The Topological Mu-Calculus: completeness and decidability

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Abstract—We study the topological μ-calculus, based on both Cantor derivative and closure modalities, proving completeness, decidability and FMP over general topological spaces, as well as over $T_0$ and $T_D$ spaces. We also investigate relational μ-calculus, providing general completeness results for all natural fragments of μ-calculus over many different classes of relational frames. Unlike most other such proofs for μ-calculus, ours is model-theoretic, making an innovative use of a known Modal Logic method (~the 'final' submodel of the canonical model), that has the twin advantages of great generality and essential simplicity.

I. INTRODUCTION

The modal μ-calculus is one of the most powerful extensions of modal logic, of great use in computer science applications. It is decidable, but very expressive, embedding many modal/temporal logics, such as PDL, CTL and CTL*, that are widely applied in program specification and verification.

The completeness of the modal μ-calculus was a difficult problem and remained open for many years. Even today, there are very few completeness results for axiomatic systems for μ-calculus with respect to standard classes of Kripke models (e.g., [1], [2], [3], see also a more recent proof theoretic approach [4]). Prior to our work, there seemed to be no general model-theoretic method to establish completeness for various natural fragments of μ-calculus over various classes of models.

An alternative interpretation of modal logic is based not on Kripke frames, but on topological spaces. This semantics is in fact older and can be traced back to McKinsey and Tarski [5]. When the modal ◊ is interpreted as topological closure and the modal □ as topological interior, one obtains a semantics for the modal logic S4 and its extensions, generalizing Kripke semantics over transitive, reflexive frames. The logic of all topological spaces in this semantics is S4. We refer to [6] for an overview of the rich landscape of results on topological completeness of modal logics above S4.

McKinsey and Tarski also suggested a second topological semantics, obtained by interpreting the modal ◊ as Cantor derivative.1 Esakia [7], [8] showed that the derivative logic of all topological spaces is the modal logic wK4 = K + (◊◊p → ◊p ∨ p). This is also the modal logic of all weakly transitive frames, i.e. those for which the reflexive closure of the accessibility relation is transitive. It is well-known that the modal logic of transitive frames is K4 [9], [10], which moreover corresponds to a natural class of topological spaces denoted $T_D$. Another natural class are $T_0$ spaces, whose modal logic is also finitely axiomatizable; we discuss $T_0$ spaces and $T_D$ spaces in the context of modal logic in Section III. Modal logics axiomatizing well-known classes of spaces also include the Gödel–Löb logic GL, which is complete with respect to the class of scattered spaces [11], [12].

Topological structures are of great interest to Computer Science. As noticed by Vickers [13] and Abramsky [14], the notion of observability and its logic require a topological setting. Abstract notions of computability also involve topological structures, and a famous example is Scott topology. More recently, developments in Formal Learning Theory [15], [16], Distributed Computing [17] and Epistemic Logic in Multi-Agent Systems [18], [19], [20], have taken a topological turn. In particular, recent epistemic work [20], [19] on modelling and reasoning about evidence and knowability uses topological structures. Research on spatial reasoning, in both topological and metric incarnations, is also of significant interest for AI. The addition of fixed point operators allows us to reason about non-trivial properties of topological spaces: for example, the well-known Cantor-Bendixson theorem states that any topological space has a perfect core, i.e. a maximal subset equal to its own derivative. The perfect core is not modally definable (in terms of derivative or closure modalities), but it is definable in the μ-calculus with the derivational semantics. Parikh [21] showed the relevance of Cantor derivative and the perfect core for multi-agent epistemic puzzles and applications.2

Our main aim in this paper is to investigate the topological μ-calculus based on the Cantor derivative modality, as well as its weaker version based on the closure modality. As a secondary aim, we explore (various fragments of) the relational μ-calculus, on (various classes of) weakly transitive frames. As such, our contribution in this paper is two-fold. First, we develop a new model-theoretic method of proving completeness for systems of μ-calculus over weakly transitive frames. This method applies to a wide range of logics, including many well-known ones. Concretely, we show that if a modal logic A is a canonical cofinal subframe logic, then its modal μ-variant,

1Recall that the derivative $d(A)$ of a set $A$ consists of all limit points of $A$.

2In on-going work, we show that the perfect core and its logic have deep connections with the topic of learnability from observations, as well as with epistemic paradoxes, such as the Surprise Examination.
obtained by adding the fixed-point axiom and induction rule, is Kripke complete and enjoys the finite model property. This implies that the modal $\mu$-variants of the well known modal logics wK4, wKT0, K4, KD4, K4.1, K4.2, K4.3, S4, S4.1, S4.2, and S4.3 have the FMP\(^3\) and are decidable. Second, we show that the derivational $\mu$-calculus is completely axiomatized on all topological spaces, all $T_0$ spaces, and all $T_D$ spaces, by the $\mu$-variants of the logics wK4, wKT0, and K4, respectively. We also give a new proof of the known fact that the weaker $\mu$-calculus based on topological closure is completely axiomatized by the $\mu$-variant of the modal logic S4.

Our model-theoretic proof is based on restricting the canonical model to the set of final theories, i.e. theories which satisfy a natural maximality condition. A similar construction has been employed by Fine [22] to prove FMP for subframe logics over K4 (without fixed point operators). Zakharyaschev [23] generalized this to show FMP for cofinal subframe logics over K4, and [24] extended this result to cofinal subframe logics over wK4. Our Kripke-completeness results apply to the $\mu$-variants of the same class of logics. The crucial new insight is that the truth lemma extends to the full $\mu$-calculus over such spaces and states our main completeness result. In Section IV, we show that the tangled fragment is not expressively complete in this setting. Section V investigates truth-preserving maps and relations between derivative spaces. Section VI presents the stepping stones of the main completeness proof. Section VII generalizes this to an infinite class of fixed-point logics, while Section VIII extends it to $T_0$ and $T_D$ spaces. We end in Section IX with some concluding remarks and a comparison with related work. All the proof details are in the Appendix.

II. DERIVATIVE SPACES

Although our primary focus in this paper is the derivational $\mu$-calculus on topological spaces, for technical reasons it is useful to consider a slightly more general class of structures.

Definition II.1. A derivative space is a pair $(X,d)$, where $X$ is a set of ‘points’, and $d : \mathcal{P}(X) \to \mathcal{P}(X)$ is an operator on subsets of $X$, satisfying the following properties, for all $X,Y \subseteq X$:

- $d(\emptyset) = \emptyset$;
- $d(X \cup Y) = d(X) \cup d(Y)$;
- $d(d(X)) \subseteq X \cup d(X)$.

The conjunction of the first two conditions above is known as normality, while the third condition is known as weak idempotence.

The notion of derivative space is the concrete set-theoretic instantiation of the more abstract concept of derivative algebra, introduced by Esakia [8] (as a generalization of a notion with the same name introduced by McKinsey and Tarski [5]).

Example II.2 (topological closure spaces). A special case of derivative spaces is given by closure spaces: these are derivative spaces $(X,c)$ that additionally satisfy $X \subseteq c(X)$ (and, a fortiori, $c(c(X)) \subseteq c(X)$). These strengthened conditions are known as the Kuratowski axioms, that define topological spaces in terms of their closure operator. When considered as a special case of derivative spaces, with $d(X) := c(X)$ given by topological closure, topological spaces will be called topological closure spaces.

Example II.3 (topological derivative spaces). Our main example of derivative spaces in this paper are structures $(X,d)$, based on an underlying topological (closure) space $(X,c)$ (satisfying the Kuratowski axioms), but with the derivative operator given by the so-called Cantor derivative, i.e. by taking $d(X)$ to be the set of limit points of $X$:

$$d(X) := \{ y \in X : y \in c(X - \{ y \}) \} = \{ y \in X : \forall U \in \mathcal{N}(y) \ X \cap (U - \{ y \}) \neq \emptyset \},$$

where $\mathcal{N}(y)$ is the family of (open) neighborhoods of $y$ in the space $(X,c)$. It is easy to see that $(X,d)$ is a derivative space, which we’ll refer to as a topological derivative space. The closure operator can be recovered as $c(X) = X \cup d(X)$.

\(^3\)In fact, there are continuum-many such logics [10], so our results apply to uncountably many classes of frames.
So, every topological space gives rise to a derivative space in at least two different ways (as a closure space, and as a topological derivative space), though we are mostly interested in the second one. The converse is also true:

Closure and interior in derivative spaces Given a derivative space \((X, d)\), we define the closure and interior operators \(c, i : \mathcal{P}(X) \rightarrow \mathcal{P}(X)\) by putting

\[
c(X) := X \cup d(X), \quad i(X) := X - c(X - X).
\]

It is easy to see that these satisfy all the Kuratowski axioms.

This means that every derivative space induces a topological space. Moreover, in a topological derivative space (with Cantor derivative over some topological space), the induced closure operator (as defined above) coincides with the underlying topological closure. But in general, this matching does not work the other way around: given an arbitrary derivative space, its derivative does not necessarily coincide with the Cantor derivative in the induced topology (given by the above-defined closure operator). It follows that not every derivative space is a topological derivative space. A counterexample is given by the next special case.

Example II.4 (weakly transitive Kripke frames). A weakly transitive frame (or wK4 frame) is a Kripke structure \((W, \rightarrow)\) (also known as a 'transition system'), consisting of a set of 'states' (or 'possible worlds') \(W\), together with a binary relation \(\rightarrow \subseteq W \times W\) (known as an 'accessibility' or 'transition' relation), assumed to be weakly transitive: i.e., for all states \(w, s, t \in W\), if \(w \rightarrow s \rightarrow t\) then either \(w \rightarrow t\) or \(w \rightarrow t\). We denote by \(\rightarrow^*\) the reflexive closure \(\mathcal{I}d \cup \rightarrow\) of \(\rightarrow\), which (due to weak transitivity) coincides with its transitive-reflexive closure \(\mathcal{I}d \cup \bigcup_{n \geq 1} \rightarrow^n\).

We also denote by \(\mathcal{C}\) the strict part of \(\rightarrow\), i.e. \(w \mathcal{C} v\) if \(w \rightarrow v\) and \(w \neq v\); and write \(w \leftrightarrow v\) if \(w \rightarrow v\) and \(w \rightarrow v\). For any state \(w \in W\), we put \(w^+ := \{s \in W : w \rightarrow s\}\) for the set of its successors, and also put \(w^* := \{s \in S : w \rightarrow^* s\}\); more generally, for any set \(X \subseteq W\), we put \(X^+ := \{s \in W : x \rightarrow s\) for some \(x \in X\} = \bigcup_{x \in X} x^+\), and similarly put \(X^* := \{s \in W : x \rightarrow^* s\) for some \(x \in X\} = X \cup X^*\). By applying the same definitions to the converse \(\leftarrow\), we obtain the corresponding notions of down-closure \(w_\downarrow, w_\downarrow^*, X_\downarrow, X_\downarrow^*\).

It is easy to see that every weakly transitive frame gives rise to a derivative space \((X, d\rightarrow)\), obtained by taking \(X := W\), and taking the derivative \(d\rightarrow\) to be usual modal 'Diamond' operator:

\[
d\rightarrow(X) := X_\downarrow = \{w \in W : X \cap w^+ \neq \emptyset\}
\]

Moreover, the induced closure \(c\rightarrow(X)\) (as defined above in arbitrary derivative spaces) is given by \(c\rightarrow(X) = X_\downarrow^*\).

In general, weakly transitive frames are not topological derivative spaces. But the intersection of the two classes is of independent interest, as shown by the next two examples:

Example II.5 (Alexandroff closure spaces as S4 Kripke frames). A topological space \((X, c)\) is Alexandroff if its closure operator distributes over arbitrary unions: \(c(\bigcup_i X_i) = \bigcup_i c(X_i)\). Given \(x, y \in X\), define \(x \rightarrow y\) if \(x \in c\{y\}\). Then, it is not hard to check that if \(X\) is Alexandroff, then \(\rightarrow\) is a reflexive-transitive relation, i.e. \((X, \rightarrow)\) is an S4 Kripke frame, and moreover the relational derivative coincides in this case with the topological closure: \(d\rightarrow = c\). As it is well-known, the converse also holds: every S4 frame \((X, \rightarrow)\) gives rise to an Alexandroff closure space, by putting \(c\rightarrow(X) := X \downarrow = X_\downarrow^*\) for the closure/derivative operator. This time, the equivalence is complete: starting from either side, and applying successively these two transformations, we obtain the original structure. So Alexandroff topological closure spaces are essentially the same as S4 Kripke frames.

Example II.6 (Alexandroff derivative spaces as irreflexive wK4 frames). Another way to convert an Alexandroff space \((X, c)\) into a relational structure is to define \(x \rightarrow y\) if \(x \in d\{y\} = c\{y\} - \{y\}\) for all \(x, y \in X\). Then \(\rightarrow\) is weakly transitive and irreflexive, and the relational derivative \(d\rightarrow\) coincides in this case with the Cantor derivative induced by \(c\). Conversely, every irreflexive wK4 frame \((X, \rightarrow)\) gives rise to an Alexandroff derivative space \((X, d)\), by putting \(c\rightarrow(X) := X _\downarrow^*\) for the topological closure, and taking \(d\rightarrow\) to be induced Cantor derivative in the resulting topology (for which one can check that \(d(X) = X_\downarrow\)). Once again, the equivalence is complete: by applying successively these transformations, we obtain the original structure. So Alexandroff topological derivative spaces are essentially the same as irreflexive wK4 frames.

D-neighborhoods For every point \(x \in X\) in a derivative space \((X, d)\), we can define the family of \(d\)-neighborhoods of \(x\):

\[
N_d(x) := \{X \subseteq X : x \notin d(X - X)\}
\]

Note that, in general, \(d\)-neighborhoods are not neighborhoods of \(x\) in the topology given by the closure \(c(X)\) induced by \(d\). In fact, in a topological derivative space (where derivative means Cantor derivative), a \(d\)-neighborhood \(X \in N_d(x)\) is just a 'punctured neighborhood' of \(x\), i.e. a set with the property that \(U - \{x\} \subseteq X\) for some open neighborhood \(U\ni x\). On the other hand, in a topological closure space (where the 'derivative' is just the topological closure), \(d\)-neighborhoods coincide with standard topological neighborhoods. Finally, in a weakly transitive frame \((W, \rightarrow)\), a set \(X \subseteq W\) is a \(d\)-neighborhood of a state \(x \in W\) iff \(x^+ \subseteq X\).

We can now characterize the derivative in terms of \(d\)-neighborhoods, in a way that generalizes the definition of Cantor derivative in topological spaces:

Lemma II.7. For every set \(X \subseteq X\) in a derivative space \((X, d)\), we have

\[
d(X) = \{y \in X : \forall U \in N_d(y) U \cap X \neq \emptyset\}.
\]

This leads to an equivalent presentation of derivative spaces as a special case of monotonic neighborhood structures [28]: a
neighborhood derivative space is a pair \((X, N(x))\), where \(X\) is a set of points, and \(N:\ X \to \mathcal{P}(\mathcal{P}(X))\) is a map that assigns to each point \(x \in X\) a family \(N(x) \subseteq \mathcal{P}(X)\) of ‘neighborhoods’ of \(x\), satisfying the following conditions:

1) \(X \in N(x)\);
2) if \(X \in N(x)\) and \(X \subseteq Y\), then \(Y \in N(x)\);
3) if \(X, Y \in N(x)\), then \(X \cap Y \in N(x)\);
4) if \(x \in X \in N(x)\), then \(\{y \in X : X \in N(y)\} \in N(x)\).

Each derivative space \((X, d)\) gives rise to a neighborhood derivative space by taking 

\[ N(x) := N_d(x) = \{X \subseteq X : x \notin d(X - X)\} \]

to be the set of all \(d\)-neighborhoods. Conversely, every neighborhood derivative space \((X, N)\) gives rise to a derivative space, via the following generalization of Cantor derivative:

\[ d(X) := \{y \in X : \forall U \in N(y) U \cap X \neq \emptyset\}. \]

This is a full equivalence between derivative spaces and neighborhood derivative spaces: starting from either side, and applying the above two transformations, we obtain the original structure.

III. MU CALCULUS ON DERIVATIVE SPACES: MAIN RESULTS

For reasons having to do with our intended applications, as well as to simplify some proof details, in this paper we take the greatest fixed point operator \(\nu x.\phi\) as primitive, and define the least fixed point \(\mu x.\phi\) as an abbreviation.\(^5\)

**Syntax:** Let \(P\) be a set of propositional variables. We recursively define the set \(L_\mu\) of formulas, together with a map \(\text{Free} : L_\mu \to \mathcal{P}(P)\), associating to each formula \(\phi \in L_\mu\) its set of free variables \(\text{Free}(\phi) \subseteq P\). The definition is by simultaneous recursion, with formulas \(\phi \in L_\mu\) given by

\[
\phi ::= \top \mid x \mid \neg \phi \mid \phi \land \phi' \mid \Diamond \phi \mid \nu x. \phi
\]

where: \(x \in P\); in the construct \(\phi \land \phi'\), no variables occur free in \(\phi\) and bound in \(\phi'\), or vice versa; and in the construct \(\nu x. \phi\), formula \(\phi\) is positive in \(x\) (i.e. whenever \(x\) occurs in \(\phi\), we have that \(x \in \text{Free}(\phi)\) and \(x\) occurs only in the scope of an even number of negations). The set \(\text{Free}(\phi)\) of free variables of a formula \(\phi\) is simultaneously defined by recursion:

\[
\text{Free}(\top) := \emptyset, \quad \text{Free}(x) := \{x\},
\]

\[
\text{Free}(\phi \land \phi') := \text{Free}(\phi) \cup \text{Free}(\phi'),
\]

\[
\text{Free}(\neg \phi) := \text{Free}(\Diamond \phi) := \text{Free}(\phi),
\]

\[
\text{Free}(\nu x. \phi) := \text{Free}(\phi) - \{x\}.
\]

A variable is **bound** in \(\phi\) if it occurs in \(\phi\) but is not in \(\text{Free}(\phi)\). For any set of variables \(P \subseteq P\), we denote by \(L_\mu^P\) the set of all formulas \(\phi \in L_\mu\) having \(\text{Free}(\phi) \subseteq P\). Note in particular that \(L_\mu = L_\mu^P\).

We use the notation \(\mathcal{T} = (x_1, \ldots, x_n)\) to denote finite strings of variables \(x_1, \ldots, x_n \in P\), and denote by \(\lambda\) the empty string.

\(^5\)This setting is of course equivalent to the more standard presentation, that takes \(\mu x.\phi\) as primitive.

When we want to make explicit some of the free variables, we write \(\phi(\mathcal{T})\) for a formula in which all variables in the string \(\mathcal{T}\) are free (if occurring at all).

**Subformulas** The subformula relation \(\sqsubseteq\) is the smallest transitive relation on formulas satisfying the following properties: \(\phi \sqsubseteq (\neg \phi), (\Diamond \phi), (\nu x. \phi),\) and \(\phi \sqsubseteq (\phi \land \phi')\). The set \(\text{Sub}(\phi)\) of all (improper) subformulas of \(\phi\) is defined as \(\text{Sub}(\phi) := \{\phi' : \phi' \sqsubseteq \phi\} \cup \{\phi\}\).

**Semantics.** An **atomic valuation** on a derivative space \((X, d)\) is a map \(\parallel \cdot \parallel : \mathcal{P}(X) \to \mathcal{P}(\mathcal{P}(X))\) associating to each propositional atom \(x \in P\) some set of states \(\{x\} \subseteq X\). For each atomic valuation \(\parallel \cdot \parallel : \mathcal{P}(X) \to \mathcal{P}(\mathcal{P}(X))\), tuple \(\mathcal{F} = (x_1, \ldots, x_n)\) of variables and corresponding tuple \(X = (X_1, \ldots, X_n)\) of sets of points \(X_i \subseteq X\), we denote by \(\parallel \cdot \parallel_{\mathcal{F}=X}\) the valuation that assigns to each variable \(x_i\) the set \(X_i\) and agrees with the original valuation \(\parallel \cdot \parallel\) on all the other atoms.

A **derivative model** \(\mathcal{M} = (X, d, \parallel \cdot \parallel)\) consists of a derivative space \((X, d)\), together with an atomic valuation \(\parallel \cdot \parallel : \mathcal{P}(X) \to \mathcal{P}(\mathcal{P}(X))\). The semantics is given by extending the atomic valuation to a map \(\parallel \cdot \parallel : L_\mu \to \mathcal{P}(\mathcal{P}(X))\), which we call extended valuation (and for which we use the same notation \(\parallel \cdot \parallel\) as for the corresponding atomic valuation). The definition of the extended valuation is by recursion on subformulas; for propositional variables \(x\) this is already given by the atomic valuation map of the model \(\mathcal{M}\), while in the rest we put

\[
\parallel \top \parallel = X, \quad \parallel \neg \phi \parallel = X - \parallel \phi \parallel, \quad \parallel \phi \land \phi' \parallel = \parallel \phi \parallel \cap \parallel \phi' \parallel, \quad \parallel \Diamond \phi \parallel = d(\parallel \phi \parallel), \quad \parallel \nu x. \phi \parallel = \bigcup\{X \subseteq X : X \subseteq \parallel \phi \parallel_{x:=x}\}.
\]

For formulas \(\phi = \phi(\mathcal{F})\) and corresponding tuples of sets \(X\), we will sometimes write \(\parallel \phi(\mathcal{F}) \parallel\) instead of \(\parallel \phi \parallel_{\mathcal{F}=X}\), in order to avoid subscript overload. With this notation, e.g., the clause for \(\nu x. \phi\) becomes:

\[
\parallel \nu x. \phi \parallel = \bigcup\{X \subseteq X : X \subseteq \parallel \phi(\mathcal{F}) \parallel\}.
\]

Whenever \(x \in \parallel \phi \parallel\) for some point \(x \in X\), we also write \(x \models_\mathcal{M} \phi\), and say that \(\phi\) is true (or satisfied) at point \(x\) in the model \(\mathcal{M}\). As usual, when the model is understood, we skip the subscript, writing \(x \models \phi\). Conversely, we may write \(\parallel \phi \parallel_{\mathcal{M}}\) instead of \(\parallel \cdot \parallel\) when we wish to specify the relevant model.

We say that \(\phi\) is valid on the model \(\mathcal{M}\) if \(\parallel \phi \parallel_{\mathcal{M}} = X\), i.e. \(\phi\) is true at all points of \(\mathcal{M}\); similarly, \(\phi\) is satisfied on the model \(\mathcal{M}\) if \(\parallel \phi \parallel_{\mathcal{M}} \neq \emptyset\). By abstracting away from the specific valuation, we say that \(\phi\) is valid on the space \((X, d)\) if for every valuation \(\parallel \cdot \parallel\) on \(X\), \(\phi\) is valid on the model \((X, d, \parallel \cdot \parallel)\); and \(\phi\) is satisfied on the space \((X, d)\) if there exists a valuation \(\parallel \cdot \parallel\) on \(X\), s.t. \(\phi\) is satisfied on the model \((X, d, \parallel \cdot \parallel)\). Finally, \(\phi\) is valid (on a class \(\mathcal{C}\) of derivative models, or of derivative spaces) if it is valid on all models/spaces (in the class \(\mathcal{C}\)).

Note that in the special case of wK4 relational models \((W, \rightarrow)\), the above semantics of \(\Diamond\) coincides with the standard Kripke semantics. As a consequence, on relational frames our semantics for \(\mu\)-calculus coincides with the standard one.

**Abbreviations.** We have the usual abbreviations \(\bot, \phi \lor \psi, \phi \Rightarrow \psi, \phi \Leftrightarrow \psi, \square \phi\). The least fixed-point formula \(\mu x. \phi(x, \mathcal{F})\) can be defined as \(\neg \nu x. \phi(-x, \mathcal{F})\). Finally, we define closure
and interior modalities, as well as tangled derivative \(\Diamond^\infty\Gamma\) and tangled closure \(\langle*\rangle^\infty\Gamma\) (for finite sets of formulas \(\Gamma\)), with the perfect core modality \(\Diamond^\infty\varphi\) as a special case:

\[
\langle*\rangle\varphi := \varphi \lor \Diamond\varphi, \quad [\varphi] := \varphi \land \Box\varphi, \\
\Diamond^\infty\Gamma := \nu x. \bigwedge_{\gamma \in \Gamma} \Diamond(x \land \gamma), \quad \langle*\rangle^\infty\Gamma := \nu x. \bigwedge_{\gamma \in \Gamma} \langle*\rangle(x \land \gamma), \\
\Diamond^\infty\varphi := \Diamond^\infty\{\varphi\}.
\]

Note that the definitions of \(\langle*\rangle\varphi\) and \([\varphi]\) do not use any fixed points. But, to justify these notations, one can easily check that, in the special case of weakly transitive frames, \([\varphi]\) and \(\langle*\rangle\varphi\) are the standard Kripke modalities for the reflexive-transitive closure \(\rightarrow^*\) of the accessibility relation (which, as already mentioned, coincides on these frames with its reflexive closure). More generally, in derivative spaces, \([\varphi]\) is equivalent to \(\nu x. (\varphi \land \Box x)\), while \(\langle*\rangle\varphi\) is equivalent to \(\neg[\varphi] \land \Box \varphi\) and thus to \(\mu x. (\varphi \lor \Box x)\). In fact, \(\langle[\varphi]\rangle\) and \(\langle*\rangle\varphi\) coincide with the interior \(i([\varphi])\) and respectively the closure \(c([\varphi])\), as defined in derivative spaces. In particular, in the case of topological derivative spaces (where \(d\) is Cantor derivative), these coincide with the underlying topological interior and closure operators. As for \(\Diamond^\infty\Gamma\) and \(\langle*\rangle^\infty\Gamma\), they are variants of the tangle modality introduced in a relational setting by Dawar and Otto [27], who showed that \(\mu\)-calculus over transitive frames collapses to tangle logic based on \(\Diamond^\infty\Gamma\). Their topological interpretations were developed by Fernandez-Duque [29], who distinguished between the tangled derivative \(\Diamond^\infty\Gamma\) and tangled closure \(\langle*\rangle^\infty\Gamma\), and axiomatized the logic of tangled closure.

More recently, Goldblatt and Hodkinson [26] axiomatized the logic of tangled derivative \(\Diamond^\infty\Gamma\) over transitive frames, and showed that it is equivalent to the logic over \(T_D\) spaces. Finally, the perfect core modality \(\Diamond^\infty\varphi\) is a special case of tangle, that captures Cantor’s perfect core: the largest subset of the state space that is equal to its own Cantor derivative.

**Substitution and natural sublanguages** Given a formula \(\varphi = \varphi(\pi)\) and a tuple of formulas \(\varphi(\overline{\theta}) = (\theta_1, \ldots, \theta_n)\), we denote by \(\varphi(\overline{\theta})\) the result of substituting every variable in \(\pi\) by the corresponding formula in \(\overline{\theta}\). Note that we have

\[
\|\varphi(\overline{\theta})\| = \|\varphi(\overline{\theta(\overline{\theta})})\|
\]

(where on the right hand we used an instance of the above-mentioned simplified notation \(\|\varphi(\overline{\theta})\|\) for \(\|\varphi(\overline{\theta(\overline{\theta})})\|\)). A natural sublanguage of \(\mathcal{L}_\mu\) is any set \(\mathcal{L} \subseteq \mathcal{L}_\mu\), which contains \(\top\), is closed under substitution, and such that if \(\varphi, \psi \in \mathcal{L}\), then \(\neg\varphi, \varphi \land \psi, \psi \in \mathcal{L}\). The basic modal language is a natural sublanguage and will be denoted \(\mathcal{L}_0\).

The following characterization of \(\|\nu y.\varphi\|\) is also well-known in the literature:

**Proposition III.1.** Let \((\mathcal{X}, d, \|\cdot\|)\) be any derivative model and \(\varphi = \varphi(y, \pi)\) be a formula that is positive in \(y\). Then we have the following:

1) the unary operator \(Y \mapsto \|\varphi(Y, \pi)\|\) is monotonic: if \(Y \subseteq Y'\) then \(\|\varphi(Y, \pi)\| \leq \|\varphi(Y', \pi)\|\):

2) \(\|\nu y.\varphi(y, \pi)\|\) is the greatest fixed point of the operator \(Y \mapsto \|\varphi(Y, \pi)\|\), i.e. the largest set \(Y \subseteq X\) s.t. \(Y = \|\varphi(Y, \pi)\|\):

3) \(\|\nu y.\varphi(y, \pi)\| = \bigcap_{\alpha \in \Omega} \varphi^\alpha(\pi)\), where \(\Omega\) is the class of ordinals and the transfinite sequence of sets \(\varphi^\alpha(\pi)\) \(\subseteq X\) is defined by ordinal recursion: \(\varphi^\alpha(\pi) = \bigcap_{\beta < \alpha} \|\varphi^\beta(\pi, \pi)\|\)

**Proof.** Well known (and easy to check). □

**Definition III.2 (\(\mu\)-wK4).** We define the logic \(\mu\)-wK4 to be the least set of formulas of \(\mathcal{L}_\mu\) containing the following axioms and closed under the following rules (for all formulas \(\varphi, \psi, \theta\), and formulas \(\theta = \theta(x)\) that are positive in \(x\)):

- All the instances of the Axioms and Rules of Propositional Logic.
- Necessitation Rule: From \(\varphi\), infer \(\Box\varphi\).
- Distribution Axiom (=Kripke’s Axiom K):
  \(\Box(\varphi \land \psi) \Rightarrow (\Box \varphi \land \Box \psi)\).
- Weak Transitivity: \(\Box \varphi \Rightarrow (\varphi \lor \Box \varphi)\).
- Fixed Point Axiom: \(\nu x. \theta \Rightarrow \theta(\nu x. \theta)\).
- Induction Rule: From \(\varphi \Rightarrow \theta(\varphi)\), infer \(\varphi \Rightarrow \nu x. \theta\).

We will also be interested in variants of \(\mu\)-wK4. If \(\Lambda\) is any normal modal logic over \(\mathcal{L}_0\) (in the sense of [9]) that extends wK4, then \(\mu\)-\(\Lambda\) is the extension of \(\mu\)-wK4 with all axioms of \(\Lambda\), closed under uniform substitution with arbitrary formulas in \(\mathcal{L}_\mu\). If \(\Lambda\) is a natural sublanguage of \(\mathcal{L}_\mu\), then \(\mu\)-\(\Lambda\) is the restriction of \(\mu\)-\(\Lambda\) to \(\mathcal{L}\), in the sense that all axioms and rules may only be applied when all formulas belong to \(\mathcal{L}\).

**Proposition III.3 (Soundness).** The logic \(\mu\)-wK4 is sound for the class of derivative spaces, and so in particular for the class of weakly transitive frames. If \(\Lambda\) is any extension of wK4, then \(\mu\)-\(\Lambda\) is sound for the class of \(\Lambda\)-spaces, i.e. the class of spaces validating all theorems of \(\Lambda\).

Our goal is to show that this system is also (weakly) complete, and that the logic is decidable. But for this, we need first look at some theorems of the above axiomatic system.

**Proposition III.4.** The following schemas are provable in the logic \(\mu\)-wK4 (for all formulas \(\varphi, \psi\), and formulas \(\theta = \theta(x)\) that are positive in \(x\)):

1) \(\nu x. \theta \leftrightarrow \theta(\nu x. \theta)\)

2) \([\star\varphi \land \theta(\psi)] \Rightarrow \theta([\varphi \land \psi])\)

3) \([\varphi \Rightarrow \theta(\psi)] \Rightarrow (\theta(\varphi) \Rightarrow \theta(\psi))\)

4) \([\varphi \Rightarrow \theta(\varphi)] \Rightarrow (\theta \Rightarrow \nu x. \theta)\)

We are now ready to state the first of our main results. Below, recall that \(\Lambda\) has the finite model property if for any formula \(\varphi\), \(\varphi\) is a theorem of \(\Lambda\) iff \(\varphi\) is valid over the class of finite \(\Lambda\)-models. The logic \(\Lambda\) has the strong finite model property if the size of a finite countermodel for \(\varphi\) can be bounded by a function computable from the length of \(\varphi\).
Moreover, discuss the tangled derivative in the context of lent modal logic, while ours is about the full language of all topological spaces, and thus also for the class of Alexandroff spaces (irreflexive weakly transitive frames). Hence, $\mu$-$\text{wK4}^L$ is complete for the class of all topological spaces, and thus also for the larger class of all derivative spaces. The logic has the strong finite model property (wrt all the above classes), and so its satisfiability problem is decidable.

This will be proven in Section VI, while in Section VII we generalize this result to many other classes of relational structures and the corresponding logics.

We conclude this section by discussing two extensions of $\text{wK4}$ that are of interest in the context of topological semantics. Recall that a topological space $(X, c)$ is $T_0$ if given $x, y \in X$ with $x \neq y$, either $x \notin c(y)$ or $y \notin c(x)$ (i.e., the two do not have the same set of neighborhoods). It is known (see [30]) that the derivational modal logic of $T_0$ spaces is the system

$$\text{wK4}_0 := \text{wK4} + p \land (q \land \neg p) \rightarrow \neg p \lor (q \land \neg q).$$

Moreover, $\text{wK4}_0$ corresponds to the class of $\text{wK4}$ frames $(W, \rightarrow)$ so that $w \rightarrow v \rightarrow w$ implies that $w \rightarrow w$ or $v \rightarrow v$. Frames satisfying this property are weakly reflexive.

If we define the cluster of $w \in W$ to be the set of points $v$ so that $v \rightarrow^* w \rightarrow^* v$ (equivalently: s.t. either $w \leftrightarrow v$ or $w = v$), then a weakly transitive frame $(W, \rightarrow)$ is weakly reflexive iff every cluster has at most one irreflexive point.

The second extension we are interested in is $K4$, given by

$$\text{wK4}_0 + \neg \neg p \rightarrow p.$$ 

It is well known (see, e.g., [10], [8], [30]) that this is the logic of all transitive frames, and that it is also the logic of all $T_0$ spaces. These are topological spaces $(X, c)$ such that every point is isolated in its own closure; i.e., if $x \in X$, there is an open set $U$ with $\{x\} = U \cap c(x)$. These results readily extend to the derivative $\mu$-calculus.

**Theorem III.5 (Completeness, FMP and Decidability).** Let $L$ be a natural sublanguage of $\mathcal{L}_\mu$. The logic $\mu$-$\text{wK4}^L$ is (weakly) complete for the class of all weakly transitive frames, as well as for the class of Alexandroff spaces (irreflexive weakly transitive frames). Hence, $\mu$-$\text{wK4}^L$ is complete for the class of all topological spaces, and thus also for the larger class of all derivative spaces. The logic has the strong finite model property (wrt all the above classes), and so its satisfiability problem is decidable.

IV. **EXPRESSION INCOMPLETENESS OF TANGLE LOGIC**

A natural question is whether topological $\mu$-calculus collapses to a simpler modal fragment; if so, then a complete axiomatization of the simpler fragment would in principle suffice, and might be easier to prove than for the full calculus. As mentioned in the Introduction, this is exactly what happened on $T_D$ spaces. Dawar and Otto [27] showed that the full $\mu$-calculus is expressively equivalent to the so-called tangled derivative logic $\mathcal{L}_{\omega,\omega}$ over the class of (finite) $K4$ frames, and thus also over $T_D$ spaces; while Goldblatt and Hodkinson [26] completely axiomatized $\mathcal{L}_{\omega,\omega}$ over these classes.6

In this section, we show that the Dawar-Otto result does not hold for general spaces, and in fact not even for $T_0$ spaces: the tangled derivative logic is no longer expressively equivalent to the $\mu$-calculus over the class of $\text{wK4}T_0$ frames, and hence over the class of all $T_0$ spaces.

For each finite set of formulas $\Gamma$, consider the tangled derivative $\diamond^{\infty} \Gamma$ and tangled closure $\langle \ast \rangle^{\infty} \Gamma$ of $\Gamma$, as defined in Section III. Let $\mathcal{L}_{\ast,\omega}$ and $\mathcal{L}_{\ast,\omega}$ be the natural sublanguages of the $\mu$-calculus whose only fixed points are of the respective forms above. To prove that $\mathcal{L}_{\ast,\omega}$ is not expressively complete for $\mu$-calculus over $\text{wK4}T_0$ frames, we will show that $\langle \ast \rangle^{\infty}$ is not definable in $\mathcal{L}_{\ast,\omega}$.

For this, we define a ‘spine’ model $S$ based on the ordinal $\omega + 3$. We briefly recall that $\omega$ denotes the first infinite ordinal, and follow the set-theoretic convention that each ordinal is identified with its set of predecessors. We moreover use interval notation on the ordinals: $(\alpha, \beta)$ is the set of ordinals $\xi$ with $\alpha < \xi < \beta$.

With this in mind, we set $S = (\omega + 3, \rightarrow, \| \cdot \|)$, where

1) $\alpha \rightarrow \beta$ if one of the following occurs:
   a) $\alpha > \beta$;
   b) $\alpha = \beta$ and $\alpha$ is odd (including $\omega + 1$), or
   c) $\alpha = \omega + 1$ and $\beta = \omega + 2$.

2) $\alpha \in \| p \|$ iff $\alpha$ is odd, $\| q \| = \emptyset$ for all $q \neq p$.

**Lemma IV.1.** $S$ is a $\text{wK4}_0$ model.

The goal is to show that over $S$, no $\mathcal{L}_{\omega,\omega}$ formula is equivalent to $\langle \ast \rangle^{\infty} \{ p, \neg p \}$. First, we evaluate the latter.

**Lemma IV.2.** Over $S$, $\| \langle \ast \rangle^{\infty} \{ p, \neg p \} \| = \{ \omega + 1, \omega + 2 \}$.

The following can then be checked by induction on the complexity of $\varphi$. In fact, it suffices to take $n_\varphi = 2|\varphi|$.

**Lemma IV.3.** If $\varphi$ is any formula of $\mathcal{L}_{\omega,\omega}$ then there exists $n_\varphi < \omega$ such that for every $\alpha, \beta \in (n_\varphi, \omega + 3)$ which are either both even or both odd, $\alpha \in \| \varphi \|$ iff $\beta \in \| \varphi \|$.

Since $n < \omega, \omega + 2$ for all $n < \omega$ and the two are even, we obtain the following special case:

6On the other hand, Goldblatt and Hodkinson [26] showed that $\diamond^{\infty}$ is not definable in $\mathcal{L}_{\ast,\omega}$ over the class of $K4$ frames, and hence over the class of $T_D$ spaces. It follows that $\mathcal{L}_{\ast,\omega}$ is not expressively complete, even over the class of $K4$ frames.
Corollary IV.4. In $S$, $\omega$ and $\omega + 2$ satisfy the same formulas of $L_{\omega}$. 

However, we have seen that $\omega + 2 \in \{p, \neg p\}$, but $\omega \notin \{p, \neg p\}$. We may thus conclude that $\{p, \neg p\}$ is not definable.

Theorem IV.5. The formula $\varphi = \{p, \neg p\}$ is not definable in $L_{\omega}$, even by an infinite set of formulas.

Given that $\{p, \neg p\}$ is definable in the $\mu$-calculus but not in $L_{\omega}$, we obtain the following result.

Corollary IV.6. Not every formula of the $\mu$-calculus is definable in $L_{\omega}$ over the class of wK4T0 frames.

For this reason, in this paper we will work over the full $\mu$-calculus, rather than the tangled derivative fragment.

V. TRUTH-PRESERVING MAPS

In this section, we focus on the relational semantics, and review and generalize some well-known properties of $\mu$-calculus [31]: locality and invariance under bounded morphisms.

Definition V.1 (D-morphisms and P-morphisms). A d-morphism between derivative spaces $(X, d)$ and $(X', d')$ is a function $\pi: X \to X'$ such that $d^{-1}d'(X') = d\pi^{-1}(X')$ for all sets $X' \subseteq X'$.

If $\pi$ is surjective, we say that the space $X'$ is a d-morphic image of the space $X$.

For any set $P \subseteq \mathbb{P}$, a P-morphism between derivative models $M = (X, d, || \cdot ||)$ and $M' = (X', d', || \cdot ||')$ is a d-morphism $\pi: X \to X'$ such that $\pi^{-1}d'(X') = d\pi^{-1}(X')$ for all atoms $x \in P$.

If $\pi$ is surjective, we say that the model $M'$ is a P-morphic image of the model $M$.

Remark V.2. The notion of P-morphism is a generalization to derivative spaces of the well-known concept of p-morphism [9], albeit relativized to a set of variables $P$. The restriction to such a set of variables (particularly, when $P$ is finite) will be essential in many of our proofs.

Lemma V.3. If $\pi: X \to X'$ is a P-morphism between derivative models $M = (X, d, || \cdot ||)$ and $M' = (X', d', || \cdot ||')$, then for all $\mu$-calculus formulas $\varphi = \varphi(\pi) \in L_{\mu}^P$ and tuples of sets $X_i$, $X'_i$ s.t. $X_i = \pi^{-1}X'_i$ for all $i$, we have:

$$||\varphi(X)|| = \pi^{-1}||\varphi(X')||'.$$

It is useful to keep in mind the special case where the tuple of substitution variables is empty.

Corollary V.4 (Invariance under P-morphisms). If $\pi: X \to X'$ is a P-morphism between derivative models $M = (X, d, || \cdot ||)$ and $M' = (X', d', || \cdot ||')$, then for all $\varphi \in L_{\mu}^P$ we have:

$$||\varphi|| = \pi^{-1}||\varphi'||.$$

In practice, P-morphisms are most useful when they are surjective, as they then preserve validity of formulas.

Corollary V.5.

1) If a derivative model $M' = (X', d', || \cdot ||')$ is a P-morphic image of a model $M = (X, d, || \cdot ||)$, then the two models validate (satisfy) the same formulas of $L_{\mu}^P$.

2) If a derivative space $(X', d')$ is a d-morphic image of space $(X, d)$, then every formula that is satisfiable on $(X', d')$ is also satisfiable on $(X, d)$; equivalently: every formula that is valid on $(X, d)$ is also valid on $(X', d')$.

It is useful to have a more ‘bisimulation-like’ characterization of d-morphisms. Using the equivalence between derivative spaces and neighborhood derivative spaces, we can characterize d-morphisms in terms of d-neighborhoods:

Lemma V.6. Let $\pi: X \to X'$ be a map between derivative spaces $(X, d)$ and $(X', d')$. Then the following are equivalent:

1) $\pi$ is a d-morphism;

2) the conjunction of the following back-and-forth conditions holds for all points $x \in X$ and all sets $X \subseteq X$ and $X' \subseteq X'$:

- (back) $X' \subseteq N_{d'}(\pi(x))$ implies $\pi^{-1}(X') \subseteq N_d(x)$, and

- (forth) $X \subseteq N_d(x)$ implies $\pi(X) \subseteq N_{d'}(\pi(x));$

3) $\pi^{-1}(X') \subseteq N_d(x)$ iff $X' \subseteq N_{d'}(\pi(x))$, for all $x \in X$ and $X' \subseteq X'$.

This follows from the general theory of bounded morphisms in monotonic neighborhood models [32], [28]; indeed, the third equivalent statement is exactly the definition of a bounded morphism in monotonic neighborhood semantics. When both spaces are topological derivative spaces, the back-and-forth conditions refer to punctured neighborhoods. When both are topological closure spaces, we obtain the usual notion of interior map. The case where $X$ is a topological space and $X'$ a Kripke frame is of a special interest:

Corollary V.7. Let $\pi: X \to X'$ be a map between a topological derivative space $(X, d)$ and a weak transitve frame $(X', \rightarrow)$. Then the following are equivalent:

1) $\pi$ is a d-morphism;

2) the conjunction of the following back-and-forth conditions holds for all points $x \in X$:

- $\pi(U \setminus \{x\}) \subseteq \pi(U \setminus \{x\})$ for some open neighborhood $U$ of $x$, and

- $\pi(U \setminus \{x\}) \subseteq \pi(U \setminus \{x\})$, for all open neighborhoods $U$ of $x$.

Finally, when both spaces are weakly transitve frames, we recover the standard notion of bounded frame morphism:

P-morphisms and P-bisimulations between relational models. When both $X$ and $X'$ are weakly transitve frames, it is easy to see that our notion of P-morphism matches the standard modal notion of p-morphisms (also known as “bounded P-morphisms”), i.e. functional P-bisimulations.

Definition V.8. Let $M_1 = (W_1, \rightarrow_1, || \cdot ||_1)$ and $M_2 = (W_2, \rightarrow_2, || \cdot ||_2)$ be relational models. A relation $B \subseteq W_1 \times W_2$ is a P-bisimulation if: 
W₂ is a P-bisimulation if, for all states \( w_1 \in W₁, w_2 \in W₂, (w₁, w₂) \in B \) implies three conditions: (a) \( w_1 \in \|p\| \) iff \( w₂ \in \|p\| \) (Atomic Preservation); (b) if \( w₁ \rightarrow s₁ \), then there exists some \( s₂ \in W₂ \) with \( w₂ \rightarrow s₂ \) and \( (s₁, s₂) \in B \) (Forth Condition); (c) if \( w₂ \rightarrow s₂ \) then there exists some \( s₁ \in W₁ \) with \( w₁ \rightarrow s₁ \) and \( (s₁, s₂) \in B \) (Back Condition).

Then, a bounded P-morphism is just a functional P-bisimulation. It is well known that relational P-bisimulations between weakly transitive relational models \( M₁ = (X₁, \rightarrow₁, \cdot, \emptyset₁) \) and \( M₂ = (X₂, \rightarrow₂, \cdot, \emptyset₂) \) are exactly the relations of the form \( \pi₁^−¹ \circ \pi₂ \), where \( \pi₁ : M \rightarrow M₁ \) and \( \pi₂ : M \rightarrow M₂ \) are P-morphisms from some other weakly transitive model \( M \) into the two models, and \( ; \) is relational composition.²

Invariance under bisimilarity The relation of P-bisimilarity \( \simeq_P \) on a given model \( M = (W, \rightarrow, \| \cdot \|) \) is the largest P-bisimulation relation \( \simeq_P \subseteq W \times W \). When \( P = \mathbb{P} \), we drop the subscript, writing e.g. \( s \simeq w \) and talking simply of ‘bisimulation’ and ‘bisimilarity’. It is easy to see that P-bisimilarity is an equivalence relation on \( W \). The following fact is a widely known feature of µ-calculus:

**Proposition V.9 (Invariance under Bisimilarity).** The valuation \( \| \varphi \| \) of every formula \( \varphi \in \mathcal{L}_P^P \) is closed under P-bisimilarity: for all \( s, w \in W \), if \( s \simeq_P w \) and \( s \in \| \varphi \| \), then \( w \in \| \varphi \| \).

**Proof.** This is well-known (and easy to verify directly). □

**Locality** Another known fact is that \( \mu \)-calculus is “local” — the truth value of a formula \( \varphi \) at a state depends only on the accessible part of the model (i.e., the so-called generated submodel). This can be generalized as follows:

**Lemma V.10.** Let \( \varphi = \varphi(\overline{X}, \overline{Y}) \) be a formula. Then we have the following:

1) \( \| \varphi(\overline{X}, \overline{Y}) \| \cap w^⁺ = \| \varphi(\overline{X}(w), \overline{Y}) \| \cap w^⁺ \), for all states \( w \in W \) and tuples of sets of states \( \overline{X}, \overline{Y} \);

2) If \( \overline{Y} = (\overline{Z}, y) \) and \( \varphi = \varphi(\overline{X}, \overline{Y}) = \varphi(\overline{X}, \overline{Z}, y) \) is positive in \( y \), then

\[ \varphi^\alpha_y(\overline{X}, \overline{Z}) \cap w^⁺ = \varphi^\alpha_y(\overline{X}, \overline{Z}(w), w^⁺) \cap w^⁺ \]

for all states \( w \in W \), ordinals \( \alpha \in \Omega \) and tuples of sets of states \( \overline{X}, \overline{Z} \). (Here, \( \varphi^\alpha_y \) is the sequence introduced in Proposition III.1(3).)

**Asserting properties locally above a point** Given a point \( w \in W \), and a property \( P(X₁, \ldots, Xₙ) \) involving sets \( X₁, \ldots, Xₙ \subseteq W \), we say that \( P(X₁, \ldots, Xₙ) \) holds above \( w \) if we have \( P(X₁ \cap w^⁺, \ldots, Xₙ \cap w^⁺) \). In particular, for two sets \( X, Y \subseteq W \), we say that \( X = Y \) holds above \( w \) iff \( X \cap w^⁺ = Y \cap w^⁺ \).

**Depth of a point in a model** Recall that \( \rightarrow₂ \) is the strict preorder induced by \( \rightarrow \). Given a weakly transitive model \( M = (W, \rightarrow, \| \cdot \|) \), and a point \( w \in W \), a strict (finite) w-chain is a finite sequence of points of the form \( w = w₀ \rightarrow₂ \cdots \rightarrow₂ wₙ \). The number \( n \) is called the length of our finite chain. The depth \( \mathrm{dpt}(w) \) of the point \( w \in W \) is the supremum of the lengths of all strict w-chains. In general, we have \( \mathrm{dpt}(w) \geq 0 \), with \( \mathrm{dpt}(w) = 0 \) iff for every \( s \in W \), \( w \rightarrow s \) implies \( s \rightarrow w \); and \( \mathrm{dpt}(w) = \omega \) iff there exist w-chains of every length \( n \in \mathbb{N} \). The depth \( \mathrm{dpt}(M) \) of the model \( M \) is the supremum of the depths of all points of the model:

\[ \mathrm{dpt}(M) := \sup \{ \mathrm{dpt}(w) : w \in W \} \]

**Lemma V.11.** Let \( M = (W, \rightarrow, \| \cdot \|) \) be a weakly transitive model, and \( w, s \in W \) be two points. Then we have the following:

1) if \( w \rightarrow s \), then \( \mathrm{dpt}(w) \geq \mathrm{dpt}(s) \);

2) if \( w \leftarrow s \), then \( \mathrm{dpt}(w) = \mathrm{dpt}(s) \);

3) if \( w \rightarrow s \) and \( \mathrm{dpt}(w) = \mathrm{dpt}(s) < \omega \), then \( w \leftarrow s \);

4) if \( w \rightarrow s \) and \( \mathrm{dpt}(s) \) is finite, then \( \mathrm{dpt}(w) > \mathrm{dpt}(s) \).

**Proof.** Easy verification. □

Our goal in the next section is to prove Proposition III.5, in particular the completeness of our axiomatization with respect to irreflexive, weakly transitive frames. But for this, recall first that modal logic cannot express irreflexivity. The following result allows us to drop the irreflexivity condition:

**Lemma V.12.** For every weakly-transitive model \( M \), there exists some irreflexive weakly-transitive model \( \tilde{M} \) that validates/satisfies the same \( \mu \)-calculus formulas as \( M \). Moreover, if \( M \) is finite, then \( \tilde{M} \) can be taken to be finite as well.

So, to prove Proposition III.5, it is enough to show completeness and FMP for weakly transitive frames. This is topic of the next section.

**VI. PROOF OF THE MAIN COMPLETENESS/FMP RESULT**

In this section, we prove our main completeness result (Theorem III.5). Throughout the section, we fix a consistent formula \( \varphi₀ \), and let \( P₀ = \mathrm{Free}(\varphi₀) \). We also fix some finite set \( \Sigma \subseteq \mathcal{L}_P^P \), with the following properties: \( \varphi₀ \in \Sigma \); \( \Sigma \) is closed under subformulas; \( \Sigma \) is closed up to logical equivalence (in our axiomatic system) under negation \( \neg \varphi \) and under \( (\ast) \varphi \) operators. The existence of such a finite set \( \Sigma \) (for every formula \( \varphi₀ \)) follows from the fact that \( (\ast) \) is provably an S4-type modality, together with the well-known fact that there are only finitely many non-equivalent modalities in the modal system S₄ [10, Ch. 3]. Note that \( \varphi₀ \in \Sigma \), and \( P₀ \subseteq \Sigma \) is finite.

**Plan of the Proof** We start with the canonical model \( \Omega \) (comprising all maximally consistent theories), a standard construction in modal logic. But we should stress that the canonical model is not our intended model. Indeed, the usual Truth Lemma fails for the \( \mu \)-calculus in the canonical model: consistent \( \mu \)-calculus formulas are not necessarily satisfied in
the canonical model by the theories that contain them. In fact, the notion of truth in the canonical model will play no role in this paper: we never evaluate our formulas in $\Omega$. Instead, we only use a few basic syntactic properties of this model.

Next, we select a special submodel of the canonical model $\Omega^\Sigma$ (called the $\Sigma$-final model). Essentially, this consists of the theories whose cluster is locally definable by some formula in $\Sigma$. Our goal will be to show that the Truth Lemma does hold in $\Omega^\Sigma$ for $\Sigma$-formulas. It is easy to show that $\Omega^\Sigma$ satisfies the usual $\diamond$-Witness Lemma for formulas in $\Sigma$, but extending this to fixed points requires some work.

An important role will be played by the notion of $\Sigma$-bisimilarity, a strengthening of the standard notion of bisimilarity, in which the Atomic Permanence clause is replaced by the requirement that $\Sigma$-bisimilar theories agree on $\Sigma$-formulas. Since it is stronger than usual $P_0$-bisimilarity, $\Sigma$-bisimilarity still preserves the truth values of $\mu$-calculus formulas, as long as their free variables belong to $\Sigma$.

Another key ingredient in our proof is the fact that $\Sigma$-final theories are attained at some fixed finite stage of the iterative process from Proposition III.1(3).

We will then use these ingredients to prove our Truth Lemma for the final model $\Omega^\Sigma$. The inductive step for the fixed-point formulas uses the fact that the valuation of these formulas is locally definable by some $\Sigma$-formula.

Once completeness is obtained in this way, we will prove the finite model property by taking the quotient of the final model $\Omega^\Sigma$ modulo $\Sigma$-bisimilarity.

**Canonical Model**

The standard ‘canonical model’ construction provides an (infinite) weakly transitive model. A *theory* is a maximally consistent set of formulas in $\mathcal{L}_\mu$ (i.e. a set $T \subseteq \mathcal{L}_\mu$ that is consistent and has no proper consistent extension). We denote by $\Omega$ the family of all theories. The canonical accessibility relation $\rightarrow$ between two such theories $T, T' \in \Omega$ is given as usual by putting

$$T \rightarrow T' \text{ iff } \forall \varphi (\text{ if } \square \varphi \in T \text{ then } \varphi \in T'),$$

and the canonical valuation is given by

$$||x|| := \{ T \in \Omega : x \in T \}.$$

The canonical model is the structure $(\Omega, \rightarrow, || \cdot ||)$. Since the weak-transitivity condition is Sahtlqvist, it immediately follows that the canonical model is weakly transitive (though not irreflexive); see [9], [10] for details on Sahtlqvist formulas and their properties. As a consequence, the reflexive closure, which we denote $\rightarrow^*$, of the canonical relation coincides with its reflexive-transitive closure.

We will make use of a few well-known properties of the canonical model, given by the next four results (see, e.g., [9]).

**Lemma VI.1** (Lindenbaum Lemma). Every consistent set $\Phi$ of formulas can be extended to a maximal consistent set $T \in \Omega$ s.t. $\Phi \subseteq T$.

**Lemma VI.2** (Canonical $\diamond$-Witness Lemma). For every theory $T \in \Omega$ and formula $\varphi \in \mathcal{L}_\mu$, we have that $\diamond \varphi \in T$ iff there exists some theory $T' \in \Omega$ s.t. $T \rightarrow T'$.

We also have an equivalent statement in $\square$-form:

$$\square \varphi \in T \text{ iff } \forall T' \in \Omega (\text{ if } T \rightarrow T' \text{ then } \varphi \in T').$$

The left-to-right implication in the first statement above is known as the (Canonical) $\diamond$-Existence Lemma. The proofs are well-known (see, e.g., [9, Ch. 4]), and these results imply that the so-called Truth Lemma holds in the canonical model for the $\diamond$-fragment of our logic.

In fact, we can extend this to a Canonical $(*)$-Witness Lemma, using the following result

**Lemma VI.3.** For theories $T, T' \in \Omega$, we have:

$$T \rightarrow^* T' \text{ iff } \forall \varphi (\text{ if } [\ast] \varphi \in T \text{ then } \varphi \in T').$$

As a consequence of Lemma VI.1, we immediately get:

**Lemma VI.4** (Canonical $(*)$-Witness Lemma). For every formula $\varphi$ and theory $T \in \Omega$, we have that $(*) \varphi \in T$ iff there exists some theory $T' \in \Omega$ s.t. $T \rightarrow^* T' \varphi$.

**Final Theories**

Given a formula $\theta$, a theory $T \in \Omega$ is $\theta$-final if we have: $\theta \in T$, and for all theories $S \in \Omega$, if $T \rightarrow S$ and $\theta \in S$ then $S \rightarrow T$ (hence $T \rightarrow S$). Given a set $\Sigma$ of formulas, a theory $T \in \Omega$ is $\Sigma$-final (or ‘final’, for short) if it is $\theta$-final for some formula $\theta \in \Sigma$.

**Final Model**

Let $\Sigma$ be any set of formulas. The final model is the canonical submodel $\Sigma$ determined by the set $\Omega^\Sigma := \{ T \in \Omega : T \text{ is } \Sigma\text{-final} \}$ of all final theories.

The final model may be infinite, but we can show that it has finite depth:

**Lemma VI.5** (Finite Depth Lemma). The final model $\Omega^\Sigma$ has depth bounded by $|\Sigma| - 1$. In other words: for every chain of $\Sigma$-final theories $T_0 \not\rightarrow^* T_1 \not\rightarrow^* \ldots T_n$, we have that $n \leq |\Sigma| - 1$.

In order to prove completeness with respect to the final model, we first need to show that every consistent formula belongs to some final theory. This is achieved by combining the Lindenbaum Lemma with the following.

**Lemma VI.6** (Final Lemma). If $\varphi \in T \in \Omega$, then there exists some $\varphi$-final theory $T^* \in \Omega$ such that $T \rightarrow^* T^*$ (and obviously, $\varphi \in T^*$, by finality).

---

8To see this, consider atoms $(p_i)_{i \in \omega}$ and check that for every $n$, the set $\Phi_n := \{ p_i : \square \varphi \in T \} \cup \{ \varphi : \varphi \in T \} : i < \omega \}$ is consistent (since all finite subsets are satisfiable). Use the Canonical Truth Lemma for Basic Modal Logic (and the fact that $[\ast]$ is definable in it) to construct $(T_n)_{n < \omega}$ with $\Phi_n \subseteq T_n$ and $T_n \rightarrow T_{n+1} \rightarrow \ldots$, Thus, $T_0 \not\rightarrow^* \ldots$ although $(\not\square \varphi \in T)$ in $T_0$.

9Any subset $X' \subseteq X$ of the set of worlds of a relational model $M = (X, =, \rightarrow, \ast, || \cdot ||)$ determines a unique submodel, obtained by taking: $X'$ as its set of worlds; the restriction of $\rightarrow$ to $X'$ as its accessibility relation; and the valuation given by $|| \cdot || \cap X'$.
The proof proceeds by a fairly standard application of Zorn’s lemma. Using similar reasoning, we may establish an analogue of the ◇-Witness Lemma for final theories:

**Lemma VI.7** (Final ◇-Witness Lemma). For any theory \( T \subseteq \Omega \) and formula \( \varphi \), we have that \( \varphi \in T \) if and only if \( \varphi \)-final theory \( T' \) such that \( T \rightarrow T' \). (Obviously, we have \( \varphi \in T' \) in this case, by finality.)

It will be useful to observe that \( \theta \)-final theories are closely related to \( \{\ast\}\)-\( \theta \)-final theories.

**Lemma VI.8.** Let \( T \subseteq \Omega \) be a \( \theta \)-final theory. Then:
1) \( T \) is also \( \{\ast\}\)-\( \theta \)-final.
2) For every \( S \subseteq \Omega \) s.t. \( T \rightarrow^* S \), we have \( \{\ast\}\)-\( \theta \) in \( T \) if either \( T = S \) or \( T \leftrightarrow^* S \).
3) All theories \( S \subseteq \Omega \) satisfying \( T \leftrightarrow^* S \) are \( \{\ast\}\)-\( \theta \)-final.

**Locality in the final model** For the rest of this section, whenever we talk about ‘locality’, we refer to the final model \( \Omega^\Sigma \). In particular, for \( T \subseteq \Omega^\Sigma \), we use the notations \( T^\uparrow := \{ S \subseteq \Omega^\Sigma : T \rightarrow S \} \), \( T^\downarrow := \{ S \subseteq \Omega^\Sigma : T \rightarrow^* S \} \), and whenever we write that a property holds locally “above \( T \)”, we mean that it holds above \( T \) in \( \Omega^\Sigma \).

**Notation.** It is useful to introduce the notation
\[
\hat{\varphi} := \{ T \subseteq \Omega^\Sigma : \varphi \in T \}
\]
for all formulas \( \varphi \in \mathcal{L}_\mu \). From the definition of the canonical valuation on (the canonical model, and hence on) the final submodel, it is obvious that we have \( \| x \|_{\Omega^\Sigma} = \bar{x} \), for all atoms \( p \in P \). Our goal is to extend this observation to all sentences in \( \Sigma \).

**Σ-Bisimilarity in the Final Model** We can apply the concept \( P \)-bisimilarity \( \simeq_P \) to the final model \( \Omega^\Sigma \) (for any set of variables \( P \subseteq \mathbb{P} \)), and in fact the special case of \( P_0 \)-bisimilarity \( \simeq_{P_0} \) will be relevant for our proof. But it is convenient to introduce a stronger notion: a relation \( B \subseteq \Omega^\Sigma \times \Omega^\Sigma \) is a \( \Sigma \)-bisimulation if it satisfies the same back-and-forth clauses as a usual \( P \)-bisimulation, but the Atomic Preservation clause is replaced by the requirement that: \( (T, T') \in B \) implies \( T \cap \Sigma = T' \cap \Sigma \). The relation \( \simeq_{\Sigma} \subseteq \Omega^\Sigma \times \Omega^\Sigma \) of \( \Sigma \)-bisimilarity is defined as the largest \( \Sigma \)-bisimulation relation on \( \Omega^\Sigma \).

It is easy to see that \( \simeq_{\Sigma} \) is an equivalence relation on \( \Omega^\Sigma \), and that it is stronger than \( P_0 \)-bisimilarity: if \( T \simeq_{\Sigma} T' \) then \( T \simeq_{P_0} T' \). Using this and the above-mentioned well-known fact about invariance of \( \mu \)-calculus under standard bisimilarity, we immediately obtain the following:

**Lemma VI.9.** All the formulas \( \varphi \in \mathcal{L}_{\mu}^{P_0} \) are invariant under \( \Sigma \)-bisimilarity, i.e. if \( T, T' \subseteq \Omega^\Sigma \) satisfy \( T \simeq_{\Sigma} T' \), then for all \( \varphi \in \mathcal{L}_{\mu}^{P_0} \) we have \( T \in \| \varphi \| \) iff \( T' \in \| \varphi \| \).

It is useful to introduce a more “local” version of closure under bisimilarity.

**Closure under \( \Sigma \) bisimilarity above a point** This is just a special case of the general notion of asserting a property locally: a set \( X \subseteq \Omega^\Sigma \) is closed under \( \Sigma \)-bisimilarity above \( T \subseteq \Omega^\Sigma \) if \( X \cap T \uparrow \) is closed under \( \Sigma \)-bisimilarity.

Of course, global closure implies local closure: if a set \( X \subseteq \Omega^\Sigma \) is closed under \( \Sigma \)-bisimilarity, then it is also closed under \( \Sigma \)-bisimilarity above every \( T \subseteq \Omega^\Sigma \). Note also that: \( X \) is closed under \( \Sigma \)-bisimilarity above \( T \) iff \( X \cap T \uparrow \) is.

**Convention on global/local versions** Sometimes we want to assert that both the global and the local version of a statement hold in the final model \( \Omega^\Sigma \); e.g. if a certain premise holds, either globally or locally, then a certain conclusion holds, either globally or locally. To do this in a compact manner, we will state the global version, but adding in brackets the words “above \( T \)”, to include the local version as well. An example is the following result:

**Proposition VI.10.** If \( \bar{X} = (X_1, \ldots, X_n) \) is a tuple of sets \( X_i \subseteq \Omega^\Sigma \) that are closed under \( \Sigma \)-bisimilarity (above some \( T \subseteq \Omega^\Sigma \)), and \( \varphi = \varphi(\bar{x}) \in \Sigma \) is a \( \Sigma \)-formula, then \( \| \varphi(\bar{X}) \| \) is also closed under \( \Sigma \)-bisimilarity (above \( T \)).

Next, we will use the following easy observation:

**Lemma VI.11.** If \( T, T' \subseteq \Omega^\Sigma \) are such that \( T \leftrightarrow T' \) and \( T \cap \Sigma = T' \cap \Sigma \), then \( T \simeq_{\Sigma} T' \).

**Notations: (sets of) \( \Sigma \)-bisimilarity classes.** It is convenient to introduce a notation for \( \Sigma \)-bisimilarity classes over the final model: for every final theory \( T \subseteq \Omega^\Sigma \), we put
\[
T_{\Sigma} := \{ S \subseteq \Omega^\Sigma : T \simeq_{\Sigma} S \}
\]
for the \( \Sigma \)-bisimilarity class of \( T \). For every set \( S \subseteq \Omega^\Sigma \) of final theories, we put
\[
S_{\Sigma} := \{ S_{\Sigma} : S \subseteq S \}
\]
for the set of \( \Sigma \)-bisimilarity classes of theories in \( S \). In particular, for the case of the set \( \Omega^\Sigma \) of all final theories, we simplify the notation, writing
\[
\Omega_{\Sigma} := (\Omega^\Sigma)_{\Sigma} = \{ S_{\Sigma} : S \subseteq \Omega^\Sigma \}
\]
for the set of \( \Sigma \)-bisimilarity classes of all \( \Sigma \)-final theories. Similarly, for each number \( n \), we put
\[
\Omega^\Sigma_N := \{ T \subseteq \Omega^\Sigma : \text{dpt}(T) \leq n \} \subseteq \Omega_{\Sigma} \quad \Omega^\Sigma_N := \{ T \subseteq \Omega^\Sigma : \text{dpt}(T) \leq n \}
\]
for the set of all \( \Sigma \)-bisimilarity classes of theories of depth no larger than \( n \). By the Finite Depth Lemma VI.5, we have \( \Omega_{\Sigma} = \Omega^\Sigma_N \) for some natural number \( N \).

**Proposition VI.12.** There are only finitely many distinct bisimilarity classes in the final model \( \Omega^\Sigma \).

**Proof sketch.** By elementary counting arguments (whose details are in the Appendix), one can show that
1) \( |\Omega^\Sigma_{\Sigma}| \leq 2^{|\Sigma|} \cdot 2^{2|\Sigma|} \cdot 2^{2^{|\Sigma|}} \cdot 2^{2^{2^{|\Sigma|}}} \);
2) \( |\Omega^\Sigma_N| \leq 2^{|\Sigma|} \cdot 2^{2|\Sigma|} \cdot 2^{2^{2^{|\Sigma|}}} \cdot 2^{2^{2^{|\Sigma|}}} \cdot 2^{2^{2^{2^{|\Sigma|}}}} \) for all \( n > 0 \).

It follows by induction on \( n \) that each \( \Omega^\Sigma_n \) is finite. Since \( \Omega_{\Sigma} = \Omega^\Sigma_N \) for some \( N \), the proposition follows. \( \Box \)
Corollary VI.13. For every fixed-point formula $\nu y. \varphi(y, \pi)$ where the values of $\pi$ are closed under $\Sigma$-bisimilarity, the iterative process in Proposition III.1(3) reaches its fixed point on the final model (above some $T \in \Omega_\Sigma$) at some finite stage. More precisely: for all tuples $\bar{X}$ of subsets of $\Omega_\Sigma$ that are closed under $\Sigma$-bisimilarity (above some $T \in \Omega_\Sigma$), there exists some $N$ s.t.

$$\|\nu y. \varphi(y, \bar{X})\| = \bigcap_n \varphi^n_\pi(\bar{X}) = \varphi^n_{\pi}(\bar{X}) \text{ holds (above $T$)},$$

where $\varphi^n_\pi(\bar{X}) := \Omega_\Sigma$, $\varphi^{n+1}_\pi(\bar{X}) = \|\varphi(\varphi^n_\pi(\bar{X}), \bar{X})\|$ (and all the formulas are interpreted in the final model $\Omega_\Sigma$).

Lemma VI.14 (Functional Truth Lemma). For every formula $\varphi = \varphi(\bar{y}) \in \Sigma$ in which the variables in the string $\bar{y} = (y_1, \ldots, y_n)$ are free (or do not occur), every $\Sigma$-final theory $T \in \Omega_\Sigma$, and every tuple $\bar{\theta} = (\theta_1, \ldots, \theta_n)$ of formulas $\theta_i \in L^\mu$ s.t.

$$\|\varphi(\bar{\theta})\| \text{ iff } T \in \varphi(\bar{\theta});$$

(1) $T \ni \|\varphi(y)\|$ if $T \ni \varphi(y)$;

(2) if $\varphi(y) = \varphi(z, \bar{y})$ is positive in $z$, then for all natural numbers $n \in N$, we have:

- $\varphi^n_\pi(\bar{\theta})$ holds above $T$;
- moreover, $\varphi^n_\pi(\bar{\theta})$ is closed under $\Sigma$-bisimilarity above $T$;

where: $\|\varphi\|$ is the interpretation of $\varphi$ in the final model $\Omega_\Sigma$;

$\varphi^n_\pi(\bar{\theta})$ is an instance of the sequence of sets in Corollary VI.13 (i.e. it is recursively defined by putting $\varphi^n_\pi(\bar{\theta}) := \Omega_\Sigma$, $\varphi^{n+1}_\pi(\bar{\theta}) := \|\varphi(\varphi^n_\pi(\bar{\theta}), \bar{\theta})\|$); and $\varphi^n_\pi(\bar{\theta})$ is a sequence of formulas, recursively defined by putting $\varphi^n_\pi(\bar{\theta}) := \top$, $\varphi^{n+1}_\pi(\bar{\theta}) := \varphi(\varphi^n_\pi(\bar{\theta}), \bar{\theta})$.

Lemma VI.15 (Truth Lemma). For every formula $\varphi \in \Sigma$, we have:

$$\|\varphi\|_{\Omega_\Sigma} = \varphi.$$

Proof. For $\varphi = \varphi(y_1, \ldots, y_n) \in \Sigma$, apply the Functional Truth Lemma VI.14 to formula $\theta_i := y_i$.

Weak completeness for wK4 frames follows immediately from Lemma VI.15 (cf. Appendix). By Lemma VI.12, this also applies to irreflexive wK4 frames, hence to topological derivative spaces, and thus to arbitrary derivative spaces.

As for finite model property, this can be shown by taking the quotient of $\Omega_\Sigma$ modulo $\Sigma$-bisimilarity:

Final Quotient The final quotient ($\Omega_\Sigma, \rightarrow_\Sigma, \|\|_\Sigma$) is defined as the ‘strongly extensional $\Sigma$-quotient’ of the final model $\Omega_\Sigma$; i.e. the set of worlds $\Omega_\Sigma$ consists of all equivalence classes $T_\Sigma = \{ S \in \Omega_\Sigma : T \rightarrow_\Sigma S \}$, the accessibility relation is given by putting $T_\Sigma \rightarrow_\Sigma S_\Sigma$ if there are $T' \in T_\Sigma$, $S' \in S_\Sigma$ s.t. $T \rightarrow S$, and the valuation is given by putting, for each $p \in P$, $T_\Sigma \in \|p\|_\Sigma$ if there is $T' \in T_\Sigma$ s.t. $T' \in \|p\|$.

Proposition VI.16 (Finite Model Property). 1) The final quotient $\Omega_\Sigma$ is finite (with an upper bound given by a computable function of $|\Sigma|$);

2) for every formula $\varphi \in L^\mu$, we have that: $\varphi$ is true in the final model at some final theory $T \in \Omega_\Sigma$ iff $\varphi$ is true in the final quotient at the $\Sigma$-bisimilarity class $T_\Sigma$;

3) $\mu$-calculus has FMP (wrt relational, topological and derivative-space semantics).

Axiomatization of natural sublanguages The proofs in this section can all be carried out within any natural sublanguage $\mathcal{L}$ of $L^\mu$. In particular, natural sublanguages are closed under all operations used to define $\Sigma$, and moreover the formulas considered in the proof are all built from elements of $\Sigma$ using substitution, Booleans or applications of modalities. This finishes the proof of Theorem III.5, establishing completeness and FMP for all logics $\mu$-wK4.$^\mathcal{L}$.

Next, we get similar results for many logics above $\mu$-wK4.

VII. GENERALIZATION TO COFINAL SUBFRAME LOGICS

Our completeness and finite model property uses only a handful of properties of the logic wK4, and can readily be extended to a wide class of related logics. To be precise, we will now show that FMP holds for any canonical cofinal subframe logic above wK4 enriched with fixed-points.

Definition VII.1. ([10, Ch. 9]) Let $(W, \rightarrow)$ be a weakly transitive frame. A subset $X \subseteq W$ is called cofinal if $X \uparrow \subseteq X \downarrow$. That is, for every $x \in W$, if there is $y \in X$ such that $y \rightarrow x$, then there is $z \in X$ with $x \rightarrow^* z$.

Let $\Lambda$ be any normal modal logic over $\mathcal{L}_\emptyset$ that extends wK4. Recall that a Kripke frame $(W, \rightarrow)$ is called a $\Lambda$-frame if it validates all the formulas in $\Lambda$, and that a modal logic $\Lambda$ is canonical if the underlying frame of the canonical model for $\Lambda$ is a $\Lambda$-frame. Every logic axiomatized by Sahlqvist formulas is canonical [9]. Recall also that a canonical logic $\Lambda$ is cofinal subframe if and only if for every $\Lambda$-frame $F = (W, \rightarrow)$ and every cofinal $U \subseteq X$, the restriction of $F$ to $U$ is also a $\Lambda$-frame [10].

Examples of canonical cofinal subframe logics above wK4 are wKT$\emptyset$, K4, K4D = K4 + $\Diamond$ $\top$, K4.1 = K4 + $\Box$ $\Diamond$ $\top$ $\rightarrow$ $\Diamond$ $\Box$ $\top$, K4.2 = K4 + $\Diamond$ $\Box$ $\rightarrow$ $\Box$ $\Diamond$ $\top$, K4.3 = K4 + $\Box$ $\Box$ $\Box$ $\Diamond$ $\top$ $\rightarrow$ $\Box$ $\Box$ $\Box$ $\Diamond$ $\top$, S4 = K4 + $\Box$ $\rightarrow$, S4.1 = S4 + $\Box$ $\Diamond$ $\rightarrow$, S4.2 = S4 + $\Diamond$ $\Box$ $\rightarrow$, S4.3 = S4 + $\Box$ $\Box$ $\Box$ $\Diamond$ $\rightarrow$, S5 = S4 + $\Box$ $\Box$ $\Box$ $\Diamond$ $\rightarrow$, etc. (see [10, Chapter 9]).

We are ready to state Theorem III.5 in full generality. The following is proven by checking that our original proof carries over verbatim in this setting, given that the canonical model is based on a $\Lambda$-frame, and that the final submodel is a cofinal submodel of the canonical model. Indeed, from the assumption that $T \in \Sigma$ and Lemma VI.7, we easily see that above any theory $T$ there is a $T$-final theory $T'$.

Theorem VII.2. Let $\Lambda$ be a canonical cofinal subframe logic over wK4, and $\mathcal{L}$ be a natural sublanguage of $L^\mu$. Then, $\mu$-$\Lambda^\mathcal{L}$ is sound for the class of $\Lambda$-frames, and complete for the class of finite $\Lambda$-frames.

There exist continuum many canonical cofinal subframe logics above K4 ([10, Theorem 11.28 and Exercise 11.14]).
Hence the above theorem covers continuum many logics. Of course, only countable many of them have a recursively enumerable set of axioms: for those logics, decidability follows from Theorem VII.2. Next, we single out some important ones.

**Corollary VII.3.** The logics $\mu$-wKT$_0$, $\mu$-K4, $\mu$-K4D, $\mu$-K4.1, $\mu$-K4.2, $\mu$-K4.3, $\mu$-S4, $\mu$-S4.1, $\mu$-S4.2, $\mu$-S4.3 have the FMP and are decidable.

**VIII. Complete-ness for $T_0$ and $T_D$ Spaces**

The simple world-duplication construction underlying the last step of the topological completeness proof in Section III.5 does not work in the case of $T_0$ and $T_D$ spaces. So we will use $d$-morphisms to prove topological completeness for these cases. In the process we give an alternative to the above-mentioned proof of completeness for arbitrary spaces, although this new proof has the disadvantage that it does not yield the finite model property in this setting.

For this it suffices, given a $wK4$ frame $F = (W, \rightarrow)$, to construct a topological space $(X, \tau)$ and a $d$-morphism $\pi: X \rightarrow W$ as characterized by Lemma VII.7, in such a way that if $F$ is a $wK4T_0$ frame then $X$ will be $T_0$, and if $F$ is a $K4$ frame, then $X$ will be $T_D$.

**Definition VIII.1.** Let $F = (W, \rightarrow)$ be a $wK4$ frame. We build a topological space $(X, \tau) = (X_F, \tau_F)$ and a map $\pi: X \rightarrow W$ as follows. Let $W^\uparrow$ be the set of reflexive points of $W$ and $W^\uparrow$ be the set of irreflexive points. Then, set $X = (W^\uparrow \times \mathbb{N}) \cup (W^\uparrow \times \{\omega\})$, and say that $U \subseteq X$ is open if whenever $(w, \alpha) \in U$, the following two properties are satisfied:

1. There is $n \in \mathbb{N}$ such that for all $(v, \beta) \in X$, $v \leftrightarrow w$ and $\beta \geq n$ implies that $(v, \beta) \in U$.
2. If $(v, \beta) \in X$ and $w \not\rightarrow^* v$ then $(v, \beta) \in U$.

Finally, set $\pi(w, \alpha) = w$.

In other words, if an open set contains $(w, \alpha)$ then it contains all copies of $v$ whenever $w \rightarrow^* v$, except possibly for cofinitely many in the case that $v \leftrightarrow w$.

**Lemma VIII.2.** If $F = (W, \rightarrow)$ is any $wK4$ frame then $\tau_F$ is a topology on $X_F$ and $\pi: X_F \rightarrow W$ is a $d$-morphism.

**Lemma VIII.3.** If $F = (W, \rightarrow)$ is any $wK4$ frame then:

1. If $F$ is a $wK4T_0$ frame then $X_F$ is $T_0$.
2. If $F$ is a $K4$ frame then $X_F$ is $T_D$.

We can now proceed to prove topological completeness for $T_0$ and $T_D$ spaces. In fact, the proof also works for the $wK4$

**Theorem VIII.4.** 1) $\mu$-wK4 is sound and complete for the class of all topological spaces.
2) $\mu$-wK4T$_0$ is sound and complete for the class of all $T_0$ topological spaces.
3) $\mu$-K4 is sound and complete for the class of all $T_D$ topological spaces.

**IX. Conclusion and Comparison with Other Work**

In this paper, we have studied the $\mu$-calculus over arbitrary topological spaces, as well as some natural subclasses, and obtained a general soundness and completeness result for the standard axiomatization.

Our results are novel for several reasons. First, in the setting of Kripke semantics, neither completeness nor the FMP for weakly transitive frames were known, nor they follow from known results. Moreover, our completeness proof appears to be the first such result for a variant of $\mu$-calculus that simultaneously applies to infinitely many logics and their respective classes of frames.

From the topological perspective, neither completeness nor decidability for non-$T_D$ spaces were known, nor they follow from known results. Unlike the transitive/$T_D$ case, our logics do not embed into standard $\mu$-calculus, or any of its known decidable extensions. This is in sharp contrast to the $T_D$-transitive case, where FMP and decidability follow via a simple encoding into standard $\mu$-calculus. Moreover, we showed that the tangled derivative is not expressively complete over the class of all topological (or even $T_0$) spaces, so we had to give a completeness proof that applies to the full language of $\mu$-calculus.

But note that even on $T_D$ spaces, our proof is the first to directly establish completeness over such spaces of a Kozen-type axiomatization for full $\mu$-calculus (rather than for some semantically equivalent modal logic). Prior work on $T_D$ spaces, mainly by Goldblatt and Hodkinson [26], had focused only on the tangled fragment. Though this fragment is known to be co-expressive with $\mu$-calculus over $T_D$ spaces (and transitive frames), completeness for the full $\mu$-calculus over these spaces only follows if we combine the results in [26] with Walukiewicz’s proof of Kripke completeness for $\mu$-calculus. In contrast, our proof of completeness is self-contained (for both the $T_D$ and the non-$T_D$ case), taking advantage of the weak transitivity to give a streamlined proof tailored for the topological setting.

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10As far as we are aware, this is a first non-trivial example of a completeness result for modal fixed-point logics that covers so many logical systems.

11Topological completeness of $wK4$, $wKT_0$ and $K4$ has already been discussed in Section 3. We also recall that $S4.1$ is the logic of spaces whose dense sets form a filter, that $S4.2$ is the logic of extremally disconnected spaces [6, Sec. 2.6] and that $S4.3$ is the logic of hereditarily extremally disconnected spaces [33].

12On the other hand, neither $wK4T_0$ nor $K4$ have the finite topological model property, so the next result cannot be improved upon.

13As already mentioned, the transitive closure of a relation can be encoded in $\mu$-calculus (and thus the decidability of $\mu$-calculus over transitive frames follows immediately from Kozen’s result on the decidability of $\mu$-calculus over arbitrary frames). In contrast, the weakly-transitive closure of a relation does not seem to be definable in $\mu$-calculus, and not even in its recent hybrid extension [34]. Weakly-transitive closure is definable only if one adds the binding operator from hybrid logic. But this increases the complexity of hybrid $\mu$-calculus, and the resulting logic is no longer known to have FMP (or to be even decidable).
Furthermore, our results are based on an innovative use of the proof techniques using final submodels (due to Fine and Zakharyaschev). This method has not been applied previously in a setting with fixed points, and provides a novel, general and relatively simple approach to dealing with fixed point logics over wK4 frames (for which the filtration method, used in [26] and elsewhere in the study of fixed point logics, does not seem to work). In fact, even for the much easier case of topological closure logics, our method provides a simpler and more uniform way to reprove existing results: while Goldblatt and Hodkinson [35] had to do a lot of work to show the FMP for S4-tangle logic (and thus also for the semantically equivalent µ-S4), in Corollary VII.3 we get this result essentially for free from our general methods.

There are many open questions left within the context of topological fixed point logics. The problem of finding a simple but expressively complete fragment of the µ-calculus over wK4, in analogy to the tangled fragment for logics over K4, remains open. But we conjecture that topological µ-calculus does indeed collapse to a simpler natural fragment, possibly the alternation-free fragment, with a proof along the lines of the similar argument for transitive frames in [36]. Anticipating such a development, we have set up our main completeness result in a modular fashion so that, if such a fragment 𝐿 is ever found, the completeness for its natural axiomatization µ-wK4-L will follow immediately from Theorem III.5.

Another line of inquiry that we leave open here is the problem of extending our methods to classes of spaces which enjoy topologically natural properties that do not correspond to any cofinal subframe logic. The prime example here is that of connected spaces, whose modal logic is well understood in presence of the universal modality [37]. We believe that our methods can be extended to such settings, but some non-trivial modifications would be required.

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