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**Title:** Non-Hermitian laser arrays with tunable phase locking

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The Hamiltonian $H_{OBC}'$ of the laser array in the linear geometry shown in Fig.1(c) of the main manuscript is described by the $N \times N$ non-Hermitian matrix

$$H_{OBC}' = \begin{pmatrix} -i\gamma + \delta\omega_1 & \kappa \exp(h) & 0 & \ldots & 0 & 0 & 0 \\ \kappa \exp(-h) & -i\gamma & \kappa \exp(h) & \ldots & 0 & 0 & 0 \\ 0 & \kappa \exp(-h) & -i\gamma & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \kappa \exp(-h) & -i\gamma & \kappa \exp(h) \\ 0 & 0 & 0 & \ldots & 0 & \kappa \exp(-h) & -i\gamma + \delta\omega_N \end{pmatrix}$$

where $\gamma$ is the (Hermitian) coupling constant between adjacent resonators, $h$ is the synthetic imaginary gauge field, $\gamma \geq 2\kappa \sinh h$ is the loss rate (i.e. inverse of the quality factor $Q$) of the passive resonators, and $\delta\omega_1, \delta\omega_N$ describe suitable shifts of the complex frequencies of the two edge resonators $n = 1$ and $n = N$ of the chain, with respect to the value $\omega_0 - i\gamma$ of all other resonators. In writing $H_{OBC}'$, without loss of generality we dropped the real part $\omega_0$ of the resonator mode frequency. When $\delta\omega_1 = \delta\omega_n = 0$, $H_{OBC}'$ reduces to $H_{OBC}$ and all supermodes have the same decay rate $\gamma$ and are exponentially localized toward the edge of the array (skin supermodes). Our aim is to strategically tailor the complex frequency offsets $\delta\omega_1$ and $\delta\omega_N$ to meet the following conditions:

(i) All supermodes of $H_{OBC}'$, except one, remain skin supermodes with the same decay rate $\gamma$ (like for $H_{OBC}$).

(ii) One supermode of $H_{OBC}'$ becomes extended over the entire array, i.e. it is a Bloch supermode, with uniform field amplitudes $E_n = \exp(iQ_0n)$ and with a phase locking condition defined by the angle $Q_0$.

(iii) The decay rate $\gamma_s$ of the Bloch supermode is smaller than $\gamma$ and the difference $\gamma - \gamma_s$ is independent of system size $N$.

Under such conditions, the linearized dynamics of the laser at the onset of threshold, as described by the Hamiltonian $H_{OBC}'$, clearly favors laser oscillation on the extended Bloch supermode, regardless of the size $N$ of the array, with a phase locking condition $Q_0$ that can be tuned by proper change of $\delta\omega_1$ and $\delta\omega_N$. The main result that we are going to prove is that the above conditions are satisfied by letting

$$\delta\omega_1 = \kappa \exp(-h - iQ_0), \quad \delta\omega_N = \kappa \exp(h + iQ_0)$$

with the decay rate of the Bloch supermode given by

$$\gamma_s = \gamma - 2\kappa \sinh(h) \sin(Q_0).$$

In fact, the eigenvalues $E$ of $H_{OBC}'$, satisfying the determinantal equation

$$\det(E - H_{OBC}') = 0,$$

are given by

$$E = -i\gamma + \kappa X$$

where $X$ are the roots of the algebraic equation

$$\left(X - \frac{\delta\omega_1}{\kappa}\right) \left(X - \frac{\delta\omega_N}{\kappa}\right) P_{N-2}(X) - \left(2X - \frac{\delta\omega_1}{\kappa} - \frac{\delta\omega_N}{\kappa}\right) P_{N-3}(X) + P_{N-4}(X) = 0.$$
of a $N \times N$ tridiagonal matrix, with elements $X$ in the main diagonal and $-\exp(\pm h)$ on the upper and lower second diagonals. To calculate $P_N(X)$, let us observe that $P_N$ satisfies the recurrence relation

$$P_N = XP_{N-1} - P_{N-2}$$  \hspace{1cm} (S-8)

which can be readily solved with the initial conditions $P_0 = 1$, $P_1 = X$, yielding

$$P_N(X) = \frac{\sin[(N + 1)\theta]}{\sin \theta}.$$  \hspace{1cm} (S-9)

In the above equation, the complex angle $\theta$ is defined by the relation

$$2 \cos \theta = X.$$  \hspace{1cm} (S-10)

Substitution of Eq.(S-9) into (S-6) and using Eqs.(S-2) and (S-10), one readily obtains

$$(X^2 - \sigma X + 1) \sin[(N - 1)\theta] - (2X - \sigma) \sin[(N - 2)\theta] + \sin[(N - 3)\theta] = 0$$  \hspace{1cm} (S-11)

where we have set

$$\sigma = \exp(h + iQ_0) + \exp(-h - iQ_0) = 2 \cos(Q_0 - ih)$$  \hspace{1cm} (S-12)

with the constraint $\sin \theta \neq 0$. Taking into account that

$$\sin[(N - 2)\theta] = \cos \theta \sin[(N - 1)\theta] - \sin \theta \cos[(N - 1)\theta] = \frac{X}{2} \sin[(N - 1)\theta] - \sin \theta \cos[(N - 1)\theta]$$  \hspace{1cm} (S-13)

and

$$\sin[(N - 3)\theta] = \cos(2\theta) \sin[(N - 1)\theta] - \sin(2\theta) \cos[(N - 1)\theta] = \left(\frac{X^2}{2} - 1\right) \sin[(N - 1)\theta] - X \sin \theta \cos[(N - 1)\theta]$$  \hspace{1cm} (S-14)

after some straightforward calculations Eq.(S-11) takes the simple form

$$(X - \sigma) \sin(N\theta) = 0$$  \hspace{1cm} (S-15)

which is solved by letting $X = X_0 = \sigma$ or $\theta = \theta_l = l\pi/N$ ($l = 1, 2, ..., N - 1$), i.e. $X = X_l = 2 \cos(l\pi/N)$. Finally, from Eq.(S-5) it follows that the eigenvalues $E$ of $\mathcal{H}'_{OBC}$ are given by

$$E_0 = -i\gamma + 2\kappa \cos(Q_0 - ih) , \ E_l = -i\gamma + 2\kappa \cos(l\pi/N) \ (l = 1, 2, ..., N - 1)$$  \hspace{1cm} (S-16)

which proves the main statement. It can be readily shown by direct substitution that the supermode corresponding to the eigenvalue $E_0$ of $\mathcal{H}'_{OBC}$ is the extended Bloch supermode $\mathcal{E}_n = \exp(iQ_0n)$ with Bloch wave number (phase locking angle) $Q_0$.

**S.2. LINEAR STABILITY ANALYSIS**

The semiconductor laser rate equations, given by Eqs.(3) and (4) of the main manuscript, read

$$\tau_p \frac{d\mathcal{E}_n}{dt} = (1 - i\alpha)\mathcal{E}_n Z_n - i\gamma \sum_{l=1}^{N} \mathcal{H}_{nl} \mathcal{E}_l$$  \hspace{1cm} (S-17)

$$\tau_s \frac{dZ_n}{dt} = p - Z_n - (1 + 2Z_n)|\mathcal{E}_n|^2$$  \hspace{1cm} (S-18)

$(n = 1, 2, ..., N)$, where $\mathcal{E}_n$ is the normalized electric field amplitude in the $n$-th resonator of the array, $Z_n$ is the normalized excess carrier density, $\mathcal{H} = \mathcal{H}'_{OBC}$ is the $N \times N$ matrix Hamiltonian defined by Eqs.(S-1) and (S-2), $\tau_p$ is the spontaneous carrier lifetime, $\alpha$ is the linewidth enhancement factor, and $p$ is the normalized excess pump current in the uniformly-pumped cavities.

The steady-state solution to Eqs.(S-17) and (S-18), corresponding to the laser array oscillating in the extended Bloch
supermode with the lowest lasing threshold, reads
\[ \mathcal{E}_n = E_0 \exp(iQ_0 n - i\Omega t), \quad Z_n = \tau_p \gamma_s = Z_0 \quad (S-19) \]
where
\[ E_0 = \sqrt{\frac{p - Z_0}{1 + 2Z_0}} \quad (S-20) \]
is the field amplitude and
\[ \Omega = \alpha \gamma_s + 2\kappa \cosh(h) \cos(Q_0) \quad (S-21) \]
is the oscillation frequency, shifted from the linear value due to the linewidth enhancement factor \( \alpha \). Note that the steady-state supermode solution does exist for \( p \geq p_{th} \), where the threshold value of pump parameter reads
\[ p_{th} = Z_0 = \tau_p \gamma_s = \tau_p \{ \gamma - 2\kappa \sin(Q_0) \sinh(h) \}. \quad (S-22) \]

The stability of the steady-state supermode solution can be studied by a standard linear stability analysis. After setting
\[ \mathcal{E}_n(t) = \mathcal{E}_n + \delta \mathcal{E}_n \exp(-i\Omega t), \quad Z_n(t) = Z_0 + \delta Z_n(t) \quad (S-23) \]
where \( \delta \mathcal{E}_n(t) \) and \( \delta Z_n(t) \) are small perturbations, the linearized equations describing the dynamics of the perturbations can be cast in the compact matrix form
\[ \tau_p \frac{d}{dt} \begin{pmatrix} \delta \mathcal{E}_n \\ \delta \mathcal{E}_n^* \\ \delta Z_n \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta \mathcal{E}_n \\ \delta \mathcal{E}_n^* \\ \delta Z_n \end{pmatrix} \quad (S-24) \]
where the \( 3N \times 3N \) Jacobian matrix \( \mathcal{M} \) has the following block form
\[ \mathcal{M} = \begin{pmatrix} \Delta & 0 & \mathcal{G} \\ 0 & \Delta^* & \mathcal{G}^* \\ \Theta & \Theta^* & \Gamma \end{pmatrix} \quad (S-25) \]

In the above equation, the elements of the \( N \times N \) matrices \( \Delta, \Gamma, \mathcal{G} \) and \( \Theta \) are given by
\[ \Delta_{n,l} = -i\tau_p \mathcal{H}_{n,l} + i\Omega \tau_p \delta_{n,l} + (1 - i\alpha)Z_0 \delta_{n,l} \quad (S-26) \]
\[ \Gamma_{n,l} = -\frac{\tau_p}{\tau_s} \left( 1 + 2 |E_0|^2 \right) \delta_{n,l} \quad (S-27) \]
\[ \mathcal{G}_{n,l} = (1 - i\alpha)E_0 \exp(iQ_0 n) \delta_{n,l} \quad (S-28) \]
\[ \Theta_{n,l} = -\frac{\tau_p}{\tau_s} (1 + 2Z_0) E_0 \exp(-iQ_0 n) \delta_{n,l} \quad (S-29) \]

where \( \delta_{n,l} \) is the Kronecker-\( \delta \) symbol. The \( 3N \) eigenvalues \( \lambda \) of the matrix \( \mathcal{M} \) can be determined numerically. Note that, owing to the phase invariance of the steady-state supermode, there is always one vanishing eigenvalue of \( \mathcal{M} \). An instability arises whenever the imaginary part of any other eigenvalue \( \lambda \) of \( \mathcal{M} \) becomes positive.

### S.3. Far-Field Intensity Pattern

Assuming that all elements in the array are identical, the far-field radiation intensity pattern \( I(\theta) \) in a linear geometry of the laser array [Fig.S1(a)] can be written as \( I(\theta) = I_0(\theta)G(\theta) \), where \( I_0(\theta) \) is the far-field intensity pattern of a single element and \( G(\theta) \) is the array intensity factor, given by [1,2]
\[ G(\theta) = \sum_{l=1}^{N} \mathcal{E}_l \exp \left( \frac{2\pi i l d}{\lambda_0 \sin \theta} \right) \quad (S-30) \]

where \( \theta \) is the polar angle, \( \mathcal{E}_l \) is the complex field amplitude in the \( l-th \) laser of the array \( (l = 1, 2, ..., N) \), \( \lambda_0 \) is the
Fig. S1: Far-field pattern of a laser array emitting in a given supermode. (a) Schematic of the laser array and far-field angle $\theta$. (b) Far-field radiation intensity pattern $I(\theta)$ of an array with $N = 6$ elements for $d/\lambda_0 = 2$ and for a single-emitter normalized intensity pattern $I_0(\theta) = \exp(-\theta^2/\theta_0^2)$ with diffraction angle $\theta_0 = \pi/10$ (dashed curve). The solid curves correspond to the far-field intensity curves for the supermode with complex amplitudes $E_l = \exp(iQ_0l)$ for a few increasing values of $Q_0$. Curve 1: $Q_0 = 0$; curve 2: $Q_0 = \pi/4$; curve 3: $Q_0 = \pi/2$; curve 4: $Q_0 = \pi$. Laser wavelength in vacuum and $d$ is space separation between adjacent elements in the array. Typically, the intensity $I_0(\theta)$ of the far-field of a single element can be taken Gaussian-shaped with some characteristic diffraction angle $\theta_0$ [1,2], i.e.

$$I_0(\theta) = I_0 \exp(-\theta^2/\theta_0^2).$$  \hspace{1cm} \text{(S-31)}$$

On the other hand, the array intensity factor $G(\theta)$ greatly depends on the amplitude and phase distributions $E_l$ of the oscillating supermode. In our case, the lasing supermode corresponds to the same amplitude in the various elements and phase locking $Q_0$, i.e. $E_l = \exp(iQ_0l)$. In this case the array intensity factor $G(\theta)$ can be calculated in a closed form and reads

$$G(\theta) = \frac{\sin^2(N\psi/2)}{\sin^2(\psi/2)}$$ \hspace{1cm} \text{(S-32)}$$

where we have set

$$\psi = Q_0 + 2\pi \frac{d}{\lambda_0} \sin \theta.$$ \hspace{1cm} \text{(S-33)}$$

The highest efficiency in the far-field intensity pattern, with cooperative emission of all elements as in superradiance, is obtained when $Q_0 = 0$ (all elements emit in-phase), corresponding to a typical high-intensity single lobe far-field pattern centered at $\theta = 0$ with the peak intensity factor $N^2$, as shown by curve 1 in Fig.S1(b). As the phase locking angle $Q_0$ is increased toward $\pi$ (all elements emit out-of-phase), the far-field patterns smoothly deform to display two lower-intensity lobes [see curves 2,3 and 4 in Fig.S1(b)]. Note that, even for a relatively large value of $Q_0$, of order $\sim \pi/4$, the far-field pattern retains a high-intensity single-lobe, like in the ideal $Q_0 = 0$ phase locking condition [compare curves 1 and 2 in Fig.S1(b)].

S.4. ROBUSTNESS OF PHASE LOCKING

In the main manuscript (Fig.4) we showed by numerical simulations the robustness of the phase locking condition for the extended Bloch supermode in the presence of disorder in the cavity resonance frequencies, which typical arises from fabrication imperfections of cavity design leading to a broadening of the cavity resonance frequency from the
Fig. S 2: Robustness of phase locking of the extended Bloch supermode in the presence of disorder of both real and imaginary parts of cavity resonance frequencies with uniform probability distributions in the ranges $(-\sigma/2, \sigma/2)$ and $(-\rho/2, \rho/2)$, respectively. The plots show the behavior of the mean phase difference $\langle \theta \rangle$ (upper panels) and variance $\langle \Delta\theta^2 \rangle$ (lower panels) for 100 different realizations of disorder of the cavity resonance frequencies $\delta\Omega_n = \kappa\sigma_n + i\kappa\rho_n$ for a few increasing values of disorder strength: (a) $\sigma = \rho = 0.2$, (b) $\sigma = \rho = 0.5$, and (c) $\sigma = \rho = 1$. Parameter values are as in Fig.3(a) of the main manuscript. The dashed horizontal line in the upper panels corresponds to the ideal phase locking condition $Q_0 = \pi/4$ in the disorder-free array.

reference value $\omega_0$. Other kinds of disorder could arise from e.g. inhomogeneous pumping of the cavities and/or different quality factors of the cavities, which can be accounted for by some disorder in the imaginary part of the cavity resonance frequency $\omega_0$. Extended numerical simulations showed that the phase locking condition is robust against mild-to-moderate disorder in both kinds of disorder of real and imaginary parts of resonance frequency. Let us assume that the cavity resonance frequency $\omega_n$ of the $n$-th resonator of the array deviates from the reference value $\omega_0$ by a quantity $\delta\Omega_n = \sigma_n \kappa + i\rho_n \kappa$, where $\sigma_n$ and $\rho_n$ are independent random variables with the same probability distribution. Specifically, we assume for $\sigma_n$ and $\rho_n$ a uniform distribution in the range $(-\sigma/2, \sigma/2)$ and $(-\rho/2, \rho/2)$, respectively, where the dimensionless parameters $\sigma$ and $\rho$ measure the strength of disorder, with respect to the coupling constant $\kappa$, of real and imaginary parts of the resonance frequency. Figure S.2 shows the behavior of the mean phase difference $\langle \theta \rangle$ (upper panels) and variance $\langle \Delta\theta^2 \rangle$ (lower panels) for 100 different realizations of disorder of the cavity resonance frequencies for a few increasing values of disorder strength. A large value of the variance $\langle \Delta\theta^2 \rangle$ is a signature of failure of the phase locking regime. Note that only for strong disorder [Fig.S2(c)] phase locking is prevented in a non-negligible number of disorder realizations, yet in less than 10% realizations.

S.5. EXTENSION OF NON-HERMITIAN ENGINEERING TO THE TWO-DIMENSIONAL ARRAY

The non-Hermitian engineering method discussed in the main text and Sec.S.1 for a one-dimensional linear array can be readily extended to the case of a two-dimensional square lattice. Let us consider a square array comprising $N \times N$ passive cavities with asymmetric left/right and up/down coupling constants $\kappa_1 = \kappa \exp(h)$ and $\kappa_2 = \kappa \exp(-h)$, which are uniformly pumped by a linear gain $g$. The linearized dynamics of the system is described by the coupled-mode equations

$$i \frac{d\xi_{n,m}}{dt} = \kappa_1 \xi_{n+1,m} + \kappa_2 \xi_{n-1,m} + \kappa_1 \xi_{n,m+1} + \kappa_2 \xi_{n,m-1} + (ig - i\gamma + \delta\omega_{n,m})\xi_{n,m}$$

(S-34)

where $\gamma$ is the loss rate of the passive resonators and $\delta\omega_{n,m}$ describe suitable shifts of the complex frequencies of the
resonators with respect to the value $\omega_0 - i\gamma$. We assume OBC, so that $E_{n,m} = 0$ for $n \leq 0$, $n \geq N$ and $m \leq 0$, $m \geq N$. The complex frequencies $\omega_{n,m}$ are assumed to factorize as $\delta\omega_{n,m} = \delta\omega_n^{(x)} + \delta\omega_m^{(y)}$, with all $\delta\omega_n^{(x,y)}$ vanishing unless $n = 1, N$. This means that engineering of the complex frequencies is required along the edge of the square lattice.

The energy spectrum of the two-dimensional lattice can be readily obtained by the method of separation of variables. After introduction of the Ansatz $E_{n,m}(t) = E_n^{(x)} e^{(x)} \exp(-iEt)$, one obtains the eigenvalue equations

$$E^{(x)} E_n^{(x)} = \kappa_1 E_{n+1}^{(x)} + \kappa_2 E_{n-1}^{(x)} + \frac{i}{2} (g - \gamma) E_n^{(x)} + \frac{i}{2} \delta\omega_n^{(x)} E_n^{(x)} \equiv \sum_{l=1}^N (H_{OBC}^x)_{n,l} E_l^{(x)} + \frac{i}{2} (g - \gamma) E_n^{(x)} \quad (S-35)$$

$$E^{(y)} E_m^{(y)} = \kappa_1 E_{m+1}^{(y)} + \kappa_2 E_{m-1}^{(y)} + \frac{i}{2} (g - \gamma) E_m^{(y)} + \frac{i}{2} \delta\omega_m^{(y)} E_m^{(y)} \equiv \sum_{l=1}^N (H_{OBC}^y)_{m,l} E_l^{(y)} + \frac{i}{2} (g - \gamma) E_m^{(y)} \quad (S-36)$$

with the eigenenergy

$$E = E^{(x)} + E^{(y)} \quad (S-37).$$

The matrix Hamiltonians $H_{OBC}^x$ and $H_{OBC}^y$ entering in Eqs.(S-35) and (S-36) are of the same form as the matrix Hamiltonian $H_{OBC}^0$ given by Eq.(S-1), and therefore the non-Hermitian engineering problem for the square lattice array reduces to the problem discussed in Sec.S.1. Specifically, assuming the complex frequencies

$$\delta\omega_n^{(x,y)} = \kappa \exp(-h - iQ_{x,y}) , \quad \delta\omega_n^{(x,y)} = \kappa \exp(h + iQ_{x,y}) \quad (S-38)$$

with locking phases $Q_x$ and $Q_y$ along the $x$ and $y$ directions, the array sustains the extended Bloch supermode $E_{n,m} = \exp(iQ_x n + iQ_y m - iEt)$ with eigenenergy $E = \kappa_1 \exp(iQ_x) + \kappa_2 \exp(-iQ_x) + \kappa_1 \exp(iQ_y) + \kappa_2 \exp(-iQ_y) + i(g - \gamma)$, while all other array supermodes are skin edge states localized near one corner of the lattice.

References

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