Unimodality of the freely selfdecomposable probability laws

TAKAHIRO HASEBE AND STEEN THORBJØRNSEN

May 22, 2014

Abstract

We show that any freely selfdecomposable probability law is unimodal. This is the free probabilistic analog of Yamazato’s result in [Ann. Probab. 6 (1978), 523-531].

1 Introduction

The class \( \mathcal{L} \) of probability measures was introduced by A. Ya. Khintchine as the class of limit distributions of certain independent triangular arrays. It plays an important role in statistics and mathematical finance, mainly as a consequence of the following characterization established by P. Lévy in 1937: A (Borel-) probability measure \( \mu \) belongs to \( \mathcal{L} \), if and only if there exists, for any constant \( c \) in \((0, 1)\), a probability measure \( \mu_c \) on \( \mathbb{R} \), such that \( \mu = D_c \mu \ast \mu_c \). Here \( D_c \mu \) is the scaling of \( \mu \) by \( c \), i.e. \( D_c \mu(B) = \mu(c^{-1}B) \) for any Borel-set \( B \). Moreover, \( \ast \) denotes (classical) convolution of probability measures, and to distinguish from the corresponding class in free probability (described below) we shall henceforth write \( \mathcal{L}(\ast) \) instead of just \( \mathcal{L} \). As a result of Lévy’s characterization the measures in \( \mathcal{L}(\ast) \) are called selfdecomposable. The class \( \mathcal{L}(\ast) \) contains in particular the class \( \mathcal{S}(\ast) \) of stable probability measures on \( \mathbb{R} \) as a proper subclass (see e.g. \[Sa99\]).

A probability measure \( \mu \) on \( \mathbb{R} \) is called unimodal, if, for some \( a \) in \( \mathbb{R} \), it has the form

\[
\mu(dx) = \mu(\{a\})\delta_a(dx) + f(x)\,dx,
\]

where \( f \) is increasing on \((-\infty, a)\) and decreasing on \((a, \infty)\), and where \( \delta_a \) denotes the Dirac measure at \( a \). The problem of unimodality of the measures in \( \mathcal{L}(\ast) \) emerged in the 1940’s. Already in the original 1949 Russian edition of the fundamental book \[GnKo68\] by B.V. Gnedenko and A.N. Kolmogorov it was claimed that all selfdecomposable distributions are unimodal. However, as explained in the English translation \[GnKo68\] (by K. L. Chung) there was an error in the proof, and it took almost 30 years before a correct proof was obtained by M. Yamazato in 1978 (see \[Ya78\]). In the appendix to the paper \[BP99\] from 1999 it was proved by P. Biane that all measures in the class \( \mathcal{S}(\boxplus) \) of stable measures with respect to free additive convolution \( \boxplus \) (see Section 2) are unimodal. In the present paper we extend this result to the class \( \mathcal{L}(\boxplus) \) of all selfdecomposable distributions with respect to \( \boxplus \); thus establishing a full free probability analog of Yamazato’s result.

In the paper \[HaTh\] it was proved by U. Haagerup and the second named author that the free analogs of the Gamma distributions (which are contained in \( \mathcal{L}(\boxplus) \setminus \mathcal{S}(\boxplus) \)) are unimodal, and the present paper is based in part on techniques from that paper.
Let us also point out that several results from Section 3 in the present paper (most notably Lemma 3.5) may be extracted from the more general and somewhat differently oriented theory developed in the papers [Hu1]-[Hu2] by H.-W. Huang. We prefer in the present paper to give a completely self-contained and elementary exposition in the specialized setup considered here. In particular our approach does not depend upon the rather deep complex analysis considered in Huang’s papers and originating in the work of S.T. Belinschi and H. Bercovici (see e.g. [BB05]).

The remainder of the paper is organized as follows: In Section 2 we provide background material on \(\boxplus\)-infinite divisibility, the Bercovici-Pata bijection, selfdecomposability and unimodality. In Section 3 we establish unimodality for probability measures in \(L(\boxplus)\) satisfying in particular that the corresponding Lévy measure has a bounded \(C^2\)-density with bounded support. In Section 4 we extend the unimodality result from such measures to general measures in \(L(\boxplus)\), using that unimodality is preserved under weak limits.

2 Background

2.1 Free and classical infinite divisibility

A (Borel-) probability measure \(\mu\) on \(\mathbb{R}\) is called infinitely divisible, if there exists, for each positive integer \(n\), a probability measure \(\mu^{1/n}\) on \(\mathbb{R}\), such that

\[
\mu = \mu^{1/n} \ast \mu^{1/n} \ast \cdots \ast \mu^{1/n},
\]

where \(\ast\) denotes the usual convolution of probability measures (based on classical independence). We denote by \(\text{ID}(\ast)\) the class of all such measures on \(\mathbb{R}\). We recall that a probability measure \(\mu\) on \(\mathbb{R}\) is infinitely divisible, if and only if its characteristic function (or Fourier transform) \(\hat{\mu}\) has the Lévy-Khintchine representation:

\[
\hat{\mu}(u) = \exp \left[ i \eta u - \frac{1}{2} a u^2 + \int_{\mathbb{R}} \left( e^{i u t} - 1 - i u t \mathbf{1}_{[-1,1]}(t) \right) \rho(dt) \right], \quad (u \in \mathbb{R}),
\]

where \(\eta\) is a real constant, \(a\) is a non-negative constant and \(\rho\) is a Lévy measure on \(\mathbb{R}\), meaning that

\[
\rho(\{0\}) = 0, \quad \text{and} \quad \int_{\mathbb{R}} \min\{1, t^2\} \rho(dt) < \infty.
\]

The parameters \(a, \rho\) and \(\eta\) are uniquely determined by \(\mu\) and the triplet \((a, \rho, \eta)\) is called the characteristic triplet for \(\mu\). Alternatively the Lévy-Khintchine representation may be written in the form:

\[
\hat{\mu}(u) = \exp \left[ i \gamma u + \int_{\mathbb{R}} \left( e^{i u t} - 1 - i u t \mathbf{1}_{[-1,1]}(t) \right) \frac{1 + t^2}{t^2} \sigma(dt) \right], \quad (u \in \mathbb{R}),
\]

where \(\gamma\) is a real constant, \(\sigma\) is a finite measure on \(\mathbb{R}\) and \((\gamma, \sigma)\) is called the generating pair for \(\mu\). The relationship between the representations (2.3) and (2.2) is as follows:

\[
a = \sigma(\{0\}),
\]

\[
\rho(dt) = \frac{1 + t^2}{t^2} \cdot 1_{\mathbb{R}\setminus\{0\}}(t) \sigma(dt),
\]

\[
\eta = \gamma + \int_{\mathbb{R}} t \left( 1_{[-1,1]}(t) - \frac{1}{1 + t^2} \right) \rho(dt).
\]
For two probability measures $\mu$ and $\nu$ on $\mathbb{R}$, the free convolution $\mu \boxplus \nu$ is defined as the distribution of $x + y$, where $x$ and $y$ are freely independent (possibly unbounded) self-adjoint operators on a Hilbert space with spectral distributions $\mu$ and $\nu$, respectively (see [BV93] for further details). The class $\mathcal{ID}(\boxplus)$ of infinitely divisible probability measures with respect to free convolution $\boxplus$ is defined by replacing classical convolution $\ast$ by free convolution $\boxplus$ in (2.1).

For a (Borel-) probability measure $\mu$ on $\mathbb{R}$ with support $\text{supp}(\mu)$, the Cauchy (or Stieltjes) transform is the mapping $G_\mu: \mathbb{C} \setminus \text{supp}(\mu) \to \mathbb{C}$ defined by:

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - t} \mu(dt), \quad (z \in \mathbb{C} \setminus \text{supp}(\mu)). \quad (2.5)$$

The free cumulant transform $\mathcal{C}_\mu$ of $\mu$ is then given by

$$\mathcal{C}_\mu(z) = zG_\mu(-1)(z) - 1 \quad (2.6)$$

for all $z$ in a certain region $R$ of $\mathbb{C}^-$ (the lower half complex plane), where the (right) inverse $G_\mu(-1)$ of $G_\mu$ is well-defined. Specifically $R$ may be chosen in the form:

$$R = \{z \in \mathbb{C}^- \mid \frac{1}{2} \in \Delta_{\eta,M}\}, \quad \text{where} \quad \Delta_{\eta,M} = \{z \in \mathbb{C}^+ \mid |\text{Re}(z)| < \eta \text{Im}(z), \text{Im}(z) > M\}$$

for suitable positive numbers $\eta$ and $M$, where $\mathbb{C}^+$ denotes the upper half complex plane. It was proved in [BV93] (see also [Ma92] and [Vo86]) that $\mathcal{C}_\mu$ constitutes the free analog of $\log \hat{\mu}$ in the sense that it linearizes free convolution:

$$\mathcal{C}_{\mu \boxplus \nu}(z) = \mathcal{C}_\mu(z) + \mathcal{C}_\nu(z)$$

for all probability measures $\mu$ and $\nu$ on $\mathbb{R}$ and all $z$ in a region where all three transforms are defined. The results in [BV93] are presented in terms of a variant, $\varphi_\mu$, of $\mathcal{C}_\mu$, which is often referred to as the Voiculescu transform, and which is again a variant of the $R$-transform $\mathcal{R}_\mu$ introduced in [Vo86]. The relationship is the following:

$$\varphi_\mu(z) = \mathcal{R}_\mu(\frac{1}{z}) = z\mathcal{C}_\mu(\frac{1}{z}) \quad (2.7)$$

for all $z$ in a region $\Delta_{\eta,M}$ as above. In [BV93] it was proved additionally that $\mu \in \mathcal{ID}(\boxplus)$, if and only if $\varphi_\mu$ extends analytically to a map from $\mathbb{C}^+$ into $\mathbb{C}^- \cup \mathbb{R}$, in which case there exists a real constant $\gamma$ and a finite measure $\sigma$ on $\mathbb{R}$, such that $\varphi_\mu$ has the free Lévy-Khintchine representation:

$$\varphi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1 + tz}{z - t} \sigma(dt), \quad (z \in \mathbb{C}^+). \quad (2.8)$$

The pair $(\gamma, \sigma)$ is uniquely determined and is called the free generating pair for $\mu$. In terms of the free cumulant transform $\mathcal{C}_\mu$ the free Lévy-Khintchine representation may be written as

$$\mathcal{C}_\mu(z) = \eta z + az^2 + \int_{\mathbb{R}} \left( \frac{1}{1 - tz} - 1 - tz1_{[-1,1]}(t) \right) \rho(dt), \quad (2.9)$$

where the relationship between the free characteristic triplet $(a, \rho, \eta)$ and the free generating pair $(\gamma, \sigma)$ is again given by (2.3).

In [BP99] Bercovici and Pata introduced a bijection $\Lambda$ between the two classes $\mathcal{ID}(\ast)$ and $\mathcal{ID}(\boxplus)$, which may formally be defined as the mapping sending a measure $\mu$ from $\mathcal{ID}(\ast)$ with generating pair $(\gamma, \sigma)$ onto the measure $\Lambda(\mu)$ in $\mathcal{ID}(\boxplus)$ with free generating pair $(\gamma, \sigma)$. It is then obvious that $\Lambda$ is a bijection, and it turns out that $\Lambda$ further enjoys the following properties (see [BP99] and [B-NT02]):
(a) If $\mu_1, \mu_2 \in \mathcal{ID}(\ast)$, then $\Lambda(\mu_1 \ast \mu_2) = \Lambda(\mu_1) \boxplus \Lambda(\mu_2)$.

(b) If $\mu \in \mathcal{ID}(\ast)$ and $c \in \mathbb{R}$, then $\Lambda(D_c \mu) = D_c \Lambda(\mu)$, where e.g. $D_c \mu$ is the transformation of $\mu$ by the mapping $x \mapsto cx : \mathbb{R} \to \mathbb{R}$.

(c) For any constant $c$ in $\mathbb{R}$ we have $\Lambda(\delta_c) = \delta_c$, where $\delta_c$ denotes Dirac measure at $c$.

(d) $\Lambda$ is a homeomorphism with respect to weak convergence.

The property (d) is equivalent to the free version of Gnedenko’s Theorem: Suppose $\mu, \mu_1, \mu_2, \mu_3, \ldots$ is a sequence of measures from $\mathcal{ID}(\boxplus)$ with free generating pairs: $(\gamma, \sigma), (\gamma_1, \sigma_1), (\gamma_2, \sigma_2), (\gamma_3, \sigma_3), \ldots$, respectively. Then

$$\mu_n \xrightarrow{w} \mu \iff \gamma_n \longrightarrow \gamma, \text{ and } \sigma_n \xrightarrow{w} \sigma.$$ (2.10)

(cf. Theorem 3.8 in [B-NT02])

### 2.2 Selfdecomposability and Unimodality

In analogy with the class $\mathcal{L}(\ast)$ of selfdecomposable probability laws in classical probability (see Section 1), a probability measure $\mu$ on $\mathbb{R}$ is called $\boxplus$-selfdecomposable, if there exists, for any $c$ in $(0, 1)$, a probability measure $\mu_c$ on $\mathbb{R}$, such that $\mu = D_c \mu \boxplus \mu_c$. Denoting by $\mathcal{L}(\boxplus)$ the class of such measures, it follows from the properties of $\Lambda$ that

$$\Lambda(\mathcal{L}(\ast)) = \mathcal{L}(\boxplus)$$ (2.11)

(see [B-NT02]). The measures in $\mathcal{L}(\ast)$ may alternatively be characterized as those measures in $\mathcal{ID}(\ast)$ whose Lévy measure (cf. (2.2)) has the form

$$\rho(dt) = \frac{k(t)}{|t|} dt,$$ (2.12)

where $k : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$ (see [Sa99]). By the definition of $\Lambda$ and (2.11) we have the exact same characterization of the measures in $\mathcal{L}(\boxplus)$, if we let the term “Lévy measure” refer to the free Lévy-Khintchine representation (2.9) rather than the classical one (2.2).

The definition of a unimodal probability measure $\mu$ given in Section 1 is equivalent to the existence of a real number $a$, such that the distribution function $t \mapsto \mu((-\infty, t])$ is convex on $(-\infty, a)$ and concave on $(a, \infty)$. From this characterization it follows that for any sequence $(\mu_n)$ of unimodal probability measures on $\mathbb{R}$ we have the implication:

$$\mu_n \xrightarrow{w} \mu \implies \mu \text{ is unimodal}$$ (2.13)

for any probability measure $\mu$ on $\mathbb{R}$ (see e.g. [GnKo68, §32, Theorem 4]).

### 3 The case of compactly supported Lévy measures

Throughout this section we consider a bounded, continuous function $k : \mathbb{R} \setminus \{0\} \to [0, \infty)$, which is increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$ and not identically 0. Note that $k(t) dt$ is then automatically a Lévy measure, and that $\mathrm{supp}(k) = [a, b]$ for some $a$ in $(-\infty, 0]$ and $b$ in $[0, \infty)$. We assume that $(a, b) \neq (0, 0)$. Note then that $k(t) > 0$ for all $t$ in $(a, b) \setminus \{0\}$. 


Next we define
\[
\tilde{k}(t) = \text{sign}(t)k(t), \quad (t \in \mathbb{R}),
\]
\[
\tilde{G}_k(z) = \int_a^b \frac{\tilde{k}(t)}{z - t} \, dt, \quad (z \in \mathbb{C} \setminus [a, b]),
\]
\[
H_k(z) = z + z\tilde{G}_k(z), \quad (z \in \mathbb{C} \setminus [a, b]).
\]

We note for later use that
\[
H_k(z) = z + z\int_a^b \frac{\tilde{k}(t)}{z - t} \, dt = z + \int_a^b \left( 1 + \frac{t}{z - t} \right) \tilde{k}(t) \, dt = z + \gamma_k + \int_a^b \frac{t|k(t)|}{z - t} \, dt, \quad (3.1)
\]
where we have introduced \(\gamma_k = \int_a^b \tilde{k}(t) \, dt\).

3.1 Lemma. Let \(\nu_k\) be the measure in \(\mathcal{MD}(\mathbb{H})\) with free characteristic triplet \((0, \frac{k(0)}{|t|}, \int_a^1 \tilde{k}(t) \, dt)\). Then the Cauchy transform \(G_{\nu_k}\) of \(\nu_k\) satisfies the identity:
\[
G_{\nu_k}(H_k(z)) = \frac{1}{z}
\]
for all \(z\) in \(\mathbb{C}^+\) such that \(H_k(z) \in \mathbb{C}^+\).

Proof. Let \(\mathcal{C}_{\nu_k}\) denote the free cumulant transform of \(\nu\). For any \(u\) in \((-\infty, 0)\) we then find (cf. formula (2.9)) that
\[
\mathcal{C}_{\nu_k}(iu) = iu \int_{-1}^1 \tilde{k}(t) \, dt + \int_a^b \left( \frac{1}{1 - iut} - 1 - iut1_{[-1,1]}(t) \right) \frac{\tilde{k}(t)}{|t|} \, dt
\]
\[
= \int_a^b \left( \frac{1}{1 - iut} - 1 \right) \frac{\tilde{k}(t)}{|t|} \, dt
\]
\[
= iu \int_a^b \frac{t}{1 - iut} \frac{\tilde{k}(t)}{|t|} \, dt = iu \int_a^b \frac{\tilde{k}(t)}{1 - iut} \, dt.
\]
Setting \(u = -\frac{1}{y}\) it follows for any \(y\) in \((0, \infty)\) that
\[
\mathcal{C}_{\nu_k}(\frac{1}{iy}) = \frac{1}{iy} \int_a^b \frac{\tilde{k}(t)}{1 - \frac{t}{iy}} \, dt = \int_a^b \frac{\tilde{k}(t)}{iy - t} \, dt = \tilde{G}_k(iy).
\]

By analytic continuation we may thus conclude that \(\mathcal{C}_{\nu_k}(\frac{1}{z}) = \tilde{G}_k(z)\) for all \(z\) in \(\mathbb{C}^+\). By definition of the free cumulant transform it therefore follows that
\[
\frac{1}{z} G_{\nu_k}^{(-1)}(\frac{1}{z}) - 1 = \mathcal{C}_{\nu_k}(\frac{1}{z}) = \tilde{G}_k(z),
\]
and hence that
\[
G_{\nu_k}^{(-1)}(\frac{1}{z}) = z\tilde{G}_k(z) + z = H_k(z) \quad \text{for all } z \in \mathbb{C}^+.
\]
For any \(z\) in \(\mathbb{C}^+\) such that also \(H_k(z) \in \mathbb{C}^+\), we may thus conclude that
\[
\frac{1}{z} = G_{\nu_k}(H_k(z)),
\]
as desired. \(\blacksquare\)
Next we introduce the function $J: \mathbb{R} \to [0, \infty]$ given by

$$J(x) = \int_a^b \frac{|t|k(t)}{(x-t)^2} \, dt, \quad (x \in \mathbb{R}), \quad (3.2)$$

and we further define the constants $c, d$ in $\mathbb{R}$ by the identities:

$$c = \inf \{ x \in \mathbb{R} \mid J(x) > 1 \}, \quad \text{and} \quad d = \sup \{ x \in \mathbb{R} \mid J(x) > 1 \}.$$ 

Note that $J(x) = \infty$ for all $x$ in $(a, b)$, and that $J(x) \to 0$ as $x \to \pm \infty$. It is therefore apparent that $-\infty < c \leq a \leq 0 \leq b \leq c < \infty$.

In the following lemma we collect some basic observations about $c, d$ and $J$.

**3.2 Lemma.** With $J, c$ and $d$ as defined above, we have that

(i) $c < 0 < d$.

(ii) $J(x) > 1$, if $x \in (c, d)$, and $J(x) < 1$, if $x \in \mathbb{R} \setminus [c, d]$. Moreover $J(c) \leq 1$, with equality if $c < a$, and $J(d) \leq 1$, with equality if $b < d$.

(iii) $H_k$ is always well-defined at $c$ and $d$.

**Proof.** (i) Recall that $c \leq a \leq 0 \leq b \leq d$. If $a = 0$, we have assumed that $b > 0$, and therefore $\int_0^b \frac{|k(t)|}{t} \, dt = \infty$. This implies, by monotone convergence, that $J(x) > 1$ for all $x$ in $(-\infty, 0)$ close enough to 0, and in particular $c < 0$. Similarly $d > 0$, if $b = 0$.

(ii) It follows from (3.2) that $J$ is strictly increasing on $(-\infty, a]$, and strictly decreasing on $[b, \infty)$. Hence the definitions of $c$ and $d$ imply that $J(x) > 1$, if $x \in (c, a] \cup [b, d)$, and also that $J(x) < 1$ for all $x$ in $\mathbb{R} \setminus [c, d]$. In addition, $J(x) = \infty$ for all $x$ in $(a, b)$. By monotone convergence $J(x) \nearrow J(c)$, as $x \nearrow c$, and hence we must have that $J(c) \leq 1$. If $c < a$, then $J$ is continuous at $c$ (by dominated convergence) and hence necessarily $J(c) = 1$. Similarly $J(d) \leq 1$, with equality if $b < d$.

(iii) This is clear if $c < a$, and $d > b$, since $H_k$ is analytical on $\mathbb{C} \setminus [a, b]$. If $a = c$, then necessarily $\int_a^b \frac{|k(t)|}{(t-a)^2} \, dt \leq 1$, and hence also $\int_a^b \frac{|k(t)|}{t-a} \, dt < \infty$, so that $H_k(c)$ is well-defined via Formula (3.1). Similarly $H_k(d)$ is well-defined even if $d = b$.  

In the following we shall consider additionally the function $F_k: \mathbb{C}^+ \to \mathbb{R}$ given by

$$F_k(x + iy) = \int_a^b \frac{|t|k(t)}{(x-t)^2 + y^2} \, dt, \quad (x + iy \in \mathbb{C}^+). \quad (3.3)$$

**3.3 Lemma.** (i) For all $x$ in $(c, d)$ there exists a unique number $y = v_k(x)$ in $(0, \infty)$ such that

$$F_k(x + iv_k(x)) = \int_a^b \frac{|t|k(t)}{(x-t)^2 + v_k(x)^2} \, dt = 1. \quad (3.4)$$

(ii) We have that

$$\mathcal{G} := \{ z \in \mathbb{C}^+ \mid H_k(z) \in \mathbb{R} \} = \{ x + iv_k(x) \mid x \in (c, d) \}.$$ 

(iii) If we define $v_k(x) = 0$ for all $x$ in $\mathbb{R} \setminus (c, d)$, then we have that

$$\mathcal{G}^+ := \{ z \in \mathbb{C}^+ \mid H_k(z) \in \mathbb{C}^+ \} = \{ x + iy \mid x \in \mathbb{R}, \ y > v_k(x) \}.$$ 

6
(iv) The function \( v_k : \mathbb{R} \to [0, \infty) \) defined in (ii) and (iii) is continuous on \( \mathbb{R} \) and differentiable on \( \mathbb{R} \setminus \{c, d\} \).

(iv) For any \( x \in \mathbb{R} \) we have that \( x + iv_k(x) \neq 0 \).

**Proof.** (i) For any \( x \in \mathbb{R} \), the function

\[
y \mapsto \int_a^b \frac{|t| k(t)}{(x - t)^2 + y^2} \, dt, \quad (y \in (0, \infty))
\]

takes values in \( [0, \infty) \) and is continuous (by dominated convergence) and strictly decreasing in \( y \). In addition

\[
\lim_{y \to \infty} \int_a^b \frac{|t| k(t)}{(x - t)^2 + y^2} \, dt = 0, \quad \text{and} \quad \lim_{y \to 0} \int_a^b \frac{|t| k(t)}{(x - t)^2 + y^2} \, dt = \int_a^b \frac{|t| k(t)}{(x - t)^2} \, dt
\]

by dominated and monotone convergence. If \( x \in (c, d) \) it follows from Lemma 3.2(ii) that \( \int_a^b \frac{|t| k(t)}{(x - t)^2 + y^2} \, dt > 1 \), and hence there is a unique \( y = v_k(x) \in (0, \infty) \), such that \( \int_a^b \frac{|t| k(t)}{(x - t)^2 + y^2} \, dt = 1 \).

(ii) For any \( x, y \in \mathbb{R} \), such that \( y > 0 \), we note that

\[
\text{Im}(H_k(x + iy)) = y + \text{Im} \left( \int_a^b \frac{x + iy - t}{x + iy - t} k(t) \, dt \right) = y + \int_a^b \frac{y(x - t) - y x}{(x - t)^2 + y^2} k(t) \, dt
\]

\[
= y \left( 1 - \int_a^b \frac{k(t)}{(x - t)^2 + y^2} \, dt \right) = y \left( 1 - \int_a^b \frac{|t| k(t)}{(x - t)^2 + y^2} \, dt \right).
\]

Hence it follows that

\[
\text{Im}(H_k(x + iy)) = 0 \iff \int_a^b \frac{|t| k(t)}{(x - t)^2 + y^2} \, dt = 1. \tag{3.6}
\]

If \( x \in (c, d) \), then the right hand side of (3.6) holds, if and only if \( y = v_k(x) \). If \( x \in (c, d) \), then the right hand side of (3.6) is false for all \( y \in (0, \infty) \), since \( \int_a^b \frac{|t| k(t)}{(x - t)^2} \, dt \leq 1 \) according to Lemma 3.2(ii). Altogether we conclude that \( \mathcal{G} = \{ x + iv_k(x) \mid x \in (c, d) \} \).

(iii) Put \( v_k(x) = 0 \) for all \( x \in (c, d)^c \). From (i) and Lemma 3.2(ii) it is then apparent that \( \int_a^b \frac{|t| k(t)}{(x - t)^2 + y^2} \, dt < 1 \) for any \( x \in \mathbb{R} \) and all \( y \in (v_k(x), \infty) \). If \( x \in (c, d) \) we have additionally that \( \int_a^b \frac{|t| k(t)}{(x - t)^2 + y^2} \, dt > 1 \) for all \( y \in (0, v_k(x)) \). In combination with (3.5) this shows that \( \mathcal{G}^+ = \{ x + iy \mid x \in \mathbb{R}, \ y > v_k(x) \} \) as desired.

(iv) Consider the function \( \Psi_k : \mathbb{R} \times (0, \infty) \to \mathbb{R} \) given by \( \Psi_k(x, y) = F_k(x + iy) \), where \( F_k \) is given by (3.3). By differentiation under the integral sign \( \Psi_k \) is a \( C^\infty \)-function. In particular

\[
\frac{\partial}{\partial y} \Psi_k(x, y) = \frac{\partial}{\partial y} \left( \int_a^b \frac{|t| k(t)}{(x - t)^2 + y^2} \, dt \right) = -2y \int_a^b \frac{|t| k(t)}{((x - t)^2 + y^2)^2} \, dt < 0
\]

for all \( (x, y) \) in \( \mathbb{R} \times (0, \infty) \). It follows then from the Implicit Function Theorem that the function \( v_k \) given by \( \Psi_k \) is a \( C^1 \)-function on \( (c, d) \). In particular \( v_k \) is continuous on \( (c, d) \), and it remains to show that \( v_k(x) \to 0 \) as \( x \searrow c \) and as \( x \nearrow d \). From Lemma 3.2(ii) we know that \( \int_a^b \frac{|t| k(t)}{(x - t)^2} \, dt \leq 1 \).

For any positive \( \epsilon \) it follows thus by dominated convergence that

\[
\int_a^b \frac{|t| k(t)}{(x - t)^2 + \epsilon^2} \, dt \to \int_a^b \frac{|t| k(t)}{(x - t)^2 + \epsilon^2} \, dt < 1,
\]
as \( x \searrow c \). Therefore the defining property (3.3) implies that \( v_k(x) \) must be smaller than \( \epsilon \), when \( x \) is sufficiently close to \( c \). Similarly \( v_k(x) \to 0 \) as \( x \searrow d \).

(v) It suffices to note that \( v_k(0) \neq 0 \), and this follows from (i) and Lemma 3.2(i).

In the following we consider the function \( P_k : \mathbb{R} \to \mathbb{R} \) defined by

\[
P_k(x) = H_k(x + iv_k(x)), \quad (x \in \mathbb{R}).
\]

(3.7)

Note that \( P_k \) is well-defined by Lemma 3.2(iii) and since \( H_k \) is well-defined on \( \mathbb{C} \setminus [a, b] \supseteq \mathbb{C} \setminus [c, d] \).

3.4 Proposition. For any \( x \) in \( \mathbb{R} \) we have that

\[
G_{v_k}(z) \to \frac{1}{x + iv_k(x)} \quad \text{as } z \to P_k(x) \text{ non-tangentially from } \mathbb{C}^+.
\]

Proof. We consider first the case where \( x \in \mathbb{R} \setminus \{c, d\} \). For any \( s \) in \([0, 1]\) we put \( w_s = x + iv_k(x) + s \), so that \( w_s \in \mathcal{G}^+ \) for all \( s \) in \((0, 1]\) according to Lemma 3.3(iii). Moreover, since \( H_k \) is analytical on \( \mathbb{C} \setminus [a, b] \supseteq \mathbb{C} \setminus [c, d] \), and \( w_s \in \mathbb{C} \setminus [c, d] \) for all \( s \) in \([0, 1]\), it follows that

\[
H_k(w_s) \to H_k(w_0) = H_k(x + iv_k(x)) = P_k(x) \in \mathbb{R} \quad \text{as } s \to 0.
\]

In addition it follows from Lemma 3.1 that

\[
G_{v_k}(H_k(w_s)) = \frac{1}{w_s} = \frac{1}{x + iv_k(x) + s} \to \frac{1}{x + iv_k(x)} \quad \text{as } s \searrow 0
\]

(cf. Lemma 3.3(v)). Thus \( G_{v_k}(z) \) has the limit \( \frac{1}{x + iv_k(x)} \) as \( z \to P_k(x) \) along the curve \( s \mapsto H_k(w_s) \). It follows then from Theorem 2.2 in [CoLo] that in fact \( G_{v_k}(z) \to \frac{1}{x + iv_k(x)} \) as \( z \to P_k(x) \) non-tangentially from \( \mathbb{C}^+ \), as desired.

We consider next the case where \( x = c \). If \( c < a \), then \( H_k(c + is) \to H_k(c) \) as \( s \searrow 0 \), since \( H_k \) is analytical on \( \mathbb{C} \setminus [a, b] \). If \( c = a \), then as noted in the proof of Lemma 3.2(iii) \( \int_a^b \frac{|k(t)|}{t-a} \, dt < \infty \). By dominated convergence and formula (3.1) it follows therefore that

\[
H_k(a + is) = a + is + \gamma_k + \int_a^b \frac{|k(t)|}{a-t+is} \, dt \to a + \gamma_k + \int_a^b \frac{|k(t)|}{a-t} \, dt = H_k(a),
\]

as \( s \searrow 0 \). Thus in both cases \( H_k(c + is) \to H_k(c) = P_k(c) \) as \( s \searrow 0 \). At the same time it follows from Lemma 3.1 and Lemma 3.3(iii) that

\[
G_{v_k}(H_k(c + is)) = \frac{1}{c + is} \to \frac{1}{c} = \frac{1}{c + iv_k(c)} \quad \text{as } s \searrow 0.
\]

Another application of Theorem 2.2 in [CoLo] then shows that in fact \( G_{v_k}(z) \to \frac{1}{c + iv_k(c)} \) as \( z \to P_k(c) \) non-tangentially from \( \mathbb{C}^+ \). It follows similarly that \( G_{v_k}(z) \to \frac{1}{c + iv_k(d)} \) as \( z \to P_k(d) \) non-tangentially from \( \mathbb{C}^+ \).

3.5 Lemma. The function \( P_k \) is a strictly increasing homeomorphism of \( \mathbb{R} \) onto \( \mathbb{R} \).
Proof. It follows immediately from (3.7) and (3.1) that \( P_k(x) \to \pm \infty \) as \( x \to \pm \infty \). Hence it suffices to show that \( P_k \) is injective and continuous on \( \mathbb{R} \), since these properties are then automatically transferred to the inverse \( P_k^{-1} \).

We show first that \( P_k \) is injective on \( \mathbb{R} \). Indeed, if \( x, x' \in \mathbb{R} \) such that \( P_k(x) = P_k(x') \), then Proposition \( 3.4 \) shows that

\[
\frac{1}{x + iv_k(x)} = \lim_{z \to P_k(x)} G_{v_k}(z) = \lim_{z \to P_k(x')} G_{v_k}(z) = \frac{1}{x' + iv_k(x')},
\]

where “\( \to \)” denotes non-tangential limits. Clearly the above identities imply that \( x = x' \).

Regarding continuity we note first that since \( H_k \) is analytical on \( \mathbb{C} \setminus [a, b] \supseteq \mathbb{C} \setminus [c, d] \), it follows immediately from (3.7) and the continuity of \( v_k \) (cf. Lemma 3.3) that \( P_k \) is continuous on each of the intervals \((-\infty, c)\), \((c, d)\) and \((d, \infty)\). Since \( P_k(x) = H_k(x) \), whenever \( x \in (c, d) \), it follows moreover from (3.1) and monotone convergence that \( P_k \) is left-continuous at \( c \) and right continuous at \( d \). To verify that \( P_k(x) \to P_k(c) \) as \( x \searrow c \), it suffices by (3.1) and continuity of \( v_k \) to show that

\[
\int_a^b \frac{|t|k(t)}{(x - t) + iv_k(x)} \, dt \to \int_a^b \frac{|t|k(t)}{c - t} \, dt \quad \text{as} \quad x \searrow c. \tag{3.8}
\]

For this we consider for each \( x \) in \([c, d]\) the function \( h_x : (a, b) \to \mathbb{C} \) given by

\[
h_x(t) = \frac{1}{(x - t) + iv_k(x)}, \quad (t \in (a, b)),
\]

and we note that by (3.4)

\[
\int_a^b |h_x(t)|^2 |t|k(t) \, dt = \int_a^b \frac{|t|k(t)}{(x - t)^2 + v_k(x)^2} \, dt = 1
\]

for all \( x \) in \((c, d)\). This implies that the family \( \{h_x : x \in (c, d)\} \) is uniformly integrable with respect to the finite measure \(|t|k(t) \, dt\), and since also \( h_x(t) \to h_c(t) \) as \( x \searrow c \) for all \( t \) in \((a, b)\), it follows that \( h_x \to h_c \) in the \( L^1 \)-space of the measure \(|t|k(t) \, dt\). Clearly this implies (3.8), and similarly it follows that \( P_k(x) \to P_k(d) \) as \( x \nearrow d \).

3.6 Corollary. The measure \( v_k \) is absolutely continuous with respect to Lebesgue measure with a continuous density \( f_{v_k}(\xi) \) given by

\[
f_{v_k}(P_k(x)) = \frac{v_k(x)}{\pi(x^2 + v_k(x)^2)}, \quad (x \in \mathbb{R}).
\]

In particular the support of \( v_k \) is the compact interval \([P_k(c), P_k(d)]\).

Proof. This follows by Stieltjes-Inversion and Proposition 3.4. Indeed, for any \( x \) in \( \mathbb{R} \) we have that

\[
\lim_{y \searrow 0} G_{v_k}(P_k(x) + iy) = \frac{1}{x + iv_k(x)}.
\]

Recalling (see e.g. Chapter XIII in [RS78]) that the singular part of \( v_k \) is concentrated on the set

\[
\{\xi \in \mathbb{R} \mid \lim_{y \searrow 0} |G_{v_k}(\xi + iy)| = \infty\};
\]
it follows in particular that \( \nu_k \) has no singular part. For any \( x \) in \( \mathbb{R} \) we find furthermore by the Stieltjes Inversion Formula that
\[
f_{\nu_k}(P_k(x)) = \frac{-1}{\pi} \lim_{y \to 0} \text{Im}(G(P_k(x) + iy)) = \frac{-1}{\pi} \text{Im} \left( \frac{1}{x + iv_k(x)} \right) = \frac{v_k(x)}{\pi(x^2 + v_k(x)^2)}.
\]
In particular we see that \( f_{\nu_k}(\xi) > 0 \), if and only if \( \xi \in (P_k(c), P_k(d)) \). Denoting by \( P_k^{(-1)} \) the inverse of \( P_k \), we note finally that
\[
f_{\nu_k}(\xi) = \frac{v_k(P_k^{(-1)}(\xi))}{\pi(P_k^{(-1)}(\xi)^2 + v_k(P_k^{(-1)}(\xi))^2)} \quad (\xi \in \mathbb{R}),
\]
which via the continuity of \( P_k^{(-1)} \) shows that \( f_{\nu_k} \) is continuous too. \( \blacksquare \)

**3.7 Lemma.** Consider a function \( k: \mathbb{R} \setminus \{0\} \to [0, \infty) \) which is not identically 0. Assume that \( k \) satisfies the following conditions:

(a) \( k \) has bounded support.

(b) \( k \in C^2(\mathbb{R} \setminus \{0\}) \), and \( k, k', k'' \) are bounded.

(c) \( k \) is increasing on \((-\infty, 0)\) and decreasing on \((0, \infty)\).

Consider in addition the function \( F_k \) defined by \( (3.3) \). Then for any \( r \) in \((0, \infty)\) there exists a number \( \theta_r \), in \((0, \pi)\), such that the function \( \theta \mapsto F_k(r \sin(\theta)e^{i\theta}) := F_k(r \sin(\theta) \cos(\theta), r \sin^2(\theta)) \) is strictly decreasing on \((0, \theta_r)\] and strictly increasing on \([\theta_r, \pi)\].

**Proof.** Let \( u \) be a new variable defined by \( t = (r \sin \theta)u \). Then
\[
F_k(r \sin(\theta)e^{i\theta}) = \int_{\mathbb{R}} \frac{|u|k(ru \sin \theta)}{1 - 2ru \cos \theta + u^2} \, du, \quad (\theta \in (0, \pi)).
\]
Now take any decreasing function \( h: (0, \infty) \to [0, \infty) \) from \( C^2((0, \infty)) \) with bounded support and such that \( h, h', h'' \) are bounded. Then let
\[
\psi_h(x) := \int_0^\infty \frac{u}{1 - 2ru + u^2} h(u\sqrt{1 - x^2}) \, du, \quad (x \in (-1, 1)).
\]
Note then that if we define \( k_r^\pm(u) := k(\pm ru) \) for \( u > 0 \), and
\[
\Psi_r(x) = \psi_{k_r^+}(x) + \psi_{k_r^-}(-x), \quad (x \in (-1, 1)), \quad (3.9)
\]
then it holds that
\[
F_k(r \sin(\theta)e^{i\theta}) = \Psi_r(\cos \theta), \quad (\theta \in (-\pi, \pi)). \quad (3.10)
\]
Assuming that \( h \) is not identically 0, we are going to show that

(1) \( \psi_h'(x) > 0 \) for \( x \) in \((0, 1)\),

(2) \( \psi_h'(x) < 0 \) for \( x \) in \((-1, -\sqrt{3})\)

(3) \( \psi_h''(x) > 0 \) for \( x \) in \([-\sqrt{3}/2, \sqrt{3}/2]\).

10
For any \( x \) in \((-1, 1)\) we note first by differentiation under the integral sign that

\[
\psi'_h(x) = \int_0^\infty \frac{2u^2}{(1-2ux+u^2)^2} h(u\sqrt{1-x^2}) \, du \\
- \int_0^\infty \frac{u^2}{1-2ux+u^2} \cdot \frac{x}{\sqrt{1-x^2}} h'(u\sqrt{1-x^2}) \, du,
\]

and so (1) holds. Moreover, by integration by parts,

\[
\psi'_h(x) = \int_0^\infty \frac{2u^2}{(1-2ux+u^2)^2} h(u\sqrt{1-x^2}) \, du \\
+ \int_0^\infty \frac{\partial}{\partial u} \left( \frac{u^2}{1-2ux+u^2} \cdot \frac{x}{\sqrt{1-x^2}} \right) \, du \\
- \int_0^\infty \frac{u^2}{1-2ux+u^2} \left( \frac{1}{\sqrt{1-x^2} + \frac{x^2}{(1-x^2)^{3/2}}} \right) h'(u\sqrt{1-x^2}) \, du \\
+ \int_0^\infty \frac{u^2}{1-2ux+u^2} \cdot \frac{x^2}{1-x^2} h''(u\sqrt{1-x^2}) \, du,
\]

Hence we have the property [2].

Finally, we proceed to compute \( \psi''_h(x) \). Using Leibniz’ formula we find that

\[
\psi''_h(x) = \int_0^\infty \frac{8u^3}{(1-2ux+u^2)^3} h(u\sqrt{1-x^2}) \, du \\
- \int_0^\infty \frac{4u^3}{(1-2ux+u^2)^2} \cdot \frac{x}{\sqrt{1-x^2}} h'(u\sqrt{1-x^2}) \, du \\
- \int_0^\infty \frac{u^2}{1-2ux+u^2} \left( \frac{1}{\sqrt{1-x^2} + \frac{x^2}{(1-x^2)^{3/2}}} \right) h'(u\sqrt{1-x^2}) \, du \\
+ \int_0^\infty \frac{u^3}{1-2ux+u^2} \cdot \frac{x^2}{1-x^2} h''(u\sqrt{1-x^2}) \, du \\
= \int_0^\infty \frac{8u^3}{(1-2ux+u^2)^3} h(u\sqrt{1-x^2}) \, du \\
- \int_0^\infty \frac{4u^3}{(1-2ux+u^2)^2} \cdot \frac{x}{\sqrt{1-x^2}} h'(u\sqrt{1-x^2}) \, du \\
- \int_0^\infty \frac{u^2}{1-2ux+u^2} \left( \frac{1}{\sqrt{1-x^2} + \frac{x^2}{(1-x^2)^{3/2}}} \right) h'(u\sqrt{1-x^2}) \, du \\
+ \int_0^\infty \frac{u^3}{1-2ux+u^2} \cdot \frac{x^2}{1-x^2} h''(u\sqrt{1-x^2}) \, du,
\]

In the resulting expression above the first three integrals are positive for any \( x \) in \((-1, 1)\), since \(-h', h \geq 0\) and \(u^2 + 2ux + 1 = (u + x)^2 + 1 - x^2 \geq 0\). By integration by parts, the last integral can be written as follows:

\[
\int_0^\infty \frac{u^3}{1-2ux+u^2} \cdot \frac{x^2}{1-x^2} h''(u\sqrt{1-x^2}) \, du \\
= -\frac{x^2}{(1-x^2)^{3/2}} \int_0^\infty \frac{\partial}{\partial u} \left( \frac{u^3}{1-2ux+u^2} \right) \cdot h'(u\sqrt{1-x^2}) \, du \\
= -\frac{x^2}{(1-x^2)^{3/2}} \int_0^\infty \frac{u^2}{(1-2ux+u^2)^2} \left( (u - 2x)^2 + 3 - 4x^2 \right) h'(u\sqrt{1-x^2}) \, du
\]

and is therefore positive as well for any \( x \) in \([-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}]\). Hence the property [3] is proved.
Recalling now formula (3.9), note that since at least one of the functions $k_+^r, k_-^r$ is not identically 0, it follows from (1)-(3) that $\Psi_r'(x) = \psi_{k_+}^r(x) - \psi_{k_-}^r(-x) > 0$, if $x \geq \frac{\sqrt{3}}{2}$, $\Psi_r'(x) < 0$, if $x \leq -\frac{\sqrt{3}}{2}$, and $\Psi_r''(x) = \psi_{k_+}''(x) + \psi_{k_-}''(-x) > 0$, if $|x| \leq \frac{\sqrt{3}}{2}$. Hence, $\Psi_r'$ is strictly increasing in $(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$ and there exists a unique zero of $\Psi_r'$ at some $x_r$ in $(-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2})$. Therefore $\Psi_r$ is strictly decreasing on $(-1, x_r)$ and strictly increasing on $[x_r, 1)$, and the lemma now follows readily from formula (3.10).

3.8 Proposition. Consider a function $k: \mathbb{R} \setminus \{0\} \to [0, \infty)$, which is not identically 0, and assume that $k$ satisfies conditions (a)-(c) in Lemma 3.7. Then the associated measure $v_k$ (described in Lemma 3.1) is unimodal. In fact there exists a number $\omega$ in $(P_k(c), P_k(d))$, such that the density $f_{v_k}$ (cf. Corollary 3.6) is strictly increasing on $[P_k(c), \omega]$ and strictly decreasing on $[\omega, P_k(d)]$.

Proof. We show first that for any number $\rho$ in $(0, \infty)$, the equality $f_{v_k}(\xi) = \rho$ has at most two solutions in $\xi$. Since $P_k$ is a bijection of $\mathbb{R}$ onto itself, this is equivalent to showing that the equality

$$\rho = f_{v_k}(P_k(x)) = \frac{v_k(x)}{\pi(x^2 + v_k(x))^2}$$

has at most two solutions in $x$. For this we note first that

$$\{x + iy \in \mathbb{C}^+ \mid \frac{v}{\pi(x^2 + r^2)} = \rho\} = C_\rho \setminus \{0\},$$

where $C_\rho$ is the circle in $\mathbb{C}$ with center $\frac{1}{2\pi\rho}$ and radius $\frac{1}{2\pi\rho}$. Writing $x + iy$ as $re^{i\theta}$ ($r > 0$, $\theta \in (-\pi, \pi]$) we find that $C_\rho$ is given by

$$C_\rho = \left\{ \frac{1}{\pi\rho} \sin(\theta)e^{i\theta} \mid \theta \in (0, \pi) \right\}$$

in polar coordinates. We need to show that $C_\rho$ intersects the graph $S$ of $v_k$ in at most two points. By the defining property (3.4) of $v_k$, this is equivalent to showing that the equality

$$F_k\left(\frac{1}{\pi\rho} \sin(\theta)e^{i\theta}\right) = 1$$

has at most two solutions for $\theta$ in $(0, \pi)$. But this follows immediately from Lemma 3.7.

It is now elementary to check that $v_k$ is unimodal. Since $f_{v_k}$ is continuous, strictly positive on $(P_k(c), P_k(d))$ and 0 on $\mathbb{R}\setminus(P_k(c), P_k(d))$, it attains a strictly positive maximum at some point $\omega$ in $(P_k(c), P_k(d))$. If $f_{v_k}$ was not increasing on $[P_k(c), \omega]$, then we could choose $\xi_1, \xi_2$ in $(P_k(c), \omega)$ such that $\xi_1 < \xi_2$, and $f_{v_k}(\xi_1) > f_{v_k}(\xi_2) > 0$. Choosing any number $\rho$ in $(f(\xi_2), f(\xi_1))$, it follows then from the continuity of $f_{v_k}$, that each of the intervals $(P_k(c), \xi_1)$, $(\xi_1, \xi_2)$ and $(\xi_2, \omega)$ must contain a solution to the equation $f_{v_k}(\xi) = \rho$, which contradicts what we established above. Subsequently the argumentation given above also implies that $f_{v_k}$ is in fact strictly increasing on $[P_k(c), \omega]$. Similarly it follows that $f_{v_k}$ must be strictly decreasing on $[\omega, P_k(d)]$, and this completes the proof.

4 The general case

In this section we extend Proposition 3.8 to general measures $\nu$ from $\mathcal{L}(\mathbb{R})$. The key step is the following approximation result.

12
4.1 Lemma. Let $k : \mathbb{R} \setminus \{0\} \to [0, \infty)$ be a function which is increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$ and such that $\frac{k(t)}{|t|} 1_{\mathbb{R}\setminus\{0\}}(t) dt$ is a Lévy measure. Let further $a$ be a non-negative number.

Then there exists a sequence $(k_n)$ of functions $k_n : \mathbb{R} \setminus \{0\} \to [0, \infty)$ satisfying the conditions (a)-(c) in Lemma 3.7 and such that

$$\frac{|t|k_n(t)}{1 + t^2} \frac{dt}{dt} \to a\delta_0 + \frac{|t|k(t)}{1 + t^2} dt$$

as $n \to \infty$.

Proof. For each $n$ in $\mathbb{N}$ we introduce first the function $k_n^0 : \mathbb{R} \to [0, \infty)$ defined by

$$k_n^0(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0], \\ k\left(\frac{t}{n}\right), & \text{if } t \in (0, \frac{1}{n}), \\ k(t), & \text{if } t \in \left[\frac{1}{n}, n\right], \\ 0, & \text{if } t \in (n, \infty), \end{cases}$$

and we note that $k_n^0 \leq k_{n+1}^0$ for all $n$. Next we choose a non-negative function $\varphi$ from $C^\infty_c(\mathbb{R})$, such that $\text{supp}(\varphi) \subseteq [-1, 0]$, and $\int_{-1}^0 \varphi(t) dt = 1$. We then define the function $\hat{R}_n : \mathbb{R} \to [0, \infty)$ as the convolution

$$\hat{R}_n(t) = n \int_{-1/n}^0 k_n^0(t-s)\varphi(ns) ds \to \int_0^1 k_n^0(t + \frac{u}{n})\varphi(-u) du, \quad (t \in \mathbb{R}), \quad (4.1)$$

and let $R_n$ be the restriction of $\hat{R}_n$ to $(0, \infty)$. Note also that

$$\hat{R}_n(t) = n \int_0^n \varphi(n(t-s))k_n^0(s) ds, \quad (t \in \mathbb{R}).$$

Since $k_n^0$ as well as the derivatives of $\varphi$ and $\varphi$ itself is a bounded function, it follows then by differentiation under the integral sign that $R_n$ is a bounded $C^\infty$-function on $\mathbb{R}$ with bounded derivatives, and so its restriction $R_n$ on $(0, \infty)$ has bounded derivatives too. Since $k_n^0$ is decreasing in $(0, \infty)$, it follows immediately from 4.1 that so is $R_n$. Moreover, $\text{supp}(R_n) \subseteq (0, n]$ by the definition of $k_n^0$.

For any $t$ in $(0, \infty)$ and $n$ in $\mathbb{N}$ note next that

$$R_n(t) \leq \int_0^1 k_{n+1}^0(t + \frac{u}{n})\varphi(-u) du \leq \int_0^1 k_{n+1}^0(t + \frac{n}{n+1})\varphi(-u) du = R_{n+1}(t).$$

Moreover, the monotonicity assumptions imply that $k$ is continuous at almost all $t$ in $(0, \infty)$ (with respect to Lebesgue measure). For such a $t$ we may further consider $n$ so large that $t + \frac{u}{n} \in \left[\frac{1}{n}, n\right]$ for all $u$ in $[0, 1]$. For such $n$ it follows then that

$$R_n(t) = \int_0^1 k(t + \frac{u}{n})\varphi(-u) du \to \int_0^1 k(t)\varphi(-u) du = k(t)$$

by monotone convergence. We conclude that $R_n(t) \nearrow k(t)$ as $n \to \infty$ for almost all $t$ in $(0, \infty)$.

Applying the considerations above to the function $\kappa : (0, \infty) \to [0, \infty)$ given by $\kappa(t) = k(-t)$, it follows that we may construct a sequence $(L_n)_{n \in \mathbb{N}}$ of non-negative functions defined on $(-\infty, 0)$ and with the following properties:
• For all $n$ in $\mathbb{N}$ the function $L_n$ has bounded support.
• For all $n$ in $\mathbb{N}$ we have that $L_n \in C^\infty((-\infty, 0))$, and $L_n^{(p)}$ is bounded for all $p$ in $\mathbb{N}_0$.
• For all $n$ in $\mathbb{N}$ the function $L_n$ is increasing on $(-\infty, 0)$.
• $L_n(t) \not> k(t)$ as $n \to \infty$ for almost all $t$ in $(-\infty, 0)$ (with respect to Lebesgue measure).

Next choose a non-negative function $\psi$ from $C_c^\infty(\mathbb{R}) \setminus \{0\}$ such that $\text{supp}(\psi) \subseteq [-1, 1]$, and such that $\psi$ is increasing on $[-1, 0]$ and decreasing on $[0, 1]$. By scaling we may assume in addition that $\int_{-1}^{1} |\psi|^2 \, dt = 1$.

We are now ready to define $k_n: \mathbb{R} \setminus \{0\} \to [0, \infty)$ by

$$k_n(t) = \begin{cases} 
    \alpha^2 \psi(nt) + R_n(t), & \text{if } t > 0, \\
    \alpha^2 \psi(nt) + L_n(t), & \text{if } t < 0.
\end{cases}$$

It is then apparent from the argumentation above that $k_n$ satisfies the conditions (a)-(c) in Lemma 3.7, and it remains to show that $\int k_n(t) \, dt \to a\delta_0 + \int k(t) \, dt$ as $n \to \infty$. For any bounded continuous function $g: \mathbb{R} \to \mathbb{R}$ we find that

$$\int_{\mathbb{R}} g(t) \frac{|t| k_n(t)}{1 + t^2} \, dt = \alpha^2 \int_{\mathbb{R}} g(t) \frac{|t| \psi(nt)}{1 + t^2} \, dt + \int_{-\infty}^{0} g(t) \frac{|t| L_n(t)}{1 + t^2} \, dt + \int_{0}^{\infty} g(t) \frac{t R_n(t)}{1 + t^2} \, dt$$

$$= a \int_{-1}^{1} g(\frac{u}{n}) \frac{|u| \psi(u)}{1 + (\frac{u}{n})^2} \, du + \int_{-\infty}^{0} g(t) \frac{|t| L_n(t)}{1 + t^2} \, dt + \int_{0}^{\infty} g(t) \frac{t R_n(t)}{1 + t^2} \, dt$$

$$\to a \int_{-1}^{1} g(0) |u| \psi(u) \, du + \int_{-\infty}^{0} g(t) \frac{|t| k(t)}{1 + t^2} \, dt + \int_{0}^{\infty} g(t) \frac{t k(t)}{1 + t^2} \, dt$$

$$= a g(0) + \int_{\mathbb{R}} g(t) \frac{|t| k(t)}{1 + t^2} \, dt$$

where, when letting $n \to \infty$, we used dominated convergence on each of the integrals; note in particular that $\int \frac{|t| L_n(t)}{1 + t^2} \, dt$ and $\int \frac{t R_n(t)}{1 + t^2} \, dt$ are dominated almost everywhere by $\int \frac{|t| k(t)}{1 + t^2} \, dt$ on the relevant intervals, and here $\int \frac{|t| k(t)}{1 + t^2} \, dt < \infty$, since $\frac{k(t)}{|t|}$ is a Lévy measure. This completes the proof.

4.2 Theorem. Any measure $\nu$ in $\mathcal{L}(\mathbb{R})$ is unimodal.

Proof. We note first that for any probability measure $\mu$ on $\mathbb{R}$ and any constant $a$ in $\mathbb{R}$, the free convolution $\mu \ast a \delta_0$ is the translation of $\mu$ by the constant $a$, and hence $\mu$ is unimodal, if and only if $\mu \ast a \delta_0$ is unimodal for some (and hence all) $a$ in $\mathbb{R}$. For $\mathbb{R}$-infinitely divisible measures this means that the measure with free generating pair $(\gamma, \sigma)$ (cf. (2.8)) is unimodal, if and only if the measure with free generating pair $(\gamma + a, \sigma)$ is unimodal for some (and hence all) $a$ in $\mathbb{R}$. In other words, unimodality depends only on the measure $\sigma$ appearing in the free generating pair.

Now let $\nu$ be a measure from $\mathcal{L}(\mathbb{R})$ with free characteristic triplet $(a, \frac{k(t)}{|t|} \, dt, \eta)$, where $a \geq 0, \eta \in \mathbb{R}$ and $k: \mathbb{R} \setminus \{0\} \to [0, \infty)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. According to the discussion above, it suffices then to show that the measure $\nu^0$ with free generating pair $(0, a\delta_0 + \frac{k(t)}{|t|} \, dt)$ is unimodal (cf. (2.4)). If $k \equiv 0$, then $\nu^0$ is a semi-circular distribution and in particular unimodal. We may thus henceforth assume that
\(k\) is not identically 0. By application of Lemma 4.1 we may choose a sequence \((k_n)\) of non-negative functions, satisfying (a)-(c) in Lemma 3.7, and such that
\[
\frac{|t|k_n(t)}{1 + t^2} \, dt \overset{w}{\to} a\delta_0 + \frac{|t|k(t)}{1 + t^2} \, dt \quad \text{as } n \to \infty. \tag{4.2}
\]
In particular \(k_n\) is not identically 0 for all sufficiently large \(n\). For such \(n\) it follows then from Proposition 3.8 and (2.4) that for some real number \(\gamma_n\) the measure with free generating pair \((\gamma_n, \frac{|t|k_n(t)}{1 + t^2} \, dt)\) is unimodal. Hence this is also true for the measure \(\nu_n^0\) with free generating pair \((0, \frac{|t|k_n(t)}{1 + t^2} \, dt)\). From (4.2) and the free version of Gnedenko’s Theorem (cf. (2.10)) it follows that \(\nu_n^0 \overset{w}{\to} \nu^0\) as \(n \to \infty\), and hence (2.13) implies that \(\nu^0\) is unimodal, as desired.

4.3 Remark. A non-degenerate classically selfdecomposable probability measure is absolutely continuous with respect to Lebesgue measure (see [Sa99, Theorem 27.13]). In the free case it was proved by N. Sakuma (see [SL]) that freely selfdecomposable measures have no atoms. By definition, a unimodal measure does not have a continuous singular part, and via Theorem 4.2 we may thus conclude that also freely selfdecomposable measures are absolutely continuous with respect to the Lebesgue measure, unless they are degenerate. Using Huang’s general density formula [Hu2, Theorem 3.10 (6)], one can also show that the density function of a freely selfdecomposable measure is continuous on \(\mathbb{R}\), which was proved in a restricted case in Corollary 3.6 of the present paper. By contrast, a classical selfdecomposable measure may have a single point of discontinuity (see [Sa99, Theorem 28.4]).

It is known that the absolutely continuous part of a freely infinitely divisible law has a real analytic density inside its support [BB04, Theorem 3.4]. Hence, any non-degenerate freely selfdecomposable measure is of the form \(f(x) \, dx\), where \(f \geq 0\) is continuous on \(\mathbb{R}\) and real analytic inside its support. By contrast, a general freely infinitely divisible measure can have a density discontinuous on the boundary of its support, even if it is absolutely continuous. For instance the free Poisson law
\[
\frac{1}{2\pi} \sqrt{\frac{4 - x}{x}} \, 1_{[0,4]}(x) \, dx
\]
has a density which is discontinuous at 0.

Acknowledgements

This paper was initiated during the “Workshop on Analytic, Stochastic, and Operator Algebraic Aspects of Noncommutative Distributions and Free Probability” at the Fields Institute in July 2013. The authors would like to express their sincere gratitude for the generous support and the stimulating environment provided by the Fields Institute.

TH is supported by Marie Curie International Incoming Fellowship at Université de Franche-Comté.

ST was partially supported by The Thiele Centre for Applied Mathematics in Natural Science at The University of Aarhus.

References

[B-NT02] O.E. Barndorff-Nielsen and S. Thorbjørnsen, Self-decomposability and Lévy processes in free probability, Bernoulli 8(3) (2002), 323-366.
[BB04] S.T. Belinschi and H. Bercovici, Atoms and regularity for measures in a partially defined free convolution semigroups, Math. Z. 248 (2004), 665-674.

[BB05] S.T. Belinschi and H. Bercovici, Partially defined semigroups relative to multiplicative free convolution, Int. Math. Res. Notices, No. 2 (2005), 65-101.

[BP99] H. Bercovici and V. Pata, Stable Laws and Domains of Attraction in Free Probability Theory, Ann. Math. 149 (1999), 1023-1060.

[BV93] H. Bercovici and D.V. Voiculescu, Free Convolution of Measures with Unbounded Support, Indiana Univ. Math. J. 42 (1993), 733-773.

[CoLo] E.F. Collingwood and A.J. Lohwater, The Theory of Cluster Sets, Cambridge Tracts in Mathematics and Mathematical Physics 56, Cambridge University Press (1966).

[GnKo68] B.V. Gnedenko and A.N. Kolmogorov, Limit Distributions for Sums of Independent Random Variables, Addison-Wesley Publishing Company, Inc. (1968).

[HaTh] U. Haagerup and S. Thorbjørnsen, On the free Gamma distributions, Indiana University Math. Journal (To appear).

[Hu1] Hao-Wei Huang, Supports of measures in a free additive convolution semigroup, arXiv:1205.5542v1.

[Hu2] Hao-Wei Huang, Supports, Regularity and $\boxplus$-Infinite Divisibility for Measures of the form $(\mu \boxplus p)^{\boxplus q}$, arXiv:1209.5787v1.

[Ma92] H. Maassen, Addition of freely independent random variables, J. Funct. Anal. 106, (1992), 409-438.

[Sa99] K. Sato, Lévy Processes and Infinitely Divisible Distributions, Cambridge studies in advanced math. 68 (1999).

[RS78] M. Reed and B. Simon, Methods of modern mathematical physics IV. Analysis of operators. Academic Press (1978).

[S11] N. Sakuma, On Free Selfdecomposable Distributions, in: Problems on Infinitely Divisible Distributions 275, 30-33, The Institute of Statistical Mathematics (2011).

[Vo86] D.V. Voiculescu, Addition of certain non-commuting random variables, J. Funct. Anal. 66, (1986), 323-346.

[Ya78] M. Yamazato, Unimodality of infinitely divisible distribution functions of class L, Ann. Probab. 6 (1978), 523-531.

Laboratoire de Mathématiques Université de Franche-Comté
16 route de Gray 25030 Besançon cedex France
thasebe@univ-fcomte.fr

Department of Mathematics University of Aarhus
Ny Munkegade 118 8000 Aarhus C Denmark
steenth@imf.au.dk