Theoretical construction of Morris-Thorne wormholes compatible with quantum field theory

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Abstract
This paper completes and extends some earlier studies by the author to show that Morris-Thorne wormholes are compatible with quantum field theory. The strategy is to strike a balance between reducing the size of the unavoidable exotic region and the degree of fine-tuning of the metric coefficients required to achieve this reduction, while simultaneously satisfying the constraints from quantum field theory. The fine-tuning also serves to satisfy various traversability criteria such as tidal constraints and proper distances through the wormhole. The degree of fine-tuning turns out to be a generic feature of the type of wormhole discussed.

PAC numbers: 04.20.Jb, 04.20.Gz

1 Introduction
Wormholes are handles or tunnels in the spacetime topology linking two separate and distinct regions of spacetime. These regions may be part of our Universe or of different universes altogether. The pioneer work of Morris and Thorne [1] has shown that macroscopic wormholes may be actual physical objects. Furthermore, such wormholes require the use of exotic matter to prevent self-collapse. Such matter is confined to a small region around the throat, a region in which the weak energy condition is violated. Since exotic matter is rather problematical, it is desirable to keep this region as small as possible. However, the use of arbitrarily small amounts of exotic matter leads to severe problems, as discussed by Fewster and Roman [2, 3]. The discovery by Ford and Roman [4, 5] that quantum field theory places severe constraints on the wormhole geometries has shown that most of the “classical” wormholes
could not exist on a macroscopic scale. The wormholes described by Kuhfittig [6, 7] are earlier attempts to strike a balance between two conflicting requirements, reducing the amount of exotic matter and fine-tuning the values of certain parameters. The purpose of this paper is to refine and solidify the earlier ideas, particularly the use of the quantum inequalities of Ford and Roman, here slightly extended, all with the aim of demonstrating that wormholes, which are based on Einstein’s theory, are compatible with quantum field theory. The models discussed will therefore (1) satisfy all the constraints imposed by quantum field theory, (2) strike a reasonable balance between a small proper thickness of the exotic region and the degree of fine-tuning of the metric coefficients, Eq. (1) below, (3) minimize the assumptions on these metric coefficients, and (4) satisfy certain traversability criteria.

Problems with arbitrarily small amounts of exotic matter are also discussed in Ref. [8], but the author states explicitly that the issues discussed here and in Ref. [6] are beyond the scope of his paper.

2 The problem

Consider the general line element

\[ ds^2 = -e^{2\beta(r)}dt^2 + e^{2\alpha(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \]

where \( \beta(r) \to 0 \) and \( \alpha(r) \to 0 \) as \( r \to \infty \). (We are using units in which \( G = c = 1 \).) The function \( \beta \) is called the redshift function; this function must be everywhere finite to avoid an event horizon at the throat. The function \( \alpha \) is related to the shape function \( b = b(r) \):

\[ e^{2\alpha(r)} = \frac{1}{1 - b(r)/r}. \]

(The shape function determines the spatial shape of the wormhole when viewed, for example, in an embedding diagram.) It now follows that

\[ b(r) = r(1 - e^{-2\alpha(r)}). \]

The minimum radius \( r = r_0 \) is the throat of the wormhole, where \( b(r_0) = r_0 \). As a result, \( \alpha \) has a vertical asymptote at the throat \( r = r_0 \):

\[ \lim_{r \to r_0^+} \alpha(r) = +\infty. \]

So \( \alpha(r) \) is assumed to be monotone decreasing near the throat. The qualitative features (again near the throat) of \( \alpha(r) \), \( \beta(r) \), and \( -\beta(r) \), the reflection of \( \beta(r) \) in the horizontal axis, are shown in Fig. 1. It is assumed that \( \beta \) and \( \alpha \) are twice differentiable with \( \beta'(r) \geq 0 \) and \( \alpha'(r) < 0 \).
The next step is to list the components of the Einstein tensor in the orthonormal frame. From Ref. [7],

\[ G_{\hat{t}\hat{t}} = \frac{2}{r} e^{-2\alpha(r)} \alpha'(r) + \frac{1}{r^2} (1 - e^{-2\alpha(r)}), \]  
\[ G_{\hat{r}\hat{r}} = \frac{2}{r} e^{-2\alpha(r)} \beta'(r) - \frac{1}{r^2} (1 - e^{-2\alpha(r)}), \]

and

\[ G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = e^{-2\alpha(r)} \left[ \beta''(r) + \alpha'(r) \beta'(r) + [\beta'(r)]^2 + \frac{1}{r} \beta'(r) - \frac{1}{r^2} \alpha'(r) \right]. \]

Now recall that since the Einstein field equations \( G_{\hat{\alpha}\hat{\beta}} = 8\pi T_{\hat{\alpha}\hat{\beta}} \) in the orthonormal frame imply that the stress-energy tensor is proportional to the Einstein tensor, the only nonzero components are \( T_{\hat{t}\hat{t}} = \rho, T_{\hat{r}\hat{r}} = -\tau, \) and \( T_{\hat{\theta}\hat{\theta}} = T_{\hat{\phi}\hat{\phi}} = p, \) where \( \rho \) is the energy density, \( \tau \) the radial tension, and \( p \) the lateral pressure. The weak energy condition (WEC) may now be stated as follows: the stress-energy tensor \( T_{\hat{\alpha}\hat{\beta}} \) must obey

\[ T_{\hat{\alpha}\hat{\beta}} \mu^\alpha \mu^\beta \geq 0 \]

for all time-like vectors and, by continuity, all null vectors. Using the radial outgoing null vector \( \mu^\alpha = (1, 1, 0, 0) \), the condition becomes \( T_{\hat{t}\hat{t}} + T_{\hat{r}\hat{r}} = \rho - \tau \geq 0 \).
0. So if the WEC is violated, then $\rho - \tau < 0$. The field equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$ now imply that

$$
\rho - \tau = \frac{1}{8\pi} \left( \frac{2}{r} e^{-2\alpha(r)} [\alpha'(r) + \beta'(r)] \right).
$$

(8)

Sufficiently close to the asymptote, $\alpha'(r) + \beta'(r)$ is clearly negative. (Recall that $\alpha' < 0$ and $\beta' \geq 0$.) According to Ford and Roman [4, 5], the exotic matter must be confined to a thin band around the throat. To satisfy these constraints, we would like the WEC to be satisfied outside of some small interval $[r_0, r_1]$. In other words,

$$
|\alpha'(r_1)| = \beta'(r_1),
$$

(9)

$$
\alpha'(r) + \beta'(r) < 0 \quad \text{for} \quad r_0 < r < r_1,
$$

(10)

and

$$
\alpha'(r) + \beta'(r) \geq 0 \quad \text{for} \quad r \geq r_1.
$$

(11)

(See Fig. 1.) Condition (11) implies that $|\alpha'(r)| \leq \beta'(r)$ for $r \geq r_1$. So if $\beta(r) \equiv$ constant, then $\alpha'(r) \equiv 0$ for $r \geq r_1$. In the neighborhood of $r = r_1$, we also require that $\alpha''(r) > 0$, $\beta''(r) < 0$, and $\alpha''(r) > |\beta''(r)|$. We now have the minimum requirements for constructing the type of wormhole that we are interested in.

Using the components of the stress-energy tensor allows us to restate Eqs. (5) and (6) in terms of $b = b(r)$:

$$
T_{\hat{t}\hat{t}} = \rho = \frac{b'(r)}{8\pi r^2}
$$

(12)

and

$$
T_{\hat{r}\hat{r}} = -\tau = -\frac{1}{8\pi} \left[ \frac{b(r)}{r^3} - \frac{2\beta'(r)}{r} \left( 1 - \frac{b(r)}{r} \right) \right].
$$

(13)

Because of Eq. (12), we require that $b'(r) > 0$. Eq. (2) implies that $\alpha(r) = -\frac{4}{3\pi} \ln(1 - b(r)/r)$. From

$$
\alpha'(r) = \frac{1}{2} \frac{1}{1 - b(r)/r} \frac{b'(r) - b(r)/r}{r},
$$

(14)

we conclude that $b'(r_0) \leq 1$ to keep $\alpha'(r)$ negative near the throat. In fact, $\lim_{r \to r_0+} \alpha'(r) = -\infty$. (The condition $b'(r_0) \leq 1$ is called the flare-out condition in Ref. [1].)

2.1 The quantum inequality

The sought-after compatibility with quantum field theory is based on the so-called quantum inequality in Ref. [5], applied to different situations. This inequality deals with an inertial Minkowski spacetime without boundaries. If
$u^\mu$ is the observer’s four-velocity (i.e., the tangent vector to a timelike geodesic), then $\langle T_{\mu\nu}u^\mu u^\nu \rangle$ is the expectation value of the local energy density in the observer’s frame of reference. It is shown that

$$\frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{\langle T_{\mu\nu}u^\mu u^\nu \rangle d\tau}{\tau^2 + \tau_0^2} \geq -\frac{3}{32\pi^2 r_0^3},$$

(15)

where $\tau$ is the observer’s proper time and $\tau_0$ the duration of the sampling time. (See Ref. [5] for details.) Put another way, the energy density is sampled in a time interval of duration $\tau_0$ which is centered around an arbitrary point on the observer’s worldline so chosen that $\tau = 0$ at this point. It is shown in Ref. [5] that the inequality can be applied in a curved spacetime as long as $\tau_0$ is small compared to the local proper radii of curvature, as illustrated in Ref. [5] by several examples. To obtain an estimate of the local curvature, we need to list the nonzero components of the Riemann curvature tensor in the orthonormal frame. From Ref. [7]

$$R_{\hat{r}\hat{r}} = e^{-2\alpha(r)} \left( \beta''(r) - \alpha'(r)\beta'(r) + [\beta'(r)]^2 \right),$$

(16)

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = \frac{1}{r} e^{-2\alpha(r)} \beta'(r),$$

(17)

$$R_{\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}} = \frac{1}{r^2} e^{-2\alpha(r)} \alpha'(r),$$

(18)

and

$$R_{\hat{\theta}\hat{\phi}} = \frac{1}{r^2} \left( 1 - e^{-2\alpha(r)} \right).$$

(19)

Still following Ref. [5], we need to introduce the following length scales over which various quantities change:

$$r_m \equiv \min \left[ r, \frac{b(r)}{\beta'(r)}, \frac{1}{|\beta'(r)|}, \left| \frac{\beta'(r)}{\beta''(r)} \right| \right].$$

(20)

The reason is that the above components of the Riemann curvature tensor can be reformulated as follows:

$$R_{\hat{r}\hat{r}} = \left( 1 - \frac{b(r)}{r} \right) \frac{1}{\beta'(r)} \left( \frac{1}{\beta'(r)} - \frac{1}{r \beta'(r)} \right) - \frac{b(r)}{2r} \left( \frac{1}{\beta'(r)} - \frac{1}{r \beta'(r)} \right) + \left( 1 - \frac{b(r)}{r} \right) \frac{1}{r \beta'(r)},$$

(21)

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{\phi}\hat{\phi}} = \left( 1 - \frac{b(r)}{r} \right) \frac{1}{r \beta'(r)},$$

(22)
\[ R_{\dot{\theta}\dot{\theta}e} = R_{\dot{\phi}\dot{\phi}e} = \frac{b(r)}{2r} \left( \frac{1}{r} \frac{b'(r)}{b(r)} - \frac{1}{r^2} \right), \]  

(23)

and

\[ R_{\dot{\phi}\dot{phi}e} = \frac{1}{r^2} \frac{b(r)}{r}. \]  

(24)

When it comes to curvature, we are going to be primarily interested in magnitudes. So we let \( R_{\text{max}} \) denote the magnitude of the maximum curvature. We know that the largest value of \( (1 - b(r)/r) \) and of \( b(r)/r \) is unity; it follows from Eqs. (20)-(24) that \( R_{\text{max}} \leq 1/r^2 \) (disregarding the coefficient \( \frac{1}{2} \)). So the smallest radius of curvature \( r_c \) is

\[ r_c \approx \frac{1}{\sqrt{R_{\text{max}}}} \geq r_m. \]  

(25)

The point is that working on this scale, the spacetime is Minkowskian (at least approximately), so that inequality (15) can be applied with an appropriate \( \tau_0 \).

As noted earlier, we assume that \( b'(r) \) and hence \( \rho \) are positive. Being nonnegative, it is suggested in Ref. [5] that a bound can be obtained by Lorentz transforming to the frame of a radially moving geodesic observer who is moving with velocity \( v \) relative to the static frame. In this “boosted frame”

\[ r_c' \approx \frac{1}{\sqrt{R_{\text{max}}'}} \geq r_m \gamma, \]  

(26)

where \( \gamma = (1 - v^2)^{-1/2} \), so that the spacetime should be approximately flat. The suggested sampling time is

\[ \tau_0 = \frac{f r_m}{\gamma} \ll r_c', \]  

(27)

where \( f \) is a scale factor such that \( f \ll 1 \). The energy density in the boosted frame is

\[ T_{\dot{\theta}\dot{\theta}e} = \rho' = \gamma^2 (\rho + v^2 p_r), \]  

(28)

where \( v \) is the velocity of the boosted observer. It is stated in Ref. [5] that in this frame the energy density does not change very much over the short sampling time and is therefore approximately constant:

\[ \frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \langle T_{\mu\nu} u^\mu u^\nu \rangle d\tau = \frac{\tau_0}{\pi} \int_{-\infty}^{\infty} d\tau \]  

\[ = \langle T_{\mu\nu} u^\mu u^\nu \rangle = \rho' \geq -\frac{3}{32\pi^2 r_0^4}. \]  

(29)

From Eqs. (12) and (13),

\[ \rho' = \frac{\gamma^2}{8\pi r^2} \left[ b'(r) - v^2 \frac{b(r)}{r} + v^2 r (2\beta'(r)) \left( 1 - \frac{b(r)}{r} \right) \right]. \]
In order for $\rho'$ to be negative, $v$ has to be sufficiently large:

$$v^2 > \frac{b'(r)}{\frac{b(r^2)}{r} - 2r' \beta'(r)(1 - \frac{b(r)}{r})}; \quad (30)$$

(observe that $v^2$ is dimensionless.) In particular, at the throat, $v^2 > b'(r_0)$. Given $b(r)$, inequality (30) places a restriction on $\beta'(r)$. We will return to this point in Sec. 5.

Next, from

$$32\pi^2 \tau_0^4 \frac{g}{c^2 \times 2m} \approx 10^8 \text{ m}^{-2}, \quad (31)$$

we have

$$32\pi^2 \tau_0^4 \frac{g}{c^2 \times 2m} \leq \frac{8\pi r^2}{\gamma^2} \left[ v^2 \frac{b(r)}{r} - b'(r) - v^2 r (2\beta'(r)) \left( 1 - \frac{b(r)}{r} \right) \right]^{-1}. \quad (31)$$

Using $\tau_0 = f r_m / \gamma$ and dividing both sides by $r^4$, we have (disregarding a small coefficient)

$$\frac{f^4 r_m^4}{r^4 \gamma^4} \leq \frac{1}{v^2 - b'(r)} \left[ v^2 \frac{b(r)}{r} - b'(r) - 2v^2 r \beta'(r) \left( 1 - \frac{b(r)}{r} \right) \right]^{-1}$$

and, after inserting $l_p$ to produce a dimensionless quantity,

$$\frac{r_m}{r} \leq \left( \frac{1}{v^2 - b'(r)} \right)^{1/4} \sqrt{\gamma} \left( \frac{l_p}{r} \right)^{1/2}. \quad (31)$$

At the throat, where $b(r_0) = r_0$, inequality (31) reduces to Eq. (95) in Ref. [5]:

$$\frac{r_m}{r_0} \leq \left( \frac{1}{v^2 - b'(r_0)} \right)^{1/4} \sqrt{\gamma} \left( \frac{l_p}{r_0} \right)^{1/2}. \quad (32)$$

Observe that inequality (32) is trivially satisfied if $b'(r_0) = 1$ but not if $b'(r_0) < 1$. In view of inequality (31) and the tidal constraints in the next subsection, we would like $b'(r)$ to be close to unity in the exotic region. (The need for $b'(r_0)$ to be close to 1 is also pointed out in Ref. [5].)

2.2 The tidal constraints

Much of what follows is based on the discussion in Ref. [1]. In particular, we have for the radial tidal constraint

$$|R_{1\phi 1\phi}| = |R_{\phi\phi\phi\phi}| \quad (33)$$

$$= e^{-2\alpha(r)} \left[ \beta''(r) - \alpha'(r) \beta'(r) + [\beta'(r)]^2 \right] \leq \frac{g_{\mu}}{c^2 \times 2m} \approx (10^8 \text{ m})^{-2}, \quad (33)$$
that is, assuming a traveler with a height of 2 m. This constraint is trivially satisfied if \( \beta(r) \equiv \text{constant} \), referred to as the zero-tidal-force solution in Ref. [1]. The lateral tidal constraints are (reinserting \( c \))

\[
|R_{2\hat{\theta}2\hat{\theta}}| = |R_{3\hat{\theta}3\hat{\theta}}| = \gamma^2 |R_{\hat{\theta}\hat{\theta}\hat{\ell}}| + \gamma^2 \left( \frac{v}{c} \right)^2 \left| R_{\hat{\theta}\hat{\ell}\hat{r}} \right|
\]

\[
= \gamma^2 \left( \frac{1}{r} e^{-2\alpha(r)\ell'(r)} \right) + \gamma^2 \left( \frac{v}{c} \right)^2 \left( \frac{1}{r} e^{-2\alpha(r)\alpha'(r)} \right) \leq (10^8 \text{m})^{-2}; \quad (34)
\]

here \( \gamma^2 = 1/ \left[ 1 - (v/c)^2 \right] \).

Returning to Eq. (3), we have for the shape function,

\[
b'(r_0) = \frac{d}{dr} \left[ r(1 - e^{-2\alpha(r)}) \right]_{r=r_0} = 2r_0 e^{-2\alpha(r_0)\alpha'(r_0)} + 1 - e^{-2\alpha(r_0)}. \quad (35)
\]

In order for \( b'(r_0) \approx 1 \), we require that

\[
\lim_{r \to r_0^+} e^{-2\alpha(r)\alpha'(r)} = 0.
\]

As a consequence, the radial tidal constraint \( \olon{33} \) is satisfied at the throat, while the lateral tidal constraints \( \olon{34} \) merely constrain the velocity of the traveler in the vicinity of the throat.

One of the consequences of the condition \( b'(r_0) \approx 1 \) is that the wormhole will flare out very slowly, so that the coordinate distance from \( r = r_0 \) to \( r = r_1 \) will be much less than the proper distance.

### 2.3 The exotic region

We saw in the last section that \( \alpha \) has to go to infinity fast enough so that \( \lim_{r \to r_0^+} e^{-2\alpha(r)\alpha'(r)} = 0 \). At the same time, \( \alpha \) has to go to infinity slowly enough so that the proper distance

\[
\ell(r) = \int_{r_0}^{r} e^{\alpha(r')} dr'
\]

is finite. Then by the mean-value theorem, there exists a value \( r = r_2 \) such that

\[
\ell(r) = e^{\alpha(r_2)}(r - r_0), \quad r_0 < r_2 < r.
\]

In particular, \( \ell(r_0) = 0 \) and

\[
\ell(r_1) = e^{\alpha(r_2)}(r_1 - r_0). \quad (36)
\]

With this information we can examine the radial tidal constraint at \( r = r_1 \). From Eq. \( \olon{10} \)

\[
|R_{\hat{r}\hat{r}\hat{r}\hat{r}}| = e^{-2\alpha(r_1)} \left| \beta''(r_1) - \alpha'(r_1)\beta'(r_1) + \beta'(r_1) \right|^2
\]

\[
= e^{-2\alpha(r_1)} \left| \beta''(r_1) - \alpha'(r_1)[-\alpha'(r_1)] + [\alpha'(r_1)]^2 \right|
\]

8
by Eq. (9). So by inequality (33),

\[ |̂R_{ij}^2| = e^{-2\alpha(r)} \left| \beta''(r_1) + \alpha'(r_1)\alpha'(r_1) + [\alpha'(r_1)]^2 \right| \leq (10^8 \text{m})^{-2}. \]

Since \( e^{-2\alpha(r)} \) is strictly increasing, it follows that

\[ e^{-2\alpha(r_2)} \left| \beta''(r_1) + 2 [\alpha'(r_1)]^2 \right| < 10^{-16} \text{m}^{-2}. \]

From Eq. (36), we now get the following:

\[ \frac{(r_1 - r_0)^2}{[ℓ(r_1)]^2} \left| \beta''(r_1) + 2 [\alpha'(r_1)]^2 \right| < 10^{-16} \text{m}^{-2} \]

and

\[ \left| \beta''(r_1) + 2 [\alpha'(r_1)]^2 \right| < \frac{[ℓ(r_1)]^2}{10^{16}(r_1 - r_0)^2} . \] \( (37) \)

As a consequence,

\[ \beta''(r_1) + 2 [\alpha'(r_1)]^2 < \frac{[ℓ(r_1)]^2}{10^{16}(r_1 - r_0)^2} \] \( (38) \)

or

\[ \beta''(r_1) + 2 [\alpha'(r_1)]^2 > -\frac{[ℓ(r_1)]^2}{10^{16}(r_1 - r_0)^2} . \] \( (39) \)

So if either condition (38) or condition (39) is satisfied, then so is condition (37).

3 A class of models; fine-tuning

To estimate the size of the exotic region, we need some idea of the magnitude of \( \alpha(r) \), which depends on the specific model chosen. The only information available is that \( \alpha(r) \) increases slowly enough as \( r \to r_0^+ \) to keep \( \int_{r_0}^r e^{\alpha(r')} dr' \) finite. (We have not made any assumptions regarding \( \beta(r) \), except for some of the basic requirements.)

First we need to recall that Morris-Thorne wormholes are not just concerned with traversability in general but more specifically with humanoid travelers. According to Ref. [1], the space station should be far enough away from the throat so that

\[ 1 - \frac{b(r)}{r} = e^{-2\alpha(r)} \approx 1, \] \( (40) \)

making the space nearly flat. Another condition involves the redshift function: at the station we must also have

\[ |\beta'(r)| \leq g_\oplus / \left( c^2 \sqrt{1 - b(r)/r} \right) . \] \( (41) \)
It will be seen below that for our wormhole, the first condition, Eq. (40), is easily satisfied. By condition (11), as well as Fig. 1, \( |\alpha'(r)| \leq |\beta'(r)| \) for \( r \geq r_1 \). So if \( 1 - b(r)/r \approx 1 \), then we have

\[
|\alpha'(r)| < 10^{-16} \text{ m}^{-1}
\]  

(42)

at the station. This inequality should give us at least a rough estimate of the distance to the station: for large \( r \), \( |\alpha(r)| \sim |\beta(r)| \), since both \( \alpha \) and \( \beta \) go to zero. It must be kept in mind, however, that the inequality \( |\alpha'(r)| \leq \beta'(r) \) implies that this procedure does underestimate the distance. The main reason for using \( \alpha \) in the first place is to avoid making additional assumptions involving \( \beta \). Instead, \( \beta \) can be left to its more obvious role, helping to meet the tidal constraints and the quantum inequality, Eq. (31). We will return to this point after discussing \( \alpha \).

Consider next a class of models based on the following set of functions:

\[
\alpha(r) = a \ln \left( \frac{1}{(r-r_0)^b} + \sqrt{\frac{1}{(r-r_0)^{ab}} + 1} \right).
\]  

(43)

(Other models are discussed in Ref. (11).) For convenience let us concentrate for now on the special case \( n = 2 \) and return to Eq. (43) later. For \( n = 2 \), the equation becomes

\[
\alpha(r) = a \sinh^{-1} \left( \frac{1}{(r-r_0)^b} \right), \quad b > \frac{1}{2a}.
\]  

(44)

The need for the assumption \( b > 1/(2a) \) comes from the shape function

\[
b(r) = r \left( 1 - e^{-2a \sinh^{-1}[1/(r-r_0)^b]} \right);
\]

\[
b'(r) = 1 - e^{-2a \sinh^{-1}[1/(r-r_0)^b]} \]

\[
+ r \left( -e^{-2a \sinh^{-1}[1/(r-r_0)^b]} \right) \frac{2ab}{(r-r_0) \sqrt{(r-r_0)^{2b} + 1}};
\]

\( b'(r) \to 1 \) as \( r \to r_0 \), as long as \( b > 1/(2a) \). To see this, it is sufficient to examine

\[
e^{-2a \sinh^{-1}[1/(r-r_0)^b]} \frac{1}{r-r_0}
\]
as \( r \to r_0 \):

\[
\frac{1}{(r-r_0)^2 + \sqrt{(r-r_0)^2 + 1}}^{2a} \frac{1}{r-r_0} = \frac{1}{(r-r_0)^{2a}} \left[ 1 + (r-r_0)^b \sqrt{\frac{1}{(r-r_0)^{2a} + 1}} \right]^{2a} \frac{1}{r-r_0} = \frac{1}{(r-r_0)^{2a-1} \left[ 1 + \sqrt{1 + (r-r_0)^{2b}} \right]^{2a}}. 
\]

So if \( 2ab - 1 > 0 \), then the second factor in the denominator becomes negligible for \( r \approx r_0 \). The result is

\[
e^{-2a \sinh^{-1}{1/(r-r_0)^b}} \frac{1}{r-r_0} \sim (r-r_0)^{2ab-1} \to 0. 
\]

For computational purposes, however, we will simply let \( b = 1/(2a) \). Consider next,

\[
\alpha'(r) = -\frac{ab}{(r-r_0)^{2b} + 1}, \quad r > r_0, \quad (45) 
\]

and

\[
\alpha''(r) = \frac{ab [(1+b)(r-r_0)^{2b} + 1]}{(r-r_0)^2 [(r-r_0)^{2b} + 1]^{3/2}}. \quad (46) 
\]

We know that the wormhole flares out very slowly at the throat, which suggests assigning a small coordinate distance to the exotic region, at least initially. A good choice is \( r - r_0 = 0.000001 \) m, as in Ref. [7]. Then from Eqs. (45) and (46), we get

\[
\alpha'(r_1) \approx -\frac{ab}{r_1} \quad \text{and} \quad \alpha''(r_1) \approx \frac{ab}{(r_1)^2}. \quad (47) 
\]

For future reference, let us replace \( ab \) by \( A \):

\[
\alpha'(r_1) \approx -\frac{A}{r_1} \quad \text{and} \quad \alpha''(r_1) \approx \frac{A}{(r_1)^2}. \quad (48) 
\]

Since we also want \( \alpha''(r_1) > |\beta''(r_1)| \) [or \( \alpha''(r_1) > -\beta''(r_1) \)], we have in view of inequality (38),

\[
\frac{A}{(r_1-r_0)^2} > -\beta''(r_1) > \frac{2A^2}{(r_1-r_0)^2} - \frac{|\ell(r_1)|^2}{10^{16}(r_1-r_0)^2}. \quad (49) 
\]

Conversely, the inequality

\[
\frac{2A^2}{(r_1-r_0)^2} - \frac{|\ell(r_1)|^2}{10^{16}(r_1-r_0)^2} = 2|\alpha'(r_1)|^2 - \frac{|\ell(r_1)|^2}{10^{16}(r_1-r_0)^2} < -\beta''(r_1) 
\]
implies condition \(38\). Since \(-\beta''(r_1) < \alpha''(r_1)\), we conclude that inequality \(49\) is valid if, and only if, condition \(38\) is met.

Inequality \(49\) now implies that

\[
2A^2 - A - \frac{[\ell(r_1)]^2}{10^{16}} < 0.
\]

The critical values are

\[
A = 1 \pm \sqrt{1 + \frac{8[\ell(r_1)]^2}{10^{16}}}
\]

Hence

\[
\frac{1 - \sqrt{1 + \frac{8[\ell(r_1)]^2}{10^{16}}}}{4} < A < \frac{1 + \sqrt{1 + \frac{8[\ell(r_1)]^2}{10^{16}}}}{4}
\]

Returning to the condition \(b'(r_0) \leq 1\) for a moment, note that \(ab\) and hence \(A\) must exceed \(1/2\). It follows that

\[
\frac{1}{2} < A < \frac{1 + \sqrt{1 + \frac{8[\ell(r_1)]^2}{10^{16}}}}{4}
\]

and, replacing \(A\),

\[
\frac{1}{2} < ab < \frac{1 + \sqrt{1 + \frac{8[\ell(r_1)]^2}{10^{16}}}}{4}
\]

The left inequality confirms that \(b > 1/(2a)\). This solution shows that considerable fine-tuning is required. We will return to this point in Sec. 4.

Finally, observe that with the extra condition \(|\beta''(r_1)| < \alpha''(r_1)|\), the qualitative features in Fig. 1 are retained, so that no additional assumptions are needed.

Letting \(b = 1/(2a)\) once again for computational purposes, we now have

\[
\ell(r_1) = \int_{r_0}^{r_0 + 0.000001} e^{a \sinh^{-1}(1/(r-r_0)^{(1/(2a)})} dr.
\]

These values change very little with \(a\). For example, if \(a\) ranges from 0.1 to 0.5, then \(\ell(r_1)\) ranges from 0.0021 m to 0.0028 m; \(\ell(r_1)\) is much larger than \(r_1 - r_0\), a consequence of the slow flaring out. From inequality \(12\) we can estimate the distance \(r_s\) to the space station: if \(|\alpha'(r_s)| = 10^{-16} \text{ m}^{-1}\), then \(r_s = 70,000\) km. Of course, we can always reduce the coordinate distance. Thus for \(r_1 - r_0 = 0.000000001\) m and \(a = 0.5\), we get \(\ell(r_1) = 0.0000089 < 0.1\) mm.

A good alternative is to use Eq. \(43\), subject to the condition

\[
nabla b + \frac{1}{2} nb > 1.
\]
(As before, this condition comes from the requirement that $b'(r_0) \leq 1$; in fact, if $n = 2$, we are back to $2ab > 1$.) For example, retaining $r_1 - r_0 = 0.000001$ m, if $a = 0.2$ and $b = 1$, then $ab = 2.857$. These values yield $\ell(r_1) \approx 0.0000725 \text{ m} < 0.1 \text{ nm}$. The corresponding distance $r_s$, obtained from $\alpha'(r)$ [now referring to Eq. (43)], is about 45,000 km. Both $\ell(r_1)$ and $r_s$ are relatively small.

Using the equation $nab - b + \frac{1}{2}nb = 1$ to eliminate $n$ in Eq. (43) shows that further reductions in $\ell(r_1)$ are only significant if $a$ and $b$ get unrealistically close to zero. So practically speaking, a further reduction in the proper distance $\ell(r_1)$ requires a reduction in the coordinate distance $r_1 - r_0$.

Returning to the radial tidal constraint, based on experience with specific functions (as in Ref. [6]), $|R_{ij}|$ is likely to reach its peak just to the right of $r = r_1$. The simplest way to handle this problem is to tighten the constraint in Eq. (33) at $r = r_1$ by reducing the right side. This change increases the degree of fine-tuning in condition (51).

A final consideration is the time dilation near the throat. Denoting the proper distance by $\ell$ and the proper time by $\tau$, as usual, we let $\gamma v = d\ell/d\tau$, so that $d\tau = d\ell/(\gamma v)$. Assume that $\gamma \approx 1$. Since $d\ell = e^{\alpha(r)}dr$ and $d\tau = e^{\beta(r)}dt$, we have for any coordinate time interval $\Delta t$:

$$\Delta t = \int_{t_a}^{t_b} dt = \int_{t_a}^{t_b} e^{-\beta(r)} \frac{d\ell}{v} = \int_{r_a}^{r_b} \frac{1}{v} e^{-\beta(r)} e^{\alpha(r)} dr.$$

From Eq. (43), we have on the interval $[r_0, r_1]$

$$\Delta t = \int_{r_0}^{r_1} \frac{1}{v} e^{-\beta(r)} \left( \frac{1}{(r-r_0)^b} + \sqrt{\frac{1}{(r-r_0)^3} + 1} \right)^a dr.$$

Since $\beta(r)$ is finite, the small size of the interval $[r_0, r_1]$ implies that $\Delta t$ is going to be relatively small for a wide variety of choices for $a$ and $b$.

### 4 The fine-tuning problem in general

The forms of inequalities (51) and (52) suggest that the degree of fine-tuning encountered is a general property of the type of wormhole being considered, namely wormholes for which $b'(r_0) \leq 1$ and $\alpha(r) = A \ln f(r - r_0)$, where (generalizing from earlier cases) $f(r - r_0)|_{r=r_0}$ is undefined ($+\infty$) and $f(r - r_0)|_{r=r_0}$ is a constant (possible zero). If we also assume that $g(r - r_0) = f(\frac{1}{r - r_0})$ can be expanded in a Maclaurin series, then we have for $r \approx r_0$,

$$f \left( \frac{1}{r - r_0} \right) = g(r - r_0) = a_0 + a_1(r - r_0) + a_2(r - r_0)^2 + a_3(r - r_0)^3 + \cdots \approx a_0 + a_1(r - r_0).$$
It follows that

\[ f(r - r_0) = a_0 + \frac{a_1}{r - r_0} \]

near the throat. So

\[ \alpha(r) = A \ln \left( a_0 + \frac{a_1}{r - r_0} \right), \]

\[ \alpha'(r_1) = \frac{-AA_1}{a_0 + \frac{a_1}{r_1-r_0}} \frac{1}{(r_1 - r_0)^2} \sim -\frac{A}{r_1 - r_0}, \tag{53} \]

and

\[ \alpha''(r_1) \sim \frac{A}{(r_1 - r_0)^2}. \tag{54} \]

To show that \( b'(r_0) \leq 1 \), we need to show that \( e^{-2\alpha(r)} \alpha'(r) \to 0 \) as \( r \to r_0 \):

\[
e^{-2A \ln[a_0 + a_1/(r-r_0)]} \frac{-AA_1}{a_0 + \frac{a_1}{r-r_0}} \frac{1}{(r-r_0)^2}
\]

\[
= \frac{1}{(a_0 + \frac{a_1}{r-r_0})^{2A}} \frac{1}{a_0 + \frac{a_1}{r-r_0}} \frac{1}{(r-r_0)^2} = \frac{-AA_1}{a_0 + \frac{a_1}{r-r_0}}^{2A+1} \frac{1}{(r-r_0)^2}
\]

\[
= \frac{-AA_1}{(a_0 + \frac{a_1}{r-r_0})^{2A+1} (r-r_0)^2} = \frac{-AA_1}{a_0(r-r_0) + a_1}^{2A+1} \frac{1}{(r-r_0)^2}
\]

The first factor in the denominator becomes negligible for \( r \approx r_0 \) as long as \( 2A - 1 > 0 \) and \( A > \frac{1}{2} \). We obtain

\[ e^{-2\alpha(r)} \alpha'(r) \sim (r-r_0)^{2A-1} \to 0. \]

Comparing Eqs. (53) and (54) to Eq. (48), we conclude that

\[ \frac{1}{2} < A < 1 + \frac{1 + 8(\ell(r_1))^2}{4A}. \tag{55} \]

So the amount of fine-tuning required appears to be a general property of wormholes of the present type. (Exactly which parameter needs fine-tuning depends on the precise form of \( f(r - r_0) \).) While the degree of fine-tuning considered so far is quite severe, it is considerably milder than most of the cases discussed in Ref. [2].

5 The solution

The discussion of Morris-Thorne wormholes in Ref. [1] is concerned not just with traversability but, more specifically, with traversability by humanoid travelers. So the length of the trip, possible time dilations, and the tidal constraints are important considerations.
The first part of this paper deals with the size of the unavoidable exotic region around the throat. It was found that the size can be reduced almost indefinitely by carefully fine-tuning \( \alpha = \alpha(r) \) or, equivalently, the shape function \( b = b(r) \). The degree of fine-tuning required of some parameter turns out to be a general property of the type of wormhole considered. To achieve this fine-tuning, it is necessary to assume that \( b'(r) \) is close to unity near the throat. This assumption proved to be sufficient to satisfy the tidal constraints.

Concerning the quantum inequalities, if \( b'(r_0) = 1 \), then inequality (32) is trivially satisfied at or near the throat. Away from the throat that may not be the case. Fortunately, we have made no assumptions on \( \beta = \beta(r) \) beyond the basic requirements, no event horizon and \( \beta'(r) \geq |\alpha'(r)| \) for \( r \geq r_1 \). For convenience, we restate inequalities (30) and (31),

\[
v^2 > \frac{b'(r)}{\frac{b(r)}{r} - 2r \beta'(r) \left( 1 - \frac{b(r)}{r} \right)},
\]

\[
\frac{r_m}{r} \leq \left( \frac{1}{v^2 \frac{b(r)}{r} - b'(r) - 2v^2 r \beta'(r) \left( 1 - \frac{b(r)}{r} \right)} \right)^{1/4} \sqrt{\frac{G}{F}} \left( \frac{1}{r} \right)^{1/2},
\]

where \( v \) is the velocity of the radially moving geodesic observer. Since \( \beta'(r) > 0 \), it now becomes evident that \( \beta'(r) \) can be adjusted (or constructed “by hand”) to become part of the fine-tuning strategy: according to Fig. 2, for a typical shape function \( b = b(r) \), the slope of the tangent line at \( (r, b(r)) \) is less than the slope \( b(r)/r \) of the chord extending from the origin to \( (r, b(r)) \). This allows us to construct (or adjust) \( \beta'(r) \) so that

\[
\frac{b(r)}{r} - b'(r) - 2r \beta'(r) \left( 1 - \frac{b(r)}{r} \right)
\]

is 0 or very nearly 0. (According to Eq. (14), \( \beta'(r) \) is large enough for \( r > r_1 \).) Hence the right-hand side of inequality (56) is 1 or very nearly 1, thereby forcing \( v \) to be 1 or very nearly 1. As a consequence, the denominator on the right-hand side of inequality (57) is 0 or very nearly 0; so the inequality is satisfied for any \( r_m \).

Remark: As noted earlier, at \( r = r_0 \), inequality (32) is trivially satisfied. Similarly, at \( r = r_1 \), expression (56) is zero since \( \beta'(r_1) = |\alpha'(r_1)| \). To the right of \( r_1 \), \( \beta'(r) \) is large enough to overtake \( b(r)/r - b'(r) \) and can therefore be adjusted to produce 0 or very nearly 0. Inside the small interval \([r_0, r_1]\), however, it may be necessary to fine-tune \( b(r) \) to keep \( b'(r) \) close to 1 inside the interval, or, which amounts to the same thing, \( \alpha(r) \) must turn sharply upward after crossing \( r = r_1 \) from the right. (Recall the qualitative features in Fig. 1.)

Observe that, given any particular \( b = b(r) \), the choice \( \beta \equiv \text{constant} \) is not likely to work, basically in agreement with the analysis in Ref. 5, since the
original wormhole models in Ref. [1] all assumed a constant \( \beta \), at least near the throat.

Since inequality (57) is satisfied, the radius of the throat, \( r = r_0 \), is macro-
scopic since \( r_m \) includes \( r_0 \). The wormholes satisfy the various traversability
criteria for humanoid travelers. All the while the exotic region is made as small
as possible while keeping the degree of fine-tuning within reasonable bounds.
The models discussed have led to the following promising results: approximately
0.1 mm for the proper thickness of the exotic region, corresponding to a distance
much less than 100 000 km to the space station. By decreasing the coordinate
distance, it is theoretically possible to decrease the proper thickness
of the exotic region indefinitely. While the decrease may be thought of as an
engineering challenge, the fact remains that the concomitant increase in the
degree of fine-tuning would eventually exceed any practical limit.

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