VARIATION AND RIGIDITY OF QUASI-LOCAL MASS

SIYUAN LU AND PENGZI MIAO

ABSTRACT. Inspired by the work of Chen-Zhang [5], we derive an evolution formula for the Wang-Yau quasi-local energy in reference to a static space, introduced by Chen-Wang-Wang-Yau [4]. If the reference static space represents a mass minimizing, static extension of the initial surface Σ, we observe that the derivative of the Wang-Yau quasi-local energy is equal to the derivative of the Bartnik quasi-local mass at Σ.

Combining the evolution formula for the quasi-local energy with a localized Penrose inequality proved in [10], we prove a rigidity theorem for compact 3-manifolds with nonnegative scalar curvature, with boundary. This rigidity theorem in turn gives a characterization of the equality case of the localized Penrose inequality in 3-dimension.

1. Introduction

The purpose in this paper is twofold. We derive a derivative formula for the integral

\[ \int_{\Sigma_t} N (H - \bar{H}) \, d\sigma \]

along a family of hypersurfaces \( \{\Sigma_t\} \) evolving in a Riemannian manifold \((M, g)\) with an assumption that \( \Sigma_t \) can be isometrically embedded in a static space \((N, \bar{g})\) as a comparison hypersurface \( \bar{\Sigma}_t \). Here \( H, \bar{H} \) are the mean curvature of \( \Sigma_t \), \( \bar{\Sigma}_t \) in \((M, g), (N, \bar{g})\), respectively, and \( N \) is the static potential on \((N, \bar{g})\). When \( \{\Sigma_t\} \) is a family of closed 2-surfaces in a 3-manifold \((M, g)\), integral (1.1) represents the Wang-Yau quasi-local energy in reference to the static space \((N, \bar{g})\), introduced by Chen-Wang-Wang-Yau [4]. In this case, if \((N, \bar{g})\) represents a mass minimizing, static extension of the initial surface \( \Sigma_0 \), we find that the derivative of the quasi-local energy agrees with the derivative of the Bartnik quasi-local mass at \( \Sigma_0 \) (see (2.8) in Section 2).

We also apply the derivative formula of (1.1) to prove a rigidity theorem for compact 3-manifolds with nonnegative scalar curvature, with boundary. Precisely, we have

**Theorem 1.1.** Let \((\Omega, \bar{g})\) be a compact, connected, orientable, 3-dimensional Riemannian manifold with nonnegative scalar curvature, with boundary \( \partial \Omega \). Suppose \( \partial \Omega \) is the disjoint union of two pieces, \( \Sigma_o \) and \( \Sigma_H \), where

(i) \( \Sigma_o \) has positive mean curvature \( H \); and
(ii) \( \Sigma_H \) is a minimal hypersurface (with one or more components) and there are no other closed minimal hypersurfaces in \((\Omega, \bar{g})\).

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Let $\mathbb{M}^3_m$ be a 3-dimensional spatial Schwarzschild manifold with mass $m > 0$ outside the horizon. Suppose $\Sigma_O$ is isometric to a convex surface $\Sigma \subset \mathbb{M}^3_m$ which encloses a domain $\Omega_m$ with the horizon $\partial \mathbb{M}^3_m$. Suppose $\overline{\text{Ric}}(\nu, \nu) \leq 0$ on $\Sigma$, where $\overline{\text{Ric}}$ is the Ricci curvature of the Schwarzschild metric $\bar{g}$ on $\mathbb{M}^3_m$ and $\nu$ is the outward unit normal to $\Sigma$. Let $H_m$ be the mean curvature of $\Sigma$ in $\mathbb{M}^3_m$ and $|\Sigma_H|$ be the area of $\Sigma_H$ in $(\Omega, \bar{g})$.

If $H = H_m$ and $\sqrt{\frac{|\Sigma_H|}{16\pi}} = m$, then $(\Omega, \bar{g})$ is isometric to $(\Omega_m, \bar{g})$.

Theorem 1.1 gives a characterization of the equality case of a localized Penrose inequality proved in [10].

**Theorem 1.2 ([10])**. Let $(\Omega, \bar{g})$ be a compact, connected, orientable, 3-dimensional Riemannian manifold with nonnegative scalar curvature, with boundary $\partial \Omega$. Suppose $\partial \Omega$ is the disjoint union of two pieces, $\Sigma_O$ and $\Sigma_H$, where

(i) $\Sigma_O$ has positive mean curvature $H$; and

(ii) $\Sigma_H$, if nonempty, is a minimal hypersurface (with one or more components) and there are no other closed minimal hypersurfaces in $(\Omega, \bar{g})$.

Let $\mathbb{M}^3_m$ be a 3-dimensional spatial Schwarzschild manifold with mass $m > 0$ outside the horizon. Suppose $\Sigma_O$ is isometric to a convex surface $\Sigma \subset \mathbb{M}^3_m$ which encloses a domain $\Omega_m$ with the horizon $\partial \mathbb{M}^3_m$. Suppose $\overline{\text{Ric}}(\nu, \nu) \leq 0$ on $\Sigma$, where $\overline{\text{Ric}}$ is the Ricci curvature of the Schwarzschild metric $\bar{g}$ on $\mathbb{M}^3_m$ and $\nu$ is the outward unit normal to $\Sigma$. Then

\begin{equation}
 m + \frac{1}{8\pi} \int_{\Sigma} N(H_m - H) \, d\sigma \geq \sqrt{\frac{|\Sigma_H|}{16\pi}}.
\end{equation}

Here $N$ is the static potential on $\mathbb{M}^3_m$, $H_m$ is the mean curvature of $\Sigma$ in $\mathbb{M}^3_m$, and $|\Sigma_H|$ is the area of $\Sigma_H$ in $(\Omega, \bar{g})$. Furthermore, equality in (1.2) holds if and only if

\begin{equation}
 H = H_m, \quad \sqrt{\frac{|\Sigma_H|}{16\pi}} = m.
\end{equation}

By Theorems 1.1 and (1.3), we have the following rigidity statement concerning the equality case of (1.2).

**Theorem 1.3.** Equality in (1.2) in Theorem 1.2 holds if and only if $(\Omega, \bar{g})$ is isometric to $(\Omega_m, \bar{g})$.

Our motivation to consider the evolution of (1.1) and the proof of Theorem 1.1 are inspired by a recent paper of Chen and Zhang [5]. In [5], Chen-Zhang proved the global rigidity of a convex surface $\Sigma$ with $\overline{\text{Ric}}(\nu, \nu) \leq 0$ among all isometric surfaces $\Sigma'$ in $\mathbb{M}^3_m$ having the same mean curvature and enclosing the horizon. As a key step in their proof, they computed the first variation of the quasi-local energy of $\Sigma'$ in reference to $\mathbb{M}^3_m$. Such a variational consideration is made possible by the openness result of solutions to the isometric embedding problem into warped product space, which is due to Li and Wang [8]. Combining the variation formula with inequality (1.2), Chen-Zhang established the rigidity of $\Sigma$ in $\mathbb{M}^3_m$.

This paper may be viewed as a further application of the method of Chen-Zhang. In Section 2, we compute the derivative of (1.1) (see Formula 2.1) and relate it to the
derivative of the Bartnik quasi-local mass. In Section 3 we prove Theorem 1.1 by applying Formula 2.1 and Theorem 1.2. In Section 4 we discuss the implication of (2.8) on the relation between the Bartnik mass and the Wang-Yau quasi-local energy.

2. Evolution of quasi-local mass

In this section we derive a formula that is inspired by [5, Lemma 2]. First we fix some notations. Let \((M, g)\) be an \((n + 1)\)-dimensional Riemannian manifold and \(\Sigma\) be an \(n\)-dimensional closed manifold. Consider a family of embedded hypersurfaces \(\{\Sigma_t\}\) evolving in \((M, g)\) according to

\[
F : \Sigma \times I \rightarrow M, \quad \frac{\partial F}{\partial t} = \eta \nu.
\]

Here \(F\) is a smooth map, \(I\) is some open interval containing 0, \(\Sigma_t = F_t(\Sigma)\) with \(F_t(\cdot) = F(\cdot, t)\), \(\nu\) is a chosen unit normal to \(\Sigma_t = F_t(\Sigma)\), and \(\eta\) denotes the speed of the evolution of \(\{\Sigma_t\}\).

Let \((N, \bar{g})\) denote an \((n + 1)\)-dimensional static Riemannian manifold. Here \((N, \bar{g})\) is called static (cf. [6]) if there exists a nontrivial function \(N\) such that

\[
(\bar{\Delta} N) \bar{g} - D^2 N + N \bar{Ric} = 0,
\]

where \(\bar{Ric}\) is the Ricci curvature of \((N, \bar{g})\), \(D^2 N\) is the Hessian of \(N\) and \(\bar{\Delta}\) is the Laplacian of \(N\). The function \(N\) is called a static potential on \((N, \bar{g})\).

In what follows, we consider another family of embedded hypersurfaces \(\{\bar{\Sigma}_t\}\) evolving in \((N, \bar{g})\) according to

\[
\bar{F} : \Sigma \times I \rightarrow N
\]

with \(\bar{\Sigma}_t = \bar{F}_t(\Sigma)\) and \(\bar{F}_t(\cdot) = \bar{F}(\cdot, t)\). We will make an important assumption:

\[
(\bar{N}) \bar{g} = \bar{F}_t^*(g), \quad \forall \ t \in I.
\]

In particular, this means that \(\bar{\Sigma}_t\) is assumed to be isometric to \(\Sigma_t\) for each \(t\).

Remark 2.1. We emphasize that, when \(n = 2\), given any \(\{\Sigma_t\}\) in \((M, g)\), if \(\Sigma_0\) admits an isometric embedding into \((N, \bar{g})\), there exists a family of \(\{\bar{\Sigma}_t\}\) in \((N, \bar{g})\) satisfying condition (2.2). This is guaranteed by the openness result of solutions to the isometric embedding problem, which is due to Li and Wang [8, 9].

We will compute

\[
\frac{d}{dt} \int_{\Sigma_t} N_t(\bar{H}_t - H_t) \, d\sigma_t,
\]

where \(N_t = \bar{F}_t^*(N)\) is the pull back of the static potential \(N\) on \((N, \bar{g})\); \(H_t, \bar{H}_t\) are the mean curvature of \(\Sigma_t, \bar{\Sigma}_t\) in \((M, g), (N, \bar{g})\), respectively; and \(d\sigma_t\) is the area element of the pull back metric \(\gamma_t = \bar{F}_t^*(\bar{g}) = F_t^*(g)\). For simplicity, the lower index \(t\) is omitted below.
Formula 2.1. Given \( \{\Sigma_t\}, \{\bar{\Sigma}_t\} \) evolving in \((M, g), (N, \bar{g})\) as specified above,

\[
\frac{d}{dt} \int_{\Sigma} N(\bar{H} - H) \, d\sigma
\]

(2.3)

\[
= \int_{\Sigma} N \left[ \frac{1}{2} |A - \bar{A}|^2 - \frac{1}{2} |H - \bar{H}|^2 + \frac{1}{2} (R - \bar{R}) \right] \eta \, d\sigma
+ \int_{\Sigma} \left[ (f - \eta) \frac{\partial N}{\partial \bar{\nu}} + \langle \nabla N, Y \rangle \right] (\bar{H} - H) \, d\sigma.
\]

Here \( A, \bar{A} \) are the second fundamental forms of \( \Sigma_t, \bar{\Sigma}_t \) in \((M, g), (N, \bar{g})\), respectively; \( R, \bar{R} \) are the scalar curvature of \((M, g), (N, \bar{g})\), respectively; \( f \) and \( Y \) are the lapse and the shift associated to \( \partial \bar{\nu} / \partial t \), i.e. \( \partial \bar{F} / \partial t = f \bar{v} + Y \), where \( f \) is a function and \( Y \) is tangential to \( \bar{\Sigma}_t \); and \( \nabla \) denotes the gradient on \((\bar{\Sigma}_t, \gamma)\).

Remark 2.2. Suppose \((M, g)\) and \((N, \bar{g})\) both are \( \mathbb{M}^3 \) and suppose \( H = \bar{H} \) at \( t = 0 \), (2.3) becomes

\[
\left|_{t=0} \frac{d}{dt} \int_{\Sigma} N(\bar{H} - H) \, d\sigma = \frac{1}{2} \int_{\Sigma} N |A - \bar{A}|^2 \eta \, d\sigma.
\]

This is the formula in [5, Lemma 2].

Remark 2.3. If \( Y = 0 \) and \( R = \bar{R} \), (2.3) reduces to

\[
\left|_{t=0} \frac{d}{dt} \int_{\Sigma} N(\bar{H} - H) \, d\sigma = \frac{1}{2} \int_{\Sigma} N |A - \bar{A}|^2 \eta \, d\sigma.
\]

This is the formula in [10, Proposition 2.2].

We now comment on the physical meaning of (2.3). Suppose \( n = 2 \). In [4], Chen, Wang, Wang and Yau introduced a notion of quasi-local energy of a 2-surface \( \Sigma \) in reference to the static spacetime \( \mathcal{S} = (\mathbb{R}^1 \times N, -N^2 dt^2 + \bar{g}) \). The notion is a generalization of the Wang-Yau quasi-local energy [15, 16] for which the reference \( \mathcal{S} \) is the Minkowski spacetime \( \mathbb{R}^{3,1} \). For this reason, we denote this quasi-local energy of \( \Sigma \) by \( E_{wY}^S (\Sigma, \mathcal{S}, X) \), where \( X : \Sigma \to \mathcal{S} \) is an associated isometric embedding. When \( \Sigma \) lies in a time-symmetric slice in the physical spacetime, one may focus on the case \( X \) embeds \( \Sigma \) into a constant \( t \)-slice of \( \mathcal{S} \), i.e. \( X : \Sigma \to (N, \bar{g}) \). In this case, setting \( \tau = 0 \) in equation (2.10) in [4], one has

\[
E_{wY}^S (\Sigma, \mathcal{S}, X) = \frac{1}{8\pi} \int_{\Sigma} N(\bar{H} - H) \, d\sigma.
\]

(2.6)

Therefore, up to a multiplicative constant, (2.3) is a formula of

\[
\frac{d}{dt} E_{wY}^S (\Sigma_t, \mathcal{S}, X_t),
\]
where $X_t = \tilde{F}_t \circ F_t^{-1}$ is the isometric embedding of $\Sigma_t$ in $(N, \bar{g})$ as $\bar{\Sigma}_t$.

Next, we tie (2.3) with the evolution formula of the Bartnik quasi-local mass $m_B(\cdot)$. We defer the detailed definition of the Bartnik mass $m_B(\cdot)$ to Section 4. For the moment, we recall the following evolution formula of $m_B(\cdot)$ derived in [12, Theorem 3.1] under a stringent condition.

**Formula 2.2 ([12]).** Suppose $\Sigma_t$ has a mass minimizing, static extension $(M_t^s, g_t^s)$ such that $\{(M_t^s, g_t^s)\}$ depends smoothly on $t$. One has

$$\frac{d}{dt}|_{t=0} m_B(\Sigma_t) = \frac{1}{16\pi} \int \Sigma N \left( |A - \bar{A}|^2 + R \right) \eta \, d\sigma. \tag{2.7}$$

To relate (2.3) to (2.7), we assume that $(N, \bar{g})$ represents a mass minimizing, static extension of the surface $\Sigma_0 \subset (M, g)$. Then, by assumption, $H = \bar{H}$ at $t = 0$. It follows from (2.3), (2.6) and (2.7) that

$$\frac{d}{dt}|_{t=0} E_{SWY}(\Sigma_t, \mathcal{S}, X_t) = \frac{1}{16\pi} \int \Sigma N \left[ |A - \bar{A}|^2 + (R - \bar{R}) \right] \eta \, d\sigma = \frac{d}{dt}|_{t=0} m_B(\Sigma_t). \tag{2.8}$$

We will reflect more on this relation in Section 4.

In the remainder of this section, we give a proof of **Formula 2.1**.

**Proof of Formula 2.1.** By the evolution equations $\frac{\partial F}{\partial t} = \eta \nu$ and $\frac{\partial \bar{F}}{\partial t} = f \bar{\nu} + Y$, we have

$$\gamma' = 2\eta A, \quad \partial_t d\sigma = \eta H \, d\sigma \tag{2.9}$$

and

$$\gamma' = 2f \bar{A} + L_Y \gamma, \quad \partial_t d\sigma = (f \bar{H} + \text{div}Y) \, d\sigma, \tag{2.10}$$

where $\text{div}Y$ is the divergence of $Y$ on $(\Sigma, \gamma)$. Thus,

$$2\eta A = 2f \bar{A} + L_Y \gamma, \quad \eta H = f \bar{H} + \text{div}Y. \tag{2.11}$$

We first compute

$$\frac{d}{dt} \int \Sigma N \bar{H} \, d\sigma = \int \Sigma (N' \bar{H} + N \bar{H}') \, d\sigma + N \bar{H} \partial_t d\sigma. \tag{2.12}$$

Let $\bar{\nabla}$ denote the gradient on $(N, \bar{g})$. We have

$$N' = \langle \bar{\nabla} N, \frac{\partial \bar{F}}{\partial t} \rangle = \langle \bar{\nabla} N, f \bar{\nu} + Y \rangle = f \left( \frac{\partial N}{\partial \bar{\nu}} + \langle \nabla N, Y \rangle \right). \tag{2.13}$$

Hence,

$$\int \Sigma N' \bar{H} \, d\sigma = \int \Sigma \left( f \left( \frac{\partial N}{\partial \bar{\nu}} + \langle \nabla N, Y \rangle \right) \right) \bar{H} \, d\sigma. \tag{2.14}$$
Recall that
\begin{equation}
\bar{A}'_{\alpha\beta} = f \bar{A}_{\alpha\delta} \bar{A}^\delta_{\beta} + (L_Y \bar{A})_{\alpha\beta} - (\nabla^2 f)_{\alpha\beta} + f \langle \bar{R}(\bar{\nu}, \partial_\alpha) \bar{\nu}, \partial_\beta \rangle,
\end{equation}
where \( \nabla^2 \) denotes the Hessian on \((\Sigma, \gamma)\). Hence,
\begin{equation}
\bar{H}' = \langle \gamma', \bar{A} \rangle + f |\bar{A}|^2 + \langle \gamma, L_Y \bar{A} \rangle - \Delta f - \bar{Ric}(\bar{\nu}, \bar{\nu}) f. \tag{2.16}
\end{equation}
By (2.10),
\begin{equation}
\langle \gamma', \bar{A} \rangle = 2 f \bar{A} + \langle L_Y \gamma, \bar{A} \rangle = 2 f |\bar{A}|^2 + \langle L_Y \gamma, \bar{A} \rangle. \tag{2.17}
\end{equation}
Thus,
\begin{equation}
\bar{H}' = \langle L_Y \gamma, \bar{A} \rangle + \langle \gamma, L_Y \bar{A} \rangle - \Delta f - f |\bar{A}|^2 - \bar{Ric}(\bar{\nu}, \bar{\nu}) f. \tag{2.18}
\end{equation}
One checks that
\begin{equation}
\langle \gamma', \bar{A} \rangle = \langle Y, \nabla \bar{H} \rangle. \tag{2.19}
\end{equation}
Hence,
\begin{equation}
\bar{H}' = -\Delta f - f |\bar{A}|^2 - \bar{Ric}(\bar{\nu}, \bar{\nu}) f + \langle Y, \nabla \bar{H} \rangle. \tag{2.20}
\end{equation}
Here we have used
\begin{equation}
\Delta N + \bar{Ric}(\bar{\nu}, \bar{\nu}) N = -\bar{H} \frac{\partial N}{\partial \nu}, \tag{2.21}
\end{equation}
which follows from the static equation (2.1).
By (2.14) and (2.20),
\begin{equation}
\int_\Sigma N' \bar{H} + N \bar{H}' \ d\sigma = \int_\Sigma \left( -\Delta N - \bar{Ric}(\bar{\nu}, \bar{\nu}) N \right) f + N \left[ -f |\bar{A}|^2 + \langle Y, \nabla \bar{H} \rangle \right] \ d\sigma \tag{2.22}
\end{equation}
On the other hand, by (2.10),
\begin{equation}
\int_\Sigma N \bar{H} \partial_t \ d\sigma = \int_\Sigma N \bar{H}(f \bar{H} + \text{div}Y) \ d\sigma. \tag{2.23}
\end{equation}
Therefore, it follows from (2.21) and (2.22) that
\begin{equation}
\frac{d}{dt} \int_\Sigma N \bar{H} \ d\sigma = \int_\Sigma 2f \frac{\partial N}{\partial \nu} \bar{H} + N f (\bar{H}^2 - |\bar{A}|^2) \ d\sigma. \tag{2.24}
\end{equation}
To proceed, we note that by (2.10),
\[(2.24) \quad 2f(\bar{H}^2 - |\bar{A}|^2) = \langle \bar{H} \gamma - \bar{A}, 2f \bar{A} \rangle = \langle \bar{H} \gamma - \bar{A}, \gamma' \rangle - \langle \bar{H} \gamma - \bar{A}, L_Y \gamma \rangle.\]

Thus,
\[(2.25) \quad 2\int_{\Sigma} NF(\bar{H}^2 - |\bar{A}|^2) d\sigma = \int_{\Sigma} N(\bar{H} \gamma - \bar{A}, \gamma') - N(\bar{H} \gamma - \bar{A}, L_Y \gamma) d\sigma.\]

Integrating by parts, we have
\[(2.26) \quad \int_{\Sigma} N\langle \bar{H} \gamma - \bar{A}, L_Y \gamma \rangle d\sigma = -2 \int_{\Sigma} (\bar{H} \gamma - \bar{A})(\nabla N, Y) - 2 \int_{\Sigma} N(d\bar{H} - \text{div} \bar{A})(Y) d\sigma.\]

By the Codazzi equation and the static equation,
\[(2.27) \quad N(\text{div} \bar{A} - d\bar{H})(Y) = N\bar{Ric}(Y, \bar{\nu}) = \bar{D}^2 N(Y, \bar{\nu}).\]

Here
\[\bar{D}^2 N(Y, \nu) = -\bar{A}(\nabla N, Y) + Y \left( \frac{\partial N}{\partial \bar{\nu}} \right).\]

Hence,
\[(2.28) \quad \int_{\Sigma} N(\bar{H} \gamma - \bar{A}, L_Y \gamma) d\sigma = \int_{\Sigma} -2\bar{H} \langle \nabla N, Y \rangle + 2Y \left( \frac{\partial N}{\partial \bar{\nu}} \right) d\sigma.\]

Therefore, (2.23) can be rewritten as
\[(2.29) \quad \frac{d}{dt} \int_{\Sigma} NH d\sigma = \int_{\Sigma} f \frac{\partial N}{\partial \bar{\nu}} H + \bar{H} \langle \nabla N, Y \rangle - Y \left( \frac{\partial N}{\partial \bar{\nu}} \right) + \frac{1}{2} N(\bar{H} \gamma - \bar{A}, \gamma') d\sigma.\]

We now turn to the term \(\int_{\Sigma} NH d\sigma\). We have
\[(2.30) \quad \frac{d}{dt} \int_{\Sigma} NH d\sigma = \int_{\Sigma} N'H + NH' + NH \eta H d\sigma\]
\[= \int_{\Sigma} f \left( \frac{\partial N}{\partial \bar{\nu}} + \langle \nabla N, Y \rangle \right) H + N \left[ -\Delta \eta - (|A|^2 + \text{Ric}(\nu, \nu))\eta \right] + NH^2 \eta d\sigma.\]

Here
\[(2.31) \quad -\int_{\Sigma} N\Delta \eta d\sigma = -\int_{\Sigma} (\Delta N)\eta d\sigma = \int_{\Sigma} \left( H \frac{\partial N}{\partial \bar{\nu}} + \text{Ric}(\bar{\nu}, \bar{\nu})N \right) \eta.\]

Therefore,
\[(2.32) \quad \frac{d}{dt} \int_{\Sigma} NH d\sigma = \int_{\Sigma} f \frac{\partial N}{\partial \bar{\nu}} H + \langle \nabla N, Y \rangle H + H \frac{\partial N}{\partial \bar{\nu}} \eta + N \left[ \text{Ric}(\bar{\nu}, \bar{\nu}) - (|A|^2 + \text{Ric}(\nu, \nu)) + H^2 \right] \eta d\sigma.\]
We group the zero order terms of $N$ in $\frac{d}{dt} \int_{\Sigma} N(\bar{H} - H) \, d\sigma$ first. Using $\gamma' = 2\eta A$, we have
\[ (2.33) \quad \frac{1}{2} N(\bar{H} \gamma - \bar{A}, \gamma') = N(\bar{H} \gamma - \bar{A}, A) \eta. \]

Thus, omitting the terms $\eta$ and $N$, using the Gauss equation, we have
\[ (2.34) \quad \langle \bar{H} \gamma - \bar{A}, A \rangle + \frac{1}{2} (R - \bar{R}) - \frac{1}{2} (H^2 - |A|^2) - \frac{1}{2} (\bar{H}^2 - |\bar{A}|^2) \]
\[ = \frac{1}{2} |A - \bar{A}|^2 - \frac{1}{2} |H - \bar{H}|^2 + \frac{1}{2} (R - \bar{R}). \]

Integrating by part and using the fact $\eta H = f \bar{H} + \text{div} Y$, we conclude
\[
\frac{d}{dt} \int_{\Sigma} N(\bar{H} - H) \, d\sigma
= \int_{\Sigma} N \left[ \frac{1}{2} |A - \bar{A}|^2 - \frac{1}{2} |H - \bar{H}|^2 + \frac{1}{2} (R - \bar{R}) \right] \eta \, d\sigma
+ \int_{\Sigma} \left( 2f \bar{H} - f H - \eta \bar{H} + \text{div} Y \right) \frac{\partial N}{\partial \bar{\nu}} + (\bar{H} - H) \langle \nabla N, Y \rangle \, d\sigma
\]
\[ (2.35) \quad = \int_{\Sigma} N \left[ \frac{1}{2} |A - \bar{A}|^2 - \frac{1}{2} |H - \bar{H}|^2 + \frac{1}{2} (R - \bar{R}) \right] \eta \, d\sigma
+ \int_{\Sigma} \left[ (f - \eta) \frac{\partial N}{\partial \bar{\nu}} + \langle \nabla N, Y \rangle \right] (\bar{H} - H) \, d\sigma. \]

\[ \square \]

3. Equality case of the localized Penrose inequality

In this section, we apply Formula 2.1, the openness result of the isometric embedding problem [8], and Theorem 1.2 to prove Theorem 1.1.

Proof of Theorem 3.1. Let $A, \bar{A}$ be the second fundamental form of $\Sigma_0$, $\Sigma$ in $(\Omega, \bar{g})$, $\mathbb{M}^3_m$, respectively. Viewing $\bar{A}$ as a tensor on $\Sigma_0$ via the surface isometry, we want to show $A = \bar{A}$.

In $(\Omega, \bar{g})$, consider a smooth family of 2-surfaces $\{\Sigma_t\}_{-\epsilon \leq t \leq 0}$ such that $\Sigma_0 = \Sigma_0$ and $\Sigma_t$ is $|t|$-distance away from $\Sigma_0$. We can parametrize $\{\Sigma_t\}$ so that, as $t$ increases, $\Sigma_t$ evolves in a direction normal to $\Sigma_t$ and has constant unit speed. Applying the openness result of the isometric embedding problem in [8], we obtain a smooth family of 2-surfaces $\{\Sigma_t\}_{-\epsilon \leq t \leq 0}$ in $\mathbb{M}^3_m$ so that $\Sigma_0 = \Sigma$ and condition (2.2) is satisfied by $\{\Sigma_t\}$ and $\{\Sigma_t\}$. By (2.3) and the assumption $H = H_m$, we have
\[ (3.1) \quad \frac{d}{dt} |_{t=0} \int_{\Sigma_t} N(\bar{H} - H) \, d\sigma = \frac{1}{2} \int_{\Sigma_0} N(|A - \bar{A}|^2 + R) \, d\sigma. \]
Here $N$ is the static potential on $M^3_m$, which is positive away from the horizon, and $R$ is the scalar curvature of $(\Omega, \tilde{g})$.

Suppose $A \neq \tilde{A}$. Then, by (3.1) and the assumption $R \geq 0$,

$$\frac{d}{dt} \left|_{t=0} \right. \int_{\Sigma_t} N(\tilde{H} - H) \, d\sigma > 0.$$  

Thus, for small $t < 0$,

$$\int_{\Sigma_t} N(\tilde{H} - H) \, d\sigma < 0.$$  

We claim (3.3) contradicts Theorem 1.2. To see this, we can first consider the case $\text{Ric}(\nu, \nu) < 0$ on $\Sigma$. By choosing $\epsilon$ small, we may assume $\text{Ric}(\nu, \nu) < 0$ on each $\bar{\Sigma}_t$. Hence, we can apply Theorem 1.2 to the region in $\Omega$ enclosed by $\Sigma_t$ and $\Sigma_H$. It follows from (1.2) and the assumption $m = \sqrt{\frac{\text{vol}(\Sigma_H)}{16\pi}}$ that

$$\int_{\Sigma_t} N(\tilde{H} - H) \, d\sigma \geq 0.$$  

This is a contradiction to (3.3).

To include the case $\text{Ric}(\nu, \nu) \leq 0$ on $\Sigma$, we point out that this assumption was imposed in [10] only to guarantee that the flow in $M^3_m$, which starts from $\Sigma$ and satisfies equation (4.2) in [10], has the property that its leaves have positive scalar curvature (see Lemma 3.8 in [10]). Now, if $\Sigma$ is slightly perturbed to a nearby surface $\Sigma'$ in $M^3_m$, though $\Sigma'$ may not satisfy $\text{Ric}(\nu, \nu) \leq 0$, the flow to (4.2) in [10] starting from $\Sigma'$ remains to have such a property. (More precisely, this follows from estimates in Lemmas 3.6, 3.7 and 3.11 of [10].) Therefore, for small $t < 0$, we can still apply Theorem 1.2 to conclude (3.4), which contradicts (3.3).

Thus we have $A = \tilde{A}$. For the same reason, we also know $R = 0$ along $\Sigma_O$ in $(\Omega, \tilde{g})$.

Next, we consider the manifold $(\tilde{M}, \tilde{g})$ obtained by gluing $(\Omega, \tilde{g})$ and $(M^3_m \setminus \Omega_m, \tilde{g})$ along $\Sigma_O$ that is identified with $\Sigma$. Since $A = \tilde{A}$, the metric $\tilde{g}$ on $\tilde{M}$ is $C^{1,1}$ across $\Sigma_O$ and is smooth up to $\Sigma_O$ from its both sides in $\tilde{M}$. To finish the proof, we check that the rigidity statement of the Riemannian Penrose inequality holds on this $(\tilde{M}, \tilde{g})$.

We apply the conformal flow used by Bray [2] in his proof of the Riemannian Penrose inequality. Since $\tilde{g}$ is $C^{1,1}$, equations (13) - (16) in [2] which define the flow hold in the classical sense when $g_0$ is replaced by $\tilde{g}$. Existence of this flow with initial condition $\tilde{g}$ follows from Section 4 in [2]. The difference is that, along the flow which we denote by $\{\tilde{g}(t)\}$, the outer minimizing horizon $\Sigma(t)$ is $C^{2,\alpha}$ and the green function in Theorems 8 and 9 in [2] is $C^{2,\alpha}$, for any $0 < \alpha < 1$. These regularities are sufficient to show Theorem 6 in [2] holds, i.e. the area of $\Sigma(t)$ stays the same; and the results on the mass and the capacity in Theorems 8 and 9 in [2] remain valid. Moreover, at $t = 0$, by the proof of Theorem 10 in [2], i.e. equation (113), we have

$$\frac{d}{dt} \left|_{t=0} \right. m(t) = \mathcal{E}(\Sigma_H, \tilde{g}) - 2m \leq 0,$$  

where

$$\int_{\Sigma_H} N(\tilde{H} - H) \, d\sigma = \mathcal{E}(\Sigma_H, \tilde{g}) - 2m \leq 0,$$  

Here $N$ is the static potential on $M^3_m$, which is positive away from the horizon, and $R$ is the scalar curvature of $(\Omega, \tilde{g})$. 

Suppose $A \neq \tilde{A}$. Then, by (3.1) and the assumption $R \geq 0$,
where $\mathcal{E}(\Sigma_\mu, \hat{g})$ is the capacity of $\Sigma_\mu$ in $(\hat{M}, \hat{g})$ and the inequality in (3.5) is given by Theorem 9 in [2].

Now, if $\frac{d}{dt} m(t)|_{t=0} < 0$, then for $t$ small, we would have

$$m(t) < m = \sqrt{\frac{\mu}{16\pi}} = \sqrt{\frac{|\Sigma(t)|}{16\pi}}.$$  

where $m(t)$ is the mass of $\hat{g}(t)$. But (3.5) violates the Riemannian Penrose inequality (for metrics possibly with corner along a hypersurface, cf. [11]). Thus, we must have

$$\frac{d}{dt} m(t)|_{t=0} = \mathcal{E}(\Sigma_\mu, \hat{g}) - 2m = 0.$$  

Since Theorem 9 in [2] holds on $(\hat{M}, \hat{g})$, by its rigidity statement we conclude that $(\hat{M}, \hat{g})$ is isometric to $\mathbb{M}_m^3$. □

Remark 3.1. As mentioned in [10, Remark 5.1], Theorem 1.1 would also follow if one could establish the rigidity statement for the Riemannian Penrose inequality on manifolds with corners along a hypersurface (cf. [11, Proposition 3.1]). Results along this direction can be found in [14].

4. Bartnik mass and Wang-Yau quasi-local energy

In (2.8) of Section 2 we have observed that, if $(N, \bar{g})$ represents a mass minimizing, static extension of $\Sigma_0 \subset (M, g)$, then

$$\frac{d}{dt}|_{t=0} \mathcal{E}_{SWY}^S(\Sigma_t, S, X_t) = \frac{d}{dt}|_{t=0} m_B(\Sigma_t).$$  

This observation was based on (2.7), which requires a stringent assumption that mass minimizing, static extensions of $\{\Sigma_t\}$ exist and depend smoothly on $t$. In this section, we will give a rigorous proof that (2.7) is true whenever the Bartnik data of $\Sigma_0$ corresponds to that of a surface in a spatial Schwarzschild manifold. We will also discuss the implication, suggested by (4.1), on the relation between the Bartnik mass and the Wang-Yau quasi-local energy.

First, we recall the definition of $m_B(\cdot)$. Given a closed 2-surface $\Sigma$, which bounds a bounded domain, in a 3-manifold $(M, g)$ with nonnegative scalar curvature, $m_B(\Sigma)$ is given by

$$m_B(\Sigma) = \inf \left\{ m(\hat{g}) \mid (\hat{M}, \hat{g}) \text{ is an admissible extension of } \Sigma \right\}.$$  

Here $m(\hat{g})$ is the mass of $(\hat{M}, \hat{g})$, which is an asymptotically flat 3-manifold with nonnegative scalar curvature, with boundary $\partial M$. $(\hat{M}, \hat{g})$ is called an admissible extension of $\Sigma$ if $\partial \hat{M}$ is isometric to $\Sigma$ and the mean curvature of $\partial \hat{M}$ equals the mean curvature $H$ of $\Sigma$. Moreover, it is assumed that $(\hat{M}, \hat{g})$ satisfies certain non-degeneracy condition that prevents $m(\hat{g})$ from becoming trivially small. For instance, one often assumes that $(\hat{M}, \hat{g})$ contains no closed minimal surfaces or $\partial M$ is outer minimizing in $(\hat{M}, \hat{g})$ (cf. [11, 2, 3, 7]).
Theorem 4.1. Let \( \Sigma \) be a 2-surface with positive mean curvature in a 3-manifold \((M, g)\) of nonnegative scalar curvature. Suppose \( \Sigma \) is isometric to a convex surface \( \bar{\Sigma} \) with \( \text{Ric}(\nu, \nu) \leq 0 \) in a spatial Schwarzschild manifold \((M^3_m, \bar{g})\) of mass \( m > 0 \). Suppose \( \bar{\Sigma} \) encloses a domain \( \Omega_m \) with the horizon of \((M^3_m, \bar{g})\).

(i) Let \( X : \Sigma \to (M^3_m, \bar{g}) \) be an isometric embedding such that \( X(\Sigma) = \bar{\Sigma} \). Let \( N \) be the static potential on \( M^3_m \) and let \( S_m \) denote the Schwarzschild spacetime, i.e. \( S_m = (\mathbb{R}^{1} \times \mathbb{R}^3_m, -N dt^2 + \bar{g}) \). Then

\[
m_\beta(\Sigma) \leq m + E^S_{\text{wY}}(\Sigma, S_m, X).
\]

Moreover, equality holds if and only if \( H = \bar{H} \) and \( m_\beta(\Sigma) = m \). Here \( H, \bar{H} \) are the mean curvature of \( \Sigma, \bar{\Sigma} \) in \((M, g), (M^3_m, \bar{g})\), respectively.

(ii) Suppose \( H = \bar{H} \). Let \( \{\Sigma_t\}_{|t|<\epsilon} \) be a smooth family of 2-surfaces evolving in \((M, g)\) according to \( \frac{\partial E}{\partial t} = \eta \nu \) and satisfying \( \Sigma_0 = \Sigma \). If \( m_\beta(\Sigma_t) \) is differentiable at \( t = 0 \), then

\[
\frac{d}{dt}|_{t=0} m_\beta(\Sigma_t) = \frac{1}{16\pi} \int_{\Sigma_0} N(|A - \bar{A}|^2 + R)\eta \, d\sigma.
\]

Here \( A, \bar{A} \) are the second fundamental form of \( \Sigma, \bar{\Sigma} \) in \((M, g), (M^3_m, \bar{g})\), respectively, and \( R \) is the scalar curvature of \((M, g)\).

Proof. Part (i) was proved in [10, Theorem 5.1]. To show part (ii), we first note that the assumption \( H = \bar{H} \) implies \( m_\beta(\Sigma) = m \). This is because, if \((\bar{M}, \bar{g})\) is any admissible extension of \( \Sigma \) by gluing \((\bar{M}, \bar{g})\) with \( \Omega_m \) along \( \bar{\Sigma} \) and applying the Riemannian Penrose inequality, one has \( m(\bar{g}) \geq m \). On the other hand, \( M^3_m \setminus \Omega_m \) is an admissible extension of \( \Sigma \). Hence, \( m_\beta(\Sigma) = m \).

Next, we proceed as in the proof of Theorem [11]. By the result of Li-Wang [8], for small \( \epsilon \), there exists a smooth family of embeddings \( \{X_t\}_{|t|<\epsilon} \) which isometrically embeds \( \Sigma_t \) in \((M^3_m, \bar{g})\) such that \( X_0 = X \). By (i), for each small \( t \), we have

\[
m_\beta(\Sigma_t) \leq m + E^S_{\text{wY}}(\Sigma_t, S_m, X_t).
\]

Note that \( m_\beta(\Sigma_0) = m \) and \( E^S_{\text{wY}}(\Sigma_0, S_m, X_0) = 0 \). Hence, it follows from (4.3) that

\[
\frac{d}{dt}|_{t=0} m_\beta(\Sigma_t) = \frac{d}{dt}|_{t=0} E^S_{\text{wY}}(\Sigma_t, S_m, X_t)
\]

\[
= \frac{1}{16\pi} \int_{\Sigma_0} N(|A - \bar{A}|^2 + R)\eta \, d\sigma.
\]

Here in the last step we have used (2.3). \( \square \)

We propose a conjecture that is inspired by Theorem [11].

Conjecture 4.1. Let \( \Sigma \) be a 2-surface, bounding some finite domain, in a 3-manifold \((M, g)\) of nonnegative scalar curvature. Let \((N, \bar{g})\) be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature such that \((N, \bar{g})\) is static outside a compact set \( K \). Suppose the static potential \( N \) on \( N \setminus K \) is positive. Let \( S \) be the static spacetime generated by \((N \setminus K, \bar{g})\), i.e. \( S = (\mathbb{R}^1 \times (N \setminus K), -N^2 dt^2 + \bar{g}) \). Suppose there exists an isometric embedding \( X : \Sigma \to (N, \bar{g}) \) such that \( \Sigma = X(\Sigma) \) encloses \( K \). Let \( H, \bar{H} \)
be the mean curvature of $\Sigma$, $\bar{\Sigma}$ in $(M, g)$, $(N, \bar{g})$, respectively. Then, under suitable conditions on $\Sigma$ and $\bar{\Sigma}$,

\[(4.5)\quad m_B(\Sigma) \leq m(\bar{g}) + E_{SWY}(\Sigma, S, X).\]

Moreover, equality holds if and only if $H = \bar{H}$ and $m_B(\Sigma) = m(\bar{g})$, in which case $(N, \bar{g})$, outside $\bar{\Sigma}$, is a mass minimizing, static extension of $\Sigma$.

By results in [13], Conjecture 4.1 is true when $(N, \bar{g})$ is $\mathbb{R}^3$. By Theorem 4.1 (i), Conjecture 4.1 is also true when $(N \setminus K, \bar{g})$ is an exterior region in $(M^3, \bar{g})$.

If Conjecture 4.1 is valid and if a mass minimizing, static extension of $\Sigma$ exists, then it would follow that

\[m_B(\Sigma) = \inf_{(N, \bar{g})} \left\{ \inf_X \left( m(\bar{g}) + E_{SWY}(\Sigma, S, X) \right) \right\}.\]

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Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA.
E-mail address: siyuan.lu@math.rutgers.edu

Department of Mathematics, University of Miami, Coral Gables, FL 33146, USA.
E-mail address: pengzim@math.miami.edu