A NON-DIAGONALIZABLE PURE STATE

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ABSTRACT. We construct a pure state on the C*-algebra $\mathcal{B}(\ell_2)$ of all bounded linear operators on $\ell_2$ which is not diagonalizable, i.e., it is not of the form $\lim_u (T(e_k), e_k)$ for any orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of $\ell_2$ and an ultrafilter $u$ on $\mathbb{N}$. This constitutes a counterexample to Anderson’s conjecture without additional hypothesis and improves results of C. Akemann, N. Weaver, I. Farah and I. Smythe who constructed such states making additional set-theoretic assumptions.

It follows from results of J. Anderson and the positive solution to the Kadison-Singer problem due to A. Marcus, D. Spielman, N. Srivastava that the restriction of our pure state to any atomic masa $D((e_k)_{k \in \mathbb{N}})$ of diagonal operators with respect to an orthonormal basis $(e_k)_{k \in \mathbb{N}}$ is not multiplicative on $D((e_k)_{k \in \mathbb{N}})$.

1. Introduction

Recall that a pure state on a C*-algebra is a positive linear functional of norm one, i.e., a state, which is not a convex combination of other states. Pure states on the algebras of all operators on finite dimensional Hilbert spaces $\mathcal{B}(\ell^2(n))$ for $n \in \mathbb{N}$ are known all to be vector states, i.e., of the form $\phi(T) = \langle T(v), v \rangle$, where $v \in \ell^2(n)$ is a unit vector. Vector states are also pure states in the case of the algebra $\mathcal{B}(\ell_2)$ of all linear bounded operators on an infinite dimensional Hilbert space $\ell_2$.

There are many other pure states on $\mathcal{B}(\ell_2)$ whose existence is usually proved by means of the Hahn-Banach theorem starting from a pure state on a maximal abelian self-adjoint subalgebra (masa) of $\mathcal{B}(\ell_2)$. If the masa is atomic, that is of the form $D((e_k)_{k \in \mathbb{N}})$ of all diagonal operators with respect to an orthonormal basis $(e_k)_{k \in \mathbb{N}}$, then the general form of the initial pure state $\phi$ for $T \in D((e_k)_{k \in \mathbb{N}})$ is

$$\lim_u (T(e_k), e_k)$$

where $u$ is an ultrafilter on $\mathbb{N}$. J. Anderson showed in [3] that (D) defines a pure state on the entire $\mathcal{B}(\ell_2)$ and conjectured in what became known as Anderson’s conjecture [5] that every pure state on $\mathcal{B}(\ell_2)$ is of the above form for some orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of $\ell_2$ and an ultrafilter $u$ on $\mathbb{N}$. Our main result (Theorem 12) is the construction of a pure state that is non-diagonalizable, that is a counterexample to Anderson’s conjecture.

Much of the research concerning the relations between pure states on $\mathcal{B}(\ell_2)$ and pure states on masas of $\mathcal{B}(\ell_2)$ has been motivated by a seminal paper [9] of Kadison and Singer. The positive solution of one of the problems stated in the paper and known as the Kadison-Singer problem due to A. Marcus, D. Spielman, N.

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Srivastava implies that a non-diagonalizable pure state on $B(\ell_2)$ necessarily cannot have multiplicative restriction to any atomic masa (because such restrictions extend to pure states on $B(\ell_2)$ of the form (D) but the extensions are unique by the positive solution to the Kadison-Singer problem).

Another problem from the paper [9] is whether any pure state on $B(\ell_2)$ has a multiplicative restriction to some masa of $B(\ell_2)$. Having multiplicative restriction in this case is equivalent to having the restriction equal to a pure state on the masa. In [1] C. Akemann and N. Weaver provided a negative solution to this problem assuming the continuum hypothesis $\textup{CH}$. This, in particular, already showed that Anderson’s conjecture is consistently false, but as suggested in [1] it could still be consistent that any pure state on $B(\ell_2)$ has a multiplicative restriction to a masa. This additional hypothesis in the case of Anderson’s conjecture was weakened to $\textup{MA}([14])$ or to $\textup{cov}(\mathcal{M}) = \mathfrak{c}$ or to $\mathfrak{d} \leq p^*$ (12.5 of [7]), or to another one in [13].

Our counterexample to Anderson’s conjecture shows that the additional hypothesis in the result of Akemann and Weaver is not needed when we limit ourself to atomic masas. However, we do not know if our non-diagonalizable pure state can have multiplicative restriction to any atomic masa (because such restrictions extend to pure states on $B(\ell_2)$ of the form (D) but the extensions are unique by the positive solution to the Kadison-Singer problem).

Our construction is entirely different than that of Akemann and Weaver which used properties of separable $C^*$-subalgebras of $B(\ell_2)$ and a well-ordering of all masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments. Let us describe the main idea of our construction here. In a sense, instead of using separable approximations we obtain masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments. Let us describe the main idea of our construction here. In a sense, instead of using separable approximations we obtain masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments. Let us describe the main idea of our construction here. In a sense, instead of using separable approximations we obtain masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments. Let us describe the main idea of our construction here. In a sense, instead of using separable approximations we obtain masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments. Let us describe the main idea of our construction here. In a sense, instead of using separable approximations we obtain masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments. Let us describe the main idea of our construction here. In a sense, instead of using separable approximations we obtain masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments. Let us describe the main idea of our construction here. In a sense, instead of using separable approximations we obtain masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments. Let us describe the main idea of our construction here. In a sense, instead of using separable approximations we obtain masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments. Let us describe the main idea of our construction here. In a sense, instead of using separable approximations we obtain masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments. Let us describe the main idea of our construction here. In a sense, instead of using separable approximations we obtain masas in the first uncountable type $\omega_1$ based on the continuum hypothesis to approximate the desired pure state with separable fragments.
for $m \in \mathbb{N}$ defining
\[ P_{\alpha, \nu} = \bigoplus_{m \in \mathbb{N}} (I \otimes \ldots \otimes R_{x_m} \otimes \ldots \otimes I). \]

Under the identifications $(I_1)$ and $(I_2)$ the operator $P_{\alpha, \nu}$ is a projection in $\mathcal{B}(\ell_2)$. It follows from $(P)$ that for any choices $\nu^\alpha$ for $\alpha \in \{0, 1\}^N$ any finite product formed by the projections $(P_{\alpha, \nu} : \alpha \in \{0, 1\}^N)$ dominates a nonzero projection because eventually $\alpha_1|m, \ldots, \alpha_m|m \in \{0, 1\}^m$ are all distinct if $\alpha_1, \ldots, \alpha_m \in \{0, 1\}^N$ are distinct. This guarantees that for any choices $\nu^\alpha$ for $\alpha \in \{0, 1\}^N$ there is a pure state $\phi$ on $\mathcal{B}(\ell_2)$ such that $\phi(P_{\alpha, \nu^\alpha}) = 1$ for all $\alpha \in \{0, 1\}^N$.

To make sure that $\phi$ is not diagonalized by any orthonormal basis we need to show that there is a constant $0 < c < 1$ such that for every orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of $\ell_2$ there is $\alpha \in \{0, 1\}^N$ such that
\begin{equation}
|\langle P_{\alpha, \nu^\alpha}(e_k), e_k \rangle| < c
\end{equation}
for every $k \in \mathbb{N}$. To obtain the above property we manipulate the choice of $\nu^\alpha$. Here we exploit the fact that if $f(d)$ points on $d$-dimensional real sphere form an $\varepsilon$-net on the sphere for $\varepsilon < 1$, then $f(d)$ must grow exponentially in the dimension $d \in \mathbb{N}$. Using this with the choice of $d$ satisfying for each $m \in \mathbb{N}$ (i) $d(m) \geq 2^m$, (ii) $32m^3(d(m)(2^m))^2d(m)(2^m-1) < (100/91)^{d(m)}$ we can obtain $\nu^\alpha$ satisfying $(ND)$ for $c = 19/20$ and a fixed orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of $\ell_2$. As there are as many orthogonal bases in $\ell_2$ as elements $\alpha \in \{0, 1\}^N$, we can make sure that $\phi$ is not diagonalized by any basis.

The structure of the paper is as follows. In the second section we discuss the preliminaries including the above mentioned tensor products of finite dimensional Hilbert spaces and the exponential growth of the above mentioned function. In the third section we construct the required family of projections (Theorem 10) and include the final argument. The main result is Theorem 12. The last section contains additional remarks.

The notation should be standard. When $X$ is a set, then $\ell_2(X)$ denotes the Hilbert space whose orthonormal basis is labeled by elements of $X$. All norms are $\ell_2$-norms or operator norms on Hilbert spaces. $|X|$ denotes the cardinality of a set and $|z|$ denotes the absolute value of a complex number $z$, it should be always clear from the context which meaning of $||$ is used. We also often identify $n \in \mathbb{N}$ with the set $\{0, \ldots, n-1\}$. For sets $A, B$ by $B^A$ we mean the set of all functions from $A$ into $B$. The restriction $\sigma = x|m$ of an infinite sequence $x \in \{0, 1\}^N$ for $m \in \mathbb{N}$ is a sequence $\sigma \in \{0, 1\}^m$ of length $m$ such that $\sigma(k) = x(k)$ for all $k < m$.

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2. Preliminaries

2.1. Projections and the inner product.

**Lemma 1.** Suppose that $P$ is an orthogonal projection in $\mathcal{B}(\ell_2)$ and $x \in \ell_2$. Then
\[ \langle P(x), x \rangle = \|P(x)\|^2. \]

**Proof.** Using the facts that $P = P^2 = P^*$ we obtain $\langle P(x), x \rangle = \langle P^2(x), x \rangle = \langle P(x), P(x) \rangle = \|P(x)\|^2$. \(\square\)
Lemma 2. Suppose that \((e_k)_{k \in \mathbb{N}}\) is an orthonormal basis of \(\ell_2\) and \(F \subseteq \ell_2\) is an \(n\)-dimensional linear subspace of \(\ell_2\). Let \(\varepsilon > 0\). There is \(X \subseteq \mathbb{N}\) of cardinality not bigger than \(n^2/\varepsilon\) such that \(\|P_F(e_k)\|^2 < \varepsilon\) for every \(k \in \mathbb{N} \setminus X\).

Proof. Let \(\{e'_0, \ldots, e'_{n-1}\}\) be an orthonormal basis of \(F\). We have \(1 = \|e'_j\|^2 = \sum_{k \in \mathbb{N}} |\langle e'_j, e_k \rangle|^2\) for each \(j < n\). So there are \(A_j \subseteq \mathbb{N}\) of cardinality not bigger than \(n/\varepsilon\) such that \(|\langle e'_j, e_k \rangle|^2 \leq \varepsilon/n\) for every \(k \in \mathbb{N} \setminus A_j\). Let \(X = \bigcup_{j<n} A_j\). Then \(|X| \leq n^2/\varepsilon\) and for \(k \in \mathbb{N} \setminus X\) we have
\[
\|P_F(e_k)\|^2 = \sum_{j<n} |\langle e'_j, e_k \rangle|^2 \leq \varepsilon/n = \varepsilon.
\]

2.2. Obtaining an inclined vector. The purpose of this subsection is to prove Lemma 5 which roughly says that there is an absolute constant such that if in \(d\)-dimensional Hilbert space we have less than “exponentially in \(d\)”-many directions, then there is another direction whose inclination to all the original ones is at least the constant.

Lemma 3. Suppose that \(X\) is a collection of unit vectors in \(\mathbb{R}^d\) for \(d \geq 2^7\), such that for every unit vector \(y \in \mathbb{R}^d\) there is \(x \in X\) with \(|x - y| \leq 9/10\). Then the cardinality of \(X\) is at least \((100/91)^d/2\).

Proof. Let \(B_r(a)\) denote the ball of radius \(r > 0\) with the center \(a \in \mathbb{R}^d\). Let \(V_d(r)\) denote the \(d\)-dimensional volume of \(B_r(a)\) for any \(a \in \mathbb{R}^d\). Recall that \(V_d(r)\) is equal to \(r^d V_d(1)\) which follows from the formula for integration by substitution with the substitution sending \(a \in \mathbb{R}^d\) to \(r a\).

The hypothesis on \(X\) implies that the sphere in \(\mathbb{R}^d\) is covered by \(\bigcup \{B_{9/10}(x) : x \in X\}\). So whenever \(99/100 \leq \|y\| \leq 1\) for \(y \in \mathbb{R}^d\), then there is \(x \in X\) such that \(d(x, y) \leq d(x, y/\|y\|) + 1/100 \leq 9/10 + 1/100 = 91/100\) and so set \(\bigcup \{B_{91/100}(x) : x \in X\}\) covers \(B_1(0^d) \setminus B_{99/100}(0^d)\), where \(0^d\) denotes the origin in \(\mathbb{R}^d\).

The latter set has volume \(V_d(1) - V_d(99/100) = (1 - (99/100)^d) V_d(1)\) and the union which covers it has volume not bigger than \(|X|(99/100)^d V_d(1)\). It follows that \((1 - (99/100)^d) V_d(1) \leq |X|(99/100)^d V_d(1)\) and so \((100/91)^d (1 - (99/100)^d) \leq |X|\). As \((99/100)^d \approx 0.276251668\) we have that \((99/100)^d \leq 1/2\) for \(d \geq 2^7\) and so \(|X| \geq (100/91)^d/2\) for such \(d\), as required.

The above argument is a version of well known fact concerning \(\varepsilon\)-nets of the \(n\)-dimensional ball, e.g. Proposition 15.1.3 of [10].

Lemma 4. Suppose that \(d \in \mathbb{N} \setminus \{0\}\) and \(x, y \in \mathbb{C}^d\) are unit vectors. Suppose that \(\varepsilon > 0\) and \(|x \pm y|, |x \pm iy| \geq \varepsilon\). Then
\[
|\langle x, y \rangle| \leq \sqrt{2}(1 - \varepsilon^2/2).
\]

Proof. Let \(\alpha \in \{1, -1, i, -i\}\). By the parallelogram law \(2(\|v\|^2 + \|w\|^2) = \|v + w\|^2 + \|v - w\|^2\) we conclude that \(|x + \alpha y|^2 = 4 - \|x - \alpha y\|^2\) and \(|x + i\alpha y|^2 = 4 - \|x - i\alpha y\|^2\).

Using the above and the polarization identity \(\langle u, v \rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2 - i\|u - iv\|^2 + i\|u + iv\|^2)\) we obtain
\[
|\langle x, y \rangle| = |\langle x, \alpha y \rangle| = \sqrt{(1 - \|x - \alpha y\|^2/2)^2 + (1 - \|x - i\alpha y\|^2/2)^2} \leq \sqrt{2}(1 - \varepsilon^2/2)
\]
for \( \varepsilon^2/2, \|x-\alpha y\|^2/2, \|x-i\alpha y\|^2/2 \leq 1 \) since the real variable function \((1-\beta)^2\) is decreasing below \(\beta = 1\) and we have \(\varepsilon^2/2 \leq \|x-\alpha y\|^2/2, \|x-i\alpha y\|^2/2\). The rest of the proof consist of noting that under our hypothesis that \(\|x\| = \|y\| = 1\) there is \(\alpha \in \{1, -1, i, -i\}\) such that \(\|x-\alpha y\|, \|x-i\alpha y\| \leq \sqrt{2}\).

By rotating the sphere in \(\mathbb{C}^d\) we may assume that \(y = (1, 0, ..., 0)\). For any \(x_1 \in \mathbb{C}\) such that \(|x_1| \leq 1\) there is \(\alpha \in \{1, -1, i, -i\}\) such that \(|x_1-\alpha|, |x_1-i\alpha| \leq 1\). Now

\[
\|x-\alpha y\|^2 = |x_1-\alpha|^2 + \sum_{1 < k < d} |x_k|^2 \leq 2.
\]

\[
\|x-i\alpha y\|^2 = |x_1-i\alpha|^2 + \sum_{1 < k < d} |x_k|^2 \leq 2.
\]

\[\square\]

Let us note that considering the points \(\pm iy\) in the above lemma is necessary in the complex case as already for \(d = 1\) we have \(\|i + 1\| = \|i - 1\| = \sqrt{2}\) but \(i\) and \(1\) are not inclined, i.e., \(|(i, 1)| = 1\) as \(i\) and \(1\) lie on the same “complex straight line” as they are linearly dependent over \(\mathbb{C}\).

**Lemma 5.** Suppose that \(d, n \in \mathbb{N}\) satisfy \(d \geq 2^7\) and \(n < (100/91)^d/8\) and that \(X = \{x^j: j < n\}\) is a collection of vectors in \(\mathbb{C}^d\). Then there is a unit vector \(x \in \mathbb{C}^d\) such that \(|\langle x, x\rangle| \leq (9/10)|x_j|\) for every \(j < n\). In particular \(\|R_x(x_j)\|^2 \leq (9/10)|x_j|^2\) for every \(j < n\), where \(R_x\) is the orthogonal projection onto the direction of \(x\).

**Proof.** First assume that all \(x_j\)s are unit vectors. Identifying \(\mathbb{R}^2\) with \(\mathbb{C}\) we can consider \(Y(l) = \{y(l): j < n\} \subseteq \mathbb{R}^{2d}\) for \(l \in \{1, 2, 3, 4\}\), satisfying

\[
x^j_k = y^j_{2k}(1) + iy^j_{2k+1}(1),
\]

\[
-x^j_k = y^j_{2k}(2) + iy^j_{2k+1}(2),
\]

\[
ix^j_k = y^j_{2k}(3) + iy^j_{2k+1}(3),
\]

\[
-ix^j_k = y^j_{2k}(4) + iy^j_{2k+1}(4)
\]

for all \(k < d\) and \(j < n\) and \(1 \leq l \leq 4\). It is clear that \(y_j(l)\) are unit vectors for all \(j < n\) and \(1 \leq l \leq 4\).

As \(|Y(1) \cup Y(2) \cup Y(3) \cup Y(4)| = 4n < (100/91)^d/2 < (100/91)^{2d}/2\) Lemma 3 implies that there is a unit \(z \in \mathbb{R}^{2d}\) such that \(\|z-y(l)\| \geq 9/10\) for all \(j < n\) and \(1 \leq l \leq 4\).

Consider \(x \in \mathbb{C}^d\) whose coordinates are complex numbers whose real and imaginary parts are formed from the \(2d\) real coordinates of \(z\), i.e.,

\[
x_k = z_{2k} + iz_{2k+1}
\]

for any \(k < d\). It is clear that \(x\) is a unit vector. We have

\[
\|x-x^j\| = \sqrt{\sum_{k<d} |x_k - x^j_k|^2} = \sqrt{\sum_{k<d} |z_{2k} + iz_{2k+1} - y^j_{2k}(1) - iy^j_{2k+1}(1)|^2} = \sqrt{\sum_{k<d} (|z_{2k} - y^j_{2k}(1)|^2 + |z_{2k+1} - y^j_{2k+1}(1)|^2)} = \|z - y(l)\|
\]
and analogously $\|x + x^j\| = \|y^j(2)\|$, $\|x - iy^j\| = \|y^j(3)\|$ and $\|x + ix^j\| = \|z - y^j(4)\|$. So $\|x - x^j\|, \|x - iy^j\|, \|x + x^j\|, \|x + ix^j\| \geq 9/10$ for all $j < n$.

It follows from Lemma 1 that for any $j < n$ we have $|\langle x, x^j \rangle| \leq \sqrt{2} (1 - (9/10)^2/2)$. We also have $\sqrt{2} (1 - (9/10)^2/2) \leq \sqrt{\frac{0.0719}{300}} \leq \frac{128.5}{200} = 9/10$, so $|\langle x, x^j \rangle| \leq 9/10$ for every $j < n$.

If $x_j$ have arbitrary norms, we have

$$|\langle x, x_j \rangle| = \|x_j\| |\langle x, x_j/\|x_j\|\rangle| \leq (9/10) \|x_j\|.$$  

Also by Lemma 1

$$\|R_{x}(x_j)\|^2 = \langle R_{x}(x_j), x_j \rangle = \langle \langle x, x \rangle x, x_j \rangle = \langle x_j, x \rangle \langle x, x^j \rangle = |\langle x, x^j \rangle|^2 \leq (9/10)^2 \|x_j\|^2 \leq (9/10) \|x_j\|^2.$$

\[\square\]

2.3. Obtaining inclined intersecting subspaces in tensor products. For sets $A, B$ as usual $B^A$ denotes the set of all functions from $A$ to $B$. $\{(a, b)\}$ will stand for a function whose domain is $\{a\}$ and which assumes value $b$ at $a$. So any $t \in B^A$ can be written uniquely as $t = s \cup \{(a, b)\}$, where $s \in B^{A \setminus \{a\}}$. We will view the Hilbert space $\ell_2(B^A)$ as the tensor product of Hilbert spaces $\bigotimes_{a \in A} \ell_2(B^{\{a\}})$, where $\langle \otimes_{a \in A} x_a, \otimes_{a \in A} y_a \rangle = \prod_{a \in A} \langle x_a, y_a \rangle$ [22, 6.3.1]. This notation will allow us to handle many-fold tensor products with precision and a relatively modest amount of indices. For example $e_{(a, b)} \otimes e_s = e_{s \cup \{(a, b)\}} = e_s \otimes e_{\{a, b\}}$ and we do not need to worry about the order of factors in tensor products of Hilbert spaces. However in the case of tensors of operators we will be using a more standard notation $S \otimes \ldots \otimes T \otimes \ldots \otimes S$ to indicate with a letter above $T$ at which coordinate we put the operator $T$. Recall that $R_v$ denotes the rank one orthogonal projection onto the direction of a nonzero vector $v$.

**Definition 6.** Suppose that $A, B$ are nonempty sets and $v \in \ell_2(B^{\{a\}})$, then we define the orthogonal projection $R_{a, v} \in B(\ell_2(B^A))$ onto the subspace $\ell_2(B^{A \setminus \{a\}}) \otimes \mathbb{C} v$ of dimension $|B|^{B \setminus \{a\} - 1}$ by

$$P_{a, v} = I \otimes \ldots \otimes R_v \otimes \ldots \otimes I.$$  

More explicitly for each $x = \sum_{t \in B^A} x_t e_t \in \ell_2(B^A)$, $a \in A$ and $s \in B^{A \setminus \{a\}}$ we define $x(s) = \sum_{t \in B^A \setminus \{a\}} x(s)(a, b) e_{(a, b)} \in \ell_2(B^{\{a\}})$ by

$$x(s)(a, b) = x_{s \cup \{(a, b)\}}.$$  

That is we arrange the coordinates of $x$ into $|A|^{B \setminus \{a\} - 1}$ blocks $x(s)$ for $s \in B^{A \setminus \{a\}}$. Then given $v = \sum_{b \in B} v^b e_{\{a, b\}} \in \ell_2(B^{\{a\}})$ we define $P_{a, v} : \ell_2(B^A) \to \ell_2(B^A)$ by

$$P_{a, v}(\sum_{t \in B^A} x_t e_t) = \sum_{s \in B^{A \setminus \{a\}}} \sum_{b \in B} \langle x(s), v \rangle v^b e_{s \cup \{(a, b)\}}.$$  

That is to each block $x(s)$ of the coordinates of $x$ we apply the projection $R_v$ onto the direction of $v$. To check that this corresponds to Definition 6 one can check this for basic vectors $e_t = e_{s \cup \{(a, b)\}}$, namely

$$\sum_{b \in B} \langle e_{\{a, b\}}, v \rangle v^b e_{s \cup \{(a, b)\}} = \langle R_v(e_{\{a, b\}}), v \rangle e_{s \cup \{(a, b)\}}.$$
Lemma 7. Let $A, B$ be finite sets and let $v_a \in \ell_2(B^{\{a\}})$ be nonzero for each $a \in A$. Then for any nonzero choice of $v_a \in \ell_2(B^{\{a\}})$ for $a \in A$ the product $\prod_{a \in A} P_{a,v_a}^{A,B} \leq P_{a,v_a}^A$ is a nonzero projection.

Proof.

$$\prod_{a \in A} P_{a,v_a}^{A,B} = \prod_{a \in A} (I \otimes \ldots \otimes \frac{a}{R_{v_a} \otimes \ldots \otimes I}) = \bigotimes_{a \in A} R_{v_a}.$$ \hfill \Box

More explicitly if $v_a = \sum v_{(a,b)} e_{\{(a,b)\}}$ for $a \in A$ then we consider

$$v = \sum_{t \in B^A} \prod_{a \in A} v_{a,t(a)} e_t \in \ell_2(B^A).$$

It is enough to show that each of the projections $P_{a,v_a}^{A,B}$ for $a \in A$ leaves $v$ intact. Indeed by (1) and (2) we have $v(s) = (\prod_{a'} A \setminus \{a\} \cap v_{a',s(a')}) v_a$, so

$$P_{a,v_a}^{A,B}(v) = P_{a,v_a}^{A,B} \big( \sum_{t \in B^A} \prod_{a \in A} v_{a,t(a)} e_t \big) =$$

$$= \sum_{s \in B^A \setminus \{a\}} \sum_{b \in B \cap A \setminus \{a\}} \left( \prod_{a \in A \setminus \{a\}} v_{a',s(a')} \right) (v_a, v_{a,b}) = \prod_{a \in A \setminus \{a\}} v_{a,t(a)} e_t = v.$$

Lemma 8. Suppose that $A, B$ are finite sets such that $|A| = m > 1$, $|B| = d \geq 2^7$ and $\{x^j : j < n\}$ are vectors of $\ell_2(B^A)$ for some $n \in \mathbb{N}$. Moreover let us assume that $nd^{m-1} < (100/91)^d/8$. Then for every $a \in A$ there is a nonzero $v_a \in \ell_2(B^{\{a\}})$ such that $\|P_{v_a}^{A,B}(x^j)\|^2 \leq (9/10)\|x^j\|^2$ for each $j < n$.

Proof. Fix $A, B, a$ and $\{x^j : j < n\}$ as in the lemma. Let $x^j = \sum_{t \in B^A} x^j_t e_t$. As in (1) we can write it as

$$x^j = \sum_{s \in B^{A \setminus \{a\}}} x^j(s) \otimes e_s.$$ Apply Lemma 7 to the collection $\{x^j(s) : j < n, s \in B^{A \setminus \{a\}}\}$ of cardinality $nd^{m-1}$ and obtain a unit vector

$$v = \sum_{b \in B} b^b e_{(a,b)} \in \ell_2(B^{\{a\}})$$

such that

$$\|R_v(x^j(s))\|^2 \leq (9/10)\|x^j(s)\|^2$$

for all $s \in B^{A \setminus \{a\}}$ and $j < n$. For each $s \in B^{A \setminus \{a\}}$ we have

$$\|P_{v_a}^{A,B}(x^j)\|^2 = \left\| (I \otimes \ldots \otimes \frac{a}{R_v} \otimes \ldots \otimes I) \left( \sum_{s \in B^{A \setminus \{a\}}} x^j(s) \otimes e_s \right) \right\|^2 =$$

$$= \left\| \sum_{s \in B^{A \setminus \{a\}}} R_v(x^j(s)) \otimes e_s \right\|^2 = \sum_{s \in B^{A \setminus \{a\}}} \|R_v(x^j(s))\|^2 \leq$$

$$\leq (9/10) \sum_{s \in B^{A \setminus \{a\}}} \|x^j(s)\|^2 = (9/10)\|x^j\|^2.$$ \hfill \Box
More explicitly using (1) and (2)
\[
\| P_{v,a}^A(x^j) \| \| = \sum_{s \in B^A \setminus \{s \}} \| \sum_{b \in B} \langle x^j(s), v \rangle v^b e_{a \cup \{(a,b)\}} \| \| = \\
\sum_{s \in B^A \setminus \{s \}} \| R_v(x^j(s)) \| \leq (9/10) \sum_{s \in B^A \setminus \{s \}} \| x^j(s) \| = (9/10) \| x^j \| .
\]

3. A FAMILY OF PROJECTIONS AND THE PURE STATE

For this section we fix \( d : \mathbb{N} \setminus \{0\} \to \mathbb{N} \) such that for any \( m \in \mathbb{N}, m > 0 \) we have
- \( d(m) \geq 2^7 \),
- \( 32m^2(d(m)^{2m})^2d(m)^{(2m-1)} < (100/91)^{d(m)} \).

Such \( d \) can be easily constructed as for each \( m \in \mathbb{N} \) the polynomial
\[
p_m(x) = 32m^2(x^{(2m)})^2x^{(2m-1)}
\]
is smaller than the exponential function \((100/91)^x\) for sufficiently big \( x \in \mathbb{R} \).

In the rest of this section we will identify \( d(m) \) with the set \( \{0, \ldots, d(m) - 1\} \).

Define
\[
\mathcal{O} = \bigcup_{m>0} d(m)^{(0,1)^m}.
\]

Note that the summands of this union are pairwise disjoint as they consist of functions with different domains \( \{0,1\}^m \) for \( m \in \mathbb{N} \). In this section instead of the usual \( \ell_2 = \ell_2(\mathbb{N}) \) we will work with \( \ell_2(\mathcal{O}) \). For \( m > 0 \) let \( Q_m : \ell_2(\mathcal{O}) \to \ell_2(\mathcal{O}) \) be the orthogonal projection onto \( \ell_2(d(m)^{(0,1)^m}) \) considered as a subspace of \( \ell_2(\mathcal{O}) \) consisting of vectors whose coordinates in \( \mathcal{O} \setminus d(m)^{(0,1)^m} \) are zero. We will be dealing with algebras \( \mathcal{B}_m = Q_m \mathcal{B}(\ell_2(\mathcal{O}))Q_m \) for \( m > 0 \), they will be identified with \( \mathcal{B}(\ell_2(d(m)^{(0,1)^m})) \). The projections we will be constructing will be elements of
\[
\bigoplus_{m>0} \mathcal{B}_m
\]
So for operators \( T_m \in \mathcal{B}(\ell_2(d(m)^{(0,1)^m})) \) for \( m > 0 \) we will have
\[
\bigoplus_{m \in \mathbb{N}} T_m \in \mathcal{B}(\ell_2).
\]

**Lemma 9.** Let \( (e_k : k \in \mathbb{N}) \) be an orthonormal basis of \( \ell_2(\mathcal{O}) \) and let \( \sigma_m \in \{0,1\}^m \) for each \( m > 0 \). For each \( m > 0 \) there is \( v_m \in \ell_2(d(m)^{(0,m)}) \) such that for each \( k \in \mathbb{N} \) we have
\[
|\langle \bigoplus_{m>0} P_{\sigma,m,v_m}^{0,1,m} d(m) \rangle (e_k), e_k \rangle| \leq 19/20.
\]

**Proof.** For \( m > 0 \) let \( X_m = \{k \in \mathbb{N} : \|Q_m(e_k)\|^2 > 3/\pi^2 m^2 \} \). As \( \sum_{m>0} \frac{1}{m} = \frac{\pi^2}{6} \), note that for every \( k \in \mathbb{N} \) we have
\[
(\ast) \sum_{\{m > 0 : k \notin X_m \}} \|Q_m(e_k)\|^2 \leq \frac{3}{\pi^2} \sum_{m>0} \frac{1}{m} \leq 1/2.
\]

For \( m \in \mathbb{N} \) by Lemma \( \ast \) applied for \( \varepsilon = 3/\pi^2 m^2 \) knowing that the dimension of \( \mathcal{B}_m \) is \( d(m)^{2m} \) we have that \( |X_m| \leq \pi^2 m^2 (d(m)^{2m})^2 / 3 \leq 4m^2(d(m)^{2m})^2 \).

Now for \( m > 0 \) consider \( \{Q_m(e_k) : k \in X_m \} \). Since \( 8|X_m|d(m)^{(2m-1)} < (100/91)^{d(m)} \) using Lemma \( \ast \) we can find \( v_m \in \ell_2(d(m)^{(0,m)}) \) such that
\[
\| P_{\sigma,m,v_m}^{0,1,m} d(m) \rangle (e_k) \| = \| P_{\sigma,m,v_m}^{0,1,m} \rangle (Q_m(e_k)) \| \leq (9/10) \| Q_m(e_k) \|^2
\]
for each $k \in X_m$ since the range of $P_{\sigma_m, v_m}^{(0,1)m,d(m)}$ is included in the range of $Q_m$.

For each $k \in \mathbb{N}$ we have
\[
\| \bigoplus_{m \in \mathbb{N}} P_{\sigma_m, v_m}^{(0,1)m,d(m)}(e_k) \|^2 \leq \sum_{\{m > 0 : k \in X_m\}} \| P_{\sigma_m, v_m}^{(0,1)m,d(m)}(e_k) \|^2 + \sum_{\{m > 0 : k \not\in X_m\}} \| Q_m(e_k) \|^2 \leq (9/10) \sum_{\{m > 0 : k \in X_m\}} \| Q_m(e_k) \|^2 + \sum_{\{m > 0 : k \not\in X_m\}} \| Q_m(e_k) \|^2.
\]
Putting $\alpha = \sum_{\{m > 0 : k \in X_m\}} \| Q_m(e_k) \|^2$ we have that $\sum_{\{m > 0 : k \in X_m\}} \| Q_m(e_k) \|^2 = 1 - \alpha$ as $e_k = \sum_{m > 0} Q_m(e_k)$. So
\[
\| \bigoplus_{m > 0} P_{\sigma_m, v_m}^{(0,1)m,d(m)}(e_k) \|^2 \leq (9/10)(1 - \alpha) + \alpha
\]
However $\alpha \in [0, 1/2]$ by (*) and $(9/10)(1 - \alpha) + \alpha = (1/10)\alpha + (9/10)$ assumes its maximum on $[0, 1/2]$ at $\alpha = 1/2$. The maximum is $19/20$ and so for each $k \in \mathbb{N}$ by Lemma 7 we have
\[
|\langle \bigoplus_{m > 0} P_{\sigma_m, v_m}^{(0,1)m,d(m)}(e_k), e_k \rangle| = \| \bigoplus_{m > 0} P_{\sigma_m, v_m}^{(0,1)m,d(m)}(e_k) \|^2 \leq 19/20,
\]
as required.

\[\square\]

**Theorem 10.** There is a collection $(P_\alpha : \alpha \in \{0,1\}^\mathbb{N})$ of infinite dimensional orthogonal projections in $\mathcal{B}(\ell_2)$ such that for any $\alpha_1, \ldots, \alpha_n \in \{0,1\}^\mathbb{N}$ and $n \in \mathbb{N}$ there is a nonzero projection $P_{\alpha_1, \ldots, \alpha_n} \leq P_{\alpha_1}, \ldots, P_{\alpha_n}$ and for every orthonormal basis $(e_k)_{k \in \mathbb{N}}$ of $\ell_2$ there is $\alpha \in \{0,1\}^\mathbb{N}$ such that for each $k \in \mathbb{N}$ we have
\[
|\langle P_\alpha(e_k), e_k \rangle| \leq 19/20.
\]

**Proof.** We will prove the theorem for $\ell_2(\mathcal{O})$ instead of $\ell_2(\mathbb{N})$. Since $\mathcal{O}$ is countably infinite, this makes no difference. Enumerate all orthonormal bases of $\ell_2(\mathcal{O})$ as $\{e_k^\alpha \in \mathbb{N} : \alpha \in \{0,1\}^\mathbb{N}\}$. This is possible since by the cardinal equality $(2^{\aleph_0})^\omega = 2^{\alpha \omega}$ both $\{0,1\}^\mathbb{N}$ and the collection of all orthonormal bases of $\ell_2$ have the same cardinality equal to the continuum. For $\alpha \in \{0,1\}^\mathbb{N}$ consider
\[
P_\alpha = \bigoplus_{m \in \mathbb{N}} P_{\alpha|\{m, v_m\}^{(0,1)m,d(m)}},
\]
where $v_m$ are chosen according to Lemma 9 for the basis $(e_k^\alpha)_{k \in \mathbb{N}}$ and $v_m = \alpha|m$ which is an element of $\{0,1\}^m$ formed by the first $m$ terms of $\alpha$. Hence we have $|\langle P_\alpha(e_k^\alpha), e_k^\alpha \rangle| \leq 19/20$ for each $k \in \mathbb{N}$ and each $\alpha \in \{0,1\}^\mathbb{N}$.

Now let $\alpha_1, \ldots, \alpha_n \in \{0,1\}^\mathbb{N}$. Let $m \in \mathbb{N}$ be such that $\alpha_j|m \neq \alpha_j'|m$ for any two $1 \leq j < j' \leq m$. By Lemma 7
\[
\prod_{1 \leq j \leq n} P_{\alpha_j|\{m, v_m\}^{(0,1)m,d(m)}}
\]
is a projection dominated by each $P_{\alpha_j|\{m, v_m\}^{(0,1)m,d(m)}}$ for $1 \leq j \leq m$ hence the same is true for the projections $P_{\alpha_1, \ldots, P_{\alpha_n}}$.

To construct our pure state we a result relating certain collections of projections in $\mathcal{B}(\ell_2)$ and pure states on $\mathcal{B}(\ell_2)$. Based on Chapter 6 of [6] it seems that the following result is due to N. Weaver. We provide the proof for the convenience of the reader.
Lemma 11. Suppose that \((P_j)_{j \in J}\) is a collection of projections in \(\mathcal{B}(\ell_2)\) such that for any \(j_1, \ldots, j_n \in J\) there is a nonzero projection \(P\) such that \(P \leq P_{j_i}\) for each \(1 \leq i \leq n\). Then there is a pure state \(\phi\) on \(\mathcal{B}(\ell_2)\) such that \(\phi(P_j) = 1\) for all \(j \in J\).

Proof. Let \(S\) denote the set of states on \(\mathcal{B}(\ell_2)\). Let \(\mathcal{P}\) denote the family of all finite subsets of \(J\) and let \(P_a = \bigcap_{j \in J} P_{j_i}\) be the projection as in the lemma for \(a = \{j_1, \ldots, j_n\}\). As \(\|P_a\| = 1\) for every \(a \in \mathcal{P}\), there are states \(\phi_a \in S\) such that \(\phi_a(P_a) = 1\) (5.1.11) which satisfy \(\phi(P_j) = 1\) for each \(j \in a\) as \(\phi(P_a) \leq \phi(P_j) \leq 1\). For \(a \in \mathcal{P}\) consider

\[F_a = \{\phi \in S : \phi(P_j) = 1\} \text{ for all } j \in a\]

\(F_a\)s are convex, nonempty, weak* closed and form a centered family as \(F_{a \cup a'} \subseteq F_a \cap F_{a'}\) for all \(a, a' \in \mathcal{P}\), so by the compactness of the dual ball of \(\mathcal{B}(\ell_2)\) in the weak* topology we have \(\bigcap_{a \in \mathcal{P}} F_a \neq \emptyset\). Moreover \(\bigcap_{a \in \mathcal{P}} F_a \neq \emptyset\) is convex as the intersection of convex sets. By the Krein-Milman theorem \(\bigcap_{a \in \mathcal{P}} F_a\) has an extreme point \(\phi\). We claim that \(\phi\) is the desired pure state. If \(\phi = \alpha \psi + (1 - \alpha)\psi'\) for some \(\psi, \psi' \in S\) and \(\alpha \in (0, 1)\), we would have \(\alpha \psi(P_a) + (1 - \alpha)\psi'(P_a) = \phi(P_a) = 1\) for any \(a \in \mathcal{P}\). But this implies that \(\psi(P_a) = \psi'(P_a) = 1\) for all \(a \in \mathcal{P}\), and so \(\psi, \psi' \in \bigcap_{a \in \mathcal{P}} F_a\). However, in such a case, \(\psi = \psi' = \phi\) as \(\phi\) was an extreme point of \(\bigcap_{a \in \mathcal{P}} F_a\). \(\Box\)

Theorem 12. There is a non-diagonalizable pure state in \(\mathcal{B}(\ell_2)\).

Proof. Let \((P_\alpha : \alpha \in \{0, 1\}^\mathbb{N})\) be the collection of orthogonal projections from Theorem 11. By Lemma 11 there is a pure state \(\phi\) on \(\mathcal{B}(\ell_2)\) such that \(\phi(P_\alpha) = 1\) for each \(\alpha \in \{0, 1\}^\mathbb{N}\).

However, by Theorem 10 for every orthonormal basis \((e_k)_{k \in \mathbb{N}}\) of \(\ell_2\) there is \(\alpha \in \{0, 1\}^\mathbb{N}\) such that

\[|\lim_{u} \langle P_\alpha(e_k), e_k \rangle| \leq 19/20 \neq 1 = \phi(P_\alpha)\]

which shows that \(\phi\) is not diagonalizable. \(\Box\)

4. Remarks.

4.1. For any nonprincipal ultrafilter \(u\) on \(\mathbb{N}\) one can construct a pure state \(\phi\) as in Theorem 12 which additionally satisfies \(\phi(\bigoplus_{m \in X} Q_m) = 1\) for all \(X \in u\). This is because the projections \(\bigoplus_{m \in X} Q_m\) can be added to the family of projections from Theorem 11 maintaining the hypothesis of Lemma 11. It follows that such states can be multiplicative on a big abelian subalgebras of \(\mathcal{B}(\ell_2)\) of the form \(\mathcal{A}[K]\) of Section 12.5 of [7]. Here \(\mathcal{A}[K]\) is the von Neumann subalgebra of \(\mathcal{B}(\ell_2)\) generated by a pairwise orthogonal collection of finite dimensional orthogonal projections in \(\ell_2\) whose supremum is the identity.

I. Farah and N. Weaver showed that under an additional set-theoretic hypothesis \(\mathcal{D} \leq p\) (12.5.10 of [7]) there is a pure state whose restriction to any algebra \(\mathcal{A}[K]\) is not multiplicative. I. Farah conjectures (p. 336 of [7]) that it is consistent that any pure state has a multiplicative restriction to a subalgebra of the form \(\mathcal{A}[K]\). So our pure state is compatible with this conjecture.
4.2. One can see that the commutators $[P_{\alpha_1}, P_{\alpha_2}]$ of projections from Theorem 10 for distinct $\alpha_1, \alpha_2 \in \{0,1\}^\mathbb{N}$ are finite dimensional, namely they belong to $\bigoplus_{0<j<m} B_j$, where $m \in \mathbb{N}$ is minimal such that $\alpha_1|m \neq \alpha_2|m$. It follows that the image under the quotient map in the Calkin algebra of $\{P_\alpha : \alpha \in \{0,1\}^\mathbb{N}\}$ is commutative. So it is another example of an uncountable collection of commuting projections in the Calkin algebra which does not lift to a simultaneously diagonalizable collection of projections in $B(\ell_2)$. Such first examples were constructed by J. Anderson in [4] assuming CH. Other constructions that do not require additional set-theoretic assumptions are based on different combinatorial arguments than ours: Akemann and Weaver’s construction is based on the counting argument (II) and Farah’s construction is based on a combinatorial argument due to Luzin (Theorem 14.3.2 of [7]).

4.3. A somewhat similar use of finite (but two-fold) tensor products to construct a non-separable object in $B(\ell_2)$ was employed in [8].

References

1. C. Akemann, N. Weaver, $B(H)$ has a pure state that is not multiplicative on any masa. Proc. Natl. Acad. Sci. USA 105 (2008), no. 14, 5313–5314.
2. J. Anderson, Extensions, restrictions, and representations of states on $C^*$-algebras. Trans. Amer. Math. Soc. 249 (1979), no. 2, 303–329.
3. J. Anderson, Extreme points in sets of positive linear maps on $B(H)$. J. Functional Analysis 31 (1979), no. 2, 195–217.
4. J. Anderson, Pathology in the Calkin algebra. J. Operator Theory 2 (1979), no. 2, 159–167.
5. J. Anderson, A conjecture concerning the pure states of $B(H)$ and a related theorem. Topics in modern operator theory (Timișoara/Herculane, 1980), pp. 27–43, Operator Theory: Adv. Appl., 2, Birkhäuser, Basel-Boston, Mass., 1981.
6. I. Farah, E. Wofsey, Set theory and operator algebras. Appalachian set theory 2006–2012, 63–119, London Math. Soc. Lecture Note Ser., 406, Cambridge Univ. Press, Cambridge, 2013.
7. I. Farah, Combinatorial Set Theory of $C^*$-algebras, Springer Monographs in Mathematics, 2019.
8. S. Ghasemi, P. Koszmider, A non-stable $C^*$-algebra with an elementary essential composition series To appear in Proc. Amer. Math. Soc.
9. R. Kadison, I. Singer, Extensions of pure states. Amer. J. Math. 81 (1959), 383–400.
10. G. Lorentz, M. Golitschek, Y. Makovoz, Constructive approximation. Advanced problems. Springer-Verlag, Berlin, 1996.
11. A. Marcus, D. Spielman, S. Srivastava, Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem. Ann. of Math. (2) 182 (2015), no. 1, 327–350.
12. G. Murphy, $C^*$-algebras and operator theory. Academic Press, Inc., Boston, MA, 1990.
13. I. Smythe, A local Ramsey theory for block sequences. Trans. Amer. Math. Soc. 370 (2018), no. 12, 8859–8893.
14. N. Weaver, Forcing for mathematicians. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2014.