A Lower Bound on WAFOM

Takehito Yoshiki ∗†‡

December 16, 2014

Abstract

We give a lower bound on Walsh figure of merit (WAFOM), which is a parameter to estimate the integration error for quasi-Monte Carlo (QMC) integration by a point set called a digital net. This lower bound is optimal because the existence of point sets attaining the order was proved in [K. Suzuki, An explicit construction of point sets with large minimum Dick weight, Journal of Complexity 30, (2014), 347-354].

1 Introduction

We explain the relation between quasi-Monte Carlo (QMC) integration and the Walsh figure of merit (WAFOM) (see [3] for details). QMC integration is one of the methods for numerical integration (see [2], [4] and [7] for details). Let $Q$ be a point set in the $s$-dimensional cube $[0,1)^s$ with finite cardinality $\#(Q) = N$, and $f : [0,1)^s \to \mathbb{R}$ be a Riemann integrable function. The QMC integration by $Q$ is the approximation of $I(f) := \int_{[0,1)^s} f(x) dx$ by the average $I_Q(f) := \frac{1}{\#(Q)} \sum_{x \in Q} f(x)$.

WAFOM bounds the error of QMC integration for a certain class of functions by a point set $P$ called a digital net, which is defined by the following identification (see [3] and [5] for details): Let $\mathcal{P}$ be a subspace of $s \times n$ matrices over the finite field $\mathbb{F}_2$ of order two. We define the function $\varphi : \mathcal{P} \ni X = (x_{i,j}) \mapsto x = (\sum_{j=1}^n x_{i,j} \cdot 2^{-j})_{i=1}^s \in \mathbb{R}^s$, where $x_{i,j}$ is considered to be 0 or 1 in $\mathbb{Z}$ and the sum is taken in $\mathbb{R}$. The digital net $P$ in $[0,1)^s$ is defined by $\varphi(P)$. We identify the digital net $P$ with a linear space $\mathcal{P}$. If $\mathcal{P}$ is an $m$-dimensional space, the cardinality of $P$ is $2^m$.

Let $f$ be a function whose mixed partial derivatives up to order $\alpha \geq 1$ in each variable are square integrable (see [1, 3] for details). We say that such a function $f$ is an $\alpha$-smooth function or the smoothness of a function $f$ is $\alpha$ here. By using ‘$n$-digit discretization $f_n$’ (see [3] for details), we approximate $I(f)$ by $I_P(f_n) := \frac{1}{\#(P)} \sum_{x \in P} f_n(x)$ for an $n$-smooth function $f$, that is, we can

∗Tokyo 153-8914 Graduate School of Mathematical Sciences
†Email: yosiki@ms.u-tokyo.ac.jp
‡This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.
evaluate the integration error by the following Koksma-Hlawka type inequality of WAFOM:

\[ |I(f) - IP(f_n)| \leq C_{s,n} ||f||_n \times \text{WAFOM}(P), \]

where \( ||f||_n \) is the norm of \( f \) defined in [1] and \( C_{s,n} \) is a constant independent of \( f \) and \( P \). If the difference between \( IP(f_n) \) and \( IP(f) \) is negligibly small, we see that \( |I(f) - IP(f)| \leq C_{s,n} ||f||_n \times \text{WAFOM}(P) \) approximately holds (see [3] for details). In [4], we proved that there is a digital net \( P \) of size \( 2^m \) with \( \text{WAFOM}(P) < 2^{-Cm^2/s} \) for sufficiently large \( m \) by a probabilistic argument. (Suzuki [8] gave a constructive proof.) In this paper, we prove that \( \text{WAFOM}(P) > 2^{-C'm^2/s} \) holds for large \( m \) and any digital net \( P \) with \( \#(P) = 2^m \) (see Theorem 3.1 for a precise statement, which is formulated for a linear subspace \( P \), instead of a digital net \( P \)). Thus this order is optimal.

This paper is organized as follows: We introduce some definitions in Section 2. We prove a lower bound on WAFOM in Section 3.

2 Definition and notation

In this section, we introduce WAFOM and the minimum weight which will be needed later on.

Let \( s \) and \( n \) be positive integers. \( M_{s,n}(F_2) \) denotes the set of \( s \times n \) matrices over the finite field \( F_2 \) of order 2. We regard \( M_{s,n}(F_2) \) as an \( sn \)-dimensional inner product space under the inner product \( A \cdot B = (a_{i,j}) \cdot (b_{i,j}) = \sum_{i,j} a_{i,j} b_{i,j} \in F_2 \).

WAFOM is defined using a Dick weight in [3].

**Definition 2.1.** Let \( X = (x_{i,j}) \) be an element of \( M_{s,n}(F_2) \). The Dick weight of \( X \) is defined by

\[ \mu(X) := \sum_{1 \leq i \leq s, 1 \leq j \leq n} j \cdot x_{i,j}, \]

where we regard \( x_{i,j} \in \{0, 1\} \) as the element of \( \mathbb{Z} \) and take the sum in \( \mathbb{Z} \), not in \( F_2 \).

**Definition 2.2.** Let \( \mathcal{P} \) be a subspace of \( M_{s,n}(F_2) \). WAFOM of \( \mathcal{P} \) is defined by

\[ \text{WAFOM}(\mathcal{P}) := \sum_{X \in \mathcal{P}^{\perp}\setminus\{O\}} 2^{-\mu(X)}, \tag{1} \]

where \( \mathcal{P}^{\perp} \) denotes the orthogonal space to \( \mathcal{P} \) in \( M_{s,n}(F_2) \) and \( O \) denotes the zero matrix.

In order to estimate a lower bound on WAFOM, we use the minimum weight introduced in [4].

**Definition 2.3.** Let \( \mathcal{P} \) be a proper subspace of \( M_{s,n}(F_2) \). The minimum weight of \( \mathcal{P}^{\perp} \) is defined by

\[ \delta_{\mathcal{P}^{\perp}} := \min_{X \in \mathcal{P}^{\perp}\setminus\{O\}} \mu(X). \tag{2} \]
3 A lower bound on WAFOM

Now we state a lower bound on WAFOM. The theorem is mentioned for a linear subspace identified with a digital net (see Section 1).

**Theorem 3.1.** Let \( n, s \) and \( m \) be positive integers such that \( m < ns \), and let \( C' \) be an arbitrary real number greater than \( 1/2 \). If \( m/s \geq (\sqrt{C' + 1/16} + 3/4)/(C' - 1/2) \), then for any \( m \)-dimensional subspace \( P \) of \( M_{s,n}(F_2) \) we have

\[
\text{WAFOM}(P) \geq 2^{-C'm^2/s}.
\]

**Proof.** Let \( n, s, m \) and \( C' \) be defined as above. The following inequality immediately results from (1), (2) in Section 2:

\[
\text{WAFOM}(P) = \sum_{X \in P^\perp \setminus \{O\}} 2^{-\mu(X)} \geq 2^{-\delta_{P^\perp}}. \tag{3}
\]

By an upper bound on \( \delta_{P^\perp} \) in Lemma 3.1 (b) below and the inequality (3), for any \( m \)-dimensional subspace \( P \) of \( M_{s,n}(F_2) \), we have

\[
\text{WAFOM}(P) = \sum_{X \in P^\perp \setminus \{O\}} 2^{-\mu(X)} \geq 2^{-\delta_{P^\perp}} \geq 2^{-C'm^2/s}.
\]

Thus Theorem 3.1 follows. \( \square \)

We prove an upper bound on the minimum weight \( \delta_{P^\perp} \) to complete the proof of Theorem 3.1.

**Lemma 3.1.** Let \( n, s \) and \( m \) be positive integers such that \( m < ns \). Then we have the following statements:

(a) Let \( q \) and \( r \) be non-negative integers satisfying \( q = (m-r)/s \) and \( r < s \). Then we obtain

\[
\delta_{P^\perp} \leq \frac{sq(q+1)}{2} + (q+1)(r+1)
\]

for any \( m \)-dimensional subspace \( P \) of \( M_{s,n}(F_2) \).

(b) Let \( C' \) be an arbitrary positive real number greater than \( 1/2 \). If \( m/s \geq (\sqrt{C' + 1/16} + 3/4)/(C' - 1/2) \), then we have

\[
\delta_{P^\perp} \leq C'm^2/s
\]

for any \( m \)-dimensional subspace \( P \) of \( M_{s,n}(F_2) \).

**Proof.** (a) If there exists a subspace \( W \) of \( M_{s,n}(F_2) \) such that for any \( m \)-dimensional subspace \( P \) of \( M_{s,n}(F_2) \) we have \( P^\perp \cap W \neq \{O\} \), then \( \delta_{P^\perp} \leq \max_{X \in W} \mu(X) \) holds. Therefore in order to obtain a sharp upper bound on
The subspace $X$.

Let us estimate $\max_{X \in W} \mu(X)$ small. We can construct $W$ as follows:

$$W := \left\{ X = (x_{i,j}) \in M_{s,n}(\mathbb{F}_2) \mid \begin{array}{l}
x_{i,j} = 0 \\
(i \leq r + 1 \text{ and } q + 2 \leq j) \\
or \\
(r + 2 \leq i \text{ and } q + 1 \leq j)
\end{array} \right\},$$

that is, $W$ consists of the following type of matrices:

$$X = \begin{pmatrix}
x_{1,1} & \cdots & x_{1,q} & x_{1,q+1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 \\
x_{r+1,1} & \cdots & x_{r+1,q} & x_{r+1,q+1} & 0 & \cdots & 0 \\
x_{r+2,1} & \cdots & x_{r+2,q} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 \\
x_{s,1} & \cdots & x_{s,q} & 0 & 0 & \cdots & 0
\end{pmatrix} \ (x_{i,j} \in \mathbb{F}_2). \quad (4)$$

The subspace $W$ satisfies $\mathcal{P}^\perp \cap W \neq \{O\}$ for any $m$-dimensional subspace $\mathcal{P}$ of $M_{s,n}(\mathbb{F}_2)$. Indeed we can see that

$$\dim(\mathcal{P}^\perp \cap W) \geq \dim\mathcal{P}^\perp + \dim W - \dim M_{s,n}(\mathbb{F}_2)$$

$$= (sn - m) + (sq + r + 1) - sn = 1.$$

Hence there exists a non-zero matrix $X_{\mathcal{P}} \in W \cap \mathcal{P}^\perp$. This yields

$$\delta_{\mathcal{P}^\perp} = \min_{X \in \mathcal{P}^\perp \setminus \{O\}} \mu(X) \leq \mu(X_{\mathcal{P}}) \leq \max_{X \in W} \mu(X).$$

Let us estimate $\max_{X \in W} \mu(X)$ of $W$. Let $X_{\max}$ of $W$ be a matrix whose entries $x_{i,j}$ in $(4)$ are all 1. The function $\mu$ attains its maximum at $X_{\max}$ in $W$. Thus it follows that

$$\max_{X \in W} \mu(X) = \mu(X_{\max}) = \frac{sq(q + 1)}{2} + (q + 1)(r + 1).$$

We obtain that

$$\delta_{\mathcal{P}^\perp} = \min_{X \in \mathcal{P}^\perp \setminus \{O\}} \mu(X) \leq \mu(X_{\mathcal{P}}) \leq \max_{X \in W} \mu(X) = \frac{sq(q + 1)}{2} + (q + 1)(r + 1),$$

where $\mathcal{P}$ is an arbitrary $m$-dimensional subspace of $M_{s,n}(\mathbb{F}_2)$.

(b) Let $C'$ be a real number greater than 1/2 and assume $m/s \geq (\sqrt{C'} + 1/16 + 3/4)/(C' - 1/2)$. By combining $r + 1 \leq s, q \leq m/s$ and the assertion (a), we have

$$\delta_{\mathcal{P}^\perp} \leq \frac{m}{2} \left( \frac{m}{s} + 1 \right) + \left( \frac{m}{s} + 1 \right) \cdot s = \frac{m^2}{s} \left( \frac{1}{2} + \frac{3s}{2m} + \frac{s^2}{m^2} \right) \leq C'm^2/s,$$

where the last inequality follows from the assumption by completing the square with respect to $s/m$. \qed
Remark 3.1. This remark is to clarify relations between the above result and existing results. Fix $\alpha$, and consider the space of $\alpha$-smooth functions. For this (and even a larger) function class, Dick \cite{1} Corollary 5.5 and the comment after its proof gave digital nets for which the QMC integration error is bounded from above by the order of $2^{-\alpha m}m^{\alpha+1}$. This is optimal, since for any point set of size $2^n$, Sharygin \cite{6} constructed an $\alpha$-smooth function whose QMC integration error is at least of this order.

Since WAFOM gives only an upper bound of the QMC integration error, our lower bound $2^{-C'nm^2/s}$ on WAFOM in Theorem \ref{theo:main} implies nothing on the lower bound of the integration error.

A merit of WAFOM is that the value depends only on the point set, not on the smoothness $\alpha$ such as \cite{1}. On the other hand, WAFOM depends on the degree $n$ of discretization. Thus, it seems not easy to compare directly the upper bound on the integration error given in \cite{1} and that by WAFOM. However, we might consider that our lower bound $2^{-C'nm^2/s}$, which is independent of $n$ and $\alpha$, shows a kind of limitation of the method in bounding the integration error in \cite{1} in the limit $\alpha \to \infty$.

References

\begin{enumerate}
\item J. Dick, Walsh spaces containing smooth functions and Quasi-Monte Carlo rules of arbitrary high order. SIAM J. Numer. Anal., 46 (2008), 1519-1553.
\item J. Dick and F. Pillichshammer, Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration. Cambridge University Press, Cambridge, 2010.
\item M. Matsumoto, M. Saito, and K. Matoba, A computable figure of merit for Quasi-Monte Carlo point sets. Math. Comp., 83 (2014), 1233-1250.
\item M. Matsumoto and T. Yoshiki, Existence of Higher Order Convergent Quasi-Monte Carlo Rules via Walsh Figure of Merit. Monte Carlo and Quasi-Monte Carlo Methods 2012, Springer, Berlin, (2013), 569-579.
\item H. Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods. CBMS-NSF, Philadelphia, Pennsylvania, 1992.
\item I. F. Sharygin, A lower estimate for the error of quadrature formulas for certain classes of functions. Zh. Vychisl. Mat. i Mat. Fiz., 3 (1963), 370-376.
\item I. H. Sloan and S. Joe. Lattice Methods for Multiple Integration. Clarendon Press, Oxford, 1994.
\item K. Suzuki, An explicit construction of point sets with large minimum Dick weight, Journal of Complexity 30, (2014), 347-354.
\end{enumerate}