ON THE NUMBER OF SUBREPRESENTATIONS OF A GENERAL QUIVER REPRESENTATION

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1. Introduction

It is well-known that the intersection multiplicities of Schubert classes in the Grassmanian are Littlewood-Richardson coefficients. For a partition $\lambda$ inside a $r \times (n - r)$ rectangle, let $Y_\lambda$ be the Schubert variety inside the Grassmanian $\text{Grass}(r, n)$ corresponding to $\lambda$ and let $[Y_\lambda]$ be its cohomology class. We have

$$ [Y_\lambda] \cdot [Y_\mu] = \sum_\nu c_{\lambda, \mu}^\nu [Y_\nu] $$

where $c_{\lambda, \mu}^\nu$ is a Littlewood-Richardson coefficient and the sum runs over all partitions $\nu$ inside a $r \times (n - r)$ rectangle. This result can be translated into the language of quivers as follows. Consider the triple flag quiver

$$ x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_n = y_n = z_n \rightarrow \cdots \rightarrow y_2 \rightarrow y_1. $$

Suppose that $\lambda, \mu, \nu$ are partitions whose Young diagram fit into a $r \times (n - r)$ rectangle, such that $|\lambda| + |\mu| + |\nu| = r(n - r)$. The intersection of the classes $[Y_\lambda]$, $[Y_\mu]$ and $[Y_\nu]$ is zero-dimensional. Define dimension vectors $\alpha, \beta$ by

$$ 1 \ 2 \ \cdots \ n \ \cdots \ 2 \ 1 $$

$$ \alpha = \begin{cases} \vdots \ \\ 2 \ \\ 1 \end{cases} $$

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and
\[
\beta(x_{n-r-\lambda_i+1}) = \cdots = \beta(x_{n-r-\lambda_{i+1}+1}) = i \\
\beta(y_{n-r-\mu_i+1}) = \cdots = \beta(y_{n-r-\mu_{i+1}+1}) = i \\
\beta(z_{n-r-\nu_i+1}) = \cdots = \beta(z_{n-r-\nu_{i+1}+1}) = i
\]
for \(i = 1, 2, \ldots, r\) with the ad-hoc conventions \(\lambda_{r+1} = \mu_{r+1} = \nu_{r+1} = 0\) and \(\lambda_0 = \mu_0 = \nu_0 = n - r - 1\).

The number of \(\beta\)-dimensional representations of a general \(\alpha\)-dimensional representation is finite and equal to the multiplicity of \([Y_\nu]\) inside \([Y_\lambda] \cdot [Y_\mu]\). Here
\[
\nu = (n - r - \nu_r, n - r - \nu_{r-1}, \ldots, n - r - \nu_1)
\]
is the complementary partition of \(\nu\) inside the \(r \times (n - r)\) rectangle.

Following the calculation in [3], the dimension of the space of semi-invariants of weight \(\langle \beta, \cdot \rangle\) on the space \(\text{Rep}(Q, \alpha - \beta)\) of \((\alpha - \beta)\)-dimensional representations is the Littlewood-Richardson coefficient \(c_{\lambda, \mu, \nu}^\nu\), where \(\langle \cdot, \cdot \rangle\) is the Euler form defined in the next section.

In this paper we generalize the connection between Schubert calculus and Littlewood-Richardson coefficients to quivers without oriented cycles and arbitrary dimension vectors. For a quiver \(Q\) without oriented cycles and two dimension vectors \(\alpha, \beta\) we define \(N(\beta, \alpha)\) as the number \(\beta\)-dimensional subrepresentations of a general \(\alpha\)-dimensional representation, and \(M(\beta, \alpha)\) as the dimension of the space of semi-invariants polynomials of weight \(\langle \beta, \cdot \rangle\) on the representation space of dimension \(\gamma := \alpha - \beta\).

**Theorem 1.** If \(\langle \beta, \gamma \rangle = 0\), then we have \(N(\beta, \alpha) = M(\beta, \alpha)\).

The proof compares the calculations of \(N(\beta, \alpha)\) and \(M(\beta, \alpha)\). The calculation of \(N(\beta, \alpha)\) comes from Intersection Theory and was done by Crawley-Boevey ([1]). The calculation of \(M(\beta, \alpha)\) comes from standard calculations in the coordinate ring of a representation space involving the Littlewood-Richardson rule. In this note we explain why both calculations are the same. For this we will make use of ([1]).

Suppose that a general representation of dimension \(\alpha\) has infinitely many \(\beta\)-dimensional subrepresentations. In Section 6 we will see that in that case, the cohomology class of the variety of \(\beta\)-dimensional subrepresentations of a \(\alpha\)-dimensional representation in general position is given by a formula whose coefficients can be interpreted as multiplicities of isotypic components in the coordinate ring of \(\text{Rep}(Q, \gamma)\).

Suppose that \(\langle \beta, \gamma \rangle = 0\). Given a general representation of dimension \(\alpha\), we will construct in Section 7 a basis of the semi-invariant polynomials on \(\text{Rep}(Q, \gamma)\) of weight \(\langle \beta, \cdot \rangle\).

2. **Basic Notation**

A quiver is a pair \(Q = (Q_0, Q_1)\) where \(Q_0\) is the set of vertices and \(Q_1\) is the set of arrows. Each arrow \(a\) has head \(ha\) and tail \(ta\), both in \(Q_0\):

\[ta \xrightarrow{a} ha.\]
An oriented cycle is a sequence of arrows \( a_1, a_2, \ldots, a_r \in Q_1 \) such that \( ta_i = ha_{i+1} \) for \( i = 1, 2, \ldots, r - 1 \) and \( ha_1 = ta_r \). We will assume that \( Q \) has no oriented cycles.

We fix an algebraically closed base field \( K \). A representation \( V \) of \( Q \) is a family of finite dimensional \( K \)-vector spaces

\[
\{ V(x) \mid x \in Q_0 \}
\]

together with a family of \( K \)-linear maps

\[
\{ V(a) : V(ta) \to V(ha) \mid a \in Q_1 \}.
\]

The dimension vector of a representation \( V \) is the function \( d(V) : Q_0 \to \mathbb{Z} \) defined by \( d(V)(x) := \dim V(x) \). The dimension vectors lie in the space \( \Gamma = \mathbb{Z}^{Q_0} \) of integer-valued functions on \( Q_0 \). A morphism \( \phi : V \to W \) of two representations is a collection of \( K \)-linear maps

\[
\{ \phi(x) : V(x) \to W(x) \mid x \in Q_0 \}
\]

such that for each \( a \in Q_1 \) we have \( W(a)\phi(ta) = \phi(ha)V(a) \), i.e., the diagram

\[
\begin{array}{ccc}
V(ta) & \xrightarrow{V(a)} & V(ha) \\
\downarrow & & \downarrow \\
W(ta) & \xrightarrow{W(a)} & W(ha)
\end{array}
\]

commutes. We denote the vector space of morphisms from \( V \) to \( W \) by \( \text{Hom}_Q(V, W) \).

The category of representations of \( Q \) is hereditary, i.e., a subobject of a projective object is projective. This implies that if \( V \) and \( W \) are representations, then \( \text{Ext}^i_Q(V, W) = 0 \) for all \( i \geq 2 \). We will write \( \text{Ext}_Q(V, W) \) instead of \( \text{Ext}^1_Q(V, W) \).

The spaces \( \text{Hom}_Q(V, W) \) and \( \text{Ext}_Q(V, W) \) can be calculated as the kernel and cokernel of the following linear map

\[
d^V_W : \bigoplus_{x \in Q_0} \text{Hom}(V(x), W(x)) \to \bigoplus_{a \in Q_1} \text{Hom}(V(ta), W(ha)).
\]

where the map \( d^V_W \) restricted to \( \text{Hom}(V(x), W(x)) \) is equal to

\[
\sum_{a, \, ta = x} \text{Hom}(V(x), W(a)) - \sum_{a, \, ha = x} \text{Hom}(V(a), W(x)).
\]

In other words,

\[
d^V_W (\{ \phi(x) \mid x \in Q_0 \}) = \{ W(a)\phi(ta) - \phi(ha)V(a) \mid a \in Q_1 \}.
\]

For a dimension vector \( \beta \) we denote by

\[
\text{Rep}(Q, \beta) = \bigoplus_{a \in Q_1} \text{Hom}(K^{\beta(ta)}, K^{\beta(ha)}).
\]
the vector space of representations of $Q$ of dimension vector $\beta$. The groups

$$\text{GL}(Q, \beta) := \prod_{x \in Q_0} \text{GL}(\beta(x))$$

and its subgroup

$$\text{SL}(Q, \beta) = \prod_{x \in Q_0} \text{SL}(\beta(x))$$

act on $\text{Rep}(Q, \beta)$ as follows. If

$$A = \{A(x) \mid x \in Q_0\} \in \text{GL}(Q, \beta)$$

and

$$V = \{V(a) \mid a \in Q_1\} \in \text{Rep}(Q, \beta),$$

then we define

$$A \cdot V := \{A(ha)V(a)A(ta)^{-1} \mid a \in Q_1\}.$$

The group $\text{GL}(Q, \beta)$ acts on the coordinate ring $K[\text{Rep}(Q, \beta)]$ as follows. If $f \in K[\text{Rep}(Q, \beta)]$, $A \in \text{GL}(Q, \beta)$ then

$$(A \cdot f)(V) = f(A^{-1} \cdot V), \quad V \in \text{Rep}(Q, \beta).$$

We are interested in the ring of semi-invariants

$$\text{SI}(Q, \beta) = K[\text{Rep}(Q, \beta)]^{\text{SL}(Q, \beta)}.$$ 

To each $\sigma \in \Gamma = \mathbb{Z}^{Q_0}$ we can associate a character of $\text{GL}(\beta)$ defined by

$$A = \{A(x) \mid x \in Q_0\} \in \text{GL}(\beta) \mapsto \prod_{x \in Q_0} \text{det}(A(x))^\sigma(x).$$

By abuse of notation this character is also denoted by $\sigma$. The 1-dimensional representation corresponding to this multiplicative character will be denoted by $\text{det}^\sigma$. For any two characters $\sigma, \tau \in \mathbb{Z}^{Q_0}$ we have $\text{det}^{\sigma+\tau} = \text{det}^\sigma \otimes \text{det}^\tau$. We can write

$$\text{det}^\sigma = \bigotimes_{x \in Q_0} \text{det}_x^{\sigma(x)}$$

where $\text{det}_x^k$ is the 1-dimensional representation of $\text{GL}(\beta(x))$ corresponding to the multiplicative character $A \mapsto \text{det}(A)^k$.

The ring $\text{SI}(Q, \beta)$ has a weight space decomposition

$$\text{SI}(Q, \beta) = \bigoplus_{\sigma} \text{SI}(Q, \beta)_\sigma$$

where $\sigma$ runs through the characters of $\text{GL}(Q, \beta)$ and

$$\text{SI}(Q, \beta)_\sigma = \{f \in K[\text{Rep}(Q, \beta)] \mid g(f) = \sigma(g)f \quad \forall g \in \text{GL}(Q, \beta) \}.$$ 

Let $\alpha, \beta$ be two elements of $\Gamma$. We define the Euler inner product

$$(3) \quad \langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$
It follows from (2) and (3) that
\[ \langle d(V), d(W) \rangle = \dim_K \text{Hom}_Q(V, W) - \dim_K \text{Ext}_Q(V, W). \]

3. The computation of \( N(\beta, \alpha) \)

If \( r, n \) are nonnegative integers with \( r \leq n \) then \( \text{Grass}(r, n) \) denotes the Grassmanian of \( r \)-dimensional subspaces of \( K^n \). Let \( \alpha, \beta, \gamma \) be two dimension vectors such that \( \alpha = \beta + \gamma \). We define
\[ \text{Grass}(\beta, \alpha) = \prod_{x \in Q_0} \text{Grass}(\beta(x), \alpha(x)). \]

A point \( W = \{ W(x) \mid x \in Q_0 \} \in \text{Grass}(\beta, \alpha) \) is a collection of subspaces with \( \dim W(x) = \beta(x) \) for all \( x \in Q_0 \). Consider the incidence variety
\[ Z(Q, \beta, \alpha) = \{ (V, W) \in \text{Rep}(Q, \alpha) \times \text{Grass}(\beta, \alpha) \mid \forall a \in Q_1 V(a)(W(ta)) \subseteq W(ha) \}. \]

There are two projections:
\[
x \quad Z(Q, \beta, \alpha) \quad q \quad p \quad \text{Rep}(Q, \alpha) \quad \text{Grass}(\beta, \alpha)
\]

The following proposition was proved in [6].

**Proposition 2.**

(a) The first projection \( q : Z(Q, \beta, \alpha) \to \text{Rep}(Q, \alpha) \) is proper.
(b) The second projection \( p : Z(Q, \beta, \alpha) \to \text{Grass}(\beta, \alpha) \) gives \( Z(Q, \beta, \alpha) \) the structure of a vector bundle over \( \text{Grass}(\beta, \alpha) \).
(c) \( \dim Z(Q, \beta, \alpha) - \dim \text{Rep}(Q, \alpha) = \langle \beta, \gamma \rangle \).

We will use a result of Crawley-Boevey ([1]) who proved a formula for the cohomology class of the general fiber of \( q \) in terms of Intersection Theory.

In order to formulate this result we need some notation. For a variety \( X \), its associated cohomology ring will be denoted by \( A^*(X) \) (the Chow ring or the singular cohomology ring). We recall that for the Grassmannian \( \text{Grass}(r, n) \) the ring \( A^*(\text{Grass}(r, n)) \) is spanned by classes of Schubert varieties \( Y_\lambda \) where \( \lambda \) denotes a partition contained in the rectangle \((n-r)r)\), i.e., \( \lambda = (\lambda_1, \ldots, \lambda_r) \) with \( n - r \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_1 \geq 0 \). The sum \( |\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_r \) of the parts of \( \lambda \) is equal to the codimension of \( Y_\lambda \).

In our setup we have
\[ A^*(\text{Grass}(\beta, \alpha)) = \bigotimes_{x \in Q_0} A^*(\text{Grass}(\beta(x), \alpha(x))). \]

We denote by \([\lambda]_x\) the cohomology class of the Schubert variety \( Y_\lambda \) in the factor \( A^*(\text{Grass}(\beta(x), \alpha(x))) \). We use the convention \([\lambda]_x = 0 \) if \( \lambda \) is not contained in the rectangle \((\gamma(x))_{\beta(x)}\).
Proposition 3 \([\square]\). For general \(V \in \text{Rep}(Q, \alpha)\) the cycle of \(q^{-1}(V)\) in \(A^*(\text{Grass}(\beta, \alpha))\) is equal to

\[
[q^{-1}(V)] = \prod_{a \in Q_1} \left( \sum_{\lambda} [\lambda]_{ta} [\lambda]_{ha} \right)
\]

where in the sum \(\sum_{\lambda} [\lambda]_{ta} [\lambda]_{ha}\), \(\lambda\) runs over all partitions which fit inside a \(\beta(ta) \times \gamma(ha)\) rectangle. In \([\square]\), \(\lambda\) denotes the complement of \(\lambda\) inside a \(\beta(ta) \times \gamma(ha)\) rectangle.

We exchange the sum and the product in \([\square]\). Clearly the summands in the formula correspond to functions \(\lambda : Q_1 \to P\) where \(P\) is a set of partitions and \(\lambda(a)\) is contained in a \(\beta(ta) \times \gamma(ha)\) rectangle for all arrows \(a\). Now \([\square]\) can be rewritten as

\[
[q^{-1}(V)] = \sum_{\Delta : Q_1 \to P} \prod_{x \in Q_0} \prod_{a \in Q_1} \frac{[\lambda(a)]_x}{[\lambda(a)]_x} \prod_{a \in Q_1} \frac{[\lambda(a)]_x}{[\lambda(a)]_x}.
\]

Suppose that \(\langle \beta, \gamma \rangle = 0\). Then we have \(\dim Z(Q, \beta, \alpha) = \dim \text{Rep}(Q, \alpha)\) by Proposition 2(c). This means that the general fiber of \(q\) is finite. Such a general fiber is reduced, even in positive characteristic (see \([\square]\) Corollary 3]). Let \(N(\beta, \alpha)\) be the cardinality of a general fiber. So a general representation \(V\) of dimension \(\alpha\) has \(N(\beta, \alpha)\) subrepresentations of dimension \(\beta\). We translate \([\square]\) into the language of Schur functors

\[
N(\beta, \alpha) = |q^{-1}(V)| = \sum_{\Delta : Q_1 \to P} \prod_{x \in Q_0} \prod_{a \in Q_1} \frac{\text{mult}(S_{\gamma(x)}^{\beta(x)}, \bigotimes_{a \in Q_1} S^{\lambda(a)}_{\beta(a)} \bigotimes_{a \in Q_1} S^{\overline{\lambda(a)}}_{\overline{\beta(a)}})}
\]

Here \(\text{mult}(S^\lambda; T)\) denotes the multiplicity of \(S^\lambda\) in \(T\).

4. The computation of \(M(\beta, \alpha)\)

Let us calculate the dimension of a weight space \(SI(Q, \gamma)_{\langle \beta, \cdot \rangle}\). For now we assume that the base field has characteristic 0. This is sufficient, because we will show in the next section that the number \(M(\beta, \alpha)\) does not depend on the (algebraically closed) base field. The space \(\text{Rep}(Q, \gamma)\) can be identified with

\[
\prod_{a \in Q_1} \text{Hom}_K(W(ta), W(ha)).
\]

where \(W(x)\) is a vector space of dimension \(\gamma(x)\) for all \(x \in Q_0\). The coordinate ring can now be identified with the symmetric algebra on the dual space

\[
K[\text{Rep}(Q, \gamma)] = \bigotimes_{a \in Q_1} \text{Sym}(W(ta) \otimes W(ha)^*).
\]

Here \(\text{Sym}\) denotes the symmetric algebra on a vector space. By Cauchy’s formula we can rewrite this in terms of Schur functors. We use the exterior power notation for
Schur functors, i.e., for a partition $\mu$ we write $\bigwedge^{\mu} W := S^{\mu'} W$ where $\mu'$ is the conjugate partition of $\mu$. In particular, $\bigwedge^{(m)}$ is the $m$-th exterior power. We have

$$K[\text{Rep}(Q, \gamma)] = \bigoplus_{\Delta: Q_1 \rightarrow P} \bigotimes_{a \in Q_1} (\bigwedge^{\Delta(a)} W(ta) \otimes \bigwedge^{\Delta(a)} W(ha)^*)$$

This can be rewritten as

$$K[\text{Rep}(Q, \gamma)] = \bigoplus_{\Delta: Q_1 \rightarrow P} \bigotimes_{x \in Q_0} \bigotimes_{a \in Q_1, ta = x} (\bigwedge^{\Delta(a)} W(x)) \otimes \bigotimes_{a \in Q_1, ha = x} (\bigwedge^{\Delta(a)} W(x)^*)$$

Let us calculate the dimension of the space of semi-invariants of weight $\langle \beta, \cdot \rangle$.

The partition $\Delta(a)$ has parts $\leq \gamma(ha)$, because $\dim W(ha) = \gamma(ha)$. We need additional restriction to match the one in Proposition 3. This is provided by the following lemma.

**Lemma 4.** If a summand in (8) corresponding to the function $\Delta: Q_1 \rightarrow P$ contains a nonzero semi-invariant of weight $\langle \beta, \cdot \rangle$, then for each $a \in Q_1$ the partition $\Delta(a)$ is contained in $(\gamma(ha))^\beta(ta)$.

**Proof.** Let us look at the space of semi-invariants $SI(Q, \gamma)_{\langle \beta, \cdot \rangle}$. By Theorem 1 of [3] (see also [8]) this space is spanned by the semi-invariants $c^V$ defined by the formula $c^V(W) := \det d^V_W$, where $V \in \text{Rep}(Q, \beta)$ and $d^V_W$ is the differential in (2). Let us investigate the contribution of the coefficients of the matrix $W(a)$ to such a semi-invariant. The only block of $d^V_W$ where this matrix $W(a)$ occurs is the block $\text{Hom}(\text{id}_{V(x)}, W(a))$:

$$
\begin{pmatrix}
W(a) & 0 & \cdots & 0 \\
0 & W(a) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & W(a)
\end{pmatrix}
$$

with $\beta(ta)$ blocks $W(a)$. Now any multihomogeneous semi-invariant will come from exhibiting a determinant $c^V(W)$ as a polynomial in coefficients of matrices from $V$ and taking a coefficient of some monomial. Such a semi-invariant has to be a linear combination of minors of the above block, multiplied by polynomials depending on other matrices $W(b)$. But the minors of the above block matrix are products of minors of $W(a)$ with at most $\beta(ta)$ factors in each summand. By the straightening law (compare for example [2]) we know that the products of $\beta(ta)$ minors of $W(a)$ are contained in the space

$$\sum_{\nu} \bigwedge^{\nu} W(ta) \otimes \bigwedge^{\nu} W(ha)^*$$

in $\text{Sym}(W(ta) \otimes W(ha)^*)$, where $\nu$ runs over partitions with at most $\beta(ta)$ parts.  \(\square\)
Let us calculate the dimension of the space of semi-invariants of weight $\sigma := \langle \beta, \cdot \rangle$.

We have

$$\sigma(x) = \beta(x) - \sum_{a \in Q_1 \atop ha = x} \beta(ta)$$

for all $x \in Q_0$. By duality, we have the following $GL(W(ha))$-isomorphism

$$\bigwedge^{\mu(a)} W(ha)^* = \bigwedge^{\overline{\mu}(a)} W(ha) \otimes det^{-\beta(ta)}_{ha}$$

where $\overline{\mu}(a)$ is the complement of $\mu(a)$ inside an $\beta(ta) \times \gamma(ha)$ rectangle.

From (10) and (8) follows that the dimension $M(\beta, \alpha)$ of $SI(Q, \gamma)_{\langle \beta, \cdot \rangle}$ is equal to

$$\sum_{\lambda} \prod_{x \in Q_0} \text{mult} (\bigwedge^x_{\alpha} W(x) \otimes \bigwedge^x_{\beta} W(x))$$

The equations (9) and (11) imply

$$M(\beta, \alpha) = \sum_{\Delta: Q_1 \to P} \prod_{x \in Q_0} \text{mult} \left( \bigwedge^x_{\alpha} \bigwedge^x_{\beta} W(x) \otimes \bigwedge^x_{\alpha} W(x) \right)$$

In view of Lemma 4, we only need to sum over those functions $\Delta: Q_1 \to P$ for which $\Delta(a)$ lies in a $\beta(ta) \times \gamma(ha)$ rectangle. We write (12) in terms of Schur functors:

$$M(\beta, \alpha) = \sum_{\Delta} \prod_{x \in Q_0} \text{mult} \left( \bigwedge^\langle \gamma(x) \rangle_{\alpha} \bigwedge^\langle \beta(x) \rangle_{\beta} \right)$$

Proof of Theorem 1. The number $N(\beta, \alpha)$ does not depend on the base field, and neither does $M(\beta, \alpha)$ by Proposition 8 in the next section. We may assume that the base field $K$ is algebraically closed and has characteristic 0. Note that $S^\lambda = \bigwedge^\langle \lambda \rangle$ where $\lambda'$ is the conjugate partition. Also, the Littlewood-Richardson coefficients $c^\nu_{\lambda, \mu}$ and $c^{\nu'}_{\lambda', \mu'}$ are the same. From these observations it follows that the righthand-sides in (6) and (11) are identical. \[\square\]

5. Good filtrations

We work over an algebraically closed field $K$ of arbitrary characteristic. Suppose that $V$ is an $n$-dimensional vector space. We use the convention that a Schur module $L_\lambda V$ is denoted by the partition of its highest weight, i.e., $\bigwedge^i V = L_{i, V}$, $S^i V = L_{i, V}$.

Let us recall that a good filtration for a rational finite dimensional $GL(n) := GL(V)$-module $W$ is a filtration

$$0 = W_0 \subset W_1 \subset \ldots \subset W_{s-1} \subset W_s = W$$

for which each factor $W_{i+1}/W_i$ is isomorphic to some Schur module $L_\lambda(z)V$. Such filtration, if it exists, may not be unique. The number of factors $L_\lambda V$ in any good filtration of $W$ does not depend on the choice of a good filtration, and will be denoted by $n_\lambda(W)$. 
Proposition 5. If the modules $W_1$ and $W_2$ have good filtrations, then the tensor product $W_1 \otimes W_2$ has a good filtration.

Proof. See [9, (4.2) Theorem] or [4, Theorem 4.3.1]. □

Proposition 6. Let $W$ be a module with good filtration. Then there exists a good filtration

$$0 = W'_0 \subset W'_1 \subset \ldots \subset W'_{s-1} \subset W'_s = W$$

such that the submodule of the $SL(V)$-invariants $W^{SL(V)}$ in $W$ is equal to $W'_t$ for some $t$. In particular, the dimension of $W^{SL(V)}$ is equal to the number of factors $W_i/W_{i-1}$ isomorphic to the trivial representation.

Proof. We use the results of [4] freely. Let us order the highest weights $\lambda$ by saying that $\lambda < \mu$ if $\lambda - \mu$ is a sum of positive roots for $SL(V)$.

Let $W$ be a module with a good filtration. We can assume without loss of generality that $W$ is a polynomial homogeneous representation of degree $d$. Then by [4, Proposition 3.2.6] there exists a filtration

$$0 = W'_0 \subset W'_1 \subset \ldots \subset W'_{s-1} \subset W'_s = W$$

with $W'_t/W'_{t-1} = L_{\lambda(t)}V$, for which $\lambda(1) \leq \lambda(2) \leq \ldots \leq \lambda(s)$. We notice that for existence of $SL(V)$-invariants it is necessary that $d = ne$ for some $e$. Now among all the possible highest weights $\lambda$ that correspond to partitions of $d$ with $\leq n$ parts, the smallest one is $\lambda = (e^n)$. Let $t$ be maximal such that $\lambda(1) = \ldots = \lambda(t) = (e^n)$. All composition factors of $W'_t$ have are isomorphic to the trivial $SL(V)$-representation $L(e^n)V$. Hence $SL(V)$ acts trivially on $W'_t$. This proves the first part of the proposition.

It remains to show that the opposite is true. Then the module $W/W'_t$ is a polynomial homogeneous representation of $GL(V)$ which has a good filtration with no factors isomorphic to $L(e^n)V$ and with a nontrivial submodule of $SL(V)$-invariants. Let $u$ be the smallest number for which the factor module $W'_{t+u}/W'_t$ has a nonzero module of $SL(V)$-invariants. Then the factor $W'_{t+u}/W'_{t+u-1} = L_{\lambda(t+u)}$ contains a nonzero $SL(V)$-invariant. This is impossible, because the unique irreducible submodule of $L_{\lambda(t+u)}$ has highest weight $\lambda(t+u)$ and is not one-dimensional. □

Corollary 7. Let $W$ be a polynomial homogeneous $GL(V)$-module of degree $d = en$ with a good filtration. Then the dimension of the $W^{SL(V)}$-invariants in $W$ is equal to $n(e^n)(W)$.

We continue with the application to quiver representations. Assume that $Q$ is a quiver without oriented cycles. The above reasoning generalizes directly to products of general linear groups, as the Schur modules for products of general linear groups are just tensor products of the Schur modules for the factors. In dealing below with the coordinate rings $K[\text{Rep}(Q, \alpha)]$ and their good filtrations, notice we are really dealing with their homogeneous components which are finite dimensional representations.
Proposition 8. Let $Q$ be a quiver with no oriented cycles and let $\alpha$ be a dimension vector. The coordinate ring $K[\text{Rep}(Q, \alpha)]$ has a good filtration as a $\text{GL}(Q, \alpha)$-module. Thus the dimension of the spaces of semi-invariants $SI(Q, \alpha)_\sigma$ can be calculated as a multiplicity of the corresponding tensor products of Schur functors in the coordinate ring $K[\text{Rep}(Q, \alpha)]$. In particular this dimension does not depend on the characteristic of $K$.

Proof. The coordinate ring $K[\text{Rep}(Q, \alpha)]$ has the following decomposition.

$$K[\text{Rep}(Q, \alpha)] = \otimes_{a \in Q_0} \text{Sym}(V(ta) \otimes V(ha)^*)$$

Now, using the straightening law (comp. [2] or [10, Theorem (2.3.2)]) we have that $\text{Sym}(V(ta) \otimes V(ha)^*)$ has a characteristic free filtration with associated graded object $\oplus L(\lambda(a))V(ta) \otimes L(\lambda(a))V(ha)^*$. This is a good filtration. Applying Proposition 5 we get that $K[\text{Rep}(Q, \alpha)]$ has a good filtration as a $\text{GL}(Q, \alpha)$-module. Now Proposition 6 (for a product of general linear groups) gives the result. \qed

6. A generalization to Covariants

The statement of Theorem 1 generalizes from semi-invariants to covariants. Let us assume that $\langle \beta, \gamma \rangle \geq 0$. Assume that the cycle $[q^{-1}(V)]$ for generic $V \in \text{Rep}(Q, \alpha)$ decomposes as follows.

$$[q^{-1}(V)] = \sum_{\Delta \to \rho} N(\beta, \alpha, \Delta) \prod_{x \in Q_0} [\bar{\Delta}(x)]_x. \tag{14}$$

Here $\bar{\Delta}(x)$ is the complement of $\Delta(x)$ in a $\beta(x) \times \gamma(x)$ rectangle.

Let $W(x)$ be a $\gamma(x)$-dimensional $K$-vector space for all $x \in Q_0$. We can identify $\text{Rep}(Q, \gamma)$ with $\bigoplus_{a \in Q_0} \text{Hom}(W(ta), W(ha))$ and $\text{GL}(Q, \gamma)$ with $\prod_{x \in Q_0} \text{GL}(W(x))$.

We define $M(\beta, \alpha, \Delta)$ as the multiplicity of $\det^\sigma$ in

$$K[\text{Rep}(Q, \gamma)] \otimes \bigotimes_{x \in Q_0} \wedge^{\mu(x)} W(x).$$

Proposition 9. Assume that $\sum_{x \in Q_0} |\mu(x)| = \langle \beta, \gamma \rangle$. Then

$$N(\beta, \alpha, \mu) = M(\beta, \alpha, \mu).$$

We will reduce Proposition 9 to Theorem 1. Let us write

$$\overline{\mu}(x) = (\gamma(x)^{b_1(x)}, (\gamma(x) - 1)^{b_2(x)}, \ldots, 1^{b_{\gamma(x)}(x)})$$

for all $x$, where $\overline{\mu}(x)$ is the complement of $\mu(x)$ inside a $\beta(x) \times \gamma(x)$ rectangle. We introduce the quiver $\hat{Q}$ with

$$\hat{Q}_0 = Q_0 \cup \bigcup_{x \in Q_0} \{y_{1,x}, y_{2,x}, \ldots, y_{\gamma(x),x}\}.$$
and
\[ \hat{Q}_1 = Q_1 \cup \bigcup_{x \in Q_1} \{a_{1,x}, a_{2,x}, \ldots, a_{\gamma(x),x}\} \]

where
\[ a_{i,x} : y_{i-1,x} \to y_{i,x} . \]
for all \( i \) and \( x \). We use the convention \( y_{0,x} = x \).

We define dimension vectors \( \hat{\beta}, \hat{\gamma} \) by
\[ \hat{\beta}(x) = \beta(x), \quad x \in Q_0, \]
\[ \hat{\beta}(y_{i,x}) = b_1(x) + b_2(x) + \cdots + b_{\gamma(x)-i+1}, \quad i = 1, 2, \ldots, \gamma(x) \]
and
\[ \hat{\gamma}(x) = \gamma(x), \quad x \in Q_0, \]
\[ \hat{\gamma}(y_{i,x}) = \gamma(x) - i + 1, \quad i = 1, 2, \ldots, \gamma(x) . \]

**Lemma 10.** We have
\[ N(\beta, \alpha, \mu) = N(\hat{\beta}, \hat{\alpha}) . \]

**Proof.** From (5) follows that
\[ N(\hat{\beta}, \hat{\alpha}) = \sum_{\Delta: Q_1 \to \mathcal{P}} \prod_{x \in Q_0} \prod_{\substack{a \in Q_1 \\text{ta} = x \\text{ha} = x}} [\Delta(a)]_x \prod_{a \in \hat{\Delta} \setminus \hat{Q}_1} [\lambda(a)]_x . \]

To get the class of a point at vertex \( y_{\gamma(x),x} \) we must have
\[ \Delta(a_{\gamma(x),x}) = (\gamma(y_{\gamma(x),x})^{\beta(y_{\gamma(x),x})}) = (1^{b_1}) . \]
Now \( \Delta(a_{\gamma(x),x}) \) and \( \Delta(a_{\gamma(x)-1,x}) \) fit together into a \( \beta(y_{\gamma(x)-1,x}) \times \gamma(y_{\gamma(x),x}) = (b_1 + b_2) \times 1 \)
rectangle. It follows that
\[ \Delta(a_{\gamma(x)-1,x}) = (1^{b_2}) . \]
To get a nonzero summand, \( \Delta(a_{\gamma(x),x}) \) and \( \Delta(a_{\gamma(x)-1,x}) \) have to fit together into a \( \beta(y_{\gamma(x)-1,x}) \times \gamma(y_{\gamma(x)-1,x}) = (b_1 + b_2) \times 2 \) rectangle. We see that
\[ \Delta(a_{\gamma(x)-1,x}) = (2^{b_1}, 1^{b_2}) . \]
Continuing by induction, we see that
\[ \Delta(a_{1,x}) = (\gamma(x)^{b_1(x)}), \ldots, 1^{b_{\gamma(x)}(x)}) = \mu(x) . \]
We have
\[ N(\hat{\beta}, \hat{\alpha}) = \prod_{a \in Q_1} (\sum_{\lambda} [\lambda]_{ta[x]}[\lambda]_{ha}) = \]
\[ = (\prod_{a \in Q_1} (\sum_{\lambda} [\lambda]_{ta[x]}[\lambda]_{ha})) (\prod_{a \in \hat{Q}_1 \setminus Q_1} (\sum_{\lambda} [\lambda]_{ta[x]}[\lambda]_{ha})) . \]
Using our calculations for $\Lambda(a_{i,x})$ above, we see that this is equal to
\[
\left( \prod_{a \in Q_1} \left( \sum_{\lambda} [\lambda]_{ta} [\lambda]_{ha} \right) \right) \prod_{x \in Q_0} [\mu(x)]_x.
\]

From (13) and (14) follows that
\[
N(\beta, \alpha, \lambda) = \left( \sum_{\Delta_{Q_0} \rightarrow P} N(\beta, \alpha, \lambda) \prod_{x \in Q_0} [\lambda(x)]_x \right) \prod_{x \in Q_0} [\mu(x)]_x = N(\beta, \alpha, \mu).
\]

\[\square\]

**Lemma 11.** We have
\[
M(\beta, \alpha, \mu) = M(\beta, \alpha, \lambda).
\]

**Proof.** The proof goes similar to the proof of the previous lemma. From (13) follows that
\[
M(\beta, \alpha, \lambda) = \sum_{\Delta_{Q_1} \rightarrow P} \prod_{x \in Q_0} \text{mult} \left( \bigwedge^{(a(x) \cdot \beta(x))} W(x); \left( \bigotimes_{a \in Q_1} \bigwedge^{(a)} W(x) \right) \otimes \left( \bigotimes_{h \in Q_2} \bigwedge^{\beta(h)} W(x) \right) \right).
\]

To get a nonzero summand, we get the same conditions for $\Lambda(a_{i,x})$ as in the previous lemma. We obtain
\[
M(\beta, \alpha, \lambda) = \sum_{\Delta_{Q_1} \rightarrow P} \prod_{x \in Q_0} \text{mult} \left( \det^\beta(x); \left( \bigotimes_{a \in Q_1} \bigwedge^{(a)} W(x) \right) \otimes \left( \bigotimes_{h \in Q_2} \bigwedge^{\beta(h)} W(x) \right) \right) = \sum_{\Delta_{Q_1} \rightarrow P} \prod_{x \in Q_0} \text{mult} \left( \det^\beta(x); \left( \bigotimes_{a \in Q_1} \bigwedge^{(a)} W(x) \right) \otimes \left( \bigotimes_{h \in Q_2} \bigwedge^{\beta(h)} W(x) \right) \right) = \text{mult} \left( \det^\beta; \bigotimes_{a \in Q_1} \bigwedge^{(a)} W(x) \otimes \bigwedge^{\beta(h)} W(x) \otimes \bigwedge^{\mu(x)} W(x) \right) = \text{mult} \left( \det^\beta; \text{Rep}(Q, \gamma) \otimes \bigwedge^{\mu(x)} W(x) \right) = M(\beta, \alpha, \mu).
\]

\[\square\]

**7. Applications**

Theorem 11 also allows us to exhibit an explicit basis of the weight space $SI(Q, \gamma)_{(\beta, \cdot)}$.

**Corollary 12.** Let $Q, \alpha, \beta, \gamma$ be as in Theorem 7. Assume that the general representation $V$ of dimension $\alpha$ has $k$ subrepresentations of dimension $\beta$. There exists a nonempty Zariski open set $U$ in $Rep_K(Q, \alpha)$ such that for $V \in U$ the semi-invariants $c^{V_1}, \ldots, c^{V_k}$ form a basis in $SI(Q, \gamma)_{(\beta, \cdot)}$, where $V_1, V_2, \ldots, V_k$ are the subrepresentations of $V$ of dimension $\beta$. 
Proof. Let us choose $V \in \text{Rep}_K(Q, \alpha)$ such that $q^{-1}(V)$ consists of $k$ points. Let $V_1, \ldots, V_k$ be the corresponding subrepresentations of $V$ of dimension $\beta$. It is enough to prove that $e^{V_1}, \ldots, e^{V_k}$ are linearly independent in $\text{SI}(Q, \gamma)_{(\beta, \cdot)}$. Let us consider the exact sequences

$$0 \to V_i \to V \to W_i \to 0$$

for $i = 1, \ldots, k$. It is clear that for $i \neq j$ we have $\text{Hom}_Q(V_i, W_j) \neq 0$. Therefore $e(V_i)(W_j) = 0$ for $i \neq j$. Therefore it is enough to show that $\text{Hom}_Q(V_i, W_i) = 0$ for $i = 1, \ldots, k$. In [6] it was proved that $\text{Hom}_Q(V_i, W_i)$ is the tangent space to the fiber of the map $q : Z(Q, \beta, \alpha) \to \text{Rep}_K(Q, \alpha)$ at the point $(V, V_i)$. The map $q$ is dominant and generically it is $k : 1$. Moreover, it is shown in [1] that the differential $Dq$ is generically surjective. Therefore it is generically an isomorphism because $Z(Q, \beta, \alpha)$ is smooth. Therefore for $V$ in some nonempty Zariski open set we have $\text{Hom}_Q(V_i, W_i) = 0$ for $i = 1, \ldots, k$. □

Proposition 13. Let $Q, \alpha, \beta, \gamma$ be as in Theorem 7. Let $U_\beta \subseteq \text{Rep}(Q, \beta)$ and $U_\gamma \subseteq \text{Rep}(Q, \gamma)$ be nonempty Zariski open subsets, stable under $\text{GL}(\beta)$ and $\text{GL}(\gamma)$ respectively. Then there exists a nonempty Zariski open subset $U$ of $\text{Rep}(Q, \alpha)$ such that for all $V \in U$ we have that every $\beta$-dimensional subrepresentation of $V$ lies in $U_\beta$ and every $\gamma$-dimensional factor representation of $V$ lies in $U_\gamma$.

Proof. Let

$$D \subset Z(Q, \beta, \alpha) \subset \text{Rep}(Q, \alpha) \times \text{Grass}(\beta, \alpha)$$

be the subset of pairs $(V, W)$ such that $W$ does not lie in $U_\beta$ or $V/W$ does not lie in $U_\gamma$. We claim that $D$ is a Zariski closed strict subset of $Z(Q, \beta, \alpha)$. For every $z \in \text{Grass}(\beta, \alpha)$ we can choose local sections $e_1, e_2, \ldots, e_{\alpha(x)}$ and an open neighborhood $X$ of $z$ such that $e_1(W), \ldots, e_{\beta(x)}(W)$ are a basis of $W$ and $e_1(W), \ldots, e_{\alpha(x)}(W)$ are a basis of $K^\alpha(x)$ for all $W \in X$. With respect to this basis, $V$ has the form

$$V(a) = \begin{pmatrix} V'(a) & * \\ 0 & V''(a) \end{pmatrix}$$

where $V' \in \text{Rep}(Q, \beta), V'' \in \text{Rep}(Q, \gamma)$ and $*$ is an arbitrary matrix.

We define

$$r_X : p^{-1}(X) \to \text{Rep}(Q, \beta) \times \text{Rep}(Q, \gamma) \times X$$

by

$$(V, W) \mapsto (V', V'', W),$$

It is clear that

$$D \cap p^{-1}(X) = r_X^{-1}((D_\beta \times \text{Rep}(Q, \gamma) \times X) \cup (\text{Rep}(Q, \beta) \times D_\gamma \times X))$$

where $D_\beta$ and $D_\gamma$ are the complements of $U_\beta$ and $U_\gamma$ respectively. Therefore we have that $D \cap p^{-1}(X)$ is a Zariski closed proper subset of $p^{-1}(X)$. Since such $p^{-1}(X)$ cover $Z(Q, \beta, \alpha)$, we conclude that $D$ is Zariski closed proper subset of $Z(Q, \beta, \alpha)$. For some $U' \subset \text{Rep}(Q, \alpha)$ open nonempty, $q^{-1}(V)$ is finite for all $V \in U'$. Take
$U = U' \setminus q(D)$. The map $q$ is proper so $q(D)$ is closed and $U$ is therefore open. We also claim that $U$ is nonempty. Indeed, the restriction $q : q^{-1}(U') \to U'$ is quasi-finite, so $q(q^{-1}(U') \cap D) \subseteq U' \cap q(D) \neq U'$ because the dimension of $q^{-1}(U') \cap D$ is strictly smaller than $\dim q^{-1}(U') = \dim U'$. If $V \in U$ then $q^{-1}(V)$ is finite and for every $(V, W) \in q^{-1}(V)$ we have that $W$ and $V/W$ lie in $U_{\beta}$ and $U_{\gamma}$ respectively. □

The proposition can be roughly reformulated as follows. If $V$ is a general representation of dimension $\alpha = \beta + \gamma$ with $\langle \beta, \gamma \rangle = 0$, then all subrepresentations of dimension $\beta$ and all factor representation of dimension $\gamma$ are in general position as well.

**Corollary 14.** Suppose that $\beta, \gamma, \delta$ are dimension vectors such that $\langle \beta, \gamma \rangle = \langle \beta, \delta \rangle = \langle \gamma, \delta \rangle = 0$. Then we have the following equality

$$N(\beta, \beta + \gamma)N(\beta + \gamma, \beta + \gamma + \delta) = N(\beta, \beta + \gamma + \delta)N(\gamma, \gamma + \delta).$$

**Proof.** With the previous proposition this is now a simple counting argument. Choose $V \in \text{Rep}(Q, \beta + \gamma + \delta)$ in general position. We count the number of pairs $(V_1, V_2)$ such that $V_1$ is a $\beta$-dimensional subrepresentation of $V_2$ and $V_2$ is a $(\beta + \gamma)$-dimensional subrepresentation of $V$. On the one hand, $V$ has $N(\beta + \gamma, \beta + \gamma + \delta)$ $(\beta + \gamma)$-dimensional subrepresentations $V_2$ and each such subrepresentation (since it is again in general position) has exactly $N(\beta, \beta + \gamma)$ subrepresentations $V_1$ of dimension $\beta$.

On the other hand $V$ has $N(\beta, \beta + \gamma + \delta)$ $\beta$-dimensional subrepresentations $V_1$. For each $V_1$, $V/V_1$ is again in general position and $V/V_1$ has exactly $N(\gamma, \gamma + \delta)$ subrepresentations of dimension $\gamma$. Note also that there is a 1–1 correspondence between $\gamma$-dimensional subrepresentations of $V/V_1$ and $\beta + \gamma$ dimensional subrepresentations $V_2$ of $V$ containing $V_1$. Comparison of the two computations completes the proof. □

**Example 15.** Let $Q = \theta(m)$ be the quiver with two vertices $x, y$ and $m$ arrows $a_1, \ldots, a_m$, with $ta_i = x, ha_i = y$ for $i = 1, \ldots, m$. Assume $m = 2r$ is even. Let $\beta$ be the dimension vector $\alpha(x) = \alpha(y) = r + 1$. Consider the subdimension vector $\beta$ with $\beta(x) = 1, \beta(y) = r$. Then $\langle \beta, \alpha - \beta \rangle = 0$. The Littlewood-Richardson calculation shows that $N(\beta, \alpha) = \binom{2r}{r}$.

In particular, for $r = 2$, we have $Q = \theta(4)$ is the quiver with two vertices $x, y$ and four arrows $a, b, c, d$ from $x$ to $y$. Consider the dimension vector $\alpha$ with $\alpha(x) = \alpha(y) = 3$. Consider the map

$$C : \text{Rep}(Q, \alpha) \to \mathbb{A}^{20}$$

where we identify $\mathbb{A}^{20}$ with the space of quaternary cubics. The map $C$ sends a representation $V$ to the point $\det (X_0V(a) + X_1V(b) + X_2V(c) + X_3V(d))$.

Consider the subrepresentations of dimension $\beta$ where $\beta(x) = 1, \beta(y) = 2$. Our calculation tells us that a generic representation of dimension $\alpha$ has 6 subrepresentations of dimension $\beta$. They correspond to six lines on the cubic surface defined by $C(V)$. These lines represent the cubic surface as a projective plane with 6 points blown up. Given a subrepresentation $W$ of dimension $\beta$, the corresponding line is constructed as follows. Choose bases of $V(x)$ and $V(y)$ such that $W(x)$ is spanned by the first basis
vector and $W(y)$ is spanned by the first 2 basis vectors. Then the cubic surface is defined by
\[
\det \begin{pmatrix}
  l_{1,1} & l_{1,2} & l_{1,3} \\
  l_{2,1} & l_{2,2} & l_{2,3} \\
  0 & l_{3,2} & l_{3,3}
\end{pmatrix} = 0,
\]
where the $l_{i,j}$ are linear functions. Now $l_{1,1} = l_{2,1} = 0$ defines a line on the surface.

References

[1] W. Crawley-Boevey, Subrepresentations of general representations of quivers, Bull. London Math. Soc. 28 (1996), 363–366.
[2] C. DeConcini, D. Eisenbud, C. Procesi, Young diagrams and determinantal varieties, Inv. Math. 56 (1980), 129–165.
[3] H. Derksen, J. Weyman, Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients, J. Amer. Math. Soc. 13 (2000), no. 3, 467–479.
[4] S. Donkin, Rational Representations of Algebraic Groups, Lecture Notes in Mathematics 1140, Springer Verlag, 1985.
[5] I. G. MacDonald, Symmetric functions and Hall polynomials, Second edition, Oxford Mathematical Monographs, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
[6] A. Schofield, General Representations of Quivers, Proc. London Math. Soc. (3) 65 (1992), 46–64.
[7] A. Schofield, Semi-invariants of Quivers, J. London Math. Soc. 43 (1991), 383–395.
[8] A. Schofield, M. Van den Bergh, Semi-invariants of quivers for arbitrary dimension vectors, Indag. Math. (N.S.) 12 (2001), no. 1, 125–138.
[9] , J.-P. Wang, Sheaf cohomology on $G/B$ and tensor products of Weyl modules, J. Algebra 77 (1982), 162–185.
[10] J. Weyman, Cohomology of Vector Bundles and Syzygies, Cambridge Tracts in Mathematics 149, Cambridge University Press, 2003.