Stability of the vortex lattice in a rotating superfluid

Gordon Baym
Loomis Laboratory of Physics
University of Illinois at Urbana-Champaign
1110 W. Green St.
Urbana, IL 61801, U.S.A.

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We analyze the stability of the vortex lattice in a rotating superfluid against thermal fluctuations associated with the long-wavelength Tkachenko modes of the lattice. Inclusion of only the two-dimensional modes leads formally to instability in infinite lattices; however, when the full three-dimensional spectrum of modes is taken into account, the thermally-induced lattice displacements are indeed finite.

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1 Introduction

A rotating $^4$He superfluid accommodates angular momentum by forming an array of vortex lines, each having quantized circulation $2\pi \hbar / m$, where $m$ is the $^4$He mass. In a classic series of papers [1], Tkachenko showed that the lowest energy configuration of an infinite vortex array is a two-dimensional triangular lattice, and that the lattice supports collective elastic modes; perpendicular to the rotation axis the modes travel with velocity $(\hbar \Omega / 4m)^{1/2}$, where $\Omega$ is the rotational velocity of the fluid. Finite arrays, as Campbell and Ziff predicted [2], should exhibit distortions from triangular, structure seen experimentally by Packard and co-workers [3]. The question we address in this paper is whether the infinite vortex lattice remains stable in the presence of thermal fluctuations associated with Tkachenko shear modes.
As in a two-dimensional system, where long-wavelength modes forbid strict long-range translational order, the long-wavelength Tkachenko motions of the lattice, when limited to the plane perpendicular to the rotation axis, lead to a weak logarithmic divergence of the displacements characteristic of a two-dimensional solid. However, when one includes the full three-dimensional spectrum of excitations of the vortex lattice, with line-bending contributions to the Tkachenko modes, stability of the lattice is restored. The vortex lattice provides an instructive realization of how addition of a third dimension leads to stability of the ordering even if the order parameter varies in only two dimensions; the results are consistent with the usual Landau-Peierls stability arguments in a three-dimensional system, which forbid an order parameter varying in only one dimension, but do not rule out stability for the two-dimensional variations encountered in the superfluid vortex lattice. Compared to a normal lattice, however, the vortex system has the peculiarity that the effective energy associated with lattice displacements depends on the displacements directly, rather than simply on their derivatives; this dependence, arising, as seen below, from the intrinsic lack of Galilean invariance of a rotating system, acts to reduce the contribution of long-wavelength excitations to the excursions of the vortices from their equilibrium sites. Understanding how stability is achieved in the vortex lattice is also instructive for the closely related issue of whether thermally-excited elastic shear modes drive instability of the Abrikosov magnetic-flux lattice in clean Type II superconductors and in the layered high Tc materials in particular. Extension of the present analysis to flux lattices will be given in a subsequent paper.

To study the stability we apply the familiar methodology of calculating the contributions of the thermal fluctuations to the displacements of the vortices from their equilibrium sites by constructing the displacement autocorrelation functions in terms of the spectrum of modes. In the following section we review the equations of motion of the vortex lattice; in Sec. 3 we derive the displacement autocorrelation functions of the lattice, and discuss the consequences for lattice stability and phase coherence. Appendix A derives the relation between the macroscopically averaged phase of the order parameter, flow velocity and vortex lattice displacements, and in Appendix B we analyze how the non-vanishing commutation relations obeyed by the components of the lattice displacements arise as one takes the limit in which the inertial mass of the vortex lines vanishes.
2 Equations of motion and long-wavelength modes

We consider an incompressible, non-dissipative superfluid in a bucket rotating about the $z$-axis at angular velocity $\vec{\Omega}$, and work in the frame rotating with the bucket; we follow the formulation given in Refs. [7] and [8] of the complete non-linear dissipative hydrodynamics of a rotating superfluid, including the elasticity of the vortex lattice. The basic degrees of freedom are the macroscopic velocity field $\vec{v}(\vec{r}, t)$ and the macroscopic field $\vec{\epsilon}(\vec{r}, t)$ that describes long wavelength displacements of the vortices in the transverse $(x, y)$ direction from their equilibrium locations. (In general, the local vortex line velocity in the transverse direction, $\vec{\epsilon}$, does not equal $\vec{v}_L$, the superfluid velocity in the plane.) Prior to calculating the thermal fluctuations of the lattice we summarize the equations of motion obeyed by $\vec{v}$ and $\vec{\epsilon}$, and the long wavelength modes of the vortex lattice.

In order to bring out the character of the equations for the modes, we assume, as in Ref. [7], that the vortex lines have an effective mass per unit length, and thus carry a normal fluid density $\rho^*$, which we assume to be $\ll \rho$, the mass density of the fluid [14]; in the end we take $\rho^* \to 0$. The mass current, $\vec{j}$, is given by

$$\vec{j} = \rho \vec{v} + \rho^* (\dot{\vec{\epsilon}} - \vec{v}_L); \quad (1)$$

for incompressible flow,

$$\nabla \cdot \vec{j} = 0. \quad (2)$$

The dynamics of the system are specified by the superfluid acceleration equation and the law of conservation of momentum. The linearized superfluid acceleration equation in the rotating frame is [7]

$$\frac{\partial \vec{v}}{\partial t} + 2 \vec{\Omega} \times \dot{\vec{\epsilon}} = -\nabla \mu', \quad (3)$$

where $\mu' = \mu - (\vec{\Omega} \times \vec{r})^2 / 2$ and $\mu$ the chemical potential per unit mass.

This equation is the time derivative of the relation between the macroscopically averaged phase of the condensate wave function and the macroscopically averaged flow velocity and vortex displacements; as we show in Appendix A, for small displacements of the vortex lattice,

$$\vec{v} + 2 \vec{\Omega} \times \vec{\epsilon} = \frac{\hbar}{m} \nabla \phi. \quad (4)$$
Equation (4) generalizes the usual relation, $\vec{v} = \frac{\hbar}{m} \vec{\nabla} \phi$, to the rotating system, where the flow velocity can have non-zero vorticity as a consequence of local variations of the vortex density. Equation (4) is the analog in the rotating system of the relation between the flow velocity and phase in a superconductor in the presence of a vector potential,

$$\vec{v} + \frac{e}{mc} \vec{A} = \frac{\hbar}{m} \vec{\nabla} \phi.$$  

Comparing with Eq. (3), we see that $\mu'$ is related to the phase in the usual way,

$$\mu' (\vec{r}, t) = - \frac{\hbar}{m} \frac{\partial \phi (\vec{r}, t)}{\partial t}.$$  

The linearized equation for conservation of momentum in the rotating frame is

$$\frac{\partial \vec{j}}{\partial t} + 2 \vec{\Omega} \times \vec{j} + \vec{\nabla} P' = - \vec{\sigma}_{el} - \vec{\zeta},$$  

where $P' = P - \rho (\vec{\Omega} \times \vec{r})^2 / 2$ and $P$ is the pressure; $-\vec{\sigma}_{el}$ is the elastic force density (directed in the transverse plane) arising from deformations of the vortex lattice, given in terms of the elastic energy density, $E_{el}$, of the vortex lattice by

$$\vec{\sigma}_{el} = \frac{\delta E_{el}}{\delta \vec{\varepsilon}} = \frac{\hbar \Omega \rho}{4m} [2 \vec{\nabla}_\perp (\vec{\nabla} \cdot \vec{\varepsilon}) - \vec{\nabla}_\perp^2 \vec{\varepsilon}] - 2 \Omega \lambda \frac{\partial^2 \vec{\varepsilon}}{\partial z^2}.$$  

Here $2 \Omega \lambda = n_v E_v$ is the vortex line energy measured per unit volume, where $E_v$ is the energy per unit length of a vortex line, and $n_v = m \Omega / \pi \hbar$ is the number of lines per unit area. (The detailed elastic constant multiplying $\nabla^2 \vec{\varepsilon}$ in the right side of (8) is specific to a triangular lattice.) We include in (4) an external driving force $-\vec{\zeta} (\vec{r}, t)$ acting on the lattice, derived from an external perturbation $H' = \vec{\zeta} \cdot \vec{\varepsilon}$, to facilitate calculation of the displacement autocorrelation functions.

At low temperatures one may neglect the normal fluid mass density associated with the bulk excitations (phonons and rotons) of the fluid. Then in Eq. (7), $\vec{\nabla} P' = \rho \vec{\nabla} \mu'$, which we eliminate using (3); keeping $\rho^*$ only in the inertial term we find

$$\rho^* \frac{\partial}{\partial t} (\vec{v} - \vec{v}_\perp) - 2 \rho \vec{\Omega} \times (\vec{v} - \vec{v}_\perp) = - \vec{\sigma}_{el} - \vec{\zeta}.$$  

To compute the superfluid velocity in terms of $\vec{\varepsilon}$, we assume a plane-wave spatial dependence with wave vector $\vec{k}$, and from (2) and (4) find, for $\rho^* \ll \rho$, that

$$\vec{v} + 2 \vec{\Omega} \times \vec{\varepsilon} = \vec{k} \cdot (2 \vec{\Omega} \times \vec{\varepsilon}),$$  

4
where \( \hat{k} \) is the unit vector in the \( \vec{k} \) direction. This equation implies that a displacement of the vortex lattice, even uniform, induces a flow velocity, a consequence of the lack of Galilean invariance in a rotating system. The quantity on the right side of (10) is the variation of the macroscopically averaged phase of the order parameter.

The equations of motion are most simply written in terms of the longitudinal and transverse displacements defined by \( \epsilon_L = \hat{q} \cdot \vec{\epsilon} \) and \( \epsilon_T = \hat{z} \cdot (\hat{q} \times \vec{\epsilon}) \), where \( \hat{q} \) is the unit vector in the \( x,y \) plane along the direction of the projection of \( \vec{k} \) in the plane. (The limit of \( \vec{k} \) along the rotation axis presents no ambiguities.) Substituting \( \vec{\zeta} \) from Eq. (10) into (9) and taking longitudinal and transverse components, we derive the coupled equations of motion for \( \epsilon_L \) and \( \epsilon_T \):

\[
\begin{align*}
\rho^* \ddot{\epsilon}_L + 2\rho\Omega\dot{\epsilon}_T + \alpha_L\epsilon_L &= -\hat{q} \cdot \vec{\zeta} \equiv -\zeta_L, \\
\rho^* \ddot{\epsilon}_T - 2\rho\Omega\dot{\epsilon}_L + \alpha_T\epsilon_T &= -\hat{z} \cdot (\hat{q} \times \vec{\zeta}) \equiv -\zeta_T,
\end{align*}
\]

(11)

where

\[
\begin{align*}
\alpha_L(\vec{k}) &= 4\Omega^2 \rho - \frac{\hbar\Omega \rho}{4m} k_z^2 + 2\Omega\lambda k_z^2, \\
\alpha_T(\vec{k}) &= 4\Omega^2 \rho k_z^2 + \frac{\hbar\Omega \rho}{4m} k_z^2 + 2\Omega\lambda k_z^2.
\end{align*}
\]

(12)

Equations (11) agree in content with those given in Ref. [7]. Note that they imply that the energy

\[
E_V = \sum_{\vec{k}} \frac{1}{2} [\rho^*(\dot{\epsilon}_L^2 + \dot{\epsilon}_T^2) + \alpha_L \epsilon_L^2 + \alpha_T \epsilon_T^2]
\]

(13)

is conserved, in the absence of external perturbations. The fact that the energy depends directly on the displacements, not only on their derivatives, as is the situation in a normal lattice, is a consequence of a vortex displacement inducing a flow velocity, Eq. (10).

The frequencies \( \omega \) of the normal modes of the system, determined by the four roots of the secular equation

\[
D(\vec{k}, \omega) \equiv (\rho^* \omega^2 - \alpha_L)(\rho^* \omega^2 - \alpha_T) - 4\Omega^2 \rho^2 \omega^2 = 0,
\]

(14)

correspond to a high frequency inertial mode, of frequency given for small \( k \) by

\[
\omega_I^2 = \left(2\Omega \rho / \rho^* \right)^2,
\]

(15)
and a generalized Tkachenko mode of frequency given for small $k$ by

$$
\omega_T^2 = \frac{\alpha_L \alpha_T}{4 \Omega^2 \rho^2} = (2\Omega \cos \theta)^2 + \left[ \frac{2\Omega \lambda}{\rho} \cos^2 \theta (1 + \cos^2 \theta) + \frac{\hbar \Omega}{4m} \sin^4 \theta \right] k^2,
$$

(16)

where $\theta$ is the angle between the wavevector $\vec{k}$ and the rotation axis. At $\theta = \pi/2$, Eq. (16) reduces to $\omega_T^2 = (\hbar \Omega/4m) k^2$, Tkachenko’s original result.

## 3 Displacement correlation functions and lattice stability

To calculate the effect of thermal excitation of Tkachenko modes on the displacements of the lattice, we first construct the space-time Fourier transform of the retarded-commutator correlation function from the relation

$$
\langle \epsilon_i \epsilon_j \rangle = \hbar \delta(\epsilon_i) \delta(\epsilon_j).
$$

(17)

Solution of Eqs. (11) for $\epsilon$ in terms of $\zeta$,

$$
\left( \begin{array}{c} \epsilon_L \\ \epsilon_T \end{array} \right) = \frac{1}{D(\vec{k}, \omega)} \left( \begin{array}{cc} \rho^* \omega^2 - \alpha_T & -2i\Omega \rho \omega \\ 2i\Omega \rho \omega & \rho^* \omega^2 - \alpha_L \end{array} \right) \left( \begin{array}{c} \zeta_L \\ \zeta_T \end{array} \right),
$$

(18)

yields the correlation functions

$$
\langle \epsilon_L \epsilon_L \rangle (\vec{k}, \omega) = \frac{\hbar (\rho^* \omega^2 - \alpha_T)}{D(\vec{k}, \omega)}, \\
\langle \epsilon_T \epsilon_T \rangle (\vec{k}, \omega) = \frac{\hbar (\rho^* \omega^2 - \alpha_L)}{D(\vec{k}, \omega)}, \\
\langle \epsilon_L \epsilon_T \rangle (\vec{k}, \omega) = \langle \epsilon_T \epsilon_L \rangle (\vec{k}, \omega)^* = -\frac{2i\hbar \Omega \rho \omega}{D(\vec{k}, \omega)}.
$$

(19)

The correlation functions (19) are conveniently written in terms of their spectral weights, defined by

$$
\langle \epsilon_i \epsilon_j \rangle (\vec{k}, \omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} C_{ij}(\vec{k}, \omega') \omega - \omega'.
$$

(20)
The weight functions obey the symmetries $C_{LL}(\vec{k}, -\omega) = -C_{LL}(\vec{k}, \omega)$, and $C_{TT}(\vec{k}, -\omega) = -C_{TT}(\vec{k}, \omega)$, while $C_{LT}(\vec{k}, -\omega) = C_{TL}(\vec{k}, \omega)$; as we find from (19), the weights are given, for $\omega > 0$, by

\[
C_{LL}(\vec{k}, \omega) = \frac{\pi \hbar}{2\Omega \rho} \delta(\omega - \omega_I) + \frac{\pi \hbar \omega_T}{\alpha_L} \delta(\omega - \omega_T),
\]

\[
C_{TT}(\vec{k}, \omega) = \frac{\pi \hbar}{2\Omega \rho} \delta(\omega - \omega_I) + \frac{\pi \hbar \omega_T}{\alpha_T} \delta(\omega - \omega_T),
\]

\[
C_{LT}(\vec{k}, \omega) = \frac{\pi \hbar}{2i\Omega \rho} [\delta(\omega - \omega_T) - \delta(\omega - \omega_T)].
\]

The equal-time spatial correlations of the displacements of the vortex lines in the rotating fluid at temperature $T = 1/\beta$ are given by

\[
\langle \epsilon_i(\vec{r}, t)\epsilon_j(\vec{r}', t) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} e^{i\vec{k} \cdot \vec{R}} C_{ij}(\vec{k}, \omega)(1 + f(\omega)),
\]

where $i, j$ denotes the transverse or longitudinal components, $\vec{R} \equiv \vec{r} - \vec{r}'$, and $f(\omega) = (e^{\beta \hbar \omega} - 1)^{-1}$. Because $\alpha_L$ is non-zero for $\vec{k} \to 0$, the longitudinal correlation function $\langle \epsilon_L(\vec{r})\epsilon_L(\vec{r}') \rangle$ has a finite range of order several vortex spacings; however, since $\alpha_T$ vanishes for in-plane wavevectors approaching zero, the transverse correlation function $\langle \epsilon_T(\vec{r})\epsilon_T(\vec{r}') \rangle$ falls as $1/R_\perp$ in the $x,y$ plane. Since for general $\vec{k}$, $\langle \epsilon^2 \rangle = \langle \epsilon_T^2 + \epsilon_L^2 \rangle$, we see that in the limit $\rho^* \to 0$, at equal times,

\[
\langle (\epsilon(\vec{r}) - \bar{\epsilon}(\vec{r}'))^2 \rangle = \int \frac{d^3k}{(2\pi)^3} (1 - \cos \vec{k} \cdot \vec{R}) \hbar \omega_T \left( \frac{1}{\alpha_T} + \frac{1}{\alpha_L} \right) (1 + 2f(\omega_T)).
\]

\[
\langle (\epsilon(\vec{r}) - \bar{\epsilon}(\vec{r}'))^2 \rangle = \int \frac{d^2k}{(2\pi)^2} \frac{1}{Z} (1 - \cos \vec{k} \cdot \vec{R}) \hbar \omega_T \left( \frac{1}{\alpha_T} + \frac{1}{\alpha_L} \right) (1 + 2f(\omega_T)),
\]

where $Z$ is the container thickness in the $z$ direction ($Z \rho$ is the mass per unit area). Equation (24) implies that Tkachenko modes in the transverse plane should, formally, prevent
formation of a stable two-dimensional lattice. The infrared contribution to the integral in the limit of large separation $R = |\vec{r} - \vec{r}'|$ is

$$\langle (\vec{\epsilon}(\vec{r}) - \vec{\epsilon}(\vec{r}'))^2 \rangle \sim \frac{2T}{Z\rho} \int_0 \frac{d^2k}{(2\pi)^2} \frac{(1 - \cos \vec{k} \cdot \vec{R})}{\omega_T^2} \sim \int \frac{dk}{R^3},$$

(25)

which diverges as $\ln R$. The instability in two dimensions is driven by the softness of the energy associated with shearing, $\sim \alpha_T$, for $\vec{k}$ lying in the $x,y$ plane (see Eq. (13)), leading to a $\ln R$ divergence in the transverse correlation function. As in three dimensions, the longitudinal displacement correlation function has finite range. The mean square displacement of a vortex from its equilibrium site similarly diverges as $\ln R$, where $R$ is the container radius.

But one should note that such an instability could, in fact, manifest itself only in films no more than tens of atomic layers thick. The square of the displacement as a fraction of the area per vortex becomes, to within a constant in the logarithm,

$$\langle \epsilon_T (\vec{r})^2 \rangle_{n_v} \approx \frac{3T}{\Theta_D k_D Z} \ln N_v,$$

(26)

where $k_D$ is the Debye wavevector, defined by $\rho/m = k_D^3/6\pi^2$; $\Theta_D = \hbar^2 k_D^2/2m$, and $N_v$ is the total number of vortex lines in the system. For $(T/\Theta_D) \ln N_v$ of order unity, the mean displacements can become comparable to the intervortex spacing only for $k_D Z < 10^2$.

When the full three-dimensional degrees of freedom of the vortex excitations are taken into account, we find, instead, a finite infrared contribution,

$$\langle \epsilon(\vec{r})^2 \rangle \sim T \int_0 \frac{d^3k}{(2\pi)^3} \frac{1}{\alpha_T} + \frac{1}{\alpha_L} = T \int_0 \frac{d^3k}{(2\pi)^3} \frac{1 + \cos^2 \theta}{\rho \omega_T^2},$$

$$\to \frac{T}{\rho} \int_{-1}^1 \frac{d\cos \theta}{4\pi^2} \int_0 \frac{k^2 dk}{4\Omega^2 \cos^2 \theta + (\hbar k^2 / 4m) \sin^4 \theta}.$$

(27)

The two terms in $(1 + \cos^2 \theta)$ arise respectively from transverse and longitudinal displacements. Inclusion of the full three-dimensional spectrum of long-wavelength fluctuations thus stabilizes the vortex lattice. The excursions of the vortices from their equilibrium positions are made finite by the smaller phase space for excitations of small $k$ in three dimensions, and the non-vanishing of both the transverse and longitudinal coefficients, $\alpha_L$ and $\alpha_T$, in the energy (13), as $\vec{k} \to 0$ with $\cos \theta > 0$. Detailed evaluation of $\langle \epsilon(\vec{r})^2 \rangle$ requires calculation over all $k$, not just the long-wavelength modes.
What are the implications of the thermal fluctuations of the lattice vibrations for the macroscopically averaged phase $\phi(\vec{r},t)$ of the condensate wave function? Since the system preserves a global (gauge) invariance allowing changes of the phase by an additive constant, only relative phases have physical meaning. The invariant correlations of the order parameter are given at equal time, for Gaussianly-distributed Fourier components of the phase, by

$$\langle e^{i\phi(\vec{r})} e^{-i\phi(\vec{r}')} \rangle = e^{-\frac{1}{2}((\phi(\vec{r})-\phi(\vec{r}'))^2)}.$$  
(28)

For plane-wave spatial dependence Eq. (10) implies that

$$\phi = \frac{im}{\hbar} \vec{k} \cdot (\hat{k} \times \hat{\epsilon}) = \frac{2im\Omega}{\hbar k} \sin \theta \epsilon_\parallel;$$  
(29)

transverse fluctuations of the lattice positions give rise to phase fluctuations of longer range (owing to the extra power of $k$ in the denominator). Calculating the correlation of phase differences from Eqs. (29) and (19) we have,

$$\langle (\phi(\vec{r}) - \phi(\vec{r}'))^2 \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} \left( 1 - \cos \vec{k} \cdot \vec{R} \right) \left( \frac{2m\Omega \sin \theta}{\hbar k} \right)^2 C_{TT}(\vec{k},\omega)(1 + f(\omega))$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi} \left( \frac{1 - \cos \vec{k} \cdot \vec{R}}{k^2} \right) \left( \frac{2m\Omega \sin \theta}{\hbar \alpha_T} \right)^2 \omega_T(1 + 2f(\omega_T))$$

$$\sim \sqrt{n_v} \frac{T}{\Theta_D} \ln(\pi R^2 n_v).$$  
(30)

Formally, (30) diverges logarithmically as $R \to \infty$, leading to the correlation function $\langle e^{i\phi(\vec{r})} e^{-i\phi(\vec{r}')} \rangle$ falling to zero as a power of $R$, as discussed originally by Moore [15]. Numerically, this falloff is insignificant in any finite system, since the vortex lattice spacing $\propto 1/\sqrt{n_v}$ is always huge compared with the interparticle spacing $\propto k_D^{-1}$, while $\ln(\pi R^2 n_v)$ is less than $\ln N_v$, where $N_v$ is the total number of vortices in the system.

In general though, vanishing of the correlation function $\langle e^{i\phi(\vec{r})} e^{-i\phi(\vec{r}')} \rangle$ for large $R$ does not indicate a loss of superfluidity. As Josephson stressed, by analogy to a very floppy “slinky,” such correlations do not tell one the fluctuations in the winding of the phase angle around a closed loop. Note also that correlations of the gradients of the phase remain finite as $R \to \infty$.

The existence of transverse excitations of the velocity field causes the normal mass density, as measured by the moment of inertia, to equal to the full mass density. The normal mass
density, \( \rho_n \), is related to the transverse current autocorrelation function by (see, e.g., [16]).

\[
\rho_n = -\lim_{k \to 0} \langle \tilde{j}_T \tilde{j}_T \rangle(\vec{k}, \omega = 0).
\]  

(31)

Taking \( \rho^* = 0 \), whereupon \( \tilde{j} = \rho \tilde{v} \), and calculating the transverse component of the velocity from Eq. (10), we find

\[
v_T = -(2\Omega \times \vec{\epsilon}) = -2\Omega \epsilon_L,
\]

and from Eqs. (19) and (21) that

\[
\rho_n = -\lim_{k \to 0} \rho^2 \langle v_T v_T \rangle(\vec{k}, \omega = 0) = -\lim_{k \to 0} (2\Omega \rho)^2 \langle \epsilon_L \epsilon_L \rangle(\vec{k}, \omega = 0) = \rho.
\]  

(32)

Tkachenko excitations replenish the “sum rule” (31), reflecting the fact that in the limit of a dense vortex lattice the moment of inertia of the container of superfluid takes on its normal fluid value. However, this result does not imply that the superfluid density measured dynamically, e.g., in a second sound experiment, vanishes; rather, in the non-Galilean invariant rotating system, the two measures become independent.

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**Appendix A. Macroscopically-averaged phase of the order parameter**

In this Appendix we derive the relation (4) between the macroscopic averages of the phase of the order parameter, the flow velocity and the vortex displacements. Imagine displacing a single vortex line, initially along the \( z \)-axis, by a small amount \( \tilde{\epsilon}(z) \) in the transverse plane. [We denote the local velocity, displacement and potential by \( \tilde{V}, \tilde{\epsilon}, \) and \( \phi \), to distinguish them from the long-wavelength averaged quantities, \( \bar{v}, \bar{\epsilon}, \) and \( \bar{\phi} \).] The flow velocity, \( \tilde{V} \), of the fluid about the vortex line obeys

\[
\tilde{\nabla} \times \tilde{V}(\vec{r}) = \frac{h}{m} \int d\tilde{l} \delta(\tilde{r} - \tilde{r}(\tilde{l})),
\]  

(33)

where \( l \) is the distance along the line. For small \( \epsilon, d\tilde{l}/dz = (d\tilde{\epsilon}(z)/dz, 1) \) (the final component being in the \( z \)-direction); thus

\[
\tilde{\nabla} \times \tilde{V}(\vec{r}) = \frac{h}{m} \left( \frac{d\tilde{\epsilon}(z)}{dz}, 1 \right) \delta(x - \tilde{\epsilon}_x) \delta(y - \tilde{\epsilon}_y).
\]  

(34)
Taking the curl of (34), expanding the right side to first order in \( \varepsilon \), and assuming an incompressible fluid, \( \nabla \cdot \vec{V} = 0 \), we find that the first variation of the velocity obeys

\[
\nabla^2 \left( \delta \vec{V} - \frac{h}{m} \vec{\varepsilon} \times \hat{z} \delta(x) \delta(y) \right) = \frac{h}{m} \nabla \left( \left( \varepsilon_x \frac{\partial}{\partial y} - \varepsilon_y \frac{\partial}{\partial x} \right) \delta(x) \delta(y) \right),
\]

with solution

\[
\delta \vec{V}(\vec{r}) = \frac{h}{m} \vec{\varepsilon} \times \hat{z} \delta(x) \delta(y) + \frac{h}{m} \nabla \delta \Phi,
\]

where

\[
\nabla^2 \delta \Phi = 2\pi \left( \varepsilon_x \frac{\partial}{\partial y} - \varepsilon_y \frac{\partial}{\partial x} \right) \delta(x) \delta(y).
\]

Since at points away from the line, the velocity is given by \( \hbar/m \) times the gradient of the phase of the order parameter, we identify \( \delta \Phi \) as the first variation of the phase of the order parameter due to displacement of the line.

We now sum (36) over a lattice of vortices, and carry out a long wavelength average. Since the average of \( \vec{V} \) at \( \vec{\varepsilon} = 0 \) is the uniform rotational velocity, \( \vec{\Omega} \times \vec{r} \), the long wavelength average of \( \delta \vec{V} \) is the flow velocity, \( \vec{v} \), in the rotating frame; similarly the long-wavelength average of \( \delta \Phi \) is the macroscopically-averaged phase, \( \phi \), in the rotating frame. Using \( \hbar \nu/m = 2\Omega \) we derive Eq. (4),

\[
\vec{v} + 2\vec{\Omega} \times \vec{\epsilon} = \frac{\hbar}{m} \nabla \phi.
\]

Equation (37) is readily integrated in terms of the Bessel function \( K_0 \); for a given wavevector \( q \) in the \( z \)-direction,

\[
\delta \Phi(\vec{r}) = \frac{\hbar}{m} \left( \varepsilon_y x - \varepsilon_x y \right) \frac{1}{r_\perp} \frac{\partial}{\partial r_\perp} K_0(qr_\perp),
\]

where \( r_\perp = (x^2 + y^2)^{1/2} \). At short distances from the line, \( qr_\perp \ll 1 \), \( \delta \Phi(\vec{r}) \) is given by \( (h/m)\delta \tan^{-1}((y - \varepsilon_y(z))/(x - \varepsilon_x(z))) \), i.e., the variation of the potential for a straight vortex line evaluated at the locally displaced position of the line. On the other hand, far from the line, \( qr_\perp \gg 1 \), the variation \( \Phi(\vec{r}) \) vanishes exponentially as \( \exp(-qr_\perp)/r_\perp^{1/2} \).
Appendix B. Commutation relations of the vortex displacements

The coefficients of $1/\omega^2$ in the high frequency limits of the correlation functions (19) imply, as expected, that at equal time, all components of the displacements commute $\langle [\epsilon_i, \epsilon_j] (\vec{k}) \rangle \equiv 0$, while the displacements and their time rates of change obey the equal-time commutation relations

$$\langle [\epsilon_L, \dot{\epsilon}_L] (\vec{k}) \rangle = \langle [\epsilon_T, \dot{\epsilon}_T] (\vec{k}) \rangle = \frac{\hbar}{\rho^*}, \quad \text{(39)}$$

and

$$\langle [\epsilon_L, \dot{\epsilon}_T] (\vec{k}) \rangle = 0. \quad \text{(40)}$$

On the other hand, if we assume *ab initio* that the inertial mass density $\rho^*$ associated with the vortex lines vanishes, then the commutation relations take on a quite different structure. The correlation functions are given by

$$\langle \epsilon_L \epsilon_L \rangle (\vec{k}, \omega) = -\frac{\hbar \alpha_T}{D'(\vec{k}, \omega)},$$

$$\langle \epsilon_T \epsilon_T \rangle (\vec{k}, \omega) = -\frac{\hbar \alpha_L}{D'(\vec{k}, \omega)},$$

$$\langle \epsilon_L \epsilon_T \rangle (\vec{k}, \omega) = \frac{2 \alpha L \alpha T - 4 \Omega^2 \rho^2 \omega^2}{D'(\vec{k}, \omega)} \frac{2 \hbar \Omega \rho \omega}{D'(\vec{k}, \omega)}, \quad \text{(41)}$$

where $D'(\vec{k}, \omega) = \alpha_L \alpha_T - 4 \Omega^2 \rho^2 \omega^2$. At equal time, $\langle [\epsilon_L, \epsilon_L] (\vec{k}) \rangle = 0 = \langle [\epsilon_T, \epsilon_T] (\vec{k}) \rangle$, as before; now, however, the longitudinal and transverse displacements no longer commute, but rather obey

$$\langle [\epsilon_L, \epsilon_T] (\vec{k}) \rangle = \frac{i \hbar}{2 \Omega \rho}. \quad \text{(42)}$$

The corresponding spectral weights are, for $\omega > 0$,

$$C_{LL}(\vec{k}, \omega) = \frac{\pi \hbar \omega T}{\rho \alpha_L} \delta(\omega - \omega_T), \quad C_{TT}(\vec{k}, \omega) = \frac{\pi \hbar \omega T}{\rho \alpha_T} \delta(\omega - \omega_T), \quad \text{(43)}$$

and

$$C_{LT}(\vec{k}, \omega) = -\frac{\pi \hbar}{2 \Omega \rho} \delta(\omega - \omega_T). \quad \text{(44)}$$

12
How does the non-vanishing commutation relation (42) develop? By letting \( \rho^* \) go to 0, or never introducing it in the first place, one is using an effective low frequency theory valid on frequency scales \( \ll \omega_I \). Indeed, the exact vanishing of \( \langle [\epsilon_L, \epsilon_T] \rangle (\vec{k}) \) comes about through a cancellation of the weight of the two delta functions in \( C_{LT} \) in (21). In the effective theory, the delta functions at \( \omega_I \) are absent in the spectral weight functions (43) and (44), and the cancellation in \( C_{LT} \) no longer takes place, leaving the effective commutation relation (42).

The operators of the low frequency theory are essentially time-averaged over several periods \( 2\pi/\omega_I \).

In the limit \( \rho^* \rightarrow 0 \), the effective Hamiltonian of the system becomes [cf. Eq. (13)],

\[
\hat{H} = \frac{1}{2} \left( \alpha_L \epsilon_L^2 + \alpha_T \epsilon_T^2 + \zeta_L \epsilon_L + \zeta_T \epsilon_T \right),
\]

and the equations of motion (11) (at \( \rho^* = 0 \)) follow directly from (45) using the commutation relation (42). The system is structurally that of an ensemble of harmonic oscillators, but unlike in a normal lattice, the energy depends on the displacements directly, rather than only on their derivatives. Note that the first integral in the contribution (27) of the long-wavelength modes to the displacements follows directly from (45).

It is instructive to compare this situation with the analogous one of two-dimensional motion of a particle in a magnetic field, \( \vec{H} \), in the guiding center approximation [18]. Consider \( \vec{H} \) in the \( z \) direction and motion only in the transverse plane. In the gauge in which the Hamiltonian is \( H = [p_x^2 + (p_y - eHx/c)^2]/2m \), the operator equations of motion have as integrals,

\[
x(t) = \frac{p_y(0)}{m\omega_L} + \left( x(0) - \frac{p_y(0)}{m\omega_L} \right) \cos \omega_L t + \frac{p_x(0)}{m\omega_L} \sin \omega_L t,
\]

\[
y(t) = y(0) - \left( x(0) - \frac{p_y(0)}{m\omega_L} \right) \sin \omega_L t + \frac{p_x(0)}{m\omega_L} (\cos \omega_L t - 1),
\]

where \( \omega_L = eH/mc \). These solutions preserve, as they must, the commutation relation \( [x(t), y(t)] = 0 \).

In studying motions varying slowly on the scale of a Larmor period, it is useful to average over several Larmor periods, the equivalent of averaging over several periods of the high-frequency inertial mode in the vortex problem. From Eqs. (46) we find the averaged
coordinate operators,
\[
\bar{x}(t) = \frac{p_y(0)}{m\omega_L}, \quad \bar{y}(t) = y(0) - \frac{p_x(0)}{m\omega_L}.
\] (47)

These averaged operators do not, in fact, commute, but rather obey
\[
[\bar{x}(t), \bar{y}(t)] = -\frac{i\hbar}{m\omega_L},
\] (48)

the analog of the commutator (42) of the displacement operators in the low-frequency effective theory [19].

In the presence of a driving potential \(V(x, y)\), the equation of motion is \(m\ddot{\vec{r}} = (e/c)(\vec{r} \times \vec{H}) - \nabla V\). For slowly varying phenomena, one can similarly neglect the inertia term here and replace the operators by their averages over several Larmor periods, in which case the equation of motion reduces, in the limit \(m \to 0\), to the requirement that the net force on the particle vanish:
\[
\frac{e}{c}(\vec{r} \times \vec{H}) - \nabla V(\bar{x}(t), \bar{y}(t)) = 0,
\] (49)

the analog of Eqs. (11) in the limit \(\rho^* \to 0\). This equation can be derived directly from the effective Hamiltonian, \(\bar{H} = V(\bar{x}(t), \bar{y}(t))\), using the commutation relation (48).

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[13] It should be emphasized that the arguments in this paper, based on the linear behavior of the vortex lattice, do not tell one whether the vortex lattice is capable of undergoing a first order melting transition, as in type II superconductors.

[14] A first estimate of the effective mass of a vortex line is that of a solid cylinder of radius $\xi$, the coherence length, moving uniformly through the fluid. The classical effective mass, even in the presence of the azimuthal vortex flow, equals the mass of the fluid displaced. Then $\rho^*/\rho \sim \Omega/\Omega_c$, where $\Omega_c$ is the upper critical rotational velocity above which the fluid becomes normal. See J.-M. Duan and A. J. Leggett, Phys. Rev. Letters 68, 1216 (1992), for calculations of the inertial mass of a vortex line in fermion superfluids.

[15] Equation (30) agrees qualitatively with the result of Moore [10] who studied effects of phase fluctuations arising from lattice oscillations in Type II superconductors and neutral superfluids; in charged superconductors he finds a critical dimension for destruction of phase coherence of four, and for a neutral superfluid a critical dimension of three. Inclusion here of the full structure of the Tkachenko modes in evaluating the fluctuations does not modify the infrared behavior found by Moore, who assumed an effective energy dependent only on gradients of the lattice displacements.

15
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