Interleaving by Parts: Join Decompositions of Interleavings and Join-Assemblage of Geodesics

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Abstract

Metrics of interest in topological data analysis (TDA) are often explicitly or implicitly in the form of an interleaving distance \(d_I\) between poset maps (i.e. order-preserving maps), e.g. the Gromov-Hausdorff distance between metric spaces can be reformulated in this way.

We propose a representation of a poset map \(F: \mathcal{P} \rightarrow \mathcal{Q}\) as a join (i.e. supremum) \(\bigvee_{b \in B} F_b\) of simpler poset maps \(F_b\) (for a join dense subset \(B \subset \mathcal{Q}\)) which in turn yields a decomposition of \(d_I\) into a product metric. The decomposition of \(d_I\) is simple, but its ramifications are manifold: (1) We can construct a geodesic path between any poset maps \(F\) and \(G\) with \(d_I(F, G) < \infty\) by assembling geodesics between all \(F_b\)s and \(G_b\)s via the join operation. This construction generalizes at least three constructions of geodesic paths that have appeared in the literature. (2) We can extend the Gromov-Hausdorff distance to a distance between simplicial filtrations over an arbitrary poset with a flow, preserving its universality and geodesicity. (3) We can clarify equivalence between several known metrics on multiparameter hierarchical clusterings. (4) We can illuminate the relationship between the erosion distance by Patel and the graded rank function by Betthauser, Bubenik, and Edwards, which in turn takes us to an interpretation on the representation \(\bigvee_b F_b\) as a generalization of persistence landscapes and graded rank functions.

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1 Introduction

Persistent homology and interleaving distances. Persistent homology plays a central role in topological data analysis (TDA) \[20, 36, 41\]. The most basic construction in persistent homology consists of applying the homology functor to an \(\mathbb{R}\)-indexed nested family of topological spaces or simplicial complexes such as the Vietoris-Rips filtration on a metric space. By utilizing homology with coefficients in a field \(\mathbb{F}\), we obtain so-called persistence modules, which are \(\mathbb{R}\)-indexed functors valued in the category \(\text{vect}\) of vector spaces and linear maps over \(\mathbb{F}\). Generalizing this notion, a poset-indexed functor valued in a certain category \(\mathcal{C}\) is called a generalized persistence module with values in \(\mathcal{C}\) \[15\].

One of the most prevalent metrics for quantifying the dissimilarity between two persistence modules is the interleaving distance \(d_I\). Since \(d_I\) was first introduced in order to compare \(\mathbb{R}\)-indexed persistence modules \[25\], it has been generalized to various different settings \[11, 10, 12, 32, 34, 35, 54, 71\]. One of the main uses of \(d_I\) is for comparing \(\mathbb{R}^n\)-indexed persistence modules, where its computation is known to be NP-hard for \(n \geq 2\) \[7\].

While poly-time computable lower bounds for \(d_I\) have been studied for \(n = 2\) \[8, 23, 44, 52\], its extension to the case \(n \geq 3\) is not much known. The erosion distance introduced by Patel \[65\] is an attractive alternative in this respect and will be subsequently further discussed.

Interleavings between poset maps. Partially ordered sets are simply called posets. An order-preserving map \(\mathcal{P} \to \mathcal{Q}\) between posets \(\mathcal{P}\) and \(\mathcal{Q}\) is called a poset map. By viewing the target poset \(\mathcal{Q}\) as a category (each point \(p \in \mathcal{Q}\) is an object and a unique arrow \(p \to q\) exists whenever \(p \leq q\) in \(\mathcal{Q}\)), a poset map can be viewed as a generalized persistence module. Poset maps are omnipresent in TDA, e.g. simplicial filtrations (indexed by arbitrary posets), hierarchical clusterings (indexed by arbitrary posets), and (generalized) rank functions of persistence modules. Poset maps have also been utilized in discrete Morse theory, cf. \[51, \text{Thm. 11.4}\].

When \(\mathcal{P}\) is equipped with a notion of flow \[12, 35, 71\], we can define an interleaving distance \(d_I\) between two poset maps \(\mathcal{P} \to \mathcal{Q}\). Examples of such include the following.

1. Interleavings between simplicial filtrations. A distance between \(\mathbb{R}\)-indexed simplicial filtrations was proposed by Mémoli \[58\]. It turned out that this distance is a generalization of the Gromov-Hausdorff distance between finite metric spaces and thus called the Gromov-Hausdorff distance and denoted by \(d_{GH}\) \[59\]. This distance was proved to upper bound the bottleneck distance between persistence diagrams \[58, \text{Thm. 4.2}\] and even the homotopy interleaving distance by Blumberg and Lesnick \[9\]; see \[71\]. A certain variant of \(d_{GH}\) also appears in the study of metrics on Reeb graphs \[2\].

2. Interleavings between hierarchical clusterings over a poset. Hierarchical clustering methods on a metric space \(X\) uncovers multiscale clustering features of \(X\) \[21\], yielding a dendrogram (Fig. 1 (A)), a poset map from \([0, \infty)\) to the lattice of partitions of \(X\). Dendrograms are widely generalized for density-sensitive hierarchical clustering \[18, 22, 68\], for hierarchical clustering on an asymmetric network \[73\], for summarizing clustering features in dynamic networks \[16, 46\], Fig. 1 (B)). These structures also arise in phylogenetic trees \[6, 74, 77\] and phylogenetic networks \[42, 43, 56, 64\]. We remark that these structure are finer than merge trees \[40, 60\] or Reeb graphs \[2, 4, 24, 34, 75\], as addressed in \[49\].

3. Interleavings between poset maps into a Grothendieck group. The erosion distance \(d_E\) was introduced for comparing generalized persistence diagrams for \(\mathbb{R}\)-indexed persistence modules
Figure 1: (A) A dendrogram derived from a hierarchical clustering method on a metric space (Ex. 2.15). The dendrogram captures multiscale clustering features of the given metric space. (B) The formigram* derived from a dynamic network. The formigram tracks the evolution of connected components in the dynamic network. *The name formigram is a combination of the words formicarium (a.k.a. ant farm) and diagram. Synthetic flocking behaviors have been successfully classified via a certain lower bound for a distance between formigrams (Defn. 2.22) [50].

valued in certain categories \( C \) beyond vect [65]. Generalized persistence diagrams can be encoded as certain poset maps, called rank functions, whose target is the Grothendieck group of \( C \) [76], equipped with a natural order. \( d_E \) is actually an interleaving distance between rank functions. Even though \( d_E \) has been adapted to more abstract settings [47, 66], its most basic use is as a tractable lower bound for \( d_I \) between \( \mathbb{R}^n \)-indexed persistence modules for any \( n \) [48 Section 5]. Indeed, \( d_E \) has been utilized for classifying spatiotemporal persistent homology (encoded as \( \mathbb{R}^3 \)-indexed persistence modules) of time-varying data [31].

Other related work. The categorification of persistent homology has provided a fertile interpretation of persistence theory and the interleaving distance [10, 12, 13, 15, 35, 71]. Among others, Scoccola [71] introduced a notion of a locally persistent category, which is a category with a notion of approximate morphism. This enables us to define an interleaving distance between objects in the category, which encompasses many distances in TDA and facilitates a uniform treatment of those distances. For example, sufficient conditions under which an interleaving distance is geodesic have been found [71, Thms. 4.5.2 and 4.5.16].

Our contributions. Let \((\mathcal{P}, \leq, \Omega)\) be a poset with a flow (Defn. 2.7) and let \( \mathcal{D} \) be a poset with a join-dense subset \( B \subseteq \mathcal{D} \) (which always exists; see Sec. 2.1). Let \( F, G : (\mathcal{P}, \leq, \Omega) \to (\mathcal{D}, \leq) \) be any two poset maps and let \( d_I(F, G) \) be their interleaving distance (Defn. 2.9).

(i) We identify join representations \( F = \bigvee_{b \in B} F_b \) and \( G = \bigvee_{b \in B} G_b \) such that

\[
d_I(F, G) = \sup_{b \in B} d_I(F_b, G_b),
\]

where each of the \( F_b \)s and \( G_b \)s is structurally simple. These join representations can be seen as a rendition of the algebraic decomposition of persistence modules (Rmks. 3.8, 3.11) as well as a generalization of persistence landscapes [5, 14] (Rmk. 4.4(i)).
We harness item (i) in order to establish all of the following:

(ii) We show that \( d_I \) is an \( \ell^\infty \)-product of multiple copies of a distance between upper sets in \( \mathcal{P} \) (Thm. 3.10).

(iii) We show that \( d_I \), as a distance between poset maps \( \mathbf{F}, \mathbf{G} : \mathcal{P} \to \mathcal{Q} \), is geodesic under the assumption that \( \mathcal{Q} \) is a complete lattice (Thm. 3.13). More specifically, we obtain a geodesic path between \( \mathbf{F} \) and \( \mathbf{G} \) by assembling geodesic paths between all \( \mathbf{F}_b \)s and \( \mathbf{G}_b \)s via the join operation.

All the metrics mentioned in subsequent items can be incorporated into the framework of interleaving distances between poset maps. This enables us to prove in a uniform way that all metrics in the items below are geodesic.

(iv) We show that computing the erosion distance between rank functions of persistence modules amounts to computing a finite number of Hausdorff distances between certain geometric signatures of graded rank functions [5] (Thm. 4.3). An analogous statement holds when comparing multiparameter hierarchical clusterings (Thm. 4.17).

(v) We generalize the Gromov-Hausdorff distance between metric spaces to a distance between simplicial filtrations over \( \mathcal{P} \). This distance inherits a universal property of the original Gromov-Hausdorff distance (Thm. 4.11), of which the celebrated Vietoris-Rips filtration stability theorem becomes a consequence (Thm. 4.14 and Rmk. 4.15 (ii)).

(vi) We establish the equivalence between several known metrics on multiparameter hierarchical clusterings (Defn. 2.22, Thm. 4.23, Rmk. 4.24).

(vii) We elucidate the computational complexity of the interleaving distance between formigrams (Thm. 4.29).

Organization. Sec. 2 reviews the notions of lattices, subpartitions, interleaving distances, and formigrams. Sec. 3 addresses items (i)–(iii) above and Sec. 4 addresses the rest of the items. Sec. 5 discusses open questions.

2 Preliminaries

We review the notions of lattices (Sec. 2.1), subpartitions (Sec. 2.2), interleaving distances (Sec. 2.3), and formigrams (Sec. 2.4).

2.1 Posets, lattices, and poset maps.

In this section, we recall basic terminology from the theory of ordered sets and lattices [39, 69].

A poset \( \mathcal{P} = (\mathcal{P}, \leq) \) is a nonempty set \( \mathcal{P} \) equipped with a partial order, i.e. a reflexive, antisymmetric, and transitive relation \( \leq \) on \( \mathcal{P} \). An element \( 0 \in \mathcal{P} \) is said to be a zero element if \( 0 \leq p \) for all \( p \in \mathcal{P} \). If a zero element exists in \( \mathcal{P} \), then it is unique. Thus we refer to 0 as the zero element in \( \mathcal{P} \). An element \( 1 \in \mathcal{P} \) is said to be a unit element if \( p \leq 1 \) for all \( p \in \mathcal{P} \). For \( p, q \in \mathcal{P} \) with \( p \leq q \), we write \([p, q] \) for the set \( \{ r \in \mathcal{P} : p \leq r \leq q \} \). Also, we write \( p^\uparrow \) for the set

\[ \bigcup \{ r : r \leq p \} \]

\[ \bigcap \{ r : r \geq p \} \]

Given any metric spaces \((M_i, d_i)\), \( i \in I \), the \( \ell^\infty \)-product metric is defined to be the metric \( \sup_{i \in I} d_i \) on \( \prod_{i \in I} M_i \).

Some of those distances are already known to be geodesic, but some are not. Known results will be cited at suitable places in the paper.
any join-dense subset must contain all irreducible elements. Therefore, when a join and a meet of $p_1, \ldots, p_n$ exist, then they are unique. Hence, whenever they exist, we refer to them as the join (denoted by $\lor \{p_i\}_{i=1}^n$) and the meet (denoted by $\land \{p_i\}_{i=1}^n$), respectively. The poset $\mathcal{P}$ is said to be a join-semilattice (resp. meet-semilattice) if $\mathcal{P}$ allows all finite joins (resp. meets). If $\mathcal{P}$ is both join- and meet-semilattice, then $\mathcal{P}$ is said to be a lattice. $\mathcal{P}$ is called a complete lattice if the meet and join of any subset (possibly infinite) $A \subset \mathcal{P}$ exist.

If a poset $\mathcal{P}$ has a zero element 0, then any nonzero $p \in \mathcal{P}$ such that $\{0, p\} = \{0, p\}$ is called an atom of $\mathcal{P}$. A nonzero element $p$ of a lattice $\mathcal{P}$ is (join-)irreducible if $p$ is not the join of two smaller elements, that is, if $p = q \land r$, then $p = q$ or $p = r$. Note that every atom is join-irreducible. For example, for the ordered set $\mathcal{Q} = \{p < q < r\}$, $p$ is the zero element, $q$ is the unique atom, and $q$ and $r$ are join-irreducible. A join representation of $p \in \mathcal{P}$ is an expression $\lor A$ which evaluates to $p$ for some $A \subset \mathcal{P}$. When $\mathcal{P}$ includes a zero element, the zero element has the join representation $\lor \emptyset$. A join representation $\lor A$ of $p$ is irredundant of $\lor A' < \lor A$ for each proper subset $A' \subset A$. We say that a subset $A \subset \mathcal{P}$ join-refines another subset $B \subset \mathcal{P}$ if, for each $a \in A$, there exists some element $b \in B$ such that $a \leq b$. Join-refinement defines a preorder $\leq$ on the subsets of $\mathcal{P}$. Fix $p \in \mathcal{P}$ and let $ijr(p)$ be the set of irredundant join representations of $p$. If $(ijr(p), \leq)$ has a unique minimum element $A \subset \mathcal{P}$, then $A$ (or $\lor A$) is called the canonical join representation of $p$.

A subset $B \subset \mathcal{P}$ is said to be join-dense if every element of $\mathcal{P}$ is the join of a subset of $B$. Trivially, $\mathcal{P}$ itself is join-dense. The poset $\mathcal{P}$ is said to be $\lor$-irreducibly generated if the set of all join-irreducible elements of $\mathcal{P}$ is join-dense. Every finite poset is $\lor$-irreducibly generated.

**Remark 2.1.** Any join-dense subset must contain all irreducible elements. Therefore, when $\mathcal{P}$ is finite, the set of irreducible elements in $\mathcal{P}$ is the smallest join-dense subset of $\mathcal{P}$.

Given any two posets $\mathcal{P}$ and $\mathcal{Q}$, a map $F : \mathcal{P} \to \mathcal{Q}$ is called an order-preserving map or a poset map if $p \leq q$ in $\mathcal{P}$ implies $F(p) \leq F(q)$. The collection of all poset maps from $\mathcal{P}$ to $\mathcal{Q}$ will be denoted by $[\mathcal{P}, \mathcal{Q}]$.

**Remark 2.2.** We regard $[\mathcal{P}, \mathcal{Q}]$ as a poset equipped with the partial order inherited from $\mathcal{Q}$, i.e. $F \leq G$ in $[\mathcal{P}, \mathcal{Q}]$ if $F(p) \leq G(p)$, for all $p \in \mathcal{P}$.

### 2.2 Lattice of subpartitions

In this section we review the notion of subpartition and show that the collection of all subpartitions of a set is a lattice.

Let us fix a nonempty finite set $X$. A partition of $X$ is a collection $P$ of nonempty disjoint $B \subset X$, called blocks, such that the union of all blocks $B$ is equal to $X$. Every partition $P$ of $X$ induces the equivalence relation $\sim_p$ given by: $x \sim_p x' \iff x, x' \in B$ for some block $B \in P$. Reciprocally, any equivalence relation $\sim$ of $X$ induces the partition $X/\sim$. A subpartition $Q$ of $X$ is a partition $Q$ of
Figure 2: The figure shows the embedding of $\text{Part}(Y)$ into $\text{SubPart}(Y)$ for $Y = \{x, y, z\}$ (subdiagram in shaded region). For simplicity, distinct blocks in a partition are separated by $|$ instead of curly brackets. In $\text{Part}(Y)$, the join-irreducible elements are the atoms $\{xy \mid z\}, \{yz \mid x\}, \{zx \mid y\}$. On the other hand, in $\text{SubPart}(Y)$, the join-irreducible elements are $\{x\}, \{y\}, \{z\}, \{xy\}, \{yz\}, \{zx\}$, and the first three singletons are in particular its atoms.

some $X' \subset X$. The set $X'$ is said to be the underlying set of $Q$. The equivalence relation on $X'$ induced by $Q$ is said to be a subequivalence relation on $X$.

**Definition 2.3** (Poset of (sub)partitions). By $\text{Part}(X)$ (resp. $\text{SubPart}(X)$), we denote the set of all partitions (resp. subpartitions) of $X$. Let $P_1, P_2 \in \text{SubPart}(X)$. $P_1$ is said to refine $P_2$ and write $P_1 \leq P_2$, if for any block $B_1 \in P_1$, there exists a block $B_2 \in P_2$ such that $B_1 \subset B_2$ (see Fig. 2 for an example).

Note that for any $P \in \text{SubPart}(X)$, we have $\emptyset \leq P \leq \{X\}$, i.e. $\emptyset$ and $\{X\}$ are the zero and unit elements of $\text{SubPart}(X)$, respectively. It is well-known that $\text{Part}(X)$ is a lattice [69]. We show that $\text{Part}(X)$ is a sublattice of $\text{SubPart}(X)$: Given any $P_1, P_2 \in \text{SubPart}(X)$, let $X_1$ and $X_2$ be the underlying sets of $P_1$ and $P_2$, respectively.

(i) The join $P_1 \lor P_2$ is the quotient set $(X_1 \cup X_2)/\sim$, where $\sim$ is the smallest equivalence relation on $X_1 \cup X_2$ containing $\sim_{P_1}$ and $\sim_{P_2}$. The join $P_1 \lor P_2$ is also called the finest common coarsening of $P_1$ and $P_2$.

(ii) The meet $P_1 \land P_2$ is the quotient set $(X_1 \cap X_2)/((\sim_{P_1} \cap \sim_{P_2}))$. The meet $P_1 \land P_2$ is also called the coarsest common refinement of $P_1$ and $P_2$.

If $X = X_1 = X_2$, then $P_1 \lor P_2$ and $P_1 \land P_2$ clearly belong to $\text{Part}(X)$. Hence:

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$^3$See [69] Sec. 4 for properties of $\text{Part}(X)$. 

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**Proposition 2.4.** SubPart$(X)$ equipped with the refinement relation is a lattice. In particular, Part$(X)$ is a sublattice of SubPart$(X)$.

**Remark 2.5.** (i) The atoms of SubPart$(X)$ are $\{x\}$ for $x \in X$. In what follows, let us assume that $|X| \geq 2$. (ii) The join-irreducible elements of SubPart$(X)$ consist of all the atoms and all sets of the form $\{x, x'\}$ for different $x, x' \in X$.

**Proposition 2.6.** Let $X$ be any nonempty finite set. (i) SubPart$(X)$ is $\lor$-irreducibly generated. (ii) Let $P \in$ SubPart$(X)$ be a nonzero element. If $P$ includes a block $B$ with $|B| \geq 3$, then $P$ has no canonical join representation.

**Proof.** (i) Let $~_p$ be the subequivalence relation on $X$ corresponding to $P$. Then $P = \lor A$ where $A := \{(x, x') : x \sim_p x'\}$ (when $x = x'$, the set $\{x, x'\}$ is a singleton). By Rmk. 2.5(ii) $P = \lor A$ is a join representation by irreducible elements.

(ii) Without loss of generality, assume that $(x, y, z) \in P$. Observe that the following three are minimal join representations of the single block partition $\{xyz\}$ and hence a minimum does not exist:

$$\lor \{\{xy\}, \{yz\}\}, \lor \{\{yz\}, \{zx\}\}, \lor \{\{zx\}, \{xy\}\}.$$  

This implies that, given a minimal join representation $\lor A$ of $P$ by irreducible elements, exactly one of the three subsets $\{\{xy\}, \{yz\}\}$, $\{\{yz\}, \{zx\}\}$, $\{\{zx\}, \{xy\}\}$ is a subset of $A$. Whatever the case is, the corresponding subset can be replaced by any of the other two, and thus there is no canonical join representation of $P$.

### 2.3 Posets with a flow and interleaving distances

We review the notion of poset with a flow and its associated interleaving distance [12, 35].

**Flows and interleavings** For a poset $\mathcal{P}$, let $I_\mathcal{P}$ be the identity map on $\mathcal{P}$.

**Definition 2.7.** A (strict) flow on a poset $\mathcal{P}$ is a family $\Omega := \{\Omega_\varepsilon : \mathcal{P} \to \mathcal{P}\}_{\varepsilon \in [0, \infty)}$ of poset maps on $\mathcal{P}$ such that (i) $\Omega_s \leq \Omega_t$ for all $s \leq t$, (ii) $\Omega_s \circ \Omega_t = \Omega_{s+t}$, for all $s, t \in [0, \infty)$, and (iii) $I_\mathcal{P} = \Omega_0$. We call the triple $(\mathcal{P}, \leq, \Omega)$ a poset with a flow.\[4\]

We define an extended pseudometric between point in a poset with a flow [35] as follows:

**Definition 2.8** (Interleaving of poset elements). Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow. For $\varepsilon \in [0, \infty)$, any $p, q \in \mathcal{P}$ are said to be $\varepsilon$-interleaved if $p \leq \Omega_\varepsilon(q)$ and $q \leq \Omega_\varepsilon(p)$. The interleaving distance between $p$ and $q$ is defined as

$$d_\Omega(p, q) := \inf \{\varepsilon \in [0, \infty) : p, q \text{ are } \varepsilon\text{-interleaved}\}.$$  

If $p, q$ are not $\varepsilon$-interleaved for any $\varepsilon \in [0, \infty)$, then we declare that $d_\Omega(p, q) = \infty$.

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[4] By weakening conditions (ii) and (iii), we obtain the notion of coherent flow [35] (or superlinear family of translations [12]). This level of generality is not required for the purpose of this paper.
By [35, Lem. 3.7] and [12, Thm. 3.21], we know that \( d_\Omega \) is an extended pseudometric on \( \mathcal{P} \). For example, let \( \mathbb{R}^n \) be equipped with the product order given as \((x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) \) if \( x_i \leq y_i \) for each \( i = 1, \ldots, n \). Then, the supremum norm distance \( \| - \|_\infty \) in \( \mathbb{R}^n \) coincides with the interleaving distance with the flow \( \Omega \) given as
\[
\Omega := \{ \Omega_\varepsilon : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n) + \varepsilon (1, \ldots, 1) \}_{\varepsilon \in [0, \infty)}.
\]

Next, we introduce two special examples of Defn. 2.8.

**Interleaving distance between poset maps.** Let \((\mathcal{P}, \leq, \Omega)\) be a poset with a flow, and let \((\mathcal{Q}, \leq)\) be another poset. Recall that \([\mathcal{P}, \mathcal{Q}]\) is a poset (Rmk. 2.2). The flow \( \Omega \) on \((\mathcal{P}, \leq)\) yields the flow \( - \cdot \Omega \) given by pre-composition with \( \Omega \), on \([\mathcal{P}, \mathcal{Q}]\). Thus, Defn. 2.8 is specialized to:

**Definition 2.9.** The **interleaving distance between poset maps** \( F, G : (\mathcal{P}, \leq, \Omega) \to (\mathcal{Q}, \leq) \) is defined as:
\[
d_I(F, G) := \inf \{ \varepsilon \in [0, \infty) : F, G \text{ are } \varepsilon\text{-interleaved w.r.t. the flow } - \cdot \Omega \}.
\]

**Interleaving distance between upper sets.** Let \((U(\mathcal{P}), \subset)\) be the poset of upper sets of \( \mathcal{P} \) with the inclusion relation. Then, the flow \( \Omega \) on \( \mathcal{P} \) gives rise to a family \( \tilde{\Omega} = (\tilde{\Omega}_\varepsilon)_{\varepsilon \in [0, \infty)} \) of poset maps \( U(\mathcal{P}) \to U(\mathcal{P}) \) given, for each \( A \in U(\mathcal{P}) \), as
\[
\tilde{\Omega}_\varepsilon(A) := \{ p \in \mathcal{P} : \Omega_\varepsilon(p) \in A \}.
\]
Indeed \( \tilde{\Omega}_\varepsilon(A) \) is an upper set. To see this let \( x \in \tilde{\Omega}_\varepsilon(A) \) and let \( x \leq y \) in \( \mathcal{P} \). Then \( \Omega_\varepsilon(x) \in A \) and \( \Omega_\varepsilon(x) \leq \Omega_\varepsilon(y) \). Since \( A \) is an upper set, \( \Omega_\varepsilon(y) \in A \), implying that \( y \in \tilde{\Omega}_\varepsilon(A) \).

**Proposition 2.10.** \( \tilde{\Omega} \) is a flow on \( U(\mathcal{P}) \).

**Proof.** Let \( A \in U(\mathcal{P}) \). The equality \( A = \tilde{\Omega}_0(A) \) is clear. Let \( t \leq s \) in \([0, \infty)\). Then,
\[
\tilde{\Omega}_t(A) = \{ p \in \mathcal{P} : \Omega_t(p) \in A \} \\
\subset \{ p \in \mathcal{P} : \Omega_s(p) \in A \} \\
\quad \text{since } \Omega_t(p) \leq \Omega_s(p) \text{ and } A \text{ is an upper set.} \\
\quad = \tilde{\Omega}_s(A).
\]

Also, for any \( s, t \in [0, \infty) \),
\[
\tilde{\Omega}_t(\tilde{\Omega}_s(A)) = \{ p \in \mathcal{P} : \Omega_t(p) \in \tilde{\Omega}_s(A) \} = \{ p \in \mathcal{P} : \Omega_s(\Omega_t(p)) \in A \} = \{ p \in \mathcal{P} : \Omega_{t+s}(p) \in A \} = \tilde{\Omega}_{t+s}(A).
\]

**Definition 2.11** (Interleaving of upper sets). Let \((\mathcal{P}, \leq, \Omega)\) be a poset together with a flow. Then, \( d_\tilde{\Omega} \) will denote the induced interleaving distance on the poset \((U(\mathcal{P}), \subset, \tilde{\Omega})\) of upper sets of \( \mathcal{P} \).

The following remark and example will be useful in later sections.

**Remark 2.12.** (i) Arbitrary unions and intersections of upper sets in \( \mathcal{P} \) yield upper sets, implying that \( U(\mathcal{P}) \) is a complete lattice.
Example 2.13. Consider the poset \( \text{Int} := \{(a, b) \in \mathbb{R}^2 : a \leq b\} \), the upper-half plane in \( \mathbb{R}^2 \) above the diagonal line \( y = x \), equipped with the order \((a, b) \leq (a', b') \iff a' \leq a < b \leq b'\). Let us define the flow \( \Omega \) on \( \text{Int} \) by

\[
\Omega := \left\{ \Omega_\varepsilon : (a, b) \mapsto (a - \varepsilon, b + \varepsilon) \right\}_{\varepsilon \in (0, \infty)}.
\]

(2)

For the poset \((U(\text{Int}), c, \hat{\Omega})\) of upper sets in \(\text{Int}\), the distance \(d_\hat{\Omega}\) coincides with the Hausdorff distance \(d_H\) in \((\text{Int}, \|\cdot\|_\infty)\) (Defn. C.1). This follows from the observation that for any \(A \in U(\text{Int})\), the \(\varepsilon\)-thickened set

\[
A^\varepsilon := \{(a, b) \in \text{Int} : \exists (a', b') \in A, \text{ such that } \| (a, b) - (a', b') \|_\infty \leq \varepsilon\},
\]

(3)

coincides with \(\hat{\Omega}_\varepsilon(A)\).

2.4 Formigrams and their interleavings

In this section we review the notion of formigrams and their specialized interleaving and Gromov-Hausdorff distances [46, 49] (note: Secs. 3, 4.1, and 4.2 can be read without reading this section).

Formigrams. We begin by reviewing the definition of dendrograms. Let us fix a nonempty finite set \(X\).

Definition 2.14 ([21]). A dendrogram over a finite set \(X\) is any function \(\theta : \mathbb{R}_{\geq 0} \to \text{Part}(X)\) such that the following properties hold: (1) \(\theta(0) = \{x : x \in X\}\), (2) if \(t_1 \leq t_2\), then \(\theta(t_1) \subseteq \theta(t_2)\), (3) there exists \(T > 0\) such that \(\theta(t) = \{X\}\) for \(t \geq T\), (4) for all \(t\) there exists \(\varepsilon > 0\) s.t. \(\theta(s) = \theta(t)\) for \(s \in [t, t + \varepsilon]\) (right-continuity) (see Fig. 1 (A)). The ultrametric induced by \(\theta\) is the distance function \(u_\theta : X \times X \to \mathbb{R}_{\geq 0}\) defined as

\[
u_\theta(x, x') := \min\left\{ t \in \mathbb{R} : x \sim_{\theta(t)} x' \right\}.
\]

Note that we have the ultra-triangle inequality: \(u_\theta(x, x'') \leq \max\{u_\theta(x, x'), u_\theta(x', x'')\}\) for every \(x, x', x'' \in X\).

Example 2.15. The single linkage hierarchical clustering method on a finite metric space \((X, d_X)\) induces the dendrogram \(\theta : \mathbb{R}_{\geq 0} \to \text{Part}(X)\) given as \(t \mapsto X/\sim_t\), where \(\sim_t\) is the transitive closure of the relation \(R_t := \{(x, x') \in X \times X : d_X(x, x') \leq t\}\) (Fig. 1 (A)). The mapping \((X, d_X) \mapsto (X, u_\theta)\) is known to be 1-Lipschitz with respect to the Gromov-Hausdorff distance (Defn. C.2 [21 Prop.2]), i.e.

\[
d_{GH}((X, u_\theta_X), (Y, u_\theta_Y)) \leq d_{GH}((X, d_X), (Y, d_Y)).
\]

(4)
Formigrams, a mathematical model for time-varying clusters in dynamic networks \[46\], are a generalization of dendrograms. Formigrams are defined as constructible cosheaves over \(\text{SubPart}(X)\) with values in \(\text{SubPart}(X)\), which amounts to a (costalk-)function \(\mathbb{R} \rightarrow \text{SubPart}(X)\) described as follows.

**Definition 2.16.** A formigram over \(X\) is a function\(^5\) \(\theta : \mathbb{R} \rightarrow \text{SubPart}(X)\) satisfying the following: there exists a finite set \(\text{crit}(\theta) = \{t_1, \ldots, t_n\} \subset \mathbb{R}\) of critical points s.t. (i) \(\theta\) is constant on \((t_i, t_{i+1})\) for each \(i = 0, \ldots, n\), where \(t_0 = -\infty\) and \(t_{n+1} = \infty\). (ii) At each critical point, \(\theta\) is locally maximal, i.e.

\[
\text{for } t_i \in \text{crit}(\theta) \text{ and for } \varepsilon \in [0, \min_{j \in [i, i+1]} (t_j - t_{j-1})], \quad \theta(t_i - \varepsilon) \leq \theta(t_i) \leq \theta(t_i + \varepsilon). \quad (5)
\]

See Fig. 1 (B), Fig. 3, and Fig. 5 (A) for illustrative examples. Note that \(\text{crit}(\theta)\) is not necessarily unique nor nonempty. We also remark that, given a dendrogram \(\bar{\theta} : \mathbb{R}_{\geq 0} \rightarrow \text{Part}(X), \bar{\theta}\) can be seen as a formigram by trivially extending its domain to \(\mathbb{R}\), i.e. \(\bar{\theta}(t) := \emptyset\) for \(t \in (-\infty, 0)\).

Since \(\text{SubPart}(X)\) is a poset, the collection of all formigrams on \(X\) can be regarded as a poset in its own right when endowed with the partial order defined as \(\theta \leq \theta' \iff \theta(t) \leq \theta'(t)\), for all \(t \in \mathbb{R}\).

**Definition 2.17.** By \(\text{Formi}(X)\), we denote the poset of all formigrams over \(X\).

**Interleaving distance on \(\text{Formi}(X)\).** For \(t \in \mathbb{R}\) and \(\varepsilon \in [0, \infty)\) we denote the closed interval \([t - \varepsilon, t + \varepsilon]\) of \(\mathbb{R}\) by \([t]^{\varepsilon}\).

**Definition 2.18** \((49)\). Let \(\theta : \mathbb{R} \rightarrow \text{SubPart}(X)\) be a formigram and let \(\varepsilon \in [0, \infty)\). We define the \(\varepsilon\)-smoothing \(S_\varepsilon(\theta) : \mathbb{R} \rightarrow \text{SubPart}(X)\) of \(\theta\) as \(S_\varepsilon(\theta)(t) := \bigvee \{\theta(s) : s \in [t]^{\varepsilon}\}\), for \(t \in \mathbb{R}\).

The family \(S = (S_\varepsilon)_{\varepsilon \in [0, \infty)}\) is a flow on the poset \(\text{Formi}(X)\) (Defn. 2.7):

**Proposition 2.19** \((49)\). Let \(\varepsilon \in [0, \infty)\), and let \(\theta\) be a formigram over \(X\). Then, (i) the \(\varepsilon\)-smoothing \(S_\varepsilon(\theta)\) of \(\theta\) is also a formigram over \(X\) and \(S_0(\theta) = \theta\). (ii) Also, for any \(a, b \in [0, \infty)\), we have: \(S_a(S_b(\theta)) = S_{a+b}(\theta)\). (iii) Let \(\theta'\) be another formigram over \(X\). For any \(\varepsilon \in [0, \infty)\), we have: \(\theta \leq \theta' \Rightarrow S_\varepsilon(\theta) \leq S_\varepsilon(\theta')\).

By Defn. 2.8 and Prop. 2.19, we have:

**Definition 2.20.** The **interleaving distance between** \(\theta, \theta' \in \text{Formi}(X)\) is

\[
d_\ell(\theta, \theta') := \inf\{\varepsilon \in [0, \infty) : \theta, \theta' \text{ are } \varepsilon\text{-interleaved w.r.t. the flow } S\}.
\]

Since Defn. 2.20 is a special case of Defn. 2.8, we readily know that \(d_\ell\) is an extended pseudometric. In fact, \(d_\ell\) is an extended metric, not just a pseudometric. This can be proved by exploiting the fact that a formigram has finitely many critical points; similar ideas can be found in \[49\] Thm. 4.5. We remark that when restricted to treegrams \[73\] (cf. Rmk. 4.22 (ii)) \(d_\ell\) agrees with what’s been called the \(\ell^\infty\)-cophenetic metric on phylogenetic trees \[19\] \[61\].

**Example 2.21.** Let \(\delta > 0\). Consider the formigrams \(\theta, \theta' : \mathbb{R} \rightarrow \text{SubPart}([x, y])\) defined as:

\[\text{But not necessarily a poset map.}\]

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Figure 3: The two formigrams in Example 2.21

\[ \theta(t) = \{x, y\}, \text{ for all } t \in \mathbb{R}, \quad \theta'(t) = \begin{cases} \{x, y\}, & \text{if } t \leq -\delta \\ \{x\}, & \text{if } -\delta < t < \delta \\ \{x, y\}, & \text{if } t \geq \delta \end{cases} \]

(see Fig. 3). We prove \( d_F(\theta, \theta') = \delta \). Let \( \varepsilon \in [0, \infty) \). Because the partition \( \{x, y\} \) refines the partition \( \{x, y\} \), we have \( \theta' \leq S_\varepsilon(\theta) \). Let us assume \( \varepsilon \in [0, \delta) \): Then, we have \( S_\varepsilon(\theta')(0) = \bigcup_{s \in [0, \varepsilon]} \theta'(s) = \{x\} \) and \( \theta(0) = \{x, y\} \). Thus, \( \theta' \not\leq S_\varepsilon(\theta) \), and \( \theta, \theta' \) are not \( \varepsilon \)-interleaved. On the other hand, \( S_\varepsilon(\theta') = \theta \) for \( \varepsilon \in [\delta, \infty) \). and thus \( d_F(\theta, \theta') = \delta \).

**Gromov-Hausdorff distance between formigrams.** We may wish to compare formigrams over different sets. To this end, we revisit the Gromov-Hausdorff distance \( d_{GH} \) between formigrams \([49, 46]\); the naming of the distance is based on the fact that it generalizes the Gromov-Hausdorff distance between finite ultrametric spaces \([46]\) (see Rmk. 4.24 (ii)). See Defn. C.2 for the original definition of the Gromov-Hausdorff distance between compact metric spaces. (Note: All the Gromov-Hausdorff distances in this paper will turn out to be special instances of the generalized Gromov-Hausdorff distance from Defn. 4.9, as proved in Prop. D.7 in the appendix.)

Let \( X, Z \) be two nonempty sets, let \( P \in \text{SubPart}(X) \), and let \( \varphi : Z \to X \) be a surjective map. The **pullback** of \( P \) via \( \varphi \) is the subpartition of \( Z \) defined as \( \varphi^*P := \{\varphi^{-1}(B) \subset Z : B \in P\} \), implying:

\[ z, z' \in Z \text{ belong to the same block of } \varphi^*P \iff \varphi(z), \varphi(z') \in X \text{ belong to the same block of } P. \quad (6) \]

Let \( \theta_X \) be a formigram over \( X \). The pullback of \( \theta_X \) via \( \varphi \) is the formigram

\[ \varphi^*\theta_X : \mathbb{R} \to \text{SubPart}(Z) \text{ such that } (\varphi^*\theta_X)(t) := \varphi^*(\theta_X(t)). \]

Let \( X \) and \( Y \) be any two nonempty sets. A **tripod** \( R \) between \( X \) and \( Y \) is a pair of surjections \( R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y \) from any set \( Z \) onto \( X \) and \( Y \) \([58]\). For \( x \in X \) and \( y \in Y \), we write \( (x, y) \in R \) when there exists \( z \in Z \) such that \( x = \varphi_X(z) \) and \( y = \varphi_Y(z) \).

**Definition 2.22.** Let \( \theta_X, \theta_Y \) be any two formigrams over \( X \) and \( Y \), respectively. The **Gromov-Hausdorff distance between** \( \theta_X \) and \( \theta_Y \) is defined as:

\[ d_{GH}(\theta_X, \theta_Y) := \frac{1}{2} \min_R d_F(\varphi_X^*\theta_X, \varphi_Y^*\theta_Y), \]
where the minimum is taken over all tripods between $X$ and $Y$.\(^6\)

It directly follows from Defn.\(^2.20\) and \(^2.22\) that, given any two formigrams $\theta_X$ and $\theta'_X$ over the same underlying set $X$, we have $2 \, d_{GH}(\theta_X, \theta'_X) \leq d_F(\theta_X, \theta'_X)$.\(^7\)

### 3 Interleaving by parts and geodesicity

We decompose the interleaving distance $d_I$ between poset maps (Sec.\(^3.1\)) and harness it to show the geodesicity of $d_I$ in certain settings (Sec.\(^3.2\)).

#### 3.1 Join representations of interleavings between poset maps

The goal of this section is to establish Thms.\(^3.7\) and \(^3.10\). Join representations for poset maps. Recall that any poset $P$ contains at least one trivial join-dense subset ($P$ itself). In this section, we adhere to:

**Convention 3.1.** $(P, \leq)$ denotes a poset and $(Q, \leq)$ denotes a poset containing a zero element and with a distinguished join-dense subset $B \subset Q$.

**Proposition 3.2.** Let $q \in Q$. Then, $q = \bigvee (B \cap [0, q])$.

**Proof.** Clearly, we have $\bigvee [0, q] = q$. As $B$ is join-dense, there exists $B' \subset B$ s.t. $\bigvee B' = q$. Since $B' \subset B \cap [0, q] \subset [0, q]$, we have $\bigvee (B \cap [0, q]) = q$, as desired. \(\Box\)

The following proposition is straightforward.

**Proposition 3.3.** Given any subfamily $\{F_a : a \in A\}$ of $[P, Q]$, assume that for all $p \in P$, the join $\bigvee_{a \in A} F_a(p)$ exists in $Q$. Then the join $\bigvee \{F_a : a \in A\}$ exists in the poset $[P, Q]$, which is given by

\[
\bigvee \{F_a : a \in A\}(p) := \begin{cases} q, & p \in U \\ 0, & \text{otherwise}. \end{cases}
\]

**Definition 3.4.** Let $F \in [P, Q]$ and let $b \in B$. For the upper set $F^{-1}(b) := \{ p \in P : b \leq F(p) \}$, let us define the $b$-part of $F$ as $F_b := F^{-1}(b)$.

---

\(^6\)Because $X$ and $Y$ are finite, the minimum is always achieved by a certain tripod $R$. In fact, it suffices to consider subsets $Z \subset X \times Y$ which project onto $X$ and $Y$ via the canonical projections with $\varphi_X, \varphi_Y$ being the canonical projections.

\(^7\)The Gromov-Hausdorff distance between formigrams in this paper is actually called the formigram interleaving distance in \([48], [49]\, \text{Defn. 4.11}\). In this paper, we reserve the name interleaving distance for Defn.\(^2.20\) because $d_{GH}$ is not the interleaving distance in the sense of Defn.\(^2.8\). A close relationship between the Gromov-Hausdorff distance and the interleaving distance is highlighted in \([13, 68, 71]\).

\(^8\)If, a priori, $\mathcal{D}$ does not contain a zero element, one can simply add one to $\mathcal{D}$. Namely, replace $\mathcal{D}$ by $\mathcal{D} \cup \{0\}$ where $0$ is forced to be the smallest element in $\mathcal{D} \cup \{0\}$, by definition.
We introduce a certain join representation of a poset map which is the source of many results in this paper.

**Proposition 3.5** (Join representation via indicator maps). For any poset map \( F : \mathcal{P} \to \mathcal{Q} \),

\[
F = \bigvee \{ F_b : b \in B \}. \tag{8}
\]

**Proof.** Fix \( p \in \mathcal{P} \). Let us define \( A_p, B_p \subset \mathcal{Q} \) as 

\[
A_p = B \cap [0, F(p)] \quad \text{and} \quad B_p = \{ F_b(p) : b \in B \}.
\]

Then,

\[
F(p) = \bigvee A_p \quad \text{by Prop. 3.2}
\]

\[
= \bigvee (A_p \cup \{0\}) \quad \text{since } A_p \cup \{0\} = B_p \cup \{0\}
\]

\[
= \bigvee B_p \quad \text{by Prop. 3.3}
\]

\[
= \left( \bigvee \{ F_b : b \in B \} \right)(p)
\]

which proves the equality in Eqn. (8).

\[ \square \]

**Remark 3.6.**  
(i) The join representation in (8) is functorial in the sense that \( F \leq G \) in \([\mathcal{P}, \mathcal{Q}]\) ⇔ for all \( b \in B \), \( F_b \leq G_b \).  
(ii) One can establish “duals” of Conv. 3.1, Props. 3.2, 3.3 and Defn. 3.4 which permit representing \( F \) as a meet of \( F \)'s “dual” \( b \)-parts, instead of the join representation given in (8). We have not found any significant use of this dual statement though for the purpose of this paper.  
(iii) The \( F_b \)'s in Eqn. (8) are not necessarily join-irreducible in \([\mathcal{P}, \mathcal{Q}]\). However, there is a special case: Let \( \mathcal{P} \) be a totally ordered set and assume that \( B \) consists solely of join-irreducible elements of \( \mathcal{Q} \) (cf. Rmk. 2.1). Then, each \( F_b \) is join-irreducible and thus, in that case, equality (8) could be regarded as a join decomposition of \( F \).

**Interleaving by parts.** We show that the interleaving distance between poset maps admits a join representation which is compatible with the join representation in (8).

**Theorem 3.7** (Interleaving by parts). Let \((\mathcal{P}, \leq, \Omega)\) be a poset with a flow, and let \( \mathcal{Q} \) be a poset with zero and a join-dense subset \( B \). For any \( F, G : (\mathcal{P}, \leq) \to (\mathcal{Q}, \leq) \),

\[
d_I(F, G) = \sup_{b \in B} d_I(F_b, G_b). \tag{9}
\]

Note that, in the standard ordered set \((\mathbb{R}, \leq)\), we have \( \bigvee A = \sup A \) for any \( A \subset \mathbb{R} \). Therefore, invoking Eqn. (8), we can rewrite Eqn. (9) as:

\[
d_I\left( \bigvee_{b \in B} F_b, \bigvee_{b \in B} G_b \right) = \bigvee_{b \in B} d_I(F_b, G_b),
\]

which shows that the join operation of Eqn. (8) commutes with \( d_I \).

**Remark 3.8.** Eqn. (9) is strongly analogous to the well-known decomposability of the interleaving distance between persistence modules \( \mathbb{R} \to \mathbf{vect} \) (Defn. B.1 in the appendix).
Assume that $B$ is finite for simplicity. Then, one can check that the RHS of (9) is equal to

$$\min_{\tau:B\to B} \max_{b \in B} d_l(F_b, G_{\tau(b)}) \quad (10)$$

where the minimum is taken over all bijections $\tau:B\to B$. Now consider two any persistence modules $M, N: \mathbb{R} \to \text{vect}$ with indecomposable direct sum decompositions $M \equiv \bigoplus_{i \in I} M_i$ and $N \equiv \bigoplus_{j \in J} N_j$ where $|I|, |J| < \infty$. Let $K := I \sqcup J$. Extend $M$ and $N$ to $K$ as follows: $M_j := 0$ for $j \in J$ and let $N_i := 0$ for $i \in I$. Then $d_l(M, N)$ is equal to the following which is in the same form as (10):

$$\min_{\tau:K\to K} \max_{k \in K} d_l(M_{\tau(k)}, N_{\tau(k)}).$$

**Proof of Thm. 3.7.** Let $\varepsilon \in [0, \infty)$.

- $F, G$ are $\varepsilon$-interleaved
  - $F \preceq G \Omega_\varepsilon$ and $G \preceq F \Omega_\varepsilon$.
  - For any $b \in B$, $F_b \preceq (G \Omega_\varepsilon)_b$ and $G_b \preceq (F \Omega_\varepsilon)_b$ by Rmk. 3.6(ii)
  - (*) For any $b \in B$, $F_b \preceq G_b \Omega_\varepsilon$ and $G_b \preceq F_b \Omega_\varepsilon$ see below
  - For any $b \in B$, $F_b, G_b$ are $\varepsilon$-interleaved by Def. 2.9

For (*), it suffices to show that $(F \Omega_\varepsilon)_b = F_b \Omega_\varepsilon$. To this end, we prove $p \in (F \Omega_\varepsilon)^{-1}(b^1) \Leftrightarrow \Omega_\varepsilon(p) \in F^{-1}(b^1)$. Indeed, for $p \in \mathcal{P}$,

$$p \in (F \Omega_\varepsilon)^{-1}(b^1) \Leftrightarrow b \leq (F \Omega_\varepsilon)(p) \Leftrightarrow b \leq F(\Omega_\varepsilon(p)) \Leftrightarrow \Omega_\varepsilon(p) \in F^{-1}(b^1).$$

\[\square\]

**Interleaving distance between upper set indicator maps.** We represent the RHS of (9) as the interleaving distance between upper sets of $\mathcal{P}$ (Defn. 2.11). Recall Defn 3.4.

**Proposition 3.9.** Let $(\mathcal{P}, \preceq, \Omega)$ be a poset with flow. Let $U, V \in U(\mathcal{P})$ and let $x \in \mathcal{Q}$. Then, we have:

$$d_l(I^U_x, I^V_x) = d_{\hat{\Omega}}(U, V).$$

**Proof.**

- $I^U_x, I^V_x$ are $\varepsilon$-interleaved.
  - $I^U_x \preceq I^V_x \Omega_\varepsilon$ and $I^V_x \preceq I^U_x \Omega_\varepsilon$.
  - For any $p \in \mathcal{P}$, $I^U_x(p) \preceq I^V_x(\Omega_\varepsilon(p))$ and $I^V_x(p) \preceq I^U_x(\Omega_\varepsilon(p))$.
  - For any $p \in \mathcal{P}$, $(p \in U \Rightarrow \Omega_\varepsilon(p) \in V)$ and $(p \in V \Rightarrow \Omega_\varepsilon(p) \in U)$.
  - $(\text{For any } p \in \mathcal{P} : p \in U \Rightarrow \Omega_\varepsilon(p) \in V)$ and $(\text{For any } p \in \mathcal{P} : p \in V \Rightarrow \Omega_\varepsilon(p) \in U)$.
  - $U \subseteq \hat{\Omega}_\varepsilon(V)$ and $V \subseteq \hat{\Omega}_\varepsilon(U)$
  - $U, V$ are $\varepsilon$-interleaved.

\[\square\]
By Thm. 3.7 and Prop. 3.9, $d_1$ coincides with the $\ell^\infty$-product metric of $|B|$ copies of $d_\Omega$ on $U(\mathcal{P})$.

**Theorem 3.10 (Decomposition of $d_1$).** Let $(\mathcal{P},\leq,\Omega)$ be a poset with a flow, and let $\mathcal{Q}$ be a poset with zero and a join-dense subset $B$. For any $F,G:(\mathcal{P},\leq)\to(\mathcal{Q},\leq)$ we have

$$d_1(F,G) = \sup_{b\in B} d_\Omega(F^{-1}(b^{1}),G^{-1}(b^{1})).$$

**Remark 3.11.** Thm. 3.10 is analogous to the celebrated isometry theorem in TDA [3, 23, 54] in the following sense: By Thm. 3.10, we can make use of the collections

$$B(F) := \{F^{-1}(b^{1})\}_{b\in B} \text{ and } B(G) := \{G^{-1}(b^{1})\}_{b\in B}$$

for computing $d_1(F,G)$. Analogously, for any persistence modules $M,N: \mathbb{R} \to \text{vect}$ (Defn. A.1), their barcodes (or equivalently persistence diagrams) are utilized for computing the interleaving distance between $M$ and $N$ via the bottleneck distance [3, 25].

### 3.2 Geodesicity of interleavings between poset maps

The goal of this section is to prove that the interleaving distance between poset maps $\mathcal{P} \to \mathcal{Q}$ is geodesic when $\mathcal{Q}$ is a complete lattice (Thm. 3.13). This proves in a uniform way that many known metrics that will be discussed in later sections are all geodesic.

In the rest of the paper, any extended pseudometric space $(M,d)$ is simply referred to as a metric space. Any $x \in M$ will be identified with the class $[x] := \{y : d(x,y) = 0\}$. Let $x,y \in M$ with $d(x,y) < \infty$. A continuous map $g:[0,1] \to M$ is called a path from $x$ to $y$ if $g(0) = x$ and $g(1) = y$. The path $g$ is called geodesic if

$$d(g(s), g(t)) = |s-t| \cdot d(x,y)$$

for all $s,t \in [0,1]$. If there exists a geodesic path between every pair of points in $M$ within a finite distance, $M$ is called geodesic.

**Proposition 3.12.** For any poset $(\mathcal{P},\leq,\Omega)$ with a flow, the distance $d_\Omega$ on $U(\mathcal{P})$ (Defn. 2.11) is geodesic.

**Proof.** Let $A,B \in U(\mathcal{P})$ with $d_\Omega(A,B) = \rho \in (0, \infty)$. For $t \in [0,1]$, let $A_t := \hat{\Omega}_\rho t(A) \cap \hat{\Omega}_\rho(1-t)(B)$. We claim that $t \mapsto A_t$ for $t \in [0,1]$ is a geodesic path from $A$ to $B$. First, we claim $d_\Omega(A,A_0) = 0$ (the equality $d_\Omega(B,B_0) = 0$ is proved similarly). To this end, we show that, for any $\delta > 0$, $A_0 \subset \hat{\Omega}_\delta(A)$ and $A \subset \hat{\Omega}_\delta(A_0)$. Let $\delta > 0$. By construction we have $A_0 \subset A \subset \hat{\Omega}_\delta(A)$. Next,

$$A \subset \hat{\Omega}_\delta(A) \cap \hat{\Omega}_{\delta+\rho}(B) \subset \hat{\Omega}_\delta(A \cap \hat{\Omega}_\rho(B))$$

by Rmk. 2.12(ii)

$$= \hat{\Omega}_\delta(A_0)$$

by definition.

Now fix any $s < t$ in $[0,1]$ and we show that $d_\Omega(A_s, A_t) = (t-s)\rho$. We claim that $d_\Omega(A_s, A_t) \leq (t-s)\rho$. By Rmk. 2.12(ii)

$$\hat{\Omega}_{\rho(t-s)}(A_s) = \hat{\Omega}_\rho t(A) \cap \hat{\Omega}_{\rho(1+t-2s)}(B) \supset \hat{\Omega}_\rho t(A) \cap \hat{\Omega}_{\rho(1-t)}(B) = A_t.$$  

Similarly, one has $\hat{\Omega}_{\rho(t-s)}(A_t) \supset A_s$. This shows that $d_\Omega(A_s, A_t) \leq (t-s)\rho$. \hfill $\square$

\footnote{This construction is similar to the construction of Hausdorff geodesic paths given in [72].}
Theorem 3.13. Let \((\mathcal{P}, \leq, \Omega)\) be a poset with a flow and \((\mathcal{Q}, \leq)\) is a complete lattice. The interleaving distance \(d_I\) on \([\mathcal{P}, \mathcal{Q}]\) is geodesic.

We actually do not need the completeness of \(\mathcal{Q}\) with respect to meets in the assumption. However, it is automatically guaranteed under the assumption that \(\mathcal{Q}\) is complete with respect to joins [69, Thm. 3.24], which we need in the proof below.

Proof. Let \(B\) be any join-dense subset of \(\mathcal{P}\). By Prop. 3.12, for each \(b \in B\), there exists a geodesic path \(g_b : [0, 1] \to U(\mathcal{P})\) from \(F^{-1}(b^\uparrow)\) to \(G^{-1}(b^\uparrow)\). We embed these paths into \([\mathcal{P}, \mathcal{Q}]\) and assemble the embedded paths via \(\sqcup\): For each \(t \in [0, 1]\), define \(H_t : (\mathcal{P}, \leq) \to (\mathcal{Q}, \leq)\) by

\[
p \mapsto \bigvee_{b \in B} g_b(t)(p),
\]

which is well-defined since \(\mathcal{Q}\) is a complete lattice.

Now we claim that \(t \mapsto H_t\) is a geodesic path from \(F\) to \(G\). It is clear that \(H_0 = F\) and \(H_1 = G\). For every \(s, t \in [0, 1]\):

\[
d_I(H_s, H_t) = d_I\left(\bigvee_{b \in B} g_b(s), \bigvee_{b \in B} g_b(t)\right) \quad \text{by definition}
\]

\[
= \sup_{b \in B} d_{\sqcup}(g_b(s), g_b(t)) \quad \text{by Thm. 3.10}
\]

\[
= \sup_{b \in B} |s - t| \cdot d_{\sqcup}(F^{-1}(b^\uparrow), G^{-1}(b^\uparrow)) \quad \text{since } g_b \text{ is a geodesic path}
\]

\[
= |s - t| \cdot \sup_{b \in B} d_{\sqcup}(F^{-1}(b^\uparrow), G^{-1}(b^\uparrow)) \quad \text{Thm. 3.10}
\]

\[
= |s - t| \cdot d_I(F, G)
\]

Remark 3.14. The construction of the geodesic path above is analogous to the construction of a geodesic path between persistence diagrams of persistence modules \(\mathbb{R} \to \text{vect}\) via a linear interpolation guided by an optimal matching [29]. In the proof above, the collection \(\{F^{-1}(b^\uparrow) : b \in B\}\) can be viewed as a proxy for the “persistence diagram” of \(F\).

4 Consequences

This section describes a number of consequences of Thms. 3.7 and 3.10.

In Sec. 4.1 we establish a connection between the erosion distance [65] and the graded rank functions [5]. In Sec. 4.2 we show that the Gromov-Hausdorff distance can be recast within the framework of interleaving distances and thereby obtain a far reaching generalization of this distance. Therein, basic properties of this distance are established including its universality and geodesicity. In Sec. 4.3 and 4.4 we prove that computing the interleaving distance
between multiparameter hierarchical clusterings reduces to computing Hausdorff distances in Euclidean spaces. This establishes an equivalence between known metrics for comparing multiparameter hierarchical clusterings. In Sec. 4.4 we determine the computational complexity of the interleaving distance between formigrams in Defn. 2.20.

**Convention 4.1.** In the rest of the paper, the posets \( \mathbb{R}_{\geq 0} \) and \( \mathbb{R}^n \) (for any \( n \in \mathbb{N} \)) are equipped with the flow in Eqn. 1. Also, the poset \( \text{Int} \) is equipped with the flow in Ex. 2.13.

### 4.1 Erosion distance and graded rank functions

The goal of this section is to establish a connection between the erosion distance and the graded rank function \[5\] and to provide an interpretation of the join representation given in (8).

Although the erosion distance has a fairly general form \[47, 65, 66\], its most basic use is for comparing two monotonically decreasing integer-valued maps as in \[48, \text{Sec. 5}\]. We restrict ourselves to this basic setting. Let \( \mathbb{Z}_{\geq 0}^{\text{op}} \) be the opposite poset of the nonnegative integers, i.e.

\[
a \leq b \text{ in } \mathbb{Z}_{\geq 0}^{\text{op}} \iff b \leq a \text{ in } \mathbb{Z}_{\geq 0}.
\]

**Definition 4.2.** Let \((\mathcal{P}, \leq, \Omega)\) be a poset with a flow. Given \( F, G : (\mathcal{P}, \leq, \Omega) \rightarrow \mathbb{Z}_{\geq 0}^{\text{op}} \), the interleaving distance \( d_I(F, G) \) is called the \((\mathcal{P})\text{-erosion distance}\).

In \( \mathbb{Z}_{\geq 0}^{\text{op}} \), we have \( \lor\{m, n\} = \min\{m, n\} \) and each element of \( \mathbb{Z}_{\geq 0}^{\text{op}} \) is join-irreducible. This implies that \( \mathbb{Z}_{\geq 0}^{\text{op}} \) itself is the unique join-dense subset of \( \mathbb{Z}_{\geq 0} \). By virtue of Thm. 3.10 we have

\[
d_I(F, G) = \sup_{n \in \mathbb{Z}_{\geq 0}} d_{\text{H}}(F^{-1}[0, n], G^{-1}[0, n]).
\]

Recall the Hausdorff distance in \((\text{Int}, \|\cdot\|_\infty)\) (Ex. 2.13 and Defn. C.1). We establish the following relationship between the erosion distance and the graded rank function \[5\], which directly follows from the above equality and Ex. 2.13.

**Theorem 4.3.** The erosion distance between \( F, G : (\text{Int}, \leq) \rightarrow \mathbb{Z}_{\geq 0}^{\text{op}} \) is equal to

\[
d_I(F, G) = \sup_{n \in \mathbb{Z}_{\geq 0}} d_{\text{H}}(F^{-1}[0, n], G^{-1}[0, n]).
\]

**Remark 4.4.**

(i) If \( F \) is the rank function of a persistence module \( M : \mathbb{R} \rightarrow \text{vect} \) (Defns. A.1 and A.2), then \( F^{-1}[n+1, \infty) = \text{Int} \setminus F^{-1}[0, n] \) is the support of the \((n+1)\)-th graded rank function of \( M [5, \text{Defn. 4.2}] \) (Fig. 4). In light of this, graded rank functions (or persistence landscapes \[14\]) are special instances of the join-representation (of rank functions) described in Prop. 3.5. Stability properties of graded rank functions have been discussed in \[5\].

(ii) The interleaving distance between persistence modules \( M, N : \mathbb{R}^d \rightarrow \text{vect} \) (Defn. B.1) is known to be bounded from below by the erosion distance between the rank functions of \( M \) and \( N \); see \[14, \text{Thm. 17}\] \[65, \text{Thm. 8.2}\] for \( d = 1 \) and \[48, \text{Thm. 6.2}\] for arbitrary \( d \).

(iii) An optimal algorithm for computing \( d_I(F, G) \) in Eqn. 11 and more general results are given in \[48, \text{Sec. 5}\].
Figure 4: Illustration of $F^{-1}[0, n]$ and $F^{-1}[n + 1, \infty)$ for the case when $F$ is the rank function of a generic constructible persistence module (Defn. A.1 and A.2). The boundary line between $F^{-1}[0, n]$ and $F^{-1}[n + 1, \infty)$ is a visual representation of the $(n + 1)$-th persistence landscape.

4.2 Generalization of the Gromov-Hausdorff distance

In this section we show the following:

(i) The Gromov-Hausdorff distance $d_{GH}$ (between metric spaces \cite{17} or between $\mathbb{R}$-indexed simplicial filtrations \cite{58, 59}) can be incorporated into the framework of interleaving distances of Sec. 2.3 Thm. 4.8. This leads to the next item.

(ii) We obtain a far reaching generalization of $d_{GH}$, which will be also denoted by $d_{GH}$ (Defn. 4.9); although the instance of $d_{GH}$ described in Defn. 4.9 is a distance between simplicial filtrations over a poset with a flow, there is a precise sense in which it generalizes all the other Gromov-Hausdorff distances mentioned in this paper (Rmk. 4.10).

(iii) This generalized $d_{GH}$ inherits a universal property satisfied by the original Gromov-Hausdorff distance (Thm. 4.11), of which the celebrated Vietoris-Rips filtration stability theorem \cite{26, 28} becomes a consequence (Thm. 4.14 and Rmk. 4.15 (ii)).

(iv) Using Thm. 3.13 we show that the generalized $d_{GH}$ is also geodesic. Interestingly, our construction of geodesic paths generalizes a known one between compact metric spaces \cite{30} in a precise sense; see Rmk. 4.13.

Some basic properties of the generalized $d_{GH}$ are deferred to the appendix.

Throughout this section, $X$, $Y$, and $Z$ will denote nonempty finite sets. By $\text{Simp}(X)$, we denote the collection of abstract simplicial complexes \cite{62} over a vertex set $A \subset X$ ordered by inclusion. $\text{Simp}(X)$ is a complete lattice whose joins are unions and meets are intersections. By $\text{pow}_{\geq 1}(X)$, we denote the the collection of nonempty subsets of $X$ ordered by inclusion.

**Definition 4.5** (Simplexization). Let us define the poset map $s : \text{pow}_{\geq 1}(X) \to \text{Simp}(X)$ as $\sigma \mapsto \text{pow}_{\geq 1}(\sigma)$. In words, nonempty $\sigma \subset X$ is sent to the simplicial complex consisting solely of the $(|\sigma| - 1)$-simplex $\sigma$.

**Remark 4.6.** Let $\Delta(X)$ be the image of the map $s$. The collection of all join-irreducible elements of $\text{Simp}(X)$ equals $\Delta(X)$. Hence, $\Delta(X)$ is join-dense in $\text{Simp}(X)$ (Rmk 2.1).

A poset map $F_X : (\mathcal{P}, \leq) \to \text{Simp}(X)$ is said to be an ($\mathcal{P}$-indexed simplicial) filtration (over $X$). The filtration is called **full** if there exists $p \in \mathcal{P}$ such that $X \in F_X(p)$.
Assume that \((\mathcal{P}, \leq) = (\mathbb{R}, \leq)\). For any \(\sigma \in \text{pow}_{\geq 1}(X)\), the \textbf{birth time of} \(\sigma\) is defined as
\[
\text{b}_{\text{F}_X}(\sigma) := \inf \{ t \in \mathbb{R} : \sigma \in \text{F}_X(t) \}.
\]
If \(\sigma\) does not belong to \(\text{F}_X(t)\) for any \(t \in \mathbb{R}\), then \(\text{b}_{\text{F}_X}(\sigma)\) is defined to be \(\infty\).

We review the Gromov-Hausdorff distance between \(\mathbb{R}\)-indexed filtrations (introduced by Mémoli [58] and further studied by Mémoli and Okutan [59]):

**Definition 4.7** ([58], p.4]). Given any \(\text{F}_X : (\mathbb{R}, \leq) \to \text{Simp}(X)\) and \(\text{G}_Y : (\mathbb{R}, \leq) \to \text{Simp}(Y)\), the \textbf{Gromov-Hausdorff distance} between \(\text{F}_X\) and \(\text{G}_Y\) is defined by
\[
d_{\text{GH}}(\text{F}_X, \text{G}_Y) := \frac{1}{2} \min_{R} \max_{\sigma \in \text{pow}_{\geq 1}(Z)} |\text{b}_{\text{F}_X}(\varphi_X(\sigma)) - \text{b}_{\text{G}_Y}(\varphi_Y(\sigma))|,
\]
where the minimum is taken over all tripods \(R : X \leftarrow Z \rightarrow Y\).

At first glance, this distance may not appear to be related to an interleaving type distance between poset maps. However, we will see that such a relation exists.

Given a surjective map \(\varphi_X : Z \to X\) and a simplicial filtration \(\text{F}_X : (\mathcal{P}, \leq) \to \text{Simp}(X)\), we define the \textit{pullback} \(\varphi_X^* \text{F}_X\) of \(\text{F}_X\) via \(\varphi_X\) as the filtration \((\mathcal{P}, \leq) \to \text{Simp}(Z)\) as follows. The filtration \(\varphi_X^* \text{F}_X\) sends each \(p \in \mathcal{P}\) to the smallest simplicial complex \(K \in \text{Simp}(Z)\) that contains all the preimages \(\varphi_X^{-1}(\tau)\) for \(\tau \in \text{F}_X(p)\).

\[
\begin{align*}
\text{Simp}(Z) & \quad \xrightarrow{\varphi_X^* \text{F}_X} \quad \text{Simp}(X) \\
\mathcal{P} & \quad \xrightarrow{\varphi_X} \quad \text{F}_X \quad \text{Simp}(X)
\end{align*}
\]

Let \(d^\text{Simp}(Z)\) be the interleaving distance on \([\mathbb{R}, \text{Simp}(Z)]\). By Thm. 3.10 we can reformulate Defn. 4.7 as follows.

**Theorem 4.8.** Given any filtrations \(\text{F}_X : (\mathbb{R}, \leq) \to \text{Simp}(X)\) and \(\text{G}_Y : (\mathbb{R}, \leq) \to \text{Simp}(Y)\),
\[
d_{\text{GH}}(\text{F}_X, \text{G}_Y) = \frac{1}{2} \min_R d^\text{Simp}(Z)(\varphi_X^* \text{F}_X, \varphi_Y^* \text{G}_Y),
\]
where the minimum is taken over all tripods \(R : X \leftarrow Z \rightarrow Y\).

**Proof.** Let \(d_H\) be the Hausdorff distance in \(\mathbb{R}\) (Defn. C.1). We have:
\[
d^\text{Simp}(Z)(\varphi_X^* \text{F}_X, \varphi_Y^* \text{G}_Y) = \max_{K \in \Delta(Z)} d_H \left( (\varphi_X^* \text{F}_X)^{-1}(K^1), (\varphi_Y^* \text{G}_Y)^{-1}(K^1) \right) \quad \text{by Thm. 3.10 and Rmk. 4.6}
\]
\[
(*) \quad \max_{\sigma \in \text{pow}_{\geq 1}(Z)} \left[ b_{\text{F}_X}(\varphi_X(\sigma)), \infty \right), \left[ b_{\text{G}_Y}(\varphi_Y(\sigma)), \infty \right) \right) \quad \text{see below}
\]
\[
\quad = \max_{\sigma \in \text{pow}_{\geq 1}(Z)} |b_{\text{F}_X}(\varphi_X(\sigma)) - b_{\text{G}_Y}(\varphi_Y(\sigma))|,
\]
where \((*)\) follows from the bijection \(s : \text{pow}_{\geq 1}(X) \to \Delta(X)\) (Defn. 4.5 and Rmk. 4.6). The desired equality directly follows. \qed
In the rest of this section we fix a poset with a flow \((\cal P, \leq, \Omega)\). In light of Thm. 3.10 and 4.8, we obtain the following generalization of \(d_{GH}\) which we still denote by the same symbol:

**Definition 4.9.** Given any filtrations \(F_X : (\cal P, \leq) \rightarrow \text{Simp}(X)\) and \(G_Y : (\cal P, \leq) \rightarrow \text{Simp}(Y)\), the generalized Gromov-Hausdorff distance between them is defined by:

\[
\hat{d}_{GH}(F_X, G_Y) := \frac{1}{2} \min_R \max_{\sigma \in \text{pow}_{\leq}(Z)} d_{\Omega}((\varphi_X^* F_X)^{-1}(\sigma^1), (\varphi_Y^* G_Y)^{-1}(\sigma^1)),
\]

where the minimum is taken over all tripods \(R : X \leftarrow Z \rightarrow Y\).

That \(\hat{d}_{GH}\) above is an extended pseudometric is shown in the appendix (Prop. D.1). We remark that (1) a sufficient condition for \(d_{GH}(F_X, G_Y)\) to be finite is that \(F_X\) and \(G_Y\) are full, and \(\hat{d}_{GH}\) is finite for every pair of upper sets in \(\cal P\). (2) When \(\cal P = \mathbb{R}^n\), \(\hat{d}_{GH}\) above boils down to the Hausdorff distance between upper sets in \(\mathbb{R}^n\).

**Remark 4.10.** All instances of \(d_{GH}\) mentioned in this paper can be viewed as special instances of \(d_{GH}\) in Defn. 4.9; see Prop. D.7 in the appendix for the precise statement.

**Universal property of \(d_{GH}\)**. Let \(\text{Met}\) be the space of finite pseudometric spaces. It is known that the Gromov-Hausdorff distance \(d_{GH}\) between finite pseudometric spaces is the largest distance \(D\) satisfying the following two conditions: (i) If there is a surjection \(\varphi : (Z, d_Z) \twoheadrightarrow (X, d_X)\) such that \(d_Z(z, z') = d_X(\varphi(z), \varphi(z'))\) for all \(z, z' \in Z\), then \(D((X, d_X), (Z, d_Z)) = 0\). (ii) For any two metrics \(d_1, d_2\) on a set \(X\), we have:

\[
2 \cdot D((X, d_1), (X, d_2)) \leq \max_{x, x' \in X} \left| d_1(x, x') - d_2(x, x') \right|.
\]

(see [71] Theorem F, p.11 for a more general statement). We extend this universal property to:

**Theorem 4.11** (Universality). Let \(d_{GH}\) the one in Defn. 4.9. Let \(D\) be any metric on \(\cal P\)-indexed finite simplicial filtrations with the following properties.

(i) Given any \(F_X : (\cal P, \leq) \rightarrow \text{Simp}(X)\) and a surjection \(\varphi_X : Z \twoheadrightarrow X\), we have \(D(F_X, \varphi_X^* F_X) = 0\).

(ii) Given any \(F_X, G_X : (\cal P, \leq) \rightarrow \text{Simp}(X)\), we have \(2 \cdot D(F_X, G_X) \leq d_{\text{Simp}}^\ast(X)(F_X, G_X)\).

Then, \(D \leq d_{GH}\).

In this theorem, by restricting \(d_{GH}\) to the Vietoris-Rips filtrations of finite pseudo metric spaces, this proposition boils down to the aforementioned universal property of \(d_{GH}^\ast\), this is a consequence of the equality in Prop. D.7(ii) of the appendix.

Our proof of Thm. 4.11 is similar to the one in [2].

**Proof of Thm. 4.11**. Given any two filtrations \(F_X : (\cal P, \leq) \rightarrow \text{Simp}(X)\) and \(G_Y : (\cal P, \leq) \rightarrow \text{Simp}(Y)\), pick any tripod \(R : X \leftarrow Z \rightarrow Y\). Then, by invoking the two conditions in order, we have:

\[
D(F_X, G_Y) = D(\varphi_X^* F_X, \varphi_Y^* G_Y) \leq d_{\text{Simp}}^\ast(Z)(\varphi_X^* F_X, \varphi_Y^* G_Y).
\]

Since \(R\) is arbitrary, we have \(D(F_X, G_Y) \leq 2 \cdot d_{GH}(F_X, G_Y)\). \(\Box\)

Now we generalize the fact that \(d_{GH}^\ast\) is geodesic [30], [58], [71], Section 6.8]:
**Theorem 4.12.** The generalized Gromov-Hausdorff distance is geodesic.

**Proof.** Let $F_X : (\mathcal{P}, \leq) \to \text{Simp}(X)$ and $F_Y : (\mathcal{P}, \leq) \to \text{Simp}(Y)$ be any two filtrations with $d_{\text{GH}}(F_X, F_Y) =: \rho < \infty$. Let us take any tripod minimizer $R : X \xrightarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y$ of $d_{\text{GH}}(F_X, F_Y)$. The existence of $R$ is guaranteed by the fact that $X$ and $Y$ are finite. Note that $d_{\text{GH}}(F_X, \varphi_X^* F_X) = 0$ by taking the tripod $X \xrightarrow{\varphi_X} Z \xrightarrow{\text{id}_Z} Z$. Similarly, $d_{\text{GH}}(F_Y, \varphi_Y^* F_Y) = 0$. Hence, $2\rho = d_{\text{GH}}(\varphi_X^* F_X, \varphi_Y^* F_Y)$ and it suffices to construct a geodesic path between $\varphi_X^* F_X$ and $\varphi_Y^* F_Y$ with respect to $d_1^\text{Simp}(Z)$. Invoking that $\text{Simp}(Z)$ is a complete lattice, a geodesic path between $\varphi_X^* F_X$ and $\varphi_Y^* F_Y$ exists in $[(\mathcal{P}, \leq), \text{Simp}(Z)]$, as desired. \hfill $\square$

By $\text{Simp}$ we denote the category of abstract simplicial complexes and simplicial maps. By $\text{Simp}^\mathcal{P}$, we denote the category of $\mathcal{P}$-indexed simplicial filtrations (i.e. functors $\mathcal{P} \to \text{Simp}$) and natural transformations between them.

**Remark 4.13** (Generalization). The construction of a geodesic path in the proof above generalizes two constructions of geodesic paths existing in the literature: A geodesic path $g : [0, 1] \to (\text{Met}, d_{\text{GH}})$ between any $(X, d_X)$ and $(Y, d_Y)$ was constructed in [30]. The corresponding path $\text{VR}(g) : [0, 1] \to (\text{Simp}^\mathcal{R}, d_{\text{GH}})$ coincides with the geodesic between $\text{VR}(X, d_X)$ and $\text{VR}(Y, d_Y)$ given in [58]. Both are special cases of the construction described in the proof above.

For $k \in \mathbb{Z}_{\geq 0}$, $H_k$ will denote the $k$-th simplicial homology with coefficients in a field $\mathbb{F}$. Given an arbitrary category $\mathcal{C}$, the interleaving distance $d^\mathcal{C}_k$ between two functors $(\mathcal{P}, \leq) \to \mathcal{C}$ is recalled in Defn. B.1 of the appendix. Recall that $\text{vect}$ denotes the category of finite dimensional vector spaces over a field $\mathbb{F}$. We have:

**Theorem 4.14.** Given any two filtrations $F_X : (\mathcal{P}, \leq) \to \text{Simp}(X)$ and $G_Y : (\mathcal{P}, \leq) \to \text{Simp}(Y)$, for any $k \in \mathbb{Z}_{\geq 0}$, we have:

$$d^\text{vect}_k(H_k(F_X), H_k(G_Y)) \leq 2 \ d_{\text{GH}}(F_X, G_Y).$$

**Proof.** Consider the metric $D := \frac{1}{2} \cdot d^\text{vect}_k(H_k(-), H_k(-))$ on $\text{Simp}^\mathcal{P}$. Quillen’s Theorem A [67] implies that, given any surjection $\varphi : Z \to X$, we have the homotopy equivalence $F_X \simeq \varphi^* F_X$. This implies $H_k(\varphi^* F_X) \cong H_k(F_X)$ and in turn that $D$ satisfies condition (i) in Thm. 4.11. Functoriality of $H_k$ guarantees condition (ii) in Thm. 4.11. Now the claim follows from Thm. 4.11. \hfill $\square$

**Remark 4.15.**

(i) The theorem above subsumes [58] Thm. 4.2 which addresses the case $\mathcal{P} = \mathbb{R}$. Our proof relying on the universality of $d_{\text{GH}}$ gives an alternative proof of [58] Thm. 4.2.

(ii) When $F_X$ and $G_Y$ are the (spatiotemporal) Vietoris-Rips filtrations of (dynamic) metric spaces $X$ and $Y$, the inequality in (12) coincides with the (spatiotemporal) Vietories-Rips filtration stability theorems [26, 28, 48]. This is a corollary of Prop. D.7 (i) and (iii).

(iii) The distance $d_{\text{GH}}(-, -)$ is more discriminative than $\max_{k \in \mathbb{Z}_{\geq 0}} d^\text{vect}_k(H_k(-), H_k(-))$ as can be seen through the following example.

**Example 4.16.** Let $X = \{x_1\}$ and let $Y = \{y_1, y_2\}$. Define $F_X : \mathbb{R} \to \text{Simp}(X)$ and $F_Y : \mathbb{R} \to \text{Simp}(Y)$ by $F_X(t) = \begin{cases} x_1, & t \in [0, \infty) \\ \emptyset, & \text{otherwise,} \end{cases}$ and $F_Y(t) := \begin{cases} \{y_1\}, & t \in [0, 1) \\ \{y_1, \{y_2\}, \{y_1, y_2\}\}, & t \in [1, \infty) \end{cases}$ respectively. Note $\emptyset$, otherwise,
that \( \max_{k \in \mathbb{Z}_{\geq 0}} d^\text{rect}_k (H_k(F_X), H_k(F_Y)) = 0 \), but \( d_{GH}(F_X, F_Y) = 1 \). Noting that \( F_X \) and \( F_Y \) are homotopy equivalent, this example shows that \( d_{GH} \) between homotopy equivalent filtrations can be strictly positive.

We further remark that \( d_{GH} \) between \( \mathbb{R} \)-indexed simplicial filtrations can be arbitrarily larger than the homotopy interleaving distance \([9]\). e.g. when comparing the Vietoris-Rips filtrations of metric spaces that are (almost) homotopy equivalent but are far from being isometric, such as a pair of a circle and a circle with long flares, cf. \([59, \text{Fig. 1}]\).

In the appendix we provide other upper and lower bounds for \( d_{GH} \); see Sec. D.

### 4.3 Hierarchical clusterings over posets

Let \( X \) be a nonempty finite set. A \((\mathcal{P}, \leq)\)-indexed hierarchical clustering (over \( X \)) is any poset map \((\mathcal{P}, \leq) \rightarrow \text{SubPart}(X)\). Standard examples include the case of \( \mathcal{P} = \mathbb{R}^n \) with the product order, often referred to as hierarchical clustering (when \( n = 1 \)) or multiparameter hierarchical clustering (when \( n \geq 2 \) \([18, 21, 22, 48, 68, 73]\)). From Rmk. \([2.5, \text{ii}]\) and Prop. \([2.6, \text{ii}]\) recall that \( \text{SubPart}(X) \) is \( \sqcup \)-irreducibly generated and the join-irreducible elements of \( \text{SubPart}(X) \) are either singleton or doubleton blocks of \( X \).

Recall that given any point \( q \) in a poset \( \mathcal{Q} \), \( q^\uparrow := \{ r \in \mathcal{Q} : q \leq r \} \). Invoking Thm. \( 3.10 \) we have:

**Theorem 4.17.** Let \((\mathcal{P}, \leq, \Omega)\) be a poset with a flow. For any \( F, G : (\mathcal{P}, \leq, \Omega) \rightarrow \text{SubPart}(X) \),

\[
d_I(F, G) = \max_{x, x' \in X} d^\Omega_\Omega \left( \left( F^{-1} \left( \{ x, x' \} \right), G^{-1} \left( \{ x, x' \} \right) \right) \right).
\]

(Note: when \( x = x' \), the set \( \{ x, x' \} \) is the singleton \( \{ x \} \)).

If \( \mathcal{P} = \mathbb{R}^n \) (resp. \( \text{Int} \)), the equality above implies that computing the interleaving distance between multiparameter hierarchical clusterings into computing the Hausdorff distance between upper sets of \( \mathbb{R}^n \) (resp. \( \text{Int} \)) a finite number of times. In the next section, we discuss the computational complexity of the RHS when \( \mathcal{P} = \text{Int} \) (Thm. \( 4.29 \)). Also we will discuss the comparison of two \((\mathcal{P}, \leq)\)-indexed hierarchical clusterings over different underlying sets.

### 4.4 Computing distances between formigrams

In this section we elucidate the structure of the two distances \( d_F \) and \( d_{GH} \) between formigrams introduced in Sec. \( 2.4 \) (Thms. \( 4.20 \) and \( 4.23 \)). Thereby we find equivalences between several known metrics for comparing hierarchical clusterings (Rmk. \( 4.24, \text{ii} \)). Also, we clarify the computational costs of \( d_F, d_{GH} \) and other related metrics (Rmk. \( 4.24, \text{ii} \) and Thm. \( 4.29 \)). \( d_{GH} \) between formigrams will be extended to a distance between any poset-indexed hierarchical clusterings (Defn. \( 4.25 \)).

**Formigrams can be viewed as poset maps.** The poset \( \text{Int} \) in Ex. \( 2.13 \) is isomorphic to the poset of nonempty closed intervals of \( \mathbb{R} \) whose partial order is inclusion. Each \((a, b) \in \text{Int} \) will be identified with the closed interval \([a, b] \subset \mathbb{R} \). Let us fix a nonempty finite set \( X \). A poset map
(Int, ≤) → (SubPart(X), ≤) will be simply denoted by Int → SubPart(X). Any formigram θ over X induces a map Int → SubPart(X):

**Definition 4.18.** For θ ∈ Formi(X), define  \( \hat{\theta} : \text{Int} \to \text{SubPart}(X) \) as \( I \mapsto \vee_{s \in I} \theta(s) \).

See Fig. 5 (A) and (B) for an illustrative example of Defn. 4.18

Recall the flow Ω on Int in Eqn. (2). Defn. 2.9 can be specialized as follows: Given \( \alpha, \alpha' : \text{Int} \to \text{SubPart}(X) \):

\[
\hat{d}_F(\alpha, \alpha') := \inf \{ \varepsilon \in [0, \infty) : \alpha, \alpha' \text{ are } \varepsilon\text{-interleaved w.r.t. } -\cdot \Omega \}.
\]

It is not difficult to check that \( d_F \) in Defn. 2.20 coincides with \( \hat{d}_F \):

**Proposition 4.19 (19, Defn. 4.11).** For any \( \theta, \theta' \in \text{Formi}(X) \), we have: \( d_F(\theta, \theta') = \hat{d}_F(\hat{\theta}, \hat{\theta}') \).

In view of Defn. 4.18 and Prop. 4.19 in what follows, any formigram over X will be identified with a poset map Int → SubPart(X) and \( d_F \) will be identified with \( \hat{d}_F \).

**d_F via interleaving by parts.** Let \( d_{\|s\|} \) be the Hausdorff distance (Defn. C.1) in (Int, \( \| - \|_\infty \)) (Ex. 2.13). As a corollary of Thm. 4.17, we have:

**Theorem 4.20.** For any two formigrams \( \theta \) and \( \theta' \) over X,

\[
d_F(\theta, \theta') = \max_{x, x' \in X} d_{\|s\|}(\theta^{-1}(\{x, x'\}\downarrow), \theta'^{-1}(\{x, x'\}\downarrow)).
\]  

(14)

We utilize Thm. 4.20 for elucidating both the computational complexity of \( d_F \) (Prop. 4.28). Inspired by Thm. 4.20 we define:

**Definition 4.21.** For \( \theta \in \text{Formi}(X) \), we call \( B(\theta) := \{ \theta^{-1}(\{x, x'\}\downarrow) \}_{x, x' \in X} \) the cosheaf-code of \( \theta \) (see Fig. 5 for an example).

See Fig. 6 and 7 for an illustrative example of an application of Thm. 4.20

**Remark 4.22.**

(i) Thm. 4.20 is analogous to the isometry theorem for zigzag modules [11], which says that a certain interleaving distance between vect-valued zigzag modules is equal to the bottleneck distance between their block barcodes.

(ii) Recall the notions of dendrograms and the ultrametrics induced by dendrograms (Defn. 2.14). In Thm. 4.20, let us assume that \( \theta \) and \( \theta' \) are dendrograms over X. Then, for each \( x, x' \in X \),

\[
\theta^{-1}(\{x, x'\}\downarrow) = \{(a, b) \in \text{Int} : b \in [u_\theta(x, x'), \infty)\}
\]

and similarly for \( \theta'^{-1}(\{x, x'\}\downarrow) \). Therefore,

\[
d_{\|s\|}\left(\theta^{-1}(\{x, x'\}\downarrow), \theta'^{-1}(\{x, x'\}\downarrow)\right) = |u_\theta(x, x') - u_{\theta'}(x, x')|
\]

and in turn \( d_F(\theta, \theta') = \max_{x, x' \in X} |u_\theta(x, x') - u_{\theta'}(x, x')| \).
Figure 5: (A) A formigram $\theta$ over $\{x, y, z\}$ such that $\theta(t) = \emptyset$ for $t \notin [0, 6]$. $x, y, z$ are colored in red, green and blue, respectively. (B) The corresponding map $\hat{\theta} : \text{Int} \to \text{SubPart}(X)$ (Defn. 4.18). (C) The cosheaf-code of $\theta$ (Defn. 4.21).

Figure 6: (A) A formigram $\theta$ over $\{x, y, z\}$. $x, y, z$ are colored in red, green and blue, respectively. (B) The formigram $\theta' := S_{\delta/2}(\theta)$ (cf. Defn. 2.18). The cosheaf-codes of $\theta$ and $\theta'$ are illustrated in Fig. [7].
Consider the formigrams $\theta$ and $\theta'$ in Fig. 6. Observe that $d_{\mathcal{H}}(\theta^{-1}({\{x, y\}^\uparrow}), \theta'^{-1}({\{x, y\}^\uparrow})) = d_{\mathcal{H}}(\theta^{-1}({\{y, z\}^\uparrow}), \theta'^{-1}({\{y, z\}^\uparrow})) = d_{\mathcal{H}}(\theta^{-1}({\{x, z\}^\uparrow}), \theta'^{-1}({\{x, z\}^\uparrow})) = \delta/2$. Also, for any $w \in \{x, y, z\}$, $d_{\mathcal{H}}((\theta^{-1}({\{w\}^\uparrow}), \theta'^{-1}({\{w\}^\uparrow})) = d_{\mathcal{H}}(\text{Int}, \text{Int}) = 0$. By Thm. 4.20 we obtain $d_F(\theta, \theta') = \delta/2$.

Structure of $d_{\mathcal{GH}}$ and related metrics. We can reformulate the Gromov-Hausdorff distance between formigrams (Defn. 2.22) via the cosheaf-codes of formigrams:

**Theorem 4.23.** Let $\theta_X, \theta_Y$ be any two formigrams over $X$ and $Y$, respectively. Then, we have:

$$d_{\mathcal{GH}}(\theta_X, \theta_Y) = \frac{1}{2} \min_R \max_{(x, y) \in R} \max_{(x', y') \in R} d_{\mathcal{H}}\left(\theta^{-1}_X({\{x, x'\}^\uparrow}), \theta^{-1}_Y({\{y, y'\}^\uparrow})\right),$$

where the minimum is taken over all tripods $R: X \xrightarrow{\phi_X} Z \xrightarrow{\phi_Y} Y$ between $X$ and $Y$.

**Proof.** This follows from Thm. 4.20 and the equivalence in (6). Details are omitted. □

**Remark 4.24.**

(i) Thm. 4.23 shows that $d_{\mathcal{GH}}$ in Eqn. (15) has exactly the same structure as the distance $d_{\mathcal{Q}}$ introduced in [22, page 69], and the distance $d_{\mathcal{CI}}$ in [68, Defn. 2.14]. The only difference is that $d_{\mathcal{GH}}$, $d_{\mathcal{Q}}$ and $d_{\mathcal{CI}}$ compare respectively $\text{Int}$-indexed hierarchical clusterings, $\mathbb{R}^2$-indexed hierarchical clusterings, and $\mathbb{R}^n$-indexed hierarchical clusterings.

(ii) In Thm. 4.23 assume that $\theta_X$ and $\theta_Y$ are dendrograms over $X$ and $Y$, respectively (Defn. 2.14). Then, by Rmk. 4.22 (ii) we have that

$$d_{\mathcal{GH}}(\theta_X, \theta_Y) = \frac{1}{2} \min_R \max_{(x, y) \in R} \max_{(x', y') \in R} |u_{\theta_X}(x, x') - u_{\theta_Y}(y', y)|.$$

Note that the RHS coincides with the Gromov-Hausdorff between (ultra)metric spaces (Defn. C.2), which is known to be NP-hard to compute [70]. Therefore, computing $d_{\mathcal{GH}}$ between formigrams is also NP-hard [49]. It is not difficult to see that the previous item implies that computing $d_{\mathcal{Q}}$ and $d_{\mathcal{CI}}$ is also NP-hard.

All distances mentioned in Rmk. 4.24 can now be seen as specializations of the following (cf. Thm. 4.17):
Definition 4.25. Let \((\mathcal{P}, \leq, \Omega)\) be a poset with a flow. Given any \(\theta_X : (\mathcal{P}, \leq, \Omega) \to \text{SubPart}(X)\) and \(\theta_Y : (\mathcal{P}, \leq, \Omega) \to \text{SubPart}(Y)\), the \textbf{Gromov-Hausdorff} distance between them is defined by:

\[
\delta_{\text{GH}}(\theta_X, \theta_Y) = \frac{1}{2} \min_R d_l(\varphi_X^* \theta_X, \varphi_Y^* \theta_Y) = \frac{1}{2} \min_R \max_{(x,y) \in R} d_H(\theta_X^{-1}(\{x,x'\}), \theta_Y^{-1}(\{y,y'\}))
\]

where the minimum is taken over all tripods \(R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y\) between \(X\) and \(Y\).

4.4.1 Computational cost for the calculation of \(d_F\)

In this section we elucidate the computational complexity of the formigram interleaving distance \(d_F\) (Thm. 4.29). We do this by clarifying the complexity of each preliminary step for the calculation of \(d_F\). Many ideas in this section can be adapted to the case of the interleaving distance between multiparameter hierarchical clustering (cf. Eqn. (13).

Let \(\theta\) be a formigram over \(X\) (Defn. 2.16). Let \(n := |X|\) and \(m := \text{crit}(\theta)\).

Proposition 4.26. Computing the corresponding poset map \(\hat{\theta} : \text{Int} \to \text{SubPart}(X)\) (Defn. 4.18) requires time \(O(n^2 m^2)\).

Proposition 4.27. Given \(\hat{\theta} : \text{Int} \to \text{SubPart}(X)\), computing the cosheaf-code of \(\theta\) takes time \(O(n^4 m^2)\) on average.

Proposition 4.28. Assume that the cosheaf-codes of two formigrams \(\theta\) and \(\theta'\) over \(X\) are given where \(n := |X|, m := \text{crit}(\theta)\) and \(m' := \text{crit}(\theta')\). Computing \(d_F(\theta, \theta')\) requires time \(O(n^2(m + m'))\).

In sum:

Theorem 4.29. \textit{Given two formigrams \(\theta\) and \(\theta'\), computing \(d_F(\theta, \theta')\) requires \(O(n^4 \ell^2)\) in expectation where \(\ell := \max(m, m')\).}

As we already saw in Sec. 2.4, any formigram \(\theta\) over \(X\) can be visualized as a topological graph over the real line, annotated by elements in \(X\). This graph is called the \textbf{underlying Reeb graph} of \(\theta\). A rigorous definition is given in [49]. We can significantly reduce the complexity \(O(n^4 \ell^2)\) mentioned above to \(O(n^2 \cdot \ell^{1.5} \log \ell)\) by restricting ourselves to formigrams whose underlying Reeb graphs do not contain any loops\(^\text{10}\); see Thm. E.3 in the appendix. To prove this claim, we utilize a special relationship between the bottleneck distance and the Hausdorff distance on the real line which may be of independent interest (Thm. E.1).

Proof of Prop. 4.26. We begin with the following lemma:

Lemma 4.30. Let \(P_1\) and \(P_2\) be any two subpartitions of \(X\), with \(n := |X|\). Computing \(P_1 \lor P_2\) requires at most time \(O(n^2)\).

\(^{10}\)For example, the formigram depicted in Fig. 5(A) contains a loop, whereas the ones in Fig. 6(A) and (B) do not.
Figure 8: (A) For a given formigram $\theta$, let us assume that $\text{crit}(\theta) = \{s_1 < s_2 < s_3 < s_4 < s_5\}$. The main diagonal line stands for the real line via the bijection $(r, r) \in \mathbb{R}^2 \leftrightarrow r \in \mathbb{R}$. We assign a subpartition of $X$ to each colored point in the grid as follows: For each point $v$ in the line 1, assign $\theta(s_i) \lor \theta(s_{i+1})$ where $s_i$ and $s_{i+1}$ are adjacent to $v$ in the grid. For each point in line $i$, assign $P_1 \lor P_2$ where $P_1$ and $P_2$ are the subpartitions assigned to two points in line $i-1$ which are adjacent to $v$. Observe that, given any $I \in \text{Int}$ with $I \cap \text{crit}(\theta) \neq \emptyset$, $\hat{\theta}(I)$ is equal to the subpartition assigned to the maximal point $v = (v_1, v_2)$ in the grid ($\subset \mathbb{R}^{op} \times \mathbb{R}$) s.t. $v_1, v_2 \in I$. (B) An illustration of $\theta_X^{-1}([x, x']^1)$ of which the number of its corner points is maximal, $2m - 1$.

**Proof.** For $i = 1, 2$, let $X_i \subset X$ be the underlying set of $P_i$. Let us consider the undirected simple graph $G_i = (X_i, E_i)$ derived from $P_i$, where $\{x, y\} \in E_i$ if and only if $x$ and $y$ belong to the same block of $X_i$. Note that $P_1 \lor P_2$ is the partition of $X_1 \cup X_2$ where each block of $P_1 \lor P_2$ constitutes a connected component of the graph $G_1 \cup G_2 = (X_1 \cup X_2, E_1 \cup E_2)$. Therefore, computing $P_1 \lor P_2$ is equivalent to computing the connected components of $G_1 \cup G_2$. One needs $O(|X_1 \cup X_2| + |E_1 \cup E_2|)$ in time to partition $X_1 \cup X_2$ according to the components of $G_1 \cup G_2$ [37, Section 5]. Since $|X_1 \cup X_2| \leq |X| = n$ and $|E_1 \cup E_2| \leq \binom{n}{2}$, at worst time $O(n^2)$ will be necessary. \hfill \Box

**Proof of Prop. 4.26** The claim directly follows from Lem. 4.30 and the observation that, in order to compute $\hat{\theta}$, it suffices to compute $O\left(\binom{n}{2}\right) = O(m^2)$ different join operations between two (sub)partitions of $X$. See Fig. 8 (A) for an illustrative example. \hfill \Box

**Proof of Prop. 4.27** Let $X = \{x_1, \ldots, x_n\}$, and let $P = \{C_1, \ldots, C_k\}$ be a subpartition of $X$. This $P$ can be encoded as the $n \times k$ membership matrix $M_P = (m_{ij})$ where the $i$-th row and the $j$-th column correspond to $x_i \in X$ and $C_j \in P$ respectively, and

\[
m_{ij} := \begin{cases} 
1, & \text{if } x_i \text{ belongs to } C_j \\
0, & \text{otherwise.}
\end{cases}
\]

Note that the $(i, j)$-entry of $M_P(M_P)^t$ is 1 iff $x_i$ and $x_j$ belong to the same block, and 0 otherwise. Let us assume that all subpartitions of $X$ are equally likely choices for $P$. Then, since $k$ cannot exceed $n$, computing $M_P(M_P)^t$ is expected to take at most in time $O(n^2)$ [63, Thm. 3.1].
Now recall from Def. [4.21] that the cosheaf-code of a formigram $\theta$ over an $n$-point set consists of \( \binom{n}{2} \) upper sets $\hat{\theta}_{(x,x')}$, $x, x' \in X$ of $\text{Int}$. Assuming that $m := |\text{crit}(\theta)|$, by the previous argument, computing each $\hat{\theta}_{(x,x')}$ requires at most in time $O\left(n^2\binom{n}{2}\right) = O(n^2m^2)$. Therefore, we directly have Prop. 4.27

**Proof of Prop. 4.28.** Recall from Thm. 4.20 that computing $d_F$ reduces to the calculation of the Hausdorff distance between upper sets of $(\text{Int}, \|\cdot\|_\infty)$. The lemma below provides an insight into computing $d_F$:

**Lemma 4.31.** If $A, B$ are upper sets of $(\text{Int}, \|\cdot\|_\infty)$, then their Hausdorff distance is given by the formula

$$d_H(A, B) = \sup_{\ell} d_H(A \cap \ell, B \cap \ell),$$

where $\ell$ ranges over all lines of slope $-1$ in $\mathbb{R}^2$.

**Proof.** ($\geq$) Pick any line $\ell$ of slope $-1$. Let $\epsilon := d_H(A, B)$. Let $x = (x_1, x_2) \in A \cap \ell$. Then, there exists $y \in B$ such that $\|x - y\|_\infty \leq \epsilon$. Since $B$ is an upper set, $y \leq (x_1 - \epsilon, x_2 + \epsilon) \in B$. Since $\ell$ is of slope $-1$, $(x_1 - \epsilon, x_2 + \epsilon)$ lies on the line $\ell$, and thus $(x_1 - \epsilon, x_2 + \epsilon) \in B \cap \ell$. In the same way, one can prove that for any $x \in B \cap \ell$, there exists a $y \in A \cap \ell$ such that $\|x - y\|_\infty \leq \epsilon$.

($\leq$) Assume the RHS is less or equal to $\epsilon$. Pick $a \in A$ and a line $\ell$ of slope $-1$ which passes through $a$. By assumptions, there is a $b \in B \cap \ell$ such that $\|a - b\|_\infty \leq \epsilon$. This $b$ also belongs to $B$. By symmetry, for all $b \in B$, there exists an $a \in A$ such that $\|a - b\|_\infty \leq \epsilon$.

**Proof of Prop. 4.28.** For $x, x' \in X$, the upper sets $A_{(x,x')} := \theta_X^{-1}(\{x, x'\})$ and $B_{(x,x')} := (\theta_X')^{-1}(\{x, x'\})$ have at most $\max(2m - 1, 2m' - 1)$ corner points, respectively (see Fig. 8 (B)). By Thm. 4.20, computing $d_F(\theta, \theta')$ reduces to computing $d_H(A_{(x,x')}, B_{(x,x')})$ for all $x, x' \in X$. By Lem. 4.31, computing $d_H(A_{(x,x')}, B_{(x,x')})$ requires at most $O((2m - 1) + (2m' - 1)) = O(m + m')$ computations of $d_H(A_{(x,x')} \cap \ell, B_{(x,x')} \cap \ell)$ where $\ell$ are (-1)-slope-lines passing through at least one of $O(m + m')$ corner points of $A_{(x,x')}$ or $B_{(x,x')}$. For any $\ell$, let $p, q \in \mathbb{R}^\text{op} \times \mathbb{R}$ be the unique minimums of $A_{(x,x')} \cap \ell$ and $B_{(x,x')} \cap \ell$, respectively. Then, it is not difficult to check that $d_H(A_{(x,x')} \cap \ell, B_{(x,x')} \cap \ell) = \|p - q\|_\infty$. Hence the claim follows.

5 Discussions

Some open questions are the following.

(1) **What is the relationship between $d_{GH}$ in Defn. 4.9 and the edit distance $d_{edit}$ in [57] between lattice-indexed simplicial filtrations?** Both distances are a generalization or a rendition of certain distances that satisfy universality; For $d_{GH}$, see Thm. 4.11 (also [71] Prop.6.2.21). $d_{edit}$ is a rendition of the edit distance on Reeb graphs which satisfies another universal property [2] (and is itself inspired on $d_{GH}$). Currently we know that $d_{GH}$ and $d_{edit}$ cannot be strongly equivalent; it is not difficult to find a pair of simplicial filtrations such that $d_{GH}$ vanishes, but $d_{edit}$ does not.

(2) **Realization of erosion geodesics.** From Thm. 3.13, we know that any two poset maps $F, G : \text{Int} \to \mathbb{Z}_{\geq 0}^{\text{op}}$ at finite erosion distance can be joined by a geodesic path in $[\text{Int}, \mathbb{Z}_{\geq 0}^{\text{op}}]$. Assume
that \( F \) and \( G \) are the rank functions of two persistence modules \( M \) and \( N \), respectively. The geodesic path \( g : [0, 1] \to [\text{Int}, \mathbb{Z}_{\geq 0}] \) between \( F \) and \( G \) constructed in the proof of Thm. 3.13 is actually not always realizable by a path in \( \text{vect}^\mathbb{R} \), i.e. there is sometimes no continuous map \( h : [0, 1] \to (\text{vect}^\mathbb{R}, d_\mathbb{E} \circ \text{rk}) \) such that \( \text{rk} \circ h = g \). This implies that, we do not know at this point whether or not \( d_\mathbb{E}(\text{rk}(-), \text{rk}(-)) \) is a geodesic distance on the space of \( \mathbb{R} \)-indexed persistence modules. We believe that studying this can potentially be useful for clarifying the relationship between the bottleneck distance and \( d_\mathbb{E}(\text{rk}(-), \text{rk}(-)) \).

### A Persistence modules and rank functions

Let \( \text{vect} \) be the category of finite dimensional vector spaces and linear maps over a field \( \mathbb{F} \).

**Definition A.1.** Any functor \( M : (\mathbb{R}, \leq) \to \text{vect} \) is said to be a (standard) **persistence module**, i.e. each \( r \in \mathbb{R} \) is sent to a vector space \( M(r) \) and each pair \( r \leq s \in \mathbb{R} \) is sent to a linear map \( M(r \leq s) \). In particular, for all \( r \leq s \leq t \), we have:

\[
M(s \leq t) \circ M(r \leq s) = M(r \leq t).
\]

\( M \) is called **constructible** if there exists a finite set \( \{c_1, \ldots, c_n\} \subset \mathbb{R} \) such that (i) for \( i = 1, \ldots, n \), whenever \( r, s \in \{c_i, c_{i+1}\} \) with \( r \leq s \), \( M(r \leq s) \) is the identity map (let \( c_{n+1} := \infty \)), (ii) for \( r \in (-\infty, a_1) \), \( M(r) = 0 \).

By replacing the indexing poset \( (\mathbb{R}, \leq) \) by \( (\mathbb{R}^d, \leq) \) for \( d \geq 2 \), we obtain a **multiparameter persistence module**.

Let \( M : (\mathbb{R}, \leq) \to \text{vect} \). For any \( r \leq r' \leq s' \leq s \) in \( \mathbb{R} \), we have

\[
M(r \leq s) = M(s' \leq s) \circ M(r' \leq s') \circ M(r \leq r'),
\]

which implies \( \text{rank}(M(r \leq s)) \leq \text{rank}(M(r' \leq s')) \).

**Definition A.2.** Let \( M : (\mathbb{R}, \leq) \to \text{vect} \). The **rank function** \( \text{rk}(M) : (\text{Int}, \subset) \to \mathbb{Z}_{\geq 0} \) of \( M \) is defined as \( [a, b] \mapsto \text{rank}(M(a \leq b)) \).

### B Interleaving distances in general

We review the general notion of interleaving distance in the language of \[35\] (Defn. 2.9 is a special instance of the definition below). Consult \[55\] for general definitions related to category theory.

Let \( (\mathcal{P}, \leq, \Omega) \) be a poset with a flow, and \( \mathcal{P} \) is viewed as a category. Then, for each \( \epsilon \in [0, \infty) \), \( \Omega_\epsilon \) is an endofunctor on \( \mathcal{P} \) and we have \( 1_\mathcal{P} \leq \Omega_\epsilon \). We view this inequality as a natural transformation \( \eta_\epsilon : 1_\mathcal{P} \Rightarrow \Omega_\epsilon \). Let \( \mathcal{C} \) be any category and let \( M : \mathcal{P} \to \mathcal{C} \) be any functor. Then, we have a natural transformation \( M\eta_\epsilon : M \to M\Omega_\epsilon \).

\[11\] Example: For \( a, b \in \mathbb{R} \) with \( a < b \), let \( I(a, b) : \mathbb{R} \to \text{vect} \) the **interval module** with support \([a, b]\) \[27\]. Let \( M, N : \mathbb{R} \to \text{vect} \) be defined as \( M = \{0, 10\} \) and \( N = \{4, 14\} \). Now define \( F, G \) to be \( \text{rk}(M) \) and \( \text{rk}(N) \) respectively.
Definition B.1. Let $(\mathcal{P}, \leq, \Omega)$ be a poset with a flow. Any two functors $M, N : \mathcal{P} \to \mathcal{C}$ are called \(\Omega \varepsilon\)-interleaved if there exists a pair of natural transformations \(\phi : M \to N \Omega \varepsilon\) and \(\psi : N \to M \Omega \varepsilon\) such that the diagram below commutes.

\[
\begin{array}{ccc}
M & \xrightarrow{M \eta \varepsilon} & M \Omega \varepsilon \\
\downarrow{\psi} & & \downarrow{\psi \Omega \varepsilon} \\
N & \xrightarrow{N \eta \varepsilon} & N \Omega \varepsilon
\end{array}
\quad \begin{array}{ccc}
M \Omega \varepsilon & \xrightarrow{M \eta \varepsilon \Omega \varepsilon} & M \Omega \varepsilon \Omega \varepsilon \\
\downarrow{\phi} & & \downarrow{\phi \Omega \varepsilon} \\
N \Omega \varepsilon & \xrightarrow{N \eta \varepsilon \Omega \varepsilon} & N \Omega \varepsilon \Omega \varepsilon
\end{array}
\]

The interleaving distance (with respect to \(\Omega\)) is:

\[
d_{I}^{\Omega}(M, N) = \inf\{\varepsilon \geq 0 : M, N \text{ are } \Omega \varepsilon \text{-interleaved}\}
\]

where \(d_{I}^{\Omega}(M, N) := \infty\) if there is no \(\varepsilon\)-interleaving for any \(\varepsilon \geq 0\).

When \(\mathcal{C}\) is a poset \(\mathcal{D}\), this definition reduces to Defn. 2.9. When \(\mathcal{P} = \mathbb{R}^n\) with the flow in [1] and \(\mathcal{C} = \text{vect}\), the distance \(d_{I}^{\Omega}\) is the standard interleaving distance [25, 54].

C (Gromov-)Hausdorff and Bottleneck distances.

We recall the definitions of the Hausdorff distance, Gromov-Hausdorff distance [17, Section 7.3.3] and the Bottleneck distance [53] in that order.

Definition C.1 (Hausdorff distance). Let \(A\) and \(B\) be closed subsets of a metric space \((M, d)\). The Hausdorff distance between \(A\) and \(B\) is defined as \(d_{H}(A, B) = \inf\{r \in [0, \infty) : A \subset B^r \text{ and } B \subset A^r\}\), where \(A^r := \{m \in M : \exists a \in A, \ d(a, m) \leq r\}\).

Let \((X, d_X)\) and \((Y, d_Y)\) be any two metric spaces and let \(R : X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y\) be a tripod (i.e. a pair of surjective maps) between \(X\) and \(Y\). Then, the distortion of \(R\) is defined as

\[
\text{dis}(R) := \sup_{z, z' \in Z} |d_X(\phi_X(z), \phi_X(z')) - d_Y(\phi_Y(z), \phi_Y(z'))|.
\]

The Gromov-Hausdorff distance \(d_{GH}\) measures how far two metric spaces are from being isometric.

Definition C.2 (Gromov-Hausdorff distance). The Gromov-Hausdorff distance between compact metric spaces \((X, d_X)\) and \((Y, d_Y)\) is defined as

\[
d_{GH}((X, d_X), (Y, d_Y)) := \frac{1}{2} \inf_{R} \text{dis}(R),
\]

where the infimum is taken over all tripods \(R\) between \(X\) and \(Y\).
**Bottleneck distance.** Bottleneck distance is an extensively studied metric. We adopt notation in [3] to describe it. Partial bijections are referred to as matchings. Given two nonempty sets $A$ and $B$, we use $\sigma : A \rightarrow B$ to denote a matching $\sigma \subset A \times B$. The canonical projections of $\sigma$ onto $A$ and $B$ are denoted by $\text{coim}(\sigma)$ and $\text{im}(\sigma)$, respectively. By \langle a, b \rangle for $a < b$ in $\mathbb{R}$, we denote one of the real intervals $(a, b)$, $[a, b]$, $(a, b]$, and $[a, b]$.

Letting $\mathcal{A}$ be a multiset of intervals in $\mathbb{R}$ and $\varepsilon \geq 0$,

$$\mathcal{A}^\varepsilon := \{\langle b, d \rangle \in \mathcal{A} : b + \varepsilon < d\} = \{I \in \mathcal{A} : [t, t + \varepsilon] \subset I \text{ for some } t \in \mathbb{R}\}.$$ 

Note that $\mathcal{A}^0 = \mathcal{A}$.

**Definition C.3** (Bottleneck distance). Let $\mathcal{A}$ and $\mathcal{B}$ be multisets of intervals in $\mathbb{R}$. We define a $\delta$-matching between $\mathcal{A}$ and $\mathcal{B}$ to be a matching $\sigma : \mathcal{A} \rightarrow \mathcal{B}$ such that $\mathcal{A}^{2\delta} \subset \text{coim}(\sigma)$, $\mathcal{B}^{2\delta} \subset \text{im}(\sigma)$, and if $\sigma\langle b, d \rangle = \langle b', d' \rangle$, then

$$\langle b, d \rangle \subset \langle b' - \delta, d' + \delta \rangle, \quad \langle b', d' \rangle \subset \langle b - \delta, d + \delta \rangle.$$ 

with the convention $+\infty + \delta = +\infty$ and $-\infty - \delta = -\infty$. We define the bottleneck distance $d_B$ by

$$d_B(\mathcal{A}, \mathcal{B}) := \inf(\delta \in [0, \infty) : \exists \delta\text{-matching between } \mathcal{A} \text{ and } \mathcal{B}).$$

We declare $d_B(\mathcal{A}, \mathcal{B}) = +\infty$ when there is no $\delta$-matching between $\mathcal{A}$ and $\mathcal{B}$ for any $\delta \in [0, \infty)$.

### D On the (generalized) Gromov-Hausdorff distance

The goal of this section is to establish basic properties of $d_{GH}$ in Defn. [4.9 Props. D.1, D.2, D.5, D.6 and D.7] Throughout this section, $X$, $Y$, $Z$, and $W$ will denote nonempty finite sets.

**Proposition D.1.** $d_{GH}$ in Defn. [4.9] is an extended pseudometric.

**Proof.** Symmetry and non-negativity are clear. We prove the triangle inequality. Consider any three filtrations $F_X : (\mathcal{P}, \leq, \Omega) \rightarrow \text{Simp}(X)$, $G_Y : (\mathcal{P}, \leq, \Omega) \rightarrow \text{Simp}(Y)$, and $H_W : (\mathcal{P}, \leq, \Omega) \rightarrow \text{Simp}(W)$. Assume that, for $\eta_1, \eta_2 > 0$, we have:

$$d_{GH}(F_X, G_Y) < \eta_1 \quad \text{and} \quad d_{GH}(G_Y, H_W) < \eta_2.$$ 

Then, there exist tripods $R_1 : X \xleftarrow{\varphi_X} Z_1 \xrightarrow{\varphi_Y} Y$ and $R_2 : Y \xleftarrow{\psi_Y} Z_2 \xrightarrow{\psi_W} W$ such that

$$d_{\text{Simp}(Z_1)}(\varphi_X^* F_X, \varphi_Y^* G_Y) < \eta_1 \quad \text{and} \quad d_{\text{Simp}(Z_2)}(\psi_Y^* G_Y, \psi_Y^* G_Y) < \eta_2.$$ 

Consider the set $Z := \{(z_1, z_2) \in Z_1 \times Z_2 : \varphi_Y(z_1) = \psi_Y(z_2)\}$ and let $\pi_1 : Z \rightarrow Z_1$ and $\pi_2 : Z \rightarrow Z_2$ be the canonical projections to the first and the second coordinate, respectively. We define the composite tripod $R_2 \circ R_1$ as follows:

$$R_2 \circ R_1 : X \xleftarrow{\omega_X} Z \xrightarrow{\omega_W} W, \quad \text{where} \quad \omega_X := \varphi_X \circ \pi_1, \quad \omega_W := \psi_W \circ \pi_2.$$ 

(16)

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The desired inequality follows by letting \( \eta \).

Next we show that \( d_{\text{GH}} = 0 \) implies a homotopy equivalence between filtrations.

**Proposition D.2.** Let \( F_X : (\mathcal{P}, \leq, \Omega) \to \text{Simp}(X) \) and \( F_Y : (\mathcal{P}, \leq, \Omega) \to \text{Simp}(Y) \) be any two filtrations. Suppose that \( d_{\text{GH}}(F_X, F_Y) = 0 \). Then \( F_X \simeq F_Y \) (the converse does not hold; see Ex. [4.16]).

This proposition can be proved in a similar way to [58, Cor. 2.1].

**Proof.** Let \( R : X \xleftarrow{\varphi_X} Z \xrightarrow{\varphi_Y} Y \) be a minimizer for \( d_{\text{GH}} \) (cf. the first footnote in Defn. 2.22). Then, \( d_1(\varphi_X^* F_X, \varphi_Y^* F_Y) = 0 \). By Defn. 4.9 we obtain

\[
\max_{\sigma \in \text{pow}_{\downarrow}(Z)} d_{\Omega}(\varphi_X^* F_X)^{-1}(\sigma^1), (\varphi_Y^* F_Y)^{-1}(\sigma^1) = 0.
\]

This implies \( \varphi_X^* F_X = \varphi_Y^* F_Y \). By Quillen's Theorem A [67], we have that \( F_X \simeq \varphi_X^* F_X \) and \( \varphi_Y^* F_Y \simeq F_Y \), completing the proof. \( \square \)

**Upper bound for \( d_{\text{GH}} \).** We aim at obtaining a coarse upper bound for \( d_{\text{GH}} \) (Prop. D.5).

**Lemma D.3** (Weak join-preserving property of \( \Omega \)). Let \( (\mathcal{P}, \leq, \Omega) \) be a poset with a flow. Let \( (A_j) \) be a family of upper sets in \( \mathcal{P} \) and let \( B \) be another upper set in \( \mathcal{P} \). Then,

\[
d_{\Omega}(\bigcup_j A_j, B) \leq \sup_{j \in J} d_{\Omega}(A_j, B)
\]

(17)

It is not difficult to find an example that shows the inequality in (17) can be strict.

**Proof.** Let \( \epsilon > 0 \) such that \( d_{\Omega}(A_j, B) < \epsilon \) for all \( j \in J \). Then for each \( j \) we have \( A_j \subset \tilde{\Omega}_c(B) \) and thus \( \bigcup_j A_j \subset \tilde{\Omega}_c(B) \). Also, for every \( j \), we have \( B \subset \tilde{\Omega}(A_j) \), which implies \( B \subset \bigcup_j \tilde{\Omega}(A_j) \). By Rmk. 2.12(ii), we have \( B \subset \tilde{\Omega}_c(\bigcup_j A_j) \). Therefore, the left-hand side of inequality (17) is at most \( \epsilon \).

---

12When \( \mathcal{P} = \mathbb{R} \), a more general statement can be found in [71, Rmk. 6.8.9] in connection with the homotopy interleaving distance [9].
For the singleton set $\{\ast\}$, note that $\text{Simp}(\{\ast\})$ includes only the two simplicial complexes; the empty complex and $\{\ast\}$. Defn. 4.9 directly implies:

**Lemma D.4.** Given any $F_X : (\mathcal{P}, \leq, \Omega) \rightarrow \text{Simp}(X)$ and $H_{\{\ast\}} : (\mathcal{P}, \leq, \Omega) \rightarrow \text{Simp}(\{\ast\})$, we have

\[
d_{\text{GH}}(F_X, H_{\{\ast\}}) = \max_{\sigma \subseteq X, \sigma \neq \emptyset} d_{\Omega}^{-1}(F_X^{-1}(\sigma), H^{-1}(\{\ast\})).
\]

By invoking the triangle inequality of $d_{\text{GH}}$ and the lemma above, we have:

**Proposition D.5 (Coarse upper bound for $d_{\text{GH}}$).** Given any filtrations $F_X : (\mathcal{P}, \leq, \Omega) \rightarrow \text{Simp}(X)$ and $G_Y : (\mathcal{P}, \leq, \Omega) \rightarrow \text{Simp}(Y)$, we have

\[
d_{\text{GH}}(F_X, G_Y) \leq \min_{A \in U(\mathcal{P})} \left( \max_{\tau \subseteq Y, \tau \neq \emptyset} d_{\Omega}^{-1}(F_X^{-1}(\tau), A) + \max_{\sigma \subseteq X, \sigma \neq \emptyset} d_{\Omega}^{-1}(F_X^{-1}(\sigma), A) \right).
\]

**Lower bound for $d_{\text{GH}}$.** Let $K \in \text{Simp}(X)$ and let $K^{(0)}$ be the vertex set of $K$. For $x, x' \in K^{(0)}$, we write $x \sim_{\pi_0(K)} x'$ if there exists a sequence of 1-simplices in $K$ connecting $x$ and $x'$, i.e. there exist $x_1, \ldots, x_n \in K^{(0)}$ with $\{x, x_1\}, \{x_1, x_2\}, \ldots, \{x_n, x'\} \in K$ (if the sequence is empty, then $x = x'$). This defines an equivalence relation on $K^{(0)}$ and thus $\pi_0(K) := K^{(0)}/\sim_{\pi_0(K)} \in \text{SubPart}(X)$.

Then note that $\pi_0$ serves as a poset map $\text{Simp}(X) \rightarrow \text{SubPart}(X)$. Recall $d_{\text{GH}}$ in Defn. 4.25

**Proposition D.6.** For any $F_X : (\mathcal{P}, \leq, \Omega) \rightarrow \text{Simp}(X)$ and $G_Y : (\mathcal{P}, \leq, \Omega) \rightarrow \text{Simp}(Y)$,

\[
d_{\text{GH}}(\pi_0 \circ F_X, \pi_0 \circ G_Y) \leq d_{\text{GH}}(F_X, G_Y).
\]

We remark that the LHS is a better lower bound for $d_{\text{GH}}$ than $d_1(H_0(F_X), H_0(G_Y))$ in Thm. 4.14. For example, if $F_X$ and $G_Y$ are the Rips filtrations of metric spaces $(X, d_X)$ and $(Y, d_Y)$, then the inequality above coincides with the inequality in Eqn. 4. The LHS of Eqn. 4 is known to be a better lower bound than the interleaving distance between the zeroth homology of the Rips filtrations of $(X, d_X)$ and $(Y, d_Y)$.

**Ubiquity of $d_{\text{GH}}$.** Definitions in the following three items are relevant to the three statements in Prop. D.7 respectively in order.

(i) Given a pseudometric space $(X, d_X)$, let VR($X, d_X$) : $\mathbb{R} \rightarrow \text{Simp}(X)$ be the *Vietoris-Rips filtration* of $(X, d_X)$, i.e. each $r \in \mathbb{R}$ is sent to:

VR($X, d_X$)($r$) = $\{\sigma \subset X : |\sigma| < \infty$ and $d_X(x, x') \leq r$ for all $x, x' \in X\}$.

(ii) Let us recall the simplexization map $s$ in Defn. 4.5. For any nonempty finite set $X$ and any $P \in \text{SubPart}(X)$, let $s(P) := \{s(B) : B \in P\} \in \text{Simp}(X)$. Then $s$ serves as a poset map $\text{Simp}(X) \rightarrow \text{SubPart}(X)$.

(iii) Let $\lambda > 0$. The $\lambda$-flow on the product poset $\text{Int} \times \mathbb{R}_{\geq 0}$ is defined by

\[
\Omega^\lambda := [\Omega^\lambda_{\varepsilon} : ((a, b), r) \mapsto (a - \varepsilon, b + \varepsilon), r + \lambda \varepsilon]_{\varepsilon \geq 0}.
\]
Proposition D.7 (Ubiquity of $d_{GH}^{Simp}$). Let $d_{GH}^{Met}$, $d_{GH}^{HC}$, and $d_{GH}^{Simp}$ be the Gromov-Hausdorff distances between metric spaces, between hierarchical clusterings, and between simplicial filtrations, respectively (Defns. C.2, 4.25, 4.7). We have:

(i) [59, Prop. 2.8] For any finite pseudometric spaces $(X, d_X)$ and $(Y, d_Y)$,
\[ d_{GH}^{Met}((X, d_X), (Y, d_Y)) = d_{GH}^{Simp}(VR(X, d_X), VR(Y, d_Y)). \]

(ii) For any $(\mathcal{P}, \leq)$-indexed hierarchical clusterings $\theta_X$ and $\theta_Y$ over $X$ and $Y$ respectively,
\[ d_{GH}^{HC}(\theta_X, \theta_Y) = d_{GH}^{Simp}(s \circ \theta_X, s \circ \theta_Y). \]

(iii) Let $\lambda > 0$. For any dynamic metric spaces $\gamma_X$ and $\gamma_Y$,
\[ d_{GH}^{\lambda}(\gamma_X, \gamma_Y) = d_{GH}^{Simp}(VR^\lambda(\gamma_X), VR^\lambda(\gamma_Y)), \]
where the LHS is the $\lambda$-slack interleaving distance [48, Defn. 2.10].

Since the respective proofs of (ii) and (iii) directly follow from the definitions of the involved metrics, we omit them.

E Specialization of Thm. 4.29

We specialize Thm. 4.29 by restricting ourselves to formigrams whose underlying Reeb graphs do not contain any loops; Thm. E.3. This theorem is a rather direct consequence of Thm. E.1 and Cor. E.2 below.

Recall the Hausdorff and bottleneck distances in Sec. C. Recall that, by $\langle a, b \rangle$ for $a < b$ in $\mathbb{R}$, we denote one of the real intervals $(a, b)$, $(a, b]$, $[a, b)$, and $[a, b]$. Given any real intervals $\langle a, b \rangle$ and $\langle c, d \rangle$, we write $\langle a, b \rangle < \langle c, d \rangle$ if $b < c$.

Theorem E.1. Let $\mathcal{A} := \{I_i := \langle a_i, b_i \rangle : 1 \leq i \leq m\}$ and $\mathcal{B} := \{J_j := \langle c_j, d_j \rangle : 1 \leq j \leq n\}$. Assume that $I_i < I_{i+1}$ for $i = 1, \ldots, m-1$ and $J_j < J_{j+1}$ for $j = 1, \ldots, n-1$. We have:
\[ d_H(C(\mathcal{A}), C(\mathcal{B})) = d_B(\mathcal{A}, \mathcal{B}), \]
where $C(\mathcal{A}) := \mathbb{R} \setminus \bigcup \mathcal{A}$.

We prove this theorem at the end of the section. By Thm. E.1 and [45, Thm. 3.1], we obtain:

Corollary E.2. Let $\ell := \max(m, n)$. Computing $d_H(C(\mathcal{A}), C(\mathcal{B}))$ takes $O(\ell^{1.5} \log \ell)$.

Theorem E.3. Assume that the underlying Reeb graphs of $\theta, \theta' \in \text{Formi}(X)$ do not contain any loops. Then, computing $d_F(\theta, \theta')$ at worst requires time $O(n^2 \ell^{1.5} \log \ell)$, where $n := |X|$ and
\[ \ell := \max(|\text{crit}(\theta_X)|, |\text{crit}(\theta')|). \]
Proof. Fix \(x, x' \in X\) and let
\[
\theta_{(x,x')} := \{ t \in \mathbb{R} : x \text{ and } x' \text{ belong to the same block in } \theta(t) \}.
\]
We claim that \(d_{t}^{\text{int}}(\theta^{-1}([x,x']^\dagger), \theta^{-1}([x,x']^\dagger))\) in Eqn. (14) is equal to \(d_{H}^{R}(\theta_{(x,x')}, \theta_{(x,x')})\). To show this, it suffices to show that for any \(\eta > 0\) one distance is upper bounded by \(\eta\) implies the other is too. Indeed, the no-loop assumption implies that, for any \(\eta > 0\), the both distances are upper bounded by \(\eta\) if and only if the following holds; if \(x \text{ and } x' \text{ belong to } \theta(t)\) (resp. \(\theta'(t)\)), then there exists \(s \in [t-\varepsilon, t+\varepsilon]\) such that \(x \text{ and } x' \text{ belong to } \theta(s)\) (resp. \(\theta'(s)\)).

By Cor. E.2 computing \(d_{H}^{R}(\theta_{(x,x')}, \theta'_{(x,x')})\) requires \(O(\ell^{1.5} \log \ell)\). Since there are \(n^2\) singleton and doubleton subsets of \(X\), by the equality in Eqn. (14), we compute \(d_{t}(\theta, \theta')\) in \(O(n^2 \cdot \ell^{1.5} \log \ell)\).

\(\square\)

**Proof of Thm. E.1.** We prove Thm. E.1. To avoid trivialities, assume that \(\mathcal{A} \neq \mathcal{B}\). Let \(E_{\mathcal{A}}\) be the collection of \(I_i\)'s endpoints, i.e. \(\mathcal{A} = \{a_i\}_{i=1}^{m} \cup \{b_i\}_{i=1}^{m}\). Letting \(b_0 = -\infty\) and \(a_{m+1} = \infty\), we have \(C(\mathcal{A}) = \mathbb{R} \setminus \cup \mathcal{A} = \bigcup_{i=0}^{m} (b_i, a_{i+1})\). Similarly, we define \(E_{\mathcal{B}}\) and \(C(\mathcal{B})\).

**Lemma E.4.** \(d_{H}(C(\mathcal{A}), C(\mathcal{B})) = \max \left\{ \max_{a \in C(\mathcal{A})} \min_{b \in C(\mathcal{B})} |a-b|, \max_{b \in C(\mathcal{B})} \min_{a \in C(\mathcal{A})} |a-b| \right\} \).

**Proof.** Since
\[
d_{H}(C(\mathcal{A}), C(\mathcal{B})) = \max \left\{ \max_{a \in C(\mathcal{A})} \min_{b \in C(\mathcal{B})} |a-b|, \max_{b \in C(\mathcal{B})} \min_{a \in C(\mathcal{A})} |a-b| \right\},
\]
by symmetry, it suffices to prove that
\[
\max_{a \in C(\mathcal{A})} \min_{b \in C(\mathcal{B})} |a-b| = \max_{a \in C(\mathcal{A})} \min_{b \in E_{\mathcal{B}}} |a-b|.
\]
For \(a \in C(\mathcal{A})\), let \(\varphi(a) := \min_{b \in C(\mathcal{B})} |a-b|\). If \(a \in C(\mathcal{B})\), then clearly \(\varphi(a) = 0\). Hence, restricting the domain \(C(\mathcal{A})\) of \(\varphi\) to the intersection of \(C(\mathcal{A})\) and \(\mathbb{R} \setminus C(\mathcal{B}) = \bigcup \mathcal{B}\) does not affect the maximum of \(\varphi\), which implies:
\[
\max_{a \in C(\mathcal{A})} \min_{b \in C(\mathcal{B})} |a-b| = \max_{a \in C(\mathcal{A}) \cap \bigcup \mathcal{B}} \min_{b \in C(\mathcal{B})} |a-b| \tag{18} \]
Next, fix an arbitrary \(a \in \bigcup \mathcal{B}\). Then the closest point in \(\mathbb{R} \setminus \bigcup \mathcal{B} = C(\mathcal{B})\) to \(a\) is obviously located on the boundary of \(C(\mathcal{B})\), the set of endpoints \(E_{\mathcal{B}}\), which implies \(\min_{b \in C(\mathcal{B})} |a-b| = \min_{b \in E_{\mathcal{B}}} |a-b|\). Therefore, the RHS of (18) coincides with the RHS of (19).

**Proof of** \(d_{H}(C(\mathcal{A}), C(\mathcal{B})) \leq d_{B}(\mathcal{A}, \mathcal{B})\).

Let \(\sigma : \mathcal{A} \rightarrow \mathcal{B}\) be a \(\delta\)-matching. By Lem. E.4 and symmetry, it suffices to prove that
\[
\max_{a \in C(\mathcal{A}) \cap \bigcup \mathcal{B}} \min_{b \in E_{\mathcal{B}}} |a-b| \leq \delta.
\]

\footnote{Since we assumed \(\mathcal{A} \neq \mathcal{B}\), the set \(C(\mathcal{A}) \cap \bigcup \mathcal{B}\) cannot be empty.}
Figure 9: An illustration for Case 2, (a).

Fix any \( a \in C(\mathcal{A}) \cap \bigcup \mathcal{B} \). Then there are \( 0 \leq i \leq m \) and \( 1 \leq j \leq n \) such that

\[
a \in \langle b_i, a_{i+1} \rangle \cap \langle c_j, d_j \rangle.
\]  

(20)

**Case 1.** Assume that \( \text{length}(c_j, d_j) \leq 2\delta \). Since \( a \in \langle c_j, d_j \rangle \), we have:

\[
\min_{b \in E_{\mathcal{B}}} |a - b| \leq \min\{|a - c_j|, |a - d_j|\} \leq \delta.
\]

**Case 2.** Assume that \( \text{length}(c_j, d_j) > 2\delta \). Then there exists \( 1 \leq k \leq m \) such that \( \langle a_k, b_k \rangle \) is matched with \( \langle c_j, d_j \rangle \) via the matching \( \sigma \). Note that the intersection in (20) can possibly be expressed as follows:

\[
\langle b_i, a_{i+1} \rangle \cap \langle c_j, d_j \rangle = \begin{cases} 
\langle c_j, d_j \rangle & \text{Case (a)} \\
\langle b_i, a_{i+1} \rangle & \text{Case (b)} \\
\langle b_i, d_j \rangle & \text{Case (c)} \\
\langle c_j, a_{i+1} \rangle & \text{Case (d)}
\end{cases}
\]

Assume Case (a), i.e. \( J_j = \langle c_j, d_j \rangle \subset \langle b_i, a_{i+1} \rangle \) (See Fig. 9). Given any intervals \( \langle a, b \rangle \) and \( \langle c, d \rangle \), let \( \|\langle a, b \rangle - \langle c, d \rangle\|_\infty := \max\{|a - c|, |b - d|\} \). Note that the closest intervals to \( \langle c_j, d_j \rangle \) in \( \mathcal{A} \) in the metric \( \|\cdot\|_\infty \) are \( I_i = \langle a_i, b_i \rangle \) and \( I_{i+1} = \langle a_{i+1}, b_{i+1} \rangle \). However, both \( \|I_i - J_j\|_\infty \) and \( \|I_{i+1} - J_j\|_\infty \) are greater than \( \delta \) because \( 2\delta \leq |d_j - c_j| \leq |b_i - d_j| \leq \|I_i - J_j\|_\infty \) and \( 2\delta \leq |d_j - c_j| = |a_{i+1} - c_j| \leq \|I_{i+1} - J_j\|_\infty \). This contradicts the fact that \( \sigma \) is a \( \delta \)-matching. Therefore, Case (a) cannot happen.

Assume Case (b). Also, assume that \( J_j = \langle c_j, d_j \rangle \) is matched with \( I_k \) for \( k \leq i \) via \( \sigma \). Then since \( b_k \leq b_i \leq a \leq d_j \),

\[
|a - d_j| \leq |b_k - d_j| \leq \|I_k - J_j\|_\infty \leq \delta.
\]

Now, suppose that \( \langle c_j, d_j \rangle \) is matched with \( I_k \) for \( k > i \). Then since \( c_j \leq a \leq a_{i+1} \leq a_k \),

\[
|a - c_j| \leq |a_k - c_j| \leq \|I_k - J_j\|_\infty \leq \delta.
\]

Therefore, we have

\[
\min_{b \in E_{\mathcal{B}}} |a - b| \leq \min\{|a - c_j|, |a - d_j|\} \leq \delta
\]

as desired.

Assume Case (c), i.e., \( c_j \leq b_i \leq d_j \leq a_{i+1} \). Note that \( I_k \) cannot be matched with \( J_j \) for \( k > i \) via \( \sigma \) because \( c_j < d_j \leq a_{i+1} \) and in turn

\[
\delta < 2\delta < d_j - c_j \leq a_{i+1} - c_j \leq a_k - c_j \leq \|I_k - J_j\|_\infty.
\]
Hence, \( J_j \) must be matched with \( I_k \) for some \( k \leq i \). Take \( k \leq i \) such that \( I_k \) is matched with \( J_j \) via \( \sigma \). Since \( b_k \leq b_j \leq a < d_j \), we have

\[
|a - d_j| \leq |d_j - b_k| \leq \|J_j - I_k\|_\infty \leq \delta.
\]

Therefore, we have

\[
\min_{b \in E_{\mathcal{R}}} |a - b| \leq |a - d_j| \leq \delta.
\]

Assume Case (d). By a similar argument to Case (c), \( J_j \) must be matched with \( I_k \) for some \( k > i \) and this in turn implies \( |a - c_j| \leq \delta \). Hence again

\[
\min_{b \in E_{\mathcal{R}}} |a - b| \leq |a - c_j| \leq \delta.
\]

We have shown that \( \min_{b \in E_{\mathcal{R}}} |a - b| \leq \delta \) for all \( a \in C(\mathcal{A}) \cap \bigcup \mathcal{B} \) as desired.

**Proof of** \( d_{\mathcal{H}}(C(\mathcal{A}), C(\mathcal{B})) \geq d_{\mathcal{R}}(\mathcal{A}, \mathcal{B}) \). Let \( \varepsilon > 0 \). Define \( \mathcal{A}^\varepsilon \) to be the collection of intervals in \( \mathcal{A} \) whose length is at least \( \varepsilon \). Also, given any interval \( I = (a, b) \), let

\[
I^{-\varepsilon} := \begin{cases} 
\emptyset & \text{if } b - a \leq 2\varepsilon \\
(a + \varepsilon, b - \varepsilon) & \text{otherwise.}
\end{cases}
\]

Let \( (C(\mathcal{A}))_\eta \) be the \( \eta \)-thickening of \( C(\mathcal{A}) \), i.e. \( \{r \in \mathbb{R} : \exists p \in C(\mathcal{A}), |p - r| \leq \eta\} \). We have:

**Lemma E.5.** \( \mathbb{R} \setminus (C(\mathcal{A}))_\eta = \bigcup_{I \in \mathcal{A}^{\eta}} I^{-\eta} \) (the proof is elementary but rather tedious so we omit it).

Suppose that \( d_{\mathcal{H}}(C(\mathcal{A}), C(\mathcal{B})) \leq \eta \) for some \( \eta > 0 \). We wish to construct an \( \eta \)-matching \( \sigma : \mathcal{A} \to \mathcal{B} \). Note that \( C(\mathcal{B}) \subset (C(\mathcal{A}))_\eta \) by assumption and thus \( \bigcup_{j=1}^\eta J_j = \mathbb{R} \setminus C(\mathcal{B}) = \mathbb{R} \setminus (C(\mathcal{A}))_\eta = \bigcup_{I \in \mathcal{A}^{\eta}} I^{-\eta} \). This implies that there exists \( j \) such that \( I^{-\eta}_i \subset J_j \), equivalently \( I_i \subset J_j^{\eta} \), for each \( I_i \in \mathcal{A}^{\eta} \) since the union \( \bigcup_{j=1}^\eta J_j \) is disjoint. We already have shown the following proposition.

**Proposition E.6.** Assume that \( \eta \geq d_{\mathcal{H}}(C(\mathcal{A}), C(\mathcal{B})) \) for some \( \eta > 0 \). Then, there exist functions \( f : A^{2\eta} \to B \) and \( g : B^{2\eta} \to A \) such that

\[
I_i \subseteq (J_{f(i)})^{\eta} \text{ for all } i \in A^{2\eta} \text{ and } J_j \subseteq (I_{g(j)})^{\eta} \text{ for all } j \in B^{2\eta}
\]

where \( A^{2\eta} = \{1 \leq i \leq m : I_i \in \mathcal{A}^{2\eta}\} \) and \( B^{2\eta} = \{1 \leq j \leq n : J_j \in \mathcal{B}^{2\eta}\} \).

Let \( f, g \) be as in the proposition. We construct an \( \eta \)-matching between \( \mathcal{A} \) and \( \mathcal{B} \). We write \( A^{2\eta} = A^{2\eta}_0 \cup A^{2\eta}_* \) where \( A^{2\eta}_0 := \{i \in A^{2\eta} : f(i) \notin B^{2\eta}\} \) and \( A^{2\eta}_* := \{i \in A^{2\eta} : f(i) \in B^{2\eta}\} \). Similarly, we write \( B^{2\eta} = B^{2\eta}_0 \cup B^{2\eta}_* \) using the function \( g \).

**Proposition E.7.** \( g \circ f |_{A^{2\eta}_*} = \text{id}_{A^{2\eta}_*} \) and \( f \circ g |_{B^{2\eta}_*} = \text{id}_{B^{2\eta}_*} \).
Proof. We only show the first equality. Take any \( i \in A^2_{x} \). We know that
\[
I_i \subseteq (J f(i))^{\eta} \subseteq \left( (I g(f(i)))^{\eta} \right)^{\eta} = (I g(f(i)))^{2\eta}.
\]
Let \( j = g(f(i)) \). The above equation means that \( \langle a_i, b_i \rangle \subseteq \langle a_j - 2\eta, b_j + 2\eta \rangle \). However, since length \( \langle a_i, b_i \rangle \geq 2\eta \), this is impossible unless either \( \langle a_i, b_i \rangle \) and \( \langle a_j, b_j \rangle \) share one of their endpoints or have nonempty intersection. Since the intervals in \( \mathcal{A} \) are disjoint and do not share their endpoints, we have \( i = j \).

Notice two important implications of the above claim: The first is that \( f(A^2_{x}) \subseteq B^2_{x} \) and \( g(B^2_{x}) \subseteq A^2_{x} \). The second is that both \( f|_{A^2_{x}} \) and \( g|_{B^2_{x}} \) are injective. Now we are going to show that \( f \) and \( g \) are injective on \( A^2_{0} \) and \( B^2_{0} \) respectively as well.

Claim E.8. The functions \( f|_{A^2_{0}} \) and \( g|_{B^2_{0}} \) are injective.

Proof. We prove the claim for \( f \). Assume that \( i, j \in A^2_{0} \), and \( f(i) = f(j) = k \), which means \((J_k)^{\eta} \supseteq I_i \) and \((J_k)^{\eta} \supseteq I_j \) and hence \((J_k)^{\eta} \equiv I_i \cup I_j \). Therefore,
\[
4\eta \geq 2\eta + \text{length}(J_k) = \text{length}(J_k)^{\eta} \geq \text{length}(I_i \cup I_j)
\]
This enforces \( I_i \) and \( I_j \) to have nonempty intersection since each of them has the length\( \geq 2\eta \). Thus \( i = j \) because the intervals in \( \mathcal{A} \) are disjoint.

We are now ready to define an \( \eta \)-matching \( \sigma : \mathcal{A} \to \mathcal{B} \). For the sake of simplicity, we would regard \( \sigma \) as a matching between index sets \( A \) and \( B \) of \( \mathcal{A} \) and \( \mathcal{B} \) respectively by identifying elements in \( \mathcal{A} \) and \( \mathcal{B} \) to their indexes. First, we define
\[
\text{coim}(\sigma) = A^2_{0} \cup A^2_{x} \cup g(B^2_{0}), \quad \text{and} \quad \text{im(}\sigma\text{)} = f(A^2_{0}) \cup B^2_{x} \cup B^2_{0}.
\]
Then, \( \text{coim}(\sigma) \supseteq A^2_{x} \) and \( \text{im}(\sigma) \supseteq B^2_{x} \). Now, define \( \sigma : A \to B \) as follows:
\[
\sigma(i) = \begin{cases} 
  f(i) & \text{if } i \in A^2_{x} = A^2_{x} \cup A^2_{0} \\
  g^{-1}(i) & \text{if } i \in g(B^2_{x}).
\end{cases}
\]
By Claim E.8, \( \sigma \) is well-defined. The following diagram depicts the construction of the matching:

\[
\begin{array}{ccc}
A^2_{0} & \cup & A^2_{x} & \cup & g(B^2_{0}) \\
| & f & | & f & | & g \\
| & | & | & | & | & g \\
| & | & | & | & | & \\
| & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
| & f(A^2_{0}) & \cup & B^2_{x} & \cup & B^2_{0} \\
\end{array}
\]

It remains to show that \( \|I_i - J_{\sigma(i)}\|_{\infty} \leq \eta \), i.e., \( I_i \subseteq (J_{\sigma(i)})^{\eta} \) and \( J_{\sigma(i)} \subseteq (I_i)^{\eta} \) for all \( i \in \text{coim}(\sigma) \). Recall that \( I_i = \langle a_i, b_i \rangle \) and \( J_j = \langle c_j, d_j \rangle \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).
Case 1. Pick $i \in A_{0}^{2\eta}$ and let $\sigma(i) = f(i) = j$ so that length $I_i \geq 2\eta$ whereas length $J_j < 2\eta$. We wish to verify that $I_i \subseteq (J_j)^{\eta}$ and $J_j \subseteq (I_i)^{\eta}$. But, the first inclusion follows automatically from the definition of $f$ and this implies that (1) $a_i \geq c_j - \eta$ and (2) $b_i \leq d_j + \eta$. So we are going to prove $J_j \subseteq (I_i)^{\eta}$ only, which amounts to show that (3) $c_j \geq a_i - \eta$ and (4) $d_j \leq b_i + \eta$. Suppose that (3) is false, i.e., $c_j < a_i - \eta$. Then we have

$$d_j = c_j + \text{length} J_j$$
$$< c_j + 2\eta$$
$$< a_i + \eta$$
$$\therefore c_j < a_i - \eta$$
$$\leq b_i - \eta$$
$$\therefore a_i = b_i - \text{length} I_i \leq b_i - 2\eta.$$

This contradicts (2) and thus (3) must hold. Similarly, the negation of (4) deduce the contradiction to the inequality (1) and thus both (3) and (4) should hold as desired. This strategy works for the case of $i \in g(B_0^{2\eta})$ as well since $g$ has the same property as $f$.

Case 2. Pick $i \in A_{0}^{2\eta}$ and let $\sigma(i) = f(i) = j$. Again by the definition of $f$, we know $I_i \subseteq (J_j)^{\eta}$. Further, $J_j \subseteq (I_{g(j)})^{\eta}$ by the definition of $g$ but recalling $g(j) = g(f(i)) = i$ by Claim E.7 we have $\| I_i - J_{g(i)} \|_\infty \leq \eta$.

Assuming $\eta \geq d_H(\mathcal{C}(\mathcal{A}), \mathcal{C}(\mathcal{B}))$ for some $\eta > 0$, we have constructed $\eta$-matching between $\mathcal{A}$ and $\mathcal{B}$. Therefore, we have inequality $d_H(\mathcal{C}(\mathcal{A}), \mathcal{C}(\mathcal{B})) \geq d_B(\mathcal{A}, \mathcal{B})$ as desired.

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