EIGENVECTORS OF A MATRIX UNDER RANDOM PERTURBATION

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Abstract. In this text, based on elementary computations, we provide a perturbative expansion of the coordinates of the eigenvectors of a Hermitian matrix of large size perturbed by a random matrix with small operator norm whose entries in the eigenvector basis of the first one are independent, centered, with a variance profile. This is done through a perturbative expansion of spectral measures associated to the state defined by a given vector.

1. Introduction

This paper is devoted to the study of the sensitivity of the eigenvectors of a given operator under small perturbations. In the previous paper [4] we studied the effect of a perturbation on the spectrum of a diagonal matrix by a random matrix with small operator norm and whose entries in the eigenvector basis of the first one were independent, centered, with a variance profile. We provided a perturbative expansion of the empirical spectral distribution, but did not consider the deformation of the eigenvectors basis with respect to the canonical basis. In the present paper, to complete this first study, we deal with the spectral measure of our matrix associated to the state defined by a given vector.

To define this measure, let us introduce some notations. We consider a real diagonal matrix $D_n = \text{diag}(\lambda_1, \ldots, \lambda_n)$ (the eigenvalue $\lambda_i$ implicitly depends on $n$), as well as a Hermitian random matrix $X_n = \frac{1}{\sqrt{n}} [x^{n}_{i,j}]_{1 \leq i,j \leq n}$ such that the $x_{ij}$ are independent (up to the symmetry), centered, with a variance profile. The normalizing factor $n^{-1/2}$ and our hypotheses below ensure that the operator norm of $X_n$ is of order one. We then define, for $\varepsilon > 0$,

$$D_n^{\varepsilon} := D_n + \varepsilon X_n.$$ 

If the perturbing matrix belongs to the GOE or GUE, then its law is invariant under this change of basis, hence all the results of this paper apply to any self-adjoint matrix $D_n$.

In contrast with [4], where we studied the empirical spectral measure $\mu^{\varepsilon}_n$ of the matrix $D_n^{\varepsilon}$, we consider here the spectral measure $\mu^{\varepsilon}_{n,e_i}$ of $D_n^{\varepsilon}$ over a vector $e_i$ of the canonical basis, defined

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through an orthonormal eigenbasis \((u_j^\varepsilon)_{j \in \{1, \ldots, n\}}\) of \(D_n^\varepsilon\) and the related eigenvalues \((\lambda_j^\varepsilon)_{j \in \{1, \ldots, n\}}\) by

\[
\mu_n^\varepsilon, e_i := \sum_{j=1}^n |\langle u_j^\varepsilon, e_i \rangle|^2 \delta_{\lambda_j^\varepsilon}.
\]

The interest of these measures is that they give information on the eigenvector basis of \(D_n^\varepsilon\), while being tractable since they satisfy, for any test function \(\varphi\), the key identity

\[
\int \varphi(x) \, d\mu_n^\varepsilon, e_i(x) = \sum_{j=1}^n |\langle u_j^\varepsilon, e_i \rangle|^2 \varphi(\lambda_j^\varepsilon) = (\varphi(D_n^\varepsilon))_{i,i}.
\] (1)

Our main result, Theorem 1, gives a perturbative expansion of \(\mu_n^\varepsilon, e_i\). More precisely, using a resolvent expansion and the Helffer-Sjöstrand formula, we give an asymptotic expansion of

\[
\int_{\mathbb{R}} \varphi(t) \, d\mu_n^\varepsilon, e_i(t)
\]

for any \(C^5\) test function \(\varphi\). From that, we deduce Theorem 2 which establishes the convergence of the average of the square of coordinates of a mesoscopic sequence of consecutive eigenvectors.

It would be indeed tempting to generalize this analysis to the non-diagonal entries of the matrix \(\varphi(D_n^\varepsilon)\). For \(1 \leq l, k \leq n\) the entry \(\varphi(D_n^\varepsilon)_{k,l}\) would give access to the measure \(\sum_{j=1}^n |\langle u_j^\varepsilon, e_l \rangle| |\langle u_j^\varepsilon, e_k \rangle| \delta_{\lambda_j^\varepsilon}\). A result on its asymptotic behavior as the one we have for \(k = l\) can not lead to more information than the mere intensity of \(\langle u_j^\varepsilon, e_l \rangle\) and \(\langle u_j^\varepsilon, e_k \rangle\) separately in terms of the distances \(|j - l|\) and \(|j - k|\). Information on correlations is beyond what we can get with our method.

Some other works, on models closed to our one or contained in it, are devoted to the sensitivity to perturbations of the eigenvectors. Some of them, as [18, 19, 21, 22], provide bounds on the deviations of these eigenvectors under perturbation, while some other, as [1, 2, 3, 7], provide explicit perturbative expansions. This is what we do here, our Theorem 2 shows that the overlaps \(|\langle u_j^\varepsilon, e_l \rangle|^2\) have order \(\varepsilon^2(\lambda_j - \lambda_l)^{-2}n^{-1}\). We cannot prove it for all indices \(i, j\) individually but only in average over some mesoscopic windows. The size of window we have to take is larger than \(n^{11/12}\) which is certainly not optimal as suggested by the recent work of Benigni [7] which has very refined and non perturbative in \(\varepsilon\) results in the special case when \(X_n\) is a Wigner matrix. He makes use, among others, of the sophisticated method of Bourgade and Yau called the eigenvector moment flow [8]. In addition to the fact that it only relies on short and elementary computations, one of the interests of the present paper is to consider rather general perturbations, since we do not suppose that all entries of \(X_n\) have the same variance nor that they are Gaussian. Another interest is to provide, with the functional \(\Xi_s(\varphi)\) from (5) and (9), an expression for the first order expansion of the measure \(\mu_n^\varepsilon, e_i\) from (1), which, up to our knowledge, did not appear so far.

The paper is organized as follows. Statement of Theorem 1 and comments are given in Section 2, whereas its proof is given in Section 4. Section 3 is devoted to the consequence of Theorem 1 on the eigenvectors, namely to Theorem 2, some comments on this result and some figures. Theorem 2 is proved in Section 5.
Notations. For \( u = u_n, v = v_n \) some sequences, \( u \ll v \) means that \( u_n/v_n \) tends to 0.

For a given sequence \( u_n \), we denote by \( O_{L^2}(u_n) \) any sequence \( U_n \) of random variables whose
\( L^2 \) norm \( \mathbb{E}(U_n^2)^{1/2} \) is uniformly bounded by \( Cu_n \) for some \( C > 0 \).

Finally, we denote by \( P_{n \to \infty} \) the convergence in probability for sequences of random variables.

2. Main result

We consider a real diagonal matrix \( D_n = \text{diag}(\lambda_1, \ldots, \lambda_n) \) (the eigenvalue \( \lambda_i \) implicitly depend on \( n \)), as well as a Hermitian random matrix
\[ X_n = \frac{1}{\sqrt{n}} \begin{bmatrix} x_{i,j}^n \end{bmatrix}_{1 \leq i,j \leq n} \]
and define, for \( \varepsilon = \varepsilon_n > 0 \),
\[ D_{\varepsilon}^n := D_n + \varepsilon X_n. \]

We make the following hypotheses:

(a) the entries \( x_{i,j}^n \) of \( \sqrt{n}X_n \) are independent (up to symmetry) random variables, centered, with variance denoted by \( \sigma_n^2(i,j) \), such that \( \mathbb{E}|x_{i,j}^n|^6 \) is bounded uniformly on \( n, i, j \),

(b) there are two bounded real functions, \( f \) and \( \sigma \), defined respectively on \([0, 1]\) and \([0, 1]^2\) such that, denoting \( \lambda_i \) by \( \lambda_{n,i} \) to emphasize the implicit dependence in \( n \), the error bound
\[
\eta_n := \sup_{x \in [0,1]} |\lambda_{n,\lfloor nx \rfloor} - f(x)| + \sup_{(x,y) \in [0,1]^2} |\sigma_n^2(\lfloor nx \rfloor, \lfloor ny \rfloor) - \sigma^2(x,y)|
\]

satisfies
\[
\eta_n \xrightarrow{n \to \infty} 0.
\]

Let us now make some assumptions on the limiting functions \( \sigma \) and \( f \):

(c) the push-forward of the uniform measure on \([0,1]\) by the function \( f \) has a density \( \rho \) with respect to the Lebesgue measure on \( \mathbb{R} \) and a compact support denoted by \( S \),

(d) the variances of the entries of \( X_n \) essentially depend on the eigenspaces of \( D_n \), namely, there exists a symmetric function \( \tau(\cdot, \cdot) \) on \( \mathbb{R}^2 \) such that for all \( x \neq y \), \( \sigma^2(x,y) = \tau(f(x), f(y)) \).

Remark 1. We refer the reader to the end of Section 3 for matrix models satisfying these hypotheses.

Remark 2. The part of assumption (a) concerning the existence of the sixth moment of \( x_{i,j} \) is due to our aim at giving a Taylor type expansion of the Stieltjes transform of the spectral measure. In this respect it is very likely to be optimal. Assumption (c) prevents us considering the case when \( D_n \) is a scalar matrix since its limiting empirical measure has no density.
Remark 3. We cannot generalize our result to the case when $D_n$ is Hermitian non diagonal, unless the perturbations belong to GOE or GUE. The reason is mainly due to assumption (a) of independence of the entries of $X_n$. It seems challenging to study the more general problem assuming the existence of a limiting correlation profile. Looking carefully at the proof of Claim 2 shows that correlations of order $\frac{1}{n}$ do not change the magnitude of our error terms and that we can maintain a statement as long as the correlations are of order $o(1)$.

Let $\mu_{n,e_i}^\varepsilon$ denote the probability measure defined, for any test function $\varphi$, by

$$\int \varphi(t) d\mu_{n,e_i}^\varepsilon(t) := (\varphi(D_n^\varepsilon))_{ii}.$$  

(3)

One can equivalently define $\mu_{n,e_i}^\varepsilon$ by

$$\mu_{n,e_i}^\varepsilon := \sum_{j=1}^n |\langle u_j^\varepsilon, e_i \rangle|^2 \delta_{\lambda_j^\varepsilon},$$

(4)

where $e_i$ denotes the $i$-th vector of the canonical basis, the $\lambda_j^\varepsilon$'s denote the eigenvalues of $D_n^\varepsilon$ and the $u_j^\varepsilon$'s denote the associated eigenvectors.

We now introduce a functional which is central in the statement of our result. This functional admits another expression, given in Proposition 1 below.

Let, for $s \in \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{C}$ a $C^2$ function,

$$\Xi_s(\varphi) := \int_{\mathbb{R}} \tau(s,t) \rho(t) \frac{(\varphi(t) - \varphi(s) - (t-s)\varphi'(s))}{(t-s)^2} dt$$

(5)

Theorem 1. Let us suppose that $\varepsilon = \varepsilon_n \ll n^{-\frac{1}{2}}$. Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a compactly supported $C^7$ function. For $x \in [0,1]$, set $i = i(n,x) = \lfloor nx \rfloor$. Then we have

$$\int_{\mathbb{R}} \varphi(t) d\mu_{n,e_i}^\varepsilon(t) = \varphi\left(\lambda_i + \frac{\varepsilon}{\sqrt{n}} x_{ii}\right) + \varepsilon^2 \Xi_f(\varphi) + \varepsilon^2 O_{L^2}\left(\|\varphi\|_{C^7} \left(\eta_n + \frac{1}{\sqrt{n}} + \varepsilon \sqrt{n}\right)\right)$$

(6)

for $\eta_n$ as in (2).

Remark 4 (Leading order transition). Note that for any $C^2$ test function $\varphi$,

$$\varphi\left(\lambda_i + \frac{\varepsilon}{\sqrt{n}} x_{ii}\right) = \varphi(\lambda_i) + \frac{\varepsilon}{\sqrt{n}} x_{ii} \varphi'(\lambda_i) + O_{L^2}\left(\frac{\varepsilon^2}{n} \|\varphi''\|_{C^7}\right).$$

Thus the previous theorem allows to expand the measure $\mu_{n,e_i}^\varepsilon$ around $\delta_{\lambda_i}$ as follows. With the notations and the hypothesis of the theorem,

$$\int \varphi(t) d\mu_{n,e_i}^\varepsilon(t) = \varphi(\lambda_i) + \frac{\varepsilon}{\sqrt{n}} x_{ii} \varphi'(\lambda_i) + \varepsilon^2 \Xi_f(\varphi)$$

$$+ O_{L^2}\left(\varepsilon^2 \|\varphi\|_{C^6} (\varepsilon n^{\frac{1}{2}} + n^{-\frac{1}{2}} + \eta_n) + \varepsilon^2 \|\varphi''\|_{C^7}\right).$$

(6)

If $\varphi'(\lambda_i) \neq 0$ then the assumption $\varepsilon \ll n^{-\frac{1}{2}}$ implies that the term $\frac{\varepsilon}{\sqrt{n}} x_{ii} \varphi'(\lambda_i)$ prevails over the term $\varepsilon^2 \Xi_f(\varphi)$ but in the following, we will apply Theorem 1 to test functions whose support avoids $\lambda_i$, so that $\varepsilon^2 \Xi_f(\varphi)$ will be the dominant term of the expansion.
Remark 5. Strikingly, the image of a function \( \varphi \) by the operator \( \Xi_f(x) \) is not changed if one adds an affine function to \( \varphi \). This can be understood because the measure \( \mu_{n,e_i}^\varepsilon - \delta_{\lambda_i} \) is of null mass and with first moment of order \( o(\varepsilon^2) \) since by (1),

\[
\int_{\mathbb{R}} x \, d (\mu_{n,e_i}^\varepsilon - \delta_{\lambda_i}) = (D_n)^{\varepsilon}_{ii} - \lambda_i = \frac{\varepsilon}{\sqrt{n}} x_{ii} = o(\varepsilon^2).
\]

Note that when both \( \varphi(f(x)) \) and \( \varphi'(f(x)) \) are null, the function \( \Xi_f(x)(\varphi) \) boils down to the integral

\[
\Xi_f(x)(\varphi) = \int_{\mathbb{R}} \tau(f(x),t) \rho(t) \frac{\varphi(t)}{(t - f(x))^2} dt.
\]

We will use this fact in Section 3 for test functions \( \varphi \) whose support does not contain \( f(x) \).

Proposition 1. Let us define, for any \( s \in \mathbb{R} \), the function \( \zeta_s \) defined on \( \mathbb{R} \) by

\[
\zeta_s(y) := \int_{1}^{+\infty} \frac{r - 1}{r^2} \tau(s, s + r(y - s)) \rho(s + r(y - s)) dr.
\]

Then for any \( C^2 \) function \( \varphi \) and any \( s \in \mathbb{R} \), the functional \( \Xi_s \) defined at (5) rewrites

\[
\Xi_s(\varphi) = \int_{\mathbb{R}} \varphi''(y) \zeta_s(y) dy.
\]

Proof. Taylor’s formula yields

\[
\varphi(t) - \varphi(s) - (t - s)\varphi'(s) = \int_{s}^{t} \varphi''(x)(t - x) dx = (t - s)^2 \int_{u=0}^{1} \varphi''(s + u(t - s))(1 - u) du.
\]

Hence,

\[
\Xi_s(\varphi) = \int_{t \in \mathbb{R}} \int_{u=0}^{1} \varphi''(s + u(t - s))(1 - u) du \, \tau(s,t) \rho(t) dt
\]

We now perform the change of variable \( (r,y) = \Psi_s(u,t) \) with

\[
\Psi_s : (u,t) \in (0,1) \times \mathbb{R} \mapsto (r,y) = \left( \frac{1}{u}, u(t - s) + s \right) \in (1,\infty) \times \mathbb{R}
\]

which gives the result \( \square \)

3. Consequence for the eigenvectors

The purpose of this section is to use the previous results to obtain information on the projection of the eigenvectors on the canonical basis (via moving averages of course, as seeking to obtain a result about eigenvectors one by one would be unrealistic at this level of generality).

Theorem 2. For all sequences \( \alpha_n \) converging to zero and satisfying \( \alpha_n^8 \gg \max \left\{ n^{\frac{1}{2}} \varepsilon, \eta_n, n^{-\frac{1}{2}} \right\} \), for all \( x, x_0 \in [0,1] \) with \( x \neq x_0 \), the following convergence in probability holds,

\[
\frac{n \varepsilon^{-2}}{\text{Card}\{j : |\lambda_j^\varepsilon - f(x)| < \alpha_n\}} \sum_{\{j : |\lambda_j^\varepsilon - f(x)| < \alpha_n\}} |(u_j^\varepsilon, e_{[nx_0]}^\varepsilon)|^2 \overset{P}{\to} \frac{\tau(f(x_0), f(x))}{(f(x) - f(x_0))^2}.
\]
Remark 6. This is a local result since the window where we take our average contains $o(n)$ eigenvectors. However, this $o(n)$ is at least $n^{15/16}$, which is for sure not optimal, as suggested by the recent work of Benigni [7] who gets a very refined result in the special case where the perturbing matrix is Wigner (which implies, among other, that $\tau \equiv 1$). He proves actually that the components of the eigenvectors are asymptotically independent and normal and gets therefore the convergence in probability for any size of window converging to infinity.

We present now two simulations (displayed in Figures 1 and 2) which show a good matching with this theoretical prediction. First we consider the case where the deterministic matrix $D_n$ is perturbed by a Gaussian Wigner matrix, $X_n$. More precisely, we take for $D_n$ the diagonal matrix with $i/n$ as $i^{th}$ entry, so that $f(x) = x$ and the density $\rho$ is equal $x \mapsto 1_{[0,1]}(x)$. The entries of the perturbating matrix $X_n$ are all Gaussian and independent with variance one. Then, we consider the case where the same matrix $D_n$ is perturbed by a band matrix. In other words, we consider now that $\sigma(x,y) = 1_{|x-y| \leq \ell}$, where $\ell \in [0,1]$ is the relative width of the band. Note that in this second example, even though there is absolutely no deterministic reason why $\langle u_{\lfloor ny \rfloor}^\epsilon, e_{\lfloor nx \rfloor} \rangle$ would vanish when $|y-x| > \ell$, we see that at first order, it is actually almost zero (Figure 2). This is related to the question of the localization of the eigenvectors of band random matrices (see e.g. [9, 10, 11, 12, 13, 14, 15, 20]).

![Figure 1. Uniform measure perturbation by a Wigner matrix.](image1)

The red curve represents a moving average of the function $t \in [0,1] \mapsto e^{-2n} |\langle u_{\lfloor n(t-1) \rfloor}^\epsilon, e_{\lfloor nx \rfloor} \rangle|^2$, over a window of length $\frac{1}{\sqrt{n}}$. The blue curve represents our theoretical prediction $t \mapsto |t - x_0|^{-2}$. Here $n = 10^4$, $\epsilon = n^{-0.7}$ and $x_0 = \frac{1}{2}$.

4. Proof of Theorem 1

The proof is divided into two parts. We shall first prove a convergence result for test functions $\varphi$ of the type $\varphi_z := \frac{1}{z-x}$. This is the purpose of Subsection 4.1. It will be obtained by writing an expansion of the resolvent of $D_n^\epsilon$.  

![Figure 2. Band measure perturbation by a Wigner matrix.](image2)
Figure 2. Uniform measure perturbation by a band matrix. The red curve represents a moving average of the function \( t \in [0,1] \mapsto \varepsilon^{-2} n \| \langle u_{[n,t^{-1}(t)]}, e_{[nx_0]} \rangle \|^2 \), over a window of length \( \frac{1}{\sqrt{n}} \). The blue curve represents our theoretical prediction \( t \mapsto \varepsilon \| t - f(x_0) \| \leq \ell \| t - x_0 \| - 2 \). Here \( n = 10^4 \), \( \ell = 0.1 \), \( \varepsilon = n^{-0.7} \) and \( x_0 = \frac{1}{2} \).

Once we have proved that such a convergence holds for the resolvent of \( D_n^\varepsilon \), we will be able to extend it to the class of compactly supported \( C^7 \) functions on \( \mathbb{R} \), by using the Helffer-Sjöstrand formula (see [16] or [5]) which expresses a regular function \( \varphi \) on \( \mathbb{R} \) as an integral against functions \( \varphi_z \) of the previous type. This is done in Subsection 4.2.

4.1. Stieltjes transform. Let us introduce the Banach space \( C^2_{b} \) of bounded \( C^2 \) functions on \( \mathbb{R} \) with bounded first and second derivatives, endowed with the norm \( \| \varphi \|_{C^2_b} := \| \varphi \|_\infty + \| \varphi' \|_\infty + \| \varphi'' \|_\infty \).

On this space, let us define, for \( x \in [0,1] \) and \( i = [nx] \), the random continuous linear form

\[
\Pi_n(\varphi) := \varepsilon^{-2} \left( \int \varphi(t) d\mu_{n,ei}(t) - \varphi(\lambda_i + \frac{\varepsilon}{\sqrt{n}} x_{ii}) \right) - \Xi_f(x)(\varphi).
\]

Lemma 1. There exists a constant \( C > 0 \) such that for all \( z \in \mathbb{C} \setminus \mathbb{R} \),

\[
\mathbb{E}[|\Pi_n(\varphi_z)|^2] \leq C \left( \frac{n^2}{|\Im(z)|^6} + \frac{n\varepsilon^2}{|\Im(z)|^8} + \frac{\varepsilon^4}{n^2|\Im(z)|^{10}} + \frac{\varepsilon^6}{n^3|\Im(z)|^{12}} \right).
\] (10)

Remark 7. This result implies that \( \forall z \in \mathbb{C} \setminus \mathbb{R}, \lim_{n \to \infty} \Pi_n(\varphi_z) = 0 \).

Let us prove the above lemma. We denote, for short, \( x^n_{i,j} \) by \( x_{ij} \) and introduce the diagonal matrix

\[
\widetilde{D}_n^\varepsilon := \text{diag} \left( \left( \lambda_i + \frac{\varepsilon}{\sqrt{n}} x_{ii} \right)_{i=1,...,n} \right)
\] (11)

which is the diagonal part of the matrix \( D_n^\varepsilon \). Note that with this notation and by using identity (1), the quantity we are interested in can be written:

\[
\Pi_n(\varphi_z) = \varepsilon^{-2} \left( (z - D_n^\varepsilon)^{-1} - (z - \widetilde{D}_n^\varepsilon)^{-1} \right)_{ii} - \Xi_f(x)(\varphi_z).
\] (12)
To deal with this quantity we introduce the null diagonal matrix
\[ \tilde{X}_n := \varepsilon^{-1}(D_n^\varepsilon - \tilde{D}_n^\varepsilon) = X_n - n^{-1/2} \text{diag}((x_{ii})_{i=1,\ldots,n}) \]

obtained by vanishing the diagonal of the matrix \( X \).

A perturbative expansion of the resolvent of \( D_n^\varepsilon = \tilde{D}_n^\varepsilon + \varepsilon \tilde{X}_n \) yields
\[
(z - D_n^\varepsilon)^{-1} - (z - \tilde{D}_n^\varepsilon)^{-1} = \varepsilon(z - \tilde{D}_n^\varepsilon)^{-1}\tilde{X}_n(z - \tilde{D}_n^\varepsilon)^{-1} + \varepsilon^2(z - \tilde{D}_n^\varepsilon)^{-1}\tilde{X}_n(z - \tilde{D}_n^\varepsilon)^{-1}\tilde{X}_n(z - \tilde{D}_n^\varepsilon)^{-1} + \varepsilon^3(z - \tilde{D}_n^\varepsilon)^{-1}\tilde{X}_n(z - \tilde{D}_n^\varepsilon)^{-1}\tilde{X}_n(z - \tilde{D}_n^\varepsilon)^{-1}. \tag{13}\]

We now want to analyze the corresponding expansion of \( (z - D_n^\varepsilon)^{-1} - (z - \tilde{D}_n^\varepsilon)^{-1} \) \( \equiv \).

**Claim 1.** For all \( i \in [1,n] \), \( (z - \tilde{D}_n^\varepsilon)^{-1}\tilde{X}_n(z - \tilde{D}_n^\varepsilon)^{-1}) \equiv 0. \)

**Proof.** This comes from the fact that the matrix \( \tilde{X}_n \) has a null diagonal. \(\square\)

**Claim 2.** If, for all \( i \in [1,n] \), we denote
\[
B_n(z,i) := (z - \tilde{D}_n^\varepsilon)^{-1}\tilde{X}_n(z - \tilde{D}_n^\varepsilon)^{-1}\tilde{X}_n(z - \tilde{D}_n^\varepsilon)^{-1},
\]

then, for all \( x \in [0,1] \),
\[
B_n(z,\lfloor nx \rfloor) - \Xi(f(x))(\varphi_x) = O_{L_2}(\eta_n + \frac{1}{\sqrt{n}3|m(z)|^3} + \frac{\varepsilon}{n|3|m(z)|^4} + \frac{\varepsilon^2}{n^2|3|m(z)|^5} + \frac{\varepsilon^3}{n^3|3|m(z)|^6}).
\]

**Proof.** With the notations of (11), the term \( B_n(z,i) \) writes
\[
B_n(z,i) = \frac{1}{n} \sum_{j=1}^{n} \frac{|x_{ij}|^2}{(z - \lambda_n^\varepsilon(i))^2(z - \lambda_n^\varepsilon(j))},
\]

and, for \( x \in [0,1] \),
\[
\Xi(f(x))(\varphi_x) = \int_{t \in \mathbb{R}} \frac{\tau(f(x),t)\rho(t)}{(t - f(x))^2} \left( \frac{1}{z - t} - \frac{1}{z - f(x)} - \frac{t - f(x)}{(z - f(x))^2} \right) dt
\]
\[
= \int_{t \in \mathbb{R}} \frac{\tau(f(x),t)}{(z - f(x))^2(z - t)} \rho(t) dt + \int_{y \in [0,1]} \frac{\sigma^2(x,y)}{(z - f(x))^2(z - f(y))} dy. \tag{14}\]
\[= \int_{y \in [0,1]} \frac{\sigma^2(x,y)}{(z - f(x))^2(z - f(y))} dy. \tag{15}\]
The difference of these quantities writes,

\[
B_n(z, \lfloor nx \rfloor) - \Xi_f(x)(\varphi_z)
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \frac{|x_{\lfloor nx \rfloor}, j|^2 - \sigma_n^2(\lfloor nx \rfloor), j}{(z - \lambda_\epsilon^x(\lfloor nx \rfloor))^2 (z - \lambda_\epsilon^x(j))} + \frac{1}{n} \sum_{j=1}^{n} \frac{\sigma_n^2(\lfloor nx \rfloor), j}{(z - \lambda_\epsilon^x(\lfloor nx \rfloor))^2 (z - \lambda_\epsilon^x(j))} - \frac{\sigma_n^2(\lfloor nx \rfloor), j}{(z - \lambda_{\lfloor nx \rfloor})^2 (z - \lambda_j)} + \int_{y \in [0,1]} \frac{\sigma_n^2(\lfloor nx \rfloor), |ny|}{(z - \lambda_{\lfloor nx \rfloor})^2 (z - \lambda_j)} dy - \int_{y \in [0,1]} \frac{\sigma(x, y)}{(z - f(x))^2 (z - f(y))} dy.
\]

Observe that the second-to-last integral coincides with the discrete sum \(\frac{1}{n} \sum_{j=1}^{n} \frac{\sigma_n^2(\lfloor nx \rfloor), j}{(z - \lambda_{\lfloor nx \rfloor})^2 (z - \lambda_j)}\) since it concerns step functions.

Using the key assumption about the independence of the variables \((x_{i,j})\), the \(L_2\) norm of the first line of the right hand side of the previous equality writes

\[
\left\| \frac{1}{n} \sum_{j=1}^{n} \frac{|x_{i,j}|^2 - \sigma_n^2(i,j)}{(z - \lambda_{\lfloor nx \rfloor})^2 (z - \lambda_j)} \right\|_{L^2} = \frac{1}{n} \left( \sum_{j=1}^{n} \frac{E[(|x_{i,j}|^2 - \sigma_n^2(i,j))^2]}{|z - \lambda_i|^4 |z - \lambda_j|^2} \right)^{\frac{1}{2}} = O\left( \frac{1}{\sqrt{n} |\Re z|^2} \right),
\]

to analyze the \(L^2\) norm of the second line, we write

\[
\frac{1}{n} \sum_{j=1}^{n} \frac{1}{(z - \lambda_{\lfloor nx \rfloor})^2 (z - \lambda_j)} + \frac{1}{n} \sum_{j=1}^{n} \frac{\sigma_n^2(\lfloor nx \rfloor), j}{(z - \lambda_\epsilon^x(\lfloor nx \rfloor))^2 (z - \lambda_\epsilon^x(j))} - \frac{\sigma_n^2(\lfloor nx \rfloor), j}{(z - \lambda_{\lfloor nx \rfloor})^2 (z - \lambda_j)} + \int_{y \in [0,1]} \frac{\sigma_n^2(\lfloor nx \rfloor), |ny|}{(z - \lambda_{\lfloor nx \rfloor})^2 (z - \lambda_j)} dy - \int_{y \in [0,1]} \frac{\sigma(x, y)}{(z - f(x))^2 (z - f(y))} dy.
\]

hence,

\[
\left\| \frac{1}{n} \sum_{j=1}^{n} \frac{1}{(z - \lambda_{\lfloor nx \rfloor})^2 (z - \lambda_j)} - \frac{1}{(z - \lambda_{\lfloor nx \rfloor})^2 (z - \lambda_j)} \right\|_{L^2} \leq \frac{\varepsilon}{\sqrt{n}} \frac{2|x_{\lfloor nx \rfloor}, j|}{|\Im z|^4} + \frac{\varepsilon^2}{n} \frac{2|x_{\lfloor nx \rfloor}, |nx| |x_{j,j}|^2}{|\Im z|^5} + \frac{\varepsilon^3}{n^2} \frac{|x_{\lfloor nx \rfloor}, |nx| |x_{j,j}|^2}{|\Im z|^6}.
\]

Therefore,
the entry is equal to 1, the Cauchy-Schwarz inequality yields

\[ \left\| \frac{1}{n} \sum_{j=1}^{n} \frac{\sigma_n^2((nx)_j)}{(z-\lambda_n((nx)_j))^2} - \frac{\sigma_n^2((nx)_j)}{(z-\lambda_n((nx)_j))^2(z-\lambda_j)} \right\|_{L^2} = O \left( \frac{\epsilon}{\sqrt{n|\Im(z)|^4}} + \frac{\epsilon^2}{n|\Im(z)|^6} + \frac{\epsilon^3}{n^2|\Im(z)|^6} \right). \]

and, finally, from assumption (b), the third line is \( O(\eta_n|\Im z|^{-3}) \).

Claim 3. For all \( i \in [1, n] \),

\[ \left( \frac{z - \tilde{D}_n^\epsilon}{n} \right)^{-1} \tilde{X} \left( \frac{z - \tilde{D}_n^\epsilon}{n} \right)^{-1} \tilde{X} \left( \frac{z - D_n^\epsilon}{n} \right)^{-1} = O_{L^2}(n^{\frac{3}{2}}|\Im z|^{-4}) \]

Proof. If we denote \( E_{i,i} \) the matrix with null entries everywhere except in position \((i, i)\) where the entry is equal to 1, the Cauchy-Schwarz inequality yields

\[
\mathbb{E} \left[ \left| \left( \frac{z - \tilde{D}_n^\epsilon}{n} \right)^{-1} \tilde{X} \left( \frac{z - \tilde{D}_n^\epsilon}{n} \right)^{-1} \tilde{X} \left( \frac{z - D_n^\epsilon}{n} \right)^{-1} \right|_{ii} \right]^2 \leq \mathbb{E} \left[ \text{Tr} \left( E_{i,i} \left( \frac{1}{z - \tilde{D}_n^\epsilon} \tilde{X} \frac{1}{z - \tilde{D}_n^\epsilon} \tilde{X} \frac{1}{z - D_n^\epsilon} \right) \right) \right]^2 \leq \mathbb{E} \left[ \text{Tr} \left| E_{i,i} \left( \frac{1}{z - \tilde{D}_n^\epsilon} \tilde{X} \right) \right|^2 \right] \cdot \mathbb{E} \left[ \text{Tr} \left| \frac{1}{z - D_n^\epsilon} \right|^2 \right].
\]

Let us now observe that since the spectra of \( \tilde{D}_n^\epsilon \) and \( D_n^\epsilon \) are real, the moduli of the entries of \( (z - \tilde{D}_n^\epsilon)^{-1} \) and \( (z - D_n^\epsilon)^{-1} \) are smaller than \( |\Im(z)|^{-1} \), which implies that,

\[ \text{Tr} \left| \frac{1}{z - D_n^\epsilon} \right|^2 \leq \frac{n}{|\Im(z)|^2}. \]

Hence, the right hand side of (16) is bounded by

\[
\frac{n^{\frac{3}{2}}}{|\Im(z)|} \mathbb{E} \left[ \sum_{l=1}^{n} \left( \frac{1}{z - \tilde{D}_n^\epsilon} \tilde{X}_n \right)_{i,l} \left( \frac{1}{z - \tilde{D}_n^\epsilon} \tilde{X}_n \right)_{i,l} \right]^\frac{1}{2} \leq \frac{n^{\frac{3}{2}}}{|\Im(z)|} \mathbb{E} \left[ \sum_{j,k,l,m,p=1}^{n} \frac{(\tilde{X}_n)_{i,j} (\tilde{X}_n)_{j,k} (\tilde{X}_n)_{k,l} (\tilde{X}_n)_{i,m} (\tilde{X}_n)_{m,p} (\tilde{X}_n)_{p,l}}{|z - \tilde{\lambda}_n(i)| \left| (z - \tilde{\lambda}_n(j)) \right| \left| (z - \tilde{\lambda}_n(k)) \right| \left| (z - \tilde{\lambda}_n(m)) \right| \left| (z - \tilde{\lambda}_n(p)) \right|}^\frac{1}{2}.
\]

Recall that the diagonal of the matrix \( \tilde{X}_n \) is null, hence the denominators of the terms of the previous sum are independent from the numerators. Moreover the expectation of the numerators are null except when the set of indices \( \{(i, j), (j, k), (k, l)\} \) are equal to the set \( \{(i, m), (m, p), (p, l)\} \). Therefore, the complexity of the previous sum is \( O(n^3) \).
Moreover, for all indices \( j, k, l, m, p \),

\[
E \left[ \frac{1}{|z - \tilde{\lambda}^\varepsilon(i)|^2 \left( z - \tilde{\lambda}^\varepsilon(j) \right) \left( z - \tilde{\lambda}^\varepsilon(k) \right) \left( z - \lambda_n^\varepsilon(m) \right) \left( z - \lambda_n^\varepsilon(p) \right)} \right] \leq \frac{1}{|\text{Im}(z)|^6}
\]

and since the \( L^6 \) norm of the entries of \( \sqrt{n}X \) is finite, we get that, uniformly in the indices \( j, k, l, m, p \),

\[
E \left[ (\tilde{X}_n)_{i,j} (\tilde{X}_n)_{j,k} (\tilde{X}_n)_{k,l} (\tilde{X}_n)_{i,m} (\tilde{X}_n)_{m,p} (\tilde{X}_n)_{p,l} \right] = O(n^{-3}).
\]

Hence,

\[
E \left[ \sum_{j,k,l,m,p=1}^{n} \frac{(\tilde{X}_n)_{i,j} (\tilde{X}_n)_{j,k} (\tilde{X}_n)_{k,l} (\tilde{X}_n)_{i,m} (\tilde{X}_n)_{m,p} (\tilde{X}_n)_{p,l}}{|z - \tilde{\lambda}^\varepsilon(i)|^2 \left( z - \tilde{\lambda}^\varepsilon(j) \right) \left( z - \tilde{\lambda}^\varepsilon(k) \right) \left( z - \lambda_n^\varepsilon(m) \right) \left( z - \lambda_n^\varepsilon(p) \right)} \right]^\frac{1}{2} \leq \frac{C}{|\text{Im}(z)|^3}.
\]

Therefore,

\[
E \left[ \left( (z - D_n^\varepsilon)^{-1} \tilde{X}_n (z - D_n^\varepsilon)^{-1} \tilde{X}_n (z - D_n^\varepsilon)^{-1} \tilde{X}_n (z - D_n^\varepsilon)^{-1} \right)_{ii} \right]^\frac{1}{2} \leq \frac{Cn^{\frac{1}{2}}}{|\text{Im}(z)|^4}.
\]

Gathering Formulas (12), (13) and Claims 1, 2 and 3, we prove Lemma 1.

4.2. From Stieltjes transform to \( C^7 \) functions. Now, let \( \varphi \) be a \( C^7 \) function on \( \mathbb{R} \) with bounded seventh derivative and let us introduce the almost analytic extension of degree 7 of \( \varphi \) defined by

\[
\forall z = x + iy \in \mathbb{C}, \quad \tilde{\varphi}_6(z) := \sum_{k=0}^{6} \frac{1}{k!} (iy)^k \varphi^{(k)}(x).
\]

An elementary computation gives, by successive cancellations, that

\[
\bar{\partial} \tilde{\varphi}_6(z) = \frac{1}{2} (\partial_x + i\partial_y) \tilde{\varphi}_6(x + iy) = \frac{1}{2 \times 6!} (iy)^6 \varphi^{(7)}(x). \tag{17}
\]

Furthermore, by Helffer-Sjöstrand formula [4, Propo. 9], for \( \chi \in C^\infty_c(\mathbb{C}; [0, 1]) \) a smooth cutoff function with value one on the support of \( \varphi \),

\[
\varphi(\cdot) = -\frac{1}{\pi} \int_\mathbb{C} \frac{\bar{\partial}(\tilde{\varphi}_6(z)(\chi(z))}{y^6} y^6 \varphi_z(\cdot) \, d^2z
\]

where \( d^2z \) denotes the Lebesgue measure on \( \mathbb{C} \).

Note that by (17), \( z \mapsto \mathbb{1}_{y \neq 0} \frac{\partial(\tilde{\varphi}_6(z)(\chi(z))}{y^6} y^6 \varphi_z \) is a continuous compactly supported function and that \( z \in \mathbb{C} \mapsto \mathbb{1}_{y \neq 0} y^6 \varphi_z \in C^1_0 \) is continuous, hence,

\[
\Pi_n(\varphi) = \frac{1}{\pi} \int_\mathbb{C} \frac{\bar{\partial}(\tilde{\varphi}_6(z)(\chi(z))}{y^6} y^6 \Pi_n(\varphi_z) \, d^2z.
\]
Therefore, using the Cauchy-Schwarz inequality and the fact that \( \chi \) has compact support at the second step, for a certain constant \( C \), we have

\[
\mathbb{E} \left( |\Pi_n(\varphi)|^2 \right) = E \left( \left| \frac{1}{\pi} \int_{\mathcal{C}} \frac{\partial (\varphi_{\circ}(z))}{y^6} y^6 \Pi_n(\varphi_z) \, d^2z \right|^2 \right) \\
\leq C \mathbb{E} \left( \int_{\mathcal{C}} \left| \frac{\partial (\varphi_{\circ}(z))}{y^6} \right| y^6 \Pi_n(\varphi_z) \, d^2z \right) \\
= C \int_{\mathcal{C}} \left| \frac{\partial (\varphi_{\circ}(z))}{y^6} \right|^2 y^{12} \mathbb{E} \left( |\Pi_n(\varphi_z)|^2 \right) \, d^2z.
\]

By (17), the function \( \left| \frac{\partial (\varphi_{\circ}(z))}{y^6} \right|^2 \) is continuous and compactly supported and bounded by \( C\|\varphi^{(7)}\|_\infty^2 \) for some constant \( C \). Besides, by Lemma 1, uniformly in \( z \),

\[
y^{12} \mathbb{E} \left( |\Pi_n(\varphi_z)|^2 \right) = O \left( (1 + y^6) \left( \eta_n + \frac{1}{n} + n\varepsilon^2 \right) \right).
\]

We deduce that

\[
\mathbb{E} \left( |\Pi_n(\varphi)|^2 \right) \leq C \int_{\mathcal{C}} \left| \frac{\partial (\varphi_{\circ}(z))}{y^6} \right|^2 y^{12} \mathbb{E} \left( |\Pi_n(\varphi_z)|^2 \right) \, d^2z = O \left( \|\varphi^{(7)}\|_\infty^2 \left( \eta_n + \frac{1}{n} + n\varepsilon^2 \right) \right),
\]

which closes the proof of Theorem 1.

5. Proof of Theorem 2

Let us start with the study of the term \( \text{Card}\{j : |\lambda_j^\varepsilon - f(x)| < \alpha_n\} \). By Weyl's inequalities on the eigenvalues of sum of operators (see [17, Cor. 4.3.15]), the ordered eigenvalues of the \( D_n^\varepsilon \) and \( D_n \) do not differ by more than \( \varepsilon \|X_n\|_{\text{op}} \). Therefore, we have, with probability tending to one,

\[
\text{Card}\{j : |\lambda_j - f(x)| < \alpha_n - \varepsilon \|X_n\|_{\text{op}}\} \leq \text{Card}\{j : |\lambda_j^\varepsilon - f(x)| < \alpha_n\} \leq \text{Card}\{j : |\lambda_j - f(x)| < \alpha_n + \varepsilon \|X_n\|_{\text{op}}\}.
\]

Since the variances of \( x_{i,j}^n \) are uniformly bounded by \( C \) for some \( C > 0 \), and that the entries \( x_{i,j}^n \) have finite moment of order 6, the assumptions of Lemma 2 of the Appendix are satisfied by Tchebychev inequality and there exists a constant \( C \) such that \( \mathbb{P}(\|X_n\|_{\text{op}} > C) \) converges to zero. Hence, since \( \varepsilon \ll \alpha_n \), the cardinality of \( \{j : |\lambda_j - f(x)| < \alpha_n - \varepsilon \|X_n\|_{\text{op}}\} \) and of \( \{j : |\lambda_j - f(x)| < \alpha_n + \varepsilon \|X_n\|_{\text{op}}\} \) are asymptotically equal to \( 2n\alpha_n \rho(x)(1 + o(1)) \).

Henceforth, for any measure \( \mu \) and integrable function \( \varphi \), we use the convenient notation, \( \mu(\varphi) := \int \varphi \, d\mu \).

Let us turn to the estimation of the sum \( \sum_{\{j : |\lambda_j - f(x)| < \alpha_n\}} |\langle u_j^\varepsilon, e_{\lfloor nx \rfloor} \rangle|^2 \). Denoting \( \varphi_{x,\alpha_n}(t) := 1_{t \in [f(x) - \alpha_n, f(x) + \alpha_n]} \), the previous sum is nothing but \( \mu_{\varepsilon, e_{\lfloor nx \rfloor}}(\varphi_{x,\alpha_n}) \). We want to apply Theorem 1, but since \( \varphi_{x,\alpha_n} \) is not smooth, we bound it from above and below after introducing some \( \omega_n \ll \alpha_n \) we will calibrate further.
With the use of a decreasing smooth function satisfying $\psi|_{\mathbb{R}^-} = 1$ and $\psi|_{[1, \infty)} = 0$, we can bound $\varphi_{x,\alpha_n}$ by two smooth functions $\varphi^-_{x,\alpha_n,\omega_n}$ and $\varphi^+_{x,\alpha_n,\omega_n}$ defined by

$$\varphi^-_{x,\alpha_n,\omega_n}(t) := \psi \left( 1 + \frac{t - f(x) - \alpha_n}{\omega_n} \right) \left( 1 - \frac{t - f(x) + \alpha_n}{\omega_n} \right),$$

$$\varphi^+_{x,\alpha_n,\omega_n}(t) := \psi \left( \frac{t - f(x) - \alpha_n}{\omega_n} \right) \psi \left( \frac{t - f(x) + \alpha_n}{\omega_n} \right).$$

The properties of these functions are illustrated on the picture below.

Since $\mu_{n,e_{[nx_0]}}^\varepsilon$ is a positive measure our quantity of interest $\mu_{n,e_{[nx_0]}}^\varepsilon(\varphi_{x,\alpha_n})$ is bounded respectively from below and above by $\mu_{n,e_{[nx_0]}}^\varepsilon(\varphi^-_{x,\alpha_n,\omega_n})$ and by $\mu_{n,e_{[nx_0]}}^\varepsilon(\varphi^+_{x,\alpha_n,\omega_n})$. Therefore, we just have to prove that each of them is asymptotically equal in probability to

$$2\varepsilon^2 \alpha_n \frac{\tau(f(x_0), f(x))}{(f(x) - f(x_0))^2} \mu(f(x)).$$

We examine all the quantities of Theorem 1. Obviously, since $\omega_n \ll \alpha_n$ and since the support of $\varphi^-_{x,\alpha_n,\omega_n}$ and $\varphi^+_{x,\alpha_n,\omega_n}$ both avoid $f(x_0)$, the deterministic quantities $\varepsilon^2 \Xi f(x_0)(\varphi^-_{x,\alpha_n,\omega_n})$ and $\varepsilon^2 \Xi f(x_0)(\varphi^+_{x,\alpha_n,\omega_n})$ are asymptotically equal to the desired quantity announced before.

Now, since $\lambda_{[nx_0]} + \frac{\varepsilon}{\sqrt{n}} x_{[nx_0], [nx_0]}$ converges in probability to $f(x_0)$ which is outside the support of $\varphi^-_{x,\alpha_n,\omega_n}$ and $\varphi^+_{x,\alpha_n,\omega_n}$ the quantity $\frac{\varepsilon^2}{\alpha_n} \varphi^-_{x,\alpha_n,\omega_n}(\lambda_{[nx_0]} + \frac{\varepsilon}{\sqrt{n}} x_{[nx_0], [nx_0]})$ converges in probability to zero.

Finally, the error term in Theorem 1 is, in $L^2$, of order $\varepsilon^2 \|\varphi^\pm_{x,\alpha_n,\omega_n}\|_{\infty} (n^{1/2} \varepsilon + n^{-1/2} + \eta_n)$ which in turn is of order $\frac{\varepsilon^2}{\omega_n} \left( n^{1/2} \varepsilon + n^{-1/2} + \eta_n \right)$. It is now time to calibrate $\omega_n$ such that $\omega_n \ll \alpha_n$ and $\frac{\varepsilon^2}{\omega_n} \left( n^{1/2} \varepsilon + n^{-1/2} + \eta_n \right) \ll \varepsilon^2 \alpha_n$. This is possible if and only if $\alpha_n \gg \max\{n^{1/2} \varepsilon, n^{-1/2}, \eta_n\}$. This closes the proof of Theorem 2.
6. Appendix

This appendix is devoted to the control of the operator norm of the random matrix $X_n$ that we use in the proof of Theorem 2. We did not find any reference for the following lemma in the literature, so we give a proof.

**Lemma 2.** Let $H = (H_{ij})_{1 \leq i,j \leq n}$ be an $n \times n$ random Hermitian matrix satisfying:

- The random variables $(H_{ij})_{1 \leq i \leq j \leq n}$ are independent, centered and satisfy $n \mathbb{E} |H_{ij}|^2 \leq 1$.
- For some constants $C_0 > 0$, $\alpha > 4$, we have, for any $t > 0$,
  $$\mathbb{P}(\sqrt{n} |H_{ij}| \geq t) \leq C_0 t^{-\alpha}.$$

Then for any $\varepsilon > 0$, there is $C$ depending only on $\varepsilon$, $C_0$ and $\alpha$ such that
  $$\mathbb{P}(\|H\| \geq 2 + \varepsilon) \leq C n^{-\frac{\alpha-4}{4}}.$$

**Proof.** Let $H' := (H_{ij} \mathbb{1}_{[H_{ij}] < n^{-\beta}})_{1 \leq i,j \leq n}$ for $\beta := \frac{\alpha-4}{4\alpha}$. Note that by the union bound,
  $$\mathbb{P}(H \neq H') \leq n^2 C_0 n^{-\alpha(1/2-\beta)} = C_0 n^{-\frac{\alpha-4}{4}},$$
so that it is enough to prove the result for $H'$ instead of $H$. The matrix $H'$ satisfies the assumptions of [6, Th. 2.6] for $\kappa = 1$ and $q := n^{\max\{1/3,\beta\}}$, so, by this theorem and [6, Eq. (2.4)], we know that for some universal positive constants $C_1, c$, for any $t \geq 0$,
  $$\mathbb{P}(\|H'\| \geq 2 + C_1 \sqrt{\log n/q} + t) \leq 2e^{-cq^2 t^2},$$
which allows to conclude. \qed

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