“ON TRANSFORMATION FORMULAE AND CERTAIN TRANSFORMATIONS”

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Abstract: In this paper an attempt has been made to generate certain transformation formulae involving various transform.

Keywords: Bailey’s transform, basic hypergeometric functions, Mellin transforms, Fourier transform, Laplace transform

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1. INTRODUCTION

In 1846, Eduard Heine first considered the Basic Hypergeometric series. The general Basic Hypergeometric Series is defined as

\[ \Phi_A \left[ a_1, a_2, a_3, \ldots, a_A; b_1, b_2, b_3, \ldots, b_B; q, x \right] \]

\[ = \sum_{n=0}^{\infty} \frac{(a_1 q)_n (a_2 q)_n \ldots (a_A q)_n x^n}{(b_1 q)_n (b_2 q)_n \ldots (b_B q)_n (q)_n} \]  

\[ = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_A q)_n x^n}{(q, q, q, \ldots, q)_n} \] \hspace{1cm} (1.1)

In which there are always A of the \(a\) parameters and B of the \(b\) parameters. In such a case, the product of products

\[ (a_1 q)_n (a_2 q)_n \ldots (a_A q)_n \]

Can be shortened still further to \((a, q)_n\) \hspace{1cm} (1.2)

Where it is understood that there are always A of the \(a\) parameters.

In 1944, Bailey’s W.N. introduced a very interesting and simple but it’s very valuable identity
If
\[
\beta_n = \sum_{r=0}^{n} u_{n-r} v_{n+r} w_r, \quad (1.3)
\]
\[
y_n = \sum_{r=n}^{\infty} u_{r-n} v_{r+n} w_r, \quad (1.4)
\]

where \( w_r, \delta_r, u_r, \) and \( v_r \) are any functions of \( r \) only such that the series \( y_n \) exists.
\[
\sum_{n=0}^{\infty} a_n y_n = \sum_{n=0}^{\infty} \beta_n \delta_n \quad (1.5)
\]

The Fourier transform is an expansion of the Fourier series resulting in the lengthening of the span of the represented feature and allowing it to reach infinity. For any \( f(x) \) function, which is usually real-valued in astronomy, but \( f(x) \) can be complex. The Fourier transformation can be denoted as \( F(s) \), where \( x \) and \( s \) product in dimensionless. The Fourier transform is defined as

\[
F(s) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} \, dx. \quad (1.6)
\]

The transformation of Mellin is an integral transformation that can be viewed as a transformation of multiplicative Laplace. Such integral transformation is strongly connected to the Dirichlet series theory but is widely used in numerical analysis, mathematical statistics and asymptotic theory of expansion. The transformation of Mellin is the integral transform defined as

\[
\Phi(x) = \int_{0}^{\infty} z^{x-1} f(z) \, dz \quad (1.7)
\]
\[
f(z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-s} \Phi(x) \, dx \quad (1.8)
\]

It is introduced as as Mellin transform in the wolfram language. The \( \Phi(x) \) transform is exists If the integral

\[
\int_{0}^{\infty} f(y) |y|^{k-1} \, dy \quad (1.9)
\]

Is bounded to some \( k > 0 \), in which case the inverse \( f(z) \) with \( c > k \) exists.

Let \( F(t) \) is a real valued function defined over the interval \((0, \infty)\) such that \( F(t)=0 \). The Laplace transform of \( F(t) \), denoted by \( L\{F(t)\} \), is defined as

\[
L\{F(t)\} = \int_{0}^{\infty} e^{-st} F(t) \, dt \quad (1.10)
\]

Also write as

\[
L\{F(t)\} = f(s) = \int_{0}^{\infty} e^{-st} F(t) \, dt \quad (1.11)
\]

Here \( L \) is called Laplace transformation operator. The parameter \( s \) is a real or complex number. In general, the parameter \( s \) is taken to be a real positive number. We use symbol \( p \) for parameter \( s \).

Therefore \( L\{F(t)\} = f(p) = \int_{0}^{\infty} e^{-pt} F(t) \, dt \quad (1.12)\)

The Laplace transform is said to exist if the integral \((1)\) is convergent for some value of \( s \). The operation of multiplying \( F(t) \) by \( e^{-st} \) and integrating from \( 0 \) to \( \infty \) is called Laplace transformation.

2. NOTATION AND DEFINITION

Some useful Notations and results are:

\[
(a_1, q)_n = (a_2, q)_n \ldots (a_n, q)_n = (a_1, a_2, a_3, \ldots, a_n, q)_n \quad (2.1)
\]
\[
(a; q)_n = \prod_{j=0}^{n-1} (1 - a q^j) = (1 - a)(1 - a q)(1 - a q^2) \ldots \ldots (1 - a q^{n-1}) \quad (2.2)
\]

is the q-shifted factorial.
\[
[a; q]_0 = 1 \quad (2.3)
\]
\[
[a; q]_n = \prod_{j=0}^{n-1} (1 - a q^j) \quad (2.4)
\]
\[
[a_1, a_2, a_3, \ldots, a_r; q]_\infty = (a_1; q)_\infty (a_2; q)_\infty \ldots \ldots (a_r; q)_\infty \quad (2.5)
\]
The Mellin transform of \( f(x) \) is defined as
\[
M[f(x)] = \tilde{f}(p) = \int_0^\infty x^{p-1} f(x) \, dx \quad \text{Over the interval } (0, \infty). \quad (2.6)
\]
The finite Fourier sine transform of \( F(x) \) is defined as,
\[
F_s\{F(x)\} = f_s(s) = \int_0^l F(x) \sin \frac{\pi n x}{l} \, dx, \quad \text{over the interval } 0 < x < l \text{ Where } s \text{ is a positive integer. (2.7)}
\]
The function \( F(x) \) is known as inverse finite Fourier sine transform of \( f_s(s) \) and is defined by
\[
F^{-1}_s\{f_s(s)\} = F(x) = \frac{1}{l} \sum_{n=1}^{\infty} f_s(s) \sin \frac{\pi n x}{l} \quad (2.8)
\]
And Fourier sine series is also written as in this form
\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{\pi n x}{l} \quad (2.9)
\]
The finite Fourier cosine transform of \( F(x) \) is defined as,
\[
F_c\{F(x)\} = f_c(s) = \int_0^l F(x) \cos \frac{\pi n x}{l} \, dx, \quad \text{over the interval } 0 < x < l. \text{ Where } s \text{ is a positive or zero integer. (2.10)}
\]
The function \( F(x) \) is known as inverse finite Fourier sine transform of \( f_c(s) \) and is defined by
\[
F^{-1}_c\{f_c(s)\} = F(x) = \frac{1}{l} \sum_{n=1}^{\infty} f_c(s) \cos \frac{\pi n x}{l} \quad (2.11)
\]
And Fourier cosine series is also written as in this form
\[
F(x) = \sum_{n=0}^{\infty} b_n \cos \frac{\pi n x}{l} \quad (2.12)
\]
Let \( L\{F(t)\} = f(s) \) and \( c_1, c_2, c, a \) be any constants
\[
f(s) = \int_0^{\infty} e^{-st} F(t) \, dt \quad (2.13)
\]
\[
L\{F(at)\} = \frac{1}{a} f \left( \frac{s}{a} \right) \quad (2.14)
\]
\[
L\{e^{at} F(t)\} = f(s - a) \quad (2.15)
\]
The Gamma function. If \( n > 0 \), we define the Gamma function by
\[
\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} \, dx \quad (2.16)
\]
\[
\Gamma(n + 1) = n \Gamma(n) \quad \text{If } n > 0 \quad (2.17)
\]
\[
\Gamma(n) = n! \text{ if } n \text{ is a positive integer} \quad (2.18)
\]
\[
\Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} \quad (2.19)
\]
\[
\Gamma(n) \Gamma(1 - n) = \frac{\pi}{\sin \pi n}, \quad 0 < n < 1 \quad (2.20)
\]

### 3. MAIN RESULTS

\[
\sum_{n=0}^{\infty} \left[ \frac{\Gamma(n) \sin \frac{\pi n}{2}}{\left( \sum_{r=0}^{\infty} \Gamma(r + n) \cos \frac{(r+n)\pi}{2} \right)} \right] = \sum_{n=0}^{\infty} \left[ \frac{\Gamma(n) \sin \frac{\pi n}{2}}{\left( \sum_{r=0}^{\infty} \Gamma(r) \sin \frac{r\pi}{2} \right)} \Gamma(n) \cos \frac{\pi n}{2} \right] \quad (3.1)
\]
\[
\sum_{n=0}^{\infty} \left( F_s^{-1}\{f_s(n)\} \right) \left( \sum_{r=0}^{\infty} F_c^{-1}\{f_c(r + n)\} \right) \right) = \sum_{n=0}^{\infty} \left( \sum_{r=0}^{\infty} F_s^{-1}\{f_s(r)\} \right) \left( F_c^{-1}\{f_c(n)\} \right) \quad (3.2)
\]
\[
\sum_{n=0}^{\infty} \left[ \left( \frac{1}{a} f_s \left( \frac{n}{a} \right) \right) \left( \frac{1}{a} \sum_{n=0}^{\infty} f_c \left( \frac{r+n}{a} \right) \right) \right] = \sum_{n=0}^{\infty} \left[ \left( \frac{1}{a} \sum_{r=0}^{\infty} f_s \left( \frac{r}{a} \right) \right) \left( \frac{1}{a} f_c \left( \frac{n}{a} \right) \right) \right] \quad (3.3)
\]
4. PROOF OF MAIN RESULTS

Now, choose \( u_r = v_r = 1 \) in the equation (1.3) and (1.4), then we have

\[
\beta_n = \sum_{r=0}^{n} \alpha_r \tag{4.1}
\]

\[
\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} \tag{4.2}
\]

Then under suitable convergence conditions

\[
\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{4.3}
\]

(I) let us assume,

\[
\alpha_r = \int_0^\infty x^{r-1} (\sin x) \, dx = \Gamma(r) \sin \frac{rn\pi}{2} \tag{4.4}
\]

And

\[
\delta_r = \int_0^\infty x^{r-1} (\cos x) \, dx = \Gamma(r) \cos \frac{rn\pi}{2} \tag{4.5}
\]

Now, put value of \( \alpha_r \) and \( \delta_r \) in (4.1) & (4.2), we get

\[
\beta_n = \sum_{r=0}^{n} \int_0^\infty x^{r-1} (\sin x) \, dx = \sum_{r=0}^{n} \Gamma(r) \sin \frac{rn\pi}{2} \tag{4.6}
\]

\[
\gamma_n = \sum_{r=0}^{\infty} \int_0^\infty x^{r+n-1} (\cos x) \, dx = \sum_{r=0}^{\infty} \Gamma(r + n) \cos \frac{(r+n)n\pi}{2} \tag{4.7}
\]

Substitution the value of \( \alpha_r, \delta_r, \beta_n, \gamma_n \) in (4.3), then we have

\[
\sum_{n=0}^{\infty} \left[ \Gamma(n) \sin \frac{nn\pi}{2} \left( \sum_{r=0}^{n} \Gamma(r+n) \cos \frac{(r+n)n\pi}{2} \right) \right] = \sum_{n=0}^{\infty} \left[ \left( \sum_{r=0}^{n} \Gamma(r) \sin \frac{rn\pi}{2} \right) \Gamma(n) \cos \frac{nn\pi}{2} \right] \tag{4.8}
\]

(II) Let us assume that,

\[
\alpha_r = \frac{2}{l} \sum_{r=1}^{\infty} f_s(r) \sin \frac{rn\pi}{l} = F_s^{-1} \{ f_s(r) \} \tag{4.9}
\]

And

\[
\delta_r = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{r=1}^{\infty} f_c(r) \cos \frac{rn\pi}{l} = F_c^{-1} \{ f_c(r) \} \tag{4.10}
\]

Now, put value of \( \alpha_r \) and \( \delta_r \) in (4.1) & (4.2), we get

\[
\beta_n = \sum_{r=0}^{n} \left[ \frac{2}{l} \sum_{r=1}^{\infty} f_s(r) \sin \frac{rn\pi}{l} \right] = \sum_{r=0}^{n} F_s^{-1} \{ f_s(r) \} \tag{4.11}
\]

\[
\gamma_n = \sum_{r=0}^{\infty} \left[ \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{r=1}^{\infty} f_c(r+n) \cos \frac{(r+n)n\pi}{l} \right] = \sum_{r=0}^{\infty} F_c^{-1} \{ f_c(r+n) \} \tag{4.12}
\]

Substitution the value of \( \alpha_r, \delta_r, \beta_n, \gamma_n \) in (4.3), then we have

\[
\sum_{n=0}^{\infty} (F_s^{-1} \{ f_s(n) \}) (\sum_{r=0}^{\infty} F_c^{-1} \{ f_c(r+n) \}) = \sum_{n=0}^{\infty} ((\sum_{r=0}^{n} F_s^{-1} \{ f_s(r) \}) (F_c^{-1} \{ f_c(n) \})) \tag{4.13}
\]

(III) Let us assume that,

\[
\alpha_r = \int_0^\infty e^{-rx} \sin ax \, dx = \frac{1}{a} f_s \left( \frac{r}{a} \right) \tag{4.14}
\]

And

\[
\delta_r = \int_0^\infty e^{-rx} \cos ax \, dx = \frac{1}{a} f_c \left( \frac{r}{a} \right) \tag{4.15}
\]

Now, put value of \( \alpha_r \) and \( \delta_r \) in (4.1) & (4.2), we get

\[
\beta_n = \sum_{r=0}^{n} \int_0^\infty e^{-rx} \sin ax \, dx = \frac{1}{a} \sum_{r=0}^{n} f_s \left( \frac{r}{a} \right) \tag{4.16}
\]
\[
\gamma_n = \sum_{r=0}^{\infty} \int_{0}^{\infty} e^{-\left(r+n\right)} \cos \alpha x \, dx = \frac{1}{\alpha} \sum_{r=0}^{\infty} f_c \left(\frac{r+n}{\alpha}\right) 
\]  (4.17)

Substitution the value of \( \alpha, \delta, \beta, \gamma \) in (4.3), then we have
\[
\sum_{n=0}^{\infty} \left[ \left( \frac{1}{\alpha} f_s \left( \frac{n}{\alpha} \right) \right) \left( \frac{1}{\alpha} \sum_{r=0}^{\infty} f_c \left(\frac{r+n}{\alpha}\right) \right) \right] = \sum_{n=0}^{\infty} \left[ \left( \frac{1}{\alpha} \sum_{r=0}^{n} f_s \left(\frac{r}{\alpha}\right) \right) \left( \frac{1}{\alpha} f_c \left(\frac{n}{\alpha}\right) \right) \right] 
\]  (4.18)

5. APPLICATIONS OF CERTAIN TRANSFORMATIONS

5.1) Fourier Transform
A Fourier arrangement may be a particular sort of unbounded scientific arrangement including trigonometric functions. Fourier arrangements are utilized in connected arithmetic and particularly in material science and hardware to specific intermittent capacities such as those that include communications signal waveforms. Fourier series are applied within the determination of fractional differential conditions, which shows up in numerous mechanical building issues such as Heat dissemination, wave Engendering and liquid mechanics issues. Too, the Fourier transform, which is exceptionally related to the Fourier series, is utilized within the range investigation of signals. Simple waves will break down a complicated signal. This breakdown is the Fourier transform, and how much of each wave is required. Fourier transform obtain a signal and transmit it in terms of the wave frequencies that make up the signal. The most advantage of Fourier investigation is that exceptionally small data is misplaced from the signal amid the change. The Fourier transform keeps up data on adequacy, sounds and employments all parts of the waveform to interpret the signal into the recurrence space.

5.2) Laplace Transform
The Laplace transform can moreover be utilized to illuminate differential equations and is utilized broadly in mechanical engineering and electrical designing. The Laplace transform decreases a straight differential condition to a logarithmic condition, which can at that point be illuminated by the formal rules of the algebra. Laplace Transform is intensely utilized in signal preparing. Utilizing Laplace or Fourier transform, you will consider a signal within the recurrence space. This will be a capable tool. The first primary utilize for Laplace transforms was to illuminate introductory esteem issues for direct standard and partial differential equations. Laplace transform could be an effective instrument for investigation and plan of ceaseless time signals and frameworks. The Laplace transform contrasts from Fourier transform since it covers a broader lesson of CT signals and frameworks which may or may not be steady frameworks. Laplace transform has a few applications in nearly all building disciplines. Laplace transform is utilized to rearrange calculations in framework modeling. Where large differential conditions are utilized. In electrical circuits, a Laplace transform is utilized for the investigation of straight time invariant frameworks. The Laplace transform is especially valuable in understanding straight standard differential conditions such as those emerging within the investigation of electronic circuits, control framework etc. Laplace change is intensely utilized in signal handling. Utilizing Laplace or Fourier transform, ready to study a signal within the recurrence space. Laplace transform may be a subset of the Fourier transform which is utilized within the handling of information signals amid their transmission. Additionally, the concept of sifting signal/data is based on a recurrence space elucidation. That’s catching and cleaning of mistakes created amid transmission of computer information.

5.3) Mellin transform
The Mellin change is broadly utilized in computer science for the examination of calculations since of its scale invariance property. The Mellin transform is utilized in investigation of the prime-counting work and happens in discourses of the Riemann Zeta work. The Mellin transform can be utilized in sound timescale-pitch modification.
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