THE MOORE COMPLEX OF A SIMPLICIAL COCOMMUTATIVE HOPF ALGEBRA

KADİR EMİR

ABSTRACT. We introduce the Moore complex of a simplicial cocommutative Hopf algebra through Hopf kernels. The most striking result to emerge from this construction is the coherent definition of 2-crossed modules of cocommutative Hopf algebras. This unifies the 2-crossed module theory of groups and of Lie algebras when we take the group-like and primitive functors into consideration.

Contents
1 Introduction 1
2 Quick Review of Hopf Algebras 4
3 The Moore Complex 8
4 Iterated Peiffer Pairings 13
5 2-Crossed Modules of Hopf Algebras 17
6 Conclusion 22
References 23

1. Introduction

A simplicial group $G = (G_n, d_i^n, s_i^n)$ is a simplicial object [May, 1992] in the category of groups. It is given by a collection of groups $G_n$, together with group homomorphisms $d_i^n: G_n \rightarrow G_{n-1}$ and $s_i^n: G_n \rightarrow G_{n+1}$ for $i = 0, \ldots, n$ called faces and degeneracies respectively, satisfying the well known simplicial identities. The Moore complex [Moore, 1955] of a simplicial group is a chain complex

$$N(G) = \left( \ldots \xrightarrow{d_{n+1}} N(G)_n \xrightarrow{d_n} \ldots \xrightarrow{d_2} N(G)_2 \xrightarrow{d_1} N(G)_1 \xrightarrow{d_0} G_0 \right)$$

of groups, where

$$N(G)_n = \bigcap_{i=0}^{n-1} \ker(d_i)$$

at level $n$, and the boundary morphisms $d_n: N(G)_n \rightarrow N(G)_{n-1}$ are the restrictions of the $d_n: G_n \rightarrow G_{n-1}$. Moreover, $N(G)$ defines a normal chain complex of groups, namely $d_n(N(G)_n) \leq N(G)_{n-1}$, for all $n \geq 1$. Thus, the Moore complex can be considered as

2010 Mathematics Subject Classification: 16T05, 55U10, 55U15, 18D05.
Key words and phrases: Hopf algebra, simplicial object, Moore complex, 2-crossed module.
© Kadir Emir, 2019. Permission to copy for private use granted.
the normalized chain complex of a simplicial group. The Moore complex has many roles in the fields of category theory and algebraic topology. It is well-known that the $n^{th}$ homotopy group of $G$ is the $n^{th}$ homology of $N(G)$. On the other hand, Moore complexes also appear in the Dold-Kan correspondence for the case of abelian groups [Kan, 1958], and abelian categories [Bourn, 2007]. For more relations, see [Goerss and Jardine, 1999]. Furthermore, it is shown in [Carrasco, 1995] that the Moore complex of a simplicial group is a hypercrossed complex that captures crossed modules and 2-crossed modules.

A crossed module is a group homomorphism $\partial: E \to G$, together with an action $\triangleright$ of $G$ on $E$, satisfying $\partial(g \triangleright e) = g \partial(e) g^{-1}$ and $\partial(e) \triangleright e' = e e'e^{-1}$, for all $e, e' \in E$ and $g \in G$. The notion is introduced in [Whitehead, 1949] as an algebraic model of connected homotopy 2-types. From another point of view, a crossed module can be considered as an encoded strict 2-group [Brown and Spencer, 1976]. Categorically, crossed modules are equivalent to the cat$^1$-groups (internal categories in the category of groups) [Loday, 1982]. For more details, see [Porter, 1982, Brown, 1987, Brown, 1999, Baez and Lauda, 2004, Faria Martins and Picken, 2010, Morton and Picken, 2015]. Also a thorough discussion of crossed modules from the topological and algebraic point of view is given in [Brown, 2018].

For a given simplicial group $G$, we call the Moore complex of length $n$ if $N(G)_i$ is trivial for all $i > n$. In the case that $n = 1$ it yields a crossed module $N(G)_1 \xrightarrow{\partial_1} G_0$, where the action of $G_0$ on $N(G)_1$ is defined using the conjugate action via $s_0$. Inspired by this functorial relationship between simplicial groups and crossed modules, Conduché introduced 2-crossed modules of groups in [Conduché, 1984]. Namely, for a given simplicial group $G$ with Moore complex of length two, it is shown that the first three level of the Moore complex $N G_2 \xrightarrow{\partial_2} N G_1 \xrightarrow{\partial_1} G_0$ leads to the 2-crossed module definition. Clearly, a 2-crossed module of groups is a complex $L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} G$ of groups together with left actions $\triangleright$ of $G$ on $L, E$, and on itself by conjugation; and a $G$-equivariant function $\{.\}: E \times E \to L$ called Peiffer lifting, satisfying certain properties (Definition 5.2). An alternative way to obtain a 2-crossed module from a simplicial group without considering the length of the Moore complex is given in [Mutlu and Porter, 1998]. Another analogy from crossed modules is that 2-crossed modules are also algebraic models for connected homotopy 3-types, that is, pointed CW-complexes $X$ such that $\pi_i(X) = 0$ for $i \geq 3$ [Carrasco and Cegarra, 1991]. Additionally, there are some other algebraic models of homotopy 3-types such as braided crossed modules [Brown and Gilbert, 1989], neat crossed squares [Faria Martins, 2011] and Gray 3-groups [Kamps and Porter, 2002] (these three are equivalent to the 2-crossed modules); crossed squares [Ellis, 1993a] and quadratic modules [Baues, 1991] (being the particular case of 2-crossed modules). Furthermore, as a generalization, 2-quasi crossed modules are introduced [Carrasco and Porter, 2016] in which some conditions are relaxed.

Regarding to all group theoretic terminology given above, and in the light of the close analogy between general algebraic properties of groups and Lie algebras; the Lie algebraic case of the whole 2-crossed module theory is given in [Ellis, 1993b], based on [Kassel and Loday, 1982] in which the crossed modules are introduced in the context
of Lie algebras. However, the analogy between groups and Lie algebras becomes more powerful in the category of Hopf algebras that allows us to unify both of these group and Lie algebraic theories in a functorial way, which was the main motivation of this study.

Hopf algebras [Sweedler, 1969] can be thought as a unification of groups and of Lie algebras as being the group algebra of a group, and the universal enveloping algebra of a Lie algebra. In other words, we have the functors \( Gl \) and \( Prim \) from the category of Hopf algebras to the category of groups and of Lie algebras which assigned group-like and primitive elements, respectively [Majid, 1995]. There exist some well-known variations of Hopf algebras in the literature that are obtained by either relaxing some properties or adding extra structure. For instance, quasi-Hopf algebras [Bulacu et al., 2019, Drinfel’d, 1990], quasi-triangular Hopf algebras [Majid, 1995], quantum groups [Majid, 2006, Daele, 2019], Leibniz-Hopf algebras [Hazewinkel, 1996], Steenrod algebras [Mihor, 1958], Hopfish algebras [Tang et al., 2007]. Crossed modules of Hopf algebras (Definition 2.8) are introduced by Majid in [Majid, 2012] as encoding strict 2-quantum groups, and examined for the cocommutative case in the sense of cat\(^1\)-Hopf algebras in [Fernández Vilaboa et al., 2007]. A more general notion is given in [Frégier and Wagemann, 2011] and there is no agreement as to the unique crossed module definition for Hopf algebras; see [Alonso Alvarez et al., 2018] for the discussion. At this point, we follow [Majid, 2012]. Moreover, it is proven in [Faria Martins, 2016] that the crossed module structure is also preserved under the functors \( Gl \) and \( Prim \). Therefore, crossed modules of Hopf algebras can be seen as a unification of crossed modules of groups and of Lie algebras.

The major outcome of this paper is to define 2-crossed modules of cocommutative Hopf algebras, which extend crossed modules to one level further. From a categorical point of view, this notion will unify the theory of 2-crossed modules of groups and of Lie algebras when we take the functors \( Gl \) and \( Prim \) into consideration. As for the group and Lie algebra case, we find out the functorial relationship between simplicial objects and 2-crossed modules in the category of cocommutative Hopf algebras. For this aim, we first give the explicit definition of a Moore complex of a simplicial cocommutative Hopf algebra, which will be constructed via Hopf kernels. No doubt this definition again unifies the Moore complex of groups and Lie algebras in the sense of the same functors. Then we obtain a 2-crossed module structure from a simplicial cocommutative Hopf algebra with Moore complex of length 2 with the aid of iterated Peiffer pairings. Consequently, we obtain the functor \( \text{SimpHopf}_{\leq 2} \longrightarrow X_2\text{Hopf} \). On the other hand, we already have the functor \( \text{SimpGrp}_{\leq 2} \longrightarrow X_2\text{Grp} \) from the category of simplicial groups with Moore complex of length two, to 2-crossed modules of groups [Mutlu and Porter, 1998] and similarly \( \text{SimpLie}_{\leq 2} \longrightarrow X_2\text{Lie} \) for the category of Lie algebras [Ellis, 1993b]. Finally, these two functors meet in the same diagram that proves the coherence of our 2-crossed module definition as being:
in which the horizontal arrows are extended from $Gl$ and $Prim$, respectively.

This study is the first step towards enhancing our understanding of Hopf algebras in terms of category theory and algebraic topology for higher dimensions, where the crossed modules can be considered as one dimensional categorical objects. As it stands, there already exist many higher dimensional categorical objects defined in the categories of groups and of Lie algebras. We are confident that the results of the present paper will serve as a base from which to unify these other higher dimensional structures and their properties in the category of (cocommutative) Hopf algebras.

Acknowledgement

The author expresses his gratitude to João Faria Martins for kindly suggesting the problem and for his invaluable mentorship throughout the study. The author is also thankful to John Bourke for his comments to improve the paper; and to Roger Picken (TQFT Group, Instituto Superior Técnico), Centro de Matemática e Aplicações (Universidade Nova de Lisboa) for their hospitalities between 2014-2017. This paper was supported by the project MUNI/A/1186/2018 of Masaryk University, Czech Republic.

2. Quick Review of Hopf Algebras

All Hopf algebras will be defined over a field $\kappa$. Towards the end of the paper, we will mainly work in the cocommutative setting.

2.1. Hopf algebraic conventions. Let $H$ be a Hopf algebra [Majid, 1995]. In full $H = (H, \mu, \eta, \Delta, \epsilon, S)$ where $H$ is a $\kappa$-vector space. And:

- $(H, \mu, \eta)$ is a unital associative algebra. Thus
  
  - $\mu: H \otimes H \longrightarrow H$ is an associative product, making $H$ into an associative algebra. In short the product in $H$ induces a map $\mu: H \otimes H \longrightarrow H$, where $x \otimes y \mapsto xy$.
  
  - $\eta: \kappa \longrightarrow H$ is an algebra map endowing $H$ with a unit. In short $\eta: \lambda \in \kappa \mapsto \lambda 1_H \in H$. (Here $1_H$ is the identity element of $H$.)

- $(H, \Delta, \epsilon)$ is a counital coassociative coalgebra. Thus
  
  - $\Delta: H \longrightarrow H \otimes H$ is a coassociative coproduct. We use Sweedler’s notation [Sweedler, 1969] for coproducts:

$$\Delta(x) = \sum_{\{x\}} x' \otimes x'', \text{ where } x \in H.$$
- $\epsilon : H \xrightarrow{\kappa} \kappa$ is the counit. So, for all $x \in H$, we have:

$$x = \sum_{(x)} \epsilon(x')x'' = \sum_{(x)} x'\epsilon(x'').$$

- $H = (H, \mu, \eta, \Delta, \epsilon)$ is a bialgebra. Thus

  - $\eta$ and $\mu$ are coalgebra morphisms,
  - $\epsilon$ and $\Delta$ are algebra morphisms.

- There exists an (inverse-like) anti-homomorphism $S : H \xrightarrow{H}$ at the level of algebra and coalgebra -called antipode-, satisfying:

$$\sum_{(x')} S(x')x'' = \sum_{(x)} x'S(x'') = \epsilon(x)1_H.$$ 

Moreover:

- A Hopf algebra $H$ is said to be “cocommutative” if, $\forall x \in H$, we have:

$$\sum_{(x)} x' \otimes x'' = \sum_{(x)} x'' \otimes x'.$$

- A Hopf algebra morphism is a bialgebra morphism such that compatible with antipode.

- Let $H$ be a Hopf algebra. An element $x \in H$ is said to be:

  - primitive, if $\Delta(x) = x \otimes 1 + 1 \otimes x$,
  - group like, if $\Delta(x) = x \otimes x$.

If $x$ is group-like then $\epsilon(x) = 1$ and $S(x) = x^{-1}$; and if $x$ is primitive then $\epsilon(x) = 0$ and $S(x) = -x$. The set of primitive elements $\text{Prim}(H)$ defines a Lie algebra, and the set of group-like elements $\text{Gl}(H)$ defines a group. Thus we have the functors:

$$\text{Lie} \xrightarrow{\text{Prim}} \text{Hopf} \xrightarrow{\text{Gl}} \text{Grp}$$

- Let $H$ be a cocommutative Hopf algebra. A sub-Hopf algebra $A \subset H$ is a subvector space $A$, such that $\mu(A \otimes A) \subset A$, $\Delta(A) \subset A \otimes A$ and $\eta(\kappa) \subset A$. Clearly a sub-Hopf algebra inherits a (cocommutative) Hopf algebra structure from $H$.

- A (cocommutative) sub-Hopf algebra $A$ of $H$ is called normal [Vespa and Wambst, 2018], if $x \!\triangleright_{\text{ad}} a \in A$ for all $x \in H$ and $a \in A$. Here we put:

$$x \!\triangleright_{\text{ad}} a = \sum_{(x)} x'aS(x'')$$

which is called the “adjoint action”.
2.2. Hopf algebra modules and smash products. Let $H$ be a Hopf algebra and $I$ be a bialgebra. $I$ is said to be an $H$-module algebra with an action $\rho: H \times I \rightarrow I$, explicitly $\rho: (x, v) \in H \times I \mapsto x \triangleright_{\rho} v \in I$, satisfying:

- $1_H \triangleright_{\rho} v = v$ and $x \triangleright_{\rho} 1_I = \epsilon(x)1_I$, for all $x \in H, v \in I$,
- $x \triangleright_{\rho} (uv) = \sum_{(x)} (x' \triangleright_{\rho} u)(x'' \triangleright_{\rho} v)$, for all $x \in H, u, v \in I$.

$I$ is said to be an $H$-module coalgebra, if there exists an algebra action $\rho: H \otimes I \rightarrow I$, such that:

- $\Delta(x \triangleright_{\rho} v) = \sum_{(x)} (x' \triangleright_{\rho} v') \otimes (x'' \triangleright_{\rho} v'')$, for all $x \in H, v \in I$,
- $\epsilon(x \triangleright_{\rho} v) = \epsilon(x)\epsilon(v)$, for all $x \in H, v \in I$.

The following is well known with a proof from [Majid, 1995].

2.3. Definition. Let $I, H$ be cocommutative Hopf algebras, where $I$ is an $H$-module algebra and coalgebra (one can call it $H$-module bialgebra) under the action $\rho: H \otimes I \rightarrow I$, then we have a cocommutative Hopf algebra $I \otimes_\rho H$ called the “smash product”, with the underlying vector space $I \otimes H$ such that:

- $(u \otimes x)(v \otimes y) = \sum_{(x)} \left((u' \triangleright_{\rho} v') \otimes x''\right) y$,
- $\Delta(u \otimes x) = \sum_{(u)(x)} (u' \otimes x') \otimes (u'' \otimes x'')$,
- $S(u \otimes x) = \left(1_I \otimes S(x)\right)(S(u) \otimes 1_H)$,

with the identity $1_I \otimes 1_H$ and the co-identity $\epsilon(u \otimes x) = \epsilon(u)\epsilon(x)$.

2.4. Proposition. If $H$ is a cocommutative Hopf algebra, then $H$ itself has a natural $H$-module algebra and coalgebra structure given by the adjoint action:

$$\rho: (x, y) \in H \otimes H \mapsto x \triangleright_{ad} y = \sum_{(x)} x'yS(x'') \in H.$$ 

However, this is not true in non-cocommutative case since

$$\Delta(x \triangleright_{ad} y) = \sum_{(x \triangleright_{ad} y)} (x \triangleright_{ad} y)' \otimes (x \triangleright_{ad} y)''' = \sum_{(x)(y)} x'y'S(x''') \otimes x''y'S(x'''')$$

$$\neq \sum_{(x)(y)} (x' \triangleright_{ad} y') \otimes (x'' \triangleright_{ad} y'').$$

2.5. Proposition. If $H$ is cocommutative, the antipode $S: H \rightarrow H$ becomes an idempotent. We therefore obtain $S(x \triangleright_{ad} y) = x \triangleright_{ad} S(y)$. 
2.6. Hopf kernels. Some categorical properties of the cocommutative Hopf algebras [Vespa and Wambst, 2018] we need in this paper are given below.

- The zero object is defined by $\kappa$, with the obvious structure maps. Given Hopf algebras $A$ and $B$, the zero map $z_{A,B}: A \rightarrow B$ is $\eta_A \epsilon_B$.

- The equaliser of two maps $f, g: A \rightarrow B$ is:
$$eq(f, g) = \{ x \in A: \sum_{(x)} x' \otimes f(x'') = \sum_{(x)} x' \otimes g(x'') \}.$$

- The (Hopf) kernel of a map $f: A \rightarrow B$ is:
$$eq(f, z_{A,B}) = \{ x \in A: \sum_{x} x' \otimes f(x'') = x \otimes 1 \},$$
which will be called $Lker(f)$ for short. Remark that:
$$Lker(f) \cong eq(f, z_{A,B}) \cong eq(z_{A,B}, f) \cong Rker(f),$$
because of the cocommutativity. Hence, two different possible Hopf kernel notions coincide in the category of cocommutative Hopf algebras. However, none of them defines the Hopf kernel for non-cocommutative case.

- Consider the Hopf kernel $Lker(f: A \rightarrow B)$. If we apply $\mu(\epsilon \otimes id)$ in (1), we obtain:
$$x \in Lker(f) \implies f(x) = \epsilon(x)1_B,$$
that means Hopf kernels are the specific cases of linear kernels.

- The functors $Gl$ and $Prim$ preserve kernels. In other words, we have:
$$Lker(f) \xrightarrow{Gl} ker(f),$$
for the corresponding categories.

- For any Hopf algebra map $f: A \rightarrow B$, Hopf kernel $Lker(f)$ defines a normal sub-Hopf algebra.

2.7. Crossed modules of cocommutative Hopf algebras. The following definition is well established as a crossed module of cocommutative Hopf algebras introduced in [Majid, 2012].
2.8. Definition. A crossed module of cocommutative Hopf algebras is given by a Hopf algebra map \( \partial: I \rightarrow H \) where \( I \) is an \( H \)-module bialgebra satisfying:

- \( \partial(x \triangleright_\rho v) = x \triangleright_{\text{ad}} \partial(v) \),
- \( \partial(u) \triangleright_\rho v = u \triangleright_{\text{ad}} v \),

for all \( x \in H \) and \( u, v \in I \).\(^1\)

2.9. Remark. Group-like and primitive elements preserve crossed module structures. Consequently, we can write down the crossed module conditions of groups and Lie algebras in the sense of the functors \( \text{Gl} \) and \( \text{Prim} \) as follows:

- A crossed module of groups is given by a group homomorphism \( \partial: E \rightarrow G \), together with an action \( \triangleright \) of \( G \) on \( E \), such that:

\[
\partial(g \triangleright e) = g \partial(e) g^{-1}, \quad \partial(e) \triangleright f = e f e^{-1},
\]

for all \( e, f \in E \) and \( g \in G \).

- A crossed module of Lie algebras (aka differential crossed module) is given by a Lie algebra homomorphism \( \partial: \mathfrak{e} \rightarrow \mathfrak{g} \), together with an action \( \triangleright \) of \( \mathfrak{g} \) on \( \mathfrak{e} \), such that:

\[
\partial(g \triangleright e) = [g, \partial(e)], \quad \partial(e) \triangleright f = [e, f],
\]

for all \( e, f \in \mathfrak{e} \) and \( g \in \mathfrak{g} \).

Therefore we have the functors:

\[
\begin{array}{ccc}
\text{XLie} & \overset{\text{Prim}}{\leftarrow} & \text{XHopf} \\
\text{Gl} & \rightarrow & \text{XGrp},
\end{array}
\]

between the categories of crossed modules of Lie algebras, cocommutative Hopf algebras, and groups, respectively. See [Faria Martins, 2016], for more details.

3. The Moore Complex

From now on, all Hopf algebras will be considered cocommutative, and we just use the term “Hopf algebra” for the sake of simplicity.

3.1. Simplicial Hopf algebras.

\(^1\)In general, there is an extra crossed module condition called “compatibility” that automatically holds for cocommutative setting.
3.2. Definition. A chain complex \( \{A_n, \partial_n\}^{+\infty}_{n=0} \) of Hopf algebras is given by a sequence of Hopf algebra maps

\[
A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \rightarrow \cdots \rightarrow A_1 \xrightarrow{\partial_1} A_0,
\]

such that: \( \forall n \geq 1 \), we have \( \partial_{n+1} \partial_n = \eta A_{n-1} \epsilon_{A_{n+1}} \), and the latter is the zero morphism \( x \in A_{n+1} \mapsto \epsilon(x)1_{A_{n-1}} \in A_{n-1} \). Given a chain complex of Hopf algebras, then \( \partial_n(A_n) \) is, a priori, only a sub-Hopf algebra of \( A_{n-1} \). We say that a chain complex of Hopf algebras is normal if \( \partial_n(A_n) \) is a normal sub-Hopf algebra of \( A_{n-1} \), for each positive integer \( n \).

3.3. Definition. A simplicial Hopf algebra is a simplicial set [Kauffman, 2018] in the category of Hopf algebras. In other words, a simplicial Hopf algebra

\[
\mathcal{H} = \ldots \xrightarrow{d_i} H_3 \xrightarrow{d_i} H_2 \xrightarrow{d_0} H_1 \xrightarrow{d_i} H_0 \xleftarrow{s_j} H_1 \xrightarrow{s_i} H_0 \xrightarrow{d_i} H_0 \xrightarrow{d_i} H_0 \xrightarrow{d_i} H_0
\]

is given by a collection of Hopf algebras \( H_n \) \( (n \in \mathbb{N}) \) together with Hopf algebra maps called faces and degeneracies

\[
d_i^n: H_n \xrightarrow{} H_{n-1}, \quad 0 \leq i \leq n
\]

\[
s_j^{n+1}: H_n \xrightarrow{} H_{n+1}, \quad 0 \leq j \leq n
\]

which are to satisfy the following simplicial identities

\[
(i) \quad d_id_j = d_{j-1}d_i \quad \text{if} \quad i < j \\
(ii) \quad s_is_j = s_{j+1}s_i \quad \text{if} \quad i \leq j \\
(iii) \quad d_is_j = s_{j-1}d_i \quad \text{if} \quad i < j \\
\quad \quad d_js_j = d_{j+1}s_j = id \\
\quad \quad d_is_j = s_jd_{i-1} \quad \text{if} \quad i > j + 1
\]

We denote the category of simplicial Hopf algebras by SimpHopf.

3.4. The Moore complex.

3.5. Lemma. Given a simplicial Hopf algebra (6), we have the chain complex of Hopf algebras \( \{NH_n, \partial_n\}^{+\infty}_{n=0} \), given by:

- \( NH_0 = H_0 \),
- \( NH_n = \bigcap_{i=0}^{n-1} \ker(d_i) \), for \( n \geq 1 \),
- \( \partial_n: NH_n \rightarrow NH_{n-1} \) as being the restriction of \( d_n \) to \( NH_n \).
Proof. We know that the intersection of any number of sub-Hopf algebras is again a sub-Hopf algebra. Thus $NH_i$ defines a sub-Hopf algebra due to (1). Moreover $\partial_n$ is well defined, since for all $x \in NH_n$ and $i < n$, we have

\[
(d_i \otimes \text{id})\Delta(d_n(x)) = \sum_{(x)} (d_i d_n)(x') \otimes d_n(x'')
\]

\[
= \sum_{(x)} (d_{n-1} d_i)(x') \otimes d_n(x'')
\]

\[
= (d_{n-1} \otimes d_n) \sum_{(x)} d_i(x') \otimes x''
\]

\[
= (d_{n-1} \otimes d_n)(1 \otimes x)
\]

\[
= 1 \otimes d_n(x).
\]

that yields $\text{im}(\partial_n) \in \text{Lker}(d_i)$ for all $i < n$; therefore $\partial_n(x) \in NH_{n-1}$.

On the other hand, to have a chain complex, we need to prove

\[
(\partial_n \partial_{n+1})(x) = \epsilon(x)1_{N_{n-1}},
\]

for all $x \in NH_{n+1}$. For this aim, we get

\[
(\partial_n \otimes \text{id})\Delta(\partial_{n+1}(x)) = \sum_{(x)} (\partial_n \partial_{n+1})(x') \otimes \partial_{n+1}(x'')
\]

\[
= \sum_{(x)} (\partial_n \partial_{n+1})(x') \otimes \partial_{n+1}(x'')
\]

\[
= (\partial_n \otimes \partial_{n+1}) \sum_{(x)} \partial_n(x') \otimes x''
\]

\[
= (\partial_n \otimes \partial_{n+1})(1 \otimes x)
\]

\[
= 1 \otimes \partial_{n+1}(x).
\]

Thus $\text{im}(\partial_{n+1}) \in \text{Lker}(\partial_n)$ that implies $(\partial_n \partial_{n+1})(x) = \epsilon(x)1_{N_{n-1}}$ from (2).

3.6. Definition. [Moore Complex] For a given simplicial Hopf algebra $\mathcal{H}$, the chain complex $\{NH_n, \partial_n\}_{n=0}^{+\infty}$ will be called the “Moore complex of $\mathcal{H}$.”

The Moore complex of groups is also known as a normalized chain complex of simplicial groups in the literature [Mutlu and Porter, 1998]. The following lemma proves that the Moore complex of a simplicial Hopf algebra has the same property.

3.7. Lemma. The Moore complex $\{NH_n, \partial_n\}_{n=0}^{+\infty}$ of a simplicial Hopf algebra $\mathcal{H}$ is a normal chain complex.
Proof. For any \( \partial_{n+1} : NH_{n+1} \to NH_n \), we have to prove that \( \text{im}(\partial_{n+1}) \) is a normal sub-Hopf algebra of \( NH_n \).

It is already clear that \( \text{im}(\partial_{n+1}) \) is a sub-Hopf algebra. So we only need to show that it is normal, i.e. \( x \triangleright_{ad} b \in NH_n \), for all \( x \in NH_n \) and \( b = \partial_{n+1}(a) \in \text{im}(\partial_{n+1}) \). For this aim, we get

\[
(d_n \otimes \text{id}) \Delta(x \triangleright_{ad} b) = (d_n \otimes \text{id}) \sum_{(x)} x' b' S(x'') \otimes x'' b'' S(x''')
\]

\[
\vdash (d_n \otimes \text{id}) \sum_{(x)} x' \partial_{n+1}(a') S(x'') \otimes x'' \partial_{n+1}(a'') S(x''')
\]

\[
= \sum_{(x)} d_n(x') (d_n(d_{n+1}(a'))) d_n(S(x'')) \otimes x'' d_{n+1}(a'') S(x''')
\]

\[
= \sum_{(x)} d_n(x') d_n(d_n(a')) d_n(S(x'')) \otimes x'' d_{n+1}(a'') S(x''')
\]

\[
= \sum_{(x)} d_n(x') d_n(S(x'')) \otimes x'' d_{n+1}(a) S(x''')
\]

\[
= 1 \otimes (x \triangleright_{ad} b)
\]

Note that we used the Hopf kernel property of \( a \in NH_{n+1} \) for cancellation.

\[\blacksquare\]

3.8. Definition. We say that a simplicial Hopf algebra has Moore complex of length \( n \), if \( NH_i \) is zero object, for all \( i > n \).

We denote the corresponding category by \( \text{SimpHopf}_{\leq n} \).

3.9. Proposition. Following the property (3), one can say that the functors \( \text{Gl} \) and \( \text{Prim} \) preserve the Moore complex definition, as well as the length of it. We therefore have the functors

\[
\text{SimpLie}_{\leq n} \xrightarrow{\text{Prim}} \text{SimpHopf}_{\leq n} \xrightarrow{\text{Gl}} \text{SimpGrp}_{\leq n},
\]

with referring to [Conduché, 1984, Ellis, 1993b] for the corresponding categories.

3.10. The simplicial decomposition. The following theorem is due to [Majid, 1994, Radford, 1985].

3.11. Theorem. [Majid/Radford] Let \( (\partial : I \to H, i) \) be a Hopf algebra projection. In other words, \( i : H \to I \) is also a Hopf algebra map and \( \partial i = \text{id}_H \). Then, \( I \) is an \( H \)-module bialgebra where the action \( \rho : H \otimes I \to I \) is the adjoint action via \( i \).

Moreover, we have an isomorphism of Hopf algebras \( I \cong \text{Lker}(\partial) \otimes_{\rho} H \), where the maps below are mutually inverse:

\[
\Psi : v \in I \mapsto \sum_{(v')} f(v') \otimes \partial(v'') \in \text{Lker}(\partial) \otimes_{\rho} H,
\]

with \( f(x) = \sum_{(x)} x' i \partial(S(x'')) \), for all \( x \in I \); and

\[
\Phi : a \otimes x \in \text{Lker}(\partial) \otimes_{\rho} H \mapsto a i(x) \in I.
\]
3.12. Remark. By using simplicial identities, for each \( n \), we can obtain Hopf algebra projections \((d_i: H_{n+1} \rightarrow H_n, s_i)\) in a simplicial Hopf algebra (6). Therefore one can adapt Theorem (3.11) to simplicial Hopf algebras as follows.

3.13. Proposition. In a simplicial Hopf algebra \( \mathcal{H} \), there exists an action

\[
\rho: s_i(H_{n-1}) \otimes \text{Lker}(d_i) \rightarrow \text{Lker}(d_i)
\]

\[
(a, x) \mapsto a \triangleright_{\rho} x = a \triangleright_{ad} x,
\]

for all \( i \leq n - 1 \). Consequently, we have the smash product Hopf algebra

\[
\text{Lker}(d_i) \otimes_{\rho_i} s_i(H_{n-1}),
\]

from Definition 2.3.

3.14. Theorem. For any simplicial Hopf algebra \( \mathcal{H} \), we have an isomorphism

\[
H_n \cong \text{Lker}(d_i) \otimes_{\rho_i} s_i(H_{n-1}),
\]

of Hopf algebras, for all \( n \in \mathbb{N} \), \( i \leq n - 1 \).

Proof. The map

\[
\phi: H_n \rightarrow \text{Lker}(d_i) \otimes_{\rho_i} s_i(H_{n-1})
\]

\[
x \mapsto \sum_{(x)} f_i(x') \otimes s_i d_i(x'')
\]

gives the isomorphism, where “Hopf kernel generator map” \( f_i: H_n \rightarrow \text{Lker}(d_i) \) is a linear map defined by \( f_i: x \mapsto \sum_{(x)} x' s_i d_i(S(x'')) \).

3.14.1. Construction. Consider the simplicial Hopf algebra

\[
\mathcal{H} = \begin{array}{c}
1 & 2 & 3 & \cdots & n & n + 1 \\
\begin{array}{c}
H_1 \\
H_2 \\
H_3 \\
\vdots \\
H_n \\
\end{array}
\end{array}
\]
3.16. Decomposing. If we apply Theorem 3.14 to $H_1$ in (8), we get
\[
H_1 \cong \ker(d_0) \otimes_{\rho_0} s_0(H_0)
= NH_1 \otimes_{\rho_0} s_0(NH_0).
\]
Similarly, considering $H_2$, we first get
\[
H_2 \cong \ker(d_0) \otimes_{\rho_0} s_0(H_1)
= \ker(d_0) \otimes_{\rho_0} s_0(NH_1 \otimes_{\rho_0} s_0(NH_0)),
\]
and by applying Theorem (3.14) in (9), we further have
\[
H_2 \cong \left( \ker(d_1) \cap \ker(d_0) \right) \otimes_{\rho_1} s_1(H_1) \otimes_{\rho_0} s_0(NH_1 \otimes_{\rho_0} s_0(NH_0))
= (\ker(d_1) \cap \ker(d_0)) \otimes_{\rho_1} s_1(NH_1) \otimes_{\rho_0} s_0(NH_1 \otimes_{\rho_0} s_0(NH_0)).
\]
One level further, by using (10), we have
\[
H_3 \cong \left( NH_3 \otimes_{\rho_1} s_2(NH_2) \right) \otimes_{\rho_1} s_1(NH_2 \otimes_{\rho_1} s_2s_1(NH_1))
\]
\[
\otimes_{\rho_0} \left( s_0(NH_2) \otimes_{\rho_1} s_2s_0(NH_1) \otimes_{\rho_0} s_1s_0(NH_1) \otimes_{\rho_0} s_2s_1s_0(NH_0) \right).
\]
By iteration, we get the general formula:

3.17. Theorem. [Simplicial Decomposition] Let $\mathcal{H}$ be a simplicial Hopf algebra. We have the decomposition of $H_n$, for any $n \geq 0$ as follows:
\[
H_n \cong \left( \cdots (NH_n \otimes_{\rho_{n-1}} s_{n-1}NH_{n-1}) \otimes \cdots s_{n-2} \cdots s_1NH_1 \right) \otimes
\]
\[
\left( \cdots (s_0NH_{n-1} \otimes s_1s_0NH_{n-2}) \otimes \cdots \otimes s_{n-1}s_{n-2} \cdots s_0NH_0 \right).
\]

4. Iterated Peiffer Pairings

The following notation and terminology is derived from [Carrasco, 1995, Carrasco and Cegarra, 1991] where it is used for both simplicial groups and simplicial algebras.

4.1. The Poset of Surjective Maps. Consider the ordered set $[n] = \{0 < 1 < \cdots < n\}$, let $\alpha_i = \alpha_i^n : [n + 1] \rightarrow [n]$ be the non-decreasing surjective map given by
\[
\alpha_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j - 1 & \text{if } j > i \end{cases}
\]
Let $S(n, n-l)$ be the set of all non-decreasing surjective maps from $[n]$ to $[n-l]$. They can be generated from the various $\alpha_i^n$ by composition. The composition of these generating maps satisfies the property $\alpha_j \circ \alpha_i = \alpha_{i-1} \circ \alpha_j$ with the condition $j < i$. This implies
that each element $\alpha \in S(n, n - l)$ has a unique expression as $\alpha = \alpha_{i_1} \circ \alpha_{i_2} \circ \cdots \circ \alpha_{i_l}$ with $0 \leq i_1 \leq i_2 < \cdots < i_l < n$, where the indices $i_k$ are the elements of $[n]$ for which $\alpha(i) = \alpha(i + 1)$. Clearly $S(n, n - l)$ is canonically isomorphic to the set $\{(i_1, \ldots, i_l) : 0 \leq i_1 < i_2 < \cdots < i_l \leq n - 1\}$. For instance, the single element of $S(n, n)$ defined by the identity map on $[n]$, corresponds to the empty 0-tuple ( ) denoted by $\emptyset$. Similarly the only elements of $S(n, 0)$ is $(n - 1, n - 2, \cdots, 0)$. For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq l \leq n} S(n, n - l).$$

Any element of $S(n)$ is of the form $(i_1, \cdots, i_l)$ where $0 \leq i_1 \leq i_2 < \cdots < i_l < n$. If $\alpha = (i_1, \cdots, i_l)$, then we say $\alpha$ has length $l$ and will write $\# \alpha = l$. Consider the lexicographic order on $S(n)$, making it a totally ordered set. For instance, the orders of $S(2), S(3)$ and $S(4)$ are respectively:

$$S(2) = \{\emptyset < (1) < (0) < (1, 0)\}$$
$$S(3) = \{\emptyset < (2) < (1) < (2, 1) < (0) < (2, 0) < (1, 0) < (2, 1, 0)\}$$
$$S(4) = \{\emptyset < (3) < (2) < (3, 2) < (1) < (3, 1) < (2, 1) < (3, 2, 1) < (0) < (3, 0)$$
$$< (2, 0) < (3, 2, 0) < (1, 0) < (3, 1, 0) < (2, 1, 0) < (3, 2, 1, 0)\}$$

If $\alpha, \beta \in S(n)$, we define $\alpha \cap \beta$ to be the set of indices which belong to both $\alpha$ and $\beta$.

4.2. Iterated Peiffer pairings. Let $H$ be a simplicial Hopf algebra and $\{NH_n, \partial_n\}_{n=0}^{+\infty}$ be its Moore complex. We define the set $P(n)$ consisting of pairs of elements $(\alpha, \beta)$ from $S(n)$ with $\alpha \cap \beta = \emptyset$ and $\beta < \alpha$, with respect to lexicographic ordering in $S(n)$ where $\alpha = (i_1, \cdots, i_l)$, $\beta = (j_m, \cdots, j_1) \in S(n)$. The pairings

$$\{F_{\alpha, \beta} : NH_{n-\#\alpha} \times NH_{n-\#\beta} \longrightarrow NH_n \mid (\alpha, \beta) \in P(n), n \geq 0\}$$

are defined as composites in the diagram

$$\begin{array}{ccc}
NH_{n-\#\alpha} \times NH_{n-\#\beta} & \xrightarrow{F_{\alpha, \beta}} & NH_n \\
\downarrow s_\alpha \times s_\beta & & \downarrow f_n \\
H_n \times H_n & \xrightarrow{\triangleright_{ad}} & H_n
\end{array}$$

such that

$$s_\alpha = s_{i_1} \cdots s_{i_l} : NH_{n-\#\alpha} \rightarrow H_n, \quad s_\beta = s_{j_m} \cdots s_{j_1} : NH_{n-\#\beta} \rightarrow H_n,$$

and $f_n : H_n \rightarrow NH_n$ is defined by the composition $f_n = f_{n-1} \circ \cdots \circ f_0$, where $f_i$ is the Hopf kernel generator map defined in Theorem 3.14.
4.3. Remark. In the prerequisite for the above idea, one can also consider the case \( \alpha > \beta \) which creates same type of Peiffer elements, namely the same elements under the antipode. Moreover, lexicographic order guarantees the Peiffer elements to be non-trivial in the sense of simplicial identities.

4.4. Calculating Peiffer Pairings. In this subsection, we obtain the Peiffer pairings that needed in the sequel.

4.4.1. \( n = 2 \) Case. We have unique type of element for this case, by taking \( \alpha = (0), \beta = (1) \). Then, \( F_{(0)(1)}(x, y) \in NH_2 \) for \( x, y \in NH_1 \) is calculated as follows:

\[
F_{(0)(1)}(x, y) = f_1f_0 \left[ (s_0(x) \triangleright_{ad} s_1(y)) \right] \\
= f_1 \left[ \sum_{(s_0(x) \triangleright_{ad} s_1(y))} [s_0(x) \triangleright_{ad} s_1(y)]' s_0d_0(S(s_0(x) \triangleright_{ad} s_1(y)))'' \right] \\
= f_1 \left[ \sum_{(x)(y)} [s_0(x') \triangleright_{ad} s_1(y')] S(s_0(x'') \triangleright_{ad} s_1(y'')) \right] \\
= f_1 \left[ \sum_{(x)(y)} [s_0(x') \triangleright_{ad} s_1(y')] S(s_0(x'') \triangleright_{ad} s_0d_0(y'')] \right] \\
= f_1 \left[ \sum_{(x)} [s_0(x') \triangleright_{ad} s_1(y)] S(s_0(x'') \triangleright_{ad} 1) \right] \\
= f_1 \left( s_0(x) \triangleright_{ad} s_1(y) \right) \\
= \sum_{(x)(y)} [s_0(x') \triangleright_{ad} s_1(y')] s_1d_1 [S(s_0(x'') \triangleright_{ad} s_1(y''))] \\
= \sum_{(x)(y)} [s_0(x') \triangleright_{ad} s_1(y')] S(s_1(x'') \triangleright_{ad} s_1(y'')) \tag{12}
\]

4.4.2. \( n = 3 \) Case. The possible six Peiffer elements belonging to \( NH_3 \) are

\[
F_{(1,0)(2)} , F_{(2,0)(1)} , F_{(0)(2,1)} , F_{(0)(2)} , F_{(1)(2)} , F_{(0)(1)} .
\]

If we calculate these elements, we get

a) for all \( x \in NH_1 \) and \( y \in NH_2 \),

- \( F_{(1,0)(2)}(x, y) = \sum_{(x)(y)} [s_1s_0(x') \triangleright_{ad} s_2(y')] S(s_2s_0(x'') \triangleright_{ad} s_2(y'')) \)

- \( F_{(2,0)(1)}(x, y) \)

\[
= \sum_{(x)(y)} (s_2s_0(x') \triangleright_{ad} s_1(y')) S(s_2s_1(x'') \triangleright_{ad} s_1(y'')) \\
= S[(s_2s_0(x'') \triangleright_{ad} s_2(y'')) S(s_2s_1(x'''') \triangleright_{ad} s_2(y'''))] 
\]
b) for all $x \in NH_2$ and $y \in NH_1$,

- $F_{(0)(2,1)}(x, y)$
  
  $$= \sum_{(x)(y)} (s_0(x') \triangleright_{ad} s_2s_1(y')) S (s_1(x'') \triangleright_{ad} s_2s_1(y''))
  
  S [s_2s_1(y''') S (s_2(x''') \triangleright_{ad} s_2s_1(y''')))$$

  
  c) for all $x, y \in NH_2$,

- $F_{(0)(1)}(x, y)$
  
  $$= \sum_{(x)(y)} (s_0(x') \triangleright_{ad} s_1(y')) S (s_1(x'') \triangleright_{ad} s_1(y''))
  
  S [s_2(y'') S (s_2(x'') \triangleright_{ad} s_2(y'')))$$

- $F_{(0)(2)}(x, y) = \sum_{(y)} (s_0(x) \triangleright_{ad} s_2(y')) S(s_2(y''))$

- $F_{(1)(2)}(x, y) = \sum_{(x)(y)} [s_1(x') \triangleright_{ad} s_2(y')] S [s_2(x'') \triangleright_{ad} s_2(y'')]$

4.5. From simplicial Hopf algebras to crossed modules.

4.6. Lemma. Let $H$ be a simplicial Hopf algebra with the Moore complex of length one. We have the crossed module

$$\partial_1: NH_1 \longrightarrow H_0,$$

where the action $\rho: H_0 \otimes NH_1 \longrightarrow NH_1$ is defined by

$$k \triangleright_{\rho} x = s_0(k) \triangleright_{ad} x,$$

for all $k \in H_0, x \in NH_1$.

Proof. The first condition of (4) follows immediately. Let us prove the second condition. Since the length of the Moore complex of $H$ is one, we have

$$d_2(F_{(0)(1)}(x, y)) = \epsilon(x)\epsilon(y)1_H,$$

from (12). Clearly, it is

$$\sum_{(x)(y)} (d_2s_0(x') \triangleright_{ad} y') S (x'' \triangleright_{ad} y'') = \epsilon(x)\epsilon(y)1_H. \quad (13)$$

Let us define $f: NH_1 \otimes NH_1 \longrightarrow NH_1$ by

$$f(x \otimes y) = \sum_{(x)(y)} (d_2s_0(x') \triangleright_{ad} y') S (x'' \triangleright y''). \quad (14)$$
Hence we have \( f(x \otimes y) = \epsilon(x)\epsilon(y)1_H \). Furthermore, we have

\[
\sum_{(x)(y)} f(x' \otimes y') \otimes x'' \otimes y'' = \sum_{(x)(y)} \epsilon(x')\epsilon(y')1_H \otimes x'' \otimes y'' ,
\]

and consequently

\[
\sum_{(x)(y)} f(x' \otimes y') (x' \triangleright_{ad} y'') = \sum_{(x)(y)} \epsilon(x')\epsilon(y')1_H (x'' \triangleright_{ad} y'')
= x \triangleright_{ad} y.
\]

The left hand-side of the previous equation is

\( d_2s_0(x) \triangleright_{ad} y \),

from (14). We therefore already proved for all \( x, y \in NH_1 \) that

\[
\partial_1(x) \triangleright_{\rho} y = s_0d_1(x) \triangleright_{ad} y
= d_2s_0(x) \triangleright_{ad} y
= x \triangleright_{ad} y.
\]

Summarily, we used the following diagram (where \( H = NH_1 \)) for equation (13):

\[
\begin{array}{c}
H \otimes H \\
\downarrow \Delta \otimes \Delta \\
H \otimes H \otimes H \otimes H \\
\downarrow \text{id} \otimes \tau \otimes \text{id} \\
H \otimes H \otimes H \otimes H \\
\downarrow f \otimes \text{id} \otimes \text{id} \\
H \otimes H \otimes H \otimes H \\
\downarrow \mu(\epsilon \otimes \epsilon) \otimes \text{id} \otimes \text{id} \\
H \otimes H \otimes H \otimes H \\
\downarrow \mu(\text{id} \otimes \triangleright_{ad}) \\
H \otimes H \\
\end{array}
\]

This diagrammatic approach will be referred in further calculations.

4.7. PROPOSITION. This construction yields a functor \( \text{SimpHopf}_{\leq 1} \rightarrow \text{XHopf} \).

5. 2-Crossed Modules of Hopf Algebras

For completeness, we first recall the 2-crossed modules of groups and of Lie algebras. We note that, a lot of different conventions appear in the literature for 2-crossed modules. We follow Conduché, [Conduché, 1984] in defining group 2-crossed modules, and from that we derived in [Faria Martins and Picken, 2011] a corresponding definition of 2-crossed module of Lie algebras (also called differential 2-crossed modules).
5.1. 2-crossed modules of groups, and Lie algebras.

5.2. Definition. A 2-crossed module of groups is given by a chain complex

\[ L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} G \]

of groups, together with left actions \( \triangleright \) of \( G \) on \( E, L \); and with a \( G \)-equivariant\(^2\) bilinear map called Peiffer lifting

\[ \{ , \} : E \times_G E \rightarrow L \]

satisfying the following axioms, for all \( l, m \in L \) and \( e, f, g \in E \):

1) \( L \xrightarrow{\partial_2} E \xrightarrow{\partial_1} G \) is a complex of \( G \)-modules \( G \) acts on itself by conjugation,
2) \( \partial_2\{e, f\} = (ef^{-1}) (\partial_1(e) \triangleright f^{-1}) \),
3) \( \{\partial_2(l), \partial_2(m)\} = lm^{-1}m^{-1} \),
4) \( \{e, fg\} = \{e, f\} (\partial_1(e) \triangleright f) \triangleright \{e, g\} \),
5) \( \{ef, g\} = \{e, fgf^{-1}\} \partial_1(e) \triangleright \{f, g\} \),
6) \( \{\partial_2(l), e\} \{e, \partial_2(l)\} = l (\partial_1(e) \triangleright l^{-1}) \).

In the fourth condition, we put the action

\[ e \triangleright' l \doteq l \{\partial_2(l^{-1}), e\} \]  \quad (16)

for each \( e \in E, l \in L \) that turns \( (\partial_2 : L \rightarrow E, \triangleright') \) into a crossed module.

5.3. Definition. A 2-crossed module of Lie algebras (aka differential 2-crossed module) is given by a chain complex of Lie algebras

\[ l \xrightarrow{\partial_2} e \xrightarrow{\partial_1} g \]

with left actions \( \triangleright \) of \( g \) on \( e, l, g \), on the latter via the adjoint representation, together with a \( g \)-equivariant\(^3\) bilinear map called Peiffer lifting

\[ \{ , \} : e \times_g e \rightarrow l \]

satisfying the following axioms, for all \( x, y \in l \) and \( u, v, w \in e \):

1) \( l \xrightarrow{\partial_2} e \xrightarrow{\partial_1} g \) is a complex of \( g \)-modules,
2) \( \partial_2\{u, v\} = [u, v] - \partial_1(u) \triangleright_v v \),

\(^2\)For groups, \( G \)-equivariance means that \( g \triangleright \{u, v\} = \{g \triangleright u, g \triangleright v\} \).
\(^3\)For Lie algebras, \( g \)-equivariance means that \( g \triangleright \{u, v\} = \{g \triangleright u, v\} + \{u, g \triangleright v\} \).
3) \( \{ \partial_2(x), \partial_2(y) \} = [x, y], \)

4) \( \{ u, [v, w] \} = \{ \partial_2\{ u, v \}, w \} - \{ \partial_2\{ u, w \}, v \}, \)

5) \( \{ [u, v], w \} = \partial_1(u) \triangleright_{\rho} \{ v, w \} + \{ u, [v, w] \} - \partial_1(v) \triangleright_{\rho} \{ u, w \} - \{ v, [u, w] \}, \)

6) \( \{ \partial_2(x), v \} + \{ v, \partial_2(x) \} = -\partial_1(v) \triangleright_{\rho} x. \)

When we put

\[ v \triangleright_{\rho}^l x = -\{ \partial_2(x), v \} \tag{17} \]

for each \( x \in I, v \in e, \) that turns \( (\partial_2: I \longrightarrow e, \triangleright_{\rho}^l) \) into a differential crossed module.

5.4. 2-crossed modules of Hopf algebras.

5.5. Definition. A 2-crossed module of Hopf algebras is given by a chain complex

\[ K \xrightarrow{\partial_2} H \xrightarrow{\partial_1} I \]

of Hopf algebras (i.e. \( \partial_1\partial_2(x) \) is zero morphism) with actions \( \triangleright_{\rho} \) of \( I \) on \( H, K, \) and also on itself by adjoint action; together with an \( I \)-equivariant\(^4\) bilinear map called Peiffer lifting

\[ \{ , \} : H \times_I H \longrightarrow K \]

satisfying the following axioms, for all \( x, y, z \in H \) and \( k, l \in K: \)

1) \( K \xrightarrow{\partial_2} H \xrightarrow{\partial_1} I \) is a complex\(^5\) of \( I \)-module bialgebras,

2) \( \partial_2\{ x, y \} = \sum_{(x)(y)} (x' \triangleright_{ad} y') \partial_1(x'') \triangleright_{\rho} S(y''), \)

3) \( \{ \partial_2(k), \partial_2(l) \} = \sum_{(l)} (k \triangleright_{ad} l') S(l'), \)

4) \( \{ x, yz \} = \sum_{(x)(y)} \{ x', y' \} (\partial_1(x'') \triangleright_{\rho} y'') \triangleright_{\rho}^l \{ x'', z \}, \)

5) \( \{ xy, z \} = \sum_{(x)(y)(z)} \{ x', y' \triangleright_{ad} z' \} \partial_1(x'') \triangleright_{\rho} \{ y'', z'' \}, \)

6) \( \sum_{(k)(x)} \{ \partial_2(k', x') \{ x'', \partial_2(k'') \} \} = \sum_{(k)} k' (\partial_1(x) \triangleright_{\rho} S(k'')), \)

We put the action

\[ x \triangleright_{\rho}^l k = \sum_{(k)} k' \{ \partial_2(S(k'')), x \} \tag{18} \]

in the fourth condition that makes \( (\partial_2: K \longrightarrow H, \triangleright_{\rho}^l) \) a crossed module.

We denote the category of 2-crossed modules of Hopf algebras by \( X_2^{Hopf} \) in which the morphisms defined in a similar way to group case [Gohla and Faria Martins, 2013].

\(^4\)Here \( I \)-equivariance means \( a \triangleright_{\rho} \{ x, y \} = \sum_{(a)} \{ a' \triangleright_{\rho} x, a'' \triangleright_{\rho} y \}, \forall a \in I \) and \( x, y \in H. \)

\(^5\)Namely, \( \partial_2(a \triangleright_{\rho} k) = a \triangleright_{\rho} \partial_2(k), \) and \( \partial_1(a \triangleright_{\rho} x) = a \triangleright_{ad} \partial_1(x), \forall a \in I, x \in H, k \in K. \)
5.6. Coherence with \textit{Prim} and \textit{Gl}.

5.7. Theorem. The functor \textit{Prim} preserves 2-crossed module structure.

\textbf{Proof.} Suppose that $K \xrightarrow{\partial_2} H \xrightarrow{\partial_1} I$ is a 2-crossed modules of Hopf algebras. We already know that the primitive elements preserve the actions [Faria Martins, 2016]. Therefore, we have the complex of \textit{Prim}(I)-modules of Lie algebras. Since the rest of the proof consists in routine calculations by setting $\Delta(x) = 1 \otimes x + x \otimes 1$ and \textit{Prim}(x $\triangleright_{ad}$ y) = $[x, y]$, we only prove that the derived crossed module action 18 is compatible with 17, namely

$$h \triangleright'_p k = \sum_{(k)} k' \{\partial_2(S(k'')) \otimes h\}$$

$$= 1 \{-\partial_2(k) \otimes h\} + k \{\partial_2(1) \otimes h\}$$

$$= -\{\partial_2(k) \otimes h\}$$

$$= h \triangleright' k.$$  

5.8. Theorem. The functor \textit{Gl} preserves 2-crossed module structure.

\textbf{Proof.} Follows immediately by letting $\Delta(x) = x \otimes x$ and \textit{Gl}(x $\triangleright_{ad}$ y) = $xyx^{-1}$.

5.9. Proposition. Consequently, we have the functors

$$X_2\text{Lie} \xleftarrow{\text{Prim}} X_2\text{Hopf} \xrightarrow{\text{Gl}} X_2\text{Grp},$$

extending (5) to 2-crossed module level.

5.10. From simplicial Hopf algebras to 2-crossed modules.

5.11. Theorem. Let $\mathcal{H}$ be a simplicial Hopf algebra with Moore complex of length two. Consider

$$NH_2 \xrightarrow{\partial_2} NH_1 \xrightarrow{\partial_1} H_0.$$  \hfill (19)

Then, we have the actions

- of $H_0$ on $NH_1$ given by $s_0(n) \triangleright_{ad} m$,
- of $H_0$ on $NH_2$ given by $s_1s_0(n) \triangleright_{ad} l$,

and the Peiffer lifting \{ , \} : $NH_1 \otimes_{NH_0} NH_1 \longrightarrow NH_2$ as being

$$\{x, y\} = \sum_{(x)(y)} [s_1(x') \triangleright_{ad} s_1(y')] S[s_0(x'') \triangleright_{ad} s_1(y'')] ,$$  \hfill (20)

that makes (19) into a 2-crossed module of Hopf algebras.
Proof. The key point of this proof is: since the length of the Moore complex of \( H \) is two, \( \partial_3(x) = \epsilon(x) 1_{H_2}, \) for all \( x \in NH_3. \) Then we use the Peiffer pairings obtained in section 4.4.2 to prove the conditions:

1) Follows from the definition of the Moore complex.

2) We straightforwardly have

\[
\partial_2\{x \otimes y\} = d_2 \left( \sum_{(x) (y)} [s_1(x') \triangleright_{ad} s_1(y')] S [s_0(x'') \triangleright_{ad} s_1(y'')] \right)
\]
\[
= \sum_{(x) (y)} [d_2s_1(x') \triangleright_{ad} d_2s_1(y')] S [d_2s_0(x'') \triangleright_{ad} d_2s_1(y'')]
\]
\[
= \sum_{(x) (y)} [x' \triangleright_{ad} y'] [s_0d_1(x'') \triangleright_{ad} S(y'')]
\]
\[
= \sum_{(x) (y)} (x' \triangleright_{ad} y') \partial_1(x'') \triangleright_\rho S(y'').
\]

3) We get

\[
\{\partial_2(k), \partial_2(l)\} = \sum_{(x) (y)} [s_1d_2(x') \triangleright_{ad} s_1d_2(y') S [s_0d_2(x'') \triangleright_{ad} s_1d_2(y'')]
\]
\[
= \sum_{(l)} (k \triangleright_{ad} l') S(l').
\]

4) It can be proven by direct calculations without using any Peiffer pairing.

5) We have

\[
\partial_3 \left( F_{(1,0)(2)}(x, y) \right) = \epsilon(x)\epsilon(y) 1_H,
\]
for all \( x \in H \) and \( y \in K. \) If we calculate the left hand side, we obtain

\[
\sum_{(x) (y)} [s_1s_0d_1(x') \triangleright_{ad} y'] S [s_0(x'') \triangleright_{ad} y''] = \epsilon(x)\epsilon(y) 1_H,
\]

which implies

\[
(s_1s_0d_1(x) \triangleright_{ad} y) = (s_0(x) \triangleright_{ad} y).
\]

By using this conclusion, we have

\[
\{xy, z\} = \sum_{(xy) (z)} [s_1(xy') \triangleright_{ad} s_1(z')] S [s_0(xy'') \triangleright_{ad} s_1(z'')]
\]
\[
\because \text{ direct calculations}
\]
\[
= \sum_{(x) (y) (z)} \{x', y' \triangleright_{ad} z'\} d_1(x'') \triangleright_\rho \{y'', z''\}.
\]
6) We have
\[
\{\partial_2(k), x\} = \sum_{(k)(x)} \left[ s_1 d_2(k') \triangleright_{ad} s_1(x') \right] S \left[ s_0 d_2(k'') \triangleright_{ad} s_1(x'') \right]
\]
and
\[
\{x, \partial_2(k)\} = \sum_{(k)(x)} \left[ s_1(x') \triangleright_{ad} s_1 d_2(k') \right] S \left[ s_0(x'') \triangleright_{ad} s_1 d_2(k'') \right]
\]
that imply
\[
\sum_{(k)(x)} \{\partial_2(k'), x'\}\{x'', \partial_2(k'')\}
\]
that correspond to the functors appear in Propositions
\[
\text{SimpGrp}_{\leq 2} \xrightarrow{\text{Gl}} \text{SimpHopf}_{\leq 2} \xrightarrow{\text{Prim}} \text{SimpLie}_{\leq 2}
\]
that all fitting into the diagram
\[
\text{X}_2\text{Grp} \xrightarrow{\text{Gl}} \text{X}_2\text{Hopf} \xrightarrow{\text{Prim}} \text{X}_2\text{Lie}
\]
where we refer [Conduché, 1984, Mutlu and Porter, 1998] for SimpGrp\(\leq 2\) \(\rightarrow\) Grp; and also [Ellis, 1993b] for SimpLie\(\leq 2\) \(\rightarrow\) Lie. One level fewer, one can analogously obtain the crossed module version of the diagram (23) that yields a new connections considering [Casas et al., 2014, Casas et al., 2017] in which they examine the relations between the category of crossed modules of groups, of Lie algebras, of Leibniz algebras and of associative algebras.

All in all, we have unified the 2-crossed module notions of groups and of Lie algebras as well as their relationships with Moore complex, in the category of (cocommutative) Hopf algebras.

References

[Alonso Alvarez et al., 2018] Alonso Alvarez, J. N., Fernández Vilaboa, J. M., and González Rodríguez, R. (2018). Crossed products of crossed modules of Hopf monoids. *Theory Appl. Categ.*, 33:868–897.

[Baez and Lauda, 2004] Baez, J. C. and Lauda, A. D. (2004). Higher-dimensional algebra. V: 2-Groups. *Theory Appl. Categ.*, 12:423–491.

[Baues, 1991] Baues, H. J. (1991). *Combinatorial homotopy and 4-dimensional complexes*. Berlin etc.: Walter de Gruyter.

[Bourn, 2007] Bourn, D. (2007). Moore normalization and Dold-Kan theorem for semi-abelian categories. In *Categories in algebra, geometry and mathematical physics. Conference and workshop in honor of Ross Street’s 60th birthday, Sydney and Canberra, Australia, July 11–16/July 18–21, 2005*, pages 105–124. Providence, RI: American Mathematical Society (AMS).

[Brown, 1987] Brown, R. (1987). From groups to groupoids: A brief survey. *Bull. Lond. Math. Soc.*, 19:113–134.

[Brown, 1999] Brown, R. (1999). Groupoids and crossed objects in algebraic topology. *Homology Homotopy Appl.*, 1:1–78.

[Brown, 2018] Brown, R. (2018). Modelling and computing homotopy types: I. *Indag. Math.*, New Ser., 29(1):459–482.

[Brown and Gilbert, 1989] Brown, R. and Gilbert, N. D. (1989). Algebraic models of 3-types and automorphism structures for crossed modules. *Proc. Lond. Math. Soc. (3)*, 59(1):51–73.

[Brown and Spencer, 1976] Brown, R. and Spencer, C. (1976). G-groupoids, crossed modules and the fundamental groupoid of a topological group. *Nederl. Akad. Wet., Proc., Ser. A*, 79:296–302.
[Bulacu et al., 2019] Bulacu, D., Caenepeel, S., Panaite, F., and Van Oystaeyen, F. (2019). Quasi-Hopf algebras. A categorical approach (to appear), volume 171. Cambridge: Cambridge University Press.

[Carrasco, 1995] Carrasco, P. (1995). Complejos hiper cruzados, cohomologías y extensiones. PhD. Thesis, Universidad de Granada.

[Carrasco and Cegarra, 1991] Carrasco, P. and Cegarra, A. (1991). Group-theoretic algebraic models for homotopy types. Journal of Pure and Applied Algebra, 75(3):195–235.

[Carrasco and Porter, 2016] Carrasco, P. and Porter, T. (2016). Coproduct of 2-crossed modules: applications to a definition of a tensor product for 2-crossed complexes. Collect. Math., 67(3):485–517.

[Casas et al., 2017] Casas, J. M., Casado, R. F., Khmaladze, E., and Ladra, M. (2017). More on crossed modules in Lie, Leibniz, associative and diassociative algebras. J. Algebra Appl., 16(6):17.

[Casas et al., 2014] Casas, J. M., Inassaridze, N., Khmaladze, E., and Ladra, M. (2014). Adjunction between crossed modules of groups and algebras. J. Homotopy Relat. Struct., 9(1):223–237.

[Conduché, 1984] Conduché, D. (1984). Modules croisés généralisés de longeur 2. J. Pure Appl. Algebra, 34:155–178.

[Daele, 2019] Daele, A. V. (2019). From hopf algebras to topological quantum groups. a short history, various aspects and some problems. arXiv:1901.04328.

[Drinfel’d, 1990] Drinfel’d, V. (1990). Quasi-Hopf algebras. Leningr. Math. J., 1(6):1419–1457.

[Ellis, 1993a] Ellis, G. (1993a). Crossed squares and combinatorial homotopy. Math. Z., 214(1):93–110.

[Ellis, 1993b] Ellis, G. J. (1993b). Homotopical aspects of Lie algebras. J. Aust. Math. Soc., Ser. A, 54(3):393–419.

[Faria Martins, 2011] Faria Martins, J. (2011). The fundamental 2-crossed complex of a reduced CW-complex. Homology Homotopy Appl., 13(2):129–157.

[Faria Martins, 2016] Faria Martins, J. (2016). Crossed modules of Hopf algebras and of associative algebras and two-dimensional holonomy. J. Geom. Phys., 99:68–110.

[Faria Martins and Picken, 2010] Faria Martins, J. and Picken, R. (2010). On two-dimensional holonomy. Trans. Am. Math. Soc., 362(11):5657–5695.
[Faria Martins and Picken, 2011] Faria Martins, J. and Picken, R. (2011). The fundamental Gray 3-groupoid of a smooth manifold and local 3-dimensional holonomy based on a 2-crossed module. *Differ. Geom. Appl.*, 29(2):179–206.

[Fernández Vilaboa et al., 2007] Fernández Vilaboa, J., López López, M., and Villanueva Novoa, E. (2007). Cat¹-Hopf algebras and crossed modules. *Commun. Algebra*, 35(1):181–191.

[Frégier and Wagemann, 2011] Frégier, Y. and Wagemann, F. (2011). On Hopf 2-algebras. *Int. Math. Res. Not.*, 2011(15):3471–3501.

[Goerss and Jardine, 1999] Goerss, P. G. and Jardine, J. F. (1999). *Simplicial homotopy theory*. Basel: Birkhäuser.

[Gohla and Faria Martins, 2013] Gohla, B. and Faria Martins, J. (2013). Pointed homotopy and pointed lax homotopy of 2-crossed module maps. *Adv. Math.*, 248:986–1049.

[Hazewinkel, 1996] Hazewinkel, M. (1996). The Leibniz-Hopf algebra and Lyndon words. *CWI Report: AM-R 9612*.

[Kamps and Porter, 2002] Kamps, K. H. and Porter, T. (2002). 2-groupoid enrichments in homotopy theory and algebra. *K-Theory*, 25(4):373–409.

[Kan, 1958] Kan, D. M. (1958). Functors involving c. s. s. complexes. *Trans. Am. Math. Soc.*, 87:330–346.

[Kassel and Loday, 1982] Kassel, C. and Loday, J.-L. (1982). Extensions centrales d’algèbres de Lie. *Ann. Inst. Fourier*, 32(4):119–142.

[Kauffman, 2018] Kauffman, L. H. (2018). Simplicial homotopy theory, link homology and Khovanov homology. *J. Knot Theory Ramifications*, 27(7):33.

[Loday, 1982] Loday, J. (1982). Spaces with finitely many non-trivial homotopy groups. *J. Pure Appl. Algebra*, 24(2):179 – 202.

[Majid, 1994] Majid, S. (1994). Algebras and Hopf algebras in braided categories. In *Advances in Hopf algebras. Conference, August 10-14, 1992, Chicago, IL, USA*, pages 55–105. New York, NY: Marcel Dekker.

[Majid, 1995] Majid, S. (1995). *Foundations of quantum group theory*. Cambridge: Cambridge Univ. Press.

[Majid, 2006] Majid, S. (2006). What is .. a quantum group? *Notices Am. Math. Soc.*, 53(1):30–31.

[Majid, 2012] Majid, S. (2012). Strict quantum 2-groups. arXiv:1208.6265.
[May, 1992] May, J. (1992). *Simplicial Objects in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press.

[Milnor, 1958] Milnor, J. (1958). The Steenrod algebra and its dual. *Ann. Math. (2)*, 67:150–171.

[Moore, 1955] Moore, J. C. (1954-1955). Homotopie des complexes monoïdaux, i. *Séminaire Henri Cartan*, 7(2):1–8. talk:18.

[Morton and Picken, 2015] Morton, J. C. and Picken, R. (2015). Transformation double categories associated to 2-group actions. *Theory Appl. Categ.*, 30:1429–1468.

[Mutlu and Porter, 1998] Mutlu, A. and Porter, T. (1998). Applications of Peiffer pairings in the Moore complex of a simplicial group. *Theory Appl. Categ.*, 4:148–173.

[Porter, 1982] Porter, T. (1982). Internal categories and crossed modules. In Kamps, K. H., Pumplün, D., and Tholen, W., editors, *Category Theory*, pages 249–255, Berlin, Heidelberg. Springer Berlin Heidelberg.

[Radford, 1985] Radford, D. E. (1985). The structure of hopf algebras with a projection. *Journal of Algebra*, 92(2):322 – 347.

[Sweedler, 1969] Sweedler, M. (1969). *Hopf algebras*. Mathematics lecture note series. W. A. Benjamin.

[Tang et al., 2007] Tang, X., Weinstein, A., and Zhu, C. (2007). Hopfish algebras. *Pac. J. Math.*, 231(1):193–216.

[Vespa and Wambst, 2018] Vespa, C. and Wambst, M. (2018). On some properties of the category of cocommutative Hopf algebras. *North-West. Eur. J. Math.*, 4:21–37.

[Whitehead, 1949] Whitehead, J. (1949). Combinatorial homotopy. ii. *Bull. Amer. Math. Soc.*, 55(5):453–496.

*Department of Mathematics and Statistics,*  
*Masaryk University,*  
*Brno, Czech Republic.*

*Email: emir@math.muni.cz*