Equational Reasoning for MTL Type Classes

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Abstract

Ability to use definitions occurring in the code directly in equational reasoning is one of the key strengths of functional programming. This is impossible in the case of Haskell type class methods unless a particular instance type is specified, since class methods can be defined differently for different instances. To allow uniform reasoning for all instances, many type classes in the Haskell library come along with laws (axioms), specified in comments, that all instances are expected to follow (albeit Haskell is unable to force it). For the type classes introduced in the Monad Transformer Library (MTL), such laws have not been specified; nevertheless, some sets of axioms have occurred in the literature and the Haskell mailing lists. This paper investigates sets of laws usable for equational reasoning about methods of the type classes MonadReader and MonadWriter and also reviews analogous earlier proposals for the classes MonadError and MonadState. For both MonadReader and MonadWriter, an equivalence result of two alternative axiomatizations in terms of different sets of operations is established. As a sideline, patterns in the choice of methods of different classes are noticed which may inspire new developments in MTL.

CCS Concepts: • Software and its engineering → Semantics; Software libraries and repositories; Designing software; Formal software verification; • Theory of computation → Formalisms; Equational logic and rewriting; Program semantics.

Keywords: monad transformers, equational reasoning

1 Introduction

Equational reasoning is often presented as one of the key benefits of functional programming. Definitions in the source code provide us basic equalities to rely on, and referential transparency in pure functional languages like Haskell allows one to safely replace terms by equal terms within any context. Definitional equality works like in mathematics.

As Haskell type class methods are defined newly for every instance type, equational reasoning relying on method definitions in the code is type specific. In order to create uniform proofs for all instances of a class, one must use assumptions in the form of equations, called axioms or laws, which are not grounded on the source code. Many type classes defined in the Haskell library come along with such axioms which every instance of the class is supposed (though not forced) to satisfy. For example, the method fmap of all instances of the class Functor is supposed to preserve composition and identities, the methods of the class Monad should satisfy the standard monad laws of category theory, etc.

The class Monad offers a uniform interface for effectful computations of various kinds. The Monad Transformer Library (MTL) which is a part of GHC standard libraries provides a lot of subclasses of Monad with specific operation interfaces for different effects. It implements a classic approach dating back to Liang et al. [14], Jones [12] and Hutton and Meijer [10], yet its type classes have no universally accepted laws for equational reasoning. Gibbons and Hinz [5] advocate equational reasoning for effectful computations, in particular in the case of some MTL classes, but without analyzing the choice of the laws or studying alternatives.

Recently, laws for MTL type classes have gained some attention in research [1, 15, 18] and in Haskell Libraries mailing list [20] which tells about rising interest in this topic. This paper studies several equational axiomatizations of monadic computations with various effects—exceptions, environment, logging (writer) and mutable state. For environment and logging, we propose a few alternative sets of laws, prove their equivalence, and prove correctness of these laws for monads built up using the exception, reader, writer and state monad transformers. For exceptions and state, we review similar results of previous work. The axiomatics considered here address one effect each; axiomatizing of combinations of different effects might be an interesting topic of future work.

We avoid a premature conclusion to have found “the right axiomatizations” for the four type classes. Firstly, validity on four types of effects might not provide sufficient evidence for declaring the laws universal enough. Secondly, some of our axioms assume operations with types that are impossible in MTL. Nevertheless, such sets of axioms can be useful for proving equivalences of legal programs. Providing a means for writing such proofs is the primary goal of developing the axiomatizations. Solving problems by going beyond the bounds imposed by the problem setting is neither unsound nor original. Quite analogously to our case, Hutton and Fulger [9] lift type restrictions imposed by the context in their proofs of equivalences of effectful programs.

Like in [5], we ignore partiality that may break the laws (see Jeuring et al. [11] for proof that the state monad transformer does not preserve monad laws for bottom cases, and
Huffman [8] for similar results for the exception and writer monad transformer). Danielsson et al. [4] show that ignoring partiality in equational reasoning is justified. We use notation from category theory throughout the paper mainly for achieving more concise formulae rather than for generality. Most results are established for category Set only.

In Sect. 2, we help the reader with preliminaries from category theory; Sect. 3 gives a short introduction to MTL. In Sect. 4, we review previous work on laws of exceptions. In Sect. 5, we study axiomatics for equational reasoning about reader monad operations. Section 6 is devoted to writer monads. We develop an abstract treatment of monads equipped with some additional operations in general category theoretical setting, which luckily applies to the writer case. The set of operations most natural for investigating at the abstract level does not coincide with the actual method set specified in MTL but, as the latter is directly expressible in terms of the former, the results are applicable to Haskell. Section 7 contains a brief review of previously proposed axiomatics of stateful computations. Section 8 concludes.

2 Preliminaries from Category Theory

In mathematics, a category consists of a collection of objects, along with a collection of morphisms working between every ordered pair of objects, satisfying the following properties:

- For every object X, there exists a distinguished morphism id_X from X to X called the identity of X;
- For every triple (X, Y, Z) of objects and morphisms f from X to Y and g from Y to Z, there exists a morphism g ∘ f from X to Z called the composition of f and g;
- The composition as an operation is associative, and each identity morphism id_X works as both a left and right unit of composition.

The claim that f is a morphism from object X to object Y is denoted by f : X → Y, where X → Y is sometimes called the type of f. The object in the subscript of id can be omitted if determined by the context. Assume the binding precedence of ∘ being higher than that of any other binary operator.

It is common to use notions of category theory when talking about Haskell. The data types are playing the role of objects and definable functions from one type to another are the morphisms between these types as objects. Identity functions and function composition play the corresponding role. Laziness of Haskell, along with the presence of seq, does not allow all properties of category to be fully satisfied, nor do the standard categorical constructions in Haskell (some of which are introduced below) fully meet their definition given in category theory; however, as shown by Danielsson et al. [4], it is justified to ignore the deviations for practical purposes. We will use both the notation of Haskell and that of category theory throughout the paper, depending on which is more concise and readable at a particular place.

Pair types (x, y) of Haskell correspond to the binary products in category theory where they are denoted by cross. In category theory, a product X × Y of objects X and Y comes along with morphisms exl : X × Y → X, exr : X × Y → Y and an operator ∆ mapping every pair of morphisms f : Z → X and g : Z → Y to a morphism f ∆ g : Z → X × Y. Thereby, they must meet the laws exl ∘ (f ∆ g) = f, exr ∘ (f ∆ g) = g and exl ∘ h ∆ exr ∘ h = h. In Haskell, the projection functions fst and snd stand for exl and exr; for f ∆ g one can take the function that applies both f and g to its argument and returns the results as a pair.

Similarly, the types Either x y of Haskell correspond to binary sums of category theory where they are written by plus. A sum X + Y of objects comes along with morphisms inl : X → X + Y, inr : Y → X + Y and an operator ⊔ that maps every pair of morphisms f : X → Z and g : Y → Z to a morphism f ⊔ g : X + Y → Z. Thereby, they must satisfy the equations (f ⊔ g) ∘ inl = f, (f ⊔ g) ∘ inr = g and h ∘ inl ⊔ inr ∘ h = h. In Haskell, inl and inr are written as Left and Right, whereas the intended behavior of ⊔ is captured by the library function either.

The Functor type class in Haskell is introduced by

class Functor f where

  fmap :: (a → b) → f a → f b

  {-... other stuff omitted ... -}

In category theory, a functor F is a structure-preserving mapping between categories, i.e., a mapping of objects of one category to objects of the other category and morphisms of type X → Y for any objects X, Y of the first category to morphisms of type FX → FY in the second category, satisfying the laws F id = id and F (g ∘ f) = F g ∘ F f. The Haskell class method fmap corresponds to the mapping of morphisms while the mapping of objects is implemented by the type constructor f. The two functor laws are expected to be fulfilled by every instance of the Functor type class.

The library of Haskell defines also the class Bifunctor analogous to Functor but applying to binary type constructors:

class Bifunctor p where

  bimap :: (a → b) → (c → d) → p a c → p b d

  {-... other stuff omitted ... -}

The functor laws are assumed to hold for both argument types. In category theory, bifunctors can be seen as functors whose domain is the direct product of two categories. Both the binary product and binary sum, considered above as operations on objects, can be extended to morphisms by defining f × g = f ∘ exl ∆ g ∘ exr and f + g = inl ∘ f ⊔ inr ∘ g, as the result of which one obtains bifunctors.

A transformation between functors F and G is a family τ of morphisms τ_X : FX → GX for every object X. In the programming point of view, transformations are polymorphic
functions. A transformation is called natural if it preserves structure embedded in the functors, i.e., satisfies for every morphism \( f \) the equation \( \tau \circ F f = G f \circ \tau \). Like in the case of identity morphisms (which altogether constitute, of course, a natural transformation between \( I \) and \( I \) where \( I \) is the identity functor leaving everything in place), the subscript of \( \tau \) is omitted when it is clear from context.

The \textit{Monad} type class which is a subclass of \textit{Functor} declares methods

\[
(\gg) :: m a \to (a \to m b) \to m b
\]

\[
(\gg) :: m a \to m b \to m b
\]

\[
\text{return} :: a \to m a
\]

along with the default definition \( x \gg k = x \gg \_ \to k \); the type variable \( m \) stands for arbitrary instance type of \textit{Monad}. The operator \( \gg \) is pronounced bind, whereas \textit{return} is called unit of the monad. In category theory, bind is usually denoted by \( (\_)* \) and the order of its arguments is reversed; so if \( M \) is a monad then \((\_)*\) maps morphisms of type \( X \to MY \) to morphisms of type \( MX \to MY \). The Haskell operator \( \gg \) is a special case of bind with constant function as argument.

In category theory, monad operations must satisfy the unit laws \( k^* \circ \text{return} = k \) and \( \text{return}^* = \text{id} \) along with associativity \( I^* \circ k^* = (I^* \circ k^*)^* \), and the functor must be expressible via monad operations by \( M f = (\text{return} \circ f)^* \). The same axioms are expected to be met by all instances of the \textit{Monad} type class. Category theorists often define monads via joint \( MX \to MX \) instead of bind. The join and bind operations are expressed in terms of each other by \( \text{join} = \text{id}^* \) and \( k^* = \text{join} \circ M k \). The axiom set for this approach equivalent to the one given above consists of:

- The two functor laws for \( M \);
- The naturality laws \( \text{return} \circ f = M f \circ \text{return} \) and \( \text{join} \circ M M f = M f \circ \text{join} \);
- The coherence axioms \( \text{join} \circ M \text{join} = \text{join} \circ \text{join} \), \( \text{join} \circ M \text{return} = \text{id} \) and \( \text{join} \circ \text{return} = \text{id} \).

The morphisms \textit{return} and \textit{join} are usually denoted by \( \eta \) and \( \mu \), respectively. \textit{Identity monad} \( I \) is the simplest monad where the mapping of objects, mappings of morphisms, monad unit, bind and join are all identities.

\textit{Relative monads on a base functor} \( J \) were introduced by Altenkirch et al. [2, 3]. Relative monads are pairs of functors \((J, M)\) equipped with bind and unit operations whose types are more general than those of the corresponding monad operations: namely, the unit must have type \( J X \to MX \), and bind takes morphisms of type \( JX \to MY \) to morphisms of type \( MX \to MY \). The functor \( M \) must be expressible via these operations by \( M f = (\text{return} \circ J f)^* \). Other relative monad laws look the same as the monad laws of return and \((\_)*\). Monads are relative monads where \( J = I \). Due to the different types, the alternative representation via \textit{join} is impossible for relative monads in general.

A \textit{monad morphism} from \( M \) to \( M' \) is a structure-preserving transformation between these monads, i.e., \( \sigma : MX \to M'X \) such that \( \sigma \circ \text{return} = \text{return} \) and \( \sigma \circ k^* = (\sigma \circ k)^* \circ \sigma \), where return and \((\_)*\) in the left-hand sides belong to \( M \) and those in the right-hand sides belong to \( M' \). Analogously, one can define relative monad morphisms.

The term \textit{pointed functor} is sometimes used for denoting functors \( F \) equipped with a unit return : \( X \to FX \) (for any \( X \)) but without monad bind. The unit is assumed to meet the naturality law \( f \circ \text{return} = F f \circ \text{return} \). The term dates back to at least Lenisa et al. [13]. We prefer to call return of a pointed functor its point rather than unit, since calling something a unit normally assumes a certain relationship with another (binary) operation (like the unit laws relating unit and bind in the case of monads) which is missing in the general case of pointed functors. We will occasionally call the unit of a relative monad also point or relative point.

### 3 A Brief Introduction to MTL Classes

Monads as a framework suitable for embedding computational effects were discovered and advocated by Moggi [16].

The same work studied constructs in category theory that we now know as monad transformers. Using monad transformers in Haskell was proposed by Liang et al. [14], Jones [12] and Hutton and Meijer [10]. The Haskell MTL has been built upon ideas of those papers. It defines the exception, reader, writer, and state monad transformers as follows:

\[
\text{newtype ExceptT} \ e \ m a = \text{ExceptT} (m (\text{Either} e a))
\]

\[
\text{newtype ReaderT} \ r \ m a = \text{ReaderT} (r \to m a)
\]

\[
\text{newtype WriterT} \ w m a = \text{WriterT} (m (a, w))
\]

\[
\text{newtype StateT} \ s m a = \text{StateT} (s \to m (a, s))
\]

(There are more transformers but we only study these four.)

Every transformer assumes a type constructor as its second parameter \( m \); provided that it is a monad, application of the transformer produces a new monad. Substituting the identity monad for \( m \), we obtain the \textit{error}, \textit{reader}, \textit{writer}, and \textit{state monads}, respectively. Using the language of category theory, we can define these monads as \textit{Except} \( (E, X) = E + X \), \textit{Reader} \( (R, X) = R \to X \), \textit{Writer} \( (W, X) = X \times W \), and \textit{State} \( (S, X) = S \to X \times S \). (Still we use the Haskell notation \( A \to B \) for function space which deviates from the notations of exponential objects used in category theory.)

Every transformer adds new effects to the monad while keeping the previously existing effects in force and available:

- The exception monad transformer adds capability of dealing with errors, where errors are values of the type given as the first parameter of the transformer;
- The state monad transformer enables one to use a hidden mutable state for computation, where states have type given as the first parameter of the transformer;
• The reader monad transformer makes it possible to use hidden “environments” of the type given as the first parameter of the transformer;

• The writer monad transformer introduces the ability of information logging throughout the computation, where the data items written into the log are of the type determined by the first parameter of the transformer.

We omit the details of defining the relevant operations and their propagation through chains of monad transformer applications; they are not needed for understanding the paper.

It is convenient to have every effect introduced by some monad transformer accessible via a fixed interface irrespective of other monad transformers applied. For that reason, MTL defines type classes MonadError, MonadReader, MonadWriter and MonadState (and others for effects not considered here). For instance, MonadError defines the interface of exception throwing and handling, MonadState specifies the interface for stateful computation, etc. Each of the following sections 4–7 discusses one of these type classes, providing also the definition details.

4 Laws of Exception Handling

MTL defines the MonadError class along with methods for throwing and catching of exceptions as follows:

```haskell
class Monad m => MonadError e m | m -> e where
  throwError :: e -> a
  catchError :: m a -> (e -> m a) -> m a
```

Here e denotes the type of exceptions; it is a parameter of the class but not of m. Gibbons and Hinze [3] consider axioms of exception handling in a narrower setting that does not involve an exception type (it is equivalent to MonadError with e = ()). Their axioms state that catch is associative, whereas failure (throwing the exception) is its unit and a left zero of ⇒ as well. Both Malakhovski [15] and the author [18] assume a wider setting with the type e being an additional parameter of the functor m, replacing the latter with a bifunctor F. The error throwing function has type E → F (E, A) and return obtains type A → F (E, A). The catch function in [18] has type (E → F (E’, A)) → F (E, A) → F (E’, A) similarly to bind which has type (A → F (E’, A’)) → F (E, A) → F (E, A’); in [15], the order of arguments of these functions is reversed.

Following [18], we denote the bind and catch operations by (⇒) and (⇒) ↩, respectively, and denote the error throwing function by throw. That work proposes the following axioms for bind, among which the first three are standard monad axioms, the fourth one is a generalization of the usual monad zero law by introducing an exception parameter, and the last one establishes that any mapping of exceptions by the bifunctor is a bind homomorphism:

\[
\begin{align*}
&k^{\Rightarrow} \circ \text{return} = k \\
&(\text{return} \circ f)^{\Rightarrow} = F (\text{id}, f) \\
&l^{\Rightarrow} \circ k^{\Rightarrow} = (l^{\Rightarrow} \circ k)^{\Rightarrow} \\
&k^{\Rightarrow} \circ \text{throw} = \text{throw} \\
&F (h, \text{id}) \circ k^{\Rightarrow} = (F (h, \text{id}) \circ k)^{\Rightarrow} \circ F (h, \text{id})
\end{align*}
\]

For catch, the dual laws are proposed:

\[
\begin{align*}
&k^{\Rightarrow} \circ \text{throw} = k \\
&(\text{throw} \circ h)^{\Rightarrow} = F (\text{h}, \text{id}) \\
&l^{\Rightarrow} \circ k^{\Rightarrow} = (l^{\Rightarrow} \circ k)^{\Rightarrow} \\
&k^{\Rightarrow} \circ \text{return} = \text{return} \\
&F (\text{id}, f) \circ k^{\Rightarrow} = (F (\text{id}, f) \circ k)^{\Rightarrow} \circ F (\text{id}, f)
\end{align*}
\]

All but the last axiom of both blocks occur also in [15]. All axioms introduced so far are meaningful also in the standard MTL setting. In addition, [18] considers the following law for interchanging bind and catch, where ρ = throw ∨ return:

\[
\begin{align*}
&k^{\Rightarrow} \circ \rho^{\Rightarrow} = (\text{throw} \lor \text{catch} \lor \text{return}) \circ (F (\text{inl}, \text{id}) \circ k)^{\Rightarrow}
\end{align*}
\]

This law inherently exploits the two-parameter setting since ρ changes the exception type of the computation.

The following is established by [18], where propagation of throw and catch through applications of transformers are assumed to be defined similarly to MTL:

• Any monad obtained by applying the error monad transformer to another monad (and considered as a bifunctor) satisfies all the given laws;

• Applying the error, reader, writer, or state monad transformer preserves all the given laws.

The paper [18] defines the joint handle (\_)∗ via (\_) ↩ and (\_) ↩ by k∗ = k⇒ ∘ ρ⇒ ∘ F (inl ∘ inl, inr) and finds an axiomatics for it that is equivalent to the set of 11 axioms described above. Namely, it observes that ρ : E + A → F (E, A) and (\_)∗ : (E + A → F (E’, A’)) → F (E, A) → F (E’, A’), the types coinciding with those of relative monad unit and bind where the sum bifunctor is in the role of the base functor J. The axiomatics contains the relative monad laws k∗ ∘ ρ = k, ρ∗ = id, and (ρ ∘ (h + f))∗ = F (h, f); the remaining law, associativity l∗ ∘ k∗ = (l∗ ∘ k∗)∗, is not valid in general, wherefore it is replaced with three special cases which establish the desired equivalence. For details, please see [18].

We can classify all operations considered in [18], including those not discussed here, into three levels:

1. **Point** operations that create structured objects from pure values: throw, return, and the joint unit ρ. (In [18], the joint unit is denoted by η. We preferred ρ for the sake of uniform notation throughout this paper.)
2. Mixmap $\phi : (E + A \to E' + A') \to F (E, A) \to F (E', A')$ and its special cases e.g. fusel : $F (E + A, A) \to F (E, A)$ and fuser : $F (E, E + A) \to F (E, A)$ given by equations fusel = $\phi (\text{inl} \lor \text{id})$ and fuser = $\phi (\text{id} \lor \text{inr})$. These functions change, without side effects, the output value returned or thrown by their argument computation, but are more general than standard $fmap$ and $bimap$ as they are able to “mix” the bifunctor arguments $E$ and $A$.

3. Handle functions. This level contains functions that execute effectful computations in sequence, i.e., bind, catch and the joint handle. This level subsumes the previous level as one can define mixmap in its general form in terms of the joint handle by $\phi (g) = (\rho \circ g)^*$. The documentation of the source code specifies that the reader monad transformer. It turns out that the exception, reader, writer and state monad transformers preserve the axioms. This axiomatics uses $\rho$ as a primitive; in Subsect. 5.2, we consider an equivalent axiomatics that uses $\text{ask}$ as a primitive and defines $\rho$ in terms of it.

5 Reader Monads Equationally

The class $\text{MonadReader}$ is designed for encoding computations in an implicit environment. Computations in an environment can be seen as stateful computations with the state being immutable. The class is defined in MTL as follows, where $r$ stands for the type of the environment:

```haskell
class Monad m => MonadReader r m | m -> r where
  ask :: m r
  ask = reader id

  local :: (r -> r) -> m a -> m a

  reader :: (r -> a) -> m a

  reader f = do { r <- ask; return (f r); }
```

The documentation of the source code specifies that the method $\text{ask}$ should return the environment of the computation. We assume that $\text{ask}$ has no side effect (though the documentation leaves it unspecified). The method $\text{reader}$ generalizes it by allowing to apply an arbitrary function to the environment. Up to isomorphism, this method embeds the reader monad (defined in Sect. 3) into the monad $m$. This method is analogous to $\rho$ of Sect. 4 that similarly embeds the exception monad into $m$. In the hierarchy described in Sect. 4, both $\text{ask}$ and $\text{reader}$ belong to the first level as point functions. The method $\text{local}$ sets up a modified environment for a local computation. It is a functor application by nature but the Haskell type does not reflect this because the environment type is not a parameter of $m$.

We abandon some Haskell function names in favour of mathematical notation. Similarly to what we saw in the case of exceptions, we here treat the environment type as an extra (first) parameter of the monad and denote the obtained bifunctor by $F$. Hence $\text{local}$ $h$ is written as $F (h, \text{id})$. Note that $F$ is contravariant in its first argument, i.e., if $h : R' \to R$ then $F (h, \text{id}) : F (R, X) \to F (R', X)$. The method $\text{reader}$ is denoted by $\rho : (R \to X) \to F (R, X)$. The arrow that constructs function spaces (like in $R \to X$) can be made a bifunctor by defining, for any $h : R' \to R$ and $f : X \to X'$, a new function $h \circ f : (R \to X) \to (R' \to X')$ by the equation $(h \circ f) g = f \circ g \circ h$. We will use this notation occasionally in this section.

In Subsection 5.1, we propose an axiomatization of computations in environment which directly implies its every model being isomorphic to a monad obtained by an application of the reader monad transformer. It turns out that the exception, reader, writer and state monad transformers preserve the axioms. This axiomatics uses $\rho$ as a primitive; in Subsect. 5.2, we consider an equivalent axiomatics that uses $\text{ask}$ as a primitive and defines $\rho$ in terms of it.

5.1 Reduction to Reader Transformer Applications

Recall that we assume computations in environment being described by a bifunctor $F$ whose first parameter is the environment type and second parameter is the type of the return value. The functor laws are $F (id, id) = id$ and $F (h \circ h', f' \circ f) = F (h', f') \circ F (h, f)$; note the change in the order of the composed morphisms in the first argument due to contravariance. As in the case of exceptions, denote the monad unit and bind by return and $(_\^2$, respectively; their types here are return : $X \to F (R, X)$ and $(_\^2 : (X \to F (R, X')) \to F (R, X) \to F (R, X')$.

Developing an axiomatics that would imply its models being isomorphic to reader transformer applications requires introducing operations that have no counterpart in Haskell MTL. Before doing it, consider the laws to be required that encode, in some way, environment-dependent monadic computations. The dependency does not have to occur in the form of a function dependencies.

Next, $\rho$ being a monad morphism from the underlying reader monad $R \to _\_$ to the monad $F (R, _)$:

$$\rho \circ \text{const} = \text{return} \quad \text{(Rdr-UniHom)}$$

$$\rho \circ k^* = (\rho \circ k)^\# \circ F (\text{id}) \quad \text{(Rdr-BndHom)}$$

Here, $(_\#$ denotes the bind operation of the underlying reader monad; note that $\text{const}$ is its unit. And lastly, $\rho$ being a natural transformation between the power bifunctor and $F$:

$$\rho \circ (h \to f) = F (h, f) \circ \rho \quad \text{(Rdr-Nat)}$$

Although valid in all monads considered in this paper, these axioms are not as powerful as we could do by widening our point of view. The underlying assumption of our approach is that types of the form $F (R, X)$ encode, in some way, environment-dependent monadic computations. The dependency does not have to occur in the form of a function.
whose argument type is \( R \), because applying, for instance, the state monad transformer with state type \( S \) to a member of MonadReader with the environment type \( R \) produces functions with argument type \( S \) (i.e., not \( R \)) inheriting the dependency on \( R \) from the member of MonadReader. Therefore, we introduce functions apply and abstr establishing an isomorphism between types \( F (R, X) \) and \( R \to MX \) for a monad \( M \). More precisely:

\[
\begin{align*}
\text{apply} & : F (R, X) \to R \to MX \\
\text{abstr} & : (R \to MX) \to F (R, X)
\end{align*}
\]

So a computation of the form \( \text{apply} t r \) fixes the environment of the environment-dependent computation \( t \) to be \( r \), which intuitively is an application of a hidden function, and \( \text{abstr} f \) "abstracts" the parameter of its argument function \( f \) by hiding it into the functor computation.

Table 1 presents precise definitions of the exception, reader, writer and state monad transformers (denoted by \( E_E, R_Q, W_W \) and \( S_S \), respectively) along with the corresponding bifunctor transformers (which are denoted similarly) and specifies propagation of \( \rho \), apply and abstr through the transformers; we omit definitions of morphism mappings of the functors and monad operations as they are standard. (In the defining equations for \( \rho \), apply and abstr, the occurrences of these operations in the l.h.s. are those of the bifunctor constructed by the transformer while the occurrences in the r.h.s. belong to the original bifunctor \( F \).) By sequential application of these transformers in all possible orders, we achieve a set of infinitely many structures each either having \( F, M \) and the related operations defined in terms of those of a simpler member structure of this set or being a base case, for which we can take \( F (R, X) = R \to MX \) with \( \rho = \text{id} \to \text{return} \) and both apply and abstr defined as identities. Note that propagation of \( \rho \) is for all transformers defined via composing from the left with the lift operation of the particular transformer; this matches the definition of reader in MTL in the case of exception, writer, and state monad transformer.

Denote the unit and bind of \( M \) by return and \( (\_)^\vee \) like those of \( F (R, \_). \) We specify apply equationally as a function translating the operations of \( F \) to operations of \( M \):

\[
\begin{align*}
\text{apply} \circ F (h, f) & = (h \to MF) \circ \text{apply} \quad \text{(App-Nat)} \\
\text{apply} \circ \text{return} & = \text{const} \circ \text{return} \quad \text{(App-UnitHom)} \\
\text{apply} \circ \text{const} & = (\text{apply} \circ \text{const})^\vee \circ \text{apply} \quad \text{(App-BndHom)} \\
\text{apply} \circ \rho & = \text{id} \to \text{return} \quad \text{(App-Rdr)}
\end{align*}
\]

Here, \( (\_)^\vee \) in the r.h.s. of \( \text{App-BndHom} \) denotes the monad bind of \( M \) lifted to functions. Note that \( \text{const} \circ \text{return} \) in \( \text{App-UnitHom} \) similarly lifts return to functions. We also assume that apply and abstr are inverses of each other:

\[
\begin{align*}
\text{apply} \circ \text{abstr} & = \text{id} \quad \text{(App-Abs)} \\
\text{abstr} \circ \text{apply} & = \text{id} \quad \text{(Abs-App)}
\end{align*}
\]

As a consequence, proving properties of \( F \) and \( \rho \) are reduced to proving properties of monad \( M \). The following theorems hold in \( \text{Set} \):

**Theorem 5.1.** Let \( M \) be a monad. Let \( F \) be a type-preserving, contravariant in its first argument, binary mapping of objects and morphisms. Let \( \rho \), apply, abstr have their right types and meet axioms \( \text{App-Nat, App-UnitHom, App-BndHom, App-Rdr, App-Abs and Abs-App} \). Then \( F \) meets the functor laws, its left section for every type \( R \) is a monad w.r.t. return and \( (\_)^\vee \), and the equations \( \text{Bifun-UnitHom, Bifun-BndHom, Rdr-UnitHom, Rdr-BndHom and Rdr-Nat} \) are all valid.

**Theorem 5.2.** Let \( M \) be a monad. Then:
- The bifunctor obtained by applying the reader monad transformer with an environment type \( R \) to \( M \), with apply and abstr defined as identities and other operations defined like in the Haskell MTL, satisfies the axioms \( \text{App-Nat, App-UnitHom, App-BndHom, App-Rdr, App-Abs and Abs-App}; \)
- Applying the exception, reader, writer and state bifunctor (and monad) transformers preserve these axioms.

Proofs are straightforward.

5.2 **Axioms of ask**

The definition of class MonadReader provides mutual definitions of reader and ask, looking in our language as follows:

\[
\begin{align*}
\text{ask} & = \rho (\text{id}) \quad \text{(Ask-Rdr)} \\
\rho (f) & = F (\text{id}, f) \text{ ask} \quad \text{(Rdr-Ask)}
\end{align*}
\]

In order to axiomatize ask instead of \( \rho \), we replace \( \text{App-Rdr} \) with

\[
\text{apply ask} = \text{return} \quad \text{(App-Ask)}
\]

That is, asking of the environment as a computation of type \( F (R, R) \), when presented as a function of type \( R \to MR \), equals the monad unit of \( M \) that immediately returns the argument environment.

This gives an equivalent axiomatics indeed, as established by the following theorem:

**Theorem 5.3.** Let \( M \) be a monad. Let \( F \) be a type-preserving, contravariant in its first argument, binary mapping of objects and morphisms. Let \( \rho \), apply, abstr have their right types and meet axioms \( \text{App-Nat, App-UnitHom, App-BndHom, App-Abs and Abs-App} \). Then the set of equations \( \{ \text{App-Rdr, Ask-Rdr} \} \) is equivalent to the set of equations \( \{ \text{App-Ask, Rdr-Ask} \} \).

Proof. Straightforward but we present it fully for illustrating equational reasoning in our axiomatics. If we assume \( \text{App-Rdr} \), the premises of Theorem 5.1 are fully met which
Table 1. Definitions of functions ρ, apply and abstr for different monad transformers.

| Exception transformer: | Reader transformer: | Writer transformer: |
|------------------------|---------------------|---------------------|
| E_{E}MX = M(E + X)    | R_{Q}MX = Q → MX   | W_{W}MX = M(X × W) |
| E_{E}F(R, X) = F(R, E + X) | R_{Q}F(R, X) = Q → F(R, X) | W_{W}F(R, X) = F(R, X × W) |
| ρ = F(id, idm)ρ       | ρ = const ◦ ρ       | ρ = F(id, id ◦ const) ◦ ρ |
| apply tr = apply tr    | apply tr = ρ q • apply (t q) r | apply tr = apply tr    |
| abstr f = abstr f      | abstr f = ρ q • abstr(ρ r, f q r) | abstr f = abstr f      |

State transformer:

| State transformer: |
|---------------------|
| S_{S}M X = S → M(X × S) |
| S_{S}F(R, X) = S → F(R, X × S) |
| ρ = (λ ts. F(id, id ◦ const s) t) ◦ ρ |
| apply tr = λ s. apply(t s) r |
| abstr f = λ s. abstr(λ r, f s r) |

allows us to use Rdr-Nat in the proof of Rdr-Ask. In the proof of Ask-Rdr, we can use functor laws since their proof does not need App-Rdr.

- `{App-Rdr, Ask-Rdr} ⊢ {App-Ask, Rdr-Rdr}

- `{App-Ask, Rdr-Rdr} ⊢ {App-Rdr, Ask-Rdr}

6 An Abstract View and its Application to MonadWriter

The class MonadWriter is defined in MTL as follows:

class (Monoid w, Monad m) ⇒ MonadWriter w m

| m → w where |
| writer :: (a, w) → m a |
| tell :: w → m () |
| listen :: m a → m (a, w) |
| pass :: m (a, w → w) → m a |
| writer ~(a, w) = do { tell w; return a; } |
| tell w = writer ((), w) |

The method writer, similarly to the method reader in the class MonadReader, embeds the writer monad into the monad m. The method tell logs the value given as argument and immediately returns the value (). It is a special case of writer. Both belong to the first level in the hierarchy of Sect. 4.

The method listen applies to a monadic computation and copies the whole log of this computation into its return value. The method pass applies to a monadic computation, the return value of which contains a function, and modifies the log of this computation by applying this function. Note that both listen and pass modify a monadic computation in a way that can be encoded as a transformation of values of the writer monad, i.e., in the form of a function of type X × W → X' × W: for listen, the corresponding function is ⃗(a, w), and for pass, the function is (λ ((a, f), w), f w). According to Sect. 4, these methods are mixmaps and belong to the second level of the method hierarchy. If the class MonadWriter contained a general method mixmap :: (λ (a, w) → (a', w)) → m a → m a', we could define listen and pass as

listen = mixmap (λ (a, w) → ((a, w), w))

pass = mixmap (λ ((a, f), w) → (a, f w))

On the other hand, mixmap can be defined in terms of listen and pass by

□
mixmap \( g = \text{pass} \circ \text{fmap} \ (\text{bimap} \ \text{id} \ \text{const} \circ g) \circ \text{listen} \)

Unlike in the case of exceptions and environments, we do not consider the monoid parameter of the class as an extra parameter of the monad. Keeping the monoid fixed enables us to derive the structure of the WriterMonad class methods as some kind of reflection of the structure of the underlying writer monad. This is not to say that generalizing the approach by letting the monoid vary would be pointless. In the rest of this section, we denote the monad under consideration by \( M \) but also use the bifunctor notation occasionally in Subsections 6.2 and 6.4 (with fixed monoid).

6.1 Pointed Functors with Mixmap

Analogously to the case of exceptions, denote the relative mixmap as one of the two functors that are particularly interesting because, as shown in Sect. 4, it does not hold in the axiomatics of exceptions we saw in Sect. 4. In this sense, MonadWriter behaves more nicely than MonadError.

For finding properties of \textit{tell}, \textit{listen} and \textit{pass}, one can rely on the mixmap laws and the previously seen definitions of these methods in terms of mixmap. We will not search for an equivalent axiomatics using these methods as primitives but we define two simpler functions and find an equivalent axiomatics for them. To this end, note that \textit{listen} and \textit{pass} serve dual purposes in the sense that \textit{listen} copies the log into the return value while \textit{pass} moves information from the return value to the log. Instead of \textit{listen} and \textit{pass}, we consider \textit{shift} : \( MX \to M (X \times W) \) and \textit{fuse} : \( M (X \times W) \to MX \), defined by equations \textit{shift} = \( \phi(\lambda (a, w), ((a, w), I)) \) and \textit{fuse} = \( \phi(\lambda ((a, w'), w), (a, w \cdot w')) \), that achieve the same aims in a cleaner way. The function \textit{shift} copies the log into the return value but, unlike \textit{listen}, replaces the original with the monoid unit. The function \textit{fuse} uses the monoid multiplication to join a monoid element in the return value with the current log. In terms of shift and fuse, the general mixmap can be expressed as

\[
\phi(g) = \text{fuse} \circ M \ g \circ \text{shift} \quad (\text{Mixmap-FuseShift})
\]

Note that \( \lambda p.I \) and \( \lambda ((a,w'),w). (a,w \cdot w') \) are the unit and join, respectively, of the writer monad. Denoting the unit and join by \( \eta \) and \( \mu \), respectively, we can abstract from the underlying monad and specify \textit{shift} and \textit{fuse} by

\[
\begin{align*}
\text{shift} & = \phi(\eta) \quad (\text{Shift-Mixmap}) \\
\text{fuse} & = \phi(\mu) \quad (\text{Fuse-Mixmap})
\end{align*}
\]

The types in the abstract view are \( \text{shift} : MX \to M (JX) \) and \( \text{fuse} : M (JX) \to MX \) where \( J \) denotes the underlying monad. The mapping of morphisms by \( M \) can be given via \( \phi \):

\[
M f = \phi(fj) \quad (\text{Fun-Mixmap})
\]

Assuming this as definition, one can prove the two functor laws for \( M \) using Mixmap-Id, Mixmap-Comp and the functor laws for \( f \). Moreover, the unit of \( M \) can be expressed by

\[
\text{return} = \rho \circ \eta \quad (\text{Point-RPoint})
\]

Proof of naturality of return is straightforward using naturality of \( \eta \) along with RPoint-RNat and Fun-Mixmap.

We are now going to form an alternative axiom set that uses shift and fuse instead of \( \phi \) as the underlying operations. Firstly, we include the following laws that resemble the coherence conditions of monad join:

\[
\begin{align*}
\text{fuse} \circ M \mu & = \text{fuse} \circ \text{fuse} \quad (\text{Fuse-Fuse}) \\
\text{fuse} \circ M \eta & = \text{id} \quad (\text{Fuse-FunPoint}) \\
\text{fuse} \circ \text{shift} & = \text{id} \quad (\text{Fuse-Shift})
\end{align*}
\]

Secondly, we include the following two homomorphism laws for \textit{shift}:

\[
\begin{align*}
\text{shift} \circ \rho & = \text{return} \quad (\text{Shift-PointHom}) \\
\text{shift} \circ \phi(g) & = M g \circ \text{shift} \quad (\text{Shift-RNat})
\end{align*}
\]

Note that Shift-PointHom and Shift-RNat uniquely determine \( \rho \) and \( \phi \). Hence there is no need to include direct definitions of \( \rho \) and \( \phi \); but if we did it, \( \rho \) would be given by

\[
\rho = \text{fuse} \circ \text{return} \quad (\text{RPoint-Point})
\]

and \( \phi \) by Mixmap-FuseShift (for proof, compose Shift-PointHom and Shift-RNat with fuse from the left and apply Fuse-Shift). Then we could replace Shift-PointHom and Shift-RNat with axioms not mentioning \( \rho \) and \( \phi \) to obtain an axiomatization expressed fully in terms of shift and fuse. We prefer the homomorphism laws for brevity and elegance.

Altogether, we have the following result that establishes equivalence of the axiomatics of \( \phi \) and the axiomatics of shift and fuse in every category:

\begin{theorem}
Let \((J, \eta, \mu)\) be a monad. Suppose that \( M \) consists of a mapping of objects to objects and a type-preserving mapping of morphisms to morphisms. (This is to say that \( M \) is an endofunctor without assuming functor laws.) Furthermore, assume transformations \( \rho : JX \to MX, \text{return} : X \to MX, \) shift : \( MX \to M (JX) \) and fuse : \( M (JX) \to MX \) along with \( \phi \) that maps morphisms of type \( JX \to JX' \) to morphisms of type \( MX \to MX' \) being given. Then the set of laws consisting of RPoint-RNat, Mixmap-Id, Mixmap-Comp,
Shift-Mixmap, Fuse-Mixmap, Fun-Mixmap and Point-RPoint is equivalent to the set of laws consisting of:
- Two functor laws for $M$;
- Naturality of return and fuse;
- The coherence laws Fuse-Fuse, Fuse-FunPoint, and Fuse-Shift;
- The homomorphism laws Shift-PointHom and Shift-RNat.

The proofs are straightforward.
We have not mentioned naturality of $\rho$ and shift; both are implied by the axioms considered. One can also deduce the following dual of Fuse-Fuse law from the axioms:
\[
M \eta \circ \text{shift} = \text{shift} \circ \text{shift} \quad \text{(Shift-Shift)}
\]

6.2 Two-Story Monads
So far, bind operation of $M$ was not involved into our study. At the third level of the hierarchy defined in Sect. 4 for exceptions, we also had a "joint handle" function of type (E + A → F (E', A')) → F (E, A) → F (E', A') generalizing the monad bind; we noted that its type equals the type of bind of a relative monad on + but it unfortunately does not meet all relative monad laws. One can also define a similar function of type (X × W → M X') → M X → M X' that takes the current log along with the return value into account when binding two computations with writer effects. All relative monad laws turn out to be satisfied for all structures constructed via the four monad transformers we consider in this paper. Therefore we start axiomatizing of the third level from the relative monad laws (with, again, $J$ replacing the writer monad):
\[
k^\ast \circ \rho = k \quad \text{(RBnd-UnitL)}
\]
\[
\rho^\ast = \text{id} \quad \text{(RBnd-Id)}
\]
\[
l^\ast \circ k^\ast = (l^\ast \circ k)^\ast \quad \text{(RBnd-Assoc)}
\]
\[
M f = (\rho \circ f^\ast)^* \quad \text{(Fun-RBnd)}
\]

In order to be able to express the usual monad bind of type (X → M X') → M X → M X' in terms of the relative monad bind whose type in the case of an abstract base functor $J$ is (J X → M X') → M X → M X', we consider a pseudobind $(\_)^\wedge$ of type (X → M X') → J X → M X'. Then one can define bind of $M$, denoted by $(\_)^\wedge$, by
\[
k^\wedge = k^\ast \ast \quad \text{(Bnd-PBndRBnd)}
\]
(We use the half-star notation for bind of $M$ and leave the standard notation $(\_)^\ast$ for bind of the monad $J$. The term pseudobind was chosen after Steele Jr. [19]; we will discuss Steele’s work in Subsect. 6.3.) In the writer case, we can take
\[
k^\wedge = \chi (a, w) \cdot f ((w \cdot), \text{id}) (a) \quad \text{(PBnd-Bifun)}
\]
where the section notation of Haskell is used in the first argument of $F$, which itself is the bifunctor obtained from $M$ by treating $W$ as its (first) parameter. (The type $W$ is still fixed. The domain of the additional parameter of $F$ is the category consisting of a singleton object $W$ and its endomorphisms.) So $k^\wedge$ takes a pair $(a, w)$, applies $k$ to $a$ and multiplies the log of the computation by $w$ from the left. We could not use $\phi (\text{id} \times (w \cdot))$ instead as the exception monad transformer does not preserve the equality $F(f, \text{id}) = \phi (\text{id} \times f)$.

We will study $F$ more in Subsect. 6.4.
For $(\_)^\wedge$, we use the following monad-like axioms:
\[
k^\wedge \circ \eta = k \quad \text{(PBnd-UnitL)}
\]
\[
(\rho \circ f)^\wedge = \rho \circ f^\ast \quad \text{(PBnd-RPoint)}
\]
\[
l^\wedge \circ k^\wedge = (l^\wedge \circ k)^\wedge \quad \text{(PBndBnd-Assoc)}
\]

But how to express the relative monad bind in terms of the bind of $M$? The functions shift and fuse studied in connection with mixmap can help. Firstly, we can define $\phi$ via $(\_)^\wedge$ by generalizing Fun-RBnd:
\[
\phi (g) = (\rho \circ g)^\ast \quad \text{(Mixmap-RBnd)}
\]
Then shift and fuse are expressible via $\phi$ by Shift-Mixmap and Fuse-Mixmap; hence these functions are definable in our "two-story monad" framework. Now we achieve an equivalent axiomatics in terms of monad $M$, shift and fuse if we add the following axioms, the first two of which are for defining $(\_)^\wedge$ and $(\_)^\wedge$, to the union of monad axioms for $M$ and the previously seen axioms of shift and fuse:
\[
\text{shift} \circ k^\ast = (\text{shift} \circ k)^\wedge \circ \text{shift} \quad \text{(Shift-BndHom)}
\]
\[
k^\wedge = k^\ast \circ \rho \quad \text{(PBnd-Bnd)}
\]
\[
\rho^\wedge = \text{fuse} \quad \text{(Bnd-RPoint)}
\]

The previously established axiomatics for shift and fuse declares shift to be a homomorphism between the relative point and point, as well as between mixmap and functor; Shift-BndHom extends this pattern also to the third level. It enables to express $(\_)^\wedge$ via $(\_)^\wedge$ as
\[
k^\wedge = k^\ast \circ \text{shift} \quad \text{(PBnd-Bnd)}
\]

Indeed:
\[
k^\ast = \{\text{Fuse-Shift, identity}\}
\]
\[
\text{fuse} \circ \text{shift} \circ k^\ast = \{\text{Bnd-RPoint, Shift-BndHom}\}
\]
\[
\rho^\wedge \circ (\text{shift} \circ k)^\wedge \circ \text{shift} = \{\text{monad}\}
\]
\[
(\rho^\wedge \circ \text{shift} \circ k)^\wedge \circ \text{shift} = \{\text{Bnd-RPoint}\}
\]
\[
(\text{fuse} \circ \text{shift} \circ k)^\wedge \circ \text{shift} = \{\text{Fuse-Shift, identity}\}
\]
\[
k^\wedge \circ \text{shift}
\]
Theorem 6.2. Let \((J, \eta, \mu)\) be a monad, let \(M\) be given as in Theorem 6.1, and let \(\rho\), return, shift, fuse and \(\phi\) be given with the same types as in Theorem 6.1. Moreover, assume:

\[
\begin{align*}
(\_)^\chi & : (X \to MX') \to MX \to M\chi X' \\
(\_)^* & : (JX \to MX') \to MX \to M\chi X' \\
(\_)^7 & : (X \to MX') \to JX \to M\chi X'
\end{align*}
\]

Then the set consisting of the laws \(\text{RBnd-UnitL}, \text{RBnd-Id}, \text{RBnd-Assoc}, \text{PBnd-UnitL}, \text{PBnd-RPoint}, \text{PBnBnd-Assoc}, \text{Mixmap-RBnd}, \text{Fun-RBnd}\) (or: \(\text{Fun-Mixmap}\)), \(\text{Fuse-Mixmap}, \text{Point-RPoint}\) and \(\text{Bnd-PBnBnd}\) is equivalent to the axiomatics consisting of:

- Naturality of fuse;
- Three coherence laws of fuse;
- Three homomorphism laws of shift (\text{Shift-PointHom}, \text{Shift-RNAT} and \text{Shift-BdHom});
- Four monad laws of \(M\) (including the definition of the morphism mapping of \(M\));
- \(\text{Bnd-RPoint}\) and \(\text{PBnd-Bnd}\).

The proofs are straightforward. In the light of the ability of expressing the operations used in this paper and the \text{MonadWriter} class methods in terms of each other, we also have the following result:

Theorem 6.3. Let \(M\) be any monad in the category \text{Set}. Then:

- The monad obtained by applying the writer monad transformer to \(M\), with methods defined as in MTL, satisfies all laws mentioned in Theorem 6.2 and also \(\text{PBnd-Bifun}\);
- Applying the exception, reader, writer, and state monad transformers preserves the laws.

6.3 Connections to Steele’s Pseudomonads

Our two-story monads are close to the pseudomonad towers studied by Steele Jr. [19]. That classic paper aims to find ways to join different monadic effects by “composing” the carrier monads of each particular effect. As monads are not always behaving nicely under composition, it introduces pseudomonads that generalize monads by allowing the target type of the function under bind to differ from the source type of the function produced by bind (the target types of the function under bind and of that produced by bind still coincide). Unit and bind of a pseudomonad are called pseudounit and pseudobind. In this sense, our operation \((\_)^7\) along with the unit \(\eta\) of monad \(J\) are pseudobind and pseudounit of a pseudomonad. The notion of monad itself as treated in [19] is wider than standard and subsumes also relative monads.

That paper modifies the standard monad axioms to be applicable to pseudomonads. Here are Steele’s axioms, written in the language of our paper:

\[
\begin{align*}
(\eta \cdot \eta)^k &= k & (\text{Steele-UnitL}) \\
(h \cdot \eta)^k &= h & (\text{Steele-UnitR}) \\
(\eta \cdot (\eta \cdot k)) &= (\eta \cdot \eta)^k & (\text{Steele-Assoc})
\end{align*}
\]

Of these, \(\text{Steele-UnitL}\) coincides with our axiom \(\text{PBnd-UnitL}\). But the next axiom, \(\text{Steele-UnitR}\), is not valid in general. It implies that every function of type \(JX \to MX'\) is a result of pseudobind—and in particular, every function of type \(JX \to JX'\) is a result of bind, which is clearly not true. The last axiom, \(\text{Steele-Assoc}\), is a theorem in our axiomatics, provable as follows:

\[
\begin{align*}
(\eta \cdot (\eta \cdot k))^\ell &= (\eta \cdot \eta)^\ell \cdot (\eta \cdot k)^\ell \\
(\eta \cdot (\eta \cdot k)) &= (\eta \cdot \eta)^\ell \cdot (\eta \cdot k)^\ell = (\eta \cdot (\eta \cdot k))^\ell
\end{align*}
\]

We were not able to prove \(\text{PBnBnd-Assoc}\) from \(\text{Steele-Assoc}\), so it seems that \(\text{Steele-Assoc}\) is strictly weaker than \(\text{PBnBnd-Assoc}\). (This does not mean a flaw in [19]. The weaker axiom might be perfect for the purposes of that paper which aims to involve cases where the monad obtained by composition does not satisfy associativity. In our axiomatics, associativity of monad \(M\) is forced by the laws of its constituent pseudomonad and relative monad.)

6.4 Some Corollaries and Non-Corollaries

In this subsection, we are working in the category \text{Set}. According to the class definition given at the beginning of Sect. 6, one can express

\[\text{tell} = \rho \circ (\text{const}(\_ \triangle \text{id})).\]

Using the theory developed above, any expression of the form \(\text{tell} w \Rightarrow t\) can be rewritten as \((\text{const} t)^\ell ((\_), w)\). (Rewrite \(\text{tell} w\) and \(\Rightarrow\) by their meaning and apply \(\text{PBnd-Bnd}\).) On the other hand, \(\text{PBnBnd-Bifun}\) allows to conclude that

\[
(\text{const} t)^\ell ((\_), w) = F ((\_ \cdot w), \text{id}) t \quad (\text{Bifun-PBnd})
\]

Therefore, bifunctor applications of the form \(F ((\_ \cdot w), \text{id}) t\) are equivalent to \(\text{tell} w \Rightarrow t\). Using \(\text{Bifun-PBnd}\) as the definition of such bifunctor applications, we can prove that they
satisfy the functor laws. For identity:

\[
\text{id} = (\text{id, constant})\\
\text{\land t \cdot const t}() = (\text{PBND-UNITL})\\
\text{\land t \cdot (const t)^2(\eta())} = (\text{PBND, extensionality})\\
F((I\cdot), id) = (\text{monoid unit})\
\]

For composition,

\[
F((w\cdot), \text{id}) \circ F((w\cdot), \text{id}) = (\text{composition})\\
\text{\land t \cdot F((w\cdot), \text{id}) F((w\cdot), \text{id}) t} = (\text{BIFUN-PBND, twice})\\
\text{\land t \cdot ((const t)^2(((), w)))^2(((), w'))} = (\text{Steele-Assoc})\\
\text{\land t \cdot ((const t)^2 \circ (id \triangle const w))^2(((), w'))} = (\text{composition})\\
\text{\land t \cdot (const t)^2(((), w))} = (\text{writer})\\
\text{\land t \cdot (const t)^2((id \times (w\cdot)))((id \triangle const w)())} = (\text{product, constant})\\
\text{\land t \cdot (const t)^2(((), w\cdot \cdot w))} = (\text{BIFUN-PBND, extensionality})\\
F((w\cdot \cdot w\cdot), \text{id}) = (\text{monoid associativity})\\
F((w\cdot\cdot), (w\cdot), \text{id})\
\]

The latter implies the equation

\[
\text{tell w \Rightarrow tell w'} = \text{tell (w \cdot w')}
\]

(for proof, rewrite \text{tell w'} = \text{tell w'} \Rightarrow \text{return()} and later the same for \text{w \cdot w'}).

We leave the proofs of the following two corollaries as an exercise:

\[
\rho \circ (id \times (w \cdot)) = F((w\cdot), \text{id}) \circ \rho = (\text{RPOINT-BINAT})\\
F((w\cdot), \text{id}) \circ k^\triangledown = k^\triangledown \circ F((w\cdot), \text{id}) = (\text{BIFUN-BND-Comm})\
\]

Finally, there are laws that cannot be deduced from the developed theory but are valid in all monads constructible by applying the writer monad transformer to any monad and preserved by the exception, reader, writer and state monad transformers. For instance:

- \text{RPOINT-BINAT} stays true after replacing \((w \cdot)\) with arbitrary \(f : W \rightarrow W\);
- For any monoid endomorphism \(h\), mapping of the computation log by \(h\) is a monad homomorphism:

\[
F(h, \text{id}) \circ \text{return} = \text{return} (\text{BIFUN-UNITHom})\\
F(h, \text{id}) \circ k^\triangledown = (F(h, \text{id}) \circ k)^\triangledown \circ F(h, \text{id}) (\text{BIFUN-BNDHom})
\]

### 7 Stateful Computations

In MTL, the class of stateful monads is introduced as follows:

**class Monad m \Rightarrow MonadState s m | m → s where**

**get** :: m s

**get** = state (\(\lambda s \rightarrow (s, s)\))

**put** :: s → m ()

**put s** = state (\(\lambda s \rightarrow (((), s))\))

**state** :: (s → (a, s)) → m a

**state f** = do

s ← get

let \(s' = f s\)

put \(s'\)

return \(a\)

The method **get**, by intention, returns the current state without modifying it, and the method **put** replaces the current state with the one given as argument while returning the trivial value \(()\). The method **state**, analogously to **reader** and **writer** seen previously, embeds a stateful operation represented in the form of a pure function in the monad.

If we denote by \(J\) the state monad \(J X = S \rightarrow X \times S\), where \(S\) is an arbitrary fixed type, and by \(M\) any monad of the **MonadState** class, then the method **state** has type \(J X \rightarrow MX\) and is in principle the point \(\rho\) of the functor \(M\) relatively pointed on \(J\). Thus all methods of **MonadState** classify as first-level in the hierarchy defined in Sect. 4 as get and put are special cases of \(\rho\). Like in Sect. 6, we keep the second type fixed, i.e., functors have only one argument.

Compared to the other classes of monads, there exists relatively much previous research considering some equational axiomatization of **MonadState**. Gibbons and Hinze [5] assume the following set of axioms:

\[
\text{put s} \gg \text{put s'} = \text{put s'} = (\text{PUT-PUT})\\
\text{put s} \gg \text{get} = \text{put s} \gg \text{return s} = (\text{PUT-GET})\\
\text{get} \gg \text{put} = \text{return}() = (\text{GET-PUT})\\
\text{get} \gg \text{\lambda s}, \text{get} \gg k s = \text{get} \gg \text{\lambda s} \cdot k s s = (\text{GET-GET})
\]
These axioms, which can now be called classic, are valid for all monads obtained by application of the state monad transformer to any monad and subsequent applications of error, reader, writer and state monad transformers. Almost all other work that we are aware of relies on the same axioms with irrelevant modifications. Exceptions are Harrison and Hook [7] and Harrison [6] that formalize their axioms in terms of get and update rather than get and put, where update corresponds to the MTL function \texttt{modify} defined by

\[
\begin{align*}
\text{modify} & : \text{MonadState} \ s \ m \Rightarrow (s \rightarrow s) \rightarrow m () \\
\text{modify} \ f & = \text{state} \ (\lambda s \rightarrow (((), f \ s))
\end{align*}
\]

(The early papers on transformers [10, 12, 14] had a singleton method \texttt{update} in class \texttt{MonadState} ; this method behaved similarly to \texttt{modify} but returned the state.) The axioms used by [6, 7] seem to be weaker than that of [5] and our experience suggests that any set of laws for \texttt{modify} equivalent to the Gibbons-Hinze axiomatics of \texttt{get} and \texttt{put} is probably less elegant.

Sometimes also the following unit law is included:

\[
\text{get} \gg t = t 
\]  \hspace{1cm} \text{(Get-UnitR)}

The law \texttt{Get-UnitR} follows from the axioms above. More surprisingly, the last among the above axioms, \texttt{Get-Get}, turns out to be implied by \texttt{Put-Put}, \texttt{Put-Get} and \texttt{Get-Put}. Given the translations between the methods of class \texttt{MonadState} in the class declaration, the set of axioms \texttt{Put-Put}, \texttt{Put-Get} and \texttt{Get-Put} is equivalent to the statement that \( \rho : J X \rightarrow M X \) is a monad morphism. All the facts listed in this paragraph were noted by Li-yao Xia in the post [20] (we have checked that these claims are correct indeed). As we have found no research papers mentioning them, despite the numerous authors using the state monad axioms, it seems that these facts are not widely known to the Haskell research community.

\section{8 Conclusion}

We have investigated equational axiomatizations for the \texttt{MonadError}, \texttt{MonadReader}, \texttt{MonadWriter}, and \texttt{MonadState} type classes defined in Haskell MTL, along with reviewing previous related work. For each class, we have (or the previous work referred to has) proposed at least two alternative axiomatizations that are proven to be equivalent. In the case of \texttt{MonadError} and \texttt{MonadReader}, the proposed axiomatizations assume a wider setting that goes beyond the limits imposed by the Haskell MTL. We think that this should not be considered as a shortcoming, as the opportunity to extend the world can facilitate proving theorems or perhaps even prove more theorems about the usual Haskell world. For instance, it is not easy to find a proof of the analogous to \texttt{Get-UnitR} law

\[
\text{ask} \gg t = t 
\]  \hspace{1cm} \text{(Ask-UnitR)}

using only the Haskell world laws of \texttt{MonadReader} from Subsect. 5.1 and the definition of \texttt{ask}. When switching to the axiomatics that involves apply and \texttt{abstr}, proving \texttt{Ask-UnitR} becomes straightforward.

One more contribution of this work is a classification of all methods of the four MTL classes into three levels (points, mixmaps and handles) which have similar categorical interpretation for all classes and based of which axiomatizations of different classes can be compared. For instance, a lot of laws are common to the axiomatics of \texttt{MonadError} studied in our previous work [18] and the axiomatics of \texttt{MonadWriter} considered in this paper; some of the common laws may look different. One can observe that unit and join of the exception monad \( X = E + X \) are \texttt{inr} and \texttt{inl} \texttt{\_\_id}, respectively, so one could define shift = \( \phi (\text{inr}) \) and fuse = \( \phi (\text{inl} \_\_id) \) for exceptions like we did for writer effects. In [18], \( \phi (\text{inl} \_\_id) \) was denoted fusel and \( \phi (\text{inr}) \) is equivalent to \( \rho ^ { - 1} \circ F (\text{inr} \_\_id, \text{inr}) \) in the axiomatics of [18]. Hence our two-story monad laws \texttt{Fuse-FunPoint} and \texttt{RBND-BND} occurred in [18] in a form that did not help to recognize them as phenomenons of such an abstract level. We hope that this uniform view of different effects may conduce to deeper understanding of the MTL class methods.

The paper might also inspire future development of MTL. Using bifunctors rather than monads or replacing MTL methods with functions more convenient in the theory is not meant to suggest making the corresponding changes in Haskell. For instance, the method \texttt{listen} is likely to be more useful in practice than our shift would be, and we do not know a way to redefine all MTL classes simultaneously to apply to binary type constructors instead of unary ones without severe penalties in usability. (Our earlier work [17] makes steps towards the latter, with the aim of increasing the expressive power of exception handling in the applicative style, but the only type that are added to monads as an extra parameter is the type of exceptions.) However, adding some new methods to the existing classes would be reasonable. In particular, the class \texttt{MonadWriter} would benefit from including the general mixmap as class method, as it would provide the shortest way to define various functions of the mixmap level one might desire besides the existing \texttt{listen} and \texttt{pass} methods. The same addition could be useful in class \texttt{MonadError} (with fixed error type).

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