DENSITY OF STATES UNDER NON-LOCAL INTERACTIONS II.
SIMPLIFIED POLYNOMIALLY SCREENED INTERACTIONS

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ABSTRACT. Following [5], we analyze regularity properties of single-site probability distributions of the random potential and of the Integrated Density of States (IDS) in the Anderson models with infinite-range interactions. In the present work, we study in detail a class of polynomially decaying interaction potentials of rather artificial (piecewise-constant) form, and give a complete proof of infinite smoothness of the IDS in an arbitrarily large finite domain subject to the fluctuations of the entire, infinite random environment. A variant of this result, based as in [5] on the harmonic analysis of probability measures, results in a proof of spectral and dynamical Anderson localization in the considered models.

1. INTRODUCTION

This text is a follow-up of [5], where the reader can find the main motivations, a historical review, and a number of bibliographical references.

The main model is an Anderson Hamiltonian with an infinite-range alloy-type random potential. This corresponds to the physical reality where the fundamental interactions are most certainly not compactly supported. By interactions we mean here those between the quantum particle (e.g., an electron), evolving in a sample of a disordered media, with surrounding ions. Physically speaking, other mobile electrons present in the media also make a non-negligible contribution to the potential field affecting the main particle under consideration, especially from the point of view of the screening phenomena, substantially attenuating the "naked" Coulomb potentials, but the full-fledged many-body spectral problem is beyond the scope of the present paper. In this particular physical metaphor, the interaction potential means the screened Coulomb potential. As discussed in [5], the rate of its decay is not universal and varies from one physical model to another.

A large class of interaction potentials was considered in [5], but in the present, relatively short paper, we focus on the particular case of interaction potentials featuring a summable power-law decay, and provide all technical details that were missing or only briefly outlined in a general discussion of long-range Anderson models in [5].

Let be given a function $x \mapsto \sum_{y \in \mathbb{Z}^d} q_y u(x - y)$, which we will always assume absolutely summable on $\mathbb{Z}^d$; more precisely, we assume that $0 \leq u(r) \leq Cr^{-A}$ for some $A > d$. Then one can define a linear transformation $U$,
well-defined on any bounded function \( q : \mathbb{Z}^d \to \mathbb{R} \):

\[
U : q \mapsto U[q] = V, \quad V : \mathbb{Z}^d \to \mathbb{R},
\]

where

\[
V(x) = (U[q])(x) = \sum_{y \in \mathbb{Z}^d} u(y-x)q_y.
\]

To clarify the main ideas of [7] and simplify some technical aspects, the interaction potential \( u : \mathbb{R}_+ \to \mathbb{R} \) is assumed to have the following form. Introduce a growing integer sequence \( r_k = \lfloor k^{\frac{1}{\kappa}} \rfloor, \kappa > 1, k \geq 0 \), and let

\[
u(r) = \sum_{k=1}^{\infty} r_k^{-A} 1_{[r_k, r_{k+1})}(r),
\]

Making \( u(\cdot) \) piecewise constant will allow us to achieve, albeit in a somewhat artificial setting, an elementary derivation of infinite smoothness of the DoS from a similar property of single-site probability distributions of the potential \( V \). We refer to \( V \) as the cumulative potential in order to distinguish it from the interaction potential \( u \) (which is a functional characteristics of the model) and from the local potential amplitudes \( \{ q_y, y \in \mathbb{Z}^d \} \). The notation \( q_y \) will be used in formulae and arguments pertaining to general functional aspects of the model, while in the situation where the latter amplitudes are random we denote them by \( \omega_y \).

We always assume the amplitudes \( q_y \) and \( \omega_y \) to be uniformly bounded. In the case of random amplitudes, one should either to assume this a.s. (almost surely, i.e., with probability one) or to construct from the beginning a product measure on \( [0, 1]^{\mathbb{Z}^d} \) rather than on \( \mathbb{R}^{\mathbb{Z}^d} \) and work with samples \( \omega \in [0, 1]^{\mathbb{Z}^d} \), which are thus automatically bounded. It is worth mentioning that boundedness is not crucial to most of the key properties established here, but results in a streamlined and more transparent presentation. On the other hand, as pointed out in [5], there are interesting models with unbounded amplitudes \( \omega \) such that \( \mathbb{E} \left[ (\omega_x - \mathbb{E}[\omega_x])^2 \right] < \infty \). It is readily seen that single-site probability distribution of the cumulative potential \( V(x, \omega) \), necessarily compactly supported when \( \omega \) are uniformly bounded and the series (1.2) (with \( q_y \) replaced with \( \omega_y \)) converges absolutely, cannot have an analytic density, for it would be compactly supported and not identically zero, which is impossible. However, in some class of marginal measures of \( \omega \) with unbounded support, considered long ago by Wintner [23] in the framework of Fourier analysis of probability measures, the single-site density of \( V(\cdot, \omega) \) can be analytic on \( \mathbb{R} \).

We also always assume that \( \omega_y \) are IID. Extensions to dependent random fields \( (\omega_x)_{x \in \mathbb{Z}^d} \) with rapidly decaying correlations do not really pose any serious problem, as limit theorems for normalized sums of random variables (r.v.) are well-known to hold true for a large class of dependent random fields. Following the program outlined in [5], we plan to address such models in a separate work, in a general context of Gibbs measures on the samples \( \omega \).
2. Main results

2.1. Infinite smoothness of single-site distributions.

**Theorem 1.** Consider the potential \( u(r) \) of the form (1.3), with \( A > d \) and let \( d \geq 1 \). Then the characteristic functions of the random variables \( V(x, \omega) \) of the form (1.2) obey the upper bound

\[
|\varphi_{\omega}(t)| \leq \text{Const} e^{-c|t|^{d/A}}.
\]

Consequently, for any \( d \geq 1 \) the r.v. \( V_x \) have probability densities \( p_x \in C^\infty(\mathbb{R}) \).

2.2. Infinite smoothness of the DoS.

**Theorem 2.** Fix a bounded connected subset \( \Lambda \subset \mathbb{Z}^d \).

(A) There exists a \( \sigma \)-algebra \( \mathcal{B}_\Lambda \), an \( \mathcal{B}_\Lambda \)-measurable self-adjoint random operator \( \tilde{H}_\Lambda(\omega) \) acting in \( l^2(\Lambda) \), and a \( \mathcal{B}_\Lambda \)-independent real-valued r.v. \( \xi_\Lambda \) such that

\[
H_\Lambda(\omega) = \tilde{H}_\Lambda(\omega) + \xi_\Lambda(\omega)1_\Lambda.
\] (2.1)

(B) The characteristic function \( \varphi_{\xi_\Lambda}(t) \) fulfills the decay bound

\[
|\varphi_{\xi_\Lambda}(t)| \leq C e^{-c|t|^{d/A}}.
\] (2.2)

2.3. Wegner estimate.

**Theorem 3** ("Frozen bath" Wegner estimates). Fix real numbers \( \tau > 1 \) and \( \theta \in (0, \tau - 1) \), consider a ball \( B = B_L(u) \), and let

\[
R_L = R_L(\tau) = L^\tau, \quad \varepsilon_L = R_L^{-1+\theta};
\] (2.3)

here \( \theta > 0 \) can be chosen arbitrarily small, for \( L \) large enough. Next, consider the Hamiltonian \( H_B \) and a larger set \( \mathcal{A} = \mathcal{A}_L(\tau) = B_{R_L(u)} \setminus B_L(u) \), and introduce the the product probability space \( (\Omega_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}}, \mathbb{P}_{\mathcal{A}}) \) generated by the r.v. \( \omega_x \) with \( y \in \mathcal{A} \). Then

\[
\mathbb{P}_{\mathcal{A}} \{ \text{dist}(\Sigma_B, E) \leq \varepsilon_L \} \leq C |B| \varepsilon_L \leq C |B| e^\beta,
\] (2.4)

with

\[
\beta = 1 - \frac{1+\theta}{\tau} \in (0, 1).
\] (2.5)

**Remark 2.1.** The constant \( C \) in (2.4) can be absorbed in the exponent \( \beta \) (in (2.5)), by taking a slightly smaller \( \theta > 0 \) and letting \( L \) be large enough.

2.4. Localization. Below we denote by \( \mathcal{B}_1(\mathbb{R}) \) the set of all bounded Borel functions \( \phi : \mathbb{R} \rightarrow \mathbb{C} \) with \( \| \phi \|_\infty \leq 1 \).

**Theorem 4.** Consider the potential \( u(r) \) of the form (1.3), with \( A > d \), and let \( d \geq 1 \). For any \( m \geq 0 \) there exist \( L_* \in \mathbb{N} \) and \( g_0 > 0 \) such that for all \( g \) with \( |g| \geq g_0 \) with probability one, the random operator \( H(\omega) = -\Delta + gV(x, \omega) \) has pure point spectrum with exponentially decaying eigenfunctions, and for any \( x, y \in \mathbb{Z}^d \) and any connected subgraph \( G \subseteq \mathbb{Z}^d \) containing \( x \) and \( y \) one has

\[
\mathbb{E} \left[ \sup_{\phi \in \mathcal{B}_1(\mathbb{R})} \|1_x \phi(H_G(\omega))1_y\| \right] \leq \frac{C'}{(1+|x-y|)^{C'}}.
\] (2.6)
Theorem 5. Consider the potential $u(r)$ of the form (1.3), with $A > d$, and let $d > 1$. There exist an energy interval $I = [E_0, E_0 + \eta]$, $\eta > 0$, near the a.s. lower edge of spectrum $E_0$ of the random operator $H(\omega) = -\Delta + V(x, \omega)$ such that with probability one, $H(\omega)$ has in $I$ pure point spectrum with exponentially decaying eigenfunctions, and for any $x, y \in \mathbb{Z}^d$ and any connected subgraph $G \subseteq \mathbb{Z}^d$ containing $x$ and $y$ one has

$$
\mathbb{E} \left[ \sup_{\phi \in \mathcal{H}_1(\mathbb{R})} \| 1_x \mathbb{P}(H_G(\omega) \psi(H_G(\omega)) 1_y) \| \right] \leq \frac{C'}{(1 + |x - y|)^C}.
$$

(2.7)

3. Fourier analysis

3.1. The Main Lemma.

Lemma 3.1. Let be given a family of IID r.v.

$$X_{n,k}(\omega), \ n \in \mathbb{N}, \ 1 \leq k \leq K_n, \ K_n \asymp n^{d-1},$$

and assume that their common characteristic function $\phi_X(t) = \mathbb{E} \left[ e^{iX} \right]$ fulfills

$$\ln \left| \phi_X(t) \right|^{-1} \geq C_X t^2, \ |t| \leq t_0. \quad (3.1)$$

Let

$$S(\omega) = \sum_{n \geq 1} \sum_{k=1}^{K_n} a_n X_{n,k}(\omega), \ a_n \asymp n^{-A},$$

$$S_{M,N}(\omega) = \sum_{n=M}^{N} \sum_{k=1}^{K_n} a_n X_{n,k}(\omega), \ M \leq N.$$

The following holds true.

(A) There exists $C = C(C_X, t_0, A, d) \in (0, +\infty)$ such that

$$\forall t \in \mathbb{R} \quad \left| \phi_S(t) \right| \leq C e^{-|t|^{d/A}}.$$

(B) For any $\varepsilon > 0$, $N \geq (1 + c)M \geq 1$ with $c > 0$, and $t$ with $|t| \leq N^{A-\varepsilon}$,

$$S_{M,N} := \ln \left( \mathbb{E} \left[ e^{iS_{M,N}(\omega)} \right] \right)^{-1} \geq CM^{-2A+d} t^2.$$  

(C) Let $I \subseteq \mathbb{R}$ be an interval of finite length $|I|$. Then for any r.v. $Y$ independent of $S_{N,2N}$, one has

$$|I| \geq N^{-1/4} \quad \Longrightarrow \quad \mathbb{P} \{ Y(\omega) + S_{M,N}(\omega) \in I \} \leq CM^A |I|,$$

where $\theta > 0$ can be chosen arbitrarily small, provided $N$ is large enough: $N \geq N_*(\theta)$.

Proof. By the IID property of the family $\{ \omega_x, x \in \mathbb{Z} \}$, we have

$$\phi_S(t) = \mathbb{E} \left[ \exp \left( i t \sum_{n \geq 1} \sum_{k=1}^{K_n} a_n X_{n,k}(\omega) \right) \right] = \prod_{n \geq 1} \prod_{k=1}^{K_n} \mathbb{E} \left[ e^{i a_n X_{n,k}} \right]$$

$$= \prod_{n \geq 1} \prod_{k=1}^{K_n} \phi_X(t a_n) = \prod_{n \geq 1} \left( \phi_X(t a_n) \right)^{K_n}.$$
so for the logarithm we have the lower bound
\[
\ln |\varphi_S(t)|^{-1} = \sum_{n \geq 1} K_n \ln |\varphi_X(a_n t)|^{-1} \geq C_1 \sum_{n \geq 1} n^{d-1} \ln |\varphi_X(a_n t)|^{-1} \\
\geq C_1 \left( \sum_{n=1}^{n \geq N_t} n^{d-1} \ln |\varphi_X(a_n t)|^{-1} \right) =: S_1(t) + S_2(t),
\]
where all terms in $S_1$ and in $S_2$ are non-negative, since $|\varphi_X(t)| \leq 1$ for any $t$. In particular, this implies that $\ln |\varphi_S(t)|^{-1} \geq \min [S_1(t), S_2(t)]$. Now focus on $S_2(t)$ and recall that, by definition of the threshold $N_t$ (cf. assertion (B)),
\[
\forall n \geq N_t \quad n^{-A}|t| \leq N_t^{-A}|t| \in [0, t_0],
\]
hence, by hypothesis (3.1),
\[
S_2(t) = C_1 \sum_{n > N_t} \ln |\varphi_X(a_n t)|^{-1} \geq C_2 t^2 \sum_{n > N_t} n^{d-1} a_n^2 \geq C_3 t^2 \sum_{n > N_t} n^{-2A+d-1} \geq C_4 t^{2} N_t^{-2A+d} \geq C_5 |t|^\frac{d-A}{A},
\]
which proves assertion (A).

In Section 3.4 we comment on Wintner's approach [23] to the estimation of $S_1(t)$, based on an elementary lemma by Pólya and Szegő [18]. The final result is similar: $S_1(t) \geq C|t|^\frac{d}{A}$.

Assertion (B) is obtained in the same way:
\[
\sum_{n=M}^{N} \ln |\varphi_X(a_n t)|^{-1} \geq C_1 t^2 \sum_{n=M}^{N} n^{-2A+d-1} \geq C_2 t^2 \int_{n=N/(1+c)}^{N} s^{-2A+d-1} ds \\
\geq C_3(A, d, c) t^2 N^{-2A+d} \geq C_4 |t|^\frac{d}{A}.
\]
Observe that for $N = M$, or close to $M$, we would have a weaker lower bound by $C'|t|^\frac{d-1}{A}$, but with $d > 1$ this is still good enough for the proof of infinite derivability. This also works when $d > 1$ is non-integer and arbitrarily close to 1. One possible setting where this observation can be useful is a subset of $\mathbb{Z}^d$ with the rate of growth of balls $r \mapsto r^{1+\delta}$, $\delta > 0$.

Now we turn to the proof of assertion (C). It suffices to consider the case where $Y(\omega) = \text{const}$ a.s., otherwise one can first condition on $Y$ (independent of $S_{M,N}$ by hypothesis), thus rendering $Y$ constant. We need to assess the integrals of the probability measure of $S(\omega)$ on intervals $I_\varepsilon$ of length $O(\varepsilon)$. It will be clear from the calculations given below that it suffices to consider the case where $I_\varepsilon$ is centered at origin; a shift results in factors of unit modulus, so we stick to $I_{[-\varepsilon,\varepsilon]}$ to have less cumbersome formulae. Further, since the main estimate will be achieved in the Fourier representation, it is more convenient to
work with a smoothed indicator function $\chi_\varepsilon$:

$$1_{I_\varepsilon} \leq 1_{[-2\varepsilon,2\varepsilon]} \leq 1_{[-4\varepsilon,4\varepsilon]} \frac{1}{2\varepsilon} 1_{[-\varepsilon,\varepsilon]}$$

$$\leq \chi_\varepsilon := 1_{[-4\varepsilon,4\varepsilon]} \cdot \frac{e^{-\sigma_\varepsilon^2 t^2/2}}{\sqrt{2\pi \sigma_\varepsilon}},$$

with $\sigma_\varepsilon = a\varepsilon$, $a \approx 1.2$. The last inequality is easily validated by an elementary numerical calculation: due to monotone decay of the Gaussian density on the positive half-axis, it suffices to check its lower bound by $1$ on $[0,4\varepsilon]$. The aim is of course to secure a much faster decay at infinity for the Fourier transform than in the case of a discontinuous indicator function. By the Parseval identity, for any $\mathcal{F}_\varepsilon > 0$,

$$\mu_S(I_\varepsilon) = \int_\mathbb{R} 1_{I_\varepsilon}(x) dF_S(x) \leq \int_\mathbb{R} \chi_\varepsilon(x) dF_S(x) \leq \int_{|t| \leq \mathcal{F}_\varepsilon} |\hat{\chi}_\varepsilon(t)| \varphi_S(t) |dt| + \int_{|t| > \mathcal{F}_\varepsilon} |\hat{\chi}_\varepsilon(t)| \varphi_S(t) |dt| =: J_1 + J_2,$$

where

$$\hat{\chi}_\varepsilon(t) = \varepsilon \frac{\sin(4\varepsilon t)}{\varepsilon t} e^{-\sigma_\varepsilon^2 t^2/2}.$$

Now define $T_N$ by

$$T_N = \inf \{ t > 0 : a_N t \leq t_0 \} = \inf \{ t > 0 : [N^{-A}] t \leq t_0 \} \sim (C + o(1))N^A, \quad (3.4)$$

(here $o(1)$ refers to $N \to \infty$) with some $C = C(t_0) \in (0, +\infty)$.

- **Bound on $J_2$.** Further, assume that $\varepsilon > 0$ is such that

$$\mathcal{F}_\varepsilon := \varepsilon^{-1} \ln^2 \varepsilon^{-1} \leq T_N, \quad (3.5)$$

(we shall see in a moment that it means $\varepsilon$ is not too small), then

$$J_2 \leq 2(1 - \Phi(\sigma_\varepsilon \mathcal{F}_\varepsilon)) \leq \varepsilon(-C \ln^2 \varepsilon^{-1}) \leq \varepsilon^2, \quad (3.6)$$

hence $\mu(I_\varepsilon) \leq J_1 + J_2 \leq C\varepsilon$, under the condition (3.5). Considering $\varepsilon^{-1} \ln^2 \varepsilon^{-1} = T$ as definition of an implicit function $T \mapsto \varepsilon$, and taking the logarithm of both sides, we see that $\ln \varepsilon(T) \propto \ln T^{-1}$ as $T \to +\infty$, so (3.5) would follow from a more explicit condition

$$\varepsilon^{-1} \leq T_N \ln^{-c} T, \quad c \in (0, +\infty), \quad (3.7)$$

hence with $T_N \sim N^A$, (3.5) is fulfilled, whenever

$$\varepsilon \geq \tilde{\varepsilon}(\theta, T_N) = N^{-A} \ln^c N = N^{-A} \left(1 - \frac{\ln \ln N}{\ln N} \right) \sim N^{-A} \left(1 - \frac{1}{\ln \ln N} \right). \quad (3.8)$$

It suffices that, with an arbitrarily small $\theta > 0$ and $N$ large enough, viz. $N \geq N_\varepsilon(\theta)$,

$$\varepsilon \geq N^{-1+\theta}, \quad (3.9)$$

but actually here $\theta = \theta(N) = o(1)$ as $N \to \infty$. 


Lemma 3.3. Assume that \( \varphi_{X_1, \ldots, X_n}(t) \) of length \( n \geq N \). Thus 

\[
\ln \left| \frac{\varphi_{X_1, \ldots, X_n}(t)}{\varphi_{X_1, \ldots, X_n}(0)} \right| \leq C \sum_{l=0}^{n-1} a_{n-l} t^{n-l} \geq C t^2 N^{-2A+d} \geq C_2 |t|^{d/A}.
\]

Thus

\[
J_1 = \int_{|t| \leq T} |\mathcal{X}_e(t) \varphi_S(t)| \, dt + \int_{|t| \leq T} \mathcal{X}_e(t) \varphi_S(t) \, dt, =: J_1^- + J_1^+.
\]

where for \( J_1^- \) we can only use a trivial upper bound, replacing \( \varphi_{X_1, \ldots, X_n}(t) \) by 1:

\[
J_1^- \leq \int_{-T}^{T} |\mathcal{X}_e(t) \varphi_S(t)| \, dt \leq |\mathcal{X}_e(t)| \, dt \\
\leq C_3 \varepsilon \int_{-T}^{T} dt < C_3 T \varepsilon \leq C_4 M^A \varepsilon.
\]

Collecting (3.6) and (3.10) completes the proof of assertion (D) for the intervals \( I = I_\varepsilon \) of length \( |I_\varepsilon| \geq N^{-1} \varepsilon^\Gamma \):

\[
\mu_{X_1, \ldots, X_n}(I_\varepsilon) \leq C_4 M^A \varepsilon + \varepsilon^2 \leq C_5 M^A \varepsilon.
\]

Clearly, the estimate (D) becomes efficient for \( N \gg M \), in view of the restriction (3.9).

3.2. Auxiliary estimates for the characteristic functions. The following simple inequality, easily proved by induction in \( n \geq 1 \), allows one to avoid exponential moments in estimation of the characteristic functions of real-valued r.v. Once again, it is to be stressed that the key bounds presented below are valid not only for a.s. bounded r.v., but in a substantially larger class of probability measures with just a few finite moments.

Lemma 3.2. For any integer \( n \geq 1 \),

\[
\forall s \in \mathbb{R} \quad \left| e^{is} - \sum_{k=0}^{n} \frac{s^k}{k!} \right| \leq \frac{|s|^{n+1}}{(n+1)!}.
\]

Lemma 3.3. Assume that \( m_3 := \mathbb{E}[|X|^3] < \infty \) and let \( \sigma^2 := \mathbb{E}[X^2] \). Then

\[
|1 - \varphi(t)| \leq \frac{\sigma^2 t^2}{m_3} < \frac{1}{2},
\]

and

\[
\left| \ln \varphi(t) + \frac{\sigma^2 t^2}{m_3} \right| \leq \frac{1}{6} m_3 |t|^3 + \frac{1}{4} \sigma^4 t^4 \leq \frac{5}{12} m_3 |t|^3.
\]
and, consequently,
\[ |t| \leq \frac{3}{5} \sigma^{1/2} \quad \implies \quad \ln |\varphi(t)|^{-1} \geq \frac{1}{4} \sigma^2 t^2. \] (3.14)

**Proof.** From the moment inequality, valid whenever the moments involved are finite,
\[ (E[|X|^a])^{1/a} \leq \left( E\left[ |X|^b \right] \right)^{1/b}, \quad 0 < a \leq b, \]
it follows, by taking \( a = 2 \) and \( b = 3 \), that \( \sigma^6 \leq m_3^2 \). Since \( E[|X|] = 0 \), by (3.11) applied

\[ \varphi(t) = E[\exp(iX)] = 1 + itE[X] + \cdots \text{ with } n = 2 \text{ and } s = tX, \]

we have
\[ \forall |t| \leq \frac{\sigma^2}{m_3} \quad |1 - \varphi(t)| \leq \frac{1}{2} \sigma^2 t^2 \leq \frac{1}{2} m_3^{2/3} \left( \frac{1}{m_3^{1/3}} \right)^2 \leq \frac{1}{2}. \]

Next, by the Taylor expansion for the logarithm, valid whenever \(|1 - \varphi(t)| < 1\),
\[ -\ln \varphi(t) - \frac{1}{2} \sigma^2 t^2 = 1 - \varphi(t) - \frac{1}{2} \sigma^2 t^2 + \sum_{k \geq 2} \frac{1}{k} (1 - \varphi(t))^k, \]
and since \(|1 - \varphi(t)| \leq \frac{1}{2}\), thus for all \( k \geq 2 \)
\[ \frac{1}{k} |1 - \varphi(t)|^k \leq 2^{-k}, \]
the claim (3.13) follows by summing the series \( \sum_{k \geq 2} 2^{-k} \):
\[ \left| \ln \varphi(t) + \frac{1}{2} \sigma^2 t^2 \right| \leq \frac{1}{6} m_3 |t|^3 + \frac{1}{4} \sigma^4 t^4 \leq \frac{5}{12} m_3 |t|^3. \]

Under an additional assumption \(|t| \leq \frac{3}{5} \sigma^{1/2}\), we obtain
\[ \ln |\varphi(t)|^{-1} \geq \frac{1}{2} \sigma^2 t^2 - \frac{5}{12} m_3 |t|^3 \geq \frac{1}{2} \sigma^2 t^2 \left( 1 - \frac{5m_3}{6\sigma^2} |t| \right) \]
\[ \geq \frac{1}{2} \sigma^2 t^2 \left( 1 - \frac{5m_3}{6\sigma^2} \cdot \frac{3\sigma^2}{5m_3} \right) = \frac{1}{4} \sigma^2 t^2, \]
owing to the same moment inequality as above, used again in the last line. \( \square \)

The above general results will be used in the situation where \( X(\omega) = a_{|x|} \omega_x \) with \( P \{ |\omega_x| \leq 1 \} = 1 \) for all \( x \), so \( m_3 = E\left[ a^3_{|x|} |\omega_x| \right] \leq a^3_{|x|} \overline{m}_3 \), where
\[ \overline{m}_3 := E\left[ |\omega_x|^3 \right] \in (0, 1]. \] (3.15)

Notice that that the key ration used in Lemma 3.3 reads as
\[ \frac{\sigma^2}{m_3} = \left( \frac{E\left[ a^2_{|x|} |\omega_x|^2 \right]}{E\left[ a^3_{|x|} |\omega_x|^3 \right]} \right)^2 \geq \frac{\overline{\sigma}^2}{a^3_{|x|} \overline{m}_3}, \quad \overline{\sigma}^2 := E\left[ |\omega_x|^2 \right] \in (0, 1], \]
and $\sigma^2 / \mathbb{E}_3 \in (0, +\infty)$ is a fixed parameter characterizing the common probability distribution of the IID scatterer amplitudes $\omega_x$. For example, $\sigma_b = \mathbb{E}_3 = 1$ for the Bernoulli distribution with values $\{-1, +1\}$.

3.3. **Thermal bath estimate for the cumulative potential.**

**Lemma 3.4.** Consider a random field $V(x, \omega)$ on $\mathbb{Z}^d$ of the form

$$V(x, \omega) = \sum_{y \in \mathbb{Z}^d} u(y - x) \omega_y,$$

where $u$ is given by (1.3) and $\{\omega_x, x \in \mathbb{Z}^d\}$ are bounded IID r.v. with nonzero variance. Then the following holds true:

(A) The common characteristic function $\varphi_V(\cdot)$ of the identically distributed r.v. $V(x, \omega)$, $x \in \mathbb{Z}^d$, obeys an upper bound

$$\forall t \in \mathbb{R} \quad |\varphi_V(t)| \leq Ce^{-|t|^{d/A}}. \tag{3.16}$$

(B) Consequently, the common probability distribution function $F_V(\cdot)$ of the cumulative potential at sites $x \in \mathbb{Z}^d$ has the derivative $\rho_V \in C(\mathbb{R})$.

(C) Let $v^* := \inf \text{supp} \rho_V$, then $F_V(v^* + \lambda) = o(|\lambda|^{-a})$.

**Proof.** The claim will be derived from the Main Lemma 3.1, so we only need to check the validity of its assumptions. Denote

$$X_n := \{x \in \mathbb{Z} : |x| \in (r_{n-1}, r_n]\}, \quad n \in \mathbb{N},$$

$$K_n := |X_n|,$$

$$a_n := u(r_n) \equiv r_n^{-A}. \tag{3.17}$$

The r.v. $X_{n,k}$ figuring in Lemma 3.1 are now $\omega_x$ with $x \in X_n$, numerated arbitrarily by $k \in [1, K_n]$.

Next, note that Lemma 3.3 applies here, since $\omega_x$ are a.s. bounded, thus have finite absolute moments of all orders. Let $N_t = \left\lfloor C t^{1/A} \right\rfloor$, where $C$ is chosen so that for any $n \geq N_t$ one has

$$a_n |t| \leq a_{N_t} |t| \sim N_t^A \sim C^A t \leq \frac{3}{5} \sigma^{1/2},$$

hence by Lemma

$$\ln |\varphi_{\mu}(a_n t)|^{-1} \geq \frac{\sigma^2}{4} r_n^{-2A} t^2.$$  

Now the claim follows from Lemma 3.1: $|\varphi_S(t)| \leq \text{Const} |t|^{-d/A}$. □

For the proof of a fractional-exponential decay of the characteristic function at infinity, there was no need to assess, in Main lemma, the ”ripple” sum $S_1(t)$ which is in general a more delicate task. However, for a particular (but rather rich) class of measures, a lower bound for $S_1(t)$ that one can obtain has the same order of magnitude as the above one for $S_2(t)$. Specifically, it suffices to assume the so-called Cramér’s condition (a.k.a. Condition (C); cf. [9]) widely used in the theory of asymptotic expansions for the limiting probability distribution (often probability density, in fact) for the sums of IID r.v.
**Lemma 3.5.** Let $\mu$ be a probability measure satisfying Cramèr’s condition (C):

$$\limsup_{|t| \to \infty} |\varphi_{\mu}(t)| \leq \zeta < 1.$$  

Then the sum $S_1(t)$ from Lemma 3.1 obeys for some $c > 0$

$$S_1(t) \geq c \ln(\zeta^{-1}) R_t = c \zeta |t|^{d/A}.$$  

The proof is obvious, as each term in $S_1(t)$ is trivially lower-bounded by $\ln(\zeta^{-1})$.

Without Cramèr’s Condition (C), one needs in general more subtle equidistribution arguments in order to show that, pictorially, a typical term of the sum $S_1$ brings a nonzero average contribution. Quite fortunately, for the potentials $u(r) = r^{-A}$, this can be done with the help of an elementary lemma due to Pólya and Szegő, as explained in the next Section 3.4.

Observe that while Condition (C) is of course violated for Bernoulli distributions, it is incomparably weaker than the Rajchman property (cf. [19, 20]), i.e., the condition on the Fourier transform of a measure $\mu$,

$$\lim_{|t| \to \infty} |\varphi_{\mu}(t)| = 0,$$

and the latter is substantially weaker than the assumptions used by Campanino and Klein [4] and Klein et al. [15]. In the latter work, the authors expressed their hope that the conditions (1.1)–(1.2) from their Main Theorem (power-law decay at infinity of the single-site characteristic function) would be sufficient for a regularity of the IDS enabling one to prove Anderson localization in dimension higher than one with rather singular probability distribution of the IID random potential. The question of whether this is true for the models on higher-dimensional lattices (or more general graphs) with short-range interaction potential remains wide open and challenging, so it is rather curious to see how simple becomes the analysis of regularity (and, as a result, of Anderson localization) in discrete models under Cramèr’s condition (C).

Although the onset of Anderson localization at low energies is no longer an open problem for Anderson Hamiltonians in $\mathbb{R}^d$, $d \geq 1$, with an alloy-type potential and arbitrary nontrivial distribution of scatterers’ amplitudes, owing to the deep works by Bourgain and Kenig [3], Aizenman, Germinet, Klein and Warzel [1], and Germinet and Klein [13], the proofs for short-range interaction potentials are rather complex, as one can judge already by the considerable size of the paper [13] summarizing the required techniques and results achieved for arbitrarily singular single-site measures. Unless some new breakthrough is made in this direction, it seems that any regularity weaker than log-Hölder continuity is essentially as hard to treat as the Bernoulli case, indeed even harder, as evidence the efforts made in [1] precisely in order to extend the ideas and techniques by Bourgain and Kenig to singular measures which are barely less singular than Bernoulli, yet not exactly Bernoulli (possibly s.c.).

It seems, therefore, that in the framework of physically realistic, viz. nonlocal interaction potentials the ”regularity threshold”, separating the ”easy” models from those where the analysis of IDS and of localization phenomena requires more involved methods, is
brought much lower than for the local interaction models: instead of the log-Hölder continuity, one can have a comfortable setup under a significantly weaker Cramér’s condition.

3.4. Comments on the proof of Assertion (A) of Main Lemma. The proof of the lower bound of the tidal sum $S_2(t)$ was quite straightforward, but the reader may wonder if the final estimate based on it is optimal, or it can be improved by using $S_1(t)$ instead. In general, assessing the “ripple” sum $S_1(t)$ may prove to be a delicate task, requiring fine results on equidistribution properties of ergodic dynamical systems, which in turn are related to algebraic properties of some parameters of those systems; see a discussion in [5]. However, in the particular case of the potentials admitting an asymptotic

$$u(r) = r^\alpha F(r),$$

where $F$ is a slowly varying function, viz. satisfies

$$\forall c \in (0, +\infty) \lim_{r \to +\infty} \frac{F(cr)}{F(r)} = 1,$$

Wintner [23] proved a lower bound essentially equivalent (in notations of the present paper) to

$$S_1(t) \geq CF(t) |t|^{1/\alpha}.$$

with the help of a result by Pólya and Szegö [18].

**Lemma 3.6** (Pólya and Szegö, [18, Section II.4.1, Problem 155]). Let be given a monotone increasing real sequence $r = (r_n)_{n \geq 1}$ such that its counting function

$$N_t = N_t(r) := \sum_{n: r_n \leq t} \equiv \text{card} \{ n : r_n \leq t \}$$

is varying regularly with exponent $\lambda > 0$ at infinity, i.e., one has a representation $N_t = t^\lambda S(t)$ where $S(t)$ is a so-called slowly varying function at infinity:

$$\forall c > 0 \lim_{t \to \infty} \frac{S(ct)}{S(t)} = 1.$$

Next, consider a Riemann-integrable\(^1\) function $f : (0, c] \to \mathbb{R}$. Then

$$\lim_{t \to +\infty} \frac{1}{N_t} \sum_{n: r_n \leq t} f\left(\frac{r_n}{t}\right) = C(r, f, \lambda, c) := \int_0^{e^\lambda} f\left(s^{1/\lambda}\right) ds. \quad (3.19)$$

Wintner used $f(s) = \ln \max \left[ \cos \left(s^{-1}\right), \frac{1}{2} \right]$. By identification of parameters used in [23] and here, one can see that for power-law potentials $u$, the "ripple" sum $S_1(t)$ has the same order of magnitude as the tidal sum $S_2(t)$.

Several problems formulated in [18, Section II.4.1] are close in spirit to the equidistribution statements from the neighboring Sections II.4.2–II.4.5 in [18]. It is therefore no surprise that Lemma 3.6 appears in Wintner’s work [23] at the place where, for general potentials $u$, one would need some sort of equidistribution arguments.

---

\(^1\)This condition is important for the proof given in [18]. The prototypical case is where $f$ is piecewise constant, and for the summation formula to be asymptotically sharp, the function $f$ must admit two-sided bounds $\psi \leq f \leq \Psi$ by piecewise constant functions $\psi, \Psi$ with arbitrarily high accuracy.
Notice that the fairly general Lemma 3.6 provides an accurate, asymptotically sharp summation formula, but Wintner’s application to the analysis of characteristic functions proceeds by a one-sided bound, in order to fulfill the required conditions. Specifically, taking \( \max[\cdot, \frac{1}{2}] \) in the definition of \( f \), one loses the benefit of small values of \(|\cos(\cdot)|\), hence of large values of \( \ln(|\cos(\cdot)|^{-1}) \).

This ultimately leads to an upper bound for the characteristic function of the infinite convolution measure at hand. In a subsequent paper on the same subject \([24]\), Wintner complemented his result by making use of the terms with small values of \(|a_n|\); the reader can see that his original argument from the one-page paper \([24]\) is much more straightforward and does not require any technical work and several approximations used in the proof of \([18, Section 2.4.1, Problem 155]\). In Section 3.3 we merely adapt his elementary proof. Both estimates – for \( S_1(t) \) and for \( S_2(t) \) – give qualitatively similar results, yet these are only upper bounds and not an asymptotic formula.

4. Infinite smoothness of the DoS and Wegner estimates

4.1. DoS in a thermal bath.

Proof of Theorem 2. The claim follows easily from the Main Lemma 3.1; we only need to identify the key ingredients of the latter:

\[
\mathcal{X}_n := \left\{ x \in \mathbb{Z}^d : \text{dist}(x, \Lambda_L) \in [r_n, r_{n+1}) \right\}, \quad K_n := |\mathcal{X}_n|, \\
\{ \omega_\ell, x \in \mathcal{X}_n \} \leftrightarrow \{ X_{n,k}, k = 1, \ldots, K_n \} \\
M := L, \quad N = +\infty, \\
S_{M,N}(\omega) = \sum_{n=M}^{\infty} \sum_{k=1}^{K_n} a_n X_{n,k} \equiv \sum_{x : |x| \geq L} u(|x|) \omega_x
\]

In some situations, one may want for technical reasons to take instead of \( \mathcal{X}_n \) only a suitable subset thereof, and include the potential induced by the remaining amplitudes \( \omega_\ell \) in a r.v. \( Y(\omega) \). Then the latter can be effectively ignored in the calculations, without invalidating them, by conditioning first on \( Y \), i.e. on the unused random potentials. All one needs is that the cardinality of the reduced \( \mathcal{X}_n \) remain sufficiently large for the desired final bound. To be more precise, the conclusion on infinite smoothness remains valid as long as \(|\mathcal{X}_n| \geq Cn^\kappa \) for some \( \kappa > 0 \), no matter how small; it would only affect the decay exponent \( a(\kappa, A) > 0 \) in \(|\varphi(t)| \leq e^{-C|t|^a(\kappa, A)} \).

By assertion (B) of Lemma 3.1,

\[
\forall t \in \mathbb{R} \quad |\varphi_{\xi_\Lambda}(t)| \leq e^{-CM^{-2A+d}|t|^{d/A}}.
\]

This implies the existence of the probability density \( \rho_{\xi_\Lambda} \in C^\infty(\mathbb{R}) \) of the measure \( dF_{\xi_\Lambda}(E) \). Due to the representation (2.1) with scalar random operator \( \xi_\Lambda(\omega)1_A \), one can label all EVs \( \lambda_j(\omega) \) of \( H_\Lambda(\omega) \) in a measurable way so that

\[
\lambda_j(\omega) = \hat{\lambda}_j(\omega) + \xi_\Lambda(\omega), \quad j = 1, \ldots, |\Lambda|,
\]
where all r.v. $\hat{\lambda}_j$ are independent of $\xi_A$, so the infinite smoothness of the IDS (hence, of the DoS) in $\Lambda$ easily follows. \hfill \Box

4.2. **Wegner estimates.** Aiming to the applications to Anderson localization, we now have to operate with a restricted, annular "bath" of finite size, the complement of which is "frozen". This is necessary for obtaining a satisfactory replacement for the IAD property very valuable in the short-range interaction models.

In the following theorem appear two important parameters, $\theta > 0$ which has the same meaning as in Main Lemma 3.1 (cf. (3.9)), and $\tau > 1$, which will be used in Section 7 and can be chosen arbitrarily large.

**Proof of Theorem 3.** The required bound follows from assertion (C) of Lemma 3.1 which was tailored specifically to suit the Wegner bound in a finite annular "bath". By translation invariance of the random field $\omega = (\omega_x)_{x \in \mathbb{Z}^d}$, we can assume w.l.o.g. that $u = 0$.

Identification of the principal ingredients of Lemma 3.1 is as follows:

$$X_n := \left\{ x \in \mathbb{Z}^d : |x| = n \right\}, \quad K_n := |X_n|,$$

$$\{ \omega_x, x \} \leftrightarrow \left\{ X_{n,k}, k = 1, \ldots, K_n \right\}$$

$$M := L, \quad N = R_L = R_L(\tau),$$

$$S_{M,N}(\omega) = \sum_{n=M}^{N} \sum_{k=1}^{K_n} a_n X_{n,k} \equiv \sum_{x : |x| \in [L.R_L]} u(|x|) \omega_x,$$

$$Y(\omega) = \sum_{n=0}^{N} \sum_{k=1}^{K_n} a_n X_{n,k} \equiv \sum_{x : |x| < L} u(|x|) \omega_x.$$

Proceeding as in Theorem 2, we obtain the representation

$$H_B(\omega) = \tilde{H}_B(\omega) + \xi_B(\omega) \mathbf{1}_B,$$  \hspace{1cm} (4.1)

where the random operator $\tilde{H}_B(\omega)$ is independent of the r.v. $\xi_B(\omega)$, and the latter is generated from the amplitudes $\omega_x$ with $x$ in the annulus $\mathcal{A} = B_{R_L}(0) \setminus B_L(0)$, modulating the "plateaus" of the respective interaction potentials $y \mapsto u(|y-x|)$, $y \in B_L(0)$, covering entirely the ball $B = B_L(0)$. By Lemma 3.1, $\xi_B$ fulfills, for any interval $I$ of length $\epsilon_L = N^{-\frac{1}{M+\theta}} \equiv L^{-\frac{1}{M+\theta}}$ \hspace{1cm} (4.2)

the concentration estimate (cf. (3.2))

$$\mathbb{P}\{\xi_B(\omega) \in I_{\epsilon} \} \leq CM^\theta |I_{\epsilon}| \equiv C L^\theta \epsilon,$$  \hspace{1cm} (4.3)

with $\theta > 0$ arbitrarily small, provided $M = L$ is large enough.

Now we make use of this freedom and pick $\theta \in (0, \tau - 1)$, then solving (4.2) for $L$ as implicit function of $\epsilon_L$, we see that

$$\mathbb{P}\{\xi_B(\omega) \in I_{\epsilon} \} \leq CM^\theta |I| \equiv C L^\theta \epsilon_L^{1+\frac{\theta}{\tau}} = \epsilon_L^{1+\frac{\theta}{\tau}} = \epsilon_L^\beta,$$  \hspace{1cm} (4.4)

where

$$0 < \beta = 1 - \frac{1+\theta}{\tau} \xrightarrow{\tau \to +\infty} 1.$$  \hspace{1cm} (4.5)
This proves the EVC estimate (2.4), since $H_B(\omega)$ acts in the Hilbert space $\ell^2(B)$ of dimension $|B|$.

\[ \square \]

**Remark 4.1.** It is readily seen that the same argument proves a direct analog of the EVC estimate (2.4) for the Anderson Hamiltonians in $L^2(B)$, $B \subset \mathbb{R}^d$, with various boundary conditions, due to the crucial decomposition (4.1) which has exactly the same form in discrete and continuous configuration spaces; idem for the Anderson Hamiltonians on quantum graphs. The only difference comes from the Weyl asymptotics for the EVs, responsible for an extra factor in the RHS depending upon the position of $I$ (say, of length $|I| \leq 1$) in the energy axis.

5. ILS estimates at low energies via "thin tails"

Due to non-negativity of the Laplacian $H_0$,

\[
\mathbb{P}\left\{ E_0^A(\omega) < \lambda \right\} \leq \mathbb{P}\left\{ \min_{x \in A} V((x, \omega) < \lambda) \right\}
\]

\[
\leq |A| \min_{x \in A} \mathbb{P}\{ V((x, \omega) < \lambda) \}
\]

\[
\leq |A| F_V((\lambda), \omega) = |A| \circ (\lambda^\infty),
\]

where the last equality is due to assertion (C) of Lemma 3.4. This strong form of decay of the EV distribution at the bottom of spectrum is, however, unsuitable for the application to the ILS (initial length scale) estimate in the course of the MSA, for it lacks independence in the setting where several volumes $B_{L_0}(u_j)$, $j = 1, 2, \ldots, n$ are considered simultaneously. This can be remedied as follows.

Given a ball $B = B_{L_0}(u)$, denote $B^+ = B_{2L_0}(u)$, and for any given sample $\omega \in [0, 1]^{\mathbb{Z}^d}$, introduce a sample measurable with respect to a sub-sigma-algebra $\mathcal{F}_{B^+} = \mathcal{F}[\omega, y \in B^+]$:

\[
\omega^+_x = \begin{cases} \omega_x, & \text{if } x \in B^+, \\ 1 \equiv \sup \text{ supp } \mu, & \text{otherwise}. \end{cases}
\]

Then, obviously,

\[
\forall x \in B \quad V(x, \omega) < V(x, \omega^+).
\]

Depending on the reader’s point of view, the above inequality can be understood as holding a.s. (a traditional probabilists’superstition: never to be "sure" but at best "almost sure"), or everywhere – if the basic probability space is defined from the very beginning as $[0, 1]^{\mathbb{Z}^d}$ rather than $\mathbb{R}^{\mathbb{Z}^d}$ (analysts are likely to prefer this variant). Just like $\omega^+$, the modified potential $V(\cdot, \omega^+)$ is $\mathcal{F}_{B^+}$-measurable. Therefore, for any family of balls $B_{L_0}(u_i)$, $i = 1, 2, \ldots, n$, with pairwise distances $|u_i - u_j| > 4L_0$, and any measurable functionals of $V|_{B^+(u_i)}$, the family of random variables

\[
\zeta_i(\omega) = f_i\left(V\left(\cdot, \omega^{(+,i)}\right)\right)|_{B^+(u_i)}
\]

is independent. Here $\omega^{(+,i)}$ is obtained from $\omega$ in the same way as $\omega^+$ for a given ball $B_{L_0}(u)$ in the general construction explained above, hence the samples $\left\{ \omega^{(+,i)}, i = 1, 2, \ldots \right\}$ are independent, yielding the same property for the r.v. $\zeta_i$. 
Theorem 6. Fix any $L_0 > 1$ and consider the Hamiltonian $H_{B_{L_0}(u)}(\omega)$ with an arbitrary $u \in \mathbb{Z}^d$. Assume that the interaction potential decays as $u(r) = r^{-A}$, $A > d$, and introduce a larger ball $B^+ = B_{L_0}(u)$ and the sigma-algebras

- $\mathcal{F}_{B^+}$ generated by all scatterers’ amplitudes $\omega_y$ with $y \in B^+$,
- $\mathcal{F}_{B^+}^{-}$ generated by all scatterers’ amplitudes $\omega_y$ with $y \in \mathbb{Z}^d \setminus B^+$.

Then for any $\theta \in (0, 1)$ there exists some $C_\theta > 0$ such that

$$\mathbb{P}\left\{ E_{0}^A(\omega) \leq L_0^{-\theta} \right\} \leq e^{-C_\theta L_0^d}. \tag{5.5}$$

Proof. Fix any $x \in B_{L_0}(u)$, let $R_0 = L_0^\theta$, $\theta > 0$, and denote $Q_x = B_{2R_0}(x) \setminus B_{R_0}(x)$; note that $|Q_x| > R_0^d$. For any $\kappa > 0$,

$$\max_{y \in Q_x} \omega_y > \kappa \Rightarrow V(x, \omega) \geq C_\kappa R_0^{-A} > L_0^{-\theta},$$

whence

$$\forall x \in B_{L_0}(u) \quad \mathbb{P}\left\{ V((x, \omega) < \lambda \right\} \leq \mathbb{P}\left\{ \forall y \in Q_x \quad \omega_y \leq \kappa \right\} = \prod_{y \in Q_x} \mathbb{P}\{ \omega_y \leq \kappa \} = \epsilon_{|Q_x|}^{|Q_x|} = e^{\ln \epsilon \kappa^{-1} R_0^d} \leq e^{-C(\kappa,d) L_0^d},$$

yielding

$$\mathbb{P}\left\{ \min_{x \in B_{L_0}(u)} V((x, \omega) < \lambda \right\} \leq \left| B_{L_0}(u) \right| e^{-C(\kappa,d) L_0^d} \leq e^{-C' L_0^d},$$

for some $C' = C'(\kappa,d) > 0$. By non-negativity of the kinetic energy operator $H_0$, this proves the claim (cf. (5.1)).

The first scenario leading to the onset of Anderson localization is more universal and robust than the one considered in the next subsection; here we do not make any assumption on the magnitude of the potential and do not attempt to achieve a global bound on the entire spectrum (which is usually possible in discrete systems and/or in one dimension). This will result in ILS estimates easily adapted to the continuous alloy models in $\mathbb{R}^d$, $d \geq 1$, as well as in a large class of quantum graphs, with tempered underlying combinatorial graphs of coupling vertices.

6. ILS ESTIMATES UNDER LARGE DISORDER

6.1. Individual EV concentration estimates. We start with the weakest estimate which has, on the other hand, the general form closest to the usual strong-disorder ILS bound used in MSA. It suits to the scaling analysis based on scale-free probability estimates at the initial scale (cf. [21, 12]).

An ILS bound is most efficiently used via the Combes–Thomas estimate [8], and the reader familiar with this technique can see that in applications to the MSA, it suffices to set $\epsilon = 1$ (or any other fixed positive number) in the inequality (6.3).
Theorem 7. Let $\Sigma_{B,g}(\omega) = \Sigma(H_B(\omega)) = \left\{ E_j^{(B,g)}(\omega), j = 1, \ldots, |B| \right\}$ be the set of random eigenvalues of $H_{B,g}(\omega) = -\Delta_B + gV(\omega)$, numbered in a measurable way, counting multiplicity. Then for any $\eta \in (0,1]$ and $\varepsilon > 0$ there exist $L_0 \in \mathbb{N}$ and $g > 0$ such that

$$\sup_{E \in \mathbb{R}} \mathbb{P}\left\{ \text{dist}\left( \Sigma_{B,g}(\omega), E \right) \leq \varepsilon \right\} \leq \eta. \quad (6.1)$$

Proof. We will actually establish a slightly stronger bound, with the scale-free threshold $\eta > 0$ replaced with $L_0^{-\kappa}$ with some $\kappa \in (0, A - d)$.

By Main Lemma 3.1, for any $x \in B$ and any interval $I_\delta$ of length $\delta \geq L^{-A+\theta}$ with arbitrarily small $\theta > 0$ and $L$ large enough (depending on $\theta$),

$$\mathbb{P}\left\{ V(x, \omega) \in I_\delta \right\} \leq C\delta,$$

hence with $\delta = L^{-A+\theta}$, $\theta := A - d - \kappa$, $\kappa \in (0, A - d)$, and sufficiently large $L$,

$$\mathbb{P}\left\{ \exists x \in B : V(x, \omega) \in I_\delta \right\} \leq CL^d \delta = CL^{-A+\theta+d} < L_0^{-\kappa}. \quad (6.2)$$

Fix $\varepsilon > 0$ and let $\omega$ be such that $\text{dist}(\Sigma_{B,g}(\omega), E) < \varepsilon$, then there exists at least one $\text{EV} E_j^{(B,g)}(\omega) \in J_\varepsilon(E) = (E - \varepsilon, E + \varepsilon)$. Since $H_{B,g}(\omega) = gV(\omega) - \Delta_B$ and $\|\Delta_B\| \leq 4d$, where the EVs of $gV(\omega)$ form the set $\{gV(x, \omega), x \in B\}$, there exists at least one $x \in B$ such that

$$|gV(x, \omega) - E| \leq \varepsilon + 4d \implies |V(x, \omega) - g^{-1}E| \leq \frac{\varepsilon + 4d}{|g|}.$$

Equivalently, $V(x, \omega) \in I_\delta := \left[ g^{-1}E - \frac{\delta}{2}, g^{-1}E + \frac{\delta}{2} \right]$ with

$$\delta = \delta(g) = \frac{\varepsilon + 4d}{|g|}.$$

Let $|g| = (\varepsilon + 4d)L^{-A+\theta}$, then $\omega$ must be contained in the event

$$\mathcal{R}_\delta := \left\{ \exists x \in B : V(x, \omega) \in I_\delta \right\},$$

with $\mathbb{P}\{\mathcal{R}_\delta\} < L_0^{-\kappa}$, according to (6.2), so the claim follows by taking $L_0 \geq \eta^{-1/\kappa}$. □

The next result gives a stronger, power-law decay of the ILS probability estimate for the MSA induction. The price to pay for this improvement is possibly large "isolation zone” around the balls $B_{l_0}(u_j), j = 1, \ldots, S$, in the scaling scheme where $S \geq 1$ bad balls should be tolerated in the decay analysis of the Green functions (cf. [12]).

Theorem 8. Let $\Sigma_{B,g}(\omega) = \Sigma(H_B(\omega)) = \left\{ E_j^{(B,g)}(\omega), j = 1, \ldots, |B| \right\}$ be the set of random eigenvalues of $H_{B,g}(\omega) = -\Delta_B + gV(\omega)$, numbered in a measurable way counting multiplicity. Given $R \in \mathbb{N}$, denote by $\mathcal{F}_R$ the sigma-algebra generated by the amplitudes $\omega_k$ with $|x| > R$, given $\mathcal{F}_R$. Fix any $b > 0$. Then for any (viz., arbitrarily large) $\varepsilon > 0$ there exist $L_0 \in \mathbb{N}$ and $g > 0$ such that

$$\sup_{E \in \mathbb{R}} \mathbb{P}\left\{ \text{dist}\left( \Sigma_{B,g}(\omega), E \right) \leq \varepsilon \right\} \mathcal{F}_R \leq L_0^{-b}. \quad (6.3)$$
**Proof.** Denote for brevity\(^2\) \(\mathbb{P}_R \{ \cdot \} \equiv \mathbb{P} \{ \cdot \mid \mathcal{F}_R \}.\) By assertion (D) of Main Lemma 3.1, with \(M = 1,\) for any \(x \in B\) and any interval \(I_\delta\) with \(|I_\delta| = \delta \geq R^{-A+\theta},\) arbitrarily small \(\theta > 0\) and \(L\) large enough (depending on \(\theta\)),

\[
\mathbb{P}_R \{ V(x, \omega) \in I_\delta \} \leq C \delta,
\]

hence with

\[
R = L^\tau, \quad \tau > \frac{b+d}{A}, \quad \delta = R^{-A+\theta}, \quad \theta := A - d - \kappa, \quad \kappa \in (0, A-d),
\]

and sufficiently large \(L,\)

\[
\mathbb{P}_R \{ \exists x \in B : V(x, \omega) \in I_\delta \} \leq CL^d \delta = CL^{-\tau A + \theta + d} < L_0^{-b}.
\]

Fix \(\varepsilon > 0\) and let \(\omega\) be such that \(\text{dist}(\Sigma_{B,g}(\omega), E) < \varepsilon,\) then there exists at least one EV \(E_i^{B,g}(\omega) \in J_\varepsilon(E) = (E-\varepsilon, E+\varepsilon).\) Since \(H_{B,g}(\omega) = gV(\omega) - \Delta_B\) and \(|\Delta_B| \leq 4d,\) where the EVs of \(gV(\omega)\) form the set \(\{gV(x, \omega), x \in B\},\) there exists at least one \(x \in B\) such that

\[
|gV(x, \omega) - E| \leq \varepsilon + 4d \implies |V(x, \omega) - g^{-1}E| \leq \frac{\varepsilon + 4d}{|g|}.
\]

Equivalently, \(V(x, \omega) \in I_\delta := [g^{-1}E - \delta, g^{-1}E + \delta] \) with

\[
\delta = \delta(g) = \frac{\varepsilon + 4d}{|g|}.
\]

To satisfy (6.5), let \(|g| = (\varepsilon + 4d)L^{-A+\theta},\) then \(\omega\) must be contained in the event

\[
\mathcal{R}_\delta := \{ \exists x \in B : V(x, \omega) \in I_\delta \},
\]

with \(\mathbb{P}\{\mathcal{R}_\delta\} < L_0^{-b},\) according to (6.7). This proves the claim. \(\square\)

7. PROOF OF LOCALIZATION

For brevity, we concentrate on the case of strong disorder and assume that the GFs on some initial scale \(L_0\) decay exponentially, with exponent \(m_0 = (1 + L_0^{-1/8})m, m \geq 1.\) It is well-known that only a minor adaptation is required in the case of "extreme energies", near spectral edge(s), where the the decay exponent \(m_0\) may be as small as \(m_0 = L_0^{-\theta},\) \(\theta \in (0, 1);\) cf., e.g., [12, Section 5.4]), [22, Section 3.2]).

It was shown in Section 5 that such a bound can be established with the help of the "thin tails" argument, replacing its "Lifshitz tails" counterpart, with high probability, viz. \(p_0 \geq 1 - e^{-c_1L_0^\delta}, c, c_1 > 0,\) which is more than sufficient to start the MSA induction. The path laid down in the spectral theory of random operators several decades ago was based on the

\(^2\)In other words, computing the probability \(\mathbb{P}_R \{ \cdot \},\) we can make a good use of the random fluctuations of the cumulative potential \(V\) coming from \(\omega_x\) with \(x \in B_R(u),\) while the remaining amplitudes generate a background potential considered as "frozen", thus useless for the regularity analysis.
observation that IID or IAD (Independent At Distance) random potentials in Anderson-type Hamiltonians lead to a simplification of mathematical analysis of the localization problem; obviously, such a simplification of the model was in contradiction with the physical reality where interactions have infinite range. It was only later, in 2005, that one fully assessed the level of mathematical difficulties brought up by that simplification in a general context of singular marginal distributions of the underlying disorder in discrete models used in physics for modeling continuous solid-state systems in the so-called tight-binding approximation which, curiously, was intended for making analysis simpler. In essence, this paper addresses a different problem of mathematical physics, which is closer to the physics and, quite fortunately, in many aspects less hard on the technical level.

♦ Needless to say, the crucial fact that makes the usual MSA machinery to work here is Theorem 3 establishing a comfortable Wegner estimate. This is precisely what allows one to avoid a radical re-writing the MSA induction, performed by Bourgain and Kenig [3] for short-range potentials with Bernoulli distribution of the local random amplitudes, and and by Germinet and Klein [13] for arbitrary nontrivial IID random amplitudes (again, for short-range potentials).

7.1. Deterministic analysis. We adapt the strategy from [14].

Working with a Hamiltonian $H_{B_L(u)} = -\Delta_{B_L(u)} + gV$ in a given ball $B_L(u)$, it will be necessary to know the values of the amplitudes $\omega_y$ with $y$ in a larger ball $B_{R_L}(u) \supset B_L(u)$, where the specific choice of $R_L$ depends upon the decay rate $r \mapsto r^{-A}$ of the interaction potential $u(r)$, along with some other parameters of the model and of the desired rate of decay of EFCs to be proved. Below we set $R_L = L^r$, $r > 1$.

Definition 7.1. Let be given a ball $B = B_L(u)$. A configuration $q \in \Omega_{Z^d}$ is called

1. $(E, \epsilon, B)$-non-singular iff the resolvent $G_B(E)$ of the operator

   $H_{B_L(u)} = -\Delta_{B_L(u)} + U[q]_{B_L}$

   (cf. the definition of $U[q]$ in (1.2)) is well-defined and satisfies

   $\max_{x \in B_{L/3}(u)} \max_{y \in B_{L}(u)} \|G_B(x, y; E)\| \leq \epsilon$;  

   (7.1)

2. $(E, \gamma, B)$-non-resonant iff

   $\text{dist}(\Sigma(H_B), E) \geq \gamma$.  

(7.2)

When the condition (7.1) (resp., (7.2)) is violated, $q$ will be called $(E, \epsilon)$-singular (resp., $(E, \gamma)$-resonant). We will be using obvious shortcuts $(E, \epsilon, B)$-NS, $(E, \epsilon, B)$-S, $(E, \gamma, B)$-NR and $(E, \gamma, B)$-R.

Definition 7.2. Let be given a ball $B = B_L(u)$ and a real number $\tau > 1$. A configuration $q_{B_{L, \tau}} \in \Omega_{B_{L, \tau}}$ is called

1. $(E, \epsilon, B)$-SNS (strongly non-singular, or stable non-singular) iff for any configuration of amplitudes $q_{B^c} \in \Omega_{B^c}$ the extension of $q_{B_{L, \tau}}$ to the entire lattice, $q = (q_{B_{L, \tau}}, q_{B^c_{L, \tau}})$ is $(E, \epsilon, B)$-NS;

2. $(E, \gamma, B)$-SNR (strongly NR, or stable NR) iff for the configuration $\Omega_{B^c} \ni q_{B^c} \equiv 0$ the function $V_B = U[q_{B_{L, \tau}} + q_{B^c_{L, \tau}}]|_{B_{L, \tau}} = U[q_{B_{L, \tau}}]|_{B_{L, \tau}}$ is $(E, \gamma)$-CNR.
In subsection 7.2 we work in the situation where the potential \( V : \mathbb{Z}^d \rightarrow \mathbb{R} \) is fixed, and perform a deterministic analysis of finite-volume Hamiltonians. It will be convenient to use a slightly abusive but fairly traditional terminology and attribute the non-singularity and non-resonance properties to various balls \( B \) rather than to a configuration \( \mathbf{q} \) or a cumulative potential \( V = U[\mathbf{q}] \), which will be fixed anyway. Therefore, we will refer, for example, to \((E, \varepsilon)-\text{NS}\) balls instead of \((E, \varepsilon, B)-\text{NS}\) configurations \( \mathbf{q} \). Similarly, we use the notions of \((E, \gamma)-\text{NR}, (E, \varepsilon)-\text{SNS}\) or \((E, \gamma)-\text{SNR}\) balls.

### 7.2. Scaling scheme

Fix \( \mathbb{N} \ni d \geq 1, A > d \) and the interaction potential \( u(r) (\sim r^{-A}) \) of the form (1.3). Further, fix an arbitrary number \( b > d \), which will represents the desired polynomial decay rate of the key probabilities in the MSA induction, and let

\[
\alpha > \tau > \frac{b}{A-d}, \quad \mathbb{N} \ni S > \frac{b \alpha}{b - \alpha d}, \quad L_{k+1} = \left\lfloor L_k^\alpha \right\rfloor, \quad k \geq 0, \tag{7.3}
\]

with \( L_0 \) large enough, to be specified on the as-needed basis. A direct analog of the well-known deterministic statement [10, Lemma 4.2] is the following statement adapted to long-range interactions essentially as in [14]. Introduce useful notation:

\[
m_k := \left(1 + L_k^{-1/8}\right) m, \quad \varepsilon_k := 4L_k^{-\varepsilon A + \theta}, \quad \theta \in (0, 1). \tag{7.4}
\]

The value of \( \theta \) can be chosen as small as one pleases.

**Lemma 7.1** (Conditions for strong non-singularity). Consider a ball \( B = B_{L_{k+1}}(u), k \geq 0, \) and suppose that

(i) \( B \) is \((E, \varepsilon_k)-\text{SNR};\)

(ii) \( B \) contains no collection of balls \( \{B_{L_i}(x_i), 1 \leq i \leq S + 1\} \), with pairwise \( 2L_k^\varepsilon \)-distant centers, neither of which is \((E, e^{-m_k L_k})-\text{SNS}.\)

Then \( B \) is \((E, m)-\text{SNS}.\)

**Proof.** Derivation of the NS property can be done essentially in the same way as in [10] and in numerous subsequent papers, with minor adaptations. See for example [6, proof of Lemma 7] where the singular balls are also supposed to be pairwise \( L_k^\varepsilon \)-distant, \( \tau > 1 \). This does not actually require any significant modification of the original argument from the work by von Dreifus and Klein [10]. What is important here, is that \((E, m)-\text{NS}\) property of \( B_{L_{k+1}}(u) \) (not the strong NS) is derived from weaker versions of the hypotheses (i)–(ii), where SNS and SNR conditions are replaced with their NS and NR counterparts. The proof refers only to the potential \( V \) in \( B_{L_{k+1}}(u) \), which is fixed, so the notion of stability (hence variation of \( V \)) just does not appear in the proof.

To show that the strong (stable) non-singularity property also holds true, one can use induction on scales \( L_k \). We have to show that the NS property of the larger ball \( B_{L_{k+1}}(u) \) is stable with respect to arbitrary fluctuations of the random amplitudes \( \omega_y \) with \( y \notin B_{L_{k+1}}(u) \). According to what has just been said in the previous paragraph, it suffices to check the stability of the properties

(i') \( B_{L_{k+1}}(u) \) is \((E, \gamma_k)-\text{NR},\)
(ii′) \( B_{k+1}(u) \) contains no collection of balls \( \{ B_k(x_i), 1 \leq i \leq S+1 \} \), with pairwise 
2L_k-distant centers, neither of which is \((E,m)\)-NS.

under the hypotheses (i)–(ii).

There is nothing to prove for the stability of (i′), as it is asserted by (i).

Stability of the NS property of balls \( \{ B_k(x_i) \} \) can be derived recursively, with the help
of the arguments from [10], [6, proof of Lemma 7]. The non-singularity property (even
its conventional, non-stable variant) is quite implicit; in fact, the whole point of making
it a sort of ”black box” in [10] was a realization that it is extremely difficult, if realistic
at all, to trace the unwanted events, susceptible to prevent the onset of localization in a
ball at some induction step \( k \gg 1 \), down to the multitude of smaller balls of size \( L_0 \) inside
\( B_{L_k}(x) \). Since it is derived inductively, we have to make sure that the fluctuations of \( \omega_y \)
with \( y \notin B_{L_{k+1}}(u) \) cannot destroy the non-resonance and non-singularity properties in the
relevant balls \( B_{L_j}(x) \subset B_{L_{k+1}}(u) \), \( j = 0, \ldots, k \).

On the scale \( L_0 \) the non-singularity is derived from non-resonance, with a comfortable
gap between an energy \( E \) and the spectrum in the ball of radius \( L_0 \), which provides the
base of induction. Evidently, given any ball \( B_{L_j}(x) \subset B_{L_{k+1}}(u) \) one has
\[
\forall j = 0, \ldots, k \quad B^c_{L_{k+1}}(u) \subset B^c_{L_j}(x). 
\]

In other words, stability encoded in the SNS or SNR properties of smaller balls \( B_{L_j}(x) \subset
B_{L_{k+1}}(u) \) is stronger than what is required for the stability w.r.t. fluctuations \( \omega_y \)
outside a much larger ball \( B_{L_{k+1}}(u) \). We conclude that the claim follows indeed from the the
hypotheses (i)–(ii). \( \square \)

7.3. Probabilistic analysis. It follows directly from Definition 7.2 that any event of the form
\[
\mathcal{A}(B_L(x), E, m) = \left\{ V_q(\cdot; \omega) \Bigr|_{B_L(x)} \text{ is } (E,m)\text{-SNS} \right\}
\]
is measurable w.r.t. the sigma-algebra \( \mathcal{F}^{B_L(x)} \).

Lemma 7.2 (Factorization of the probability of a bad cluster). Suppose that
\[
P\{B_{L_k}(u) \text{ is not } (E,m)\text{-SNS}\} \leq p_k.
\]

Let \( S_{k+1} \) be the maximal cardinality of a collection of balls \( B_{L_k}(u_i), i = 1, 2, \ldots, \), with
pairwise 2L_k-distant admissible centers, of which neither is \((E,m)\)-SNS. Then for any
integer \( S \geq 0 \)
\[
P\{S_{k+1} > S\} \leq C_d Y^{(S+1)d}_{k+1} p_k^{S+1}.
\]

Proof. Using induction on \( j \in [1, S] \), it suffices to prove that, with
\[
\mathcal{A}_j = \bigcup_{i=1}^{j} B_{2L_k(u_i)}, \quad j = 1, \ldots, S,
\]
one has
\[
P \left\{ B_{L_k}(u_{j+1}) \text{ is not } (E,m)\text{-SNS} \big| \mathcal{F}^Z \setminus \mathcal{A}_j \right\} \leq p_k. \quad (7.5)
\]
By Lemma 7.1 the event \( \{ B_{L_k}(u) \text{ is not } (E,m)\text{-SNS} \} \) is \( \mathcal{F}_{B_{L_k}(u)} \)-measurable, so (7.5) holds true. Hence the claim follows by induction, since the number of all collections of \((S+1)\) admissible centers \( u_i \in B_{L_k+1}(u) \), distant or not, is bounded by \( C_d Y_{k+1}^{(S+1)d} \).

A reader familiar with the paper [12] can see that main ingredients of the proof of the next lemma follow closely the respective arguments from the Germinet–Klein analysis.

**Lemma 7.3** (Scaling of probabilities). Assume that
\[
\sup_{u \in \mathbb{Z}^d} \mathbb{P}\left\{ B_{L_k}(u) \text{ is not } (E,m)\text{-SNS} \right\} \leq p_k \leq L_k^{-b}
\]
Then
\[
\sup_{u \in \mathbb{Z}^d} \mathbb{P}\left\{ B_{L_{k+1}}(u) \text{ is not } (E,m)\text{-SNS} \right\} \leq p_{k+1} \leq L_{k+1}^{-b}.
\]

**Proof.** By Lemma 7.1, if \( B_{L_{k+1}}(u) \) is not \((E,m)\)-SNS, then either it is not \((E,\mathbf{y}_{k+1})\text{-CNR}\) or it contains a collection of at least \(S+1\) balls \( B_{L_k}(x_i)\) neither of which is \((E,m)\)-SNS, with admissible and pairwise \(2L_k^\tau\)-distant centers.

The probability of the former event is assessed with the help of the Wegner-type estimate from Theorem 3, relying on the disorder in the balls \( B_{L_k} \subseteq B \). Even the largest among them, \( B_{L_{k+1}} \), is surrounded by a belt of width \( L_{k+1}^\tau \) where the random amplitudes are not fixed hence can contribute to the Wegner estimate with \( \varepsilon = \varepsilon_{R_{k+1}} \), \( R_{k+1} = L_{k+1}^\tau \), hence the same is true for all of these balls: for any \( j \), we have (cf. (2.3) and (2.4))
\[
\mathbb{P}\left\{ B_{L_j} \text{ is not } (E,\varepsilon_{R_j})\text{-SNR} \right\} \leq (L_{k+1}^\tau)^{-A+d+\theta}
\]
where we are free to choose \( \theta > 0 \) as small as we please (the actual correction to the power \( A-d \) is logarithmic). Since \( \tau > (b+1)/(A-d) \), we can pick \( \theta \) so small that \( (A-d-\theta)\tau > b+1 \), hence the RHS of (7.6) is bounded by \( \frac{1}{2} L_{k+1}^{-b-1} \) with \( b' > b \).

The total number of such balls is \( Y_{k+1} \leq L_k^{\alpha^{-1}} = L_{k+1}^{1-\alpha^{-1}} \), with \( 1-\alpha^{-1} < 1 \), therefore,
\[
\mathbb{P}\left\{ B_{L_{k+1}}(u) \text{ is not } (E,\varepsilon_{R_j})\text{-CNR} \right\} \leq \frac{1}{2} L_{k+1}^{-(b+1)+1} < \frac{1}{2} L_{k+1}^{-b}.
\]

By Lemma 7.2
\[
\mathbb{P}\left\{ S_{k+1} > S \right\} \leq \frac{Y_{k+1}^{S+1}}{(S+1)!} P_k^{S+1} \leq \frac{1}{2} L_k^{-(S+1)b} \leq \frac{1}{2} L_{k+1}^{-b},
\]
whence
\[
\mathbb{P}\left\{ B_{L_{k+1}}(u) \text{ is not } (E,m)\text{-SNS} \right\} \leq \frac{1}{2} L_{k+1}^{-b} + \frac{1}{2} L_{k+1}^{-b} = L_{k+1}^{-b}.
\]

By induction on \( k \), we come to the conclusion of the fixed-energy MSA under a polynomially decaying interaction.

**Theorem 9.** Suppose that the ILS estimate
\[
\sup_{u \in \mathbb{Z}^d} \mathbb{P}\left\{ B_{L_0}(u) \text{ is not } (E,m)\text{-SNS} \right\} \leq L_0^{-b}
\]
holds for some $L_0$ large enough, uniformly in $E \in I \subset \mathbb{R}$. Then for all $k \geq 0$ and all $E \in I$

$$\sup_{u \in \mathbb{Z}^d} \mathbb{P}\{B_{L_k}(u) \text{ is not } (E,m)\text{-SNS} \} \leq L_k^{-b}.$$  

In turn, the required ILS estimate is established in Section 6 for strongly disordered systems, and in Section 5 for an arbitrary nonzero amplitude of the potential, with the help of the "thin tails" argument. Therefore, the fixed-energy bound on the decay of Green functions in the balls of radii $L_k$, $k \in \mathbb{N}$, is proved under either of these conditions.

This concludes the fixed-energy MSA for the long-range Anderson Hamiltonians on $\mathbb{Z}^d$ with arbitrary nontrivial probability distribution of the IID amplitudes $\omega_k$ and polynomially decaying interaction potential $u(r) = r^{-A}$, $A > d$.

8. ON DERIVATION OF SPECTRAL AND DYNAMICAL LOCALIZATION

Here we follow closely [7, Section 5].

Proposition 10 is an adaptation of a result by Elgart et al. [11], and Proposition 11 is essentially a reformulation of an argument by Germinet and Klein (cf. [12, proof of Theorem 3.8]) which substantially simplified the derivation of strong dynamical localization from the energy-interval MSA bounds, compared to [?].q

Introduce the following notation: given a ball $B_L(z)$ and $E \in \mathbb{R}$,

$$F_{z,L}(E) := |B_L(z)| \max_{|y-z|} |G_{B_L(z)}(z,y;E)|,$$

with the convention that $|G_{B_L(z)}(z,y;E)| = +\infty$ if $E$ is in the spectrum of $H_{B_L(z)}$. Further, for a pair of balls $B_L(x), B_L(y)$ set

$$F_{x,y,L}(E) := \max \{F_{x,L}(E), F_{y,L}(E)\}.$$

The fixed-energy MSA in an interval $E \in I \subset \mathbb{R}$ provides probabilistic bounds on the functional $F_{x,L}(E)$ of the operator $H_{B_L(x)}(\omega)$; as a rule, they are easier to obtain than those on $\sup_{E \in I} F_{x,y,L}(E)$ (referred to as energy-interval bounds). Martinelli and Scoppola [16] were apparently the first to notice a relation between the two kinds of bounds, and used it to prove a.s. absence of a.c. spectrum for Anderson Hamiltonians obeying suitable fixed-energy bounds on fast decay of their Green functions. Elgart, Tautenhahn and Veselić [11] improved the Martinelli–Scoppola technique, so that energy-interval bounds implying spectral and dynamical localization could be derived from the outcome of the fixed-energy MSA.

**Proposition 10** (Cf. [7, Proposition 1], [11]). Let be given a bounded interval $I \subset \mathbb{R}$, an integer $L \geq 0$ and disjoint balls $B_L(x), B_L(y)$. Assume that for some $a_L, q_L \in (0,1]$

$$\sup_{E \in I} \max_{z \in \{x,y\}} \mathbb{P}\{F_{z,L}(E) > a_L\} \leq q_L,$$

and for some function $f : (0,1] \to \mathbb{R}_+$,

$$\forall \varepsilon \in (0,1] \quad \mathbb{P}\left\{ \text{dist}(\Sigma(H_{B_L(x)}), \Sigma(H_{B_L(y)})) \leq \varepsilon \right\} \leq f(\varepsilon).$$
Then
\[
\mathbb{P}\left\{ \sup_{E \in I} F_{x,y,L}(E) > \max \left[ a_L, q_L^{1/2} \right] \right\} \leq |I| q_L^{1/4} + f \left( 2q_L^{1/4} \right)
\]}

For any interval \( I \subset \mathbb{R} \), denote by \( \mathcal{B}_1(I) \) the set of bounded Borel functions \( \phi : \mathbb{R} \to \mathbb{C} \) with \( \text{supp} \phi \subset I \) and \( \| \phi \|_{\infty} \leq 1 \).

The next result is based on the Germinet–Klein techniques.

**Proposition 11** (Cf. [7, Theorem 3], [12]). Assume that the following bound holds for some \( \varepsilon > 0 \), \( h_L > 0 \), \( L \in \mathbb{N} \) and a pair of balls \( B_L(x), B_L(y) \) with \( |x - y| \geq 2L + 1 \):
\[
\mathbb{P}\left\{ \sup_{E \in I} F_{x,y,L}(E) > \varepsilon \right\} \leq h_L.
\]

Then for any ball \( B \supset (B_{L+1}(x) \cup B_{L+1}(y)) \)
\[
\mathbb{E} \left[ \sup_{\phi \in \mathcal{B}_1(I)} \left| \langle 1_x \vert \phi(H_B) \vert 1_y \rangle \right| \right] \leq 4\varepsilon + h_L.
\]

Now the assertions of Theorems 4 and 5 follow from the fixed-energy localization analysis carried out in Section 7 with the help of Propositions 10 and 11.

9. Concluding remarks

9.1. What’s beyond the piecewise-constant, toy models of interaction? Evidently, the analytic form (1.3) of \( u \) is by far too artificial to be physically realistic, so the question is, to what extent an approximation of a slowly decaying function \( u : r \mapsto r^{-A} \) (i.e., with the derivative \( u' \) decaying faster than \( u \) itself) by larger and larger plateaus on \( [t_k, t_{k+1}] \) preserves the regularity properties:

(i) of the single-site distributions of \( V(x, \cdot) \) subject to an infinite thermal bath,

(ii) of the IDS/DoS in finite balls, again in an infinite thermal bath,

(iii) of the finite-ball EV concentration in a ”frozen bath”.

Of course, in (iii), it is out of question to get any universal continuity property for an arbitrary nontrivial marginal measure of the random amplitudes \( \omega_x \). For example, in the Bernoulli case, having at our disposal only a finite configuration \( \{ \omega_x, x \in B_{L+R}(u) \} \), \( L, R < \infty \), we get a finite number of samples of the EVs, hence no bona fide continuity of the IDS. On the other hand, one only needs here a descent EVC bound, not continuity of the EV distribution.

Curiously, when we replace a random constant potential on a ball \( B_L(u) \) by an almost constant function \( x \mapsto \omega_x u(|x - y|) \), the resulting effect on the variation of the EVs, which are therefore no longer linear functionals of the r.v. \( \omega_x \) (not exactly, anyway), is akin to introducing a weak dependence between the variations of any given EV \( \lambda_j(\omega) \) due to distinct amplitudes, say, \( \omega_y \) and \( \omega_z \). This is a subtle effect on which I plan to elaborate in a forthcoming work in the present series devoted to long-range interactions, but informally it can be explained as follows.

Fix an EV \( \lambda_j(\omega) \) of \( H_{B_L(u)}(\omega) \), and two amplitudes \( \omega_y, \omega_z \) with \( y \neq z \) far away from \( B_L(u) \). For notational brevity, let \( \omega_x \) take values 0 and 1 (Bernoulli). Denote \( \lambda_j(a,b) \) the
value of $\lambda_j$ with $\omega_x = a$, $\omega_y = b$, $a, b \in \{0, 1\}$. If the interaction, modulated by $\omega_x$ and $\omega_y$, were constant ($= C \neq 0$) on $B_L(u)$, we would have

$$\lambda_j(1,0) - \lambda_j(0,0) = \lambda_j(1,1) - \lambda_j(0,1) = C \omega_x,$$

which is thus independent of the r.v. $\omega_y$. This is precisely the comfortable setting in which we have been working in the present paper. However, if the interaction is not constant on $B_L(u)$, the variations

$$\lambda_j(1,0) - \lambda_j(0,0) \quad \text{and} \quad \lambda_j(1,1) - \lambda_j(0,1)$$

are no longer identical, generally speaking, hence we have effectively a dependence between the variations of the EVs induced by independent random amplitudes $\omega_x$, $\omega_y$.

Naturally, by the min-max principle, the nonlinear effects (hence, stochastic dependencies) are as weak as the sup-norm deviation of the potential $B_L(u) \ni z \mapsto u(|z-x|)$ from an approximating constant as $z$ runs through the ball $B_L(u)$.

The bottom line is that the simplified analysis carried out in this work should be extended in fairly similar ways

- to the random media with weak stochastic dependence between the source potentials $x \mapsto \omega_x u(|x-y|)$, and
- to the slowly decaying interaction potentials $u$, including $r \mapsto r^{-A}$.

In particular, the Gaussian asymptotics should remain valid in a large class of correlated random fields \( \{ \omega_x, x \in \mathbb{Z}^d \} \). Different approaches have been developed in the probability theory to prove limit theorem for sums of weakly dependent r.v.; historically, the first fairly general method was proposed by S. N. Bernstein [2] in 1927. It has the advantage to be quite straightforward and flexible. Pictorially, the key idea is that the asymptotic probability distribution for sequences of random variables $X_n$ and $Y_n$ is the same, provided that $X_n - Y_n$ has variance relatively small as compared to $\sigma_n^2 = \mathbb{E} \left[ (X_n - \mathbb{E}[X_n])^2 \right]$. Bernstein’s method applied both to the analysis of the limiting measures and to their Fourier transforms relies on the linear/bilinear/quadratic functionals of the measures in question, which makes the analysis explicit and computational. Certainly, this program requires a number of accurate perturbation estimates.

### 9.2. Bernstein-type approximation of the characteristic functions

We start with auxiliary statements relative to possibly dependent r.v. $X_1, \ldots, X_n$ satisfying the following hypotheses which will be referred to but not repeated every time again:

$$\mathbb{E} [X_k] = 0, \quad m_k = \mathbb{E} \left[ |X_k|^3 \right] < +\infty, \quad \mathbb{E} [X_k^2] =: \sigma_k^2,$$

$$S_n = \sum_{k=1}^n X_k, \quad B_n := \mathbb{E} \left[ S_n^2 \right], \quad \beta'_n := \sum_{k=1}^n \sigma_k^2$$

(9.1)

$$\max_k \quad \text{ess sup} \left\{ \mathbb{E} \left[ X_k \big| \mathcal{F} < k \right] - \mathbb{E} [X_k] \right\} \equiv \max_k \quad \text{ess sup} \left\{ \mathbb{E} [X_k \big| \mathcal{F} < k] \right\} \leq \alpha_k,$$

$$\max_k \quad \text{ess sup} \left\{ \mathbb{E} \left[ X_k^2 \big| \mathcal{F} < k \right] - \mathbb{E} [X_k^2] \right\} \leq \beta_k,$$

(9.2)

$$\max_k \quad \mathbb{E} \left[ X_k^3 \big| \mathcal{F} < k \right] \leq c_k.$$
Here esssup(…) refers to the dependence of the respective conditional expectations (which are, by definition, random variables) upon the conditions.

**Lemma 9.1** (Cf. [2, Section 9. Eqns. (65)–(66)]). Consider $X_k, k = 1, \ldots, n$ satisfying (9.1).

Let $Y_k := X_k/B_n^{1/2}$. For any $T < \infty$ there exists a constant $C_T$ determined solely by $T$ such that for all $|t| \leq T$,

$$
\mathbb{E} \left[ e^{itY_k} \right] = 1 - \frac{\mathbb{E} [X_k^2]}{B_n} t^2 + \delta_k(t), \\
|\delta_k(t)| \leq C_T \left( \alpha_k B_n^{-1/2} + \beta_k B_n^{-1} + c_k B_n^{-3/2} \right) =: \eta_k(T).
$$

The next result is an adaptation of [2, Lemme Fondamental, p.21], where we focus on the characteristic functions rather on the respective probability measures.

**Lemma 9.2.** Consider $X_k, k = 1, \ldots, n$ satisfying (9.1), let $S_n = X_1 + \cdots + X_n$, and assume that

$$
B_n^{-1/2} \sum_{k=1}^n \alpha_k + B_n^{-1} \sum_{k=1}^n \beta_k + B_n^{-3/2} \sum_{k=1}^n c_k \longrightarrow 0. \tag{9.3}
$$

Denote, for $k = 1, \ldots, n$,

$$
\phi_k(t) = \mathbb{E} \left[ e^{it\sum_{j=1}^k Y_j} \right], \quad \Psi_k(t) := \prod_{j=1}^k \left( 1 - \sigma_j^2 B_n^{-1} t^2 \right). \tag{9.4}
$$

Then

$$
\sup_{|t| \leq T} |\phi_k(t) - \Psi_k(t)| \leq \sum_{k=1}^n \eta_k(T). \tag{9.5}
$$

**Proof.** Use Lemma 9.1. For any pair of r.v., $X$ and $Z$, such that

$$
\mathbb{E} \left[ Z \mid X \right] = \mathcal{R} + \delta(X),
$$

for some non-random $\mathcal{R} \in \mathbb{R},$

$$
\mathbb{E} [XZ] = \mathcal{R} \mathbb{E} [XZ] + \mathbb{E} [X \delta(X)],
$$

thus if $|X| \leq C$ and $|\delta(X)| < \varepsilon$, then

$$
|\mathbb{E} [XZ] - \mathcal{R} \mathbb{E} [X] | < C \varepsilon.
$$

Hence

$$
\phi_m = \mathbb{E} \left[ e^{it\sum_{k=1}^m Y_k} \cdot e^{itY_m} \right] = \phi_{m-1} \cdot \left( 1 - \tilde{B}_n^{-1/2} \mathbb{E} \left[ X_m^2 \right] t^2 \right) + \gamma_m, \quad \text{with } |\gamma_m| \leq \eta_m.
$$

By recurrence,

$$
\phi_m = \Psi_m + \Psi_m \sum_{k=1}^m \frac{\gamma_k}{E_k}, \quad \Psi_k := \prod_{k=1}^m \left( 1 - \frac{\mathbb{E} \left[ X_k^2 \right]}{B_n} t^2 \right).
$$
On note that $|\Psi_m/\Psi_k| \leq 1$, and $\gamma_k \to 0$ as $k \to \infty$. So

$$|\varphi_m - \Psi_m| < \sum_{k=1}^{m} |\gamma_k| \longrightarrow 0.$$  

This proves the claim; note that in the original statement from [2], one derives from the above estimates a perturbation bound on the respective probability distribution and concludes that it is asymptotically Gaussian (Bernstein focuses in [2] on the proof of a CLT for weakly dependent r.v.).

Now it becomes clear that, in view of the discussion in the previous subsection, the artificial, piecewise-constant interaction potentials $u$ can be effectively used as convenient approximants for more realistic ones, and such an approximation preserves the qualitative (as well as some quantitative) estimates resulting in infinite derivability of the single-site distributions and of the finite-volume DoS. There would be basically no point in carrying out this program exclusively for the IID random fields $\{\omega_x, x \in \mathbb{Z}^d\}$, as Bernstein’s techniques had been specifically designed to address the (weakly) dependent systems, and I plan to do so in a separate text.

9.3. Extremely slow decay of interaction: making use of embedded dipoles. As was already mentioned in [5], an infinite range of interaction is after all a double-edged sword, and the regularization effects beneficial to the EVC analysis come with a price of infinite-range stochastic correlations, particularly inconvenient in the case of weakly screened, barely summable interaction potentials. A spectacular example is provided by 1D systems where one can have an asymptotic behaviour (cf., e.g., [17]) $u(r) \sim 1/r \ln r$ for $r \gg 1$. In fact, we have seen that the problem with very slowly decaying interactions is two-fold: not only the analysis of the effects of correlations becomes more tedious, but also the ”frozen bath” EVC estimates, relying only on disorder in a ball of limited size, become weaker. One can wonder, therefore, if there is a way to circumvent these difficulties, particularly the latter one.

Indeed, the situation is not hopeless, as one can benefit from an effect which can be pictorially described as ”screening within screening”. We will now consider the case of a realistic potential $u$ not approximated by a piecewise constant function, e.g., $u(r) = r^{-A}$ or $r^{-1} \ln^{-A} r, A > 0$, and Bernoulli amplitudes $\omega_x = \pm 1$. If for a pair of neighboring sites $x, y$ one has $\omega_x = -\omega_y$, then this pair generates a dipole type potential

$$z \mapsto \pm (u(|z - x|) - u(|z - y|))$$

which behaves at large distances ($|x - z| \gg |x - y|$) like the derivative of $u$ in the direction of the vector $x - y$, hence decays slower than the $u(|x - z|)$. Observe that $u$ is already a screened potential, and restricting ourselves to the particular case of a dipole pair, we acquire in the derivative an additional power $r^{-A}$ in the decay, compared to $r^{-A}$ or, respectively, $r^{-1} \ln^{-A} r$.

Naturally, an occurrence of a dipole pair is a random event, but working for simplicity on the Bernoulli model and partitioning non-randomly a set of sites $\Lambda$ of large cardinality into nearest-neighbor pairs $(x_i, y_i), i = 1 \ldots, |\Lambda|/2$, we shall have by the usual CLT roughly half of the pairs in the dipole configurations, and among these dipoles, roughly a half will
be of the type \((1, -1)\) and half of the type \((-1, 1)\) – again by the CLT. Moreover, the standard Large Deviations Estimates (LDE) theory provides exponentially strong probabilistic bounds on the events where the deviation of the number of desired configurations from their expected value is \(\geq c|\Lambda|\), with an arbitrarily small \(c > 0\). Thus unwanted, non-typical situations can be ruled out with probabilistic precision exceeding by far what we need in the analysis of regularity of the DoS.

We conclude that the efficiency of the regularity analysis can be improved by considering an embedded random dipole model, where the lattice is partitioned from the beginning into dipoles and non-dipole pairs. Thanks to the LDE, we can safely assume the quantity of dipoles suffices to repeat our regularity analysis relying only on dipoles and conditioning on the remaining pairs \((x_i, y_i)\); by doing so, we effectively replace the potential \(u\) by its faster-decaying derivatives.

Naturally, one can consider a random dipole (rather than random charge) problem from the beginning. Technically speaking, it is quite close to a random displacements problem, where not the amplitude \(\omega\) but the position of each charge is random and takes at least two distinct values; in this case the respective amplitude \(\omega_x\) can be fixed, as the main contribution to the regularizing convolution mechanism would come from randomization of the position.

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