Saha Ionization Equation in the Early Universe

Aritra Das1, Ritesh Ghosh2, and S. Mallik2

1 HENPP Division, Saha Institute of Nuclear Physics, HBNI, 1/AF Bidhan Nagar, Kolkata 700064, India; aritra.das@saha.ac.in
2 Theory Division, Saha Institute of Nuclear Physics, HBNI, 1/AF Bidhan Nagar, Kolkata 700064, India; ritesh.ghosh@saha.ac.in samir.mallik@saha.ac.in

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Abstract

The Saha equation follows from thermal equilibrium of matter and radiation. We discuss this problem of equilibrium in the early universe, when matter consists mostly of electrons, protons, and hydrogen atoms. Taking H atoms in their ground state only and applying the real-time formulation of thermal field theory, we calculate the difference of ionization and recombination rates, which controls the equilibrium of H atoms. By comparing with realistic calculations including the excited states of the H atom, we conclude that the presence of excited states lowers the equilibrium temperature from 5000 to 4000 K.

Key words: cosmology; theory – early universe

1. Introduction

A century ago Saha (1920) used the thermodynamics of chemical equilibrium to find the degree of ionization of atoms in a thermal bath. Since then it has been vigorously applied to investigate the spectrum of the Sun and other stars. More recently, it has been used to find the epoch of hydrogen recombination involving the reaction

$$\gamma + H \rightleftharpoons p^+ + e^−,$$

in the early universe, leading to an understanding of the cosmic microwave background (CMB) radiation observed today (Dicke et al. 1965; Penzias & Wilson 1965). (There was an earlier epoch of helium recombination, which can be treated separately in a first approximation).

In the early universe the equilibrium condition is not guaranteed a priori; it depends on the reaction rate and cosmic expansion rate. Also the Saha equation neglects the excited states of the H atom, which is a very good approximation as long as thermal equilibrium prevails. But away from equilibrium the excited states may be important in the process. Accordingly, a number of authors (Peebles 1968; Zeldovich & Sunyaev 1969) have investigated the hydrogen ionization and recombination in the realistic case, without assuming equilibrium and including the excited states (2s, 2p) of the H atom along with the ground (1s) state. The discovery of CMB anisotropies prompted a resurgence of the calculation of recombination including nonleading effects. Thus, the earlier (effective) three-level calculation was replaced with a multi-level one (Seager et al. 1999) including helium and its higher excited states. The effect of Raman scattering along with the related two-photon emission (Chluba & Thomas 2011) was incorporated in the calculation. A review of all such processes is contained in the Karl Schwarzschild lecture by Sunyaev & Chluba (2009).

In this paper, we attempt an indirect but easy way to study the effect of the excited states in attaining equilibrium. We calculate the ionization and recombination rates in a simplified model, where we exclude the excited states, taking only the H atom in the ground state. Comparing these reaction rates with the cosmic expansion rate, we may know at which temperature the equilibrium is lost in the simplified model. On the other hand, comparing the fractional hydrogen ionization from the Saha equation with that of the realistic calculation (including the excited states), we can find when equilibrium is lost in the real world. Now comparing the two results for the loss of equilibrium, we may see the role of the excited states of H atom.

We use the reaction rates to write the Boltzmann equation for an arbitrary (nonequilibrium) distribution of H atoms. It has a simple analytic solution consisting of two terms (Weldon 1983). The first term gives the equilibrium distribution, while the second term vanishes exponentially with time. It is the coefficient of time in the exponential of this term, which is identified with the reaction rate tending the distribution to equilibrium. This rate must be compared with the expansion rate of the universe, given by the then Hubble parameter.

Though the problem is nonrelativistic, we shall use relativistic expressions to evaluate the rates and then apply a nonrelativistic approximation. We use units with $h(=h/2\pi)$ and $c$ taken to be unity, where $h$ and $c$ are Planck’s constant and the velocity of light, respectively. Also, we write $\beta = 1/k_B T$, where $k_B$ is the Boltzmann constant and $T$ the temperature.

2. Ionization and Recombination Rates

In atomic physics the processes represented by (1) are well known. Recombination (or recapture) is the capture of an electron in the continuum by the atomic nucleus with the emission of a photon. Ionization (or photoeffect) is the inverse process, where a photon is absorbed by an atom accompanied by ejection of an electron. These transitions are caused by the electric dipole operator in quantum mechanics (Bethe & Salpeter 2008; Weinberg 2013). It is given by the potential function, $V = -(e/m) A \cdot p$, where $e$, $m$, and $p$ are the electric charge, mass of the electron, and its momentum operator and $A$ is the radiation field. The transition amplitudes for the above processes are given by the matrix elements of $V$ between two atomic states, where one is discrete and the other in the continuum.

Instead of using the above method to calculate the reaction rates, we use the elegant method of thermal quantum field theory for the problem (Semenoff & Umezu 1983; Niemi & Semenoff 1984; Mallik & Sarkar 2016), where we directly get the recombination and ionization probabilities multiplied by appropriate factors involving distribution functions for particles in the medium. We first construct the Lagrangian interaction
involving all particles in (1). Let the photon, electron, and proton fields be respectively \( A_\mu(x), \psi(x), \) and \( \bar{\psi}_\mu(x) \). We take H atoms in the ground state only (ignoring its excited states), when it can be represented by an elementary scalar field \( \Phi(x) \) (Dashen & Rajaraman 1974; Weinberg 1995). Then the required effective Lagrangian interaction is

\[
L_{\text{int}} = g(\Phi \bar{\psi} \gamma^\mu \psi + h.c.) A_\mu.
\]

It describes an electromagnetic interaction and \( g \) must have mass dimension \(-1\). So we take

\[
g = \frac{e}{m},
\]

to within some uncertainty.

The calculation of the relevant reaction rate in thermal field theory can be best approached by considering the self-energy graph for the H atom shown in Figure 1. It includes in the intermediate state all other particles that appear in (1). The elements of real-time thermal field theory is sketched in Appendix A. These elements are used in Appendix B to calculate this graph, from which we obtain the different imaginary parts, which are collected in (48) in a compact form. Any particular imaginary part may be isolated by integrating the variables \( k_{0i}^I, k_{0i}^L, \) and \( k_{0i}^m \) in (48) over the appropriate delta functions in the spectral functions, such as in (27).

Before we write the desired imaginary part, we establish our notation. We shall not use the four-momentum notation anymore; instead, we now denote the magnitudes of the three momenta of the H atom (of mass \( M_{\text{H}} \)), photon, electron (of mass \( m \)), and proton (of mass \( M \)) by \( q, k_i(i = 1, 2, 3) \) and energies by \( \omega = \sqrt{q^2 + M_{\text{H}}^2}, \omega_1 = k_1, \omega_2 = \sqrt{k_2^2 + m^2}, \) and \( \omega_3 = \sqrt{k_3^2 + M^2} \), respectively. We are interested in the imaginary part shown in Figure 2, corresponding to processes in (1) with the photon (and H atom) incoming and electron and proton outgoing for ionization, when the reverse process (recombination) will automatically be given by the second term in brackets in (48). Accordingly, we choose the delta functions in \( k_{0i}^I, k_{0i}^L, \) and \( k_{0i}^m \) variables as \( \delta(k_{01}^I + \omega_1), \delta(k_{02}^I - \omega_2), \) and \( \delta(k_{03}^I - \omega_3) \). On using (30) and (35) to convert \( f \) and \( \bar{f} \) to \( n \) and \( \bar{n} \), we get for the required processes (denoted by subscript 1) from (48)

\[
\text{Im} \Sigma^{(1)} = 16 \pi g^2 M_{\text{H}} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{1}{2\omega_3} \left[ n_1(1 - \bar{n}_2)(1 - \bar{n}_3) - (1 + n_1)\bar{n}_2 \bar{n}_3 \right] \times \delta(\omega + \omega_1 - \omega_2 - \omega_3),
\]

where \( n_1(\omega_1), \bar{n}_2(\omega_2), \) and \( \bar{n}_3(\omega_3) \) are the equilibrium distribution functions for the photon, electron, and proton, respectively. It resembles the unitarity relation for the \( S \) matrix in vacuum for the two-particle states of the photon and H atom. Compared to that relation, we now have the difference of two terms. Dividing it by \( \omega \), we convert it to a rate of (Weldon 1983)

\[
\frac{\text{Im} \Sigma^{(1)}}{\omega} = \Gamma_d - \Gamma_i,
\]

where \( \Gamma_d \) and \( \Gamma_i \) are the first and the second terms in (4), representing respectively the decay and inverse decay rates of the H atom.

### 3. Boltzmann Equation

The rates \( \Gamma_d \) and \( \Gamma_i \) are related. To see this, we write the equilibrium distribution functions explicitly as

\[
n_i = \frac{\exp(-\beta\omega_i/2)}{\exp(\beta\omega_i/2) - \exp(-\beta\omega_i/2)},
\]

\[
1 + n_i = \frac{\exp(\beta\omega_i/2)}{\exp(\beta\omega_i/2) - \exp(-\beta\omega_i/2)}
\]

and similarly for factors involving \( \bar{n}_1 \) and \( \bar{n}_2 \). Taking into account the energy conserving delta function in (4), we get

\[
\Gamma_d; \Gamma_i = 16 \pi g^2 M_{\text{H}} \left\{ \frac{\exp[(\beta(\omega - \mu_2 - \mu_3)/2)]}{\omega} \right\} \times \exp[-\beta(\omega - \mu_2 - \mu_3)/2] \times L,
\]

where \( \mu_2 \) and \( \mu_3 \) are the chemical potentials for the electron and proton and \( L \) is given by

\[
L = \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{1}{2\omega_3} \times \frac{\delta(\omega + \omega_1 - \omega_2 - \omega_3)}{\prod \left\{\exp[\beta(\omega_i - \mu_i)/2] + \exp[-\beta(\omega_i - \mu_i)/2]\right\}},
\]

where the product runs over \( i = 1, 2, 3 \). For \( i = 1, \mu_1 = 0 \) and we take the upper sign, and for \( i = 2, 3 \), we take the lower sign. So we get the ratio

\[
\frac{\Gamma_d}{\Gamma_i} = \exp[\beta(\omega - \mu_2 - \mu_3)].
\]

So far we have treated the H atom as a single particle without any distribution in the medium. Let us now assume an arbitrary (nonequilibrium) distribution \( n(\omega, t) \) of these particles. We can write a Boltzmann equation for \( n(\omega, t) \), noting that it decreases at the rate \( n\Gamma_d \) and increases at the rate \( (1 + n)\Gamma_i \),

\[
\frac{dn(\omega, t)}{dt} = (1 + n)\Gamma_i - n\Gamma_d.
\]
whose solution is \((\text{Weldon }1983; \text{Le Bellac }2000)\)

\[
n(\omega, t) = \frac{\Gamma_i}{\Gamma_d - \Gamma_i} + c(\omega)e^{-(\Gamma_d - \Gamma_i)t} \\
= \frac{1}{\exp\beta(\omega - \mu_2 - \mu_3) - 1} + c(\omega)e^{\Gamma t},
\]

\[
\Gamma = \Gamma_d - \Gamma_i, \tag{10}
\]

where \(c(\omega)\) is an arbitrary function and we use (8). If \(\mu\) is the chemical potential of the H atom, the first term in \(n(\omega, t)\) satisfies the condition

\[
\mu = \mu_2 + \mu_3, \tag{11}
\]

which is the condition of chemical equilibrium (Reif 1985), as can be read from (1). Observe that this condition arises automatically in our calculation as a result of using the equilibrium thermal propagators. So the distribution function approaches the equilibrium value exponentially in time, irrespective of its initial distribution, and its rate is governed by the reaction rate \(\Gamma\).

So far the formulae are exact. Let us now make two simplifications appropriate for the problem at hand. First, we write the energies in a nonrelativistic approximation:

\[
\omega = M_H + \frac{q^2}{2M_H}, \quad \omega_1 = k_t, \\
\omega_2 = m + \frac{k_2^2}{2m}, \quad \omega_3 = M + \frac{k_3^2}{2M}, \tag{12}
\]

with \(M_H = M + m - \epsilon_0\), where \(\epsilon_0(=13.6 \text{ eV})\) is the binding energy of the H atom in the ground state, which we neglect except in the exponential. Our second simplification results from the particle densities being dilute. We replace Fermi–Dirac and Bose–Einstein distributions with those of Maxwell–Boltzmann:

\[
\bar{n}_2 = \exp\left[-\beta\left(\frac{k_2^2}{2m} - \mu_2'\right)\right], \\
\bar{n}_3 = \exp\left[-\beta\left(\frac{k_3^2}{2M} - \mu_3'\right)\right], \\
n = \exp\left[-\beta\left(\frac{q^2}{2M} - \epsilon_0 - \mu_2' - \mu_3'\right)\right], \tag{13}
\]

where we define nonrelativistic chemical potentials by \(\mu_2' = \mu_2 - m\) and \(\mu_3' = \mu_3 - M\).

The total number of electrons in volume \(V\) is

\[
N_e = 2V\int \frac{dk}{(2\pi)^3}\bar{n}_2(k) = 2V\left(\frac{m}{2\pi\beta}\right)^{3/2} \exp(\beta \mu_2'). \tag{14}
\]

Similarly, the total numbers of protons and H atoms are

\[
N_p = 2V\left(\frac{M}{2\pi\beta}\right)^{3/2} \exp(\beta \mu_2'), \tag{15}
\]

\[
N_H = 4V\left(\frac{M_H}{2\pi\beta}\right)^{3/2} \exp[\beta(\epsilon_0 + \mu_2' + \mu_3')]. \tag{16}
\]

which incorporates the equilibrium condition (11). We then get the Saha equation (Weinberg 2008)

\[
\frac{N_eN_p}{N_H} = V\left(\frac{m}{2\pi\beta}\right)^{3/2} \exp(-\beta \epsilon_0). \tag{17}
\]

4. Early Universe

We now examine the equilibrium condition in the early universe. We first estimate the reaction rate \(\Gamma\). As in the previous section, we reduce the distributions to that of Maxwell–Boltzmann by retaining only the positive exponentials in the products in the denominator of (7). Also, we ignore terms of \(O(1/M)\) compared to \(O(1/m)\) and write \(\omega_2\omega_3 \simeq mM\) in this denominator to get \(L\) as

\[
L = \left(\frac{4\pi}{(2\pi)^3}\right)^2 \frac{1}{8Mm} \int_0^{\infty} dk_1 k_1 dk_2 k_2^2 \\
\times \exp\left(-\beta\left(k_1 + k_2^2 - \mu_2' - \mu_3'\right)\right) \delta\left(-\epsilon_0 + k_1 + k_2^2 - 2m\right) \tag{18}
\]

We remove the \(k_1\) integral with the delta function, when the \(k_2\) integral reduces to a Gamma function, giving

\[
L = \left(\frac{4\pi}{(2\pi)^3}\right)^2 \frac{1}{8Mm} \exp[-\beta(\epsilon_0 - \mu_2' - \mu_3')/2] \\
\times \frac{\sqrt{\pi}}{4} \left(\frac{m}{\beta}\right)^{3/2} \left(\epsilon_0 + \frac{3}{4\beta}\right). \tag{19}
\]

As the H-atom concentration is dilute, we set \(1 + n \simeq 1\) in (9), where the solution (10) shows the equilibrium distribution to be of Maxwell–Boltzmann type and the reaction rate becomes \(\Gamma \simeq \Gamma_d\). From (6) and (19) we thus get

\[
\Gamma = \frac{g^2\sqrt{\pi}}{8\pi^3M} \left(\frac{m}{\beta}\right)^{3/2} \epsilon_0 \exp(-\beta \epsilon_0) \left(1 + \frac{3}{4\beta \epsilon_0}\right). \tag{20}
\]

Noting (3) we may rewrite it as

\[
\Gamma \simeq \frac{e^2}{4\pi^2} \left(\frac{m}{\beta}\right)^{3/2} \frac{1}{M} \left(\frac{k_B T}{m}\right)^{3/2} \epsilon_0 \exp(-\epsilon_0/k_B T). \tag{21}
\]

Next, the expansion rate of the universe is given by the Hubble parameter \(H = \dot{a}/a\), where \(a(t)\) is the scale factor in the metric. In the era of interest to us (\(T > 1000\) K), the constant vacuum energy is utterly negligible and we consider the energy density of matter, both nonrelativistic \((\rho_M)\) and relativistic \((\rho_R)\). Also, we do not include the curvature term, which, assuming a prior inflationary epoch, is driven to 0. Denoting the present values by the subscript (0), we then write the total energy density as

\[
\rho(t) = \rho_0 \left(\Omega_M \left(\frac{T}{T_0}\right)^4 + \Omega_R \left(\frac{T}{T_0}\right)^4\right), \tag{22}
\]

where \(\Omega_M\) and \(\Omega_R\) are fractions of the present critical energy density \(\rho_0\). Applying the Einstein equation with the spatially
Including the contributions of photons and neutrinos in the universe at temperatures in the region of interest. We only estimate the role of excited states of the H atom in calculating the reaction rate. Because the Saha equation assumes the equilibrium condition, this condition should prevail as long as the complete thermal trace is over a complete set of states. Another advantage of thermal field theory is that the polarization sums are done automatically in reaction probabilities.

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**Appendix A**

Equilibrium Thermal Field Theory

Here we recall the basic elements of equilibrium thermal field theory (Mallick & Sarkar 2016). Compared to vacuum field theory, it differs essentially in the time path and hence the propagator. To bring out this difference let us take a scalar field \( \phi(x) \), with \( x^i = (\tau, x) \), where the time variable \( \tau \) may be complex, and consider the time ordered propagator, which arises in perturbative calculations. While for the vacuum propagator

\[
\Delta_{\tau}(x, x') = i \langle 0 | \tau \phi(\tau, x) \phi(\tau', x') | 0 \rangle,
\]

the time variables run over the real-time axis, the corresponding thermal propagator

\[
D(x, x') = i \text{Tr}[\exp(-\beta H)\phi(\tau, x)\phi(\tau', x')]/\text{Tr}[\exp(-\beta H)]
\]

has the time variables running over an interval in the complex time plane. Here \( H \) is the Hamiltonian of the system and the thermal trace is over a complete set of states.

The time path at a finite temperature may be broadly chosen in two different ways. It may be the imaginary segment from 0 to \(-i\beta\) in the complex \( \tau \) plane, giving the imaginary time formulation. In the real-time formulation, which we shall use here, the time path must traverse the real axis. Then it must end at a point with \( \text{Im } \tau = -\beta \). There are different ways to complete this path; we shall choose the one shown in Figure 3. Only the two horizontal lines contribute to the propagator, making the propagator a \( 2 \times 2 \) matrix.

It is possible to write a spectral representation for the propagator of fields of any spin. For the problem at hand, we have the scalar field \( \phi(x) \) to represent the H atom, the vector field \( A_\mu(x) \) for the photon, and the Dirac fields \( \psi_e(x) \) and \( \psi_p(x) \) for the electron and proton. The form of the spectral representation depends on the bosonic or fermionic nature of the field. The \( 2 \times 2 \) matrix propagator for the scalar field is

\[
D_\phi(k_0, k) = \int_{-\infty}^{+\infty} \frac{dk_0'}{2\pi} \rho(k_0', k) \Delta_\phi(k_0', k_0), \quad (i, j = 1, 2),
\]
where the spectral function is
\[
\rho(k_0', k) = 2\pi \epsilon(k_0') \delta(k_0'^2 - \omega^2) = 2\pi \epsilon(k_0') \frac{\delta(k_0' - \omega) + \delta(k_0' + \omega)}{2\omega},
\]
with \( \omega = \sqrt{k^2 + m^2} \).

(27)

Here \( \epsilon(k_0) \) is the sign function defined as +1 for \( k_0 > 0 \) and -1 for \( k_0 < 0 \). As we shall see, only the 11-component of the propagator appears in our calculation, for which

\[
\Lambda_1(k_0', k_0) = \frac{1 + f(k_0')}{k_0' - k_0 - i\eta} - \frac{f(k_0')}{k_0' - k_0 + i\eta},
\]

(28)

where \( f(k_0') \) is a distribution-like function

\[
f(k_0') = \frac{1}{\exp[\beta(k_0' - \mu)] - 1}.
\]

(29)

We shall use the \( \delta \) function in the spectral function to express it in terms of the true distribution function \( n(\omega) \),

\[
f(\omega) = \frac{1}{\exp[\beta(\omega - \mu)] - 1} \equiv n(\omega),
f(-\omega) = -(1 + n(\omega)).
\]

(30)

For the vector field \( A_\mu(x) \), the propagator is again given by the one for the scalar propagator with the spectral function

\[
\rho_{\mu\nu}(k_0', k) = 2\pi \epsilon(k_0') g_{\mu\nu} \delta(k^2).
\]

(31)

For the spin \( \frac{1}{2} \) (fermion) propagator, we have a similar representation:

\[
S_{ij}(p_0, p) = \int \frac{dk_0'}{2\pi} \sigma(p_0', p) \Omega(p_0', p_0), (i, j = 1, 2),
\]

(32)

with the spectral function

\[
\sigma(p_0', p) = 2\pi \epsilon(p_0')(p + m) \delta(p^2 - m^2)
\]

and the 11-component of the \( \Omega \) matrix is

\[
\Omega_{11}(p_0', p_0) = \frac{1 - \tilde{f}(p_0')}{p_0' - p_0 - i\eta} + \frac{\tilde{f}(p_0')}{p_0' - p_0 + i\eta}.
\]

(34)

Again \( \tilde{f}(p_0') \) can be written in terms of the fermion distribution function as

\[
\tilde{f}(\omega) = \frac{1}{\exp[\beta(\omega - \mu)] + 1} \equiv \tilde{n}(\omega).
\]

(35)

(Also \( \tilde{f}(-\omega) = 1 - (\exp[\beta(\omega + \mu)] + 1)^{-1} \), but we shall not need it.)

The above free propagators and the interacting (complete) ones and hence the self-energies may be diagonalized. Here we are primarily interested in (the imaginary part of) the self-energy of the \( \Phi(x) \) field representing the H atom, which diagonalizes as

\[
\Sigma(q) = U^{-1}(q) \left( \begin{array}{cc} \Sigma(q) & 0 \\ 0 & -\Sigma(q) \end{array} \right) U^{-1}(q),
\]

(36)

where the diagonalizing matrix is

\[
U^{-1}(q) = \left( \begin{array}{cc} \sqrt{1 + n} & -\sqrt{n} \\ -\sqrt{n} & \sqrt{1 + n} \end{array} \right) n = \frac{1}{\exp[\beta(q_0)] - 1}.
\]

(37)

From (36) we get

\[
\text{Im} \Sigma(q) = \epsilon(q_0) \tanh \left( \frac{\beta q_0}{2} \right) \text{Im} \Sigma_{11}(q),
\]

(38)

so that we may evaluate only the 11-component of the \( \Sigma \) matrix to get the imaginary part of the diagonalized matrix.

**Appendix B**

**Evaluation of the Self-energy Graph**

Here we shall evaluate the two-loop thermal self-energy graph (Figure 1) of the H atom. As the vacuum and medium calculations differ only in the propagators, we can first conveniently find it in vacuum:

\[
\Sigma(q) = -2g^2 \int \frac{dk_1}{(2\pi)^2} \frac{dk_2}{(2\pi)^2} \frac{dk_3}{(2\pi)^2} \frac{dk_4}{(2\pi)^2} \times \delta^3(q - k_1 - k_2 - k_3) D^{\mu\nu}(k_1) \\
\times \text{tr}[S^{(e)}(k_2) \gamma_0 S^{(p)}(k_3) \gamma_0],
\]

(39)

where \( \text{tr} \) indicates trace over \( \gamma \) matrices and \( D^{\mu\nu}, S^{(e)}, \) and \( S^{(p)} \) are the vacuum propagators for the photon, electron, and proton, respectively. Then the 11-component of the thermal self-energy matrix is immediately obtained by replacing the vacuum propagators with the corresponding 11-component of the thermal propagator matrices:

\[
\Sigma_{11}(q) = -2g^2 \int \frac{dk_1}{(2\pi)^2} \frac{dk_2}{(2\pi)^2} \frac{dk_3}{(2\pi)^2} \frac{dk_4}{(2\pi)^2} D_{11}^{\mu\nu}(k_1) \text{tr} \\
\times [S_{11}^{(e)}(k_2) \gamma_0 S_{11}^{(p)}(q - k_1 - k_2) \gamma_0],
\]

(40)

where we keep only the independent loop momenta. We write the propagator in their spectral representations (27) and (32). The tensor and spinor factors in the spectral functions can be collected to give

\[
g^{\mu\nu} \text{tr}[(\not{q} + m) \gamma_\nu(q - \not{k}_1 - \not{k}_2) \gamma_0] \simeq 8 Mm.
\]

(41)

Removing these factors, we get the three spectral functions as

\[
\rho(k_1) = 2\pi \epsilon(k_0) \delta(k_1^2),
\sigma(k_2) = 2\pi \epsilon(k_0) \delta(k_2^2 - m^2),
\sigma(k_3) = 2\pi \epsilon(k_0) \delta(k_3^2 - M^2),
\]

(42)

where \( k_3 = q - k_1 - k_2 \). Also, we segregate the integrals in energy components of \( k_1 \) and \( k_2 \) over the energy denominators of propagators. We thus write (40) as

\[
\Sigma_{11} = -16g^2 M \int \frac{dk_1}{(2\pi)^2} \frac{dk_2}{(2\pi)^2} \frac{dk_3}{(2\pi)^2} \int \frac{dk_0'}{(2\pi)^2} \frac{dk_0''}{(2\pi)^2} \frac{dk_0'''}{(2\pi)^2} \\
\times \rho(k_0', \not{k}_1) \sigma_0(k_0'', \not{k}_1) \sigma_{\mu\nu}(k_0''', \not{q} - \not{k}_1 - \not{k}_2) \cdot \mathbf{K},
\]

(43)
where
\[ K = \int \frac{dk_{10} dk_{20}}{2\pi} \frac{1 + f'}{k_{10} - k_{10} - i\eta} - \frac{f'}{k_{0} - k_{10} + i\eta} \]
\[ \times \left( \frac{1 - \tilde{f}''}{k_{0}'' - k_{20} - i\eta} + \frac{\tilde{f}''}{k_{0}'' - k_{20} + i\eta} \right) \]
\[ \times \left( \frac{1 - \tilde{f}'''}{k_{0}''' - (q_{0} - k_{10} - k_{20}) - i\eta} + \frac{\tilde{f}'''}{k_{0}''' - (q_{0} - k_{10} - k_{20}) + i\eta} \right). \] (44)

with \( f' = f(k_{0}'), \tilde{f}'' = \tilde{f}(k_{0}''), \tilde{f}''' = \tilde{f}(k_{0}'''). \)

Let us work out the integral over \( k_{10} \) first, noting that this variable appears in the first and third factors in (44). When these two factors are multiplied out, these result in four terms. As the integral converges in both the upper and lower halves of the \( k_{0} \) plane, we can evaluate it closing the integration contour in either half. So only two of these terms, having poles in both the upper and lower halves of \( k_{10} \) plane, can contribute to the integral. Thus, we evaluate the \( k_{10} \) integral in (44) to get
\[ K = \left( \frac{(1 + f')(1 - \tilde{f}'')}{k_{20} + k_{0}'' + k_{0}''' - q_{0} + i\eta} \right) \]
\[ + \left( \frac{f''\tilde{f}'''}{k_{0} + k_{10} + k_{20}'' - q_{0} + i\eta} \right). \] (45)

Next we carry out the \( k_{20} \) integral in the same way over the second factor in (44) and the one just obtained to get
\[ K = \left( (1 + f')(1 - \tilde{f}'')(1 - \tilde{f}''') \right) \]
\[ - \left( \frac{f''\tilde{f}'''}{k_{0} + k_{10}'''' - q_{0} + i\eta} \right) \] (46)
giving its imaginary part as
\[ \text{Im} K = -\pi [(1 + f')(1 - \tilde{f}'')(1 - \tilde{f}''') \]
\[ + f''\tilde{f}''\tilde{f}'''\delta(q_{0} - k_{0}'''' - k_{0}'''' - k_{0}''')] \]
\[ = -\pi \coth \left( \frac{3\eta}{2} \right) \]
\[ \times [(1 + f')(1 - \tilde{f}''')(1 - \tilde{f}'')] \]
\[ - f''\tilde{f}''\tilde{f}'''\delta(q_{0} - k_{0}'''' - k_{0}'''' - k_{0}'''). \] (47)

Comparing with (38) we finally get the imaginary part of the diagonalized self-energy as
\[ \text{Im} \Sigma(q) = 16g^{2}mM\pi\epsilon(q_{0}) \int \frac{d^{3}k_{0}'}{2\pi} \frac{d^{3}k_{2}}{2\pi} \sigma_{p}(k_{0}'', k_{2}) \]
\[ \times \int \frac{d^{3}k_{0}'''}{2\pi} \rho(k_{0}''', k_{1}) \sigma_{p}(k_{0}'', k_{2}) \]
\[ \times [(1 + f')(1 - \tilde{f}'')(1 - \tilde{f}''') - f''\tilde{f}''\tilde{f}'''] \]
\[ \times \delta(q_{0} - k_{0}'''' - k_{0}'''' - k_{0}'''). \] (48)

Integrating over \( k_{10}', k_{10}''', \) and \( k_{10}''' \) with the delta functions contained in the three spectral functions, we get eight terms corresponding to different particles in the initial and final states. In Section 2 we get one of these terms representing the ionization and recombination probabilities.

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