Classification of Zero Divisor Graphs of a Commutative Ring With Degree Equal 7 and 8

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ABSTRACT

In 2005 J. T Wang investigated the zero divisor graphs of degrees 5 and 6. In this paper, we consider the zero divisor graphs of a commutative rings of degrees 7 and 8.

Keywords: Zero-divisor, Ring, Zero-divisor graph.

1. Introduction

The concept of zero divisor graph of a commutative ring was introduced by Beck in [3]. He let all the elements of the ring be vertices of the graph. In [1] Anderson and Livingston introduced and studied the zero divisor graph whose vertices are the non-zero zero divisors.

Throughout this paper, all rings are assumed to be commutative rings with identity, and Z(R) be the set of zero divisors. We associate a simple graph Γ(R) to a ring R with vertices Z(R)∗ = Z(R)−{0}, the set of all non-zero zero divisors of R. For all distinct x,y ∈ Z(R)∗, the vertices x and y are adjacent if and only if xy=0. (R,m) and |S| will stand respectively for the local ring with maximal ideal m and cardinal numbers of a set S.

In [1] Anderson and Livingston proved that for any commutative ring R Γ(R) is connected.

In 2005 J. T Wang [5] investigated the zero divisor graphs of degrees 5 and 6. In this paper, we extend this results to consider the zero divisor graphs of commutative rings of degrees 7 and 8.

The main result when |Z(R)∗|=7 is given in Theorem 2.7,while when |Z(R)∗|=8 the main result is given in Theorem 3.4. We also extend Wang’s result concerning local rings (Theorem 2.2).

2. Rings with |Z(R)∗|=7

It is known that if R is a ring then Γ(R) is connected. In this section, we find all possible graphs of Γ(R) with Γ(R)∗=7.

Recall that if R is finite ring, then every element of R is either a unit or a zero divisor [2]. In [5] Wang proved the following result.
Lemma 2.1:
Let \((R_1,m_1)\) and \((R_2,m_2)\) are local rings, then \(|Z(R_1 \times R_2)| = |R_1|m_2 + |R_2|m_1 - m_1|m_2| - 1. ■

Now, we shall prove the following theorem which extends Wang's result.

Theorem 2.2:
If \((R_1,m_1), (R_2,m_2)\) and \((R_3,m_3)\) are finite local rings, then \(|Z(R_1 \times R_2 \times R_3)| = |R_1|m_2 + |R_2|m_1 - |m_1|m_2| - 1\) where \(|Z(R_1 \times R_2)| = |R_1|m_2 + |R_2|m_1 - m_1|m_2| - 1\). Let \(R_{(1)(2)} = R_1 \times R_2\) then \(|R_{(1)(2)}| = |R_1|2\) and \(|Z(R_{(1)(2)})| = Z(R_1 \times R_2)\).

For any non-zero-divisor \((a,b)\) in \(R_{(1)(2)}\), we have the following cases:
1. If \(a\) is a non-zero divisor of \(R_{(1)(2)}\), then a must be a unit element. If \(b\) is a zero divisor of \(R_3\), then there are \(|Z(R_{(1)(2)})| = |R_1| 3\) elements of this type.
2. If \(a\) is a non-zero divisor of \(R_{(1)(2)}\) and \(b\) any element in \(R_3\), then \(|Z(R_{(1)(2)})| = |R_1| 3\) elements of this type.
3. If \(a = 0\) and \(b\) is a non-zero element in \(R_3\), then there are \(|R_3| - 1\).

Now, sum up these three types of elements; there are as follows:
\(|R_{(1)(2)}| = |R_1|m_2 + |R_2|m_1 - |m_1|m_2| - 1\) where \(|Z(R_{(1)(2)})|=Z(R_1 \times R_2)\).

As a direct consequence to Theorem 2.2, we obtain the following:

Corollary 2.3:
If \(R_1, R_2\) and \(R_3\) are finite fields, then \(|Z(R_1 \times R_2 \times R_3)| = |R_1|R_2| + |R_1|R_3| + |R_2|R_3| - |R_1|R_2| - |R_1|R_3| - |R_2|R_3| ■

Corollary 2.4:
If \(R\) finite and \(R \cong R_1 \times R_2 \times R_3\), then \(|Z(R)| \geq 13\) for some local rings \(R_i\) but not field.

Proof:
Suppose that \(R_3\) is local which is not a field, then clearly \(|R_3| \geq 4\) and \(|m_3| \geq 2\) and since \(|R_1|, |R_2| \geq 2\) and \(|m_1|, |m_2| \geq 1\), then \(Z(R_1 \times R_2) \geq 3\), therefore \(|Z(R)| \geq 2.2.2 + 3(4-2)-1 = 13\). ■

Next, we prove two fundamental lemmas

Lemma 2.5:
If \(R\) is a ring with \(|Z(R)| = 7\), then is either \(R\) local ring or \(R\) is isomorphic to a product of two local rings.

Proof:
Since \(|Z(R)| = 7\), then \(R\) is finite and hence \(R \cong R_1 \times R_2 \times \ldots \times R_n\) where \(R_i, i = 1,2,\ldots,n\) are local rings. If \(n \geq 4\), then by [5, Lemma 4.7], \(|Z(R)| \geq 14\) this is a contradiction.

Now, consider \(n = 3\), if \(R_i\) local, but not field for some \(1 \leq i \leq 3\), then by Corollary 2.4, \(|Z(R)| \geq 13\) which is a contradiction. Hence \(R_i\) are fields for all \(1 \leq i \leq 3\). Applying Corollary 2.3, \(|Z(R_1 \times R_2 \times R_3)| = |R_1|R_2| + |R_1|R_3| + |R_2|R_3| - |R_1|R_2| - |R_1|R_3| - |R_2|R_3| = 7\). If \(|R_1| = |R_2| = 2\,
then $|R_i|=7/3$ which is also a contradiction. Finally, if $|R_i|\geq 3$ for some $i$, then by [5,Lemma 4.5], $|Z(R)^*|\geq 9$ which is also a contradiction. Therefore, $n=1$ or 2.

**Lemma 2.6 :**

Let $R$ be a ring which is not local and $|Z(R)^*|=7$, then $R \cong Z_4 \times Z_3$ or $Z_2[X]/(X^2)x Z_3$ or $Z_2 \times Z_7$ or $F_4 \times Z_5$.

**Proof:**

Suppose that $R$ is a ring which is not local, then by Lemma 2.5 $R \cong R_1 \times R_2$. If $R_1$ and $R_2$ are local, but not a field, then by [5, Corollary 4.4], $|Z(R)^*| \geq 11$ which is a contradiction. If $R_1$ local, but not a field, $R_2$ field, then we have $|Z(R)^*|=|R_1|+|R_2|-2=7$, this yields to $|R_1|+|m_1|(|R_2|-1)-8=0 \ldots (1)$

Now, if $|m_1|=p$ where $p$ is prime number, then by [5, Lemma 4.2], $|R_1|=|m_1|^2=p^2$, so from equation (1) we have $p^2+kp-8=0 \ldots (2)$, where $k=|R_2|-1$ this implies that $p=\frac{-k+\sqrt{k^2+32}}{2}$, so the only solution for $p$ to be prime is $k=2$, and hence $p=2$, and this implies $|R_1|=4$ and $|R_2|=3$. Then, by [4,pp.687] $R_1 \cong Z_4$ or $Z_2[X]/(X^2)$ and $R_2 \cong Z_3$. Hence, $R \cong Z_4 \times Z_3$ or $Z_2[X]/(X^2) \times Z_3$. Now if $R_1$ and $R_2$ are fields, then $|Z(R)^*|=|R_1|+|R_2|-2=7$, this yields to $|R_1|+|R_2|=9$. Therefore, $|R_1|=2$, $|R_2|=7$ or $|R_1|=4$, $|R_2|=5$. Thus, $R \cong Z_2 \times Z_7$ or $F_4 \times Z_5$. ■

Now, we shall prove the main result of this section.

**Theorem 2.7 :**

Let $R$ be a ring which is not local and $|Z(R)^*|=7$, then the following graph can be realized as $\Gamma(R)$.

![Figure (1)](image1)

![Figure (2)](image2)

![Figure (3)](image3)

**Proof:**

By Lemma 2.6, $R \cong Z_4 \times Z_3$ or $Z_2[X]/(X^2) \times Z_3$ or $Z_2 \times Z_7$ or $F_4 \times Z_5$. In Figure (1), can be realized as $\Gamma(Z_4 \times Z_3)$ or $\Gamma(Z_2[X]/(X^2) \times Z_3)$, Figure (2) can be realized as $\Gamma(Z_2 \times Z_7)$ and Figure (3) can be realized as $\Gamma(F_4 \times Z_5)$. ■

**3. Rings with $|Z(R)^*|=8$**

The main aim of this section is to find all possible zero divisor graphs of 8 vertices and rings which correspond to them.

We shall start this section with following lemmas which play a central role in the sequel.

**Lemma 3.1 :**

Let $R$ be a ring with $|Z(R)^*|=8$, then $R$ is local or $R$ is isomorphic to a product of two local rings.

**Proof:**

Since $|Z(R)^*|=8$, then $R$ is finite and hence, $R \cong R_1 \times R_2 \times \ldots \times R_n$ where $R_i$, $i=1,2,\ldots,n$ are local rings.

If $n \geq 4$, then by [5, Lemma 4.7 ], $|Z(R)^*| \geq 14$; this is a contradiction.
Now, consider \( n=3 \), if \( R_i \) local but not field for some \( 1 \leq i \leq 3 \), then by Corollary 2.4, \(|Z(R)^*| \geq 13\) which is a contradiction. So \( R_i \) is a field for all \( 1 \leq i \leq 3 \). Then, by Corollary 2.3

\[ |Z(R_1 \times R_2 \times R_3)^*| = |R_1||R_2||R_3|+|R_2||R_3|-|R_1|-|R_2|-|R_3| = 8. \]

If \(|R_1|=|R_2|=2\) and \(|R_3|=8/3\) which is a contradiction. If \(|R_i| \geq 3\) for some \( i \), then by [5, Lemma 4.5], \(|Z(R)^*| \geq 9\) which is a contradiction. Therefore, \( n=1 \) or \( 2 \). ■

**Lemma 3.2:**

Let \( R \) be a ring which is not local and \(|Z(R)^*|=8\), then \( R \cong F_1 \times F_2 \), where \( F_1 \) and \( F_2 \) are fields.

**Proof:**

Since \( R \) not local, then by Lemma 3.1 \( R \cong R_1 \times R_2 \), where \( R_1, R_2 \) are local rings. If \( R_1 \) and \( R_2 \) local, but not field, then by [5, Corollary 4.4], \(|Z(R)^*| \geq 11\) which is a contradiction.

If \( R_1 \) field and \( R_2 \) local not field, then \(|m_1|=1\) if \(|m_2|=p\) is prime number, then by [5, Lemma 4.8], \(|R_2|=p^2\) and applied [5, Lemma 4.2], we have \( p^2+kp-9=0 \) where \( k=|R_2|-1 \), so that \( p = \frac{-k + \sqrt{k^2 + 36}}{2} \) (3), since \( p \) is prime, then we have a contradiction. If \(|m_1|\) not prime then \(|m_1| \geq 4\) and since \(|R_2| \geq 2\), then \(|R_1|=9-|m_1||R_2|-1|<9-4(2-1)=5\) which is a contradiction. Therefore, \( R_1 \) and \( R_2 \) are fields. Hence, \( R \cong F_1 \times F_2 \), where \( F_1 \) and \( F_2 \) are fields. ■

**Lemma 3.3:**

Let \( R \) be a ring which is not local and \(|Z(R)^*|=8\), then \( R \cong Z_2 \times F_8 \) or \( Z_3 \times Z_7 \) or \( Z_5 \times Z_5 \).

**Proof:**

By Lemma 3.2, \( R \cong F_1 \times F_2 \), where \( F_1, F_2 \) are fields, we have \(|F_1|+|F_2|=2|=8\) which implies that \(|F_1|+|F_2|=10\), so that \(|F_1|=2, |F_2|=8\) or \(|F_1|=3, |F_2|=7\) or \(|F_1|=5, |F_2|=5\). Therefore, \( R \cong Z_2 \times F_8 \) or \( Z_3 \times Z_7 \) or \( Z_5 \times Z_5 \). ■

Now, we are in a position to give the main result of this section

**Theorem 3.4:**

Let \( R \) be a ring which is not local and \(|Z(R)^*|=8\), then the following graph can be realized as \( \Gamma(R) \).

\[
\text{Figure (1)} \quad \text{Figure (2)} \quad \text{Figure (3)}
\]

**Proof:**

By Lemma 3.3, then \( R \cong Z_2 \times F_8 \) or \( Z_3 \times Z_7 \) or \( Z_5 \times Z_5 \). In Figure (1), can be realized as \( \Gamma(Z_2 \times F_8) \). Figure (2), can be realized as \( \Gamma(Z_3 \times Z_7) \). Figure (3), can be realized as \( \Gamma(Z_5 \times Z_5) \). ■
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