ROW-FINITE EQUIVALENTS EXISTS ONLY FOR GRAPHS HAVING NO UNCOUNTABLE EMITTERS

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Abstract. If $E$ is a not-necessarily row-finite graph, such that each vertex of $E$ emits at most countably many edges, then a desingularization $F$ of $E$ can be constructed as described in [3] or [7]. The desingularization process has been effectively used to establish various characteristics of the Leavitt path algebras of not-necessarily row-finite graphs. Such a desingularization $F$ of $E$ has the properties that: (1) $F$ is row-finite, and (2) the Leavitt path algebras $L(E)$ and $L(F)$ are Morita equivalent. We show here that for a given graph $E$, a graph $F$ having properties (1) and (2) exists (we call such a graph a row-finite equivalent) if and only if $E$ contains no vertex $v$ for which $v$ emits uncountably many edges.

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Originally, the notion of a Leavitt path algebra was defined and investigated for row-finite graphs (i.e., graphs for which each vertex emits at most finitely many edges). Subsequently, the Leavitt path algebras of more general graphs were investigated in [3]; to wit, those graphs for which the vertices emit at most a countable number of edges. One of the methods used in [3] to establish various results in this more general situation is as follows: associate with the given graph $E$ a row-finite graph $F$ for which the Leavitt path algebras $L(E)$ and $L(F)$ are closely related (specifically, are Morita equivalent), then apply known results about the Leavitt path algebras of row-finite graphs to conclude some structural property of $L(F)$, then transfer this information back to $L(E)$.

Recently the notion of a Leavitt path algebra has been investigated in settings where there are no restrictions placed on the cardinality of either the vertex set or edge set of the underlying graph $E$ (see e.g. [1], [2], and [6]). In many contexts, results which have been established for graphs having infinite (but at most countable) emitters remain true verbatim in this general case.

In contrast, however, we show here that the method used in [3] to pass information from countable graphs to row-finite graphs does not apply to graphs having unrestricted cardinality on the edge sets. Specifically, in our main result (Theorem 14), we show that for a given graph $E$, there exists a row-finite graph $F$ for which $L(E)$ is Morita equivalent to $L(F)$ if and only if $E$ contains no uncountable emitters.

We recall that a graph $E = (E^0, E^1, r, s)$ has vertex set $E^0$, edge set $E^1$, and source and range functions $r, s$ respectively. We call a vertex $v \in E^0$ regular in case $1 \leq |s^{-1}(v)| < \infty$; otherwise, $v$ is called singular. The singular vertices consist of the sinks (i.e., vertices which emit no edges) and the infinite emitters (i.e., vertices which emit infinitely many edges).

Additional germane definitions and various notation may be found in the four aforementioned works.
Definition 1. Let $E = (E^0, E^1, r, s)$ be an arbitrary directed graph (i.e., there is no restriction placed on the cardinalities of the vertex set $E^0$ or the edge set $E^1$). By a row-finite equivalent of $E$ we mean a directed graph $F$ for which:

1. $F$ is row-finite, and
2. the Leavitt path algebras $L(E)$ and $L(F)$ are Morita equivalent.

An historical comment is in order here. For the graph $C^*$-algebraists, the non-existence of sinks in a graph seems to play an important role. Thus the analog of the aforementioned “trading-in” process in the context of $C^*$-algebras seeks to trade in an arbitrary graph for a graph that is not only row-finite, but contains no sinks as well; in other words, a graph which contains no singular vertices. Rephrased, the $C^*$-algebraists are interested in a desingularized equivalent of a graph, which for notational convenience is simply referred to as a desingularization of a graph.

In our main result, we will show that there exists a row-finite equivalent of a graph $E$ if and only if there exists a desingularization of $E$, if and only if $E$ contains no uncountable emitters.

For any edge $e \in E^1$ it is always the case that $ee^*$ is an idempotent in $L(E)$, and that if $e \neq f \in E^1$ then $ee^*$ and $ff^*$ are orthogonal. The following is thereby straightforward.

Lemma 2. Suppose $w$ is an uncountable emitter in $E$. Let the edges being emitted at $w$ be denoted by $\{e_\alpha | \alpha \in A\}$. Then $\{ee^*_\alpha | \alpha \in A\}$ is an uncountable set of pairwise orthogonal idempotents in $\nu L(E)w$. Rephrased, the set $\{ee^*_\alpha | \alpha \in A\}$ is an uncountable set of pairwise orthogonal idempotents in $\text{End}_{L(E)}(L(E)w)$.

We seek to show that if $F$ is a row-finite graph ($F$ is allowed to have uncountably many vertices and / or edges), then there is no finitely generated projective left $L(F)$-module whose endomorphism ring contains an uncountable set of pairwise orthogonal idempotents. The conclusion will then be that if $E$ contains an uncountable emitter, then $L(E)$ cannot be Morita equivalent to $L(F)$, since otherwise we would contradict Lemma 2.

Proposition 3. Let $F$ be row-finite (but possibly with uncountably many vertices and / or edges). Then there is an isomorphism of semigroups $\varphi : M_F \to V(L(F))$, where $M_F$ denotes the semigroup described in [4].

Proof. Since any row-finite graph is the direct limit of its finite complete subgraphs, the proof is identical to that given in [4, Theorem 3.5]. □

Proposition 4. If $\Phi : L(E)\text{Mod} \to L(F)\text{Mod}$ is a Morita equivalence, and $P \in V(L(E))$, then $\Phi(P) \in V(L(F))$.

Proof. This is established in [6, Corollary 5.6]. □

Proposition 5. If $\Phi : L(E)\text{Mod} \to L(F)\text{Mod}$ is a Morita equivalence, then for each $w \in E^0$ we have an isomorphism of left $L(F)$-modules $\Phi(L(E)w) \cong \oplus_{i=1}^{n} L(F)v_i$ for some (not necessarily distinct) $\{v_i | 1 \leq i \leq n\} \subseteq F^0$.

Proof. By Proposition [4, Proposition 3] $\Phi(L(E)w)$ is in $V(L(F))$. But by Proposition [3, each object in $V(L(F))$ is isomorphic to an $L(F)$-module of the indicated type. □
Corollary 6. If $E$ is a graph which contains an uncountable emitter, and $F$ is a graph with the property that for any finite set $\{v_i|1 \leq i \leq n\} \subseteq F^0$ the ring $\text{End}_{L(F)}(\oplus_{i=1}^n L(F)v_i)$ does not contain an uncountable set of pairwise orthogonal idempotents, then $L(E)$ cannot be Morita equivalent to $L(F)$.

Proof. To the contrary, suppose $\Phi : L(E)\text{Mod} \rightarrow L(F)\text{Mod}$ is a Morita equivalence. Let $w$ denote an uncountable emitter in $E$. Then, by Proposition 5 $\Phi(L(E)w) \cong \oplus_{i=1}^n L(F)v_i$ for some $\{v_i|1 \leq i \leq n\} \subseteq F^0$. But Morita equivalence preserves endomorphism rings, so this would yield $\text{End}_{L(E)}(L(E)w) \cong \text{End}_{L(F)}(\oplus_{i=1}^n L(F)v_i)$. But as noted in Lemma 2 $\text{End}_{L(E)}(L(E)w)$ contains an uncountable set of orthogonal idempotents, while by hypothesis $\text{End}_{L(F)}(\oplus_{i=1}^n L(F)v_i)$ does not.

Proposition 7. Let $F$ be row-finite. Let $v, v' \in F^0$. Then there are at most countably many distinct expressions of the form $pq^*$ in $L(F)$ for which $s(p) = v, r(q^*) = v'$, and $r(p) = r(q)$.

Proof. Because $F$ is row-finite, for any positive integer $N$ and any vertex $v$ there exists at most finitely many distinct paths of length $N$ which emanate from $v$. So there are at most countably many distinct paths in $F$ which emanate from $v$. Similarly there are at most countably many distinct (real) paths which emanate from $v'$, so that there are at most countably many ghost paths of the form $q^*$ having $r(q^*) = s(q) = v'$. Now any nonzero expression of the form $pq^*$ corresponds to a pair of directed paths $p$ and $q$ for which $s(p) = v, r(q^*) = v'$, and $r(p) = r(q)$, and the result follows.

Corollary 8. Let $F$ be row-finite. Let $v, v' \in F^0$. Then $\dim_K(vL_K(F)v')$ is at most countable.

Proof. As a $K$-space, $L_K(F)$ is spanned by expressions of the form

$$\{pq^*|p, q \text{ are paths in } F \text{ with } r(p) = r(q)\}.$$

(This set is typically not linearly independent, but that is not of concern here.) Then $vL_K(F)v'$ is spanned by expressions of the form

$$\{pq^*|p, q \text{ are paths in } F \text{ with } s(p) = v, s(q) = r(q^*) = v' \text{ and } r(p) = r(q)\}.$$

The result now follows from Proposition 7.

Corollary 9. Let $F$ be row-finite, and let $\{v_i|1 \leq i \leq n\} \subseteq F^0$. Then the $K$-dimension of the $K$-algebra $\text{End}_{L(F)}(\oplus_{i=1}^n L(F)v_i)$ is at most countable.

Proof. Since each $v_i$ is idempotent, it is standard that as a ring we have $\text{End}_{L(F)}(\oplus_{i=1}^n L(F)v_i) \cong R$, where $R$ is the $n \times n$ matrix ring having $R_{i,j} = v_i L(F) v_j$ for each pair $1 \leq i, j \leq n$. But this isomorphism is clearly seen to be a $K$-algebra map as well. Since $\dim_K(R_{i,j})$ is at most countable for each pair $i, j$ by Corollary 8 we are done.

Lemma 10. If $B$ is any $K$-algebra, and $B$ contains a set $S$ of nonzero orthogonal idempotents, then $\dim_K(B) \geq \text{card}(S)$.

Proof. Suppose $\sum_{i=1}^n k_i e_i = 0$ with $k_i \in K$ and $e_i \in S$. Then by hypothesis each $e_i \neq 0$, and $e_i e_j = \delta_{ij} e_i$ for all $i, j$. So multiplying the given equation on the right by $e_i$ gives $k_i e_i = 0$, whence $k_i = 0$ and we are done.

Putting all the pieces of the puzzle together, we now have the tools to conclude
Proposition 11. Suppose $E$ contains an uncountable emitter. Then $E$ admits no row-finite equivalent. (In particular, $E$ admits no desingularization.)

Proof. Let $F$ be a row-finite graph. By Corollary $\|$, $\text{End}_{L(F)}(\oplus_{i=1}^{n} L(F)v_i)$ has at most countable $K$-dimension, so that, by Lemma $\|$, $\text{End}_{L(F)}(\oplus_{i=1}^{n} L(F)v_i)$ cannot contain an uncountable set of nonzero orthogonal idempotents. Corollary $\|$ now gives the result. 

Proposition 11 establishes one direction of our main result. We now review the appropriate constructions which allow us to build row-finite equivalents. Indeed, we will show that these row-finite equivalents can be chosen in such a way that they are in fact desingularizations. The germane ideas appear in [5] and [3]. If $v_0$ is a sink in $E$, then by adding a tail at $v_0$ we mean attaching a graph of the form 

$$
\bullet^{v_0} \longrightarrow \bullet^{v_1} \longrightarrow \bullet^{v_2} \longrightarrow \bullet^{v_3} \longrightarrow \cdots
$$

We remove the edges in $s^{-1}(v_0)$, and for every $e_j \in s^{-1}(v_0)$ we draw an edge $g_j$ from $v_{j-1}$ to $r(e_j)$.

Example 12. If we consider the infinite rose graph $R_{\infty}$ having one vertex and countably many loops, then a desingularization of $R_{\infty}$ is the following graph

Remark 13. Obviously a desingularization of a graph is always row-finite and has no sinks. In general, as noted in [5], there might be different graphs $\mathcal{F}$ that are desingularizations of $\mathcal{E}$. In fact, different orderings of the edges of $s^{-1}(v_0)$ may give rise to nonisomorphic graphs via the desingularization process.

Theorem 14. Let $\mathcal{E}$ be an arbitrary graph. The following are equivalent:

1. $\mathcal{E}$ admits a row-finite equivalent.
2. $\mathcal{E}$ admits a desingularization.
3. $\mathcal{E}$ contains no uncountable emitters.

Proof. (2) implies (1) is trivial, while (1) implies (3) is (the contrapositive of) Proposition $\|$. For (3) implies (2), suppose that $\mathcal{E}$ contains no uncountable emitters. We show that we can desingularize $\mathcal{E}$. To do so we mimic the process described in [3, Theorem 5.2]; we describe the process here for completeness.

We construct a graph $\mathcal{F}$ by adding a tail at each sink and countable emitter of $\mathcal{E}$ as described above. Note in particular that $E^0 \subseteq F^0$. By [3, Proposition 5.1], there exists

$$
\phi : L(E) \hookrightarrow L(F)
$$
a monomorphism of algebras.

Recall that \( L(E) \) has the collection of sums of distinct vertices as a set of local units. In other words, if we label the vertices \( E^0 = \{v_\alpha | \alpha \in A \} \), then the set

\[
T = \{ \sum_{j \in A_x} v_j | A_x \text{ is a finite subset of } A \}
\]

is a set of local units for \( L(E) \). Since \( E^0 \subseteq F^0 \), we may view the elements of \( T \) as elements of \( L(F) \) as well.

We pick an arbitrary idempotent \( t \in T \). We claim that \( tL(E)t \cong tL(F)t \). Suppose \( t = \sum_{j \in A_x} v_j \) for the finite subset \( A_x \) of \( A \). We consider the restriction \( \phi|_{tL(E)t} : tL(E)t \hookrightarrow L(F) \). Since \( \phi(t) = t \) (where we identify a singular vertex \( v \in E \) with its corresponding \( v_0 \) in \( F \)), we have that \( \phi|_{tL(E)t} \) is indeed a monomorphism from \( tL(E)t \) to \( tL(F)t \), so that we only need to see that this restriction is onto.

Recall that \( tL(F)t \) is the linear span of the monomials of the form \( pq^* \) where \( r(p) = r(q) \) and both \( p \) and \( q \) are paths in \( F \) that begin at any vertex \( v_l \) with \( l \in A_x \). Note that any path \( p \) in the previous conditions must be of the form \( p_1 \ldots p_r f_1 \ldots f_{j-1} \) where \( p_n \) are either edges already in \( E \) or new paths in \( F \) of the form \( \overrightarrow{f_1} \ldots \overrightarrow{f_{h-1}}\overrightarrow{f_h} \), and \( f_m \) are edges along a tail. Any of the \( p_i \)'s is obviously in the image of \( \phi \). So it is enough to show that

\[
(f_1 \ldots f_{j-1})((f')^*_{j-1} \ldots (f')^*_1)
\]

is in the image of \( \phi \).

But this is done exactly as in the proof of \([3]\) Propostion 5.2]. Indeed, the rest of that proof remains valid verbatim in this more general setting, so we are done. (Alternately, we could also use the construction given in \([8]\) Lemma 6.7 to reach the same conclusion as well.) \( \square \)

**Remark 15.** Let \( E \) be a graph with no uncountable emitters. Then one may form the row-finite graph \( G \) by adding a tail at each countable emitter of \( E \), but, in contrast to the previous construction, not adding a tail at each sink of \( E \). Then \( G \) is clearly row-finite, and it can be shown, using the same proof as given in Theorem \([14]\) that \( G \) is a row-finite equivalent of \( E \). The graph \( G \) gives a perhaps more “efficient” row-finite equivalent of \( E \) than does the desingularization process.

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