Duopoly Investment Problems with Minimally Bounded Adjustment Costs

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Abstract

In this paper, we study two-player investment problems with investment costs that are bounded below by some fixed positive constant. We seek a description of optimal investment strategies for a duopoly problem in which two firms invest in advertising projects to abstract market share from the rival firm. We show that the problem can be formulated as a stochastic differential game in which players modify a jump-diffusion process using impulse controls. We prove that the value of the game may be represented as a solution to a double obstacle quasi-variational inequality and derive a PDE characterisation (HJBI equations) of the value of the game. We characterise both the saddle point equilibrium and a Nash equilibrium for the zero-sum and non-zero-sum payoff games.

Keywords: Impulse control, Stochastic Differential Games, Optimal Stopping, Jump-diffusion, Dynkin Games, Verification Theorem, Duopoly, Advertising.

1 Introduction

Over the past three decades, a considerable amount of attention has been dedicated towards modelling the duopolistic advertising problem. The problem is one of finding the optimal dynamic investment strategy for a firm that seeks to maximise cumulative profits over some given time horizon. Each firm uses strategic advertising investments to increase market share. This paper is concerned with an investment problem in which two competing firms make strategic investments over time in order to maximise their cumulative profits. The paper studies a duopoly environment with future uncertainty. In order to accurately model firm behaviour, we introduce the notion of fixed minimal investment costs so that the competing firms incur at least some fixed minimum cost for each investment. This leads to a new description of the advertising investment problem in terms of a non-zero-sum stochastic differential game involving impulse controls.

Early versions of the advertising oligopoly problem were formulated as single-player optimal control models in which a controller maximises a payoff extracted from a system whose evolution is governed by some deterministic process. Thus, in the early models of the advertising problem, the influence of competing firms and the effect of future uncertainty derived from market fluctuations and exogenous shocks were neglected (see for example the surveys conducted in [rge82, Eri94]). To augment the model description, more recent models include a larger repertoire of modelling features, firstly, by modelling the problem as a (two-player) differential game framework the influence of a rival firm can be incorporated into the system - this approach has yielded considerable descriptive success in modelling the strategic interactions between firms. Following that, [PS04] (among others) adopts a stochastic differential game approach to model the problem which accounts for future uncertainty and random market fluctuations, the inclusion of which, has further increased

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modelling accuracy. We refer the reader to [Eri94] for exhaustive discussions on duopolistic advertising models and to [PS04] for a stochastic differential game approach.

In each of the above models, the firms’ investment modifications are modelled using continuous controls - in particular, it is assumed that the competing firms are able to make infinitesimally fine adjustments to their investment positions that incur arbitrarily small costs. In reality however, advertising investment projects have fixed minimal costs which eliminates the possibility of continual investment since such a strategy would result in singular costs and hence, immediate firm ruin. The presence of fixed minimal costs produces adjustment stickiness (rigidities) since firms now adjust their investment positions at discrete points over irregular time intervals. Consequently, the set of feasible investment strategies consists of those in which the firm makes a sequence of investments at selected times along the firm’s lifetime. Despite the relevance of minimally bounded adjustment costs, the literature concerning multiplayer strategic environments with bounded costs remains scarce. Indeed with the exception of the formal mathematical treatment of the zeros-sum case presented in [Cos12], models of multiplayer strategic interactions with bounded costs remain limited to environments in which one of the players is allowed to modify the system dynamics continuously (e.g. [Yon94; TY93; Zha11]).

To account for this, we construct a duopoly investment model in which each firm incurs at least some fixed minimal cost for each advertising investment. To model this, we introduce a non-zero-sum stochastic differential game in which both players use impulse controls to modify the system dynamics. Moreover, in contrast to the models of advertising described above, in order to embed into the description future uncertainty and exogenous economic shocks, we construct a game in which the underlying diffusion process which is allowed to have jumps.

The game we study is one in which two players modify a jump-diffusion process using impulse controls in order to maximise some given payoff criterion; we give a PDE characterisation of the value for the game for both zero-sum and non-zero-sum games. Though the paper focuses on addressing the duopoly advertising investment problem, the framework studied and subsequent results derived are general and therefore apply to modelling competitive multiplayer environments with future uncertainty in which players face fixed adjustment costs.

We show that the solutions (optimal investment strategies) to the problem can be represented as a solution to a double obstacle problem for a stochastic differential game in which players use impulse controls to modify the system dynamics and give a PDE characterisation of the optimal investment strategies.

**Background Material**

Problems that involve strategic modifications of a controlled dynamic system in competitive environments have attracted much attention over recent years both in theoretical and applied settings. In particular, there is a notable amount of literature on models of this kind in which two players use continuous controls to modify the system dynamics to satisfy some performance criterion. In the deterministic case, it was shown in [EK72a; EK72b] and [ES84] that the deterministic differential game admits a value and in fact, the value of the game is a unique solution to a HJBI equation in the viscosity sense. Following on from this, in [FS89], the corresponding result was proven for the case in which the system dynamics are stochastic. Indeed, building on the successes of the deterministic cases, the study of stochastic differential game theory has produced significant results and has been successfully applied in various settings within finance and economics.

Stochastic differential game theory underpins theoretical models used to prescribe optimal portfolio strategies in a Black-Scholes market (see e.g. [MK07; BC00]), descriptions of pursuer-invader dynamics (see e.g. [PY81]) and investment games in competitive advertising (see e.g. [rge82; Eri94; PS04]) amongst others.

If in the differential game, associated to the controllers’ modifications to the system dynamics is some fixed minimal cost, the appropriate mathematical framework is a differential game in which the controllers use impulse controls to modify the system dynamics. In the single player case, impulse control problems are stochastic control models in which the cost of control is bounded below by some fixed positive constant which prohibits continuous control, thus the problem is augmented to one of finding both an optimal sequence of times to apply the control policy, in addition to determining optimal control magnitudes.

Impulse control frameworks therefore underpin the description of financial environments with transaction costs and liquidity risks and more generally, applications of optimal control theory in which the system dynamics are modified by a sequence of discrete actions. We
Duopoly Investment Problems

refer the reader to [BL82] as a general reference to impulse control theory and to [CG14; PS10] for articles on applications.

Despite the fundamental relevance of fixed minimum costs within economics and financial systems, modelling multi-player competitive economic and financial systems has yet to incorporate the use of impulse control theory. Indeed, unlike in the case of continuous controls for which there is a plethora of studies, with the exception of the zero-sum game studied in [Cos12], stochastic games in which the players use impulse controls remain largely uninvestigated. Deterministic versions of this game were first studied by [Yon94; TY93] - in the model presented in [Yon94], impulse controls are restricted to use by one player and the other uses continuous control. Similarly, in [Zha11], stochastic differential games in which one player uses impulse control and the other uses continuous controls were studied. Using a verification argument, the conditions under which the value of the game (with a single impulse controller) is a solution to a HJBI equation is also shown in [Zha11]. In [Cos12], Cosso was the first to study a stochastic differential game in which both players use impulse control using viscosity theory. Thus, in [Cos12] it is shown that the game admits a value which is a unique viscosity solution to a double obstacle quasi-variational inequality. In ?? a nonzero-sum formulation of the game was studied from which a verification theorem that characterises the value function of the game was derived. These results however do not cover the general case of jump-diffusions.

Contribution

To our knowledge, this is the first paper to study the duopolistic advertising problem using a differential game approach with impulsive controls. The paper also makes several theoretical contributions to stochastic differential game theory involving impulse controls. We extend the analyses in [Cos12] where stochastic differential games in which both controllers use impulse controls were introduced, to consider i) system dynamics in which the uncontrolled state process is allowed to have jumps ii) non-zero-sum payoff structures.

We prove verification theorems for both the zero-sum case and the non-zero-sum case in which the appropriate equilibrium concept is a Nash equilibrium. A central component of the proof of the verification theorem is the analysis the players’ non-intervention regions. In the zero-sum case, the opponent’s actions produce two changes in the value function: Firstly, each impulse action performed by the opponent produces an immediate shift in the value of the state process, this in turn causes indirect changes to the value function since the state process enters as one of its inputs. Secondly, at each intervention, the opponent incurs an intervention cost which, in the zero-sum case represents a transfer of wealth from the opponent to the player - this produces direct instantaneous changes to the value function. To capture the two effects on the dynamics of the value function, it is necessary to reformulate the impulse control system as a singular control system which has minimally bounded adjustment costs (Lemma 5.3).

We then generalise the zero-sum payoff structure in the game to a non-zero-sum payoff structure wherein we appeal to a Nash equilibrium as the appropriate equilibrium concept. Owing to the fact that we use a verification theoretic approach, our proofs are markedly different to [Cos12]. Both non-zero-sum payoff structures and the inclusion of jumps in the underlying system dynamics serve as important modelling features for applications within finance and economics.

Organisation

The paper is organised as follows: In section 2, we introduce our dynamic duopoly model; here we elucidate the key features of the problems and show that the underlying structures are stochastic differential games. In section 3, we give a technical description of the stochastic differential game in which impulse controls are used to modify the state process. In section 4, we prove some preliminary results that underpin the main analysis which is performed in sections 6 and 7. In section 5, we give a precise statement of the optimal investment strategies for the problem and give some relevant technical results. In section 6, we prove a verification result for zero-sum stochastic differential games with impulse controls. In section 7, we characterise the Nash equilibrium for the stochastic differential game in which the payoff structure is no longer zero-sum. We then give some concluding remarks and lastly, the appendix contains some of the technical proofs from sections 4-6.

We start by providing a description of a generalised duopolistic advertising problem.
2 A Duopoly Investment Problem: Dynamic Competitive Advertising

The Model

The problem is a firm advertising problem in which two firms compete for market share in a duopoly market.

We develop our model by considering firstly two cases of advertising investment models and then lastly present our model as a third case. To fix ideas, as our first case we consider an environment in which both firms may only make continuous modifications to their investment positions - this approach reproduces the stochastic differential game version of the Vidale-Wolfe model of advertising.

In case II we consider environments in which advertising investments incur at least some fixed minimal cost. We relax the zero-sum payoff structure and include a description of cross-over effects from exogenous shocks within each firm’s market share process. In doing so we construct a model of dynamic competitive advertising which encapsulates some of the key features of the continuous control model. The current model however, has system dynamics that evolve according a jump-diffusion process and, associated to each investment is some fixed minimal cost.

We refer the reader to [Dea79, MDG11] for exhaustive discussions on duopoly advertising investment models and to [PS04] for a stochastic differential game approach.

We now give a detailed analysis of each case. The results for the model in case II are presented in section 8.

Overview

A overview of the three cases is as follows:

Consider two firms who compete for share of a single market. We will refer to the firms as Firm 1 and Firm 2. Both firms seek to maximise profits over some (possibly fixed) horizon by investing in advertising activities in order to increase market share and raise revenue from sales.

2.1 Case I: Duopoly with Continuous Investments (Review)

As our first case, consider an duopoly in which each firm modifies the advertising investment positions continuously. This approach provides a (generalised) description of a model which reproduces the key features of the Vidale-wolfe model as considered in [Dea79].

At time $t \in [t_0, \tau_S]$ each Firm $i$ has a revenue stream $S_i(t)$ which is a stochastic process, where $\tau_S \in \mathbb{R}^+ / \{\infty\}$ is some (possibly random) time horizon of the firm’s problem. Denote by $U_i$ the set of admissible investments for Firm $i$, $i \in \{1, 2\}$; at any point, Firm $i$ makes costly investments of size $u_i \in U_i$ i.e. $u_i$. Denote by $M \in \mathbb{R}^+$ the potential market size and by $b_i \in [0, 1]$ the response rate to advertising, then the revenue stream for Firm $i$ is given by the following expression:

$$dS_i(t) = b_iu_i(t^-)[M - S_i^{t_0,s_0}(t) - S_j^{t_0,s_0}(t)]M^{-1}dt - r_iS_i^{t_0,s_0}(t)dt + \sigma(t, S_i^{t_0,s_0}(t))dB(t).$$

P-a.s., where $s_0 \in \mathbb{R}$ are the initial sales, $r_i \in \mathbb{R}$ are constants and $i, j \in \{1, 2\} (i \neq j)$. The term $B(t)$ is Brownian motion which captures the random component of the system.

The cumulative profit for each Firm $i$ is denoted by $\Pi_i$, each firm seeks to maximise its cumulative profit which consists of its revenue due to sales $h : [t_0, \tau_S] \times \mathbb{R} \to \mathbb{R}$ minus its running advertising costs $c_i : [t_0, \tau_S] \times \mathbb{R} \to \mathbb{R}$ over the horizon $\tau_S < \infty$ and lastly, a function of the firm’s terminal market share $G : [t_0, \tau_S] \times \mathbb{R} \to \mathbb{R}$. The profit function for each Firm $i, \Pi_i$ is then described by the following:

$$\Pi_i(t_0, s_i; u_i, u_j) = \mathbb{E}\left[\int_{t_0}^{\tau_S} (h(S_i^{t_0,s_i,u_i}(t)) - c_i(t, u_i) - c_j(t, u_j))dt + G(S_i^{t_0,s_i,u_i}(\tau_S))\right],$$

Moreover, since the market is duopolistic, the payoff structure between the two firms is zero-sum which gives the following condition:

$$\Pi_i + \Pi_j = 0$$
We can now write the dynamic (zero-sum) duopoly problem as:
Find \( \phi \) and \((\tilde{u}_1, \tilde{u}_2) \in \mathcal{U}_1 \times \mathcal{U}_2 \) s.t. for all \((t, s_0) \in [t_0, \tau_S] \times \mathbb{R}^2\):
\[
\phi(t, s_0) = \sup_{u_1 \in \mathcal{U}_1} \left( \inf_{u_2 \in \mathcal{U}_2} \Pi^{u_1, u_2}(t, s_0) \right) = \inf_{u_1 \in \mathcal{U}_1} \left( \sup_{u_2 \in \mathcal{U}_2} \Pi^{u_1, u_2}(t, s_0) \right) = \Pi^{u_1, u_2}(t, s_0).
\]
where \( \Pi^{u_1, u_2}(t, s_0) \equiv \Pi(t, s_0; u_1, u_2) = \Pi_1(t, s_0; u_1, u_2) = -\Pi_2(t, s_0; u_1, u_2) = -\Pi_2(t, s_0; u_1, u_2).

We recognise \( \Pi \) as the problem associated to a (zero-sum) stochastic differential game and is a general version of the Vidale-Wolfe advertising model (see for example the differential game extension of the Vidale-wolfe model in [Dec79]). Case I is a stochastic differential game in which both players modify the state process using continuous controls, this approach was used (for example [PSO4]) to model the advertising duopoly investment problem. Thus, we observe the behaviour of the firms in the advertising problem can be characterised by computing the optimal policies a stochastic differential game.

We note finally that since the investment adjustments of each firm are described using continuous controls, here firms are permitted to make arbitrarily small adjustments to their investment positions which in turn, incur arbitrarily small costs.

Having conducted a review of the standard method of modelling the advertising problem, we now adapt the Vidale-Wolfe framework to now accommodate fixed minimal investment costs. Before constructing the main model of the section:

We now present the main model of the section:

### 2.2 Case II: Non-Zero-Sum Payoff with Impulse Controls with Jumps

In the above model, each firm can make investments of arbitrary size and with no minimal cost. Additionally, the zero-sum payoff structure implies that a transfer of wealth occurs between firms whenever an advertising investment is made. We wish to firstly remove this feature and to secondly account for the effect that both firms have on the market which we assume undergoes exogenous shocks.

Suppose now that each firm’s sales process experiences exogenous economic shocks and that the firms now incur a fixed minimal cost for an advertising investment. Denote by \( c : [t_0, \tau_S] \times \mathcal{Z} \to \mathbb{R} \) and \( \chi : [t_0, \tau_S] \times \mathcal{Z} \to \mathbb{R} \) the cost function associated to the advertising investments of Firm 1 and Firm 2 respectively where \( \mathcal{Z} \subseteq \mathbb{R}^2 \) is a given set. In this case, since the firms advertising investments incur minimal costs, we have that there exists positive constants \( \lambda_1 \in \mathbb{R}^+ \) and \( \lambda_2 \in \mathbb{R}^+ \) s.t. \( c(t, \cdot) \geq \lambda_1 \) and \( \chi(t, \cdot) \geq \lambda_2 \). Since now continuous investment in the firm’s advertising projects would result in immediate bankruptcy, the firm must now modify its advertising investments position in a discretised fashion. Each firm therefore performs a sequence of advertising investments over the horizon of the problem.

Denote by \( \mathcal{U} \) the set of admissible investments for Firm 1, then the sequence of investments \( \{\xi_k\}_{k \in \mathbb{N}} \) for Firm 1 is performed over a sequence of times \( \{\tau_k\}_{k \in \mathbb{N}} \). The investment strategy for Firm 1 is therefore given by a double sequence:
\[
\Pi(t, s_0; u_1, u_2) = \Pi_1(t, s_0; u_1, u_2) = -\Pi_2(t, s_0; u_1, u_2).
\]

where \( \Pi^{u_1, u_2}(t, s_0) \equiv \Pi(t, s_0; u_1, u_2) = \Pi_1(t, s_0; u_1, u_2) = -\Pi_2(t, s_0; u_1, u_2) \).

We now adapt the Vidale-Wolfe framework to now accommodate fixed minimal investment costs. Before constructing the main model of the section:

The rate of sales \( S_i \) for each Firm \( i \in \{1, 2\} \) evolve according to the following expressions:
\[
dS_1(t) = \mu_1(S_1^{t_1, s_1, u_1, v_1}(r))dr + \sum_{j \geq 1} \xi_j \cdot 1_{[\tau_j \leq \tau_S]}(t) + \sigma_{11}(S_1^{t_1, s_1, u_1, v_1}(r))dB_1(r) + \theta_{11}(S_1^{t_1, s_1, u_1, v_1}(t), z)d\Lambda_1 + \sigma_{12}(S_1^{t_1, s_1, u_1, v_1}(r))dB_2(r) + \theta_{12}(S_1^{t_1, s_1, u_1, v_1}(r), t, z)d\Lambda_2,
\]
\[
S_1^{t_1, s_1, u_1, v_1}(t_1) = s_1 \in \mathbb{R}.
\]
\[
dS_2(t) = \mu_2(S_2^{t_2, s_2, u_2, v_2}(r))dr + \sum_{m \geq 1} \eta_m \cdot 1_{[\tau_m \leq \tau_S]}(t) + \sigma_{22}(S_2^{t_2, s_2, u_2, v_2}(r))dB_2(r) + \theta_{22}(S_2^{t_2, s_2, u_2, v_2}(t), z)d\Lambda_2 + \sigma_{21}(S_2^{t_2, s_2, u_2, v_2}(r))dB_1(r) + \theta_{21}(S_2^{t_2, s_2, u_2, v_2}(r), t, z)d\Lambda_1.
\]
\[
S_2^{t_2, s_2, u_2, v_2}(t_2) = s_2 \in \mathbb{R}.
\]
where \( \Lambda_i = \int_{t_0}^{\tau_S} \int_{\mathbb{R}} \tilde{N}(ds, dz) \) and where \( B_1 \) and \( B_2 \) are Wiener processes and \( \tilde{N} \) is a compensated Poisson random measure.
Unlike in the competitive advertising models that appeal to differential games in which the players’ modifications of their investment positions are modelled using continuous controls (in which firms may make arbitrarily small advertising investments), the above model now assumes that each advertising investment requires at least some fixed minimal cost. Hence here, the firms undertake marketing and advertising projects at discrete points in time in such a way that maximises their cumulative profit.

Our next modification is to relax the zero-sum payoff structure \( [\alpha] \), we therefore decouple the payoff criterion \( [\alpha] \) into two profit functions \( J_i \) for each firm \( i \in \{1, 2\} \). Hence now Firm \( i \) seeks to maximise its running profits over the fixed horizon \( \tau_S < \infty \) plus a valuation of its terminal market share hence we may write Firm \( i \)’s objective as:

\[
\Pi_i(t_0, s_1, s_2; u, v) = E^x_{[s_1, s_2]} \left[ \int_{t_0}^{\tau_S} \exp(-r_t) \left( \alpha_1 S_1^{t_1, s_1, u, v}(r) - \beta_1 S_2^{t_1, s_1, u, v}(r) \right) dr - \sum_{j \geq 1} c_1(\tau_j, s_j) \cdot 1(\tau_j \leq \tau_S) + \gamma_1 \exp(-r_{\tau_S}) \left[ S_1^{t_1, s_1, u, v}(\tau_S) \right]^2 \right].
\]

(7)

\[
\Pi_2(t_0, s_1, s_2; u, v) = E^x_{[s_1, s_2]} \left[ \int_{t_0}^{\tau_S} \exp(-r_t) \left( \alpha_2 S_2^{t_1, s_1, u, v}(r) - \beta_2 S_1^{t_1, s_1, u, v}(r) \right) dr - \sum_{m \geq 1} c_2(\rho_m, \eta_m) \cdot 1(\rho_m \leq \tau_S) + \gamma_2 \exp(-r_{\tau_S}) \left[ S_2^{t_1, s_1, u, v}(\tau_S) \right]^2 \right].
\]

(8)

The above model has the following interpretation: the market share \( S_i \) of Firm \( i \) determines the size of the revenue \( \alpha_i S_i \) generated from sales, the parameters \( \alpha_i \) and \( \beta_i \) may be interpreted as representing the Firm \( i \) profit margin and the sensitivity of Firm \( i \)’s sales on Firm \( j \)’s market share (respectively). In order to increase its revenue stream, each firm may use advertising investments \( (\xi_j)_{j \in \mathbb{N}} \) for Firm \( 1 \), \( (\eta_m)_{m \in \mathbb{N}} \) for Firm \( 2 \) to abstract market share which reduces the rival firm’s revenue stream. However, increases in either firm’s market size expands the economy leading to a higher terminal valuation for both firms which is proportional to the square of the terminal cost, (this term is often included in models of duopoly with finite horizon - see for example [3,4]).

The dynamic duopoly problem is now to characterise the policies \( (\hat{u}, \hat{v}) \in \mathcal{U} \times \mathcal{V} \) and to find the functions \( \phi_i \in C([t, \tau_S], \mathbb{R}) \) for \( i \in \{1, 2\} \) s.t.:

\[
\phi_1 = \sup_{u \in \mathcal{U}} \Pi_1(t_0, s_1, s_2; u, \hat{v}) = \Pi_1(t_0, s_1, s_2; \hat{u}, \hat{v})
\]

(9)

\[
\phi_2 = \sup_{v \in \mathcal{V}} \Pi_2(t_0, s_1, s_2; \hat{u}, v) = \Pi_2(t_0, s_1, s_2; \hat{u}, \hat{v}),
\]

(10)

\( \forall (t_0, s_1, s_2) \in [t, \tau_S] \times \mathbb{R} \times \mathbb{R} \).

The stochastic differential game therefore involves the use of impulse controls exercised by both players who modify system dynamics to maximise some given payoff function.

In section 6 and 7, we provide a complete characterisation of the solution to the zero-sum cases and non-zero-sum cases in terms of classical solution to a PDE (respectively). In section 8 we apply the results of section 7 to characterise the firm’s policies for case II.

We now provide a formal description of the game. We begin by providing a description and the relevant concepts for the zero-sum game which, in section 7 we shall adapt for the non-zero-sum game:

### 3 Description of The Game

Stochastic differential games are environments in which a number of players interact by altering the dynamics of a stochastic system by strategically selected magnitudes. The players modify the system dynamics in order to maximise some state-dependant payoff criterion over some time horizon where the space of payoffs is described by the sample space and the players’ payoffs evolve according to some stochastic process.

**Canonical Description**

Let \( \mathcal{C}(U; \mathcal{G}) \) be the set of continuous functions from some set \( U \subseteq \mathbb{R} \) to a field \( \mathcal{G} \). The index \( s \in [t, \tau_S] \) is time which runs continuously over some random and possibly finite time horizon \( \tau_S \). We denote the coordinate mapping on \( \mathcal{C}([t, \tau_S]; \mathbb{R}^p) \) by \( B_s(\omega_B) = \omega_B(s) \) and
denote also by $F = \{ F_s \}_{s \in [t, \tau_S]}$ the completed natural filtration and define $\{ F_{t,s} \}_{s \in [t, \tau_S]}$ to be $\{ F_s \}_{s \in [t, \tau_S]}$ restricted to the interval $[t, s]$ (uncompleted natural filtration). Correspondingly, we also denote by $W_k$ a $\sigma$-algebra generated by the paths in $C([t, \tau_S]; \mathbb{R}^p)$ up to time $t'$ and let $B(s) \in \mathbb{R}^p$ be a $p$-dimensional standard Brownian motion with state space $S$. $\tilde{N}(ds, dz) = N(ds, dz) - \nu(dz) dt$ is a $F-$Poisson random measure with $\nu(\cdot) := \mathbb{E}[N(1, \cdot)]$ is a Lévy measure; both $\tilde{N}(ds, dz)$ and $B(s)$ are supported by the filtered probability space and $F$ is the filtration of the probability space $(\Omega, \mathbb{P}, F = \{ F_s \}_{s \in [t, \tau_S]})$. We assume that $N$ and $B$ are independent.

As in [CG14], we note that the above specification of the filtration ensures stochastic integration and hence, the controlled jump diffusion is well defined (this is proven in [SV02]).

We suppose then that the uncontrolled passive state $X \in S \subseteq \mathbb{R}^p (p \in \mathbb{N})$, evolves according to a (jump-)diffusion on $(C([t, \tau_S]; \mathbb{R}^p), (F_{t,s})_{s \in [t, \tau_S]}), \mathbb{P}, 0)$ that is to say for $s \in [t, \tau_S]$ the state process obeys the following SDE:

$$
dX_{s,x}^t = \mu(s, X_{s,x}^t) ds + \sigma(s, X_{s,x}^t) dB(s) + \int \gamma(X_{s-}, z) \tilde{N}(ds, dz), \quad X_{t,x}^t := x_0. \quad (11)$$

$\forall s \in [t, \tau_S], (t, x_0) \in [t, \tau_S] \times \mathbb{R}^p; \mathbb{P}$-a.s.

We assume that the functions $\mu : [t, \tau_S] \times \mathbb{R}^p \rightarrow \mathbb{R}^p, \sigma : [t, \tau_S] \times \mathbb{R}^p \rightarrow \mathbb{R}^{p \times m}$ and $\gamma : \mathbb{R}^p \times \mathbb{R} \rightarrow \mathbb{R}^{p \times 1}$ satisfy the usual assumptions so as to ensure the existence of $\mathbb{H}$ (see assumptions A.1.1 & A.2).

The generator of $X$ (the uncontrolled process) is:

$$
\mathcal{L} \phi(x) = \sum_{i=1}^{p} \mu_i(x) \frac{\partial \phi}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{p} (\sigma \sigma^T)_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + I \phi(x) \quad (12)
$$

$\forall x \in [t, \tau_S] \times \mathbb{R}^p$, where $I$ is the integro-differential operator defined by:

$$
I \phi(x) := \sum_{j=1}^{l} \int_{\mathbb{R}^p} \{ \phi(x + \gamma^j(x, z_j)) - \phi(x) - \nabla \phi(x) \gamma^j(x, z_j) \} \nu_j(dz_j), \quad (13)
$$

$\forall x \in [t, \tau_S] \times \mathbb{R}^p$.

The state process is influenced by a pair of impulse controls $u \in \mathcal{U}, v \in \mathcal{V}$ exercised by each player where $u(s) = \sum_{j \geq 1} \xi_j \cdot 1_{\{ \tau_j \leq \tau_S \}}(s)$ for all $s \in [t, \tau_S]$, with impulses $\{ \xi_j \} \in \mathbb{Z} \subset S$ are exercised by player I who intervenes at $F$-measurable stopping times $\{ \tau_j \}$ where $t \leq \tau_1 < \tau_2 < \cdots < \tau_S$ and where $v(s) = \sum_{m \geq 1} \eta_m \cdot 1_{\{ \rho_m \leq \tau_S \}}(s)$ for all $s \in [t, \tau_S]$, with impulses $\{ \eta_m \} \in \mathbb{Z} \subset S$ are exercised by player II who intervenes at $F-$measurable stopping times $\{ \rho_m \}$ where $t \leq \rho_1 < \rho_2 < \cdots < \rho_{\rho_S}$ where $S \subseteq \mathbb{R}^p$ is a given set. Thus, we interpret $\tau_n$ (resp., $\rho_n$) as the $n^{th}$ time at which player I (resp., player II) modifies the system dynamics with an impulse intervention $\xi_n$ (resp., $\eta_n$).

We assume $\mathcal{U} \subseteq \mathbb{R}^p$ and $\mathcal{V} \subseteq \mathbb{R}^p$ are convex cones which are the set of admissible control actions for player I and player II (resp.) and $\mathcal{V}$ is the set of admissible impulse values. Indeed, let us suppose that an impulse $\zeta \in Z$ determined by some admissible policy $w$ is applied at some $F$-measurable stopping time $\tau \in [t, \tau_S]$ when the state is $x' = X_{t,x_0}^\tau(\tau^-)$, then the state immediately jumps from $x' = X_{t,x_0}^\tau, (\tau^-)$ to $X_{t,x_0}^\tau(\tau) = \Gamma(x', \zeta)$ where $\Gamma : \mathbb{R}^p \times \mathcal{Z} \rightarrow \mathbb{R}^p$ is called the impulse response function.

We assume that the impulses $\xi_j$ (resp., $\eta_m$) are $\mathcal{U}-$ valued (resp., $\mathcal{V}-$ valued) and are $F-$measurable for all $j$ (resp., $m$).

The evolution of the state process with interventions is described by the equation:

$$
X(r) = x + \int_{t}^{r} \mu(s, X_{t,x}^{u,v}(s)) ds + \int_{t}^{r} \sigma(s, X_{t,x}^{u,v}(s)) dB(s) + \sum_{j \geq 1} \xi_j \cdot 1_{\{ \tau_j \leq \tau_S \}}(r) + \sum_{m \geq 1} \eta_m \cdot 1_{\{ \rho_m \leq \tau_S \}}(r) + \int_{t}^{r} \int \gamma(X_{t,x}^{u,v}(s), z) \tilde{N}(ds, dz). \quad (14)
$$

for all $s \in [t, \tau_S]; \mathbb{P}$-a.s., where the player I impulse control $u \in \mathcal{U}$ player II impulse control $v \in \mathcal{V}$ is given by the following expression respectively:

$$
u(s) = \sum_{m \geq 1} \eta_m \cdot 1_{\{ \rho_m \leq \tau_S \}}(s), \quad u(s) = \sum_{j \geq 1} \xi_j \cdot 1_{\{ \tau_j \leq \tau_S \}}(s) \quad (15)$$
Player I has a gain (or profit) function \( J \) which player I maximises given by the following expression for all \( u \in U, v \in V, (t, x) \in [t, \tau_s] \times \mathbb{R}^p \):

\[
J(t, x; u, v) = \mathbb{E}\left[ \int_t^{\tau_s} f(s, X^i_s, x, u, v) ds + \sum_{m \geq 1} c(\tau_m, \xi_m) \cdot 1_{\{\tau_m \leq \tau_s\}} - \sum_{l \geq 1} \chi(\rho_l, \eta_l) \cdot 1_{\{\rho_l \leq \tau_s\}} + G(\tau_s, X^{*i}_{\tau_s}, x, u, v) \right].
\]

(16)

where the functions \( f : [t, \tau_s] \times \mathbb{R}^p \rightarrow \mathbb{R}, G : [t, \tau_s] \times \mathbb{R}^p \rightarrow \mathbb{R} \) are deterministic, measurable, uniformly continuous functions which we shall refer to as the running cost function and the bequest function respectively.

In the zero-sum game, the payoff function \( J \) is also the player II cost function which player II minimises.

**Standing Assumptions**

We assume that the state parameters \( \mu : [t, \tau_s] \times \mathbb{R}^p \rightarrow \mathbb{R}^p, \sigma : [t, \tau_s] \times \mathbb{R}^p \rightarrow \mathbb{R}^{p \times m} \) and \( \gamma : \mathbb{R}^p \times \mathbb{R}^l \rightarrow \mathbb{R}^{p \times l} \) are deterministic, measurable functions that are Lipschitz continuous and satisfy a growth condition. That is we assume the following conditions are satisfied:

**A.1. Lipschitz Continuity**

We assume there exist real-valued constants \( c_\mu, c_\sigma > 0 \) and \( c_\gamma(\cdot) \in L^1 \cap L^2(\mathbb{R}^l, \nu) \) s.t. \( \forall s \in [t, \tau_s], \forall x, y \in \mathbb{R}^p \) and \( \forall z \in \mathbb{R}^l \) we have:

\[
|\mu(s, x) - \mu(s, y)| \leq c_\mu |x - y|
\]

\[
|\sigma(s, x) - \sigma(s, y)| \leq c_\sigma |x - y|
\]

\[
\int_{|z| \geq 1} |\gamma(x, z) - \gamma(y, z)| \leq c_\gamma(z) |x - y|.
\]

A.1.2. Lipschitz Continuity

We also assume the Lipschitzianity of the running functions \( h, g, \psi \) and \( \phi \) so that we assume the existence of real-valued constants \( c_h, c_g, c_\psi, c_\phi > 0 \) s.th. \( \forall s \in [t, \tau_s], \forall (x, y) \in \mathbb{R}^p \) we have for \( R \in \{ h, g, k, l, \psi, \phi \} \):

\[
|R(s, x) + R(s, y)| \leq c_R |x - y|.
\]

**A.2. Growth Conditions**

We assume the existence of a real-valued constants \( d_\mu, d_\sigma > 0 \) and \( d_\gamma(\cdot) \in L^1 \cap L^2(\mathbb{R}^l, \nu), \rho \in [0, 1) \) s.t. \( \forall (s, x) \in [t, \tau_s] \times \mathbb{R}^p \) and \( \forall z \in \mathbb{R}^l \) we have:

\[
|\mu(s, x)| \leq d_\mu([1 + |x|^{\rho}])
\]

\[
|\sigma(s, x)| \leq d_\sigma([1 + |x|^{\rho}])
\]

\[
\int_{|z| \geq 1} |\gamma(x, z)| \leq d_\gamma([1 + |x|^{\rho}]).
\]

**A.3.**

We also assume that there exists constants \( \lambda_c > 0 \) and \( \lambda_\chi > 0 \) s.th. the following conditions hold \( \inf_{s \in \mathbb{R}} c(s, \xi) \geq \lambda_c, \forall \) and \( \inf_{s \in \mathbb{R}} \chi(\cdot, \xi) \geq \lambda_\chi \) where \( \xi \in \mathbb{Z} \) is a measurable impulse intervention.

**A.4.**

We assume that the cost functions \( \chi \) and \( c \) are quasi-linear in the impulse inputs. That is to say for all \( \mathcal{F}_\tau \) measurable stopping times \( \tau \in [t, \tau_s] \) and for all \( \mathcal{F}_\tau \) measurable impulse interventions \( z \in \mathbb{Z} \) we assume the functions \( \chi \) and \( c \) take the following form:

\[
\chi(\tau, z) \equiv a_2(\tau) z + \kappa_2 \text{ and } c(\tau, z) \equiv a_1(\tau) z + \kappa_1
\]

for some constants \( \kappa_1, \kappa_2 > 0 \) and some functions \( a_i : [t, \tau_s] \rightarrow \mathbb{R}, i \in \{ 1, 2 \} \).

In the following exposition we will set \( a_2(\tau) \equiv \lambda_2 \in \mathbb{R} \) and \( a_1(\tau) \equiv \lambda_1 \) for some \( \lambda_1, \lambda_2 \in \mathbb{R}^+ \). We will also refer to \( \lambda_1 \) and \( \kappa_i \) as the player i proportional intervention cost and fixed intervention cost parts (resp.) for \( i \in \{ 1, 2 \} \).

Assumptions A.1.1 and A.2 ensure the existence and uniqueness of a solution to (11) (c.f. [1W81]). Assumption A.1.2 is required to ensure the regularity of the value function (see for example [Cos12] and for the single-player case, see for example [MDG10]). Assumption A.3 is integral to the definition of the impulse control problem. Lastly, assumption A.4 is necessary to derive an equivalent singular control representation of the problem.
Throughout the script we adopt the following standard notation (e.g. \[TY93; Zha11\]):

**Notation**

Let \( \Omega \) be a bounded open set on \( \mathbb{R}^{p+1} \). Then we denote by: \( \Omega - \) The closure of the set \( \Omega \).

\( Q(s, x; R) = s'(s', x') \in \mathbb{R}^{p+1} : \max |s' - s|^2 + |x' - x| < R, s' < s, \)

\( \partial \Omega - \) the parabolic boundary \( \Omega \) i.e. the set of points \((s, x) \in \mathbb{S} \) s.t. \( R > 0, Q(s, x; R) \not\subseteq \Omega \).

\( C^{(1,2)}([t, \tau], \Omega) = \{ h \in C^{(1,2)}(\Omega) : \partial_t h, \partial_{s,t}, \partial_{s,x}, h \in \mathcal{C}(\Omega) \} \), where \( \partial_t \) and \( \partial_{s,x} \) denote the temporal differential operator and second spatial differential operator respectively.

\( \nabla \phi = (\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_p}) - \) The gradient operator acting on some function \( \phi \in C^{1}(]t, \tau[ \times \mathbb{R}^p) \).

\( C^\prime([a, b]) - \) The set of càdlàg functions that map \([a, b] \mapsto U \) for some set \( \mathbb{U} \subseteq \mathbb{R}^p \).

\( | \cdot | - \) The Euclidean norm to which \( (x, y) \) is associated scalar product acting between two vectors belonging to some finite dimensional space.

As in \[CG14\], we will use the notation \( u = \{ \tau_j, \xi_j \}_{j \geq 1} \) to denote the control policy \( u = \sum_{j \geq 1} \xi_j \cdot 1_{\{\tau_j \leq \tau_S\}}(s) \in U \) which consists of \( \mathcal{F} \)-measurable stopping times \( \{ \tau_j \}_{j \in \mathbb{N}} \) and \( \mathcal{F} \)-measurable impulse interventions \( \{ \xi_j \}_{j \in \mathbb{N}} \).

The following definitions will be useful:

**Definition 3.1.**

Denote by \( \mathcal{T}_t(\tau) \) the set of all \( \mathcal{F} \)-measurable stopping times in the interval \([t, \tau] \), where \( \tau' \leq \tau_S \), if \( \tau' = \tau_S \) then we will denote by \( \mathcal{T} = \mathcal{T}_t(\tau_S) \).

Let \( u = \{ \tau_j, \xi_j \}_{j \in \mathbb{N}} \) be a control policy where \( \{ \tau_j \}_{j \in \mathbb{N}} \) and \( \{ \xi_j \}_{j \in \mathbb{N}} \) are \( \mathcal{F}_{\tau_j} \)-measurable stopping times and interventions respectively. We denote by \( \mu_{\tau}(u) \) the number of impulses executed within the interval \([t, \tau] \) under the control policy \( u \) for some \( \tau \in \mathcal{T} \).

**Definition 3.2.**

Let \( u \) a control policy. We say that an impulse control is admissible on \([t, \tau_S] \) if the number of impulse interventions is finite \( \mathbb{P} \)-a.s. that is to say we have that: \( E[\mu_t(\tau_S)(u)] < \infty \).

We shall hereon use the symbol \( \mathcal{U} \) (resp., \( \mathcal{V} \)) to denote the set of admissible controls for player I (resp., player II).

For controls \( u \in \mathcal{U} \) and \( u' \in \mathcal{U} \), we interpret the notation \( u \equiv u' \) on \([t, \tau_S] \) iff \( \mathbb{P}(u = u' \text{ a.e. on } [t, \tau_S]) = 1 \). For two player II controls \( v \) and \( v' \), we interpret the notation \( v \equiv v' \) on \([t, \tau_S] \) analogously.

**Definition 3.3.**

Let \( u(s) = \sum_{j \geq 1} \xi_j \cdot 1_{\{\tau_j \leq \tau_S\}}(s) ; u \in \mathcal{U} \) a player I impulse control defined over \([t, \tau_S] \), further suppose that \( \tau \in [t, \tau_S] \) and \( \tau' \in [t, \tau_S] \) are two \( \mathcal{F} \)-measurable stopping times with \( \tau \geq s > \tau' \), then we define the restriction \( u_{\tau', \tau}(s) \) to be \( u(s) = \sum_{j \geq 1} \xi_{j, \tau, \tau'} \cdot 1_{\{\Sigma_{j \geq 1} \xi_j \geq \Sigma_{j \geq 1} \xi_{j, \tau, \tau'} \}}(s) \).

Analogously, we define the restriction for the player II control \( v(s) = \sum_{j \geq 1} \eta_{m, \tau, \tau'} \cdot 1_{\{\Sigma_{j \geq 1} \eta_j \geq \Sigma_{j \geq 1} \eta_{j, \tau, \tau'} \}}(s) ; v \in \mathcal{V} \) defined over \([t, \tau_S] \) so that we define the restriction \( v_{\rho, \rho'}(s) \) to be \( v(s) = \sum_{j \geq 1} \eta_{m, \tau, \tau'} \cdot 1_{\{\Sigma_{j \geq 1} \eta_j \geq \Sigma_{j \geq 1} \eta_{j, \tau, \tau'} \}}(s) \) where \( \rho \in [t, \tau_S] \) and \( \rho' \in [t, \tau_S] \) are two \( \mathcal{F} \)-measurable stopping times s.t. \( \rho \geq s > \rho' \).

**Strategies**

A player strategy is a map from the other player’s set of controls to the player’s own set of controls. An important feature of the players’ strategies is that they are non-anticipative - neither player may guess in advance, the future behaviour of other players given his current information. We formalise this condition by constructing non-anticipative strategies which were used in the viscosity solution approach to differential games. Non-anticipative strategies were introduced by \[EK72b; EK72a; Rox69; Var67\]. Hence, in this game, one of the players chooses his control and the other player responds by selecting a corresponding strategy.

**Definition 3.4.**

A non-anticipative strategy on \([t, \tau_S] \) for Player I is a measurable mapping which we shall denote by \( A \) s.t. \( A : [t, \tau_S] \times \mathcal{X} \times \mathcal{V}(t, \tau_S) \rightarrow \mathcal{U}(t, \tau_S) \) and for any stopping time \( \tau : \mathcal{T} \rightarrow [t, \tau_S] \) and any \( v_1, v_2 \in \mathcal{V}(t, \tau_S) \) with \( v_1 \equiv v_2 \) on \([t, \tau] \) we have that \( A(v_1) \equiv A(v_2) \) on \([t, \tau] \). We define Player II non-anticipative strategy \( B : [t, \tau_S] \times \mathcal{X} \times \mathcal{U}(t, \tau_S) \rightarrow \mathcal{V}(t, \tau_S) \) analogously. Hence, \( A \) and \( B \) are Elliott-Kalton strategies.

Following the notation in \[Cos12\], we denote the set of all non-anticipative strategies for Player I (Player II) by \( A(t, \tau_S) \) (resp., \( B(t, \tau_S) \)).

**Remark 3.5.**

The intuition behind definition 3.4 is as follows: suppose player I uses the strategy \( u_1 \in \mathcal{U} \) and the system follows a path \( w \) and that player II employs the strategy \( B \in \mathcal{B}(t, \tau_S) \) against the control \( u_1 \). If in fact player II cannot distinguish between the control \( u_1 \) and some other
player I control \( u_2 \in \mathcal{U} \) then controls \( u_1 \) and \( u_2 \) induce the same response from the player II strategy that is to say \( \mathcal{B}(u_1) \equiv \mathcal{B}(u_2) \).

Note that when either \( \mathcal{U} \) or \( \mathcal{V} \) is a singleton, the game collapses into a classical stochastic impulse control problem with only one player with a value function and solution as that in [kS07].

**Definition 3.6.**

Suppose we denote the space of measurable functions by \( \mathcal{H} \), suppose also that the function \( \phi : [t, \tau_S] \times \mathbb{R}^p \rightarrow \mathbb{R}^p \) s.th. \( \phi \in \mathcal{H} \).

Let \( \tau \) be some \( \mathcal{F} \)-measurable stopping time and let \( A \) (resp., \( B \)) be a non-anticipative strategy on \([t, \tau_S]\) for Player I (resp., player II). We define the [non-local] Player I-intervention operator \( M_1 : \mathcal{H} \rightarrow \mathcal{H} \) acting at some \( \mathcal{F} \)-measurable stopping time \( \tau \) by the following expression:

\[
M_1[\phi] := \inf_{z \in \mathcal{Z}} [\phi(\tau, \Gamma(X(\tau-), z)) + c(\tau, z) \cdot 1_{\{\tau \leq \tau_S\}}].
\]

We analogously define the [non-local] Player II-intervention operator \( M_2 : \mathcal{H} \rightarrow \mathcal{H} \) at some \( \mathcal{F} \)-measurable stopping time \( \rho \) by:

\[
M_2[\phi] := \sup_{z \in \mathcal{Z}} [\phi(\rho, \Gamma(X(\rho-), z)) - \chi(\rho, z) \cdot 1_{\{\rho \leq \tau_S\}}].
\]

where \( \Gamma : \mathbb{R}^p \times \mathcal{Z} \rightarrow \mathbb{R}^p \) is the impulse response function defined earlier.

Given the remarks of section 3, we now define the value functions of the game. As in [FS89, Cos12], we define the value functions in terms of Elliot-Kalton strategies introduced in [EK72b]:

**Definition 3.7.**

For any \( x \in [t, \tau_S] \times \mathbb{R}^p \) and \( (t, x_0) \in [t, \tau_S] \times \mathbb{R}^p \) the two value functions associated to the game are given by the following expressions:

\[
V^{-}(x) = \inf_{u \in \mathcal{U}} \sup_{v \in \mathcal{V}} J(t, x; u, v);
\]

\[
V^{+}(x) = \sup_{v \in \mathcal{V}} \inf_{u \in \mathcal{U}} J(t, x; u, v)
\]

where we will refer to \( V^- \) and \( V^+ \) as the upper and lower value functions respectively.

We say that the value of the game exists if \( \forall x \in [t, \tau_S] \times \mathbb{R}^p \) we can compute the supremum and infimum operators in definition 3.7 wherein we can deduce the existence of a function \( V \in \mathcal{C}^{1,2}([t, \tau_S]; \mathbb{R}^p) \) with \( V \equiv V^-(x) = V^+(x) \).

**Remark 3.8.**

Suppose the value of the game exists. We may then note, with particular reference to the duopoly problem in section 2, according to the law of supply and demand, the quantity \( \partial_x V(x) \) represents the revenue associated with a marginal change in the components thus, \( \partial_x V(x) \) is the market price when the quantity available is \( x \).

**Remark 3.9.**

Suppose the value of the game exists. Then the term \( M_i V(\cdot, x), i \in \{1, 2\} \) that is, the non-local intervention operator \( M_i \) acting the value function associated to the game represents the value of the player \( i \) strategy that consists of performing the best possible intervention at some given time \( s \in [t, \tau_S] \) when the state is at \( x \in \mathbb{R}^p \), then performing optimally thereafter.

We note however, that an immediate intervention may not be optimal, hence \( \forall (s, x) \in [t, \tau_S] \times \mathbb{R}^p \) the following inequalities hold:

\[
M_1 V(s, x) \geq V(s, x)
\]

\[
M_2 V(s, x) \leq V(s, x)
\]

Statements [20] and [21] in fact follow as a direct consequence of the dynamic programming principle. A formal statement and proof for the single impulse controller case is given as Lemma 3.5 in [Zha11] for which both proof and results are entirely applicable in our case.

### 4 Main Results

We prove two key theoretical results for the game that characterise the conditions for a HJBI equation in both zero-sum and non-zero-sum games.
We prove a verification theorem (Theorem 6.1) for stochastic differential games with a jump-diffusion process and in which the players use impulse controls. In doing so, we prove the following statement

Theorem 4.1.

Suppose that the value of the zero-sum game \( V \) exists and that \( V \in C^{1,2}([t, \tau_S]; \mathbb{R}^p) \) then the value of the game satisfies the following double obstacle quasi-variational inequality \( \forall y \in \mathbb{R}^p, x \in [t, \tau_S] \times \mathbb{R}^p: \)

\[
\begin{align*}
\max \{ \min &[-(\partial_x V(x) + \mathcal{L}V(x) + f(x)), V(x) - M_2 V(x)], V(x) - M_1 V(x) \} = 0 \\
V(\tau_S, y) = \Gamma(\tau_S, y)
\end{align*}
\]

(22)

Moreover, denote by \( \hat{u} := [\hat{r}_j, \hat{\xi}_j]_{j \in \mathbb{N}} \) and \( \hat{v} := [\hat{\rho}_m, \hat{\eta}_m]_{m \in \mathbb{N}} \) the equilibrium controls, then \( V \) satisfies the following expressions:

\[
\Delta \hat{u} V(X(\tau_j)) = -c(\tau_j, \hat{\xi}_j).
\]

(23)

where \( \Delta \hat{u} \phi(X(\tau)) := \phi(\Gamma(X(\tau), z)) - \phi(X(\tau - e)) + \Delta \xi X(\tau) \) and for some \( \mathcal{F}_\tau \)-measurable stopping intervention \( z \in \mathcal{Z} \) and \( \Delta \xi X(\tau) \) denotes a jump at some \( \mathcal{F}_\tau \)-measurable time \( \tau \in [t, \tau_S] \) due to \( \hat{N} \).

For the non-zero-sum payoff case, we have the following result:

Theorem 4.2.

Denote by \( \phi_i \) the Firm \( i \) value function for the non-zero-sum game for \( i \in \{1, 2\} \), then the value functions \( \phi_i \) satisfy the following quasi-variational inequalities \( \forall y \in \mathbb{R} \times \mathbb{R}, x \in [t, \tau_S] \times \mathbb{R} \times \mathbb{R}: \)

\[
\begin{align*}
\max \{ -(\partial_x \phi_i(x) + \mathcal{L}\phi_i(x) + f_i(x)), \phi_i(x) - M_i \phi_i(x) \} &= 0, \\
\phi_i(\tau_S, y) &= \phi_i(\tau_S, y).
\end{align*}
\]

(24)

The following results characterise the optimal investment policies for the duopoly problem described in section 2:

Theorem 4.3.

Suppose that the market share \( X_i \) of Firm \( i \), \( i \in \{1, 2\} \) evolves according to \( \{5\} - \{6\} \) and let the firm payoff functions be given by \( \{7\} - \{9\} \), then the sequence of optimal investments \( \hat{u} = [\hat{r}_j, \hat{\xi}_j]_{j \in \mathbb{N}} \equiv \{ \hat{\xi}_j \cdot 1_{\{\hat{\xi}_j \leq \tau_S\}} \} \) for Firm 1 is characterised by the investment times \( \{\hat{\tau}_j\}_{j \in \mathbb{N}} \) and investment magnitudes \( \{\hat{\xi}_j\}_{j \in \mathbb{N}} \) where \( [\hat{r}_j, \hat{\xi}_j]_{j \in \mathbb{N}} \) are constructed via the following expressions:

i. \( \hat{\tau}_0 \equiv t_0 \) and \( \hat{\tau}_{j+1} = \inf \{ s > \tau_j; X_1^{u,v}(s) < x_1^* \} \wedge \tau_S \) \( \forall v \in \mathcal{V} \)

ii. \( \hat{\xi}_j = \hat{x}_1 - X_1(\hat{r}_j) \).

Similarly for Firm 2, the optimal sequence of investments \( \hat{v} := [\hat{\rho}_m, \hat{\eta}_m]_{m \in \mathbb{N}} = \sum_{m \geq 1} \hat{\eta}_m \cdot 1_{\{\hat{\rho}_m \leq \tau_S\}}(s) \) is given by:

i. \( \hat{\rho}_0 \equiv t_0 \) and \( \hat{\rho}_{m+1} = \inf \{ s > \rho_m; X_2^{u,v}(s) < x_2^* \} \wedge \tau_S \) \( \forall u \in \mathcal{U} \)

ii. \( \hat{\eta}_m = \hat{x}_2 - X_2(\hat{\rho}_m) \).

where the quadruplet \( (x_1^*, x_2^*, \hat{x}_1, \hat{x}_2) \) is determined by the following equations \( i \in \{1, 2\} \):

\[
\begin{align*}
C_1 r_1 e^{r_1 : x_1^*} + C_2 r_2 e^{r_2 : x_1^*} + \frac{\alpha_i}{\epsilon} &= \lambda_i \\
C_1 r_1 e^{r_1 : \hat{x}_1} + C_2 r_2 e^{r_2 : \hat{x}_1} + \frac{\alpha_i}{\epsilon} &= \lambda_i \\
C_1 (e^{r_1 : x_1^*} - e^{r_1 : \hat{x}_1}) + C_2 (e^{r_2 : x_1^*} - e^{r_2 : \hat{x}_1}) &= -\kappa_i + \left( \lambda_i - \frac{\alpha_i}{\epsilon} \right) (x_1^* - \hat{x}_1),
\end{align*}
\]

(25) - (27)

where \( C_1 \) and \( C_2 \) are endogenous constants whose values are determined by \( \{25\} - \{27\} \), \( \lambda_i \) and \( \kappa_i \) are the Firm \( i \) proportional and fixed intervention costs (respectively), \( \alpha_i \) is the Firm \( i \) margin parameter, \( \epsilon \) is the discount rate and the values \( r_1,i \) and \( r_2,i \) are roots of the equation:

\[
q(r_{k,i}) := \frac{1}{2} \sigma_{i,k}^2 r_{k,i}^2 + \mu_i r_{k,i} - \epsilon + \int_\mathbb{R} \{ e^{r_k,i \theta_{ij} z} - 1 - \theta_{ij} r_{k,i} z \} \nu_j(dz).
\]

(28)
for $i, j, k \in \{1, 2\}$. 

Theorem 4.3 says that each firm performs a sequence of investments over the time horizon of the problem. The decision to invest is determined by the firm’s market share process - in particular, at the point at which Firm $i$’s share of the market falls below the level $x^*_i$, then the firm performs an investment in order to raise its market share to $\hat{x}_i$, where the fixed values $\hat{x}_i$ and $x^*_i$ are determined by the given parameters $\lambda_i, \lambda_j, \kappa_i, \kappa_j, \alpha_i, \epsilon$ via (25) - (27). In particular, each Firm $i$ seeks to retain a market share of at least $x^*_i$, where $x^*_i$ is a quantity determined by the size and influence of both firms. At any point Firm $i$’s market share falls below $x^*_i$ the firm immediately reacts by performing an investment in order to raise its market share to $\hat{x}_i$ and in doing so abstracting market share from its competitors. Thus if $S$ is the total size of the market the value $S - x^*_i$ represents the maximum level of market share that Firm $i$ is prepared to cede to other firms.

In summary, each firm observes its own market share and only intervenes at the points at which the firm’s market share has fallen below some fixed level. At this point, the firm performs an advertising investment in order to raise the market share to within some prefixed levels. For each firm, both the minimum market share and the investment magnitudes are determined by the firm’s size and the responsiveness of the market to advertising investments of both firms. Each firm’s intervention policy is reactant to the investment and subsequent market acquisition of the other firm, each firm therefore reacts by performing the best sequence of response investments to the other firm’s investment strategy.

The following corollary follows directly from Theorem 4.3 and establishes when each of the firm performs investments under the optimal Nash equilibrium strategy:

**Corollary 4.4.**

For the duopoly advertising problem, the sample space splits into three regions: a region in which Firm $i$ performs an advertising investment - $I_1$, a region in which Firm 2 performs an advertising investment - $I_2$ and a region in which no action is taken by either firm $I_3$. Moreover, the three regions are characterised by the following expressions for:

$I_1 = \{x < x^*_i | x, x^*_i \in \mathbb{R}\}, \quad i, j \in \{1, 2\}$

$I_2 = \{x \geq x^*_i \wedge x^*_j | x, x^*_i, x^*_j \in \mathbb{R}\}$.

where the $x^*_i, x^*_j, i, j \in \{1, 2\}$ are determined by (25) - (27).

To our knowledge, this paper is the first to deal with a jump-diffusion process within a stochastic differential game in which the players use impulse controls to modify the state process.

5 Preliminaries

**Lemma 5.1.**

Let $(\tau, x) \in [t, \tau_s] \times \mathbb{R}^p$ where $\tau$ is some $\mathcal{F}$-measurable stopping time, then the set $\Xi(\tau, x)$ defined by:

$$
\Xi(\tau, x) := \{\xi \in \mathcal{Z} : MV(\tau, x) = V(\tau, x + \xi) + c(\tau, \xi) \cdot 1_{\{\tau \leq \tau_s}\}}
$$

is non-empty.

The proof of the lemma is essentially that given as the proof of Lemma 3.7 in [CG14] with little adaptation - we therefore omit the proof of lemma here.

**Definition 5.2.** [GT08]

Suppose $x \in I$, then an admissible singular control is a pair $(\nu^+_s, \nu^-_s)_{s \geq 1}$ of $\mathcal{F}$-adapted, non-decreasing càdlàg processes s.th. $\nu^+(t) = \nu^-(t) = t, X(t) := x + \nu^+_s - \nu^-_s$ and $d\nu^+, d\nu^-$ are supported on disjoint subsets.

The following result demonstrates that general impulse control problems can be represented as a singular control problem, in particular it shows that the game (14) can be represented as a game of singular control.

**Lemma 5.3.**

Let $(\nu_1(s), \nu_2(s)) \in \mathbb{R}^p \times \mathbb{R}^p$ be a pair of adapted finite variation càdlàg processes with increasing components. Then the impulse control problem can be written as the following
equivalent singular control problem \( \forall (t, x) \in [t, \tau_S] \times \mathbb{R}^p, \mathbb{P}\text{-a.s.} \):

\[
\inf_{\nu_1} \sup_{\nu_2} J(t, x; \nu) = \mathbb{E} \left[ \int_t^{\tau_S} f(s, X_s^{t, x, \nu}) ds + \int_t^{\tau_S} \Theta_1(s) d\nu_1(s) - \int_t^{\tau_S} \Theta_2(s) d\nu_2(s) + G(\tau_S, X_{\tau_S}^{t, x, \nu}) \right]
\]

and given some admissible player I (resp., player II) control policy \( u = [\tau_j, \xi_j]_{j \in \mathbb{N}} \) (resp., \( v = [\rho_m, \eta_m]_{m \in \mathbb{N}} \)) where the cost functions \( c \) and \( \chi \) for player I and player II respectively are given by the following expressions \( \forall \xi, \eta \in \mathbb{Z} \):

\[
c(\tau_k, \xi_k) \equiv \lambda_1 \xi_k + \kappa_1 \quad \text{and} \quad \chi(\rho_m, \eta_m) \equiv \lambda_2 \eta_m + \kappa_2,
\]

where \( \lambda_i \in \mathbb{R}^+ \) and \( \kappa_i \in \mathbb{R}^+ \) are the player \( i \) proportional intervention cost and fixed intervention cost parts (resp.) for \( i \in \{1, 2\} \). The state process \( X \) evolves follows the SDE:

\[
X(r) = x + \int_t^r \mu(s, X_s^{t, x, \nu}(s)) ds + \int_t^r \sigma(s, X_s^{t, x, \nu}(s)) dB(s) + \nu(r) + \int_t^r \int \gamma(X_s^{t, x, \nu}(s-), z) \tilde{N}(ds, dz)
\]

\( \mathbb{P} - \text{a.s.}, \forall (t, x) \in [t, \tau_S] \times \mathbb{R}^p. \)

We will use Lemma 5.3 in order to prove a verification theorem for the stochastic differential game which has a zero-sum payoff structure.

We defer the proof of Lemma 5.3 to the appendix.

6 A HJBI Equation for Zero-Sum Stochastic Differential Games with Impulse Controls.

In this section, we give a verification theorem for the value of the game therefore giving conditions under which the value of the game is a solution to the HJBI equation.

To accommodate the influence of impulses exercised by player II on the value function, it is necessary to reformulate the problem as a singular impulse control problem for which we appeal to Lemma 5.3. The following theorem characterises the conditions in which the value of the game satisfies a HJBI equation:

**Theorem 6.1 [Verification Theorem for Zero-Sum Games with Impulse Control]**

Let \( \tau \) be some \( \mathcal{F}_\tau \)-measurable stopping time and denote by \( \bar{X}(\tau) = X(\tau-) + \Delta_\tau X(\tau) \) where \( \Delta_\tau X(\tau) \) denotes a jump at some \( \mathcal{F}_\tau \)-measurable time \( \tau \) due to \( N \). Denote also by \( \Delta_\tau \phi(X(\tau)) := \phi(\Gamma(X(\tau-), z)) - \phi(X(\tau-)) + \Delta_\tau X(\tau) \) where \( \tau \in [t, \tau_S] \) and \( z \in \mathbb{Z} \) is some \( \mathcal{F}_\tau \)-measurable stopping time and intervention (resp.).

Suppose that the value of the game exists and that \( V \in \mathcal{C}^{1, 2}([t, \tau_S], S) \cap \mathcal{C}([t, \tau_S], \bar{S}) \). In the following we will use the shorthand and denote by \( X(s) \equiv (s, X) \forall (s, X) \in [t, \tau_S] \times \mathbb{R}^p \) and \( \phi \equiv \phi(X, \cdot) \equiv \phi(X) \forall X \in S \).

Suppose also that there exists a function \( \phi \) that satisfies technical conditions (T1) - (T4) and the following conditions:

i. \( \phi \in \mathcal{C}([t_0, \tau_S], S) \cap \mathcal{C}([t_0, \tau_S], \bar{S}) \)

ii. \( \phi \leq M_1 \phi \) and \( \phi \geq M_2 \phi \) in \( S \) where \( D_1 \) and \( D_2 \) are defined by: \( D_1 = \{ X \in S, s \in [t, \tau_S]; \phi(X) < M_1 \phi(X) \} \) and \( D_2 = \{ X \in S, s \in [t, \tau_S]; \phi(X) > M_2 \phi(X) \} \) where we refer to \( D_1 \) (resp., \( D_2 \)) as the player I (resp., player II) continuation region.

iii. \( \frac{\partial \phi}{\partial s} + \mathcal{L}\phi(X^{t, \cdot, \cdot}(\cdot)) + f(X^{t, \cdot, \cdot}(\cdot)) \leq 0 \forall v \in \mathcal{V}, X \in S \setminus \partial D_2. \)

iv. \( \frac{\partial \phi}{\partial s} + \mathcal{L}\phi(X^{t, \cdot, \cdot}(\cdot)) + f(X^{t, \cdot, \cdot}(\cdot)) \geq 0 \forall u \in \mathcal{U}, X \in S \setminus \partial D_1. \)

v. \( \frac{\partial \phi}{\partial s} + \mathcal{L}\phi(X^{t, \cdot, \cdot}(\cdot)) + f(X^{t, \cdot, \cdot}(\cdot)) = 0 \) in \( D \equiv D_1 \cap D_2 \forall X \in S. \)

vi. \( X^{t, \cdot, \cdot}(\tau_S) \in \mathcal{D} \mathbb{P}-\text{a.s.} \) on \( \{ \tau_S < \infty \} \) and \( \phi(X^{t, \cdot, \cdot}(\cdot)) \rightarrow G(X^{t, \cdot, \cdot}(\tau_S)) \cdot 1_{\{\tau_S < \infty\}} \) as \( s \rightarrow \tau_S \) \( \mathbb{P}-\text{a.s.}, \forall X \in S, u \in \mathcal{U}, v \in \mathcal{V} \).
vii. \( \hat{\xi}_k \in \arg\inf_{z \in Z} \{ \phi(\Gamma(X(\tau_k - \cdot), z)) + c(\tau_k, z) \} \) is a Borel measurable selection and similarly, \( \hat{\eta}_j \in \arg\sup_{z \in Z} \{ \phi(\Gamma(X(\tau_j - \cdot), z)) - \chi(\rho_j, z) \} \) is a Borel measurable selection for all \( X \in S \).

Put \( \hat{\tau}_0 \equiv t \) and define \( \hat{u} := [\hat{\tau}_j, \hat{\xi}_j]_{j \in \mathbb{N}} \) inductively by:
\[
\hat{\tau}_{j+1} = \inf \{ s > \tau_j; X^{[\cdot, u, v]}(s) \notin D_1 \} \land \tau_S \forall x \in S, v \in V \text{ similarly, put } \hat{\rho}_0 \equiv t \text{ and define } \\
\hat{\rho}_m, \hat{\eta}_m \in \mathbb{N} \text{ inductively by } \hat{\rho}_{m+1} = \inf \{ s > \rho_m; X^{[\cdot, u, v]}(s) \notin D_2 \} \land \tau_S \forall x \in S, u \in U.
\]

xiii. \( \Delta(\phi(\hat{\rho}_m)) = \chi(\hat{\rho}_m, z) \) and \( \Delta(\phi(\hat{\tau}_j)) = -c(\hat{\tau}_j, z) \)
\( \forall z \in Z \) and the intervention times \( \{ \hat{\rho}_m \}_{m \in \mathbb{N}}, \{ \hat{\tau}_j \}_{j \in \mathbb{N}} \)

Then
\[
\phi(x) = J(t, x; \hat{u}, \hat{v}) = \inf_{u \in U} \sup_{v \in V} J(t, x; u, v) = \inf_{v \in V} \sup_{u \in U} J(t, x; u, v) \quad (34)
\]
for all \( x \in [t, \tau_S] \times \mathbb{R}^p \)

Theorem 6.1 says that if a sufficiently smooth function can be found then the value function of the game is characterised in terms of a PDE.

From Theorem 6.1 we also see that the sample space aligns into three regions that consist of a continuation region, in which neither player performs an intervention and intervention regions for each player within which the players perform an impulse execution. That is we have the following corollary:

**Corollary 6.2.**

The sample space aligns into three regions that represent a region for player I interventions \( I_1 \) a region for player II interventions \( I_2 \), and a region \( I_3 \) in which no action is taken by neither player; moreover the three regions are characterised by the following expressions:

\[
I_1 = \{ x \in [t, \tau_S] \times \mathbb{R}^p : V(x) = M_1 V(x), L V(x) + f(x) \geq 0 \},
\]

\[
I_2 = \{ x \in [t, \tau_S] \times \mathbb{R}^p : V(x) = M_2 V(x), L V(x) + f(x) \geq 0 \},
\]

\[
I_3 = \{ x \in [t, \tau_S] \times \mathbb{R}^p : V(x) < M_1 V(x), V(x) > M_2 V(x); L V(x) + f(x) = 0 \}.
\]

**Proof of Theorem 6.1.**

In the following, we make the distinction between the jumps due to the players’ impulse controls and the jumps due to \( \hat{N} \). Indeed, let \( \tau \in [t, \tau_S] \) be some \( F^- \)-measurable stopping time we denote by \( X(\tau) = X(\tau^-) + \Delta_N X(\tau) \) where \( \Delta_N X(\tau) \) is the jump at some \( F^- \)-measurable time \( \tau \) due to \( \hat{N} \) where \( \Delta_N X(s) = \int \gamma(x_s - z) \hat{N}(ds, dz) \) and \( \hat{N}(ds, dz) = \hat{N}(s, dz) - \hat{N}(s-, dz) \).

Similarly, given an impulse \( \xi \in Z \) (resp., \( \eta \in Z \)) exercised by player I (resp., player II), we denote the jump induced by the player I (resp., player II) impulse by \( \Delta_\xi \) (resp., \( \Delta_\eta \)).

That is, if we suppose that \( \tau \in [t, \tau_S] \) is some \( F^- \)-measurable intervention time for player I, then we define
\[
\Delta_\xi(\phi(X^{[\cdot, u, v]}(\tau^-))) := \phi(\Gamma(X^{[\cdot, u, v]}(\tau^-), \xi)) - \phi(X^{[\cdot, u, v]}(\tau^-)) + \Delta_N \phi(X^{[\cdot, u, v]}(\tau^-))
\]
and similarly, \( \Delta_\eta(\phi(X^{[\cdot, u, v]}(\tau^-))) := \phi(\Gamma(X^{[\cdot, u, v]}(\tau^-), \eta)) - \phi(X^{[\cdot, u, v]}(\tau^-)) + \Delta_N \phi(X^{[\cdot, u, v]}(\tau^-)) \) is the change in \( \phi \) due to the player II impulse \( \eta \in Z \) at some \( F^- \)-measurable intervention time \( \rho \in [t, \tau_S] \).

To prove the theorem, we will use a singular control representation of the combined impulse controls for each player. For our first case, we define \( \nu(s) = \eta(s) + \xi(s) \) where \( \nu \) is a process consisting of the combined player I and player II controls. Note that by Lemma 5.3 we have the following equivalences \( \forall r \in [t, \tau_S] \):

\begin{align*}
a. \quad & \xi(r) = \sum_{m=1}^{H[r, t]} \xi_j \cdot 1_{\{\tau_j \leq \tau_S\}} \\
b. \quad & \eta(r) = \sum_{m=1}^{H[r, t]} \eta_m \cdot 1_{\{\rho_m \leq \tau_S\}} \\
c. \quad & \int_r^{r'} d\xi(s) = \sum_{j=1}^{H[r, r'](s)} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \tau_S\}} \\
d. \quad & \int_r^{r'} d\eta(s) = \sum_{m=1}^{H[r, r'](s)} \chi(\rho_m, \eta_m) \cdot 1_{\{\rho_m \leq \tau_S\}}
\end{align*}

We now fix \( \hat{u} = [\hat{\rho}_m, \hat{\eta}_m]_{m \geq 1} \in V \) and hence using (a) and (b) we find that \( \nu(s) \) is now given by \( \nu(s) = \sum_{m \geq 1} \hat{\eta}_m(s) \cdot 1_{\{\rho_m \leq \tau_S\}} + \sum_{j \geq 1} \xi(s) \cdot 1_{\{\tau_j \leq \tau_S\}} \).
By Itô’s formula for càdlàg semi-martingale (jump-diffusion) processes (see for example theorem II.33 of \cite{Pro03} in conjunction with Theorem 1.24 of \cite{kS07}), we have that:

\[
\mathbb{E}[(\hat{X}^t,\tau_k)(\eta_j)] - \mathbb{E}[\hat{X}^t,\tau_{k+1}](\hat{\rho}_m)] \\
= -\mathbb{E}\left[\int_{t_j}^{t_{j+1}} \frac{d\phi}{ds} + \mathcal{L}(\hat{X}^t,x_0,u,\hat{\nu})(s)ds + \sum_{m \leq \nu_j} \mathcal{L}(\hat{X}^t,x_0,u,\hat{\nu})(\hat{\rho}_m)) \right].
\]

(35)

We note firstly that by definition of the intervention times \(\{\tau_j\}_{j \in \mathbb{N}}\) we have that \(\mu(\tau_j,\tau_{j+1})(u) = 0\) since no player I interventions occur in the interval \([\tau_j,\tau_{j+1})\). Hence, on the interval \([\tau_j,\tau_{j+1})\) we have that \(\Delta \hat{\nu} = \Delta \hat{\eta}\) in particular, \(\Delta \nu = \Delta \phi \) so that \(\Delta \nu(\hat{X}^t,x_0,u,\hat{\nu})(\hat{\rho}_m) = \Delta \phi(\hat{X}^t,x_0,u,\hat{\nu})(\hat{\rho}_m)).\)

Hence, by \cite{xii} we have that:

\[
\mathbb{E}[\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)] - \mathbb{E}[\hat{X}^t,x_0,u,\hat{\nu}(\tau_{j+1})] \\
= -\mathbb{E}\left[\int_{t_j}^{t_{j+1}} \frac{d\phi}{ds} + \mathcal{L}(\hat{X}^t,x_0,u,\hat{\nu})(s)ds + \sum_{m \leq \nu_j} \chi(\hat{\nu}_m,\hat{\eta}_m) \right].
\]

(36)

Summing both sides from \(j = 0\) to \(j = k < \infty\), we obtain the following:

\[
\phi(x) + \sum_{j=1}^{k} \mathbb{E}[\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)] - \phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_{j+1})) - \mathbb{E}[\hat{X}^t,x_0,u,\hat{\nu}(\tau_{k+1})] \\
\leq \mathbb{E}\left[\int_t^{t_{k+1}} f(\hat{X}^t,x_0,u,\hat{\nu})(s)ds - \sum_{m \leq \nu_j} \chi(\hat{\nu}_m,\hat{\eta}_m) \right].
\]

(37)

Now by definition of the non-local intervention operator \(\mathcal{M}_1\), we have that

\[
\phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) = \phi(\Gamma(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j),\xi_j)) \geq \mathcal{M}_1(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j) - c(\tau_j,\xi_j),
\]

hence,

\[
\phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) - \phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) - \mathcal{M}_1(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j) - c(\tau_j,\xi_j),
\]

and by \cite{vii} we readily observe that

\[
\phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_S)) - \phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_S)) = 0.
\]

(39)

After plugging \cite{iii} into \cite{vii} we obtain the following:

\[
\phi(x) + \sum_{j=1}^{k} \mathbb{E}[\mathcal{M}_1(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) - \phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) - c(\tau_j,\xi_j) \cdot 1_{\{\tau_j < \tau_S\}}] \\
\leq \mathbb{E}\left[\int_t^{t_{k+1}} f(\hat{X}^t,x_0,u,\hat{\nu})(s)ds - \sum_{m \leq \nu_j} \chi(\hat{\nu}_m,\hat{\eta}_m) \right].
\]

(40)

Hence,

\[
\phi(x) + \sum_{j=1}^{k} \mathbb{E}[\mathcal{M}_1(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) - \phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) \cdot 1_{\{\tau_j \leq \tau_S\}}] \\
\leq \mathbb{E}\left[\int_t^{t_{k+1}} f(\hat{X}^t,x_0,u,\hat{\nu})(s)ds + \phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) + \sum_{j=1}^{\infty} c(\tau_j,\xi_j) \cdot 1_{\{\tau_j \leq \tau_S\}} - \sum_{m \leq \nu_j} \chi(\hat{\nu}_m,\hat{\eta}_m) \right].
\]

(41)

using the fact that \(\sum_{j=1}^{k} c(\tau_j,\xi_j) \cdot 1_{\{\tau_j \leq \tau_S\}} \equiv \sum_{j=1}^{\infty} c(\tau_j,\xi_j) \cdot 1_{\{\tau_j \leq \tau_S\}}\). Now

\[
\lim_{k \to \infty} \sum_{j=1}^{k} \mathbb{E}[\mathcal{M}_1(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) - \phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j))] = 0
\]

since by \cite{vii} we have that \(\phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) - \phi(\hat{X}^t,x_0,u,\hat{\nu}(\tau_j)) = 0,\mathbb{P} - a.s.\) when \(\tau_j = \tau_S\) we can then deduce the statement by Lemma 3.10 in \cite{CG14} i.e. using the \(\frac{1}{2}\)-Hölder continuity of the non-local operator \(\mathcal{M}_1\).
Similarly we have by (xiii) that $\phi(X \cdot (s)) \rightarrow G(X \cdot (\tau_S)) \cdot 1_{\{\tau_S < \infty\}}$ as $s \to \tau_S$ P–a.s. Hence, letting $k \to \infty$ in (11) gives:

$$
\phi(x) \leq E\left[\int_t^{\tau_S} f(X^{t,x_0,\hat{u},\hat{v}}(s))ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \tau_S\}} - \sum_{n \geq 1} \chi(\hat{\rho}_n, \hat{\eta}_n) \cdot 1_{\{\hat{\rho}_n \leq \tau_S\}} + G(X^{t,x_0,\hat{u},\hat{v}}(\tau_S))\right].
$$

Since this holds for all $u \in \mathcal{U}$, we have that:

$$
\phi(x) \leq \inf_{u \in \mathcal{U}} E\left[\int_t^{\tau_S} f(X^{t,x_0,u,\hat{v}}(s))ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \tau_S\}} - \sum_{n \geq 1} \chi(\hat{\rho}_n, \hat{\eta}_n) \cdot 1_{\{\hat{\rho}_n \leq \tau_S\}} + G(X^{t,x_0,u,\hat{v}}(\tau_S))\right] = V^+(x).
$$

In particular, we have that:

$$
\phi(x) \leq \sup_{v \in \mathcal{V}} \inf_{u \in \mathcal{U}} E\left[\int_t^{\tau_S} f(X^{t,x_0,u,v}(s))ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \tau_S\}} - \sum_{n \geq 1} \chi(\hat{\rho}_n, \hat{\eta}_n) \cdot 1_{\{\hat{\rho}_n \leq \tau_S\}} + G(X^{t,x_0,u,v}(\tau_S))\right] = V^-(x).
$$

Using an analogous arguments, namely replacing $\hat{v}$ with $\hat{\rho}$ in (35), then performing similar steps (using condition (vi)) we can similarly prove that:

$$
\phi(x) \geq \sup_{u \in \mathcal{U}} \inf_{v \in \mathcal{V}} E\left[\int_t^{\tau_S} f(X^{t,x_0,u,v}(s))ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \tau_S\}} - \sum_{n \geq 1} \chi(\hat{\rho}_n, \hat{\eta}_n) \cdot 1_{\{\hat{\rho}_n \leq \tau_S\}} + G(X^{t,x_0,u,v}(\tau_S))\right] = V^-(x).
$$

Let us now fix the pair of controls $(\hat{u}, \hat{v}) \in \mathcal{U} \times \mathcal{V}$, using the definition of $\Delta_z$ and by (xiii) we have that:

$$
0 = \Delta_x \phi(X(\hat{\rho}_m)) - \chi(\hat{\rho}_m, z)
= \phi(\Gamma(X(\hat{\rho}_m, z))) - \phi(X(\hat{\rho}_m)) - \Delta_N X(\hat{\rho}_m) - \chi(\hat{\rho}_m, z)
= \phi(\Gamma(\hat{X}(\hat{\rho}_m, z))) - \phi(X(\hat{\rho}_m)) - \chi(\hat{\rho}_m, z).
$$

Now since (14) holds for all $z \in \mathcal{Z}$, after applying $\sup$ to both sides of (14) we find that:

$$
0 = \sup_{z \in \mathcal{Z}} [\phi(\Gamma(\hat{X}(\hat{\rho}_m, z))) - \chi(\hat{\rho}_m, z)] - \phi(\hat{X}(\hat{\rho}_m)) = M_2 \phi(\hat{X}(\hat{\rho}_m)) = \phi(\hat{X}(\hat{\rho}_m)),
$$

from which we immediately deduce the statement:

$$
M_2 \phi(\hat{X}(\hat{\rho}_m)) = \phi(\hat{X}(\hat{\rho}_m)).
$$

From which we see that an immediate impulse intervention at $\hat{\rho}_m$ is indeed optimal for player II.

Using analogous arguments we can deduce that:

$$
M_1 \phi(\hat{X}(\hat{\tau}_m^-)) = \phi(X(\hat{\tau}_m^-)).
$$

We hereafter straightforwardly observe using (15) and (T4) we find the following equality:

$$
\phi(x) = E\left[\int_t^{\tau_S} f(X^{t,x_0,\hat{u},\hat{v}}(s))ds + \sum_{j \geq 1} c(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \tau_S\}} - \sum_{m \geq 1} \chi(\hat{\rho}_m, \hat{\eta}_m) \cdot 1_{\{\hat{\rho}_m \leq \tau_S\}} + G(X^{t,x_0,\hat{u},\hat{v}}(\tau_S))\right].
$$

Hence, we can deduce the following statement:

$$
V^+(x) \geq \phi(x) \geq V^-(x).
$$
Now, by definition of $V^+$ and $V^-$ we always have that $V^-(s, x) \geq V^+(s, x)$ \(\forall (s, x) \in [t, \tau_S] \times \mathbb{R}^p\), hence we deduce that:

$$V^+(x) = V^-(x) = V(x).$$

and

$$V(x) = \phi(x).$$

\(\forall x \in [t, \tau_S] \times \mathbb{R}^p\),

after which we deduce the thesis.

7 A HJBI Equation for Non-Zero-Sum Stochastic Differential Games with Impulse Controls

In this section, we study the games as described in section 3, however we now extend the results to non-zero-sum stochastic differential games. The results of this section are loosely based on [Mk07] where we make the necessary adjustments to accommodate impulse controls. We prove a non-zero-sum verification theorem for the game in which both players use impulse controls to modify the state process.

Suppose firstly that the uncontrolled passive state $X \in S \subset \mathbb{R}^p (p \in \mathbb{N})$, evolves according to a (jump-)diffusion (i.e. 11).

Suppose also that player I (resp., player II) uses the impulse controls $u \in \mathcal{U}$ (resp., $v \in \mathcal{V}$ ) drawn from a set of admissible controls $\mathcal{U}$ (resp., $\mathcal{V}$ ) to modify the state process also as described in section 3. As in [Mk07], we decouple the objective performance functions so that we now consider the following payoff functions:

$$J_1^{(\tilde{u}, \tilde{v})}(x) = \mathbb{E} \left[ \int_t^{\tau_S} f_1(X^{t,x,\tilde{u},\tilde{v}}(s))ds \right]$$

and

$$J_2^{(\tilde{u}, \tilde{v})}(x) = \mathbb{E} \left[ \int_t^{\tau_S} f_2(X^{t,x,\tilde{u},\tilde{v}}(s))ds \right]$$

where \((t, x) \in [t, \tau_S] \times \mathbb{R}^p\). \(\tilde{u} = [\tilde{\tau}_j, \tilde{\xi}_j]_{j \geq 1}\) and \(\tilde{v} = [\tilde{\rho}_m, \tilde{\eta}_m]_{m \geq 1}\) are admissible controls for player I and \(\tilde{u} = [\tilde{\tau}_j, \tilde{\xi}_j]_{j \geq 1}\) and \(\tilde{v} = [\tilde{\rho}_m, \tilde{\eta}_m]_{m \geq 1}\) are admissible controls for player II.

We note that the function $J_1^{(\tilde{u}, \tilde{v})}(x)$ (resp., $J_2^{(\tilde{u}, \tilde{v})}(x)$) defines the payoff received by the player I (resp., player II) uses the control $\tilde{u} \in \mathcal{U}$ (resp., $\tilde{v} \in \mathcal{V}$) and player II (resp., player I) uses the control $\tilde{v} \in \mathcal{V}$ (resp. $\tilde{u} \in \mathcal{U}$).

Since we are now handling a game with a non-zero-sum payoff structure, we must adapt the definitions of the non-local intervention operators (definition 3.6) to the following:

**Definition 7.1.**

Let $\phi : [t, \tau_S] \times \mathbb{R}^p \to \mathbb{R}$ s.th. $\phi \in C([t, \tau_S]; \mathbb{R}^p)$. Let $\tau$ be some $\mathcal{F}$–measurable stopping time and $\mathcal{A}$ (resp., $\mathcal{B}$) let be a non-anticipative strategy on $[t, \tau_S]$ for player $i$. We define the [non-local] Player I-intervention operator $\mathcal{M}_i : H \to \mathcal{H}$ acting at some $\mathcal{F}$–measurable stopping time $\tau \in [t, \tau_S]$ by the following expression:

$$\mathcal{M}_i[\phi] := \sup_{z \in \mathcal{Z}} \phi(\Gamma(X(\tau-), z)) - c_i(\tau, z) \cdot 1_{\{\tau \leq \tau_S\}}$$

where $\Gamma : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ is the impulse response function.

**Definition 7.2. [Nash Equilibrium]**

We say that a pair $\tilde{u} \in \mathcal{U} \times \mathcal{V}$ is a Nash equilibrium of the stochastic differential game with impulse controls $\tilde{u} = [\tilde{\tau}_j, \tilde{\xi}_j]_{j \in \mathcal{E}} \in \mathcal{U}$, $\tilde{v} = [\tilde{\rho}_m, \tilde{\eta}_m]_{m \in \mathcal{E}} \in \mathcal{V}$ if the following statements hold:

$$J_1^{(\tilde{u}, \tilde{v})}(x) \geq J_1^{(\tilde{u}, \tilde{v})}(x) \text{ for all } u \in \mathcal{U} \text{ and } \forall x \in [t, \tau_S] \times \mathbb{R}^p$$

and

$$J_2^{(\tilde{u}, \tilde{v})}(x) \geq J_2^{(\tilde{u}, \tilde{v})}(x) \text{ for all } v \in \mathcal{V} \text{ and } \forall x \in [t, \tau_S] \times \mathbb{R}^p$$

Condition (i) states that given some fixed player II control policy $\tilde{v} \in \mathcal{V}$ player I cannot profitably deviate from playing the control policy $\tilde{u}$. Analogously, condition (ii) is the equivalent statement given the player I’s control policy is fixed as $\tilde{u}$, player II cannot profitably deviate from $\tilde{v}$. We therefore see that $(\tilde{u}, \tilde{v})$ is an equilibrium in the sense of a Nash
equilibrium since for both players, given their opponent plays the equilibrium policy, neither player has an incentive to deviate.

We generalise our zero-sum Theorem 6.1 to cover non-zero-sum payoff structure with the use of a Nash Equilibrium solution concept.

**Theorem 7.3.** [Verification Theorem for Non-Zero-Sum Games with Impulse Control]

Let us suppose that conditions (i), (iii)-(iv) of Theorem 6.1 hold and that the value of the game exists and that there exists a functions \( \phi_i, i \in \{1, 2\} \) s.th. \( \phi_i \) that satisfy technical conditions (T1) - (T4) and the following conditions:

(i') \( \phi_i \geq M_i \phi_i \) on \( S \) and the regions \( D_i \) are defined by:

\[
D_i = \{ X \in S; \phi_i(X) > M_i \phi_i(X) \}, i \in \{1, 2\} \text{ where we refer to } D_1 \text{ (resp., } D_2 \text{) as the player I (resp., player II) continuation region.}
\]

(ii') \[
\frac{\partial \phi_i}{\partial s} + \mathcal{L}_1 (X \cdot u, \tilde{v}(\cdot)) + f_1 (X \cdot \hat{u}, \hat{v}(\cdot)) \geq 0 \quad \forall u \in U, X \in S \setminus D_1.
\]

(iii') \[
\frac{\partial \phi_i}{\partial s} + \mathcal{L}_2 (X \cdot \hat{u}, \hat{v}(\cdot)) + f_2 (X \cdot \hat{u}, \hat{v}(\cdot)) \geq 0 \quad \forall v \in \mathcal{V}, X \in S \setminus D_2.
\]

(iv') \[
\frac{\partial \phi_i}{\partial s} + \mathcal{L}_i (X \cdot \hat{u}, \hat{v}(\cdot)) + f_i (X \cdot \hat{u}, \hat{v}(\cdot)) = 0 \quad \forall X \in D_1 \cap D_2.
\]

(v') \( \hat{\xi}_i \in \text{argsup}_{x \in \mathcal{Z}} \phi_i (\Gamma (\tau_{k_i} - \cdot), z) - c (\tau_{k_i}, z) \) is a Borel Measurable selection and similarly, \( \hat{\eta}_j \in \text{argsup}_{x \in \mathcal{Z}} \phi_j (\Gamma (\tau_{k_j} - \cdot), z) - c (\rho_{j_i}, z) \) is a Borel Measurable selection.

Put \( \hat{\tau}_0 \equiv t \) and define \( \hat{u} := [\hat{\tau}_j, \hat{\xi}_j]_{j \in \mathbb{N}} \) inductively by \( \hat{\tau}_{j+1} = \inf \{ s > \hat{\tau}_j; X^{u, \hat{v}(\cdot)}(\cdot) \notin D_1 \} \lor \tau_S \forall v \in \mathcal{V} \) similarly, put \( \hat{\rho}_0 \equiv t \) and define \( \hat{\nu} := [\hat{\rho}_{m_i}, \hat{\eta}_{m_i}]_{m \in \mathbb{N}} \) inductively by \( \hat{\rho}_{m+1} = \inf \{ s > \hat{\rho}_m; X^{u, \hat{v}(\cdot)}(\cdot) \notin D_2 \} \lor \tau_S \forall u \in \mathcal{U} \).

Then \((\hat{u}, \hat{v})\) is a Nash equilibrium for the game, that is to say the following statements hold:

\[
\phi_1 (x) = \sup_{u \in \mathcal{U}} J_1^{(u, \hat{v})} (x) = J_1^{(\hat{u}, \hat{v})} (x), \quad (53)
\]

and

\[
\phi_2 (x) = \sup_{v \in \mathcal{V}} J_2^{(u, \hat{v})} (x) = J_2^{(\hat{u}, \hat{v})} (x).
\]

\( \forall x \in [t, \tau_S] \times \mathbb{R}^p \).

The proof of Theorem 7.3 follows a similar path to that of Theorem 6.1. We therefore defer the proof of the theorem to the appendix.

In an analogous manner to Corollary 6.2, we can readily arrive at the following corollary to Theorem 7.3:

**Corollary 7.4.**

The sample space aligns into three regions that represent a region in which player I intervenes \( I_1 \), a region in which player II intervenes \( I_2 \), and a region in which no action is taken by either player I or player II; moreover the three regions are characterised by the following expressions for \( j \in \{1, 2\} \):

\[
I_j = \{ x \in [t, \tau_S] \times \mathbb{R}^p : V_j (x) = \mathcal{M}_j V_j (x); \mathcal{L} V_j (x) + f_j (x) \geq 0 \}.
\]

\[
I_3 = \{ x \in [t, \tau_S] \times \mathbb{R}^p : V_j (x) \geq \mathcal{M}_j V_j (x); \mathcal{L} V_j (x) + f_j (x) = 0 \}.
\]

**8 The Duopoly Investment Problem Revisited**

In this section we apply the results of section 7 to prove Theorem 4.3. Let us denote by \( Y \) the process \( Y(s) = (s + t_0, X_1(s), X_2(s)) \), where \( X_1, X_2 \in \mathcal{C}([t, \tau_S], \mathbb{R}^p) \) are processes which represent the market share processes for Firm 1 and Firm 2 respectively and whose evolution is described by (53) - (54). We wish to fully characterise the optimal investment strategies for each firm, in order to do this we will invoke 7.3. We will restrict ourselves to the case when \( \theta_{ij}(s) \equiv \theta_{ij} \in \mathbb{R} \) and \( \sigma_{ij}(s) \equiv \sigma_{ij} \in \mathbb{R} \).

We firstly make the following important observations:

Given some test function \( \phi \in \mathcal{C}^1 \mathcal{L} ([t, \tau_S], \mathbb{R}) \) the infinitesimal generator \( \mathcal{L} \) associated to \( Y \) is
for some $\epsilon > 0$.

Indeed (61) is immediately observed using (iv') of 7.3. This implies that:

$$
\pi \forall y \equiv (t, x_1, x_2) \in [t, \tau_s] \times \mathbb{R} \times \mathbb{R}.
$$

Given an admissible Firm 1 (resp., Firm 2) investment policy $u = [\tau_j, \xi_j]_{j \in \mathbb{N}} \in \mathcal{U}$ (resp., $v = [\rho_m, \eta_m]_{m \in \mathbb{N}} \in \mathcal{V}$) we note that the following identities hold:

$$X_1(\tau_j) = \Gamma(X_1(\tau_j -) + \Delta_N X_1(\tau_j), \xi_j) = \tilde{X}_1(\tau_j) + \xi_j. \quad (55)$$

$$X_2(\rho_m) = \Gamma(X_2(\rho_m -) + \Delta_N X_2(\rho_m), \eta_m) = \tilde{X}_2(\rho_m) + \eta_m. \quad (56)$$

The Firm 1 and Firm 2 investments are given by the following expressions $\forall x \in \mathbb{R}$:

$$\mathcal{M}_1 \psi = \sup_{x \in \mathbb{R}} \{\psi(\tau, x + \xi) - (\lambda_1 \xi + \kappa_1)\}. \quad (57)$$

$$\mathcal{M}_2 \psi = \sup_{x \in \mathbb{R}} \{\psi(\tau, x + \eta) - (\lambda_2 \eta + \kappa_2)\}. \quad (58)$$

where $\tau \in [t, \tau_s]$ is some $\mathcal{F}$-measurable stopping time.

Recall that the Firm 1 and the Firm 2 profit functions are given by:

$$\Pi_1(y; u, v) = \mathbb{E}[y] \left[ \int_{t_0}^{\tau_s} e^{-\epsilon(t - r)}[\alpha_1 X_1(t_1 -) + \alpha_2 X_2(t_2 -)] e^{-\epsilon(t - r)} dr + \beta \right]$$

$$- \sum_{j \geq 1} c_1(\tau_j, \xi_j) \cdot 1_{\{\tau_j \leq \tau_s\}} + \gamma_1 e^{-\epsilon \tau_s} [X_1(\tau_s)]^2 [X_2(\tau_s)]^2. \quad (59)$$

$$\Pi_2(y; u, v) = \mathbb{E}[y] \left[ \int_{t_0}^{\tau_s} e^{-\epsilon(t - r)}[\alpha_2 X_1(t_1 -) + \alpha_2 X_2(t_2 -)] e^{-\epsilon(t - r)} dr + \beta \right]$$

$$- \sum_{m \geq 1} c_2(\rho_m, \eta_m) \cdot 1_{\{\rho_m \leq \tau_s\}} + \gamma_2 e^{-\epsilon \tau_s} [X_1(\tau_s)]^2 [X_2(\tau_s)]^2 \right]. \quad (60)$$

where $x_i := X_i(t_i) \in \mathbb{R}$ and $t_i \in \mathbb{R}^+$ is the starting point for Firm $i$.

Given the setup of Theorem 7.3., at time $s \in [t_0, \tau_s]$ our running cost $f_i$ is now given by:

$$e^{-\epsilon(t - s)}[\alpha_i X_i(t_i -) + \beta_i X_i(t_i -)]; i, j \in \{1, 2\};$$

the player $i$ intervention costs are given by: $c_i(\tau, \xi) = \lambda_i \xi + \kappa_i$ for some intervention time $\tau \in [t_0, \tau_s]$ and intervention $\xi \in \mathbb{Z}$ and the player $i$ terminal reward is given by: $G_i(Y(\tau_s)) = \pi_i \gamma_i e^{-\epsilon \tau_s} [X_i(\tau_s)]^2 [X_i(\tau_s)]^2$.

We can now apply the conditions of 7.3., to show that the value function is a solution to the following Stefan problem:

$$\mathcal{L} \phi_i(y) + f_i = 0, \quad \forall y \in D = D_1 \cap D_2 \quad (61)$$

$$\frac{\partial}{\partial x}(x_1 + x_2) = e^{-\epsilon t} \lambda_1, \quad \forall y \notin D = D_1 \cap D_2 \quad (62)$$

Indeed (61) is immediately observed using (iv') of 7.3. This implies that:

$$0 = \alpha_1 e^{-\epsilon t} x_1 - \beta_1 e^{-\epsilon t} x_2 + \frac{\partial \phi_1}{\partial t} + \sum_{j=1}^{2} \mu_j \frac{\partial \phi_1}{\partial x_j} + \frac{1}{2} \sum_{i,j=1}^{2} \sigma_{ij} \frac{\partial^2 \psi_1(x_1, x_2)}{\partial x_i \partial x_j}$$

$$+ \int_{R} \{\phi_1(s, x_1 + \theta x_2 + \theta x_2 + \theta x_2) - \phi_1(s, x_1, x_2) - \theta x_1 \frac{\partial \phi_1}{\partial x_1} - \theta x_2 \frac{\partial \phi_1}{\partial x_2} \} \nu_j(dz). \quad (63)$$

We now try a candidate for the function, i.e. we specify the form:

$$\phi_1(y) = e^{-\epsilon t} \psi_1(x_1, x_2). \quad (64)$$

for some $\epsilon > 0$. 

19
We wish to determine the value $z$.

We now derive (62) and in doing so we shall determine $x$ by the following expression:

\[
G_{i,j,k} \xi \forall \theta_{1j} z, x_2 + \theta_{2j} z) - \psi(x_1, x_2) - z \theta_{ij} \frac{\partial \psi_1(x_1, x_2)}{\partial x_1} - z \theta_{2j} \frac{\partial \psi_1(x_1, x_2)}{\partial x_2} \psi_j(dz) = 0.
\]

(65)

Let us now suppose that $\psi(x_1, x_2) \equiv \psi_1(x_1) + \psi_2(x_2)$ hence using (65) we deduce that:

\[
\alpha_1 x_1 - \beta_1 x_2 + \sum_{i=1}^{2} \left\{ - \epsilon \psi_i(x_i) + \mu_i \frac{\partial \psi_i(x_i)}{\partial x_i} + \frac{1}{2} \sum_{i=1}^{2} \sigma_{ij} \frac{\partial^2 \psi_i(x_i)}{\partial x_i} \right\} + \sum_{i,j=1}^{2} \int_{\mathbb{R}} \left[ \psi_i(x_i + \theta_{ij} z) - \psi_i(x_i) - z \theta_{ij} \frac{\partial \psi_i(x_i)}{\partial x_i} \right] \psi_j(dz) = 0.
\]

(66)

After which we find that $\psi_1$ is a solution to

\[
h(y) = A_1 e^{r_1 x_1} + A_2 e^{r_2 x_1} + \frac{\alpha_1}{e} x_1 + B_1 e^{r_1 x_2} + B_2 e^{r_2 x_2} - \frac{\beta_1}{\epsilon} x_2 + \frac{1}{2 \epsilon^2} (\mu_1 \alpha_1 - \mu_2 \beta_1),
\]

(67)

where $A_1, A_2, B_1, B_2 \in \mathbb{R}$ are unknown constants and $r_1$ and $r_2$ are roots of the equation:

\[
q(r_k) := \frac{1}{2} r_k^2 + \mu_i r_k - \epsilon + \int_{\mathbb{R}} \{ e^{r_k \theta_{ij}} - 1 - \theta_{ij} r_k z \} \psi_j(dz),
\]

(68)

for $i, j, k \in \{1, 2\}$.

W.log. let us set $r_1 < r_2$. Now since $\lim_{r \to -\infty} q(r) = \infty$ and $q(0) = -\epsilon < 0$ and since $\forall r, z$ we have that: $\{ e^{r_k \theta_{ij}} - 1 - \theta_{ij} r_k z \} \psi_j(dz) > 0$ we find that:

\[
|r_1| > r_2.
\]

(69)

and

\[
r_1 < 0 < r_2.
\]

(70)

Our ansatz for the continuation region $D_1$ is that it takes the form:

\[
D_1 = \{ x_1 > x_1^\ast | x_1, x_1^\ast \in \mathbb{R} \}.
\]

(71)

We now derive (62) and in doing so we shall determine $x_1^\ast$.

Now for all $x_1 \leq x_1^\ast$ we have that:

\[
\psi_1(x_1, x_2) = M_1 \psi_1(x_1, x_2) = \sup_{z \in \mathbb{Z}} \{ \psi_1(x_1 + z, x_2) - (\kappa_1 + \lambda_1 z) \}.
\]

(72)

We wish to determine the value $z$ that maximises (72), hence let us now define the function $G$ by the following expression:

\[
G(\xi) = \psi_1(x_1 + \xi, x_2) - (\kappa_1 + \lambda_1 \xi)
\]

(73)

$\forall \xi \in \mathbb{Z}, x_1, x_2 \in \mathbb{R}$.

We now seek to evaluate the maxima of (73) hence when:

\[
G'(\xi) = 0.
\]

(74)

We see that the following expression holds $\forall x_1, x_2 \in \mathbb{R}, \xi \in \mathbb{Z}$:

\[
\psi_1'(x_1 + \xi, x_2) = \lambda_1.
\]

(75)

Let us now consider a unique point $\hat{x}_1 \in (0, x_1^\ast)$ then:

\[
\psi_1'(\hat{x}_1, x_2) = \lambda_1.
\]

(76)
Hence, we have that
\[ \dot{x}_1 = x_1 + \dot{\xi}(x_1) \text{ or } \dot{\xi}(x_1) = \dot{x}_1 - x_1. \]  
(77)

We therefore deduce that for \( x \in (0, x_1^* \) we have that:
\[ \psi_1(x_1, x_2) = \psi_1(\dot{x}_1, x_2) - \kappa_1 + \lambda_1(x_1 - \dot{x}_1). \]  
(78)

Using (75) - (76) and (78) and inserting (67) we can construct the following system of equations:
\[ A_1 r_1 e^{r_1^* z^*} + A_2 r_2 e^{r_2 z^*} + \frac{\alpha_1}{\epsilon} = \lambda_1, \]  
(79)
\[ A_1 r_1 e^{r_1^* \dot{x}_1} + A_2 r_2 e^{r_2 \dot{x}_2} + \frac{\alpha_1}{\epsilon} = \lambda_1, \]  
(80)
\[ A_1 (e^{r_1^* x_1} - e^{r_1^* \dot{x}_1}) + A_2 (e^{r_2^* x_2} - e^{r_2^* \dot{x}_2}) = -\kappa_1 + \left( \lambda_1 - \frac{\alpha_1}{\epsilon} \right) (x_1^* - \dot{x}_1). \]  
(81)

Repeating the above steps for \( \phi_2 \) leads to an analogous set of equations as (79) - (81) with \((A_1, A_2, x_1^*, \dot{x}_1, \alpha_1, \lambda_1) \) replaced by \((B_1, B_2, \dot{x}_2, \alpha_2, \lambda_2) \).

Now, since the system (59) - (60) is invariant under the transformations \( \{ 1 \leftrightarrow 2 \} \) then we must have \( A_1 = B_1 (: = C_1) \) and \( A_2 = B_2 (: = C_2) \) (since (67) must still be a solution to (iv) after the transformation \( \{ 1 \leftrightarrow 2 \} \)). Hence, we are left with a system of 6 unknowns \((C_1, C_2, x_1^*, \dot{x}_1, x_2^*, \dot{x}_2) \) and 6 equations. We can therefore uniquely determine the values \((C_1, C_2, x_1^*, \dot{x}_1, x_2^*, \dot{x}_2) \) - this proves Theorem 4.3.

9 Conclusion

Using standard assumptions, we proved a verification theorem for a stochastic differential game with impulse controls, then generalising the results to cover a non-zero-sum payoff structure where the appropriate equilibrium concept is a Nash equilibrium. Having characterised the value for the stochastic differential game, we then applied the results to characterise optimal investment strategies for the dynamic advertising duopoly problem described in section 2.

An interesting question for future research is investigating the above framework (in which the controllers use impulse controls to modify the state process) when either or both of the players only has access to partial information. Of particular interest is partial state information- that is, a system in which the state process is adapted to some subset of the canonical filtration. A single player impulse controller version of a framework in which players have partial state information was studied in [kS08], here the controller’s actions are subject to some execution delay so that there is some non-zero lag between the decision to apply an impulse intervention and the execution being carried out.

In [kS08] it is shown that the delayed reaction problem can be transformed into a sequence of no-delay optimal stopping problems and thus forming an equivalence between non-delay impulse control problems and delay impulse control problems. Naturally, investigating whether such an equivalence relationship between delay and non-delay problems in the controller-stopper framework serves as an interesting area of discussion.

Another form of partial information is that of incomplete information about the payoff criterion - continuous controller-stopper games in which the players have asymmetric information about some fixed payoff index is investigated in [Gru12].

10 Appendix

Technical Conditions for (T1) - (T4).

(T1) Assume that \( E[\int_t^{\tau_S} 1_{\partial D} (X^{=u}(s))ds] = 0 \) for all \( X \in S, u \in U \) where \( D \equiv D_1 \cup D_2 \).

(T2) \( \partial D \) is a Lipschitz surface - that is to say that \( \partial D \) is locally the graph of a Lipschitz continuous function: \( \phi \in C^2(S, \partial D) \) with locally bounded derivatives.

(T3) The sets \( \{ \phi^-(X^{=u}(\tau_m)) ; \tau_m \in [t, \tau_S], \forall m \in N \} \) and \( \{ \phi^-(X^{=u}(\rho)) ; \rho \in T \} \) are uniformly integrable \( \forall x \in S, u \in U. \)

(T4) \( E[|\phi(X^{=u}(\tau_m))| + |\phi(X^{=u}(\rho))| + \int_t^{\tau_S} |L\phi(X^{=u}(s))|ds] < \infty, \forall \) intervention times \( \tau_m \in [t, \tau_S], \rho \in T, u \in U. \)
Proof of Lemma 5.3.
Firstly, for \( s \in [t, \tau_S] \), let us set \( \Theta_i(s) \equiv \lambda_i, i \in \{1, 2\} \) and suppose that \( \xi(s) \equiv \nu^+_i(s) - \nu^-_i(s) \) and \( \eta(s) \equiv \nu^+_i(s) - \nu^-_i(s) \) for player I and player II controls (resp.) where \( \nu^+_i \) and \( \nu^-_i \), \( i \in \{1, 2\} \) are given by the using expressions:

\[
\nu^+_i(s) = \frac{1}{2} \left[ \sum_{j \geq 1} (\xi_j \cdot 1_{\{\xi_j > 0\}} + \lambda^{-1}_i \kappa_1 \cdot 1_{\{\tau_j \leq s\}} - \lambda^{-1}_i \kappa_1) \cdot 1_{\{\tau_j \leq s\}} \right],
\]

and similarly for the player II control:

\[
\nu^-_i(s) = \frac{1}{2} \left[ \sum_{j \geq 1} (\xi_j \cdot 1_{\{\xi_j < 0\}} + \lambda^{-1}_i \kappa_1 \cdot 1_{\{\tau_j \leq s\}} - \lambda^{-1}_i \kappa_1) \cdot 1_{\{\tau_j \leq s\}} \right]
\]

We do the proof for the player II impulse controls, the proof for the player I part is analogous. Hence, using (82) - (83) we readily deduce that:

\[
d\eta(s) = d\nu^+_i(s) - d\nu^-_i(s)
\]

\[
= \frac{1}{2} \sum_{m \geq 1} (\eta_m \cdot 1_{\{\eta_m > 0\}} + 2\lambda^{-1}_m \kappa_2 \cdot 1_{\{\rho_m \leq s\}} - \lambda^{-1}_m \kappa_2) \cdot \delta_{\rho_m}(s)
\]

Hence using the properties of the Dirac-delta function and by Fubini’s theorem we find,

\[
\int_t^{\tau_S} \Theta_2(s)d\eta(s) = \sum_{m \geq 1} \int_t^{\tau_S} (\lambda_2 \eta_m + (2 \cdot 1_{\{\rho_m \leq s\}} - 1)\kappa_2) \cdot \delta_{\rho_m}(s)
\]

\[
= \sum_{m \geq 1} \int_t^{\tau_S} (\lambda_2 \eta_m + (2 \cdot 1_{\{\rho_m \leq \infty\}} - 1)\kappa_2) \cdot \delta_{\rho_m}(s)
\]

\[
= \sum_{m \geq 1} \lambda_2 \eta_m + \kappa_2 = \sum_{m \geq 1} \chi(\rho_m, \eta_m) \cdot 1_{\{\rho_m \leq \tau_S\}},
\]

Lastly, we compute \( \eta(s) \), indeed we observe that:

\[
\eta(s) = \nu^+_2(s) + \nu^-_2(s)
\]

\[
= \sum_{m \geq 1} (\eta_m + \lambda^{-1}_m \kappa_2 \cdot 1_{\{\rho_m \leq s\}} - \lambda^{-1}_m \kappa_2) \cdot 1_{\{\eta_m > 0\}}
\]

\[
+ \sum_{m \geq 1} (\eta_m + \lambda^{-1}_m \kappa_2 \cdot 1_{\{\rho_m \leq s\}} - \lambda^{-1}_m \kappa_2) \cdot 1_{\{\rho_m \leq s\}} \cdot 1_{\{\eta_m < 0\}}.
\]

Now, since \( 1_{\{\rho_m \leq s\}} \cdot 1_{\{\rho_m \leq s\}} = 1_{\{\rho_m \leq s\}} \) we find that:

\[
\eta(s) = \sum_{m \geq 1} (\eta_m + \lambda^{-1}_m \kappa_2 - \lambda^{-1}_m \kappa_2) \cdot 1_{\{\rho_m \leq s\}} = \sum_{m \geq 1} \eta_m \cdot 1_{\{\rho_m \leq s\}} = \sum_{m \geq 1} \eta_m.
\]

Hence, after repeating the exercise for the player I controls (using that \( \xi(s) \equiv \nu^+_i(s) - \nu^-_i(s) \) and setting \( \nu(s) = \nu^+_i(s) - \nu^-_i(s) \) where \( \nu^+_i(s) \equiv \nu^+_i(s) + \nu^+_i(s) \), \( \nu^-_i \equiv \nu^-_i + \nu^-_i \) we recover the impulse control game.

Proof of Theorem 7.3.
We prove the theorem for player I with the proof for player II being the same up to a trivial modification.
As in the proof of Theorem 6.1, let us fix the player II control \( \hat{\tau} \in V \); we firstly appeal to Dynkin’s formula for jump diffusion processes hence for \( X = X^{u,\hat{v}} \) we have the following:
\[
\mathbb{E}[\phi_1(X^{t,x_0,0,\hat{v}}(T))] - \mathbb{E}[\phi_1(X^{t,x_0,0,\hat{v}}(T_{j+1}^-))] = -\mathbb{E} \left[ \int_{T_j}^{T_{j+1}} \frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,x_0,0,\hat{v}}(s)) \, ds \right].
\]

Summing from \( j = 0 \) to \( j = k \) implies that:
\[
\phi_1(x) + \sum_{j=1}^{k} \mathbb{E}[\phi_1(X^{t,x_0,0,\hat{v}}(T_j))] - \phi_1(X^{t,x_0,0,\hat{v}}(T_k^-)) = -\mathbb{E} \left[ \int_{t}^{T_{k+1}} \frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,x_0,0,\hat{v}}(s)) \, ds \right].
\]

Now, similarly reasoning as in the zero-sum case (c.f. \( 38 \)), we have that:
\[
\mathcal{M}_1 \phi_1(X^{t,x_0,0,\hat{v}}(T_j^-)) - \phi_1(X^{t,x_0,0,\hat{v}}(T_j^-)) + c_1(T_j, \xi_j)
\geq \phi_1(X^{t,x_0,0,\hat{v}}(T_j^-)) - \phi_1(X^{t,x_0,0,\hat{v}}(T_j^-)).
\]

Inserting \( 38 \) into \( 37 \) implies that:
\[
\phi_1(x) + \sum_{j=1}^{k} \mathbb{E}[\mathcal{M}_1 \phi_1(X^{t,x_0,0,\hat{v}}(T_j^-))] - \phi_1(X^{t,x_0,0,\hat{v}}(T_k^-))
\geq -\mathbb{E} \left[ \int_{t}^{T_{k+1}} \frac{\partial \phi_1}{\partial s} + \mathcal{L} \phi_1(X^{t,x_0,0,\hat{v}}(s)) \, ds \right].
\]

Additionally, by (II) we have that:
\[
\frac{\partial \phi}{\partial s} + \mathcal{L} \phi(X^{t,x_0,0,\hat{v}}(s))
\geq \frac{\partial \phi}{\partial s} + \mathcal{L} \phi(X^{t,x_0,0,\hat{v}}(s)) + f_1(X^{t,x_0,0,\hat{v}}(s)) - f_1(X^{t,x_0,0,\hat{v}}(s))
\geq -f_1(X^{t,x_0,0,\hat{v}}(s)).
\]

Or
\[
\left( \frac{\partial \phi}{\partial s} + \mathcal{L} \phi(X^{t,x_0,0,\hat{v}}(s)) \right) \leq f_1(X^{t,x_0,0,\hat{v}}(s)).
\]

Hence, inserting \( 39 \) into \( 38 \) yields:
\[
\phi_1(x) + \sum_{j=1}^{k} \mathbb{E}[\mathcal{M}_1 \phi_1(X^{t,x_0,0,\hat{v}}(T_j^-))] - \phi_1(X^{t,x_0,0,\hat{v}}(T_k^-)) + c(T_j, \xi_j) \cdot 1_{\{T_j \leq T_{\hat{v}}\}}
- \mathbb{E}[\phi_1(X^{t,x_0,0,\hat{v}}(T_{k+1}^-))]
\geq \mathbb{E} \left[ \int_{t}^{T_{k+1}} f_1(X^{t,x_0,0,\hat{v}}(s)) \, ds \right].
\]

Now, as in the proof for the zero-sum case, we have, using (ii) that
\[
\lim_{s \to T_{\hat{v}}} [\mathcal{M}_1 \phi_1(X^{t,x_0,0,v}(s)) - \phi_1(X^{t,x_0,0,v}(s))] = 0, \quad i \in \{1, 2\}
\lim_{s \to T_{\hat{v}}} \phi(X^{t,v,u}(s)) = G(X^{t,v,u}(T_{\hat{v}})) \quad \forall u \in U, v \in V \mathcal{P}-a.s.
\]
Hence, after taking the limit \( k \to \infty \), we recognize:
\[
\phi_1(x) \geq \mathbb{E} \left[ \int_{t}^{T_{\hat{v}}} f_1(X^{t,x_0,0,v}(s)) \, ds \right] - \sum_{j \geq 1} c(T_j, \xi_j) \cdot 1_{\{T_j \leq T_{\hat{v}}\}} + G(X^{t,v,u}(T_{\hat{v}})).
\]

or
\[
\phi_1(x) \geq J_1^{u,\hat{v}}(x).
\]

Since this holds for all \( u \in U \) we have for all \( x \in [t, T_{\hat{v}}] \times \mathbb{R}^p \)
\[
\phi_1(x) \geq \sup_{u \in U} J_1^{u,\hat{v}}(x).
\]
Now, applying the above arguments and fixing the pair of controls \((\hat{u}, \hat{v}) \in U \times V\) yields the following equality:

\[
\phi_1(x) = \sup_{u \in U} J_1^{(u, \hat{v})}(x) = J_1^{\hat{u}, \hat{v}}(x).
\] (94)

\(\forall x \in [t, \tau_S] \times \mathbb{R}^p\). After using an analogous argument for the player II policy \(v \in V\) (fixing the player I control as \(\hat{u}\)), we deduce that \(\forall x \in [t, \tau_S] \times \mathbb{R}^p\):

\[
\phi_2(x) = \sup_{v \in V} J_2^{(\hat{u}, v)}(x) = J_2^{\hat{u}, \hat{v}}(x).
\] (95)

after which we observe the following statements:

\[
\phi_2(x) = \sup_{v \in V} J_2^{(\hat{u}, v)}(x) = J_2^{\hat{u}, \hat{v}}(x).
\] (96)

\[
\phi_1(x) = \sup_{u \in U} J_1^{(u, \hat{v})}(x) = J_1^{\hat{u}, \hat{v}}(x).
\] (97)

from which we deduce that \((\hat{u}, \hat{v})\) is a Nash equilibrium and hence the thesis is proven.

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