On the Possibility of a Relativistic Anyon Beam

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(Dated: February 5, 2019)

In this paper we propose the construction of a relativistic anyon beam. Following Jackiw and Nair [18] we derive explicit form of relativistic plane wave solution of a single anyon. Subsequently we construct the planar anyon beam by superposing these solutions. Explicit expressions for the conserved anyon current are evaluated. Finally, we provide expressions for the anyon beam current using the superposed waves and discuss its features pictorially. We also comment on the possibility of experimental observation of anyon beam.

In this work we propose to construct a relativistic beam of anyons in a plane. Anyons are planar excitations with arbitrary spin and statistics. The procedure is, in spirit, similar to optical (spin 1, bosonic) and electron (spin 1/2, fermionic) vortex beams in three space dimensions.

Vortex beams carrying intrinsic orbital angular momentum have created an enormous amount of interest both in theoretical and experimental aspects in physics. These are essentially non-diffracting wave packets in motion. In three space dimensions existence of such optical beams was envisaged in the pioneering work of Durnin et al. [1]. This idea was successfully extended to electron vortex beams by Bliokh et al. [2] in non-relativistic regime and by Bliokh et al. [3] in relativistic regime (see Ref. [4] for an exhaustive review). In the former scalar electron (without spin) and in the latter the spinor electron were considered that satisfied Schrödinger equation and Dirac equation, respectively. Wave packets were constructed out of free plane wave solutions that possess the vorticity property or intrinsic orbital angular momentum. In three dimensional space, the spin one case of photon (in optical beam) and spin half case of electrons (in electron vortex beam) are the only two possibilities. In this work we propose a novel way of generalizing this construction to two space dimensions where we can construct anyon beams, that is directed localized anyon wave packets. Planar physics allows the existence of anyons, excitations with arbitrary spin and statistics, was discovered by Wilczek [5] (see Ref. [6] for a comprehensive collection of papers).

Anyons have potential applications in fractional quantum Hall effect [7], in high-$T_c$ superconductivity [8, 9], and in the description of physical processes in presence of cosmic strings [10]. The Graphene system also involves anyons [11] with the recent exciting possibility of non-Abelian anyons [12], that are touted as the theoretical building blocks for topological quantum computers. An exact chiral spin liquid with non-Abelian anyons has been reported in [13]. Theoretical models of these new class of quantum spin liquids [13] show that many of them support anyonic excitations. The direct observational status of anyons has shown promising development in recent years [14].

There are different theoretical models for anyons: solitons of the $O(3)$-model with a Hopf term are localized objects with arbitrary spin and statistics [13]; point particles minimally coupled to a $U(1)$ Chem-Simons (statistical) gauge field [16] become anyonic upon removal of the gauge field. However these models do not provide a minimal description of anyon. In [15], there are many other integral spin states, whereas removal of the statistical gauge field in [16] is necessary to generate the anyon in fact produces interactions [17].

The minimal field theoretic and relativistic model of a single anyon was constructed by Jackiw and Nair (JN) [18]. Coincidentally, this JN anyon happens to be the most suitable one that serves our purpose since, being first order in spacetime derivatives, it closely resembles the Dirac equation used by [2] for the electron vortex beam. However, a major difference is that unitary representation for arbitrary spin JN anyon requires an infinite number of independent components to a single one [18]. Alternative 2 + 1-dimensional relativistic generalized point particle (or mechanical) models simulating arbitrary spin are discussed in Ref. [19].

Our flowchart for anyon beam construction is as follows: (i) We derive an explicit form of the single anyon solution of the JN anyon equation [18] (which is a new result) [20]. (ii) We find the anyon conserved current (again a new result). (iii) We construct the anyon wave packet in the two-dimensional plane. (iv) We introduce the anyon wave packet in the conserved current that constitutes the anyon beam. (v) After that we plot the anyon charge and current densities for the anyon beam numerically since closed analytic expressions could not be obtained, unlike the optical [1] or electron vortex beam [3]. (v) Then, we briefly touch upon the experimental scenario and comment on the possibility of a laboratory setup for observing anyon beam. (vi) Finally, we summarize and discuss about future prospects.

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(i) Jackiw-Nair anyon equation and its solution: We start with a familiar system, free spin one particle in \(2+1\)-dimensions [18]. The dynamical equation in co-ordinate and momentum space \((i\partial_a = p_a)\) is given by
\[
\partial_a e^{abc} F_c = m F^b = 0; \quad (p_j)^a_b F^b + ms F^a = 0.
\] (1)
The solution of three vector \(F^a, a = 0, 1, 2\) (we use Minkowski metric \(\eta_{\mu\nu} = \text{diag}(1, -1, -1)\)) is given by
\[
F^a(p) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & p^1 + ip^2 \\ m(E + m) & p^1 \\ p^2 & \end{bmatrix} \psi(p)
\] (2)
where \(\psi(p)\) is an arbitrary momentum dependent function that is fixed by normalization. The same \(F^a\) can also be expressed as a Lorentz boosted form
\[
F^a(p) = B^a_0(p) N^b_c F^c_0(p), \quad N^a_c = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & i \end{bmatrix}
\] (3)
where the boost is expressed in terms of the generators \(j^a\)
\[
B(p) = e^{i\eta_a(p)j^a}; \quad [B(p)]^a_0 = [B^{-1}(p)]^a_0 = \delta^a_0 - \frac{(p^a + \eta^a m)(p_0 + \eta_0 m) + 2 p^a \eta_0}{m}
\] (4)
with \(\eta_a = 1, 0, 0\). It is straightforward to check that \(F^a\) describes a spin one particle \(s = 1\) of mass \(m\).

This construction has been extended in an elegant way to the JN anyon equation [18] to describe an anyon of the same numerical matrix as in (3).

An explicit matrix representation of \(K^a\) \[18\] is given by
\[
P(K + j)_{an} a' n' f^{a'}_{n'} + ms f_{an} = 0, \quad (D_a f^a)_n = 0,
\] (5)
\([D^a_{an}, \epsilon^b_{kc} P^b K^c_{an}]\) where the second equation is the subsidiary (constraint) relation. For \(\lambda = 0\) the anyon reduces to spin one model discussed earlier [18]. The actions of \(j^a, K^a\) are given by
\[
P K_{an} a' n' = P K_{nn'} \delta_{aa'}, \quad P K_{an} a' n' = P j_{an'} \delta_{n'n'}; \quad K^a_{an} = <\lambda, n | K^a | \lambda, n' >, \quad (j^a)_{a' a''} = i \epsilon^a_{a' a''}.
\]
An explicit matrix representation of \(K^a\) is [18]
\[
K^0 | \lambda, n >= (\lambda + n) | \lambda, n + 1 >
\]
\[
K^+ | \lambda, n >= \sqrt{(2\lambda + n)(n+1)} | \lambda, n + 1 >; \quad K^- | \lambda, n >= \sqrt{(2\lambda + n-1)n} | \lambda, n - 1 >.
\] (6)
Generalizing the spin-1 case [20], the free anyon solution is formally given by [18]
\[
f^{a(\pm)}_n(p) = B_{n0}(p) B^a_0(p) N^b_c f^{c(\pm)}(p),
\] (7)
where \(B_{n0}(p), B^a_0(p)\) are the spin \(\lambda\) and spin 1 representations of the boost transformation respectively and \(N^a_c\) is the same numerical matrix as in [20].

We have constructed explicit form of \(B_{n0}(p)\) [20] (for details see Appendix A) to write down the free anyon solution,
\[
f^{a+}_n = \left(\frac{2m}{E + m}\right)^{\lambda} \frac{1}{\Gamma(2\lambda + n)n!} \frac{\Gamma(2\lambda + n)}{(E + m)^n} \frac{m}{E} f^a_0(p) e^{-ip \cdot x}
\] (8)
where \(f^a_0(p)\) is same as the spin-1 case defined in [20]. This is our primary result that we exploit to construct the wave packet and subsequent anyon beam.

(ii) Conserved current for single anyon: Next we derive the conserved probability current for anyon \(\partial^\mu j^{(s=1-\lambda)}_\mu = 0\) where \(j^{(s=1-\lambda)}_\mu\) is the probability density. Since the anyon model of [18] is an extension of the spin-1 case we can take a cue from the latter where the conservation law \(\partial^\mu j^{(s=1)}_\mu = 0\) reads
\[
\partial^\mu j^{(s=1)}_\mu = \partial^0 [f^{00} f^0 + f^{11} F^1 + f^{21} F^2] - \partial^1 [f^{00} f^1 + f^{11} f^0] - \partial^2 [f^{00} f^2 + f^{21} f^0] = 0
\] (9)
Exploiting the (position space equivalent of the anyon equation of motion [6]) it is straightforward but tedious to derive the conserved free (single) anyon current \( j_{\mu}^{(1-\lambda)} \),

\[
\partial^0 \sum_{n=0}^{\infty} \left[ (f_n^0 f_n^0 + f_n^1 f_n^1 + f_n^2 f_n^2) - i(f_n^2 K_{nn'} f_n^1 - f_n^1 K_{nn'} f_n^2) \right]
\]

\[
- \partial^1 \sum_{n=0}^{\infty} \left[ (f_n^0 f_n^0 + f_n^1 f_n^1) - i(f_n^2 K_{nn'} f_n^1 - f_n^1 K_{nn'} f_n^2) \right]
\]

\[
- \partial^2 \sum_{n=0}^{\infty} \left[ (f_n^0 f_n^0 + f_n^1 f_n^1) - i(f_n^2 K_{nn'} f_n^1 - f_n^1 K_{nn'} f_n^2) \right] = 0
\]

(10)

where we have explicitly shown the summation over \( n \), the anyonic index. For \( \lambda = 0 \) the current \( j_{\mu}^{(1)} \) reduces to the spin 1 current \( j_{\mu}^{(1)} \) of [9]. A nontrivial check of the consistency of the expressions for anyon current (10) is to substitute \( f_n^0 \) from (8) to yield

\[
\begin{align*}
  j^0 &= (1 - \lambda); \\
  j^x &= (1 - \lambda) p^x / E; \\
  j^y &= (1 - \lambda) p^y / E \to j^\mu = (1 - \lambda) \frac{p^\mu}{(p.\eta)} = s \frac{p^\mu}{(p.\eta)} 
\end{align*}
\] (11)

In deriving the anyon current we have used the matrix representations of \( K^a \)-matrices in Ref. [18],

\[
K_{nn'}^0 = (\lambda + n) \delta_{nn'}, \quad K_{nn'}^1 = \frac{1}{2} \left( \sqrt{(2\lambda + n - 1)n\delta_{n,n+1} + (2\lambda + n)(n+1)\delta_{n,n'-1}} \right), \\
K_{nn'}^2 = \frac{i}{2} \left( \sqrt{(2\lambda + n - 1)n\delta_{n,n+1} - (2\lambda + n)(n+1)\delta_{n,n'-1}} \right),
\]

(12)

and the identity

\[
\sum_{n=0}^{\infty} \frac{\Gamma(a+n)s^n}{n!} = (1-s)^{(-a)}\Gamma(a); \quad | s | < 1.
\] (13)

(iii) **Anyon wave packet:** Our aim is to construct the anyon current, not for a single anyon as done above, but for an anyon wave packet which can be amenable to experimental verification. Let us now construct the anyon wave packet that we want to move towards, say, \( +x \)-direction. Since we have superposed plane waves, later figures will reveal that the current density has a sharply peaked profile with the \( y \)-component of current density having a comparatively reduced value. Note an important difference in geometry between our construction and that of the three-dimensional vortex beam [3]. In case of the later the free mono-energetic plane wave solutions (to be superposed) are distributed over the surface of a right circular cone with identical momentum amplitude in the propagation direction. However, for our anyon wave packet, in a planar geometry the above is not possible. Instead we use the superposition scheme as shown in Fig. [1] where the red arrows show momenta of the plane wave and \( \phi \) is integrated symmetrically from \( \phi_0 = -\pi/2 \) to \( \phi_0 = +\pi/2 \). In Fig. [2] we have shown profiles for charge density of anyon beam, \( J_0^x \), for \( \lambda = 0.2 \to s = 1 - \lambda = 0.8 \) for \( \phi_0 = \pm \pi/2 \). In Fig. [3] this is repeated for \( \lambda = 0.6 \to s = 1 - \lambda = 0.2 \). In

\[ \text{FIG. 1: (Color online). Superposition of anyon plane waves.} \]
energy under beam probability density, in polar coordinates in the expression of the anyon current (10). Since the current components are quadratic in the packet wavefunctions, visualizing the anyon beam: Conserved current for anyon wave packet:

\[ F_n^a(p, x) = A_n \int_{-\pi/2}^{\pi/2} \left[ \left( 1 + \frac{p^2 m^2}{i^2} \cos \alpha \ e^{i \alpha} \right) e^{i \alpha} \left( e^{i (X_1 \cos \alpha + X_2 \sin \alpha)} + e^{i (X_1 \cos \alpha - X_2 \sin \alpha)} \right) \right] d\alpha \]  

(14)

where \( A_n = \frac{1}{2} \left( \frac{2m}{E + m} \right)^{\lambda} \frac{1}{\Gamma(2\lambda)} \sqrt{\frac{\Gamma(2\lambda + n)}{n!}} \left( \frac{p}{E + m} \right)^n \)  

(15)

We have used the notation \( X^1 = px, X^2 = py; \ M = m(E + m) \). The expression is manifestly symmetric separately under \( x \to -x \) and \( y \to -y \).

\textbf{Conserved current for anyon wave packet:} The final analytical task is to substitute the anyon wave packet (14) in the expression of the anyon current (10). Since the current components are quadratic in the packet wavefunctions \( F_n^a \), the final expressions are quite long and involved. We have shown only the expression for probability density \( J^0 \) and have relegated \( J^x \) and \( J^y \) to Appendix B together with a few computational steps. The cherished form of anyon beam probability density, in polar coordinates \( x = \rho \cos \theta, y = \rho \sin \theta \), is

\[ J^0(\rho, \theta) = \left( \frac{2m}{E + m} \right)^{2\lambda} \frac{1}{\Gamma(2\lambda)} \int_{-\pi/2}^{\pi/2} d\alpha \int_{-\pi/2}^{\pi/2} d\beta \left[ e^{-2ip \sin(\theta - (\alpha + \beta)/2) \sin((\alpha - \beta)/2)} + e^{-2ip \sin(\theta - (\alpha - \beta)/2) \sin((\alpha + \beta)/2)} \right] \left[ 1 - e^{-i(\alpha - \beta) \sigma^2} \right]^{-2\lambda} \]

- \[ \left\{ \left( \frac{p}{2m} \right)^2 e^{-i(\alpha - \beta)} + 1 \left( 1 + \lambda \right) \left( 2 + \frac{p^2}{M^2} \right) + \frac{p^4}{M^2} e^{-i(\alpha - \beta)} (\cos(\alpha - \beta) + i \lambda \sin(\alpha - \beta)) \right\} \]

\[ - \frac{\lambda}{2} \left\{ \left[ 1 - e^{-i(\alpha - \beta) \sigma^2} \right]^{-2\lambda - 1} \left( 2 + \frac{p^2}{M^2} + \frac{p^4}{M^2} \sin(\alpha - \beta) e^{-i(\alpha - \beta)} \right) \right\} \]

\[ \left[ 1 + \lambda \right] \left( 2 + \frac{p^2}{M^2} \right) + \frac{p^4}{M^2} \sin(\alpha - \beta) e^{-i(\alpha - \beta)} \]

(16)

where, \( \sigma = \left( \frac{p}{E + m} \right) \) and \( M = m(E + m) \).

(iv) \textbf{Visualizing the anyon beam:} Unfortunately, closed form expressions for the anyon beam current components \( J^0, J^x \), and \( J^y \) are not possible to obtain. Hence we show the features of the anyon beam profile with numerical plots. We have shown \( J^0 \) for two values of \( \lambda \), (or spin \( s = 1 - \lambda \)) for \( \lambda = 0.2 \) in Fig. 2 and \( \lambda = 0.6 \) in Fig. 3 both for \( \phi_0 = \pm \pi/2 \). Again in Fig. 4 we have plotted \( J^0 \) for \( \lambda = 0.6 \) for superposition angles having limiting values of \( \phi_0 = \pm \pi/3, \pi/6 \). Subsequently, in all the figures (Fig. 2, Fig. 3, Fig. 5 and Fig. 6) for each case \( s, \phi_0 \) we have plotted the profiles of \( J^a \) in three ways: a two-dimension plot of \( J^a \) against the polar angle \( \theta \) for a few values of the (planar) radial distance \( \rho \), panel (a) in each group of figures. Another two-dimensional graph of the same data as panel (a) with magnitude of \( J^a \) against \( \theta \) for the same values of \( \rho \) is shown in panel (b). A density plot in co-ordinate plane \( x-y \) is given in panel (c). Finally, a three-dimensional plot of \( J^a \) in co-ordinate plane \( x-y \) is provided in panel (d). Note that in panel (a), each continuous curve represents a fixed polar distance in coordinate plane \( x-y \) with the height being a measure of \( J^a \) whereas in panel (b), the radial distance is a measure of the intensity of \( J^a \). Hence the curves that are further away from the centre in panel (b) represent points that are closer the coordinate plane \( x-y \).

As expected, all the wave packet profiles are symmetric about the abscissa since the packets are superposition of plane waves, that are symmetrically placed about the \( x \)-axis. Contrasting with the three-dimensional wave packets \( J^0, J^x \) it is clear that the planar anyon beams do not possess a vortex nature since the axial symmetry is manifestly broken while constructing a propagating anyon beam. This is also corroborated in the figures that do not have any destructive interference at the origin, a characteristic feature of vortex beams \( J^x \). Hence the anyon beams are characterized by the spin value \( s \) of the wave packet, which is same as that of individual plane wave single anyon component.

An important observation is that in the cases we have considered, \( J^0 \) is always positive, which has to be the case since it is the probability density. But \( J^x \) is also positive throughout whereas \( J^y \) has positive and negative values in equal amount. Furthermore, maximum value of \( J^y \) is far lower than each of \( J^0 \) and \( J^x \). These reflect the nature of
FIG. 2: (Color online). (a) $J^\theta$ as a function of $\theta$ for different values of $\rho$. (b) Polar plot for $J^\theta$ with $\theta$ for fixed $\rho$ where the radial distance corresponds to the magnitude of $J^\theta$. (c) Density plot of $J^\theta$. (d) 3D plot of $J^\theta$. The results are computed for $\lambda = 0.2$ considering the integration range for superposition, from $-\pi/2$ to $+\pi/2$.

FIG. 3: (Color online). Same as Fig. 2 with $\lambda = 0.6$.

Our construction of the anyon beam where all the plane waves have positive velocity components along $x$-direction but have pairwise opposite (both positive and negative) velocity components along $y$-direction. Hence, the anyon beam will predominantly move in the positive $x$-direction with the $y$-component effectively canceled out.

(v) Experimental possibilities: The first direct observation of fractionally charged quasiparticles were done in quantum antidot experiments, where quasiperiodic resonant conductance peaks were observed when the occupation of the antidot is incremented by one quasiparticle [21]. Detection of anyonic statistics of Laughlin quasiparticles has been reported [22] in experiments on a Laughlin quasiparticle interferometer.
FIG. 4: (Color online). Density plot and the 3D plot of $J^0$ under two different integration ranges for superposition, considering $\lambda = 0.6$. In (a) and (b) we choose the integration range from $-\pi/3$ to $\pi/3$, while in (c) and (d) the range is taken from $-\pi/6$ to $\pi/6$.

FIG. 5: (Color online). Characteristics of $J^x$ setting $\lambda = 0.6$ and selecting the integration range from $-\pi/2$ to $+\pi/2$, where different spectra correspond to the similar meaning as described in Fig. 2.

Charged anyons have played a crucial role ever since Laughlin suggested that charged anyon fluid can simulate high $T_c$ superconductivity [8]. Further results in this context are discussed in Ref. [23]. In a recent work [24], a dynamical property of anyons i.e., their Josephson frequency has been observed in two-dimensional electron systems in high magnetic field.

One of the most effective way of experimentally observing anyonic behavior is to consider the effect of external
FIG. 6: (Color online). Characteristics of $J^y$ setting $\lambda = 0.6$ and selecting the integration range from $-\pi/2$ to $+\pi/2$, where different spectra correspond to the similar meaning as described in Fig. 2.

FIG. 7: (Color online). Experimental setup: A thin slice of a three-dimensional convex lens in the $x$-$y$ plane (left diagram) is taken out and shown separately in the right diagram. The black lines show the superposing wave vectors in $x$-$y$ plane.

electromagnetic field on charged anyon, that carries a magnetic moment given by

$$\vec{M} = e \int dV (\vec{r} \times \vec{j}_a)/\int dV j^0$$  \hspace{1cm} (17)

for single anyon, where the single anyon current $j^a_a$ is given in (14) using (8) and $e$ is the charge parameter. The expression (17) follows from the generic form given in Ref. [3]. For the anyon packet we just replace $j^a_a$ by $J^a$ using the wave packet (14). One way to create a wave packet is to excite matter coherently with an ultrafast laser pulse. The nonstationary quantum superposition state, or wave packet, thus created, is composed of the eigenstates spanned by the frequency bandwidth of the laser pulse [25].

Finally, generation of anyons in experimentally controllable setup is indeed non-trivial and an area of immense activity. We list some of the diverse procedures mentioned in Ref. [14]: collective excitations behaving as anyons constructed from electrons in the fractional quantum Hall systems or from atoms in one-dimensional optical lattices, creation of fractional quantum Hall effect for photons, using one-dimensional or two-dimensional cavity array to construct anyons from photons, using a nonlinear resonator lattice subject to dynamic modulation for creating anyons from photons and among others.
Let us suggest in a tentative way, a possible laboratory construction of the anyon beam. This is schematically depicted in Fig. 7. The left panel of Fig. 7 shows a convex lens in three-dimensions $x, y, z$ with parallel rays shown only along $x$-$y$ plane that converge on $x$-axis. We consider an extremely thin slice of the lens in the $x$-$y$ plane which can be thought as the shaded area in the left panel. The same slice is drawn separately in the right panel that shows the planar superposition that we have considered in our work.

(vi) **Summary and future prospects:** In this paper we have proposed the possibility of constructing a relativistic anyon beam that is a symmetrical superposition of single anyon solutions of Jackiw-Nair form. We have provided explicit forms of the wave packets and have numerically plotted profiles of the anyon beam current. We have discussed significant features of the anyon beam profile. Lastly, a laboratory model of anyon beam construction has been suggested. The new ingredients in this paper that can be utilized elsewhere are (i) explicit relativistic single anyon solution, (ii) conserved anyon current, (iii) planar superposition of anyon plane waves to construct the anyon beam.

**Appendix A: Explicit form of the free anyon wavefunction**

In order to compute the free anyon wavefunction from (7) we require explicit form of $B_{n0}(p)$ sector of the infinite component matrix $B_{nm}(p)$ [20]. The best strategy is to use coherent states, given in Ref. [18], replacing $z$ by a complex variable $\omega$. The coherent states are obtained by quantization of the canonical one-form

$$dA = -i\lambda \left( \frac{\omega \dot{\omega} - \dot{\omega} \omega}{1 - \omega \bar{\omega}} \right)$$  \hspace{1cm} (18)

This one-form defines a symplectic structure on $SU(1, 1)/U(1)$, where

$$g = \begin{pmatrix} 1 - \omega & \omega \\ \bar{\omega} & 1 \end{pmatrix} \frac{1}{\sqrt{1 - \omega \bar{\omega}}} \times \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$$  \hspace{1cm} (19)

The $\phi$-part of $g$ being irrelevant in $dA$, is not shown. The quantization of the canonical structure in (18) gives the coherent states

$$g_{n0} = \sqrt{\frac{\Gamma(2\lambda + n)}{n!\Gamma(2\lambda)}} (1 - \omega \bar{\omega})^\lambda \omega^n$$  \hspace{1cm} (20)

To construct $B_{n0}(p)$, we start with Eq.(3.6) of Ref. [18],

$$B(p) = \frac{1}{\sqrt{2m}} \left[ \sqrt{p_\eta + m + i} \frac{1}{\sqrt{p_\eta + m}} e^{abc} p_a \eta_b \gamma_c \right]$$  \hspace{1cm} (21)

with the convention

$$\gamma^a = \{-\sigma^3, -i\sigma^2, i\sigma^1\}; \quad \gamma_a = \{-\sigma^3, i\sigma^2, -i\sigma^1\}; \quad \eta^a = \eta_a = \{1, 0, 0\},$$  \hspace{1cm} (22)

which gives $B_{n0}$ in the $2 \times 2$ matrix representation.

We use this $B(p)$ in place of $g$ in

$$B(p) \equiv g = \sqrt{\frac{E + m}{2m}} \left( \frac{1}{p_+} - \frac{p_-}{E + m} \right)$$  \hspace{1cm} (23)

where $p_\pm = p_x \pm ip_y$. The above is written as

$$B = \sqrt{\frac{1}{1 - \omega \bar{\omega}}} \begin{pmatrix} 1 & \omega \\ \bar{\omega} & 1 \end{pmatrix}$$  \hspace{1cm} (24)

where $\omega = \frac{p_+}{E + m}, \quad \bar{\omega} = \frac{p_-}{E + m}$.

Substituting $\omega$ in (20)

$$g_{n0} = \sqrt{\frac{\Gamma(2\lambda + n)}{n!\Gamma(2\lambda)}} (1 - \omega \bar{\omega})^\lambda \omega^n = \sqrt{\frac{\Gamma(2\lambda + n)}{n!\Gamma(2\lambda)}} \left( \frac{2m}{E + m} \right)^\lambda \left( \frac{p e^{i\phi}}{E + m} \right)^n$$  \hspace{1cm} (25)

where $p_x = p \cos \phi, \quad p_y = p \sin \phi$ and $p = \sqrt{p_x^2 + p_y^2}$. 


The spatial components of the anyon current built from the wavefunctions are:

$$J^z(\rho, \theta) = \left(\frac{2m}{E + m}\right)^{2\lambda} \frac{1}{\Gamma(2\lambda)} \int \int d\alpha d\beta \left[ e^{-i\rho p \sin(\theta - (\alpha + \beta)/2) \sin((\alpha - \beta)/2)} + e^{-i\rho p \sin((\theta - (\alpha - \beta)/2) \sin((\alpha + \beta)/2))} \right.
\left. + e^{-i\rho p \sin(\theta + (\alpha - \beta)/2) \sin((\alpha - \beta)/2))} + e^{-i\rho p \sin(\theta + (\alpha - \beta)/2) \sin((\alpha + \beta)/2))} \right]
\left[ \frac{1}{2} \frac{p}{2m} \left\{ 1 - e^{-i(\alpha - \beta)\sigma^2} \right\} - 2\lambda \left( e^{-i\frac{\pi}{2} + i\frac{\pi}{2} + \frac{p^2}{M} e^{-i(\alpha - \beta)\cos(\alpha + \beta/2) \cos(\alpha - \beta/2)}} \right) \right]
- \frac{i\alpha\lambda}{4} \left\{ 1 - e^{-i(\alpha - \beta)\sigma^2} \right\} - 2\lambda - 1 \left( e^{-i\frac{\pi}{2} - e^{i\frac{\pi}{2}}} \right) \left( -2i - 2i \frac{p^2}{M} + \frac{p^4}{M^2} e^{-i(\alpha - \beta)\sin(\alpha - \beta)} \right) \right]\] (27)

and

$$J^\theta(\rho, \theta) = \left(\frac{2m}{E + m}\right)^{2\lambda} \frac{1}{\Gamma(2\lambda)} \int \int d\alpha d\beta \left[ e^{-i\rho p \sin(\theta - (\alpha + \beta)/2) \sin((\alpha - \beta)/2)} + e^{-i\rho p \sin((\theta - (\alpha - \beta)/2) \sin((\alpha + \beta)/2))} \right.
\left. + e^{-i\rho p \sin(\theta + (\alpha - \beta)/2) \sin((\alpha - \beta)/2))} + e^{-i\rho p \sin(\theta + (\alpha - \beta)/2) \sin((\alpha + \beta)/2))} \right]
\left[ \frac{1}{2} \frac{p}{2m} \left\{ 1 - e^{-i(\alpha - \beta)\sigma^2} \right\} - 2\lambda \left( e^{-i\frac{\pi}{2} + i\frac{\pi}{2} + \frac{p^2}{M} e^{-i(\alpha - \beta)\cos(\alpha + \beta/2) \cos(\alpha - \beta/2)}} \right) \right]
+ \frac{\sigma\lambda}{4} \left\{ 1 - e^{-i(\alpha - \beta)\sigma^2} \right\} - 2\lambda - 1 \left( e^{-i\frac{\pi}{2} - e^{i\frac{\pi}{2}}} \right) \left( -2i - 2i \frac{p^2}{M} + \frac{p^4}{M^2} e^{-i(\alpha - \beta)\sin(\alpha - \beta)} \right) \right]\] (28)

where, \( \sigma = p/(E + m) \).

Below we provide a few computational steps leading to \( J^0 \) given in (10). \( J^0 \) is given by

$$J^0 = \sum_{n=0}^{\infty} \left[ F^0_n F^\dagger_n + F^1_n F^\dagger_n + F^{2\dagger}_n F^1_n - i \left( F^{2\dagger}_n K^{0\dagger}_{nn'} F^1_{n'} - F^0_n K^{0\dagger}_{nn'} F^{2\dagger}_{n'} \right) \right] \] (29)

Consider the first term in the RHS where we have used expressions for the wave packet given in (14)

$$\sum_{n=0}^{\infty} F^0_n F^0_n = \left(\frac{2m}{E + m}\right)^{2\lambda} \frac{1}{\Gamma(2\lambda)} \int \int d\alpha d\beta e^{-i\alpha} e^{-i\alpha} \left[ e^{-i(X_1 \cos \alpha + X_2 \sin \alpha)} + e^{-i(X_1 \cos \alpha - X_2 \sin \alpha)} \right] \] (30)

Now we substitute \( X_1 = \rho \rho \cos(\theta) \) and \( X_2 = \rho \rho \sin(\theta) \) and after using some well known trigonometric identity and the relation \( \sum_{n=0}^{\infty} \frac{\Gamma(2\lambda + n)}{n!} x^n = (1 - x)^{-2\lambda} \Gamma(2\lambda) \), we finally arrive at

$$\sum_{n=0}^{\infty} F^0_n F^0_n = \left(\frac{2m}{E + m}\right)^{2\lambda} \frac{1}{\Gamma(2\lambda)} \int \int d\beta \left[ e^{-2i\rho \sin(\theta - (\alpha + \beta)/2) \sin((\alpha - \beta)/2)} + e^{-2i\rho \sin((\theta - (\alpha - \beta)/2) \sin((\alpha + \beta)/2))} \right.
\left. + e^{-2i\rho \sin(\theta + (\alpha - \beta)/2) \sin((\alpha - \beta)/2))} + e^{-2i\rho \sin(\theta + (\alpha - \beta)/2) \sin((\alpha + \beta)/2))} \right]
\left[ 1 + e^{-i(\alpha - \beta)\sigma^2} \right]^{-2\lambda} \left(\frac{p}{2m}\right)^2 e^{-i(\alpha - \beta)} \] (31)
Similarly we can calculate other terms and we recover $J^0$ given in \cite{16}.

Acknowledgement: It is indeed a pleasure to thank Professor V. Parameswaran Nair for actively helping us in this project, (in particular Appendix A), and for providing many helpful suggestions.

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