Research Article

An Exponential Spline Difference Scheme for Solving a Class of Boundary Value Problems of Second-Order Ordinary Differential Equations

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In this paper, we mainly study an exponential spline function space, construct a basis with local supports, and present the relationship between the function value and the first and the second derivative at the nodes. Using these relations, we construct an exponential spline-based difference scheme for solving a class of boundary value problems of second-order ordinary differential equations (ODEs) and analyze the error and the convergence of this method. The results show that the algorithm is high accurate and conditionally convergent, and an accuracy of \((1/240)h^6\) was achieved with smooth functions.

1. Introduction

In physics, chemistry, biology, sociology, and many other disciplines, there are tremendous problems that can be described by differential equations (DEs), but it is difficult to get their explicit expressions. So, people began to seek the numerical solutions of these problems, which can also be applied to scientific research and engineering practice if their accuracy satisfies the needs. Especially the advent of computers makes it possible to quickly carry out a large number of calculations, which also makes the numerical solution method of DEs become one of the most important branches of computational mathematics. Due to its high smoothness, low power, and easy calculation, the spline function has been widely used in computer graphics, data interpolation and fitting, shape control, and numerical solutions of DEs. There are two main schemes in numerical solutions of DEs using spline functions: the spline finite element method and the spline difference method. The first has a wide range of application and can be applied to many types of equations, but it requires a large amount of calculation. While the second is simple process with a small amount of calculation and high accuracy, but it can only be applied to specific types of equations.

In this paper, we mainly focus a class of second-order ordinary differential equations (ODEs):

\[
    u'' + q(x)u' + p(x)u = g(x), \quad x \in [a, b],
\]

which meets one of the following boundary conditions.

(1) First boundary condition:

\[
    u(a) = \mu_0, \quad u(b) = \mu_1.
\]

(2) Second boundary condition:

\[
    u^{(1)}(a) = \mu_0, \quad u^{(1)}(b) = \mu_1.
\]

(3) Third boundary condition:

\[
    u(a) = \mu_0, \quad u^{(1)}(b) = \mu_1.
\]
where \( \mu_0 \) and \( \mu_1 \) are constants and \( p(x) \), \( q(x) \), and \( g(x) \) are continuous functions in the interval of \([a, b]\).

Many scholars have been studying such two-point boundary value problems. Albasiny and Hoskins [1] and Raghavarao et al. [2] used cubic polynomials to solve such problems. Blue [3] used quintic polysplines to solve such problems. Caglar et al. [4] used cubic B-splines in their solving scheme. Chawla and Shiva Kumar [5] extended the problem to semi-infinite regions.

In recent years, with the deepening of research, people have begun to use nonpolynomial splines to solve such problems. Zahra [6], Rao and Kumar [7], Tirmizi et al. [8], Ramadan et al. [9], Sural and Stojanović [10], Jha [11, 12], and Kadalbajoo and Patidar [13] have carried out a lot of research in this area and achieved very high computational accuracy.

However, there are still many theoretical problems to be broken in the study of nonpolynomial splines. Due to the diversity of nonpolynomial splines, it is crucial to choose the basis and parameters in solving the problem. However, there is still no reference in this regard. In this paper, a selected set of spline basis functions was used to deduce the relationship between the derivative and the function value and then to obtain the second-order difference scheme for solving second-order ODEs, which provides a method for solving such problems.

2. Exponential Spline Function Space

Exponential spline refers to a type of spline in which the nonpolynomial factors of spline basis functions contain only exponential functions. The exponential spline in this sense is not very specific; it can contain many forms of exponential spline, which can produce substantially different splines, and is inconvenient to study. Therefore, the exponential spline refers to that with a specific form in the rest of this work.

Next, we define an exponential spline function space. Let

\[
s_i(x) = a_i + b_i(x-x_i) + c_i e^{\tau_1(x-x_i)} + d_i e^{\tau_2(x-x_i)},
\]

\[i = 1, 2, \ldots, n,
\]

where \( a_i, b_i, c_i, \) and \( d_i \) are coefficients and \( \tau_1 \) and \( \tau_2 \) are parameters with \( \tau_1 \neq \tau_2 \).

\[
\mathcal{E}_3^r(\mathcal{D}_n) = \{ s: s(x) = s_i(x), x \in I_i, i = 1, 2, \cdots, \}
\]

\[ns(x) \in C^r[a, b], r < 3, \]

is called the cubic \( r \)-order exponential spline function space. Obviously, the function \( s(x) \) in \( \mathcal{E}_3^r(\mathcal{D}_n) \) must meet

\[
s_i^{(r)}(x_i) = s_i^{(r)}(x_i), \quad r = 0, 1, 2; \quad i = 1, 2, \cdots, n - 1.
\]

The dimension of \( \mathcal{E}_3^2(\mathcal{D}_n) \) is \( n + 3 \). Then, we find a set of basic functions with local supports for \( \mathcal{E}_3^2(\mathcal{D}_n) \).

Assume \( s(x) \in \mathcal{E}_3^2(\mathcal{D}_n) \), for given \( j, 2 \leq j \leq n - 2 \), let

\[
\left\{
\begin{array}{l}
s_{j-1}^{(r)}(x_{j-2}) = 0, \quad r = 0, 1, 2, \\
\end{array}
\right.
\]

\[
\left\{
\begin{array}{l}
s_{j+2}^{(r)}(x_{j+2}) = 0, \quad r = 0, 1, 2, \\

s_{j+k}^{(r)}(x_{j+k}) = s_{j+k+1}^{(r)}(x_{j+k}), \quad k = -1, 0, 1, \quad r = 0, 1, 2, \\
\end{array}
\right.
\]

\[
s_j(x_j) = 1.
\]

We can obtain

\[
a_{i-1} = \frac{|A_i|}{|A|}, \quad b_{i-1} = \frac{|A_2|}{|A|}, \quad c_{i-1} = \frac{|A_3|}{|A|}, \quad d_{i-1} = \frac{|A_4|}{|A|}, \]

\[
a_i = \frac{|A_0|}{|A|}, \quad b_i = \frac{|A_1|}{|A|}, \quad c_i = \frac{|A_5|}{|A|}, \quad d_i = \frac{|A_6|}{|A|}, \]

\[
a_{i+1} = \frac{|A_0|}{|A|}, \quad b_{i+1} = \frac{|A_{10}|}{|A|}, \quad c_{i+1} = \frac{|A_{11}|}{|A|}, \quad d_{i+1} = \frac{|A_{12}|}{|A|}, \]

\[
a_{i+2} = \frac{|A_{13}|}{|A|}, \quad b_{i+2} = \frac{|A_{14}|}{|A|}, \quad c_{i+2} = \frac{|A_{15}|}{|A|}, \quad d_{i+2} = \frac{|A_{16}|}{|A|}.
\]

where
\[
A = [A_1', A_2']^T,
\]

\[
A_1' = \begin{bmatrix}
1 & -h_{i-1} & e(-\theta_{i-1}) & e(-\eta_{i-1}) & 0 & 0 & 0 & 0 \\
0 & 1 & \tau_1 e(-\theta_{i-1}) & \tau_2 e(-\eta_{i-1}) & 0 & 0 & 0 & 0 \\
0 & 0 & \tau_1^2 e(-\theta_{i-1}) & \tau_2^2 e(-\eta_{i-1}) & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & -h_i & -e(-\theta_i) & -e(-\eta_i) \\
0 & 1 & \tau_1 & \tau_2 & 0 & -1 & -\tau_1 e(-\theta_i) & -\tau_2 e(-\eta_i) \\
0 & 0 & \tau_1^2 & \tau_2^2 & 0 & 0 & -\tau_1^2 e(-\theta_i) & -\tau_2^2 e(-\eta_i) \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\]

\[
A_2' = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & h_{i+1} & -e(-\theta_{i+1}) & -e(-\eta_{i+1}) & 0 & 0 & 0 & 0 \\
0 & -1 & -\tau_1 e(-\theta_{i+1}) & -\tau_2 e(-\eta_{i+1}) & 0 & 0 & 0 & 0 \\
0 & 0 & -\tau_1^2 e(-\theta_{i+1}) & -\tau_2^2 e(-\eta_{i+1}) & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & -h_{i+2} & -e(-\theta_{i+2}) & -e(-\eta_{i+2}) \\
0 & 1 & \tau_1 & \tau_2 & 0 & -1 & -\tau_1 e(-\theta_{i+2}) & -\tau_2 e(-\eta_{i+2}) \\
0 & 0 & \tau_1^2 & \tau_2^2 & 0 & 0 & -\tau_1^2 e(-\theta_{i+2}) & -\tau_2^2 e(-\eta_{i+2}) \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\]

\[
A_k(k = 1, 2, \ldots, 16) \text{ is the matrix obtained by replacing the } k\text{ column of } A \text{ with } [0, \ldots, 0, 1]^T, \ \theta_i = \tau_1 h_i, \ \text{and } \eta_i = \tau_2 h_i.
\]

For even splitting, i.e., \( h_i = h, i = 1, 2, \ldots, n \), and \( \tau_2 = -\tau_1 = \tau \neq 0 \), the results obtained are

\[
\begin{align*}
\text{(11)}
\end{align*}
\]
\[ a_{j-1} = \frac{\omega \theta}{\sigma}, \]
\[ b_{j-1} = \frac{\omega \theta}{h \sigma}, \]
\[ c_{j-1} = \frac{1}{2} \frac{\omega (-\cosh (\theta) + \sinh (\theta))}{\sigma}, \]
\[ d_{j-1} = \frac{1}{2} \frac{\omega (\cosh (\theta) + \sinh (\theta))}{\sigma}, \]
\[ a_j = \frac{\theta (-4 \sinh (3\theta) + \sinh (4\theta) - 4 \sinh (\theta) + 6 \sinh (2\theta))}{\sigma}, \]
\[ b_j = \frac{\theta (-3 \sinh (3\theta) + 2 \sinh (2\theta) + \sinh (4\theta) + \sinh (\theta))}{h \sigma}, \]
\[ c_j = \frac{1}{2\sigma} (-5 + 4 \cosh (2\theta) - 2 \sinh (2\theta) - 4 \cosh (3\theta) + 3 \sinh (3\theta) + \cosh (4\theta) - \sinh (4\theta) + 4 \cosh (\theta) - \sinh (\theta)), \]
\[ d_j = \frac{1}{2\sigma} (\cosh (4\theta) + \sinh (4\theta) - 5 - 4 \cosh (3\theta) - 3 \sinh (3\theta) + 4 \cosh (\theta) + \sinh (\theta) + 4 \cosh (2\theta) + 2 \sinh (2\theta)), \]
\[ a_{j+1} = -\frac{\omega \theta}{\sigma}, \]
\[ b_{j+1} = -\frac{\theta (-3 \sinh (3\theta) + 2 \sinh (2\theta) + \sinh (4\theta) + \sinh (\theta))}{h \sigma}, \]
\[ c_{j+1} = \frac{1}{4\sigma} (-5 + 4 \cosh (2\theta) + 10 \sinh (2\theta) - 4 \cosh (3\theta) + \cosh (4\theta) - \sinh (4\theta) + 4 \cosh (\theta) - 16 \sinh (\theta)), \]
\[ d_{j+1} = \frac{1}{4\sigma} (\cosh (4\theta) + \sinh (4\theta) - 5 - 4 \cosh (3\theta) + 4 \cosh (\theta) + 16 \sinh (\theta) + 4 \cosh (2\theta) - 10 \sinh (2\theta)), \]
\[ a_{j+2} = 0, \]
\[ b_{j+2} = \frac{\omega \theta}{h \sigma}, \]
\[ c_{j+2} = \frac{\omega}{2\sigma}, \]
\[ d_{j+2} = -\frac{\omega}{2\sigma}. \]

where
\[ \omega = 5 \sinh (\theta) - 4 \sinh (2\theta) + \sinh (3\theta), \]
\[ \sigma = 5 + \theta \sinh (4\theta) - 4 \theta \sinh (\theta) + 6 \theta \sinh (2\theta) - 4 \theta \sinh (3\theta) \]
\[ + 4 \cosh (3\theta) - 4 \cosh (\theta) - \cosh (4\theta) - 4 \cosh (2\theta), \]
\[ \theta = \tau h. \]  

(13)

Define a function
\[ B_j(x) = \begin{cases} 
  s_{j-1}(x), & x \in I_{j-1}, \\
  s_j(x), & x \in I_j, \\
  s_{j+1}(x), & x \in I_{j+1}, \\
  s_{j+2}(x), & x \in I_{j+2}, \\
  0, & \text{otherwise}, 
\end{cases} \]
\[ j = 2, 3, \ldots, n-2, \]

(14)

where the coefficients of \( s_{j-1}(x), s_j(x), s_{j+1}(x), \) and \( s_{j+2}(x) \) are given by the solution of function (9). Besides, for
j = -1, 0, 1, n - 1, n, n + 1, an interval expansion will be conducted, i.e., (x0, x1, ..., xn) will be extended to be
(x0, x1, x2, x3, x4, ..., xn-1, xn, xn+1), Proposition 1.

Thus, j in (11) can take the values of -1, 0, 1, ..., n, n + 1.

Of course, the function domain is still [x0, xn].

For the space basis of \( \mathcal{E}_j^2(\Delta_n) \), we have the following proposition.

**Proposition 1.** When \( \tau_1 \neq \tau_2 \), the function set of \( \{B_j(x)\}_{j=1}^{n+1} \) is a set of basis of \( \mathcal{E}_j^2(\Delta_n) \).

For special cases, we can prove the following.

**Theorem 1.** When \( \tau_2 = -\tau_1 = \tau \neq 0 \) and \( h_i = h, i = -2, -1, \ldots, n + 3 \), the function set of \( \{B_j(x)\}_{j=1}^{n+1} \) is a set of basis of \( \mathcal{E}_j^2(\Delta_n) \).

**Proof.** Obviously, \( B_j(x) \in \mathcal{E}_j^2(\Delta_n) \), \( j = -1, 0, 1, \ldots, n, n + 1 \), and the number of functions equals the dimension of \( \mathcal{E}_j^2(\Delta_n) \). Therefore, we need to show that \( \{B_j(x)\}_{j=1}^{n+1} \) is linearly independent.

For \( x = x_i, i = -1, 0, \ldots, n, n + 1 \), let \( B = (B_{ij}) = (B_j(x_i))_{(n+3) \times (n+3)} \), if matrix \( B \) is invertible, then \( \{B_j(x)\}_{j=1}^{n+1} \) is linearly independent. It is easy to know that \( B \) is a tridiagonal matrix, and

\[
\begin{align*}
  b_{j+1} &= a_{j-1} + c_{j-1} + d_{j-1}, \\
  b_{j+1} &= a_j + c_j + d_j = 1, \\
  b_{j+1} &= a_{j+1} + c_{j+1} + d_{j+1}, \\
\end{align*}
\]

where \( i = -1, 0, \ldots, n + 1 \). Because

\[
|b_{j+1}| = |b_j| = |a_{j-1} + c_{j-1} + d_{j-1} + a_{j+1} + c_{j+1} + d_{j+1}|
\]

\[
= \left| \frac{8}{\sigma} \left( (\cosh(\theta) - 1)^2 (1 - \cosh(\theta))^2 + \theta \sinh(\theta) \right) \right|
\]

\[
= \left| \frac{1 - \cosh(\theta)^2 + \theta \sinh(\theta)}{1 - \cosh(\theta)^2 + \theta \sinh(\theta) \cosh(\theta)} \right|
\]

\[
\leq \frac{1}{2} < |b_i|,
\]

Figure 1: Basis function set \( \{B_j(x)\}_{j=1}^{n+1} \) of the exponential spline function space.

\[
B \text{ is a strictly tridiagonal matrix. Thus, } B \text{ is invertible.}
\]

This proves that \( \{B_j(x)\}_{j=1}^{n+1} \) is linearly independent.

When \( h_i = h, i = -2, -1, \ldots, n + 3 \), \( \{B_j(x)\}_{j=1}^{n+1} \) has the following properties.

**Proposition 2.** For any \( x \in [a, b] \)

\[
\sum_{j=1}^{n+1} B_j(x) = C,
\]

where \( C \) is not related to \( x \), and

\[
C = \frac{\tau_1 \tau_2 h (\tau_2 - \tau_1) (e^{-\tau_1 h} + e^{-\tau_2 h} - e^{-h(\tau_2 + \tau_1)}) - 1}{(\tau_2^2 - \tau_1^2) e^{-h(\tau_1 + \tau_2)} + (\tau_1^2 + \tau_2^2 + \tau_1 \tau_2 h - \tau_1^2 \tau_2 h) e^{-h(\tau_1)} - (\tau_1^2 + \tau_2^2 - \tau_1 \tau_2 h + \tau_1^2 \tau_2 h) e^{-h(\tau_2)}}.
\]
The basis function $B_j(x)$ has a local supporting set of $[x_{i-2}, x_{i+2}]$. Figure 1 shows all functions in the function set of $\{B_j(x)\}_{j=1}^{n+1}$ at the same coordinate.

3. Properties of Exponential Spline Functions

The relationship between the function value and the first derivative and that between the function value and the second derivative are commonly used in numerically solving DEs. Next, we will derive some properties of the functions in $\mathcal{C}_2^2(\Delta_n)$, and these relationships will be used in numerically calculating DEs.

Let $s(x) \in \mathcal{C}_2^2(\Delta_n)$, then,

$$s(x) = s_i(x), \quad x \in I_i, \quad i = 1, 2, \ldots, n. \quad (20)$$

Denote

$$S_i = s(x_i), \quad D_i = s'(x_i), \quad M_i = s'' \quad (21)$$

since $s_i(x_i) = s_{i+1}(x_i), s_i'(x_i) = s_{i+1}'(x_i)$, the factors of $a_i, b_i, c_i$, and $d_i$ can be written as

$$a_i = \frac{1}{\tau_1 \tau_2 \rho_i}(\tau_1^2 - \tau_2^2)M_{i-1} + (\tau_2^2 e^{-\tau_2 \rho_i} - \tau_1^2 e^{-\tau_1 \rho_i})M_i + \tau_1 \tau_2 \rho_i S_i.$$

$$b_i = \frac{1}{\rho_i}(\tau_2^2 e^{-\tau_2 \rho_i} - \tau_1^2 e^{-\tau_1 \rho_i} + \tau_2 \tau_1 \rho_i S_i).$$

$$c_i = \frac{1}{\tau_1 \rho_i}(M_{i-1} - e^{-\tau_1 \rho_i}M_i),$$

$$d_i = \frac{1}{\tau_2 \rho_i}(M_{i-1} - e^{-\tau_2 \rho_i}M_i),$$

where $\rho_i = e^{-\tau_i \rho_i} - e^{-\tau_i \rho_{i+1}}$, $i = 1, 2, \ldots, n$ (the same applies hereinafter). Using the continuous condition of the first derivative

$$s_i'(x_i) = s_{i+1}'(x_i), \quad i = 1, 2, \ldots, n - 1, \quad (23)$$

we obtain

$$\alpha_{i+1} + \alpha_2 \alpha_{i+1} + \alpha_3 \alpha_{i+1} - \frac{1}{h_i} S_{i+1} + \frac{h_i h_{i+1} + h_{i+1} h_{i+2} - 1}{h_{i+1}} S_{i+1} = 0, \quad (24)$$

where $i = 1, 2, \ldots, n - 1$, with $h_i = h > 0, i = 1, 2, \ldots, n$ and $\tau_2 = -\tau_1 = \tau$, then the corresponding coefficients become

$$\alpha_{i+1} = \frac{h_i \tau_1 \tau_2 - h_i \tau_2^2 - \tau_1^2 e^{-\tau_1 \rho_i} + \tau_2^2 e^{-\tau_2 \rho_i} + \tau_1 \tau_2 \rho_i}{\tau_1^2 \tau_2 \rho_i h_i},$$

$$\alpha_{2i} = \frac{1}{\tau_1^2 \tau_2 \rho_i h_{i+1}} \left( (h_{i+1} \tau_1^2 - h_i \tau_1 \tau_2) e^{-\tau_1 \rho_i} - (h_i \tau_1 \tau_2 - h_i \tau_2^2) e^{-\tau_2 \rho_i} + (h_{i+1} \tau_2^2 - h_i \tau_1 \tau_2) e^{-\tau_2 \rho_i} - (h_i \tau_1 \tau_2 - h_i \tau_2^2) e^{-\tau_1 \rho_i} \right),$$

$$\alpha_{3i} = \frac{\tau_2^2 e^{-\tau_2 \rho_i} - \tau_1 \tau_2 \rho_i S_i}{\tau_1^2 \tau_2 \rho_i h_{i+1}}.$$
\[ \alpha_{1i} = \frac{(e^{z_{ih}} - 2r \theta^{z_{ih}})}{h^2 (e^{z_{ih}} - 1)} \]
\[ \alpha_{2i} = \frac{2(e^{z_{ih}} + 2r \theta^{z_{ih}} + 1 + r \theta)}{h^2 (e^{z_{ih}} - 1)} \]
\[ \alpha_{3i} = \frac{(e^{z_{ih}} - 2r \theta^{z_{ih}})}{h^2 (e^{z_{ih}} - 1)} \]
\[ \alpha_{4i} = \frac{1}{h} \]
\[ \alpha_{5i} = \frac{2}{h} \]
\[ \alpha_{6i} = \frac{1}{h} \]

Let \( \tau \to 0 \); then, (24) is equivalent to
\[ h^2 (M_{i+1} + 4M_i + M_{i+1}) - 6S_{i+1} + 12S_i - 6S_{i+1} = 0. \]  (27)

In a similar way, we can obtain the relationship between the function value and the first derivative:
\[ \beta_{1i}D_{i+1} + \beta_{2i}D_i + \beta_{3i}D_{i+1} + \beta_{4i}S_{i+1} + \beta_{5i}S_i + \beta_{6i}S_{i+1} = 0, \]  (28)

where
\[
\beta_{1i} = \frac{1}{K_i K_{i+1}} \left[ (\tau_1 - \tau_2)^2 (-\tau_1 + \tau_1 \tau_2 h_i - \tau_2) e^{-h_i (\tau_1 + \tau_2)} - \tau_2^2 (\tau_1 - \tau_2) e^{-h_i (\tau_1 + \tau_2)} - \tau_2^2 (\tau_1 + \tau_2 h_i + \tau_2) e^{-h_i (\tau_1 + \tau_2)} \right. \\
\left. + \tau_1^2 (\tau_1 - \tau_2) e^{-h_i (\tau_1 + \tau_2)} - \tau_2^2 (\tau_1 + \tau_2 h_i + \tau_2) e^{-h_i (\tau_1 + \tau_2)} \right] \\
- \tau_1^2 (\tau_1 - \tau_2) e^{-h_i (\tau_1 + \tau_2)} + \tau_1^2 (\tau_1 h_i + \tau_2) e^{-h_i (\tau_1 + \tau_2)} + \tau_2^2 (\tau_1 h_i + \tau_2) e^{-h_i (\tau_1 + \tau_2)} \\
- \tau_1^2 - \tau_2^2 + \tau_1^2 \tau_2 + \tau_1^2 \tau_2 h_i + \tau_2^2 \tau_2 h_i + \tau_1 \tau_2 h_i + \tau_1 \tau_2 h_i + \tau_2^2 h_i + \tau_2^2 h_i + \tau_1^2 h_i + \tau_1^2 h_i + \tau_2^2 h_i + \tau_2^2 h_i + \tau_1 h_i + \tau_2 h_i + \tau_2 h_i + \tau_2 h_i + \tau_1 + \tau_2 \]
\[ \beta_{2i} = \frac{1}{K_i K_{i+1}} \left[ -\tau_1^2 (\tau_1 - \tau_2) e^{-h_i (\tau_1 + \tau_2)} + \tau_1^2 \tau_2 h_i + \tau_1^2 \tau_2 h_i + \tau_2^2 \tau_2 h_i + \tau_2^2 \tau_2 h_i + \tau_1 \tau_2 h_i + \tau_1 \tau_2 h_i + \tau_2^2 h_i + \tau_2^2 h_i + \tau_1 h_i + \tau_2 h_i + \tau_2 h_i + \tau_2 h_i + \tau_1 + \tau_2 \right. \\
\left. - \tau_1^2 (\tau_1 - \tau_2) e^{-h_i (\tau_1 + \tau_2)} + \tau_1^2 \tau_2 h_i + \tau_1^2 \tau_2 h_i + \tau_2^2 \tau_2 h_i + \tau_2^2 \tau_2 h_i + \tau_1 \tau_2 h_i + \tau_1 \tau_2 h_i + \tau_2^2 h_i + \tau_2^2 h_i + \tau_1 h_i + \tau_2 h_i + \tau_2 h_i + \tau_2 h_i + \tau_1 + \tau_2 \right] \\
- \tau_1^2 \tau_2 h_i + \tau_1^2 \tau_2 h_i + \tau_2^2 \tau_2 h_i + \tau_2^2 \tau_2 h_i + \tau_1 \tau_2 h_i + \tau_1 \tau_2 h_i + \tau_2^2 h_i + \tau_2^2 h_i + \tau_1 h_i + \tau_2 h_i + \tau_2 h_i + \tau_2 h_i + \tau_1 + \tau_2 \]
\[
\beta_{3j} = \frac{1}{K_iK_j} \left[ (\tau_1 - \tau_2)^2 (\tau_1 \tau_2 h_{i+1} + \tau_1 + \tau_2) e^{-h_i(\tau_1 + \tau_2)} + \tau_1^2 (\tau_2 - \tau_1) e^{-h_i(\tau_1 - \tau_2)} \right]
\]
\[
- \tau_2^2 (\tau_2 - \tau_1) e^{-h_i \tau_1 - h_i \tau_2} - (\tau_2 - \tau_1) (\tau_2 + \tau_1 h_j - \tau_1) (\tau_2 \tau_1 h_{i+1} + \tau_1 + \tau_2) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
- (\tau_1 - \tau_2) (\tau_1 \tau_2 h_{i+1} + \tau_1 - \tau_2) (\tau_1 \tau_2 h_{i+1} + \tau_1 + \tau_2) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
- \tau_1^2 (\tau_1 + \tau_2 h_{i+1} - \tau_1) e^{-h_i(\tau_1 + \tau_2)} + (\tau_1 + \tau_2)^2 (\tau_1 \tau_2 h_{i+1} + \tau_1 + \tau_2) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
- \tau_1^2 (\tau_1 + \tau_2 h_{i+1} + \tau_2 - \tau_1) e^{-h_i \tau_1 - h_i \tau_2} + \tau_1 (\tau_1 \tau_2 h_{i+1} + \tau_2 - \tau_1) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
+ \tau_2 (\tau_1 + \tau_2 h_{i+1} - \tau_1) e^{-h_i \tau_1 - h_i \tau_2} + (\tau_1 + \tau_2)^2 (\tau_1 \tau_2 h_{i+1} - \tau_1 - \tau_2) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
\beta_{4j} = \frac{\tau_1 \tau_2}{K_iK_j} \left[ (\tau_1 - \tau_2)^2 e^{-h_i(\tau_1 + \tau_2)} + \tau_1 (\tau_1 - \tau_2) e^{-h_i \tau_1 - h_i \tau_2} \right]
\]
\[
- \tau_1 (\tau_1 - \tau_2) e^{-h_i \tau_1 - h_i \tau_2} + \tau_1 (\tau_1 + \tau_2 h_{i+1} + \tau_2 - \tau_1) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
+ \tau_1 (\tau_1 + \tau_2 h_{i+1} - \tau_1) e^{-h_i \tau_1 - h_i \tau_2} + \tau_2 (\tau_1 - \tau_2) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
- \tau_1 (\tau_1 - \tau_2) e^{-h_i \tau_1 - h_i \tau_2} + \tau_1 (\tau_1 + \tau_2 h_{i+1} + \tau_2 - \tau_1) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
- \tau_2 (\tau_1 + \tau_2 h_{i+1} - \tau_1) e^{-h_i \tau_1 - h_i \tau_2} + (\tau_1 + \tau_2)^2 (\tau_1 \tau_2 h_{i+1} - \tau_1 - \tau_2) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
\beta_{6j} = \frac{\tau_1 \tau_2}{K_iK_j} \left[ (\tau_1 - \tau_2)^2 e^{-h_i(\tau_1 + \tau_2)} + \tau_1 (\tau_1 - \tau_2) e^{-h_i \tau_1 - h_i \tau_2} \right]
\]
\[
- \tau_2 (\tau_1 - \tau_2) e^{-h_i \tau_1 - h_i \tau_2} - (\tau_1 - \tau_2) (\tau_1 + \tau_2 h_{i+1} + \tau_2 - \tau_1) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
+ (\tau_1 - \tau_2) (\tau_1 + \tau_2 h_{i+1} + \tau_2 - \tau_1) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
- (\tau_1 - \tau_2)^2 e^{-h_i \tau_1 - h_i \tau_2} + \tau_2 (\tau_1 + \tau_2 h_{i+1} + \tau_2 - \tau_1) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
+ (\tau_1 + \tau_2)^2 (\tau_1 \tau_2 h_{i+1} + \tau_2 - \tau_1) e^{-h_i \tau_1 - h_i \tau_2} + \tau_1 (\tau_1 - \tau_2) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
- \tau_2 (\tau_1 + \tau_2 h_{i+1} - \tau_1) e^{-h_i \tau_1 - h_i \tau_2} + \tau_1 (\tau_1 + \tau_2 h_{i+1} + \tau_2 - \tau_1) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
+ \tau_2 (\tau_1 + \tau_2 h_{i+1} - \tau_1) e^{-h_i \tau_1 - h_i \tau_2} + (\tau_1 + \tau_2)^2 (\tau_1 \tau_2 h_{i+1} - \tau_1 - \tau_2) e^{-h_i \tau_1 - h_i \tau_2}
\]
\[
K_i = (-\tau_1 + \tau_2 h_{i+1} + \tau_2) e^{-h_i \tau_1} + (-\tau_1 - \tau_2 h_{i+1} + \tau_2) e^{-h_i \tau_2} + (\tau_1 + \tau_2) e^{-h_i \tau_1 - h_i \tau_2} + (\tau_1 - \tau_2) e^{-h_i \tau_1 + h_i \tau_2} + \tau_1 - \tau_2.
\]

If \( \tau_2 = -\tau_1 = \tau \) and \( h_i = h, i = 1, 2, \ldots, n \), then
\[
\beta_{3i} = \frac{-2 \tau e^{\rho h} + e^{\rho h^2}}{4 e^{\rho^2 h^2} - 2 e^{\rho h^2} + \tau e^{\rho h^2} - 2 \tau h^2 - 1 - \tau h} \]
\[
\beta_{4i} = \frac{2 (\tau h e^{\rho h^2} - e^{\rho h^2} + \tau h + 1 - \tau h)}{4 e^{\rho h^2} - 2 e^{\rho h^2} + \tau e^{\rho h^2} - 2 \tau h^2 - 1 - \tau h} \]
\[
\beta_{5i} = \frac{2 e^{\rho h^2} - 2 e^{\rho h^2} + \tau e^{\rho h^2} - 2 \tau h^2 - 1 - \tau h}{4 e^{\rho h^2} - 2 e^{\rho h^2} + \tau e^{\rho h^2} - 2 \tau h^2 - 1 - \tau h} \]
\[
\beta_{6i} = \frac{2 e^{\rho h^2} - 2 e^{\rho h^2} + \tau e^{\rho h^2} - 2 \tau h^2 - 1 - \tau h}{4 e^{\rho h^2} - 2 e^{\rho h^2} + \tau e^{\rho h^2} - 2 \tau h^2 - 1 - \tau h} \]
\[
\beta_{6i} = 0 \]
\[
\beta_{6i} = \frac{1 + e^{\rho h^2} - 2 e^{\rho h^2} + e^{\rho h^2} h^2 + 2 + \tau h}{4 e^{\rho h^2} - 2 e^{\rho h^2} + \tau e^{\rho h^2} - 2 \tau h^2 - 1 - \tau h}
\]
Let \( \tau \to 0 \); then, (28) is equivalent to
\[
h(D_{i+1} + 4D_i + D_{i+1}) + 3S_{i+1} - 3S_i = 0.
\]

This is consistent with the cubic 2nd-order polynomial spline function relationship.

Besides, using

\[
\text{we can obtain}
\]
\[
a_i = \frac{(\tau_1^2 - \tau_2^2) M_{i+1} + (\tau_2^2 e^{-\tau_2 h} - \tau_1^2 e^{-\tau_1 h}) M_i + \tau_1^2 \tau_2^2 \rho_i S_i}{\tau_1^2 \tau_2^2 \rho_i}
\]
\[
b_i = \frac{(\tau_1 - \tau_2) M_{i+1} + (\tau_2 e^{-\tau_2 h} - \tau_1 e^{-\tau_1 h}) M_i + \tau_1 \tau_2 \rho_i D_i}{\tau_1^2 \tau_2 \rho_i}
\]
\[
c_i = \frac{-M_{i+1} + e^{-\tau_2 h} M_i}{\tau_1 \tau_2 \rho_i}
\]
\[
d_i = \frac{-M_{i+1} + e^{-\tau_1 h} M_i}{\tau_1 \tau_2 \rho_i}
\]
\[
\text{Then, using continuous condition of } s_i^{'}(x_i) = s_{i+1}^{'}(x_i), s_i^{''}(x_i) = s_{i+1}^{''}(x_i), \text{ we obtain}
\]

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\[ - \left( h_{i+1} r_1^2 - h_{i+1} r_1^2 r_2 + r_2^2 e^{-\tau_i h_{i+1}} - r_2^2 e^{-\tau_2 h_{i+1}} + r_2^2 - r_2^2 \right) M_i \]
\[ + \left( r_2^2 e^{-\tau_i h_{i+1}} - r_2^2 e^{-\tau_2 h_{i+1}} + h_{i+1} r_1^2 e^{-\tau_i h_{i+1}} - h_{i+1} r_1^2 r_2 e^{-\tau_i h_{i+1}} + r_2^2 e^{-h_{i+1} (\tau_i + \tau_2)} - r_2^2 e^{-h_{i+1} (\tau_i + \tau_2)} \right) M_{i+1} \]
\[ + S_i - S_{i+1} + h_{i+1} D_{i+1} = 0, \]
\[ (39) \]

\[ \frac{\tau_2 e^{-\tau_i h_{i+1}} - \tau_1 e^{-\tau_i h_{i+1}} M_{i+1}}{\tau_1 \tau_2 \rho_i} \]
\[ \frac{\tau_2 e^{-\tau_i h_{i+1}} - \tau_1 e^{-\tau_i h_{i+1}} + \tau_1 - \tau_2 M_i + D_i - D_{i+1} = 0.} \]
\[ (40) \]

\[
D_{i+1} = - \left( \frac{r_1^2 e^{-\tau_i h_i} - r_2^2 e^{-\tau_i h_i}}{\tau_1 \tau_2 \rho_i} h_{i+1} r_1^2 r_2 \right) M_{i+1} \]
\[ + \left( \frac{-h_{i+1} r_1^2 + r_1^2 + h_{i+1} r_1^2 r_2}{\tau_1 \tau_2 \rho_i} e^{-\tau_i h_i} + \left( r_1^2 + r_2^2 \right) e^{-\tau_i h_i} \right) M_i \]
\[ + \left( \frac{-h_{i+1} r_1^2 + r_2^2}{\tau_1 \tau_2 \rho_i} \right) \left( \frac{-h_{i+1} r_1^2 + r_1^2 r_2}{\tau_1 \tau_2 \rho_i} e^{-\tau_i h_i} \right) M_i \]
\[ (41) \]

Meanwhile, by eliminating \( D_{i+1} \) with (39) and (40) and rearranging, we obtain

\[
D_i = \left( \frac{(-r_1^2 - h_{i+1} r_1^2 r_2) e^{-\tau_i h_{i+1}} + (r_1^2 + h_{i+1} r_1^2 r_2) e^{-\tau_2 h_{i+1}} - r_2^2 + r_2^2}{\tau_1 \tau_2 \rho_{i+1}} \right) M_i \]
\[ \left( -\frac{r_1^2 + h_{i+1} r_1^2 r_2 - h_{i+1} r_1^2 + r_2^2}{\tau_1 \tau_2 \rho_{i+1}} e^{-\tau_i h_{i+1}} + r_1^2 e^{-\tau_i h_{i+1}} - r_2^2 e^{-\tau_2 h_{i+1}} \right) M_{i+1}, \]
\[ (42) \]
\[
\frac{S_i}{h_{i+1}} + \frac{S_{i+1}}{h_{i+1}} \]
\[ (43) \]

We can also use \( s_{i+1} (x_{i+1}) = S_{i+1}, s_i (x_i) = S_i, s_{i+1} (x_{i+1}) = D_{i+1}, \) and \( s_{i+1} (x_i) = D_i \) to obtain
\[ a_i = \frac{1}{\varrho_i} \left( (\tau_1 h_i - \tau_2 h_i + \rho_i)D_{i-1} + (h_i \tau_2 e^{-\tau_2 h_i} - \tau_1 e^{-\tau_1 h_i} h_i - \rho_i)D_i \right) \\
+ \left( \tau_2 e^{-\tau_2 h_i} + \tau_1 - \tau_2 - \tau_1 e^{-\tau_1 h_i} \right)S_{i-1} \\
+ \left( (-\tau_1 - \tau_1^2 h_i \tau_2) e^{-\tau h_i} + (\tau_1 h_i \tau_2 + \tau_2) e^{-\tau_1 h_i} + (\tau_1 - \tau_2) e^{-h_i (\tau_1 + \tau_2)} \right)S_i, \]

\[ b_i = \frac{1}{\varrho_i} \left( (\tau_1 - \tau_2 - \tau_1 e^{-\tau_1 h_i} + \tau_2 e^{-\tau_1 h_i})D_{i-1} + (\tau_2 e^{-\tau_1 h_i} - \tau_1 e^{-\tau_1 h_i} + (\tau_1 - \tau_2) e^{-h_i (\tau_1 + \tau_2)})D_i \right) \\
- \tau_1 \tau_2 \rho_i S_{i-1} + \tau_1 \tau_2 \rho_i S_i, \]

\[ c_i = \frac{1}{\varrho_i} \left( (e^{-\tau_1 h_i} - 1 + \tau_2 h_i)D_{i-1} + (e^{-\tau_1 h_i} + 1 - h_i \tau_2 e^{-\tau_1 h_i})D_i \right) \\
+ \left( -\tau_2 e^{-\tau_1 h_i} + \tau_2 \right)S_{i-1} + (-\tau_2 + \tau_2 e^{-\tau_1 h_i})S_i, \]

\[ d_i = \frac{1}{\varrho_i} \left( (-\tau_1 h_i - e^{-(\tau_1 h_i) + 1})D_{i-1} + (\tau_1 e^{-\tau_1 h_i} h_i + e^{-\tau_1 h_i} - 1)D_i \right) \\
+ \tau_1 e^{-\tau_1 h_i} + \tau_1 S_{i-1} + (-\tau_1 e^{-\tau_1 h_i} + \tau_1)S_i, \]

where \( \varrho_i = (-\tau_1 h_i \tau_2 + \tau_2 - \tau_1) e^{-\tau_1 h_i} + (\tau_1 h_i \tau_2 + \tau_2 - \tau_1) e^{-\tau_1 h_i} + (\tau_1 - \tau_2) e^{-h_i (\tau_1 + \tau_2)} + \tau_1 - \tau_2. \) Fast Hermite interpolation can be achieved by using this set of relations.

4. Exponential Spline Difference Method

The spline difference method uses the relationship between the spline function and its derivative to construct the differential expression to numerically solve ODEs, by which the numerical solution at nodes can be obtained, and that within the subintervals can also be calculated by using the spline function expressions. It is the advantage of this method compared with the general difference schemes. In fact, it can be said that the approximate analytical solution using the splines is obtained.

\[ t_i = (\alpha_{11}u''_{i-1} + \alpha_{12}u'' + \alpha_{13}u''_{i+1}) - \frac{u_{i-1}}{h_i} + \frac{(h_i + h_{i+1})u_i}{h_i h_{i+1}} - \frac{u_{i+1}}{h_{i+1}}, \]

where \( i = 1, \ldots, n - 1, u_i = u(x_i) \) and \( u''_i = u''(x_i) \). \( t_i \) is the local truncation error at \( x_i \). By substituting (46) into (47) and rearranging, we obtain

\[ \left( -\frac{1}{h_i} - \alpha_{11} \rho_{i-1} \right)u_{i-1} + \left( h_i + h_{i+1} - \alpha_{12} \rho_i \right)u_i \\
+ \left( -\frac{1}{h_{i+1}} - \alpha_{13} \rho_{i+1} \right)u_{i+1} + \alpha_{11} g_{i-1} + \alpha_{12} g_i + \alpha_{13} g_{i+1} = t_i, \]

where \( i = 1, 2, \ldots, n - 1. \) Thus, we get \( n - 1 \) equations about \( u_0, u_1, \ldots, u_n \).

4.1. Differential Expression. The following presents a spline difference method for solving (1) which satisfies one of the boundary conditions (2)–(5) for the boundary value problems of ODEs. Due to the limitation of this method, we only consider the case of \( q(x) = 0 \) in this section. For convenience of description, we first consider the boundary condition (2). From (1), we can obtain

\[ u'' = g(x) - p(x)u, \quad x \in [a, b]. \]

By discretization the above equation, we obtain

\[ u''_i = g_i - p_i u_i, \quad i = 0, 1, 2, \ldots, n, \]

where \( g_i = g(x_i) \) and \( p_i = p(x_i) \). Substitute \( S \) with \( u \) and \( M \) with \( u'' \) in (24), and we obtain

\[ U = \varpi, \]

where \( A = B - W, \) with
\[
B = \begin{bmatrix}
1 & 0 & 0 \\
\frac{1}{h_1} & 1 + h_1 & 1 \\
\frac{1}{h_2} & h_1 + h_2 & h_2 \\
\frac{1}{h_3} & h_1 + h_3 & h_1h_3 \\
\cdot & \cdot & \cdot \\
\frac{1}{h_{n-2}} & h_{n-2} + h_{n-1} & h_{n-2}h_{n-1} \\
\frac{1}{h_{n-1}} & h_{n-1} + h_n & h_{n-1}h_n \\
0 & 0 & 1
\end{bmatrix}
\]

\[
W = \begin{bmatrix}
a_{11}p_0 & a_{12}p_1 & a_{13}p_2 & \cdots & a_{1n-2}p_{n-3} & a_{1n-1}p_{n-2} & a_{1n}p_n \\
a_{21}p_0 & a_{22}p_1 & a_{23}p_2 & \cdots & a_{2n-2}p_{n-3} & a_{2n-1}p_{n-2} & a_{2n}p_n \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]

\[
F = \begin{bmatrix}
\mu_0, \\
\left( -\alpha_{2i}g_i + \alpha_{2i+1} + \alpha_{2i+2}g_{i+1} \right) \\
\left( -\alpha_{2i-1}g_{i-1} + \alpha_{2i} + \alpha_{2i+1}g_i \right) \\
\left( -\alpha_{2i-2}g_{i-2} + \alpha_{2i-1} + \alpha_{2i}g_i \right)
\end{bmatrix}^T.
\]

\[
T = \begin{bmatrix}
[t_i]_{i=0}^n \\
[t_i]_{i=0}^n \\
[t_i]_{i=0}^n
\end{bmatrix}^T,
\]

\[
U = \begin{bmatrix}
[u_i]_{i=0}^n \end{bmatrix}^T.
\]

If \( A \) is nonsingular, then equation (50) has a unique solution. Solving (50) yields the approximate values of \( u_i \) in \( u(x) \) at the splitting points \( x_i, i = 0, 1, \ldots, n \). Since \( A \) is a tridiagonal matrix, the catch-up method can be used to reduce the calculation in practice.

To obtain the spline function expression, we can calculate \( u_i' \) from (2) after finding \( u_i \) and substitute them into (22) to obtain \( a_i, b_i, c_i \) and \( d_i \) so that the approximate solution of the spline over the entire interval will be found.

The case of other boundary conditions will be discussed below. For the second boundary conditions, from (41), (42), and (3), we obtain

\[
\begin{align*}
\alpha_{20}M_0 + \alpha_{30}M_1 + \frac{S_0}{h_1} - \frac{S_1}{h_1} &= -\mu_0, \\
\alpha_{40}M_{n-1} + \alpha_{2n}M_n - \frac{S_{n-1}}{h_n} + \frac{S_n}{h_n} &= \mu_1,
\end{align*}
\]

where

\[
F_n = \mu_1 - (\alpha_{2n}g_n + \alpha_{3n}g_n).
\]

And the third and the fourth boundary conditions should also be modified accordingly.

4.2 Error Estimation. Suppose \( u(x) \) is sufficiently smooth in \([a, b]\), by Taylor expanding \( u_{i-1}, u_{i+1}, u''_{i-1}, u''_{i+1} \) in difference equation (47) at \( x_i \), we obtain

\[
t_i = \sum_{j=2}^{6} \xi_ju_j^{(j)} + O(h_i^j),
\]

where
\[ \xi_2 = \alpha_{11} + \alpha_{21} + \alpha_{31} - \frac{1}{2} (h_i + h_{i+1}), \]
\[ \xi_3 = -\alpha_i h_i + \alpha_i h_{i+1} + \frac{1}{6}(h_i^2 - h_{i+1}^2), \]
\[ \xi_4 = \frac{1}{2} \alpha_i h_i^3 + \frac{1}{2} \alpha_i h_{i+1}^3 - \frac{1}{24} (h_i^3 - h_{i+1}^3), \]
\[ \xi_5 = \frac{1}{6} \alpha_i h_i^5 + \frac{1}{6} \alpha_i h_{i+1}^5 + \frac{1}{120} (h_i^5 + h_{i+1}^5), \]
\[ \xi_6 = \frac{1}{24} \alpha_i h_i^7 + \frac{1}{24} \alpha_i h_{i+1}^7 - \frac{1}{720} (h_i^7 + h_{i+1}^7). \] (59)

Let \( r_1 \to 0 \) and \( r_2 \to 0 \), we find \( \xi_2 = \xi_3 = 0 \) and \( \xi_4 = (1/24) (h_i^3 + h_{i+1}^3) \). When
\[ \alpha_{11} = \frac{1}{12} h_i + h_{i+1} - h_i^2, \]
\[ \alpha_{21} = \frac{1}{12} h_i^2 + 4 h_i h_{i+1} + 4 h_i^2 h_{i+1} + h_{i+1}^3, \]
\[ \alpha_{31} = \frac{1}{12} h_i^3 - h_{i+1}^3, \]
we obtain \( \xi_2 = \xi_3 = \xi_4 = 0 \),
\[ \xi_5 = \frac{1}{360} (-h_i + h_{i+1}) (2 h_i + h_{i+1}) (h_i + 2 h_{i+1}) (h_i + h_{i+1}). \] (60)

\[ \xi_5 = 0 \] for an even splitting, i.e., \( h_i = h_i \), \( i = 1, 2, \ldots, n \), and \( \xi_6 = (1/240) h_i^5 \). This indicates that the even splitting has higher accuracy if \( u(x) \) is sufficiently smooth.

4.3. Convergence Analysis. We mainly discuss the convergence of equation (47) and the differential expression of (49) in the sense of \( \|E\|_\infty \).

**Lemma 1** (see [14]). If the \( n \)-order matrix \( B \) satisfies one of the following two conditions:

1. \( B \) is a strictly diagonally dominant matrix

\( B \) is a second half strong diagonally dominant matrix

Then, \( B \) is nonsingular and \( \rho(1 - D^{-1} B) < 1 \), where \( D = \text{diag}(B) \) and \( I \) is the identity matrix.

By (14),
\[ AS = F. \] (62)

If \( A \) is reversible, combining (50), we obtain
\[ E = A^{-1} T = (B - W)^{-1} T = (I - B^{-1} W)^{-1} B^{-1} T, \] (63)
where \( E = (e_i) = U - S \) and \( B, W, \) and \( T \) are given by (51)–(53). The reversibility of \( B \) can be proved by 1. From the boundary condition (2), we can get \( c_0 = c_n = 0 \). Thus, discussing the convergence of \( \|E\|_\infty \) is consistent with that of \( \|e_i\|_\infty \). So, in the following convergence discussion, the first and the last row and the first and the last column will be removed from the original matrices of \( A, B, W \) to obtain the \( n - 1 \)-order ones, keeping the subscript value unchanged.

If
\[ \|B^{-1}\|_\infty \|W\|_\infty < 1, \]
then
\[ \|E\|_\infty \leq \frac{\|B^{-1}\|_\infty \|T\|_\infty}{1 - \|B^{-1}\|_\infty \|W\|_\infty} \leq \|B^{-1}\|_\infty \|T\|_\infty. \] (65)

To calculate \( B^{-1} \), the following lemma is needed:

**Lemma 2** (see [15]). Let the square matrix \( A \) be an \( n \)-order tridiagonal one, with the following expression:
\[ A = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 \\ 0 & c_2 & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots & b_{n-1} \\ 0 & \cdots & \cdots & c_{n-1} & a_n \end{bmatrix}, \] (66)
and \( b_i c_i \neq 0, i = 1, 2, \ldots, n - 1, a_i \neq 0, i = 1, 2, \ldots, n \); then, the expression of \( A^{-1} \) can be written as
\[ a_{ij}^{(-1)} = \begin{cases} (-1)^{i+j} \left( \prod_{k=i}^{j-1} b_k \right) \frac{\det(A[1, \ldots, i-1]) \det(A[j+1, \ldots, n])}{\det(A)}, & i < j, \\ (-1)^{i+j} \left( \prod_{k=j}^{i-1} c_k \right) \frac{\det(A[1, \ldots, i-1]) \det(A[j+1, \ldots, n])}{\det(A)}, & i \geq j, \end{cases} \] (67)
where \( A[i_1, i_2, \ldots, i_k] \) denotes the matrix consisting of the elements of \( i_1, i_2, \ldots, i_k \) rows crossed with the columns in \( A \). Specifically, \( A[1, \ldots, i-1] = 1 \) for \( i = 1 \).

Using (2), \( B^{-1} = (b_{ij}^{(-1)}) \) in (65) can be written as (with \( \sum_{k=1}^{n} h_k = b - a \))
Table 1: The maximum error $\|E\|_{\infty}$ in solving (72) with the exponential spline difference method.

|        | $n = 32$ | $n = 64$ | $n = 128$ | $n = 256$ | $n = 512$ |
|--------|----------|----------|-----------|-----------|-----------|
| Ref. [7] | $1.84e-4$ | $4.61e-5$ | $1.15e-5$ | $2.88e-6$ | $7.21e-7$ |
| Our method ($\tau = [n^{1/4}, -n^{1/4}]$) | $8.11e-6$ | $2.03e-6$ | $5.09e-7$ | $1.28e-7$ | $4.25e-8$ |

![Figure 2](image1)

**Figure 2:** Results of the boundary value problem (72) for $\varepsilon = 0.1$, and $n = 5, 10, 20$, respectively.

Table 2: The maximum error $\|E\|_{\infty}$ and root mean square error for problem (74).

|        | $n = 10$ | $n = 20$ | $n = 40$ | $n = 80$ |
|--------|----------|----------|----------|----------|
| Ref. [11] | $1.02e-3$ | $2.96e-5$ | $3.69e-6$ | $5.11e-8$ |
| Our method ($\tau = [10^{1/4}, -0.012]$) | $8.57e-7$ | $8.18e-7$ | $8.03e-7$ | $7.97e-7$ |

![Figure 3](image2)

**Figure 3:** Results of the boundary value problem (74) for $\varepsilon = 0.01$, and $n = 5, 10, 20$, respectively.

![Figure 4](image3)

**Figure 4:** Results of the boundary value problem (76) for $\varepsilon = 1/32$, and $n = 5, 10, 20$, respectively.
According to (27), we find

\[
\int_{\mathbb{R}} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(a + \frac{i}{n} \Delta x) \Delta x.
\]

So far, we have proved that

\[
\sum_{i=1}^{n} h_k \leq 2\Delta x.
\]

Our method

\[
\tau = \left[ \frac{1}{256}, \frac{1}{8} \right].
\]

Table 3: The maximum error \(\|E\|_{\infty}\) and root mean square error for problem (76).

| \(\varepsilon\) | Method | \(n = 16\) | \(n = 32\) | \(n = 64\) | \(n = 128\) | \(n = 256\) | \(n = 512\) | \(n = 1024\) |
|---|---|---|---|---|---|---|---|---|
| 1/2 | Ref. [16] | 2.6e−4 | 6.5e−5 | 1.6e−5 | 4.1e−6 | 1.0e−6 | 2.6e−7 | 6.4e−8 |
| | Our method | 2.34e−4 | 6.37e−5 | 1.65e−5 | 4.20e−6 | 1.06e−6 | 2.72e−7 | 5.21e−8 |
| | \((\tau = [1.18, -12.75])\) | | | | | | | |
| 1/4 | Ref. [16] | 1.3e−3 | 4.7e−4 | 1.2e−4 | 2.9e−5 | 7.4e−6 | 1.8e−6 | 4.6e−7 |
| | Our method | 3.04e−3 | 8.19e−4 | 2.11e−4 | 5.38e−5 | 1.35e−5 | 3.36e−6 | 1.94e−7 |
| | \((\tau = [1.91, -11.77])\) | | | | | | | |
| 1/8 | Ref. [16] | 7.2e−2 | 4.2e−2 | 4.7e−3 | 1.2e−3 | 2.9e−4 | 7.3e−5 | 1.8e−5 |
| | Our method | 8.23e−2 | 2.39e−2 | 6.13e−3 | 1.54e−3 | 3.93e−4 | 9.66e−5 | 6.74e−7 |
| | \((\tau = [13, -1])\) | | | | | | | |
| 1/16 | Ref. [16] | 2.9e−2 | 7.1e−3 | 1.8e−3 | 4.4e−4 | 1.1e−4 | 2.8e−5 | 6.9e−6 |
| | Our method | 1.61e−2 | 4.03e−3 | 1.04e−3 | 2.67e−4 | 6.80e−5 | 1.66e−5 | 3.74e−6 |
| | \((\tau = [13, -1])\) | | | | | | | |
| 1/32 | Ref. [16] | — | 3.8e−2 | 9.8e−3 | 2.5e−3 | 6.2e−4 | 1.5e−4 | 3.9e−5 |
| | Our method | 7.52e−2 | 1.99e−2 | 5.0e−3 | 1.30e−3 | 3.25e−4 | 7.93e−5 | 1.12e−5 |
| | \((\tau = [1, -13])\) | | | | | | | |
| 1/64 | Ref. [16] | — | 3.4e−2 | 8.7e−3 | 2.2e−3 | 5.5e−4 | 1.4e−4 | 3.4e−5 |
| | Our method | 8.54e−2 | 2.33e−2 | 6.39e−3 | 1.63e−3 | 4.10e−4 | 1.03e−4 | 1.85e−5 |
| | \((\tau = [12.51, -1.48])\) | | | | | | | |
| 1/128 | Ref. [16] | — | — | 2.8e−2 | 7.2e−3 | 1.8e−3 | 4.5e−4 | 1.1e−4 |
| | Our method | — | — | 6.46e−2 | 1.69e−2 | 4.33e−3 | 1.09e−3 | 2.73e−4 | 5.70e−5 |
| | \((\tau = [11.53, -2.46])\) | | | | | | | |
| 1/256 | Ref. [16] | — | — | — | — | 6.3e−2 | 1.6e−2 | 3.9e−3 |
| | Our method | — | — | — | — | 3.21e−2 | 8.09e−3 | 9.13e−4 |
| | \((\tau = [13, -1])\) | | | | | | | |
| 1/512 | Ref. [16] | — | — | — | — | 4.2e−2 | 1.0e−2 | 2.5e−3 |
| | Our method | — | — | — | — | 2.12e−2 | 5.35e−3 | 6.92e−4 |
| | \((\tau = [11.53, -2.46])\) | | | | | | | |
| 1/1024 | Ref. [16] | — | — | — | — | 3.4e−2 | 8.5e−3 |
| | Our method | — | — | — | — | 1.92e−2 | 2.55e−3 |
| | \((\tau = [3.93, -10])\) | | | | | | | |

\[
\|E\|_{\infty} = O\left(\frac{h_{\max}^2}{h_{\min}}\right).
\]  

(71)

that is, the differential expression of (47) is convergent if 

\[ h_{\max} = O(h_{\min}). \]

5. Example

**Example 1.** Solve the following singular boundary value problem (see [7]):

\[
\begin{align*}
-\varepsilon u'' + p(x)u = g(x), & \quad 0 \leq x \leq 1, \\
u(0) = u(1) = 0.
\end{align*}
\]

(72)

where \( p(x) = 1 + x (1 - x) \) and \( g(x) = 1 + x (1 - x) + [2 \sqrt{\varepsilon} - x] e^{-x/\sqrt{\varepsilon}} + [2 \sqrt{\varepsilon} - x^2 (1 - x)] e^{-x(1-x)/\sqrt{\varepsilon}} \), and its analytical solution is

\[
u(x) = 1 + (x - 1) e^{-x(1-x)/\sqrt{\varepsilon}} - xe^{-x(1-x)/\sqrt{\varepsilon}}.
\]

(73)

Table 1 lists the results of the exponential spline difference method for \( \varepsilon = 0.1 \) and that of the method proposed in [7].

Figure 2 shows the results for \( \varepsilon = 0.1 \), and \( n = 5 \) and 10, respectively. To show the difference between the exact
solution and the numerical one, we deliberately use fewer
split points and simply connect two adjacent solutions with
straight lines. A smoother exponential spline function which
is closer to the exact solution can be constructed using these
obtained numerical solutions.

\[
\begin{align*}
u(x) &= 1 + \frac{e^{(1-\sqrt{(1+4)/\lambda})(2/\lambda)-1}e^{\sqrt{(1+4)/\lambda}(2/\lambda)x} + (1 - e^{(1+\sqrt{(1+4)/\lambda})(2/\lambda)})e^{-(1-\sqrt{(1+4)/\lambda})(2/\lambda)}}{(1 + \sqrt{(1 + 4)/\lambda})(2/\lambda)) - (1 - \sqrt{(1 + 4)/\lambda})/2/\lambda} \end{align*}
\]

The computational results are shown in Table 2 for \(\varepsilon = 0.01\) and various values of \(n\) (Figure 3).

Example 3. Consider the convection-dominated equation
(see [16–19]) Figure 4:

\[
\begin{align*}
u'' + \varepsilon \nu' + \nu &= 0, \quad 0 < x < 1, \\
u(0) &= 0, \nu(1) = 1, \\
\nu(x) &= \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})} \quad \varepsilon 
eq (n\pi)^2.
\end{align*}
\]

The computational results are shown in Table 3 for \(\varepsilon = 1/2\) and various values of \(n\).

6. Conclusions
Exponential spline, which is a generalization of polynomial
spline, is an ideal function approximation tool due to its
excellent curve fitting ability. High accuracy can be
achieved in solving second-order ODEs using the exponen-
tial spline scheme. The spline difference method is an
ideal scheme because it can give not only the numerical
results but also the spline function expressions by reusing
these numerical results at the same time, whereas it is only
suitable for solving certain types of equations and does not
have generality. And the selection of the appropriate pa-
rameters is also needed for this method, but there is no
better guideline for the selecting.

Data Availability
The data used to support the findings of this study are in-
cluded within the article.

Conflicts of Interest
The authors declare that there are no conflicts of interest
regarding the publication of this paper.

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