ON THE BURNSIDE-BRAUER-STEINBERG THEOREM

BENJAMIN STEINBERG

Abstract. A well-known theorem of Burnside says that if \( \rho \) is a faithful representation of a finite group \( G \) over a field of characteristic 0, then every irreducible representation of \( G \) appears as a constituent of a tensor power of \( \rho \). In 1962, R. Steinberg gave a module theoretic proof that simultaneously removed the constraint on the characteristic, and allowed the group to be replaced by a monoid. Brauer subsequently simplified Burnside's proof and, moreover, showed that if the character of \( \rho \) takes on \( r \) distinct values, then the first \( r \) tensor powers of \( \rho \) already contain amongst them all of the irreducible representations of \( G \) as constituents. In this note we prove the analogue of Brauer’s result for finite monoids. We also prove the corresponding result for the symmetric powers of a faithful representation.

1. Introduction

A famous result of Burnside [3] states that if \( K \) is a field of characteristic 0, \( G \) is a finite group and \( V \) is a finite dimensional \( KG \)-module affording a faithful representation of \( G \), then each simple \( KG \)-module is a composition factor of a tensor power \( V \otimes^i \) of \( V \). Burnside’s original proof [3] was via characters and formal power series. This result was vastly generalized by R. Steinberg in 1962 [20]. He showed that if \( K \) is any field, \( M \) is any monoid (possibly infinite) and \( V \) is a \( KM \)-module affording a faithful representation of \( M \), then the tensor algebra \( T(V) = \bigoplus_{i=0}^{\infty} V \otimes^i \) is a faithful \( KM \)-module (i.e., its annihilator in \( KM \) is 0). This easily implies that if \( M \) is finite and \( V \) is finite dimensional, then every simple \( KM \)-module is a composition factor of some tensor power of \( V \) (in fact one of the first \( |M| \)). Rieffel extended this result even further to bialgebras [19]; see also [15][16].

In 1964, Brauer gave a simpler character-theoretic proof of Burnside’s theorem and at the same time refined it [2]. Namely, he showed that if \( G \) is a finite group, \( K \) is a field of characteristic 0 and \( V \) is a finite dimensional \( KG \)-module affording a faithful representation of \( G \) whose character takes on \( r \) distinct values, then every simple \( KG \)-module is a composition factor of

\[ Date: \] October 7, 2014.
\[ 2010 \ Mathematics \ Subject \ Classification. \] 20M30,20C15,16G99,16T10.
\[ Key \ words \ and \ phrases. \] monoids, representation theory, characters, tensor products, symmetric powers.

This work was partially supported by a grant from the Simons Foundation (#245268 to Benjamin Steinberg), the Binational Science Foundation of Israel and the US (#2012080 to Benjamin Steinberg) and by a CUNY Collaborative Incentive Research Grant.
one of the first $r$ tensor powers of $V$. Because of this refinement, Burnside’s result is often referred to as the Burnside-Brauer theorem.

It is natural to ask whether R. Steinberg’s theorem can be similarly refined: is it true that if $V$ is a finite dimensional $KM$-module affording a faithful representation of a finite monoid $M$ over a field $K$ of characteristic 0 and that the character of $V$ takes on only $r$ distinct values, then every simple $KM$-module is a composition factor of one of $V^0, \ldots, V^{(r-1)}$?

This note answers the above question affirmatively. On the other hand, we also show that the minimal $k$ such that $\bigoplus_{i=0}^k V^i$ is a faithful $KM$-module cannot be bounded as a function of solely the number of distinct values assumed by the character of $V$, as is the case for finite groups.

Brauer’s proof [2] relies on the orthogonality relations for group characters. The irreducible characters of a finite monoid do not form an orthogonal set with respect to the natural inner product on mappings $M \to K$. So we have to adopt a slightly different tactic. Instead of using the orthogonality relations, we apply the character of $V^i$ to carefully chosen primitive idempotents. To make Brauer’s argument work, we also need to apply at a key moment a small part of the structure theory of irreducible representations of finite monoids, cf. [9][12][15] and [4] Chapter 5.

A detailed study of the minimal degree a faithful representation of a finite monoid was undertaken by the author and Mazorchuk in [13].

It is also known that if $V$ is a finite dimensional $KG$-module affording a faithful representation of a finite group $G$ over a field of characteristic 0, then every simple $KG$-module is a composition factor of a symmetric power $S^n(V)$ of $V$, cf. [7]. We prove the corresponding result for monoids and give a bound on how many symmetric powers are needed in terms of dim$V$ and the number of distinct characteristic polynomials of the linear operators associated to elements of $M$ acting on $V$. These kinds of results for representations of finite monoids over finite fields can be found in [10][11].

2. Tensor powers

We follow mostly here the terminology of the book of Curtis and Reiner [5], which will also serve as our primary reference on the representation theory of finite groups and finite dimensional algebras.

Let $K$ be a field, $A$ a finite dimensional $K$-algebra, $S$ a simple $A$-module and $V$ a finite dimensional $A$-module. We denote by $(V : S)$ the multiplicity of $S$ as a composition factor of $V$. Recall that $S \cong Ae/Re$ where $R$ is the radical of $A$ and $e \in A$ is a primitive idempotent, cf. [5], Corollary 54.13. (An idempotent $e$ is primitive if whenever $e = e_1 + e_2$ with $e_1, e_2$ orthogonal idempotents, then either $e_1 = 0$ or $e_2 = 0$.) To prove the main result, we need two lemmas about finite dimensional algebras. The first is the content of [5], Theorem 54.12.

**Lemma 1.** Let $K$ be a field and $A$ a finite dimensional $K$-algebra with radical $R$. Let $S$ be a simple $A$-module, $e \in A$ a primitive idempotent with
$S \cong Ae/Re$ and $V$ a finite dimensional $A$-module. Then $(V : S) > 0$ if and only if $eV \neq 0$.

The second lemma on finite dimensional algebras concerns the connection between primitive idempotents for an algebra and its corners. We recall that if $A$ is a finite dimensional algebra with radical $R$ and $e \in A$ is an idempotent, then $eRe$ is the radical of $eAe$ [5, Theorem 54.6].

**Lemma 2.** Let $A$ be a finite dimensional $K$-algebra with radical $R$ and let $e \in A$ be an idempotent. Suppose that $S$ is a simple $A$-module such that $eS \neq 0$. Then $eS$ is a simple $eAe$-module and, moreover, if $f \in eAe$ is a primitive idempotent with $eAef/eRef \cong eS$, then $f$ is a primitive idempotent of $A$ and $Af/Rf \cong S$.

**Proof.** If $v \in eS$ is a nonzero vector, then $eAev = eAv = eS$ because $S$ is a simple $A$-module. Thus $eS$ is a simple $eAe$-module. Let $f \in eAe$ be as above. If $f = e_1 + e_2$ with $e_1, e_2$ orthogonal idempotents in $A$, then $ee_ie = ef_ie = fe_i = e_i$ for $i = 1, 2$ and so $e_1, e_2 \in eAe$. Thus one of $e_1, e_2$ is 0 by primitivity of $f$ in $eAe$ and hence $f$ is primitive in $A$. Finally, since $(eS : eAef/eRef) = 1$, we have by Lemma [1] that $0 \neq feS = fS$ and so $(S : Af/Rf) > 0$ by another application of Lemma [1]. Since $S$ is simple, we deduce that $S \cong Af/Rf$, as required. □

Next we need a lemma about idempotents of group algebras.

**Lemma 3.** Let $G$ be a finite group and $K$ a field of characteristic 0. Suppose that $e = \sum_{g \in G} c_g g$ in $KG$ is a nonzero idempotent. Then $c_1 \neq 0$.

**Proof.** Because $e \neq 0$, we have $\dim eKG > 0$. Let $\theta$ be the character of the regular representation of $G$ over $K$, which we extend linearly to $KG$. Then

$$\dim eKG = \theta(e) = \sum_{g \in G} c_g \theta(g) = c_1 \cdot |G|$$

since

$$\theta(g) = \begin{cases} |G|, & \text{if } g = 1 \\ 0, & \text{else.} \end{cases}$$

Therefore, $c_1 = (\dim eKG)/|G| \neq 0$. □

Let $M$ be a finite monoid and $K$ a field. If $V$ is a finite dimensional $KM$-module, then $\theta_V : M \to K$ will denote the character of $V$. Sometimes it will be convenient to extend $\theta_V$ linearly to $KM$. Note that $V^{\otimes i}$ is a $KM$-module by defining

$$m(v_1 \otimes \cdots \otimes v_i) = mv_1 \otimes \cdots \otimes mv_i$$

for $m \in M$. By convention $V^{\otimes 0}$ is the trivial $KM$-module. One has, of course, that $\theta_V^{\otimes 0} = \theta_V \cdot \theta_W$ and that the character of the trivial module is identically 1. Therefore, $\theta_V^{\otimes i} = \theta_V^{i}$ for all $i \geq 0$. The following is a monoid analogue of a well-known fact for groups.
Lemma 4. Let $M$ be a finite monoid, $K$ a field of characteristic 0 and $\rho: M \to M_n(K)$ a representation affording the character $\theta$. Then $\rho(m) = I$ if and only if $\theta(m) = n$.

Proof. If $\rho(m) = I$, then trivially $\theta(m) = n$. Suppose that $\theta(m) = n$. Because $M$ is finite, there exist $r, s > 0$ such that $m^r = m^{r+s}$. Then the minimal polynomial of $\rho(m)$ divides $x^r(x^s - 1)$ and so each nonzero eigenvalue of $\rho(m)$ is a root of unity (in an algebraic closure of $K$). Now the proof proceeds analogously to the case of finite groups, cf. [5] Corollary 30.11. That is, $\theta(m)$ is a sum of at most $n$ roots of unity and hence can only be equal to $n$ if all the eigenvalues of $\rho(m)$ are 1. But then $\rho(m)$ is both unipotent and of finite order, and hence $\rho(m) = I$ as $K$ is of characteristic 0.

We shall now need to apply a snippet of the structure theory for irreducible representations of finite monoids. Details can be found in [4 Chapter 5] or [18]; a simpler approach was given in [9]. Let $M$ be a finite monoid and $e \in M$ an idempotent. Denote by $G_e$ the group of units of the monoid $eMe$. It is well known that $I_e = eMe \setminus G_e$ is an ideal of $eMe$, i.e., $(eMe)I_e(eMe) = I_e$; see, for instance, [21] Proposition 1.2 in Eilenberg [6].

Lemma 5. Let $M$ be a monoid and $K$ a field. Let $e \in M$ be an idempotent and let $V$ be a finite dimensional $KM$-module. Then $(\theta_V)|_{eMe} = \theta_{eV}$.

Proof. There is a vector space direct sum decomposition $V = eV \oplus (1-e)V$. As $eMe$ annihilates $(1-e)V$ and preserves $eV$, the result follows.

Let $S$ be a simple $KM$-module with $K$ a field. An idempotent $e \in M$ is called an apex for $S$ if $eS \neq 0$ and $I_eS = 0$. By classical results of Munn [14] and Ponizovsky [17], each simple $KM$-module has an apex; see [9] Theorem 5] or [4] Theorem 5.33]. The apex is unique up to $\mathcal{J}$-equivalence of idempotents, although this fact is not relevant here. We are now ready to prove our refinement of R. Steinberg’s theorem [20].

Theorem 6. Let $M$ be a finite monoid and $K$ a field of characteristic 0. Let $V$ be a finite dimensional $KM$-module affording a faithful representation of $M$. Suppose that the character $\theta$ of $V$ takes on $r$ distinct values. Then every simple $KM$-module is a composition factor of $V^\otimes i$ for some $0 \leq i \leq r - 1$.

Proof. Let $S$ be a simple $KM$-module and let $e \in M$ be an apex for $S$. Put $A = KM$ and let $R$ be the radical of $A$. Observe that $eAe = K[eMe]$. As $eS \neq 0$, there is a primitive idempotent $f$ of $eAe$ such that $f$ is primitive in $A$ and $S \cong Af/Rf$ by Lemma [2]. Write

$$f = \sum_{m \in eMe} c_m m.$$ 

By definition of an apex $I_eS = 0$. On the other hand, $fS \neq 0$ by Lemma [1]. Thus $f \notin Ke$. Define a homomorphism $\varphi: eAe \to KG_e$ by

$$\varphi(m) = \begin{cases} m, & \text{if } m \in G_e \\ 0, & \text{if } m \in I_e \end{cases}$$
for \( m \in eMe \) and note that \( \ker \varphi = KI_e \). Therefore,

\[
\varphi(f) = \sum_{g \in G_e} c_g g
\]

is a nonzero idempotent of \( KG_e \) and hence \( c_e \neq 0 \) by Lemma 3.

Let \( \theta_1, \ldots, \theta_r \) be the values taken on by \( \theta \) and let

\[
M_j = \{ m \in eMe \mid \theta(m) = \theta_j \}.
\]

Without loss of generality assume that \( \theta_1 = \theta(e) \). Put

\[
b_j = \sum_{m \in M_j} c_m.
\]

Suppose now that \( (V^\otimes i : S) = 0 \) for all \( 0 \leq i \leq r - 1 \). We follow here the convention that \( \theta^0_j = 1 \) even if \( \theta_j = 0 \). Then by Lemma 1, we have that

\[
0 = \dim fV^\otimes i = \theta_{V^\otimes i}(f) = \sum_{m \in eMe} c_m \theta^i(m) = \sum_{j=1}^r \theta^i_j \sum_{m \in M_j} c_m = \sum_{j=1}^r \theta^i_j b_j
\]

for all \( 0 \leq i \leq r - 1 \). By nonsingularity of the Vandermonde matrix, we conclude that \( b_j = 0 \) for all \( 1 \leq j \leq r \). By Lemma 5 we have that \( M_1 = \{ m \in eMe \mid \theta_{eV}(m) = \dim eV \} \). Because \( V \) affords a faithful representation of \( M \), it follows that \( eV \) affords a faithful representation of \( eMe \). Lemma 4 then implies that \( M_1 = \{ e \} \). Thus \( 0 = b_1 = c_e \neq 0 \). This contradiction concludes the proof.

\[\square\]

**Remark 1.** We need to include the trivial representation \( V^\otimes 0 \) because if \( M \) is a monoid with a zero element \( z \) and if \( zV = 0 \), then \( zV^\otimes i = 0 \) for all \( i > 0 \) and so the trivial representation is not a composition factor of any positive tensor power of \( V \). The proof of Theorem 6 can be modified to show that if \( S \) is not the trivial module, or if \( M \) has no zero element, then \( S \) appears as a composition factor of \( V^\otimes i \) with \( 1 \leq i \leq r \). The key point is that only the trivial representation can have the zero element of \( M \) as an apex and so in either of these two cases, \( \theta(e) \neq 0 \).

**Remark 2.** If \( G \) is a finite group, \( K \) is a field of characteristic 0 and \( V \) is a finite dimensional \( KG \)-module affording a faithful representation of \( G \) whose character takes on \( r \) distinct values, then \( \bigoplus_{i=0}^{r-1} V^\otimes i \) contains every simple \( KG \)-module as a composition factor by Brauer’s theorem and hence is a faithful \( KG \)-module because \( KG \) is semisimple. We observe that the analogous result fails in a very strong sense for monoids. Let \( N_t = \{ 0, 1, \ldots, t \} \) where 1 is the identity and \( xy = 0 \) for \( x, y \in N_t \setminus \{ 1 \} \). Define a faithful two-dimensional representation \( \rho: N_t \to M_2(\mathbb{C}) \) by

\[
\rho(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(j) = \begin{bmatrix} 0 & j \\ 0 & 0 \end{bmatrix}, \quad \text{for } 2 \leq j \leq t.
\]

Let \( V \) be the corresponding \( \mathbb{C}N_t \)-module. The character of \( \rho \) takes on 2 values, 0 and 1. However, \( V^\otimes 0 \oplus V^\otimes 1 \) is 3-dimensional and so cannot be a
faithful $CN_t$-module for $t \geq 9$ by dimension considerations. In fact, given any integer $k \geq 0$, we can choose $t$ sufficiently large so that $\bigoplus_{i=0}^{k} V^\otimes i$ is not a faithful $CN_t$-module (again by dimension considerations). Thus, the minimum $k$ such that $\bigoplus_{i=0}^{k} V^\otimes i$ is a faithful $CN_t$-module cannot be bounded as a function of only the number of values assumed by the character $\theta_V$ (independently of the monoid in question).

Remark 3. A monoid homomorphism $\varphi: M \to N$ is called an LI-morphism if $\varphi$ separates $e$ from $eMe \setminus \{e\}$ for all idempotents $e \in M$. The proof of Theorem 6 only uses that the representation $\rho: M \to \text{End}_K(V)$ afforded by $V$ is an LI-morphism, and not that it is faithful. Hence one could obtain the conclusion of Theorem 6 under the weaker hypothesis that the representation afforded by $V$ is an LI-morphism. However, if $\varphi: M' \to M''$ is a surjective LI-morphism of finite monoids and $K$ is a field of characteristic 0, then the induced algebra homomorphism $\varphi: KM' \to KM''$ has nilpotent kernel [1] and hence each simple $KM'$-module is lifted from a simple $KM''$-module. Thus applying Theorem 6 to $\rho(M)$ allows one to recover the result under the weaker hypothesis from the original result.

3. **Symmetric powers**

Let $K$ be a field of characteristic 0 and $V$ a vector space over $K$. Then the symmetric group $S_d$ acts on the right of $V^\otimes d$ by twisting, e.g.,

$$(v_1 \otimes \cdots \otimes v_d)\sigma = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}.$$  

The $d^{th}$-symmetric power is the coinvariant space

$$S^d(V) = V^\otimes d \otimes_{KS_d} K$$

where $K$ is the trivial $KS_d$-module. In characteristic 0, one can identify $S^d(V)$ with the symmetric tensors (the tensors fixed by $S_d$). If $V$ is a $KM$-module, where $M$ is a monoid, then $S^d(V)$ is naturally a $KM$-module due to the $KM$-$KS_d$-bimodule structure on $V^\otimes d$. It is well known that if $\rho: M \to \text{End}_K(V)$ is the representation afforded by $V$, then

$$\theta_{S^d(V)}(m) = h_d(\lambda_1, \ldots, \lambda_n)$$

where $h_d(x_1, \ldots, x_n)$ is the complete symmetric polynomial of degree $d$, $\dim V = n$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\rho(m)$ (in a fixed algebraic closure of $K$) with multiplicities, cf. [8, Page 77]. We shall also need the well-known identity [8, Appendix A]

$$\sum_{i=0}^{\infty} h_i(x_1, \ldots, x_n)t^i = \prod_{j=1}^{n} \frac{1}{1-tx_i}. \quad (1)$$

**Theorem 7.** Let $K$ be a field of characteristic 0, let $M$ be a finite monoid and let $V$ be a finite dimensional $KM$-module affording a faithful representation $\rho: M \to \text{End}_K(V)$. Then every simple $KM$-module is a composition
factor of one of $\mathcal{S}^0(M), \ldots, \mathcal{S}^{r-1}(M)$ with $r = \dim V \cdot s$ where $s$ is the number of distinct characteristic polynomials of the elements $\rho(m)$ with $m \in M$.

Proof. We proceed as in the proof of Theorem \textup{[14]} Let $S$ be a simple $KM$-module and let $e \in M$ be an apex for $S$. Since $\mathcal{S}^0(V)$ is the trivial module, we may assume that $S$ is not the trivial module. Then $e$ is not the zero of $M$ (if it has one) and so $eV \neq 0$ because $\rho$ is faithful. Put $A = KM$ and let $R$ be the radical of $A$. As $eS \neq 0$, there is a primitive idempotent $f$ of $eAe$ such that $f$ is primitive in $A$ and $S \cong Af/Rf$ by Lemma \textup{[2]}. Write

$$f = \sum_{m \in eMe} c_mm.$$  

The proof of Theorem \textup{[14]} shows that $c_e \neq 0$.

Let $a_i = \dim f \mathcal{S}^i(V)$ and let $g(t) = \sum_{i=0}^{\infty} a_it^i$ be the corresponding generating function. We prove that $g(t)$ is a non-zero rational function with denominator of degree at most $r$ by establishing a Molien type formula.

Let $n = \dim V$ and let $p_m(t)$ be the characteristic polynomial of $\rho(m)$ for $m \in M$. Let $q_1(t), \ldots, q_s(t)$ be the $s$ characteristic polynomials of the endomorphisms $\rho(m)$ with $m \in M$.

Notice that $e\mathcal{S}^i(V) = \mathcal{S}^i(eV)$ as an $eAe$-module because $eV^{\otimes i} = (eV)^{\otimes i}$. Let $\rho': eMe \to \text{End}_K(eV)$ be the representation afforded by $eV$. Note that if $m \in eMe$, then

$$t^n p_m(1/t) = \det(I - t\rho(m)) = \det(I - t\rho'(m))$$  

(2)  

because if we write $V = eV \oplus (1-e)V$ and choose a basis accordingly, we then have the block form

$$I - t\rho(m) = \begin{bmatrix} I - t\rho'(m) & 0 \\ 0 & I \end{bmatrix}.$$  

Let $M_j = \{ m \in eMe \mid p_m(t) = q_j(t) \}$ and assume that $q_1(t) = p_e(t)$. Let

$$b_j = \sum_{m \in M_j} c_m.$$  

Note that if $M_j = \emptyset$, then $b_j = 0$. Observe that

$$t^n q_1(1/t) = \det(I - t\rho'(e)) = \det(I - tI) = (1-t)^k$$  

where $k = \dim V$. On the other hand, since $\rho'$ is faithful if $m \in eMe \setminus \{e\}$, by Lemma \textup{[11]} not all eigenvalues of $\rho'(m)$ are 1. Therefore, $t^n p_m(1/t) = \det(I - t\rho'(m))$ is a degree $k$ polynomial whose roots are not all equal to 1. In particular, $M_1 = \{ e \}$ and so $b_1 = c_e \neq 0$.

Let $m \in eMe$ and let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of $\rho'(m)$ with multiplicities in a fixed algebraic closure of $K$. Then, using \textup{[11]}, we have that

$$\sum_{i=0}^{\infty} \theta_{\mathcal{S}^i(eV)}(m)t^i = \sum_{i=0}^{\infty} h_i(\lambda_1, \ldots, \lambda_k)t^i = \prod_{j=1}^{k} \frac{1}{1 - t\lambda_i} = \frac{1}{\det(I - t\rho'(m))}.$$
Therefore, applying \([2]\),

\[
g(t) = \sum_{i=0}^{\infty} a_i t^i = \sum_{i=0}^{\infty} \theta_{S^i(V)}(f) t^i = \sum_{m \in eMe} c_m \sum_{i=0}^{\infty} \theta_{S^i(V)}(m) t^i
\]

\[
= \sum_{m \in eMe} \frac{c_m}{\det(I - t\rho(m))} = \sum_{j=1}^{s} \frac{b_j}{t^n q_j(1/t)} = \frac{b_1}{(1-t)^k} + \sum_{j=2}^{s} \frac{b_j}{t^n q_j(1/t)}
\]

Since, for all \(j = 2, \ldots, s\) with \(b_j \neq 0\), the polynomial \(t^n q_j(1/t)\) has degree \(k\) and not all roots equal to 1 and since \(b_1 = c_e \neq 0\), we conclude that \(g(t) \neq 0\) and \(g(t) = h(t)/q(t)\) where \(\deg q(t) \leq ks \leq \dim V \cdot s = r\). Thus the sequence \(a_i\) is not identically zero and satisfies a recurrence of degree \(r\), and hence there exists \(0 \leq i \leq r - 1\) such that \(a_i \neq 0\). By Lemma 1 we conclude that \(S\) is a composition factor of one of \(S^0(V), \ldots, S^{r-1}(V)\). \(\square\)

Remark 4. Using Newton’s identities, the characteristic polynomial of \(\rho(m)\) is determined by \(\theta_{V}(m), \ldots, \theta_{V}(mn-1)\) where \(n = \dim V\), and hence \(s\) can be bounded in terms of the number of values assumed by \(\theta_{V}\).

Remark 5. Let \(V\) and \(N_t\) be as in Remark 2. Then there are only two distinct characteristic polynomials for elements of \(N_t\) acting on \(V\) because every non-identity element of \(N_t\) acts as a nilpotent operator. But, for any fixed \(k\), \(\bigoplus_{i=0}^{k} S^i(V)\) cannot be a faithful \(\mathbb{C}N_t\)-module for \(t\) sufficiently large by dimension considerations. Thus the smallest \(k\) giving a faithful module for the monoid algebra cannot be bounded in terms of just \(\dim V\) and the number of different characteristic polynomials, as is the case for finite groups.

Acknowledgments

Thanks are due to Nicholas Kuhn, who pointed out to me his results [10, 11], which led me to consider symmetric powers.

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Department of Mathematics, City College of New York, Convent Avenue at 138th Street, New York, New York 10031, USA

E-mail address: bsteinberg@ccny.cuny.edu