Deciding Disjunctive Linear Arithmetic
with SAT  

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Abstract. Disjunctive Linear Arithmetic (DLA) is a major decidable theory that is supported by almost all existing theorem provers. The theory consists of Boolean combinations of predicates of the form $\sum_{j=1}^{n} a_j \cdot x_j \leq b$, where the coefficients $a_j$, the bound $b$ and the variables $x_1 \ldots x_n$ are of type Real ($\mathbb{R}$). We show a reduction to propositional logic from disjunctive linear arithmetic based on Fourier-Motzkin elimination. While the complexity of this procedure is not better than competing techniques, it has practical advantages in solving verification problems. It also promotes the option of deciding a combination of theories by reducing them to this logic. Results from experiments show that this method has a strong advantage over existing techniques when there are many disjunctions in the formula.

1 Introduction

Disjunctive Linear Arithmetic (DLA) is a major decidable theory that is supported by almost all existing theorem provers, and is used frequently when proving infinite state systems. The theory consists of Boolean combinations of predicates of the form $\sum_{j=1}^{n} a_j \cdot x_j \leq b$, where the coefficients $a_j$, the bound $b$ and the variables $x_1 \ldots x_n$ are of type Real ($\mathbb{R}$).

Decision procedures for this theory typically handle disjunctions by ‘case-splitting’, i.e., transforming the formula to Disjunctive Normal Form (DNF) and then solving each clause separately. Naive case-splitting procedures explicitly transform the formula to DNF, and are therefore very restricted in the size of the formula that they can handle (the number of clauses in the resulting formula can be exponential in the size of the original formula). More sophisticated implementations split the formula only ‘as needed’, which increases in many cases the capacity of these procedures, although there can still be an exponential number of cases to solve.

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** An early version of this article appeared in [20].
Recently a different approach was introduced almost simultaneously by three different groups [8,11,23]. The procedure is based on a combination of a SAT procedure and an arithmetic solver, and is now implemented by tools such as CVC, MATHSAT and ICS-SAT 1. The procedure works roughly as follows. The linear predicates are encoded with Boolean variables, and then the encoded Boolean formula is solved with a SAT solver. If the SAT instance is unsatisfiable, then the procedure terminates and declares the formulas unsatisfiable. Otherwise, it checks whether the given assignment is consistent with respect to the linear constraints. This step amounts to solving a conjunction of predicates or negation of predicates, which is possible by using any number of procedures (see below). If a satisfying assignment is found, then the procedure terminates and declares the formula to be satisfiable. Otherwise, it backtracks in order to find a different assignment, while typically (depending on the specific system) applying a learning mechanism, i.e. adding a Boolean conflict clause that prevents a repetition of the bad assignment. Although this approach can still be seen as case splitting, as it still may call the arithmetic solver an exponential number of times, the learning and pruning power of the SAT solver makes it far more robust than naive case-splitting methods. We will further discuss the advantages and disadvantages of these techniques in section 4.3.

The lower-bound complexity of solving each DNF clause, i.e., a conjunction of linear constraints, is polynomial [13]. When considering small to medium size problems, as the ones that are typically encountered in formal verification, the existing polynomial procedures are rarely better in practice comparing to some exponential methods like Simplex [7] and the various variable-elimination techniques. For this reason, as far as we know, no automated theorem prover uses a polynomial procedure for linear arithmetic.

The most commonly used method by theorem provers is the Fourier-Motzkin (FM) variable elimination method [3], which is used in popular tools such as PVS [17], ICS [10], SVC [2], IMPS [9] and others. We describe the FM method in detail in section 2. Although FM has a worst-case super-exponential complexity, it is popular because it is frequently faster than competing methods for the size of instances encountered in practice. Hence, the current practice in solving DLA is to solve, in the worst case, an exponential number of FM instances. Theoretically this is not the best possible, as explained above, but experience has showed that for the type of formulas encountered in verification, it is adequate.

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1 ICS-SAT is the name we call the version of ICS that works according to this combined approach. The distinction between the two versions is important in this article, as ICS works with case-splitting.
The procedure described in this paper solves one FM instance in order to generate a SAT instance, and then solves this instance with a standard SAT solver. It has a similar complexity to what we just described as the common practice, but we expect it to be better in practice because of reasons that we will later discuss. SAT solvers are generally far more efficient than case splitting in handling propositional combinations of formulas, although both have the same theoretical complexity. Propositional SAT checkers apply techniques like learning, pruning and guidance (‘guidance’ refers to heuristics for prioritizing the internal steps of the decision procedure) that can not be easily imitated by case-splitting. We refer the reader to [22] where an elaborated discussion of this distinction is given. Based on this observation, our suggested procedure is expected to be more efficient than case-splitting methods in deciding formulas where the case-splitting itself is the bottleneck of the procedure, i.e., formulas that their equivalent DNF has many clauses, but each one of them is relatively small.

An efficient reduction of DLA to propositional logic not only enables to (potentially) solve them faster, but also to integrate them with other theories on the propositional logic level. Many other decidable theories that are frequently encountered in verification (e.g. bit-vector arithmetic [12]) already have such reductions to propositional logic. Solving mixed theories by reducing them to a common logic facilitates the application of various learning techniques between sub-expressions that originate from different theories. Furthermore, current popular techniques for integrating theories such as Nelson-Oppen [16] invoke different procedures for deciding each theory, and propagate equalities between them in order to decide the combined theory. The overhead of this mutual updating can become significant. This overhead is avoided if only one procedure (SAT in this case) is used.

The rest of the article is structured as follows. In the next section we briefly describe the FM method. In section 3 we present a propositional version of the same procedure and explain how it can be used to reduce DLA to SAT. In section 4 we present a method called ‘conjunctions matrices’, which is useful for reducing the complexity of the procedure described in section 3. In section 5 we summarize our experiments with this method on both real examples and random instances.

## 2 Fourier-Motzkin Elimination

A linear inequality predicate over \(n\) variables has the form \(\sum_{j=1}^{n} a_j \cdot x_j \leq b\). A conjunction of \(m\) such constraints is conveniently described by \(C : AT \leq \overline{b}\) where \(A\) is an \(m \times n\) real-valued coefficient matrix, \(T = x_1...x_n\) is a vector of \(n\) variables, and \(\overline{b}\) is a vector of real-valued bounds. Given a variable order \(x_1...x_n\) the FM method eliminates (existentially quantifies) them in decreasing order. Each variable is eliminated by
projecting its constraints on the rest of the system. The procedure works as follows: at
each elimination step, the list of constraints is partitioned to three segments, according
to the sign of the coefficient of \( x_n \) in each constraint. Let \( a_{i,n} \) denote the coefficient of
\( x_n \) in constraint \( i \), for \( i \in [1..m] \).

The three segments are:

1. For all \( i \) s.t. \( a_{i,n} > 0 \):
   \[ a_{i,n} \cdot x_n \leq b_i - \sum_{j=1}^{n-1} a_{i,j} \cdot x_j \]
2. For all \( i \) s.t. \( a_{i,n} < 0 \):
   \[ \sum_{j=1}^{n-1} a_{i,j} \cdot x_j - b_i \leq -a_{i,n} \cdot x_n \]
3. For all \( i \) s.t. \( a_{i,n} = 0 \):
   \[ \sum_{j=1}^{n-1} a_{i,j} \cdot x_j \leq b_i \]

The first and second segments correspond to upper and lower bounds on \( x_n \), respectively. To eliminate \( x_n \), FM replaces each pair of lower and upper bound constraints
\( L \leq c_l \cdot x_n \) and \( c_u \cdot x_n \leq U \), where \( c_l, c_u > 0 \), with the new constraint \( c_u \cdot L \leq c_l \cdot U \).

If, in the process of elimination, the procedure derives the constraint \( c \leq 0 \) where \( c \) is
a constant greater than 0, it terminates and indicates that the system is unsatisfiable.

Note that it is possible that variables are not bounded from both ends. In this case it
is possible to simplify the system by removing these variables from the system together
with all the constraints to which they belong. This can make other variables unbounded.
Thus, this simplification stage iterates until no such variables are left.

The FM method can result in the worst case in \( m^2n \) constraints, which is the rea-
son that it is only suitable for a relatively small set of constraints with small number
of variables. There are various heuristics for choosing the elimination order. A stan-
dard greedy criteria gives priority to variables that their elimination produces less new
constraints.

*Example 1.* Consider the following formula:

\[
\varphi = x_1 - x_2 \leq 0 \quad \land \quad x_1 - x_3 \leq 0 \quad \land \quad -x_1 + 2x_3 + x_2 \leq 0 \quad \land \quad -x_3 \leq -1
\]

The following table demonstrates the elimination steps following the variable order
\( x_1, x_2, x_3 \):

| Eliminated var | Lower bound | Upper bound | New constraint |
|----------------|-------------|-------------|---------------|
| \( x_1 \)      | \( x_1 - x_2 \leq 0 \) | \( -x_1 + 2x_3 + x_2 \leq 0 \) | \( 2x_3 \leq 0 \) |
| \( x_1 - x_3 \leq 0 \) | \( -x_1 + 2x_3 + x_2 \leq 0 \) | \( x_2 + x_3 \leq 0 \) |
| \( x_2 \)      | no lower bound |             |               |
| \( x_3 \)      | \( 2x_3 \leq 0 \) | \( -x_3 \leq -1 \) | \( 2 \leq 0 \) |

The last line results in a contradiction, which implies that this system is unsatisfiable.
The extension of FM to handle a combination of strict ($<$) and weak ($\leq$) inequalities is simple. If either the lower or upper bound are a strict inequality, then so is the resulting constraint.

In the next section we present a Boolean version of the FM method.

3 A Boolean version of Fourier-Motzkin

Given a DLA formula $\varphi$, we now show how to derive a propositional formula $\varphi'$ s.t. $\varphi$ is satisfiable iff $\varphi'$ is satisfiable. The procedure for generating $\varphi'$ emulates the FM method.

1. Normalize $\varphi$:
   (a) Rewrite equalities as conjunction of inequalities.
   (b) Transform $\varphi$ to Negation Normal Form (negations are allowed only over atomic constraints).
   (c) Eliminate negations by reversing inequality signs.
2. Encode each inequality $i$ with a Boolean variable $e_i$. Let $\varphi'$ denote the encoded formula.
3. (a) Perform FM elimination on the set of all constraints in $\varphi$, while assigning new Boolean variables to the newly generated constraints.
   (b) At each elimination step, for every pair of constraints $e_i, e_j$ that result in the new constraint $e_k$, add the constraint $e_i \land e_j \rightarrow e_k$ to $\varphi'$.
   (c) If $e_k$ represents a contradiction (e.g., $1 \leq 0$), replace $e_k$ by FALSE.

We refer to this procedure from here on as Boolean Fourier Motzkin (BFM).

Example 2. Consider the following formula:

$$\varphi = 2x_1 - x_2 \leq 0 \land (2x_2 - 4x_3 \leq 0 \lor x_3 - x_1 \leq -1)$$

By Assigning an increasing index to the predicates from left to right we initially get $\varphi' = e_1 \land (e_2 \lor e_3)$.

Let $x_1, x_2, x_3$ be the elimination order. The following table illustrates the process of updating $\varphi'$:

| Eliminated var | Lower bound | Upper bound | New constraint | Encoding | Add to $\varphi'$ |
|----------------|-------------|-------------|----------------|---------|------------------|
| $x_1$          | $x_3 - x_1 \leq -1$ | $2x_1 - x_2 \leq 0$ | $2x_3 - x_2 \leq -2$ | $e_4$ | $e_3 \land e_1 \rightarrow e_4$ |
| $x_2$          | $2x_3 - x_2 \leq -2$ | $2x_2 - 4x_3 \leq 0$ | $4 \leq 0$ | FALSE | $e_4 \land e_2 \rightarrow$ FALSE |
Thus, the resulting satisfiable formula is:

\[ \varphi' = (e_1 \land (e_2 \lor e_3)) \land (e_1 \land e_3 \rightarrow e_4) \land (e_4 \land e_2 \rightarrow \text{FALSE}) \]

Example 2 demonstrates the main drawback of this method. Since in step 2 we consider all inequalities, regardless of the Boolean connectives between them, the number of constraints that the FM procedure adds is potentially larger than those that we would add if we considered each case separately (where a 'case' corresponds to a conjoined list of inequalities). In the above example, case splitting would result in two cases, none of which results in added constraints. Since the complexity of FM is the bottleneck of this procedure, this drawback may significantly worsen the overall run time and risk its usability.

As a remedy, we will suggest in section 4 a polynomial method that bounds the number of constraints to the same number that would otherwise be added by solving the various cases separately.

**Complexity of deciding \( \varphi' \).** The encoded formula \( \varphi' \) has a unique structure that makes it easier to solve comparing to a general propositional formula of similar size. Let \( m \) be the set of encoded predicates of \( \varphi \) and \( n \) be the number of variables.

**Proposition 1.** \( \varphi' \) can be decided in time bounded by \( O(2^{|m|} \cdot |m|^2n) \).

**Proof.** SAT is worst case exponential in the number of decided variables and linear in the number of clauses. The Boolean value assigned to the predicates in \( m \) imply the values of all the generated predicates. Thus, we can restrict the SAT solver to split only on \( m \). Hence, in the worst case the SAT procedure is exponential in \( m \) and linear in the number of clauses, which in the worst case is \(|m|^2n\). \( \square \)

### 4 Conjunctions matrices

Case splitting can be thought of as a two step procedure, where in the first step the formula is transformed to DNF, and in the second each clause, which now includes a conjunction of constraints, is solved separately. In this section we show how to predict, in polynomial time, whether a given pair of predicates would share a clause if the formula was transformed to DNF. It is clear that there is no need to generate a new constraint from two predicates that do not share a clause.

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\(^2\) Note that the constraints added in step 4 are Horn clauses. This means that for a given assignment to the predicates in \( m \), these constraints are solvable in linear time.
4.1 Joining operands

We assume that $\varphi$ is normalized, as explained in step 1. Let $\varphi'_f$ denote the encoded formula after step 2 and $\varphi'_c$ denote the added constraints of step 3 (thus, after step 3 $\varphi' = \varphi'_f \land \varphi'_c$). All the internal nodes of the parse tree of $\varphi'_f$ correspond to either disjunctions or conjunctions. Consider the lowest common parent of two leaves $e_i, e_j$ in the parse tree. We call the Boolean operand represented by this node the joining operand of these two leaves and denote it by $J(e_i, e_j)$.

Example 3. In the formula $\varphi'_f = e_1 \land (e_2 \lor e_3)$, $J(e_1, e_2) = \lor$ and $J(e_2, e_3) = \land$.

For simplicity, we first assume that no predicates appear in $\varphi$ more than once. In section 4.2 we solve the more general case. Denote by $\varphi^D$ the DNF representation of $\varphi$. The following proposition is the basis for the prediction technique:

**Proposition 2.** Two predicates $e_i, e_j$ share a clause in $\varphi^D$ iff $J(e_i, e_j) = \land$.

**Proof.** Recall that $\varphi'_f$ does not contain negations and no predicate appears more than once. $(\Rightarrow)$ Let node denote the node joining $e_i$ and $e_j$, and assume it represents a disjunction ($J(e_i, e_j) = \lor$). Transform the right and left branches descending from node to DNF. A disjunction of two DNF formulas is a DNF, and therefore the formula under node is now a DNF expression. If node is the root or if there are only disjunctions on the path from node to the root, we are done. Otherwise, the distribution of conjunction only adds elements to each of the clauses under node but does not join them into a single clause. Thus, $e_i$ and $e_j$ do not share a clause if their joining operand is a disjunction. $(\Leftarrow)$ Again let node denote the node joining $e_i$ and $e_j$, and assume it represents a conjunction ($J(e_i, e_j) = \land$). Transform the right and left branches descending from node to DNF. Transforming a conjunction of two DNF sub formulas back to DNF is done by forming a clause for each sequence of literals from the different clauses. Thus, at least one clause contains $e_i \land e_j$. Since there are no negations in the formula, the literals in this clause remain together in $\varphi^D$ regardless of the Boolean operands above node.

For a given pair of predicates, it is a linear operation (in the height of the parse tree $h$) to check whether their joining operand is a conjunction or disjunction. If there are $m$ predicates in $\varphi$, constructing the initial $m \times m$ conjunctions matrix $M_{\varphi}$ of $\varphi$ has the complexity of $O(m^2 h)$. $M_{\varphi}$ is a binary, symmetric matrix, where $M_{\varphi}[e_i, e_j] = 1$ if and
only if \( J(e_i, e_j) = \lor \). For example, \( M_\varphi \) corresponding to \( \varphi'_f \) of example \( \ref{example3} \) is given by

\[
M_\varphi = \begin{pmatrix}
e_1 & e_2 & e_3 \\
e_1 & 0 & 1 & 1 \\
e_2 & 1 & 0 & 0 \\
e_3 & 1 & 0 & 0
\end{pmatrix}
\]

Given proposition \( \ref{proposition2} \) this means that these predicates share at least one clause in \( \varphi^D \).

New entries are added to \( M_\varphi \) when new constraints are generated, and other entries, corresponding to constraints with non-zero coefficients over eliminated variables, are removed. The entry for a new predicate \( e_k \) that was formed from the predicates \( e_i, e_j \) is updated as follows:

\[
\forall l \in [1..k-1]. M_\varphi[e_k, e_l] = M_\varphi[e_i, e_l] \land M_\varphi[e_j, e_l]
\]

This reflects the fact that the new predicate is relevant only to predicates that share a clause with both \( e_i \) and \( e_j \).

4.2 Handling repeating predicates

Practically most formulas contain predicates that appear more than once, in different parts of the formula. We denote by \( e_i^k, k \geq 1 \) the \( k \) instance of the predicate \( e_i \) in \( \varphi' \). It is possible that the same pair of predicates has different joining operands, e.g. \( J(e_i^1, e_j^1) = \land \) but \( J(e_i^1, e_j^2) = \lor \). There are two possible solutions to this problem:

1. Represent each predicate instance as a separate predicate.
2. Assign \( M_\varphi[e_i, e_j] = 1 \) if there exists an instance of \( e_i \) and of \( e_j \) s.t. \( J(e_i, e_j) = \land \).

The first option leads to a higher complexity of constructing the initial conjunctions matrix, because it is determined by the number of predicate instances rather than the number of unique predicates. More specifically, if \( m' \) denotes the number of predicate instances, then the complexity of constructing the initial matrix \( M_\varphi \) is \( O(m'^2 h) \).

The second option has a more concise representation, but may result in redundant constraints, as the example below demonstrates.

**Example 4.** Let \( \varphi'_f = e_1 \land (e_2 \lor e_3) \lor (e_2 \land e_3) \). According to option 2, \( \varphi' \) contains only three predicates \( e_1 \ldots e_3 \) and therefore \( M_\varphi \) is a \( 3 \times 3 \) matrix with an entry ‘1’ in all its cells. Thus, \( M_\varphi \) does not contain the information that the three predicates never appear together in the same clause, which potentially results in redundant constraints.

Conjunctions matrices can be used to speed up many of the other decision procedures that were published in the last few years for subset of linear arithmetic \([11, 16, 15, 18, 22]\). We refer the reader to a technical report \([21]\) for a detailed description of how this can be done.
4.3 A revised decision procedure and its complexity

Given the initial conjunctions matrix $M_\varphi$, we now change step 3 as follows:

3. (a) Perform FM elimination on the set of all constraints in $\varphi$, while assigning new Boolean variables to the newly generated constraints.

(b) At each elimination step consider the pair of constraints $e_i, e_j$ only if $M_\varphi[e_i, e_j] = 1$. In this case let $e_k$ be the new predicate.
   i. Add the constraint $e_i \land e_j \rightarrow e_k$ to $\varphi'$.
   ii. If $e_k$ represents a contradiction (e.g., $1 \leq 0$), replace $e_k$ by $\text{FALSE}$.
   iii. Otherwise update $M_\varphi$ as follows:
   $\forall l \in [1..k-1]. M_\varphi[e_k, e_l] = M_\varphi[e_i, e_l] \land M_\varphi[e_j, e_l]$.

The main difference between this procedure and the previous one is that now step 3(b) is restricted to pairs of predicates that are conjoined in the DNF of the formula.

Given the revised procedure, we now compare the number of constraints that it generates comparing to the case-splitting methods, and the combined SAT/FM method [8,1,23] that was described in the introduction. Let $bfm$, $split$ and $comb$ be the number of constraints that are generated by these three techniques, respectively.

**Claim 1** For unsatisfiable formulas, BFM generates less or equal number of constraints to the accumulated number of constraints that are generated by case splitting ($bfm \leq split$).

This claim can be easily justified with the observation that due to conjunctions matrices, no constraint is generated in BFM that is not a resolvent of two constraint in a DNF clause. This means that the same resolvent is generated by case-splitting methods. In satisfiable instances, the number of constraints generated by case splitting depends on the location of the first satisfiable clause. While case splitting terminates after finding the first such clause, $bfm$ generates all constraints.

**Claim 2** In most cases in which the formula is unsatisfiable, $bfm \ll split$.

The reason for the big difference between the two procedures is that constraints that are repeated in many separate cases resolve in a single new constraint in BFM. For example, naive case splitting over the formula $\varphi' = e_1 \land e_2 \land (e_3 \lor e_4)$ generates the resolvent of $e_1$ and $e_2$ twice, while BFM only generate it once. As states above, the

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3 Smarter implementation of case splitting can identify, in this simple example, that the resolvent has to be generated once. But in the general case redundant constraints can be generated.
comparison of the two methods is harder in the case of satisfiable formulas, since the number of constraints generated by case splitting procedures depends on the location of the first satisfiable clause.

The value of \( \text{comb} \) is harder to compare to \( \text{bfm} \) and \( \text{split} \), because in practice it strongly depends on the success of the heuristics in the SAT procedure to prune the search space. By guiding the search, the SAT solver may eventually call the arithmetic procedure for only a small subset of the possible combinations of predicates. In the worst case, however, \( \text{comb} \) can be larger than \( \text{split} \), because it may generate resolvents of constraints that belong to different DNF clauses (adding conjunctions matrices to this method can solve this problem. Such an optimization was not described, though, in the literature [8, 1, 23]).

Conjunctions matrices is not the only reason for the potentially larger number of constraints that are generated by the SAT/FM combined procedure. Unlike BFM, this algorithm may generate the same constraint more than once. Such repeated resolution can occur, for example, if a pair of consistent predicates appear in many satisfying assignments. When each of these assignments is checked for consistency, the resolvent of this pair is potentially regenerated. Although saving this information in a hash table may save some of this repeated work, it may introduce a new source of complexity because of the possibly exponential number of resolvents.

A third source for a large number of redundant constraints in the combined procedure, which does not occur in BFM, is the following. Given a set of predicates \( p_1 \ldots p_n \), assume that only \( p_1 \) and \( p_2 \) are contradictory. Once the conflict in the set \( p_1 \ldots p_n \) is identified, a conflict clause of size \( n \) is added, which prevents a repetition of these assignments. This clause does not, however, prune the other \( 2^n - 2 \) contradictory assignments to this set. There are several solutions to this problem, all of which are either computationally expensive or not optimal. CVC tries to overcome this problem by identifying a small (yet not necessarily minimal) subset of these literals that actually cause the conflict. In our example, ideally it identifies that \( p_1 \) and \( p_2 \) alone cause the conflict. Consequently it adds a conflict clause of size two, pruning away the redundant assignments as well as the corresponding resolvents and conflict clauses. The ICS-SAT tool \cite{3} copes with this problem by following a trial-and-error approach, in which in each step it tries to remove a predicate and see whether the conflict still occurs. If the answer is affirmative - it removes the reference to this predicate from the conflict clause. The success of this approach naturally depends on the order in which the predicates are removed, and in general does not detect a minimal subset.
5 Experiments

To test the efficiency of BFM, we implemented a tool called BFM on top of PORTA [19]. We then randomly generated formulas in 2-CNF style (that is, a 2-CNF where the literals are linear inequalities) with different number of clauses and variables. The coefficients were chosen randomly in the range \(-10..10\). The time it takes to generate the SAT instance with BFM is summarized in Fig. 1. The time it takes Chaff [15] to solve each of the instances that we are able to generate is relatively negligible. Normally it is less than a second, with the exception of 3 instances that take 10-20 seconds each to solve. All experiments were run on a 1.5 GHz AMD Athlon machine with 1.5 G memory, on top of Linux.

| # vars | 10  | 30  | 50  | 70  | 90  | 110 | 130 | 150 |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|
| 10     | 1   | 2   | 1.1 | 56  | 103 | 208 | 254 |
| 30     | 1   | 2   | 2.5 | 61.1| 68  | 618 | *   |
| 50     | 0.1 | 1   | 0.2 | 4.9 | 8   | 173 | 893 |
| 70     | 0.1 | 0.2 | 0.4 | 13.4| 108 | *   | *   |
| 90     | 0.2 | 0.2 | 0.3 | 0.5 | 1   | 14  | 181 |
| 110    | 0.3 | 0.5 | 8.2 | 396 | 594 | *   | *   |
| 130    | 0.3 | 0.4 | 0.7 | 2.9 | 195 | 2658| *   |
| 150    | 0.2 | 0.3 | 0.8 | 18.4| 334 | 1227| *   |

Fig. 1. Time, in seconds, required for generating a SAT instance for random 2-CNF style linear inequalities with a varying number of clauses and variables. ‘*’ indicates running time exceeding 2 hours.

We also ran these instances with ICS and CVC. ICS solves these type of formulas with FM combined with case-splitting, while CVC implements a combined SAT/FM procedure, as described in the introduction. Both tools can solve only one of these instances (the 10 x 10 instance) in the specified time bound. They either run out of memory or out of time in all other cases. This is not very surprising, because in the worst case \(2^c\) separate cases need to be solved, where \(c\) is the number of clauses.

The CNF style formulas are harder not only for ICS and CVC, but also for BFM because they make conjunctions matrices ineffective. Each predicate in \(\varphi\) appears with all other predicates in some clause of \(\varphi^D\), except those predicates it shares a clause with in \(\varphi\). Thus, almost all the entries of \(M_\varphi\) are equal to ‘1’. In general, conjunctions matrices only prevent \(bfm\) from adding redundant constraints, and in CNF formulas only little redundancy is created in the first place. In order to check the effectiveness
of these matrices and experiment with a larger set of formulas, we ran another batch of examples, where this time the Boolean connectives (conjunction or disjunction) between the linear constraints is chosen randomly. That is, a formula with \( n \) variables and \( m \) clauses has the form \( \land_{1...m}(p(n) \lor p(n)) \) where \( \land \) denotes either a conjunction or a disjunction, and \( p(n) \) is a linear predicate with \( n \) variables and randomly chosen coefficients. For each cell in the table of figure 2, we generated six random instances (a total of 384 random formulas). The numbers in the table represent the average time it takes to generate the SAT instance with BFM without conjunctions matrices. For comparison, the time it takes to generate the corresponding SAT instances with conjunctions matrices is almost negligible (a few seconds to generate the entire set). The reason for this performance can be attributed to the random construction which apparently results in very few concurrent constraints. As before, solving the generated SAT formulas does not consume a significant amount of time. We also ran CVC on this batch of examples. CVC can solve 18 formula out of the 384 rather rapidly (the longest took about three minutes), but exceeds the time bound or, more frequently, runs out of memory in all other cases.

There are several interesting things to note about the results in figure 2. First, the results tend to be worse when the ratio between the number of clauses to number of variables is high. This is not surprising because FM is sensitive to the product of upper and lower bounds on each variable. The higher the ratio is, the larger this product is on average. Second, although not listed here, there seems to be a very large variance between the different samples, in particular when the formulas are large. For example, the standard deviation of the results in each of the cells in the right-most column is around 400. The reason for these extreme differences is not the different Boolean structures (to which BFM is insensitive if conjunctions matrices is inactive), rather it is the different number of lower and upper bounds on each variable, which is determined by the randomly selected sign of the coefficients.

Next, we ran BFM, ICS and CVC on several real examples. The results, which are not as conclusive as with the random instances (many of them can be solved easily by all three tools), are summarized in figure 3. As in the random instances, here too there seems to be an extreme variation in the performance of the tools with respect to the different formulas, which can probably be attributed to the FM method. If the number of constraints starts to grow exponentially, it is typically impossible to solve the instance in a short time. The examples shown in the table are the following. The first batch includes seven formulas resulting from symbolic simulation of hardware designs. The second batch includes four formulas resulting from scheduling problems. The third batch of examples contains three standard timed-automata verification problems, namely the verification of a railroad crossing controller. The first three sets of
Fig. 2. Average time, in seconds, required for generating a SAT instance for a formula with random Boolean structure, without conjunctions matrices. With conjunctions matrices the time is almost negligible.

| # vars | 10 | 30 | 50 | 70 | 90 | 110 | 130 | 150 |
|--------|----|----|----|----|----|-----|-----|-----|
| 10     | < 1| < 1| 0.1| 0.5| 2.4| 2.8 | 385.0| 719.8|
| 30     | < 1| < 1| 0.1| 0.7| 0.3| 174.2| 534.4| 672.0|
| 50     | < 1| < 1| 0.2| 1.6| 3.9| 114.3| 393.3| 696.0|
| 70     | < 1| < 1| 0.2| 4.2| 1.2| 10.2 | 542.3| 446.1|
| 90     | < 1| 0.1| 0.4| 0.6| 285.2| 103.4| 425.4|
| 110    | < 1| 0.1| 0.7| 0.3| 8.27 | 107.4| 171.0|
| 130    | < 1| 0.1| 0.7| 0.7| 1.37 | 13.8 | 166.6|
| 150    | < 1| 0.1| 0.3| 0.5| 0.55 | 0.7  | 0.8 \\

Fig. 3. Results achieved by the three tested solvers on several realistic examples from different origins. ‘*’ indicates running time exceeding 2 hours.

| Source   | Instance | BFM | ICS | CVC |
|----------|----------|-----|-----|-----|
| Hardware designs | 1-5      | < 1 | < 1 | < 1 |
| Scheduling problems | 6-7      | < 1 | *  | < 1 |
| Timed Automata | 1-2      | < 1 | < 1 | < 1 |
| Random (Conjoined) | 3    | 90  | *  | < 1 |
|          | 4        | 3   | 952 | 221 |
|          | 3        | 3   | 35  | < 1 |
|          | 1        | *   | 2   | *   |
|          | 2        | *   | 7   | *   |
examples consist of a Boolean combination of *separation predicates* rather than full linear arithmetic, i.e. predicates of the form $x < y + c$, where $c$ is a constant. This is obviously a special case of linear arithmetic. We also examined two standard ICS benchmarks, ‘linsys-035’ and ‘linsys-100’, which consist of 35 and 100 variables and linear inequalities, respectively. The results corresponding to these examples appear as the last batch in the table. Note that while ICS solves these instances in a few seconds, both BFM and CVC cannot solve them in the specified time limit. The reason for this seemingly inconsistency is that the ICS benchmark formulas consist of a conjunction of linear equalities, and therefore no case splitting is required. The better performance of ICS can be attributed to the higher quality of implementation of FM comparing to that of PORTA, on top of which BFM is built, and CVC.

Our conclusion from the experiments is that the advantage of BFM, as stated in the introduction, is in solving formulas that have a large number of disjunctions and hence are hard for any method that is based on solving the various cases separately. The results in figures 1 and 2 prove this observation. The results shown in figure 3, however, are not conclusive. BFM has recently been integrated in the theorem prover C-PROVER [14], which means that in the long run additional data concerning the performance of this technique when solving real verification problems will be gathered.

Finally, as direction for future research, we note that since both DLA and SAT are NP-complete, there is no complexity argument to rule out the option of finding a polynomial reduction of DLA to SAT. Finding such a reduction will enable to solve larger formulas than can be solved by BFM.

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