A Dynamic S-Procedure for Dynamic Uncertainties

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Abstract: We show how to compose robust stability tests for uncertain systems modeled as linear fractional representations and affected by various types of dynamic uncertainties. Our results are formulated in terms of linear matrix inequalities and rest on the recently established notion of finite-horizon integral quadratic constraints with a terminal cost. The construction of such constraints is motivated by an unconventional application of the S-procedure in the frequency domain with dynamic Lagrange multipliers. Our technical contribution reveals how this construction can be performed by dissipativity arguments in the time-domain and in a lossless fashion. This opens the way for generalizations to time-varying or hybrid systems.

Keywords: Robustness Analysis, Dynamic Uncertainties, Integral Quadratic Constraints, Linear Matrix Inequalities.

1. INTRODUCTION

The S-procedure (also referred to as S-lemma or S-method) was developed more than 90 years ago and has applications in optimization, control and many other areas in mathematics. Naturally, by now there exists a multitude of variations of the S-procedure, some of which can be found, e.g., in the books by Boyd et al. (1994), Ben-Tal and Nemirovski (2001) or by Boyd and Vandenberghe (2004). Detailed surveys on results related to the S-procedure can be found in (Uhlig, 1979), (Pölk and Terlaky, 2007) and in (Gusev and Likhtarnikov, 2006).

Traditionally, the S-procedure guarantees the positivity of a quadratic function on a set defined by quadratic inequalities through the positivity of a related quadratic function. The latter function is constructed by Lagrange relaxation and, hence, involves sign-constrained Lagrange multipliers that are also called scalings in the context of control. In the book by Boyd et al. (1994), for example, it is shown how this standard variant can be used for the robustness analysis of uncertain dynamical systems. However, this approach is limited since it only permits the construction of robustness tests with scalings that are real-valued and structured. In contrast, e.g., $\mu$-theory (Packard and Doyle, 1993) offers the use of frequency-dependent scalings for dynamic and parametric uncertainties in order to substantially reduce the conservatism of such tests. Similarly, the so-called full-block variant of the S-procedure in (Scherer, 1997, 2001) permits the reduction of conservatism by relying on unstructured scalings, even if the uncertainty itself is structured.

Robust stability and performance tests based on the full-block S-procedure and most variants thereof can be viewed as an application of integral quadratic constraint (IQC) analysis with static (frequency-independent) multipliers (Megretsky and Rantzer, 1997). Also in this context, it is well-established that dynamic (frequency-dependent) multipliers often lead to less conservative results. Recently, Scherer (2021) has hinted at how to construct finite time-horizon IQCs with dynamic multipliers for one dynamic uncertainty and with frequency domain arguments.

Based on these ideas, the purpose of this paper is to propose a dynamic version of the S-procedure which has not found much attention in the literature. In particular, we provide a new alternative and simplified proof of the aforementioned result of Scherer (2021) without relying on frequency domain arguments. This is beneficial, e.g., for its generalization to hybrid systems, where no suitable notion of a frequency domain exists. Moreover, we demonstrate that our conceptual proof allows for deriving several novel variations and extensions that permit us, e.g., to analyze the robustness of a system against dynamic uncertainties with a Nyquist plot known to be located in a given LMI region. This substantially extends, for example, the inspiring results by Balakrishnan (2002).

Outline. The remainder of the paper is organized as follows. After a short paragraph on notation, we specify the considered robust analysis problem and recall some preliminary results in Section 2. In Section 3, we provide and elaborate on our main analysis result for systems affected by a class of dynamic uncertainties. In Section 4, we demonstrate how to modify our approach in order to handle dynamic uncertainties whose frequency response is located in regions of the complex plane beyond disks and half-spaces. We conclude with a numerical example and some further remarks in Sections 5 and 6, respectively. Finally, most proofs are moved to the appendix.
Notation. Let $L^2_x := \{ x \in L^2_x : \int_0^\infty |x(t)|^2 dt < \infty \}$ where $L^2_x$ is the space of locally square integrable functions $x : [0, \infty) \to \mathbb{R}^n$. Moreover, $\mathbb{S}^n$ denotes the set of symmetric $n \times n$ matrices, and $\text{RH}_{\infty}^{m \times n}$ ($\text{RL}_{\infty}^{m \times n}$) is the space of real rational $m \times n$ matrices without poles in the extended closed right half-plane (imaginary axis) and is equipped with the maximum norm $\| \cdot \|_{\infty}$. For a transfer matrix $G(s) = D + C(sI - A)^{-1}B$, we take $(-A^T, C^T, -B^T, D^T)$ as a realization of $G^*(s) = G(-s)^T$. Finally, we use the abbreviation 
\[ \text{diag}(X_1, \ldots, X_N) := \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_N \end{pmatrix} \]
for matrices $X_1, \ldots, X_N$, utilize the Kronecker product $\otimes$ as defined in (Horn and Johnson, 1991) and indicate objects that can be inferred by symmetry or are not relevant with the symbol $\ast$.

2. PRELIMINARIES

For real matrices $A, B, C, D$ and some initial condition $x(0) \in \mathbb{R}^n$, let us consider the uncertain feedback interconnection 
\[ \begin{pmatrix} \dot{x}(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} + \Delta(z)(t) \]  
for $t \geq 0$. The uncertainty $\Delta$ is assumed to be a stable LTI system described by its input-output map $\Delta : L^2_x \to L^2_{zx}$, 
\[ \Delta(z)(t) = \int_0^t C \Delta e^{A \Delta(t-s)} B \Delta s + D \Delta z(t) \] 
for some real matrices $A, B, C, D, \Delta$. We slightly abuse the notation by using the same symbol for the map (1) and the transfer matrix $\Delta(s) = C \Delta(sI - A \Delta)^{-1}B + D \Delta$. We suppose that $\Delta$ satisfies some structural constraints, which are imposed on its frequency response as 
\[ \Delta(i \omega) \in \mathbf{V} \] 
for all $\omega \in \mathbb{R} \cup \{ \infty \}$ (1c) for some given value set $\mathbf{V} \subset \mathbb{C}^{s,k}$. We denote the set of all such dynamic uncertainties by $\Delta(\mathbf{V})$.

Definition 1. The interconnection (1) is said to be robustly stable if it is well-posed, i.e., if $\det(I - DV) \neq 0$ holds for all $V \in \mathbf{V}$, and if there exists a constant $M > 0$ such that the state-trajectories $x(\cdot)$ satisfy 
\[ \lim_{t \to \infty} x(t) = 0 \quad \text{and} \quad \|x(t)\| \leq M \|x(0)\| \] 
for all $t \geq 0$ for all initial conditions $x(0) \in \mathbb{R}^n$ and any $\Delta \in \Delta(\mathbf{V})$.

The following robust analysis result is a consequence of the (full-block) S-procedure as proposed by Scherer (1997).

Lemma 2. The interconnection (1) is robustly stable if there exist some (Lyapunov) certificate $X \in \mathbb{S}^n$ and a multiplier $P \in \mathbb{S}^{n+1}$ satisfying the LMIs 
\[ X > 0, \] 
\[ (\bullet)^T \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} + (\bullet)^T P \begin{pmatrix} C & D \\ D^T & I \end{pmatrix} P > 0, \] 
\[ \begin{pmatrix} I \\ V \end{pmatrix}^T P \begin{pmatrix} I \\ V \end{pmatrix} \preceq 0 \] 
and 
\[ \begin{pmatrix} I \\ 0 \end{pmatrix}^T P \begin{pmatrix} I \\ 0 \end{pmatrix} \preceq 0 \] 
for all $\mathbf{V} \in \mathbf{V}$.

The inequalities (2) are not directly numerically tractable since (2d) usually involves an infinite number of LMIs. As a remedy one typically does not search for a suitable multiplier $P$ with (2d) in the full set $\mathbb{S}^{n+1}$, but one confines the search to a subset $\mathbf{P}^{(V)} \subset \mathbb{S}^{k+l}$ which admits an LMI representation. This means that there exist affine symmetric-valued maps $F$ and $G$ on $\mathbb{R}^s$ with $\mathbf{P}^{(V)} = \{ F(\nu) | \nu \in \mathbb{R}^s \}$ and $G(\nu) > 0$. In order to counteract the conservatism introduced by this restriction while maintaining computational efficiency, one should aim for subsets that capture the properties of $V$ as well and as simple as possible. A detailed discussion and several choices for concrete instances of $\mathbf{V}$ are found, e.g., in (Scherer and Veenman, 2018).

First, let us assume that $\Delta$ is a single repeated dynamic uncertainty, i.e., $\Delta = \delta I_k$ for some $\delta \in \mathbb{R}^{1 \times 1}$. Moreover, we suppose that its values on the imaginary axis are confined to the set 
\[ \mathbf{V} := \left\{ vI_k \mid v \in \mathbb{C} \text{ and } \left( \begin{pmatrix} I \\ v \end{pmatrix}^* P_0 \begin{pmatrix} I \\ v \end{pmatrix} \right) \right\} \] 
for some given matrix $P_0 = (\frac{q}{r}) \in \mathbb{S}^2$ with $r \leq 0$. This means that the Nyquist plot of $\delta$ is contained in the disk or half-plane corresponding to the matrix $P_0$.

Note that $r \leq 0$ is a technical requirement in our proof, and it implies that the set $\mathbf{V}$ is convex; this restriction and the related inequality (2c) in Lemma 2 can be dropped if the uncertainty $\Delta$ affecting the system (1a) is known to be parametric. In view of the KYP lemma (Rantzer, 1996), the constraint (1c) with (3) can equivalently be expressed in the time domain by means of a nonstrict dissipation inequality; in the sequel, we rather use the frequency domain formulation for brevity.

The concrete description (3) leads to the following tractable robust stability test.

Corollary 3. The interconnection (1) with (3) is robustly stable if there exist some certificate $X \in \mathbb{S}^n$ and a multiplier 
\[ P \in \mathbf{P}^{(V)} := \{ P_0 \otimes M \mid M \in \mathbb{S}^k, M > 0 \} \] 
satisfying (2a) and (2b).

Proof. We just have to observe that, by the rules of the Kronecker product, 
\[ 0 \preceq \left( \begin{pmatrix} I \\ v \end{pmatrix}^* P_0 \begin{pmatrix} I \\ v \end{pmatrix} \right) \otimes M = (\bullet)^* (P_0 \otimes M) \left( \begin{pmatrix} I \\ v \end{pmatrix} \otimes I_k \right) \] 
holds for any $V = vI_k \in \mathbf{V}$ and any $P = P_0 \otimes M \in \mathbf{P}^{(V)}$. □

Note that neither Lemma 2 nor Corollary 3 takes into account that the uncertainty $\Delta$ entering (1a) is an LTI system (1b). Consequently, both analysis criteria can be rather conservative, as emphasized for example by Poolla and Tzikos (1995). This can be resolved, e.g., by employing IQCs with dynamic multipliers instead of the static ones used in Lemma 2 and Corollary 3.

3. MAIN RESULT

Our main robustness result for the interconnection (1) is based on the IQC theorem in (Scherer and Veenman, 2018)
involving dynamic multipliers. To this end, we construct such multipliers described as $\Pi = \Psi^*P\Psi$ with a fixed stable dynamic outer factor $\Psi \in RH_{\infty}^{m \times (k+1)}$ and a real middle matrix $P \in S^n$ which serves as a variable and is subject to suitable constraints. Moreover, let us recall the following notion introduced by Scherer and Veenman (2018), which involves some state-space description of the filter $\Psi$. In fact, the output $y$ of $\Psi$ in response to the input $u \in L_2^{+\infty}$ is supposed to be given by

$$
\begin{pmatrix}
\xi(t) \\
y(t)
\end{pmatrix} = \begin{pmatrix} A_{\psi} & B_{\psi} \\
C_{\psi} & D_{\psi}\end{pmatrix} \begin{pmatrix}
\xi(t) \\
n(t)
\end{pmatrix}
$$

(4)

for $t \geq 0$ and with a zero initial condition, i.e., $\xi(0) = 0$.

Definition 4. The uncertainty $\Delta : L_2^p \to L_2^p$ satisfies a finite-horizon IQC with terminal cost matrix $Z = Z^T$ for the multiplier $\Pi = \Psi^*P\Psi$ if the dissipation inequality

$$
\int_0^T y(t)^TPy(t)dt + \xi(T)^TZ\xi(T) \geq 0 \quad \text{for all} \quad T \geq 0
$$

holds for the trajectories of the filter (4) driven by the input $u = (\Delta x)$ with any $x \in L_2^p$.

If $\Delta \in \Delta(V)$ satisfies a finite-horizon IQC with terminal cost $Z$, then standard arguments involving Parseval’s theorem lead to the frequency domain inequality (FDI)

$$
\left( \frac{I}{i\omega} \right)^* \Pi(i\omega) \left( \frac{I}{i\omega} \right) \succeq 0 \quad \text{for all} \quad \omega \in \mathbb{R}
$$

since $\Psi$ and $\Delta$ are assumed to be stable. Note that this is the frequency dependent analogue of (2d). A detailed analysis of links with the classical IQCs in (Megretsky and Rantzer, 1997) (and the corresponding stability tests) can be found in (Scherer and Veenman, 2018; Scherer, 2021). In particular, this notion only involves a partial storage function defined by the filter’s state and $Z$ and does not rely on any information about “internal properties” of $\Delta$.

Now we address how to systematically construct finite-horizon IQCs with a nontrivial terminal cost for $\Delta(V)$ and the value set (3) inspired by the proof of Corollary 3. To this end, we pick a dynamic MIMO scaling $\psi^*M\psi$ with $\psi \in RH_{\infty}^{n \times k}$ and $M \in S^n$ satisfying $\psi(\omega)^*M\psi(\omega) > 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

Then the concrete choice of $\Psi$ in (3) naturally implies

$$
\begin{pmatrix}
1 \\
\delta
\end{pmatrix}^*P_0 \begin{pmatrix}
1 \\
\delta
\end{pmatrix} \otimes \psi^*M\psi \succeq 0 \quad \text{on} \quad i\mathbb{R}
$$

for any $\Delta = \delta I_k \in \Delta(V)$. Hence, the identity

$$
\begin{pmatrix}
1 \\
\delta
\end{pmatrix} \otimes \psi = \begin{pmatrix}
\psi \\
\psi \delta
\end{pmatrix} = \begin{pmatrix}
\psi & 0 \\
0 & \psi
\end{pmatrix} \begin{pmatrix}
I \\
\delta I_k
\end{pmatrix}
$$

(6a)

on the imaginary axis yields the inequality

$$
0 \preceq \begin{pmatrix}
1 \\
\delta
\end{pmatrix}^*P_0 \begin{pmatrix}
1 \\
\delta
\end{pmatrix} \otimes \psi^*M\psi = (\psi)^*(P_0 \otimes M) \begin{pmatrix}
1 \\
\delta
\end{pmatrix} \otimes \psi = (\psi)^*(P_0 \otimes M) \begin{pmatrix}
1 \\
0
\end{pmatrix} \begin{pmatrix}
I \\
\delta I_k
\end{pmatrix} = (\psi)^*\Pi \begin{pmatrix}
I \\
\Delta
\end{pmatrix}
$$

(6b)

on $i\mathbb{R}$ for any $\Delta = \delta I_k \in \Delta(V)$. Observe that this a typical $S$-procedure argument, but now much less commonly applied in the context of frequency domain inequalities. In particular, the dynamic scaling $\psi^*M\psi$ plays the role of the Lagrange multiplier appearing in such arguments.

This brings us to the main technical result of this paper, a lossless time-domain formulation of the FDI (6b), which is proved (in the appendix) by dissipativity arguments only.

Theorem 5. Suppose that $\psi \in RH_{\infty}^{n \times k}$ has the realization $(A_{\psi}, B_{\psi}, C_{\psi}, D_{\psi})$ where $A_{\psi} \in R^{n \times n_{\psi} + k}$ is Hurwitz. If

$$
(\psi)^T \begin{pmatrix}
0 & Y \\
Y & 0
\end{pmatrix} + (\psi)^TM(C_{\psi}D_{\psi}) > 0
$$

(7)

holds for $Y \in S^{n_{\psi} + k}$ and $M \in S^n$, then any $\Delta \in \Delta(V)$ satisfies a finite-horizon IQC with terminal cost matrix $P_0 \otimes Y$ for $\Pi = \Psi^*(P_0 \otimes M)\Psi$ where $\Psi := \text{diag}(\psi, \psi)$.

Note that the LMI (7) results from applying the KYP lemma to the FDI (5). Moreover, $P_0 \otimes Y$ encodes how the information about the uncertainty is affecting the IQC.

We stress that this result can also be found in (Scherer, 2021). However, there the proof heavily relies on frequency domain arguments and, in particular, on the following commutation property for all $\omega \in \mathbb{R} \cup \{\infty\}$:

$$
\psi(\omega)\Delta(\omega) = \psi(\omega)\delta(\omega) = \delta(\omega)\psi(\omega).
$$

(8)

We only make use of this property in (6) for the purpose of motivation. In contrast and as a major benefit, the new proof in this paper only employs time domain arguments. This means that it can be extended, e.g., to hybrid systems where no suitable notion of a frequency domain exists (Holicki and Scherer, 2019). Naturally, the proof evolves around a time domain analogue of (8) that is explicitly stated in Lemma 12 and relies on Fubini’s theorem.

With Theorem 5 at hand, we can now formulate our main robust stability test for the interconnection (1). This is a consequence of the general IQC robust stability result as stated in Theorem 4 of Scherer and Veenman (2018).

Theorem 6. Let $(A_{\psi}, B_{\psi}, C_{\psi}, D_{\psi}) \in \mathbb{R}^{(n_{\psi}+m_{\psi}) \times (n_{\psi}+k)}$ with stable $A_{\psi}$ be given and abbreviate $J_{\omega} := \text{diag}(J_{\omega}, J_{\omega})$ for $J \in \{A, B, C, D\}$. Then the interconnection (1) with $V$ as in (3) is robustly stable if there exist $X \in S^{2(n_{\psi}+m_{\psi})}$, $Y \in S^{n_{\psi}}$, and $M \in S^{m_{\psi}}$ satisfying (7).

$$
(\psi)^T \begin{pmatrix}
0 & X \\
X & 0
\end{pmatrix} \begin{pmatrix}
P_0 & 0 \\
0 & M
\end{pmatrix} \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix} < 0
$$

(9a)

and

$$
X - (P_0 \otimes Y) > 0 \quad \text{on} \quad i\mathbb{R}.
$$

(9b)

First note that we recover Corollary 3 by choosing $n_{\psi} = 0$ and $D_{\psi} = I_k$, i.e., by restricting the filter (4) to be static and of the form $\Psi = I_k$. We also recover the analysis results from Scherer and Köse (2012) with $P_0 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$: then $\Delta(V)$ is the set of repeated dynamic uncertainties whose $H_\infty$-norm is bounded by one. Further restricting $Y$ to $Y = 0$ and replacing (7) with $M > 0$ leads back to the stability criteria proposed by Balakrishnan (2002). Since the latter test relies on a vanishing terminal cost, it is typically more conservative as seen in (Scherer, 2021).

Let us recall the benefit of the time-domain formulation in Theorem 6 from (Fetzer et al., 2018). In fact, the
Theorem 7. Let \( \begin{bmatrix} A_v & B_v \\ C_v & D_v \end{bmatrix} \in \mathbb{R}^{(n_v+u_v) \times (n_v+1)} \) with stable \( A_v \) be given and abbreviate \( J_v := \text{diag}(I_k \otimes J_v, I_t \otimes J_v) \) for \( J_v \in \{A, B, C, D\} \). Then the interconnection (1) with \( V \) as in (10) is robustly stable if there exist \( X \in \mathbb{S}^{(k+l)u_v+n} \), \( Y \in \mathbb{S}^{n_v} \) and \( M \in \mathbb{S}^{n_v} \) satisfying (7), (9a) and (9b).

4. VARIATIONS

The flexibility of the approach leading to Theorem 5 and Theorem 6 permits us to adequately treat other types of dynamic uncertainties with minor adjustments of the involved arguments only. This is illustrated next.

4.1 Dynamic Full-Block Uncertainties

Suppose that the interconnection (1) involves dynamic full-block uncertainties, i.e., uncertainties in \( \Delta(V) \) with

\[
V := \left\{ V \in \mathbb{C}^{k \times k} \mid \begin{pmatrix} I_k & \psi \end{pmatrix} P_0 \begin{pmatrix} I_k \\ \psi \end{pmatrix} \succeq 0 \right\} \tag{10}
\]

for some given matrix \( P_0 = \begin{pmatrix} Q & S \end{pmatrix} \in \mathbb{R}^{k+l} \) with \( R. \in \mathbb{R} \). In order to construct a corresponding finite-horizon IQC with terminal cost, this time, we pick a dynamic SISO scaling \( \psi^* M \psi \) since the value set \( \mathcal{V} \) is taken as in (10). Next, we proceed analogously as in (6) and observe that the identity

\[
\begin{pmatrix} I_k \\ \psi \end{pmatrix} \Delta \psi = \begin{pmatrix} I_k \otimes \psi \\ \Delta \otimes \psi \end{pmatrix} = \begin{pmatrix} I_k \otimes \psi \\ (I_t \otimes \psi) \Delta \otimes I_1 \end{pmatrix} = \begin{pmatrix} I_k \otimes \psi \\ 0 \end{pmatrix} \begin{pmatrix} I_k \otimes \psi \\ I_t \otimes \psi \end{pmatrix} = \begin{pmatrix} I_k \otimes \psi \\ I_t \otimes \psi \end{pmatrix}
\]

on the imaginary axis yields the inequality

\[
0 \preceq \begin{pmatrix} I_k \\ \psi \end{pmatrix} P_0 \begin{pmatrix} I_k \\ \psi \end{pmatrix} \psi^* M \psi = \begin{pmatrix} \bullet \end{pmatrix} (P_0 \otimes M) \begin{pmatrix} I_k \otimes \psi \\ 0 \end{pmatrix} = \begin{pmatrix} \bullet \end{pmatrix} (P_0 \otimes M) \begin{pmatrix} I_k \otimes \psi \\ 0 \end{pmatrix} \begin{pmatrix} I_k \otimes \psi \\ I_t \otimes \psi \end{pmatrix} \tag{11}
\]

for any \( \Delta \in \Delta(V) \). This constitutes an alternative S-procedure argument and motivates the following robust stability test for the interconnection (1).

Theorem 8. Let \( \begin{pmatrix} A_v & B_v \\ C_v & D_v \end{pmatrix} \in \mathbb{R}^{(n_v+u_v) \times (n_v+1)} \) with stable \( A_v \) be given and abbreviate \( J_v := \text{diag}(I_k \otimes J_v, I_t \otimes J_v) \) for \( J_v \in \{A, B, C, D\} \). Then the interconnection (1) with \( V \) as in (10) is robustly stable if there exist \( X \in \mathbb{S}^{(k+l)u_v+n} \), \( Y \in \mathbb{S}^{n_v} \) and \( M \in \mathbb{S}^{n_v} \) satisfying (7), (9a) and (9b).

4.2 Dynamic Repeated Uncertainties in Intersections of Disks and Half-Planes

Next, let us suppose again that the interconnection (1) involves a dynamic repeated uncertainty, but now with its Nyquist plot being contained in the value set

\[
V := \left\{ v I_k \mid v \in \mathbb{C}, \begin{pmatrix} \bullet \end{pmatrix} P \begin{pmatrix} 1 \\ v \end{pmatrix} \succeq 0 \right\} \tag{11}
\]

for given matrices \( P_1, \ldots, P_\nu \in \mathbb{S}^2 \) with nonpositive (2, 2) entries. Note that \( V \) is an intersection of discs and half-planes in \( \mathbb{C} \), which permits us to capture joint gain- and phase constraints on the uncertainties \( \Delta = \delta I_k \in \Delta(V) \).

A corresponding finite-horizon IQC with terminal cost is motivated by the following observation. If recalling (6) for each summand, we infer that

\[
0 \preceq \sum_{i=1}^\nu \begin{pmatrix} 1 \\ \delta \end{pmatrix} P_i \begin{pmatrix} \psi \end{pmatrix} \psi^* M_i \psi
\]

holds on the imaginary axis for any \( \Delta = \delta I_k \in \Delta(V) \) and all dynamic scalings \( \psi^* M_i \psi, \psi^* M_\mu \psi \) which are positive definite on \( i \mathbb{R} \cup \{\infty\} \). Therefore, by repeating the arguments leading to Theorem 6 almost verbatim, we arrive at the following robustness result.

Theorem 9. Let \( J_v \) and \( J_v \) for \( J_v \in \{A, B, C, D\} \) be given as in Theorem 6. Then the interconnection (1) with \( V \) as in (11) is robustly stable if there exist \( X \in \mathbb{S}^{2(n+u)+n} \), \( Y_1, \ldots, Y_\nu \in \mathbb{S}^{n_v} \) and \( M_1, \ldots, M_\nu \in \mathbb{S}^{n_v} \) satisfying

\[
\begin{pmatrix} \bullet \end{pmatrix} T \begin{pmatrix} 0 & Y_i \\ Y_i & A_v \end{pmatrix} + \begin{pmatrix} \bullet \end{pmatrix} T M_i \begin{pmatrix} C_v \\ D_v \end{pmatrix} \succeq 0
\]

for all \( i = 1, \ldots, \nu \), \( \psi \) as well as (9a) and (9b) with \( P_0 \otimes M \) and \( P_0 \otimes Y \) respectively replaced by

\[
\sum_{i=1}^\nu P_i \otimes M_i \quad \text{and} \quad \sum_{i=1}^\nu P_i \otimes Y_i.
\]

Note that we make use of dynamical scalings with identical filters \( \psi \). It is not too difficult to work with different filters \( \psi, \ldots, \psi_\nu \), but then one will in general face a large Lyapunov matrix \( X \in \mathbb{S}^d \) with \( d = 2 \times \sum_{i=1}^\nu n_\psi + n \).

4.3 Dynamic Repeated Uncertainties in LMI Regions

Next, let us assume that the interconnection (1) involves a dynamic repeated uncertainty with a Nyquist plot located in an LMI region. Hence, we suppose that

\[
V := \left\{ v I_k \mid v \in \mathbb{C} \text{ and } \begin{pmatrix} L_v \\ vL_v \end{pmatrix} P_0 \begin{pmatrix} L \nu \nu L \end{pmatrix} \succeq 0 \right\} \tag{12}
\]

for a given matrix \( P_0 = \begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \in \mathbb{S}^{2u} \) satisfying \( R. \in \mathbb{R} \). For example in (Chilali and Gahinet, 1996) and (Peaucelle et al., 2000), it is shown that LMI regions can describe a wide range of practically relevant complex convex sets that are symmetric with respect to the real axis. This offers even more flexibility in imposing gain- and phase constraints on \( \Delta \in \Delta(V) \) if compared to (11). Observe that (11) is recovered by enforcing \( Q, S \) and \( R \) in (12) to be diagonal.
For didactic reasons, let us begin with criteria involving static multipliers. Note that, to the best of our knowledge, even these have not appeared in the literature so far.

**Theorem 9.** The interconnection (1) with $V$ as in (12) is robustly stable if there exist $X \in S^n$ and $M \in S^{n \times k}$ satisfying $M > 0$, (2a) and (2b) with $P$ replaced by

$$P(P_0, M) := (\bullet)^T (P_0 \otimes M) (I_k \otimes \text{vec}(I_\nu) \otimes I_k).$$

Here, vec($S$) denotes the vectorization of the matrix $S$.

**Proof.** Note at first that we have, by the rules of the Kronecker product and for any $V = vI_k \in V$,

$$0 \preceq \left( I_{1 \nu} \frac{v}{vI_\nu} \right)^\ast P_0 \left( I_{1 \nu} \frac{v}{vI_\nu} \right) \otimes M = (\bullet)^\ast (P_0 \otimes M) \left( \left( I_{1 \nu} \frac{v}{vI_\nu} \right) \otimes I_{kv} \right).$$

Next, observe that

$$\left( I_{1 \nu} \frac{v}{vI_\nu} \right) \otimes I_{kv} = \left( I_{k_v} \otimes I_{kv} \right) = \left( I_{kv} \otimes I_{kv} \right) \otimes I_k$$

and that, hence, a multiplication with vec($I_\nu$) \otimes I_k yields

$$\left( \text{vec}(I_\nu) \otimes I_k \right) \cdot \left( \text{vec}(I_\nu) \otimes I_k \right) = \left( \text{vec}(I_\nu) \otimes I_k \right).$$

In order to construct a suitable finite-horizon IQC with terminal cost, we pick again a dynamic scaling $\psi^*M\psi$ for $x \in RH_{m \times k}^m$ and $M \in S^{m \times n}$ satisfying (5) for the inequality in (14) and another dynamic scaling $\psi^*N\psi$ with the same filter $\psi$ and a matrix $N \in RH_{m \times k}^m$ satisfying

$$0 = (\psi^*N\psi)^* + \psi^*N\psi = \psi^*(NT + N)\psi$$

on $i\mathbb{R} \cup \{\infty\}$ for the equation constraint. The particular choice for $\psi^*N\psi$ permits us to conclude

$$0 = \left( \frac{1}{\delta} \right)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes (-i \cdot \psi^*N\psi)$$

$$= (\bullet)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes (-i \cdot N)$$

$$= (\bullet)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes (NT + N)$$

$$= (\bullet)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes (NT + N)$$

on the imaginary axis and for any $\Delta = \delta I_k \in \Delta(V)$. Together with (6), we get

$$0 \preceq (\bullet)^* P_0 \left( \frac{1}{\delta} \right)^* \otimes \psi^* M \psi + (\bullet)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes (-i \cdot \psi^* N \psi)$$

$$= (\bullet)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes \psi^* M \psi + (\bullet)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes \psi^* N \psi$$

$$= (\bullet)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes \psi^* M \psi + (\bullet)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes \psi^* N \psi$$

$$= (\bullet)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes \psi^* M \psi + (\bullet)^* \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes \psi^* N \psi$$

on $i\mathbb{R}$ and for any $\Delta = \delta I_k \in \Delta(V)$. Note that the choice for $\psi^*N\psi$ assures that $-i \cdot \psi^*N\psi$ is Hermitian on $i\mathbb{R}$ and that $\Pi < RH_{2k \times 2k}^m$ has a symmetric middle matrix.

Another variation of the proof of Theorem 5 then leads to the following result, which is related to $\mu$-analysis via dynamic D-G-scalings if choosing $P_0 = \left( \begin{array}{c} 1 \delta \\ 0 \end{array} \right)$. **Theorem 11.** Let $J_\nu$ and $J_\nu$ for $j \in \{A, B, C, D\}$ be given as in Theorem 6. Then the interconnection (1) with $V$ as in (14) is robustly stable if there exist $X \in S^{2n \times n}$, $Y \in S^{m \times n}$, $Z \in S^{n \times v}$, $M \in S^{m \times n}$ and $N \in RH_{m \times k}^m$ satisfying (7),

$$(\bullet)^T \left( \begin{array}{c} 0 & 1 \\ -i & 0 \end{array} \right) \otimes (NT + N) (C_\psi D_\psi) = 0$$

respectively.

Note that if the matrices $Q, S, R$ characterizing the set (12) are diagonal, then we recover Theorem 8 by restricting $Y$ and $M$ to be block diagonal matrices.
as well as (9a) and (9b) with $P_0 \otimes M$ and $P_0 \otimes Y$ replaced by

$$P_0 \otimes M + \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix} \quad \text{and} \quad P_0 \otimes Y + \frac{1}{2} \begin{pmatrix} 0 & Z \\ Z & 0 \end{pmatrix},$$

respectively.

Note that this result can be related to the generalized KYP lemma proposed by Iwasaki and Hara (2005), but this aspect is not further explored here.

5. EXAMPLE

As an illustration let us consider a networked system composed of $N = 20$ identical subsystems with description

$$
\begin{pmatrix}
\dot{x}_k(t) \\
\dot{z}_k(t) \\
\dot{e}_k(t)
\end{pmatrix} =
\begin{pmatrix}
\frac{A}{2} & B & B_2 \\
\frac{C}{2} & D & D_{12} \\
C_2 & D_{21} & D_{22}
\end{pmatrix}
\begin{pmatrix}
x_k(t) \\
w_k(t) \\
d_k(t)
\end{pmatrix}
$$

(15a)

for $t \geq 0$ and $k \in \{1, \ldots, N\}$, where $d_1, \ldots, d_N \in L_2$ are input disturbances and $e_1, \ldots, e_N$ are error output signals. We suppose that the matrices describing the subsystems are given by

$$
\begin{pmatrix}
A & B & B_2 \\
C & D & D_{12} \\
C_2 & D_{21} & D_{22}
\end{pmatrix}
= \begin{pmatrix}
-13 & -12 & 10 \\
1 & 0 & 0 \\
1 & 1 & 0
\end{pmatrix}
$$

(15b)

and that the subsystems are coupled as

$$w_k(t) = \sum_{j=1}^{N} a_{kj} (z_k(t) - z_j(t))$$

(15c)

for $t \geq 0$ and $k \in \{1, \ldots, N\}$. More concretely, we suppose that the link strengths $a_{kj}$ are uncertain and satisfy

$$a_{kj} \in \{0, 0.75, 1\}$$

if $j = k + 1$ or $(j, k) = (N, 1)$

(15d)

otherwise.

for all $k, j \in \{1, \ldots, N\}$. This means that we consider a directed cyclic interconnected network as depicted in Fig. 1 with uncertain link strengths.

For example on the basis of the results from Wieland (2010), it is not difficult to show that the networked system (15) is robustly stable and admits a robust energy gain smaller than $\gamma$ if and only if the uncertain interconnection

$$
\begin{pmatrix}
\dot{x}(t) \\
\dot{z}(t) \\
\dot{e}(t)
\end{pmatrix} =
\begin{pmatrix}
\frac{A}{2} & B & B_2 \\
\frac{C}{2} & D & D_{12} \\
C_2 & D_{21} & D_{22}
\end{pmatrix}
\begin{pmatrix}
x(t) \\
w(t) \\
d(t)
\end{pmatrix}, \quad w(t) = \delta z(t)
$$

(16)

is robustly stable and admits a robust energy gain smaller than $\gamma$. Here, $\delta$ is a single repeated parametric uncertainty satisfying

$$\delta \in \tilde{\psi} := \bigcup_{(a_{kj})_{k,j=1}^N \text{ with (15d)}} \text{eig} \left( \mathcal{L}'((a_{kj})_{k,j=1}^N) \right)$$

(17)

where $\mathcal{L}'((a_{kj})_{k,j=1}^N) =: \mathcal{L}(a)$ denotes the Laplacian matrix corresponding to the graph in Fig. 1 for the instance $a$ of link strengths satisfying (15d). Precisely, this matrix is given by

$$
\mathcal{L}(a_{kj}) := \begin{cases}
\sum_{j=1}^{N} a_{kj} & \text{if } j = k, \\
-a_{kj} & \text{otherwise.}
\end{cases}
$$

The set $\tilde{\psi}$ in (17) is somewhat complicated, but we can find a suitable superset for applying our robust analysis tools. Indeed, one can show that

$$\tilde{\psi} \subset \psi := \left\{ v \in \mathbb{C} : |v - 1| \leq 1 \right\} \cap \left\{ v \in \mathbb{C} : |v - \frac{3}{4}| \geq \frac{3}{4} \right\}.$$

(18)

The boundary of the set $\psi$ and the eigenvalues of $\mathcal{L}(a)$ for several selected and randomly chosen values of $a$ with (15d) are depicted in Fig. 2.

Note that a robust analysis of the system (16) based on covering $\tilde{\psi}$ with a single disk $\{ v \in \mathbb{C} : |v - 1| \leq 1 \}$ is doomed to fail, since $\delta = 0.5$ is contained in this disk and since (16) is unstable for this particular value of $\delta$.

However, since we can express the set $\psi := \{ v I_2 : v \in \psi \}$ equivalently as (11) with

$$P_1 := \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad P_2 := \begin{pmatrix} 0 & -0.75 \\ -0.75 & 1 \end{pmatrix},$$

we can freely apply the standard extension of Theorem 8, 9 or 10 for analyzing uncertain systems with a performance channel. Recall that we do not require convexity of $\psi$ since (16) only involves parametric uncertainties. Applying the extension of Theorem 8 for $n_v = 0$ (static scaling) and for $n_v = 2$, $n_e = 4$ (dynamic scaling) assures robust stability of the network (15) and yields the optimal upper bounds

0.654, 0.572 and 0.572

on its robust energy gain, respectively; we did choose the filter $\psi$ as in (Scherer and Köse, 2012) with $\alpha = 2$ and $\nu = 0$, $\nu = 1$ and $\nu = 2$, respectively. Similarity as in (Poolla and Tikk, 1995), this demonstrates that there is a benefit of using dynamic scalings over static ones even if their McMillian degree is small.

6. CONCLUSION

We provide an alternative proof of one of the results in (Scherer, 2021) for analyzing systems affected by dynamic
uncertainties by means of IQCs with dynamic multipliers. In contrast to Scherer (2021), we do not rely on frequency domain arguments. Moreover, we provide several interesting variations that permit us, e.g., to analyze the robustness of a system against dynamic uncertainties with a Nyquist plot known to be located in a given LMI region.

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Appendix A. AUXILIARY RESULTS AND TECHNICAL PROOFS

Lemma 12. Let \( g : L_{2e}^1 \to L_{2e}^1 \) and \( H : L_{2e}^1 \to L_{2e}^1 \) be convolution maps represented as in (1b). Then we have

\[
H \circ (g_I) = (g_I) \circ H
\]

with \( g_I : L_{2e}^1 \to L_{2e}^1, w \mapsto (g(w_1), \ldots, g(w_j))^T \) for \( j \in \{k, l\} \).

Proof. Let \( g \) and \( H \) be realized by \( (A_g, B_g, C_g, D_g) \) and \((A, B, C, D)\), respectively. Further, let us abbreviate the functions \( \tilde{g}(s) := C_g e^{A_g s} B_g \) and \( \tilde{H}(s) := C e^{A s} B \). Since \( g \) is SISO, note that we have in particular \( \tilde{H} \tilde{g} = \tilde{g} \tilde{H} \).

Now, let \( w \in L_{2e}^1 \) and \( t \geq 0 \) be arbitrary. Then we have via integration by substitution

\[
\int_0^t \tilde{H}(t-s) \left( \int_0^s \tilde{g}(s-r) w(r) \; dr \right) \; ds
\]

\[
\int_0^t \tilde{H}(s) \left( \int_0^{t-s} \tilde{g}(t-s-r) w(r) \; dr \right) \; ds
\]

\[
\int_0^t \tilde{H}(s) \tilde{g}(r) w(t-s) \chi(r, t-s) \; dr \; ds
\]
here $\chi(a,b) = 1$ if $a \leq b$ and $\chi(a,b) = 0$ otherwise. Using $\tilde{H}g = \tilde{g}\tilde{H}$ and Fubini’s theorem, the last term equals
\begin{align*}
\int_0^t \left( \int_0^s \tilde{g}(r)\tilde{H}(s)w(t-s-r)\chi(r,t-s)\,dr \right) \,ds \\
= \int_0^t \left( \int_0^r \tilde{g}(r)\tilde{H}(s)w(t-s-r)\chi(r,t-s)\,dr \right) \,ds \\
= \int_0^t \left( \int_0^{r-t} \tilde{g}(r)\tilde{H}(s)w(s)\,dr \right) \,ds \\
= \int_0^t \tilde{g}(r-t)\left( \int_0^r \tilde{H}(r-s)w(s)\,ds \right) \,dr \nend{align*}
which yields the statement for $D = 0$ and $D_g = 0$. The general case is obtained by using linearity and (A.1)

**Proof of Theorem 5.** Let $\Delta \in \Delta(V)$ be arbitrary. Then, there exists some $\delta \in RH^{1,\infty}_1$ with $\Delta = \delta I_k$. Let $(A_3,B_3,D_3)$ be a minimal realization of $\delta$ and recall that $A_3$ is Hurwitz. Further, note that we can work with $(A_3 \otimes I_k, B_3 \otimes I_k, D_3 \otimes I_k)$ as a realization of $\Delta$.

Since the value set $V$ is chosen as in (3), the FDI
\begin{equation}
\left( \begin{array}{c} 1 \\ \delta(\omega) \end{array} \right) \begin{pmatrix} \frac{1}{\beta} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\delta(\omega)} \geq 0 \quad \text{for all} \quad \omega \in \mathbb{R} \cup \{\infty\}
\end{equation}

Consequently, we can infer the existence of a symmetric matrix $W$ satisfying
\begin{equation}
(\bullet)^T \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A_3 & B_3 \end{pmatrix} + (\bullet)^T P_0 \begin{pmatrix} I & 0 \\ C_3 & D_3 \end{pmatrix} \gtrless 0 \quad (A.2)
\end{equation}
by the KYP lemma. Since $A_3$ is Hurwitz, $r \leq 0$ and since the left upper block of (A.2) is a standard Lyapunov inequality, we can conclude that $W \nless 0$ holds.

Next, let $K \in S^d$ with $d := n_\delta + k$ denote the whole left hand side of (7) and observe that we can merge the LMIs $K \succ 0$ and (A.2) to obtain
\begin{equation}
(\bullet)^T \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A_3 & B_3 \end{pmatrix} + (\bullet)^T P_0 \begin{pmatrix} 0 & 1 \\ C_3 & D_3 \end{pmatrix} \otimes K \gtrless 0
\end{equation}
by standard properties of $\otimes$. By using further rules of this product, we get the identities
\begin{align*}
(\bullet)^T \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A_3 & B_3 \end{pmatrix} \otimes K &= (\bullet)^T \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix} \begin{pmatrix} I \\ 0 \end{pmatrix} \otimes K \\
&= (\bullet)^T \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix} \begin{pmatrix} A_3 & 0 \\ 0 & B_3 \end{pmatrix} \otimes K \\
&= (\bullet)^T \begin{pmatrix} 0 & W \otimes K \\ W \otimes K & 0 \end{pmatrix} \begin{pmatrix} I \otimes A_3 & I \otimes B_3 \\ I \otimes A_3 & I \otimes B_3 \end{pmatrix} \otimes (L \otimes I_d)
\end{align*}
and, with analogous intermediate steps,
\begin{equation}
(\bullet)^T P_0 \begin{pmatrix} 0 & I \\ C_3 & D_3 \end{pmatrix} \otimes K = (\bullet)^T (P_0 \otimes K) \begin{pmatrix} 0 & I \\ C_3 \otimes I_d & D_3 \otimes I_d \end{pmatrix} \cdot
\end{equation}
These permit us to conclude
\begin{equation}
(\bullet)^T \begin{pmatrix} 0 & W \otimes K \\ W \otimes K & 0 \end{pmatrix} \begin{pmatrix} I \otimes A_3 & I \otimes B_3 \\ I \otimes A_3 & I \otimes B_3 \end{pmatrix} \otimes (L \otimes I_d) \gtrless 0
\end{equation}
As a next step toward showing the validity of the IQC in Definition 4, we define $u = \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) = \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right)$ for some arbitrary $z \in L_2^\infty$. Next to this signal, we also work with the output $y = \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right)$ and the state $\xi = \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right)$ of the filter (4) in response to the input $u$. Here, the partitions are induced by the block diagonal structure of the realization matrices. Finally, we define the auxiliary signals
\begin{equation}
\tilde{y}_i := \left( \begin{array}{c} \xi_i \\ u_i \end{array} \right) \in L_2^\infty \quad \text{for} \quad i \in \{1,2\}
\end{equation}
By the definition of $K \in S^d$ as the left-hand side of (7), we infer for all $t \geq 0$ and $i,j \in \{1,2\}$ that
\begin{equation}
y_i(t)^T M y_j(t) + \frac{d}{dt} \xi_i(t)^T Y \xi_j(t) = \tilde{y}_i(t)^T K \tilde{y}_j(t)
\end{equation}
holds. As a consequence of direct calculations, we get
\begin{equation}
(\bullet)^T (P_0 \otimes M) y(t) + \frac{d}{dt} (\bullet)^T (P_0 \otimes Y) \xi(t) = (\bullet)^T (P_0 \otimes K) \tilde{y}(t)
\end{equation}
for all $t \geq 0$.

Now recall that $u_2 = \Delta(u) = (\delta I_k)(u_1)$ holds and that $\tilde{y}_i$ is the response of an LTI filter with zero initial condition to the input $u_i$ for $i \in \{1,2\}$. Therefore, Lemma 12 implies
\begin{equation}
\tilde{y}_2 = (\delta I_d)(\tilde{y}_1)
\end{equation}
Note that we can equivalently express this identity as
\begin{equation}
\begin{pmatrix} \xi(t) \\ \tilde{y}_2(t) \end{pmatrix} = \begin{pmatrix} A_3 \otimes I_d & B_3 \otimes I_d \\ C_3 \otimes I_d & D_3 \otimes I_d \end{pmatrix} \begin{pmatrix} \xi(t) \\ \tilde{y}_1(t) \end{pmatrix}
\end{equation}
for $t \geq 0$ and with $\xi(0) = 0$.

At this point we exploit (A.3). By standard dissipation arguments, the function $\eta : t \mapsto \xi(t)^T (W \otimes K) \xi(t)$ satisfies
\begin{equation}
\eta(t) = (\bullet)^T \begin{pmatrix} 0 & W \otimes K \\ W \otimes K & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ A_3 \otimes I_d & B_3 \otimes I_d \end{pmatrix} \begin{pmatrix} \xi(t) \\ \tilde{y}_1(t) \end{pmatrix}
\end{equation}
\begin{equation}
\geq - (\bullet)^T (P_0 \otimes K) \begin{pmatrix} 0 & I \\ C_3 \otimes I_d & D_3 \otimes I_d \end{pmatrix} \begin{pmatrix} \xi(t) \\ \tilde{y}_1(t) \end{pmatrix} = - (\bullet)^T (P_0 \otimes K) \tilde{y}(t)
\end{equation}
for all $t \geq 0$. By integration, we obtain
\begin{equation}
\int_0^T (\bullet)^T (P_0 \otimes K) \tilde{y}(t) \,dt \geq - (\eta(T) - \eta(0)) \geq 0 \quad (A.5)
\end{equation}
for all $T \geq 0$, were we exploited $\eta(0) = 0$ and $\eta(T) \leq 0$. The latter two properties of $\eta$ are consequences of $\xi(0) = 0$ as well as of $W \nless 0$ and of $K \succ 0$.

Finally, integrating (A.4) and using $\xi(0) = 0$ as well as (A.5) yields
\begin{equation}
\int_0^T y(t)^T (P_0 \otimes M) y(t) \,dt + \xi(T)^T (P_0 \otimes Y) \xi(T)
\end{equation}
\begin{equation}
= \int_0^T \tilde{y}(t)^T (P_0 \otimes K) \tilde{y}(t) \,dt \geq 0
\end{equation}
for all $T \geq 0$ as desired. ■