PSEUDODIFFERENTIAL OPERATORS ON $L^p$, WIENER AMALGAM AND MODULATION SPACES

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Abstract. We give a complete characterization of the continuity of pseudodifferential operators with symbols in modulation spaces $M^{p,q}$, acting on a given Lebesgue space $L^r$. Namely, we find the full range of triples $(p, q, r)$, for which such a boundedness occurs. More generally, we completely characterize the same problem for operators acting on Wiener amalgam space $W(L^r, L^s)$ and even on modulation spaces $M^{r,s}$. Finally the action of pseudodifferential operators with symbols in $W(\mathcal{F}L^1, L^\infty)$ is also investigated.

1. Introduction

A pseudodifferential operator in $\mathbb{R}^d$ with symbol $a \in \mathcal{S}'(\mathbb{R}^d)$ is defined by the formula

$$a(x, D)f(x) = \int_{\mathbb{R}^d} a(x, \omega) \hat{f}(\omega) e^{2\pi i x \omega} d\omega, \quad f \in \mathcal{S}(\mathbb{R}^d),$$

where $\hat{f}(x) = \mathcal{F}f(x) = \int_{\mathbb{R}^d} e^{-2\pi i x \omega} f(x) dx$ is the Fourier transform of $f$. Hence $a(x, D)f$ is well-defined as a temperate distribution.

Pseudodifferential operators arise at least in three different frameworks: partial differential equations (PDEs), quantum mechanics and engineering. In PDEs they were introduced independently in [22] and [24]. Since then, many symbol classes have been considered, according to several applications to PDEs. In particular, a deep analysis of such operators has been carried on for Hörmander’s classes $S^{m}_{\rho,\delta}$, $m \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$, of smooth functions $a(x, \omega)$ satisfying the estimates $|\partial_x^\alpha \partial_\omega^\beta a(x, \omega)| \leq C_{\alpha,\beta}(1 + |\omega|)^{m+\delta|\alpha|-\rho|\beta|}$.

Boundedness results on $L^p$-based Sobolev spaces for those operators are of special interest because they imply regularity results for the solutions of the corresponding PDEs.

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The basic result in this connection is the boundedness on $L^2$ of operators in the above classes, with $\delta < \rho$, which can be achieved by means of the symbolic calculus. Indeed, $L^2$-boundedness still holds for $0 \leq \delta = \rho < 1$ and even for symbols in $C^{2d+1}(\mathbb{R}^{2d})$, which is the classical Calderón-Vaillancourt theorem [4, 5]. Boundedness on $L^p$, $1 < p < \infty$, holds for symbols in $S^0_{1,\delta}$, $0 \leq \delta < 1$, but generally fails for $\rho < 1$ and a loss of derivatives may then occur. We refer the reader to [23, 31, 35] and the references therein for a detailed account. There are also many results for symbols which are smooth and behave as usual with respect to $\omega$, but less regular with respect to $x$, e.g. just belonging to some Hölder class. In this connection see the books [35, 36], where important applications to nonlinear equations are presented as well.

The smoothness of the symbol or the boundedness of all derivatives of the symbol are not necessary for the boundedness of pseudodifferential operators on $L^2(\mathbb{R}^d)$. Being motivated by this argument, many authors (see, e.g., [2, 3, 25, 27]) contributed to investigate the minimal assumption on the regularity of symbols for the corresponding operators to be bounded on $L^2$. In particular, Sugimoto [32] showed that symbols in the Besov space $B^{(\infty,\infty),(1,1)}_{d/2, d/2}$ imply $L^2$-boundedness (see also [33] and the references therein for extensions to the $L^p$ framework). In 1994/95 Sjöstrand introduced a new symbol class, larger than $S^0_{0,0}$, which was then recognized to be the modulation space $M^{\infty,1}(\mathbb{R}^d)$, first introduced in time-frequency analysis by Feichtinger [10, 11, 12]. For $1 \leq p, q \leq \infty$, the modulation space $M^{p,q}(\mathbb{R}^d)$ consists of the temperate distributions $f$ such that the function $\mathcal{F}(g(\cdot - x)f)(\omega)$ belongs to the mixed-norm space $L^{p,q}$ (see (2.2) below), where $g$ (the so-called window) is any non-zero Schwartz function. The role of the factor $g(\cdot - x)$ is that of localizing $f$ near the point $x$. Roughly speaking, distributions in $M^{p,q}$ have therefore the same local regularity as a function whose Fourier transform is in $L^q$, but decay at infinity like a function in $L^p$ (see [17] and Section 2 below for details). In [29, 30] Sjöstrand proved that symbols in $M^{\infty,1}$ give rise to $L^2$-bounded operators. In view of the inclusion $C^{2d+1}(\mathbb{R}^{2d}) \subset M^{\infty,1}(\mathbb{R}^d)$, this result represented an important generalization of the classical Calderon-Vaillancourt Theorem. Since then, several extensions appeared, mostly due to Gröchenig and collaborators. In particular, in [17, 19], symbols in $M^{\infty,1}$ were proved to produce bounded operators on all $M^{p,q}$. Further refinements appeared in [18, 26, 37, 38].

We now come more specifically to the results of the present paper. Examples show that symbols merely on $L^\infty$ generally do not produce bounded operator in $L^2$, but some additional regularity condition should be assumed. The above Sjöstrand’s result is just an instance of this. There is a space larger than $M^{\infty,1}$ which still consists of bounded functions having locally the same regularity as a function whose Fourier transform is integrable. It is the so-called Wiener amalgam space $W(\mathcal{F}L^1, L^\infty)$, the sub-space of temperate distributions $f$ such that the function
\[ \mathcal{F}(g(-x)f)(\omega) \] belongs to \( L^\infty L^1_\omega \) (see (2.3) below). A natural question which arises is whether pseudodifferential operators with symbols in \( W(\mathcal{F}L^1, L^\infty) \) are \( L^2 \)-bounded. Fourier multipliers with symbols in \( W(\mathcal{F}L^1, L^\infty) \) are indeed bounded on \( L^2 \) and the same holds, more generally, for symbols in \( W(\mathcal{F}L^1, L^\infty) \) of the type \( a(x, \omega) = a_1(x)a_2(\omega) \) (see Proposition 6.1 below). However, contrary to what these special cases could suggest, we shall show in Proposition 6.3 that, for more general symbols in that class, boundedness on \( L^2 \) may fail.

Another natural question is which modulation spaces give rise to bounded operators on \( L^p \), \( p \neq 2 \). We do not know results in this connection in the existent literature. We give here a complete answer to this problem, in the following form (see Corollary 3.7, Proposition 4.7 and Figure 1).

Let \( 1 \leq p, q, r \leq \infty \) such that

\[
\frac{1}{p} \geq \frac{1}{r} - \frac{1}{2} + \frac{1}{q}, \quad q \leq \min\{r, r'\}.
\]

Then every symbol \( a \in M^{p,q} \) gives rise to a bounded operator \( a(x, D) \) on \( L^r \).

Viceversa, if this conclusion holds true, then the constraints in (1.2) must be satisfied.

\[\text{Figure 1: The triples } (1/r, 1/q, 1/p) \text{ inside the convex polyhedron are exactly those for which every symbol in } M^{p,q} \text{ produces a bounded operator on } L^r.\]

To avoid technicalities, we only consider the action of \( a(x, D) \) on Schwartz functions, so that the definition of boundedness which is relevant here requires a small subtlety when \( r = \infty \); see Section 4.

Actually, we address to the more general problem of boundedness on the so-called Wiener amalgam spaces \( W(L^p, L^q) \), \( 1 \leq p, q \leq \infty \), which generalize the
Lebesgue spaces. We recall that a measurable function \( f \) belongs to \( W(L^p, L^q) \) if the following norm

\[
\|f\|_{W(L^p, L^q)} = \left( \sum_{n \in \mathbb{Z}^d} \left( \int_{\mathbb{R}^d} |f(x)T_n\chi_Q(x)|^p \right)^{\frac{q}{p}} \right)^\frac{1}{q},
\]

where \( Q = [0,1)^d \) (with the usual adjustments if \( p = \infty \) or \( q = \infty \) is finite (see [21] and Section 2 below). In particular, \( W(L^p, L^p) = L^p \). For heuristic purposes, functions in \( W(L^p, L^q) \) may be regarded as functions which are locally in \( L^p \) and decay at infinity like a function in \( L^q \). In this connection, our results read as follows (see Theorem 3.6 and Proposition 4.7).

Let \( 1 \leq p, q, r, s \leq \infty \) such that

\[
\frac{1}{p} \geq \left| \frac{1}{r} - \frac{1}{2} \right| + \frac{1}{q'}, \quad q \leq \min\{r, r', s, s'\}.
\]

Then every symbol \( a \in M^{p,q} \) gives rise to a bounded operator \( a(x, D) \) on \( W(L^r, L^s) \). Vice versa, if this conclusion holds true, then the constraints in (1.4) must be satisfied.

Finally, we investigate the boundedness of \( a(x, D) \) on modulation spaces. We wonder whether there are results other than those which follow by interpolation from the known ones. It turns out that this is not the case, as shown by the following result (see Theorem 5.2 and Proposition 5.4).

Let \( 1 \leq p, q, r, s \leq \infty \) such that

\[
\frac{1}{p} \leq q', \quad q \leq \min\{r, r', s, s'\}.
\]

Then every symbol \( a \in M^{p,q} \) gives rise to a bounded operator \( a(x, D) \) on \( M^{r,s} \). Vice versa, if this conclusion holds true, then the constraints in (1.5) must be satisfied.

This last result generalizes [20], where the above necessary conditions were proved in the case \( r = s = 2 \) (i.e. for \( L^2 \)-boundedness).

So far we considered pseudodifferential operators in the form (1.1), which is usually referred to as the Kohn-Nirenberg correspondence. However, as shown in Section 4, all the above results concerning symbols in modulation spaces apply to the Weyl quantization as well, defined in terms of the the cross-Wigner distribution \( W(f, g) \) in (2.9) by \( \langle L_\sigma f, g \rangle = \langle \sigma, W(g, f) \rangle \), \( f, g \in \mathcal{S}(\mathbb{R}^d) \), or directly as

\[
L_\sigma f(x) = \int e^{2\pi i(x-y)\omega} \sigma \left( \frac{x+y}{2}, \omega \right) f(y) \, dy \, d\omega.
\]

Instead, we shall prove in Section 6 that, contrary to what happens for modulation spaces, Wiener amalgam spaces \( W(FL^p, L^q) \), for \( p \neq q \) are not invariant under the
action of the mateplectic operator which switches the Kohn-Nirenberg and Weyl symbol of a pseudodifferential operator, so that the above mentioned counterexamples for symbols in $W(FL^1, L^\infty)$ will be provided for both Kohn-Nirenberg and Weyl operators.

The paper is organized as follows. Section 2 is devoted to preliminary definitions and properties of the involved function spaces. In Sections 3 and 4 we study sufficient and necessary conditions, respectively, for the boundedness on Wiener amalgam spaces (results for the Lebesgue spaces are attained there as a particular case). Section 5 provides necessary and sufficient conditions for the boundedness on modulation spaces. Finally Section 6 is devoted to some result for operators with symbols in $W(FL^p, L^q)$.

Notation. To be definite, let us fix some notation we shall use later on (and have already used in this Introduction). We define $xy = x \cdot y$, the scalar product on $\mathbb{R}^d$. We define by $C_0^\infty(\mathbb{R}^d)$ the space of smooth functions on $\mathbb{R}^d$ with compact support. The Schwartz class is denoted by $S(\mathbb{R}^d)$, the space of tempered distributions by $S'(\mathbb{R}^d)$. We use the brackets $\langle f, g \rangle$ to denote the extension to $S(\mathbb{R}^d) \times S'(\mathbb{R}^d)$ of the inner product $\langle f, g \rangle = \int f(t)g(t)dt$ on $L^2(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t)e^{-2\pi i t \omega}dt$. Moreover we set $f^*(x) = f(-x)$. Throughout the paper, we shall use the notation $A \lesssim B$, $A \gtrsim B$ to indicate $A \leq cB$, $A \geq kB$ respectively, for a suitable constant $c > 0$, whereas $A \asymp B$ if $A \leq cB$ and $B \leq kA$, for suitable $c, k > 0$.

2. Preliminary results

2.1. Function Spaces. For $1 \leq p \leq \infty$, recall the $FL^p$ spaces, defined by

$$FL^p(\mathbb{R}^d) = \{ f \in S'(\mathbb{R}^d) : \exists h \in L^p(\mathbb{R}^d), \hat{h} = f \};$$

they are Banach spaces equipped with the norm

$$\|f\|_{FL^p} = \|h\|_{L^p}, \quad \text{with} \ \hat{h} = f. \quad (2.1)$$

The mixed-norm space $L^{p,q}(\mathbb{R}^{2d})$, $1 \leq p, q \leq \infty$, consists of all measurable functions on $\mathbb{R}^{2d}$ such that the norm

$$\|F\|_{L^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \omega)|^p dx \right)^{\frac{q}{p}} d\omega \right)^{\frac{1}{q}} \quad (2.2)$$

(with obvious modifications when $p = \infty$ or $q = \infty$) is finite.
The function spaces $L^p L^q (\mathbb{R}^{2d})$, $1 \leq p, q \leq \infty$, consists of all measurable functions on $\mathbb{R}^{2d}$ such that the norm
\begin{equation}
\|F\|_{L^p L^q} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \omega)|^q d\omega \right)^{\frac{p}{q}} dx \right)^{\frac{1}{q}}
\end{equation}
(with obvious modifications when $p = \infty$ or $q = \infty$) is finite. Notice that, for $p = q$, we have $L^p L^p (\mathbb{R}^{2d}) = L^p(\mathbb{R}^{2d})$.

**Wiener amalgam spaces.** We briefly recall the definition and the main properties of Wiener amalgam spaces. We refer to [10, 12, 16, 21] for details.

Let $g \in C_0^\infty$ be a test function that satisfies $\|g\|_{L^2} = 1$. We will refer to $g$ as a window function. Let $B$ one of the following Banach spaces: $L^p, \mathcal{F}L^p, L^{p,q}, L^p L^q$, $1 \leq p, q \leq \infty$. Let $C$ be one of the following Banach spaces: $L^p, L^{p,q}, L^p L^q$, $1 \leq p, q \leq \infty$. For any given temperate distribution $f$ which is locally in $B$ (i.e. $gf \in B$, $\forall g \in C_0^\infty$), we set $f_B(x) = \|f_T g\|_B$.

The *Wiener amalgam space* $W(B, C)$ with local component $B$ and global component $C$ is defined as the space of all temperate distributions $f$ locally in $B$ such that $f_B \in C$. Endowed with the norm $\|f\|_{W(B, C)} = \|f_B\|_C$, $W(B, C)$ is a Banach space. Moreover, different choices of $g \in C_0^\infty$ generate the same space and yield equivalent norms.

If $B = \mathcal{F}L^1$ (the Fourier algebra), the space of admissible windows for the Wiener amalgam spaces $W(\mathcal{F}L^1, C)$ can be enlarged to the so-called Feichtinger algebra $W(\mathcal{F}L^1, L^1)$. Recall that the Schwartz class $\mathcal{S}$ is dense in $W(\mathcal{F}L^1, L^1)$.

The following properties of Wiener amalgam spaces will be frequently used in the sequel.

**Lemma 2.1.** Let $B_i, C_i$, $i = 1, 2, 3$, be Banach spaces such that $W(B_i, C_i)$ are well defined. Then,

(i) Convolution. If $B_1 \ast B_2 \hookrightarrow B_3$ and $C_1 \ast C_2 \hookrightarrow C_3$, we have
\begin{equation}
W(B_1, C_1) \ast W(B_2, C_2) \hookrightarrow W(B_3, C_3).
\end{equation}

(ii) Inclusions. If $B_1 \hookrightarrow B_2$ and $C_1 \hookrightarrow C_2$,
\[ W(B_1, C_1) \hookrightarrow W(B_2, C_2). \]

Moreover, the inclusion of $B_1$ into $B_2$ need only hold “locally” and the inclusion of $C_1$ into $C_2$ “globally”. In particular, for $1 \leq p_i, q_i \leq \infty$, $i = 1, 2$, we have
\begin{equation}
p_1 \geq p_2 \text{ and } q_1 \leq q_2 \implies W(L^{p_1}, L^{q_1}) \hookrightarrow W(L^{p_2}, L^{q_2}).
\end{equation}

(iii) Complex interpolation. For $0 < \theta < 1$, we have
\[ [W(B_1, C_1), W(B_2, C_2)]_{[\theta]} = W([B_1, B_2]_{[\theta]}, [C_1, C_2]_{[\theta]}), \]
if $C_1$ or $C_2$ has absolutely continuous norm. The same holds if every Wiener amalgam space is replaced by the closure of the Schwartz space into itself.

(iv) Duality. If $B', C'$ are the topological dual spaces of the Banach spaces $B, C$ respectively, and the space of test functions $C_0^\infty$ is dense in both $B$ and $C$, then

$$(2.6) \quad W(B, C)' = W(B', C').$$

(v) Pointwise products. If $B_1 \cdot B_2 \hookrightarrow B_3$ and $C_1 \cdot C_2 \hookrightarrow C_3$, we have

$$(2.7) \quad W(B_1, C_1) \cdot W(B_2, C_2) \hookrightarrow W(B_3, C_3).$$

Finally, recall the following result, proved in [8, Proposition 2.7].

**Lemma 2.2.** Let $1 \leq q \leq p \leq \infty$. For every $R > 0$, there exists a constant $C_R > 0$ such that, for every $f \in S'(\mathbb{R}^d)$ whose Fourier transform is supported in any ball of radius $R$, it turns out

$$\|f\|_{W(L^p, L^q)} \leq C_R \|f\|_q.$$ 

### 2.2. Short-Time Fourier Transform (STFT) and Wigner distribution.

The time-frequency representations needed for our results are the short-time Fourier transform and the Wigner distribution.

The short-time Fourier transform (STFT) of a distribution $f \in S'(\mathbb{R}^d)$ with respect to a non-zero window $g \in S(\mathbb{R}^d)$ is

$$(2.8) \quad V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} dt,$$ 

whereas the cross-Wigner distribution $W(f, g)$ of $f, g \in L^2(\mathbb{R}^d)$ is defined to be

$$(2.9) \quad W(f, g)(x, \omega) = \int f(x + \frac{t}{2}) \overline{g(x - \frac{t}{2})} e^{-2\pi i \omega t} dt.$$ 

The quadratic expression $Wf = W(f, f)$ is usually called the Wigner distribution of $f$.

Both the STFT $V_g f$ and the Wigner distribution $W(f, g)$ are defined on many pairs of Banach spaces. For instance, they both map $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ and $S(\mathbb{R}^d) \times S(\mathbb{R}^d)$ into $S(\mathbb{R}^{2d})$. Furthermore, they can be extended to a map from $S'(\mathbb{R}^d) \times S(\mathbb{R}^d)$ into $S'(\mathbb{R}^{2d})$. We first list some crucial properties of the STFT (for proofs, see, e.g., [17, Ch. 3].

**Lemma 2.3.** Let $f, g \in L^2(\mathbb{R}^d)$, then we have

(i) $V_g f(x, \omega) = \langle f \cdot T_x \bar{g} \rangle(\omega)$.

(ii) (STFT of time-frequency shifts) For $y, \xi \in \mathbb{R}^d$, we have

$$(2.10) \quad V_g(M_\xi T_y f)(x, \omega) = e^{-2\pi i \omega \cdot \xi} (V_g f)(x-y, \omega - \xi),$$ 

The following result was proved in [17, Lemma 14.5.1].
Lemma 2.4. Let $\Phi = W(\varphi, \varphi) \in \mathcal{S}(\mathbb{R}^{2d})$. Then the STFT of $W(g, f)$ with respect to the window $\Phi$ is given by

$$V_\Phi(W(g, f))(z, \zeta) = e^{-2\pi i z \cdot \zeta} V_\varphi f(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2}) V_\varphi g(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2}),$$

where $z = (z_1, z_2) \in \mathbb{R}^{2d}$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

Modulation spaces. For their basic properties we refer to [11, 17].

Given a non-zero window $g \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq p, q \leq \infty$, the modulation space $M^{p,q}(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that the STFT, defined in (2.8), fulfills $V_g f \in L^{p,q}(\mathbb{R}^{2d})$. The norm on $M^{p,q}$ is

$$\|f\|_{M^{p,q}} = \|V_g f\|_{L^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p dx \right)^{q/p} d\omega \right)^{1/p}.$$ 

If $p = q$, we write $M^p$ instead of $M^{p,p}$.

$M^{p,q}$ is a Banach space whose definition is independent of the choice of the window $g$. Moreover, if $g \in M^1 \setminus \{0\}$, then $\|V_g f\|_{L^{p,q}}$ is an equivalent norm for $M^{p,q}(\mathbb{R}^d)$.

Among the properties of modulation spaces, we record that $M^{2,2} = L^2$, and we list the following results.

Lemma 2.5. We have

(i) $M^{p_1,q_1} \hookrightarrow M^{p_2,q_2}$, if $p_1 \leq p_2$ and $q_1 \leq q_2$.

(ii) If $1 \leq p, q < \infty$, then $(M^{p,q})' = M^{p',q'}$.

(iii) For $1 \leq p, q, p_i, q_i \leq \infty$, $i = 1, 2$, with $q_1 < \infty$ or $q_2 < \infty$, and

$$\frac{1}{p} = \frac{1}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{\theta}{q_2},$$

we have

$$[M^{p_1,q_1}, M^{p_2,q_2}]_\theta = M^{p,q}.$$ 

The same holds if every modulation space is replaced by the closure of the Schwartz space into itself.

(iv) If $1 \leq p_i, q_i \leq \infty$, $i = 1, 2, 3$, with

$$\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p_3}, \quad \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3}$$

then

$$M^{p_1,q_1} * M^{p_2,q_2} \hookrightarrow M^{p_3,q_3}.$$ 

Modulation spaces and Wiener amalgam spaces are closely related: for $p = q$, we have

$$\|f\|_{W(FL^p, L^p)} = \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_g f(x, \omega)|^p m(x, \omega)^p dx d\omega \right)^{1/p} \asymp \|f\|_{M^p}.$$
More generally, from comparing the definitions of $M^{p,q}$ and $W(\mathcal{F}L^p, L^q)$, it is obvious that $M^{p,q} = \mathcal{F}W(\mathcal{F}L^p, L^q)$.

To prove the boundedness results for pseudodifferential operators, we shall write their symbols as superposition of time-frequency shifts. Namely, we shall use the following STFT inversion formula (see, e.g., [17, 19]).

**Theorem 2.6.** If $g \in S(\mathbb{R}^d)$ and $\|g\|_2 = 1$, then

$$a = \int_{\mathbb{R}^4d} V_g a(\alpha, \beta) M_\beta T_{\alpha} gd\alpha d\beta. \tag{2.13}$$

If $a \in M^{p,q}$, with $1 \leq p, q < \infty$, then this integral converges in the norm of this space. If $p = \infty$ or $q = \infty$ then this integral converges weakly.

We also recall the following well-known result (see, e.g., [14, 28]).

**Lemma 2.7.** Let $1 \leq p, q \leq \infty$.

(i) For every $u \in S'(\mathbb{R}^d)$, supported in a compact set $K \subset \mathbb{R}^d$, we have $u \in M^{p,q} \iff u \in \mathcal{F}L^q$, and

$$C_K^{-1} \|u\|_{M^{p,q}} \leq \|u\|_{\mathcal{F}L^q} \leq C_K \|u\|_{M^{p,q}}, \tag{2.14}$$

where $C_K > 0$ depends only on $K$.

(ii) For every $u \in S'(\mathbb{R}^d)$, whose Fourier transform is supported in a compact set $K \subset \mathbb{R}^d$, we have $u \in M^{p,q} \iff u \in L^p$, and

$$C_K^{-1} \|u\|_{M^{p,q}} \leq \|u\|_{L^p} \leq C_K \|u\|_{M^{p,q}}, \tag{2.15}$$

where $C_K > 0$ depends only on $K$.

3. **Boundedness on Wiener amalgam spaces: sufficient conditions**

To avoid the fact that $S(\mathbb{R}^d)$ is not dense in $W(L^r, L^s)$, if $r = \infty$ or $s = \infty$, we use the following definition of the boundedness of pseudodifferential operators $a(x, D)$ on Wiener amalgam spaces: we say that $a(x, D)$ is bounded from $W(L^r, L^s)(\mathbb{R}^d)$ to $W(L^r, L^s)(\mathbb{R}^d)$ if there exists a constant $C > 0$ such that $\|a(x, D)f\|_{W(L^r, L^s)} \leq C \|f\|_{W(L^r, L^s)}$, for all $f \in S(\mathbb{R}^d)$.

Let us recall the following result (see, e.g., [17 Theorem 14.3.5]).

**Theorem 3.1.** Let $T$ be a continuous linear operator mapping $S(\mathbb{R}^d)$ into $S'(\mathbb{R}^d)$. Then there exist tempered distributions $K, \sigma, a \in S'((\mathbb{R}^d)^2)$, such that $T$ has the following representations:

(i) as an integral operator $\langle Tf, g \rangle = \langle K, g \otimes f \rangle$, for $f, g \in S(\mathbb{R}^d)$;

(ii) as a pseudodifferential operator $T = L_\sigma$, with Weyl symbol $\sigma$ and $T = a(x, D)$. 
with Kohn-Nirenberg symbol $a$.

The relations between $K, \sigma, a$ are given by

$$\sigma = \mathcal{F}_2 \tau_s K, \quad a = U \sigma, \quad \sigma = U^{-1} a$$

where $\mathcal{F}_2$ is the partial Fourier transform in the second variable, $\tau_s$ is the symmetric coordinate transformation $\tau_s K(x, y) = K(x + \frac{s}{2}, x - \frac{s}{2})$ and the operator $U$ is defined by $(U \sigma)(\omega_1, \omega_2) = e^{\pi i \omega_1 \omega_2} \tilde{\sigma}(\omega_1, \omega_2)$.

**Remark 3.2.** Since $a(x, D) = L_{U^{-1} a}$, a straightforward modification of [17, Corollary 14.5.5] shows that the modulation spaces $M^{p,q}$, $1 \leq p, q \leq \infty$, are invariant under $U^{-1}$, so that boundedness results for pseudodifferential operators with symbols in modulation spaces can be obtained using either the Weyl or the Kohn-Nirenberg form. In the sequel, we shall adopt the operator form which is more convenient.

We also need the following useful remark.

**Remark 3.3.** Observe that, if $1 \leq r < \infty$, then

$$\left( S^{W(L^r, L^{\infty})} \right)' = W(L', L^1),$$

see, e.g., [13, Theorem 2.8].

**Theorem 3.4.** If $\sigma \in M^{\infty,1}(\mathbb{R}^d)$, then the Weyl operator $L_\sigma$ is bounded on $W(L^2, L^s)(\mathbb{R}^d)$, for every $1 \leq s \leq \infty$, with the uniform estimate

$$\|L_\sigma f\|_{W(L^2, L^s)} \lesssim \|\sigma\|_{M^{\infty,1}} \|f\|_{W(L^2, L')}.$$

**Proof.** Let us show the estimate

$$|\langle L_\sigma f, g \rangle| \lesssim \|\sigma\|_{M^{\infty,1}} \|f\|_{W(L^2, L')} \|g\|_{W(L^2, L')}, \quad \forall f, g \in S(\mathbb{R}^d),$$

where $1/s + 1/s' = 1$. This will give at once the desired result if $s > 1$, whereas the case $s = 1$ follows by Remark 3.3.

Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ and set $\Phi = W(\varphi, \varphi) \in S(\mathbb{R}^d)$. By the definition of the Weyl operator via cross-Wigner distribution and Hölder’s inequality,

$$|\langle L_\sigma f, g \rangle| = |\langle \sigma, W(g, f) \rangle| = |\langle V_\phi \sigma, V_\phi W(g, f) \rangle| \leq \|V_\phi \sigma\|_{L^{\infty,1}} \|V_\phi W(g, f)\|_{L^{1,\infty}}$$

$$\lesssim \|\sigma\|_{M^{\infty,1}} \|W(g, f)\|_{M^{1,\infty}}.$$

Then, the result is proved if we show that $\|W(g, f)\|_{M^{1,\infty}} \lesssim \|f\|_{W(L^2, L^s)} \|g\|_{W(L^2, L')}$. If $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^d$, we write $\tilde{\zeta} = (\zeta_2, -\zeta_1)$. Then Lemma 2.4 says that

$$|V_\Phi(W(g, f))(z, \zeta)| = |V_\Phi f(z + \frac{\tilde{\zeta}}{2})| |V_\Phi g(z - \frac{\tilde{\zeta}}{2})|.$$

Consequently

$$\|W(g, f)\|_{M^{1,\infty}} \simeq \sup_{\zeta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |V_\Phi f(z + \frac{\tilde{\zeta}}{2})| |V_\Phi g(z - \frac{\tilde{\zeta}}{2})| dz.$$
We set $\pi(\tilde{\zeta}) = M_{\tilde{\zeta}}T_{\hat{\zeta}}$. In what follows, we make the change of variables $z \mapsto \tilde{z}/2$, and use Lemma 2.3 (i) and (ii), the Cauchy-Schwarz’s and Parseval’s inequalities with respect to the $z_2$ variable, so that
\[
\|W(g, f)\|_{M^{1, \infty}} \lesssim \sup_{\zeta \in \mathbb{R}^{2d}} \int_{\mathbb{R}^d} |V_\varphi f(z)| |V_\varphi g(z - \tilde{\zeta})| \, dz
\]
\[
= \sup_{\zeta \in \mathbb{R}^{2d}} \int_{\mathbb{R}^d} |V_\varphi f(z)| |V_\varphi (\pi(\tilde{\zeta})g)(z)| \, dz
\]
\[
= \sup_{\zeta \in \mathbb{R}^{2d}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |fT_{z_1} \varphi(z_2)||\pi(\tilde{\zeta})gT_{z_1} \varphi(z_2)| \, dz_1 \, dz_2
\]
\[
\lesssim \sup_{\zeta \in \mathbb{R}^{2d}} \int_{\mathbb{R}^d} \|fT_{z_1} \varphi\|_2 \|\pi(\tilde{\zeta})gT_{z_1} \varphi\|_2 \, dz_1
\]
\[
\lesssim \sup_{\zeta \in \mathbb{R}^{2d}} \|f\|_{W(L^2, L^s)} \|\pi(\tilde{\zeta})g\|_{W(L^2, L^{s'})}
\]
\[
= \|f\|_{W(L^2, L^s)} \|g\|_{W(L^2, L^{s'})},
\]
where we have used Hölder’s inequality in the last-but-one step and the invariance of the $W(L^2, L^s)$ spaces under time-frequency shifts $\pi(\tilde{\zeta})$ in the last one. 

If we choose symbols with a stronger decay, namely symbols $a \in M^{p, 1} \subset M^{\infty, 1}$, $1 \leq p \leq 2$, then the corresponding pseudodifferential operators $a(x, D)$ are bounded on every Wiener amalgam spaces $W(L^r, L^s)$, as shown in the following result.

**Theorem 3.5.** If $a \in M^{p, 1}(\mathbb{R}^{2d})$, $1 \leq p \leq 2$, then the operator $a(x, D)$ is bounded on $W(L^r, L^s)(\mathbb{R}^d)$, for every $1 \leq r, s \leq \infty$, with the uniform estimate
\[
\|a(x, D)f\|_{W(L^r, L^s)} \lesssim \|a\|_{M^{p, 1}} \|f\|_{W(L^r, L^s)}.
\]

**Proof.** By the inclusion relations for modulation spaces we can just consider the case $a \in M^{2, 1}$. We shall show that the integral kernel
\[
K(x, y) = (\mathcal{F}_2 a)(x, y - x)
\]
($\mathcal{F}_2$ stands for the partial Fourier transform with respect to the second variable) of $a(x, D)$ can be controlled from above by $|K(x, y)| \leq F(x - y)$, where $F$ is a positive function in $L^1(\mathbb{R}^d)$. If it is so, then $|a(x, D)f(x)| \leq (F * |f|)(x)$ and the convolution relations for Wiener amalgam spaces in Lemma 2.1 (i) give the desired result.

We use the inversion formula (2.13) for the symbol $a$. Namely, for any window $g \in \mathcal{S}(\mathbb{R}^{2d})$, with $\|g\|_2 = 1$, we have
\[
(3.2) \quad a(x, \omega) = \int_{\mathbb{R}^{4d}} (V_g a)(\alpha, \beta)(M_\beta T_\alpha g)(x, \omega) \, d\alpha d\beta.
\]
Hence, if \( \alpha = (\alpha_1, \alpha_2); \beta = (\beta_1, \beta_2); \) it turns out
\[
K(x, y) = \int_{\mathbb{R}^{2d}} e^{2\pi i \alpha_2 \beta_2}(V_g a)(\alpha_1, \alpha_2, \beta_1, \beta_2)M(\alpha_1, \alpha_2)T_{(\alpha_1 - \beta_2)}(\mathcal{F}_2^{-1}g)(x, y)d\alpha_1d\alpha_2d\beta_1d\beta_2.
\]
Setting
\[
H(\alpha_1, t; \beta_1, \beta_2) = \int_{\mathbb{R}^d} (V_g a)(\alpha, \beta)e^{2\pi i (t - \beta_2)\alpha_2}d\alpha_2,
\]
and using the Cauchy-Schwarz’s inequality with respect to the \( \alpha_2 \) variable, we obtain
\[
|K(x, y)| \leq \int_{\mathbb{R}^{2d}} |H(\alpha_1, x - y + \beta_2; \beta_1, \beta_2)\mathcal{F}_2^{-1}g(x - \alpha_1, x - y + \beta_2)|d\alpha_1d\beta_1d\beta_2
\]
\[
\leq \int_{\mathbb{R}^{2d}} \|H(\cdot, x - y + \beta_2; \beta_1, \beta_2)\|_2\|T_{(0, -\beta_2)}\mathcal{F}_2^{-1}g(\cdot, x - y)\|_2d\beta_1d\beta_2.
\]
For simplicity, let us set
\[
F(t) := \int_{\mathbb{R}^{2d}} \|H(\cdot, t + \beta_2; \beta_1, \beta_2)\|_2\|T_{(0, -\beta_2)}\mathcal{F}_2^{-1}g(\cdot, t)\|_2d\beta_1d\beta_2,
\]
so that \( |K(x, y)| \leq F(x - y) \).
We are left to estimate \( \|F\|_1 \). Using the Cauchy-Schwarz’s inequality with respect to the \( t \) variable,
\[
\|F\|_1 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{2d}} \|H(\cdot, t + \beta_2; \beta_1, \beta_2)\|_2\|T_{(0, -\beta_2)}\mathcal{F}_2^{-1}g(\cdot, t)\|_2d\beta_1d\beta_2dt
\]
\[
\leq \int_{\mathbb{R}^{2d}} \|H(\cdot, \cdot; \beta_1, \beta_2)\|_2\|T_{(0, -\beta_2)}\mathcal{F}_2^{-1}g\|_2d\beta_1d\beta_2
\]
\[
= \|g\|_2 \int_{\mathbb{R}^{2d}} \|H(\cdot, \cdot; \beta_1, \beta_2)\|_2d\beta_1d\beta_2
\]
\[
= \|H\|_{L^2,1} = \|V_g a\|_{L^2,1} \asymp \|a\|_{M^{2,1}},
\]
where in the last row we used Parseval’s formula and the assumption \( \|g\|_2 = 1 \).
This concludes the proof. \( \square \)

**Theorem 3.6.** Let \( 1 \leq p, q, r, s \leq \infty \) such that
\[
\frac{1}{p} = \frac{1}{r} - \frac{1}{2} + \frac{1}{q'}, \quad q \leq \min\{r, r', s, s'\}.
\]
Then every symbol \( a \in M^{p,q} \) gives rise to a bounded operator \( a(x, D) \) on \( W(L^r, L^s) \) with the uniform estimate
\[
\|a(x, D)f\|_{W(L^r, L^s)} \lesssim \|a\|_{M^{p,q}} \|f\|_{W(L^r, L^s)}.
\]
Proof. We first make the complex interpolation between the estimates of Theorem 3.4 and Theorem 3.5 (which deal with the cases in which \( q = 1 \)). Using the interpolation relations for Wiener amalgam and modulation spaces of Lemma 2.1 (iii) and Lemma 2.5 (iii), we obtain, for every \( 1 \leq s < \infty \),

\[
\|a(x, D)f\|_{W(L^r, L^s)} \lesssim \|a\|_{M^{p,1}} \|f\|_{W(L^r, L^s)},
\]

where

\[
\frac{1}{p} \geq \left| \frac{1}{r} - \frac{1}{2} \right| + \frac{1}{q}. \]

The remaining cases, when \( s = \infty \), \( p > 2 \) (and therefore \( r > 1 \)), follow by duality, for \( W(L^r, L^\infty) = W(L^{r'}, L^1)' \) (Lemma 2.1 (iv)) and, considering the Weyl form \( L_\sigma \) of \( a(x, D) \), we have \( (L_\sigma)^* = L_{\bar{\sigma}} \).

Finally, by interpolation between what we just proved and the well-known case \( p = q = r = s = 2 \) (pseudodifferential operators with symbols in \( M^2(R^{2d}) = L^2(R^{2d}) \) are bounded on \( W(L^2, L^2)(R^d) = L^2(R^d) \); see \cite[Theorem 14.6.1]{17}), we obtain the claim. □

Recalling that, for \( s = r \), we have \( W(L^r, L^r) = L^r \), the above boundedness result can be rephrased for pseudodifferential operators acting on \( L^p \) spaces as follows (see Figure 1 in Introduction).

**Corollary 3.7.** Let \( 1 \leq p, q, r \leq \infty \) such that

\[
\frac{1}{p} \geq \left| \frac{1}{r} - \frac{1}{2} \right| + \frac{1}{q'}, \quad q \leq \min\{r, r'\}.
\]

Then every symbol in \( a \in M^{p,q} \) gives rise to a bounded operator \( a(x, D) \) on \( L^r \) with the uniform estimate

\[
\|a(x, D)f\|_r \lesssim \|a\|_{M^{p,q}} \|f\|_r.
\]

4. Boundedness on Wiener amalgam spaces: necessary conditions

In this section we show the optimality of Theorem 3.6 (and Corollary 3.7). We need the following auxiliary results.

**Lemma 4.1.** Suppose that for some \( 1 \leq p, q, r, s \leq \infty \) the following estimate holds:

\[
\|L_\sigma f\|_{W(L^r, L^s)} \leq C\|\sigma\|_{M^{p,q}} \|f\|_{W(L^r, L^s)}, \quad \forall \sigma \in S(R^{2d}), \forall f \in S(R^d).
\]

Then the same estimate is satisfied with \( r, s \) replaced by \( r', s' \) (even if \( r = \infty \) or \( s = \infty \)).

**Proof.** Indeed, observe that \( \langle L_\sigma f, g \rangle = \langle f, L_{\bar{\sigma}} g \rangle \), \( \forall f, g \in S(R^d) \). Hence, by Lemma 2.1 (iv) and the assumptions written for \( L_{\bar{\sigma}} \) (observe that \( \|\bar{\sigma}\|_{M^{p,q}} = \|\sigma\|_{M^{p,q}} \)), we have

\[
\|\langle L_\sigma f, g \rangle\| \leq C\|\sigma\|_{M^{p,q}} \|f\|_{W(L^{r'}, L^{s'})} \|g\|_{W(L^r, L^s)}, \quad \forall f \in S(R^d), \forall g \in S(R^d).
\]
Since \( L_\sigma f \) is a Schwartz function, it belongs to \( W(L^{r'}, L^{s'}) \subset W(L^r, L^s)' \), and \( \|L_\sigma f\|_{W(L^r, L^s)'} = \|L_\sigma f\|_{W(L^{r'}, L^{s'})}' \), because \( W(L^{r'}, L^{s'}) \) is isometrically embedded in \( W(L^r, L^s)' \). Hence it suffices to prove that the estimate in (4.1) holds for every \( g \in W(L^r, L^s) \). This follows by a density argument. Namely, consider, for a given \( g \in W(L^r, L^s) \), a sequence \( g_n \) of Schwartz functions, with \( g_n \to g \) in \( S'(\mathbb{R}^d) \) and \( \|g_n\|_{W(L^r, L^s)} \leq \|g\|_{W(L^r, L^s)}\). Letting \( n \to \infty \) in the above estimate (written with \( g_n \) in place of \( g \)) gives the desired conclusion. \( \square \)

**Lemma 4.2.** Let \( h \in C_0^\infty(\mathbb{R}^d) \), and consider the family of functions

\[
h_\lambda(x) = h(x)e^{-\pi x_\lambda |x|^2}, \quad \lambda \geq 1.
\]

Then, for \( 1 \leq q \leq \infty \),

\[
\|\widehat{h_\lambda}\|_q \lesssim \lambda^{\frac{d}{q} - \frac{d}{2}}.
\]

**Proof.** The result is known and outlined, e.g., in [34, Exercise 2.34]. We report on a sketch of the proof for the sake of completeness.

Let \( c > 0 \) be such that \( h(x) \) vanishes for \( |x| > c \). First one shows the estimate \( |\widehat{h_\lambda}(\omega)| \leq C_N \langle \omega \rangle^{-N} \lambda^{-N} \), for every \( N > 0 \) and \( \omega \in \mathbb{R}^d \) such that \( |\omega| \geq 2c\lambda \). To this end, we observe that by rotational symmetry we can assume \( \omega = (\omega_1, 0, ..., 0) \). The claim then follows by applying the Non-stationary Phase Theorem [31, Proposition 1, page 331] with the asymptotic parameter \( \omega_1 \) and the phase \( \phi(x_1) := -2\pi x_1 - \pi \lambda x_1^2 \) (the assumptions being satisfied for \( |x_1| \leq c \), uniformly with respect to the parameter \( \lambda/\omega_1 \)).

In the region \( |\omega| < 2c\lambda \) we have the estimate \( |\widehat{h_\lambda}(\omega)| \leq C\lambda^{-d/2} \), as a consequence of the Stationary Phase Theorem (see [31, 5.13 (a), page 363]) with the phase given by the quadratic polynomial \( \phi(x) := -\pi |x|^2 \). One hence obtains the upper bound \( \|\widehat{h_\lambda}\|_q \lesssim \lambda^{d(1/q - 1/2)} \). Since

\[
\|h_\lambda\|_2 = \|\widehat{h_\lambda}\|_2 \leq \|\widehat{h_\lambda}\|_q \|\widehat{h_\lambda}\|_{q'} \lesssim \lambda^{d(1/q' - 1/2)} \|\widehat{h_\lambda}\|_q,
\]

the lower bound follows as well. \( \square \)

We now establish a version of the upper bound in Lemma 4.2 for Wiener amalgam spaces.

**Lemma 4.3.** With the notation of Lemma 4.2 we have, for \( 1 \leq q \leq \infty \),

\[
\|\widehat{h_\lambda}\|_{W(L^p, L^q)} \lesssim \lambda^{\frac{d}{q} - \frac{d}{2}}, \quad \lambda \geq 1.
\]

**Proof.** When \( p \geq q \) the desired result follows from Lemmata 2.2 and 4.2. When \( p < q \) the result follows from the inclusion \( L^q \hookrightarrow W(L^p, L^q) \) and Lemma 1.2. \( \square \)

\(^1\)For example, take \( g_n(x) = n^d \varphi_1(x/n) (g * \varphi_2(n \cdot))(x) \), with \( \varphi_1, \varphi_2 \in C_0^\infty(\mathbb{R}^d), \varphi_1(0) = 1, \|\varphi_2\|_1 = 1.\)
Proposition 4.4. Let $\chi \in C_0^\infty(\mathbb{R}^d)$, $\chi \geq 0$, $\chi(0) = 1$. Suppose that, for some $1 \leq p, q, r, r_1, r_2 \leq \infty$, $C > 0$, the estimate

$$\|\chi a(x, D)f\|_r \leq C\|a\|_{M^{p,q}}\|f\|_{W(L^{r_1}, L^{r_2})}, \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), \ f \in \mathcal{S}(\mathbb{R}^d),$$

holds. Then $q \leq r_2'$ and

$$\frac{1}{p} \geq \frac{1}{2} - \frac{1}{r} + \frac{1}{q'}.$$

Proof. First we prove the constraint $q \leq r_2'$. Let $h \in C_0^\infty(\mathbb{R}^d)$, $h \geq 0$, $h(0) = 1$. Let $h_\lambda$ be as in Lemma 4.2. We test (4.3) on the family of symbols

$$a_\lambda(x, \omega) = h(x)h_\lambda(\omega) = h(x)h(\omega)e^{-\pi i \lambda |\omega|^2},$$

and functions $f_\lambda = \mathcal{F}^{-1}(\overline{h_\lambda}) \in \mathcal{S}(\mathbb{R}^d)$. An explicit computation shows that

$$\chi(x)a_\lambda(x, D)f_\lambda(x) = \int e^{2\pi i x \omega} \chi(x)h(x)h^2(\omega) \, d\omega,$$

which is a non-zero Schwartz function independent of $\lambda$. On the other hand, by Lemma 4.2 we have

$$\|a_\lambda\|_{M^{p,q}} \lesssim \|a_\lambda\|_{\mathcal{F}L^q} \lesssim \|h_\lambda\|_{\mathcal{F}L^q} \lesssim \lambda^{\frac{d}{4} - \frac{d}{q}}.$$

Similarly, by Lemma 4.3

$$\|f_\lambda\|_{W(L^{r_1}, L^{r_2})} \lesssim \lambda^{\frac{d}{2} - \frac{d}{q}}.$$

Taking into account these estimates and letting $\lambda \to +\infty$, (4.3) then gives $q \leq r_2'$.

Let us now prove (4.4). Let $h_\lambda$ be as above. We now test the estimate (4.3) on the family of symbols

$$a_\lambda'(x, \omega) = e^{-\pi \lambda |x|^2} \overline{h_\lambda(\omega)},$$

and functions $f_\lambda' = \overline{h_\lambda}$. The operator $a_\lambda'(x, D)$ has integral kernel

$$K_\lambda(x,y) = e^{-\pi \lambda |x|^2} h_\lambda(x-y),$$

so that

$$\chi(x)|a_\lambda'(x, D)f_\lambda'(x)| = \left| \int e^{-\pi \lambda |x|^2 + 2\pi i \lambda xy} h(x-y) \chi(x)h(y) \, dy \right| \geq \Re \int e^{-\pi \lambda |x|^2 + 2\pi i \lambda xy} h(x-y) \chi(x)h(y) \, dy.$$

Now, $h(y)$ has compact support, say, in the ball $|y| \leq C$. Moreover, if $|x| \leq \lambda^{-1}$ for $\lambda \geq \lambda_0$ large enough, and $|y| \leq C$ we have $\Re \left( e^{-\pi \lambda |x|^2 + 2\pi i \lambda xy} \right) \geq \frac{1}{2}$. Hence we deduce

$$\chi(x)|a_\lambda'(x, D)f_\lambda'(x)| \gtrsim 1, \quad \text{for } |x| \leq \lambda^{-1},$$

PSEUDODIFFERENTIAL OPERATORS ON LEBESGUE SPACES 15
which implies  
\[ \| \chi a'_\lambda(x, D) f'_\lambda \|_r \gtrsim \lambda^{-\frac{d}{2}}. \]
On the other hand, \( \| f'_\lambda \|_{W(L^{r_1}, L^{r_2})} \) is clearly independent of \( \lambda \). Moreover, by [7, Lemma 3.2],
\[ \| e^{-\pi \lambda^2 |\cdot|^2} \|_{M^{p,q}} \lesssim \lambda^{-\frac{d}{p}} \]
and by Lemmata 2.7 (ii) and 4.2 we have
\[ \| \hat{h}_\lambda \|_{M^{p,q}} \lesssim \| \hat{h}_\lambda \|_p \lesssim \lambda^{\frac{d}{p} - \frac{d}{2}}. \]
Hence
\[ \| a'_\lambda \|_{M^{p,q}} = \| e^{-\pi \lambda^2 |\cdot|^2} \|_{M^{p,q}} \| \hat{h}_\lambda \|_{M^{p,q}} \lesssim \lambda^{\frac{d}{p} - \frac{d}{2} - \frac{d}{4}}. \]
Putting all together and letting \( \lambda \to \infty \) we obtain (4.4). 

**Proposition 4.5.** Suppose that, for some \( 1 \leq p, q, r, s, r_1, r_2 \leq \infty, C > 0 \), the estimate
\[ \| a(x, D) f \|_{W(L^{r_1}, L^{r_2})} \leq C \| a \|_{M^{p,q}} \| f \|_{W(L^{r'}, L^s)}, \quad \forall a \in S(\mathbb{R}^{2d}), \ f \in S(\mathbb{R}^d), \]
holds. Then \( q \leq r \).

**Proof.** Let \( h_1, h_2 \) be two Schwartz functions in \( \mathbb{R}^d \) such that \( h_1 \) and \( \hat{h}_2 \) are real valued, with \( h_1(0) = 1, \hat{h}_2(0) = 1 \), and satisfying
\[ \text{supp } \hat{h}_1 \subset B(0, 1), \quad \text{supp } \hat{h}_2 \subset B(0, 1). \]
Consider then, for every \( N \geq 1 \), the finite lattice
\[ \Lambda_N = \{ n = (n_1, \ldots, n_d) \in 4^d : 0 \leq n_j \leq 4(N - 1), \ j = 1, \ldots, d \}. \]
Observe that \( \Lambda_N \) has cardinality \( N^d \), and
\[ |n| \leq d^{1/2}(N - 1), \ \forall n \in \Lambda_N \text{ and } |n - m| \geq 4, \ \forall n, m \in \Lambda_N, n \neq m. \]
Moreover, let \( h \) be a smooth real-valued function, \( h \geq 0, \ h(0) = 1 \), supported in a ball \( B(0, \epsilon) \), for a small \( \epsilon \) to be chosen later. We test the estimate (4.5) on the family of functions \( f_N(x) = h(Nx) \) and symbols
\[ a_N(x, \xi) = \sum_{n \in \Lambda_N} b_n(x, \omega), \text{ where } b_n(x, \omega) = (M_n h_1)(x) T_n h_2(\omega). \]
The integral kernel of the operator \( a_N(x, D) \) is given by
\[ K_N(x, y) = (\mathcal{F}_2^{-1} a_N)(x, x-y) = \sum_{n \in \Lambda_N} \mathcal{F}_2^{-1}(b_n)(x, x-y) = \sum_{n \in \Lambda_N} e^{-2\pi i n y} h_1(x) \hat{h}_2(y-x). \]
We now show that, for a suitable \( \delta > 0 \),
\[ |a_N(x, D) f_N(x)| \gtrsim 1, \text{ for } x \in B(0, \delta), \]

\[ \]
which implies

\begin{equation}
\|a_N(x, D)f_N\|_{W(L^{r_1}, L^{r_2})} \gtrsim 1. 
\end{equation}

In order to prove (4.10) observe that, by the above computation,

\[ a_N(x, D)f_N(x) = \int_{\mathbb{R}^d} \left( \sum_{n \in \Lambda_N} e^{-2\pi i y_n} \right) h_1(x) \widehat{h}_2(y-x) h(Ny) \, dy. \]

Now, as a consequence of the first condition in (4.8), we see that on the support of $h(Ny)$, hence where $|y| \leq \epsilon N^{-1}$, we have

\[ \text{Re} \left( e^{-2\pi i y_n} \right) \geq \frac{1}{2}, \quad \forall n \in \Lambda_N, \]

if $\epsilon \leq d^{-1/2}/24$. This implies that

\[ |a_N(x, D)f_N(x)| \geq \frac{N^d}{2} \int_{\mathbb{R}^d} h_1(x) \widehat{h}_2(y-x) h(Ny) \, dy. \]

Since $h_1(0) = \widehat{h}_2(0) = 1$, if $\delta$ and $\epsilon$ are small enough, so as $h_1(x) \geq 1/2$ and $\widehat{h}_2(y-x) \geq 1/2$ for $|y| \leq \epsilon$, $|x| \leq \delta$. It turns out that

\[ |a_N(x, D)f_N(x)| \geq \frac{N^d}{8} \int_{\mathbb{R}^d} h(Ny) \, dy = \frac{\|h\|_1}{8}, \quad \text{for } |x| \leq \delta, \]

which implies (4.10).

We now prove that

\begin{equation}
\|a_N\|_{M^{p,q}} \lesssim N^{d/q}.
\end{equation}

To see this, observe that

\begin{equation}
\widehat{b}_n(\zeta_1, \zeta_2) = (T_{-n}\widehat{h}_1)(\zeta_1)(M_{-n}\widehat{h}_2)(\zeta_2).
\end{equation}

We choose a window $\Phi = \varphi \otimes \varphi$, where $\varphi$ is a Schwartz function supported in $B(0, 1)$. It follows from (4.13), the second condition in (4.8) and (4.6), that the functions $V_{\varphi} b_n(z_1, z_2, \omega_1, \omega_2) = \left( \widehat{b}_n * M_{-z} \Phi^\ast \right)(\omega)$ (with $z = (z_1, z_2)$, $\omega = (\omega_1, \omega_2)$, $\Phi^\ast(z) = \Phi(-z)$), vanish unless $\omega_1 \in B(-n, 2)$, $\omega_2 \in B(0, 2)$. Hence, when $n$ varies in $\Lambda_N$, they have pairwise disjoint supports, as well as the functions $\|V_{\varphi} b_n(\cdot, \cdot, \omega_1, \omega_2)\|_p$. 
We deduce that
\[
\|a_N\|_{M^{p,q}} \lesssim \left( \int_{\mathbb{R}^d} \left( \sum_{n \in \Lambda_N} \| V_\Phi b_n (\cdot, \cdot, \omega_1, \omega_2) \|^q_{p} d\omega_1 d\omega_2 \right)^{q/p} \right)^{1/q}
\]
\[
= \left( \int_{\mathbb{R}^d} \left( \sum_{n \in \Lambda_N} \| V_\Phi b_n (\cdot, \cdot, \omega_1, \omega_2) \|^q \right) d\omega_1 d\omega_2 \right)^{1/q}
\]
\[
= \left( \sum_{n \in \Lambda_N} \int_{B(-n,2)} \int_{B(0,2)} \| V_\Phi b_n (\cdot, \cdot, \omega_1, \omega_2) \|^q d\omega_1 d\omega_2 \right)^{1/q}.
\]
(4.14)

On the other hand, since \( V_\Phi b_n(z, \omega) = e^{-2\pi i z \omega} (b_n * M_\omega \Phi^*) (z) \), by Young's inequality, we have
\[
\| V_\Phi b_n (\cdot, \cdot, \omega_1, \omega_2) \|_p \leq \| b_n \|_p,
\]
and the expression for \( b_n \) in (4.9) shows that \( \| b_n \|_p \) is in fact independent of \( n \). Hence the expression in (4.14) is
\[
\lesssim \left( \sum_{n \in \Lambda_N} \int_{B(-n,2)} \int_{B(0,2)} dz_1 dz_2 \right)^{1/q} = C_d N^{d/q},
\]
which gives (4.12).

Finally, since the functions \( f_N \) are supported in a fixed compact subset, we have
\[
\| f_N \|_{W^{r', L^s}} \leq \| f_N \|_r = \| h \|_r N^{-d/r}.
\]
Combining this estimate with (4.11), (4.12) and (4.5), and letting \( N \to \infty \) yields the desired constraint \( q \leq r \).

**Theorem 4.6.** Suppose that, for some \( 1 \leq p, q, r, s \leq \infty \), the estimate
\[
\| a(x, D)f \|_{W^{r, L^s}} \leq C \| a \|_{M^{p,q}} \| f \|_{W^{r, L^s}}, \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), \ f \in \mathcal{S}(\mathbb{R}^d),
\]
holds. Then \( q \leq \min\{r, r', s, s'\} \) and
\[
\frac{1}{p} \geq \left| \frac{1}{2} - \frac{1}{r} \right| + \frac{1}{q'}.
\]
(4.17)

**Proof.** We already know from Proposition 4.5 that \( q \leq r \). The constraints \( q \leq r' \) follows by duality arguments, namely by Lemma 4.1.
Now, we shall prove that
\begin{equation}
q \leq s', \quad \frac{1}{p} \geq \frac{1}{2} - \frac{1}{r} + \frac{1}{q'}.
\end{equation}
The remaining constraints will follow by duality as above.

Let \( \chi \in C_0^\infty(\mathbb{R}^d) \), with \( 0 \leq \chi \leq 1, \chi(0) = 1 \). Then (4.16) implies
\[
\|\chi a(x, D)f\|_{W(L^r, L^s)} \leq C\|a\|_{M^{p,q}}\|f\|_{W(L^r, L^s)}, \quad \forall f \in S(\mathbb{R}^d), \forall a \in S(\mathbb{R}^d).
\]
Since for functions \( u \) supported in a fixed compact subset we have \( \|u\|_{W(L^r, L^s)} \asymp \|u\|_r \), we deduce
\[
\|\chi a(x, D)f\|_r \leq C\|a\|_{M^{p,q}}\|f\|_{W(L^r, L^s)} \quad \forall f \in S(\mathbb{R}^d), \forall a \in S(\mathbb{R}^d).
\]
Then Proposition 4.7 implies (4.18). \qed

**Proposition 4.7.** Suppose that, for some \( 1 \leq p, q, r, s \leq \infty \), every symbol \( a \in M^{p,q} \) gives rise to an operator \( a(x, D) \) bounded on \( W(L^r, L^s) \). Then the constraints \( q \leq \min\{r, r', s, s'\} \) and (4.17) must hold.

**Proof.** Let \( W(L^r, L^s) \) be the closure of \( S(\mathbb{R}^d) \) in \( W(L^r, L^s) \). By assumption the map
\[
M^{p,q} \ni a \mapsto a(x, D) \in B(W(L^r, L^s), W(L^r, L^s))
\]
is well defined. By an application of the Closed Graph Theorem and Theorem 4.6 we see that the desired conclusion follows if we prove that this map has closed graph.
To this end, let \( a_n \to a \) in \( M^{p,q} \), with \( a_n(x, D) \to A \) in \( B(W(L^r, L^s), W(L^r, L^s)) \). We have to prove that \( A = a(x, D) \), i.e. \( \langle Af, g \rangle = \langle a(x, D)f, g \rangle \forall f, g \in S(\mathbb{R}^d) \).
Now, clearly \( \langle a_n(x, D)f, g \rangle \to \langle Af, g \rangle \). On the other hand \( \langle a_n(x, D)f, g \rangle = \langle a_n, G \rangle \), where \( G(x, \omega) = e^{-2\pi i x \omega}f(\omega)g(x) \) is a fixed Schwartz function. Hence \( \langle a_n(x, D)f, g \rangle \) tends to \( \langle a, G \rangle = \langle a(x, D)f, g \rangle \), which concludes the proof. \qed

5. **Boundedness on modulation spaces**

In the present section we show the full range of exponents \( 1 \leq p, q, r, s \leq \infty \) such that every symbol in \( M^{p,q} \) gives rise to a bounded operator on \( M^{r,s} \). Again, to avoid the fact that \( S(\mathbb{R}^d) \) is not dense in \( M^{r,s}(\mathbb{R}^d) \) if \( r = \infty \) or \( s = \infty \), we say that a pseudodifferential operator \( a(x, D) \) is bounded on \( M^{r,s}(\mathbb{R}^d) \) if there exists a constant \( C > 0 \) such that \( \|a(x, D)f\|_{M^{r,s}} \leq C\|f\|_{M^{r,s}}, \forall f \in S(\mathbb{R}^d) \).

We need the following auxiliary result.

**Lemma 5.1.** Suppose that for some \( 1 \leq p, q, r, s \leq \infty \) the following estimate holds:
\[
\|L_\sigma f\|_{M^{r,s}} \leq C\|\sigma\|_{M^{p,q}}\|f\|_{M^{r,s}}, \quad \forall \sigma \in S(\mathbb{R}^{2d}), \forall f \in S(\mathbb{R}^d).
\]
Then the same estimate is satisfied with \( r, s \) replaced by \( r', s' \) (even if \( r = \infty \) or \( s = \infty \)).
Proof. The proof goes exactly as that of Lemma 4.1. It suffices to replace everywhere the spaces $W(L^r, L^s)$ with $M^{r,s}$. 

**Theorem 5.2.** Let $1 \leq p, q, r, s \leq \infty$ such that

$$p \leq q', \quad q \leq \min\{r, r', s, s'\}. \tag{5.1}$$

Then, every symbol $a \in M^{p,q}$ gives rise to a bounded operator on $M^{r,s}$, with the uniform estimate

$$\|a(x, D)f\|_{M^{r,s}} \lesssim \|a\|_{M^{p,q}} \|f\|_{M^{r,s}}. \tag{5.2}$$

**Proof.** The desired conclusion follows at once by interpolation (see Lemma 2.5 (iii)) from the known cases $(p, q) = (\infty, 1)$, $1 \leq r, s \leq \infty$ (see [17, Corollary 14.5.5]), and $p = q = r = s = 2$ (see [17, Theorem 14.6.1]).

**Proposition 5.3.** Suppose that, for some $1 \leq p, q, r, s \leq \infty$, $C > 0$ the estimate

$$\|a(x, D)f\|_{M^{r,s}} \leq C\|a\|_{M^{p,q}} \|f\|_{M^{r,s}} \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), \quad f \in \mathcal{S}(\mathbb{R}^{d}) \tag{5.2}$$

holds. Then the constraints in (5.1) must hold.

**Proof.** We first prove that $q \leq \min\{r, r'\}$. The estimate $q \leq r'$ follows by testing (5.2) on the same families of symbols and functions as in the first part of the proof of Proposition 4.4, taking into account that, since the functions $f_\lambda$ considered there have Fourier transform supported in a fixed compact set, it turns out $\|f_\lambda\|_{L^r} \asymp \|f_\lambda\|_r$. Precisely, using Lemma 4.2,

$$\|f_\lambda\|_{L^r} \asymp \|f_\lambda\|_r \asymp \lambda^\frac{d}{r} - \frac{d}{2}.$$  

Moreover, $\|a_\lambda\|_{M^{p,q}} \lesssim \|f_\lambda\|_r \asymp \lambda^\frac{d}{r} - \frac{d}{2}$.  

which is a non-zero Schwartz function independent of $\lambda$, so that, letting $\lambda \to +\infty$ in (5.2), we get $q \leq r'$. The constraint $q \leq r$ is obtained using duality arguments, i.e. by Lemma 5.1.

Let us now prove that $q \leq \min\{s, s'\}$. Again, we can consider the Weyl quantization $a(x, D) = L_\sigma$ in place of the Kohn-Nirenberg one. Then we conjugate the operator $L_\sigma$ with the Fourier transform. An explicit computation shows that $\mathcal{F}^{-1} L_\sigma \mathcal{F} = L_{\sigma \chi}$, where $\chi(x, \omega) = (\omega, -x)$. On the other hand, the map $\sigma \mapsto \sigma \circ \chi$ is an isomorphism of $M^{p,q}$, so that (5.2) is in fact equivalent to

$$\|a(x, D)f\|_{W(L^r, L^s)} \lesssim \|a\|_{M^{p,q}} \|f\|_{W(L^r, L^s)} \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), \quad f \in \mathcal{S}(\mathbb{R}^{d})$$

where we came back to the Kohn-Nirenberg quantization. Then one can test this last estimate again on the same families of symbols and functions as in the first part of the proof of Proposition 4.4 (and in the first part of this proof), taking into
account that, since the functions $f_\lambda$ have Fourier transform supported in a fixed compact set, it turns out $\|f_\lambda\|_{W(\mathcal{F}L^r, L^s)} \asymp \|f_\lambda\|_s$ (in fact, this amounts to saying $\|\hat{f}_\lambda\|_{M^{r,s}} \asymp \|\hat{f}_\lambda\|_{\mathcal{F}L^s}$; see Lemma 2.7 (i)). Hence we get $q \leq s'$. By duality as above then $q \leq s$ follows.

We finally prove the constraint $p \leq q'$. Let $\phi(t) = e^{-\pi|t|^2}$, $t \in \mathbb{R}^d$. We test (5.2) on the family of symbols

$$a'_\lambda(x, \omega) = \phi(\lambda x)\phi(\lambda^{-1}\omega),$$

and functions

$$f'_\lambda(x) = \phi(\lambda x),$$

with $\lambda \geq 1$. By [7, Lemma 3.2] we have

$$\|a'_\lambda\|_{M^{p,q}} \asymp \lambda^{\frac{d}{p} - \frac{d}{q}},$$

and

$$\|f_\lambda\|_{M^{r,s}} \asymp \lambda^{-\frac{d}{q}}.$$  

On the other hand, an explicit computation gives

$$a'_\lambda(x, D)f'_\lambda(x) = (a'_1(x, D)\phi)(\lambda x),$$

where $a_1(x, D)\phi$ is still a Gaussian function. Hence, again by [7, Lemma 3.2]

$$\|a'_\lambda(x, D)f'_\lambda\|_{M^{r,s}} \asymp \lambda^{-\frac{d}{q}}.$$ 

Taking into account these estimates and letting $\lambda \to +\infty$ we obtain the desired constraint $p \leq q'$. \[\square\]

**Proposition 5.4.** Suppose that, for some $1 \leq p, q, r, s \leq \infty$, every symbol in $M^{p,q}$ gives rise to a bounded operator on $M^{r,s}$. Then the constraints in (5.1) must hold.

**Proof.** The proof is essentially the same as that of Proposition 4.7, relying on the Closed Graph Theorem and Proposition 5.3. We leave the details to the reader. \[\square\]

### 6. Symbols in Wiener amalgam spaces

A natural question which can be arisen is whether the boundedness of pseudodifferential operators on $L^2$ or, more generally, on $M^{p,q}$, can be attained by widening the symbol Sjöstrand class $M^{\infty,1}$ to the Winer amalgam space $W(\mathcal{F}L^1, L^\infty)$. An example of a function belonging to $W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \setminus M^{\infty,1}(\mathbb{R}^d)$ is the chirp $\varphi(x) = e^{\pi|x|^2}$, $x \in \mathbb{R}^d$, see Theorem 14 of [1] and Proposition 3.2 of [6]. The multiplier operator $a(x, D)f(x) = e^{\pi|x|^2}f(x)$ is a pseudodifferential operator with symbol $a(x, \omega) = (e^{\pi|x|^2} \otimes 1)(x, \omega) \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^{2d}) \setminus M^{\infty,1}(\mathbb{R}^{2d})$ and it is bounded on $M^{p,q}$ if and only if $p = q$, see Proposition 7.1 of [2].

The subsequent Proposition 6.3 shows that generally symbols in $W(\mathcal{F}L^1, L^\infty)$ do not produce bounded operator even on $L^2(\mathbb{R}^d)$. However, symbols expressed as
tensor products \( a(x, \omega) = a_1(x)a_2(\omega) \), are bounded on \( M^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \) (and hence on \( L^2(\mathbb{R}^d) \)), as shown in the next result.

**Proposition 6.1.** If \( a(x, \omega) = a_1(x)a_2(\omega) \), \( a_i \in W(\mathcal{F}L^1, L^\infty) \), \( i = 1, 2 \), then \( a(x, D) \) is bounded on \( M^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \).

**Proof.** We have

\[
a(x, D)f(x) = a_1(x)\mathcal{F}^{-1}(a_2\hat{f})(x) = a_1(x)[\mathcal{F}^{-1}(a_2) * f](x).
\]

If \( a_2 \in W(\mathcal{F}L^1, L^\infty)(\mathbb{R}^d) \), then \( \mathcal{F}^{-1}(a_2) \in M^{1, \infty}(\mathbb{R}^d) \) and \( \mathcal{F}^{-1}(a_2) * f \in M^{1, \infty}(\mathbb{R}^d) \ast M^p(\mathbb{R}^d) \hookrightarrow M^p(\mathbb{R}^d) \); see Lemma 2.5 (iv). It remains to show that the multiplier \( U_a f(x) = a_1(x)f(x) \) is bounded on \( M^p \). This immediately follows by the pointwise products for Wiener amalgam spaces of Lemma 2.5 (v). Indeed, \( \mathcal{F}L^1 \cdot \mathcal{F}L^p = \mathcal{F}(L^1 \ast L^p) \hookrightarrow \mathcal{F}L^p, L^\infty \cdot L^p \hookrightarrow L^p \), so that, for \( M^p = W(\mathcal{F}L^p, L^p) \), we have \( W(\mathcal{F}L^1, L^\infty) \cdot W(\mathcal{F}L^p, L^p) \hookrightarrow W(\mathcal{F}L^p, L^p) \), as desired. \( \square \)

Observe that the previous result does not hold if the space \( M^p \) is replaced by \( M^{p,q}, p \neq q \), the counterexample being given by the multiplier operator \( a(x, D)f(x) = e^{\pi |x|^2}f(x) \), which is not bounded on \( M^{p,q}, p \neq q \), as discussed above.

We now come to a necessary condition.

**Proposition 6.2.** Suppose that, for some \( 1 \leq p, q, r, s, r_1, r_2 \leq \infty \), \( C > 0 \), either the estimate

\[
\|a(x, D)f\|_{W(\mathcal{F}L^p, L^q)} \leq C\|a\|_{W(\mathcal{F}L^p, L^r)}\|f\|_{W(L^s, L^t)}, \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), \ f \in \mathcal{S}(\mathbb{R}^d),
\]

or the estimate

\[
\|L_\sigma f\|_{W(\mathcal{F}L^p, L^q)} \leq C\|\sigma\|_{W(\mathcal{F}L^p, L^r)}\|f\|_{W(L^s, L^t)}, \quad \forall a \in \mathcal{S}(\mathbb{R}^{2d}), \ f \in \mathcal{S}(\mathbb{R}^d),
\]

holds. Then \( q \leq r \).

**Proof.** We first suppose that (6.1) holds true. We test that estimate on the same families of functions and symbols as in the proof of Proposition 4.5 but with (4.6) replaced by

\[
(6.3) \quad \text{supp } h_1 \subset B(0, 1), \quad \text{supp } h_2 \subset B(0, 1),
\]

(the other conditions being unchanged).

Then, (4.11) and (4.15) still hold, because in their proof we did not use (4.6). On the other hand, we can prove that

\[
(6.4) \quad \|a_N\|_{W(\mathcal{F}L^p, L^q)} \lesssim N^{d/q},
\]

by using the same arguments in the proof of (4.12), with the roles of \( h_1, h_2 \) replaced by \( \hat{h}_2 \) and \( \hat{h}_1 \), respectively. Precisely, we now choose a window \( \Phi = \varphi \otimes \varphi \) with \( \varphi \in \mathcal{S}(\mathbb{R}^d) \) supported in \( B(0, 1) \). Then the functions \( V_\varphi b_n(z_1, z_2, \omega_1, \omega_2) \) vanish
unless \( z_1 \in B(-n, 2), z_2 \in B(0, 2) \), so that they have disjoint supports, as well as 
\( \|V b_n(z_1, z_2, \cdot, \cdot)\|_p \). Hence one deduces that

\[
\|a_n\|_{W(FL^p, L^q)} \simeq \left( \int_{\mathbb{R}^d} \left\| \sum_{n \in \Lambda_N} V b_n(z_1, z_2, \cdot, \cdot) \right\|_p^q \, dz_1 \, dz_2 \right)^{1/q}
\]

\[
= \left( \sum_{n \in \Lambda_N} \int_{B(-n,2)} \int_{B(0,2)} \|V b_n(z_1, z_2, \cdot, \cdot)\|_p^q \, dz_1 \, dz_2 \right)^{1/q}.
\]

On the other hand, by Young’s inequality, we have

\[
\|V b_n(z_1, z_2, \cdot, \cdot)\|_p \leq \|\hat{b}_n\|_p,
\]

which is independent of \( n \). Hence (6.4) follows.

We now suppose that the estimate (6.2) holds. Since the arguments are similar to those just used, we only sketch the main point of the proof. We test the estimate (6.2) on the family of functions 
\( f_N(x) = h(Nx) \), where \( h \) is a smooth real-valued function, with \( h \geq 0, h(0) = 1 \), supported in a ball \( B(0, \varepsilon) \), for a sufficiently small \( \varepsilon \), and symbols

\[
\sigma_N(x, \omega) = \sum_{n \in \Lambda_N} b_n(x, \omega), \text{ where } b_n(x, \omega) = (M_{-2n}h_1)(x)(T_nh_2)(\omega).
\]

The lattice \( \Lambda_N \) is defined in (4.7) and \( h_1, h_2 \) are two Schwartz functions in \( \mathbb{R}^d \) such that \( h_1 \) and \( \hat{h}_2 \) are real valued, with \( h_1(0) = 1, \hat{h}_2(0) = 1 \), and satisfying (6.3). By using the definition (1.6) we see that \( L_\sigma \) has integral kernel

\[
K(x, y) = \mathcal{F}^{-1} \sigma \left( \frac{x + y}{2}, x - y \right) = \sum_{n \in \Lambda_N} e^{-4\pi iny}h_1 \left( \frac{x + y}{2} \right) \hat{h}_2(y - x).
\]

Hence, by arguing as in the proof of (4.10) one obtains, for a suitable \( \delta > 0 \),

\[
|L_N f_N(x)| \gtrsim 1, \text{ for } x \in B(0, \delta),
\]

which implies

\[
\|L_N f_N\|_{W(L^{r_1}, L^{r_2})} \gtrsim 1.
\]

The arguments in the first part of the present proof, with essentially no changes, show that

\[
\|\sigma_N\|_{W(FL^p, L^q)} \lesssim N^{d/q}.
\]

Combining these estimates with \( \|f_N\|_{W(L^r, L^s)} \lesssim N^{-d/r} \) and letting \( N \to +\infty \) give the desired conclusion.
Proposition 6.3. Suppose that, for some $1 \leq p, q, r, s \leq \infty$, every symbol $a \in W(\mathcal{F}L^p, L^q)$ gives rise to a bounded operator on $W(L^r, L^s)$. Then $q \leq r$. The same happens if one replaces the Kohn-Nirenberg operator $a(x, D)$ by the Weyl one $L^a$.

Proof. The result follows from Proposition 6.2 by arguing as in the proof of Proposition 4.7. 

As anticipated in the Introduction, we address now to the problem of the invariance of the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)$ under the action of the operator $U$ in Theorem 3.1, which expresses the Kohn-Nirenberg symbol of an operator in terms of the Weyl one. The lack of invariance, expressed by the following result, justifies the fact that the necessary conditions in this section were proved for both Kohn-Nirenberg and Weyl quantizations.

Proposition 6.4. Let $1 \leq p, q \leq \infty$. If $p \neq q$, the operator $U$ in Theorem 3.1 does not map $W(\mathcal{F}L^p, L^q)$ into itself.

Proof. Consider the symmetric matrix $B = \frac{1}{2} \begin{bmatrix} 0 & I_d \\ I_d & 0 \end{bmatrix}$, and the symplectic matrix $A = \begin{bmatrix} I_{2d} & B \\ 0 & I_{2d} \end{bmatrix}$. It follows from Theorem 4.51 of [15] that the operator $U$ is exactly the metaplectic operator associated with the matrix $A$. We write $U = \mu(A)$. Hence, as a consequence of Theorem 4.1 of [7], we deduce that $U$ and $U^{-1} = \mu(A^{-1})$ map $W(\mathcal{F}L^q, L^p)$ into $W(\mathcal{F}L^p, L^q)$ continuously for every $1 \leq p \leq q \leq \infty$.

Now, assume $p < q$. Let $f$ be any distribution in $W(\mathcal{F}L^q, L^p)$; therefore, by the boundedness result we have just recalled, $U^{-1}f \in W(\mathcal{F}L^p, L^q)$. Suppose, by contradiction, that $U$ maps $W(\mathcal{F}L^p, L^q)$ into itself. Then one would obtain $f = UU^{-1}f \in W(\mathcal{F}L^p, L^q)$, and therefore the inclusion $W(\mathcal{F}L^q, L^p) \subseteq W(\mathcal{F}L^p, L^q)$, which is false.

Suppose now $p > q$. Assume, by contradiction, that for every $f \in W(\mathcal{F}L^p, L^q)$ it turns out that $Uf \in W(\mathcal{F}L^q, L^p)$. Then we would have $f = U^{-1}Uf \in W(\mathcal{F}L^q, L^p)$, and therefore the inclusion $W(\mathcal{F}L^p, L^q) \subseteq W(\mathcal{F}L^q, L^p)$, which is false. 

Remark 6.5. A concrete example of a Weyl symbol $\sigma \in W(\mathcal{F}L^1, L^\infty)$ such that the corresponding Kohn-Nirenberg symbol $a = U\sigma$ does not belong to $W(\mathcal{F}L^1, L^\infty)$ is provided by $\sigma = U^{-1}\delta$; therefore $a = \delta$. To see this, observe, first of all, that the Dirac distribution $\delta$ belongs to $W(\mathcal{F}L^\infty, L^1)$. Indeed, for a given window $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$,

$$\|\delta\|_{W(\mathcal{F}L^\infty, L^1)} = \int \|\hat{\delta}T_x g\|_{\infty} dx = \int |g(-x)| dx < \infty.$$
Hence, by Theorem 4.1 of [7], we infer $\sigma \in W(\mathcal{F}L^1, L^\infty)$. On the other hand, it is clear that $\delta$ does not belong to $W(\mathcal{F}L^1, L^\infty)$ (which consists only of continuous functions).

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