On Curvature-Squared Corrections for D-brane Actions

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Abstract
Curvature squared corrections for D-brane actions in type II string theory were derived by Bachas, Bain and Green. Here we write down a generalisation of these corrections to all orders in $F$, the field strength of the U(1) gauge field on the brane. Some of these terms are needed to restore consistency with T-duality.
The low energy effective action for D-branes carries interesting information about the physics of D-branes as well as the space-time it lives in. The massless fields usually include a Yang-Mills multiplet, and the action for these fields to lowest order is then given by the standard Yang-Mills action. Many higher order corrections have been found, and in general we should expect every possible correction to be present unless it is forbidden by symmetries (or shown not to exist by calculation). There are however some corrections that can be calculated to all orders with relative ease and that carry interesting information about D-brane physics. The prime example of these are the $F^n$ corrections where $F$ is the field strength of the worldvolume gauge field. At tree level, these sum into the (Dirac)-Born-Infeld action, as has been established by several different arguments.

In this note we would like to write down tree level corrections for D-branes in type II string theory of the form $R^2 F^n$ to all orders in $F$. Here $R$ denotes the pull-back of the space-time curvature and we are referring to terms in the CP-even part of the action. There are similar terms in the Chern-Simons part of the action which are well known. In the limit $F \to 0$ we reduce to the curvature-squared corrections found in [2] (see also [3]). Since other corrections in $F$ to curvature-squared terms will involve derivatives, these terms can still be trusted for uniform near-critical field strengths.

While there may be several reasons for wanting to know such corrections, our interest arose from an apparent inconsistency with T-duality. When a D-brane is wrapped on a K3 surface, the curvature–squared term produces a correction to the tension of the brane after integration over the K3. In [4] the known terms in the Lagrangian

$$e^{-\phi} \sqrt{\det(1 + F)} + e^{-\phi} R^2$$

(1)

were used to get relations between the worldvolume gauge theory and parameters in the supergravity solution for such wrapped branes. It was pointed out in [5] that these formulae are not consistent with the $SO(20,4; Z)$ T-duality group of the K3. The resolution of this puzzle was also mentioned in [5]: in the presence of non-constant $F$, the $R^2$ corrections to the action should take the form

$$e^{-\phi} R^2 \to e^{-\phi} \sqrt{\det(1 + F)} R^2 + \ldots$$

(2)

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1 For some history and references, see [1].

2 We will be schematic here, more precise expressions appear below. Also, a factor of $2\pi\alpha'$ will be absorbed in $F$. 

It is not too hard to verify the presence of such terms; the factor of $\sqrt{\det(1 + F)}$ in front of $R^2$ comes from the partition function on the disk when there is a constant $F$ background. The ellipses stand for corrections involving $R$ and $F$ with cross-contractions. Now when a D-brane is wrapped on K3 with volume $V$ and the $R^2$ term is integrated, the reduced action is of the form

$$\left(\frac{V}{(2\pi\sqrt{\alpha'})^4} - 1\right) \int e^{-\varphi} \sqrt{\det(1 + F)}$$

and this restores consistency with T-duality. Thus the $R^2$ term, besides correcting the tension, also corrects a host of other couplings, such as the effective Yang-Mills coupling on the unwrapped part of the D-brane.

In the remainder we would like to try to fix the remaining $R^2$ corrections which involve cross-contractions with $F$ in (3). We will do this by trying to write down vertices that reproduce the string amplitude for the scattering of a graviton off a D-brane to order $\alpha'^2$.

The string amplitude that we will use is for the scattering of a graviton from a D-brane with constant electric-magnetic field on it. It was first written down in [6] but we have checked it independently. In order to write down the string amplitude, we need to introduce some notation. The graviton polarisation tensors will be denoted by $\epsilon_i$ and the graviton momenta by $p_i$. The polarisation tensors are symmetric and on-shell

$$\epsilon_{i\mu\nu} = \epsilon_{i\nu\mu}, \quad \epsilon_{i\mu} = 0,$$

$$\epsilon_{i\mu\nu} p_i^\mu = \epsilon_{i\mu\nu} p_i^\nu = 0 \forall i.$$  \hspace{1cm} (4)

Similarly the momenta satisfy $p_i^2 = 0$. This is compatible with turning on a constant $F$, since the curvature of the field $B + F$ is still identically zero and does not source the bulk Ricci tensor. In this paper we will also assume that the second fundamental form vanishes identically. We will comment on relaxing this assumption later.

We are interested in computing a disk amplitude with two graviton vertex operators inserted in the interior of the disk and an arbitrary number of photon vertex operators on the boundary of the disk. The boundary of the disk is restricted to lie along the worldvolume of a Dp-brane, and so are the photon momenta and polarisations. The polarisation tensors and momenta of the gravitons however have no such restrictions.

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3 For the similar computation of graviton scattering with $F = 0$, see [6].
fig. 1: (A): Relevant disk diagram. The following pictures arise in the low energy limit. (B): t-channel diagram. (C): s-channel diagram. (D): Contact interactions.

In principle one would need to compute infinitely many disk diagrams, with various numbers of photon vertex operators inserted on the boundary. However since we are assuming the photon field strength to be uniform, we may effectively sum over all the contractions between the gravitons and the photons at once by introducing an $F$ dependent propagator \[8\]. So we only need a single disk diagram with two graviton insertions in the interior. Let us put $\alpha' = 2$ in the next few equations. The amplitude is of the form

$$A \sim \int d^2z_1 d^2z_2 \langle V_1(z_1, \bar{z}_1)V_2(z_2, \bar{z}_2) \rangle.$$ 

We will take the first graviton vertex operator in the (-1,-1) picture and the second in the (0,0) picture:

$$V_1(z_1, \bar{z}_1) = \epsilon_{1\mu\nu} : e^{-\phi(z_1)}\psi^{\mu}(z_1)e^{ip_1 \cdot X(z_1)} : \epsilon^{-\phi(\bar{z}_1)}\bar{\psi}^{\nu}(\bar{z}_1)e^{ip_1 \cdot \bar{X}(\bar{z}_1)} :$$

$$V_2(z_2, \bar{z}_2) = \epsilon_{2\rho\sigma} : (\partial X^{\rho} + ip_2 \cdot \psi \psi^{\rho})e^{ip_2 \cdot X} : (\bar{\partial} \bar{X}^{\sigma} + ip_2 \cdot \bar{\psi} \bar{\psi}^{\sigma})e^{ip_2 \cdot \bar{X}} :$$

The contractions between leftmovers are the usual ones:

$$\langle X(z)^{\mu}X^{\nu}(z') \rangle = -\eta^{\mu\nu} \log(z - z')$$

$$\langle \bar{\psi}^{\mu}(z)\psi^{\nu}(z') \rangle = -\frac{\eta^{\mu\nu}}{z - z'}$$

$$\langle \phi(z)\bar{\phi}(z') \rangle = -\log(z - z')$$

and similarly for the rightmovers. The contractions between left- and rightmovers with indices will be modified due to the Dirichlet and $F$-dependent boundary conditions on the
propagators, so we will have
\[ \langle X(z)^\mu \bar{X}^\nu(z') \rangle = -D_F^{\mu\nu} \log(z - z') \]
\[ \langle \psi^\mu(z) \bar{\psi}^\nu(z') \rangle = -\frac{D_F^{\mu\nu}}{z - z'} \]
\[ \langle \phi(z) \bar{\phi}(z') \rangle = -\log(z - z') \] (8)

where
\[ D(F)_{\mu\nu} = \begin{cases} \frac{1-F}{1+F} & \text{for tangent indices} \\ -\delta^{\mu\nu} & \text{for normal indices} \end{cases} \] (9)

Notice that \( D(F)^T = D(F)^{-1} = D(-F) \), i.e. \( D(F) \) is an orthogonal matrix.

Let us also introduce the kinematic variables \( p_1 \cdot p_2 = -t/2 \) where \( t \) is interpreted as the momentum transfer between the incoming graviton and the brane, and \( p_1 \cdot D_F \cdot p_1 = p_2 \cdot D_F \cdot p_2 = 2q_F^2 \equiv -2s \). Then one finds the following amplitude (with a slight rearranging compared to [3]):
\[ A \sim T_p \frac{\Gamma(2q_F^2)\Gamma(-t/2)}{\Gamma(1+2q_F^2-t/2)} \sqrt{\det(\eta_{\mu\nu} + F_{\mu\nu})} \left( 2q_F^2 a_1^F - \frac{t}{2} a_2^F \right). \] (10)

with
\[ a_1^F = -\Tr(\epsilon_1 \cdot D_F)(p_1 \cdot \epsilon_2 \cdot p_1) - \Tr(\epsilon_2 \cdot D_F)(p_2 \cdot \epsilon_1 \cdot p_2) - 2q_F^2 \Tr(\epsilon_1 \cdot \epsilon_2) \\
+ (p_2 \cdot \epsilon_1 \cdot D_F \cdot \epsilon_2 \cdot p_1) + (p_2 \cdot \epsilon_1 \cdot D_F^T \cdot \epsilon_2 \cdot p_1) \\
+ (p_2 \cdot D_F \cdot \epsilon_1 \cdot \epsilon_2 \cdot D_F^T \cdot p_1) + (p_2 \cdot D_F^T \cdot \epsilon_1 \cdot \epsilon_2 \cdot D_F \cdot p_1) \\
- (p_1 \cdot D_F \cdot \epsilon_1 \cdot \epsilon_2 \cdot D_F^T \cdot p_2) - (p_1 \cdot D_F^T \cdot \epsilon_1 \cdot \epsilon_2 \cdot D_F \cdot p_2) \] (11)

and
\[ a_2^F = -\Tr(\epsilon_1 \cdot D_F)(p_1 \cdot \epsilon_2 \cdot p_1) - \Tr(\epsilon_2 \cdot D_F)(p_2 \cdot \epsilon_1 \cdot p_2) \\
+ \Tr(\epsilon_1 \cdot D_F)(p_1 \cdot D_F \cdot \epsilon_2 \cdot D_F \cdot p_1) + \Tr(\epsilon_2 \cdot D_F)(p_2 \cdot D_F \cdot \epsilon_1 \cdot D_F \cdot p_2) \\
- 2q_F^2 \Tr(\epsilon_1 \cdot \epsilon_2) - (2q_F^2 - t/2)\Tr(\epsilon_1 \cdot D)\Tr(\epsilon_2 \cdot D) + 2q_F^2 \Tr(\epsilon_1 \cdot D_F \cdot \epsilon_2 \cdot D_F) \\
+ (p_1 \cdot D_F^T \cdot \epsilon_1 \cdot \epsilon_2 \cdot D_F^T \cdot p_2) + (p_1 \cdot D_F \cdot \epsilon_1 \cdot \epsilon_2 \cdot D_F \cdot p_2) \\
- (p_1 \cdot D_F \cdot \epsilon_1 \cdot \epsilon_2 \cdot D_F^T \cdot p_2) - (p_1 \cdot D_F^T \cdot \epsilon_1 \cdot \epsilon_2 \cdot D_F \cdot p_2). \] (12)

\( T_p \) is the D-brane tension, excluding the factor of \( 1/g_c \). The amplitude is invariant under the replacement \( F \to -F \). The factor of \( \sqrt{\det(\eta_{\mu\nu} + F_{\mu\nu})} \) comes from the partition function on the disk with \( F \)-dependent boundary conditions, which multiplies the amplitude. Notice that the positions of the open string poles are changed when \( F \neq 0 \). This
indicates a change in the effective tension of an open string depending on direction. Since $F$ is uniform and translation symmetry is unbroken along the brane, linear momentum in the worldvolume directions is conserved, and one makes heavy use of this as well as the identities (4) in the course of the derivation.

Next we turn to effective actions. In the low energy limit we can expand the string amplitude, with $\alpha'$ restored, as

$$A \sim T_p \sqrt{\det(\eta_{\mu\nu} + F_{\mu\nu})} \left( -2s a_1^F - \frac{t}{2} a_2^F \right) \left( \frac{\alpha'}{2} \right)^2 \left( \frac{1}{\alpha'^2 st} - \frac{\pi^2}{6} + \mathcal{O}(\alpha') \right).$$

(13)

Now the leading term in this string amplitude can be reproduced by the usual Dirac-Born-Infeld action plus the $\sqrt{g} R$ bulk supergravity Lagrangian. This was essentially shown in [9]; this paper deals with the bosonic case, but the leading order terms in the string amplitudes (equations (22)-(24) in this paper) coincide with the superstring and so the calculation carries over. The next order contribution for the superstring has two extra powers of $\alpha'$. Here we expect no t-channel contributions from the effective action, since the relevant 3-point vertices in the bulk receive no higher order corrections (as one can check from the string amplitude for three massless NS bosons [10]). Similarly one can show there are no new contributions from the s-channel. As in the computations for the lowest order contributions to graviton scattering, all order contributions in $F$ to graviton-ripple mixing as well as propagators on the brane can be grouped together and described by the standard DBI action, so that the claim becomes that there are no new contributions beyond these. We can check this by computing a disk diagram with one graviton and one massless open string vertex operator in a constant $F$ background. The amplitude coincides with the leading order terms of the bosonic computation in equation (66) of [9] while the higher order terms are absent, and so we can again use the calculations of this paper to show that there are indeed no new contributions from the s-channel. Therefore all the contributions to the subleading term in (13) should be given by contact interactions. The usual DBI action does not contain the needed interactions at order $\alpha'^2$, so we are going to have to add them.

4 By leading order here we mean we have included all order contributions in $F$ but treat any additional corrections as higher order. For this reason we still keep a factor of $2\pi\alpha'$ absorbed in $F$. 
There is a vast amount of available interactions one could add. To try to simplify the task somewhat, let us consider the limit $F \to 0$. In this case the new interactions were first derived in [2]:

$$\Delta \mathcal{L} \sim (R^\alpha_{\gamma\delta} R_{\alpha\beta} R^\beta \gamma R - R_{\alpha \beta} R^\alpha \beta + 2 \hat{R}_{ab} R^{ab}).$$

(14)

The notation in this expression from [2] is: lower alphabet Greek letters for tangent indices, Roman letters for normal indices and a hat for contractions over the tangent indices while excluding the normal indices. Using vanishing of the Ricci tensor this may be rewritten in the following form:

$$R_{\alpha \beta \gamma \delta} R_{\alpha \beta} R_{\gamma \delta} - R_{\alpha \beta} R^\alpha \beta = \frac{1}{8} R_{\mu \nu \kappa \lambda} R_{\pi \rho \sigma \tau} g^{\mu \pi} D^{\nu \rho} (g + D)^{\kappa \sigma} (g + D)^{\lambda \tau}$$

$$-2 \hat{R}_{\alpha \beta} \hat{R}^{\alpha \beta} + 2 \hat{R}_{ab} \hat{R}^{ab} = -\frac{2}{8} R_{\mu \nu \kappa \lambda} R_{\pi \rho \sigma \tau} g^{\mu \pi} D^{\kappa \sigma} D^{\nu \lambda} D^{\rho \tau}.$$ 

(15)

By $D$ we mean $D(F = 0)$. This suggests one might be able to build the new interactions out of similar looking tensors by replacing $D$ by $D_F$ or $D_F^T$ and writing down all possible orderings. Since the string amplitude is symmetric under $F \to -F$ of course we want to keep the contact terms symmetric too. This yields the following combinations:

$$R^2_{(1)} = R_{abcd} R_{efgh} g^{ae} D_{F}^{bd} D_{F}^{fh} D_{F}^{cg} + \{F \to -F\}$$

$$R^2_{(2)} = R_{abcd} R_{efgh} g^{ae} D_{F}^{bd} D_{F}^{fh} D_{F}^{Tcg} + \{F \to -F\}$$

$$R^2_{(3)} = R_{abcd} R_{efgh} g^{ae} D_{F}^{Tbd} D_{F}^{fh} D_{F}^{cg} + \{F \to -F\}$$

$$R^2_{(4)} = R_{abcd} R_{efgh} g^{ae} D_{F}^{bd} D_{F}^{Tfh} D_{F}^{cg} + \{F \to -F\}$$

$$R^2_{(5)} = R_{abcd} R_{efgh} g^{ae} D_{F}^{Tbf} (g + D_F)^{cg} (g + D_F)^{dh} + \{F \to -F\}$$

$$R^2_{(6)} = R_{abcd} R_{efgh} g^{ae} D_{F}^{Tbf} (g + D_F)^{cg} (g + D_F)^{dh} + \{F \to -F\}$$

$$R^2_{(7)} = R_{abcd} R_{efgh} g^{ae} (g + D_F)^{bf} D_{F}^{cg} (g + D_F)^{dh} + \{F \to -F\}$$

$$R^2_{(8)} = R_{abcd} R_{efgh} g^{ae} (g + D_F)^{bf} D_{F}^{cg} (g + D_F)^{dh} + \{F \to -F\}$$

$$R^2_{(9)} = R_{abcd} R_{efgh} g^{ae} (g + D_F)^{bf} D_{F}^{Tcg} (g + D_F)^{dh} + \{F \to -F\}$$

$$R^2_{(10)} = R_{abcd} R_{efgh} g^{ae} (g + D_F)^{bf} D_{F}^{Tcg} (g + D_F)^{dh} + \{F \to -F\}$$

$$R^2_{(11)} = R_{abcd} R_{efgh} g^{ae} (g + D_F)^{bf} D_{F}^{Tcg} (g + D_F)^{dh} + \{F \to -F\}.$$ 

(16)

There are many ways to rewrite the above expressions, but we can always ask for the first index on each curvature tensor to be contracted with the metric. It turns out these expressions contain all the needed contact interactions. Notice that we included some
orderings that do not reduce to either of the terms in (15); however one cannot reproduce
the string amplitude without them. Leaving out the horrendous computation and simply
quoting the answer, we find that the effective Lagrangian is given by

\[ L = T_p e^{-\varphi} \sqrt{\det(g + F)} \left[ 1 - \frac{1}{24} \frac{(4\pi^2\alpha')^2}{32\pi^2} \times \right. \\
\left. \times \left( R_{(1)}^2 + R_{(2)}^2 + R_{(3)}^2 - R_{(4)}^2 - \frac{1}{2} R_{(5)}^2 - \frac{1}{2} R_{(6)}^2 + R_{(7)}^2 - R_{(8)}^2 \right) + \ldots \right] \] (17)

Given the way we have arrived at these corrections we cannot be absolutely sure
that there are no additional corrections of type \( R^2 \) that do not contribute to the string
amplitude. Indeed even for the \( R^2 \) corrections in [2] there was an additional Gauss-Bonnet
term for which the coefficient could not be fixed by the string amplitude and was fixed (to
zero) by other means. Given that our corrections reduce to the \( R^2 \) terms in [2] in the limit
\( F \to 0 \), any additional corrections must vanish in this limit and cannot contribute to the
amplitude. We have tried many generalisations other than (16) and these requirements
seem hard to satisfy, but we haven’t proven it is impossible.

There are other corrections to the D-brane action at order \( \alpha'^2 \). In reference [11]
corrections involving four derivatives of \( F \) where determined to all orders in \( F \), using
boundary state formalism. This implicitly contains corrections of type \( \Omega^4 \) where \( \Omega \) is the
second fundamental form, by applying T-duality. To our knowledge, other possible terms
such as \( R \cdot \Omega^2 \) have not been determined to all orders in \( F \).

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