I will sketch a proof that the short distance behavior of the even scaling functions for the Ising model that arise by taking the scaling limit of the Ising correlation functions from below the critical temperature is given by the Luther–Peschel formula [6] (see below). The fact that the Luther–Peschel formula is consistent with conformal field theory insights into the correlations for the large scale limit of the critical Ising model has lead to the conviction in the Physics community that this formula is right [4]. The critical scaling limit for the Ising model is obtained by sitting right at $T_c$ (the critical temperature) and extracting the large scale limit of the correlations at this fixed temperature. However, as far as I am aware there is no mathematical proof that the large scale limit of the critical Ising model even exists (with the exception of the asymptotics of the diagonal two point correlation at the critical point [15]). In this note we will not address the question of the critical (massless) scaling limit for the Ising model but instead we will consider the massive scaling limit obtained by looking at the lattice correlations at the scale of the correlation length as the temperature approaches the critical temperature. This scaling limit was first considered in [14] and [11] and a mathematical proof of its existence was established in [7]. What we will sketch a proof of here is that the short distance asymptotics of this scaling limit for the even scaling function from below $T_c$ is given by the Luther–Peschel formula. There is a conjecture in the Physics literature called the scaling hypothesis which suggests that the large scale asymptotics at the critical temperature should agree with the short distance behaviour of the massive scaling theory. A rather detailed version of this scaling hypothesis has been proved for the two point function [12],[13]. Modulo the fact that this scaling hypothesis can only be literally correct for the scaling limit from above $T_c$ (since the critical Ising correlations vanish for an odd number of spins on the lattice and the odd scaled correlations from below $T_c$ are non-zero) this hypothesis does suggest a connection between the result we consider here and the conjectured result for the critical scaling limit. Incidentally, the technique to be explained below should extend to an analysis of the short distance asymptotics of the odd correlations in the $T < T_c$ scaling limit.

In [11] M. Sato, T. Miwa and M. Jimbo showed that the scaled correlations for the Ising model have logarithmic derivatives that can be expressed in terms of the solution to a monodromy preserving deformation problem. In one version of this connection the log derivative is expressed in terms of the solution to non-linear monodromy preserving deformation equations. Of more technical interest for us is an alternative connection with the linear differential equation which gives rise to the monodromy preserving deformation. Much recent work on the asymptotics of non-linear isomonodromic equations exploits in a precisely analogous manner the connection with an associated linear problem (see [3] and references for an account of modern Riemann–Hilbert techniques in the analysis of such
problems).

I will begin by recalling the Sato, Miwa, Jimbo characterization [11] for the logarithmic derivative of the scaled correlations for the two dimensional Ising model (a more detailed proof of this result appears in [8]). The two dimensional Ising model on the lattice $\mathbb{Z}^2$ is a classical statistical mechanical system of $\pm 1$ spins with ferromagnetic nearest neighbor interactions. The Boltzmann weight for a configuration of spins $\sigma_i$ at sites $i \in \Lambda$ with $\Lambda$ a finite subset of $\mathbb{Z}^2$ is given by

$$e^K \sum_{<ij>} \sigma_i \sigma_j,$$

where the sum is over nearest neighbors $<ij>$ in $\Lambda$ ($K > 0$ favors nearest neighbor alignment and is referred to as the ferromagnetic case). To be a little more precise if the sum is restricted to nearest neighbors $<ij>$ both of which are in $\Lambda$ we will refer to open boundary conditions. If the nearest neighbors $<ij>$ are allowed one element (say $i$) on the boundary of $\Lambda$ (but not in $\Lambda$) and we extend the spin configuration to the boundary by setting $\sigma_i = +1$ then we will refer to plus boundary conditions. Write

$$<\sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_n} >_T,$$

(1.1)

for the thermodynamic (infinite volume) limit of the spin correlation functions at temperature $T > 0$. For $T$ less than the critical temperature $T_c$ we consider the limit of the finite volume correlations with plus boundary conditions and for $T > T_c$ we consider the limit with open boundary conditions (these are the limits discussed in [8]). Next we introduce a scaling parameter $\delta > 0$ and a temperature $T(\delta) < T_c$ so that the correlation length (the exponential rate of clustering for (1.1)) is $\delta^{-1}$ and we define for $a_j \in \mathbb{R}^2$,

$$\tau_-(a_1, a_2, \ldots, a_n) = \lim_{\delta \to 0} \frac{<\sigma_{\delta^{-1}a_1} \sigma_{\delta^{-1}a_2} \cdots \sigma_{\delta^{-1}a_n} >_{T(\delta)}}{<\sigma >_{T(\delta)}},$$

(1.2)

where $<\sigma >_{T(\delta)}$ is the translation invariant one point function (the spontaneous magnetization). The existence of this limit and the subtleties associated with the fact that $\delta^{-1}a_j$ is not always a lattice point in $\mathbb{Z}^2$ are discussed in [7]. We will refer to $\tau_-$ as the scaling function for the Ising model from below $T_c$. We define a scaling function, $\tau_+$, for the Ising model from above $T_c$ by

$$\tau_+(a_1, a_2, \ldots, a_n) = \lim_{\delta \to 0} \frac{<\sigma_{\delta^{-1}a_1} \sigma_{\delta^{-1}a_2} \cdots \sigma_{\delta^{-1}a_n} >_{T(\delta)^*}}{<\sigma >_{T(\delta)^*}},$$

(1.3)

where $T(\delta)^*$ is the Kramers-Wannier dual temperature $T(\delta)^* > T_c$ (see [7]). Note that in the denominator it is $T(\delta) < T_c$ which appears and not the dual temperature; this is significant since $<\sigma >_{T(\delta)^*} = 0$ for $T > T_c$.

In [11] Sato, Miwa and Jimbo introduced a characterization of $d_n \log \tau_\pm(a)$ which we will now describe for $\tau_-$. For later developments it will be useful to introduce a parameter $m > 0$ which will scale the dependence on the $a_j$ in (1.2) and (1.3). For Sato, Miwa and Jimbo the functions $\tau_\pm$ are tau functions for monodromy preserving deformations of the Euclidean Dirac equation on $\mathbb{R}^2$. The appropriate Dirac operator is,

$$mI - \partial = \begin{bmatrix} m & -2\partial \\ -2\partial & m \end{bmatrix},$$

2
where

\[
\partial = \frac{1}{2} \left( \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2} \right),
\]

\[
\bar{\partial} = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right).
\]

Although we will not be working exclusively with holomorphic functions, the presence of \(\partial\) and \(\bar{\partial}\) in the Dirac operator makes it very convenient to identify \(\mathbb{R}^2\) with \(\mathbb{C}\) in the usual way, \((x_1, x_2) \rightarrow z = x_1 + ix_2\) and of course, \(\bar{z} = x_1 - ix_2\). Henceforth we will use this identification and work over \(\mathbb{C}\) instead of \(\mathbb{R}^2\). For brevity we will write \(f(z)\) for a function of the two real variables \(x_1\) and \(x_2\), even though it is customary to use a notation like \(f(z, \bar{z})\) to avoid the temptation to regard \(f(z)\) as a holomorphic function of \(z\).

Let \(a = \{a_1, a_2, \ldots, a_n\}\) denote a collection of \(n\) distinct points in \(\mathbb{C}\). Let \(\mathcal{R}(\mathbb{C} \setminus a)\) denote the 2-fold ramified covering space of \(\mathbb{C} \setminus a\) with base point \(a_0\) covering \(a_0 \in \mathbb{C}\) with \(a_0 \neq a_j\) for \(j = 1, 2, \ldots, n\). Let \(\gamma_j\) denote a simple closed loop based at \(a_0\) surrounding \(a_j\) but no other point \(a_k\) for \(k \neq j\). Let \(\Gamma_j\) denote the deck transformation of \(\mathcal{R}(\mathbb{C} \setminus a)\) associated with \(\gamma_j\).

The differential operator \(mI - \phi\) acts naturally on smooth functions \(f : \mathcal{R}(\mathbb{C} \setminus a) \rightarrow \mathbb{C}^2\). We will call a smooth function \(w : \mathcal{R}(\mathbb{C} \setminus a) \rightarrow \mathbb{C}^2\) an Ising type solution to the Dirac equation if and only if

\[
(mI - \phi)w = 0,
\]

and \(w \circ \Gamma_j = -w\) for \(j = 1, \ldots, n\) (that is \(w\) changes by a sign “going around” the loop \(\gamma_j\)). An Ising type solution to the Dirac equation has local expansions (essentially Fourier expansions) about each of the points \(a_\nu\) given by

\[
w(z) = \sum_{k \in \mathbb{Z}} c_k(w)w_k(z_\nu) + c_k^{*}(w)\bar{w}_k(z_\nu),
\]

where \(z_\nu = z - a_\nu\), the quantities \(c_k(w)\) and \(c_k^{*}(w)\) are complex numbers which we refer to as local expansion coefficients, and the functions \(w_k(z)\) and \(\bar{w}_k(z)\) are multivalued solutions to the Dirac equation defined by (see [11]),

\[
w_k(z) = \begin{bmatrix} v_k(z) \\ v_{k+1}(z) \end{bmatrix} = \begin{bmatrix} m^{k+\frac{1}{2}}z^{k+\frac{1}{2}} \\ 2^{k+\frac{1}{2}}\Gamma(k+\frac{1}{2}) \end{bmatrix} + O(m^{k+\frac{1}{2}}),
\]

(1.4)

\[
\bar{w}_k(z) = \begin{bmatrix} \bar{v}_{k+1}(z) \\ \bar{v}_k(z) \end{bmatrix} = \begin{bmatrix} m^{k+\frac{1}{2}}z^{k+\frac{1}{2}} \\ 2^{k+\frac{1}{2}}\Gamma(k+\frac{1}{2}) \end{bmatrix} + O(m^{k+\frac{1}{2}}).
\]

(1.5)

Where \(v_k(z) = e^{i(k-\frac{1}{2})\theta}I_{k-\frac{1}{2}}(mr)\), \(z = re^{i\theta}\) is the polar form for \(z\), \(I_s\) is the modified Bessel function of order \(s\), and \(\Gamma\) is the usual gamma function. The functions \(w_k(z)\) and \(\bar{w}_k(z)\) are simultaneously solutions to the Dirac equation \((mI - \phi)w_k^{*} = 0\) and eigenfunctions for the infinitesimal rotation that commutes with the Dirac operator. Observe that the
functions $w_k(z)$ and $w_k^*(z)$ become less and less locally singular as functions of $z$ as the index $k$ increases. Whenever we write down a local expansion that explicitly exhibits only a few terms it is understood that the remaining terms involve less locally singular wave functions $w_k^{(*)}$ (i.e., higher values of the index $k$) than the terms which make an explicit appearance.

If we define $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then it is easy to see that $w_k^*$ is obtained from $w_k$ by the following conjugation,

$$w_k^*(z) = C w_k(z).$$

The conjugation defined on $f : \mathbb{C} \rightarrow \mathbb{C}^2$ by $f \rightarrow f^* := C f$ will have special significance for us as it commutes with the Dirac operator and defines a real structure on the space of solutions to the Dirac equation. This real structure plays a role in the Ising model application we are looking at; observe that because the monodromy multiplier -1 is real the conjugation * also defines a real structure on the space of Ising type solutions to the Dirac equation.

Now we recall some results from [11]. Write $x \cdot y = x_1 y_1 + x_2 y_2$ for the standard bilinear form on $\mathbb{C}^2$, so that the standard Hermitian form is given by $\overline{x} \cdot y$. If $u$ and $v$ are Ising type solutions to the Dirac equation then the inner product $\overline{\pi} \cdot v$ descends to a function on $\mathbb{C} \setminus \mathfrak{a}$ since the monodromy multipliers cancel out. We will say that such a solution, $w$, is in $L^2(\mathbb{C})$ if and only if

$$\frac{i}{2} \int_C \overline{w} \cdot w \, dzd\overline{z} < \infty.$$ 

It is a result of SMJ [11] the space of Ising type solutions to the Dirac equation that are in $L^2(\mathbb{C})$ is an $n$ dimensional vector space. The local expansions for such a solution, $w$, are restricted,

$$w(z) = \sum_{k=0}^{\infty} c_k^{(w)} w_k(z_\nu) + c_k^{(w)} w_k^*(z_\nu) \text{ for } \nu = 1, \ldots, n,$$

and there is a formula for the inner product of two such $L^2$ solutions in terms of the leading local expansion coefficients,

$$\frac{i}{2} \int_C \overline{\pi} \cdot v \, dzd\overline{z} = -\frac{4}{m^2} \sum_{\nu} \overline{c_0^{(w)}(u)c_0^{(v)}(v)} = -\frac{4}{m^2} \sum_{\nu} \overline{c_0^{(w)}(u)c_0^{(v)}(v)}, \quad (1.6)$$

which follows from the identities,

$$\frac{i}{2} \overline{u} \cdot v \, dzd\overline{z} = -\frac{i}{m} d(\overline{u}_2 v_1 \, dz) = \frac{i}{m} d(\overline{u}_1 v_2 \, d\overline{z}), \quad (1.7)$$

that are a consequence of $u$ and $v$ satisfying the Dirac equation, and the use of Stokes’ theorem to localize the integrals of the differentials on the right of (1.7) to the singularities
Equation (1.6) shows that $L^2$ Ising type solutions are determined uniquely by the low order expansion coefficients $c_0^* \nu = 1, \ldots, n$. We will now characterize the $L^2$ Ising type solution we are interested in by specifying the behavior of its low order expansion coefficients. This solution will satisfy a "reality" condition with respect to the conjugation * so it will be convenient to work with a real basis constructed from $w_k$ and $w_k^*$. Define $\Re w_k = \frac{1}{2}(w_k + w_k^*)$ and $\Im w_k = \frac{1}{2i}(w_k - w_k^*)$. Then for each $\nu = 1, \ldots, n$ there exists a unique Ising type solution $\mathcal{W}_\nu$ to the Dirac equation which is real ($\mathcal{W}_\nu^* = \mathcal{W}_\nu$), is in $L^2$, and which has leading order local expansions,

$$\mathcal{W}_\nu(z) = \delta_{\nu \nu} \Im w_0(z_\mu) + T_{\mu \nu} \Re w_0(z_\mu) + \cdots,$$

for $z$ near $a_\mu$. The coefficient matrix $T_{\mu \nu}$ is real since $\mathcal{W}_\nu$ is real but as we shall see it is not necessary to specify anything further about it to achieve uniqueness (it will turn out to be skew symmetric). We will consider the existence and uniqueness for $\mathcal{W}_\nu$ in a moment but first we note that the connection with the scaling function $\tau_-$ for the Ising model is,

$$d_a \log \tau_-(ma) = \frac{m}{2i} \sum_\nu c_1^\nu(\mathcal{W}_\nu) da_\nu - c_1^\nu(\mathcal{W}_\nu^*) da_\nu.$$

This result is due to SMJ [11] (see also the proof of theorem 5.1 in [8]) and will be the foundation of the analysis we make of the $L^2$ Ising type solutions to the Dirac equation. This result is due to SMJ [11] (see also the proof of theorem 5.1 in [8]) and will be the foundation of the analysis we make of the $L^2$ Ising type solutions to the Dirac equation. This result is due to SMJ [11] (see also the proof of theorem 5.1 in [8]) and will be the foundation of the analysis we make of the $L^2$ Ising type solutions to the Dirac equation. This result is due to SMJ [11] (see also the proof of theorem 5.1 in [8]) and will be the foundation of the analysis we make of the $L^2$ Ising type solutions to the Dirac equation.

We now recall the construction of the wave function $\mathcal{W}_\nu$ and the proof that it is uniquely determined by the conditions given above. In [11], SMJ prove that there exists a canonical basis $\{w_\nu\}$ for the space of $L^2$ Ising type solutions to the Dirac equation. This basis is characterized by local expansions with leading order terms,

$$w_\nu(z) = \delta_{\mu \nu} w_0(z_\mu) + c_0^\nu(w_\nu) w_0^*(z_\mu) + \cdots,$$

for $z$ near $a_\mu$. Applying the conjugation * to (1.10) we get what is called by SMJ the dual canonical basis,

$$w_\nu^*(z) = \delta_{\mu \nu} w_0^*(z_\mu) + c_0^{\nu*}(w_\nu) w_0^*(z_\mu) + \cdots.$$  

The formula for the inner product (1.6) shows that $L^2$ Ising type solutions are uniquely determined by the values of the $n$ linear functionals $c_0^\nu$ for $\nu = 1, \ldots, n$. Hence we find,

$$w_\nu(z) = \sum_\alpha c_0^{\alpha*}(w_\nu) w_\alpha^*(z),$$

and

$$\sum_\alpha c_0^{\alpha*}(w_\nu) c_0^{\alpha*}(w_\alpha) = \delta_{\mu \nu}.$$  

If we use the formula (1.6) to calculate the inner product of two elements of the canonical basis we find,

$$<w_\mu, w_\nu> = \frac{i}{2} \int_C w_\mu \cdot w_\nu \; d\alpha d\tau = -\frac{4}{m^2} c_0^{\mu*}(w_\nu) = -\frac{4}{m^2} c_0^{\nu*}(w_\mu).$$
Now write $c_0^*$ for the matrix with $\mu \nu$ matrix element $c_0^{\mu *}(w_\nu)$ and we can summarize (1.13) and (1.14) by saying that $c_0^*$ is an hermitian symmetric, negative definite matrix with $\overline{c_0^*}c_0^* = I$. If we now rewrite the local expansion result for $W_\nu$ in terms of $w_0$ and $w_0^*$ instead of $\Im w_0$ and $\Re w_0$ we find by comparing local expansions that,

$$
W_\nu = \frac{1}{2} \sum_\mu (T_{\mu \nu} - i\delta_{\mu \nu})w_\mu,
$$

$$
W_\nu = \frac{1}{2} \sum_\mu (T_{\mu \nu} + i\delta_{\mu \nu})w_\mu^*.
$$

Comparing these two results with (1.12) we find,

$$
c_0^*(T - iI) = (T + iI),
$$

where $T$ denotes the matrix with $\mu \nu$ matrix element $T_{\mu \nu}$. We can solve this for $T$ to get

$$
T = i(c_0^* - I)^{-1}(c_0^* + I).
$$

The fact that $c_0^*$ is negative definite implies that $c_0^* - I$ is invertible so we can define $T$ by (1.17). Since $c_0^*$ is Hermitian symmetric it follows that $T^* = -T$. Since $\overline{c_0^*}c_0^* = I$ it follows that $\overline{T} = T$, that is $T$ is real. Thus $T$ must be (real) skew symmetric as well. With this definition for $T$ we can define $W_\nu$ by (1.15) and uniqueness follows from the developments leading up to (1.17).

We are interested in computing the $m \to 0$ limit of the right hand side of (1.9). In order to do this we will first introduce a slightly different characterization of $W_\nu$ which will introduce the principal player in our understanding of the $m \to 0$ limit; this is an appropriate Green function for the Dirac operator $mI - \partial \partial$ acting on a suitable space of “multivalued” functions (or more precisely it is an explicit formula for the $m = 0$ limit of this Green function). We begin by noting a uniqueness result which will suggest the definition of the Green function we are interested in. Suppose that $w$ is an $L^2$ Ising type solution to the Dirac equation which has leading order local expansions,

$$
w(z) = c^\mu(w)\Re w_0(z_\mu) + \cdots,
$$

for $z$ near $a_\mu$. Then $w$ is identically 0. To see this observe that for such a $w$ one has $c_0^\mu(w) = c_0^{\mu *}(w) = c_0^{\mu *}(w)$. Substituting this result in the formula for $<w, w>$ obtained from (1.6) one finds that $<w, w>$ must be negative unless $c^\mu(w) = 0$ in which case $<w, w> = 0$.

Roughly speaking our strategy in dealing with $W_\nu$ can now be described as follows. We will make a local subtraction $\phi_\nu$ from $W_\nu$ which will kill off the term $\delta_{\mu \nu}\Im w_0$ in the expansion (1.8) for $W_\nu$. The difference $W_\nu - \phi_\nu$ then satisfies the inhomogenous Dirac equation,

$$
(mI - \partial)(W_\nu - \phi_\nu) = -(mI - \partial)\phi_\nu := f_\nu,
$$

and $W_\nu - \phi_\nu$ is uniquely characterized by this equation and the fact that it is in $L^2(C)$ with local expansions of type (1.18) at each of the points $a_\mu$. We will demonstrate the
convergence of the solution to (1.19) in the limit $m \to 0$ with sufficient control to say what happens to (1.9) in this limit.

We will now sketch the existence theory needed to make this possible (we follow the existence theory in SMJ [11] quite closely). Let $C_0^\infty(\mathcal{R}(\mathbb{C}\setminus a))$ denote the space of complex valued $C^\infty$ functions, $f$, on $\mathcal{R}(\mathbb{C}\setminus a)$ such that $f \circ \Gamma_j = -f$, and such that the projection of the support of $f$ in $\mathcal{R}(\mathbb{C}\setminus a)$ onto $\mathbb{C}\setminus a$ is compact. Let $H_{m,0}^{1,0}$ denote the Hilbert space completion of $C_0^\infty(\mathcal{R}(\mathbb{C}\setminus a))$ with respect to the norm determined by the inner product,

$$
<f, g>_1 = \frac{i}{2} \int_{\mathbb{C}} \left( \overline{\partial f(z)} \partial g(z) + m^2 f(z) g(z) \right) dzd\overline{z}.
$$

(1.20)

Let $H_{m,1}^{0,1}$ denote the Hilbert space completion of $C_0^\infty(\mathcal{R}(\mathbb{C}\setminus a))$ with respect to the inner product

$$
<f, g>_0 = \frac{i}{2} \int_{\mathbb{C}} \left( \overline{\partial f(z)} \partial g(z) + m^2 f(z) g(z) \right) dzd\overline{z}.
$$

(1.21)

Now suppose that $f \in C_0^\infty(\mathcal{R}(\mathbb{C}\setminus a))$ and consider the linear functional,

$$
C_0^\infty(\mathcal{R}(\mathbb{C}\setminus a)) \ni g \to \frac{i}{2} \int_{\mathbb{C}} \overline{f(z)} g(z) dzd\overline{z}.
$$

For $m \neq 0$ it is clear that this linear functional extends to a continuous linear functional on both $H_{m,0}^{1,0}$ and $H_{m,1}^{0,1}$. Thus by the Riesz representation theorem there exists an $F^{i,j} \in H_{m}^{i,j}$ so that,

$$
<F^{i,j}, g>_{i,j} = \frac{i}{2} \int_{\mathbb{C}} \overline{f(z)} g(z) dzd\overline{z},
$$

(1.22)

where $i, j = 1, 0$ or 0,1. It follows from this last equation that away from the branch points $F^{i,j}$ is a distribution solution to

$$(m^2 - \Delta)F^{i,j} = f.$$

Elliptic regularity implies that $F^{i,j}$ is $C^\infty$ “away from the branch points” (i.e., on the covering space $\mathcal{R}(\mathbb{C}\setminus a)$). Near the branch point $z = a_\nu$, $F^{i,j}$ is an $L^2$ branched solution to the Helmholtz equation $(m^2 - \Delta)F^{i,j} = 0$ and so has a local expansion,

$$
F^{i,j}(z) = \sum_{k \geq 0} a_k^{\nu,i,j} v_k(z - a_\nu) + \sum_{k \geq 0} b_k^{\nu,i,j} \overline{v}_k(z - a_\nu),
$$

(1.22)

Since $\partial F^{1,0}$ is locally in $L^2$ it follows that $a_0^{\nu,1,0} = 0$ and since $\overline{\partial} F^{0,1}$ is locally in $L^2$ it follows that $b_0^{\nu,0,1} = 0$ for $\nu = 1, \ldots, n$.

Now we apply this to solve the Dirac equation,

$$(m - \phi) F = f,$$

(1.23)
where \( f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \) and \( f_j \in C^\infty_0(\mathcal{R}(\mathbb{C}\backslash \mathbb{a})) \). Let \( F^{0,1} \in \mathcal{H}^{0,1}_m \) be the solution to \((m^2 - \Delta)F^{0,1} = f_1\) constructed above and let \( F^{1,0} \in \mathcal{H}^{1,0}_m \) be the similar solution to \((m^2 - \Delta)F^{1,0} = f_2\). Define,

\[
F := \left[ mF^{0,1} + 2\partial F^{1,0} \right].
\]  

(1.24)

It is easy to check that \( F \) is a solution to the Dirac equation (1.23) and furthermore that because \( F^{i,j} \) lies in \( \mathcal{H}^{i,j}_m \) it follows that the solution \( F \) is locally in \( L^2 \). This is not quite the solution to (1.23) which we want. Instead, notice that because \( F \) is locally in \( L^2 \) about each point \( a_\nu \) we can subtract a linear combination of the canonical wave functions \( w_\nu \) from \( F \) to produce a new solution \( F \) to (1.23) with the property that,

\[
c^\nu_0(F) = 0 \text{ for } \nu = 1, \ldots, n
\]  

(1.25)

Observe that these boundary conditions uniquely determine an \( L^2 \) solution to (1.23) since the formula (1.6) shows that any \( L^2 \) solution to the homogenous equation killed by all the linear functionals \( c^\nu_0 \) is necessarily 0. Now that we know existence and uniqueness for \( L^2 \) solutions to (1.23) with the boundary conditions (1.25) we can use this enlarge the class of suitable boundary conditions to include the conditions relevant to the Ising model. Let \( N \) denote the space of \( L^2 \) Ising type solutions, \( w \), to \((m - \bar{\phi})w = 0\). Let \( \mathcal{N} \) denote the image of \( N \) under the map,

\[
N \ni w \rightarrow (c^0_0(w), c_0^*(w)) \in \mathbb{C}^n \oplus \mathbb{C}^n,
\]

where \( c_0^*(w) \) denotes the \( n \) tuple \((c_0^{1*(w)}, \ldots, c_0^{n*(w)})\) in \( \mathbb{C}^n \). We know that \( \mathcal{N} \) is an \( n \) dimensional subspace of \( \mathbb{C}^n \oplus \mathbb{C}^n \) and that the intersection of \( \mathcal{N} \) with \( \{0\} \oplus \mathbb{C}^n \) is \( \{0\} \). Since \( \{0\} \oplus \mathbb{C}^n \) is \( n \) dimensional it follows that \( \mathcal{N} \) and \( \{0\} \oplus \mathbb{C}^n \) are transverse subspaces in \( \mathbb{C}^n \oplus \mathbb{C}^n \). Now let \( \mathcal{I} \) denote any subspace of \( \mathbb{C}^n \oplus \mathbb{C}^n \) which is transverse to \( \mathcal{N} \). Then we claim that for any \( f \in C^\infty_0(\mathcal{R}(\mathbb{C}\backslash \mathbb{a})) \) there exists a unique \( L^2 \) Ising type solution, \( F \), to \((m - \bar{\phi})F = f\), with,

\[
(c_0(F), c^*_0(F)) \in \mathcal{I}.
\]  

(1.26)

Existence is simple. We know that the differential equation \((m - \bar{\phi})F = f\) has an \( L^2 \) Ising type solution with \( c_0(F) = 0 \). By transversality there is a unique element \( w \in \mathcal{N} \) so that,

\[
0 \oplus c_0^*(F) - (c_0(w), c_0^*(w)) \in \mathcal{I}.
\]

It is clear that \( F - w \) is a solution satisfying the boundary condition (1.26). Uniqueness is obvious.

Now we introduce the boundary condition that is relevant to the Ising model. Henceforth, let \( \mathcal{I} \) denote the subspace of \( \mathbb{C}^n \oplus \mathbb{C}^n \) which consists of all vectors of the form \((v, v)\) with \( v \in \mathbb{C}^n \). This corresponds to the boundary condition,

\[
c_0(F) = c^*_0(F),
\]  

(1.27)

which the reader should note is the “complexification” of condition (1.18). Clearly \( \mathcal{I} \) is \( n \) dimensional and the formula (1.6) for the inner product of \( L^2 \) solutions shows that any
element \( w \in \mathcal{N} \) which has boundary values in \( I \) has an \( L^2 \) norm which is negative unless the linear functional \( c_0 \) vanishes on it. This is enough to show that the function must be 0 and hence that \( I \cap \mathcal{N} = \{0\} \). Thus \( I \) is transverse to \( \mathcal{N} \) and there is an existence and uniqueness result for solutions to (1.23) satisfying (1.27). This is the complexification of the result we really want. If \( f \) is a real function, that is, \( f = f^* \), it is easy to see that the solution, \( F \), satisfying (1.27) must also be real (i.e., \( F^* = F \)).

In this case both \( c_0(F) \) and \( c_0^*(F) \) will be real numbers.

It is straightforward to describe the subtraction \( \phi_\nu \) that we make from \( W_\nu \). Let \( \varphi_\nu(x) \) denote a real valued \( C^\infty \) function on \( C \) which is identically 1 in a small ball centered at \( a_\nu \) and which vanishes outside a slightly larger ball about \( a_\nu \) but still small enough so that \( \varphi_\nu \) is zero in a neighborhood of all the other points \( a_\mu \) for \( \mu \neq \nu \). Define

\[
\phi_\nu(z) = \varphi_\nu(z) \Im w_0(z_\nu).
\]

Consulting (1.8) we see that \( W_\nu - \phi_\nu \) will have local expansions at each point \( a_\mu \) of type (1.18). Write \( \delta W_\nu := m^{\frac{1}{2}}(W_\nu - \phi_\nu) \). Then using the fact that \( W_\nu \) and \( \Im w_0 \) both satisfy the massive Dirac equation we find,

\[
(m - \partial) \delta W_\nu = \begin{bmatrix}
0 & -2\partial \varphi_\nu \\
-2\partial \varphi_\nu & 0
\end{bmatrix}
\]

\[
m^{\frac{1}{2}} \Im w_0(z_\nu) := f_\nu.
\]  

The scale factor \( m^{\frac{1}{2}} \) is introduced here so that following limit exits,

\[
\lim_{m \to 0} m^{\frac{1}{2}} \Im w_0(z_\nu) = \frac{1}{\sqrt{2\pi i}} \begin{bmatrix}
\frac{1}{z_\nu^{\frac{1}{2}}} \\
-\frac{1}{z_\nu^{\frac{1}{2}}}
\end{bmatrix}.
\]  

In terms of \( \delta W_\nu \) the coefficient \( mc_1^\nu(W_\nu) \) which appears in the formula for \( d \log \tau_- \) is given by,

\[
mc_1^\nu(W_\nu) = \left( \frac{m^{\frac{1}{2}}}{I_\frac{1}{2}(m\epsilon)} \right) \frac{1}{2\pi} \int_0^{2\pi} (\delta W_\nu)_1 (\epsilon e^{i\theta_\nu}) e^{-i\theta_\nu} d\theta_\nu.
\]  

The function \( (\delta W_\nu)_1 \) is the first component of \( \delta W_\nu \) and the integral on the right of (1.30) calculates a (half integer) Fourier coefficient of this function on the circle of radius \( \epsilon \) about \( a_\nu \). Since

\[
\lim_{m \to 0} m^{\frac{1}{2}} I_\frac{1}{2}(m\epsilon) = \sqrt{\frac{2}{\pi}} \Gamma \left( \frac{3}{2} \right),
\]

it will suffice for our purposes to control the \( m \to 0 \) convergence of \( \delta W_\nu \) in \( L^p(C_\epsilon(a_\nu)) \) for any \( p \geq 1 \) and all \( \nu \). Here \( C_\epsilon(a_\nu) \) is the circle of radius \( \epsilon \) about \( a_\nu \). A device that will be useful for this purpose will be to “excise” small neighborhoods of the branch points \( a_\nu \). Let \( D_\nu \) denote a small open disk centered at \( a_\nu \) and suppose that the radii of these
disks are small enough so that they are mutually disjoint. We also want $D_\nu$ chosen small enough so that the $\varphi_\nu$ is identically 1 on $D_\nu$. The function $f_\nu$ will then have its support outside the union of the $D_\nu$. Write $C_\nu = \partial D_\nu$ for the *clockwise* oriented circle that is the boundary of $D_\nu$. Let $D_\infty$ denote a disk centered at $a_0$ with a radius that is large enough so that all the disks $D_\nu$ are contained in the interior of $D_\infty$. We will write $D_\infty^c$ for the complement in $\mathbb{C}$ of $D_\infty$. Write $C_\infty$ for the *counterclockwise* oriented circle $\partial D_\infty$. Let $\mathcal{D}$ denote the bounded domain which is the complement of the union of the closed disks $D_\nu$ for $\nu = 1, \ldots, n$ in $D_\infty$. We now rephrase the problem we need to solve to find $\delta W_\nu$. We wish to find a solution to 

$$(m - \partial)\delta W_\nu(x) = f_\nu(x),$$

for $x \in \mathcal{D}$ with the restriction of $\delta W_\nu$ to $C_\mu$ for $\mu = 1, \ldots, n$ equal to the boundary value of a solution to the Dirac equation in $D_\mu$ with local expansion of type (1.18) (we will refer to this subspace of boundary values as $W^{(m)}_\mu$). We also require that the restriction of $\delta W_\nu$ to $C_\infty$ belongs to the space of boundary values of solutions to the Dirac equation which are in $L^2(D_\infty^c)$ (we refer to this subspace as $W^{(m)}_\infty$). Of course, we really need to work in a two fold covering space for $\mathcal{D}$ and two fold covers of the circles $C_\alpha$, but for simplicity of exposition in the remainder of this note we will ignore this (the details of the argument we only sketch here will appear in a paper that is still in preparation). Our principal tool in understanding the $m \to 0$ limit of the solution to this problem is a “guess” for the $m = 0$ Green function. Here is the guess,

$$G_0(x, y) = -\frac{1}{4\pi i} \left[ \sum_\nu \frac{u_\nu(x)v_\nu(y)}{g(x, y)} \frac{g(x, y)}{\sum_\nu u_\nu(x)v_\nu(y)} \right],$$

where

$$u_\nu(x) = (x - a_\nu)^{-\frac{1}{2}} \prod_{\mu \neq \nu} \frac{(x - a_\mu)^{\frac{1}{2}}}{(a_\nu - a_\mu)^{\frac{1}{2}}},$$

$$g(x, y) = \sum_{|\epsilon| = 0} c(\epsilon) \prod_\nu \left( x - a_\nu \right)^{\epsilon_\nu} \left( y - a_\nu \right)^{-\epsilon_\nu},$$

with $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ and each $\epsilon_\nu = \pm \frac{1}{2}$. Also

$$|\epsilon| := \sum_\nu \epsilon_\nu,$$

$$c(\epsilon) := \frac{\prod_{\mu < \nu} |a_\mu - a_\nu|^{2\epsilon_\mu \epsilon_\nu}}{\sum_{|\epsilon| = 0} \prod_{\mu < \nu} |a_\mu - a_\nu|^{2\epsilon_\mu \epsilon_\nu}},$$

and

$$v_\nu(x) = (x - a_\nu)^{-\frac{1}{2}} \sum_{|\epsilon(\nu)| = 0, \epsilon_\nu = \frac{1}{2}} c(\epsilon) \prod_{\mu \neq \nu} \frac{(x - a_\mu)^{\epsilon_\mu}}{(a_\nu - a_\mu)^{\epsilon_\mu}}.$$

A possible surprise in this guess are the “chiral symmetry breaking terms” on the diagonal in $G_0$. The $m = 0$ Dirac operator $\partial$ is purely off diagonal and one might expect
the same for its Green function. However, the boundary conditions we are interested in force the off diagonal terms even in the \( m = 0 \) limit. Indeed the principal ingredient that went into this guess was setting the boundary conditions for the \( m = 0 \) limit equal to the limits, \( W[^{(0)}]_\nu \), of the subspaces \( W[^{(m)}]_\nu \) as \( m \to 0 \). It is not difficult to check that \( G_0 \) is a Green function for \( -\delta \) on \( \mathcal{D} \) essentially using the fact that \( \frac{1}{x-y} \) is a Green function for \( \overline{\mathcal{D}} \). As a first approximation to \( \delta W_\nu \) we invert \( m - \delta \) according to the following scheme,

\[
(m - \delta)^{-1} = -\delta^{-1}(1 - m\delta^{-1})^{-1} = G_0(1 + mG_0)^{-1}. \tag{1.31}
\]

It is a classical result [5] that the Cauchy kernel \( \frac{1}{x-y} \) defines a bounded operator on \( L^p(\mathcal{D}) \) for all \( p > 2 \). This result carries over without difficulty to \( G_0(x,y) \) (modulo the difference that we should be working on a covering of \( \mathcal{D} \)). The Neuman series for \( (1 + mG_0)^{-1} \) then converges in the strong operator topology on \( L^p(\mathcal{D}) \) and we can use this to show that (1.31) does invert \( m - \delta \) for all small enough \( m \). Our first approximation to \( \delta W_\nu \) we take to be

\[
\delta_1 W_\nu := G_0(1 + mG_0)^{-1} f_\nu.
\]

This solves the appropriate inhomogeneous Dirac equation but the boundary values are in the subspaces \( W[^{(0)}]_\nu \) instead of \( W[^{(m)}]_\nu \). The \( m \to 0 \) limit of \( \delta_1 W_\nu \) is straightforward to compute.

The Calderon projector [2] associated with the Green function \( G_0(x,y) \) maps the direct sum \( \oplus W[^{(m)}]_\nu \) onto \( \oplus W[^{(0)}]_\nu \) along the subspace, \( \mathbf{N} \), of boundary values of solutions to \( \delta \psi = 0 \). For small \( m \) one can show that this projection determines an isomorphism of \( \oplus W[^{(m)}]_\nu \) with \( \oplus W[^{(0)}]_\nu \) with a complementary projection on \( \mathbf{N} \) which is \( O(m) \). The fact that this is true for the Green function \( G_0 \) is precisely what makes our choice of \( G_0 \) suitable in this application. Thus we can adjust \( \delta_1 W_\nu \) by an element \( \delta_2 W_\nu \in \mathbf{N} \) which is \( O(m) \) so that \( \delta_1 W_\nu + \delta_2 W_\nu \) has boundary values in \( W[^{(m)}]_\nu \). This is not quite the solution we want but since \( \delta_2 W_\nu \in \mathbf{N} \) it follows that

\[
(m - \delta)(\delta_1 W_\nu + \delta_2 W_\nu) = f_\nu + m\delta_2 W_\nu.
\]

Thus the adjustment we need to get \( \delta W_\nu \) from \( \delta_1 W_\nu + \delta_2 W_\nu \) is given by the solution to the Dirac equation with right hand side \( m\delta_2 W_\nu = O(m^2) \) and boundary values in \( \oplus W[^{(m)}]_\nu \).

We don’t have very detailed control of the solution to this problem but because the right hand side is \( O(m^2) \) a simple apriori estimate shows that this last adjustment is \( O(m) \) as well. Making use of the explicit form for the \( m \to 0 \) limit of \( \delta_1 W_\nu \) we are now in a position to calculate the limiting form of the logarithmic derivative of \( \tau_- \). We find,

\[
\lim_{m \to 0} d \log \tau_- = \frac{1}{2} d \log \sum_{|\epsilon|=0} \prod_{\mu<\nu} |a_{\mu} - a_{\nu}|^{2\epsilon_{\mu}\epsilon_{\nu}}. \tag{1.32}
\]

This is the Luther–Peschel formula for the \( 2n \) point function.

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