The twisted Wilson fermion for the Standard Model on a lattice

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We present a preliminary study of a chiral-gauge-invariant Wilson term for the Standard Model on a lattice. In the strong coupling limit for doublers, this Wilson term shows that the doublers of leptons (baryons) are anti-baryons (anti-leptons), and they are decoupled by getting gauge invariant masses. We discuss that the possibility of achieving the low-energy Standard Model could be realized by the “three-fermion-cuts” at the weak coupling limit for normal modes, where baryons and anti-baryons dissolve into their constituents.

1. Twisted pairing phenomenon in the SM

The first generation of the SM contains 15 Weyl fermions. The left-handed sector comprises of 8 Weyl fermions:

\[ \nu_L, \ e_L; \ u_L, \ \bar{d}_L \]
\[ \nu^c_L, \ \bar{e}^c_L; \ u^c_L, \ \bar{d}^c_L, \]

where the subscript “c” indicates anti-particles that are right-handed, and the color index is dropped. A right-handed neutrino \( \nu_R \) is added into the right-handed sector, \( \nu_R \rightarrow L \) in eqs. (1,2) and henceforth for all equations.

By using the 8 left-handed and 8 right-handed elementary Weyl particles and their anti-particles, we can construct 8 right-handed and 8 left-handed Weyl three-fermion-states (3FS) called as “proton”, “neutron” and leptoquarks. In the symmetries of the SM, these 8+8 3FS carry the exactly same quantum numbers as 8+8 elementary Weyl fermions, however, their chiralities are just opposite.

We take the “proton” and “neutron” as an example to show how these 3FS are constructed (color index is anti-symmetrized),

\[ P_L(u^c, u, d) \sim (\bar{u}^c_L \cdot u_L)d_L + \cdots, \]
\[ N_L(d^c, d, u) \sim (\bar{d}^c_L \cdot d_L)u_L + \cdots. \]

One can check that in the gauge symmetries of the SM, the following pairs:

\[ P^c_L \leftrightarrow e_L; \quad N^c_L \leftrightarrow \nu_L, \]
\[ P_L \leftrightarrow \bar{e}^c_L; \quad N_L \leftrightarrow \nu^c_L, \]

have the same quantum numbers and opposite chiralities.

This is what we call the twisted pairing phenomenon in the SM. In fact, this twisted pairing phenomenon is the vector-like phenomenon required by the “no-go” theorem of Nielsen-Ninomiya. This is just the consequence of the anomaly-free in the SM.

Via the Eichten-Preskill mechanism, these 3FS are bound states due to strong multi-fermion vertices that we give as

\[ V_L = \bar{e}_L \cdot P^c_L + \bar{P}_L \cdot e^c_L + \bar{\nu}_L \cdot N^c_L + \bar{N}_L \cdot \nu^c_L + h.c., \]

which preserve the gauge symmetries of the SM and violate the \( U_B + L(1) \) global symmetry.

These vertices couple 16+16 Weyl fermions for giving rise to 8+8 mass terms of Majorana-type. However, these mass terms are not only \( SU_L(2) \otimes U_Y(1) \) invariant, but also \( U_{em}(1) \) and \( SU_c(3) \) invariant. In the Domain-wall fermion, we study these vertices to show this is indeed the case.

2. Twisted Wilson terms for doublers

Going onto the lattice regularization of the SM, we have the doubling problem for each species of the 16 Weyl fermions (3). We introduce the twisted Wilson terms for doublers,

\[ V_L = \bar{e}_L \Delta P^c_L + \bar{P}_L \Delta e^c_L + \bar{\nu}_L \Delta N^c_L + \bar{N}_L \Delta \nu^c_L + h.c., \]

where \( \Delta \) is the ordinary second order derivatives in the lattice with appropriate gauge fields.
to preserve the $SU_L(2) \otimes U_Y(1)$ symmetries. Analogous to the Wilson term, we expect that the twisted Wilson terms give different gauge-invariant masses to different doublers.

In the strong coupling limit $g \gg 1$, we can calculate the two-point functions for doublers ($p \simeq \pi_A$). As an example, we show the result for the sector of electron and anti-proton,

$$\int_x \langle \psi_e(x), \bar{\psi}_{\bar{e}}(x) \rangle = \frac{1}{2} \frac{\gamma_{\mu} \sin p_{\mu} P_R}{\sin^2 p_{\mu} + g^2 w^2(p)}, \quad (9)$$

$$\int_x \langle \psi_e(x), \bar{\psi}_{\bar{e}}(x) \rangle = \frac{1}{2} \frac{\gamma_{\mu} \sin p_{\mu} P_L}{\sin^2 p_{\mu} + g^2 w^2(p)}, \quad (10)$$

$$\int_x \langle \psi_e(x), \bar{\psi}_{\bar{e}}(x) \rangle = \frac{1}{2} \frac{\gamma_{\mu} \sin p_{\mu} P_L}{\sin^2 p_{\mu} + g^2 w^2(p)}. \quad (11)$$

where $w(p) = \sum_{\mu}(1 - \cos p_{\mu})$ and $\int_x \frac{1}{2} \frac{1}{\sin^2 p_{\mu} + g^2 w^2(p)}$.

The anti-proton ($P_{\bar{e}}^L$) indeed behaves as a simple pole ($p \simeq \pi_A$) in its propagator (10). Eq. (11) shows that the electron ($\psi_e$) and anti-proton ($P_{\bar{e}}^L$) mix up to form a massive “Dirac” fermion $\psi_{\pi_A}(x) = \psi_e(x) + P_{\bar{e}}^L(x)$, and its propagator is ($p \simeq \pi_A$),

$$\int_x \langle \psi_{\pi_A}(x), \bar{\psi}_{\pi_A}(x) \rangle = \frac{1}{2} \frac{\gamma_{\mu} \sin p_{\mu} + g w(p)}{\sin^2 p_{\mu} + g^2 w^2(p)}. \quad (12)$$

The doublers are thus decoupled.

In the basis of the twisted Wilson terms and the spectrum for doublers ($p \simeq \pi_A$), we find that the doublers of the electron ($\psi_e$) are the anti-proton ($P_{\bar{e}}^L$) at $p \simeq \pi_A$. Actually, the doublers of the electron ($\psi_e$) are the anti-quark $d^c_L$ at $p \simeq \pi_A$, since the anti-proton is given by

$$P_{\bar{e}}^L(p) \sim \int_{p_1, p_2} [u_L(p_1) \cdot u_L(p_2)] d^c_L(p_3) \cdot \cdots \cdot (13)$$

which is composed by the anti-quark $d^c_L$ at $p \simeq \pi_A$ and a soft $(p_1 - p_2 \sim 0)$ pair of the quark $u_L(p_1)$ and anti-quark $u_L(p_2)$, where the $p_1$ and $p_2$ must be in the same Brillouin zone.

On the other hand, the doublers of the quark $d_L$ are the leptoquark $[u_L \cdot u_L] e^c_L$ at $p \simeq \pi_A$, and this means that the doublers of the quark $d_L$ are the anti-electron ($\bar{e}^c_L$) at $p \simeq \pi_A$. Similar discussions can be applied to the neutrino-quitark sector. In the Brillouin zones ($p \simeq \pi_A$), we have,

$$e_{L,R} \rightarrow d^c_{L,R} \quad \text{and} \quad \bar{e}^c_{L,R} \rightarrow d_{L,R} \quad (15)$$

$$\nu_{L,R} \rightarrow u^c_{L,R} \quad \text{and} \quad \bar{\nu}_{L,R} \rightarrow u_{L,R} \quad (16)$$

In the twisted Wilson terms, the doublers of leptons and quarks (together with an appropriate pair of quark and anti-quark) are twisted to preserve the chiral gauge symmetries of the SM. This is in agreement with the “no-go” theorem.

3. The shift-symmetries of $e_{L,R}$ and $\nu_{L,R}$

The vertices $[\bar{e}^c_L \gamma_{\mu} e^c_L]$ possess the shift symmetries $[\bar{e}^c_L \gamma_{\mu} e^c_L] \sim [\bar{e}^c_L \gamma_{\mu} e^c_L]$ of $e_{L,R}$ and $\nu_{L,R}$. The Ward identities corresponding to these symmetries are, e.g.,

$$i \frac{\gamma_{\mu} \partial_{\mu} e^c_L(x) + g a w^c L(x) - \frac{\delta_{\mu} \partial_{\mu} w^c L(x)}{\delta e^c_L(x)} = 0; \quad (17)$$

$$i \frac{\gamma_{\mu} \partial_{\mu} w^c L(x) + g a w^c L(x) - \frac{\delta_{\mu} \partial_{\mu} w^c L(x)}{\delta w^c L(x)} = 0, \quad (18)$$

where $\Gamma$ is the vacuum functional and $\langle \cdots \rangle$ is the vacuum expectation value. The primed fields $\psi' = \langle \psi \rangle$. The Ward identities (17,18) are helpful to find 1PI vertices that describe the dynamics we are looking for in the phase diagram.

Based on these Ward identities, we can obtain the various identities,

$$\int_x \frac{\delta^2 \Gamma}{\delta e^c_L(x) \delta e^c_L(0)} = \frac{1}{a} \gamma_{\mu} \sin p_{\mu} P_L; \quad (19)$$

$$\int_x \frac{\delta^2 \Gamma}{\delta \nu_{L,R}(x) \delta \nu_{L,R}(0)} = \frac{1}{a} \gamma_{\mu} \sin p_{\mu} P_L, \quad (20)$$

which show that lepton fields do not receive the wave-function renormalization $Z_{\nu_e}$, and

$$\int_x \frac{\delta^2 \Gamma}{\delta \nu_{L,R}(x) \delta \nu_{L,R}(0)} = \frac{1}{2} \Sigma_{\nu}(p) = 0, \quad (21)$$

$$\int_x \frac{\delta^2 \Gamma}{\delta \nu_{L,R}(x) \delta \nu_{L,R}(0)} = \frac{1}{2} \Sigma_{\nu}(p) = 0. \quad (22)$$

These equations are independent of the coupling $g$. Analogously, we can also obtain the vacuum expectation values $\langle \bar{q}_{R} \cdot \Gamma \rangle = 0$, where $q = u, d$, and $L \rightarrow R, R \rightarrow L$.

4. Hard spontaneous symmetry breaking ?

As for the vacuum expectation values $\langle \bar{q}_{L} \cdot q_{R} \rangle$, where $q = u, d$ and $q' = u, d$, we expect that they...
are zero, since they are the vacuum expectation values of connections between the left-vertex \( V_L \) and the right vertex \( V_R \),

\[
\langle \bar{q}_L \cdot q'_R \rangle \sim \langle \bar{l}_L \cdot l'_R \rangle = 0,
\]

(23)

where \( l, l' = \nu, e \). This is due to the fact that the left-handed fields and the right-handed fields are completely separated and the shift-symmetry protects such separation.

However, for the weak multi-fermion coupling \( "g" \), a spontaneous symmetry breaking is expected. The vacuum expectation values of fields within the L-(or R-) sector, e.g. \( \langle \bar{q}_L \cdot q'_L \rangle \), break symmetries. We need to do dynamical calculations to find where the broken phase is.

We expect a symmetric segment \( (g_c < g < \infty) \) in the strong coupling \( "g" \) phase, where normal fermion modes could get around the broken phase and not form the 3FS. Analytical and numerical work is needed to verify that such a symmetric “segment” indeed exists.

5. Chiral fermions at three-fermion-cuts

The most important question is what the spectrum for normal modes \( (p \sim 0) \) in the continuum limit would be in this symmetric segment. There are two possible cases.

- the spectrum is vector-like;
- at the scale \( \epsilon \) \((250\text{GeV} < \epsilon \ll \frac{\pi}{a})\) for the continuum limit, the 3FS dissolve into the three-fermion-cuts that are the virtual states of three free chiral fermions and have the same gauge quantum numbers of the 3FS.

We are probably ended up with the first case, in which 3FS are simple poles for \( p \sim 0 \). A plausible study for this case to happen bases on the locality of the theory that leads to the continuation of the simple poles, i.e. 3FS, in the whole Brillouin zone.

Nevertheless, we try to argue for the second case. In the continuum limit \( p \rightarrow 0 \), the effective multi-fermion coupling \( g(p) \) goes to zero, the 3FS are no longer simple poles, but dissolve into three-fermion-cuts on the physical sheet, where the “no-go” theorem does not apply. This is indeed physically feasible.

Let us look at the wave-function renormalization \( Z_3 \)'s of the 3FS, which characterize the 3FS propagating as particles (simple poles), e.g.,

\[
\int_{x} \langle P^c_L(0), P^c_L(x) \rangle \sim \frac{Z_3^2}{p + m(p)} P_R.
\]

(24)

\( Z_3 \)'s can be calculated by the Ward identities.

\[
Z_3^p(p) = \int_{x} \frac{\delta^2 \Gamma}{\delta P^c_L(x) \delta e^c_L(0)} = gw(p),
\]

(25)

\[
Z^n_3(p) = \int_{x} \frac{\delta^2 \Gamma}{\delta N^c_L(x) \delta \nu^c_L(0)} = gw(p).
\]

(26)

And \( Z_3^p(p) \rightarrow 0, Z_3^n(p) \rightarrow 0 \) for \( p \rightarrow 0 \). This indicates that the 3FS at \( p \sim 0 \) increase their size, and would turn into cuts.

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