Hierarchical joint remote state preparation in noisy environment

Chitra Shukla\textsuperscript{1} · Kishore Thapliyal\textsuperscript{2} · Anirban Pathak\textsuperscript{2}

Abstract A novel scheme for quantum communication having substantial applications in practical life is designed and analyzed. Specifically, we have proposed a hierarchical counterpart of the joint remote state preparation (JRSP) protocol, where two senders can jointly and remotely prepare a quantum state. One sender has the information regarding amplitude, while the other one has the phase information of a quantum state to be jointly prepared at the receiver’s port. However, there exists a hierarchy among the receivers, as far as powers to reconstruct the quantum state are concerned. A 5-qubit cluster state has been used here to perform the task. Further, it is established that the proposed scheme for hierarchical JRSP (HJRSP) is of enormous practical importance in critical situations involving defense and other sectors, where it is essential to ensure that an important decision/order that can severely affect a society or an organization is not taken by a single person, and once the order is issued, all the receivers do not possess an equal right to implement it. Further, the effect of different noise models (e.g., amplitude damping (AD), phase damping (PD), collective noise and Pauli noise models) on the HJRSP protocol proposed here is investigated. It is found that in AD and PD noise models a higher-power agent can reconstruct the quantum state to be remotely prepared with higher fidelity than that done by the

\textsuperscript{1} Graduate School of Information Science, Nagoya University, Furo-cho 1, Chikusa-ku, Nagoya 464-8601, Japan

\textsuperscript{2} Jaypee Institute of Information Technology, A-10, Sector-62, Noida, UP 201307, India
lower-power agent(s). In contrast, the opposite may happen in the presence of collective noise models. We have also proposed a scheme for probabilistic HJRSP using a non-maximally entangled 5-qubit cluster state.

**Keywords** Joint remote state preparation · Hierarchical quantum communication · Amplitude damping noise · Phase damping noise · Collective noise · Pauli noise · Quantum communication

### 1 Introduction

There are two major facets of quantum information science: quantum computing [1–3] and quantum communication [3–6]. In the last three decades, both of these have attracted major attraction of the scientific community for their ability to perform computational or communication tasks beyond the capacity of their classical counterparts. Specifically, a quantum algorithm may perform a computational task much faster than its classical counterpart. For example, Grover’s algorithm [1] can search unsorted database quadratically faster than its best known classical counterpart, and Shor’s algorithm [2] can factorize large numbers in a speed not attainable in a classical computer. Similarly, in the domain of quantum communication, protocols have been proposed for teleportation [5], which does not have any classical analog, and unconditional security of quantum key distribution [4] has been established [7]. Unconditional security is a desirable feat not achievable in classical world. The non-classical nature of quantum communication schemes drew considerable attention of the scientific community. Particularly, teleportation (where the state to be transmitted from the sender to the receiver never exists in the channel joining the receiver and the sender) drew major attention because of its magical characteristics. In the original teleportation scheme proposed by Bennett et al. [5], the sender Alice used to transmit/teleport an unknown single-qubit quantum state to the receiver Bob using a shared entangled state and two bits of classical communication. Subsequently, several variants of teleportation have been proposed. For example, schemes were proposed for quantum information splitting (QIS) or controlled teleportation (CT) [8,9], quantum secret sharing (QSS) [10], hierarchical quantum information splitting (HQIS) [11,12], remote state preparation (RSP) [13], etc. (see Ref. [6] for a review). All these schemes can be viewed as variants of teleportation.

Recently, a few hierarchical versions of already existing aspects of quantum communication (variants of teleportation) have been proposed. Specifically, hierarchical quantum information splitting (HQIS) [11,12,14,15], hierarchical quantum secret sharing (HQSS) [12], hierarchical dynamic quantum secret sharing (HDQSS) [16], etc. have been proposed. It is also shown that these schemes have enormous practical importance (for a detailed discussion on the interesting applications of these schemes, see Sect. 1 of Refs. [12,16]). In these protocols, there is a hierarchy among the powers of receivers (agents) to reconstruct a quantum state sent by the sender, i.e., the agents are graded in accordance with their power for the reconstruction of an unknown quantum state. Specifically, in HQIS the receivers can reconstruct the teleported quantum state with the help of other receivers (as in QIS [8,9]), where the power of a partic-
ular receiver is decided by the minimum number of receivers required to cooperate with him to enable him to reconstruct the state [12]. In a HQSS scheme, the sender wishes to send information in pieces to all the receivers who can reconstruct it with the help of either some or all other agents. Later, the scheme was extended to propose a HDQSS scheme, and in HDQSS scheme, an additional feature to add and drop an agent was included [16], and that made HDQSS most practical hierarchical scheme proposed until now. In all the existing hierarchical schemes, there is only one sender, but in many practical purposes, we need more than one sender for the secrecy of the initial message/state to be transmitted in a hierarchical manner. This paper aims to address this particular issue and design a new type of scheme for hierarchical quantum communication, which we would refer to as hierarchical joint remote state preparation (HJRSP) scheme in analogy with the well-known joint remote state preparation (JRSP) schemes [17–20].

Here, it would be apt to note that there exists a variant of teleportation known as RSP, where a known quantum state is remotely prepared at the receiver’s end. Thus, RSP may be viewed as teleportation of a known quantum state. The first RSP scheme was proposed by Pati [13] using Bell states, in 2000. This probabilistic scheme required 1 bit of classical communication and a shared entangled state (1 ebit). As the standard teleportation scheme requires 2 cbit and 1 ebit, Pati’s scheme of RSP was able to teleport a known quantum state with reduced resources compared to the unknown qubit case. Later, a deterministic counterpart of the RSP scheme was proposed by An et al. [21], where the required resources become equal to that of teleportation [5]. Subsequently, a large amount of work has been carried out on RSP. In these works, RSP has been implemented in probabilistic, deterministic, controlled and controlled bidirectional manner using different quantum states, such as $n$-level, 4-qubit GHZ state, multi-qubit GHZ state, 4-qubit cluster-type state, arbitrary two qubit state and $W$ state [17–20,22–27]. Interestingly, a few experimental realizations of some of the RSP schemes have also been reported in the past [28–33]. Recently, a QSS scheme has been proposed exploiting the ideas of RSP, and it is also subjected to noisy environment [34].

In 2008, the RSP scheme was modified to a three-party scheme of the joint RSP (JRSP) [17]. This one is a unique quantum communication scheme, where two senders jointly prepare a known quantum state at the remote port. The state to be prepared remotely is completely known neither to sender 1 nor to sender 2, but they jointly know the state to be teleported. It is worth stressing that this scheme has no analog in teleportation, as an unknown quantum state cannot be teleported in this way by more than one sender. Since then, several schemes for JRSP have been proposed [18–20]. In fact, the JRSP [35] scheme was investigated under the amplitude damping (AD) and phase damping (PD) noise models in the recent past. Interestingly, practical applications of the hierarchical quantum communication schemes, described in the context of HQIS [12] and HDQSS [16], motivated us to investigate the possibility of designing a hierarchical version of JRSP (HJRSP) scheme with the hope of using it in some of the practical purposes of daily life. Here, it would be apt to note that hierarchical communications play important roles in our day-to-day life. Several examples of practical situations where hierarchical communication is essential have been discussed in our earlier works [12,16]. Specifically, its relevance is discussed in context of HQSS [12],
HQIS [11] and HDQSS [16]. In all those cases, the information or quantum state to be shared was in possession of a single person (whom we referred to as Alice or Sender). In contrast, here the initial state is jointly possessed by two senders. In many practical scenarios, it allows an avenue for joint decision and thus reduces risk associated with policy decisions taken by a single person. Let us elaborate this particular feature through a specific example.

Consider that there exists a code to unlock a nuclear weapon. Because of the fatal effect it may cause, this particular code cannot be given to a single authorized person. Thus, the information (code) has to be distributed among at least two authorized persons, so that none of the authorized persons can misuse the code. Now, consider that Alice1 and Alice2 (two authorized persons) are the Prime Minister and President of a country, Bob2 is the defense minister, and Bob1 and Bob3 are the defense secretary and the chief of the armed forces of that country, respectively. If and only if the Prime Minister and President together wish to permit the use of the nuclear weapon at a suitable time, then they jointly distribute the information (the code which is required to unlock the nuclear weapon) among the defense minister, the defense secretary and the chief of the armed forces in such a way that the minister can unlock the weapon if either the defense secretary or the chief of the armed forces agrees and cooperates with him. However, if the chief of the armed forces or the defense secretary wants to unlock the weapon, they would require the cooperation of each other and that of the defense minister. Thus, the defense minister is more powerful than the chief of the armed forces and the defense secretary, but even he is not powerful enough to unlock the weapon alone, and the senders (i.e., the Prime Minister and the President) are not powerful enough to issue an individual order that allows the receivers to unlock the weapon. This type of joint responsibility is essential and routinely exercised in a democracy. However, earlier proposed hierarchical schemes of quantum communication did not contain this particular feature. We can discuss a large number of similar practical situations where HJRSP or a variant of it is essential, but it is not our purpose to provide a long list of practical situation. Rather, because of its applicability in various practical situations, we are interested in designing HJRSP scheme. Further, recent investigations on RSP schemes [22,35] under the noisy environment, such as AD and PD channels, motivated us to simulate a similar study for the proposed HJRSP protocol.

In what follows, we have proposed a scheme for HJRSP using a 5-qubit cluster state of the form

$$|C\rangle = \frac{1}{2} [ |00000\rangle + |00111\rangle + |11010\rangle + |11101\rangle ]_{S_1S_2R_1R_2R_3} \quad (1)$$

with the qubits $S_1$, $S_2$ belonging to the senders Alice1 and Alice2; and $R_1$, $R_2$ and $R_3$ corresponding to the qubits of the three receivers, who are referred to as Bob1, Bob2 and Bob3, respectively. The preparation of the cluster state (1) by Alice1 and distribution of qubits to all other parties is illustrated through the schematic diagram shown in Fig. 1. More detail of the scheme follows in the forthcoming section.

Before we proceed further, it would be required to establish that at least 5-qubit quantum channel is required for HJRSP and the fact that cluster state is good to perform the task in hand. Historically, the motivation of RSP led to studies of the joint RSP
Fig. 1  HJRSP scheme illustrated through a schematic diagram. The quantum and classical communications involved between all the parties are shown using the *smooth* and *dashed lines*, respectively. It is shown that Bob$_2$ can reconstruct the quantum state with the help of only one of the lower-power agents (in the diagram it is shown as Bob$_1$). In contrast, Bob$_3$ requires help of both Bob$_1$ and Bob$_2$. Here, we have not shown the case when Bob$_1$ reconstructs the quantum state as it is similar to the case when Bob$_3$ does so.
(JRSP) schemes with at least 3-qubit entanglement [20]. Further, it is known that a
multiparty quantum state sharing scheme requires entanglement shared between the
sender and each receiver [6]. Independently, it has been established that a hierarchical
quantum information splitting (HQIS) scheme requires at least three receivers with
one sender, and entanglement is to be shared between all four parties [11,12]. Thus,
any scheme for HQIS would require at least 4-qubit entanglement. A JRSP scheme is
where the amplitude and phase information of the desired quantum state is with two
independent senders, and a HJRSP scheme is one where we have at least five users:
two senders and three receivers. Thus, 5-qubit entanglement is necessary. However,
the use of a cluster state is not necessary. HJRSP may be implemented using a set of
other 5-qubit entangled states, too. We have specifically discussed the possibility of
implementing HJRSP using 5-qubit cluster state because of the fact that 5-qubit cluster
state can be generated using the current technology. Specifically, in 2008, experimental
generation of the 5-qubit cluster state has been reported by Lu et al. [36]. Interestingly,
this was followed by experimental generation of the cluster state with higher number
of qubits. A brief review on the same can be found at [37].

In the following sections, the paper is organized as follows. In Sect.2, we propose
a protocol of deterministic HJRSP using 5-qubit cluster state of the form (1). This
is followed by a protocol of probabilistic HJRSP using a non-maximally entangled
counterpart of the 5-qubit cluster state (1). Subsequently, in Sect.3, we study the effect
of a set of noise models on the proposed deterministic HJRSP scheme. Finally, we
conclude the paper in Sect. 4.

2 Hierarchical joint remote state preparation with 5-qubit cluster state

In any scheme of JRSP (independent of whether the scheme is hierarchical or not and
whether it is deterministic or probabilistic), two senders jointly prepare a quantum state
at the end of receiver(s). Naturally, it is required that the information that characterizes
the quantum state to be prepared remotely is distributed among the senders, and it is a
convention to consider that amplitude information is available with one sender and the
phase information is available with the other user. This convention is followed in the
existing schemes of JRSP (see [19,20]). The same convention is followed here, and
it is assumed that Alice$_1$ and Alice$_2$ wish to jointly prepare a known quantum state
$|\xi\rangle = a|0\rangle + be^{i\phi}|1\rangle: a^2 + b^2 = 1$, at one of the three receivers’ end, where the values
of the amplitude parameters $a = \cos \theta$ and $b = \sin \theta$ are decided by Alice$_1$ and the
phase parameter $\phi$ is chosen by Alice$_2$. Alice$_1$ and Alice$_2$ encode their information
of amplitude $\theta$ and phase $\phi$, respectively, by performing the measurements where
they project their qubits in their known basis to send the values of state parameters
$\theta$ and $\phi$. Further, we require at least three receivers to design a hierarchical scheme
for joint remote state preparation. Here, the three receivers are referred to as Bob$_1$,
Bob$_2$ and Bob$_3$, respectively. The hierarchy among the receivers will be established
in what follows. To accomplish the task, we have chosen a 5-qubit cluster state of
the form (1) where each of the participants possesses one qubit of the 5-qubit cluster
state. Subsequently, a probabilistic version of HJRSP scheme is introduced, and it is
shown that the probabilistic HJRSP scheme appears when we use a non-maximally
entangled state of the form (1) as a quantum channel.
2.1 Deterministic HJRSP

We may now describe a protocol of deterministic HJRSP using a quantum state of the form (1) in the following steps:

**Step 1** Alice prepares the 5-qubit cluster state (1). She keeps S1 (1st) qubit and sends S2 (2nd) qubit to Alice2 and R1, R2, R3 qubits (3rd, 4th and 5th qubits) to the three receivers Bob1, Bob2, Bob3, respectively.

**Step 2** Alice measures her qubit in \(|u_0, u_1\rangle\) basis, where \(|u_0\rangle = a|0\rangle + b|1\rangle\) and \(|u_1\rangle = b|0\rangle - a|1\rangle\), and announces the measurement outcome. As the cluster state (1) can be expressed as

\[
|C\rangle = \frac{1}{2} \left( |u_0\rangle_{S1} \left( a(|0000\rangle + |0111\rangle) + b(|1010\rangle + |1101\rangle) \right)_{S2R1R2R3} + |u_1\rangle_{S1} \left( b(|0000\rangle + |0111\rangle) - a(|1010\rangle + |1101\rangle) \right)_{S2R1R2R3} \right),
\]

the measurement of Alice1 would reduce \(|C\rangle\) to \(|\psi_0\rangle\) (\(|\psi_1\rangle\)) if her measurement outcome is \(|u_0\rangle\) (\(|u_1\rangle\)), where

\[
|\psi_0\rangle = \frac{1}{\sqrt{2}} \left( a(|0000\rangle + |0111\rangle) + b(|1010\rangle + |1101\rangle) \right)_{S2R1R2R3},
\]

\[
|\psi_1\rangle = \frac{1}{\sqrt{2}} \left( b(|0000\rangle + |0111\rangle) - a(|1010\rangle + |1101\rangle) \right)_{S2R1R2R3}.
\]

**Case 1: In Step 2, Alice1’s measurement yields \(|u_0\rangle\)**

**Step 3** If Alice1’s measurement yields \(|u_0\rangle\), then Alice2 applies a single-qubit phase gate \(P(2\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\phi} \end{pmatrix}\) on her qubit and that transforms \(|\psi_0\rangle\) to

\[
|\psi_0'\rangle = \left( P(2\phi) \otimes I^{\otimes 3}_2 \right) |\psi_0\rangle = \frac{1}{\sqrt{2}} \left( a(|0000\rangle + |0111\rangle) + be^{2i\phi}(|1010\rangle + |1101\rangle) \right)_{S2R1R2R3} = \frac{1}{2} \left[ |v_0\rangle_{S2} \left( a(|000\rangle + |111\rangle) + be^{i\phi}(|010\rangle + |101\rangle) \right)_{R1R2R3} + e^{i\phi} |v_1\rangle_{S2} \left( a(|000\rangle + |111\rangle) - be^{i\phi}(|010\rangle + |101\rangle) \right)_{R1R2R3} \right] = \frac{1}{\sqrt{2}} \left( |v_0\rangle_{S2} |\phi_0\rangle_{R1R2R3} + |v_1\rangle_{S2} |\phi_1\rangle_{R1R2R3} \right),
\]

where

\[
|v_0\rangle = \frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}, \quad |v_1\rangle = \frac{e^{-i\phi}|0\rangle - |1\rangle}{\sqrt{2}}, \quad |\phi_0\rangle = \frac{1}{\sqrt{2}} \left( a(|000\rangle + |111\rangle) + be^{i\phi}(|010\rangle + |101\rangle) \right)_{R1R2R3}, \quad \text{and} \quad |\phi_1\rangle = \frac{1}{\sqrt{2}} \left( a(|000\rangle + |111\rangle) - be^{i\phi}(|010\rangle + |101\rangle) \right)_{R1R2R3}.
\]

**Step 4** Alice2 measures her qubit \(S_2\) in \(|v_0, v_1\rangle\) basis and announces the outcome obtained by her.
From Eq. (4), we can easily observe that if Alice$_2$ obtains $v_0$ ($v_1$) then the composite state shared by the receivers reduces to $|\phi_0\rangle$ ($|\phi_1\rangle$).

**Case 2: In Step 2, Alice$_1$’s measurement yields $|u_1\rangle$**

If Alice$_1$’s measurement yields $|u_1\rangle$, then **Step 3** and the following steps described above will be modified to **Step 3-1** and so on as described in the following steps:

**Step 3-1** If Alice$_1$’s measurement outcome is $|u_1\rangle$, then Alice$_2$ need not apply the phase gate $P(2\phi)$, and the state can be written as

$$|\Psi_1\rangle = \frac{1}{\sqrt{2}} \left[ b(|0000\rangle + |0111\rangle) - a(|1010\rangle + |1101\rangle) \right]_{S_2R_1R_2R_3}$$

$$= \frac{1}{2} \left[ |v_0\rangle_{S_2} \left( b(|000\rangle + |111\rangle) - ae^{-i\phi}(|010\rangle + |101\rangle) \right)_{R_1R_2R_3} + |v_1\rangle_{S_2} \left( be^{i\phi}(|000\rangle + |111\rangle) + a(|010\rangle + |101\rangle) \right)_{R_1R_2R_3} \right]$$

$$= \frac{1}{\sqrt{2}} \left[ |v_0\rangle_{S_2} |\phi_0'\rangle_{R_1R_2R_3} + |v_1\rangle_{S_2} |\phi_1'\rangle_{R_1R_2R_3} \right]. \quad (5)$$

Subsequently, Alice$_2$ measures her qubit $S_2$ in $\{v_0, v_1\}$ basis, where basis elements are already described in Step 3. The reduced state at the end of the three receivers can be deduced from Eq. (5).

At the end of this step, the state among Bobs is in one of four states $|\phi_i\rangle_{R_1R_2R_3}$ or $|\phi_i'\rangle_{R_1R_2R_3}$ for $i \in \{0, 1\}$.

**Case 1.1: The receivers decide that Bob$_2$ will reconstruct the state**

**Step 5** If the agents decide that Bob$_2$ will reconstruct the unknown quantum state $|\xi\rangle$, and Alice$_1$’s measurement yields $|u_0\rangle$, then $|\phi_0\rangle$ and $|\phi_1\rangle$ can be decomposed as

$$|\phi_0\rangle = \frac{1}{\sqrt{2}} \left\{ |00\rangle_{R_1R_3} (a|0\rangle + be^{i\phi}|1\rangle)_{R_2} + |11\rangle_{R_1R_3} (a|1\rangle + be^{i\phi}|0\rangle)_{R_2} \right\},$$

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} \left\{ |00\rangle_{R_1R_3} (a|0\rangle - be^{i\phi}|1\rangle)_{R_2} + |11\rangle_{R_1R_3} (a|1\rangle - be^{i\phi}|0\rangle)_{R_2} \right\}. \quad (6)$$

From Eq. (6), it is clear that Bob$_2$ can reconstruct the quantum state $|\xi\rangle$ with the collaboration of either Bob$_1$ or Bob$_3$ by applying Pauli operations $I, X, Z$ and $iY$, respectively, as given in Table 1. It is important to note that Bob$_2$ needs the collaboration of either the receiver Bob$_1$ or Bob$_3$. He does not require collaboration of both as the measurement taken by the receivers Bob$_1$ and Bob$_3$ in computational basis (i.e., $\{|0\rangle, |1\rangle\}$) always yields the same outcome (cf. Column 3 of Table 1). Thus, the communication of the measurement outcome from anyone of the receivers Bob$_1$ or Bob$_3$ is sufficient for the receiver Bob$_2$. Consequently, the receiver Bob$_2$ is the higher-power agent in the proposed HJRSP scheme.

**Step 4-1** When Bob$_2$ is supposed to reconstruct the unknown quantum state $|\xi\rangle$, and Alice$_1$’s measurement yields $|u_1\rangle$, the reduced quantum state can be decomposed as
Table 1  The relation between the measurement outcomes of the senders and receivers with the unitary operations to be applied by the receiver Bob₂ to recover the quantum state jointly sent by the senders Alice₁ and Alice₂

| Alice₁’s measurement outcome in \{u₀, u₁\} basis | Alice₂’s measurement outcome in \{v₀, v₁\} basis | Bob₂’s measurement outcome (in \{0, 1\} basis), where \(i \in \{1, 3\}\) | Pauli operation to be applied by Bob₂ |
|-------------------------------------------------|---------------------------------|---------------------------------|---------------------------------|
| \(|u₀\rangle\)                                  | \(|v₀\rangle\)                   | \(|0\rangle\)                    | \(I\)                           |
|                                                 |                                 | \(|1\rangle\)                    | \(X\)                           |
| \(|u₁\rangle\)                                  | \(|v₀\rangle\)                   | \(|0\rangle\)                    | \(iY\)                          |
|                                                 |                                 | \(|1\rangle\)                    | \(Z\)                           |
| \(|u₁\rangle\)                                  | \(|v₁\rangle\)                   | \(|0\rangle\)                    | \(X\)                           |
|                                                 |                                 | \(|1\rangle\)                    | \(I\)                           |

Bob₂ needs the collaboration of either of the agents Bob₁ or Bob₃

\[
|\phi₀⟩ = \frac{1}{\sqrt{2}} e^{-iφ} \left\{ |00⟩_{R₁ R₃} (be^{iφ}|0⟩ - a|1⟩)_{R₂} + |01⟩_{R₁ R₃} (be^{iφ}|1⟩ - a|0⟩)_{R₂} + |10⟩_{R₁ R₃} (be^{iφ}|0⟩ + a|1⟩)_{R₂} + |11⟩_{R₁ R₃} (be^{iφ}|1⟩ + a|0⟩)_{R₂} \right\},
\]

(7)

where the global phase \(e^{-iφ}\) can be ignored, which is consistent with the quantum mechanics.

From Eq. (7), it is clear that Bob₂ can reconstruct the quantum state \(|ξ⟩\) by applying suitable Pauli operations with the help of measurement results of either Bob₁ or Bob₃ as both of them have the same measurement outcomes (cf. Column 3 of Table 1). Consequently, the receiver R₂ is accredited the higher-power agent in the proposed hierarchical scheme.

Case 1.2: The receivers decide that Bob₃ will reconstruct the state

Step 5’ In case the agents wish Bob₃ to reconstruct \(|ξ⟩\), and Alice₁’s measurement yields \(|u₀⟩\), then \(|φ₀⟩\) and \(|φ₁⟩\) can be decomposed as

\[
|φ₀⟩ = \frac{1}{2} \left[ \left(|+⟩|0⟩\right)_{R₁ R₂} \left(a|0⟩ + be^{iφ}|1⟩\right)_{R₃} + \left(|−⟩|0⟩\right)_{R₁ R₂} \left(a|0⟩ - be^{iφ}|1⟩\right)_{R₃} + \left(|+⟩|1⟩\right)_{R₁ R₂} \left(a|1⟩ + be^{iφ}|0⟩\right)_{R₃} \right].
\]
$|\phi_1\rangle = \frac{1}{2} \left[ (|+\rangle|0\rangle)_{R_1 R_2} \left( a|0\rangle - be^{i\phi}|1\rangle \right)_{R_3} + (|\rangle|0\rangle)_{R_1 R_2} \left( a|0\rangle + be^{i\phi}|1\rangle \right)_{R_3} + (|+\rangle|1\rangle)_{R_1 R_2} \left( a|1\rangle - be^{i\phi}|0\rangle \right)_{R_3} - (|\rangle|1\rangle)_{R_1 R_2} \left( a|1\rangle + be^{i\phi}|0\rangle \right)_{R_3} \right].$ (8)

From Eq. (8), it is clear that Bob$_3$ can reconstruct the quantum state $|\xi\rangle$ by applying the Pauli operations as given in Table 2 if both the receivers Bob$_1$ and Bob$_2$ cooperate simultaneously by measuring their qubits (in $\{|+,|\rangle\}$ basis and computational basis, respectively) and sharing the measurement outcomes. This fact can be established mathematically by tracing over $R_1$ and $R_2$ qubits, which gives a completely mixed state for the receiver Bob$_3$. The measurement outcomes of the receivers Bob$_1$ and Bob$_2$ are summarized in column 3 of Table 2 with the corresponding Pauli operations Bob$_3$ has to apply in the next column. Here, it is important to note that Bob$_3$ needs the collaboration of both Bob$_1$ and Bob$_2$, whereas in Case 1.1, we have seen that Bob$_2$ can reconstruct the quantum state with the help of either Bob$_1$ or Bob$_3$. Thus, Bob$_2$ is a higher-power agent. A similar analysis would reveal that Bob$_1$ is also a lower-power agent as to reconstruct the unknown quantum state $|\xi\rangle$ sent by the senders he will also require the help of the remaining two Bobs.

**Step 4-1’** When Bob$_3$ has to reconstruct the quantum state, and Alice$_1$’s measurement yields $|u_1\rangle$, the reduced quantum state can be written (corresponding to measurement outcomes of Alice$_2$) as follows:

If the measurement outcome of Alice$_2$ is $|v_0\rangle$, the reduced state becomes

$|\phi_0\rangle = \frac{e^{-i\phi}}{2} \left[ (|+\rangle|0\rangle)_{R_1 R_2} \left( be^{i\phi}|0\rangle - a|1\rangle \right)_{R_3} + (|\rangle|0\rangle)_{R_1 R_2} \left( be^{i\phi}|0\rangle + a|1\rangle \right)_{R_3} + (|+\rangle|1\rangle)_{R_1 R_2} \left( be^{i\phi}|1\rangle - a|0\rangle \right)_{R_3} - (|\rangle|1\rangle)_{R_1 R_2} \left( be^{i\phi}|1\rangle + a|0\rangle \right)_{R_3} \right],$ (9)

and if the measurement outcome of the sender Alice$_2$ is $|v_1\rangle$, the state can be written as

$|\phi'_1\rangle = \frac{1}{2} \left[ (|+\rangle|0\rangle)_{R_1 R_2} \left( be^{i\phi}|0\rangle + a|1\rangle \right)_{R_3} + (|\rangle|0\rangle)_{R_1 R_2} \left( be^{i\phi}|0\rangle - a|1\rangle \right)_{R_3} + (|+\rangle|1\rangle)_{R_1 R_2} \left( be^{i\phi}|1\rangle + a|0\rangle \right)_{R_3} - (|\rangle|1\rangle)_{R_1 R_2} \left( be^{i\phi}|1\rangle - a|0\rangle \right)_{R_3} \right].$ (10)

From Eqs. (9) and (10), it can be inferred that Bob$_3$ can reconstruct $|\xi\rangle$ by applying an appropriate Pauli operation only if both the remaining receivers help him by
Table 2  The measurement outcomes of all the senders and receivers, and corresponding Pauli operations Bob3 has to apply to recover the quantum state

| Alice1’s measurement outcome in \{u_0, u_1\} basis | Alice2’s measurement outcome in \{v_0, v_1\} basis | Bob1’s and Bob2’s joint measurement outcome (in \{\ket{+}, \ket{-}\} and \{\ket{0}, \ket{1}\} bases, respectively) | Pauli operation to be applied by Bob3 |
|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|----------------------------------|
| \ket{u_0}                                        | \ket{v_0}                                        | \ket{+}\ket{0}                                   | I                                |
|                                                  |                                                  | \ket{-}\ket{0}                                   | Z                                |
|                                                  |                                                  | \ket{+}\ket{1}                                   | X                                |
|                                                  |                                                  | \ket{-}\ket{1}                                   | Y                                |
| \ket{v_1}                                        | \ket{+}\ket{0}                                   |                                                  | Z                                |
|                                                  | \ket{-}\ket{0}                                   |                                                  | I                                |
|                                                  | \ket{+}\ket{1}                                   |                                                  | Y                                |
|                                                  | \ket{-}\ket{1}                                   |                                                  | X                                |
| \ket{u_1}                                        | \ket{v_0}                                        | \ket{+}\ket{0}                                   | IY                               |
|                                                  |                                                  | \ket{-}\ket{0}                                   | X                                |
|                                                  |                                                  | \ket{+}\ket{1}                                   | Z                                |
|                                                  |                                                  | \ket{-}\ket{1}                                   | I                                |
| \ket{v_1}                                        | \ket{+}\ket{0}                                   |                                                  | X                                |
|                                                  | \ket{-}\ket{0}                                   |                                                  | iY                               |
|                                                  | \ket{+}\ket{1}                                   |                                                  | I                                |
|                                                  | \ket{-}\ket{1}                                   |                                                  | Z                                |

The receiver Bob3 needs the joint collaboration of all other agents conveying their measurement results after measuring their qubits in suitable basis. In Table 2, it is explicitly shown that Bob1 and Bob2 measure their qubits in \{\ket{+}, \ket{-}\} and \{\ket{0}, \ket{1}\} bases, respectively. In the table, Pauli operations corresponding to each possible measurement outcome are also clearly mentioned. Hence, Bob3 is the lower-power agent in the proposed HJRSP scheme. Interestingly, a similar situation is obtained for Bob1; consequently, he is at the same level of hierarchy as Bob3.

2.2 Probabilistic HJRSP

In the last subsection, we have proposed a deterministic HJRSP scheme using maximally entangled 5-qubit cluster state. In this section, we aim to propose a scheme for probabilistic HJRSP using non-maximally entangled 5-qubit cluster state of a specific form. To do so, we assume that Alice1 prepares and shares a non-maximally entangled 5-qubit cluster state of the form

$$|C'\rangle = \frac{1}{\sqrt{2}} [\alpha (|00000\rangle + |00111\rangle) + \beta (|11010\rangle + |11101\rangle)]_{S1S2R1R2R3}$$

$$= \frac{1}{\sqrt{2}} \left[ |u_0\rangle_{S1} \{\alpha a(|0000\rangle + |0111\rangle) + \beta b(|1010\rangle + |1101\rangle)\} + |u_1\rangle_{S1} \{\alpha b(|0000\rangle + |0111\rangle) - \beta a(|1010\rangle + |1101\rangle)\} \right]_{S2R1R2R3}, \quad (11)$$
where \(|\alpha|^2 + |\beta|^2 = 1\) with \(|\alpha| \neq \frac{1}{\sqrt{2}}\) and \(|u_0\rangle\) and \(|u_1\rangle\) are already defined in the context of the previous protocol. From Eq. (11), it can be observed that if Alice_1 measures her qubit \(S_1\) in \(|u_0\rangle, |u_1\rangle\) basis, then the reduced quantum state corresponding to measurement outcome \(|u_0\rangle\) and \(|u_1\rangle\) is given as

\[
|\psi_0\rangle = \frac{1}{\sqrt{2 (a^2|\alpha|^2 + b^2|\beta|^2)}} [aa(|0000\rangle + |0111\rangle) + \beta b(|1010\rangle + |1101\rangle)] S_2 R_1 R_2 R_3, \\
|\psi_1\rangle = \frac{1}{\sqrt{2 (b^2|\alpha|^2 + a^2|\beta|^2)}} [\alpha b(|0000\rangle + |0111\rangle) - \beta a(|1010\rangle + |1101\rangle)] S_2 R_1 R_2 R_3,
\]

(12)

respectively.

**Case 1:** Alice_1’s measurement outcome is \(|u_0\rangle\)

Then Alice_2 applies a phase operator \(P (2\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\phi} \end{bmatrix}\) on \(|\psi_0\rangle\), as in the deterministic HJRSP scheme defined above, and the state can be written as

\[
|\psi'_0\rangle = (P (2\phi) \otimes I^{\otimes 3}) |\psi_0\rangle = \frac{1}{\sqrt{2 (a^2|\alpha|^2 + b^2|\beta|^2)}} [aa(|0000\rangle + |0111\rangle) + \beta be^{2i\phi}(|1010\rangle + |1101\rangle)] S_2 R_1 R_2 R_3.
\]

(13)

As the transformed state can be written as

\[
|\psi'_0\rangle = \frac{1}{2\sqrt{2 (a^2|\alpha|^2 + b^2|\beta|^2)}} \left[|v_0\rangle S_2 [aa(|0000\rangle + |1111\rangle) + \beta be^{i\phi}(|0101\rangle + |1010\rangle)] R_1 R_2 R_3 \\
+ e^{i\phi} |v_1\rangle S_2 [aa(|0000\rangle + |1111\rangle) - \beta be^{i\phi}(|0101\rangle + |1010\rangle)] R_1 R_2 R_3 \right] \\
= \frac{1}{\sqrt{2}} \left[|v_0\rangle S_2 |\Phi_0\rangle R_1 R_2 R_3 + |v_1\rangle S_2 |\Phi_1\rangle R_1 R_2 R_3 \right],
\]

(14)

where \(|\Phi_0\rangle = \frac{1}{\sqrt{2 (a^2|\alpha|^2 + b^2|\beta|^2)}} [aa(|0000\rangle + |1111\rangle) + \beta be^{i\phi}(|0101\rangle + |1010\rangle)] R_1 R_2 R_3\) and \(|\Phi_1\rangle = \frac{1}{\sqrt{2 (a^2|\alpha|^2 + b^2|\beta|^2)}} [aa(|0000\rangle + |1111\rangle) - \beta be^{i\phi}(|0101\rangle + |1010\rangle)] R_1 R_2 R_3\).

Further, Alice_2 measures her qubit \(S_2\) in \(|v_0, v_1\rangle\) basis, where \(|v_0\rangle\) and \(|v_1\rangle\) are as defined in the previous subsection.

**Case 2:** Alice_1’s measurement outcome is \(|u_1\rangle\)
Then Alice$_2$ need not apply a phase operator $P(2\phi)$ on $|\psi_1\rangle$ in Eq. (12). Subsequently, Alice$_2$ measures her qubit $S_2$ in $\{v_0, v_1\}$ basis, where $|v_0\rangle$ and $|v_1\rangle$ have the same meaning as in the last subsection. As the state can be written as

$$|\psi''_1\rangle = \frac{1}{2\sqrt{b^2|\alpha|^2+a^2|\beta|^2}} \left[ |v_0\rangle_{S_2} \{ \alpha b (|000\rangle + |111\rangle) - \beta a e^{-i\phi} (|010\rangle + |101\rangle) \} R_1 R_2 R_3 \right. + |v_1\rangle_{S_2} e^{i\phi} \alpha b (|000\rangle + |111\rangle) + \beta a (|010\rangle + |101\rangle) \} R_1 R_2 R_3 \right],$$

$$= \frac{1}{\sqrt{2}} \left[ |v_0\rangle_{S_2} |\Phi'_0\rangle_{R_1 R_2 R_3} + |v_1\rangle_{S_2} |\Phi'_1\rangle_{R_1 R_2 R_3} \right].$$

from which the quantum state after measurement can be deduced to be $|\Phi'_0\rangle (|\Phi'_1\rangle)$ for Alice$_2$’s measurement outcomes $|v_0\rangle (|v_1\rangle)$.

At the end of this step, all the receivers share one of the four states $|\Phi_i\rangle_{R_1 R_2 R_3}$ for $i \in \{0, 1\}$.

**Case 1.1: The receivers decide that Bob$_2$ will recover the state**

If the agents decide that Bob$_2$ will reconstruct the unknown quantum state $|\xi\rangle$ when Alice$_1$’s measurement yields $|u_0\rangle$, and other two receivers would measure their qubits in computational basis, then the reduced quantum state can be decomposed as

$$|\Phi_0\rangle = \frac{1}{\sqrt{2(a^2|\alpha|^2 + b^2|\beta|^2)}} \{|00\rangle_{R_1 R_3} (\alpha a |0\rangle + \beta b e^{i\phi} |1\rangle)_{R_2} + |11\rangle_{R_1 R_3} (\alpha a |1\rangle + \beta b e^{i\phi} |0\rangle)_{R_2} \}.$$ 

$$|\Phi_1\rangle = \frac{1}{\sqrt{2(a^2|\alpha|^2 + b^2|\beta|^2)}} \{|00\rangle_{R_1 R_3} (\alpha a |0\rangle - \beta b e^{i\phi} |1\rangle)_{R_2} + |11\rangle_{R_1 R_3} (\alpha a |1\rangle - \beta b e^{i\phi} |0\rangle)_{R_2} \}.$$ 

(16)

From Eq. (16), it is clear that Bob$_2$ cannot directly reconstruct the quantum state just by applying the Pauli operators even if all the receivers cooperate. Therefore, he has to change the strategy as follows: Bob$_2$ prepares an ancilla qubit in $|0\rangle_{aux}$ and applies the following 2-qubit unitary operations $U_0/U_1$ on his qubits (i.e., on the combined system of his existing qubit and ancilla), where

$$U_0 = \begin{pmatrix} \frac{\beta}{\alpha} & \sqrt{1 - \frac{\beta^2}{\alpha^2}} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \sqrt{1 - \frac{\beta^2}{\alpha^2}} & -\frac{\beta}{\alpha} & 0 & 0 \end{pmatrix}.$$ 

(17)
and

\[ U_1 = U_0 (X \otimes I) = \begin{pmatrix}
0 & 0 & \frac{\beta}{\alpha} & \sqrt{1 - \frac{\beta^2}{\alpha^2}} \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & \sqrt{1 - \frac{\beta^2}{\alpha^2}} & -\frac{\beta}{\alpha}
\end{pmatrix}. \]  \quad (18)

As \( \alpha \) and \( \beta \) are known, the construction of \( U_0/U_1 \) is possible. In Eq. (16), let us consider a case that the measurement outcomes of Alice 2 and Bob1 (or Bob3 as the measurement outcomes of both lower-power agents are the same) are \(|v_0\rangle\) and \(|0\rangle\langle1|)\), respectively. Then Bob2 has to apply \( U_0(U_1) \) on his product state. For instance,

\[ |\chi\rangle_1' = |\chi\rangle_1 |0\rangle_{\text{aux}} = \frac{1}{\sqrt{(a^2|\alpha|^2 + b^2|\beta|^2)}} \left( a|0\rangle + \beta be^{i\phi}|1\rangle \right) |0\rangle_{\text{aux}}, \]

where \(|\chi\rangle_1\) is Bob2’s reduced quantum state corresponding to Bob1’s measurement outcome \(|0\rangle\). Subsequent operation of \( U_0 \) on this composite state gives

\[ U_0 |\chi\rangle_1' = \frac{1}{\sqrt{a^2|\alpha|^2 + b^2|\beta|^2}} \left\{ \beta(a|0\rangle + be^{i\phi}|1\rangle)|0\rangle + a\sqrt{a^2 - \beta^2}|1\rangle\langle1| \right\}. \]

Finally, Bob2 measures the last qubit (ancilla) in the computational basis \(\{|0\rangle, |1\rangle\}\). If his measurement yields \(|0\rangle\), then he obtains unknown state with unit fidelity, but if his measurement on ancilla yields \(|1\rangle\), then he fails to reconstruct the state. In other words, he can recover the unknown state only when he obtains a specific measurement outcome. Thus, in analogy to the probabilistic teleportation scheme we may refer to this scheme as probabilistic hierarchical joint remote state preparation scheme. Similarly, we can check the other three possibilities. All these results are summarized in Table 3.

On the other hand, if all the agents decree Bob2 to reconstruct the unknown quantum state when Alice1’s measurement yields \(|u_1\rangle\), the reduced quantum state of the receivers can be decomposed as

\[ |\Phi_0\rangle = \frac{1}{\sqrt{2 (b^2|\alpha|^2 + a^2|\beta|^2)}} \left\{ \langle 00|_{R_1,R_3} (\alpha b|0\rangle - \beta a e^{-i\phi}|1\rangle)_{R_2} + |11\rangle_{R_1,R_3} (\alpha b|1\rangle - \beta a e^{-i\phi}|0\rangle)_{R_2} \right\}, \]
\[ |\Phi_1\rangle = \frac{1}{\sqrt{2 (b^2|\alpha|^2 + a^2|\beta|^2)}} \left\{ \langle 00|_{R_1,R_3} (\alpha b e^{i\phi}|0\rangle + \beta a|1\rangle)_{R_2} + |11\rangle_{R_1,R_3} (\alpha b e^{i\phi}|1\rangle + \beta a|0\rangle)_{R_2} \right\}. \]  \quad (19)

From Eq. (19), it is concluded that Bob2 cannot directly reconstruct the quantum state unless he adopts the same strategy as was discussed for the case of Alice1’s measurement outcome \(|u_0\rangle\). In analogy of Case 1, when Alice1 obtained \(|u_0\rangle\), specific cases may be studied. Here, we have restricted ourselves only to the successful cases of probabilistic HJRSP which is summarized in Table 3.
Table 3 The probabilistic HJRSR scheme is summarized here for all the successful cases

| Alice1’s measurement outcome in {u₀, u₁} basis | Alice2’s measurement outcome in {v₀, v₁} basis | Bobi’s measurement outcome (in {0, 1} basis), where i ∈ {1, 3} | Two qubit unitary operation to be applied by Bob2 (U₀/U₁) | Pauli operation Bob2 have to apply to obtain |ξ⟩ |
|-----------------------------------------------|-----------------------------------------------|-------------------------------------------------|-------------------------------------------------|-----------------------------------------------|------------------|
| |Ψ₀⟩= 1/(a²|α|² + b²|β|²) [(|+⟩|0⟩)R₁R₂ (αa|0⟩ + βbeᵢφ|1⟩)]R₃ | | | I |
| |Ψ₀⟩= 1/(a²|α|² + b²|β|²) [(|−⟩|0⟩)R₁R₂ (αa|0⟩ + βbeᵢφ|1⟩)]R₃ | | | Z |
| |Ψ₀⟩= 1/(a²|α|² + b²|β|²) [(|+⟩|0⟩)R₁R₂ (αa|0⟩ + βbeᵢφ|1⟩)]R₃ | | | iY |
| |Ψ₀⟩= 1/(a²|α|² + b²|β|²) [(|−⟩|0⟩)R₁R₂ (αa|0⟩ + βbeᵢφ|1⟩)]R₃ | | | X |
| |Ψ₀⟩= 1/(a²|α|² + b²|β|²) [(|+⟩|0⟩)R₁R₂ (αa|0⟩ + βbeᵢφ|1⟩)]R₃ | | | |

The measurement outcomes of all the senders and receivers are summed up with the unitary and Pauli operations Bob2 needs to apply. It can be deduced from the third column that the receiver Bob2 is a higher-power agent as he needs the collaboration of either of the agents.

**Case 1.2: The receivers decide that Bob3 will recover the state**

If the agents unanimously agreed that Bob3 will reconstruct the unknown quantum state, and Alice1’s measurement outcome was |u₁⟩, the reduced quantum state of three receivers can be decomposed as follows.

When the measurement outcome of Alice2 is |v₀⟩

|Φ₀⟩= 1/(a²|α|² + b²|β|²) [(|+⟩|0⟩)R₁R₂ (αa|0⟩ + βbeᵢφ|1⟩)]R₃

When the measurement outcome of Alice2 is |v₁⟩

|Φ₀⟩= 1/(a²|α|² + b²|β|²) [(|+⟩|0⟩)R₁R₂ (αa|0⟩ + βbeᵢφ|1⟩)]R₃

From Eqs. (20) and (21), it is clear that Bob3 cannot directly reconstruct the quantum state as in the previous case. Hence, similar to the strategy followed by Bob2 above, Bob3 also applies a unitary operation U₀/U₁ on his composite system (i.e., on his
existing qubit and ancilla). Finally, he can reconstruct the quantum state by further application of an appropriate Pauli operation as given in Table 4.

When Bob3 is assigned the task to reconstruct the unknown quantum state, and Alice1’s measurement outcome was $|u_1\rangle$, the reduced quantum state can be decomposed as follows.

When the measurement outcome of Alice2 is $|v_0\rangle$

$$|\Phi_0\rangle = \frac{1}{2\sqrt{(b^2|\alpha|^2 + a^2|\beta|^2)}} \left[ (|+\rangle|0\rangle)_{R_1 R_2} \left( ab|0\rangle - \beta a e^{-i\phi}|1\rangle \right)_{R_3} 
+ (|\rangle|0\rangle)_{R_1 R_2} \left( ab|0\rangle + \beta a e^{-i\phi}|1\rangle \right)_{R_3} 
+ (|+\rangle|1\rangle)_{R_1 R_2} \left( ab|1\rangle - \beta a e^{-i\phi}|0\rangle \right)_{R_3} - (|\rangle|1\rangle)_{R_1 R_2} \left( ab|1\rangle + \beta a e^{-i\phi}|0\rangle \right)_{R_3} \right].$$

When Alice2 obtains $|v_1\rangle$

$$|\Phi_1\rangle = \frac{1}{2\sqrt{(b^2|\alpha|^2 + a^2|\beta|^2)}} \left[ (|+\rangle|0\rangle)_{R_1 R_2} \left( abe^{i\phi}|0\rangle + \beta a|1\rangle \right)_{R_3} 
+ (|\rangle|0\rangle)_{R_1 R_2} \left( abe^{i\phi}|0\rangle - \beta a|1\rangle \right)_{R_3} 
+ (|+\rangle|1\rangle)_{R_1 R_2} \left( abe^{i\phi}|1\rangle + \beta a|0\rangle \right)_{R_3} - (|\rangle|1\rangle)_{R_1 R_2} \left( abe^{i\phi}|1\rangle - \beta a|0\rangle \right)_{R_3} \right].$$

Similar to the earlier cases of probabilistic HJRSP, Bob3 will have to apply one of the unitary operations $U_0$ or $U_1$ before reconstructing the state by operating a suitable Pauli gate, which depends on the measurement outputs of Bob1 and Bob2. All the successful cases of probabilistic HJRSP are listed in Table 4.

### 3 Effect of a set of noise models on the HJRSP scheme

In the recent past, several schemes for classical and quantum communication tasks have been theoretically proposed in the ideal situations, i.e., without considering the effects of noise present in the communication channel. However, it is well understood that in any practical situation, noise would play a crucial role, and the success of a scheme would depend on the noise present in the channel. This fact motivates us to investigate the effect of different quantum noise models on the HJRSP schemes proposed here. Specifically, in this section we will investigate the effect of AD noise, PD noise, collective dephasing (CD) noise, collective rotation (CR) noise and Pauli noise.

In this section, we aim to analyze the feasibility of the implementation of the proposed deterministic HJRSP scheme in a noisy environment. To quantitatively investigate the effect of noise on a scheme of quantum communication, fidelity, which is a
Table 4 All possible successful cases in the probabilistic HJRSP scheme are summarized with the measurement outcomes of all the senders and receivers with corresponding unitary and Pauli operations to be implemented by Bob3 to recover the quantum state

| Alice1’s measurement outcome in \{u_0, u_1\} basis | Alice2’s measurement outcome in \{v_0, v_1\} basis | Bob1’s measurement outcome (in \{+\}, \{-\}) basis | Bob2’s measurement outcomes (in \{0\}, \{1\}) basis | Two qubit unitary operation to be applied by Bob3 (U_0/U_1) | Pauli operation to be applied by Bob3 to reconstruct |\xi\rangle |
|---|---|---|---|---|---|
| \uvec{0} \rangle | \uvec{0} \rangle | + \rangle | 0 \rangle | U_0 \quad I | |
| \uvec{0} \rangle | \uvec{1} \rangle | + \rangle | 1 \rangle | U_1 \quad I | |
| \uvec{1} \rangle | \uvec{0} \rangle | + \rangle | 0 \rangle | U_0 \quad Z | |
| \uvec{1} \rangle | \uvec{1} \rangle | + \rangle | 1 \rangle | U_1 \quad Z | |
| \uvec{1} \rangle | \uvec{0} \rangle | - \rangle | 0 \rangle | U_0 \quad I | |
| \uvec{1} \rangle | \uvec{1} \rangle | - \rangle | 1 \rangle | U_1 \quad I | |
| The receiver Bob3 needs the joint collaboration of all the other agents |
distance based measure, is usually used. Specifically, the fidelity of the state obtained after considering the effect of noise with the reconstructed state in the ideal case can be obtained as

$$\mathcal{F} = \langle T | \rho_k | T \rangle. \quad (24)$$

Here, $|T\rangle$ is the quantum state reconstructed after implementing the HJRSP scheme in an ideal situation, while $\rho_k$ is the density matrix of the quantum state obtained after considering the effect of an interaction with the surrounding (i.e., when noise is present). To be precise, the definition of fidelity used here (also used in Refs. [22, 35, 38–40]) is slightly different from the conventional one, $F'(\sigma, \rho) = \text{Tr} \sqrt{\sigma \frac{1}{2} \rho \sigma \frac{1}{2}}$. Further, in what follows, we will use the strategy adopted in Refs. [22,35,38–40] to study the effect of noise. It is a reasonable assumption that the qubits not traveling through the channel are hardly affected by the noisy environment. Hence, we have not considered the effect of noise on the home qubit, which Alice1 has prepared and kept for herself. Consider the initial quantum state $\rho = |C\rangle_{S_1S_2R_1R_2R_3} |C\rangle$, where $|C\rangle_{S_1S_2R_1R_2R_3}$ is the cluster state given in Eq. (1). The transformed density matrix under the effect of AD or PD noisy channel can be expressed as

$$\rho_k = \sum_{i,j,k,l} I_{2, S_1} \otimes E_{i, S_2}^p \otimes E_{j, R_1}^p \otimes E_{k, R_2}^p \otimes E_{l, R_3}^p \rho \left( I_{2, S_1} \otimes E_{i, S_2}^p \otimes E_{j, R_1}^p \otimes E_{k, R_2}^p \otimes E_{l, R_3}^p \right)^\dagger, \quad (25)$$

where $I_2$ is a $2 \times 2$ identity matrix, and its application on qubit $S_1$ corresponds to unaffected home qubit of Alice1. For the remaining qubits, $E_j^p$ are the Kraus operators for AD or PD noise channels with $p \in \{AD, PD\}$ for AD and PD noise, respectively. Here, $J \in \{0, 1\}$ for AD and $J \in \{0, 1, 2\}$ for PD noise models. The Kraus operators for AD and PD noise channels will be described in detail in the following subsections. In the subscripts of the Kraus operators, the qubit on which it operates is also mentioned. The same strategy may be used for the investigation of the effect of Pauli channels, where the four Pauli gates (including identity operator) are used to study the errors introduced due to noisy channel. Further, the evolved quantum state under the collective noise models can be described as

$$\rho_k = \left( I_{2, S_1} \otimes U_k^{\otimes 4} \right) \rho \left( I_{2, S_1} \otimes U_k^{\otimes 4} \right)^\dagger, \quad (26)$$

where the subscript $k$ is CD or CR for CD or CR noise channels and $U_k$ is a $2 \times 2$ unitary matrix for either CD or CR noise channels. The notion of collective noise was introduced as a coherent effect on all the qubits that are interacting with its environment [41]. Specifically, all the polarization-encoded photonic qubits traveling through an optical fiber may undergo the same birefringence [42]. This kind of a coherent effect is mathematically handled by a unitary evolution of all the qubits controlled by a time-
dependent noise parameter which remains the same for all the qubits at a particular time (see [38] for detail discussion).

In the following subsections, we will analyze the deterministic HJRSP scheme subjected to various noise models after briefly introducing them. Further, we will discuss the dependence of the obtained average fidelity expressions on the noise parameters, which would quantitatively illustrate the effect of noise on the scheme. Here, we will refrain from considering the effect of noise on the probabilistic HJRSP scheme, which will be discussed elsewhere. For calculating the average fidelity, we have performed averaging over all possible measurement outcomes of the senders and receivers in accordance with Ref. [43–45] by defining average fidelity as

\[
F_{\text{av}} = \sum_{i} P_i \langle I | \rho_i | I \rangle,
\]

where \(|I\rangle\) is the input state and \(\rho_i\) is the remotely prepared state corresponding to \(i\)th element of the set of projective measurement outcomes of all the senders and receivers except the one reconstructing it (as one who reconstructs it does not perform any measurement). The probability of each measurement outcome can be calculated as

\[
P_i = \text{Tr}(\rho_{\Omega} M_i)
\]

where \(\rho_{\Omega}\) is the channel in the noisy environment (initially prepared in the state \(|\Omega\rangle = |\psi_{\text{channel}}\rangle\) and \(M_i\) is the set of measurement operators applied by all the senders and receivers. In particular, when the higher- (lower-) power agent reconstructs a quantum state, we need to compute fidelity expressions for 8 (16) possible combinations of measurement outcomes of all the senders and the other collaborating receivers. In what follows, we have dropped the subscript “av” in \(F_{\text{av}}\) and simply write it as \(F\). It is also worth mentioning here that another notion of average fidelity is also present in the literature [46,47], where the \(F_{\text{av}}\) we are calculating is called fidelity, and after averaging over all possible input states, average fidelity is computed (see [47] for detail discussion). However, we have not used that notion here.

### 3.1 Amplitude damping (AD) noise

The AD noise model is represented by the following Kraus operators [3,35,48]

\[
E_{0\text{AD}} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1 - \eta_A} \end{bmatrix}, \quad E_{1\text{AD}} = \begin{bmatrix} 0 & \sqrt{\eta_A} \\ 0 & 0 \end{bmatrix},
\]

where \(\eta_A (0 \leq \eta_A \leq 1)\) is the decoherence rate and is the probability of energy loss when a travel qubit passes through an AD channel. Specifically, an AD channel simulates the interaction of a quantum system with a vacuum bath. Using Eqs. (24), (25) and (28), we obtain the average fidelity of the quantum state reconstructed by the receivers Bob_2 and Bob_3 as

\[
F_{\text{Bob}_2}^{\text{AD}} = \frac{(2\eta_A^2 - 3\eta_A + 2) \sin^4(\theta) + (2\eta_A^2 - 3\eta_A + 2) \cos^4(\theta) - 2 \sin^2(\theta) \cos^2(\theta) (\eta_A^3 + (\eta_A - 1)\eta_A^2 \cos(2\phi) - 3\eta_A^2 + 3\eta_A - 2)}{2((\eta_A - 1)\eta_A + 1)},
\]

(29)
and

\[
F_{AD}^{\text{Bob}_3} = (1 - \eta_A)^{3/2} \eta_A \sin^2(\theta) \cos^2(\theta) \cos(2\phi) \\
+ \frac{1}{2} \left( ((\eta_A - 2)\eta_A + 2) \sin^4(\theta) \\
+ ((\eta_A - 2)\eta_A + 2) \cos^4(\theta) + 2(\eta_A - 2) \\
\left( (\sqrt{1 - \eta_A} - 1) \eta_A - \sqrt{1 - \eta_A} \right) \sin^2(\theta) \cos^2(\theta) \right),
\]

(30)

respectively. It would be relevant to mention that the fidelity expressions of the quantum state obtained by other lower-power agent Bob1 are exactly the same as that for Bob3 in all the noise models discussed in this paper. Hence, the state reconstructed by both the lower-power agents will be equally affected by the noise. Henceforward, we will only report the fidelity expressions for Bob3 and the conclusions deduced from them will be automatically implementable for Bob1 as well.

It would also be worth noting here that in the absence of noise, both the lower-power agents always obtain the same measurement results. Therefore, the probability corresponding to all those measurement outcomes when they do not obtain the same results is zero. However, in a noisy environment, there may be a nonzero probability for such events to occur. All these cases would lead to failure of the proposed scheme. There is no point of calculating fidelity when the scheme fails. So the average fidelity would mean averaged over those cases where the scheme succeeds, and probability of a successful reconstruction event should be considered as the relative probability among the all possible cases where the scheme succeeds. Hence, we have normalized the probability of all the successful cases. In case of higher-power agents, it is done by dividing the probability of a successful event by the total probability of success. This approach is new and logical. Earlier, there was an attempt to calculate the average fidelity of the probabilistic teleportation scheme [49], but unlike their scheme our scheme is deterministic in the absence of noise.

The average fidelity \(F_{AD}\) depends on various parameters, such as amplitude \((\theta)\) and phase \((\phi)\) of the quantum state to be remotely prepared and the decoherence rate \((\eta_A)\). This dependence is illustrated in Fig. 2. Specifically, Fig. 2a establishes that the state reconstructed by the higher-power agent Bob2 gets less affected due to noise in comparison with the state reconstructed by the lower-power agents. Figure 2b illustrates variation of the fidelity with state parameters for the higher-power agent considering the decoherence rate \(\eta_A = 1/2\). It can be inferred from the plot that the obtained fidelity may be higher for certain choices of quantum state to be remotely prepared. These facts can also be illustrated using contour plots. Hereafter, we would only use contour plots to investigate the effect of noise on the deterministic HJRSP scheme. However, here we have shown a corresponding contour plot as well in Fig. 2c. A slightly different nature of dependence for the average fidelity of the lower-power agent on state parameters is observed in Fig. 2d.

The case reported here for a dissipative interaction with a vacuum bath, simulated as AD channel, can be extended to generalized amplitude damping [50,51] and squeezed generalized amplitude damping [50,51] noise models, where finite temperature reser-
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Fig. 2 (Color online) In (a), the fidelity $F_{AD}$ is plotted with the decoherence rate $\eta_A$ considering the state parameters $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{3}$. The smooth (blue) and dotted (red) lines correspond to the fidelities of the quantum states for the higher-power agent Bob 2 and the lower-power agent Bob 3. (b) Three-dimensional variation of the fidelity $F_{AD}$ with quantum state parameters, i.e., amplitude ($\theta$) and phase ($\phi$) for decoherence rate $\eta_A = \frac{1}{2}$. (c, d) The variation of the average fidelity of the quantum state obtained by the receivers Bob 2 and Bob 3, respectively, with the state parameters considering $\eta_A = \frac{1}{2}$.

3.2 Phase damping (PD) noise

The PD noise model is represented by the following Kraus operators [3, 35, 48]

$$E_{PD}^{0} = \sqrt{1 - \eta_P} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_{PD}^{1} = \sqrt{\eta_P} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{PD}^{2} = \sqrt{\eta_P} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (31)$$

where $\eta_P (0 \leq \eta_P \leq 1)$ is the decoherence rate. The PD noise simulates an interaction with the surroundings when energy loss is not involved. In the presence of this noise, the average fidelities of the quantum state reconstructed by the higher- and lower-power agents Bob 2 and Bob 3 can be obtained using Eqs. (24), (25) and (31) as

$$F_{PD}^{Bob_2} = \frac{1}{4} \left( \eta_P^2 - (\eta_P - 2) \eta_P \cos(4\theta) - 2\eta_P + 4 \right),$$

$$F_{PD}^{Bob_3} = \frac{1}{4} \left( \eta_P^2 - (\eta_P - 2) \eta_P \cos(4\theta) - 2\eta_P + 4 \right).$$
In (a), the average fidelity $F_{PD}$ of the quantum state reconstructed by both lower- and higher-power agents are plotted together with the decoherence rate $\eta_P$ in the presence of PD noise for $\theta = \frac{\pi}{4}$. The smooth (blue) and dotted (red) lines correspond to the fidelity for the state reconstructed by higher-power agent and the lower-power agent. In (b, c) the variation of average fidelity with state parameter $\theta$ and decoherence rate $\eta_P$ in contour plots for receivers Bob2 and Bob3, respectively. It can be observed that the contour plot for fidelity variation of Bob2’s quantum state is symmetric with that of Bob3. However, the higher-power agent obtains the state with slightly more fidelity than that of the lower-power agent.

and

$$F_{PD}^{Bob3} = \sin^4(\theta) + \cos^4(\theta) - 2\sin^2(\theta)\cos^2(\theta)(\eta_P - 1)^3,$$

respectively.

It can be observed from the expressions of average fidelity in Eq. (32) that it is phase ($\phi$) independent. Consequently, a family of states with the same value of $\theta$ undergo the same decoherence, i.e., for a particular value of $\theta$ all the states will be affected in the same way for any value of $\phi$. Here, in the HJRSP scheme, this corresponds to the fact that any choice of Alice2 will not affect the state reconstructed by the receivers. Further, we have illustrated the dependence of average fidelity on various parameters in Fig. 3. Specifically, Fig. 3a shows a similar nature of what was observed in the presence of AD noise in Fig. 2a, i.e., if the higher-power agent Bob2 reconstructs the state, then the reconstructed state gets less affected by noise in comparison with the cases where the lower-power agents reconstruct the state. In Fig. 3b, c, the contour plots of variation of the average fidelity with the amplitude ($\theta$) and decoherence rate ($\eta_P$) are shown for Bob2 and Bob3, respectively. The contour plots show that although a symmetric variation in the fidelity is observed, the state reconstructed by the higher-power agent is found to be less affected by the noise. Further, it is observed that for certain values of different parameters we may obtain a unit fidelity even in the noisy environment.

### 3.3 Collective dephasing (CD) noise

The CD noise model is represented by the following unitary (phase) operator

$$U_{CD} = \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\Phi) \end{bmatrix},$$

(33)
where $\Phi$ is the noise parameter that may change with time, but remains the same at an instant for all the qubits traveling simultaneously through the noisy channel. It should be noted here that the collective noise models consider a coherent effect of interaction on all the travel qubits. The average fidelities for the state reconstructed by Bob$_2$ and Bob$_3$ can be obtained under CD noise model as

\begin{align}
F_{\text{CD}}^{\text{Bob}_2} &= \frac{1}{16} \left( -\cos(4\theta - 2\Phi) - \cos(4\theta + 2\Phi) + 2\cos(4\theta) + 2\cos(2\Phi) + 14 \right), \\
F_{\text{CD}}^{\text{Bob}_3} &= \sin^2(\theta)\cos^2(\theta)(\cos(\Phi) + \cos(3\Phi)) + \sin^4(\theta) + \cos^4(\theta),
\end{align}

respectively.

From the above expressions, it can be observed that the average fidelity is free from the phase of the quantum state to be remotely prepared. Hence, the fidelity of the obtained state will only depend on the noise parameter and the amplitude of the state to be reconstructed. To elaborate the dependence explicitly, we have plotted the average fidelity ($F_{\text{CD}}$) as a function of amplitude ($\theta$) and the noise parameter ($\Phi$) in Fig. 4. Specifically, in Fig. 4a, the average fidelity is shown to vary with the noise parameter for a family of states with an arbitrary value of phase angle and amplitude $\theta = \frac{\pi}{4}$. Here, it is interesting to note that, unlike the previous cases when the HJRS scheme was subjected to AD and PD noise models, even the lower-power agent can acquire the quantum state with higher fidelity than that of the higher-power agent. Specifically, in Fig. 4a, in the vicinity of $\Phi = \frac{\pi}{2}$ the state reconstructed by a lower-power agent is found to be less affected by noise in comparison with the same state reconstructed by a higher-power agent. Further, when the noise parameter $\Phi = \pi$, then Bob$_2$ recovers unaffected state, i.e., the state with unit fidelity, while Bob$_3$ can reconstruct the quantum state with negligible fidelity. Figure 4b, c further illustrates the same facts through contour plots and manifests the effect of noise on all the possible quantum states that can be remotely prepared. It can be observed from the plots that unit fidelity can be obtained in some cases. Except a few values of the parameters, in general, the higher-power agent can extract higher fidelity quantum state.
3.4 Collective rotation (CR) noise

The CR noise model is represented by a unitary rotation operator

\[ U_{CR} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix}, \]  

where \( \Theta \) is the noise parameter and have the same effect and property as \( \Phi \) in the CD noise model. The average fidelity expressions of the quantum states reconstructed by the higher- and lower-power agents when the HJRSP scheme is subjected to CR noise are obtained as

\[
F_{\text{Bob}_2}^{\text{CR}} = \frac{1}{\sin^4(\Theta) + \cos^4(\Theta)} \left\{ \frac{1}{8} \sin(2\theta) \sin^2(\Theta) \cos(2\phi)(-\sin(2\theta - 4\Theta) + 2 \sin(2(\Theta - \theta)) - 5 \sin(2\theta)) \rule{0pt}{0.5em} \right. \\
\times (\cos(\Theta) - \sin(\Theta))(\sin(\Theta) + \cos(\Theta)) + \frac{3}{32} \sin^2(2\theta) \sin^4(2\Theta) + \sin(2\theta) \sin^3(\Theta) \\
\times \cos^5(\Theta)(\cos(\theta) - \sin(\theta))(\sin(\theta) + \cos(\theta)) - \frac{1}{2} \sin(4\Theta) \sin^5(\Theta) \cos^3(\Theta) \\
+ \cos^2(2\theta) \sin^6(\Theta) \cos^2(\Theta) + \frac{1}{4} (\cos(4\Theta) + 3) \sin^8(\Theta) + \cos^8(\Theta) \\
+ \sin^2(\Theta) \cos^6(\Theta) \} ,
\]

\[
F_{\text{Bob}_3}^{\text{CR}} = \frac{1}{4} (2 \sin(2\theta) \sin^3(\Theta) \cos(2\phi) \cos(2\Theta + \Theta) \\
- 2 \sin(2\theta) \sin^2(\Theta) \cos(\Theta) \cos(2\phi) \sin(2\theta - 3\Theta) \\
+ (\cos(4\Theta) + 3) \sin^2(\Theta) \cos^4(\Theta) + 4 \sin(4\Theta) \sin^3(\Theta) \cos^3(\Theta) \\
- 2(\cos(4\Theta) - 3) \times \sin^4(\Theta) \cos^2(\Theta) + (\cos(4\Theta) + 3) \sin^6(\Theta) + 4 \cos^6(\Theta)).
\]

To express the dependence of average fidelity of the state received by both higher- and lower-power agents on various parameters, we have plotted the fidelity expressions in Fig. 5. Specifically, Fig. 5a shows the variation in the average fidelity \( F_{\text{CR}} \) with the noise parameter (\( \Theta \)) for a particular quantum state with \( \theta = \pi/4 \) and \( \phi = \pi/3 \) to be remotely prepared. Interestingly, for a few specific values of the noise parameter (e.g., for \( \frac{\pi}{6} < \Theta < \frac{2\pi}{3} \)), the higher-power agent can reconstruct the state with lesser fidelity compared to the state obtained by the lower-power agent. In contrast to the CD noise case, here lower-power agent can also reconstruct a state with unit fidelity. For a particular value of noise parameter, i.e., \( \Theta = \pi/3 \), the class of states with phase angle \( \pi/2 \) shows that they are not equally affected due to noise (cf. Fig. 5b). In fact, if the state is reconstructed by the higher-power agent, then the value of the amplitude parameter matters more than the cases when lower-power agents choose to do so. Further, it also shows that for \( \theta = \pi/2 \), higher-power agent Bob_2 achieves the maximum fidelity of the received state, whereas the fidelity of the state obtained by the lower-power agent Bob_3 could never reach that limit. Figure 5c shows the fidelity variation with phase angle \( \phi \), and a complimentary nature for the fidelity of the quantum state obtained by higher- and lower-power agents is observed, i.e., it is found that if the fidelity for one increases, it decreases for the other. The contour plots in Fig. 5d–g show that some
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Fig. 5 (Color online) In (a), the average fidelity $F_{\text{CR}}$ is plotted together for both higher- and lower-power agents with the noise parameter $\Theta$ for state parameters $\theta = \pi/4$ and $\phi = \pi/3$. b The dependence of the average fidelity $F_{\text{CR}}$ of the reconstructed state for the agents on the amplitude parameter $\theta$ for $\phi = \pi/2$ and noise parameter $\Theta = \pi/3$. In (c), the average fidelity is plotted with the phase angle $\phi$ for $\theta = \pi/3$ and $\Theta = \pi/2$. In (a–c), the smooth (blue) and dotted (red) lines correspond to the fidelities of state reconstructed by higher-power agent Bob$_2$ and the lower-power agent Bob$_3$. d, e The variation of average fidelity with noise parameter and amplitude parameter considering $\phi = \pi/3$ for the receivers Bob$_2$ and Bob$_3$, respectively. Similarly, (f) and (g) The variation for $\theta = \pi/4$ with noise parameter and phase angle for the higher- and lower-power agents, respectively.

A specific choice of initial quantum states can be remotely prepared with unit fidelity or fidelity close to 1 in the CR noisy environment for particular values of the noise parameter.

The nature observed here for the collective noise model, i.e., the effect of collective noise is in contrast of that in the AD and PD channels, is consistent with a few earlier observations [38,52]. Specifically, in Ref. [38], it has been observed that single qubits perform better as decoy qubits in secure quantum communication than entangled qubits when subjected to AD and PD noise, whereas completely opposite results have been obtained in collective noise, i.e., a few Bell states were shown to be decoherence-free. Similarly, single-qubit-based secure quantum communication schemes have been found robust in AD and PD channels, while their entangled-state-based counterparts were found to perform better in the presence of collective noise [52].
3.5 Pauli (P) noise

There are four well-known Pauli matrices that are frequently used in quantum information. Just to introduce the notation followed in this section, we note that the Pauli matrices are described in the following manner:

\[
\sigma_1 \equiv \sigma_x \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 \equiv \sigma_y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 \equiv \sigma_z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\]

(37)

These operators are relevant for the present study as a quantum state described by the density operator \(\rho\) evolves under Pauli noise model (in Pauli channel) as described in Eq. (25). Specifically, in Pauli channels, a single-qubit state will evolve as

\[
\rho' = \sum_{i=1}^{4} E_i^P \rho E_i^{P\dagger},
\]

(38)

where \(E_i^P = \sqrt{p_i} \sigma_i\); \(p_i\) is the probability of \(i\)th type of error, which is modeled by Pauli operator (error operator) \(\sigma_i\) defined in Eq. (37), and \(\sum_{i=1}^{4} p_i = 1\). To visualize that \(p_i\) is the probability of the error modeled by \(\sigma_i\), we may expand Eq. (38) as

\[
\rho' = p_1\sigma_1\rho\sigma_1 + p_2\sigma_2\rho\sigma_2 + p_3\sigma_3\rho\sigma_3 + (1 - p_1 - p_2 - p_3)\rho,
\]

(39)

which clearly shows that the density operator \(\rho\) undergoes a bit-flip (modeled by \(\sigma_1\)) with the probability \(p_1\), a combined bit and phase-flip (modeled by \(\sigma_2\)) with probability \(p_2\) and a phase-flip (modeled by \(\sigma_3\)) with the probability \(p_3\). However, \(\rho\) remains unchanged with the probability \((1 - p_1 - p_2 - p_3) = p_4\), which implies an error-free channel. The Pauli channel is widely discussed in the literature [53–55], and its name originates from the error operators of the channel. Specifically, the most general Pauli channel described by (39) reduces to (i) a bit-flip channel when \(p_2 = p_3 = 0\), (ii) a bit-phase-flip channel when \(p_1 = p_3 = 0\), (iii) a phase-flip channel when \(p_1 = p_2 = 0\) and (iv) a noise-free channel or identity channel when \(p_1 = p_2 = p_3 = 0\). Similarly, we may define a depolarizing noise channel when all the bit-flip, phase-flip and bit-phase-flip errors occur with equal probability \(p_1 = p_2 = p_3 = p \leq \frac{1}{3}\), and the state remains unchanged with the remaining probability.

Above discussion shows that if we can obtain fidelity for the most general Pauli channel described by (39), we can obtain the fidelities for evolution under specific noise models as special cases of that. Keeping this in mind, we have obtained following analytic expressions for the average fidelity of the quantum state remotely prepared by the proposed HJRSP scheme when the quantum state to be prepared remotely is
evolved under most general Pauli noise model, and either Bob2 or Bob3 reconstructs the state:

\[ F_{p_{\text{Bob}}} = \frac{1}{(p_1 + p_2)^2 + (p_3 + p_4)^2} \left( 2p_1^4 + (p_3 - p_4)^2 - 2p_2^2p_1^2 \right. \\
-2(p_3 - p_4)((p_3 + p_4)^2 + p_2(p_4 - p_3))p_1 \\
+ p_2 \left( p_3^2 + (p_3 - p_4)^2 p_2 + 2(p_3 - p_4)(p_3 + p_4) \right) \sin^2(\theta) \cos^2(\theta) \cos(2\phi) \\
+ \left. \left( p_1^2 + (2p_2 - p_3 - p_4)p_1 + p_2^2 + (p_3 + p_4)^2 - 2p_2(p_3 + p_4) \right) \sin^4(\theta) \right) \\
+ \left( p_1^2 + (2p_2 - p_3 - p_4)p_1 + p_2^2 + (p_3 + p_4)^2 - 2p_2(p_3 + p_4) \right) \cos^4(\theta) \\
+ 2(-p_3 - 3p_4)p_1^3 + 3(p_3 + p_4)^2 + p_2(p_3 + 5p_4)p_1^2 + ((p_3 + p_4)^3 \\
+ 2p_2(p_3 + p_4)^2 + p_2^2(5p_3 + p_4))p_1 + p_2(p_3 + p_4)^3 + 3p_2^2(p_3 + p_4)^2 \\
+ \left( p_3^2 - p_4^2 \right) p_1^2 + \left. \left( 3p_3 - p_4 \right) \right) \sin^2(\theta) \cos^2(\theta) \right). \]

\[ F_{p_{\text{Bob}}} = 2(p_1 - p_2 - p_3 + p_4) \left( p_1^3 - p_2p_1^2 - \left( p_2^2 + (p_3 - p_4)(p_3 + 3p_4) \right) p_1 \\
+ p_2 \left( p_2^2 + (p_3 - p_4)(3p_3 + p_4) \right) \right) \\
\times \sin^2(\theta) \cos^2(\theta) \cos(2\phi) + ((p_1 + p_2)^2 + (p_3 + p_4)^2) \sin^4(\theta) \right) \\
+ \left( p_1^2 + p_2^2 + (p_3 + p_4)^2 \right) \cos^4(\theta) \\
+ 2((p_3 + 5p_4)p_1^3 + \left( 7p_4^2 + (p_2 + 4p_3)p_4 + 5p_3(p_2 + p_3) \right) p_1^2 \\
+ ((11p_3 + 7p_4)p_2^2 + \left( 2p_3 + p_4 \right)(5p_3 + 3p_4)p_2 + (p_3 + p_4) \left( 3p_2^2 + 2p_4p_3 + 3p_3^2 \right))p_1 \\
- p_2^2(p_3 - 3p_4) - (p_3 - p_4)^3(p_3 + p_4) + p_2(p_3 + p_4) \left( p_3^2 + 6p_4p_3 + p_3^2 \right) \\
+ p_2^2(p_3^2 + 12p_4p_3 + 3p_2^2)) \sin^2(\theta) \cos^2(\theta). \]  

(40)

To illustrate the effect of Pauli noise, we have plotted the average fidelity \( F_p \) under the Pauli noise model for the state parameters \( \theta = \pi/4 \) and \( \phi = \pi/3 \) in Fig. 6. Here, we have used a depolarizing channel, where the total probability of error for bit, phase and bit-phase-flip errors is \( p' \), and consequently, \( \frac{p}{3} \) is the probability for each kind of error. Specifically, in Fig. 6a, the average fidelity \( F_p \) is shown in case of bit-flip errors. It can be observed that the fidelities of the state obtained by Bob2 and Bob3 vary almost similarly and decay to the lowest value when the probability of errors \( p' \) is unity. Similarly, in Fig. 6b, the average fidelity of the state reconstructed by both higher- and lower-power agents in bit-phase-flip channel is shown. Interestingly, it has been observed that for higher probability of errors, i.e., \( p' > 0.5 \), even higher-power agent Bob2 has less fidelity of the reconstructed state than that of the lower-power agent. In fact, beyond this value of probability of errors a revival in the fidelity for the lower-power agent can be seen. In Fig. 6c, variation of the average fidelity \( F_p \) with the probability of phase-flip error is illustrated. In contrast to Fig. 6b, here the fidelity obtained by the higher-power agent is revived and reached to unity for maximum probability of error. Further, the fidelity of the state obtained by the higher-power agent is always higher than that of the lower-power agent, which shows a gradual decay with increasing probability of error (cf. red (dashed) line in Fig. 6c). Finally, in Fig. 6d, we have plotted the average fidelity \( F_p \) under the effect of depolarizing
channel, where the fidelity of states obtained by both the receivers converges for $0.6 < p' < 0.9$, and after that it decays for the higher-power agent.

Before we conclude this paper, we would like to summarize the effect of noise on the proposed scheme and thus make an effort to compare the effects of various types of noise on the proposed scheme. To do so, we may first note that the investigations performed in this section reveal that the proposed HJRSP scheme has a finite failure probability in the presence of AD, CR and Pauli noises. However, the scheme works with unit success probability for other type of noises. Thus, the proposed deterministic HJRSP scheme gets transformed into a probabilistic scheme in the presence of AD, CR and Pauli noises and remains deterministic in the presence of the other types of noise. Further, in the presence of phase damping and collective dephasing noises, the effect of noise is independent of the input of Alice$_2$, i.e., it does not depend on the value of phase angle $\phi$ (see Eqs. (32) and (34)). However, for the other type of noises, fidelity depends on $\phi$. A comparison between AD and PD noises reveals that for the same value of noise parameter ($\eta = \frac{1}{2}$), states can be reconstructed with unit fidelity by both the higher and lower power receivers in PD noise, but the same is not feasible in AD noise (this may be observed by comparing Fig. 2c, d with Fig. 3b, c). In both type of collective noises, for certain choices of noise and state parameters a
lower-power agent can reconstruct the state with a higher fidelity in comparison with the higher-power agent (see in Figs. 4a, 5a–c, there are regions where the red lines depicting fidelity obtained by a lower-power agent are above the blue lines depicting fidelity obtained by a higher-power agent). This feature is not observed in AD and PD noises (cf. Figs. 2a, 3). In a similar manner, in the presence of Pauli noises, bit and bit-phase-flip allow a lower-power agent to reconstruct the state with higher fidelity in comparison with a higher-power agent for certain values of noise and state parameters (cf. Fig. 6a, b). However, phase-flip does not allow this possibility (cf. Fig. 6c), and depolarizing channels allow it to happen only for very large values of probability of errors (cf. Fig. 6d). Further, in case of collective noises, the values of noise parameters for which states can be reconstructed with unit fidelity by a lower- or a higher-power agent are found to be different (see Figs. 4b, c, 5d–g).

4 Conclusions

In the present paper, we have introduced the notion of HJRSP and have proposed a protocol for deterministic HJRSP using 5-qubit cluster state of the form (1). The scheme is also illustrated in Fig. 1. It is unique in nature because it allows joint preparability (more than one sender), a feature which is desirable in practical situations, but not present in any of the existing schemes of hierarchical quantum communication. Further, in the present work, the effect of various noise models on the proposed HJRSP scheme has been investigated in detail. In contrast, in none of the existing proposals for hierarchical quantum communication, the effect of noise has been discussed. However, in a practical situation, it is impossible to circumvent the existence of noise. Thus, the present study not only makes it more realistic, it also yields an extremely relevant and essential feature of hierarchical quantum communication. Further, in addition to the deterministic HJRSP, a protocol for probabilistic HJRSP using a non-maximally entangled cluster state is also introduced here. Usually, a reduction of required quantum resources is expected in probabilistic RSP. However, it was not obtained here due to the use of a non-maximally entangled state.

The study of the effect of different noise models has led to many interesting results. Specifically, it is observed that the quantum state that the higher-power agent and the lower-power agent can reconstruct may get decohered due to interaction with its surroundings, which have been analyzed and discussed in detail in the previous section. In brief, the higher-power agent can reconstruct the quantum state less affected due to AD and PD noise than that of the lower-power agent. However, when the travel qubits are subjected to collective or Pauli noise, lower-power agent may also get better performance than that of the higher-power agent. Interestingly, a few specific quantum states can be remotely prepared in an unaffected manner, even in the presence of noise. Here, we have restricted ourselves to Markovian noise. The effect of non-Markovian type of noise will be reported elsewhere.

It would be interesting to study controlled versions of the hierarchical quantum communication protocols proposed here and in the recent past, as a variety of applications of the various controlled quantum communication protocols have been widely discussed [39]. We expect that due to the wide applicability of the hierarchical schemes
the proposed scheme may be of interest to experimentalists and in the near future the results reported here will be experimentally verified.

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