THE FUNCTOR $\tilde{F}^G_P$ AT THE LEVEL OF $K_0$

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ABSTRACT. Let $G$ be a $p$-adic Lie group with reductive Lie algebra $g$. Denote by $D(G)$ the locally analytic distribution algebra of $G$. Orlik-Strauch and Agrawal-Strauch have studied certain exact functors defined on various categories of $g$-representations with image in the category of locally analytic $G$-representations or $D(G)$-modules. In this paper we prove that for suitably defined categories of $D(G)$-modules, this functor gives rise to injective homomorphisms at the level of Grothendieck groups. We also explain how this functor interacts with translation functors at the level of Grothendieck groups.

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1. Introduction

Let $F$ be a finite extension of $\mathbb{Q}_p$ and consider a split reductive group $G$ over $F$. Let $P \supset B \supset T$ be a parabolic subgroup, a Borel subgroup, and a maximal torus respectively. Denote by $L_P$ the Levi subgroup of $P$ containing $T$, and by $U_P$ the unipotent radical of $P$. Let $g, p, b$ and $t$ be the Lie algebras of the corresponding algebraic groups. Let $G = G(F), P = P(F), B = B(F), T = T(F), L_P = L_P(F), \text{ and } U_P = U_P(F)$ be the corresponding groups of $F$-valued points. Furthermore, we fix once and for all a finite extension $E$ of $F$ which will be our field of coefficients. The base change of an $F$-vector space to $E$ will always be denoted by the subscript $E$, for example $g_E = g \otimes_F E$ and $b_E = b \otimes_F E$ etc. A weight $\lambda \in t^*$ is said to be algebraic if it is in the image of the canonical map $\text{im}(X^*(T) \to t^*)$. We denote the set of algebraic weights as $t^*_{\text{alg}}$.

We are interested in locally analytic representations of $G$ on $E$-vector spaces, cf. [14, 15]. However, instead of working directly with locally analytic representations, we will be working in the framework of $D(G)$-modules where $D(G) := C^{\text{fin}}(G, E)'$ is the locally analytic distribution algebra, which is the strong dual of the space of $E$-valued locally $F$-analytic functions on $G$, cf. [14, Cor. 3.3].
In the spirit of [1, 12], we study the functor \( \mathcal{F}_P^G \) defined as follows.

\[
\mathcal{F}_P^G : \mathcal{O}_{\text{alg}}^P \times \text{Rep}_E^\text{adm}(L_P)^{\text{adm}} (M, V) \longrightarrow D(G)\text{-mod} \cong D(G) \otimes_{D(\mathfrak{g}, P)} (M \otimes_E V').
\]

Here \( \mathcal{O}_{\text{alg}}^P \) is the full subcategory of the Bernstein-Gelfand-Gelfand category \( \mathcal{O} \) for the Lie algebra \( \mathfrak{g}_E \), consisting of modules \( M \) with algebraic weights and on which the action of \( \mathfrak{p}_E \) is locally finite. Furthermore, \( \text{Rep}_E^\text{adm}(L_P)^{\text{adm}} \) is the category of smooth strongly admissible representations on \( E \)-vector spaces of the subgroup \( L_P \). Here \( D(\mathfrak{g}, P) \) is the subring of \( D(G) \) generated by the universal enveloping algebra \( U(\mathfrak{g}_E) \) and \( D(P) \). Functors of this form have turned out to be very useful towards various aspects of locally analytic representations and the \( p \)-adic local Langlands program, see [3, 16] for example.

The functor \( \mathcal{F}_P^G \) is exact in both arguments, cf. [1, Thm. 4.2.4]. When the second argument of \( \mathcal{F}_P^G \) is the trivial representation \( 1 \) of \( P \), we write \( \mathcal{F}_P^G(M) \) instead of \( \mathcal{F}_P^G(M, 1) \). This gives an exact functor \( \mathcal{F}_P^G : \mathcal{O}_{\text{alg}}^P \longrightarrow D(G)\text{-mod} \), defined by \( M \mapsto \mathcal{F}_P^G(M) \).

The image of \( \mathcal{F}_P^G \) lies inside the category \( D(G)\text{-mod}^{\text{road}}_{\text{f.i.l.}} \), which is the category of coadmissible \( D(G) \)-modules of finite length, cf. 4.3. We consider the induced map on Grothendieck groups, which we also denote by \( \mathcal{F}_P^G \).

\[
\mathcal{F}_P^G : K_0(\mathcal{O}_{\text{alg}}^P) \big( \frac{[M]}{} \big) \longrightarrow K_0(D(G)\text{-mod}^{\text{road}}_{\text{f.i.l.}})[\mathcal{F}_P^G(M)].
\]

Our main result is the following.

**Theorem 1.1** (4.6). The induced map \( \mathcal{F}_P^G : K_0(\mathcal{O}_{\text{alg}}^P) \rightarrow K_0(D(G)\text{-mod}^{\text{road}}_{\text{f.i.l.}}) \) is injective.

Our result follows from an application of [3, Cor. 2.7] (cf. 4.5) and the irreducibility result in [12, 5.8].

We also prove a version of Theorem 1.1 for various categories indexed by weights (or rather the dot orbits of weights, see 5.1) as explained below. Denote by \( z_E \) the center of \( U(\mathfrak{g}_E) \). For \( \lambda \in z_E \), let \( \chi \) be the central character of \( z_E \) associated to \( \lambda \). Following the notation of [9], we denote by \( U(\mathfrak{g}_E)\text{-mod}_{\lambda} \) and \( D(G)\text{-mod}_{\lambda} \) the full subcategory of \( U(\mathfrak{g}_E)\text{-mod} \) (and \( D(G)\text{-mod} \)) consisting of modules \( M \) such that each \( m \in M \) is annihilated by sufficiently large powers of \( \ker(\chi) \). Let \( \mathcal{O}_{\text{alg}, \lambda} \) be the full subcategory of \( \mathcal{O} \) consisting of modules with algebraic weights which lie in \( U(\mathfrak{g}_E)\text{-mod}_{\lambda} \). We show that the image of \( \mathcal{O}_{\text{alg}, \lambda} \) under \( \mathcal{F}_B^G \) (for a Borel subgroup \( B \subset G \)) lies in \( \overline{\mathcal{O}}_{\text{alg}}(G, B)_{\lambda} \). Here \( \overline{\mathcal{O}}_{\text{alg}}(G, B)_{\lambda} \) denotes the smallest full subcategory of \( D(G)\text{-mod}_{\lambda} \) consisting of modules \( M \in D(G)\text{-mod}_{\lambda} \) with the following properties:

(i) It contains all modules \( \mathcal{F}_B^G(M) \), where \( M \) is in \( \mathcal{O}_{\text{alg}} \).

(ii) It is closed under taking subquotients and extensions.

With this notation in place we have the following result which is a consequence of 1.1.

**Theorem 1.2** (5.11). The induced map \( \mathcal{F}_B^G : K_0(\mathcal{O}_{\text{alg}, \lambda}) \rightarrow K_0(\overline{\mathcal{O}}_{\text{alg}}(G, B)_{\lambda}) \) is injective.

As an application of Theorem 1.2, in Section 5, we explain how the functor \( \mathcal{F}_B^G \) interacts with translation functors at the level of Grothendieck groups. We consider the translation functors \( T_{\chi}^a \) and \( T_{\chi}^b \) (cf. [9, 2.4.1] and [9, 2.4.5]). In that direction we have the following result.
Theorem 1.3 (5.12). Let $\lambda, \mu \in t^*_E$ be such that $\mu - \lambda$ is algebraic. Then the functors $\tilde{F}_B^G$, $T_\lambda^\mu$ and $T_\mu^\lambda$ give rise to the following commutative diagram of abelian groups.

\[
\begin{array}{ccc}
K_0(O_{\text{alg}, |\lambda|}) & \xrightarrow{\tilde{F}_B^G} & K_0(\overline{O}_{\text{alg}}(G, B)|_\lambda) \\
\downarrow{T_\lambda^\mu} & & \downarrow{T_\mu^\lambda} \\
K_0(O_{\text{alg}, |\mu|}) & \xrightarrow{\tilde{F}_B^G} & K_0(\overline{O}_{\text{alg}}(G, B)|_\mu)
\end{array}
\]

where the horizontal maps are injective. Additionally, if $\lambda, \mu$ satisfy the conditions (ii) and (iii) in 5.9, then the vertical maps are isomorphisms.

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2. Jordan-Hölder series for coadmissible modules

In this section we consider a Fréchet-Stein algebra $A$ over $E$ and a ring extension $A \subset B$, where $B$ is an associative unital $E$-algebra. We call a $B$-module coadmissible, if it coadmissible as an $A$-module. Recall that a coadmissible $A$-module carries a canonical topology [15, before 3.6].

Definition. Let $M$ be a coadmissible $B$-module.

(i) We say that a $B$-module $M$ is topologically simple if it has no closed submodule other than 0 and $M$.

(ii) A topological composition series of $M$ is a finite chain of closed $B$-submodules

\[0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M\]

such that $M_i/M_{i-1}$ is topologically simple for $1 \leq i \leq n$.

Convention 2.1. From now on, if not said otherwise, we will call a topological composition series simply a composition series. This should not lead to confusion with the concept of a composition series in the sense of the theory of (abstract) modules over a ring.

Lemma 2.2. Let $M$ be a coadmissible $B$-module. If $M$ has a composition series, then so does any closed submodule $N \subset M$.

Proof. Take any composition series for $M$, say

\[0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M\]

Taking intersections with the submodule $N$ yields a chain of closed $B$-submodules of $N$,

\[0 = M_0 \cap N \subsetneq M_1 \cap N \subsetneq \cdots \subsetneq M_n \cap N = N.\] (2.3)

It is possible for the submodules in the chain in 2.3 to be equal, so it may not be a composition series. We have

\[
\frac{(M_i \cap N)}{(M_{i-1} \cap N)} = \frac{(M_i \cap N)}{(M_{i-1} \cap (M_i \cap N))} \cong \frac{((M_i \cap N) + M_{i-1} \cap (M_i \cap N))}{M_{i-1}} \subseteq M_i/M_{i-1}.
\]

Since $M_i/M_{i-1}$ is simple, the modules $(M_i \cap N)/(M_{i-1} \cap N)$ occurring in 2.4 are simple or zero. \qed
**Definition.** Let $M$ be a coadmissible $B$-module. Consider the following two composition series for $M$

\begin{equation}
0 = M_0 \subsetneq M_1 \cdots \subsetneq M_n = M
\end{equation}

\begin{equation}
0 = M'_0 \subsetneq M'_1 \cdots \subsetneq M'_m = M
\end{equation}

The composition series in 2.5 and 2.6 are said to be equivalent if the following conditions hold.

(i) $m = n$.

(ii) There is a permutation $\sigma$ of $\{1, \ldots, n\}$ such that $M_i/M_{i-1} \cong M'_{\sigma(i)}/M'_{\sigma(i)-1}$

The goal of this section is to prove that any two composition series of a coadmissible $D$-module are equivalent. We closely follow the arguments for abstract $B$-modules given in [6, 3.2].

**Lemma 2.7.** Let $M$ be a coadmissible $B$-module. Consider the following two composition series

\begin{equation}
0 = M'_0 \subsetneq M'_1 \cdots \subsetneq M'_m = M
\end{equation}

Suppose $M'_{m-1} \neq M_{n-1}$ and consider $S = M'_{m-1} \cap M_{n-1}$. Then the inclusion maps $M_{n-1} \hookrightarrow M$ and $M'_{m-1} \hookrightarrow M$ induce isomorphisms

$$M_{n-1}/S \cong M/M'_{m-1} \quad \text{and} \quad M'_{m-1}/S \cong M/M_{n-1}$$

and hence both of these quotients are topologically simple.

**Proof.** Note first that $N := M_{n-1} + M'_{m-1}$ is again a coadmissible $A$-module, by [15, 3.4 (iii)], and hence closed by [15, 3.6]. If $N$ would be equal to $M_{n-1}$, then $M'_{m-1}$ would be contained in $M_{n-1}$, and hence would be properly contained in $M_{n-1}$, since we assume that $M_{n-1} \neq M'_{m-1}$. This shows that $N$ properly contains $M_{n-1}$, and thus $N = M$ since $M_{n-1}$ is maximal among the closed proper $B$-submodules of $M$.

Now we have

$$M/M'_{m-1} = (M_{n-1} + M'_{m-1})/M'_{m-1} \cong M_{n-1}/(M_{n-1} \cap M'_{m-1}) = M_{n-1}/S.$$

Similarly, we show that $M'_{m-1}/S \cong M/M_{n-1}$. \hfill \Box

**Theorem 2.10.** Let $M$ be a coadmissible $B$-module. Consider any two composition series for $M$

\begin{equation}
0 = M_0 \subsetneq M_1 \cdots \subsetneq M_n = M
\end{equation}

\begin{equation}
0 = M'_0 \subsetneq M'_1 \cdots \subsetneq M'_m = M
\end{equation}

Both of these series are equivalent.

**Proof.** We will proceed by induction on $n$, which is the length of the chain in 2.11. If $n = 0$, then $M = 0$ and there is nothing to prove.

Now suppose that $n > 0$. By the induction hypothesis, the theorem holds for modules which have composition series of length $\leq n - 1$. \hfill \Box
First let us consider the case when $M_{n-1} = M'_{n-1} =: T$ say. Then $T$ inherits a composition series of length $n - 1$ from 2.11. By the induction hypothesis, any two composition series for $T$ have the same length. Thus the composition series inherited from 2.12 is also of length $n - 1$. Hence $m - 1 = n - 1$ and thus $m = n$. By the induction hypothesis we also have a permutation $\sigma$ of $\{1, \ldots, n - 1\}$ such that $M_i/M_{i-1} \cong M'_{\sigma(i)}/M'_{\sigma(i-1)}$. We also have $M_n/M_{n-1} = M/T = M'_n/M'_{n-1}$. So if we view $\sigma$ as a permutation of $\{1, \ldots, n\}$ fixing $n$, we have the required permutation.

Now assume that $M_{n-1} \neq M'_{m-1}$. We define $S := M_{n-1} \cap M'_{m-1}$. Consider a composition series of $S$, which exists by 2.2

$$0 = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_t = S.$$ 

By 2.7, the modules $M_{n-1}/S$ and $M'_{m-1}/S$ are topologically simple. So we get the following two composition series of $M$

\begin{align}
0 = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_t = S \subsetneq M_{n-1} \subsetneq M \quad \text{(2.13)}
\end{align}

and

\begin{align}
0 = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_t = S \subsetneq M'_{m-1} \subsetneq M. \quad \text{(2.14)}
\end{align}

The composition factors appearing in both 2.13 and 2.14 are the same till the the module $S$. But the last two composition factors are also going to be same, thanks to 2.7. So both the composition series 2.13 and 2.14 are equivalent.

Now we claim that $m = n$. The module $M_{n-1}$ inherits a composition series of length $n - 1$ from 2.11. So by the induction hypothesis all composition series of $M_{n-1}$ have length $n - 1$. The composition series inherited from 2.13 has length $t + 1$ and hence $t + 1 = n - 1$. Similarly the module $M'_{m-1}$ inherits a composition series of length $m - 1$ from 2.12. Again, by induction hypothesis, any two composition series of $M'_{m-1}$ have the same length, and so $t + 1 = m - 1$ from 2.14. Therefore $m = n$.

Next we show that the composition series 2.11 and 2.13 are equivalent. By the induction hypothesis the composition series of $M_{n-1}$ inherited from 2.11 and 2.13 are equivalent. So there is a permutation $\gamma$ of $\{1, \ldots, n - 1\}$ such that

$$S_i/S_{i-1} \cong M_{\gamma(i)}/M_{\gamma(i-1)}, (i \neq n - 1) \quad \text{and} \quad M_{n-1}/S \cong M_{\gamma(n-1)}/M_{\gamma(n-1)}.$$ 

We view $\gamma$ as a permutation of $\{1, \ldots, n\}$ which fixes $n$. Then

$$M/M_{n-1} = M_n/M_{n-1} \cong M_{\gamma(n)}/M_{\gamma(n)}$$

which proves that 2.11 and 2.13 are equivalent. Similarly we can show that 2.12 and 2.14 are equivalent. Thus it follows that 2.11 and 2.12 are equivalent. \(\square\)

If a coadmissible $B$-module has a topological composition series of length $n$, then we call $n$ the topological length of $M$. We will usually just say “length” instead of “topological length”.

**Lemma 2.15.** Let $M$ be a coadmissible $B$-module of finite length and $N \subset M$ be a coadmissible $B$-submodule. Then $M/N$ is a coadmissible $B$-module of finite length.

**Proof.** By [15, 3.6], $M/N$ is coadmissible as an $A$-module and hence coadmissible as a $B$-module. We construct a composition series of $M/N$ from a composition series of $M$. Consider the following composition series of $M$

\begin{align}
0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M. \quad \text{(2.16)}
\end{align}

Then we have a series of submodules of $M/N$

\begin{align}
0 = (M_0 + N)/N \subsetneq (M_1 + N)/N \subsetneq \cdots \subsetneq (M_n + N)/N = M/N. \quad \text{(2.17)}
\end{align}
It is possible that some of the terms in 2.17 are equal. Each $M_i + N$ is coadmissible as an $A$-module by [15, 3.4 (iii)], and hence closed by [15, 3.6]. The quotient $(M_i + N)/N$ is coadmissible by [15, 3.6]. We have

\[(M_i + N)/((M_{i-1} + N)/N) \cong (M_i + N)/(M_{i-1} + N)\]

(2.18)

\[= (M_{i-1} + N + M_i)/(M_{i-1} + N)\]

\[\cong M_i/((M_{i-1} + N) \cap M_i)\]

where the equality in the second step holds because $M_{i-1} \subset M_i$. We also have $M_{i-1} \subset M_{i-1} + N$ and therefore $M_{i-1} \subset (M_{i-1} + N) \cap M_i \subset M_i$. But $M_{i-1}$ is a maximal closed submodule of $M_i$. Hence the quotient $M_i/((M_{i-1} + N) \cap M_i)$ is simple or zero. If we omit all the terms that are zero in 2.17, we end up with a composition series of $M/N$. \hfill \Box

3. Coadmissible modules of finite length

Define $B\text{-mod}^{\text{coad}}$ to be the full subcategory of the category $B\text{-mod}$ of all left $B$-modules which consists of the coadmissible modules. Furthermore, we denote by $B\text{-mod}^{\text{coad}}_{\text{f.g.}}$ the full subcategory of $B\text{-mod}^{\text{coad}}$ consisting of modules of finite length.

**Lemma 3.1.** (i) $B\text{-mod}^{\text{coad}}$ is an abelian category.

(ii) $B\text{-mod}^{\text{coad}}_{\text{f.g.}}$ is an abelian category.

**Proof.** (i) Consider a morphism $f : M \to N$ of coadmissible $B$-modules. Then $f$ is also a morphism of coadmissible $A$-modules. The kernel and cokernel of $f$ as a morphism of $A$-modules coincides with the respective kernel and cokernel as a morphism of $B$-modules. The category of coadmissible $A$-modules is abelian by [15, 3.5] and hence $B\text{-mod}^{\text{coad}}$ is also abelian.

(ii) $B\text{-mod}^{\text{coad}}_{\text{f.g.}}$ is a full subcategory of $B\text{-mod}^{\text{coad}}$. Consider any $M \in B\text{-mod}^{\text{coad}}_{\text{f.g.}}$ and a coadmissible submodule $N \subset M$. Since $B\text{-mod}^{\text{coad}}$ is abelian, it is enough to show that $N \in B\text{-mod}^{\text{coad}}_{\text{f.g.}}$ and $M/N \in B\text{-mod}^{\text{coad}}_{\text{f.g.}}$. But that follows from 2.2 and 2.15 respectively. \hfill \Box

**Definition.** As defined in [13, Appendix E] we say that a category $\mathcal{C}$ is *skeletally small* if the isomorphism classes of objects in $\mathcal{C}$ is a set.

**Lemma 3.2.** (i) For any ring $R$, the category of finitely generated $R$-modules is skeletally small.

(ii) The category $B\text{-mod}^{\text{coad}}$ is skeletally small.

(iii) The category $B\text{-mod}^{\text{coad}}_{\text{f.g.}}$ is skeletally small.

**Proof.** (i) Let $Q^{\text{f.g.}}(R^N)$ be the set of quotients $N$ of the countably free $R$-module $R^N$, and which have the property that the quotient map $R^N \to N$ factors through some $R^m$ (which is a quotient of $R^N$ by sending all components indexed by $k > m$ to zero). Any such $N$ is finitely generated. It is clear that $Q^{\text{f.g.}}(R^N)$ is a set and any finitely generated module is isomorphic to one in $Q^{\text{f.g.}}(R^N)$. Let $I^{\text{f.g.}}(R)$ be the set of isomorphism classes in $Q^{\text{f.g.}}(R^N)$. So every finitely generated module is isomorphic to a module in $Q^{\text{f.g.}}(R^N)$ which in turn determines a unique element in $I^{\text{f.g.}}(R)$.

(ii) Consider a coadmissible $B$-module $M$. Write $A$ as the projective limit

\[A \cong \lim_{\to} A_n\]
of a countable projective system \((A_n)_{n \geq 0}\) of (left) Noetherian Banach algebras \(A_n\), where for each \(n \geq 0\) the ring \(A_n\) is a flat right \(A_{n+1}\)-module via the transition map \(A_{n+1} \to A_n\), cf. [15, sec. 3]. We consider the finitely generated \(A_n\)-module \(M_n\) defined as follows

\[ M_n \cong A_n \otimes_A M, \]

cf. [15, Cor. 3.1]. Let \(I^f(A_n)\) be the collection of isomorphism classes of finitely generated \(A_n\)-modules. Then \(I^f(A_n)\) is a set by part (i). Given two isomorphic coadmissible \(B\)-modules \(M\) and \(N\), we have \(M_n \cong N_n\) for all \(n\). Thus we have the equality \(\text{cl}_{A_n}(M_n) = \text{cl}_{A_n}(N_n)\) in \(I^f(A_n)\), where \(\text{cl}_{A_n}(M_n)\) and \(\text{cl}_{A_n}(N_n)\) are the isomorphism classes of \(M_n\) and \(N_n\), respectively. It follows that if \(M\) and \(N\) are isomorphic coadmissible \(B\)-modules, then the sequences \((\text{cl}_{A_n}(M_n))_n\) and \((\text{cl}_{A_n}(N_n))_n\) are identical. It follows that the collection of isomorphism classes of coadmissible \(B\)-modules is a set.

(iii) Since \(B\)-mod\textsuperscript{coad} is skeletally small, any subcategory of \(B\)-mod\textsuperscript{coad} is also skeletally small. Hence \(B\)-mod\textsuperscript{coad}\textsubscript{1,1} is skeletally small. \hfill \Box

**Notation 3.3.** Given any \(M \in B\)-mod\textsuperscript{coad}\textsubscript{1,1}, we denote by \([M]\) its class in the Grothendieck group \(K_0(B\text{-mod}^{\text{coad}})\).

**Lemma 3.4.** The Grothendieck group \(K_0(B\text{-mod}^{\text{coad}})\) is a free abelian group with basis \([M]\) \(M\) is simple.

**Proof.** This follows from 3.2 and [5, Theorem C]. \hfill \Box

**Remark 3.5.** We do not know if a topologically simple coadmissible \(B\)-module is monogenic in the algebraic sense. Similarly, we also do not know if a coadmissible \(B\)-module of topologically finite length is finitely generated in the algebraic sense.

### 4. The map induced by the functor \(\hat{F}_G^G\) on the Grothendieck groups

We recall our notation from Section 1. Let \(F\) be a finite extension of \(\mathbb{Q}_p\), and \(G\) be a split reductive group over \(F\). Let \(P \supset B \supset T\) be a parabolic subgroup, a Borel subgroup, and a maximal torus respectively. Denote by \(g, p, b\) and \(t\) their respective Lie algebras. Let \(G = G(F), P = P(F), B = B(F)\) and \(T = T(F)\) be the corresponding groups of \(F\)-valued points. We fix once and for all a finite extension \(E\) of \(F\) which will be our field of coefficients. The base change of an \(F\)-vector space to \(E\) will always be denoted by the subscript \(E\), for example \(g_E = g \otimes_F E\) and \(b_E = b \otimes_F E\) etc. In this section closely follow the notation of [12, 1].

#### 4.1. Category \(\mathcal{O}_{\text{alg}}^p\) and the functor \(\hat{F}_G^G\)

Given a representation \(\phi : t_E \to \text{End}_E(M)\), a weight \(\lambda \in t_E^*\), and a positive integer \(i \geq 1\) we set

\[ M_\lambda = \{m \in M \mid \forall \mathbf{r} \in t_E : (\phi(\mathbf{r}) - \lambda(\mathbf{r}) \cdot \text{id})m = 0\}. \]

The category \(\mathcal{O}_p\) for the pair \((g,p)\) and the coefficient field \(E\) is defined to be the full subcategory of all \(U(g_E)\)-modules \(M\) which satisfy the following properties:

1. \(M\) is finitely generated as a \(U(g_E)\)-module.
2. \(M = \bigoplus_{\lambda \in I_G} M_\lambda\).
3. The action of \(p_E\) on \(M\) is locally finite, i.e. for every \(m \in M\), the subspace \(U(p_E)m \subset M\) is finite-dimensional over \(E\).
When $\mathfrak{p} = \mathfrak{b}$, the category $\mathcal{O}^\mathfrak{p}$ coincides with the Bernstein-Gelfand-Gelfand category $\mathcal{O}$. We denote by $\mathcal{O}_{\text{alg}}^\mathfrak{p}$ the full subcategory $\mathcal{O}^\mathfrak{p}$ consisting of modules $M$ for which the weights lie in the image $t_{\text{alg}}^*$ of the differential $X^*(T) \to \mathfrak{t}^*$.

Let $D(G) := C^{an}(G, E)_1^0$ be the locally analytic distribution algebra, which is the strong dual of the space of $E$-valued locally $F$-analytic functions on $G$, cf. [14, Cor. 3.3]. Denote by $D(\mathfrak{g}, P)$ the subring of $D(G)$ generated by the universal enveloping algebra $U(\mathfrak{g}_E)$ and $D(P)$.

**Definition.** Denote by $\text{Rep}^\text{sm}_E(\mathcal{L}_P)^{s\text{-adm}}$ the category of smooth strongly admissible representations on $E$-vector spaces of the subgroup $\mathcal{L}_P$. In the spirit of [1] we consider the functor

$$\hat{\mathcal{F}}_P^G : \mathcal{O}_{\text{alg}}^\mathfrak{p} \times \text{Rep}^\text{sm}_E(\mathcal{L}_P)^{s\text{-adm}}(M, V) \to D(G)\text{-mod} \to D(G) \otimes_{D(\mathfrak{g}, P)} (M \otimes_E V'),$$

cf. [1, 4.2.1] (where $\text{Lift}(M, \log)$ is the canonical lift of $M$ to a module over $D(\mathfrak{g}, P)$, as explained in [12, 3.2, 3.6]).

Let $\text{Rep}^{\text{la}}_E(G)$ be the category of locally analytic $G$-representations on $E$-vector spaces. We would like to point out that there is an analogue of $\hat{\mathcal{F}}_P^G$, with image in $\text{Rep}^{\text{la}}_E(G)$ rather than $D(G)\text{-mod}$ as described in [12, 11]. This functor is denoted by $\mathcal{F}_P^G$. The functors $\hat{\mathcal{F}}_P^G$ and $\mathcal{F}_P^G$ are related in the following way

$$\hat{\mathcal{F}}_P^G(M, V) = \mathcal{F}_P^G(M, V)'_b$$

for $M \in \mathcal{O}_{\text{alg}}^\mathfrak{p}$ and $V \in \text{Rep}^\text{sm}_E(\mathcal{L}_P)^{s\text{-adm}}$. Here $\mathcal{F}_P^G(M, V)'_b$ denotes the dual of $\mathcal{F}_P^G(M, V)$ equipped with the strong topology.

**4.2. $\hat{\mathcal{F}}_P^G$ at the level of $K_0$.** In accordance with the notation introduced in Section 3, we will study the category $B\text{-mod}_{\text{coad}}^{\text{la}}$ where $B = D(G)$. So the category $D(G)\text{-mod}_{\text{coad}}^{\text{la}}$ is abelian by 3.1(ii) and skeletally small by 3.2(iii).

**Lemma 4.3.** $\hat{\mathcal{F}}_P^G(M) \in D(G)\text{-mod}_{\text{coad}}^{\text{la}}$ for any $M \in \mathcal{O}$.

**Proof.** Coadmissibility holds by [1, 4.2.3]. Since $M$ has finite length, and because $\hat{\mathcal{F}}_P^G$ is an exact functor, it suffices to show that $\hat{\mathcal{F}}_P^G(M)$ has finite length when $M$ is simple as a module over the enveloping algebra. If $\mathfrak{q}$ is maximal for $M$ in the sense of [12, 5.2], then we have $\hat{\mathcal{F}}_P^G(M) = \hat{\mathcal{F}}_Q^G(M, \text{ind}_{L_P(L_Q \cap U_P)}^L(1)\mathfrak{g})$ by [1, 4.3.3], where $Q$ is the standard parabolic subgroup with Lie algebra $\mathfrak{q}$. Since $\text{ind}_{L_P(L_Q \cap U_P)}^L(1)$ has finite length, it suffices to show that $\hat{\mathcal{F}}_Q^G(M, \pi)$ is topologically simple as $D(G)$-module, when $\pi$ is an irreducible smooth $L_P$-representation. This is a consequence of $\mathcal{F}_Q^G(M, \pi)$ being topologically irreducible, by [12, 5.8].

**Notation 4.4.** Given an exact functor $F : \mathcal{C} \to \mathcal{D}$ between two abelian categories, we get an induced homomorphism between Grothendieck groups

$$K_0(\mathcal{C}) \to K_0(\mathcal{D})$$

$$[M] \mapsto [FM].$$

By abuse of notation we will also denote this map as $F : K_0(\mathcal{C}) \to K_0(\mathcal{D})$.

We recall the following notion of maximality from [12]. For $M \in \mathcal{O}^\mathfrak{p}$, we say that the parabolic subalgebra $\mathfrak{p}$ is maximal for $M$ if $M$ does not lie in $\mathcal{O}^\mathfrak{q}$ for any parabolic subalgebra $\mathfrak{q}$ properly containing $\mathfrak{p}$. A crucial input in the proof of our main result is the following theorem of C. Breuil [3, Cor. 2.7].
Theorem 4.5. Let $M_1$ and $M_2$ be two simple objects of $\mathcal{O}_{\text{alg}}^\mathbb{P}$, $Q_1$ and $Q_2$ be their maximal parabolics, $\pi_{Q_1}$ and $\pi_{Q_2}$ be two smooth admissible representations of finite length of $L_{Q_1}(F)$ and $L_{Q_2}(F)$ respectively on $E$. Then we have $\mathcal{F}^G_{Q_1}(M_1, \pi_{Q_1}) \cong \mathcal{F}^G_{Q_2}(M_2, \pi_{Q_2})$ if and only if $Q_1 = Q_2$ and $M_1 \cong M_2$ and $\pi_{Q_1} \cong \pi_{Q_2}$.

Consider $\lambda \in \mathfrak{t}^*_E$. Following [7], we will denote by $L(\lambda)$ the simple highest weight module of weight $\lambda$ in $\mathcal{O}$. Thus $L(\lambda)$ is the unique irreducible quotient of the Verma module $U(\mathfrak{g}_E) \otimes_{U(k_E)} 1_\lambda$. Now we present our main result.

Theorem 4.6. The induced map $\hat{\mathcal{F}}^G_P : K_0(\mathcal{O}_{\text{alg}}^\mathbb{P}) \to K_0(D(G)\text{-mod}_{\text{lad}}^\text{coad})$ is injective.

Proof. The set $\{[L(\lambda)] \mid L(\lambda) \in \mathcal{O}_{\text{alg}}^\mathbb{P}\}$ forms a $\mathbb{Z}$-basis of $K_0(\mathcal{O}_{\text{alg}}^\mathbb{P})$, cf. [7, Sec. 1.11]. So any arbitrary element $\xi \in K_0(\mathcal{O})$ is of the form

$$\xi = \sum_{i=1}^n m_i[L(\lambda_i)]$$

where $L(\lambda_i)$ are pairwise nonisomorphic simple modules in $\mathcal{O}$ and $m_i \in \mathbb{Z}$. We need to show that if $\hat{\mathcal{F}}^G_P(\xi) = 0$, then $\xi = 0$. So assume that $\hat{\mathcal{F}}^G_P(\xi) = 0$. Then

$$0 = \hat{\mathcal{F}}^G_P \left( \sum_{i=1}^n m_i[L(\lambda_i)] \right) = \sum_{i=1}^n m_i \hat{\mathcal{F}}^G_P ([L(\lambda_i)])$$

It follows that

$$\sum_{m_i > 0} m_i \hat{\mathcal{F}}^G_P ([L(\lambda_i)]) = \sum_{m_i < 0} -m_i \hat{\mathcal{F}}^G_P ([L(\lambda_i)])$$

which is equivalent to

$$\sum_{m_i > 0} m_i [\mathcal{F}^G_P(L(\lambda_i))] = \sum_{m_i < 0} -m_i [\mathcal{F}^G_P(L(\lambda_i))].$$

Let $q_i$ be the standard parabolic subalgebra which is maximal for $L(\lambda_i)$ in the sense of [12]. Denote by $Q_i$ the subgroup which corresponds to $q_i$. Now we have

$$\mathcal{F}^G_P(L(\lambda_i)) = \mathcal{F}^G_P(L(\lambda_i), 1) \cong \mathcal{F}^G_{Q_i}(L(\lambda_i), \text{ind}_{B_i}^{L_{Q_i}}(1)) \cong \mathcal{F}^G_{Q_i}(L(\lambda_i), \text{ind}_{B_i\cap L_{Q_i}}^{L_{Q_i}}(1)).$$

where the isomorphism in the second step follows from [1, 4.3.3]. Let $\pi_{i,1}, \pi_{i,2}, \ldots, \pi_{i,k_i}$ be the distinct Jordan-Hölder factors of $\text{ind}_{B_i\cap L_{Q_i}}^{L_{Q_i}}(1)$ with multiplicities $\mu_{i,1}, \mu_{i,2}, \ldots, \mu_{i,k_i}$ respectively. Since $\mathcal{F}^G_{Q_i}$ is exact, it follows that

$$\left[ \mathcal{F}^G_{Q_i}(L(\lambda_i), \text{ind}_{B_i\cap L_{Q_i}}^{L_{Q_i}}(1)) \right] = \sum_{j=1}^{k_i} \mu_{i,j} \cdot [\mathcal{F}^G_{Q_i}(L(\lambda_i), \pi_{i,j})].$$

Combining 4.10 and 4.11 we conclude that

$$\left[ \mathcal{F}^G_P(L(\lambda_i)) \right] = \sum_{j=1}^{k_i} \mu_{i,j} \cdot [\mathcal{F}^G_{Q_i}(L(\lambda_i), \pi_{i,j})].$$
Now it follows from 4.9 and 4.12
\[
\sum_{m_i>0} \left( \sum_{j=1}^{k_i} m_i \mu_{i,j} \cdot \left[ F^G_{Q_i}(L(\lambda_i), \pi_{i,j}) \right] \right) = \sum_{m_i<0} \left( \sum_{j=1}^{k_i} -m_i \mu_{i,j} \cdot \left[ F^G_{Q_i}(L(\lambda_i), \pi_{i,j}) \right] \right).
\]
Recall that the set \( S = \{ [M] \mid M \in D(G)\text{-mod}_{\text{radd}}^\coad \} \) forms a basis for \( K_0(D(G)\text{-mod}_{\text{radd}}^\coad) \) by 3.4. By [12, 5.8], the representations \( F^G_{Q_i}(L(\lambda_i), \pi_{i,j}) \) appearing in 4.13 are all irreducible. So the terms \( \left[ F^G_{Q_i}(L(\lambda_i), \pi_{i,j}) \right] \) are all elements of the basis \( S \). Furthermore, since the \( L(\lambda_i) \)'s are pairwise nonisomorphic, the terms \( \left[ F^G_{Q_i}(L(\lambda_i), \pi_{i,j}) \right] \) are all pairwise distinct by 4.5. So the equality in 4.13 holds only if the product \( m_i \mu_{i,j} = 0 \) for all \( i, j \). But \( \mu_{i,j} \) is nonzero by definition. Hence we must have \( m_i = 0 \) for all \( i \). Thus \( \xi = \sum_{i=1}^n m_i [L(\lambda_i)] = 0 \). \( \square \)

5. RELATION WITH TRANSLATION FUNCTORS

Our goal in this section is to explain how the functor \( \tilde{F}^G_B \) (for a fixed Borel subgroup \( B \subset G \)) interacts with translation functors at the level of Grothendieck groups (see 5.12). We give a brief overview of translation functors in 5.1. Translation functors for Lie algebra representations have been studied in [2, 8, 7]. Translation functors for locally analytic representations were introduced in [9]. We closely follow the notation of [9].

5.1. Translation functors. Let \( \mathfrak{z}_E \) be the center of \( U(\mathfrak{g}_E) \) and let \( \text{Max}(\mathfrak{z}_E) \) be the set of maximal ideals of \( \mathfrak{z}_E \). Denote by \( U(\mathfrak{g}_E)\text{-mod}_{\text{fin}} \) the full subcategory of \( U(\mathfrak{g}_E)\text{-mod} \) consisting of modules \( M \) such that each \( m \in M \) is annihilated by an ideal of finite codimension in \( \mathfrak{z}_E \). Modules \( M \) in this category have a direct sum decomposition of the form
\[
M = \bigoplus_{m \in \text{Max}(\mathfrak{z}_E)} M_m
\]
where \( M_m \) is the submodule of \( M \) consisting of elements which are annihilated by powers of \( \mathfrak{m} \). Denote by \( U(\mathfrak{g}_E)\text{-mod}_m \) the subcategory of modules \( M \) with the property that \( M = M_m \). The projection \( M \to M_m \) is denoted by \( \text{pr}_m \). Translation functors are endo-functors of \( U(\mathfrak{g}_E)\text{-mod}_{\text{fin}} \) of the form
\[
M \rightsquigarrow \text{pr}_n (L \otimes_E \text{pr}_m (M)),
\]
where \( L \) is a finite-dimensional irreducible representation of \( \mathfrak{g}_E \), and \( \mathfrak{m}, \mathfrak{n} \in \text{Max}(\mathfrak{z}_E) \).

For a weight \( \lambda \in t_E^* \), let \( \chi_\lambda \) be the central character of \( \mathfrak{z}_E \) associated to \( \lambda \). Let \( |\lambda| \) be the orbit of \( \lambda \) under the dot action of the Weyl group \( W \) of \((\mathbf{G}, \mathbf{T})\). Consider the maximal ideal \( \text{ker}(\chi_\lambda) \). In accordance with the notation introduced in [9], we write \( \text{pr}_{|\lambda|} \) and \( U(\mathfrak{g}_E)\text{-mod}_{|\lambda|} \) instead of \( \text{pr}_{\text{ker}(\chi_\lambda)} \) and \( U(\mathfrak{g})\text{-mod}_{\text{ker}(\chi_\lambda)} \), respectively. Notice that \( \text{pr}_{|\lambda|} = \text{pr}_{|\theta|} \) and \( U(\mathfrak{g}_E)\text{-mod}_{|\lambda|} = U(\mathfrak{g}_E)\text{-mod}_{|\theta|} \) for any \( \theta \in |\lambda| \) by the Harish-Chandra Theorem for reductive Lie algebras, cf. [10, 4.115].

Definition. Given \( \lambda, \mu \in t_E^* \) such that \( \nu := \mu - \lambda \) is integral, the translation functor \( T^\mu_\lambda \) is the defined by
\[
T^\mu_\lambda : U(\mathfrak{g}_E)\text{-mod}_{\text{fin}} \to U(\mathfrak{g}_E)\text{-mod}_{\text{fin}}
\]
\[
M \rightsquigarrow \text{pr}_{|\mu|}(L(\overline{\nu}) \otimes_E \text{pr}_{|\lambda|}(M)),
\]
where \( \overline{\nu} \) is the dominant weight in the (linear) Weyl orbit of \( \nu \) and \( L(\overline{\nu}) \) the irreducible finite-dimensional module with highest weight \( \overline{\nu} \).
Now we review the construction of translation functors for $D(G)$-modules. There is a canonical injective algebra homomorphism $U(g_E) \hookrightarrow D(G)$ under which $\bar{\mathfrak{g}}_E$ is mapped into the center of $D(G)$. As a consequence, the categories $D(G)\text{-}\text{mod}_{\text{fin}}$ and $D(G)\text{-}\text{mod}_{|\lambda|}$ can be readily defined.

**Definition.** The translation functor

\[
T^\mu_{\lambda} : D(G)\text{-}\text{mod}_{\text{fin}} \to D(G)\text{-}\text{mod}_{\text{fin}}
\]

\[
M \rightsquigarrow \text{pr}_{|\mu|}(L(\mathfrak{g}) \otimes_E \text{pr}_{|\lambda|}(M))
\]

is then defined exactly as in (5.2), except that we require that $\mathfrak{g}$ lifts to an algebraic character of $\mathfrak{T}$, which in turn ensures that $L(\mathfrak{g})$ lifts to an algebraic representation of $G$.

5.4. **Induced maps on Grothendieck groups.** We denote by $\mathcal{O}_{\text{alg}}$ the full subcategory of $\mathcal{O}$ consisting of modules with algebraic weights.

**Definition.** We define $\overline{\mathcal{O}}_{\text{alg}}(G, B)$ to be the smallest full subcategory of $D(G)\text{-}\text{mod}_{\text{fin}}$ which has the following properties.

(i) It contains all modules $\tilde{\mathcal{F}}^G_B(M)$, where $M$ is in $\mathcal{O}_{\text{alg}}$.

(ii) It is closed under taking subquotients and extensions.

For $\lambda \in \mathcal{t}_{\text{alg}}$, we define $\overline{\mathcal{O}}_{\text{alg}}(G, B)_{|\lambda|}$ to be the full subcategory of $\overline{\mathcal{O}}_{\text{alg}}(G, B)$ consisting of modules $M$ which lie in $D(G)\text{-}\text{mod}_{|\lambda|}$.

We recall the following result from [9, 2.3.4] which we will use in 5.6.

**Lemma 5.5.** Let $L$ be a finite-dimensional locally $F$-analytic representation of $G$ over $E$. Consider a $D(\mathfrak{g}, P)$-module $M$. Then we have the isomorphism of $D(G)$-modules

\[
L \otimes_E D(G) \otimes_D(\mathfrak{g}, P) M \longrightarrow D(G) \otimes_{D(\mathfrak{g}, P)} (L \otimes_E M)
\]

**Lemma 5.6.** Let $L \in \mathcal{O}_{\text{alg}}^0$ be finite-dimensional. Then $L$ lifts canonically to an algebraic representation of $G$ which we also denote by $L$. For any $N \in \overline{\mathcal{O}}_{\text{alg}}(G, B)$, the module $L \otimes_E N$ also lies in $\overline{\mathcal{O}}_{\text{alg}}(G, B)$.

**Proof.** Our result will be a consequence of three claims that we will prove. Our first claim is that any object of the form $L \otimes_E \tilde{\mathcal{F}}^G_B(M)$ lies in $\overline{\mathcal{O}}_{\text{alg}}(G, B)$.

\[
L \otimes_E D(G) \otimes_{D(\mathfrak{g}, P)} M \overset{5.5}{=} D(G) \otimes_{D(\mathfrak{g}, P)} (L \otimes_E M)
\]

where $L \otimes_E M \in \mathcal{O}_{\text{alg}}$ by [7, Thm 1.1(d)]. This completes the proof of our first claim.

Our second claim is the following: if an object $N$ of $\overline{\mathcal{O}}_{\text{alg}}(G, B)$ which has the property that $L \otimes_E N$ is again in $\overline{\mathcal{O}}_{\text{alg}}(G, B)$, then the same is true for any subquotient $N'$ of $N$. Write $N' = N_1 / N_2$ with submodules $N_2 \subset N_1 \subset N$. Then we have the isomorphism of $D(G)$-modules

\[
L \otimes_E N' = L \otimes_E (N_1 / N_2) \cong (L \otimes_E N_1) / (L \otimes_E N_2).
\]

Now $(L \otimes_E N_1) / (L \otimes_E N_2)$ is a subquotient of $L \otimes_E N$ which lies in $\overline{\mathcal{O}}_{\text{alg}}(G, B)$. The module $L \otimes_E N'$ lies then in $\overline{\mathcal{O}}_{\text{alg}}(G, B)$ since $\overline{\mathcal{O}}_{\text{alg}}(G, B)$ is closed under taking subquotients.

Now consider a $D(G)$-module $N$ which is part of an exact sequence of the following form

\[
0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2 \longrightarrow 0
\]
where $N_1$ and $N_2$ are objects in $\mathcal{O}_{\text{alg}}(G, B)$ which have the property that $L \otimes_E N_1$ and $L \otimes_E N_2$ are in $\mathcal{O}_{\text{alg}}(G, B)$. Our third claim is that $L \otimes_E N$ lies in $\mathcal{O}_{\text{alg}}(G, B)$. The functor $L \otimes_E (\cdot)$ is an exact endofunctor on $D(G)$-mod. So we have the exact sequence of $D(G)$-modules

\[(5.8) \quad 0 \rightarrow L \otimes_E N_1 \rightarrow L \otimes_E N \rightarrow L \otimes_E N_2 \rightarrow 0.\]

Since $\mathcal{O}_{\text{alg}}(G, B)$ is closed under taking extensions, it follows that $L \otimes_E N$ lies in $\mathcal{O}_{\text{alg}}(G, B). \quad \Box$

**Theorem 5.9.** Let $\lambda, \mu \in \mathfrak{t}^*_E$ satisfy the following conditions:

(i) $\lambda$ and $\mu$ are compatible.

(ii) $\lambda$ and $\mu$ are both anti-dominant.

(iii) $W^\lambda_\mu = W^\mu_\lambda$.

Then the functors

$$T^\mu_\lambda : \mathcal{O}_{\text{alg}}(G, B)|_{\lambda|} \rightarrow \mathcal{O}_{\text{alg}}(G, B)|_{\mu|}$$

and

$$T^\lambda_\mu : \mathcal{O}_{\text{alg}}(G, B)|_{\mu|} \rightarrow \mathcal{O}_{\text{alg}}(G, B)|_{\lambda|}$$

induce an equivalence of categories. One has natural isomorphisms $T^\mu_\lambda \circ T^\lambda_\mu \cong \text{id}$ and $T^\mu_\lambda \circ T^\mu_\lambda \cong \text{id}$.

**Proof.** First we claim that $T^\mu_\lambda$ preserves the category $\mathcal{O}_{\text{alg}}(G, B)$. The functor $T^\mu_\lambda$ is a composition of projections and tensoring with $L(\mathfrak{g})$. By 5.6, the category $\mathcal{O}_{\text{alg}}(G, B)$ is closed under taking tensor product with $L(\mathfrak{g})$. Since $\mathcal{O}_{\text{alg}}(G, B)$ is closed under taking submodules, it is preserved by projection functors. Hence $T^\mu_\lambda$ preserves $\mathcal{O}_{\text{alg}}(G, B)$.

Now $\mathcal{O}_{\text{alg}}(G, B)|_{\lambda|}$ is a subcategory of $D(G)$-mod$_{\lambda|}$. That $\mathcal{O}_{\text{alg}}(G, B)|_{\lambda|}$ is mapped to $\mathcal{O}_{\text{alg}}(G, B)|_{\mu|}$ by $T^\mu_\lambda$ follows from the fact that $T^\mu_\lambda$ maps $D(G)$-mod$_{\lambda|}$ to $D(G)$-mod$_{\mu|}$, cf. [9, 3.2.1]. Similarly, $T^\lambda_\mu$ maps $\mathcal{O}_{\text{alg}}(G, B)|_{\mu|}$ to $\mathcal{O}_{\text{alg}}(G, B)|_{\lambda|}$. That $T^\mu_\lambda \circ T^\lambda_\mu \cong \text{id}$ and $T^\mu_\lambda \circ T^\mu_\lambda \cong \text{id}$ and that these functors induce equivalence of categories follows from [9, 3.2.1]. \hfill \Box

**Theorem 5.10.** The image of $\mathcal{O}_{\text{alg},|\lambda|}$ under $\mathcal{F}^G_B$ is contained in $\mathcal{O}_{\text{alg}}(G, B)|_{\lambda|}$.

**Proof.** It is clear that the image of $\mathcal{O}_{\text{alg},|\lambda|}$ under $\mathcal{F}^G_B$ is contained in $\mathcal{O}_{\text{alg}}(G, B)$, by definition of $\mathcal{O}_{\text{alg}}(G, B)$. So it is enough to prove that the image is contained in $D(G)$-mod$_{\lambda|}$. For $M \in \mathcal{O}_{\text{alg},|\lambda|}$ we have

$$\mathcal{F}^G_B(M) = D(G) \otimes_{D(G, B)} M = D(G) \otimes_{D(G, B)} \text{pr}_{|\lambda|}(M) = \text{pr}_{|\lambda|} \left( D(G) \otimes_{D(G, B)} M \right)$$

where the last equality holds by [9, 2.2.2 (iv)]. Thus $\mathcal{F}^G_B(M)$ lies in $D(G)$-mod$_{\lambda|}$. \hfill \Box

**Theorem 5.11.** The induced map $\mathcal{F}^G_B : K_0(\mathcal{O}_{\text{alg},|\lambda|}) \rightarrow K_0(\mathcal{O}_{\text{alg}}(G, B)|_{\lambda|})$ is injective.

**Proof.** The homomorphism of the statemset is the upper horizontal arrow in the commutative diagram

$$\begin{array}{ccc}
K_0(\mathcal{O}_{\text{alg},|\lambda|}) & \longrightarrow & K_0(\mathcal{O}_{\text{alg}}(G, B)|_{\lambda|}) \\
\downarrow & & \downarrow \\
K_0(\mathcal{O}_{\text{alg}}) & \longrightarrow & K_0(D(G)-\text{mod}^\text{coad}_{\mu_1})
\end{array}$$
where the lower horizontal arrow is injective by 4.6. The vertical arrow on the left is injective, because \( \mathcal{O}_{alg} = \bigoplus_{t \in \mathfrak{t}^\ast} W \mathcal{O}_{alg,t} \), where the direct sum is over the orbits of \( W \) for the dot-action on \( t^\ast_{alg} \). Hence the composition of these maps is injective, which implies that the upper horizontal arrow is injective. \( \square \)

**Theorem 5.12.** Let \( \lambda, \mu \in \mathfrak{t}^\ast_{B} \) be such that \( \mu - \lambda \) is algebraic. Then the functors \( \mathcal{F}^G_B, \mathcal{T}^\mu_\lambda \) and \( \mathcal{T}^\mu_\lambda \) give rise to the following commutative diagram of abelian groups.

\[
\begin{array}{ccc}
K_0(\mathcal{O}_{alg,|\lambda|}) & \xrightarrow{\mathcal{F}^G_B} & K_0(\mathcal{O}_{alg}(G, B)|_{\lambda}) \\
\downarrow{\mathcal{T}^\mu_\lambda} & & \downarrow{\mathcal{T}^\mu_\lambda} \\
K_0(\mathcal{O}_{alg,|\mu|}) & \xrightarrow{\mathcal{F}^G_B} & K_0(\mathcal{O}_{alg}(G, B)|_{\mu})
\end{array}
\]

where the horizontal maps are injective. Additionally, if \( \lambda, \mu \) satisfy the conditions (ii) and (iii) in 5.9, then the vertical maps are isomorphisms.

**Proof.** By 5.10, the images of \( K_0(\mathcal{O}_{alg,|\lambda|}) \) and \( K_0(\mathcal{O}_{alg,|\mu|}) \) under \( \mathcal{F}^G_B \) are contained in \( K_0(\mathcal{O}_{alg}(G, B)|_{\lambda}) \) and \( K_0(\mathcal{O}_{alg}(G, B)|_{\mu}) \) respectively. The commutativity of the diagram follows from [9, 4.1.12]. The injectivity of the horizontal arrows follows from 5.11. Now assume that \( \lambda \) and \( \mu \) satisfy the conditions (ii) and (iii) in 5.9. Then the left vertical map is an isomorphism by [7, Propn. 7.8]. The right vertical map is an isomorphism by 5.9. \( \square \)

**Example 5.13.** Let \( G = SL_2 \) and \( G = SL_2(F) \). Then \( \mathfrak{g} = sl_2(F) \). Let \( B \subset \mathfrak{g} \) be a Borel subalgebra, \( \Delta = \Phi^+ = \{ 2\rho \} \). Then \( t^\ast_{alg} = \mathbb{Z} \cdot \rho \). A weight \( \lambda = c \cdot \rho \) where \( c \in \mathbb{Z} \), is dot-regular if and only if \( c \neq 1 \).

Let \( \lambda = -\rho \) and consider \( \mathcal{O}_{alg,|\lambda|} = \mathcal{O}_{alg,|-\rho|} \). The category \( \mathcal{O}_{alg,|\lambda|} \) is a semisimple category and every object is isomorphic to a finite direct sum \( M(-\rho) = L(-\rho) \). In this case \( \mathcal{F}^G_B(M(-\rho)) \) is a topologically simple \( D(G) \)-module. Thus \( K_0(\mathcal{O}_{alg}(G, B)|_{\lambda}) \) is a free \( \mathbb{Z} \)-module of rank 1, generated by \( [\mathcal{F}^G_B(M(-\rho))] \). In this case the map

\[
\mathcal{F}^G_B : K_0(\mathcal{O}_{alg,|\lambda|}) \rightarrow K_0(\mathcal{O}_{alg}(G, B)|_{\lambda})
\]

is surjective, since both the domain and codomain are of rank 1. Thus \( \mathcal{F}^G_B \) is an isomorphism.

Now consider a weight \( \lambda = c \cdot \rho \), where \( c < -1 \). Then \( \lambda \) is antidominant and \( M(\lambda) \cong L(\lambda) \). The dot-orbit of \( \lambda \) is \( W \cdot \lambda = \{ \lambda, -\lambda - 2\rho \} \). Now \(-\lambda - 2\rho \) is dominant so that \( L(-\lambda - 2\rho) \) is finite dimensional. In this case the Grothendieck group \( K_0(\mathcal{O}_{alg,|\lambda|}) \) is given by

\[
K_0(\mathcal{O}_{alg,|\lambda|}) = \mathbb{Z}[M(\lambda)] \oplus \mathbb{Z}[L(-\lambda - 2\rho)].
\]

Now \( \mathcal{F}^G_B(M(\lambda)) \) is topologically simple as a \( D(G) \)-module. We have the following exact sequence of smooth \( G \)-representations

\[
0 \rightarrow 1 \rightarrow \text{ind}^G_B(1) \rightarrow \text{St} \rightarrow 0
\]

where \( \text{St} \) denotes the Steinberg representation and \( 1 \) denotes the trivial 1-dimensional representation and \( \text{ind}^G_B(1) \) denotes the smooth induction from \( B \) to \( G \). Taking duals, we get the exact sequence of \( D(G) \)-modules

\[
0 \rightarrow \text{St}' \rightarrow \text{ind}^G_B(1)' \rightarrow 1 \rightarrow 0.
\]

Tensoring with \( L(-\lambda - 2\rho) \) we get the following exact sequence of \( D(G) \)-modules

\[
(5.14) \quad 0 \rightarrow L(-\lambda - 2\rho) \otimes_E \text{St}' \rightarrow L(-\lambda - 2\rho) \otimes_E \text{ind}^G_B(1)' \rightarrow L(-\lambda - 2\rho) \rightarrow 0.
\]
Now $\mathcal{F}_B^G(L(-\lambda - 2\rho)) \cong L(-\lambda - 2\rho) \otimes E \operatorname{ind}_B^G(1)'$. Thus from 5.14 it follows that

$$[\mathcal{F}_B^G(L(-\lambda - 2\rho))] = [L(-\lambda - 2\rho)] + [L(-\lambda - 2\rho) \otimes E \operatorname{St}'].$$

Furthermore, $L(-\lambda - 2\rho)$ is absolutely irreducible, and hence $\operatorname{End}_{E[\mathbf{SL}_2]}(L(-\lambda - 2\rho)) = E$. This implies that $L(-\lambda - 2\rho) \otimes E \operatorname{St}'$ is irreducible by [4, 4.2.8]. It follows that

$$K_0(\mathcal{O}_{\operatorname{alg}}(G, B)_{|\lambda}) = \mathbb{Z}[\mathcal{F}_B^G(M(\lambda))] \oplus \mathbb{Z}[L(-\lambda - 2\rho)] \oplus \mathbb{Z}[L(-\lambda - 2\rho) \otimes E \operatorname{St}']\]$$

Hence the image of the map

$$\mathcal{F}_B^G : K_0(\mathcal{O}_{\operatorname{alg}}(G, B)_{|\lambda}) \longrightarrow K_0(\mathcal{O}_{\operatorname{alg}}(G, B)_{|\lambda})$$

has codimension 1.

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