Measurement of Stochastic Gravitational Wave Background with a Single Laser Interferometer

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Abstract

Laser interferometer gravitational wave detectors can be operated at their free spectral range frequency. We show that in this case and when the interferometer is well understood one could detect a stochastic background using a single detector.

1. Introduction

We define the stochastic gravitational wave background (sgwb) by a dimensionless strain $h(t)$ which is a real random variable, stationary in time. Namely, it is Gaussian distributed with zero mean

$$< h(t) > = 0 \quad (1)$$

and finite “power”

$$< h^2(t) > = P \quad (2)$$

Eqs.(1,2) do not suffice to completely describe the random process. To do so we must also specify how the power $P$ is distributed in frequency\(^1\), what is referred to as the power spectral density (PSD).

We therefore assume that $h(t)$ can be represented, at a given point in space, as a Fourier integral

$$h(t) = \int_{-\infty}^{\infty} \tilde{h}(f)e^{-2\pi ift}df \quad (3)$$

Our assumptions about the properties of $h(t)$ imply that

$$< \tilde{h}(f) > = 0 \quad (4)$$

and that [1]

$$< \tilde{h}^*(f)\tilde{h}(f') > = \frac{1}{2}\delta(f - f')S(f) \quad 0 < f < \infty \quad (5)$$

\(^1\)For instance whether the random process is like “white noise”, “1/f noise”, resonant at a given frequency, etc.
The function $S(f)$ is the (one-sided) power spectral density per unit time. The proof of Eq.(5) and the subtleties in the definition of $\tilde{h}(f)$ are discussed in the Appendix. Throughout this note we have adopted the nomenclature and conventions of reference [2] as much as possible.

The $\delta$-function can be removed from Eq.(5) by integrating over $f'$ in the narrow interval $f - \Delta f / 2 < f' < f + \Delta f / 2$, to obtain

$$S(f) = \lim_{\Delta f \to 0} 2 \int_{f - \Delta f / 2}^{f + \Delta f / 2} \tilde{h}(f) \tilde{h}(f') > df' \approx 2 |\tilde{h}(f)|^2 > \Delta f$$  \hspace{1cm} (6)

The integration over the small interval $\Delta f$ expresses a kind of smoothing of $\tilde{h}(f)$ [3,4]. If $\tilde{h}(f)$ is obtained from a stretch of data of length $T$, then $\Delta f = 1 / T$ and Eq.(6) takes the form

$$S(f) = \frac{2}{T} < |\tilde{h}(f)|^2 > \hspace{1cm} 0 < f < \infty$$  \hspace{1cm} (7)

It follows from Parceval’s theorem that

$$\int_0^\infty S(f) df = \frac{1}{T} \int_{-\infty}^{\infty} \tilde{h}(f)|^2 > df = \frac{1}{T} \int_{-T/2}^{T/2} |h(t)|^2 dt = \int_{-\infty}^{\infty} < |h(t)|^2 > = P$$  \hspace{1cm} (8)

Eqs.(7,8) are our central result. Experimentally we wish to determine $S(f)$. We will introduce the notation

$$\hat{h}(f) = \sqrt{S(f)}$$  \hspace{1cm} (9)

and refer to $\hat{h}(f)$ as the sgwb amplitude (density) per square root of frequency, i.e. $\hat{h}(f)$ represents strain /$\sqrt{\text{Hz}}$.

2. Single noiseless detector

In an experiment we always deal with finite time intervals and with time series (rather than functions of time). This is well suited to the form introduced in Eqs.(6,7). We consider a linear detector that in response to g.w. strain $h(t)$ outputs a signal $a(t)$. To include the response of the detector we must work in the frequency domain

$$A(f) = H(f)\tilde{h}(f)$$  \hspace{1cm} (10)

$$= \frac{2}{T} < |\tilde{h}(f)|^2 > \hspace{1cm} 0 < f < \infty$$  \hspace{1cm} (7)

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$$A(f) = H(f)\tilde{h}(f)$$  \hspace{1cm} (10)
where \( H(f) \) is a deterministic transfer function\(^2\) peaked at some value \( f_0 \) and decreasing below and above \( f_0 \). The detector output is given by

\[
a(t) = \int_{-\infty}^{\infty} A(f) e^{-2\pi if t} df
\]

(11)

The output \( a(t) \) is sampled at a rate \( 1/\Delta = 2f_c \), where \( f_c \) is referred to as the Nyquist critical frequency. We examine a record consisting of \( N \) data points; the length of the record is \( \tau = N\Delta \) and the resulting time series is

\[
a_n = a(t_n) \quad \text{with} \quad t_n = n\Delta \quad n = 0, 1, \ldots, (N-1)
\]

(12)

We can form the Discrete Fourier Transform (DFT) of this time series [2].

\[
A_k = \sum_{n=0}^{N-1} a_n e^{2\pi i kn/N} \quad k = 0, 1, \ldots, (N-1)
\]

(13)

This is a mapping of the \( N \) complex numbers \( a_n \) (of course in our application the \( a_n \) are real) onto \( N \) complex numbers \( A_k \). The mapping is invertible

\[
a_n = \frac{1}{N} \sum_{k=0}^{N-1} A_k e^{-2\pi i kn/N}
\]

(14)

and obeys Parceval’s theorem

\[
\sum_{n=0}^{N-1} |a_n|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |A_k|^2
\]

(15)

The elements \( A_k \) are related to \( A(f_k) \), the Fourier transform of \( a(t) \), through

\[
A(f_k) \approx \Delta A_k
\]

(16)

We can obtain an estimate of the PSD by forming the periodogram of \( a(t) \). The frequency is defined only for zero and for \( N/2 \) positive values

\[
f_k = \frac{k}{N\Delta} = 2f_c \frac{k}{N} \quad k = 0, 1, \ldots, N/2
\]

(17)

The bandwidth \( B \) is the frequency spacing between \( k \) and \( k + 1 \), or

\[
B = 1/N\Delta = 1/\tau
\]

(18)

\(^2\)We take \( H(f) \) to be dimensionless, that is the transfer function from g.w. strain to strain recorded by a calibrated detector.
The terms of the periodogram are

\[ P(0) = \frac{1}{N^2} |A_0|^2 \]

\[ P(f_k) = \frac{1}{N^2} \left[ |A_k|^2 + |A_{-k}|^2 \right] \quad k = 1, 2, \ldots, (N/2 - 1) \]  \hspace{1cm} (19)

\[ P(f_c) = \frac{1}{N^2} |A_{N/2}|^2 \]

For \( a(t) \) real \( |A_k|^2 = |A_{-k}|^2 \) and therefore

\[ P(f_k) = \frac{2}{N^2} |A_k|^2 \]  \hspace{1cm} (20)

We now use the approximate form of Eq.(16) in Eq.(7) to obtain for the PSD per unit time

\[ S_A(f_k) = \frac{2}{\tau} < |A(f_k)|^2 > \approx \frac{2\Delta^2}{\tau} < |A_k|^2 > = \]

\[ = \tau \frac{2}{N^2} < |A_k|^2 > = \frac{1}{B} P(f_k) \]  \hspace{1cm} (21)

Returning to Eq.(10) we can write for the discrete case

\[ \Delta A_k \approx A(f_k) = H(f_k) \tilde{h}(f_k) \]  \hspace{1cm} (22)

and therefore

\[ S_A(f_k) = \frac{2}{\tau} |H(f_k)|^2 < |\tilde{h}(f_k)|^2 > \]  \hspace{1cm} (23)

If \( \tilde{h}(f) \) is approximately constant in the region where \( H(f) \) peaks then the detected spectral density (per unit time) \( S_A(f_k) \) will follow the spectrum of \( |H(f_k)|^2 \).

3. Single noisy detector

Suppose now that the detector introduces noise so that the output signal is

\[ c(t) = q(t) + a(t) \]  \hspace{1cm} (24)

Here \( q(t) \) is a stochastic variable with zero mean and standard deviation \( \sigma \). After sampling \( c(t) \) and forming the DFT we obtain as in Eq.(13).
Since \( q(t) \) and \( a(t) \) are uncorrelated, the PSD/time, upon averaging, will contain only two terms

\[
S_C(f_k) = \frac{2\tau}{N^2} \left[ < |Q_k|^2 > + < |A_k|^2 > \right]
\]

(26)

If the noise spectrum is flat in frequency, one can extract \( < |A_k|^2 > \) by fitting \( S_C(f_k) \) with the expected spectral shape, which according to Eq.(23) is that of \( |H(f_k)|^2 \). In this way one can determine \( < |\hat{h}(f_k)|^2 > \), in the region \( f_k = f_0 \), with a single detector, provided the spectrum of \( < |Q_k|^2 > \) is known.

In fact, one can work directly with the magnitude of \( A_k \) which is related to \( \hat{h}(f) \), defined by Eq.(9). To extract a meaningful signal, \( |A_k| \) must be larger or equal than the fluctuations of \( Q_k \) (rather than \( Q_k \) itself). For a single measurement the fluctuations of \( |Q_k| \) are equal to \( \left[ < |Q_k|^2 > \right]^{1/2} \) since \( q(t) \) was assumed to have zero mean. After averaging \( N_a \) different DFT spectra

\[
\sigma_{Q_k}(N_a) = \frac{1}{\sqrt{N_a}} |Q_k|
\]

(27)

Further, in fitting the data to extract \( |A_k|^2 \) we have at our disposal the \( n \) frequency bins in the bandwidth \( \Delta f \), typically a few times the FWHM of \( H(f) \),

\[
n = \frac{\Delta f}{B} = \tau \Delta f
\]

(28)

The significance of the fit can be expressed as a signal to noise ratio

\[
\frac{S}{N}|_{N_a} = \sqrt{\frac{< |A_k|^2 >}{< |Q_k|^2 >}} \sqrt{n} \frac{1}{\sqrt{N_a}} = \sqrt{T \Delta f} \frac{\sqrt{< |A_k|^2 >}}{\sqrt{< |Q_k|^2 >}} = \sqrt{T \Delta f} |H_k| \left[ \hat{h}_s \right] \frac{1}{\hat{h}_N}
\]

(29)

In Eq.(29) we have set \( T = \tau N_a \) for the total measurement time and \( |H_k| \) is the magnitude of the transfer function averaged over the frequency band \( \Delta f \). We also used Eqs.(25,26) and the definition of Eq.(9) to relate \( < |A_k| > \) and \( < |Q_k| > \) to
the amplitude (densities) per unit frequency for the sgwb signal and the noise in the detector.

Eq.(29) shows that when using a single interferometer, the $S/N$ ratio for $\hat{h}_s/\hat{h}_N$ improves as $\sqrt{T \Delta f}$, in contrast to the correlation method [see Eq.(37)] where the $S/N$ for the same ratio improves only as $(T \Delta f)^{1/4}$. This is particularly true if no signal is detected and the measurement yields only an upper limit on $\hat{h}_s$. Computer simulations [5] of a sgwb signal using for noise the actual time series from LIGO test data confirm this conclusion.

4. Correlated detectors

The “standard” approach to the measurement of a stochastic signal is to correlate the output of two detectors. For the sgwb this method has been discussed extensively in [6,7] and has been applied to data from the LIGO interferometers in [8]. Here we only briefly outline the analysis procedure and present the conclusions.

Assuming that the two detectors 1,2 are co-located and co-aligned, the stochastic signal, $a(t)$ will be the same in both detectors whereas the noise signals $q_1(t)$ and $q_2(t)$ are treated as uncorrelated. Thus we write

$$
c_1(t) = q_1(t) + a(t)
$$
$$
c_2(t) = q_2(t) + a(t)
$$

We form the cross-correlation (at zero lag) of the two signals and integrate over a time interval $\tau$, to obtain a statistic [1]

$$
C = \int_0^\tau c_1(t)c_2(t)dt
$$

The mean value $\mu$, of this statistic obtained from repeated cross-correlation spectra (samples) is a measure of the stochastic signal because

$$
\mu = \langle C \rangle = \tau \langle a^2(t) \rangle = \tau \langle a^2(t) \rangle
$$

We can express $\mu$ in terms of frequency domain variables by using the definition of Eq.(8)
\[
\mu = \tau \int_0^\infty S_A(f) df
\]  
(33)

where \( S_A(f) \) is the spectral density of the stochastic signal.

The fluctuations in \( \mu \) are given by the variance of \( C \)

\[
\sigma_C^2 = < C^2 > - < C >^2
\]  
(34)

and if \( a(t) \ll q_i(t) \), \( \sigma_C^2 \) is dominated by the noise spectrum

\[
\sigma_C^2 = \int_0^\tau < q_1^2(t) > < q_2^2(t) > d\tau = \tau \int_0^\infty S_{N_1}(f) S_{N_2}(f) df
\]  
(35)

where \( S_{N_i}(f) \) are the spectral densities of the noise. If we use \( N_S \) samples to determine the mean, then \( < \mu > \) will be Gaussian distributed around its true value with standard deviation \( \sigma_C/\sqrt{N_S} \). Thus the signal to noise ratio is

\[
\frac{S}{N}\bigg|_{N_S} = \frac{< \mu >}{\sigma_C} \sqrt{N_S} = \sqrt{N_S \tau} \frac{\int_0^\infty S_A(f) df}{\int_0^\infty S_{N_1}(f) S_{N_2}(f) df}^{1/2}
\]  
(36)

If we further assume that \( S_A(f) \) is reasonably constant in the region of integration, and so are \( S_{N_1}(f) \simeq S_{N_2}(f) \) we can use the notation of Eq.(9) to simplify Eq.(36)

\[
\frac{S}{N}\bigg|_{N_S} = \sqrt{T \Delta f} \left[ \frac{\hat{h}_s(f)}{\hat{h}_N(f)} \right]^2
\]  
(37)

where we set \( T = N_S \tau \) for the total measurement time. Eq.(37) is to be compared with Eq.(29) obtained for measurements with a single detector.

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APPENDIX A

Spectral Density of a Random Variable

We expressed a stochastic signal \( h(t) \) by Eq. (3)

\[
h(t) = \int_{-\infty}^{\infty} \tilde{h}(f) e^{-2\pi i ft} df \tag{A1}
\]

It is not possible to invert Eq. (A1) because the integral

\[
\int_{-\infty}^{\infty} h(t) e^{2\pi i ft} dt \tag{A2}
\]

diverges for a stationary signal.

Instead, following [4] we can introduce the truncated function

\[
h_T(t) = h(t) \quad - T/2 < t < T/2
\]

\[
= 0 \quad \text{elsewhere} \tag{A3}
\]

Thus

\[
\tilde{h}_T(f) = \int_{-\infty}^{\infty} h_T(t) e^{2\pi i ft} dt \tag{A4}
\]

and we form

\[
\tilde{h}_T^*(f) \tilde{h}_T(f') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_T^*(t) h_T(t') e^{-2\pi i ft} e^{2\pi i f' t'} dt dt' \tag{A5}
\]

We set \( t' = t + \tau \) so that

\[
\tilde{h}_T^*(f) \tilde{h}_T(f') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_T^*(t) h_T(t + \tau) e^{2\pi i (f' - f) \tau} e^{2\pi i f' \tau} d\tau d\tau \tag{A6}
\]

We now take the ensemble average of both sides, interchange the order of averaging and integration on the right-hand side, and define the \textit{autocorrelation} function [9, 3]

\[
R_T(\tau) = \langle h_T^*(t) h_T(t + \tau) \rangle \tag{A7}
\]

The function \( R_T(\tau) \) vanishes for \( |\tau| > T \), and Eq. (A6) reads

\[
\langle \tilde{h}_T^*(f) \tilde{h}_T(f') \rangle = \delta(f - f') \int_{-\infty}^{\infty} R_T(\tau) e^{2\pi i f \tau} d\tau \tag{A8}
\]
We therefore define the power spectral density (per unit time) as the Fourier transform of the autocorrelation function

\[ S_T(f) = 2 \int_{-\infty}^{\infty} R_T(\tau) e^{2\pi if\tau} d\tau \quad 0 < f < \infty \quad (A9) \]

Thus Eq.(A8) is recast as

\[ \langle \tilde{h}_T^*(f)\tilde{h}_T(f') \rangle = \frac{1}{2} \delta(f - f')S_T(f) \quad (A10) \]

By letting \( T \to \infty \) we obtain Eq.(5) of the main text.

To prove Eq.(7) of the main text we proceed as follows. By definition, [Eq.(A7)],

\[ R_T(0) = \langle |h_T(t)|^2 \rangle = \frac{1}{T} \int_{-T/2}^{T/2} |h_T(t)|^2 dt = \frac{1}{T} \int_{-\infty}^{\infty} |h_T(t)|^2 dt = \frac{1}{T} \int_{-\infty}^{\infty} |\tilde{h}_T(f)|^2 df \quad (A11) \]

Integrating Eq.(A9) over frequency we obtain

\[ \int_{0}^{\infty} S_T(f) df = \frac{1}{2} \int_{-\infty}^{\infty} 2R_T(\tau) e^{2\pi if\tau} d\tau d\tau = \int_{-\infty}^{\infty} R_T(\tau) \delta(\tau) d\tau = R_T(0) = \]

\[ = \frac{1}{T} \int_{-\infty}^{\infty} |\tilde{h}_T(f)|^2 df = \frac{2}{T} \int_{0}^{\infty} |\tilde{h}_T(f)|^2 df \quad (A12) \]

where we used Eq.(A11) for \( R_T(0) \). Equating the integrands we establish

\[ S_T(f) = \frac{2}{T} |\tilde{h}_T(f)|^2 \quad (A13) \]

which is analogous to Eq.(7) of the text but for the Fourier transforms \( \tilde{h}_T(f) \) of the truncated functions \( h_T(t) \). It would seem that by letting \( T \to \infty \) we could recover the relation shown in Eq.(7). This is not the case [10] because \( S_T(f) \) does not converge to a single value as \( T \to \infty \). It can however be “smoothed” by taking the ensemble average of \( |h_T(f)|^2 \) before letting \( T \to \infty \), so that

\[ S(f) = \lim_{T \to \infty} \frac{2}{T} \langle |\tilde{h}_T(f)|^2 \rangle \quad (A14) \]

which is the mathematically precise statement of Eq.(7).
It is of some interest to express Eq.(A10) in the time domain. Using the inverse of Eq.(A4) we write

$$h^*_T(t)h_T(t') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{h}^*(f) e^{2\pi if t} \tilde{h}(f') e^{-2\pi if t'} df df'$$

(A15)

Ensemble average, interchange the order of integration and averaging, and use Eq.(A10)

$$< h^*_T(t)h_T(t') > = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(f - f') S_T(f) e^{2\pi i f(t - t')} df' df$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{2\pi i f(t - t')} S_T(f) df$$

(A16)

When $t' = t$, Eq.(A16) reduces to Eq.(8) of the main text in the limit $T \to \infty$.

The physical interpretation of the autocorrelation function, Eq.(A7), implies that if the signal $h(t)$ “repeats” itself, on the average, with period $\tau_0$, then it must contain a frequency component $f = 1/\tau_0$. For example if $h(t) = \cos \omega_0 t = \cos(2\pi f_0 t)$, then

$$R_T(\tau) = \cos(2\pi f_0 \tau) \quad \text{and} \quad S_T(f) = \delta(f - f_0)$$

(A17)

If $R_T(\tau)$ drops off for $\tau > \tau_0$, then the highest frequency contained in $h(t)$ is $f < 1/\tau_0$. For white noise we can write $R_T(\tau) = \tau_c \delta(\tau)$ where $\tau_c$ is the correlation time, defined for instance by our sampling interval. Then the spectral density is flat up to a frequency $f_c = 1/\tau_c$. 
References and Notes

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