General Type IIB Fluxes with SU(3) Structures

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ABSTRACT

A supersymmetric vacuum has to obey a set of constraints on fluxes as well as first order differential equations defined by the G-structures of the internal manifold. We solve these equations for type IIB supergravity with SU(3) structures. The 6-dimensional internal manifold has to be complex, the axion/dilaton is in general non-holomorphic and a cosmological constant is only possible if the SU(3) structures are broken to SU(2) structures. The general solution is expressed in terms of one function which is holomorphic in the three complex coordinates and if this holomorphic function is constant, we obtain a flow-type solution and near poles and zeros we find the so-called type-A and type-B vacuum.
1 Introduction

A string theory vacuum has to meet a number of requirements to be consistent with 4-dimensional phenomenology. Not only the moduli, which are abundant in string theory compactifications, have to be fixed, but also chiral fermions charged under the gauge group of the standard model have to present. Most difficult however, is to achieve this in a space time with a small positive cosmological constant. A good starting point are however, the supersymmetric ground states with a negative or zero cosmological constant. A crucial ingredients for lifting the moduli space is the presence of background fluxes, which generate a potential for the scalar fields and there are basically two approaches to these flux compactifications. If one neglects the back reaction of the fluxes on the geometry, one can simply perform a KK-reduction on a Calabi-Yau space yielding the explicit form of the potential. This is especially reliable as long as one has only RR or only NS fluxes, see [1, 2, 3]. The other approach discusses background fluxes as specific torsion components of the internal geometry and classifies the vacua with respect to the corresponding $G$-structures [4, 5, 6, 7, 8, 9, 10, 11]. This approach takes the back reaction of the fluxes on the geometry into account and gives not only the deformed geometry but identifies also the embedding of the fluxes. It does however not give an expression for the potential and a discussion of deSitter vacua is not possible a priori.

To consider type IIB string theory is especially interesting due to its role for providing supergravity duals of RG-flows and for providing the basis on most discussions on string cosmology [12]. From the technical viewpoint, fluxes on the IIB side do not generate the severe back-reaction onto the geometry as it is known on the type IIA side [13, 14, 15, 16, 17] (see [18] for type IIA fluxes that also preserve the Calabi-Yau property). In fact, as we will verify, the internal geometry is always given by a complex space and it is known that 3-form fluxes can be turned on while still preserving the Calabi-Yau space [19, 20, 21, 22]. On the other hand, type IIB supergravity yields generically a no-scale model related to at least one un-fixed modulus (typically the volume modulus) and the supersymmetric vacuum is a flat 4-dimensional Minkowski space. The no-scale structure can be lifted by non-perturbative effects due to brane-instantons that typically generate a (negative) cosmological constant. Since branes are sources for fluxes it would be interesting to see whether AdS vacua can also be obtained from flux compactifications and if so, they are promising candidates for vacua.
without (closed string) moduli. Having fixed the moduli, one can address the other important questions: Are there chiral fermions and a proper gauge group supported on these vacua? On the type IIA side, a promising vacuum was presented in [13, 17] and one might ask whether this vacuum has a mirror dual on the IIB side. It is also interesting to ask: If we consider general fluxes, is there still a landscape of string vacua as discussed by Douglas [23] in traditional CY compactifications? The fluxes on the type IIA side single out a rather unique vacuum, can this also happen on the type IIB side?

In the discussion of fluxes, the Ansatz for the Killing spinor plays an important role and there is a direct link between specific solutions (as e.g. supergravity brane solutions) and the assumptions for the spinor. The type-A(ndy) vacuum, for example, is obtained if the Killing spinor is Majorana-Weyl and describes the NS-type vacuum of [24] and the prototype supergravity solutions are NS5-branes. The S-dual configuration is called type-C vacuum [25]. For the so-called type-B(cker) vacuum, one assumes the Killing spinor to be a direct product of a chiral internal and external spinor. This was originally introduced in [26] and adopted in the analysis of Grana and Polchinski [19, 21]. The prototype supergravity solutions in this class are 3- and 7-branes. The most general Ansatz is of course an interpolation, which we will consider later-on. We should also mention the type-D(all’Agata) vacuum related to SU(2) structures [27].

The flow type solutions have been discussed most intensively over the years and some aspects can be found, for example in [14, 16, 28, 25, 29, 30]. Interestingly, Frey [30] was able to solve the equations explicitly in this class, whereas other explicit solutions are restricted to a sub-class of fields or torsion classes. In this paper, we could solve all equations, imposed by supersymmetry and SU(3) structures, without making any assumptions. The solution preserves four supercharges, i.e. yields $\mathcal{N} = 1$ supersymmetry upon compactification to four dimensions, the internal space is complex and a cosmological constant cannot be generated as long as we keep SU(3) structures. The fields are expressed in terms of one holomorphic function $f = f(z)$, which can be chosen freely. If it is constant we recover the flow type solutions whereas zeros and poles of this function can be related to the type-B and type-A vacua. We solved only the supersymmetry equations and the solution contains, besides the holomorphic function, one un-fixed field which, in the spirit of [29], is the “master function” and has to be fixed by the equations of motion or Bianchi identities. This is nothing
specific to our solution, but a well-known feature of BPS vacua.

The paper is organized as follows. In the next Section, we setup our conventions and discuss especially the spinor Ansatz and the Killing spinor equations. In Section 3 we derive the constraints on the fluxes and differential equations from the supersymmetry variations. The main part is Section 4, where we first solve the differential equations explicitly, followed by a discussion of cases related to special assumptions for the holomorphic function $f(z)$. We discuss three cases in more detail: (i) RG-flow for $f = \text{const.}$, (ii) special Hermitian and Kähler manifolds and (iii) we expand the solution around zeros and poles of $f(z)$. In the last section we also comment on the cosmological constant. Recall, SU(3) structures forbid a cosmological constant, but as we will show, for SU(2) structures a cosmological constant can be generated. In allowing for SU(2) structures, one has however to keep in mind, that there may appear an issue in obtaining chiral fermions [similar to the situation on the type IIA side \cite{31]}.

2 Warp Compactification and Fluxes

2.1 Bosonic Fields

We use the standard convention of type IIB supergravity \cite{32} (see also \cite{33, 19}) and combine the bosonic fields into a complex 1-form $P$, a complex 3-form $G$ and the 5-form $F$

\[
\begin{align*}
P_M &= \frac{\partial M}{1-|B|^2} B = \frac{1+i\tau}{1-i\tau} B, \\
G_{(3)} &= \frac{F_{(3)} - BF_{(3)}^*}{\sqrt{1-|B|^2}}, \\
F_{(3)} &= d(B(2) + iC(2)), \\
F_{(5)} &= dA(4) - \frac{1}{8} \text{Im}\left[\left(B(2) + iC(2)\right) \wedge F_{(3)}^*\right]
\end{align*}
\]

where $\tau = \tau_1 + i\tau_2 = C + ie^{-\phi}$ combines the axion $C$ and the dilaton $\phi$. The equations of motions read

\[
\begin{align*}
\mathcal{D}^MP_M &= \frac{1}{24} G_{MNP} G^{MNP}, \\
\mathcal{D}^M G_{MNP} &= P^M G_{MNP} - \frac{2i}{3} F_{MNPQR} G^{MQR}, \\
R_{MN} &= P_M P_N + \bar{P}_M P_N + \frac{1}{6} F_{MLPQR} F_N^{LPQR} \\
&\quad+ \frac{1}{8} \left( G_M^{LP} \bar{G}_{NLP} + G_N^{LP} \bar{G}_{MLP} - \frac{1}{6} g_{MN} G_{LPQ} \bar{G}^{LPQ} \right)
\end{align*}
\]
where the covariant derivative $\mathcal{D}_M = D_M - iqQ_M$ with $D_M$ as the usual covariant derivative to the Levi-Civita connection and $q$ the local $U(1)$ charge ($G_{MNP}$ has charge $q = 1$, and $P_M$ has the charge $q = 2$) with respect to the $U(1)$ connection

$$Q_M = \frac{\text{Im}(B\partial_MB^*)}{1-|B|^2}.$$  

(2.5)

By employing differential forms, the equations of motion can also be written in a more compact way as

$$d^*G = iQ \wedge *G + P \wedge *\tilde{G} - 4iG \wedge *F,$$

$$d^*P = 2iQ \wedge *P + \frac{1}{3}G \wedge *G,$$

and the Bianchi identities for the forms are given by

$$dQ = -iP \wedge \tilde{P}, \quad dP = 2iQ \wedge P,$$

$$dF = \frac{5}{12}iG \wedge \tilde{G}, \quad dG = iQ \wedge G - P \wedge \tilde{G}. $$

(2.6)

(2.7)

Defining a phase $e^{2i\theta} = \frac{1+i\tau}{1-i\tau}$, one finds moreover

$$Q_M = \partial_M \theta - \frac{\partial_M \tau_1}{2\tau_2}, \quad P_M = i e^{2i\theta} \frac{\partial_M \tau}{2\tau_2}, \quad G_{(3)} = i \frac{e^{i\theta}}{\sqrt{\tau_2}} (dA_{(2)} - \tau dB_{(2)}).$$

(2.8)

The phase $\theta$ drops out from the equations of motion as well as from the Bianchi identities and can be set to zero yielding the string theory convention, but note, we are working in the Einstein frame and thus $G_3$ has the pre-factor $\tau_2^{-1/2} = e^{\phi/2}$. There is a local $U(1)$ symmetry related to this phase, which we will discuss below at eq. (2.12).

We are interested in vacua, which preserve four supercharges with a maximal symmetric external space, ie. either flat or anti de Sitter space. The 10-dimensional geometry is therefore a warped product of the 4-dimensional space-time $X_{1,3}$ and the 6-dimensional internal space $Y_6$ and we write the metric as

$$ds^2 = e^{-2V(y)} g_{\mu\nu} dx^\mu dx^\nu + e^{2U(y)} h_{mn}(y) dy^m dy^n$$

(2.9)

where $(g_{\mu\nu}, h_{mn})$ are the metrics on $(X_{1,3}, Y_6)$ and the warp factors depend only on the internal coordinates. To preserve 4-d Poincarè invariance, the 3-form field strength $G_{(3)}$ and the one-form $P$ have non-zero components only inside $Y_6$. On the other hand, the self-dual 5-form $F_{(5)}$ has to have also components along the external space,
but they are proportional to the four dimensional Poincaré invariant volume form, ie.\(^1\)
\[
F_{\mu\nu\lambda\rho m} = 5 e^{4V} \epsilon_{[\mu\nu\lambda\rho} \partial_{m]} Z(y) ,
\]
\[
F_{mnpqr} = -e^{4V} \epsilon_{mnpqrs} \partial^s Z(y) .
\]

### 2.2 The Supersymmetry Variations

The fermionic fields of Type IIB supergravity consists of a complex Weyl gravitino \(\Psi_M\) \((\Gamma^{11} \Psi_M = -\Psi_M)\) and a complex Weyl dilatino \(\lambda\) \((\Gamma^{11} \lambda = \lambda)\). A bosonic vacuum, with unbroken supersymmetry requires that the supersymmetry variations of the fermionic fields vanish yielding

\[
0 = \delta \lambda = \frac{1}{\kappa} \Gamma^M P_M \epsilon^* - \frac{i}{24} \Gamma^{MNP} G_{MNP} \epsilon ,
\]
\[
0 = \delta \Psi_M = \frac{1}{\kappa} \left( D_M - \frac{i}{2} Q_M \right) \epsilon + \frac{i}{480} \Gamma^{M_1...M_5} F_{M_1...M_5} \Gamma_M \epsilon + \frac{1}{96} \left( \Gamma_M^{\ PQR} G_{PQR} - 9 \Gamma^{PQ} G_{MPQ} \right) \epsilon^* .
\]

We combined here both Majorana-Weyl supercharges of the same chirality into one complex Weyl spinor \(\epsilon\). Any non-trivial solution of these eqs. corresponds to an unbroken supercharge and \(\epsilon\) is called a Killing spinor. These equations are invariant under the local U(1) gauge transformation

\[
\epsilon \to e^{ig} \epsilon , \quad \theta \to \theta + g
\]

where the phase \(\theta\) was introduced at equation (2.8) and we denoted the corresponding U(1) charge by \(q\). This local symmetry is due to the coset \(SL(2, R)/U(1)\) which is parameterized by the scalar fields of type IIB supergravity and it implies that the phase \(\theta\) can be chosen freely, one can take \(\theta = 0\) (string theory convention) or \(e^{2i\theta} = \frac{1+i\tau}{1-i\tau}\) (supergravity convention) or any other value.

As we have done it for the metric and the fluxes, we have also split the spinors into external and internal components, ie. we have to distinguish between internal and external spinors. Globally well-defined supersymmetry requires that Killing spinors are singlets under the structure group of the internal manifold and depending on the group this provides a classification of supersymmetric vacua. If the structure group is

\(^1\)Throughout this paper, we use \(\epsilon_{\mu_1\mu_2...\mu_p}\) for the antisymmetric tensor and \(\epsilon_{\mu_1\mu_2...\mu_p}\) for the corresponding tensor density. In other words, \(\epsilon_{\mu\nu\lambda\rho} = e^{-4V} \sqrt{-g} \epsilon_{\mu\nu\lambda\rho}\) and \(\epsilon_{mnpqrs} = e^{6U} \sqrt{h} \epsilon_{mnpqrs}\).
SU(3), there can exist only one internal spinor and for SU(2) we can have two singlet spinors. The maximal number would be four corresponding to a trivial structure group. In the following, we are only interested in SU(3) structures and hence we consider for the 10-dimensional spinor the general Ansatz

\[ \epsilon = a \zeta_1 \otimes \chi + b^* \zeta_2 \otimes \chi^* \]  \hspace{1cm} (2.13)

where \( \chi \) is the internal Weyl spinor and \( \zeta_{(1/2)} \) are two 4-dimensional spinors of opposite chirality (so that \( \epsilon \) is chiral). We are primarily be interested in vacua preserving only four supercharges yielding \( \mathcal{N} = 1, D = 4 \) supersymmetry upon compactification and it is worth pointing out that the internal SU(3)-structure alone does not guarantee that the vacuum has only \( \mathcal{N} = 1 \) in four dimensions. In fact, one generically gets \( \mathcal{N} = 2 \) in four dimensions with the two spinors \( \zeta_{(1/2)} \). Only if these two spinors are not independent and obey a projector, the supersymmetry is reduced to \( \mathcal{N} = 1 \) or four supercharges. Due to the chirality of these spinors, we can only impose

\[ \zeta_2 = \zeta_1^* = \zeta . \]  \hspace{1cm} (2.14)

Of course, any non-trivial factor appearing in this relation can be absorbed into the two complex functions \( a \) and \( b \) which are introduced for the most general SU(3) invariant Killing spinor. There are some special cases for the spinor Ansatz, which played an important role in the literature and which we mentioned already shortly in the introduction. The type-A vacuum is given by \( a = b^* \) so that \( \epsilon \) becomes a Majorana-Weyl spinor. This Ansatz yields especially NS-type vacua and the S-dual configuration is described by the type-C vacuum. On the other hand, the type-B vacuum corresponds to \( b = 0 \) (or \( a = 0 \)) and the importance of this Ansatz is due to the fact, that it yields always an internal space that is Kähler, or for constant axion/dilaton Calabi-Yau\(^2\).

To be concrete, we will use the following chirality conventions

\[ \Gamma^u \epsilon = -\epsilon \] , \hspace{1cm} \( \gamma^5 \zeta = \zeta \) , \hspace{1cm} \( \gamma^7 \chi = -\chi \)  \hspace{1cm} (2.15)

with \( \chi \) normalized to \( \chi^\dagger \chi = 1 \). Actually, without loss of generality, we can take \( \chi = e^{-\beta} \chi_0 \) with \( \chi_0 \) as constant spinor and the phase \( \beta \) will drop out in most of our calculations (any coordinate dependence can be absorbed in \( a \) and \( b \)). We now

\(^2\)We refer to Kähler spaces as spaces with U(3) holonomy, in contrast to Calabi-Yau spaces that have SU(3) holonomy.
decompose the $\Gamma$-matrices as usual

\[
\hat{\Gamma}^\mu = \hat{\gamma}^\mu \otimes 1 \quad , \quad \hat{\Gamma}^\dot{m} = \gamma^5 \otimes \hat{\gamma}^\dot{m} \quad , \quad \Gamma^{11} = \gamma^5 \otimes \gamma^7 ,
\]

where hatted $\hat{\Gamma}^\mu$ and $\hat{\gamma}^\mu$ have flat tangent space indices and $\hat{\epsilon}$ denotes tensors with respect to the un-warped metric. We use the Majorana representation so that $\hat{\Gamma}^\mu$, $\hat{\Gamma}^{\dot{m}}$, $\Gamma^{11}$, $\gamma^5$, $\gamma^7$, $\hat{\gamma}^\mu$ are real and $\gamma^5$, $\gamma^7$, $\hat{\gamma}^{\dot{m}}$ are imaginary and antisymmetric.

The covariant derivative $D_M$ in the gravitino variation refers to the ten dimensional warped metric $G_{MN} = (e^{-2V} g_{\mu\nu}, e^{2U} h_{mn})$, which is related to the covariant derivatives $(\nabla_\mu, \nabla_m)$ with respect to $(g_{\mu\nu}, h_{mn})$ by

\[
D_\mu = (\nabla_\mu \otimes 1) - \frac{1}{2} e^{-V-U} (\gamma_\mu \gamma^5 \otimes \partial V) , \\
D_m = 1 \otimes (\nabla_m + \frac{1}{2} \gamma_m \partial U) .
\]

Since we do not assume, that the external space is flat, also the 4-dimensional spinor is not necessarily covariantly constant. Instead we use

\[
\nabla_\mu \zeta = \gamma_\mu \bar{W} \zeta^* \tag{2.18}
\]

and $W = W_1 - i W_2$ is related to the 4-dimensional superpotential or more exactly, its value in the vacuum. The integrability constraint for this spinor equations requires that the vacuum is anti deSitter space. Actually, from 4-dimensional supergravity we infer that our $W$ is related to the holomorphic superpotential by $W \sim e^{K/2} W_{hol}$, where $K$ is the Kähler potential.

Now, inserting our spinor Ansatz into the variations \[(2.11)\] yields

\[
\delta \lambda = i e^{-U} \zeta \otimes (bP - \frac{b^*}{24} e^{-2U} G) \chi \\
- i e^{-U} \zeta^* \otimes (a^* P - \frac{b^*}{24} e^{-2U} G) \chi^* \tag{2.19}
\]

where $P = \gamma^m P_m$ and $G = \gamma^{mnp} G_{mnp}$. In evaluating the supersymmetry variation of the gravitino, we need the field strength $F_{(3)}$ contracted with $\Gamma$-matrices and find

\[
\frac{i}{480} \Gamma^{M_1 \ldots M_5} F_{M_1 \ldots M_5} = \frac{1}{2} e^{-U/4} (1 \otimes \gamma^m \partial_m Z - \gamma^5 \otimes \gamma^7 \gamma^m \partial_m Z) \\
= \frac{1}{2} e^{-U/4} (1 \otimes \partial Z) \tag{2.20}
\]
where \( \partial = \gamma^m \partial_m \). Then, the external supersymmetry variation of the gravitino becomes

\[
\delta \Psi_\mu = \gamma_\mu \zeta \otimes \left[ W b^* \chi^* - \frac{1}{2} e^{-V-U} (a \partial V - a e^4 \partial Z - \frac{b}{38} e^{-2U} G) \chi \right] +
\gamma_\mu \zeta^* \otimes \left[ W a \chi + \frac{1}{2} e^{-V-U} (b^* \partial V + b e^4 \partial Z - a^* e^{-2U} G) \chi^* \right].
\]

(2.21)

Finally, for the internal gravitino variation we get

\[
\delta \Psi_m = \zeta \otimes \left[ \nabla_m + \frac{1}{2} (\gamma^m \partial_n U - i Q_m + e^4 \partial Z \gamma_m) + \frac{b}{96a} e^{-2U} (\gamma_m G - 12 G_m) \right] a \chi +
\zeta^* \otimes \left[ \nabla_m + \frac{1}{2} (\gamma^m \partial_n U - i Q_m - e^4 \partial Z \gamma_m) + \frac{a^*}{96b} e^{-2U} (\gamma_m G - 12 G_m) \right] b^* \chi^*.
\]

where \( G_m = \gamma^{np} G_{mnp} \). Setting these variations to zero gives us constraints on the background field strengths and in addition differential equations for the internal Killing spinor, which in turn require constraints on the geometry of the internal space expressed by specific non-vanishing torsion components.

### 3 Constraints on \( \mathcal{N} = 1 \) Fluxes and Intrinsic Torsion

We will start with the flux constraints, that come from the dilatino (2.19) as well as from the external gravitino variation (2.21).

#### 3.1 Constraints of Fluxes

Using the above relations and the one that we collected in the appendix, one gets from \( \delta \lambda = 0 \) (recall, \( \zeta \) and \( \zeta^* \) have opposite chirality)

\[
\Omega_{mnp} G^{mnp} = \bar{\Omega}_{mnp} G^{mnp} = 0, \tag{3.1}
\]

\[
(P_r - i J_{rs} P^s) + i \frac{a}{8b} e^{-2U} (J_{[mn} h_{p]r} + i J_{[mn} J_{p]r}) G^{mnp} = 0, \tag{3.1}
\]

\[
(P_r + i J_{rs} P^s) - i \frac{b}{8a} e^{-2U} (J_{[mn} h_{p]r} - i J_{[mn} J_{p]r}) G^{mnp} = 0.
\]

The constraints in the first line imply that \( G \wedge \Omega = G \wedge \bar{\Omega} = 0 \) and hence \( G_{(3,0)} = G_{(0,3)} = 0 \). By using holomorphic indices these constraints can be written as

\[
G_{ijk} = G_{\bar{i}\bar{j}k} = 0, \tag{3.2}
\]

\[
P_i = \frac{b^*}{4a^*} e^{-2U} G_{ij}^j, \tag{3.2}
\]

\[
P_i = \frac{a}{4b} e^{-2U} G_{ij}^j.
\]
With these constraints $\delta \Psi_\mu = 0$ yields
\[
a W = b^* W = 0 ,
\]
\[
a^* 2 P_i = 2 b^* (\partial_i V + e^{4V} \partial_i Z) ,
\]
\[
b^2 P_i = 2 a^2 (\partial_i V - e^{4V} \partial_i Z) .
\]

This has the important consequence that
\[
W = 0 \quad (3.4)
\]
and hence the 4 dimensional cosmological constant is identically zero and the external space time is always flat Minkowskian. Note, it does not mean that there is no superpotential, it only implies that the supersymmetric vacuum is given by: $dW = W = 0$. Below we will see, that in order to generate a 4-dimensional cosmological constant one has to consider SU(2) structures.

### 3.2 Killing Spinor and Intrinsic Torsion

Finally, the internal variation $\delta \Psi_m = 0$ gives differential equations for the Killing spinor fixing the torsion components of the internal manifold. Using the same relations as before, eq. (2.22) simplifies to

\[
\hat{\nabla}_m (a \chi) = -\frac{i}{2} \gamma_{mn} (\partial^n U + \partial^n V - 2 e^{4V} \partial^n Z) a \chi + \frac{b}{8a} e^{-2U} G_m a \chi ,
\]
\[
\hat{\nabla}_m (b^* \chi^*) = -\frac{i}{2} \gamma_{mn} (\partial^n U + \partial^n V + 2 e^{4V} \partial^n Z) b^* \chi^* + \frac{a^*}{8b^*} e^{-2U} G_m b^* \chi^*
\]
with $\hat{\nabla}_m = \nabla_m - \frac{i}{2} Q_m + \frac{1}{2} \partial_m V$. These two equations are consistent if

\[
\partial_m \log \left( \frac{b}{a} \right) = i Q_m - 2 i e^{4V} J_m \partial_n Z + \frac{i a^*}{8b^*} e^{-2U} J_{np} G_m np - \frac{i b^*}{8a^*} e^{-2U} J_{np} \bar{G}_m np . \quad (3.6)
\]

And notice that $\nabla_p (\chi^T \chi^*) = 0$ requires

\[
\partial_m \log |a| = -\frac{1}{2} \partial_m V - \frac{ib}{16a} e^{-2U} J_{np} G_m np + \frac{i b^*}{16a^*} e^{-2U} J_{np} \bar{G}_m np . \quad (3.7)
\]

The 3-form flux can be replaced by using the relations (3.2) and (3.3) and hence we find in complex coordinates

\[
\partial_i \log \left( \frac{b}{a} \right) = i Q_i + 2 e^{4V} \partial_i Z ,
\]
\[
\partial_i \log \left( \frac{b}{a} \right)^* = i Q_i + 2 e^{4V} \partial_i Z + 2(\frac{b}{a}^2 - |\frac{b}{a}|^2) \partial_i V + 2(\frac{b}{a}^2 + |\frac{a}{b}|^2) e^{4V} \partial_i Z , \quad (3.8)
\]
\[
\partial_i \log |a| + \frac{1}{2} \partial_i V = -\frac{1}{2} \frac{b}{a}^2 (\partial_i V + e^{4V} \partial_i Z) + (\partial_i V - e^{4V} \partial_i Z) .
\]
Combining the first two equations, we find
\[\partial_i \log \left| b^a \right|^2 = 2 \left| b^a \right|^2 + 2 \left| a^b \right|^2 + 2 \partial_i V, \]
\[\partial_i \log \left( \frac{a^b}{ab^a} \right) = -2iQ_i + 2 \left| b^a \right|^2 - 2 \left| a^b \right|^2 \partial_i V. \]

The first equation can be re-written as
\[\partial_i \log \left( \left| b^a \right|^2 + 1 \right) = 2 \left| b^a \right|^2 + 1 \partial_i V + 2 \left( \left| b^a \right|^2 - \left| a^b \right|^2 \right) \partial_i V, \]
and with the third equation in (3.8), we get
\[\partial_i \log (a^0 + b^0) = -\frac{1}{2} \partial_i \log (|b|^2 + 1) \Rightarrow |a|^2 + |b|^2 = e^{-V}. \]

The integration constant for the warp factor \(V\) can be absorbed into a coordinate rescaling in the metric Ansatz. Thus our spinor becomes equivalent to \(30\)
\[\epsilon = e^{-\frac{V+i\omega}{2}} \left( \sin \alpha \left[ \zeta \otimes \chi \right] + \cos \alpha \left[ \zeta^* \otimes \chi^* \right] \right) \]
or
\[a = e^{-\frac{V+i\omega}{2}} \sin \alpha , \quad b = e^{-\frac{V-i\omega}{2}} \cos \alpha . \]

Note, we absorbed the common phase of \(a\) and \(b\) into the spinor \((\chi = e^{i\beta} \chi_0)\), because this phase drops out in most of our calculations. In addition, using the U(1) symmetry \(2.12\), one can always set \(\omega = 0\) (or fix the phase \(\theta\)). We will explore these equations further in the next section.

Since the internal spinor is covariantly constant only for specific fluxes, we have to compensate the fluxes by intrinsic torsion components of the internal manifold. In other words, the fluxes deform the internal geometry in a way dictated by the non-vanishing torsion components. A real 6-dimensional space with \(SU(3)\) structures \(J\) and \(\Omega\) is classified by five torsion classes \(\mathcal{W}_{1,2,3,4,5}\), which are defined by \(31\)
\[dJ = \frac{3i}{4} (\mathcal{W}_1 \bar{\Omega} - \bar{\mathcal{W}}_1 \Omega) + \mathcal{W}_3 + J \wedge \mathcal{W}_4, \]
\[d\Omega = \mathcal{W}_1 J \wedge J + J \wedge \mathcal{W}_2 + \Omega \wedge \mathcal{W}_5 \]
with
\[\mathcal{W}_1 \leftrightarrow (dJ)^{(3,0)} , \quad \mathcal{W}_2 \leftrightarrow (d\Omega)^{(2,2)} , \quad \mathcal{W}_3 \leftrightarrow (dJ)^{(2,1)} , \quad \mathcal{W}_4 = \frac{1}{2} J \wedge dJ \leftrightarrow (dJ)^{(1,0)} , \quad \mathcal{W}_5 = \frac{1}{2} (Re \Omega) \wedge d(Re \Omega) \leftrightarrow (d\Omega)^{(3,1)} . \]
The subscript 0 denotes the irreducible $SU(3)$ representation with the trace part removed and the operation $\cdot$ is defined as:

$$(\alpha \cdot \beta) = \frac{1}{p!} (\alpha \cdot \beta)$$

[here we take $\alpha$ to be the lower form of rank $p$]. Due to the differential equations (3.5) $J$ and $\Omega$ are not closed and we find

$$\partial_{[m} J_{np]} = J_{[mn} M_{p]} + \frac{ib}{4a} e^{-2U} \left( J_{[m} \overset{r}{J} G_{p]}^{rs} - G_{mpn} - 2i J_{[m} \overset{l}{G}_{l]}_{np]} \right) + c.c. ,$$

$$\partial_{[m} \Omega_{npq]} = \Omega_{[mpn} N_{p]} ,$$

with the two vectors:

$$M = -d \log |a| - \frac{5}{2} dV - 2dU + 4e^{4V} dZ - \frac{ib}{8a} e^{-2U} G \cdot J$$

$$N = 2d[\log |a| + 2V + \frac{i}{2}(\omega + \beta)] - iQ + 3(dU - 2e^{4V} dZ) + \frac{ib}{4a} e^{-2U} G \cdot J$$

And therefore the torsion classes read

$$W_1 = 0 ,$$

$$W_2 = 0 ,$$

$$W_3_{mn} = \frac{ib}{4a} e^{-2U} \left( G_{mpn} - J_{[m} \overset{r}{J} G_{p]}^{rs} - \frac{i}{2} J_{rs}^{l} J_{lmn} G_{p]}^{rs} \right) + c.c. ,$$

$$W_4_{p} = M_{p} - \frac{b}{8a} e^{-2U} J_{p}^{l} G_{lmn} J^{mn} + c.c. ,$$

$$W_5_{p} = \frac{1}{2} \left[ \delta_{p}^{l} - i J_{p}^{l} \right] N_{l} + cc .$$

Because $W_1 = W_2 = 0$, the internal space has to be a complex manifold, i.e. the almost complex structure $J_{mn}$ is integrable. This result has also been found in [14], but since they assumed a flat Minkowski vacuum, it was not clear if a non-zero cosmological constant could have been compensated by $W_1$ for example, which is exactly the case on the type IIA side [13, 17]. For generic background fluxes, the intrinsic torsion lies therefore in $(W_3 \oplus W_4 \oplus W_5)$, with $W_3$ purely supported by the (2,1) primitive$^4$ components of $G_{(3)}$ and $W_{4/5}$ are related to a non-trivial axion/dilaton and warping. One can easily check that our expressions reduces to the well known results in the corresponding special limits. For example, when $ab = 0$, $W_3$ and $W_4$ vanish if we let $U = V$, and $W_5 = -i(Q + d\beta)$ characterizes a Kähler space. If $b = \pm a,$

$^3$Also notice that $(d\alpha)_{m_1...m_{p+1}} = (p+1)\partial_{[m_1} \alpha_{m_2...m_{p+1}]}.$

$^4$Primitive means $J \cdot G_{(3)} = 0.$

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on the other hand, $2W_4 + W_5 = 0$ if we let $U + V = 0$. Let us also remark, that the combination $3W_4 + 2W_5$ is independent of $U$ as it should be, because $U$ is only a conformal rescaling.

In complex coordinates the expressions simplify to

\[
W_{\bar{3}ijk} = \frac{i\theta}{\theta} e^{-2U} (G_{\bar{ijkl}} - \frac{3}{2} h_{|jkl|}^l) ,
\]

\[
W_4^i = \partial_i \log(\cos \alpha) - 2(\partial_i U + \partial_i V - 2 e^{4V} \partial_i Z) ,
\]

\[
W_5^i = -i (Q_i + \partial_i \omega + \partial_i \beta) + \partial_i \log \cos^2 \alpha + 3 \partial_i U + 7 \partial_i V - 10 e^{4V} \partial_i Z
\]

where we used eqs. (3.2), (3.3), (3.10) and (3.12).

Observe that all constraints required by supersymmetry do not involve the internal warp factor $U$ and we may thus simplify our torsion classes by imposing relations between $U$ and the other functions. Remember however a change in $U$ does change the solution by a conformal rescaling of the internal metric. In what follows, by internal geometry we always mean the non-warped metric, so a Calabi-Yau space really means the internal space equipped with the warped metric is conformal to a Calabi-Yau and so on.

## 4 Solutions of SUSY Constraints

To simplify the calculations we use the symmetry (2.12) to set in the following

\[
\omega = 0
\]

(equivalently, one could also fix the phase $\theta$). We will start by solving the supersymmetry constraint equations followed by a discussion of special cases/limits.

### 4.1 Solving the supersymmetry equations

Using the parameterization from (3.13) we find from the first equation in (3.9)

\[
4 \partial_i Z = \partial_i \left( e^{-4V} \cos 2\alpha \right)
\]

and hence

\[
Z - Z_0 = \frac{1}{4} e^{-4V} \cos 2\alpha .
\]
To keep the notation simple, we set in the following: $Z_0 = 0$. The second equation in (3.9) can be written as

\[ i Q_i = 2 \cos 2\alpha \partial_i V + \left( \frac{2}{\sin 2\alpha} - \sin 2\alpha \right) \partial_i \alpha \]

\[ = -\cos 2\alpha \partial_i \log \sqrt{Z} + \partial_i \log \cot \alpha . \quad (4.2) \]

Next, the equations in (3.3) become

\[ P_i = 4 \cos^2 \alpha \partial_i V + 2 \frac{\cos^3 \alpha}{\sin \alpha} \partial_i \alpha , \]

\[ P_i = 4 \sin^2 \alpha \partial_i V - 2 \frac{\sin^3 \alpha}{\cos \alpha} \partial_i \alpha \]

and therefore

\[ (P + P^\star)_i = -\partial_i \log |\tilde{Z}| , \quad (P - P^\star)_i = -\cos 2\alpha \partial_i \log |\tilde{Z}| + \partial_i \cos 2\alpha = 2i Q_i \]

where we used the expression for $Q_i$ as found in (4.2) and introduced

\[ \tilde{Z} = Z \frac{\sin^2 2\alpha}{\cos 2\alpha} . \quad (4.4) \]

The vector $P$ can now be written as

\[ P = -\frac{1}{2} d \log |\tilde{Z}| + i Q \]

and from the Bianchi identities in (2.7) we get

\[ dQ \equiv -i P \wedge \tilde{P} = -Q \wedge d \log |\tilde{Z}| \quad (4.6) \]

which can formally be solved by: $Q = \tilde{Z}^{-1} Q_0$ where $Q_0$ can be any closed 1-form. With this expression for $dQ$, one can also verify that the Bianchi identity for $P$, $dP = 2iQ \wedge P$, is identical fulfilled for $P$ as given in (4.5).

On the other hand, using the expression for the vector $P$ from eq. (2.8), we find

\[ P = \frac{1}{2} \left( -e^\phi \sin 2\theta d\tau_1 + \cos 2\theta d\phi \right) + \frac{i}{2} \left( e^\phi \cos 2\theta d\tau_1 + \sin 2\theta d\phi \right) . \]

In comparing these two expressions for $P$ we get two equations. Taking $Q$ from eq. (2.8), we find from equating the imaginary parts

\[ 0 = e^\phi \cos \theta \partial_i \tau_1 + \sin \theta \partial_i \phi - \frac{\partial_i \theta}{\cos \theta} . \]
whose integral is
\[ \tau_1 = c_0 + \sin \theta \cos \phi e^{-\phi}. \] (4.7)

Then, from equating the real parts we find
\[ e^{-(\phi - \phi_0)} = \tilde{Z} \cos^2 \theta. \] (4.8)

We have now everything expressed in terms of \( \theta, \alpha \) and \( Z \) and these functions have to solve the eqs.

\[ 2i Q_i \equiv 2i \left( \partial_i \theta - \frac{\partial_i \tau_1}{2\tau_2} \right) = -\cos 2\alpha \partial_i \log |\tilde{Z}| + \partial_i \cos 2\alpha. \]

Inserting our result for \( \tau \) we find
\[ 2i Q_i = i \partial_i \tan \theta - i \tan \theta \partial_i \log |\tilde{Z}| \]
and thus
\[ \partial_i \left( \frac{\tilde{Z}}{\cos 2\alpha - i \tan \theta} \right) = \partial_i f^* = 0. \] (4.9)

By complex conjugation, we infer that the complex function
\[ f = \frac{\tilde{Z}}{\cos 2\alpha + i \tan \theta} \]
has to be holomorphic in the three complex coordinates parameterizing the internal space. Separating the complex function \( f \) into real and imaginary part, this equation can then also be written as
\[ 0 = \Re f \tan \theta + \Im f \cos 2\alpha, \]
\[ \tilde{Z} = \frac{|f|^2}{\Im f} \tan \theta = -\frac{|f|^2}{\Re f} \cos 2\alpha, \] (4.10)

and the final solution reads
\[ \tau = c_0 + i e^{-\phi_0} \frac{|f|^2 \cos 2\alpha}{f \sin^2 \alpha + f^* \cos^2 \alpha}, \]
\[ e^{4W} = \frac{\Re f}{4|f|^2} \frac{\sin^2 2\alpha}{\cos 2\alpha}, \]
\[ Z = \frac{|f|^2}{\Re f} \frac{\cos^2 2\alpha}{\sin^2 2\alpha}. \] (4.11)

Recall, we dropped the constant \( Z_0 \), which can be re-introduced by \( Z \to Z - Z_0 \) and similarly, by replacing \( \theta \to \theta + \omega \) we can again write it in a \( U(1) \)-gauge invariant form [with respect to local symmetry (2.12)].
So, we were able to integrate the first order differential equations required by
supersymmetry and the solution is parameterized by one holomorphic function \( f \).
But we have to keep in mind, that in addition, also the Bianchi identities and the
equations of motion for the form fields have to be satisfied in order to have really
a BPS solution. Note, in total we have to fix five function, the four bosonic fields
on the lhs of (4.11) plus \( \alpha \) which fixes the spinor (\( \omega \) can be gauged away). The eqs.
(4.11) give four algebraic relations and hence one function remains free (the “master
function” as discussed in [29]), which has to be fixed by the equations of motion – as
it is very typical for supersymmetric constraints.

4.2 The \( SU(3) \) Flow

The simplest case is to assume that all functions depend only on one real coordinate,
say \( r \), which is the case for a flow-type solution which have been widely discussed
already in the literature [35, 29, 30, 16]. Since there are no holomorphic functions
depending only on one real coordinate we have to set

\[
f = \text{constant}
\]

providing two real integration constants. It is straightforward to verify that in this
case our solution as given in (4.11) is equivalent to the results obtained in [30]. There
are two special cases: (i) \( \text{Im} f = 0 \) or (ii) \( \text{Re} f = 0 \), where some fields become trivial:
for case (i) it is the axion whereas for (ii) the 5-form flux vanishes, ie.

\[
\text{Im} f = 0 : \quad \tau_1 = c_0 ,
\]

\[
\text{Re} f = 0 : \quad Z = 0 .
\]

Note, for this choice of \( f \) one cannot add, in addition to the fluxes, 7-branes or 3-
branes and in the following we will assume that \( f \) is a generic complex constant. Since
all fields depend only on one real coordinate, we find trivially

\[
dQ = -iP \wedge \bar{P} = 0 , \quad Q \wedge P = 0
\]

and using the \( U(1) \) symmetry (2.12) we can also set \( Q \equiv 0 \). In addition, this case
requires \( dZ \wedge dV = 0 \) and hence we can always impose \( \mathcal{W}_4 = 0 \) by choosing an
appropriate conformal factor of the internal metric, ie. to fix the function \( U \).

Recall, the (algebraic) relations in (4.11) fix four of the five independent function
and the remaining one has to be fixed by imposing the Bianchi identities and/or
equations of motion. For this it is suggestive to take the angle $\alpha$, but we can also take the 5-form flux function $Z$ as the “master function”, because this function can be fixed by the 5-form Bianchi identity (the equations for the 3-form are trivially solved for this case)

$$dF = \frac{5i}{12} G \wedge \bar{G}.$$  \hspace{1cm} (4.12)

The conformal factor of the internal metric can now be chosen in a way that the lhs of this equation becomes a (curved space) Laplacian on $Z$. This is the case for $U = V$ and we can now distinguish the two possibilities whether the rhs is zero or not.

If $G \wedge \bar{G} = 0$, the function $Z$ has to be harmonic on the 6-d internal space. If $Z$ is constant also $\alpha$ is constant and the solution becomes trivial, ie. the internal space is Calabi-Yau. If it is not constant, the real harmonic function $Z$ has to have a singularity. Since $e^{-4V} \to \infty$ at this point, the warp factor blows up and from the AdS/CFT perspective this point corresponds to the UV regime. On the other hand, if $Z$ vanishes (or $Z = Z_0$ if we re-introduce the integration constant), also the warp factor goes to zero and we reach the IR regime. Depending on the concrete internal space, these can be regular fixed points corresponding to an $AdS_5 \times X_5$ geometry, with $X_5$ as a 5-dimensional Einstein space (or products thereof). Thus, the UV-regime corresponds $\alpha \simeq 0$ and $e^{-4V} \simeq 4Z \simeq \frac{|f|^2}{\text{Re} f \alpha^2}$. Going back to the spinor equation (3.5), in this regime the spinor $\chi$ becomes covariantly constant and the internal space becomes Calabi-Yau. Since $U = V$, the point $\alpha \simeq 0$ can only be regular, if the Calabi-Yau has a conical singularity with $e^{2U} \simeq \alpha r^2 = \text{constant}$., where $r$ is the radial coordinate of the cone. This yields a regular UV fixed point and $X_5$ has to be an Einstein-Sasaki space as required by supersymmetry; for an explicit example see [36]. A prototype example is the $AdS_5 \times S^5$ geometry appearing in the near horizon limit of D3-branes and we reach this geometry in our setup in the limit

$$f \to \lambda^2 f , \quad \alpha \to \lambda \alpha , \quad e^{\phi_0} \to \lambda^2 e^{\phi_0} \quad \text{and} \quad \lambda \to 0 .$$  \hspace{1cm} (4.13)

As consequence, only one chirality of the internal spinor contributes to the 10-d Killing spinor ($a = 0$ and $b = 1$) and $\tau = c_0 + i e^{-\phi_0} f$ (and $\theta$ becomes the phase of $f$), which is the type-B vacuum discussed in [19, 20]. Note, the 3-form flux does not need to vanish, the primitive components of (1,2)-type can still be non-zero fixing the complex structure moduli of the Calabi-Yau space. Therefore, in the UV regime, one can always approximate the internal geometry by a Calabi-Yau space. In the IR regime on the other hand, the situation is more involved. It is Calabi-Yau only if
we take the limit (4.13). If we do not re-scale $f$, it corresponds to $\cos 2\alpha \simeq 0$ and hence $\sin^2 \alpha \simeq \cos^2 \alpha (a = \pm b^*)$ and the spinor does not become covariantly constant. From (4.11) follows, that we are in a strongly coupled regime ($e^\phi \to \infty$) and far off a Calabi-Yau approximation. Actually, only for the Calabi-Yau case, we have control over the IR regime\(^5\). See also section 4.4 for a related discussion.

Let us also comment on the case if: $G \wedge \bar{G} \neq 0$. This is only possible if the 3-form $G$ has primitive components, because all non-primitive components of $G$, given by $G \wedge J$, are along $dr$ and hence do not contribute to $G \wedge \bar{G}$. In the case of $\mathcal{W}_3 \neq 0$ the space cannot be Calabi-Yau. If one can find, for a given space, an appropriate 3-form which satisfies the Bianchi identities and the equations of motion, the function $Z$ can have a regular maximum (ie. no poles anymore), which also represents the endpoint of the flow ($dZ = dV = d\tau = 0$) and is reminiscent to the transgression mechanism discussed in [37]. Since the warp factor is finite at this point, this flow terminates on a 5-dimensional flat space time (instead of an $AdS_5$ fixed point discussed before) and using results from AdS/CFT, we expect a confining gauge theory dual at the endpoint of this flow (because Wilson loop calculation show the area law behavior in this case).

4.3 Special Hermitian or Kähler spaces

Next, we will allow for a general holomorphic function $f$ and comment on the examples of special Hermitian manifolds. Special Hermitian manifolds are complex half-flat manifolds\(^6\) and type IIB compactified on half-flat manifolds arises as mirror symmetric partner of type IIA on Calabi-Yau manifolds with NS-NS $H(3)$ fluxes [5]. Both compactifications usually retain $N = 2$ supersymmetries in four dimensions.

Observe the condition $\mathcal{W}_4 = 0$ implies that $dV \wedge dZ = 0$ and we can solve the equation for the function $U$. Then, we find in the gauge $\omega = 0$

$$0 = \mathcal{W}_5 = iQ - \frac{7}{2} \partial \log (\sin \alpha) - 4 \partial V + 4 e^{4V} \partial Z \quad (4.14)$$

where we have taken here only the holomorphic component. Inserting for $Q$ the expression as given in eq. (4.2) and using the solution for $Z$ in (4.1), we get a differential

\(^5\)Although, in the spirit of F-theory, this may also be an indication of a de-compactification and the higher-dimensional theory does not need to be singular.

\(^6\)These are 6-dimensional manifolds with $SU(3)$-structure and intrinsic torsion satisfying $\mathcal{W}_4 = \mathcal{W}_5 = 0$ and $d(\text{Im}\Omega) = 0$. They can be lifted to seven dimensional manifolds with $G_2$-holonomy by Hitchin’s construction, where only the special Hermitian manifolds are also complex.
equation for \( V = V(\alpha) \). The solution \( V = V(\alpha) \) has to be compared with the one given before in eq. (4.11) and hence yields a relation between \( \alpha \) and the holomorphic function \( f \).

If we do not impose eq. (4.14), but instead require the vanishing of \( \mathcal{W}_3 \), the internal space is Kähler with \( \mathcal{W}_5 \) as the Kähler connection. The simplest possibility has been discussed by Grana and Polchinski [21], where the 10-dimensional spinor factorizes and only one chirality of the internal spinor contributes \( (a = 0) \). This exactly corresponds to the limit that we discuss at eq. (4.13) with the axion/dilaton

\[
\tau = c_0 + i e^{-\phi_0} f
\]

i.e. it is a holomorphic function of the internal coordinates (recall \( f = f(z^i) \)) and the 3-form can only have primitive components of \( (1,2) \) type. The warp factor and the 5-form flux become in this limit

\[
e^{-4V} = 4Z = \frac{|f|^2}{\text{Re} \alpha^2}
\]

which ensures that \( \mathcal{W}_4 = 0 \) if we take \( U = V \) (recall \( \sin \alpha \to 0 \) in this limit) and the internal space becomes conformal to a Kähler space with the Kähler connection (in the gauge \( \omega = 0 \))

\[
\mathcal{W}_5 = -i (Q + d\beta).
\]

Since the Kähler form factorizes \( (dQ = -iP \wedge \bar{P}) \), these can only be very specific Kähler spaces and one may wonder, whether more general spaces with non-holomorphic dilaton are possible. We do not want to discuss this question here in detail, let us only mention that, imposing \( \mathcal{W}_4 = 0 \) implies that the rhs of (4.14) can be written as

\[
\mathcal{W}_5 = i \partial K
\]

with \( K \) as the Kähler potential and \( \mathcal{W}_5 \) the Kähler connection (recall \( dV \wedge dZ = 0 \) in order to have \( \mathcal{W}_4 = 0 \)).

4.4 Special limits of the general solution

Most explicitly known solutions have an internal space, where all possible torsion classes are non-trivial and the mathematical classification of spaces with respect to non-vanishing torsion classes, has no obvious physical interpretation. In physics literature the solutions are typically classified with respect to the spinor Ansatz, e.g.
whether the 10-dimensional Killing spinor is only Weyl (e.g. \( \sin \alpha = 0 \)) or if the spinor is Majorana-Weyl (e.g. \( \sin \alpha = \cos \alpha \)). Different spinor Ansätze correspond to different solutions, as e.g. different brane solutions in supergravity.

The general solution, which we found, is parameterized by a holomorphic function \( f \) and we discussed already in detail the case of constant \( f \), which is the only regular case if the internal space is compact. Other solutions can be classified with respect to the singularities and zeros of \( f \). If we write

\[
f(z^i) = \frac{p(z^i)}{q(z^i)}
\]

where \( p \) and \( q \) are some polynomials in the complex coordinates \( z^i \) of different degree. To get a regular solution around a zero of \( p \) or \( q \), the other fields have to behave appropriately.

Let us first discuss the solution around zeros of \( f \). Denoting a zero of the polynomial \( p(z^i) \) by \( z_p \), a (regular) expansion around \( z = z_p \) can be identified with the limit \((4.13)\) if we take\(^7\): \( \lambda^2 = |z - z_p| \). This (vanishing) parameter drops out and the fields are given in eqs. \((4.15)\) and \((4.16)\). Recall, the solution becomes the B-type vacuum and the supergravity solution in this class describes intersecting 7-branes in the background of a 5-form flux and a primitive 3-form flux of type \((1, 2)\). The resulting geometry is a (special) Kähler space. If the fluxes vanish, i.e. especially for \( Z = \text{const.} \) yielding \( \alpha^2 \sim |f|^2/Re f \), the only non-trivial field is the axion/dilaton and one can follow the discussion in \([38, 39]\), where the singularities of the holomorphic function \( f \) are related to 7-branes, that generate a deficit angle and if one has 24 branes, the space becomes smooth and compact. By considering \( D_4 \) singularities, one can also setup an orientifold with \( D7 \)-branes and coincident \( O7 \)-planes; see \([40]\). Note, if there is no 5-form flux, there is also no warping (\( V \) and \( U \) become constant), which one should expect for a 7-brane background in the Einstein frame.

In the same way we can also explore the solution around zeros of the polynomial \( q(z^i) \) giving poles for the holomorphic function \( f \). As before, we are asking for a regular expansion around this point, but this time we consider

\[
\begin{align*}
f &\to \lambda^{-2} f, \quad \alpha \to \left( \frac{\pi}{4} - \lambda^2 \right) \alpha, \quad \theta \to \lambda^2 \theta \quad \text{and} \quad \lambda \to 0 \\
\end{align*}
\]

\(^7\)The corresponding phase gives a shift to \( \theta \) and can be gauged away due to the symmetry \((2.12)\).
and obtain
\[ e^4V = \frac{\text{Re} f}{8|f|^2} \frac{1}{\alpha} , \]
\[ Z = 0 , \tag{4.18} \]
\[ \tau = c_0 + \frac{i}{8} e^{-(\phi_0 + 4V)} . \]

Since now \( a = b^* \) we get the A-type vacuum, which has been analyzed in \cite{24}. All RR fields are trivial up to the primitive 3-form flux of \((1,2)\) type, which one can always turn on. The dual \((2,1)\)-forms have to vanish because the Bianchi identity for the 5-form requires \( G \wedge \bar{G} = 0 \) and the non-primitive part is of NS-type, since \( G \wedge J \) is real in this limit. A prototype supergravity solution would be the near-horizon geometry of NS5-branes with a constant warp factor in the string frame metric \( (ds^2_{\text{str}} = e^{\phi/2}ds^2 \sim e^{2V}ds^2) \), but the dilaton is not stabilized.

In summary, around zeros of the holomorphic function, the solution is equivalent to the type-B vacuum, whereas close to a pole of \( f \) one reaches the type-A vacuum.

5 Comments on the 4-d Cosmological Constant

The value of the superpotential in the vacuum gives a mass term for the gravitino and hence enters also the spinor equation \((2.18)\). We have seen that supersymmetry preserving background fluxes cannot contribute to \( W \), as long as the internal manifold has \( \text{SU}(3) \) structures. The external 4-d space time will always be Minkowski. In order to generate a cosmological constant, one has to consider an internal manifold with \( \text{SU}(2) \) structures and let us briefly discuss some aspects here; see also \cite{31,27}. In this case, the internal spinors have to be singlets only under \( \text{SU}(2) \) and hence there are two independent Weyl spinors \( \chi_1 \) and \( \chi_2 \). Equivalently, one can define a real unit vector \( v \) globally, which relates the two spinors by
\[ \chi_2 = \gamma_v \chi_1 = v_m \gamma^m \chi_1 . \tag{5.19} \]

Without loss in generality, we set \( \gamma^7 \chi_1 = -\chi_1 \), and then \( \gamma^7 \chi_2 = \chi_2 \) and \( \chi_1 \) can be taken as the \( \text{SU}(3) \) singlet spinor that we used before [of course, \( \chi_2 \) transforms then under \( \text{SU}(3) \) and is a singlet only under \( \text{SU}(2) \)]. With the notation of \( \chi = (\chi_1 \chi_2) \), this is \( \gamma^7 \chi_i = -(\sigma_3 \chi)_i \), where \( \sigma_k \) the usual Pauli matrices. Decomposing the chiral 10-d supersymmetry transformation parameters, we get two 4-d Weyl spinors \( \zeta_1 \) and \( \zeta_2 \), or \( \zeta = (\xi \zeta_2) \), with chiralities \( \gamma^5 \xi = (\sigma_3 \zeta)_i \). In order to get \( N = 1, D=4 \) supersymmetric
backgrounds, we project out half of the spinor components of the two 4-d spinors by imposing: $\zeta_i = (\sigma_1 \zeta^*_i)$; [due to the chiral choice, this is the only possibility]. Now, the 10-d Killing spinor decomposes as

$$\epsilon = f_1 \zeta_1 \otimes \chi_1 + f_2 \zeta_2 \otimes \chi_2 + f_3 \zeta_1 \otimes \chi_2^* + f_4 \zeta_2 \otimes \chi_1^* \quad (5.20)$$

where $f_i$ are globally defined complex functions, which are the analogs of the function $a$ and $b$ used before and the spinors are normalized to $\chi_i^\dagger \chi_j = \zeta^\dagger_i \zeta_j = \delta_{ij}$. Now the superpotential is in general a $2 \times 2$ matrix denoted $\hat{W}$,

$$\nabla_\mu \zeta_i = \gamma_\mu \hat{W}_{ij} \zeta^*_j \quad (5.21)$$

However, since $\zeta_i$ are chiral and $\zeta_1^* = \zeta_2^*$, we find that $\hat{W}$ is diagonal and $\hat{W}_{11} = \hat{W}_{22} = \tilde{W}$. Here as before we defined $\tilde{W} = W^* = W_1 - iW_2$.

Now, the calculation can be repeated in the same way as we were doing it before. Since this is very involved, let us only show here that now $W$ can be non-zero in the vacuum. For this, it is enough to consider the simple case, which is the analog of the type-B vacuum, i.e. where the spinor is a direct product

$$\epsilon = \zeta \otimes \chi$$

with $\zeta_1 \equiv \zeta$ and $\chi = f_1 \chi_1 + f_3 \chi_2^*$ (or $f_2 = f_4 = 0$). Using the above conventions and taking the spinor product $(\chi^T \delta \Psi_\mu)$, we find from the vanishing of this expression that

$$\tilde{W} = \frac{1}{16} e^{-V/3U} (\chi^T G \chi) = \frac{1}{8} e^{-V/3U} f_1 f_3 (\chi_2^T G \chi_1) .$$

Inserting $\chi_2 = v_m \gamma^m \chi_1$ and taking $\chi_1$ as the SU(3) singlet spinor, satisfying the relations from the appendix, one gets an explicit form of $\tilde{W}$. This is straightforward and we do not want to discuss further details here.

Note the importance of the vector field, which implies that the internal space has to admit a complex fibration over a 4-d base space. As long as one does not take into account brane sources, this vector field has to be globally defined, which severely constraints the geometry of the internal space. On the other hand, if we allow for singularities, e.g., related to 7-branes wrapping the 4-d base, also the spinors and the vector field do not need to be globally defined and to make the supersymmetry nevertheless well-defined one has to add explicitly the brane sources. This is consistent with the statements, how a cosmological constant can be generated by wrapped branes [40].
Appendix

On a real 6-dimensional space, an $SU(3)$ structure is defined by a 2-form $J$ and a 3-form $\Omega$ which are globally well-defined and are in one-to-one correspondence to an $SU(3)$ singlet spinor $\eta$. If this spinor (and therewith the forms) is covariantly constant, the space has $SU(3)$ holonomy, which is however in general not the case and the deviation is measured by non-zero torsion components. As consequence of the $SU(3)$ singlet property, this spinor satisfies

\[(\gamma_m - iJ_{mn}\gamma^n)\chi = 0 ,\]

\[(\gamma_{mn} + iJ_{mn})\chi = i e^{2i\beta} \Omega_{mnp}\gamma^p \chi^* ,\]  \hspace{1cm} (A.22)

\[(\gamma_{mnp} + 3iJ_{[mn}\gamma_{p]}\chi = i e^{2i\beta} \Omega_{mnp}\chi^* .\]

The first relation is a set of three projector equations for the $SU(3)$-invariant spinor. Only two of them are independent and, together with $\zeta_2 = \zeta_1$ in (2.14), reduce the number of supersymmetries to $N = 1$ in four dimensions. The fundamental 2-form $J$ and the 3-form $\Omega$ are defined as spinor bilinears by

\[\chi^T \gamma_{mn} \chi^* = iJ_{mn} , \hspace{1cm} \chi^T \gamma_{mnp} \chi = i e^{2i\beta} \Omega_{mnp} \chi^* .\]  \hspace{1cm} (A.23)

Both are $SU(3)$ singlets and these are the only non-vanishing forms constructed from $\chi$, because any other form, especially a vector, would transform under $SU(3) \subset SO(6)$. They satisfy in addition the following relations (which follow directly from Fierz identities)

\[J \wedge \Omega = 0 ,\]

\[\Omega \wedge \tilde{\Omega} = -\frac{4i}{3} J \wedge J \wedge J = -8i \, \text{vol}_6\]

The almost complex structure on the internal manifold is given by $J_m^n = h^{ln}J_{nl}$ ($J^2 = -1$) and can be used to define holomorphic projectors so that $\Omega$ is a (3,0)-form with respect to $J$. Note, this does not require $J$ to be integrable or the manifold to be complex.
Some further useful identities are

\[ \chi^T \gamma_{mnpq} \chi^* = -3 J_{[mn} J_{pq]} , \]

\[ \bar{\Omega}^{mnp} \Omega_{pqr} = 4 \delta_{qr}^{mn} - 4 J_{[q} [m J_{r]}^{n]} + 8 i \delta_{[q}^{[m} J_{r]}^{n]} , \]

\[ \bar{\Omega}^{mnp} \Omega_{mpq} = 8 \delta_q^m + 8 i J_q^m , \]

\[ \bar{\Omega}^{mnp} \Omega_{mnp} = 48 , \]

\[ J^{mn} J_{[mn} T_p] = \frac{4}{3} T_p , \]

\[ \bar{\Omega}^{mnp} \Omega_{[mnp} T_q] = 6 (\delta_q^r - i J_q^r) T_r . \]

(A.24)

The last three can also be obtained from realizing \( \Omega_{ijk} = i \varepsilon_{ijk} \) and \( \varepsilon_{123} = \varepsilon_{123} = \sqrt{8} . \)

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