Supersymmetric solutions and Borel singularities for $N = 2$ supersymmetric Chern-Simons theories

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In supersymmetric (SUSY) field theory, there exist configurations which formally satisfy SUSY conditions but are not on original path integral contour. We refer to such configurations as complexified supersymmetric solutions (CSS). In this paper we discuss that CSS provide important information on large order behavior of weak coupling perturbative series in SUSY field theories. We conjecture that CSS with a bosonic (fermionic) free parameter give poles (zeros) of Borel transformation of perturbative series whose locations are uniquely determined by actions of the solutions. We demonstrate this for various SUSY observables in 3d $\mathcal{N} = 2$ SUSY Chern-Simons matter theories on sphere. First we construct infinite number of CSS in general $3d \mathcal{N} = 2$ SUSY theory with Lagrangian where adjoint scalar in vector multiplet takes a complex value and matter fields are nontrivial. Then we compare their actions with Borel transformations of perturbative expansions by inverse Chern-Simons levels for the observables and see agreement with our conjecture. It turns out that the CSS explain all the Borel singularities for this case.

I. INTRODUCTION

Supersymmetric (SUSY) field theories have provided good laboratories to find and sharpen ideas in theoretical physics. In SUSY field theories, configurations preserving a part of SUSY play prominent roles because of their stabilities, restricted quantum corrections and so on. While most of attention has been paid to SUSY solutions which take values on original path integral contour, there also exist configurations which formally satisfy SUSY conditions but are not on the original contour. We refer to such configurations as complexified supersymmetric solutions (CSS). Compared to standard SUSY solutions, CSS have been much less appreciated. Purpose of this paper is to point out a physical role of CSS.

Since SUSY conditions are often sufficient to satisfy saddle point equation, CSS are also often complex saddle points which are saddle points but not on original path integral contour. Therefore it is worth to note historical situations on complex saddle points in quantum field theory (QFT). While saddle points along original contour always contribute to path integral, complex saddles may or may not contribute since this is determined by details on steepest descents associated with them. Historically most physicists have not taken this possibility into account in saddle point analysis of path integral despite its importance was pointed out long time ago [1]. Namely, people have usually considered only saddles on the original contour and not discussed whether or not complex saddles contribute. Recently there appeared relatively more examples in the context of resurgence where some complex saddles actually give important contributions to path integral, and are necessary to obtain unambiguous answer from resummation of weak coupling perturbative series [2]. Although the examples are much simpler than typical interacting QFT, which are either quantum mechanics, two dimensional or topological, there is no a priori reason to doubt that similar things happen in more general setup. However, it is usually difficult to check this explicitly since we need to construct complex saddles and find large order behaviors of perturbative series around all the contributing saddles. Therefore SUSY field theories would provide good next steps to understand roles of complex saddles in QFT.

The above facts suggest that CSS may importantly contribute to path integral. Indeed recent works on SUSY localization [3] imply that CSS sometimes give important contributions [4–6]. In this paper we do not discuss whether CSS are contributing saddles or not. Here we propose that independently of this question, CSS provide important information on large order behavior of weak coupling perturbative series in SUSY field theories.

Perturbative series in QFT is typically non-convergent [3]. One of standard ways to resum non-convergent series is Borel resummation. Given a formal series $P(g) = \sum_{\ell=0}^{\infty} c_{\ell} g^{\ell+\ell}$, its Borel resummation along the direction $\theta$ is defined by

$$S_\theta P(g) = \int_0^\infty e^{it} \, dt \, e^{-\frac{\theta}{2} BP(t)},$$

where $BP(t)$ is analytic continuation of the formal Borel transformation $\sum_{\ell=0}^{\infty} \frac{2\ell}{(\alpha + \ell)} g^{\ell+\ell-1}$. In general Borel transformation have singularities in complex $t$-plane and their locations contain important information. Technically they determine whether or not the perturbative series is Borel summable given $\theta$, and large order behavior of perturbative series as they give radius of convergence of the Borel transformation. Physically, according to Lipatov’s argument [20], Borel singularities are interpreted as non-trivial saddle points and their actions determine locations of the singularities.

In this paper we conjecture a relation between CSS
and analytic properties of Borel transformation in SUSY field theories. Let us consider weak coupling perturbative expansion \( P(g) \) around a saddle point with certain topological numbers. In general there exist two types of CSS with the same topological numbers: one has a bosonic parameter while the other has a fermionic one. We refer to them as bosonic and fermionic complexified supersymmetric solutions, respectively. We propose that bosonic CSS basically give poles of Borel transformation of \( P(g) \) while fermionic CSS give zeroes. More precisely if there are \( n_B \) bosonic and \( n_F \) fermionic solutions with the topological number and the action \( S = S_c/g \), then we conjecture that the Borel transformation has a pole for \( n_B \geq n_F \) and zero for \( n_B \leq n_F \) with degree \( |n_B - n_F| \) at \( t = S_c \). Namely the Borel transformation includes the following factor

\[
BP^{(f)}(t) \supset \prod_{\text{solutions}} \frac{1}{(t - S_c)^{n_B - n_F}}. \tag{2}
\]

Note that CSS gives information not only on locations of singularities but also on their degrees. This leads us to further insights into non-perturbative structures of path integrals.

In next sections we demonstrate the above story for various SUSY observables in 3d \( \mathcal{N} = 2 \) SUSY Chern-Simons (CS) matter theory on sphere \([26]\). In sec. III we construct infinite number of CSS in general 3d \( \mathcal{N} = 2 \) theory with Lagrangian. They are described by zero-modes of differential operators appearing in SUSY transformations of chiral multiplets \([26]\). We cannot have such zero-modes for most case when adjoint scalar \( \sigma \) in vector multiplet is real i.e. on the original path integral contour, but relaxing the reality condition allows such zero-modes. In sec. IV we compare their actions with explicit Borel transformation found in \([9]\) (see also \([11, 12]\)) and see agreement with our conjecture.

II. CONSTRUCTION OF COMPLEXIFIED SUPERSYMMETRIC SOLUTIONS

Let us construct complexified supersymmetric solutions in general 3d \( \mathcal{N} = 2 \) SUSY gauge theories with Langragians on a squashed \( S^3 \). We choose the squashed \( S^3 \) to be ellipsoid \( \mathbb{S}^3_b \) defined as the hypersurface in \( \mathbb{R}^4 \):

\[
(x_1^2 + x_2^2 + b^2(x_3^2 + x_4^2)) = \text{Const..}
\]

The round \( S^3 \) corresponds to the limit \( b \to 1 \). As we will see, the relation between CSS and Borel singularities are more transparent for generic \( b \) rather than the \( b \to 1 \) case because actions of many solutions become coincident as \( b \to 1 \).

A. Vector multiplet

First we find CSS for the 3d \( \mathcal{N} = 2 \) vector multiplet, which consists of gauge field \( A_\mu \), adjoint scalar \( \sigma \), gaugino \( (\lambda, \bar{\lambda}) \) and auxiliary field \( D \). For this purpose, we need only the SUSY transformations \([27]\):

\[
\delta A_\mu = \frac{-1}{2} \bar{\lambda} \gamma_\mu \epsilon - \frac{1}{2} \bar{\epsilon} \gamma_\mu \lambda, \quad \delta \sigma = \frac{i}{2} \bar{\lambda} \epsilon + \frac{i}{2} \bar{\epsilon} \lambda, \\
\delta \lambda = (\frac{1}{2} \epsilon \mu \rho \epsilon \nu \rho - D_\mu \sigma) \gamma^{\mu \nu} \epsilon - iD \epsilon - \frac{i}{f(\bar{\sigma})} \sigma \epsilon, \\
\delta \bar{\lambda} = (\frac{1}{2} \epsilon \mu \rho \epsilon \nu \rho + D_\mu \sigma) \gamma^{\mu \nu} \bar{\epsilon} + iD \bar{\epsilon} + \frac{i}{f(\sigma)} \bar{\sigma} \epsilon, \\
\delta D = \frac{1}{2} \bar{\epsilon} \gamma^{\mu} D_\mu \lambda \bar{\lambda} - \frac{1}{2} \bar{D} \lambda \bar{\lambda} + \frac{i}{2} [\lambda, \sigma] + \frac{1}{2} [\bar{\lambda}, \bar{\sigma}] \\
- \frac{i}{4f(\bar{\sigma})} (\epsilon \lambda + \bar{\sigma} \epsilon), \tag{3}
\]

where we take the coordinate \((x_1, x_2, x_3, x_4) = (\cos \vartheta \cos \varphi, \cos \vartheta \sin \varphi, \sin \vartheta \cos \chi, \sin \vartheta \sin \chi)\) and \( f(\bar{\sigma}) = \sqrt{b^2 \sin^2 \vartheta + b^2 \cos^2 \vartheta} \). Our SUSY condition is simply vanishing of all the transformations \([27]\). As a sufficient condition, we can solve the SUSY condition by \([28]\):

\[
F_{\mu \nu} = 0, \quad \sigma = \text{const.}, \quad D = -\frac{\sigma}{f(\bar{\sigma})}, \quad \lambda = \bar{\lambda} = 0. \tag{4}
\]

This is just Coulomb branch solution for real \( \sigma \) but the SUSY condition can be formally satisfied even for complex \( \sigma \). Next we find CSS for chiral multiplet under the configuration \([11]\). It turns out that \( \sigma \) is fixed to a particular complex value by SUSY condition for chiral multiplet.

B. Chiral multiplet

The 3d \( \mathcal{N} = 2 \) chiral multiplet consists of scalars \((\phi, \bar{\phi})\), fermions \((\psi, \bar{\psi})\) and auxiliary fields \((F, \bar{F})\), whose SUSY transformations are

\[
\delta \phi = i \bar{\psi} \epsilon, \quad \delta \bar{\phi} = i \psi \epsilon, \\
\delta \psi = -\gamma^\mu \epsilon D_\mu \phi - \epsilon \sigma \phi - \frac{i \Delta}{f(\bar{\sigma})} \epsilon \phi + i \bar{\epsilon} F, \\
\delta \bar{\psi} = -\gamma^\mu \epsilon D_\mu \bar{\phi} - \bar{\sigma} \bar{\epsilon} \bar{\psi} - \frac{i \Delta}{f(\bar{\sigma})} \bar{\epsilon} \bar{\psi} + i \bar{F} \epsilon, \\
\delta F = \epsilon (\gamma^\mu D_\mu \psi + \sigma \psi + \lambda \phi) + \frac{i(2\Delta - 1)}{2f(\bar{\sigma})} \epsilon \psi, \\
\delta \bar{F} = \epsilon (\gamma^\mu D_\mu \bar{\psi} + \bar{\sigma} \bar{\psi} - \bar{\lambda} \bar{\phi}) + \frac{i(2\Delta - 1)}{2f(\bar{\sigma})} \epsilon \bar{\psi}, \tag{5}
\]

where we have assigned the \( U(1)_R \)-charges \((-\Delta, \Delta, 1 - \Delta, 1 - 2\Delta, -\Delta, -2\Delta)\) to \((\phi, \bar{\phi}, \psi, \bar{\psi}, F, \bar{F})\) respectively. Under the vector multiplet configuration \([11]\), we find two types of CSS where one of them has a bosonic parameter while the other has a fermionic parameter. Below we take \( \sigma \) to be Cartan part by using the gauge symmetry.

1. Bosonic solutions

First we look for solutions with trivial \((\psi, \bar{\psi}, F, \bar{F})\) but nontrivial \((\phi, \bar{\phi})\), namely

\[
\psi = \bar{\psi} = F = \bar{F} = 0, \tag{6}
\]
which leads us to \( \delta \phi = \delta \bar{\phi} = \delta F = \delta \bar{F} = 0 \). A non-trivial condition for \( \phi \) comes from \( \delta \psi = 0 \). Decomposing the fields into components associated with weight vector \( \rho \) of gauge group representation, the condition for the component \( \phi^\rho \) is

\[
\gamma^\mu \epsilon D_\mu \phi^\rho + \epsilon (\rho \cdot \sigma) \phi^\rho + \frac{i\Delta}{f(\bar{\theta})} \epsilon \phi^\rho = 0. \tag{7}
\]

We can easily find solutions for this condition because eigenvalue problem of the differential operator above is already solved in [13] in the context of SUSY localization of \( S^3 \) partition function. In order to compute its one-loop determinant for the chiral multiplet, [13] studied the eigenvalue problem

\[
\gamma^\mu \epsilon D_\mu \Phi + \epsilon (\rho \cdot \sigma) \Phi + \frac{i\Delta}{f(\bar{\theta})} \epsilon \Phi = M e^\Phi, \tag{8}
\]

where \( M \) is the eigenvalue given by

\[
M = M_{m,n} = \rho \cdot \sigma + i \left( nb + nb^{-1} + \frac{Q}{2} \Delta \right),
\text{ with } m, n \in \mathbb{Z}_{\geq 0}, \ Q = b + b^{-1}. \tag{9}
\]

The product of the eigenvalues roughly gives “denominator” of the one-loop determinant. Our SUSY condition is satisfied for \( M = 0 \) but real \( \sigma \) cannot satisfy this for generic \((b, \Delta)\). If we relax this, denoting eigenmode with the eigenvalue \( M_{m,n} \) by \( \Phi_{m,n} \), we have the SUSY solutions

\[
\phi = \Phi_{m,n}, \quad \rho \cdot \sigma = \rho \cdot \sigma^\text{bos}_{m,n} = -i \left( nb + nb^{-1} + \frac{Q}{2} \Delta \right). \tag{10}
\]

Note that these conditions do not uniquely determine value of \( \phi \) since we have freedom to change an overall constant.

2. Fermionic solutions

Next we find solutions with trivial \((\phi, \bar{\phi}, F, \bar{F})\) but non-trivial \((\psi, \bar{\psi})\), namely

\[
\phi = \bar{\phi} = F = \bar{F} = 0, \tag{11}
\]

which gives \( \delta \psi = \delta \bar{\psi} = 0 \). Then nontrivial SUSY conditions for \( \psi \) come from vanishing of \( \delta \phi \) and \( \delta F \):

\[
\epsilon \psi^\rho = 0, \quad \epsilon (-\gamma^\mu D_\mu + \rho \cdot \sigma) \psi^\rho + \frac{i(2\Delta - 1)}{2f(\bar{\theta})} \epsilon \psi^\rho = 0. \tag{12}
\]

Again we can easily find solution of this condition because [13] also solved the eigenvalue problem

\[
\epsilon (-\gamma^\mu D_\mu \Psi + \rho \cdot \sigma \Psi) + \frac{i(2\Delta - 1)}{2f(\bar{\theta})} \epsilon \Psi = M e^\Psi,
\]

\[
M = M_{m,n} = \rho \cdot \sigma - i \left( nb + nb^{-1} - \frac{Q(\Delta - 2)}{2} \right), \tag{13}
\]

to compute “numerator” of the one-loop determinant. The SUSY condition is satisfied for \( M = 0 \) but again we need to take complex \( \sigma \). Thus, denoting eigenmode with the eigenvalue \( M_{m,n} \) by \( \Psi_{m,n} \), we have the SUSY solutions

\[
\epsilon \psi^\rho = 0, \quad \psi^\rho = \Psi_{m,n}, \quad \rho \cdot \sigma = \rho \cdot \sigma^\text{fer}_{m,n} = i \left( mb + nb^{-1} - \frac{Q(\Delta - 2)}{2} \right). \tag{14}
\]

So far we have focused on the particular component \( \rho \) of single chiral multiplet. In general we have various chiral multiplets with various representations and each of those has multiple components associated with weights. We can always construct a CSS where only one component of a chiral multiplet is \((10)\) or \((14)\) and all the others are trivial. This solution forces \( \sigma \) to be in a \((\text{rank}(G) - 1)\)-dimensional hypersurface in \( \sigma \)-space. Hence \( \sigma \) is not completely determined for \( \text{rank}(G) > 1 \) and we can also take nontrivial configuration \((10)\) or \((14)\) for other components. In general there are solutions with multiple nontrivial components unless \( \sigma \) is “overdetermined”, namely SUSY conditions set more than \( \text{rank}(G) \) linearly independent hypersurfaces in the \( \sigma \)-space.

III. COMPLEXIFIED SUPERSYMMETRIC SOLUTIONS AND ANALYTIC PROPERTY OF BOREL TRANSFORMATION

Let us consider 3d \( \mathcal{N} = 2 \) SUSY CS matter theory coupled to chiral multiplets of representations \( \{ R_a \} \) with \( R \)-charges \( \{ \Delta_a \} \). We take gauge group to be a product of semi-simple gauge groups: \( G = G_1 \times \cdots \times G_n \) with CS levels \( k_1, \cdots, k_n \) respectively. We are interested in perturbative expansion by \( g_p \propto 1/|k_p| \) and a relation between its Borel transformation and the CSS.

A. Squashed sphere

In [13], the author found explicit finite dimensional integral representations of Borel transformations for \( S^3 \) partition functions. As we briefly review below, the Borel transformations are somehow hidden in localization formula given by [13] and

\[
Z_{S^3}(g) = \int_{-\infty}^{\infty} d^{\text{rank}(G)} \sigma Z_{\text{cl}}(\sigma) Z_{\text{1loop}}(\sigma), \tag{15}
\]
where $Z_{cl}$ is the classical contribution and $Z_{1\text{loop}}$ is the one-loop determinant in the localization procedure [29]:

$$Z_{cl}(\sigma) = \exp \left[ \sum_{p=1}^{n} \frac{\text{sgn}(k_p)}{g_p} \text{tr}(\sigma(p)^2) \right],$$

$$Z_{1\text{loop}}(\sigma) = \prod_{\alpha \in \text{root}+} 4 \sinh(\pi b \alpha \cdot \sigma) \sinh(\pi b^{-1} \alpha \cdot \sigma),$$

$$s_b(z) = \prod_{m=0}^{\infty} \prod_{n=0}^{\infty} \frac{m b + n b^{-1} + Q/2 - i z}{m b + n b^{-1} + Q/2 + i z},$$

The basic idea for finding the Borel transformation is quite simple [10]. First we take polar coordinate for each gauge group $G_p$: $\sigma_i^{(p)} = \sqrt{t_p} x_i^{(p)}$ with $t_p = \text{sgn}(k_p) g_p$ and $x_i^{(p)} \in S^{\text{rank}(G_p)-1}$ [30]. Then the partition function takes the form

$$Z_{S^3} = \prod_{p=1}^{n} \text{sgn}(k_p) \int_{0}^{\text{dim}(G_p)} dt_p e^{-t_p} f(\text{sgn}(k)p),$$

$$f(\tau) = \prod_{p=1}^{n} \frac{\text{dim}(G_p)}{2^\text{dim}(G_p)} \int_{\text{spheres}} d\hat{x} h(\tau, \hat{x}),$$

$$h(\tau, \hat{x}) = Z_{1\text{loop}}(\sigma) \prod_{p=1}^{n} \left. \frac{\text{dim}(G_p) - \text{rank}(G_p)}{\tau_p} \sigma_i^{(p)} \right|_{\sigma_i^{(p)} = \sqrt{t_p} x_i^{(p)}}.$$

This is similar to the form of Borel resummation [1] for multiple couplings and it is natural to ask whether the function $f(\tau)$ is related to Borel transformation. We can actually prove that $f(\tau)$ is related to the Borel transformation of the perturbative series by [6]

$$\prod_{p=1}^{n} \text{sgn}(k_p) f(\{\tau_p\}) = BZ_{S^3} BZ_{S^3}(\{ -\text{sgn}(k)p\}).$$

Now we compare analytic property of the Borel transformation with the CSS in the last section. When gauge group is product of rank-1 gauge groups, we can easily see this since $f(\tau)$ is no longer integral representation. For example, let us consider a U(1) CS theory with CS level $k > 0$ coupled to charged $\{q_a\}$ chiral multiplets with $R$-charges $\{\Delta_a\}$. Borel transformation of perturbative expansion of $Z_{S^3}$ by $g = 1/\alpha$ in this theory is

$$BZ_{S^3}(t) = \frac{1}{2\sqrt{-i t} \prod_{a=1}^{N_f} s_b \left( q_a \sqrt{t} - iQ(1-\Delta_a) \right)}.$$

This has simple poles and zeroes at

$$t_{\text{pole}}^{m,n} = -\frac{i}{q_a^2} \left( mb + nb^{-1} + \frac{Q}{2} \Delta_a \right)^2,$$

$$t_{\text{zero}}^{m,n} = -\frac{i}{q_a^2} \left( mb + nb^{-1} - \frac{Q}{2} \Delta_a - 2 \right)^2,$$

which are parameterized by two integers $m, n \in \mathbb{Z}_{>0}$. Let us compare these with actions of the CSS. Although the theory has various terms in general, only relevant term for us is the SUSY CS term

$$S_{\text{CS}} = \frac{i k}{4\pi} \int d^4x \sqrt{g} \text{tr} \left[ \epsilon^{\mu
u\rho\sigma} \left( A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho \right) - \bar{\lambda} \lambda + 2D\sigma \right],$$

because the CSS have vanishing Yang-Mills and matter actions [31]. In this theory, the bosonic and fermionic CSS associated with $a$-th chiral multiplet have

$$\sigma_{m,n}^{\text{bos}} = -\frac{i}{q_a} \left( mb + nb^{-1} + \frac{Q}{2} \Delta_a \right),$$

$$\sigma_{m,n}^{\text{fer}} = \frac{i}{q_a} \left( mb + nb^{-1} - \frac{Q}{2} \Delta_a - 2 \right),$$

respectively. Therefore their actions are given by

$$S_{\text{bos}} = \frac{i\pi k}{q_a^2} \left( mb + nb^{-1} + \frac{Q}{2} \Delta_a \right)^2 = \frac{t_{\text{pole}}^{m,n}}{g},$$

$$S_{\text{fer}} = \frac{i\pi k}{q_a^2} \left( mb + nb^{-1} - \frac{Q}{2} \Delta_a - 2 \right)^2 = \frac{t_{\text{zero}}^{m,n}}{g}.$$

Thus single bosonic (fermionic) CSS with the action $S = S_g$ gives a simple pole (zero) of the Borel transformation at $t = S_g$.

For general rank, it is more involved since there are integrals for $f(\tau)$ and not easy to find precise locations of poles and zeroes of the Borel transformation. However, it is always true that origins of all Borel singularities are the CSS because all the singularities are technically coming from the poles of the one-loop determinant whose locations are the same as the values of $\sigma$ on the CSS. Although we have focused on the partition function, the same result holds also for SUSY Wilson loops [10] because its effect on Borel transformation is insertion of a function without poles and zeroes into $h(\tau, \hat{x})$ [9].

### B. Round sphere limit

For $b = 1$, the building block $s_b(z)$ of the one-loop determinant becomes a product over single integer [17]:

$$s_1(z) = \prod_{n=1}^{I} \left( \frac{n}{n + i z} \right)^n.$$

Because of this, poles and zeroes of Borel transformation become degenerate. For example in the U(1) CS theory considered for the squashed case, the Borel transformations have the poles and zeroes with degree-$n$ at

$$t_{\text{pole}}^{n} = -\frac{i(n + \Delta_a - 1)}{q_a^2}, \quad t_{\text{zero}}^{n} = -\frac{i(n - \Delta_a + 1)}{q_a^2}.$$
with $n \in \mathbb{Z}_+$. Hence the poles and zeroes parametrized by the two integers for general $b$ become coincident and are described by single integer for $b = 1$. This degeneration is reflected as the change of the degrees of the poles and zeroes.

Correspondingly, the eigenvalues [9] and [13] become degenerate for $b \to 1$ and the CSS with the same $m + n$ have the same actions. Therefore we have $n$ bosonic and fermionic solutions with the actions

$$S_{\text{bos}} = \frac{t^n_{\text{pole}}}{g}, \quad S_{\text{fer}} = \frac{t^n_{\text{zero}}}{g},$$

(26)

respectively. Thus the $n$-bosonic (fermionic) solutions with the coincident action $S = S_c/g$ give a degree-$n$ pole (zero) of the Borel transformation at $t = S_c$. The same result holds also for various observables studied in [8]. Especially, for superconformal case, one can obtain the same conclusion for Bremsstrahlung function [18], two point functions [19] of stress tensor and $U(1)$ flavor symmetry current on $\mathbb{R}^3$. Therefore we expect that for superconformal case (not limited to 3d $\mathcal{N} = 2$ theory) CSS on sphere provide information on analytic property of Borel transformation even for flat space case.

C. Hyper multiplets on round sphere

When matters are only hyper multiplets, the one-loop determinant becomes much simpler:

$$Z_{\text{1-loop}}(\sigma) = \prod_{a \in \text{root}+} 4 \sinh^2 (\pi a \cdot \sigma)$$

$$\prod_a \prod_{\rho_a \in \mathbb{R}^3} 2 \cosh (\pi \rho_a \cdot \sigma),$$

(27)

because of the identity

$$\frac{1}{s_1 (z - i/2) s_1 (z - i/2)} = \frac{1}{2 \cosh (\pi z)}. \quad (28)$$

This simplification comes from cancellations of bosonic and fermionic contributions in a pair of conjugate representations with the $R$-charges 1/2.

For example, if we consider a $U(1)$ CS theory with CS level $k > 0$ coupled to charge $\{q_a\}$ hyper multiplets, then the Borel transformation does not have zeroes but has simple poles at

$$t^n_{\text{pole}} = \frac{i(2n - 1)^2}{4q_a^2}, \quad n \in \mathbb{Z}_+.$$ \quad (29)

Correspondingly we have $n_B$ bosonic and $n_F$ fermionic solutions with the following actions

$$S_{\text{bos}} = \frac{i(2n_B - 1)^2}{4g^2 y}, \quad S_{\text{fer}} = \frac{i(2n_F + 1)^2}{4g^2 y},$$

(30)

with $n_B, n_F \in \mathbb{Z}_+$. For solutions with the action $t^n_{\text{pole}}/g$, we have $n$ bosonic and $(n - 1)$ fermionic solutions and hence always $n_B - n_F = 1$ for any $n$. Thus the CSS predict that the degree-$n$ poles at $t = t^n_{\text{pole}}$ are canceled by the degree $(n - 1)$-zeroes and becomes the simple poles. This agrees with our conjecture.

D. Planar limit

It is usually expected that the perturbative series by ’t Hooft coupling in the planar limit is convergent as the number of Feynman diagrams do not grow as factorial. Hence, if there are no renormalons, we expect that Borel transformation in the planar limit does not have singularities. We can easily see that our CSS are indeed consistent with this expectation. In the planar limit $N \to \infty$ with fixed $N/k$, the actions of the CSS become infinite unless we have additional factor of order $1/N$. Thus in typical “matrix type” theory like large-$N$ CS theory coupled to chiral multiplets with up to two index representations, Borel singularities go away and we do not have Borel singularities in the planar limit.

IV. CONCLUSION AND DISCUSSIONS

In this paper we have pointed out physical importance of complexified supersymmetric solutions which satisfy SUSY conditions but are not on original path integral contour. We have conjectured that if there are $n_B$ bosonic and $n_F$ fermionic CSS with the action $S = S_c/g$, then Borel transformation has a pole for $n_B \geq n_F$ and zero for $n_B \leq n_F$ with degree $|n_B - n_F|$ at $t = S_c$. We have shown this statement for various SUSY observables in the 3d $\mathcal{N} = 2$ SUSY CS matter theories on sphere. We have constructed the CSS in general Langrangian 3d $\mathcal{N} = 2$ theory and compared their actions with analytic property of the Borel transformations to see agreement with our conjecture.

We have seen that the fermionic CSS give information on the zeroes of the Borel transformation. To our knowledge physical implication of Borel zeroes have not been explored in literature. Naively Borel zeroes do not seem to be related to large order behavior of perturbative series unless they are ends of branch cuts. There might be general argument for a relation between Borel zeroes and fermionic saddle points as for the relation between Borel singularities and saddle points [20, 21].

We have not discussed whether the CSS are contributing saddle points or not. In order to answer this question, it is appropriate to perform Lefschetz thimble analysis for 3d $\mathcal{N} = 2$ theories. More generally, since most of the analyzes of SUSY localization in literature have not performed serious complex saddle point analysis to our knowledge, there is a logical possibility that we are missing contributions from complex saddles in most setups. Perhaps the setups in [4, 5, 15, 22] would be good starting points to address this issue.

It is known that in the (Coulomb branch) localization formula for $Z_{S^2}$, picking up poles of the one-loop determinant in a half complex plane gives rise to Higgs branch representation of the partition function including a product of vortex and anti-vortex partition functions for some theories [4, 5, 15, 22]. Since we have seen the correspondence between the poles and bosonic CSS, presumably
the CSS are related to the Higgs branch formula or vortices. It is interesting to make this intuition more precise.

Recently there appeared multi-cut solutions in matrix models obtained by localization of the $S^3$ partition function of 3d $\mathcal{N} = 2$ CS theories. In the solutions, poles and zeroes in the one-loop determinant play important roles to gather eigenvalues of the matrix models. It would be illuminating to study connection between the solutions and our CSS.

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