THE NO-DEFECT CONJECTURE IN COSMIC CRYSTALLOGRAPHY

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The topology of space is usually assumed simply connected, but could be multi-connected. We review in the latter case the possibility that topological defects arising at high energy phase transitions might still be present and find that either they are very unlikely to form at all, or space is effectively simply connected on scales up to the horizon size.

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I. INTRODUCTION

The occurrence of topological defects (TD) during the early universe phase transitions depends only on the topology of the vacuum manifold \( G/H \) when a large symmetry \( G \) is broken down to a smaller one \( H \) (including \( SO(3) \times U(1) \) if the model is to be valid at low energies). If \( G/H \) is disjoint \( \pi_0(G/H) \not\sim \{1\} \) then domain walls must form, while strings or monopoles appear respectively in the cases \( \pi_1(G/H) \not\sim \{1\} \) and \( \pi_2(G/H) \not\sim \{1\} \) \( \mathbb{Z}_2 \), with \( \pi_n \) the \( n \)th homotopy group of the vacuum manifold seen as a topological space. If \( G \) is compact and simply connected, which is usually assumed to be the case for Grand Unified Theories (GUT) in order to have only one single coupling constant, then the experimental fact that electromagnetism \( [U(1)] \) is unbroken leads to the prediction that monopoles must have formed \( \mathbb{Z}_2 \), and with a number density much too large to be compatible with the evidence that the universe still exists at all \( \mathbb{R} \). This observation, together with the horizon problem, led to the idea of inflation \( \mathbb{R} \). Besides, cosmic strings and domain walls have been shown to have the ability to generate density fluctuations that could lead to large scale structure formation, leaving an observable imprint in the microwave background in so doing \( \mathbb{R} \). Hence, the possible existence of TDs is not a mere speculative idea but is seen on the contrary to have cosmological consequences that are worth investigating (see Ref. \( \mathbb{R} \) for a review). So the question of whether they exist or not demands an answer.

On the other hand, the standard framework in which cosmology is studied is that of connected universe, an hypothesis essentially based on a principle of simplicity: since general relativity, being a differential and therefore local theory, says nothing about the global, topological aspect of space, it is natural to consider the universe as endowed with the simplest possible structure and therefore it is considered simply connected \( \mathbb{R} \). However, it has been argued that since quantum gravity possibly allows changes of topology \( \mathbb{R} \) and because the probability for creation of a universe decreases with the volume of the universe \( \mathbb{R} \), a multi connected space with less volume than a simply connected one is at least not less probable and could therefore be actually realized \( \mathbb{R} \). These models are for the time being constrained but not excluded \( \mathbb{R} \). It is the purpose of this letter to show that a definite observation of a single TD would provide a very strong constraint on cosmological theories not based on a simply connected universe. Conversely, if by some other means one was able to prove the multi-connectedness of the universe, then our result states that the formation of stable topological defects should be seriously reconsidered, the probability that we ever observe one being considerably reduced. This gives an additional insight into the mechanism for the symmetry breaking, including the dynamics of the phase transition itself.

It appears therefore that testing the topology of the universe, i.e. its properties on large, cosmological scales, provides some information about the nature of particle physics, and in particular reveals aspects of the possible phenomenology otherwise inaccessible.

The article is organized as follows: in the next section, we derive the argument according to which TDs are incompatible with a multi-connected space. More precisely, we argue that long-lived TDs have a very low probability of being formed, the simplest possible configurations involving at least two of them in a definite state. Then, we specialize the discussion to more physical arguments to conclude on the actual probability that we ever observe a TD in a multi connected universe.

II. DEFECTS IN NONTRIVIAL TOPOLOGICAL SPACES.

We shall examine in turn the cosmic string, monopole and domain wall cases, and for strings and wall, we consider only those lying along incontractible directions, the other ones being unstable against decay into elementary particles. We discuss them in Sec. III.

Let us first consider the topology of space in some details in order to fix the notation. A multi connected space is conveniently described by its fundamental polyhedron \( \mathcal{P} \) which is convex with a finite number of faces \( \{F\} \) identified by pairs, together with the holonomy group \( \Gamma \) con-
sisting in the collection of transformations $\gamma$ which carry a face to its homologous face. An important point concerning the group $\Gamma$ is that none of its generator (except the identity) can have any fixed point $[1]$. Moreover, we shall consider that the phase transition is driven by a Higgs field $\Phi$ taking values in $G/H$. For a domain wall resulting of the breaking of a discrete symmetry, we shall assume $\Phi$ real and taking values in $\{-1, 1\}$. For a string $\Phi$ is a complex field whose phase $\sigma$ should wind $n$ times around some line and for a monopole we assume $\Phi$ to be a vector in an internal three dimensional space. With these notations in mind, we can now turn to the different cases.

\section{A. Cosmic Strings and monopoles.}

We begin with the case of a cosmic string lying along an incontractible direction $L$ of the fundamental polyhedron. We assume in this case, as in the monopole case to be seen later, that the spatial section of the universe is orientable. This restriction is necessary for the proof, and can be justified on the ground that one wants CPT to be conserved while CP to be possibly violated (as is observed to be the case experimentally) $[12]$. Besides, it is also necessary in order to have well-defined spinors in all space as is needed in particle physics where almost all the particle are fermions $[13]$. Note this restricts our analysis to non-twisted field theories $[14]$. Let us consider the intersection $L$ of the fundamental polyhedron $P$ with an arbitrary 2-surface $\Pi$ such that the phase of the Higgs field responsible for the potential compactness of the space, must vanish. Hence there can be only an even number of monopoles trapped in the fundamental cell; and there should be in fact an equal number of monopoles and anti-monopole. We discuss the implication of this fact in Sec. III.

\section{B. Domain Walls.}

Even though domain walls are conceptually simpler to represent than the two previous TDs, the proof of their inexistence in multi connected spaces is slightly more involved and we now turn to it. Also as in the previous cases, we do not consider walls entirely contained within the volume of the fundamental polyhedron, since those can trivially exist and anyway will decay in less than a Hubble time after their formation.

$\Phi$ has values on each face of the fundamental polyhedron, according to which we can classify the various faces into subsets:

$$S^\pm \equiv \{F; \forall M \in F, \langle \Phi \rangle (M) = \pm 1\}$$

and

$$S^0 \equiv P - (S^+ \cup S^-).$$

Then because $\forall \gamma \in \Gamma$, one must impose $\langle \Phi \rangle (\gamma M) = \langle \Phi \rangle (M)$, it should be clear that the spaces $S^{\pm, 0}$ are stable under $\Gamma$ and are connex.

We now define $S^1$ as the subset of $S^+$ including only those faces having a boundary in common with a face in $S^0$. Since $S^0$ is stable under $\Gamma$, so is $S^1$, and it is not empty. Let us now define $S^{1+} = S^+ - S^1$. Then $S^{1+} \subset S^+$ and $\forall \gamma \in \Gamma, \gamma S^{1+} = S^{1+}$. Similarly we define $S^2$ as the subset of $S^{1+}$ which is the set of faces having boundaries with faces in $S^1$. Again, $S^2$ is stable under the action of $\Gamma$ and is not empty. By induction, it is then possible to define a series $S^{n+}$ such that

$$\begin{cases} S^{1+} \subset S^{(n-1)+} \\ \text{Card } S^{n+} \neq 0 \end{cases}$$

since $\text{Card } S^+ < \infty$. Thus, either $\Gamma \sim \{\text{Id}\}$ and space is simply connected, or $\exists \gamma \in \Gamma$ with a fixed point which is impossible since $\Gamma$ is a holonomy group.

Let us turn now to the more complicated case of two domain walls in the volume. If the two walls cross the polyhedron on different faces, then the previous analysis applies on two disjoint subsets and the result is the same. This is also true if they cross the same faces of the polyhedron while not crossing each other on the face. However the situation is different when the intersections of the walls and the face have a common point. In this case, let us define the subsets.
\[ S^{00} \equiv \{ \mathcal{F} \in \mathcal{P}; \Pi_1 \cap \mathcal{F} \neq \emptyset \text{ and } \Pi_2 \cap \mathcal{F} \neq \emptyset \} , \]

\[ S^0 \equiv \{ \mathcal{F} \in \mathcal{P}; \Pi_1 \cap \mathcal{F} \neq \emptyset \text{ or } \Pi_2 \cap \mathcal{F} \neq \emptyset \} , \]

and

\[ S^{\pm} \equiv \{ \mathcal{F}; \forall M \in \mathcal{F}, (\Phi)(M) = \pm 1 \} . \]

Then the previous proof applying for one wall applies straightforwardly upon the substitution

\[
\begin{align*}
S^0 & \longrightarrow S^{00} \\
S^+ & \longrightarrow S^0.
\end{align*}
\]

A generalization to an arbitrary number of walls turned out not to be feasible in a simple way, so we can only conjecture this result to apply also in the case of arbitrary many walls. Fig. 2 gives an intuitive understanding in the case of one domain wall in a 3-torus.

### III. PHYSICAL CONSIDERATIONS.

The basic assumption regarding the possibility of a nontrivial topology for the universe is that it is made up of a collection of multi-connected spatial sections, all of them obeying Einstein’s equations as well as the cosmological principle of homogeneity and isotropy. The observable universe itself is identified with the universal covering of the spatial section \( \mathcal{S} \), so that to each point in \( \mathcal{S} \) corresponds infinitely many points in \( \mathcal{U} \) for a multi-connected \( \mathcal{S} \). This property is the one that has been used intensively \cite{11,15} to try and detect multi-connectedness.

The spatial section has a characteristic length scale \( L \) that scales like the metric. In the case where topology comes from the quantum to classical gravity transition, one may expect \( L \) to be of the order of the Planck length \( \ell_P \) at the Planck time \( t_P \) (note however the possibility of some other length scale, namely the inverse square root of the cosmological constant, for the actual spatial extension of the fundamental cell \cite{16}; this possibility is not very well established yet and we shall therefore not consider it any further, although it should be kept in mind for definite conclusion to be drawn). Then the leading behavior of \( L \) is given by the scale factor \( a(t) \) [normalized to unity at the Planck time]:

\[ L \sim \ell_P a(t) , \]

independently of the epoch considered.

For topological defects to be produced at a phase transition, one needs to investigate the values of the field responsible for the symmetry breaking over distances larger than its correlation length \( \xi \) which is of the order of the inverse temperature \( T_{PT} \) at the phase transition:

\[ \xi \sim T_{PT}^{-1} \sim \ell_P a(t) , \]

so that the ratio scales like

\[ \frac{\xi}{L} \sim 1 , \]

which is valid as long as the defect forming phase transition takes place at a time where temperature scales like \( a(t)^{-1} \). This is true in particular before the inflationary phase (if any) and/or prior to reheating. Hence we expect in this case only a few TDs to be formed at the phase transition.

Monopoles can only be formed pairwise with vanishing total index, which means a quite special field configuration. Thus, it should be clear that the probability to have monopole forming at the phase transition is in fact much smaller than that of not forming monopoles at all. In standard cosmology, one considers various correlation volumes in which all the possible states are physically realized, and therefore one expects roughly one monopole per correlation volume on average. In the compact case however, there is only one such realization and therefore no ergodic principle can apply. The conclusion is this case can be based on the anthropic principle \cite{17}; as monopoles are a cosmological nuisance in order for the universe to exist as such still now, it can be deduced with a good confidence level that monopoles were simply not formed and the highest probability was realized. If ever we observe one (and at any energy scale), then we will have to conclude that either the universe is simply connected, or the phase transition leading to their appearance took place after reheating (in which case it is necessary that the correlation length \( L_{PT} \) be much smaller than the cell size \( L \)), or the length scale at the Planck time was much greater that the Planck length.

Let us turn to extended defects, namely cosmic strings and domain walls. Still in the hypothesis of \( L = \ell_P \) at \( t = t_P \), then inflation is necessary in multi-connected universe models, for otherwise the size of the universe now would be, assuming scaling of \( L \), roughly 350 km, in obvious contradiction with the constraint \cite{11} \( L \gtrsim 350 \text{ Mpc} \). But extended defects can be of two different kinds: either they are contractible in the spatial topology, in which case they will decay very rapidly and thus have very little cosmological relevance, or they are aligned along incontractible directions, so they are topologically preserved as the universe expands. The latter, as we have seen, are quite improbable for topological reasons (and we even conjecture walls to be actually excluded).

Let us see however the implication of an unambiguous observation of a cosmic string. If inflation was such that the cell size now is much greater than the horizon, and if the condition of Eq. \cite{12} was fulfilled, then we expect at most a few strings in the entire universe and the probability of observing one almost vanishes. Thus, either we are incredibly lucky, or one of our hypothesis is wrong. For instance, it could be that \( L \) is much greater than the correlation length at the time of symmetry breaking so that the ergodic hypothesis applied. However, even in that case, one can convince oneself that it is not only the field distribution on scales of the order of the correlation...
length that matters because the total configuration must be such that the total winding number vanishes. For a large number of strings $N$, this requires a phase distribution whose probability goes like some inverse power of $N$ (depending on the topology of the spatial section). Hence, we are led back to the conclusion that space is simply connected.

The last possibility is then that the characteristic length $L$ now exceeds the horizon so that TDs smaller than $L$ did not yet all decay. Then multiconnectedness is an irrelevant hypothesis.

IV. CONCLUSIONS.

We have proved that a single topological defect, i.e. a domain wall, cosmic string or a monopole, cannot appear at a phase transition in a multi-connected universe. This is based on purely topological considerations and is therefore completely model-independent. The only possibility left for TDs to appear are pairwise for cosmic strings and monopoles, and we conjecture walls cannot form at all. We believe this is linked with the fact that walls appear as a result of the breaking of a finite symmetry group, just like the homology group $\Gamma$, whereas monopoles and strings are created when a continuous group is broken.

Extended defects can exist however if they are completely inside the volume of the spatial section. Because this spatial section is small at the phase transition, we expect only a few such TDs to be formed anyway, and therefore, if ever created, they decayed almost instantaneously so that they can’t be of any cosmological relevance. An observation of a TD would thus be a strong indication that the universe is simply connected.

Another point worth mentioning is the question of monopoles: we have seen that they are not expected to form, independently of the model that gives them birth. However, the vacuum manifold in GUT supports their creation, which is in fact a problem for GUT models, at least all these models having the standard (and observed) $SU(3) \times U(1)$ as a low energy invariance limit. Inflation is usually invoked to get rid of these monopole. We see here that a multi-connected spatial topology does the same work.

Yet, we have implicitly supposed that the spatial characteristic length $L$ is of the order of the Planck length at the Planck time, an hypothesis justified by quantum gravity. However, it could be that this is not a correct hypothesis and that the spatial section can be anything, in particular the inverse square root of the cosmological constant has been suggested as long as it scales with expansion. In this case, our result states that we still can observe TD if $L$ is at least greater than the horizon size. But in this case, this means that the universe is simply connected up to the scale of the horizon, and there is very little chance we can test multiconnectedness observationally.

Finally we should like to mention yet another possibility, namely that if by some other means one proves the universe to be multi-connected. That would mean that TD are very unlikely to exist. However, it would not imply any constraint on particle physics models. A final point to mention in this case is that such a multi-connected universe would not contain any monopole. This is a way out for the monopole problem not constraining GUT models.

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Figure 1: Cosmic strings on a 3–torus. Identical faces are shown with identical patterns, such as (ABCD) and (A’B’C’D’), and two different kinds of strings are present, namely $\Gamma$ which is entirely contained in the volume (and represents a decaying configuration because of the string tension), and the line connecting $\mathbf{M}$ to $\mathbf{M}'$ (both points being identified). That the long string configuration is not possible is seen as follows: the phase variation between $\mathbf{A}$ and $\mathbf{B}$ is the same as between $\mathbf{D}$ and $\mathbf{C}$ given that these points are identified by pairs, and therefore minus the phase variation between $\mathbf{C}$ and $\mathbf{D}$. The same applies between $\mathbf{B}$ $\mathbf{C}$ and $\mathbf{A}$ $\mathbf{D}$, so the total phase winding around the up face vanishes. And this is true on any face, so no string can cross the entire volume.
Figure 2: The wall separates the fundamental polyhedron of the 3–torus. $<\phi>=+1$ on (DCC'D') and $-1$ on (ABB'A'). It is obvious that this two faces cannot be homologous to any of the other faces.