An emergent force in a bilayer superfluid Bose-Fermi mixture

Mehmet Günay

Institute of Nuclear Sciences, Hacettepe University, 06800 Ankara, Turkey

We investigate a system of two-atomic species in mixed dimensions, in which one species is spread in a three-dimensional space and the other species is confined in two parallel layers. The presence of atoms in 3-dimensions creates an induced potential for the ones confined in layers. Depending on the effective scattering length and the layer separation, the formation of p-wave pairing within the same layer or s-wave pairing between different layers has been suggested. It is shown that these pairs cannot coexist when time-reversal symmetry (TRS) is on, and there appears a transition from p-wave to s-wave as the ratio of the layer separation and the effective scattering length decreases. With the formation of the inter-layer pairing, we find an emergent force to be present at the critical point and show that it can be derived from the thermodynamic potential. This result offers a tool for experimentally realizing such transitions, and can find notable potential in the field of quantum-thermodynamics.

The mechanism of Cooper pairing beyond the conventional BCS spin singlet state remains one of the most intriguing problem in condensed matter physics. After the discoveries of new phases of $^3$He [1, 2], the examples of the unconventional pairings have been seen in heavy fermion [3] and high temperature [4] superconductors. In recent years, the mechanisms of the unconventional pairing has been a strong focus of attention in the context of superconductors having nontrivial topological properties [5]. These materials are important for the device applications due to having gapless edge state modes [6].

There are several techniques to reveal the symmetry of the order parameters experimentally. These are mostly related to the measurements of the thermodynamical quantities such as temperature dependence of the superfluid density, the specific heat, etc. In most of these observations, the existence of nodal structure in the superfluid gap play crucial role. For instance, $T^3$-dependence of the specific heat is considered as the existence of the point node in the superfluid gap, whereas $T^2$-dependence is the mark for the line nodes [7]. Although, one can have an idea about the nature of the superfluid gap, it is difficult to derive the underlying mechanism of the pairing from these measurements.

At this point, atomic gases provide an ideal platform for simulating exotic physical systems due to having the ability to control the system parameters externally. Very recently, experimental observations of the pairing in two-dimensional Fermi gases have been reported through the BEC-BCS crossover [8–12]. In addition to these two-dimensional systems, there are some promising setups in mixed-dimensions for the observation of superfluid gap with different symmetries [13]. The long-range interaction in obtaining unconventional pairing with such symmetries is a crucial ingredient. In this direction, it has been shown that in a system of two-atomic species in mixed dimensions, in which one species is confined in two-dimensions with keeping the other species in a three-dimensional space, the long-range correlations among the confined particles can be induced through the interaction with the background particles in a three-dimensional space [13][19].

In this paper, we study the thermodynamics of a system of the two-atomic species in mixed-dimensions, where identical fermions are confined in two parallel layers and placed in a three-dimensional space occupied by the BEC particles. In Ref. [19], it is demonstrated that this structure can hold at least four distinct quantum phases. Moreover, in Ref. [18], it is shown that the controllable topological phase transition can be observed by changing the interlayer distance and (or) the BEC coherence length. In both of these studies, it is found that when the layers are far apart, the superfluid gap can be obtained with $p$-wave symmetry. On the other hand, with decreasing layer separation, interlayer pairing comes into play with $s$-wave symmetry. When TRS is manifested, the first-order phase transition in the superfluid gap ($s$-$p$ switching) at a critical layer separation is observed. Here, we focus on this solution to investigate the effects of such a phase transition in the superfluid gap on the thermodynamical quantities. We find that the change of the entropy is not continuous at the critical point. We also discuss that the similar behavior can be observed in the measurements of the specific heat and the density of state. These results can be used as a tool for the experimental detection of the symmetry of the superfluid gap and possible phase transitions.

Next, we study the internal energy of the system, in which we clearly show the interlayer distance dependence of the thermodynamical potential. By using this relation, we define a force emerging with the formation of the interlayer pairing. Since the phase transition is of first order, this force appear like the delta-function at the critical point, and its long-range nature is observed for the small interlayer separations. This result can be realized in excitonic graphene structures [22]. Finally, we discuss that the work done by this force can be used in the realization of the quantum heat-engines.

We consider the structure as studied in Refs. [18] [19].
where identical fermions are confined in two parallel layers locating at \( z = 0 \) and \( z = D \) with an equal number of particles. The layer structures are immersed in a 3-dimensional space occupied by bosons (Fig. 1) with mass \( m_B \) and density \( n_B \). The BEC particles in a 3-dimensional space behaves like the interaction-center for the fermions confined in layers and make intra- and inter-layer coupling possible. The resulting induced interaction for the fermions is given by [14]–[17]

\[
V_{\text{ind}}^{\nu'\nu}(p) = - \frac{2g^2 n_B m_B}{\sqrt{p^2 + 2/\xi_B^2}} e^{-D|\nu' - \nu|/\sqrt{p^2 + 2/\xi_B^2}},
\]

where \( g \) represents Bose-Fermi coupling and \( \xi_B = (8\pi n_B m_B)^{-1/2} \) is the coherence length. Since the induced interaction is long-range and attractive, it opens a gate for the fermions to form Cooper pairing both in between the layers and within the each layer. This can be observed from the form of the induced potential, which is labeled with the layer index \( \nu = 1, 2 \). This layer index introduces an extra degree of freedom and makes it possible to observe the superfluid gap with different symmetries [18]–[19].

With the formation of such pairs, the Hamiltonian in the basis: \( \hat{\Psi}_k = (\hat{c}_{k1}^\dagger \hat{c}_{k2}^\dagger \hat{c}_{-k1} \hat{c}_{-k2}) \) can be given by

\[
H = \sum_k \hat{\Psi}_k^\dagger H_k \hat{\Psi}_k, \quad H_k = \begin{pmatrix} \epsilon_k \sigma_0 & \Delta_k \\ \Delta_k^\dagger & -\epsilon_k \sigma_0 \end{pmatrix},
\]

where \( \hat{c}_{kj}^\dagger (\hat{c}_{kj}) \) is the creation (annihilation) operator for the \( j \)th layer with \( j=1,2 \). Here \( \epsilon_k = \hbar^2 k^2 / 2m - \mu, \) \( m \) is the band mass, \( \mu \) is the chemical potential. Throughout this paper, we assume that each layer has the equal number of particles and we ignore the effects of the fermions on the BEC particles by considering the weak coupling between bosons and fermions [14]–[17]. The spin-dependent full \( 2 \times 2 \) matrix \( \Delta_k \) is given by

\[
\Delta_k = \begin{pmatrix} \Delta_{11}(k) & \Delta_{12}(k) \\ \Delta_{21}(k) & \Delta_{22}(k) \end{pmatrix},
\]

where we obtain the self consistent components with the mean field approach as

\[
\Delta_{\nu'\nu}(k) = -\frac{1}{V} \sum_q V_{\text{ind}}^{\nu'\nu}(k - q) \langle \hat{c}_{q,\nu'} \hat{c}_{-q,\nu} \rangle.
\]

Here, we focus on the solutions of the gap equation when TRS is manifested. Moreover, the intralayer (triplet) pairs are connected with this symmetry as: \( \Delta_{11}(k) = \Delta_{22}(-k) \) and we define: \( \Delta_{12}(k) = \Delta_{21}(k) e^{i(\phi_k - \phi_0)} \propto (k_x + ik_y) \), where \( \Delta_{11}(k) \) is a real and even function with \( k = |k| \) and \( \phi_0 \) is the phase difference between the particles residing in the upper and lower layers. We take the interlayer pairing to be \( s \)-wave so that \( \Delta_{12}(k) = -\Delta_2(k) \). Here, \( \Delta_2(k) \) is a even function due to the Fermi antisymmetry and the presence of TRS dictates that it can only take real values. After some algebra, we obtain the corresponding self-consistent equations for the triplet and the singlet amplitudes as

\[
\Delta_2(k) = -\frac{1}{V} \sum_{k',\lambda} V_{\nu}(k, k') \frac{\Delta_{\nu',\lambda}}{\epsilon_{k',\lambda}} \tanh \left( \frac{E_{k',\lambda}}{2k_B T} \right),
\]

\[
\Delta_{\nu}(k) = -\frac{1}{V} \sum_{k',\lambda} V_{\nu}(k, k') \frac{\lambda \Delta_{\nu',\lambda}}{\epsilon_{k',\lambda}} \tanh \left( \frac{E_{k',\lambda}}{2k_B T} \right),
\]

and for the total number

\[
N = \frac{1}{2} \sum_{k,\lambda} \left[ 1 - \epsilon_k \frac{\lambda}{2E_{k,\lambda}} \tanh \left( \frac{E_{k,\lambda}}{2k_B T} \right) \right].
\]

Here \( \lambda = \pm \) is the branch index, \( V_{\nu}(k, k') = \langle V_{\text{ind}}^{\nu}(|k - k'|) \cos \phi \rangle_\phi \) represents the corresponding interaction channel for the triplet and \( V_{\nu}(k, k') = \langle V_{\text{ind}}^{\nu}(|k - k'|) \rangle_\phi \) for the singlet pairing amplitudes respectively with the angular average of the relative phase \( \phi = \phi_{k'} - \phi_k \). Here, one can observe that the short-range interaction (i.e. \( V(|k - k'|) = V_0 \)) is insufficient to create pairing in the triplet channel due to the appearance of \( \cos \phi \) term. And, the existence of such terms, in addition to induced potential, will be eliminated through the angular average. The eigen-energies with the new pairing fields, \( \Delta_{\nu\lambda} = \Delta_{\nu}(k) + \lambda \Delta_{\nu}(k) \), are given by

\[
E_{k,\lambda} = \sqrt{\epsilon_k^2 + \Delta_{\nu\lambda}^2}.
\]
The ratio of the layer separation and the coherence length \( d = D/\xi_B \) has decisive influence on the symmetry of the superfluid gap. When this ratio is large \( D \to \infty \), the induced interaction for the singlet pairing is suppressed. Therefore, in this limit, triplet solutions are favored and the superfluid gap is expected to have \( p \)-wave symmetry [23]. On the other hand, when \( D \to 0 \), it was shown that the symmetry of the gap can only be \( s \)-wave if the TRS is manifested [18] [19]. In Fig. 2 we demonstrate this phase transition (\( s-p \) crossing) as a function of \( d \). The similar result is also obtained in Ref. [18] (see Fig. 3(b) in Ref. [18]), where they investigate the topological phase transition of such crossing. Here, we show that the appearance of the inter-layer pairing costs to the thermodynamical quantities. For instance, in Fig. 3, we demonstrate the reflection of the \( s-p \) crossing in the entropy of the system, where we calculate it from \([24]\)

\[
S_{cS} = -2k_B \sum_k \left[ (1 - f_k) \ln(1 - f_k) + f_k \ln(f_k) \right]. \tag{9}
\]

Here \( f_k = (1 + e^{-\beta E_k})^{-1} \) is the Fermi-Dirac factor with \( \beta = 1/k_B T \). Since the pairing fields cannot coexist due to TRS, the energy branches become degenerate, i.e., \( E_{k\lambda} = E_k \). It can be seen from Fig. 3 that there appears a jump in the entropy of the system at critical layer separation \( D_c \), which supports our theory. The similar jumps can also be observed in the specific heat, density of state and entropy of the system, where we calculate it from \([24]\) shown that the symmetry of the gap can only be

\[
\frac{\Delta}{\epsilon_F} = \frac{\theta(D - D_c)}{\sqrt{\frac{4 \pi n_F}}}, \tag{11}
\]

where \( \theta(D - D_c) \) is the unit step-function. And, tuning the distance between layers around \( D_c \) can lead to dramatic changes in the energy of the system due to altering symmetry of the superfluid gap. This can be done in an experiment by adiabatic changing the scattering length \( \lambda \) around \( \delta_c \) via Feshback resonances, which is more feasible than the moving layers up and down.

An essential consequence of the relation between the energy and the distance is the force. Therefore, one can speculate appearing an emergent force engaged with these variations in the thermodynamic potential, as such observations are related to the distance between the layers. This type of force, which emerges with the formation of inter-layer pairing, can be derived by taking the derivative of the thermodynamic potential with respect to layer separation. It is defined by

\[
F_\Delta = -\frac{\partial \Omega_\Delta}{\partial D}. \tag{12}
\]

In Fig. 4, we demonstrate the results of the Eq. (10) and Eq. (12) by varying layer separation. The behavior of the thermodynamic potential (straight line in Fig. 4) around \( D_c \) is similar to the entropy of the system, where a jump can be observed. The effects of the phase transition on the thermodynamic potential at \( D_c \) ends up with the delta-potential like behaviour in the value of the force (dotted line in Fig. 4). This is due to the step-like behavior of the superfluid gap in Eq. (11) [Fig. 1]. Moreover, the nature of the force, in a region \( D < D_c \), is long-range and decays with increasing \( D \) and vanishes when the layer separation exceeds the critical value. This result is expected from the form of the induced interaction in Eq. (11).

In obtaining results, we use similar parameters with Ref. [18] and scale momenta with Fermi momentum \( k_F = \sqrt{4 \pi n_F} \), where \( n_F \) is the fermion density in each
layer. We consider a weak Bose-Fermi coupling $g = 2\pi a/\sqrt{m_r m_f}$, where $a_{KF} = 0.12$ is the scaled-scattering length [27], which is a tunable parameter via Feshbach resonance and $m_r$ is the reduced mass.

The presence of the symmetries considered in this work lead to have a first-order phase transition in the superfluid gap, which enhances the results of this paper. However, in a system where the second-order phase transition is the case, the effects of the layer separation (the force) will also be present, as there always be a critical $D_c$ (see, for instance, Fig. 2(a) in Ref. [13]). In analogy with the relation, for instance, between the temperature and the specific heat, where a jump can be observed in specific heat at the critical temperature, the formation of the inter-layer pairing will leave a mark in the thermodynamic potential. And, by following these signs, this force can help to identify the nature of the superfluid gap.

Finally, let us briefly discuss how the results obtained in this paper can be used in the realization of the quantum-heat engines. In its simplest form, work done by this force at zero temperature, which can be called as the interlayer-pairing work, can be given as

$$W = \Delta \Omega = \Omega_\Delta(d \to \infty) - \Omega_\Delta(d \to 0) \approx \Omega_\Delta(d_c^+) - \Omega_\Delta(d_c^-)$$

where $d_c^\pm = D_c/\xi_B^\pm$ is the small deviations from the critical ratio for the s-p crossing. In a temperature gradient environment, constraint to $T \ll T_c^{\text{SEC}}$ to keep the BEC structure in its condensed phase, with the use of Jarzynski fluctuation theorem [28], the bilayer superfluid structures offer a promising setup for the quantum heat-engine applications with long-range correlations. Such solutions in these structures are, in general, considered in terms of the Casimir force or pressure [29, 30]. The addition of the inter-layer force can enrich the problem and lead to explore more exotic structures.

In summary, we study the bilayer superfluid Bose-Fermi mixture in a mixed dimension and show that it is possible to reveal the nature of the superfluid gap by following the thermodynamical signatures. Moreover, it is found that the formation of the inter-layer pairing creates an additional force. This force, actually, will be present for any bilayer system as long as the sufficient pairing between different layers is observed. We also discussed that, besides the fundamental interest, the work done by this force can be used in the quantum-heat engine applications.

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*gunaymehmt@gmail.com