On the Existence and Long-Term Stability of Voltage Equilibria in Power Systems with Constant Power Loads

Alexey S. Matveev, Juan E. Machado, Romeo Ortega, Fellow, IEEE, Johannes Schiffer and Anton Pyrkin, Member, IEEE

Abstract—Voltage instability is a major threat in power system operation. The growing presence of constant power loads significantly aggravates this issue, hence motivating the development of new analysis methods for both existence and stability of voltage equilibria. Formally, this problem can be cast as the analysis of solutions of a set of nonlinear algebraic equations of the form \( f(x) = 0 \), where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \), and the associated differential equation \( \dot{x} = f(x) \). By invoking advanced concepts of dynamical systems theory and effectively exploiting its monotonicity, we exhibit all possible scenarios for existence, uniqueness and stability of its equilibria. We prove that, if there are equilibria, there is a distinguished one that is locally stable and attractive, and we give some physically-interpretable conditions such that it is unique. Moreover, a simple on-line procedure to decide whether equilibria exist or not, and to compute the distinguished one is proposed. In addition, we show how the proposed framework can be applied to long-term voltage stability analysis in AC power systems, multi-terminal high-voltage DC systems and DC microgrids.

Index Terms—Power systems, existence of equilibria, constant power loads.

I. INTRODUCTION

A \textit{sine qua non} condition for the correct operation of power systems is the existence of a steady-state behavior that, moreover, should be robust in the presence of perturbations [1]. Viewed as dynamical systems, described with differential equations, this requirement translates into the existence of equilibria, which should also be stable and attractive. The accurate description of modern power systems necessarily incorporates “strong” nonlinear effects, complicating the task of analysis of its equilibria.

Variables of particular importance in both AC and DC power systems are the voltage magnitudes at the different nodes of the system. In fact, during the past decades an increasing number of incidents can be attributed to fast and slow voltage variations [2], [3]. Hence, voltage stability analysis has significantly gained in relevance in AC power systems [1], [2]. In DC power systems the voltage magnitudes can be considered even more relevant, since—in the absence of a system frequency—variations in the system loading always have a direct impact on the DC voltages [5].

In this paper we derive a methodological approach, which permits to determine existence and stability properties of voltage equilibria in a broad range of power system applications. More precisely, we show that our proposed approach is applicable to analyze the steady-state voltage behavior of traditional AC power systems [1], [8] as well as of two emerging power system concepts, namely multi-terminal high-voltage (MT-HV) DC networks [6], [5] and DC microgrids [7], [8].

In addition, if stationary voltage solutions exist our method also allows to identify the solution with the highest voltage magnitudes as well as to assert its long-term stability properties. Following standard practice [9], [10], [11], [12], the latter notion is defined in terms of the eigenvalues of the Jacobian of the algebraic power system equations evaluated at a stationary solution.

In all the examples mentioned above, the key problem is the study of a nonlinear algebraic equation \( f(x) = 0 \in \mathbb{R}^n \) in \( x \in \mathbb{R}^n \), where only solutions \( x \) with positive components are of interest. The approach adopted in the paper to tackle these problems is to associate to \( f(x) \) the ordinary differential equation (ODE) \( \dot{x} = f(x) \), and to apply to it tools of dynamical systems [13] to study existence and stability of its equilibria, which are nothing but the solutions of the primal algebraic equation.

The main contributions of our work are the proofs of the following properties of the ODE.

C1. If there are no equilibria (stable or unstable) then, in all solutions of the ODE, one or more components converge to zero in finite time.

C2. If equilibria exist, there is a distinguished equilibrium, say \( \bar{x}_{\text{max}} \), among them that dominates component-wise all the other ones. This equilibrium \( \bar{x}_{\text{max}} \) is locally stable and attracts all trajectories that start in a certain well-defined domain.

C3. By solving a system of \( n \) convex algebraic inequalities in \( n \) positive unknowns we explicitly identify a set of initial states with the following characteristics: (i) all trajectories starting there monotonically decay in all components; (ii) they either have at least one component that converges to zero in finite time or none of them does. Moreover,
in the latter case, the trajectory is forward complete and converges to \( \bar{x}_{\text{max}} \).

Clearly, the contribution C3 suggests a simple on-line computational procedure to answer the questions raised in the paper: find some solution of the convex inequalities mentioned in C3, run a simulation of \( \dot{x} = f(x) \) starting from this set, and check whether there is a component of the trajectory that converges to zero in finite time and, if not, find the limit state \( \bar{x}_{\text{max}} \) of the trajectory, which is an asymptotically stable equilibrium. An additional contribution is to give physically-interpretable conditions on the problem data that ensure \( \bar{x}_{\text{max}} \) is the only stable equilibrium.

The remainder of the paper is organized as follows. Section II describes the ODE \( \dot{x} = f(x) \) of interest and gives the main theoretical results pertaining to it. In Section III we illustrate these results with three canonical power systems examples. Section IV presents some numerical simulation results. The paper is wrapped-up with concluding remarks in Section V.

To enhance readability, all proofs of the technical results are exhaustively described in Figure 1.

**Notation** \((\cdot)^{\top}\) denotes transposition, \(\mathbb{R}\) is the real line, \(\mathbb{R}^n\) is the Euclidean space of vector columns \(x = (x_1, \ldots, x_n)^{\top}\), its positive orthant is denoted as \(\mathbb{K}_+^n := \{x \in \mathbb{R}^n : x > 0\}\), stack \((p_i) \in \mathbb{R}^{r_1 + \cdots + r_N}\), denote stacking \(p_i \in \mathbb{R}^{r_i}, i \in \{1, \ldots, N\}\) on top of one another, \(\text{diag}(A_1, \ldots, A_k)\) is the block-diagonal matrix composed of the listed square blocks \(A_i\). Inequalities between vectors \(x, y \in \mathbb{R}^n\) are meant component-wise. All mappings are assumed smooth. Given a mapping \(f : \mathbb{R}^n \to \mathbb{R}^n\) we denote its Jacobian by \(\nabla f(x) := \frac{\partial f(x)}{\partial x}\). The operator \((\cdot)^{\ast}\) denotes the clipping function \((a) = \max\{a, 0\}\).

II. ANALYSIS OF THE ODE OF INTEREST

As indicated in the introduction, in this paper we are interested in the steady-state voltage solutions of AC power systems (under the common decoupling assumption I). MT-HVDC networks as well as DC microgrids. In Section III it is shown that this study boils down to the analysis of solutions of the following algebraic equation

\[
A \bar{x} + \text{stack} \left( \frac{b_i}{x_i} \right) - w = 0
\]

(1)

where \(\bar{x} \in \mathbb{K}_+^n\). Here \(A \in \mathbb{R}^{n \times n}\), \(b_i \in \mathbb{R}\), and \(w \in \mathbb{R}^n\) are given and satisfy the following.

**Assumption 2.1:** The matrix \(A\) is symmetric and positive definite, all its off-diagonal elements are non-positive and \(b_i \neq 0\) for all \(i\).

To study the solutions of (1) we consider the following ODE

\[
\dot{x} = f(x) := -Ax - \text{stack} \left( \frac{b_i}{x_i} \right) + w,
\]

(2)

and we are interested in studying the existence, and stability, of the equilibria of (2). In particular, we will provide answers to the following questions.

**Q1** When do equilibria exist? Is it possible to offer a simple test to establish their existence?

**Q2** If there are equilibria, is there a distinguished element among them?

**Q3** Is this equilibrium stable and/or attractive?

**Q4** If it is attractive, can we estimate its domain of attraction?

**Q5** Is it possible to propose a simple procedure to compute this special equilibrium using the system data \((A, b, w)\)?

**Q6** Are there other stable equilibria?

Instrumental to provide answers to the questions Q1—Q6 is the fact that the system (2) is monotone. That is, for any two solutions \(x_a(t), x_b(t)\) of (2), defined on a common interval \([0, T]\), the inequality \(x_a(0) \leq x_b(0)\) implies that \(x_a(t) \leq x_b(t) \forall t \in [0, T]\). This can be easily verified by noticing that equation (2) satisfies the necessary and sufficient condition for monotonicity [13] Proposition 1.1 and Remark 1.1, Ch. III

\[
\frac{\partial f_i(x)}{\partial x_j} \geq 0, \forall i \neq j, \forall x \in \mathbb{K}_+^n.
\]

In the sequel, we denote by \((x(t), x_0)\) the solution of (2) with initial conditions \(x(0) = x_0 > 0\), and use the following.

**Definition 2.1:** An equilibrium \(\bar{x} > 0\) of (2) is said to be globally attractive from the right if for any \(x_0 \geq \bar{x}\), the solution \(x(t, x_0)\) is defined on \([0, \infty)\) and converges to \(\bar{x}\) as \(t \to \infty\). The equilibrium is said to be hyperbolic if the Jacobian matrix \(\nabla f(\bar{x})\) has no eigenvalue with zero real part [13].

A. The simplest example

To gain an understanding of some key traits of possible results, it is instructive to start with the simplest case \(n = 1\).

Then, \(x \in \mathbb{R}\) and (2) is the scalar equation

\[
\dot{x} = -ax - b\frac{x}{x} + w,
\]

(3)

where \(a > 0, b \neq 0\). Feasible behaviors of the system are exhaustively described in Figure 1.

The following can easily be inferred from this figure:

**p.1** The system has no equilibria, it has finitely many equilibria, or a single equilibrium.

**p.2** If the system has equilibria, the rightmost of them is globally attractive from the right.

**p.3** Non-hyperbolic equilibria may be globally attractive from the right but are not locally stable; apart from such equilibria, there may be no other ones.

**p.4** Hyperbolic and globally attractive from the right equilibria are locally stable.

**p.5** If \(b > 0\), globally stable equilibria do not exist.

We will show below that several of the traits mentioned above are inherited by the \(n\)-th order ODE (2).

B. A generic assumption

Situation p.3 above is, clearly, undesirable. Since this can happen in the general case—e.g., considering a diagonal matrix \(A\)—it is reasonable to exclude its possible appearance.

**Assumption 2.2:** There are no non-hyperbolic equilibria of the system (2). This is, clearly, equivalent to assuming that the following set identity holds

\[
\{x \in \mathbb{K}_+^n \mid \det \begin{bmatrix} A - \text{diag} \left( \frac{b_i}{x_i} \right) \end{bmatrix} = 0, w = Ax + \text{stack} \left( \frac{b_i}{x_i} \right) \} = \emptyset.
\]

(4)
Case $b < 0$

- Unique globally attractive equilibrium $\bar{x}_s$.
- All solutions converge to zero in a finite time.
- Two equilibria, the smallest of which $\bar{x}_u$ is unstable, whereas the larger one $\bar{x}_s$ is locally stable and globally attractive from the right.

Case $b > 0$

- $w - 2\sqrt{ab} < 0$:
  - $\lim_{t \to t_f} x(t) = \bar{x}_u$.
  - $\lim_{t \to \infty} x(t) = \bar{x}_s$.

- $w - 2\sqrt{ab} > 0$:
  - $\lim_{t \to t_f} x(t) = \bar{x}_u$.
  - $\lim_{t \to \infty} x(t) = \bar{x}_s$.

Fig. 1: Feasible behaviors of the one-dimensional system (5).

The lemma below proves that Assumption 2.2 is almost surely true, hence it is done without loss of generality. The proof of the lemma is given in Appendix A.

Lemma 2.1: For any given $A$ and $b_i \neq 0$, the set of all $w \in \mathbb{R}^n$ for which Assumption 2.2 does not hold has zero Lebesgue measure and is nowhere dense.

C. Main results on system (2)

The first proposition contains a qualitative analysis of the system.

Proposition 2.1: Consider the system (2), verifying Assumptions 2.1 and 2.2. One and only one of the following two mutually exclusive statements holds.

1. There are no equilibria $\bar{x}$, either stable or unstable, and any solution $x(\cdot)$ is defined only on a finite time interval $[0, t_f) \subset [0, \infty)$, since for any of them, there exists at least one coordinate $x_i$ such that $x_i(t) \to 0$, $\dot{x}_i(t) \to -\infty$ as $t \to t_f$. Such a coordinate is necessarily associated with $b_i > 0$.

2. There exist one or finitely many equilibria $\bar{x}_k$. One of them $\bar{x}_{\text{max}} > 0$ verifies $\bar{x}_{\text{max}} \geq \bar{x}_k$, $\forall k$, and this equilibrium is locally stable and attractive from the right. If all $b_i$’s are of the same sign, then in the case s.2, there are no other locally stable equilibria apart from $\bar{x}_{\text{max}}$.

The proof of this proposition is given in Appendix D.

The next proposition provides a constructive test to identify which of the cases s.1) or s.2) holds, as well as a method to find $\bar{x}_{\text{max}}$ in the case s.2). To articulate the result, we introduce the following.

Definition 2.2: A solution $x(\cdot)$ of the differential equation (2) is said to be characteristic if its initial condition lives in the set

$$E := \left\{ x \in \mathbb{R}^n_+ \mid Ax > \text{stack} \left( \langle w_i \rangle + \left\langle \frac{-b_i}{x_i} \right\rangle \right) \right\}. \quad (5)$$

If all coefficients $b_i > 0 \forall i$, the set $E$ reduces to the (convex open polyhedral) cone $\{ x \in \mathbb{R}^n_+ \mid Ax > \text{stack}(\langle w_i \rangle) \}$. □□□

Proposition 2.2: Consider the system (2) verifying Assumptions 2.1 and 2.2.

I) The set $E$ is non-empty, consequently there are characteristic solutions.

II) All characteristic solutions $x(\cdot)$ strictly decay, in the sense that $\dot{x}(t) < 0$, for all $t$ in the domain of definition of $x(\cdot)$.

III) One and only one of the following two mutually exclusive statements holds for all characteristic solutions $x(\cdot)$:

(i) For a finite time $t_f \in (0, \infty)$, some coordinate $x_i(\cdot)$ approaches zero:

$$x_i(t) \to 0 \quad \text{as} \quad t \to t_f, \quad (6)$$

and the solution $x(\cdot)$ is defined only on the finite time interval $[0, t_f)$.

(ii) There is no coordinate approaching zero, the solution is defined on $[0, \infty)$, and the following limit exists and verifies

$$\lim_{t \to \infty} x(t) > 0. \quad (7)$$

This limit is the same for all characteristic solutions.

IV) If the case (i) holds for a characteristic solution, the situation s.1) from Proposition 2.1 occurs.

V) If the case (ii) holds for a characteristic solution, the situation s.2) from Proposition 2.1 occurs, and the dominant equilibrium $\bar{x}_{\text{max}}$ is equal to the limit (7).

The proof of this proposition is given in Appendix D.

D. A procedure to verify Propositions 2.1 and 2.2

Proposition 2.2 suggests a computational procedure to verify whether the system has equilibria and, if they do exist, to find the dominant one $\bar{x}_{\text{max}}$ among them, which is necessarily stable (and is the only stable equilibrium if all $b_i$’s are of the same sign). Specifically, it suffices to find an element of the set $E$ defined in (5), to launch the solution of the differential equation (2) from this vector, and to check whether—as the solution decays—there is a coordinate approaching zero or, conversely, all of them remain separated from zero. In the last case, the solution will have a limit, which is precisely the stable equilibrium of the system.

1So the case s.1) does not occur if $b_j < 0, \forall j$. 

The proof of this proposition is given in Appendix D.
The statement I of Proposition 2.2 ensures that the first step of this algorithm, i.e., generating an element of the set \( \mathcal{E} \) defined in (5), is feasible. Technically, this step consists in solving the following system of feasible convex inequalities:

\[
\langle w_i \rangle + \frac{-b_i}{x_i} - \sum_{j=1}^{n} a_{ij} x_j < 0, \quad \forall i.
\]

This problem falls within the area of convex programming and so there is an armamentarium of effective tools to solve it. Nevertheless, this problem can be further simplified via transition from nonlinear convex inequalities to linear ones, modulo closed-form solution of finitely many scalar quadratic equations. The basis for this is given by the following lemma, whose proof is given in Appendix B.

**Lemma 2.2:** Pick any vector \( z \) in the set \( \{ x \in \mathcal{K}^+_n | Ax > 0 \} \). Define the scaled vector \( x := \mu z \), where

\[
\mu > \frac{\langle w_i \rangle + \sqrt{\langle w_i \rangle^2 + 4(Az)_i z_i}}{2(Az)_i}, \quad \forall i.
\]

Then, \( x \in \mathcal{E} \).

**E. Some additional properties of system (2)**

**P1** In III(i), there may be several coordinates \( x_i \) with the described property, all coordinates do not necessarily possess it, and different solutions \( x(\cdot) \) may have distinct sets of coordinates with this trait.

**P2** The claim s.1 in Proposition 2.1 and IV in Proposition 2.2 yield that (6) is necessarily associated with \( b_i > 0 \) and \( \dot{x}_i(t) \to -\infty \) as \( t \to t_f \).

**P3** Regarding the claim s.2 in Proposition 2.1 the basin of attraction of the equilibrium \( \bar{x}_{\text{max}} \) is open and has the property that it contains all states \( x \geq \bar{x}_{\text{max}} \).

**P4** The linear programming problem of finding elements in the set \( \{ x \in \mathcal{K}^+_n | Ax > 0 \} \) has been widely studied in the literature [12], [16], [17]. There is a whole variety of computationally efficient methods to solve this problem, including the Fourier-Motzkin elimination, the simplex method, interior-point/bARRIER-like approaches, and many others; for a recent survey, we refer the reader to [18].

**P5** For any \( i \) with \( b_i > 0 \), the inequality (8) clearly simplifies into

\[
\mu > \frac{\langle w_i \rangle}{(Az)_i}.
\]

**III. LONG-TERM VOLTAGE STABILITY ANALYSIS OF SOME CANONICAL POWER SYSTEMS**

In this section we apply the results of Section II to three different types of power systems. These comprise standard conventional AC power systems as well as MT-HVDC networks and DC microgrids—two promising emerging power system concepts. These dynamical systems admit equilibrium points satisfying algebraic constraints that, under standard assumptions, can be written in the form (1) and verifying Assumption 2.1. This permits the use of Propositions 2.1 and 2.2 to study the existence and stability of equilibrium points. Moreover, we can also try the numerical procedure proposed in Subsection II-D to verify the claims of the propositions.

In all these examples, \( x \) represents the vector of voltage magnitudes of the system. Following standard definitions and classifications of voltage stability in AC power systems [9], [10], [11], [12], [1], [2], we introduce the following notion of long-term voltage stability for the system (1), which relates the objectives stated above to standard power system practice.

**Definition 3.1:** A positive root \( \bar{x} \) of the system (1) is long-term voltage stable if the Jacobian \( \nabla f(x) |_{x=\bar{x}} \) with \( f \) given in (2) is Hurwitz, i.e., all its eigenvalues have a negative real part.

Definition 3.1 originates from a sensitivity analysis of the voltage magnitudes with respect to changes in the reactive power flows in AC networks, see [10], [12], [1] and the more recent work [1].

Lemma C.2 in the Appendix implies that the Jacobian of the dynamics (2) evaluated at any stable equilibrium point is Hurwitz. Hence, if case V) of Proposition 2.2 applies then the dominant equilibrium is long-term voltage stable in the sense of Definition 3.1. Consequently, Proposition 2.2 provides a constructive procedure to evaluate the existence of a unique dominant and long-term stable voltage solution in power systems with constant power loads.

**A. Long-term voltage stability in AC power systems**

Consider a high-voltage AC power network with \( n \geq 1 \) nodes. Denote by \( V_i > 0 \) and \( Q_i \) the voltage and the reactive power load demand at the node \( i \), respectively. Under the standard decoupling assumption [1], for each \( i = 1, \ldots, n \), the decoupled reactive power flow, is given by [1], [3], [12]

\[
Q_{\text{ZIP},i} = V_i \sum_{j=1}^{n} |B_{ij}|(V_j - V_k),
\]

where \( B_{ij} < 0 \) if nodes \( i \) and \( j \) are connected via a power line and \( B_{ij} = 0 \) otherwise. The reactive power demand \( Q_{\text{ZIP},i} \) at the \( i \)-th node is described by a so-called, ZIP model, i.e.,

\[
Q_{\text{ZIP},i} := (Y_i V_i^2 + k_i V_i + Q_i).
\]

The term ZIP load refers to a parallel connection of a constant impedance \( Y_i \in \mathbb{R} \), a constant current \( k_i \in \mathbb{R} \), and a constant power \( Q_i \in \mathbb{R} \) load. Then, we obtain the (algebraic) reactive power balance equation

\[
(Y_i V_i^2 + k_i V_i + Q_i) = V_i \sum_{j=1}^{n} |B_{ij}|(V_i - V_j), \quad i = 1, \ldots, n,
\]

(9)
which by defining $x := \text{stack} \left( V_i \right) \in \mathbb{K}_+^n$, $A \in \mathbb{R}^{n \times n}$ with
\[
A_{ii} = \sum_{j=1}^n |B_{ij}| - Y_i, \quad A_{ij} = -|B_{ij}|,
\]
\[
w = \text{stack} \left( k_i \right), \quad b_i = -Q_i,
\]
can be rewritten as (1). If we make the reasonable assumption that $\alpha_i < 0$ for at least one node, $A$ satisfies Assumption 2.1. The reactive power balance (9) has been recently employed in (4) to study long-term voltage stability.

We bring to the readers attention the fact that the coefficients $-b_i$ are the constant reactive powers extracted or injected into the network, being positive (capacitive) in the former case, and negative (inductive) in the latter. As indicated in Section II sharper results—are available if the signs of the coefficients $b_i$ are known. Hence, the proposed conditions have a direct interpretation in terms of reactive power demand.

Another observation is that the solution $\overline{x}_{\text{max}}$ for the system (9) represents the physically admissible steady state for the network with the highest values of voltage magnitudes at each node, which is the usually desired high-voltage operating point.

### B. Multi-terminal HVDC transmission networks with constant power devices

An MT-HVDC network with $n$ power-controlled nodes ($P$-nodes) and $s$ voltage-controlled nodes ($\mathcal{V}$-nodes), interconnected by $m$ RL transmission lines, can be modeled by (20):
\[
\tau \dot{I}_i = -I_i - h(V), \quad L \dot{I} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} I + \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} V, \quad (10)
\]
where $I \in \mathbb{R}^n$, $V \in \mathbb{K}_+^n$, $L \in \mathbb{R}^{m \times m}$ and $R$ and $G$ are diagonal, positive definite matrices of appropriate sizes. The physical meaning of each state variable and of every matrix of parameters is given in Table I. Furthermore, $B = \text{stack} \left( B_V, B_P \right) \in \mathbb{R}^{(s+n) \times m}$ denotes the appropriately split, node-edge incidence matrix of the network. The open-loop current injection at the power terminals is described by
\[
h(V) = \text{stack} \left( P_i, V_i \right),
\]
where $P_i \in \mathbb{R}$ denotes the power setpoint.

As done for the model (12), it can be shown by simple calculations that (10) admits an equilibrium if and only if the system
\[
0_n = -h(\overline{V}) - (B_P R^{-1} B_P^T + G) \overline{V} - B_P R^{-1} B_V^T V, \quad (11)
\]
has real solutions for $\overline{V} \in \mathbb{K}_+^n$. Notice that (11) is equivalent to the right hand side of (2) if we define
\[
x := \overline{V}, \quad A := B_P R^{-1} B_P^T + G, \quad b_i := P_i, \quad w := -B_P R^{-1} B_V^T V.
\]

Note that $B_P$ is an incidence matrix and $R$ and $G$ are diagonal positive definite matrices. Hence, the term $B_P R^{-1} B_P^T$ is a Laplacian matrix and thus it is positive semidefinite. Consequently, $A = A^T$ is positive definite. Hence, Assumption 2.1 is satisfied and the results of Section II can be used to analyze the existence of equilibria of the dynamical system (10). This, through the computation of the solutions of $\dot{x} = f(x)$, taking $f$ as the right hand side of (11).

In this scenario, the coefficients $-b_i$ are the powers extracted or injected into the network, being negative in the former case and positive in the latter.

### C. DC microgrids with constant power loads

A standard Kron-reduced model of a DC microgrid, with $n \geq 1$ converter-based distributed generation units, interconnected by $m \geq 1$ RL transmission lines, can be written as (21):
\[
L_t \dot{I}_t = -R_t I_t - V + u, \quad C_t \dot{V} = I_t + B_I - I_{\text{ZIP}}(V), \quad (12)
\]
where $I_t \in \mathbb{R}^n$, $V \in \mathbb{K}_+^n$, $u \in \mathbb{K}_+^n$ and $I \in \mathbb{R}^m$ as well as $R_t$, $R$, $L_t$, $L$ and $C_t$ are diagonal, positive definite matrices of appropriate size. The physical meaning of each term appears in Table II. We denote by $B \in \mathbb{R}^{n \times n}$, with $B_{ij} \in \{-1, 0, 1\}$, the node-edge incidence matrix of the network. The load demand is described by a ZIP model, i.e.,
\[
I_{\text{ZIP}}(V) = \gamma V + k + \text{stack} \left( \frac{P_i}{\nu} \right),
\]
where $V \in \mathbb{R}^{n \times n}$ is a diagonal positive semi-definite matrix, $k \in \mathbb{R}^n$ is a constant vector, and $P_i \in \mathbb{R}$.

Some simple calculations show that, for a given $u = \bar{u}$ constant, the dynamical system (12) admits a real steady state if and only if, the system
\[
0_n = R_t^{-1} (\bar{u} - V) - B R^{-1} B^T V - I_{\text{ZIP}}(V), \quad (13)
\]

\begin{table}[h]
\centering
\caption{Nomenclature for the model (10).}
\begin{tabular}{|c|c|}
\hline
\textbf{State variables} & \\
\hline
$T$ & $P$-nodes injected currents \\
$V$ & $P$-nodes voltages \\
$I$ & Line currents \\
\hline
\textbf{Parameters} & \\
$L$ & Line inductances \\
$C$ & $P$-nodes shunt capacitances \\
$R$ & Line resistances \\
$G$ & $P$-nodes shunt conductances \\
$\tau$ & Converter time constants \\
$V$ & $\mathcal{V}$-nodes voltages \\
\hline
\end{tabular}
\end{table}
TABLE II: Nomenclature for the model (12).

| State variables | Parameters | External variables |
|-----------------|------------|--------------------|
| \( I \)         | \( L_1 \)  | \( u \)            |
| \( V \)         | \( L \)    | \( \bar{v}_i \)   |
| \( I \)         | \( C \)    | \( k_i \)          |
| \( L \)         | \( R \)    | \( P \)            |
| \( C \)         | \( R \)    | \( P_i \)          |

TABLE III: Simulation Parameters of the multi-port network of Fig. 2

| Parameter | Value |
|-----------|-------|
| \( E \)  | 24    |
| \( r_1 \) | 0.04  |
| \( L_1 \) | 78    |
| \( C_1 \) | 2     |
| \( r_2 \) | 0.06  |
| \( L_2 \) | 98    |
| \( C_2 \) | 1     |

Fig. 2: DC Linear RLC circuit with two CPLs.

This set can also be written, in the simpler form

\[
\mathcal{E} = \left\{ x \in \mathbb{K}_+^n \mid E < x_1 < x_2 < \frac{(r_1 + r_2)}{r_1} x_1 - \frac{r_2 E}{r_1} \right\}.
\]

A portion of this set, for the the values of the parameters given in Table III, is shown in Fig. 3a, together with a characteristic solution for \( \dot{x} = f(x) \).

Next, we verify numerically the procedure to test the existence of equilibria of the system (5) suggested in Subsection II-D. Namely, taking an initial condition from the set \( \mathcal{E} \), we integrate the ODE to test whether one on the components of the state converges to zero in finite time, in which case there are no equilibria. On the other hand, if no component goes to zero, there are equilibria, and the trajectory will asymptotically converge to \( \bar{x}_{\text{max}} \). Notice that, according to Proposition 2.1, since the coefficients \( b_i > 0 \) this is the only equilibrium of the system.

Now, we recall that in [22, Proposition 1 and 3], an LMI characterization for the existence of real solutions for (14) is given. Using this test, we obtain the set of (positive) values for \( (b_1, b_2) \) for which there exists an equilibrium for the network— for any pair \( (b_1, b_2) \) outside this region the equilibrium does not exist. The set of admissible powers is indicated by the shadowed region shown in Fig. 3b.

Next, we compute the solutions of the ODE (2) in two scenarios. In the first case, we take \( (b_1, b_2) = (500, 450) \), which belongs to a feasible set according to Fig. 3a, then, the network has an equilibrium. We take the initial condition \( x_0 = (25.01, 25.77) \in \mathcal{E} \), and notice that none of the components of \( x(t, x_0) \) approach zero—hence, we have the case III.(ii) of Proposition 2.2 and \( x(t, x_0) \) converges asymptotically to the unique equilibrium \( \bar{x}_{\text{max}} = (22.24, 20.95) \), as shown in Fig. 3c.

On the other hand, in Fig. 3d, we show the evolution of the same characteristic solution \( x(t, x_0) \), but now taking \( (b_1, b_2) = (3000, 1000) \), which is outside the darkened region of the Fig. 3b, implying that the network admits no equilibria. Clearly, \( x_2(t, x_0) \) converges to zero in finite time, as predicted by the case III.(i) of Proposition 2.2.

Lastly, in Fig. 3e we present the plot of the characteristic solution \( x(t, x_0) \) for the two scenarios just described, i.e., with \( (b_1, b_2) = (500, 450) \), which is feasible, and with \( (b_1, b_2) = (3000, 1000) \) which is infeasible.

A. An RLC circuit with constant power loads

Consider the electrical network shown in Fig. 2 which has been previously studied in [22] as a benchmark example. Its steady state is described by the system of quadratic equations

\[
\begin{align*}
0 &= -Y u + \bar{v}_1, \\
0 &= v_i z_i - P_i, \quad i = 1, 2, \\
0 &= I_c z_i - P_i, \quad i = 1, 2,
\end{align*}
\]

where \( z_i \) is the current of the inductor \( L_i \) and \( v_i \) is the voltage of the capacitor \( C_i \) and

\[
Y = \begin{bmatrix}
\frac{1}{r_1} + \frac{1}{r_2} & -\frac{1}{r_2} \\
-\frac{1}{r_1} & \frac{1}{r_2}
\end{bmatrix}, \quad u = \begin{bmatrix}
\frac{E}{r_1} \\
0
\end{bmatrix}.
\]

Defining

\[
x := \begin{bmatrix}
\bar{v}_1 \\
\bar{v}_2
\end{bmatrix}, \quad A := Y, \quad b_i := P_i, \quad w := u,
\]

the algebraic equations (14) can be equivalently written in the form (1).

First, we compute the set \( \mathcal{E} \), given in (5), of initial conditions of the characteristic solutions as

\[
\mathcal{E} = \left\{ x \in \mathbb{K}_+^n \mid \left( \frac{1}{r_2} + \frac{1}{r_1} \right) x_1 - \frac{1}{r_2} x_2 > \frac{E}{r_1}, \quad -\frac{1}{r_2} x_1 + \frac{1}{r_2} x_2 > 0 \right\}.
\]
Fig. 3: Simulation results for the RLC circuit of Fig. 2: (a) plot of a a portion of the set $\mathcal{E}$ and a characteristic solution converging to $x_{\text{max}}$. (b) Set of positive values (shadowed region) for $(b_1, b_2)$ for which the network admits an equilibrium. (c) Characteristic solution $x(t, x_0)$, with $b = (500, 450)$, converging to the equilibrium point $x_{\text{max}}$. (d) Characteristic solution $x(t, x_0)$, taking $b = (3000, 1000)$, with one of its components converging to zero in finite time, the system has no equilibrium points. (e) Phase-space plot of the characteristic solution $x(t, x_0)$ for two different values of $b$: one feasible and another one infeasible. Convergence to $x_{\text{max}}$ is observed in the former (solid curve), and convergence of the second component to zero is visualized in the latter (dashed curve).

TABLE IV: Numerical parameters associated with the edges for the network in Fig. 4

| Transmission line | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|------------------|------|------|------|------|------|
| $r_1$ (Ω)        | 0.9576 | 1.4365 | 1.9153 | 1.9153 | 0.9576 |

Fig. 4: Associated graph for the HVDC network studied in [20, Section V].

B. An HVDC transmission system

In this subsection we numerically evaluate the existence (and approximation) of equilibrium points for the particular HVDC system presented as an example in [20, Fig. 5]. The network, whose associated graph is shown in Fig. 4, consists in four nodes $\mathcal{N} = \{\mathcal{V}_1, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$, where $\mathcal{V}_1$ is a voltage controlled node with voltage $V_{\mathcal{V}_1}^{(1)} = E$, and $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ are power-controlled nodes with power $P_1$, $P_2$, and $P_3$, respectively. The network edges, representing the RL transmission lines, are $c = \{c_1, c_2, ..., c_5\}$, with each $c_i$ having an associated pair of parameters $(r_i, L_i)$. If we assign arbitrary directions to the edges of the graph, then we can define an incidence matrix $B = \text{stack}(B_v, B_p)$, where

$$B_v = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \end{bmatrix},$$

and

$$B_p = \begin{bmatrix} -\frac{1}{r_1} & 0 & 0 \\ 0 & -\frac{1}{r_2} & 0 \\ 0 & 0 & -\frac{1}{r_3} \\ 0 & 0 & 0 & -\frac{1}{r_4} \end{bmatrix}.$$  

Then, the elements of the algebraic system (1), which is codified by $f(x) = 0$, are given by

$$A = \begin{bmatrix} \gamma_1 + \frac{1}{r_1} + \frac{1}{r_5} & 0 \\ -\frac{1}{r_5} & \gamma_2 + \frac{1}{r_2} + \frac{1}{r_4} \\ -\frac{1}{r_5} & \frac{1}{r_3} + \frac{1}{r_2} + \frac{1}{r_4} + \frac{1}{r_5} \end{bmatrix},$$

$b = \text{stack}(P_i)$, $w = \text{stack}(E, E, E)$, with each $r_i$ and $\gamma_i$ are the diagonal elements of the matrices $R$ and $G$, respectively.

Taking the numerical values shown in Tables V and IV we compute—through Lemma 2.2—an initial condition $x_0 \in \mathcal{E}$ given by

$$x_0 = 10^5 \cdot \text{stack}(6.66, 4.66, 5.99).$$

The particular solution $x(t, x_0)$ of $\dot{x} = f(x)$ is shown in Fig. 5. Clearly, none of its components converges to zero. Then, by Proposition 2.2, we establish that the limit of this solution is the dominant equilibrium point, $\bar{x}_{\text{max}}$, of the system. Its value is given by

$$\bar{x}_{\text{max}} = 10^5 \cdot \text{stack}(4.0054, 3.9991, 4.0043).$$

V. CONCLUSIONS

We have shown in the paper that the steady-state equations of several conventional and emerging power systems architectures satisfy a set of nonlinear algebraic constraints with a particular structure, denoted in the manuscript by $f(x) = 0$. 

It was established that the associated ODE \( \dot{x} = f(x) \) is a monotone dynamical system, for which we have described all possible scenarios for existence, uniqueness and stability of its equilibria. It was proven that if an equilibria exist, then, there is a distinguished one, denoted by \( \bar{x}_{\text{max}} \), which dominates—all the other ones and attracts all the ODE trajectories starting from a well-defined domain. We have further provided an algorithm to establish whether solutions of the ODE will converge to \( \bar{x}_{\text{max}} \) or not. By using the above-mentioned motivating correspondence, we have shown that if \( x \) represents the voltage magnitudes in an AC or (HV)DC power system, then \( \bar{x}_{\text{max}} \) corresponds to its unique long-term stable voltage equilibrium.

Finally, we have demonstrated via supporting numerical experiments on two benchmark power system models that our methodology performs very satisfactorily for realistic power system parametrizations.

### Appendix A

#### Proof of Lemma 2.1

The set \( \Xi := \{z \in \mathbb{R}^n : \det (A - \text{diag}(z_i)) = 0 \} \) is clearly closed and for any \( i \) and given \( z_j \)‘s with \( j \neq i \), its section \( \{z_i \in \mathbb{R} : \text{stack} (z_1, \ldots , z_n) \in \Xi \} \) has no more than \( n \) elements. So the Lebesgue measure of \( \Xi \) is zero by the Fubini theorem. The function \( x \in K^n_+ \mapsto g(x) := \text{stack} (b_i x_i^{-2}) \) diffeomorphically maps \( K^n_+ \) onto an open subset of \( \mathbb{R}^n \). Hence the inverse image \( \Xi_L := g^{-1}(\Xi) \) is closed, has the zero Lebesgue measure and, due to these two properties, is nowhere dense.

Let \( C \) be the set of all critical points of the semi-algebraic map \( \mathbf{23} \) \( x \in K^n_+ \mapsto h(x) := Ax + \text{stack} (b_i x_i^{-2}) \in \mathbb{R}^n \), \( i.e., \) points \( x \) such that the Jacobian matrix \( h'(x) \) is singular. By the extended Sard theorem \( \mathbf{24} \), the set of critical values \( h(C) \) has the zero Lebesgue measure and is nowhere dense. Meanwhile, the restriction \( h_{|\Xi_L \setminus C} \) is a local diffeomorphism and so the image \( h(\Xi_L \setminus C) \) is nowhere dense and has the zero Lebesgue measure. It remains to note that the set of \( w \)’s for which Assumption 2.2 does not hold lies in \( h(\Xi_L \setminus C) \cup h(C) \).

### Appendix B

#### Proof of Lemma 2.2

The following system of linear inequalities is feasible

\[
Az > 0, \quad z > 0. \tag{15}
\]

**Proof:** Suppose that the system \( \mathbf{15} \) is infeasible. Then two open convex cones \( A K^n_+ \) and \( K^n_+ \) are disjoint and so can be separated by a hyperplane: there exists

\[
\tau \in \mathbb{R}^n, \quad \tau \neq 0
\]

such that

\[
\tau^T x \geq 0 \quad \forall x \in K^n_+, \quad \tau^T x \leq 0 \quad \forall x \in AK^n_+. \tag{16}
\]

By continuity argument, these inequalities extend on the closures of the concerned sets:

\[
\tau^T x \geq 0 \quad \forall x \in \overline{K^n_+} = \{x : x_i \geq 0\},
\]

\[
\tau^T x \leq 0 \quad \forall x \in \overline{AK^n_+} > \overline{AK^n_+}.
\]

Here the first relation implies that \( \tau \in \overline{K^n_+} \) and so \( A \tau \leq 0 \) by the second one. Hence \( \tau^T A \tau \leq 0 \). Since \( A \) is positively definite by Assumption 2.1, the last inequality yields that \( \tau = 0 \), in violation of the second relation from (16). This contradiction completes the proof.

Based on any solution \( z \) of \( \mathbf{15} \), a solution of (5) can be built in the form \( x := \mu z \) by picking \( \mu > 0 \) so that for all \( i \),

\[
\mu(Az)_i > \langle w_i \rangle + \frac{(b_i)_i}{\mu z_i} \quad \iff \quad \mu^2 (Az)_i - \mu (w_i) - \frac{(b_i)_i}{z_i} > 0
\]

\[
\iff \quad \mu > \frac{\langle w_i \rangle}{(Az)_i} \quad \forall i = j.
\]

Clearly, in the case \( b_j > 0 \) for some \( j \), then, the above inequality can be simplified as

\[
\mu > \frac{\langle w_i \rangle}{(Az)_i} \quad i = j.
\]

This concludes the proof.

### Appendix C

#### Technical facts needed to prove Propositions 2.1 and 2.2

In this section, we consider a \( C^1 \)-map \( g : K^n_+ \rightarrow \mathbb{R}^n \) and provide a general study of the ODE

\[
\dot{x} = g(x), \quad x \in K^n_+, \tag{17}
\]

under the following.

**Assumption C.1:** For any \( x \in K^n_+ \), the off-diagonal elements of the Jacobian matrix \( \nabla g(x) \) are nonnegative.

**Assumption C.2:** For any \( x \in K^n_+ \), the Jacobian matrix \( \nabla g(x) \) is symmetric.

For the convenience of the reader, we first recall several facts that will be instrumental in our study. The first group of
them reflects that the system (17) is monotone (see [14] for a definition).

**Proposition C.1:** Let Assumption C.1 hold and let the order $\triangleright$ in $\mathbb{R}^n$ be either $\geq$ or $>$ for any solutions $x_1(t), x_2(t), x(t)$ of (17) defined on $[0, \tau), \tau > 0$, the following implications hold

\[ x_2(0) \triangleright x_1(0) \Rightarrow x_2(t) \triangleright x_1(t) \quad \forall t \in [0, \tau), \quad (18) \]

\[ x(0) \triangleright 0 \Rightarrow x(t) \triangleright 0 \quad \forall t \in [0, \tau), \quad (19) \]

\[ (x_+ > 0 \land \varsigma = \pm 1 \land \varsigma(g(x_+)) > 0) \Rightarrow \quad \]  

the domain $\Upsilon_\varsigma := \{ x : \varsigma(x - x_+) > 0 \} \cap K_\nu$ is positively invariant.  

**Proof:** Relation (13) is given by Proposition 1.1 and Remark 1.1 in Chapter 3 of [14], whereas (19) is due to [14] Prop. 2.1, Ch. 3. When proving on $\varsigma = 1$; the case $\varsigma = -1$ is treated likewise. Let $x_1(t), t \in [0, \theta)$ stand for the maximal solution of (17) starting from $x_1(0) = x_+$. Since $\dot{x}_1(0) = g(x_+)$, (19) guarantees that $x_1(t)$ constantly increases $x_1(t) > 0 \quad \forall t \in [0, \theta)$ and so $x_1(t) > x_+ \quad \forall t \in (0, \theta)$. Now let solve a solution $x(t), t \in [0, \tau], \tau > 0$. Let $x(0) > x_+(0)$ and $x(t) > x_+(t)$ into $\Upsilon_\varsigma$. Then $x(t) > x_+(t)$ and $x(t) > x_+(t)$ by (18). So $x(t) \in \Upsilon_\varsigma$ for any $t \in [0, \theta)$ and $\Upsilon_\varsigma$. It suffices to show that $\tau < \theta$ if for $\theta < \infty$.

Suppose to the contrary that $\tau \geq \theta$. Letting $t \rightarrow -\theta$, we see that $\|x(t)\| \rightarrow \infty$ by [13] Th. 3.1, Ch. II since $x_1(t) \triangleright x_+ > 0$, and so $x(t) \triangleright x_+(t) \Rightarrow \|x(t)\| \rightarrow \infty$. However, $\|x(t)\| \rightarrow \|x(\tau)\| < \infty$. This contradiction completes the proof.  

Let $x(t, a), t \in [0, \tau_a)$ stand for the maximal solution of (17) that starts at $t = 0$ with $a > 0$. The distance $\inf_{x' \in A} \|x - x'\|$ from point $x \in \mathbb{R}^n$ to a set $A \subset \mathbb{R}^n$ is denoted by $\text{dist}(x, A)$

**Corollary C.1:** Whenever $0 < \alpha_1 \leq \alpha_2$, we have $\tau_a \geq \min(\tau_1, \tau_2)$.  

Claims similar to the following lemma can be inferred from the equivalences $G_{20}$ and $I_{27}$ in [25] Th. 2.3, Ch. VI and (1.1) $\Leftrightarrow (1.2)$ in [26] Prop. 1.

**Lemma C.1:** A nonsingular matrix $A = A^T$ with nonnegative off-diagonal elements is Hurwitz if $Ah > 0$ $\Rightarrow$ $h \leq 0$.  

For $h := \text{stack}( \{\text{sgn}(x_\lambda_j \times h_j) \}_{j=1}^k )$, we have $Ah = \text{stack}( \{ |x_\lambda_j \times h_j| \}_{j=1}^k ) > 0$. So (11) yields that $\{\text{sgn}(x_\lambda_j \times h_j) \neq 0 \Rightarrow \text{sgn}(x_\lambda_j) = -1 \Rightarrow \lambda_j < 0 \forall j\}$. Since the eigenvalue $\lambda_j$ is dominant, all eigenvalues of any block $A_j$ are negative.  

**Lemma C.2:** Let Assumptions C.1 and C.2 hold. Suppose that a solution $x(t), t \in [0, \infty)$ of (17) decays $x(t) \rightarrow 0$ and converges to $\bar{x} > 0$ as $t \rightarrow \infty$. Then $\bar{x}$ is an equilibrium of the ODE (17). If this equilibrium is hyperbolic, it is locally asymptotically stable.

**Proof:** The first claim is given by [14] Prop. 2.1, Ch. 3. By Lemma C.1 it suffices to show that $A := \nabla g(\bar{x})$ meets (21) to prove the second claim. Suppose to the contrary that there exists $x \in \mathbb{R}^n$ such that $Ah > 0$ and $hi > 0$ for some $i$. For $x_0 := \bar{x} + \varepsilon h$ and small enough $\varepsilon > 0$, we have $g(x_0') = g(\bar{x}) + \varepsilon Ah + o(\varepsilon) = \varepsilon Ah + o(\varepsilon) > 0$, $x_0' > 0$, and $x_0'' > \bar{x}, x(0) \in \Upsilon_\varsigma$. Since the set $\Upsilon_\varsigma$ is positively invariant by (19), we infer that $x(t) \in \Upsilon_\varsigma \Rightarrow x(t) \triangleright x_+, t \rightarrow \bar{x}$, in violation of $x(t) \rightarrow \bar{x}$ as $t \rightarrow \infty$. This contradiction completes the proof.  

For any $x' \leq x'' \in \mathbb{R}^n$, we denote $\{ x', x'' \} := \{ x \in \mathbb{R}^n : x' \leq x \leq x'' \}$.

**Lemma C.3:** Suppose that Assumption C.1 holds and $\bar{x} > 0$ is a locally asymptotically stable equilibrium. Its domain of attraction $A(\bar{x}) \subset K_\nu$ is open and

\[ a_1, a_2 \in A(\bar{x}) \cap a_1 \leq a_2 \Rightarrow \{ a_1, a_2 \} \subset A(\bar{x}). \quad (22) \]

**Proof:** Let $B(r, x)$ stand for the open ball with a radius of $r > 0$ centered at $x$.

For any $a \in A(\bar{x})$, we have $\tau_a = \infty$ and $x(t, a) \rightarrow \bar{x}$ as $t \rightarrow \infty$, whereas $B(\bar{x}, \varepsilon) \subset A(\bar{x})$ for a sufficiently small $\varepsilon > 0$ thanks to local stability of $\bar{x}$. Hence there is $\theta > 0$ such that $x(\theta, a) \in B(\varepsilon, \bar{x})$. By [13] Th. 3.1, Ch. VI, there exists $\delta > 0$ such that whenever $\|a' - a\| < \delta$, the solution $x(\cdot, a')$ is defined at least on $[0, \theta]$ and $\|x(\theta, a') - x(\theta, a)\| < \varepsilon$. It follows that $x(\theta, a') \in B(\bar{x}, \varepsilon)$ and so $x(\cdot, a')$ is in fact defined on $[0, \infty)$ and converges to $\bar{x}$ as $t \rightarrow \infty$. Thus we see that $\|a' - a\| < \delta \Rightarrow a' \in A(\bar{x})$, i.e., the set $A(\bar{x})$ is open.

Let $a \in \{ a_1, a_2 \}$. By Corollary C.1 and (13), $\tau_a = \infty$ and $x(t, a) \leq x(t, a_2) \leq 0$. Letting $t \rightarrow \infty$ shows that $x(t, a) \rightarrow \bar{x}$ and so $a \in A(\bar{x})$.

**Lemma C.4:** Let $\tau_1 \leq \tau_2$ and let $D \subset \mathbb{E} := \{ \tau_1, \tau_2 \}$ be an open (in $\mathbb{E}$) set such that (i) $\forall x', x'' \subset \mathbb{E} \Rightarrow x', x'' \subset D$; (ii) either $\tau_1 \in D$ or $\tau_2 \in D$; (iii) $D \neq \mathbb{E}$. Then there exists a continuous map $M : \mathbb{E} \rightarrow \mathbb{E}$ such that $M(\mathbb{E}) \subset \mathbb{E} \setminus D$ and $M[x] = x$ $\forall x \in \mathbb{E}$.  

**Proof:** Let $\pi_2 \in D$ for the definiteness; then $\pi_1 \subset D$ by (i) and (iii). It can be evidently assumed that $0 = \pi_1 \subset \pi_2$. We denote $\chi_\delta(x) := \max(x - \delta, 0)$, where $\delta := \text{stack}(1, \ldots, 1)$ and the max is meant component-wise. Evidently, $\Theta(x) := \{ \theta \geq 0 : \chi_\delta(x) \subset D \} = [0, \tau(x)) \times D$, where $0 < \tau(x) < \infty$. For $x \notin D$, we put $\tau(x) := 0$. We are going to show first that the function $\tau(x)$ is continuous on $\mathbb{E}$. To this end, it suffices to prove that $\tau(x) = \tau_\varsigma$ whenever

\[ \bar{x} = \lim_{k \rightarrow \infty} x_k, \quad x_k \in \mathbb{E}, \quad \tau_\varsigma = \lim_{k \rightarrow \infty} \tau(x_k). \]

\[ ^* \text{In brief, this lemma says that } \mathbb{E}_\varsigma \text{ is a retract of the convex set } \mathbb{E}. \]
Passing to a subsequence ensures that either \( x_k \notin D \forall k \) or \( x_k \in D \forall k \). In the first case, \( \bar{x} \notin D \) since \( D \) is open. Then \( \tau(\bar{x}) = 0 = \tau(x_k) = \tau_x \). Let \( x_k \in D \forall k \). Since \( \chi_{x_k}[\tau(x_k)] \notin D \) and \( D \) is open, letting \( k \to \infty \) yields \( \chi_{x_k}[\tau_x] \notin D \Rightarrow \tau(\bar{x}) \leq \tau_x \). So the claim holds if \( \tau_x = 0 \). If \( \tau_x > 0 \), we pick \( 0 < \theta < \tau_x \). Then \( \theta < \tau(x_k) \) for \( k \approx \infty \), i.e., \( \chi_{x_k}(\theta) \in D \). Let \( x_{i,t} \) be the \( i \)th component of \( x_t \in \mathbb{R}^\mathbb{P} \). Then

\[
\tau'_k := \max \left\{ \tau \geq 0 : \chi_{x_k}(\tau) \geq \chi_{x_k}(\theta) \right\} = \max_{i:x_k,i \geq \theta} \left[ x_{i,t} - k \tau_x + \theta \right].
\]

Here the second max is over a nonempty set since \( \chi_{x_k}(\theta) \in D \neq \emptyset \). Thus \( \tau'_k \to \theta \) as \( k \to \infty \). By (ii), \( \chi_{x_k}(\tau'_k) \in D \) and so \( \tau(\bar{x}) \geq \tau'_k \to \theta \) if \( \theta < \tau_x \Rightarrow \tau(\bar{x}) \geq \tau_x \Rightarrow \tau(\bar{x}) = \tau_x \). Thus the function \( \tau(\cdot) \) is continuous indeed. The needed map \( M \) is given by \( M(x) := \chi_{x}[\tau(x)] \).

**Lemma C.5:** Let Assumption \( C.1 \) hold and \( 0 < \tau_1 \leq \tau_2, \tau_1 \neq \tau_2 \) be two locally asymptotically stable equilibria. Then there exists a third equilibrium \( \bar{x} \) in between them \( \tau_1 \leq \bar{x} \leq \tau_2, \bar{x} \neq \tau_1, \tau_2 \).

**Proof:** By Lemma \( \ref{lem:nonemptyset} \), the set \( D_i := A(\tau_i) \cap \Xi, i = 1, 2 \) meets the assumptions of Lemma \( \ref{lem:existencemap} \), which associates this set with a map \( M_i \). Since the sets \( D_i \) are open and disjoint, they do not cover the connected set \( \Xi \). So the set \( \Xi := \Xi \setminus (D_1 \cup D_2) \) of all fixed points of the map \( M = M_1 \circ M_2 \) is non-empty and compact.

For all \( a \in \Xi \), the solution \( x(-a) \) is defined on \([0, \infty)\) by Corollary \( \ref{cor:nonemptyset} \) and \( x(t, a) \in \Xi \) by (18). So the flow \( \{\Phi_t(a) : x(t, a)\}_{t \geq 0} \) is well defined on \( \Xi \), acts from \( \Xi \) into \( \Xi \), and is continuous by (13) Th. 2.1, Ch. V). The sets \( D_i \) are positively and negatively invariant with respect to it:

\[
a \in D_i \Rightarrow \Phi_t(a) \in D_i \forall t \geq 0, \\
a \in \Xi \wedge \exists t \geq 0 : \Phi_t(a) \in D_i \Rightarrow a \in D_i.
\]

It follows that \( \Xi \subseteq \Xi \) is positively invariant with respect to this flow. By the Brouwer fixed point theorem, the continuous map \( \Phi_t \circ M : \Xi \to \Xi \subseteq \Xi \) has a fixed point \( a_0 = \Phi_t \circ M(a) \in \Xi \).

Since \( M(a_k) \in \Xi \) and \( \Phi_t(\Xi) \subseteq \Xi \), we see that \( a_k \in \Xi \) and so \( M(a_0) = a_0 \) and \( a_1 = \Phi_t(a_0) \).

Since \( \Xi \) is compact, there exists a sequence \( \{t_k \to 0\}_{k=1}^{\infty} \) such that \( t_k \to 0 \) and \( a_k \to \bar{x} \) as \( k \to \infty \) for some point \( \bar{x} \in \Xi \). Since \( \tau_1, \tau_2 \notin \Xi \), we have \( \bar{x} \neq \tau_1, \tau_2 \); meanwhile \( \bar{x} \in \Xi \subseteq \Xi \Rightarrow \tau_1 \leq \bar{x} \leq \tau_2 \). Furthermore,

\[
0 = t_k^{-1} [\Phi_{t_k}(a_k) - a_k] = t_k^{-1} \int_{t_k}^{t_k} g(x(t, a_k)) \, dt \xrightarrow{k \to \infty} g(\bar{x}).
\]

Thus we see that \( g(\bar{x}) = 0 \), i.e., \( \bar{x} \) is an equilibrium.

**APPENDIX D**

**Proofs of Propositions \ref{prop:2.1} and \ref{prop:2.2}**

Now we revert to study of the system (2) under the Assumptions \( \ref{assump:2.1} \) and \( \ref{assump:2.2} \).

**Lemma D.1:** Suppose that \( y \) belongs to the set (5). There exists \( \theta \in (0, 1) \) such that the domain \( \Xi_{-}(\theta) := \{x : 0 < x \leq \theta y\} \) is globally absorbing, i.e., the following statements hold:

(i) This domain is positively invariant: if a solution starts in \( \Xi_{-}(\theta) \), it does not leave \( \Xi_{-}(\theta) \);
(ii) Any solution defined on \([0, \infty)\) eventually enters \( \Xi_{-}(\theta) \) and then never leaves this set.

**Proof:** Thanks to (5), there exists \( \delta > 0 \) such that

\[
Ay > \text{stack} \left\{ \langle w_i \rangle + \langle -b_i \rangle \frac{b_i}{y_i} + 3\delta \right\}.
\]

We also pick \( \theta \in (0, 1) \) so close to 1 that

\[
\begin{align*}
\theta - 1 \langle w_i \rangle_{+} + \delta \theta & \geq 0, \\
\theta - 1 \langle -b_i \rangle_{-} + \delta \theta & \geq 0 \quad \forall i.
\end{align*}
\]

Let \( x(\cdot) \) be a solution of (2). By the Danskin theorem (28), the function \( \rho(t) := \max_{i=x(0)} \int_{t_1}^{t} w_i(t) / y_i \) is absolutely continuous and for almost all \( t \), the following equation holds

\[
\dot{\rho}(t) = \max_{i=x(t)} \dot{x}(t)/y_i, \quad \text{where}
\]

\[
I(t) := \{i : x(t)/y_i = \rho(t)\}.
\]

For any \( i \in I(t) \) and \( j \), we have \( x(t,i) = y_i \rho(t), x_j(t) \leq y_j \rho(t), \) and

\[
\dot{x}_i(t) \geq -a_{i,i} x_i(t) + \sum_{j \neq i} \left[ -a_{i,j} x_j(t) - \frac{b_i}{y_i} + w_i \right]
\]

\[
\leq - \rho(t) \left( a_{i,i} y_i + \sum_{j \neq i} a_{i,j} y_j \right) - \rho(t)^{-1} \rho(t) + w_i
\]

\[
\leq - \rho(t) \left[ \langle w_i \rangle + \langle -b_i \rangle + 3\delta \right] + \rho(t)^{-1} \left[ \rho(t) + \langle w_i \rangle \right] - \delta \rho(t) - \left[ \rho(t) - 1 \langle w_i \rangle + \delta \rho(t) \right] + \cdots
\]

\[
\leq \left[ \rho(t) - \rho(t)^{-1} \langle -b_i \rangle \right] \frac{\rho(t)}{y_i} + \delta \rho(t).
\]

Hence whenever \( \rho(t) \geq 0 \),

\[
\dot{x}_i(t) \leq - \rho(t) - \left[ \rho(t) - 1 \langle w_i \rangle + \delta \rho(t) \right] - \delta \rho(t)
\]

\[
\cdots - \left[ \rho(t) - \rho(t)^{-1} \langle -b_i \rangle \right] \frac{\rho(t)}{y_i} + \delta \rho(t).
\]

So by invoking (25), we infer that \( \rho(t) \geq 0 \Rightarrow \dot{\rho}(t) \leq -\delta \rho(t) \leq -\delta \rho(t) \) [Claim (i)].

**Lemma D.2:** Claim II of Proposition \( \ref{prop:2.2} \) holds.

**Proof:** This is immediate from (22) since for any characteristic solution \( x(\cdot) \) and \( y := x(0) \),

\[
\dot{x}(0) \geq -Ay + \text{stack} \left\{ \langle w_i \rangle - \frac{b_i}{y_i} + w_i \right\}
\]

\[
\leq -Ay + \text{stack} \left\{ \langle -b_i \rangle \frac{b_i}{y_i} + \langle w_i \rangle \right\} = 0.
\]

**Lemma D.3:** Suppose that a solution \( x(\cdot) \) of (2) is defined on \([0, \tau) \) with \( \tau < \infty \) but cannot be extended to the right. Then there is \( i \) such that \( b_i > 0 \) and \( x_i(t) \to 0, \dot{x}_i(t) \to -\infty \) as \( t \to \tau^- \).

\footnote{In fact, this implication holds for almost all \( t \) such that the premises are true.}
Proof: By Lemma 2.2 there exists a solution \( y > 0 \) of (5). Via multiplying \( y \) by a large enough factor, we ensure that \( y \sim x(0) \). Let \( x_t(\cdot) \) be the characteristic solution starting with \( x_0(0) = y \). By Lemma D.2 \( x_t(\cdot) \leq y \) for \( t \geq 0 \), whereas \( x(t) \leq x_t(\cdot) \) on the intersection of the domains of definitions of \( x(\cdot) \) and \( x_t(\cdot) \) by (18). Then [13] Th. 3.1, Ch. II] ensures that \( x(t) \) converges to the boundary of \( \mathcal{K}^n_+ \) as \( t \to \tau - \) and is bounded. In other words, 
\[
\min_{i} x_i(t) \to 0 \quad \text{as} \quad t \to \tau - , \quad c := \sup_{t \in [0, \tau]} \|x(t)\| < \infty.
\]

Meanwhile putting \( W := \max_{i} \{ |w_i| + c \sum_{j} |a_{ij}| \} \), we see that 
\[
\dot{x}_i(t) \geq - \frac{b_i}{W} x_i(t) + w_i 
\]
(27)

so is negatively definite as well.

Proof: Suppose to the contrary that \( x^{(0)} \neq x^{(1)} \). By Lemma C.5 and (i) of Lemma D.4 there exists one more equilibrium \( x^{(1/2)} \) in between \( x^{(0)} \) and \( x^{(1)} \), i.e., \( x^{(0)} < x^{(1/2)} \leq x^{(1)} \) and \( x^{(1/2)} \neq x^{(0)}, x^{(1)} \). By (ii) of Lemma D.4 this newequilibrium \( x^{(1/2)} \) is stable. This permits us to repeat the foregoing arguments first for \( x^{(0)} \) and \( x^{(1/2)} \) and second for \( x^{(1/2)} \) and \( x^{(1)} \). As a result, we see that there exist two more stable equilibria \( x^{(1/4)} \in [x^{(0)}, x^{(1/2)}] \) and \( x^{(3/4)} \in [x^{(1/2)}, x^{(1)}] \) that differ from all previously introduced equilibria. This permits us to repeat the foregoing arguments once more to show that there exist stable equilibria \( x^{(1/8)}, x^{(3/8)}, x^{(5/8)}, x^{(7/8)} \) such that \( x^{(1/8)} \leq x^{(1/8)} \forall i \leq i \leq j \leq 8 \) and \( x^{(1/8)} \neq x^{(1/8)} \forall i \leq i \leq 8, i \neq j \).

By continuing likewise, we assign a stable equilibrium \( x^{(r)} \) to any number \( r \in [0, 1] \) whose representation in the base-2 numeral system is finite (i.e., number representable in the form \( r = 2^{-k} \) for some \( k = 1, 2, \ldots \) and \( j = 0, \ldots, 2^k \) and ensure that these equilibria are pairwise distinct and depend on \( r \) monotonically: \( x^{(r)} \leq x^{(r')} \) whenever \( 0 \leq r < r' \leq 1 \).

Since all they lie in the compact set \( [x^{(0)}, x^{(1)}] \), there exists a sequence \( \{r_k\}_{k=1}^\infty \) of pairwise distinct numbers \( r_k \) for which \( \exists \bar{x} \lim_{k \to \infty} x^{(r_k)} \). Then \( \bar{x} \in (x^{(0)}, x^{(1)}) \) and so \( \bar{x} > 0 \) and \( f(\bar{x}) = \lim_{k \to \infty} f(x^{(r_k)}) = 0 \), i.e., \( \bar{x} \) is an equilibrium. Then the Jacobian matrix \( \nabla f(\bar{x}) \) is nonsingular, as was remarked just after (30). However, this implies that in a sufficiently small vicinity \( V \) of \( \bar{x} \), the equation \( f(x) = 0 \) has no roots apart from \( \bar{x} \) in violation of \( x^{(r_k)} \in V \forall k \approx \infty \) and \( x^{(r_k)} \neq x^{(r_l)} \forall k \neq l \). This contradiction obtained completes the proof.

Proof of Proposition 2.2: Claim I) is justified by Lemma 2.2.

Claim II) is justified by Lemma 2.2. By II), the limit \( \bar{x} \) from (7) exists and \( \bar{x} \geq 0 \).

Claim III) Let \( x(t), t \in [0, t_f] \) be a characteristic solution. If \( t_f < \infty \), then III.i) of Proposition 2.2 holds by Lemma 2.3.

Suppose that \( t_f = \infty \). Then the limit \( \bar{x} \) from (7) exists due to II) of Proposition 2.2 and \( \bar{x} \geq 0 \). We are going to show that in fact \( \bar{x} > 0 \).

Suppose the contrary that \( \bar{x}_i = 0 \) for some \( i \). Then \( x_i(t) \to 0 \) as \( t \to \infty \), (28) means that \( b_i > 0 \), and (29) (where \( \tau = \infty \) now) implies that \( \|x(\theta)\|^2 \) assumes negative values for large enough \( \theta \). This assures that \( \bar{x} > 0 \) and so (7) does hold. By Lemma C.2 \( \bar{x} \) is an equilibrium.

Now suppose that III.i) holds for a characteristic solution \( x_t(\cdot) \).

Suppose that there is another characteristic solution \( \bar{x}(\cdot) \) for which III.i) is not true. Then \( x(\cdot) \) is defined on \( [0, \infty) \) by Lemma 2.3 and also \( \exists \bar{x} = \lim_{t \to \infty} x(t) > 0 \) by the foregoing. By (ii) of Lemma D.1 (with \( y := x(0) \)), \( x(\sigma) \leq \bar{x}x(0) \leq x(0) \) for large enough \( \sigma \). By applying (18) to \( x(t) := x(t + \sigma) \) and \( x(2\sigma) = x(t) \), we see that \( x(t + \sigma) \leq x_1(t) \) and so \( x_1(t) \) goes to zero in a finite time, in violation of \( \bar{x} \). This contradiction proves that III.i) holds simultaneously for all characteristic solutions.

Since III.i) and III.ii) are mutually exclusive and complementary, we see that either III.i) holds for all characteristic solutions, or III.ii) holds for all of them.

Finally, suppose that III.ii) holds. As was shown in the penultimate paragraph, \( x(t + \sigma) \leq x_1(t) \) for any two char-
characteristic solutions $x(\cdot)$ and $x_1(\cdot)$. Hence $\lim_{t \to \infty} x(t) \leq \lim_{t \to \infty} x_1(t)$. By flipping $x(\cdot)$ and $x_1(\cdot)$ here, we see that these limit coincide, i.e., the limit (7) is the same for all characteristic solutions.

**Claim IV** is straightforward from Lemmas [D.1] and [D.3] since any equilibrium is associated with a constant solution defined on $[0, \infty)$.

**Claim V** Suppose that III.ii) holds. Let $\bar{x}_{\max}$ stand for the limit (7). By (II) and Lemmas [C.2] and [D.4] $\bar{x}_{\max}$ is a locally asymptotically stable equilibrium. Let us consider a solution $x(\cdot)$ defined on $[0, \infty)$ and a characteristic solution $x_1(\cdot)$. By retracing the above arguments based on (ii) of Lemma [D.1] we see that $x(\xi + t) \leq x_1(t)$ $\forall t \geq 0$ for some $\xi \geq 0$. By considering here a constant solution $x(\cdot)$ and letting $t \to \infty$, we see that $\bar{x}_{\max}$ dominates any other equilibrium.

Now suppose that $x(0) \geq \bar{x}_{\max}$. By (18), $x(t) \geq \bar{x}_{\max}$ on the domain $\Delta$ of definition of $x(\cdot)$ and so $\Delta = [0, \infty)$ by Lemma [D.3]. Thus we see that $x_{\max} \leq x(\xi + t) \leq x_1(t) \forall t \geq 0$ for some $\xi \geq 0$. It follows that $x(t) \to x_{\max}$ as $t \to \infty$, i.e., the equilibrium $x_{\max}$ is attractive from the right by Definition [2.1].

It remains to show that there exist only finitely many equilibria $\bar{x}_{k}$. Suppose the contrary. Since all equilibria lie in the compact set $\{ x : 0 \leq x \leq \bar{x}_{\max}\}$, there exists an infinite sequence $\{\bar{x}_{k}\}_{k=1}^{\infty}$ of pairwise different equilibria that converges $\bar{x}_{k} \to \bar{x}$ as $t \to \infty$ to a point $\bar{x} \geq 0$. The estimates (28), (29) applied to any equilibrium solution $x(\cdot)$ assure that $x_{1} \geq |b_{1}|/(2W)$ on it, where $W := \max_{i} \{ |w_{1}| + c \sum_{j} |a_{ij}| \}$ and $c$ is any upper bound on $|x(t)|$. For the solutions related to the convergent and so bounded sequence $\{\bar{x}_{k}\}_{k=1}^{\infty}$, this bound can be chosen common. As a result, we infer that $\bar{x} > 0$ and so $f(\bar{x}) = \lim_{t \to \infty} f(\bar{x}_{k}) = 0$, i.e., $\bar{x}$ is an equilibrium. Then the Jacobian matrix $\nabla f(\bar{x})$ is nonsingular, as was remarked just after (30). This implies that in a sufficiently small vicinity $V$ of $\bar{x}$, the equation $f(x) = 0$ has no roots apart from $\bar{x}$, in violation of $x_{k} \in V \forall s \approx \infty$ and $x_{k} \neq x_{k} \forall s \neq \tau$. This contradiction completes the proof.

**Proof of Proposition 2.1** This proposition is immediate from Proposition 2.2.

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Alexey S. Matveev was born in Leningrad, Russia, in 1954. He received the M.S. and Ph.D. degrees in applied mathematics and engineering cybernetics both from the Leningrad University, St. Petersburg, Russia, in 1976 and 1980, respectively. He is currently a Professor in the Department of Mathematics and Mechanics, Saint Petersburg University. His research interests include control over communication networks, hybrid dynamical systems, and navigation and control of mobile robots.

Juan E. Machado received the B.Sc. degree in electromechanical engineering in 2012 from Instituto Tecnológico de La Paz, La Paz, México and the M.Sc. degree in applied mathematics in 2015 from Centro de Investigación en Matemáticas, Guanajuato, México. Currently, he is a Ph.D student at Université Paris Sud - Centrale Supélec, Gif-Sur-Yvette, France. His interests include modeling and control of electromechanical systems.

Romeo Ortega (S’81, M’85, SM’98, F’99) was born in Mexico. He obtained his BSc in Electrical and Mechanical Engineering from the National University of Mexico, Master of Engineering from Polytechnical Institute of Leningrad, USSR, and the Docteur D'Etat from the Polytechnical Institute of Grenoble, France in 1974, 1978 and 1984 respectively. He then joined the National University of Mexico, where he worked until 1989. He was a Visiting Professor at the University of Illinois in 1987-88 and at the McGill University in 1991-1992, and a Fellow of the Japan Society for Promotion of Science in 1990-1991. He has been a member of the French National Researcher Council (CNRS) since June 1992. Currently he is in the Laboratoire de Signaux et Systemes (SUPELEC) in Gif-sur-Yvette. His research interests are in the fields of nonlinear and adaptive control, with special emphasis on applications. Dr Ortega has published three books and more than 290 scientific papers in international journals, with an h-index of 79. He has supervised more than 30 PhD thesis. He has served as chairman in several IFAC and IEEE committees and participated in various editorial boards.

Anton Pyrkin (M’11) was born in Zaozerniy, USSR, in 1985. He received the B.S. degree in 2006, the M.S. degree in 2008, the Ph.D. degree in 2010, and the Doctor of Science (habilitation thesis) degree in system analysis, data processing and control (in technical systems) in 2015, all from ITMO University, St. Petersburg, Russia.

He is a leading Scientist and Docent at the Department of Control Systems and Informatics, ITMO University. He has created the company Robotronica Ltd. main purpose of which is the research, developing, and assembling the mechatronic and robotic models of real technical plants for experimental approval of designed control systems. He is a coauthor of more than 100 publications in science journals and proceedings of conferences. His research interests include adaptive and robust control, frequency estimation, disturbance cancellation, time-delay systems, mechatronic and robotic systems, and autopilot and dynamic position systems for vessels. He is a Member of the International Public Association Academy of Navigation and Motion Control.

Johannes Schiffer received the Diploma degree in engineering cybernetics from the University of Stuttgart, Stuttgart, Germany, in 2009 and the Ph.D. degree (Dr.-Ing.) in electrical engineering from Technische Universität (TU) Berlin, Berlin, Germany, in 2015.

He currently holds the chair of Control Systems and Network Control Technology at Brandenburgische Technische Universität Cottbus-Senftenberg, Cottbus, Germany. Prior to that, he has held appointments as Lecturer (Assistant Professor) at the School of Electronic and Electrical Engineering, University of Leeds, Leeds, U.K. and as Research Associate in the Control Systems Group and at the Chair of Sustainable Electric Networks and Sources of Energy both at TU Berlin. In 2017 he and his co-workers received the Automatica Paper Prize over the years 2014-2016. His current research interests include distributed control and analysis of complex networks with application to microgrids and power systems.