Analytical Representation of Characteristic Modes Decomposition

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Abstract—Aspects of the theory of characteristic modes, based on their variational formulation, are presented and an explicit form of a related functional, involving only currents in a spatial domain, is derived. The new formulation leads to deeper insight into the modal behavior of radiating structures as demonstrated by a detailed analysis of three canonical structures: a dipole, an array of two dipoles and a loop, cylinder and a sphere. It is demonstrated that knowledge of the analytical functional can be utilized to solve important problems related to the theory of characteristic modes decomposition such as the resonance of inductive modes or the benchmarking of method of moments code.

Index Terms—Antenna theory, eigenvalues and eigenfunctions, electromagnetic theory.

I. INTRODUCTION

THE theory of characteristic modes (CMs), formally developed by Garbacz [1] and Harrington and Mautz [2], has become very popular in recent years as this theory constitutes a general approach to characterizing the modal resonant behavior of arbitrarily shaped antennas and scatterers [3]. In its original form, which is considered here, the CM assumes perfect electric conductors (PEC) in a vacuum. Academic interest and a number of publications dealing with CMs continue to grow. However, most papers focus only on the application character, such as [4]–[6]. Excluding the first attempt to summarize CMs in a book [7], there are also related chapters to be found in older books [8] and [9].

This paper briefly reviews characteristic mode decomposition and what constitutes the necessary theoretical background. An analytical form of the functional, composed of reactive and radiated power, is derived, based on previous research [10], [11]. This relation has to be satisfied for each mode but is not restricted to the characteristic basis. Hence, it is possible to specify arbitrary current distribution (the CM can be predicted, see [12]) and compare it with real CMs. Based on this result, properties of canonical shapes are investigated, including inductive modes. Analogically, if the modes are analytically known, they can be substituted into a derived functional instead of using an approximative solution given by the numerical spectral decomposition of an underlying operator.

II. DERIVATION OF THE FUNCTIONAL

Based on previous work by Garbacz [1], Harrington [2] reduced the CMs into the following generalized eigenvalue problem (GEP, [13])

\[ \mathcal{X} (J_n) = \lambda_n \mathcal{R} (J_n), \]  

(1)

where \( \mathcal{R} \) and \( \mathcal{X} \) are real and symmetric operators forming the impedance operator

\[ Z (J_n) = \mathcal{R} (J_n) + j \mathcal{X} (J_n) = n_0 \times (j \omega A + \nabla \phi), \]

(2)

\( A \) and \( \phi \) are corresponding magnetic and electric time-harmonic potentials in Lorenz gauge [14]. \( J_n \) is the modal current density, and \( n_0 \) is the unit vector tangential to the PEC boundary of a radiator. The continuous operator \( Z \) is usually discretised by the method of moments (MoM, [15]), utilizing a proper set of basis functions

\[ J_n (r) \approx \sum_{m=1}^{N} I_{mn} f_{mn} (r), \]

(3)

where \( I_{mn} \) are (modal) expansion coefficients and \( f_{mn} (r) \) are frequency-independent basis functions, e.g., RWG basis functions [16]. Consequently, the MoM procedure leads to an impedance matrix \( Z = R + jX \), which is the discrete representation of the analytical operator \( Z \). Finally, the CMs can be defined in (common) algebraic form [2]

\[ X I_n = \lambda_n R I_n, \]

(4)

which is, in comparison to [1], numerically solvable for an arbitrary radiator since it is based on real and symmetric matrices of size \( N \times N \), where \( N \) is the number of basis functions.

The solution of the GEP produces the characteristic basis \( \{ J_n, \lambda_n \} \) of eigencurrents \( J_n \) and associated eigenvalues \( \lambda_n \), and, due to the properties of the impedance matrix, all eigenvalues are real with all eigencurrents equiphasal (they can also be selected as real, [17]). Furthermore, the CMs minimize the ratio of the net reactive power \( \omega \left( W_m - W_e \right) \) to radiated power \( P_r \). Note that the extremal value of radiated to stored power is considered for the basis as a whole.

It is known [7] that the GEP (1) minimizes a power functional

\[ F (J_n) = \frac{\langle J_n, X J_n \rangle}{\langle J_n, R J_n \rangle} = \frac{2(\omega W_m - W_e)}{P_r} = \lambda_n, \]

(5)

\(^1\)Through this paper, the following notation is used \( \langle f, g \rangle = \int_\Omega f \cdot g \, d\Omega \) and \( \langle f, g \rangle_r = \int_\Omega f \cdot g \, d\Omega. \)
where \( W_m^n \) and \( W_e^n \) are modal magnetic and electric potentials-based energies, defined here as

\[
W_m^n = \frac{1}{2} \Re \int_V \mathbf{A} \cdot \mathbf{J}_n^* \, dV, \tag{6}
\]

\[
W_e^n = \frac{1}{2} \Re \int_V \varphi \rho_n^* \, dV, \tag{7}
\]

with \( P^n_m \) as modal radiated power which is commonly normalized as \( P^n_m = 1 W \). It should be noted that energies (6) and (7) are not equal to true electric (\( \int_V \varphi \lVert \mathbf{E} \rVert^2 dV/2 \)) and magnetic (\( \int_V \mu \lVert \mathbf{H} \rVert^2 dV/2 \)) energy [18]. However, a clear advantage of (6) and (7) is that they can be calculated easily and directly from the (characteristic) currents if they are prescribed analytically or calculated numerically. The paradigm used, and its further extension towards the stored energy, is briefly discussed in Section II-A.

A particular form of the above mentioned functional (5), established directly for the sources (currents/charges) on the antenna, is derived using (2) and it reads

\[
\mathcal{F}(\mathbf{J}_n) = \frac{\langle \mathbf{J}_n, X \mathbf{J}_n \rangle}{\langle \mathbf{J}_n, \mathbf{R} \mathbf{J}_n \rangle} = -\frac{3}{\Re} \int_V \left( \mathbf{A} \cdot \mathbf{J}_n^* - \varphi \rho_n^* \right) dV, \tag{8}
\]

where \( V \) is the volume of an antenna and \( \rho_n \) is the charge density. Inserting the continuity equation [14], \( \rho = -\nabla \cdot \mathbf{J}/\omega \), the functional involves only currents and reads

\[
\mathcal{F}(\mathbf{J}_n) = \frac{\int_{V'} \int_{V'} \mathcal{J}(\mathbf{J}_n) \cos(kR) dV' \, dV}{\int_{V'} \int_{V'} \mathcal{J}(\mathbf{J}_n) \sin(kR) dV' \, dV} = \kappa_n, \tag{9}
\]

where \( \mathcal{J}(\mathbf{J}_n) = (k^2 \mathbf{J}_n(r) \cdot \mathbf{J}_n^*(r') - \nabla \cdot \mathbf{J}_n(r) \nabla \cdot \mathbf{J}_n^*(r')) \), \( R = |r - r'| \) is Euclidean distance, \( k \) is the wavenumber and \( \kappa_n \) is the Rayleigh quotient [19], which is equal to characteristic number \( \lambda_n \) when the true characteristic current \( \mathbf{J}_n \) enters into (9).

Thanks to the “source” formulation (9), arbitrary current distribution can be studied and its properties with true CMs can be compared. This formulation extends the understanding of the original definition in [2], since, as will be shown later, we can study the separated component of (9).

It is important to stress that the functional is minimized by characteristic currents, i.e. solutions of (4). Such a (eigen) basis maximizes the radiated power and minimizes the net reactive power, indicating external resonances of the radiator. Hence, the extremum of (9) is given by characteristic basis \{\( \mathbf{J}_n \)\} with associated eigenvalues \( \lambda_n \).

An exact analytical solution for characteristic currents is exceedingly complicated with only two bodies of finite extent already known, one of them being a spherical shell [23]. However, the expression (9) permits the definition of an arbitrary current distribution \( \mathbf{J} \) without the necessity of numerically computing the impedance matrix \( \mathbf{Z} \) and its decomposition in [4]. In addition, if we analytically try to test a basis \( \mathbf{J} \) that is similar to the true CM basis, we can precisely analyze its behaviour and estimate how close the selected current distribution is to the optimal solution [12].

### A. Relation Between Characteristic Modes and Stored Energy

There is an interesting relationship between decomposition into CMs and the evaluation of the modified stored electromagnetic energy, proposed by Vandenbosch in [10] as

\[
\tilde{W}_{sto} = \frac{1}{4} \Re \left( \mathbf{J}, \frac{\partial \mathcal{X}}{\partial \omega} \right), \tag{10}
\]

in which the structure of \( \mathcal{X} \) is obvious from (2) and \( \mathbf{J} \) is the current density which, in the context of this paper, can be composed as

\[
\mathbf{J} \approx \sum_n \alpha_n \mathbf{J}_n, \tag{11}
\]

where \( \alpha_n \) is given in [2]. If (3) is substituted, (10) can be represented in a useful matrix form as proposed by Gustafsson et al. [24]

\[
\tilde{W}_{sto} \approx \frac{1}{4} \Re \left( \mathbf{H}, \frac{\partial \mathbf{X}}{\partial \omega} \right), \tag{12}
\]

and anticipated much earlier by Harrington and Mautz [25].

It is argued in [25] that in the vicinity of \( n \)-th modal resonances the quality factor \( Q_n \), defined as

\[
Q_n = \frac{\omega}{\partial \lambda_n}, \tag{13}
\]

is approximately equal to the quality factor rigorously derived by Vandenbosch and later reformulated by Gustafsson, i.e.,

\[
Q_{X,n} = \frac{\omega \tilde{W}_{sto}}{\tilde{T}} \approx Q_n. \tag{14}
\]

The argumentation is based on the assumption that the dominant frequency variation is due to the imaginary part of the impedance matrix [25]. Interestingly, the relationship between these two quality factor\(^3\) can be expressed rigorously as

\[
Q_n = Q_{X,n} - \lambda_n Q_{R,n}, \tag{15}
\]

in which \( Q_{R,n} \) is defined in the same way as \( Q_{X,n} \), although \( \mathcal{R} \) or \( \mathbf{R} \) is used instead of \( \mathcal{X} \) or \( \mathbf{X} \). For the exact derivation of (15), see Appendix A. Moreover, the above-mentioned assumption is not needed since the equality \( Q_n = Q_{X,n} \) is based on definition (5) where the eigenvalues are zero at the modal resonances.

Equality between (13), (15) and (10), (12) establishes explicit link between frequency behavior of eigenvalues \( \lambda_n \) and modified modal stored energies [27]. This connection is possible thanks to the modal potential-based energies (6), (7), which occur both in definition of eigenvalues (5) and in (10) through (26).

\(^3\)Please, keep in mind that there are number of quality factor \( Q \) definitions through the literature with possible different meaning [26].
III. Elementary Radiators – Case Studies

In certain (simple) cases the CM basis can be sufficiently approximated by analytical currents. We inspect three canonical examples:

- a thin-strip dipole (Section III-A),
- two parallel coupled dipoles, separated by distance \( h \) with in-phase and out-of-phase modes (Section III-B),
- a loop with uniform mode (Section III-C).

These examples establish a direct way to understand stationary inductive modes. It will be seen that these fulfil \( \nabla \cdot \mathbf{J}(r) = 0 \), i.e., they have no charge. Observations denoted in this section introduce material which is to be developed in Section IV.

A. Thin-strip dipole

Let us consider a thin-strip dipole of length \( L \) and width \( w = L/100 \). Since the dipole is thin the inductive modes are not considered and the current has to fulfill the Dirichlet boundary condition at its ends. It is significant that the choice of any mode from the basis predestinates the basis, as a whole, as the modes are orthogonal. We consider the natural first-order current basis.

\[
\tilde{\mathbf{J}}_n(z) = \mathbf{z}_0 K_0 \delta(y) \sin \left( \frac{\pi n z}{L} \right), x \in \left( -\frac{w}{2}, \frac{w}{2} \right), z \in (0, L), \quad (16)
\]

where the surface current density

\[
K_0 = \frac{1}{h}
\]

is assumed. The divergence of (16) is

\[
z_0 \cdot \frac{\partial \tilde{\mathbf{J}}_n(z)}{\partial z} = K_0 \delta(y) \frac{\pi n z}{L} \cos \left( \frac{\pi n z}{L} \right). \quad (17)
\]

Due to the complexity of (16), (16) and (18) were inserted in [29] and solved numerically in MATLAB [29]. First, three modes, \( n = \{1, 2, 3\} \), are considered. Figure 1 shows the \( \kappa_n \) quotients, together with exact eigenvalues \( \lambda_n \), obtained by solving (1) in CST-MWS software [30]. A good match is attained, even for such a simple basis [16].

It can be seen from Fig. 2 that the agreement between the CM current and its approximation is good, especially for the dominant mode. The analytical current in (16) is, in fact, exact for a non-radiating 1D resonator, while, in turn, the real CMs maximize radiation and, thus, the shape slightly deviates from the sine basis [16, 31].

B. Two thin-strip dipoles

The next scenario involves two closely spaced collinear thin-strip dipoles with length \( L \), separation \( h = L/50 \) and strip width of \( w = L/100 \). There are, depending on the actual orientation of currents, two possible modes: in-phase and out-of-phase. Currents are considered in the form of fundamental distribution \( \tilde{\mathbf{J}}_1 \) from (16).

For the in-phase mode [32], the course of the \( \kappa_1 \) quotient (light-blue line at Fig. 3) is similar to that of the dominant mode on a single dipole. It radiates well and the two in-phase currents may be interpreted as one, flowing along a thicker dipole in a manner similar to a folded dipole. This is not the case for the out-of-phase mode, where the radiated power is much lower. Consequently, the orange line in Fig. 3 shows extremely steep resonance for this mode. Other properties, especially those regarding radiated Q factors, have been discussed in [27] and analytically treated in [33].

Using (9), it is possible to investigate the hypothetical situation where the currents on the dipoles are out-of-phase but with the charge density eliminated (\( \nabla \cdot \tilde{\mathbf{J}}_1 \equiv 0 \)). It strongly resembles the situation where the ends of the dipoles are connected to form a loop. The dark-blue line in Fig. 3 reveals that this mode does not resonate because the “charge” part in (9) is missing and the mode, thus, exhibits pure inductive character. In the next section we show that this behaviour is similar to the uniform zero-order mode on a loop.
C. A loop

A loop is an elementary radiator on which the uniform (also termed static or inductive) mode with $\nabla \cdot \vec{J} = 0$ exists and its behaviour is similar to the modified out-of-phase mode previously analysed. Current distribution on a thin-wire loop with surface density (17), which simplifies (9) to coordinates $(r, \varphi, z)$ as

$$\vec{J}_0(\varphi, r, z) = \varphi_0 K_0(z) \delta(r - \chi), \quad z \in \left(-\frac{h}{2}, \frac{h}{2}\right)$$

(19)

with surface density (17), which simplifies (9) to

$$\kappa_0 = \frac{\int_0^{2\pi} \cos(\varphi) \frac{\cos(k\chi \varphi)}{k\chi} d\varphi}{\int_0^{2\pi} \cos(\varphi) \frac{\sin(k\chi \varphi)}{k\chi} d\varphi}.$$  

(20)

The pure inductive character ($\kappa_0 > 0$) can be clearly seen in Fig. 3. The results presented in this section were quite insensitive to the choice of (17) or (21), thus, the distribution physically closer to reality (21) was used.

Uniform modes do not contribute significantly to far field, but they are important when evaluating near field, input impedance and stored energies.

IV. ON THE UTILIZATION OF THE ANALYTICAL FUNCTIONAL

The usefulness of the analytical functional (8) is investigated in a series of examples involving two surface bodies, a cylinder and a spherical shell. It is important to note that the purpose of this section is to demonstrate the potential applications and not to provide a comprehensive treatment of all issues mentioned.

A. Can inductive modes resonate?

The first example deals with the same topology introduced in Section III-C the only difference being the variable height $h$ of the loop. For a significant height, we obtain a cylinder and we need to integrate in $z$-dimension as well. As mentioned already, the uniform mode can occur on the loop-like topology and it is often claimed that this inductive mode, i.e., a mode with $\lambda_n \to \infty$ for $ka \to 0$, cannot resonate [3]. This question can easily be investigated using tools presented in this paper.

The same current (19) is assumed for the cylindrical shell. Both the uniform (17) and the Maxwellian surface current distribution

$$K_0(z) = \frac{2}{\pi \sqrt{h^2 - 4z^2}},$$

(21)

were tested. In both cases the current was normalized as

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} K_0(z) \, dz = 1.$$  

(22)

The results presented in this section were quite insensitive to the choice of (17) or (21), thus, the distribution physically closer to reality (21) was used.

It can easily be seen that the analytical current (19) has no charge, i.e.,

$$\nabla \cdot \vec{J}_0(\varphi, r, z) = \frac{1}{r} \frac{\partial J_\varphi}{\partial \varphi} = 0.$$  

(23)

As a consequence, the $\phi \rho^* \rho$ terms in (8) are identically zero, which, in conjunction with (7), immediately leads to $W^0_m = 0$. Inspecting (8), it seems that such a current cannot resonate, however this is only true when $W^0_m$ is always positive, which is not the case here.

The uniform mode for a cylinder of various $\chi/h$ is depicted in Fig. 5 in terms of eigenvalues (24)

$$\delta_n = \frac{180}{\pi} \left(\pi - \tan^{-1}(\lambda_n)\right).$$  

(24)
The characteristic eigenangles normalize the eigenvalues and indicate the electromagnetic behaviour of CMs. Modes are capacitive for \( \delta_n > 180^\circ \), inductive for \( \delta_n < 180^\circ \) and resonate for \( \delta_n = 180^\circ \). We can see in Fig. 5 that the uniform mode of the sufficiently tall cylinder can cross the resonance even if it lacks \( W_e \) energy (charge). This observation is verified in Fig. 6 in which the eigenvalues were calculated using the AToM package \( [35] \) (solid lines), in CST-MWS (cross markers) and finally evaluated according to \( [17] \) with \( [19] \) substituted

\[
\kappa_0 = \frac{2\pi}{\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} K_0(z_1) K_0(z_2) \cos(\varphi) \frac{\cos(kR)}{R} \, dz_1 \, dz_2 \, d\varphi}
\]

\[
\int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} K_0(z_1) K_0(z_2) \sin(\varphi) \frac{\sin(kR)}{R} \, dz_1 \, dz_2 \, d\varphi
\]

where \( R = \sqrt{2\chi^2 (1 - \cos(\varphi)) + (z_1 - z_2)^2} \) and the axial symmetry of the cylinder have been utilized as in \( [20] \) in order to reduce one of integrals in \( \varphi \) direction.

The fact that the uniform mode can resonate, even when \( W_e^0 = 0 \), clearly indicates that the term \( W_m^0 \) can be negative. Finally, using the formula for modified stored energy \( [10] \) from

\[
\tilde{W}_{\text{sto}} = W_m + W_e + W_{\text{rad}}
\]

and evaluating it according to formulas (63) and (64) in \( [10] \), we obtain the values of quality factor \( Q \). The results are depicted in Fig. 7. The uniform mode on the tall cylinder has a negative value of modified stored energy \( \tilde{W}_{\text{sto}} \), which means that \( W_m^0 < W_{\text{rad}} \) since \( W_e^0 = 0 \). This is equivalent to the negative slope of eigenvalue \( \lambda_0 \) in \( [13] \) and both observations lead to the negative value of the quality factor \( Q \).

The same behaviour has already been described in \( [36] \), and, so far, only loop-like, divergence-free currents were found, which supports the reasoning in \( [36] \). Using another kind of analysis, the characteristic modes, we hypothesize that the problem is caused by extraordinary uniform currents with \( W_e \approx 0 \) which, in reality, cannot exist independently (it can be shown that there is no realistic feeding that can excite only \( J_0 \) mode).

B. Numerical analysis of CMs as GEP – Benchmarks utilizing a spherical shell

Bearing in mind the results of the previous sections, we can perform a series of benchmarks, employing the knowledge of characteristic modes and numbers in analytical form. To do this, we need to find a scatterer whose characteristic modes are known analytically. The perfect candidate is a spherical shell, whose characteristic fields coincide with properly normalized spherical harmonics \( [1] \). The characteristic numbers can be evaluated analytically if the characteristic currents are substituted into \( [8] \). This becomes of interest when dealing with the numerical solvers which are encumbered with rounding (and
other) numerical errors.

The characteristic numbers $\lambda_n$, obtained using decomposition (4) of the impedance matrices $Z$ from different commercial and in-house packages, are compared with exact radiation coefficients $\epsilon_{kl}^{\text{TE/TM}}$ calculated via (5) for spherical harmonics $J_{kl}^{\text{TM/TE}}$, see Fig. 8 and Fig. 9, respectively. The software packages have been used to generate impedance matrices and, in all cases, the eigen-decomposition has been performed in Matlab. The exact characteristic numbers are depicted by solid black lines and constitute known references. It can be seen that even state-of-the-art commercial simulators are capable of finding only the first four TM$_{kl}$ and TE$_{kl}$ modes. This is caused mainly by the $2k + 1$ degeneracy (the number of degenerated modes increases rapidly) and by the limited (double) numerical precision. Surprisingly, the number of well-defined modes is not influenced by the number of discretization triangles $N_\Delta$. On the contrary, the relative error between analytically and numerically calculated characteristic numbers is a function of $N_\Delta$ which is confirmed by Fig. 10. While the relative error of dominant TM$_{4l}$ and TE$_{4l}$ modes is a few percentage points, it quickly reaches about 10% for groups of TM$_{4l}$ and TE$_{4l}$ modes. The overall results, presented in Figs. 8-10 favour the in-house Matlab tool AToM. However, FEKO [38] and CEM One [39] packages reach comparable results. The routines available for free in [40] suffer from non-symmetry of produced impedance matrices. This issue can be resolved manually during pre-processing to reduce the relative error significantly. Notice that CST is not depicted since the symmetry of produced impedance matrices. This issue can be resolved manually during pre-processing to reduce the relative error significantly. Notice that CST is not depicted since the symmetric matrices have been found correctly since all modes are closer to their resonance.

Other tests, e.g., those involving modal currents, can be performed as well. For example, the numerically calculated characteristic modes on the spherical shell can be compared with their analytical forms via

$$\epsilon_{nkl} = \left\langle J_n^*, J_{kl}^{\text{TE/TM}} \right\rangle.$$  \hspace{1cm} (27)

However, that study goes beyond the scope of this paper.

V. CONCLUSION

The paper discusses specific advances of the theory of characteristic modes as introduced by Garbacz, Harrington and Mautz, but expressed here in terms of a particular functional, which is minimized by eigencurrents. This novel formula provides a different perspective on characteristic mode decomposition.

The usefulness of the functional is illustrated by three canonical examples: a dipole, two closely spaced dipoles and a loop. It was shown that the functional formulation is better suited to be analysed than the original formulation because there is no impedance matrix involved. A deeper investigation of the modes on a dipole reveals the limitations of the approximation of the zero-order current distribution expressed as a sin function.

Knowledge of the analytical functional is important for a few significant topics dealing with various issues of antenna analysis and design. In particular, any method of moment code or characteristic mode solver can be benchmarked using analytical results for a spherical shell.
APPENDIX

RELATIONSHIP BETWEEN $Q_n$ AND $Q_{X,n}$

The purpose of this appendix is to derive (15). To simplify the underlying mathematical nomenclature, the derivation is done for all quantities in their matrix forms. First, modal quality factor $Q$ (13) is expressed in terms of characteristic currents using the matrix form of (5) as

$$Q_n = \frac{\omega}{2} \left[ \frac{\partial}{\partial \omega} \left( \begin{bmatrix} \mathbf{H}^\text{H} \mathbf{I}_n \mathbf{X}^\text{H} \mathbf{R}_n \end{bmatrix} \right) \right],$$

then the differentiation is performed

$$Q_n = Q_{X,n} + \frac{\partial}{\partial \omega} \left( \begin{bmatrix} \mathbf{H}^\text{H} \mathbf{X} \mathbf{I}_n \mathbf{H}^\text{H} \mathbf{R}_n \end{bmatrix} - \frac{1}{2} \mathbf{H}^\text{H} \mathbf{X} \mathbf{I}_n \mathbf{H}^\text{H} \mathbf{R}_n \right) \left( \begin{bmatrix} \mathbf{H}^\text{H} \mathbf{I}_n \mathbf{R}_n \end{bmatrix} \right)^2,$$

(28)

in which the quality factor (15) has been substituted and the following identity has been employed

$$\omega \frac{\partial}{\partial \omega} \left( \begin{bmatrix} \mathbf{H}^\text{H} \mathbf{X} \mathbf{I}_n \mathbf{H}^\text{H} \mathbf{R}_n \end{bmatrix} \right) + \mathbf{H}^\text{H} \mathbf{X} \mathbf{I}_n \omega \frac{\partial}{\partial \omega} \left( \begin{bmatrix} \mathbf{H}^\text{H} \mathbf{R}_n \end{bmatrix} \right) = 2 \omega \frac{\partial}{\partial \omega} \left( \begin{bmatrix} \mathbf{H}^\text{H} \mathbf{X} \mathbf{I}_n \mathbf{H}^\text{H} \mathbf{R}_n \end{bmatrix} \right)$$

(29)

since the modal currents are supposed to be purely real. Then, the RHS of (4) is substituted into the last two terms on the RHS of (29), which yields

$$Q_n = Q_{X,n} + \lambda_n \left( \begin{bmatrix} \mathbf{H}^\text{H} \mathbf{R}_n \end{bmatrix} - \frac{1}{2} \mathbf{H}^\text{H} \mathbf{I}_n \mathbf{R}_n \right),$$

(30)

Finally, performing the differentiation in the last term on the RHS of (30) and using identity (30), we get

$$Q_n = Q_{X,n} - \lambda_n \left( \begin{bmatrix} \mathbf{I}_n \mathbf{H}^\text{H} \mathbf{R}_n \end{bmatrix} - \frac{1}{2} \mathbf{I}_n \mathbf{H}^\text{H} \mathbf{R}_n \right) = Q_{X,n} - \lambda_n Q_{R,n}.$$

(31)

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