RNA Number of Some Parity Signed Generalized Petersen

Graphs

Deepak Sehrawat
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati, India - 781039
Email: deepakmath55555@iitg.ac.in

Bikash Bhattacharjya
Department of Mathematics
Indian Institute of Technology Guwahati
Guwahati, India - 781039
Email: b.bikash@iitg.ac.in

Abstract. A signed graph \( \Sigma = (G, \sigma) \) is said to be parity signed if there exists a bijection \( f : V(G) \to \{1, 2, \ldots, |V(G)|\} \) such that \( \sigma(uv) = + \) if and only if \( f(u) \) and \( f(v) \) are of same parity, where \( uv \) is an edge of \( G \). The RNA number of a graph \( G \), denoted \( \sigma^{-}(G) \), is the minimum number of negative edges among all possible parity signed graphs over \( G \). The RNA number is also equal to the minimum cut size that has nearly equal sides.

In this paper, for generalized Petersen graph \( P(n,k) \), we prove that \( 3 \leq \sigma^{-}(P(n,k)) \leq n \) and these bounds are sharp. The exact value of \( \sigma^{-}(P(n,k)) \) is determined for \( k = 1, 2 \). Some famous generalized Petersen graphs namely, Petersen graph \( P(5,2) \), Dürer graph \( P(6,2) \), Möbius-Kantor graph \( P(8,3) \), Dodecahedron \( P(10,2) \), Desargues graph \( P(10,3) \) and Nauru graph \( P(12,5) \) are also treated.

We show that the minimum order of a \((4n-1)\)-regular graph having RNA number one is bounded above by \( 12n - 2 \). The sharpness of this upper bound is also shown for \( n = 1 \). We also show that the minimum order of a \((4n+1)\)-regular graph having RNA number one is \( 8n + 6 \). Finally, for any simple
connected graph of order \( n \), we propose an \( O(2^n + n^{\frac{3}{2}}) \) time algorithm for computing its \textit{rna} number.

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\textbf{Keywords:} generalized Petersen graph; parity signed labeling; parity signed graph; edge cut

\section{Introduction}

All graphs and signed graphs considered in this paper are simple, connected and undirected. For all the graph theoretic terms that are used here, we refer the reader to [3].

A signed graph is a graph whose edges are either positive or negative. Harary [5] introduced the concept of signed graphs and since then signed graphs have been considered to be a natural generalization of ordinary graphs.

Recently, Acharya and Kureethara [1] introduced a special type of signed graph called the parity signed graph. In [2], Acharya, Kureethara, and Zaslavsky have shown that parity signed graphs also have sociological aspects. This concept is based on the assignment of integers \( \{1, 2, ..., |V(G)|\} \) to the vertices of a graph \( G \). It is equivalent to a partition of the vertex set of a graph into two subsets, \( A \) and \( B \), such that \( ||A| - |B|| \leq 1 \). In [2], the authors characterized some families of parity signed graphs, namely, signed stars, bistars, cycles, paths and complete bipartite graphs.

The \textit{rna} number of a graph \( G \), denoted \( \sigma^- (G) \), is the minimum number of negative edges among all the possible parity signed graphs over \( G \). It is equal to the size of a minimum cut whose sides are nearly equal. The \textit{rna} number of some families of graphs such as stars, wheels, paths, cycles and complete graphs have been examined. For details, see [1, 2].

This paper is organized as follows. In Section 2, we give necessary definitions and existing results. In Section 3, generalized Petersen graphs and their cuts with equal sides are discussed. In Section 4, we consider \textit{rna} number of generalized Petersen graphs and we show that

1. the \textit{rna} number for the class of all generalized Petersen graphs \( P(n, k) \) lies between 3 and \( n \).

2. \( \sigma^- (P(n, 1)) = 5 \) for odd \( n \geq 5 \), and \( \sigma^- (P(n, 1)) = 4 \) for even \( n \geq 4 \); and

3. \( \sigma^- (P(n, 2)) = 7 \) for odd \( n \geq 7 \), and \( \sigma^- (P(n, 2)) = 6 \) for even \( n \geq 8 \).

In Section 5, we show that the \textit{rna} number of Petersen graph, Dürer graph, Möbius-Kantor graph, Dodecahedron, Desargues graph and Nauru graph are 5, 4, 6, 6, 8 and 8, respectively. In Section 6, we show that the minimum order of a \((4n - 1)\)-regular graph having \textit{rna} number one is bounded above by \( 12n - 2 \). Also, a unique cubic graph is constructed having \textit{rna} number one that reach this bound.
also show that the minimum order of a \((4n + 1)\)-regular graph having \textit{rna} number one is \(8n + 6\). In Section 7, we propose an exponential time algorithm for computing the \textit{rna} number \(\sigma^-(G)\) of \(G\). In Section 8, we conclude the paper and propose a conjecture which states that the \textit{rna} number of a graph can be computed in polynomial time.

\section{Preliminaries}

A graph \(G = (V(G), E(G))\) is an ordered pair, where \(V(G)\) and \(E(G)\) represent the vertex set and the edge set of \(G\), respectively. By \(|V(G)|\) and \(|E(G)|\), we denote the order and the size of \(G\), respectively. An edge joining the vertices \(x\) and \(y\) is denoted by \(xy\). The length of a shortest path joining the vertices \(x\) and \(y\), denoted \(d_G(x, y)\), is called the \textit{distance} between \(x\) and \(y\). The \textit{k-th power} of a simple graph \(G\) is the graph \(G^k\) whose vertex set is \(V(G)\), and two distinct vertices are adjacent in \(G^k\) if and only if their distance in \(G\) is at most \(k\).

An edge \(e\) of a graph \(G\) is said to be a \textit{cut edge} of \(G\) if deletion of \(e\) results in a disconnected graph. An \textit{edge cut} (or simply a \textit{cut}) of \(G\) is a set of edges whose deletion results in a disconnected graph. In \(G\), a cut whose edges lie between vertices of \(A\) and \(A^c\) for some \(A \subseteq V(G)\) is denoted by \([A : A^c]\). The \textit{size} of the cut \([A : A^c]\) is the number of edges in \([A : A^c]\) and is denoted by \(|[A : A^c]|\). A cut of odd size is said to be an \textit{odd cut} and of even size is said to be an \textit{even cut}. The numbers \(|A|\) and \(|A^c|\) are called the sides of the cut \([A : A^c]\). The \textit{edge-connectivity} \(\kappa'(G)\) of a graph \(G\) is the minimum size of cut. For a connected graph \(G\) with minimum degree \(\delta\), it is well known that \(1 \leq \kappa'(G) \leq \delta\).

A \textit{signed graph} \(\Sigma = (G, \sigma)\) consists of a graph \(G = (V, E)\) and a sign function \(\sigma\) which labels each edge of \(G\) by \(+\) or \(-\) sign. The graph \(G\) is called the \textit{underlying graph} of \(\Sigma\). An edge is called \textit{positive} if \(\sigma(e) = +\), and \textit{negative} otherwise. The set of negative edges of \(\Sigma\) is \(E^-(\Sigma)\) and the set of positive edges is \(E^+(\Sigma)\). A signed graph \(\Sigma\) is said to be \textit{all-positive} if \(E^-(\Sigma) = \emptyset\) and \textit{all-negative} if \(E^+(\Sigma) = \emptyset\). A signed graph is \textit{homogeneous} if it is either all-positive or all-negative, and \textit{heterogeneous} otherwise.

Now we give some necessary definitions and results.

\textbf{Definition 1.} \cite{2} Given a graph \(G\) of order \(n\) and a bijection \(f : V(G) \rightarrow \{1, 2, ..., n\}\), define \(\sigma_f : E(G) \rightarrow \{+, -\}\) such that \(\sigma_f(uv) = +\) if \(f(u)\) and \(f(v)\) are of same parity and \(\sigma_f(uv) = -\) if \(f(u)\) and \(f(v)\) are of different parity, where \(uv\) is an edge in \(G\). We define \(\Sigma_f\) to be the signed graph \((G, \sigma_f)\).

\textbf{Definition 2.} \cite{1} A signed graph \(\Sigma = (G, \sigma)\) is called a \textit{parity signed graph}, if there exists a bijection \(f : V(G) \rightarrow \{1, 2, ..., n\}\) such that \(\sigma = \sigma_f\).

A signed graph is said to be \textit{balanced} if every cycle in it has an even number of negative edges. Harary introduced this idea in \cite{5}. In \cite{1}, the authors proved that a parity signed cycle is always balanced. Consequently, every parity signed graph is balanced, see \cite[Theorem 2.1]{2}.

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Definition 3. [1] The rna number of a graph $G$, denoted $\sigma^{-}(G)$, is the minimum number of negative edges among all possible parity signed graphs over $G$.

Note that finding the minimum number of negative edges among all parity signed graphs over a graph $G$ is equivalent to finding the size of a minimum cut of $G$ with nearly equal sides. More precisely, if $G$ is of even order then it is equivalent to find the size of a minimum cut whose sides are equal and if $G$ is of odd order then it is equivalent to find the size of a minimum cut whose sides differ by exactly one.

We now mention some known results about the rna number of some graphs, viz., paths, cycles, stars, wheels and complete graphs.

Theorem 1. [1] For any path $P_n$ of order $n$, $\sigma^{-}(P_n) = 1$.

Theorem 2. [1] For any cycle $C_n$ with $n$ vertices, $\sigma^{-}(C_n) = 2$.

Theorem 3. [1] For a star $K_{1,n}$ with $n+1$ vertices, $\sigma^{-}(K_{1,n}) = \left\lceil \frac{n}{2} \right\rceil$.

A wheel $W_n$ is the edge-disjoint union of $C_{n-1}$ and $K_{1,n-1}$. The rna number of a wheel $W_n$ is determined in [2, Theorem 12].

Theorem 4. [2] For a wheel $W_n$, $\sigma^{-}(W_n) = \left\lceil \frac{n+4}{2} \right\rceil$.

Theorem 5. [1] For a complete graph $K_n$ with $n \geq 2$ vertices, $\sigma^{-}(K_n) = \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$.

3 Generalized Petersen Graphs and Their Forbidden Cuts

The family of generalized Petersen graphs was introduced by Coxeter [4] in 1950 and was given its name by Watkins [8] in 1969.

Definition 4. For any $n \geq 3$ and $k \geq 1$ with $2k < n$, the generalized Petersen graph $P(n,k)$ has vertex set $V(P(n,k)) = \{u_i, v_i \mid i = 0, 1, ..., n-1\}$ and edge set $E(P(n,k)) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} \mid i = 0, 1, ..., n-1\}$, where subscripts are read modulo $n$.

From the definition, it is clear that $P(n,k)$ is a cubic graph and $P(5,2)$ is the well-known Petersen graph. The vertices $u_0, u_1, ..., u_{n-1}$ are called $u$-vertices and the vertices $v_0, v_1, ..., v_{n-1}$ are called $v$-vertices. The edges $u_iv_i$, for $i \in \{0, 1, ..., n-1\}$, are called spokes and the set of all spokes is denoted by $S$. We call the vertex $u_i$ ($v_i$) is the partner of the vertex $v_i$ ($u_i$), for each $i \in \{0, 1, ..., n-1\}$.

The cycle induced by all $u$-vertices is called the outer cycle of $P(n,k)$ and is denoted by $C_o$. The cycle(s) induced by all $v$-vertices is(are) called the inner cycle(s) of $P(n,k)$. If $\gcd(n,k) = d$ then the subgraph induced by all $v$-vertices consists of $d$ pairwise disjoint $\frac{n}{d}$-cycles. If $d > 1$ then no two vertices
among \(v_0, v_1, ..., v_{d-1}\) can be in the same \(d\)-cycle. For \(d = 1\), \(P(n, k)\) has only one inner cycle, and in this case the inner cycle is denoted by \(C_I\). If \(d > 1\), then \(P(n, k)\) has \(d\) inner cycles, and these inner cycles are denoted by \(C_1, C_2, ..., C_d\) such that \(v_i \in V(C_{i+1})\), for \(i \in \{0, 1, 2, ..., d-1\}\).

**Definition 5.** For any subset \(A\) of vertices of \(G\), the induced subgraph \(G[A]\) is the subgraph of \(G\) whose vertex set is \(A\) and whose edge set consists of all edges of \(G\) having both end points in \(A\).

**Lemma 1.** For any \(n \geq 4\) and \(k \geq 1\), \(P(n, k)\) cannot have a cut of size three with equal sides.

*Proof.* We analyse two cases depending upon whether \(n\) is odd or even.

**Case 1.** Let \(n = 2l\), for some \(l \geq 2\). Let there exist a subset \(A \subset V(P(2l, k))\) such that \(|A| = 2l\) and \(|[A : A^c]| = 3\). Denote the degree of a vertex \(a\) in \(P(2l, k)[A]\) by \(d_A(a)\). We have

\[
\sum_{a \in A} d_A(a) = 3(2l) - 3, \text{ an odd number.}
\]

This shows that \(P(2l, k)[A]\) does not satisfy the handshaking lemma. Hence no such \(A\) is possible.

**Case 2.** Let \(n = 2l + 1\), for some \(l \geq 2\). Let there exist a subset \(A \subset V(P(2l + 1, k))\) such that \(|A| = 2l + 1\) and \(|[A : A^c]| = 3\). If \(A\) contains either all \(u\)-vertices or all \(v\)-vertices then all the spokes will be in \([A : A^c]\). This contradicts the fact that \(|[A : A^c]| = 3\). Therefore \(A\) must contain \(u\)-vertices as well as \(v\)-vertices. Consequently \([A : A^c]\) contains at least two edges of \(C_o\), since \(u\)-vertices lie in both \(A\) and \(A^c\).

Now we consider two sub-cases.

**Subcase 2(i).** Let \(gcd(2l + 1, k) = 1\), so that \(P(2l + 1, k)\) has exactly one inner cycle \(C_I\). The condition that \(v\)-vertices lie in both \(A\) and \(A^c\) will insist \([A : A^c]\) to contain at least two edges of \(C_I\). Thus we have \(|[A : A^c]| \geq 4\), a contradiction to the fact that \(|[A : A^c]| = 3\).

**Subcase 2(ii).** Let \(gcd(2l + 1, k) = d \geq 2\) so that \(P(2l + 1, k)\) has \(d\) inner cycles \(C_1, C_2, ..., C_d\) each of length \(\frac{2l+1}{d}\). Note that each \(C_i\) lies completely either in \(A\) or in \(A^c\), otherwise we will get a contradiction on the size of \(|[A : A^c]|\). Therefore at least one cycle among \(C_1, C_2, ..., C_d\) lies in \(A\) and at least one lies in \(A^c\). Hence both \(A\) and \(A^c\) contain at least three \(u\)-vertices.

It is easy to see that if \(|[A : A^c]| = 3\) then exactly two edges of \([A : A^c]\) must be edges of \(C_o\) and the third edge must be a spoke. Let this spoke be \(u_jv_j\), for some \(j \in \{0, 1, 2, ..., 2l\}\). Without loss of generality, let \(u_j \in A\) and \(v_j \in A^c\). As exactly one spoke lies across \(A\) and \(A^c\), the remaining \(u\)-vertices of \(A\) must have their partners in \(A\). Thus the number of \(u\)-vertices and \(v\)-vertices in \(A\) are \(l + 1\) and \(l\), respectively. As \([A : A^c]\) has exactly two edges of \(C_o\), there exists a path of length \(l\) induced by the \(u\)-vertices of \(A\). Let the end points of this path be \(u_r\) and \(u_{r+l}\), for some \(r \in \{0, 1, 2, ..., 2l\}\). Consequently, the set of \(v\)-vertices in \(A\) is \(\{v_r, v_{r+1}, ..., v_{r+l}\}\) and the subgraph induced by these \(v\)-vertices must be edge-disjoint union of some inner cycle(s) of length \(\frac{2l+1}{d}\).

The condition \(2k < 2l + 1\), together with \(gcd(2l + 1, k) = d \geq 2\), implies that \(3 \leq k \leq l\). Also there must be an inner cycle \(v_rv_{r+k}v_{r+2k}...v_{r+(2l+1)-2k}v_{r+(2l+1)-k}v_r\) containing the vertex \(v_r\). Now if \(r \leq k\)
then \( r + (2l + 1) - k \geq r + (l + 1) \), since \( k \leq l \). If \( r > k \) then \( r + (2l + 1) - k = r - k < r \). Thus in both cases, the vertex \( v_{r+(2l+1)-k} \) does not lie in \( A \). Therefore at least two edges of the inner cycle containing \( v_r \) must lie in \([A : A^c]\). This gives \(|[A : A^c]| \geq 5\), a contradiction. Note that if \( r = j \), then we can consider the inner cycle containing \( v_{r+1} \) and get a similar contradiction.

This completes this proof. \( \square \)

**Lemma 2.** For any even \( n \geq 4 \), \( P(n,k) \) cannot have an odd cut with equal sides.

**Proof.** Take \( n = 2l \) for some \( l \geq 2 \). On the contrary, let \( P(n,k) \) have an odd cut with equal sides. Thus there exists a subset \( A \subset V(P(2l,k)) \) such that \(|A| = 2l\) and \(|[A : A^c]| = 2r + 1\), for some \( r \geq 1 \).

If \( d_A(a) \) is the degree of vertex \( a \) in \( P(2l,k)[A] \), then we have

\[
\sum_{a \in A} d_A(a) = 3(2l) - (2r + 1) , \text{ an odd number.}
\]

This is a contradiction to the handshaking lemma. Hence no odd cut with equal sides is possible in \( P(2l,k) \). This completes the proof. \( \square \)

**Lemma 3.** For any odd \( n \geq 5 \), \( P(n,k) \) cannot have an even cut with equal sides.

**Proof.** Take \( n = 2l + 1 \) for some \( l \geq 2 \). Let, if possible, \( P(2l + 1,k) \) have an even cut with equal sides. Thus there exists a subset \( A \subset V(P(2l + 1,k)) \) such that \(|A| = 2l + 1\) and \(|[A : A^c]| = 2r\), for some \( r \geq 2 \). Here \( r \) cannot be one, since no edge cut of size less than three is possible due to the edge-connectivity of generalized Petersen graphs.

If \( d_A(a) \) is the degree of vertex \( a \) in \( P(2l + 1,k)[A] \), then we have

\[
\sum_{a \in A} d_A(a) = 3(2l + 1) - 2r , \text{ an odd number,}
\]

a contradiction. Hence no even cut with equal sides is possible in \( P(2l + 1,k) \). This completes the proof. \( \square \)

### 4 Main Results

A simple but important result is the following.

**Theorem 6.** Let \( G \) be a graph with edge-connectivity \( k \). Then \( \sigma^-(G) \geq k \).

**Proof.** Clearly, no cut of \( G \) with nearly equal sides can have less than \( k \) edges since \( \kappa'(G) = k \). Hence \( \sigma^-(G) \geq k \). \( \square \)

**Theorem 7.** For any \( n \geq 3 \) and \( k \geq 1 \), the \textit{rna} number of \( P(n,k) \) satisfies

\[
3 \leq \sigma^-(P(n,k)) \leq n .
\]
**Proof.** The lower bound directly follows from Theorem 6 as the edge-connectivity of the generalized Petersen graph is three.

Define \( f : V(P(n, k)) \rightarrow \{1, 2, ..., 2n\} \) such that \( f(u_i) = 2i + 1 \) and \( f(v_i) = 2i + 2 \), for \( 0 \leq i \leq n - 1 \). This labeling \( f \) induces the parity signed graph \((P(n, k), \sigma_f)\). It is clear that the number of negative edges in \((P(n, k), \sigma_f)\) is \( n \). Hence \( \sigma^-(P(n, k)) \leq n \). \( \square \)

It is proved (in Lemma 4 and Example 5.1, respectively) that \( \sigma^-(P(3, 1)) = 3 \) and \( \sigma^-(P(5, 2)) = 5 \). Thus bounds of Theorem 7 are sharp.

**Theorem 8.** Let \( n \geq 5 \) and \( k \geq 2 \) be such that \( \gcd(n, k) = 1 \). Then the rna number of \( P(n, k) \) satisfies

\[
5 \leq \sigma^-(P(n, k)) \leq n.
\]

**Proof.** The upper bound is given by Theorem 7.

For odd \( n \), the lower bound simply follows from Lemma 1 and Lemma 3. Let \( n \) be an even number. By Lemma 1, \( \sigma^-(P(n, k)) \geq 4 \). Now we show that the rna number of \( P(n, k) \) can not be four.

By contradiction, let \( P(n, k) \) has a cut of size four with equal sides. Then there exists a subset \( A \subset V(P(n, k)) \) such that \( |A| = n \) and \( ||A : A^c|| = 4 \). If \( A \) contains either all \( u \)-vertices or all \( v \)-vertices then \( [A : A^c] \) has precisely \( n \) spokes of \( P(n, k) \). Hence \( ||[A : A^c]| = n \), a contradiction to the fact that \( ||[A : A^c]| = 4 \) and \( n \geq 5 \). Therefore, \( A \) must contain \( u \)-vertices as well as \( v \)-vertices.

Since \( \gcd(n, k) = 1 \), there is only one inner cycle \( C_I \) in \( P(n, k) \). Note that \( [A : A^c] \) has to contain exactly two edges of \( C_o \) and two edges of \( C_I \) because both \( A \) and \( A^c \) contain vertices of \( C_o \) and \( C_I \). Also \( A \) contains as many \( u \)-vertices as \( v \)-vertices. Otherwise, \( [A : A^c] \) will contain at least one spoke and we will get a contradiction on \( ||[A : A^c]| \). Further, the condition “\( [A : A^c] \) contains exactly two edges of \( C_I \)” insists \( u \)-vertices of \( A \) to induce a path of order \( \frac{n}{2} \). Let this path be \( P : v_rv_{r+k}...v_{r+(\frac{n}{2})} \), for some \( r \in \{0, 1, 2, ..., n-1\} \).

Similarly, all \( u \)-vertices of \( A \) induce a path of order \( \frac{n}{2} \) and let this path be \( P' : u_ju_{j+1}...u_{j+(\frac{n}{2}-1)} \), for some \( j \in \{0, 1, 2, ..., n-1\} \). As \( k \geq 2 \), all these \( u \)-vertices and \( v \)-vertices of the paths \( P \) and \( P' \) cannot be partners of each other, for any \( r, j \in \{0, 1, 2, ..., n-1\} \). Hence at least two spokes lie across \( A \) and \( A^c \). Therefore, \( ||[A : A^c]| \geq 6 \), a contradiction. This completes the proof. \( \square \)

**Remark 1.** From Theorem 7 and Lemma 1, it is easy to see that \( \sigma^-(P(n, k)) \geq 4 \), for \( n \geq 4 \).

**Theorem 9.** Let \( n \geq 8 \) be even and \( k \geq 3 \) be such that \( \gcd(n, k) = 1 \). Then the rna number of \( P(n, k) \) satisfies

\[
6 \leq \sigma^-(P(n, k)) \leq n.
\]

**Proof.** The result follows from Theorem 8 and Lemma 2. \( \square \)
4.1 The rna Number of $P(n, 1)$

Lemma 4. The rna number of $P(3, 1)$ is three.

Proof. Since the edge-connectivity of $P(3, 1)$ is three, Theorem 6 gives $\sigma^- (P(3, 1)) \geq 3$. We now label the vertices of $P(3, 1)$ through a bijection $f : V(P(3, 1)) \rightarrow \{1, 2, ..., 6\}$ such that the number of negative edges in the induced $(P(3, 1), \sigma_f)$ is exactly three.

Define $f : V(P(3, 1)) \rightarrow \{1, 2, ..., 6\}$ such that $f(u_i) = 2i + 1$ and $f(v_i) = 2i + 2$, for $0 \leq i \leq 2$. Let $A = \{u_0, u_1, u_2\}$ and $B = \{v_0, v_1, v_2\}$. Clearly all the spokes of $P(3, 1)$ are negative and all the edges of the $C_o$ and $C_I$ are positive in $(P(3, 1), \sigma_f)$. Thus $\sigma^- (P(3, 1)) = 3$. \hfill \Box

In the subsequent discussion, we consider $n \geq 4$.

Theorem 10. For $n \geq 4$, we have

$$\sigma^- (P(n, 1)) = \begin{cases} 4 & \text{if } n \text{ is even}, \\ 5 & \text{if } n \text{ is odd}. \end{cases}$$

Proof. We analyse two cases depending upon whether $n$ is even or odd.

Case 1. Let $n = 2l$, for some $l \geq 2$. By Remark 1, we have $\sigma^- (P(2l, 1)) \geq 4$. Thus to show that $\sigma^- (P(2l, 1)) = 4$, we produce an induced parity signed $P(2l, 1)$ which contains exactly four negative edges. Let $f : V(P(2l, 1)) \rightarrow \{1, 2, ..., 4l\}$ be defined by

$$f(u_i) = \begin{cases} 2i + 2 & \text{for } 0 \leq i \leq l - 1, \\ 2i - 2l + 1 & \text{for } l \leq i \leq 2l - 1; \end{cases}$$

and

$$f(v_i) = \begin{cases} 2i + 2l + 2 & \text{for } 0 \leq i \leq l - 1, \\ 2i + 1 & \text{for } l \leq i \leq 2l - 1. \end{cases}$$

Let $A = \{u_l, u_{l+1}, ..., u_{2l-1}, v_l, ..., v_{2l-1}\}$ and $B = \{u_0, u_1, ..., u_{l-1}, v_0, ..., v_{l-1}\}$ so that $|A| = |B| = 2l$. Hence, every edge of $P(2l, 1)$ having both end points either in $A$ or in $B$ is positive in the induced parity signed labeling. Consequently, all edges of $P(2l, 1)$ lying between $A$ and $B$ get negative sign in the induced parity signed labeling. Clearly, $|A : B| = \{u_{2l-1}u_0, u_{l-1}u_l, v_{2l-1}v_0, v_{l-1}v_l\}$. That is, there are exactly four edges between vertices of $A$ and $B$. Hence $\sigma^- (P(2l, 1)) = 4$.

Case 2. Let $n = 2l + 1$, for some $l \geq 2$. By Remark 1 and Lemma 3, we have $\sigma^- (P(2l + 1, 1)) \geq 5$. Now we produce an induced parity signed graph over $P(2l + 1, 1)$ which has exactly five negative edges.

Let $f : V(P(2l + 1, 1)) \rightarrow \{1, 2, ..., 4l + 2\}$ be defined by

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } 0 \leq i \leq l, \\ 2i - 2l & \text{for } l + 1 \leq i \leq 2l; \end{cases}$$

and
and
\[ f(v_i) = \begin{cases} 2i + (2l + 3) & \text{for } 0 \leq i \leq l - 1, \\ 2i + 2 & \text{for } l \leq i \leq 2l. \end{cases} \]

Let \( A = \{u_0, u_1, ..., u_v, v_0, ..., v_{l-1}\} \) and \( B = \{u_{l+1}, u_{l+2}, ..., u_{2l}, v_l, ..., v_{2l}\} \) so that \( |A| = |B| = 2l + 1 \). Note that \( [A : B] = \{u_l u_{l+1}, u_{2l} u_0, u_l v_1, u_{l-1} v_1, v_2 v_0\} \). Each edge of \( P(2l + 1, 1) \), except the five edges of \( [A : B] \), is positive in the parity signed graph \( (P(2l + 1, 1), \sigma_f) \). Consequently, the number of negative edges in \( (P(2l + 1, 1), \sigma_f) \) is exactly five. Hence, \( \sigma^-(P(2l + 1, 1)) = 5 \). This completes the proof.

\[ \square \]

### 4.2 The RNA Number of \( P(n, 2) \)

In this section, our aim is to prove the following two theorems.

**Theorem 11.** For any \( l \geq 3 \), \( \sigma^-(P(2l + 1, 2)) = 7 \).

**Theorem 12.** For any \( l \geq 4 \), \( \sigma^-(P(2l, 2)) = 6 \).

In light of Lemma 3, it is clear for \( l \geq 3 \) that the RNA number of \( P(2l + 1, 2) \) cannot be 4 and 6.

**Lemma 5.** For any \( l \geq 3 \), the RNA number of \( P(2l + 1, 2) \) cannot be 5.

**Proof.** We prove that no edge cut of size five with equal sides is possible in \( P(2l + 1, 2) \).

On the contrary, let \( P(2l + 1, 2) \) has a cut of size five with equal sides. Then there exists a subset \( A \subset V(P(2l + 1, 2)) \) such that \( |A| = 2l + 1 \) and \( |[A : A^c]| = 5 \). If \( A \) contains either all \( u \)-vertices or all \( v \)-vertices then the set of all spokes of \( P(2l + 1, 2) \) will constitute \( [A : A^c] \). This gives \( |[A : A^c]| \geq 7 \), a contradiction. Therefore, \( A \) must contain some \( u \)-vertices as well as \( v \)-vertices.

Clearly, \( P(2l + 1, 2) \) has only one inner cycle \( C_l \) induced by \( v \)-vertices, since \( \gcd(2l + 1, 2) = 1 \). Since \( A \) contains both \( u \)-vertices and \( v \)-vertices, \( [A : A^c] \) must consist of two edges of \( C_o \), two edges of \( C_l \) and one spoke. Let this spoke be \( u_j v_j \), for some \( j \in \{0, 1, ..., 2l\} \). Without loss of generality, let \( u_j \in A \) and \( v_j \in A^c \). The conditions: (i) \( |A| = 2l + 1 \), (ii) \( u_j \in A \) and (iii) \( [A : A^c] \) has exactly one spoke, together imply that the number of \( u \)-vertices and \( v \)-vertices in \( A \) are \( l + 1 \) and \( l \), respectively.

Consequently, there exist two paths \( P \) and \( P' \) of order \( l + 1 \) and \( l \), respectively, induced by \( u \)-vertices and \( v \)-vertices of \( A \). Let these paths be \( P \) and \( P' \) be \( u_r u_{r+1} ... u_{r+l-1} u_{r+l} \) and \( v_s v_{s+2} ... v_{s+(2l-2)} \), for some \( r, s \in \{0, 1, ..., 2l\} \), and \( u_j \) be one of the vertices of \( P \).

It is easy to check that all vertices of \( P' \) cannot have their partners in the vertices of \( P \), for any \( r, s \in \{0, 1, ..., 2l\} \). This means at least one \( v \)-vertex of \( A \) has its partner in \( A^c \), and consequently two spokes lie in \( [A : A^c] \). This gives \( |[A : A^c]| \geq 6 \), a contradiction to the assumption \( |[A : A^c]| = 5 \). This establishes the lemma.  

\[ \square \]
Proof of Theorem 11. By Theorem 8, Lemma 3 and Lemma 5, we have $\sigma^-(P(2l + 1, 2)) \geq 7$. To complete the proof, we produce a parity signed graph over $P(2l + 1, 2)$ with exactly seven negative edges.

Define $f : V(P(2l + 1, 2)) \to \{1, 2, ..., 4l + 2\}$ by

$$f(u_i) = \begin{cases} 2i + 1 & \text{for } i = 0, 1, ..., l, \\ 2i - 2l & \text{for } i = l + 1, l + 2, ..., 2l; \end{cases}$$

and

$$f(v_i) = \begin{cases} 4l + 2 & \text{for } i = 0, \\ 2l + (2i + 1) & \text{for } i = 1, 2, ..., l, \\ 2i & \text{for } i = l + 1, l + 2, ..., 2l. \end{cases}$$

Let $A$ and $B$ be set of all vertices of $P(2l + 1, 2)$ labeled with odd and even integers, respectively. Thus $A = \{u_0, u_1, ..., u_l, v_1, ..., v_l\}$ and $B = \{u_{l+1}, u_{l+2}, ..., u_{2l}, v_0, v_{l+1}, ..., v_{2l}\}$ with $|A| = |B| = 2l + 1$. Only the edges of $P(2l + 1, 2)$ between vertices of $A$ and $B$ are negative in $(P(2l + 1, 2), \sigma_f)$. Note that $[A : B] = \{u_0v_0, u_0u_2l, u_{l}u_{l+1}, v_0v_2l, v_2lv_1, v_1v_1l+l, v_1v_{l+2}\}$ with $|[A : B]| = 7$. Hence the number of negative edges in $(P(2l + 1, 2), \sigma_f)$ is seven. This proves that $\sigma^-(P(2l + 1, 2)) = 7$. $\square$

Lemma 6. For any $l \geq 4$, $P(2l, 2)$ cannot have a cut of size four with equal sides.

Proof. On the contrary, let if possible, $P(2l, 2)$ have a cut of size four with equal sides. Then there exists a subset $A \subset V(P(2l, 2))$ such that $|A| = 2l$ and $|[A : A^c]| = 4$. Obviously, $A$ must contain some $u$-vertices as well as $v$-vertices.

Since $\gcd(2l, 2) = 2$, $P(2l, 2)$ has two inner cycles, $C_1 = v_0v_2v_4...v_{2l-2}v_0$ and $C_2 = v_1v_3v_5...v_{2l-1}v_1$. Note that $[A : A^c]$ contains at least two edges of $C_o$, since $u$-vertices lie in both $A$ and $A^c$. Thus the followings are the only possibilities for the edges of $[A : A^c]$.

1. All four edges of $[A : A^c]$ are edges of $C_o$.

2. Two edges of $[A : A^c]$ are edges of $C_o$ and remaining two are spokes.

3. Two edges of $[A : A^c]$ are edges of $C_o$ and remaining two edges are of one of the $C_1$ and $C_2$.

We discuss the above possibilities one by one.

Case 1. Let $[A : A^c]$ have four edges of $C_o$. Thus all the vertices of one of the $C_1$ and $C_2$ lie in $A$ and other’s vertices lie in $A^c$. Without loss of generality, assume that all the vertices of $C_1$ are contained in $A$. Since no spoke is lying across $A$ and $A^c$, we have $A = \{u_0, u_2, u_4, ..., u_{2l-2}, v_0, v_2, v_4, ..., v_{2l-2}\}$. Consequently, $A^c = \{u_1, u_3, ..., u_{2l-1}, v_1, v_3, ..., v_{2l-1}\}$. Clearly, $[A : A^c]$ contains all the edges of $C_o$. That is, $|[A : A^c]| = 2l \geq 8$, a contradiction.
Case 2. Let \([A : A^c]\) have two edges of \(C_o\) and two spokes. Since \([A : A^c]\) contains no edge of inner cycles, without loss of generality, assume that \(A\) contains all the vertices of \(C_1\). That is, \(\{v_0, v_2, \ldots, v_{2l-2}\} \subseteq A\).

Consequently, the set of \(u\)-vertices in \(A\) is \(\{u_r\} \cup \{u_0, u_2, \ldots, u_{2l-2}\} \setminus \{u_j\}\) for some \(r \in \{1, 3, \ldots, 2l-1\}\) and \(j \in \{0, 2, \ldots, 2l-2\}\) because \([A : A^c]\) has exactly two spokes. Equivalently, the set of \(u\)-vertices in \(A\) is \(\{u_r\} \cup \{u_{j+2}, u_{j+4}, \ldots, u_{j+(2(2l-1))}\}\) for some \(r \in \{1, 3, \ldots, 2l-1\}\) and \(j \in \{0, 2, \ldots, 2l-2\}\). As \(l \geq 4\), it is easy to check that for any \(r \in \{1, 3, \ldots, 2l-1\}\) and for any \(j \in \{0, 2, \ldots, 2l-2\}\), \([A : A^c]\) will contain at least four edges of \(C_o\), a contradiction.

Case 3. Let \([A : A^c]\) have two edges of \(C_o\) and two edges of one of the inner cycles. Without loss of generality, assume that two edges of \(C_1\) lie in \([A : A^c]\). Thus both \(A\) and \(A^c\) contain at least one vertex of \(C_1\). Clearly, all the vertices of \(C_2\) lie in either \(A\) or in \(A^c\). Hence either \(A\) or \(A^c\) contains at least \(l + 1\) \(u\)-vertices. If \(A\) contains at least \(l + 1\) \(u\)-vertices then the number of \(u\)-vertices in \(A\) will be at most \(l - 1\). This observation shows that at most \(l - 1\) \(u\)-vertices of \(A\) can have their partners in \(A\). Hence at least two spokes will lie across \(A\) and \(A^c\), contradicting our assumption that \([A : A^c]\) contains no spokes. These contradictions establish the lemma.

Proof of Theorem 12. By Lemma 2 and Lemma 6, we have \(\sigma^- (P(2l, 2)) \geq 6\). Let us define \(f : V(P(2l, 2)) \to \{1, 2, 3, \ldots, 4l\}\) by

\[
f(u_i) = \begin{cases} 2i + 1 & \text{for } i = 0, 1, \ldots, l - 1, \\ 2i - (2l - 2) & \text{for } i = l, l + 1, \ldots, 2l - 1; \end{cases}
\]

and

\[
f(v_i) = \begin{cases} 2l + (2i + 1) & \text{for } i = 0, 1, \ldots, l - 1, \\ 2i + 2 & \text{for } i = l, l + 1, \ldots, 2l - 1. \end{cases}
\]

Let \(A = \{u_0, u_1, \ldots, u_{l-1}, v_0, v_1, \ldots, v_{l-1}\}\) and \(B = \{u_l, u_{l+1}, \ldots, u_{2l-1}, v_l, v_{l+1}, \ldots, v_{2l-1}\}\) so that \(|A| = |B| = 2l\). Consequently, all edges of \(P(2l, 2)\) between the vertices of \(A\) and \(B\) are negative in \((P(2l, 2), \sigma_f)\). Note that \([A : B] = \{u_0u_{2l-1}, u_{l-1}u_l, v_{2l-2}v_0, v_{2l-1}v_l, v_{l-2}v_l, v_{l-1}v_{l+1}\}\) with \(|[A : B]| = 6\). Hence the number of negative edges in \((P(2l, 2), \sigma_f)\) is six. This proves that \(\sigma^- (P(2l, 2)) = 6\).

5 Some Famous Generalized Petersen Graphs

In this section, we compute the RNA number of some well known generalized Petersen graphs such as Petersen graph, Dürer graph, Möbius-Kantor graph, Dodecahedron, Desargues graph and Nauru graph.

Example 5.1. The generalized Petersen graph \(P(5, 2)\) is well known as the Petersen graph. Since gcd(5, 2) = 1, Theorem 8 gives \(\sigma^- (P(5, 2)) \geq 5\). Label the vertices of \(P(5, 2)\) via \(f : V(P(5, 2)) \to \{1, 2, 3, \ldots, 10\}\) defined by \(f(u_i) = 2i + 1\) and \(f(v_i) = 2i + 2\), for \(0 \leq i \leq 4\). Thus all the spokes of \(P(5, 2)\)
are negative while other edges of \( P(5, 2) \) are positive in \((P(5, 2), \sigma_f)\). Hence the \textit{rna} number of Petersen graph is five. That is, \( \sigma^{-}(P(5, 2)) = 5 \).

![Figure 1: A parity signed Dürer graph with four negative edges. Solid lines denote positive edges and dashed lines denote negative edges.](image)

**Example 5.2.** The generalized Petersen graph \( P(6, 2) \) is known as the Dürer graph. It is depicted in Figure 1. By Remark 1, we have \( \sigma^{-}(P(6, 2)) \geq 4 \). Let \( f : V(P(6, 2)) \to \{1, 2, ..., 12\} \) be defined by \( f(v_i) = i + 7 \), for \( i = 0, 1, 2, 3, 4, 5 \) and

\[
f(u_i) = \begin{cases} 
2i + 1 & \text{for } i = 0, 1, 2, \\
2i - 4 & \text{for } i = 3, 4, 5.
\end{cases}
\]

This vertex labeling of \( P(6, 2) \) is described in Figure 1. Clearly \((P(6, 2), \sigma_f)\) have four negative edges as shown in Figure 1. Hence the \textit{rna} number of Dürer graph is four.

**Example 5.3.** The generalized Petersen graph \( P(8, 3) \) is known as the \textit{Möbius-Kantor} graph. A parity signed Möbius-Kantor is depicted in Figure 2. By Theorem 9, we have \( \sigma^{-}(P(8, 3)) \geq 6 \). Let \( f : V(P(8, 3)) \to \{1, 2, ..., 16\} \) be defined by \( f(v_0) = 9, \ f(v_1) = 15, \ f(v_2) = 10, \ f(v_3) = 11, \ f(v_4) = 12, \ f(v_5) = 14, \ f(v_6) = 13, \ f(v_7) = 16 \) and

\[
f(u_i) = \begin{cases} 
2i + 1 & \text{for } i = 0, 1, 2, 3, \\
2i - 6 & \text{for } i = 4, 5, 6, 7.
\end{cases}
\]

The vertex labeling \( f \) is shown in Figure 2. We see that \((P(8, 3), \sigma_f)\) has six negative edges. Hence the \textit{rna} number of Möbius-Kantor graph is six.
Example 5.4. The generalized Petersen graph $P(10, 2)$ is known as the Dodecahedron graph. A parity signed Dodecahedron with six negative edges is depicted in Figure 3. This labeling is the one described in the proof of Theorem 12. Clearly, the rna number of Dodecahedron graph is six.

Example 5.5. The generalized Petersen graph $P(10, 3)$ is known as the Desargues graph. See Figure 4 for a parity signed Desargues graph. Since $\gcd(10, 3) = 1$, Theorem 9 gives $\sigma^{-}(P(10, 3)) \geq 6$. Let the labeling $f : V(P(10, 3)) \rightarrow \{1, 2, \ldots, 20\}$ be defined by $f(v_0) = 11$, $f(v_1) = 13$, $f(v_2) = 12$, $f(v_3) =$
15. \( f(v_4) = 17, \) \( f(v_5) = 14, \) \( f(v_6) = 16, \) \( f(v_7) = 19, \) \( f(v_8) = 18, \) \( f(v_9) = 20, \) and

\[
f(u_i) = \begin{cases} 
2i + 1 & \text{for } i = 0, 1, 2, 3, 4, \\
2i - 8 & \text{for } i = 5, 6, 7, 8, 9.
\end{cases}
\]

This vertex labeling is shown in Figure 4. It is clear that \((P(10, 3), \sigma_f)\) has six negative edges. This gives \(\sigma^{-}(P(10, 3)) = 6.\) Hence the rna number of Desargues graph is six.

![Figure 4: A parity signed Desargues graph with six negative edges](image)

The generalized Petersen graph \(P(12, 5)\) is known as Nauru graph. A parity signed Nauru graph is depicted in Figure 5.

![Figure 5: A parity signed Nauru graph with eight negative edges](image)
Lemma 7. The rna number of Nauru graph is at least 8.

Proof. Theorem 9 gives that \( \sigma^{-}(P(12, 5)) \geq 6 \). Also, Lemma 2 shows that the rna number of Nauru graph cannot be seven. Thus it remains to prove that \( \sigma^{-}(P(12, 5)) \) cannot be six.

By contradiction, let \( P(12, 5) \) have a cut of size six with equal sides. Then there exists a subset \( A \subset V(P(12, 5)) \) such that \( |A| = 12 \) and \( |[A : A^c]| = 6 \). Note that \( A \) must contain \( u \)-vertices as well as \( v \)-vertices. Clearly, there is only one inner cycle \( C_1 \) in \( P(12, 5) \) since \( \gcd(12, 5) = 1 \). Thus the cut \( [A : A^c] \) will contain even number of edges of both \( C_o \) and \( C_1 \), because \( A \) and \( A^c \) contain \( u \)-vertices as well as \( v \)-vertices. Hence we analyse two cases.

Case 1. Let \( [A : A^c] \) contain four edges of \( C_o \) (or \( C_I \)) and two edges of \( C_I \) (or \( C_o \)). Assume that \( [A : A^c] \) contains four edges of \( C_o \) and two edges of \( C_I \). The case that \( [A : A^c] \) contains two edges of \( C_o \) and four edges of \( C_I \) can be treated similarly. Since \( [A : A^c] \) contains no spoke, \( A \) has to have six \( u \)-vertices and six \( v \)-vertices.

Note that all \( u \)-vertices of \( A \) induce a path of order six, since exactly two edges of \( C_I \) are lying across \( A \) and \( A^c \). Thus the set of \( u \)-vertices in \( A \) is \( \{v_r, v_{r+5}, v_{r+10}, v_{r+3}, v_{r+8}, v_{r+1}\} \), for some \( r \in \{0, 1, ..., 11\} \). Consequently, the set of \( u \)-vertices in \( A \) is \( \{u_r, u_{r+5}, u_{r+10}, u_{r+3}, u_{r+8}, u_{r+1}\} \), as no spoke of \( P(12, 5) \) lies in \( [A : A^c] \). Now for any \( r \in \{0, 1, ..., 11\} \), the edge-cut \( [A : A^c] \) contains the following edges of \( C_o \): \( u_{r+1}u_{r+2}, u_{r+2}u_{r+3}, u_{r+3}u_{r+4}, u_{r+4}u_{r+5}, u_{r+5}u_{r+6}, u_{r+7}u_{r+8}, u_{r+8}u_{r+9}, u_{r+9}u_{r+10}, u_{r+10}u_{r+11} \) and \( u_{r+11}u_r \). Thus \( |[A : A^c]| \geq 12 \), a contradiction.

Case 2. Let \( [A : A^c] \) contain two edges of \( C_o \), two edges of \( C_I \) and two spokes. In this case, \( A \) (or \( A^c \)) cannot have more than seven \( u \)-vertices or \( v \)-vertices, otherwise the number of spokes in \( [A : A^c] \) will exceed two and a contradiction will occur. Hence we consider two sub-cases depending upon whether \( A \) has seven \( u \)-vertices or six \( u \)-vertices.

Subcase 2(i). Let \( A \) have seven \( u \)-vertices (or seven \( v \)-vertices) and five \( v \)-vertices (or five \( u \)-vertices). Without loss of generality, assume that \( A \) has seven \( u \)-vertices and five \( v \)-vertices. These seven \( u \)-vertices and five \( v \)-vertices induce paths of order seven and five, respectively. Therefore set \( A \) must be of form \( \{u_j, u_{j+1}, u_{j+2}, u_{j+3}, u_{j+4}, u_{j+5}, u_{j+6}, v_r, v_{r+5}, v_{r+10}, v_{r+3}, v_{r+8}\} \), for some \( j, r \in \{0, 1, ..., 11\} \). It is easy to check that all five \( v \)-vertices of \( A \) cannot be adjacent to five \( u \)-vertices of \( A \), for any \( j \) and \( r \). This means at most four \( v \)-vertices of \( A \) can have their partners in \( A \). Thus \( [A : A^c] \) will contain at least four spokes, a contradiction.

Subcase 2(ii). Let \( A \) have six \( u \)-vertices and six \( v \)-vertices. Since \( [A : A^c] \) has only two edges of each \( C_o \) and \( C_I \), the \( u \)-vertices and \( v \)-vertices of \( A \) form two paths of order six. Thus \( A \) must be \( \{u_j, u_{j+1}, u_{j+2}, u_{j+3}, u_{j+4}, u_{j+5}, v_r, v_{r+5}, v_{r+10}, v_{r+3}, v_{r+8}, v_{r+1}\} \), for some \( j, r \in \{0, 1, ..., 11\} \). For any \( j \) and \( r \), it is a simple checking that at most four \( v \)-vertices of \( A \) can have their partners in \( A \). Therefore \( [A : A^c] \) will contain at least four spokes, a contradiction. This completes the proof.
Example 5.6. By Lemma 7, we have $\sigma^{-}(P(12,5)) \geq 8$. Let $f : V(P(12,5)) \to \{1, 2, ..., 24\}$ be defined by

$$f(u_i) = \begin{cases} 
2i + 1 & \text{for } i = 0, 1, 2, 6, 7, 8, \\
2i - 4 & \text{for } i = 3, 4, 5, 9, 10, 11;
\end{cases}$$

and

$$f(v_i) = \begin{cases} 
2i + 7 & \text{for } i = 0, 1, 2, 6, 7, 8, \\
2i + 2 & \text{for } i = 3, 4, 5, 9, 10, 11.
\end{cases}$$

This vertex labeling is shown in Figure 5. Clearly, $(P(12,5), \sigma_f)$ has eight negative edges. Hence the rna number of Nauru graph is eight.

6 Regular Graphs with rna Number One

For a given graph $G$, distribution of odd and even integers to the vertices of $G$ is the crucial aspect of determining the rna number of $G$. However, obvious lower and upper bounds for the rna number of $G$ are 1 and $m$, respectively, where $m$ is the size of $G$. It is shown in [1, Proposition 4] that the rna number of a path of order $n \geq 2$ is one. In [2], it is shown that a graph with rna number one must have a cut-edge. More precisely, we have the following theorem.

Theorem 13. [2, Theorem 3.5] Let $G$ be a connected graph. Then $\sigma^{-}(G) = 1$ if and only if $G$ has a cut-edge joining two graphs whose orders differ by at most one.

Note that having only a cut-edge is not enough for a graph to have rna number one. For example, consider the graph $G$ obtained by adding an edge $e$ to a vertex of the complete graph $K_3$. It is easy to check that $\sigma^{-}(G) = 2$.

An even regular graph is a regular graph in which every vertex has even degree. Similarly, an odd regular graph is a regular graph in which every vertex has odd degree. Since an even regular graph cannot have a cut-edge, in light of Theorem 13, the rna number of an even graph is at least two. Therefore, the following problem is worth exploring.

Problem 1. For an odd $k \geq 3$, what is the minimum order of a $k$-regular graph whose rna number is one?

In order to address this problem, first we construct a $(4n - 1)$-regular graph with a cut-edge joining two graphs of order $6n - 1$ each.

Lemma 8. There exists a $(4n - 1)$-regular graph on $12n - 2$ vertices with a cut-edge joining two graphs of order $6n - 1$ each.
Proof. Let $C_{6n-1}$ be the cycle with $V(C_{6n-1}) = \{v_0, v_1, v_2, \ldots, v_{6n-2}\}$ and $E(C_{6n-1}) = \{v_iv_{i+1} \mid 0 \leq i \leq 6n-2\}$, where subscripts are read modulo $6n-1$. Consider the power graph $C_{6n-1}^2$ of $C_{6n-1}$. Note that the degree of each vertex in $C_{6n-1}^2$ is $4n-2$. Now for each $1 \leq i \leq 3n-1$, insert an edge between $v_i$ and $v_{i+(3n-1)}$ in $C_{6n-1}^2$, and denote the graph so obtained by $G_r$. Clearly, the order of $G_r$ is $6n-1$ and the degree of $v_0$ is $4n-2$, whereas the degree of all other vertices in $G_r$ is $4n-1$. Now take two disjoint copies of $G_r$ and join their $v_0$-vertices by an edge. This resulting graph is the required graph. 

By Theorem 13, the $(4n-1)$-regular graph constructed in Lemma 8 has rna number one.

Lemma 9. A cubic graph of order four cannot have rna number one.

Proof. The only cubic graph on four vertices is $K_4$, which does not have a cut-edge. Hence the result follows by Theorem 13. 

Lemma 10. A cubic graph of order six cannot have rna number one.

Proof. Non-isomorphic cubic graphs of order six are those as shown in Figure 6. Clearly, none of these graphs contain a cut-edge. Thus, in light of Theorem 13, the result follows.

![Figure 6: Non-isomorphic cubic graphs of order six](image)

Lemma 11. A cubic graph of order eight cannot have rna number one.

Proof. There are five non-isomorphic cubic graphs of order eight. These eight graphs are depicted in Figure 7. It is clear that none of these graphs contain a cut-edge. Hence by Theorem 13, the result follows.

Lemma 12. There exists a parity signed cubic graph of order ten with rna number one.

Proof. Let $\Sigma$ be the parity signed cubic graph as shown in Figure 8. Clearly, it is a cubic graph of order 10 and it has a cut-edge joining two graphs of same order. Thus by Theorem 13, we have $\sigma^{-1}(\Sigma) = 1$. This completes the proof.
Theorem 14. The smallest order of a \((4n - 1)\)-regular graph having \(r\)na number one is bounded above by \(12n - 2\). Moreover, this bound is sharp for \(n = 1\).

Proof. The proof of the theorem follows from Lemma 8 and Lemma 12. □

Now we construct a \((4n + 1)\)-regular graph on \(8n + 6\) vertices with a cut-edge joining two graphs of order \(4n + 3\) each.

Lemma 13. There exists a \((4n + 1)\)-regular graph on \(8n + 6\) vertices with a cut-edge joining two graphs of order \(4n + 3\) each.

Proof. Let \(C_{4n+3}\) be the cycle with \(V(C_{4n+3}) = \{v_0, v_1, v_2, ..., v_{4n+2}\}\) and \(E(C_{4n+3}) = \{v_i, v_{i+1} \mid 0 \leq i \leq 4n + 2\}\), where subscripts are read modulo \(4n + 3\). Consider the power graph \(C_{4n+3}^{2n}\) of \(C_{4n+3}\). Note that the degree of each vertex in \(C_{4n+3}^{2n}\) is \(4n\). Now for each \(1 \leq i \leq 2n + 1\), insert an edge between \(v_i\) and \(v_{i+(2n+1)}\) in \(C_{4n+3}^{2n}\), and denote the graph so obtained by \(G_s\). Clearly, the order of \(G_s\) is \(4n + 3\) and the degree of \(v_0\) is \(4n\), whereas the degree of all other vertices in \(G_s\) is \(4n + 1\). Now take two disjoint copies of \(G_s\) and join their \(v_0\)-vertices by an edge. This resulting graph is the required graph. □

Observe that both sides of the graph constructed in Lemma 13 are of order \(4n + 3\). Also, except one vertex, each vertex of these sides have degree \(4n + 1\). These sides are the smallest such sides of same
order whose joining by a cut-edge produce a \((4n + 1)\)-regular graph. Hence this graph is the \((4n + 1)\)-regular graph of smallest order having \text{rna} number one. Thus the following theorem is immediate from Lemma 13.

**Theorem 15.** The minimum order of \((4n + 1)\)-regular graphs having \text{rna} number one is \(8n + 6\).

### 7 Time complexity of computing \text{rna} number

For basic terminologies related to algorithm and its time complexity, we refer the reader to [6]. Recall that the edge-connectivity \(\kappa'(G)\) of a graph \(G\) is the size of minimum cut. For any graph \(G\) of order \(n\) and size \(m\), the best time bound for edge-connectivity is \(O(m + \min\{\kappa'(G)n^2, mn + n^2\log(n)\})\) due to Nagamochi and Ibaraki [7]. Hence the edge-connectivity \(\kappa'(G)\) of a graph \(G\) can be computed in polynomial time.

For a given graph \(G\) of order \(n\), we define a family \(\mathcal{A}\) of subsets of \(V(G)\) as follows

\[
\mathcal{A} = \{A \subset V(G) : |A| = \left\lfloor \frac{n}{2} \right\rfloor}\.
\]  

(1)

Clearly the cardinality of \(\mathcal{A}\) is \(C(n, \left\lfloor \frac{n}{2} \right\rfloor)\). Let the collection \(\mathcal{B}\) be defined by

\[
\mathcal{B} = \{|[A : A^c]| : A \in \mathcal{A}\}.
\]  

(2)

If \(|\mathcal{B}|\) denotes the cardinality of \(\mathcal{B}\) counting the multiplicities of its elements then \(|\mathcal{B}| = |\mathcal{A}|\).

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**Algorithm 1:** The \text{rna} number \(\sigma^-(G)\)

- **Input:** A graph \(G\) of order \(n\).
- **Output:** The \text{rna} number \(\sigma^-(G)\) of \(G\)

1: obtain the family \(\mathcal{A}\)
2: for \(A \in \mathcal{A}\) do
3: compute \(|[A : A^c]|\)
4: end for
5: obtain the family \(\mathcal{B}\) by the numbers obtained in step 3
6: find the smallest number in \(\mathcal{B}\) and denote it by \(\sigma^-(G)\)
7: return \(\sigma^-(G)\).

**Lemma 14.** The number returned by Algorithm 1 is the \text{rna} number of \(G\).

**Proof.** Clearly each \(A \in \mathcal{A}\) generates a cut \([A : A^c]\) whose sides differ by at most one. Thus \(\mathcal{B}\) is the collection of sizes of all possible cuts in \(G\) whose sides are nearly equal. Consequently, the smallest element of \(\mathcal{B}\) will be the \text{rna} number of \(G\). Thus the lemma follows. 

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**Theorem 16.** The running time of Algorithm 1 is $O(2^n + n^{|\frac{n}{2}|})$.

**Proof.** It is well known that, using the inclusion-exclusion principle, all possible subsets of a given size of a set of $n$ elements can be computed in $O(2^n)$ time. Thus for step 1, we spend $O(2^n)$ time. Since $|A| = C(n, \lfloor \frac{n}{2} \rfloor)$, step 2 to step 4 can be done in $O(n^{|\frac{n}{2}|})$ time. Clearly step 5 can be done in constant time.

It is also known that the smallest element in a set of $n$ numbers can be computed in $O(n)$ time. Thus step 6 takes $O(n^{|\frac{n}{2}|})$ time, because $|B| = C(n, \lfloor \frac{n}{2} \rfloor)$. Hence the overall running time of Algorithm 1 is $O(2^n + n^{|\frac{n}{2}|})$. This completes the proof. \qed

8 Concluding Remarks

We have determined the rna number of $P(n, k)$ for $k = 1, 2$. For $k \geq 3$, the distribution of odd and even integers to the vertices of $P(n, k)$ to obtain the exact value of $\sigma^-(P(n, k))$ seems to be hard. Thus it would be nice if one can solve the following problem.

**Problem 2.** Let $n \geq 7$ and $k \geq 3$ be any given positive integers. What is the value of $\sigma^-(P(n, k))$?

We have proved that the minimum order of a $(4n+1)$-regular graph having rna number one is $8n+3$. We have also proved that the minimum order of a $(4n-1)$-regular graph having rna number one is bounded above by $12n - 2$. We could prove the sharpness of this bound only for $n = 1$. For $n \geq 2$, it is not known if this bound is sharp. From these, we also see that the best possible lower bound for the rna number of odd regular graphs is 1. For each odd $k \geq 3$, best possible upper bound for the rna number of $k$-regular graphs is unknown. Hence the following problems are also interesting.

**Problem 3.** For $n \geq 2$, what is the minimum order of a $(4n-1)$-regular graph having rna number one?

**Problem 4.** What is the best possible upper bound for the rna number of odd regular graphs?

We have proposed an exponential time algorithm to find the rna number of a graph $G$. However, there is a minor difference between the concept of the edge-connectivity and the rna number of a graph. So, we propose the following conjecture:

**Conjecture 1.** The rna number of a graph can be computed in polynomial time.

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