A note on the time evolution of generalized coherent states

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Abstract

I consider the time evolution of generalized coherent states based on non-standard fiducial vectors, and show that only for a restricted class of fiducial vectors does the associated classical motion determine the quantum evolution of the states. I discuss some consequences of this for path integral representations.

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I. INTRODUCTION

Coherent states were originally introduced into physics by Schrödinger \[1\] in order to reconcile Heisenberg’s abstract solution of the quantum harmonic oscillator with the classical picture of a swinging pendulum. Schrödinger’s minimum-uncertainty wave-packets maintain their shape under the quantum time evolution and their “centre-of-mass” motion coincides with that of the pendulum. There are various generalizations of the original harmonic oscillator coherent states, but most of them exhibit the following classical-quantum correspondence:

- There is a family of states parameterized by some classically interpretable variables.
- For simple Hamiltonians, a state originally in this family remains in it.
- The parameters evolve according to the classical equations of motion.
- In addition to the parameter evolution, the states accumulate a phase equal to the exponential of the classical action.

The last property was pointed out by van Hove \[2\]. If all four of these features are present, the classical dynamics completely determines the quantum time evolution.

One such extension of the Schrödinger states is the class of “generalized coherent states” introduced by Perelomov \[3,4\]. Perelomov’s states are obtained by selecting a fiducial vector in some representation of a Lie group, and considering its orbit under the action of the group. For compact groups (and for some representations of non-compact groups) this construction guarantees the existence of a resolution of unity, and so suggests a path integral representation of the dynamics \[5,6\].

Usually the Perelomov coherent states are constructed by taking a greatest (or least) weight state as the fiducial vector. This generates minimum uncertainty states and, for Hamiltonians that are elements of the Lie algebra, the states evolve classically in the manner described above \[7\]. In recent articles, however, M. Matsumoto \[8,9\] has considered
coherent states built on arbitrary elements of the representation space. His motivation is that such states are useful in quantum optics and that novel phase interference effects may be measurable.

The purpose of this note is to point out what must be well known—although I have not been able to find an explicit statement in the literature—that these “general” generalized coherent states do not necessarily inherit the classical-quantum correspondence property. If we insist on the traditional picture, then we must restrict our choice of fiducial vectors. The essential requirement is that two distinct notions of the isotropy group associated with the fiducial state must, in the end, define the same group.

In the next section I first review some basic facts about Perelomov coherent states and define the two isotropy groups associated with a fiducial vector. In the third section I show that only if these two isotropy groups coincide does the classical motion provide complete information about the quantum dynamics. Finally I make some brief remarks on the implication of these issues for path integrals.

II. INFORMATIVE FAMILIES

Suppose $G$ is a compact Lie group and $|0\rangle$ a vector in an irreducible representation space, $V$, of $G$. Since $G$ is compact, we may assume that $V$ comes equipped with an inner product with respect to which the representation is unitary. For $g \in G$ an element of the group we will write its action on a vector $|v\rangle \in V$ by $|v\rangle \rightarrow g|v\rangle$. Thus we make no notational distinction between the elements of the group and the corresponding operators $g : V \rightarrow V$ in the representation. We will also write $|g\rangle = g|0\rangle$. Following Perelomov [4], we will call the set $\{|g\rangle\}$ a family of generalized coherent states. The starting $|0\rangle$ is the fiducial vector of the family.

If $d\mu$ denotes the invariant measure on the group then

$$B = \int d\mu |g\rangle \langle g|$$

\[ (2.1) \]
commutes with all matrices $g$, and so, by Shur's lemma, is proportional to the identity operator. Thus any $|0\rangle$ provides a resolution of the identity

$$I = \text{const.} \int d\mu|g\rangle\langle g|.$$  

(2.2)

This is probably the most important property of such sets of coherent states.

Although any state in the representation produces a family of states with resolution of the identity, the families are not all equivalent, nor are all equally useful. The most commonly seen are those built on highest (or lowest) weight vectors, such as the state $|j,j\rangle$ in the spin $j$ representation of $SU(2)$. As with the Schrödinger wave-packets, these families are composed of minimum uncertainty states, and have other nice mathematical properties. In particular they are naturally complex homogeneous manifolds with a Kähler structure [7]. Here, however, we are interested in a broader class of fiducial vectors.

For any $|0\rangle$ consider the following sets:

- $H_{|0\rangle} = \{ h \in G \mid h|0\rangle = (\text{phase})|0\rangle \} \quad \text{i.e. the set of elements of } G \text{ for which } |0\rangle \text{ is a common eigenvector.}$

- $H_0 = \{ h \in G \mid \langle h|\hat{\lambda}|h\rangle = \langle 0|\hat{\lambda}|0\rangle \forall \hat{\lambda} \in \text{Lie}(G) \} \quad \text{i.e. the set of elements of } G \text{ which stabilizes } f_0 = \langle 0|\ldots|0\rangle \text{ considered as an element of } (\text{Lie}(G))^*.$

Clearly both sets are subgroups and $H_{|0\rangle} \subseteq H_0$.

The first subgroup, $H_{|0\rangle}$, is the isotropy group of the family of coherent states. In other words, the physically distinct states in the family are in one-to-one correspondence with the quotient space $G/H_{|0\rangle}$. States in any particular coset differ only by an overall phase.

The second group, $H_0$, is the isotropy group of the linear functional $f_0 : \text{Lie}(G) \rightarrow \mathbb{C}$, where $f_0(\hat{\lambda}) = \langle 0|\hat{\lambda}|0\rangle$, under the co-adjoint action of $G$ i.e.

$$f_0(\hat{\lambda}) \rightarrow f_g(\hat{\lambda}) = f_0(g^{-1}\hat{\lambda}g).$$  

(2.3)

As we will see, the degrees of freedom of the associated classical system live in the co-adjoint orbit $G/H_0$.  

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Since we wish the classical and quantum evolutions to be related, it would be nice if the two coset spaces \(G/H_{(0)}\) and \(G/H_0\) were the same. Unfortunately the two subgroups \(H_{(0)}\) and \(H_0\) do not necessarily coincide.

As an example, consider one of the families discussed by Matsumoto \[9\]. We take as fiducial vector the state

\[
|0\rangle = \sqrt{\frac{2}{3}}|1,1\rangle + \sqrt{\frac{1}{3}}|1,-1\rangle
\] (2.4)

in the spin-1 representation of \(SU(2)\). We have

\[
\langle 0|\hat{J}_1|0\rangle = 0
\]
\[
\langle 0|\hat{J}_2|0\rangle = 0
\]
\[
\langle 0|\hat{J}_3|0\rangle = \frac{1}{3}.
\] (2.5)

The isotropy group \(H_0\) consists of rotations about the “3” axis. The group \(H_{(0)}\) must be a subgroup of this, but under such rotations

\[
\sqrt{\frac{2}{3}}|1,1\rangle + \sqrt{\frac{1}{3}}|1,-1\rangle \rightarrow e^{i\theta} \sqrt{\frac{2}{3}}|1,1\rangle + e^{-i\theta} \sqrt{\frac{1}{3}}|1,-1\rangle.
\] (2.6)

Thus \(H_0 = \{e^{i\theta}\hat{J}_3\}\), while \(H_{(0)}\) contains only the identity element.

Suppose now that \(H_0\) is equal to \(H_{(0)}\). In this case the cosets \(G/H_{(0)}\) and \(G/H_0\) coincide and consequently \(|g\rangle\) is determined (up to a phase) by the values of the expectations \(\langle g|\hat{\lambda}_i|g\rangle = \lambda_i\) as \(\hat{\lambda}_i\) ranges over the Lie algebra of \(G\). This latter property is a reasonable one to require, since then we can determine (up to a phase) the quantum state of the system from measurement of the these expectations. There may be a standard name for fiducial vectors for which \(H_0 = H_{(0)}\), but I am unaware of it. For want of a suitable word, I will call such vectors, and the resulting family of coherent states, informative.

**III. DYNAMICS**

If we restrict our attention to Hamiltonians that are elements of the Lie algebra, then a state that starts as a coherent state will remain one under the quantum time evolution.
To associate a classical dynamical process with the quantum one, let \( \hat{H} \in \text{Lie}(G) \) be the quantum Hamiltonian, \(|0\rangle\) be our selected fiducial vector, and define the action

\[
S = \int dt \left\{ i\langle 0|g^{-1}\dot{g}|0\rangle - \langle 0|g^{-1}\hat{H}g|0\rangle \right\}.
\]

(3.1)

The first term is the canonical (or Berry) phase which determines the symplectic structure on the phase space \([10–12]\). The second term serves as the classical Hamiltonian.

The variation of \( S \) is

\[
\delta S = \int dt \left\{ \langle 0|[ig^{-1}\dot{g} - g^{-1}\hat{H}g, g^{-1}\delta g]|0\rangle \right\}.
\]

(3.2)

The corresponding equation of motion for \( g \) is therefore

\[
g^{-1}\dot{g} = -ig^{-1}\hat{H}g + i\dot{\lambda}(t),
\]

(3.3)

where \( \dot{\lambda}(t) \) is any element of the Lie algebra obeying \( \langle 0|[i\dot{\lambda}(t), g^{-1}\delta g]|0\rangle = 0 \) for all \( g^{-1}\delta g \in \text{Lie}(G) \). This condition means that \( \dot{\lambda}(t) \) lies in the Lie algebra of \( H_0 \). Since \( g^{-1}\dot{g} \) is indeterminate up to an element of \( \text{Lie}(H_0) \), we must consider the classical trajectories as living in the co-adjoint orbit \( G/H_0 \), rather than in \( G \). In general, knowing the classical trajectory in this space is not enough to determine the evolution of the quantum state. For example, the evolution of the fiducial state \((2.4)\) with \( \hat{H} = \hat{J}_3 \) alters the relative phase of the two components as in \((2.6)\). This phase-shift is invisible in the classical motion since the expectation values of the Lie algebra generators do not change.

We can make further progress, however, if we take \(|0\rangle\) to be informative. This means \( \dot{\lambda}(t) \) is also in \( H_{|0\rangle} \), so

\[
\dot{\lambda}(t)|0\rangle = \lambda(t)|0\rangle
\]

(3.4)

for some (real) number \( \lambda(t) \).

Given this information we note that

\[
S = -\int \lambda(t)dt.
\]

(3.5)
Next we observe that the solution of the equation of motion for \( g(t) \) is

\[
g(t) = T \left\{ e^{-i \int_0^t \hat{H} dt} \right\} g(0) T \left\{ e^{i \int_0^t \hat{\lambda} dt} \right\},
\]

(3.6)

where \( T \) denotes anti-time-ordering. Therefore

\[
|g(t)\rangle = g(t)|0\rangle = T \left\{ e^{-i \int_0^t \hat{H} dt} \right\} g(0) e^{i \int_0^t \hat{\lambda} dt} |0\rangle.
\]

(3.7)

In other words

\[
T \left\{ e^{-i \int_0^t \hat{H} dt} \right\} |g(0)\rangle = e^{iS} |g(t)\rangle.
\]

(3.8)

The left hand side of this equation is the quantum time-evolved coherent state, while the right hand side is, up to a phase, the coherent state corresponding to the classically evolved variable \( g(t) \).

At first sight the action \( S \) appearing in the above expression is arbitrary. This is because in equation (3.3) \( \hat{\lambda} \) could have been any element of Lie \( (H_0) \). The ambiguity is removed however, when we select a specific representative, \( |g\rangle \), from each ray in \( G/H|0\rangle \). This is what is normally done when we define a family of coherent states. For informative states, we have therefore recovered the traditional picture dating back to van Hove [2]. This result, that the quantum evolution may be found by solving a purely classical problem, is essentially equivalent to the Wei-Norman disentangling procedure [13].

**IV. DISCUSSION**

Because any fiducial vector gives rise to a resolution of unity, we can use the coherent states constructed on it to write down an exact discrete-time path integral for any transition amplitude. From this, by taking a formal limit of infinitely many intermediate steps, we may “derive” a continuous-time path integral representation of the quantum dynamics. The action appearing in this path integral is (3.1), together with some boundary terms that serve to make the initial and final value problem well defined. We might reasonably expect the classical paths, those with stationary variations, to play an important role in evaluating this
path integral. Unfortunately we have seen that for the general fiducial vector these paths do not capture the full quantum dynamics. We must expect, therefore, substantial analytic difficulties in making rigorous the continuous-time limit of the discrete path integral.

As a symptom of these problems, consider the formal path integral that comes from taking (2.4) as fiducial vector. The canonical phase term in the classical action is then

$$\int \frac{1}{3} (\cos \theta - 1) \dot{\phi} dt. \quad (4.1)$$

The coefficient, 1/3, violates the condition required to make the “Dirac string” at the south pole invisible. Only integers and half integers are allowed as coefficients if the path integral is to be well-defined [14].

For the spin coherent states of $SU(2)$, it has recently been shown [15] that, when the fiducial vector is taken to be a highest weight vector, the formal semi-classical expansion about the continuous-time classical paths does yield correct answers. It would be a salutary exercise to trace exactly what goes wrong with the continuous-time limit in the more general case.

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