ON THE DENSITY FUNCTION OF THE DISTRIBUTION OF REAL ALGEBRAIC NUMBERS

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Abstract. In this paper we study the distribution of real algebraic numbers. Given an interval \((a, b)\), a positive integer \(n\) and \(Q > 1\), define the counting function \(\Phi_n(Q; a, b)\) to be the number of algebraic numbers in \((a, b)\) of degree \(n\) and height \(\leq Q\). The distribution function is defined to be the limit (as \(Q \to \infty\)) of \(\Phi_n(Q; a, b)\) divided by the total number of real algebraic numbers of degree \(n\) and height \(\leq Q\). We prove that the distribution function exists and is continuously differentiable. We also give an explicit formula for its derivative (to be referred to as the distribution density) and establish an asymptotic formula for \(\Phi_n(Q; \alpha, \beta)\) with upper and lower estimates for the error term in the asymptotic. The estimates are shown to be exact for \(n \geq 3\). One consequence of the main theorem is the fact that the distribution of real algebraic numbers of degree \(n \geq 2\) is non-uniform.

1. Introduction and main results

This paper was inspried by two famous results: the equidistribution of Farey fractions and the fact that real algebraic numbers form a regular system. So we briefly describe the background of our investigation.

The classical Farey sequence \(\mathcal{F}_Q\) of order \(Q\) is formed by irreducible rational fractions in \([0, 1]\) having denominators at most \(Q\) and arranged in increasing order:

\[\mathcal{F}_Q := \left\{ \frac{a}{b} : a, b \in \mathbb{N}, 1 \leq a \leq b \leq Q, \gcd(a, b) = 1 \right\}.\]

The cardinality of \(\mathcal{F}_Q\) has the following asymptotics \([30]\) p. 144, Satz 1):

\[\#\mathcal{F}_Q = \frac{3}{\pi^2} Q^2 + O(Q(\ln Q)^{2/3}(\ln \ln Q)^{4/3}).\]

The fact that this sequence is uniformly distributed in \([0, 1]\) is well-known, that is, for any interval \(I \subseteq [0, 1]\)

\[\lim_{Q \to \infty} \frac{\#(\mathcal{F}_Q \cap I)}{\#\mathcal{F}_Q} = |I|.\]
There are several proofs of this fact (see, e.g. [12], [25], [26]). The discrepancy of the Farey sequence is defined as
\[
D_Q := \sup_{\alpha \in [0,1]} \left| \frac{\#(\mathcal{F}_Q \cap [0,\alpha])}{\#\mathcal{F}_Q} - \alpha \right|.
\]

In 1973, Niederreiter [26] established the true order of the discrepancy: \( D_Q \propto Q^{-1} \). In 1999, Dress [9] found its true value:
\[
D_Q = \frac{1}{Q}.
\]

For more bibliography, one can see a nice book [22, Section 2.4]. In 1924, Franel [14] showed relation between the distribution of Farey series and the Riemann hypothesis. Additional interesting facts and many references can be found in a survey [7].

In 1970, Baker and Schmidt [2] introduced the concept of a regular system and proved that the set of algebraic numbers of degree at most \( n \) forms a regular system, that is, there exists a constant \( c_n \) depending on \( n \) only such that for any interval \( I \) for all sufficiently large \( Q \in \mathbb{N} \) there exist at least
\[
c_n |I| \frac{Q^{n+1}}{(\ln Q)^{3n^2}}
\]

algebraic numbers \( \alpha_1, \ldots, \alpha_k \) of degree at most \( n \) and height at most \( Q \) satisfying
\[
|\alpha_i - \alpha_j| \geq \frac{(\ln Q)^{3n^2}}{Q^{n+1}}, \quad 1 \leq i < j \leq k.
\]

In 1983, their results were improved by Bernik [4]. In 1999, concerning the regularity of the set of real algebraic numbers, Beresnevich [3] showed that the logarithmic factors can be omitted. In the theory of Diophantine approximation, regular systems arose as useful tool for calculation the Hausdorff dimension of sets of transcendental numbers allowing approximation by algebraic numbers with a given precision. These problems go back to Mahler’s investigations and the Khintchine theorem. For a more detailed discussion of the literature we refer to a monograph by Bugeaud [6].

In 1971, H. Brown and K. Mahler [5] proposed a natural generalization of the Farey sequences for algebraic numbers of higher degrees. However, till recently, an important question about the distribution of these generalized Farey sequences remained unanswered. In 1985, in a letter to V. G. Sprindžuk, K. Mahler noticed that it is unknown, even for the second degree, what is the distribution of algebraic numbers. In a private talk, V. I. Bernik conjectured that real algebraic numbers are distributed uniformly, he also brought some natural heuristic arguments in favour of his conjecture.

This all motivated the author to start his own investigation that finally resulted in this paper. A short schema [19] of a proof for an arbitrary degree was published in 2012. For algebraic numbers of the second degree this question was solved in [20]. Here, we present a full proof for arbitrary degrees based on the schema [19].

Note that the generalized Farey sequence [5] is based on so called “naive” height, which is used in our paper too. This height is not the only height function used in Number Theory. In many applications, multiplicative Weil height is extensively used. There is a number
of papers concerning the following problem: over a fixed number field, one needs to count all elements of degree $n$ having multiplicative height at most $T$ as $T$ tends to infinity. For example, results of such type are obtained by Masser and Vaaler in [24], [23].

The results of this paper are closely related to the setting of random polynomials. Most of results in this area concern problems, where the degrees of the polynomials grow to infinity. In the context of the paper, two types of results are of interest. Going in the first direction (see, e.g., [17], [18], [16]), the average number of real roots of random polynomials is estimated for different conditions on the distribution of polynomial coefficients. The second direction is to study the distribution of zeros of random polynomials on the complex plane, and the landmark result by Erdős and Turán [13] states that the arguments of complex roots of random polynomials are uniformly distributed as the degree tends to infinity. For some general conditions on polynomial coefficients, these roots are clustered near the unit circle [15].

Turning back to the subject of the article, we say that the Bernik conjecture about uniformity is now disproved: it turned out that real algebraic numbers of higher degrees are nonuniformly distributed in contrast to rational numbers. However, the fact of equidistribution of rational numbers can be obtained as a particular case of the degree one using the way proposed here. Now, we describe the main result.

Let $p(x) = a_n x^n + \ldots + a_1 x + a_0$ be a polynomial of degree $n$, and let $H(p)$ be its height defined as $H(p) = \max_{0 \leq i \leq n} |a_i|$. Let $\alpha \in \mathbb{C}$ be an algebraic number with its corresponding minimal polynomial $p \in \mathbb{Z}[x]$, i.e., a polynomial with integer coefficients such that $p(\alpha) = 0$, and its degree $\deg(p)$ is minimal, and the greatest common divisor of its coefficients equals to 1. For an algebraic number $\alpha$ its degree $\deg(\alpha)$ and its height $H(\alpha)$ are defined as the degree and the height of the corresponding minimal polynomial. Let $\mathbb{A}_\mathbb{R}_n(Q)$ denote the set of real algebraic numbers $\alpha$ of degree $\deg(\alpha) = n$ and height $H(\alpha) \leq Q$.

Everywhere $\#M$ denotes the number of elements in a set $M$, and $\text{mes}_k M$ denotes the $k$–dimensional Lebesgue measure of a set $M \subset \mathbb{R}^d$ ($k \leq d$). The length of an interval $I$ will be denoted as $|I|$. To denote asymptotic relations between functions, Vinogradov’s symbol $\ll$ is going to be used: the expression $f \ll g$ implies $f \leq c_1 g$, where $c_1$ is a constant depending only on the degree $n$ of the studied algebraic numbers. The notation $f \asymp g$ is used for asymptotically equivalent functions, i.e. $g \ll f \ll g$. The relation $f \ll_{x_1,x_2,\ldots} g$ means that implicit constants depend only on quantities $x_1, x_2, \ldots$, and asymptotic equivalence $f \asymp_{x_1,x_2,\ldots} g$ is defined similarly.

Throughout the paper, we consider polynomials with real coefficients as vectors in Euclidean space. So the usual Lebesgue measure becomes applicable to sets of polynomials with a fixed degree.

In the case $n = 2$, an extra factor $\log Q$ appears in formulas. Therefore, for conciseness, we use the following notation

$$
\ell(n) := \begin{cases} 
1, & n = 2, \\
0, & n \geq 3.
\end{cases}
$$

Let $n \in \mathbb{N}$, and $Q > 1$. In order to describe the distribution of algebraic numbers, we introduce the following quantities.
Definition 1. For real algebraic numbers of degree $n$ and heights at most $Q$, the counting function $\Phi_n(Q, x)$ is defined as
\[ \Phi_n(Q, x) = \#\{\alpha \in \mathbb{A}R_n(Q) : \alpha < x\} . \]

Definition 2. For the same class of algebraic numbers, the distribution function $F_n(x)$ is defined by
\[ F_n(x) = \lim_{Q \to \infty} \frac{\Phi_n(Q, x)}{#\mathbb{A}R_n(Q)} \]
if this limit exists.

Definition 3. If the function $F_n(x)$ is differentiable, its derivative
\[ \rho_n(x) = F_n'(x) \]
will be called the distribution density.

The asymptotic behavior of the function $\Phi_n(Q, x)$ is described by the following theorem.

Theorem 1. There exists a continuous positive function $\phi_n(x)$ such that for all $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$,
\[ \Phi_n(Q, \beta) - \Phi_n(Q, \alpha) = \frac{Q^{n+1}}{2\zeta(n+1)} \int_{\alpha}^{\beta} \phi_n(x) \, dx + O\left(Q^n (\ln Q)^{\ell(n)}\right) , \]
where the implicit constant in the big-O notation depend only on the degree $n$. For infinitely many intervals $[a, b)$, the true order of the remainder term is $O(Q^n)$.

The function $\phi_n(x)$ has the following properties:
\[ \phi_n(-x) = \phi_n(x), \quad \forall x \in \mathbb{R}, \]
\[ x^2 \phi_n(x) = \phi_n\left(\frac{1}{x}\right), \quad \forall x \in \mathbb{R}\setminus\{0\}, \]
and it can be written explicitly as
\[ \phi_n(t) = \int_{\Delta_n(t)} \left| \sum_{k=1}^{n} k p_k t^{k-1} \right| dp_1 \ldots dp_n, \quad t \in \mathbb{R}, \]
where $\zeta(x)$ is the Riemann zeta function, and integration is performed over the region
\[ \Delta_n(t) = \left\{ (p_n, \ldots, p_1) \in \mathbb{R}^n : \max_{1 \leq i \leq n} |p_i| \leq 1, \ |p_n t^n + \ldots + p_1 t| \leq 1 \right\} . \]

Note that for $n = 2$ the true order of the remainder term in (4) is $O(\arctan t \log Q + Q^2)$, where the implicit constant in the big-O notation is absolute (see [21] for a proof). But here we use $O(Q^2 \log Q)$ to omit particular details and give a common proof for all degrees $n \geq 2$.

Definition 4. The function $\phi_n(x)$ will be called the counting density.
Clearly, the counting density exists if and only if the distribution density exists. The counting density is only introduced to simplify the proofs.

2. Auxiliary lemmas

Lemma 1. The counting function $\Phi_n(Q, x)$ has the following properties:

\begin{align*}
(8) \quad & \Phi_n(Q, -x) + \Phi_n(Q, x) = \#\mathbb{AR}_n(Q) - 1_n(Q, x), \quad \forall x \in \mathbb{R}, \\
(9) \quad & \Phi_n \left( Q, \frac{1}{x} \right) + \Phi_n(Q, x) = \frac{2 + \text{sgn}(x)}{2} \#\mathbb{AR}_n(Q) - 1_n(Q, x), \quad \forall x \in \mathbb{R} \setminus \{0\},
\end{align*}

where $1_n(Q, x)$ is the indicator function of the set $\mathbb{AR}_n(Q)$:

$$1_n(Q, x) = 1_{\mathbb{AR}_n(Q)}(x) := \begin{cases} 1, & x \in \mathbb{AR}_n(Q), \\ 0, & x \notin \mathbb{AR}_n(Q). \end{cases}$$

We define $1_n(Q, \pm \infty) := 0$.

Proof. The counting function can be written as

$$\Phi_n(Q, x) = \sum_{\alpha \in \mathbb{AR}_n(Q)} 1 = \sum_{\alpha \in \mathbb{AR}_n(Q)} 1 - 1_n(Q, x).$$

To prove the identity (8) it is enough to observe that $\alpha \in \mathbb{AR}_n(Q)$ if and only if $-\alpha \in \mathbb{AR}_n(Q)$ since $P(x) \in \mathbb{Z}[x]$ is equivalent to $P(-x) \in \mathbb{Z}[x]$, where $\deg P(x) = \deg P(-x)$ and $H(P(x)) = H(P(-x))$. Thus,

$$\Phi_n(Q, -x) = \sum_{\alpha \in \mathbb{AR}_n(Q)} 1 = \sum_{\alpha \in \mathbb{AR}_n(Q)} 1 - \sum_{\alpha \in \mathbb{AR}_n(Q)} 1 = \#\mathbb{AR}_n(Q) - \Phi_n(Q, x) - 1_n(Q, x).$$

The property (9) follows from a similar argument: $\alpha \in \mathbb{AR}_n(Q)$ if and only if $1/\alpha \in \mathbb{AR}_n(Q)$ since $P(x) \in \mathbb{Z}[x]$ is equivalent to $x^n P \left( \frac{1}{x} \right) \in \mathbb{Z}[x]$, where $n = \deg P(x)$ with $H \left( x^n P \left( \frac{1}{x} \right) \right) = H(P(x))$. Since $P(x)$ is an irreducible polynomial with $\deg(P) \geq 2$, we have $P(0) \neq 0$ and thus $\deg x^n P \left( \frac{1}{x} \right) = \deg P(x)$.

To prove (9), let us consider the positive and negative values of $x$ separately. For $x > 0$ the inequality $\alpha < 1/x$ is equivalent to $1/\alpha > x$ or $1/\alpha < 0$. Hence,

$$\Phi_n \left( Q, \frac{1}{x} \right) = \sum_{\alpha \in \mathbb{AR}_n(Q)} 1 = \sum_{\alpha \in \mathbb{AR}_n(Q)} 1 + \sum_{\alpha \in \mathbb{AR}_n(Q)} 1 = \Phi_n(Q, 0) + \#\mathbb{AR}_n(Q) - \Phi_n(Q, x) - 1_n(Q, x).$$
Now let $x < 0$. Then the inequality $\alpha < 1/x$ is equivalent to $x < 1/\alpha < 0$. This yields
\[
\Phi_n\left(\frac{Q}{x}\right) = \sum_{\alpha \in \mathbb{R}_n(Q) \atop \alpha < 1/x} 1 = \sum_{\alpha \in \mathbb{R}_n(Q) \atop x < \alpha < 0} 1 = \Phi_n(Q, 0) - \Phi_n(Q, x) - 1_n(Q, x).
\]

Note that $1_n(Q, x) = 0$ for all $x \in \mathbb{Q}$. From the property (8), we have $\Phi_n(Q, 0) = \frac{1}{2}\#\mathbb{R}_n(Q)$, which completes the proof. \hfill \Box

**Lemma 2** ([28] and [10]). Let $R_n(Q)$ be the set of polynomials $p$, where $\deg p = n$, $H(p) \leq Q$, and each $p$ is reducible over $\mathbb{Q}$. For $Q \to \infty$, the cardinality of $R_n(Q)$ can be asymptotically estimated as
\[
\#R_n(Q) \approx_n Q^n (\ln Q)^{\ell(n)}.
\]

**Proof.** For the reader’s convenience, we give a concise proof here.

Let $p(x) = f(x)g(x)$, where $f$ and $g$ are integral polynomials with $\deg f = n_1$, $\deg g = n_2$, $H(f) = H_1$, $H(g) = H_2$, $n_1 + n_2 = n$, $n_1 \leq n_2$.

It is known (see e.g. [28], Theorem 4.2.2, p. 144) that
\[
\left(\frac{2n_1 + 2n_2 - 2}{n_1 + n_2 + 1}\right)^{-1} H(f)H(g) \leq H(p) \leq (1 + n_1)H(f)H(g).
\]

Hence, the number of reducible polynomials in $R_n(Q)$ must not exceed the number of pairs $(f, g)$ with the heights bounded by the condition $H_1H_2 \leq c_1(n)Q$.

Setting $n_1$ and $n_2$ as fixed, define the following set of polynomial pairs:
\[
\mathcal{P}_{n_1,n_2}(Q) := \{(p_1, p_2) \in \mathbb{Z}[x] \times \mathbb{Z}[x] : \deg p_i = n_i, H(p_i)H(p_2) \leq c_1(n)Q\}.
\]

Now the proof can be concluded by writing
\[
\#R_n(Q) \leq \# \bigcup_{1 \leq n_1 \leq n_2 \atop n_1 + n_2 = n} \mathcal{P}_{n_1,n_2}(Q) \ll_n \sum_{1 \leq n_1 \leq n_2 \atop n_1 + n_2 = n} \left( \sum_{1 \leq H_1, H_2 \atop H_1H_2 \leq c_1(n)Q} H_1^{n_1}H_2^{n_2} \right).
\]

The lower bounds can be obtained in a similar way. \hfill \Box

**Lemma 3** ([8]). For a finite system of inequalities
\[
F_i(x_1, \ldots, x_d) \geq 0, \quad 1 \leq i \leq k,
\]
where each $F_i$ is a polynomial with real coefficients of degree $\deg F_i \leq m$, let $\mathcal{D} \subset \mathbb{R}^d$ be a bounded set of its solutions. Let
\[
\Lambda(\mathcal{D}) = \mathcal{D} \cap \mathbb{Z}^d.
\]

Then
\[
|\#\Lambda(\mathcal{D}) - \text{mes}_d \mathcal{D}| \leq C \max(\tilde{V}(\mathcal{D}), 1),
\]
where the constant $C$ depends only on $d$, $k$, $m$, and $\tilde{V}(\mathcal{D})$ is the maximal $r$-dimensional measure of projections of $\mathcal{D}$ obtained by equating $d - r$ coordinates of the points in $\mathcal{D}$ to
zero, where \( r \) takes all values from 1 to \( d - 1 \), i.e.,
\[
\bar{V}(\mathcal{D}) := \max_{1 \leq r < d} \{ \bar{V}_r(\mathcal{D}) \}, \quad \bar{V}_r(\mathcal{D}) := \max_{J \subset \{1, \ldots, d\} \backslash \{r\} \text{ #}} \{ \text{mes, Proj}_J \mathcal{D} \},
\]
where \( \text{Proj}_J \mathcal{D} \) is an orthogonal projection of \( \mathcal{D} \) onto a coordinate subspace formed by coordinates with indices in \( J \).

For a bounded set \( \mathcal{D} \subset \mathbb{R}^d \), let \( Q \cdot \mathcal{D} \) denote the set \( Qx \in \mathbb{R}^d : x \in \mathcal{D} \).

The following lemma estimates the number of primitive vectors \( x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \), i.e., vectors such that \( \gcd(x_1, \ldots, x_d) = 1 \) lying within bounded regions in \( \mathbb{R}^d \).

\[ \text{Lemma 4.} \quad \text{For a finite set of algebraic inequalities } \]
\[ F_i(x_1, \ldots, x_d) \geq 0, \quad 1 \leq i \leq k, \]
where each \( F_i \) is a polynomial with real coefficients of degree \( \deg F_i \leq m \), let \( \mathcal{D} \subset [-1, 1]^d \) be a bounded set of its solutions.

Let \( \lambda(\mathcal{D}, Q) \) be the number of primitive vectors of the lattice \( \mathbb{Z}^d \) in the region \( Q \cdot \mathcal{D} \).

Then asymptotically we have
\[ \lambda(\mathcal{D}, Q) = Q^d \cdot \frac{\text{mes}_d \mathcal{D}}{\zeta(d)} + \mathcal{O}\left(Q^{d-1}\left(\ln Q\right)^{\ell(d)}\right), \]
where \( \zeta(x) \) is the Riemann zeta function. The implicit constant in the big-O notation depends only on the dimension \( d \), the size of the system \( k \) and the maximal degree \( m \) of the algebraic inequalities.

Remark. Results of such type are well-known and can be found e.g. in a classical monograph by Paul Bachmann [1, pp. 436–444] (see formulas (83a) and (83b) on pages 441–442). For the reader’s convenience, we give a short proof here.

\[ \text{Proof.} \quad \text{Without loss of generality, we can exclude the point } 0 = (0, \ldots, 0) \text{ from counting.} \]

For a positive integer \( \nu \), let us count the number of integral points \( x = (x_1, \ldots, x_d) \in \mathcal{D} \) such that \( \nu \) divides \( \gcd(x_1, \ldots, x_d) \). All this points are contained in the lattice \( \nu \cdot \mathbb{Z}^d \), and their number in the region \( Q \cdot \mathcal{D} \) equals \( \#\Lambda(Q \cdot \mathcal{D}) \). Applying Lemma 3 we have
\[ \#\Lambda(Q \cdot \mathcal{D}) = Q^d \cdot \frac{\text{mes}_d \mathcal{D}}{\zeta(d)} + \mathcal{O}\left(Q^{d-1}\left(\ln Q\right)^{\ell(d)}\right). \]

By applying the inclusion–exclusion principle, we obtain
\[ \lambda(\mathcal{D}, Q) = \#\Lambda(Q \cdot \mathcal{D}) - \sum_{q_1} \#\Lambda\left(\frac{Q}{q_1} \cdot \mathcal{D}\right) + \sum_{q_1 < q_2} \#\Lambda\left(\frac{Q}{q_1 q_2} \cdot \mathcal{D}\right) - \ldots + \]
\[ + (-1)^n \sum_{q_1 < q_2 < \ldots < q_n} \#\Lambda\left(\frac{Q}{q_1 q_2 \ldots q_n} \cdot \mathcal{D}\right) + \ldots = \sum_{n=1}^{\infty} \mu(n) \#\Lambda\left(\frac{Q}{n} \cdot \mathcal{D}\right), \]
where the sums are taken over prime numbers \( q_1, q_2, \ldots, q_n \); \( \mu(x) \) is the Möbius function.
Clearly, for \( n > Q \), the lattice \( n \cdot \mathbb{Z}^d \) doesn’t contain any non-zero points lying in \( Q \cdot \mathbb{D} \), i.e. \( \# \Lambda ( \frac{Q}{n} \cdot \mathbb{D} ) = 0 \). Therefore, from (12) and (13), we have

\[
\lambda(\mathbb{D}, Q) = Q^d \cdot \text{mes}_d \mathbb{D} \sum_{n=1}^{Q} \frac{\mu(n)}{n^d} + O \left( Q^{d-1} \sum_{n=1}^{Q} \frac{1}{n^{d-1}} \right).
\]

Now applying the well-known facts that for \( d \geq 2 \)

\[
\left| \sum_{n=Q+1}^{\infty} \frac{\mu(n)}{n^d} \right| \leq \sum_{n=Q+1}^{\infty} \frac{1}{n^d} \sim \frac{1}{Q^{d-1}}
\]

as \( Q \to \infty \), and that \( \sum_{n=1}^{\infty} \mu(n)n^{-d} = (\zeta(d))^{-1} \), where \( \zeta(d) \) is the Riemann zeta function, completes the proof. \( \square \)

**Lemma 5.** Consider a polynomial relation

\[
a_n x^n + \ldots + a_1 x + a_0 = (x-\alpha)(x-\beta)(b_{n-2}x^{n-2} + \ldots + b_1 x + b_0).
\]

Then for the vectors \((a_n, \ldots, a_1, a_0)\) and \((b_{n-2}, \ldots, b_1, b_0, \alpha, \beta)\), we can write

\[
\begin{pmatrix}
  a_n \\
  a_{n-1} \\
  a_{n-2} \\
  \vdots \\
  a_1 \\
  a_0 
\end{pmatrix}
=
\begin{pmatrix}
  1 & \alpha \beta & \cdots & \alpha \beta \\
  -\alpha - \beta & 1 & \cdots & \alpha + \beta \\
  \alpha \beta & -\alpha - \beta & \cdots & 1 \\
  \vdots & \alpha \beta & \cdots & \alpha \beta \\
  0 & \alpha \beta & \cdots & \alpha \beta 
\end{pmatrix}
\cdot
\begin{pmatrix}
  b_{n-2} \\
  b_{n-3} \\
  \vdots \\
  b_1 \\
  b_0 
\end{pmatrix},
\]

and the Jacobian \( |\det J| = \left| \frac{\partial (a_n, a_{n-2}, a_1, a_0)}{\partial (b_{n-2}, \ldots, b_0, \alpha, \beta)} \right| \) satisfies:

\[
|\det J| = |\alpha - \beta| \cdot |g(\mathbf{b}, \alpha)g(\mathbf{b}, \beta)|,
\]

where \( g(\mathbf{b}, x) = b_{n-2}x^{n-2} + \ldots + b_1 x + b_0 \).

**Proof.** By definition, we have

\[
\det J = \begin{vmatrix}
  1 & 0 & 0 \\
  -\alpha - \beta & 1 & 0 \\
  \alpha \beta & -\alpha - \beta & 1 \\
  \vdots & \alpha \beta & \cdots & 1 \\
  \alpha \beta & \cdots & -\alpha - \beta & 1 \\
  \alpha \beta & \cdots & \alpha \beta & \alpha \beta \\
\end{vmatrix}
\]

Using Laplace’s formula along the first row reduces the dimension of this determinant by 1. Subtracting the \((n-2)\)-th column from the \((n-1)\)-th and dividing the result by \((\alpha - \beta)\)
Any polynomial in $M$ yields

$$
\begin{vmatrix}
1 & -b_{n-2} & 0 \\
-(\alpha + \beta) & 1 & -b_{n-3} & b_{n-2} \\
\alpha \beta & -(\alpha + \beta) & \ddots & \ddots \\
\alpha \beta & \ddots & 1 & -b_1 \\
\alpha \beta & \ddots & -(\alpha + \beta) & -b_0 & b_1 \\
\alpha \beta & 0 & b_0 & 0
\end{vmatrix}.
$$

(17) $\det J = (\alpha - \beta)$.

It is easy to see that the determinant in the left-hand side of (17) is equal to the resultant $R(f, g)$ of the polynomials $f(x) = (x - \alpha)(x - \beta)$ and $g(x) = b_{n-2}x^{n-2} + \ldots + b_1x + b_0$, up to a sign. This proves the lemma since $R(f, g) = g(\alpha)g(\beta)$ [29, section 5.9].

**Lemma 6.** Let $I = [a, b) \subset \mathbb{R}$ be a bounded interval, $|I| \leq 1$. Let $\mathcal{M}_n(I)$ be the set of polynomials $p \in \mathbb{R}[x]$ with $\deg p = n$ and $H(p) \leq 1$ that have at least two roots in $I$. Then

$$\mes_{n+1}\mathcal{M}_n(I) \leq \frac{c_2(n)}{\rho^n} |I|^3,$$

where $\rho = \rho(I) = \max(1, |a + b|/2)$, and $c_2(n)$ is a constant that depends only on $n$.

**Proof.** Let us find an upper bound for

$$\mes_{n+1}\mathcal{M}_n(I) = \int_{\mathcal{M}_n(I)} da.$$

Any polynomial in $\mathcal{M}_n(I)$ can be written in the form (14) with $\alpha, \beta \in I$. So we use the substitution (15) to evaluate this integral. The condition $a \in \mathcal{M}_n(I)$ is equivalent to the system of inequalities

$$
\begin{cases}
|a_n| = |b_{n-2}| \leq 1, \\
|a_{n-1}| = |b_{n-3} - (\alpha + \beta)b_{n-2}| \leq 1, \\
|a_k| = |b_{k-2} - (\alpha + \beta)b_{k-1} + \alpha \beta b_k| \leq 1, & k = 2, \ldots, n-2, \\
|a_1| = |-(\alpha + \beta)b_0 + \alpha \beta b_1| \leq 1, \\
|a_0| = |\alpha \beta b_0| \leq 1, \\
a \leq \alpha < b, \\
a \leq \beta < b.
\end{cases}
$$

(18)

Using Lemma 5 we obtain

$$\mes_{n+1}\mathcal{M}_n(I) \leq \int_{\mathcal{M}_n(I)} |\alpha - \beta| \cdot |g(b, \alpha)g(b, \beta)| \, db \, d\alpha \, d\beta,$$

(19)

where $\mathcal{M}^*_n(I)$ is new domain of integration defined by (18) and $g(b, x) = b_{n-2}x^{n-2} + \ldots + b_1x + b_0$.

This expression cannot be written as an equality since polynomials can have three or more roots in the interval $I$, and then several representations of the form (14) will exist. If
a polynomial has \( k > 2 \) different roots on \( I \), then there exist \( \binom{k}{2} \) different representations of this type.

Let \( I \subset [-2, 2] \). Then for all \( \alpha, \beta \in I \) we can write

\[
|\alpha + \beta| \leq 4, \quad |\alpha \beta| \leq 4,
\]

and therefore for any \( (b_{n-2}, \ldots, b_1, b_0, \alpha, \beta) \in \mathcal{M}_n^*(I) \) we have

\[
\begin{align*}
|b_{n-2}| & \leq 1, \\
|b_{n-3}| & \leq 1 + 4|b_{n-2}|, \\
|b_k| & \leq 1 + 4|b_{k-1}| + 4|b_k|, \quad k = 2, \ldots, n-2, \\
|\alpha| & \leq 2, \\
|\beta| & \leq 2.
\end{align*}
\]

(20)

In other words, for \( I \subset [-2, 2] \) the domain \( \mathcal{M}_n^*(I) \) is enclosed within some box, whose dimensions are determined by \( n \) only.

Let us rewrite the multiple integral in (19) as follows:

\[
\text{mes}_{n+1} \mathcal{M}_n(I) \leq \int_{I^* \times I} |\alpha - \beta| \, d\alpha \, d\beta \int_{\mathcal{M}_n^*(\alpha, \beta)} |g(\mathbf{b}, \alpha)g(\mathbf{b}, \beta)| \, d\mathbf{b},
\]

where \( \mathcal{M}_n^*(\alpha, \beta) = \{ \mathbf{b} \in \mathbb{R}^{n-1} : \|A(\alpha, \beta)\mathbf{b}\|_\infty \leq 1 \} \), and \( A(\alpha, \beta) \) is the \( (n+1) \times (n-1) \) matrix from (15).

Let us denote \( G(\mathbf{b}, \alpha, \beta) = g(\mathbf{b}, \alpha)g(\mathbf{b}, \beta) \), and

\[
\psi(\alpha, \beta) = \int_{\mathcal{M}_n^*(\alpha, \beta)} |G(\mathbf{b}, \alpha, \beta)| \, d\mathbf{b}.
\]

As stated above, for \( \alpha, \beta \in [-2, 2] \) the domain \( \mathcal{M}_n^*(\alpha, \beta) \) lies within a box of dimensions determined only by \( n \). The function \( G(\mathbf{b}, \alpha, \beta) \) is a polynomial, and its values cannot exceed a certain constant determined only by \( n \) for all \( \mathbf{b} \in \mathcal{M}_n^*(\alpha, \beta) \) and all \( \alpha, \beta \in I \). Hence, there exists a constant \( c_2(n) \) which depends only on \( n \) such that \( 0 < \psi(\alpha, \beta) \leq c_2(n) \) for all \( \alpha, \beta \in I \). Thus, we obtain for \( I \subset [-2, 2] \)

\[
\text{mes}_{n+1} \mathcal{M}_n(I) \leq c_2(n)|I|^3.
\]

Let \( I \subset \mathbb{R} \setminus [-1, 1] \). In this case, we substitute the polynomial \( p(x) = a_nx^n + \ldots + a_1x + a_0 \) by the polynomial \( q(x) = x^np(1/x) = a_0x^n + \ldots + a_{n-1}x + a_n \), and the interval \( I \) by \( I^* = (1/b, 1/a) \subset [-1, 1] \), \( ab > 0 \). Clearly, under these substitutions the number of roots remains invariant.

Now we can apply the substitution (15) to the vector \( (a_0, a_1, \ldots, a_n) \), which leads to

\[
\text{mes}_{n+1} \mathcal{M}_n(I) = \text{mes}_{n+1} \mathcal{M}_n(I^*) \leq c_2(n)|I^*|^3 = \frac{c_2(n)}{|ab|^2}|I|^3,
\]

proving the lemma. □
Lemma 7. Let \( x_0 = a/b \) with \( a \in \mathbb{Z}, b \in \mathbb{N} \) and \( \gcd(a, b) = 1 \). Then there are no algebraic numbers \( \alpha \) of degree \( \deg \alpha = n \) and height \( H(\alpha) \leq Q \) in an interval \( |x - x_0| \leq r_0 \), where

\[
r_0 = r_0(x_0, Q) = \frac{c_3(n)}{b^n Q},
\]

and \( c_3(n) \) is an effective constant depending on \( n \) only.

A similar fact can be stated for a neighborhood of infinity: no algebraic number \( \alpha \) of degree \( \deg(\alpha) = n \) and height \( H(\alpha) \leq Q \) lies in the set \( \{ x \in \mathbb{R} : |x| \geq Q + 1 \} \).

Note that the statement of the lemma implies the following equations: \( \Phi_n(Q, x) = 0 \) for \( x \leq -Q - 1 \), and \( \Phi_n(Q, x) = \#\text{AR}_n(Q) \) for \( x \geq Q + 1 \).

Proof. Let \( p(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x] \) with \( H(p) \leq Q \).

We develop \( p(x) \) into the Taylor series:

\[
p(x) = p(x_0) + \sum_{k=1}^{n} \frac{p^{(k)}(x_0)}{k!} (x - x_0)^k.
\]

Let \( p(x_0) \neq 0 \). Then

\[
|p(x_0)| \geq \frac{1}{b^n}.
\]

Assuming \( |x_0| \leq 1 \), we have \( |p^{(k)}(x_0)| \ll_n H(p) \) for \( k = 0, \ldots, n \), and thus

\[
\left| \sum_{k=1}^{n} \frac{p^{(k)}(x_0)}{k!} (x - x_0)^k \right| \ll_n H(p) |x - x_0| \sum_{k=0}^{n-1} |x - x_0|^k.
\]

Development (21) and estimates (22), (23) imply that any integral polynomial of degree \( n \) and height at most \( H \) has no roots \( x \neq x_0 \) in the circle

\[
|x - x_0| < \frac{c_3(n)}{b^n H},
\]

where \( c_3(n) \) is an effective constant.

The case \( |x_0| > 1 \) can be reduced to the case \( |x_0| < 1 \) with the use of mapping \( z \rightarrow z^{-1} \). Let \( |\beta| \geq Q + 1 \). Then

\[
|p(\beta)| \geq |\beta|^n - Q|\beta|^{n-1} - \ldots - Q|\beta| - Q \geq 1.
\]

Thus, the number \( \beta \) cannot be a root of the polynomial \( p(x) \). The lemma is proved. \( \square \)

3. The proof of the main theorem

Recall that \( I = [\alpha, \beta] \) is a bounded interval. Let prime polynomials be defined as irreducible primitive polynomials with positive leading coefficients. Clearly, the distribution of algebraic numbers can be expressed in terms of prime polynomials. Let \( N_n(Q, k, I) \) denote
the number of prime polynomials $p$ of degree $\deg p = n$ and height $H(p) \leq Q$ which have exactly $k$ roots in the set $I$. Clearly, we have

$$\Phi_n(Q, \beta) - \Phi_n(Q, \alpha) = \sum_{k=1}^{n} kN_n(Q, k, I).$$

Let $G_n(k, S)$ denote the set of polynomials $p \in \mathbb{R}[x]$ with $\deg p = n$ and $H(p) \leq 1$ that have exactly $k$ roots in a set $S$. From Lemmas 2 and 4, we have:

$$N_n(Q, k, I) = \text{mes}_{n+1} G_n(k, I) \cdot Q^{n+1} + O(Q^n (\ln Q)^{\ell(n)}),$$

where the implicit constants in the big-O notation depend only on $n$.

**Lemma 8.** The function $\Psi_n(S)$ defined as

$$\Psi_n(S) = \sum_{k=1}^{n} k \text{mes}_{n+1} G_n(k, S)$$

is additive and bounded for all $S \subseteq \mathbb{R}$.

**Proof.** Indeed, if the intersection of $S_1$ and $S_2$ is empty, $S_1 \cap S_2 = \emptyset$, then

$$\Psi_n(S_1) + \Psi_n(S_2) = \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} k \text{mes}_{n+1} (G_n(k, S_1) \cap G_n(m, S_2)) +$$

$$+ \sum_{m=1}^{\infty} \sum_{m=0}^{\infty} m \text{mes}_{n+1} (G_n(k, S_1) \cap G_n(m, S_2)) =$$

$$= \sum_{\nu=1}^{\infty} \nu \sum_{k=0}^{\nu} \text{mes}_{n+1} (G_n(k, S_1) \cap G_n(\nu - k, S_2)) = \Psi_n(S_1 \cup S_2).$$

Here we have used the facts that $G_n(k, S) = \emptyset$ for $k > n$ and that

$$G_n(\nu, S_1 \cup S_2) = \bigcup_{k=0}^{\nu} (G_n(k, S_1) \cap G_n(\nu - k, S_2)).$$

For any $S \subseteq \mathbb{R}$, we have

$$\Psi_n(S) \leq n \sum_{k=1}^{n} \text{mes}_{n+1} G_n(k, S) \leq n \text{mes}_{n+1}([-1, 1]^{n+1}) = n2^{n+1}.$$

The lemma is proved. \qed

Let us introduce the function

$$\hat{\Phi}_n(x) := \Psi_n((-\infty, x)).$$

Clearly, we have $\hat{\Phi}_n(x) \leq n2^{n+1}$. 

---

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From the additivity of $\Psi_n(S)$, we have

\[
\hat{\Phi}_n(\beta) - \hat{\Phi}_n(\alpha) = \sum_{k=1}^{n} k \operatorname{mes}_{n+1} G_n(k, I).
\]

Let us prove that $\hat{\Phi}_n(x)$ is differentiable. Define

\[
D(I) = \{ p \in \mathbb{R}^{n+1} : p(\alpha)p(\beta) < 0, \ H(p) \leq 1 \},
\]

where $p = (p_n, \ldots, p_1, p_0)$ is the vector form of a polynomial $p(x) = p_n x^n + \ldots + p_1 x + p_0$. Clearly, for any vector in $D(I)$ the corresponding polynomial has an odd number of roots lying in the interval $I$.

From Lemma 6, we have

\[
\hat{\Phi}_n(\beta) - \hat{\Phi}_n(\alpha) = \operatorname{mes}_{n+1} D(I) + O(|I|^3),
\]

where the implicit constant in the big-O notation depends only on the degree $n$.

Now let us calculate $\operatorname{mes}_{n+1} D(I) = \int_{D(I)} dp$. The domain $D(I)$ can be defined by the following system of inequalities

\[
\begin{cases}
|p_i| \leq 1, & 0 \leq i \leq n, \\
f_*(p_1, \ldots, p_n) \leq p_0 \leq f^*(p_1, \ldots, p_n),
\end{cases}
\]

where

\[
f_*(p_1, \ldots, p_n) = \min \left\{ \sum_{k=1}^{n} p_k \alpha_k, \sum_{k=1}^{n} p_k \beta_k \right\},
\]

\[
f^*(p_1, \ldots, p_n) = \max \left\{ -\sum_{k=1}^{n} p_k \alpha_k, -\sum_{k=1}^{n} p_k \beta_k \right\}.
\]

To simplify notation, define a function $h$ as

\[
h(p_1, \ldots, p_n) := f^*(p_1, \ldots, p_n) - f_*(p_1, \ldots, p_n) = (\beta - \alpha) \left| \sum_{k=1}^{n} p_k \sum_{i=0}^{k-1} \alpha^i \beta^{k-i-1} \right|.
\]

Consider regions

\[
\Delta_* := \Delta_n(\alpha) \cap \Delta_n(\beta), \quad \Delta^* := \Delta_n(\alpha) \cup \Delta_n(\beta),
\]

where

\[
\Delta_n(t) = \left\{ (p_n, \ldots, p_1) \in \mathbb{R}^n : \max_{1 \leq i \leq n} |p_i| \leq 1, \ \left| \sum_{k=1}^{n} p_k t^k \right| \leq 1 \right\}.
\]

For all $(p_1, \ldots, p_n) \in \Delta_*$, the inequalities $|f_*(p_1, \ldots, p_n)| \leq 1$ and $|f^*(p_1, \ldots, p_n)| \leq 1$ hold, and thus, the bound $|p_0| \leq 1$ gives no effect in (29). For any $(p_1, \ldots, p_n) \not\in \Delta^*$, the system
of inequalities (29) is inconsistent for \( \alpha \) and \( \beta \) being close enough. Hence, we have an estimate
\[
\int_{\Delta_*} h(p_1, \ldots, p_n) \, dp_1 \ldots dp_n \leq \text{mes}_{n+1} D(I) \leq \int_{\Delta_*} h(p_1, \ldots, p_n) \, dp_1 \ldots dp_n.
\]
From above, it follows that
\[
\left| \text{mes}_{n+1} D(I) - \int_{\Delta_{n}(\alpha)} h(p_1, \ldots, p_n) \, dp_1 \ldots dp_n \right| \leq \int_{\Delta^* \setminus \Delta_*} h(p_1, \ldots, p_n) \, dp_1 \ldots dp_n.
\]
It is easy to show that the difference of \( \Delta^* \) and \( \Delta_* \) has a small measure
\[
\text{mes}_n (\Delta^* \setminus \Delta_*) = O(\beta - \alpha), \quad \beta \to \alpha,
\]
where the implicit constant in the big-O notation depends on the degree \( n \) only.

Therefore, as \( \beta \) tends to \( \alpha \), we obtain for any \( \alpha \in \mathbb{R} \)
\[
(30) \quad \lim_{\beta \to \alpha} \frac{\text{mes}_{n+1} D(I)}{(\beta - \alpha)} = \int_{\Delta_1(\alpha)} \left| k p_k \alpha^{k-1} \right| \, dp_n \ldots dp_1.
\]
Thus, (28) can be rewritten as
\[
\hat{\Phi}_n(\beta) - \hat{\Phi}_n(\alpha) = \phi_n(\alpha)(\beta - \alpha) + o(\beta - \alpha),
\]
where \( \phi_n(\cdot) \) is defined by (7). This proves that the function \( \hat{\Phi}_n(x) \) is differentiable and \( \phi_n(x) \) is its derivative. Therefore, we have
\[
\hat{\Phi}_n(\beta) - \hat{\Phi}_n(\alpha) = \int_{\alpha}^{\beta} \phi_n(x) \, dx.
\]
The statements (4) and (7) have been proved. It is also easy to see that \( \phi_n(x) \) is a continuous positive function on \( \mathbb{R} \).

The upper bound for the remainder term in (4) is obtained from the error term in (11) and the bound (10) for the number of reducible polynomials.

Lemma 7 gives a lower bound for the remainder term of (4). The set \( Q \) is everywhere dense in \( \mathbb{R} \). Thus, Lemma 7 implies that in every interval there exist infinitely many subintervals with midpoints \( \alpha \in Q \) and lengths \( |I| \geq Q^{-1} \) containing no algebraic numbers \( \alpha \) with \( \deg(\alpha) = n \) and \( H(\alpha) \leq Q \). Hence, the remainder term in (4) is bounded from below by \( O(Q^n) \) for infinitely many intervals.

Note that
\[
\hat{\Phi}(x) = 2\zeta(n+1) \lim_{Q \rightarrow \infty} \frac{\Phi_n(Q, x)}{Q^{n+1}},
\]
and the equations (3) and (9) from Lemma 1 translate to the following equations for $\hat{\Phi}_n(x)$:

\[
\hat{\Phi}_n(-x) + \hat{\Phi}_n(x) = \gamma_n,
\]

\[
\hat{\Phi}_n\left(\frac{1}{x}\right) + \hat{\Phi}_n(x) = \frac{2 + \text{sgn}(x)}{2} \cdot \gamma_n,
\]

where $\gamma_n := 2\zeta(n+1) \lim_{Q \to \infty} \frac{\# \mathbb{A}_{\mathbb{Q}}(Q)}{Q^{n+1}}$. Differentiating w.r.t. $x$ yields (3) and (5), completing the proof.

4. Final remarks

**Remark 1.** The counting density on the interval $|x| \leq \frac{\sqrt{2} - 1}{\sqrt{2}} \approx 0.29$ can be expressed as

\[
\phi_n(x) = \frac{2^{n-1}}{3} \left(3 + \sum_{k=1}^{n-1} (k+1)^2 2^{2k}\right).
\]

**Proof.** To simplify the calculation of (7), let us place the following restriction on $t$: $|p_n t^n + \ldots + p_2 t^2 + p_1 t| \leq 1$ for all $p_i$ such that $\max_{1 \leq i \leq n} |p_i| \leq 1$. This is equivalent to the inequality $|t| \leq t_0(n)$, where $t_0(n)$ is the positive solution of the equation $t^n + \ldots + t^2 + t = 1$. For $|t| \leq t_0(n)$, we have

\[
\phi_n(t) = 2 \int_{\tilde{\Delta}_n(t)} \left(\sum_{k=1}^{n} kp_k t^{k-1}\right) dp_1 \ldots dp_n,
\]

where

\[
\tilde{\Delta}_n(t) = \left\{(p_n, \ldots, p_1) \in \mathbb{R}^n : \max_{1 \leq i \leq n} |p_i| \leq 1, \, np_n t^{n-1} + \ldots + 2p_2 t + p_1 \geq 0\right\}.
\]

Let $f_n(t; p_n, \ldots, p_2)$ denote the function $np_n t^{n-1} + \ldots + 2p_2 t$. The variable $p_1$ assumes the values between $\max(-f_n(t; p_n, \ldots, p_2), -1)$ and $1$. Assume $f_n(t; p_n, \ldots, p_1) \leq 1$ for all $p_i$ such that $\max_{2 \leq i \leq n} |p_i|$. This restriction is equivalent to $|t| \leq t_1(n)$, where $t_1(n)$ is the positive solution of $nt^{n-1} + \ldots + 2t = 1$. Clearly, $t_1(n) \leq t_0(n)$ and $\lim_{n \to \infty} t_1(n) = \frac{\sqrt{2} - 1}{\sqrt{2}}$.

For $|t| \leq t_1(n)$ we have

\[
\phi_n(t) = 2 \int_{|p_1| \leq 1} \left(\int_{-f_n(t;p_n,\ldots,p_2)}^{1} (f_n(t;p_n,\ldots,p_2) + p_1) \, dp_1\right) dp_2 \ldots dp_n =
\]

\[
= \int_{|p_1| \leq 1} \left(\int_{-f_n(t;p_n,\ldots,p_2)}^{1} (f_n(t;p_n,\ldots,p_2) + 1)^2 \, dp_2\right) \ldots dp_n.
\]

From the symmetry, integrals of all non-square terms are equal to zero, which leads to (31).
Now it is easy to see that the distribution of algebraic numbers of degree \( n \), \( n \geq 2 \), is non-uniform. However, rational numbers, i.e., algebraic numbers of the first degree, are distributed uniformly in the interval \([-1, 1]\) with the counting density \( \phi_1(x) = 1 \). In general, for \( n = 1 \) we can write
\[
\phi_1(x) = \begin{cases} 1, & |x| \leq 1, \\ \frac{1}{2\pi}, & |x| > 1. \end{cases}
\]
This result obtained from the general formula (7) agrees with the well-known result on the distribution of the Farey sequence:
\[
\Phi_1(Q, \beta) - \Phi_1(Q, \alpha) = \frac{3}{\pi^2} (\beta - \alpha) Q^2 + O(Q \ln Q),
\]
where \(-1 < \alpha < \beta < 1\).

For \( \phi_2(x) \), a piecewise formula involving rational functions only is obtained in [21].

**Remark 2.** It is possible to generalize Theorem 1 to height functions different from \( H(p) \). In general, a height function is defined as follows.

**Definition 5.** A function \( h : \mathbb{R}^{n+1} \to [0, +\infty) \) satisfying the conditions

1. \( h(tx) = |t| h(x) \) for all \( t \in \mathbb{R} \) and all \( x \in \mathbb{R}^{n+1} \);
2. the set \( \{ x \in \mathbb{R}^{n+1} : h(x) \leq 1 \} \) is a convex body;
3. \( h(x) = 0 \) if and only if \( x = 0 \);
4. \( h(x_n, -x_{n-1}, \ldots, (-1)^{n-1}x_1, (-1)^n x_0) = h(x_n, x_{n-1}, \ldots, x_1, x_0) \) for all \( x \in \mathbb{R}^{n+1} \);
5. \( h(x_0, x_1, \ldots, x_{n-1}, x_n) = h(x_n, x_{n-1}, \ldots, x_1, x_0) \) for all \( x \in \mathbb{R}^{n+1} \),

is called a **height function**.

Note that the last two conditions in the definition correspond to (5) and (6) respectively.

Now we can define the respective height function for algebraic numbers. Let \( p \) be the minimal polynomial of an algebraic number \( \alpha \). Clearly, for any given algebraic \( \alpha \), the same minimal polynomial \( p \) is obtained for any height function \( h \), and the height \( h(\alpha) \) can be defined as \( h(p) \).

Let us define the distribution of algebraic numbers with respect to the height function \( h(\alpha) \). In the general case, the counting density with respect to \( h(x) \) assumes the form
\[
\phi_n(h; t) = \int_{\Delta_n(h; t)} \left| \sum_{k=1}^{n} k p_k t^{k-1} \right| dp_1 \ldots dp_n,
\]
where
\[
\Delta_n(h; t) = \left\{ (p_n, \ldots, p_1) \in \mathbb{R}^n : h\left(p_n, \ldots, p_2, p_1, -\sum_{k=1}^{n} p_k t^n \right) \leq 1 \right\}.
\]
Naturally, this function behaves differently for different height functions \( h \).

To illustrate this fact, let us calculate the counting density with respect to the spherical norm \( \| \cdot \|_2 \). The region \( D(I) \) defined by (27) is then replaced by two \( (n+1) \)-dimensional
spherical wedges of radius 1 with an acute angle $\theta(\alpha, \beta)$ formed by two planes with normal vectors $n_1 = (\alpha^n, \ldots, \alpha, 1)$ and $n_2 = (\beta^n, \ldots, \beta, 1)$. The volume of such wedge equals
\[
\frac{\pi^{\frac{n+1}{2}} \theta}{\Gamma\left(\frac{n+3}{2}\right) 2\pi},
\]
and thus the formula \((30)\) can be written as
\[
\phi_n(\| \cdot \|; \alpha) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} \lim_{\beta \to \alpha} \frac{\theta(\alpha, \beta)}{\beta - \alpha}.
\]
The angle $\theta(\alpha, \beta)$ tends to zero as $\beta$ tends to $\alpha$, allowing us to substitute $\theta$ by $\sin \theta$, which can be calculated from the scalar product of $n_1$ and $n_2$:
\[
\sin \theta = \sqrt{1 - \frac{(n_1 \cdot n_2)^2}{n_1^2 n_2^2}}
\]
Note that if a function $f : \mathbb{R}^2 \to \mathbb{R}$ is three times continuously differentiable and $f(x, y) = f(y, x)$ for any $x, y \in \mathbb{R}$, then by Taylor development of $f(\cdot, \cdot)$ at $(x, x)$ as $y$ tends to $x$, one can obtain
\[
f(x, x) f(y, y) - (f(x, y))^2 = \left( f(x, x) \frac{\partial f^2}{\partial x \partial y} \bigg|_{y=x} - \left( \frac{\partial f}{\partial x} \bigg|_{y=x} \right)^2 \right) (y - x)^2 + O((y - x)^3).
\]
Taking $f(x, y) = \sum_{k=0}^{n}(xy)^k = \frac{(xy)^{n+1} - 1}{xy - 1}$, we obtain after transformations
\[
n_1^2 n_2^2 - (n_1 \cdot n_2)^2 = \frac{(\alpha^{2n+2} - 1)^2 - (n + 1)^2 \alpha^{2n}(\alpha^2 - 1)^2}{(\alpha^2 - 1)^4} (\beta - \alpha)^2 + O((\beta - \alpha)^3).
\]
Finally, we have
\[
\phi_n(\| \cdot \|; t) = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+3}{2}\right)} \sqrt{\frac{1}{(t^2 - 1)^2} - \frac{(n + 1)^2 t^{2n}}{(t^{2n+2} - 1)^2}}.
\]
Clearly, this function is quite different from \((31)\). Up to the constant factor, the function $\phi_n(\| \cdot \|; \alpha)$ coincides with the density function of zeros of random polynomials of $n$-th degree with independent identically normally distributed coefficients (see \cite{17}). In an excellent paper by Edelman and Kostlan \cite{11} §§2.2–2.4, one can find an interesting geometrical interpretation of this case.

**Remark 3.** The main result can be reformulated in terms of sequences.

**Corollary 1.** Let all real algebraic numbers of $n$-th degree form a sequence $(\alpha_i)_{i=1}^\infty$ using the following ordering: if $k < m$, then $H(\alpha_k) \leq H(\alpha_m)$. Let $\mathcal{N}_n(S, N)$ be the number of elements of the truncated sequence $(\alpha_i)_{i=1}^N$ lying in a set $S$. Then for any interval $I = [a, b]$ we have
\[
\mathcal{N}_n(I, N) = N \int_a^b \rho_n(x) \, dx + O\left(N^{\frac{n}{n+1}} \ln N \right),
\]
where the implicit constant in the big-O notation depends only on the degree \( n \).

**Proof.** Let us denote \( H(\alpha_N) \) by \( Q_N \). By the definition of the sequence \( (\alpha_i)_{i=1}^{\infty} \), we have

\[
\# \mathbb{A} \mathbb{R}_n(Q_N - 1) \leq N \leq \# \mathbb{A} \mathbb{R}_n(Q_N),
\]

(32)

\[
\Phi_n(Q_N - 1, b) - \Phi_n(Q_N - 1, a) \leq \mathfrak{g}_n(I, N) \leq \Phi_n(Q_N, b) - \Phi_n(Q_N, a).
\]

(33)

Theorem 1 yields

\[
\# \mathbb{A} \mathbb{R}_n(Q) = \frac{\gamma_n Q^{n+1}}{2\zeta(n+1)} + O(Q^n (\ln Q)^{\ell(n)}),
\]

where

\[
\gamma_n = \int_{-\infty}^{+\infty} \phi_n(x) \, dx.
\]

Thus, from (32), we obtain

\[
Q^{n+1}_N = \frac{N}{\gamma_n} + O(N^\frac{\pi}{2\pi} (\ln N)^{\ell(n)}).
\]

Applying (1), (33) and the identity \( \rho_n(x) = \gamma_n^{-1} \phi_n(x) \) proves the corollary. \( \square \)

It is easy to construct an example of such ordering: if \( \alpha \) and \( \beta \) are algebraic numbers of degree \( n \), let \( \alpha \) precede \( \beta \) if and only if \( H(\alpha) < H(\beta) \) or \( H(\alpha) = H(\beta) \) and \( \alpha < \beta \):

\[
\alpha < \beta \iff \begin{cases} H(\alpha) < H(\beta), \\ H(\alpha) = H(\beta), \quad \alpha < \beta. \end{cases}
\]

Note that the ordering imposed on algebraic numbers with identical degrees and heights can be arbitrary: the count of algebraic numbers \( \alpha \) with \( \deg(\alpha) = n \) and \( H(\alpha) = Q \) equals \( O(Q^n) \), whereas for \( H(\alpha) \leq Q \) it is \( O(Q^{n+1}) \).

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