Algebraic Techniques for Gaussian Models

Mathias Drton

Abstract: Many statistical models are algebraic in that they are defined by polynomial constraints or by parameterizations that are polynomial or rational maps. This opens the door for tools from computational algebraic geometry. These tools can be employed to solve equation systems arising in maximum likelihood estimation and parameter identification, but they also permit to study model singularities at which standard asymptotic approximations to the distribution of estimators and test statistics may no longer be valid. This paper demonstrates such applications of algebraic geometry in selected examples of Gaussian models, thereby complementing the existing literature on models for discrete variables.

MSC 2000: 62H05, 62H12
Key words: Algebraic statistics, multivariate normal distribution, parameter identification, singularities

1 Introduction

Algebraic statistics applies algebraic geometry to gain insight in structure and properties of statistical models, and to tackle computational problems arising in tasks of statistical inference. Work in this field has addressed, for example, exact tests in contingency tables, experimental design, phylogenetic trees, maximum likelihood estimation under multinomial sampling, and Bayesian networks; cf. [8, 9]. Algebraic geometry typically enters the playing field in one of two ways. On one hand, statistical models are sometimes derived from a simple saturated model by imposing constraints. These constraints may, in particular, be motivated by considerations of (conditional) independence, stationarity or homogeneity. If the constraints are polynomial constraints on the parameters of the saturated model, then the model corresponds to the intersection of an algebraic variety and the saturated parameter space. An algebraic variety is the solution set of a system of polynomial equations. On the other hand, many statistical models are defined via a parameterization rather than via constraints. However, if this parameterization is a polynomial, or more generally a rational map, then the model, which can be identified with the image of the parameterization map, is naturally embedded in an algebraic variety. This algebraic description of the model is often useful because it can reveal insights about the model that are not as readily obtained from the parameterization alone.

Virtually all work in algebraic statistics considers purely discrete variables; see [9] for an exception. The sample space is then finite, and the objects of interest are algebraic varieties over a probability simplex. However, the philosophy of algebraic

Research supported by the US National Science Foundation (DMS-0505612).
statistics applies, regardless of the distributional setting, whenever a statistical model is an “algebraic” submodel of some natural supermodel. A particularly interesting case occurs when the supermodel is a regular exponential family, because algebraic submodels may inherit desirable statistical properties at points at which the submodel’s local geometry is sufficiently “regular.”

Discussing simple problems from parameter identification and likelihood ratio testing, this paper demonstrates algebraic techniques for Gaussian models, i.e., families of (non-singular) multivariate normal distributions. Section 2 reviews the normal distribution and introduces the algebraic point of view. Section 3 treats the problem of identification of a graphical model with hidden variables. Section 4 is devoted to model singularities, at which standard $\chi^2$-approximations to the distribution of the likelihood ratio test statistic may no longer be valid.

2 Algebraic Gaussian models

Let $\mathbb{R}^{p \times p}_{pd}$ and $\mathbb{R}^{p \times p}_{psd}$ be the cones of positive definite and positive semi-definite symmetric $p \times p$-matrices, respectively. The multivariate normal distribution $\mathcal{N}_p(\mu, \Sigma)$ with mean vector $\mu = (\mu_1, \ldots, \mu_p)^t \in \mathbb{R}^p$ and covariance matrix $\Sigma = (\sigma_{ij}) \in \mathbb{R}^{p \times p}_{pd}$ is the probability distribution on $\mathbb{R}^p$ that has Lebesgue density function

$$f_{\mu, \Sigma}(x) = \frac{1}{\sqrt{(2\pi)^p \det(\Sigma)}} \exp\left\{ -\frac{1}{2} (x - \mu)^t \Sigma^{-1} (x - \mu) \right\}, \quad x \in \mathbb{R}^p.$$  

A Gaussian (statistical) model with mean parameter space $M \subseteq \mathbb{R}^p \times \mathbb{R}^{p \times p}_{pd}$ is the family of multivariate normal distributions $\{\mathcal{N}_p(\mu, \Sigma) \mid (\mu, \Sigma) \in M\}$.

**Proposition 2.1** ([1, p. 194]). The saturated Gaussian model, that is, the family of all multivariate normal distributions on $\mathbb{R}^p$, which has mean parameter space $M = \mathbb{R}^p \times \mathbb{R}^{p \times p}_{pd}$, forms a regular exponential family with sufficient statistics $x \in \mathbb{R}^p$ and $xx^t \in \mathbb{R}^{p \times p}_{psd}$. The natural parameters are $\Sigma^{-1} \mu \in \mathbb{R}^p$ and $\Sigma^{-1} \in \mathbb{R}^{p \times p}_{pd}$.

Statistical modelling in the Gaussian framework involves hypotheses about structural relationships among the components of the mean parameters $\mu$ and $\Sigma$. In many interesting cases, such a relationship comes from a parameterization. If this parameterization is rational, as detailed in the following definition, then the resulting model can be studied taking an algebraic point of view. Recall that a set $\Theta \subseteq \mathbb{R}^d$ is semi-algebraic if it is a union of sets of points satisfying polynomial equalities and inequalities; compare Chapter 2 in [2].

**Definition 2.2.** A Gaussian model is a parametric algebraic model if its mean parameter space $M = f(\Theta)$, where $\Theta \subseteq \mathbb{R}^d$ is an open semi-algebraic set, and

$$f : \Theta \rightarrow \mathbb{R}^p \times \mathbb{R}^{p \times p}_{pd}$$

$$\theta \mapsto \left( \frac{g_1}{h_1}(\theta), \ldots, \frac{g_p}{h_p}(\theta), \frac{g_{11}}{h_{11}}(\theta), \ldots, \frac{g_{pp}}{h_{pp}}(\theta) \right)$$
is a rational map defined everywhere on $\Theta$. In other words, the functions $g_k, h_k, g_{ij}$ and $h_{ij}$ are polynomial functions such that $0 \notin h_k(\Theta)$ for all $k \in [p] := \{1, \ldots, p\}$ and $0 \notin h_{ij}(\Theta)$ for all $(i, j) \in [p]^2$.

Not all statistical models of interest are specified in terms of a parameterization; instead the model may be specified implicitly in the form of constraints on the mean parameters $\mu$ and $\Sigma$. One important class of constraints arises from conditional independence, which in the multivariate normal distribution corresponds to well-known polynomial conditions on the covariance matrix $\Sigma$.

**Proposition 2.3.** Let $X$ be a random vector in $\mathbb{R}^p$ that follows a multivariate normal distribution $N_p(\mu, \Sigma)$, in symbols, $X \sim N_p(\mu, \Sigma)$. For three pairwise disjoint index sets $A, B, C \subseteq [p] := \{1, \ldots, p\}$, it holds that

$$X_A \perp \perp X_B \mid X_C \iff \det(\Sigma_{(i) \cup C \times (j) \cup C}) = 0 \forall i \in A, j \in B.$$  

Here, $X_A \perp \perp X_B \mid X_C$ denotes conditional independence of $X_A$ and $X_B$ given $X_C$, and $X_A \perp \perp X_B \mid X_\emptyset$ denotes marginal independence of $X_A$ and $X_B$.

The fact that conditional independence is an algebraic condition motivates the next definition, in which $\mathbb{R}[\mu_k, \sigma_{ij} \mid i, j, k \in [p], i \leq j]$ denotes the ring of polynomials in the entries $\mu_k$ and $\sigma_{ij}$ of the mean vector $\mu$ and the covariance matrix $\Sigma$.

**Definition 2.4.** A Gaussian model is an implicit algebraic model if its mean parameter space $M$ is equal to the intersection of an algebraic variety $V$ and the Cartesian product $\mathbb{R}^p \times \mathbb{R}^{pd}$. In other words, there exist polynomials $f_1, \ldots, f_t \in \mathbb{R}[\mu_k, \sigma_{ij} \mid i, j, k \in [p], i \leq j]$ such that

$$M = \left\{ (\mu, \Sigma) \in \mathbb{R}^p \times \mathbb{R}^{pd} \mid f_1(\mu, \Sigma) = \cdots = f_t(\mu, \Sigma) = 0 \right\}.$$

The next fact is a consequence of the Tarski-Seidenberg Theorem [2, Thm. 2.3.4].

**Proposition 2.5.** The mean parameter space of an algebraic Gaussian model, parametric or implicit, is a semi-algebraic set.

An immediate consequence of Proposition 2.5 is that an algebraic Gaussian model always has a well-defined dimension, namely, the dimension $\dim(M)$ of the semi-algebraic mean parameter space [2, Def. 2.5.3]. In the parametric case, $\dim(M) = \dim(f(\Theta))$ can be determined as the maximal rank of the Jacobian of the rational parameterization map $f$. In the implicit case, $\dim(M)$ can be computed based on Gröbner basis techniques [8, Thm. 3.7]. An implicit algebraic model need not be a parametric algebraic model, and vice versa. Nevertheless, a technique known as implicitization, cf. [3, §3.3] and [8, §3.2], permits to find a unique smallest implicit model whose mean parameter space $M$ contains the mean parameter space $f(\Theta)$ of a given parametric model while satisfying $\dim(M) = \dim(f(\Theta))$. 


3 Identifiability

When specifying a parametric statistical model, one of the first concerns is whether the model is identifiable, that is, whether the parameters uniquely specify probability distributions in the model.

**Definition 3.1.** Consider a parametric Gaussian model with mean parameter space $M = f(\Theta)$ given as the image of a map $f : \Theta \rightarrow \mathbb{R}^p \times \mathbb{R}^{pd}$. The model is

(i) *globally identifiable* at $\theta_0 \in \Theta$ if $f^{-1}(f(\theta_0)) = \{\theta_0\}$;

(ii) *locally identifiable* at $\theta_0 \in \Theta$ if there exists a ball $B_\varepsilon(\theta_0)$ with center $\theta_0$ and radius $\varepsilon > 0$ such that $f^{-1}(f(\theta_0)) \cap B_\varepsilon(\theta_0) = \{\theta_0\}$.

If the model is globally identifiable at all points in $\Theta$, that is, if the map $f : \Theta \rightarrow M$ is a bijection, then we say that the model is *identifiable*.

For parametric algebraic models, global and local identifiability at a given point $\theta_0 \in \Theta$ can be investigated by studying whether a system of polynomial equations deduced from the possibly rational equation system $f(\theta_0) = f(\theta)$ has a (locally) unique solution $\theta \in \Theta$. We illustrate this in the following example.

Consider the directed graphical Gaussian model with one hidden variable depicted in Figure 1, which shows the relationship between $p = 4$ observed variables (shaded nodes) and one hidden variable $H$. For simplicity we consider the model comprising only centered distributions. This model is a parametric algebraic model with mean parameter space $M = f(\Theta)$ given by a polynomial map.

The points in the parameterization domain $\Theta = \mathbb{R}^4 \times (0, \infty)^4$ are vectors $\theta = (\beta_1, \beta_2, \beta_3, \beta_4, \omega_1, \omega_2, \omega_3, \omega_4)$. Here, the four regression coefficients $\beta_i \in \mathbb{R}$ appear in conditional means, namely $E[H \mid X_1, X_2] = \beta_1 X_1 + \beta_2 X_2$ and $E[X_i \mid H] = \beta_i H$ if $i = 3, 4$. The variances $\omega_i > 0$ are either marginal or conditional variances, $\omega_i = \text{Var}[X_i]$ for $i = 1, 2$, and $\omega_i = \text{Var}[X_i \mid H]$ for $i = 3, 4$. The parameterization map is

$$f : \Theta \rightarrow \mathbb{R}^4 \times \mathbb{R}^{4 \times 4}$$

$$\theta \mapsto [0, f_C(\theta)],$$

Figure 1: Acyclic digraph for a hidden variable model.
where \( f_\Sigma(\theta) \) is the symmetric covariance matrix

\[
\begin{pmatrix}
\omega_1 & 0 & \beta_3 \beta_1 \omega_1 & \beta_4 \beta_1 \omega_1 \\
\omega_2 & \beta_3 \beta_2 \omega_2 & \beta_4 \beta_2 \omega_2 \\
\omega_3 + \beta_3^2 (\beta_1^2 \omega_1 + \beta_2^2 \omega_2 + 1) & \beta_4 \beta_3 \beta_1 \omega_1 + \beta_2^2 \omega_2 + 1) & \omega_4 + \beta_1^2 (\beta_1^2 \omega_1 + \beta_2^2 \omega_2 + 1)
\end{pmatrix}.
\]

(1)

Note that we set the conditional variance \( \text{Var}[H \mid X_1, X_2] = 1 \) because the image of the parameterization map remains unchanged if a free parameter \( f \) or this variance is introduced. For details on such parameterizations see e.g. [10, §8].

Is this parametric hidden variable model globally (or locally) identifiable at \( \theta_0 \in \Theta \)? We can answer this question by studying the system of equations \( f_\Sigma(\theta_0) = f_\Sigma(\theta) \) in which the components \( \beta_i \) and \( \omega_i \) of \( \theta_0 \) are fixed numbers and the components \( \beta_i \) and \( \omega_i \) of \( \theta \) are indeterminants. From (1), it is apparent that if \( f(\theta) = f(\theta_0) \) then \( \omega_1 = \omega_{10} \) and \( \omega_2 = \omega_{20} \). Additional consequences can be worked out by hand, but we can also let the computer do this for us.

Running the code in Table 1 in the computer algebra system Singular [7] informs us that the model is not globally identifiable at generic \( \theta_0 \in \Theta \) because the solution set \( f^{-1}(f(\theta_0)) \) generally contains \( \text{mult}(J) = 2 \) isolated points (\( \text{dim}(J) = 0 \)). The computed Gröbner basis (see [3, 8] for the relevant background)

\[
> J;
\]

\[
J[1]=w4+(-w40)
J[2]=w3+(-w30)
J[3]=w2+(-w20)
J[4]=w1+(-w10)
J[5]={(b40)*b3+(-b30)*b4}
J[6]=(-b40)*b2+(b20)*b4
J[7]=(-b40)*b1+(b10)*b4
J[8]=b4^2+(-b40^2)
\]

suggests that for \( \beta_{40} \neq 0 \), it holds that

\[
f^{-1}(f(\theta_0)) = \{(\beta_{10}, \ldots, \beta_{40}, \omega_{10}, \ldots, \omega_{40}), (-\beta_{10}, \ldots, -\beta_{40}, \omega_{10}, \ldots, \omega_{40})\}.
\]

(2)

However, in the Gröbner basis computation simplifications are made that are only valid if \( b10, \ldots, w40 \) are generic. In other words, during the computation a (finite) number of polynomial expressions in \( b10, \ldots, w40 \) are assumed to be non-zero. So while we can conclude that [2] holds for almost every \( \theta_0 \in \Theta \), it may and does fail at certain points in the parameter domain \( \Theta \). For example, [2] does not hold if \( \beta_{40} = 0 \), in which case it is possible that \( \text{dim}(J) \in \{1, 2\} \) indicating failure of local identifiability. In conclusion, the computation shows that the model is locally identifiable at almost every \( \theta_0 \in \Theta \). In more complicated models, computations treating \( \theta_0 \) as symbolic quantity may become prohibitive. However, solving the system \( f(\theta_0) = f(\theta) \) for a particular numeric vector \( \theta_0 \) may still be feasible and informative.
LIB "linalg.lib"; option(redSB);
ring R = (0,b10,b20,b30,b40,w10,w20,w30,w40),
      (b1,b2,b3,b4,w1,w2,w3,w4),dp;
// b1,...,w4 are indeterminants; b10,...,w40 are symbolic parameters
matrix B[5][5] = 1,0,0,0,0,
                  0,1,0,0,0,
                  0,0,1,0,-b3,
                  0,0,0,1,-b4,
                  -b1,-b2,0,0,1;
matrix W[5][5] = w1,0,0,0,0,
                0,w2,0,0,0,
                0,0,w3,0,0,
                0,0,0,w4,0,
                0,0,0,0,1;
matrix B0[5][5] = 1,0,0,0,0,
                 0,1,0,0,0,
                 0,0,1,0,-b30,
                 0,0,0,1,-b40,
                -b10,-b20,0,0,1;
matrix W0[5][5] = w10,0,0,0,0,
                0,w20,0,0,0,
                0,0,w30,0,0,
                0,0,0,w40,0,
                0,0,0,0,1;
matrix f[4][4] = submat(inverse(B)*W*inverse(transpose(B)),1..4,1..4);
matrix f0[4][4] = submat(inverse(B0)*W0*inverse(transpose(B0)),1..4,1..4);
ideal I=0; int i,j;
for(i=1; i<=4; i++) { for(j=i; j<=4; j++){
    I = I + ideal(f0[i,j]-f[i,j]); // identifiability equations
} }
ideal J = std(I); // Groebner basis for ideal I
dim(J); mult(J);

Table 1: Code from session in computer algebra system Singular.

4 Singularities of Gaussian models

The saturated Gaussian model is a regular exponential family (Proposition 21). Therefore, under “regularity” conditions, results about asymptotic distributions of MLE and likelihood ratio test statistic can be transferred to submodels. Suppose the submodel is an algebraic model with mean parameter space $M$ containing the true distribution $\mathcal{N}(\mu_0, \Sigma_0)$. If $M$ is a smooth manifold, in which case the submodel is a curved exponential family, then regardless of where the true parameter $(\mu_0, \Sigma_0)$ is located, the MLE is asymptotically normal as the sample size tends to infinity. Moreover, the likelihood ratio test statistic for testing the submodel against the
saturated model has an asymptotic $\chi^2$-distribution with degrees of freedom equal to the codimension of $M$, that is, the difference between the dimension of the saturated mean parameter space and $\dim(M)$.

These standard results need no longer be true if one leaves the realm of curved exponential families. For example, if inequality constraints are imposed on the mean parameter space of a curved exponential family, boundary effects may be created. More subtly, the regularity conditions may be violated at points that are “singularities” in the mean parameter space of an algebraic Gaussian model. For a rigorous definition of singularities of algebraic varieties, see e.g. [2, §3.2]. In the examples in this section the singularities are obvious and intuitive. However, this will not necessarily be the case in larger models, in which case computer algebra software is very helpful for locating singular points. In particular, the software Singular offers the command slocus for computation of singular loci.

4.1 Simple bivariate examples under independence

Issues with singularities can be illustrated nicely with bivariate normal distributions. For a closed set $C \subseteq \mathbb{R}^2$, let $M_C = C \times \{I_2\}$ be the mean (and natural) parameter space of the model of all bivariate normal distributions with mean vector $\mu \in C$ and covariance matrix $\Sigma$ equal to the identity matrix $I_2 \in \mathbb{R}^{2 \times 2}$. In this case the MLE $\hat{\mu}$ for the model with mean parameter space $M_C$ is the point in $C$ that is closest to the sample mean vector $\bar{X}$ in Euclidean distance. The likelihood ratio test statistic $\lambda_C$ for testing $\mu \in M_C$ versus $\mu \in M_{\mathbb{R}^2}$ is equal to the product of the sample size $n$ and the squared Euclidean distance of $\bar{X}$ and $C$.

Example 4.1 (Folium of Descartes). Let $C = \{\mu \in \mathbb{R}^2 \mid \mu_2^3 = \mu_1^3 + \mu_1^2\}$, which is a curve that can be parameterized as $f(\theta) = [\theta^2 - 1, \theta(\theta^2 - 1)]$. The curve is shown in the left plot in Figure 2. The algebraic model with mean parameter space $M_C$ is not a curved exponential family due to the singularity at the point of self-intersection, which is $\mu = 0$. The dashed lines $\mu_2 = \pm \mu_1$ in the plot are orthogonal to each other and indicate the tangent cone at $\mu = 0$. If the true parameter point is $\mu = 0$, then the asymptotic distribution of the likelihood ratio test statistic $\lambda_C$ is
given by the squared Euclidean distance between a draw from \( N(0, I_2) \) and the two orthogonal lines. This asymptotic distribution is the distribution of the minimum of two independent \( \chi^2_1 \)-random variables.

**Example 4.2 (Neil’s parabola).** Let \( C = \{ \mu \in \mathbb{R}^2 \mid \mu_2^2 = \mu_1^3 \} \) be the curve with parameterization \( f(\theta) = (\theta^2, \theta^3) \), which is shown in the right-most picture of Figure 2. The algebraic model with mean parameter space \( M_C \) is again not a curved exponential family due to the singularity at the cusp point \( \mu = 0 \). For true parameter point \( \mu = 0 \), the likelihood ratio test statistic \( \lambda_C \) has an asymptotic distribution that is the mixture of a \( \chi^2_1 \)- and a \( \chi^2_2 \)-distribution. This mixture distribution is the distribution of the squared Euclidean distance between a draw from \( N(0, I_2) \) and the (dashed) half-ray \( \{ \mu \mid \mu_1 \geq 0, \mu_2 = 0 \} \).

Examples 4.1 and 4.2 demonstrate non-standard asymptotics at model singularities. At regular points in the respective mean parameter spaces the usual asymptotics apply. However, if the true parameter forms a regular point that is close to the singular locus then a very large sample size may be required in order for the usual asymptotics to provide good approximations to the distributions of estimators and test statistics.

### 4.2 A conditional independence model with singularities

Many conditional independence models, in particular graphical models, form curved exponential families. However, singularities may arise from combining arbitrary independence constraints. Consider, for example, the model of trivariate normal distributions under which a random vector satisfies \( X_1 \perp \perp X_2 \) and simultaneously \( X_1 \perp \perp X_2 \mid X_3 \). By Proposition 2.3, the model is an implicit algebraic model with mean parameter space

\[
M = \left\{ (\mu, \Sigma) \in \mathbb{R}^3 \times \mathbb{R}^{3 \times 3}_{\text{pd}} \mid \sigma_{12} = 0, \det(\Sigma_{\{1,3\} \times \{2,3\}}) = \sigma_{12}\sigma_{33} - \sigma_{13}\sigma_{23} = 0 \right\}
\]

The set \( M \) is defined by the joint vanishing of the two polynomials \( f_1 = \sigma_{12} \) and \( f_2 = \sigma_{13}\sigma_{23} \). We see that

\[
M = M_{13} \cup M_{23} := \{ (\mu, \Sigma) \in M \mid \sigma_{12} = \sigma_{13} = 0 \} \cup \{ (\mu, \Sigma) \in M \mid \sigma_{12} = \sigma_{23} = 0 \}.
\]

This reflects the well-known fact that

\[
[X_1 \perp \perp X_2 \land X_1 \perp \perp X_2 \mid X_3] \iff [X_1 \perp \perp (X_2, X_3) \lor X_2 \perp \perp (X_1, X_3)]
\]

which also holds for distributions other than the multivariate normal; compare [4, Thm. 8.3]. The singular locus of \( M \) is the intersection

\[
M_{\text{sing}} = M_{13} \cap M_{23} = \{ (\mu, \Sigma) \in M \mid \sigma_{12} = \sigma_{13} = \sigma_{23} = 0 \},
\]

which corresponds to diagonal \( \Sigma \), i.e., complete independence \( X_1 \perp \perp X_2 \perp \perp X_3 \).
The likelihood ratio test statistic for testing the model with mean parameter space $M$ against the saturated model can be expressed as

$$\lambda = n \cdot \min \left\{ \log \left( \frac{s_{11} \det(S_{(2,3) \times (2,3)})}{\det(S)} \right), \log \left( \frac{s_{22} \det(S_{(1,3) \times (1,3)})}{\det(S)} \right) \right\}. \quad (3)$$

If $(\mu, \Sigma) \in M \setminus M_{\text{sing}}$, then $\lambda$ converges to a $\chi^2$-distribution for $n \to \infty$; over the singular locus the limiting distribution is non-standard.

**Proposition 4.3.** Let $(\mu, \Sigma) \in M_{\text{sing}}$, and let $W_{12}$, $W_{13}$, $W_{23}$ be three independent $\chi^2_1$-random variables. As $n \to \infty$, the likelihood ratio test statistic $\lambda$ converges to the minimum of two dependent $\chi^2_2$-distributed random variables, namely,

$$\lambda \overset{d}{\longrightarrow} \min(W_{12} + W_{13}, W_{12} + W_{23}) = W_{12} + \min(W_{13}, W_{23}).$$

**Proof.** For $i \in \{1, 2, 3\}$ and $A \subset \{1, 2, 3\}$, let

$$s_{ii.A} = s_{ii} - S_{\{i\} \times A}S_{A \times A}^{-1}S_{A \times \{i\}}.$$

The likelihood ratio test statistic can be rewritten as

$$\lambda = n \log \left( \frac{s_{11}s_{22}}{s_{11}s_{22} - s_{12}^2} \right) + n \cdot \min \left\{ \log \left( \frac{s_{33,2}}{s_{33,12}} \right), \log \left( \frac{s_{33,12}}{s_{33,12}} \right) \right\}. \quad (4)$$

Recall that

$$\sqrt{n} \left[ (s_{11}, s_{12}, s_{13}, s_{22}, s_{23}, s_{33})^t - (\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33})^t \right] \overset{d}{\longrightarrow} \mathcal{N}(0, \sigma_{ik}\sigma_{jm} + \sigma_{im}\sigma_{jk})_{ij,km}. \quad (5)$$

Since $(\mu, \Sigma) \in M_{\text{sing}}$ implies that $\Sigma$ is diagonal, the covariance matrix of the normal distribution in (5), known as the Isserlis matrix of $\Sigma$, is diagonal. Using an expansion up to the Hessian in the delta-method [11, §3.3], we can show that the three logarithmic terms in (4) converge to three independent $\chi^2_1$-random variables. \qed

## 5 Conclusion

The goal of this paper was to demonstrate the usefulness of algebraic geometry for studying properties of statistical (Gaussian) models. In order to keep intuition alive, the examples in this paper were chosen to be rather simple, but algebraic geometry can also provide useful insights in larger, less tractable models.

Two particular problems were visited in this paper. First, parameter identifiability often gives rise to polynomial equation systems, the structure of which becomes more transparent when the equations are presented in Gröbner basis form (Section 3). Second, model singularities can result into non-standard asymptotics (Section 4). Locating singularities and working out the associated asymptotics are
the first steps towards solving the challenging problem of divising sensible statistical procedures for models with singularities. Finally, we remark that methods combining Gröbner basis techniques with numerical solving can also be used to compute all solutions to interesting likelihood equations, compare e.g. [5].

References

[1] T. W. Anderson. *An Introduction to Multivariate Statistical Analysis*, 3rd ed. Wiley, 2003.

[2] R. Benedetti and J.-J. Risler. *Real Algebraic and Semi-algebraic Sets*. Actualités Mathématiques, Hermann, Paris, 1990.

[3] D. Cox, J. Little, and D. O’Shea. *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 2nd ed. Springer-Verlag, New York, 1997.

[4] A. P. Dawid. Conditional independence for statistical operations. *Annals of Statistics*, 8:598–617, 1980.

[5] M. Drton. Computing all roots of the likelihood equations of seemingly unrelated regressions. *Journal of Symbolic Computation*, 41:245–254, 2006.

[6] M. Drton, B. Sturmfels, and S. Sullivant. Algebraic factor analysis: Tetrads, pentads and beyond. [arXiv:math.ST/0509390](http://arxiv.org/abs/math.ST/0509390) 2005.

[7] G.-M. Greuel and G. Pfister. *A Singular Introduction to Commutative Algebra*. Springer-Verlag, New York, 2002.

[8] L. Pachter and B. Sturmfels. *Algebraic Statistics for Computational Biology*. Cambridge University Press, 2005.

[9] G. Pistone, E. Riccomagno, and H. P. Wynn. *Algebraic Statistics. Computational Commutative Algebra in Statistics*. Chapman & Hall, 2001.

[10] T. S. Richardson and P. Spirtes. Ancestral graph Markov models. *Annals of Statistics*, 30:962–1030, 2002.

[11] A. W. van der Vaart. *Asymptotic Statistics*. Cambridge University Press, 1998.

Mathias Drton: The University of Chicago, Department of Statistics, 5734 S. University Ave, Chicago, Illinois, 60637, U.S.A., drton@galton.uchicago.edu