ON MAXIMAL REGULARITY AND SEMIVARIATION OF $\alpha$-TIMES RESOLVENT FAMILIES

FU-BO LI AND MIAO LI

Abstract. Let $1 < \alpha < 2$ and $A$ be the generator of an $\alpha$-times resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ on a Banach space $X$. It is shown that the fractional Cauchy problem $D_\alpha^\mu u(t) = Au(t) + f(t)$, $t \in [0, r]$; $u(0), u'(0) \in D(A)$ has maximal regularity on $C([0, r]; X)$ if and only if $S_\alpha(\cdot)$ is of bounded semivariation on $[0, r]$.

1. Introduction

Many initial and boundary value problems can be reduced to an abstract Cauchy problem of the form

$$
\begin{align*}
    u'(t) &= Au(t) + f(t), \quad t \in [0, r] \\
    u(0) &= x \in D(A)
\end{align*}
$$

where $A$ is the generator of a $C_0$-semigroup. One says that (1.1) has maximal regularity on $C([0, r]; X)$ if for every $f \in C([0, r]; X)$ there exists a unique $u \in C^1([0, r]; X)$ satisfying (1.1). From the closed graph theorem it follows easily that if there is maximal regularity on $C([0, r]; X)$, then there exists a constant $C > 0$ such that

$$
\|u'\|_{C([0, r]; X)} + \|Au\|_{C([0, r]; X)} \leq \|f\|_{C([0, r]; X)}.
$$

Travis [5] proved that the maximal regularity is equivalent to the $C_0$-semigroup generated by $A$ being of bounded semivariation on $[0, r]$.

Chyan, Shaw and Piskarev [2] gave similar results for second order Cauchy problems. More precisely, they showed that the second order Cauchy problem

$$
\begin{align*}
    u''(t) &= Au(t) + f(t), \quad t \in [0, r] \\
    u(0) &= x, \quad u'(0) = y, \quad x, y \in D(A)
\end{align*}
$$

has maximal regularity on $[0, r]$ if and only if the cosine operator function generated by $A$ is of bounded semivariation on $[0, r]$.

In this paper we will consider the maximal regularity for fractional Cauchy problem

$$
\begin{align*}
    D_\alpha^\mu u(t) &= Au(t) + f(t), \quad t \in [0, r] \\
    u(0) &= x, \quad u'(0) = y, \quad x, y \in D(A)
\end{align*}
$$

where $\alpha \in (1, 2)$, $A$ is the generator of an $\alpha$-times resolvent family (see Definition 2.2 below) and $D_\alpha^\mu u$ is understood in the Caputo sense. We show that (1.3) has maximal regularity on $C([0, r]; X)$ if and only if the corresponding $\alpha$-times resolvent family is of bounded semivariation on $[0, r]$.

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2. Preliminaries

Let $1 < \alpha < 2$, $g_0(t) := \delta(t)$ and $g_\beta(t) := \frac{\alpha-1}{\Gamma(\alpha)}(\beta > 0)$ for $t > 0$. Recall the Caputo fractional derivative of order $\alpha > 0$

$$D^\alpha_t f(t) := \int_0^t g_{2-\alpha}(t-s)\frac{d^2}{ds^2}f(s)ds, \quad t \in [0, r]$$

for $f \in C^2([0, r]; X)$. The condition that $f \in C^2([0, r]; X)$ can be relaxed to $f \in C^1([0, r]; X)$ and $g_{2-\alpha} * (f - f(0) - f'(0)g_\alpha) \in C^2([0, r]; X)$, for details and further properties see [1] and references therein. And in the above we denote by

$$(g_\beta * f)(t) = \int_0^t g_\beta(t-s)f(s)ds$$

the convolution of $g_\beta$ with $f$. Note that $g_\alpha * g_\beta = g_{\alpha+\beta}$.

Consider a closed linear operator $A$ densely defined in a Banach space $X$ and the fractional evolution equation (1.3).

**Definition 2.1.** A function $u \in C([0, r]; X)$ is called a strong solution of (1.3) if

$$u \in C([0, r]; D(A)) \cap C^1([0, r]; X), \quad g_{2-\alpha} * (u(t) - x - ty) \in C^2([0, r]; X)$$

and (1.3) holds on $[0, r]$. $u \in C([0, r]; X)$ is called a mild solution of (1.3) if $g_\alpha * u \in D(A)$ and

$$u(t) - x - ty = A(g_\alpha * u)(t) + (g_\alpha * f)(t)$$

for $t \in [0, r]$.

**Definition 2.2.** Assume that $A$ is a closed, densely defined linear operator on $X$. A family $\{S_\alpha(t)\}_{\alpha \geq 0} \subset B(X)$ is called an $\alpha$-times resolvent family generated by $A$ if the following conditions are satisfied:

(a) $S_\alpha(\cdot)$ is strongly continuous on $\mathbb{R}_+$ and $S_\alpha(0) = I$;

(b) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$;

(c) For all $x \in D(A)$ and $t \geq 0$, $S_\alpha(t)x = x + (g_\alpha * S_\alpha)(t)x$.

**Remark 2.3.** Since $A$ is closed and densely defined, it is easy to show that for all $x \in X$, $(g_\alpha * S_\alpha)(t)x \in D(A)$ and $A(g_\alpha * S_\alpha)(t)x = S_\alpha x - x$.

The alpha-times resolvent families are closely related to the solutions of (1.3). It was shown in [1] that if $A$ generates an $\alpha$-times resolvent family $S_\alpha(\cdot)$, then (1.3) has a unique strong solution given by $S_\alpha(t)x + \int_0^t S_\alpha(s)gds$.

Next we recall the definition of functions of bounded semivariation (see e.g. [3]). Given a closed interval $[a, b]$ of the real line, a subdivision of $[a, b]$ is a finite sequence $d : a = d_0 < d_1 < \cdots < d_n = b$. Let $D[a, b]$ denote the set of all subdivisions of $[a, b]$.

**Definition 2.4.** For $G : [a, b] \to B(X)$ and $d \in D[a, b]$, define

$$SV_d[G] = \sup\{\|\sum_{n=1}^n [G(d_i) - G(d_{i-1})]x_i\| : x_i \in X, \|x_i\| \leq 1\}$$

and $SV[G] = \sup\{SV_d[G] : d \in D[a, b]\}$. We say $G$ is of bounded semivariation if $SV[G] < \infty$. 
3. Main results

We begin with some properties on \( \alpha \)-times resolvent families which will be needed in the sequel.

**Proposition 3.1.** Let \( 1 < \alpha < 2 \) and \( \{S_\alpha(t)\}_{t \geq 0} \) be the \( \alpha \)-times resolvent family with generator \( A \). Define

\[
P_\alpha(t)x = (g_{\alpha-1} \ast S_\alpha)(t)x = \int_0^t g_{\alpha-1}(t-s)S_\alpha(s)x ds, \quad x \in X,
\]

then the following statements are true.

(a) For every \( x \in X \), \( \int_0^t P_\alpha(s)dx \in D(A) \) and

\[
A \int_0^t P_\alpha(s)dx = S_\alpha(t)x - x;
\]

(b) For every \( x \in X \), \( 0 \leq a, b \leq t \), \( \int_a^b sP_\alpha(t-s)dx \in D(A) \) and

\[
A \int_a^b sP_\alpha(t-s)dx = aS_\alpha(t-a)x - bS_\alpha(t-b)x + \int_a^b S_\alpha(t-s)dx;
\]

(c) For every \( x \in X \), \( \int_0^t g_\alpha(t-s)P_\alpha(s)dx \in D(A) \) and

\[
A \left( \int_0^t g_\alpha(t-s)P_\alpha(s)dx \right) = -\alpha(g_\alpha \ast S_\alpha)(t)x + tP_\alpha(t)x;
\]

(d) If \( f \in C([0, r]; X) \), then \( g_\alpha \ast S_\alpha \ast f \in D(A) \) and

\[
A(g_\alpha \ast S_\alpha \ast f) = (S_\alpha - 1) \ast f.
\]

**Proof.** (a) follows from the fact that \( \int_0^t P_\alpha(s)dx = (g_1 \ast g_{\alpha-1} \ast S_\alpha)(t)x = (g_\alpha \ast S_\alpha)(t)x \in D(A) \) and

\[
A(g_\alpha \ast S_\alpha)(t)x = S_\alpha(t)x - x \quad \text{by Remark 2.3}
\]

(b) By integration by parts we have

\[
\int_a^b sP_\alpha(t-s)dx = \int_a^b sd_a \left[ \int_0^s P_\alpha(t-\tau)x d\tau \right]
\]

\[
= \int_a^b sd_a [g_\alpha \ast S_\alpha](t-s)x]
\]

\[
= -s(g_\alpha \ast S_\alpha)(t-s)x \bigg|_a^b + \int_a^b (g_\alpha \ast S_\alpha)(t-s)dx
\]

\[
= a(g_\alpha \ast S_\alpha)(t-a)x - b(g_\alpha \ast S_\alpha)(t-b)x + \int_a^b (g_\alpha \ast S_\alpha)(t-s)dx,
\]

since \( (g_\alpha \ast S_\alpha)(t)x \in D(A) \) by Remark 2.3, operating \( A \) on both sides of the above identity gives (b).
(c) follows from the fact that
\[
\int_0^t g_\alpha(t-s)sP_\alpha(s) ds = \int_0^t g_\alpha(t-s)(s-t)P_\alpha(s) ds + t \int_0^t g_\alpha(t-s)P_\alpha(s) ds
\]
\[
= -\alpha \int_0^t g_{\alpha+1}(t-s)P_\alpha(s) ds + t(g_\alpha \ast P_\alpha)(t)x
\]
\[
= -\alpha(g_{\alpha+1} \ast P_\alpha)(t)x + t(g_\alpha \ast (\alpha \ast S_\alpha))(t)x
\]
\[
= -\alpha(g_\alpha \ast \alpha \ast S_\alpha)(t)x + t(g_\alpha \ast P_\alpha)(t)x
\]
belongs to \(D(A)\) and
\[
A(\int_0^t g_\alpha(t-s)P_\alpha(s) ds) = -\alpha(g_\alpha \ast A(g_\alpha \ast S_\alpha))(t)x + t(g_\alpha \ast (\alpha \ast S_\alpha))(t)x
\]
\[
= -\alpha(g_\alpha \ast (S_\alpha - 1))(t)x + t(g_\alpha \ast (S_\alpha - 1))(t)x
\]
\[
= -\alpha(g_\alpha \ast S_\alpha)(t)x + t(g_\alpha \ast P_\alpha)(t)x + t(g_\alpha \ast S_\alpha)(t)x - t\alpha g_\alpha(t)x
\]
\[
= -\alpha(g_\alpha \ast S_\alpha)(t)x + tP_\alpha(t)x.
\]

(d) (3.1) is true for step functions, and then for continuous functions by the closedness of \(A\).

The following two lemmas can be proved similarly as that in [2 5].

**Lemma 3.2.** If \(f \in C([0,r]; X)\) and the \(\alpha\)-times resolvent family \(S_\alpha(t)\) is of bounded semivariation on \([0,r]\), then \((P_\alpha \ast f)(t) \in D(A)\) and
\[
A(P_\alpha \ast f)(t) = -\int_0^t d_s[S_\alpha(t-s)]f(s).
\]

**Lemma 3.3.** If \(f \in C([0,r]; X)\) and the \(\alpha\)-times resolvent family \(S_\alpha(t)\) is of bounded semivariation on \([0,r]\), then \(\int_0^t d_s[S_\alpha(t-s)]f(s)\) is continuous in \(t\) on \([0,r]\).

We next turn to the solution of
\[
D_\alpha^\circ u(t) = Au(t) + f(t), \quad t \in [0,r],
\]
\[
u(0) = 0, \quad u'(0) = 0,
\]
where \(A\) is the generator of an \(\alpha\)-times resolvent family. If \(v(t)\) is a mild solution of (3.2), then by Definition 2.1 \((g_\alpha \ast v)(t) \in D(A)\) and \(v(t) = A(g_\alpha \ast v)(t) + (g_\alpha \ast f)(t)\). It then follows from the properties of \(\alpha\)-times resolvent family that
\[
1 \ast v = (S_\alpha - A(g_\alpha \ast S_\alpha)) \ast v = S_\alpha \ast v - S_\alpha \ast A(g_\alpha \ast v) = S_\alpha \ast (v - (g_\alpha \ast v)) = S_\alpha \ast g_\alpha \ast f,
\]
which implies that \(g_\alpha \ast S_\alpha \ast f\) is differentiable and
\[
v(t) = \frac{d}{dt}(g_\alpha \ast S_\alpha \ast f)(t) = (g_\alpha \ast S_\alpha \ast f)(t) = (P_\alpha \ast f)(t).
\]
Therefore, the mild solution of (1.3) is given by
\[
u(t) = S_\alpha(t)x + \int_0^t S_\alpha(s)yds + (P_\alpha \ast f)(t).
\]
Proposition 3.4. Let $A$ be the generator of an $\alpha$-times resolvent family $S_\alpha(\cdot)$, and let $f \in C([0,r];X)$ and $x,y \in D(A)$. Then the following statements are equivalent:
(a) (1.3) has a strong solution;
(b) $(S_\alpha * f)(\cdot) \in C^1([0,r];X)$;
(c) $(P_\alpha * f)(t) \in D(A)$ for $0 \leq t \leq r$ and $A(P_\alpha * f)(t)$ is continuous in $t$ on $[0,r]$.

Proof. (a) If $u(t)$ is a strong solution of (1.3), then $u$ is given by (3.3) since every strong solution is a mild solution. Therefore, by the definition of strong solutions, $g_{2-\alpha} * P_\alpha * f = g_1 * S_\alpha * f \in C^2([0,r];X)$; it then follows that $S_\alpha * f \in C^1([0,r];X)$, this is (b).

(b) $\Rightarrow$ (c). Suppose that $S_\alpha * f \in C^1([0,r];X)$. Since $g_1 * P_\alpha * f = g_\alpha * S_\alpha * f$, by Proposition 3.4 (d), $g_1 * P_\alpha * f \in D(A)$ and
\begin{align*}
A(g_1 * P_\alpha * f) &= A(g_\alpha * S_\alpha * f) = (S_\alpha - 1) * f.
\end{align*}
Since $A$ is closed and $S_\alpha * f \in C^1([0,r];X)$, we have $P_\alpha * f \in D(A)$ and $A(P_\alpha * f) = (S_\alpha * f)' - f$ is continuous.

(c) $\Rightarrow$ (a). By (3.4), $g_1 * A(P_\alpha * f) = A(g_1 * P_\alpha * f) = (S_\alpha - 1) * f$, therefore $S_\alpha * f$ is differentiable and thus $g_{2-\alpha} * P_\alpha * f = g_1 * S_\alpha * f$ in $C^2([0,r];X)$. It is easy to check that $u(t)$ defined by (3.3) is a strong solution of (1.3).

Now we are in the position to give the main result of this paper. The proof is similar to that of Proposition 3.1 in [5] or Theorem 4.2 in [2], we write it out for completeness.

Theorem 3.5. Suppose that $A$ generates an $\alpha$-times resolvent family $\{S_\alpha(t)\}_{t \geq 0}$. Then the function (2.3) is a strong solution of the Cauchy problem (1.3) for every pair $x,y \in D(A)$ and continuous function $f$ if and only if $S_\alpha(\cdot)$ is of bounded semivariation on $[0,r]$.

Proof. The sufficiency follows from Lemmas 3.2 and 3.3.

Conversely, suppose that for $x,y \in D(A)$ and continuous function $f$, $u(t)$ given by (3.3) is a strong solution for (1.3). Define the bounded linear operator $L : C([0,r];X) \rightarrow X$ by $L(f) = (P_\alpha * f)(r)$. By Proposition 3.4 (c) $Lf \in D(A)$, it thus follows from the closeness of $A$ that $AL : C([0,r];X) \rightarrow X$ is bounded.

Let $\{d_i\}_{i=0}^n$ be a subdivision of $[0,r]$ and $\epsilon > 0$ such that $\epsilon < \min_{1 \leq i \leq n} \{d_i - d_{i-1}\}$. For $x_i \in X$ with $\|x_i\| \leq 1$ ($i = 1,2,\ldots,n+1$), define $f_{d,\epsilon} \in C([0,r];X)$ by
\begin{align*}
f_{d,\epsilon}(\tau) &= \begin{cases} x_i, & d_{i-1} - \epsilon \leq \tau \leq d_i - \epsilon, \\
x_{i+1} + \frac{\tau - d_i}{\epsilon}(x_{i+1} - x_i), & d_i - \epsilon \leq \tau \leq d_i, \end{cases}
\end{align*}
then $\|f_{d,\epsilon}\|_{C([0,r];X)} \leq 1$. By Proposition 3.1
\begin{align*}
AL(f_{d,\epsilon}) &= A \int_0^r P_\alpha(r - s)f_{d,\epsilon}(s)ds \\
&= \sum_{i=1}^n \left[ A \int_{d_{i-1}}^{d_i} P_\alpha(r - s)x_ids + \frac{d_i}{\epsilon} \int_{d_{i-1}}^{d_i} P_\alpha(r - s)(x_{i+1} - x_i)dx ight]
\end{align*}
\[
\sum_{i=1}^{n} \left\{ [S_\alpha(r - d_{i-1}) x_i - S_\alpha(r - d_i + \epsilon) x_i] + [S_\alpha(r - d_i + \epsilon) x_{i+1} - S_\alpha(r - d_i) x_{i+1}] - \frac{d_i}{\epsilon} [S_\alpha(r - d_i + \epsilon)(x_{i+1} - x_i) - S_\alpha(r - d_i)(x_{i+1} - x_i)] + \frac{1}{\epsilon} \int_{d_i - \epsilon}^{d_i} S_\alpha(r - s)(x_{i+1} - x_i) ds \right\}
\]

By letting \( \epsilon \to 0 \), we obtain that \( S_\alpha \) is of bounded semivariation on \([0, r]\). \(\square\)

**Corollary 3.6.** Suppose that \( \{S_\alpha(t)\}_{t \geq 0} \) is an \( \alpha \)-times resolvent family with generator \( A \) and \( S_\alpha(\cdot) \) is of bounded semivariation on \([0, r]\) for some \( r > 0 \). Then \( R(P_\alpha(t)) \subset D(A) \) for \( t \in [0, r] \) and \( \|tAP_\alpha(t)\| \) is bounded on \([0, r]\).

**Proof.** For \( x \in X \), consider \( f(t) = \alpha S_\alpha(t)x \). By Proposition 3.1, \( tP_\alpha(t)x \) is a mild solution of (3.2). Moreover, it follows from Proposition 3.4 that \( P_\alpha * f \) is a strong solution of (3.2). Since a strong solution must be a mild solution, we have \( (P_\alpha * f)(t) = tP_\alpha(t)x \). Thus our claim follows from Proposition 3.4. \(\square\)

**Remark 3.7.** Let \( \alpha = 1 \). If \( A \) generates a \( C_0 \)-semigroup \( T(\cdot) \), then the condition that \( tAT(\cdot) \) is bounded on \([0, r]\) implies that \( T(\cdot) \) is analytic (see [4]). When \( \alpha = 2 \) and \( A \) generates a cosine function \( C(\cdot) \), then the condition that \( tAC(\cdot) \) is bounded on \([0, r]\) implies that \( A \) is bounded (2). However, since there is no semigroup properties for \( \alpha \)-times resolvent family, it is not clear that one can get the analyticity of \( S_\alpha(\cdot) \) from the local boundedness of \( tAP_\alpha(t) \).

**References**

[1] E.G. Bajlekova, Fractional Evolution Equations in Banach Spaces, Dissertation, Eindhoven University of Technology, 2001.
[2] D.K. Chyan, S.Y. Shaw and S. Piskarev, *On maximal regularity and semivariation of cosine operator functions*, J. London Math. Soc. 59 (1999), 1023-1032.

[3] C.S. Hönig, Volterra Stieltjes Integral Equations, North-Holland, Amsterdam, 1975.

[4] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Math. Series 44, Springer, New-York, 1984.

[5] C.C. Travis, Differentiability of weak solutions to an abstract inhomogeneous differential equation, Proc. Amer. Math. Soc. 82 (1981), 425-430.

Department of Mathematics, Sichuan University, Chengdu, Sichuan 610064, P.R. China

E-mail address: lifuboscu.edu.cn; mli@scu.edu.cn