bosonic of type II strings, and heterotic models with bosonic \( G_{-k}/H_{-k} \times \text{(Gauge)} \) right movers and \( N = 1 \) \( G_{-k}/H_{-k+g-h} \) left movers (for a review, see [8]).

It is convenient to write (2.4) as a Klein-Gordon equation in the exact metric, upon rescaling \( \Psi \rightarrow e^{-\Phi} \Psi \):

\[ \left\{ -\nabla_g^2 + \widetilde{M}_f^2 \right\} \Psi_f \equiv \left\{ -\nabla_g^2 + M_f^2 + (\nabla_g \Phi)^2 - \nabla_g^2 \Phi \right\} \Psi_f = 0 \]  

(2.7)

In this way we summarize all “stringy” modifications to minimal coupling in the effective mass \( \widetilde{M}_f \), which includes an additive renormalization depending on the exact dilaton. For black hole applications we are interested in rotationally invariant situations, where the exact metric is parametrized as:

\[ ds^2 = -\lambda(r)dt^2 + \frac{dr^2}{\mu(r)} + r^2d\Omega_{d-2}^2 \]
One can then factorize the angles and rewrite (2.4) as a Schrödinger equation for the frequencies $\omega_f$:

$$\left\{ -\frac{1}{2} \frac{d^2}{dx^2} + V^f_\ell (x) \right\} \Psi_{n\ell}(x) = \frac{\omega_{n\ell}^2}{2} \Psi_{n\ell}(x) \quad (2.8)$$

with the following effective potential:

$$V^f_\ell = \frac{d - 2}{4} \gamma(r) \left( \frac{\gamma'(r)}{r} + \frac{d - 4}{2r^2} \gamma(r) \right) + \frac{\lambda(r)}{2} \left( \frac{C_\ell}{r^2} + \tilde{M}^2_f \right) \quad (2.9)$$

here $\gamma(r) = \sqrt{\lambda(r)\mu(r)}$, $\ell$ denotes the angular momenta, $C_\ell$ is the quadratic Casimir of the $SO(d - 1)$ rotation group, and prime means $d/dr$. The variable $x$ appearing in (2.8) becomes simply a “tortoise” radial coordinate $x = \int dr/\gamma(r)$. In the presence of regular horizons, $\lambda(r) \sim \lambda_0 (r - r_0)$, $\mu(r) \sim \mu_0 (r - r_0)$, and for non-singular dilaton fields, we have the usual properties [4]: the horizon appears at $x = -\infty$ and the potential vanishes exponentially there, $V_\ell (x) \sim C \exp(4\pi T H x)$. As a result, the positive spectrum $\omega_f^2$ remains continuous even after enclosing the system in a large box (located at $x_+ \gg 1$), and a second “wall” (at $x_- \ll 0$ near the horizon), is needed to avoid infrared problems. We thus conclude that the string black hole still allows a continuous spectrum of states as long as the exact metric has horizons and the dilaton field is well behaved there.

Some comments are in order regarding Dirichlet walls in String Theory. It is known that a Dirichlet condition on the full coordinate field $X^\mu (z, \bar{z})$ leads to a divergent vacuum energy [9]. This is not of direct relevance here because we are conventionally subtracting vacuum energies in discussing thermodynamics. On the other hand, $X^\mu (z, \bar{z})$ is not a bona fide physical operator and it is not surprising that pathologies appear. Instead, we impose Dirichlet conditions on the equation (2.4). The result is a certain projection (discretization) of the spectrum of zero modes, similar to the case of toroidal compactifications, although more complicated. In this way we preserve the conformal structure and world-sheet current algebra.

So far all formulas were exact, but the WKB approximation proceeds from (2.8) by simply using the semiclassical formula for the level density when computing the space-time trace:

$$\text{Tr} \mathcal{H}_f \rightarrow \frac{1}{\pi} \sum_\ell \int d \left( \int dx \sqrt{\omega^2 - 2V^f_\ell (x)} \right)$$

For example, the result for the vacuum energy integrand per field degree of freedom (2.3) reads:

$$\Lambda^{WKB}_f (s) = \frac{(-)^{F_f + 1}}{(2\pi)^2} \sum_\ell \int_{x_-}^{x_+} dx \ e^{-s \pi V^f_\ell (x)}$$
In this form the infrared problems become obvious when the effective potential vanishes in any of the limits $x_{\pm} \to \pm\infty$, and this occurs at the horizon for any mass (in terms of the vertex operators $\Psi_f$, they oscillate wildly as the target fields approach the horizon). Since $x$ is interpreted as a radial coordinate, we see that the WKB method naturally produces integrated densities over space. To be more explicit, if we further approximate the angular momentum sum by a smooth integral we obtain

$$\Lambda_{f}^{WKB}(s) = \frac{(-)^{F_f+1}}{\pi^{d/2} 2^{1+d/2}} \int d(\text{Vol}) \ (\sqrt{\lambda})^{1-d} \ s^{1-d/2} \ e^{-s \pi V^f_f} \ (2.10)$$

where $d(\text{Vol})$ is the spatial volume element and $V^f_f$ denotes the s-wave effective potential that results from (2.9) by putting $C_{\ell} = 0$. This expression resembles the flat space one with a position dependent “mass” given by $\sqrt{2V^f_f}$. The red-shift factor $(\sqrt{\lambda})^{1-d}$, when combined with the appropriate power of $\beta$, yields the natural scaling on the red-shifted temperature, and also blows up at the horizon. So, it seems that the horizon instabilities add up for the different fields in the string spectrum. In the next section we argue that this is not exactly true.

3. Local Hagedorn singularity

According to equation (2.7), red-shift arguments in the exact background should give good estimates of the thermodynamic densities, up to the dilaton terms that deviate from minimal coupling to the metric. It is easy to see that WKB is the formal counterpart of red-shift arguments. By directly plugging the WKB ansatz in (2.1) one obtains after some manipulations,

$$\beta F^{WKB}_{f} = -\frac{\Gamma(d/2)}{\pi^{d/2} \Gamma(d)} \int d(\text{Vol}) \ \tilde{\beta}^{1-d} \sum_{f} \int_{\beta \sqrt{2V^f_f}}^{\infty} \frac{dz (z^2 - 2\beta^2 V^f_f)^{d-1}}{e^{z} - (-)^{F_f}} \ (3.1)$$

where we have estimated again the discrete sum over angular momenta ($\ell$) by a smooth integral. We see that, up to the back scattering effects (first term in formula (2.9)), all local dependence goes in the combination $\tilde{\beta} M_f$, for $\tilde{\beta} = \beta \sqrt{\lambda}$ the red-shifted inverse temperature. Each term in (3.1) has now the form of the flat space free energy density of free bosons/fermions with a local red-shifted temperature and a local mass. Formally, the WKB approximation is good as long as the effective potential (2.3) is a smooth function (it is exact in flat backgrounds). However, for a fixed brick-wall cutoff $x_-$, the WKB
approximation gets worse as we consider higher masses or angular momenta in the sum (3.1), because the effective potential becomes rather steep near \( x^− \). Nevertheless, the WKB method gives correctly the leading exponential supression at large masses, since this follows basically from scaling and dimensional analysis (for a discussion of these issues see [10]).

Equation (3.1) is very appropriate for a large mass expansion. Assuming that the dilaton field is regular and bounded, and defining the density of levels \( \rho(M) = \sum_f \delta(M - M_f) \) we have the asymptotic estimate,

\[
\beta F_{WKB} \to \frac{-\Gamma(d/2)}{\pi^{d/2}} \int d(Vol) \tilde{\beta}^{1-d} \sum_M \rho(M) e^{-\tilde{\beta}M} + O\left(\frac{1}{M}\right)
\]

(as previously stated, the constants in front may be wrong, but this will not change the conclusion). On general grounds, we expect the density of states to diverge exponentially for string theories above two dimensions, \( \rho(M) \sim \exp(M \beta_{Hag}) \). This is a physical requirement because the string spectrum of any black hole solution should approach the flat space spectrum in the asymptotically flat region. As a result, there is a local Hagedorn transition where the local red-shifted temperature reaches the critical value (the same singularity follows form (2.2) and (2.10) as a pole at \( s = \infty \)). This means that we are unable to discuss the horizon physics of “stringy” black holes in interesting dimensions, unless we understand first the old Hagedorn problem.

An interesting example is Rindler space, which corresponds formally to extremelly massive black holes. We can apply the formalism of the previous section because the world-sheet conformal field theory is related to the Minkowkian one by a field redefinition. This means that current algebra dependent quantities like the mass formula stay the same and the mass-shell equation is given by (2.4) with \( g_{\alpha \beta} \) the Rindler metric and \( \Phi = 0 \). The associated Schrödinger problem (2.8) becomes a Bessel equation with effective potential \( V^f(x) = (k_\perp^2 + M_f^2)\lambda(x)/2 \), where \( \lambda(x) = e^{4\pi T_H x} \) and \( k_\perp \) is the transverse momentum. The horizon regularization is taken again as a reflecting wall, which discretizes the frequency spectrum (note that our horizon cutoff differs from the one in [11], where a deformation in the Rindler mapping was used instead, resulting in a deformed mass formula). Integrating the transverse momenta we get for the free energy:

\[
\beta F_{WKB}^{Rin} = -\int_0^\infty \frac{ds}{s^{1+d/2}} \sum_f e^{-\tilde{\beta}M_f^2} \int d(Vol) \tilde{\beta} \sum_{n \neq 0} \frac{(-)^{(n+1)}F_f}{2(2\pi)^{d/2}} e^{-\frac{\tilde{\beta}^2 n^2}{2s}}
\]
This equals the Minkowski formula with $\beta$ replaced by the local red-shifted inverse temperature $\tilde{\beta}(x) = \beta \sqrt{\lambda(x)}$. Since $M_f$ is the Minkowskian mass, a modular invariant expression follows easily using the tricks in [3]. Focusing on the bosonic case to simplify notation, the WKB one-loop free energy of strings in Rindler space takes the form

$$\beta F^{WK}_B = \int_{\mathcal{F}} \frac{d^2 \tau}{(\text{Im} \tau)^2} \Lambda(\tau) \int d(\text{Vol}) \left( \Theta_{\tilde{\beta}}(\tau) - 1 \right) \tilde{\beta}$$

(3.2)

Here $\Lambda(\tau)$ is the flat space modular invariant cosmological constant integrand, $\mathcal{F}$ is the genus one fundamental region of the modular group, and $\Theta_{\tilde{\beta}}(\tau)$ is the well known thermal theta function, coming from the winding modes of the torus cycles onto the thermal circle of length $\beta \sqrt{\lambda(x)}$ at each point:

$$\Theta_{\tilde{\beta}}(\tau) = \sum_{m,n = -\infty}^{+\infty} e^{-\frac{\tilde{\beta}^2}{2\pi \text{Im} \tau} |m \tau + n|^2}$$

Possible extra compactified dimensions are already taken into account in $\Lambda(\tau)$, and supersymmetric generalizations are straightforward by changing $\beta \to \tilde{\beta}$ in the Minkowskian densities. Formula (3.2) is one of the main results of this paper. It shows that modular invariant, U.V. finite expressions can be derived in Rindler space in a well defined approximation. It also shows the Hagedorn problem as the space integral approaches the horizon, although in this case we can speculate about the high temperature phase using the duality properties of the theta function. The free energy density close to the horizon ($\tilde{\beta} \to 0$) behaves as

$$\beta f(x \to -\infty) \to \text{const.} \times \tilde{\beta}^{-1} \int_{\mathcal{F}} \frac{d^2 \tau}{(\text{Im} \tau)^2} \Lambda(\tau)$$

We see the typical two-dimensional Atick-Witten form with a coefficient given by the vacuum energy density of the string in flat space. For bosonic strings this is I.R. divergent due to the zero temperature tachyon, and for supersymmetric strings we expect the cosmological constant to vanish. As a result, perturbative String Theory is regular at the horizon precisely in the supersymmetric case, assuming of course that the high temperature phase given by duality has any physical significance (which is questionable, for example, the canonical entropy is negative in this phase). This behavior is specific of the duality properties of String Theory, since supersymmetric field theories are still ill defined at the horizon: using the identity $\beta F_F(\beta) = \beta F_B(\beta) - 2\beta F_B(2\beta)$ between the fermionic/bosonic free energies per degree of freedom, and the asymptotics in (3.1), it is easy to see that the horizon divergence proportional to $\int d(\text{Vol}) \tilde{\beta}^{1-d}$ never cancels between a finite number of bosons and fermions.
4. Two dimensional black hole

The $SL(2, R)/U(1)$ black hole is an interesting example for many reasons. The only propagating degree of freedom is a single tachyon mode (discrete states are global modes and we do not expect them to drive horizon divergences). In addition, the WKB approximation is reliable because it only affects the massless tachyon, and there is no Hagedorn transition. The interesting question to answer is whether the “stringy” dilaton effects of manage to smooth out the horizon instability. The exact metric and dilaton fields read:

$$ds^2 = \frac{k-2}{2} (dr^2 - \gamma^2(r) dt^2)$$

$$\Phi(r) = \frac{1}{2} \log \frac{\gamma(r)}{\sinh r} , \quad \gamma(r) = 2 \left( \coth \frac{r}{2} - \frac{2}{k} \right)^{-1/2}$$

Tachyon vertex operators are given in terms of hypergeometric functions. In the axially gauged model they can be expressed as matrix elements $\Psi_{\ell,\omega}(r, t) = \langle \ell, \omega | g(r, t) | \ell, -\omega \rangle$ in the continuous representations of $SL(2, R)$, $\ell = -1/2 + i \lambda$. The mass-shell condition $L_0 = 1$ leads to $\omega^2 = 9 \lambda^2$ at the critical value $k = 9/4$, and $\lambda$ may be interpreted as a radial momentum. After the appropriate rescalings, the differential representation of the mass-shell equation is,

$$\left\{ -\frac{1}{2} \frac{d^2}{dx^2} + \frac{k-2}{2} \gamma^2(x) \left[ (\nabla \Phi)^2 - \nabla^2 \Phi - 2 \right] \right\} \Psi_{\omega} = 2\omega^2 \Psi_{\omega}$$

where we have changed variables to tortoise coordinates $x = \int dr/\gamma(r)$ and $\omega$ differs from the previous sections by a factor of 2 (note that this eigenvalue problem is not the same as those in $[7]$). The effective potential is completely determined by dilaton effects, and has the form:

$$V_{\text{eff}}(x) = \frac{\gamma^2(x)}{4k} \left( \frac{1}{\sinh^2 r} \frac{\gamma^2(x)}{4k} \left( 1 + \frac{\gamma^2(x)}{4k} \right) + \frac{k-2}{8} R(x) \right)$$

here $R(x)$ is the scalar curvature (the only remaining term in the “low energy” $k \to \infty$ limit). Again, this potential is regular at the horizon and vanishes exponentially with the tortoise coordinate $V_{\text{eff}} \sim C e^{4\pi T_H x}$. Thus, the dilaton fails to provide a “wall” at the horizon and the $x_-$ cutoff must be imposed by hand in order to prevent infrared problems. This is achieved by setting $\Psi_{\omega_n}(r_\pm, t) = 0$, which produces the desired discretization of the spectrum. An interesting point is that a generic choice of the inner and outer walls $r_\pm$...
projects out the discrete states, because in general $\omega_n(r_{\pm})$ falls out of the discrete states list. This is a very natural phenomenon, which occurs in the $c = 1$ compactified model as well (at non-rational multiples of the self-dual radius $R = 1/\sqrt{2}$ the spectrum reduces to tachyons and the zero momentum discrete states such as $\partial X\overline{X}$).

The WKB analysis along the lines of the previous sections is now straightforward. The free energy density near the horizon ($x \ll 0$) is given by

$$\beta f(x) \simeq -\frac{\pi}{6} \frac{1}{\beta \gamma(x)}$$

and the entropy diverges linearly with the “tortoise distance” to the horizon: $S \sim \pi \Delta x_-/3\beta$. At the Hawking temperature we get the well known universal result

$$S_{\text{boundary}} = \frac{1}{6} \log \frac{1}{\epsilon}$$

where $\epsilon$ is now a proper distance cutoff. In general, as long as the dilaton is regular in the horizon region, the local geometry is still asymptotically of Rindler type, and we will get (4.1). However, in some situations the dilaton effects may be important. For example, if we consider the canonical ensemble defined in the dual background (region V of the black hole), we simply replace $\gamma(r) \rightarrow \tilde{\gamma}(r)$ which is obtained by the substitution $\coth \frac{r}{2} \rightarrow \tanh \frac{r}{2}$. The red-shifted local temperature tends to zero at the singularity $r_0 = 2 \tanh^{-1} \sqrt{2/k}$, but the dilaton contribution is singular and negative, and produces an effective potential unbounded from below.

5. Conclusions

We have used semiclassical methods to study some thermal properties of free strings in the “brick-wall” model of ’t Hooft. We find that, in this approximation, “stringy” effects amount to work in the exact sigma-model metric, with a local mass renormalization due to the dilaton. The perturbative spectrum in the horizon region is still continuous like in Field Theory, but the horizon physics is strongly dimension-dependent. For higher dimensional models the horizon problem is shadowed by the the Hagedorn problem and the stretched horizon starts abruptly at the Hagedorn radius where $T/\sqrt{-g_{00}} = T_{\text{Hag}}$. It is interesting to note that a “soft” Atick-Witten high temperature phase (13): $\beta f(r) \sim T/\sqrt{-g_{00}}$, leads to a logarithmic divergence of the type (4.1), unless the coefficient in front vanishes,
as is the case for supersymmetric strings. Generally speaking, the Hagedorn transition opens the possibility for new string effects beyond the one-loop thermal physics, but also puts them beyond the reach of elementary methods. It is doubtful that a perturbative treatment based on a fixed background will suffice. This conclusion readily applies to the two-dimensional case, where the absence of Hagedorn singularity is compatible with the idea that there is no “stretching effect” for two-dimensional strings. In fact, the target space effective theory is renormalizable in two dimensions and should provide a pure field-theoretical resolution of the logarithmic singularity that appears in this case.

It would be very interesting to have a path integral computation (along the lines of point (ii) in the introduction), with an euclidean understanding of the local Hagedorn transition. If, on the other hand, such calculation gives a finite result, one should conclude that the “analog model” and the euclidean path integral are no longer equivalent definitions of the string thermodynamics in curved backgrounds (this possibility was suggested in ref. [5] on the basis of the first quantized path integral representations in field theory).

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