EQUIVARIANT UNFOLDINGS OF G-STRATIFIED PSEUDOMANIFOLDS

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To Rodolfo Ricabarra, in memoriam.

Abstract. For any abelian compact Lie group \( G \), we introduce a family of \( G \)-stratified pseudomanifolds, whose main feature is the preservation of the orbit spaces in the category of stratified pseudomanifolds. Which generalize a previous definition given in \[8\]. We also find a sufficient condition for the existence of equivariant unfoldings, so we have Intersection Cohomology with differential forms, as defined in \[9\]. Moreover, if \( G \) act on a manifold \( M \), we find a equivariant unfolding of \( M \) which induce a canonical unfolding on the \( k \)-orbits space for every closed subgroup \( K \) of \( G \).

Introduction

Let \( X \) be a Thom-Mather stratified space with depth \( d(X) = n \). The De Rham Intersection Cohomology of \( X \) with differential forms was defined in \[3\] by means of an auxiliar construction called unfolding, which is a continuous map \( \tilde{L} : \tilde{X} \rightarrow X \) where \( \tilde{X} \) is a smooth manifold obtained through a finite composition

\[
L : \tilde{X} = X_n \xrightarrow{L_n} X_{n-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{L_1} X_0 = X
\]

of topological operations \( X_i \xrightarrow{L_i} X_{i-1} \), called elementary unfoldings. This iterative construction is possible because the stratification of \( X \) is controlled by the existence of a family of conical fiber bundles over the singular strata. Later in \[9\] we find a more abstract definition of unfoldings, which impose some conditions of transversality over the singular strata. For instance, if the depth of \( X \) is 1 then the first elementary unfolding of \( X \) is an unfolding in the new sense.

Now let \( G \) be a compact Lie group. We introduce the definition of a \( G \)-stratified pseudomanifold in the category of stratified pseudomanifolds. Our definition is related to a previous one given in \[8\]. A \( G \)-stratified pseudomanifold is a stratified pseudomanifold in the usual sense together with a continuous action preserving the strata, and whose local model near each singular strata is given by a conical slice. We also give a definition of equivariant unfoldings, which is a suitable adaptation

\begin{itemize}
  \item \textbf{Date}: July 2003.
  \item 1991 \textit{Mathematics Subject Classification}. 35S35; 55N33.
  \item \textit{Key words and phrases}. Intersection Cohomology, Stratified Pseudomanifolds.
\end{itemize}
of the usual definition of unfolding to the family of $G$-stratified pseudomanifolds. We give a sufficient condition for the existence of equivariant unfoldings, which is related to the choice of a good family of tubular neighborhoods and a sequence of equivariant elementary unfoldings of $X$. Each elementary unfolding induces an elementary unfolding on each orbit space together with a factorization diagram.

The content of this paper is as follows:

In §1 we introduce the category of stratified pseudomanifolds.

In §2 we define stratified $G$-pseudomanifolds and study their corresponding $K$-orbit spaces, for a closed subgroup $K$ of $G$.

In §3 we study equivariant tubular neighborhoods, which are equivariant versions of the usual ones.

In §4 we introduce the concept of an equivariant explosion of a $G$-pseudomanifold, and present elementary explosions as given in [1].

In §5 we define $G$-pseudomanifolds $X$ stratified with a transversal Thom-Mather structure whose main feature is that the sequence of elementary explosions determine an equivariant explosion of the space $X$, which project naturally onto an equivariant explosion of $X/K$, for a closed subgroup $K$ of $G$.

1. Stratified Pseudomanifolds

In this section we review the usual definitions of stratified spaces, stratified morphisms and stratified pseudomanifolds. For a more detailed introduction see [5], [7].

1.1. Stratified spaces Let $X$ be a Hausdorff, locally compact and 2nd countable space. A stratification of $X$ is a locally finite partition $S_X$ satisfying:

(i) Each element $S \in S_X$ is a connected manifold with the induced topology, which a stratum of $X$.

(ii) If $S' \cap \overline{S} \neq \emptyset$ then $S' \subset \overline{S}$ for any two strata $S, S' \in S_X$. In this case we write $S' \leq S$ and we say that $S$ incides on $S'$. We say that $(X, S_X)$ is a stratified space whenever $S_X$ is a stratification of $X$.

With the above conditions, the incidence relationship is a partial order on $S_X$. Moreover, since $S_X$ is locally finite, any strictly ordered chain

$$S_0 < S_1 < \cdots < S_m$$

in $S_X$ is finite. The depth of $X$ is by definition the supremum (possibly infinite) of the integers $m$ such that there is a strictly ordered chain as above. We write this as $d(X)$.

The maximal (resp. minimal) strata in $X$ are open (resp. closed) in $X$. A singular stratum is a non-maximal stratum in $X$. The union of the singular strata is the singular part of $X$, denoted by $\Sigma \subset X$, which is closed in $X$. Its complement $X - \Sigma$ is open and dense in $X$. The family of minimal strata will often be denoted
by $S_X^{\text{min}}$, while the union of minimal strata will be denoted by $\Sigma^{\text{min}}$, which we call the \textbf{minimal part} of $X$.

1.2. \textbf{Examples} Here there are some examples of stratified spaces.

1. For any manifold $M$ the trivial stratification of $M$ is the family
$$S_M = \{ C : C \text{ is a connected component of } M \}$$

2. For any connected manifold $M$, the space $M \times X$ is a stratified space, with the stratification
$$S_{M \times X} = \{ M \times S : S \in S_X \}$$

Notice that $d(M \times X) = d(X)$.

3. The \textbf{cone} of a compact stratified space $L$ is the quotient space
$$c(L) = L \times [0, \infty) / L \times \{0\}$$

We write $[p,r]$ for the equivalence class of $(p, r) \in L \times [0, \infty)$. The symbol $*$ will be used for the equivalence class of $L \times \{0\}$, this is the \textbf{vertex} of the cone. The family
$$S_{c(L)} = \{ * \} \cup \{ S \times (0, \infty) : S \in S_L \}$$

is the canonical stratification of $c(L)$. Notice that $d(c(L)) = d(L) + 1$.

1.3. \textbf{Stratified subspaces and morphisms} Let $(X, S_X)$ be a stratified space. For each subset $Z \subset X$ the \textbf{induced partition} is the family
$$S_{Z/Y} = \{ C : C \text{ is a connected component of } Z \cap S, S \in S_X \}$$

We will say that $Z$ is a \textbf{stratified subspace} of $X$, whenever the induced partition on $Z$ is a stratification of $Z$.

Now let $(Y, S_Y)$ be another stratified space. A \textbf{morphism} (resp. \textbf{isomorphism}) is a continuous map $f : X \to Y$ (resp. homeomorphism) which smoothly (resp. diffeomorphically) sends strata into strata. In particular, $f$ is a \textbf{embedding} if $f(X)$ is a stratified subspace of $Y$ and $f : X \to f(X)$ is an isomorphism.

Henceforth, we will write $\text{Iso}(X, S_X)$ for the group of isomorphisms of a stratified space $X$. The following statement will be used later, we leave the proof to the reader.

\textbf{Lemma 1.4.} \textit{Let $(X, S_X)$ be a stratified space, and $\mathcal{F} \subset S_X$ a subfamily of equidimensional strata. The connected components of $M = \bigcup_{S \in \mathcal{F}} S$ are the strata in $\mathcal{F}$.}

Stratified pseudomanifolds were used by Goresky and MacPherson in order to introduce the Intersection Homology and extend the Poincaré duality to the family of stratified spaces. For a brief introduction the reader can see [5].

1.5. \textbf{Stratified pseudomanifolds} The definition of a stratified pseudomanifold is made by induction on the depth of the space. More precisely:
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(1) A stratified pseudomanifold of depth 0 is a manifold with the trivial stratification.
(2) An arbitrary stratified space \((X, S_X)\) is a stratified pseudomanifold if, for any singular stratum \(S \in S_X\), there is a compact stratified pseudomanifold \(L_S\) depending on \(S\) (called a link of \(S\)) such that each point \(x \in S\) has a coordinate neighborhood \(U \subset S\) and an embedding onto an open subset of \(X\).

\[
\varphi : U \times c(L_S) \rightarrow X
\]

such that \(x \in \text{Im}(\varphi)\). The pair \((U, \varphi)\) is a chart of \(x\) modelled on \(L_S\).

1.6. Examples Here are some examples of stratified pseudomanifolds.
(1) If \(X\) is a stratified pseudomanifold, then any open subset \(A \subset X\) is also a stratified pseudomanifold. Also the product \(M \times X\) (with the canonical stratification) is a stratified pseudomanifold, for any manifold \(M\).
(2) If \(L\) is a compact stratified pseudomanifold, then \(c(L)\) is a stratified pseudomanifold.

2. G-Stratified Pseudomanifolds

From now on, we fix an abelian compact Lie group \(G\). We will study the family of actions of \(G\) which preserve the strata. Our definition is strongly related to the previous one given in [8]. Also some easy proofs in this section can be seen in [6].

Given a stratified space \((X, S_X)\) and an effective action \(\Phi : G \times X \rightarrow X\); we write \(\Phi(g, x) = gx\) for any \(g \in G, x \in X\). We denoted \(X/K\) by the \(K\)-orbit space for every \(K\) closed subgroup of \(G\), and by \(\pi : X \rightarrow X/K\) the orbit map. The group of \(G\)-equivariant isomorphisms of \(X\) will be denoted by \(\text{Iso}_G(X, S_X)\).

2.1. G-stratified spaces We say that \(X\) is G-stratified whenever:
(1) For each stratum \(S \in S_X\) the points of \(S\) all have the same isotropy group, denoted by \(G_S\).
(2) Each \(g \in G\) induces an isomorphism \(\Phi_g : X \rightarrow X \in \text{Iso}_G(X, S_X)\).

The orbit space \(X/K\) inherits a canonical stratification given by the family

\[
S_{X/K} = \{\pi(S) : S \in S_X\}
\]

Notice also that \(d(X) = d(X/K)\).

2.2. Examples Here are some examples of G-stratified spaces:
(1) Each \(G\)-manifold \(M\) has a natural structure of \(G\)-stratified space, when \(M\) is endowed with the stratification given by orbit types.
(2) If \(X\) is a \(G\)-stratified space, then \(M \times X\) is a \(G\)-stratified space with the action \(g(m, x) = (m, gx)\); for any manifold \(M\).
(3) If \(L\) is a compact \(G\)-stratified space then \(c(L)\), with the action \(g[x, r] = [gx, r]\), is a \(G\)-stratified space.
Lemma 2.3. Let $G$ be an abelian compact Lie group, $K \subset G$ a closed subgroup. Then for any $G$-stratified space $X$ the orbit space $X/K$ is a $G/K$-stratified space.

Proof. Write $[g] \in G/K$ for the equivalence class of $g \in G$. Consider the quotient action

$$\overline{\Phi} : G/K \times X/K \to X/K \quad \overline{g} : \pi(x) = \pi(gx)$$

This action is well defined because $G$ is abelian. So:

- The isotropy groups are constant over the strata of $X/K$: This is straightforward, since for each stratum $S \in S_X$ we have

$$(G/K)_{\pi(S)} = KG_S/K$$

Hence $\pi(S)$ has constant isotropy.

- Each $\overline{g}$ induces an isomorphism $\overline{\Phi}_g : G/K \in Iso(X/K, S_{X/K})$: For each $g \in G$ we have a $K$-equivariant isomorphism $\Phi_g \in Iso(X, S_X)$. Passing to the quotients we obtain an isomorphism $\overline{\Phi}_g : G/K \in Iso(X/K, S_{X/K})$. The differentiability of this map on $\pi(S)$ is immediate from the following commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\Phi_g} & gS \\
\downarrow \pi & & \downarrow \pi \\
\pi(S) & \xrightarrow{\overline{\Phi}_g} & \pi(gS)
\end{array}
$$

Now we introduce the definition of a $G$-stratified pseudomanifold.

2.4. $G$-stratified pseudomanifolds A $G$-stratified pseudomanifold is a stratified pseudomanifold in the usual sense, endowed with a structure of $G$-stratified space (i.e. $G$ acts by isomorphisms) and whose local model is described through conical slices. Conical slices were introduced in [8] in order to state a sufficient condition on any continuous action of a compact Lie group (abelian or not) on stratified pseudomanifold so that the corresponding orbit space would remain in the same class of spaces.

Let $(X, S_X)$ be a $G$-stratified space. Take a singular stratum $S \in S_X$ a point $x \in S$. A conical slice of $x$ in $X$ is a slice $S_x$ in the usual sense of [2], with a conical part transverse to the stratum $S$. In other words:

1. $S_x$ is an invariant $G_S$-space containing $x$.
2. For any $g \in G$, if $gS_x \cap S_x \neq \emptyset$ then $g \in G_S$.
3. $GS_x$ is open in $X$. And
4. There is a $G_S$-equivalence $\beta : \mathbb{R}^i \times c(L) \to S_x$ where $i \geq 0$ and $L$ is a compact $G_S$-stratified space. Here the action of $G_S$ on $\mathbb{R}^i$ is trivial (notice that $\beta$ induces on $S_x$ a structure of $G_S$-stratified space).
The definition of a $G$-stratified pseudomanifold is made by induction on the depth of the space. A $G$-stratified pseudomanifold with depth 0 is a manifold with a smooth free action of $G$. In general, we will say that $X$ is a $G$-stratified pseudomanifold if, for each singular stratum $S \in S_X$, there is a compact $G_S$-stratified pseudomanifold $L_S$ such that each point $x \in S$ has a conical slice
$$\beta : \mathbb{R}^i \times c(L_S) \to S_x$$
and the usual map on the twisted product
$$\alpha : G \times_{G_S} S_x \to X \quad \alpha([g, y]) = gy$$
is an equivariant (stratified) embedding on an open subset of $X$. We say that the triple $(S_x, \beta, L_S)$ is a distinguished slice of $x$.

2.5. Examples

Here there are some examples of $G$-stratified pseudomanifolds.

(1) Take a smooth effective action $\Phi : G \times M \to M$ with fixed points on a manifold $M$ endowed with the stratification by orbit types. By the Equivariant Slice Theorem, $M$ is a $G$-stratified pseudomanifold.

(2) If $X$ is a $G$-stratified pseudomanifold then $M \times X$ is a $G$-stratified pseudomanifold with the obvious action.

(3) If $L$ is a compact $G$-stratified pseudomanifold, then $c(L)$ is a $G$-stratified pseudomanifold with the obvious action.

(4) Any invariant open subspace of a $G$-stratified pseudomanifold is itself a $G$-stratified pseudomanifold.

♣ Remark 2.6. Each $G$-stratified pseudomanifold is a stratified pseudomanifold in the previous sense.

To see this, proceed by induction on the depth. Take a $G$-stratified pseudomanifold $X$. For $d(X) = 0$ the statement is trivial. Assume the inductive hypothesis and suppose that $d(X) > 0$. Take a singular stratum $S \in S_X$, a point $x \in S$ and a distinguished slice $(S_x, \beta, L_S)$ of $x$. The isotropy subgroup $G_S$ acts on $G$ by the restriction of the group operation. We fix a slice $S_e$ of the identity element $e \in G$ with respect to this action. Since $G_S S_e$ is open in $G$, the composition
$$(S_e \times \mathbb{R}^i) \times c(L_S) \to S_e \times (\mathbb{R}^i \times cL_S) \to S_e \times S_x \to$$
$$\to S_e \times (G_S \times_{G_S} S_x) \to (G_S S_e) \times_{G_S} S_x \to X$$
is an embedding. Notice that $L_S$ is a stratified pseudomanifold by induction. Since $S_e \times \mathbb{R}^i \simeq S_e G_S (S \cap S_x)$ is open in $S$. We have obtained a chart of $x$ modelled on $L_S$.

♣ Remark 2.7. If $X$ is a $G$-stratified pseudomanifold and $K$ is any closed subgroup of $G$, then $X$ is also a $K$-stratified pseudomanifold.

It is straightforward that $X$ is a $K$-stratified space. For any singular stratum $S$ and any $x \in S$, in order to choose a distinguished slice in $x$ we proceed as follows: Take a distinguished slice $\beta : \mathbb{R}^i \times c(L_S) \to S_x$ in $x$ with respect to the action of $G$. Take also a slice $V_e$ of the identity element $e \in G$ with respect to the action of...
GSK in G. Then \( \iota \times \beta : (V \times \mathbb{R}^i) \times c(L_S) \to VS_x \) is a distinguished slice of \( x \) with respect to the action of \( K \).

Now we study the factorization of a \( G \)-stratified pseudomanifold when considered as a \( K \)-stratified pseudomanifold for any closed subgroup \( K \subset G \).

**Proposition 2.8.** Let \( G \) be a compact, abelian Lie group; \( K \subset G \) a closed subgroup. If \( X \) is a \( G \)-stratified pseudomanifold then \( X/K \) is a \( G/K \)-stratified pseudomanifold.

**Proof.** As before, write \( \pi: X \to X/K \) for the orbit map induced by the action of \( K \) on \( X \). Proceed by induction on \( l = d(X) \). For \( l = 0 \) it is straightforward, since \( d(X/K) = d(X) = 0 \). Assume the inductive hypothesis and suppose that \( d(X) > 0 \). By §2.3 \( X/K \) is a \( G/K \)-stratified space, so we must verify the existence of conical slices.

Take a singular stratum \( S \in S_X \), fix a point \( x \in S \) and a distinguished slice \( (S_x, \beta, L_S) \) of \( x \). The \( K \)-equivariant isomorphism \( \beta : \mathbb{R}^i \times c(L_S) \to S_x \) induces an isomorphism on the orbit spaces

\[
\beta : \mathbb{R}^i \times c(L_S/G_S \cap K) \to \pi(S_x) \quad \beta(b, [l, r]) = \pi(\beta(b, [l, r]))
\]

Now we will show that the triple \( (\pi(S_x), \beta, L_S/G_S \cap K) \) is a distinguished slice of \( \pi(x) \in X/K \). We do it in three steps.

- **\( \pi(S_x) \) is a slice of \( \pi(x) \)**: This is straightforward, since \( (G/K)_{\pi(x)} = KG_S/K \), the quotient \( \pi(S_x) \) is a \( (G/K)_{\pi(x)} \)-space with the quotient action and the orbit map \( \pi \) is an open map.

- **\( \beta \) is a \( KG_S/K \)-equivalence**: This is immediate, since \( \beta \) is an \( H \)-equivalence. Notice that, by induction on the depth, \( L_S/G_S \cap K \) is a \( KG_S/K \)-stratified pseudomanifold.

- **The induced map \( \overline{\alpha} : (G/K) \times_{(G_S/G_S \cap K)} \pi(S_x) \to X/K \) is an embedding**: This \( \overline{\alpha} \) is given by the rule \( \overline{\alpha}(g, \pi(z)) = \overline{g} \pi(z) \), and is a homeomorphism. We consider the following commutative diagram

\[
\begin{array}{ccc}
G \times_{G_S} S_x & \xrightarrow{\alpha} & X \\
\downarrow \pi & & \downarrow \pi \\
(G/K) \times_{(G_S/G_S \cap K)} \pi(S_x) & \xrightarrow{\overline{\alpha}} & X/K
\end{array}
\]

Since the vertical arrows are submersions, and \( \alpha \) is an embedding, we obtain that \( \overline{\alpha} \) is an embedding. \( \Box \)

### 3. Tubular neighborhoods

Henceforth we fix a compact, abelian Lie group \( G \), and a \( G \)-stratified pseudomanifold \( X \). In this section we will study the family of equivariant tubular neighborhoods, which are equivariant version of the usual ones. Given a singular stratum \( S \) in \( X \), a tubular neighborhood is just a locally trivial fiber bundel over a \( S \), whose
fiber is $c(L_S)$, the cone of the link of $S$; and whose structure group is $\text{Iso}_{G_S}(L_S, S_{LS})$.

We will clarify these ideas immediately, considering a previous step in our way: the definition of a $G$-stratified fiber bundle. The reader will find in [10] a detailed introduction to the fiber bundles, while [11] provides the usual definition of a tubular neighborhood in the stratified context (see also [2] for the smooth case).

### 3.1. $G$-stratified fiber bundles

Let $\xi = (E, p, B, F)$ be a locally trivial fiber bundle with (maximal) trivializing atlas $\mathcal{A}$. We will say that $\xi$ is a $G$-stratified whenever:

1. The total space $E$ is a $G$-stratified space.
2. The base space $B$ is a manifold, endowed with a smooth action $\Psi : G \times B \to B$ and with constant isotropy $H \subset G$ at all its points.
3. The fiber $F$ is a $H$-stratified space.
4. The projection $p : E \to B$ is $G$-equivariant.
5. The group $G$ acts by isomorphisms. In other words, each chart $\varphi : U \times F \to p^{-1}(U) \in \mathcal{A}$ is $H$-equivariant; and for any two charts $(U, \varphi), (U', \varphi') \in \mathcal{A}$ such that $U' \cap g^{-1}U \neq \emptyset$ for some $g \in G$, there is a map $g_{\varphi, \varphi'} : U' \cap g^{-1}U \to \text{Iso}_H(F, S_F)$ such that $\varphi^{-1}g\varphi'(b, z) = (gb, g_{\varphi, \varphi'}(b)z)$.

**Lemma 3.2.** Let $\xi = (E, p, B, F)$ be a $G$-stratified fiber bundle, $H$ the isotropy of $B$. If $F$ is an $H$-stratified pseudomanifold, then $E$ is a $G$-stratified pseudomanifold.

**Proof.** Fix a singular stratum $S$ in $E$ and a point $x \in S$. We must prove the existence of a link $L_S$ depending only on $S$ and, a distinguished slice $(S_x, \beta, L_S)$ in $x$. For this purpose, let’s take a trivializing chart $\varphi : U \times F \to p^{-1}(U) \in \mathcal{A}$ such that $x \in p^{-1}(U)$. Take $z = p(x)$ and a $G$-slice $V_z$ in $B$. Since $V_z$ is contractible, we assume that $V_z \cong \mathbb{R}^k$ and $V_z$ is contained in $U$.

Write $\varphi^{-1}(x) = (z, y) \in V_z \times F$ and take $S'$ the stratum in $F$ containing $y$. Since $F$ is an $H$-stratified pseudomanifold, we can choose a distinguished slice $S_y$ in $y$; say $\beta_0 : S_y \to \mathbb{R}^i \times c(L_{S'})$.

Consider the following composition

$$\varphi(V_z \times S_y) \xrightarrow{\varphi^{-1}} V_z \times S_y \xrightarrow{1 \times \beta_0} V_z \times \mathbb{R}^i \times c(L_{S'}) \cong \mathbb{R}^{i+k} \times c(L_{S'})$$

We will show that $(S_x, \beta, L_S) = (\varphi(V_z \times S_y), (1 \times \beta_0) \circ \varphi^{-1}, L_{S'})$.
is a distinguished slice in $x$. We proceed in three steps.

- **$L_S$ only depends on $S$**: If $(U', \psi) \in \mathcal{A}$ is another trivializing chart covering $x$, $\psi^{-1}(x) = (z, y') \in V_z \times F$ and $\beta'(g, y') : S_{y'} \to \mathbb{R}^i \times c(L_{S'})$ is a distinguished slice in $y'$; then the composition $\beta' \beta^{-1}$ induces an $H$-isomorphism $L_S \cong L_{S'}$.

- **$S_x$ is a conical slice**: We verify the conditions (1) to (4) of [2,4] (1) Since $V_z$ is a slice of $z \in B$, we have $gp(x) = p(gx) = p(x) \in V_z$ for any $g \in G$. So $G_S = H \cap G_S = H_S$, but $\varphi$ is $H$-equivariant, hence $G_S = H_S = H\mathbb{S}$. Again, since $\varphi$ is $H$-equivariant and $S_y$ is $H\mathbb{S}' = G_S$ invariant, we obtain that $S_x$ is $G_S$-invariant.

(2) Take $g \in G$, $x' \in S_x$ such that $g x' \in S_x$. Then $gp(x') = p(gx') \in V_z$, so $g \in H$ and $gp(x') = p(x')$. Since $\varphi$ is $H$-equivariant, if $x' = \varphi(p(x'), y)$ then $g x' = \varphi(p(x'), g y)$, and $g y \in S_y$; hence $g \in H\mathbb{S}' = G_S$.

(3) Take a slice $S_e$ of the identity element $e \in G$ with respect to the action of $H$. Since $S_e$ is contractible, we can assume that $S_e V_z \subset U$. Notice that $S_e H$ is open in $G$. Since $G S_x = \bigcup_{g \in G} g (S_e H) S_x$, we only have to show that $(S_e H) S_x$ is open in $X$. But $\varphi$ is $H$-equivariant and the action of $H$ on $V_z$ is trivial, so we get the following equality
\[
(S_e H) S_x = S_e (H \varphi(V_z \times S_y)) = S_e \varphi(V_z \times HS_y).
\]
Since $HS_y$ is open in $F$ we deduce that $S_e \varphi(V_z \times HS_y)$ is open in $S_e \varphi(V_z \times F)$. Finally we show that $S_e \varphi(V_z \times F) = S_e p^{-1}(V_z)$ is open in $X$: Since $p$ is equivariant and $S_e V_z$ is open in $U$ the set $S_e p^{-1}(V_z) = p^{-1}(S_e V_z) = p^{-1}(S_e HV_z)$ is open in $p^{-1}(U)$ (and so in $X$).

(4) It is straightforward that the map $\beta$ is a $G_S$-equivalence.

- **$S_x$ is a distinguished slice**: We will show that usual the map
\[
\alpha : G \times_{G_S} S_x \to X
\]
is a (stratified) embedding.

(a) **$\alpha$ preserves the strata**: Take a stratum $S^0$ in $S_x$. We will prove that $G' S^0$ is an open subset in some stratum of $X$, for any connected component $G' \subset G$. It is enough to prove this for the connected component $G_0$ of the identity element $e \in G$. Let $H_0$ be the connected component of the identity element $e \in H$. The set $S_e H_0$ is a connected open subset in $S_e H$, so is also connected and open in $G_0$. Since $G_0 S^0$ is connected, we need to prove that $S_e H_0 S^0$ is open in some stratum of $X$. But $S_e H S_x$ is contained in $p^{-1}(S_e V_z)$ and $\varphi$ is a stratified embedding, and so we only have to show that $\varphi^{-1}(S_e H_0 S^0)$ is open in some stratum of $(S_e V_z) \times F$. Consider the map
\[
\Delta : S_e H \times V_z \times S_y \to (S_e V_z) \times F
\]
\[\Delta: (gh,b,l) \mapsto (ghb,(gh)_{\varphi}(b)(z)) = (gb_{\varphi},(b)(hz))\]
Let $S^1$ be the stratum of $S_y$ such that $S^0 = \varphi(V_z \times S^1)$. By hypothesis $S_y$ is a distinguished slice of $y$ in $F$, and there is a stratum $S^2$ in $F$ such that $H_0 S^1$ is open.
in \(S^2\). Notice that
\[
\varphi^{-1}(S_eH_0S^0) = \Delta(S_eH_0 \times V_z \times S^1) = \Delta(S_e \times V_z \times H_0S^1)
\]
Also, since \(\varphi\) is \(H\)-equivariant, we have
\[
p(\varphi^{-1}(S_eH_0S^0)) = S_eV_z
\]
Hence the projection \(pr_2: U \times F \rightarrow F\) sends \(\varphi^{-1}(S_eH_0S^0)\) on some open subset of \(S^2\). Notice that \(\varphi^{-1}(S_eH_0S^0)\) is connected, so
\[
pr_2(\varphi^{-1}(S_eH_0S^0)) = \bigcup_{(g,b) \in S_e \times V_z} g_{\varphi}(b)(H_0S^1)
\]
is a connected subset of \(F\). Each \(g_{\varphi}(b)\) is an \(H\)-equivariant stratified isomorphism; hence \(g_{\varphi}(b)(H_0S^1)\) is open is some stratum of \(F\) with the same dimension of \(S^2\). Since \(e_{\varphi}(b)(H_0S^1) = H_0S^1 \subset S^2\), by \([1.4](2)\) the set \(\bigcup_{(g,b) \in S_e \times V_z} g_{\varphi}(b)(H_0S^1)\) is contained in \(S^2\).

(b) \(\alpha\) is smooth on each stratum: Since \(G \times_{G_x} S_x\) has the quotient stratification induced on \(G \times S_x\) by the action of \(H\), the stratification of \(S_x\) is induced by \(X\) and the action of \(G\) is smooth on each stratum of \(G \times X\). We conclude that the restriction of \(\alpha\) to each stratum is smooth. \(\square\)

3.3. Equivariant tubular neighborhoods An equivariant tubular neighborhood is a conical locally trivial fiber bundle. For a detailed introduction the reader can see \([7, 11]\). In \([11]\), the tubular neighborhoods are used in order to show the existence of an unfolding for any manifold endowed with a Thom-Mather structure. We will provide an equivariant version of this fact for any \(G\)-stratified pseudomanifold.

Let \(X\) be a \(G\)-stratified pseudomanifold with \(d(X) > 0\). Let’s take a singular stratum \(S\) in \(X\). An equivariant tubular neighborhood of \(S\) is a \(G\)-stratified fiber bundle \((T_S, \tau_S, S, c(L_S))\) with (maximal) trivializing atlas \(\mathcal{A}\), verifying

(1) \(T_S\) is an open invariant neighborhood of \(S\) and the inclusion \(S \rightarrow T_S\) is a section of \(\tau_S: T_S \rightarrow S\).

(2) \(G\) preserves the conical radium: For any two charts \((U, \varphi), (U', \varphi') \in \mathcal{A}\) such that \(U' \cap g^{-1}U \neq \emptyset\) for some \(g \in G\), there is a map
\[
g_{\varphi, \varphi'}: U' \cap g^{-1}U \rightarrow \text{Iso}_{G_S}(L_S, \mathcal{S}_L)
\]
such that
\[
\varphi^{-1}g_{\varphi'}(b, [l, r]) = (gb, [g_{\varphi, \varphi'}(b)l, r])
\]
This allows us to define a (global) radium on \(T_S\), as the map \(\rho_S: T_S \rightarrow [0, \infty)\) satisfying
\[
\rho_S(\varphi(z, [l, r])) = r \quad \forall (z, [l, r]) \in U \times c(L_S); (U, \varphi) \in \mathcal{A}
\]
We also define the radial action \(\delta_S: \mathbb{R}^+ \times T_S \rightarrow T_S\) as follows
\[
\delta_S(r, x) = \varphi(z, [l, rt]) \quad \forall (z, [l, t]) \in U \times c(L_S); (U, \varphi) \in \mathcal{A} \quad (\text{for } x = \varphi(z, [l, t])).
\]
We will write \( rx \) instead of \( \delta_S(r,x) \) in the future. These functions satisfy

(a) \( \rho_S(rx) = r \rho_S(x) \) and \( \rho_S(gx) = \rho_S(x) \) for any \( r \in \mathbb{R}^+ \), \( x \in T_S \), \( g \in G \).

(b) \( S \cap \rho_S^{-1}(0,\infty) = \emptyset \)

(c) The radial action commutes with the action of \( G \) on \( T_S \).

3.4. Thom-Mather spaces (see [12, 13]): A Thom-Mather \( G \)-stratified pseudomanifold is a pair \( (X,\mathcal{T}) \) where \( X \) is a \( G \)-stratified pseudomanifold and \( \mathcal{T} = \{T_S : S \in \mathcal{S}_X^{\text{sing}}\} \) is a family of equivariant tubular neighborhoods satisfying the following condition:

\[
T_S \cap T_R \neq \emptyset \iff R \leq S \text{ or } S \leq R
\]

for any two singular strata \( R, S \) in \( X \). We will usually omit the family \( \mathcal{T} \) if there is no possible confusion.

3.5. Examples Here are some examples of \( G \)-stratified tubular neighborhoods.

(1) Following [2, p.306], for any manifold \( M \) endowed with a smooth action \( \Phi : G \times M \to M \) there is a riemannian metric \( \mu \) such that \( G \) acts by \( \mu \)-isometries. By the local properties of the exponential map, each singular stratum \( S \) of \( M \) has a smooth \( G \)-equivariant tubular neighborhood which can be realized as the normal fiber bundle \( N_{\mu}(S) \) over \( S \) with respect to \( \mu \). The cocycles of this bundle are orthogonal actions. Hence, this tubular neighborhood is actually a \( G \)-stratified tubular neighborhood.

(2) If \( L \) is a compact \( G \)-stratified pseudomanifold, the map \( c(L) \to \{\ast\} \) is a \( G \)-stratified tubular neighborhood of the vertex.

(3) If \( \xi = (T_S, \tau_S, S, c(L_S)) \) is a \( G \)-stratified tubular neighborhood of \( S \) in \( X \), then \((M \times T_S, \iota_M \times \tau_S, M \times S, c(L_S)) \) is a \( G \)-stratified tubular neighborhood of \( M \times S \) in \( M \times X \); for any connected manifold \( M \).

(4) If \( f : Y \to X \) is a \( G \)-equivariant isomorphism, then for any \( G \)-stratified tubular neighborhood \( \xi = (T_S, \tau_S, S, c(L_S)) \) of a stratum \( S \) in \( X \); the pull-back \( f^*(\xi) = (f^{-1}(T_S), f^{-1}\tau_Sf, f^{-1}(S), c(L_S)) \) is a \( G \)-stratified tubular neighborhood of \( f^{-1}(S) \) in \( Y \).

Proposition 3.6. Let \( X \) be a \( G \)-stratified pseudomanifold, \( K \) a closed subgroup of \( G \). Write \( \pi : X \to X/K \) for the orbit map induced by the action of \( K \). Let \( \xi = (T_S, \tau_S, S, c(L_S)) \) be an equivariant tubular neighborhood of \( S \) in \( X \) and write

\[
\tau_S : \pi(T_S) \to \pi(S)
\]

for the induced quotient map. Then \( \xi/K = (\pi(T_S), \tau_S, \pi(S), c(L_S/G_S \cap K)) \) is an equivariant tubular neighborhood of \( \pi(S) \) in \( X/K \).

Proof. Since \( \pi \) is an open map, \( \pi(T_S) \) is an open neighborhood of \( \pi(S) \) in \( X/K \). Also the inclusion \( \pi(S) \to \pi(T_S) \) is a section of \( \tau_S : \pi(T_S) \to \pi(S) \). In order to prove that \( \xi/K \) is a \( G \)-stratified tubular neighborhood we should first verify that it is a \( G \)-stratified fiber bundle, but the conditions \([3,1](1)\) to \( (4) \) are straightforward.

Now we will prove \([3,3](2)\), which implies \([3,1](5)\). We will show that the trivializing atlas \( A = \{(U, \varphi)\} \) of \( \xi \) induces a trivializing atlas \( A/K = \{(V, \psi)\} \) of \( \xi/K \).
Write $\pi' : L_S \to L_S/G_S \cap K$ for the orbit map induced by the action of $G_S \cap K$ in $L_S$.

- **Trivializing charts:** Take a chart $(U, \varphi) \in \mathcal{A}$ and a point $x \in U$. Take also a $K$-slice $V$ of $x$ in $S$, we assume that $V \subset U$. Since $G_S$ acts trivially on $V$ and $KV$ is open in $S$ we deduce that

$$V = V/G_S \cap K = \pi(KV)$$

is open in $\pi(S)$. Since $\varphi$ is $G_S$-equivariant, the function

$$(1) \quad \psi : V \times c(L_S/G_S \cap K) \to \pi(T_S) \quad \psi(b, [\pi'(l), r]) = \pi(\varphi(b, [l, r]))$$

is well defined. Moreover, $\psi$ is injective because $G$ acts by isomorphisms and $V$ is a $K$-slice in $S$. Notice that $W = KV \cap U$ is open in $U$; since $G$ also preserves the radium in $T_S$,

$$\text{Im}(\psi) = \pi(\varphi(W \times c(L_S)))$$

Hence $\text{Im}(\psi)$ is open in $X/K$. It is straightforward that $\psi$ sends smoothly strata onto strata, so actually $\psi$ is an embedding.

- **Atlas and cocycles:** We consider the family $\mathcal{A}/K = \{V, \psi\}$ of all the pairs $(V, \psi)$ as in (1). We will show that $\mathcal{A}/K$ is a trivializing atlas of $\xi/K$. Take two charts $(V, \psi); (V', \psi') \in \mathcal{A}/K$ respectively induced by $(U, \varphi); (U', \varphi') \in \mathcal{A}$. Assume that there is some $g_0 \in G/K$ such that $g_0^{-1}V \cap V' \neq \emptyset$; so $g^{-1}U \cap U' \neq \emptyset$ for some $g \in g_0K$. By (3.3) (2), there is a map

$$g_{\varphi\varphi'} : g^{-1}U \cap U' \to \text{Iso}_{G_S}(L_S, S_{L_S})$$

satisfying

$$g\varphi(b, [l, r]) = \varphi(gb, [g_{\varphi\varphi'}(b)(l), r]) \quad (b, [l, r]) \in (g^{-1}U \cap U') \times c(L_S)$$

Passing to the orbit space $L_S/G_S \cap K$ we obtain the induced map

$$\overline{g_0\psi} : \overline{g_0^{-1}V} \cap V' \to \text{Iso}_{(G_S/G_S \cap K)}(L_S/G_S \cap K, S_{L_S/G_S \cap K})$$

satisfying

$$\overline{g_0\psi}(b, [\pi'(l), r]) = \psi(\overline{g_0b}, [\overline{g_0\psi\varphi'}(b)(\pi'(l)), r]); \quad (b, [\pi'(l), r]) \in (\overline{g_0^{-1}V} \cap V') \times c(L_S/G_S \cap K)$$

Notice that, by definition, $G/K$ preserves the radium of $\pi(T_S)$.

4. **Equivariant unfoldings**

An unfolding of a stratified pseudomanifold is an auxiliary construction which allows us to define the intersection cohomology from the point of view of differential forms [11,12]. For a detailed introduction to unfoldings, the reader can see [11,12]. In this section we introduce equivariant unfoldings, these are a suitable adaptation of the usual unfoldings to the equivariant category. We also show that for any $G$-manifold, considered as a $G$-stratified pseudomanifold, there is always an equivariant unfolding which induces a canonical unfolding on the orbit space.
4.1. Equivariant unfoldings

Broadly speaking, an unfolding of a stratified pseudomanifold $X$ is a manifold $\tilde{X}$ and a surjective continuous map $L : \tilde{X} \to X$ such that $L^{-1}(X - \Sigma)$ is a union of finitely many disjoint copies of $X - \Sigma$, and which smoothly unfolds the singular part so that the restriction $L : L^{-1}(S) \to S$ is a submersion, for any singular stratum $S$.

As for the usual unfoldings, the definition of an equivariant unfolding is made by induction on the depth. Let $X$ be a $G$-stratified pseudomanifold. An equivariant unfolding of $X$ is a manifold $\tilde{X}$ together with a smooth free action $\tilde{\Phi} : G \times \tilde{X} \to \tilde{X}$; a surjective, continuous, equivariant map $L : \tilde{X} \to X$ and a family of equivariant unfoldings $\{L_{L_S} : \tilde{L}_S \to L_S\}_{S \in S^{\text{sing}}_X}$ where $S$ runs on the singular strata of $X$; satisfying:

1. The restriction $L : L^{-1}(X - \Sigma) \to X - \Sigma$ is a smooth finite trivial covering.
2. For each singular stratum $S$ and each $x \in S$, there is a liftable modelled chart, i.e.; a commutative square

\[
\begin{array}{ccc}
U \times \tilde{L}_S \times \mathbb{R} & \xrightarrow{\tilde{\varphi}} & \tilde{X} \\
\downarrow{L_c} & & \downarrow{L} \\
U \times c(L_S) & \xrightarrow{\varphi} & X
\end{array}
\]

such that

(a) $(U, \varphi)$ is a $G_S$-equivariant chart of $x$ modelled on $L_S$.
(b) $\tilde{\varphi}$ is a $G_S$-equivariant smooth embedding on an open subset of $\tilde{X}$.
(c) The map $L_c$ is given by the rule $L_c(u, z, t) = (u, [L_{L_S}(z), |t|])$.

A $G$-stratified pseudomanifold $X$ is said to be unfoldable whenever it has an equivariant unfolding.

4.2. Examples

Here are some examples of equivariant unfoldings.

1. For any free smooth action $\Phi : G \times M \to M$ the identity $\iota : M \to M$ is an equivariant unfolding.
2. If $L : \tilde{X} \to X$ is an equivariant unfolding, then for any manifold $M$ the product $\iota : M \times \tilde{X} \to M \times X$ is also an equivariant unfolding.
3. For any equivariant unfolding $L : \tilde{L} \to L$ over a compact $G$-stratified pseudomanifold $L$, the map $L_c : \tilde{L} \times \mathbb{R} \to c(L)$ defined above is also an equivariant unfolding.

4.3. Elementary unfolding of a $G$-stratified pseudomanifold

The elementary unfolding of a Thom-Mather space is essentially the resolution of singularities given in [4] for the smooth case. This topological operation can be done because the stratification is controlled through a family of tubular neighborhoods. Under certain conditions, after the iterated composition of finitely many elementary unfoldings,
one obtains an equivariant unfolding as defined above. We follow the exposition of [1].

Henceforth we fix a Thom-Mather $G$-stratified pseudomanifold $X$, a closed (hence minimal) stratum $S$ in $X$ and an equivariant tubular neighborhood $(T_S, \tau_S, S, c(L_S))$ of $S$. Define the **unitary sub-bundle** as the set $E_S = \rho^{-1}_S(1)$; this is by construction a $G$-invariant stratified subspace of $X$. The restriction $\tau_S : E_S \to S$ is a $G$-stratified fiber bundle with fiber $L_S$. Consider the map

$$L_{T_S} : E_S \times \mathbb{R} \to T_S \quad L_{T_S}(x, t) = \begin{cases} |t| \cdot x & \text{si } t \neq 0 \\ \tau_S(x) & \text{si } t = 0 \end{cases}$$

Each chart $(U, \varphi)$ in the trivializing atlas provides a local description of $L_{T_S}$ through the following commutative square

$$
\begin{array}{ccc}
U \times L_S \times \mathbb{R} & \xrightarrow{\hat{\varphi}} & E_S \times \mathbb{R} \\
1_U \times \mathbb{R} & \downarrow & \downarrow L_{T_S} \\
U \times cL_S & \xrightarrow{\varphi} & T_S
\end{array}
$$

where $\hat{\varphi}(x, l, t) = (\varphi(x, [l, 1], t))$ and $L_C(l, t) = [l, |t|]$. We also obtain the following properties:

(a) The map $\hat{\varphi}$ is a $G_S$-equivariant embedding.

(b) The composition $\tau_S \circ L_{T_S} : E_S \times \mathbb{R} \to S$ is a locally trivial fiber bundle with fiber $L_S \times \mathbb{R}$ and structure group $\text{Iso}_{G_S}(L_S, S_{L_S})$.

(c) $d(E_S \times \mathbb{R}) = d(E_S) = d(T_S) - 1$.

Now take a disjoint family of equivariant tubular neighborhoods $\{T_S : S \in S_X^{\text{min}}\}$ of the minimal strata. The **elementary unfolding** of $X$ with respect to the family $\{T_S : S \in S_X^{\text{min}}\}$ is the pair $(\hat{X}, L)$ constructed as follows: First $\hat{X}$ is the amalgamated sum

$$\hat{X} = \bigsqcup_{S \in S_X^{\text{min}}} E_S \times \mathbb{R} \bigcup_{\theta} [(X - \Sigma_X^{\text{min}}) \times \{\pm 1\}]$$

where $\theta$ is the map

$$\theta : \bigsqcup_{S \in S_X^{\text{min}}} E_S \times \mathbb{R}^* \to [X - \Sigma_X^{\text{min}}] \times \{\pm 1\} \quad \theta(x, t) = (|t| \cdot x, |t|^{-1}t)$$

Second, $L$ is the continuous map given by the rule

$$L : \hat{X} \to X \quad L(x) = \begin{cases} L_{T_S}(x) & x \in E_S \times \mathbb{R} \\ y & x = (y, j) \in (X - \Sigma_X^{\text{min}}) \times \{\pm 1\} \end{cases}$$

Here there are some properties of the elementary unfoldings.

**Proposition 4.4.** Let $L : \hat{X} \to X$ be the elementary unfolding of a Thom-Mather $G$-stratified pseudomanifold $X$. Then
(1) $\hat{X}$ is a $G$-stratified pseudomanifold, whose stratification is the family $S_{\hat{X}}$ consisting of all the following sets

$$\hat{R} = \bigsqcup_{S \in S_X^{\text{min}}} (E_S \cap R) \times \mathbb{R} \cup (R \times \{\pm 1\})$$

where $R$ runs over the non closed strata in $X$. Moreover, $\hat{X}$ satisfies the Thom-Mather condition.

(2) The map $L$ is a $G$-equivariant morphism. The restriction

$$L : L^{-1}(X - \Sigma_{\text{min}}) \to X - \Sigma_{\text{min}}$$

is a (trivial) double covering.

(3) $d(\hat{X}) = d(X) - 1$. In particular, if $d(X) = 1$ then $L : \hat{X} \to X$ is an equivariant unfolding.

(4) If $X$ is compact, then so is $\hat{X}$.

(5) For any closed subgroup $K \subset G$, the induced map $\overline{T} : \hat{X}/K \to X/K$ is an elementary unfolding.

Proof. (1) The stratification of $\hat{X}$ can be seen in [1]. Since each equivariant tubular neighborhood is a $G$-stratified pseudomanifold (because they are invariant open subsets of $X$); so are the unitary sub-bundles (see §3.2), and hence $\hat{X}$ is a $G$-stratified pseudomanifold. Now we verify the Thom-Mather condition: Take a family $\{T_S : S \in S_X\}$ of equivariant tubular neighborhoods in $X$. Take also a stratum $\hat{R}$ in $\hat{X}$ induced by a non closed stratum $R$ in $X$. Define

$$T_{\hat{R}} = \bigsqcup_{S \in S_X^{\text{min}}} (E_S \cap T_R) \times \mathbb{R} \cup (T_R \times \{\pm 1\}) = L^{-1}(T_R)$$

where $\theta$ is the map given in the equation (4) of §4.3. This $T_{\hat{R}}$ is an equivariant tubular neighborhood of $\hat{R}$ in $\hat{X}$; we leave the details to the reader.

(2) and (3) are straightforward, see again [1] for more details. The last observation of (3) is a consequence of def.4.1

(4) Since $X$ is compact, $S_X^{\text{min}}$ is finite. But $\hat{X}$ is the quotient of the finite family of compact spaces $\bigsqcup_{S \in S_X^{\text{min}}} (E_S \times [-1,1])$ and $[X - \bigsqcup_{S \in S_X^{\text{min}}} \rho_S^{-1}[0,1/2]] \times \{-1,1\}$. Then we get the result.

(5) This is a consequence of §3.6 \hfill \square

\textit{Remark} 4.5. With tubular neighborhood of §3.3, $\mathcal{M} \times \hat{X} = M \times \hat{X}$, for any manifold $M$.

5. Iteration of elementary unfoldings

From now on, we fix a Thom-Mather $G$-stratified pseudomanifold $X$. We will study the composition of finitely many elementary unfoldings, starting at $X$. As we have already seen, for any elementary unfolding $L : \hat{X} \to X$, the space $\hat{X}$ is again
a Thom-Mather $G$-stratified pseudomanifold and satisfies $d(\hat{X}) = d(X) - 1$. This allows us to ask for the behavior of a chain
\begin{equation}
X_l \xrightarrow{L_l} X_{l-1} \xrightarrow{L_{l-1}} \ldots \xrightarrow{L_2} X_1 \xrightarrow{L_1} X
\end{equation}
of elementary unfoldings, where $l = d(X)$. As we shall see, under certain conditions on the tubular neighborhoods, this iterative process leads us to an equivariant unfolding

$L : \tilde{X} \to X$

where $\tilde{X} = X_l$ and $L = L_1 \ldots L_d$.

Recall the definition of a saturated subspace \[\Box\]. Let $Y \subset X$ be a stratified subspace of $X$. We say that $Y$ is saturated whenever

$Y \cap T_S = \tau_S^{-1}(Y \cap S) \quad \forall S \in S_X$

For instance, if $S$ is a singular stratum and $U \subset S$ is open, then $Y = \tau_S^{-1}(U)$ is a saturated. Also the unitary sub-bundle $Y = E_S$ is saturated.

5.1. Transverse morphisms

Now we introduce the family of transverse morphisms, whose main feature is the preservation of the tubular neighborhoods. Let $H \subset G$ be a closed subgroup, $Y$ a Thom-Mather $H$-stratified pseudomanifold and $M$ be a connected manifold. A morphism

$\psi : M \times Y \to X$

is transverse whenever:

1. $\text{Im}(\psi)$ is a saturated open subspace of $X$.
2. If $\psi(M \times S) \subset R$ then $\psi^{-1}(T_R) = M \times T_S$, for any $R \in S_X, S \in S_Y$.

Now let $\psi : M \times Y \to X$ be a transverse morphism. The lifting of $\psi$ is, by definition, the map

$\hat{\psi} : M \times \hat{Y} \to \hat{X}$

$\hat{\psi}(m,z,t) = \begin{cases} 
(\psi(m,z),t) & (m,z,t) \in M \times E_S \times \mathbb{R} \\
(\psi(m,z),t) & (m,z,t) \in M \times (Y - \Sigma^{\min}) \times \{\pm 1\}
\end{cases}$

This is the unique morphism such that the diagram

\begin{align*}
M \times \hat{Y} &\xrightarrow{\hat{\psi}} \hat{X} \\
\downarrow \iota_{M \times Y} &\downarrow L_X \\
M \times Y &\xrightarrow{\psi} X
\end{align*}

commutes.

5.2. Examples

For any smooth effective action of $G$ in a manifold, the trivializing charts of the tubular neighborhoods are transverse morphisms. In order to see this, take a manifold $M$ endowed with a smooth action $\Phi : G \times M \to M$ and an invariant metric $\mu$ in $M$. Recall that $M$ has a natural structure of Thom-Mather $G$-stratified pseudomanifold, where $S_M$ is the stratification induced by the orbit types of the
action. For any singular stratum $S$ with codimension $\text{codim}(S) = q + 1 > 0$, the equivariant tubular neighborhood $T_S = N_\mu(S)$ is the normal fiber bundle over $S$ induced by $\mu$ (see §3.5). Take also a trivializing chart 

\[ \varphi : U \times c(S^q) \to \tau_S^{-1}(U) \]

We claim that $\varphi$ is transverse. First notice that $\text{Im}(\varphi)$ is a saturated open subspace in $M$, so we only have to verify §5.1-(2). Let $S'$ be a stratum in $c(S^q), R$ a stratum in $M$. Suppose that $\varphi(U \times S') \subset R$. We consider the following cases:

- $S' = \{\star\}$ is the vertex: It is straightforward, since $R = S$ and $T_{S'} = c(S^q)$.
- $S' = S'' \times \mathbb{R}^+$ for some stratum $S''$ in $S^q$: Then $S < R$. We consider in $T_S$ the following decomposition of the metric:

\[ \mu|_{T_S} = \mu_H + \mu_V \]

corresponding to the the orthogonal decomposition of the tangent $T(T_S)$ in the horizontal and vertical subfiber bundles. Hence

\[ \varphi^{-1}(T_R) = \varphi^*(N_\mu(R)) = N_{\varphi^*(\mu)}(U \times S') = U \times N_\mu(S') = U \times T_{S'} \]

Now we show two easy properties of the transverse morphisms.

**Proposition 5.3.** Let $K, H$ a closed subgroups of $G$, $L$ a Thom-Mather $H$-stratified pseudomanifold, $\psi : M \times L \to X$ a transverse morphism. Then

1. The lifting $\hat{\psi} : M \times \hat{L} \to \hat{X}$ is transverse.
2. The induced quotient map $\overline{\psi} : M \times (L/H \cap K) \to X/K$ is transverse.

**Proof.** (1) is straightforward from the definition. (2) is a consequence of §3.6. □

Finally, we provide a sufficient condition for the existence of an equivariant unfolding, depending on the transversality of the tubular neighborhoods.

**Theorem 5.4.** Let $X$ be a Thom-Mather $G$-stratified pseudomanifold. Suppose that for any singular stratum $S$, each trivializing chart

\[ \varphi : U \times c(L_S) \to T_S \]

is transverse. Then

1. The composition of the $l$ elementary unfoldings of starting at $X$ induces an equivariant unfolding $L : \tilde{X} \to X$ where $\tilde{X}$ is the last (non trivial) elementary unfolding and $L = L_1L_2\ldots L_l$ (see eq. (6) at the begining of this section).
2. For any closed subgroup $K \subset G$, the induced map $\overline{L} : \tilde{X}/K \to X/K$ is an unfolding.

**Proof.** (1) Take a family of equivariant tubular neighborhoods in $X$ with transverse trivializing charts. Let

\[ X_l \xrightarrow{L_l} X_{l-1} \xrightarrow{L_{l-1}} \ldots \xrightarrow{L_2} X_1 \xrightarrow{L_1} X \]

be the chain of elementary unfoldings induced by this family of tubular neighborhoods. We proceed by induction on $l = d(X)$; for $l = 1$ the statements are trivial.
Suppose that \( l > 1 \) and assume the inductive hypothesis, so \( L' : \widetilde{X} \to X_1 \) is an equivariant unfolding, for \( \widetilde{X} = X_l \) and \( L' = L_2 \ldots L_l \). Take a closed stratum \( S \) and a transverse trivializing chart \( \varphi : U \times c(L_S) \to \tau^{-1}_S(U) \subset T_S \).

Apply the chain of elementary unfoldings and use \( \S.5.3 \) you will get the following commutative diagram:

\[
\begin{array}{ccc}
U \times \tilde{L}_S \times \mathbb{R} & \xrightarrow{\psi_1} & \widetilde{X} \\
\downarrow & & \downarrow \L' \\
U \times L_S \times \mathbb{R} & \xrightarrow{\psi_1} & X_1 \\
\downarrow & & \downarrow \L_1 \\
U \times c(L_S) & \xrightarrow{\psi} & X
\end{array}
\]

We conclude that \( L = L_1 L' : \widetilde{X} \to X \) is an equivariant unfolding.

(2) This is a consequence of \( \S.4.3(6) \). \( \square \)

**Corollary 5.5** (Unfolding of a \( G \)-manifold).

*Let \( M \) be a manifold, \( \Phi : G \times M \to M \) a smooth effective action, possibly with fixed points. Endow \( M \) with the stratification induced by the orbit types and the usual structure of a Thom-Mather \( G \)-stratified pseudomanifold. Then there is an equivariant unfolding \( \tilde{L} : \widetilde{M} \to M \).*

*Proof.* Apply the above theorem to the transverse charts obtained in \( \S.5.2 \). \( \square \)

**Acknowledgments**

I would like to thank M. Saralegi for some helpful conversations, and R. Popper for some accurate remarks. While writing this work, I received the hospitality of the staff in the Math Department, Université D’Artois; and the financial support of the ECOS-Nord project and the CDCH-Universidad Central de Venezuela.

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