OPEN SETS IN COMPUTABILITY THEORY AND REVERSE MATHEMATICS

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Abstract. In computational approaches to mathematics, open sets are generally studied indirectly via countable representations. For instance, an open set of real numbers with discontinuous characteristic function can be represented -or ‘coded’- by a sequence of open balls with rational center and radius. It is then a natural question whether the introduction of such codes changes the logical and computational properties of basic theorems pertaining to open sets. As we will see, sequential compactness seems unaffected by the use of codes, while (countable) open cover compactness is greatly impacted. Indeed, we identify numerous theorems for which the Main Question of Reverse Mathematics, namely which set existence axioms are necessary for a proving the theorem, does not have an (unique/unambiguous) answer when using the aforementioned characteristic functions in the stead of codes of open sets. In particular, we establish this for the Heine-Borel theorem (for countable covers) and the Heine, Urysohn, and Tietze theorems. We establish similar differences for the computational properties, in the sense of Kleene’s S1-S9, of these theorems, namely a shift from ‘computable’ to ‘not computable in any type two functional’. A finer study of representations of open sets leads to the new ‘Δ-functional’ which has unique computational properties. Finally, we also study the computational properties of Baire category theorem, resulting in similar results that however require very different proofs.

1. Introduction

1.1. Aim and motivation. It is a commonplace that the notion of open set is central to topology and fundamental to large parts of mathematics in ways that few notions can boast. Historical analysis dates back the concept of open set to Baire’s 1899 doctoral thesis, while Dedekind already considered this and related concepts twenty years earlier; the associated paper was published much later (9125).

In this paper, we study open sets in computability theory (Kleene’s S1-S9; see Section 2.2) and Reverse Mathematics (RM hereafter; see Section 2.1 for an introduction). Our motivation -in a nutshell- is that lots of extra data and structure is assumed on open sets in the various ‘computational’ approaches to mathematics, as detailed in Remark 1.1. For both foundational and mathematical reasons, it is then a natural question, and part of Shore’s [43, Problem 5.1], what the influence of this extra data and structure is.

As discussed in detail in Section 1.2, the addition of this extra data and structure has huge consequences for (countable) open-cover compactness, but not for sequential compactness. For instance, the Heine-Borel theorem for countable covers of closed sets in the unit interval (HBC hereafter) is rather ‘mundane’ when working
with open sets represented via sequences of open balls: \( \text{HBC} \) is provable from *weak König’s lemma* while the finite sub-cover in \( \text{HBC} \) is outright computable (via an unbounded search) in terms of the other data (see [41, IV.1]).

By contrast, working with (higher-order/possibly discontinuous) characteristic functions for open sets, the minimal comprehension axiom needed to prove \( \text{HBC} \) implies full second-order arithmetic, while the finite sub-cover is no longer computable in any type two functional. By contrast, weak König’s lemma plus *countable choice* still suffices to prove \( \text{HBC} \). As discussed in detail in Section 1.2 we may conclude that the *Main Question of RM*, namely which set existence axioms are necessary for a proving the theorem, does not have an (unique/unambiguous) answer for \( \text{HBC} \) formulated with characteristic functions, but rather depends (greatly) on the presence of countable choice. We obtain similar results for other basic theorems, like the Heine, Urysohn, and Tietze theorems, and the Baire category theorem.

We motivate our study of characteristic functions of open sets by the observation that e.g. \( \mathbb{R} \setminus \{0\} \) has an obvious representation via a sequence of open balls, but also a discontinuous characteristic function. In general, open sets are given in RM by formulas involving an existential numerical quantifier, and Kohlenbach has established the intimate connection between these formulas and discontinuous functions (see [21, §3]). In this light, the change from codes to characteristic functions is only a small step, yet has an immense impact on \( \text{HBC} \) and related theorems.

By the previous, a slightly different representation of open sets can have a huge effect on the associated theorems. It is then a natural question how strong the ‘coding principle’ is that expresses every characteristic function of an open set can be represented by a sequence of open balls, as well as how hard it is to compute (in the sense of Kleene’s S1-S9 schemes) this representation. In both cases, one needs a functional of which the existence implies full second-order arithmetic.

Moreover, a finer study of representations of open sets shall give rise to the new ‘\( \Delta \)-functional’. In a nutshell, the *unique \( \Delta \)-functional* converts between certain natural representations of open sets; \( \Delta \) also has unique computational properties, discussed below, in that it is natural, genuinely type 3, but does not add any computational strength to \( \exists^2 \), or equivalently, to Feferman’s \( \mu \), when it comes to computing functions from functions.

We discuss the aforementioned results in detail in Section 1.2. We finish this section with a remark on the use of representations of open sets.

**Remark 1.1 (Open sets and representations).** A set is *open* if it contains a neighbourhood around each of its points, and an open set can be written as a *countable* union of such neighbourhoods in separable spaces. In computational approaches to mathematics, open sets come with various constructive enrichments, as follows.

For instance, the neighbourhood around a point of an open set is often assumed to be given together with this point (see e.g. [3, p. 69]). This is captured by our representation (R.2) in Section 5. Alternatively, open sets are simply represented as countable unions (called ‘codes’ in [41 II.5.6], ‘names’ in [51 §1.3.4], and ‘presentations’ in [10]) of open neighbourhoods, i.e. a non-deterministic search yields the aforementioned neighbourhood of a point.

Moreover, there are a number of ‘effective’ results pertaining to such coded open sets, including the Urysohn lemma and Tietze theorems (see [41 II.7] and [51 §6.2]) in which the object claimed to exist can also be computed (in the sense of Turing)
Moreover, since WKL but much stronger comprehension axioms are needed in the absence of the latter.

1.2. Overview. We discuss the results to be obtained in Section 3 in some detail. We assume basic familiarity with RM and computability theory, while we refer to Section 2 for an introduction and the necessary technical details.

First of all, the equivalence between weak König’s lemma and the Heine-Borel theorem for countable covers of the unit interval, is one of the early results in RM, announced in [12] and to be found in [41 IV.1.2]. The same equivalence holds if we generalise the latter theorem to closed subsets of the unit interval by [5 Lemma 3.13]. Now, closed sets are the complements of open sets in RM, and an open set \( U \subset \mathbb{R} \) is represented as in (1.1). We then write the following for any \( x \in \mathbb{R} \):

\[
x \in U \text{ if and only if } (\exists n \in \mathbb{N}) (|x - a_n| < r_n).
\]

Secondly, the open set \( U_0 = \mathbb{R} \setminus \{0\} \) can (trivially) be represented as in (1.1), although \( U_0 \) has a discontinuous characteristic function. One also readily proves that given a discontinuous function on \( \mathbb{R} \), every open set \( U \) as in (1.1) has a characteristic function (see [21 §3]). In this light, working with (possibly discontinuous) characteristic functions for open sets seems to stay rather close to the representation (1.1) standard in RM, leading to the following definition; see Section 2.1 for RCA.

**Definition 1.2.** [Open sets in \( \text{RCA}_0^\omega \)] We let \( Y : \mathbb{R} \to \mathbb{R} \) represent open subsets of \( \mathbb{R} \) as follows: we write ‘\( x \in Y \)’ for ‘\( |Y(x)| > 0 \)’ and call a set \( Y \subset \mathbb{R} \) ‘open’ if for every \( x \in Y \), there is an open ball \( B(x, r) \subset Y \) with \( r^0 > 0 \). A set \( Y \) is called ‘closed’ if the complement, denoted \( Y^c = \{ x \in \mathbb{R} : x \notin Y \} \), is open.

Note that for open \( Y \) as in the previous definition, the formula ‘\( x \in Y \)’ has the same complexity (modulo higher types) as (1.1). Hereafter, an ‘open set’ refers to Definition 1.2 while ‘RM-open set’ refers to (1.1). By [41 II.7.1], one can effectively convert between RM-open sets and (RM-codes for) continuous characteristic functions, i.e. (1.1) is included in Definition 1.2.

Thirdly, we let HBC be the higher-order theorem that for a closed set \( C \subset [0,1] \) (as in Definition 1.2) every countable cover of \( C \) has a finite sub-cover. We let \( \text{HBC}_{cm} \) be the associated second-order theorem based on closed sets as in (1.1). We now have the following, where \( \mathbb{Z}_2^\omega \) is a higher-order version of \( \mathbb{Z}_2 \) and HBU is the Heine-Borel theorem for uncountable covers (see Section 2.2 for definitions).

(a) The system \( \text{RCA}_0 \) proves \( \text{HBC}_{cm} \iff \text{WKL} \) (see [5 Lemma 3.13]).

(b) The finite sub-cover in \( \text{HBC}_{cm} \) is computable in terms of the closed set and the countable cover (see [23 §7.3.4]).

(c) The system \( \mathbb{Z}_2^\omega \) cannot prove HBC while \( \text{RCA}_0 + \text{QF-AC}_0^{0,1} \) proves HBC \( \iff \text{WKL} \) and \( \text{RCA}_0 \) proves \( (\exists^3) \rightarrow \text{HBU}_{\text{closed}} \rightarrow \text{HBC} \) (Section 2.1).

(d) The finite sub-cover in HBC is not computable in terms of the closed set, the countable cover, and any type two object (Section 3.1).

Note that by the final part of item (c), HBC is provable \textit{without countable choice}, but much stronger comprehension axioms are needed in the absence of the latter. Moreover, since \( \text{WKL} + \text{QF-AC}_0^{0,1} \) and \( \text{HBU}_{\text{closed}} \) are independent but have the same first-order strength, there is no unique set of minimal (comprehension) axioms that
prove HBC. Thus, the so-called Main Question of RM (see e.g. [11, p. 9]) does not have a unique/unambiguous answer for HBC. We identify a number of similar theorems in Sections 3.1 and 3.3, including those by Heine, Urysohn, and Tietze, that have the same properties.

We stress that the above ‘non-standard’ behaviour does not apply to sequential compactness: the latter property for closed sets as in Definition 1.2 in the unit interval is equivalent to \( \text{ACA}_0 \) over \( \text{RCA}_0 \) by Theorem 3.1. In other words, changing the coding of open sets does not seem to affect sequential compactness, but does greatly affect (countable) open-cover compactness.

Fourth, the previous results suggest that Definition 1.2, while quite close to the original 1.1, does provide a stronger notion of open set than 1.1. It is then a natural question how hard it is to prove the ‘coding principle’ \( \text{Open} \) (see Section 3.2) which expresses that every characteristic function as in Definition 1.2 has a representation in terms of basic open balls as in 1.1. In Section 3.3 we also show that \( \text{Open} \) is equivalent to the Urysohn and Tietze theorems for closed sets as in Definition 1.2. Moreover, one wonders how hard it is to compute (Kleene S1-S9; see Section 2.2) this representation in terms of the other data. As show in Section 3.2, in both cases we need a functional of which the existence implies full second-order arithmetic, while \( \text{Open} \) together with (higher-order) comprehension axioms yields a hierarchy parallel to the (inclusion based/higher-order) Gödel hierarchy (see 1.2).

Fifth, another prominent theorem about open sets is the Baire category theorem, studied in all the computational approaches to mathematics from Remark 1.1. We therefore study the computational properties of the Baire Category theorem in Section 4, based on the concept of open set as in Definition 1.2. In this way, a realiser for the Baire category theorem cannot be computed by any type two functional, while \( \text{Open} \) together with (higher-order) comprehension axioms yields a hierarchy parallel to the (inclusion based/higher-order) Gödel hierarchy (see 1.2).

Sixth, there are of course representations of open sets other than the ones provided by 1.1 and Definition 1.2. We study two such representations, called (R.2) and (R.3) in Section 5. Intuitively, realisers of the ‘\( \forall \exists \)'-definition of open sets are the representations as in (R.2), while in (R.3) an open set is given by the function showing that the complement is located. In this context, we study the unique functional \( \Delta \) which converts the former to the latter representation. The \( \Delta \)-functional has surprising computational properties (see Sections 2 and 5 for definitions):

(P1) \( \Delta \) is not computable in any type 2 functional, but computable in any Pincherle realiser, a class weaker than \( \Theta \)-functionals.

(P2) \( \Delta \) is unique, genuinely type 3, and adds no computational strength to \( \exists^2 \) in terms of computing functions from functions.

In Section 5.2 we also briefly discuss the computational complexity related to the Baire category theorem and HBC under the representation (R.2) from Section 5.1. We finish this section with a discussion of some of our previous results from [30], which were the starting point of this paper.

**Remark 1.3** (The Pincherle phenomenon). Pincherle’s theorem is one of the first local-global principles, originally proved around 1882 in [33, p. 67], and expresses that a locally bounded function, say on Cantor space, is bounded. We have shown in [30] that Pincherle’s theorem is closely related to (open cover) compactness, but
has fundamentally different logical and computational properties. For instance, Pincherle’s theorem, called PIT, in [30], satisfies the following:

- The system $Z_2^\omega$ cannot prove PIT, while $RCA_0^\omega + QF-AC^{0,1}$ proves $WKL \leftrightarrow PIT$ and $RCA_0^\omega$ proves $(\exists^7) \rightarrow HBU \rightarrow PIT$.
- A realiser $M$ for PIT cannot be computed (Kleene S1-S9) in terms of any type two functional.

Clearly, Pincherle’s theorem exhibits the same properties as HBC described in items (c) and (d) above, and we shall therefore say that HBC exhibits the Pincherle phenomenon, due to PIT being the first theorem identified as exhibiting the above behaviour, namely in [30].

Another way of interpreting the Pincherle phenomenon is as follows: it is claimed in [19] that ‘disasters’ happen in topology in absence of the Axiom of Choice. It is no leap of the imagination to claim that such disasters already happen for Pincherle’s theorem and HBC, i.e. in ordinary mathematics. Indeed, Pincherle’s theorem and HBC ‘should’ be equivalent to weak König’s lemma, or at least provable from relatively weak axioms, but this can only be guaranteed in the presence of countable choice.

2. Preliminaries

We introduce Reverse Mathematics in Section 2.1 as well as its generalisation to higher-order arithmetic, and the associated base theory $RCA_0^\omega$. We introduce some essential axioms in Section 2.2.

2.1. Reverse Mathematics. Reverse Mathematics is a program in the foundations of mathematics initiated around 1975 by Friedman ([11, 12]) and developed extensively by Simpson ([41]). The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics.

We refer to [44] for a basic introduction to RM and to [10][41] for an overview of RM. We expect basic familiarity with RM, but do sketch some aspects of Kohlenbach’s higher-order RM ([21]) essential to this paper, including the base theory $RCA_0^\omega$ (Definition 2.1). As will become clear, the latter is officially a type theory but can accommodate (enough) set theory via Definition 2.4.

First of all, in contrast to ‘classical’ RM based on second-order arithmetic $Z_2$, higher-order RM uses $L_\omega$, the richer language of higher-order arithmetic. Indeed, while the latter is restricted to natural numbers and sets of natural numbers, higher-order arithmetic can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, et cetera. To formalise this idea, we introduce the collection of all finite types $T$, defined by the two clauses:

(i) $0 \in T$ and (ii) If $\sigma, \tau \in T$ then $(\sigma \to \tau) \in T$,

where 0 is the type of natural numbers, and $\sigma \to \tau$ is the type of mappings from objects of type $\sigma$ to objects of type $\tau$. In this way, $1 \equiv 0 \to 0$ is the type of functions from numbers to numbers, and where $n + 1 \equiv n \to 0$. Viewing sets as given by characteristic functions, we note that $Z_2$ only includes objects of type 0 and 1.

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1 A realiser $M$ for PIT, called weak Pincherle realiser in [30], takes as input a functional $F^2$ that is locally bounded on $2^N$ together with a functional $G^2$ such that $G(f)$ is an upper bound for $F$ in the neighbourhood $[fG(f)]$ of $f \in 2^N$, and outputs an upper bound $M(F,G)$ for $F$ on $2^N$. 

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Secondly, the language \( L_\omega \) includes variables \( x^\rho, y^\rho, z^\rho, \ldots \) of any finite type \( \rho \in T \). Types may be omitted when they can be inferred from context. The constants of \( L_\omega \) include the type 0 objects 0, 1 and \( +_0, \times_0, =_0 \) which are intended to have their usual meaning as operations on \( \mathbb{N} \). Equality at higher types is defined in terms of ‘\( =_0 \)’ as follows: for any objects \( x^\tau, y^\tau \), we have
\[
[x =_\tau y] \equiv ((\forall z^i_1 \ldots z^i_k)[xz_1 \ldots z_k =_0 yz_1 \ldots z_k]),
\]
if the type \( \tau \) is composed as \( \tau \equiv (\tau_1 \to \ldots \to \tau_k \to 0) \). Furthermore, \( L_\omega \) also includes the \textit{recursor constant} \( R_\sigma \) for any \( \sigma \in T \), which allows for iteration on type \( \sigma \)-objects as in the special case \( (2.2) \). Formulas and terms are defined as usual. One obtains the sub-language \( L_{n+2} \) by restricting the above type formation rule to produce only type \( n + 1 \) objects (and related types of similar complexity).

\textbf{Definition 2.1.} The base theory \( \text{RCA}_0^{\omega} \) consists of the following axioms.

\text{(a)} Basic axioms expressing that 0, 1, \( +_0, \times_0, =_0 \) form an ordered semi-ring with equality \( =_0 \).

\text{(b)} Basic axioms defining the well-known \( \Pi \) and \( \Sigma \) combinators (aka \( K \) and \( S \) in \( [2] \)), which allow for the definition of \( \lambda \)-abstraction.

\text{(c)} The \textit{defining axiom of the recursor constant} \( R_0 \): For \( m^0 \) and \( f^1 \):
\[
R_0(f, m, 0) := m \quad \text{and} \quad R_0(f, m, n + 1) := f(n, R_0(f, m, n)). \tag{2.2}
\]

\text{(d)} The \textit{axiom of extensionality}: for all \( \rho, \tau \in T \), we have:
\[
(x^\rho \equiv y^\rho \rightarrow \varphi(x) =_\tau \varphi(y)). \tag{E_{\rho, \tau}}
\]

\text{(e)} The \textit{induction axiom for quantifier-free} \( m \)-formulas of \( L_\omega \).

\text{(f)} \text{QF-AC}^{1,0}: The \textit{quantifier-free Axiom of Choice} as in Definition \( (2.2) \).

\textbf{Definition 2.2.} The axiom \( \text{QF-AC} \) consists of the following for all \( \sigma, \tau \in T \):
\[
(\forall x^\sigma)(\exists y^\tau)A(x, y) \rightarrow (\exists y^\sigma)(\forall x^\sigma)A(x, y), \tag{QF-AC}^{\sigma, \tau}
\]
for any quantifier-free formula \( A \) in the language of \( L_\omega \).

We let \( \text{IND}^\omega \) be the induction axiom for all formulas in \( L_\omega \).

As discussed in \( [21] \), \( \text{RCA}_0^{\omega} \) and \( \text{RCA}_0 \) prove the same sentences ‘up to language’ as the latter is set-based and the former function-based. Recursion as in \( (2.2) \) is called \textit{primitive recursion}; the class of functionals obtained from \( R_\rho \) for all \( \rho \in T \) is called \textit{Gödel’s system} \( T \) of all (higher-order) primitive recursive functionals.

We use the usual notations for natural, rational, and real numbers, and the associated functions, as introduced in \( [21] \) p. 288-289].

\textbf{Definition 2.3} (Real numbers and related notions in \( \text{RCA}_0^{\omega} \)).

\text{(a)} Natural numbers correspond to type zero objects, and we use ‘\( n^0 \)’ and ‘\( n \in \mathbb{N} \)’ interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ‘\( q \in \mathbb{Q} \)’ and ‘\( <_\mathbb{Q} \)’ have their usual meaning.

\text{(b)} Real numbers are coded by fast-converging Cauchy sequences \( q(\cdot) : \mathbb{N} \rightarrow \mathbb{Q} \), i.e. such that \( (\forall n^0, i^0)(|q_n - q_{n+1}| <_\mathbb{Q} \frac{1}{2^i}) \). We use Kohlenbach’s ‘hat function’ from \( [21] \) p. 289 to guarantee that every \( q^1 \) defines a real number.

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\(^2\)To be absolutely clear, variables (of any finite type) are allowed in quantifier-free formulas of the language \( L_\omega \): only quantifiers are banned.
(c) We write ‘$x \in \mathbb{R}$’ to express that $x^1 := (q_1^1)$ represents a real as in the
previous item and write $[x](k) := q_k$ for the $k$-th approximation of $x$.

(d) Two reals $x, y$ represented by $q_1$ and $r_1$ are equal, denoted $x =_R y$, if
$(\forall n)((q_n - r_n) \leq 2^{-n+1})$. Inequality ‘$<_R$’ is defined similarly. We some-
times omit the subscript ‘$R$’ if it is clear from context.

(e) Functions $F : \mathbb{R} \to \mathbb{R}$ are represented by $\Phi^{1\to 1}$ mapping equal reals to equal
reals, i.e. extensionality as in $(\forall x, y \in \mathbb{R})(x =_R y \to \Phi(x) =_R \Phi(y))$.

(f) The relation ‘$x \leq_R y$’ is defined as in $(\ref{36})$ but with ‘$\leq_0$’ instead of ‘$=_0$’.
Binary sequences are denoted ‘$f^1, g^1 \leq_1 1$’, but also ‘$f, g \in C$’ or ‘$f, g \in 2^{\omega}$’.
Elements of Baire space are given by $f^1, g^1$, but also denoted ‘$f, g \in \mathbb{N}^\omega$’.

(g) For a binary sequence $f^1$, the associated real in $[0, 1]$ is $\tau(f) := \sum_{n=0}^{\infty} \frac{f(n)}{2^n}$.

(h) Sets of type $\rho$ objects $X^{\rho\to 0}, Y^{\rho\to 0}, \ldots$ are given by their characteristic
functions $F_{\rho}^{\omega\to 0} \leq_\rho 1$, i.e. we write ‘$x \in X$’ for $F_\rho(x) =_0 1$.

The following special case of item $(h)$ is singled out, as it will be used frequently.

**Definition 2.4.** [RCA$_0^\rho$] A ‘subset $D$ of $\mathbb{N}^\omega$ is given by its characteristic function
$F_{\rho}^D \leq_\rho 1$, i.e. we write ‘$f \in D$’ for $F_D(f) = 1$ for any $f \in \mathbb{N}^\omega$. Assuming extension-
ality on the reals as in item $(a)$, we obtain characteristic functions that represent
subsets of $\mathbb{R}$. Using pairing functions, it is clear we can also represent sets of finite
sequences (of reals), and relations thereon.

Next, we mention the highly useful ECF-interpretation.

**Remark 2.5** (The ECF-interpretation). The (rather) technical definition of ECF
may be found in $(\ref{47})$ p. 138, §2.6. Intuitively, the ECF-interpretation $[A]_{ECF}$ of a
formula $A \in \mathcal{L}_\omega$ is just $A$ with all variables of type two and higher replaced by count-
ably representable representations of continuous functionals. Such representations are also (equi-
valently) called ‘associates’ or ‘RM-codes’ (see $(\ref{20})$, §4). The ECF-interpretation
connects RCA$_0^\rho$ and RCA$_0$ (see $(\ref{21})$, Prop. 3.1) in that if RCA$_0^\rho$ proves $A$, then RCA$_0$
proves $[A]_{ECF}$ again ‘up to language’, as RCA$_0$ is formulated using sets, and $[A]_{ECF}$
is formulated using types, namely only using type zero and one objects.

In light of the widespread use of codes in RM and the common practise of
identifying codes with the objects being coded, it is no exaggeration to refer to
ECF as the canonical embedding of higher-order into second-order arithmetic. For
completeness, we list the following notational convention for finite sequences.

**Notation 2.6** (Finite sequences). We assume a dedicated type for ‘finite sequences
of objects of type $\rho$’, namely $\rho^*$. Since the usual coding of pairs of numbers goes
through in RCA$_0^\rho$, we shall not always distinguish between 0 and 0*. Similarly, we
do not always distinguish between ‘$s^\rho$’ and ‘$(s^\rho)^*$’, where the former is the ‘object
$s$ of type $\rho$’, and the latter is the ‘sequence of type $\rho^*$’ with only element $s^0$. The
empty sequence for the type $\rho^*$ is denoted by ‘$\emptyset_{\rho}$’, usually with the typing omitted.

Furthermore, we denote by ‘$|s| = n$’ the length of the finite sequence $s^\rho =
\langle s_0^\rho, s_1^\rho, \ldots, s_{n-1}^\rho \rangle$, where $|\emptyset| = 0$, i.e. the empty sequence has length zero. For
sequences $s^\rho, t^\rho$, we denote by ‘$s*t$’ the concatenation of $s$ and $t$, i.e. $(s*t)(i) = s(i)$
for $i < |s|$ and $(s*t)(j) = t(|s| - j)$ for $|s| \leq j < |s| + |t|$. For a sequence $s^{\rho^0}$, we define
$\pi N := \langle s(0), s(1), \ldots, s(N - 1) \rangle$ for $N^0 < |s|$. For a sequence $\alpha^{0\to \rho^0}$, we also write
$\pi N = \langle \alpha(0), \alpha(1), \ldots, \alpha(N - 1) \rangle$ for any $N^0$. By way of shorthand, $(\forall q^\rho \in Q^\rho)A(q)$
abbreviates $(\forall q^\rho < |Q|)A(Q(q))$, which is (equivalent to) quantifier-free if $A$ is.
2.2. Higher-order computability theory. As noted above, our main results will be proved using techniques from computability theory. Thus, we first make our notion of ‘computability’ precise as follows.

(I) We adopt ZFC, i.e. Zermelo-Fraenkel set theory with the Axiom of Choice, as the official metatheory for all results, unless explicitly stated otherwise.

(II) We adopt Kleene’s notion of higher-order computation as given by his nine clauses S1-S9 (see [23, 31]) as our official notion of ‘computable’.

Similar to [27, 31], one main aim of this paper is the study of functionals of type 3 that are natural from the perspective of mathematical practise. Our functionals are genuinely of type 3 in the sense that they are not computable from any functional of type 2. The following definition is then standard in this context.

Definition 2.7. A functional $\Phi^3$ is countably based if for every $F^2$ there is a countable set $X$ such that $\Phi(F) = \Phi(G)$ for every $G$ that agrees with $F$ on $X$.

Now, if $\Phi^3$ is computable in a functional of type 2, then it is countably based, but the converse does not hold. However, Hartley proves in [15] that, assuming the Axiom of Choice and the Continuum Hypothesis, if $\Phi^3$ is not computable, then there is some $F^2$ such that $\exists^3$ (see below) is computable in $\Phi$ and $F$. In other words, stating the existence of $\Phi$ brings us ‘close to’ $\mathbb{Z}^2_2$ (see below). In the sequel, we shall explicitly point out where we use the notion of countably based functional.

For the rest of this section, we introduce some existing functionals which will be used below. In particular, we introduce some functionals which constitute the counterparts of second-order arithmetic $\mathbb{Z}_2$, and some of the Big Five systems, in higher-order RM. We use the formulation from [21, 29].

First of all, $\text{ACA}_0$ is readily derived from:

$$\exists \mu^2(\forall f^1)(\exists n)(f(n) = 0) \rightarrow [(f(\mu(f)) = 0) \land (\forall i < \mu(f))f(i) \neq 0] \land (\forall n)(f(n) \neq 0 \rightarrow f(\mu(f)) = 0)],$$

and $\text{ACA}^\omega_0 \equiv \text{RCA}^\omega_0 + (\mu^2)$ proves the same sentences as $\text{ACA}_0$ by [18, Theorem 2.5]. The (unique) functional $\mu^2$ in $(\mu^2)$ is also called Feferman’s $\mu$ ([2]), and is clearly discontinuous at $f = 1, 1, \ldots$; in fact, $(\mu^2)$ is equivalent to the existence of $F: \mathbb{R} \rightarrow \mathbb{R}$ such that $F(x) = 1$ if $x > 0$, and 0 otherwise ([21, §3]), and to

$$\exists \varphi^2 \le_2 1)(\forall f^1)(\exists n)(f(n) = 0) \leftrightarrow \varphi(f) = 0].$$

(32)

Secondly, $\Pi^1_1-\text{CA}_0$ is readily derived from the following sentence:

$$\exists S^2 \le_2 1)(\forall f^1)(\exists g^1)(\forall n^0)(f(\varphi n) = 0) \leftrightarrow S(f) = 0],$$

and $\Pi^1_1-\text{CA}_0^\omega = \text{RCA}^\omega_0 + (S^2)$ proves the same $\Pi^1_1$-sentences as $\Pi^1_1-\text{CA}_0$ by [35, Theorem 2.2]. The (unique) functional $S^2$ in $(S^2)$ is also called the Suslin functional ([21]). By definition, the Suslin functional $S^2$ can decide whether a $\Sigma^1_2$-formula as in the left-hand side of $(S^2)$ is true or false. We similarly define the functional $S^2_k$ which decides the truth or falsity of $\Sigma^1_k$-formulas; we also define the system $\Pi^1_k-\text{CA}_0$ as $\text{RCA}^\omega_0 + (S^2_k)$, where $(S^2_k)$ expresses that $S^2_k$ exists. Note that we allow formulas with function parameters, but not functionals here. In fact, Gandy’s Superjump ([13]) constitutes a way of extending $\Pi^1_k-\text{CA}_0^\omega$ to parameters of type two. We identify the functionals $\exists^2$ and $S^2_0$ and the systems $\text{ACA}^\omega_0$ and $\Pi^1_k-\text{CA}_0^\omega$ for $k = 0$. 
Thirdly, full second-order arithmetic $\mathbb{Z}_2$ is readily derived from $\cup_k \Pi^1_k \text{-} \text{CA}_0^\omega$, or from:

$$\exists E^3 \leq 1 \exists Y^2 \exists f^1 Y(f) = 0 \iff E(Y) = 0,$$

and we therefore define $\mathbb{Z}^\Omega_2 \equiv \text{RCA}_0^\omega + (\exists^2)$ and $\mathbb{Z}^\omega_2 \equiv \cup_k \Pi^1_k \text{-} \text{CA}_0^\omega$, which are conservative over $\mathbb{Z}_2$ by [18 Cor. 2.6]. Despite this close connection, $\mathbb{Z}^\omega_2$ and $\mathbb{Z}^\Omega_2$ can behave quite differently, as discussed in e.g. [29, §2.2]. The functional from $(\exists^2)$ is also called ‘$\exists^3$’, and we use the same convention for other functionals.

Fourth, the Heine-Borel theorem states the existence of a finite sub-cover for an open cover of certain spaces. Now, a functional $\Psi : \mathbb{R} \to \mathbb{R}^+$ gives rise to the canonical cover $\cup_{x \in I} I^\Psi_x$ for $I \equiv [0, 1]$, where $I^\Psi_x$ is the open interval $(x - \Psi(x), x + \Psi(x))$. Hence, the uncountable cover $\cup_{x \in I} I^\Psi_x$ has a finite sub-cover by the Heine-Borel theorem; in symbols:

$$\forall \Psi : \mathbb{R} \to \mathbb{R}^+(\exists y_1, \ldots, y_k \in I)(\forall x \in I)(\exists i \leq k)(x \in I^\Psi_{y_i}).$$

(Note that HBU is almost verbatim Cousin’s lemma (see [8, p. 22]), i.e. the Heine-Borel theorem restricted to canonical covers. The latter restriction does not make much of a big difference, as studied in [37]. By [29, 30]. $\mathbb{Z}^\omega_2$ proves HBU but $\mathbb{Z}^\omega_2 + \text{QF-AC}^0$ cannot, and many basic properties of the gauge integral ([20, 44]) are equivalent to HBU.

Fifth, since Cantor space (denoted $C$ or $2^\omega$) is homeomorphic to a closed subset of $[0, 1]$, the former inherits the same property. In particular, for any $G^2$, the corresponding ‘canonical cover’ of $2^\omega$ is $\cup_{f \in 2^\omega} [\overline{\{G(f)\}}]$ where $[\sigma^0]$ is the set of all binary extensions of $\sigma$. By compactness, there is a finite sequence $\langle f_0, \ldots, f_n \rangle$ such that the set of $\cup_{i \leq n} \overline{\{G(f_i)\}}$ still covers $2^\omega$. By [29, Theorem 3.3], HBU is equivalent to the same compactness property for $C$, as follows:

$$\forall G^2(\exists f_1, \ldots, f_k \in C)(\forall f \in C)(\exists i \leq k)(f \in \overline{\{G(f_i)\}}).$$

We now introduce the specification SCF$(\Theta)$ for a (non-unique) functional $\Theta$ which computes a finite sequence as in HBU. We refer to such a functional $\Theta$ as a realiser for the compactness of Cantor space, and simplify its type to ‘$3’.

$$(\forall G^2)(\forall f^1 \leq 1 \exists g \in \Theta(G))(f \in \overline{\{G(g)\}}).$$

Clearly, there is no unique such $\Theta$ (just add more binary sequences to $\Theta(G)$) and any functional satisfying the previous specification is referred to as ‘a $\Theta$-functional’. As to its provenance, $\Theta$ was introduced as part of the study of the Gandy-Hyland functional in [36, §2] via a slightly different definition. These definitions are identical up to a term of Gödel’s $T$ of low complexity by [28, Theorem 2.6]. As shown in [29, §3], one readily obtains a realiser $\Theta$ from HBU if the latter is given; in fact, it is straightforward to establish HBU $\iff (\exists \Theta)\text{SCF}(\Theta)$ over $\text{ACA}_0^\omega + \text{QF-AC}^{0,1}$.

Sixth, a number of higher-order axioms are introduced in [39] including the following comprehension axiom (see also Remark [23]):

$$(\forall Y^2)(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})(n \in X \iff (\exists f^1)(Y(f, n) = 0)).$$

(BOOT)

We only mention that this axiom is equivalent to e.g. the monotone convergence theorem for nets indexed by Baire space (see [39, §3]). As it turns out, the coding principle Open (see Section [32]) is closely related to BOOT and fragments, as shown in Section [3.2.1]. We also mention some historical remarks related to BOOT.
**Remark 2.8** (Historical notes). Zeroth of all, BOOT is called the ‘bootstrap’ principle as it is weak in isolation (equivalent to ACA₀ under ECF, in fact), but becomes much stronger when combined with comprehension axioms: \( \Pi^1_k \text{-CA}_{\omega}^0 \) + BOOT readily proves \( \Pi^1_{k+1} \text{-CA}_{\omega}^0 \).

First of all, BOOT is definable in Hilbert-Bernays’ system \( H \) from the Grundlagen der Mathematik (see 17, Supplement IV)). In particular, one uses the functional \( \nu \) from 17, p. 479 to define the set \( X \) from BOOT. In this way, BOOT and subsystems of second-order arithmetic can be said to ‘go back’ to the Grundlagen in equal measure, although such claims may be controversial.

Secondly, after the completion of 39, it was observed by the second author that Feferman’s axiom (Proj1) from 10 is similar to BOOT. The former is however formulated using sets, which makes it more ‘explosive’ than BOOT in that full \( Z_2 \) follows when combined with (\( \mu^2 \)), as noted in 10, I-12).

3. The Heine-Borel theorem

We establish the results sketched in Section 1.2 for the (countable) Heine-Borel theorem and related theorems. We use the notion of open (and closed) set as outlined in Definition 1.2, unless explicitly stated otherwise.

3.1. **Sequential and open-cover compactness.** We show that sequential compactness behaves ‘as normal’ for our notion of closed sets, but that Heine-Borel compactness for countable covers behaves quite out of the ordinary. In particular, we show that this notion of compactness suffers from the Pincherle phenomenon.

We now establish the results in item (3) from Section 1.2 pertaining to HBC, and related results. First of all, Theorem 3.1 is a sanity check for Definition 1.2: our closed sets have the same properties as RM-codes for closed sets, as follows.

(a) RM-closed sets are sequentially closed, i.e. if a sequence in an RM-closed set converges to some limit, the latter is also in the set (trivial in RCA₀).

(b) RM-closed sets in \([0, 1]\) are sequentially compact in ACA₀ (5, Lemma 3.14).

(c) Given a sequence in an RM-closed set in \([0, 1]\), \( \exists^2 \) computes the limit (35).

The following theorem shows that our closed sets mirror these three items perfectly.

**Theorem 3.1.** The system RCA₀\( \omega \) proves that a closed set is sequentially closed. The system \( \text{RCA}_0^\omega \) proves the equivalence between \( \text{ACA}_0 \) and the statement a closed set in \([0, 1]\) is sequentially compact. The functional \( \exists^2 \) computes an accumulation point of a sequence in a closed set in \([0, 1]\).

Proof. For the first part, if a sequence \( x_n \) in a closed set \( C \subset \mathbb{R} \) converges to \( y \in \mathbb{R} \), but \( y \notin C \), then there is \( N^0 \) such that \( B(y, \frac{1}{2^n}) \subset C^c \), as the complement of \( C \) is open by definition. However, \( x_n \) is eventually in \( B(y, \frac{1}{2^n}) \) by definition, a contradiction.

For the second part, the reversal follows from considering the unit interval and 11, III.2.2]. For the forward direction, the usual proof of the Bolzano-Weierstrass theorem as in 11, III.2.1] goes through, modulo using (\( \exists^2 \)) to decide elementhood of closed sets. In case \( \neg(\exists^2) \), all \( \mathbb{R} \to \mathbb{R} \)-functions are continuous by 21, §3 and closed sets in \([0, 1]\) reduce to the usual RM-definition by 11, II.5.7 and 20, Prop. 4.10]. The second-order proof from 15, Lemma 3.14] now finishes this case. The law of excluded middle (\( \exists^2 \) \( \lor \neg(\exists^2) \)) finishes the proof. \( \square \)
The previous theorem can be generalised to other theorems pertaining to sequential compactness (see e.g. [11 III]), like the monotone convergence theorem for sequences in closed subsets of the unit interval.

Secondly, the previous theorem shows that our notion of closed sets has the usual properties when it comes to sequential compactness. We now show that the situation is markedly different for Heine-Borel compactness: Theorems 3.3 and 3.4 show that HBC, defined as follows, suffers from the Pincherle phenomenon.

**Definition 3.2.** [HBC] Let \( C \subseteq [0,1] \) be a closed set and let \( a_n, b_n \) be sequences of reals such that \( C \subseteq \bigcup_{n \in \mathbb{N}} (a_n, b_n) \). Then there is \( n_0 \) such that \( C \subseteq \bigcup_{n \leq n_0} (a_n, b_n) \).

We let \( \text{HBC}_m \) be HBC with \( C \) represented by RM-codes. By Theorem 3.3, HBC is provable without countable choice and has weak first-order strength. Indeed, \( \text{HBU}_{\text{closed}} \) is HBU generalised to closed sets \( C \subseteq [0,1] \), and both have the first-order strength of \( \text{WKL} \); this follows from applying ECF and noting [5, Lemma 3.13]. Furthermore, \( Z^2_2 \) proves \( \text{HBU}_{\text{closed}} \) in the same way as in [31 Theorem 4.2].

**Theorem 3.3.** Either \( \text{RCA}_0 + \text{WKL} + \text{QF-AC}^{0,1} \) or \( \text{RCA}_0^\omega + \text{HBU}_{\text{closed}} \) proves HBC.

**Proof.** For the first part, in case \( \neg (\exists^2) \), all functions on \( \mathbb{R} \) are continuous by [21 §3]. Following the results in [20 §4], continuous functions have an RM-code, i.e. our definition of open set reduces to an \( L_2 \)-formula in \( \Sigma^0_1 \), which (equivalently) defines a code for an open set by [11 II.5.7]. In this way, HBC is merely \( \text{HBC}_m \), which follows from WKL by [5 Lemma 3.13]. In case \( (\exists^2) \), let \( C \subseteq [0,1] \) be a closed set and let \( a_n, b_n \) be as in HBC. If there is no finite sub-cover, then \( (\forall m^0) (\exists x \in C) [x \notin \bigcup_{n \leq m} (a_n, b_n)] \). Apply QF-AC\(^{0,1} \) and \( (\exists^2) \) to obtain a sequence \( x_n \) of reals in \( C \) with this property. Since \( (\exists^2) \rightarrow \text{ACA}_0 \), any sequence in \( [0,1] \) has a convergent sub-sequence \( y_n \) by [11 III.2]. If \( y_n \) converges to \( y \notin C \), then there is \( N^0 \) such that \( B(y, \frac{1}{2^N}) \subseteq C^c \), as the complement of \( C \) is open by definition. However, \( x_n \) is eventually in \( B(y, \frac{1}{2^N}) \) by definition, a contradiction. Hence, \( \lim_{n \to \infty} y_n = y \in C \) but if \( y \in (a_k, b_k) \), then \( y_n \) is eventually in this interval, which contradicts the definition of \( x_n \) (and \( y_n \)). The law of excluded middle now finishes the proof.

For the second part, let \( C \subseteq [0,1] \) be a closed set and let \( a_n, b_n \) as in HBC. Similar to the first case, we may assume \( (\exists^2) \). Apply QF-AC\(^{1,0} \) and \( (\exists^2) \) to \( (\forall x \in C)(\exists n^0)(x \in (a_n, b_n)) \) to obtain \( \Psi^2 \) yielding \( n^0 \) from \( x \in C \). Then \( \bigcup_{x \in C} (a_{\Psi(x)}, b_{\Psi(x)}) \) is a cover of \( C \) that is readily converted to a canonical cover. We now obtain HBC from applying \( \text{HBU}_{\text{closed}} \).

In contrast to sequential compactness and Theorem 3.1, countable Heine-Borel compactness as in HBC either requires countable choice or lots of comprehension, namely by the following theorem.

**Theorem 3.4.** The system \( Z^2_2 \) cannot prove HBC.

**Proof.** We will modify the method used to prove [29 Theorem 3.4] and [30 Theorem 4.8]. As in those proofs, let \( A = \bigcup_{k \in \mathbb{N}} A_k \) be a countable set such that

(i) We have that \( \Pi^1_n \)-formulas are absolute for \( (A, \mathbb{N}^R) \) for all \( n \).
(ii) For all \( k \), we have \( A_k \subseteq A_{k+1} \) and there is \( f_k \in A_{k+1} \) enumerating \( A_k \).
(iii) For all \( k \), there is a sequence \( y_0, y_1, \ldots \) and a number \( n_k \) such that \( A_k \) is the computational closure of \( \{y_0, \ldots, y_{n_k}\} \) relative to \( S^k_2 \).
Since $A_k$ can be enumerated in $A_{k+1}$, there is an open set $O_k$ containing $A_k$ with an RM-code in $A_{k+1}$ and with measure bounded by $2^{-(k+1)}$. We let $U_k = A \cap \bigcup_{m \leq k} O_m$, and we consider the type structure $\mathcal{M} = \{M_n\}_{n \in \mathbb{N}}$ where we let:

- $M_0 = \mathbb{N}$ and $M_1 = A$,
- for $n > 1$, $\Phi \in M_n$ when $\Phi : M_{n-1} \to \mathbb{N}$ and for some $k \in \mathbb{N}$, $\Phi$ is computable in $\{U_i\}_{i \in \mathbb{N}}$, $S_k^2$ and $g_0, \ldots, g_{mk}$.

Note that Kleene-computations as in S1-S9 are interpreted inside $\mathcal{M}$.

Clearly, $\mathcal{M}$ satisfies that $\{U_k\}_{k \in \mathbb{N}}$ is an open covering of $A$. Now, each $U_k$ has a code in $A_{k+1}$, but there is no universal code for the whole sequence of open sets. Nonetheless, $(A, \{U_k\}_{k \in \mathbb{N}})$ satisfies

$$\forall f^1(\exists k)(x \in U_k) \land \forall f^1(\exists k)(f \in U_k \to (\exists n)(\forall g)(\forall n))(\forall \bar{g}n = \bar{g}n \to g \in U_k)),$$

i.e. an open covering of $A$ is indeed provided by $\bigcup_{k \in \mathbb{N}} U_k$.

To establish the theorem, we prove the following two key facts:

1. Each $U_k$ is a proper subset of $A$ for each $k$.
2. If $f$ is computable in $\{U_k\}_{k \in \mathbb{N}}$, some $S^2_k$, and a finite set from $A$, then $f \in A$.

For item (1), we use that $U_k$ has a code in $A_{k+1}$ witnessing that the measure of $\cup_{m \leq k} O_m$ is less than 1. In this case, there is an element outside $\cup_{m \leq k} O_m$ that is arithmetical in this code, and thus in $A_{k+1}$.

For item (2), it suffices to observe that $\{U_n\}_{n \in \mathbb{N}}$ restricted to $A_k$ is arithmetical in elements in $A_k$. To prove this observation, fix $g \in A_k$ and do the following to decide $g \in U_n$. If $n \geq k$, the answer is simply yes, since then $A_k \subseteq O_k \cap A \subseteq U_n$. If $n < k$, we use that $U_n$ has a code in $A_k$, and we can decide membership from this code. Given $k$, there are only finitely many codes to consider, so we are through.

To finish the proof, note that inside $\mathcal{M}$, $\cup_{m \leq n} U_m$ has measure strictly below 1, i.e. there is no finite sub-cover according to $\mathcal{M}$.

The previous results are not an isolated incident, as witnessed by the following.

Note that items (3) to (6) are studied in [11 VI.2] for RM-codes of closed sets, while e.g. items (7), (8), and (9) for RM-codes are studied in [6 §4].

**Theorem 3.5.** The following theorems imply HBC over $\text{RCA}_0^\omega$:

(a) Pincherle’s theorem for the unit interval.
(b) If $F^2$ is continuous on a closed set $D \subseteq \mathbb{N}$, it is bounded on $D$.
(c) If $F^2$ is continuous on a closed set $D \subseteq \mathbb{N}$, it is uniformly cont. on $D$.
(d) If $F$ is continuous on a closed set $D \subseteq [0, 1]$, it is bounded on $D$.
(e) If $F$ is continuous on a closed set $D \subseteq [0, 1]$, it is uniformly cont. on $D$.
(f) If $F$ is continuous on a closed set $D \subseteq [0, 1]$, it attains a maximum on $D$.
(g) If $F$ is continuous on a closed set $D \subseteq [0, 1]$, it is Riemann integrable there.
(h) If $F$ is continuous on a closed set $D \subseteq [0, 1]$, then for every $\varepsilon > 0$ there is a polynomial $p(x)$ such that $|p(x) - F(x)| < \varepsilon$ for all $x \in D$.

An equivalence holds for e.g. items (3) and (4) if additionally given $\text{ACA}_0$.

**Proof.** By the results in [30 §4] and [11 IV], all items from the theorem imply WKL. Moreover, in case $\neg(\exists^2)$, closed sets reduce to RM-codes for closed sets, i.e. HBC is just $\text{HBC}_{\text{rm}}$, which follows from WKL by [5, Lemma 3.13]. Similarly, items (1) and (2) reduce to their second-order counterparts, equivalent to $\text{ACA}_0$ by [11 IV.2.11]. Hence, we may assume $(\exists^2)$ in the following. For the first item, let
Proof. It is easily seen that \( \forall x \in C \exists n \forall y \in C \) \( x \in (a_n, b_n) \). Applying QF-AC\(^{1,0}\) and \( (\exists^2) \) (to decide whether \( x \in C \) or not), one obtains \( \Phi^2 \) such that \( \Phi(x) \) is the least such \( n \) if \( x \in C \). By definition, \( \Phi \) satisfies \( \Phi(y) \leq \Phi(x) \) for any \( x \in C \), \( y \in C \cap B(x, r) \), and small enough \( r > 0 \). Hence, the function \( f : R \to R \) defined as \( \Phi(x) \) if \( x \in C \) and 1 otherwise, is locally bounded on \([0, 1]\). By Pincherle’s theorem, \( f \) is bounded on \([0, 1]\), implying that \( \Phi \) is bounded on \( C \) and immediately yielding a finite sub-cover for \( \cup_{n \in N}(a_n, b_n) \).

For the second item, since Cantor space is homeomorphic to a closed subset of \([0, 1]\), HBC is equivalent to HBC for Cantor space. Let \( D \subseteq 2^N \) be a closed and let \( \sigma_n^0 \to_0^2 \) be a sequence of finite binary sequence covering \( D \), i.e. \( \forall f \in D \exists n \in [\sigma_n] \). Applying QF-AC\(^{1,0}\) and \( (\exists^2) \) (to decide whether \( x \in D \) or not), one obtains \( \Phi^2 \) such that \( \Phi(x) \) is the least such \( n \) if \( x \in D \). Define \( G(f) := |\sigma_\Phi(f)| \) and note:

\[
(\forall f, g \in 2^N)(\exists G(f) = G(f) \to \Phi(f) = \Phi(g)),
\]

i.e. \( \Phi \) is continuous with modulus of continuity \( G \). Item \([\text{I}]\) proves that \( \Phi \) is bounded on \( D \), yielding a finite sub-cover of \( \cup_{n \in N}[\sigma_n] \). Item \([\text{II}]\) now also readily implies HBC. For items \([\text{II}1]\) and \([\text{II}2]\), since closed sets in Cantor space are also closed sets in \([0, 1]\), these items follows from the items \([\text{I}1]\) and \([\text{I}2]\).

For the second part, any continuous function on \( R \) has a continuous modulus of continuity by \([20, \S 4] \) given WKL. Using \( (\exists^2) \), one similar defines such a modulus, say \( G^2 \), for \( F \) continuous on closed \( D \subseteq 2^N \). Since \( G \) is continuous, \( \cup f \in D[fG(f)] \) has a countable sub-cover, and HBC yields a finite sub-cover. Thus, there are only finitely many values for \( F \) on \( D \), and item \([\text{II}1]\) follows. Now apply the latter item to \( G^2 \) and obtain item \([\text{II}2]\).

Note that item \([\text{I}1]\) and \([\text{I}2]\) immediately imply item \([\text{I}3]\), while item \([\text{I}2]\) implies the latter with minimal effort.

Regarding \([3,1]\), Kohlenbach shows in \([20, \S 4] \) that over RCA\(^{\omega}_0\) the existence of a modulus of continuity is equivalent to the existence of an RM-code, i.e. the exact formulation of continuity does not matter in the previous theorem.

Finally, we establish the results in item \([\text{I}4]\) from Section \([1.2]\) pertaining to HBC, and related results. We define a realiser for HBC as follows.

**Definition 3.6.** A functional \( \beta^3 \) is called a realiser for HBC if for closed \( C \subseteq [0, 1] \) and a sequence of rationals \( a_n, b_n \) such that \( C \subseteq \cup_{n \in N}(a_n, b_n) \), we also have \( C \subseteq \cup_{n \leq \beta(C,a_n,b_n)}(a_n, b_n) \).

The following theorem is not that surprising in light of some of our previous results. We shall establish a more impressive result in Theorem \([5,7]\).

**Theorem 3.7.** No type two functional can compute a realiser \( \beta^3 \) for HBC.

**Proof.** It is easily seen that \( \beta \) cannot be countably based. Now put \( D = \{\frac{1}{2}, \frac{3}{4}\} \), \( a_n = \frac{1}{n} \) and \( b_n = 1 \). If \( X \) is a countable base for the value \( \beta(D, a_n, b_n) = k \), we may choose \( x \notin X \) in the interval \((0, \frac{1}{k}) \) and obtain a contradiction by considering the set \( D' = D \cup \{x\} \).

3.2. **Coding open sets.** We study the questions raised in Section \([1.2]\) concerning representations of open sets. In Section \([3.2.1]\) we study the coding principle \( \text{Open} \) expressing that every characteristic function of an open set has an RM code. In Section \([5.2.2]\) we study how hard it is to compute such a code in terms of the other
data. In each case, we need a functional of which the existence implies full second-order arithmetic. The aforementioned theorem also yields a non-trivial hierarchy parallel to the usual comprehension based hierarchy, i.e. the medium range of the G"{o}del hierarchy based on higher types and inclusion (see [42]).

3.2.1. Reverse Mathematics. We study the following theorem pertaining to the countable representation of open sets. Let \((q_n, r_n)_{n \in \mathbb{N}}\) be a fixed enumeration of all non-trivial open balls \(B(q_n, r_n)\) with rational center and radius. Note that we use ‘open set’ in the sense of Definition 1.2.

**Definition 3.8.** [Open] For every open set \(Y \subseteq \mathbb{R}\), there is \(X \subseteq \mathbb{N}\) such that \((\forall n \in \mathbb{N})(n \in X \iff B(q_n, r_n) \subseteq Y)\).

Note that for \(X, Y\) as in Open, we have \(x \in Y \iff (\exists m \in \mathbb{N})(m \in X \land x \in B(q_m, r_m))\), i.e. Open endows open sets in \(\mathbb{R}\) with a countable representation. According to Bourbaki ([41, p. 222]), Cantor first proved that open sets can be written as countable unions of open intervals, i.e. Open also carries some historical interest. However, Aczel states in [1] p. 134] that constructive set theory cannot prove Open.

We need the following comprehension principle, which is a special case of the comprehension principle \(\text{BOOT}\) from Section 2.2.

**Definition 3.9.** [\(\text{BOOT}^-\)] For \(Y^2\) such that \((\forall n \in \mathbb{N})(\exists \text{ at most one } f^1(Y(f, n) = 0))\), we have \((\exists X \subseteq \mathbb{N})(\forall n \in \mathbb{N})(n \in X \iff (\exists f^1)(Y(f, n) = 0))\).

The name of the previous principle is derived from the verb ‘to bootstrap’, as combining the relatively weak\(^4\) (in isolation) principles \((\exists^2)\) and \(\text{BOOT}^-\), gives rise to the much stronger principle of arithmetical transfinite recursion, as in Theorem 3.10. A more general result is proved in Corollary 3.13 below.

**Theorem 3.10.** The system \(\text{ACA}_0\) + \(\text{BOOT}^-\) implies \(\text{ATR}_0\).

**Proof.** We recall [41, V.5.2] which states that \(\text{ATR}_0\) is equivalent to the following second-order version of \(\text{BOOT}^-\): for every arithmetical \(\varphi(n, X)\) such that \((\forall n \in \mathbb{N})(\exists \text{ at most one } X^1)\varphi(n, X)\), there is \(Z \subseteq \mathbb{N}\) such that 
\[
(\forall n \in \mathbb{N})(n \in Z \iff (\exists X \subseteq \mathbb{N})\varphi(n, X)).
\]

(3.2)

To prove the latter principle, we consider the Kleene normal form lemma as in [41, V.5.4] which expresses that an arithmetical formula \(\psi(X)\) is equivalent to \((\exists f^1)(\forall n^0)\theta_0(Xn, f_n)\), where the formula \(\theta_0\) is bounded and we also have that \((\forall X)(\exists at most one f^1)\theta_0(\forall n^0)\theta_0(Xn, f_n)\). In this light, for arithmetical \(\varphi(n, X)\), we have that \((\forall n \in \mathbb{N})(\exists at most one X^1)\varphi(n, X)\) is equivalent to the formula \((\forall n \in \mathbb{N})(\exists at most one f^1)Y(f, n) = 0\) for some \(Y^2\) defined in terms of \(\exists^2\). Applying \(\text{BOOT}^-\) then yields the required set \(Z\) as in (3.2).

The importance of \(\text{BOOT}^-\) is illustrated by Theorem 3.12. We also need the following principle from [48], which is used in [29, §3] to derive e.g. HBU.

**Definition 3.11.** [\(\text{NFP}\)] For any \(\Pi^1_2\)-formula \(A\) with any parameter:
\[
(\forall f^1)(\exists n^0)A(\overrightarrow{f}n) \rightarrow (\exists \gamma^1 \in K_0)(\forall f^1)A(\overrightarrow{f}\gamma(f)).
\]

\(^4\)Note that \(\text{ACA}_0\) is conservative over \(\text{ACA}_0\) by [13, Cor. 2.5], while the ECF-translation of \(\text{BOOT}\) is provable in \(\text{ACA}_0\), i.e. \(\text{RCA}_0 + \text{BOOT}\) is not stronger than \(\text{ACA}_0\) in terms of second-order consequences, while \(\text{ECF}\) translates \(\text{BOOT}^-\) to a triviality.
Here, \( \gamma^1 \in K_0 \) expresses that \( \gamma^1 \) is an \textit{associate}, which is the same as a \textit{code} from RM by \cite[Prop. 4.4]{20}. Formally, \( \gamma^1 \in K_0 \) is the following formula:
\[
(\forall f^1)(\exists n^0)(\gamma(\overline{f} n) > 0) \land (\forall n^0, m^0, f^1, )(m > n \land \gamma(\overline{f} n) > 0 \rightarrow \gamma(\overline{f} m) = 0 \gamma(\overline{f} m)).
\]
The value \( \gamma(f) \) for \( \gamma \in K_0 \) is defined as the unique \( \gamma(\overline{f} n) - 1 \) for \( n \) large enough.

We now have the following theorem, where \( \text{BOOT} \) was introduced in Section 2.

**Theorem 3.12.** The system \( \text{RCA}_0^2 \) proves \( \text{BOOT} \rightarrow [\text{Open} + \text{ACA}_0] \rightarrow \text{BOOT}^{-} \) and \( \text{RCA}_0^2 + \text{IND}^\omega \) proves \( \text{NFP} \rightarrow \text{BOOT} \rightarrow \text{HBU}_{\text{closed}}. \)

**Proof.** The implication \( \text{NFP} \rightarrow \text{BOOT} \) follows from the proof of \cite[Theorem 3.5]{38} combined with \cite[Theorem 3.6]{38}. A sketch is as follows: assume that \( \text{BOOT} \) is false for some \( Y_0 \). The formula expressing this has the form \( (\forall X \in \mathbb{N})(\exists n \in \mathbb{N})A(X, n), \) and a trivial modification yields \( (\forall X \in \mathbb{N})(\exists n \in \mathbb{N})B(\overline{X} n) \), i.e. only the first \( n \) digits of \( X \) are used. Let \( \gamma \in K_0 \) be as provided by \( \text{NFP} \) and use \( \text{WKL} \) to obtain an upper bound on \( C \) on \( \gamma \). However, \( \text{IND}^\omega \) proves \textit{`bounded comprehension'} (see \cite[II.3.9]{41}) for arbitrary formulas, yielding a contradiction.

For the third implication, in case \( \neg(\exists^2) \), all functions on Baire space are continuous by \cite[§3]{21}. Thus, \( (\exists f^1)(Y(\langle f, n \rangle) = 0) \) is equivalent to \( (\exists \sigma^0)(Y(\langle \sigma \star 00 \ldots, n \rangle) = 0) \), and \( \text{ACA}_0 \) provides the set required by \( \text{BOOT}^{-} \). In case \( \exists^2 \), let \( Y^2 \) satisfy \( (\forall n \in \mathbb{N})(\exists \gamma \in \mathbb{N})(\exists n \in \mathbb{N})B(\overline{X} n) \). The formula \( (\exists f^1)(Y(\langle f, n \rangle) = 0) \) is equivalent to \( (\exists X \in \mathbb{N}^2)(Y(F(X), n) = 0) \), where \( F(X)(n) := (\mu m)((n, m) \in X) \). Hence, \( (\exists f^1)(Y(\langle f, n \rangle) = 0) \) is equivalent to a formula \( (\exists f \in \mathbb{C})(Y(\langle f, n \rangle) = 0) \), where \( \tilde{Y} \) is defined explicitly in terms of \( Y \) and \( \exists^2 \). Now define \( Z : \mathbb{R} \rightarrow \mathbb{R} \) as:
\[
Z(x) := \begin{cases} 
0 & n < \overline{r} \| x \| \leq \overline{r} \ n + 1 \land \tilde{Y}(\eta(x)(0), n) \times \tilde{Y}(\eta(x)(1), n) = 0, \\
1 & \text{otherwise},
\end{cases}
\]
where \( \eta(x) \) provides a pair consisting of the binary expansions of \( x - \lfloor x \rfloor \); the pair consists of identical elements if there is a unique such expansion. Note that \( \exists^2 \) can define such functionals \( Z \) and \( \eta^{1-(1 \times 1)} \). By definition, for each \( n \in \mathbb{N} \) there is at most one real \( y \in (n, n + 1) \) such that \( Z(y) = 0 \). Hence, \( Z \) is open and we may apply \( \text{Open} \) to obtain \( X \in \mathbb{N} \) such that \( (\forall n \in \mathbb{N})(n \in X \iff B(q_n, r_n) \subset Z) \). Now note that for any \( m \in \mathbb{N} \), we have
\[
\overline{B}(m + \frac{1}{2}, \text{\frac{1}{2}}) \subset Z \iff (\forall f^1)(Y(\langle f, m \rangle) > 0),
\]
which is sufficient to obtain \( \text{BOOT}^{-} \) in this case. The law of excluded middle now finishes this part of the proof. For the implication \( \text{BOOT} \rightarrow \text{Open} \), we use \( (\exists^2) \lor \neg(\exists^2) \) as follows: in the former case \( \text{BOOT} \) readily yields the set \( X \) as in \( \text{Open} \), while \( \text{ACA}_0 \) does the job in the latter case as quantifiers over the reals may be replaced by quantifiers over the rationals if all functions on \( \mathbb{R} \) are continuous (as they are by \cite[§3]{21}).

For the final implication, \( \text{BOOT} \rightarrow \text{Open} \) means that \( \text{HBC} \) reduces to \( \text{HBC}_m \), and the latter is equivalent to \( \text{WKL}_0 \) by \cite[Lemma 3.13]{5}. We therefore have access to \( \text{HBC} \), as well as \( \exists^2 \) in the same way as in the previous paragraphs. Now let \( C \subseteq [0, 1) \) be closed and fix \( \Psi : I \rightarrow \mathbb{R}^+ \). Use \( \text{BOOT} \) to define \( (a_n, b_n) \) as \( B(q_n, r_n) \) in case \( (\exists x \in C)(x \in B(q_n, r_n) \subseteq I_n^2) \), and \( 0 \) otherwise. Then clearly \( \cup_{n \in \mathbb{N}}(a_n, b_n) \) covers \( C \), and \( \text{HBC} \) yields a finite sub-cover. Now use \( \text{IND}^\omega \) to conclude that this finite sub-cover yields a finite sub-cover for \( \cup_{x \in C}I_n^2 \) ‘by definition’. \( \square \)

We now have the following corollary, yielding a parallel hierarchy.
Corollary 3.13. The system $\Pi^1_k\text{-CA}_0 + \text{Open}$ implies $\Pi^1_k\text{-TR}_0$.

Proof. The case $k = 0$ follows from the theorem and Theorem 3.10. The general case follows in the same way: one notes that the proof of $'2 \to 1'$ from [41, V.5.2] relativises to $S^k_1$ for $k^0 > 0$.

Thus, we have established that Open is hard to prove and moreover is ‘explosive’ as it becomes much stronger when combined with (higher-order) comprehension.

Finally, we show that $\text{BOOT}^-$ shows up in the study of the Cantor-Bendixson theorem and located sets, both involving closed sets as in Definition 1.2. Now, the former theorem states that a closed set can be expressed as the union of a perfect closed set and a countable set of isolated points; this theorem for RM-closed sets is equivalent to $\Pi^1_1\text{-CA}_0$ ([41, VI.1.6]). We define a version based on Definition 1.2.

**Principle 3.14 (CBT).** For any closed set $C \subseteq \mathbb{R}$, there exist $P, S \subseteq C$ such that $C = P \cup S$, $P$ is perfect, and $S^{0\to 1}$ lists the isolated points of $C$.

To be absolutely clear, the countable set of isolated points $S$ is given as a sequence of real numbers, just like in second-order RM. Furthermore, a set $C$ in a metric space is located if $d(x, C) = \inf_{y \in C} d(x, y)$ exists as a continuous function. By [14, Theorem 1.2], ACA$_0$ is equivalent to the locatedness of non-empty closed sets in the unit interval.

**Principle 3.15 (CLO).** Any non-empty closed set $C \subseteq \mathbb{R}$ is located.

Let QF-AC$_1^{0,1}$ be QF-AC$_1^{0,1}$ restricted to $Y$ such that $\langle \forall m \rangle (\exists ! f^1) (Y (f, n) = 0)$, i.e. unique existence. We have the following theorem.

**Theorem 3.16 (RCA$_0^s$).** Either CLO or CBT implies $[\text{BOOT}^- + \text{QF-AC}_1^{0,1}]$.

Proof. The proof is trivial in case $\neg (\exists \overline{\mu})$, in the same way as in the proof of Theorem 3.12. Thus, we may assume $\exists \overline{\mu}$. Let $Y^2$ be as in $\text{BOOT}^-$ and let $C$ be the complement of the open set $Z$ defined in (3.3) in the proof of Theorem 3.12. By definition, for each $n \in \mathbb{N}$ there is at most one real $y \in (n, n + 1]$ such that $Z (y) = 0$.

To obtain $\text{BOOT}^-$, we proceed as follows.

First assume CLO and note that to check whether $\exists ! f^1 (Y (f, n) = 0)$, it suffices to: (i) check $Y (n + \frac{1}{2}, n) = 0$, (ii) if $Y (n + \frac{1}{2}, n) \neq 0$, then consider $d (n + \frac{1}{2}, C)$; the latter is inside $(n, n + 1]$ if and only if $\exists ! f^1 (Y (f, n) = 0)$.

Inequalities of real numbers can be decided by $\exists \overline{\mu}$ and $\text{BOOT}^-$ readily follows.

Secondly, assume CBT and note that the set $C$ by assumption consists of isolated points. Hence, $\langle \exists ! f^1 \rangle (Y (f, n) = 0)$ is equivalent to an isolated point of $C$ being in $(n, n + 1]$. Since $\text{CBT}$ provides a list $S$ of isolated points of $C$, $\text{BOOT}^-$ readily follows using $\exists \overline{\mu}$.

Thirdly, to prove QF-AC$_1^{0,1}$, assume $\langle \forall m \rangle (\exists ! f^1) (Y (f, n) = 0)$ and again consider the aforementioned set $C$. By assumption, for every $n$ there is exactly one $y \in \mathbb{R}$ such that $y \in (n, n + 1] \cap C$. In the same way as in the previous two paragraphs, the set $S$ from $\text{CBT}$ and the function $d (x, C)$ from CLO allow one to find this unique real. The theorem now follows.

As a corollary, we show that the perfect set theorem for our notion of closed set as in Definition 1.2 also gives rise to $\text{BOOT}^-$. The second-order version of the perfect set theorem is equivalent to ATR$_0$ by [41, V.5.5]. Note that a set is uncountable in RM if there is no sequence that lists its elements (see e.g. [41] p. 193).
**Principle 3.17** (PST). For any closed and uncountable set $C \subseteq [0,1]$, there exist $P \subseteq C$ such that $P$ is perfect and closed.

**Corollary 3.18.** The system $\text{RCA}_0^\omega$ proves $\text{PST} \rightarrow \text{BOOT}^-$. 

**Proof.** The set $C$ from the proof of the theorem does not have perfect subsets, and hence $\text{PST}$ provides a sequence that lists the elements of $C$. The proof of the theorem now yields $\text{BOOT}^-$. □

In conclusion, we have shown that the Cantor-Bendixson theorem, the perfect set theorem, and located sets give rise to $\text{BOOT}^-$ when using closed sets as in Definition 1.2. Hence, a slight change to the aforementioned theorems makes them much harder to prove by the above. Similar results no doubt exist for other theorems pertaining to closed sets from RM. However, it is not clear whether (nice) equivalences can be obtained based on Theorem 3.16. A different notion of open set, namely given by uncountable unions of open balls, does yield nice equivalences involving the Cantor-Bendixson theorem, perfect set theorem, and located sets, as explored in [39]. Moreover, the results pertaining to $\text{CBT}$ and $\text{PST}$ are nice, but seem to depend on the particular notion of ‘(un)countable’ used. As also shown in [39], this problem does not occur for open sets given by uncountable unions of open balls, i.e. the usual definition of ‘countable’ can be used.

### 3.2.2. Computability theory

In this section, we study the computational properties of $\text{Open}$, i.e. how hard is it to compute (Kleene S1-S9) the representation provided by this coding principle?

First of all, we introduce two functionals based on Definition 1.2 as follows.

**Definition 3.19.** [Open sets]

1. Let $\Phi^3$ be such that for every open set $Y \subseteq \mathbb{R}$ and real $x \in Y$, $\Phi(Y,x)$ equals a rational $r^0 > 0$ such that $B(x,r) \subseteq Y$.
2. Let $\Psi^3$ be such that for every open set $Y \subseteq \mathbb{R}$, $\Psi(Y)$ equals two sequences of rationals $a_n, r_n$ such that $Y = \bigcup_{n \in \mathbb{N}} B(a_n, r_n)$.

It goes without saying that these functionals are not unique. We allow $r_n = 0$ and note that those instances can be removed using $\mu^2$ in case $Y \neq \emptyset$. Note that one can test for the latter condition for open sets using $\mu^2$.

Note that $\Phi^3$ is (obviously) computable in $\mu^2$ and $\exists^1$, while the same holds for $\Psi^3$ by Theorem 3.23 a better result cannot be expected by the following.

**Theorem 3.20.** No functional $\Phi^3$ as above is countably based.

**Proof.** Assume that $\Phi(Y,0) = r$, where $0 \in Y$, and where $Y$ is the constant 1, i.e. it represents $\mathbb{R}$. If $\Phi$ is countably based, there is a countable set $X$ such that if $Y'$ agrees with $Y$ on $X$ and defines an open set, then $\Phi(Y,0) = \Phi(Y',0)$. However, there will be a real $r'$ such that $0 < r' < r$ and such that the oracle call $r'$ was not used in the computation $r' \notin X$. Then consider $Y' = Y \setminus \{r'\}$. Since $r$ is an unacceptable value for $\Phi(Y',0)$, we obtain a contradiction. □

**Corollary 3.21.** The functional $\Psi^3$ is not countably based.

**Proof.** Note that $\Psi$ computes $\Phi$ modulo $\mu^2$: the latter can be used to find the right open ball in the sequence provided by $\Psi$. The class of countably based functionals is closed under Kleene computability (see [15]. □
Realisers for \( \text{HBU}_c \) are called \textit{special fan functionals}, or simply \( \Theta \)-functionals, and compute the finite sub-cover in \( \text{HBU}_c \) in terms of \( G^2 \). We have the following.

**Theorem 3.22.** A realiser \( \Theta_{\text{closed}} \) for \( \text{HBU}_{\text{closed}} \) together with \( \mu^2 \) computes an instance of \( \Psi \).

**Proof.** Fix some open set \( U \subset [0,1] \) and let \( C \) be its complement in \([0,1]\). Define \( \Phi_n(x) \) as \( \frac{1}{b} \) if \( x \in C \), and 0 otherwise. Clearly, for fixed \( n^0 \), \( \lambda x.\Phi_n(x) \) yields a closed set, these theorems even hold recursively, i.e. the object claimed to exists may be computed (in the sense of Turing) from the data by \([41, II.7\])

We now show that either \( \exists \) is a continuous function such that \( \forall n \exists x \in C (x \in \cup_{i|\leq k} I_{y_i}^{\Psi_n}) \),

where \( w_n := \Theta_{\text{closed}}(\lambda x.\Psi_n(x)) \) provides the finite sequence of \( y_i \)'s. With minor modification, \( [0,1] \setminus \bigcup_{n \in N} \left( \frac{1}{n} \cup \frac{1}{2^n} \right) \) yields a countable union of open intervals \((a_n, b_n)\) such that \( U = \bigcup_{n \in N}(a_n, b_n) \).

Finally, the ‘standard’ proof that an open set in \( \mathbb{R} \) is the union of countable many open balls, goes through modulo \( (\exists^3) \).

**Theorem 3.23.** \( \text{The system } Z^2 \text{ proves Open. The associated functional } \Psi^3 \text{ can be computed from } \exists^3 \text{ via a term from Gödel’s } T. \)

**Proof.** Let \( Y \subset \mathbb{R} \) be open and define for \( x \in Y \) the (non-empty by definition) sets \( A_x := \{ a \in \mathbb{R} : [a, x) \subset Y \} \) and \( B_x := \{ b \in \mathbb{R} : (b, y) \subset Y \} \) using \( \exists^3 \), letting them be open sets if \( x \notin Y \). If the set \( A_x \) (resp. \( B_x \)) has a lower (resp. upper) bound (which is decidable assuming \( \exists^3 \)), then \( a_x := \inf A_x \) (resp. \( b_x := \sup B_x \)) exists thanks to \( \exists^3 \), using the usual interval-halving technique. In case such a bound is missing, we use a default value for \( a_x \) (resp. \( b_x \)) meant to represent \(-\infty \) (resp. \( +\infty \)). We define \( J_x := (a_x, b_x) \) and will show that \( Y = \bigcup_{q \in \mathbb{Q}} J_q \), thus establishing the theorem. By the definition of \( J_x \), we must have \( J_x \subset Y \) for all \( x \in Y \). Since also \( Y \subset \bigcup_{x \in Y} J_x \) due to \( x \in J_x \) if \( x \in Y \), we actually have \( Y = \bigcup_{x \in Y} J_x \), and note that \( \exists^3 \) guarantees this union actually exists. We now show that either \( J_x \cap J_y = \emptyset \) for \( x, y \in Y \). Hence, for \( x \in Y \), \( J_x = J_q \) for any \( q \in J_x \cap \mathbb{Q} \), and the theorem follows. Suppose \( z \in J_x \cap J_y \) for \( x, y \in Y \) (implying \( x \neq y \)). Then if \( x < y \), we have \( a_y < z < b_y \), and if \( y < x \), we have \( a_x < z < b_x \). In the first case, we have \( a_x < w < x \) such that \((w, y] \subset Y \), as \((w, y] = (w, x] \cup [x, z] \cup [z, y]) \) and the latter three intervals are in \( Y \) by assumption. However, this implies that \( a_x = a_y \), and \( b_x = b_y \) follows in the same way. Hence, \( J_x = J_y \) if \( x < y \), and the other case is treated in the same way; thus, we are done.

### 3.3. The Urysohn and Tietze theorems

The Urysohn and Tietze extension theorems are basic results of topology that are well-known in RM: for RM-codes of closed sets, these theorems even hold recursively, i.e. the objects claimed to exists may be computed (in the sense of Turing) from the data by \([11, 11.7]\). We now show that the situation is dramatically different for our notion of closed sets.

First of all, we study the Urysohn lemma for \( \mathbb{R} \) and Definition 1.2 as follows.

**Definition 3.24.** [URY] For closed disjoint sets \( C_0, C_1 \subset \mathbb{R} \), there is a continuous function \( g : \mathbb{R} \rightarrow [0,1] \) such that \( x \in C_i \leftrightarrow g(x) = i \) for any \( x \in \mathbb{R} \) and \( i \in \{0,1\} \).

We let \( \zeta \) be a realiser for URY, i.e. for disjoint closed sets \( C_0, C_1 \subset \mathbb{R} \), \( \zeta(C_0, C_1) \) is a continuous function such that \( \zeta(C_0, C_1)(x) = i \) whenever \( x \in C_i \) for \( i = 0,1 \). In case one works with \textit{codes} rather than actual sets, the Urysohn lemma is 'recursively
true’ by [11 II.7.3], i.e. one can compute (in the sense of Turing) a code for the required continuous function from a code for the sets $C_i$, over $\text{RCA}_0$. How different things are at the higher-order level: we namely have the following RM and relative computability result. Recall the functional $\Phi$ from Definition 3.19.

**Theorem 3.25.** The functional $\Phi$ can be computed from $\zeta$ and $\mu^2$. The functional $\zeta$ can be computed from $\Psi$ and $\mu^2$. The system $\text{RCA}_0^\omega$ proves $\text{URY} \iff \text{Open}$.

**Proof.** For the first part, fix an open set $U \subseteq \mathbb{R}$, define disjoint closed sets $C_0 = U^c$ and $C_1 = \emptyset$, yielding $x \in U \iff \zeta(U^c, \emptyset)(x) > 0$ for all $x \in \mathbb{R}$. Since $\lambda x. \zeta(U^c, \emptyset)$ is continuous, $\mu^2$ provides a modulus of (pointwise) continuity by (the proof of) [20] Prop. 4.7]. Using this modulus (and $\mu^2$), we may find $r > 0$ such that for all $y \in B(x, r)$, we have $\zeta(U^c, \emptyset)(y) > 0$, which is exactly as required for $\Phi$. Note that $\text{URY} \rightarrow \text{Open}$ also follows, over $\text{ACA}_0^\omega$.

For the implication $\text{Open} \rightarrow \text{URY}$ over $\text{ACA}_0^\omega$, define $h(x)$ as $i$ for $x \in C_i$ and $i = 0, 1$. In case $x \in Z = C_0^c \cap C_1^c$, we define $h(x)$ as follows: first note that $Z$ is open (by definition), and hence $Z = \bigcup_{n \in \mathbb{N}} B(a_n, r_n)$ by $\text{Open}$. Note that $Z$ cannot be empty by assumption. Using $\mu^2$, we find $m^0$ such that $x \in B(a_m, r_m)$ and we may test if $a_m \pm r_m$ belong to $C_0$ or $C_1$. If such a point is in neither, it is in $Z$, and hence in $B(a_k, r_k)$ for some $k \neq m$, and we can repeat the previous process. There are three possible outcomes:

(a) This procedure ends after finitely many steps, say with $a_{k_0} - r_{k_0} < x < a_{k_1} + r_{k_1}$ and the former (resp. latter) rational is in $C_0$ (resp. $C_1$).

(b) This procedure only ends after finitely many steps in one direction, say with $a_{k_0} - r_{k_0} < x$ and this rational is in $C_0$.

(c) This procedure never ends in both directions.

If item (a) is the case, then $C_0 = C_1 = \emptyset$, and $h(x) = 0$ everywhere. If item (b) is the case, define $h(x)$ as the (increasing) straight line connecting $(a_{k_0} - r_{k_0}, 0)$ and $(a_{k_1} + r_{k_1}, 1)$ for $a_{k_0} - r_{k_0} < x < a_{k_1} + r_{k_1}$. If $C_1$ (resp. $C_0$) is eventually met on the left (resp. right), the modification to $h$ is obvious. If item (c) is the case, then define $h(x) := 0$ for $a_{k_0} - r_{k_0} < x$. If $C_1$ is eventually met on the left, or if the unbounded area is on the left, the modification to $h$ is obvious.

Finally, an indirect proof of $\text{Open} \rightarrow \text{URY}$ follows from replacing the closed sets in the latter by RM-codes and applying the RM-version of the Urysohn lemma, namely [11 II.7.3]. Since $\text{URY}$ and $\text{Open}$ are provable in case $\neg(\exists^2)$, the law of excluded middle now finishes the proof. 

Secondly, we study the Tietze extension theorem, which expresses that a continuous function on a closed set can be extended to a continuous function on the whole space, while if the original function is bounded, so is the extended function with the same bound. Lebesgue ([22]), de la Vallée-Poussin ([49]), and Carathéodory ([7]) prove special cases of this theorem not involving boundedness conditions. Furthermore, Tietze explicitly mentions that the given function can be discontinuous (outside the closed set; see [46 p. 10]), while he also states in [46 p. 10, Footnote *) that the boundedness condition may be dropped. We therefore consider the following version of the Tietze extension theorem.

**Definition 3.26.** ([TIE]) For $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous on the closed $D \subseteq [0, 1]$, there is $g : \mathbb{R} \rightarrow \mathbb{R}$, continuous on $[0, 1]$ such that $f(x) =_R g(x)$ for $x \in D$. 
Theorem 3.27. The system RCA_0\^\omega proves [TIE + WKL] \to HBC. The system RCA_0\^\omega + QF-AC^{0,1} proves TIE \iff URY.

Proof. For the implication URY \to TIE, we have URY \to Open by Theorem 3.25 and we may therefore use the RM-proof of the Tietze extension theorem (see [41 II.7.5]). The latter applies to bounded functions and we can guarantee boundedness in case WKL (using HBC in particular). In case \neg WKL, we also have \neg (\exists^2), i.e. all functions \( f : \mathbb{R} \to \mathbb{R} \) are continuous by \([21] \S3\), rendering TIE a triviality.

For TIE \to URY, in case \neg (\exists^2), open sets reduce to RM-codes and the usual proof of URY from \([41] II.7\) goes through. In case (\exists^2), let \( C_i \) be as in URY for \( i = 0, 1 \) and define \( f \) on \( C_2 := C_0 \cup C_1 \) as follows: \( f(x) = 0 \) if \( x \in C_0 \) and 1 otherwise. If \( f \) is continuous on \( C_{2x} \), then its extension \( g \) provided by TIE is as required for URY. To show that \( f \) is continuous on \( C \), we prove that

\[
(\forall N^0)(\exists n^0)(\forall x \in C_0, y \in C_1)(x, y \in [-N, N] \to |x - y| \geq \frac{1}{n}). \tag{3.5}
\]

If (3.5) is false, QF-AC^{0,1} yields a double sequence \( x_n, y_n \) in \([-N, N]\) such that for all \( n \), we have \( x_n \in C_0, y_n \in C_1 \), and \( |x_n - y_n| < \frac{1}{n^2} \). As \( C_0, C_1 \) are closed and the sequences bounded, there are \( x \in C_0, y \in C_1 \) such that \( x_n \to x \) and \( y_n \to y \). However, \((\forall n^0)(|x_n - y_n| < \frac{1}{n^2})\) implies that \( x = y \), a contradiction since \( C_0 \cap C_1 = \emptyset \). Finally, since (3.5) provides a positive ‘distance’ between \( C_0 \) and \( C_1 \) in every interval \([-N, N]\), we can always chose a small enough neighbourhood to exclude points from one of the parts of \( C \), thus guaranteeing continuity for \( f \) everywhere on \( C_2 \).

For \([TIE + WKL] \to HBC\), let \( F \) and \( D \) be as in item (3) of Theorem 3.25 and consider the extension \( g \) provided by TIE. By WKL, \( g \) is bounded on \([0, 1]\), and \( f \) is therefore bounded on the closed set \( D \). Theorem 3.25 now finishes the theorem. Note that we could also use the equivalence TIE \iff Open, together with the RM-equivalence between countable Heine-Borel and WKL.

A reversal in the first implication is possible if one additionally assumes that \( f : \mathbb{R} \to \mathbb{R} \) in TIE has a modulus of continuity on \( C \).

4. The Baire category theorem

4.1. Introduction. The Baire category theorem expresses that a sequence of dense open sets has an intersection that is also dense; this theorem can be found in Baire’s 1899 doctoral thesis and Osgood’s paper [32]. This theorem is studied (in various guises) in the computational approaches to mathematics mentioned in Remark 4.1. It is therefore a natural question what the computational properties of the Baire category theorem are when using Definition 4.2. Our main results are as follows.

(a) A realiser \( \xi \) for the Baire category theorem (Definition 4.1) can be computed from \( \exists^2 \) and IND, the inductive definition operator (Definition 4.2).

(b) There is no Baire-realiser \( \xi \) computable in a functional of type 2.

Thus, we observe that the Baire category theorem also (partially) exhibits the Pincherele phenomenon. Note that by Theorem 4.102, ZF proves that separable completely metrisable spaces have the property of Baire, i.e. the Axiom of Choice is not the cause of the (computational) hardness of the Baire category theorem. The aforementioned notion of realiser is defined as follows.
**Definition 4.1.** A realiser for the Baire Category Theorem is any \( \xi^3 \) such that whenever \( \{Y_n\}_{n \in \mathbb{N}} \) is a sequence of dense open sets of reals, then

\[
\xi(\{Y_n\}_{n \in \mathbb{N}}) \in \bigcap_{n \in \mathbb{N}} Y_n.
\]

For the previous definition, we assume a standard coding of the reals in \( \mathbb{N}^\mathbb{N} \), and that a set \( Y \subset \mathbb{R} \) is given in in the form of its characteristic function as in Definition 3.2. So, technically we are working inside the full type-structure over \( \mathbb{N} \) up to level 3. All our “algorithms” are relative to \( \exists^3 \) and any other objects specified in the argument.

As noted above, the Baire category theorem for separable spaces can be proved in ZF, yielding the existence of a Baire realiser \( \xi \); a direct translation of that argument however requires access to computations relative to \( \exists^3 \). Our first result involves a construction of a Baire realiser that terminates on all countable sequences of sets, and gives an element in the intersection when the sets are all open and dense, working within the realm of the countably based functionals. By contrast, other 'basic' theorems about open sets, like every open set is a union of open intervals with rational endpoints and for every closed set \( C \), the distance \( d(x, C) \) is continuous, only have realisers that are not countably based.

**Definition 4.2.** For \( \Gamma : 2^\mathbb{N} \to 2^\mathbb{N} \) satisfying the monotonicity condition \((\forall A \subseteq \mathbb{N}) (\Gamma(A) \supseteq A)\), define the well-ordered sequence of sets \( \Gamma_{\alpha} \) as follows:

\[
\Gamma_{\alpha} := \Gamma\left(\bigcup_{\beta < \alpha} \Gamma_{\beta}\right).
\]  

(4.1)

For any such \( \Gamma \), there is an ordinal \( \alpha_0 \) with \( \Gamma_{\alpha_0+1} = \Gamma_{\alpha_0} \), where the latter is called the inductive closure of \( \Gamma \). Finally, for any \( F : 2^\mathbb{N} \to 2^\mathbb{N} \) we define \( \text{IND}(F) \) as the inductive closure \( \Gamma_{\alpha_0} \) for the functional \( \Gamma \) defined as \( \Gamma(A) := A \cup F(A) \).

Regarding the previous definition, \( \text{IND} \) denotes a type three functional, while \( \text{IND}^3 \) is the induction axiom for all \( \text{L}_{\omega_1} \)-functionals.

4.2. Computational properties. We prove the results sketched in the previous section. To this end, we first fix some notation, as follows.

**Definition 4.3.**

- A tag is a pair \((r, \epsilon)\) from \( \mathbb{Q} \) where \( \epsilon > 0 \). We let \((r, \epsilon)_o = (r - \epsilon, r + \epsilon)\) and \((r, \epsilon)_c\) be the corresponding closed interval \([r - \epsilon, r + \epsilon]\).
- If \((r_1, \epsilon_1)\) and \((r_2, \epsilon_2)\) are tags, we let \((r_1, \epsilon_1) \prec (r_2, \epsilon_2)\) if \( r_1 = r_2 \) and \( \epsilon_1 \geq 2\epsilon_2 \).
- An attempt is a sequence \( s = [(r_1, \epsilon_1), \ldots, (r_k, \epsilon_k)] \) of tags, where we for \( i < k \) have that \((r_{i+1}, \epsilon_{i+1})_c \subseteq (r_i, \epsilon_i)_o\) and that \( 2\epsilon_{i+1} \leq \epsilon_i \).
- If \( s \) and \( t \) are attempts, we let \( s \prec t \) if either \( s \) is an initial sub-sequence of \( t \) or if \( s \) comes before \( t \) in the lexicographical ordering on attempts based on \( \prec \) on tags.

The ordering ‘\( \prec \)’ is not the Kleene-Brouwer ordering, but a partial ordering all the same. Intuitively, an attempt will be an attempt to find a shrinking sequence of closed intervals whose single point in the intersection will be in the intersection of all \( Y_n \). We have limited access to information about \( Y_n \), we do not know if it is open and dense. If it is, we have no way to say that a tag \((r, \epsilon)\) represents a subset of \( Y_n \). So certain attempts may lead to failure, and we have to try again with better attempts. This is what will be captured by our inductive definition, which will be defined from a given sequence \( \{Y_n\}_{n \in \mathbb{N}} \) in a uniform way, as follows.
Let the sets $Y_n$ be given, and let $A$ be a set of attempts. We define $\Gamma(A)$ in cases as follows, where we order $\mathbb{Q}$ according to a standard enumeration.

- If $A$ is not totally ordered by $\prec$, we let $\Gamma(A) = A$. For the rest of the cases we will assume that $A$ is totally ordered.
  - In case $A = \emptyset$ and if $Y_1 \cap \mathbb{Q} = \emptyset$, put $\Gamma(A) = A$. If not, $\Gamma(A) := \{(r_1, 1)\}$, where $r_1$ is the first rational number in $Y_1$.
  - If the set $A$ has a $\preceq$-maximal element $[(r_1, \epsilon_1), \ldots, (r_k, \epsilon_k)]$ and if $(r_k, \epsilon_k)_o \cap Y_{k+1} \cap \mathbb{Q} = \emptyset$ we let $\Gamma(A) = A$. If not, we define $\Gamma(A) = A \cup \{(r_1, 1), (r_k, \epsilon_k), (r_{k+1}, \epsilon_{k+1})\}$, where $r_{k+1}$ is the first rational number in $(r_k, \epsilon_k)_o \cap Y_{k+1}$, and then $\epsilon_{k+1}$ is the first positive rational number such that $(r_{k+1}, \epsilon_{k+1}) \subseteq (r_k, \epsilon_k)_o$ and such that $2\epsilon_{k+1} \leq \epsilon_k$.
  - If there is $k \in \mathbb{N}$ with infinitely many attempts $[(r_1, \epsilon_1), \ldots, (r_k, \epsilon_k)]$ in $A$, we put $\Gamma(A) = A$.
  - If none of the above apply, we proceed as follows: for each $k$, let $[(r_1, \epsilon_1), \ldots, (r_k, \epsilon_k)]$ be the $\preceq$-maximal attempt in $A$ of length $k$. It is easy to see that for $i \leq k$, the tag $(r_i, \epsilon_i)$ will be independent of the choice of $k$. Let $x$ be the unique element in $\bigcap_{i \in \mathbb{N}} (r_i, \epsilon_i)_c$. We consider the following sub-cases:
    1. If $x \in \bigcap_{i \in \mathbb{N}} Y_i$, let $\Gamma(A) = A$.
    2. Otherwise, let $k$ be minimal such that $x \notin Y_k$ and we define $\Gamma(A) = A \cup \{(r_1, 1), \ldots, (r_{k-1}, \epsilon_{k-1}), (r_k, \frac{2k}{k+1})\}$.

Intuitively, in all cases where $A$ is totally ordered by $\prec$, we either define $\Gamma(A) = A$ or we add one new element on top of $A$. This means that $\Gamma$, seen as an inductive definition, will generate a well-ordered set $A_\infty$ of attempts such that $\Gamma(A_\infty) = A_\infty$. Moreover, we define $\xi(\{Y_n\}_{n \in \mathbb{N}}) = x$ as in case iv). If this is the ‘stopping’ case. Otherwise, we define $\xi(\{Y_n\}_{n \in \mathbb{N}}) = 0$.

The resulting functional $\xi$ is clearly computable in $\text{IND}$ and $\exists^2$. It remains to show that iv). If $\{Y_n\}_{n \in \mathbb{N}}$ will be the ‘stopping’ case when each $Y_n$ is open and dense. We consider the other alternatives, and show that they are impossible in this situation.

- Since we generate a totally ordered set, case o) will never be relevant for any input $\{Y_n\}_{n \in \mathbb{N}}$.
- Since $Y_1$ is open and dense, we start the recursion by adding an element in case i). Since $Y_{k+1}$ is open and dense, we will also continue the recursion when we are in case ii).
- The remaining alternative is case iii). In this case, there will be a least $k$ for which this is possible. Since the only way to develop $A$ sideways (using $<$ on tags) is via case iv).2, there will be a maximal attempt $[(r_1, \epsilon_1), \ldots, (r_{k-1}, \epsilon_{k-1})]$ of length $k - 1$, and tags $(r_k, \frac{2k}{k+1})$ such that $[(r_1, \epsilon_1), \ldots, (r_{k-1}, \epsilon_{k-1}), (r_k, \frac{2k}{k+1})] \in A$ for all $n$. By the construction, $r_k \in Y_k$, and $Y_k$ is open, so there is an $n \in \mathbb{N}$ such that $(r_k, \frac{2k}{k+1})_c \subseteq Y_k$. But then, whenever we are employing case iv).2 after the attempt $[(r_1, \epsilon_1), \ldots, (r_{k-1}, \epsilon_{k-1}), (r_k, \frac{2k}{k+1})]$ enters $A$, we will ask if some $x \in (r_k, \frac{2k}{k+1})_c$ is in the intersection of all $Y_m$, and if the answer is that it is not, we will not find $Y_k$ to be the guilty one. So, when all $Y_n$ are open, the tree of attempts that we are constructing will be finitely branching.

We have now proved the following theorem.
Theorem 4.5. There is a total type 3 functional $\xi$ computable in $\text{IND}$ and $\exists^2$ such that whenever $\{Y_n\}_{n \in \mathbb{N}}$ is a sequence of open, dense sets, then

$$\xi(\{Y_n\}_{n \in \mathbb{N}}) \in \bigcap_{n \in \mathbb{N}} Y_n.$$ 

By contrast, we also have the following negative result.

Theorem 4.6. There is no Baire-realiser $\xi$ computable in a functional of type 2.

We will prove this theorem by contradiction. We first have to develop some machinery and associated lemmas; then we will prove the theorem by reference to the machinery and the lemmas.

First of all, assume that there is an index $d$ and a type 2 functional $F$ such that for all sequences $\{Y_n\}_{n \in \mathbb{N}}$ of subsets of $\mathbb{N}^\mathbb{N}$ we have that the following:

(i) the function $f := \lambda a \in \mathbb{N}.\{d\}F, a, \{Y_n\}_{n \in \mathbb{N}}$ is total,

(ii) if each $Y_n$ is open and dense, then $f \in \bigcap_{n \in \mathbb{N}} Y_n$.

We will show that this assumption leads to a contradiction, by constructing, from $d$ and $F$, a sequence $\{Y_n\}_{n \in \mathbb{N}}$ for which (i) and (ii) fail. The construction is based on Moschovakis’ definition of computation trees from [21], but we provide most details. Let us first fix some notation, where we write $Y$ for $\{Y_n\}_{n \in \mathbb{N}}$.

Notation 4.7.

- A computation tuple is a sequence $\langle e, \bar{a}; c \rangle$ indicating the terminating computation $\{e\}(F, Y, \bar{a}) = c$, where $\bar{a}$ is a finite sequence of numbers and we modify Kleene’s S8 as follows:
  - If $e = (8, 0, d)$, then $\{e\}(F, Y, \bar{a}) := F(f)$.
  - If $e = (8, n + 1, d)$, then
    $$\{e\}(F, Y, \bar{a}) := \begin{cases} 0 & \text{if } f \in Y_n \\ 1 & \text{if } f \notin Y_n \end{cases},$$
    where in both cases $f = \lambda a.\{d\}(F, Y, a, \bar{a})$.
- In an incomplete computation tuple $\langle e, \bar{a} \rangle$ we leave out the final $c$, indicating that the value of the computation is unknown (possibly forever).
- The set of incomplete computation tuples $\langle e, \bar{a} \rangle$ is enumerated via a standard sequence numbering, and we let $n(\langle e, \bar{a} \rangle)$ be the corresponding number.

Let $\varepsilon$ denote the empty sequence of integers. We assume throughout the construction that $e_0 = \langle 8, 0, \bar{d} \rangle$ is such that we for all $Y$ have that

$$\{e_0\}(F, Y, \varepsilon) \downarrow.$$

Let us first consider the well-understood case where $Y$ is fixed and $\{e_0\}(F, Y, \varepsilon) \downarrow$. Then we can find the value $c$ by building the computation tree for the computation by transfinite induction. We start with the top node $\langle e_0, \varepsilon \rangle$, i.e. an incomplete computation tuple, and then in a combined top-down and bottom-up procedure construct a tree of incomplete and complete as explained below. In the process, we may add new incomplete computation tuples and we may turn incomplete ones to complete ones. We give a semi-formal description of he inductive process:

- In the case of composition as follows:
  $$\{e\}(F, Y, \bar{a}) = \{e_1\}(F, Y, \{e_2\}(F, Y, \bar{a}), \bar{a})$$
  we first fill in $\langle e_2, \bar{a} \rangle$ as an incomplete sub-computation of $\langle e, \bar{a} \rangle$. When we later observe that $\langle e_2, \bar{a}; b \rangle$ is the proper sub-computation, we can also fill
in ⟨e_1, b, \vec{a}⟩ as an incomplete sub-computation. When we then at an even later stage realise that ⟨e_1, b, \vec{a}; c⟩ is the proper sub-computation, we can make ⟨e, \vec{a}⟩ complete as ⟨e, \vec{a}; c⟩.

- Primitive recursion can be seen as iterated composition, and is therefore handled in a similar way.
- For the rest of the schemes, it is obvious what is going on: either the incomplete computation tuple at hand is one of an initial computation, and we can fill in the correct value right away, or the set of incomplete tuples for the immediate sub-computations is uniquely given, we have to wait for the process to complete these, and then we can find the right value of the one at hand.

The whole process can be seen as a simultaneous inductive definition of the construction of the tree of incomplete computation tuples and the completion of these.

The above describes the construction when Y is fixed, but in order to obtain the desired contradiction we will have to construct Y and the computation tree simultaneously, which adds complications. The major problem is that we do not know Y = \{Y_n\}_{n \in \mathbb{N}} when we construct the tree, but we have to make a decision what to answer whenever the procedure for constructing the computation tree requests an answer to Y_n(\lambda a.\{d\}(F, Y, a, \vec{a})). Our solution to this problem is that the first time f in the form of \lambda a.\{d\}(F, Y, a, \vec{a}) is needed in our computation tree, as an input to F or to some Y_n, we define f /∈ Y_n exactly when n = n(⟨d, \vec{a}⟩). One useful feature of this strategy is that Y_n then either is all of \mathbb{N}^N or just \mathbb{N}^N with one point missing, so Y_n is open and dense. Another useful feature is that f is not an acceptable value of ξ(Y) since f is left out of one Y_n. One complication is that we have to convert the tree we are constructing into a well-ordering in order to talk about e.g. ‘the first occurrence’; another complication is that we have to ensure that whenever we want to give the correct value to an S8-computation tuple, we already know which functions are used in earlier S8-computations.

We will now give the details of the construction. First some conventions and some intuition are needed.

**Definition 4.8 (Computation paths).**

- A computation path will be a finite sequence (t_0, ..., t_k) of computation tuples such that t_{i+1} is a sub-computation of t_i as defined below. Each t_i may be complete or incomplete, but if t_i is complete and j > i, then t_j must also be complete. This reflects that we cannot give a value to a computation without knowing the values of all sub-computations.
- In a complete computation path, all computation tuples are complete.
- If t is a computation tuple, we will order the possible sub-computations. In this ordering, we will not discriminate between an incomplete sub-computation and its possible completion. In the process we are about to define, certain incomplete computation tuples will be turned into complete ones, and we do not want to change the position in the overall ordering.
- When we construct our tree by transfinite recursion, we will refer to the Kleene-Brouwer ordering based on the node-wise ordering of the sub-computations, meaning that if we extend a computation path to a longer one, we move down in the ordering.
Based on Definition 4.8, we now introduce the tree of computation paths. We establish below that this tree must be well-founded.

**Definition 4.9 (Tree of computation paths).** By recursion on the ordinal \( \alpha \), we construct a tree \( T_\alpha \) of computation paths as follows.

First of all, if \( \alpha = 0 \), we let \( T_\alpha \) consist of the single computation-path \( (t_0) \), where \( t_0 \) is the incomplete computation tuple \( \langle e_0, \varepsilon \rangle \), the computation the process aims to find the value of.

Secondly, if \( \alpha \) is a limit ordinal, we let \( T_\alpha = \lim_{\beta < \alpha} T_\beta \). Since we at each step described below either will add some incomplete sub-computation tuples at the end of a computation path that has been introduced at an earlier stage, or turn one incomplete computation tuple in the tree into a complete one, this limit makes sense.

Thirdly, if \( \alpha = \beta + 1 \), and all computation paths in \( T_\beta \) are complete, we stop.

From now on, assume that the latter is not the case, and also assume that \( T_\beta \) is well founded, and thus well ordered by the Kleene-Brouwer ordering we introduce in the process. Let \( (t_0, \ldots, t_k) \) be the least element of \( T_\beta \) in this Kleene-Brouwer ordering consisting of entirely incomplete computation tuples. What to do, is split into several cases, as follows.

- **If** \( t_k \) is a computation for S1, S2 or S3, i.e. an initial computation. Turn \( t_k \) into the correct complete version and let \( T_\alpha \) be the resulting tree.
- **If** \( t_k = \langle e, \vec{a} \rangle \) where \( e \) is an index for
  \[
  \{e\}(F, \vec{X}, \vec{a}) = \{e_1\}(F, \vec{X}, \{e_2\}(F, \vec{X}, \vec{a})
  \]
  By the choice of \( (t_0, \ldots, t_k) \) there will be no incomplete extension in the tree \( T_\beta \). Thus there will be three subcases:
  1. **There is no extension of** \( (t_0, \ldots, t_k) \) **to** \( T_\beta \) **at all:** Then add \( (t_0, \ldots, t_{k+1}) \) to \( T_\beta \), where \( t_{k+1} = \langle e_2, \vec{a} \rangle \).
  2. **There is an extension** \( (t_0, \ldots, t_{k+1}) \) **to** \( T_\beta \), **where** \( t_{k+1} = \langle e_2, \vec{a}; b \rangle \), **but no extension of the form** \( (t_0, \ldots, t_k, t'_{k+1}) \) **where** \( t'_{k+1} = \langle e_1, \vec{a}; c \rangle \). Then add \( (t_0, \ldots, t_k, t''_{k+1}) \) to \( T_\beta \), where \( t''_{k+1} = \langle e_1, \vec{a} \rangle \). In forming the ordering of \( T_\alpha \), we let this sub-computation will be above the first one in our ordering of sub-computations.
  3. **There is an extension** \( (t_0, \ldots, t_k, t_{k+1}) \) **of** \( (t_0, \ldots, t_k) \) **to** \( T_\beta \) **where** \( t_{k+1} = \langle e_2, \vec{a}; b \rangle \), **and an extension** \( (t_0, \ldots, t_k, t'_{k+1}) \) **in** \( T_\beta \) **where** \( t'_{k+1} = \langle e_1, \vec{a}; c \rangle \). Then obtain \( T_\alpha \) by replacing \( t_k \) with \( (e, \vec{a}; c) \) in the computation path at hand.
- The cases where \( t_k \) is a computation for one of the schemes S5 (primitive recursion), S6 (permutation) or S9 (enumeration) are left for the reader as they just will be similar to, or simpler than, the case for S4. In our case S6 does not come to use, but it will in any case be an initial computation if we allow for function arguments.
- **If** \( t_k \) is an S8-computation, \( t_k = \langle e, \vec{a} \rangle \) where
  \[
  \{e\}(F, \vec{X}, \vec{a}) = H(\lambda a. \{d\}(F, \vec{X}, \vec{a}))
  \]
  where \( H = F \) or \( H = Y_n \) for some \( n \). There will be two subcases:
  1. **If** \( (t_0, \ldots, t_k) \) **has no extensions in** \( T_\beta \), we extend \( T_\beta \) to \( T_\alpha \) by adding all computation paths \( (t_0, \ldots, t_k, t_{k+1, a}) \) for each \( a \in \mathbb{N} \), where \( t_{k+1, a} = \langle d, a, \vec{a} \rangle \). We well-order these extensions by the value of \( a \).
(2) If \((t_0, \ldots, t_k)\) has extensions in \(T_\beta\), the added computation tuple in all such extensions must be complete, by choice of \((t_0, \ldots, t_k)\). Moreover, since item (1) is the only way we add extensions to an S8-computation, there is a function \(f\) such that we have an extension with \(t_{k+1,a} = \langle d, a, \vec{a}; f(a) \rangle\) for each \(a \in \mathbb{N}\). Now, by choice of \((t_0, \ldots, t_k)\) again, if there is any computation path \((s_0, \ldots, s_j)\) in \(T_\beta\) below \((t_0, \ldots, t_k)\) in the Kleene-Brouwer ordering, where \(s_j\) is an S8-computation, \(s_j\) has to be complete, and some function \(g\) has been introduced. If \(f\) has already been introduced as some \(g\) this way, we know the value of \(H(f)\) from before, and use this to make \(t_k\) complete. If \(f\) is introduced for the first time while we replace \(T_\beta\) with \(T_\alpha\), we let \(n = \alpha(\langle d, \vec{a} \rangle)\) as defined above, we let \(f \notin Y_n\), and \(f \in Y_m\) for \(m \neq n\), and use this to turn \(t_k\) into a complete computation tuple for all cases of \(H\).

This ends the construction and Definition 4.9.

We have not said what to do in the case when \(T_\alpha\) is not well-founded, in which case we cannot identify the least \((t_0, \ldots, t_k)\) where all \(t_i\) are incomplete. Lemma 4.11 show that this is never the case. The argument is based on Lemma 4.10 which has an easy proof, also under the assumption that the recursion stops when the tree \(T_\alpha\) is not well-founded.

**Lemma 4.10.** For each ordinal \(\alpha\) and \((t_0, \ldots, t_k) \in T_\alpha\), if \(t_k\) is complete, then \(t_k\) is the computation tuple of a terminating computation, where \(Y\) is interpreted as the sequence of partial sets defined at stage \(\alpha\).

**Proof.** Trivial, by induction on \(\alpha\). \(\square\)

**Lemma 4.11.** For each ordinal \(\alpha\), \(T_\alpha\) is a well founded tree.

**Proof.** We obviously need a limit ordinal \(\alpha\) to introduce an infinite descending sequence. Assume that there is one, and let \((t_0, t_1, \ldots)\) be the leftmost one. Let \(Y\) be a total extension of the sequence of partial sets \(Y_\alpha\) constructed at level \(\alpha\). By Lemma 4.10, the sequence will consist only of incomplete computation tuples, where each extension represents a sub-computation. This will be a so called Moschovakis witness, witnessing that \(\{e_0\}(F, Y, \varepsilon)\), which again contradicts the assumption. We need Lemma 4.10 in order to verify that the sequence is a Moschovakis witness when passing an instance of composition (or primitive recursion). Note that in the presence of S9, we do not need the scheme S5, primitive recursion, so for the understanding, one may ignore this case. \(\square\)

We can now complete the proof of Theorem 4.6.

**Proof.** Assume that the theorem is false. Then there is an \(F\) and an index \(d\) such that for all \(Y\) we have \(\xi(Y) = \lambda a.\{d\}(F, Y, a)\). Let \(e_0\) be an index such that

\[
\{e_0\}(F, Y, \varepsilon) = F(\lambda a.\{d\}(F, Y, a)).
\]

When we apply our construction above to this \(e_0\), we construct a \(Y\) where each \(Y_n\) is open and dense, but where every function \(f\) appearing in the computation tree of \(\{e_0\}(F, Y, \varepsilon)\) is left out of exactly one \(Y_n\). This will in particular be the case for \(\lambda a.\{d\}(F, Y, a)\), so this is not an acceptable value for \(\xi(Y)\) after all. \(\square\)

We finish this section with a remark on future research.
Remark 4.12. The proof of Theorem 4.6 is very different from known proofs of theorems expressing that certain type 3 functionals are not computable in any functional of type 2; we refer to [27–30] for the latter kind of proofs. Baire realisers are not unique, but as shown in Theorem 4.5, there is a specimen computable in IND. This begs the question of the necessary complexity of Baire realisers, and how they compare to realisers for HBU and Pincherle’s Theorem, as studied in [27–30]. We have no answer to this, and offer it as a research problem.

5. A finer computational study of open sets

5.1. Introduction. In the previous, we have considered two different representations of open sets, namely the standard (RM) one as in (1.1) and the approach via characteristic functions as in Definition 1.2. There are of course other possible representations, namely as part of (R.1)-(R.4) below; in this section, we show that (R.3) and (R.4) are computationally equivalent, and study the computational properties of the ‘conversion’ functional $\Delta$ that converts a representation as in (R.2) to a representation as in (R.3). The $\Delta$-functional has interesting properties, as follows.

(P1) $\Delta$ is not computable in any type 2 functional, but computable in any Pincherle realiser, a class weaker than $\Theta$-functionals (Theorem 5.5).

(P2) $\Delta$ is unique, genuinely type 3, and adds no computational strength to $\exists^2$ in terms of computing functions from functions (Corollary 5.6).

Prior to the study of (R.2) and (R.3), we believed that the only way to find a functional with properties (P1) and (P2) would be through some ad hoc construction and that there would be no natural examples.

For the sake of simplicity, we restrict the attention to $[0,1]$, though the $\sigma$-compactness of $\mathbb{R}$ makes it easy to extend all results to $\mathbb{R}$. Thus, we consider the following four ways of representing an open set $O$ in $[0,1]$. For the sake of notational simplicity, we let $(a,b)$ denote $(a,b) \cap [0,1]$.

(R.1) The set $O$ is represented by itself, its characteristic function or as the set of points where a function $Y : [0,1] \to \mathbb{R}$ takes a positive value. We just have the extra information that $O$ is open, i.e. as in Definition 1.2.

(R.2) The set $O$ is represented by a function $Y : [0,1] \to [0,1]$ such that
(i) we have $O = \{x \mid Y(x) > 0\}$,
(ii) if $Y(x) > 0$, then $(x - Y(x), x + Y(x)) \cap [0,1] \subseteq O$.

(R.3) The set $O$ is represented by the continuous function $Y$ where
(i) $Y(x)$ is the distance from $x$ to $[0,1] \setminus O$ if the latter is nonempty,
(ii) $Y$ is constant 1 if $O = [0,1]$.

(R.4) The set $O$ is given as the union of a sequence of open rational intervals $(a_i, b_i)$, the sequence being a representation of $O$.

Assuming $\exists^2$, it is clear that the information given by a representation increases when going down the list. For completeness, we prove that (R.3) and (R.4) are the same from the computational point of view.

Theorem 5.1. Items (R.3) and (R.4) are computationally equivalent modulo $\exists^2$.

Proof. Let $Y$ be continuous as in (R.3). Then $Y$ has an RM code computable in $\exists^2$ by [20, §4]. From this representation we can decide if $Y$ is constant 1 or if $Y(x) = 0$ for at least one $x$. Let $\alpha$ be this representation for $Y$. Then $x \in O$ if and only if there is some $((a,b), (c,d)) \in \alpha$ such that $c > 0$ and $x \in (a,b)$, and the
set of intervals \((a,b)\) where \(\{(a,b),(c,d)\} \in \alpha\) for some \((c,d)\) with \(c > 0\) will be a representation of \(O\) in the sense of (R.4).

Now assume that we have a set \(A\) of open rational intervals defining \(O\) as in (R.4). Then \(O = [0,1]\) if and only if \(A\) contains a finite sub-covering of \([0,1]\). If it does, we let \(Y\) be the constant 1. If not, let \(x\) be given. If \(x \notin (a,b)\) for all \((a,b) \in A\), we let \(Y(x) = 0\). If \(x \in (a,b)\) for some \((a,b) \in A\), we let \(Y(x)\) be the supremum of the set of rationals \(r\) such that \(A\) contains a finite sub-covering of \([x-r,x+r]\). □

In RM, \(\text{ACA}_0\) is equivalent to the fact that closed sets are located ([11, Theorem 1.2]). The previous theorem similarly expresses that a set is open if and only if the complement is located. Next, we study the computational relation between the representations defined by (R.2) and (R.3).

5.2. Converting between representations. In this section, we study the complexity of operators that produce a representation as in (R.3), or equivalently by Theorem 5.1: as in (R.4), from a representation as in (R.2). We choose to study (R.3) as this representation is unique for each open set, resulting in a unique functional (in more ways than one), as follows.

**Definition 5.2.** Let \(\Delta^3\) be the functional such that \(\Delta(Y)\) represents an open set \(O\) as in (R.3) whenever \(Y\) represents \(O\) as in (R.4).

The following proof is straightforward in light of similar proofs in [27–30], and we therefore only provide a sketch.

**Lemma 5.3.** The functional \(\Delta\) is not computable in any type 2 functional.

**Proof.** Given \(F^2\), we construct \(Y^2\) with the following properties.

- The value \(Y(f)\) is defined if \(f\) represents a fast-converging sequence of rational numbers in \([0,1]\),
- If the sequence represented by \(f\) is equivalent to a sequence represented by some \(g\) computable in \(F\), we use Gandy selection for \(F\) to find an index \(e\) for one such \(g\) as computable in \(F\). Note that the Gandy-search is such that the resulting \(g\), and index for it, respects equivalence between representations of reals. We then define \(Y(f) = 2^{-(e+2)}\) for the aforementioned index \(e\).

The crux of the previous construction is as follows: the functional \(Y\) represents \(O = [0,1]\) but no Kleene-algorithm relative to \(F\) and \(Y\) is able to recognise this. Indeed, since \(Y\) is partially computable in \(F\) when restricted to functions computable in \(F\), it only covers a subset of measure below \(\frac{1}{2}\) in this situation. □

Next, we show that \(\Delta\) is computable from \(\exists^2\) and a Pincherle realiser, i.e. a realiser for Pincherle’s theorem, a concept first introduced in [30]. The latter theorem expresses that a locally bounded functional on \(C\) is also bounded (see [33, p. 67]); a Pincherle realiser (PR for short) computes this upper bound in terms of some of the other data. We consider the following equivalent form of Pincherle realisers, going back to an equivalent formulation by Pincherle himself (see [30, 33]). Note that we assume that \(Y\) is extensional on the reals.

**Definition 5.4.** A PR is any functional \(M_{\exists}^3\) such that for any \(Y : [0,1] \to \mathbb{R}^+\), the number \(M_{\exists}(Y) >_R 0\) is a lower bound for all \(Z : [0,1] \to \mathbb{R}\) locally bounded away from zero by \(Y\), i.e. \((\forall x, y \in [0,1])(|x - y| < Y(x) \rightarrow Z(y) > Y(x))\).
Noe that the functional \( Y \) is a realiser for ‘\( Z \) is locally bounded from zero’. As discussed in [30], Pincherle assumes the existence of such realisers (for local boundedness) in [34]. The following theorem is interesting, as PRs are strictly weaker than \( \Theta \)-functionals (realisers for HBU), as shown in [30].

**Theorem 5.5.** The functional \( \Delta \) is computable in any PR and \( \exists^2 \).

**Proof.** Let \( Y \) represent the open set \( O \) as in (R.2). We first show that any PR \( M_u \) and \( \exists^2 \) allows us to decide if \( O = [0, 1] \) or not.

Define \( Y_n(x) \) as \( Y(x) \) if \( Y(x) > 0 \) and \( 2^{-n} \) otherwise. If \( O = [0, 1] \) then \( Y_n = Y \) for all \( n \), and \( M_u(Y_n) \) is positive and independent of \( n \). If \( x_0 \notin O \), there is no \( x \) such that \( Y(x) > 0 \) and \( x_0 \) is in the neighbourhood around \( x \) defined by \( Y(x) \), so \( 2^{-n} \) will be the lower bound on \( Z(x_0) \) induced by \( Y_n \). Thus \( 0 < M_u(Y_n) \leq 2^{-n} \).

Hence, we can decide if \( O = [0, 1] \) or not using the sequence \( \lambda n.M_u(Y_n) \).

Next consider \( x \in [0, 1] \) and assume that there is some (unknown) \( z \notin O \). For each rational \( r > 0 \), we define the following set:

\[
O_{x,r} = O \cup \{ y \in [0, 1] : |x - y| > r \},
\]

and we let \( Y_{x,r} \) be the representation of \( O_{x,r} \) provided by (R.2). Now let \( Z(x) \) be the distance from \( x \) to the complement of \( O \). Then \( Z(x) \leq r \) if and only if \( O_{x,r} \neq [0, 1] \), and we can use \( M_u \) as above deciding this for each \( x \in [0, 1] \) and \( r \).

Since we can decide if \( Z(x) \leq r \) uniformly in \( r \), we can use \( \exists^2 \) to compute \( Z(x) \). \( \square \)

We have the following corollary, establishing the above claims concerning \( \Delta \).

**Corollary 5.6.**

(a) For each \( f \in \mathbb{N}^\mathbb{N} \), all functions computable in \( \Delta, \exists^2 \) are also hyperarithmetical in \( f \) alone.

(b) There is no PR \( M_u \) computable in \( \Delta \) and \( \exists^2 \).

**Proof.** Since any \( \Theta \)-functional computes some PR, \( \Delta \) is uniformly computable in any \( \Theta \)-functional, and the only functions that are uniformly computable in any \( \Theta \) and \( \exists^2 \) are the hyperarithmetical ones (see [29, 30]). This readily relativises to any function \( f \), i.e. the first item follows.

Since each \( M_u \) computes, not uniformly, a function that is not hyperarithmetical, while \( \Delta \) computes only hyperarithmetical ones, no \( M_u \) can be computable in \( \Delta \). \( \square \)

5.3. **Representations, Heine-Borel, and Baire.** We finish this paper with two theorems on the computational complexity of the countable Heine-Borel theorem and the Baire Category Theorem when formulated using the representation (R.2).

First of all, \( \Delta + \exists^2 \) suffices to compute the finite sub-cover from HBC when formulated using (R.2); no type two functional can replace \( \Delta \) here.

**Theorem 5.7.** Let \( Y_n \) be a representation of the open set \( O_n \subseteq [0, 1] \) as in (R.2) and assume \( [0, 1] \subseteq \bigcup_{n \in \mathbb{N}} O_n \). Then \( \Delta + \exists^2 \) computes \( k \in \mathbb{N} \) such that \( [0, 1] \subseteq \bigcup_{i \leq k} O_i \).

The functional \( \Delta \) cannot be replaced by any functional of type 2 here.

**Proof.** Using \( \Delta \) on each \( Y_n \), and the equivalence between (R.3) and (R.4), we obtain a standard RM-covering of the unit interval, and then we only need a realiser for WKL, computable in \( \exists^2 \), to prove the first claim.

For the second claim, we use one of our standard techniques: given \( F^2 \), assume that the realiser \( \beta \) for HBC (formulated with (R.2)) is computable in \( F + \exists^2 \). We
let each $Y_n$ be the same: it represent $[0, 1]$ as an open set but in such a way that $Y_n$ restricted to the reals computable in $F + \exists^2$ is partially computable in $F + \exists^2$ and only defines an open set of measure $\leq \frac{1}{2}$. Let $k = \beta(\{Y_n\})$, and let $Y_n'$ be derived from $Y_i$ for $i \leq k + 1$ by removing one, and the same, point that is outside this set. Let $Y_i' = Y_i$ for $i > k + 1$. Then $\beta(\{Y_n\}) = \beta(\{Y_n'\})$, contradicting what $\beta$ is supposed to achieve.

The first part of this proof also implies that the optimal realiser for HBC (formulated using (R.2)), the one selecting the least $k$ as in Theorem 5.7, is equivalent to $\Delta$, given $\exists^2$. We leave the proof of this to the reader.

Secondly, the Baire category theorem becomes a lot ‘tamer’ from the computational point of view upon the introduction of (R.2).

**Theorem 5.8.** Assuming a representation as in (R.2) is given for all the $Y_n$, we can compute a Baire realiser relative to $\exists^2$. If we know that each $Y_n$ represents a dense open set, we can even avoid $\exists^2$.

**Proof.** Let $Y_n$ be given for each $n$ and first assume that each $Y_n$ is an (R.2) representation of the open, dense set $O_n$. Then the school-book proof of the Baire Category Theorem can be transformed to an algorithm by using $Y_n$ restricted to the rational numbers to find the shrinking sequence of open-and-then-closed intervals, and iterated search over $\mathbb{Q}$ for the sequence that will converge to a point in the intersection. We need to search for $q \in \mathbb{Q}$ within an open interval such that $Y_n(q) > 0$, but since the latter relation is $\Sigma^0_1$, this is effective. If one $Y_n$ does not represent an open, dense set, the only thing that might prevent this algorithm from terminating is that the search goes on for ever, and whether this will be the case can be decided using $\exists^2$, so we can output a value even then. In this case, it does not matter what the value is, we are only required to have one. □

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