STRONG SOLUTION OF 3D-NSE WITH EXPONENTIAL DAMPING

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Abstract. In this paper we prove the existence and uniqueness of strong solution of the incompressible Navier-Stokes equations with damping $\alpha(e^{\beta|u|^2} - 1)u$.

1. Introduction

The classical Navier-Stokes equation is an area wish has received some attention during the last period, where as the study of this equation has become classical see [2],[6]. Despite, the solution of 3D Navier-Stokes equations is still a big open problem although. In 2008, the modified Navier-Stokes equations with damping $\alpha|u|^\beta - 1 u$, was studied by Cai and Jiu [7], they proved the global existence of a weak solution if $u^0$ in $L^2(\mathbb{R}^3)$ and they proved the global existence and uniqueness of a strong solution if the initial condition $u^0$ is in $H^1(\mathbb{R}^3) \cap L^{\beta+1}(\mathbb{R}^3)$, with $\beta \geq 7/2$. To construct a global solution Cai and Jiu used the Galerkin approximations. There is a large of literature dealing the classical Navier-Stokes equations in different spaces. Recently, Benameur in [4] has considered a new model of the Navier-stokes equation called Navier-Stokes equations with exponential damping, where he proved the global existence of weak solution. In this paper we study the global existence

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of strong solution to the incompressible Navier-Stokes equations with exponential damping in three spatial dimensions

\[
\begin{align*}
(\text{NS}) & \begin{cases}
\partial_t u - \nu_h \Delta_h u - \nu_\beta \partial_t^2 u + u.\nabla u + \alpha (e^{\beta|u|^2} - 1)u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u_0(x) = u^0(x) & \text{in } \mathbb{R}^3,
\end{cases}
\end{align*}
\]

where \(\nu_h > 0\) and \(\nu_\beta \geq 0\) are respectively the horizontal and vertical viscosity of fluid, \(u = u(t, x) = (u_1, u_2, u_3)\) and \(p = p(t, x)\) denote respectively the unknown velocity and the unknown pressure of the fluid at the point \((t, x) \in \mathbb{R}^+ \times \mathbb{R}^3\), and \(\alpha, \beta > 0\). The terms \((u, \nabla u) := u_1 \partial_1 u + u_2 \partial_2 u + u_3 \partial_3 u\), while \(u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))\) is an initial given velocity. If \(u^0\) is quite regular, the divergence free condition determines the pressure \(p\). First, we study the isotropic case \(\nu_h = \nu_\beta = 1\):

\[
(\text{NS}_1) \begin{cases}
\partial_t u - \Delta u + u.\nabla u + \alpha (e^{\beta|u|^2} - 1)u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u_0(x) = u^0(x) & \text{in } \mathbb{R}^3,
\end{cases}
\]

and the associated space:

\[
\mathcal{H}_\beta = \{ f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3 \text{ measurable; } (e^{\beta|f|^2} - 1)|f|^2, (e^{\beta|f|^2} - 1)|\nabla f|^2, e^{\beta|f|^2} |\nabla |f|^2|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3) \}. \]

Our first result is the following.

**Theorem 1.1.** Let \(u^0 \in H^1(\mathbb{R}^3)\) be a divergence free vector fields, then there is a unique global solution of (NS) \(u \in L^\infty(\mathbb{R}^+, H^1(\mathbb{R}^3) \cap C(\mathbb{R}^+, H^{-2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, H^2(\mathbb{R}^3)) \cap \mathcal{H}_\beta\). Moreover, for all \(t \geq 0\)

\[
(1.1) \quad \| u(\cdot) \|_{L^2}^2 + 2 \int_0^t \| \nabla u(t) \|_{L^2}^2 + 2 \alpha \int_0^t \| (e^{\beta|u|^2} - 1)|u|^2 \|_{L^1} \leq \| u^0 \|_{L^2}^2,
\]

\[
(1.2) \quad \| \nabla u(t) \|_{L^2}^2 + \int_0^t \| \Delta u \|_{L^2}^2 + \alpha \beta \int_0^t \| (e^{\beta|u|^2} - 1)|u|^2 \|_{L^1} + \alpha \int_0^t \| (e^{\beta|u|^2} - 1)|u|^2 \|_{L^1} \leq \| \nabla u^0 \|_{L^2}^2 \frac{e^{\frac{1}{\alpha\beta}}}},
\]

\[
(1.3) \quad \| \nabla u(t) \|_{L^2}^2 + \int_0^t \| \Delta u \|_{L^2}^2 + \alpha \beta \int_0^t \| (e^{\beta|u|^2} - 1)|u|^2 \|_{L^1} + \alpha \int_0^t \| (e^{\beta|u|^2} - 1)|u|^2 \|_{L^1} \leq M_{\alpha, \beta}(u^0),
\]

where \(M_{\alpha, \beta}(u^0) = \| \nabla u^0 \|_{L^2}^2 + \frac{\| u^0 \|_{H^2}^2}{\alpha\beta}\).}

**Remark 1.2.**

1. The fact \((e^{\beta|u|^2} - 1)|u|^2 \in L^1(\mathbb{R}^+, L^1(\mathbb{R}^3))\) implies \(u \in \cap_{1 \leq p < \infty} L^p(\mathbb{R}^+, L^p(\mathbb{R}^3))\).

   Indeed: we have

\[
\int_0^\infty \| (e^{\beta|u(t)|^2} - 1)|u(t)|^2 \|_{L^1} dt = \sum_{k=1}^\infty \frac{\beta_k}{k!} \int_{0}^\infty \| u(t) \|_{L^{2k+2}}^{2k+2} dt.
\]

2. By interpolation between \(H^{-2}(\mathbb{R}^3)\) and \(H^1(\mathbb{R}^3)\), we obtain: For all \(s < 1\), we have \(u \in C(\mathbb{R}^+, H^s(\mathbb{R}^3))\).
(3) \( u \in C_t(\mathbb{R}^+, H^1(\mathbb{R}^3)) \). Indeed: By equations (1.1)-(1.2) we get \( \limsup_{t \to t_0^+} \|u(t)\|_{H^1} \leq \|u^0\|_{H^1} \). Applying Proposition 2.1, we get the continuity of \( u \) at 0. For \( t_0 > 0 \), consider the following system
\[
(S(t_0)) \begin{cases}
\partial_t v - \Delta v + v \cdot \nabla v + \alpha(e^{|v|} - 1)v = -\nabla q \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } v = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\
v(0, x) = u(t_0, x) \text{ in } \mathbb{R}^3.
\end{cases}
\]
By the uniqueness given by Theorem 1.1, we obtain \( v(t, x) = u(t_0 + t, x) \) is the global solution of \((S(t_0))\). Then \( v \) is right continuous at 0, which implies the right continuity of \( u \) at \( t_0 \).

(4) The continuity set of \( u \) solution of \((NS_1)\) in \( H^1(\mathbb{R}^3)\): Put the following subset of \( \mathbb{R}^+ \)
\[
A = \{ t \in \mathbb{R}^+; \text{ } u \text{ discontinuous at } t \text{ in } H^1(\mathbb{R}^3) \}.
\]
\( A \) is at most countable. Indeed: Let \( f(t) = e^{-\alpha t^2} \|\nabla u(t)\|_{L^2}^2 \), \( g(t) = \|\nabla u(t)\|_{L^2}^2 \) and
\[
B = \{ t \in \mathbb{R}^+; f \text{ discontinuous at } t \} = \{ t \in \mathbb{R}^+; g \text{ discontinuous at } t \}.
\]
Let \( 0 \leq t_1 < t_2 \). Combining the uniqueness of strong solution of \((NS_1)\) and inequality (1.2), we get
\[
\|\nabla u(t_2)\|_{L^2}^2 \leq \|\nabla u(t_1)\|_{L^2}^2 e^{\frac{t_2-t_1}{\alpha^2}}
\]
and
\[
\|\nabla u(t_2)\|_{L^2}^2 e^{\frac{t_2-t_1}{\alpha^2}} \leq \|\nabla u(t_1)\|_{L^2}^2 e^{\frac{t_2-t_1}{\alpha^2}}.
\]
Thus, \( f \) is a decreasing function. According Lemma 2.8, \( B \) is at most countable. By using Proposition 2.1 and Remark 2.2, we obtain \( B = A \) and the desired result is proved.

Secondly, we study the anisotropic Navier-Stokes case \( \nu_1 = 1, \nu_3 = 0 \) with the same damping:
\[
(NS_2) \begin{cases}
\partial_t u - \Delta_h u + u \cdot \nabla u + \alpha(e^{|u|} - 1)u = -\nabla p \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\
\text{div } u = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0, x) = u^0(x) \text{ in } \mathbb{R}^3,
\end{cases}
\]
we refer the reader to [1]. Clearly, when \( \alpha = 0 \) or \( \beta = 1 \) it corresponds to the classical anisotropic Navier-Stokes equation for more details the reader is referenced to the book [9]. The second purpose of this paper is to study the system \((NS_2)\) in the anisotropic Sobolev space \( H^{0,1}(\mathbb{R}^3) \). Before stating the second main result, we define the space corresponding to the system:
\[
\mathcal{G}_\beta = \{ f : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3 \text{ measurable}; e^{\beta |f|^2} - 1)\|f\|^2, e^{\beta |f|^2} (\partial_3 (|f|^2))^2, (e^{\beta |f|^2} - 1)|\partial_3 f|^2 \in L^1_{\text{loc}}(\mathbb{R}^+ \times H^1(\mathbb{R}^3)) \}.
\]

**Theorem 1.3.** Let \( u^0 \in H^{0,1}(\mathbb{R}^3) \) be a divergence free vector fields, then there is a unique global solution of \((NS_2)\): \( u \in L^\infty_{\text{loc}}(\mathbb{R}^+, H^{0,1}(\mathbb{R}^3)) \cap C(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2_{\text{loc}}(\mathbb{R}^+, H^{1,1}(\mathbb{R}^3)) \cap \mathcal{G}_\beta \). Moreover, for all \( t \geq 0 \)
\[
(1.4) \quad \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla_h u\|_{L^2}^2 + 2\alpha \int_0^t \|(e^{\beta |u|^2} - 1)|u|^2\|_{L^1} \leq \|u^0\|_{L^2}^2,
\]
\( (1.5) \quad \| \partial_3 u(t) \|_{L^2}^2 + \int_0^t \| \nabla \partial_3 u \|_{L^2}^2 dz + \alpha \int_0^t \| (e^{\beta|u|^2} - 1) \|_{L^1}^2 \| \partial_3 u \|_{L^1}^2 \leq \| \partial_3 u \|_{L^1}^2 e^{\alpha|u|^2}. \)

Remark 1.4. (1) Combining the above result \( u \in C(\mathbb{R}^+, L^2(\mathbb{R}^3)) \) and the fact \( u \in L_\text{loc}^\infty(\mathbb{R}^+, H^{0,1}(\mathbb{R}^3)) \) with the interpolation result, we get: For all \( s < 1 \), we have \( u \in C(\mathbb{R}^+, H^{0,s}(\mathbb{R}^3)) \).

(2) By inequalities (1.4)-(1.5) and Proposition 2.1, we get
\[ \lim_{t \to 0} \| u(t) - u^0 \|_{H^{0,1}} = 0. \]

(3) By using the same idea of Remark 1.2-(3), we get : \( u \in C_r(\mathbb{R}^+, H^{0,1}(\mathbb{R}^3)) \).

(4) The continuity set of \( u \) solution of \((NS_2)\) in \( H^{0,1}(\mathbb{R}^3) \): Put the following subset of \( \mathbb{R}^+ \)
\[ A' = \{ t \in \mathbb{R}^+; \ u \text{ discontinuous at } t \text{ in } H^{0,1}(\mathbb{R}^3) \}. \]

\( A' \) is at most countable.

The remainder of our paper is organized as follows. In the second section we give some notations, definitions and preliminary results. Throughout Section 3, we will study the uniqueness and global existence of solution of Cauchy problem \((NS_1)\) in \( H^1(\mathbb{R}^3) \). In section 4, we will interest to anisotropic case: We will also prove the uniqueness and global existence of solution in \( H^{0,1}(\mathbb{R}^3) \).

2. Notations and preliminary results

2.1. Notations. In this section, we collect some notations and definitions that will be used later.

- The Fourier transformation is normalized as
\[ \mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix.\xi)f(x)dx, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3. \]

- The inverse Fourier formula is
\[ \mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \exp(i\xi.x)g(\xi)d\xi, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3. \]

- The convolution product of a suitable pair of function \( f \) and \( g \) on \( \mathbb{R}^3 \) is given by
\[ (f * g)(x) := \int_{\mathbb{R}^3} f(y)g(x - y)dy. \]

- If \( f = (f_1, f_2, f_3) \) and \( g = (g_1, g_2, g_3) \) are two vector fields, we set
\[ f \otimes g := (g_1f, g_2f, g_3f), \]
and
\[ \text{div } (f \otimes g) := \text{div } (g_1f), \text{ div } (g_2f), \text{ div } (g_3f). \]

Moreover, if \( \text{div } g = 0 \) we obtain
\[ \text{div } (f \otimes g) := g_1 \partial_1 f + g_2 \partial_2 f + g_3 \partial_3 f := g.\nabla f. \]
Proposition 2.1. Let \((B, ||\cdot||)\), be a Banach space, \(1 \leq p \leq \infty\) and \(T > 0\). We define \(L^p_T(B)\) the space of all measurable functions \([0, t] \ni t \mapsto f(t) \in B\) such that \(t \mapsto ||f(t)|| \in L^p([0, T])\).

- The Sobolev space \(H^s(\mathbb{R}^3) = \{ f \in S'(\mathbb{R}^3); (1 + |\xi|^2)^{s/2} \hat{f} \in L^2(\mathbb{R}^3) \}\).
- The homogeneous Sobolev space \(\dot{H}^s(\mathbb{R}^3) = \{ f \in S'(\mathbb{R}^3); \hat{f} \in L^1_{loc} \text{ and } |\xi|^s \hat{f} \in L^2(\mathbb{R}^3) \}\).
- For \(R > 0\), the Friedrich operator \(J_R\) is defined by

\[
J_R(D)f = \mathcal{F}^{-1}(1_{|\xi| < R} \hat{f}).
\]

- The Leray projector \(\mathbb{P}: (L^2(\mathbb{R}^3))^3 \to (L^2(\mathbb{R}^3))^3\) is defined by

\[
\mathcal{F}(\mathbb{P}f) = \hat{f}(\xi) - (\hat{\phi}(\xi) \frac{\xi}{|\xi|}) \hat{\phi}(\xi); M(\xi) = (\delta_{k,l} - \frac{\xi_k \xi_l}{|\xi|^2})_{1 \leq k,l \leq 3},
\]

- The Sobolev space \(L^p_{s^s}(\mathbb{R}^3) = \{ f \in (L^2(\mathbb{R}^3))^3; \text{div } f = 0 \}\).
- For \(s_1, s_2 \in \mathbb{R}\), the anisotropic Sobolev spaces are defined by:

\[
H^{s_1,s_2}(\mathbb{R}^3) = \{ f \in S'(\mathbb{R}^3); (1 + |\xi|^2)^{s_1/2}(1 + \xi_3^2)^{s_2/2} \hat{f}(\xi) \in L^2(\mathbb{R}^3) \},
\]

\[
\dot{H}^{s_1,s_2}(\mathbb{R}^3) = \{ f \in S'(\mathbb{R}^3); |\xi|^s_1 |\xi|^{s^2} \hat{f}(\xi) \in L^2(\mathbb{R}^3) \}.
\]

- Let \((B, ||\cdot||)\), be a Banach space and \(I\) be nonempty interval. We define \(C^r(I, B)\) the space of all right continuous functions : \(I \ni t \mapsto f(t) \in B\).
- We often use the convex inequality: For \(p, q \in (1, \infty)\) such that \(\frac{1}{p} + \frac{1}{q} = 1\), we have

\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \forall a, b \in \mathbb{R}^+.
\]

2.2. Preliminaries. In this section, we recall some classical results and we give new technical lemmas.

Proposition 2.1. ([3]) Let \(H\) be a Hilbert space.

1. If \((x_n)\) is a bounded sequence of elements in \(H\), then there is a subsequence \((x_{\varphi(n)})\) such that

\[
(x_{\varphi(n)}|y) \to (x|y), \forall y \in H.
\]

2. If \(x \in H\) and \((x_n)\) is a bounded sequence of elements in \(H\) such that

\[
(x_n|y) \to (x|y), \forall y \in H.
\]

Then \(|x| \leq \lim \inf_{n \to \infty} ||x_n||\).

3. If \(x \in H\) and \((x_n)\) is a bounded sequence of elements in \(H\) such that

\[
(x_n|y) \to (x|y), \forall y \in H
\]

\[
\lim \sup_{n \to \infty} ||x_n|| \leq ||x||,
\]

then \(\lim_{n \to \infty} ||x_n - x|| = 0\).
Remark 2.2. Combining Proposition 2.1-(1) and (2), we get: If \((x_n)\) is a bounded sequence of elements in \(H\) such that

\[
(x_n|y) \to (x|y), \forall y \in H,
\]

then

\[
\lim_{n \to \infty} \|x_n - x\| = 0 \iff \lim_{n \to \infty} \|x_n\| = \|x\|.
\]

Lemma 2.3. ([6]) Let \(s_1, s_2\) be two real numbers and \(d \in \mathbb{N}\).

1. If \(s_1 < d/2\) and \(s_1 + s_2 > 0\), there exists a constant \(C_1 = C_1(d, s_1, s_2)\), such that: if \(f, g \in \dot{H}^{s_1}(\mathbb{R}^d) \cap \dot{H}^{s_2}(\mathbb{R}^d)\), then \(f, g \in \dot{H}^{s_1 + s_2 - \frac{d}{2}}(\mathbb{R}^d)\) and

\[
\|fg\|_{\dot{H}^{s_1 + s_2 - \frac{d}{2}}} \leq C_1(\|f\|_{\dot{H}^{s_1}} \|g\|_{\dot{H}^{s_2}} + \|f\|_{\dot{H}^{s_2}} \|g\|_{\dot{H}^{s_1}}).
\]

2. If \(s_1, s_2 < d/2\) and \(s_1 + s_2 > d\) there exists a constant \(C_2 = C_2(d, s_1, s_2)\) such that: if \(f \in \dot{H}^{s_1}(\mathbb{R}^d)\) and \(g \in \dot{H}^{s_2}(\mathbb{R}^d)\), then \(f, g \in \dot{H}^{s_1 + s_2 - 1}(\mathbb{R}^d)\) and

\[
\|fg\|_{\dot{H}^{s_1 + s_2 - 1}} \leq C_2 \|f\|_{\dot{H}^{s_1}} \|g\|_{\dot{H}^{s_2}}.
\]

Lemma 2.4. Let \(\alpha > 0\) and \(d \in \mathbb{N}\). Then, for all \(x, y \in \mathbb{R}^d\), we have

\[
(|x|^\alpha x - |y|^\alpha y).(x - y) \geq C_\alpha(|x|^\alpha + |y|^\alpha)|x - y|^2,
\]

with \(C_\alpha = \min\left(\frac{1}{18}, \frac{1}{2^{\alpha+1}}\right) > 0\).

Proof. In all the proof we suppose that \(|x| \geq |y|\). Particularly, we have

\[
|x|^\alpha = \frac{|x|^\alpha + |x|^\alpha}{2} \geq \frac{|y|^\alpha + |x|^\alpha}{2}.
\]

First case: we suppose that \((x, y)\) is related. We treat two subcases:

- Suppose that \(x, y \leq 0\), then \(x, y = -|x|, |y|\) and \(|x - y| = |x| + |y| \leq 2|x|\). We have

\[
(|x|^\alpha x - |y|^\alpha y).(x - y) = |x|^{\alpha+2} + |y|^{\alpha+2} - (|x|^\alpha + |y|^\alpha)x.y
\]

\[
= |x|^{\alpha+2} + |y|^{\alpha+2} + (|x|^\alpha + |y|^\alpha)|x|.|y|
\]

\[
= (|x|^\alpha + |y|^\alpha)(|x| + |y|)
\]

\[
\geq |x|^\alpha + |y|^\alpha|y - x|
\]

\[
\geq |x|^\alpha |x| |x - y|
\]

\[
\geq \frac{|y|^\alpha + |x|^\alpha}{2} \cdot \frac{|x - y|^2}{2}
\]

\[
\geq \frac{1}{4}(|x|^\alpha + |y|^\alpha)|x - y|^2.
\]

- Suppose that \(xy > 0\), then \(x, y = |x|, |y|\) and \(|x - y| = |x| - |y|\). We have

\[
(|x|^\alpha x - |y|^\alpha y)(y - x) = (|x|^{\alpha+1} - |y|^{\alpha+1})|x - y|
\]

\[
= (|x|^{\alpha+1} - |y|^{\alpha+1})(|x| - |y|)
\]

\[
= |x|^{\alpha+2}(1 - \left(\frac{|y|}{|x|}\right)^{\alpha+1})(1 - \frac{|y|}{|x|}).
\]
Let \( \theta = \frac{|x|}{|y|} \in [0, 1] \), then

\[
(|x|^\alpha x - |y|^\alpha y). (y - x) \geq |x|^{\alpha+2}(1 - \theta^{\alpha+1})(1 - \theta) \\
\geq |x|^{\alpha+2}(1 - \theta)^2 \\
\geq |x|^\alpha(|x| - |x|\theta)^2 \\
\geq |x|^\alpha(|x| - |y|)^2 \\
\geq |x|^\alpha|x - y|^2 \\
\geq \frac{1}{2}(|x|^\alpha + |y|^\alpha)|y - x|^2.
\]

**Second case:** Suppose that \((x, y)\) are **linearly independent** elements. There are two subcases:

- If \(x.y \leq 0\), then

\[
|x - y|^2 = |x|^2 + |y|^2 - 2x.y \Rightarrow |x - y|^2 \leq 2(|x|^2 + |y|^2) \leq 4|x|^2.
\]

We have

\[
(|x|^\alpha x - |y|^\alpha y).(x - y) = |x|^{\alpha+2} + |y|^{\alpha+2} - |x|^\alpha x.y - |y|^\alpha x.y \\
\geq |x|^{\alpha+2} \\
\geq |x|^\alpha |x|^2 \\
\geq \frac{|x|^\alpha + |x|^\alpha}{2} \frac{|x - y|^2}{4} \\
\geq \frac{1}{8}(|x|^\alpha + |y|^\alpha)|x - y|^2.
\]

- If \(x.y > 0\), and suppose that \(|y| \leq \frac{|x|}{2}\). We have

\[
|x - y| \leq |x| + |y| \leq \frac{3}{2}|x| \Rightarrow \frac{4}{9}|x - y|^2 \leq |x|^2.
\]

Put the following vectors:

\[
v = \frac{x}{|x|}, \quad w = \frac{y}{|x|}.
\]

Clearly, we get

\[
|v| = 1, \quad |w| = \frac{|y|}{|x|} \in [0, \frac{1}{2}], \quad v.w > 0.
\]
We have

\[
(x^\alpha - y^\alpha)(x - y) = |x|^\alpha(v - |w|^\alpha w). (v - w) \\
= |x|^\alpha(1 + |w|^\alpha - (1 + |w|^\alpha)v.w) \\
\geq |x|^\alpha(1 + |w|^\alpha - (1 + |w|)|v||w|) \\
\geq |x|^\alpha(1 + |w|^\alpha - \frac{3}{2}|v||w|) \\
\geq |x|^\alpha(1 + |w|^\alpha - \frac{3}{2}|w|) \\
\geq |x|^\alpha(1 + |w|^\alpha - \frac{3}{4}) \\
\geq \frac{1}{4}|x|^\alpha |x|^2 \\
\geq \frac{1}{4} \left( |x|^\alpha + |y|^\alpha \right) \frac{4}{9} |x - y|^2 \\
\geq \frac{1}{18} (|x|^\alpha + |y|^\alpha) |x - y|^2.
\]

- If \(x.y > 0\), and suppose that \(\frac{|x|}{2} < |y| \leq |x|\).

Put the plan \(P = \text{Span}\{x, y\}\) and \(B = (u_1 = \frac{x}{|x|}, u_2)\) a normalized basis of \(P\). We start by noting the following relations

\[
x = |x|u, \\
y = a_1u_1 + a_2u_2, \\
a_1 = x.y > 0, \\
|y| = \sqrt{a_1^2 + a_2^2}.
\]

Then, if we put \(z = \frac{y}{|x|} = b_1u_1 + b_2u_2\), we obtain

\[
b_1 = \frac{a_1}{|x|} > 0, \\
\frac{1}{2} \leq |z| = \sqrt{b_1^2 + b_2^2} \leq 1.
\]
Then

\[(|x|^\alpha x - |y|^\alpha y).(x - y) = |x|^\alpha+2[(u - |z|^\alpha z). (u - z)]\]

\[= |x|^\alpha+2[(u - |z|^\alpha (b_1 u + b_2 v)). (u - (b_1 u + b_2 v))]\]

\[= |x|^\alpha+2[((1 - |z|^\alpha b_1)u - |z|^\alpha b_2 v).((1 - b_1) u - b_2 v)]\]

\[= |x|^\alpha+2[(1 - |z|^\alpha b_1)(1 - b_1) + |z|^\alpha b_2^2]\]

\[\geq |x|^\alpha+2[(1 - b_1)^2 + (\frac{1}{2})^\alpha b_2^2], \quad (b_1 > 0 \text{ and } \frac{1}{2} \leq |z| \leq 1)\]

\[\geq (\frac{1}{2})^\alpha|x|^\alpha+2[(1 - b_1)^2 + b_2^2]\]

\[\geq (\frac{1}{2})^\alpha|x|^\alpha+2|u - z|^2\]

\[\geq (\frac{1}{2})^\alpha|x|^\alpha|x|u - |x|z|^2\]

\[\geq (\frac{1}{2})^\alpha|x|^\alpha|x - y|^2\]

\[\geq (\frac{1}{2})^\alpha(|x|^\alpha + |y|^\alpha)|x - y|^2\]

\[\geq (\frac{1}{2})^{\alpha+1}(|x|^\alpha + |y|^\alpha)|x - y|^2.\]

So, the real \(C_\alpha = \min(\frac{1}{18}, b_\alpha = \frac{1}{2^{k+1}})\) answers the question.

**Remark 2.5.** It’s easy to see that \(b_\alpha\) is decreasing to 0. By a straightforward computation we get, for \(\alpha \geq 3, b_\alpha < \frac{1}{18}\). If \(k \in \mathbb{N}\), then \(C_{2k} = \left\{ \begin{array}{ll} \frac{1}{18}, & \text{if } k = 1 \\ b_{2k} = \frac{1}{2^{2k+1}}, & \text{if } k \geq 2. \end{array} \right. \)

Then

\[\forall k \in \mathbb{N}; \quad C_{2k} \geq \frac{8}{18} b_{2k} = \frac{2}{9} \frac{1}{2^{2k}} = \frac{2}{9} \frac{1}{4^k}.\]

Combining Lemma 2.4 and Remark 1.2, we get the following result.

**Lemma 2.6.** If \(\beta > 0\), then, for all \(x, y \in \mathbb{R}^d\), we have

\[\left((e^{\beta|x|^2} - 1)x - (e^{\beta|y|^2} - 1)y\right).(x - y) \geq \frac{2}{9} \left( (e^{\frac{4}{9}|x|^2} - 1) + (e^{\frac{4}{9}|y|^2} - 1) \right) |x - y|^2.\]

In the following we give another version of Gronwall’s lemma that we often use:

**Lemma 2.7.** Let \(A, T > 0\) and \(f, g, h : [0, T] \rightarrow \mathbb{R}^+\) three continuous functions such that

\[(2.1) \quad \forall t \in [0, T]; \quad f(t) + \int_0^t g(z)dz \leq A + \int_0^t h(z)f(z)dz.\]

Then

\[\forall t \in [0, T]; \quad f(t) + \int_0^t g(z)dz \leq A \exp(\int_0^t h(z)dz).\]
Proof. By Gronwall Lemma, we get
\[ \forall t \in [0, T]; \quad f(t) \leq A \exp(\int_0^t h(z)dz). \]

Put this inequality in (2.1) we obtain
\[ f(t) + \int_0^t g(z)dz \leq A + A \int_0^t h(z) \exp(\int_0^z h(r)dr)dz \]
\[ \leq A + A \int_0^t (\exp(\int_0^z h(r)dr))'dz \]
\[ \leq A + A \left( \exp(\int_0^t h(r)dr) - 1 \right) \]
\[ \leq A \exp(\int_0^t h(r)dr), \]
which ends the proof.

To prove the right continuity of strong solutions of \((NS_1)\) and \((NS_2)\), I need the following classical result:

**Lemma 2.8.** Let \( f : I \to \mathbb{R} \) be a monotonic function on an interval \( I \). Then there is \( A \subset \mathbb{R} \) at most countable family such that \( f \) is continuous on \( I \setminus A \).

**Proof.** Suppose that \( f \) is increasing (if \( f \) is decreasing, we can consider \( g = -f \)). Then, for \( t \in \text{int}(I) \), we have
\[ f \text{ is discontinuous at } t \iff \lim_{t^-} f < \lim_{t^+} f. \]
Let \( A = \{ t \in \mathbb{R}; \ f \text{ discontinous at } t \} \) and \( a \in A \cap \text{int}(I) \), then we have \( \lim_{a^-} f < \lim_{a^+} f \).
So \( (\lim_{a^-} f, \lim_{a^+} f) \cap \mathbb{Q} \neq \emptyset \), and we can choose \( r_a \in (\lim_{a^-} f, \lim_{a^+} f) \cap \mathbb{Q} \). Then, the following function
\[ \varphi : A \to \mathbb{Q} \]
\[ a \mapsto r_a \]
is well defined. For \( a, b \in A \) such that \( a < b \), we have
\[ r_a < \lim_{a^+} f \leq \lim_{b^-} f < r_b \]
so \( \varphi(a) < \varphi(b) \) which implies that \( \varphi \) is injective function. Therefore \( A \) is at most countable family.

3. Proof of Theorem 1.1

3.1. A priori estimates. We start by taking the \( L^2 \) scalar product of the first equation of \((NS_1)\) with \( u \), we get
\[ \|u(t)\|_{L^2}^2 + 2 \int_0^t \|
abla u\|_{L^2}^2 + 2\alpha \int_0^t (e^{\beta}|u|^2 - 1)|u|^2 \|_{L^1} \leq \|u^0\|_{L^2}^2. \]

Also, taking the \( H^1 \) scalar product of the first equation of \((NS_1)\) with \( u \), we obtain
\[ \langle \partial_t \nabla u, \nabla u \rangle_{L^2} - \langle \Delta \nabla u, \nabla u \rangle_{L^2} + \alpha(\nabla((e^{\beta}|u|^2 - 1)u), \nabla u \rangle_{L^2} \leq |\langle u \nabla u, \Delta u \rangle_{L^2}|. \]
By using the following identities,
\[
\alpha (\nabla ((e^{\beta|u|^2} - 1)u), \nabla u)_{L^2} = \frac{\alpha \beta}{2} \|e^{\beta|u|^2}|\nabla(|u|^2)|^2\|_{L^1} + \alpha \|(e^{\beta|u|^2} - 1)|\nabla u|^2\|_{L^1}
\]
\[
|\langle u \nabla u, \Delta u \rangle_{L^2}| \leq \int_{\mathbb{R}^3} |u| |\nabla u| |\Delta u|
\leq \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\Delta u|^2,
\]
we obtain
\[
\frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + 2\alpha \beta \|e^{\beta|u|^2}|\nabla(|u|^2)|^2\|_{L^1} + 2\alpha \|(e^{\beta|u|^2} - 1)|\nabla u|^2\|_{L^1} \leq \int_{\mathbb{R}^3} |u|^2 |\nabla u|^2.
\]
Moreover, the elementary inequalities
\[
\alpha (e^{\beta|u|^2} - 1) \geq \alpha \left( \frac{\beta|u|^2}{2!} \right)^2 \geq \frac{\alpha \beta^2}{2} |u|^4
\]
\[
|u|^2 = \left( \frac{\sqrt{\alpha \beta} |u|^2}{2} \right)^2 \geq \frac{\alpha \beta^2}{16} |u|^4 + \frac{1}{\alpha \beta^2}
\]
imply
\[
\frac{d}{dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \alpha \beta \|e^{\beta|u|^2}|\nabla(|u|^2)|^2\|_{L^1} + \alpha \|(e^{\beta|u|^2} - 1)|\nabla u|^2\|_{L^1} \leq \frac{1}{\alpha \beta^2} \|\nabla u\|_{L^2}^2.
\]
Integrate on [0, t], we get
\[
\|\nabla u\|_{L^2}^2 + \int_0^t \|\Delta u\|_{L^2}^2 + \alpha \beta \int_0^t \|e^{\beta|u|^2}|\nabla(|u|^2)|^2\|_{L^1} + \int_0^t \alpha \|(e^{\beta|u|^2} - 1)|\nabla u|^2\|_{L^1} \leq \|\nabla u^0\|_{L^2}^2 + \frac{1}{\alpha \beta^2} \int_0^t \|\nabla u\|_{L^2}^2.
\]
This inequality implies two results, the first by applying Lemma 2.7:
\[
(3.2) \|\nabla u\|_{L^2}^2 + \int_0^t \|\Delta u\|_{L^2}^2 + \alpha \beta \int_0^t \|e^{\beta|u|^2}|\nabla(|u|^2)|^2\|_{L^1} + \int_0^t \alpha \|(e^{\beta|u|^2} - 1)|\nabla u|^2\|_{L^1} \leq \|\nabla u^0\|_{L^2}^2 e^{\alpha \beta t},
\]
and the second by using inequality (3.1):
\[
(3.3) \|\nabla u\|_{L^2}^2 + \int_0^t \|\Delta u\|_{L^2}^2 + \alpha \beta \int_0^t \|e^{\beta|u|^2}|\nabla(|u|^2)|^2\|_{L^1} + \alpha \int_0^t \|(e^{\beta|u|^2} - 1)|\nabla u|^2\|_{L^1} \leq \|\nabla u^0\|_{L^2}^2 + \frac{\|u^0\|_{L^2}^2}{\alpha \beta^2}.
\]
Absolutely, these bounds come from the approximate solutions via the Friedrich’s regularization procedure. Hence it remains to pass to the limit in the sequence of solutions of approximate schema. The passage to the limit follows using classical argument by combining Ascoli’s theorem and the Cantor diagonal process (see [4]). Moreover, inequalities (1.1)-(1.2)-(1.3) are given by (3.1)-(3.2)-(3.3). Finally,
\( u \) is in the space \( C(\mathbb{R}^+, H^{-2}(\mathbb{R}^3)) \) as an interpolation between \( C(\mathbb{R}^+, H^{-4}(\mathbb{R}^3)) \) and \( L^\infty(\mathbb{R}^+, H^1(\mathbb{R}^3)) \).

### 3.2. Uniqueness.

Let \( u, v \) two solutions of \((NS)_1\) and put \( w = u - v \). We make the difference of two following equations

\[
\begin{align*}
\partial_t u - \Delta u + u.\nabla u + \alpha (|u|^2 - 1)u &= -\nabla p \\
\partial_t v - \Delta v + v.\nabla v + \alpha (|v|^2 - 1)v &= -\nabla \tilde{p},
\end{align*}
\]

we get

\[
\partial_t w - \Delta w + w.\nabla u + v.\nabla w + \alpha (|u|^2 - 1)u - \alpha (|v|^2 - 1)v = -\nabla (p - \tilde{p}).
\]

Taking the \( L^2 \) scalar product with \( w \), we obtain:

\[
\langle \partial_t w, w \rangle_{L^2} - \langle \Delta w, w \rangle_{L^2} + \langle \alpha (|u|^2 - 1)u - \alpha (|v|^2 - 1)v, w \rangle_{L^2} \leq |\langle w.\nabla u, w \rangle_{L^2}|.
\]

Using Lemma 2.6, we get

\[
\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + \frac{2\alpha}{9} \int_{\mathbb{R}^3} (e^\frac{\beta}{4}|u|^2 - 1)|w|^2 \leq \|wu\|_{L^2} \|\nabla w\|_{L^2}
\]

and

\[
\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \frac{1}{2} \|\nabla w\|_{L^2}^2 + \frac{2\alpha}{9} \int_{\mathbb{R}^3} (e^\frac{\beta}{4}|u|^2 - 1)|w|^2 \leq \frac{1}{2} \int_{\mathbb{R}^3} |w|^2 |u|^2 dz.
\]

By the elementary inequalities

\[
\frac{2\alpha}{9} (e^\frac{\beta}{4}|u|^2 - 1) \geq \frac{2\alpha}{9} \frac{1}{2!} (\frac{\beta}{4}|u|^2)^2 = \frac{\alpha \beta^2}{36} |u|^4
\]

\[
|u|^2 = (\frac{\sqrt{\alpha \beta}}{\sqrt{18}} |u|^2) \leq \frac{\alpha \beta^2}{36} |u|^4 + \frac{9}{\alpha \beta^2},
\]

we get

\[
\frac{1}{2} \frac{d}{dt} \|w\|_{L^2}^2 + \frac{1}{2} \|\nabla w\|_{L^2}^2 \leq \frac{18}{\alpha \beta^2} \int_{\mathbb{R}^3} |w|^2.
\]

According to Gronwall Lemma, we obtain

\[
\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 e^{\frac{18}{\alpha \beta^2}t}.
\]

But \( w(0) = 0 \), so \( u = v \), which ends the proof.

### 4. Proof of Theorem 1.3.

In this section, we do the same procedure as the previous proof.
4.1. A priori estimates. We start by taking the scalar product in $L^2(\mathbb{R}^3)$

\begin{equation}
\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla_h u\|_{L^2}^2 + 2\alpha \int_0^t \| (e^{\beta |u|^2} - 1) |u|^2 \|_{L^1} \leq \|u_0\|_{L^2}^2.
\end{equation}

At present, taking $\dot{H}^{0,1}$ scalar product of the system with the solution $u$, we get:

\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\partial_3 u\|_{L^2}^2 + \|\nabla_h \partial_3 u\|_{L^2}^2 + \frac{\alpha \beta}{2} \|e^{\beta |u|^2} |\partial_3 (|u|^2)|^2 \|_{L^1} \\
+ \alpha \| (e^{\beta |u|^2} - 1) |\partial_3 u|^2 \|_{L^1} = \langle \partial_3 u \nabla u, \partial_3 u \rangle_{L^2}.
\end{align*}

By following the process of [2] (see pages 1819-1820), we get two universal constants $C_0, C_1$ and

\begin{equation}
|\langle \partial_3 u \nabla u, \partial_3 u \rangle_{L^2}| \leq \frac{1}{2} \|\nabla_h \partial_3 u\|_{L^2}^2 + C_0 \|\partial_3 u\|_{L^2}^2 + C_1 \||u|^2 \partial_3 u\|_{L^2}^2.
\end{equation}

Then, we obtain

\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\partial_3 u\|_{L^2}^2 + \|\nabla_h \partial_3 u\|_{L^2}^2 + \frac{\alpha \beta}{2} \|e^{\beta |u|^2} |\partial_3 (|u|^2)|^2 \|_{L^1} \\
+ \alpha \| (e^{\beta |u|^2} - 1) |\partial_3 u|^2 \|_{L^1} \leq C_0 \|\partial_3 u\|_{L^2}^2 + C_1 \||u|^2 \partial_3 u\|_{L^2}^2.
\end{align*}

Moreover, the elementary inequalities

\begin{align*}
\frac{\alpha}{2} (e^{\beta |u|^2} - 1) & \geq \frac{\alpha (\beta |u|^2)^2}{2 \cdot 3!} \\
& \geq \frac{\alpha \beta}{12} |u|^6 \\
C_1 |u|^2 & = \left( \frac{\alpha^{1/3} |u|^2}{4^{1/3}} \right) \cdot \left( \frac{C_1 4^{1/3}}{\alpha^{1/3} \beta} \right) \\
& \leq \frac{\alpha \beta}{12} |u|^6 + C(\alpha, \beta), \quad C(\alpha, \beta) = \frac{256 C_1^{4/3}}{\alpha^{4/3} \beta^{4/3}}
\end{align*}

imply

\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\partial_3 u\|_{L^2}^2 + \|\nabla_h \partial_3 u\|_{L^2}^2 + \frac{\alpha \beta}{2} \|e^{\beta |u|^2} |\partial_3 (|u|^2)|^2 \|_{L^1} \\
+ \frac{\alpha}{2} \| (e^{\beta |u|^2} - 1) |\partial_3 u|^2 \|_{L^1} \leq a_{\alpha, \beta} \|\partial_3 u\|_{L^2}^2,
\end{align*}

where $a_{\alpha, \beta} = 2(C_0 + C(\alpha, \beta))$. Integrate on $[0, t]$ and using Lemma 2.7, we get

\begin{align*}
\|\partial_3 u\|_{L^2}^2 + \int_0^t \|\nabla_h \partial_3 u\|_{L^2}^2 + \alpha \beta \int_0^t \|e^{\beta |u|^2} |\partial_3 (|u|^2)|^2 \|_{L^1} \\
+ \alpha \int_0^t \| (e^{\beta |u|^2} - 1) |\partial_3 u|^2 \|_{L^1} \leq \|\partial_3 u_0\|_{L^2}^2 \exp(a_{\alpha, \beta} t).
\end{align*}
4.2. **Approximate system.** In this step we construct a global solution of \((NS_2)\), where we use a method inspired by \([4]-[6]\). For this, consider the approximate system with the parameter \(n \in \mathbb{N}\):

\[
(S_n) \begin{cases}
\partial_t u - \Delta_h J_n u + J_n(J_n u \nabla J_n u + \alpha J_n[(e^{\beta|J_n u|^2} - 1)J_n u]) = -\nabla p_n \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\
n_p = (-\Delta)^{-1} \left( \text{div} J_n(J_n u \nabla J_n u + \alpha \text{div} J_n[(e^{\beta|J_n u|^2} - 1)J_n u] \right) \\
\text{div} u = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^3 \\
u(0,x) = J_n u^0(x) \quad \text{in} \quad \mathbb{R}^3.
\end{cases}
\]

By Cauchy-Lipschitz Theorem, we obtain a unique solution \(u_n \in C^1(\mathbb{R}^+, L^2_0(\mathbb{R}^3))\) of \((S_n)\). Moreover, \(J_n u_n = u_n\). By the last section, we get

\[
\|u_n(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla_h u_n\|_{L^2}^2 \, dt + 2\alpha \int_0^t \|\partial_t u_n\|_{L^1}^2 \, dt \leq \|u^0\|_{L^2}^2.
\]

\[
\|\partial_3 u_n(t)\|_{L^2}^2 + \int_0^t \|\nabla_h \partial_3 u_n\|_{L^2}^2 \, dt + \alpha \beta \int_0^t \|\partial_3[u_n]^2\|_{L^1} \, dt
\]

\[
\leq \|\partial_3 u^0\|_{L^2}^2 \exp(a_{\alpha,\beta}t).
\]

Let \((T_q)_q \in (0,\infty)^\mathbb{N}\) such that \(T_q < T_{q+1}\) and \(T_q \to \infty\) as \(q \to \infty\). Let \((\theta_q)_q \in \mathbb{N}\) be a sequence in \(C^\infty(\mathbb{R}^3)\) such that: for all \(q \in \mathbb{N}\)

\[
\begin{cases}
\theta_q(x) = 1, \quad \forall x \in B(0, q + 1 + \frac{1}{q}) \\
\theta_q(x) = 0, \quad \forall x \in B(0, q + 2)^c \\
0 \leq \theta_q \leq 1.
\end{cases}
\]

Using \((4.2)-(4.3)\) and classical argument by combining Ascoli’s theorem and the Cantor diagonal process, we get a nondecreasing \(\varphi : \mathbb{N} \to \mathbb{N}\) and \(u \in L^\infty(\mathbb{R}^+, H^{0,1}(\mathbb{R}^3)) \cap C(\mathbb{R}^+, H^{-3}(\mathbb{R}^3))\) such that: for all \(q \in \mathbb{N}\), we have

\[
\lim_{n \to \infty} \|\theta_q (u_{\varphi(n)} - u)\|_{L^\infty([0,T_q],H^{-4})} = 0,
\]

and

\[
\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla_h u\|_{L^2}^2 + 2\alpha \int_0^t \|\partial_t u\|_{L^1}^2 \, dt \leq \|u^0\|_{L^2}^2.
\]

\[
\|\partial_3 u(t)\|_{L^2}^2 + \int_0^t \|\nabla_h \partial_3 u\|_{L^2}^2 + \alpha \beta \int_0^t \|\partial_3[u]^2\|_{L^1} \, dt
\]

\[
\leq \|\partial_3 u^0\|_{L^2}^2 \exp(a_{\alpha,\beta}t).
\]

- We have \(u \in C(\mathbb{R}^+, L^2(\mathbb{R}^3))\) (for the proof see Appendix). Then, by using interpolation inequality, we get \(u \in C(\mathbb{R}^+, H^{0,s}(\mathbb{R}^3))\), for all \(s < 1\).
4.3. **Uniqueness.** This proof is inspired by [2]. Let \( u, v \) two solutions of \((NS_2)\) and denote by \( w = u - v \). We make the difference of the following equations
\[
\partial_t u - \Delta_h u + u.\nabla u + \alpha(e^{\beta|u|^2} - 1) u = -\nabla p
\]
\[
\partial_t v - \Delta_h v + v.\nabla v + \alpha(e^{\beta|v|^2} - 1) v = -\nabla \tilde{p}
\]
we get
\[
\partial_t w - \Delta_h w + w.\nabla u + v.\nabla w + \alpha(e^{\beta|u|^2} - 1) u - \alpha(e^{\beta|v|^2} - 1) v = -\nabla (p - \tilde{p}).
\]
Taking the \( L^2 \) scalar product with \( w \), we obtain
\[
\langle \partial_t w, w \rangle_{L^2} - \langle \Delta_h w, w \rangle_{L^2} + \alpha(\langle e^{\beta|u|^2} - 1 \rangle u - \langle e^{\beta|v|^2} - 1 \rangle v, w \rangle_{L^2} \leq |\langle w, \nabla u, w \rangle_{L^2}|
\]
Using Lemma 2.6, we get:
\[
\frac{1}{2} \frac{d}{dt} \| w \|_{L^2}^2 + \| \nabla_h w \|_{L^2}^2 + \frac{2\alpha}{9} \int_{\mathbb{R}^3} \left( (e^{\frac{\beta}{2}|u|^2} - 1) + (e^{\frac{\beta}{2}|v|^2} - 1) \right) |w|^2 \leq \left| \int_{\mathbb{R}^3} (w.\nabla u).wdx \right|
\]
and
\[
\frac{1}{2} \frac{d}{dt} \| w \|_{L^2}^2 + \| \nabla_h w \|_{L^2}^2 \leq \left| \int_{\mathbb{R}^3} (w.\nabla u).wdx \right|
\]
The second member of this inequality can be written as follows
\[
\int_{\mathbb{R}^3} (w.\nabla u).wdx = F_1 + F_2
\]
with
\[
F_1 = \sum_{i=1}^{2} \sum_{j=1}^{3} \int_{\mathbb{R}^3} w_i \partial_i u_j w_j
\]
\[
F_2 = \sum_{j=1}^{3} \int_{\mathbb{R}^3} w_3 \partial_3 u_j w_j.
\]
By Hölder inequality, we have
\[
F_1 \leq \sum_{i=1}^{2} \sum_{j=1}^{3} \| \partial_i u_j \|_{L^\infty(L^2_h)} \| w_i \|_{L^2(L^2_h)} \| w_j \|_{L^2(L^2_h)}
\]
Since \( \dot{H}^\frac{1}{2}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2) \) and by interpolation we have :
\[
\| w_j (., x_3) \|_{L^6_h(\mathbb{R}^2)} \leq \| w_j (., x_3) \|_{L^2(\mathbb{R}^2)} \| \nabla_h w_j (., x_3) \|_{L^\frac{1}{2}(\mathbb{R}^2)}
\]
so
\[
\| w_j \|_{L^2(L^2_h)} \leq \| w_j \|_{L^{\frac{1}{2}}(\mathbb{R}^2)} \| \nabla_h w_j \|_{L^2}.
\]
To estimate \( \| \partial_i u_j \|_{L^\infty(L^2_h)} \), we write
\[
\| \partial_i u_j (., x_3) \|_{L^2_h(\mathbb{R}^2)}^2 = \int_{-\infty}^{x_3} \frac{d}{dz} \| \partial_i u_j (., z) \|_{L^2_h(\mathbb{R}^2)}^2 dz
\]
\[
= 2 \int_{-\infty}^{x_3} \langle \partial_i \partial_i u_j (., z), \partial_i u_j (., z) \rangle_{L^2_h(\mathbb{R}^2)} dz
\]
\[
\leq \| \partial_3 \partial_i u_j \|_{L^2} \| \partial_i u_j \|_{L^2}.
\]
Then
\[
F_1 \leq c \| \partial_3 \nabla_h u \|_{L^2} \| \nabla_h u \|_{L^2} \| w \|_{L^2} \| \nabla_h w \|_{L^2}.
\]
By Young inequality, we obtain
\[ (4.7) \quad F_1 \leq \frac{1}{4} \|\nabla_h w\|^2_{L^2} + \frac{c}{4} (\|\partial_3 \nabla_h u\|^2_{L^2} + \|\nabla_h u\|^2_{L^2}) \|w\|^2_{L^2} \]
The same procedure for $F_2$ we have:
\[ F_2 \leq \|w_3\|_{L^\infty(L^2_h)} \|\nabla_h w\|^\frac{1}{2}_{L^2} \|\partial_3 \nabla_h u\|^\frac{1}{2}_{L^2} \|\nabla_h u\|_{L^2} \|w\|_{L^2}. \]
Since
\[ \|w_3(\cdot, x)\|^2_{L^2_h(\mathbb{R}^2)} = 2 \int_{-\infty}^{x_3} \int_{\mathbb{R}^2} w_3(x_h, z) \partial_3 w_3(x_h, z) dx_h dz. \]
Using the fact that $\nabla \cdot w = 0$ so $\text{div}_h w_h = -\partial_3 w_3$ and
\[ \|w_3(\cdot, x)\|^2_{L^2_h(\mathbb{R}^2)} = -2 \int_{-\infty}^{x_3} \int_{\mathbb{R}^2} w_3(x_h, z) \text{div}_h w_h(x_h, z) dx_h dz \leq 2 \|\text{div}_h w_h\|_{L^2(\mathbb{R}^3)} \|w_3\|_{L^2(\mathbb{R}^3)}. \]
By Young inequality:
\[ (4.8) \quad F_2 \leq \frac{1}{4} \|\nabla_h w\|^2_{L^2} + \frac{c}{4} (\|\partial_3 \nabla_h u\|^2_{L^2} + \|\partial_3 u\|^2_{L^2}) \|w\|^2_{L^2} \]
Hence, according to (4.7) and (4.8):
\[ \frac{1}{2} \frac{d}{dt} \|w\|^2_{L^2} + \frac{1}{2} \|\nabla_h w\|^2_{L^2} \leq c (\|\partial_3 \nabla_h u\|^2_{L^2} + \|\partial_3 u\|^2_{L^2} + \|\nabla_h u\|^2_{L^2}) \|w\|^2_{L^2} \]
Integrate on $[0, t]$, we have:
\[ \|w(t)\|^2_{L^2} + \int_0^t \|\nabla_h w\|^2_{L^2} \leq \|w(0)\|^2_{L^2} + c \int_0^t (\|\partial_3 \nabla_h u\|^2_{L^2} + \|\partial_3 u\|^2_{L^2} + \|\nabla_h u\|^2_{L^2}) \|w\|^2_{L^2} \]
Now, by Gronwall Lemma:
\[ \|w(t)\|^2_{L^2} \leq \|w(0)\|^2_{L^2} \exp \left( c \int_0^t (\|\partial_3 \nabla_h u\|^2_{L^2} + \|\partial_3 u\|^2_{L^2} + \|\nabla_h u\|^2_{L^2}) \right). \]
Using inequalities (1.4) and (1.5), we get
\[ (4.9) \quad \|w(t)\|^2_{L^2} \leq \|w(0)\|^2_{L^2} \exp \left( 2c \|\partial_3 u^0\|^2_{L^2} e^{\frac{ct}{2}} + c \|u^0\|^2_{L^2} \right). \]
But $w(0) = u(0) - v(0)$, then $u = v$, which ends the proof of Theorem 1.3.

5. Appendix

In this section, we give a simple proof of $u \in C(\mathbb{R}^+, L^2(\mathbb{R}^3))$, where $u$ is a solution of $(NS_2)$ given by Friedrich approximation. We point out that we can use this method to show the same results in the case [1].

• By inequality (1.4) we get
\[ \limsup_{t \to 0} \|u(t)\|_{L^2} \leq \|u^0\|_{L^2}. \]
Then, Proposition 2.1-(3) implies that
\[ \limsup_{t \to 0} \|u(t) - u^0\|_{L^2} = 0, \]
which ensures continuity at 0.

• Let $t_0 > 0$. For $\varepsilon \in (0, t_0/2)$ and $n \in \mathbb{N}$, put the following function
  
v_{n, \varepsilon}(t) = u_{\varphi(n)}(t + \varepsilon).

Applying the same method to prove the uniqueness to $u_{\varphi(n)}$ and $v_{n, \varepsilon}$, and using (5.1) we get
  \[ \|u_{\varphi(n)}(t + \varepsilon) - u_{\varphi(n)}(t)\|_{L^2}^2 \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \exp\left(cF_n(t)\right), \]
where
  \[ F_n(t) = \int_0^t \left( \|\partial_3 \nabla_h u_{\varphi(n)}\|_{L^2}^2 + \|\partial_3 u_{\varphi(n)}\|_{L^2}^2 + \|\nabla_h u_{\varphi(n)}\|_{L^2}^2 \right). \]

By using inequalities (4.2)-(4.3), we get
  \[ F_n(t) \leq 2 \|\partial_3 u_0\|_{L^2} e^{2\alpha_0 \beta t} + \|\partial_3 u_0\|_{L^2} \left( \int_0^t e^{\alpha_0 \beta z} dz \right) + \frac{\|u_0\|_{L^2}^2}{2} \]
  \[ \leq 2 \|\partial_3 u_0\|_{L^2} e^{2\alpha_0 \beta t} + \frac{\|u_0\|_{L^2}^2}{2}. \]

For $t \in [0, 2t_0]$, we have
  \[ F_n(t) \leq 2 \|\partial_3 u_0\|_{L^2} e^{2\alpha_0 \beta t_0} + \frac{\|u_0\|_{L^2}^2}{2} = M_{\alpha_0, \beta}(t_0, u_0). \]

Then, for $t = t_0$ and $t = t_0 - \varepsilon$, we get
  \[ F_n(t_0) \leq \|u_{\varphi(n)}(t_0)\|_{L^2}^2 - 2Re\langle u_{\varphi(n)}(\varepsilon), u_{\varphi(n)}(0)\rangle_{L^2}. \]

The idea is to lower the terms on the left and increase the terms on the right of the inequalities (5.1) and (5.2).

For the right term, we write
  \[ \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 = \|u_{\varphi(n)}(\varepsilon)\|_{L^2}^2 + \|u_{\varphi(n)}(0)\|_{L^2}^2 - 2Re\langle u_{\varphi(n)}(\varepsilon), u_{\varphi(n)}(0)\rangle_{L^2}. \]

By using inequality (4.2) we obtain
  \[ \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \leq 2 \|u_0\|_{L^2}^2 - 2Re\langle u_{\varphi(n)}(\varepsilon), u_{\varphi(n)}(0)\rangle_{L^2} \]
  \[ \leq 2 \|u_0\|_{L^2}^2 - 2Re\langle u_{\varphi(n)}(\varepsilon), u_0\rangle_{L^2} - 2Re\langle u_{\varphi(n)}(\varepsilon), u_{\varphi(n)}(0) - u_0\rangle_{L^2}. \]

But
  \[ |\langle u_{\varphi(n)}(\varepsilon), u_{\varphi(n)}(0) - u_0\rangle_{L^2}| \leq \|u_{\varphi(n)}(\varepsilon)\|_{L^2} \|u_{\varphi(n)}(0) - u_0\|_{L^2} \leq \|u_0\|_{L^2} \|u_{\varphi(n)}(0) - u_0\|_{L^2}, \]
then
  \[ \lim_{n \to \infty} |\langle u_{\varphi(n)}(\varepsilon), u_{\varphi(n)}(0) - u_0\rangle_{L^2}| = 0. \]

On the other hand, and by using that $u_{\varphi(n)}(\varepsilon)$ converge weakly in $L^2(\mathbb{R}^3)$ to $u(\varepsilon)$, we get
  \[ \lim_{n \to \infty} \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \leq 2 \|u_0\|_{L^2}^2 - 2Re\langle u(\varepsilon); u_0\rangle_{L^2}. \]

For the left term, we have, for all $q, N \in \mathbb{N}$,
  \[ \|J_N(\theta_q(u_{\varphi(n)}(t_0 + \varepsilon) - u_{\varphi(n)}(t_0)))\|_{L^2} \leq \|\theta_q(u_{\varphi(n)}(t_0 + \varepsilon) - u_{\varphi(n)}(t_0))\|_{L^2} \leq \|u_{\varphi(n)}(t_0 + \varepsilon) - u_{\varphi(n)}(t_0))\|_{L^2}^2. \]
Using (4.4) we get, for $q$ big enough,
\[
\|J_N\left(\theta_q, (u(t_0 \pm \varepsilon) - u(t_0))\right)\|_{L^2}^2 \leq \liminf_{n \to \infty} \|u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2.
\]
Then
\[
\|J_N\left(\theta_q, (u(t_0 \pm \varepsilon) - u(t_0))\right)\|_{L^2}^2 \leq 2\left(\|u^0\|_{L^2}^2 - \text{Re}\langle u(\varepsilon); u^0 \rangle_{L^2}\right) \exp(cM_{\alpha, \beta}(t_0, u^0)).
\]
By applying the Monotonic Convergence Theorem in the order $N \to \infty$ and $q \to \infty$, we get
\[
\|u(t_0 \pm \varepsilon) - u(t_0)\|_{L^2}^2 \leq 2\left(\|u^0\|_{L^2}^2 - \text{Re}\langle u(\varepsilon); u^0 \rangle_{L^2}\right) \exp(cM_{\alpha, \beta}(t_0, u^0)).
\]
Using the continuity at $0$ and make $\varepsilon \to 0$, we get the continuity at $t_0$, which ends the proof.

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