Abstract The global time is defined in covariant form under the condition of a constant mean curvature slicing of spacetime. The background static metric is taken in the tangent space. The global intrinsic time is identified with the logarithmic function of the mean value of the ratio of the square root of the metric determinants. The procedures of the Hamiltonian reduction and deparametrization of dynamical systems are implemented. The Hamiltonian system appears to be non-conservative. The Hamiltonian equations of motion of gravitational field in the global time are written.
The intrinsic time interval $\delta D$ was implemented in the symplectic 1-form \([9]\). The extended phase space, Lie-dragged along the coordinate time evolution vector, can be introduced
\[
f := f_b(x)dx^k \otimes dx^k.
\]

The Minkowskian metric as a background one was used in \([8]\) for description of gravitational problems in asymptotically flat spacetimes. We claim that the background metric has to be chosen from the physical point of view. The mapping of the Riemannian space with metric $\gamma$ \([2]\) to the background space with the metric $f$ \([3]\) should be bijective. For this reason we suggest to consider a closed manifold. The conformal metric \([3]\)
\[
\tilde{\gamma} := \tilde{\gamma}_{ik}(t,x)dx^i \otimes dx^k
\]
is a tensor field, i.e., it is transformed according to the tensor representation of the group of diffeomorphisms. The scaling factor $(\gamma/f)$ is a scalar field, i.e., it is invariant under the diffeomorphisms. To the conformal variables
\[
\tilde{\gamma}_{ij} := \frac{\gamma_{ij}}{\gamma^{1/2}/f}, \quad \tilde{\pi}^{ij} := \frac{\sqrt{\gamma}}{f} \left( \pi^{ij} - \frac{1}{3} \gamma^{ij} \right),
\]
we add the canonical pair: a local intrinsic time $D$ and a trace of momentum density $\pi$:
\[
D := -\frac{2}{3} \ln \frac{\sqrt{f}}{\tilde{\gamma}^{1/2}}, \quad \pi = 2K\sqrt{\gamma}.
\]

Formulae \([9]\) and \([10]\) define (the scaled Dirac’s mapping) \([7]\)
\[
(\gamma_{ij}, \pi^{ij}) \rightarrow (D, \pi; \tilde{\gamma}_{ij}, \tilde{\pi}^{ij}).
\]
The intrinsic time interval $\delta D$ that is a scalar under diffeomorphisms, but without adding an auxiliary metric, was implemented in the symplectic 1-form \([9]\).

The Poisson brackets between new variables in the extended phase space $\Gamma_D$ are
\[
\{D(t,x), \pi(t,x')\} = -\delta(x-x'),
\]
\[
\{\tilde{\gamma}_{ij}(t,x), \tilde{\pi}^{kl}(t,x')\} = \delta_{ij}^{kl} \delta(x-x'),
\]
\[
\{\tilde{\pi}^{ij}(t,x), \tilde{\pi}^{kl}(t,x')\} = \frac{1}{3}(\tilde{\gamma}_{ij}^{kl} - \tilde{\gamma}_{ij}^{kl}) \delta(x-x')(11)
\]
where
\[
\delta_{ij}^{kl} := \delta_{ik}^{ij} + \delta_{il}^{ij} - \frac{1}{3} \tilde{\gamma}_{ij}^{kl}
\]
is the conformal Kronecker symbol.

The Hamiltonian constraint in the new variables yields the Lichnerowicz–York differential equation
\[
\left( \tilde{\Delta} - \frac{1}{8} \tilde{R} \right) \phi + \frac{1}{8} \tilde{\pi}_{ij} \tilde{\pi}^{ij} \phi^{-7} - \frac{1}{12} K^2 \phi^5 + \frac{1}{8} \tilde{T}_{+++} \phi^3 = 0.
\]

Here $\tilde{\nabla}_k$ is the conformal connection associated with the conformal metric $\tilde{\gamma}_{ij}$; quantity $\tilde{R}$ is the conformal Ricci scalar expressed through the standard Ricci scalar $R$:
\[
R = \frac{1}{\phi^4} \tilde{R} - \frac{8}{\phi^4} \tilde{\Delta} \phi.
\]
The matter density is transformed according to
\[
\tilde{T}_{++} := \phi^4 T_{++}.
\]

### 3 Hamiltonian reduction and deparametrization

The momentum density $\pi$ enters into the conformal Hamiltonian constraint quadratically \([12]\) as usual for relativistic theories. So, with the plus sign, it can be expressed from the constraint
\[
\pi[\tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; \phi; x]
\]
\[
= 4\sqrt{3}\sqrt{\gamma} \left( \frac{1}{\phi^5} \left( \tilde{\Delta} - \frac{1}{8} \tilde{R} \right) \phi + \frac{1}{8\phi^{12}} \tilde{\pi}_{ij} \tilde{\pi}^{ij} + \frac{1}{8} \tilde{T}_{+++} \right)^{1/2}.
\]

Substituting the extracted $\pi$ \([13]\) into the ADM action \([5]\), we obtain the functional presimplectic 1-form with dilaton field $D$ playing the role of a local time \(10\)
\[
\omega = \int_{\Sigma_D} d^3x \left[ \tilde{\pi}^{ij}(\tilde{\gamma}_{ij} - \pi[\tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; D; x])dD - N^i \mathcal{H}_i \right]
\]

where $\mathcal{H}_i$ are supermomenta.

Taking in the role of the Hamiltonian $H(x)$ the integral over the hypersurface of $\pi(x)$
\[
H = \int_{\Sigma} d^3x \left( \tilde{\pi}^{ij}(\tilde{\gamma}_{ij} - \pi[\tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; D; x])dD - N^i \mathcal{H}_i \right)
\]
one should find the canonically conjugated global time $T$. Let us extract the zero mode out of the scalar field that is the square root of ratio of the determinants of metric tensors $\sqrt{\gamma/f}$
\[
\sqrt{f^{ij}}(x) = \sqrt{\gamma^{ij}} + \frac{\tilde{\gamma}^{ij}}{\gamma^{ij}}(x),
\]
where the mean value of it over the hypersurface \(\Sigma_t\) is
\[
<\sqrt{\gamma}/f> := \frac{\int_{\Sigma_t} d^3y \sqrt{\gamma(y)} \sqrt{\gamma/f(y)}}{\int_{\Sigma_t} d^3y \sqrt{\gamma(y)}}.
\] (19)

The second term in (18) is its residue with a zero mean value
\[
\int_{\Sigma_t} d^3y \sqrt{\gamma(y)} \sqrt{\gamma/f(y)} = 0.
\] (20)

The Poisson bracket between the mean value (19) and the Hamiltonian (17) reads
\[
\{<\sqrt{\gamma}/f>, H[\gamma ij, \pi ij]\} = \frac{3}{2} <\sqrt{\gamma}/f> .
\] (21)

Now we can define the global time
\[
T(t) := \frac{2}{3} \ln <\sqrt{\gamma}/f>
\] (22)
as the logarithmic function of the mean value over a hypersurface \(\Sigma_t\) at every instant \(t\). Thus, one gets the canonical pair (17) and (22) that commutes to minus unit:
\[
\{T, H\} = -1.
\] (23)

The application of the perfect cosmological principle to formula (22) for the Friedmann–Robertson–Walker interval yields the ratio of Universe radius \(a(t)\) to the present day one \(a_0\) from the standard cosmological conception
\[
\frac{a(t)}{a_0} = e^{-(3/2)T}.
\]
Here, for the background space a sphere of the present day radius \(a_0\) was taken. Thus, the global time is related to the observable redshift
\[
z(t) = \frac{a_0 - a(t)}{a(t)}.
\]
The constant mean curvature slicing allows to obtain the global time by integration over the manifold \(\Sigma_t\)
\[
\int_{\Sigma_t} d^3x \pi \frac{dD}{dt} = H \frac{dT}{dt}.
\]
Then, we extract the zero mode of the field \(\pi(x)\), that is a scalar density
\[
\pi(x) = \sqrt{\gamma}(y) <\pi > + \bar{\pi}(x),
\] (24)
where the mean value of \(\pi(x)\) over a hypersurface \(\Sigma_t\) is
\[
<\pi> := \frac{\int_{\Sigma_t} d^3y \pi(y)}{\int_{\Sigma_t} d^3y \sqrt{\gamma(y)}}.
\] (25)

Two third of the average of \(\pi\) over \(\Sigma_t\) is the global York time and the conjugated variable (the Hamiltonian) is the volume of the hypersurface in [11]. Our approach is different because of these geometric characteristics possess in our case quite another essence. In general, canonical momenta are not defined within the hypersurface. They refer to motion in time of the original \(\Sigma_t\). Intrinsic time is the variable constructed entirely out of the metric of the hypersurface. In a sense, the roles of the Hamiltonian and global time are interchanged in [11] in contradiction with the general principles.

The second term in (24) is the residue with zero mean value over a hypersurface \(\Sigma_t\)
\[
\int_{\Sigma_t} d^3x \bar{\pi}(x) = 0.
\] (26)

Thus, the mapping of phase space \(\Gamma_D\) to the phase space \(\Gamma\) after extraction of the global variables \(T\) and \(H\) is executed:
\[
(D, \pi; \tilde{\gamma} ij, \tilde{\pi} ij) \rightarrow (T, H, \bar{\pi}, \sqrt{\gamma}/f; \tilde{\gamma} ij, \tilde{\pi} ij).
\]

The Poisson brackets of the residues with the Hamiltonian are
\[
\sqrt{\gamma}/f, H = \frac{3}{2} \left(e^{-(3/2)T} + \sqrt{\gamma}/f\right),
\] (27)
\[
\{\pi, H\} = 0.
\] (28)

The residues commute with the global time
\[
\sqrt{\gamma}/f, T = 0,
\]
other commutation relations \(\{\bar{\pi}, \bar{T}\}, \{\sqrt{\gamma}/f, \pi\}\) are not required further. Then, the integral Hamiltonian reduction to the phase space \(\tilde{\Gamma}_D\) and the deparametrization procedure are performed
\[
(D, \pi; \tilde{\gamma} ij, \tilde{\pi} ij) \rightarrow (\tilde{\gamma} ij, \tilde{\pi} ij).
\]

That yields the action
\[
S = \int_{\Sigma_t} dT \int d^3x \left[\tilde{\pi} ij d\tilde{\gamma} ij + H \bar{\pi}\right] - \int_{\Sigma_t} H dT -
\] (29)
\[
\int_{\Sigma_t} dT \int d^3x N^i \mathcal{H}_i
\] (30)

with the Hamiltonian depending on the global time \(T\)
\[
H[T(t), \sqrt{\gamma}/f(x); \tilde{\pi} ij(x), \tilde{\gamma} ij(x)] := \int_{\Sigma_t} d^3x \mathcal{H}(x)
\] (31)
with the Hamiltonian density
\[ H(x) = 4\sqrt{3}\sqrt{\gamma} \left[ \frac{1}{\phi^2} \left( \tilde{\Delta} - \frac{1}{8} \tilde{R} \right) + \frac{1}{8\phi^2} \tilde{\pi}_{ij} \tilde{\pi}^{ij} + \frac{1}{8} \tilde{T}_{11} \right]^{1/2}. \]

According to the commutation relations \((27), (28)\), and the Hamiltonian \((31)\), the residue function \(\sqrt{\gamma/\mathcal{H}}(x)\) does not depend on time \(T\). Functions of \(\phi(x)\) and \(\sqrt{\gamma}\) in Eq. \((32)\) can be expressed via the global time and the residues, e.g., function \(\phi^{12}(x)\) is a sum of the time-dependent function and the space-dependent one:
\[ \phi^{12}(x) = \exp \left( -\frac{3}{2} \frac{1}{\xi} \right) + \frac{\sqrt{\gamma}}{T(x)}. \] \(4)\) Equations of motion

The momentum constraints generate spatial diffeomorphisms by the last term in action \((30)\). Below, studying dynamics, we are not interested in changes of coordinates of the hypersurface. The energy of the universe is not conserved, it exponentially increases in time \(T\). The Hamiltonian flow is governed by Hamiltonian \((31)\) with the Poisson brackets \((10)\) and \((11)\):
\[ \frac{d}{dt} \tilde{\gamma}_{ij}(x) = \int_{\Sigma_t} d^3x' \{ \tilde{\gamma}_{ij}(x), \tilde{\pi}^{kl}(x') \} \frac{\delta}{\delta \tilde{\pi}^{kl}(x')} H, \]
\[ \frac{d}{dt} \tilde{\pi}^{ij}(x) = \int_{\Sigma_t} d^3x' \{ \tilde{\pi}^{ij}(x), \tilde{\pi}^{kl}(x') \} \frac{\delta}{\delta \tilde{\pi}^{kl}(x')} H. \]

The functional derivative with respect to the momentum density reads
\[ \frac{\delta}{\delta \tilde{\pi}^{kl}(x')} H[\phi; \tilde{\pi}^{ij}, \tilde{\gamma}_{ij}] = \frac{6\gamma(x')}{\phi^{12}(x') H[\phi; \tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; x]} \tilde{\pi}^{kl}(x'). \]

The derivative of the conformal metric with respect to the global time \((34)\) after application of \((10)\) and \((36)\) becomes
\[ \frac{d}{dt} \tilde{\gamma}_{ij}(x) = \frac{12\gamma(x') \tilde{\pi}_{ij}(x)}{\phi^{12}(x') H[\phi; \tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; x]} \]

Thus, the relation between the derivative of the generalized coordinates \(\tilde{\gamma}_{ij}\) with respect to the global time \(T\) with the conjugate generalized momenta \(\tilde{\pi}^{ij}\) is obtained.

The first term in \((38)\) is calculated easily
\[ \int d^3x' \{ \tilde{\pi}^{ij}(x), \tilde{\pi}^{kl}(x') \} \frac{\delta}{\delta \tilde{\pi}^{kl}(x')} H \]
\[ = -2\gamma(x) \tilde{\gamma}^{ij}(x) \tilde{\pi}^{kl}(x) \tilde{\pi}_{kl}(x) \frac{\phi^{12}(x) H[\phi; \tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; x]}{\phi^{12}(x) H[\phi; \tilde{\pi}^{ij}, \tilde{\gamma}_{ij}; x]} \]

One can write the functional derivative of the Hamiltonian density \(H\) with respect to the conformal metric components \(\gamma_{kl}\) as
\[ \frac{\delta}{\delta \gamma_{kl}(x)} H[\phi; \tilde{\gamma}_{ij}, \tilde{\pi}^{ij}] = \int_{\Sigma_t} d^3y \left( -3\gamma(y) \frac{\phi^{12}(y)}{\phi^{4}(y) H(y)} \right. \]
\[ \times \frac{\delta}{\delta \gamma_{kl}(x)} \tilde{R}[\tilde{\gamma}_{ij}, y] + \frac{24\gamma(y)}{\phi^{4}(y) H(y)} \frac{\delta}{\delta \gamma_{kl}(x)} \tilde{R}[\tilde{\gamma}_{ij}, y] \left. \right). \]

The functional derivative of the Ricci scalar with respect to the metric coefficients reads
\[ \frac{\delta}{\delta \gamma_{kl}(x)} \tilde{R}[\tilde{\gamma}_{ij}, y] = \left( -\tilde{R}^{kl}[\tilde{\gamma}_{ij}, y] + \tilde{\gamma}^{kl}(y) \tilde{\Delta} y \right. \]
\[ - \nabla^k \nabla^l \right) \delta(x - y). \]

By summing up these four terms we obtain the functional derivative
\[ \frac{\delta}{\delta \gamma_{kl}(x)} H[\phi; \tilde{\gamma}_{ij}, \tilde{\pi}^{ij}] = \frac{3\gamma(x)}{\phi^{4}(x) H(x)} \tilde{R}^{kl} \]
\[ + 3 \left( \tilde{\nabla}^k \tilde{\nabla}^l - \tilde{\gamma}^{kl}(x) \tilde{\Delta} x \right) \left[ \frac{\gamma(x)}{\phi^{4}(x) H(x)} \right] \]
\[ + 12(2\tilde{\gamma}^{km} \tilde{\gamma}^{ln} - \tilde{\gamma}^{kl} \tilde{\gamma}^{mn}) \tilde{\nabla}^m \left[ \frac{\gamma(x)}{\phi^{4}(x) H(x)} \right] \tilde{\nabla}^n \phi(x). \]

Finally, the derivative of the conformal momentum density with respect to the global time \((35)\) with application of the commutation relations between conformal phase variables, taking into account \((38)\) and \((39)\), becomes
\[ \frac{d}{dt} \tilde{\pi}^{ij} = -\frac{6\gamma(x)}{\phi^{4}(x) H(x)} \left( \tilde{R}^{ij} - \frac{1}{6} \tilde{\gamma}^{ij} \tilde{R} \right) \]
\[ - \frac{2\gamma(x)}{\phi^{4}(x) H(x)} \tilde{\gamma}^{ij}(x) \tilde{\pi}_{kl} \]
\[ - 3 \left( \tilde{\nabla}^i \tilde{\nabla}^j + \tilde{\nabla}^j \tilde{\nabla}^i - \frac{3}{4} \tilde{\gamma}^{ij} \nabla^k \nabla_k \right) \left[ \frac{\gamma(x)}{\phi^{4}(x) H(x)} \right] \]
\[ - 24 \left( \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} + \tilde{\gamma}^{ik} \tilde{\gamma}^{jl} - \frac{3}{2} \tilde{\gamma}^{ijkl} \right) \nabla_i \nabla_j \left[ \frac{\gamma(x)}{\phi^{4}(x) H(x)} \right] . \]

5 Conclusions and discussion

We demonstrated that in Geometrodynamics of closed manifolds, it is possible to generalize the Misner approach and introduce the global time. After application
of the Hamiltonian reduction procedure, one yields differential evolution equations for conformal metric components and conformal momentum densities. In opposite to the case of asymptotically flat, for closed manifolds we get non-conservative Hamiltonian systems.

The deviation of the mean value of the global times appears as a classical scalar field, it deserves additional attention to be physically interpreted. Note that it emerged without any modification of the Einstein’s theory. The Wheeler’s thin sandwich conjecture in General Relativity under positive lapse and some restriction is valid [13]. Thus, the lapse function can be found from the Hamiltonian constraint. It was supposed that the deviations from the mean value of the global time can play the role of static gravitational potentials [12].

Earlier to construct a scalar, the Minkowskian metric as a background one was used for asymptotically flat spaces [8]. The intrinsic time interval $\delta D$ as a scalar was implemented in the symplectic 1-form [9]. For splitting one degree of freedom the average of the trace of the momentum density was used as York time in the shape dynamics [11]. The key difference of our study is the consideration closed manifolds without the asymptotically flat space condition. Our choice is motivated by cosmological applications.

For interpretation of the latest data of the Hubble diagram, the global time as the scale factor of the Friedmann–Robertson–Walker model was successfully implemented in refs. [14][15]. The choice of conformal variables allows to suggest a new interpretation of the redshift of distant stellar objects. Both the changing volume of the Universe in standard cosmology and the changing or masses of elementary particles in conformal cosmology [16] can serve as the measure of time. Above we have shown that the observed expansion of the Universe can be directly related to the global time of the Universe.

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