ALGEBRAIC INVARIANTS OF TRUNCATED FOURIER MATRICES

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Abstract. A partial Hadamard matrix $H \in M_{M \times N}(\mathbb{C})$ is called of “classical type” if the associated quantum semigroup $G \subseteq \tilde{S}_N^+$ is classical. In combinatorial terms, if $H_1, \ldots, H_M \in T^N$ are the rows of the matrix, the vectors $H_i/H_j \in T^N$ must be pairwise proportional, or orthogonal. We propose here of definition for the algebraic (or quantum) invariants of such matrices. For the truncated Fourier matrices, which are all of classical type, we obtain certain Bernoulli laws, that we compute in the $N >> M$ regime.

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Introduction

A complex Hadamard matrix is a matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle, and whose rows are pairwise orthogonal. The basic example is the Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$. For a number of results, and for potential applications of such matrices, mainly to quantum physics questions, see [5], [7], [14], [16].

The Hadamard matrices are also known to parametrize the orthogonal MASA in the simplest von Neumann algebra, $M_N(\mathbb{C})$. See [9], [13]. As a consequence, each such matrix $H \in M_N(\mathbb{C})$ produces an irreducible subfactor $M \subset R$ of the Murray-von Neumann hyperfinite factor $R$, having index $[R : M] = N$. The associated planar algebra $P = (P_k)$ has a direct description in terms of $H$, and a key problem is that of computing the corresponding Poincaré series, $f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$. See [8], [10], [11].

An alternative approach to this latter question is via quantum groups. The idea is that associated to $H \in M_N(\mathbb{C})$ is a certain quantum permutation group $G \subseteq S_N^+$, and the
problem is to compute the spectral measure \( \mu \in \mathcal{P}(\mathbb{R}_+) \) of the main character \( \chi : G \to \mathbb{C} \). This is the same problem as above, because \( f \) is the Stieltjes transform of \( \mu \):

\[
f(z) = \int_G \frac{1}{1-z\chi}
\]

The computation of \( f \) is a difficult question. One idea that emerged in recent years is that \( f \) might be related to some other invariants of \( H \), of geometric or probabilistic nature. There is some ongoing comparison work between these invariants, for certain uniform families of matrices \( H_N \in M_N(\mathbb{C}) \), in the \( N \to \infty \) limit. See [1], [2].

Another idea comes from the partial Hadamard matrices \( H \in M_{M \times N}(\mathbb{C}) \). Indeed, while an Hadamard matrix remains an "exceptional" object, a partial Hadamard one, in the \( N >> M \) regime, is more of a "random matrix" type object. So, one hope would be that the various invariants can be computed and compared, in the \( N \to \infty \) limit.

Perhaps most convincive here are the results in [12]. In that paper, de Launey and Levin studied the probability \( P_M \) for a random \( H \in M_{M \times N}(\pm 1) \) to be partial Hadamard, and established the following formula, in the \( N \in 4N, N \to \infty \) limit:

\[
P_M \simeq \frac{2^{(M-1)^2}}{\sqrt{(2\pi N)^{\binom{M}{2}}}}
\]

The proof in [12], perhaps more important than the result itself, uses the fact that the partial Hadamard matrices \( H \in M_{M \times N}(\pm 1) \) correspond to certain random walks, and so their number in the \( N \to \infty \) limit can be estimated by using the Fourier inversion formula. Thus, in a certain sense, [12] not only tells us how many partial Hadamard matrices \( H \in M_{M \times N}(\pm 1) \) there are, but also tells us what these matrices are.

It would be of course of great interest if such \( N \to \infty \) methods in the rectangular setting can ultimately lead to results about the square matrices \( H \in M_N(\mathbb{C}) \).

One problem with the partial Hadamard matrices is that there is no analogue for them of Popa’s MASA result in [13], or of Jones’ subfactor results in [10]. The hope, however, comes from quantum groups. Indeed, as explained in [4], associated to \( H \in M_{M \times N}(\mathbb{C}) \) is a representation \( \pi_H : C(\tilde{S}_M^+) \to M_N(\mathbb{C}) \), and hence is a semigroup of quantum partial permutations \( G \subset \tilde{S}_M^+ \), obtained via the Hopf image construction:

\[
\begin{tikzcd}
C(\tilde{S}_M^+) \arrow{r}{\pi_H} \arrow{dr} & M_N(\mathbb{C}) \arrow{d}
\end{tikzcd}
\]

\[
\begin{tikzcd}
C(G) \arrow{ur}
\end{tikzcd}
\]

Thus, in a certain sense, we have here some serious hope for the above-mentioned \( N \to \infty \) program in the rectangular matrix setting to be viable.

The present paper is a continuation of [4], mainly dealing with the "classical" case. Indeed, let us call \( H \in M_{M \times N}(\mathbb{C}) \) of classical type if the associated quantum semigroup
$G \subset \tilde{S}_N^+$ is classical. In combinatorial terms, if $H_1, \ldots, H_M \in \mathbb{T}^N$ are the rows of the matrix, the vectors $H_i/H_j \in \mathbb{T}^N$ must be pairwise proportional, or orthogonal. We propose here of definition for the algebraic invariants of such matrices, generalizing the measure $\mu = \text{law}(\chi)$. For the truncated Fourier matrices, which are all of classical type, we obtain in this way certain Bernoulli laws, that we compute in the $N \gg M$ regime.

The paper is organized as follows: 1 is a preliminary section, in 2 discuss the basic examples of classical PHM, in 3 we construct the algebraic invariants of such matrices, and in 4 we discuss the computation of the invariants of truncated Fourier matrices.

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1. Matrices and permutations

A partial Hadamard matrix (PHM) is a matrix $H \in M_{M \times N}(\mathbb{T})$, where $\mathbb{T}$ is the unit circle in the complex plane, having its rows pairwise orthogonal. The basic examples are the full-width submatrices of the complex Hadamard matrices $\tilde{H} \in M_N(\mathbb{T})$.

As explained in [4], the PHM have something to do with the partial permutations. In this section we briefly discuss this phenomenon. Our starting point will be:

Definition 1.1. A partial permutation of $\{1 \ldots, N\}$ is a bijection $\sigma : X \simeq Y$, with $X, Y \subset \{1, \ldots, N\}$. We denote by $\tilde{S}_N$ the set formed by such partial permutations.

Observe that we have $S_N \subset \tilde{S}_N$. The embedding $u : S_N \subset M_N(0,1)$ given by permutation matrices can be extended to an embedding $u : \tilde{S}_N \subset M_N(0,1)$, as follows:

$$u_{ij}(\sigma) = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases}$$

By looking at the image of this embedding, we see that $\tilde{S}_N$ is in bijection with the matrices $M \in M_N(0,1)$ having at most one 1 entry on each row and column.

Proposition 1.2. The number of partial permutations is given by

$$|\tilde{S}_N| = \sum_{k=0}^N k! \left(\frac{N}{k}\right)^2$$

that is, 1, 2, 7, 34, 209, ..., and with $N \to \infty$ we have $|\tilde{S}_N| \simeq N! \sqrt{\frac{\exp(4\sqrt{N}-1)}{4\pi\sqrt{N}}}$.

Proof. Indeed, for $\sigma : X \simeq Y$ we can set $k = |X| = |Y|$, and this leads to the formula in the statement. For the asymptotic formula, see the OEIS sequence A002720. \qed

Let us discuss now the “quantum version” of the above notions. In what follows we denote by $A$ an arbitrary $C^*$-algebra. That is, $A$ is a complex algebra with a norm and an involution, which is complete, and satisfies $||aa^*|| = ||a||^2$ for any $a \in A$. 
Definition 1.3. A submagic matrix is a matrix $u \in M_N(A)$ whose entries are projections $(p^2 = p^* = p)$, which are pairwise orthogonal on rows and columns. We let $C(S^+_N)$ be the universal $C^*$-algebra generated by the entries of a $N \times N$ submagic matrix.

If in addition the sum is 1 on each row and column, we say that $u$ is magic. Observe that we have a quotient map $C(S^+_N) \to C(S^+_N)$, where at right we have Wang’s algebra $C(S^+_N)$, generated by the entries of the universal $N \times N$ magic matrix [15].

The algebra $C(S^+_N)$ has a comultiplication given by $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$, and a counit given by $\varepsilon(u_{ij}) = \delta_{ij}$. Thus $S^+_N$ is a quantum semigroup, and we have maps as follows, with the bialgebras at left corresponding to the quantum semigroups at right:

$$
\begin{array}{ccc}
C(\tilde{S}^+_N) & \to & C(S^+_N) \\
\downarrow & & \downarrow \\
C(S_N) & \to & C(S_N)
\end{array}
$$

The relation with the PHM is immediate, coming from the following fact:

Proposition 1.4. If $H \in M_{M \times N}(T)$ is a PHM, with rows denoted $H_1, \ldots, H_M \in T^N$, then the matrix of rank one projections $P_{ij} = \text{Proj}(H_i/H_j)$ is submagic. Thus $H$ produces a $C^*$-algebra representation $\pi_H : C(S^+_N) \to M_N(\mathbb{C})$, given by $u_{ij} \to P_{ij}$.

Proof. We have indeed the following computation:

$$
\langle H_i/H_j, H_k \rangle = \sum_l H_{il}/H_{jl} \cdot H_{kl}/H_{jl} = \sum_l H_{kl} \cdot H_{jl} = \langle H_k, H_j \rangle
$$

A similar computation works for the columns of $P$, and this gives the result. \quad \Box

We can further build on this simple observation, in the following way:

Definition 1.5. The minimal semigroup $G \subset \tilde{S}^+_M$ producing a factorization of type

$$
\begin{array}{ccc}
C(S^+_M) & \overset{\pi_H}{\longrightarrow} & M_N(\mathbb{C}) \\
\downarrow & & \downarrow \\
C(G) & \longrightarrow & C(G)
\end{array}
$$

with a bialgebra map at left, is called quantum semigroup associated to $H$.

Here the existence and uniqueness of the bialgebra $C(G)$, with the universal property as above, can be shown by dividing $C(S^+_M)$ by a suitable ideal. See [4].

As is usually the case when dealing with “quantum” constructions, the very first problem is that of deciding under which exact assumptions our construction is in fact “classical”. In order to explain the answer here, we will need the following notion:
**Definition 1.6.** A pre-Latin square is a matrix $L \in M_M(1, \ldots, N)$ having the property that its entries are distinct, on each row and each column.

Given such a matrix $L$, to any $x \in \{1, \ldots, N\}$ we can associate the partial permutation $\sigma_x \in \tilde{S}_M$ given by $\sigma_x(j) = i \iff L_{ij} = x$. We denote by $G \subset \tilde{S}_M$ the semigroup generated by $\sigma_1, \ldots, \sigma_N$, and call it semigroup associated to $L$.

Also, given an orthogonal basis $\xi = (\xi_1, \ldots, \xi_N)$ of $\mathbb{C}^N$, we can construct a submagic matrix $P \in M_M(M_N(\mathbb{C}))$, according to the formula $P_{ij} = \text{Proj}(\xi L_{ij})$.

With these notations, we have the following result, from [4]:

**Theorem 1.7.** If $H \in M_{N \times M}(\mathbb{C})$ is a PHM, the following are equivalent:

1. The semigroup $G \subset \tilde{S}_M$ is classical, i.e. $G \subset \tilde{S}_M$.
2. The projections $P_{ij} = \text{Proj}(H_i/H_j)$ pairwise commute.
3. The vectors $H_i/H_j \in \mathbb{T}^N$ are pairwise proportional, or orthogonal.
4. The submagic matrix $P = (P_{ij})$ comes for a pre-Latin square $L$.

In addition, if so is the case, $G$ is the semigroup associated to $L$.

**Proof.** Here (1) $\iff$ (2) is clear, (2) $\iff$ (3) comes from the fact that two rank 1 projections commute precisely when their images coincide, or are orthogonal, (3) $\iff$ (4) is clear again, and the last assertion comes from Gelfand duality. See [4].

We call “classical” the matrices in Theorem 1.7. Further studying such matrices, and computing the corresponding semigroups $G$, will be the main goal of this paper.

## 2. Truncated Fourier matrices

We discuss in this section some basic examples of classical PHM. Let us begin with a complete study at $M = 2$. With $\tilde{S}_2 = \{\text{id}, \tau, 11, 12, 21, 22, \emptyset\}$, where $\tau$ is the transposition, $ij$ is the partial permutation $i \rightarrow j$, and $\emptyset$ is the null map, we have:

**Proposition 2.1.** A partial Hadamard matrix $H \in M_{2 \times N}(\mathbb{T})$, in dephased form

$$H = \begin{pmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_N \end{pmatrix}$$

is of classical type when one of the following happens:

1. Either $\lambda_i = \pm w$, for some $w \in \mathbb{T}$, in which case $G = \{\text{id}, \tau\}$.
2. Or $\sum_i \lambda_i^2 = 0$, in which case $G = \{\text{id}, 11, 12, 21, 22, \emptyset\}$

**Proof.** With $1 = (1, \ldots, 1)$ and $\lambda = (\lambda_1, \ldots, \lambda_N)$, the matrix formed by the vectors $H_i/H_j$ is $(\frac{1}{\lambda})$. Since $1 \perp \lambda, \bar{\lambda}$ we just have to compare $\lambda, \bar{\lambda}$, and we have two cases:

1. Case $\lambda \sim \bar{\lambda}$. This means $\lambda^2 \sim 1$, and so $\lambda_i = \pm w$, for some $w \in \mathbb{T}$. In this case the associated pre-Latin square is $L = (\frac{1}{\lambda})$, the partial permutations $\sigma_x$ associated to $L$ are $\sigma_1 = \text{id}$ and $\sigma_2 = \tau$, and we have $G = \langle \text{id}, \tau \rangle = \{\text{id}, \tau\}$, as claimed.
(2) Case $\lambda \perp \bar{\lambda}$. This means $\sum_i \lambda_i^2 = 0$. In this case the associated pre-Latin square is $L = (\frac{1}{3} \frac{1}{3})$, the associated partial permutations $\sigma_x$ are given by $\sigma_1 = id, \sigma_2 = 21, \sigma_3 = 12$, and so we obtain $G = < id, 21, 12 > = \{id, 11, 12, 21, 22, \emptyset\}$, as claimed. □

The matrices in (1) are, modulo equivalence, those which are real. As for the matrices in (2), these are parametrized by the solutions $\lambda \in T^N$ of the following equations:

$$\sum_i \lambda_i = \sum_i \lambda_i^2 = 0$$

In general, it is quite unclear on how to deal with these equations. Observe that, as a basic example here, we have the upper $2 \times N$ submatrix of $F_N$, with $N \geq 3$.

Let us discuss now in detail the truncated Fourier matrix case. First, we have:

**Proposition 2.2.** The Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$, is of classical type, and the associated group $G \subset S_N$ is the cyclic group $\mathbb{Z}_N$.

**Proof.** Since $H = F_N$ is a square matrix, the associated semigroup $G \subset \tilde{S}_N$ must be a quantum group, $G \subset \tilde{S}_N^+$. We must prove that this quantum group is $G = \mathbb{Z}_N$.

With $\rho = (1, w, w^2, \ldots, w^{N-1})$ the rows of $H$ are given by $H_i = \rho^i$, and so we have $H_i/H_j = \rho^{i-j}$. We conclude that $H$ is indeed of classical type, coming from the Latin square $L_{ij} = j - i$ and from the orthogonal basis $\xi = (1, \rho^{-1}, \rho^{-2}, \ldots, \rho^{1-N})$.

We have $G = < \sigma_1, \ldots, \sigma_N >$, where $\sigma_x \in S_N$ is given by $\sigma_x(j) = i \iff L_{ij} = x$. From $L_{ij} = j - i$ we obtain $\sigma_x(j) = j - x$, and so $G = \{\sigma_1, \ldots, \sigma_N\} \simeq \mathbb{Z}_N$, as claimed. □

Let $F_{M,N}$ be the upper $M \times N$ submatrix of $F_N$, and $G_{M,N} \subset \tilde{S}_M$ be the associated semigroup. The simplest case is that when $M$ is small, and here we have:

**Theorem 2.3.** In the $N > 2M - 2$ regime, $G_{M,N} \subset \tilde{S}_M$ is formed by the maps

$$\sigma = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}$$

that is, $\sigma : I \simeq J$, $\sigma(j) = j - x$, with $I, J \subset \{1, \ldots, M\}$ intervals, independently of $N$.

**Proof.** Since for $\tilde{H} = F_N$ the associated Latin square is circulant, $\tilde{L}_{ij} = j - i$, the pre-Latin square that we are interested in is:

$$L = \begin{pmatrix}
0 & 1 & 2 & \cdots & M - 1 \\
N - 1 & 0 & 1 & \cdots & M - 2 \\
N - 2 & N - 1 & 0 & \cdots & M - 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
N - M + 1 & N - M + 2 & N - M + 3 & \cdots & 0
\end{pmatrix}$$

Observe that, due to our $N > 2M - 2$ assumption, we have $N - M + 1 > M - 1$, and so the entries above the diagonal are distinct from those below the diagonal.
Let us compute now the partial permutations $\sigma_x \in \tilde{S}_M$ given by $\sigma_x(j) = i \iff L_{ij} = x$. We have $\sigma_0 = id$, and then $\sigma_1, \sigma_2, \ldots, \sigma_{M-1}$ are as follows:

\[
\begin{array}{cccc}
\sigma_1 &=& \vdots & \sigma_2 &=& \vdots & \ldots & \sigma_{M-1} &=& \vdots \\
\end{array}
\]

Observe that $\sigma_2 = \sigma_1^2, \sigma_3 = \sigma_1^3, \ldots, \sigma_{M-1} = \sigma_1^{M-1}$. As for the remaining partial permutations, these are given by $\sigma_{N-1} = \sigma_1^{-1}, \sigma_{N-2} = \sigma_2^{-1}, \ldots, \sigma_{N-M+1} = \sigma_{M-1}^{-1}$:

\[
\begin{array}{cccc}
\sigma_{N-1} &=& \vdots & \sigma_{N-2} &=& \vdots & \ldots & \sigma_{N-M+1} &=& \vdots \\
\end{array}
\]

Thus $G'_{M,N} = \langle \sigma_1 \rangle$. Now if we denote by $G''_{M,N}$ the semigroup in the statement, we have $\sigma_1 \in G''_{M,N}$, so $G_M,N \subseteq G''_{M,N}$. The reverse inclusion can be proved as follows:

1. Assume first that $\sigma \in G'_{M,N}$, $\sigma : I \simeq J$ has the property $M \in I, J$:

\[
\sigma = \vdots
\]

Then we can write $\sigma = \sigma_{N-k}\sigma_k$, with $k = M - |I|$, so we have $\sigma \in G_{M,N}$.

2. Assume now that $\sigma \in G'_{M,N}$, $\sigma : I \simeq J$ has just the property $M \in I$ or $M \in J$:

\[
\begin{array}{cc}
\sigma' &=& \vdots \\
\sigma'' &=& \vdots \\
\end{array}
\]

In this case we have as well $\sigma \in G_{M,N}$, because $\sigma$ appears from one of the maps in (1) by adding a “slope”, which can be obtained by composing with a suitable map $\sigma_k$.

3. Assume now that $\sigma \in G'_{M,N}$, $\sigma : I \simeq J$ is arbitrary:

\[
\sigma = \vdots
\]

Then we can write $\sigma = \sigma'\sigma''$ with $\sigma' : L \simeq J$, $\sigma'' : I \simeq L$, where $L$ is an interval satisfying $|L| = |I| = |J|$ and $M \in L$, and since $\sigma', \sigma'' \in G_{M,N}$ by (2), we are done.

Summarizing, we have so far complete results at $N = M$, and at $N > 2M - 2$. In the remaining regime, $M < N \leq 2M - 2$, the semigroup $G_{M,N} \subseteq S_M$ looks quite hard to compute, and for the moment we only have some partial results regarding it.

For a partial permutation $\sigma : I \simeq J$ with $|I| = |J| = k$, set $\kappa(\sigma) = k$. We have then the following result, to be of use later on, in section 4 below:
Proposition 2.4. The components $G_{M,N}^{(k)} = \{ \sigma \in G_{M,N} | \kappa(\sigma) = k \}$ with $k > 2M - N$ are, in the $M < N \leq 2M - 2$ regime, the same as those in the $N > 2M - 2$ regime.

Proof. In the $M < N \leq 2M - 2$ regime the pre-Latin square that we are interested in has as usual 0 on the diagonal, and then takes its entries from the set $S = \{1, \ldots, N - M\} \cup \{N - M + 1, \ldots, M - 1\} \cup \{M, \ldots, N - 1\}$, in a uniform way from each of the 3 components of $S$. Here is an illustrating example, at $M = 6, N = 8$:

$$L = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 \\
7 & 0 & 1 & 2 & 3 & 4 \\
6 & 7 & 0 & 1 & 2 & 3 \\
5 & 6 & 7 & 0 & 1 & 2 \\
4 & 5 & 6 & 7 & 0 & 1 \\
3 & 4 & 5 & 6 & 7 & 0
\end{pmatrix}$$

The point now is that $\sigma_1, \ldots, \sigma_{N-M}$ are given by the same formulae as those in the proof of Theorem 2.3, then $\sigma_{N-M+1}, \ldots, \sigma_{M-1}$ all satisfy $\kappa(\sigma) = 2M - N$, and finally $\sigma_M, \ldots, \sigma_{N-1}$ are once again given by the formulae in the proof of Theorem 2.3.

Now since we have $\kappa(\sigma\rho) \leq \min(\kappa(\sigma), \kappa(\rho))$, adding the maps $\sigma_{N-M+1}, \ldots, \sigma_{M-1}$ to the semigroup $G_{M,N} \subset \tilde{S}_M$ computed in the proof of Theorem 2.3 won’t change the $G_{M,N}^{(k)}$ components of this semigroup at $k > 2M - N$, and this gives the result. \(\square\)

3. Fixed points, invariants

In this section we propose a definition for the algebraic invariants of the PHM of classical type. Let us first recall the theory in the square case, $M = N$:

**Definition 3.1.** Associated to a complex Hadamard matrix $H \in M_N(\mathbb{C})$ is the spectral measure $\mu \in \mathcal{P}(\mathbb{R}_+)$ of the main character $\chi = \sum_i u_{ii}$ of the corresponding quantum group $G \subset S_N^+$, with respect to the Haar integration on this quantum group.

Here we use the fact, due to Woronowicz [17], that the compact quantum groups have indeed a Haar integration functional. Let us also mention that, once again due to the results of Woronowicz in [17], the following key Peter-Weyl type formula holds:

$$\int_G \chi^k = \dim(Fix(u^{\otimes k}))$$

In the quantum semigroup case no Haar integration or Peter-Weyl theory is available, at least so far, and we will restrict attention in what follows to the classical case. In order to do so, let us first go back to the group case, $G \subset S_N$. Here $\chi = \sum_i u_{ii}$ is the number of fixed points, and in the case $G = S_N$, the following key result is available:

**Proposition 3.2.** For $G = S_N$, the number of fixed points $\chi : G \to \mathbb{N}$ becomes a Poisson variable, in the $N \to \infty$ limit.
Proof. This is well-known. First, the probability for a random $\sigma \in S_N$ to have no fixed points can be computed by using the inclusion-exclusion principle, and is
\[ \sum_{r=0}^{N} \frac{(-1)^r}{r!} \approx \frac{1}{e}. \]
With a little more care we obtain $P(\chi = k) \approx \frac{1}{ek!}$ for any $k \in \mathbb{N}$, as claimed. $\square$

As explained in [3], a “free” analogue of the above result holds for $S_N^+$, with the Poisson law replaced by the Marchenko-Pastur law, or free Poisson law. This $S_N/S_N^+$ phenomenon is in fact a key instance of the Bercovici-Pata bijection [6].

In the semigroup case now, the counting measure on $\tilde{S}_N$ is of course available, but any global fixed point computation over $\tilde{S}_N$ stumbles, in the $N \to \infty$ limit, into the quite subtle formula
\[ |\tilde{S}_N| \approx N! \sqrt{\frac{\exp(\sqrt{N}-1)}{4\pi N}}, \]
mentioned in Proposition 1.2 above.

However, we have here a solution, as follows. Let $\kappa : \tilde{S}_N \to \mathbb{N}$ be the cardinality of the domain/range, and let $\chi : \tilde{S}_N \to \mathbb{N}$ be the number of fixed points. Then:

**Theorem 3.3.** For $G = \tilde{S}_N$ we have the formula
\[ \text{law}(\chi | \kappa = k) = \sum_{k=0}^{s} \binom{k}{s} \frac{(N-s)!}{N!} \left( \delta_1 - \delta_0 \right)^s \]
and this law coincides with that of the truncated character $u_{11} + \ldots + u_{kk} \in C(S_N)$.

Proof. We proceed as in the proof of Proposition 3.2. By using the inclusion-exclusion principle, in the general counting framework of Proposition 1.2 above, we obtain:
\[ P(\chi = p | \kappa = k) = \frac{1}{k! \binom{N}{k}^2} \left( \frac{N}{p} \right)^k \sum_{r=0}^{k} (-1)^r \binom{k}{p} \binom{N-p}{r} (k-p-r)! \binom{N-p-r}{k-p-r}^2 \]
\[ = \sum_{r=0}^{k-p} (-1)^r \binom{k}{p} \binom{k-p-r}{r} \frac{(N-p-r)!}{N!} \]

We can now compute the law in the statement, as follows:
\[ \text{law}(\chi | \kappa = k) = \sum_{p \geq 0} \sum_{r=0}^{k-p} (-1)^r \binom{k}{p} \binom{k-p-r}{r} \frac{(N-p-r)!}{N!} \delta_p \]
\[ = \sum_{s=0}^{k} \sum_{r=0}^{s} (-1)^r \binom{k}{s} \binom{k-s+r}{r} \frac{(N-s)!}{N!} \delta_{s-r} \]
\[ = \sum_{s=0}^{k} \binom{k}{s} \frac{(N-s)!}{N!} \sum_{r=0}^{s} (-1)^r \binom{s}{r} \delta_{s-r} \]

Now by using the binomial formula at right, this gives the formula in the statement.

Regarding now the second assertion, this follows either directly, or from the fact that the law of $u_{11} + \ldots + u_{kk} \in C(S_N)$, computed in [3], is given by exactly the same formula. $\square$
Here are now a few consequences of the above result, in various asymptotic regimes:

**Proposition 3.4.** The measure $\mu_k = \text{law}(\chi|\kappa = k)$ over $G = \tilde{S}_N$ is as follows:

1. With $N \to \infty$ we have $\mu_k \simeq B\left(k, \frac{1}{N}\right)$.
2. With $k = N \to \infty$ we have $\mu_k = \text{Poisson}(1)$.
3. With $k = \lfloor tN \rfloor \to \infty$ we have $\mu_k = \text{Poisson}(t)$.

**Proof.** We use Theorem 3.3, as well as the various formulae from its proof.

1. In the $N \to \infty$ limit, we have:
   
   \[ P(\chi = p|\kappa = k) \simeq \binom{k}{p} \frac{1}{N^p} \left(1 - \frac{1}{N}\right)^{k-p} \]

   We conclude that have indeed a binomial law, as in the statement.

2. This is clear from the formula in Theorem 3.3.

3. Once again, this is clear from the formula in Theorem 3.3. \qed

Summarizing, the measures $\mu_k = \text{law}(\chi|\kappa = k)$ have very nice properties, and appear to be the good generalization of the measure $\mu = \text{law}(\chi)$ from the group case.

We can therefore state the following definition, in tune with Definition 3.1:

**Definition 3.5.** The algebraic invariants of a semigroup $G \subset \tilde{S}_N$ (and in particular, of the semigroup associated to a classical PHM) are the measures $\mu_k = \text{law}(\chi|\kappa = k)$.

Observe that in the case of a classical PHM, $H \in M_{M \times N}(\mathbb{T})$, the algebraic invariants depend only on the associated pre-Latin square $L \in M_M(1, \ldots, N)$.

As a basic example, at $M = 2$, in the context of Proposition 2.1, we have $\mu_1 = 0, \mu_2 = \frac{1}{2}(\delta_0 + \delta_2)$ in the real case, and $\mu_1 = \frac{1}{2}(\delta_0 + \delta_1), \mu_2 = \delta_2$ in the purely complex case.

4. **Bernoulli laws, conclusion**

In this section we discuss the computation of the algebraic invariants constructed in section 3 above, for the truncated Fourier matrices. First, we have:

**Proposition 4.1.** Let $\Lambda \subset \{1, \ldots, N\}$, and consider the corresponding submatrix $H \in M_{\Lambda \times N}(\mathbb{T})$ of the Fourier matrix $F_N$. Then $H$ is of classical type, and its algebraic invariants $\mu_k$ are all Bernoulli laws, $\mu_k = (1 - c_k)\delta_0 + c_k\delta_k$, with $c_k \in [0, 1]$.

**Proof.** Since the Latin square associated to $\tilde{H} = F_N$ is circulant, $\tilde{L}_{ij} = j - i$, the semigroup $G \subset S_{|\Lambda|}$ associated to $H$ satisfies $G \subset T_{|\Lambda|}$, where $T_{|\Lambda|} \subset \tilde{S}_{|\Lambda|}$ is the semigroup of all partial permutations of type $\sigma : I \simeq J, \sigma(j) = j - x$, with $I, J \subset \Lambda$. Now since such a partial permutation has either $0$ or $k = |I| = |J|$ fixed points, this gives the result. \qed

More generally, given $\Gamma = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$, the Fourier matrix $F_\Gamma = F_{N_1} \otimes \ldots \otimes F_{N_k}$ is known to be classical, the permutation group $G \subset S_{|\Gamma|}$ associated to it being $\Gamma$ itself. Now if we truncate this matrix by using an arbitrary set of row indices $\Lambda \subset \Gamma$, we obtain a
classical PHM, whose invariants are once again Bernoulli laws. Indeed, the same argument as in the proof of Proposition 4.1 applies, with $\bar{L}_{ij} = j - i$ taken this time inside $\Gamma$.

For arbitrary classical PHM, we can in principle obtain quite complicated laws, and in particular non-Bernoulli ones. We don’t have, however, an explicit example here.

Now let us go back to the computations in section 3 above. In terms of algebraic invariants, the consequences of the results there are as follows:

**Theorem 4.2.** Let $F_{M,N}$ be the upper $M \times N$ submatrix of the Fourier matrix $F_N$. Then the algebraic invariants of $F_{M,N}$ are as follows:

1. At $N = M$, $\mu_0 = \ldots = \mu_{M-1} = \delta_0$ and $\delta_M = (1 - \frac{1}{M})\delta_0 + \frac{1}{M}\delta_M$.
2. At $M < N \leq 2M - 2$, $\mu_k = \left(1 - \frac{1}{M-k+1}\right)\delta_0 + \frac{1}{M-k+1}\delta_k$, $\forall k > 2M - N$.
3. At $N > 2M - 2$ we have $\mu_k = \left(1 - \frac{1}{M-k+1}\right)\delta_0 + \frac{1}{M-k+1}\delta_k$, for any $k$.

**Proof.** We use the various results in section 3 above:

1. This is clear from Proposition 2.2.
2. This follows from (3) below, and from Proposition 2.4.
3. The cardinality of the semigroup computed in Theorem 2.3 is:

$$|G| = 1 + \sum_{k=1}^{M} (M - k + 1)^2$$

Here, with notations in Theorem 2.3, we have set $k = |I| = |J|$, and the two $M - k + 1$ factors correspond to the choices of the intervals $I, J$ inside $\{1, \ldots, M\}$. Now by looking at the fixed points, this gives the formula in the statement. □

Observe that the measures $\mu_k$ computed above, while constant in the $N > 2M - 2$ regime, are subject to an obvious phase transition at $N = M$. We do not know what happens in the $M < N \leq 2M - 2$ regime, at values $k \leq 2M - N$.

As a conclusion, the computation of the algebraic invariants, for the truncated Fourier matrices, or for more general classical PHM, looks like a quite interesting question. In addition, we have the question of constructing the invariants in the non-classical case.

Generally speaking, all this can be regarded as a “complement”, perhaps bringing a useful degree of flexibility, to the computations of the quantum invariants of (square) complex Hadamard matrices, where, while some general algebraic machinery is available from [10], [11], only few complete results are available so far [2], [8].

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