Non-degenerate graded Lie algebras with a degenerate transitive subalgebra

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Abstract

The property of degeneration of modular graded Lie algebras, first investigated by B. Weisfeiler, is analyzed. Transitive irreducible graded Lie algebras $L = \sum_{i \in \mathbb{Z}} L_i$, over algebraically closed fields of characteristic $p > 2$, with classical reductive component $L_0$ are considered. We show that if a non-degenerate Lie algebra $L$ contains a transitive degenerate subalgebra $L'$ such that $\dim L'_1 > 1$, then $L$ is an infinite-dimensional Lie algebra.

0 Introduction

One of the most important steps the program of classifying the simple finite-dimensional Lie algebras of characteristic $p > 0$, a program developed by A.I. Kostrikin and I.R. Shafarevich [10], is the investigation of non-contractible filtrations of simple Lie algebras. Let $L_0$ be a maximal subalgebra in a simple Lie algebra $L$ such that the nilradical of the adjoint representation of $L_0$ on $L$ is nontrivial, let $L_{-1}$ be an $L_0$-submodule of $L$ such that $L_0 \subset L_{-1}$, and suppose that $L_{-1}/L_0$ is an irreducible $L_0$-submodule. The non-contractible filtration of $L$ corresponding to the pair $(L_{-1}, L_0)$ is constructed by induction:

$$L_{-i} = [L_{-1}, L_{-i+1}] + L_{-i+1}, L_i = \{l \in L_{i-1} | [l, L_{-1}] \subset L_{i-1}\}, i > 0.$$ 

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It follows from the definition of a non-contractible filtration that the associated graded Lie algebra \( g = \bigoplus_{i=0}^r g_i \) possesses the following properties:

1. (irreducibility) \( g_{-1} \) is an irreducible \( g_0 \)-module;
2. (transitivity) for any \( j \geq 0 \) if \( x \in g_j \) and \([x, g_{-1}] = 0\), then \( x = 0\);
3. \( g_{-i} = [g_{-i+1}, g_{-1}] \) for any \( i > 1\);

A graded Lie algebra \( g = \bigoplus_{i \in \mathbb{Z}} g_i \) satisfying conditions (1) - (3) is called a transitive irreducible Lie algebra. A graded subalgebra \( g' \subset g \) is called transitive if \( g'_{-1} = g_{-1} \). In [14] and [15], B. Weisfeiler investigated the properties of finite-dimensional transitive irreducible graded Lie algebras \( g \) over algebraically closed fields of characteristic \( p > 0 \) and obtained results of fundamental importance for the theory of modular graded Lie algebras. He showed ([15]) that the subalgebra \( g^- = \sum_{i<0} g_i \) contains a unique maximal ideal \( M = M(g) \) of \( g \) called the Weisfeiler radical of \( g \), such that the factor algebra \( g/M \) is a semisimple Lie algebra with a unique minimal ideal \( I \). The centroid of the ideal \( I \) is a truncated polynomial algebra.

Lie algebras of the form \( g \) fall into two classes: non-degenerate and degenerate Lie algebras. In the non-degenerate case, the centroid of the ideal \( I \) has zero degree, and the grading of \( g/M \) is determined by the grading of a simple Lie algebra, namely the core of the differentially simple ideal \( I \). In the degenerate case, the grading in \( g \) is determined by a nontrivial grading of the centroid; in addition, \( g_2 = 0 \) and \([g_{-1}, g_1], g_1] = 0\). Weisfeiler showed ([15], Proposition 3.2.1) that the last property is a criterion for degeneration in finite-dimensional transitive irreducible Lie algebras \( g \).

When investigating the transitive irreducible graded Lie algebras generated by the local part \( g_{-1} + g_0 + g_1 \), it is not \( a \) priori clear whether the algebra \( g \) is finite dimensional or not. Therefore, we assume Weisfeiler’s criterion of degeneration of finite-dimensional Lie algebras to be the definition of a degenerate (resp. non-degenerate) transitive irreducible \( \mathbb{Z} \)-graded Lie algebra \( g \). Properties (1) - (3) are asymmetric with respect to \( g_{-1} \) and \( g_1 \). It is therefore natural to consider the subalgebra \( g' \) in \( g \) generated by the local part \( g_{-1} + g_0 + g_1' \) where \( g_1' \) is an irreducible \( g_0 \)-submodule of \( g_1 \). (Note that \( g' \) satisfies the above definition of a transitive subalgebra.) However, even if \( g \) is non-degenerate, \( g' \) might nonetheless be degenerate. We investigate this problem for the case in which \( g_0 \) is a classical reductive Lie algebra. This case is of particular interest for the classification theory of simple modular...
Lie algebras. The Recognition Theorem describing such Lie algebras satisfying the additional condition of transitivity with respect to \( g_1 \), was obtained by V. Kac [8] for \( p > 5 \) under assumption that \( g_{-1} \) is a restricted \( g_0 \)-module. G. M. Benkart, T.B. Gregory and A. A. Premet [2] extended the Recognition Theorem for the case \( p > 3 \). When \( p = 3 \), there exist Lie algebras which are unlike any in characteristics \( p > 3 \) yet satisfy the conditions of the Recognition Theorem (see [3], [5], [12]). In the present paper, a theorem (Theorem 2.1) is proved that is of great importance for the classification of graded Lie algebras with a classical reductive component \( g_0 \) and nonrestricted \( g_0 \)-module \( g_{-1} \) when \( p > 2 \).

**Theorem 0.1** Let \( L = \bigoplus_{i \in \mathbb{Z}} L_i \) be a non-degenerate transitive irreducible graded Lie algebra with classical reductive component \( L_0 \) over an algebraically closed field \( F \) of characteristic \( p > 2 \). If \( L \) contains a degenerate transitive subalgebra \( L' \) such that \( \dim L'_1 > 1 \), then \( L \) is an infinite-dimensional Lie algebra.

By transitivity, the representation of \( L'_0 \) on \( L_1 \) is restricted when and only when the representation of \( L'_0 \) on \( L_{-1} \) is restricted. Since no non-restricted representation of \( L'_0 \) can have dimension 1, we have the following corollary.

**Corollary 0.2** Let \( L \) be as in the above theorem, and suppose that the representation of \( L'_0 \) on \( L_{-1} \) is not restricted. Then \( L \) is an infinite-dimensional Lie algebra.

It should be pointed out that both the case in which \( \dim L'_1 = 1 \) and the case of even characteristic are currently being investigated elsewhere.

The proof of Theorem 0.1 is given in Section 2. Section 1 contains needed definitions, notations, and results obtained by B. Weisfeiler.

## 1 Preliminaries

Recall that a classical Lie algebra over a field \( F \) of characteristic \( p > 0 \) can be obtained from a \( \mathbb{Z} \)-form (the “Chevalley basis”) of a complex simple Lie algebra by reducing the scalars modulo \( p \) and extending them to \( F \). This process may result in a Lie algebra with a non-zero center; such a Lie algebra is still referred to as “classical”, as is the quotient of such a Lie algebra by its center. For example, the Lie algebras \( \mathfrak{sl}(pk) \) and \( \mathfrak{psl}(pk) \) are both considered
to be classical Lie algebras. It could also happen that a classical Lie algebra
has a noncentral ideal, as does the Lie algebra $G_2$ if $p = 3$. The Lie algebras
$\mathfrak{gl}(pk)$ and $\mathfrak{pgl}(pk)$ are also considered classical.

A classical reductive Lie algebra $\mathfrak{g}$ is the sum of commuting ideals $\mathfrak{g}_j$
which are classical Lie algebras, and an at-most-one-dimensional center $\mathfrak{z}(\mathfrak{g})$:

$$\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_k + \mathfrak{z}(\mathfrak{g})$$

Let $\mathcal{O}_n = F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ be a truncated polynomial algebra.
Denote by $\mathfrak{W}_n$ the Lie algebra of vector fields $\text{Der}\mathcal{O}_n$. The following theorem
of B. Weisfeiler [15] plays a fundamental rôle in the study of graded Lie
algebras.

**Theorem 1.1 (Weisfeiler’s Theorem)** Let $L = L_{-q} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus$
$\cdots \oplus L_r$ be a transitive irreducible graded Lie algebra over an algebraically
closed field $F$ of characteristic $p > 0$, and let $M(L)$ be the largest ideal of $L$
contained in $L_{-q} \oplus \cdots \oplus L_{-1}$. Then

(i) $L/M(L)$ is semisimple and contains a unique minimal ideal $I = S \otimes$
$\mathcal{O}_n$, where $S$ is a simple Lie algebra and $n$ is a non-negative integer. The
ideal $I$ is a graded ideal, and $I_i = (L/M(L))_i$ for any $i < 0$.

(ii) (Degenerate case) If $I_1 = (0)$, then there exists an integer $k$, $1 \leq k \leq$
n such that the grading of $\mathcal{O}_n$, induced by the grading of $I$, is given by setting
deg$(x_i) = -1$ for $1 \leq i \leq k$, and deg$(x_i) = 0$ for $k < i \leq n$. Moreover, $I_i =$
$S \otimes \mathcal{O}_{n,i}$ for any $i$; $L_0 = (0)$; $I_0 = [L_{-1}, L_1]$; $L_1 \subseteq \{D \in 1 \otimes \mathcal{W}_n|\text{deg}(D) = 1\}$;
$[[L_{-1}, L_1], L_1] = 0$; and

$$L_0 \subset \text{Der}(S \otimes \mathcal{O}_{n-k}) + 1 \otimes \mathcal{O}_{n-k} \otimes \mathcal{W}_{k,0},\mathcal{W}_{k,0} \cong \mathfrak{gl}(\mathcal{O}_{k-1})$$

(iii) (Non-degenerate case) If $I_1 \neq (0)$, then $S$ is a graded Lie algebra, and
$I_i = S_i \otimes \mathcal{O}_n$ for any $i$. Moreover, $(0) \neq [L_{-1}, L_1] \subseteq I_0$. If $\text{Der}S = \oplus(\text{Der}S)_i$
is the grading of the Lie algebra $\text{Der}S$ induced by the grading of $S$, then

$$L_0 \subset (\text{Der}S)_0 \otimes \mathcal{O}_n + 1 \otimes \mathcal{W}_n$$

for $i > 0 G_i \subset (\text{Der}S)_i \otimes \mathcal{O}_n$.

A transitive graded Lie algebra $L = \oplus_{i \in \mathbb{Z}}L_i$ is called degenerate if $[[L_{-1},$
$L_1], L_1] = 0$. The following proposition proved by B. Weisfeiler ([15], Propo-
sition 3.2.1) motivates this definition.
Proposition 1.2 Let \( L \) be a finite-dimensional transitive irreducible graded Lie algebra, and let \( V \) be a \( L_0 \)-submodule of \( L_1 \). Suppose that \( L_0 \) is not faithful on \( V \). Then

(i) \([V, V] = 0\), and \([L_{-1}, V], V] = 0\).

(ii) Let \( L' \) be the subalgebra of \( L \) generated by \( L_{-1} + L_0 + V \). Then \( L'/M(L') \) satisfies the conditions and conclusions of the degenerate case of Theorem 1.1.

Set \( L^{-} = \oplus_{i<0} L_i \) and \( L^{+} = \oplus_{i>0} L_i \). Let \( L = L_{-q} + \ldots + L_{-1} + L_0 + L_1 + \ldots \) be a \( q \)-graded Lie algebra and set \( G = \oplus_{i \in \mathbb{Z}} L_i \). Then \( G \) is a graded subalgebra of \( L \) which can be considered to be a 1-graded Lie algebra if we set \( G_i = L_i \). Denote by \( T \) the largest ideal of \( G \) contained in \( G_0 + G^{+} \). The factor algebra \( G/T \) is denoted by \( B(L_{-q}) \). Evidently, \( B(L_{-q}) \) is a transitive 1-graded Lie algebra. The design of \( B(L_{-q}) \) is a particular case of the construction given in [1].

2 The proof of the Theorem

To simplify notation, we formulate Theorem 0.1 another way.

Theorem 2.1 Let \( L = \oplus_{i \in \mathbb{Z}} L_i \) be a non-degenerate transitive irreducible graded Lie algebra over an algebraically closed field \( F \) of characteristic \( p > 2 \) with classical reductive component \( L_0 \). If \([L_{-1}, V], V] = 0\) for some \( L_0 \)-submodule \( V \subset L_1 \) such that \( \dim V > 1 \), then \( \dim L = \infty \).

The proof of the Theorem consists of several steps. We suppose that \( \dim L < \infty \) and obtain a contradiction. In \( \text{Aut} L \), denote by \( T \) the one-dimensional torus which defines the \( \mathbb{Z} \)-grading of \( L \). All algebras constructed below will be \( T \)-invariant, and thus inherit the \( \mathbb{Z} \)-grading. Note that the symbols \( L^{\dagger} \), \( L_i^{\dagger} \), etc., used below, retain their meanings only within a particular proof segment (i.e., (a), (b), (c), etc.)

(a) If \( V_1 \) is an \( L_0 \)-submodule in \( L_1 \), \( V \subset V_1 \) and \([L_{-1}, V_1], V_1] = 0\), then \( V_1 = V \).

Let \( L^{\dagger} \) be the subalgebra of \( L \) generated by the local Lie algebra \( L_{-1} + L_0 + V_1 \). By Weisfeiler’s Theorem, \( L^{\dagger} \) is a degenerate Lie algebra and \([L_{-1}, V_1] \) is a minimal ideal of \( L_0 \). It follows from Theorem 1.1 (ii) that \( S = [L_{-1}, V_1] \) is a simple ideal of \( L_0 \) and that \( L_{-1} = S \otimes \mathcal{O}_{n,-1} \), where \( \mathcal{O}_{n,-1} = \langle x_1, \ldots, x_n \rangle \) is an irreducible \( L_0/S \)-module. Since \( V_1 = \langle \partial_1, \ldots, \partial_n \rangle = \langle x_1, \ldots, x_n \rangle^{*} \),
it follows that $V_1$ is an irreducible $L_0/S$-module and, therefore, an irreducible $L_0$-module. Hence, $V_1 = V$, so (a) is proved.

(b) Here the problem is reduced to the case where $L_1/V$ is an irreducible $L_0$-module, $L^+$ is generated by $L_1$ and $M(L) = 0$. The subalgebra $\mathcal{L}_0$ generated by $L_{-1} + L_0 + V$ is described.

Since by assumption $L$ is non-degenerate, i.e., $[[L_{-1}, L_1], L_1] \neq 0$, it follows that $L_1 \neq V$. Let $\mathcal{Y}$ be an $L_0$-submodule of $L_1$ such that $V < \mathcal{Y}$ and $0 \neq \mathcal{Y}/V$ is an irreducible $L_0$-module. It follows from (a) that $[[L_{-1}, \mathcal{Y}], \mathcal{Y}] \neq 0$. We will consider the subalgebra of $L$ generated by the local part $L_{-1} + L_0 + \mathcal{Y}$. Since we are assuming that $L$ is finite dimensional, this subalgebra is finite dimensional, also, and it satisfies the hypotheses of Theorem 2.1. Therefore, we will prove Theorem 2.1 by replacing the original $L$ (if necessary) by this subalgebra, which we will henceforth refer to as $L$. Now, out of this new $L$ we can factor its Weisfeiler radical $M(L)$. Thus, we will consider a non-degenerate finite-dimensional Lie algebra $L$ satisfying the conditions of Theorem 2.1 such that $L$ is generated by its local part, $L_1/V$ is an irreducible $L_0$-module, and $M(L) = 0$.

Denote by $\mathcal{L}_0$ the subalgebra of $L$ generated by $L_{-1} + L_0 + V$. As noted in the proof of part (a) above, it follows from Weisfeiler’s Theorem and the assumption that the null component is classical reductive that $S = [L_{-1}, V]$ is a simple ideal of $L_0$. According to (a), $V$ is an irreducible $L_0$-module. Evidently, $M(\mathcal{L}_0)$ is both a maximal $V$-invariant ideal of $L^-$ and $T$-invariant. By Weisfeiler’s Theorem, $[V, V] = 0$ and

$$\mathcal{L}_0/M(\mathcal{L}_0) = L^\dagger_{-|\delta|} + L^\dagger_{-|\delta|+1} + \ldots + L^\dagger_{-1} + L_0 + V,$$

where

$$L^\dagger_{-1} = L_{-1}, \text{ and } L^\dagger_{-i} = S \otimes \mathcal{O}_{n,-i}, i > 0;$$

here $\delta$ is the $n$-tuple $(p - 1, \ldots, p - 1)$ and $\mathcal{O}_n = \oplus_{i \leq 0} \mathcal{O}_{n,i}$ is the grading of $\mathcal{O}_n = F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ opposite to the standard grading. Since $\mathcal{O}_{n,-1} = < x_1, \ldots, x_n > \cong V^*$, it follows that $n = \dim V > 1$. Futhermore, $L_{-1} = S \otimes \mathcal{O}_{n,-1}$, and $V = id_S \otimes < \partial_1, \ldots, \partial_n > \subset \text{Hom}_F(L_{-1}, L_0)$. We will identify $V$ with $< \partial_1, \ldots, \partial_n >$. By Weisfeiler’s Theorem, $S \subset L_0 \subset (\text{Der} S) \otimes 1 + 1 \otimes W_{n,0} \subset \text{Hom}_F(L_{-1}, L_0)$, where $W_{n,0} = < x_i \partial_j, i, j = 1, \ldots, n >$ and $1$ denotes the identity map. Let $\pi : (\text{Der} S) \otimes 1 + 1 \otimes W_{n,0} \rightarrow$
Let \( W_{n,0} \) be the projection along the ideal \((\text{Der} S) \otimes 1\). Since \( L_{-1} \) is an irreducible \( L_0 \)-module, it follows that \( < x_1, \ldots, x_n > \) is an irreducible \( \pi(L_0) \)-module.

\[(c) \] Here it is shown that \( \mathcal{L}_0 \) is a maximal subalgebra of \( L \) and that \( L/\mathcal{L}_0 \) contains a unique nontrivial irreducible \( \mathcal{L}_0 \)-submodule \( \overline{\mathcal{L}_1} = \mathcal{L}_{-1}/\mathcal{L}_0 \). The non-contractible filtration of \( L \) corresponding to \( \mathcal{L}_1 \) and \( \mathcal{L}_0 \) is constructed, as is its associated graded Lie algebra \( g \).

It follows from Weisfeiler’s Theorem that for \( i > 1 \), \( \{ l \in L_i | (\text{ad} L_{-1})^{i-1} l \subset V \} = 0 \), since otherwise there would exist a degenerate Lie algebra \( L_q + \ldots + L_{-1} + L_0 + V_1 + V_2 + \ldots \) in which \( V_1 = V \) and \( V_2 = \{ l \in L_2 | (\text{ad} L_{-1}) l \subset V \} \neq 0 \). Since \( L \) is a transitive graded Lie algebra and \( L_1/V \) is an irreducible \( L_0 \)-module, it follows that any nontrivial \( \mathcal{L}_0 \)-submodule of \( L/\mathcal{L}_0 \) contains the unique irreducible \( \mathcal{L}_0 \)-submodule of \( L/\mathcal{L}_0 \), namely, the intersection of all \( \mathcal{L}_0 \)-submodules of \( L/\mathcal{L}_0 \) which contain \( (L_1 + \mathcal{L}_0)/\mathcal{L}_0 \). Since \( L^+ \) is generated by \( L_1 \), it follows that \( \mathcal{L}_0 \) is a maximal subalgebra of \( L \).

Denote by \( \mathcal{L}_{-1} \) the \( \mathcal{L}_0 \)-submodule of \( L \) such that \( \mathcal{L}_0 \subset \mathcal{L}_{-1} \) and \( \mathcal{L}_{-1}/\mathcal{L}_0 \) is the unique irreducible \( \mathcal{L}_0 \)-submodule of \( L/\mathcal{L}_0 \) described in the preceding paragraph. Note that \( L, \mathcal{L}_0, L/\mathcal{L}_0 \) and, therefore, \( \mathcal{L}_{-1} \) are invariant under the torus \( T \), so \( \mathcal{L}_{-1} \) is a \( \mathbb{Z} \)-graded subspace of \( L \). Let

\[
L = \mathcal{L}_{-s} \supset \mathcal{L}_{-s+1} \supset \cdots \supset \mathcal{L}_{-1} \supset \mathcal{L}_0 \supset \mathcal{L}_1 \supset \cdots
\]

be the non-contractible filtration of \( L \) corresponding to the pair \( (\mathcal{L}_{-1}, \mathcal{L}_0) \) where \( \mathcal{L}_{-i} = \mathcal{L}_{-i+1} + \mathcal{L}_{-i+1} \) for \( i > 1 \) and \( \mathcal{L}_j = \{ l \in \mathcal{L}_{j-1} | [\mathcal{L}_{-1}, l] \subset \mathcal{L}_{j-1} \} \) for \( j > 0 \). Obviously, the filtration \( \{ \mathcal{L}_i \} \) is invariant with respect to the torus \( T \).

Let \( g = grL = \bigoplus_{i=-s}^{r} g_i \) be the graded Lie algebra associated with the filtration \( \{ \mathcal{L}_i \} \). Since the filtration is \( T \)-invariant, it follows that \( T \) acts on \( g \) by homogeneous automorphisms. Thus, \( g_i = \bigoplus_{j \in \mathbb{Z}} g_{ij} \) and \( g = \bigoplus_{i,j \in \mathbb{Z}} g_{ij} \) is a bigrading of \( g \). The second index of \( g_{ij} \) refers to its elements’ weight with respect to the torus \( T \). Evidently, \( \mathcal{L}_1 \) is a maximal \( \text{ad} \)-nilpotent ideal of \( \mathcal{L}_0 \); therefore, \( M(\mathcal{L}_0) \subset \mathcal{L}_1 \), where \( M(\mathcal{L}_0) \) is the Weisfeiler radical of the degenerate graded Lie algebra \( \mathcal{L}_0 \). By Weisfeiler’s Theorem \( \mathcal{L}_0/M(\mathcal{L}_0) \) is a semisimple Lie algebra, so that \( M(\mathcal{L}_0) = \mathcal{L}_1 \), and

\[
g_0 = \mathcal{L}_0/\mathcal{L}_1 = \mathcal{L}_0/M(\mathcal{L}_0) = S \otimes \mathcal{O}_n + L_0 + V.
\]

The \( \mathbb{Z} \)-gradings in \( g_0 \) and in \( \mathcal{O}_n \) corresponding to the torus \( T \) coincide with the grading corresponding to the degenerate graded Lie algebra \( \mathcal{L}_0/M(\mathcal{L}_0) \) in Weisfeiler’s Theorem.
(d) Here we show that \( \mathfrak{g}_{-1} \) has the following \( \mathbb{Z} \)-grading corresponding to the torus \( T \):

\[
\mathfrak{g}_{-1} = \mathfrak{g}_{-1,1} + \ldots + \mathfrak{g}_{-1,|\delta|+1}, |\delta| = n(p - 1),
\]

where \( \mathfrak{g}_{-1,i} = (\text{ad} V)^{i-1}(L_1/V) \). Moreover, \( \mathfrak{g}_{-1} \) is a graded \( O_n \)-module where the grading in \( O_n \) is induced by the torus \( T \), \( \deg x_i = -1, i = 1, \ldots, n \), the bracket operation of \( S \otimes O_n \subset \mathfrak{g}_{-1} \) is \( O_n \)-bilinear, and \( \mathfrak{g}_{-1,|\delta|+1} \) is both an irreducible \( \mathfrak{g}_{0,0} \)-module and a nontrivial \( S \)-module. Here \( \mathfrak{g}_{0,0} \) is the 0-term of the \( \mathbb{Z} \)-grading of \( \mathfrak{g}_0 \) corresponding to the torus \( T \), and \( S = S \otimes 1 \subset L_0 = \mathfrak{g}_{0,0} \).

Denote by \( \mathfrak{h} \) the subalgebra of \( \mathfrak{g}_0 \) equal to \( S \otimes O_n + L_0 \). Then \( \mathfrak{g}_0 = V \oplus \mathfrak{h} \).

Note that \( \text{ad}_L V \) consists of nilpotent elements and that \( (\text{ad}_L l)^p = 0 \) for any \( l \in V \). Since \( \mathfrak{g}_{-1} \) is an irreducible \( \mathfrak{g}_0 \)-module, it follows that \( (\text{ad}_L l)^p = 0 \) for any \( l \in V \). Let \( A = U(\mathfrak{g}_0)/\langle l^p, l \in V \rangle \) be the quotient of the universal enveloping algebra of \( \mathfrak{g}_0 \) by the ideal generated by the set \( \{l^p, l \in V \} \), and set \( B = U(\mathfrak{h}) \). Since \( L_1/V \) is an irreducible \( L_0 \)-module, it is therefore also an irreducible \( \mathfrak{h} \)-module; consequently, \( \mathfrak{g}_{-1} \) is covered by the induced \( \mathfrak{g}_0 \)-module \( A \otimes_B (L_1 + L_0)/L_0 = F[V] \otimes_F L_1/V \) where \( F[V] = F[\partial_1, \ldots, \partial_n]/(\partial_1^p, \ldots, \partial_n^p) \) is a truncated polynomial algebra. Here \( \{\partial_1, \ldots, \partial_n\} \) is a basis of \( V \). It follows that

\[
\mathfrak{g}_{-1} = \sum_{i=0}^{|\delta|} (\text{ad} V)^i(L_1/V) = L_1/V \oplus \text{ad} V(L_1) \oplus \ldots \oplus (\text{ad} V)^{|\delta|}(L_1).
\]

In particular, the \( \mathbb{Z} \)-grading of \( \mathfrak{g}_{-1} \) is as follows

\[
\mathfrak{g}_{-1} = \bigoplus_{i=1}^{|\delta|+1} \mathfrak{g}_{-1,i}, \mathfrak{g}_{-1,i} = (\text{ad} V)^{i-1}(L_1/V).
\]

We now show that \( \mathcal{T}_1 = L_1/V \) is a nontrivial \( S \)-module. Since \( \mathcal{T}_1 \) is an irreducible \( L_0 \)-module and \( S \) is an ideal of \( L_0 \), it follows that \( [S, \mathcal{T}_1] \) is equal to \( \mathcal{T}_1 \) or zero. In the latter case \( \text{ad}_S L_1 \subset V \), and since \( [S, V] = 0 \), it follows that \( (\text{ad}_{L_1} s)^2 = 0 \) for any \( s \in S \), so \( \text{ad}_{L_1} S \) is nilpotent by Engel’s Theorem. But \( S \) is simple, so, by a version of Schur’s Lemma, either \( \text{ad}_{L_1} S \cong S \) or \( \text{ad}_{L_1} S = 0 \); therefore, being nilpotent, \( \text{ad}_{L_1} S = 0 \). Hence, \( \text{ad}_{L_1} : L_{-1} = S \otimes O_{n-1} \longrightarrow L_0 \) is a nontrivial morphism of \( S \)-modules for any \( 0 \neq l_1 \in L_1 \); thus, \( [L_{-1}, L_1] \) is an isotypical \( S \)-submodule of \( L_0 \) of type \( S \). Therefore, \( [L_{-1}, L_1] = S \) and \( [[L_{-1}, L_1], L_1] = [S, L_1] = 0 \). We have
obtained a contradiction, since $L$ is a nondegenerate graded Lie algebra. Thus, $ad (\mathcal{L}_1) = \mathcal{L}_1$. Since $[S, V] = 0$, it follows that for any $i$

\[ [S, g_{-1,i}] = [S, (ad V)^{i-1}(\mathcal{L}_1)] = (ad V)^{i-1}[S, \mathcal{L}_1] = g_{-1,i}. \quad (2.1) \]

Since $S \otimes \mathcal{O}_n$ is a minimal ideal of $g_0$, it follows that $(\text{Ann}_{g_0} g_{-1}) \cap (S \otimes \mathcal{O}_n) = 0$. Therefore, since $\text{deg} S \otimes x^\delta = -|\delta|$ and $[S \otimes x^\delta, g_{-1}] \neq 0$, it follows, in view of the $\mathbb{Z}$-grading in $g_{-1}$, that

\[ [S \otimes x^\delta, g_{-1,|\delta|+1}] \neq 0. \quad (2.2) \]

Hence, $g_{-1,|\delta|+1} \neq 0$. Since $g_{-1,|\delta|+1} = (ad V)^{|\delta|} L_1 = \text{ad}(\partial_1^{p-1} \cdots \partial_n^{p-1}) L_1$, it follows that the $L_0$-module $g_{-1,|\delta|+1}$ is isomorphic to the twisted $L_0$-module $L_1 \otimes F_\sigma$ where $F_\sigma$ is a one-dimensional $L_0$-module corresponding to a morphism $\sigma : L_0 \rightarrow F$. Thus, $g_{-1,|\delta|+1}$ is an irreducible $L_0$-module.

For any $B$-module $U$ denote by $c(U)$ the truncated coinduced $g_0$-module $\text{coind}_B U = \text{Hom}_B (A, U)$, where $A$ and $B$ are as above. Let $\tilde{g}_{-1} = \sum_{i=1}^{\delta} g_{-1,i}$. Obviously, $\tilde{g}_{-1}$ is an $\mathfrak{h}$-submodule of the $g_0$-module $g_{-1}$ and $\tilde{g}_{-1}/\tilde{g}_{-1} \cong g_{-1,|\delta|+1}$. According to the theory of truncated coinduced modules ([11], [13]), there exists a nonzero morphism of $g_0$-modules from $g_{-1}$ to $c(g_{-1,|\delta|+1})$ which is injective since $g_{-1}$ is an irreducible $g_0$-module. Furthermore, $\mathcal{O}_n = c(F)$ where $F$ is a trivial $\mathfrak{h}$-module, and $S \otimes \mathcal{O}_n = c(S \otimes \mathcal{O}_n/S \otimes n) = c(S)$ where $n$ is the maximal ideal of $\mathcal{O}_n$. In addition, any module $c(U)$ is a free $\mathcal{O}_n$-module. The bracket operation $[,]$ in $g$, $[,] : S \times g_{-1,|\delta|+1} \rightarrow g_{-1,|\delta|+1}$ induces the $\mathcal{O}_n$-bilinear mapping

\[ \mu : c(S) \times c(g_{-1,|\delta|+1}) \rightarrow c(g_{-1,|\delta|+1}), \]

$\mu(\phi, \psi) = [ , ] \circ \phi \otimes \psi \circ \delta$, where $\delta$ is the coproduct in $A$. As in Proposition 2.2 of [7], it may be shown that the bracket operation in the Lie algebra $g,[,] : S \otimes \mathcal{O}_n \times g_{-1} \rightarrow g_{-1}$ coincides with the restriction of the mapping $\mu$. Note that a basis of the space $g_{-1,|\delta|+1}$ is a basis of the free $\mathcal{O}_n$-module $c(g_{-1,|\delta|+1})$.

Since the bracket operation of $S \otimes \mathcal{O}_n$ with $g_{-1}$ is $\mathcal{O}_n$-bilinear and $[S, g_{-1,|\delta|+1}] = g_{-1,|\delta|+1}$, it follows that

\[
c(g_{-1,|\delta|+1}) = \mathcal{O}_n g_{-1,|\delta|+1} \]
\[
= \mathcal{O}_n [S, g_{-1,|\delta|+1}] = [S \otimes \mathcal{O}_n, g_{-1,|\delta|+1}] \]
\[
\subseteq g_{-1} \subseteq c(g_{-1,|\delta|+1}).
\]
Thus, $g_{-1} = c(g_{-1,\delta+1}) = g_{-1,\delta+1} \otimes \mathcal{O}_n, g_{-1,i} = g_{-1,\delta+1} \otimes \mathcal{O}_{n,i-\delta-1}$.

**Remark.** It may be inferred from the results of R. Block [4] that $g_{-1}$ is a free $\mathcal{O}_n$-module and that the bracket operation of $g_{-1}$ with $S \otimes \mathcal{O}_n$ is $\mathcal{O}_n$-bilinear.

(e) Here it is shown that $g_1 \neq 0$.

According to (e), $\mathcal{L}_1 = M(\mathcal{L}_0)$, where $\mathcal{L}_0 = L_{-q} + \ldots + L_{-1} + L_0 + V$. Suppose that $M(\mathcal{L}_0) = 0$. Then

$$\mathcal{L}_0 = g_0 = S \otimes \mathcal{O}_n + L_0 + V,$$

$q = |\delta|, L_{-|\delta|} = S \otimes x^d$, and $L_{-|\delta|-1} = 0$. It follows from (d) that the subspaces $g_{-1,i}, i = 2, \ldots, |\delta| + 1$ may be identified with subspaces $L_i \subset L_i$; moreover,

$$[L_{-1}, L_{|\delta|+1}^\dagger] = [S \otimes \mathcal{O}_{n-1}, g_{-1,|\delta|+1}] = \mathcal{O}_{n-1} g_{-1,|\delta|+1} = g_{-1,|\delta|} = L_{|\delta|}^\dagger.$$

We now show that $[[L_{-|\delta|}, L_{|\delta|}^\dagger], L_{|\delta|}^\dagger] \neq 0$ and that $[[L_{-|\delta|}, L_{|\delta|}^\dagger], L_{-|\delta|}] \neq 0$. Bearing (2.2) in mind, let $U \overset{\text{def}}{=} [L_{-|\delta|}, L_{|\delta|+1}] \subset L_1$. Since the bracket operation of $g_{-1}$ with $S \otimes \mathcal{O}_n \subset g_0$ is $\mathcal{O}_n$-bilinear, it follows that $U/U \cap V = L_1/V = g_{-1,1}$ (see (d)). By the assumptions on $L, V$ and $L_1/V$ are irreducible $L_0$-modules, so either $V \subset U$, or $L_1 = V \oplus U$ is a direct sum of $L_0$-modules. Furthermore,

$$[L_{-|\delta|}, L_{|\delta|}^\dagger] = [L_{-|\delta|}, [L_{-1}, L_{|\delta|+1}^\dagger]] = [L_{-1}, [L_{-|\delta|}, L_{|\delta|+1}^\dagger]] = [L_{-1}, U]. \quad (2.3)$$

We next show that $S \subset [L_{-1}, U]$. Suppose $[L_{-1}, U] \cap S = 0$. Since $[L_{-1}, U]$ and $S$ are ideals of $L_0$, and $S$ is simple, it follows that $[[L_{-1}, U], S] = 0$. Since $[L_{-1}, V] = S$, it follows that $V \cap U = 0$; therefore, $L_1 = U \oplus V$.

Consider the subalgebra $L^\dagger$ in $L$ generated by $L_{-1} + L_0 + U$,

$$L^\dagger = L_{-|\delta|} + L_{-|\delta|+1} + \ldots + L_{-1} + L_0 + U + \ldots.$$

Now $M(L^\dagger)$ is an ideal of $S \otimes \mathcal{O}_n = L_{-|\delta|} + \ldots + L_{-1} + S$; therefore, $M(L^\dagger) = S \otimes J$ where $J$ is an ideal of $\mathcal{O}_n$. Inasmuch as $M(L^\dagger) \subset L_{-|\delta|} + \ldots + L_{-2}$, we have $J \neq m$ where $m$ is the maximal ideal of $\mathcal{O}_n$. Evidently,

$$\overline{L_i} = L_i/M(L^\dagger) = T_{-k} + \ldots + T_{-2} + L_{-1} + L_0 + U + \ldots,$$

$$\overline{L_i} = S \otimes (\mathcal{O}_n/J)_i, i = -2, \ldots, -k.$$

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Suppose $\overline{L}^{\dag}$ is a nondegenerate Lie algebra. Since $L_0$ is a reductive classical Lie algebra, it follows from Weisfeiler’s Theorem that $\overline{L}^{\dag}$ contains a simple graded Lie algebra

$$S^{\dag} = \bigoplus_i S^{\dag}_i, S^{\dag} \subset \overline{L}^{\dag} \subset \text{Der} S^{\dag},$$

$$\overline{L}^{\dag}_i = S^{\dag}_i, i = -1, \ldots, -k, S^{\dag}_j \subset L^{\dag}_j \subset \text{Der}_j S^{\dag}, j = 0, 1, \ldots.$$ 

Note that $S^{\dag}_{-k}$ is an irreducible $S^{\dag}_0$-module and

$$S^{\dag}_0 = [S^{\dag}_{-1}, S^{\dag}_1] = [S^{\dag}_{-1}, U] = [L_{-1}, U].$$

Let $\rho : L_0 \to \mathfrak{gl}(S^{\dag}_{-1})$ be the restriction of the adjoint representation. By Schur’s Lemma, $\dim C_{\mathfrak{gl}(S^{\dag}_{-1})}(\rho(S^{\dag}_0)) = 1$. On the other hand $S^{\dag}_{-k} = \overline{L}^{\dag}_{-k} = S \otimes (O_{n}/J)_{-k}$. Inasmuch as $S \cap S^{\dag}_0 = S \cap [L_{-1}, U] = (0)$, we have $S \cong \rho(S) \subset C_{\mathfrak{gl}(S^{\dag}_{-1})}(\rho(S^{\dag}_0))$ and we obtain a contradiction.

If $\overline{L}^{\dag}$ is a degenerate Lie algebra, then by Weisfeiler’s Theorem, $\overline{L}^{\dag}$ contains a differentially simple ideal $S^{\dag} \otimes O_m, O_m = F[y_1, \ldots, y_m]/(y_1^p, \ldots, y_m^p)$. $\overline{L}^{\dag}_{-i} = S^{\dag} \otimes O_{m-i}, i = -1, \ldots, -k$ where $O_m = \oplus O_{m-i}$ is the grading of $O_m$ opposite to the standard grading and $S^{\dag} = [L_{-1}, U]$. In particular, $\overline{L}^{\dag}_{-k} = S^{\dag} \otimes y^\delta$, where $\delta = (p-1, \ldots, p-1)$ is an $m$-tuple. Therefore, since we determined above that $[[L_{-1}, U], S] = 0$, it follows that $\overline{L}^{\dag}_{-k}$ is a trivial $S$-module. On the other hand, $\overline{L}^{\dag}_{-k} = S \otimes (O_{n}/J)_{-k}$ is an isotypical $S$-module of the type $S$, and we obtain a contradiction. Thus, $S \subset [L_{-1}, U]$.

Since $L_{-[\delta]} = S \otimes x^\delta$, it follows (from (2.3)) that

$$[[L_{-[\delta]}, L^{\dag}_{[\delta]}], L_{-[\delta]}] = [[L_{-1}, U], L_{-[\delta]}] \supset [S, L_{-[\delta]}] = L_{-[\delta]} \neq 0.$$ 

According to (d), $L_1/V$ is a nontrivial $S$-module and $L^{\dag}_{[\delta]} = [[\ldots [L_1/V, V], \ldots], V]_{[\delta-1]}$, whence $[S, L^{\dag}_{[\delta]}] = L^{\dag}_{[\delta]} \neq 0$. Therefore,

$$[[L_{-[\delta]}, L^{\dag}_{[\delta]}], L_{-[\delta]}] = [[L_{-1}, U], L^{\dag}_{[\delta]}] \supset [S, L^{\dag}_{[\delta]}] = L^{\dag}_{[\delta]}.$$ 

Consider the subalgebra $B(L_{-[\delta]})$ (see Section 1). Since

$$[[L_{-[\delta]}, L^{\dag}_{[\delta]}], L_{-[\delta]}] = L_{-[\delta]} \text{ and } [[L_{-[\delta]}, L^{\dag}_{[\delta]}], L^{\dag}_{[\delta]}] = L^{\dag}_{[\delta]},$$

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it follows that the Lie algebra $B(L_{-|\delta|})$ is a one-graded transitive irreducible Lie algebra,

$$B(L_{-|\delta|}) = B_{-1} + B_0 + B_1 + \ldots, B_1 \neq 0.$$ 

Since $B_{-1} = L_{-|\delta|} = S \otimes x^\delta$, it follows that $S \subset B_0 \subset \text{Der}S + \mathfrak{z}(B_0)$, $B_{-1} \cong S$. According to the theorem of Kostrikin-Ostrik ([9]) such an algebra does not exist when $p > 2$.

Thus, $M(L_0) \neq 0$ and, therefore, $g_1 \neq 0$.

(f) Let $L = \ldots \supset L_{-1} \supset L_0 \supset L_1 \supset \ldots$ be the non-contractible filtration of $L$ corresponding to the maximal subalgebra $L_0 = L_{-q} + \ldots + L_{-1} + L_0 + V$ and the $L_0$-module $L_{-1}$,

$$L_{-1} = L_0 + L_1 + \sum_{i=1}^{[\delta]} (adV)^i L_1$$

(2.4)

(see (c), (e)). Here the following statements are proved:

1) let $l \in L_1$ and suppose that $[l, V] = 0$; then $l \in L_2$ if and only if $[l, L_1] \subset L_1$;

2) let $g_1 = \oplus_{i-k} g_{1-i}, g_{1-k} \neq 0$ be the $\mathbb{Z}$-grading of $g_1$ corresponding to the torus $T$; then $k = \min\{i| M(L_0) \cap L_{-i} \neq 0\}$, and we have $2 \leq k \leq |\delta| + 1$;

$$L_1 \subset L_{-q} + \ldots + L_{-k}, L_2 \subset L_{-q} + \ldots + L_{-k+1},$$

$g_{1,-k} \cong M(L_0) \cap L_{-k}$;

3) $[g_1, g_{-1}] = S \otimes O_n \subset g_0$.

According to (c), $L_1 = M(L_0) \subset L_{-q} + \ldots + L_{-2}$. Evidently, if $l \in L_2$, then $[l, L_1] \subset [l, L_{-1}] \subset L_1$. Let $l \in L_1$, and suppose that $[l, V] = 0$. According to (2.4),

$$[l, L_{-1}] = [l, L_0] + [l, L_1] + \sum_{i=1}^{[\delta]} (adV)^i ([l, L_1]).$$

Therefore, if $[l, L_1] \subset L_1$, then $[l, L_{-1}]$ is contained in $L_1$; that is, $l \in L_2$, and 1) is proved.
To prove 2), set \( s = \min\{ i | M(L_0) \cap L_{-i} \neq 0 \} \). Then \( k \geq s \geq 2 \). Let \( L^+ = M(L_0) \cap L_{-s} \) and \( L^{++} = \{ l \in L^+ | [l, L_1] = 0 \} \). Obviously, \( L^{++} \) is an \( L_0 \)-submodule of \( L^+ \). Since \( L^+ \) is generated by \( L_1 \), it follows that the ideal \( J \) of \( L \) generated by \( L^{++} \) is contained in \( L_- \). By the assumptions on \( L_0 \), \( M(L) = 0 \); therefore, \( J = 0 \) and \( L^{++} = 0 \). Hence, \( 0 \neq [l, L_1] \subset L_{-s+1} \) for any \( 0 \neq l \in L^+_s \subset L_1 \). By the definition of \( s \), \( [l, V] = M(L_0) \cap L_{-s+1} = 0 \) for any \( l \in L^+_s \); therefore, according to 1), \( l \notin L_2 \) for any \( 0 \neq l \in L^+_s \). Thus, \( k = s \) and \( L_{1-k} = M(L_0) \cap L_{-k} \). Since the \( \mathbb{Z} \)-grading of the semisimple Lie algebra \( L_0/L_1 = L_0/M(L_0) \) is equal to \( |\delta| \), it follows that \( L_{-|\delta|-1} \subset M(L_0) \). Therefore, \( k \leq |\delta| + 1 \).

3) Since \( V \subset L_0 \) and \( L_1 \) is a \( L_0 \)-module, we have from (2.4) that

\[
[L_1, L_{-1}] \subset L_1 + [L_1, L_1] + \sum_{i>0} (adV)^i [L_1, L_1].
\]

As \( L_1 \subset L_{-q} + \ldots + L_{-2} \), we have that \( [L_1, L_1] \subset L_{-q} + \ldots + L_{-1} \) and

\[
(adV)^i [L_1, L_1] \subset L_{-q} + \ldots + L_{-1} + S,
\]

\[
([L_1, L_{-1}] + L_1)/L_1 \subset S \otimes O_n \subset g_0 = L_0/L_1;
\]

that is, \( [g_1, g_{-1}] \subset S \otimes O_n \). Since \( S \otimes O_n \) is a minimal ideal of \( g_0 \) and \( [g_{1-k}, g_{-1}] \approx [L^+_{-k}, L_1] \neq 0 \), it follows that \( [g_1, g_{-1}] = S \otimes O_n \).

\((g)\) Let \( k \) be as in (f). We show that \([g_{-1}, g_1] \neq 0 \) and that if \( k = |\delta| + 1 \), then \( g_{1-k} \) is a nontrivial \( S \)-module, and \( g_{1-|\delta|-1} \approx L_{-|\delta|-1} \).

Note that if \( [S, g_{1-k}] \neq 0 \), then according to (f), 3) \([g_{-1}, g_1], g_1] \neq 0 \). Suppose that \([S, g_{1-k}] = 0 \).

Let \( k \leq |\delta| \). As in (f), we set \( L^+_{-k} = L_1 \cap L_{-k} \) and show that \( L^+_{-k-1} = [L_{-1}, L^+_{-k}] \neq 0 \), \( L^+_{-k-1} \not\subset L_2 \). Since \( L^+ \approx g_{1-k} \), \( L_{-1} = S \otimes O_{n-1} \approx g_{0-1} \subset [g_{-1}, g_1] \), it will follow that \( [S \otimes O_{n-1}, g_{1-k}] \neq 0 \); that is, \([g_{-1}, g_1], g_1] \neq 0 \).

Since \( [V, L^+_{-k}] = 0 \), it follows that \( [V, L^+_{-k-1}] = [V, L_{-1}, L^+_{-k}] = [V, L_{-1}], L^+_{-k}] = [S, L^+_{-k}] \approx [S, g_{1-k}] = 0 \). Furthermore,

\[
[L_1, [L_{-1}, L^+_{-k}]] \subset [[L_1, L_{-1}], L^+_{-k}] + [L_{-1}, [L_1, L^+_{-k}]].
\]

As \( [L_0, L^+_{-k}] \subset L^+_{-k} \subset L_1 \) and \( 0 \neq [L_1, L^+_{-k}] \subset L_{-k+1} = S \otimes O_{n-(k-1)} \), we have that

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\[0 \neq ([L_{-1}, [L_1, L_{-k}^\dagger]] + \mathcal{L}_1)/\mathcal{L}_1 = ([L_{-1}, [L_1, L_{-k}^\dagger]] + L_{-k}^\dagger)/L_{-k}^\dagger \subset [S \otimes \mathcal{O}_{n,-1}, S \otimes \mathcal{O}_{n,-k+1}] = S \otimes \mathcal{O}_{n,-k}.\]

So \([L_{-1}, L_{-k}^\dagger], L_1 \not\subset \mathcal{L}_1\) and by (f), 1) \([L_{-1}, L_{-k}^\dagger] = L_{-k-1}^\dagger \not\subset \mathcal{L}_2\).

Now let \(k > |\delta|\). Then \(k = |\delta| + 1, L_{-k}^\dagger = L_{-k} \cong g_{1,-k}, L_{-i} = S \otimes \mathcal{O}_{n,-i}, i = 1, \ldots, |\delta|, L_{-k} = [L_{-1}, L_{-|\delta|}] = [S \otimes \mathcal{O}_{n,-1}, S \otimes x^\delta]\). By assumption, \([S, L_{-k}] = 0\). For \(x \in \mathcal{O}_{n,-1}\), the bracket operation in \(L, [\cdot, \cdot] : S \otimes x \times S \otimes x^\delta \longrightarrow L_{-k}\) gives us a mapping of the \(S\)-module \(S \otimes S\) into the trivial \(S\)-submodule \([S \otimes x, S \otimes x^\delta]\). Since \(S\) is a classical simple Lie algebra, \(S \cong S^*\) as \(S\)-modules. This fact follows, in particular, from Curtis’s Theorem. (See [6] for \(p > 7\), and [9] for any \(p > 0\).) Therefore, the quotients of \(S \otimes S\) which are trivial over \(S\), are in one-to-one correspondence with the trivial \(S\)-submodules in \(S^* \otimes S \cong (S \otimes S^*)^* = \text{Hom}(S, S)\). By Schur’s Lemma \(\text{Hom}(S, S)\) has the unique nonzero trivial \(S\)-module \(\text{Hom}_S(S, S), \dim \text{Hom}_S(S, S) = 1\). So, \([S \otimes x, S \otimes x^\delta] = \langle \hat{x} \rangle\) is a one-dimensional trivial \(S\)-submodule in \(L_{-|\delta|-1}\) and

\[\varphi_x(s_1, s_2) = \varphi_x(s_1, s_2)\hat{x},\]

where \(\varphi_x(s_1, s_2)\) is an invariant bilinear form on \(S\). Since \(\Phi_0 = \{x \in \mathcal{O}_{n,-1}|\varphi_x = 0\}\) is a \(\pi(L_0)\)-submodule of \(\mathcal{O}_{n,-1}\) and \(\mathcal{O}_{n,-1}\) is an irreducible \(\pi(L_0)\)-module, it follows that \(\Phi_0 = 0\). (See the description of \(\pi(L_0)\) in (b).)

Let \(\{x_1, \ldots, x_n\}\) be a basis of \(\mathcal{O}_{n,-1}\). The above discussion shows that the space of invariant bilinear forms on \(S\) is one-dimensional. So, the elements \(\hat{x}_i\) may be chosen in such a way that \([s_1 \otimes x_i, s_2 \otimes x^\delta] = \hat{\phi}(s_1, s_2)\hat{x}_i\) for a fixed nonzero invariant form \(\phi\) on \(S\).

Obviously, \(L_{-|\delta|-1}\) is isomorphic to \(\mathcal{O}_{n,-1}\) as a \(\pi([L_0, L_0])\)-module, and is therefore an irreducible \(L_0\)-module. Let \(L_{|\delta|+1}^\dagger = (\text{ad} V)^{|\delta|} L_1\). According to (d), \(L_{|\delta|+1}^\dagger\) is not only not zero but also isomorphic to the irreducible \(g_{0,0}\)-module \(g_{-1,|\delta|+1}\). Note that \(g_{0,0} = L_0\). Since \(S\) is an ideal of \(L_0\), it follows that \(L_{|\delta|+1}^\dagger\) is an isotypical \(S\)-module. Since \([L_{-|\delta|-1}, L_1] = L_{-|\delta|} = S \otimes x^\delta\) and \([L_{-|\delta|-1}, V] = 0\), it follows that
Therefore, 

\[ [L_{-|\delta|-1}, L_{|\delta|+1}^1] = [L_{-|\delta|-1}, (\text{ad} V)^{|\delta|} L_1] \]

\[ = (\text{ad} V)^{|\delta|} [L_{-|\delta|-1}, L_1] = (\text{ad} V)^{|\delta|} S \otimes x^\delta = S. \]

Now, as \( L_{-|\delta|-1} \) is an irreducible \( L_0 \)-module which is trivial over \( S \), the mapping \( \text{ad} \ l, l \in L_{-|\delta|-1} \) is a nonzero \( S \)-morphism from \( L_{|\delta|+1}^1 \) to \( S \). Therefore, \( L_{|\delta|+1}^1 \) is an isotypical \( S \)-module of type \( S \) and may be represented as \( S \otimes U \) where \( U \) is an irreducible \( L_0 \)-module which is trivial over \( S \); that is, \( U \) is an irreducible \( \pi(L_0) \)-module. The bracket operation in \( L \) of elements \( \hat{x} \in L_{-|\delta|-1} \) and \( s \otimes u \in L_{|\delta|+1}^1 = S \otimes U \) may be written as \([\hat{x}, s \otimes u] = <\hat{x}, u> s \) where \(<,> \) is a nondegenerate pairing, \( <,>: L_{-|\delta|-1} \times L_{|\delta|+1}^1 \rightarrow \mathbb{F} \).

Let \( \alpha \) be a root of the classical simple Lie algebra \( S \), let \( e_\alpha, e_{-\alpha} \) be root vectors, and let \( h_\alpha = [e_\alpha, e_{-\alpha}] \), so that \( sl(2) = <e_\alpha, e_{-\alpha}, h_\alpha> \) is the corresponding three-dimensional simple subalgebra of \( S \). Evidently, \( \varphi(e_{-\alpha}, e_\alpha) \neq 0 \). For \( e_\alpha \otimes x^\delta \in L_{-|\delta|} \) and \( e_\alpha \otimes u \in S \otimes U = L_{|\delta|+1}^1 \), let \( [e_\alpha \otimes x^\delta, e_\alpha \otimes u] \) be the corresponding coclass in \( L_1/V = g_{-1,1} \). Since the bracket of \( S \otimes \mathcal{O}_n \subset g_0 \) with \( g_{-1} \) is \( \mathcal{O}_n \)-bilinear (see (c)),

\[
\overline{[e_\alpha \otimes x^\delta, e_\alpha \otimes u]} = [e_\alpha, e_\alpha \otimes u]x^\delta = [e_\alpha, e_\alpha] \otimes ux^\delta = 0.
\]

Consequently, in \( L \) we have \( [e_\alpha \otimes x^\delta, e_\alpha \otimes u] \in V \subset L_1 \). Inasmuch as \( [S, V] = 0 \), we have \([h_\alpha, [e_\alpha \otimes x^\delta, e_\alpha \otimes u]] = 0 \). However,

\[
[h_\alpha, [e_\alpha \otimes x^\delta, e_\alpha \otimes u]] = 2\alpha(h_\alpha)[e_\alpha \otimes x^\delta, e_\alpha \otimes u].
\]

Thus, \( [e_\alpha \otimes x^\delta, e_\alpha \otimes u] = 0 \).

Let \( <\hat{x}, u> \neq 0 \). Then in \( L \) we have

\[
0 = [e_{-\alpha} \otimes x, [e_\alpha \otimes x^\delta, e_\alpha \otimes u]]
\]

\[
= \varphi(e_{-\alpha}, e_\alpha)[\hat{x}, e_\alpha \otimes u] + [e_\alpha \otimes x^\delta, [e_{-\alpha}, e_\alpha] \otimes ux]
\]

\[
= \varphi(e_{-\alpha}, e_\alpha) <\hat{x}, u> e_\alpha - [e_\alpha \otimes x^\delta, h_\alpha \otimes ux].
\]

Therefore,

\[
0 \neq [e_\alpha \otimes x^\delta, h_\alpha \otimes ux] = \varphi(e_{-\alpha}, e_\alpha) <\hat{x}, u> e_\alpha \in S \subset L_0.
\]
Thus, $S \subset [L_{-|\delta|}, L^+_{|\delta|}]$ where $L^+_{|\delta|} = g_{-1,|\delta|}$ (see (d)). Note that $L_{-|\delta|} = S \otimes x^\delta$. It now follows from (2.1) that

$$[[L_{-|\delta|}, L^+_{|\delta|}], L^+_{|\delta|}] \supset [S, L^+_{|\delta|}] = L_{\pm|\delta|},$$

where $L^+_{|\delta|} = L_{-|\delta|}$.

Consider the subalgebra $h$ in $g$ generated by the local part $h_{-1} + h_0 + h_1$, where $h_{-1} = g_{0, -|\delta|} = S \otimes x^\delta \cong L_{-|\delta|}, h_0 = g_{0,0} = L_0$, and $h_1 = g_{-1,|\delta|} \cong L^+_{|\delta|} = (S \otimes U) \otimes O_{n-1}$. Evidently, $h$ is a one-graded Lie algebra with respect to the $\mathbb{Z}$-grading corresponding to the torus $T$. The transitive one-graded Lie algebra $B = B(h_{-1})$ satisfies the conditions of the Kostrikin-Ostrik Theorem [9]. Obviously, $B = B_{-1} + B_0 + B_1 + \ldots$, where $B_1 \neq 0, S \subset B_0 \subset \text{Der} S + 3(B_0)$, and $B_{-1} \cong S$. It follows from the Kostrikin-Ostrik Theorem that such an algebra does not exist. We have arrived at a contradiction.

Therefore, $g_{1, -k} \cong L_{-k}$ is a nontrivial $S$-module in the case $k = |\delta| + 1$. Therefore, $[[g_{-1}, g_1], g_1] \neq 0$.

(h) In this section of the proof, it is proved that there exists a simple graded Lie algebra $\hat{S} = \oplus_{i \in \mathbb{Z}} \hat{S}_i$ such that $g_{-1} = \hat{S}_{-1} \otimes O_n, \hat{S}_0 \otimes O_n \subset g_0 \subset (\text{Der}_0 \hat{S}) \otimes O_n + 1 \otimes W_n, \hat{S}_1 \otimes O_n \subset g_1 \subset (\text{Der}_1 \hat{S}) \otimes O_n$, and $\hat{S}_0 = S$. The bracket operation in $g, [\cdot, \cdot] : g_{-1} \times g_1 \longrightarrow [g_{-1}, g_1] = S \otimes O_n$ is the restriction of the $O_n$-bilinear bracket operation of the Lie algebra $(\text{Der} \hat{S}) \otimes O_n; [g_{-1}, g_1] = \hat{S}_0 \otimes O_n$.

Consider the subalgebra $\hat{g}$ of $g$ generated by $g_{-1}, g_0, g_1$. The torus $T$ acts by automorphisms on the subalgebra $\hat{g}$; therefore, $\hat{g}$ has a $\mathbb{Z}$-grading corresponding to the torus $T$. Let $\tilde{g} = \hat{g}/M(\hat{g})$ where $M(\hat{g})$ is the Weisfeiler radical of $\hat{g}$. According to (g), $[[g_{-1}, g_1], g_1] \neq 0$. So $\tilde{g}$ is a nondegenerate graded Lie algebra. By Weisfeiler’s Theorem, $\tilde{g}$ contains a minimal ideal $A(\tilde{g}) = \tilde{S} \otimes \tilde{O}_m$ where $\tilde{S}$ is a graded simple Lie algebra, $\tilde{S} = \oplus \tilde{S}_i, \tilde{g}_i = \tilde{S}_i \otimes \tilde{O}_m$ for $i < 0, \tilde{g}_{-1} = g_{-1}, \tilde{g}_0 = g_0, \tilde{g}_1 = g_1, \tilde{S} \otimes O_m \subset g_0 \subset (\text{Der}_0 \tilde{S}) \otimes O_m + 1 \otimes W_m = \text{Der}_0(\tilde{S} \otimes O_m), and \tilde{S}_1 \otimes O_m \subset g_1 \subset (\text{Der}_1 \tilde{S}) \otimes O_m$. Since $A(\tilde{g})$ is the unique minimal ideal of $\tilde{g}$, it is invariant with respect to the torus $T$.

According to (f), 3), $[g_{-1}, g_1] = S \otimes O_n$. On the other hand, since $[\tilde{S}_{-1}, \tilde{S}_1] = \tilde{S}_0$ in the graded simple Lie algebra $\tilde{S}$, it follows that $[g_{-1}, g_1] = \tilde{S}_0 \otimes O_m$; the bracket operation

$$[g_{-1}, g_1] = \tilde{S}_0 \otimes O_m = S \otimes O_n$$

is $O_m$-bilinear; and $O_m$ is naturally contained in the centroid $\mathcal{C}$ of the Lie algebra $S \otimes O_n$. Inasmuch as $\mathcal{C}$ is isomorphic to $O_n$, we have $O_m \subset O_n$. 

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Since $\mathcal{O}_m$ is isomorphic to the centroid of the $T$-invariant ideal $\hat{S} \otimes \mathcal{O}_m$, it follows that $\mathcal{O}_m$ is a graded subalgebra in $\mathcal{O}_n$ with respect to the $\mathbb{Z}$-grading corresponding to $T$.

Let $m$ be the maximal ideal of $\mathcal{O}_m$, and let $n$ be the maximal ideal of $\mathcal{O}_n$. We have the following series of natural isomorphisms:

$$\hat{S}_0 \cong (\hat{S}_0 \otimes_F \mathcal{O}_m) \otimes_{\mathcal{O}_m} \mathcal{O}_m / m\mathcal{O}_m = (S \otimes_F \mathcal{O}_n) \otimes_{\mathcal{O}_n} \mathcal{O}_m / m\mathcal{O}_m$$

Therefore, $\hat{S}_0 \otimes_F \mathcal{O}_m \cong S \otimes_F B \otimes_F \mathcal{O}_m$ where $B = \mathcal{O}_n / m\mathcal{O}_n$. Hence, $\mathcal{O}_n \cong B \otimes_F \mathcal{O}_m$ and so $B \cong \mathcal{O}_l$, $\mathcal{O}_n \cong \mathcal{O}_l \otimes \mathcal{O}_m$. The ring $\mathcal{O}_l$ is the trivial ring, $\mathcal{O}_n = l + m$, $\hat{S}_0 = S \otimes \mathcal{O}_l$. According to (b) and (c), $S \otimes \mathcal{O}_n \subset g_0 \subset (\text{Der}S) \otimes \mathcal{O}_n + V + W_{n,0}$, where $V = 1 \otimes W_{n,1}$. Recall that $\mathcal{O}_n = F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ and $\deg x_i = -1$ with respect to the grading corresponding to the torus $T$. Therefore, we can choose the additional subalgebra $\mathcal{O}_l$ to be graded. In such a case, $n_{-1} = l_{-1} \oplus m_{-1}$ where $l$ is the maximal ideal of $\mathcal{O}_l$.

According to (b) and (c), $S \otimes \mathcal{O}_n \subset g_0 \subset (\text{Der}S) \otimes \mathcal{O}_n + 1 \otimes W_m$. To compare it with the first description, consider the adjoint representation $\rho$ of $g_0$ on the minimal ideal $S \otimes \mathcal{O}_n$. Note that $\rho$ is a faithful representation of $g_0$. On the other hand, $\hat{S}_0 \otimes \mathcal{O}_m \subset g_0 \subset (\text{Der}S) \otimes \mathcal{O}_m + 1 \otimes W_m$.

Inasmuch as $g_0 \subset \rho(\mathfrak{g})$, we can, by taking into account the $\mathbb{Z}$-grading of $g_0$, obtain that

$$g_0 \subset (\text{Der}S) \otimes \mathcal{O}_n + 1 \otimes W_{n,1} + 1 \otimes W_{n,1} \otimes l_{-1} + 1 \otimes W_{n,0} \otimes 1 + 1 \otimes 1 \otimes W_{m,0}.$$
\[ \pi(L_0) \subset W_{1,1} \otimes m_{-1} + W_{1,0} \otimes 1 + 1 \otimes W_{m,0}. \]

This means that \( 1 \otimes m_{-1} \) is invariant with respect to \( \pi(L_0) \). Thus, either \( m_{-1} = 0 \) or \( m_{-1} = n_{-1} \); that is, either \( m = 0 \) or \( m = n \).

Suppose that \( m = 0 \). Then

\[ A(\hat{g}) = \hat{S} = \hat{S}_{-u} + \ldots + \hat{S}_{-1} + \hat{S}_0 + \hat{S}_1 + \ldots, \hat{S}_0 = S \otimes \mathcal{O}_n. \]

Since \( \hat{S} \) is simple, it follows that \( \hat{S}_{-u} \) is an irreducible \( \hat{S}_0 \)-module. Furthermore, as the nilradical \( S \otimes n \) of \( \hat{S}_0 = S \otimes \mathcal{O}_n \) has the \( \mathbb{Z} \)-grading \( S \otimes n = \bigoplus_{i>0} S \otimes \mathcal{O}_{n_{-i}} \) defined by the torus \( T \), it follows that it acts as nilpotent elements on \( \hat{g} \). Therefore, \( [S \otimes n, \hat{S}_{-u}] = 0 \). Since \( \hat{S}_{-u} = \hat{g}_{-u} \) is a \( \hat{g}_0 \)-module, \( \hat{g}_0 = g_0 \) and \( S_0 = S \otimes \mathcal{O}_n \) is the minimal ideal of \( g_0 \), it follows that \( S_0 \) is contained in the kernel of the adjoint representation of \( g_0 \) on \( \hat{S}_{-u} \); that is, \( [\hat{S}_0, \hat{S}_{-u}] = 0 \). Since \( \hat{S} \) is a simple Lie algebra, it follows that \( [\hat{S}_{-1}, \hat{S}_1] = \hat{S}_0 \).

Then

\[ [\hat{S}_{-1}, \hat{S}_1, \hat{S}_{-u}] = [[\hat{S}_{-1}, \hat{S}_1], \hat{S}_{-u}] = [\hat{S}_0, \hat{S}_{-u}] = 0. \]

Inasmuch as \( g_{-1} = \hat{S}_{-1} \) and \( \hat{g} \) is a transitive Lie algebra, we have by Lemma 6 of [1] that if \( x \in \hat{g}, i > -u \) and \([\hat{g}_{-1}, x] = 0 \), then \( x = 0 \). Thus, \( [\hat{S}_{1}, \hat{S}_{-u}] = 0 \).

Set \( \hat{S}^- \equiv \bigoplus_{i \leq 0} \hat{S}_i \), and \( \hat{S}^+ \equiv \bigoplus_{i > 0} \hat{S}_i \). Since \( \hat{S} \) is simple, it follows that it is covered by the induced module

\[ \text{ind} \hat{S}_{-u} = U(\hat{S}) \otimes_{U(\hat{S}^-)} \hat{S}_{-u} = U(\hat{S}^+) \otimes_{\mathbb{F}} \hat{S}_{-u}. \]

Hence, \( \hat{S}_{-u+1} = [\hat{S}_1, \hat{S}_{-u}] = 0 \). This contradiction shows that \( m = n \). Thus,

\[ A(\hat{g}) = \hat{S} \otimes \mathcal{O}_n, \hat{S}_0 = S, \hat{g}_{-1} = g_{-1} = \hat{S}_{-1} \otimes \mathcal{O}_n \]

and \( \hat{S}_1 \otimes \mathcal{O}_n \subset \hat{g}_1 = g_1 \subset \text{Der}_1(\hat{S}) \otimes \mathcal{O}_n \). Therefore, the bracket operation in \( g, [\cdot, \cdot] : g_{-1} \times g_1 \longrightarrow [g_{-1}, g_1] \) is the restriction of the \( \mathcal{O}_n \)-bilinear bracket operation in \( \text{Der}(\hat{S}) \otimes \mathcal{O}_n \),

\[ \hat{S}_{-1} \otimes \mathcal{O}_n \times \text{Der}_1(\hat{S}) \otimes \mathcal{O}_n \longrightarrow \hat{S}_0 \otimes \mathcal{O}_n = S \otimes \mathcal{O}_n = [g_{-1}, g_1]. \]

(i) Conclusion of the proof of the Theorem.
Let \( g_1 = \bigoplus_{i \geq k} g_{1,-i} \). Suppose that \( k \leq |\delta| = n(p - 1) \). By (h) the bracket operation \([g_1, g_{-1}]\) is a restriction of the \( \mathcal{O}_n \)-bilinear bracket operation. Therefore, \( 0 \neq [g_1, g_{-1}] = [g_{1,-k}, \hat{S}_{-1} \otimes \mathcal{O}_n] \) is an \( \mathcal{O}_n \)-submodule of \([g_1, g_{-1}] = S \otimes \mathcal{O}_n \). On the other hand, by (d),

\[
[g_{1,-k}, g_{-1}] = [g_{1,-k}, g_{-1,1} + \ldots + g_{-1,|\delta|+1}] 
\subset g_{0,-k+1} + g_{0,-k+2} + \ldots 
= S \otimes \mathcal{O}_{n,-k+1} + \ldots + S \otimes 1.
\]

Hence, \( S \otimes \mathcal{O}_{n,-|\delta|} \cap [g_{1,-k}, g_{-1}] = 0 \). However, any nonzero \( \mathcal{O}_n \)-submodule of \( S \otimes \mathcal{O}_n \) has a nontrivial intersection with \( S \otimes x^\delta = S \otimes \mathcal{O}_{n,-|\delta|} \). We have arrived at a contradiction. Consequently, \( k = |\delta| + 1 \). By (g), \( g_{1,-|\delta|-1} = L_{-|\delta|-1} \) and \([S, L_{-|\delta|-1}] \neq 0 \).

We now show that if \( U \subset L_{-|\delta|-1} \) and \([S, U] \neq 0 \), then \([S \otimes x, U] \neq 0 \) in \( L \). Here \( x \in \mathcal{O}_{n,-1} \), and \( S \otimes x \subset S \otimes \mathcal{O}_{n,-1} = L_{-1} \).

Let \( 1 \otimes \partial_\xi \in 1 \otimes < \partial_{i_1}, \ldots, \partial_{i_n} > = V \subset L_1, \xi(x) \neq 0 \). By (f), 2), \( M(L_0) = L_{-|\delta|-1} + L_{-|\delta|-2} + \ldots \). Hence \([V, L_{-|\delta|-1}] = 0 \) and

\[
[1 \otimes \partial_\xi, [S \otimes x, U]] = [[1 \otimes \partial_\xi, S \otimes x], U] = [S, U] \neq 0.
\]

Therefore, \([S \otimes x, U] \neq 0 \) for any \( x \in \mathcal{O}_{n,-1} \). Since \([S, L_{-|\delta|-1}] \neq 0 \) and \( L_{-|\delta|-1} = [L_{-1}, L_{-|\delta|}] = \sum_{i=1}^n [S \otimes x_i, L_{-|\delta|}] \), there exists an \( i \) such that \([S, [S \otimes x_i, L_{-|\delta|}]] \neq 0 \). Set \( U = [S \otimes x_i, L_{-|\delta|}] \subset L_{-|\delta|-1} \). Then \([S, U] \neq 0 \). Thus, as above, \([S \otimes x_j, U] = [S \otimes x_j, [S \otimes x_i, L_{-|\delta|}]] \neq 0 \). We may renumber the variables and suppose \( i = 1 \). Since by assumption \( \dim V = n > 1 \), it follows that \([S \otimes x_2, [S \otimes x_1, L_{-|\delta|}]] \neq 0 \) in \( L \). For \( 1 \otimes \partial_2 \in V \subset L_1 \), we have

\[
[[S \otimes x_2, [S \otimes x_1, L_{-|\delta|}]], 1 \otimes \partial_2] = [[S \otimes x_2, 1 \otimes \partial_2], [S \otimes x_1, L_{-|\delta|}]] 
= [S, [S \otimes x_1, L_{-|\delta|}]] \neq 0.
\]

On the other hand, \([S \otimes x_1, 1 \otimes \partial_2] = 0 \) and

\[
[[S \otimes x_2, L_{-|\delta|}], 1 \otimes \partial_2] \subset [L_{-|\delta|-1}, V] = 0.
\]

Inasmuch as \( M(L_0) = L_{-|\delta|-1} + L_{-|\delta|-2} + \ldots \), we have that \( L_{-i} = S \otimes \mathcal{O}_{n,-i}, i = 1, \ldots, |\delta| \) and the bracket operation in \( L, L_{-i} \times L_{-j} \rightarrow L_{-i-j} \) coincides with the bracket operation in \( S \otimes \mathcal{O}_n \) for any \( i, j \) such that \( 1 \leq i, j, i + j \leq |\delta| \). Hence, in \( L \),

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\[ [S \otimes x_1 x_2, S \otimes x^\delta] = [S \otimes x_1 x_2, [S \otimes x_1^{p-1}, S \otimes x_2^{p-1} \ldots x_n^{p-1}]] \]
\[ = [[S \otimes x_1 x_2, S \otimes x_1^{p-1}], S \otimes x_2^{p-1} \ldots x_n^{p-1}] \]
\[ + [S \otimes x_1^{p-1}, [S \otimes x_1 x_2, S \otimes x_2^{p-1} \ldots x_n^{p-1}]] \]
\[ = 0. \]

Now, as \([S \otimes x_1, 1 \otimes \partial_2] = 0\) and \([[S \otimes x_2, L_{-[\delta]}], 1 \otimes \partial_2] \subset [L_{-[\delta]-1}, V] = 0\), we have

\[ [[S \otimes x_2, [S \otimes x_1, L_{-[\delta]}]], 1 \otimes \partial_2] = [[[S \otimes x_2, S \otimes x_1], L_{-[\delta]}], 1 \otimes \partial_2] \]
\[ + [[S \otimes x_1, [S \otimes x_2, L_{-[\delta]}]], 1 \otimes \partial_2] \]
\[ = [[S \otimes x_1 x_2, L_{-[\delta]}], 1 \otimes \partial_2] \]
\[ = [[[S \otimes x_1 x_2, S \otimes x^\delta]], 1 \otimes \partial_2] = 0. \]

The contradiction obtained completes the proof of the theorem. \(\Box\)

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