RANDOM FUSION FRAMES ARE NEARLY EQUIANGULAR AND TIGHT

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ABSTRACT. This paper demonstrates that random, independently chosen equi-dimensional subspaces with a unitarily invariant distribution in a real Hilbert space provide nearly tight, nearly equiangular fusion frames. The angle between a pair of subspaces is measured in terms of the Hilbert-Schmidt inner product of the corresponding orthogonal projections. If the subspaces are selected at random, then a measure concentration argument shows that these inner products concentrate near an average value. Overwhelming success probability for near tightness and equiangularity is guaranteed if the dimension of the subspaces is sufficiently small compared to that of the Hilbert space and if the dimension of the Hilbert space is small compared to the sum of all subspace dimensions.

1. Introduction

A collection of closed subspaces in a Hilbert space is a fusion frame if a weighted sum of the corresponding orthogonal projections provides an approximate identity. Research on fusion frames has enjoyed a rapid growth in the frame theory literature of the last decade, see [12] and references therein. There are many applications of this field, driven by demands from distributed sensing [9], parallel processing [3], communication theory [1], quantum computing [6] and even neuroscience [34]. Fusion frames are important for these applications because they model linear, distributed signal representation strategies. A primary goal in many of these settings is to make the representation robust to partial data loss when a signal to be stored or transmitted is projected onto the subspaces given by a fusion frame. If the components in the subspaces are lost then this constitutes an erasure. The performance of a fusion frame is typically measured by its ability to compensate for such lost components, either in a deterministic, adversarial erasure regime or in an averaged sense with additional statistical assumptions on the signal and on the erasures. Earlier work on fusion frame design and erasures has shown that typical notions of optimality have a geometric characterization: If equi-isoclinic fusion frames exist, then they are optimal for the worst-case error for recovery based on the canonical dual when up to two subspace components are removed, and if equi-distant fusion frames exist, then they are optimal for the mean-squared error after applying a Wiener filter to suppress the effect of erasures, see also a simpler setting in which they provide optimality [5]. The existence and construction of such subspace collections depends on the dimensions and has remained a challenge despite many efforts [32, 29, 28, 25, 26, 27, 8, 14]. The motivation for this paper is to show that a random choice of subspaces is with high probability nearly tight and nearly equiangular. In a certain regime, this was already accomplished in an earlier paper by the construction of nearly tight frames with nearly isoclinic subspaces from matrices with the restricted isometry [7], but this only resulted in nearly orthogonal subspaces. It somehow seems natural that a random selection favors mutual orthogonality which is in a way the simplest case of equiangularity. A generalized form of the Welch bound shows that an equiangular, equi-dimensional tight fusion frame minimizes the maximum value for the Hilbert-Schmidt inner product.

The research presented in this paper was supported by NSF DMS 1109545 and by AFOSR FA9550-11-1-0245.
product of any pair of orthogonal projections \( P \) and \( P' \) corresponding to two subspaces, resulting in the value of the inner product \[ \text{tr}[PP'] = \frac{s(Ks - N)}{(K - 1)N} . \]

As \( N, K, s \to \infty \), \( N \text{tr}[PP']/s^2 \to 1 \). We establish these asymptotics for a random choice of subspaces. We cover this case of non-orthogonal equiangular subspaces by further developing the partial orthonormalization strategy applied to frames that can be partitioned into nearly tight Riesz sequences from [7]. Such frames are implicitly constructed in the compressed sensing literature as the column vectors of sensing matrices that have the restricted isometry property. Many of the standard construction methods rely on randomization, meaning they pick a random sensing matrix which is shown to have the restricted isometry property with high probability. In the present paper, we use a similar strategy with a specific choice of random frames and investigate the properties of the corresponding fusion frames, given by unitarily invariant, independently selected subspaces. We conclude that these are with overwhelming probability nearly tight and nearly equidistant, and hence near-optimal for many applications.

The main results are summarized as follows, see the next section for notation:

If \( \{V_j\}_{j=1}^K \) are \( s \)-dimensional subspaces that are chosen independently at random in a \( N \)-dimensional Hilbert space, with a unitarily invariant distribution, and \( \epsilon, \delta > 0 \), \( 1 + \epsilon = (1 + \delta)^6 \), then with probability at least

\[
1 - 2(1 + \frac{4}{\delta})^N e^{-M\delta^2/4+M\delta^3/3} - 2K(1 + \frac{4}{\delta})^s e^{-N\delta^2/4+N\delta^3/3}
\]

they form an \( \epsilon \)-nearly \( Ks/M \)-tight fusion frame.

Under the same assumptions, letting each \( P_j \) denote the orthogonal projection onto \( V_j \), and with \( 1 + \epsilon = (1 + \delta)^3 \), the subspaces form a nearly equiangular fusion frame, meaning for all \( j \neq l \),

\[
\frac{1}{1 + \epsilon} - \epsilon \sqrt{\frac{(1 + \epsilon)N}{s}} \leq \frac{N}{s^2} \text{tr}[P_j P_l] \leq 1 + \epsilon \sqrt{\frac{(1 + \epsilon)N}{s}} + \frac{N\epsilon^2}{4s}
\]

where the failure probability is bounded by the sum of

\[
K(K + 1)s e^{(1+\delta)s/2-s(s-1)(\delta^2/2-\delta^3/3)/2}
\]

and

\[
K(K + 1)((M + K(1 + \frac{4}{\delta}))^s e^{-N\delta^2/4+N\delta^3/3} + (1 + \frac{4}{\delta})^N e^{-M\delta^2/4+M\delta^3/3}) .
\]

Examining the terms in these error bounds shows that as \( N, s, K \to \infty \) and \( s/N \to c > 0 \) then the failure probability decays exponentially in \( N \) if

\[
\frac{3\ln (K + 1)}{N} + \frac{s}{N} \ln(1 + \frac{4}{\delta}) < \delta^2/4 - \delta^3/3
\]

and

\[
\frac{N}{Ks} \ln(1 + \frac{4}{\delta}) < \delta^2/4 - \delta^3/3 .
\]

This paper is organized as follows: In the next section, we fix notation and recall elementary results. Section 3 demonstrates that random subspaces lead to nearly tight fusion frames. The final section is dedicated to showing that the subspaces are nearly equiangular.
2. Frames, Fusion Frames and Riesz sequences

We briefly review frames, fusion frames and Riesz sequences.

Definition 2.1. A family of vectors \( \{ \varphi_j \}_{j \in J} \) in a real or complex Hilbert space \( H \) is a frame for \( H \) if there are constants \( 0 < A \leq B < \infty \) so that for all \( x \in H \) we have
\[
A \| x \|^2 \leq \sum_{j \in J} |\langle x, \varphi_j \rangle|^2 \leq B \| x \|^2.
\]
If \( A = B \) then we say that the frame is \( A \)-tight, and if \( A = B = 1 \), it is a Parseval frame.

If there is \( c > 0 \) and \( \| \varphi_j \| = c \) for all \( j \in J \) then it is called an equal norm frame, and if \( c = 1 \) it is a unit norm frame. The analysis operator of the frame \( T : H \to \ell_2(J) \) is given by \( (Tx)_j = \langle x, \varphi_j \rangle \). The synthesis operator \( T^* \) is the (Hilbert) adjoint of \( T \) and the frame operator is the positive self-adjoint invertible operator \( S = T^*T \).

We recall that if \( \{ \varphi_j \}_{j \in J} \) is a frame with frame operator \( S \) then \( \{ S^{-1/2} \varphi_j \}_{j \in J} \) is a Parseval frame for \( H \).

While frames assign scalar coefficients to a vector, fusion frames map it to its components in subspaces [17].

Definition 2.2. Given a real or complex Hilbert space \( H \) and a family of closed subspaces \( \{ W_i \}_{i \in K} \) with associated positive weights \( 0 < v_i, i \in K \), then \( \{ W_i, v_i \}_{i \in K} \) is a fusion frame for \( H \) if there exist constants \( 0 < A \leq B < \infty \) such that
\[
A \| x \|^2 \leq \sum_{i \in K} v_i^2 \| P_i x \|^2 \leq B \| x \|^2
\]
for any \( x \in H \), where each \( P_i \) is the orthogonal projection onto \( W_i \). A fusion frame is called tight if \( A \) and \( B \) can be chosen to be equal, and Parseval if \( A = B = 1 \). For \( \epsilon > 0 \), the fusion frame is \( \epsilon \)-nearly tight if there is a constant \( C \) so that \( A = \frac{1}{1+\epsilon} C \), \( B = (1+\epsilon)C \). The fusion frame is equi-dimensional if all its subspaces \( W_i \) have the same dimension. If \( \{ W_i \}_{i \in K} \) are closed subspaces of \( H \), we define the space
\[
\bigoplus_{i \in K} W_i = \{ \psi = (\psi_i)_{i \in K} | \psi_i \in W_i, \langle \psi, \psi \rangle < \infty \},
\]
with the inner product given by
\[
\langle (\psi_i)_{i \in K}, (\phi_i)_{i \in K} \rangle = \sum_{i \in K} \langle \psi_i, \phi_i \rangle.
\]

The analysis operator of the fusion frame is the operator
\[
T : H \to \bigoplus_{i \in K} W_i,
\]
given by
\[
Tx = (v_i P_i x)_{i \in K}.
\]
The synthesis operator of the fusion frame is \( T^* \) and is given by
\[
T^*(\psi_i)_{i \in K} = \sum_{i \in K} v_i \psi_i.
\]
In analogy with frames, the fusion frame operator is the positive, self-adjoint and invertible operator \( S = T^*T \).

We also need the notion of \( \epsilon \)-Riesz sequences. We choose the convention from [7] for tight Riesz sequences which is convenient for orthonormalization.
Definition 2.3. A family of vectors $\{\varphi_i\}_{i=1}^{N}$ in a Hilbert space $H$ is a Riesz basic sequence with lower (resp. upper) Riesz bounds $0 < A \leq B < \infty$ if for all scalars $\{a_i\}_{i=1}^{N}$ we have

$$A \sum_{i=1}^{N} |a_i|^2 \leq \|\sum_{i=1}^{N} a_i \varphi_i\|^2 \leq B \sum_{i=1}^{N} |a_i|^2.$$ 

This family of vectors is $\epsilon$-Riesz basic if for all scalars $\{a_i\}_{i=1}^{N}$ we have

$$\frac{1}{1+\epsilon} \sum_{i=1}^{N} |a_i|^2 \leq \|\sum_{i=1}^{N} a_i \varphi_i\|^2 \leq (1+\epsilon) \sum_{i=1}^{N} |a_i|^2.$$ 

3. Random fusion frames are nearly tight

We first recall that random Gaussian vectors form a frame that can be partitioned into nearly orthonormal systems.

Lemma 3.1. Let $X$ be a random $N \times M$ matrix whose entries are independent, standard-normal distributed random variables, and let $X_j$ denote the vector containing the entries of the $j$th row. Let $u \in \mathbb{R}^M$, $\|u\| = 1$, and $Z = \frac{1}{N} \sum_{j=1}^{N} |\langle u, X_j \rangle|^2$ then

$$\mathbb{P}(Z \geq 1 + \delta) \leq e^{-N\delta^2/4 + N\delta^3/6}$$

and

$$\mathbb{P}(Z \leq \frac{1}{1+\delta}) \leq e^{-N\delta^2/4 + N\delta^3/3}$$

Proof. The distribution of the row vectors is unitarily invariant, so without loss of generality, we can set $u = e_1$, the first vector of the canonical basis for $\mathbb{R}^M$. The sum $Z = \frac{1}{N} \sum_{j=1}^{N} X_j^2$ is up to the normalization factor $1/N$ chi-squared distributed with $N$ degrees of freedom. The usual combination of the Laplace transform and the Chernoff bound gives measure concentration. We have

$$\mathbb{P}(Z \geq 1 + \delta) \leq (1 - t)^{-N/2} e^{-tN(1+\delta)/2}$$

for any $0 \leq t < 1$, so after choosing $t = \delta/(1+\delta)$ and truncating the Taylor expansion of the exponential

$$\mathbb{P}(Z \geq 1 + \delta) \leq e^{N \ln(1+\delta)/2 - N\delta/2} \leq e^{-N\delta^2/4 + N\delta^3/6}.$$

Similarly, for $t \geq 0$,

$$\mathbb{P}(Z \leq \frac{1}{1+\delta}) \leq e^{-N \ln(1+t)/2 + tN(1+\delta)^{-1}/2}$$

so setting $t = \delta$ and comparing the terms of the Taylor series in the exponent gives the desired bound. 

Choosing $u$ among the $M$ canonical basis vectors in $\mathbb{R}^M$ together with a union bound shows that the norms of all the columns are nearly a constant.

Corollary 3.2. With the random $N \times M$ matrix $X$ as above,

$$\mathbb{P}(\frac{1}{1+\delta} \leq \frac{1}{N} \sum_{j=1}^{N} X_{j,l}^2 \leq 1 + \delta \text{ for all } l) \geq 1 - 2M e^{-N\delta^2/4 + N\delta^3/3}.$$ 

We derive a stronger consequence with an argument similar to the exposition in Baraniuk et al. [2], as presented in [5]: With high probability, sufficiently small subsets of the column vectors of $X$ have nearly tight upper and lower Riesz bounds.
Lemma 3.3. Let \( W = \text{span}\{e_{j_1}, e_{j_2}, \ldots, e_{j_s}\} \) in \( \mathbb{R}^M \), let \( \{X_j\}_{j=1}^N \) be a random family of vectors in \( \mathbb{R}^M \) as above and let \( 0 < \delta < 2 \), then the set
\[
\mathcal{X}_\leq = \{X : Z(y) \leq \frac{1}{(1 + \delta)^3} \|y\|^2 \text{ for all } y \in W\}
\]
defined by the random variables \( Z(y) = \frac{1}{N} \sum_{j=1}^N |\langle y, X_j \rangle|^2 \) for \( y \in W \) has probability
\[
P(\mathcal{X}_\leq) \leq (1 + \frac{4}{\delta})^s e^{-N\delta^2/4 + N\delta^3/3}.
\]
Moreover, if \( \delta < 1 \), then
\[
\mathcal{X}_\geq = \{X : Z(y) \geq (1 + \delta)^3 \|y\|^2 \text{ for all } y \in W\}
\]
has the same upper bound for its probability as \( \mathcal{X}_\leq \).

Proof. Using the triangle inequality and Lipschitz continuity of the norm we can bootstrap from a net \( S \) with \( \min_{w \in S} \|y - w\| \leq \frac{\delta}{2} \) for all \( y \in W, \|y\| = 1 \). By a volume inequality for sphere packings, we know there is such an \( S \) with cardinality
\[
|S| \leq \left(1 + \frac{4}{\delta}\right)^s.
\]
Applying a union bound for the probability of the complement of
\[
\mathcal{X}_S = \{X : Z(w) \geq \frac{1}{1 + \delta} \|w\|^2 \text{ for all } w \in S\}
\]
we get
\[
P(\mathcal{X}_S) \geq 1 - (1 + \frac{4}{\delta})^s e^{-N\delta^2/4 + N\delta^3/3}.
\]
Now let \( a \) be the smallest number such that \( Z^{1/2}(y) \geq \frac{1}{(1 + a)^{1/2}} \|y\| \) holds for all \( y \in W \). We show \( 1 + a \leq (1 + \delta)^3 \). To see this, let \( y \in W, \|y\| = 1 \) and pick \( w \in S, \|x - w\| \leq \frac{\delta}{2} \). Then, using the triangle inequality yields
\[
Z^{1/2}(y) \geq |Z^{1/2}(w) - Z^{1/2}(y - w)| \geq \frac{1}{(1 + \delta)^{1/2}} - \frac{\delta/2}{(1 + a)^{1/2}}.
\]
Since the right hand side of the inequality chain is independent of \( y \), according to the definition of \( a \) we obtain
\[
\frac{1}{(1 + a)^{1/2}} \geq \frac{1}{(1 + \delta)^{1/2}} - \frac{\delta/2}{(1 + a)^{1/2}}.
\]
Solving for \( (1 + a)^{-1} \) and further estimating gives
\[
\frac{1}{1 + a} \geq \frac{1}{(1 + \delta)^3}.
\]
For the second inequality, choose
\[
\mathcal{X}_S' = \{X : Z(w) \leq (1 + \delta) \|w\|^2 \text{ for all } w \in S\}
\]
and establish the same bound for its probability as for \( \mathcal{X}_S \). Let \( a \) be smallest such that
\[
Z^{1/2}(y) \leq (1 + a)^{1/2} \|y\|, y \in W.
\]
We again use the triangle inequality to obtain that if \( X \in \mathcal{X}_S' \) then for any \( y \in W \) with \( \|y\| = 1 \)
\[
Z^{1/2}(y) \leq (1 + \delta)^{1/2} + (1 + a)^{1/2} \frac{\delta}{2}.
\]
Again by definition
\[
(1 + a)^{1/2} \leq (1 + \delta)^{1/2} + (1 + a)^{1/2}\frac{\delta}{2}
\]
so
\[
1 + a \leq \frac{1 + \delta}{(1 - \delta/2)^2}
\]
and if \(\delta < 1\) then \((1 - \delta/2)^{-1} \leq 1 + \delta\), which implies
\[
1 + a \leq (1 + \delta)^3.
\]
\(\square\)

A union bound gives a lower bound for the probability that all subsets in a partition of the column vectors of \(X\) have good Riesz bounds.

**Corollary 3.4.** Let \(X\) be a random \(N \times M\) matrix whose entries are independent, standard-normal distributed random variables, \(0 < \delta < 1\), and let \(\{1, 2, \ldots, M\}\) be partitioned into sets \(\{J_k\}_{k=1}^K\) of maximal size \(\max_k |J_k| \leq s\), then with probability
\[
1 - 2K(1 + 4\delta)s e^{-N\delta^2/4 + N\delta^3/3}.
\]
for each \(k\) the set of rescaled column vectors \(\{(X_{j,l}/\sqrt{N})_{j=1}^N\}_{l \in J_k}\) forms an \(\epsilon\)-Riesz sequence with \(\epsilon = (1 + \delta)^3 - 1\).

**Proof.** If the matrix is multiplied by the normalization factor, then a fixed set of columns \(\{(X_{j,l}/\sqrt{N})_{j=1}^N\}_{l \in J_k}\) indexed by \(J_k\) in the partition with \(|J_k| \leq s\) is by the preceding lemma \(\epsilon\)-Riesz with \(1 + \epsilon = (1 + \delta)^3\) with probability of at least \(1 - 2(1 + 4\delta)s e^{-N\delta^2/4 + N\delta^3/3}\). Applying the union bound for all \(K\) sets in the partition gives the claimed estimate. \(\square\)

The same proof allows us to establish frame bounds if we think of \(X\) as the analysis operator. If we choose a trivial partition with \(s = M \leq N\) then the above lemma states that the family of row vectors \(\{X_j\}_{j=1}^N\) with standard normal entries forms with high probability a nearly tight frame for \(\mathbb{R}^M\). We swap \(N\) and \(M\) for later notational convenience.

**Corollary 3.5.** Let \(X\) be a random \(M \times N\) matrix whose entries are independent, standard-normal distributed random variables, then with probability at least
\[
1 - 2(1 + 4\delta)^N e^{-M\delta^2/4 + M\delta^3/3}.
\]
the row vectors \(\{X_j\}_{j=1}^M\) of \(X\) form a frame for \(\mathbb{R}^N\) with lower and upper frame bounds
\[
\frac{M}{(1 + \delta)^3} ||x||^2 \leq \sum_{j=1}^M |\langle x, X_j \rangle|^2 \leq M(1 + \delta)^3 ||x||^2, \quad x \in \mathbb{R}^N.
\]

**Theorem 3.6.** Given a family of random, independent subspaces \(\{V_k\}_{k=1}^K\) of \(\mathbb{R}^N\), whose distribution is invariant under unitaries and whose dimensions \(s_k = \dim V_k\) are bounded by \(s_k \leq s\) for all \(k\), and let \(M = \sum_{k=1}^K s_k\) then with probability at least
\[
1 - 2(1 + 4\delta)^N e^{-M\delta^2/4 + M\delta^3/3} - 2K(1 + 4\delta) s e^{-N\delta^2/4 + N\delta^3/3}
\]
the fusion frame \(\{V_j, 1\}_{j=1}^K\) has upper and lower frame bounds \(M/(N(1 + \delta)^6)\) and \(M(1 + \delta)^6/N\).
The right hand side is minimized with high probability the analysis operator of a frame that is nearly $M/N$-tight, because

$$M/(N(1+\delta)^2)\|x\|^2 \leq \frac{1}{N} \sum_{j=1}^{M} |\langle x, X_j \rangle|^2 \leq M(1+\delta)^3/N$$

for all $x \in \mathbb{R}^N$ except for a set of probability $2(1+\delta)^N e^{-M\delta^2/4+M\delta^3/3}$.

Moreover, let $\{1, 2, \ldots, M\}$ be partitioned into subsets $\{J_k\}_{k=1}^{K}$ of size $\max_i |J_i| \leq s$, then a union bound shows that with probability bounded below by

$$1 - 2K(1 + \frac{4}{\delta})^s e^{-N\delta^2/4 + N\delta^3/3}$$

each set of rescaled row vectors $\{X_j/\sqrt{N}\}_{j \in J_k}$ has upper and lower Riesz bounds $(1+\delta)^3$ and $(1+\delta)^{-3}$, respectively.

Orthonormalizing the Riesz sequences then changes the frame bounds by at most a factor of $(1+\delta)^{\pm 3}$. The resulting frame is by construction partitioned into orthonormal systems, and thus equivalent to a fusion frame with the same frame bounds. Combining the probabilities for the failure of the frame bounds and of the Riesz bounds gives the stated bound.

\[ \square \]

4. Random fusion frames are nearly equiangular

Using a variation of the strategy by Dasgupta and Gupta [24], we obtain measure concentration for an average of projected norms, which implies that with a proper scaling of dimensions, random subspaces become nearly equiangular.

**Lemma 4.1.** Given $\{X_{j,l} : 1 \leq j \leq s, 1 \leq l \leq N\}$, independent identically standard-normal distributed random variables, $N \geq s$, and $0 < \beta < 1$ then

$$\mathbb{P}_{\leq} \equiv \mathbb{P}(N \sum_{j=1}^{s} \sum_{l=1}^{s} X_{j,l}^2 \leq \beta s^2 \sum_{l=1}^{N} X_{1,l}^2) \leq e^{s(\beta s(1/(\ln \beta))^{2-s/2+s(1-\beta)s^2/2}}.$$

Similarly, if $\beta > 1$, then

$$\mathbb{P}_{\geq} \equiv \mathbb{P}(N \sum_{j=1}^{s} \sum_{l=1}^{s} X_{j,l}^2 \geq \beta s^2 \sum_{l=1}^{N} X_{1,l}^2) \leq e^{s(\beta s(1/(\ln \beta))^{2-s/2+s(1-\beta)s^2/2}}.$$

**Proof.** The first probability under consideration is equal to

\[
\mathbb{P}(e^{\beta s^2 \sum_{l=1}^{N} X_{1,l}^2 - N \sum_{j,l=1}^{s} X_{j,l}^2} \geq 1)
\leq \mathbb{E}[e^{t(\beta s^2 \sum_{l=1}^{N} X_{1,l}^2 - N \sum_{j,l=1}^{s} X_{j,l}^2)}]
= \mathbb{E}[e^{t(\beta s^2 X_{1,1}^2) N - s}] \mathbb{E}[e^{t(\beta s^2 X_{1,1}^2) s}] \mathbb{E}[e^{-tNX_{1,1}^2}]^{s(s-1)}
\leq \mathbb{E}[e^{t(\beta s^2 X_{1,1}^2) N}] \mathbb{E}[e^{-tNX_{1,1}^2}]^{s(s-1)} = (1 - 2t\beta s^2)^{-N/2}(1 + 2tN)^{-s(s-1)/2}
\]

The right hand side is minimized with

$$t = \frac{(1-\beta)s-1}{2\beta s(N+s(s-1))}$$
and inserting this in the expression

\[ \mathbb{P}_{\leq} \leq \left( \frac{N + \beta s^2}{N + (s - 1)s} \right)^{-N/2} \left( \frac{(s - 1)(N + \beta s^2)}{\beta s(N + (s - 1)s)} \right)^{-s(s-1)/2} \]

\[ = \left( 1 + \frac{(1 - \beta)s^2}{N + \beta s^2} \right)^{s(s-1)/2 + N/2} \left( \frac{s - 1}{\beta s} \right)^{-s(s-1)/2} \]

\[ \leq (1 + 2t\beta s^2)^{-N/2} e^{s/2} e^{(1-\beta)s^2(s(s-1)/2 + N/2)/(N + \beta s^2)} \leq (1 + 2t\beta s^2)^{-N/2} e^{s/2} e^{(1-\beta)s^2/2} \]

If \( \beta > 1 \), then we get

\[ \mathbb{P}(e^{\beta s^2 \sum_{i=1}^{N} X_{i,i}^2 - N \sum_{j,i=1}^s X_{j,l}^2} \leq 1) \]

\[ \leq (1 + 2t\beta s^2)^{-N/2} (1 - 2tN)^{-s(s-1)/2} \]

Now the optimal choice for \( t \) is

\[ t = \frac{(1 - \beta)s - 1}{2\beta s(N + s(s - 1))} \]

and the result follows

\[ \mathbb{P}_{\geq} \leq \left( \frac{N + \beta s^2}{N + (s - 1)s} \right)^{-N/2} \left( \frac{(s - 1)(N + \beta s^2)}{\beta s(N + (s - 1)s)} \right)^{-s(s-1)/2} \]

\[ \leq (1 + 2t\beta s^2)^{-N/2} e^{s/2} e^{(1-\beta)s^2(s(s-1)/2 + N/2)/(N + \beta s^2)} \leq e^{s(s-1)(\ln\beta)/2 + s/2 + (1-\beta)s^2/2} \]

\[ \square \]

Setting \( t = 1 + \delta \) or \( t = (1 + \delta)^{-1} \) gives the following estimate.

**Corollary 4.2.** Given \( \{X_{j,l} : 1 \leq j \leq s, 1 \leq l \leq N\} \), independent identically standard-normal distributed random variables, \( N \geq s \) and \( \epsilon > 0 \), then

\[ \mathbb{P}(N \sum_{j=1}^{s} \sum_{l=1}^{s} X_{j,l}^2 \leq \frac{1}{1 + \delta} \sum_{l=1}^{N} X_{j,l}^2) \leq e^{-s(s-1)(\ln(1+\delta))/2 + s/2 + s/2 + \delta s^2/2} \]

\[ \leq e^{(1+\delta)s/2 - s(s-1)(\delta^2/2 - \delta^3/3)/2} \]

and

\[ \mathbb{P}(N \sum_{j=1}^{s} \sum_{l=1}^{s} X_{j,l}^2 \geq (1 + \delta)s^2 \sum_{l=1}^{N} X_{j,l}^2) \leq e^{s(s-1)(\ln(1+\delta))/2 + s/2 - \delta s^2/2} \]

\[ \leq e^{(1+\delta)s/2 - s(s-1)(\delta^2/2 - \delta^3/3)/2} \]

For the next result it is sometimes more convenient to work with random matrices whose column vectors are normalized. We will alternate between the random \( N \times M \) matrix \( X \) with standard normal entries and the random matrix \( x \) which is obtained by

\[ x_{j,l} = \frac{X_{j,l}}{\left( \sum_{i=1}^{N} X_{i,l}^2 \right)^{1/2}} \]
and consequently has column vectors that are independent, uniformly distributed on the unit sphere in \(\mathbb{R}^N\).

**Theorem 4.3.** Let \(\{x_1, x_2, \ldots, x_s\}\) be vectors in \(\mathbb{R}^N\), drawn independently according to a uniform distribution on the unit sphere. If \(V\) is a fixed subspace of dimension \(s < N\) and \(P_V\) is the orthogonal projection onto \(V\), then
\[
\mathbb{P}(\frac{1}{1+\delta} \leq \sum_{i=1}^s \frac{N}{s^2} \|P_V x_i\|^2 \leq 1 + \delta) \geq 1 - 2se^{(1+\delta)s/2-s(s-1)(\delta^2/2-\delta^3/3)/2}.
\]

**Proof.** This follows from the fact that mapping a standard normal Gaussian random vector to the unit vector in its span leads to a uniform distribution on the sphere. The preceding probability estimates are unchanged by scaling the vectors on both sides. Let \(Z_j = \sum_{i=1}^N X_{j,l}^2\), then the theorem we wish to prove is equivalent to
\[
\mathbb{P}(\frac{1}{1+\delta} \leq \frac{N}{s^2} \sum_{j=1}^s \frac{1}{Z_j} \sum_{l=1}^s X_{j,l}^2 \leq (1 + \delta)) \geq 1 - 2se^{(1+\delta)s/2-s(s-1)(\delta^2/2-\delta^3/3)/2}.
\]

To deduce this, we use a union bound again, which implies with the preceding estimates that
\[
\mathbb{P}(\frac{1}{1+\delta} \max_j Z_j \leq \frac{N}{s^2} \sum_{j=1}^s \sum_{l=1}^s X_{j,l}^2 \leq (1 + \delta) \min_j Z_j) \geq 1 - 2se^{(1+\delta)s/2-s(s-1)(\delta^2/2-\delta^3/3)/2}.
\]

Now \(\sum_{l} X_{l,j}^2 \leq Z_j \leq \max_i \sum_{l} X_{i,l}^2\) establishes the bound. \(\square\)

Next, we prepare the result on the equiangularity of random fusion frames.

**Lemma 4.4.** Let \(M = Ks, M \geq N \geq s, 0 < \delta < 1, \epsilon = (1+\delta)^3 - 1\) and \(\{x_i\}_{i=1}^M\) be independent random vectors in \(\mathbb{R}^N\) that are uniformly distributed on the unit sphere. Let \(\{P_k\}_{k=1}^K\) be the orthogonal projection onto the span of \(\{x_i\}_{i \in J_k}\), where \(\{J_k\}\) partitions the index set and each \(J_k\) has size \(s\), then for fixed \(k \neq l\),
\[
\mathbb{P}(\frac{1}{1+\epsilon} - \epsilon \sqrt{(1+\epsilon)N} s \leq \frac{N}{s^2} \text{tr}[P_l P_k] \leq 1 + \epsilon(1 + \sqrt{(1+\epsilon)N} s) + \frac{Ne^2}{4s}) \geq 1 - R_1 - R_2
\]
where the failure probability is bounded by the sum of
\[
R_1 = 2se^{(1+\delta)s/2-s(s-1)(\delta^2/2-\delta^3/3)/2}.
\]

and
\[
R_2 = 2(M + K(1 + \frac{4}{\delta})^s)e^{-N\delta^2/4+N\delta^3/3} + 2(1 + \frac{4}{\delta}) Ne^{-M\delta^2/4+M\delta^3/3}.
\]

**Proof.** A random, uniformly distributed \(s\)-dimensional subspace is realized by taking the span of independent, identically uniformly distributed random unit vectors \(\{x_i\}_{i=1}^s\). Correspondingly, the orthogonal projection is obtained via orthonormalizing these random vectors with the square root of the pseudoinverse \(S^t\) of \(S = \sum_{j=1}^s x_j \otimes x_j^*\). As a consequence, the trace is identical to
\[
\text{tr}[P_l P_k] = \sum_{j=1}^s \text{tr}[P_l (S^t)^{1/2} x_j \otimes x_j^* (S^t)^{1/2}] = \sum_{j=1}^s \|P_l (S^t)^{1/2} x_j\|^2.
\]
With the triangle inequality, we split this expression into three parts that we estimate separately,

\[ Q_1 - Q_2 + Q_3 \leq \sum_{i \in J_k} \| P_i (S^\dagger)^{1/2} x_i \|^2 \leq Q_1 + Q_2 + Q_3 \]

with

\[ Q_1 = \sum_{i \in J_k} \| P_i x_i \|^2, \quad Q_2 = 2 \sum_{i \in J_k} \| P_i x_i \| \| P_i ((S^\dagger)^{1/2} x_i - x_i) \| \]

and

\[ Q_3 = \sum_{i} \| P_i ((S^\dagger)^{1/2} x_i - x_i) \|^2. \]

The Cauchy-Schwarz inequality gives \( Q_2 \leq 2(Q_1 Q_3)^{1/2} \) so it is enough to control \( Q_1 \) and \( Q_3 \).

The quantity \( Q_1 \) is concentrated near \( s^2/N \) by Theorem 4.3, which gives a lower probability bound of \( 1 - R_1 \) for the set with \( \frac{1}{1+\epsilon} \leq \frac{1}{1+\delta} \leq NQ_1/s^2 \leq 1 + \delta \leq 1 + \epsilon \). The third quantity is with probability \( 1 - R_2 \) small by Theorem 3.4, because if \( \epsilon = (1 + \delta)^3 - 1 \) then \( \frac{1}{1+\epsilon} \leq S \leq 1 + \epsilon \) and if \( \| x \| = 1 \), then

\[ \| (S^\dagger)^{1/2} x - x \|^2 \leq (\frac{1}{\sqrt{1+\epsilon}} - 1)^2 \leq (\sqrt{1+\epsilon} - 1)^2 \leq \epsilon^2/4. \]

Thus, apart from a set of probability given in Theorem 3.4, \( Q_3 \leq s\epsilon^2/4 \) and \( 2(Q_1 Q_3)^{1/2} \leq (Q_1 s)^{1/2}\epsilon \).

Next, if \( \frac{1}{1+\epsilon} \leq Q_1 \leq (1 + \epsilon)\frac{s^2}{N} \) then

\[ \frac{1}{1+\epsilon} \leq \frac{1}{1+\delta} \leq \frac{1}{1+\epsilon} \frac{s^2}{N} \leq (1 + \epsilon)^{1/2} s^{3/2}\epsilon \sqrt{N} \leq Q_1 \]

and

\[ Q_1 - Q_2 \leq \sum_{i \in J_k} \| P_i (S^\dagger)^{1/2} x_i \|^2 \]

\[ \leq \sum_{i \in J_k} \| P_i (S^\dagger)^{1/2} x_i \|^2 \leq Q_1 + Q_2 + Q_3 \leq (1 + \epsilon)\frac{s^2}{N} + (1 + \epsilon)^{1/2} s^{3/2}\epsilon \sqrt{N} + s\epsilon^2/4. \]

\[ \square \]

By a union bound over all pairings of subspaces, we get the following, more qualitative estimate:

**Theorem 4.5.** Let \( K, s, N \in \mathbb{N}, M = KS \geq N \geq s \). For any \( c > 1 \) there exists \( \epsilon_0 > 0 \) such that for all \( 0 < \epsilon < \epsilon_0 \), if \( \{ V_k \}_{k=1}^K \) is a family of \( s \)-dimensional subspaces selected independently at random according to the unitarily invariant distribution in \( \mathbb{R}^N \) and \( \{ P_k \}_{k=1}^K \) the corresponding orthogonal projections, then the set

\[ \mathcal{X} = \{ 1 - cc(1 + \sqrt{\frac{N}{s}}) \leq \frac{N}{s} \text{ tr}[P_k P_l] \leq 1 + cc(1 + \sqrt{\frac{N}{s}}) + \frac{N\epsilon^2}{4s} \text{ for all } k \neq l \} \]

has probability

\[ \mathbb{P}(\mathcal{X}) \geq 1 - (R_1 + R_2)K(K-1)/2 \]

with \( R_1 \) and \( R_2 \) as in the preceding lemma.

**Proof.** If \( \epsilon \) is sufficiently small, then the remainder for the series expansion, \( 1/(1+\epsilon) - 1 + \epsilon + (\sqrt{1+\epsilon} - 1 - \epsilon/2)\sqrt{N/s} \) is bounded by \( (c-1)\epsilon + c\epsilon \sqrt{N/s} \). A union bound for the \( K(K-1)/2 \) pairings gives the probability bound. \( \square \)
Corollary 4.6. Let $K, N, s$ and $\delta$ be as above. If $N, s \to \infty$, $s/N \to c > 0$, and $K$ is such that

$$\frac{3 \ln(K + 1)}{N} + \frac{s}{N} \ln(1 + \frac{4}{\delta}) < \frac{\delta^2}{4} - \frac{\delta^3}{3}$$

and

$$\frac{N}{Ks} \ln(1 + \frac{4}{\delta}) < \frac{\delta^2}{4} - \frac{\delta^3}{3}$$

then the upper bound $R_1 + R_2$ for the probability of $X$ in the preceding theorem decays exponentially in $N$.

Proof. As $s/N$ remains bounded away from zero as $N \to \infty$, $R_1$ decays exponentially. Examining the exponents of the terms in $R_2$ shows that if the conditions on $K$, $s$ and $N$ are satisfied then it decays exponentially as $N \to \infty$ as well. $\Box$

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