Spectral instability for some Schrödinger operators

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Received October 9, 1998 / Revised version received September 13, 1999 / Published online 16 March 2000 – © Springer-Verlag 2000

Summary. We show that the condition numbers of isolated eigenvalues of typical non-self-adjoint differential operators such as the harmonic oscillator may be extremely large. We describe a stable procedure for computing the condition numbers for Schrödinger operators in one dimension, and apply it to the complex resonances of a typical operator with a dilation analytic potential.

1. Introduction

In some earlier papers we showed that typical non-self-adjoint Schrödinger operators $H$ exhibit spectral instability in the following sense. For any $\varepsilon > 0$ there exist many $\lambda \in \mathbb{C}$ and $f \in \text{Dom}(H)$ such that

$$\| H f - \lambda f \| \leq \varepsilon \| f \|$$

even though $\lambda$ is not near the spectrum of $H$. This behaviour occurs for the harmonic oscillator with a nonreal coupling constant as well as for many non-self-adjoint anharmonic oscillators. There is a rapidly growing literature on pseudospectral theory, which was invented to explore just such possibilities, [5, 6, 8–10, 17–19, 21–23].

In this paper we return to the same type of operator, but measure spectral instability by a method which provides more precise information about the

* The authors thank the Engineering and Physical Sciences Research Council for support under grant No. GR/L75443

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instability of individual eigenvalues. We have computed the condition numbers of the first 100 eigenvalues $\lambda_n$ of the harmonic oscillator, and see that they appear to increase exponentially with $n$. We have carried out a similar exercise for the resonances of a typical Schrödinger operator with dilation analytic potential, and report our conclusions.

The condition number of an isolated simple eigenvalue $\lambda$ of $H$ is defined by

$$\kappa(\lambda) := \frac{\|f\| \|f^*\|}{|\langle f, f^* \rangle|}$$

where $f$ is the eigenfunction of $H$ associated with $\lambda$ and $f^*$ is the eigenfunction of $H^*$ associated with its eigenvalue $\overline{\lambda}$. This concept is extensively investigated in many texts on numerical linear algebra (see, for instance, [11, 14, 24, 25]). If $H := -\Delta + V$ acts in $L^2(\mathbb{R}^N)$ where $V$ is a complex-valued potential then $H^* = -\Delta + \overline{V}$ and $f^* = \overline{f}$. Hence for any isolated eigenvalue we have

$$\kappa(\lambda) = \frac{\int_{\mathbb{R}^N} |f|^2}{\int_{\mathbb{R}^N} f^2}$$

(1)

It is well known in numerical linear algebra that if $\kappa(\lambda)$ is large the eigenvalue is very unstable under small perturbations of the operator and hence also unstable because of rounding errors in the computation. What we have shown is that these problems arise in a very extreme form for the simplest non-self-adjoint differential operators. Proving this involves computing the eigenvalue and its condition number accurately in situations in which the denominator of (1) is very small because the complex-valued eigenfunction $f$ is oscillating rapidly.

Because the spectral instability develops so rapidly as $n$ increases, we have had to take great care to use computational methods which are reliable. Fortunately for our first problem there are independent methods of checking the values which we have obtained. In the second case we use the experience gained by the first problem, and have checked the reliability of the conclusions under the variation of several different parameters in the computational method. In Sect. 6 we summarize the conclusions of our investigation.

2. The condition number

If $\lambda$ is an isolated point of the spectrum of a closed operator $H$ in a Hilbert space $\mathcal{H}$, the spectral projection $P$ associated with $\lambda$ is defined by

$$P\phi := \frac{1}{2\pi i} \int_{\gamma} (z - H)^{-1}\phi \, dz$$